A probabilistic Takens theorem

Krzysztof Barański1, Yonatan Gutman2,3 and Adam Spiewak1

1 Institute of Mathematics, University of Warsaw, ul. Banacha 2, 02-097 Warszawa, Poland
2 Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-656 Warszawa, Poland

E-mail: baranski@mimuw.edu.pl, y.gutman@impan.pl and a.spiewak@mimuw.edu.pl

Received 22 August 2019, revised 15 April 2020
Accepted for publication 4 May 2020
Published 31 July 2020

Abstract
Let $X \subset \mathbb{R}^N$ be a Borel set, $\mu$ a Borel probability measure on $X$ and $T : X \to X$ a locally Lipschitz and injective map. Fix $k \in \mathbb{N}$ strictly greater than the (Hausdorff) dimension of $X$ and assume that the set of $p$-periodic points of $T$ has dimension smaller than $p$ for $p = 1, \ldots, k - 1$. We prove that for a typical polynomial perturbation $\tilde{h}$ of a given locally Lipschitz function $h : X \to \mathbb{R}$, the $k$-delay coordinate map $x \mapsto (\tilde{h}(x), \tilde{h}(Tx), \ldots, \tilde{h}(T^{k-1}x))$ is injective on a set of full $\mu$-measure. This is a probabilistic version of the Takens delay embedding theorem as proven by Sauer, Yorke and Casdagli. We also provide a non-dynamical probabilistic embedding theorem of similar type, which strengthens a previous result by Alberti, Bölcskei, De Lellis, Koliander and Riegler. In both cases, the key improvements compared to the non-probabilistic counterparts are the reduction of the number of required coordinates from $2 \dim X$ to $\dim X$ and using Hausdorff dimension instead of the box-counting one. We present examples showing how the use of the Hausdorff dimension improves the previously obtained results.

Keywords: Takens delay embedding theorem, probabilistic embedding, Hausdorff dimension, box-counting dimension

Mathematics Subject Classification numbers: 37C45 (dimension theory of dynamical systems), 28A78 (Hausdorff and packing measures), 28A80 (fractals).

1. Introduction

Consider an experimentalist observing a physical system modelled by a discrete time dynamical system $(X, T)$, where $T : X \to X$ is the evolution rule and the phase space $X$ is a subset of the
Euclidean space \( \mathbb{R}^N \). It often happens that, for a given point \( x \in X \), instead of an actual sequence of \( k \) states \( x, Tx, \ldots, T^{k-1}x \), the observer’s access is limited to the values of \( k \) measurements \( h(x), h(Tx), \ldots, h(T^{k-1}x) \), for a real-valued observable \( h : X \to \mathbb{R} \). Therefore, it is natural to ask, to what extent the original system can be reconstructed from such sequences of measurements and what is the minimal number \( k \), referred to as the number of delay-coordinates, required for a reliable reconstruction. These questions have emerged in the physical literature (see e.g. [PCFS80]) and inspired a number of mathematical results, known as Takens-type delay embedding theorems, stating that the reconstruction of \( (X, T) \) is possible for certain observables \( h \), as long as the measurements \( h(x), h(Tx), \ldots, h(T^{k-1}x) \) are known for all \( x \in X \) and large enough \( k \).

The possibility of performing measurements at every point of the phase space is clearly unrealistic. However, such an assumption enables one to obtain theoretical results which justify the validity of actual procedures used by experimentalists (see e.g. [HGLS05, KY90, SGM90, SM90]). Note that one cannot expect a reliable reconstruction of the system based on the measurements of a given observable \( h \), as it may fail to distinguish the states of the system (e.g. if \( h \) is a constant function). It is therefore necessary (and rather realistic) to assume that the experimentalists are able to perturb the given observable. The first result obtained in this area is the celebrated Takens delay embedding theorem for smooth systems on manifolds [Tak81, theorem 1]. Due to its strong connections with actual reconstruction procedures used in the natural sciences, the Takens theorem has been met with great interest among mathematical physicists (see e.g. [HBS15, SYC91, Vos03]). Let us recall its extension due to Sauer, Yorke and Casdagli [SYC91]. In this setting, the number \( k \) of delay-coordinates should be two times larger than the upper box-counting dimension of the phase space \( X \) (denoted by \( \text{dim}_b X \); see section 2 for the definition), and the perturbation is a polynomial of degree \( 2k \). The formulation of the result given here follows [Rob11].

**Theorem 1.1** ([Rob11, theorem 14.5]). Let \( X \subset \mathbb{R}^N \) be a compact set and let \( T : X \to X \) be Lipschitz and injective. Let \( k \in \mathbb{N} \) be such that \( k > 2 \dim_b X \) and assume \( 2 \dim_b \{ x \in X : T^p x = x \} < p \) for \( p = 1, \ldots, k - 1 \). Let \( h : \mathbb{R}^N \to \mathbb{R} \) be a Lipschitz function and \( h_1, \ldots, h_m : \mathbb{R}^N \to \mathbb{R} \) a basis of the space of polynomials of degree at most \( 2k \). For \( \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m \) denote by \( h_\alpha : \mathbb{R}^N \to \mathbb{R} \) the map

\[
h_\alpha(x) = h(x) + \sum_{j=1}^m \alpha_j h_j(x).
\]

Then for Lebesgue almost every \( \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m \), the transformation

\[
\phi^T_\alpha : X \to \mathbb{R}^k, \quad \phi^T_\alpha(x) = (h_\alpha(x), h_\alpha(Tx), \ldots, h_\alpha(T^{k-1}x))
\]

is injective on \( X \).

The map \( \phi^T_\alpha \) is called the delay-coordinate map. Note that theorem 1.1 applies to any compact set \( X \subset \mathbb{R}^N \), not necessarily a manifold. This is a useful feature, as it allows one to consider sets with a complicated geometrical structure, such as fractal sets arising as attractors in chaotic dynamical systems, see e.g. [ER85]. Moreover, the upper box-counting dimension of \( X \) can be smaller than the dimension of any smooth manifold containing \( X \), so theorem 1.1 may require fewer delay-coordinates than its smooth counterpart in [Tak81].

As it was noted above, usually an experimentalist may perform only a finite number of observations \( h(x_1), \ldots, h(T^{k-1}x_l) \) for some points \( x_j \in X, j = 1, \ldots, l \). We believe it is realistic to assume there is an (explicit or implicit) random process determining which initial states \( x_j \) are accessible to the experimentalist. In this paper we are interested in the question of
reconstruction of the system in presence of such process. Mathematically speaking, this corresponds to fixing a probability measure $\mu$ on $X$ and asking whether the delay-coordinate map $\phi^T_\alpha$ is injective almost surely with respect to $\mu$. Since in this setting we are allowed to neglect sets of probability zero, it is reasonable to ask whether the minimal number of delay-coordinates sufficient for the reconstruction of the system can be smaller than $2 \dim X$. Our main result states that this is indeed the case, and the number of delay-coordinates can be reduced by half for any (Borel) probability measure.

The problem of determining the minimal number of delay-coordinates required for reconstruction has been already considered in the physical literature. In [PCFS80], the authors analysed an algorithm which may be interpreted as an attempt to determine this number in a probabilistic setting. Our work provides rigorous results in this direction. The following theorem is a simplified version of our result.

**Theorem 1.2 (Probabilistic Takens delay embedding theorem).** Let $X \subset \mathbb{R}^N$ be a Borel set, $\mu$ a Borel probability measure on $X$ and $T : X \to X$ an injective, locally Lipschitz map. Take $k \in \mathbb{N}$ such that $k > \dim X$ and assume that for $p = 1, \ldots, k - 1$ we have
\[
\dim \left( \{ x \in X : T^p x = x \} \right) < p \mu \left( \{ x \in X : T^p x = x \} \right) = 0.
\]
Let $h : X \to \mathbb{R}$ be a locally Lipschitz function and $h_1, \ldots, h_m : \mathbb{R}^N \to \mathbb{R}$ a basis of the space of real polynomials of $N$ variables of degree at most $2k - 1$. For $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$ denote by $h_\alpha : \mathbb{R}^N \to \mathbb{R}$ the map
\[
h_\alpha(x) = h(x) + \sum_{j=1}^m \alpha_j h_j(x).
\]
Then for Lebesgue almost every $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$, there exists a Borel set $X_\alpha \subset X$ of full $\mu$-measure, such that the delay-coordinate map
\[
\phi^T_\alpha : X \to \mathbb{R}^k, \quad \phi^T_\alpha(x) = \left( h_\alpha(x), h_\alpha(Tx), \ldots, h_\alpha(T^{k-1}x) \right)
\]
is injective on $X_\alpha$.

In the above theorem, the dimension $\dim$ can be chosen to be any of $\dim_H, \dim_L, \dim_B$ (Hausdorff, lower and upper box-counting dimension; for the definitions see section 2). Recall that for any Borel set $X$ one has
\[
\dim_H X \leqslant \dim_L X \leqslant \overline{\dim}_B X
\]
(see e.g. [Fal14, proposition 3.4]). Since the inequalities in (1.1) may be strict, using the Hausdorff dimension instead of the box-counting one(s) may reduce the required number of delay-coordinates. In particular, there are compact sets $X \subset \mathbb{R}^N$ with $\dim_H X = 0$ and $\overline{\dim}_B X = N$, hence theorem 1.2 can reduce significantly the number of required delay-coordinates compared to theorem 1.1 (in a probabilistic setting).

Notice that in theorem 1.2 we do not assume that the measure $\mu$ is $T$-invariant. However, the invariance of $\mu$ provides some additional benefits, as shown in the following remark.

**Remark 1.3 (Invariant measure case).** Suppose that the measure $\mu$ in theorem 1.2 is additionally $T$-invariant, i.e. $\mu(Y) = \mu(T^{-1}(Y))$ for every Borel set $Y \subset X$. Then the set $X_\alpha$ can be chosen to satisfy $T(X_\alpha) = X_\alpha$. Moreover, if $\mu$ is $T$-invariant and ergodic (i.e. $T^{-1}(Y) = Y$ can occur only for sets $Y$ of $0$ or full $\mu$-measure), then the assumption on the periodic points of $T$ in theorem 1.2 can be omitted.

Note that in the case when the measure $\mu$ is $T$-invariant, theorem 1.2 and remark 1.3 show that for a suitable choice of $X_\alpha$, the map $\phi^T_\alpha$ is injective on the invariant set $X_\alpha$, which
implies that the dynamical system $(\dot{X}, \dot{T})$ for $\dot{X} = \phi^T_{\alpha}(X, \alpha), \dot{T} = \phi^T_{\alpha} \circ (\phi^T_{\alpha})^{-1}$, is a model of the system $(X, T)$ embedded in $\mathbb{R}^k$.

An extended version of theorem 1.2 and remark 1.3 are presented and proved in section 4 as theorem 4.3 and remark 4.4, respectively. Theorem 4.3 shows that the assumption $k > \text{dim} X$ can be slightly weakened, and in addition to locally Lipschitz functions $h$, one can consider locally $\beta$-Hölder functions for suitable $\beta \in (0, 1]$. Moreover, one can replace the probabilistic measure $\mu$ by any Borel $\sigma$-finite measure on $X$. For details, see section 4.

Notice that to eliminate the assumption on the periodic points of $T$ in theorem 1.2, one can also consider systems with ‘few’ or no periodic points. For instance, as proved in [Yor69], a flow on a subset of Euclidean space given by an autonomous differential equation $\dot{x} = F(x)$, where $F$ is Lipschitz with a constant $L$, has no periodic orbits of period smaller than $\frac{2\pi}{L}$. It follows that if $T$ is a $t$-time map for such a flow with $t < \frac{2\pi}{L \text{dim} X}$, then it has no periodic points of periods smaller than $\text{dim} X$ and therefore the assumption on periodic points in theorem 1.2 can be omitted (compare also [Gut16, remark 1.2]). The same holds if the number of periodic points of a given period is finite, which by the Kupka–Smale theorem is a generic condition in the space of $C^r$-diffeomorphisms ($r \geq 1$) of a compact manifold equipped with the uniform $C^r$-topology.4

As has been mentioned already, Takens theorems are used in order to justify actual (approximate) delay map procedures based on real experimental data (see e.g. [MGNS18, HGLS05, SGM90, SM90]). Note, however, that in the cited papers the dimension of the phase space $X$ is deduced a posteriori from the properties of the time series (orbits of the delay coordinate map for a given observable). It would be very interesting to know whether in the literature it has been observed for some experimental data originating from a space $X$ with known dimension that it is sufficient to have $k \approx \text{dim} X$ (instead of $k \approx 2 \text{dim} X$) delay-coordinates (in other words, time series of length $k$) in the framework of such procedures.

In this paper we focus our attention to the case when the space $X$ is a subset of a finite-dimensional Euclidean space. Takens-type delay embedding theorems have also been extended to finite-dimensional subsets of Banach spaces (see e.g. [Rob05]). It is a natural question, whether our probabilistic embedding theorems can also be transferred into the infinite-dimensional setup. This problem will be considered in a subsequent work.

Takens-type delay embedding theorems can be seen as dynamical versions of embedding theorems which specify when a finite-dimensional set can be embedded into a Euclidean space. Indeed, under the assumptions of theorem 1.1, the delay-coordinate map $\phi^T_{\alpha}$ is an embedding of $X$ into $\mathbb{R}^k$ for typical $\alpha$. Embedding theorems in various categories have been extensively studied in a number of papers (see section 3 for a more detailed discussion). Recently, Alberti, Bölcskei, De Lellis, Koliander and Riegler [ABDL+18] proved a probabilistic embedding theorem involving the modified lower box-counting dimension of the set (see theorem 3.6). We are able to improve this result by considering the Hausdorff dimension. Below we present a simplified version of our theorem, which can be seen as a non-dynamical counterpart of theorem 1.2.

