A Stability Criterion for Nonparametric Minimal Submanifolds

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Abstract

An \( n \) dimensional minimal submanifold \( \Sigma \) of \( \mathbb{R}^{n+m} \) is called non-parametric if \( \Sigma \) can be represented as the graph of a vector-valued function \( f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^m \). This note provides a sufficient condition for the stability of such \( \Sigma \) in terms of the norm of the differential \( df \).

1 Introduction

A minimal submanifold is stable if the second derivative of the volume functional with respect to any compact supported normal variational field is non-negative. A non-parametric minimal hypersurface in the Euclidean space \( \mathbb{R}^{n+1} \) is always stable. This is no longer true when the codimension is greater than one. A non-stable non-parametric minimal surface in four dimension was constructed by Lawson and Osserman in [5]. It seems very little is known about the stability of higher codimension minimal submanifolds except for calibrated ones. Recall a submanifold \( \Sigma \) is calibrated by a calibrating form \( \Omega \) if \( \Omega|_\Sigma \) is the volume form of \( \Sigma \), or \( *\Omega = 1 \) where \(*\) is the Hodge star operator. In particular, a non-parametric hypersurface in \( \mathbb{R}^{n+1} \) is calibrated by the \( n \) form \( i(N)dx^1 \cdots dx^{n+1} \) where \( N \) is a extension of the unit normal vector field of \( \Sigma \).

In [8], the second author constructs solutions to the Dirichlet problem of minimal surface systems in higher dimension and codimension and the solutions satisfies \( *\Omega > \frac{1}{2} \) for \( \Omega \) the volume form of an \( n \)-dimensional subspace. When \( \Sigma \) is the graph of \( f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^m \) and \( \Omega \) is the volume form of the domain \( \mathbb{R}^n \) extending to the whole \( \mathbb{R}^{n+m} \), we have the relation

\[ (*) \]
\[ *\Omega = \frac{1}{\sqrt{\det(I + (df)^T df)}}. \]  
\(*\Omega\) is actually the Jacobian of the projection map \(\pi: \Sigma \mapsto D\). In particular, a lower bound on \(*\Omega\) implies an upper bound on the norm of \(df\). In this paper, we discover a criterion for the stability of minimal submanifolds in terms of such condition.

**Theorem A.** Let \(\Sigma\) be the graph of \(f: D \subset \mathbb{R}^n \mapsto \mathbb{R}^m\). If \(\Sigma\) is minimal and \(||df|| \leq \frac{1}{\sqrt{1+c}}\), then \(\Sigma\) is stable. Here the norm \(||df||\) is defined to be \(\sup_{|V|=1} |df(V)|\) and \(c\) is a constant which is 1 if \(m \leq 2\) or \(n \leq 2\), and is \(\min{(m - 1, n - 1)}\) in other cases.

Since \(*\Omega = \frac{1}{\sqrt{\det(I + (df)^T df)}} = \frac{1}{\sqrt{\prod(1+\lambda_i^2)}}\), it is not hard to see the following consequence.

**Corollary.** Let \(c\) be the constant as in Theorem A. If \(\Sigma\) is minimal and \(*\Omega \geq \frac{c}{2(c+1-\sqrt{1+c})}\), then \(\Sigma\) is stable.

In particular, when \(m \leq 2\) or \(n \leq 2\), if \(\Sigma\) is minimal and \(*\Omega \geq \frac{2+\sqrt{2}}{4}\), then \(\Sigma\) is stable. The condition \(*\Omega > \frac{c}{2(c+1-\sqrt{1+c})}\) corresponds a region in the Grassmannian. For minimal surfaces in \(\mathbb{R}^3\), Barbosa and Do Carmo [1,2] proved if the area of the image of the Gauss map is less than \(2\pi\) then \(D\) is stable. Their result was also obtained by D. Fischer-Colbrie and R. Schoen in [4] by a different method. This raises the general question of how to characterize stability by the Gauss map of a minimal submanifold. We remark the condition is not sharp in view of the codimension one case.

The proof of Theorem A utilizes a second variation formula of R. Mclean [6] for calibrated submanifolds. The corresponding formula for complex submanifolds of Kähler manifolds was derived by J. Simons [7].

