Dimensional Reduction of the Generalized DBI

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Abstract

We study the generalized Dirac-Born-Infeld (DBI) action of a $q$-brane ending on a $p$-brane with a $(q+1)$-form background. This action has the equivalence of commutative and non-commutative description, which can be understood from the generalized metric and Nambu-Sigma model. This theory reduces to the usual DBI action when $q = 1$. We also discuss the structure of the dimensional reduction of the generalized DBI.

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1 Introduction

In string theory, T-duality shows the equivalence of two theories that seem different by exchanging radius $R$ and radius $\frac{\alpha'}{R}$. For closed string, T-duality exchanges winding and momentum modes. In the case of open string, it exchanges the Dirichlet and Neumann boundary conditions. The low energy effective description of an open string ending on a single D-brane can be described by the DBI model [1]. We can observe T-duality in the DBI model. T-duality can also be seen in string field theory [2].

One interesting problem of brane theory is to construct action of a single M5-brane. An important observation is the equivalence of commutative and non-commutative description [3, 4] by studying the DBI, non-abelian DBI and Nambu-Poisson M5 [5] model. Hopefully, the equivalence of commutative and non-commutative description gives a strong constraint on the final form of a single M5-brane action. From this equivalence, we can deduce the suitable form of brane theory. The related closed-open string relations can also be obtained from the Nambu-Sigma model, which is the generalization of the Poisson-Sigma model. The description of this theory is a $q$-brane ending on a $p$-brane. This theory is called the generalized DBI which can be consistently reduced to DBI by considering 2-form background. One nice thing of the generalized DBI is that we can find the same form for the M5-brane as given in [6] up to the second order when considering the 5-brane theory in 3-form background. This implies that the DBI-part of the M5-brane theory can be obtained from the generalized DBI [7]. More evidences on the validity of the generalized DBI can be found by looking into the generalized metric as done in [8].

On the other hand, the recent interesting developments of T-duality are double field theory [9] and generalized geometry [10]. We can see manifest $O(D, D)$ by doubling coordinates in the formulation of the double field theory. The meaning of manifest $O(D, D)$ is to embed Busher’s rule in the $O(D, D)$ structure. Then we can exchange coordinates to obtain Busher’s rule. The current stage of double field theory is established on closed strings. Currently, we only have few understanding on open strings. The most important thing is the “stringy geometry” [11, 12] constructed in double field theory in understanding the non-geometric flux [13]. We already understood how to generalize the standard 10-dimensional supergravity (NS-NS) to the new 10-dimensional supergravity as shown in [12, 14]. We also expect that this different structure can give inspirations.
to other fields in analogy to the famous example that the quantum correction of string
theory inspires new gravity model [15]. Under the strong constraint (that is a constraint
for removing the additional coordinate), double field theory will reduce to the generalized
geometry. The study of relaxing constraint is in [16]. Other interesting features of double
field theory are $\alpha'$ geometry [17], exceptional field theory [18], D-brane [19] and others
[20]. The recent reviews are in [21]. The related discussions of generalized geometry
are curvature, torsion [22], Courant algebra [23], reduction [24], exceptional generalized
geometry [25] and supergravity [26].

The main task of this paper is to carry out the dimensional reduction of the general-
ized DBI. We perform dimensional reduction from a $(q + 1)$-brane ending on a $(p + 1)$-
brane to a $q$-brane ending on a $p$-brane. In this work, we only consider flat spacetime,
constant background and $(q+1)$-form gauge field only exists in $(q+1)$-dimensional world-
volume directions (no time direction) in $q$-$p$ system. The non-trivial result is that the
appearance of the $2(q + 1)$-th root can be shown by the equivalence of commutative and
non-commutative description, which is inherited under the consistent dimensional reduc-
tion. The most interesting case is a 2-brane ending on a 5-brane. It can reduce to a
1-brane ending on a 4-brane. It shows that the non-trivial 2-brane ending on a 5-brane
can reduce to DBI theory in our simple consideration, and gives an interesting interpre-
tation to our calculations. The 2-5 system possibly inspires to the M2-M5 system, which
can go to F1-D4 system by dimensional reduction. We also explore the possibility of
extending the generalized DBI. We show that this theory is possible to include one-form
gauge field based on the consistency of dimensional reduction.

The plan of this paper is to first review the generalized DBI in Sec. 2. Then we
show the discussion of dimensional reduction without scalar fields in Sec. 3. Dimensional
reduction with scalar fields is in Sec. 4. Finally, we conclude in Sec. 5.

2 Review of the Generalized DBI

In this section, we review the generalized DBI. At first, we show the closed-open string
relations from string sigma model. Secondly, we generalize Poisson-Sigma model to
Nambu-Sigma model. We can also obtain generalized closed-open relations from the
Nambu-Sigma model. Thirdly, we introduce membrane action. We find that the action
is equivalent to Nambu sigma model under the gauge fixing. We also introduce worldsheet
We define our notations as follows. We denote the index $A$ to be worldvolume direction. While $a, b = 1, 2, \cdots, p$ are reserved for the spatial components of worldvolume coordinates. $\mu, \nu = 0, 1, \cdots, D - 1$ denote target space indices and $w = 0, 1$ denote worldsheet indices. In addition, we use $I$ to denote transverse direction and $i, j$ to denote antisymmetric indices, $i = (i_1, i_2, \cdots, i_r)$ with $0 \leq i_1 < i_2 < \cdots < i_r \leq (r + 1)$, where $r$ is the dimension of $i$.

### 2.1 Closed-Open Relations

We first introduce action of the Poisson-Sigma model \[ S_P = \int_{\Sigma} \left( A_\mu \wedge dX^\mu - \frac{1}{2} \Pi^{\mu \nu} A_\mu \wedge A_\nu \right), \quad \Pi \equiv \frac{1}{2} \Pi^{\mu \nu} (X) \partial_\mu \wedge \partial_\nu, \] where $X : \Sigma \rightarrow M$, $\Sigma$ is the two dimensional world-sheet and $M$ is the target space manifold. The one-form field $A(\sigma)$ is on $\Sigma$ and $\Pi$ is an antisymmetric tensor. From the equations of motion

$$
\begin{align*}
    dX^\mu - \Pi^{\mu \nu} A_\nu &= 0, \\
    dA_\mu + \frac{1}{2} \partial_\mu \Pi^{\nu \rho} A_\nu \wedge A_\rho &= 0,
\end{align*}
$$

we can show that the bi-vector $\Pi$ satisfy the Jacobi identity. The first equation is the equation of motion for $A_\mu$. The other one is the equation of motion for $X^\mu$. We can add a metric term in the Poisson-Sigma model to obtain the non-topological generalized Poisson-Sigma model

$$
S_P = \int_{\Sigma} \left( A_\mu \wedge dX^\mu - \frac{1}{2} \Pi^{\mu \nu} A_\mu \wedge A_\nu - \frac{1}{2} (G^{-1})^{\mu \nu} A_\mu \wedge *A_\nu \right),
$$

where $*A_\nu$ is the Hodge dual of $A_\nu$. The signature of world-sheet is $(-, +)$ and volume form $d^2 \sigma \equiv d\sigma^0 \wedge d\sigma^1$. The $A_\mu \equiv A_{\mu w}(\sigma) d\sigma^w$ are auxiliary fields. By using the equation of motion of $A_\mu$, the action \[ (3) \] can be rewritten as the string sigma model action,

$$
S_S = - \int_{\Sigma} \frac{1}{2} (g_{\mu \nu} dX^\mu \wedge *dX^\nu + B_{\mu \nu} dX^\mu \wedge dX^\nu),
$$

where the $g$ and $B$ are defined by the closed-open string relations \[ (3) \]

$$
\frac{1}{g + B} = G^{-1} + \Pi \Rightarrow \quad G = g - B g^{-1} B, \quad \Pi = -G^{-1} B g^{-1} = -g^{-1} B G^{-1}.
$$
The action (3) can also be rewritten in terms of the components of $\eta_\mu \equiv -A_\mu(\sigma)$ and $\tilde{\eta}_\nu \equiv A_{\nu 0}(\sigma)$, the action is

$$S_P = \int d^2 \sigma \left[ -\frac{1}{2} (G^{-1})^{\mu\nu} \eta_\mu \eta_\nu + \frac{1}{2} (G^{-1})^{\mu\nu} \tilde{\eta}_\mu \tilde{\eta}_\nu + \eta_\mu \partial_0 X^\mu + \tilde{\eta}_\mu \partial_1 X^\mu - \Pi^{\mu\nu} \eta_\mu \tilde{\eta}_\nu \right].$$

