DWORK CRYSTALS II
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1. Introduction

This paper is a continuation of [3], which we will refer to as Part I. We start with recalling our notations and definitions.

Let $p$ be a prime and $R$ a $p$-adically complete characteristic zero domain such that $\cap_p p^s R = \{0\}$. Let $f \in R[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be a Laurent polynomial and $\Delta \subset \mathbb{R}^n$ be its Newton polytope. A subset $\mu \subset \Delta$ is said to be open if its complement $\Delta \setminus \mu$ is a union of faces of any dimensions. For such a subset we consider the $R$-module of rational functions $\Omega_f(\mu) = \left\{ \frac{g(x)}{f(x)^k} \mid k \geq 1, g \in R[x_1^{\pm 1}, \ldots, x_n^{\pm 1}], \text{supp}(g) \subset k\mu \right\}$.

When $\mu = \Delta$ we tend to omit it from the notation, e.g. $\Omega_f(\Delta)$ is simply $\Omega_f$. The submodule of derivatives $d\Omega_f \subset \Omega_f$ is defined as the $R$-span of all $x_i \frac{\partial}{\partial x_i} \omega$ with $\omega \in \Omega_f$ and $1 \leq i \leq n$. In Part I we constructed, for every Frobenius lift $\sigma$ on $R$, an $R$-linear Cartier operator on the $p$-adic completions $\hat{\Omega}_f(\mu) \rightarrow \hat{\Omega}_{f^s}(\mu)$.

This operator commutes with the derivations of $R$ and satisfies $\mathcal{C}_p \circ x_i \frac{\partial}{\partial x_i} = p x_i \frac{\partial}{\partial x_i} \circ \mathcal{C}_p$ for $1 \leq i \leq n$. It is then immediate that the Cartier operator preserves $d\Omega_f$. We consider submodules $U_f(\mu) = \{ \omega \in \hat{\Omega}_f(\mu) \mid \mathcal{C}_p^s(\omega) \equiv 0 \mod p^s \hat{\Omega}_{f^s}(\mu) \}$ for all $s \geq 1$.

It follows from the above mentioned commutation relations that $d\Omega_f \cap \Omega_f(\mu) \subset U_f(\mu)$. Denote by $\mu_\mathbb{Z} = \mu \cap \mathbb{Z}^n$ the set of integral points in $\mu$. The main result of Part I states that if the Hasse–Witt matrix

$$\beta_p(\mu) = \left( \text{coefficient of } x^{v \cdot u} \text{ in } f(x)^{p-1} \right)_{u,v \in \mu_\mathbb{Z}}$$

is invertible then the quotient

$$Q_f(\mu) = \frac{\hat{\Omega}_f(\mu)}{U_f(\mu)}$$

is a free $R$-module of rank $h = \# \mu_\mathbb{Z}$ where the images of $\omega_u = \frac{x^u}{f(x)}$, $u \in \mu_\mathbb{Z}$

can be taken as a basis. In this case, for every Frobenius lift $\sigma$ and every derivation $\delta$ on $R$ we define matrices $\Lambda_\sigma, N_\delta \in R^{h \times h}$ by the conditions

$$\mathcal{C}_p(\omega_u) \equiv \sum_{v \in \mu_\mathbb{Z}} (\Lambda_\sigma)_{u,v} \omega_v^\sigma \mod U_{f^s}(\mu),$$

$$\delta(\omega_u) \equiv \sum_{v \in \mu_\mathbb{Z}} (N_\delta)_{u,v} \omega_v \mod U_f(\mu).$$

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One has $\Lambda_\sigma \equiv \beta_p(\mu) \pmod{p}$ and hence $\mathcal{C}_p : Q_f(\mu) \to Q_{f^p}(\mu)$ is invertible. In this paper we shall give explicit formulas for the matrices $\Lambda_\sigma, N_\delta$ in a number of situations. One $p$-adic approximation was already given in Part I:

$$
\begin{align*}
\Lambda_\sigma & \equiv \beta_{p^r}(\mu) \cdot \sigma (\beta_{p^{r-1}}(\mu))^{-1} \pmod{p^s}, \\
N_\delta & \equiv \delta (\beta_{p^r}(\mu)) \cdot \beta_{p^r}(\mu)^{-1} \pmod{p^s},
\end{align*}
$$

where $\beta_p(\mu) \in R^{h \times h}$ is given by the same formula as the above Hasse–Witt matrix with $p$ replaced by a positive integer $m$.

Let us say that a formal series $q(t) = \sum_{k \geq 0} b_k t^k \in \mathbb{Z}_p[[t]]$ with $b_0 = 1$ satisfies Dwork’s congruences if one has

$$
\frac{q(t)}{q(t^p)} = \frac{\sum_{k=0}^{p^s-1} b_k t^k}{\sum_{k=0}^{p^{s-1}} b_k t^k} \pmod{p^s \mathbb{Z}_p[[t]]}
$$

for every $s \geq 1$. In [4] Dwork proved this congruence for a class of hypergeometric series. His result was generalized in [5] for the generating series of sequences

$$
\beta = \text{constant term of } g(x)^k,
$$

where $g(x)$ is a multivariable Laurent polynomial such that its Newton polytope $\Delta$ contains $0$ as its only internal integral point. In Sections 2, 3 and 4 we shall apply our methods to give an alternative proof of the main result of [5]. Namely, with $f(x) = 1 - tq(x)$ and $\mu = \Delta^o$ the module $Q_f(\mu)$ has rank 1 and we will see that $\Lambda_\sigma = g(t)/g(t^p)$. Dwork’s congruence then follows from a $p$-adic approximation similar to [10], where $\beta_{p^r} = \sum_{k=0}^{p^{r-1}} (-1)^k (p^r - 1) b_k t^k$ are substituted with the truncations $\gamma_{p^r} = \sum_{k=0}^{p^{r-1}} b_k t^k$. In Section 4 we explore the relation between truncations and $\text{periods modulo } m$ used in Part I; this relation is the key fact in our proof of Dwork’s congruences. The main result of this paper is Theorem 12. It generalizes Dwork’s congruences to the $A$-hypergeometric setting.

