On retardation, radiation and Liénard-Wiechert type potentials in electrodynamics and elastodynamics

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Abstract

The aim of this paper is to investigate the fundamental problems of retardation and radiation caused by non-uniformly moving point sources using the theories of electrodynamics and elastodynamics. This paper investigates and compares the retarded electromagnetic fields and the retarded elastodynamic fields. For the non-uniform motion of a general point source, the Liénard-Wiechert type potentials and the radiation and nonradiation fields are derived for a point charge and for a point force.

Keywords: non-uniform motion; electrodynamics; elastodynamics; radiation; retardation.

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1 Introduction

A lot of research has been devoted to problems in electrodynamics and elastodynamics. Often the investigations use some analogies, the main differences are due to the vector character of the electromagnetic fields and the tensor character of elastodynamic fields. Other important differences are that electrodynamics possesses only one characteristic velocity (velocity of light), and the fundamental solution of the field equations is a scalar-valued generalized function. On the other hand, elastodynamics of isotropic materials has two characteristic velocities (longitudinal and transversal velocity of the elastic waves) and the fundamental solution of the field equation is a tensor-valued generalized function (see Table 1). These differences may be significant for the explicite expressions, but often the analysis is very similar from the mathematical point of view. However, the dynamics of elastodynamic waves is more complicated than the dynamics of electromagnetic waves.

In this paper we compare the fundamentals and waves in electrodynamics and elastodynamics, and this research is motivated by analogies. We review the retarded causal fields and the non-uniform motion of a point charge in electrodynamics. Motivated by straightforward analogies, we have investigated the retarded fields and the non-uniform motion of a point force in elastodynamics.

In this paper we give the answers to the fundamental questions:

• What is the elastodynamic Liénard-Wiechert potential of a point force?

• What are the equations in analogy to the Liénard-Wiechert form and the Heaviside-Schott-Feynman form of the electromagnetic field strengths?

• What is the form of the elastodynamic radiation fields?

One important property of the dynamics of waves is the retardation, which is a consequence of the finite speed of the propagating fields. There is always a time delay, since an effect observed by the receiver at the present position and present time was caused by the sender at some earlier time (retarded time) and at the retarded position.

This paper is organized as follows. In Section 2, we review the theory of electrodynamics. First of all, the retarded fields are presented. Especially, the Jefimenko equations are presented as the retarded electromagnetic field strengths, which are the causal solutions of the Maxwell equations. Later, we consider the non-uniform motion of a point charge with time-dependent magnitude. The so-called Heaviside-Schott-Feynman equations will be discussed as well as the Liénard-Wiechert form of the electromagnetic field strengths.

Table 1: Comparison between electrodynamics and elastodynamics

| Electrodynamics       | Elastodynamics       |
|-----------------------|-----------------------|
| Green function $G(r, t)$ | Green tensor $G_{ij}(r, t)$ |
| 1 velocity of light: $c$ | 2 velocities of sound: $c_T, c_L$ |
| 1 retarded time: $t_r = t - R/c$ | 2 retarded times: $t_T = t - R/c_T, t_L = t - R/c_L$ |
The electromagnetic radiation produced by the point charge will be given explicitly. In Section 3, we consider the theory of elastodynamics and we investigate the retarded displacement, velocity and distortion fields. The non-uniform motion of a point force is studied. We calculate the elastodynamic radiation and non-radiation parts. We show that the fields of a non-uniform moving point force are the generalization of the Stokes solution towards non-uniform motion. Also we consider the case of time-harmonic forces. All the mathematical expressions concerning the derivative with respect to the retarded time are given in an Appendix.

2 Electrodynamics

In this section we investigate the retarded fields and the radiation of electromagnetic fields using the theory of electrodynamics.

2.1 Basic framework and retarded fields

The basic electromagnetic field laws are represented by the inhomogeneous and homogeneous Maxwell equations [1, 2, 3]

\[ \nabla \cdot D = \rho, \quad \nabla \times H - \partial_t D = J, \]

\[ \nabla \cdot B = 0, \quad \nabla \times E + \partial_t B = 0, \]

where \( D \) is the electric displacement vector (electric excitation), \( H \) is the magnetic excitation vector, \( B \) is the magnetic field strength vector, \( E \) is the electric field strength vector, \( J \) is the electric current density vector, and \( \rho \) is the electric charge density. In addition, the electric current density vector and the electric charge density fulfill the continuity equation

\[ \nabla \cdot J + \partial_t \rho = 0. \]

The constitutive equations for the fields in a vacuum read

\[ D = \epsilon_0 E, \quad H = \frac{1}{\mu_0} B, \]

where \( \epsilon_0 \) is the vacuum permittivity and \( \mu_0 \) is the vacuum permeability. The speed of light in vacuum is given by

\[ c^2 = \frac{1}{\epsilon_0 \mu_0}. \]

The electromagnetic field strengths can be expressed in terms of the electromagnetic gauge potentials \( \phi \) and \( A \) (scalar potential \( \phi \) and vector potential \( A \)):

\[ E = -\nabla \phi - \partial_t A, \]

\[ B = \nabla \times A. \]
Using the Lorentz gauge condition:
\[
\frac{1}{c^2} \partial_t \phi + \nabla \cdot A = 0,
\]
the electromagnetic gauge potentials fulfill the following inhomogeneous wave equations
\[
\Box \phi = \frac{1}{\varepsilon_0} \rho
\]
and
\[
\Box A = \mu_0 J,
\]
where the d’Alembert operator is defined by
\[
\Box := \frac{1}{c^2} \partial_{tt} - \Delta \quad \text{with} \quad \Delta = \nabla \cdot \nabla.
\]

For zero initial conditions, the solutions of Eqs. (9) and (10) may be written as space-time convolution integrals
\[
\phi(r, t) = \frac{1}{4\pi\varepsilon_0} \int_{-\infty}^{t} \int_{-\infty}^\infty G(r - r', t - t') \rho(r', t') \, dr' \, dt'
\]
\[
A(r, t) = \mu_0 \int_{-\infty}^{t} \int_{-\infty}^\infty G(r - r', t - t') J(r', t') \, dr' \, dt'.
\]

Using the 3D Green function of the wave equation (e.g. [4])
\[
G(r, t) = \frac{1}{4\pi r} \delta(t - r/c),
\]
we obtain the retarded electromagnetic potentials, which were originally introduced by Lorenz [5], and they read [1, 2]:
\[
\phi(r, t) = \frac{1}{4\pi\varepsilon_0} \int_{\mathcal{V}} \frac{\rho(r', t - R/c)}{R} \, dr',
\]
\[
A(r, t) = \frac{1}{4\pi\varepsilon_0 c^2} \int_{\mathcal{V}} \frac{J(r', t - R/c)}{R} \, dr',
\]
where \(R = r - r', R = |r - r'|, r \in \mathbb{R}^3, t \in \mathbb{R}\) and \(\mathcal{V}\) denotes the whole three-dimensional space. The idea of a retarded scalar potential was first developed by Lorenz [6] in 1861 while studying waves in the theory of elasticity. The retarded potentials fulfill the Lorentz gauge condition (see, e.g., [2, 7]).