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2 Second Variation Formula

We first recall the second variation formula for minimal submanifolds. Let \( F_0 : \Sigma \hookrightarrow \mathbb{R}^{n+m} \) be a minimal submanifold and let \( F : \Sigma \times [0,1) \hookrightarrow \mathbb{R}^{n+m} \) be a one-parameter family of immersions with \( F(\cdot,0) = F_0 \). We may assume the variation field \( V = \bar{F}_s(\frac{\partial}{\partial s}) \) is normal and of compact support. For simplicity, we will identify \( F_0(\Sigma) \) with \( \Sigma \) and denote \( F_s(\cdot,s) \) by \( F_s \). A coordinate system \( \{x^i\} \) in a neighborhood of \( p \in \Sigma \) is fixed. Let \( g_{ij}^s \) be the induced metric and \( dv_s = \sqrt{\det g_{ij}} dX \) be the volume form on \( F_s(\Sigma) \). At \( s = 0 \), the volume form will be written as \( dv \) instead.

We recall the second variation formula from [3]:

\[
\frac{d^2}{ds^2} \bigg|_{s=0} \int_\Sigma dv_s = \int_\Sigma \left( ||\nabla^N V||^2 - B(V,V) \right) dv
\]

where \( \nabla^N V \) is the covariant derivative of \( V \) as a section of the normal bundle and \( B(V,V) = \sum_{ijkl} g^{ik} g^{jl} \langle \partial^2 F/\partial x^i \partial x^j, V \rangle \langle \partial^2 F/\partial x^k \partial x^l, V \rangle \). The minimal submanifold \( \Sigma \) is stable if and only if

\[
\int_\Sigma ||\nabla^N V||^2 dv \geq \int_\Sigma B(V,V) dv
\]

for any normal vector field with compact support. Since the second variation formula does not depend on \( \partial^2 F/\partial s^2 \), we may consider only the case \( \partial^2 F/\partial s^2 \) is zero at \( s = 0 \). In this case, the following equation holds at every point.

\[
\frac{\partial^2}{\partial s^2} \bigg|_{s=0} \sqrt{\det g_{ij}}(s) = (||\nabla^N V||^2 - B(V,V)) \sqrt{\det g_{ij}} \tag{2.1}
\]

In [3], a different second variation formula is derived in the presence of a calibrating form \( \Omega \). In the following, we derive the formula for completeness. We shall assume \( \Omega \) is locally an exact form.

Now \( \int_{F_s(\Sigma)} \Omega = \int_{\Sigma} F_s^* \Omega \) is a constant. Write \( F_s^* \Omega = \ast \Omega(s) \sqrt{\det g_{ij}}(s) dX \), where \( \ast \Omega = \frac{1}{\sqrt{\det g_{ij}}}(\frac{\partial F}{\partial x^1}, \ldots, \frac{\partial F}{\partial x^n}) \).

Since

\[
0 = \frac{d^2}{ds^2} \int_\Sigma F_s^* \Omega
= \int_\Sigma [(\frac{\partial^2}{\partial s^2} \ast \Omega) \sqrt{\det g_{ij}}(s) + 2(\frac{\partial}{\partial s} \ast \Omega)(\frac{\partial}{\partial s} \sqrt{\det g_{ij}}(s)) + \ast \Omega \frac{\partial^2}{\partial s^2} \sqrt{\det g_{ij}}(s)] dX
\]

3
and \( \frac{\partial}{\partial s}|_{s=0} \sqrt{\det g_{ij}(s)} = 0 \) by the minimal condition, at \( s = 0 \) we have

\[
\int_{\Sigma} *\Omega \frac{\partial^2}{\partial s^2} \sqrt{\det g_{ij}(s)} \, dX = - \int_{\Sigma} \left( \frac{\partial^2}{\partial s^2} *\Omega \right) \, dv
\]

That is,

\[
\int_{\Sigma} *\Omega \left( ||\nabla^N V||^2 - B(V, V) \right) \, dv = - \int_{\Sigma} \left( \frac{\partial^2}{\partial s^2} \right)\bigg|_{s=0} *\Omega \, dv \quad (2.2)
\]

We shall compute \( \frac{\partial^2}{\partial s^2} *\Omega \) using the formula \( *\Omega = \frac{1}{\sqrt{\det g_{ij}}} \Omega \left( \frac{\partial F}{\partial x^1}, \cdots, \frac{\partial F}{\partial x^n} \right) \).

