Some comments on spacelike minimal surfaces with null polygonal boundaries in $AdS_m$

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Abstract

We discuss some geometrical issues related to spacelike minimal surfaces in $AdS_m$ with null polygonal boundaries at conformal infinity. In particular for $AdS_4$, two holomorphic input functions for the Pohlmeyer reduced system are identified. This system contains two coupled differential equations for two functions $\alpha(z, \bar{z})$ and $\beta(z, \bar{z})$, related to curvature and torsion of the surface. Furthermore, we conjecture that, for a polynomial choice of the two holomorphic functions, the relative positions of their zeros encode the conformal invariant data of the boundary null $2n$-gon.

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1 Introduction

The $N$-point MHV gluon scattering amplitude at strong coupling in $\mathcal{N} = 4$ super Yang-Mills is related to a string worldsheet in $AdS_5$, approaching a $N$-sided polygon spanned by the lightlike momenta of the scattering process on the conformal boundary of $AdS_5\ [1]$. To find the corresponding minimal surface, is a difficult Plateau-like problem, and not much is known for the case of a generic null polygonal boundary. For the tetragon the surface has been constructed explicitly and, calculating the regularized area, the conjecture has been checked for the case of the four-point amplitude $[1]$. There has been also an interesting discussion of the limit of a large number of polygon sides, which led to the conclusion that the BDS ansatz $[2]$ for gluon amplitudes breaks down $[3]$.

By inspection, this tetragon surface is spacelike and flat. In our previous work $[4]$ we have proven, that besides isometry transformations, there are no other flat spacelike minimal surfaces. The flatness was an issue, because there exist flat timelike surfaces beyond the Wick rotated version of the tetragon surface, and because flatness would have simplified the explicit construction of the wanted surfaces.

Besides numerical work $[5]$ and a screening of some solutions of the relevant differential equations with respect to their boundary behaviour $[6]$, the only systematic progress has been made in $[7]$. Seen geometrically, the crucial new insight of this work is the fact, that in $AdS_3$ null $N$-gons ($N > 4$) only can arise as boundaries, if the second fundamental form has zeros on the surface. In more detail, it has been shown, that the conformal invariant data of the wanted boundary null polygon are in one to one correspondence to the relative position of zeros of a holomorphic polynomial parameterizing the second fundamental form. This includes a certain boundary condition for the scale factor of the induced metric in conformal coordinates. Although the surfaces could not be constructed explicitly, the authors of $[7]$ where able to calculate the regularized area of the minimal surface related to an octagon in 2-dimensional Minkowski space. In this way an explicit formula for the remainder function, which describes the part not fixed by anomalous dual conformal Ward identities, has been found for the 8-point amplitude.

Taking place in a $\mathbb{R}^{1,1}$ subspace of physical 4-dimensional Minkowski space is of course a degenerated case for a $N$ particle scattering. The aim of the present paper is to add some observations which could be helpful to extend the strategy of $[7]$ to less degenerated or even the generic kinematics.

The paper is organized as follows. To fix the notation, we summarize in section 2 some necessary formulae from $[4]$. Furthermore, this section contains a more elaborate discussion of the scalar invariants characterizing minimal surfaces in $AdS_m$ and a counting of the number of independent cross ratios formed out of the position of the vertices of the null $N$-gons in arbitrary dimensional Minkowski space. Section 3 describes in some detail geometrical issues in $AdS_3$, related to the identification of the one holomorphic polynomial carrying all the information about the boundary. In section 4 we turn to $AdS_4$ and identify two holomorphic functions which serve in the Pohlmeyer reduced system as an input for a coupled system of two differential equations controlling curvature and torsion of the minimal surface. The structural
similarity to the AdS$_3$ case and the matching of the numbers of parameters allows us to formulate in section 5 the conjecture, that now the two holomorphic functions of section 4 carry all information about the boundary. In section 6 we make some remarks on the more complicated full problem in AdS$_5$.

2 The general framework for minimal surfaces in AdS$_m$

Realizing AdS$_m$ (with coordinates $X^k$) as a hyperboloid in $\mathbb{R}^{2,m-1}$ (coordinates $Y^N$) and choosing conformal coordinates on the surface, one gets as the minimal surface condition

$$\partial \bar{\partial} Y^N(X(z)) - \partial Y^K \bar{\partial} Y^N = 0 .$$

(1)

The choice of conformal coordinates gives the additional condition

$$\partial Y^N \partial Y^N = \bar{\partial} Y^N \bar{\partial} Y^N = 0 ,$$

(2)

where $\partial$, $\bar{\partial}$ are defined by $\partial = \partial_\sigma + \partial_\tau$, $\bar{\partial} = \partial_\sigma - \partial_\tau$ for timelike surfaces and by $\partial = \partial_\sigma - i\partial_\tau$, $\bar{\partial} = \partial_\sigma + i\partial_\tau$ for spacelike surfaces.

