Rethinking the Model of Moving Cosmic Strings
Producing Anisotropy in the Microwave Background

Francine R. Marleau, Charles C. Dyer and Jody H. Palmer
Department of Astronomy, University of Toronto,
60 St. George Street, Toronto, Ontario, Canada M5S 1A7

ABSTRACT

We re-analyze the issue of redshifts induced by a moving cosmic string by looking at moving sources and observers on a conical spacetime in a fully relativistic approach. By replacing the concept of a moving spacetime with the more clearly defined concept of moving sources and observers in the string spacetime, we show that there is no effect: the only redshift is a Doppler shift due to the motion of the source or observer.

Subject headings: cosmic strings – cosmology: theory – cosmic microwave background

1. Introduction

Cosmic strings are topological defects that might be created in the early universe during the phase transition/symmetry breaking that occurs in the classical field theory describing the universal forces at that time (Kibble 1981; Vilenkin 1985). These strings evolve into infinitely long vortices or small loops that decay over time by radiating gravitational waves. According to the literature (Vilenkin 1985), infinitely long cosmic strings have a gravitational field associated with a conical Minkowski spacetime and have acquired relativistic motion after their formation. It is commonly postulated that cosmic strings might provide the seeds for the formation of inhomogeneities in the universe, and that their presence will be revealed by inhomogeneities in the cosmic microwave background (CMB) (Kaiser & Stebbins 1984; Stebbins 1988). Because of the similarity of a conical spacetime to a topologically flat Minkowski spacetime, the effects of strings on the microwave background have been treated in a nonrelativistic fashion in the literature.
Here, we use a relativistic formulation to analyze physics on a string spacetime and to comment on the effects of a string on the microwave background.

When modeling the presence of physical strings in the universe, one must cope with a particularly difficult problem: a string is a topological formation and there is no accepted mechanism for the topological evolution of a spacetime. Therefore, concepts such as the creation of a string spacetime, and the motion of strings through the universe must be dealt with through simplified models. The greatest simplification comes by studying the spacetime associated with a single infinitely long string. This is not without its drawbacks. In particular, it has been shown that an infinitely long string is not embeddable in a homogeneous and isotropic cosmology (Dyer, Oattes & Starkman 1988), and thus is of limited use when discussing cosmological issues.

The spacetime associated with a self-gravitating cosmic string was introduced in 1981 by Vilenkin. In the weak-field limit, he found that the spacetime was identically conical Minkowski. Using numerical methods, it has been recently shown (Dyer & Marleau 1995; Laguna-Castillo & Matzner 1987) that static cosmic strings described as infinite cylinders in a U(1) model are self-gravitating objects with a spacetime that is not conical Minkowski. These two results are reconciled by realizing that the spacetime is curved near the string, and asymptotically approaches conical Minkowski. By recalculating Vilenkin’s result, we find that the cause of the discrepancy is an invalid coordinate transformation in going from his equation (27) to his equation (31). This can be seen by calculating the curvature associated with the two metrics. Before the transformation, the Riemann curvature tensor is non vanishing; after the transformation, the Riemann tensor is zero. Fortunately, the impact of this error has been slight because the field is asymptotically conical Minkowski. While more complete solutions have been investigated, the conical solution is commonly used in the analysis of strings in cosmology.

The simplicity of the conical solution has lead to a non-relativistic treatment of the lensing and redshift properties of cosmic strings. In particular, the effects of a moving string have been modeled by moving a wedge representing the deficit angle with respect to otherwise “stationary” observers. This model is difficult to apply self consistently since it is not known how to move a string and calculate the back-reaction of the rest of the universe; moving the string is not equivalent to moving the source and observer. Kaiser and Stebbins use this picture of a conical spacetime as “Minkowski minus a wedge” to calculate the effect of a moving string on temperature fluctuations of the microwave background. In this approach the spacetime is modeled by imagining the
spacelike slices with constant $z$ as flat pieces of paper with a wedge (with apex at the string) removed. Then a source of radiation and an observer are set on the paper such that they are not moving with respect to each other. Finally, the motion of the string with respect to the source and observer is introduced by dragging the wedge across the paper between the source and observer.