Substituting the retarded potentials (15) and (16) into the definition of the electromagnetic field strengths (6) and (7), and using the relations
\[
\nabla \rho(r', t - R/c) = -\frac{R}{cR} \partial_t \rho(r', t - R/c), \quad \nabla J(r', t - R/c) = -\frac{R}{cR} \partial_t J(r', t - R/c),
\]

we obtain the retarded electromagnetic field strength vectors:

\[
E(r, t) = \frac{1}{4\pi\varepsilon_0} \int_V \left( \rho(r', t - R/c) \frac{R}{R^3} + \frac{\partial_t \rho(r', t - R/c)}{cR^2} R - \frac{\partial_t J(r', t - R/c)}{c^2 R} \right) \, dr',
\]

(18)

\[
B(r, t) = \frac{\mu_0}{4\pi} \int_V \left( \frac{J(r', t - R/c)}{R^3} + \frac{\partial_t J(r', t - R/c)}{cR^2} \right) \times R \, dr'.
\]

(19)

Eq. (18) is the time-dependent generalized Coulomb-Faraday law and Eq. (19) is the time-dependent generalized Biot-Savart law (see also [8]). Eqs. (18) and (19) express the electromagnetic fields in terms of their retarded sources \(\rho, J, \partial_t \rho\) and \(\partial_t J\) with full generality. They were originally derived by Jefimenko [7] (see also [9]). They also appear in the book of Clemmow and Dougherty [3] and in the third edition of Lorrain, Corson and Lorrain [10]. An equivalent representation was given by Panofsky and Phillips [11] (see also [9]). Nowadays both equations are called the Jefimenko equations in standard books on electrodynamics (e.g. [1, 2, 8]). They are fundamental, elegant, and very useful equations.

2.2 A non-uniformly moving point charge

Now we consider a non-uniformly moving point charge carrying the time-dependent charge \(q(t)\) at the position \(s(t)\). The electric charge density and the electric current density vector are given by

\[
\rho(r, t) = q(t) \delta(r - s(t)), \quad J(r, t) = q(t) V(t) \delta(r - s(t)),
\]

(20)

where \(V(t) = \partial_t s(t) = \dot{s}(t)\) is the arbitrary velocity of the non-uniform motion. We consider the case that the velocity of the point charge is less than the speed of light: \(|V| < c\). Substitution of Eq. (20) in Eqs. (15) and (16) gives

\[
\phi(r, t) = \frac{1}{4\pi\varepsilon_0} \int_V q(t - R/c) \frac{\delta(r' - s(t - R/c))}{R} \, dr',
\]

(21)

\[
A(r, t) = \frac{1}{4\pi\varepsilon_0 c^2} \int_V q(t - R/c) V(t - R/c) \frac{\delta(r' - s(t - R/c))}{R} \, dr'.
\]

(22)

Here remain the integrals of the delta functions, which are done by changing the variable of integration from \(r'\) to \(z = r' - s(t - R/c)\) with (see, e.g., [12, 13, 14])

\[
dz = J \, dr'
\]

(23)

and using the Jacobian \(J\) of this transformation

\[
J = \det \left( \frac{\partial z}{\partial r'} \right) = 1 - \frac{V(t - R/c) \cdot (r - r')}{c |r - r'|},
\]

(24)

we obtain

\[
\int F(r') \delta(r' - s(t - R/c)) \, dr' = \int F(r') \delta(z) \frac{1}{J} \, dz = \frac{F(r')}{J} \bigg|_{z=0} = \frac{F(r')}{1 - \frac{V(t-R/c) \cdot (r-r')}{c |r-r'|}} \bigg|_{r'=s(t_\varepsilon)}.
\]

(25)
From the argument of the \( \delta \)-function we get \( r' = s(t_r) \). Therefore, now \( R(t_r) = r - s(t_r) \) and \( R(t_r) = |r - s(t_r)| \) are time-dependent and the retarded time is now given by

\[
t_r = t - |r - s(t_r)|/c = t - R(t_r)/c,
\]

where \( s(t_r) \) is the retarded position of the moving source point. For a point charge moving with velocity less than the speed of light in free space (\( |V| < c \)), there is only one retarded time \( t_r \) that satisfies Eq. (26) for each time \( t \). In addition, we define

\[
P(t') = R(t') - V(t') \cdot R(t')/c,
\]

after the integration in \( r' \), Eqs. (21) and (22) transform into the Liénard-Wiechert potentials (scalar potential \( \phi \) and vector potential \( A \)) of a point charge [2, 15]:

\[
\phi(r, t) = \frac{1}{4\pi\epsilon_0} \left[ \frac{q(t')}{P(t')} \right]_{t' = t_r}, \quad A(r, t) = \frac{1}{4\pi\epsilon_0 c^2} \left[ \frac{q(t')V(t')}{P(t')} \right]_{t' = t_r}.
\]

These are the Liénard-Wiechert potentials [16, 17] for a point charge with time-dependent magnitude. Of course, the Liénard-Wiechert potentials fulfill the Lorentz gauge condition (see, e.g., [1, 18]).

Substituting Eq. (20) into the Jefimenko equations (18) and (19) gives the so-called Heaviside-Schott-Feynman formulae [19, 20, 21] for the electric field strength \( E \) and the magnetic field strength \( B \) of a non-uniformly moving point charge [1, 8]:

\[
E(r, t) = \frac{1}{4\pi\epsilon_0} \left( \left[ \frac{q(t')R(t')}{R^2(t')P(t')} \right]_{t' = t_r} + \frac{1}{c} \partial_t \left[ \frac{q(t')R(t')}{P(t')} \right]_{t' = t_r} - \frac{1}{c^2} \frac{1}{P(t')} \partial_{r'} \left[ \frac{q(t')V(t')}{P(t')} \right]_{t' = t_r} \right),
\]

\[
B(r, t) = \frac{1}{4\pi\epsilon_0 c^2} \left( \left[ \frac{q(t')V(t') \times R(t')}{R^2(t')P(t')} \right]_{t' = t_r} + \frac{1}{c} \partial_t \left[ \frac{q(t')V(t') \times R(t')}{R(t')P(t')} \right]_{t' = t_r} \right).
\]

We continue with the Liénard-Wiechert form of the electromagnetic field strengths for the point charge with time-dependent magnitude. Since the time derivative in Eqs. (29) and (30) is involved, it is clear that the electromagnetic fields (29) and (30) will be functions not only of the velocity \( V \), but also of the acceleration \( V \), and of the time-derivative of the magnitude of the point charge \( \dot{q} \). We may therefore separate \( E \) and \( B \) into two parts each, one of which involves the acceleration and the time-derivative of the charge and goes to zero for \( V = 0 \) and \( \dot{q} = 0 \), and one of which involves only the velocity \( V \):

\[
E(r, t) = E^{\text{nonr}}(r, t) + E^{\text{rad}}(r, t),
\]

\[
B(r, t) = B^{\text{nonr}}(r, t) + B^{\text{rad}}(r, t).
\]

