Peculiarity of the Entanglement for $\text{Lif}^{(2)}_4 \times S^1 \times S^5$
Spacetime with String Excitations

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Abstract

The (F1,D2,D8) brane configuration with $\text{Lif}^{(2)}_4 \times S^1 \times S^5$ geometry is a known Lifshitz vacua supported by massive $B_{\mu\nu}$ field in type IIA theory. This system allows exact IR excitations which couple to massless modes of the fundamental string. Due to these massless modes the solutions have a flow to a dilatonic $\text{Lif}^{(3)}_4 \times S^1 \times S^5$ vacua in IR. We study the entanglement entropy on the boundary of this spacetime for the strip and the disc subsystems. To our surprise net entropy density of the excitations at first order is found to be independent of the typical size of subsystems. We interpret our results in the light of first law of entanglement thermodynamics.
1 Introduction

The gauge-gravity correspondence \[1, 2, 3\] has got a nonrelativistic version where strongly coupled quantum theories at critical points can be studied \[4-22\]. Some of these quantum systems involve strongly coupled fermions at finite density or it may simply be a gas of ultra-cold atoms \[4, 5\]. In the studies involving ‘nonrelativistic’ Schrödinger spacetimes the 4-dimensional spacetime geometry generally requires supporting Higgs like field such as massive vector field \[6, 9, 4\] or a tensor field. The spacetimes possessing a Lifshitz symmetry provide similar holographic dual description of nonrelativistic quantum theories living on their boundaries \[10\], also see \[22\].

In this work we shall mainly study entanglement entropy of the excitations in asymptotically $\text{Lif}^{(a=2)}_{4} \times S^1 \times S^5$ background. The latter is a Lifshitz vacua in massive type IIA (mIIA) theory \[19, 20\] with dynamical exponent of time being $a = 2$. The massive type IIA theory \[37\] is a ten-dimensional maximal supergravity where the antisymmetric tensor field is explicitly massive. The theory also includes a positive cosmological constant related to mass parameter. Due to this structure the mIIA theory provides a unique setup to study Lifshitz solutions. Particularly the $\text{Lif}^{(2)}_{4} \times S^1 \times S^5$ solution is a background generated by the bound state of $(F1, D2, D8)$ branes \[19\]

$$
\begin{align*}
\text{ds}^2 &= L^2 \left( \frac{-dt^2}{z^4} + \frac{dx_1^2 + dx_2^2}{z^2} + \frac{dz^2}{q^2} + \frac{dy^2}{q^2} + d\Omega_5^2 \right), \\
e^\phi &= g_0, \\
C_{(3)} &= -\frac{1}{g_0} \frac{L^3}{z^4} dt \wedge dx_1 \wedge dx_2, \\
B_{(2)} &= \frac{L^2}{qz^2} dt \wedge dy
\end{align*}
$$

(1)

The metric and the form fields have explicit invariance under constant scalings (dilatation); $z \to \lambda z$, $t \to \lambda^2 t$, $x_i \to \lambda x_i$, $y \to y$. The dynamical exponent of time is 2 here. The background describes a strongly coupled nonrelativistic quantum theory at the UV critical point. \[\dagger\]

It is worthwhile to study excitations of the $\text{Lif}^{(2)}_{4} \times S^1 \times S^5$ vacua as it immediately provides us a prototype $\text{Lif}^{(2)}_{4}$ background in four dimensions which is holographic dual to 3-dimensional Lifshitz theory on its boundary. The excitations would tell us how this Lifshitz theory behaves near its critical point. Particularly we shall study a class of string like excitations which themselves form solutions of massive IIA sugra and explicitly involve $B$-field \[20\]. These also induce running of dilaton

\[\dagger\] Analogous T-dual solution do also exist in type IIB theory with constant axion flux switched on \[13\].
as well. It is observed that the resulting RG flow in the deep IR can be described simply by ordinary type IIA theory. The reason for this is due to the fact that the contributions of massive stringy modes decouple from the low energy dynamics of the theory in the IR, far away from UV critical point [20].

In this report we aim to study holographic entanglement entropy [23 - 36] of the excited Lifshitz subsystems which are either disc or a strip in a perturbative framework. A critical observation is that for small size systems the entanglement entropy density remains constant at first order. That is, the first order contributions to the entropy density remain independent of the size \((l)\) of the subsystem. This is a peculiarity and quite unlike relativistic CFTs where usually the entropy density (of excitations) is linearly proportional to the typical size of the subsystem [27]. We discover that the resolution lies in the nature of the chemical potential \((\mu_E)\) for the Lifshitz system. We gather evidence that suggests that energy density (of the excitations) falls off with the size of system as \(\propto 1/l^2\). Furthermore the \(1/l^2\) dependence is exactly same as the entanglement temperature behaviour in the Lifshitz theory. Notwithstanding these peculiarities, the entropy of excitations consistently follows the first law of entanglement thermodynamics [27, 28] up to first order.

