The Bulk-Edge Correspondence for Disordered Chiral Chains

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Abstract

We study one-dimensional insulators obeying a chiral symmetry in the single-particle picture. The Fermi level is assumed to lie in a mobility gap. Topological indices are defined for infinite (bulk) or half-infinite (edge) systems, and it is shown that for a given Hamiltonian with nearest neighbor hopping the two indices are equal. We also give a new formulation of the index in terms of the Lyapunov exponents of the zero energy Schrödinger equation, which illustrates the conditions for a topological phase transition occurring in the mobility gap regime.

1 Introduction

Topological materials come in classes differing by symmetry type and by the dimension of the physical space. The classification table [17, 16, 10, 8] associates an index group to each class or actually a concrete index [13, 9], the values of which separate topological phases of materials within the same class. The fairly general model to be analyzed here obeys chiral symmetry (class AIII of the table) in dimension one, and exhibits moreover strong disorder. The symmetry of the Hamiltonian is matched by that of the state, which is at half-filling. The prototypical model in the same class, yet lacking disorder, is the Su-Schrieffer-Heeger model of polyacetylene [18]: This is an alternating chain of sites or, in other words, a bipartite lattice, along which electrons hop between sub-lattices, either to the right or to the left, but without experiencing an on-site potential. As a result, the Hamiltonian $H$ and its opposite, $-H$, are unitarily conjugate. In particular the energy zero is special, being the fixed point under the sign flip, and it singles out half-filling. If that energy lies in a spectral gap of $H$, the model exhibits topological properties which depend on the (constant) ratio of the amplitudes for hopping in the two directions (from a given sub-lattice). How much of this survives when the hopping changes randomly from bond to bond? And what if the disorder is actually so strong as to close the spectral gap about zero? At first sight, disorder seems to induce
localization throughout the spectrum, as it certainly is the case for on-site randomness [1] which corresponds to the class A, and is topologically trivial. The truth for class AIII however is that localization may fail, but need not, at the one special energy, i.e. zero. This is enough to rescue the topological features; in fact Hamiltonians may be loosely viewed as belonging to a same topological phase as long as they can be deformed while preserving localization (mobility gap) at zero energy. Put differently: The closing of the mobility gap about zero defines the phase boundaries.

More precisely, we will cast the crucial assumption of a mobility gap in precise technical terms, which relate to known signatures of localization. We then consider two quantities associated to the bulk and the edge of the material respectively, and show that they are well-defined and integer-valued, whence they serve as indices. We show that they agree (bulk-edge correspondence) and finally that the index can be characterized in terms of the Lyapunov spectrum of the time-independent Schrödinger equation.

The paper is organized as follows. We start in Section 2 by describing the mathematical setting, defining chiral symmetry and its features, including the bulk and edge invariants. We define the notion of a mobility gap and state the main result about bulk-edge correspondence in that context. We also reformulate the index in terms of Lyapunov exponents. Section 3 is an aside about the more restrictive case of a spectral gap and the resulting simplifications. For completeness the even more special, translation invariant case is addressed there, too. In Section 4 we return to the general case by reformulating the edge index, so as to conclude the proof of the main result in Section 5. The Appendix contains a few technical lemmas, as well as a discussion of more general boundary conditions.

In concluding this section we comment on literature on related models formulated in the framework of stochastically translation invariant Hamiltonians [2]. In [12] a similar model has been discussed in the strong disorder regime and its phases explored numerically; bulk-edge correspondence is shown in [13] for the case of the spectral gap. The appropriate bulk index was introduced in [14] and moreover shown to be well-defined and continuous w.r.t. the Hamiltonian in the case of a mobility gap. Finally we note that in [4] the role of the Lyapunov exponents at zero energy is addressed, including that of a zero exponent in some model.

2 The model and the results

Figure 1: The lattice underlying the model is an alternating chain. The hopping amplitudes $A_n, B_n \in GL_N(C)$ are in direction of the arrows. In the opposite direction the adjoint matrices apply.
In this section we shall specify the setting of chiral one-dimensional systems, define the relevant indices, and formulate the main result on bulk-edge duality.

2.1 One-dimensional chiral systems

The lattice underlying the model is an alternating chain, where particles perform nearest-neighbor hopping (see Figure 1). The single-particle Hilbert space of a tight-binding model is

\[ \mathcal{H} = \mathcal{K} \otimes \mathbb{C}^2 \ni (\psi_{n}^+, \psi_{n}^-)_{n \in \mathbb{Z}}, \]

with \( \mathcal{K} := l^2(\mathbb{Z}, \mathbb{C}^N) \), where \( \mathbb{C}^N \) stands for the internal degrees of freedom of each site and \( \mathbb{C}^2 \) for their grouping into dimers. The Hamiltonian is

\[ H = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix} \tag{2.1} \]

with \( S \) acting on \( \mathcal{K} \) as

\[ (S\psi^+)_n := A_n\psi^+_{n-1} + B_n\psi^+_n; \tag{2.2} \]

hence

\[ (S^*\psi^-)_n = A^*_n\psi^-_{n+1} + B^*_n\psi^-_n. \tag{2.3} \]

We assume \( A_n, B_n \in GL_N(\mathbb{C}) \), whence solutions to \( S\psi^+ = 0 \) are determined by \( \psi^+_n \) for any \( n \). Otherwise, i.e. if some matrices were singular, the corresponding bonds would be effectively broken; put differently, the model would have edges within.

The chiral symmetry

\[ \Pi := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

is a symmetry of the Hamiltonian, in the sense that

\[ \{H, \Pi\} = HP + PH = 0. \]

It implies

\[ f(H)\Pi = \Pi f(-H) \tag{2.4} \]

for any Borel bounded function \( f = f(\lambda) \).

The many-particle state is the Fermi sea at half-filling, meaning that the Fermi level is at \( \lambda = 0 \). Its single-particle density matrix thus is the Fermi projection \( P := \chi_{(-\infty,0)}(H) \), where \( \chi_I \) is the characteristic function of the set \( I \subseteq \mathbb{R} \).
We assume localization for $H$ at the Fermi level and formulate that condition deterministically. In a companion paper [7] it will be shown that it holds either with probability zero or one, depending on the details of the model. This means that no further recourse to probabilistic arguments will be made in proofs, and that the indices are properties of the individual system, and not just of the statistical ensemble.

**Assumption 1.** For some $\mu, \nu > 0$ we have
\[
\sum_{n, n' \in \mathbb{Z}} \| P(n, n') \| (1 + |n|)^{-\nu} e^{\mu |n - n'|} \leq C < +\infty ,
\]
where $(\delta_n)_{n \in \mathbb{Z}}$ is the canonical (position) basis of $\ell^2(\mathbb{Z})$ and the map $P(n, n') = \langle \delta_n, P\delta_{n'} \rangle : \mathbb{C}^{2N} \to \mathbb{C}^{2N}$ acts between the internal spaces of dimers of $n'$ and $n$. Here, $\| \cdot \|$ is the trace norm of such maps. Moreover, the same bound applies to the Fermi projections of the edge Hamiltonians introduced below.

