Low energy singularities in the ground state of fermionic superfluids

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We analyze the effects of order parameter fluctuations on the ground state of fully gapped charge-neutral fermionic superfluids. The Goldstone mode associated with the spontaneously broken symmetry leads to a problem of coupled singularities in $d \leq 3$ dimensions. We derive a minimal set of one-loop renormalization group equations which fully captures the interplay of the singularities. The flow equations are based on a symmetry conserving truncation of a scale dependent effective action. We compute the low energy behavior of longitudinal, transverse and mixed order parameter correlations, and their impact on the fermionic gap. We demonstrate analytically that cancellations protecting the Goldstone mode are respected by the flow, and we present a numerical solution of the flow equations for the two-dimensional attractive Hubbard model.

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I. INTRODUCTION

Spontaneously broken continuous symmetries in condensed matter and high energy physics give rise to emergent collective excitations, well known as Goldstone modes. Superfluidity in interacting Fermi and Bose systems is associated with a spontaneously broken $U(1)$ symmetry corresponding to particle number conservation. In charged superfluids the Goldstone modes are gapped by the coupling to the electromagnetic field via the Anderson-Higgs mechanism. In superfluids made from charge neutral constituents such as superfluid liquid helium or ultracold atomic gases, the Goldstone modes are gapless and therefore have a significant impact on the low-energy behavior.

While the Goldstone modes themselves are protected by symmetry, they strongly affect longitudinal order parameter fluctuations, leading to drastic deviations from mean-field theory in dimensions $d \leq 3$ even at zero temperature. This effect was intensively studied for the interacting Bose gas with important contributions scattered over several decades. The Goldstone modes lead to infrared divergences in perturbation theory. These divergences partially cancel each other due to symmetries, and the remaining singularities can be treated by flow equations derived from a suitable renormalization group.

In this work we analyze low energy singularities in the ground state of charge neutral fermionic superfluids. We assume spin-singlet pairing with $s$-wave symmetry, so that the fermionic degrees of freedom are fully gapped in the superfluid phase. The attractive Hubbard model serves as a prototypical microscopic model for this case. We decouple the fermionic interaction by introducing a bosonic pairing field via a Hubbard-Stratonovich transformation, and study the resulting coupled fermion-boson system by flow equations derived from the functional renormalization group (fRG).

In its one-particle irreducible version, the fRG yields an exact flow equation for the effective action $\Gamma$, which interpolates smoothly between the bare action $S$ at the highest scale $\Lambda_0$ and the final effective action $\Gamma$ for $\Lambda \rightarrow 0$. Truncations of the exact flow equation have been established as a valuable source of powerful approximation schemes in quantum field theory and quantum many-body physics. Synchronous symmetry breaking in interacting Fermi systems can be treated either by a fermionic flow with anomalous propagators and interactions, or by a coupled flow involving the fermions and a bosonic Hubbard-Stratonovich field for the order parameter fluctuations. For fermionic flows there is a relatively simple one-loop truncation with the attractive feature that it solves mean-field models for symmetry breaking exactly, although the effective two-particle interaction diverges at a finite critical scale $\Lambda_c$. The same truncation captures also several fluctuation effects, as has been shown in applications to the superfluid ground state of the attractive Hubbard model. However, the singular renormalization of longitudinal fluctuations by the Goldstone mode is captured only in a two-loop truncation of the fermionic flow.

The treatment of order parameter fluctuations is facilitated by introducing a bosonic field via a Hubbard-Stratonovich transformation. Fluctuation effects in the superfluid phase of attractively interacting Fermi systems have been studied by fRG flows with a Hubbard-Stratonovich field already in several works. The interplay between longitudinal and transverse (Goldstone) order parameter fluctuations was addressed by Strack et al. In particular, it was shown that the singular renormalization of the longitudinal fluctuations known from the interacting Bose gas is captured already in a simple one-loop truncation. However, the cancellations protecting the Goldstone mode were implemented by hand, by discarding fluctuation contributions to the transverse mass and wave function renormalization. In the present work we complete the treatment of singular renormalizations in a fermionic superfluid by deriving flow equations from an improved ansatz for the effective action $\Gamma$, which captures the renormalization of longitudinal and transverse fluctuations in full agreement with the behavior of the interacting Bose gas.

This article is structured as follows. In Sec. II we define the bare action and introduce the Hubbard-Stratonovich field. The ansatz for the effective action is formulated in Sec. III, and the corresponding truncated flow equations are derived in Sec. IV. In Sec. V we discuss cancellations and the asymptotic low-energy behavior, and we present numerical results for the flow in two dimensions. A short conclusion in Sec. VI closes the presentation.
II. BARE ACTION

We consider a system of spin-$\frac{1}{2}$ fermions with an attractive interaction leading to s-wave pairing. The precise form of the interaction is not important for the qualitative low-energy behavior. We therefore assume a local interaction for simplicity. The bare action of the system is then given by

$$S[\psi, \bar{\psi}] = - \int_{k\sigma} \bar{\psi}_{k\sigma} (i k_0 - \xi_k) \psi_{k\sigma} + \int_{k, k', q} \bar{U} \bar{\psi}_{-k + \frac{\sigma}{2}} (i k_0 + \xi_k) \psi_{k + \frac{\sigma}{2}} + \bar{U} \bar{\psi}_{-k - \frac{\sigma}{2}} (i k_0 - \xi_k) \psi_{k - \frac{\sigma}{2}} ,$$

where $\xi_k = \epsilon_k - \mu$ is the single-particle energy relative to the chemical potential, and $U < 0$ parametrizes the attractive interaction; $\psi_{k\sigma}$ and $\bar{\psi}_{k\sigma}$ are fermionic (Grassmann) fields corresponding to creation and annihilation operators. The variables $k = (k_0, \mathbf{k})$ and $q = (q_0, \mathbf{q})$ collect Matsubara energies and momenta, and $\sigma$ is the spin orientation. We use the shorthand notation $\int_k = \int_{k_0} \int_{\mathbf{k}} = \int_{-\infty}^{\infty} \frac{d k_0}{2\pi} \int_{0}^{\pi} \frac{d^d k}{(2\pi)^d}$ for momentum and energy integrals, and $\int_{k\sigma}$ includes also a spin sum. We analyze only ground state properties, such that the energy variables are continuous.

In the absence of a lattice the dispersion relation is $\epsilon_k = k^2/2m$, and a suitable regularization of the action at large momenta is required. For lattice fermions, Eq. (1) is the action of the attractive Hubbard model. For nearest neighbor hopping on a $d$-dimensional simple cubic lattice with an amplitude $-t$, the dispersion relation has the form $\epsilon_k = -2t \sum_{i=1}^{d} \cos k_i$.

The attractive interaction causes spin-singlet pairing with s-wave symmetry and a spontaneous breaking of the global U(1) particle number symmetry. We decouple the interaction in the pairing channel by introducing a complex bosonic Hubbard-Stratonovich field $\phi_q$ conjugate to the bilinear composite of fermionic fields $\tilde{\phi}_q = U \int_k \bar{\psi}_{k+\frac{\sigma}{2}} \psi_{k-\frac{\sigma}{2}}$. The system is then described by a functional integral over the fields $\psi$, $\bar{\psi}$, and $\phi$, with the fermion-boson action

$$S[\psi, \bar{\psi}, \phi] = - \int_{k\sigma} \bar{\psi}_{k\sigma} (i k_0 - \xi_k) \psi_{k\sigma} - \int_q \phi_q^* \frac{1}{U} \phi_q + \int_{k, q} \left( \bar{\psi}_{-k + \frac{\sigma}{2}} (i k_0 + \xi_k) \phi_q + \psi_{k + \frac{\sigma}{2}} (i k_0 - \xi_k) \phi_q^* \right),$$

where $\phi^*$ is the complex conjugate of $\phi$, while $\psi$ and $\bar{\psi}$ are algebraically independent Grassmann variables.

