Singular statistics revised

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Abstract. In this paper, we analyze the ‘singular statistics’ of pseudointegrable Šeba billiards, i.e. billiards perturbed by zero-range perturbations. We have shown that the computation of a spectrum is reduced to the calculation of the uniquely defined renormalized Green’s function. We relate a spectrum of the billiard to the scattering length, which is the only parameter describing the perturbation. We show that taking into account the growing number of resonances, one observes a transition from ‘semi-Poissonian’-like statistics to Poissonian. This observation is in agreement with the argument that a classical particle does not feel a point perturbation.

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1. Introduction

The singular perturbed square billiard (i.e. the billiard perturbed by a zero-range perturbation; also called Šeba billiard by several authors) is one of the key models for quantum chaotic systems. Whereas the unperturbed square billiard with a typical irrational (like ‘golden mean’) side ratio shows Poissonian level-spacing statistics \[2\] in accordance with the Berry–Tabor conjecture \[3\] and the Sinai billiard, proved to be a fully chaotic system \[4\], exhibits GOE statistics \[5\], the ‘intermediate’ case of a singular perturbed billiard is expected to demonstrate some transient behavior \[1\], \[6\]–\[8\]. However, it has been reported that the billiard with a point perturbation can exhibit ‘fully developed quantum chaos’ \[1, 7\]. Then, according to the Bohigas–Giannoni–Schmit conjecture \[9\], the perturbed billiard should show GOE-statistics instead of Poissonian statistics. This may seem strange, since a point perturbation has almost no influence on the classical phase space of the billiard. Thus, it is natural to expect that in the semiclassical limit the billiard with a point perturbation shows similar statistics to an unperturbed billiard.

The proposed explanation for the given paradox was based on the argument that for zero-range perturbation for any wavelength, one can never reach the limit of the classical billiard with a point perturbation, since the wavelength is finite while the perturbation radius is zero. Therefore, the quantum system becomes chaotic, while its classical analogue is almost integrable. This argument was seemingly justified experimentally \[7\]. Despite this observation, it was reported in \[10\] that in the semiclassical limit the statistics tend to a Poissonian. Later in \[8\], some kind of ‘semi-Poissonian’ statistics were reported.

Further progress in the theoretical study of Šeba billiards was in the semiclassical calculation of the two-point correlation form factor \[11, 12\]. This quantity was related to a family of diffractive periodic orbits. It was found that the cross terms between the contributions of classical orbits are crucial. Recently, this conclusion was justified experimentally \[23\]. However, these remarkable findings did not illuminate the spectral statistics in the limit \(k \to \infty\)!

Thus, the very important question of whether ‘one small-size impurity changes completely the
statistical properties of integrable models’ [11] remains uncertain. It was this question that aroused interest in Šeba billiards.

The state of affairs described above seems to be unsatisfactory. The aim of this paper is to clarify the situation! In this paper, we present a proper description of a point perturbation removing the ambiguities of different treatments.

Point perturbations were introduced rigorously in [13] (for a historical overview see the introductions of [14, 15]). It was shown that a point perturbation is equivalent to a boundary condition at the perturbation point. Significant consideration has been given to point perturbations in [16]. The only relevant parameter describing the boundary condition in the three three-dimensional (3D) case was found to be the scattering length of the perturbation \( \beta \).

The same findings take place in the 2D case as well [14]. Moreover, in [14] it was shown that a point perturbation may approximate not only a quantum mechanical well of small radius but a more general perturbation of a small range, corresponding to various boundary conditions. Thus, the scattering length is the only parameter describing the perturbation itself.

The ambiguities mentioned above arise from the fact that the standard representation for the Green’s function in terms of a spectral sum,

\[
G(r, R; k) = \sum_n \frac{\psi_n(r)\psi_n(R)}{k^2 - E_n},
\]

is only conditionally convergent for \( r \neq R \) and exhibits the well-known logarithmic singularity of the free Green’s function \( G_f \) in the limit \( r \to R \),

\[
G_f(r \to R, R; k) \to \frac{1}{2\pi} \ln(k|r - R|) + \text{const}.
\]

There are essentially two approaches to regularizing the sum when \( r = R \): either by regularizing each term to achieve the convergence [1, 6, 10, 17–21],

\[
\sum_n \psi_n^2(R) \left( \frac{1}{k^2 - E_n} + \frac{1}{E_n} \right)
\]

or similar, or by subtracting from the sum the corresponding integral [8, 11]. The last idea is especially transparent when all of eigenfunctions take the same value at point \( R \). Then the regularization reads

\[
\lim_{N \to \infty} \left( \sum_{n=1}^{N} \frac{1}{k^2 - E_n} - \frac{1}{\langle \Delta E \rangle} \int_{0}^{E_N} \frac{dE}{k^2 - E} \right),
\]

where \( \langle \Delta E \rangle \) is the mean level spacing, and \( \int \) means a principal value integral. In both approaches, one has to compensate for the ambiguity coming from different regularizations by a proper renormalization of the ‘coupling constant’. Although mathematically correct, the approaches have the serious disadvantage of offering no recipe to relate the renormalized coupling constant to the scattering length.

The main difference between (3) and (4) lies in the fact that the first regularization is \( k \)-independent, removing only the divergency due to the space singularity, whereas the second one removes the logarithmic singularity at \( k \to \infty \) as well. It follows from the rigorous mathematical consideration, discussed in section 2, that the proper renormalization should be \( k \)-independent. We use the term ‘renormalization’ to denote the proper regularization leading
to the correct spectrum determination. Different regularizations are discussed in detail in appendix C.

In the present paper, we discuss a convenient renormalization of the Green’s function, which has two important advantages over previous approaches. It provides a direct relationship to the scattering length of the perturbation and gives exponentially convergent series. The idea of the renormalization is based on Ewald’s resummation technique, well known in solid-state physics for computing lattice sums. Using the obtained representation, we show that the level-spacing statistics of Šeba billiards actually tend to a Poissonian when the number of taken eigenvalues tends to infinity. These findings are in agreement with [10] and the intuitive ‘classical’ argument given above.