(6)

We can use matrix notation to rewrite it. We define

$$\eta \equiv \eta_\mu, \quad \tilde{\eta} \equiv \tilde{\eta}_\nu, \quad G \equiv G^{\mu\nu}, \quad X \equiv X^\mu, \quad \Pi \equiv \Pi^{\mu\nu}.$$  

(7)

The action is

$$S_P = \int d^2 \sigma \left[ -\frac{1}{2} \eta^T G^{-1} \eta + \frac{1}{2} \tilde{\eta}^T G^{-1} \tilde{\eta} + \partial_0 X^T \eta + \partial_1 X^T \tilde{\eta} - \eta^T \Pi \eta \right].$$

(8)

where the superscript $T$ indicates transpose of matrix. By using the matrix notation, it is easier to generalize Poisson-Sigma model.

2.2 Generalized Closed-Open Relations

We introduce Nambu-Sigma model at first. It is a generalized Poisson-Sigma model. The action is

$$S_N = \int d^{q+1} \sigma \left[ -\frac{1}{2} \eta^T \tilde{G}^{-1} \eta + \frac{1}{2} \tilde{\eta}^T \tilde{G}^{-1} \tilde{\eta} + \partial_0 X^T \eta + \partial_1 X^T \tilde{\eta} - \eta^T \Pi \eta \right],$$

(9)

where

$$\tilde{G}_{ij} = \sum_\pi \text{sgn}(\pi) G_{i_{\pi(1)}j_1} \cdots G_{i_{\pi(q)}j_q},$$

(10)

$\pi$ is a permutation and the antisymmetric product of partial derivatives

$$\tilde{\partial} X^i \equiv \sum_{a_1, \ldots, a_q = 1} \epsilon^{a_1 a_2 \cdots a_q} \partial_{a_1} X^{i_1} \cdots \partial_{a_q} X^{i_q},$$

(11)

where $0 \leq i_1 < \cdots < i_q \leq (q + 1)$. There are two types of metrics $G$ and $\tilde{G}$, auxiliary fields $\eta$ and $\tilde{\eta}$, and an antisymmetric $(q + 1)$-form tensor $\Pi$. We can integrate out the fields $\eta$ and $\tilde{\eta}$. Then the resulting action is

$$S_b = \frac{1}{2} \int d^{q+1} \sigma \left[ \partial_0 X^T g \partial_0 X - \tilde{\partial} X^T \tilde{g} \tilde{\partial} X \right] - \int d^{q+1} \sigma \partial_0 X^T C \tilde{\partial} X,$$

(12)
where
\[ g \equiv g_{\mu\nu}, \quad \tilde{g} \equiv \tilde{g}_{ij}, \quad C \equiv C_{\mu i}. \] (13)

We identify \( g \) by
\[ g = (G^{-1} + \Pi \tilde{G} \Pi^T)^{-1}, \quad \tilde{g} = (\tilde{G}^{-1} + \Pi^T \tilde{G} \Pi)^{-1}, \quad C = -g \Pi \tilde{G} = -G \Pi \tilde{g}. \] (14)

For the special case \( q = 1 \), these relations reduce to the closed-open string relations \[ 5 \]. We can also rewrite the action after using Wick rotation \( (\sigma^0 \rightarrow -i\sigma^0) \) by the compact matrix form
\[ S_{bE} = \frac{1}{2} \int d^{t+1}\sigma \left[ V^+ \left( \begin{array}{cc} g & C \\ -CT & \tilde{g} \end{array} \right) V \right], \quad \left( \exp(iS_b) = \exp(-S_{bE}) \right) \] (15)
where
\[ V \equiv \left( \begin{array}{c} i\partial_\tau X^\mu \\ \frac{\partial_\sigma}{\partial X^i} \end{array} \right). \] (16)

Let \( \mathcal{G} \) denote the matrix
\[ \mathcal{G} \equiv \left( \begin{array}{cc} g & C \\ -CT & \tilde{g} \end{array} \right). \] (17)
We can find the inverse matrix
\[ \mathcal{G}^{-1} = \left( \begin{array}{cc} (g + C \tilde{g}^{-1} CT)^{-1} & -(g + C \tilde{g}^{-1} CT)^{-1} \tilde{g}^{-1} \\ \tilde{g}^{-1} CT(g + C \tilde{g}^{-1} CT)^{-1} & (g + C \tilde{g}^{-1} CT)^{-1} \tilde{g} \end{array} \right), \] (18)

by using the analytic inversion formula
\[ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)^{-1} = \left( \begin{array}{cc} a^{-1} + a^{-1}b(d-ca^{-1}b)^{-1}ca^{-1} & -a^{-1}b(d-ca^{-1}b)^{-1} \\ -(d-ca^{-1}b)^{-1}ca^{-1} & (d-ca^{-1}b)^{-1} \end{array} \right) \]
\[ = \left( \begin{array}{cc} (a - bd^{-1}c)^{-1} & -(a - bd^{-1}c)^{-1}bd^{-1} \\ -d^{-1}c(a - bd^{-1}c)^{-1} & d^{-1} + d^{-1}c(a - bd^{-1}c)^{-1}bd^{-1} \end{array} \right). \] (19)

We also consider
\[ \mathcal{H} \equiv \left( \begin{array}{cc} G & \Phi \\ -\Phi^T & \tilde{G} \end{array} \right)^{-1} + \left( \begin{array}{cc} 0 & \Pi \\ -\Pi^T & 0 \end{array} \right) \]
\[ = \left( \begin{array}{cc} (G + \Phi \tilde{G}^{-1} \Phi^T)^{-1} & -(G + \Phi \tilde{G}^{-1} \Phi^T)^{-1} \tilde{G}^{-1} + \Pi \\ \tilde{G}^{-1} \Phi^T(G + \Phi \tilde{G}^{-1} \Phi^T)^{-1} - \Pi^T & (G + \Phi \tilde{G}^{-1} \Phi^T)^{-1} \Phi \tilde{G}^{-1} \end{array} \right). \] (20)
Interestingly, we can get the generalized closed-open relations by setting $G^{-1} = H$, the results are

$$g + C\hat{g}^{-1}C^T = G + \Phi\hat{g}^{-1}\Phi^T, \quad \hat{g} + C^Tg^{-1}C = \hat{G} + \Phi^TG^{-1}\Phi,$$

$$g^{-1}C = G^{-1}\Phi - \Pi(\hat{G} + \Phi^TG^{-1}\Phi), \quad \Phi\hat{G}^{-1} = C\hat{g}^{-1} + (g + C\hat{g}^{-1}C^T)\Pi.$$  (21)

These relations imply that we can use

$$g \leftrightarrow G, \quad \hat{g} \leftrightarrow \hat{G}, \quad C \leftrightarrow \Phi, \quad \Pi \leftrightarrow -\Pi$$