III. EFFECTIVE ACTION

Our aim is to devise a simple approximation for the fermionic and bosonic correlation functions that fully captures the low energy singularities in the superfluid ground state. All properties of the system are encoded in the effective action $\Gamma[\psi, \bar{\psi}, \phi]$, which can be obtained by functional integration of the bare action in the presence of source fields and a subsequent Legendre transform with respect to these fields. Functional derivatives of $\Gamma[\psi, \bar{\psi}, \phi]$ with respect to $\psi$, $\bar{\psi}$, $\phi$ (and $\phi^*$) yield the one-particle irreducible vertex functions, from which correlation (or Green) functions can be easily obtained. To analyze $\Gamma[\psi, \bar{\psi}, \phi]$ we use the RG. In that approach, one introduces a suitable infrared cutoff to define a scale dependent effective action, $\Gamma^\Lambda[\psi, \bar{\psi}, \phi]$, which interpolates smoothly between the bare action $S[\psi, \bar{\psi}, \phi]$ and the final effective action $\Gamma[\psi, \bar{\psi}, \phi]$. The evolution of $\Gamma^\Lambda[\psi, \bar{\psi}, \phi]$ as a function of the scale $\Lambda$ is given by an exact functional flow equation.

The exact effective action is a non-polynomial functional of the fields. However, the low-energy behavior can be described by an approximate polynomial ansatz for $\Gamma^\Lambda[\psi, \bar{\psi}, \phi]$, whose flow is given by a manageable system of ordinary differential equations. We keep only the leading low-energy terms which are necessary to capture the asymptotic low-energy behavior. We make sure that the U(1) symmetry is respected by the ansatz, as is crucial for obtaining cancellations of unphysical divergences. The shape of $\Gamma^\Lambda[\psi, \bar{\psi}, \phi]$ changes qualitatively at a finite critical scale $\Lambda_c$, the scale of spontaneous symmetry breaking. We thus distinguish between the symmetric regime above $\Lambda_c$ and the symmetry-broken regime below.

A. Symmetric regime

In the symmetric regime, the ansatz for the scale dependent effective action has the form

$$\Gamma^\Lambda = \Gamma_{\bar{\psi}\phi} + \Gamma_{\bar{\psi}^*\phi} + \Gamma_{\psi\phi^*} + \Gamma_{\psi^*\phi^*} ,$$

where the subscripts indicate the field-monomials contained in each term. To simplify the notation, we do not write a superscript for the $\Lambda$-dependence of each term.

For the fermionic part, we just keep the term contained already in the bare action,

$$\Gamma_{\bar{\psi}\phi} = \int_{k\sigma} \bar{\psi}_{k\sigma} (i k_0 - \xi_0) \psi_{k\sigma} .$$

We neglect renormalizations of the fermionic propagator since these are finite and do not affect the low-energy singularities qualitatively. We also discard fermionic interaction terms which are generated by contributions of fourth order in the fermion-boson vertex in the flow. One could deal with such terms by a dynamical decoupling of the interactions at each scale but they also lead only to finite renormalizations.

The quadratic bosonic part is parameterized as

$$\Gamma_{\phi^*\phi} = \frac{1}{2} \int_q \phi_q^* \left[ m_0^2 + Z_0 (\phi_q^2 + \omega_q^2) - i W q_0 \right] \phi_q ,$$

where $m_0$, $Z_0$, and $W$ are $\Lambda$-dependent numbers and $\omega_q$ is a fixed function which is proportional to $|q|$ for small $q$. For a lattice system it is convenient to choose a periodic function for $\omega_q$. In numerical evaluations of the flow for the two-dimensional Hubbard model we will use $\omega_q^2 = 4t - 2t (\cos q_x + \cos q_y)$. The bosonic mass $m_0$ is initially (in the bare action) $\sqrt{2}/U$, and vanishes at the critical scale $\Lambda_c$. The other terms are generated by fermionic pairing fluctuations, and then renormalized also by bosonic fluctuations (see Sec. IV).
The imaginary part proportional to \( q_0 \) vanishes for particle-hole symmetric systems and was not taken into account in the ansatz used in Ref. [22]. Here we include it since this term leads to a mixing of longitudinal and transverse fluctuations in the symmetry-broken regime, and thus affects the low-energy singularities qualitatively. The quadratic frequency dependence was discarded in several earlier works\cite{19,21}. Linear and quadratic frequency terms were both taken into account in a previous fRG study of interacting Bose systems\cite{20}. Momentum and energy dependences beyond quadratic order are irrelevant.

The quartic bosonic part has the form

\[
\Gamma_{\phi^4} = \frac{1}{8} \sum_{q,p} U(p) \phi_{q-p}^* \phi_{q-p} \phi_{q} \phi_{q} .
\]  
(6)

The bosonic potential is parametrized by the ansatz

\[
U(p) = u + Y \left( p^2 + \alpha^2 \right) ,
\]  
(7)

where \( u \) and \( Y \) are scale dependent numbers. \( U(p) \) should not be confused with the Hubbard \( U \) in Eq. (1). The \( u \)-term corresponds to the standard local \( \phi^4 \) interaction. The \( Y \)-term with \( \omega_q \) expanded to quadratic order corresponds to a quartic gradient term proportional to \( \int dx (\nabla \phi(x))^2 \), where \( x = (\tau, \mathbf{r}) \) is an imaginary time and (real) space coordinate. This term is irrelevant in naive power-counting and was therefore not included in earlier truncations of the effective action. However, a term of this form is crucial in the symmetry-broken regime to implement distinct gradient terms for longitudinal and transverse fluctuation in a \( U(1) \) symmetric way, as is well known from studies of \( O(N) \) field theories\cite{22,23}. The momentum and frequency dependence of the quartic interaction has also been taken into account in recent truncations of the effective action for the interacting Bose gas\cite{22,24}.

The fermion-boson interaction is restricted to a 3-point vertex of the form already present in the bare action,

\[
\Gamma_{\phi \bar{\psi} \psi} = g \sum_{kq} \left( \bar{\psi}_{-k+q/2} \bar{\psi}_{k+q/2} \psi_{q} + \psi_{k+q/2} \bar{\psi}_{-k+q/2} \phi_{q} \right) .
\]  
(8)

The coupling constant \( g \) is equal to one in the bare action. Within the above ansatz for the effective action it remains unrenormalized in the symmetric regime of the flow\cite{22}. Higher order fermion-boson interactions (beyond three-point) and momentum or energy dependences are irrelevant for the low-energy behavior.

### B. Symmetry-broken regime

Below the critical scale \( \Lambda_b \) the \( U(1) \) symmetry of the system is spontaneously broken, so that anomalous terms emerge in the effective action. The bosonic part of the action develops a minimum at a finite \( \phi_{0} = \alpha \), where only the modulus of \( \alpha \) is fixed by the minimization condition while the phase remains arbitrary. Implementing this minimum in our ansatz for the effective action, one is led to the following form for the bosonic part,

\[
\Gamma_b = \frac{Z_b}{2} \int dx |\nabla \phi(x)|^2 + \frac{W}{2} \int dx \phi^* (x) \partial_x \phi(x) + \frac{\mu}{8} \int dx \left( |\phi(x)|^2 - |\xi|^2 \right)^2 + \frac{Y}{8} \int dx \left( \nabla |\phi(x)|^2 \right)^2 ,
\]  
(9)

where \( x = (\tau, \mathbf{r}) \). Here we have written \( \Gamma_b \) as a functional of \( \phi(x) \) because the structure of the quartic terms is more transparent in a space-time (instead of momentum-frequency) representation. All terms in \( \Gamma_b \) obviously respect the global \( U(1) \) symmetry.