Let us turn to the experimental microwave Šeba billiard [7, 22, 23]. Theoretically, in all cases, the scatterer was treated as a point scatterer, which means physically that the characteristic wavelength \( \lambda \) of the field inside the cavity is much larger than the radius \( a \) of the scatterer. At the same time, the scattering length \( \beta \) of the given scatterer is, generally speaking, a free parameter depending on the internal structure of the scatterer (e.g. on the material of the coating, the radius of the metallic core, etc). We show below that the influence of the point scatterer is significant when \( \lambda \gtrsim \beta \) and vanishes when \( \lambda \ll \beta \) in accordance with a classical limit. To treat the scatterer as a point perturbation, we have to require \( a \ll \lambda \). Combining the last two estimations, we conclude that to cover experimentally the classical limit of a Šeba billiard one needs to create a scatterer with \( a \ll \beta \).

However, experimentally one often has \( a \sim \beta \). This means that in the regime \( \lambda \sim \beta \), the corresponding billiard should be treated as a quantum Sinai billiard but not a Šeba billiard. In this case, the ‘classical’ limit with a point perturbation cannot be achieved.

The expected ‘experimental’ evolution of the level-spacing statistics computed for a given number of resonances taken at different frequencies is plotted in figure 1. It is assumed that the radius of a small scatterer \( a \ll \beta \), but remains finite. Figures 1(a) and (b) correspond to the Šeba billiard approximation when the radius of the perturbation can be neglected. Figure 1(c) shows the ‘GOE’ statistics of the Sinai billiard in the regime where the wavelength of the field is comparable to the radius of the perturbation.

In what follows, we take the limit \( a \to 0 \), which means that figure 1(c) cannot be reproduced within the framework of the considered approach. Thus, we restrict ourselves to the mathematical model of a point perturbation as was done by the authors of [1, 6, 8, 10, 11, 17–21]. In this model, the radius \( a \) of the perturbation is zero.

2. Point perturbation of the billiard

Let us now turn to the theoretical and numerical study of the Šeba billiard. Following [1, 14, 15], let us first introduce a point perturbation of the billiard at point \( \mathbf{R} \). Basically, we will construct the ‘self-adjoint extension’ of the unperturbed ‘Hamiltonian’. This approach has already been used in [24].

For the unperturbed billiard, the eigenfunctions \( \psi \) and eigenvalues \( k^2 = k^2_n \) obey the equation

\[
(\Delta + k^2_n) \psi(\mathbf{r}) = 0.
\]  

(5)

If \( \psi(\mathbf{R}) = 0 \), the corresponding states do not feel the perturbation; thus these eigenfunctions and the corresponding eigenvalues are identical for the unperturbed and perturbed billiards. Next we
Figure 1. Sketch of the level-spacing statistics evolution computed for a given fixed number of eigenvalues of a billiard with a small perturbation whose radius $a$ is significantly smaller than its scattering wavelength $\beta$. The direction from left to right corresponds to increasing eigenvalues. The ‘semi-Poissonian’ statistics (a) correspond to the range $a \ll \beta \lesssim \lambda$, the ‘Poissonian’ statistics (b) correspond to the range $a \ll \lambda \ll \beta$ and the ‘GOE’ statistics (c) correspond to the range $a \sim \lambda \ll \beta$. Here, $\lambda$ is the characteristic wavelength. The names of distributions are written in parentheses since the corresponding plotted curves keep the essentials of these statistics but may differ from real distributions.

assume that the perturbed eigenfunctions $G$ obey the equation

$$(\Delta + k^2)G(r, R; k) = 0$$

outside of the scatterer with radius $a$. In what follows, we assume $a \to 0$. We shall see below that $G$ is nothing but the Green’s function of the unperturbed system. To recover an appropriate boundary condition at the perturbation point, let us consider the asymptotics of the function $G(r, R; k)$ outside of the scatterer when $r$ tends to $R$. Rewriting (6) in cylindrical coordinates, we obtain

$$\left[\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} + k^2\right] G(r, R; k) = 0,$$

where $\rho = |r - R|$ and $\varphi$ is the angle between the vector $r - R$ and the $x$-axis going along one side of the rectangle. We require $G(r, R; k)$ to be cylindrically symmetric in the vicinity of the point $r = R$. Thus, we obtain

$$\left[\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + k^2\right] G(r \to R, R; k) = 0.$$ 

The solution of the last equation is

$$G(r \to R, R; k) = c_1(R; k)J_0(k\rho) + c_2(R; k)Y_0(k\rho),$$

where $J_0$ and $Y_0$ are Bessel functions of the first and the second kind, respectively, and $c_1$ and $c_2$ are constants parametrically depending on $R$ and $k$. The last equality should be understood only in an asymptotic sense, since $J_0$ and $Y_0$ do not obey the proper conditions at the outer boundary of the billiard.

Using the asymptotic form of $Y_0(z \to 0)$ [25],

$$Y_0(z \to 0) = \frac{2}{\pi} \left[\ln\left(\frac{z}{2}\right) + \gamma\right] + O(z^2 \ln z),$$

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Figure 2. The illustration of the logarithmic singularity of the perturbed eigenfunction.

where $\gamma$ is the Euler constant, and the ‘identity’
\[
\Delta \left( \frac{1}{2\pi} \ln(k|\mathbf{r} - \mathbf{R}|) \right) = \delta(\mathbf{r} - \mathbf{R}),
\]  
we find that in the limit $a \to 0$ the function $G(\mathbf{r}, \mathbf{R}; k)$ obeys the equation
\[
(\Delta + k^2)G(\mathbf{r}, \mathbf{R}; k) = \delta(\mathbf{r} - \mathbf{R})
\]  
if we assume that $c_2(\mathbf{R}; k) = 1/4$. Another choice of the constant $c_2(\mathbf{R}; k)$ would only lead to a different normalization. Taking into account the boundary conditions for the function $G(\mathbf{r}, \mathbf{R}; k)$ at the outer boundary of the billiard and (12), we conclude that the perturbed eigenfunctions are the Green’s functions of the unperturbed billiard.

We are now going to derive the proper boundary condition at the perturbation point. First we separate the Green’s function into its regular and singular parts, respectively.

Combining (9) and (10), we obtain
\[
G(\mathbf{r} \to \mathbf{R}, \mathbf{R}; k) = \frac{1}{2\pi} \ln \left( \frac{\rho}{b} \right) + \xi_b(\mathbf{R}; k) + O(z^2 \ln z),
\]  
where $z = k\rho$, $b$ is some arbitrary length, and
\[
\xi_b(\mathbf{R}; k) = c_1(\mathbf{R}; k) + \frac{1}{2\pi} \left[ \ln \left( \frac{kb}{2} \right) + \gamma \right].
\]  
Here, $\xi_b(\mathbf{R}; k)$ is the renormalized Green’s function. In figure 2, we illustrate the singularity of the Green’s function near the perturbation point.