Thus

\[
\frac{\partial^2}{\partial s^2} *\Omega = \frac{\partial^2}{\partial s^2} \left( \frac{1}{\sqrt{\det g_{ij}}} \right) \Omega \left( \frac{\partial F}{\partial x^1}, \cdots, \frac{\partial F}{\partial x^n} \right) + 2 \frac{\partial}{\partial s} \left( \frac{1}{\sqrt{\det g_{ij}}} \right) \frac{\partial}{\partial s} \Omega \left( \frac{\partial F}{\partial x^1}, \cdots, \frac{\partial F}{\partial x^n} \right) + \left( \frac{1}{\sqrt{\det g_{ij}}} \right) \frac{\partial^2}{\partial s^2} \Omega \left( \frac{\partial F}{\partial x^1}, \cdots, \frac{\partial F}{\partial x^n} \right) \quad (2.3)
\]

At \( s = 0 \), the minimal condition implies the second term vanishes and the first term becomes

\[
\frac{\partial^2}{\partial s^2} \bigg|_{s=0} \left( \frac{1}{\sqrt{\det g_{ij}}} \right) = -(\det g_{ij})^{-1} \frac{\partial^2}{\partial s^2} \bigg|_{s=0} \left( \sqrt{\det g_{ij}} \right)
\]

Because \( \frac{\partial^2 F}{\partial s^2} \) is zero at \( s = 0 \), the third term is

\[
\frac{\partial^2}{\partial s^2} \Omega \left( \frac{\partial F}{\partial x^1}, \cdots, \frac{\partial F}{\partial x^n} \right) = 2[\Omega \left( \frac{\partial^2 F}{\partial s \partial x^1}, \frac{\partial^2 F}{\partial s \partial x^2}, \cdots, \frac{\partial F}{\partial x^n} \right) + \cdots] \quad (2.4)
\]

Denote \( \frac{\partial^2 F}{\partial s^2} \) by \( \partial_i \), then \( \frac{\partial^2 F}{\partial s \partial x^i} = \frac{\partial^2 F}{\partial x^i s} = \nabla_{\partial_i} V \) where \( \nabla \) is the connection on \( \mathbb{R}^{n+m} \) and \( V = \frac{\partial F}{\partial s} \) is the variation field. In the following computation, \( (\cdot)^T \) and \( (\cdot)^N \) denote the tangent and normal part of vector respectively.

\[
\Omega \left( \frac{\partial^2 F}{\partial s \partial x^1}, \frac{\partial^2 F}{\partial s \partial x^2}, \cdots, \frac{\partial F}{\partial x^n} \right) = \Omega \left( (\nabla_{\partial_1} V)^T, (\nabla_{\partial_1} V)^N, (\nabla_{\partial_2} V)^T, (\nabla_{\partial_2} V)^N, \cdots, \partial_n \right) = \Omega \left( (\nabla_{\partial_1} V)^T, (\nabla_{\partial_2} V)^T, \cdots, \partial_n \right) + \Omega \left( ((\nabla_{\partial_1} V)^T)^N, (\nabla_{\partial_2} V)^N, \cdots, \partial_n \right)
\]
We can assume \( \{x^i\} \) is a normal coordinate system in a neighborhood of \( p \) with respect to the induced metric on \( \Sigma \). Hence \( g_{ij}(0) = \delta_{ij} \) at \( p \). We do the computation at point \( p \) and get

\[
\Omega((\nabla_{\partial_1} V)^T, (\nabla_{\partial_2} V)^T, \cdots \partial_n) = \ast \Omega(\langle V, \nabla_{\partial_1} \partial_1 \rangle \langle V, \nabla_{\partial_2} \partial_2 \rangle - \langle V, \nabla_{\partial_1} \partial_2 \rangle^2)
\]

Continue from equation (2.4), we derive

\[
2[\Omega(\frac{\partial^2 F}{\partial s \partial x^1}, \frac{\partial^2 F}{\partial s \partial x^2} \cdots \frac{\partial F}{\partial x^n}) + \cdots] = 2 \sum_{i<j} \Omega(\partial_1, \cdots (\nabla_{\partial_1} V)^T, \cdots (\nabla_{\partial_j} V)^N, \cdots \partial_n)
\]

\+

\[
2 \sum_{i<j} \Omega(\partial_1, \cdots (\nabla_{\partial_1} V)^N, \cdots (\nabla_{\partial_j} V)^T, \cdots \partial_n)
\]

\+

\[
2 \sum_{i<j} \Omega(\partial_1, \cdots (\nabla_{\partial_1} V)^N, \cdots (\nabla_{\partial_j} V)^N, \cdots \partial_n)
\]