(23)

to write the action in terms of $G, \Phi$ and $\Pi$. If $q = 1$, we can get

$$\frac{1}{g + B} = \frac{1}{G + \Phi} + \Pi.$$  (24)

We can use $G = H^{-1}$ to get another form of the generalized closed-open relations as well.

$$\begin{pmatrix} g & C \\ -C^T & \hat{g} \end{pmatrix} = H^{-1}.\quad  (25)$$

We determine $g$ explicitly by this way

$$g^{-1} = (G + \Phi\hat{G}^{-1}\Phi^T)^{-1}$$

$$-\left(- (G + \Phi\hat{G}^{-1}\Phi^T)^{-1}\Phi\hat{G}^{-1}(\hat{G} + \Phi^TG^{-1}\Phi) + \Pi(\hat{G} + \Phi^TG^{-1}\Phi)\right)$$

$$\cdot \left(\hat{G}^{-1}\Phi^T(G + \Phi\hat{G}^{-1}\Phi)^{-1} - \Pi^T\right)$$

$$= (G + \Phi\hat{G}^{-1}\Phi^T)^{-1} + (G + \Phi\hat{G}^{-1}\Phi^T)^{-1}\Phi\hat{G}^{-1}(\hat{G} + \Phi^TG^{-1}\Phi)\hat{G}^{-1}\Phi^T(G + \Phi\hat{G}^{-1}\Phi^T)^{-1}$$

$$- (G + \Phi\hat{G}^{-1}\Phi^T)^{-1}\Phi\hat{G}^{-1}(\hat{G} + \Phi^TG^{-1}\Phi)\Pi^T - \Pi(\hat{G} + \Phi^TG^{-1}\Phi)\hat{G}^{-1}\Phi^T(G + \Phi\hat{G}^{-1}\Phi^T)^{-1}$$

$$+ \Pi(\hat{G} + \Phi^TG^{-1}\Phi)\Pi^T.$$  (26)

Before we give the explicit answer, we show the trick for the third and fourth terms. The third term is

$$-(G + \Phi\hat{G}^{-1}\Phi^T)^{-1}\Phi\hat{G}^{-1}(\hat{G} + \Phi^TG^{-1}\Phi)\Pi^T$$

$$= -(G + \Phi\hat{G}^{-1}\Phi^T)^{-1}(\Phi + \Phi\hat{G}^{-1}\Phi^TG^{-1}\Phi)\Pi^T$$

$$= -\left((G + \Phi\hat{G}^{-1}\Phi^T)G^{-1}\Phi(G^{-1}\Phi)^{-1}\right)^{-1}(\Phi + \Phi\hat{G}^{-1}\Phi^TG^{-1}\Phi)\Pi^T$$

$$= -G^{-1}\Phi\Pi^T.$$  (27)
The fourth term is
\[
-\Pi(\tilde{G} + \Phi^T G^{-1}\Phi) G^{-1}\Phi^T(G + \Phi\tilde{G}^{-1}\Phi^T)^{-1}
\]
\[
= -\Pi(\Phi^T + \Phi^T G^{-1}\Phi\tilde{G}^{-1}\Phi^T)(G + \Phi\tilde{G}^{-1}\Phi^T)^{-1}
\]
\[
= -\Pi(\Phi^T + \Phi^T G^{-1}\Phi\tilde{G}^{-1}\Phi^T) \left( (\Phi^TG^{-1})^{-1}\Phi^TG^{-1}(G + \Phi\tilde{G}^{-1}\Phi^T) \right)^{-1}
\]
\[
= -\Pi\Phi^TG^{-1}. \quad (28)
\]

By using the same method, the first and second term are
\[
(G + \Phi\tilde{G}^{-1}\Phi^T)^{-1} + G^{-1}\Phi\tilde{G}^{-1}\Phi^T(G + \Phi\tilde{G}^{-1}\Phi^T)^{-1}
\]
\[
= (1 + G^{-1}\Phi\tilde{G}^{-1}\Phi^T)(G + \Phi\tilde{G}^{-1}\Phi^T)^{-1}
\]
\[
= G^{-1}. \quad (29)
\]

We can see explicit answer by combining all terms.
\[
g^{-1} = (1 - \Phi\Pi^T)^T G^{-1}(1 - \Phi\Pi^T) + \Pi\tilde{G}\Pi^T. \quad (30)
\]

Then \(C\) is also easy to obtain
\[
C = -\left( (1 - \Phi\Pi^T)^T G^{-1}(1 - \Phi\Pi^T) + \Pi\tilde{G}\Pi^T \right)^{-1}
\]
\[
\times \left( - (G + \Phi\tilde{G}^{-1}\Phi^T)^{-1}\Phi\tilde{G}^{-1} + \Pi \right) \left( \tilde{G} + \Phi^TG^{-1}\Phi \right)
\]
\[
= \left( (1 - \Phi\Pi^T)^T G^{-1}(1 - \Phi\Pi^T) + \Pi\tilde{G}\Pi^T \right)^{-1} \left( (1 - \Phi\Pi^T)^T G^{-1}\Phi - \Pi\tilde{G} \right). \quad (31)
\]

Explicit expression for \(\tilde{g}^{-1}\) can be shown below
\[
\tilde{g}^{-1} = (\tilde{G} + \Phi^T G^{-1}\Phi)^{-1}
\]
\[
+ \left( \tilde{G}^{-1}\Phi^T - \Pi^T(G + \Phi\tilde{G}^{-1}\Phi^T) \right) \left( (G + \Phi\tilde{G}^{-1}\Phi^T)^{-1}\Phi\tilde{G}^{-1} - \Pi \right)
\]
\[
= (\tilde{G} + \Phi^TG^{-1}\Phi)^{-1} + \tilde{G}^{-1}\Phi^T(G + \Phi\tilde{G}^{-1}\Phi^T)^{-1}\Phi\tilde{G}^{-1}
\]
\[
- \tilde{G}^{-1}\Phi^T\Pi - \Pi^T\Phi\tilde{G}^T + \Pi^T(G + \Phi\tilde{G}^{-1}\Phi^T)\Pi. \quad (32)
\]

The first term can be rewritten by
\[
(a + b)^{-1} = a^{-1} - a^{-1}b(b + ba^{-1}b)^{-1}ba^{-1}. \quad (33)
\]
The above formula can be derived from Binomial Inverse Theorem. The first term is
\[
\tilde{G}^{-1} - \tilde{G}^{-1}(\Phi^T \tilde{G}^{-1} \Phi+(\Phi^T \tilde{G}^{-1} \Phi \tilde{G}^{-1} \Phi^T \Phi \tilde{G}^{-1} \Phi^T \Phi \tilde{G}^{-1} \Phi)^{-1}) \tilde{G}^{-1} \Phi \tilde{G}^{-1}
\]
\[
= \tilde{G}^{-1} - \tilde{G}^{-1} \Phi^T (\Phi^T + \Phi^T \tilde{G}^{-1} \Phi \tilde{G}^{-1} \Phi^T)^{-1} \Phi \tilde{G}^{-1}
\]
\[
= \tilde{G}^{-1} - \tilde{G}^{-1} \Phi^T \left(1 + \tilde{G}^{-1} \Phi \tilde{G}^{-1} \Phi^T \right)^{-1} \tilde{G}^{-1} \Phi \tilde{G}^{-1}
\]
(34)

If we combine the first term and second term, we obtain \(\tilde{G}^{-1}\). Now we can combine all terms to see explicit answer.
\[
\tilde{g}^{-1} = (1 - \Phi^T \Pi) \tilde{G}^{-1} (1 - \Phi^T \Pi) + \Pi^T \Pi.
\]  
(35)