At the end of this introduction we would like to recall a detail from Part I which will be also useful for us here. When there is a vertex $b \in \Delta$ such that the coefficient of $f$ at $b$ is a unit in $R$, one can give the following description of our Cartier operator. By expanding rational functions into formal power series supported in the cone $C(\Delta - b)$, we embed $\Omega_f$ into $\Omega_{\text{formal}} = \{ \sum_{k \in C(\Delta - b)} a_k x^k \mid a_k \in R \}$. The Cartier operator on formal expansions is simply given by

$$
\mathcal{C}_p : \sum_k a_k x^k \mapsto \sum_k a_{pk} x^k
$$

and $U_f(\mu)$ coincides with the submodule of formal derivatives $\hat{\Omega}_f(\mu) \cap d\Omega_{\text{formal}}$, see [3] Prop 8.

2. Peroids

In part I we introduced the Cartier operator as operator on infinite Laurent series. However, the image of a rational function under the Cartier operator is again rational. Consider the rational function $\omega = \frac{g(x)}{f(x)^p} \in \Omega_f$. We assert that the image of $\omega$ under $\mathcal{C}_p$ is given by

$$
\frac{1}{p^n} \sum_{y \neq x} \frac{g(y)}{f(y)^k},
$$

where the summation is over all $y = (\zeta_p^{r_1}, \ldots, \zeta_p^{r_n}, \zeta_p^{r_n})$ with $0 \leq r_1, \ldots, r_n < p$, with $\zeta_p$ a primitive $p$-th root of unity. This is again a rational function, but with denominator
\[ \prod_{y: y^p = x} f(y)^k. \]

Choose a vertex \( b \) of the Newton polytope \( \Delta \) of \( f \) and expand in a Laurent series with respect to \( x^b \). The result is a Laurent series with support in the cone \( C(\Delta - b) \). Suppose it reads \( \sum_k a_k x^k \). Then application of \( \mathcal{C}_p \) yields

\[ \mathcal{C}_p(\omega) = \frac{1}{p^n} \sum_k a_k \left( \sum_{r_1, \ldots, r_n = 0}^{p-1} c_r^{k_1 + \ldots + r_n k_n} \right) x^{k/p}. \]

The summation over the integers \( r_1, \ldots, r_n \) yields something non-zero if and only if \( p \) divides \( k_i \) for \( i = 1, \ldots, n \). The summation value then equals \( p^n \). Replacing \( k \) by \( pk \) then yields

\[ \mathcal{C}_p(\omega) = \sum_k a_{pk} x^k, \]

which is precisely the Cartier operator defined in Part I.

There are also ways to produce Laurent series expansions of \( \omega \). This happens in the case when \( R \) has another non-archimedean valuation, let us call it the \( t \)-adic valuation, and one coefficient of \( f \) that dominates all the others \( t \)-adically. So let us write \( f = \sum_{w \in \Delta_2} v_w x^w \) and suppose that there exists \( v \) such that \( v_\omega \) is a unit in \( R \) and \( |v_\omega|_t > |v_w|_t \) for all \( w \neq v \). We can then expand \( \omega \) in a \( t \)-adically converging Laurent series via

\[
\begin{align*}
\omega &= \frac{g(x)}{(v_\omega x^\omega + \sum_{w \neq v} v_w x^w)^k} = \frac{g(x) x^{-k \omega}}{v_\omega^k \left( 1 + \sum_{w \neq v} (v_w/v_\omega) x^{w-\omega} \right)^k} \\
&= \frac{1}{v_\omega^k} g(x) x^{-k \omega} \sum_{r \geq 0} \left( \frac{-k}{r} \right) \left( \sum_{w \neq v} (v_w/v_\omega) x^{w-\omega} \right)^r.
\end{align*}
\]

The series expansion is \( t \)-adically convergent, but when \( v \) is not a vertex of \( \Delta \) we may end up with a Laurent series in \( x \) whose support is not a cone. It could possibly be all of \( \mathbb{Z}^n \). The coefficients are then in the completion of \( R \) with respect to \( |.|_t \). We denote this completion by \( S \) and assume that \( v_\omega \in S^x \). Suppose we get

\[ \omega = \sum_{k \in \mathbb{Z}^n} c_k x^k, \quad c_k \in S. \]

Assuming that for \( v_1, v_2 \in R \) inequality \( |v_1|_t > |v_2|_t \) implies \( |\sigma(v_1)|_t > |\sigma(v_2)|_t \), one can do analogous expansion in \( \Omega_f^* \). Then, the same argument as above yields

\[ \mathcal{C}_p(\omega) = \sum_{k \in \mathbb{Z}^n} c_{pk} x^k. \]

**Definition 1.** Let \( v \in \Delta_2 \) be such that \( |v_\omega|_t > |v_w|_t \) for all \( w \in \Delta \) distinct from \( v \) and \( v_\omega \in S^x \). Then define the period map \( p_v: \Omega_f \to S \) given by \( p_v(\omega) = c_0 \), the constant term in the Laurent series expansion of \( \omega \) with respect to \( v \).