Let us now consider the rectangle with a small pricked circle of radius $a$ whose center is situated at the point $\mathbf{R}$. We designate it by $\Omega_a$. Then we consider the linear space of functions consisting of two subspaces: (i) the subspace of functions $f^{(1)}(\mathbf{r})$ vanishing at the outer boundary and possessing the asymptotics
\[
f^{(1)}(\mathbf{r} \to \mathbf{R}) = B \left[ \frac{1}{2\pi} \ln \left( \frac{\rho}{b} \right) + \xi \right] + O(z^2 \ln z),
\]  
where $B$ and $\xi$ are constants, and (ii) the subspace of regular functions $f^{(2)}(\mathbf{r})$ vanishing at the outer boundary of the billiard such that $f^{(2)}(\mathbf{R}) = 0$. $G(\mathbf{r}, \mathbf{R}; k)$ belongs to subspace (i) because of its asymptotic behavior (13). The unperturbed eigenfunctions $\psi(\mathbf{r})$ belong to subspace (ii).
Now we study the action of the operator $-\Delta = -\nabla^2$ on the space of functions defined above. The requirement of hermicity gives
\[
\lim_{a \to 0} \left[ (f^{(i)}| - \Delta|f^{(j)})_a - (f^{(j)}| - \Delta|f^{(i)})^*_a \right] = 0,
\]
where $f^{(i)}$ and $f^{(j)}$, with $i, j = 1, 2$, are arbitrary functions taken from subspaces (i) and (ii), respectively, and
\[
(f^{(i)}| - \Delta|f^{(j)})_a = \int_{\Omega_a} d^2r f^{(i)*}(-\Delta f^{(j)}).
\]
By means of Green’s theorem, we obtain
\[
(f^{(i)}| - \Delta|f^{(j)})_a = \frac{2\pi a}{\rho} f^{(i)*}(\partial f^{(j)}/\partial \rho) - f^{(j)}(\partial f^{(i)}/\partial \rho)^*_a = 0.
\]
Equation (18) shows that (16) holds automatically if $f^{(i)}$ and $f^{(j)}$ both belong to the subspace (2). Assume now that $i = 1$ and $j = 2$, or vice versa. Using the asymptotics (13), we see that (16) again holds automatically for any $f^{(1)}$ and $f^{(2)}$. Thus the case $i, j = 1$ implies the only nontrivial condition superimposed by the hermicity requirement of the constructed operator. Let us take two functions $f^{(1)}_1$ and $f^{(1)}_2$ with asymptotic expansions,
\[
f^{(1)}_i(r) = B_i \left[ \frac{1}{2\pi} \ln \left( \frac{\rho}{b} \right) + \xi \right] + O(z^2 \ln z).
\]
Substituting (19) into (18), we find
\[
\lim_{a \to 0} \left[ (f^{(1)}_1| - \Delta|f^{(1)}_2)_a - (f^{(1)}_2| - \Delta|f^{(1)}_1)_a \right] = B_1^2 B_2 (\xi^*_1 - \xi^*_2) = 0.
\]
Equality (20) must hold for any values of $B_i, \xi_i$. This leads to the conclusion that for all functions from subspace (1), the constant $\xi$ in (15) is real and the same. Let us choose a certain value $\xi = -D$. Then the boundary condition at the perturbation point reads
\[
\xi + D = 0
\]
provided that the length $b$ is fixed. Comparing (13) with (15), we find that for the perturbed eigenfunctions $G(r, R; k)$ the constant $\xi$ is equal to $\xi_b(R; k)$. Then (21) gives
\[
\xi_b(R; k) + D = 0.
\]
Substituting (14) into (22), we obtain
\[
c_1(R; k) + \frac{1}{2\pi} \left[ \ln \left( \frac{kb}{2} \right) + \gamma \right] + D = 0.
\]
The proper boundary condition cannot depend on the arbitrary length $b$ but rather should depend on a parameter characterizing the inner structure of the perturbation. Therefore, the length $b$ should be canceled in (23) by a proper choice of $D$. This can be achieved by the following choice,
\[
D = \frac{1}{2\pi} \ln \left( \frac{\beta}{b} \right),
\]
where $\beta$ is the scattering length of the perturbation. In appendix B, we prove that the definition of a scattering length is in accordance with scattering theory. The value of the length $\beta$ cannot
be obtained from the above consideration, since we have cut out the area containing the perturbation, thereby losing the information on it. Thus, we come to the conclusion that the scattering length is the only parameter describing the perturbation if $ka \ll 1$. Substituting (24) into (22) and using (14), we find

$$\xi_{\beta}(R; k) = c_1(R; k) + \frac{1}{2\pi} \left[ \ln \left( \frac{k\beta}{2} \right) + \gamma \right] = 0$$

in agreement with [22, 23]. The perturbed part of the spectrum consists of eigenvalues $k_n^2$, where $k = k_n$ are the solutions of (25).

Replacing $b$ with $\beta$, and $k$ with $k_n$, in (13) and using the equality $\xi_{\beta}(R, k_n) = 0$, we find the asymptotic expansion of the perturbed eigenfunction corresponding to the eigenvalue $k_n^2$,

$$G(r \rightarrow R, R; k_n) = \frac{1}{2\pi} \ln \left( \frac{\rho}{\beta} \right) + O(\varepsilon^2 \ln \varepsilon).$$

The leading term of the asymptotics (26) becomes zero when $\rho = \beta$. This fact can be used to determine experimentally the scattering length of a given perturbation.

Several conclusions on the level-spacing distribution can already be drawn from (25). Indeed, from (13) and (14) we conclude that $c_1(R, k)$ has the same poles as the Green’s function of the unperturbed billiard. From (25), we determine that when $k$ tends to infinity, the eigenvalues of the perturbed billiard approach the eigenvalues of the unperturbed one. Indeed, close to the eigenvalue $k_n$, the function $c_1(R, k)$ may be approximated by $\text{const}/(k^2 - k_n^2)$ whence follows

$$\text{const} \frac{k^2 - k_n^2}{k^2} + \eta_{\beta}(k) = 0, \quad \eta_{\beta}(k) = \frac{1}{2\pi} \ln \left( \frac{k\beta}{2} \right) \gg 1.$$  

Then $k^2 - k_n^2 = -2\pi \text{ const}/\ln(k\beta/2)$. In the limit $k \rightarrow \infty$, we recover the original spectrum of the billiard! Since the statistics of inter-level spacings for the unperturbed rectangular billiard with chosen side ratio are Poissonian [2], we conclude that the same statistics for high-lying eigenvalues of the Šeba billiard are also Poissonian. This is the most important conclusion of the paper, which contradicts the prediction given in [1].