**Assumption 2.** $\lambda = 0$ is not an eigenvalue of $H$.

**Remark 1.** These two assumptions are trivially fulfilled in the spectral gap case. In this paper we are rather interested in the mobility gap regime, which is the typical one at large disorder.

In physical terms, Assumption 2 states that every state is either a particle or a hole state, thus prompting the notation
\[
P_- := P, \quad P_+ := \chi(0, \infty)(H)
\]
and the rephrasing
\[
P_- + P_+ = 1, \quad P_- \Pi = \Pi P_+ \quad (2.5)
\]
of the assumption and of the chiral symmetry.

We will define shortly a **bulk index** $\mathcal{N}$ associated to $H$, as well as an **edge index** $\mathcal{N}_a$ associated to its truncation to the half-lattice to the left of an arbitrary point $a \in \mathbb{Z}$.

**Theorem 1.** (Duality) Under Assumptions 1 and 2 we have
\[
\mathcal{N} = \mathcal{N}_a .
\]

We anticipate that $\mathcal{N}_a$ will be manifestly an integer. Hence so is $\mathcal{N}$, and $\mathcal{N}_a$ is independent of $a$. In the proof though, we will first establish the independence and then obtain the result by passing to the limit $a \to +\infty$. The two steps will be carried out in Sections 4 and 5.

**Bulk Index.** Let $\Sigma := \text{sgn} H$. Let $\Lambda : \mathbb{Z} \to \mathbb{R}$ be a switch function, i.e. $\Lambda(n) = 1$ (resp. $= 0$) for $n$ (resp. $-n$) large and positive. It defines a (multiplication) operator on $\ell^2(\mathbb{Z})$, which carries naturally to its descendant spaces $\mathcal{K}$ and $\mathcal{H}$.
Definition 1. The bulk index is

\[ \mathcal{N} := \frac{1}{2} \text{tr}(\Pi \Sigma [\Lambda, \Sigma]). \] (2.6)

The index is well-defined. In fact, \( \Sigma = P_+ - P_- \) with \( P_\pm \) as above and we have

Lemma 2. \([\Lambda, P_\pm]\) are trace class and the index can be expressed as

\[ \mathcal{N} = - \text{tr} \Pi P_+ [\Lambda, P_-] - \text{tr} \Pi P_- [\Lambda, P_+]. \] (2.7)

Edge Index. The model is truncated to \( \mathbb{Z}_a := (-\infty, a] \subset \mathbb{Z} \) with Hilbert space \( \mathcal{H}_a := \ell^2(\mathbb{Z}_a, \mathbb{C}^{2N}) \). Of course the choice of \( a \) ought not be of physical relevance. Here we keep this choice free and explicit in the notation since we shall eventually take the limit \( a \to +\infty \) which helps associating edge objects with bulk ones.

The truncation procedure can be recast algebraically as follows. Let \( \iota_a : \ell^2(\mathbb{Z}_a) \hookrightarrow \ell^2(\mathbb{Z}) \) be the natural injection, whence \( \iota_a^* : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}_a) \) is the restriction operator. Thus \( \iota_a \) is an isometry, but not a unitary: In fact

\[ \iota_a^* \iota_a = 1_{\mathcal{H}_a}, \quad \iota_a \iota_a^* = \chi_a, \] (2.8)

where \( \chi_a \) is the projection \( \chi_a : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}_a) \) associated to the subspace \( \ell^2(\mathbb{Z}_a) \subseteq \ell^2(\mathbb{Z}) \). Thus

\[ \chi_a \iota_a = \iota_a, \quad \iota_a^* = \iota_a^* \chi_a \]

and

\[ \chi_a A \chi_a : \text{im} \chi_a \to \text{im} \chi_a = \iota_a^* \iota_a \] (2.9)

for any operator \( A \) acting on \( \ell^2(\mathbb{Z}) \) or on some of its descendant spaces. In particular, letting

\[ S_a := \iota_a^* S \iota_a, \] (2.10)

we then have

\[ H_a := \iota_a^* H \iota_a = \begin{pmatrix} 0 & S_a^* \ S_a \ 0 \end{pmatrix}. \] (2.11)

More general boundary conditions will be discussed in Section 6.2.

Remark 2. As \( a \to +\infty \) an ever larger portion \( \mathbb{Z}_a \) of \( \mathbb{Z} \) is retained, resulting in the limit \( H_a \to H \) in the strong resolvent sense, as will be seen and used.
From (2.11) we still have
\[ \{H_a, \Pi\} = 0 \] (2.12)
and its consequence (2.4).

Let \( P_{0,a} := \chi(0) H_a \) be the spectral projection for \( \lambda = 0 \), i.e. its eigenprojection if it is an eigenvalue. We note that Assumption 2 generically fails for the edge system.

**Definition 2.** The edge index is
\[ N_a := \text{tr}(\Pi P_{0,a}). \] (2.13)

**Remark 3.** \( \Pi \) maps \( \text{im} P_{0,a} = \ker H_a \) into itself. Indeed, \( H_a \psi = 0 \) implies \( H_a \Pi \psi = 0 \) by (2.12).

In particular \( N_a \in \mathbb{Z} \) as anticipated, and the index may be written as
\[ N_a = \dim \ker S_a - \dim \ker S^*_a, \] (2.14)
which is finite by (2.2, 2.3). Despite appearances, this is not a Fredholm index in general, simply because \( S_a \) is not Fredholm in the mobility gap regime of Assumption 1. Indeed, \( \text{im} S_a \) is not closed then.

**Example 1.** Figure 1 should be viewed as just one example of a lattice leading to a chiral Hamiltonian (2.1). Other lattices may do so too. An example is shown in Figure 2.

