PROPER HOLOMORPHIC DISCS IN $\mathbb{C}^2$

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1. Results.

Let $U$ denote the open unit disc in $\mathbb{C}$ and $T = \partial U$ the unit circle. Let $X$ be a Stein manifold of dimension at least two. It was proved in [Glo] that for any point $p \in X$ there exists a proper holomorphic map $f: U \rightarrow X$ satisfying $f(0) = p$. We shall call such maps proper holomorphic discs in $X$. For smoothly bounded pseudoconvex domains in $\mathbb{C}^n$ this was proved earlier in [FG], and the essential addition in [Glo] was a method for crossing critical points of a strongly plurisubharmonic exhaustion function on $X$. The methods developed in [FG] and [Glo] actually show the following.

1.1 Theorem. Let $X$ be a Stein manifold with $\dim X \geq 2$, let $\rho: X \rightarrow \mathbb{R}$ be a smooth exhaustion function which is strongly plurisubharmonic on $\{ \rho > M \}$ for some $M \in \mathbb{R}$, and let $d$ be a metric on $X$. Given a continuous map $h: U \rightarrow X$ which is holomorphic on $U$ and satisfies $\rho(h(e^{i\theta})) > M$ for $e^{i\theta} \in T$, there exists for any pair of numbers $0 < r < 1$, $\epsilon > 0$, and for any finite set $A \subset U$ a proper holomorphic map $f: U \rightarrow X$ satisfying

(i) $\lim_{|\zeta| \rightarrow 1} \rho(f(\zeta)) = +\infty$,
(ii) $\rho(f(\zeta)) > \rho(h(\zeta)) - \epsilon$ for $\zeta \in U$,
(iii) $d(f(\zeta), h(\zeta)) < \epsilon$ for $|\zeta| \leq r$, and
(iv) $f(\zeta) = h(\zeta)$ for $\zeta \in A$.

We are interested to what extent does theorem 1.1 hold if $\rho$ is a (strongly) plurisubharmonic function whose sub-level sets are not necessarily relatively compact. Besides its intrinsic interest, we are motivated by the question whether it is possible to avoid any closed complex hypersurface $L$ in a Stein manifold by proper holomorphic discs. Such $L$ is the zero set of a smooth plurisubharmonic function $\rho: X \rightarrow \mathbb{R}_+$ which is strongly plurisubharmonic on $\{ \rho > 0 \} = X \setminus L$; therefore a positive answer to the first question gives proper holomorphic discs in $X$ avoiding $L$. In this paper we obtain positive results in certain model situations in $\mathbb{C}^2$. We begin with the following result.

1.2 Theorem. For each $c < 1$ and $M \in \mathbb{R}$ the conclusion of theorem 1.1 holds with $X = \mathbb{C}^2$ and the function $\rho_c: \mathbb{C}^2 \rightarrow \mathbb{R}$ given by

$$\rho_c(z_1, z_2) = \rho_c(x_1 + iy_1, x_2 + iy_2) = x_1^2 + x_2^2 - c(y_1^2 + y_2^2).$$

(1.1)

If on the other hand $c \geq 1$ then for any proper holomorphic map $f: U \rightarrow \mathbb{C}^2$ the function $\rho_c \circ f$ is unbounded from below on $U$; hence there exist no proper holomorphic discs satisfying theorem 1.1 (i) for $\rho = \rho_c$ with $c \geq 1$. 
Note that $\rho_c$ is strongly plurisubharmonic if $c < 1$, strongly plurisuperharmonic if $c > 1$, and $\rho_1(z_1, z_2) = \Re(z_1^2 + z_2^2)$ is pluriharmonic.

The second statement in theorem 1.2 (for $c \geq 1$) can be seen by applying theorem 1.5 (d) below to the function $g = f_1^2 + f_2^2$: since its range at any boundary point $e^{i\theta} \in T$ omits at most a polar set in $\mathfrak{F}$, its real part $\Re g = \rho_1(f_1, f_2)$ is unbounded from below. Since $\rho_c \leq \rho_1$ for $c \geq 1$, the same is true for $\rho_c \circ f$. The first part of theorem 1.2 (for $c < 1$) is proved in section 3.

When $c > 0$, $\rho_c$ is not an exhaustion function on $\mathfrak{F}^2$. For $0 < c < 1$ theorem 1.2 gives proper holomorphic maps $f:U \to \mathfrak{F}^2$ with images $f(U)$ contained in the real cone $\Gamma_c = \{ \rho_c > 0 \}$ with axis $\Re^2 = \{ y = 0 \}$. Moreover, when $c > 1$ we can apply theorem 1.2 with $-\rho_c(z)/c = y_1^2 + y_2^2 - \frac{1}{c}(x_1^2 + x_2^2)$ to obtain a proper holomorphic map $f:U \to \mathfrak{F}^2$ whose image avoids $\Gamma_c$. This gives proper holomorphic discs in $\mathfrak{F}^2$ avoiding relatively large real cones. On the other hand, no proper holomorphic disc (in fact, no transcendental complex curve) in $\mathfrak{F}^2$ can avoid a nonempty open complex cone; see theorem 2 in [SW] and theorem 1.5 below.

Our next result concerns discs avoiding pairs of complex lines in $\mathfrak{F}^2$.

**1.3 Theorem.** There exists a proper holomorphic map $f = (f_1, f_2):U \to \mathfrak{F}^2$ whose image $f(U)$ is contained in $(\chi^2)^2 = \mathfrak{F}^2 \setminus \{ zw = 0 \}$.

Writing $f:U \to (\chi^2)^2$ as $f = (e^{g_1}, e^{g_2}) = (e^{u_1 + iv_1}, e^{u_2 + iv_2})$, we have $|f|^2 = |f_1|^2 + |f_2|^2 = e^{2u_1} + e^{2u_2}$, and $f$ is proper as a map into $\mathfrak{F}^2$ if and only if $\max\{u_1, u_2\}$ tends to $+\infty$ at the boundary of $U$. Thus theorem 1.3 is equivalent to

**1.4 Theorem.** There exists a pair of harmonic functions $u_1, u_2$ on the disc $U$ such that

$$\lim_{|\zeta| \to 1} \max\{u_1(\zeta), u_2(\zeta)\} = +\infty.$$  

Theorem 1.3 is a special case of theorem 4.1 in section 4 below. A different proof of theorem 1.4 was shown to us by J.-P. Rosay (private communication).

It would be interesting to know whether proper discs in $\mathfrak{F}^2$ can avoid any given finite collection of complex lines. Part (d) in theorem 1.5 shows that such a disc cannot avoid a non-polar set of complex lines through the origin (or parallel complex lines) in $\mathfrak{F}^2$. The same holds if we replace the disc by any transcendental complex curve (Sibony and Wong [SW], Theorem 2). H. Alexander [Ale] proved in 1975 that for parallel lines in $\mathfrak{F}^2$ this is the only obstruction: If $E \subset \mathfrak{F}$ is a closed polar set containing at least two points, there exists a proper holomorphic map $f = (f_1, f_2):U \to \mathfrak{F}^2$ such that $f_1:U \to \mathfrak{F}\setminus E$ is a universal covering map of the disc onto $\mathfrak{F}\setminus E$. We don’t know whether an analogue of Alexander’s result holds for complex lines through the origin.

In the remainder of this section we discuss the boundary behavior of proper holomorphic maps $f = (f_1, f_2):U \to \mathfrak{F}^2$ at the circle $T = \{ |\zeta| = 1 \}$. We must recall some basic notions from the theory of cluster sets of meromorphic
functions on the disc; we refer to Chapter 8 in the monograph [CL] (see section 5 below for more details).

Let \( g \) be a meromorphic function on \( U \). A point \( e^{i\theta} \in T \) at which the (unrestricted) cluster set of \( g \) equals \( \overline{T} = \mathbb{C} \cup \{\infty\} \) is called a **Weierstrass point** of \( g \). If the restricted cluster set of \( g \) at \( e^{i\theta} \) within each conical region in \( U \) with vertex \( e^{i\theta} \) equals \( \mathbb{C} \) then \( e^{i\theta} \) is called a **Plessner point** of \( g \). A point \( e^{i\theta} \) at which \( g \) has a non-tangential limit (a limit as \( \zeta \to e^{i\theta} \) within any cone in \( U \) with vertex \( e^{i\theta} \)) is called a **Fatou point** of \( g \), and the set of all Fatou points is the **Fatou set** of \( g \). The range of \( g \) at \( e^{i\theta} \), denoted \( R(g, e^{i\theta}) \), consists of all \( \alpha \in \mathbb{C} \) such that \( g(\zeta_j) = \alpha \) for sequences \( \zeta_j \in U \) with \( \lim_{j \to \infty} \zeta_j = e^{i\theta} \).