For a differential ring \( S \) with a homomorphism \( R \to S \) which extends the derivations of \( R \), a period map is an \( R \)-linear map \( p: \Omega_f \to S \) which vanishes on \( d \Omega_f \) and commutes with derivations of \( R \). Values of a period map on elements of \( \Omega_f \) are called periods. All period maps considered in this paper satisfy an extra condition of vanishing on the submodule of formal derivatives \( U_f = \Omega_f \cap d \Omega_{f\text{form}} \).

It follows almost from the definition that \( p_v \) vanishes on \( d \Omega_f \). It is slightly less trivial to see that \( p_v \) vanishes on the formal derivatives.

**Proposition 2.** Let notation be as above. Then, for all \( \eta \in U_f \) we have \( p_v(\eta) = 0 \).
Proof. First of all notice that the constant term of $\eta$ equals the constant term of $\mathcal{C}_p^s(\eta)$ for all $s \geq 0$. Since $\eta \in U_f$ we also know that the $\mathcal{C}_p^s(\eta) \equiv 0 (\text{mod } p^s)$. In particular the constant term of $\eta$ is divisible by $p^s$ for all $s \geq 0$, hence equals 0. We conclude that $p_v(\eta) = 0$. \hfill $\square$

**Theorem 3.** Let $\mu \subseteq \Delta$ be an open set and $h = \# \mu_Z$. Consider the column vector $p_v \in S^h$ with components $p_v(\omega_u)$ for $u \in \mu_Z$.

Assume that $R$ is $p$-adically complete and the Hasse–Witt matrix $\beta_p(\mu)$ is invertible in $R$. For any Frobenius lift $\sigma$ and any derivation $\delta$ of $R$, we have

\begin{equation}
(4) \quad p_v = \Lambda_{\sigma} \sigma(p_v)
\end{equation}

and

\begin{equation}
(5) \quad \delta(p_v) = N_\delta p_v.
\end{equation}

**Proof.** Consider the equality

$$\mathcal{C}_p(\omega_u) = \sum_{w \in \mu_Z} \lambda_{u,w} \omega_u \omega_w (\text{mod } U_f(\mu)).$$

Expand all terms in a Laurent series with respect to the vertex $v$ and determine the constant coefficient. Using the fact that the constant term of elements in $U_f$ vanish (Proposition 2) we get the first statement. In a similar vein, starting with

$$\delta(\omega_u) \equiv \sum_{w \in \mu_Z} \nu_{u,w} \omega_u (\text{mod } U_f(\mu))$$

we get the second statement again by taking the constant term of the Laurent series expansions with respect to $v$. \hfill $\square$

3. **Example**

Let $g(x)$ be a Laurent polynomial in $x_1, \ldots, x_n$ with coefficients in $\mathbb{Z}_p$. Suppose that $0$ is the only lattice point in the interior of the Newton polytope $\Delta$ of $g$. We introduce another variable $t$ and define $f(x) = 1 - tg(x)$. We apply Theorem 3 to $f(x)$ with $\mu = \Delta^\circ$ and $u = v = 0$. In this case $\beta_m$ has only one entry, the constant coefficient of $f(x)^{m-1}$. Let $R$ be the $p$-adic completion of $\mathbb{Z}_p[t, 1/b_p]$. Its $t$-adic closure is $S = \mathbb{Z}_p[[t]]$. The period

$$q(t) := p_0 \left( \frac{1}{f(x)} \right)$$

reads $\sum_{k \geq 0} b_k t^k$ with $b_k$ equal to the constant term of $g(x)^k$. Take the Frobenius lift given by $t \mapsto t^p$. Then we obtain as a consequence of Theorem 3

**Corollary 4.** We have $q(t)/q(t^p) = \Lambda$ where $\Lambda \in \mathbb{Z}[t, \beta_p(t)^{m-1}]$ is the (single entry) matrix of the Cartier operation $\mathcal{C}_p : Q_f(\Delta^\circ) \rightarrow Q_{f^p}(\Delta^\circ)$.

One easily checks that

$$\beta_m(t) = \sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k} b_k t^k.$$

Define

$$\gamma_m(t) = \sum_{k=0}^{m-1} b_k t^k.$$

These can be interpreted as truncated version of the power series $q(t)$. In [5] it is shown that

**Theorem 5** (Mellit–Vlasenko, 2016). For all $s \geq 1$ we have $q(t)/q(t^p) = \gamma_{p^s}(t)/(\gamma_{p^s-1}(t^p)) (\text{mod } p^s)$. 


This is a generalization of the famous congruence in Dwork’s ‘p-adic cycles’ §4 (6.29)]. The latter can be obtained using $g(x) = x + 1/x + y + 1/y$.

Note that Theorem 4 with $\gamma_m$ replaced by $\beta_m$ is simply Corollary 4. We shall prove Theorem 5 in the next section. It will follow from our proof that in fact $\frac{\tau(t)}{\tau\phi(t)} \equiv \frac{\gamma_m(t)}{\gamma_m(t)} \pmod{m}$ with any $m \geq 1$, and a similar congruence holds for the derivatives:

$$\frac{q'(t)}{q(t)} \equiv \frac{\gamma'_m(t)}{\gamma_m(t)} \pmod{p^{\ord_p(m)}}.$$

It is a curious fact that when $g(x)$ has coefficients in $\mathbb{Z}$ then the series $q'(t)q(t)^{-1} \in \mathbb{Z}[[t]]$ is a $p$-adic analytic element for each $p$.

4. TRUNCATIONS

In this section we consider periods mod $m$ which, in a number of relevant cases, turn out to be truncations of the Laurent series solutions of a system of linear differential equations. But first we turn to general $f(x)$ with coefficients in a $p$-adic ring $R$.

