Critical Amplitudes in Two-dimensional Theories.

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Abstract

We derive exact analytical expressions for the critical amplitudes $A_\psi, A_{\text{gap}}$ in the scaling laws for the fermion condensate $\langle \bar{\psi} \psi \rangle = A_\psi m^{1/3} g^{2/3}$ and for the mass of the lightest state $M_{\text{gap}} = A_{\text{gap}} m^{2/3} g^{1/3}$ in the Schwinger model with two light flavors, $m \ll g$. $A_\psi$ and $A_{\text{gap}}$ are expressed via certain universal amplitude ratios being calculated recently in TBA technique and the known coefficient $A_{\psi \psi}$ in the scaling law $\langle \bar{\psi} \psi(x) \bar{\psi} \psi(0) \rangle = A_{\psi \psi}(g/x)$ at the critical point. Numerically, $A_\psi = -0.388 \ldots$, $A_{\text{gap}} = 2.008 \ldots$. The same is done for the standard square lattice Ising model at $T = T_c$. Using recent Fateev’s results, we get $\langle \sigma_{\text{lat}} \rangle = 1.058 \ldots (H_{\text{lat}}/T_c)^{1/15}$ for the magnetization and $M_{\text{gap}} = a/\xi = 4.010 \ldots (H_{\text{lat}}/T_c)^{8/15}$ for the inverse correlation length ($a$ is the lattice spacing). The theoretical prediction for $\langle \sigma_{\text{lat}} \rangle$ is in a perfect agreement with numerical data. Two available numerical papers give the values of $M_{\text{gap}}$ which differ from each other by a factor $\approx \sqrt{2}$. The theoretical result for $M_{\text{gap}}$ agrees with one of them.

1 Introduction.

A system with second order phase transition exhibits a critical behavior at its vicinity. For example, the order parameter $\langle \phi \rangle$, which is zero at the phase transition point $T = T_c$ when the corresponding symmetry is not explicitly broken by external field, behaves as

$$\langle \phi \rangle_h(T = T_c) = A_\phi h^{1/\delta}$$ (1.1)

when external field is present. The exponent $1/\delta$ is one of the critical exponents. Another critical exponent $\mu$ appears in the scaling behavior of the mass gap (the inverse correlation length)

$$M_{\text{gap}}^h(T = T_c) = A_{\text{gap}} h^\mu$$ (1.2)

The third critical exponent $\zeta$ we will be interested in determines the power fall-off of the correlator $\langle \phi(x)\phi(0) \rangle$ at $T = T_c$ at large distances

$$\langle \phi(x)\phi(0) \rangle_{h=0,T=T_c} = A_{\phi\phi} |x|^{-(d-2+\zeta)}, \quad |x| \to \infty$$ (1.3)

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where \( d \) is the spatial dimension. Three exponents \( \delta, \mu, \) and \( \zeta \) are not independent parameters but satisfy two scaling relations (see e.g. [1])

\[
\begin{align*}
\delta(d\mu - 1) &= 1 \\
\delta \mu(d - 2 + \zeta) &= 2
\end{align*}
\]  

(1.4)

Other critical exponents relate to the scaling behavior of the system at small non-zero \( |T - T_c| \) (see [1] for the full list).

The important fact is that many quite different physical systems can have the same values of critical exponents (they depend only on gross symmetry features). This property is usually referred to as universality. Universality is due to the fact that at small \( |T - T_c| \) and small external fields \( h \), the correlation length is high. At large distances, a critical system “forgets” about peculiarities of microscopic interactions, is scale invariant, and is described by an effective conformal field theory.

On the other hand, the values of the critical amplitudes like \( A_\phi \) in (1.3) are not universal. Really, the order parameter \( \phi \) and the external field \( h \) have distinct physical dimensions. The coefficient \( A_\phi \) also carries a dimension and depends on dimensionful constants in the microscopic Hamiltonian.

