The Confining Phase Kähler Potential in SUSY QCD

Peter Cho  
Lyman Laboratory  
Harvard University  
Cambridge, MA 02138

Abstract

We investigate the low energy structure of the Kähler potential in SUSY QCD with \(N_f = N_c + 1\) quark flavors. Since this theory’s moduli space is everywhere smooth, a systematic power series expansion of its Kähler potential can be developed in terms of confined meson and baryon fields. Perturbation theory in the supersymmetric sigma model based upon a momentum expansion consistent with naive dimensional analysis and \(1/N_f\) power counting exhibits some similarities with ordinary QCD chiral perturbation theory along with several key differences. We compute meson and baryon wavefunction renormalization as well as Kähler potential operator mixing to leading nontrivial order. We also deduce the asymptotic dependence of the lowest dimension operators’ coefficients upon moduli space location along flat directions where the theory is Higgsed down to \(N_f - 1 = (N_c - 1) + 1\) SUSY QCD. Although an exact form for the confining phase Kähler potential remains unknown, we find that some detailed Kähler sector information can nevertheless be derived from first principles.
1. Introduction

During the past few years, significant progress has been made in understanding the vacuum structure of confining $\mathcal{N} = 1$ supersymmetric gauge theories \cite{1}. Following Seiberg’s pioneering work on SUSY QCD \cite{2}, several examples of supersymmetric models that exist in a confining/Higgs phase everywhere throughout moduli space have been analyzed in detail \cite{3,4}. Massless spectra are now known in a large class of theories, and a sizable number of dynamically generated superpotentials which constrain gauge invariant moduli have been derived. The often complicated superpotential expressions succinctly summarize nonperturbative quantum deformations of classical moduli spaces. Once one identifies the exact superpotential in a particular theory, it is frequently possible to deduce many others by integrating in or out nonchiral matter fields \cite{10,11}. An intricate web of mutually consistent moduli space results from multiple gauge theory studies has thus been established.

Although the first discoveries of such nonperturbative vacua results were impressive, they have provided only limited insight into the sigma models which describe confining SUSY gauge theories at low energies. Unfortunately, one can only go so far with just ground state information. In the absence of any knowledge regarding a sigma model’s kinetic sector, it is impossible to explore a broad class of questions related to scattering processes involving nonzero energy transfer. To date, relatively little has been uncovered about Kähler potentials in $\mathcal{N} = 1$ SUSY sigma models.

In this note, we initiate an investigation into the low energy Kähler sector in SUSY QCD. We focus upon the infrared description of the microscopic theory with $N_f = N_c + 1$ quark flavors. As Seiberg has demonstrated, SUSY QCD with this particular matter content confines, and its massless degrees of freedom consist of composite meson and baryon superfields \cite{2}. Unlike the superpotential whose holomorphic form is exactly fixed by symmetry and asymptotic limit considerations, the Kähler potential involves an infinite number of \textit{à priori} undetermined coupling constants. However, a systematic perturbation theory can be developed for low energy SUSY QCD in which only a finite number of Kähler interaction terms contribute to scattering processes at any given order. As we shall see, it shares some similarities with and exhibits several key differences from ordinary chiral perturbation theory for non-SUSY QCD.

Our article is organized as follows. We first construct the leading Kähler potential terms within the effective theory built upon the vacuum at the origin of moduli space.
in section 2. We next investigate renormalization of superpotential couplings and mixing among nonrenormalizable operators in section 3. Deformations of $N_f = N_c + 1$ SUSY QCD’s Kähler potential along flat directions away from the origin are explored in section 4. Finally, we close with some thoughts on extending our Kähler sector findings in section 5, and we list our superspace conventions in the Appendix.

2. The low energy Kähler potential

Of the many $\mathcal{N} = 1$ supersymmetric gauge theories which have been studied in the past, SUSY QCD is among the simplest and best understood. A number of key insights into the nonperturbative dynamics of strongly coupled supersymmetric models were first developed within this theory. So it represents a natural starting point for exploring confining phase Kähler potentials.

We will focus upon SUSY QCD with $N_f = N_c + 1 \geq 4$ quark flavors. In the absence of any tree level superpotential, the microscopic gauge theory has continuous symmetry group

$$G = SU(N_c)_{\text{local}} \times \left[ SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_R \right]_{\text{global}},$$

(2.1)

matter content

$$Q^{ai} \sim (\bar{\mathbf{1}} : \mathbf{1} : 1 ; 1, \frac{1}{N_f})$$

$$\bar{Q}_{aI} \sim (\mathbf{1} : 1, \mathbf{1} : -1, \frac{1}{N_f})$$

(2.2)

and Wilsonian beta function coefficient $b_0 = 2N_f - 3$. Gauge invariant meson and baryon operators with quantum number assignments

$$M_i^t = \bar{Q}Q \sim (1; \bar{\mathbf{1}} ; \mathbf{1} ; 0, \frac{2}{N_f})$$

$$B_i = Q^{N_c} \sim (1; \bar{\mathbf{1}} ; 1 ; N_c, \frac{N_c}{N_f})$$

$$\bar{B}^t = \bar{Q}^{N_c} \sim (1; \mathbf{1} ; -N_c, \frac{N_c}{N_f})$$

(2.3)

act as moduli space coordinates. Anomaly matching as well as duality arguments compellingly demonstrate that these composite superfields saturate the massless spectrum at the moduli space origin and that no massive states become massless at points away from $\langle M \rangle = \langle B \rangle = \langle \bar{B} \rangle = 0$ \[2\]. Since effective theory singularities generally result from overlooked massless degrees of freedom, $N_f = N_c + 1$ SUSY QCD reduces at energies
below its intrinsic scale $\Lambda$ to an effective theory of mesons and baryons with a completely smooth moduli space.