Kaiser and Stebbins found that the results of this procedure were: First, the observer sees one image, then as the string moves across the sky between the source and observer, the observer sees two images. Second, the observer views the source with no redshift before the string passes, and with a negative redshift after the string has passed. In addition to these results, Kaiser and Stebbins applied the “moving lens method” from the theory of gravitational lenses (Birkinshaw & Gull 1983) to find the perturbation of the microwave background caused by a moving string. They found that there should be a jump discontinuity in the observed temperature of the microwave background at the location of the string.

In this paper we re-analyze these results through modeling a moving string by placing a moving source and a moving observer on a conical spacetime. Now the motion of the string is modeled by the motion of the source and observer which is understood in terms of the equations of timelike geodesics on the conical spacetime. The multiple imaging characteristics of strings are analyzed through the equations for null geodesics, and the redshift of the microwave background is calculated directly from the variation of $u_αk^α$ as in the usual definition of frequency and redshift in general relativity.

2. Geodesics on a Conical Spacetime

The motions of the source and (nonaccelerated) observer on a conical spacetime are governed by the geodesic equations. For a conical spacetime the metric has the form

$$ds^2 = -dt^2 + dr^2 + α^2 r^2 dφ^2 + dz^2 \quad (2-1)$$

in the usual coordinates $(t, r, φ, z)$. The conical nature of this spacetime is due to the deficit angle $2π(1 − α)$. A value of $α = 1$ gives the usual Minkowski spacetime.

The null geodesics on a conical spacetime are very similar to the timelike geodesics.
The geodesic tangent vector has the components:

\[
\left(-A, \pm \sqrt{C - \frac{B^2}{\alpha^2 r^2}}, \frac{B}{\alpha^2 r^2}, 0\right),
\]

(2-2)

where \(A\), \(B\), and \(C\) are constant along each geodesic. We shall choose \(A = e\), \(B = l\), and \(C = e^2\) for null geodesics, and \(A = E\), \(B = L\), and \(C = E^2 - 1\) for timelike geodesics. Each geodesic is specified by the two constants of motion, \((e, l)\) in the case of null geodesics, and \((E, L)\) in the case of timelike geodesics. The point of closest approach to the string is given by \(r_c = B/\alpha \sqrt{C}\). The resulting geodesic equations can be integrated directly to give the following relationship between \(r\) and \(\phi\) along the geodesic:

\[
r = \frac{r_c}{\cos(\alpha(\phi - \phi_c))},
\]

(2-3)

where \(\phi_c\) is the value of \(\phi\) at closest approach.

In order to view these geodesics, we will draw the spacelike slice with constant \(z\) in three different representations: (1) the “cone” representation, in which the geodesics are drawn on the surface of a cone, (2) the projection of the cone onto two dimensional \(r\) and \(\phi\) space referred to here as the “rubber” representation and (3) the “Minkowski spacetime minus a wedge” representation called the “paper” representation. This nomenclature was chosen as representative of the methods – to go from the paper representation to the cone representation, one merely closes the edges of the missing wedge, thereby lifting the paper from a plane to a cone. To go from the paper representation to the rubber representation, one deforms the paper as though it were rubber, stretching it so that the edges of the missing wedge are closed while the rubber sheet remains lying in a plane. Orbits in the rubber representation are just the orthogonal projections of the geodesics onto spacelike sections with \(t\) constant. The distinction of the paper representation is that the missing angle is all accounted for in a wedge pointing in some arbitrary direction (i.e. by restricting the range of \(\phi\)), while in the other two representations, the missing angle is accounted for uniformly throughout the diagram via the use of the metric (i.e. by scaling \(\phi\)). If carefully applied, all three representations are equivalent.