By a period map mod $m$ we mean an $R$-linear map $\rho: \Omega_f \to R$ such that $\rho(d\Omega_f) \subset mR$ and $\rho \delta = \delta \rho \pmod{mR}$ for every derivation $\delta$ on $R$. All period maps mod $m$ considered in this paper will satisfy the condition $\rho(U_f) \subset mR$ of “vanishing” on formal derivatives. Choose a vertex $b \in \Delta$ and consider Laurent series expansions with respect to $b$. We assume its coefficient $f_b$ in $f$ to be a unit in $R$. For an integer $m \geq 1$ and a Laurent polynomial $g(x) \in R[x^\pm_1, \ldots, x^\pm_n]$ the functional

$$\rho_{m,g}: \omega \mapsto \text{constant term of } g(x)^m \omega.$$

is a period map mod $m$. It is clear that on formal derivatives we also have $\rho_{m,g}(U_f) \subset mR$. These properties follow easily if one observes that, modulo $m$, $m$th powers behave like constants under derivations (see Part I, Lemma 15). In Part I we already used two particular instances of these period maps: $\tau_{mv} = \rho_{m,x-v}(f(x))$ for $v \in \Delta_Z$ and $\alpha_{mk} = \rho_{m,x-k}$ for $k \in C(\Delta - b)_Z$. We now describe their behaviour under the Cartier operator and relevant congruences in this more general context:

Proposition 6. For a Laurent polynomial $g = \sum g_w x^w$ denote $g^\sigma = \sum g_w^\sigma x^w$. For any $m \geq 1$ divisible by $p$ we have $\rho_{m,g} \equiv \rho_{m/p,g} \circ \phi^\sigma_p \pmod{p^{\ord_p(m)}}$.

Proof. Similar to the proof of Proposition 16 in Part I. 

Theorem 7. Let $\mu \subseteq \Delta$ be an open set and $h = \#\mu_Z$. For $m \geq 1$ consider column vectors $\rho_m \in R^h$ with components $\rho_{m,g}(\omega_u)$ for $u \in \mu_Z$. If $R$ is $p$-adically complete and the Hasse–Witt matrix $\beta_p(\mu)$ is invertible, then for any Frobenius lift $\sigma$ and any derivation $\delta$ of $R$ we have

$$\rho_m \equiv \Lambda_\sigma \rho_{m/p} \pmod{p^{\ord_p(m)}}$$

and

$$\delta(\rho_m) \equiv N_\delta \rho_m \pmod{p^{\ord_p(m)}}$$

for all $m \geq 1$.

Proof. Similar to the proof of Theorem 17 in Part I.

Let us choose a tuple of elements $\phi_\nu \in R$ for $\nu \in \Delta_Z$ and consider matrices of periods mod $m$ given by

$$\gamma_m = \text{constant term of } (\phi_\nu^m - (\phi_\nu - f(x)/x^\lambda)^m) \omega_u.$$
Observe that the entries of $\gamma_m$ do not depend on the choice of $b$ since they are constant terms of Laurent polynomials that are independent of $b$. For a subset $\mu \subset \Delta$ we denote by $\gamma_m(\mu)$ the submatrix given by $(\gamma_m)_{u,v \in \mu}$. We can rewrite these matrices via $\beta$-matrices as

\[
(\gamma_m)_{u,v} = \sum_{k=1}^{m} (-1)^{k+1} \binom{m}{k} \phi^m_{V}(-k)_{u,v},
\]

from which the following congruence follows trivially:

**Lemma 8.** We have $\beta_p(\mu) \equiv 2^{p}\gamma_p(\mu) (\text{mod } p)$. In particular $\beta_p(\mu)$ is invertible if and only if this holds for $\gamma_p(\mu)$.

**Corollary 9.** Let $\gamma_m(\mu)$ be as above and suppose $\gamma_p(\mu)$ is invertible. Then for any Frobenius lift $\sigma$ and any derivation $\delta$ of $R$ we have

\[
\gamma_m(\mu) \equiv \Lambda_{\sigma} \sigma (\gamma_{m/p}(\mu)) (\text{mod } p^{\text{ord}_p(m)}),
\]
\[
\delta(\gamma_m(\mu)) \equiv N_{\delta} \gamma_m(\mu) (\text{mod } p^{\text{ord}_p(m)})
\]

for all $m \geq 1$.

As it follows from the first congruence in this corollary, we have

\[
\gamma_{p^s}(\mu) \equiv \gamma_p(\mu) \cdot \sigma (\gamma_p(\mu)) \cdot \ldots \cdot \sigma^{s-1}(\gamma_p(\mu)) (\text{mod } p).
\]

Hence all $\gamma_{p^s}(\mu)$ are invertible and we obtain $p$-adic limit formulas

\[
\Lambda_{\sigma} \equiv \gamma_{p^s}(\mu) \cdot \sigma (\gamma_{p^{s-1}}(\mu))^{-1}, \quad N_{\delta} \equiv \delta(\gamma_{p^s}(\mu)) \cdot \gamma_{p^s}(\mu)^{-1} (\text{mod } p^s).
\]

**Proof of Theorem 5.** We apply Corollary 9 in the case $f(x) = 1 - ty(x), \phi = 1$ and $\mu = \Delta^0$. Then $\gamma_m(\mu)$ is the polynomial $\sum_{k=0}^{m-1} b_k t^k$. It follows from Corollary 9 with $\sigma(t) = t^p$ that $\gamma_{p^s}(t) \equiv \Lambda_{\sigma}(t^p) (\text{mod } p^s)$ for all $s \geq 1$. Theorem 5 then follows from Corollary 4 which says that $\Lambda = q(t)/q(t^p)$.