1.5 Theorem. Let \( f = (f_1, f_2): U \to \mathbb{C}^2 \) be a proper holomorphic map of the disc to \( \mathbb{C}^2 \). Let \( P, Q \) be nonconstant holomorphic polynomials on \( \mathbb{C}^2 \) whose leading order homogeneous parts have no common divisor. Denote by \( g \) any of the following (meromorphic) functions: (i) \( f_1 \) or \( f_2 \), (ii) \( f_1/f_2 \), (iii) \( P(f_1, f_2) \), (iv) \( P(f_1, f_2)/Q(f_1, f_2) \). Then
(a) the Fatou set of \( g \) has Lebesgue measure zero in \( T \),
(b) every point of \( T \) is a Weierstrass point of \( g \),
(c) almost every point of \( T \) is a Plessner point of \( g \), and
(d) for every \( e^{i\theta} \in T \) the set \( C \setminus R(g, e^{i\theta}) \) is polar.

Theorem 1.5 is proved in section 5. Part (d) can be interpreted as a result on polynomial hulls as follows. We define the polynomial hull \( \hat{K} \) of an arbitrary subset \( K \subset \mathbb{C}^n \) as the intersection of all closed sets in \( \mathbb{C}^n \) of the form \( \{\Re P \leq 0\} \) containing \( K \), where \( P \) is a holomorphic polynomial. For compact sets this coincides with the usual definition of the polynomial hull. Clearly \( \hat{K} \) is contained in the closed convex hull of \( K \). Theorem 1.5 (d) implies

1.6 Corollary. If \( f: U \to \mathbb{C}^2 \) is a proper holomorphic map then for each open set \( D \subset \mathbb{C} \) intersecting \( T \) the polynomial hull of \( f(U \cap D) \) equals \( \mathbb{C}^2 \) (and hence its closed convex hull also equals \( \mathbb{C}^2 \)).

Theorem 1.5 does not generalize directly to proper maps \( f: U \to \mathbb{C}^n \) for \( n > 2 \). Namely, if \( (f_1, f_2): U \to \mathbb{C}^2 \) is proper holomorphic and if \( f_3 \) is any holomorphic function on \( U \) then \( (f_1, f_2, f_3): U \to \mathbb{C}^3 \) is also proper holomorphic; thus the addition of the third component need not enlarge the cluster set at any boundary point.

In the Appendix we comment on the proof of theorem 1.1 in [Glo]. Let \( \rho: X \to \mathbb{R} \) be a strongly plurisubharmonic Morse exhaustion function on a Stein manifold \( X \) of dimension \( \geq 2 \). We show that one can push the boundary of an analytic disc in \( X \) over a critical level of \( \rho \) by using the gradient flow of \( \rho \). This creates a non-holomorphic contribution which can be cancelled off during a later stage of the lifting procedure (this was the crucial observation in [Glo]).

Acknowledgements. We thank D. Marshall who showed us a simple proof of Frostman’s theorem to the effect that a meromorphic function on \( U \) which
omits a set of positive capacity has bounded Nevanlinna characteristic (and hence its Fatou set has full measure in $T$; see section 5). We also thank N. Sibony for pointing out the reference [SW]. Research of the first author was supported in part by an NSF grant, by the Vilas foundation at the University of Wisconsin–Madison, and by the Ministry of Science of the Republic of Slovenia. Research of the second author was supported in part by the Ministry of Science of the Republic of Slovenia.

2. Lifting holomorphic discs.

In this section we describe a general method for lifting the boundary of an analytic disc in $\mathbb{C}^n$ to a higher level set of a strongly plurisubharmonic function $\rho: \mathbb{C}^n \to \mathbb{R}$. This method was developed in [FG], but for our present needs we need more precise estimates for the amount of possible lifting at each step of the process.

2.1 Proposition. Let $\lambda: T \times \overline{U} \to \mathbb{C}^n$ be a continuous map such that for each $\zeta \in T$ the map $\lambda_\zeta = \lambda(\zeta, \cdot): \overline{U} \to \mathbb{C}^n$ is holomorphic in $U$ and $\lambda_\zeta(0) = 0$. Given numbers $\epsilon > 0$ and $0 < r < 1$, there exists a holomorphic polynomial map $h: \mathbb{C} \to \mathbb{C}^n$ satisfying

(i) $\text{dist}(h(\zeta), \lambda_\zeta(T)) < \epsilon$ ($\zeta \in T$),
(ii) $\text{dist}(h(t\zeta), \lambda_\zeta(U)) < \epsilon$ ($\zeta \in T$, $r \leq t \leq 1$), and
(iii) $|h(\zeta)| < \epsilon$ ($|\zeta| \leq r$).

Proof. It suffices to show that $\lambda$ can be approximated uniformly on $T \times \overline{U}$ by maps of the form

$$\tilde{\lambda}(\zeta, w) = \frac{w}{\zeta^M} \sum_{j=1}^{N} A_j(\zeta)w^{j-1},$$

where the $A_j$’s are holomorphic polynomials and $M, N$ are positive integers. The polynomial map

$$h(\zeta) = \tilde{\lambda}(\zeta, \zeta^K) = \zeta^{K-M} \sum_{j=1}^{N} A_j(\zeta)\zeta^{(j-1)K}$$

then satisfies proposition 2.1 provided that the approximation of $\lambda$ by $\tilde{\lambda}$ is sufficiently close and the integer $K \geq M$ is chosen sufficiently large.

We begin by replacing $\lambda$ by $(\zeta, w) \mapsto \lambda(\zeta, sw)$ for a suitable $s < 1$ sufficiently close to 1. Denoting the new map again by $\lambda$ we may thus assume that $\lambda_\zeta$ is holomorphic in a larger disc $|w| < 1/s$ for each $\zeta \in T$. We expand $\lambda$ in Taylor series with respect to $w$ and approximate it uniformly on $bU \times T$ by a Taylor polynomial $\lambda_N(\zeta, w) = \sum_{j=1}^{N} a_j(\zeta)w^j$ with continuous coefficients $a_j: T \to \mathbb{C}^n$. (The coefficient $a_0$ is zero since $\lambda(\zeta, 0) = 0$.) Finally we approximate each $a_j$ uniformly on $T$ by a map $A_j(\zeta)/\zeta^M$ for some holomorphic polynomial $A_j$ and some integer $N$ which can be chosen to be independent of $j$. This gives the desired approximation of $\lambda$ by a map of the form (2.1). ♠
2.2 Corollary. Let $g_0: \overline{U} \to \mathbb{C}^n$ be a continuous map that is holomorphic in $U$ and let $\lambda$ be as in proposition 2.1. Suppose that $\rho: \mathbb{C}^n \to \mathbb{R}$ is a real continuous function such that for some constants $C_0 < C_1$ and $0 < r < 1$ we have

(a) $\rho(g_0(\zeta) + \lambda(\zeta, w)) = C_1$ \hspace{1mm} ($\zeta \in T$; \hspace{1mm} $w \in T$),
(b) $\rho(g_0(\zeta) + \lambda(\zeta, w)) > C_0$ \hspace{1mm} ($\zeta \in T$; \hspace{1mm} $w \in \overline{U}$), and
(c) $\rho(g_0(\zeta)) > C_0$ \hspace{1mm} ($r \leq |\zeta| \leq 1$).

Then for each $\epsilon > 0$ there exists a holomorphic polynomial map $g: \mathbb{C} \to \mathbb{C}^n$ satisfying

(i) $|\rho(g(\zeta)) - C_1| < \epsilon$ \hspace{1mm} ($\zeta \in T$),
(ii) $\rho(g(\zeta)) > C_0$ \hspace{1mm} ($r \leq |\zeta| \leq 1$), and
(iii) $|g(\zeta) - g_0(\zeta)| < \epsilon$ \hspace{1mm} ($|\zeta| \leq r$).

Proof. Take $g(\zeta) = \tilde{g}_0(\zeta) + h(\zeta)$, where $\tilde{g}_0$ is a polynomial approximation of $g_0$ and $h$ is a suitably chosen map provided by proposition 2.1.