It is well known, however, that certain dimensionless combinations of critical amplitudes exist which, like exponents, are determined by only a large–distance behavior of the system and are universal [2]. Actually, any scaling relation between exponents is associated with a certain universal ratio.

As an illustration, consider the shift in free energy density of the critical system at \( T = T_c \) due to the presence of external field \( h \). We have

\[
\Delta F(h) = b T_c M_{\text{gap}}^d
\]  

(1.5)

where \( b \) is a dimensionless numerical coefficient which is universal. Bearing in mind that \( <\phi>_h = \partial F(h)/\partial h \), we derive the first relation in (1.4) and, simultaneously, that the dimensionless ratio

\[
r_1 = \frac{A_\phi}{T_c A_{\text{gap}}^d} = b \mu d
\]  

(1.6)

is universal.

The second universal relation in (1.4) is derived considering the correlator of order parameters \( <\phi(x)\phi(0)> \) at \( T = T_c \) in the presence of external field. When \( h \) is small, the correlator exhibits first a power fall–off as in (1.3). Then at \( |x| \sim \xi = M_{\text{gap}}^{-1} \), the behavior of the correlator is modified. Asymptotically, it tends to \( <\phi>_h^2 \). Preasymptotic terms decay exponentially \( \sim \exp\{-M_{\text{gap}}|x|\} \). When the correlation length is high, the behavior of correlator at the distances \( |x| \sim \xi \) should not depend on the details of microscopic interactions but only on the dynamics of the effective conformal theory describing a critical system in the scaling regime. In other words,

\[
<\phi(|x| = \xi) \phi(0)> = c <\phi>^2
\]
where \( c \) is a universal constant. We derive thereby a second relation in (1.4) and also that the dimensionless ratio

\[
   r_2 = \frac{A_\phi^2}{A_{\phi\phi} A_{\text{gap}}} = c
\]

is universal.

In physical 3–dimensional systems, the values of critical exponents and universal critical ratios are calculated numerically as a series over the parameter \( \epsilon = 4 - d \). In many two–dimensional statistical systems, both can be determined analytically (the mathematical reason for that is that conformal group in two dimensions is much richer and imposes much stringer constraints on the behavior of the system that at \( d \geq 3 \)).

The values of exponents for the Ising model and many other exactly solved two–dimensional critical systems were known for a long time. Recently, it has become clear that many 2D critical systems in some vicinity of critical point (in particular, the Ising model at \( T = T_c \) in weak external magnetic field) are described by exactly integrable two–dimensional field theories with a known \( S \)-matrix. The ingenious Thermodynamic Bethe Ansatz (TBA) technique has been developed which allowed one to evaluate the universal ratios \( r_{1,2} \) analytically.

With the ratios \( r_{1,2} \) at hand, one only need to know one of the amplitudes \( A_\phi, A_{\text{gap}}, A_{\phi\phi} \) to determine two others. In this paper, we exploit this fact and discuss two examples of two-dimensional exactly solved models with second order phase transition — the Ising model on the square lattice and the Schwinger model with two fermion flavors. In both cases, the critical amplitude \( A_{\phi\phi} \) (which is not universal and depends on a particular form of the microscopic hamiltonian) has been determined from independent premises.

Thereby the amplitudes \( A_\phi \) and \( A_{\text{gap}} \) can also be determined.