In addition, we also carry out a calculation of entanglement entropy at second order for both disc and strip subsystems. Contributions arising at this order bestow an explicit \(l\) dependence upon the entropy. We argue how the first law can still be followed by modifying our chemical potential \((\mu_E)\) and entanglement temperature \((T_E)\). A similar argument was presented in [32] for asymptotically AdS spacetime.

The rest of the paper is organized as follows: in section 2 salient features of \(Lif^{(2)}_4 \times S^1 \times S^5\) vacua with IR excitations in mIIa theory has been highlighted. We calculate the holographic entanglement entropy for a disc subsystem on the boundary of the spacetime in section 3 and try to interpret its thermodynamic properties by introducing a chemical potential. In section 4 we carry out similar analysis for strip subsystem at first and second orders, section 5 contains the conclusion.

## 2 Lif\(_4^{(2)}\) \(\times S^1 \times S^5\) vacua and excitations

The massive type IIA supergravity theory is the only known maximal supergravity in ten dimensions which allows massive string \(B_{\mu\nu}\) field and a mass dependent cosmological constant [37]. The cosmological constant generates a nontrivial potential term for the dilaton field. The mIIA theory does not admit flat Minkowski solutions. Nonetheless the theory gives rise to well known Freund-Rubin type vacua \(AdS_4 \times S^6\) [37], the supersymmetric domain-walls or D8-branes [38, 39, 40, 41, 42], \((D6, D8)\), \((D4, D6, D8)\) bound states [43, 44] and Galilean-AdS geometries [11, 12]. In all of
these massive tensor field plays a key role. Under the ‘massive’ T-duality [39] the D8-branes can be mapped over to the axionic D7-branes of type IIB string theory and vice-versa. The $B$-field also plays important role in obtaining non-relativistic Lifshitz solutions [19, 20]. The latter solutions are of no surprise in mIIA theory, as an observed feature in four-dimensional AdS gravity theories has been that in order to obtain non-relativistic solutions one needs to include massive (Proca) gauge fields in the gravity theory [4]. Other different situations where massless vector fields can give rise to nonrelativistic vacua, involve boosted black $Dp$-branes compactified along lightcone direction [14, 15]. These latter class of solutions are also called hyperscaling (or conformally) Lifshitz vacua [18].

Particularly the $a = 2$ Lifshitz vacua with IR excitations in mIIA theory can be written as [20]

$$ds^2 = L^2 \left( -\frac{dt^2}{z^4 h} + \frac{dx_1^2 + dx_2^2}{z^2} + \frac{dz^2}{q^2 z^4 h} + d\Omega_5^2 \right),$$

$$e^\phi = g_0 h^{-1/2}, \quad C_{(3)} = -\frac{1}{g_0} \frac{L^3}{z^4} dt \wedge dx_1 \wedge dx_2,$$

$$B_{(2)} = \frac{L^2}{q^2} h^{-1} dt \wedge dy,$$  \hspace{1cm} (2)

where the harmonic function $h(z) = 1 + \frac{z^2}{z_I^2}$. The parameter $z_I$ is related to the charge of the NS-NS strings. The excitations involve $g_{tt}$ and $g_{yy}$ metric components, and leaving the $x_1, x_2$ plane (worldvolume directions of D2-branes) unaffected. The excitations do also induce a running of dilaton field. The $B_{1y}$ component of the string field is also coupled to the excitations. Since $h \sim 1$ as $z \to 0$, these excitations form normalizable modes ($z_I$ would correspond to adding relevant operators in the boundary Lifshitz theory). The solution (2) asymptotically flows to weakly coupled regime in the UV (note that the string coupling, $g_0 < 1$). While, in the deep IR region, with $z \gg z_I$ where $h \approx \frac{z^2}{z_I^2}$, the vacua is driven to another weakly coupled Lifshitz regime. For $z \gg z_I$, the IR geometry transforms to dilatonic $Lif^{(3)}_4 \times S^1 \times S^5$ solution. This solution enables us to study the effect of the excitations in $a = 2$ Lifshitz theory. Note the $z_I$ dependent excitations at zero temperature are mainly in the form of charge excitations, along with nontrivial entanglement chemical potential, as we would see next.

$^2$ Here $L = \frac{2}{g_0 m_{\text{IIA}}}$, and $m$ being the mass parameter in the mIIA action. (We would set $l_s = 1$ and $g_0 = 1$.) The constant $q$ is a free (length) parameter and $g_0$ is weak string coupling. Note $L$ is dimensionless parameter, it determines overall radius of curvature of the spacetime. Therefore Romans’ theory with $m \ll \frac{2}{g_0 l_s}$ would be preferred here so that $L \gg 1$ in the solutions (2), else these classical vacua cannot be trusted. Also, from the D8 brane/domain-wall correspondence in [39], one typically expects $m \approx \frac{2 g_0 N_{D8}}{l_s}$, a value which is definitely well within $\frac{2}{g_0 l_s}$ for a finite number of $D8$ branes, $N_{D8}$, in these backgrounds.

