Hamiltonian embedding of the massive Yang-Mills theory and the generalized Stückelberg formalism

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Abstract

Using the general notions of Batalin, Fradkin, Fradkina and Tyutin to convert second class systems into first class ones, we present a gauge invariant formulation of the massive Yang-Mills theory by embedding it in an extended phase space. The infinite set of correction terms necessary for obtaining the involutive constraints and Hamiltonian is explicitly computed and expressed in a closed form. It is also shown that the extra fields introduced in the correction terms are exactly identified with the auxiliary scalars used in the generalized Stückelberg formalism for converting a gauge noninvariant Lagrangian into a gauge invariant form.

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1 Introduction

The foundations for a Hamiltonian formulation of constrained systems were originally laid down by Dirac [1] whose work still remains fundamental to our understanding of the subject. For second class systems, however, this approach poses problems which are related to ambiguities that may arise in the transition of (classical) Dirac brackets to (quantum) commutators. During recent years a viable and powerful alternative for discussing such systems has been developed in a series of papers by Batalin, Fradkin, Fradkina and Tyutin (BFFT) [2, 3]. The basic idea is to convert, using an iterative prescription, the second class system into a first class one by enlarging the original phase space. Once this is achieved it is possible to exploit the existing machinery available for quantising first class theories [4]. The use of Dirac brackets is completely avoided.

The BFFT method, or modified versions of it [5, 6, 7, 8], have been used by several authors [9] to discuss the formulation of specific examples of second class theories. These have in turn provided an insight into the general formalism which cannot be otherwise gained. But such analyses has been mostly confined to Abelian models where the constraints are linear. For nonlinear constraints which occur naturally in non-Abelian theories, the situation is rather complicated and explicit computations [10] are quite sparse and, at best, limited to a simple group like $SU(2)$. The reason for this complication is that, contrary to the example of linear constraints, the iterative process in the BFFT approach may not terminate. Nevertheless it was shown by one of us [11] that the nonlinear constraints occurring in the nonlinear sigma models could be tackled by a clever choice of the generating matrices which forces the iterative process to stop at the first step. Unfortunately, such a simplification cannot be done in general [12]. Then the iterative scheme has to be continued indefinitely in which case it is not clear whether consistent solutions for all the steps can be obtained from which meaningful closed form expressions for the involutive constraints and Hamiltonian may be determined.

In this paper we systematically develop an algorithm within the BFFT approach by which all the iterative corrections necessary to transform the original second class nonlinear constraints into strongly involutive constraints can be explicitly computed and exhibited in a closed form. We shall discuss this in Section 3 within the context of the massive Yang-Mills theory which is a classic example of a second class system with nonlinear constraints. The iterative corrections are explicitly computed and form an infinite set which is expressed in a closed (exponential-like) form. We then use a modified prescription, which is a departure from the conventional BFFT ideas, of computing the involutive Hamiltonian. Once again the iterative terms can be combined to yield an exponential-like series. The modified prescription mentioned above provides great technical simplifications. This has been outlined in Section 2 which also contains a short review of the BFFT approach. In our entire analysis there is no restriction on either the dimensionality of space time or the specific non-Abelian gauge group employed in the model. Since the gauge invariance of the massive Yang-Mills theory is broken by the mass term, it will be clear
that our analysis is applicable to all such generic second class systems whose lack of
gauge invariance is attributed to identical reasons. In Section 4 we first reconsider
the generalized St"uckelberg formalism [13] of converting the massive Yang-Mills La-
grangian into a gauge invariant form by the introduction of auxiliary scalars. Next,
the canonical formalism of the gauge invariant Lagrangian is developed in details.
Finally, by a series of remarkable algebraic manipulations it is shown that these
scalars are exactly identified with the additional fields that occur in the correction
terms found in the BFFT approach. This establishes a one-to-one correspondence
between the Lagrangian and Hamiltonian embedding prescriptions of converting
gauge noninvariant into gauge invariant systems. Some appendices are included
for elaborating calculations relevant for Section 4. Our conclusions are given in
Section 5.

2 Brief review of the BFFT formalism

In this section we present a short review of the Hamiltonian formalism, which will
be relevant for the subsequent discussion, for converting second class systems into
true gauge (i.e. first class) systems by enlarging the phase space. Towards the
end we shall also discuss a deviation from the conventional approach that provides
considerable technical simplification.

Let us take a system described by a Hamiltonian $H_0$ in a phase-space with vari-
ables $(q^i, p_i)$ where $i$ runs from 1 to $N$. For simplicity these variables are assumed
to be bosonic (extension to include fermionic degrees of freedom and to the con-
tinuous case can be done in a straightforward way). It is also supposed that there
exists second-class constraints only since this is the case that will be investigated.
Denoting them by $T_a$, with $a = 1, \ldots, M < 2N$, we have

$$\{T_a, T_b\} = \Delta_{ab},$$

(2.1)

where $\det(\Delta_{ab}) \neq 0$.

The first objective is to transform these second-class constraints into first-class
ones. Towards this goal auxiliary variables $\eta^a$ are introduced, one for each second-
class constraint (the connection between the number of constraints and the new
variables in a one-to-one correlation is to keep the same number of the physical
degrees of freedom in the resulting extended theory), which satisfy a symplectic
algebra,

$$\{\eta^a, \eta^b\} = \omega^{ab},$$

(2.2)

where $\omega^{ab}$ is a constant quantity with $\det(\omega^{ab}) \neq 0$. The first class constraints are
now defined by,
\( \tilde{T}_a = \tilde{T}_a(q, p; \eta) \), \hspace{1cm} (2.3)

and satisfy the boundary condition

\( \tilde{T}_a(q, p; 0) = T_a(q, p) \). \hspace{1cm} (2.4)

A characteristic of these new constraints is that they are assumed to be strongly involutive, i.e.

\[ \{ \tilde{T}_a, \tilde{T}_b \} = 0. \] \hspace{1cm} (2.5)

The solution of (2.5) can be achieved by considering an expansion of \( \tilde{T}_a \), as

\[ \tilde{T}_a = \sum_{n=0}^{\infty} T_a^{(n)}, \] \hspace{1cm} (2.6)

where \( T_a^{(n)} \) is a term of order \( n \) in \( \eta \). Compatibility with the boundary condition (2.4) requires that

\[ T_a^{(0)} = T_a. \] \hspace{1cm} (2.7)

The replacement of (2.6) into (2.5) leads to a set of recursive relations, one for each coefficient of \( \eta^n \). We explicitly list the equations for \( n = 0, 1, 2 \),

\[ \{ T_a^{(0)}, T_b^{(0)} \}_{(q,p)} + \{ T_a^{(1)}, T_b^{(1)} \}_{(\eta)} = 0, \] \hspace{1cm} (2.8)

\[ \{ T_a^{(0)}, T_b^{(1)} \}_{(q,p)} + \{ T_a^{(1)}, T_b^{(0)} \}_{(q,p)} + \{ T_a^{(1)}, T_b^{(2)} \}_{(\eta)} + \{ T_a^{(2)}, T_b^{(1)} \}_{(\eta)} = 0, \] \hspace{1cm} (2.9)

\[ \{ T_a^{(0)}, T_b^{(2)} \}_{(q,p)} + \{ T_a^{(1)}, T_b^{(1)} \}_{(q,p)} + \{ T_a^{(2)}, T_b^{(0)} \}_{(q,p)} + \{ T_a^{(1)}, T_b^{(3)} \}_{(\eta)} + \{ T_a^{(2)}, T_b^{(2)} \}_{(\eta)} + \{ T_a^{(3)}, T_b^{(1)} \}_{(\eta)} = 0, \] \hspace{1cm} (2.10)

The notations \( \{ , \}_{(q,p)} \) and \( \{ , \}_{(\eta)} \) used above represent the parts of the Poisson bracket \( \{ , \} \) relative to the variables \( (q, p) \) and \( (\eta) \). \footnote{Sometimes in explicit calculations if no suffix appears in the definition of the Poisson bracket, it will imply an evaluation relative to the initial phase space variables \( (q, p) \).}

The above equations are used iteratively to obtain the corrections \( T^{(n)} (n \geq 1) \). Equation (2.8) shall give \( T^{(1)} \). With this result and (2.3), one calculates \( T^{(2)} \), and so on. Since \( T^{(1)} \) is linear in \( \eta \) we may write

\[ T^{(1)}_a = X_{ab}(q, p) \eta^b. \] \hspace{1cm} (2.11)
Introducing this expression into (2.8) and using the boundary condition (2.4), as well as (2.1) and (2.2), we get

\[ \Delta_{ab} + X_{ac} \omega^{cd} X_{bd} = 0. \]  

We notice that this equation contains two unknowns \( X_{ab} \) and \( \omega^{ab} \). Usually, first of all \( \omega^{ab} \) is chosen in such a way that the new variables are unconstrained. It is opportune to mention that it is not always possible to make such a choice \[14\]. In consequence, the consistency of the method requires an introduction of other new variables in order to transform these constraints also into first-class. This may lead to an endless process. However, it is important to emphasize that \( \omega^{ab} \) can be fixed anyway.