The fields \( E^{\text{nonr}} \) and \( B^{\text{nonr}} \) are called the nonradiation or velocity parts of the electromagnetic field strengths, and the fields \( E^{\text{rad}} \) and \( B^{\text{rad}} \) are called the radiation or acceleration parts. To obtain the Liénard-Wiechert form of the electromagnetic fields, we must carry
out the time-derivatives, which are not trivial because of the subtle relation between the present and retarded time. Using Eqs. (A.3)–(A.5), the result reads

\[ E_{\text{nonr}}(r, t) = \frac{1}{4\pi \varepsilon_0} \left[ q(t') \left( 1 - \frac{V^2(t')}{c^2} \right) \left( R(t') - \frac{V(t') R(t')}{c} \right) \right]_{t' = t_r}, \]  

\[ E_{\text{rad}}(r, t) = \frac{1}{4\pi \varepsilon_0} \left[ \left( \frac{\dot{q}(t')}{c P^2(t')} + \frac{q(t') \dot{V}(t') \cdot R(t')}{c^2 P^3(t')} \right) \left( R(t') - \frac{V(t') R(t')}{c} \right) - \frac{q(t') \dot{V}(t') R(t')}{c^2 P^2(t')} \right]_{t' = t_r}, \]

\[ B_{\text{nonr}}(r, t) = \frac{1}{4\pi \varepsilon_0 c^2} \left[ \left( 1 - \frac{V^2(t')}{c^2} \right) \frac{q(t') V(t') \times R(t')}{P^3(t')} \right]_{t' = t_r}, \]

\[ B_{\text{rad}}(r, t) = \frac{1}{4\pi \varepsilon_0 c^2} \left[ \left( \frac{\dot{q}(t')}{c P^2(t')} + \frac{q(t') \dot{V}(t') \cdot R(t')}{c^2 P^3(t')} \right) V(t') \times R(t') + \frac{q(t') \dot{V}(t') \times R(t')}{c P^2(t')} \right]_{t' = t_r}. \]

It holds: \( B = R \times E/cR \). We can see a clear separation into the near field or non-radiation field (velocity-dependent, which falls off as \( 1/R^2 \)) and the far field or radiation field (which falls off as \( 1/R \)). The nonradiation fields are identical with the ‘convective’ fields of a uniformly moving charge (see, e.g., [11]). If the charge is accelerated, the electromagnetic fields are neither static nor convective, and there is a net change in the field energy which causes radiation. The electromagnetic radiation possesses two sources, and this is caused by the time-change of the charge \( \dot{q} \) and the acceleration \( \dot{V} \). For a constant charge \( \dot{q} = 0 \), we recover in Eqs. (33)–(36) the original Liénard-Wiechert form of the electromagnetic fields (see also [3, 8, 9, 22, 23]).

For a localized point charge with time-dependent charge we obtain

\[ E(r, t) = \frac{1}{4\pi \varepsilon_0} \frac{R}{R^3} \left[ q(t - R/c) + \frac{R}{c} \dot{q}(t - R/c) \right] \]

and \( B = 0 \).

### 3 Elastodynamics

In this section we investigate the retarded fields and the radiation of elastodynamic fields using the theory of elastodynamics.

#### 3.1 Basic framework and retarded fields

In elastodynamics, the force equilibrium condition reads (e.g. [24, 25])

\[ \partial_t p_i - \partial_j \sigma_{ij} = F_i, \]  

(38)
where \( \mathbf{p}, \mathbf{\sigma}, \) and \( \mathbf{F} \) are the linear momentum vector, the force stress tensor, and the body force vector. The constitutive relations are

\[
p_i = \rho v_i = \rho \partial_t u_i, \tag{39}
\]

\[
\mathbf{\sigma}_{ij} = C_{ijkl} \beta_{kl} = C_{ijkl} \partial_{il} u_k, \tag{40}
\]

where \( \mathbf{v} = \partial_t \mathbf{u} \) is the velocity vector of the continuum (particle velocity), \( \mathbf{\beta} = (\nabla \mathbf{u})^T \) is the distortion tensor (displacement gradient), and \( \mathbf{u} \) denotes the displacement vector. Here \( \rho \) denotes the mass density, and \( C_{ijkl} \) is the tensor of elastic moduli. If we substitute the constitutive relations (39) and (40) in Eq. (38), we obtain an inhomogeneous Navier equation for the displacement vector \( \mathbf{u} \)

\[
\rho \partial_{tt} u_i - C_{ijkl} \partial_j \partial_l u_k = F_i. \tag{41}
\]

For an isotropic material, the tensor of elastic moduli is given by

\[
C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \tag{42}
\]

where \( \lambda \) and \( \mu \) are the Lamé constants.

In an unbounded medium and under the assumption of zero initial conditions, which means that \( \mathbf{u}(\mathbf{r}, t_0) \) and \( \dot{\mathbf{u}}(\mathbf{r}, t_0) \) are zero for \( t_0 \to -\infty \), the solution of \( \mathbf{u} \) reads

\[
u_i(\mathbf{r}, t) = \int_{-\infty}^t \int_{-\infty}^{\infty} G_{ij}(\mathbf{r} - \mathbf{r}', t - t') F_j(\mathbf{r}', t') \, d\mathbf{r}' \, dt'. \tag{43}
\]

When the material is isotropic and infinitely extended, the three-dimensional elastodynamic Green tensor reads \([26, 27]\)

\[
G_{ij}(\mathbf{r}, t) = \frac{1}{4\pi \rho} \left\{ \frac{\delta_{ij}}{r c_T^2} \delta(t - r/c_T) + \frac{x_i x_j}{r^3} \left( \frac{1}{c_L^2} \delta(t - r/c_L) - \frac{1}{c_T^2} \delta(t - r/c_T) \right) \right. \\
+ \left. \left( \frac{3 x_i x_j}{r^2} - \delta_{ij} \right) \frac{1}{r} \int_{1/c_L}^{1/c_T} \kappa \delta(t - \kappa r) \, d\kappa \right\}, \tag{44}
\]

where \( r = \sqrt{x_1^2 + x_2^2 + x_3^2} \) and \( \kappa \) is a dummy variable with the dimension \( 1/[\text{velocity}] \). Here \( c_L \) and \( c_T \) denote the velocities of the longitudinal and transversal elastic waves (sometimes called P- and S-waves). The two sound-velocities are given in terms of the Lamé constants \( c_T < c_L \)

\[
c_L = \sqrt{\frac{2\mu + \lambda}{\rho}}, \quad c_T = \sqrt{\frac{\mu}{\rho}}. \tag{45}
\]

Substituting the Green tensor (44) into Eq. (43) and integrating in time \( t' \), the retarded displacement vector is obtained as (see also \([28]\))

\[
u_i(\mathbf{r}, t) = \frac{1}{4\pi \rho} \int_V \left\{ \frac{1}{c_T^2} \left( \frac{\delta_{ij}}{R} - \frac{R_i R_j}{R^3} \right) F_j(\mathbf{r}', t_T) + \frac{1}{c_L^2} \frac{R_i R_j}{R^3} F_j(\mathbf{r}', t_L) \right. \\
+ \left. \left( \frac{3 R_i R_j}{R^3} - \frac{\delta_{ij}}{R} \right) \int_{1/c_L}^{1/c_T} \kappa F_j(\mathbf{r}', t_\kappa) \, d\kappa \right\} d\mathbf{r}', \tag{46}
\]
where the so-called retarded times are given by

\[ t_T = t - \frac{R}{c_T}, \quad (47) \]
\[ t_L = t - \frac{R}{c_L}, \quad (48) \]
\[ t_\kappa = t - \kappa R. \quad (49) \]

Here \( t_T \) and \( t_L \) are the transversal retarded time and the longitudinal retarded time, respectively. The retarded time \( t_\kappa \) is an effective retarded time for the \( \kappa \)-integration with the limits \((1/c_L, 1/c_T)\). Since \( c_L > c_T \), the retarded times fulfill: \( t_T > t_L \) and \( t_\kappa \in [t_L, t_T] \).

The idea of a retarded scalar potential was first developed by Lorentz \[6\] in 1861 while studying waves in the theory of elasticity. Love \[29\] introduced the retarded potentials based on the Helmholtz decomposition (see also \[24, 25\]). Our approach in this paper is more direct and straightforward, since we have introduced the retarded displacement vector \( (46) \) as the causal solution of the Navier equation \( (41) \).