## 2 Ising Model.

A classical example of an exactly solved two–dimensional critical system is the Ising model. The hamiltonian of the model reads

\[
   \mathcal{H} = -J \sum_{ij} (\sigma_{ij} \sigma_{i+1,j} + \sigma_{ij} \sigma_{i,j+1}) - H \sum_{ij} \sigma_{ij}
\]

The corresponding partition function is

\[
   Z = \text{Tr} \exp\{-\mathcal{H}/T\} = \sum_{\{\sigma_{ij}\}} \exp\left( \beta \sum_{ij} (\sigma_{ij} \sigma_{i+1,j} + \sigma_{ij} \sigma_{i,j+1}) + h \sum_{ij} \sigma_{ij} \right)
\]

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where $\beta = J/T$, $h = H/T$. It was known for a long time that at $\beta = \beta_c = \frac{1}{2}\ln(1+\sqrt{2})$

$$<\sigma>_{h} = A_\sigma \text{ sign}(h) h^{1/15}$$

$$M_{\text{gap}} = A_{\text{gap}} h^{8/15}$$

$$<\sigma(x)\sigma(0)>_{h=0} = \frac{A_{\sigma\sigma}}{|x|^{1/4}}$$  \tag{2.3}$$

The coefficient $A_{\sigma\sigma}$ was determined some time ago by direct evaluation of the lattice correlator in the theory (2.2)

$$<\sigma_{NN}\sigma_{00}> \sim \frac{C}{N^{1/4}}, \quad N \to \infty$$  \tag{2.4}$$

where $C = .645\ldots$ is a known transcendental constant \cite{3}. From this, one easily gets

$$A_{\sigma\sigma} = C 2^{1/8} = .703\ldots$$  \tag{2.5}$$

where distance is measured in the units of lattice spacing which we set to one in the subsequent discussion. The constant $A_{\sigma\sigma}$ depends on the particular form of the hamiltonian (2.1) and is not universal - its value is different, say, on a triangle lattice or in a model with not only nearest neighbors interaction.

As was already noted, the analytical determination of the universal ratios (1.6, 1.7) has become possible only recently after a beautiful A. Zamolodchikov’s work who described the Ising model at critical temperature and at weak external magnetic field (so that the correlation length is much larger than the lattice spacing) as a perturbed conformal field theory

$$S_{\text{Ising}}(h) = S_{\text{Ising}}(0) - h^{CFT} \int \sigma^{CFT}(x) \ d^2x$$  \tag{2.6}$$

where $\sigma^{CFT}$ is the conformal spin field normalized such that

$$<\sigma^{CFT}(x)\sigma^{CFT}(0)>_{h^{CFT}=0} = \frac{1}{|x|^{1/4}}$$  \tag{2.7}$$

so that $A^{CFT}_{\sigma\sigma} = 1$. Zamolodchikov found out that the model is exactly integrable involving an infinite number of conserved charges. The spectrum of the model includes 8 states with definite peculiar mass ratios. These states scatter on each other without reflection and the $S$–matrix is exactly calculable \cite{4}.

TBA technique \cite{5} allows one to find the universal ratios $r_{1,2}$. The ratio $r_1$ in the wide class of theories has been found in \cite{4}. Speaking precisely, the coefficient $b$ entering the free energy density (1.5) was evaluated. The general formula is

$$b = -\frac{1}{2\phi_{11}^{(1)}}$$  \tag{2.8}$$

where $\phi_{11}^{(1)}$ is a certain constant extracted from the high energy asymptotics of the scattering amplitude of the state with the lowest mass:

$$\phi_{11}^{(1)} = i \lim_{\theta \to \infty} e^{\theta} \frac{d}{d\theta} S_{11}(\theta)$$
\[ \theta \text{ is the rapidity). For the Ising model,} \]
\[ b = \frac{1}{16\sqrt{3}\cos(\pi/30)\sin(\pi/5)} \quad (2.9) \]