**Theorem 1.4 (Probabilistic embedding theorem).** Let $X \subset \mathbb{R}^N$ be a Borel set and let $\mu$ be a Borel probability measure on $X$. Take $k \in \mathbb{N}$ such that the $k$th Hausdorff measure of $X$ is zero (it suffices to take $k > \text{dim}_H X$) and let $\phi : X \to \mathbb{R}^k$ be a locally Lipschitz map. Then for

---

4 According to the Kupka–Smale theorem ([PdM82, chapter 3, theorem 3.6]) it is generic that the periodic points are hyperbolic and thus periodic points of a given period are isolated by the Hartman–Grobman theorem ([PdM82, chapter 2, theorem 4.1]).
Lebesgue almost every linear transformation \( L : \mathbb{R}^N \to \mathbb{R}^k \) there exists a Borel set \( X_L \subset X \) of full \( \mu \)-measure, such that \( \phi_L = \phi + L \) is injective on \( X_L \).

The extended version of the theorem is formulated and proved in section 3 as theorem 3.1. In particular, we obtain the following geometric corollary (see section 3 for details).

**Corollary 1.5 (Probabilistic injective projection theorem).** Let \( X \subset \mathbb{R}^N \) be a Borel set and let \( \mu \) be a Borel probability measure on \( X \). Then for every \( k > \dim_H X \) and almost every \( k \)-dimensional linear subspace \( S \subset \mathbb{R}^N \), the orthogonal projection of \( X \) into \( S \) is injective on a full \( \mu \)-measure subset of \( X \).

Notice that by the Marstrand–Mattila projection theorem (see [Mar54, Mat75]), if \( X \subset \mathbb{R}^N \) is Borel and \( k \geq \dim_H X \), then for almost all \( k \)-dimensional linear subspaces \( S \subset \mathbb{R}^N \), the image of \( X \) under the orthogonal projection into \( S \) has Hausdorff dimension equal to \( \dim_H X \). Note also that Sauer and Yorke proved in [SY97] that the dimension of a bounded Borel subset \( X \) of \( \mathbb{R}^N \) is preserved under typical smooth maps and typical delay-coordinate maps into \( \mathbb{R}^k \) as long as \( k \geq \dim_H X \).

In this paper we also provide several examples. Example 3.5 shows that in general the condition \( k > \dim_H X \) in theorem 1.4 cannot be replaced by \( k \geq \dim_H X \). Example 4.6 shows that linear perturbations of the observable are not sufficient for the Takens theorem. Section 5 contains a pair of examples. The first one is based on Kan’s example from the appendix to [SYC91], showing that condition \( k > 2 \dim_H X \) is not sufficient for existence of a linear transformation into \( \mathbb{R}^k \) which is injective on \( X \). As in the probabilistic setting one can work with the Hausdorff dimension, we consider a set \( X \subset \mathbb{R}^2 \) similar to the one provided by Kan, which cannot be embedded linearly into \( \mathbb{R} \), but when endowed with a natural probability measure, almost every linear transformation \( L : \mathbb{R}^2 \to \mathbb{R} \) is injective on a set of full measure. The second example provides a probability measure with \( \dim_H \mu < \dim_{MB} \mu \) (see section 2 for definitions), showing that theorem 1.4 strengthens a previous result from [ABDL+18].

**Organisation of the paper.** The paper is organised as follows. In section 2 we introduce notation, definitions and preliminary results. Section 3 contains the formulation and proof of the extended version of the probabilistic embedding theorem (theorem 3.1), while section 4 is devoted to the proof of the extended version of the probabilistic Takens delay embedding theorem (theorem 4.3). In section 5 we present examples showing how the use of the Hausdorff dimension improves the previously obtained results.

### 2. Preliminaries

Consider the Euclidean space \( \mathbb{R}^N \) for \( N \in \mathbb{N} \), with the standard inner product \( \langle \cdot, \cdot \rangle \) and the norm \( \| \cdot \| \). The open \( \delta \)-ball around a point \( x \in \mathbb{R}^N \) is denoted by \( B_\delta(x, \delta) \). By \( |X| \) we denote the diameter of a set \( X \subset \mathbb{R}^N \). We say that function \( \phi : X \to \mathbb{R}^k \), \( X \subset \mathbb{R}^N \) is locally \( \beta \)-Hölder for \( \beta > 0 \) if for every \( x \in X \) there exists an open set \( U \subset \mathbb{R}^N \) containing \( x \) such that \( \phi \) is \( \beta \)-Hölder on \( U \cap X \), i.e. there exists \( C > 0 \) such that

\[
\| \phi(x) - \phi(y) \| \leq C \| x - y \|^\beta
\]

for every \( x, y \in U \cap X \). We say that \( \phi \) is locally Lipschitz if it is locally 1-Hölder.

For \( k \leq N \) we write \( \text{Gr}(k, N) \) for the \( (k, N) \)-Grassmannian, i.e. the space of all \( k \)-dimensional linear subspaces of \( \mathbb{R}^N \), equipped with the standard rotation-invariant (Haar) measure.

\[ \text{Gr}(k, N) \]

\[ \text{Gr}(k, N) \]
see [Mat95, section 3.9] (and [FR02] for another construction of a rotation-invariant measure on the Grassmannian). By \( \eta_N \) we denote the normalised Lebesgue measure on the unit ball \( B_N(0, 1) \), i.e. 
\[
\eta_N = \frac{1}{\text{Leb}(B_N(0, 1))}\text{Leb}|_{B_N(0, 1)},
\]
where Leb is the Lebesgue measure on \( \mathbb{R}^N \).

For \( s > 0 \), the \( s \)-dimensional (outer) Hausdorff measure of a set \( X \subset \mathbb{R}^N \) is defined as
\[
\mathcal{H}^s(X) = \lim_{\delta \to 0} \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : X \subset \bigcup_{i=1}^{\infty} U_i, \ |U_i| \leq \delta \right\}.
\]
The Hausdorff dimension of \( X \) is given as
\[
\dim_H X = \inf \{ s > 0 : \mathcal{H}^s(X) = 0 \} = \sup \{ s > 0 : \mathcal{H}^s(X) = \infty \}.
\]
For a bounded set \( X \subset \mathbb{R}^N \) and \( \delta > 0 \), let \( N(X, \delta) \) denote the minimal number of balls of diameter at most \( \delta \) required to cover \( X \). The lower and upper box-counting (Minkowski) dimensions of \( X \) are defined as
\[
\overline{\dim}_B X = \lim_{\delta \to 0} \sup \frac{\log N(X, \delta)}{\log \delta} \quad \text{and} \quad \underline{\dim}_B X = \lim_{\delta \to 0} \inf \frac{\log N(X, \delta)}{\log \delta}.
\]
The lower (resp. upper) box-counting dimension of an unbounded set is defined as the supremum of the lower (resp. upper) box-counting dimensions of its bounded subsets.

The lower and upper modified box-counting dimensions of \( X \subset \mathbb{R}^N \) are defined as
\[
\overline{\dim}_{MB} X = \inf \left\{ \sup_{i \in \mathbb{N}} \dim_B K_i : X \subset \bigcup_{i=1}^{\infty} K_i, K_i \text{ compact} \right\},
\]
\[
\underline{\dim}_{MB} X = \inf \left\{ \sup_{i \in \mathbb{N}} \overline{\dim}_B K_i : X \subset \bigcup_{i=1}^{\infty} K_i, K_i \text{ compact} \right\}.
\]
With this notation, the following inequalities hold:
\[
\dim_H X \leq \overline{\dim}_{MB} X \leq \underline{\dim}_{MB} X \leq \underline{\dim}_B X,
\]
\[
\dim_H X \leq \overline{\dim}_{MB} X \leq \overline{\dim}_B X \leq \underline{\dim}_B X.
\]
(2.1)

We define dimension of a finite Borel measure \( \mu \) in \( \mathbb{R}^N \) as
\[
\dim \mu = \inf \{ \dim X : X \subset \mathbb{R}^N \text{ is a Borel set of full } \mu \text{-measure} \}.
\]
Here \( \dim \) may denote any one of the dimensions defined above. Recall that for a measure \( \mu \) on a set \( X \) and a measurable \( Y \subset X \) we say that \( Y \) is of full \( \mu \)-measure, if \( \mu(X \setminus Y) = 0 \).

For \( N, k \in \mathbb{N} \) let \( \text{Lin}(\mathbb{R}^N; \mathbb{R}^k) \) be the space of all linear transformations \( L : \mathbb{R}^N \to \mathbb{R}^k \). Such transformations are given by
\[
Lx = (l_1, x), \ldots, (l_k, x),
\]
(2.2)
where \( l_1, \ldots, l_k \in \mathbb{R}^N \). Thus, the space \( \text{Lin}(\mathbb{R}^N; \mathbb{R}^k) \) can be identified with \( (\mathbb{R}^N)^k \), and the Lebesgue measure on \( \text{Lin}(\mathbb{R}^N; \mathbb{R}^k) \) is understood as \( \bigotimes_{i=1}^{k} \text{Leb} \), where \( \text{Leb} \) is the Lebesgue
measure in \( \mathbb{R}^N \). Within the space \( \text{Lin} (\mathbb{R}^N; \mathbb{R}^k) \) we consider the space \( E^N_k \) consisting of all linear transformations \( L : \mathbb{R}^N \to \mathbb{R}^k \) of the form (2.2), for which \( l_1, \ldots, l_k \in B_N(0, 1) \). Note that by the Cauchy–Schwarz inequality,

\[
\|Lx\| \leq \sqrt{N} \|x\| \tag{2.3}
\]

for every \( L \in E^N_k \) and \( x \in \mathbb{R}^N \).

By \( \eta_{N,k} \) we denote the normalised Lebesgue measure on \( E^N_k \), i.e. the probability measure on \( E^N_k \) given by

\[
\eta_{N,k} = \prod_{j=1}^k \frac{1}{\text{Leb}(B_N(0, 1))} \text{Leb}|_{B_N(0, 1)}.
\]

The following geometrical inequality, used in [HK99] (see also [Rob11, lemma 4.1]) is the key ingredient of the proof of theorem 3.1.

**Lemma 2.1.** Let \( L : \mathbb{R}^N \to \mathbb{R}^k \) be a linear transformation. Then for every \( x \in \mathbb{R}^N \setminus \{0\}, z \in \mathbb{R}^k \) and \( \varepsilon > 0 \),

\[
\eta_{N,k}\left(\{L \in E^N_k : \|Lx + z\| \leq \varepsilon\}\right) \leq CN^{k/2} \varepsilon^k \|x\|^k,
\]

where \( C > 0 \) is an absolute constant.

For \( L \in \text{Lin}(\mathbb{R}^m; \mathbb{R}^k) \), where \( m, k \in \mathbb{N} \), denote by \( \sigma_p(L) \), \( p \in \{1, \ldots, k\} \), the \( p \)th largest singular value of the matrix \( L \), i.e. the \( p \)th largest square root of an eigenvalue of the matrix \( L^*L \). In the proof of theorem 4.3, instead of lemma 2.1 we will use the following lemma, proved as [SYC91, lemma 4.2] (see also [Rob11, lemma 14.3]).

**Lemma 2.2.** Let \( L : \mathbb{R}^m \to \mathbb{R}^k \) be a linear transformation. Assume that \( \sigma_p(L) > 0 \) for some \( p \in \{1, \ldots, k\} \). Then for every \( z \in \mathbb{R}^k \) and \( \rho, \varepsilon > 0 \),

\[
\frac{\text{Leb}\left(\{\alpha \in B_m(0, \rho) : \|L\alpha + z\| \leq \varepsilon\}\right)}{\text{Leb}(B_m(0, \rho))} \leq C_{m,k} \left(\frac{\varepsilon}{\sigma_p(L) \rho}\right)^p,
\]

where \( C_{m,k} > 0 \) is a constant depending only on \( m, k \) and \( \text{Leb} \) is the Lebesgue measure on \( \mathbb{R}^m \).

To verify the measurability of the sets occurring in subsequent proofs, we will use the two following elementary lemmas. A measure \( \mu \) on a set \( X \) is called \( \sigma \)-finite if there exists a countable collection of measurable sets \( A_n \), \( n \in \mathbb{N} \) such that \( \mu(A_n) < \infty \) for each \( n \in \mathbb{N} \) and \( \bigcup_{n=1}^\infty A_n = X \). Recall that a \( \sigma \)-compact set is a countable union of compact sets.

**Lemma 2.3.** Let \( X \subset \mathbb{R}^N \) be a Borel set and let \( \mu \) be a Borel \( \sigma \)-finite measure on \( X \). Then there exists a \( \sigma \)-compact set \( K \subset X \) of full \( \mu \)-measure.

**Proof.** Follows directly from the fact that a \( \sigma \)-finite Borel measure in a Euclidean space is regular (see e.g. [Bil99, theorem 1.1]). \( \square \)

**Lemma 2.4.** Let \( X, Z \) be metric spaces. Then the following hold.

(a) If \( K \subset X \times Z \) is \( \sigma \)-compact, then so is \( \pi_X(K) \), where \( \pi_X : X \times Z \to X \) is the projection given by \( \pi_X(x, z) = x \). In particular, \( \pi_X(K) \) is Borel.