Assume now that $\rho: \mathbb{C}^n \to \mathbb{R}$ is a function of class $C^2$. For each fixed $z$ we write

$$\rho_z(w) = \rho(z + w) - \rho(z) = \Re Q_z(w) + \mathcal{L}_z(w) + O(|w|^2),$$  \hspace{1mm} (2.2)

where

$$Q_z(w) = 2 \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(z)w_j + \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial z_k}(z)w_jw_k$$

$$\mathcal{L}_z(w) = \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial z_k}(z)w_j\overline{w_k}$$

($Q_z$ is the Levi polynomial and $\mathcal{L}_z$ is the Levi form of $\rho$ at $z$). The set

$$\Lambda_z = \{ w \in \mathbb{C}^n : Q_z(w) = 0 \}$$  \hspace{1mm} (2.3)

is a quadratic complex hypersurface in $\mathbb{C}^n$ and we have $\rho_z(w) = \mathcal{L}_\rho(z; w) + O(|w|^2)$ for $w \in \Lambda_z$. For $c > 0$ we denote by $B(z; c)$ the connected component of the sublevel set $\{ w \in \Lambda_z : \rho_z(w) < c \}$ which contains the point $0 \in \Lambda_z$. If $\rho$ is strongly plurisubharmonic near $z$ (i.e., its Levi form $\mathcal{L}_z$ at $z$ is positive definite) and if $\partial \rho(z) \neq 0$ (so that the hypersurface $\Lambda_z$ is smooth near 0), then for all sufficiently small $c > 0$ the set $B(z; c)$ is diffeomorphic to the real $(2n - 2)$-dimensional ball. Moreover, if $C > 0$ is such that the function $\rho_z|\Lambda_z$ has no critical points on $B(z; C)$ other than the point 0, Morse theory shows that for $0 < c \leq C$ the sets $B(z; c)$ are complex manifolds diffeomorphic to the $(2n - 2)$-ball. (We include the singularities of $\Lambda_z$ among the critical points of $\rho_z|\Lambda_z$.) In particular, when $n = 2$, these sets are complex one-dimensional and hence conformally equivalent to the disc. We state the next proposition only for $n = 2$ since we shall only need this case.

\hspace{1mm}
2.3 Proposition. Let \( g_0: U \to \mathbb{C}^2 \) be a continuous map that is holomorphic in \( U \) and let \( \rho: \mathbb{C}^2 \to \mathbb{R} \) be a \( C^2 \) function which is strongly plurisubharmonic in a neighborhood of \( g_0(T) \) and has no critical points on \( g_0(T) \). Suppose that \( C: T \to (0, \infty) \) is a continuous function such that the function \( \rho_{g_0(\zeta)}|\Lambda_{g_0(\zeta)} \) (2.2) has no critical points on \( B(g_0(\zeta); C(\zeta))\backslash\{0\} \) for each \( \zeta \in T \). Then for each \( \epsilon > 0 \) and \( 0 < r < 1 \) there is polynomial map \( g: \mathbb{C} \to \mathbb{C}^2 \) satisfying

(i) \( |\rho(g(\zeta)) - \rho(g_0(\zeta)) - C(\zeta)| < \epsilon \quad (\zeta \in T) \),

(ii) \( \rho(g(\zeta)) > \rho(g_0(\zeta)) - \epsilon \quad (\zeta \in \overline{U}) \), and

(iii) \( |g(\zeta) - g_0(\zeta)| < \epsilon \quad (|\zeta| \leq r) \).

Proof. We have seen above that for each \( \zeta \in T \) the set \( B(g_0(\zeta); C(\zeta)) \subset \Lambda_{g_0(\zeta)} \) is conformally equivalent to the disc \( U \). Decreasing \( C(\zeta) \) slightly (so that \( B(g_0(\zeta); r) \) is still biholomorphic to \( U \) for some \( r > C(\zeta) \)) we can obtain a parametrization \( \lambda_\zeta: \overline{U} \to \overline{B}(g_0(\zeta); C(\zeta)) \) \( (\zeta \in T) \), depending continuously on \( (\zeta, w) \in T \times \overline{U} \), such that \( \lambda_\zeta \) is holomorphic in \( U \) and \( \lambda_\zeta(0) = g_0(\zeta) \) for each \( \zeta \in T \). The result now follows from proposition 2.1 applied to the family of discs \( \lambda_\zeta \).

If \( K_0 \Subset K_1 \Subset \mathbb{C}^2 \) is a pair of compact sets such that \( \rho \) is strongly plurisubharmonic and has no critical points on \( K_1 \), there is a constant \( C > 0 \) such that \( \rho|\Lambda_\zeta \) has no critical points on \( B(z; C)\backslash\{0\} \) for each \( z \in K_0 \). Hence proposition 2.3 provides a uniform lifting of the boundary of an analytic disc (with respect to \( \rho \)) as long as the boundary remains in \( K_0 \). If the set \( A(c_0, c_1) = \{ x \in X: c_0 \leq \rho(x) \leq c_1 \} \) is compact for some \( c_0 < c_1 \) and if \( \rho \) is strongly plurisubharmonic and without critical points on this set, proposition 2.3 allows us to lift the boundary of an analytic disc in \( X \) from the level \( \rho = c_0 \) to the level \( \rho = c_1 \). Unfortunately this breaks down in general if the level sets of \( \rho \) are not compact. In this case we need a more precise analysis which we shall do for the function (1.1).

2.4 Proposition. Let \( \rho_c \) be the function (1.1). If \( c < 1 \) there exists a number \( a = a(c) > 0 \) with the following property: For each continuous map \( h: \overline{U} \to \mathbb{C}^2 \), holomorphic in \( U \), such that \( m(h) = \inf\{\rho_c(h(\zeta)): |\zeta| = 1\} > 0 \), and for each pair of numbers \( \epsilon > 0 \) and \( 0 < r < 1 \) there exists a holomorphic polynomial map \( g: \mathbb{C} \to \mathbb{C}^2 \) satisfying

(i) \( m(g) \geq (1 + a)m(h) \),

(ii) \( \rho_c(g(\zeta)) > \rho_c(h(\zeta)) - \epsilon \quad (|\zeta| \leq 1) \), and

(iii) \( |g(\zeta) - h(\zeta)| < \epsilon \quad (|\zeta| \leq r) \).

Proof. Note \( \rho_c \) is strongly plurisubharmonic when \( c < 1 \). Fix such a \( c \) and write \( \rho = \rho_c \). The only critical point of \( \rho \) is \( z_1 = z_2 = 0 \). Proposition 2.4 follows immediately from proposition 2.3 and the following

2.5 Lemma. Let \( \rho = \rho_c \) for some \( c < 1 \) be given by (1.1). There is a constant \( a = a(c) > 0 \) such that for each \( z \in \mathbb{C}^2 \) with \( \rho(z) > 0 \) the function \( \rho_z|\Lambda_z \) has no critical points on \( B(z; a\rho(z))\backslash\{0\} \).
Proof. A calculation shows that \( \rho_z(w) = \Re Q_z(w) + L_z(w) \), where

\[
Q_z(w) = 2(x_1 + icy_1)w_1 + 2(x_2 + icy_2)w_2 + \frac{1}{2}(1 + c)(w_1^2 + w_2^2)
\]

\[
L_z(w) = \frac{1}{2}(1 - c)(|w_1|^2 + |w_2|^2) = \frac{1}{2}(1 - c)|w|^2.
\]

It suffices to consider the case \( 0 < c < 1 \). If \( w \in \Lambda_z \) then

\[
\rho_z(w) = \rho(z + w) - \rho(z) = \frac{1}{2}(1 - c)|w|^2 \quad (2.4)
\]

The critical points of \( \rho_z|\Lambda_z \) are precisely those points \( w \in \Lambda_z \) at which the complex gradients \( \partial Q_z \) and \( \partial \rho_z \) (with respect to the variable \( w = (w_1, w_2) \in \mathbb{C}^2 \)) are \( \mathbb{C} \)-linearly dependent. This set will include any singular points of \( \Lambda_z \). By (2.4) we may replace \( \partial \rho_z \) by \( \partial |w|^2 \). Set \( h(x + iy) = x + icy \), so \( |h(x + iy)|^2 = x^2 + c^2y^2 \). We have

\[
\partial Q_z(w) = (2h(z_1) + (1 + c)w_1, 2h(z_2) + (1 + c)w_2), \quad \partial |w|^2 = (\overline{w}_1, \overline{w}_2).
\]

This gives the following system of two equations for \( w \), in which the first is the colinearity equation between \( \partial Q_z \) and \( \partial |w|^2 \) (after conjugation) and the second is \( Q_z(w) = 0 \):

\[
2\overline{h(z_2)}w_1 - 2\overline{h(z_1)}w_2 = -(1 + c)(w_1\overline{w_2} - \overline{w_1}w_2)
\]

\[
4h(z_1)w_1 + 4h(z_2)w_2 = -(1 + c)(w_1^2 + w_2^2) \quad (2.5)
\]