The ratio \( r_1 \) is then given by Eq. (1.6). The ratio \( r_2 \) (actually, the combination \( r_1^2/r_2 \)) has been determined recently by Fateev [8]. The results can be presented as explicit expressions for the amplitudes \( A_\sigma \) and \( A_{gap} \) when the normalization convention (2.7) is chosen (see also [9]). The result is

\[ <\sigma^{CFT}> = \frac{8}{15} \left( \frac{4\pi^2 \Gamma^2(\frac{13}{16}) \Gamma(\frac{3}{4})}{\Gamma^2(\frac{3}{10}) \Gamma(\frac{1}{4})} \right)^{8/15} \frac{\sin(\frac{\pi}{3}) \sin(\frac{8\pi}{15})}{\sin(\frac{\pi}{5}) \sin(\frac{\pi}{30})} \left( h^{CFT} \right)^{1/15} = 1.277 \ldots \left( h^{CFT} \right)^{1/15} \]

\[ M_{gap} = \left( \frac{4\pi^2 \Gamma^2(\frac{13}{16}) \Gamma(\frac{3}{4})}{\Gamma^2(\frac{3}{10}) \Gamma(\frac{1}{4})} \right)^{4/15} \frac{4\sin(\frac{\pi}{3}) \Gamma(\frac{1}{4})}{\Gamma(\frac{3}{5}) \Gamma(\frac{7}{15})} \left( h^{CFT} \right)^{8/15} = 4.404 \ldots \left( h^{CFT} \right)^{8/15} \quad (2.10) \]

To find out the critical amplitudes relating the physical spin expectation value and the physical correlation length to the physical magnetic field in the standard Ising model on square lattice, one should take into account the difference in normalizations of \( \sigma^{lat} \) and \( h^{lat} \) vs \( \sigma^{CFT} \) and \( h^{CFT} \). Comparing (2.2) and (2.4) with (2.6) and (2.7), we obtain

\[ \sigma^{lat} = A^{1/2}_\sigma \sigma^{CFT}, \quad h^{lat} = A^{-1/2}_\sigma h^{CFT} \quad (2.11) \]

From this and Eq. (2.10) a final result can be derived

\[ <\sigma^{lat}> = 1.277 \ldots (A,\sigma)^{8/15} \left( h^{lat} \right)^{1/15} = 1.058 \ldots \left( h^{lat} \right)^{1/15} \quad (2.12) \]

\[ M_{gap} = 1/\xi = 4.404 \ldots (A,\sigma)^{8/15} \left( h^{lat} \right)^{8/15} = 4.010 \ldots \left( h^{lat} \right)^{8/15} \quad (2.13) \]

The theoretical result for \(<\sigma^{lat}>\) perfectly agrees with the available numerical data \(<\sigma^{lat}> = 0.999(1)(h^{lat}/\beta_c)^{1/15} [10]\) and \(<\sigma^{lat}> = 1.003(2)(h^{lat}/\beta_c)^{1/15} [11]\). The prediction (2.13) for the mass gap agrees well with the numerical result \( \xi = 0.38(1)(h^{lat}/\beta_c)^{-8/15} [11]\) but dramatically disagrees (by the factor \( \sim \sqrt{2} \)) with the numerical result \( M_{gap} = 1.839(7)(h^{lat}/\beta_c)^{8/15} \) as given in [11]. Thereby these two numerical works contradict to each other at this point. One can further notice that the theoretical value of the universal constant \( b \) as quoted in [11] is twice as large as it should be [ the factor 8 instead of 16 in Eq. (2.9) ]. The reasons of this disagreement are not clear.
3 Schwinger model with $N_f = 2$.

Our remark is that the critical coefficients can also be determined along similar lines in another exactly solved model with critical behavior — the Schwinger model with two fermion flavors. The Euclidean lagrangian of the model is

$$\mathcal{L} = \frac{1}{2} F^2 - i \sum_{f=1,2} \bar{\psi}_f \gamma^\mu (\partial_\mu - igA_\mu) \psi_f + m \sum_{f=1,2} \bar{\psi}_f \psi_f$$

where $F = F_{01}$ and $\gamma^\mu$ are anti-hermitian. All fields live in 1+1 dimensions. The coupling constant $g$ has the dimension of mass. At $m = 0$, the theory enjoys the chiral $SU_L(2) \otimes SU_R(2)$ symmetry much like as standard QCD. The corresponding order parameter is the fermion condensate $<\bar{\psi}_1 \psi_1> = <\bar{\psi}_2 \psi_2>$. The appearance of non-zero condensate would break spontaneously chiral symmetry. Spontaneous breaking of a continuous symmetry is not possible, however, in 1+1 dimensions \[12\]. Hence the condensate is zero when $m = 0$.