After fixing \( \omega^{ab} \), we pass to consider the coefficients \( X_{ab} \). They cannot be obtained unambiguously since, even after fixing \( \omega^{ab} \), expression (2.12) leads to less equations than variables. The choice of \( X's \) has therefore to be done in a convenient way \[11\].

The knowledge of \( X_{ab} \) permits us to obtain \( T^{(1)}_a \). If \( X_{ab} \) does not depend on \((q,p)\), it is easily seen that \( T_a + T^{(1)}_a \) is already strongly involutive and we succeed in obtaining \( \tilde{T}_a \). This is what happens for systems with linear constraints. For nonlinear constraints, on the other hand, \( X_{ab} \) becomes variable dependent which necessitates the analysis to be pursued beyond the first iterative step. All the subsequent corrections must be explicitly computed, the knowledge of \( T^{(n)}_a (n = 0, 1, 2, \ldots n) \) leading to the evaluation of \( T^{(n+1)}_a \) from the recursive relations. Once again the importance of choosing the proper solution for \( X_{ab} \) becomes apparent otherwise the series of corrections cannot be put in a closed form and the expression for the involutive constraints becomes unintelligible and uninteresting.

Another point in the Hamiltonian formalism is that any dynamic function \( A(q,p) \) (for instance, the Hamiltonian) has also to be properly modified in order to be strongly involutive with the first-class constraints \( \tilde{T}_a \). Denoting the modified quantity by \( \tilde{A}(q,p; \eta) \), we then have

\[ \{ \tilde{T}_a, \tilde{A} \} = 0. \]  

(2.13)

In addition, \( \tilde{A} \) has also to satisfy the boundary condition

\[ \tilde{A}(q,p; 0) = A(q,p). \]  

(2.14)

To obtain \( \tilde{A} \) an expansion analogous to (2.6) is considered,

\[ \tilde{A} = \sum_{n=0}^{\infty} A^{(n)}, \]  

(2.15)

where \( A^{(n)} \) is also a term of order \( n \) in \( \eta's \). Consequently, compatibility with (2.14) requires that
\[ A^{(0)} = A. \]  

(2.16)

The combination of (2.6), (2.13) and (2.15) gives

\[
\begin{align*}
\{ T_a^{(0)}, A^{(0)} \}_{(q,p)} + \{ T_a^{(1)}, A^{(1)} \}_{(\eta)} &= 0, \\
\{ T_a^{(0)}, A^{(1)} \}_{(q,p)} + \{ T_a^{(1)}, A^{(0)} \}_{(q,p)} + \{ T_a^{(1)}, A^{(2)} \}_{(\eta)} \\
+ \{ T_a^{(2)}, A^{(1)} \}_{(\eta)} &= 0, \\
\{ T_a^{(0)}, A^{(2)} \}_{(q,p)} + \{ T_a^{(1)}, A^{(1)} \}_{(q,p)} + \{ T_a^{(2)}, A^{(0)} \}_{(q,p)} \\
+ \{ T_a^{(2)}, A^{(1)} \}_{(\eta)} + \{ T_a^{(2)}, A^{(2)} \}_{(\eta)} \\
+ \{ T_a^{(3)}, A^{(1)} \}_{(\eta)} &= 0,
\end{align*}
\]

(2.17)

which correspond to the coefficients of the powers \( \eta^0, \eta^1, \eta^2, \) etc., respectively. The expression (2.17) above gives us \( A^{(1)} \)

\[ A^{(1)} = -\eta^a \omega_{ab} X^{bc} \{ T_c, A \}, \]

(2.20)

where \( \omega_{ab} \) and \( X^{ab} \) are the inverses of \( \omega^{ab} \) and \( X_{ab} \).

It was earlier seen that \( T_a + T_a^{(1)} \) was strongly involutive if the coefficients \( X_{ab} \) do not depend on \( (q,p) \). However, the same argument does not necessarily apply in this case. Usually we have to calculate other corrections to obtain the final \( \tilde{A} \). Let us discuss how this can be systematically done. We consider the general case first. The correction \( A^{(2)} \) comes from equation (2.18), that we conveniently rewrite as

\[ \{ T_a^{(1)}, A^{(2)} \}_{(\eta)} = -G_a^{(1)}, \]

(2.21)

where

\[ G_a^{(1)} = \{ T_a, A^{(1)} \}_{(q,p)} + \{ T_a^{(1)}, A \}_{(q,p)} + \{ T_a^{(2)}, A^{(1)} \}_{(\eta)} . \]

(2.22)

Thus

\[ A^{(2)} = -\frac{1}{2} \eta^a \omega_{ab} X^{bc} G_c^{(1)} . \]

(2.23)

In the same way, other terms can be obtained. The final general expression reads

\[ A^{(n+1)} = -\frac{1}{n+1} \eta^a \omega_{ab} X^{bc} G_c^{(n)} , \]

(2.24)
where

\[
G^{(n)}_a = \sum_{m=0}^{n} \{T^{(n-m)}_a, A^{(m)} \}_{(q,p)} + \sum_{m=0}^{n-2} \{T^{(n-m)}_a, A^{(m+2)} \}_{(\eta)} + \{T^{(n+1)}_a, A^{(1)} \}_{(\eta)}.
\]

(2.25)

Although it is possible to convert any dynamical variable in the original phase space into its involutive form by the above method, there may be technical problems in carrying out this construction particularly if one considers non-Abelian theories. In such cases the relevant variable is already quite complicated and this process may lead to an arcane structure which would not be illuminating. We suggest the following simplification which has been employed, though in a modified form, earlier for abelian models [6, 8]. The basic idea is to obtain the involutive forms of the initial fields \(\tilde{q}\) and \(\tilde{p}\). This can be derived from the previous analysis. Denoting these by \(\tilde{q}\) and \(\tilde{p}\) so that,

\[
\{\tilde{T}, \tilde{q}\} = \{\tilde{T}, \tilde{p}\} = 0.
\]

(2.26)

Now any function of \(\tilde{q}\) and \(\tilde{p}\) will also be strongly involutive since,

\[
\{\tilde{T}, \tilde{F}(\tilde{q}, \tilde{p})\} = \{\tilde{T}, \tilde{q}\} \frac{\partial \tilde{F}}{\partial \tilde{q}} + \{\tilde{T}, \tilde{p}\} \frac{\partial \tilde{F}}{\partial \tilde{p}} = 0.
\]

(2.27)

Thus if we take any dynamical variable in the original phase space, its involutive form can be obtained by the replacement,

\[
F(q, p) \rightarrow F(\tilde{q}, \tilde{p}) = \tilde{F}(\tilde{q}, \tilde{p}).
\]

(2.28)

It is obvious that the initial boundary condition in the BFFT process, namely, the reduction of the involutive function to the original function when the new fields are set to zero, remains preserved.

### 3 Involutive constraints and Hamiltonian in the massive Yang-Mills theory

The massive Yang-Mills theory is defined by the Lagrangian density,

\[
\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \frac{1}{2} m^2 A^a_A^a A^{a\mu},
\]

(3.1)
where the following conventions and notations will be used,

\begin{align*}
F_{\mu\nu}^a &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu, \\
[T^a, T^b] &= i f^{abc} T^c, \\
\text{tr} (T^a T^b) &= \frac{1}{2} \delta^{ab}, \\
(T^a)^{bc} &= i f^{abc}.
\end{align*}

The canonical momentum conjugated to $A^a_\mu$ reads,

\begin{equation}
\pi^a_\mu = \frac{\partial L}{\partial \dot{A}^a_\mu} = -F^a_{0\mu}.
\end{equation}

Hence, $\pi^a_0$ is a primary constraint,

\begin{equation}
T^a_1 = \pi^a_0 \approx 0.
\end{equation}

In order to look for secondary constraints, we construct the canonical Hamiltonian density,

\begin{align*}
\mathcal{H}_c &= \pi^{a\mu} \dot{A}^a_\mu - L, \\
&= \pi^{a0} \dot{A}^a_0 - \frac{1}{2} \pi^{ai} \pi^a_i + \pi^{ai} \partial_i A^a_0 - g f^{abc} \pi^{ai} A^b_0 A^c_i \\
&\quad + \frac{1}{4} F^a_{ij} F^{aij} - \frac{1}{2} m^2 A^a_0 A^{a0} - \frac{1}{2} m^2 A^a_i A^{ai}. \tag{3.5}
\end{align*}

The total Hamiltonian \cite{footnote1} is given by,

\begin{equation}
H_T = \int d^3 x \left( \mathcal{H}_c + \lambda^a \pi^a_0 \right). \tag{3.6}
\end{equation}