It suffices to obtain a good lower estimate for the norm \( |w| \) of any nonzero solution of (2.5) in terms of \( |z| \). We apply Cramer’s formula to express \( w_1 \) and \( w_2 \) from the linear part in terms of the right hand side terms in (2.5). The determinant of the matrix of coefficients is \( W(z) = 8(|h(z_1)|^2 + |h(z_2)|^2) \geq c'|z|^2 \) where \( c' > 0 \) depends only on \( c \). If we replace one of the columns of the coefficient matrix by the right hand side then each term in the corresponding determinant is of the form constant times \( h(z_j)w_kw_l \) for some \( j, k, l \in \{1, 2\} \). Hence we can estimate the determinant from above by the Cauchy-Schwarz inequality and thus obtain the following estimate for the solutions of (2.5):

\[
|w_j| \leq \frac{c_2(|h(z_1)|^2 + |h(z_2)|^2)^{1/2}|w|^2}{W(z)} \leq \frac{c_3|w|^2}{|z|} \quad (j = 1, 2).
\]

This gives \( |w| \leq c_4|w|^2/|z| \) and therefore \( |w| \geq c_5|z| \) for any nonzero solution \( w \) of (2.5), where \( c_5 > 0 \) depends only on \( c \). Since \( w \in \Lambda_z \), (2.4) gives

\[
\rho(z + w) \geq \rho(z) + c_6|z|^2 \geq \rho(z) + c_7\rho(z)
\]

for some \( c_7 > 0 \). Thus any constant \( a < c_7 \) satisfies lemma 2.5. \( \spadesuit \)

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&3. Proper discs in cones in $\mathbb{C}^2$ with real axis.

In this section we prove theorem 1.2. If the constant $M$ in the theorem is negative, we first apply the procedure described in [Glo] to cross the critical point of $\rho_c$ at $(0,0)$ and thus push the boundary of the given initial analytic disc $h$ to the set $\rho_c > 0$ while changing $h$ as little as desired on $\{|\zeta| \leq r\}$. Hence it suffices to prove theorem 1.2 for $M \geq 0$. In this case the result follows immediately from the following.

**3.1 Theorem.** Let $c < 1$, $M \geq 0$, and let $\rho = \rho_c$ be the function (1.1). Given a continuous map $h: \mathbb{U} \to \mathbb{C}^2$, holomorphic in $\mathbb{U}$, such that $\rho(h(\zeta)) > M$ for $|\zeta| = 1$, there exists for each $\epsilon > 0$ and $0 < r_1 < 1$ a proper holomorphic map $f: \mathbb{U} \to \mathbb{C}^2$ satisfying

1. $\lim_{|\zeta| \to 1} \rho_c(f(\zeta)) = +\infty$,
2. $\rho(f(\zeta)) > \rho(h(\zeta)) - \epsilon$ ($|\zeta| < 1$), and
3. $|f(\zeta) - h(\zeta)| < \epsilon$ ($|\zeta| \leq r_1$).

**Proof.** It suffices to consider the case $0 < c < 1$. Fix numbers $M > 0$, $0 < r < 1$, $\epsilon > 0$ and a map $h$ as in the statement of theorem 3.1 and write $M_1 = M$, $\epsilon_1 = \epsilon$, $f_1 = h$. Let $a > 0$ be the number given by proposition 2.4 for the pair $c$ and $M_1$. Set

$$M_k = (1 + a)^{k-1} M_1, \quad \epsilon_k = \epsilon/2^{k-1}, \quad k = 2, 3, 4, \ldots$$

We inductively construct a sequence of polynomial maps $f_k: \mathbb{U} \to \mathbb{C}^2$ and a sequence of numbers $0 < r_1 < r_2 < r_3 < \ldots < 1$ with $\lim_{k \to \infty} r_k = 1$ such that the following hold for each $k \geq 2$:

1. $\rho(f_k(\zeta)) > M_k$ ($r_k \leq |\zeta| \leq 1$),
2. $\rho(f_k(\zeta)) > \rho(f_{k-1}(\zeta)) - \epsilon_{k-1}$ ($|\zeta| \leq 1$), and
3. $|f_k(\zeta) - f_{k-1}(\zeta)| < \epsilon_{k-1}$ ($|\zeta| \leq r_{k-1}$).

The construction proceeds as follows. By assumptions the condition $(a_1)$ holds for $|\zeta| = 1$. By continuity we can increase $r_1$ such that $(a_1)$ holds for $r_1 \leq |\zeta| \leq 1$. Proposition 2.4 gives a map $f_2$ such that $\rho(f_2(\zeta)) > M_2$ for $|\zeta| = 1$ and such that $(b_2)$ and $(c_2)$ hold. By continuity we can choose a number $r_2 < 1$ sufficiently close to 1 such that $(a_2)$ holds for $r_2 \leq |\zeta| \leq 1$.

This process can be continued inductively. If we already have $f_{k-1}$, proposition 2.4 gives the next map $f_k$ which satisfies $(a_k)$ initially only for $|\zeta| = 1$, and it satisfies $(b_k)$ and $(c_k)$. By continuity we can choose $r_k < 1$ sufficiently close to 1 so that $(a_k)$ holds. We can thus insure that $\lim_{k \to \infty} r_k = 1$.

Condition $(c)$ insures that $f = \lim_{k \to \infty} f_k: \mathbb{U} \to \mathbb{C}^2$ exists uniformly on compacts in $\mathbb{U}$. For $|\zeta| \leq r_1$ we have

$$|f(\zeta) - f_1(\zeta)| \leq \sum_{k=1}^{\infty} |f_{k+1}(\zeta) - f_k(\zeta)| < \sum_{k=1}^{\infty} \epsilon_{k+1} = \epsilon.$$
This proves (iii) since $h = f_1$. For a fixed $\zeta \in U$ and $k \geq 1$ we have

$$\rho(f(\zeta)) = \lim_{j \to \infty} \rho(f_j(\zeta)) = \rho(f_k(\zeta)) + \sum_{j=k}^{\infty} (\rho(f_{j+1}(\zeta)) - \rho(f_j(\zeta)))$$

$$> \rho(f_k(\zeta)) - \sum_{j=k}^{\infty} \epsilon_{j+1} = \rho(f_k(\zeta)) - \epsilon_k.$$  

For $k = 1$ we get (ii) in the theorem. For points $\zeta$ in the annulus $r_k \leq |\zeta| < 1$ we get $\rho(f(\zeta)) > \rho(f_k(\zeta)) - \epsilon_k > M_k - \epsilon$. Since $\lim_{k \to \infty} M_k = \infty$, this implies (i) and completes the proof of theorem 3.1. ⊣

\&4. Proper discs in $\mathbb{C}^2$ which omit a pair of lines.

Theorem 1.3 follows from the following more precise result.

4.1 Theorem. Let $n \geq 2$. Given a continuous map $h = (h_1, h_2, \ldots, h_n): \overline{U} \to \mathbb{C}^n$ which is holomorphic in $U$ and given a number $0 < r < 1$ such that the components $h_j$ have no zeros in $\{ \zeta : r \leq |\zeta| \leq 1 \}$, there exists for each $\epsilon > 0$ a proper holomorphic map $f = (f_1, f_2, \ldots, f_n): U \to \mathbb{C}^n$ such that the $f_j$'s have no zeros in $\{ \zeta : r \leq |\zeta| < 1 \}$ and $|f(\zeta) - h(\zeta)| < \epsilon$ for $|\zeta| \leq r$.

We shall give details only for $n = 2$. By factoring out the (finitely many) zeros of the $h_j$'s we can reduce to the case when the $h_j$'s have no zeros on $\overline{U}$. We seek a solution in the form $f = (e^{g_1}, e^{g_2}) = (e^{u_1+iv_1}, e^{u_2+iv_2})$ for some holomorphic map $g = (g_1, g_2): U \to \mathbb{C}^2$. Set

$$\rho(x_1 + iy_1, x_2 + iy_2) = \max\{x_1, x_2\}. \quad (4.1)$$

Since $|f|^2 = |f_1|^2 + |f_2|^2 = e^{2u_1} + e^{2u_2}$, $f$ is proper into $\mathbb{C}^2$ if and only if $\rho(g(\zeta)) = \max\{u_1(\zeta), u_2(\zeta)\}$ tends to $+\infty$ as $|\zeta| \to 1$. Such map $g$ will be obtained as the limit $g = \lim_{k \to \infty} g_k$ of an inductively constructed sequence $g_k$, where the inductive step from $g_{k-1}$ to $g_k$ will be furnished by corollary 2.2. To this end we need a suitable family of analytic discs which we now construct.

