SU(2) Yang-Mills Theory in Savvidy Background at Finite Temperature and Chemical Potential

R.Parthasarathy\textsuperscript{1} and Alok Kumar\textsuperscript{2}

The Institute of Mathematical Sciences
C.P.T.Road, Taramani Post
Chennai 600113, India.

Abstract

The one-loop effective energy density of a pure SU(2) Yang-Mills theory in the Savvidy background, at finite temperature and chemical potential is examined with emphasis on the unstable modes. After identifying the stable and unstable modes, the stable modes are treated in the quadratic approximation. For the unstable modes, the full expansion including the cubic and the quartic terms in the fluctuations is used. The functional integrals for the unstable modes are evaluated and added to the results for the stable modes. The resulting energy density is found to be real, coinciding with the real part of the energy density in the quadratic approximation of earlier study. There is now no imaginary part. Numerical results are presented for the variation of the energy density with temperature for various choices of the chemical potential.

Keywords: Savvidy vacuum; SU(2) Yang-Mills; background chromomagnetic; finite temperature; chemical potential; unstable modes; quartic terms in fluctuations.

\textsuperscript{1}e-mail:sarathy@imsc.res.in
\textsuperscript{2}e-mail:alok@imsc.res.in

1
I.INTRODUCTION

The ground state of $SU(2)$ Yang-Mills theory in a covariantly constant chromomagnetic field in the third color direction ($F^3_{12} = H$) had been studied by Savvidy [1]. The important result obtained was the one-loop effective energy density was lower than the perturbative vacuum and the theory has a gluon condensate, $<|F^a_{\mu\nu}F^{\mu\nu a}|> \neq 0$. However, Nielsen and Olesen [2] pointed out that the one-loop effective energy density had an imaginary part due to the lowest Landau level. Various studies were devoted to this aspect [3]. In recent times, gauge invariance arguments to exclude the imaginary part [4] and a stable vacuum by dynamically generating mass term for the off-diagonal gluons [5] have been proposed. In all these studies, only the terms quadratic in the fluctuations are kept, throwing out the cubic and quartic terms in the fluctuations. As early as 1983, the Savvidy vacuum within the framework of the background field method was examined by Flory [6] and later by Kay [7] treating the unstable modes including the cubic and quartic terms and the possibility of obtaining real effective energy density was pointed out. The present authors with Kay [8] have explicitly shown at zero temperature, the resulting energy density, when the cubic and the quartic terms in the fluctuations are included for the unstable modes, has no imaginary part and coincided with the real part of the earlier calculations. Briefly, the stable modes are treated in the quadratic approximation and for the unstable modes, the functional integral is evaluated keeping the cubic and quartic terms along with the quadratic term. This result (Eqn.30 of [8]) has no imaginary part and when added to the result of the stable modes, gives the effective energy density which is real. The calculations leading to this result are gauge invariant [8].

The Savvidy vacuum at finite temperature has been studied by many authors [9 - 15]. In all these studies, the quadratic approximation has been used. As a result, the effective energy density involved an imaginary part, dependent on the temperature. This is serious as it persists at high temperatures. It has been observed by Meisinger and Ogilvie [15] that with the introduction of a non-trivial Polyakov loop, specified by $\phi$, it is possible to stabilize the vacuum, if $\beta\sqrt{gH} < \phi < 2\pi - \beta\sqrt{gH}$, where $\beta = 1/kT$, $H$ is the chromomagnetic background in the third color direction, as for this range of $\phi$, the imaginary part becomes zero. However, the imaginary part is non-zero
at the global minimum. So, in the understanding of the finite temperature behaviour, the imaginary part severely inhibits the progress.

It is the purpose of this work to extend our method \[8\] at zero temperature, to finite temperature with chemical potential, by separating the unstable modes and including the cubic and quartic terms in these modes in the evaluation of the functional integral and adding to the contribution of the stable modes in the quadratic approximation. The motivation is to get real effective energy density. Another purpose is to resolve a discrepancy in the analytical expressions for the one-loop energy density between \[14\] and \[15\]. The discrepancy is an interchange of \(J_1\) and \(Y_1\) and also a relative sign between the two \(K_1\) functions in the energy density. Our results show that the finite part of the effective energy density is real. There is no imaginary part. The real energy density coincides with the real part of \[15\] which therefore resolves the discrepancy alluded above in favour of \[15\].

The next section develops the formalism for the evaluation of the effective energy density. The stable and the unstable modes are separately treated. Section.III provides the details of handling the unstable modes including the cubic and quartic terms in the expansion. The results are added to the contribution of the stable modes and the full expression for the effective energy density is exhibited. Section IV contains the details of the numerical evaluation of the effective energy density for various temperatures.

II. EFFECTIVE ENERGY DENSITY IN THE BACKGROUND FIELD METHOD (BFM)

The Euclidean functional integral for an \(SU(2)\) pure YM theory is

\[
Z = \int [dA^a_\mu] e^S, \tag{1}
\]

where

\[
S = \int d^4x \{-\frac{1}{4} F^{a\mu\nu} F_{\mu\nu}^{a}\}, \tag{2}
\]

and

\[
F^{a}_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g \epsilon^{abc} A^b_\mu A^c_\nu. \tag{3}
\]
Expanding $A^a_\mu = \bar{A}^a_\mu + a^a_\mu$ with $\bar{A}^a_\mu$ as the classical background field satisfying the equation of motion $\bar{D}^{ab}_\mu \bar{F}^b_{\mu\nu} = 0$, with $\bar{D}^{ab}_\mu = \bar{\partial}_\mu \delta^{ab} + g e^{abc} \bar{A}^c_\mu$ as the background covariant derivative, $\bar{F}^a_{\mu\nu}$ is same as (3) with $\bar{A}^a_\mu$ and using the background gauge