In Eq. (3.6) note that the term $\pi^{a0} \dot{A}^a_0$ that appears in (3.3) has been absorbed in the $\lambda^a \pi^a_0$ term by a redefinition of the Lagrange multiplier $\lambda^a$. Consequently, $\mathcal{H}_c$ occurring in (3.6) differs from (3.3) by this piece. The consistency condition for the primary constraint,

\begin{align*}
\{ \pi^a_0 (x), H_T \} &= \partial_i \pi^{ai} + g f^{abc} A^b_i \pi^{ci} + m^2 A^a_0, \\
&= (D_i \pi^a)^i + m^2 A^a_0 \approx 0. \tag{3.7}
\end{align*}

yields a secondary constraint,
\[ T_2^a = (D_i \pi^i)^a + m^2 A_0^a \approx 0. \] (3.8)

There are no more constraints since the Poisson algebra of the constraints \( T_1^a \) and \( T_2^a \) is noninvolutive,

\[
\{ T_1^a(x), T_1^b(y) \} = 0, \tag{3.9}
\]
\[
\{ T_1^a(x), T_2^b(y) \} = -m^2 \delta^{ab} \delta(x - y), \tag{3.10}
\]
\[
\{ T_2^a(x), T_2^b(y) \} = g f^{abc} (D_i \pi^i) c \delta(x - y). \tag{3.11}
\]

The quantities \( \Delta_{AB}^{ab}(x, y) \) \((A, B = 1, 2)\) introduced in Eq. (2.1) are therefore given by,

\[
\Delta_{11}^{ab}(x, y) = 0, \\
\Delta_{12}^{ab}(x, y) = -m^2 \delta^{ab} (x - y) - \Delta_{21}^{ab}(x, y), \\
\Delta_{22}^{ab}(x, y) = g f^{abc} (D_i \pi^i) c \delta(x - y). \tag{3.12}
\]

Let us now extend the phase space by introducing the set of new variables \((\eta^{1a}, \eta^{2a})\). We consider them as canonical, i.e.,

\[
\{ \eta^{1a}(x), \eta^{2b}(y) \} = \delta^{ab} \delta(x - y), \\
\{ \eta^{1a}(x), \eta^{1b}(y) \} = 0 = \{ \eta^{2a}(x), \eta^{2b}(y) \}. \tag{3.13}
\]

The symplectic matrix \( \omega \) defined in Eq. (2.2) therefore has the structure,

\[
\left( \omega^{aA bB}(x, y) \right) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \delta^{ab} \delta(x - y). \tag{3.14}
\]

To calculate the first correction of the constraints we have to solve Eq. (2.12), which is now written as,

\[
\Delta_{AB}^{ab}(x, y) + \int dzdz' X_{AC}^{ac}(x, z) \omega^{CD} (z, z') X_{BD}^{bd}(y, z') = 0. \tag{3.15}
\]

Using (3.12) and (3.14), three independent relations from (3.13) are obtained. \(^2\)

\(^2\)All algebra will be implemented at equal times. Furthermore, the three dimensional delta function \( \delta^3(\vec{x} - \vec{y}) \) will be written simply as \( \delta(x - y) \).

\(^3\)Detailed reference to the arguments of \( X \) and the integral occurring in (3.13) will be subsequently omitted.
\( A = 1, B = 1 \implies X_{11}^{ac}X_{12}^{bc} - X_{11}^{bc}X_{12}^{ac} = 0, \quad (3.16) \)

\( A = 1, B = 2 \implies X_{11}^{ac}X_{22}^{bc} - X_{12}^{ac}X_{21}^{bc} = m^2 \delta^{ab}\delta(x-y), \quad (3.17) \)

\( A = 2, B = 2 \implies X_{21}^{ac}X_{22}^{bc} - X_{22}^{ac}X_{21}^{bc} = -gf^{abc}(D_i\pi^i)^c\delta(x-y), \quad (3.18) \)

As already discussed the number of equations is less than the number of unknowns. Consequently there is an arbitrariness in the solutions even after \( \omega \) has been fixed. At this point it is important to carefully choose the solutions for the variables \( X \) so that the subsequent algebra is simplified. From the last equation in the above set the following choice is practically spelled out,

\[
X_{21}^{ac} = \delta^{ac}\delta(x-z), \\
X_{22}^{bc} = \frac{1}{2}gf^{bcd}(D_i\pi^i)^d\delta(x-z). \quad (3.19)
\]

Using (3.16), (3.17) and the above solutions allows us to conclude the results for the remaining variables so that the complete \( X \) matrix has the structure,

\[
\left( X_{AB}^{ab}(x,y) \right) = \begin{pmatrix} 0 & -m^2\delta^{ab} \\ \delta^{ab} & \frac{1}{2}gf^{abc}(D_i\pi^i)^c \end{pmatrix} \delta(x-y). \quad (3.21)
\]

Inserting the expressions for the elements of \( X \) in Eq. (2.11) immediately leads to the first corrections,

\[
T_1^{(1)a} = -m^2\eta^{2a}, \\
T_2^{(1)a} = \eta^{1a} + \frac{1}{2}gf^{abc}\eta^{2b}(D_i\pi^i)^c, \\
= \eta^{1a} + \frac{1}{2}g\bar{\eta}^{ac}(D_i\pi^i)^c, \quad (3.22)
\]

where a compact notation is invoked which will prove highly useful,

\[
\bar{\eta}^{ac} = f^{abc}\eta^{2b}. \quad (3.23)
\]

The next task is to evaluate the second correction. This can be done by using Eq. (2.9) which, for the present model, takes the form,

\[
\{T_A^{(1)a}, T_B^{(1)b}\}_{(A,\pi)} + \{T_A^{(1)a}, T_B^{(2)b}\}_{(\eta)} + \{T_A^{(2)a}, T_B^{(1)b}\}_{(\eta)} + \{T_A^{(2)a}, T_B^{(2)b}\}_{(\eta)} = 0. \quad (3.24)
\]

Exploiting the expressions for the original constraints (3.4), (3.8) and the first corrections (3.22) in the above equation leads to the following conditions for distinct values of \( A \) and \( B \),

\[
T_A^{(1)a}, T_B^{(1)b}_{(A,\pi)} = T_A^{(2)a}, T_B^{(2)b}_{(\eta)}. \quad (3.25)
\]
\[ A = 1, B = 1 \]
\[ \{ \eta^{2a}, T^{(2)b}_1 \}_{(\eta)} + \{ T^{(2)a}_1, \eta^{2b} \}_{(\eta)} = 0, \quad (3.25) \]

\[ A = 1, B = 2 \]
\[ m^2 \{ \eta^{2a}, T^{(2)b}_2 \}_{(\eta)} + \{ T^{(2)a}_1, \eta^{1b} - \frac{1}{2} g \bar{\eta}^{bd} (D_i \pi^i)^d \}_{(\eta)} = 0, \quad (3.26) \]

\[ A = 2, B = 2 \]
\[ \{(D_i \pi^i)^a + m^2 A^a_0, \eta^{1b} + \frac{1}{2} g \bar{\eta}^{bd} (D_j \pi^j)^d \} \\
+ \{ \eta^{1a} + \frac{1}{2} g \bar{\eta}^{ad} (D_i \pi^i)^d, (D_j \pi^j)^b + m^2 A^b_0 \} \\
+ \{ \eta^{1a} + \frac{1}{2} g \bar{\eta}^{ad} (D_i \pi^i)^d, T^{(2)b}_2 \}_{(\eta)} \\
+ \{ T^{(2)a}_2, \eta^{1b} + \frac{1}{2} g \bar{\eta}^{bd} (D_j \pi^j)^d \}_{(\eta)} = 0. \quad (3.27) \]

The above relation (3.27) can be further simplified to yield,
\[ \{ \eta^{1a} + \frac{1}{2} g \bar{\eta}^{ad} (D_i \pi^i)^d, T^{(2)b}_2 \}_{(\eta)} + \{ T^{(2)a}_2, \eta^{1b} + \frac{1}{2} g \bar{\eta}^{bd} (D_j \pi^j)^d \}_{(\eta)} \]
\[ = -\frac{1}{2} g^2 f^{abcd} \bar{\eta}^{de} (D_i \pi^i)^e \delta(x - y). \quad (3.28) \]

It is evident that by choosing the second correction of \( T^a_1 \) to be zero achieves considerable simplification. Naturally the same cannot be done for \( T^a_2 \) because of (3.28). Thus (3.25) is trivially fulfilled while (3.26) will also be satisfied provided \( T^{(2)a}_2 \) does not depend on \( \eta^{1a} \). With this hypothesis (3.28) is solved leading to the final structures,
\[ T^{(2)a}_2 = \frac{g^2}{3!} (\bar{\eta}^2)^{af} (D_i \pi^i)^f, \quad (3.29) \]
\[ T^{(2)a}_1 = 0. \quad (3.30) \]