One now extends the vectors $Y, \partial Y, \bar{\partial} Y$ to a basis of $\mathbb{R}^{2,m-1}$

$$\{e_N\} = \{Y, \partial Y, \bar{\partial} Y, B_4, \ldots, B_{m+1}\} .$$

(3)

The orthonormal vectors $B_a$ pointwise span the normal space of the surface inside AdS$_m$. Due to the hyperboloid condition, $Y$ is timelike. For timelike surfaces a further timelike vector is parallel to the surface, hence the normal space has to be positive definite. In contrast, for spacelike surfaces the second timelike vector has to be in the normal space. We choose it to be $B_4$. With $(a, b = 4, \ldots, m+1)$

$$h_{ab} = \delta_{ab} \text{ or } \eta_{ab}, \text{ for timelike or spacelike surface},$$

(4)

we require

$$(B_a, B_b) = h_{ab} , \ (B_a, Y) = (B_a, \partial Y) = (B_a, \bar{\partial} Y) = 0 .$$

(5)

Moving the basis (3) along the surface one gets

$$\partial e_N = A^K_N e_K , \quad \bar{\partial} e_N = \bar{A}^K_N e_K .$$

(6)

Introducing

$$\alpha(\sigma, \tau) = \log(\partial Y, \bar{\partial} Y)$$

$$u_a(\sigma, \tau) = (B_a, \partial Y) , \quad \bar{u}_a(\sigma, \tau) = (B_a, \bar{\partial} Y) ,$$

$$A_{ab}(\sigma, \tau) = (\partial B_a, B_b) , \quad \bar{A}_{ab}(\sigma, \tau) = (\bar{\partial} B_a, B_b) ,$$

(7)

(8)
and using (1), (5) one can give eqs. (6) a more detailed form

\begin{align*}
\partial Y &= \partial Y \\
\partial \partial Y &= \partial \alpha \partial Y + u^b B_b \\
\partial \partial Y &= e^\alpha Y \\
\partial B_a &= - e^{-\alpha} u_a \bar{\partial} Y + A_a^\ b B_b ,
\end{align*}

(9)
as well as the equations which one gets by the replacements \( \partial \leftrightarrow \bar{\partial} , \ u^a \rightarrow \bar{u}^a , \ A^a_b \rightarrow \bar{A}^a_b . \)

Indices on \( u, \bar{u} \) and \( A, \bar{A} \) are raised and lowered with the normal space metric \( h , \) see eq. (4). \( A \) and \( \bar{A} \) with both indices downstairs are antisymmetric.

Then, the integrability condition \( \partial \bar{\partial} e_N = \bar{\partial} \partial e_N \) for eq. (6) gives

\begin{align*}
\partial \bar{\partial} \alpha - e^{-\alpha} u^b \bar{u}_b - e^\alpha &= 0 , \\
\partial \bar{u}_a - A^\ a_b \bar{u}_b &= 0 , \\
e^{-\alpha} (\bar{u}_a u^b - u_a \bar{u}^b) &= F^\ b_a ,
\end{align*}

(10)-(12)

with

\[ F^\ b_a = \partial \bar{A}^\ a_b - \bar{\partial} A^\ a_b + \bar{A}^\ c_a A^\ b_c - A^\ c_a \bar{A}^\ b_c . \]

(13)

A appears as a gauge field with the related field strength \( F (A \in so(1, m-3) \) for spacelike surfaces and \( A \in so(m-2) \) for timelike surfaces).

Here, a comment on the geometrical meaning of our quantities \( \alpha , u, A \) is in order. Since we are using conformal coordinates,

\[ R = -2 e^{-\alpha} \partial \bar{\partial} \alpha \]

(14)
is the curvature scalar on our surface. \( u, \bar{u} \) parameterize the second fundamental forms \( l^\ _{\mu \nu} = (B^\ e, \partial_{\mu} \partial_{\nu} Y) \) with built in minimal surface condition \( l^\ _{\mu \nu} = 0. \)

The matrices \( A, \bar{A} \) in (11)-(12) describe the torsion of the surface (for \( AdS_m, m \geq 4 \)). Eqs. (10)-(12) are the Gauß, Codazzi-Mainardi and Ricci equations specialized to minimal surfaces in conformal coordinates. In the physics literature their described derivation is often called Pohlmeyer reduction [11].

After this general discussion with spacelike and timelike in parallel, we now restrict to spacelike minimal surfaces. For more comments on timelike surfaces see e.g. [4,9].