However, the (21) and (22) can be a possibility of the generalization of the closed-open relations. We call these relations “generalized closed-open relations”. We can also use the generalized metric to see the generalized closed-open relations. The generalized metric is exactly the matrix in the Hamiltonian. We start from
\[
S_{bE} = \frac{1}{2} \int d^{q+1}\sigma \left[ V^T \left( \begin{array}{cc} g & C \\ -C^T & \tilde{g} \end{array} \right) \right].  
\]  
(36)

Then we show the Hamiltonian
\[
H[X,P] = \int d^q\sigma \left( \partial_0 X^T P - S_{bE} \right), 
\]
\[
= \int d^q\sigma \left( \partial_0 X^T (g \partial_0 X - i C \partial X) - \frac{1}{2} \partial_0 X^T g \partial_0 X - \frac{1}{2} \partial_0 X^T \tilde{g} \partial X + i \partial_0 X^T C \partial X \right) 
\]
\[
= \int d^q\sigma \left( \frac{1}{2} \partial_0 X^T g \partial_0 X - \frac{1}{2} \partial X^T \tilde{g} \partial X \right) 
\]
\[
= -\frac{1}{2} \int d^q\sigma \left( \frac{i P}{\partial X^i} \right)^T \left( \begin{array}{cc} g^{-1} & -g^{-1} C \\ -C^T g^{-1} \tilde{g} + C^T g^{-1} C \end{array} \right) \left( \frac{i P}{\partial X^i} \right), 
\]  
(37)

where \(P\) is the canonical momenta corresponding to the fields \(X\) (\(P = g \partial_0 X - i C \partial X\)). If we consider \(q = 1\), the matrix in Hamiltonian has the same structure as the familiar generalized metric.
We can use another way to write the generalized metric
\[
\begin{pmatrix}
1 & \Pi \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-\Phi^T & 1
\end{pmatrix}
\begin{pmatrix}
G^{-1} & 0 \\
0 & \tilde{G}
\end{pmatrix}
\begin{pmatrix}
1 & -\Phi \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
\Pi^T & 1
\end{pmatrix}
\]
\[
= \begin{pmatrix}
(1 - \Pi \Phi^T)G^{-1}(1 - \Phi \Pi^T) + \Pi \tilde{G}\Pi^T & -(1 - \Pi \Phi^T)G^{-1}\Phi + \Pi \tilde{G} \\
-\Phi^T G^{-1}(1 - \Phi \Pi^T) + \tilde{G}\Pi^T & \Phi^T G^{-1}\Phi + \tilde{G}
\end{pmatrix}
\]
\[
= \begin{pmatrix}
g^{-1} & -g^{-1}C \\
-C^T g^{-1} & \tilde{g} + C^T g^{-1}C
\end{pmatrix}. \tag{38}
\]

Then we can get the generalized closed-open relations from the generalized metric.

### 2.3 Membrane Action

We start from the action
\[
S_M = -\int d^{p+1}\sigma \sqrt{-\det(g_{\mu\nu}\partial_{A}\!^{\mu}\partial_{B}\!^{\nu})} \tag{39}
\]

to introduce membrane action. We can introduce an auxiliary field \( h_{AB} \) and write the classically equivalent action
\[
S_{Mc} = -\frac{1}{2} \int d^{p+1}\sigma \sqrt{-\det h(x)} \left( g_{\mu\nu} h^{AB} \partial_{A}\!^{\mu}\partial_{B}\!^{\nu} - (q - 1) \right). \tag{40}
\]

We used equations of motion of \( h^{AB} \)
\[
\frac{1}{2} h_{AB} \left( h^{CD} \partial_{C}\!^{\mu}\partial_{D}\!^{\nu} g_{\mu\nu} - (q - 1) \right) = \partial_{A}\!^{\mu}\partial_{B}\!^{\nu} g_{\mu\nu} \tag{41}
\]

to derive the equivalence. For \( q \neq 1 \), we also have
\[
h^{AB} \partial_{A}\!^{\mu}\partial_{B}\!^{\nu} g_{\mu\nu} = q + 1. \tag{42}
\]

Then we can get \( h_{AB} = \partial_{A}\!^{\mu}\partial_{B}\!^{\nu} g_{\mu\nu} \). Even for \( q \neq 1 \), we can also show the equivalence as \( q = 1 \). After we gauge fix (by reparametrization invariance) the components \( h_{a0}, h_{0b} \) and \( h_{00} \) by choosing \( h_{a0} = h_{0b} = 0 \) and \( h_{00} = -\det(h_{ab}) \), and use the equations of motion of \( h^{ab} \)
\[
h_{ab} \left( h^{cd} \partial_{c}\!^{\mu}\partial_{d}\!^{\nu} g_{\mu\nu} - (q - 1) \right) = \partial_{a}\!^{\mu}\partial_{b}\!^{\nu} g_{\mu\nu}, \tag{43}
\]
we get a classically equivalent action with gauge fixing

\[
S_{gf} = \frac{1}{2} \int d^{q+1}\sigma \left[ g_{\mu\nu} \partial_\mu X^\nu \partial_0 X^\nu - \det(g_{\mu\nu} \partial_a X^\mu \partial_b X^\nu) \right]. \tag{44}
\]

The action (44) can also be rewritten as

\[
S_{gf} = \frac{1}{2} \int d^{q+1}\sigma \left[ \partial_0 X g \partial_0 X - \tilde{\partial} X \tilde{g} \tilde{\partial} X \right]. \tag{45}
\]

We can also add a \((q+1)\)-form background field term, \(\frac{1}{(q+1)!} C_{i_1 i_2 \ldots i_{q+1}} dx^{i_1} dx^{i_2} \ldots dx^{i_{q+1}}\). The action is

\[
S_C = - \int d^{q+1}\sigma \partial_0 X C \tilde{\partial} X. \tag{46}
\]

If we combine \(S_{gf}\) with \(S_C\), we can get the same action as the Nambu-Sigma model.

### 2.4 Generalized DBI

Before we generalize the DBI action, we first review the well-known theory, DBI theory, which is an effective action for an open string ending on a D-brane. The action is

\[
-\frac{1}{g_s} \int d^{p+1}x \sqrt{-\det [g + B + F]} = -\frac{1}{g_s} \int d^{p+1}x \left( -\det[g] \right)^{\frac{1}{4}} \left( -\det \left[ g-(B+F)g^{-1}(B+F) \right] \right)^{\frac{1}{4}}, \tag{47}
\]

where \(g_s\), \(g\) and \(B\) are closed string coupling constant, metric and background. \(F\) is the usual abelian field strength \((F = dA)\). Before we show the equivalence of commutative and non-commutative description, we discuss the relations between the closed and open string parameters. These are

\[
G_s = g_s \left( \frac{\det(G + \Phi)}{\det(g + B)} \right)^{\frac{1}{2}}, \tag{48}
\]

\[
g - Bg^{-1}B = G - \Phi G^{-1}\Phi, \quad Bg^{-1} = \Phi G^{-1} - (G - \Phi G^{-1}\Phi)\Pi. \tag{49}
\]

The meaning of the above relations is to determine the open string variables from closed string variables by choosing \(\Pi\). We can also rewrite \(G_s\) as

\[
G_s = g_s \left( \frac{\det(G + \Phi)}{\det(g + B)} \right)^{\frac{1}{2}} = g_s \left( \frac{\det G}{\det g} \right)^{\frac{1}{4}} \left( \frac{\det(G - \Phi G^{-1}\Phi)}{\det(g - Bg^{-1}B)} \right)^{\frac{1}{4}} = g_s \left( \frac{\det G}{\det g} \right)^{\frac{1}{4}}. \tag{50}
\]
Now we include the gauge field in the generalized metric
\[
\begin{pmatrix}
1 & F \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & B \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
g & 0 \\
0 & g^{-1}
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-B & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-F & 1
\end{pmatrix}
\]
\[
= \begin{pmatrix}
g - (B + F)g^{-1}(B + F) & (B + F)g^{-1} \\
g^{-1}(B + F) & g^{-1}
\end{pmatrix}
\]
\[
= \begin{pmatrix}
1 & F \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
\Pi & 1
\end{pmatrix}
\begin{pmatrix}
1 & \Phi \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
G & 0 \\
0 & G^{-1}
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-\Phi & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-F & 1
\end{pmatrix}
\]
\[
\cdot \begin{pmatrix}
1 & 0 \\
-(\Phi + F') & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
F' & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-F & 1
\end{pmatrix}
\] (51)