The absence of moduli space singularities restricts the model’s dynamically generated superpotential to a simple polynomial form \[2\]:

$$W = \lambda_0 \frac{B_0 M_0 \overline{B}_0 - \text{det} M_0}{\Lambda^{2N_f - 3}}. \quad (2.4)$$

We have labeled the hadrons appearing in this superpotential expression with zero subscripts as a reminder that they represent bare fields. The constant $\lambda_0$ similarly represents a bare coupling. Although it is often absorbed into a redefinition of $\Lambda$, we shall keep track of this dimensionless parameter for later perturbation theory purposes.

The Kähler potential in the low energy effective theory is inherently more complicated than the superpotential, for it depends upon both chiral and antichiral superfields as well as their derivatives. Although dimensional analysis and symmetry considerations are not sufficiently powerful to completely fix the nonholomorphic Kähler potential as they essentially do for the holomorphic superpotential, they provide useful constraints on its form:

$$K = \Lambda^{\dagger} \Lambda \ast k \left( \frac{M_0 \dagger M_0}{(\Lambda^{\dagger} \Lambda)^2}, \frac{B_0 \overline{B}_0 \dagger}{(\Lambda^{\dagger} \Lambda)^{N_c}}, \frac{\overline{B}_0 \overline{B}_0 \dagger}{(\Lambda^{\dagger} \Lambda)^{N_c}}, \cdots \right). \quad (2.5)$$

Since the moduli space is smooth, we can decompose the hadron fields into quantum fluctuations about classical expectation values

$$M_0 = \langle M_0 \rangle + \delta M_0 \quad B_0 = \langle B_0 \rangle + \delta B_0 \quad \overline{B}_0 = \langle \overline{B}_0 \rangle + \delta \overline{B}_0 \quad (2.6)$$

and then Taylor expand $K$ about the vacuum point $(\langle M_0 \rangle, \langle B_0 \rangle, \langle \overline{B}_0 \rangle)$:

$$K = K_{M_0 \dagger M_0} \text{Tr}(\delta M_0 \dagger \delta M_0) + K_{B_0 \overline{B}_0 \dagger} \delta B_0 \delta \overline{B}_0 \dagger + K_{\overline{B}_0 \overline{B}_0 \dagger} \delta \overline{B}_0 \delta \overline{B}_0 \dagger + \cdots \quad (2.7)$$

where $K_{H \dagger H} \equiv \partial^2 K/\partial H \dagger \partial H$. This power series need not have an infinite radius of convergence. But it should be free of singularities for all finite, complex values of the hadron fields.

---

1 In order for individual Kähler terms to be Lorentz invariant, they must contain an even number of both $D$ and $\overline{D}$ superspace derivatives which transform as $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ under $SL(2, \mathbb{C})$.

2 We drop constant and linear terms in (2.7) which vanish when integrated against $\int d^4\theta$. We also respectively regard $\delta B_0$ and $\delta \overline{B}_0$ as row and column vectors in flavor space. $\delta B_0 \delta B_0 \dagger$ and $\delta \overline{B}_0 \dagger \delta \overline{B}_0$ consequently represent $1 \times 1$ matrices.
In order to develop a systematic perturbation theory within the low energy sigma model, it is convenient to work with renormalized meson and baryon fields which have unit mass dimension rather than their bare counterparts. We relate bare and renormalized fields as follows:

\[
\left(\Lambda^\dagger \Lambda \right)^{1/2} M_0 = \Lambda Z_M^{1/2}(\mu) M(\mu) \\
\left( (\Lambda^\dagger \Lambda)^{N_c-1} K_{B_0B_0^\dagger} \right)^{1/2} B_0 = \Lambda^{N_c-1} Z_B^{1/2}(\mu) B(\mu) \\
\left( (\Lambda^\dagger \Lambda)^{N_c-1} K_{\overline{B}_0\overline{B}_0^\dagger} \right)^{1/2} \overline{B}_0 = \Lambda^{N_c-1} Z_{\overline{B}}^{1/2}(\mu) \overline{B}(\mu).
\]

Since the effective theory does not contain any states with negative norm, the Lehmann-Källen spectral decomposition guarantees that the nonperturbative bounds \(0 \leq Z_M, Z_B, \overline{Z}_{\overline{B}}(\mu) \leq 1\) are satisfied. As we shall see, the logarithmic running to zero of the wavefunction renormalization constants as \(\mu \to 0\) forms the basis for perturbation theory in low energy SUSY QCD.