To gain further understanding of the shape of the orbits, we calculate the radius of curvature of the orbits in the rubber representation. The radius of curvature at a point along the orbit is:

\[
R = r_c \frac{\left(\alpha^2 \sin^2(\alpha(\phi - \phi_c)) + \cos^2(\alpha(\phi - \phi_c))\right)^{3/2}}{(1 - \alpha^2) \cos^4(\alpha(\phi - \phi_c))}.
\]

(2-4)
The radius of curvature varies with $\phi$ along the orbit, and reaches its minimum value, $r_c/(1 - \alpha^2)$, at closest approach. For an orbital segment with endpoints $(r_i, \phi_i)$ and $(r_f, \phi_f)$, we have the relation:

$$\alpha (\phi_f - \phi_i) = \cos^{-1}\left(\frac{r_c}{r_i}\right) + \cos^{-1}\left(\frac{r_c}{r_f}\right), \quad (2-5)$$

when the segment does in fact have a closest approach. Clearly, letting the two endpoints go to infinity leads to the result that $\phi_f - \phi_i = \pi/\alpha$, independent of $r_c$. Thus the variation in curvature along an orbit occurs in such a way that the asymptotic deflection from a straight line is $\pi(1 - 1/\alpha)$. In fact, any two orbits are related by a similarity transformation corresponding to scaling the closest approach distance, or equivalently, scaling the radius of curvature. Any two orbits can be transformed into each other via rotation and scaling of $r_c$. This equivalence relation defines an equivalence class of orbits. For the Minkowski spacetime, this is equivalent to the fact that all straight lines can be obtained simply by translation and rotation of any single straight line.

In Figure 1a the cone/rubber/paper representations of timelike geodesics are presented to demonstrate that straight lines in the paper representation are bent in the cone and rubber representations and that the bending occurs nearer to the closest approach radius for smaller values of $r_c$. The diagram shows that in the rubber representation the curvature near the closest approach point decreases with increasing $r_c$, yet the angle between incoming and outgoing asymptotes is independent of $r_c$. Figure 1b shows the case for which $\alpha$ is smaller than 0.5; with such extreme deficit angles, any geodesic of sufficient length will wrap around upon itself.

It is difficult to find an observer and a source which do not move with respect to each other over a finite length of time. The only true test of the relative motion of a source and an observer is a calculation of the observed redshift, but the paper representation can easily fool one into developing an incorrect intuitive notion of comoving observers and sources. In Figure 1c we plot the curves corresponding to a geodesic source and a geodesic observer that appear in the paper representation to be moving parallel to each other. This diagram is closely related to the diagram used by Kaiser and Stebbins in discussing the onset of redshift as a string passes by. In the rubber representation, it is obvious that only the asymptotes are parallel. This representation shows that the source and observer are never moving parallel to each other, and implies that the redshift observed between them does not suddenly turn on as the string passes, but instead always exists as the string is approached. Of course the definitive answer is only achieved by looking at the quantity $u_a k^a$, proportional to the observed frequency, for a source and
observer, each moving along a geodesic and connected by a segment of a null geodesic, as we shall discuss later.

Fig. 1.— Figure 1a shows two orbits in the cone/rubber/paper representations for $\alpha = 0.7$. Similarly, Figure 1b shows the same orbits for $\alpha = 0.4$. Figure 1c shows two orbits for $\alpha = 0.8$ that are, in the paper representation, apparently parallel.

3. Double Imaging

Because of the conical nature of the spacetime, there will be regions where two null orbits will connect the same initial and final points. For a given observer any source within a region bounded by the two null orbits with $r_c$ approaching zero from either side of the string will produce two images. This is equivalent to the characteristic double imaging found by Vilenkin (1984) and later referred to by Kaiser and Stebbins.