The model is of the form (2.1, 2.2) with \( N = 2M \) upon grouping amplitudes as bispinors:
\[ \psi^-_n = \begin{pmatrix} \varphi^-_{2n-1} \\ \varphi^-_{2n} \end{pmatrix}, \quad \psi^+_n = \begin{pmatrix} \varphi^+_{2n} \\ \varphi^+_{2n+1} \end{pmatrix}. \]

Comparing Figures 1 and 2 yields
\[ A_n = \begin{pmatrix} T^+_{2n-2} & T^-_{2n-1} \\ 0 & T^+_{2n-1} \end{pmatrix}, \quad B_n = \begin{pmatrix} T^-_{2n} & 0 \\ T^+_{2n} & T^-_{2n+1} \end{pmatrix}. \]

In particular \( A_n, B_n \in GL_N(\mathbb{C}) \) iff \( T^\pm_m \in GL_M(\mathbb{C}) \).
2.2 The zero-energy Lyapunov spectrum

We conclude this section with an alternate formulation of the index. To this end we consider the equation \( S\psi^+ = 0 \) as a finite difference equation for sequences \( \psi^+: \mathbb{Z} \to \mathbb{C}^N \), foregoing normalizability. By (2.2) and \( A_n \in GL_N(\mathbb{C}) \) the equation is solved recursively,

\[
S\psi^+ = 0 \iff \psi^+_{n-1} = T_n\psi^+_n, \quad (n \in \mathbb{Z})
\] (2.15)

with \( T_n := -A_n^{-1}B_n \). The associated transfer matrix is

\[
T(n) := T_{n-1} \cdots T_0, \quad (n < 0).
\]

The Lyapunov exponent of a vector \( v \in \mathbb{C}^N \) is then given as

\[
\chi(v) := \limsup_{n \to -\infty} \frac{1}{n} \log \|T(n)v\|
\] (2.16)

with \( \chi(v) \in \bar{\mathbb{R}} \equiv \mathbb{R} \cup \{\pm \infty\} \) and \( \chi(0) = -\infty \). The set

\[
V_\chi := \{v \in \mathbb{C}^N | \chi(v) \leq \chi\}
\] (2.17)

is a linear subspace which is non-decreasing in \( \chi \in \bar{\mathbb{R}} \). Let \( \chi_N \leq \cdots \leq \chi_1 \) be the values of \( \chi \) at which \( \chi \mapsto \dim V_\chi \) jumps, listed repeatedly according to the jump in dimension.

**Assumption 3.** Let 0 not be in the Lyapunov spectrum, i.e. \( \chi_i \neq 0, (i = 1, \ldots, N) \).

**Theorem 3.** Under Assumption 3 the edge index equals the number of negative Lyapunov exponents:

\[
\mathcal{N}_a = \# \{i | \chi_i < 0\}
\] (2.18)

for any \( a \in \mathbb{Z} \).

The proof will be given in Section 4. In [7] we will give conditions such that in (2.16) \( \limsup \) can be replaced almost surely by \( \lim \) for all \( v \in \mathbb{C}^N \), actually with the limits being finite for \( v \neq 0 \) and with simple Lyapunov spectrum. Moreover, \( V_\chi \) is the spectral subspace of the self-adjoint matrix \( \Lambda := \lim_{n \to \infty} (T(n)^*T(n))^{1/2n} \) and of eigenvalues \( \leq e^\chi \). Finally, Assumptions 1 and 3 will be shown to be equivalent.

**Remark 4.** As a complement to (2.18), the edge index \( \mathcal{N}_a \) may also be expressed in terms of the equation \( S^*\psi^- = 0 \), in which case it is given by the number of positive Lyapunov exponents. In fact, introducing \( \varphi_n^- = B_n^*\psi_n^- \) that equation is \( \varphi_{n-1}^- = \tilde{T}_n\varphi_n^- \), where \( \tilde{T}_n = T_n^\circ \) and \( M^\circ = (M^*)^{-1} \) (using \( B_n \in GL_N(\mathbb{C}) \), too). Its spaces are

\[
\tilde{V}_\chi = V_{-\chi}^\perp,
\] (2.19)

provided \( \chi \) is not in the Lyapunov spectrum. In particular, \( \tilde{\chi}_i = -\chi_{N+1-i} \). Eq. (2.19) follows from \( \tilde{T}(n) = T(n)^\circ \) and \( \tilde{\Lambda} = \Lambda^{-1} \).
Remark 5. The usual scenario of a phase transition is that of a spectral gap closing on the Fermi level. [Theorem 3] gives a different scenario, whereby the Fermi level may not lie in a gap throughout the transition. More precisely, the Lyapunov spectrum associated to \((H - \lambda)\psi = 0\) consists of \(2N\) exponents \(\{\gamma_i\}_{i=1}^{2N}\) and is even under sign flip \(\gamma \mapsto -\gamma\) (counting multiplicity). For \(\lambda \neq 0\) the spectrum is moreover simple, implying that 0 is not an exponent and thus localization. For \(\lambda = 0\) however the exponents are those of \(S\psi^* = 0\) and their flips, \(\{\chi_i, -\chi_i\}_{i=1}^N\). In particular 0 may, but need not be an exponent. If it isn’t, [Theorem 3] applies, but if it becomes one, the localization length diverges at \(\lambda = 0\), signaling the topological phase transition.

3 The spectral gap case

In the case of a spectral gap the analysis simplifies, as was noted in [13] and discussed in terms of K-theory. We present here an equivalent simplification as a contrast to the general case, to be proven later. Until the end of this section we forgo definition (2.2) and allow \(S\) to be any operator \(S : \mathcal{K} \to \mathcal{K}\) for which \([\Lambda, S]\) is trace class; this being a generalization since the commutator is of finite rank in the former case. The spectral gap condition means that Assumptions 1 and 2 are now replaced by the stronger condition

\[ 0 \not\in \sigma(H). \quad (3.1) \]

Thus \(H\) is Fredholm, and so is \(S\) in view of

\[ \ker H = \ker S \oplus \ker S^*, \quad \im H = \im S^* \oplus \im S. \]

We first discuss the bulk index:

**Lemma 4.** The index (2.6) is well-defined and equals

\[ \mathcal{N} = \text{tr} U^* [\Lambda, U], \quad (3.2) \]

where \(U\) is the (unique) unitary in the polar decomposition of \(S : S = U|S|\) with \(|S| \equiv (S^*S)^{1/2}\).

**Proof.** By (3.1), \(|H|\) is invertible and we have \(\Sigma = H|H|^{-1}\). That operator is computed as

\[ \Sigma = \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix}, \]

as seen from \(S^* = |S|U^*, \ H^2 = \text{diag} (S^*S, SS^*) = \text{diag} (|S|^2, U|S|^2 U^*), \ |H| = \text{diag} (|S|, U|S|U^*). \)

We conclude that

\[ [\Lambda, \Sigma] = \begin{pmatrix} 0 & [\Lambda, U^*] \\ [\Lambda, U] & 0 \end{pmatrix} \]

and \(\Pi \Sigma[\Lambda, \Sigma] = U^*[\Lambda, U] \oplus [\Lambda, U]U^*\). The claim is now immediate, provided \([\Lambda, U]\) is trace class. This holds true by the following lemma, because \([\Lambda, S]\) already is trace class. \(\square\)
Lemma 5. Let $A : \mathcal{K} \to \mathcal{K}$ be Fredholm. If $[A, \Lambda]$ is trace class, then so is $[A|A|^{-1}, \Lambda]$.