4.2 Proposition. Let $\rho$ be the function (4.1). Given a compact set $K \subsetneq \mathbb{C}^2$ and constants $C_0, C_1 \in \mathbb{R}$ such that $C_0 < \rho(z) < C_1$ ($z \in K$), there is a continuous map $\lambda: K \times \overline{U} \to \mathbb{C}^2$ such that for each $z \in K$ the map $\lambda(z, \cdot): U \to \mathbb{C}^2$ is holomorphic and

(i) $\rho(\lambda(z, w)) = C_1$ ($z \in K, |w| = 1$),
(ii) $\rho(\lambda(z, w)) > C_0$ ($z \in K, |w| \leq 1$).

Proof. We follow the proof of Bochner’s tube theorem (see [Hör], p. 41). We first describe the model situation. Write the coordinates on $\mathbb{C}^2$ in the form
$z = x + iy$, with $x, y \in \mathbb{R}^2$, and identify $\mathbb{R}^2$ with $\{y = 0\} \subset \mathbb{C}^2$. Set

$$k = \{(x_1, 0): 0 \leq x_1 \leq 1\} \cup \{(0, x_2): 0 \leq x_2 \leq 1\}$$

$$K_\epsilon = \{x + iy \in \mathbb{C}^2: x \in k, \ |y|^2 \leq 1/\epsilon\}$$

$$ co(k) = \{(x_1, x_2): x_1 \geq 0, \ x_2 \geq 0, \ x_1 + x_2 \leq 1\}$$

$$\gamma_\epsilon = \{(x_1, x_2) \in co(k): x_1 + x_2 - \epsilon(x_1^2 + x_2^2) = 1 - \epsilon\}$$

$$\Gamma_\epsilon = \{(z_1, z_2) \in \mathbb{C}^2: (x_1, x_2) \in co(k), \ z_1 + z_2 - \epsilon(z_1^2 + z_2^2) = 1 - \epsilon\}$$

4.3 Lemma. (Notation as above) There is an $\epsilon_0 > 0$ such that for each $\epsilon$ with $0 < \epsilon < \epsilon_0$ the set $\Gamma_\epsilon$ is a holomorphic disc with boundary contained in $K_\epsilon$, $\Gamma_\epsilon \cap \mathbb{R}^2 = \gamma_\epsilon$, and $\gamma_\epsilon$ is a smooth real-analytic curve contained in the convex hull $co(k)$ of $k$. The union $\bigcup_{0 < \epsilon < \epsilon_0} \gamma_\epsilon$ contains every point in the interior of $co(k)$ and sufficiently close to the open segment $\gamma_0 = \{(x_1, 1 - x_1): 0 < x_1 < 1\}$.

Proof. Observe that $\gamma_\epsilon = \{F_\epsilon = 0\} \cap co(k)$ where

$$F_\epsilon(x_1, x_2) = x_1 + x_2 - \epsilon(x_1^2 + x_2^2) - 1 + \epsilon.$$ 

Simple calculations show that for $0 < \epsilon < 1/2$ we have $F_\epsilon(x_1, 0) < 0$ for $0 \leq x_1 < 1$, $F_\epsilon(1, 0) = F_\epsilon(0, 1) = 0$, $F(x_1, 1 - x_1) > 0$ when $0 < x_1 < 1$, and $\frac{\partial}{\partial x_2} F_\epsilon(x_1, x_2) = 1 - 2\epsilon x_2 > 0$ for $0 \leq x_2 \leq 1$. These properties imply that $\gamma_\epsilon$ is a graph $y_1 = h_\epsilon(x_1)$ of a real-analytic function $h_\epsilon$ over the segment $0 \leq x_1 \leq 1$, with with the endpoints $(1, 0)$ and $(0, 1)$. Since $\partial F_\epsilon/\partial \epsilon = 1 - (x_1^2 + x_2^2) \geq 0$ on $co(k)$ we conclude that, as $\epsilon$ decreases to 0, the functions $h_\epsilon$ increase to $h_0(x_1) = 1 - x_1$. This gives the last claim in lemma 4.3.

We will show that for sufficiently small $\epsilon > 0$ there exists a bounded, simply connected region $D_\epsilon \subset \{z_2 = 0\}$ with piecewise smooth boundary such that $\Gamma_\epsilon$ is the graph of a holomorphic function over $D_\epsilon$. The equation for $\Gamma_\epsilon$ is equivalent to

$$x_1 + x_2 - \epsilon(x_1^2 + x_2^2) + \epsilon(y_1^2 + y_2^2) = 1 - \epsilon$$

$$ (1 - 2\epsilon x_1)y_1 + (1 - 2\epsilon x_2)y_2 = 0. \quad (4.2)$$

When $y_1 = y_2 = 0$ we get the equation for $\gamma_\epsilon$, and hence $\Gamma_\epsilon \cap \mathbb{R}^2 = \gamma_\epsilon$. On $co(k)$ we have $x_1 + x_2 \geq 0$ and $x_1^2 + x_2^2 \leq 1$, with equality only at the points $(1, 0)$ and $(0, 1)$. Rewriting the first equation in (4.2) in the form

$$(x_1 + x_2) + \epsilon(y_1^2 + y_2^2) = 1 - \epsilon(1 - (x_1^2 + x_2^2)) \leq 1$$

we see that (4.2) has no solutions for $|y|^2 = y_1^2 + y_2^2 > 1/\epsilon$, and it has no solutions on $\gamma_0 + i\mathbb{R}^2$ ($\gamma_0$ was defined in lemma 4.3). Hence the boundary of $\Gamma_\epsilon$ is contained in $K_\epsilon$ and therefore $\Gamma_\epsilon \subset co(K_\epsilon)$. From the second equation in (4.2) we get

$$y_2 = -y_1 \frac{1 - 2\epsilon x_1}{1 - 2\epsilon x_2}. \quad (4.3)$$
(again this requires \( \epsilon < 1/2 \) since \( 0 \leq x_1, x_2 \leq 1 \) on \( \Gamma_\epsilon \)). Inserting this into the first equation (4.2) we get

\[
G_\epsilon(x_1, y_1, x_2) := x_1 + x_2 - \epsilon \left( x_1^2 + x_2^2 \right) + \epsilon y_1^2 \left( 1 + \frac{(1 - 2\epsilon x_1)^2}{(1 - 2\epsilon x_2)^2} \right) - 1 + \epsilon = 0. \tag{4.4}
\]

Consider first its restriction to \( x_2 = 0 \):

\[
G_\epsilon(x_1, y_1, 0) = x_1 - \epsilon x_1^2 + \epsilon y_1^2 \left( 1 + (1 - 2\epsilon x_1)^2 \right) - 1 + \epsilon = 0.
\]

Let \( a_\epsilon > 0 \) be the solution of the equation \( G(0, a_\epsilon, 0) = 2\epsilon a_\epsilon^2 - 1 + \epsilon = 0 \). Calculations show that \( G_\epsilon(0, y_1, 0) < 0 \) for \( |y_1| < a_\epsilon \), \( G_\epsilon(1, y_1, 0) \geq 0 \) (with equality only at \( y_1 = 0 \)), and \( \frac{\partial G_\epsilon}{\partial x_1}(x_1, y_1, 0) > 0 \) for \( 0 \leq x_1 \leq 1 \). This shows that the set

\[
\sigma_\epsilon = \{ x_1 + iy_1 : 0 \leq x_1 \leq 1, \ G_\epsilon(x_1, y_1, 0) = 0 \}
\]

is a smooth real-analytic curve which can be written as a graph \( x_1 = g_\epsilon(y_1) \) over the interval \( |y_1| \leq a_\epsilon \), and the set

\[
D_\epsilon = \{ x_1 + iy_1 \in \mathbb{C} : 0 < x_1 < 1, \ G_\epsilon(x_1, y_1, 0) < 0 \}
\]

\( = \{ x_1 + iy_1 : 0 < x_1 < g_\epsilon(y_1), \ |y_1| < a_\epsilon \} \)