3. Ewald’s representation of the renormalized Green’s function

For an explicit calculation of the spectrum of the perturbed billiard from (25), an expression for $\xi_{\beta}(R; k)$ is needed, which for general cases is by no means a trivial task (see appendices C and D). For the rectangle, it can be obtained by an application of Ewald’s method [26–31] (see appendix A), yielding for the renormalized Green’s function

$$\xi_{\beta}(R; k) = \sum_{n,m=-\infty}^{\infty} \sum_{s_1,s_2=0}^{1} (1 - \delta_{n,0}\delta_{m,0}\delta_{s_1,0}\delta_{s_2,0})(-1)^{s_1+s_2}G_f^{(i)}(R, R_{s_1s_2} + R_{nm}; k)$$

$$+ G^{(ii)}(R, R; k) + \frac{1}{2\pi} \ln \left( \frac{k\beta}{2} \right) + \varkappa = 0,$$

where $\delta_{n,0}$ is the Kronecker symbol, $\varkappa \simeq -0.058942$, $R = (x', y')$,

$$G_f^{(i)}(r, R; k) = -\int_0^1 dt \frac{e}{4\pi t} \exp \left( \frac{t - k^2(r - R)^2}{4t} \right).$$
Figure 3. Graphical interpretation of (28). Vertical gray dashed lines correspond to eigenwavenumbers of the unperturbed billiard. Above the first eigenwavenumber, the thin solid line shows $G^{(ii)}(R, R; k)$ as a function of $k$ and the thick solid line corresponds to the remainder in (28) taken with a minus sign. In this figure, $d_x = \pi / (\sqrt{5} - 1)$ and $d_y = \pi$, $\beta = 1$.

\[ G^{(ii)}(R, R; k) = \sum_{n,m=1}^{\infty} \psi_{nm}^2(x', y') e^{1-E_{nm}/k^2} \frac{e^{1-E_{nm}/k^2}}{k^2 - E_{nm}}, \]

\[ R_{ss} = ((-1)^{sx'}(-1)^{sy'}), \quad R_{nm} = (2nd_x, 2md_y), \]

\[ \psi_{nm}(x, y) = \frac{2}{\sqrt{d_x d_y}} \sin\left(\frac{\pi nx}{d_x}\right) \sin\left(\frac{\pi my}{d_y}\right), \]

\[ E_{nm} = \left(\frac{\pi n}{d_x}\right)^2 + \left(\frac{\pi m}{d_y}\right)^2, \]

where $d_x$ and $d_y$ are the lengths of the rectangle sides.

Now (28) resembles (23), so the main conclusions drawn above could be repeated. Equation (28) is exact and is especially suited to numerical studies, since it contains exponentially convergent series.

For large $k$, the double sum in (28) can be neglected and the rest looks very similar to the ‘$N$-poles’ approximation [24]. Indeed, in this regime the spectrum of the billiard can be found from the equation

\[ \sum_{n,m=1}^{\infty} \psi_{nm}^2(x', y') e^{1-E_{nm}/k^2} + \frac{1}{2\pi} \ln \left(\frac{k\beta}{2}\right) + \kappa = 0. \]

Equation (34) is similar to (3) of [8] except for the fact that the second term in (34) is not a polynomial as a function of $k^2$. Figure 3 shows a graphical interpretation of (28). In our calculations, we found that approximation (34) works perfectly above the first resonance already.
Figure 4. ‘Subbilliards’ corresponding to rational ratios \(x'/d_x\) and \(y'/d_y\). The point \((x', y')\) is shown by the black circle.

4. Integrated density of states

In what follows, we are interested in level-spacing statistics for the subset of perturbed eigenvalues of the Šeba billiard. There are several reasons for restricting ourselves to the statistics of the subspectrum. First of all, the influence of the perturbation is more pronounced if one considers only the perturbed part of the spectrum. That is probably the reason why in the pioneering work [1], only the statistics of the subspectrum are considered. Another reason for considering the subspectrum’s statistics is (25), which determines only the perturbed subspectrum. The graphical interpretation of (28) already gives an idea of the structure of the perturbed subspectrum (see figure 3), whereas when considering the unperturbed subspectrum as well, we lose the clearness. The last reason for considering the statistics of the perturbed subspectrum only is the direct correspondence of the perturbed subspectrum to the spectrum obtained from the reflection measurement with a single antenna introduced at the point of the perturbation. In such an experiment, the unperturbed subspectrum is not seen at all, since the corresponding eigenstates, vanishing at the perturbation point, cannot be excited.

If one considers only the perturbed subspectrum, it makes a difference whether the ratios \(x'/d_x\) and \(y'/d_y\) are rational or irrational numbers (see figure 4). The difference arises from the fact that for irrational numbers all eigenfunctions are perturbed while for rational ones part of the eigenfunctions remain unperturbed.

To compute the statistics and compare them with the GOE, semi-Poissonian and Poissonian predictions, one should first unfold the spectrum to a mean level spacing of one. This can be achieved by the following definition of the scaled eigenvalues,

\[
E_n^{(s)} = N(k_n^2),
\]  

(35)

where \(N(z)\) is a smoothened function counting the total number of eigenvalues \(k_n^2\) less than \(z\), i.e. the integrated density of states. If the spectrum of a system is known, the function \(N(z)\) can be obtained from a numerical fit. For the conventional unperturbed 2D billiard, one can use the Weyl estimation of the integrated density of states (see e. g. [32]),

\[
N_W(z) = \frac{A_1}{4\pi} z - \frac{A_2}{4\pi} \sqrt{z} + A_W,
\]

(36)
where \( A_1 \) is the area of the billiard, \( A_2 \) is its circumference and \( A_W \) is a constant. For the unperturbed rectangular billiard with the sides \( d_x \) and \( d_y \), we obtain \( A_1 = d_x d_y \) and \( A_2 = 2(d_x + d_y) \). While the Weyl estimation holds for the whole spectrum of the unperturbed billiard, it cannot be directly applied to its subspectrum as well as to the perturbed subspectrum of the Šeba billiard. However, to fit numerically the integrated density of states, one can still assume that the function to be found has the form (36) with some unknown coefficients \( A_1 \), \( A_2 \) and \( A_W \).