4
3 Entanglement of a disc subsystem

We consider a round disc of radius $l$ at the center of the $x_1, x_2$ plane with its boundary identified with the corresponding boundary of 2d Ryu-Takayanagi surface lying inside the Lifshitz bulk geometry \cite{2}. We shall assume $y \sim y + 2\pi r_y$. In radial coordinates ($r = \sqrt{x_1^2 + x_2^2}$) the Ryu-Takayanagi area functional \cite{23} for static bulk surface is given by

$$A_\gamma = 8\pi^2 L^3 y \int_\epsilon^{z_*} \frac{dz}{q z^2} r \frac{\sqrt{1 + r'^2}}{q z^2} h \frac{1}{q z^2}$$ \hspace{1cm} (3)

where $r' = \frac{dr}{dz}$, $h(z) = (1 + \frac{z^2}{z_I^2})$ and $\epsilon \ll l$ is UV cut-off of the Lifshitz theory. We need to extremize the area integral by solving the Euler-Lagrange equation for $r(z)$

$$2zr'h(z) - 4rr'^2h(z) - 4rr'h(z) - 2zr'^2h(z) - 2zh(z) - zrr'^3h'(z) - zrr'h'(z) = 0$$ \hspace{1cm} (4)

For small size subsystem, with $l \ll z_I$, we can make a perturbative expansion and obtain solutions order by order in the dimensionless ratio $\frac{l}{z_I}$; such that $r(z) = r(0) + r(1) + \cdots$, and correspondingly we would write

$$A_\gamma = A_0 + A_1 + \cdots$$

for small $l$. Our immediate interest is in calculating terms up to leading order and first order only in the $\frac{l}{z_I}$ expansion.

The equation at zeroth order is

$$zr''(0) - 2r(0)r'^3 - 2r(0)r'(0) - zr'^2 - z = 0$$ \hspace{1cm} (5)

for which $r(0) = \sqrt{L^2 - z^2}$ defines the extremal surface (half circle) \cite{23, 29}. With the boundary conditions $r(0)(0) = l$, and $r(0)(z_*) = 0$, where $z = z_*$ being the point of return that lies at $z_* = l$. One finds that the area

$$A_0 = 8\pi^2 L^3 y \int_\epsilon^{z_*} \frac{dz}{q z^2} \sqrt{1 + r(0)^2} z_0 \frac{1}{q z^2}$$

$$= \frac{8\pi^2 L^3 y}{q} \left( \frac{1}{\epsilon} - \frac{1}{l} \right)$$ \hspace{1cm} (6)

As $A_0$ being a ground state contribution it obviously remains independent of the parameter $z_I$ of the bulk geometry. This only means that there is no effect of
excitations on the leading term. The first order contributions can be evaluated using only the tree level embedding function [29] and is given by

\[ A_1 = 8\pi^2 L^3 r_y \int_{\epsilon}^{z_I} dz r_{t(0)} \sqrt{1 + r_{t(0)}^2} \frac{qz_I^2}{2qz_I^2} \]

\[ = 4\pi^2 L^3 r_y l \int_0^l dz \frac{1}{qz_I^2} \]

\[ = 4\pi^2 L^3 r_y \left( \frac{l^2}{qz_I^2} \right) \]  

(7)

From here the complete expression of entanglement entropy of a disc shaped sub-system up to first order becomes

\[ S_{E}^{\text{Disc}}[l, z_I] \equiv \frac{A_1}{4G_5} \]

\[ = S_E^{(0)} + \frac{L^3 \pi^2 r_y}{G_5 q} \left( \frac{l^2}{z_I^2} \right) \]  

(8)

where we defined \( G_5 \equiv \frac{L^5 Vol(S^5)}{G_{10}} \) as the 5-dimensional Newton’s constant. The ground state entropy contribution is

\[ S_E^{(0)} = \frac{2L^3 \pi^2 r_y l}{G_5 q} \left( \frac{1}{\epsilon} - \frac{1}{l} \right) . \]  

(9)

The eq.(8) is a meaningful expression for entanglement entropy only if we maintain \( l \ll z_I \). The first order term explicitly depends on \( z_I \), so small fluctuations of the bulk quantities, like \( \delta z_I \), would result in corresponding change in entropy also. For a fixed size \( l \), one could express these variations of the entropy density as

\[ \delta S_{E}^{\text{Disc}} = \frac{\delta S_{E}^{\text{Disc}}}{\pi l^2} = \frac{L^3 \pi r_y}{G_5 q} \delta \left( \frac{1}{z_I^2} \right) \]  

(10)

where \( \pi l^2 \) is the disc area. Equation (10) provides a complete expression up to first order. At second order the entropy will receive new \( z_I \) dependent contributions. Next, we note that the right hand side of equation (10) is actually independent of the disc size \( l \)! On first hand observation this appears very surprising because, as per the first law of entanglement thermodynamics [27], we expected that the entropy density of excitations would have had \( l^2 \) dependence, namely in the form of inverse temperature (usually entanglement temperature goes as \( T_{E}^{-1} \propto l^a \); and the dynamical exponent of time in our Lifshitz background is \( a = 2 \)). Especially this aspect of the first law has been found to remain true in a variety of relativistic CFTs, where entanglement temperature is given by \( T_E \propto \frac{1}{\pi l^a} \). There is pretty good
evidence to suggest this; see for example in [27, 28, 32, 33, 21, 35]. What, then, is so different for the Lifshitz system described by equation (10)? To understand this phenomenon we first need to get an estimate of the energy associated with the excitations in our system.