Working with the renormalized hadron fields in \(d = 4 - \epsilon\) spacetime dimensions, we can now construct the leading independent terms in the Kähler potential’s expansion about \(\langle M \rangle = \langle B \rangle = \langle \overline{B} \rangle = 0\):

\[
K = Z_M \text{Tr}(M^\dagger M) + Z_B (B B^\dagger + \overline{B}^\dagger \overline{B}) \\
+ \frac{1}{\Lambda^\dagger \Lambda} \left\{ Z_1 \text{Tr}(M^\dagger \square M) + Z_2 (B \square B^\dagger + \overline{B}^\dagger \square \overline{B}) \\
+ (4\pi\mu^{\epsilon/2}\lambda) \left[ \frac{Y_1}{\sqrt{N_f}} B (\frac{-D^2}{4} M) \overline{B} + \frac{Y_2}{\sqrt{N_f}} \left( BM (\frac{-D^2}{4} \overline{B}) + (\frac{-D^2}{4} B) M \overline{B} \right) \right] \\
+ (4\pi\mu^{\epsilon/2}\lambda)^* \left[ \frac{Y_1}{\sqrt{N_f}} \overline{B} (\frac{-D^2}{4} M^\dagger) B^\dagger + \frac{Y_2}{\sqrt{N_f}} \left( B^\dagger M^\dagger (\frac{-D^2}{4} B^\dagger) + (\frac{-D^2}{4} B^\dagger) M^\dagger B^\dagger \right) \right] \\
+ |4\pi\mu^{\epsilon/2}\lambda|^2 \left[ \frac{X_1}{N_f^2} \text{Tr}(M^\dagger M) \text{Tr}(M^\dagger M) + \frac{X_2}{N_f} \text{Tr}(M^\dagger M M^\dagger M) \\
+ \frac{X_3}{N_f} (B B^\dagger) (\overline{B}^\dagger \overline{B}) + \frac{X_4}{N_f} \left( (B B^\dagger)^2 + (\overline{B}^\dagger \overline{B})^2 \right) \\
+ \frac{X_5}{N_f^2} \text{Tr}(M^\dagger M) (B B^\dagger + \overline{B}^\dagger \overline{B}) + \frac{X_6}{N_f} (B M M^\dagger B^\dagger + \overline{B}^\dagger M^\dagger M \overline{B}) \right] \}
+ O\left( \left( \frac{1}{\Lambda^\dagger \Lambda} \right)^2 \right).
\]

(2.9)

Following superspace conventions (A.1) and (A.2) listed in the appendix, we include a \(-1/4\) prefactor with every \(D^2\) and \(\overline{D}^2\) which enters into the Kähler potential.
We also rewrite the superpotential as

\[ W = \frac{4\pi \mu^{\epsilon/2}\lambda}{\sqrt{N_f}} B M \overline{B} - \frac{(4\pi \mu^{\epsilon/2}\lambda)^{N_f-2}}{\sqrt{(N_f-1)!}} \frac{A}{\Lambda^{N_f-3}} \det M \]  

(2.10)

where the renormalized couplings are related to the bare \( \lambda_0 \) parameter as

\[
\frac{4\pi \mu^{\epsilon/2}\lambda(\mu)}{\sqrt{N_f}} = \frac{Z_B(\mu)}{K_{B_0B_0}} \left[ \begin{array}{c} Z_M(\mu) \\ K_{M_0^+M_0} \end{array} \right]^{1/2} \frac{\lambda_0}{(\Lambda^+\Lambda)^{N_f-3/2}} \\
\frac{(4\pi \mu^{\epsilon/2}\lambda(\mu))^{N_f-2}}{\sqrt{(N_f-1)!}} A(\mu) = \left[ \begin{array}{c} Z_M(\mu) \\ K_{M_0^+M_0} \end{array} \right]^{N_f/2} \frac{\lambda_0}{(\Lambda^+\Lambda)^{N_f/2}}.
\]  

(2.11)

Several points about these expressions should be noted:

i) SUSY QCD’s discrete charge conjugation symmetry is preserved at the origin of moduli space. As a result, \( K \) and \( W \) remain invariant under \( B \leftrightarrow \overline{B}^T \) and \( M \leftrightarrow M^T \). This \( \mathbb{Z}_2 \) reflection ensures baryon and antibaryon renormalization are identical. It also implies that all trilinear Kähler potential operators involving two superderivatives can be obtained via integration by parts from the \( Y_1 \) and \( Y_2 \) terms in (2.9). For example,

\[
(DB)(DM)\overline{B} + B(DM)(D\overline{B}) \longleftrightarrow -B(D^2M)\overline{B}.
\]  

(2.12)

ii) All interaction terms in \( K \) and \( W \) become weak at low energies. It is important to recall that the non-asymptotically free sigma model flows to a free field theory in the far infrared. Perturbative computations of quantum corrections within a systematic momentum expansion therefore become arbitrarily accurate at lower and lower energies. Although nonperturbative effects are critically important in generating the sigma model from the underlying microscopic gauge theory, they should be small at energies below the SUSY QCD scale in the effective theory itself.

iii) The dimensionless momentum expansion parameter is essentially given by

\[
|\lambda(p^2)|^2 \xrightarrow{p^2 \to 0} \frac{\text{constant}}{\log^2(\Lambda \Lambda/p^2)}.
\]  

(2.13)

We have factored out various powers of \( \lambda \) from the coefficients of nonrenormalizable operators in \( K \) and \( W \) so that \( |\lambda|^2 \) acts as a loop counting parameter. All contributions to squared scattering amplitudes from Feynman graphs with \( E \) external lines and \( L \) loops are proportional to \( |\lambda|^{2(E+2L-2)} \). Unlike chiral perturbation theory for ordinary QCD, the dominant renormalizable \( BM\overline{B} \) superpotential interaction does not involve
derivatives. So $|\lambda|^2$ vanishes only logarithmically as $p^2 \to 0$. In contrast, the $p^2/|\Lambda|^2$ expansion parameter in the QCD chiral lagrangian vanishes linearly.