(b) If \( X \) is \( \sigma \)-compact, \( F : X \to Z \) is continuous and \( K \subset Z \) is \( \sigma \)-compact, then \( F^{-1}(K) \) is \( \sigma \)-compact, hence Borel.
(c) If $X$, $Z$ are $\sigma$-compact, $F: X \times Z \to \mathbb{R}^k$, $k \in \mathbb{N}$, is continuous and $K \subset X$ is $\sigma$-compact, then the set

$$\{(x,z) \in X \times Z : F(x,z) = F(y,z) \text{ for some } y \in K \setminus \{x\}\}$$

is $\sigma$-compact and hence Borel.

**Proof.** The statement (a) follows from the fact that $\pi_X$ is continuous, and a continuous image of a compact set is also compact. To show (b), it is enough to notice that $F^{-1}(K)$ is a countable union of closed subsets of a $\sigma$-compact space. To check (c), let $\pi_{X \times Z}(x,y,z) = (x,z)$. Then

$$\{(x,z) \in X \times Z : F(x,z) = F(y,z) \text{ for some } y \in K \setminus \{x\}\}$$

$$= \pi_{X \times Z}\left(\{(x,y,z) \in X \times K \times Z : F(x,z) = F(y,z), \text{ } d(x,y) \neq 0\}\right)$$

$$= \bigcup_{n=1}^{\infty} \pi_{X \times Z}\left(\left\{(x,y,z) \in X \times K \times Z : F(x,z) = F(y,z), \text{ } d(x,y) \geq \frac{1}{n}\right\}\right),$$

where $d$ is the metric in $X$. Since $d$ is continuous, we can use (a) and (b) to end the proof. □

### 3. Probabilistic embedding theorem

In this section we prove an extended version of the probabilistic embedding theorem, formulated below. Obviously, theorem 1.4 follows from theorem 3.1

**Theorem 3.1** (Probabilistic embedding theorem—extended version). Let $X \subset \mathbb{R}^N$ be a Borel set and $\mu$ be a Borel $\sigma$-finite measure on $X$. Take $k \in \mathbb{N}$ and $\beta \in (0,1]$ such that $\mathcal{H}^{\beta}(X) = 0$ and let $\phi : X \to \mathbb{R}^k$ be a locally $\beta$-Hölder map. Then for Lebesgue almost every linear transformation $L : \mathbb{R}^N \to \mathbb{R}^k$ there exists a Borel set $X_L \subset X$ of full $\mu$-measure, such that the map $\phi_L = \phi + L$ is injective on $X_L$.

**Remark 3.2.** It is straightforward to notice that if $\dim_H X = 0$, then $\phi$ can be taken to be an arbitrary Hölder map.

**Proof of theorem 3.1.** Note first that it is sufficient to prove that the set $X_L$ exists for $\eta_{\mathcal{H}^{\beta}}$-almost every $L \in E_k^N$. Indeed, if this is shown, then for a given locally $\beta$-Hölder map $\phi : X \to \mathbb{R}^k$ we can take sets $L_j \subset E_k^N$, $j \in \mathbb{N}$, such that $\eta_{\mathcal{H}^{\beta}}(L_j) = 1$ and for every $L \in L_j$ the map $(\phi / j)_L = \phi / j + L$ is injective on a Borel set $X_L^{(j)} \subset X$ of full $\mu$-measure. Then the set $L = \bigcup_{j \in \mathbb{N}} \{L : L \in L_j\} \subset \text{Lin}(\mathbb{R}^N; \mathbb{R}^k)$ has full Lebesgue measure and for every $L \in L$ there exists $j$ such that $L / j \in L_j$, so $\phi(L)_{j/(\phi / j)} = (\phi + L_j) / j$ (and hence $\phi_L$) is injective on $X_L = \bigcap_{j \in \mathbb{N}} X_L^{(j)}$, which has full $\mu$-measure.

By lemma 2.3, we can assume that $X$ is $\sigma$-compact. Take $k \in \mathbb{N}$, $\beta \in (0,1]$ with $\mathcal{H}^{\beta}(X) = 0$ and a locally $\beta$-Hölder map $\phi : X \to \mathbb{R}^k$. Set

$$A = \left\{(x,L) \in X \times E_k^N : \phi_L(x) = \phi_L(y) \text{ for some } y \in X \setminus \{x\}\right\}.$$ 

By lemma 2.4, $A$ is Borel. For $x \in X$ and $L \in E_k^N$, denote by $A_x$ and $A_L$, respectively, the sections

$$A_x = \left\{L \in E_k^N : (x,L) \in A\right\}, \quad A_L = \{x \in X : (x,L) \in A\}.$$
The sets $A_x$ and $A^c$ are Borel as sections of a Borel set. Observe first that in order to prove the theorem it is enough to show $\eta_{N,k}(A_x) = 0$ for every $x \in X$, since then by Fubini’s theorem ([Rud87, theorem 8.8]), $(\eta_{N,k} \otimes \mu)(A)$ is 0 and, consequently, $\mu(A^c) = 0$ for $\eta_{N,k}$-almost every $L \in E_N^X$. Since $\phi_L$ is injective on $X \setminus A^c$, the assertion of the theorem is true.

Take a point $x \in X$. Since $\phi$ is locally $\beta$-Hölder and $X$ is separable, there exists a countable covering of $X$ by open sets $U_j \subset \mathbb{R}^N$, $j \in \mathbb{N}$, such that

$$\|\phi(y) - \phi(y')\| \leq C_j \|y - y'\|^\beta$$

for every $y, y' \in U_j \cap X$ \hspace{1cm} (3.1)

for some $C_j > 0$. Let

$$K_n = \left\{ y \in X : \frac{1}{n} \leq \|x - y\| \right\}.$$

To show $\eta_{N,k}(A_x) = 0$, it suffices to prove $\eta_{N,k}(A_{x,j,n}) = 0$ for every $j, n \in \mathbb{N}$, where

$$A_{x,j,n} = \left\{ L \in E_N^X : \phi_L(x) = \phi_L(y) \text{ for some } y \in U_j \cap K_n \right\}.$$ 

Note that by lemma 2.4, the set $A_{x,j,n}$ is Borel.

Take $j, n \in \mathbb{N}$ and fix a small $\varepsilon > 0$. Since $\mathcal{H}^{\beta}(U_j \cap K_n) \leq \mathcal{H}^{\beta}(X) = 0$, there exists a collection of balls $B_N(y_i, \varepsilon_i)$, $i \in \mathbb{N}$, for some $y_i \in U_j \cap K_n$ and $\varepsilon_i > 0$, such that

$$U_j \cap K_n \subset \bigcup_{i \in \mathbb{N}} B_N(y_i, \varepsilon_i) \quad \text{and} \quad \sum_{i=1}^{\infty} \varepsilon_i^{\beta} = \varepsilon.$$

Take $L \in A_{x,j,n}$ and $y \in U_j \cap K_n$ such that $\phi_L(x) = \phi_L(y)$. Then $y \in B_N(y_i, \varepsilon_i)$ for some $i \in \mathbb{N}$ and

$$\|L(y_i - x) + \phi(y_i) - \phi(x)\| = \|\phi_L(y_i) - \phi_L(x)\|$$

$$= \|\phi_L(y_i) - \phi_L(y)\|$$

$$\leq \|\phi(y_i) - \phi(y)\| + \|L(y_i) - y\|$$

$$\leq C_j \|y_i - y\|^\beta + \sqrt{N}\|y_i - y\|$$

$$\leq M_j^\beta \varepsilon_i^{\beta}$$

for some $M_j > 0$, by (2.3) and (3.1). This shows that

$$A_{x,j,n} \subset \bigcup_{i \in \mathbb{N}} \left\{ L \in E_N^X : \|L(y_i - x) + \phi(y_i) - \phi(x)\| \leq M_j^\beta \varepsilon_i^{\beta} \right\}. $$

By lemma 2.1, (3.2) and the fact $y_i \in K_n$, we have

$$\eta_{N,k}(A_{x,j,n}) \leq \sum_{i=1}^{\infty} \eta_{N,k} \left\{ L \in E_N^X : \|L(y_i - x) + \phi(y_i) - \phi(x)\| \leq M_j^\beta \varepsilon_i^{\beta} \right\}$$

$$\leq \frac{CN^{k/2}M_j}{1/n^{k}} \sum_{i=1}^{\infty} \varepsilon_i^{\beta} \leq CN^{k/2}M_j^\beta n^{k}\varepsilon.$$ 

Since $\varepsilon > 0$ was arbitrary, we obtain $\eta_{N,k}(A_{x,j,n}) = 0$, which ends the proof. \hfill $\square$
Remark 3.3. Note that the assumption \( \mathcal{H}^{\beta k}(X) = 0 \) is fulfilled if \( \dim_H X < \beta k \), so theorem 3.1 is indeed a Hausdorff dimension embedding theorem. Moreover, it may happen that \( \mathcal{H}^{\beta k}(X) = 0 \) and \( \dim_H X = \beta k \).

As a simple consequence of theorem 3.1, we obtain the following corollary, formulated in a slightly simplified version in section 1 as corollary 1.5.

**Corollary 3.4** (Probabilistic injective projection theorem—extended version). Let \( X \subset \mathbb{R}^N \) be a Borel set and let \( \mu \) be a Borel \( \sigma \)-finite measure on \( X \). Then for every \( k \in \mathbb{N} \), \( k \leq N \) such that \( \mathcal{H}^k(X) = 0 \) and almost every \( k \)-dimensional linear subspace \( S \subset \mathbb{R}^N \) (with respect to the standard measure on the Grassmannian \( \text{Gr}(k, N) \)), the orthogonal projection of \( X \) into \( S \) is injective on a full \( \mu \)-measure subset of \( X \) (depending on \( S \)).

**Proof of corollary 3.4.** Apply theorem 3.1 for the map \( \phi \equiv 0 \). Then we know that a linear map \( L \in \text{Lin}(\mathbb{R}^N; \mathbb{R}^k) \) of the form (2.2) is injective on a set \( X_L \subset X \) of full \( \mu \)-measure for Lebesgue almost every \((l_1, \ldots, l_k) \in (\mathbb{R}^N)^k \). We can assume that \( l_1, \ldots, l_k \) are linearly independent for all such \( L \), which also implies that the same holds for \( Ll_1, \ldots, Ll_k \). Setting

\[
S_L = \text{Span}(l_1, \ldots, l_k)
\]

and taking \( V_L \in \text{Lin}(\mathbb{R}^k; \mathbb{R}^N) \) defined by \( V_L(Ll_j) = l_j \) for \( j = 1, \ldots, k \), we have

\[
V_L \circ L = \Pi_{S_L}
\]

where \( \Pi_{S_L} \) is the orthogonal projection from \( \mathbb{R}^N \) onto \( S_L \) and \( V_L \) is injective. It follows that \( \Pi_{S_L} \) is injective on \( X_L \) for almost every \((l_1, \ldots, l_k) \), so \( \Pi_L \) is injective on a full Lin-measure subset of \( X \) for almost every \( k \)-dimensional linear subspace \( S \subset \mathbb{R}^N \). \( \square \)

Let us note that in general, the requirement \( \mathcal{H}^{\beta k}(X) = 0 \) in theorem 3.1 cannot be replaced by the weaker condition \( \dim_H(X) \leq \beta k \).

**Example 3.5.** Let \( k = \beta = 1 \), \( X = S^1 \subset \mathbb{R}^2 \) be the unit circle and let \( \mu \) be the normalised Lebesgue measure on \( S^1 \). We shall prove that there is no Lipschitz transformation \( \phi : S^1 \to \mathbb{R} \) which is injective on a set of full \( \mu \)-measure. Let \( \phi \) be such a transformation. Then \( \phi(S^1) = [a, b] \) for some compact interval. As \( \phi \) is injective on a set of full measure, the interval \([a, b]\) is non-degenerate, i.e. \( a < b \). Fix points \( x, y \in S^1 \) with \( \phi(x) = a, \phi(y) = b \). As \( x \neq y \), there are exactly two open arcs \( I, J \subset S^1 \) of positive measure joining \( x \) and \( y \) such that \( I \cap J = \{x, y\} \) and \( I \cup J = S^1 \). Clearly \( \phi(I) = \phi(J) = [a, b] \). Let \( A \subset S^1 \) be a Borel set such that \( \phi \) is injective on \( A \) and \( \mu(A) = 1 \). As Lipschitz maps transform sets of zero Lebesgue measure to sets of zero Lebesgue measure, we conclude that \( \phi(I \cap A) \) and \( \phi(J \cap A) \) are disjoint Lebesgue measurable subsets of \([a, b]\) with Lebesgue measure equal to \( b - a \). This contradiction shows that no Lipschitz transformation \( \phi : S^1 \to \mathbb{R} \) is injective on a full measure set.

Theorem 3.1 strengthens the following embedding theorem, proved recently by Alberti, Bölcskei, De Lellis, Koliander and Riegler in [ABDL⁺18].

**Theorem 3.6** ([ABDL⁺18, theorem 2.1]). Let \( \mu \) be a Borel probability measure in \( \mathbb{R}^N \) and let \( k \in \mathbb{N} \) be such that \( k > \dim_{\mu_B} \mu \). Then for Lebesgue almost every linear transformation \( L : \mathbb{R}^N \to \mathbb{R}^k \) there exists a Borel set \( X_L \subset \mathbb{R}^N \) such that \( \mu(X_L) = 1 \) and \( L \) is injective on \( X_L \).