In spite of the absence of the ordered phase, it has been shown that the second order phase transition still occurs in the massless multiflavor Schwinger model at zero critical temperature \[13\]. That means that at $T = 0$ and at small positive temperatures the system behaves much like a critical system at the phase transition point or slightly above. Also at zero temperature and at small non-zero fermion mass, the correlators, the fermion condensate, and mass gap exhibit a critical behavior

$$<\bar{\psi}_1 \psi_1>_m = -A_{\psi\psi} m^{-N_f-1} g^{N_f+1} \quad M_{\text{gap}}(m) = A_{\text{gap}} m^{-N_f+1} g^{N_f+1}$$

$$<\bar{\psi}_1 \psi_1(x) \bar{\psi}_1 \psi_1(0)>_{m=0} = A_{\psi\psi} \frac{g^{2/N_f}}{x^{2-2/N_f}}, \quad gx \gg 1$$

where $N_f$ is the number of light flavors. A small fermion mass $m \ll g$ plays here the same role as the small magnetic field $h$ in the Ising model. It breaks explicitly the chiral symmetry $SU_L(N_f) \otimes SU_R(N_f)$ of the massless theory down to $SU_V(N_f)$. The values of critical exponents in (3.2) have been calculated analytically for any $N_f$ in \[14\]–\[17\]. The critical coefficient $A_{\psi\psi}$ for the fermion correlator can also be easily determined \[17\] using the fact that the corresponding path integral has a Gaussian form and can be calculated exactly. We have

$$A_{\psi\psi} = \frac{e^{2\gamma/N_f}}{2\pi^2} \left(\frac{N_f}{4\pi}\right)^{1/N_f}$$

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1 To avoid confusion, note that the universal ratio $r_1$ is defined now without the factor $(T_c)^{-1}$ as in Eq. (1.6) here. The matter is, in statistical systems everything is usually defined via free energy density $F = -T/V \ln Z$ while in a field theory a more natural quantity is the vacuum energy density $\epsilon_{\text{vac}} = -1/V^{\text{Eucl}} \ln Z$. Non-zero temperatures in Schwinger model would correspond to the same theory defined on an Euclidean 2-dimensional cylinder whereas changing the ”statistical temperature” means changing coupling constants or adding extra terms in the Euclidean theory framework.
where $\gamma = 0.577\ldots$ is the Euler constant. The coefficient $A_{\psi\psi}$ is sensitive to the short distance region of the theory $x \sim 1/g$ (the full coefficient $A_{\psi\psi} g^{2/N_f}$ depends explicitly on the intrinsic mass scale $g$) and is not universal.

In this paper we determine the coefficients $A_\psi$ and $A_{\text{gap}}$ in the case $N_f = 2$. Our starting point is the abelian bosonization procedure of [14]. We identify

$$i\bar{\psi}_1\gamma_\mu \psi_1 \equiv \frac{1}{\sqrt{\pi}} \epsilon_{\mu\nu} \partial_\nu \phi_1, \quad m\bar{\psi}_1\psi_1 \equiv -C \cos\sqrt{4\pi}\phi_1$$

$$i\bar{\psi}_2\gamma_\mu \psi_2 \equiv \frac{1}{\sqrt{\pi}} \epsilon_{\mu\nu} \partial_\nu \phi_2, \quad m\bar{\psi}_2\psi_2 \equiv -C \cos\sqrt{4\pi}\phi_2$$

(3.4)