The time-derivative of Eq. \((46)\) gives the retarded particle velocity vector

\[ v_i(r, t) = \frac{1}{4\pi \rho} \int_V \left\{ \frac{1}{c_T^2} \left( \frac{\delta_{ij}}{R} - \frac{R_i R_j}{R^3} \right) \partial_t F_j(r', t_T) + \frac{1}{c_L^2} \frac{R_i R_j}{R^3} \partial_t F_j(r', t_L) \right. \]
\[ + \left( \frac{3R_i R_j}{R^3} - \frac{\delta_{ij}}{R} \right) \int_{1/c_L}^{1/c_T} \kappa \partial_t F_j(r', t_\kappa) \, d\kappa \} \, dr'. \quad (50) \]

The gradient of Eq. \((46)\) and using the relation

\[ \partial_k F_j(r', t - R/c) = \frac{R_k}{c \kappa} \partial_k F_j(r', t - R/c), \quad c = c_T, c_L, 1/\kappa, \quad (51) \]

lead to the retarded distortion tensor

\[ \beta_{ik}(r, t) = -\frac{1}{4\pi \rho} \int_V \left\{ \frac{1}{c_T^2} \left( \frac{\delta_{ij}}{R} - \frac{R_i R_j}{R^3} \right) \partial_t F_j(r', t_T) \right. \]
\[ + \frac{1}{c_T^2} \left( \frac{\delta_{ij} R_j - R_i R_j}{R^2} \right) \frac{R_k}{R^2} \partial_t F_j(r', t_T) - \frac{1}{c_L^2} \left( \frac{\delta_{jk} R_i + \delta_{ik} R_j}{R^3} - \frac{3R_i R_j R_k}{R^5} \right) F_j(r', t_L) \]
\[ + \frac{1}{c_L^2} \frac{R_i R_j R_k}{R^4} \partial_t F_j(r', t_L) - \left( \frac{\delta_{ij} R_k + 3 \delta_{jk} R_i + 3 \delta_{ik} R_j - 9R_i R_j R_k}{R^5} \right) \int_{1/c_L}^{1/c_T} \kappa \, F_j(r', t_\kappa) \, d\kappa \]
\[ + \left( \frac{3R_i R_j}{R^2} \right) \frac{R_k}{R^2} \left. \int_{1/c_L}^{1/c_T} \kappa \partial_t F_j(r', t_\kappa) \, d\kappa \right\} \, dr'. \quad (52) \]

Since the integrals \((50)\) and \((52)\) are evaluated at the retarded times, they are called retarded elastic fields. The sources \( F \) and \( \partial_t F \) at the position \( r' \) depend on the retarded times. Thus, the sources are retarded due to the retarded times. Although Eqs. \((50)\) and \((52)\) have some similarities with the Jefimenko equations in electrodynamics \((18)\) and \((19)\), the retarded fields in elastodynamics are more complicated than the retarded fields in electrodynamics.
3.2 Time-harmonic forces

Here the retarded fields (46), (50), and (52) are applied to the special case of time-harmonic forces. The time-harmonic force vector reads

\[ F_j(r, t) = \text{Re} \left\{ F_j(r)e^{-i\omega t} \right\}, \quad (53) \]

where \( \omega \) denotes the angular frequency, \( i = \sqrt{-1} \) and \( \text{Re} \) indicates that the real part should be taken. In Eqs. (46), (50), and (52) the force enters at the retarded times as

\[ F_j(r, t_c) = \text{Re} \left\{ F_j(r)e^{-i\omega(t-t_c)} \right\}, \quad c = c_T, c_L, 1/\kappa. \quad (54) \]

Substituting Eq. (54) into Eq. (46) and using the integral

\[ \int_{1/\kappa}^{1/ct} \kappa e^{i\omega R\kappa} d\kappa = \frac{1}{\omega^2 R^2} \left\{ \frac{1}{c_T^2} \left( \frac{\delta_{ij}}{R} - \frac{R_i R_j}{R^3} \right) e^{i\omega R/c_T} \left( 1 - \frac{i\omega R}{c_T} \right) - e^{i\omega R/c_L} \left( 1 - \frac{i\omega R}{c_L} \right) \right\}, \quad (55) \]

the displacement vector takes the form

\[ u_i(r, t) = \text{Re} \left\{ \frac{e^{-i\omega t}}{4\pi \rho} \left\{ \frac{1}{c_T^2} \left( \frac{\delta_{ij}}{R} - \frac{R_i R_j}{R^3} \right) e^{i\omega R/c_T} \left( 1 - \frac{i\omega R}{c_T} \right) - e^{i\omega R/c_L} \left( 1 - \frac{i\omega R}{c_L} \right) \right\} \right\}, \quad (56) \]

The first two terms behave like \( 1/R \) and the last term gives \( 1/R^2 \) and \( 1/R^3 \) contributions. The velocity vector is given by: \( v_i = -\text{Re} [i\omega u_i] \). Now substituting Eq. (54) into Eq. (52) and using the integrals (55) and

\[ \int_{1/\kappa}^{1/ct} \kappa^2 e^{i\omega R\kappa} d\kappa = \frac{2i}{\omega^3 R^3} \left\{ e^{i\omega R/c_T} \left( 1 - \frac{i\omega R}{c_T} - \frac{\omega^2 R^2}{2c_T^2} \right) - e^{i\omega R/c_L} \left( 1 - \frac{i\omega R}{c_L} - \frac{\omega^2 R^2}{2c_L^2} \right) \right\}, \quad (57) \]

the distortion tensor is obtained

\[ \beta_{ik}(r, t) = \text{Re} \left\{ -\frac{e^{-i\omega t}}{4\pi \rho} \left\{ \frac{1}{c_T^2} \left( \frac{\delta_{ij} R_k + \delta_{jk} R_i + \delta_{ik} R_j}{R^3} - \frac{6R_i R_j R_k}{R^5} \right) \left( \frac{1}{c_T^2} e^{i\omega R/c_T} - \frac{1}{c_L^2} e^{i\omega R/c_L} \right) \right\} \right\}, \quad (58) \]

Eq. (58) consists of \( 1/R, 1/R^2, 1/R^3 \) and \( 1/R^4 \) terms.
3.3 A non-uniformly moving point force

Now we consider the non-uniform motion of a point force of total strength \( Q(t) \) situated at the position \( s(t) \), then the non-uniformly moving point force vector is given by

\[
F_i(r, t) = Q_i(t) \delta(r - s(t)) \quad \text{for } r \in \mathbb{R}^3, \ t \in \mathbb{R}.
\]  

(59)

Moreover, only subsonic source-speeds will be admitted (\(|V| < c_T\)). Substitution of Eq. (59) in Eq. (46) and integration in \( t' \) lead to (see [30])

\[
u_i(r, t) = \frac{1}{4\pi \rho} \int_V \left\{ \frac{1}{c_T^2} \left( \frac{\delta_{ij} - \frac{R_i R_j}{R^3}}{R} \right) Q_j(t_T) \delta(r' - s(t_T)) + \frac{1}{c_L^2} \frac{R_i R_j}{R^3} Q_j(t_L) \delta(r' - s(t_L)) + \left( \frac{3R_i R_j}{R^3} - \frac{\delta_{ij}}{R} \right) \int_{1/c_L}^{1/c_T} \kappa Q_j(t_\kappa) \delta(r' - s(t_\kappa)) d\kappa \right\} dr'.
\]  

(60)