Consider some null geodesic segment with initial point $(r_i, \phi_i)$. The form of the geodesic tangent vector shows that $dr/d\phi$ at $r_i$ can be of either sign, so there are two null geodesics that can leave $r_i$. In general these two orbits will have distinct closest approach distances, $r_c$. Let the point where these two orbits cross again be $(r_f, \phi_f)$, and define the two quantities $\Delta \phi_1$ and $\Delta \phi_2$ to be the angle swept out along each of the two orbits. Letting $r_{c1}$ and $r_{c2}$ be the respective distances of closest approach, we can write equation (2-3) for each orbit to obtain:

$$\alpha \Delta \phi_1 = \cos^{-1}\left(\frac{r_{c1}}{r_i}\right) + \cos^{-1}\left(\frac{r_{c1}}{r_f}\right),$$

$$\alpha \Delta \phi_2 = \cos^{-1}\left(\frac{r_{c2}}{r_i}\right) + \cos^{-1}\left(\frac{r_{c2}}{r_f}\right).$$

From each of these equations, the closest approach distance, $r_{c1}$ or $r_{c2}$, can be found, in the form:

$$r_{c1}^2 = \frac{r_i^2 r_f^2 \sin^2(\alpha \Delta \phi_1)}{r_i^2 + r_f^2 - 2r_i r_f \cos(\alpha \Delta \phi_1)}.$$  

To ensure that the two orbits do cross at the points $(r_i, \phi_i)$ and $(r_f, \phi_f)$, we must require that there is closure in the angle swept through by the orbits, so that $\Delta \phi_1 + \Delta \phi_2 = 2\pi$.

For a source moving behind a string, the onset of double imaging occurs when one of the null geodesics passes near the string with a vanishingly small distance of
closest approach, and the other null geodesic passes on the other side of the string with a maximum distance of closest approach. We choose to take $r_{c1} = 0$ from which we have that $\alpha \Delta \phi_1 = \pi$, so that $\alpha \Delta \phi_2 = 2\pi \alpha - \pi$. Since $\sin(\alpha \Delta \phi_2) = -\sin(2\pi \alpha)$ and $\cos(\alpha \Delta \phi_2) = -\cos(2\pi \alpha)$, we find the maximum distance of closest approach for double imaging:

$$r_{c\text{ double}}^2 = \frac{r_i^2 r_f^2 \sin^2(2\pi \alpha)}{r_i^2 + r_f^2 + 2r_i r_f \cos(2\pi \alpha)}.$$  \hfill (3-4)

When $\alpha = 1$, it follows that $\Delta \phi_1 = \Delta \phi_2 = \pi$ and hence, $r_{c1} = r_{c2} = 0$; there is only one orbit joining the points $(r_i, \phi_i)$ and $(r_f, \phi_f)$, as one would expect in a non-conical Minkowski spacetime.

An observer at $(r_f, \phi_f)$ would measure an angle $\theta$ on the sky separating the image and the string. This angle can be found by taking the dot product of the spatial parts of the null vectors pointing at the source, $k^i$, and at the string, $\tilde{k}^i$,

$$\cos \theta = \frac{k^i \tilde{k}_i}{\sqrt{k^a k_a \tilde{k}^b \tilde{k}_b}} = \pm \sqrt{1 - \frac{r_{c1}^2}{r_{f2}^2}}.$$  \hfill (3-5)

When the source is in the double image region, there are two different values of $\Delta \phi$ and two different position angles, $\theta_1$ and $\theta_2$ for the two images. The observed angular separation between images is $\theta_s = \theta_1 + \theta_2$.