$$\bar{D}^{ab}_\mu a^b_\mu = 0; \quad (4)$$

we have

$$Z = \int [da^a_\mu] e^{S'}, \quad (5)$$

with

$$S' = \int d^4x \left( -\frac{1}{4} \bar{F}^a_{\mu\nu} F^a_{\mu\nu} + \frac{1}{2} a^a_\mu \Theta^{ac} a^c_\nu + g e^{acd} (D^{ae}_\nu a^e_\mu) a^c_\mu a^d_\nu ight. \right.$$

$$\left. - \frac{g^2}{4} \{(a^a_\mu a^a_\mu)^2 - a^a_\mu a^a_\nu a^a_\mu a^\nu\} - \log \det (-\bar{D}^{ab}_\mu \bar{D}_\mu^{bc}) \right), \quad (6)$$

where

$$\Theta^{ac}_{\mu\nu} = (\bar{D}^{ab}_\lambda \bar{D}_\lambda^{bc}) \delta_{\mu\nu} + 2 g e^{ace} \bar{F}^e_{\mu\nu}. \quad (7)$$

In arriving at (5) and (6), we have introduced the gauge fixing and the ghost lagrangian for the background gauge (4) and integrated the ghost fields, resulting in the last term in (6). The expansion in (6) is exact. If the cubic and the quadric terms in $a^a_\mu$ are neglected in (6), then the effective potential will be given by

$$\Gamma^{1-loop} = \frac{1}{2} Tr \log det(-\Theta^{ac}_{\mu\nu}) - Tr \log det(-\bar{D}^{ab}_\mu \bar{D}_\mu^{bc}). \quad (8)$$

The Savvidy background is chromomagnetic in the third color direction and is given by

$$\bar{A}^a_0 = 0; \quad \bar{A}^a_i = \delta^{a3} \left( -\frac{Hy}{2}, \frac{Hx}{2}, 0 \right), \quad (9)$$

which gives $\bar{F}^a_{12} = H$ and solves the classical equation of motion, $\bar{D}^{ab}_\mu \bar{F}^b_{\mu\nu} = 0$. We introduce the chemical potential ($\mu$) as a background field, following Loewe, Mendizabel and Rojas [16], namely

$$B^a_\mu = \frac{\mu}{g} v_\mu \delta^{a3}, \quad v_\mu = (1, 0, 0, 0). \quad (10)$$
This method, introduced in [16], has the advantage that as chemical potentials are not introduced as Lagrange multipliers in BFM, it is not necessary to compute the conserved charges. In [16], only the background (10) was considered. In our case we combine (9) and (10) as

\[ \bar{A}_\mu^a = \delta^a_3 \left( \frac{\mu}{g} - \frac{H y}{2}, \frac{H x}{2}, 0 \right), \] (11)

which gives again \( \bar{F}^{3}_{12} = H \) as the non-vanishing background field strength and which solves the classical equation of motion. In this method of introducing the chemical potential (10,11), it will play the role as the Polyakov loop specified by a constant \( \bar{A}_3^3 \) field in the third color direction as in [15] with the identification of \( \phi = \beta \mu \). As in [8] (eqn.10), we notice that (7) gives

\[ \Theta^{ac}_{44} = \Theta^{ac}_{33} = \bar{D}^{ab}_\lambda \bar{D}^{bc}_\lambda, \] (12)

where now

\[ \bar{D}^{ab}_\lambda = \partial_\lambda \delta^{ab} + g \epsilon^{abc} \bar{A}^c_\lambda + \mu \epsilon^{abc} v_\lambda, \] (13)

where \( \bar{A}^3_\lambda \) is given by (9) and \( v_\lambda \) in (10). The relation (12) gives the important result that the contributions of \( \Theta^{ac}_{44} \) and \( \Theta^{ac}_{33} \) to \( \Gamma^{1-loop} \) in (8), cancel the ghost contribution in (8). Using (11) in (7), it is easy to see that

\[ \Theta^{ac}_{41} = \Theta^{ac}_{42} = \Theta^{ac}_{31} = \Theta^{ac}_{32} = 0. \] (14)

The remaining eigenvalues are from \( \Theta^{ac}_{ij} \) for \( i, j = 1, 2 \) only. The eigenmodes and the corresponding eigenvalues are:

\[
\begin{align*}
(a_1^1 + ia_1^2) + i(a_2^1 + ia_2^2) & : (k_4 + \mu)^2 + k_3^2 + (2N + 1)gH - 2gH, \\
(a_1^1 + ia_1^2) - i(a_2^1 + ia_2^2) & : (k_4 + \mu)^2 + k_3^2 + (2N + 1)gH + 2gH, \\
(a_1^1 - ia_1^2) + i(a_2^1 - ia_2^2) & : (k_4 - \mu)^2 + k_3^2 + (2N + 1)gH + 2gH, \\
(a_1^1 - ia_1^2) - i(a_2^1 - ia_2^2) & : (k_4 - \mu)^2 + k_3^2 + (2N + 1)gH - 2gH, \\
(a_3^i) & : k_4^2 + k_3^2 + k_2^2 + k_1^2; \ i = 1, 2,
\end{align*}
\] (15)

where \( N = 0, 1, 2, \cdots \) is the harmonic oscillator (in the \( x - y \) plane) quantum number.
We now pass on to the finite temperature case by replacing $k_4$ by $\frac{2\pi n}{\beta}$ and $\int_{-\infty}^{\infty} dk_4$ by $\sum_{n=-\infty}^{\infty}$. With this, the effective potential in the "quadratic approximation" becomes