To form out of \( F \) scalar invariants with respect to \( SO(1, m-3) \) transformations in the normal space, one has at ones disposal the traces \( \text{tr}(F^n) \). Due to the special structure imposed for minimal surfaces by the Ricci equation (12), one finds

\[ \text{tr}(F^{2n+1}) = 0 , \quad \text{tr}(F^{2n}) = 2^{1-n} \left( \text{tr}(F^2) \right)^n , \]

(15)

which means that \( \text{tr}(F^2) \) is the only independent \( SO(1, m-3) \) invariant. To get a quantity invariant in addition with respect to conformal coordinate changes on the

\[ ^2\text{Note that for timelike surfaces } u \text{ and } \bar{u} \text{ as well as } A \text{ and } \bar{A} \text{ are real. On the other side, for spacelike surfaces } u \text{ and } A \text{ are complex, and then the bar means complex conjugation.} \]
surface $z \mapsto \zeta(z)$, one has to compensate the transformation of $\text{tr}(F^2)$ by that of a suitable power of $e^{-\alpha}$. Following [4] we introduce

$$T := \frac{1}{2} e^{-2\alpha} \text{tr}(F^2).$$

One could form invariants also directly out of $u$ and $\bar{u}$, i.e.

$$K := e^{-2\alpha} u_a \bar{u}^a, \quad L := e^{-4\alpha} u_a u^a \bar{u}_b \bar{u}^b.$$  

However, they contain no new information since due to the Ricci equation (12)

$$T = K^2 - L, \quad (18)$$

and by the Gauß equation (10)

$$R + 2K + 2 = 0.$$  

(19)

This shows that for minimal surfaces, in generic $AdS_m$, the scalar curvature $R$ and the torsion invariant $T$ are the only independent scalar invariants built out of the surfaces induced metric, curvature tensor and the torsion $F^a_b$. This of course refers to invariants without derivatives. A further outcome of this discussion is a universal inequality [4] arising from (18), (19) and the semi-definiteness of $L \geq 0$

$$\frac{(R + 2)^2}{4} - T \geq 0.$$  

(20)

Surfaces in $AdS_3$ have no torsion, i.e. then in the equations above one has to set always $T = 0$. For the discussion of the sign of $T$ for $m > 3$ we define $u_{\pm} = u_5 \pm u_4$, $\bar{u} = (u_6, u_7, \ldots)$, $\bar{u} = \bar{a} + i \bar{b}$ and get

$$(u_k \bar{u}^k)^2 - u_k u^k \bar{u}_l \bar{u}^l = -(\text{Im}(u_+ \bar{u}_-))^2 + 4 \text{Im}(\bar{u} \bar{u}_+) \text{Im}(u_+ \bar{u}_-) + 4 (\bar{a}^2 \bar{b}^2 - (\bar{a} \bar{b})^2).$$

(21)

The first term on the r.h.s. is always $\leq 0$, the last one always $\geq 0$. For $AdS_4$, there is no $\bar{u} = \bar{a} + i \bar{b}$. Hence only the first term is present and we get $T \leq 0$. For $AdS_m$, $m \geq 5$ both signs are possible, but due (20) $T$ is nevertheless bounded from above. [4]

The minimal surfaces needed in the Alday-Maldacena conjecture have to solve a Plateau-like problem, i.e. they have to extend to infinity and to approach a null polygonal at the conformal boundary of $AdS_m$. Since isometries of $AdS_m$ act as conformal transformations on the boundary, the relevant boundary data are encoded in the conformal invariants of null polygons in $(m-1)$-dimensional Minkowski space. These are given by cross ratios formed out of the positions of the cusps of the polygon. Counting the number of coordinates, taking into account the constraints set by the null condition for the sides of the polygon and subtracting the number of parameters of the conformal group one gets for the number of independent cross ratios for a null $N$-gon in $D = m - 1$ dimensions

$$C = \max \left\{ 0, \ N(D - 1) - \frac{(D + 1)(D + 2)}{2} \right\}.$$  

(22)

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*The conformal transformations of $uu$ or $\bar{u}\bar{u}$ alone cannot be compensated by a power of $e^{-\alpha}$.

*Note that for timelike surfaces one always has $T \leq 0$. If they are minimal [20] is valid, too.
### 3 Spacelike minimal surfaces in $AdS_3$

In this most simplest case there is no chance for torsion, there is only one complex valued function $u = u_4$, parameterizing the single second fundamental form. Equation (11) forces $u$ to be holomorphic and eq. (10) becomes

$$\partial \bar{\partial} \alpha + e^{-\alpha} u \bar{u} - e^\alpha = 0 .$$

(23)

$u(z)$ behaves under conformal coordinate transformations $z \mapsto \zeta(z)$ as

$$u(z) \mapsto (\zeta'(z))^{-2} u(z) .$$

(24)

If $u(z) \neq 0$ in some open set of the surface, then there one can transform $u$ to a constant, e.g. $u = 1$. Now the choice of the conformal coordinates in this open set is up to translations fixed completely, and one has to solve the sinh-Gordon equation

$$\partial \bar{\partial} \alpha - 2 \sinh \alpha = 0 .$$

(25)

Inserting the trivial solution $\alpha(z, \bar{z}) = 0$ into the linear problem (9), to reconstruct the embedding of the surface, one gets (due to good luck) the well-known tetragon solution of the Plateau-like problem under investigation [4,7,10]. This explicit solution is defined in the whole $z$-plane, and the boundary of $AdS_3$ is reached for $|z| \to \infty$. From now we assume the existence of such a globally defined system of conformal coordinates, with the property that $|z| \to \infty$ is mapped to the null N-gon on the boundary of $AdS_m$ also for other cases, i.e. both higher $N$ or/and higher $m$. One to one conformal maps of the complex plane are given by the Möbius group. Since in addition we insist on the correlation of infinite $z$ to the boundary N-gon, only translations and dilatations remain as a freedom for the choice of the global conformal coordinates. In contrast to the infinite dimensional local freedom (24), the required global property thus fixes the conformal coordinates up to the choice of the origin and up to multiplication with a constant. We also require, that in this coordinates all functions appearing in (10)-(12) are free of singularities at finite $z$.