## 5. A-hypergeometric periods

We continue the calculation of periods following the idea in Section 2. Let $f(x) = \sum_{i=1}^{N} v_i x^{a_i}$, where the $v_i$ are independent variables. This is the A-hypergeometric setting. Let $\Delta \subset \mathbb{R}^n$ be the Newton polytope of $f(x)$, which is now the convex hull of the set $\{a_1, \ldots, a_N\} \subset \mathbb{Z}^n$. Pick some integer exponent vector $u \in k\Delta$, expand $x^u f(x)^{-k}$ with respect to $a_i \in \Delta_Z$ and take the constant term. We get

\[
p_{a_i}(x^u f(x)^{-k}) := \text{constant term of } \sum_{m \geq 0} \binom{-k}{m} \sum_{\sum_{r \neq i} v_r a_r - a_i \neq 0} a_i^m v_i^{\sum_{r \neq i} v_r a_r - a_i}.
\]

Before we proceed we like to make a remark which considerably simplifies our calculation. Denote by $\tilde{a}_i \in \mathbb{Z}^{n+1}$ the exponent vector $a_i$, preceded by an extra component 1. We call the set $A = \{\tilde{a}_1, \ldots, \tilde{a}_N\} \subset \mathbb{Z}^{n+1}$ saturated when

\[
\left(\sum_{j=1}^{N} \mathbb{R}_{\geq 0} \tilde{a}_j\right) \cap \mathbb{Z}^{n+1} = \sum_{j=1}^{N} \mathbb{Z}_{\geq 0} \tilde{a}_j.
\]

When $A$ is saturated, the following Proposition can be applied to any exponent vector $u$:

\[
\left(\sum_{j=1}^{N} \mathbb{R}_{\geq 0} \tilde{a}_j\right) \cap \mathbb{Z}^{n+1} = \sum_{j=1}^{N} \mathbb{Z}_{\geq 0} \tilde{a}_j.
\]
Proposition 10. For an integral point \( u \in k\Delta \) we denote \( \tilde{u} = (k, u) \). Assume that there exist \( \alpha_1, \ldots, \alpha_N \in \mathbb{Z}_{\geq 0} \) such that \( \sum_{r=1}^{N} \alpha_r \tilde{a}_r = \tilde{u} \). Then \( p_n(\mathbf{x}^u/\mathbf{x}^{-k}) \) is equal to the application of the differential operator \( (-1)^{k-1}/(k-1)! \prod_{r=1}^{N} \partial^\alpha_r \) where \( \partial_r = \frac{\partial}{\partial v_r} \) to the universal series

\[
p_n(\log f) := \text{constant term of } \left( \log v_i + \sum_{m \geq 1} \frac{(-1)^{m-1}}{m} \left( \sum_{r \neq i} v_r \mathbf{x}^{a_r-a_i} \right)^m \right).
\]

The proof is straightforward with induction on \( k \).

We proceed with the calculation of \( p_n(\log f) \) and get

\[
\log v_i + \sum_{1}^{N} \frac{(-1)^{\ell_1+\cdots+\ell_N-1}}{\ell_1+\cdots+\ell_N} \left( \ell_1 + \cdots + \ell_N \right) \prod_{r \neq i} (v_r/v_i)^{\ell_r},
\]

where the sum is over all non-negative \( \ell_1, \ldots, \ell_N \), not all zero, such that \( \sum_{r \neq i} \ell_r (a_r - a_i) = 0 \). Here the \( \vee \) in the summation range and the sum itself means that \( \ell_i \) is to be omitted. Introduce \( \ell_i = -\sum_{r \neq i} \ell_r \). Recall our notation \( \tilde{a}_r = (1, a_r) \). Then the definition of \( \ell_i \) sees to it that the support of the resulting Laurent series (aside from the constant \( \log v_i \)) is contained in the set

\[
L_i := \{ \ell = (\ell_1, \ldots, \ell_N) \in \mathbb{Z}^N | \sum_{r=1}^{N} \ell_r \tilde{a}_r = 0, \ell_r \geq 0 \text{ if } r \neq i \}.
\]

In order to have a more compact notation, let us rewrite the multinomial coefficient as

\[
\frac{(-1)^{\ell_1+\cdots+\ell_N-1}}{\ell_1+\cdots+\ell_N} \left( \ell_1 + \cdots + \ell_N \right) \prod_{r \neq i} (v_r/v_i)^{\ell_r} = \prod_{r=1}^{N} \frac{1}{\Gamma(\ell_r+1)},
\]

where \( \Gamma^*(n) \) with \( n \in \mathbb{Z} \) is defined as \( (n-1)! \) if \( n \geq 1 \) and \( (-1)^n/|n|! \) if \( n \leq 0 \). Notice that the modified \( \Gamma^* \) satisfies \( \Gamma^*(n+1) = n\Gamma^*(n) \) for all integers \( n \neq 0 \). One also checks that \( \Gamma^*(n)\Gamma^*(1-n) = \text{sign}(n)(-1)^{n-1} \) for all integers \( n \). Here \( \text{sign}(n) = -1 \) if \( n \leq 0 \) and \( 1 \) if \( n \geq 1 \). The period now takes the shape

\[
(10) \quad p_n \left( \frac{\mathbf{x}^u}{\mathbf{x}^{-k}} \right) = \frac{(-1)^{k-1}}{\Gamma(k)} \prod_{r=1}^{N} \partial^\alpha_r \left( \log v_i + \sum_{\ell \in L_i^*} \prod_{r=1}^{N} \frac{v_r^{\ell_r}}{\Gamma^*(\ell_r+1)} \right),
\]

where \( L_i^* = L_i \setminus \{0\} \). Although we do not need this in the rest of this paper, we like to notice that this period is a Laurent series solution of the A-hypergeometric system of equations with A-matrix the matrix with columns \( \tilde{a}_1, \ldots, \tilde{a}_N \) and parameter vector \( -\tilde{u} \).