Like in Eqs. (23)–(25) we can carry out the integration in \( r' \) in Eq. (60) and we find the explicit expression for the displacement field of a non-uniformly moving point force which we call the elastodynamic Liénard-Wiechert potential of a point force (see also [30])

\[
u_i(r, t) = \frac{1}{4\pi \rho} \left\{ \frac{1}{c_T^2} \left[ \left( \frac{\delta_{ij} - \frac{R_i R_j}{R^2(t')}}{R^2(t')} \right) \frac{Q_j(t')}{P_T(t')} \right] \bigg|_{t'=t_T} + \frac{1}{c_L^2} \left[ \frac{R_i R_j}{R^2(t')} \frac{Q_j(t')}{P_L(t')} \right] \bigg|_{t'=t_L} + \int_{1/c_L}^{1/c_T} d\kappa \kappa \left[ \left( \frac{3R_i R_j}{R^2(t')} - \frac{\delta_{ij}}{R} \right) \frac{Q_j(t')}{P_\kappa(t')} \right] \bigg|_{t'=t_\kappa} \right\},
\]  

(61)

with

\[
P_\kappa(t') = R(t') - V_m(t')R_m(t')/c, \quad c = c_T, c_L, 1/\kappa,
\]  

(62)

where \( V(t') = \partial_t s(t') \) denotes the arbitrary velocity of the non-uniformly moving point force. The retarded times \( t_c = t'(r, t) \) are given now as the solution of the condition

\[
t - t' - |r - s(t')|/c = 0, \quad \text{with} \quad c = c_T, c_L, 1/\kappa.
\]  

(63)

The substitution of Eq. (59) into the retarded elastic fields (50) and (52) and after the \( r' \)-integration gives expressions for the elastic fields similar the Heaviside-Schott-Feynman equations

\[
u_i(r, t) = \frac{1}{4\pi \rho} \left\{ \frac{1}{c_T^2} \partial_t \left[ \left( \frac{\delta_{ij} - \frac{R_i R_j}{R^2(t')}}{R^2(t')} \right) \frac{Q_j(t')}{P_T(t')} \right] \bigg|_{t'=t_T} + \frac{1}{c_L^2} \partial_t \left[ \frac{R_i R_j}{R^2(t')} \frac{Q_j(t')}{P_L(t')} \right] \bigg|_{t'=t_L} \right.
\]

\[+ \left. \int_{1/c_L}^{1/c_T} d\kappa \kappa \partial_t \left[ \left( \frac{3R_i R_j}{R^2(t')} - \frac{\delta_{ij}}{R} \right) \frac{Q_j(t')}{P_\kappa(t')} \right] \bigg|_{t'=t_\kappa} \right\},
\]  

(64)
and

\[
\beta_{ik}(r, t) = -\frac{1}{4\pi\rho} \left\{ \frac{1}{c_T^2} \left[ \left( \delta_{ij} R_k(t') + \delta_{jk} R_i(t') + \delta_{ik} R_j(t') - 3R_i(t') R_j(t') R_k(t') \right) \frac{Q_j(t')}{P_T(t')} \right] \bigg|_{t'=t_T} + \frac{1}{c_T^2} \partial_t \left[ \left( \delta_{ij} R_k(t') - \frac{R_i(t') R_j(t')}{R(t')} \right) \frac{Q_j(t')}{P_T(t')} \right] \bigg|_{t'=t_T} - \frac{1}{c_L^2} \left[ \left( \delta_{jk} R_i(t') + \delta_{ik} R_j(t') - 3R_i(t') R_j(t') R_k(t') \right) \frac{Q_j(t')}{P_L(t')} \right] \bigg|_{t'=t_L} + \frac{1}{c_L^2} \partial_t \left[ R_i(t') R_j(t') R_k(t') \frac{Q_j(t')}{P_L(t')} \right] \bigg|_{t'=t_L} \right. \\
- \int_{1/c_L}^{1/c_T} \frac{d\kappa}{\kappa} \left[ \left( \delta_{ij} R_k(t') + 3\delta_{jk} R_i(t') + 3\delta_{ik} R_j(t') - 9R_i(t') R_j(t') R_k(t') \right) \frac{Q_j(t')}{R(t')} \right] \bigg|_{t'=t_\kappa} \left. + \partial_t \int_{1/c_L}^{1/c_T} \frac{d\kappa}{\kappa} \kappa^2 \left[ \left( \delta_{ij} R_k(t') + 3\delta_{jk} R_i(t') + 3\delta_{ik} R_j(t') \right) \frac{Q_j(t')}{R(t')} \right] \bigg|_{t'=t_\kappa} \right\}. \tag{65}
\]

Eqs. (64) and (65) are similar in some sense to the Heaviside-Schott-Feynman equations (29) and (30). But of course, they have a more complicated tensor structure.

Using Eqs. (A.3)–(A.7), we obtain for the time derivative of the displacement field (61) or from (64) (see also [30])

\[
v_i(r, t) = \frac{1}{4\pi\rho} \left\{ \frac{1}{c_T^2} \left[ \left( \delta_{ij} R_i(t') - \frac{R_i(t') R_j(t')}{R(t')} \right) \frac{\dot{Q}_j(t')}{P_T^2(t')} \right] \bigg|_{t'=t_T} + \frac{1}{c_L^2} \left[ \left( \delta_{ij} R_i(t') - \frac{R_i(t') R_j(t')}{R(t')} \right) \frac{\dot{Q}_j(t')}{P_L^2(t')} \right] \bigg|_{t'=t_L} \right. \\
+ \int_{1/c_L}^{1/c_T} \frac{d\kappa}{\kappa} \left[ \left( \delta_{ij} R_i(t') - \frac{R_i(t') R_j(t')}{R(t')} \right) \frac{\dot{Q}_j(t')}{P_T^2(t')} \right] \bigg|_{t'=t_\kappa} + \partial_t \int_{1/c_L}^{1/c_T} \frac{d\kappa}{\kappa} \kappa^2 \left[ \left( \delta_{ij} R_i(t') - \frac{R_i(t') R_j(t')}{R(t')} \right) \frac{\dot{Q}_j(t')}{P_L^2(t')} \right] \bigg|_{t'=t_\kappa} \left. \right\}. \tag{66}
\]

This is the velocity field (particle velocity) produced by a non-uniformly moving point force in the Liénard-Wiechert type representation. If we carry out the time-derivatives in
Eq. (65) and after arranging terms, we obtain the Liénard-Wiechert type representation of the distortion tensor of a non-uniformly moving point force (see also [30])