Now we add one new block matrix \( N \) to factorize of the generalized metric. Later we will combine them to see the equivalence of non-commutative and commutative description
\[
\begin{pmatrix}
1 & 0 \\
\Pi' & 1
\end{pmatrix}
\begin{pmatrix}
N^T & 0 \\
0 & N^{-1}
\end{pmatrix}
\begin{pmatrix}
1 & \Phi' \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
G & 0 \\
0 & G^{-1}
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-\Phi' & 1
\end{pmatrix}
\begin{pmatrix}
N & 0 \\
0 & (N^{-1})^T
\end{pmatrix}
\begin{pmatrix}
1 & -\Pi' \\
0 & 1
\end{pmatrix}
\]
\[
= \begin{pmatrix}
1 & 0 \\
\Pi' & 1
\end{pmatrix}
\begin{pmatrix}
N^T & 0 \\
0 & N^{-1}
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
\Pi' & 1
\end{pmatrix}
\begin{pmatrix}
1 & -F' \\
0 & 1
\end{pmatrix}
\]
\[
= \begin{pmatrix}
1 & F \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
\Pi & 1
\end{pmatrix}
\begin{pmatrix}
1 & -F' \\
0 & 1
\end{pmatrix}
\]
\[
= \begin{pmatrix}
1 + F\Pi & -1 + (F\Pi)F' + F \\
\Pi & -\Pi F' + 1
\end{pmatrix}
\]
\[
= \begin{pmatrix}
1 & 0 \\
\Pi' & 1
\end{pmatrix}
\begin{pmatrix}
N^T & 0 \\
0 & N^{-1}
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
\Pi & 1
\end{pmatrix}
\begin{pmatrix}
1 & -\Pi' \\
0 & 1
\end{pmatrix}
\] (52)

where \( \Phi' = \Phi + F' \). From
\[
\begin{pmatrix}
1 & 0 \\
\Pi' & 1
\end{pmatrix}
\begin{pmatrix}
N^T & 0 \\
0 & N^{-1}
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
\Pi & 1
\end{pmatrix}
\begin{pmatrix}
1 & -\Pi' \\
0 & 1
\end{pmatrix}
\]
\[
= \begin{pmatrix}
1 & \Pi \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
N^T & 0 \\
0 & N^{-1}
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
\Pi & 1
\end{pmatrix}
\begin{pmatrix}
1 & -\Pi' \\
0 & 1
\end{pmatrix}
\]
\[
= \begin{pmatrix}
1 + F\Pi & -1 + (F\Pi)F' + F \\
\Pi & -\Pi F' + 1
\end{pmatrix}
\]
\[
= \begin{pmatrix}
1 & 0 \\
\Pi' & 1
\end{pmatrix}
\begin{pmatrix}
N^T & 0 \\
0 & N^{-1}
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
\Pi & 1
\end{pmatrix}
\begin{pmatrix}
1 & -\Pi' \\
0 & 1
\end{pmatrix}
\] (53)

we can obtain
\[
\Pi' = (1 + \Pi F)^{-1}\Pi = \Pi(1 + F\Pi)^{-1},
\]
\[
F' = F(1 + \Pi F)^{-1} = (1 + F\Pi)^{-1}F,
\]
\[
N = 1 + \Pi F.
\] (54)
We can also find useful formula from
\[
\begin{pmatrix}
1 & 0 \\
\Pi' & 1
\end{pmatrix}
\begin{pmatrix}
N^T & 0 \\
0 & N^{-1}
\end{pmatrix}
\begin{pmatrix}
1 & \Phi' \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
G & 0 \\
0 & G^{-1}
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-\Phi' & 1
\end{pmatrix}
\begin{pmatrix}
N & 0 \\
0 & (N^{-1})^T
\end{pmatrix}
\begin{pmatrix}
1 & -\Pi'
\end{pmatrix}
= 
\begin{pmatrix}
g - (B + F)g^{-1}(B + F) & (B + F)g^{-1} \\
-g^{-1}(B + F) & g^{-1}
\end{pmatrix}.
\]
(55)

Then we find
\[
g - (B + F)g^{-1}(B + F) = N^T\left(G - \Phi'G^{-1}\Phi'\right)N,
\]
\[
(B + F)g^{-1} = -N^T\left(G - \Phi'G^{-1}\Phi'\right)\Pi' + N^T\Phi'G^{-1}(N^T)^{-1},
\]
\[
g^{-1} = -\Pi'N^TGN\Pi' + \left(\Pi'N^T\Phi' + N^{-1}\right)G^{-1}\left(\Phi'N\Pi' + (N^T)^{-1}\right).
\]

Thus, we have
\[
det\left(g - (B + F)g^{-1}(B + F)\right) = det^2(N) det\left(G - \Phi'G^{-1}\Phi'\right),
\]
\[
(g + B + F)^{-1} = \Pi' + \left(N^T(G + \Phi')N\right)^{-1}.
\]
(56)

The DBI action can be rewritten from the closed string parameters to the open string parameters by the above relations
\[
-\frac{1}{g_s}\left(-\det\left[g + B + F\right]\right)^{\frac{1}{2}}
= -\frac{1}{g_s}\left(-\det[g]\right)^{\frac{1}{4}}\left(-\det\left[g - (B + F)g^{-1}(B + F)\right]\right)^{\frac{1}{4}}
= -\frac{1}{g_s}\left(-\det[g]\right)^{\frac{1}{4}}\det^{\frac{1}{4}}\left(1 + \Pi F\right)\left(-\det\left[G - \Phi'G^{-1}\Phi'\right]\right)^{\frac{1}{4}}
= -\frac{1}{G_s}\det\left[1 + \Pi F\right]\left(-\det[G]\right)^{\frac{1}{4}}\left(-\det\left[G - \Phi'G^{-1}\Phi'\right]\right)^{\frac{1}{4}}
= -\frac{1}{G_s}\det\left[1 + \Pi F\right]\left(-\det\left[G + \Phi'\right]\right)^{\frac{1}{2}}.
\]
(57)

Then we perform Seiberg-Witten map to get
\[
-\int d^{p+1}x \frac{1}{g_s}\left(-\det\left[g + B + F\right]\right)^{\frac{1}{2}} = -\int d^{p+1}\hat{x} \frac{1}{G_s}\det^{\frac{1}{4}}\left(\hat{\Pi}\right)\left(-\det\left[\hat{G} + \hat{\Phi}'\right]\right)^{\frac{1}{2}},
\]
(58)
where the superscript \( ^\hat{} \) means the fields evaluated at **covariant coordinates**. When we change the coordinates, \( x \mapsto \rho^*_A(x) = \hat{x} = x + \Pi \hat{A} \) induced by a map \( \Pi \mapsto \Pi' = (1 + \Pi \cdot F)^{-1}\Pi \). The coordinate \( \hat{x}^\mu \) is called **covariant coordinate**. We used
\[
\det(1 + \Pi F) = \det \left( \frac{\Pi'}{\Pi} \right) \det^2 \left( \frac{\partial x}{\partial \hat{x}} \right)
\]
to show the equivalence of the non-commutative and commutative description in the DBI theory.