The original theory (3.1) is equivalent to the bosonic theory

$$\mathcal{L} = \frac{1}{2} F^2 + \frac{1}{2} (\partial_\mu \phi_1)^2 + \frac{1}{2} (\partial_\mu \phi_2)^2 + ig \frac{F}{\sqrt{\pi}} (\phi_1 + \phi_2)$$

$$- C \left( \cos\sqrt{4\pi}\phi_1 + \cos\sqrt{4\pi}\phi_2 \right)$$

(3.5)

in a sense that it has the same spectrum and that all correlators of fermion currents in the theory (3.3) coincide with the correlators of the corresponding bosonic currents in the theory (3.5) [18]. We can integrate now over $F$ and arrive at the following bosonic lagrangian involving only physical degrees of freedom

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi_+)^2 + \frac{1}{2} (\partial_\mu \phi_-)^2 + \frac{g^2}{2\pi} \phi_+^2 - 2C \cos(\sqrt{2\pi}\phi_+) \cos(\sqrt{2\pi}\phi_-)$$

(3.6)

where $\phi_\pm = (\phi_1 \pm \phi_2)/\sqrt{2}$.

If the original fermion theory is massless, $C = 0$ and we have the theory of two free bosonic fields. One of them ($\phi_+$) has the mass

$$\mu_+^2 = \frac{2g^2}{\pi}$$

(3.7)

and the other ($\phi_-$) is massless. The absence of the mass gap means that the correlation length of the system is infinite. That results in the power fall-off of the correlator of order parameters $C_{11}(\mathbf{x}) = <\bar{\psi}_1\psi_1(\mathbf{x}) \bar{\psi}_1\psi_1(0)> \sim g/|\mathbf{x}|$ at large distances which characterizes a critical system at the phase transition point. If $m$ is non-zero but small, $C$ is also non-zero and small and the fields begin to interact. We are interested in the dynamics of the system at large distances and small energies. Then the heavy field $\phi_+$ decouples (it freezes down at the value $\phi_+ = 0$) and the system is described by the effective lagrangian involving only the light field $\phi_-:

$$\mathcal{L}_{\text{eff}} = \frac{1}{2} (\partial_\mu \phi_-)^2 - 2C \cos(\sqrt{2\pi}\phi_-)$$

(3.8)

In this limit,

$$m\bar{\psi}_1\psi_1 \equiv m\bar{\psi}_2\psi_2 \equiv -C \cos(\sqrt{2\pi}\phi_-)$$

(3.9)
and the correlators \( C_{11}(x) \) and \( C_{12}(x) = \langle \bar{\psi}_1 \psi_1(x) \bar{\psi}_2 \psi_2(0) \rangle \) coincide.

The lagrangian (3.8) describes the sine-Gordon model which, like (2.6), can be treated as a perturbation of the conformal theory \( \mathcal{L} = (\partial_\mu \phi)^2/2 \). Like (2.6), the model (3.8) is exactly solved and can be analyzed along similar lines. Actually, this analysis is much simpler here. The sine-Gordon model is the first known example of a non-trivial non-linear theory where exact \( S \)-matrix has been constructed [19]. The spectrum and the free energy density of the sine-Gordon model have been recently found by Al. Zamolodchikov [20]. He studied the model

\[
\mathcal{L}_{SG} = \frac{1}{2} (\partial_\mu \phi)^2 - 2C \cos(\beta \phi)
\]

(3.10)
at arbitrary coupling \( \beta \) assuming the normalization

\[
\langle \cos[\beta \phi(x)] \cos[\beta \phi(0)] \rangle_{C=0} = \frac{1}{2|x|^3/2\pi}
\]

(3.11)
The spectrum of the model involves a soliton, an antisoliton and some number of the soliton-antisoliton bound states. For \( \beta = \sqrt{2\pi} \) (the case we are interested in) there are just two such bound states. One of them is has the same mass \( M_{\text{gap}} \) as the solitons so that these three states form an isotropic triplet (recall that the original fermion model (3.1) had the isotopic \( SU(2) \) symmetry and so should its bosonized version), and the other one has the mass \( M_{\text{gap}}\sqrt{3} \) and is an isotropic singlet. Actually, Sine–Gordon model with \( \beta = \sqrt{2\pi} \) is simpler than a theory with an arbitrary \( \beta \). It enjoys a pure elastic scattering matrix. Its associated Lie algebra is \( D_4^{(1)} \) (cf. e.g. Table 1 in [7]). As was pointed out in [21] and recently in [22, 23], it belongs to the same universality class as the antiferromagnetic quantum spin chain [24].