$$\Gamma^{1-\text{loop}} = 2 \times \frac{1}{2} \times \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3} \log\left(\left(\frac{4\pi^2 n^2}{\beta^2} + \vec{k}^2\right)/\Lambda^2\right)$$

$$+ \frac{1}{2} \times \frac{1}{\beta} \left(\frac{gH}{2\pi}\right) \sum_{n=-\infty}^{\infty} \sum_{N=0}^{\infty} \int_{-\infty}^{\infty} dk_3$$

$$\left(\log\frac{1}{\Lambda^2}\{(\frac{2\pi n}{\beta} + \mu)^2 + k_3^2 + (2N + 1)gH - 2gH\}\right)$$

$$+ \log\frac{1}{\Lambda^2}\{(\frac{2\pi n}{\beta} - \mu)^2 + k_3^2 + (2N + 1)gH + 2gH\}$$

$$+ \log\frac{1}{\Lambda^2}\{(\frac{2\pi n}{\beta} - \mu)^2 + k_3^2 + (2N + 1)gH - 2gH\})$$

where the first term corresponds to the last eigenvalue in (15) and the remaining correspond respectively to the first four eigenvalues in (15). The pre-factor $\frac{gH}{2\pi}$ is the harmonic oscillator degeneracy and the pre-factor 2 in the first term accounts for the two modes in the last term in (15) for $i = 1, 2$. $\Lambda$ is the dimensionful parameter to render the argument of the logarithms dimensionless. We suppress this hereafter.

This expression (16) agrees with Eqn.8 of [15] with their $\phi$ replaced by $\mu \beta$ and with Eqn.(2.16) of Ninomiya and Sakai [12] with $\mu = 0$. It can be seen from (15), that for $k_3 = 0$ and $n = 0; N = 0$, the eigenvalues first and the fourth become $\mu^2 - gH$ and therefore to avoid negative eigenvalues $\mu > \sqrt{gH}$. However, for $n = 1$, the fourth eigenvalue becomes $(\frac{2\pi}{\beta} - \mu)^2 - gH$ (with N=0 and $k_3 = 0$) and if this is to remain positive, then $\mu < \frac{2\pi}{\beta} - \sqrt{gH}$. With this, the first eigenvalue remains positive. So for $\sqrt{gH} < \mu < \frac{2\pi}{\beta} - \sqrt{gH}$, it is possible to avoid the negative eigenvalues and hence the instability. However, the instability enters at the global minimum [15]. We will return to this in the next section. Now, we proceed to evaluate (16). In order to exhibit the contribution from the unstable modes, we consider the second logarithm.

6
(without its prefactors) in (16) and write that as [8]

\[ L_2 = - \sum_{n=-\infty}^{\infty} \sum_{N=0}^{\infty} \int_{-\infty}^{\infty} \frac{dk_3}{2\pi} \int_0^\infty dt \ t^{-1} e^{-t\left(\frac{2\mu i}{\beta} + \mu^2 + k_3^2 + A\right)}, \quad (17) \]

where \( A = 2NgH - gH \). After performing the \( k_3 \) integration and the sum over \( n \), we find

\[ L_2 = - \frac{1}{2\sqrt{\pi}} \sum_{N=0}^{\infty} \int_0^\infty dt \ t^{-\frac{1}{2}} \ \theta_3\left(\frac{2\mu i}{\beta}, \frac{4\pi i t}{\beta^2}\right) e^{-t(\mu^2 + A)}, \]

where

\[ \theta_3(z, \tau) = \sum_{n=-\infty}^{\infty} e^{i\pi \tau n^2} e^{2\pi i nz}. \]

Now, using the property of the \( \theta_3 \)-function [18],

\[ \theta_3(z, i\tau) = \tau^{-\frac{1}{2}} e^{-\pi z^2/\tau} \ \theta_3\left(\frac{z}{i\tau}, \frac{i}{\tau}\right), \]

we get

\[ L_2 = - \frac{\beta}{4\pi} \sum_{N=0}^{\infty} \int_0^\infty dt \ t^{-\frac{1}{2}} \ \theta_3\left(\frac{\mu i}{2\pi}, \frac{i\beta^2}{4\pi t}\right) e^{-tA}. \]

Including the prefactor in (16), the contribution to \( \Gamma^{1-loop} \) from \( L_2 \) is

\[ \Gamma^{1-loop}(L_2) = -\frac{gH}{16\pi^2} \sum_{N=0}^{\infty} \int_0^\infty dt \ t^{-2} \ \theta_3\left(\frac{\mu i}{2\pi}, \frac{i\beta^2}{4\pi t}\right) e^{-tA}. \]

The unstable mode corresponds to \( N = 0 \). Splitting the above sum over \( N \) as for \( N = 0 \) and \( N = 1, 2, \cdots, \infty \), we find

\[ \Gamma^{1-loop}(L_2) = -\frac{gH}{16\pi^2} \int_0^\infty dt \ t^{-2} \ \theta_3\left(\frac{\mu i}{2\pi}, \frac{i\beta^2}{4\pi t}\right) \left(e^{tgH} + \frac{e^{-tgH}}{1 - e^{-2gH}}\right), \quad (18) \]

where the first term in \( \left( \cdots \right) \) is the contribution from the unstable mode.