\[
\beta_{ik}(r, t) = -\frac{1}{4\pi}\left\{ \frac{1}{c^2 T} \left[ \left( \frac{\delta_{ij} - \frac{R_i(t') R_j(t')}{R^2(t')} \frac{R_k(t') \dot{Q}_j(t')}{P^2_{ik}(t')} \right) \bigg|_{t'=t_T} + \frac{1}{c^2 L} \left[ \frac{R_i(t') R_j(t') R_k(t')}{R^2(t')} \frac{\dot{Q}_j(t')}{P^2_{ik}(t')} \right] \bigg|_{t'=t_L} \right. \\
+ \int_{1/c_L}^{1/c_T} d\kappa \kappa^2 \left[ \left( \frac{3R_i(t') R_j(t')}{R^2(t')} - \delta_{ij} \right) \frac{R_k(t') \dot{Q}_j(t')}{P^2_{ik}(t')} \right] \bigg|_{t'=t_L} \right. \\
+ \frac{1}{c^2 T} \left[ \left( \frac{R_i(t') R_j(t')}{R^2(t')} \right) \frac{\dot{V}_m(t') R_m(t')}{P^2_{ik}(t')} R_k(t') + \left( 1 - \frac{V^2(t')}{c^2 T} \right) - \frac{P_L(t')}{c_L} V_k(t') \right] \\
- \frac{Q_j(t') R_i(t') R_k(t')}{R^2(t')} P_L(t') \left( \frac{\dot{V}_i(t') R_k(t')}{c_L P_L(t')} - \frac{Q_i(t') R_j(t')}{R^2(t')} \right) \left( \frac{\dot{V}_i(t') R_k(t')}{c_L P_L(t')} \right) \\
+ \frac{2Q_j(t') R_i(t') R_j(t') R_k(t')}{R^2(t')} P_L(t') \right. \\
+ \int_{1/c_L}^{1/c_T} d\kappa \kappa \left[ \left( \frac{3R_i(t') R_j(t')}{R^2(t')} \right) \frac{\dot{V}_m(t') R_m(t')}{P^2_{ik}(t')} R_k(t') \right. \\
+ \left( 1 - \frac{V^2(t')}{c^2 L} \right) - \frac{P_L(t')}{c_L} V_k(t') \right] \\
- \frac{3Q_j(t') R_i(t')}{R^2(t')} \left( \frac{\dot{V}_i(t') R_k(t')}{P_L(t')} \right) + \frac{3Q_j(t') R_i(t')}{R^2(t')} \left( \frac{\dot{V}_i(t') R_k(t')}{P_L(t')} \right) \\
+ \frac{6Q_j(t') R_i(t') R_j(t') R_k(t')}{R^2(t')} P_L(t') \right. \left. \bigg|_{t'=t_L} \bigg. \right\}. \tag{67}
\]

The fields (66) and (67) consist of near fields which are the nonradiation parts and they fall off as \(1/R^2\), and of far fields which are the radiation parts due to \(\dot{Q}\) and \(\dot{V}\) terms and they fall off as \(1/R\). Thus, there is a clear separation of \(\beta\) and \(\mathbf{v}\) into two parts each, one which involves radiation and goes to zero for \(\mathbf{V} = 0\) and \(\dot{q} = 0\), and one which involves only the velocity, \(\mathbf{V}\), and yields to the static field for a point force having \(\mathbf{V} = 0\):

\[
\beta_{ik}(r, t) = \beta^\text{nonr}_{ik}(r, t) + \beta^\text{rad}_{ik}(r, t) \tag{68}
\]

\[
v_i(r, t) = v^\text{nonr}_i(r, t) + v^\text{rad}_i(r, t). \tag{69}
\]

\(\beta^\text{nonr}\) and \(v^\text{nonr}\) are called the nonradiation or velocity-dependent fields and \(\beta^\text{rad}\) and \(v^\text{rad}\) are called the radiation fields or the acceleration fields. Using a little algebra the
nonradiation parts read

\[
v^\text{nonr}_i(r, t) = \frac{1}{4\pi\rho} \left\{ \frac{1}{c_T^2} \left[ \left( \delta_{ij} - \frac{R_i(t')R_j(t')}{R^2(t')} \right) \frac{Q_j(t')}{P_T^3(t')} \left( 1 - \frac{V^2(t')}{c_T^2} \right) V_m(t')R_m(t') - \frac{P_T(t')}{c_T} V^2(t') \right] + \right. \\
+ \frac{Q_j(t')}{R(t')P_T^2(t')} \left( V_i(t')R_j(t') + V_j(t')R_i(t') \right) - \frac{2Q_j(t')R_i(t')R_j(t')V_m(t')R_m(t')}{R^3(t')P_T^2(t')} \right\} \\
+ \frac{1}{c_L^2} \left[ \frac{R_i(t')R_j(t')}{R^2(t')} \frac{Q_j(t')}{P_L^3(t')} \left( 1 - \frac{V^2(t')}{c_L^2} \right) V_m(t')R_m(t') - \frac{P_L(t')}{c_L} V^2(t') \right] \\
- \frac{Q_j(t')}{R(t')P_L^2(t')} \left( V_i(t')R_j(t') + V_j(t')R_i(t') \right) + \frac{2Q_j(t')R_i(t')R_j(t')V_m(t')R_m(t')}{R^3(t')P_L^2(t')} \right\} \bigg|_{t'=t_T}
\]

and

\[
\beta^\text{nonr}_{ik}(r, t) = -\frac{1}{4\pi\rho} \left\{ \frac{1}{c_T^2} \left[ \left( \delta_{ij} - \frac{R_i(t')R_j(t')}{R^2(t')} \right) \frac{Q_j(t')}{P_T^3(t')} \left( 1 - \frac{V^2(t')}{c_T^2} \right) R_k(t') - \frac{P_T(t')}{c_T} V_k(t') \right] + \right. \\
+ \frac{Q_j(t')R_j(t')}{R^2(t')P_T^2(t')} \left( \delta_{ik} + \frac{V_i(t')R_k(t')}{c_TP_T(t')} \right) + \frac{Q_j(t')R_i(t')}{R^2(t')P_T(t')} \left( \delta_{jk} + \frac{V_j(t')R_k(t')}{c_TP_T(t')} \right) \\
- \frac{2Q_j(t')R_i(t')R_j(t')R_k(t')}{R^3(t')P_T^2(t')} \right\} \bigg|_{t'=t_T}
\]

\[
+ \frac{1}{c_L^2} \left[ \frac{R_i(t')R_j(t')}{R^2(t')} \frac{Q_j(t')}{P_L^3(t')} \left( 1 - \frac{V^2(t')}{c_L^2} \right) R_k(t') - \frac{P_L(t')}{c_L} V_k(t') \right] \\
- \frac{Q_j(t')R_j(t')}{R^2(t')P_L(t')} \left( \delta_{ik} + \frac{V_i(t')R_k(t')}{c_LP_L(t')} \right) - \frac{Q_j(t')R_i(t')}{R^2(t')P_L(t')} \left( \delta_{jk} + \frac{V_j(t')R_k(t')}{c_LP_L(t')} \right) \\
+ \frac{2Q_j(t')R_i(t')R_j(t')R_k(t')}{R^3(t')P_L^2(t')} \right\} \bigg|_{t'=t_L}
\]

\[
+ \int_{1/c_L}^{1/c_T} \frac{d\kappa}{c_T} \left[ \frac{3R_i(t')R_j(t')}{R^2(t')} - \delta_{ij} \right] \frac{Q_j(t')}{P_T^3(t')} \left( 1 - \kappa^2V^2(t') \right) R_k(t') - \kappa P_T(t') V_k(t') \right) \\
- \frac{3Q_j(t')R_j(t')}{R^2(t')P_T(t')} \left( \delta_{ik} + \frac{\kappa V_i(t')R_k(t')}{P_T(t')} \right) - \frac{3Q_j(t')R_i(t')}{R^2(t')P_T(t')} \left( \delta_{jk} + \frac{\kappa V_j(t')R_k(t')}{P_T(t')} \right) \\
+ \frac{6Q_j(t')R_i(t')R_j(t')R_k(t')}{R^3(t')P_T^2(t')} \bigg|_{t'=t_\kappa} \right\} \bigg|_{t'=t_T}.
\]