The construction of [7] shows, that such global conformal coordinates exist in $AdS_3$ also for $N > 4$. Based on this, our assumption for higher dimensional $AdS_m$ is justified for null polygons in the neighbourhood of those degenerated to a location in a two-dimensional Minkowski space. For null polygonal configurations far from the degenerated ones, one should expect the possibility of branched minimal surfaces. Branched minimal surfaces come into the game also in the classical Plateau problem in $\mathbb{R}^m$ if one goes beyond $m = 3$ [12].

To handle minimal surfaces with boundary null $N$-gons with even $N > 4$ in $AdS_3$, the authors of [4] allow zeros of $u(z)$. They start with a polynomial ansatz for $u(z)$ in global conformal coordinates and are able to relate the data parameterizing the relative position of the zeros of this polynomial in a bijective manner to the cross ratios of the boundary N-gon. For $AdS_3$, i.e. $D = 2$, formula (22) gives for an

\[^5\]As a shorthand we will call them global conformal coordinates.
\((N = 2n)\)-gon \(2(n - 3)\) independent (real) cross ratios. Therefore, the polynomial \(u(z)\) has to be of degree \((n - 2)\).

In contrast to the \(2n - 6\) cross ratios for a \(2n\)-gon degenerated to live in a \(\mathbb{R}^{1,1} \subset \mathbb{R}^{1,3}\), a generic \(2n\)-gon in 4-dimensional Minkowski space via (22) has \(6n - 15\) independent cross ratios. For a partial lift of the degeneracy via a generic embedding in a \(\mathbb{R}^{1,2}\) one has to handle \(4n - 10\) cross ratios\(^6\). To make at least partial progress beyond \([7]\), a natural step is the discussion of minimal surfaces in \(AdS_4\).

We close this section by some geometrical comments, which give another motivation for the polynomial ansatz for \(u(z)\) in \([7]\). In terms of the scalar invariants, discussed in the previous section, \(u = 0\) (at finite \(\alpha\)) implies \(K = 0\) and via (19) \(R = -2\). This value of \(R\) coincides with that for the surface \(\mathbb{H}^2\). If \(u = 0\) in some open set, then there all geodesics of the surface would be also geodesics of the embedding \(AdS_3\) and the surface called totally geodesic. If \(u = 0\) at an isolated point, then there all geodesics of the surface passing this point have zero curvature in the sense of \(AdS_3\).

In addition, there is a nice descriptive argument for the necessity of zeros of \(u\) for \(N = 2n > 4\). Let us map \(AdS_3\) to half of ESU\(_3\), i.e. a cylinder in \(\mathbb{R}^3\) and consider the maximal symmetric null \(2n\)-gon on its boundary. Furthermore, we consider the geodesics on the surface connecting the middle points of the opposite sides of the polygons. In the case of the tetragon one finds by explicit calculations, that these lines are also geodesics in the sense of \(AdS_3\). Their image in the ESU\(_3\) are just the straight lines (in the sense of \(\mathbb{R}^3\)) connecting the middle points of the opposite sides of the polygons and crossing each other on the axis of the cylinder.

Due to the symmetry of the problem, one expects these straight lines (in the sense of \(\mathbb{R}^3\)) to be geodesic both in the sense of the surface and \(AdS_3\) for \(2n > 4\), too. At a point of a minimal surface with \(u \neq 0\) at most two lines, geodesic both in the sense of the surface as well in the sense of the embedding \(AdS_3\), can cross (see appendix). This shows that starting from \(n = 3\) the crossing point must be a (multiple) zero of \(u\). As shown by the analysis of \([7]\), for the generic unsymmetric configuration the multiple zero is dissolved into separated single zeros.

4 Spacelike minimal surfaces in \(AdS_4\)

Here the new degree of freedom relative to \(AdS_3\) allows minimal surfaces with torsion. Vice versa we expect torsion necessary to get surfaces for null polygons winding in full \(\mathbb{R}^{1,2}\).