In fact, in [ABDL⁺18] the authors introduced the notion of \( \dim_{\mu_B} \mu \), denoting it by \( K(\mu) \) and calling it the descriptive complexity of the measure. In particular, theorem 3.6 holds for measures \( \mu \) supported on a Borel set \( X \subset \mathbb{R}^N \) with \( \dim_{\mu_B} X < k \). By (2.1), we have
dim_Hμ ≤ dim_MHμ, and in section 5 we present an example (theorem 5.5) showing that the
inequality may be strict. Therefore, theorem 3.1 actually strengthens theorem 3.6.

Non-probabilistic embedding theorems were first obtained in topological and smooth cate-
gories. The well-known Menger–Nöbeling embedding theorem (see e.g. [HW41, theorem 5.2])
states that for a compact metric space X with Lebesgue covering dimension at most k, a
generic continuous transformation φ : X → R^{2k+1} is injective (and hence defines
a homeomorphism between X and φ(X)). Genericity means here that the set of injective
transformations φ : X → R^{2k+1} is a dense Gδ subset of C(X; R^{2k+1}) endowed with the supremum
metric. The dimension 2k + 1 is known to be optimal. The corresponding result in the category
of smooth manifolds is the Whitney embedding theorem (see [Whi36]). It states that for a given
k-dimensional C^r-manifold M, a generic C^r-transformation from M to R^{2k+1} is a C^r-embedding
(i.e. an injective immersion of class C^r).

Let us now compare theorem 3.1 to non-probabilistic embedding theorems involving
the box-counting dimension. One of the first results in this area was a theorem by Mañé
[Mn81, lemma 1.1]. We present its formulation following [SYC91, theorem 4.6] and [Rob11,
theorem 6.2] (originally, Mañé proved that topologically generic linear transformation is
injective on X).

**Theorem 3.7.** Let X ⊂ R^N be a compact set. Let k ∈ N be such that k > 2dim_H X (it suffices
to take k > dim_H (X − X)). Then Lebesgue almost every linear transformation L : R^N → R^k is
injective on X.

**Remark 3.8.** As noticed by Mañé and communicated in [ER85, p 627], his original
statement in [Mn81] is incorrect. Namely, he assumed k > 2dim_H X + 1 instead of k >
dim_H (X − X). However, this is known to be insufficient for the existence of a linear embed-
ding of X into R^k. In fact, in [SYC91, appendix A], Kan presented an example of a set
X ⊂ R^m with dim_H X = 0, such that any linear transformation L : R^m → R^{m−1} fails to be injective
on X. It turns out that the assumption k > 2dim_H X is insufficient, while k > 2dim_H X is
sufficient. This stems from the fact that the proof of theorem 3.7 actually requires the property
k > dim_H(X − X), and the upper box-counting dimension satisfies

\[ \overline{\dim}_B (A × B) ≤ \overline{\dim}_B (A) + \overline{\dim}_B (B), \]  

\[ \text{(3.3)} \]

for A, B ⊂ R^N, hence

\[ \text{dim}_H(X − X) ≤ \text{dim}_H(X × X) ≤ \overline{\dim}_B (X × X) ≤ 2 \overline{\dim}_B X \]

(note that this calculation shows that k > 2 \overline{dim}_B X is a stronger assumption than k > dim_H(X − X)).

On the other hand, (3.3) does not hold for the Hausdorff dimension (nor for the lower
box-counting dimension), and dim_H X does not control dim_H(X − X). The fact that in theorem
3.1 we can work with the Hausdorff dimension comes from the application of Fubini’s
theorem, which enables us to consider covers of the set X itself, instead of X − X. In section 5
we analyse Kan’s example from the point of view of theorem 3.1.

Theorem 3.7 is also true for subsets of an arbitrary Banach space B for a prevalent set of
linear transformations L : B → R^k (see [Rob11, chapter 6] for details).

Note that the linear embedding from theorem 3.1 need not preserve the dimension of X.
Indeed, the Hausdorff and box-counting dimensions are invariants for bi-Lipschitz transforma-
ations, yet inverse of a linear map on a compact set does not have to be Lipschitz. Therefore,
we only know that dim φ_l(X) ≤ dim_X (see [Rob11, proposition 2.8.4 and lemma 3.3.4])
and the inequality can be strict. For example, let φ ≡ 0 and X = \{(x, f(x)) : x ∈ [0, 1]\} be
Theorem 3.9. Let $X \subset \mathbb{R}^N$ be a compact set. Let $k \in \mathbb{N}$ be such that $k > 2 \dim_H X$ and let $\beta$ be such that $0 < \beta < 1 - 2 \dim_H X/k$. Then Lebesgue almost every linear transformation $L : \mathbb{R}^N \to \mathbb{R}^k$ is injective on $X$ with $\beta$-Hölder continuous inverse.

However, this is not true in the case of theorem 3.1.

Remark 3.10. In general, we cannot claim that the injective map $\phi_L|_{X_k}$ from theorem 3.1 has a Hölder continuous inverse. Indeed, it is well-known that for $n \in \mathbb{N}$ there are examples of compact sets $X \subset \mathbb{R}^N$ of Hausdorff and topological dimension equal to $n$, which do not embed topologically into $\mathbb{R}^d$ for $k \leq 2n$ (showing the optimality of the bounds in the Menger–Nöbeling embedding theorem, see [HW41, example 5.3]). Consider a probability measure $\mu$ on $X$ with $\text{supp}\mu = X$, where $\text{supp}$ denotes the topological support of the measure (the intersection of all closed sets of full measure). It is known that such measure exists for any compact set. If the map $\phi_L|_{X_k}$ from theorem 3.1 for $k = n + 1$ had a Hölder continuous inverse $f = \phi_L^{-1}$, then we could extend $f$ from $\phi_L(X_k)$ to $\mathbb{R}^{n+1}$ preserving the Hölder continuity ([Ban51, theorem 4.7.5], see also [Min70]). Then $Y = \{x \in X : f \circ \phi_L(x) = x\}$ would be a closed subset of $X$ with $\mu(Y) = 1$, hence $Y = X$, so $\phi_L$ would be homeomorphism between $X$ and $\phi_L(X) \subset \mathbb{R}^{n+1}$, which would give a contradiction.

4. Probabilistic Takens delay embedding theorem

In this section we present the proof of the extended probabilistic Takens delay embedding theorem. It turns out that linear perturbations are insufficient for Takens-type theorems, see example 4.6. As observed in [SYC91], it is enough to take perturbations over the space of polynomials of degree $2k$. This can be easily extended to more general families of functions.

Definition 4.1. Let $X$ be a subset of $\mathbb{R}^N$. A family of transformations $h_1, \ldots, h_m : X \to \mathbb{R}$ is called a $k$-interpolating family on set $X$, if for every collection of distinct points $x_1, \ldots, x_k \in X$ and every $\xi = (\xi_1, \ldots, \xi_k) \in \mathbb{R}^k$ there exists $(\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$ such that $\alpha_1 h_1(x_1) + \cdots + \alpha_m h_m(x_k) = \xi_i$ for each $i = 1, \ldots, k$. In other words, the matrix

$$
\begin{bmatrix}
h_1(x_1) & \cdots & h_m(x_1) \\
\vdots & \ddots & \vdots \\
h_1(x_k) & \cdots & h_m(x_k)
\end{bmatrix}
$$

has full row rank as a transformation from $\mathbb{R}^m$ to $\mathbb{R}^k$. Note that the same is true for any collection of $l$ distinct points with $l \leq k$.

Remark 4.2. It is known that any linear basis $h_1, \ldots, h_m$ of the space of real polynomials of $N$ variables of degree at most $k - 1$ is a $k$-interpolating family (see e.g. [GS00, sections 1.2, equation (1.9)])

For a transformation $T : X \to X$ and $p \in \mathbb{N}$ denote by $\text{Per}_p(T)$ the set of periodic points of minimal period $p$, i.e.

$$
\text{Per}_p(T) = \{x \in X : T^p x = x \text{ and } T^j x \neq x \text{ for } j = 1, \ldots, p - 1\}.
$$
Let $\mu$ and $\nu$ be measures on a measurable space $(\mathcal{X}, \mathcal{F})$. The measure $\mu$ is called singular with respect to $\nu$, if there exists a measurable set $Y \subset \mathcal{X}$ such that $\mu(\mathcal{X}\setminus Y) = \nu(Y) = 0$. In this case we write $\mu \perp \nu$. By $\mu|_A$ we denote the restriction of $\mu$ to a set $A \in \mathcal{F}$.

**Theorem 4.3** (Probabilistic Takens delay embedding theorem—extended version). Let $X \subset \mathbb{R}^N$ be a Borel set, $\mu$ be a Borel $\sigma$-finite measure on $X$ and $T : X \to X$ an injective, locally Lipschitz map. Take $k \in \mathbb{N}$ and $\beta \in (0, 1]$ such that $\mathcal{H}^\beta(X) = 0$ and assume $\mu|_{\text{Per}(T)} \perp \mathcal{H}^\beta$ for every $p = 1, \ldots, k - 1$. Let $h : X \to \mathbb{R}$ be a locally $\beta$-Hölder function and $h_1, \ldots, h_m : X \to \mathbb{R}$ a $2k$-interpolating family on $X$ consisting of locally $\beta$-Hölder functions. For $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$ denote by $h_\alpha : X \to \mathbb{R}$ the transformation

$$h_\alpha(x) = h(x) + \sum_{j=1}^m \alpha_j h_j(x).$$

Then for Lebesgue almost every $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$, there exists a Borel set $X_\alpha \subset X$ of full $\mu$-measure, such that the delay-coordinate map

$$\phi^\alpha_n : X \to \mathbb{R}^k, \quad \phi^\alpha_n(x) = (h_{\alpha}(x), h_{\alpha}(Tx), \ldots, h_{\alpha}(T^{k-1}x))$$

is injective on $X_\alpha$.

Notice that theorem 1.2 follows from theorem 4.3 by remark 4.2.

**Remark 4.4** (Invariant measure case—extended version). Under the assumptions of theorem 4.3, the following hold.

(a) If the measure $\mu$ is $T$-invariant, then the set $X_\alpha$ can be chosen to satisfy $T(X_\alpha) \subset X_\alpha$.
(b) If the measure $\mu$ is finite and $T$-invariant, then the set $X_\alpha$ can be chosen to satisfy $T(X_\alpha) = X_\alpha$.
(c) If the measure $\mu$ is $T$-invariant and ergodic, then the assumption on the periodic points of $T$ in theorem 4.3 can be omitted.

Under the notation of theorem 4.3, we first show a preliminary lemma. For $x \in X$ define its full orbit $\text{Orb}(x)$ as

$$\text{Orb}(x) = \{T^n x : n \geq 0\} \cup \{y \in X : T^n y = x \text{ for some } n \in \mathbb{N}\}.$$

Note that since $T$ is injective, all full orbits are at most countable, and any two full orbits $\text{Orb}(x)$ and $\text{Orb}(y)$ are either equal or disjoint. For $x, y \in X$ let $D_{x,y}$ be the $k \times m$ matrix defined by

$$D_{x,y} = \begin{bmatrix}
    h_1(x) - h_1(y) & \ldots & h_m(x) - h_m(y) \\
    h_1(Tx) - h_1(Ty) & \ldots & h_m(Tx) - h_m(Ty) \\
    \vdots & \ddots & \vdots \\
    h_1(T^{k-1}x) - h_1(T^{k-1}y) & \ldots & h_m(T^{k-1}x) - h_m(T^{k-1}y)
\end{bmatrix}. $$

**Lemma 4.5.** For $x, y \in X$, the following statements hold.

(i) If $y \neq x$, then $\text{rank} D_{x,y} \geq 1$.

(ii) $\text{If } y \notin \text{Orb}(x) \text{ and } y \in \text{Per}(T) \text{ for some } p \in \{1, \ldots, k-1\}, \text{ then } \text{rank} D_{x,y} \geq p$.

(iii) $\text{If } y \notin \text{Orb}(x) \text{ and } y \notin \bigcup_{p=1}^{k-1} \text{Per}(T), \text{ then } \text{rank} D_{x,y} = k$.