3.1 Energy, winding charge and chemical potential

We now turn to find the energy of excitations of the ‘massive strings’ due to which we have a configuration in eq. (2), where we can express \( B_{ty} \approx B_{ty}^{massive} + B_{ty}^{excitation} \). Note that we are treating \( y \) as a compact direction. The Scherk-Schwarz compactification [45, 46] of the Lifshitz background (2) on a circle along \( y \) gives rise to the following 1-form potential

\[
A_{(1)} = \frac{L^2}{q z^2} (1 + \frac{z^2}{z^2_f})^{-1} dt. \tag{11}
\]

It represents a gauge field in the lower dimensional supergravity whose only non-zero component is \( A_t \). It can be determined from here that due to string excitations the net change in the \( U(1) \) charge (due to winding strings) is

\[
\Delta \rho = \frac{N}{V_2} = \frac{\Delta Q}{r_y V_2} = \frac{4\pi L}{G_5 z^2_f} \tag{12}
\]

where \( V_2 \) is the area element of \( x_1, x_2 \) plane, see a calculation in the appendix. The entanglement chemical potential, with the prescription in [32], can be obtained by measuring gauge field at the turning point, namely

\[
\mu_E \equiv A_t|_{z=z_*} = \frac{L^2 r_y}{q z^2} + \cdots \tag{13}
\]

where ellipses denote subleading terms which are not required at first order. At leading order we have \( z_* \approx l \), hence essentially this thermodynamic variable gets uniquely fixed by the Lifshitz ground state [11]. So for small \( l [> 0] \) the chemical potential remains quite important, and we obtain

\[
\mu_E \cdot \Delta \rho \approx \frac{4\pi L^3 r_y}{G_5 q} \frac{1}{z^2_f l^2} \tag{14}
\]

There are no other excitations except the winding strings, the energy density due to the excitations can be estimated to be

\[
\Delta \mathcal{E} = \mathcal{E} - \mathcal{E}_0 \approx \frac{1}{2} \mu_E \Delta \rho = \frac{2\pi L^3 r_y}{G_5 q z^2_f l^2} \tag{15}
\]

where \( \mathcal{E}_0 \) is (normalized) energy of the ground state of our Lifshitz theory. This is the only meaningful deduction we can make from here, particularly in absence of a direct
method to evaluate full stress-energy tensor of the Lifshitz theory. Assuming that the entanglement temperature of the 3-dimensional $a=2$ Lifshitz system faithfully behaves as \[ T_E = \frac{4}{l^2} \tag{16} \]

we determine that the ratio
\[ \frac{\mu E}{T_E} = \frac{L^2 r_y}{4q} \]
is indeed independent of $l$. Essentially this ratio seems to get uniquely fixed by the Lifshitz ground state (11) at the leading order. Note the excitations seems to have no effect on it. The analysis also implies that the energy density and the entanglement temperature both fall off with the system size $l$ at the same rate, and the ratio
\[ \frac{\Delta E}{T_E} = \frac{\pi L^3 r_y}{2q G z I^2} = \frac{1}{2} \frac{k_E N}{V_2} \tag{17} \]
stays fixed for small discs. However this ratio does depend on the excitations namely through $z_I$. In the second equality we have preferred to view dimensionless quantity $k_E = \frac{L^2 r_y}{4q}$ as being analogous to the Boltzmann constant in usual thermodynamics. (For example, we could have expressed total energy of disc as $\Delta E = \frac{1}{2} N k_E T_E$ with out affecting anything.) \textit{Hence it can be concluded that the entanglement entropy per unit disc area is fixed for small discs of radii $l \ll z_I$.} It is also confirmed that the entropy of excitations (10) follows the first law relation:\[ \delta s_E = \frac{1}{T_E} (\delta \Delta E + \frac{1}{2} \mu E \delta \Delta \rho) \tag{18} \]
under infinitesimal changes in the bulk quantity, $\delta z_I$.

We summarize our main observations at first order;
\[ T_E \propto \frac{1}{l^2}, \quad \Delta s_E = Fixed, \quad \mu E \propto r_y T_E, \quad \Delta E \propto N T_E, \quad \Delta \rho = Fixed, \tag{19} \]
at a given entanglement temperature.