The Zamolodchikov’s result for the mass gap \( M_{\text{gap}} \) in the model (3.10) with \( \beta = \sqrt{2\pi} \) reads

\[
M_{\text{gap}} = \frac{2\pi^{1/6} \Gamma^{2/3}(\frac{3}{4}) \Gamma(\frac{4}{3})}{\Gamma^{2/3}(\frac{1}{4}) \Gamma(\frac{5}{3})} C^{2/3}
\]

(3.12)

In order to express \( M_{\text{gap}} \) via physical parameters \( m \) and \( g \), we have to fix the coefficient \( C \). Using the result (3.3) with \( N_f = 2 \) for the physical fermion correlator, the property (3.3) and the definitions (3.4), (3.11), we obtain

\[
C = \frac{mg^{1/2} e^{\gamma/2}}{2^{1/4} \pi^{5/4}}
\]

(3.13)

Substituting it in (3.12), we finally derive

\[
M_{\text{gap}} = m^{2/3} g^{1/3} 2^{5/6} e^{\gamma/3} \left[ \frac{\Gamma(\frac{3}{4})}{\pi \Gamma(\frac{1}{4})} \right]^{2/3} \frac{\Gamma(\frac{4}{3})}{\Gamma(\frac{2}{3})} = 2.008 \ldots m^{2/3} g^{1/3}
\]

(3.14)

\(^2\)Note that, in the fermion language, the correlator \( C_{11}(x) \) is saturated by topologically trivial gauge fields while the correlator \( C_{12}(x) \) — by the fields belonging to 1-instanton topological sector.
To find the coefficient $A_\psi$, we use the expression for the vacuum energy density of the model derived in [7, 25]. For $\beta = \sqrt{2\pi}$ it reads

$$\epsilon_{\text{vac}} = -\frac{M_{\text{gap}}^2}{4\sqrt{3}}$$

(3.15)

Differentiating it over fermion mass $m$, we obtain the expression for the fermion condensate

$$<\bar{\psi}_1\psi_1>_{\text{vac}} = <\bar{\psi}_2\psi_2>_{\text{vac}} = \frac{1}{2} \frac{\partial \epsilon_{\text{vac}}}{\partial m}$$

$$= -m^{1/3}g^{2/3} \frac{2^{2/3}e^{2\gamma/3}}{3^{3/4}\pi^{4/3}} \left[ \frac{\Gamma(3/4)}{\Gamma(1/4)} \right]^{1/3} \left[ \frac{\Gamma(1/6)}{\Gamma(2/3)} \right]^2 = -0.388\ldots m^{1/3}g^{2/3}$$

(3.16)

The mass gap in the Sine–Gordon model was evaluated earlier by quasiclassical methods. In [26] the lagrangian of the model was chosen in the form

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{\mu^2}{\beta^2} \mathcal{N}_\mu \cos(\beta \phi)$$

(3.17)

where $\mu$ is the meson mass in the weak coupling (small $\beta$) limit and $\mathcal{N}_\mu$ is the normalization ordering prescription with respect to that mass. Quasiclassically, the soliton mass is

$$M_{\text{sol}} = \left( \frac{8}{\beta^2} - \frac{1}{\pi} \right) \mu$$

(3.18)