Note that we have introduced a matrix notation, which will be frequently used, to denote the product among the \( \bar{\eta} \) variables. It is now useful to explicitly write the involutive constraints obtained up to the second iterative step. These are given by,
\[ \tilde{T}^a_1 = \pi^a_0 - m^2 \eta^{2a} + \cdots \quad (3.31) \]
\[ \tilde{T}^a_2 = m^2 A^a_0 + (D_i \pi^i)^a + \eta^{1a} + \frac{1}{2} g \bar{\eta}^{ac} (D_i \pi^i)^c \\
+ \frac{g^2}{3!} (\bar{\eta}^2)^{ae} (D_i \pi^i)^e + \cdots \quad (3.32) \]
It is already seen that while $\tilde{T}^a_1$ has a simple structure, the correction terms in $\tilde{T}^a_2$ generate an exponential-like series. This is further elaborated by considering the third step of the method. The equation we have to solve occurs in (2.10) which, in the present instance, reads,

$$\{T^a_A, T^b_B\} + \{T^a(A), T^b(B)\} + \{T^a(A), T^b(B)\}(\eta) = 0.$$  

We thus have the following relations for distinct values of $A$ and $B$,

$$A = 1, B = 1$$

$$\{\eta^2, T^{(3)b}_1\} + \{T^{(3)b}_1, \eta^2b\} = 0,$$  

(3.34)

$$A = 1, B = 2$$

$$m^2 \{\eta^2, T^{(3)b}_2\} + m^2 \{T^{(3)a}_1, \eta^1b + \frac{1}{2} g \tilde{\eta}^{bd} (D_i \pi^i)^d\}(\eta) = 0,$$  

(3.35)

$$A = 2, B = 2$$

$$\{\eta^1 + \frac{1}{2} g \tilde{\eta}^{ad} (D_i \pi^i)^d, T^{(3)b}_2\}(\eta) + \{T^{(3)a}_2, \eta^1b + \frac{1}{2} g \tilde{\eta}^{bd} (D_i \pi^i)^d\}(\eta) = g^3 \left( \frac{1}{6} f^{bcd} f^{def} f f a n + \frac{1}{4} f^{aced} f^{bef} f f d n \right) + \frac{1}{6} f^{acd} f^{def} f f b n \right) \eta^{2c} \eta^{2e} (D_i \pi^i)^n \delta(x - y).$$  

(3.36)

Using our algorithm of taking a vanishing correction for $T^a_1$, it is simple to observe that these equations will be satisfied by choosing,

$$T^{(3)a}_1 = 0,$$

$$T^{(3)a}_2 = \frac{g^3}{4!} (\tilde{\eta}^3)^{aj} (D_i \pi^i)^j.$$  

(3.37)

This iterative process can be extended to arbitrary orders and the final expressions for the involutive constraints are given by,

$$\tilde{T}^a_1 = \pi^a_0 - m^2 \eta^2a,$$

$$\tilde{T}^a_2 = m^2 A^a_0 + \eta^1a + \sum_{n=0}^{\infty} \left( \frac{(g \tilde{\eta})^n}{(n + 1)!} \right)^{ab} (D_i \pi^i)^b.$$  

(3.38)
This completes the first part of the analysis. It is now necessary to construct the involutive Hamiltonian. We adopt our modified prescription of first doing this construction for those phase space variables occurring in the initial canonical Hamiltonian. The calculation for \( \pi_i^a \) is now given here in details. The first step is to compute the inverses of the matrices (3.14) and (3.21), denoted by \( (\omega_{AB}^{ab}) \) and \( (X^{aA,bB}) \), respectively. These are given by,

\[
(\omega_{AB}^{ab}(x,y)) = \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix} \delta^{ab} \delta(x-y), \tag{3.39}
\]

\[
(X^{aA,bB}(x,y)) = \begin{pmatrix}
\frac{g}{2m^2} f^{abc} (D_i \pi^i)^c & \delta^{ab} \\
-\frac{1}{m^2} \delta^{ab} & 0
\end{pmatrix} \delta(x-y). \tag{3.40}
\]

Using the elements of these matrices the first correction is obtained from (2.20),

\[
\pi^{(1)a}_i = -\frac{1}{m^2} \eta^{1b} G^{(0)ba}_{1i} + \frac{g}{2m^2} f^{bcd} \eta^{2b} (D_i \pi^i)^c G^{(0)da}_{1i} - \eta^{2b} G^{(0)ba}_{2i}, \tag{3.41}
\]

where the generating functions are defined by,

\[
G^{(0)ba}_{1i} = \{T^b_1, \pi^a_i\} = 0, \tag{3.42}
\]

\[
G^{(0)ba}_{2i} = \{T^b_2, \pi^a_i\} = g f^{bad} \pi^d_i. \tag{3.43}
\]

Inserting these in Eq. (3.41) the explicit result for the first correction follows,

\[
\pi^{(1)a}_i = g \tilde{\eta}^{ac} \pi^c_i. \tag{3.44}
\]

This will be used to derive the next correction. The result for an arbitrary iteration is given from the general formula (2.24), which is further simplified by inserting the matrices (3.39) and (3.40)

\[
\pi^{(n+1)a}_i = -\frac{1}{(n+1)m^2} \eta^{1b} G^{(n)ba}_{1i} + \frac{g}{2(n+1)m^2} \tilde{\eta}^{bc} (D_i \pi^i)^b G^{(n)ca}_{1i} - \frac{1}{n+1} \eta^{2b} G^{(n)ba}_{2i}, \tag{3.45}
\]

The structure of the second correction is therefore given by,

\[
\pi^{(2)a}_i = -\frac{1}{2m^2} \eta^{1b} G^{(1)ba}_{1i} + \frac{g}{4m^2} \tilde{\eta}^{bc} (D_i \pi^i)^b G^{(1)ca}_{1i} - \frac{1}{2} \eta^{2b} G^{(1)ba}_{2i}, \tag{3.46}
\]
where the generating functions, defined in Eq. (2.25), are,

\[
G_{1_{i_{1}}}^{(1)ba} = 0 , \\ G_{2_{i_{2}}}^{(1)ba} = \{T_{2_{2}}^{(1)b}, \pi_{i_{1}}^{(1)a}\} + \{T_{2_{2}}^{(1)b}, \pi_{i_{2}}^{(2)a}\}_{(\eta)} , \\ = g^{2} f^{bcd} f^{ace} \eta^{2c} \pi_{i_{1}}^{d} + \frac{g^{2}}{2} f^{bcd} f^{dah} \eta^{2c} \pi_{i_{2}}^{h} . \quad (3.47)
\]

Using these results in Eq. (3.46), we find the explicit form of the second correction,

\[
\pi_{i_{1}}^{(2)a} = g^{2} \eta^{2} \pi_{i_{1}}^{ac} . \quad (3.49)
\]

Similarly, the expression for the third correction looks like,

\[
\pi_{i_{1}}^{(3)a} = - \frac{1}{3m^{2}} \eta^{1b} G_{1_{i_{1}}}^{(2)ba} + \frac{g}{6m^{2}} \eta^{bc} (D_{j_{2}} \pi^{j})^{b} G_{1_{i_{1}}}^{(2)ca} - \frac{1}{3} \eta^{2b} G_{2_{i_{2}}}^{(2)ba} , \quad (3.50)
\]

where,

\[
G_{1_{i_{1}}}^{(2)ba} = 0 , \\ G_{2_{i_{2}}}^{(2)ba} = \{T_{2_{2}}^{(2)b}, \pi_{i_{1}}^{a}\} + \{T_{2_{2}}^{(1)b}, \pi_{i_{1}}^{(1)a}\} + \{T_{2_{2}}^{(1)b}, \pi_{i_{2}}^{(2)a}\}_{(\eta)} + \{T_{2_{2}}^{(3)b}, \pi_{i_{2}}^{(1)a}\}_{(\eta)} , \\ = \frac{g^{3}}{6} f^{bcd} f^{def} f \eta^{2c} \eta^{2e} \{A_{j_{2}}^{a}, \pi_{i_{1}}^{a}\} \pi^{aj} + \frac{g^{3}}{2} f^{bcd} f^{afe} f^{def} \eta^{2c} \eta^{2e} \{A_{j_{2}}^{a}, \pi_{i_{1}}^{a}\} \pi^{aj} + \frac{g^{3}}{2} f^{ace} f^{def} f^{dgh} \eta^{2c} \eta^{2e} \{A_{j_{2}}^{a}, \pi_{i_{1}}^{a}\} \pi^{aj} . \quad (3.51)
\]

Using the above results in (3.50), we find,

\[
\pi_{i_{1}}^{(3)a} = - \frac{g^{3}}{3!} \eta^{3} \pi_{i_{1}}^{ac} . \quad (3.53)
\]