We now fix the phase of the superfluid order parameter by choosing \( \alpha \) real and positive, and we decompose \( \phi(x) \) as

\[
\phi(x) = \alpha + \sigma(x) + i \pi(x) ,
\]  
(10)

where \( \sigma(x) \) and \( \pi(x) \) are real fields describing longitudinal and transverse fluctuations, respectively. In Fourier space, the decomposition reads

\[
\phi_q = \alpha \delta_{q0} + \sigma_q \Omega_q ,
\]  
(11)

\[
\phi_q^* = \alpha \delta_{q0} + \sigma_{-q} - i \pi_{-q} .
\]  

Note that we have returned to the momentum representation and replaced the factors \( q_0^2 \) corresponding to spatial Laplace operators by the function \( \omega_q \) introduced in Sec. IIIA. The term with the first order time derivative in Eq. (9) leads to a mixing of \( \sigma \) and \( \pi \) fields. The longitudinal mass is determined by the order parameter \( \alpha \) and the quartic coupling \( u \) in Eq. (9) as

\[
m_{\sigma}^2 = u \alpha^2 .
\]  

The \( Z \)-factors for longitudinal and transverse fluctuations are related to \( Z_b \) and \( Y \) by

\[
Z_{\sigma} = Z_b + Y \alpha^2 ,
\]  
\[
Z_{\pi} = Z_b .
\]  
(15)
The cubic and quartic interaction terms read
\[
\Gamma_{\sigma^3} = \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} U(p) \alpha \sigma_p \sigma_q \sigma_{-q-p},
\]
\[
\Gamma_{\sigma^4} = \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} U(p) \alpha \sigma_p \pi_q \pi_{-q-p},
\]
\[
\Gamma_{\sigma^3} = \frac{1}{8} \int \frac{d^4q}{(2\pi)^4} U(p) \sigma_q \sigma_{-p-q} \sigma_q \sigma_{-p-q},
\]
\[
\Gamma_{\sigma^4} = \frac{1}{4} \int \frac{d^4q}{(2\pi)^4} U(p) \pi_q \pi_{-p-q} \pi_q \pi_{-p-q},
\]
\[
(16)
\]
where \(U(p) = u + Y(p_0^2 + \omega_p^2)\) as in Eq. (7). The above contributions to \(\Gamma_b\) are the same as for longitudinal and transverse fields in an \(O(2)\) model.\cite{10,11} The relations \(\Gamma_{\sigma^3} = \Gamma_{\sigma^4}\) are valid for the quartic ansatz.\cite{6} Expanding the exact effective action around \(\alpha\) would lead to additional terms from higher (than quartic) orders.

Spontaneous symmetry breaking leads also to anomalous fermionic contributions. The normal quadratic term \(\Gamma_{\phi^2}\), Eq. (4), is supplemented by the anomalous term
\[
\Gamma_{\phi^2} = \int \frac{d^4k}{(2\pi)^4} \left( \Delta \bar{\psi}_{-k} \bar{\psi}_{k}^T + \Delta^* \psi_{-k} \psi_{k} \right),
\]
which \(\Delta\) is the fermionic excitation gap. Furthermore, an anomalous fermion-boson interaction of the form \(\Gamma_{\phi^2} = \bar{\psi}_{-k} \bar{\psi}_{k} \psi_{k+q/2} \psi_{k-q/2} + \psi_{k+q/2} \psi_{k-q/2} \bar{\psi}_{-k} \bar{\psi}_{k} \) is generated by the flow for \(\Lambda < \Lambda_c\).\cite{24} Decomposing \(\phi\) in longitudinal and transverse fields one obtains the fermion-boson interaction in the form
\[
\Gamma_{\phi^2} = g_{\sigma} \int \frac{d^4k}{(2\pi)^4} \left( \bar{\psi}_{-k} \bar{\psi}_{k} \sigma_q + \psi_{k+q/2} \psi_{k-q/2} \sigma_{-q} \right),
\]
\[
(18)
\]
\[
\Gamma_{\phi^4} = i g_{\pi} \int \frac{d^4k}{(2\pi)^4} \left( \bar{\psi}_{-k} \bar{\psi}_{k} \pi_q - \psi_{k+q/2} \psi_{k-q/2} \pi_{-q} \right),
\]
where \(g_{\sigma} = g + \bar{g}\) and \(g_{\pi} = g - \bar{g}\). A Ward identity derived from the \(U(1)\) symmetry yields a relation between the fermionic gap and the bosonic order parameter \(\alpha\), namely
\[
\Delta = g_{\pi} \alpha.
\]
\[
(20)
\]
A derivation is given in Appendix A. A similar relation for a truncation with \(\bar{g} = 0\) was derived previously in Ref.\cite{24} For \(\bar{g} = 0\), there is only one Yukawa coupling, \(g_{\pi} = g_{\sigma} = g\), and the Ward identity can be written as \(\Delta = g \alpha\).

In summary, our ansatz for the effective action in the symmetry-broken regime has the form
\[
\Gamma^\Lambda = \Gamma_b + \Gamma_{\phi^2} + \Gamma_{\phi^4},
\]
where \(\Gamma_b\) is a sum of purely bosonic terms as listed in Eq. (12). An ansatz with distinct renormalizations for longitudinal and transverse fluctuation fields was formulated and evaluated already in Ref.\cite{23} Here we have included two additional terms. Most importantly, the quartic gradient term \(Y\) is crucial for capturing the very different behavior of \(Z_\sigma\) and \(Z_\pi\) while conserving the \(U(1)\) symmetry.\cite{10,11} Eq. (15) implies that setting \(Y = 0\) forces \(Z_\sigma = Z_\pi\) within a \(U(1)\) symmetric ansatz. Implementing \(Z_\sigma \neq Z_\pi\) at the expense of the \(U(1)\) symmetry as in Ref.\cite{22} spoils important cancellations of singularities, so that the Goldstone mode has to be protected by hand. Second, the term linear in \(q_0\) in the bosonic sector \(W\)-term leads to a mixing of \(\sigma\) and \(\pi\) fields in the symmetry-broken regime. This term is important for capturing the generic low-energy behavior of the superfluid. In particular, it determines the condensate compressibility.\cite{4}

IV. FLOW EQUATIONS

Inserting the ansatz for the effective action \(\Gamma^\Lambda\) into the exact flow equation\cite{25,20} and comparing coefficients, one obtains flow equations for the scale dependent parameters. The scale dependence is generated by regulator functions which are added to the inverse propagators. For the fermions we choose the function
\[
R_f(k) = R_f(k_0) = [i \Lambda sgn(k_0) - ik_0] \Theta(\Lambda - |k_0|),
\]
\[
(22)
\]
and for the bosons
\[
R_b(q) = R_b(q_0) = Z_b(\Lambda^2 - q_0^2) \Theta(\Lambda - |q_0|).
\]
\[
(23)
\]
Regulator functions of this form lead to relatively simple integrals.\cite{15} The \(U(1)\) symmetry is not spoiled by these regulators. The scale derivatives of the regulators are
\[
\frac{d}{d\Lambda} R_f(k) = isgn(k_0) \Theta(\Lambda - |k_0|),
\]
\[
(24)
\]
and
\[
\frac{d}{d\Lambda} R_b(q) = [2Z_b(\Lambda^2 - q_0^2) \Theta(\Lambda - |q_0|).\]
\[
(25)
\]
The scale derivative of \(Z_b\) turns out to be quite large at the initial stage of the flow. Hence, we take its contribution to the scale derivative of the regulator into account. We now present the flow equations in the symmetric and symmetry-broken regime.

A. Symmetric regime

The right hand sides of the flow equations are loop integrals over products of fermionic and bosonic propagators. The propagators are the inverse of the quadratic kernels of the effective action (including the regulator functions). In the symmetric regime (for \(\Lambda > \Lambda_c\)), the fermion propagator has the form
\[
G_f(k) = -\langle \psi_{k \sigma} \bar{\psi}_{k \sigma} \rangle = \frac{1}{ik_0 - \xi_k + R_f(k)}.
\]
\[
(26)
\]
and the boson propagator reads\[G_b(q) = \frac{1}{2}(q\phi\phi^*) = \frac{1}{m_b^2 + \omega_q^2} - iWq_0 + R_b(q).\]

The regulator functions replace $k_0$ by $\text{sgn}(k_0)\Lambda$ in the fermion propagator and $q_0^2$ by $\Lambda^2$ in the boson propagator, if $|k_0| < \Lambda$ and $|q_0| < \Lambda$, respectively. One factor in the propagator product is subject to a scale derivative acting on the regulator function, leading to the single-scale propagators

$$G_{f/b}^{r}(k) = D_X G_{f/b}(k) = -G_{f/b}^2(k) \partial_X R_{f/b}(k) \tag{28}$$

for fermions and bosons, respectively. The differential operator $D_X$ acts only on the regulator function, not on other $\Lambda$-dependencies of the propagators.