Using the relation (62), one verifies

\[
v^\text{nonr}_i(r, t) = -\left[ \beta^\text{nonr}_{ik}(r, t') V_k(t') \right] \bigg|_{t'=t_T}, \quad \text{for} \quad t_T = t_T, t_L, t_\kappa.
\]
On the other hand, the elastodynamic radiation fields are given by

\[
v_i^{\text{rad}}(r, t) = \frac{1}{4\pi\rho} \left\{ \frac{1}{c_T^2} \left[ \left( \delta_{ij} R(t') - \frac{R_i(t') R_j(t')}{R(t')} \right) \left( \frac{\dot{Q}_j(t')}{{P}_T^2(t')} + \frac{Q_j(t') \dot{V}_m(t') R_m(t')}{c_T P_T^3(t')} \right) \right] \right|_{t' = t_T} \\
+ \frac{1}{c_T^2} \left[ \frac{R_i(t') R_j(t')}{R(t')} \left( \frac{\dot{Q}_j(t')}{{P}_L^2(t')} + \frac{Q_j(t') \dot{V}_m(t') R_m(t')}{c_L P_L^3(t')} \right) \right] \right|_{t' = t_L} \\
+ \int_{1/c_L}^{1/c_T} d\kappa \left\{ \frac{3 R_i(t') R_j(t')}{R(t')} \left( \frac{\dot{Q}_j(t')}{{P}_T^2(t')} + \frac{Q_j(t') \dot{V}_m(t') R_m(t')}{c_T P_T^3(t')} \right) \right|_{t' = t} \}
\]

and

\[
\beta_i^{\text{rad}}(r, t) = \frac{1}{4\pi\rho} \left\{ \frac{1}{c_T^2} \left[ \left( \delta_{ij} R_k(t') - \frac{R_i(t') R_j(t') R_k(t')}{R^2(t')} \right) \left( \frac{\dot{Q}_j(t')}{{P}_T^2(t')} + \frac{Q_j(t') \dot{V}_m(t') R_m(t')}{c_T P_T^3(t')} \right) \right] \right|_{t' = t_T} \\
+ \frac{1}{c_T^2} \left[ \frac{R_i(t') R_j(t') R_k(t')}{R^2(t')} \left( \frac{\dot{Q}_j(t')}{{P}_L^2(t')} + \frac{Q_j(t') \dot{V}_m(t') R_m(t')}{c_L P_L^3(t')} \right) \right] \right|_{t' = t_L} \\
+ \int_{1/c_L}^{1/c_T} d\kappa \left\{ \frac{3 R_i(t') R_j(t') R_k(t')}{R^2(t')} \left( \frac{\dot{Q}_j(t')}{{P}_T^2(t')} + \frac{Q_j(t') \dot{V}_m(t') R_m(t')}{c_T P_T^3(t')} \right) \right|_{t' = t} \}
\]

The following relation holds for the radiation parts

\[
v_i^{\text{rad}}(r, t) = \left[ \beta_i^{\text{rad}}(r, t') R_k(t') / c R(t') \right] \right|_{t' = t}, \quad \text{for} \quad c = c_T, c_L, 1/\kappa.
\]

The elastodynamic radiation is caused by the time-change of the magnitude of the point force \( \dot{q} \) and the acceleration \( \ddot{V} \).

**Remark:** Some remarks concerning transonic and supersonic motions will be given. If the velocity of the moving point force is \( c_T < |\dot{V}| < c_L \), then the motion is transonic. Since the source of the elastodynamic fields travels faster than the elastodynamic waves, some sort of shock front is set up. The disturbances initiated at each point of its track by the moving point force arrive simultaneously at the surface of a right circular cone whose vertex is at the point force. The semi-vertical angle of this cone is: \( \theta_T = \sin^{-1}(c_T/|\dot{V}|) \). This cone is the Mach cone with respect to \( c_T \). The field depending on \( c_T \) at any point is zero outside this Mach cone. If the velocity of the moving point force is \( c_L < |\dot{V}| \), then the motion is supersonic. Another shock front and Mach cone with respect to \( c_L \) are built. The semi-vertical angle of this Mach cone is: \( \theta_L = \sin^{-1}(c_L/|\dot{V}|) \). Outside the Mach cones the elastodynamic fields are zero and inside the elastodynamic fields can be obtained from the Liénard-Wiechert potentials with modified retarded times. For the transonic and supersonic motions the retarded times \( t_T \) and \( t_L \), respectively, have more than one solution (at least two solutions), each. Depending on the specific motion, any number of retarded times, not just one transversal retarded time and one longitudinal retarded time, may be associated with a given time \( t \). For points outside the Mach cone no retarded time exists and the fields must be zero. For points that are inside the Mach cone many retarded times may exist (see, e.g., [3, 23]). Also we want to mention that the non-uniform motion of a supersonic screw dislocation has been analyzed in [31].
3.4 The Stokes solution as limit of a non-uniformly moving point force

If the position of the point force is fixed that means that \( s \) is time-independent and therefore \( \mathbf{V} = 0 \), we recover from the displacement (61) the famous Stokes solution \([32]\) of a concentrated point force with time-dependent magnitude (e.g. \([27, 33, 34]\))

\[
\begin{align*}
    u_i(r, t) &= \frac{1}{4\pi \rho R} \left\{ \frac{1}{c_T^2} \left( \delta_{ij} - \frac{R_i R_j}{R^2} \right) Q_j(t - R/c_T) + \frac{1}{c_L^2} \frac{R_i R_j}{R^2} Q_j(t - R/c_L) \\
    &+ \left( \frac{3R_i R_j}{R^2} - \delta_{ij} \right) \int_{1/c_L}^{1/c_T} \kappa \dot{Q}_j(t - \kappa R) \, d\kappa \right\} .
\end{align*}
\]

The first terms in Eq. (76) are usually called the far-field terms since they behave as \( 1/R \) and the last term in Eq. (76) is called the near-field term. From Eq. (66), we obtain the particle velocity vector

\[
\begin{align*}
    v_i(r, t) &= \frac{1}{4\pi \rho R} \left\{ \frac{1}{c_T^2} \left( \delta_{ij} - \frac{R_i R_j}{R^2} \right) \dot{Q}_j(t - R/c_T) + \frac{1}{c_L^2} \frac{R_i R_j}{R^2} \dot{Q}_j(t - R/c_L) \\
    &+ \left( \frac{3R_i R_j}{R^2} - \delta_{ij} \right) \int_{1/c_L}^{1/c_T} \kappa \dot{Q}_j(t - \kappa R) \, d\kappa \right\} .
\end{align*}
\]

From Eq. (67) and after some mathematical manipulations we find the corresponding displacement gradient of the Stokes solution (e.g. \([27, 33, 34]\))

\[
\beta_{ik}(r, t) = -\frac{1}{4\pi \rho} \left\{ \frac{3}{R^5} \left( 5R_i R_j R_k - \delta_{ij} R_k + \delta_{jk} R_i + \delta_{ik} R_j \right) \right\} \int_{1/c_L}^{1/c_T} \kappa \dot{Q}_j(t - \kappa R) \, d\kappa \\
+ \left( \frac{6R_i R_j R_k}{R^5} - \delta_{ij} R_k + \delta_{jk} R_i + \delta_{ik} R_j \right) \left[ \frac{1}{c_L^2} Q_j(t - R/c_L) - \frac{1}{c_T^2} Q_j(t - R/c_T) \right] \\
+ \frac{\delta_{ij} R_k}{c_T^2 R^3} \left[ Q_j(t - R/c_T) + \frac{R}{c_T} \dot{Q}_j(t - R/c_T) \right] \\
+ \frac{R_i R_j R_k}{R^4} \left[ \frac{1}{c_L^2} \dot{Q}_j(t - R/c_L) - \frac{1}{c_T^2} \dot{Q}_j(t - R/c_T) \right] \right\} .
\]

Thus, Eqs. (61) and (67) give the correct Stokes solution as limit. It can be seen that the \( Q \)-terms in Eq. (78) behave as \( 1/R \) (far-field terms or radiation terms) and the \( \dot{Q} \)-terms behave as \( 1/R^2 \) (near-field terms). Also, it can be seen that the third line in Eq. (78) is analogous to Eq. (37). Thus, the solutions (61), (66) and (67) are the generalization of the Stokes solution toward the non-uniform motion.