It is clear that the structure for the corrections to the constraints has considerably simplified the algebra. There is no correction to the first generating function while the second acquires a typical form. This leads to a correction in \( \pi_{i_{1}}^{a} \) which is easily generalized to arbitrary orders. The final result for the involutive expression is thereby given by,
\[ \tilde{\pi}_i^a = \pi_i^a + \sum_{n=1}^{\infty} \pi_i^{(n)a}, \]
\[ = \sum_{n=0}^{\infty} \frac{g^n}{n!} (\eta^n)^{ac} \pi_i^c. \quad (3.54) \]

Likewise it is possible to compute the involutive expressions for the other variables. Since same steps are employed which lead to identical simplifications, the details are omitted. The final expression for the involutive \( A_i \) is given by,

\[ \tilde{A}_i^a = A_i^a + \sum_{n=1}^{\infty} A_i^{(n)a}, \]
\[ = A_i^a - g \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (\eta^n)^{ac} (D_i \eta^2)^c. \quad (3.55) \]

The corresponding computation for \( A_0 \) is slightly tricky since the normalisation is no longer given by a simple factorial. We quote the final involutive structure,

\[ \tilde{A}_0^a = A_0^a + \sum_{n=1}^{\infty} A_0^{(n)a}, \]
\[ = A_0^a + \frac{1}{m^2} \sum_{n=1}^{\infty} \left[ \eta^{1a} \delta_{1n} - g^n b_n (\eta^n)^{ac} (D_i \pi^i)^c \right], \quad (3.56) \]

where the coefficients are found from the recursive relation,

\[ b_n = \frac{1}{n} \left[ b_{n-1} + \frac{1}{(n+1)!} \right] \quad n > 1, \]
\[ b_1 = \frac{1}{2}. \quad (3.57) \]

The involutive Hamiltonian \( \tilde{H} \) is now obtained directly from the canonical Hamiltonian that appears in Eq. (3.6) by substituting the initial phase space variables by their corresponding involutive expressions, i.e.,

\[ \tilde{H}(\tilde{A}_0, \tilde{A}_i, \tilde{\pi}_i) = H_c(A_0, A_i, \pi_i) \mid_{A_0 \to \tilde{A}_0} \]
\[ \mid_{A_i \to \tilde{A}_i} \]
\[ \mid_{\pi_i \to \tilde{\pi}_i}. \quad (3.58) \]

This completes the BFFT conversion of the second class massive Yang-Mills theory into a true (first-class) gauge theory. The constraints (3.38) and Hamiltonian (3.58) satisfy a strongly involutive algebra in the extended phase space.
4 Generalized Stückelberg formulation

It is well known that a gauge noninvariant Lagrangian can be converted into a

gauge invariant form by introducing auxiliary scalars. This is the Stückelberg [15]
formalism by which it is possible to discuss a gauge invariant formulation of
the massive Maxwell theory, usually called the Proca model. A non-Abelian generalisa-
tion of the Stückelberg mechanism was first proposed by Kunimasa and Goto [13].
It was subsequently used to analyse a gauge invariant Lagrangian formulation of
the massive Yang-Mills theory by a number of authors [16, 17, 18]. These analyses
are presented either from a geometric or path integral viewpoint. In this section
we shall first develop in details, using the generalized Stückelberg prescription, the
canonical formalism of the gauge invariant formulation of the massive Yang-Mills
theory in the coordinate language. Subsequently it will be shown that the auxiliary
scalars (and their conjugates) introduced in this case are just the canonical pairs
\((\eta_1^a, \eta_2^a)\) used in the BFFT approach.

The pure (massless) Yang-Mills Lagrangian is invariant under the gauge trans-
formations,

\[
A^a_\mu \rightarrow (A^\theta)^a_\mu = U^{ab}(\theta) A^b_\mu + B^a_\mu(\theta),
\]

where \(\theta^a(x)\) are the group parameters and,

\[
B^a_\mu(\theta) = V^{ab}(\theta) \partial_\mu \theta^b.
\]

The transformation matrices \(U\) and \(V\) are given by,

\[
U^{ab}(\theta) = \left[ \exp(-g\bar{\theta}) \right]^{ab}, \tag{4.3}
\]

\[
V^{ab}(\theta) = \sum_{n=0}^{\infty} \frac{(-g\bar{\theta})^n}{(n+1)!} \Gamma^{ab}, \tag{4.4}
\]

where,

\[
\bar{\theta}^{ab} = f^{abc} \theta^c \tag{4.5}
\]

uses a notation exploited previously in (3.23). From the above relations it is simple
to deduce the result for infinitesimal variations,

\[
\delta A^\theta_\mu = U(\theta) \delta A_\mu + D_\mu [A^\theta] (V(\theta) \delta \theta) \tag{4.6}
\]

and \(D_\mu\) is the covariant derivative already introduced in (3.7). The inclusion of a
mass term in the pure Yang-Mills Lagrangian, as has been done in Eq. (3.1), breaks
the above gauge symmetry. Nevertheless, it is straightforward to incorporate a non-Abelian gauge invariance into the massive Yang-Mills Lagrangian by extending the configuration space. This is the content of the generalized Stückelberg mechanism. Using (4.2) and (4.3) it can be shown [17] that by construction, the following Lagrangian,

\[ L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} \left( A_\mu - B_\mu(\theta) \right) \left( A^\mu - B^\mu(\theta) \right) \]  

is invariant under the infinitesimal non-Abelian gauge transformations,

\[ A_\mu^a \rightarrow A_\mu^a + D_\mu^{ab} \left( V(\theta) \lambda \right)^b, \]

\[ \theta^a \rightarrow \theta^a + \lambda^a, \]  

where \( \lambda \) is the gauge parameter.

Let us next develop the canonical formalism for the gauge invariant Lagrangian (4.7). For subsequent calculations it is useful to introduce the matrix \( W^{ab}(\theta) \) where,

\[ W^{ab}(\theta) = \sum_{n=1}^{\infty} \left[ \left( -g \theta \right)^n \right]^n^{ab}, \]

\[ = V^{ab}(\theta) - \delta^{ab}. \]  

Up to a four divergence the Lagrangian (4.7) may be written as,

\[ L = -\frac{1}{4} F_{\mu\nu}^2 + \frac{m^2}{2} \left( A_\mu^2 + B_\mu^2 \right) - m^2 A_\mu \left[ \partial^\mu \theta + W(\theta) \partial^\mu \theta \right], \]

\[ = -\frac{1}{4} F_{\mu\nu}^2 + \frac{m^2}{2} \left( A_\mu^2 + B_\mu^2 \right) - m^2 A_\mu W(\theta) \partial^\mu \theta + m^2 \theta \partial_\mu A_\mu. \]  

The canonical momenta are,

\[ \pi_0^a = \frac{\partial L}{\partial \dot{A}_0^a} = m^2 \theta^a, \]

\[ \pi_i^a = \frac{\partial L}{\partial \dot{A}_i^a} = -F_0^a, \]

\[ \pi_\theta^a = \frac{\partial L}{\partial \dot{\theta}^a} = m^2 V^{ca} V^{cd} \dot{\theta}^d - m^2 A_0^b W^{ba}. \]  

There is one primary constraint,

\[ \tilde{T}_1^a = \pi_\theta^a - m^2 \theta^a \approx 0. \]
The above manipulations of isolating the identity component from $V^{ab}(\theta)$ by defining $W^{ab}(\theta)$ (4.9) and then expressing the Lagrangian up to a four divergence now become clear. Our motivation was to obtain a constraint that could be identified with $\tilde{T}_2^a$ in (3.38). Unless this manipulation was done the primary constraint obtained from (4.7) would just be $\pi_0^a \approx 0$ and the mapping fails.

For the computation of the secondary constraint, it is necessary to obtain the expression for the canonical Hamiltonian. This is given by,

$$
H = \frac{1}{2} \int d^3x \left( \pi_0 \dot{A}_0 + \pi_i \dot{A}_i + \pi_{\theta}^a \dot{\theta}^a - \mathcal{L} \right) = H_c + \frac{m^2}{2} \int d^3x V^{cd} V^{ca} \dot{\theta}^d \dot{\theta}^a + \Delta H ,
$$

(4.15)

where $H_c$ is the canonical Hamiltonian that appears in Eq. (3.6) for the usual massive Yang-Mills theory, and,

$$
\Delta H = m^2 \int d^3x \left( \frac{1}{2} B_i^2 - \theta^a \partial_i A_i^a + A_i^a W^{ab} \partial_i \theta^b \right)
$$

(4.16)

involves terms not depending on either $A_0$ or $\dot{\theta}$ so that it does not influence the computation of the secondary constraint. In order to simplify the velocity dependent term in (4.15) the first step is to invert (4.13) so that the velocity $\dot{\theta}^a$ is expressed in terms of the momenta $\pi_{\theta}^a$. This is done in Appendix A. Next, using the results of this Appendix, the desired simplification is done in Appendix B. Using (B.4) in (4.15) the final form of the Hamiltonian is obtained,

$$
H = H_c + \Delta H + \frac{m^2}{2} \int d^3x \Gamma^a \sum_{n=0}^{\infty} (-1)^n (\Lambda^n)^{ga} .
$$