The flow equations for the bosonic mass, the bosonic $Z$-factor, and the coefficient $W$ are obtained from the flow of the bosonic self-energy $\Sigma_b(p) = [G_b(p)]^{-1} - [G_b^0(p)]^{-1}$, with $G_b^0(p) = [m_b^2 + R_b(p)]^{-1}$. The self-energy obeys the flow equation

$$\frac{d}{d\Lambda}\Sigma_b(p) = -2g^2 \int_k D_X [G_f(k)G_f(p-k)]$$

\[+ \int_q |U(q)|G_b(q) + U(p-q)G_b(p) \tag{29}\]

The first term is a fermionic one-loop contribution, while the second and third term is the bosonic Hartree and Fock term, respectively. The corresponding Feynman diagrams are shown in Fig. 1, where the Hartree and Fock terms are represented by a single diagram with a symmetrized vertex. The flow of the mass is obtained from the flow of $\Sigma_b(0)$ as

$$\frac{d}{d\Lambda}m_b^2 = -4g^2 \int_k G_f'(k)G_f(-k)$$

\[+ \int_q [U + U(q)]G_b(q) \tag{30}\]

The fermionic term reduces the mass upon lowering $\Lambda$, while the bosonic term partially compensates this reduction. The flow for the bosonic $Z$-factor is extracted by applying a second order momentum derivative to the bosonic self-energy, that is,

$$\frac{d}{d\Lambda}Z_b = \frac{d}{d\Lambda}m_b^2 \tag{31}\]

This yields

$$\frac{d}{d\Lambda}Z_b = -2g^2 \int_k G_f'(k)G_f(p-k) \bigg|_{q=0}$$

\[+ \frac{Y}{2} \int_q \omega^2_{p-q} G_b(q) \bigg|_{p=0}, \tag{31}\]

Note that the second momentum derivative at $p = 0$ does not depend on the direction, even for lattice systems. The flow of the parameter $W$ is obtained from a frequency derivative,

$$\frac{d}{d\Lambda}W = -4ig^2 \int_k G_f'(k)G_f(p-k) \bigg|_{p=0} - 2iY \int_q G_b(q) \tag{32}\]

The flow of the quartic couplings $u$ and $Y$ is determined by the flow of the bosonic four-point vertex. The Feynman diagrams corresponding to the fermionic and bosonic one-loop contributions are shown in Fig. 2. Setting all ingoing and outgoing momenta to zero, one obtains the flow equation for $u$,

$$\frac{d}{d\Lambda}u = 4g^4 \int_k D_X [G_f^2(k)G_f^2(-k)]$$

\[+ \int_q |U(q)|G_b(q)G_b(-q) \tag{33}\]

The flow equation for $Y$ is obtained by setting ingoing and outgoing momenta equal to $\pm p/2$ such that the total momentum is zero. Applying a second order derivative with respect to $p$, the flow equation for the (symmetrized) vertex yields

$$\frac{d}{d\Lambda}Y = 4g^4 \int_k D_X [G_f^2(k)G_f(-k + p/2)G_f(-k - p/2)] \bigg|_{p=0}$$

\[- \frac{1}{4} \int_q [U(0) + U(q/2)] [U(0) + U(q - p/2)] \times D_X [G_b(q)G_b(-q)] \bigg|_{p=0}$$

\[+ \frac{1}{2} \int_p G_b(q)G_b(-q) \cdot \frac{d}{d\Lambda}m_b^2 \tag{34}\]

The first contribution in Eqs. (33) and (34) is generated by fermions, while the remaining terms are due to bosonic fluctuations.

### B. Symmetry-broken regime

Spontaneous symmetry breaking leads to anomalous terms in the effective action and the form of the propagators changes.
and

\[
G_f(k) = -\langle \psi_{k\sigma} \bar{\psi}_{-k\sigma} \rangle = \frac{-i k_0 - R_f(k_0) - \xi_k}{|i k_0 + R_f(k_0)|^2 + E_k^2}, \tag{35}
\]

\[
F_f(k) = -\langle \psi_{k\uparrow} \bar{\psi}_{-k\downarrow} \rangle = \frac{\Delta}{|i k_0 + R_f(k_0)|^2 + E_k^2}, \tag{36}
\]

where \( E_k = \sqrt{\Delta^2 + \xi_k^2} \) is the fermionic excitation energy in the superfluid phase. The corresponding single-scale propagators are given by \( G_f'(k) = D_A G_f(k) \) and \( F_f'(k) = D_A F_f(k) \). Note the relation

\[
|G_f(k)|^2 + |F_f(k)|^2 = \frac{1}{\Delta} F_f(k). \tag{37}
\]

Inverting the quadratic bosonic terms yields the propagators for the \( \sigma \) and \( \pi \) fields

\[
G_{\sigma\sigma}(q) = \langle \sigma_q \sigma_{-q} \rangle = \frac{\gamma_{\pi\pi}(q)}{\gamma_{\sigma\sigma}(q) \gamma_{\pi\pi}(q) + W^2 q_0^2}, \tag{38}
\]

\[
G_{\pi\pi}(q) = \langle \pi_q \pi_{-q} \rangle = \frac{\gamma_{\sigma\sigma}(q)}{\gamma_{\pi\pi}(q) \gamma_{\pi\pi}(q) + W^2 q_0^2}, \tag{39}
\]

\[
G_{\sigma\pi}(q) = \langle \pi_q \sigma_{-q} \rangle = \frac{W q_0}{\gamma_{\pi\pi}(q) \gamma_{\pi\pi}(q) + W^2 q_0^2}, \tag{40}
\]

and \( G_{\pi\sigma}(q) = -G_{\sigma\pi}(q) \), where \( \gamma_{\sigma\sigma}(q) = m_{\sigma}^2 + Z_{\sigma}(q_0^2 + \omega_{\sigma}^2) + R_{\sigma}(q) \) and \( \gamma_{\pi\pi}(q) = Z_{\pi}(q_0^2 + \omega_{\pi}^2) + R_{\pi}(q) \). The corresponding single-scale propagators are \( G_{\sigma\sigma}'(q) = D_A G_{\sigma\sigma}(q) \) etc. The relation

\[
\gamma_{\sigma\sigma}(q) - \gamma_{\pi\pi}(q) = U(q) \alpha^2 \tag{41}
\]

following from Eqs. (14) and (15) will be useful below.

The order parameter \( \alpha \) is determined from the condition that the one-point \( \sigma \)-vertex \( \gamma_{\sigma \sigma \alpha} \) has to vanish if \( \alpha \) is a minimum of the effective action. The flow of \( \gamma_{\sigma \sigma \alpha} \) is given by

\[
\frac{d \gamma_{\sigma \sigma \alpha}}{d \Lambda} = m_{\sigma}^2 \frac{d \alpha}{d \Lambda} - 2 g_{\sigma \sigma} \int_k F_f'(k) + \frac{\alpha}{2} \int_q [u + 2 U(q)] G_{\sigma\sigma}(q) + \frac{\alpha}{2} \int_q G_{\pi\pi}(q). \tag{42}
\]

The first term on the right hand side is generated by the scale dependence of the point around which the effective action is expanded (in powers of the fields). The Feynman diagrams representing the other contributions are shown in Fig. 3. The condition \( \frac{d}{d \Lambda} \gamma_{\sigma \sigma \alpha} = 0 \) yields the flow equation for \( \alpha \),

\[
\frac{d \alpha}{d \Lambda} = \frac{2}{m_{\sigma}^2} \frac{g_{\sigma \sigma}}{\int_k F_f'(k)} - \frac{\alpha}{2 m_{\sigma}^2} \int_q [u + 2 U(q)] G_{\sigma\sigma}(q) + u G_{\pi\pi}(q). \tag{43}
\]