**Proof.** For (a), it suffices to observe that the first row of $D_{x,y}$ is non-zero as long as $x \neq y$ and therefore $\text{rank}(D_{x,y}) \geq 1$. Indeed, otherwise we would have $h_j(x) = h_j(y)$ for $j = 1, \ldots, m$ which contradicts the fact that $h_1, \ldots, h_m$ is an interpolating family.
Assume now \( y \not\in \text{Orb}(x) \), which implies \( \text{Orb}(y) \cap \text{Orb}(x) = \emptyset \). Let \( q \) (resp. \( r \)) be a maximal number from \( \{1, \ldots, k\} \) such that the points \( x, Tx, \ldots, T^{q-1}x \) (resp. \( y,Ty,\ldots,T^{r-1}y \)) are distinct. Notice that if \( y \in \text{Per}_p(T) \) for some \( p \in \{1, \ldots, k-1\} \), then \( r = p \), and if \( y \not\in \bigcup_{p=1}^{k-1}\text{Per}_p(T) \), then \( r = k \). Thus, the assertions (b) and (c) of the lemma can be written simply as one condition

\[
\text{rank}D_{x,y} \geq r. \tag{4.1}
\]

To show that (4.1) holds, denote the points \( x, Tx, \ldots, T^{q-1}x, y, Ty, \ldots, T^{r-1}y \), preserving the order, by \( z_1, \ldots, z_q \), for \( l = q + r \). By the definition of \( q, r \), we have \( 1 \leq l \leq 2k \) and the points \( z_1, \ldots, z_l \) are distinct. Thus, the matrix \( D_{x,y} \) can be written as the product

\[
D_{x,y} = J_{x,y}V_{x,y},
\]

where

\[
V_{x,y} = \begin{bmatrix}
    h_1(z_1) & \ldots & h_m(z_1) \\
    \vdots & \ddots & \vdots \\
    h_1(z_l) & \ldots & h_m(z_l)
\end{bmatrix}
\]

and \( J_{x,y} \) is a \( k \times l \) matrix with entries in \( \{-1, 0, 1\} \) and block structure of the form

\[
J_{x,y} = \begin{bmatrix}
    * & -\text{Id}_{r \times r} \\
    * & *
\end{bmatrix},
\]

where \( \text{Id}_{r \times r} \) is the \( r \times r \) identity matrix. It follows that \( \text{rank}J_{x,y} \geq r \). Moreover, since \( z_1, \ldots, z_l \) are distinct and \( h_1, \ldots, h_m \) is a \( 2k \)-interpolating family, the matrix \( V_{x,y} \) is of full rank, hence \( \text{rank}D_{x,y} = \text{rank}J_{x,y} \geq r \), which ends the proof.

**Proof of theorem 4.3.** We proceed similarly as in the proof of theorem 3.1, using lemma 2.2 instead of lemma 2.1, together with the suitable rank estimates coming from lemma 4.5. In the same way as in the proof of theorem 3.1, we show that it is enough to check that the suitable set \( X_\alpha \) exists for \( \eta_{m} \)-almost every \( \alpha \in B_m(0,1) \).

Applying lemma 2.3 to the sets \( \text{Per}_p(T) \), \( p = 1, \ldots, k-1 \) and (possibly zero) measures \( \mu|_{\text{Per}_p(T)} \), we find (possibly empty) disjoint \( \sigma \)-compact sets \( X_1, \ldots, X_{k-1} \subset X \) such that

\[
X_p \subset \text{Per}_p(T), \quad \mu(\text{Per}_p(T) \setminus X_p) = 0 \quad \mathcal{H}^{2\beta}(X_p) = 0 \quad \text{for } p = 1, \ldots, k-1.
\]

Similarly, there exists a \( \sigma \)-compact set \( X_k \subset X \setminus \bigcup_{p=1}^{k-1}\text{Per}_p(T) \) such that

\[
\mu \left( X \setminus \bigcup_{p=1}^{k-1}\text{Per}_p(T) \right) = 0 \quad \text{and} \quad \mathcal{H}^{2\beta}(X_k) = 0.
\]

Note that \( X_k \) contains both aperiodic and periodic points (with period at least \( k \)). Let

\[
\tilde{X} = \bigcup_{p=1}^{k}X_p.
\]

Then \( \tilde{X} \subset X \) is a \( \sigma \)-compact set of full \( \mu \)-measure. Define

\[
A = \{(x, \alpha) \in \tilde{X} \times B_m(0,1) : \phi_\alpha^x(y) = \phi_\alpha^y(x) \text{ for some } y \in \tilde{X} \setminus \{x\}\}.
\]
The set $A$ is Borel by lemma 2.4. For $x \in \tilde{X}$ and $\alpha \in B_m(0, 1)$, denote, respectively, by $A_x$ and $A^\alpha$, the Borel sections

$$A_x = \{ \alpha \in B_m(0, 1) : (x, \alpha) \in A \}, \quad A^\alpha = \{ x \in \tilde{X} : (x, \alpha) \in A \}.$$ 

Observe that to show the injectivity of $\phi^T_\alpha$ on a set of full $\mu$-measure, it is enough to prove $\eta_m(A_x) = 0$ for every $x \in \tilde{X}$, since then by Fubini’s theorem ([Rud87, theorem 8.8]), $(\eta_m \otimes \mu)(A) = 0$ and, consequently, $\mu(A^\alpha) = 0$ for $\eta_m$-almost every $\alpha \in B_m(0, 1)$. As $\phi^T_\alpha$ is injective on $\tilde{X}\backslash A^\alpha$ and $\tilde{X}$ has full $\mu$-measure, the proof of the claim is finished.

Fix $x \in \tilde{X}$. To show $\eta_m(A_x) = 0$, note that for $y \in \tilde{X}$,

$$\phi^T_\alpha(x) - \phi^T_\alpha(y) = D_{x,y}\alpha + w_{x,y}$$

for

$$w_{x,y} = \begin{bmatrix} h(x) - h(y) \\ h(Tx) - h(Ty) \\ \vdots \\ h(T^{k-1}x) - h(T^{k-1}y) \end{bmatrix}.$$ 

Write $A_x$ as

$$A_x = A_x^\text{orb} \cup \bigcup_{p=1}^k A_x^p,$$

where

$$A_x^\text{orb} = \{ \alpha \in B_m(0, 1) : \phi^T_\alpha(x) = \phi^T_\alpha(y) \text{ for some } y \in \tilde{X} \cap \text{Orb}(x) \backslash \{ x \} \},$$

$$A_x^p = \{ \alpha \in B_m(0, 1) : \phi^T_\alpha(x) = \phi^T_\alpha(y) \text{ for some } y \in X_p \backslash \{ x \} \}, \quad p = 1, \ldots, k.$$ 

The set $A_x^\text{orb}$ is Borel as a countable union of closed sets of the form

$$\{ \alpha \in B_m(0, 1) : \phi^T_\alpha(x) = \phi^T_\alpha(y) \text{ for some } y \in \tilde{X} \cap \text{Orb}(x) \backslash \{ x \} \},$$

while each set $A_x^p$ is Borel as a section of the set

$$\{ (x, \alpha) \in \tilde{X} \times B_m(0, 1) : \phi^T_\alpha(x) = \phi^T_\alpha(y) \text{ for some } y \in X_p \backslash \{ x \} \},$$

which is Borel by lemma 2.4. To end the proof, it is enough to show that the sets $A_x^\text{orb}$ and $A_x^p$, $p = 1, \ldots, k$, have $\eta_m$ measure zero.

To prove $\eta_m(A_x^\text{orb}) = 0$ it suffices to check that the sets of the form (4.3) have $\eta_m$ measure zero. By (4.2), we have

$$\{ \alpha \in B_m(0, 1) : \phi^T_\alpha(x) = \phi^T_\alpha(y) \} = \{ \alpha \in B_m(0, 1) : D_{x,y}\alpha = -w_{x,y} \}$$

and lemma 4.5 gives $\text{rank} D_{x,y} \geq 1$ whenever $y \neq x$, so each set of the form (4.3) is contained in an affine subspace of $\mathbb{R}^m$ of codimension at least 1. Consequently, it has $\eta_m$ measure zero.

Since $T$ is locally Lipschitz, $h, h_1, \ldots, h_m$ are locally $\beta$-Hölder and $X$ is separable, there exists a countable covering $\mathcal{V}$ of $X$ by open sets in $\mathbb{R}^N$, such that for every $V \in \mathcal{V}$, the map $T$ is Lipschitz on $V$ and $h, h_1, \ldots, h_m$ are $\beta$-Hölder on $V$. Let $\mathcal{U}$ be the collection of all sets of the form $U = V_0 \cap T^{-1}(V_1) \cap \cdots \cap T^{-(k-1)}(V_{k-1})$, where $V_0, \ldots, V_{k-1} \in \mathcal{V}$. Then $\mathcal{U}$ is
a countable covering of $X$ by open sets, and we can write $\mathcal{U} = \{ U_j \}_{j \in \mathbb{N}}$. By definition, for every $j \in \mathbb{N}$ there exists $C_j > 0$ such that
\[
\| T^{s+1}(y) - T^{s+1}(y') \| \leq C_j \| T^s(y) - T^s(y') \|,
\]
\[
\| h(T^s(y)) - h(T^s(y')) \| \leq C_j \| T^s(y) - T^s(y') \|^2,
\]
\[
\| h_r(T^s(y)) - h_r(T^s(y')) \| \leq C_j \| T^s(y) - T^s(y') \|^2
\]
for every $y, y' \in U_j \cap X, s \in \{ 0, \ldots, k - 1 \}, r \in \{ 1, \ldots, m \}$. By induction, it follows that
\[
\| T^s(y) - T^s(y') \| \leq C_j \| y - y' \|, \\
\| h(T^s(y)) - h(T^s(y')) \| \leq C_j^{s+1} \| y - y' \|^{3},
\]
\[
\| h_r(T^s(y)) - h_r(T^s(y')) \| \leq C_j^{s+1} \| y - y' \|^{3}
\]
for $y, y' \in U_j \cap X, s \in \{ 0, \ldots, k - 1 \}, r \in \{ 1, \ldots, m \}$.

To prove $\eta_m(A^p) = 0$ for $p = 1, \ldots, k$, we follow the strategy used in [SYC91] (see also [Rob11]). Fix $n \in \mathbb{N}$ and for $j \in \mathbb{N}$ define
\[
X^{p,n}_x = \left\{ y \in X_p : \sigma_p(D_{x,y}) \geq \frac{1}{n} \right\},
\]
\[
A^{p,j,n} = \left\{ \alpha \in B_m(0,1) : \phi_n^\alpha(x) = \phi_n^\alpha(y) \text{ for some } y \in U_j \cap X^{p,n}_x \{ x \} \right\},
\]
where $\sigma_p(D_{x,y})$ is the $p$th largest singular value. Note that singular values of given order depend continuously on the coefficients of the matrix, see e.g. [GVL13, corollary 8.6.2]. Hence, the set $X^{p,n}_x$ is $\sigma$-compact as a closed subset of $X_x$ and by lemma 2.4, the set $A^{p,j,n}$ is Borel.

By lemma 4.5, for every $y \in X_p \setminus \text{Orb}(x)$ we have $\text{rank}D_{x,y} \geq p$. This implies $\sigma_p(D_{x,y}) > 0$ (see e.g. [Rob11, lemma 14.2]). Hence,
\[
A^{p,j,n} \setminus A^{p,n} = \bigcup_{j=1}^{\infty} \bigcup_{n=1}^{\infty} A^{p,j,n} \setminus A^{p,n}.
\]

Consequently, it is enough to prove $\eta_m(A^{p,j,n} \setminus A^{p,n}) = 0$ for every $n \in \mathbb{N}$.

Fix $\varepsilon > 0$. Since $\mathcal{H}^{B^p}(U_j \cap X^{p,n} \setminus \text{Orb}(x)) \leq \mathcal{H}^{B^p}(X_p) = 0$, there exists a collection of balls $B_\varepsilon(y_i, \varepsilon_i)$, for $y_i \in U_j \cap X^{p,n} \setminus \text{Orb}(x)$ and $0 < \varepsilon_i < \varepsilon$, $i \in \mathbb{N}$, such that
\[
U_j \cap X^{p,n} \setminus \text{Orb}(x) \subset \bigcup_{i \in \mathbb{N}} B_\varepsilon(y_i, \varepsilon_i) \quad \text{and} \quad \sum_{i=1}^{\infty} \varepsilon_i^{\beta_p} \leq \varepsilon.
\]

Take $\alpha \in A^{p,j,n} \setminus A^{p,n}$ and let $y \in U_j \cap X^{p,n} \setminus \text{Orb}(x)$ be such that $\phi_n^\alpha(x) = \phi_n^\alpha(y)$. Then for $y_i$ with $y \in B(y_i, \varepsilon_i)$ we have
\[
\| D_{x,y} \alpha + w_{y,y} \| = \| \phi_n^\alpha(x) - \phi_n^\alpha(y) \| = \| \phi_n^\alpha(y) - \phi_n^\alpha(y_i) \|
\]
\[
\leq \sqrt{\sum_{k=0}^{k-1} \| h(T^s(y)) - h(T^s(y_i)) \|^2 + \sum_{r=1}^{m} \alpha_r \| h_r(T^s(y)) - h_r(T^s(y_i)) \|^2}
\]
\[
\leq M_j \| y - y' \|^{\beta} \leq M \varepsilon_i^{\beta}
\]

4955
for
\[ M_j = (1 + \sqrt{m}) \sqrt{\sum_{j=0}^{k-1} C_j^2(\beta+1)}, \]
by (4.4) and the fact \( \alpha \in B_\text{in}(0, 1) \). By (4.6),
\[ A^p_i \cap A \subset \bigcup_{i \in \mathbb{N}} \left\{ \alpha \in B_\text{in}(0, 1) : \| D_{x,\beta}, \alpha + w_{x,\beta} \| \leq M_j \right\}. \]

Since for every \( i \in \mathbb{N} \) we have \( \sigma_p(D_{\epsilon,\beta}) \geq 1/n \), we can apply lemma 2.2 and (4.5) to obtain
\[ \eta_m(A^p_i \cap A) \leq \sum_{i=1}^{\infty} C_{m,k} \frac{M_j^{\epsilon_0 \beta_p}}{1/n^p} \leq C_{m,k} M_j^p n^p \varepsilon. \]

Since \( \varepsilon > 0 \) was arbitrary, we conclude that \( \eta_m(A^p_i \cap A) = 0 \), so in fact \( \eta_m(A^p_i) = 0 \). This ends the proof of theorem 4.3.