We have an Abelian gauge group related to the surfaces normal space, \(SO(1, 1|\mathbb{R})\). Denoting

\[
\phi := A_4^5 = A_{45} = -A_{54} = A_5^4 , \quad u_\pm = u_5 \pm u_4
\]

the Codazzi equations in coordinates decouple

\[
\partial u_\pm \mp \phi \bar{u}_\pm = 0 , \quad \bar{\partial} u_\pm \mp \bar{\phi} u_\pm = 0 .
\]

\(^6\)For the octagon, discussed for \(\mathbb{R}^{1,1}\) in detail in \([7]\), the numbers are 2, 9 and 6.
The gauge field $A_a^b$ is related to the complex derivative $\partial$. Hence it is in $so(1,1|\mathbb{C})$, which implies $\phi \in \mathbb{C}$. Gauge transformations are described by

$$
\begin{align*}
\phi & \mapsto \phi + \partial \omega , & \bar{\phi} & \mapsto \bar{\phi} + \bar{\partial} \omega , & \omega & \in \mathbb{R} , \\
u_\pm & \mapsto e^{\pm \omega} u_\pm , & \bar{u}_\pm & \mapsto e^{\mp \omega} \bar{u}_\pm .
\end{align*}
$$
(28)

Now we parameterize

$$
\phi = \partial \eta , \quad \bar{\phi} = \bar{\partial} \bar{\eta} , \quad \eta \in \mathbb{C} .
$$
(29)

Gauge transformations then look like

$$
\eta(z, \bar{z}) \mapsto \eta(z, \bar{z}) + \omega(z, \bar{z}) , \quad \omega \in \mathbb{R} .
$$
(30)

There remains a gauge parameterization freedom

$$
\eta(z, \bar{z}) \mapsto \eta(z, \bar{z}) + \xi(z) ,
$$
(31)

with holomorphic $\xi(z)$. The inversion of (29) is then

$$
\eta(z, \bar{z}) = \frac{1}{\partial \bar{\partial}} \bar{\partial} \phi(z, \bar{z}) + \xi(z) .
$$
(32)

Introducing the gauge invariants

$$
\begin{align*}
\bar{v}_\pm & := e^{-\eta} \bar{u}_\pm , & v_\pm & := e^{\eta} u_\pm
\end{align*}
$$
(33)

and

$$
\beta := -i (\eta - \bar{\eta}) ,
$$
(34)

the equations (10)-(12) take the form

$$
\begin{align*}
\partial \bar{\partial} \alpha - \frac{e^{-\alpha}}{2} (e^{i \beta} v_- \bar{v}_+ + e^{-i \beta} v_+ \bar{v}_-) - e^\alpha &= 0 ,
\partial \bar{v}_\pm &= 0 , & \bar{\partial} v_\pm &= 0 ,
\partial \bar{\partial} \beta + \frac{e^{-\alpha}}{2 \eta} (e^{i \beta} v_- \bar{v}_+ - e^{-i \beta} v_+ \bar{v}_-) &= 0 .
\end{align*}
$$
(35)-(37)

Due to (36), the two functions $v_\pm$ have to be holomorphic. They appear as an input in the two coupled equations for the real functions $\alpha$ and $\beta$. This is similar to the situation in $AdS_3$, where one holomorphic equation appears as an input into one equation for $\alpha$.

As an aside let us mention, that with $\gamma := \alpha - i \beta$ the two equations (35) and (37) can be combined into one equation for a complex valued function $\gamma$

$$
\partial \bar{\partial} \gamma - e^{-\gamma} v_- \bar{v}_+ - e^{\frac{\gamma}{2}} (\gamma + \bar{\gamma}) = 0 .
$$
(38)

$\beta$ and $v_\pm$ are gauge invariant, but we have traded another unphysical degree of freedom, the gauge parameterization freedom (31). The two holomorphic functions $v_\pm$ transform under (31) as

$$
v_\pm(z) \mapsto e^{\mp \xi(z)} v_\pm(z) ,
$$
(39)
and under conformal coordinate transformations \( z \mapsto \zeta(z) \) as

\[
v_\pm(z) \mapsto (\zeta'(z))^{-2} v_\pm(z) .
\] (40)

We see that, although \( v_\pm(z) \) transform, the position of their zeros has an invariant meaning.

To explore the consequences for the scalar invariant quantities, we use (26), (33), (17) and (18) to get

\[
L = e^{-4\alpha} |v_-|^2 |v_+|^2 , \quad K = e^{-2\alpha} \Re(e^{i\beta} v_- \bar{v}_+) , \quad T = -e^{-4\alpha} \Im(e^{i\beta} v_- \bar{v}_+)^2 .
\] (41)

Therefore, zeros of \( v_- \) or \( v_+ \) are zeros of \( L, K \) and \( T \). However, zeros of \( T \) or of \( K \) (via (19) points with \( R = -2 \)) appear also at other points, generically on a net of lines in the \( z \)-plane. Only for \( L \) we have \( L = 0 \iff v_- = 0 \) or \( v_+ = 0 \). Note that at just these points the universal inequality (20) for \( R \) and \( T \) is saturated.

In this section our main result is twofold. At first we showed, that the function \( \bar{u}_a \bar{u}^a \), which is holomorphic for all \( AdS_m, m \geq 3 \), factorizes in the two independent holomorphic functions \( v_+ \) and \( v_- \) in the case of \( AdS_4 \). At second we formulated the equations for the Pohlmeyer reduced system for \( AdS_4 \) with these two holomorphic functions as input data. Based on this we will motivate in the next section a conjecture on the construction of minimal surfaces with null polygonal boundaries.

