Simple Construction of Elliptic Boundary $K$-matrix

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ABSTRACT

We give the infinite-dimensional representation for the elliptic $K$-operator satisfying the boundary Yang-Baxter equation. By restricting the functional space to finite-dimensional space, we construct the elliptic $K$-matrix associated to Belavin’s completely $Z$-symmetric $R$-matrix.

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The quantum $R$-matrix as solutions of the Yang-Baxter equation (YBE) has received much attention in mathematical physics. The algebraic structure reveals as quantum group. Recently the $R$-matrix has been treated as operator acting on functional space. In one sense this gives the infinite-dimensional representation for solutions of YBE. By use of operator description for $R$-matrix, the construction of $R$-matrix, especially for elliptic case, becomes much simpler. Based on the elliptic $R$-operator defined by Shibukawa and Ueno [1], Felder and Pasquier constructed Belavin’s elliptic $R$-matrix [2].

The $R$-matrix has been used to study spin chains with periodic boundary condition in terms of the quantum inverse scattering method. Besides the $R$-matrix, the other matrix called $K$-matrix is used to solve the spin chain with open boundary [3, 4]. In this letter we propose a method to construct the boundary $K$-matrix associated with Belavin’s $R$-matrix.

Throughout this paper we use the doubly periodic function $\sigma_\mu(z) \equiv \sigma_\mu(z, \tau)$,

\[
\begin{align*}
\sigma_\mu(z + 1) &= \sigma_\mu(z), \\
\sigma_\mu(z + \tau) &= e^{2\pi i \mu} \sigma_\mu(z),
\end{align*}
\]

where $\tau$ is an arbitrary complex number, satisfying $\text{Im} \tau > 0$. The function $\sigma_\mu(z)$ only has simple poles on the lattice $\mathbb{Z} + \tau \mathbb{Z}$, and the residue at origin equals to one. Note that the function $\sigma_\mu(z)$ can be explicitly written as

\[
\sigma_\mu(z) = \frac{\vartheta_1(z - \mu, \tau) \vartheta'_1(0, \tau)}{\vartheta_1(z, \tau) \vartheta_1(-\mu, \tau)},
\]

where $\vartheta_1(z, \tau)$ is the Jacobi’s theta function,

\[
\vartheta_1(z, \tau) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} \exp \left( i \pi n^2 \tau + 2\pi i n \left( z + \frac{1}{2} \right) \right).
\]

For the elliptic function $\sigma_\mu(z)$, we have following lemma;

**Lemma 1.** The elliptic function $\sigma_\mu(z)$ defined in [1] satisfies the following identities;

(a) $\sigma_\mu(z) = -\sigma_z(\mu)$ ,

(b) $\sigma_\mu(z) = -\sigma_{-\mu}(-z)$,
\(\sigma_{\mu}(z)\sigma_{-\mu}(z) = \wp(z) - \wp(\mu),\)

\(\sigma_{\lambda}(z)\sigma_{\mu}(w) - \sigma_{\lambda+\mu}(w)\sigma_{\lambda}(z-w) - \sigma_{\mu}(w-z)\sigma_{\lambda+\mu}(z) = 0,\)

\(\sigma_{\lambda}(z)\sigma_{\mu}(z) = \sigma_{\lambda+\mu}(z) \cdot \left( \zeta(z) - \zeta(\lambda) - \zeta(\mu) - \zeta(z-\lambda-\mu) \right).\)

The proof can be seen in, for example, Ref. [3]. Another property for the function \(\sigma_{\mu}(z)\) is as follows;

**Lemma 2 ([2]).**

\[\sigma_{\mu}(z, \tau) = \frac{1}{k} \sum_{a=0}^{k-1} \sigma_{\mu+a}(z, \tau/k),\]  
\[(3)\]

This identity is easy to be proved when one compares periodicity and residues of both hand sides; one can check that the both hand sides have simple poles on \(\mathbb{Z} + \tau \mathbb{Z}\). This lemma becomes useful when we construct Belavin’s completely \(\mathbb{Z}\)-symmetric \(R\)-matrix [6].

In terms of the elliptic function \(\sigma_{\mu}(z)\), Shibukawa and Ueno introduced the “infinite-dimensional” representation for \(R\)-operator as a solution of YBE.