### 3.2 Entanglement entropy of a disc at second order

Let us now consider corrections to holographic entanglement entropy at next higher order. It is somewhat easier to calculate when one chooses $z(r)$ parameterization,

\footnote{There is an early work \cite{47} but it does not include dilatonic scalar field excitations like in our background. In contrast in asymptotically AdS spacetimes one knows how to obtain stress-energy tensor by doing Fefferman-Graham expansion near AdS boundary \cite{48}. Perhaps something similar could also be done in the Lifshitz case involving dilaton field.}

\footnote{Refer to \cite{27} - \cite{35} for earlier work on entanglement thermodynamics.}

...
so let us rewrite the integral as

$$A_\gamma = 8\pi^2 L^3 r_y \int_0^1 dr \frac{r \sqrt{1 + z'^2}}{qz^2} \frac{1}{\sqrt{1 - r^2}}$$

where we rescaled $r$ and $z$ to the dimensionless variables $\tilde{r}$ and $\tilde{z}$. It suffices to obtain the embedding up to first order to get the entanglement at second order \[29, 26\]. So, we expand $z(r)$ as $z(r) = z(0) + z(1) + \cdots$, where $z(0) = \sqrt{1 - r^2}$ and $z(1)$ satisfies the equation

$$z''(1) + \frac{1 - 2r^2}{r(1 - r^2)} z'(1) - \frac{2}{(1 - r^2)^2} z(1) = \frac{1}{\sqrt{1 - r^2}}$$

with the boundary conditions: $z'(1)(0) = 0$ and $z(1)(l) = 0$. One can check that a consistent solution to equation (21) is

$$z(1) = \frac{-1 - r^2 - 2 \sqrt{1 - r^2} + 2 \ln(1 + \sqrt{1 - r^2})}{2 \sqrt{1 - r^2}}$$

Therefore, the area integral now acquires a new contribution $A_\gamma = A_0 + A_1 + A_2$ where

$$A_2 = \frac{8\pi^2 L^3 r_y}{q} \frac{l^4}{z_f^2} \left(\frac{5}{8} \ln 2\right)$$

which is negative as expected. Total entropy of the disc at this order will be

$$S_E^{(2)} = S_E^{(0)} + \frac{\pi^2 L^3 r_y}{qG_5} \frac{l^2}{z_f^2} \left(1 + \frac{l^2}{z_f^2} \left(\frac{5}{4} - 2 \ln 2\right)\right)$$

So that the variation of entropy density, at second order, becomes:

$$\delta s_E^{(2)} = \frac{\pi L^3 r_y}{qG_5} \left(1 + \frac{l^2}{z_f^2} \left(\frac{5}{2} - 4 \ln 2\right)\right) \delta(z_f^{-2})$$

As previous, we wish to express (25) as a ‘first law’ like relationship. We will follow the method of [32] and absorb all second order corrections to a modified temperature and chemical potential. To this end, we first note that the turning point $z_*$ should be corrected at $O\left(\frac{l^2}{z_f^2}\right)$ as

$$z_* \equiv z(0) = l + \frac{l^3}{z_f^2} \left(\frac{1}{2} - \ln 2\right)$$

The chemical potential, defined in equation (13), can be expressed including $O\left(\frac{l^2}{z_f^2}\right)$ corrections as

$$\mu_E^{(1)} \sim \frac{L^2 r_y}{ql^2} \left(1 + \frac{l^2}{z_f^2} \left(\frac{1}{2} - \ln 2\right)\right)^{-2} \left(1 + \frac{l^2}{z_f^2}\right)^{-1} \left(1 - \frac{l^2}{z_f^2} \left(2 - 2 \ln 2\right)\right)$$

(26)
So we get
\[
\mu_E^{(1)} \delta \Delta \rho = \frac{4\pi L^3 r_y^4}{qG_5 l^2} (1 - \frac{l^2}{z_I^2} (2 - 2 \ln 2)) \delta (z_I^{-2})
\]
while the energy remains the same as defined in \((15)\). From equation \((25)\), a bit of paperwork then leads to the following result
\[
\delta s^{(2)}_E = \frac{1}{T_E^{(2)}} \left( \delta \Delta \mathcal{E} + \frac{1}{2} \mu_E^{(1)} \delta \Delta \rho \right)
\]
(27)
where \(T_E^{(2)}\) denotes the ‘entanglement temperature’ at second order, which is given by
\[
T_E^{(2)} = \frac{4\pi L^3 r_y}{qG_5 l^2} \left[ 1 - \frac{l^2}{z_I^2} (1 + \ln 2) \right] \frac{\pi L r_y}{qG_5} \left[ 1 - \frac{l^2}{z_I^2} \left( 4 \ln 2 - \frac{5}{2} \right) \right] \\
\approx T_E^{(1)} \left[ 1 + \frac{l^2}{z_I^2} (5 \ln 2 - \frac{7}{2}) \right]
\]
(28)
where \(T_E^{(1)}\) stands for the first order temperature, defined in \((16)\). The term in parentheses is a negative number, so second order correction to ‘entanglement temperature’ results in its sharper fall. See figure 1 for an illustration of this behaviour.

Figure 1: The unbroken and dashed curves display the behaviour of the uncorrected and corrected quantities, respectively; both the entanglement temperature and chemical potential decrease due to higher order corrections. The plots were drawn by setting \(z_I^2 = 2\) and \(L = r_y = q = 1\).