The scaled level spacing corresponding to the nearest eigenvalues \( k_n^2 \) and \( k_{n+1}^2 \) is

\[
  s_n = E_n^{(x)} - E_n^{(x)} = N(k_{n+1}^2) - N(k_n^2).
\]

Thus, the constant term \( A_W \) in (36) does not influence the statistics. The mean level spacing

\[
  \langle s \rangle = \frac{1}{M} \sum_{n=1}^{M} s_n = \frac{1}{M} [N(k_{M+1}^2) - N(k_1^2)] \rightarrow 1
\]

when \( M \rightarrow \infty \) in accordance with the rescaling requirement.

Although the integrated density of states corresponding to the perturbed subspectrum can be fitted numerically, it is possible to estimate it \emph{a priori}. Indeed, in figure 3 one sees that between two successive eigenvalues of the unperturbed spectrum corresponding to poles of the function \( G^{(ii)}(\mathbf{R}, \mathbf{R}; k) \), there always exists an eigenvalue of the perturbed billiard. Thus, the number of perturbed eigenvalues of the billiard below \( z \) should coincide (up to a single eigenvalue) with a number of eigenvalues of the unperturbed billiard below \( z \) corresponding to nonvanishing eigenfunctions at the point of the perturbation. Obviously, this result does not depend on the value of the scattering length.

Following the argument given above, we can compute the expected function \( N_e(z) \) right from the unperturbed billiard, where \( N_e(z) \) is equal to the integrated density of those states whose eigenfunctions do not vanish at the perturbation point \((x', y')\). Let us assume that \( x' = d_x p_1/q_1 \), \( y' = d_y p_2/q_2 \), where \( p_1/q_1 \) and \( p_2/q_2 \) are irreducible fractions. From figure 4, one can draw the conclusion that eigenfunctions of the small hatched ‘subbilliards’ with Dirichlet conditions at all boundaries are eigenfunctions of the initial billiard and vanish at the point \((x', y')\). According to the Weyl formula, the number of eigenvalues below \( z \) can be estimated as

\[
  N_v(z) = \frac{d_x d_y}{4\pi q_1} z - \frac{d_x/q_1 + d_y}{2\pi} \sqrt{z} + A_v, \tag{39}
\]

\[
  N_h(z) = \frac{d_x d_y}{4\pi q_2} z - \frac{d_x + d_y/q_2}{2\pi} \sqrt{z} + A_h \tag{40}
\]

for the vertical and horizontal hatched billiards, respectively. Here, \( A_v \) and \( A_h \) are some constants. We have computed twice the eigenvalues of the billiard obtained as an intersection of these subbilliards. Its number of eigenvalues can be estimated as

\[
  N_{vh}(z) = \frac{d_x d_y}{4\pi q_1 q_2} z - \frac{d_x/q_1 + d_y/q_2}{2\pi} \sqrt{z} + A_{vh}. \tag{41}
\]

Finally, the number of eigenvalues corresponding to vanishing eigenfunctions is

\[
  N_v(z) + N_h(z) - N_{vh}(z) = \left( \frac{1}{q_1} + \frac{1}{q_2} - \frac{1}{q_1 q_2} \right) \frac{d_x d_y}{4\pi} z - \frac{d_x + d_y}{2\pi} \sqrt{z} + A_v + A_h - A_{vh}. \tag{42}
\]
Figure 5. Integrated density of states of the perturbed subspectrum of the Šeba billiard. Panel (b) is a zoom of panel (a). The perturbation with the scattering length $\beta = 1$ is placed at the center of the billiard with $d_x = \pi / (\sqrt{5} - 1)$, $d_y = \pi$. The stepwise line corresponds to the computed subspectrum, the dashed line corresponds to (44) and the smooth solid line corresponds to a numerical fit of the form (36).

Subtracting the last estimation from the total number of eigenvalues below $z$,

$$N_W(z) = \frac{d_x d_y}{4\pi} z - \frac{d_x + d_y}{2\pi} \sqrt{z} + A_W,$$

we obtain the following estimation for the number of eigenvalues corresponding to nonvanishing eigenfunctions,

$$N_e(z) = \left(1 - \frac{1}{q_1} - \frac{1}{q_2} + \frac{1}{q_1 q_2}\right) \frac{d_x d_y}{4\pi} z + A_e,$$  \hspace{1cm} (44)

where $A_e = A_W + A_{vh} - A_v - A_h$. Surprisingly, the surface contribution $\sim \sqrt{z}$ vanishes.

In figure 5, the estimation (44) and the numerical fit of the form (36) for the integrated density of states are shown for comparison. One can see that the estimation (44) works very well.

From (34) and (44), one can compute a shift in the resonance induced by a point perturbation as compared to the mean level spacing. Indeed, when $k \to E_{nm}$, the function $G^{(ii)}(\mathbf{R}, \mathbf{R}; k)$ (30) tends to the following expression,

$$G^{(ii)}(\mathbf{R}, \mathbf{R}; k) \to \frac{4}{d_x d_y} \frac{\sin^2(\pi n_x' / d_x) \sin^2(\pi m_y' / d_y)}{k^2 - E_{nm}}.$$

From (44), we find the mean level spacing $\langle \Delta E\rangle$,

$$\langle \Delta E \rangle = 1/N'_e(z) = \frac{4\pi Q}{d_x d_y}, \hspace{1cm} \frac{1}{Q} = 1 - \frac{1}{q_1} - \frac{1}{q_2} + \frac{1}{q_1 q_2},$$  \hspace{1cm} (46)

Substituting (45) into (34) and using (46), we find for the relative shift of the resonance

$$\frac{k^2 - E_{nm}}{\langle \Delta E \rangle} = \frac{d_x d_y}{4\pi Q} (k^2 - E_{nm}) = -\frac{4 \sin^2(\pi n_x' / d_x) \sin^2(\pi m_y' / d_y)}{Q[\ln(E_{nm} \beta^2 / 4) + \gamma + g_0(1)]}.$$  \hspace{1cm} (47)
Let us estimate the number of resonances needed to show the transition to the Poissonian level-spacing statistics. Then the relative shift should be very small for all sufficiently large numbers $n$ and $m$. The sufficient condition is

$$\frac{Q}{4} [\ln(E_{nm}/4) + \gamma + g_0(1)] \gg 1.$$  

(48)

Depending on the values of $Q$ and $\beta$, the value of $E_{nm}$ can be very large.