5 Null polygonal boundaries in the \( AdS_4 \) case

To start with, we consider the situation where \( v_\pm \) in some open subset have a finite number of zeros. The fact that they transform the same way under local conformal transformations (40), but in an inverse way under the change of gauge parameterization (39), could be used to bring them both into a polynomial form. However, having chosen global conformal coordinates as defined in the previous section, the transformation (40) is no longer available. But nevertheless, this observation supports somehow the expectation that, similar to the \( AdS_3 \) case [7], the wanted null polygonal boundaries are realized, if one starts with two polynomials in \( z \) as an input.

Let \( v_-(z) \) and \( v_+(z) \) be two polynomials of degree \( (n - 2) \), whose coefficients in front of the highest power is one. We take this as input in the coupled system of differential equations for \( \alpha(z, \bar{z}) \) and \( \beta(z, \bar{z}) \), i.e. eqs. (35) and (37). With the boundary condition specified below we expect, that the solution for \( \alpha(z, \bar{z}) \), \( \beta(z, \bar{z}) \), after solving the linear problem (9), generates a minimal surface with a null 2n-gonal boundary at \( |z| \to \infty \).

There is strong support for this guess from counting parameters. The relative position of zeros of \( v_\pm(z) \) is characterized by \( 2(n - 2) - 1 \) complex parameters, i.e. \( 4n - 10 \) real parameters. This just matches the number of independent cross ratios for a null 2n-gon in \( \mathbb{R}^{1,2} \), as identified in section 3.

The degenerated case \( v_- = v_+ =: v(z) \) leads to \( \partial \bar{\partial} \alpha - |v|^2 e^{-\alpha} \cos \beta - e^\alpha = 0 \), \( \partial \bar{\partial} \beta + |v|^2 e^{-\alpha} \sin \beta = 0 \). The second equation is then solved by \( \beta = \pi \), which puts the first equation in the form of the \( \alpha \)-equation in \( AdS_3 \).
It remains to discuss the boundary condition for \( \alpha \) and \( \beta \). We closely follow the line of reasoning used in [7] for \( AdS_3 \). There the boundary condition for \( \alpha \) is naturally found in the \( w \)-plane, related to the \( z \)-plane via \( dw = \sqrt{u(z)} \, dz \). The \( w \)-plane is no global conformal coordinate system, to cover the whole surface, one has to go in a Riemann surface over the \( w \)-plane. However, one has \( |w| \to \infty \Leftrightarrow |z| \to \infty \). The transformed entries in the Gauß equation are \( \hat{u}(w) = 1 \) and \( \hat{\alpha}(w, \bar{w}) = \alpha(z, \bar{z}) - \log|u(z)| \). Reasoning, that for \( |w| \to \infty \) the surface should behave as the known tetragon solution, one gets the boundary condition \( \hat{\alpha} \to 0 \). Translated back into the \( z \)-plane this means \( \alpha(z, \bar{z}) = \log|u(z)| + o(1) \) at \( |z| \to \infty \).

For \( AdS_4 \) we define
\[
dw = (v_+ v_-)^{\frac{1}{4}} \, dz .
\]
(42)
Then the transformed entries for (35) and (37) are
\[
\hat{v}_+(w) = \left( \frac{v_+(z)}{v_-(z)} \right)^{\frac{1}{2}}, \quad \hat{v}_-(w) = \left( \frac{v_-(z)}{v_+(z)} \right)^{\frac{1}{2}},
\]
\[
\hat{\alpha}(w, \bar{w}) = \alpha(z, \bar{z}) - \frac{1}{2} \log|v_+(z)v_-(z)|, \quad \hat{\beta}(w, \bar{w}) = \beta(z, \bar{z}).
\]
(43)
Contrary to the \( AdS_3 \) case, in the generic situation, the transformed \( \hat{v}_\pm \) are not constant equal to one. But since we have chosen \( v_\pm(z) \) to be monic polynomials of the same degree, \( \hat{v}_\pm \) converge to one for \( |w| \to \infty \). Then \( \hat{\alpha} \to 0 \) and \( \hat{\beta} \to \pi \) brings our equations for \( |w| \to \infty \) in the same form as in the \( AdS_3 \) case. Thus, to get a null 2\( n \)-gon, we expect as the appropriate boundary conditions at \( |z| \to \infty \) for the two coupled differential equations (35) and (37)
\[
\alpha(z, \bar{z}) = (n-2) \log|z| + o(1), \quad \beta(z, \bar{z}) \to \pi .
\]
(44)

6  Spacelike minimal surfaces in \( AdS_5 \)

Now the gauge group related to the normal space is non-Abelian (\( SO(1, 2|\mathbb{R}) \)), and the torsion invariant \( T \) can have both signs.