(with piecewise smooth boundary) is conformally equivalent to the disc. A calculation shows that for \( \epsilon > 0 \) sufficiently small we have \( \frac{\partial G_\epsilon}{\partial x_2}(x_1, y_1, x_2) > 0 \) on \( 0 \leq x_1 \leq 1 \) and \( y_1^2 \leq 1/\epsilon \), and \( G_\epsilon(x_1, y_1, 1) > 0 \) for \( x_1 + iy_1 \in D_\epsilon \). Since \( G_\epsilon(x_1, y_1, 0) < 0 \) for \( x_1 + iy_1 \in D_\epsilon \), it follows that (4.4) has a unique solution \( x_2 = \xi_\epsilon(x_1, y_1) \in [0, 1] \) for each \( z_1 = x_1 + iy_1 \in \overline{D_\epsilon} \) and it has no solutions for points in \( \{ 0 \leq x_1 \leq 1 \} \setminus \overline{D_\epsilon} \). From (4.3) we also calculate \( y_2 \) and thus obtain a unique analytic solution \( z_2 = f_\epsilon(z_1) \) \( (z_1 \in \overline{D_\epsilon}) \) of the system (4.2). This proves that \( \Gamma_\epsilon \) is an analytic disc with boundary in \( K_\epsilon \).

\[\top\hat\] We continue with the proof of proposition 4.2. For each \( y \in \mathbb{R}^2 \) and \( C \in \mathbb{R} \) we have

\[
\{ x \in \mathbb{R}^2 : \rho(x + iy) = C \} = \{ (x_1, C) : x_1 \leq C \} \cup \{ (C, x_2) : x_2 \leq C \}.
\]

For each point \( z = x + iy \in \mathbb{C}^2 \) with \( C_0 < \rho(z) < C_1 \) we can choose a line segment \( l_z \subset \mathbb{R}^2 + iy \) passing through \( z \) such that \( \rho > C_0 \) on \( l_z \) and the endpoints of \( l_z \) belong to \( \{ \rho = C_1 \} \). We can choose such \( l_z \) depending smoothly on \( z \) in the region \( C_0 < \rho(z) < C_1 \). The segment \( l_z \) together with the two bounded segments in the level set \( \rho = C_1 \) (in \( \mathbb{R}^2 + iy \)) determines a closed triangle \( T_z \subset \mathbb{R}^2 + iy \) which corresponds (after a rotation and dilation of coordinates) to the set \( co(k) \) in the model case. Lemma 4.3, applied to a slightly larger triangle \( \tilde{T}_z \supset T_z \) obtained by a small parallel translation of the segment \( l_z \) so as to include the point \( z \) in the interior of \( \tilde{T}_z \), gives an analytic disc \( \Gamma_z \subset \mathbb{C}^2 \) passing through \( z \) such that \( \rho > C_0 \) on \( \Gamma_z \) and \( \rho = C_1 \) on \( b\Gamma_z \). We can parametrize \( \Gamma_z \) by a map \( \lambda(z, \cdot) : \overline{U} \to \Gamma_z \), holomorphic in \( U \) and depending continuously on \( z \in K \).

\[\top\hat\] Combining proposition 4.2 an corollary 2.2 we obtain
4.4 Corollary. Let \( \rho \) be the function (4.1). Given a continuous map \( g_0: \overline{U} \to \mathbb{C}^2 \), holomorphic in \( U \), and constants \( 0 < r < 1 \), \( C_0, C_1 \in \mathbb{R} \) such that \( C_0 < \rho(g_0(\zeta)) < C_1 \) for \( r < |\zeta| \leq 1 \), there is for each \( \epsilon > 0 \) a holomorphic polynomial map \( g: \overline{U} \to \mathbb{C}^2 \) satisfying

(i) \( |\rho(g(\zeta)) - C_1| < \epsilon \) (\(|\zeta| = 1\)),
(ii) \( \rho(g(\zeta)) > C_0 \) (\( r \leq |\zeta| \leq 1 \)), and
(iii) \( |g(\zeta) - g_0(z)| < \epsilon \) (\(|\zeta| \leq r\)).

Proof of theorem 4.1. Choose a sequence \( \epsilon_k > 0 \), \( \sum_{k=1}^{\infty} \epsilon_k < 1 \). We begin by an arbitrary continuous map \( g_1: \overline{U} \to \mathbb{C}^2 \) that is holomorphic in \( U \) and a number \( 0 < r_1 < 1 \). Choose numbers \( M_0, M_1 \in \mathbb{R} \) such that \( M_0 < \rho(g_0(\zeta)) < M_1 \) for \( r_1 \leq |\zeta| \leq 1 \). Choose a number \( M_2 = M_1 + 1 \) and apply corollary 4.4 to get a polynomial map \( g_2: \mathbb{C}^2 \to \mathbb{C}^2 \) and a number \( r_2, r_1 < r_2 < 1 \), such that the following hold for \( k = 2 \):

(a) \( M_{k-1} < \rho(g_k(\zeta)) < M_k \) (\( r_k \leq |\zeta| \leq 1 \)),
(b) \( \rho(g_k(\zeta)) > M_{k-2} \) (\( r_{k-1} \leq |\zeta| \leq 1 \)), and
(c) \( |g_k(\zeta) - g_{k-1}(\zeta)| < \epsilon_{k-1} \) (\( |\zeta| \leq r_{k-1} \)).

This process can be continued inductively as follows. Suppose that we have already constructed \( g_{k-1} \) for some \( k \geq 2 \). Choose \( M_k \geq M_{k-1} + 1 \) and apply corollary 4.4 to get a map \( g_k \) which satisfies (a) for \( |\zeta| = 1 \) and it satisfies (b) and (c). By continuity we can choose \( r_k < 1 \) such that \( 1 - r_k < (1 - r_{k-1})/2 \) and such that (a) holds for \( r_k \leq |\zeta| \leq 1 \).

By construction we have \( \lim_{k \to \infty} r_k = 1 \), \( \lim_{k \to \infty} M_k = +\infty \), and \( g = \lim_{k \to \infty} g_k \) exists uniformly on compacts in \( U \) by (c). It remains to show that \( \rho(g(\zeta)) \to \infty \) as \( |\zeta| \to 1 \). Fix \( k \geq 2 \) and consider points in \( A_k = \{ \zeta : r_{k-1} \leq |\zeta| \leq r_k \} \). For \( l \geq k \) we have \( |g_{l+1}(\zeta) - g_l(\zeta)| < \epsilon_l \), so \( |g(\zeta) - g_k(\zeta)| \leq \sum_{l=k}^\infty |g_{l+1}(\zeta) - g_l(\zeta)| < \sum_{l=k}^\infty \epsilon_l < 1 \). From this and (b) we get \( \rho(g(\zeta)) > \rho(g_k(\zeta)) - 1 > M_{k-2} - 1 \) for \( \zeta \in A_k \). Since \( M_{k-2} \to \infty \) as \( k \to \infty \), the result follows.

Remark. One can give an alternative proof of theorem 1.4 as follows. One can construct a family of holomorphic maps \( F_p: \mathbb{C} \to \mathbb{C}^2 \), depending continuously on \( p \in (\mathbb{C}^*)^2 \), such that (i) \( F_p(0) = 0 \), (ii) \( |F_p(\zeta)| \geq |p| - \epsilon_p \) for all \( \zeta \in \mathbb{C} \) (where \( \epsilon_p > 0 \) can be made independent of \( p \) in any compact set \( K \subset (\mathbb{C}^*)^2 \), (iii) \( F_p(\mathbb{C}) \) misses \( zw = 0 \), and (iv) \( \lim_{|\zeta| \to \infty} |F_p(\zeta)| = +\infty \). The discs \( \zeta \to F(\zeta) \), \( |\zeta| \leq R \) with \( R \) large enough, can be taken as building blocks to construct proper holomorphic discs \( U \to \mathbb{C}^2 \) whose image avoids both coordinate axes (compare with proposition 4.2). Similar method can be used to construct proper holomorphic discs in \( \mathbb{C}^2 \) avoiding the curve \( zw = 1 \).

5. Boundary behavior of proper holomorphic discs.

In this section we prove theorem 1.5. We begin by recalling some classical results on boundary behavior of meromorphic functions on \( U = \{|\zeta| < 1\} \) (see e.g. [CL] and [Pri]). Let \( \overline{\mathbb{C}} = \mathbb{C} \cup \{0\} \) denote the Riemann sphere. For \( a \in \mathbb{C} \)
and \( r > 0 \) set \( D(a; r) = \{ \zeta \in \mathbb{C} : |\zeta - a| < r \} \). In what follows let \( f \) be a meromorphic function on \( U \). We denote by \( C(f, e^{i\theta}) \) its \textbf{unrestricted cluster set} at \( e^{i\theta} \in T \):

\[
C(f, e^{i\theta}) = \bigcap_{r>0} f(U \cap D(e^{i\theta}; r)).
\]

Equivalently, \( a \in \overline{T} \) belongs to \( C(f, e^{i\theta}) \) if and only if there exists a sequence \( \zeta_j \in U \) such that \( \lim_{j \to \infty} \zeta_j = e^{i\theta} \) and \( \lim_{j \to \infty} f(\zeta_j) = a \). If \( D \subset U \) is a subset with \( e^{i\theta} \in D \), we denote by \( C_D(f, e^{i\theta}) \) the \textbf{restricted cluster set} of \( f \) at \( e^{i\theta} \), defined as the set of limits of \( f \) along sequences \( \zeta_j \in D \) with \( \lim_{j \to \infty} \zeta_j = e^{i\theta} \).