Proof. The commutator property is inherited under taking adjoints and products; and if $A \geq \varepsilon > 0$ also under taking inverses, $[A^{-1}, \Lambda] = -A^{-1}[A, \Lambda]A^{-1}$. In the latter case the property is also passed down to $A^{-1/2}$ because of

$$A^{-1/2} = C \int_0^\infty \lambda^{-1/2}(A + \lambda)^{-1}d\lambda$$

($C^{-1} = \int_0^\infty \lambda^{-1/2}(1 + \lambda)^{-1}d\lambda$). In particular, the property applies to $A^*A = |A|^2$ and to $|A|^{-1}$, where we used that $A$ is Fredholm through $A^*A \geq \varepsilon > 0$; finally it applies to $|A|^{-1/2}$.

We notice that the index (3.2) is independent of the choice of the switch function, this being tantamount to the vanishing of the expression when $\Lambda$ is replaced by a function of compact support. Then, in fact, $\Lambda$ would already be trace class and the claim seen by expanding the commutator. In particular we may pick $\Lambda = 1 - \chi_a$, $\chi_a$ being the projection seen in (2.8). We then conclude by ([1], Theorems 6.1, 5.2) that the bulk index is that of a pair of projections:

$$N = -\text{tr}(U^*\chi_aU - \chi_a) = \text{ind}(\chi_a, U^*\chi_aU) = \text{ind}(\chi_aU\chi_a : \text{im}\chi_a \to \text{im}\chi_a) = \text{ind}(i_a^*U_i)$$

We now turn to the edge index and first state a definition: Let

$$\sigma_{\text{ess}}(A) = \{\lambda \in \mathbb{C} | A - \lambda \mathbbm{1} \text{ is not Fredholm}\}$$

be the essential spectrum of a (not necessarily self-adjoint) closed operator $A$. It enjoys stability under compact perturbations $K$, i.e. $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(A + K)$. [3].

Lemma 6. Suppose $A : \mathcal{K} \to \mathcal{K}$ is such that $[A, \Lambda]$ is compact, where $\Lambda$ is some (and hence any) switch function. Then

$$\sigma_{\text{ess}}(i_a^*\Lambda A_i) \subset \sigma_{\text{ess}}(A). \quad (3.3)$$

In particular, if $A$ is Fredholm, then so is $i_a^*\Lambda A_i$.

Proof. We have

$$A = \Lambda A \Lambda + (\mathbbm{1} - \Lambda)A(\mathbbm{1} - \Lambda) + K \quad (3.4)$$

with $K = \Lambda A(\mathbbm{1} - \Lambda) + (\mathbbm{1} - \Lambda)A\Lambda = [\Lambda, A](\mathbbm{1} - \Lambda) + (\mathbbm{1} - \Lambda)[A, \Lambda]$ compact.

The injection $i_a$ is matched by another one, $i_a$, corresponding to the complementary half-line $\mathbb{Z} \setminus \mathbb{Z}_a$. For $\Lambda = \chi_a$ the first two terms on the RHS of (3.4) are

$$\chi_a A\chi_a + (\mathbbm{1} - \chi_a)A(\mathbbm{1} - \chi_a) \cong i_a^*\Lambda A_i \oplus i_a^*\Lambda A_i$$

because of (2.8) and the unitarity of $i_a \oplus i_a$. Eq. (3.3) follows. \qed
For \( A = S \) the lemma yields that \( S_a \) is Fredholm by (2.10). Thus \( \text{im} \, S_a \) is closed, \( \ker S_a^* = \text{coker} \, S_a \), and the edge index (2.14) is a Fredholm index,

\[
N_a = \text{ind} \, S_a.
\] (3.5)

The proof of the bulk-edge duality, \( N = N_a \), is concluded by

**Lemma 7.**

\[
\text{ind} \, \iota_a^* S \iota_t = \text{ind} \, \iota_a^* U \iota_a.
\] (3.6)

**Proof.** We consider the interpolating family

\[
S_t = tU + (1 - t)S = U(t \mathbb{1} + (1 - t)|S|),
\]

\((0 \leq t \leq 1)\), with \( S_0 = S, \ S_1 = U \). The hypothesis of [Lemma 6](#) holds true for \( t = 0, 1 \), as remarked before, and thus true for \( 0 \leq t \leq 1 \). Moreover, by that lemma, \( \iota_a^* S \iota_t \) is Fredholm if \( S_t \) is. That however is immediate from

\[
S_t^* S_t = (t \mathbb{1} + (1 - t)|S|)^2 \geq \delta,
\]

for some \( \delta > 0 \). Thus (3.6) holds true by the continuity of the index. \( \square \)

**The translation invariant case.** We here assume that \( S : \mathcal{K} \to \mathcal{K} \) commutes with the shift operator, whence \( S \) is of Toeplitz form in the position basis \((\delta_n)_{n \in \mathbb{Z}}\) of \( \ell^2(\mathbb{Z}) \),

\[
\langle \delta_n, S \delta_{n'} \rangle = S_{n-n'},
\]

where the maps \( S_m : \mathbb{C}^N \to \mathbb{C}^N \) may themselves be viewed as matrices. For simplicity we assume that they rapidly decay in \( m \in \mathbb{Z} \).

**Proposition 8.** The sum

\[
S(z) := \sum_{m \in \mathbb{Z}} S_m z^{-m}
\]

is absolutely convergent for \( |z| = 1 \), i.e. for \( z \) on the unit circle \( \mathcal{C} \). Then the spectral gap condition (3.1) holds iff \( \det S(z) \) vanishes nowhere on \( \mathcal{C} \). In that case the index (3.2) is (the negative of) the winding number of \( \mathcal{C} \ni z \mapsto \det S(z) \in \mathbb{C} \) (Zak number [19]).

**Example 2.** In the translation invariant case, (2.2) reduces to \( S(z) = Az^{-1} + B \). The spectral gap condition requires that

\[
T := -A^{-1}B
\] (3.7)

has no eigenvalue of unit modulus, or equivalently that 1 is not among its singular values. Since \( z \mapsto w = z^{-1} \) reverses the orientation of \( \mathcal{C} \), the index equals the winding number of \( w \mapsto \det(Aw + B) \) and, by the argument principle, the number of zeroes in \( \mathcal{C} \), i.e. the algebraic number of eigenvalues \( w \) of \( T \) with \( |w| < 1 \); equivalently it is the number of zeroes of \( \det S(z) \) with \( |z| > 1 \).
Proof of Proposition 8. In line with the general assumptions of this section, we first verify that $[\Lambda, S]$ is trace class. This follows from
\[
\langle \delta_n, [\Lambda, S]\delta_{n'} \rangle = (\Lambda(n) - \Lambda(n')) S_{n-n'},
\]
and from (6.1) by the reasoning used in the proof of Lemma 2. Second, we discuss the gap condition (3.1): By Bloch decomposition,
\[
S = \int_C S(z) \frac{ds}{2\pi} \tag{3.8}
\]
with $ds = |dz| = -iz^{-1}dz$ and w.r.t. $K = \ell^2(\mathbb{Z}) \otimes \mathbb{C}^N$, $\ell^2(\mathbb{Z}) \cong \int_C \mathbb{C} ds/2\pi$. The isomorphism is given by
\[
\psi(z) = \sum_{n \in \mathbb{Z}} z^{-n} \psi_n
\]
with Parseval identity
\[
\sum_{n \in \mathbb{Z}} \varphi_n^* \psi_n = \int_C \varphi(z)^* \psi(z) \frac{ds}{2\pi}. \tag{3.9}
\]
Moreover $(S\psi)(z) = S(z)\psi(z)$ is readily verified, proving (3.8). Since $S(z)$ is smooth we have $\sigma(H) = \bigcup_{z \in C} \sigma(H(z))$. The claim on the spectral gap property now follows, and we assume its validity in the sequel.