iv) The sigma model becomes strongly coupled and breaks down at energies close to the SUSY QCD scale. Following the rules of naive dimensional analysis [13–17], we have inserted factors of $4\pi$ into (2.9) and (2.10) so that all tree level and loop contributions to individual scattering processes become comparable in magnitude when $p^2 \simeq |\Lambda|^2$. The dimensionless couplings in $K$ and $W$ are then expected to be $O((4\pi)^0)$ rather than $O(4\pi)$ or $O((4\pi)^{-1})$.

v) If the number of quark flavors is large, another systematic expansion can be formulated in the sigma model based upon $1/N_f$. We have factored out various powers of $N_f$ from the dimensionless couplings so that Feynman graph contributions to Green’s functions remain finite as $N_f \to \infty$. As a result, multiple iterations of weak interactions at low energies do not add together to mimic genuine strong effects at energies comparable to $\Lambda$. Since the baryon and antibaryon transform according to fundamental and antifundamental irreps under the $SU(N_f)_L \times SU(N_f)_R$ chiral symmetry group whereas the meson transforms according to a 2-index bifundamental representation, large $N_f$ power counting within the supersymmetric sigma model is qualitatively similar to large $N_c$ counting in ordinary QCD [18,19].

With the Kähler potential and superpotential expressions in hand, we can straightforwardly work out supergraph Feynman rules for hadron propagators and interaction vertices. Some representative examples are displayed in figure 1. As can be seen in the figure, we treat the $Z_1$ and $Z_2$ terms as quadratic perturbations and do not resum these nonrenormalizable quadratic operators into the meson and baryon propagators. One could choose to eliminate such noncanonical terms from $K$ via a field redefinition. But the hadron propagators would then develop ghost poles and become much more complicated. We should also note that we have annotated the Feynman rule vertices with $D^2$ and $\overline{D}^2$ symbols. These serve as reminders that each internal chiral (antichiral) line attached to a Kähler potential vertex in a supergraph is accompanied by a factor of $-\overline{D}^2/4$ ($-D^2/4$) coming from chiral (antichiral) superfield functional differentiation as indicated in (A.9) [(A.10)]. A similar rule holds for the superpotential interaction terms, except one such squared superderivative factor is used to convert a $\int d^2 \theta$ or $\int d^2 \overline{\theta}$ integral into an integration over all of superspace. After Grassmann derivatives and delta functions are manipulated according to the rules of supergraph perturbation theory [20,21], the integration over fermionic
Fig. 1. Supergraph Feynman rules for some representative propagators and interaction vertices. Solid, dashed and dotted lines respectively denote mesons, baryons and antibaryons, while dark circles represent individual nonrenormalizable Kähler potential operators. Arrows indicate chirality flow.

variables always reduces to a single $\int d^4\theta$ integral. The remaining evaluation of a sigma model supergraph then proceeds along the same lines as for any Feynman diagram in a nonsupersymmetric bosonic field theory.

As a check on the factors of $4\pi$, $\lambda$ and $N_f$ appearing in (2.9) and (2.10), we estimate the magnitudes of several supergraphs which mediate baryon-antibaryon scattering in figure 2. When the energy transfer $p^2$ is much smaller than $|\Lambda|^2$, the first tree diagram involving only the trilinear superpotential interaction dominates over the other renormalizable loop and
nonrenormalizable tree graphs. On the other hand, if $p^2 \simeq |\Lambda|^2$ and the number of quark flavors is small, the coupling $\lambda$ is expected to be of order unity, and all contributions to $B\bar{B} \rightarrow B\bar{B}$ scattering are comparable in size. Finally, we observe that the tree diagrams in the figure with nonrenormalizable operator insertions become more important than some loop graphs involving only the renormalizable superpotential interaction as $N_f \rightarrow \infty$.

It would be interesting to consider correlated combinations of the momentum and $1/N_f$ expansions which could be used to isolate certain subclasses of diagrams that contribute to a particular scattering process. However, we first need to determine how the
“fine structure constant” $|\lambda|^2$ varies with energy. We therefore turn to consider coupling constant evolution in the following section.

3. Renormalization

The low energy effective description of SUSY QCD with $N_f = N_c + 1$ flavors is basically a massless Wess-Zumino model with an infinite number of nonrenormalizable Kähler potential operators. Although we do not know the precise numerical values for these operators’ Wilson coefficients at the scale around $\Lambda$ where the microscopic gauge theory matches onto the sigma model, we can at least calculate how they evolve with energy under the action of the renormalization group. In the absence of mass terms, operators of a given dimension cannot mix down into other operators of lower dimension. As a result, composite operator mixing involves only a finite number of Kähler potential terms at any fixed order in $|\lambda|^2$. This simple but important point is illustrated in figure 3 where we schematically display tree-level and one-loop contributions to certain 1PI Green’s functions.