Some comments are in order to justify equation \((27)\), we have seen that for small enough subsystem size \((l \ll z_I)\), the change in entanglement entropy at first order in our perturbative calculation follows a relationship akin to the first law of thermodynamics. If one considers this relationship an actual ‘law’ for entanglement...
entropy, one must find a consistent way to describe new contributions at higher orders. Equation (28) proposes that at second order, the chemical potential as well as the entanglement temperature should be corrected to keep the law intact. In fact, we expect this procedure to work at all higher orders. It could be thought that a more accurate measure of these quantities are obtained as one climbs the perturbation ladder.

4 Entanglement entropy of narrow strip

We now consider a strip like subsystem with coordinate width $-l/2 \leq x_1 \leq l/2$, and the range of $x_2 \in [0, l_2]$, such that $l_2 \gg l$. The straight line boundary of the two-dimensional strip is identified with the boundary of the RT surface in the bulk at constant time. The area functional of this static surface is

$$A_\gamma = 4\pi L^3 r_y l_2 \int_{z}^{z_+} dz \frac{1 + x_1^2}{q z^2} h_*^2$$  \quad (29)

For small width $l \ll z_I$, we make a perturbative expansion of the integrand. The extremal surface satisfies the following equation

$$x_1' = \frac{z^2}{z_*^2} \frac{1}{\sqrt{h_*^2 - z_*^4}}$$  \quad (30)

where $h_* \equiv h(z_*)$. We have specific boundary conditions such that near the space-time boundary $x_1|_{z=0} = l/2$ and the turning point is given by $x_1|_{z=z_*} = 0$. This leads to the first integral of the following type

$$l = 2 \int_{0}^{z_*} dz z^2 \frac{1}{\sqrt{h_*^2 - z_*^4}}$$  \quad (31)

which gives rise to a perturbative expansion in $z_*^2$

$$l = z_*(b_0 + \frac{z_*^2}{2z_I^2} I_1 + \cdots)$$  \quad (32)

where coefficients are expressible as Beta-functions $b_0 = \frac{1}{4} B(\frac{3}{4}, \frac{1}{2})$, $I_1 = \frac{1}{4} (B(\frac{3}{4}, -\frac{1}{2}) - B(\frac{5}{4}, -\frac{1}{2}))$. The equation (32) can be inverted and expressed as a perturbative expansion of the turning point

$$z_* = z_*^{(0)} (1 - \frac{z_*^{(0)2}}{2b_0} I_1 + \cdots)$$  \quad (33)
where \( z_*^{(0)} \equiv \frac{l}{2b_0} \) is the turning point in the absence of excitations.

The leading area of strip can be evaluated using the tree level values

\[
A_0 = 4\pi L^3 r_y l_2 \int_{\epsilon}^{z_*^{(0)}} \frac{dz}{q z^2} \sqrt{1 + x_{1(0)}^2} Q z^2
\]

\[
= \frac{4\pi L^3 r_y l_2}{q z_*^{(0)}} \int_{z_*^{(0)}}^{1} d\zeta \frac{1}{\zeta^2 \sqrt{1 - \zeta^4}}
\]

\[
= \frac{4\pi L^3 r_y l_2}{q} \left( \frac{1}{\epsilon} - \frac{2(b_0)^2}{l} \right).
\] (34)

while the first order contribution is evaluated as

\[
A_1 = 4\pi L^3 r_y l_2 \int_0^{z*} dz \sqrt{1 + x_{1(0)}^2} \frac{1}{2q z^2}
\]

\[
= 2\pi L^3 r_y l_2 \left( \frac{a_1 z_*^{(0)}}{q z_*^2} \right)
\] (35)

where the coefficient \( a_1 = \frac{1}{4} B\left( \frac{1}{4}, \frac{1}{2} \right) \). The entanglement entropy of small strip up to first order is then given by

\[
S_{E}^{\text{strip}} = \frac{A_0 + A_1}{4G_5} = \frac{L^3 \pi r_y l_2}{G_5 q} \left( \frac{1}{\epsilon} - \frac{2b_0^2}{l} + \frac{a_1 l}{4b_0 z_*^2} \right)
\] (36)

Now any small change in the bulk parameter \( \delta z_I \) will necessarily effect the entanglement entropy at first order. For a fixed width \( l \), we find the change in entropy per unit area of the strip as

\[
\delta S_{E}^{\text{strip}} \equiv \frac{\delta S_{E}^{\text{strip}}}{l^2 \epsilon} = \frac{\pi L^3 r_y a_1}{4G_5 q b_0} \delta \left( z_*^2 \right)
\] (37)

which is complete expression up to first order. Once again we find that the right hand side is independent of \( l \), as it was also in the case of a disc. Following from the disc case in the previous section, the effective chemical potential for strip becomes

\[
\mu_E = \frac{L^2 r_y}{q z_*^2} \simeq \frac{4b_0^2 L^2 r_y}{q l^2}
\] (38)

From here and the eq.(12) let us define for the strip

\[
\Delta \mathcal{E} \equiv \frac{1}{2} \mu_E \Delta \rho = \frac{8\pi L^3 r_y b_0^2}{G_5 q z_*^2 l^2}
\] (39)
This is like the disc result in (15), i.e. $\Delta E \propto T_E$. Using (39) we conclude that the entanglement entropy density (37) of the strip subsystems also conforms to the first law relation

$$
\delta s_E = \frac{1}{T_E} (\delta \Delta E + \frac{1}{2} \mu_E \delta \Delta \rho)
$$

(40)

where for the strip, entanglement temperature is defined as $T_E = \frac{8b_0}{a_1}$ in 3-dimensional Lifshitz theory.