**Theorem 3 ([1]).** Let \(R\)-operator acts on the space of functions of two variables,

\[R(\xi) f(z_1, z_2) = \sigma_{\mu}(z_{12}) f(z_1, z_2) - \sigma_{\xi}(z_{12}) f(z_2, z_1).\]  
\[(4)\]

This \(R\)-operator satisfies YBE,

\[R_{12}(\xi_{12})R_{13}(\xi_{13})R_{23}(\xi_{23}) = R_{23}(\xi_{23})R_{13}(\xi_{13})R_{12}(\xi_{12}).\]  
\[(5)\]

Here and hereafter we use notations, \(z_{12} \equiv z_1 - z_2\), etc. The proof follows by using Lemma [2]. Remark that we can define rational and trigonometric \(R\)-operators as degenerate cases of elliptic operator [4];

\[\sigma_{\mu}(z) \rightarrow \begin{cases} \cot z - \cot \mu, & \text{trigonometric}, \\ \frac{1}{z} - \frac{1}{\mu}, & \text{rational}. \end{cases}\]  
\[(6)\]
These degenerate types of infinite-dimensional representations for $R$-operator were also studied in Ref. [7].

To introduce the generalized Shibukawa-Ueno’s $R$-operator, Felder and Pasquier introduced the gauge-transformation. They defined the translation operator in functional space as

$$T_k(\xi)f(z) = f(z - \frac{\xi}{k}).$$  \hfill (7)

For the translation operator $T_k(\xi)$, we have following identities;

**Lemma 4.**  \hfill (a) $T_k(\xi + \eta) = T_k(\xi)T_k(\eta),$

\hfill (b) $[R(\xi), T_k(\eta) \otimes T_k(\eta)] = 0.$

The first identity is trivial. The second one is due to the fact that $R$-operator depends only on the difference of two spectral parameters.

By use of the translation operator $T_k$, we can introduce the ‘modified’ $R$-operator as a solution of YBE.

**Theorem 5 ([2]).** Let the modified $R$-operator be

$$R_k(\xi_{12}) = \left( T_k(\xi_1 - \mu)^{-1} \otimes T_k(\xi_2)^{-1} \right) \cdot R(\xi_{12}) \cdot \left( T_k(\xi_1) \otimes T_k(\xi_2 - \mu) \right).$$  \hfill (8)

The operator $R_k(\xi)$ also satisfies YBE.

Remark that the action of the modified $R$-operator $R_k(\xi)$ on functional space is explicitly written as

$$R_k(\xi)f(z_1, z_2) = \sigma_{\mu}(z_{12} + \frac{\mu + \xi}{k})f(z_1 + \frac{\mu}{k}, z_2 - \frac{\mu}{k})$$

$$- \sigma_{\xi}(z_{12} + \frac{\mu + \xi}{k})f(z_2 - \frac{\xi}{k}, z_1 + \frac{\xi}{k}).$$  \hfill (9)

This modified $R$-operator $R_k(\xi)$ plays a crucial role in defining Belavin’s completely $\mathbb{Z}$-symmetric $R$-matrix.
On the other hand, we shall introduce the elliptic boundary $K$-operator. The $K$-matrix was originally introduced to solve the spin system with open boundary based on the quantum inverse scattering method; algebraically $K$-matrix satisfies the so-called boundary YBE (reflection equation) \cite{3,4}. Here we regard the $K$-matrix as an operator acting on the functional space.

**Theorem 6.** Let boundary $K$-operator act on the space of functions of single variable,

\begin{align}
K^I(\xi) f(z) &= \sigma_{2\xi}(z) f(z) - \sigma_{2\nu}(z) f(-z), \quad (10a) \\
K^{II}(\xi) f(z) &= \sigma_{\xi}(2z) f(z) - \sigma_{\nu}(2z) f(-z). \quad (10b)
\end{align}

These $K$-operators satisfy the boundary YBE,

\begin{equation}
R_{21}(\xi_1 - \xi_2) \left( K(\xi_1) \otimes 1 \right) R_{12}(\xi_1 + \xi_2) \left( 1 \otimes K(\xi_2) \right) = \left( 1 \otimes K(\xi_2) \right) R_{21}(\xi_1 + \xi_2) \left( K(\xi_1) \otimes 1 \right) R_{12}(\xi_1 - \xi_2), \quad (11)
\end{equation}

where $R$-operator is defined in Theorem 3.