We now turn to quantities which can be derived from the \( \sigma \)- and \( \pi \)-field self-energies. The one-loop contributions to the flow of these bosonic self-energies are represented diagrammatically in Fig. 4. The flow of \( m_{\pi}^2 \) and \( Z_{\sigma} \) can be extracted from the flow of the \( \sigma \)-field self-energy. The latter obeys the flow equation

\[
\frac{d}{d \Lambda} \Sigma_{\sigma\sigma}(p) = \left[ U(0) + 2 U(p) \right] \alpha \frac{d \alpha}{d \Lambda} - g_{\sigma \sigma}^2 \int_k \left[ D_A \left( G_f(k) G_f(p-k) - F_f(k) F_f(p-k) \right) + (p \leftrightarrow -p) \right] + \frac{1}{2} \int_q \left[ U(0) + 2 U(p+q) \right] G_{\sigma\sigma}(q) + u G_{\pi\pi}(q) - \frac{1}{2} \int_q \left[ U(p) + U(q) + U(p+q) \right]^2 \alpha^2 D_A \left[ G_{\sigma\sigma}(q) G_{\sigma\sigma}(p+q) \right] - \frac{1}{2} \int_q \left[ U(p) \right]^2 \alpha^2 D_A \left[ G_{\sigma\sigma}(q) G_{\sigma\sigma}(p+q) \right] - \int_q \left[ U(p) + U(q) + U(p+q) \right] U(p) \alpha^2 D_A \left[ G_{\sigma\sigma}(q) G_{\sigma\sigma}(p+q) \right]. \tag{44}
\]
The first term on the right hand side is generated by the scale dependence of $\alpha$. The flow of the mass is obtained from $\Sigma_{\sigma\sigma}(0)$ as

$$
\frac{d}{d\Lambda} m_r^2 = \frac{d}{d\Lambda} \Sigma_{\sigma\sigma}(0)
= 3u a \frac{d\alpha}{d\Lambda} \int d\Lambda \left[ |G_f(k)|^2 - F_f^2(k) \right]
+ \frac{1}{2} \int d\Lambda \left[ (u + 2U(q)) G'_{\sigma\sigma}(q) + u G'_{\sigma\pi}(q) \right]
- \frac{a^2}{2} \int d\Lambda \left[ (u + 2U(q))^2 G''_{\sigma\sigma}(q) + u^2 G''_{\sigma\pi}(q) \right]
+ \alpha^2 \int d\Lambda \left[ (u + 2U(q)) u D_\Lambda [G_{\sigma\sigma}(q)]^2 \right].
$$  \hspace{1cm} (45)

The flow of $Z_\sigma$ can be obtained from a second momentum derivative of the self-energy at $p = 0$, that is, $\frac{d^2}{d\Lambda^2} Z_\sigma = \frac{1}{2} \partial^2_{\rho, \Sigma} \Sigma_{\sigma\sigma}(p) |_{p=0}$. The flow equation for the $\pi$-field self-energy reads

$$
\frac{d}{d\Lambda} \Sigma_{\sigma\pi}(p) = u a \frac{d\alpha}{d\Lambda} \left[
- g_{\sigma}\int d\Lambda \left[ G_f(k)G_f(-p-k) + F_f(k)F_f(p-k) \right]
+ (p \leftrightarrow -p)\right]
+ \frac{1}{2} \int d\Lambda \left[ (U(0) + 2U(p+q)) G'_{\sigma\pi}(q) + u G'_{\sigma\sigma}(q) \right]
- \int d\Lambda \left[ U(q)^2 \alpha^2 D_\Lambda [G_{\sigma\sigma}(q)G_{\sigma\pi}(p+q)] \right]
- \int d\Lambda \left[ U(q)U(p+q) \alpha^2 D_\Lambda [G_{\sigma\sigma}(q)G_{\sigma\sigma}(p+q)] \right].
$$  \hspace{1cm} (46)

The flow of $W$ in the symmetry broken regime can be obtained from a frequency derivative of the mixed $\sigma\pi$ self-energy, that is, $\frac{d}{d\Lambda} W = \partial^2_{\rho, \Sigma} \Sigma_{\sigma\sigma}(p) |_{p=0}$. The flow of $\Sigma_{\sigma\pi}$ is determined by the flow equation

$$
\frac{d}{d\Lambda} \Sigma_{\sigma\pi}(p) = i g_{\sigma\sigma} g_{\pi}\int d\Lambda \left[ G_f(k)G_f(-p-k) - G_f(k)G_f(p-k) \right]
- \int d\Lambda \left[ U(p+q) G'_{\sigma\pi}(q) \right]
- \int d\Lambda \left[ U(q) \left[ U(p) + U(q) + U(p+q) \right] \alpha^2 \right]
\times D_\Lambda \left[ G_{\sigma\sigma}(q)G_{\sigma\pi}(p+q) \right]
- \int d\Lambda \left[ U(p)U(q) \alpha^2 D_\Lambda [G_{\sigma\sigma}(q)G_{\sigma\pi}(p+q)] \right].
$$  \hspace{1cm} (47)

Note that all contributions vanish at $p = 0$, so that no $\sigma\pi$-mixing mass term is generated. The bosonic contributions to the flow of the bosonic self-energy in Eqs. (44), (46) and (47) are equivalent to those derived previously for the interacting Bose gas by Sinner et al.\[24\]

In the fermionic sector, the gap is the only flow parameter in our truncation. Its flow is obtained from the flow of the anomalous fermionic self-energy at zero frequency and a momentum $k_F$ on the Fermi surface. This yields the flow equation

$$
\frac{d}{d\Lambda} \Delta = g_\sigma \frac{d\alpha}{d\Lambda} \left[
- \int q \left[ F_f(k_F - q) \left[ g_{\sigma\sigma}^2 G_{\sigma\sigma}(q) - g_{\sigma\pi}^2 G_{\sigma\pi}(q) \right] \right] \right],
$$  \hspace{1cm} (48)

where $k_F = (0, k_F)$. The second (fluctuation) term can be represented by the Feynman diagram in Fig. 5. Contributions involving $\sigma\pi$ mixing cancel. We recall that $D_\Lambda$ acts only on the regulator functions in the propagators. For isotropic systems, the right hand side of Eq. (48) does not depend on the choice of $k_F$. For lattice systems the dependence is generically very weak, since the bosonic fluctuation contribution is dominated by small momenta $q$, and $F_f(k_F) = \Delta/\Delta^2 + \Delta^2$ is independent of $k_F$. Considering the Ward identity Eq. (20), one may expect the coupling $g_\sigma$ instead of $g_\xi$ in the contribution proportional to $da/\Lambda$ in Eq. (48). Indeed, we will see below that the Ward identity is fulfilled within our truncation of the effective action only if one sets $g_\sigma = g_\xi$.

The set of flow equations is completed by the flow of the couplings $g_{\sigma\sigma}$ and $g_{\pi\pi}$ parametrizing the fermion-boson interaction. The flow of $g_{\sigma\sigma}$ is given by

$$
\frac{d}{d\Lambda} g_{\sigma\sigma} = g_{\sigma\sigma} \left[ F_f^2(k_F - q) + G_f(k_F - q) \right] \left[ \left[ g_{\sigma\sigma}^2 G_{\sigma\sigma}(q) - g_{\pi\pi}^2 G_{\pi\pi}(q) \right] \right]
+ 2g_{\sigma\pi} g_{\pi\pi} \int d\Lambda \left[ F_f(k_F - q) \right]
\times \left[ G_{\sigma\sigma}(q)G_{\pi\pi}(q) + G_{\sigma\sigma}(q)G_{\sigma\sigma}(q) \right].
$$  \hspace{1cm} (49)

The corresponding Feynman diagrams are shown in Fig. 6. Using the relations (57) and (41), the flow equation can be
simplified to
\[
\frac{d}{d\Lambda} m_\sigma^2 = \frac{g_\pi}{\Lambda} \int_q D_\Lambda \left[ F_j(k_F - q) \left[ g_\sigma^2 G_{\sigma\sigma}(q) - g_\pi^2 G_{\pi\pi}(q) \right] \right] \\
- \frac{2 g_\sigma g_\pi}{\alpha} \int_q D_\Lambda \left[ F_j(k_F - q) \left[ G_{\sigma\sigma}(q) - G_{\pi\pi}(q) \right] \right].
\]

(50)

The flow equation for \( g_\sigma \) will not be used for reasons that become clear below.