Next we compute the index (3.2). The fibers $U(z) = S(z)/|S(z)|$ are smooth as well, whence the sum
\[
U(z) := \sum_{m \in \mathbb{Z}} U_m z^{-m}
\]
has rapidly decaying coefficients $U_m$. Thus
\[
\mathcal{N} = \text{tr} U^* [\Lambda, U] = \sum_{n \in \mathbb{Z}} \text{tr} \{ U\delta_n, [\Lambda, U]\delta_n \}
\]
\[
= \sum_{n,m} \text{tr} |U_{m-n}|^2 (\Lambda(m) - \Lambda(n)) = \sum_{n,k} \text{tr} |U_k|^2 (\Lambda(n + k) - \Lambda(n))
\]
\[
= \sum_k k \text{tr} |U_k|^2,
\]
where we used $\sum_n \Lambda(n+1) - \Lambda(n) = 1$. Using $-z \partial_z U(z) = \sum_m m U_m z^{-m}$ and (3.9) we obtain
\[
\mathcal{N} = \frac{i}{2\pi} \int_C dz \text{ tr} U(z)^* \partial_z U(z)
\]
\[
= \frac{i}{2\pi} \int_C dz \frac{d}{dz} \frac{\det U(z)}{\det U(z)} = \frac{i}{2\pi} \int_C \frac{d \det S(z)}{\det S(z)},
\]
because $\det |S(z)| > 0$ has no winding. \hfill \qed
4 Generalized states of zero energy

Zero energy edge states will be extended to bulk states which are however not $\ell^2$ on the other side of the edge. For that purpose, let us consider the (bulk) equation $S\psi^+ = 0$ as a finite difference equation for $\psi^+ : \mathbb{Z} \to \mathbb{C}^N$. The edge index can be characterized in terms of their behavior at $-\infty$. In fact we have:

Lemma 9.

$$N_a = \dim V, \quad V := \{\psi^+ : \mathbb{Z} \to \mathbb{C}^N | S\psi^+ = 0 \text{ and } \psi^+_n \text{ is } \ell^2 \text{ at } n \to -\infty\}.$$ (4.1)

In particular $N_a$ is seen to be independent of $a$, independently of Theorem 1. Moreover every $\psi \in V$ is uniquely determined by its restriction to $\mathbb{Z}_a$ for any $a$.

Proof. By (2.14) we are led to determine the null spaces of $S_a$ and $S^*_a$ separately. The two operators act by (2.2, 2.3) with $n \leq a$; (2.2) comes without boundary conditions, whereas (2.3) is supplemented by $\psi_{a+1}^+ = 0$. By $B_n \in GL_N(\mathbb{C})$ the difference equation $S\psi^+ = 0$ can be solved recursively to the right ($\psi^+_{n-1}$ determines $\psi^+_n$) whereas $S^*\psi^- = 0$ can be solved recursively to the left. Hence $\ker S_a = V$, whereas the Dirichlet boundary condition implies $\ker S^*_a = \{0\}$.

We next show that the edge index may be computed using a finite-box truncation.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig3.png}
\caption{The finite box used in order to approximate the edge index.}
\end{figure}

Lemma 10. The common value of $N_a$, $(a \in \mathbb{Z})$ is

$$N^\sharp = \lim_{a \to +\infty} \text{tr}(\Pi a P_0,a).$$ (4.2)

Here we denoted by $\Lambda_a$ the switch function $\Lambda$ when viewed as a multiplication operator on $\ell^2(\mathbb{Z}_a)$ and its descendant spaces. We have

$$t_a \Lambda_a = \Lambda t_a, \quad \Lambda_a^* t_a^* = t_a^* \Lambda.$$ (4.3)

The switch $\Lambda_a$ roughly restricts states to $n \geq 0$ within $n \leq a$, thereby singling out a finite box growing with $a$ (see Figure 3). The lemma asserts that edge states are unaffected by this restriction for $a \to +\infty$, because they are concentrated near the edge $n = a$. Consequently the task is to show $\| (1 - \Lambda_a) P_0,a \| \to 0$, $(a \to +\infty)$. 

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Proof of Lemma 10. Let $V$ be the linear space of solutions $\psi = (\psi^+, 0)$ seen in (4.1). By
\[
\dim V \leq N < \infty
\] (4.4)
all norms on $V$ are equivalent, and we pick one, $\| \cdot \|_V$. For any $b \in \mathbb{Z}$, let $R_b : V \to H_b$ be defined by restriction. That map is injective by the conclusion of the previous lemma. Therefore and by (4.4) we have
\[
\|R_b \psi\| \geq c_b \|\psi\|
\] (4.5)
for some $c_b > 0$. Elements $\psi \in V$ cannot be $\ell^2$ at $n \to +\infty$ as well, unless $\psi = 0$, since that would imply a solution of $H\psi = 0$, which is ruled out by Assumption 2. We thus have
\[
\|R_b \psi\| \to \infty, \quad (b \to +\infty).
\] (4.6)

For $b < a$ we denote by $\iota_{ab} : \ell^2(\mathbb{Z}_b) \to \ell^2(\mathbb{Z}_a)$ the injection (extension by zero); correspondingly $\iota_{ab}^* : \ell^2(\mathbb{Z}_a) \to \ell^2(\mathbb{Z}_b)$ is the restriction operator. These operators are analogous to those seen in (2.8); in fact $\iota_b = \iota_{a \to b}$. For $b$ large enough we have $(I - \Lambda)(1 - \chi_b) = 0$ by disjointness of support. Using (4.3) we get $I - \Lambda = (I - \Lambda)\iota_{ab}^* \iota_{ab}^* = \iota_b (I - \Lambda_b)\iota_{ab}^*$ and thus, by multiplication with $\iota_{ab}^*$ and $\iota_{ab}$ from left and right,
\[
(I - \Lambda_a) = \iota_{ab} (I - \Lambda_b)\iota_{ab}^*
\] (4.7)
(b large, $a > b$).