A point \( e^{i\theta} \) for which \( C(f, e^{i\theta}) = \overline{T} \) is called a \textbf{Weierstrass point} of \( f \), and the set of all such points is the \textbf{Weierstrass set} \( W(f) \) [CL, p. 149].

For each \( e^{i\theta} \in T \) and \( 0 < \alpha < 1 \) we set

\[
\Gamma_\alpha(e^{i\theta}) = \{ \zeta \in U : |\Im(1 - \zeta e^{-i\theta})| < \alpha |\zeta - e^{i\theta}| \}.
\]

This is an angle in \( U \) with vertex at \( e^{i\theta} \) and opening \( 2 \arcsin \alpha \), bisected by the radius that terminates at \( e^{i\theta} \). If the limit

\[
f^*(e^{i\theta}) = \lim_{\Gamma_\alpha(e^{i\theta}) \ni \zeta \to e^{i\theta}} f(\zeta) \in \overline{T}
\]

exists and is independent of \( \alpha \), it is called the \textbf{nontangential limit of} \( f \) \textbf{at} \( e^{i\theta} \) and \( e^{i\theta} \) is called a \textbf{Fatou point} of \( f \). The set of all Fatou points the \textbf{Fatou set} \( F(f) \) [CL, p. 21].

A point \( e^{i\theta} \in T \) is called a \textbf{Plessner point} of \( f \) if for every angle \( \Gamma \) with vertex \( e^{i\theta} \) the partial cluster set \( C_\Gamma(f, e^{i\theta}) \) equals \( \overline{T} \) (i.e., it is total). The set of all Plessner points is the \textbf{Plessner set} \( I(f) \) [CL, p. 147]. Clearly \( I(f) \subset W(f) \).

The \textbf{Nevanlinna characteristic} of a holomorphic function \( f \) on \( U \) is defined by

\[
T(r, f) = \int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{dt}{2\pi} \quad (0 \leq r < 1);
\]

for meromorphic functions see [CL, p. 39] or [Nev].

The \textbf{range} of \( f \), denoted \( R(f) \), is the set of all \( \alpha \in \overline{T} \) for which there exists a sequence \( \zeta_j \in U \) with \( \lim_{j \to \infty} \zeta_j = 1 \) such that \( f(\zeta_j) = \alpha \) for all \( j \in \mathbb{N} \). By restricting the attention only to sequences \( z_j \in U \) with \( \lim_{j \to \infty} \zeta_j = e^{i\theta} \) we get the range of \( f \) at \( e^{i\theta} \), denoted \( R(f, e^{i\theta}) \).

The notion of \textbf{logarithmic capacity} of a Borel set \( E \subset \mathbb{C} \) can be found in [CL, p. 9]. Such a set is of capacity zero if and only if it is polar, i.e., it is contained in the \(-\infty \) level set of a non-constant subharmonic function on \( \mathbb{C} \) [Lan, Tsu]. The following summarizes some of the known results which we shall need in the proof of theorem 1.5.

\textbf{5.1 Theorem}: Let \( f \) be a meromorphic function on the disc \( U \).

(a) If \( e^{i\theta} \in T \) is not a Weierstrass point of \( f \) then there is an open arc \( \gamma \subset T \) containing \( e^{i\theta} \) such that almost every point in \( \gamma \) is a Fatou point of \( f \).
(b) If \( f \) has bounded Nevanlinna characteristic on \( U \) then almost every point of \( T \) is a Fatou point of \( f \).

(c) Almost every point in \( T \) belongs to \( F(f) \cup I(f) \).

(d) If \( f(U) \subset \overline{\mathbb{U}} \setminus E \) for some set \( E \) of positive capacity then \( f \) has bounded Nevanlinna characteristic.

(e) If \( R(f, e^{i\theta}) \) omits a set \( E \subset \overline{\mathbb{U}} \) of positive capacity then there is an open arc \( \gamma \subset T \) containing \( e^{i\theta} \) such that almost every point of \( \gamma \) is a Fatou point of \( f \).

Proof. (a) If \( e^{i\theta} \) is not a Weierstrass point of \( f \), there is a disc \( D(e^{i\theta}; r) \) such that \( f(D(e^{i\theta}; r) \cap U) \) omits a disc \( D(a; \delta) \subset \mathbb{C} \). The function \( g(\zeta) = 1/(f(\zeta) - a) \) is then bounded holomorphic in \( D(e^{i\theta}; r) \cap U \) and hence by Fatou’s theorem it has nontangential limit \( g^*(e^{it}) \) at almost every point \( e^{it} \in \gamma = T \cap D(e^{i\theta}; r) \) [CL, p. 21]. The same is then true for \( f \) and hence almost every point of \( \gamma \) belongs to the Fatou set \( F(f) \). (See also [CL, Theorem 8.4].) Part (b) follows by combining Fatou’s theorem with a theorem of R. Nevanlinna to the effect that a meromorphic function with bounded Nevanlinna characteristic on \( U \) is the quotient of two bounded holomorphic functions [CL, p. 41]. Part (c) is a classical theorem due to Plessner ([Ple], [Pri, p. 217] or [CL, p. 147]).

Part (d) is due to Frostman [Fro]; the following simple proof was shown to us by D. Marshall. After a fractional linear transformation we may assume that \( \infty \in E \subset \{|z| > 1\} \). Let \( g_0(z) \) be the Green’s function for \( \overline{\mathbb{C}} \setminus E \) with a logarithmic pole at 0 (so \( g_0(z) \to 0 \) as \( z \to E \)). Then \( \log^+ \frac{1}{|z|} \leq g_0(z) \). The function \( u(z) := g_0(z) + \log |z| \) is harmonic on \( \overline{\mathbb{C}} \setminus E \) since both summands are harmonic on \( \overline{\mathbb{C}} \setminus (E \cup \{0\}) \) and the pole at 0 cancels off. If \( f: U \to \overline{\mathbb{C}} \setminus E \) is a holomorphic function then \( u \circ f \) is harmonic on \( U \) and we have

\[
\int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi} \leq \int_0^{2\pi} \left( \log^+ \frac{1}{|f(re^{i\theta})|} + \log |f(re^{i\theta})| \right) \frac{d\theta}{2\pi} \\
\leq \int_0^{2\pi} \left( g_0(f(re^{i\theta})) + \log |f(re^{i\theta})| \right) \frac{d\theta}{2\pi} \\
= \int_0^{2\pi} u(f(re^{i\theta})) \frac{d\theta}{2\pi} = u(f(0)).
\]

Thus \( \int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{dt}{2\pi} \leq u(f(0)) \) for \( r \in (0, 1) \) which proves (d).

For (e) observe that \( R(f, e^{i\theta}) = \bigcap_{n \in \mathbb{N}} f(D_n) \), where \( D_n = D(e^{i\theta}; 1/n) \cap U \). The sets \( f(D_n) \) are decreasing with \( n \). If \( R(f, e^{i\theta}) \) omits a set \( E \) of positive capacity then \( f(D_n) \) omits a set \( E' \) of positive capacity for some sufficiently large \( n \in \mathbb{N} \). We may assume that \( \infty \in E' \). Observe that \( D_n \) is conformally equivalent to the disc. From (d) and (b) applied to the holomorphic function \( f: D_n \to \overline{\mathbb{C}} \setminus E' \) it follows that almost every point of the arc \( \gamma = D_n \cap T \) is a Fatou point of \( f \).

We shall frequently use the following uniqueness theorem due to Plessner ([Ple], [CL, p. 146]) and to Lusin and Priwalow [Pri, p. 212].

\[ \square \]
5.2 Theorem. If a meromorphic function \( f \) on \( U \) has an angular limit equal to zero at each point in a set \( E \subset T \) of positive Lebesgue measure then \( f \) is the zero function.

Remark. In theorem 5.2 we cannot replace angular limits with radial limits, see examples due to Lusin and Priwalow in [Pri], sec. IV.5. Here we use the term ‘angular limit’ rather than ‘nontangential limit’ since the latter usually means the existence of the limit within every angle with the given vertex.