5. Level-spacing statistics

In this section, we present a great deal of numerical results. Figures 6–8 show level-spacing distributions for the perturbed part of the spectrum for various situations to be discussed in detail below. For comparison, the curves corresponding to Poissonian, semi-Poissonian and

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**Figure 7.** The level-spacing statistics for the first 1484 resonances for a billiard with \(d_x = \pi / (\sqrt{5} - 1)\) and \(d_y = \pi\). The subspectrum of the unperturbed billiard corresponding to nonvanishing eigenfunctions at the position \((0.55d_x, 0.65d_y)\) of the billiard (a). Perturbed subspectrum of the Šeba billiard with a perturbation placed at point \((0.55d_x, 0.65d_y)\), \(\beta = 1\) (b). (c) The same as (b) with \(\beta = 0.1\). (d) The same as (b) with \(\beta = 0.02\). The scattering strength \(\eta_\beta(k)\) changes between 0.074 and 0.619 (b), −0.293 and 0.253 (c) and −0.549 and −0.003 (d), where \(k\) varies through the subspectrum (a).

GOE distributions are plotted by solid, dashed and solid lines, respectively. In each of the figures, panel (a) shows the level-spacing distribution for the unperturbed system to make sure that the distribution is really Poissonian, since it is well known that there may be deviations for small distances depending on the side ratio of the rectangle. Panels (b)–(d) show level-spacing distributions for \(\beta = 1, 0.1\) and 0.02, respectively. Note that a decrease in \(\beta\) means an increase in perturbation.

In figure 6, the scatterer is placed at the center, whereas in figure 7 it is at the point \((0.55d_x, 0.65d_y)\). In both cases, about 1500 lowest perturbed eigenvalues have been considered. For \(\beta = 1\), i.e. for a weak perturbation, the distribution shows a linear repulsion for small distances and an exponential tail, but not a semi-Poissonian behavior in the strict sense (dashed line). With \(\beta = 0.1\) and 0.02, there is a gradual transition to a broader distribution, resembling the GOE distribution for the scatterer at the center (figure 6(d)) and a semi-Poissonian distribution for the scatterer in the off-center position (figure 7(d)). This observation
Figure 8. The level spacings statistics for the resonances 25 000–27 000 for a billiard with \( d_x = \pi/(\sqrt{5} - 1) \) and \( d_y = \pi \). The subspectrum of the unperturbed billiard corresponding to nonvanishing eigenfunctions at the center of the billiard (a). Perturbed subspectrum of the Šeba billiard with a perturbation placed at the center, \( \beta = 1 \) (b). (c) The same as in (b) with \( \beta = 0.1 \). (d) The same as in (b) with \( \beta = 0.02 \). The scattering strength \( \eta_\beta(k) \) changes between 0.843 and 0.849 (b), 0.476 and 0.482 (c) and 0.220 and 0.226 (d), where \( k \) varies through the subspectrum (a).

deserves a more quantitative treatment, but this is beyond the scope of this paper. Qualitatively, it may be understood from the fact that at the center of the billiard all perturbed eigenfunctions have the same value, whereas for an off-center position there is a distribution of eigenfunction amplitudes giving rise to a corresponding distribution of resonance shifts.

Figure 8 finally shows the level-spacing distribution again with the perturbation at the center but now for the numbers of perturbed eigenvalues from 25 000 to 27 000. Comparison of figures 6 and 8 shows a pronounced change in the distribution towards a Poissonian with increasing eigenvalue numbers. This is particularly evident for the weaker perturbations, \( \beta = 1 \) and 0.1. Figures 8(b) and (c) demonstrate the main result of this paper: with increasing eigenvalue numbers, eventually the level-spacing distribution of the unperturbed system is recovered.

One can see that in figure 8, in contrast to figures 6 and 7, the variation in the scattering strength through the considered eigenvalue range is small as compared to one. In such a
situation, the analytical approach [8, 11] can be used, where the scattering strength is assumed to be constant, to treat the obtained distributions.

The variation in the scattering strength between \( k_1 \) and \( k_2 > k_1 \) is \( \eta_\beta(k_2) - \eta_\beta(k_1) = (2\pi)^{-1}\ln(k_2/k_1) \). The number of perturbed eigenvalues \( N_e(k_i) \) lying below \( k_i \), \( i = 1, 2 \), can be estimated from (44) and (46). Then the variation in the scattering strength can be written as \( \pi^{-1}\ln[1 + n/N_e(k_1)] \), where \( n = N_e(k_2) - N_e(k_1) \). If \( n/N_e(k_1) \ll 1 \), the variation can be estimated as

\[
\eta_\beta(k_2) - \eta_\beta(k_1) \simeq n \frac{n}{\pi N_e(k_1)} \ll 1.
\]  

(49)

It was found numerically that for good statistics one needs at least 1000 resonances. Then to obey (49), \( N_e(k_1) \) should be larger than 10 000. Even for superconducting microwave billiards, the accessible number of resonances does not exceed several thousand [33]. Therefore, the regime of a constant scattering strength is not realistic for present-day microwave studies.

6. Conclusions

First of all, we have presented here the complete solution of the spectral problem for the rectangular billiard with a single-point perturbation. We have shown that the problem is reduced to the calculation of the uniquely defined renormalized Green’s function. We have related the spectrum of a Šeba billiard to a scattering length of the perturbation. We have shown that the statistics of the Šeba billiard tends to a Poissonian when the number of levels taken into account tends to infinity. The estimation given at the end of section 5 showed, however, that the transition to Poissonian statistics appears, depending on the scattering length, only at exponentially large quantum numbers. The solution is based on Ewald’s representation of the renormalized Green’s function (28). This representation contains exponentially rapidly convergent series. Together with the Ewald’s representation of the usual Green’s function, the approach presented here is a powerful tool to analyze various experiments carried out in rectangular billiards.

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Appendix A. On Ewald’s method

In this appendix, we derive a convenient representation for the renormalized Green’s function. This is based on Ewald’s method [26]–[31]; therefore we call it the Ewald’s representation.

We start from the eigenfunction representation of the Green’s function for the free billiard

\[
G(\mathbf{r}, \mathbf{R}; k) = \sum_{n,m=1}^{\infty} \frac{\psi_{nm}(x, y)\psi_{nm}(x', y')}{k^2 - E_{nm}},
\]

(A.1)
where $R = (x', y')$,

$$\psi_{nm}(x, y) = \frac{2}{\sqrt{d_x d_y}} \sin\left(\frac{\pi n x}{d_x}\right) \sin\left(\frac{\pi m y}{d_y}\right). \quad (A.2)$$

$$E_{nm} = \left(\frac{\pi n}{d_x}\right)^2 + \left(\frac{\pi m}{d_y}\right)^2, \quad (A.3)$$

and $d_x$ and $d_y$ are the two sides of the rectangle. When $x \to x'$ and $y \to y'$, the series (A.1) diverges logarithmically. This is just another manifestation of the well-known singularity of the Green’s function for $r \to R$ (see (13)) which is a local feature and does not depend on the outer boundary conditions. This suggests that eigenfunction representation is not an appropriate choice to compute the renormalized Green’s function but that image representation [29] might be preferable.