**4.1 Strip entropy at second order**

It is instructive to find out the change in entanglement entropy at higher orders in $l^2$ and interpret its thermodynamic property, here we include the results at $O(l^4)$. The turning point $z_*$, as discussed before in (31), could be related to the strip-width $l$ as:

$$
z_* = z_*^{(0)} \frac{z_*^{(0)}}{1 + \frac{z_*^{(0)^2}}{2z_l^2} b_0 - \frac{z_*^{(0)^4}}{8z_l^2} (\frac{I_2}{b_0} + \frac{4I_1^2}{b_0^2})}
$$

(41)

where the new co-efficient $I_2$ can be expressed as: $I_2 = \frac{1}{8} (2B(\frac{3}{4}, -\frac{3}{2}) - 3B(\frac{5}{4}, -\frac{3}{2}))$.

With the help of (41), the area integral (29) now reads $A_{\gamma} = A_0 + A_1 + A_2$, where $A_0$ and $A_1$ are as obtained before. The second order contribution is

$$
A_2 = -\frac{4\pi L^3 r_y l_2 z_*^{(0)^4}}{8z_l^4} (\frac{4a_0 I_2^2}{b_0^2} + \frac{2I_1 J_1}{b_0})
$$

(42)

The new coefficients introduced in above expression are listed below:

$$
a_0 = -\frac{1}{4} B(\frac{3}{4}, -\frac{1}{2}) = -b_0
$$

$$
J_1 = \frac{1}{4} (B(\frac{3}{4}, -\frac{1}{2}) + 3B(\frac{1}{4}, -\frac{1}{2}))
$$

After some simplification the contribution to the area of the RT surface at second order turns out to be

$$
A_2 = -\frac{\pi L^3 r_y l_2 l^2}{32q} z_l^2 \frac{1}{b_0^2} (\frac{a_1^2}{b_0^2} - 1)
$$

(43)

The coefficient $a_1$ has already been defined in eq. (35). Hence, the total entanglement entropy density, at second order in perturbation theory, becomes

$$
s^{(2)}_E = s^{(0)}_E + \frac{\pi L^3 r_y}{4q G_5} \frac{1}{z_l^2 b_0} \left(1 - \frac{l^2}{32b_0^2} \left(\frac{a_1^2}{b_0^2} - 1 \right)\right)
$$

(44)
To write down the ‘first law’ we need to rewrite the expression for $s^{(2)}_E$ in terms of variation in $E$ and $\mu_E \Delta \rho$; recall that the chemical potential was defined as the value of the gauge potential at the turning point. Here, it is sufficient to compute $\mu_E$ up to first order

$$\mu_E^{(1)} \simeq \frac{L^2}{z_*^2} \left( 1 - \frac{z_*^2}{z_I^2} \right) = \frac{L^2 r_y}{q z_*^{(0)2}} \left( 1 + \frac{z_*^{(0)2}}{z_I^2} \left( I_1 \frac{I_1}{b_0^2} - 1 \right) \right)$$

So that,

$$\mu_E^{(1)} \delta \Delta \rho = \frac{L^3 r_y}{q G_5} \frac{8 b_0^2}{l^2} \left[ 1 + \frac{l^2}{z_I^2} \frac{1}{8 b_0^2} \left( \frac{a_1}{b_0} - 3 \right) \right] \delta(z_I^{-2})$$

A little effort, then, allows us to write

$$\delta s^{(2)}_E = \frac{1}{T^{(2)}_E} \left( \delta \Delta E + \frac{1}{2} \mu_E^{(1)} \delta \Delta \rho \right) \quad (45)$$

Here, $T^{(2)}_E$ stands for the entanglement temperature corrected up to $O(\frac{l^4}{z_I^4})$.

$$T^{(2)}_E = \frac{4}{l^2} \frac{8 b_0^2}{a_1} \left[ 1 + \frac{l^2}{z_I^2} \frac{1}{16 b_0^2} \left( \frac{a_1}{b_0} - 3 \right) \left( \frac{a_1^2}{b_0^2} - 1 \right) \right]$$

$$= T^{(1)}_E \left[ 1 + \frac{l^2}{z_I^2} \frac{1}{16 b_0^2} \left( \frac{a_1}{b_0} - 1 \right) \left( \frac{a_1}{b_0} + 2 \right) - 2 \right] \quad (46)$$

Where by $T^{(1)}_E$, we refer to the temperature at first order defined in (40), the numerical value of $\frac{a_1}{b_0} \approx 2.188$, so the correction at this order results in an increase of $T_E$, albeit by a tiny amount. The uncorrected and corrected temperatures are plotted in figure 2.