Equation (3-6) can be rewritten in the form $\theta_1 = \sin^{-1}(r_{c1}/r_f)$, and $\theta_2 = \sin^{-1}(r_{c2}/r_f)$ for the two image positions. We will require the derivatives:

$$\frac{d\theta_1}{dr_{c1}} = \frac{1}{\sqrt{r_f^2 - r_{c1}^2}} \quad \text{and} \quad \frac{d\theta_2}{dr_{c1}} = \frac{1}{\sqrt{r_f^2 - r_{c2}^2}} \frac{dr_{c2}}{dr_{c1}}.$$  \hfill (3-7)

Using equation (3-1) and (3-2) and the requirement of closure to obtain $dr_{c2}/dr_{c1}$, we obtain the derivative of $\theta_s$:

$$\frac{d\theta_s}{dr_{c1}} = \frac{1}{\sqrt{r_f^2 - r_{c1}^2}} - \frac{1}{\sqrt{r_f^2 - r_{c2}^2}} \left( \frac{1}{\sqrt{r_f^2 - r_{c1}^2}} + \frac{1}{\sqrt{r_f^2 - r_{c2}^2}} \right).$$  \hfill (3-8)
The only time when this expression can vanish is when \( r_{c1} = r_{c2} \), i.e. exact alignment, or when \( r_i = r_f \). Thus in general we can expect the image separation to vary as the observer or source moves.

If a point source is moving behind a distant string, a single image appears in the sky of the observer until that image moves within a critical angle, \( \theta_{\text{double}} \), of the string where

\[
\theta_{\text{double}} = \sin^{-1} \left( \frac{r_i \sin(2\pi\alpha)}{\sqrt{r_i^2 + r_f^2 + 2r_ir_f \cos(2\pi\alpha)}} \right).
\]  

(3-9)

At this point the two images appear separated by the angle \( \theta_s = \theta_{\text{double}} \), one image lying very close to the string. As the source continues to move across the sky, both images move in the same direction in such a way that their angular separation, \( \theta_s \), will vary. When the leading image is at the critical angle, the separation angle between the images has returned to \( \theta_{\text{double}} \). A moment later, only one image exists, moving away from the string.

4. Redshift and CMB anisotropy

In order to consider the effects of a cosmic string on the microwave background, we will model the microwave background as a continuous collection of point sources. Initially, we will require these sources to be at rest with respect to the string, and to each other. For any one of these sources, the four-velocity is \( u^a = (-1, 0, 0, 0) \). We introduce a general timelike observer with four velocity

\[
u^a = \left( -E, \pm \sqrt{E^2 - 1 - \frac{L^2}{\alpha^2 r^2} - \frac{L}{\alpha^2 r^2}}, 0 \right). \]

(4-1)

If the null geodesic which joins the source and observer has a tangent vector given by

\[
k^a = \left( -e, \pm \sqrt{e^2 - \frac{l^2}{\alpha^2 r^2}}, \frac{l}{\alpha^2 r^2}, 0 \right),
\]

(4-2)

then the redshift is given by:

\[
1 + z = \frac{\nu_{\text{source}}}{\nu_{\text{obs}}} = \frac{(u^a k_a)_{\text{source}}}{(u^b k_b)_{\text{obs}}}
\]

\[
= \frac{-1}{-E \pm \sqrt{E^2 - 1 - L^2/(\alpha^2 r_f^2)} \sqrt{1 - r_c^2/r_f^2 \pm r_c L/(\alpha r_f^2)}}.
\]

(4-3)

(4-4)
In terms of the image position angle on the observer’s sky, $\theta$, this is:

$$1 + z = \frac{1}{E + B \cos \theta + C \sin \theta},$$  \hspace{1cm} (4-5)

where $B = \sqrt{E^2 - 1 - L^2/(\alpha^2 r_f^2)}$ and $C = L/(\alpha r_f)$ are constants depending only upon the motion and position of the observer.

The direction of the peak of redshift $\theta_0$, found by setting $dz/d\theta = 0$, is given by $\tan \theta_0 = C/B$. Introducing a new angle $\theta' = \theta - \theta_0$ centered about $\theta_0$ gives:

$$1 + z = \frac{1}{E + B'_0 \cos \theta'},$$  \hspace{1cm} (4-6)

where $B'_0 = B_0 (\cos \theta_0 + \tan \theta_0 \sin \theta_0)$.