We expect that we can use the similar method to generalize DBI theory. In other words, we use the equivalence of non-commutative and commutative description to construct the generalized DBI theory. In the generalized DBI theory, the background field is \((q + 1)\)-form. When \( q=1 \), we can obtain the usual DBI theory.

The generalization of DBI can also be done by a similar decomposition of matrix
\[
\begin{pmatrix}
1 & -H^T \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & -C^T \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\tilde{g} & 0 \\
0 & g^{-1}
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-C & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-H & 1
\end{pmatrix}
\]
\[
= \begin{pmatrix}
\tilde{g} + (C + H)^T g^{-1}(C + H) & -(C + H)^T g^{-1} \\
-g^{-1}(C + H) & g^{-1}
\end{pmatrix}
\]
\[
= \begin{pmatrix}
1 & -H^T \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
\Pi & 1
\end{pmatrix}
\begin{pmatrix}
1 & -\Phi^T \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\tilde{G} & 0 \\
0 & G^{-1}
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-\Phi & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-H & 1
\end{pmatrix}
\]
\[
= \begin{pmatrix}
1 & -H^T \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
\Pi & 1
\end{pmatrix}
\begin{pmatrix}
1 & H^T \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & -(\Phi^T + H'^T) \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\tilde{G} & 0 \\
0 & G^{-1}
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-\Phi & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-H & 1
\end{pmatrix}
\cdot
\]
\[
\begin{pmatrix}
1 & 0 \\
\Pi' & 1
\end{pmatrix}
\begin{pmatrix}
N^T & 0 \\
0 & M^T
\end{pmatrix}
\begin{pmatrix}
1 & -\Phi^T \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\tilde{G} & 0 \\
0 & G^{-1}
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-\Phi' & 1
\end{pmatrix}
\begin{pmatrix}
N & 0 \\
0 & M
\end{pmatrix}
\begin{pmatrix}
1 & \Pi'^T \\
0 & 1
\end{pmatrix}
\]
where $\Phi' = \Phi + H'$. From
\[
\begin{pmatrix} 1 & 0 \\ \Pi' & 1 \end{pmatrix} \begin{pmatrix} N^T & 0 \\ 0 & M^T \end{pmatrix} = \begin{pmatrix} N^T & 0 \\ \Pi' N^T & M^T \end{pmatrix} \\
= \begin{pmatrix} 1 & -H^T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \Pi & 1 \end{pmatrix} \begin{pmatrix} 1 & H'^T \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - H^T \Pi & (1 - H^T \Pi) H'^T - H^T \\ \Pi & \Pi H'^T + 1 \end{pmatrix},
\]
(62)
we can obtain
\[
\Pi' = \Pi(1 - H^T \Pi)^{-1},
\]
\[
H' = H(1 - \Pi^T H)^{-1},
\]
\[
N = 1 - \Pi^T H = (1 + \Pi^T H)^{-1},
\]
\[
M = 1 + H' \Pi^T = (1 - H \Pi)^{-1}.
\]
(63)

We can also find useful formula from
\[
\begin{pmatrix} 1 & 0 \\ \Pi' & 1 \end{pmatrix} \begin{pmatrix} N^T & 0 \\ 0 & M^T \end{pmatrix} \begin{pmatrix} 1 & -\Phi'^T \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{G} & 0 \\ 0 & G^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\Phi' & 1 \end{pmatrix} \begin{pmatrix} N & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} 1 & \Pi'^T \\ 0 & 1 \end{pmatrix} \\
= \begin{pmatrix} \tilde{g} + (C + H)^T g^{-1} (C + H) & -(C + H)^T g^{-1} \\ -g^{-1} (C + H) & g^{-1} \end{pmatrix}.
\]
(64)

Then we find
\[
\tilde{g} + (C + H)^T g^{-1} (C + H) = N^T \left( \tilde{G} + \Phi'^T G^{-1} \Phi' \right) N,
\]
\[
-(C + H)^T g^{-1} = N^T \left( \tilde{G} + \Phi' G^{-1} \Phi' \right) N \Pi'^T \Pi'^T - N^T \Phi'^T G^{-1} M,
\]
\[
g^{-1} = \Pi'^T N^T \tilde{G} N \Pi'^T + \left( - \Pi' N^T \Phi'^T + M^T \right) G^{-1} \left( - \Phi' N \Pi'^T + M \right).
\]

Thus, we have
\[
det \left( \tilde{g} + (C + H)^T g^{-1} (C + H) \right) = det^2(N) det \left( \tilde{G} + \Phi'^T G^{-1} \Phi' \right) \\
= det^2 \left( 1 - \Pi'^T H \right) \left( \tilde{G} + \Phi'^T G^{-1} \Phi' \right). \quad (65)
\]
From the down right block of
\[
\begin{pmatrix}
1 & -\Pi^T \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
N^{-1} & 0 \\
0 & M^{-1}
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
\Phi' & 1
\end{pmatrix}
\begin{pmatrix}
\tilde{G}^{-1} & 0 \\
0 & G
\end{pmatrix}
\begin{pmatrix}
1 & \Phi^T \\
0 & (M^T)^{-1}
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
-P' & 1
\end{pmatrix}
\begin{pmatrix}
\tilde{g} + (C + H)^T g^{-1}(C + H) & - (C + H)^T g^{-1} \\
-g^{-1}(C + H) & g^{-1}
\end{pmatrix}^{-1},
\]
we can obtain
\[
\det \left( g + (C + H)\tilde{g}^{-1}(C + H)^T \right) = \det [M^{-1} \left( G + \Phi' \tilde{G}^{-1} \Phi^T \right) (M^{-1})^T] = \det^2 (1 - H \Pi^T) \det \left( G + \Phi' \tilde{G}^{-1} \Phi^T \right). \tag{66}
\]
We also have
\[
\det (1 - \Pi^T H) = \det (1 - H \Pi^T) = \det \left( \frac{\Pi}{\Pi} \right) \det g^{q+1} \left( \frac{\partial x}{\partial \tilde{x}} \right). \tag{67}
\]
We can get \(\frac{\partial x}{\partial \tilde{x}}\)^2(q+1) in the action. From this term, we postulate the action can be
\[
S_{GDBI} = - \int d^{p+1}x \frac{1}{g_b} \left( - \det [g] \right)^{\frac{q}{2(q+1)}} \cdot \left( - \det [g + (C + H)\tilde{g}^{-1}(C + H)^T] \right)^{\frac{1}{2(q+1)}}.
\tag{69}
\]
because the term \(\frac{\partial x}{\partial \tilde{x}}\)^2(q+1) cancel with the Jacobian which arise from coordinate transformation, such that the Lagrangian is an integral density. The coupling constant \(g_b\) is called closed brane coupling constant. We can also rewrite open brane coupling constant \(G_b\) as
\[
G_b = g_b \left( \frac{\det G}{\det g} \right)^{\frac{q}{2(q+1)}} \left( \frac{\det (G + \Phi \tilde{G}^{-1} \Phi^T)}{\det (g + C \tilde{g}^{-1} C^T)} \right)^{\frac{1}{2(q+1)}} = g_b \left( \frac{\det G}{\det g} \right)^{\frac{q}{2(q+1)}}.
\tag{70}
\]
We used
\[
G + \Phi \tilde{G}^{-1} \Phi^T = g + C \tilde{g}^{-1} C^T
\tag{71}
in the last equality. The action of the generalized DBI can be rewritten from the closed.
brane parameters to the open brane parameters.