Next we note that $P_{0,a} : \mathcal{H}_a \to \mathcal{H}_a$ induces a natural map $P_{0,a} : \mathcal{H}_a \to V$, because for any $\psi_a \in \mathcal{H}_a$ the image $P_{0,a}\psi_a$ is the left tail of a solution in $V$ which it fully determines. It satisfies
\[
R_b P_{0,a} = \iota_{ab}^* P_{0,a}.
\] (4.8)

We next claim for any $b$
\[
\|R_b P_{0,a}\| \to 0, \quad (a \to +\infty).
\] (4.9)
We have to show that $\|R_b P_{0,a}\psi_a\| \to 0$ for every sequence $\psi_a \in \mathcal{H}_a$ with $\|\psi_a\| = 1$. Clearly the sequence at hand is at least bounded in $a$ (as well as in $b > a$) and so is $\|P_{0,a}\psi_a\|$ because of (4.8) and (4.5). By compactness (dim $V < \infty$) we have $P_{0,a}\psi_a \to \hat{\psi}$, $(a \to +\infty)$ upon passing to a subsequence. Hence $R_b P_{0,a}\psi_a$ has a limit $R_b \hat{\psi}$ as $a \to +\infty$, which inherits the boundedness in $b$. This contradicts (4.6) unless $\hat{\psi} = 0$, thus proving (4.9).

That in turn implies, by taking $b$ large and using (4.7) (4.8),
\[
\|(I - \Lambda_a) P_{0,a}\| \to 0, \quad (a \to +\infty).
\]
The same then holds in trace class norm $\| \cdot \|_1$ because $\|A\|_1 \leq \|A\|$ rank $A$ and rank $P_{0,a} = \dim V$. Finally (4.2) follows by taking a (redundant) limit of (2.13).

Proof of Theorem 3. By (2.17) and the definition of $\chi$, as well as by Assumption 3 the RHS of (2.18) equals $\dim V_\chi = \dim V_0$ for some $\chi < 0$. We also recall (2.15) and Definition (4.1). We have the inclusions $V_\chi \subseteq V$ for any $\chi < 0$ and $V \subseteq V_0$, since $\ell^2 \subseteq \ell^\infty$ at $-\infty$. The conclusion now follows from Lemma 9. \qed

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5 Proof of duality

Lemma 11. As \( a \to +\infty \),

\[
\begin{align*}
\ell_a P_{\pm,a}^* - P_{\pm} & \xrightarrow{s} 0 , \quad (5.1) \\
[t_a P_{\pm,a}^* - P_{\pm}, \Lambda] & \xrightarrow{t} 0 . \quad (5.2)
\end{align*}
\]

where \( s, t \) denote strong and trace norm convergence respectively, and \( P_{\pm,a} := \chi(0, \infty)(\pm H_a) \).

Proof of Theorem 1. The operator \( \Lambda_a \) introduced in (4.3) is of finite rank. The basic identity is

\[
\text{tr}(\Pi \Lambda_a) = 0 , \quad (5.3)
\]

which follows by evaluating the trace in the position basis and by using \( \text{tr}_{C^2 N} \Pi = 0 \). We insert \( \mathbf{1} = P_{0,a} + P_{+,a} + P_{-,a} \) with \( P_{\pm,a} \equiv \chi(0, \infty)(\pm H_a) \) and obtain

\[
\text{tr}(\Pi \Lambda_a) = \text{tr}(\Pi \Lambda_a P_{0,a}) + \text{tr}(\Pi \Lambda_a P_{+,a}) + \text{tr}(\Pi \Lambda_a P_{-,a}) . \quad (5.4)
\]

The first term tends to \( N^2 \) as \( a \to +\infty \) by (4.2). The second one is

\[
\begin{align*}
\text{tr}(\Pi \Lambda_a P_{+,a}) &= \text{tr}(\Pi P_{-,a} \Lambda_a P_{+,a}) = \text{tr}(\Pi P_{-,a}[\Lambda_a, P_{+,a}]) \\
&= \text{tr}(\ell_a P_{-,a}[\Lambda_a, P_{+,a}]^*) = \text{tr}(\Pi \ell_a P_{-,a}[\Lambda, \ell_a P_{+,a}]^*) ,
\end{align*}
\]

where we used \( \Pi P_{-,a} = P_{+,a} \Pi P_{-,a} \) (see (2.4, 2.12)), \( \text{tr}_{\mathcal{H}_a} A = \text{tr}_{\mathcal{H}}(\ell_a A \ell_a^*) \) for any operator \( A \) on \( \mathcal{H}_a \), as well as (2.8, 4.3). We next use (5.1, 5.2) together with the implication

\[
X_a \xrightarrow{s} X, Y_a \xrightarrow{t} Y \implies X_a Y_a \xrightarrow{t} XY , \quad (5.5)
\]

(see e.g. [5], Eq. (56)) to conclude

\[
\lim_{a \to +\infty} \text{tr}(\Pi \Lambda_a P_{+,a}) = \text{tr}(\Pi P_{-,a}[\Lambda, P_+]) .
\]

We likewise have for the third term in (5.4)

\[
\lim_{a \to +\infty} \text{tr}(\Pi \Lambda_a P_{-,a}) = \text{tr}(\Pi P_{+,a}[\Lambda, P_-])
\]

and thus find from (2.7, 5.3) that

\[
0 = N^2 - N .
\]

In comparing the proofs of the cases of spectral and mobility gaps the following may be noted: While in the spectral gap case bulk and edge may be related at any finite \( a \), in the mobility gap case the relation emerges at \( a \to +\infty \), and this is made possible by Lemma 10.

\[ \Box \]
6 Appendix

6.1 Proofs of lemmas for the duality

Lemma 12. We have for $T$ operating on $\ell^2(\mathbb{Z})$

$$\|T\|_1 \leq \sum_{n, n'} |T(n, n')|.$$  \hspace{1cm} (6.1)

The bound is passed down to $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^{2N}$ provided $\| \cdot \|$ is interpreted as the trace norm of operators on the second factor.