\+

\[
2 \ast \Omega(\sum_{i<j} \langle V, \nabla_{\partial_1} \partial_1 \rangle \langle V, \nabla_{\partial_2} \partial_2 \rangle - \langle V, \nabla_{\partial_1} \partial_2 \rangle^2)
\]

It thus follows from (2.3) that

\[
\frac{\partial^2}{\partial s^2} |_{s=0} \ast \Omega = - \ast \Omega ||\nabla^N V||^2 + \ast \Omega \sum_{i,j} \langle V, \nabla_{\partial_i} \partial_j \rangle^2
\]

\+

\[
2 \sum_{i<j} \Omega(\partial_1, \cdots (\nabla_{\partial_1} V)^T, \cdots (\nabla_{\partial_j} V)^N, \cdots \partial_n)
\]

\+

\[
2 \sum_{i<j} \Omega(\partial_1, \cdots (\nabla_{\partial_1} V)^N, \cdots (\nabla_{\partial_j} V)^T, \cdots \partial_n)
\]

\+

\[
2 \sum_{i<j} \Omega(\partial_1, \cdots (\nabla_{\partial_1} V)^N, \cdots (\nabla_{\partial_j} V)^N, \cdots \partial_n)
\]

\+

\[
2 \ast \Omega(\sum_{i<j} \langle V, \nabla_{\partial_1} \partial_1 \rangle \langle V, \nabla_{\partial_2} \partial_2 \rangle - \langle V, \nabla_{\partial_1} \partial_2 \rangle^2)
\]

However,

\[
\ast \Omega \sum_{i,j} \langle V, \nabla_{\partial_1} \partial_j \rangle^2 + 2 \ast \Omega(\sum_{i<j} \langle V, \nabla_{\partial_1} \partial_1 \rangle \langle V, \nabla_{\partial_2} \partial_2 \rangle - \langle V, \nabla_{\partial_1} \partial_2 \rangle^2)
\]

\[
= \ast \Omega(\sum_i \langle V, \nabla_{\partial_i} \partial_i \rangle)^2 = 0
\]
Therefore,
\[
\frac{\partial^2}{\partial s^2} |_{s=0} \ast \Omega = - \ast \Omega \| \nabla^N V \|^2 + 2 \sum_{i<j} \Omega(\partial_1, \cdots, (\nabla_{\partial_i} V)^T, \cdots, (\nabla_{\partial_j} V)^N, \cdots, \partial_n)
\]
\[
+ 2 \sum_{i<j} \Omega(\partial_1, \cdots, (\nabla_{\partial_i} V)^N, \cdots, (\nabla_{\partial_j} V)^T, \cdots, \partial_n)
\]
\[
+ 2 \sum_{i<j} \Omega(\partial_1, \cdots, (\nabla_{\partial_i} V)^N, \cdots, (\nabla_{\partial_j} V)^N, \cdots, \partial_n)
\]

Combine equations (2.1) and (2.2), we obtain

**Proposition 2.1** Let \( \Omega \) be an exact parallel n-form and \( \Sigma \) be an n dimensional minimal submanifold in \( \mathbb{R}^{n+m} \). Assume that \( V \) is a normal variation field and \( \nabla_N V = 0 \) along \( \Sigma \). Then one has
\[
\int_{\Sigma} \ast \Omega(\| \nabla^N V \|^2 - B(V, V)) \, dv
\]
\[
= \int_{\Sigma} [\ast \Omega(\| \nabla^N V \|^2 - 2 \sum_{i<j} \Omega(\partial_1, \cdots, (\nabla_{\partial_i} V)^T, \cdots, (\nabla_{\partial_j} V)^N, \cdots, \partial_n) - 2 \sum_{i<j} \Omega(\partial_1, \cdots, (\nabla_{\partial_i} V)^N, \cdots, (\nabla_{\partial_j} V)^N, \cdots, \partial_n) - 2 \sum_{i<j} \Omega(\partial_1, \cdots, (\nabla_{\partial_i} V)^T, \cdots, (\nabla_{\partial_j} V)^N, \cdots, \partial_n)] \, dv
\]