The third term in (16) has no negative eigenvalue and writing this as in (17), the sum over \( N = 0, 1, 2, \cdots, \infty \) is performed to give

\[ \Gamma^{1-loop}(L_3) = -\frac{gH}{16\pi^2} \int_0^\infty dt \ t^{-2} \ \theta_3\left(\frac{\mu i}{2\pi}, \frac{i\beta^2}{4\pi t}\right) \frac{e^{-3tgH}}{1 - e^{-2gH}}. \quad (19) \]
The fourth term in (16) is the same as the third term except for $\mu$ replaced by $-\mu$. So also the fifth term and the second term are same but for this replacement. As $\theta_3(z, \tau) = \theta_3(-z, \tau)$ [18], we have

$$
\sum_{j=2}^{5} \Gamma_{\text{1-loop}}(L_j) = -\frac{gH}{8\pi^2} \int_{0}^{\infty} dt \ t^{-2} \ \theta_3\left(\frac{\mu\beta}{2\pi}, \frac{i\beta^2}{4\pi t}\right) \left(t^{-gH} + \frac{e^{-t\gamma H} + e^{-3t\gamma H}}{1 - e^{-2t\gamma H}}\right),
$$

$$
= -\frac{gH}{8\pi^2} \int_{0}^{\infty} dt \ t^{-2} \ \theta_3\left(\frac{\mu\beta}{2\pi}, \frac{i\beta^2}{4\pi t}\right) \left(e^{t\gamma H} - e^{-t\gamma H} + \frac{2e^{-t\gamma H}}{1 - e^{-2t\gamma H}}\right),
$$

(20)

Now $\theta_3\left(\frac{\mu\beta}{2\pi}, \frac{i\beta^2}{4\pi t}\right)$ can be rewritten using

$$
\theta_3(z, \tau) = \sum_{\ell=-\infty}^{\infty} e^{i\pi \tau \ell^2} \ e^{2\pi i \tau \ell},
$$
as

$$
\theta_3\left(\frac{\mu\beta}{2\pi}, \frac{i\beta^2}{4\pi t}\right) = 1 + 2 \sum_{\ell=1}^{\infty} \cos(\mu\beta\ell) \ e^{-\frac{\beta^2 \ell^2}{4t}}.
$$
The exponentials in (20) can be written as $\frac{\cosh(2t\gamma H)}{\sinh(t\gamma H)}$ and so (20) becomes

$$
-\frac{gH}{8\pi^2} \int_{0}^{\infty} dt \ t^{-2} \ \frac{\cosh(2t\gamma H)}{\sinh(t\gamma H)} \left(1 + 2 \sum_{\ell=1}^{\infty} \cos(\mu\beta\ell) \ e^{-\frac{\beta^2 \ell^2}{4t}}\right).
$$

(21)

The first term in (16) is the familiar term and its finite part is $-\frac{\pi^2}{45\beta^4}$. Thus the one-loop potential is

$$
\Gamma_{\text{1-loop}} = -\frac{\pi^2}{45\beta^4}
$$
$$
- \frac{(gH)^2}{8\pi^2} \int_{0}^{\infty} d\tau \ \tau^{-2} \ \frac{\cosh(2\tau)}{\sinh(\tau)} \left(1 + 2 \sum_{\ell=1}^{\infty} \cos(\mu\beta\ell) e^{-\frac{\beta^2 \ell^2}{4\tau}}\right),
$$

(22)

where $\tau = gHt$. The zero temperature part is reproduced. In [8], this was evaluated treating the unstable modes carefully including the cubic and the quartic terms and the effective energy density has no imaginary part. It was found to be,

$$
\mathcal{E}_{\text{1-loop}}(T = 0) = \frac{H^2}{2} + \frac{11(gH)^2}{48\pi^2} \left(\log(\frac{gH}{\Lambda^2}) - \frac{1}{2}\right).
$$

(23)
We now proceed to evaluate the finite temperature part of (22),

\[
\Gamma^{1-loop}(T) = -\frac{\pi^2}{45\beta^4} - \frac{(gH)^2}{4\pi^2} \int_0^\infty d\tau \tau^{-2} \frac{\cosh 2\tau}{\sinh^2 \tau} \sum_{\ell=1}^\infty \cos(\mu\beta\ell) e^{-\frac{\beta^2 \ell^2 aH}{4\tau}},
\]

\[
= -\frac{\pi^2}{45\beta^4} - \frac{(gH)^2}{4\pi^2} \int_0^\infty d\tau \tau^{-2} \left( e^\tau - e^{-\tau} + \frac{2e^{-\tau}}{1 - 2e^{-2\tau}} \right) \sum_{\ell=1}^\infty \cos(\mu\beta\ell) e^{-\frac{\beta^2 \ell^2 aH}{4\tau}},
\]

(24)

where we exhibit the unstable mode contribution explicitly by the first term in (\cdots). Expanding

\[
\frac{1}{1 - e^{-2\tau}} = \sum_{n=0}^\infty e^{-2n\tau} = e^{-2\tau} + \sum_{n=1}^\infty e^{-2n\tau},
\]

and introducing

\[
I_1 = \int_0^\infty d\tau \tau^{-2} e^\tau e^{-\frac{\beta^2 \ell^2 aH}{4\tau}}, \quad (25)
\]

\[
I_2 = \int_0^\infty d\tau \tau^{-2} e^{-\tau} e^{-\frac{\beta^2 \ell^2 aH}{4\tau}}, \quad (26)
\]

\[
I_3 = 2\int_0^\infty d\tau \tau^{-2} e^{-(2n+1)\tau} e^{-\frac{\beta^2 \ell^2 aH}{4\tau}}, \quad (27)
\]

the finite temperature part of (24) is written as

\[
\Gamma^{1-loop} = -\frac{\pi^2}{45\beta^4} - \frac{(gH)^2}{4\pi^2} \sum_{\ell=1}^\infty \cos(\mu\beta\ell) \left( I_1 + I_2 + I_3 \right),
\]

(28)

in which \(I_1\) is from the \(N = 0\) unstable mode. The integrals in (27) are to be evaluated.
In $I_1$ (25), we perform a Wick rotation to arrive at

$$I_1 = i^{-1} \int_0^\infty dt \ t^{-2} e^{i\left(t + \frac{\beta^2 e_H}{4}\right)},$$

$$= \frac{2\pi i}{\beta \ell \sqrt{gH}} H_1^{(1)}(\beta \ell \sqrt{gH}),$$

(29)

where $H_1^{(1)}$ is the Hankel function of the first kind [19].