**Proof of remark 4.4.** Suppose that the measure \( \mu \) is \( T \)-invariant. Then it is easy to check that the set
\[ \tilde{X}_\alpha = \bigcap_{n=0}^{\infty} T^{-n}(X_\alpha) \]
is a Borel subset of \( X_\alpha \) of full \( \mu \)-measure satisfying \( T(\tilde{X}_\alpha) \subset \tilde{X}_\alpha \). Hence, to show (a), it suffices to replace the set \( X_\alpha \) by \( \tilde{X}_\alpha \).

In the case when \( \mu \) is additionally finite, we first remark that the measure \( \mu \) is also forward invariant, i.e. \( \mu(T(Y)) = \mu(Y) \) for Borel sets \( Y \subset X \). Note that if \( Y \) is Borel, then so is \( T(Y) \) as the image of a Borel set under a continuous and injective mapping (see e.g. [Kec95, theorem 15.1]). Using this together with the invariance of \( \mu \) and the injectivity of \( T \), we check that
\[ \tilde{X}_\alpha = \bigcap_{n \in \mathbb{Z}} T^{-n}(X_\alpha) \]
is a Borel subset of \( X_\alpha \) of full \( \mu \)-measure satisfying \( T(\tilde{X}_\alpha) = \tilde{X}_\alpha \). This gives (b). Notice that the finiteness of \( \mu \) is indeed necessary, as for \( X = \mathbb{N} \), \( T(x) = x + 1 \) and \( \mu \) the counting measure, there does not exist a set \( Y \subset X \) of full \( \mu \)-measure satisfying \( T(Y) = Y \).

To show (c), suppose that \( \mu \) is \( T \)-invariant and ergodic. Obviously, we can assume that the \( \mu \)-measure of the set of all periodic points of \( T \) is positive (including \( +\infty \)). Then there exists \( p \in \mathbb{N} \) such that the measure of the set \( P \) of all \( p \)-periodic points of \( T \) is positive (including \( +\infty \)).

Suppose first that \( \mu \) restricted to \( P \) is non-atomic. Then there exists a Borel set \( Y \subset P \) with \( 0 < \mu(Y) < \mu(X)/p \). Let \( Z = Y \cup T^{-1}(Y) \cup \cdots \cup T^{-(p-1)}(Y) \). Then \( 0 < \mu(Z) < \mu(X) \) and, by the injectivity of \( T \), we have \( T^{-1}(Z) = Z \), which contradicts the ergodicity of \( \mu \).

Suppose now that \( \mu \) has an atom in \( P \). Since \( \mu \) is a Borel \( \sigma \)-finite measure in a Euclidean space, this is equivalent to the fact that \( \mu(\{x\}) > 0 \) for some \( x \in P \). Let \( \mathcal{O} \) be the periodic orbit of \( x \). Again by the injectivity of \( T \), we have \( T^{-1}(\mathcal{O}) = \mathcal{O} \), so by the ergodicity of \( \mu \), the set \( \mathcal{O} \) has full \( \mu \)-measure. This means that \( \mu \) is supported on a set of Hausdorff dimension \( 0 \), which obviously gives (c).
The original Takens delay embedding theorem states that for given finite dimensional $C^2$ manifold $M$ and generic pair of $C^2$-diffeomorphism $T : M \to M$ and $C^2$-function $h : M \to \mathbb{R}$, the corresponding delay-coordinate map $\phi : M \to \mathbb{R}^k$, $\phi(x) = (h(x), h(Tx), \ldots, h(T^{k-1}x))$ is a $C^s$-embedding (an injective immersion) as long as $k > 2 \dim M$. It was followed by the box-counting dimension version of Sauer, Yorke and Casdagli (theorem 1.1) and subsequently by the infinite-dimensional result of [Rob05] (see also [Rob11, section 14.3]). Refer to [NV18] for a version of the Takens theorem with a fixed observable and perturbation performed on the dynamics. The Takens theorem involving Lebesgue covering dimension on compact metric spaces and a continuous observable was proved in [Gut16] (see [GQS18] for a detailed proof). See also [Sta99, Cab00] for the Takens theorem for deterministically driven smooth systems and [SBDH97, SBDH03] for stochastically driven smooth systems.

**Example 4.6.** It turns out that linear perturbations are not sufficient for theorems 1.1 and 4.3, i.e. it may happen that linear perturbations are not sufficient for theorems 1.1 and 4.3, i.e. it may happen that $\phi_L = (\phi(x) + Lx, \ldots, \phi(T^{k-1}x) + LT^{k-1}x)$ is not injective for a generic linear map $L : \mathbb{R}^N \to \mathbb{R}$. As an example, let $X = B_2(0,1)$, fix $a \in (0,1)$ and define $T : X \to X$ as

$$T(x) = ax.$$ 

Then $T$ is a Lipschitz injective transformation on the unit disc $X \subset \mathbb{R}^2$ with zero being the unique periodic point. Fix $\phi \equiv 0$. We claim that there is no linear observable $L : \mathbb{R}^2 \to \mathbb{R}$ which makes the delay map injective, i.e. for every $k \in \mathbb{N}$ and every $v \in \mathbb{R}^2$ the transformation $x \mapsto \phi_L(x) = (\langle x, v \rangle, \langle Tx, v \rangle, \ldots, \langle T^{k-1}x, v \rangle) \in \mathbb{R}^k$ is not injective on $X$. This follows from the fact that for each one-dimensional linear subspace $W \subset \mathbb{R}^2$ the set $W \cap X$ is $T$-invariant, hence $\phi_L^v$ is not injective for any $v \in \mathbb{R}^2$. No we will see that it also not almost surely injective for $\mu$ being the Lebesgue measure on $X$. Note that for $v \in \mathbb{R}^2$ and $c \in \mathbb{R}$, the segment $W_c = \{z \in X : \langle z, v \rangle = c\}$ satisfies $T(W_c) \subset W_{ac}$, hence all points on $W_c$ will have the same observation vector $(\langle x, v \rangle, \langle Tx, v \rangle, \ldots, \langle T^{k-1}x, v \rangle) = (c, ac, a^2c, \ldots, a^{k-1}c)$. Therefore, a set $X_v \subset X$ on which $\phi_L^v$ is injective can only have one point on each of the parallel segments $W_c$ contained in $X$. However, such a set $X_v$ cannot be of full Lebesgue measure. Note that the above example can be easily modified to make $T$ a homeomorphism.

**5. Examples**

In this section we present two examples which illustrate the usage of theorem 3.1. Let us begin with fixing some notation. For $x \in [0,2)$ we will write

$$x = x_0.x_1x_2 \ldots,$$

where $x_0,x_1,x_2 \ldots$ is the binary expansion of $x$, i.e.

$$x = \sum_{j=0}^{\infty} \frac{x_j}{2^j}, \quad x_0, x_1, x_2, \ldots \in \{0,1\}.$$ 

For a dyadic rational we agree to choose its eventually terminating expansion, i.e. the one with $x_j = 0$ for $j$ large enough. Let $\pi : \{0,1\}^\mathbb{N} \to [0,1]$ be the coding map

$$\pi(x_1, x_2, \ldots) = \sum_{j=1}^{\infty} \frac{x_j}{2^j}.$$
5.1. A modified Kan example

In the appendix to [SYC91], Kan presented an example of a compact set $K \subset \mathbb{R}^N$ with $\dim_H K = 0$ and such that every linear transformation $L : \mathbb{R}^N \to \mathbb{R}^{N-1}$ fails to be injective on $K$ (see also remark 3.8). It follows from theorem 3.1, that whenever we are given a Borel $\sigma$-finite measure $\mu$ on such a set, then almost every linear transformation $L : \mathbb{R}^N \to \mathbb{R}$ is injective on a set of full $\mu$-measure. To illustrate this, we construct a $\sigma$-compact set $X \subset \mathbb{R}^2$ with $\dim_H X = 0$, which is a slight modification of Kan’s example, equipped with a natural Borel $\sigma$-finite measure $\mu$, such that no linear transformation $L : \mathbb{R}^2 \to \mathbb{R}$ is injective on $X$, while for almost every $L$ we explicitly show a set $X_L \subset X$ of full $\mu$-measure, such that $L$ is injective on $X_L$.

Following [SYC91, appendix], we begin with constructing compact sets $A, B \subset [0, 1]$ such that

$$\dim_H A = \dim_B A = \dim_H B = \dim_B B = 0$$

(hence $\dim_H (A \cup B) = 0$), (5.1)

and

$$\dim_B A = \dim_B B = 1, \quad \dim_B (A \cup B) = \dim_B (A \cup B) = 1.$$  (5.2)

To this aim, let $M_k, k \geq 0$, be an increasing sequence of positive integers such that $M_0 = 1$ and $M_k \to \infty$ with $\lim_{k \to \infty} M_k = \infty$. Define

$$\tilde{A} = \left\{ (x_1, x_2, \ldots) \in \{0, 1\}^\mathbb{N} : \text{for every even } k, \ x_j = 0 \text{ for all } j \in [M_k, M_{k+1}) \right\} = \{0\} \times \bigcup_{n \in \mathbb{Z}} (A + n),$$

or $x_j = 1$ for all $j \in [M_k, M_{k+1})$,

$$\tilde{B} = \left\{ (x_1, x_2, \ldots) \in \{0, 1\}^\mathbb{N} : \text{for every odd } k, \ x_j = 0 \text{ for all } j \in [M_k, M_{k+1}) \right\} = \{1\} \times \bigcup_{n \in \mathbb{Z}} (B + n),$$

or $x_j = 1$ for all $j \in [M_k, M_{k+1})$,

and set

$$A = \pi(\tilde{A}), \quad B = \pi(\tilde{B}).$$

It is a straightforward calculation to check that $A$ and $B$ satisfy (5.1) and (5.2) (see [SYC91, appendix], [Fal14, example 7.8] or [Rob11, section 6.1]). Define $X \subset \mathbb{R}^2$ as

$$X = \left\{ 0 \right\} \times \bigcup_{n \in \mathbb{Z}} (A + n) \cup \left\{ 1 \right\} \times \bigcup_{n \in \mathbb{Z}} (B + n).$$

By (5.1), we have $\dim_H X = 0$. The following two propositions describe the embedding properties of the set $X$.

**Proposition 5.1.** No linear transformation $L : \mathbb{R}^2 \to \mathbb{R}$ is injective on $X$.

**Proof.** The map $L$ has the form $L(x, y) = \alpha x + \beta y$ for $\alpha, \beta \in \mathbb{R}$. Obviously, we can assume $\beta \neq 0$. Note that the points

$$u = (0, a + n), \quad v = (1, b + m), \quad \text{for } a \in A, \ b \in B, \ n, m \in \mathbb{Z}$$

are in $X$ and

$$L(u) = L(v) \quad \text{if and only if } b - a = z,$$  (5.3)
where
\[ z = \frac{\alpha}{\beta} + n - m. \]

For given \( \alpha \) and \( \beta \), choose \( n, m \in \mathbb{Z} \) such that \( z \in [0, 1) \). Consider the binary expansion \( z = 0.z_1z_2\ldots \) and define
\[
a = 0.a_1a_2\ldots \in A, \quad b = 0.b_1b_2\ldots \in B
\]

setting
\[
a_j = 0, \quad b_j = z_j \quad \text{for } j \in [M_k,M_{k+1}), \quad \text{if } k \text{ is even,}
\]
\[
a_j = 1 - z_j, \quad b_j = 1 \quad \text{for } j \in [M_k,M_{k+1}), \quad \text{if } k \text{ is odd}
\]

(5.4)

(if all \( b_j \) are equal to 1, we set \( b = 1 \)). Then \( z = b - a \) and (5.3) implies that \( L \) is not injective on \( X \).

\[ \square \]

Let us now define a natural Borel \( \sigma \)-finite measure \( \mu \) on \( X \), starting from a pair of probability measures \( \nu_1, \nu_2 \) on \( A \) and \( B \), respectively. Let
\[
\nu_1 = \bigotimes_{k=0}^{\infty} p_k, \quad \nu_2 = \bigotimes_{k=0}^{\infty} q_k,
\]

where \( p_k \) and \( q_k \) are probability measures on \( \{0,1\}^{M_{k+1}-M_k} \) given as
\[
p_k = \begin{cases} 
\frac{1}{2} \delta_{(0,\ldots,0)} + \frac{1}{2} \delta_{(1,\ldots,1)} & \text{if } k \text{ is even} \\
\left( \frac{1}{2} \delta_0 + \frac{1}{2} \delta_1 \right)^{\otimes (M_{k+1}-M_k)} & \text{if } k \text{ is odd}
\end{cases}, \quad q_k = \begin{cases} 
\left( \frac{1}{2} \delta_0 + \frac{1}{2} \delta_1 \right)^{\otimes (M_{k+1}-M_k)} & \text{if } k \text{ is even} \\
\frac{1}{2} \delta_{(0,\ldots,0)} + \delta_{(1,\ldots,1)} & \text{if } k \text{ is odd}
\end{cases}
\]

and the symbol \( \delta_a \) denotes the Dirac measure at \( a \). Then \( \text{supp} \nu_1 = \tilde{A} \), \( \text{supp} \nu_2 = \tilde{B} \), hence defining
\[
\mu_1 = \pi_*(\nu_1), \quad \mu_2 = \pi_*(\nu_2),
\]

we obtain probability measures on \( A, B \), respectively, with \( \text{supp}\mu_1 = A \), \( \text{supp}\mu_2 = B \). Finally, let
\[
\mu = \sum_{n \in \mathbb{Z}} \delta_0 \otimes (\tau_n) \mu_1 + \sum_{n \in \mathbb{Z}} \delta_1 \otimes (\tau_n) \mu_2,
\]

where \( \tau_n : \mathbb{Z} \rightarrow \mathbb{R} \), \( \tau_n(x) = x + n, n \in \mathbb{Z} \). Clearly, \( \mu \) is a Borel \( \sigma \)-finite measure with \( \text{supp}\mu = X \).