Our ansatz is an extension of the truncation used by Strack et al.,\[23\] where the \( W \)-term was absent and the bosonic potential \( U(p) \) was constant in momentum space (that is, \( Y = 0 \)). Setting \( W = Y = 0 \), the flow equations presented above reduce essentially to the previously derived flow equations, with three corrections and extensions. First, we have obtained an extra factor \( \frac{1}{2} \) in front of the two-boson contributions to the \( \sigma \)-field self-energy, which determines the flow of \( m_\sigma^2 \) and \( Z_\sigma \). Second, the two-boson contribution to the flow of the fermion-boson vertex was overlooked in Ref.\[22\]. Third, the bosonic fluctuation contributions to \( Z_\sigma \) were discarded. The latter are finite and therefore not crucial qualitatively, but they are finite only as a consequence of rather subtle cancellations, as we shall see below.

\[ \text{C. Goldstone theorem and Ward identity for gap} \]

The Goldstone theorem implies that transverse order parameter fluctuations are massless, that is, there is no \( \pi \)-field mass \( m_\pi \). Such a mass is indeed absent in our \( U(1) \)-symmetric ansatz for the effective action. However, inserting the ansatz into the flow equations, various contributions to \( m_\pi^2 = \Sigma_{\pi\pi}(0) \) are being generated and it is not obvious that these contributions cancel. Evaluating the flow equation \([46]\) for \( \Sigma_{\pi\pi}(p) \) at \( p = 0 \) yields
\[
\frac{d}{d\Lambda} m_\pi^2 = u\alpha \frac{d\alpha}{d\Lambda} - 2 g_\pi^2 \int_k D_\Lambda \left[ \left( G_j(k) \right)^2 + F_j(k) \right] \\
+ \frac{1}{2} \int_q \left[ \left[ u + 2 U(q) \right] G_{\pi\pi}'(q) + u G_{\sigma\sigma}'(q) \right] \\
- \int_q \left[ U(q) \right]^2 \alpha^2 D_\Lambda \left[ G_{\sigma\sigma}(q) G_{\pi\pi}(q) + G_\sigma^2(q) \right].
\]

(51)

Using Eqs. \([37] \) and \([41] \), the right side can be simplified to
\[
\frac{d}{d\Lambda} m_\pi^2 = u\alpha \frac{d\alpha}{d\Lambda} - 2 g_\pi^2 \int_k F_j(k) \\
+ \frac{1}{2} \int_q \left[ u + 2 U(q) \right] G_{\sigma\sigma}'(q) + u G_{\pi\pi}'(q) \right].
\]

(52)

Inserting the flow equation for \( \alpha \), Eq. \([43] \), the bosonic fluctuation contributions cancel such that
\[
\frac{d}{d\Lambda} m_\pi^2 = 2 \left( \frac{g_\sigma}{\alpha} - \frac{g_\pi^2}{\Delta} \right) \int_k F_j(k). \]

(53)

Using the Ward identity \( \Delta = g_\pi \alpha \), the prefactor in front of the integral can be written as \( \frac{1}{\alpha^2} (g_\pi - g_\sigma) \). Hence, the contributions to the flow of \( m_\pi^2 \) vanish if (and only if) \( g_\sigma = g_\pi \). Within our truncation of the effective action, we are thus constrained to set \( g_\sigma = g_\pi \) in order to respect the Goldstone theorem.

Using the condition \( g_\sigma = g_\pi \) and the Ward identity \( \Delta = g_\pi \alpha \), the two contributions to the flow of \( g_\sigma \) in Eq. \([50] \) can be combined to
\[
\frac{d}{d\Lambda} g_\sigma = - \frac{g_\pi^2}{\alpha} \int_q D_\Lambda \left[ F_j(k_F - q) \left[ G_{\sigma\sigma}(q) - G_{\pi\pi}(q) \right] \right].
\]

(54)

Comparing this equation with the flow equation \([43] \), one can see that the flow equations for \( g_\sigma \) and \( \Delta \) are indeed consistent with the Ward identity \( \Delta = g_\pi \alpha \), if one chooses \( g_\sigma = g_\pi \). The Ward identity would not be respected by the flow if one computed \( g_\sigma \) and \( g_\pi \) from their distinct flow equations, or if both were set equal and computed from the flow equation for \( g_\sigma \).

In summary, the flow equations are consistent with the Goldstone theorem (\( m_\pi = 0 \)) and the Ward identity \( \Delta = g_\pi \alpha \), if one sets \( g_\sigma = g_\pi \), and computes the flow of this unified coupling from the flow equation for \( g_\sigma \). Since \( g_\pi \) deviates only mildly from its initial value \( g_\pi^0 = 1 \), one might also simplify the truncation further by discarding the flow of the fermion-boson vertex completely, such that \( g_\sigma = g_\pi = 1 \) and \( \Delta = \alpha \). To implement \( g_\sigma \neq g_\pi \) consistently, one would have to extend the ansatz for the effective action, as discussed briefly in the Conclusions.

\[ \text{V. FLOW} \]

We now discuss the behavior of the flow as obtained from the flow equations derived in the preceding section. We first analyze the low energy behavior in the limit \( \Lambda \to 0 \) and then present numerical results for the flow on all scales, using the two-dimensional Hubbard model as a prototypical example.

\[ \text{A. Low energy behavior} \]

The low energy behavior in the limit \( \Lambda \to 0 \) is independent of model details and can be analyzed quite generally. For \( \Lambda < \Lambda_c \), the fermionic propagator is regularized by the energy gap \( \Lambda \). Infrared singularities arise therefore solely from the bosonic propagators, where the most singular one is the propagator for transverse order parameter fluctuations \( G_{\pi\pi} \).

It is well-known that the Goldstone mode leads to a singular renormalization of longitudinal fluctuations.\[39\] For \( \Lambda \to 0 \), the flow equation \([45] \) for \( m_\pi^2 \) is dominated by the term involving a product of two \( \pi \)-propagators, such that
\[
\frac{d}{d\Lambda} m_\pi^2 \sim - \alpha^2 u^2 \int_q G_{\pi\pi}(q) G_{\pi\pi}'(q).
\]

(55)

The integral diverges as \( \Lambda^{d-4} \) for \( \Lambda \to 0 \), where \( d \) is the spatial dimensionality. Note that \( G_{\pi\pi}(q) \) vanishes for \( |k_0| > \Lambda \). Using the relation \( m_\pi^2 = u\alpha^2 \) one thus finds that \( m_\pi^2 \) and \( u \) both vanish
as $\Lambda^{-d}$ for $d < 3$, and logarithmically in three dimensions in agreement with the behavior derived by other methods.

The flow of $Z_\sigma$ is also dominated by the two-\pi-boson term [see Eq. (44)], that is,

$$
\frac{d}{d\Lambda} Z_\sigma \sim -\frac{1}{4} \partial_{p_3}^2 U(p) [G_\pi(p) G_\pi(p + q)] |_{p=0}
$$

(56)

for small $\Lambda$. $U(p)$ scales as $\Lambda^{-d}$, hence $\frac{d}{d\Lambda} Z_\sigma$ scales as $\Lambda^{-2} \Lambda^{(3-d)} \Lambda^{-d} = \Lambda^{-d}$ for $1 < d < 3$, so that $Z_\sigma$ diverges as $\Lambda^{-1}$. In three dimensions $U(p)$ vanishes only logarithmically for $\Lambda \to 0$, so that $\frac{d}{d\Lambda} Z_\sigma$ scales as $|\Lambda \log \Lambda|^{-2}$ and $Z_\sigma$ diverges as $|\Lambda \log \Lambda|^{-1}$. The scaling behavior of $m_\sigma^2$ and $Z_\sigma$ is not altered by the $\sigma-\pi$ mixing term and the momentum dependence of the bosonic potential and therefore agrees with previous results obtained for a simpler truncation where $W = Y = 0$.