$I_2$ and $I_3$ integrals are evaluated using [19]

$$K_{\nu}(xz) = \frac{z^{\nu}}{2} \int_0^\infty t^{-\nu-1} e^{-\frac{x^2}{2} (t + \frac{x^2}{t})} \ dt,$$

(30)

as

$$I_2 = \frac{4}{\beta \ell \sqrt{gH}} K_1(\beta \ell \sqrt{gH}),$$

(31)

$$I_3 = 8 \sum_{n=1}^\infty \frac{\sqrt{(2n+1)}}{\beta \ell \sqrt{gH}} K_1(\sqrt{(2n+1)\beta \ell \sqrt{gH}}),$$

(32)

so that the finite part of (28) becomes

$$\Gamma^{1-loop}(T) = \frac{\pi^2}{45} \beta^4$$

$$- \frac{(gH)^{\frac{3}{2}}}{\beta \pi^2} \sum_{\ell=0}^\infty \frac{\cos(\mu \beta \ell)}{\ell}$$

$$\times \left( \frac{i \pi}{2} H_1^{(1)}(\beta \ell \sqrt{gH}) + K_1(\beta \ell \sqrt{gH}) \right)$$

$$+ 2 \sum_{n=1}^\infty \sqrt{(2n+1)} K_1(\sqrt{2n+1} \beta \ell \sqrt{gH}).$$

(33)

Using $H_1^{(1)}(\beta \ell \sqrt{gH}) = J_1(\beta \ell \sqrt{gH}) + iY_1(\beta \ell \sqrt{gH})$, we find that our result (33) agrees with Meisinger and Ogilvie [15] who used a different method to evaluate (16). The interchange of $J_1$ and $Y_1$ and a relative sign between the two $K_1$ functions in Starinets, Vshivtsev, and Zhukovskii [14] are incorrect.
From (28) and (33), it is seen that the unstable mode ($N = 0$) contributions are contained in $I_1$ and explicitly

$$\Gamma_{\text{1-loop \ unstable}}(T) = \frac{(gH)^2}{\beta \pi^2} \sum_{\ell=1}^{\infty} \frac{\cos(\mu \beta \ell)}{\ell} \left( -\frac{\pi}{2} Y_1(\beta \ell \sqrt{gH}) + i\frac{\pi}{2} J_1(\beta \ell \sqrt{gH}) \right).$$

(34)

The imaginary part above is reminiscent of the zero temperature situation. In the later case, we [8] have treated the unstable modes including the cubic and quartic terms in (6) and showed then the contribution is real. In view of the important difficulties raising from the imaginary part at finite temperature, we, in the next section, treat the unstable modes at finite temperature including the cubic and the quartic terms in the expansion (6).

III. FINITE TEMPERATURE UNSTABLE MODES-INCLUSION OF CUBIC AND QUARTIC TERMS

There are two unstable modes when the harmonic oscillator quantum number $N = 0$, as seen from (15). They are the first and the fourth line in (15) with $N = 0$. As the expression (16) was obtained in the “quadratic approximation”, we cannot use (16) for the unstable modes when we want to include the cubic and the quartic terms in the unstable modes. So, we take the equation (5) for $Z$ but confine ourselves to the unstable modes only here. Without loss of generality we will take $\mu = 0$, zero chemical potential. The normalized unstable mode eigenfunctions are

$$\phi_{k_3,n}(x) = \sqrt{\frac{gH}{2\pi}} e^{-\frac{gH}{2}(x_1^2+x_2^2)} \frac{1}{\sqrt{L_3}} e^{-i(k_3 x_3 + \frac{2\pi n}{3} x_4)},$$

(35)

where the first exponential is the ground statet wavefunction ($N = 0$) of the harmonic oscillator in the $(x_1 - x_2)$ plane, the second exponential is the box-normalized plane wave in the $x_3$ direction and the $S^1$-harmonics in the $x_4$-direction. The index $n$ is the Matsubaro index and is not the same $n$ in (33) as the later index originated in the expansion of $\frac{1}{1-e^{-\beta \tau}}$ in the expression above (25). The unstable eigenmodes in (15) are then $c(k_3,n)\phi_{k_3,n}(x)$ and $c(k_3,n)^*\phi_{k_3,n}^*(x)$ respectively.
The unstable modes involve Lorentz indices 1 and 2 (as can be seen in (15)) and the $SU(2)$ indices 1 and 2, since the classical background in (11) is in the third isospin direction. Consequently, the cubic term $e^{acd}(\bar{D}_\nu a^c_\mu) a^a_\mu a^d_\nu$ vanishes as in [8]. The terms quartic in $a^a_\mu$ in (5) are simplified to $\frac{3}{2} a^+ a^\perp a_+ a^-$. Then, the full partition function for the unstable modes, from (5) and (6) is

$$Z_{unstable} = \int [da^a_\mu] e^{-\int d^4x \{a_u(k^2_3 + k^2_4 - gH)a_u + \frac{\pi^2}{\beta} a^2_e\}},$$