To identify gauge invariant and holomorphic objects, we adapt the procedure used for \( AdS_4 \). \( A \in so(1, 2|\mathbb{C}) \) can be parameterized by some \( M \in SO(1, 2|\mathbb{C}) \) via \(^7\)
\[
A = \partial M \, M^{-1} .
\]
(45)
Then
\[
v(z) := \overline{M^{-1}(z, \bar{z}) \, u(z, \bar{z})}
\]
(46)
is holomorphic, due to (11). Gauge transformations \( u \mapsto \Omega u, \quad M \mapsto \Omega M \) leave \( v \) invariant, since \( \Omega \in SO(1, 2|\mathbb{R}) \) is real. Again there remains a gauge parameterization freedom
\[
M(z, \bar{z}) \mapsto M(z, \bar{z}) \, \overline{Q(z)} ,
\]
(47)
\(^7\)We skip the conjugated equations. For a mathematical comment on this parameterization see the second part of the appendix.
with holomorphic $Q(z) \in SO(1,3|\mathbb{C})$. Under such a transformation $v(z)$ behaves as

$$v(z) \mapsto Q^{-1}(z) v(z).$$

We have now identified three holomorphic functions: the three components of $v(z)$, i.e. $v_4(z), v_5(z)$ and $v_6(z)$. The position of their zeros has again invariant meaning under conformal coordinate transformations. However, due to the unavoidable matrix structure in (18), there is no invariant meaning of the zeros of all three holomorphic functions with respect to a change of the gauge parameterization. One could try to use (18) to set one or even two of the components of $v(z)$ to zero. Away from the zeros of the old $v_k$ this is possible locally. But we did not find a suitable way to implement some reduction with globally holomorphic $Q$. The separate parameterizations of the gauge field used in [1] for $T < 0$ and $T > 0$ are not suitable to globally cover a situation where $T$ changes sign on the surface. We suspect that sign changes of $T$, which are not possible in $AdS_4$, are necessary to realize the most generic boundary null $N$-gon in $AdS_5$.

For sure there is one related holomorphic function whose zeros have invariant meaning, namely $v_k v^k$. Since $L = e^{-4\alpha} |v_k v^k|^2$, these zeros correspond to points, where the universal inequality (20) for $R$ and $T$ is saturated.

Irrespective of the outcome of counting the independent holomorphic functions, there must be additional parameters beyond the position of their zeros. From the zeros we get in any case an even number of real parameters. However, for the $2n$-gons in $\mathbb{R}^{1,3}$ one has $6n - 15$ parameters. A resolution of this mismatch could come from free real parameters in the non-Abelian gauge field solution, similar to scale parameters in instanton solutions.

We close this section by a reformulation of the basic equations (10)-(12) for $AdS_5$, using the mapping of the gauge group $SO(1,2)$ to $SL(2,\mathbb{R})$. This formulation so far did not lead to further insights in the problem just discussed. We present it here since it is interesting in its own right.

Choosing $(\sigma_k$ Pauli matrices) the following basis of $sl(2,\mathbb{R})$

$$\tau_0 = -i \sigma_2, \quad \tau_1 = \sigma_1, \quad \tau_2 = \sigma_3,$$

and (a shift $(4,5,6) \mapsto (0,1,2)$ understood)

$$U = u^k \tau_k, \quad A = \frac{1}{4} A^{kl} \tau_k \tau_l,$$

one gets

$$\partial \bar{\partial} \alpha - \frac{e^{-\alpha}}{2} \text{tr}(UU) - e^\alpha = 0,$$

$$\bar{\partial} U - [\bar{A},U] = 0, \quad \partial \bar{U} - [A,\bar{U}] = 0,$$

$$e^{-\alpha} [\bar{U},U] = \partial \bar{A} - \bar{\partial} A + [\bar{A}, A].$$

In this form it is even more eye-catching, that the basic equations for the reduced system in the sense of Pohlmeyer reduction, i.e. the set of Gauß, Codazzi and Ricci
equations (10)-(12), can be regarded as a Hitchin system \((A,U)\) \cite{13}, coupled to the metric parameterized by \(\alpha\). It would be interesting to find a possibility to handle \(\alpha\) as part of an enlarged gauge field, so that the whole system is equivalent to a pure Hitchin system. Such a construction has been realized for the \(AdS_3\) case in \cite{7}.

7 Conclusions

We discussed several geometrical issues related to the construction of minimal surfaces with null polygonal boundaries at conformal infinity of \(AdS_m\). There are two independent scalar invariants, curvature \(R\) and torsion \(T\). They obey the universal inequality (20). Points where this inequality is saturated, play a distinguished role for the surfaces under consideration.

For \(AdS_3\) and \(2n\)-gons with \(2n > 4\), descriptive geometrical arguments have been given for the existence of points with vanishing second fundamental form \(u\). These were based on the observation, that at points of a minimal surface with a crossing of three or more curves, which are geodesic in \(AdS_3\), \(u\) has to vanish.