The relation (15) implies that $\sigma$ is proportional to $Z_\sigma$ for $\Lambda \to 0$, provided that $Z_\sigma$ remains finite as expected (and shown below). Hence, $\sigma$ also diverges as $\Lambda^{-1-d}$ for $1 < d < 3$, and as $|\Lambda \log \Lambda|^{-2}$ in three dimensions.

The vanishing $\sigma$-mass leads to divergent prefactors in the flow equation (53) for the order parameter $\sigma$. However, in the bosonic term this is compensated by the bosonic potential, which scales exactly as $m_\sigma^2$, and the fermionic integral vanishes linearly in $\Lambda$, so that the flow of $\sigma$ always saturates at a finite value for $\Lambda \to 0$. One would run into artificially diverging flows for $\sigma$ if the fermionic cutoff were lowered too slowly compared to the bosonic one. It is easy to see that the flows for $\Delta$ and $g_\sigma$ also saturate at finite values for $\Lambda \to 0$. The singularities of the bosonic contributions on the right hand side of the corresponding flow equations are integrable.

We now show that the flow of $Z_\sigma$ saturates at a finite value for $\Lambda \to 0$, so that the Goldstone mode maintains a finite spectral weight, as expected. This is not obvious, since the right hand side of the flow equation for $Z_\sigma$ contains divergent terms. From the flow equation (46) one obtains

$$
\frac{d}{d\Lambda} Z_\sigma = \frac{1}{2} \partial_{p_3}^2 \int_q U(p + q) G_\pi^2(p) |_{p=0}
$$

$$
- \frac{1}{2} \partial_{p_3}^2 \int_q \left[ U(p) \right]^2 \partial_{p_3} D_\Lambda [G_\sigma(q) G_\pi(p + q)] |_{p=0}
$$

+ finite terms .

(57)

The first and the second term both diverge as $\Lambda^{-1}$ for $\Lambda \to 0$ for $1 < d < 3$, which suggests a logarithmic divergence of $Z_\sigma$. However, these divergences cancel each other. To see this, we use the relation (41) to replace the factor $\alpha^2 U(q)$ in the second term by $\gamma_\pi(q) - G_\pi(q)$. For $\Lambda \to 0$ one can neglect the $W$-term in the denominator of $G_\sigma$, such that $G_\sigma(q) G_\pi(q) \to 1$, yielding

$$
\frac{d}{d\Lambda} Z_\sigma \sim \frac{1}{2} \partial_{p_3}^2 \int_q U(p + q) G_\pi^2(q) |_{p=0}
$$

$$
- \frac{1}{2} \partial_{p_3}^2 \int_q U(p) D_\Lambda \left[ G_\sigma(p) G_\pi(p + q) - G_\sigma(q) \gamma_\pi(q) G_\pi(p + q) \right] |_{p=0}
$$

+ finite terms .

(58)

The divergent terms involving $G_\sigma'$ (and no other propagator) cancel, and the remaining terms are finite for $\Lambda \to 0$, so that $Z_\sigma$ indeed saturates. Note that the momentum dependence of $U(q)$, parametrized by the naively irrelevant coupling $Y$, is crucial here. Otherwise the first term in Eq. (57) would vanish, and the divergence of the second term would remain uncancelled.

We finally derive the asymptotic behavior of the $\sigma-\pi$ mixing parameter $W$. From the flow equation (47) for $Z_\sigma$, the last term yields the most singular contribution to the flow of $W$,

$$
\frac{d}{d\Lambda} W \sim \partial_{p_3} \int_q U(p) (p - q) \alpha^2 D_\Lambda \left[ G_\sigma(p - q) G_\pi(q) \right] |_{p=0},
$$

(59)

where we have shifted the integration variable so that the $p$-dependence appears in the argument of $G_\sigma$. The factor $U(p)$ can be replaced by $U(0) = u$, since $\partial_{p_3} U(p)$|$_{p=0} = 0$. Anticipating that $W$ vanishes sufficiently rapidly for $\Lambda \to 0$, we neglect the term of order $W^2$ in the denominator of $G_\sigma(p - q)$. Inserting the relation $\alpha^2 U(p - q) = \gamma_\pi(p - q) - G_\sigma(q)$ then yields

$$
\frac{d}{d\Lambda} W \sim -u \partial_{p_3} \int_q \left( \frac{W(p_0 - q_0)}{\gamma_\pi(p - q)} - \frac{W(p_0 - q_0)}{\gamma_\pi(p - q)} \right) G_\pi(q) \right) |_{p=0},
$$

(60)

where $G_\sigma(p) \sim \gamma_\pi(p)$. The second term in the bracket is sub-leading and can be neglected. Carrying out the scale derivative $D_\Lambda$, and substituting $q \to p - q$ in one of the resulting terms, the contributions can be combined to

$$
\frac{d}{d\Lambda} W \sim u \partial_{p_3} \int_q \frac{W(p_0)}{\gamma_\pi(q)} \frac{W(p_0)}{\gamma_\pi(p - q)} |_{p=0}.
$$

(61)

Only the $p_0$ dependence in the numerator yields a contribution to the frequency derivative at $p_0 = 0$. Using again $\gamma_\pi(p) \sim G_\pi(q)$, one thus obtains the simple asymptotic flow equation

$$
\frac{d}{d\Lambda} W \sim -u W \int_q G_\pi(q) G_\pi(q).\n$$

(62)

Comparing this with the asymptotic flow equation (55) for $m_\pi^2$, we see that $W^{-1} \frac{d}{d\Lambda} W \sim (m_\pi^2)^{-1} \frac{d}{d\Lambda} m_\pi^2$, so that

$$
\frac{W}{m_\sigma^2} \to C = \text{const}
$$

(63)

for $\Lambda \to 0$. In other words, $W$ vanishes with the same power of $\Lambda$ as $m_\sigma^2$. This confirms that the term $W^2 G_\sigma^2$ is indeed sub-leading in the denominator of the bosonic propagators. In an interacting Bose gas, the asymptotic ratio $C$ is indeed the condensate compressibility $d^2\rho/d\mu$, where $\mu$ is the chemical potential for the bosons.

Since $Z_\sigma$ and $W/m_\sigma^2$ tend to finite constants for $\Lambda \to 0$, the propagators containing a $\pi$-field assume the simple asymptotic
form

\begin{align}
G_{\pi\pi}(q) & \sim \frac{1}{Z_{\pi}(q_0^2 + \omega_q^2)}, \\
G_{\sigma\sigma}(q) & \sim \frac{Cq_0}{Z_{\pi}(q_0^2 + \omega_q^2)} - G_{\pi\pi}(q)
\end{align}

(64)

(65)

for small \(q\), in agreement with the behavior known for the interacting Bose gas.\(^{39}\) Only \(G_{\sigma\sigma}(q)\) exhibits anomalous scaling, with a power-law depending on the dimensionality for \(d < 3\), and logarithmic corrections in three dimensions.

B. Flows in two dimensions

We now present numerical results for the renormalization group flow of the variables parametrizing the effective action, using the attractive two-dimensional Hubbard model as a prototypical example.\(^{43}\) Thereby, the asymptotic behavior derived analytically in the preceding section is confirmed, and further interesting features at low and intermediate scales are obtained. We choose a moderate attraction \(U = -4t\), and fix the fermion density at quarter-filling, where the Fermi surface is nearly circular and far from the Van Hove points in the Brilloin zone.\(^{39}\) All results are presented in units of \(t\).

The flow of the bosonic masses, \(m_b\), for \(\Lambda\) above \(\Lambda_c\) and \(m_\sigma\) below, is shown in Fig. 7. The masses vanish at the critical scale \(\Lambda_c = 1.36\), and \(m_\sigma^2\) decreases linearly in \(\Lambda\) for \(\Lambda \to 0\), in agreement with the analytic asymptotic results.