\[- \int d^{p+1}x \frac{1}{g_b} \left( - \det [g] \right)^{\frac{q}{2(q+1)}} \cdot \left( - \det \left[ g + (C + H)\tilde{g}^{-1}(C + H)^T \right] \right)^{\frac{1}{2(q+1)}} \]

\[= - \int d^{p+1}x \frac{1}{G_b} \left( - \det [G] \right)^{\frac{q}{2(q+1)}} \cdot \left( - \det \left[ g + (C + H)\tilde{g}^{-1}(C + H)^T \right] \right)^{\frac{1}{2(q+1)}} \]

\[= - \int d^{p+1}x \frac{1}{G_b} \left( - \det [\hat{G}] \right)^{\frac{q}{2(q+1)}} \cdot \left( - \det \left[ \hat{G} + \hat{\Phi}'\tilde{G}^{-1}\hat{\Phi}^T \right] \right)^{\frac{1}{2(q+1)}}. \quad (72)\]

This action is based on the equivalence of non-commutative and commutative gauge theory. The closed-open relations can be generalized from the generalized metric. On the other hand, it can also be derived from the Nambu-Sigma model. This generalized DBI theory can also be viewed as a generalization of the DBI. If we consider 2-form background, it goes back to the usual DBI theory. If we choose 3-form background and \(p=5\), the action is

\[S_{5-\text{DBI}} = - \int d^6x \frac{1}{g_b} \sqrt{-\det(g)} \cdot \det \left[ 1 + g^{-1}(C + H)\tilde{g}^{-1}(C + H)^T \right] \]

\[\approx - \int d^6x \frac{1}{g_b} \sqrt{-\det(g)} \cdot \left[ 1 + \frac{1}{3}\text{Tr}k - \frac{1}{6}\text{Tr}k^2 + \frac{1}{18}(\text{Tr} k)^2 + \cdots \right]^{\frac{1}{2}}, \quad (73)\]

where \(k^\mu_\nu = (H + C)^\mu_\rho(H + C)^\nu_\sigma\). This action is consistent with the [6] up to the second order. This action up to the second order can be understood from the \(\kappa\)-symmetry and equivalence of non-commutative and commutative gauge description. The understanding of full order comes from the equivalence of non-commutative and commutative gauge description. The supersymmetric extension and other formulation of the membrane theory are in [28].

### 3 Consistency of Dimensional Reduction

In this section, we discuss dimensional reduction of the action (69) without scalar fields. At first, we show dimensional reduction from \((q+1) - (p+1)\) to \(q - p\). We only consider flat spacetime, constant background, and \((q+1)\)-form gauge field exists in \((q+1)\)-dimensional worldvolume directions (without time direction) in \(q-p\) system. In other
words, we will have two types worldvolume directions. We denote \( \alpha \) is the worldvolume direction without background and \( \dot{\alpha} \) is the other one direction with background. For a consistent notation, we define \((\dot{1}, \dot{2}, \cdots, \dot{q})\) \(\equiv (p - q, p - q + 1, \cdots, p - 1)\). The generalized DBI theory \((69)\) gives

\[
S_{q+1,p+1} = - \int d^{p+2}x \frac{1}{g_b} \det \frac{1}{2\pi i} \left[ \delta_A^B + H_A \tilde{g}^{ij} H_{Cj} g^{CB} \right]
\]

\[
= - \int d^{p+2}x \frac{1}{g_b} \exp \left( \frac{1}{2(q + 2)} \text{Tr} \ln(\delta_A^B + H_A \tilde{g}^{ij} H_{Cj} g^{CB}) \right)
\]

\[
= - \int d^{p+2}x \frac{1}{g_b} \exp \left( \frac{1}{2(q + 2)} \text{Tr} \left( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left( H_A \tilde{g}^{ij} H_{Cj} g^{CB} \right)^n \right) \right). \tag{74}
\]

We used

\[
\det x[I + M] = \exp[x \text{Tr} \ln(I + M)], \quad \ln(1 - x) = - \sum_{n=1}^{\infty} \frac{x^n}{n} \tag{75}
\]

in the above action. Then we calculate \( H_A \tilde{g}^{ij} H_{Cj} g^{CB} \)

\[
H_A \tilde{g}^{ij} H_{Cj} g^{CB} = \frac{1}{((q + 1)!)^2} H_{A_1 \cdots C_{q+1}} \sum_{\pi \in \sigma_{q+1}} \text{sgn}(\pi) (g_{C_{u(1)}} D_1 g_{C_{u(2)}} D_2 \cdots g_{C_{u(q+1)}} D_{q+1}) H_{E D_1 \cdots D_{q+1}} g^{EB}
\]

\[
= \frac{1}{((q + 1)!)^2} \sum_{\pi \in \sigma_{q+1}} \text{sgn}(\pi) H_A^{D_{u(1)} \cdots D_{u(q+1)}} H^B_{D_1 \cdots D_{q+1}}
\]

\[
= \frac{1}{(q + 1)!} H_A^{D_1 \cdots D_{q+1}} H^B_{D_1 \cdots D_{q+1}}
\]

\[
\equiv \frac{1}{(q + 1)!} (H^2)_A^B. \tag{76}
\]

The only non-zero components in \((H^2)_A^B\) are

\[
(H^2)_{p-q}^{p-q} = (H^2)_{p-q+1}^{p-q+1} = \cdots = (H^2)_{p+1}^{p+1} = (q + 1)! (H^2)_{p-q,p-q+1,\cdots,p+1}. \tag{76}
\]

Substituting the result and taking trace in the action \((74)\), we get

\[
S_{q+1,p+1} = - \int d^{p+2}x \frac{1}{g_m} \exp \left( \frac{1}{2(q + 2)} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (q + 2) (H_{p-q,p-q+1,\cdots,p+1})^{2n} \right)
\]

\[
= - \int d^{p+2}x \frac{1}{g_m} \sum_{m=0}^{\infty} \frac{1}{m!} \left( \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (H_{p-q,p-q+1,\cdots,p+1})^{2n} \right)^m, \tag{77}
\]
If we compactify one direction, the final expression (77) simply becomes

\[ S_{q,p} = -\int d^{p+1}x \frac{1}{g_m} \sum_{m=0}^{\infty} \frac{1}{m!} \left( \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} (H_{p-q,p-q+1,...,p})^{2n} \right)^m. \tag{78} \]

In conclusion, we start from a system of \((q + 1) - (p + 1)\), we can get an effective action for \(q-p\) system by dimensional reduction.

We want to emphasize that this is not a trivial check because the \(2(q+1)\) root in the action is so far predicted based on the equivalence of non-commutative and commutative gauge theory. Thus, the calculation of this simple example gives us a confidence to show that this theory can also be consistent with dimensional reduction.

## 4 Comments on Pull-Back

If we also require that the generalized DBI can also go from \((q + 1) - (p + 1)\) to \(q - p\) with scalar fields (by pull-back). Generalized DBI (69) needs to include a one-form gauge potential for a \(U(1)\) gauge symmetry. For the non-commutative gauge theory, we also have these similar systems \([29]\). We wish to explore the possibility by dimensional reduction.

### 4.1 Scalar Fields and Gauge Potential

When a worldvolume direction is compactified, the component of the compactified direction of a gauge potential \(A^I\) give a scalar field \(X^I\),

\[ A^I \rightarrow X^I, \quad F^{AI} \rightarrow \partial^A X^I. \tag{79} \]

The scalar fields \(X^I\) are actually the positions of a brane in transverse directions.