(4.17)

Time conserving the primary constraint yields the secondary constraint $\tilde{T}_2^a$,

$$
\{ \tilde{T}_1^a, H \} = \{ \pi_0^a - m^2 \theta^a, H \} = \tilde{T}_2^a \approx 0 ,
$$

(4.18)

where,

$$
\tilde{T}_2^a = T_2^a + \Delta T^a
$$

(4.19)

involves the sum of the secondary constraint (3.8) occurring in the usual massive Yang-Mills theory and an extra piece,

$$
\Delta T^a = - \pi_{\theta}^a - \pi_{\theta}^d \chi^{da} + m^2 A_0^d \Omega^{da} .
$$

(4.20)

The structures $\chi$ and $\Omega$ have been defined as well as simplified in Appendix C (C.13 and C.26).
We now make certain observations. An inspection shows that the constraint (4.14) gets mapped on to the first constraint in (3.38) by a simple identification of fields. However the same thing cannot be done for the other constraint. Nevertheless, as we shall now show by a series of manipulations, a correspondence between these constraints can also be achieved. Using (C.26) in (4.20) we obtain,

$$\Delta T^a = \Theta^d \Omega^{da} - \pi_0^a,$$  

(4.21)

where

$$\Theta^d = -\pi_0^d + m^2 A_0^d.$$  

(4.22)

On the constraint surface $\tilde{T}_2^a = 0$ we find the solution:

$$\Theta^a = -\left(D_i \pi^i\right)^d \sum_{n=0}^{\infty} (-1)^n \left[\Omega^n\right]^{da},$$  

$$= -\left(D_i \pi^i\right)^d \left(\delta^{da} + \Omega^{da}\right)^{-1}.$$  

(4.23)

Defining the inverse by $I$ so that,

$$\left(\delta^{ab} + \Omega^{ab}\right) I^{bc} = \delta^{ac}.$$  

(4.24)

Expressing this as a series,

$$I^{ab} = \delta^{ab} + \sum_{n=1}^{\infty} \beta_n \left[(g\bar{\theta})^n\right]^{ab}$$  

(4.25)

and inserting it in (4.24) leads to the following condition for $\beta_n$;

$$\sum_{n=1}^{\infty} \beta_n \left[(g\bar{\theta})^n\right]^{ab} + \Omega^{ab} + \Omega^{ac} \sum_{n=1}^{\infty} \beta_n \left[(g\bar{\theta})^n\right]^{cb} = 0.$$  

(4.26)

From the above relations, therefore, the extra piece $\Delta T^a$ simplifies to

$$\Delta T^a = -\left(D_i \pi^i\right)^e \Omega^{eb} I^{ba} - \pi_0^a,$$

$$= \left(D_i \pi^i\right)^e \sum_{n=0}^{\infty} \beta_n \left[(g\bar{\theta})^n\right]^{ea} - \pi_0^a.$$  

(4.27)

\[4\text{It is crucial to note that the equality between } \chi \text{ and } \Omega \text{ is essential to derive this result, which plays a central role in the forthcoming analysis.}\]
The final task is to compute $\beta_n$ by solving Eq. (4.26). This is done in Appendix D. Using Eq. (D.4) the final expression for the constraints is given,

\[\tilde{T}_1^a = T_1^a - m^2 \theta^a,\]

\[= \pi_0^a - m^2 \theta^a,\]

\[\tilde{T}_2^a = T_2^a - \pi_0^a + \sum_{n=1}^{\infty} a_n \left[ (g\bar{\theta})^n \right]^{ba} \left( D_i \pi^i \right)^b,\]

\[= m^2 A_0^a - \pi_0^a + \sum_{n=0}^{\infty} a_n \left[ (g\bar{\theta})^n \right]^{ba} \left( D_i \pi^i \right)^b,\]

\[= m^2 A_0^a - \pi_0^a + \sum_{n=0}^{\infty} \left[ \frac{(g\bar{\theta})^n}{(n+1)!} \right]^{ab} \left( D_i \pi^i \right)^b,\]

(4.28)

where (A.2) has been used to obtain the last line. If we make the following identifications with the fields introduced in the BFFT approach,

\[\theta^a \longleftrightarrow \eta^{2a}, \quad \pi_0^a \longleftrightarrow - \eta^{1a}\]

(4.29)

the canonical algebra is preserved,

\[\left\{ \theta^a(x), \pi^b(y) \right\} = - \left\{ \eta^{2a}(x), \eta^{1b}(y) \right\} = \delta^{ab} \delta(x-y).\]

(4.30)

Now the above results for $\tilde{T}_1^a$ and $\tilde{T}_2^a$ are exactly identifiable with the corresponding expressions given in (3.38). This shows that the auxiliary scalars in the generalized Stückelberg formalism are exactly mapped on to the BFFT fields.

Before closing this section it may be worthwhile to point out that the constraint in the form (4.28) (and not as it occurs in (4.20)) reveals that it is the generator of gauge transformations because,

\[\int d^3x \left\{ \tilde{T}_2^a(x) \lambda^a(x), \theta^b(y) \right\} = \lambda^b(y),\]

\[\int d^3x \left\{ \tilde{T}_2^a(x) \lambda^a(x), A_i^b(y) \right\} = D_i^{bd} \left( V^{da} \lambda^a(y) \right),\]

(4.31)

which correctly reproduces (1.8).
5 Conclusion

We have discussed a systematic method, within the Batalin, Fradkin, Fradkina and Tyutin (BFFT) \[2, 3\] approach, of converting the massive Yang-Mills theory to a gauge invariant theory by extending the phase space. Exploiting an intelligent choice for the symplectic matrix $\omega^{ab}$ and the generating matrix $X^{ab}$, the infinite number of iterative corrections necessary for obtaining the strongly involutive constraints were considerably simplified. These corrections were explicitly computed and expressed in a closed (exponential-like) form. In obtaining the strongly involutive Hamiltonian, on the other hand, the conventional BFFT approach was modified. First, the strongly involutive forms for the initial phase space variables were deduced. Once again infinite sets of iterative corrections were necessary which were computed and put in exponential-like series. Then the canonical Hamiltonian of the massive Yang-Mills theory was rewritten, replacing the original phase space variables by their corresponding involutive expressions. This directly gave the cherished form of the involutive Hamiltonian. Indeed any dynamical variable in the original theory can be converted into its involutive form by this technique.

In the latter half of the paper the generalized St"uckelberg formalism \[13\] of changing the gauge noninvariant massive Yang-Mills Lagrangian into a gauge invariant form by introducing auxiliary scalars was reconsidered. The complete canonical formalism of the gauge invariant Lagrangian was developed in the coordinate basis. The explicit structures of the involutive constraints and Hamiltonian were determined. Subsequently it was shown by a series of algebraic simplifications that the auxiliary St"uckelberg scalars and their conjugates were exactly identified with the additional canonical pairs of fields, defined in the extended phase space, invoked earlier in the context of the BFFT analysis. By this identification a mapping between the Lagrangian and Hamiltonian embedding prescriptions of transforming gauge noninvariant into gauge invariant systems based on the generalized St"uckelberg and the BFFT approaches, respectively, was established. It should be stressed that this mapping is independent of either the space-time dimensionality or the specific non-Abelian gauge group employed in the analysis. In this sense, therefore, the present paper extended and generalized similar correspondences reported earlier \[9, 10\] for abelian groups. In this instance the abelian result follows trivially by setting the structure constants to zero.

It is clear that just as the generalization of the usual St"uckelberg formalism \[13\] from Abelian to non-Abelian theories \[13\] is nontrivial, the same is true in the BFFT formalism. Indeed a distinctive feature of transforming non-Abelian second class into first class systems, in contrast to abelian theories, was that all orders of iteration in the BFFT approach were mandatory. But our approach provided an algorithm of systematically computing these corrections, and expressing them in closed forms. An interesting application of the method developed here would be to study the bosonisation and duality among non-Abelian theories in higher dimensions where the conventional master Lagrangian approach \[19\] had failed whereas, that based on the generalised St"uckelberg formalism was effective \[20\].
As final remarks we mention that the Hamiltonian formalism for the massive Yang-Mills theory, regarded as a second class system, was originally developed in [21]. The corresponding first class (canonical) interpretation in terms of the Stückelberg approach was earlier presented in [18] which, contrary to the present work, starts from an embedded Lagrangian and does not use the elaborate BFFT method. Incidentally, the idea of converting second class systems their first class forms by extending the phase space, which is the crux of this method, was initially suggested in [22].