Wavefunction renormalization in the effective theory is especially simple. Dimensional analysis ensures that the values of the meson and baryon wavefunction renormalization constants are completely insensitive to the nonrenormalizable terms in $K$. The det $M$ superpotential interaction similarly has no impact. The values for $Z_M$ and $Z_B$ can therefore be systematically calculated order by order in $|\lambda|^2$ exactly as in a Wess-Zumino model. After adopting the mass independent renormalization scheme of dimensional regularization plus modified minimal subtraction and evaluating the first one-loop supergraph displayed in figure 3, we find the divergent meson and baryon wavefunction renormalization constants

$$Z_M = 1 - \frac{|\lambda|^2}{N_f} \Delta + O(|\lambda|^4)$$

$$Z_B = 1 - |\lambda|^2 \Delta + O(|\lambda|^4)$$

(3.1)

where $\Delta = 2/\epsilon$. We may trade $2/\epsilon$ for its ultraviolet cutoff regulator analog $\log(\Lambda_{UV}/\mu)^2$ and insert $Z_M$ and $Z_B$ into (2.11) in order to explicitly verify that the superpotential couplings logarithmically vanish as $\mu \to 0$:

$$\lambda(\mu) = \frac{\lambda(|\Lambda|)}{1 + \left(1 + \frac{1}{2N_f}\right)|\lambda|^2 \log\left(\frac{|\Lambda|}{\mu}\right)^2}$$

(3.2)

$$\lambda(\mu)^{N_f-2}A(\mu) = \frac{\lambda(|\Lambda|)^{N_f-2}A(|\Lambda|)}{1 + \frac{1}{2}|\lambda|^2 \log\left(\frac{|\Lambda|}{\mu}\right)^2}.$$
Fig. 3. Some representative contributions to 1PI Green’s functions. Mesons, baryons and antibaryons are all denoted by solid lines in this figure. Curly brackets enclose supergraphs which are of the same order in $(\Lambda^\dagger\Lambda)^{-1}$.

Perturbation theory based upon $|\lambda|^2$ is consequently sensible so long as $\mu \ll |\Lambda|$.

Like all the other dimensionless Wilson coefficients in the Kähler potential, the wavefunction renormalization constants are real. So it is no surprise that they develop nonholomorphic dependence upon $|\lambda|^2$ at one-loop order. This nonanalytic behavior feeds into the superpotential couplings. We recall that $\lambda(\mu)$ and $\lambda(\mu)^{N_f-2}A(\mu)$ come from a common bare constant which multiplies the entire superpotential for $N_f = N_c + 1$ SUSY QCD. But as (3.2) demonstrates, these renormalized couplings do not run at the same rate. Therefore, the relative coefficient between the $BM\overline{B}$ and $\det M$ terms changes with
energy scale, and the exact form of Seiberg’s superpotential is lost once one works with nonholomorphic renormalized fields.

We next consider mixing among the dimension-4 operators in the Kähler potential. We assign them the names

\[
O_1 = \text{Tr}(M^\dagger M) \quad P_1 = \kappa_P B(D^2 M) \overline{B} \\
O_2 = B \overline{B} B^\dagger \quad P_2 = \kappa_P (D^2 B) M \overline{B} \\
\overline{O}_2 = B^\dagger \overline{B} \quad \overline{P}_2 = \kappa_P BM(D^2 \overline{B})
\]

\[
R_1 = \kappa_R [\text{Tr}(M^\dagger M)]^2 \quad R_2 = N_f \kappa_R \text{Tr}(M^\dagger MM^\dagger M) \quad R_3 = N_f \kappa_R (BB^\dagger)(\overline{B}^\dagger \overline{B}) \\
R_4 = N_f \kappa_R (BB^\dagger)^2 \quad R_5 = \kappa_R \text{Tr}(M^\dagger M)BB^\dagger \quad R_6 = N_f \kappa_R BMM^\dagger B \\
\overline{R}_4 = N_f \kappa_R (\overline{B}^\dagger \overline{B})^2 \quad \overline{R}_5 = \kappa_R \text{Tr}(M^\dagger M)B^\dagger \overline{B} \quad \overline{R}_6 = N_f \kappa_R \overline{B}^\dagger M^\dagger M \overline{B}
\]

with \( \kappa_P = -\frac{(4\pi \mu^\epsilon/2)\lambda}{(4\sqrt{N_f})} \) and \( \kappa_R = \frac{|4\pi \mu^\epsilon/2|2}{N_f^2} \). These operators’ coefficients flow under the renormalization group according to

\[
\mu \frac{dC_i(\mu)}{d\mu} = \sum_j (\gamma^T)_{ij} C_j(\mu)
\]

where \( \gamma \) denotes the anomalous dimension matrix which governs how all operators mix with one another. \( \gamma \) is most readily calculated by solving the renormalization group equation for 1PI Green’s functions \( \Gamma_i^{(n_M,n_B,n_{\overline{B}})} \) with \( n_M, n_B, n_{\overline{B}} \) external meson, baryon and antibaryon lines and insertions of composite operators labeled by index \( i \):

\[
\gamma_{ij} \Gamma_j^{(n_M,n_B,n_{\overline{B}})} = - \left[ \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial \lambda} + \beta^* \frac{\partial}{\partial \lambda^*} - n_M \gamma_M - (n_B + n_{\overline{B}}) \gamma_B \right] \Gamma_i^{(n_M,n_B,n_{\overline{B}})}. \]