When we vary the different periods over \( \ell_i \) and define \( L_i^* \), we see that their union also lies in a regular cone. The following result, as well as its proof, is taken from [1, Prop 2.9]. We use a different formulation however.

Lemma 11. Let \( L_i(\mathbb{R}) \) be the real positive cone generated by \( L_i \) and define \( L^*(\mathbb{R}) = \sum_{i=1}^{N} L_i(\mathbb{R}) \). Then \( L^*(\mathbb{R}) \) is a finitely generated cone with \( 0 \) as a vertex.

Proof. It suffices to show the following assertion. Let \( \ell^{(i)} \in L_i \) for \( i = 1, \ldots, N \). Then \( \sum_{i=1}^{N} \ell^{(i)} = 0 \) implies that \( \ell^{(i)} = 0 \) for each \( i \).

Denote the coordinates of \( \ell^{(i)} \) by \( l^{(i)}_k \). Suppose that \( \ell^{(i)} \neq 0 \). Then \( l^{(i)}_k < 0 \) and \( l^{(i)}_k \geq 0 \) for all \( k \neq i \). In particular

\[
\tilde{a}_i = \sum_{k \neq i} -\frac{f^{(i)}_k}{l^{(i)}_k} \tilde{a}_k,
\]
so we see that \( \tilde{a}_i \) is a (real) positive linear combination of some other \( a_k \). Define the set

\[ C = \{ \tilde{a}_k | \text{there exists } j \text{ such that } t_k(j) \neq 0 \}. \]

So \( C \) is the set of \( a_k \) that are non-trivially involved in some relation \( \ell(j) \). Suppose \( C \) is not empty. Let \( a_k \) be a vertex of the convex hull of \( C \). Suppose that \( t_k(k) < 0 \). Then \( a_k \), being a positive linear combination of other \( a_j \) in \( C \) cannot be a vertex of the convex hull of \( C \). So \( t_k(k) \geq 0 \) and fortiori, \( t_k(j) \geq 0 \) for all \( j \). Their sum should be zero, contradicting the fact that \( t_k(j) \neq 0 \) for some values of \( j \). Hence we conclude that \( C \) is empty. In particular \( \ell(j) = 0 \) for all \( j \).

Due to Lemma 11 the set of formal power series supported in \( L^o = L^o(R) \cap \mathbb{Z}^N \) is a ring. Let us denote this ring by

\[ R = \{ \sum_{\ell \in L^o} b_\ell \ell^t | b_\ell \in \mathbb{Z} \}. \]

We will also consider the bigger ring

\[ S = R[v_1^\pm 1, \ldots, v_N^\pm 1]. \]

Elements of \( S \) are power series supported in a finite number of integral translations of the cone \( L^o \). It follows from Proposition 11 and formula 10 that \( p_n(x^u f(x)^{-k}) \in (\prod_{r=1}^N v_r^{-\alpha_r})R \subset S \). Note that when \( A \) is saturated, this argument can be applied with any \( k \geq 1 \) and \( u \in k\Delta \). With a bit more effort one can also show that \( p_n(x^u f(x)^{-k}) \in S \) for any integral \( u \in k\Delta \) without the assumption. In what follows we shall not assume that \( A \) is a saturated set.

We shall be interested in the \( N \times N \) matrix \( \Psi \) with entries

\[ \Psi_{ji} = p_n(\omega_{a_j}) = v_j^{-1} \left( \delta_{ij} + \sum_{\ell \in L^o} \ell_j \prod_{r=1}^N \frac{v_r^\ell_r}{\Gamma(\ell_r + 1)} \right). \]

This formula follows from 10 with \( u = a_j \) and \( k = 1 \). It will be convenient to work with the renormalized series \( \tilde{\Psi}_{ji} := v_j \Psi_{ji} \in R \). Let us now consider their truncated versions. Define for any \( m \geq 1 \) the \( N \times N \)-matrix \( \psi_m \) with entries

\[ (\psi_m)_{ji} = \text{constant term of } \left( 1 - \left( 1 - \frac{f(x)}{v_j x^{a_j}} \right)^m \right) \omega_{a_j}. \]

A straightforward calculation shows that this is equal to the series development 9 with \( k = 1, u = a_j \) summed over \( m = 0, 1, 2, \ldots, M - 1 \). Further calculation along the same lines as earlier shows that we get

\[ v_j(\psi_m)_{ji} = \delta_{ij} + \sum_{\ell \in L^o} \ell_j \prod_{k=1}^N \frac{v_k^\ell_k}{\Gamma(\ell_k + 1)}, \]

where

\[ L_i(m) = \{ \ell \in \mathbb{Z}^N | \sum_{k=1}^N \ell_k \tilde{a}_k = 0, \ell_k \geq 0 \text{ for all } k \neq i \text{ and } \ell_i > -m \}. \]

Comparing 12 and 11 one sees that \( \tilde{\psi}_m(j) := v_j(\psi_m)_{ji} \in R \) is the truncation of the element \( \tilde{\Psi}_{ji} = v_j \Psi_{ji} \in R \). Let us consider the function \( | \cdot | : L^o \rightarrow \mathbb{Z}_{\geq 0} \) given by

\[ |\ell| := \sum_{k: \ell_k > 0} \ell_k = -\sum_{k: \ell_k < 0} \ell_k \text{ for } \ell \in L^o \]
and define truncations of elements of $\mathcal{R}$ by

$$r = \sum_{\ell \in L^a} b_{\ell} \mathbf{v}^\ell \quad \mapsto \quad r(m) := \sum_{|\ell| \leq m} b_{\ell} \mathbf{v}^\ell$$

for all $m \geq 0$. With this notation, the above computation shows that $\tilde{\Psi}(m) = \tilde{\psi}_m$. Note that the constant term $\tilde{\Psi}(0)$ is the identity matrix, and hence $\tilde{\Psi}$ and all its truncations $\tilde{\psi}_m$ are invertible over $\mathcal{R}$.