This theorem can be proved as follows. We suppose the action of the $K$-operator as

\[ K(\xi) f(z) = G(\xi, z) f(z) - H(z) f(-z). \]

Substituting the definition of $R$- and $K$-operators into the boundary YBE (11), we obtain three functional equations [3];

\begin{align}
G(\xi_2, z_2) \sigma_{\xi_1+\xi_2}(z_1 - z_2) \sigma_{\xi_1-\xi_2}(z_1 + z_2) + G(\xi_1, z_1) \sigma_\mu(z_1 + z_2) \sigma_\mu(-z_1 - z_2) \\
+ G(\xi_1, -z_2) \sigma_{\xi_1+\xi_2}(z_1 + z_2) \sigma_{\xi_1-\xi_2}(z_1 + z_2) \\
= G(\xi_1, z_1) \sigma_\mu(z_1 - z_2) \sigma_\mu(z_2 - z_1) + G(\xi_1, z_2) \sigma_{\xi_1-\xi_2}(z_1 - z_2) \sigma_{\xi_1+\xi_2}(z_1 + z_2) \\
+ G(\xi_2, -z_2) \sigma_{\xi_1-\xi_2}(z_1 - z_2) \sigma_{\xi_1+\xi_2}(z_1 + z_2), \quad (12a)
\end{align}

\begin{align}
G(\xi_1, z_1) G(\xi_2, z_2) \sigma_{\xi_1-\xi_2}(z_2 - z_1) + G(\xi_1, z_2) G(\xi_2, z_2) \sigma_{\xi_1+\xi_2}(z_1 - z_2) \\
+ H(-z_2) H(z_2) \sigma_{\xi_1+\xi_2}(z_1 + z_2) \\
= G(\xi_1, z_1) G(\xi_2, z_1) \sigma_{\xi_1+\xi_2}(z_2 - z_1) + G(\xi_1, z_2) G(\xi_2, z_1) \sigma_{\xi_1-\xi_2}(z_1 - z_2) \\
+ H(z_1) H(-z_1) \sigma_{\xi_1+\xi_2}(z_1 + z_2), \quad (12b)
\end{align}
\begin{align}
G(\xi_1, z_1) \sigma_{\xi_1 + \xi_2}(z_2 - z_1) &+ G(\xi_1, z_2) \sigma_{\xi_1 - \xi_2}(z_1 - z_2) + G(\xi_2, -z_1) \sigma_{\xi_1 + \xi_2}(z_1 + z_2) \\
= G(\xi_2, z_2) \sigma_{\xi_1 - \xi_2}(z_1 + z_2). \tag{12c}
\end{align}

One can conclude by comparing the periodicity and residues of the both hand sides that functions

I. \( G(\xi, z) = \sigma_{2\xi}(z), \quad H(z) = \sigma_{2\nu}(z), \)

II. \( G(\xi, z) = \sigma_{\xi}(2z), \quad H(z) = \sigma_{\nu}(2z), \)

solve these functional equations. In this calculation, we have used Lemma 1.

We also introduce the modified \( K \)-operator associated with modified \( R \)-operator as a solution of the boundary YBE (11).

\textbf{Theorem 7.} Let the modified \( K \)-operators be

\[ K^{I,H}_k(\xi) = T_k(\xi + \nu) K^{I,H}(\xi) T_k(\xi - \nu). \] \( (13) \)

The modified operators \( K^{I,H}_k(\xi) \) and \( R_k(\xi) \) also satisfy the boundary YBE (11).

This Theorem follows with help of Lemma 4. Note that explicit forms of action of the modified \( K \)-operator can be written as

\[ K^{I}_k(\xi) f(z) = \sigma_{2\xi}(z + \frac{\xi + \nu}{k}) f(z + \frac{2\xi}{k}) - \sigma_{2\nu}(z + \frac{\xi + \nu}{k}) f(-z - \frac{2\nu}{k}), \] \( (14a) \)

\[ K^{II}_k(\xi) f(z) = \sigma_{\xi}(2z + \frac{2\xi + 2\nu}{k}) f(z + \frac{2\xi}{k}) - \sigma_{\nu}(2z + \frac{2\xi + 2\nu}{k}) f(-z - \frac{2\nu}{k}). \] \( (14b) \)