The meson and baryon anomalous dimensions along with the beta function entering into this last formula are simply related to the wavefunction renormalization constants in (3.1):

\[
\gamma_M = \frac{1}{2} \frac{\mu}{Z_M} \frac{dZ_M}{d\mu} = \frac{|\lambda|^2}{N_f} + O(|\lambda|^4)
\]

\[
\gamma_B = \frac{1}{2} \frac{\mu}{Z_B} \frac{dZ_B}{d\mu} = |\lambda|^2 + O(|\lambda|^4)
\]

\[
\beta = \mu \frac{d\lambda}{d\mu} = \lambda \left[ -\frac{\epsilon}{2} + \gamma_M + 2\gamma_B \right] = \lambda \left[ -\frac{\epsilon}{2} + (2 + \frac{1}{N_f}) |\lambda|^2 + O(|\lambda|^4) \right].
\]

The basic forms for one-loop supergraphs with a single insertion of an \( O, P \) or \( R \) operator which contribute to 2, 3 or 4 point Green’s functions are illustrated in figure 3.
Although the coefficients of certain linear combinations of negative eigenvalues of $\gamma$ are complicated, unenlightening functions of $N_f$. But with a symbolic manipulator, one may readily check that they are real for $N_f \geq 4$ and approach the limiting set $(-4, -2, 0, 7, 4)$ with subscripts indicating eigenvalue multiplicities as $N_f \to \infty$.

After decomposing the anomalous dimension matrix as $\gamma = SDS^{-1}|\lambda|^2 + O(|\lambda|^4)$ where diagonal matrix $D$ contains the eigenvalues of $\tilde{\gamma}$ while matrix $S$ holds its eigenvectors, we can explicitly solve the differential equation (3.3) for the dimension-4 operators’ coefficients:

$$C_i(\mu) = \sum_{j,k} (S^{-1}T)_{ij} \left[ \frac{|\lambda(\mu)|^2}{|\lambda(\Lambda)|^2} \right]^{D_j/4 + 2/N_f} (S^T)_{jk} C_k(|\Lambda|).$$

(3.11)

Although the coefficients of certain linear combinations of $O$, $P$ and $R$ corresponding to negative eigenvalues of $\gamma$ are logarithmically enhanced as $\mu \to 0$, these Kähler potential
operators are still irrelevant at long distance scales provided $|\lambda|^2$ is small and perturbation theory is valid. So regardless of the signs of nonrenormalizable operator anomalous dimension eigenvalues, SUSY QCD with $N_f = N_c + 1$ flavors flows to a free field theory in the far infrared.

With only our crude $O(1)$ estimates for Wilson coefficients at the matching scale $\mu \simeq |\Lambda|$ between the gauge theory and sigma model, the practical utility of the renormalization group results encoded into (3.11) is small at present. But if precise matching conditions can someday be determined, the values for Kähler coefficients at all energies below the confinement scale will then be fixed as well.

4. Flat direction deformations

In his original study of SUSY QCD’s confining phase, Seiberg exploited the fact that the theory with $N_f$ flavors and $N_c$ colors must flow to essentially the same model with fewer flavors and colors when one or more matter fields are given arbitrarily large expectation values and heavy fields are decoupled $\mathbb{I}$. This consistency requirement plays an important role in the exact determination of the dynamically generated superpotential. As we shall see, the same recursive condition also yields nontrivial information about the dimensionless coefficients which enter into the Kähler potential.
We will focus upon the relationship between Kähler sector coefficients at the origin of moduli space and those at the point

$$
\langle M_0 \rangle = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \quad \langle B_0 \rangle = \langle \overline{B}_0 \rangle = 0
$$

with \( a \in \mathbb{R} \). We have performed a biunitary flavor rotation to bring the bare meson field into diagonal form and then frozen its \( N_f \)th component at a classical expectation value whose magnitude greatly exceeds \( \Lambda^4 \). The large vev for the last meson field may be regarded as arising from underlying quark and antiquark expectation values which break the gauge group. So at this moduli space location, \( N_f = N_c + 1 \) SUSY QCD connects onto the theory with \( N_f - 1 \) flavors and \( N_c - 1 \) colors, and the scales \( \Lambda \) and \( \Lambda_L \) in the upstairs and downstairs microscopic theories are related by the matching condition \( \Lambda^{2N_f-3} = a^2 \Lambda_L^{2N_f-5} \). This classical Higgsing interpretation makes sense only if \( a \gg |\Lambda| \).

At the point (4.1), the \( N_f \)th diagonal meson fluctuation is eaten via the Higgs mechanism, while the first \( N_f - 1 \) baryon fluctuations are linear in \( a \). The Taylor expansion (2.7) of the bare Kähler potential therefore looks like

$$
K = K_{M_0, M_0} (a^2; \Lambda; N_f) \text{Tr} (\delta \widehat{M}_0^\dagger \delta \widehat{M}_0) + a^2 K_{B_0 B_0} (a^2; \Lambda; N_f) [\delta \widehat{B}_0 \delta \widehat{B}_0^\dagger + \delta \widehat{B}_0^\dagger \delta \widehat{B}_0] \\
+ K_{B_0 B_0} (a^2; \Lambda; N_f) [\delta B_0 N_f \delta B_0^{N_f \dagger} + \delta B_0^{N_f \dagger} \delta B_0^{N_f}] + \cdots
$$

where hatted fields signify massless hadrons which survive in the low energy description of \( (N_f - 1) = (N_c - 1) + 1 \) SUSY QCD. The bare superpotential can similarly be rewritten in terms of downstairs theory hadrons:

$$
W = \lambda_0 [\frac{\delta \widehat{B}_0 \delta \widehat{M}_0 \delta \widehat{B}_0^\dagger - \text{det} \delta \widehat{M}_0}{\Lambda_L^{2N_f-5}}] + \frac{\lambda_0 a^2}{\Lambda_L^{2N_f-3}} \delta B_0 N_f \delta B_0^{N_f}.
$$