(36)

where $a_u$ stands for $a^+_u$ and $a^-_u$. From (36), the unstable modes for $k^2_3 + (\frac{2\pi n}{\beta})^2 < gH$ render the quadratic term in $a_u$ in the exponent of (36) divergent. However, the quartic term in $a_u$ in the exponent of (36) provides the necessary and the crucial convergence. Thus the overall integral over $a_u$ will be convergent. Now expanding $a_u$ in terms of the eigenfunctions (35) and carrying out the $d^4x$ integration in the exponent, the quadratic term becomes

$$\{k^2_3 + (\frac{2\pi n}{\beta})^2 - gH\} c^2(k_3, n),$$

(37)

and the quartic term becomes

$$\frac{g^2}{32} \frac{(gH)^2}{4\pi^2} \frac{\pi}{gH} c^4(k_3, n) = \frac{g^4 H}{128\pi} c^4(k_3, n),$$

(38)

where we have taken all the four $\phi_{k_3,n}(x)$ having the same $k_3$, $n$. Then, we obtain

$$Z_{unstable} = \left( \int \prod_{k_3,n} dc(k_3, n) e^{(gH-k^2_3+(\frac{2\pi n}{\beta})^2)c^2-\frac{g^4 H}{128\pi} c^4} \right)^D, \quad (39)$$

where $D$ is the degeneracy factor $D = \frac{gH}{2\pi^2} V$ with $V$ as the spatial volume. Now, introducing $c = \sqrt{gH} c$ and $\hat{k}_3 = \frac{k_3}{\sqrt{gH}}$, (39) is written as

$$Z_{unstable} = \left( \frac{1}{\sqrt{gH}} \int \prod_{k_3,n} d\hat{c} e^{(1-\hat{k}^2_3-(\frac{2\pi n}{\beta})^2)c^2-\frac{g^4 H}{128\pi} c^4} \right)^D. \quad (40)$$

The contribution of the complete unstable modes to the energy density $(-\frac{1}{3\lambda^2} \log Z_{unstable})$ is then

$$-\frac{gH\sqrt{gH}}{2\pi^2\beta} \int dk_3 \sum_n \log J(k_3, n),$$

(41)

12
where
\[ J(k_3, n) = \int_{-\infty}^{\infty} d\hat{c} e^{\{(1-\hat{k}_3^2-(\frac{2\pi n}{\beta \sqrt{gH}})^2)\hat{c}^2-\frac{\beta^2}{128\pi(gH)}\hat{c}^4\}}. \] (42)

In (42), if the quartic term is neglected, then (41) will give (34). We retain the crucial quartic term contribution. The integral over \(d\hat{c}\) is convergent irrespective of the sign of the coefficient of the \(\hat{c}^2\) term. It is evaluated using [19]
\[ \int_{0}^{\infty} e^{-(\beta^2x^4+2\gamma^2x^2)}dx = 2^{-\frac{5}{4}} \left(\frac{\sqrt{\beta}}{\gamma}\right) e^{\frac{\gamma^4}{2\beta^2}} K_{\frac{1}{4}}\left(\frac{\gamma}{2\beta^2}\right), \]
with
\[ \beta^2 = \frac{g^2}{128\pi(gH)}, \]
\[ \gamma^2 = \frac{1}{2} (\hat{k}_3^2 + (\frac{2\pi n}{\beta \sqrt{gH}})^2 - 1), \]

Then (42) becomes
\[ J(k_3, n) = \frac{8\sqrt{\pi gH}}{\sqrt{2}g} \left(\hat{k}_3^2 + (\frac{2\pi n}{\beta \sqrt{gH}})^2 - 1\right)^{\frac{1}{2}} e^{\frac{16\pi gH}{g^2} (\hat{k}_3^2 + (\frac{2\pi n}{\beta \sqrt{gH}})^2 - 1)^2} \]
\[ \times K_{\frac{1}{4}}\left(\frac{16\pi gH}{g^2} (\hat{k}_3^2 + (\frac{2\pi n}{\beta \sqrt{gH}})^2 - 1)^2\right). \] (43)

When \(\hat{k}_3^2 + (\frac{2\pi n}{\beta \sqrt{gH}})^2 < 1\) (that is where the instability arose), the above expression using [20] (since \(n\) starts from 1, the argument of \(K_{\frac{1}{4}}\) will be small)
\[ K_{\nu}(x) \rightarrow \frac{2^{\nu-1}\Gamma(\nu)}{x^\nu}; \quad \text{small} \ x, \]
becomes
\[ J(k_3, n) \approx \frac{8\sqrt{\pi gH}}{\sqrt{2}g} \left(\hat{k}_3^2 + (\frac{2\pi n}{\beta \sqrt{gH}})^2 - 1\right)^{\frac{1}{2}} \]
\[ \times \frac{2^{-\frac{3}{4}}\Gamma(\frac{1}{4})}{\sqrt{g}} \frac{\sqrt{g}}{\{\hat{k}_3^2 + (\frac{2\pi n}{\beta \sqrt{gH}})^2 - 1\}^{\frac{3}{4}}} \frac{\sqrt{g}}{2(\pi gH)^{\frac{3}{4}}}, \]
\[ = \frac{2\sqrt{2}}{\sqrt{g}} (\pi gH)^{\frac{1}{4}} 2^{-\frac{3}{4}}\Gamma(\frac{1}{4}). \] (44)

13
In particular the radical \( \{ \hat{k}_3^2 + (\frac{2\pi n}{\beta \sqrt{gH}})^2 - 1 \}^{\frac{1}{2}} \) gets cancelled. This result is real. The imaginary part coming from the radical gets cancelled by the contribution from \( \hat{K}_4 \). This is made possible by the inclusion of the quartic term.

When (44) is used in (41), the \( \hat{k}_3 \)-integration and the sum over \( n \) (constrained by \( \hat{k}_3^2 + (\frac{2\pi n}{\beta \sqrt{gH}})^2 < 1 \)) produces an uninteresting term which is omitted.