Now by use of the modified operators, \( R_k(\xi) \) and \( K^{I,H}_k(\xi) \), we shall construct Belavin’s completely \( \mathbb{Z} \)-symmetric \( R \)-matrix [6, 9, 10, 11, 12] and its \( K \)-matrix [13, 14, 15]. This can be done by restricting functional space into finite-dimensional space [2]; define \( V_k \) as the space of entire functions such that

\[ f(z + 1) = f(z), \]
\[ f(z + \tau) = e^{-2\pi ikz - \pi ik\tau} f(z). \]

The space \( V_k \) has dimension \( k \). As bases of \( k \)-dimensional functional space \( V_k \), we use the \( \theta \)-function for \( a \in \mathbb{Z}_k \equiv \mathbb{Z}/k\mathbb{Z} \) defined by

\[ \theta_a(z) = \sum_{n \in \mathbb{Z}} \exp\left( \pi in^2 \frac{\tau}{k} + 2\pi in\left( z - \frac{a}{k} \right) \right). \] \( (15) \)
We note that the functions $\theta_a(z)$ have properties,

$$\hat{S} \theta_a(z) = \theta_{a-1}(z), \quad \hat{T} \theta_a(z) = e^{2\pi i a/k} \theta_a(z),$$

where operators $\hat{S}$ and $\hat{T}$ act on the functional space as follows;

$$\hat{S} f(z) = f(z + \frac{1}{k}), \quad \hat{T} f(z) = e^{2\pi i z + \pi i \tau/k} f(z + \frac{\tau}{k}).$$

When one checks periodicity of the modified $R_k(\xi)$ and $K_k(\xi)$ operators in (9) and (14) respectively, we shall have the following proposition:

**Proposition 8.**

(a) Modified operator $R_k(\xi)$ preserves $V_k \otimes V_k$.

(b) Modified operator $K_k^{I,H}(\xi)$ preserves $V_k$.

This proposition is easy to be proved by direct calculations.

Based on above proposition, we can restrict the functional space, on which modified $R$- and $K$-operators act, to $V_k$. In this finite-dimensional functional space, we can recover Belavin’s $\mathbb{Z}$-symmetric solution from the modified $R_k$-operator.

**Theorem 9 (2).** Define matrix elements of modified $R$-operator $R_k(\xi)$ by

$$R_k(\xi) \theta_a \otimes \theta_b = \sum_{c,d \in \mathbb{Z}_k} R_k(\xi)_{ac, bd} \theta_c \otimes \theta_d.$$  \hspace{1cm} (16)

Then we get,

$$R_k(\xi)_{ac, bd} = \delta_{a+b,c+d} \frac{\vartheta_1\left(\frac{\mu - \xi - a + b}{k}, \frac{\tau}{k}\right) \vartheta_1\left(0, \frac{\tau}{k}\right)}{k \vartheta_1\left(\frac{-a + c}{k}, \frac{\tau}{k}\right) \vartheta_1\left(\frac{\xi - b + c}{k}, \frac{\tau}{k}\right)}.$$  \hspace{1cm} (17)

The proof should be done by use of Lemma 2. The key is to rewrite $R_k(\xi) \theta_a(z_1) \otimes \theta_b(z_2)$ as

$$\frac{1}{k} \sum_{c \in \mathbb{Z}_k} \left\{ \sigma_{\frac{\mu + \xi - a}{k}}(z_{12} + \frac{\mu}{k}, \frac{\tau}{k}) \theta_a(z_1 + \frac{\mu}{k}) \theta_b(z_2 - \frac{\mu}{k}) - \sigma_{\frac{\xi - b + c}{k}}(z_{12} + \frac{\mu + \xi}{k}) \theta_a(z_2 - \frac{\xi}{k}) \theta_b(z_1 + \frac{\xi}{k}) \right\}.$$
Each summand is an eigenstate of $\hat{T} \otimes 1$ with eigenvalue $e^{2n\pi i c/k}$, which proves that the summand is proportional to $\theta_c \otimes \theta_d$ with $d = a + b - c$. By setting $z_{12} = (-\xi + c - a)/k$ and using the fact, $\theta_a(z_2 - \frac{\xi}{k}) \theta_b(z_1 + \frac{\xi}{k}) = \theta_c(z_1) \theta_d(z_2)$, we can obtain above expression. For $k = 2$ it reduces to $R$-matrix of the Baxter’s eight vertex model (XYZ spin chain).