**Remark:** Ben-Menahem and Singh \([35]\) mentioned that Stokes \([32]\) had conceived with his solution the first mathematical model of an earthquake. In this sense our solution of a non-uniformly point force may serve as a mathematical model of the non-uniform motion of sources (senders) of elastodynamic waves (P- and S-seismic waves) in seismology.
The same class of problems considered in this paper for elastic waves in an unbounded medium might be more significant, but also more complicated, for point sources moving over the surface of a half-space and the interaction with external or coupled fields (see, e.g., [24]).

### 3.5 Time-harmonic point force

Here we consider the special case in which the body force is harmonic in time. For a time-harmonic point force with the spatial part

\[ F_j(r) = Q_j \delta(r - r'), \quad (79) \]

Eq. (56) gives directly

\[
\begin{align*}
u_i(r, t) &= \text{Re} \left[ \frac{Q_j e^{-i\omega t}}{4\pi \rho} \left\{ \frac{1}{c_T^2} \left( \frac{\delta_{ij}}{R} - \frac{R_i R_j}{R^3} \right) e^{i\omega R/c_T} + \frac{1}{c_L^2} \frac{R_i R_j}{R^3} e^{i\omega R/c_L} \ight. \\
&\quad \left. + \frac{1}{\omega^2 R^2} \left( 3 \frac{R_i R_j}{R^3} - \frac{\delta_{ij}}{R} \right) \left[ e^{i\omega R/c_T} \left( 1 - \frac{i\omega R}{c_T} \right) - e^{i\omega R/c_L} \left( 1 - \frac{i\omega R}{c_L} \right) \right] \right\} \right], \quad (80)
\end{align*}
\]

which is in agreement with the expression given in [27, 28]. Moreover, the velocity vector reads \( v_i = -\text{Re} \left[ i\omega u_i \right] \), and Eq. (58) gives the elastic distortion

\[
\beta_{ik}(r, t) = \text{Re} \left[ -\frac{Q_j e^{-i\omega t}}{4\pi \rho} \left\{ \left( \frac{\delta_{ij} R_k + \delta_{jk} R_i + \delta_{ik} R_j}{R^3} - 6 \frac{R_i R_j R_k}{R^5} \right) \left( \frac{1}{c_T^2} e^{i\omega R/c_T} - \frac{1}{c_L^2} e^{i\omega R/c_L} \right) \ight. \\
&\quad \left. + \frac{\delta_{ij} R_k}{c_T^2 R^5} e^{i\omega R/c_T} \left( 1 - \frac{i\omega R}{c_T} \right) + i\omega \frac{R_i R_j R_k}{R^5} \left( \frac{1}{c_T^2} e^{i\omega R/c_T} - \frac{1}{c_L^2} e^{i\omega R/c_L} \right) \ight. \\
&\quad \left. - \frac{3}{\omega^2 R^2} \left( \frac{\delta_{ij} R_k + \delta_{jk} R_i + \delta_{ik} R_j}{R^3} - 5 \frac{R_i R_j R_k}{R^5} \right) \left[ e^{i\omega R/c_T} \left( 1 - \frac{i\omega R}{c_T} \right) \ight. \\
&\quad \left. + e^{i\omega R/c_L} \left( 1 - \frac{i\omega R}{c_L} \right) \right] \right\} \right]. \quad (81)
\]

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### Appendix: Derivatives at the retarded time

Here we give some useful relations of derivatives of quantities depending on the retarded time, which is the unique solution of the relation

\[ t - t_r = |r - s(t_r)|/c = 0. \quad (A.1) \]
First of all, we carry out the time derivatives, which are not trivial because of the subtle relation between present and retarded time (see also [4]):

\[
\frac{\partial t'}{\partial t} \Big|_{t'=t_r} = \left. \frac{R(t')}{P(t')} \right|_{t'=t_r} \tag{A.2}
\]

\[
\partial_t [Q_j(t')] \Big|_{t'=t_r} = \left. \frac{\partial t'}{\partial t} \frac{\partial Q_j(t')}{\partial t'} \right|_{t'=t_r} = \left. \frac{R(t')}{P(t')} \dot{Q}_j(t') \right|_{t'=t_r} \tag{A.3}
\]

\[
\left. \partial_t \left[ \frac{1}{P(t')} \right] \right|_{t'=t_r} = \left. \frac{1}{P^3(t')} \left( \left( \dot{V}_m(t') R_m(t') - V^2(t') \right) \frac{R(t')}{c} + V_m(t') R_m(t') \right) \right|_{t'=t_r} \tag{A.4}
\]

\[
\left. \partial_t \left[ \frac{R_k(t')}{R(t') P(t')} \right] \right|_{t'=t_r} = \left. \frac{R_k(t')}{R(t') P^3(t')} \left( \left( \dot{V}_m(t') R_m(t') - V^2(t') \right) \frac{R(t')}{c} + V_m(t') R_m(t') \right) \right|_{t'=t_r}
+ \left. \frac{1}{P^2(t')} \left( \dot{V}_m(t') R_m(t') \frac{R_k(t')}{R^2(t')} - V_k(t') \right) \right|_{t'=t_r} \tag{A.5}
\]

\[
\left. \partial_t \left[ \frac{R_i(t') R_j(t')}{R^2(t') P(t')} \right] \right|_{t'=t_r} = \left. \frac{R_i(t') R_j(t')}{R^2(t') P^3(t')} \left( \left( \dot{V}_m(t') R_m(t') - V^2(t') \right) \frac{R(t')}{c} + V_m(t') R_m(t') \right) \right|_{t'=t_r}
- \left. \frac{1}{R(t') P^2(t')} \left( \dot{V}_i(t') R_j(t') + V_j(t') R_i(t') \right) \right|_{t'=t_r}
+ \left. \frac{2 R_i(t') R_j(t') V_m(t') R_m(t')}{R^3(t') P^2(t')} \right|_{t'=t_r} \tag{A.6}
\]

\[
\left. \partial_t \left[ \frac{R_i(t') R_j(t') R_k(t')}{R^3(t') P(t')} \right] \right|_{t'=t_r} = \left. \frac{R_i(t') R_j(t') R_k(t')}{R^3(t') P^3(t')} \left( \left( \dot{V}_m(t') R_m(t') - V^2(t') \right) \frac{R(t')}{c} + V_m(t') R_m(t') \right) \right|_{t'=t_r}
- \left. \frac{1}{R^2(t') P^2(t')} \left( \dot{V}_i(t') R_j(t') R_k(t') + V_j(t') R_k(t') R_i(t') + V_k(t') R_i(t') R_j(t') \right) \right|_{t'=t_r}
+ \left. \frac{3 R_i(t') R_j(t') R_k(t') V_m(t') R_m(t')}{R^4(t') P^2(t')} \right|_{t'=t_r}. \tag{A.7}
\]

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