Proof of theorem 1.5. Let \((f_1, f_2): U \to \mathbb{C}^2\) be a proper holomorphic map and let \( g \) be any of the functions as in theorem 1.5. It suffices to show that the Fatou set \( F(g) \) has measure zero. From theorem 5.1 (a) it will then follow that \( W(g) = T \), theorem 5.1 (c) will imply that the Plessner set \( I(g) \) has full measure in \( T \), and theorem 5.1 (e) will imply that the complement of the range \( R(g, e^{i\theta}) \) in \( \mathbb{C} \) has capacity zero for each \( e^{i\theta} \in T \). Since sets of capacity zero in \( \mathbb{C} \) coincide with polar sets ([Tsu], [Lan]), theorem 1.5 (d) follows.

To prove that \( F(g) \) has measure zero we consider separately each case.

Case (i). Suppose that \( f_1 \) has an angular limit \( f_1^*(e^{i\theta}) \in \overline{T} \) (5.1) at all points \( e^{i\theta} \) in a set \( A \subset T \). Then \( A \) is Lebesgue measurable and can be written as \( A = A_1 \cup A_2 \), where \( A_1 \) is the set of all \( e^{i\theta} \in A \) such that \( f_1^*(e^{i\theta}) \notin \mathbb{C} \) and \( A_2 \) is the set of all \( e^{i\theta} \in A \) with \( f_1^*(e^{i\theta}) = \infty \). Then \( 1/f_1 \) has angular limits zero at each point of \( A_2 \). If \( A_2 \) is of positive measure, theorem 5.2 implies that \( 1/f_1 \) is identically zero in \( U \), a contradiction. Thus \( A_2 \) has measure zero. Consider now \( A_1 \). Since \((f_1, f_2): U \to \mathbb{C}^2\) is proper, \( \max\{|f_1(\zeta)|, |f_2(\zeta)|\} \) tends to \( +\infty \) as \( |\zeta| \to 1 \). Since \( f_1 \) has a finite angular limit at each \( e^{i\theta} \in A_1 \), \( |f_2| \) has an angular limit \( \infty \) at each point of \( A_1 \). If \( A_1 \) is of positive measure, Plessner’s theorem, applied to \( 1/f_2 \), gives a contradiction as before. This shows that \( A_1 \) is of measure zero as well, and therefore the Fatou set of \( f_1 \) is of measure zero. The same applies to \( f_2 \).

Case (ii). Suppose that \( g = f_1/f_2 \) has an angular limit \( g^*(e^{i\theta}) \in \overline{T} \) (5.1) within an angle \( \Gamma_0 \) at each point \( e^{i\theta} \) in a set \( A \subset T \). As in part (i) we write \( A = A_1 \cup A_2 \), where \( g^* \) is finite on \( A_1 \) and equals \( \infty \) on \( A_2 \). Theorem 5.2 shows as above that \( A_2 \) must be of measure zero for otherwise \( g \) would be constant. If \( A_1 \) is of positive measure, there is a set \( A_0 \subset A_1 \) of positive measure and a number \( 0 < M < \infty \) such that \( |g^*(e^{i\theta})| < M \) for each \( e^{i\theta} \in A_0 \). Hence there is a disc \( U_\theta \) centered at \( e^{i\theta} \) such that \( |f_1(\zeta)/f_2(\zeta)| \leq M \) for \( \zeta \in \Gamma_0 \cap U_\theta \). Hence \( |f_1(\zeta)| \leq M|f_2(\zeta)| \) and therefore

\[
\max\{|f_1(\zeta)|, |f_2(\zeta)|\} \leq \max\{M|f_2(\zeta)|, |f_2(\zeta)|\} \quad (\zeta \in \Gamma_0 \cap U_\theta).
\]

Since this maximum tends to \( +\infty \) as \( \zeta \to e^{i\theta} \), it follows that \( |f_2(\zeta)| \to \infty \) as \( \zeta \to e^{i\theta} \) within \( \Gamma_0 \). Thus \( 1/f_2 \) has angular limits zero at each point of \( A_0 \), a contradiction to theorem 5.2. This proves that \( A_1 \) must be of measure zero as well.
Case (iii). This follows from case (i) by observing that for each nonconstant holomorphic polynomial $P$ on $\mathbb{C}^2$ there exists another holomorphic polynomial $Q$ such that $(P, Q): \mathbb{C}^2 \to \mathbb{C}^2$ is a proper map, and hence $P(f_1, f_2): U \to \mathbb{C}^2$ is the first component of a proper map $U \to \mathbb{C}^2$. In fact, we have

**5.3 Lemma.** Let $P$ and $Q$ be nonconstant holomorphic polynomials on $\mathbb{C}^2$ whose leading order homogeneous parts $P'$ resp. $Q'$ have no common zero on $\mathbb{C}^2 \backslash \{0\}$. Then $(P, Q): \mathbb{C}^2 \to \mathbb{C}^2$ is a proper map.

We leave out the simple proof. Observe that the zero set of $P'$ is a finite union of complex lines, so it suffices to choose $Q$ to be a linear function which does not vanish on $P' = 0$ except at the origin; the pair $(P, Q)$ then provides a proper self-map of $\mathbb{C}^2$.

Case (iv). Apply (i) and lemma 5.3 to the map $(P(f_1, f_2), Q(f_1, f_2)): U \to \mathbb{C}^2$.

**Appendix: Crossing a critical level by analytic discs.**

Let $X$ be a Stein manifold of dimension at least two and $\rho: X \to \mathbb{R}$ a strongly plurisubharmonic Morse exhaustion function. Let $p \in X$ be a critical point of $\rho$. Choose constants $c_0, c_1$ such that $c_0 < \rho(p) < c_1$ and $p$ is the only critical point of $\rho$ in $A(c_0, c_1) = \{ x \in X : c_0 \leq \rho(x) \leq c_1 \}$. Suppose that $f_0: U \to X$ is a holomorphic map such that $c_0 < \rho(f_0(e^{i\theta})) < \rho(p)$ for each $e^{i\theta} \in T$. In [Glo] the second author showed how to construct a smooth map $f_1: U \to X$ which is close to being holomorphic on $U$ such that $\rho(p) < \rho(f_1(e^{i\theta})) < c_1$ for $e^{i\theta} \in T$ and such that $f_1$ approximates $f_0$ on a smaller disc $|\zeta| \leq r < 1$. The map $f_1$ is obtained by adding to $f_0$ a small non-holomorphic contribution which can be controlled by the data. Once the boundary curve $f_1(T)$ passes the critical level of $\rho$ at $p$ we can use the procedure described in sect. 2 above (or in [FG]) to continue pushing it higher towards the next critical level of $\rho$. It was shown in [Glo] that the non-holomorphic contribution made at the initial step may be cancelled off during a later stage of the construction, once the boundary of the disc is sufficiently far above the critical level at $p$. The reason is that the modification process is a linear one, and we obtain the final solution as the sum of a convergent series. (Here it is convenient to embed $X$ into a Euclidean space $\mathbb{R}^N$.)

Here we wish to point out that the transition from $f_0$ to $f_1$ as above can also be accomplished by applying to $f_0$ the gradient flow $\theta_t$ of $\rho$ (in the direction of increasing $\rho$). Unless a point $x \in A(c_0, c_1)$ belongs to the stable manifold $W^s(p)$ of $p$ (see e.g. Shub [Sh]), we have $\rho(\theta_t(x)) > \rho(p)$ for sufficiently large $t > 0$. Thus, if $f_0(T) \cap W^s(p) = \emptyset$, we can choose a smooth positive function $a$ on $\overline{U}$ such that the map $f_1(\zeta) = \theta_{at(\zeta)}(f_0(\zeta))$ ($|\zeta| \leq 1$) satisfies $\rho(f_1(e^{i\theta})) > \rho(p)$ for $e^{i\theta} \in T$.

If the number $c_0$ is sufficiently close to $\rho(p)$ as we may assume to be the case, the set $W^s(p) \cap A(c_0, c_1)$ is a closed real submanifold of $A(c_0, c_1)$ whose dimension equals the index $i(p)$ (the number of negative eigenvalues of the Hessian) of $\rho$ at $p$. Since $\rho$ is strongly plurisubharmonic, we have $i(p) \leq \dim \mathbb{R}X$ (see [AF]) and therefore $\dim f_0(T) + \dim W^s(p) \leq 1 + \dim \mathbb{R}X < \dim \mathbb{R}X$. 

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By transversality a generic small holomorphic perturbation of $f_0$ satisfy the required condition $f_0(T) \cap W^*(p) = \emptyset$ which makes it possible to obtain $f_1$ as above. The rest of the procedure remains as in [Glo].

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