In the same manner, the restriction of modified operator $K_k(\xi)$ into $V_k$ gives the boundary $K$-matrix associated with Belavin’s $R$-matrix (17).

**Theorem 10.** Define matrix elements of the modified $K^I$-operator by

$$K_k^I(\xi) \theta_a(z) = \sum_{c \in \mathbb{Z}_k} K_k^I(\xi)_{a,c} \theta_c(z).$$

Then we get,

$$K_k^I(\xi)_{a,c} = \frac{\vartheta_1(\frac{2\nu + 2a - 2\xi}{k}, \frac{\tau}{k}) \vartheta_1'(0, \frac{\tau}{k})}{k \vartheta_1(\frac{2\nu + c + a}{k}, \frac{\tau}{k}) \vartheta_1(-\frac{2\xi + c + a}{k}, \frac{\tau}{k})} \frac{\theta_c(-\nu - \xi)}{\theta_a(-\nu - \xi)}.$$  

(19)

Outline of proof is essentially same with the case of $R$-matrix. Based on Lemma 2, we can rewrite $K_k^I(\xi) \theta_a(z)$ as

$$\frac{1}{k} \sum_{c \in \mathbb{Z}_k} \left\{ \sigma_{2\nu+c+a}(z + \frac{\xi}{k}, \frac{\tau}{k}) \theta_a(z + \frac{2\xi}{k}) - \sigma_{2\nu+c+a}(z + \frac{\xi}{k}, \frac{\tau}{k}) \theta_a(-z - \frac{2\nu}{k}) \right\}.$$  

Each summand is an eigenstate of $\hat{T}$ with eigenvalue $e^{2n\pi i c/k}$, which shows that summand is proportional to $\theta_c(z)$ for arbitrary $z$. The prefactor can be calculated by setting $z = (-\xi + \nu + c + a)/k$.

The elliptic matrix $K_k^I(\xi)$ coincides with result of Ref. [15] (for trigonometric case, Ref. [16]). For $k = 2$ case, it gives one of the boundary $K$-matrix for the Baxter’s eight vertex model studied in Ref. [13].

For the second-type $K_k^{II}(\xi)$ operator, the matrix elements can be calculated for odd-$k$ case.

**Theorem 11.** Define matrix elements of the modified $K^{II}$-operator $K_k^{II}(\xi)$ by

$$K_k^{II}(\xi) \theta_{2a}(z) = \sum_{c \in \mathbb{Z}_k} K_k^{II}(\xi)_{2a,2c} \theta_{2c}(z).$$

(20)
To choose $\theta_{2a}$ as bases of $V_k$, we must assume that dimension $k$ is odd. In this case, we obtain that the matrix elements have the form,

$$K_{k}^{II}(\xi)_{2a,2c} = \frac{\theta_1\left(\frac{\nu-\xi+2a}{k}, \frac{\tau}{k}\right)}{k \theta_1\left(\frac{\nu+c+a}{2k}, \frac{\tau}{k}\right) \theta_1\left(-\frac{\xi-c+a}{k}, \frac{\tau}{k}\right)} \cdot \frac{\theta_2\left(\frac{2\xi-\nu+c+a}{2k}\right)}{\theta_2\left(-\frac{2\xi-\nu+c+a}{2k}\right)}.$$  (21)

The key of proof is to rewrite $K_{k}^{II}(\xi)\theta_{2a}(z)$ as

$$\frac{1}{k} \sum_{c \in \mathbb{Z}_k} \left\{ \sigma_{\xi+c-a} \left(2z + \frac{2\xi + 2\nu}{k}\right) \theta_{2a}(z + \frac{2\xi}{k}) - \sigma_{\xi+c+a} \left(2z + \frac{2\xi + 2\nu}{k}\right) \theta_{2a}(-z - \frac{2\nu}{k}) \right\}.$$  

In this case, the each summand becomes an eigenstate of $\hat{T}$ with eigenvalue $e^{2\pi i 2c/k}$. By substituting $z = (-2\xi - \nu + c + a)/2k$, one obtains matrix elements. Remark that the difficulty for even-$k$ is due to Lemma 3.

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