**3 Proof of Theorem A**

The idea now is to show the right hand side of equation (2.5) is greater than or equal to
\[
\delta \int_{\Sigma} \ast \Omega(\| \nabla^N V \|^2 - B(V, V)) \, dv
\]
for some \( \delta < 1 \). We shall express the integrand in the right hand side of equation (2.5) in terms of a particular orthonormal basis. At any point \( p \), we consider the singular value decomposition of \( df : \mathbb{R}^n \mapsto \mathbb{R}^m \). We have
\[
df(a_i) = \lambda_i a_{n+i}
\]

6
where \( \lambda_i \geq 0 \) are the singular values of \( df \), or eigenvalues of \( \sqrt{(df)^T df} \). 
\( \{a_i\}_{i=1,\ldots,n} \) is an orthonormal basis of eigenvectors of \( \sqrt{(df)^T df} \). The set \( \{a_{n+i}\} \) can be completed to form an orthonormal basis \( \{a_{\alpha}\}_{\alpha=n+1,\ldots,n+m} \) for \( \mathbb{R}^m \). (In case \( m < n \), we will have \( \lambda_i = 0 \) for \( i > m \) and \( \{a_{n+i}\}_{i=1,\ldots,m} \) forms an orthonormal basis for \( \mathbb{R}^m \).) Now \( \{e_i = \frac{1}{\sqrt{1+\lambda_i^2}}(a_i + \lambda_i a_{n+i})\} \) and \( \{e_{n+i} = \frac{1}{\sqrt{1+\lambda_i^2}}(a_{n+i} - \lambda_i a_i)\} \) can be completed to give orthonormal basis of the tangent and normal space. The orthonormal basis for the normal space is denoted by \( \{e_{\alpha}\}_{\alpha=n+1,\ldots,n+m} \). In these bases we denote \( (\nabla_{e_i} V)^N = \sum_{\alpha} V_i^\alpha e_{\alpha} \) and \( (\nabla_{e_i} V)^T = -\sum_{\alpha,j} V^\alpha h_{\alpha ij} e_j \). We shall assume \( m \geq n \) in the following calculation, the other case can be treated similarly.

At the point \( p \) in these bases, the integrand of the right hand side of equation (2.3) can be written as

\[
\sum_{i,\alpha} \left( (V_i^\alpha)^2 - 2 \sum_{i<j} \lambda_i \lambda_j V_i^{n+i} V_j^{n+j} + 2 \sum_{i<j} \lambda_i \lambda_j V_i^{n+j} V_j^{n+i} \right) + 2 \sum_{a} \sum_{i<j} V^\alpha h_{a ii} V_j^{n+j} \lambda_j - 2 \sum_{a} \sum_{i<j} V^\alpha h_{a ij} V_j^{n+i} \lambda_i + 2 \sum_{a} \sum_{i<j} V^\alpha h_{a ij} V_i^{n+i} \lambda_i - 2 \sum_{a} \sum_{i<j} V^\alpha h_{a ij} V_i^{n+j} \lambda_j \]

which is the same as

\[
\sum_{i,\alpha} \left( (V_i^\alpha)^2 - 2 \sum_{i\neq j} \lambda_i \lambda_j V_i^{n+i} V_j^{n+j} + \sum_{i\neq j} \lambda_i \lambda_j V_i^{n+j} V_j^{n+i} \right) + 2 \sum_{a} \sum_{i\neq j} V^\alpha h_{a ii} V_j^{n+j} \lambda_j - 2 \sum_{a} \sum_{i\neq j} V^\alpha h_{a ij} V_j^{n+i} \lambda_i \]

By minimality, we have

\[
2 \sum_{i\neq j} V^\alpha h_{a ii} V_j^{n+j} \lambda_j = -2 \sum_{j} V^\alpha h_{a jj} V_j^{n+j} \lambda_j
\]

Define \( \Xi \) by

\[
\Xi = \sum_{i,\alpha} \left( (V_i^\alpha)^2 - \sum_{i\neq j} \lambda_i \lambda_j V_i^{n+i} V_j^{n+j} + \sum_{i\neq j} \lambda_i \lambda_j V_i^{n+j} V_j^{n+i} \right) - 2 \sum_{a} \sum_{i,j} V^\alpha h_{a ij} V_j^{n+i} \lambda_i
\]
If we can show
\[ \Xi \geq \delta \left( \sum_{i,\alpha} (V_i^\alpha)^2 - \sum_{i,j} (\sum_{\alpha} V_i^\alpha h_{ij})^2 \right) \]
for some \( \delta < 1 \), then we are done in review of equation \((2.5)\).