We introduced the notion of global conformal coordinates, which are fixed up to translations and a multiplication with a constant. The existence of such coordinates for the relevant minimal surfaces in \(AdS_3\) is guaranteed by the construction of Alday and Maldacena \cite{7}. It is natural to assume their existence also for minimal surfaces in higher dimensional \(AdS_m\) for null polygonal boundaries in the neighbourhood of those of the \(AdS_3\) type.

In \(AdS_4\) the Pohlmeyer reduced system for spacelike minimal surfaces can be reformulated such that it is described by two functions \(\alpha(z, \bar{z}), \beta(z, \bar{z})\), which obey the two coupled differential equations (35),(37). In these equations appear two holomorphic functions as input. Guided by the match of the number of parameters and the similarities to the \(AdS_3\) case, we formulated boundary conditions for \(\alpha\) and \(\beta\). We conjectured, that with a polynomial ansatz for the two holomorphic input functions the solution for the reduced system and its boundary condition, after solving the related linear problem, yields a null polygonal boundary. The conformal invariant data of the null polygon are expected to be in one to one correspondence to the relative positions of the zeros of the two holomorphic polynomials.

In \(AdS_5\) the torsion invariant for spacelike minimal surfaces can have both signs. Also in this case it is straightforward to identify holomorphic input functions. However, up to now we did not succeed in a full identification of their gauge invariant content.

Note added:

In two recent papers \cite{14, 16} only \(p(z) = \bar{u}_a u^a\) has been kept as a holomorphic function, and as the partner for \(\alpha\) in the Pohlmeyer reduced equations for \(AdS_4\), instead of our \(\beta\), a modified function has been introduced via \(\bar{u}_4 = i \sqrt{p(z)} \cos \bar{\beta}/2, \bar{u}_5 = \sqrt{p(z)} \sin \bar{\beta}/2\). Then the difference \(\bar{\beta} - \beta\) is proportional to \(\log \left(\frac{v - \bar{v}}{v + \bar{v}}\right)\) \cite{15}.
and thus contains some winding in the $z$-plane. Winding for $\tilde{\beta}$ has been observed in appendix B.2. of [16] just in a way as required by the log-term in $\tilde{\beta} - \beta$. This gives further support for our assumption that $\beta$ is regular everywhere.

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Appendix

We ask for curves on a minimal surface in $AdS_3$, which are geodesics in the sense of the embedding space. At points with nonvanishing second fundamental form at most two such curves can cross. If there is a point with three or more such curves crossing, then at this point the second fundamental form has to be zero. The analog statement for minimal surfaces in $\mathbb{R}^3$ is obvious. To be on the safe side, we sketch here some formulae proving it for our situation.

Let $Y(t)$, $t$ affine parameter, describe a curve on the surface by its coordinates in the embedding space of $AdS_3$, i.e. $\mathbb{R}^{2,2}$. Then the curve is geodesic with respect to the surface iff

$$\ddot{Y} = \rho Y + \omega B_4$$

and it is geodesic in $AdS_3$, iff in addition $\omega = 0$. Now we use $\frac{d}{dt} = \dot{z} \partial + \bar{z} \bar{\partial}$ and (9) to express $\ddot{Y}$ as a linear combination of $Y, \partial Y, \bar{\partial} Y, B_4$. For a geodesic in the sense of the surface, the coefficients in front of $\partial Y$ and $\bar{\partial} Y$ have to vanish, i.e.

$$\ddot{z} + \dot{z}^2 \partial \alpha = 0.$$  \hspace{1cm} (55)

To be in addition also geodesic in the sense of $AdS_3$, the coefficient in front of $B_4$ has to vanish, i.e.

$$\ddot{z}^2 u + \dot{z}^2 \bar{u} = 0.$$  \hspace{1cm} (56)

Expressed in terms of the real coordinates $\sigma$ and $\tau$ and using $u = a + bi$, the last equation becomes

$$(\dot{\sigma}, \dot{\tau}) \begin{pmatrix} a & -b \\ -b & -a \end{pmatrix} \begin{pmatrix} \sigma \\ \tau \end{pmatrix} = 0.$$  \hspace{1cm} (57)

The eigenvalues of the matrix are $\pm \sqrt{a^2 + b^2}$. Hence as long as $u \neq 0$, there are just two (orthogonal) directions in the $z$-plane which satisfy (57).
Finally, a comment on the parameterization (45) is in order. It is obvious that it generates elements of $so(1,2|\mathbb{C})$. Since $\partial$ is a complex derivative, it is less obvious that all elements of this Lie algebra can be represented in this manner. The differential equation (45) is equivalent to the integral equation

$$M = \frac{1}{\partial\bar{\partial}} \bar{\partial}(AM) + \mathbb{I}.$$  

(58)

A solution of this equation as an infinite series can be generated by successive approximation, starting with zeroth approximation $M^{(0)} = \mathbb{I}$. Up to now we did not find a proof, that the resulting matrix is indeed out of $SO(1,2|\mathbb{C})$.

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