Proof. Let $\mathcal{H}$ be a Hilbert space and $\{\varphi_n\}_n$ an orthonormal basis. Then

$$\|T\|_1 \equiv \sum_{n'} \langle \varphi_{n'}, T|\varphi_{n'}\rangle \leq \sum_{n'} \|T|\varphi_{n'}\| = \sum_{n'} \|T\varphi_{n'}\| \leq \sum_{n, n'} |\langle \varphi_n, T\varphi_{n'}\rangle|,$$

where we used $\|\psi\| \leq \sum_n |\langle \varphi_n, \psi \rangle|$ in the last step. \hfill $\square$

Proof of Lemma 2. We first prove the trace class property. The operator $T = [\Lambda, P]$ has kernel

$$T(n, n') = (\Lambda(n) - \Lambda(n'))P(n, n').$$

For large $|n|$ we have $\Lambda(n') = \Lambda(n)$ unless $|n - n'| > |n|/2$. For such $n$ we have

$$|\Lambda(n) - \Lambda(n')| e^{-\mu |n-n'|} \leq 2\|\Lambda\|_\infty e^{-\mu |n|/2} \leq C(1 + |n|)^{-\nu}$$

for any $\mu, \nu > 0$, suitable $C > 0$ and all $n'$; thus

$$|\Lambda(n) - \Lambda(n')| \leq C(1 + |n|)^{-\nu} e^{\mu |n-n'|}.$$  \hspace{1cm} (6.2)

For the finitely many remaining $n$ we have

$$|\Lambda(n) - \Lambda(n')| \leq 2\|\Lambda\|_\infty \leq C(1 + |n|)^{-\nu}$$

by adjusting the constant $C$, and thus (6.2) as well. The bound $\|T\|_1 < \infty$ now follows from Assumption 1 by (6.1).

For the second statement of the lemma, we have by $\Sigma = P_+ - P_-$

$$2\mathcal{N} = \text{tr} \Pi P_+ [\Lambda, \Sigma] - \text{tr} \Pi P_- [\Lambda, \Sigma].$$  \hspace{1cm} (6.3)

The first term is the sum of two equal ones,

$$\text{tr} \Pi P_+ [\Lambda, \Sigma] = \text{tr} \Pi P_+ [\Lambda, P_+] - \text{tr} \Pi P_+ [\Lambda, P_-] = -2 \text{tr} \Pi P_+ [\Lambda, P_-];$$
indeed, by $P_+ = (P_+)^2$, and (2.5) we have
\[ \text{tr } \Pi P_+ [\Lambda, P_\pm] = \text{tr } \Pi P_+ [\Lambda, P_\pm] P_- = \mp \text{tr } \Pi P_+ \Lambda P_- . \]
Likewise can be said about the last term in (6.3):
\[ \text{tr } \Pi P_- [\Lambda, \Sigma] = 2 \text{tr } \Pi P_- [\Lambda, P_+] . \]
We obtain (2.7).

**Proof of Lemma 11.** We first prove (5.1) and claim
\[ \iota_a f(H_a) \iota_a^* = f(\iota_a H_a \iota_a^*) + f(0)(1 - \iota_a \iota_a^*) \]
for any (Borel) function $f$. In fact, let us decompose $\ell^2(\mathbb{Z}) = \ell^2(\mathbb{Z}_a) \oplus \ell^2(\tilde{\mathbb{Z}}_a)$, where $\tilde{\mathbb{Z}}_a = \mathbb{Z} \setminus \mathbb{Z}_a$, as well as any descendant space such as $\mathcal{H}$. The isometries $\iota_a$ and $\tilde{\iota}_a$ (similarly defined) provide a partition of unity, $1 = \iota_a \iota_a^* + \tilde{\iota}_a \tilde{\iota}_a^*$, and a block decomposition of
\[ \iota_a H_a \iota_a^* = \iota_a H_a \iota_a^* + \tilde{\iota}_a 0 \tilde{\iota}_a^* \equiv H_a \oplus 0 . \]
Thus, by the functional calculus,
\[ f(\iota_a H_a \iota_a^*) = f(\iota_a H_a \iota_a^*) + \tilde{\iota}_a f(0) \tilde{\iota}_a^* , \] (6.4)
as claimed.

For uniformly bounded operators, like $\iota_a H_a \iota_a^*$ and $H$, strong resolvent convergence is equivalent to strong convergence (see [15], Problem VIII.28). The latter,
\[ \iota_a H_a \iota_a^* - H \xrightarrow{s} 0 , \quad (a \to +\infty) \]
is evident, because the LHS vanishes for large but finite $a$, when applied to any state $\psi \in \mathcal{H}$ from the dense subspace $\{\text{supp } \psi \subseteq \mathbb{Z} \text{ is bounded}\}$. Finally we specialize to $f = \chi_{(-\infty,0)}$. By ([15], Theorem VIII.24 (b)) and Assumption 2, the strong resolvent convergence implies $f(\iota_a H_a \iota_a^*) - f(H) \xrightarrow{s} 0 , \quad (a \to +\infty)$. The limit (5.1) now follows from (6.4) by $f(0) = 0$.

**Proof of (5.2).** We write $D_a := \iota_a P_- \iota_a^* - P_-$ for brevity. As shown in ([6], Eq. (3.20)), Assumption 1 implies
\[ \|e^{-\mu n} e^{-\varepsilon |n|} P_- e^{\mu n}\| \leq C\varepsilon , \quad (\varepsilon > 0) , \]
where $g(n)$ denotes the multiplication operator by the namesake function. The same holds true by the same assumption for $\iota_a P_- \iota_a^*$ instead of $P_-$, and thus for $D_a$ as well. The same estimate holds for $\mu$ replaced by $-\mu$.

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We pick a switch function $\Lambda$ with compactly supported variation and denote by $\Lambda^b(n) = \Lambda(n - b)$ its translate by $b \in \mathbb{N}$. We note that $\Lambda - \Lambda^b$ is of finite rank and that for fixed $\varepsilon \in (0, \mu)$ we have

$$\|(1 - \Lambda)e^{\mu n}e^{\varepsilon |n|}\| \leq C, \quad \|e^{-\mu n}\Lambda^b\|_1 \leq Ce^{-\mu b}.$$ 

The LHS of (5.2) is

$$[D_a, \Lambda] = (1 - \Lambda)D_a\Lambda - \Lambda D_a(1 - \Lambda)$$

and we claim that in the limit $a \to +\infty$ each term vanishes separately in trace norm. Indeed,

$$(1 - \Lambda)D_a\Lambda = (1 - \Lambda)D_a\Lambda^b + (1 - \Lambda)D_a(\Lambda - \Lambda^b),$$  

$$(1 - \Lambda)D_a\Lambda^b = (1 - \Lambda)e^{\mu n}e^{\varepsilon |n|} \cdot e^{-\mu n}e^{-\varepsilon |n|}D_a e^{\mu n} \cdot e^{-\mu \Lambda^b}.$$ 

Thus $\|(1 - \Lambda)D_a\Lambda^b\|_1$ can be made arbitrarily small, uniformly in $a$, by first picking $b$ large. Then $\|(1 - \Lambda)D_a(\Lambda - \Lambda^b)\|_1$ will be small for large $a$ by (5.1) (see (5.5)). The other term on the RHS of (6.5) is dealt with similarly.