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Appendix A

This Appendix is devoted to invert (4.13) so that the velocities are expressed in terms of the momenta. The product of the $V$ matrices is simplified by using (4.4),

$$V^{ca}(\theta)V^{cb}(\theta) = \sum_{n=0}^{\infty} \left[ \frac{(g\bar{\theta})^{2n}}{(n+1)(2n+1)!} \right]^{ab},$$

$$= \tilde{V}^{ab}(\theta) = \tilde{V}^{ba}(\theta),$$ (A.1)

where the symmetry follows from the definition of $\bar{\theta}$ given in Eq. (4.3) since,

$$[\tilde{\theta}^n]^{ab} = (-1)^n [\tilde{\theta}^n]^{ba}. \quad (A.2)$$

It is once again useful to separate the $n = 0$ contribution,

$$\tilde{V}^{ab}(\theta) = \delta^{ab} + \Lambda^{ab}(\theta),$$ (A.3)

$$\Lambda^{ab}(\theta) = \sum_{n=1}^{\infty} \left[ \frac{(g\bar{\theta})^{2n}}{(n+1)(2n+1)!} \right]^{ab} = \Lambda^{ba}(\theta). \quad (A.4)$$

Defining

5In a geometric language this product has the meaning of a two sided invariant metric on the group of rotations [16].
\[ \Gamma^a = \frac{\pi^a}{m^2} + A_0^b W^{ba} \]  

(A.5)

and using the above relations enables Eq. (4.13) to be expressed as,

\[ \Gamma^a = \dot{\theta}^a + \dot{\theta}^b \Lambda^{ba}. \]  

(A.6)

The solution for \( \dot{\theta}^a \) is now given by,

\[ \dot{\theta}^a = \Gamma^b \sum_{n=0}^{\infty} (-1)^n (\Lambda^n)^{ba}, \]  

(A.7)

which may be verified by directly substituting in the RHS of (A.6),

\[ \Gamma^b \sum_{n=0}^{\infty} (-1)^n (\Lambda^n)^{ba} + \Gamma^b \sum_{n=0}^{\infty} (-1)^n (\Lambda^{n+1})^{ba}. \]  

(A.8)

Changing the sum in the second term from \( n \) to \( m = n + 1 \) leads to

\[ \Gamma^b \sum_{n=0}^{\infty} (-1)^n (\Lambda^n)^{ba} - \Gamma^b \sum_{m=1}^{\infty} (-1)^m (\Lambda^m)^{ba} = \Gamma^a, \]  

(A.9)

which reproduces the LHS of (A.6).

Appendix B

The velocity dependent factor in the Hamiltonian (4.15) will be expressed here in terms of the phase space variables. This factor is written as

\[ V^{ca} V^{cb} \dot{\theta}^a \dot{\theta}^b = (\delta^{ab} + \Lambda^{ab}) \dot{\theta}^a \dot{\theta}^b, \]

\[ = (\delta^{ab} + \Lambda^{ab}) \Gamma^c \sum_{n=0}^{\infty} (-1)^n (\Lambda^n)^{ca} \Gamma^d \sum_{m=0}^{\infty} (-1)^m (\Lambda^m)^{db}. \]  

(B.1)

where we have used (A.1), (A.3) and (A.7). The RHS simplifies to,

\[ \Gamma^c \sum_{n=0}^{\infty} (-1)^n (\Lambda^n)^{cb} \Gamma^d \sum_{m=0}^{\infty} (-1)^m (\Lambda^m)^{db} \]

\[ + \Gamma^c \sum_{n=0}^{\infty} (-1)^n (\Lambda^{n+1})^{cb} \Gamma^d \sum_{m=0}^{\infty} (-1)^m (\Lambda^m)^{db}. \]  

(B.2)
Consider the second term. Replace the first sum from $n$ to $m' = n + 1$,
\[- \Gamma^c \sum_{m'=1}^{\infty} (-1)^{m'} (\Lambda^{m'})^{-c} \Gamma^d \sum_{m=0}^{\infty} (-1)^m (\Lambda^m)_{db}. \quad (B.3)\]

Adding this with the first term in Eq. (B.2) yields the final result
\[ V^{ca} V^{cb} \dot{q}^a \dot{q}^b = \Gamma^a \Gamma^b \sum_{m=0}^{\infty} (-1)^m (\Lambda^m)_{ab}. \quad (B.4) \]

Appendix C

The matrices $\chi$ and $\Omega$ appearing in the constraint (4.13) are considered here in details. We first treat $\chi$ whereas $\Omega$ is taken up from (C.17) onwards. The matrix elements of $\chi$ are given by,
\[ \chi^{ab} = \sum_{n=1}^{\infty} (-1)^n (\Lambda^n)_{ab} + W^{ac} \sum_{n=0}^{\infty} (-1)^n (\Lambda^n)_{cb}, \quad (C.1) \]
where $W$ and $\Lambda$ have been introduced in (4.9) and (A.4), respectively. Consider the expression,
\[ \sum_{n=0}^{\infty} (-1)^n (\Lambda^n)^{ab} = (\delta^{ab} + \Lambda^{ab})^{-1}. \quad (C.2) \]
Writing this as a series in $\theta$,
\[ \sum_{n=0}^{\infty} (-1)^n (\Lambda^n)^{ab} = \sum_{n=0}^{\infty} C_n [(g\bar{\theta})^n]^{ab}, \quad (C.3) \]
it is possible to determine $C_n$ using (C.2),
\[ (\delta^{ac} + \Lambda^{ac}) \sum_{n=0}^{\infty} C_n [(g\bar{\theta})^n]^{cb} = \delta^{ab}. \quad (C.4) \]

It is easy to verify that only even powers of $(g\bar{\theta})$ contribute. The final result is
\[ \sum_{n=0}^{\infty} (-1)^n (\Lambda^n)^{ab} = \sum_{n=0}^{\infty} c_n [(g\bar{\theta})^{2n}]^{ab}, \quad (C.5) \]
where
\[ c_0 = 1, \]
\[ c_n = - \sum_{m=1}^{n} A_m c_{n-m} \quad (n > 0), \]
\[ A_m = \frac{1}{(m+1)(2m+1)!}. \quad \text{(C.6)} \]

We now simplify the second term in (C.1):
\[ W^{ac} \sum_{n=0}^{\infty} (-1)^n (\Lambda^n)^{cb} = \sum_{p=1}^{\infty} a_p \left[ (g\bar{\theta})^p \right]^{ac} \sum_{n=0}^{\infty} c_n \left[ (g\bar{\theta})^{2n} \right]^{cb}, \quad \text{(C.7)} \]

with,
\[ a_p = \frac{(-1)^p}{(p+1)!}. \quad \text{(C.8)} \]

Let us express the product occurring in the RHS of (C.7) as a power series where the odd and even terms are explicitly separated,
\[ - \frac{g\bar{\theta}^{ab}}{2!} + \sum_{n=1}^{\infty} d_n \left[ (g\bar{\theta})^{2n} \right]^{ab} + \sum_{n=1}^{\infty} f_n \left[ (g\bar{\theta})^{2n+1} \right]^{ab}. \quad \text{(C.9)} \]

A straightforward comparison of terms fixes \( d_n \) and \( f_n \) to be,
\[ d_n = \sum_{m=0}^{n-1} a_{2(n-m)} c_m, \]
\[ f_n = \sum_{m=0}^{n} a_{2(n-m)+1} c_m. \quad \text{(C.10)} \]

Writing the complete expansion for \( f_n \) yields,
\[ f_n = a_{2n+1} c_0 + a_{2n-1} c_1 + \cdots + a_1 c_n, \]
\[ = a_{2n+1} c_0 + a_{2n-1} c_1 + \cdots - a_1 \left( A_1 c_{n-1} + \cdots + A_n c_0 \right), \quad \text{(C.11)} \]

where we have inserted the expansion of \( c_n \) in the last term. Now using the identity
\[ a_1 A_n = a_{2n+1}, \quad \text{(C.12)} \]
that follows from the respective definitions (C.6) and (C.8), it is seen that the last factor in the parenthesis cancels the first term in $f_n$. Likewise, there will be a pairwise cancellations among all terms so that,

$$f_n = 0.$$  \hspace{1cm} (C.13)

Collecting all pieces together, we obtain,

$$\chi^{ab} = \sum_{n=1}^{\infty} (-1)^n (\Lambda^n)^{ab} - \frac{g \tilde{\theta}^{ba}}{2!} + \sum_{n=1}^{\infty} d_n \left[ (g \tilde{\theta})^{2n} \right]^{ba}.$$  \hspace{1cm} (C.14)

Using the result (C.5) and the symmetry properties of $\tilde{\theta}^{ba}$, $\chi^{ab}$ is further simplified,

$$\chi^{ab} = \frac{g}{2!} \tilde{\theta}^{ab} + \sum_{n=1}^{\infty} \omega_n \left[ (g \tilde{\theta})^{2n} \right]^{ab},$$  \hspace{1cm} (C.15)

where,

$$\omega_n = c_n + d_n,$$

$$= \sum_{m=0}^{n} a_{2(n-m)} c_m, \hspace{1cm} (C.16)$$

which follows from the explicit expression for $d_n$ given in (C.10) and $a_0 = 1$.