This is the form of a simple dipole due to the observer’s motion through the cosmic radiation. This result contrasts with Kaiser and Stebbin’s moving wedge model in which a jump discontinuity is observed at the string. The dipole predicted by this method is essentially due to the motion of the observer and does not reveal any characteristics of the string. The same result will hold for a collection of “comoving” sources: if we modify our model of the microwave background by introducing a moving collection of sources, we will still find a dipole result. If the collection of sources is moving such that there is no redshift between each pair of sources, and yet there is a redshift between one of the sources and the string, then an observer can be boosted to have no redshift with respect to that one source. Obviously, the observer must have no redshift with respect to the other sources as well. Thus, any redshift seen by the observer can be boosted away: any redshift is simply due to the motion of the observer.

Naturally, it would be possible to choose a collection of moving sources that would be observed as a non-dipole pattern of redshifts, but the fundamental result is still the same: the only redshift which exists on this spacetime is a Doppler shift. There is no redshift due to a time varying potential as used in “the moving gravitational lens method” adopted by Kaiser and Stebbins. The spacetime is Minkowskian everywhere except at the vertex of the cone.
5. Conclusion

The implications of this work for the study of cosmic strings flow from two points: it is unknown how to move a topological structure and understand the back-reaction of the rest of the universe; and photons moving on a conical spacetime experience only Doppler shift, not gravitational redshift. Until we are capable of understanding topological dynamics, we must treat cosmic strings in a fully relativistic manner, taking seriously the conical nature of the spacetime. It is hoped that eventually the dynamics of a cosmic string will be better understood so that problems such as the evolution of cosmic strings during an early universe phase transition, and the motion of cosmic strings with respect to each other can be treated with more rigour. Since strings are created before the last scattering surface, it is possible that the changes in topology occurring during the formation of strings could cause density perturbations that would present themselves as temperature fluctuations in the microwave background. Until it is possible to understand such mechanisms, we must work with the observational consequences of microwave background radiation propagating in the vicinity of a cosmic string. The background will be largely homogeneous and isotropic, and will contain some small temperature variations. The question that must be answered is whether or not further inhomogeneities are introduced by the presence of a cosmic string. Kaiser and Stebbins developed the presently accepted means of answering this question with the “Minkowski minus a wedge representation” of a moving string and found that the string introduced further temperature fluctuations. We have taken the conical spacetime and treated it as any other spacetime in general relativity: it is the arena in which sources and observers move. Using this method, we have shown that the string does not introduce further fluctuations in the temperature of the microwave background. The temperature of the cosmic microwave background would be entirely isotropic except for a dipole due to the motion of the observer.

This work has been supported by the Natural Sciences and Engineering Research Council of Canada through a Postgraduate Scholarship (F. R. M.), a Postdoctoral Fellowship (J. H. P.) and an operating grant (C. C. D.).

REFERENCES

Birkinshaw, M. and Gull, S. 1983, Nature, 302, 315-317
Bouchet, F.R., Bennett, D.P. and Stebbins, A. 1988, Nature, 335, 410

Dyer, C., and Marleau, F. 1995, “Complete Model of a Self-gravitating Cosmic String: A New Class of Exact Solutions”, to appear in Phys. Rev. D

Dyer, C., Oattes, L.M. and Starkman, G.D. 1988, Gen.Rel.Grav., 20, 71

Gott III, J.R. 1985, ApJ, 288, 422

Kaiser, N. and Stebbins, A. 1984, Nature, 310, 391-393

Kibble, T.W.B. 1976, J.Phys.A, 9, 1387

Laguna-Castillo, P. and Matzner, R.A. 1987, Phys. Rev. D36, 3663

Stebbins, A. 1988, ApJ, 327, 584

Vilenkin, A. 1981, Phys. Rev. D23, 852

Vilenkin, A. 1984, ApJ282, L51

Vilenkin, A. 1985, Phys.Rep., 121, 263

This preprint was prepared with the AAS \LaTeX{} macros v4.0.