The image representation of the Green’s function reads

$$G(r, R; k) = \sum_{n, m = -\infty}^{\infty} \sum_{s_1, s_2 = 0}^{1} (-1)^{s_1 + s_2} G_I(r, R; s_1 s_2) + R_{nm}; k), \quad (A.4)$$

where

$$R_{s_1 s_2} = ((-1)^{s_1} x', (-1)^{s_2} y'), \quad R_{nm} = (2n d_x, 2m d_y), \quad (A.5)$$

and $G_I(r, R; k)$ is the Green’s function for the 2D plane (see figure A.1). The image representation is much better suited to compute the renormalized Green’s function, since when $r$ tends to $R$, the term $n = m = 0, s_1 = s_2 = 0$ is the only one in the representation logarithmically tending to infinity. Now the divergency can be subtracted analytically. However, the image representation does not solve the problem yet, since it converges absolutely only if $\text{Im} k > 0$. To overcome this obstacle, Ewald [26] proposed the dual representation keeping features of the image as well as the eigenmode representation.

Below, we follow the previous works [27–29]. Let us first find a convenient representation for $G_I$. To this end, we consider the following initial-value problem for the function $g(t; r, R, k)$,

$$\frac{\partial g}{\partial t} = (\Delta + k^2) g, \quad g(t = 0; r, R, k) = \delta(r - R). \quad (A.6)$$
Then $G_\ell$ can be written as

$$G_\ell(r, R; k) = -\int_C dt \, g(t; r, R, k). \quad (A.7)$$

The contour $C$ should start at $t = 0$ and tend to infinity in such a way that $g$ tends to zero. Obviously,

$$g(t; r, R, k) = e^{k^2 t} K(t; r, R), \quad (A.8)$$

where the heat kernel $K(t; r, R)$ can be found by a separation of variables

$$K(t; r, R) = \frac{1}{4\pi t} e^{-(r-R)^2/(4t)}. \quad (A.9)$$

Finally, we obtain

$$G_\ell(r, R; k) = -\int_C \frac{dt}{4\pi t} \exp\left(k^2 t - \frac{(r-R)^2}{4t}\right). \quad (A.10)$$

The simplest contour of the integration is the imaginary half-line going from zero to $i\infty$ (contour $C_1$ in figure A.2(a)). Using this contour, we conclude ([25], entry B.187(2)) that

$$G_\ell(r, R; k) = -\frac{i}{4} H_0^{(1)}(k|\rho - \rho|), \quad (A.11)$$

where $H_0^{(1)}$ is the Hankel function of the first kind. However, the asymptotic behavior (13) at $\rho \to 0$ becomes hidden in this representation. To recover the asymptotic behavior, we will use another contour of the integration. We see that for small values of $|t|$ the best convergence provides an interval lying in the real axis from zero to some positive value $t_{\text{Ew}}$ (see figure A.2(a)). We will call this value the Ewald parameter. From the other side, the best convergence for large values of $|t|$ would provide the half-line going from some negative value (we choose it to be equal to $-t_{\text{Ew}}$) to $-\infty$. What remains is to connect these parts to make a contour. We connect them by a half-circle $C_2$. Finally, the constructed contour is equivalent to $C_1$ since the integrals along the quarter-circles $C_3$ and $C_4$ (plotted by dashed lines in figure A.2(a)) tend to zero when the radius of $C_3$ tends to infinity and the radius of $C_4$ tends to zero.

Using the constructed contour, we can obtain the asymptotics of the type (13) from the representation (A.10). The developed technique will be used further to compute the renormalized Green’s function. We write

$$G_\ell(r, R; k) = G_\ell^{(i)}(r, R; k) + G_\ell^{(ii)}(r, R; k), \quad (A.12)$$
where

\[ G^{(i)}_t(r, R; k) = - \int_0^{t_{\text{Ew}}} g(t; r, R, k) dt, \quad (A.13) \]

\[ G^{(ii)}_t(r, R; k) = - \int_{C_2} g(t; r, R, k) dt - \int_{-t_{\text{Ew}}}^{-\infty} g(t; r, R, k) dt. \]

Introducing the notation \( u = (r - R)^2/(4t_{\text{Ew}}) \), \( v = k^2 t_{\text{Ew}} \), we write \( G^{(i)}_t \) as follows,

\[ G^{(i)}_t(r, R; k) = - \int_0^1 \frac{dt}{4\pi t} e^{vt-u/t}. \quad (A.14) \]

When \( r \) tends to \( R \), \( u \) tends to zero. To compute the asymptotics of \( (A.14) \) for \( u \to 0 \), we make the following transformations,

\[ G^{(i)}_t(r \to R, R; k) = - \int_0^1 \frac{dt}{4\pi t} (e^{vt} - t + 1)e^{-u/t} \]

\[ \approx \int_0^1 \frac{dt}{4\pi t} (1 - e^{vt}) - \int_u^\infty \frac{dt}{4\pi t} e^{-t} \]

\[ = \int_0^1 \frac{dt}{4\pi t} (1 - e^{vt}) - \int_u^1 \frac{dt}{4\pi t} e^{-t} - \int_1^\infty \frac{dt}{4\pi t} e^{-t} \]

\[ \approx \int_0^1 \frac{dt}{4\pi t} (1 - e^{-t}) - \int_1^\infty \frac{dt}{4\pi t} e^{-t} + \int_0^1 \frac{dt}{4\pi t} (1 - e^{vt}) + \frac{1}{4\pi} \ln u \]

\[ = \frac{1}{4\pi} [g_0(v) + \gamma + \ln u], \quad (A.15) \]

where

\[ g_0(v) = \int_0^1 \frac{dt}{t} (1 - e^{-t}), \quad \gamma = \int_0^1 \frac{dt}{t} (1 - e^{-t}) - \int_1^\infty \frac{dt}{t} e^{-t} \quad (A.16) \]

(see [25], entry 8.367.12). Since \( g_0(v) \) has no singularity at \( t = 0 \), it can be written as

\[ g_0(v) = \int_0^1 \frac{dt}{t} (1 - e^{-vt}) - \int_{C_0} \frac{dt}{t} (1 - e^{vt}), \quad (A.17) \]

where \( C_0 \) is a half-circle of unit radius.