We now consider the matrix $\Omega$ which is defined by,

$$\Omega^{ab} = -W^{ac} \sum_{n=0}^{\infty} (-1)^n (\Lambda^n)^{cb} - W^{ac} W^{bd} \sum_{n=0}^{\infty} (-1)^n (\Lambda^n)^{dc},$$

$$= -W^{ac} \left\{ \delta^{bc} - \frac{g \tilde{\theta}^{bc}}{2} + \sum_{n=1}^{\infty} \omega_n \left[ (g \tilde{\theta})^{2n} \right]^{bc} \right\},$$  \hspace{1cm} (C.17)

where the second line follows from the previous analysis in this Appendix. Putting the explicit form for $W$ leads to,

$$\Omega^{ab} = - \left\{ \sum_{p=1}^{\infty} a_p [(g \tilde{\theta})^p]^{ab} + \frac{1}{2} \sum_{p=1}^{\infty} a_p [(g \tilde{\theta})^{p+1}]^{ab} \right\} + \sum_{p=1}^{\infty} a_p [(g \tilde{\theta})^p]^{ac} \sum_{n=1}^{\infty} \omega_n \left[ (g \tilde{\theta})^{2n} \right]^{cb}.$$  \hspace{1cm} (C.18)

The first two terms are combined by a simple change in the summation $p \longrightarrow p+1$. It leads to,
\[ \Omega^{ab} = - \sum_{p=1}^{\infty} a_p [(g\bar{\theta})^p]^{ac} \sum_{n=1}^{\infty} \omega_n [(g\bar{\theta})^{2n}]^{cb} \]

\[ - \sum_{q=1}^{\infty} \frac{q}{2(q+2)} a_q [(g\bar{\theta})^{q+1}]^{ab} + \frac{g\bar{\theta}^{ab}}{2}. \quad (C.19) \]

We now prove that terms involving odd powers of \( g \) in the two summations conspire to vanish. An arbitrary odd powered term \( (g\bar{\theta})^{2n+1} \) \( (n = 1, 2, \ldots) \) has the coefficient (apart from a minus sign),

\[ p_{2n+1} = a_{2n-1} \omega_1 + a_{2n-3} \omega_2 + \cdots + a_3 \omega_{n-1} + a_1 \omega_n + \frac{n}{2n+2} a_{2n}. \quad (C.20) \]

Inserting the expansion for \( \omega_n \), given in \( (C.16) \), this yields,

\[ p_{2n+1} = a_{2n-1} (a_2 c_0 + a_0 c_1) + a_{2n-3} \left( a_4 c_0 + a_2 c_1 + a_0 c_2 \right) + \cdots + a_1 \left( a_2 c_0 + a_2 c_1 + \cdots + a_0 c_n \right) + \frac{n}{2n+2} a_{2n}. \quad (C.21) \]

The term \( a_1 a_0 c_n \) in the last parenthesis is further elaborated by decomposing \( c_n \),

\[ a_1 a_0 c_n = -a_1 a_0 \left( A_1 c_{n-1} + A_2 c_{n-2} + \cdots + A_{n-1} c_1 + A_n c_0 \right). \quad (C.22) \]

Using the identity \( (C.12) \), it is found that the penultimate term in \( (C.22) \) cancels the second factor in the first parenthesis of \( (C.21) \). Indeed, just as discussed in Appendix B, there will be pair wise cancellation except for the last factor in \( (C.22) \) and the first one in the final parenthesis of \( (C.21) \). Combining these remaining pieces,

\[ p_{2n+1} = a_1 a_2 c_0 - a_1 a_0 A_n c_0 + \frac{n}{2n+2} a_{2n} = 0, \quad (C.23) \]

which follows from the explicit expressions. Hence, as announced before, terms with odd powers of \( g \) vanish. Next, looking at even powered terms, the coefficient of \( O(g^{2n}) \) \( (n = 1, 2, \ldots) \) is given by,

\[ p_{2n} = -\{a_{2n-2} \omega_1 + a_{2n-4} \omega_2 + \cdots + a_4 \omega_{n-2} + a_2 \omega_{n-1}\} - \frac{2n-1}{2(2n+1)} a_{2n-1}. \quad (C.24) \]

By following analogous steps it may be shown \(^6\) that this simplifies to

\(^6\)We shall subsequently discuss in Appendix D (D.8 to D.11) how this result also follows from a different viewpoint.
\[ p_{2n} = \sum_{m=0}^{n} a_{2(n-m)} c_m = \omega_n . \]  

Hence we find the remarkable result,

\[ \Omega^{ab} = \frac{g\bar{\theta}^{ab}}{2} + \sum_{n=1}^{\infty} \omega_n [(g\bar{\theta})^{2n}]^{ab}, \]
\[ = \chi^{ab}. \]  

(C.26)

**Appendix D**

The explicit solution for \( \beta_n \) will be obtained by first expressing (4.26) in full form,

\[ \sum_{n=1}^{\infty} \beta_n [(g\bar{\theta})^n]^{ab} + \frac{g\bar{\theta}^{ab}}{2!} + \sum_{n=1}^{\infty} \omega_n [(g\bar{\theta})^{2n}]^{ab} \]
\[ + \left( \frac{g\bar{\theta}^{ac}}{2} + \sum_{n=1}^{\infty} \omega_n [(g\bar{\theta})^{2n}]^{ac} \right) \sum_{n=1}^{\infty} \beta_n [(g\bar{\theta})^n]^{cb} = 0. \]  

(D.1)

By equating coefficients of the first few terms, we find,

\[ \beta_1 + \frac{1}{2!} = 0, \]
\[ \beta_2 + \omega_1 + \frac{\beta_1}{2!} = 0, \]
\[ \beta_3 + \frac{\beta_2}{2!} + \omega_1 \beta_1 = 0, \]
\[ \beta_4 + \omega_2 + \frac{\beta_3}{2!} + \omega_1 \beta_2 = 0, \]
\[ \vdots \]  

(D.2)

The first equation gives \( \beta_1 \) which is used to obtain \( \beta_2 \) from the next equation. This iterative process can be continued to obtain all the \( \beta \)'s. Indeed an explicit computation, using the value of \( \omega_n \) from (C.16), shows
\[ \beta_1 = -\frac{1}{2!}, \]
\[ \beta_2 = \frac{1}{3!}, \]
\[ \beta_3 = -\frac{1}{4!}, \]
\[ \beta_4 = \frac{1}{5!}, \]
\[ : \] (D.3)

which suggests a general solution for \( \beta_n \);
\[ \beta_n = \frac{(-1)^n}{(n+1)!} = a_n. \] (D.4)

It is now simple to prove this result explicitly. Take the coefficient of the \( O(g^{2n+1}) \) term \((n = 1, 2, \cdots)\),
\[ \beta_{2n+1} + \frac{\beta_{2n}}{2!} + \sum_{m=1}^{n} \omega_m \beta_{2(n-m)+1} = 0. \] (D.5)

Setting the solution (D.4) for \( \beta_n \), the LHS of the above equation yields,
\[ a_{2n+1} + \frac{a_{2n}}{2!} + \sum_{m=1}^{n} \omega_m a_{2(n-m)+1} = \left( a_{2n+1} + \frac{a_{2n}}{2} \right) + \omega_1 a_{2n-1} + \omega_2 a_{2n-3} + \cdots + \omega_n a_1. \] (D.6)

The series separated from the parenthesis has already been evaluated in Appendix C (see C.20 and C.23). Inserting this result in (D.6) we obtain,
\[ a_{2n+1} + \frac{a_{2n}}{2} - \frac{n}{2n+2} a_{2n} = 0, \] (D.7)
obtained by using (C.8). Thus the consistency of the solution for \( \beta_n \) is verified.

It is instructive to observe the coefficients of the \( O(g^{2n}) \) term \( (n = 2, 3, \ldots) \) which is given by,
\[ \beta_{2n} + \omega_n + \frac{\beta_{2n-1}}{2!} + \sum_{m=1}^{n-1} \omega_m \beta_{2(n-m)} = 0. \] (D.8)

Inserting the solution for \( \beta_n \), it follows,
\[- \sum_{m=1}^{n-1} \omega_m a_{2(n-m)} - a_{2n} - \frac{a_{2n-1}}{2!} = \omega_n. \quad (D.9)\]

It may be easily checked that,
\[a_{2n} + \frac{a_{2n-1}}{2!} = \frac{2n-1}{2(2n+1)} a_{2n-1} \quad (D.10)\]

and we obtain,
\[- \sum_{m=1}^{n-1} \omega_m a_{2(n-m)} - \frac{2n-1}{2(2n+1)} a_{2n-1} = \omega_n, \quad (D.11)\]

which just reproduces the result (C.24) and (C.25).

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\[\text{Refer to footnote 6}\]
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