After using the superfield equations of motion

$$
\begin{align*}
\frac{\mathcal{D}^2}{4} \frac{\delta K}{\delta (\delta B_0 N_f)} - \frac{\delta W}{\delta (\delta B_0 N_f)} &= 0 \\
\frac{\mathcal{D}^2}{4} \frac{\delta K}{\delta (\delta B_0 N_f^\dagger)} - \frac{\delta W}{\delta (\delta B_0 N_f^\dagger)} &= 0
\end{align*}
$$

(4.4)
to integrate out the massive $\delta B_{0N_f}$ and $\delta \overline{B}_0^{N_f}$ fluctuations and applying superspace identities (A.1) and (A.2), we find that the kinetic terms for the $N_f$th baryon and antibaryon in (4.2) are precisely canceled. The Kähler potential and superpotential then reduce to those for $(N_f - 1) = (N_c - 1) + 1$ SUSY QCD as $a^2 \to \infty$ provided

$$
K_{M_0^+ M_0}(a^2; \Lambda; N_f) \equiv f\left(\frac{a^2}{\Lambda^2}; N_f\right) \rightarrow K_{M_0^+ M_0}(0; \Lambda_L; N_f - 1) = \frac{f(0; N_f - 1)}{\Lambda_L^2 \Lambda_L}
$$

$$
K_{B_0 B_0^+}(a^2; \Lambda; N_f) \equiv g\left(\frac{a^2}{(\Lambda^2 \Lambda)^{N_c-1}}; N_f\right) \rightarrow \frac{1}{a^2} K_{B_0 B_0^+}(0; \Lambda_L; N_f - 1) = \frac{g(0; N_f - 1)}{a^2(\Lambda_L^2 \Lambda_L)^{N_c-2}}.
$$

In this recursion relation, we have factored out inverse powers of the SUSY QCD scale from the Kähler coefficients which are trivially fixed by dimensional analysis. The asymptotic $a^2$ dependence of the dimensionless $f$ and $g$ functions is then basically set by the scale matching condition. Recalling $Z_M \propto K_{M_0^+ M_0} \propto f$ and $Z_B \propto K_{B_0 B_0^+} \propto g$, we deduce

$$
Z_M(\mu; a^2; N_f) = \left(\frac{a^2}{\Lambda^2 \Lambda}\right)^{\frac{1}{2(N_f - 1)}} \left[1 + O(|\lambda(\mu)|^2)\right]
$$

$$
Z_B(\mu; a^2; N_f) = \left(\frac{a^2}{\Lambda^2 \Lambda}\right)^{-\frac{1}{2(N_f - 1)}} \left[1 + O(|\lambda(\mu)|^2)\right].
$$

As the $N_f$th baryon fluctuations were integrated out using just classical equations of motion, the $O(|\lambda|^2)$ quantum corrections to the meson and baryon wavefunction renormalization constants are not determined by these tree-level arguments. But they could readily be found by evaluating a few massive one-loop supergraphs.

The dependence of other Kähler potential couplings upon moduli space location may be worked out along similar lines. In general, they each vary as $a^2 \to \infty$. On the other hand, all $a^2$ dependence cancels out from the superpotential couplings as can be seen in (2.11). The renormalized $\lambda(\mu)$ expansion parameter consequently remains uniform throughout the entire moduli space like its bare $\lambda_0$ progenitor.

5. Conclusion

The analysis of the low energy Kähler potential in $N_f = N_c + 1$ SUSY QCD which we have presented in this article represents only a modest first step, and many extensions of this work would be interesting to pursue. In particular, it would clearly be valuable to determine how asymptotic limits restrict Kähler sector coefficients at the origin of moduli
space. As Seiberg emphasized in his study of SUSY QCD’s confining phase superpotential, recovering classical results in weakly coupled regions of moduli space represents a key constraint which must be satisfied by the correct quantum description of the low energy theory [2]. It may be possible to determine, for instance, how the coefficients of Kähler potential terms at the moduli space origin must be adjusted so that $K \rightarrow \sqrt{M_0^† M_0}$ along the flat direction $\langle \det M_0 \rangle \neq 0$, $\langle B_0 \rangle = \langle \overline{B}_0 \rangle = 0$ within the classical domain where the gauge group is completely broken. Such partial power series information might provide nontrivial insight into the full functional form for $K$.

Other avenues would also be worth exploring. For example, it should be straightforward to derive low energy Kähler potentials in SUSY QCD with $N_f < N_c + 1$ quark flavors by giving mass to matter fields and integrating them out from (2.9) and (2.10). The results ought to be simpler than those which we have found here, for $N_f < N_c + 1$ SUSY QCD contains fewer types of hadrons than the theory with $N_c + 1$ flavors.