When \( \hat{k}_3^2 + (\frac{2\pi n}{\beta \sqrt{gH}})^2 > 1 \), the expression (43) takes the form, using [20],

\[
K_\nu(x) \rightarrow \sqrt{\frac{\pi}{2x}} e^{-x}; \quad \text{large } x,
\]

as

\[
J(\hat{k}_3, n) = \left( \hat{k}_3^2 + (\frac{2\pi n}{\beta \sqrt{gH}})^2 - 1 \right)^{-\frac{1}{2}}.
\]

(45)

Then (41) is evaluated as:

\[
\int d\hat{k}_3 \sum_n \log J(\hat{k}_3, n) = -\frac{1}{2} \int_1^\infty \sum_n \log \{ \hat{k}_3^2 + (\frac{2\pi n}{\beta \sqrt{gH}})^2 - 1 \},
\]

\[
= -\frac{1}{2} \int_1^\infty d\hat{k}_3 \log \{ \cosh(\beta \sqrt{(\hat{k}_3^2 - 1) gH}) - 1 \},
\]

\[
= -\frac{1}{2} \int_1^\infty \hat{k}_3 \left( \beta \sqrt{(\hat{k}_3^2 - 1) gH} + 2 \log(1 - e^{-\beta \sqrt{(\hat{k}_3^2 - 1) gH}}) \right),
\]

\[
= -\frac{1}{2} \beta \int_1^\infty d\hat{k}_3 \sqrt{(\hat{k}_3^2 - 1) gH} - \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n} Y_1(n \beta \sqrt{gH}) + I_1,
\]

(46)

where

\[
I_1 = -\sqrt{gH} \sum_{n=1}^{\infty} \int_0^{\frac{\pi}{2}} \cos(n \beta \sqrt{gH} \cos \theta) \cos \theta \, d\theta.
\]

(47)

The first term in (46), when used in (41), produces a \( \beta \)-independent contribution and neglected for finite temperature effects. The integral \( I_1 \) in (47) is evaluated [19] to be

\[
I_1 = -\sqrt{gH} \sum_{n=1}^{\infty} \left( -1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(n \beta \sqrt{gH})^{2k}}{1 \cdot (2^2 - 1) \cdot (4^2 - 1) \cdots ((2k)^2 - 1)} \right).
\]

14
which is not contributing to the finite part of the energy density. Thus, the unstable mode contributions to the finite $\beta$-dependent part of the energy density is found to be

$$\frac{gH\sqrt{gH}}{2e^{\alpha/\beta}} \cdot \frac{\pi^2}{2} \sum_{n=1}^{\infty} \frac{1}{n} Y_1(n\beta \sqrt{gH}),$$

(48)

which is just the real part of (34). There is no imaginary part. This is due to the inclusion of the cubic and the quartic terms in the unstable modes. Thus, the difficulties associated with the imaginary part are not due to the intrinsic property of the $SU(2)$ chromomagnetic ground state but due to the use of the gaussian approximation. This reaffirms our earlier study [8] of the same system at zero temperature.

The complete expression of the energy density of the $SU(2)$ chromomagnetic state, including the zero contribution from [8] is

$$\mathcal{E} = \frac{H^2}{2} + \frac{11(gH)^2}{48\pi^2} \left\{ \log\left(\frac{gH}{\Lambda^2}\right) - \frac{1}{2} \right\}
+ \frac{\pi^2}{45\beta^4}
+ \frac{(gH)^{3/2}}{\beta \pi^2} \sum_{\ell=1}^{\infty} \frac{\cos(\mu\beta\ell)}{\ell} \left( -\frac{\pi}{2} Y_1(\beta\ell \sqrt{gH}) \right)
+ K_1(\beta\ell \sqrt{gH}) + 2 \sum_{n=1}^{\infty} \sqrt{2n+1} K_1(\sqrt{2n+1}\beta\ell \sqrt{gH}).$$

(49)

The finite temperature part agrees with (the real part of) [15]. A similar result was found for the zero temperature case in [8].

**IV. NUMERICAL RESULTS AND DISCUSSION**

The expression for the effective energy density (49) involves summation over $\ell$ and in most studies, the high and low temperature behaviours were examined. Even then, the summation is very involved as can be seen from [14] and [15]. As the high and low temperature expressions are given in [14] and [15] (also in 12), we attempt to evaluate (49) numerically. First, it can be seen from [21], the $K_1$ functions have a fall-off to zero when the argument
is greater than 5. On the other hand the $Y_1$ function is oscillatory with decreasing amplitude. Second, we used the "polynomial approximations" for these functions given in [21] and verified that they are good by computing these functions for various values of $x$ from 0.05 to large values and comparing them with the tables of these functions. Then, we used MATLAB to find the values of $Y_1(x)$ and $K_1(x)$ for various values of $x$ which agree with the previous method. Third, we need to evaluate the sums in (49).