5 Conclusion

The Lifshitz background $Lif^{(2)}_A \times S^1 \times S^5$ of the massive type IIA theory allows exact excitations which couple to massless modes of string in the IR. We calculated the entanglement entropy of the theory at the boundary of these spacetimes, both for strip as well as disc shaped systems. At leading order, we found that the entropy density of the excitations remains fixed and does not grow with $l$, the subsystem size, so long as $l \ll z_I$. We find that this behaviour is consistent with the fact that energy density of the excitations itself behaving as $\Delta E \propto 1/l^2$, which is in agreement with $\Delta E \simeq \frac{1}{2} \mu_E \Delta \rho$. Note that the entanglement temperature itself goes as $T_E \propto \frac{1}{l^2}$.

But this entanglement behaviour is quite different in comparison to the relativistic CFTs, where the entropy density of excitations grows linearly with the
Figure 2: The unbroken and dashed curves display the behaviour of the uncorrected and corrected quantities, respectively; the entanglement temperature is found to increase due to higher order corrections while the chemical potential decreases. The plots were drawn by setting $z_I = 2$ and $L = r_y = q = G_5 = 1$.

subsystem size, while the energy density of excitations remains fixed. Nevertheless we have found that the first law of entanglement thermodynamics,

$$\delta s_E = \frac{1}{T_E} (\delta \Delta E + \frac{1}{2} \mu_E \delta \Delta \rho)$$

holds good if we accept the hypothesis that the energy of a subsystem in the Lifshitz background is given by

$$\Delta E \simeq \mu E N \simeq \frac{1}{2} k E T$$

Our results appear to indicate an equipartition nature of the entanglement thermodynamics for non-relativistic Lifshitz system. But this is perhaps true only for the high entanglement temperature regime (i.e. small $l \ll z_I$).

We also discussed how the first law could be extended up to second order by making use of appropriately modified chemical potential and entanglement temperature. We think this is necessary because otherwise, we need to look for a new quantity at each higher order to account for the corrections; while the entanglement entropy, like its thermal counterpart should depend only on the energy and charges in the theory. Such redefinition should work at all orders, thereby allowing the ‘first law of entanglement thermodynamics’ to be obeyed quite generally, irrespective of the degree of perturbation theory.

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A The winding string charge in massive Lifshitz vacua

Here we would like to know the winding number of the string excitations. The circle compactification of the background (2) along $y$ direction gives rise to following 9-dimensional fields (we set $g_0 = 1$, $\alpha' = 1$)

$$ ds^{2}_{D=9} = L^2 \left( -\frac{dt^2}{z^4 h} + \frac{dx_1^2 + dx_2^2}{z^2} + \frac{dz^2}{z^2} + d\Omega_5^2 \right), $$

$$ e^{2\tilde{\phi}} = \frac{1}{h \sqrt{G_{yy}}}, \quad A_t = \frac{L^2}{qz^2} h^{-1}, \quad (48) $$

where $G_{yy} = \frac{r^2}{q^2 h}$, $h(z) = 1 + \frac{z^2}{z_I^2}$. The $\tilde{\phi}$ is 9-dimensional dilaton field. The corresponding gauge field strength $F^{(2)} = dA$ gives rise to the winding charge

$$ Q = \frac{\pi r_y}{G_{10}} \int e^{-\frac{4\tilde{\phi}}{7} G^{yy}(\ast_9 F^{(2)})} $$

$$ = \frac{\pi L^6 \omega_5 r_y}{G_{10}} \int dx_1 dx_2 \left( \frac{2}{z^2} + \frac{4}{z_I^2} \right) $$

$$ = \frac{\pi L r_y V_2}{G_5} \left( \frac{2}{z^2} + \frac{4}{z_I^2} \right) $$

$$ \equiv Q_{\text{ground-state}} + \Delta Q \quad (49) $$

where $\omega_5$ is the size of unit 5-sphere. The total charge $Q$, of course, depends on scale $z$, because we are in asymptotically (non-flat) Lifshitz spacetime. However, the contribution purely due to string excitations is given by $\Delta Q$. The second term in (49) is not affected by $z$ and remains constant. Therefore the net contribution of string excitations is

$$ \Delta Q = Q - Q_{\text{ground-state}} = \frac{2\pi L r_y V_2}{G_5} \left( \frac{2}{z_I^2} \right) \simeq Q|_{z=\infty}. \quad (50) $$

Alternatively the charge due to string excitations can also be measured near $z \sim \infty$, where the massive mode gets completely decoupled and only massless strings survive which contribute to the charge. Net winding number of these strings is quantized in the units $N = \frac{\Delta Q}{r_y}$, where $N$ is an integer.
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