Finally, the basic ideas underlying this work can be applied to investigate Kähler sectors in many other confining supersymmetric gauge theories. It would be especially interesting to examine scattering processes in models with intricate quantum superpotentials. As we have seen, leading order kinetic operator coefficients are systematically calculable in theories with one or more renormalizable superpotential interaction terms. So exact moduli space information encoded into dynamically generated superpotentials should not be completely obscured in scattering calculations by Kähler potential uncertainties. Instead, one may be able to uncover relations among low order hadron scattering amplitudes. Such findings would represent a novel departure from vacuum structure analysis.

Acknowledgments

It is a pleasure to thank Marc Grisaru for his many patient and enlightening tutorials on supergraph perturbation theory. I am also grateful to Howard Georgi, Per Kraus, Lisa Randall and Sandip Trivedi for helpful discussions. This work was supported by the National Science Foundation under Grant #PHY-9218167.
Appendix. Superspace conventions

Since the number of different supersymmetry conventions appearing in the literature is comparable to the total number of supersymmetry publications, we list here those which we follow in this article along with some useful superspace identities:

Spacetime metric signature: \((+,-,-,-)\)

Grassmann measures and delta functions:

\[
\int d^2\theta = -\frac{1}{4} D^2 |_{\theta = \bar{\theta} = 0} \quad (A.1)
\]

\[
\int d^2\bar{\theta} = -\frac{1}{4} \bar{D}^2 |_{\theta = \bar{\theta} = 0} \quad (A.2)
\]

\[
\delta^{(4)}(\theta) = (\theta \theta)(\bar{\theta} \bar{\theta}) \quad (A.3)
\]

Superspace derivative (anti) commutator relations:

\[
\{D_\alpha, \overline{D}_\beta\} = -2i \sigma^{\mu}_{\alpha \beta} \partial_\mu \quad (A.4)
\]

\[
[D^2, \overline{D}^2] = 8i (\overline{D} \sigma^\mu D) \partial_\mu - 16 \Box \quad (A.5)
\]

\[
[\overline{D}^2, D^2] = 8i (D \sigma^\mu \overline{D}) \partial_\mu - 16 \Box \quad (A.6)
\]

\[D\text{-algebra identities:}\]

\[
\overline{D}\sigma^\mu D + D\sigma^\mu \overline{D} = -4i \partial^\mu \quad (A.7)
\]

\[
\delta^{(4)}(\theta_x - \theta_y) D^2 \overline{D}^2 \delta^{(4)}(\theta_x - \theta_y) = 16 \delta^{(4)}(\theta_x - \theta_y) \quad (A.8)
\]

Chiral superfield functional derivatives:

\[
\frac{\delta \Phi(x, \theta)}{\delta \Phi(x', \theta')} = -\frac{\overline{D}^2}{4} \delta^{(4)}(x - x') \delta^{(4)}(\theta - \theta') \quad (A.9)
\]

\[
\frac{\delta \Phi^\dagger(x, \bar{\theta})}{\delta \Phi^\dagger(x', \bar{\theta})} = -\frac{D^2}{4} \delta^{(4)}(x - x') \delta^{(4)}(\theta - \theta') \quad (A.10)
\]
References

[1] For reviews, see K. Intriligator and N. Seiberg, hep-th/9509066, Nucl. Phys. Proc. Suppl. 45BC (1996) 1; M.E. Peskin, hep-th/9702094; M. Shifman, hep-th/9704114, Prog. Part. Nucl. Phys. 39 (1997) 1.
[2] N. Seiberg, Phys. Rev. D49 (1994) 6857.
[3] K. Intriligator and P. Pouliot, Phys. Lett. B353 (1995) 471.
[4] I. Pesando, Mod. Phys. Lett A10 1995, 1871.
[5] S.B. Giddings and J.M. Pierre, Phys. Rev. D52 (1995) 6065.
[6] E. Poppitz and S. Trivedi, Phys.Lett. B365 (1996) 125.
[7] P. Cho and P. Kraus, Phys. Rev. D54 (1996) 7640.
[8] C. Csáki, M. Schmaltz and W. Skiba, Nucl. Phys. B487 (1997) 128.
[9] C. Csáki, M. Schmaltz and W. Skiba, Phys.Rev. D55 (1997) 7840.
[10] K. Intriligator, R.G. Leigh and N. Seiberg, Phys. Rev. D50 (1994) 1092.
[11] K. Intriligator, Phys. Lett. B336 (1994) 409.
[12] N. Seiberg, Nucl. Phys. B435 (1995) 129.
[13] H. Georgi and A. Manohar, Nucl. Phys. B234 (1984) 189.
[14] H. Georgi and L. Randall, Nucl. Phys. B276 (1986) 241.
[15] H. Georgi, Phys. Lett. B298 (1993) 187.
[16] M. Luty, Phys.Rev. D57 (1998) 1531.
[17] A. Cohen, D. Kaplan and A. Nelson, Phys. Lett. B412 (1997) 301.
[18] G. t’Hooft, Nucl. Phys. B72 (1974) 461.
[19] E. Witten, Nucl. Phys. B160 (1979) 57.
[20] M.T. Grisaru, W. Siegel and M. Roček, Nucl. Phys. B159 (1979) 429.
[21] S.J. Gates, M.T. Grisaru, M. Roček and W. Siegel, Superspace (Benjamin/Cummings Publishing Co., Reading, MA 1983).
[22] G. t’Hooft, Acta Phys. Austr. Suppl. 22 (1980) 531.