The flows of the gap \(\Delta\) and the bosonic order parameter \(\alpha\) are shown in Fig. 8. Both quantities saturate at finite values for \(\Lambda \to 0\), where \(\alpha\) is slightly smaller than \(\Delta\). While \(\Delta\) increases monotonically in the course of the flow, \(\alpha\) is slightly decreased in the final stage by bosonic fluctuation contributions. The final gap \(\Delta = 0.88\) is reduced compared to the BCS mean-field gap \(\Delta_{MF} = 1.16\). The reduction is partially due to the truncation of the bosonic potential at quartic order, and, of course, to fluctuations. However, the reduction is weaker than expected, which can be attributed to lacking fluctuation contributions from the fermionic particle-hole channel in our truncation. Perturbation theory and fermionic RG calculation\(^{45,46}\) yield a gap reduction to roughly one half. The fermion-boson flow computed by Strack et al.\(^{22}\) yielded an even stronger gap reduction, caused mostly by a strong suppression of the critical scale by fluctuations in the symmetric regime. A more accurate calculation of the gap within a bosonized IRG requires rebosonization of effective two-fermion interactions which are generated by the flow.\(^{22}\) We do not pay attention to these contributions, since our aim here is not an accurate estimate of the gap, but rather a comprehensive analysis of the low energy singularities. The flow of the fermion-boson coupling \(g_\pi\) shown in Fig. 9 is determined by the flow of \(\Delta\) and \(\alpha\) via the Ward identity \(\Delta = g_\pi\alpha\). The increase of \(g_\pi\) with decreasing \(\Lambda\) therefore reflects the increasing ratio \(\Delta/\alpha\) observed already in Fig. 8.

In Fig. 10 we show the flow of the prefactors of the quadratic momentum and frequency dependence of the bosonic self-energies, \(Z_\Delta\) above the critical scale, and \(Z_\pi, Z_\sigma\) below. While \(Z_\sigma\) converges rapidly to a finite value, \(Z_\pi\) first seems to saturate and falls even below \(Z_\pi\), but then takes off to diverge as \(\Delta^{-1}\) for small \(\Lambda\), in agreement with the analytic asymptotic results.

The flow of the bosonic interaction parameters \(u\) and \(Y\) is displayed in Fig. 11. The local coupling \(u\) first increases upon
lowering the scale, exhibits a weak kink at $\Lambda_c$, and finally decreases linearly in $\Lambda$ for $\Lambda \to 0$. The latter behavior is dictated by $m_F^2 \propto \Lambda$ via the relation $u = m_F^2/\alpha^2$. The non-local coupling $Y$ is initially negative, and changes sign at a scale $\Lambda$, well below $\Lambda_c$. The relation $a^2 Y = Z_\pi - Z_\sigma$ thus implies that $Z_\sigma < Z_\pi$ in the regime between $\Lambda$ and $\Lambda_c$, as is indeed observed in Fig. 10. For small $\Lambda$, the coupling $Y$ approaches $Z_\sigma/\alpha^2$ and therefore diverges as $\Lambda^{-1}$.

Finally, in Fig. 12 we show the flow of the variable $W$ parametrizing the imaginary linear frequency contribution to the bosonic self-energy, which leads to a mixing of $\sigma$- and $\pi$-fields in the symmetry-broken regime. $W$ is always positive, increases until $\Lambda$ has reached the critical scale and a bit beyond, and then decreases again with an asymptotic behavior proportional to $\Lambda$ for $\Lambda \to 0$, in agreement with the analytic analysis in the preceding section. The ratio $W/m_F^2$ converges to $C = 0.25$ for $\Lambda \to 0$.

VI. CONCLUSION

Charge-neutral fermionic superfluids exhibit rather complex low-energy behavior due to singular renormalizations generated by the Goldstone mode. In particular, the mass for longitudinal order parameter fluctuations scales to zero in dimensions $d \leq 3$. We have found a relatively simple set of one-loop renormalization group equations which fully captures these singularities, and conserves cancellations imposed by the $U(1)$ charge symmetry. The flow equations are based on a symmetry-conserving truncation of the IRG effective action. Systematic cancellations guarantee that the flow conserves the massless Goldstone boson with a finite $Z$-factor, and that it respects an important Ward identity relating the pairing gap $\Delta$ to the bosonic order parameter and the fermion-boson vertex. The truncation includes a self-energy term which mixes longitudinal and transverse order parameter fluctuations. We have shown that its renormalization is linked to the scaling of the longitudinal mass, in agreement with the behavior of an interacting Bose gas, where the mixing term is related to the condensate compressibility.

Our truncation conserves the Goldstone theorem and the lowest order Ward identity relating fermions and bosons. The $U(1)$ symmetry actually entails an infinite hierarchy of Ward identities relating vertex functions of arbitrary order. Any truncation of the effective action at finite order in the fields leads to a deviation from the exact flow and a violation of Ward identities involving vertex functions which are discarded in the truncation. For example, there is a Ward identity relating the difference between the longitudinal and transverse fermion-boson coupling, $g_\sigma - g_\pi$, to a two-fermion-two-boson vertex. If the latter is discarded, as in our truncation, one has to set $g_\sigma = g_\pi$ for the sake of consistency.

The main goal of this work was to deal with the singularities associated with the Goldstone boson. We did not pay attention to finite renormalizations such as contributions from the fermionic particle-hole channel or fermionic $Z$-factors. However, building on the insights gained in the present work, one can construct improved truncations which yield more accurate results for the gap and other non-universal quantities, and still capture the singular low-energy behavior.
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Appendix A: Ward identity

Here we describe the derivation of the Ward identity, Eq. (20), relating the fermionic gap to the bosonic order parameter and the fermion-boson vertex. From the global $U(1)$ symmetry corresponding to charge conservation one can derive the following identity for the effective action,

$$\int \left( \psi_{\rho r} \frac{\delta \Gamma}{\delta \psi_{\rho r}} - \bar{\psi}_{\rho r} \frac{\delta \Gamma}{\delta \bar{\psi}_{\rho r}} \right) = 2 \int \left( \phi_{\rho} \frac{\delta \Gamma}{\delta \phi_{\rho}} - \phi_{\rho} \frac{\delta \Gamma}{\delta \bar{\phi}_{\rho}} \right).$$

(A1)

This relation holds at any scale $\Lambda$, since the regulator does not spoil the symmetry. From this functional identity one can derive Ward identities for vertex functions by taking functional derivatives with respect to the fields at $\psi_{\rho r} = \bar{\psi}_{\rho r} = 0$ and $\phi_{\rho} = \alpha \delta \phi_{0}$. In particular, applying $\frac{\delta^2}{\delta \psi_{\rho r} \delta \bar{\psi}_{-\rho r}}$ yields

$$\frac{\delta^2 \Gamma}{\delta \psi_{\rho r} \delta \bar{\psi}_{-\rho r}} = \alpha \frac{\delta^3 \Gamma}{\delta \psi_{\rho r} \delta \bar{\psi}_{-\rho r} \delta \phi_{0}} - \alpha \frac{\delta^3 \Gamma}{\delta \psi_{\rho r} \delta \bar{\psi}_{-\rho r} \delta \bar{\phi}_{0}},$$

(A2)

that is, a relation between the gap and the fermion-boson vertices. Applying $\frac{\delta^2}{\delta \phi_{\rho} \delta \bar{\phi}_{-\rho}}$ yields the complex conjugate relation. For our ansatz, with a $k$-independent gap and a $k$-independent fermion-boson coupling, Eq. (A2) yields

$$\Delta = g \alpha - \bar{g} \alpha^*.$$

(A3)

Choosing $\alpha$ and $\Delta$ real, one obtains the Ward in the form of Eq. (20).
fermion self-energy is not taken into account.

40 A. Martín-Rodero and F. Flores, Phys. Rev. B 45, 13008 (1992).
41 B. Obert, Ph.D. thesis, University Stuttgart, 2013.

42 See, for example, chapter 12 in Ref. [7].