For the $Y_1$ function appearing in (49), we found the typical sum $\sum_{\ell=1}^{\infty} \frac{Y_1(\ell x)}{\ell}$ converged to a steady value for $\ell x$ up to 200 whereas for the sum $\sum_{\ell=1}^{\infty} \frac{K_1(\ell x)}{\ell}$, a steady value is reached when $\ell x = 20$. These allow us to choose $\ell_{\text{max}} = 200/x$ for the $Y_1$ sum while $\ell_{\text{max}} = 20/x$ for the sum involving $K_1$. Keeping $\ell_{\text{max}}$ as $20/x$, in the last $K_1$ sum, the $n$ sum was carried out from $n = 1$ to $n_{\text{max}}$ with $n_{\text{max}} = \frac{1}{2} \{ \ell_{\text{max}}^2 - 1 \}$. In order to evaluate the temperature variation of (49), we first set $\beta = \frac{a}{\sqrt{gH}}$ and $\mu = b\sqrt{gH}$. Then the temperature dependent part of (49) becomes

\[
\frac{\mathcal{E}_T}{(gH)^2} = \frac{\pi^2}{45a^4} + \frac{1}{\pi^2 a} \sum_{\ell=1}^{\infty} \frac{\cos(ab\ell)}{\ell} \left( - \frac{\pi}{2} Y_1(a\ell) \right) + K_1(a\ell) + 2 \sum_{n=1}^{\infty} \sqrt{2n + 1} K_1(a\ell \sqrt{2n + 1}) .
\]

In Fig.1, we have plotted $\frac{\mathcal{E}_T}{(gH)^2}$ with $T = \frac{\sqrt{gH}}{a}$, that is, $T$ in units of $\sqrt{gH}$, for $b = 0, 1, 2, 3$. For $b = 0$, zero chemical potential, the variation is smooth apart from small oscillatory behaviour at low temperatures. For $b = 2, 3$, the variation shows a minimum and then raising smoothly. At high temperatures, the behaviour is like that of non-interacting relativistic gas. In [15], the Polyakov loop is measured in terms of $\phi$ which in our notation is $\mu \beta$ and that is $ab$. In the sense that $a$ is varied, it is not possible to relate directly our results to [15]. However, the importance of the chemical potential is seen in Fig.1. A non-zero chemical potential or non-zero $\phi$ triggers a possible deconfinement phase transition. Our variation is qualitatively in agreement with [15] for their "real part".

Now, we wish to examine the inclusion of the cubic and quartic terms for all the modes. It can be seen from (15), the stable eigenvalues being
distinctly different from the unstable eigenvalues for a given $N$. So, it is justifiable to consider the corresponding eigenmodes as orthogonal. Then, from (15), it follows that the cubic terms will vanish for the stable modes as well. The resulting full expression can be evaluated as in (43) with explicit $N$ appearing. When the logarithm is taken, as in (41), the finite part will remain unaltered. The situation for the unstable modes is different in the sense the troublesome imaginary part does not appear in (44).

To summarize, we have considered the one-loop effective energy density of a pure $SU(2)$ Yang-Mills theory in the Savvidy background, at finite temperature and chemical potential. The unstable modes are treated by keeping the cubic and the quartic terms in the fluctuations. This result is added to the contribution from the stable modes. There is no imaginary part. The variation of the energy density for a given chromomagnetic background with temperature is studied numerically. When the chemical potential is non-zero, the variation shows a minimum which is (roughly) interpreted as indicating a deconfinement phase transition. At high temperatures, the behaviour is like that of a relativistic gas.

Acknowledgements

The authors thank Dr.G.S.Moni for helping the numerical computations. One of us (A.K) acknowledges with thanks the award of the Junior Research Fellowship by IMSc.

References

1. G.K.Savvidy, Phys.Lett. B71 (1977) 133.
2. N.K.Nielsen and P.Olesen, Nucl.Phys. B144 (1978) 376.
3. H.B.Nielsen and M.Ninomiya, Nucl.Phys. B156 (1979) 1; H.B.Nielsen and P.Olesen, Nucl.Phys. B160 (1979) 380; J.Ambjorn and P.Olesen, Nucl.Phys. B70 (1980) 60;265; R.Anishetty, J.Phys.G:Nucl.Phys. 10 (1984) 423;439.
4. Y.M. Cho, *Gauge Invariance and stability of SNO vacuum in QCD*, hep-th/0409247.

5. K.-I. Kondo, *Magnetic condensation, Abelian dominance and instability of Savvidy vacuum in Yang-Mills theory*, hep-th/0410024.

6. C.A. Flory, *Covariant constant chromomagnetic fields and elimination of the one-loop instabilities*, SLAC-PUB-3744, October 1983.

7. D. Kay, *Unstable modes, zero modes, and phase transition in QCD*, Ph.D Thesis, Simon Fraser University, August 1985.

8. D. Kay, A. Kumar and R. Parthasarathy, *Mod.Phys.Lett*. **A20** (2005) 1655.

9. J.J. Kapusta, *Nucl.Phys*. **B190 (FS3)** (1981) 425.

10. B. Muller and J. Rafelski, *Phys.Lett*. **B101** (1981) 111.

11. J. Chakrabarti, *Phys.Rev*. **D24** (1981) 2232; M. Reuter and W. Dittrich, *Phys.Lett*. **B144** (1984) 99.

12. M. Ninomiya and N. Sakai, *Nucl.Phys*. **B190 (FS3)** (1981) 316.

13. A. Cabo, O. K. Kalashnikov and A. E. Shabad, *Nucl.Phys*. **B185** (1981) 473.

14. A.O. Starinets, A.S. Vshivtsev and V.C. Zhukovsky, *Phys.Lett*. **B322** (1994) 403.

15. P.N. Meisinger and M.C. Ogilvie, *Phys.Rev*. **D66** (2002) 105006.

16. M. Loewe, S. Mendizabel and J.C. Rojas, *Phys.Lett*. **B635** (2006) 213.

17. J.J. Kapusta, *Finite temperature field theory*, Cambridge University Press, 1989.

18. Bateman Manuscript, Vol.II, McGraw-Hill, 1953.

19. I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, 1965.
20. N.N. Lebedev, *Special Functions and their Applications*, Prentice-Hall, 1965.

21. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover Publications, Inc., New York, 1968.
Figure 1: Variation of scaled energy density with scaled temperature