**Theorem 12.** Let $\mu \subseteq \Delta$ be an open set and denote $h = \#\mu_{\mathbb{Z}}$. Assume that $h \geq 1$ and $\#\{j : a_j \in \mu\} = h$. Consider the $h \times h$ submatrices with entries in $\mathcal{R}$ given by

$$\tilde{\Psi} = (\tilde{\Psi}_{ji})_{a_j, a_i \in \mu},$$

where $\tilde{\Psi}_{ji} = v_j \Psi_{ji}$ are renormalized series (11). Let $\hat{\psi}_m = \tilde{\Psi}(m)$ for $m \geq 1$ be the respective truncations. For the Frobenius lift $\sigma : \mathcal{R} \to \mathcal{R}$ that sends $v_j$ to $v_j^p$ for each $1 \leq j \leq N$ and any of the derivations $\delta = v_j \frac{\partial}{\partial m} : \mathcal{R} \to \mathcal{R}$ one has congruences

\begin{align*}
\Psi \cdot (\Psi)^{-1} &\equiv \tilde{\psi}_m \cdot (\tilde{\psi}_m)^{-1} \pmod{p^{\text{ord}_p(m)}} \\
\delta(\Psi) \cdot \Psi^{-1} &\equiv \delta(\tilde{\psi}_m) \cdot \tilde{\psi}_m^{-1} \pmod{p^{\text{ord}_p(m)}}
\end{align*}

for all $m \geq 1$.

Let $V$ be the $h \times h$ diagonal matrix with the entries $v_j$ for $a_j \in \mu$. Note that substituting $\tilde{\Psi} = V \Psi$ and $\hat{\psi}_m = V \psi_m$ into (13) and (14) shows that these congruences are equivalent to

\begin{align*}
\Psi \cdot (\Psi)^{-1} &\equiv \psi_m \cdot (\psi_m)^{-1} \pmod{p^{\text{ord}_p(m)}}, \\
\delta(\Psi) \cdot \Psi^{-1} &\equiv \delta(\psi_m) \cdot \psi_m^{-1} \pmod{p^{\text{ord}_p(m)}}
\end{align*}

Matrices in the latter congruences have entries in the bigger ring $S$. We preferred to state our theorem for the normalized matrices because truncations are more naturally defined on elements of $\mathcal{R}$ rather than $S$.

**Proof.** Consider the matrices of periods mod $m$ given by (5) with $\phi_{a_i} = \psi_i$:

$$\gamma_m \equiv \Lambda_{\sigma} \sigma(\gamma_{m/p}) \quad \text{and} \quad \delta(\gamma_m) \equiv N_\delta \gamma_m \pmod{p^{\text{ord}_p(m)}}.$$  

Their entries are in $\mathbb{Z}[v_1, \ldots, v_N]$ and we have $\gamma_m = V^{-1} \psi_m V^m$. It particular, the coefficient of the monomial $(\prod_{a_j \in \mu} v_j)^{p-1}$ in $\det(\gamma_p)$ is 1. Let $R$ be the $p$-adic completion of $\mathbb{Z}[v_1^{\pm 1}, \ldots, v_N^{\pm 1}]$. Since $\det(\gamma_p)$ is not divisible by $p$, this ring satisfies our assumption $\cap_{n \geq 1} p^n R = \{0\}$ and hence one can apply Corollary 9. It follows that there are matrices $\Lambda_{\sigma}, N_\delta \in R^{h \times h}$ such that

$$\gamma_m \equiv \Lambda_{\sigma} \sigma(\gamma_{m/p}) \quad \text{and} \quad \delta(\gamma_m) \equiv N_\delta \gamma_m \pmod{p^{\text{ord}_p(m)}}.$$  

Observe that all matrices $\gamma_m$ are invertible over $S$ because

$$\det(\gamma_m) = (\prod_{a_j \in \mu} v_j)^{m-1} \det(\tilde{\psi}_m) \in (\prod_{a_j \in \mu} v_j)^{m-1} \mathcal{R} \subset S.$$  

One of the consequences of this fact is that $R$ is a subring of the $p$-adic completion $S := \hat{S} \subset \mathbb{Z}_p[[v_1^{\pm 1}, \ldots, v_N^{\pm 1}]].$  

Working in the big ring $S$ we can invert matrices in (16) and conclude that

$$\gamma_m \cdot \sigma(\gamma_{m/p})^{-1} \equiv \Lambda_{\sigma} \quad \text{and} \quad \delta(\gamma_m) \cdot \gamma_m^{-1} \equiv N_\delta \pmod{p^{\text{ord}_p(m)}}.$$
Substituting $γ_m = V^{-1} \tilde{ψ}_m V^m$ in the left-hand sides yields
\begin{equation}
\tilde{ψ}_m \cdot σ(\tilde{ψ}_m/p)^{-1} \equiv VΛ_σ V^{-p} \pmod{p^{ord_p(m)}}
\end{equation}
\begin{equation}
\delta(\tilde{ψ}_m) \cdot \tilde{ψ}_m^{-1} \equiv VN_δ V^{-1} + δ(V) V^{-1} \pmod{p^{ord_p(m)}}.
\end{equation}

One particular consequence of these congruences is that the matrices in their right-hand side have entries in $R$. Secondly, they must coincide with the limits of the left-hand sides which, using the fact that $\tilde{ψ}_m$ is a truncation of $\tilde{Ψ}$, immediately implies that
\begin{equation}
VΛ_σ V^{-p} = \tilde{Ψ} \cdot σ(\tilde{Ψ})^{-1} \quad \text{and} \quad VN_δ V^{-1} + δ(V) V^{-1} = δ(\tilde{Ψ}) \cdot \tilde{Ψ}^{-1}.
\end{equation}