By Cauchy-Schwarz inequality, for any \( \epsilon > 0 \) to be determined,
\[ -2 \sum_{i,j,\alpha} V_i^\alpha h_{ij} V_j^{n+i} \lambda_i \geq -\epsilon \sum_{i,j} (\sum_{\alpha} V_i^\alpha h_{ij})^2 - \frac{1}{\epsilon} \sum_{i,j} (V_j^{n+i} \lambda_i)^2 \]

Therefore
\[ \Xi \geq \sum_{i,\alpha} (V_i^\alpha)^2 - \sum_{i \neq j} \lambda_i \lambda_j V_i^{n+i} V_j^{n+j} + \sum_{i \neq j} \lambda_i \lambda_j V_i^{n+j} V_j^{n+i} - \frac{1}{\epsilon} \sum_{i,j} (V_j^{n+i} \lambda_i)^2 \]
\[- \epsilon \sum_{i,j} (\sum_{\alpha} V_i^\alpha h_{ij})^2 \]

Now assume each \( \lambda_i \leq \eta \), then
\[ \Xi \geq \sum_{i,\alpha} (V_i^\alpha)^2 - \eta^2 \sum_{i \neq j} |V_i^{n+i}| |V_j^{n+j}| - \eta^2 \sum_{i \neq j} |V_i^{n+j}| |V_j^{n+i}| - \frac{\eta^2}{\epsilon} \sum_{i,j} (V_j^{n+i})^2 \]
\[- \epsilon \sum_{i,j} (\sum_{\alpha} V_i^\alpha h_{ij})^2 \]

By Cauchy-Schwarz inequality
\[ \sum_{i \neq j} |V_i^{n+i}| |V_j^{n+j}| \leq (n-1) \sum_i (V_i^{n+i})^2 \]
and
\[ \sum_{i \neq j} |V_i^{n+j}| |V_j^{n+i}| \leq \sum_{i \neq j} (V_i^{n+j})^2 \]

In a general case, the coefficient in the right hand side of the first inequality should be \( \min(m-1, n-1) \). Let \( c = 1 \) if \( m \leq 2 \) or \( n \leq 2 \) and \( c = \min(m-1, n-1) \) in other cases. Plug into equation \((3.1)\), we obtain,
\[ \Xi \geq (1 - \frac{\eta^2}{\epsilon} - c \eta^2) \sum_{i,\alpha} (V_i^\alpha)^2 - \epsilon \sum_{i,j} (\sum_{\alpha} V_i^\alpha h_{ij})^2 \]
\[ (3.2) \]
We need to have \(1 - \frac{\eta^2}{\epsilon} - c\eta^2 \geq \epsilon\) which is \(\eta^2 \leq \frac{\epsilon(1 - \frac{\sqrt{c + 1} - 1}{\sqrt{c}})}{1 + \epsilon}\). It is not hard to see \(\max_{0 < \epsilon < 1} \frac{\epsilon(1 - \frac{\sqrt{c + 1} - 1}{\sqrt{c}})^2}{1 + \epsilon} = \frac{(\sqrt{c + 1} - 1)^2}{c}\) when \(\epsilon = \frac{\sqrt{c + 1} - 1}{\sqrt{c}}\). Thus if we assume \(|df| \leq \frac{\sqrt{1 + c - 1}}{\sqrt{c}}\), then each \(|\lambda_i| \leq \frac{\sqrt{1 + c - 1}}{\sqrt{c}}\) and

\[
\Xi \geq \frac{\sqrt{1 + c - 1}}{c} \left( \sum_i (V_i^\alpha)^2 - \sum_{i,j} \sum_\alpha (\sum V^\alpha h_{\alpha ij})^2 \right)
\]

Theorem A is proved.

4 Examples

The construction in [8] supplies examples for such stable minimal submanifolds. Given any \(\phi : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^m\) defined on a convex domain \(D\), we can scale \(\phi\) so that on the graph of \(\phi\), \(\ast\Omega \geq \frac{\epsilon}{2(c + 1 - \sqrt{1 + c})}\) and the derivative of \(\phi\) satisfies the requirement in [8]. It was proved in [8] that the Cauchy-Dirichlet problem of the mean curvature flow for initial data \(\phi\) is solvable and \(\ast\Omega \geq \frac{\epsilon}{2(c + 1 - \sqrt{1 + c})}\) is preserved along the flow. The flows converges to a minimal submanifold which is stable by the Corollary.

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