The way we introduce scalar field is simply to set the metric by pull-back. In static gauge and the case of flat spacetime

\[ g_{AB} = \eta_{AB} + \partial_A X^I \partial_B X^I. \tag{80} \]

The inverse of this metric is

\[ g^{AB} = \eta^{AB} + \sum_{n=1}^{\infty} (-1)^n (\partial^A X^{I_1})(\partial_{D_1} X^{I_1})(\partial^{D_1} X^{I_2}) \cdots (\partial_{D_{n-1}} X^{I_{n-1}})(\partial^{D_{n-1}} X^{I_n})(\partial^B X^{I_n}), \tag{81} \]
which indeed satisfy the condition $g_{AB}g^{BC} = \delta^C_A$. We define
\[ \omega^{AB} = \sum_{n=1}^{\infty} (-1)^n (\partial^AX^I_1)(\partial^DX^I_2)\cdots (\partial^D_{n-1}X^I_n)(\partial^B X^I_n) \] (82)
for convenience and notice that it is symmetric, i.e. $\omega^{AB} = \omega^{BA}$.

4.2 $(q + 1)-(p + 1) \rightarrow q-p$

We show that the effective action of a $q$-$p$ brane system without one-form gauge potential can be deduced from the $(q + 1)-(p + 1)$ system up to $H^2$ order.

Again, we assume that only $\alpha$ components of $H$ and backgrounds are turned on. The action is
\[
S_{q+1,p+1} = -\int d^{p+2}x \frac{1}{g_b} \sqrt{-\det g} \text{ det}\{\delta_A^B + H_{AB}\tilde{g}^{ij}H_{CB}g^{CB}\}
\]
\[
= -\int d^{p+2}x \frac{1}{g_b} \sqrt{-\det g} \exp\left[\frac{1}{2(q + 2)} \text{ Tr}\left( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (H_{AB}\tilde{g}^{ij}H_{CB}g^{CB})^n \right) \right].
\] (83)

For $n = 1$, we can obtain
\[
\text{ Tr}(H_{AB}\tilde{g}^{ij}H_{CB}g^{CB})
\]
\[
= H_{p-q,p-q+1,\cdots,p+1} g^{q+1} \sum_{k=0}^{q+1} \sum_{k=0}^{p+1} \frac{1}{k!(q + 2 - k)!}
\]
\[
\epsilon_{\hat{\alpha}_1\hat{\beta}_2\cdots\hat{\alpha}_k\hat{\gamma}_{k+1}\cdots\hat{\gamma}_{q+2}} \omega^{\hat{\alpha}_1\hat{\beta}_1}\omega^{\hat{\alpha}_2\hat{\beta}_2}\cdots\omega^{\hat{\alpha}_k\hat{\beta}_k},
\] (84)
where $\epsilon_{\hat{\alpha}_1\hat{\beta}_2\cdots\hat{\alpha}_k\hat{\gamma}_{k+1}\cdots\hat{\gamma}_{q+2}}$ and $\epsilon_{\hat{\beta}_1\hat{\beta}_2\cdots\hat{\beta}_k\hat{\gamma}_{k+1}\cdots\hat{\gamma}_{q+2}}$ are Levi-Civita symbols. The factorials $k!$ and $(q + 2 - k)!$ are used to cancel the factor of overcounting such that the coefficients of each term in the summation is simply unity. The expression (83) becomes
\[
S_{q+1,p+1} = -\int d^{p+2}x \frac{1}{g_b} \sqrt{-\det g} \exp\left[ \frac{1}{2} H_{p-q,p-q+1,\cdots,p+1} g^{q+1} \sum_{k=0}^{q+1} \sum_{k=0}^{p+1} \frac{1}{k!(q + 2 - k)!}
\]
\[
\epsilon_{\hat{\alpha}_1\hat{\beta}_2\cdots\hat{\alpha}_k\hat{\gamma}_{k+1}\cdots\hat{\gamma}_{q+2}} \omega^{\hat{\alpha}_1\hat{\beta}_1}\omega^{\hat{\alpha}_2\hat{\beta}_2}\cdots\omega^{\hat{\alpha}_k\hat{\beta}_k} + \cdots \right].
\] (85)
The factors \((q + 2)\) in (83) and (84) cancel out each other. If we compactify one world-volume direction with background, say \(p + 1\). Then, all \(\omega^{\hat{\alpha}(p+1)}\)'s vanish and this is equivalent to ruling the \(p + 1\) out in the summation, that is

\[
\sum_{k=0}^{p+1} \to \sum_{k=0}^{p}
\]

\[
\{\hat{\alpha}_k, \hat{\beta}_k, \gamma_k = p-q\} \to \{\hat{\alpha}_k, \hat{\beta}_k, \gamma_k = p-q\}
\]

(86)

On the other hand, the Levi-Civita symbols should be modified to

\[
\epsilon_{\hat{\alpha}_1 \hat{\beta}_2 \cdots \hat{\alpha}_k \gamma_k+1 \gamma_k+2 \cdots \gamma_{q+2}} \to (q+2-k) \epsilon_{\hat{\alpha}_1 \hat{\beta}_2 \cdots \hat{\alpha}_k \gamma_k+1 \gamma_k+2 \cdots \gamma_{q+2}}
\]

As a result, the action (85) becomes

\[
S_{q,p} = - \int d^{p+1}x \frac{1}{g_b} \sqrt{-\det g} \exp\left\{ \frac{1}{2} H^2_{p-q, p-q+1, \ldots, p} \sum_{k=0}^{q} \sum_{k=0}^{p} \frac{1}{k!(q+1-k)!} \epsilon_{\hat{\alpha}_1 \hat{\beta}_2 \cdots \hat{\alpha}_k \gamma_k+1 \gamma_k+2 \cdots \gamma_{q+2}} \omega^{\hat{\alpha}_1 \hat{\beta}_1 \hat{\alpha}_2 \hat{\beta}_2 \cdots \hat{\alpha}_k \hat{\beta}_k + \cdots} \right\},
\]

(87)

which is exactly the action (85) with \(q+1\) and \(p+1\) being replaced by \(q\) and \(p\) respectively. This calculation shows that we could possibly incorporate one-form gauge field in this theory up to \(H^2\) order.

5 Conclusion

The generalized DBI is aimed for describing a \(q\)-brane ending on a \(p\)-brane. The most non-trivial feature of this action is that it contains a \(2(q+1)\) root, which is predicted by the existence of the equivalence of the commutative and non-commutative description of the \(q-p\) system. In this paper, we showed that the generalized DBI action is also consistent with dimensional reduction to all orders perturbatively in the absence of scalar fields for \(q+1)-(p+1)\) \(q-p\) in flat spacetime, constant background, and \((q+1)\)-form gauge field which only exists in \((q+1)\)-dimensional worldvolume (without time direction). It gives
more understanding on the relation between 2-5 with M2-M5 system. In addition, we also find the possibility of the extension of including one-form gauge field in the presence of scalar fields based on dimensional reduction. The full understanding of dimensional reduction for \((q+1)-(p+1) \rightarrow q-p\) leave it to the future.

In this paper, we focus on the dimensional reduction. However, the most interesting problem should be T-duality rule. Of course, we still have familiar Buscher’s rule for \(q=1\) with different \(p\). Exploring T-duality rule is a challenging and interesting problem. It should give more interesting understanding to \(q-p\) system.

One related interesting problem is to explore double field theory of the DBI. By now, we do not get any insight to put one-form gauge fields. It is still an open problem about how to consider gauge fields in the double field theory. The starting direction is to find the gauge transformation which can be related to Courant bracket. It can offer the unique structure to constrain the DBI theory in double field theory. One more interesting thing related to open string of double field theory is to understand string sigma model with manifest Buscher’s rule. It is a well-known fact that DBI model is equivalent to the calculation of the one-loop \(\beta\) function of the string sigma model. If we can include strong constraints in double field theory of the string sigma model, the one-loop \(\beta\) function would be an interesting thing. We also point out that one-loop \(\beta\) function of the Nambu-Sigma model is an important problem. So far, we only use the generalized metric and equivalence of the commutative and non-commutative description to understand the generalized DBI. We expect that one-loop \(\beta\) function of the Nambu-Sigma model should give the generalized DBI.

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