For $\beta = \sqrt{2\pi}, M_{\text{sol}} = 3\mu/\pi$. To compare it with the exact Zamolodchikov’s result (3.12) and its corollary (3.14), we have to relate $\mu$ and $C$. It can be easily done comparing Eq.(3.11) with

$$<\mathcal{N}_\mu \cos[\sqrt{2\pi}\phi(x)] \mathcal{N}_\mu \cos[\sqrt{2\pi}\phi(0)]> = \cosh[2\pi D(\mu, x)]$$

$$\sim \cosh[-\gamma - \ln(\mu x/2)] \sim e^{-\gamma} \frac{\mu x}{\mu x}$$

(3.19)

for small $\mu x$. We have

$$\mu = (2\sqrt{2\pi}e^{\gamma/2}C)^{2/3}$$

Substituting it in Eq.(3.18) and taking into account (3.13), we would get

$$A_{\text{gap}}^{WKB} = 2.07\ldots$$

(3.20)

We see that the WKB result turned out to be rather close to our exact result (3.14). The difference in just 3%. May be, there is no wonder that the accuracy of quasiclassical analysis is so high. Notice that the quasiclassical ratio $M_{\text{sol}}/\mu = 3/\pi$ is also very close to 1 — the theoretical prediction for the ratio of the soliton mass to the lowest breather mass for $\beta = \sqrt{2\pi}$. Note also that if we would estimate the mass gap as the classical mass $\mu$ of the basic Sine–Gordon boson rather than as $M_{\text{sol}}$ (that was in fact
done in [22], we would get $A^{WKB}_{\text{gap}} = 2.16\ldots$ and the agreement would be somewhat worse. In [22] also the critical amplitude for the fermion condensate was estimated by quasiclassical methods. The result

$$<\bar{\psi}\psi> = -\left(\frac{e^{4\gamma}}{2\pi^4}\right)^{1/3} m^{1/3} g^{2/3} = -0.37\ldots m^{1/3} g^{2/3}$$

agrees rather well with the exact formula (3.16).

The results (3.14), (3.16) refer to the Schwinger model with 2 flavors. The Schwinger model with larger number of flavors also exhibits a critical behavior at $T = m = 0$, and the value of the non-universal critical amplitude for the fermion correlator at any $N$ has been quoted in (3.3). The problem lies, however, in a conformal part of derivation. The effective low-energy lagrangian for the multiflavor Schwinger model is

$$L_{N}^{eff} = \sum_{i=1}^{N} \frac{1}{2}(\partial_{\mu}\phi_{i})^{2} - C \sum_{i=1}^{N} \cos \left(\sqrt{4\pi}\phi_{i}\right),$$

$$\sum_{i=1}^{N} \phi_{i} = 0$$

($C \to 0$ in the massless limit). To the best of our knowledge, the model (3.22) is not exactly solved, the exact $S$-matrix is not known, and the thermodynamic Bethe ansatz technique used in [8, 20] to derive the universal relations between the critical coefficients cannot be applied.

The exact results (3.14) and (3.16) should be confronted with numerical lattice simulations. The path integral calculations in a theory with light fermions are, of course, much more difficult than in a bosonic theory (like the Ising model), but it is exactly what is needed in standard 4-dimensional QCD. On the other hand, two-dimensional calculations are much simpler than 4-dimensional ones, and, if the exact results for the Schwinger model would be reproduced in such a numerical calculation, the methods to calculate the fermion determinant etc would be effectively checked and there would be much more trust in the lattice results for $QCD_{4}$ which are of physical interest.

We are aware of only one paper where relevant numerical computations have been done [16]. The critical exponents coincide with theoretical predictions. The numerical values for the critical amplitudes overshoot the theoretical predictions by $\sim 25\%$ for the correlation length and by $\sim 35\%$ for the condensate. It would be very interesting to repeat these calculations with better accuracy on larger lattices, with different numerical algorithms etc. Two-dimensional models are an excellent playground where all methods of the lattice gauge theory can be tested.
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