For \( a \in A, b \in B \) let
\[
A_a = \{ x \in A \setminus \{1\} : x + a = z_0z_1z_2\ldots \text{ such that the sequence } (z_0,z_1,\ldots) \text{ is constant on } [M_k,M_{k+1} - 1) \cap \mathbb{N} \text{ for every odd } k \},
\]
\[
B_b = \{ x \in B \setminus \{1\} : x + b = z_0z_1z_2\ldots \text{ such that the sequence } (z_0,z_1,\ldots) \text{ is constant on } [M_k,M_{k+1} - 1) \cap \mathbb{N} \text{ for every even } k \}.
\]

Lemma 5.2. For every \( a \in A, b \in B \), we have \( \mu_1(A_a) = \mu_2(B_b) = 0 \).
Proof. Fix $b = b_0, b_1 b_2 \ldots \in B$. We will show $\mu_2(B_b) = 0$ (the fact $\mu_1(A_u) = 0$ can be proved analogously). The proof proceeds by showing that for each even $k$, the vector $(x_{M_k}, \ldots, x_{M_k+1-2})$, where $x = x_0, x_1 x_2 \ldots \in B_b$, can assume at most four values. This will imply $\mu_2(B_b) \leq 2 \cdot 2^{-(M_{k+1}-M_k)}$ for each even $k$ and, consequently, $\mu_2(B_b) = 0$. To show the assertion, fix an even $k$ and let

$$\xi = \sum_{j=M_{k+1}-1}^{\infty} \frac{x_j + b_j}{2^j}.$$ 

Note that $\xi < 2^{-(M_{k+1}-3)}$ (as $\xi < 2$ and we exclude expansions with digits eventually equal to 1). Hence, $\xi = \xi_0 \xi_1 \xi_2 \ldots$ with $\xi_j = 0$ for $j \leq M_{k+1} - 3$. Note that, since $b$ is fixed, the values of $\xi_{M_{k+1}-2} \in \{0, 1\}$ and $(x_{M_k} + b_{M_k}, \ldots, x_{M_{k+1}-2} + b_{M_{k+1}-2}) \in \{(0, \ldots, 0), (1, \ldots, 1)\}$ determine uniquely the value of $(x_{M_k}, \ldots, x_{M_{k+1}-2})$. Therefore, $(x_{M_k}, \ldots, x_{M_{k+1}-2})$ can assume at most four values. □

Now for Lebesgue almost every linear transformation $L : \mathbb{R}^2 \to \mathbb{R}$ we will construct a set $X_L \subset X$ of full $\mu$-measure, such that $L$ is injective on $X_L$. As previously, write $L(x, y) = \alpha x + \beta y$ for $\alpha, \beta \in \mathbb{R}$. Neglecting a set of zero Lebesgue measure, we can assume $\beta \neq 0$. Let $l \in \mathbb{Z}$ be such that

$$z = -\frac{\alpha}{\beta} + l \text{ belongs to } [0, 1).$$

(5.5)

Similarly as in (5.4), we can write

$$z = a' - b', \quad z - 1 = a'' - b'' \quad \text{for some } a', a'' \in A, b', b'' \in B.$$

(5.6)

Let

$$X_L = \left(\{0\} \times \bigcup_{n \in \mathbb{Z}} (A + n)\right) \cup \left(\{1\} \times \bigcup_{n \in \mathbb{Z}} ((B \setminus (B_{\beta'} \cup B_{\beta''} \cup \{1\})) + n)\right).$$

Then $X_L \subset X$ and lemma 5.2 implies that $X_L$ has full $\mu$-measure.

Proposition 5.3. For every $\alpha \in \mathbb{R}, \beta \in \mathbb{R} \setminus \{0\}$, the linear transformation $L : \mathbb{R}^2 \to \mathbb{R}$, $L(x, y) = \alpha x + \beta y$, is injective on $X_L$.

For the proof of the proposition we will need the following simple lemma. The proof is left to the reader.

Lemma 5.4. Let $x = x_0, x_1 x_2 \ldots \in [0, 1], y = y_0, y_1 y_2 \ldots \in [0, 1], M, N \in \mathbb{N}, M < N - 1$, be such that $x + y < 2$ and sequences $(x_M, \ldots, x_N)$ and $(y_M, \ldots, y_N)$ are constant. Then $x + y = z_0, z_1 z_2 \ldots$ where the sequence $(z_M, \ldots, z_{N-1})$ is constant.

Proof of proposition 5.3. Assume, on the contrary, that there exist points $u, v \in X_L$ such that $L(u) = L(v)$. As $\beta \neq 0$, we cannot have $u, v \in \{0\} \times \mathbb{R}$ or $u, v \in \{1\} \times \mathbb{R}$. Hence, we can assume $u \in \{0\} \times \mathbb{R}, v \in \{1\} \times \mathbb{R}$. Then, following the previous notation, we have $u = (0, a + N), v = (1, b + m)$ for $a \in A, b \in B \setminus (B_{\beta'} \cup B_{\beta''} \cup \{1\})$, $n, m \in \mathbb{Z}$. Note that $b - a \in [-1, 1)$, so by (5.3), we have

$$b - a = z \quad \text{or} \quad b - a = z - 1,$$

for $z$ from (5.5), and (5.6) implies

$$b - a = a' - b' \quad \text{or} \quad b - a = a'' - b''.$$
Hence,
\[ a + a' = b + b' \quad \text{or} \quad a + a'' = b + b''. \]

This is a contradiction, as lemma 5.4 implies that the binary expansion sequences of \( a + a' \) and \( a + a'' \) are constant on \([M_k, M_{k+1} - 1] \cap \mathbb{N}\) for every even \( k \), while by the condition \( b \in B' \setminus (B_{0'} \cup B_{0''} \cup \{1\}) \), the binary expansion sequences of \( b + b' \) and \( b + b'' \) are not constant on \([M_k, M_{k+1} - 1] \cap \mathbb{N}\) for some even \( k \). \( \square \)

5.2. Measure with \( \dim_H \mu < \dim_{MB} \mu \)

To show that theorem 3.1 is an actual strengthening of theorem 3.6, we present an example of a measure \( \mu \), for which \( \dim_H \mu < \dim_{MB} \mu \). More precisely, we show the following.

**Theorem 5.5.** There exists a Borel probability measure \( \mu \) on \([0, 1]^2\), such that \( \dim_H \mu = 1 \) and \( \dim_{MB} \mu = 2 \).

To begin the construction of \( \mu \), fix an increasing sequence of positive integers \( N_k, k \in \mathbb{N} \), such that \( N_k \not
infty \) with \( \frac{S_k}{S_{k+1}} \leq \frac{1}{4} \), where \( S_k = \sum_{j=1}^{k} N_j \). Consider the probability distributions \( p_0, p_1 \) on \([0, 1]\) given by

\[ p_0(\{0\}) = 1, \quad p_1(\{1\}) = 1. \]

For \( y = 0, y_2, \ldots \in [0, 1] \) (in this subsection we assume that the binary expansion of 1 is 0.111\ldots), define the probability measure \( \nu_y \) on \([0, 1]^{\mathbb{N}}\) as the infinite product

\[ \nu_y = \bigotimes_{j=1}^{\infty} p_{y_j}. \]

Further, let \( \mu_y \) be the Borel probability measure on \([0, 1]\) given by

\[ \mu_y = \pi_y \nu_y. \]

Finally, let \( \mu \) be the Borel probability measure on \([0, 1]^2\) defined as

\[ \mu = \int_{[0, 1]} \mu_y d\text{Leb}(y), \quad \text{i.e.} \mu(A) = \int_{[0, 1]} \mu_y(A') d\text{Leb}(y) \quad \text{for a Borel set } A \subset [0, 1]^2, \]

where \( A' = \{ x \in [0, 1] : (x, y) \in A \} \). It is easy to see that \( \mu \) is well-defined, as the function \( y \mapsto \mu_y(A') \) is measurable for every Borel set \( A \subset [0, 1]^2 \).

The proof of theorem 5.5 is based on the analysis of the local dimension of \( \mu \), defined in terms of dyadic squares (rather then balls). For \( n \in \mathbb{N} \) and \( x_1, \ldots, x_n \in [0, 1] \) let \([x_1, \ldots, x_n]\) denote the dyadic interval corresponding to the sequence \((x_1, \ldots, x_n)\), i.e.

\[ \begin{align*}
[x_1, \ldots, x_n] &= \left\{ \sum_{j=1}^{n} \frac{x_j}{2^j}, \sum_{j=1}^{n} \frac{x_j}{2^j} + \frac{1}{2^n} \right\} \\
&= \left\{ \begin{array}{ll}
\left[ 0, \frac{1}{2^n} \right] & \text{if } \sum_{j=1}^{n} \frac{x_j}{2^j} + \frac{1}{2^n} < 1 \\
\left[ \frac{1}{2^n}, 1 \right) & \text{otherwise}
\end{array} \right.
\end{align*} \]
Under this notation, for \( n \in \mathbb{N} \) and \((x, y) \in [0, 1]^2\) let \( D_n(x, y) \) be the dyadic square of sidelength \( 2^{-n} \) containing \((x, y)\), i.e.

\[
D_n(x, y) = [x_1, \ldots, x_n] \times [y_1, \ldots, y_n], \quad \text{where } x = 0.x_1x_2\ldots \text{ and } y = 0.y_1y_2\ldots .
\]

Recall that the box-dimensions can be defined equivalently in terms of dyadic squares. Precisely, let \( N(X, 2^{-n}) \) be the number of dyadic squares \( D_n(x, y) \) of sidelength \( 2^{-n} \) intersecting \( X \). Then (see e.g. [Fal14, section 2.1])

\[
\dim_b(X) = \liminf_{n \to \infty} \frac{\log N(X, 2^{-n})}{n \log 2} \quad \text{and} \quad \dim_u(X) = \limsup_{n \to \infty} \frac{\log N(X, 2^{-n})}{n \log 2}.
\]

(5.7)

For a Borel finite measure \( \mu \) on \([0, 1]^2\) and \((x, y) \in [0, 1]^2\) define the lower and upper local dimension of \( \mu \) at \((x, y)\) as

\[
d_l(\mu, (x, y)) = \liminf_{n \to \infty} \frac{\log \mu(D_n(x, y))}{n \log 2}, \quad \text{and} \quad d_u(\mu, (x, y)) = \limsup_{n \to \infty} \frac{\log \mu(D_n(x, y))}{n \log 2}.
\]

It is well-known (see e.g. [Hoc14, propositions 3.10 and 3.20]) that

\[
\dim_{H\mu} = \text{ess sup}_{(x,y) \in \mu} d_l(\mu, (x, y)) \quad \text{(5.8)}.
\]

The following lemma gives estimates on the measure of dyadic squares at suitable scales.

**Lemma 5.6.** Let \( x = 0.x_1x_2\ldots \in [0, 1] \), \( y = 0.y_1y_2\ldots \in [0, 1] \), \( n \in \mathbb{N} \) and \( D = D_n(x, y) = [x_1, \ldots, x_n] \times [y_1, \ldots, y_n] \). Let \( k \in \mathbb{N} \) be such that \( S_k < n \leq S_{k+1} \). Then the following hold:

(a) If \( y_k = y_{k+1} = 1 \), then \( \mu(D) \leq 2^{-(1 - \frac{1}{k})n} \).

(b) If \( n = S_{k+1} \) and \( y_k = 0 \), then either \( \mu(D) = 0 \) or \( \mu(D) \geq 2^{-(1 + \frac{1}{k})n} \).

**Proof.** Note that for \( y' = 0.y'_1y'_2\ldots \in [0, 1] \) such that \((y'_1, \ldots, y'_n) = (y_1, \ldots, y_n)\) we have

\[
\mu(D') = \mu([x_1, \ldots, x_n]) = p_{y_1}([x_1]) \cdots p_{y_n}([x_n]) = p_{y_{k+1}}([x_{S_k+1}]) \cdots p_{y_{k}}([x_{S_k+1}]) = \cdot \cdot \cdot p_{y_2}([x_2]) = \frac{1}{2}^{n-S_k}.
\]

Moreover, as \( k < n \), the value of \( \mu(D') \) depends only on \((y_1, \ldots, y_n)\) and \((x_1, \ldots, x_n)\). Using (5.9), we can prove both assertions of the lemma, as follows.

Ad (a).