6.2 More general boundary conditions

In this section we generalize Theorem 1 to (largely) arbitrary boundary conditions. In the case of a spectral gap (see Section 3) we implement them by relaxing (2.10) to

$$S_a := \lambda^* S_{ta} + S_{BC},$$

(6.6)

where $S_{BC}$ is any compact operator. Since the Fredholm index is invariant under compact perturbations, the change does not affect the edge index (3.5) and Theorem 1 remains true.

In the mobility gap regime and in the context of the model with nearest neighbor hopping eqs. (2.1–2.3) more general boundary conditions are obtained by allowing $S_{BC}$ to affect only sites $a$ and $a - 1$; we thus allow the hopping matrices $A_n, B_n$ of the boundary $n = a$ to become singular, whereas they remain regular for $n \leq a - 1$. The edge Hamiltonian (2.11) remains defined with $S_a$ as in (2.10). Thus $S_a$ acts as in (2.2) for $n \leq a$; likewise does $S^*_a$ as in (2.3), except for $n = a$ where $(S^*_a \psi^{-})_a = B^*_a \psi^{-}_a$.

Proposition 13. The edge index $N_a$ is the same for all boundary matrices $(A_a, B_a)$. In particular it is the same as in (4.1).

Example 3. The case of regular $A_a, B_a$ corresponds to the edge Hamiltonian discussed so far. In relation to Figure 1 it amounts to breaking the thin bond between dimers $a$ and $a + 1$. To set $A_a = 0$ amounts to further remove one more dimer; to set instead $B_a = 0$ to break the thick bond of the last dimer.
Proof. By (2.14) we have \( \mathcal{N}_a = \text{dim } V^+ - \text{dim } V^- \) with

\[
V^\pm = \{ \psi^\pm : Z \rightarrow \mathbb{C}^N | \psi^\pm \text{ is } \ell^2 \text{ at } n \rightarrow -\infty \text{ and satisfies (6.7), resp. (6.8, 6.9)} \},
\]

\[
A_{n+1} \psi_n^+ + B_{n+1} \psi_{n+1}^+ = 0, \quad (n \leq a - 1) \quad (6.7)
\]

\[
A_n^* \psi_n^- + B_n^* \psi_n^- = 0, \quad (n \leq a - 1) \quad (6.8)
\]

\[
B_n^* \psi_n^- = 0. \quad (6.9)
\]

(The first equation is \((S \psi^+)_n = 0\) for \(n \leq a\) after shifting the index by one.)

Introducing \(\varphi_n^- = B_n^* \psi_n^-\), the eqs. (6.7, 6.8) are solved iteratively to the left for \(n \leq a - 2\) by

\[
\psi_n^+ = T_{n+1} \psi_{n+1}^+ , \quad \varphi_n^- = T_{n+1}^o \varphi_{n+1}^- \]

with \(T_n = -A_n^{-1} B_n\) and \(M^o = (M^*)^{-1}\). In particular,

\[
\langle \psi_n^+, \varphi_n^- \rangle = \langle T_{n+1} \psi_{n+1}^+, T_{n+1}^o \varphi_{n+1}^- \rangle = \langle \psi_{n+1}^+, \varphi_{n+1}^- \rangle ,
\]

which means that the LHS is constant in \(n \leq a - 1\). Actually we have

\[
\langle \psi_n^+, \varphi_n^- \rangle = 0 , \quad (n \leq a - 1) \quad (6.10)
\]

because of the \(\ell^2\)-condition, and not by resorting to (6.9). To sum up: Since \(A_n, B_n \in GL_N(\mathbb{C})\), \((n \leq a - 1)\) the solutions of (6.7, 6.8) with \(n \leq a - 2\) are bijectively determined by \(\psi_{a-1}^+, \varphi_{a-1}^- \in \mathbb{C}^N\); among them, those that are \(\ell^2\) correspond to subspaces (independent of \(A_n, B_n\)) of complementary dimensions. This follows from (2.19) with \(\chi = 0\).

The claim now follows by applying the following lemma to eqs. (6.7, 6.9) for \(n = a - 1\) by identifying \(A_a = A, B_a = B, \psi_{a-1}^+ = \psi^+, \psi_a^+ = \tilde{\psi}^+, B_{a-1}^* \psi_{a-1}^- = \psi^-, \psi_a^- = \tilde{\psi}^-, \) and by using (6.10) for \(n = a - 1\).

\[\square\]

Lemma 14. Let an orthogonal decomposition \(\mathbb{C}^N = V^+ \oplus V^-\) and matrices \(A, B \in \text{Mat}_N(\mathbb{C})\) be given. We consider the set of equations

\[
A \psi^+ + B \tilde{\psi}^+ = 0 , \quad (6.11)
\]

\[
A^* \tilde{\psi}^- + \psi^- = 0 , \quad B^* \hat{\psi}^- = 0 \quad (6.12)
\]

in the the unknowns \(\psi^+ \in V^+, \tilde{\psi}^+ \in \mathbb{C}^N\). Then

\[
\dim\{(\psi^+, \tilde{\psi}^+)\}_{(6.11)} - \dim\{(\psi^-, \hat{\psi}^-)\}_{(6.12)} = \text{dim } V^+ . \quad (6.13)
\]

Proof. Let \(P\) be the orthogonal projection onto \(V^+\), whence \(\text{dim } \text{im } P = \text{dim } V^+\). Then the dimensions on the LHS of (6.13) are unaffected upon supplementing (6.11, 6.12) with \(P \psi^+ = \psi^+\) and \(P \psi^- = 0\) respectively, while solving for \((\psi^+, \tilde{\psi}^+) \in \mathbb{C}^N \oplus \mathbb{C}^N = \mathbb{C}^{2N}\). We are then left computing

\[
I := \text{dim ker } T_+ - \text{dim ker } T_-
\]

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with \( T_+ : \mathbb{C}^{2N} \to \mathbb{C}^{2N} \), \( T_- : \mathbb{C}^{2N} \to \mathbb{C}^{3N} \) given by

\[
T_+ = \begin{pmatrix} 1 - P & 0 \\ A & B \end{pmatrix}, \quad T_- = \begin{pmatrix} P & 0 \\ 1 & A^* \\ 0 & B^* \end{pmatrix}.
\]

Using that

\[
T_-^* = \begin{pmatrix} P & 1 & 0 \\ 0 & A & B \end{pmatrix}, \quad \left( P - P^0 \right) A B^* = \begin{pmatrix} P & 1 - P & 0 \\ 0 & A & B \end{pmatrix}
\]

have the same range, we have

\[
\dim \text{im} T_-^* = \dim V^+ + \dim \text{im} T_+;
\]

using also that \( \dim \ker T_- = 2N - \dim \text{im} T_-^* \) we find

\[
I = \dim \ker T_+ + \dim \text{im} T_+ - 2N + \dim V^+ = \dim V^+.
\]

\[\square\]

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