Substituting these values back into (17) proves our theorem. □

The above proof is based on the ideas from Section 3. By Lemma 8, the Hasse–Witt matrix $β_p(μ)$ is congruent modulo $p$ to the matrix $γ_p$ given in (13). (In the special case $μ = Δ^o$ this was observed in [11 Proposition 3.8].) Using this fact we can conclude from the above proof that under the assumptions of Theorem 12 the determinant of the Hasse–Witt matrix is a polynomial not divisible by $p$ and there exist the respective matrices $Λ_σ, N_δ ∈ R^{h \times h}$, where $R$ is the $p$-adic completion of the ring $\mathbb{Z}[v_1^{±1}, \ldots, v_N^{±1}, det(β_p(μ))^{-1}]$. These are the same ring $R$ and the same matrices that were used in the proof. In particular, $R$ is a subring of the $p$-adic completion $S = \hat{S}$ and we have

**Corollary 13.** $Λ_σ = Ψ \cdot σ(Ψ)^{-1}, N_δ = δ(Ψ) \cdot Ψ^{-1}.$

**Proof.** Substitute $\tilde{Ψ} = VΨ$ into (18). □

A special consequence of this corollary is that the matrices $VΛ_σ V^{-p}$ and $VN_δ V^{-1} + δ(V) V^{-1}$ have their entries in $R$. Furthermore, it turns out that $N_δ$ and, in a lesser way, $Λ_σ$, are independent of the choice of $p$.

Finally, we remark that in fact there are well defined period maps
\[ p_α : \hat{Ω}_f → S. \]

As we explained in Section 2 these period maps are invariant under the Cartier operator (we have $p_α = p_α^d \circ φ^p$ where $p_α^d$ denotes the respective period map $Ω_f → S$ ) and vanish on formal derivatives. Corollary 13 is then a direct consequence of Theorem 3. Let us also mention the main result of [2], Theorem 1.4. It states that in the A-hypergeometric setting with the assumption that $Δ$ has $a_0$ as its unique interior lattice point the series $Φ(ν)/Φ(ν^p)$, where $Φ(ν) = Ψ_0(v_0, \ldots, v_N)$ is the unique entry of our matrix $Ψ$ for $μ = Δ^o$, is a $p$-adic analytic element with the set of poles determined by the Hasse invariant $β_p(Δ^o)$. Hence [2 Theorem 1.4] follows from Corollary 13.

6. Example

We continue the example from Part I, Section 7 with
\[ f(x, y) = v_1y^2 + v_2x + v_3x^3 + v_4x^2 + v_5xy. \]

We determine the entries of the matrix $Ψ$. The vectors $\hat{a}_k$ are given by the columns of
\[ \begin{pmatrix}
  1 & 1 & 1 & 1 & 1 \\
  0 & 1 & 3 & 2 & 1 \\
  2 & 0 & 0 & 0 & 1
\end{pmatrix}. \]

The supports $L_i$ lie in the null space of this matrix which can be written as
\[ (r + 2s, s, s, r, -2r - 4s), \quad r, s ∈ \mathbb{Z}. \]

In $L_1$ we have the inequalities $s, r, -2r - 4s ≥ 0$. This is only possible when $r = s = 0$. The only non-trivial series $Ψ_{j,1}$ is $v_1Ψ_{1,1} = 1$. 
In $L_2$ we have the inequalities $r + 2s, s, r, -2r - 4s \geq 0$ and we find $v_2 \Psi_{2,2} = 1$ as non-trivial series.

In $L_3$ we again get $v_3 \Psi_{3,3}$ as only non-trivial $\Psi_j$.

In $L_4$ we have the inequalities $r + 2s, s, -2r - 4s \geq 0$. Hence $r = -2s, s \geq 0$. So we get

$$v_j \Psi_{j,4} = \delta_{j,4} - \sum_{s \geq 1} m_j(s) \frac{(2s - 1)!}{s!s!} (v_2 v_3/v_4^2)^s,$$

where $m_j(s)$ is the $j$-th component of $(0, s, s, -2s, 0)$. The $m$-truncated version has the extra condition $m_4(s) = -2s > -m$, hence $s < m/2$.

In $L_5$ we have the inequalities $r + 2s, s, r \geq 0$. So we get

$$v_j \Psi_{j,5} = \delta_{j,5} - \sum_{r,s \geq 0} m_j(r, s) \frac{(2r + 4s - 1)!}{(r + 2s)!s!s!} (v_1 v_4/v_5^2)^r (v_2^2 v_3/v_5^3)^s,$$

where $m_j(r, s)$ is the $j$-th component of $(r + 2s, s, r, -2r - 4s)$. The $m$-truncated version has the extra condition $m_5(r, s) = -2r - 4s > -m$, hence $r + 2s < m/2$.

If we restrict our matrix to the index set $\Delta^5_\circ$, a computation shows that we get the $1 \times 1$-matrix with element

$$v_5 \Psi_{5,5} = \sum_{r,s \geq 0} \frac{(2r + 4s)!}{(r + 2s)!s!s!} x^r y^s = \frac{1}{\sqrt{1 - 4x}} F \left( \begin{array}{c} 1/4, 3/4, 1 \end{array} \left| \frac{64y}{(1 - 4x)^2} \right. \right),$$

where $x = v_2 v_3/v_4^2, y = v_1^2 v_2 v_3/v_5$. The other components $v_j \Psi_{j,5}$ are not so easy to express in terms of one-variable hypergeometric functions, if possible at all.

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