If \( y_k = y_{k+1} = 1 \), then for \( j \in \{S_{k-1} + 1, \ldots, n\} \) we have \( p_{y_j}(x_j) = \frac{1}{2} \), where \( l \in \{k, k+1\} \) is such that \( S_{l-1} < j \leq S_l \). Therefore, in the product (5.9) there is at least \( n - S_{k-1} \) terms equal to \( \frac{1}{2} \). Consequently,

\[
\mu(D') \leq 2^{-(n - S_{k-1})} = 2^{-(1 - \frac{1}{S_k})n} \leq 2^{-(1 - \frac{1}{S_{k+1}})n} \leq 2^{-(1 - \frac{1}{k})n},
\]

hence

\[
\mu(D) = \int_{[y_1, \ldots, y_n]} \mu(D')d\text{Leb}(y') \leq \text{Leb}([y_1, \ldots, y_n])2^{-m(1 - \frac{1}{k})} = 2^{-m(2 - \frac{1}{k})}.
\]

4962
Assume that $\mu(D) \neq 0$. Then all the terms in (5.9) have to be non-zero, so every term is equal to either $\frac{1}{2}$ or $1$. Moreover, as $y_{k+1} = 0$ and $n = S_{k+1}$, we have

$$p_{y_{k+1}} \left( \{x_{S_{k+1}}\} \right) \cdots p_{y_{k+1}} \left( \{x_n\} \right) = 1$$

and, consequently,

$$\mu(D) = 2^{-n} p_{y_1} \left( \{x_1\} \right) \cdots p_{y_1} \left( \{x_{S_1}\} \right) p_{y_2} \left( \{x_{S_1+1}\} \right) \cdots p_{y_2} \left( \{x_{S_2}\} \right) \cdots p_{y_{S_{k-1}+1}} \left( \{x_{S_{k-1}}\} \right) \cdots p_{y_{S_k}} \left( \{x_{S_k}\} \right) \geq 2^{-n - S_k} = 2^{-\left(1 + \frac{1}{S_{k+1}}\right)n} \geq 2^{-\left(1 + \frac{1}{n^2}\right)n}.$$

Now we are ready to give the proof of theorem 5.5.

**Proof of theorem 5.5.** We begin by proving $\dim_H \mu = 1$. Note that $\dim_H \mu \geq 1$, as $\mu$ projects under $[0,1]^2 \ni (x,y) \mapsto y \in [0,1]$ to the Lebesgue measure, so it is sufficient to show $\dim_H \mu \leq 1$. By (5.8), it is enough to prove that $d(\mu,(x,y)) \leq 1$ for $\mu$-almost every $(x,y) \in [0,1]$. Note that for Lebesgue almost every $y = 0, y_2, \ldots \in [0,1]$, the sequence $(y_1, y_2, \ldots)$ contains infinitely many zeros. Hence, it is sufficient to show $d(\mu,(x,y)) \leq 1$ for $\mu$-almost every $x \in [0,1]$, assuming that $y \in [0,1]$ has this property. Moreover, for $\mu_x$-almost every $x \in [0,1]$, we have $\mu(D_x(x,y)) > 0$ for all $n \in \mathbb{N}$ (see (5.9)). For such $x$, by lemma 5.6(b), we have

$$d(\mu,(x,y)) \leq \liminf_{k \to \infty} \frac{-\log \mu(D_{S_k} (x,y))}{S_n \log 2} \leq \lim_{k \to \infty} \frac{(1 + \frac{1}{m}) S_m}{S_n} = 1.$$ 

Therefore, $\dim_H \mu \leq 1$, so in fact $\dim_H \mu = 1$.

Let us prove now $\dim_M \mu = 2$. Since $\mu$ is supported on $[0,1]^2$, it suffices to show $\dim_M \mu \geq 2$. Let $A \subset [0,1]^2$ be a Borel set with $\mu(A) > 0$. We show $\dim_M A \geq 2$. Note that there exists $c > 0$ such that the set

$$B = \{ y \in [0,1] : \mu_x(A) \geq c \}$$

satisfies $\operatorname{Leb}(B) > 0$. Fix $\varepsilon \in (0,\frac{1}{2})$. By the Lebesgue density theorem (see e.g. [Hoc14, corollary 3.16]), there exists a dyadic interval $I \subset [0,1]$ such that

$$\frac{\operatorname{Leb}(B \cap I)}{|I|} \geq 1 - \varepsilon,$$

where $|I| = 2^{-N}$ is the length of $I$. Fix $k \geq N + 2$ and $n \in \{S_k + 1, \ldots, S_{k+1}\}$. Consider the collection $C_n$ of dyadic intervals of length $2^{-n}$ defined as

$$C_n = \{ [y_1, \ldots, y_n] : y_k = y_{k+1} = 1 \text{ and } [y_1, \ldots, y_n] \cap B \cap I \neq \emptyset \}.$$

By (5.11), we have

$$\operatorname{Leb} \left( B \cap \bigcup C_n \right) \geq \left( \frac{1}{4} - \varepsilon \right) 2^{-N}.$$

Let

$$A_n = A \cap \left( [0,1] \times \left( B \cap \bigcup C_n \right) \right).$$
Then \( A_n \subset A \) and (5.10) together with (5.12) imply
\[
\mu(A_n) = \int_{B \cap \bigcup C_n} \mu_y(A) \, d\text{Leb}(y) \geq c \left( \frac{1}{4} - \varepsilon \right) 2^{-N}. \tag{5.13}
\]

Note that the above lower bound does not depend on \( k \) and \( n \). Let \( N'(A_n, 2^{-n}) \) be the number of dyadic squares of sidelength \( 2^{-n} \) intersecting \( A_n \). If \( D = I_1 \times I_2 \) is a dyadic square of sidelength \( 2^{-n} \) intersecting \( A_n \), then \( I_2 \subset C_n \), hence by lemma 5.6(a) we have
\[
\mu(D) \leq 2^{-(2 - \frac{1}{m})n}.
\]
As any two dyadic squares of the same sidelength are either equal or disjoint, (5.13) gives
\[
N'(A, 2^{-n}) \geq N'(A_n, 2^{-n}) \geq c \left( \frac{1}{4} - \varepsilon \right) 2^{-N + (2 - \frac{1}{m})n}.
\]
Since \( k \) and \( n \) can be taken arbitrary large, invoking (5.7) gives \( \dim_H A \geq 2 \). Hence, \( \dim_M \mu \geq 2 \), so in fact \( \dim_M \mu = 2 \). \( \square \)

**Remark 5.7.** Note that as \( \esssup_{z \sim \mu} \|d(\mu, z)\| = \dim_H \|\mu \leq \dim_M \|\mu = \dim_P \|\mu = \esssup_{z \sim \mu} \|d(\mu, z)\| \)
(dim\( _P \) denotes the packing dimension, see e.g. [Fal14, proposition 3.9] and [Fal97, proposition 10.3]), the equality \( \dim_M \|\mu = \dim_{eb} \|\mu \) holds for all exact dimensional measures \( \mu \), i.e. the measures \( \mu \) with \( d(\mu, z) = d(\mu, z) = \text{const for \( \mu \)-almost every } z \).

**Acknowledgments**

We are grateful to Erwin Riegler for helpful discussions and to the anonymous referees for helpful comments. YG and AS were partially supported by the National Science Centre (Poland) Grant 2016/22/E/ST1/00448.

**References**

Alberti G, Bölcskei H, De Lellis C, Koliander G and Riegler E 2018 Lossless analog compression *IEEE Trans. Inform. Theory* 65 7480–513

Ben-Artzi A, Eden A, Foias C and Nicolaenko B 1993 Hölder continuity for the inverse of Mañe’s projection *J. Math. Anal. Appl.* 178 22–9

Banach S 1951 *Wstęp Do Teorii Funkcji Rzeczywistych* (Polish) [Introduction to the Theory of Real Functions] (Monografie Matematyczne. Tom XVII) (Warszawa-Wrocław: Polskie Towarzystwo Matematyczne)

Billingsley P 1999 *Convergence of Probability Measures* (Wiley Series in Probability and Statistics: Probability and Statistics) 2nd edn (New York: Wiley)

Caballero V 2000 On an embedding theorem *Acta Math. Hung.* 88 269–78

Eden A, Foias C, Nicolaenko B and Temam R M 1994 *Exponential Attractors for Dissipative Evolution Equations* (RAM: Research in Applied Mathematics vol 37) (Paris: Masson)

Eckmann J-P and Ruelle D 1985 Ergodic theory of chaos and strange attractors *Rev. Mod. Phys.* 57 617–56

Falconer K 1997 *Techniques in Fractal Geometry* (Chichester: Wiley)
Falconer K 2014 Fractal Geometry Mathematical: Foundations and Applications 3rd edn (Chichester: Wiley)

Friz P K and Robinson J C 2002 Constructing an elementary measure on a space of projections J. Math. Anal. Appl. 267 714–25

Gutman Y, Qiao Y and Szabó G 2018 The embedding problem in topological dynamics and Takens’ theorem Nonlinearity 31 597–620

Gascó M and Sauer T 2000 Polynomial interpolation in several variables Adv. Comput. Math. 12 377–410

Multivariate polynomial interpolation

Gutman Y 2016 Takens’ embedding theorem with a continuous observable Ergodic Theory (Berlin: De Gruyter) pp 134–41

Golub G H and Van Loan C F 2013 Matrix Computations (Johns Hopkins Studies in the Mathematical Sciences) 4th edn (Baltimore, MD: Johns Hopkins University Press)

Hamilton F, Berry T and Sauer T Jul 2015 Predicting chaotic time series with a partial model Phys. Rev. E 92 010902

Hsieh C-H, Glaser S M, Lucas A J and George S 2005 Distinguishing random environmental fluctuations from ecological catastrophes for the North Pacific Ocean Nature 435 336–40

Hunt B R and Kaloshin V Y 1999 Regularity of embeddings of infinite-dimensional fractal sets into finite-dimensional spaces Nonlinearity 12 1263–75

Hochman M 2014 Lectures on dynamics, fractal geometry, and metric number theory J. Mod. Dyn. 8 437–97

Hurewicz W and Henney W 1941 Dimension Theory (Princeton Mathematical Series vol 4) (Princeton, NJ: Princeton University Press)

Kechris A S 1995 Classical Descriptive Set Theory (Graduate Texts in Mathematics vol 156) (New York: Springer)

Kostelich E J and Yorke J A 1990 Noise reduction: finding the simplest dynamical system consistent with the data Phys. D 41 183–96

Marstrand J M 1954 Some fundamental geometrical properties of plane sets of fractional dimensions Proc. Lond. Math. Soc. 4 257–302

Mattila P 1975 Hausdorff dimension, orthogonal projections and intersections with planes Ann. Acad. Sci. Fenn. Ser. A 1 227–44

Mattila P 1995 Geometry of Sets and Measures in Euclidean Spaces: Fractals and Rectifiability (Cambridge Studies in Advanced Mathematics vol 44) (Cambridge: Cambridge University Press)

Munch S B, Giron-Nava A and George S 2018 Nonlinear dynamics and noise in fisheries recruitment: a global meta-analysis Fish Fish. 19 964–73

Minty G J 1970 On the extension of Lipschitz, Lipschitz-Hölder continuous, and monotone functions Bull. Am. Math. Soc. 76 334–9

Mañé R 1981 On the dimension of the compact invariant sets of certain nonlinear maps Dynamical Systems and Turbulence, Warwick 1980 (Coventry, 1979/1980) (Lecture Notes in Math. vol 898) (Berlin: Springer) pp 230–42

Navarrete R and Viswanath D 2018 Prevalence of delay embeddings with a fixed observation function (arXiv:1806.07529)

Packard N H, Crutchfield J P, Doyne Farmer J and Shaw R S 1980 Geometry from a time series Phys. Rev. Lett. 45 712–6

Palis J Jr and de Melo W 1982 Geometric Theory of Dynamical Systems (Berlin: Springer) (Engl. trans.)

Robinson J C 2005 A topological delay embedding theorem for infinite-dimensional dynamical systems Nonlinearity 18 2135–43

Robinson J C 2011 Dimensions, Embeddings, and Attractors (Cambridge Tracts in Mathematics vol 186) (Cambridge: Cambridge University Press)

Rudin W 1987 Real and Complex Analysis 3rd edn (New York: McGraw-Hill)

Stark J, Broomehead D S, Davies M E and Huke J P 1997 Takens embedding theorems for forced and stochastic systems Proc. of the 2nd World Congress of Nonlinear Analysts (Athens, 1996) vol 30 pp 5303–14

Stark J, Broomehead D S, Davies M E and Huke J P 2003 Delay embeddings for forced systems II. Stochastic forcing J. Nonlinear Sci. 13 519–77

Sugihara G, Grenfell B and May R 1990 Distinguishing error from chaos in ecological time-series Phil. Trans. R. Soc. B 330 235–51

Sugihara G and May R 1990 Nonlinear forecasting as a way of distinguishing chaos from measurement error in time series Nature 344 734–41
Stark J 1999 Delay embeddings for forced systems I. Deterministic forcing J. Nonlinear Sci. 9 255–332
Sauer T D and Yorke J A 1997 Are the dimensions of a set and its image equal under typical smooth functions? Ergod. Theor. Dynam. Syst. 17 941–56
Sauer T D, Yorke J A and Casdagli M 1991 Embedology J. Stat. Phys. 65 579–616
Takens F 1981 Detecting strange attractors in turbulence Dynamical Systems and Turbulence, Warwick 1980 (Lecture Notes in Math. vol 898) (Berlin: Springer) pp 366–81
Voss H U 2003 Synchronization of reconstructed dynamical systems Chaos 13 327–34
Whitney H 1936 Differentiable manifolds Ann. Math. 37 645–80
Yorke J 1969 Periods of periodic solutions and the Lipschitz constant Proc. Am. Math. Soc. 22 509–12