Since \( G^{(ii)}_t \) has no singularity when \( r \to R \), the leading term of its asymptotics is

\[ G^{(ii)}_t(R, R; k) = - \int_{C_0} \frac{dt}{4\pi t} e^{vt} - \int_1^\infty \frac{dt}{4\pi t} e^{-vt}. \quad (A.18) \]

Using \( (A.15), (A.17) \) and \( (A.18) \), we find

\[ G^{(i)}_t(r \to R, R; k) + G^{(ii)}_t(R, R; k) \approx \frac{1}{4\pi} \left( \ln u + \gamma + \int_0^1 \frac{dt}{t} (1 - e^{-vt}) - \int_1^\infty \frac{dt}{t} e^{-vt} \right) - \frac{i}{4}. \quad (A.19) \]

The last equality does not depend on the choice of \( t_{\text{Ew}} \). To prove it, we can take \( v \) as an independent parameter; then \( u = k^2 (r - R)^2/(4v) \). Differentiation of the last equality with
respect to $v$ gives zero. The reasonable choice of $v$ should not lead to exponentially large values of $g_0(v)$. Indeed, due to (A.19), such a large contribution is somehow artificial since it is annihilated by $G^{(i)}_t$. Thus it is natural to take $v = 1$. Then (A.19) gives

$$G_t(r \to R, R; k) \simeq \frac{1}{2\pi} \left[ \ln \left( \frac{k |r - R|}{2} \right) + \gamma \right] - \frac{i}{4}.$$  \hspace{1cm} (A.20)

This calculation has demonstrated that it is important to divide the free Green’s function into two parts: $G^{(i)}_t$ and $G^{(ii)}_t$. The first part describes the space singularity, and the second part makes the contribution to the regular part of the asymptotics.

Let us now turn to image representation (A.4) of the Green’s function of the rectangular cavity. We again write the free Green’s function $G_t$ in the form

$$G_t(r, R; k) = G^{(i)}_t(r, R; k) + G^{(ii)}_t(r, R; k),$$  \hspace{1cm} (A.21)

where $G^{(i)}_t$ is defined by (A.14) and

$$G^{(ii)}_t(r, R; k) = \frac{1}{\sqrt{4\pi t}} \exp \left( \frac{k^2 t - (r - R)^2}{4t} \right).$$  \hspace{1cm} (A.22)

The best choice of the contour $C_5$ will be discussed later. We shall see that the choice used to compute the asymptotics (A.20) does not fit. Substituting (A.21) into (A.4), we obtain

$$G(r, R; k) = G^{(i)}(r, R; k) + G^{(ii)}(r, R; k),$$  \hspace{1cm} (A.23)

where

$$G^{(i)}(r, R; k) = \sum_{n, m = -\infty}^{\infty} \sum_{s_1, s_2 = 0}^1 (-1)^{s_1 + s_2} G^{(i)}_t(r, R_{s_1 s_2} + R_{nm}; k),$$  \hspace{1cm} (A.24)

$$G^{(ii)}(r, R; k) = \sum_{n, m = -\infty}^{\infty} \sum_{s_1, s_2 = 0}^1 (-1)^{s_1 + s_2} G^{(ii)}_t(r, R_{s_1 s_2} + R_{nm}; k).$$  \hspace{1cm} (A.25)

To improve the convergency of series for $G^{(ii)}$, we use the identity

$$\sum_{n = -\infty}^{\infty} e^{-(x - 2nd_z)^2/(4t)} = \sqrt{\pi t} \sum_{n = -\infty}^{\infty} e^{\pi x / d_z - \pi^2 n^2 / d_z^2},$$  \hspace{1cm} (A.26)

which can be proved by applying the Poisson sum rule [29] to the function

$$f(x) = e^{-x^2/(4t)}.$$  \hspace{1cm} (A.27)

Performing the resummation, we obtain

$$G^{(ii)}(r, R; k) = -\frac{1}{4d_x d_y} \sum_{n, m = -\infty}^{\infty} e^{\pi x / d_x + \pi y / d_y} \sum_{s_1, s_2 = 0}^1 (-1)^{s_1 + s_2}$$

$$\times \int_{C_5} e^{k^2 - (\pi n / d_x)^2 - (\pi m / d_y)^2} dt.$$  \hspace{1cm} (A.28)

Now we see that the integral over $C_5$ should converge for positive as well as for negative values of the real part of $k^2 - (\pi n / d_x)^2 - (\pi m / d_y)^2$ provided that $\text{Im} \ k > 0$. Therefore, we
have to assume $t \to \infty$ along the contour $C_5$. This leads to the choice of the contour shown in figure A.2(b).

Performing the integration, we obtain

$$G^{(ii)}(\mathbf{r}, \mathbf{R}; k) = \frac{1}{4d_xd_y} \sum_{n,m=-\infty}^{\infty} \frac{e^{(k^2-(\pi n/d_x)^2-(\pi m/d_y)^2)t_{EW}}}{k^2 - (\pi n/d_x)^2 - (\pi m/d_y)^2} e^{\imath \pi n x + \imath \pi my} \times \sum_{s_1,s_2=0}^{1} (-1)^{s_1+s_2} e^{-\imath \pi n(1-s_1)x/d_x - \imath \pi m(1-s_2)y/d_y}.$$

Summarizing over $s_1$ and $s_2$, we obtain

$$\sum_{s_1,s_2=0}^{1} (-1)^{s_1+s_2} e^{-\imath \pi n(1-s_1)x/d_x - \imath \pi m(1-s_2)y/d_y} = -4 \sin \left( \frac{\pi nx'}{d_x} \right) \sin \left( \frac{\pi my'}{d_y} \right).$$

Now we can perform summations over $n$ and $m$,

$$G^{(ii)}(\mathbf{r}, \mathbf{R}; k) = -\frac{1}{d_xd_y} \sum_{n,m=-\infty}^{\infty} \sin \left( \frac{\pi nx}{d_x} \right) \sin \left( \frac{\pi my}{d_y} \right) \frac{e^{(k^2-(\pi n/d_x)^2-(\pi m/d_y)^2)t_{EW}}}{k^2 - (\pi n/d_x)^2 - (\pi m/d_y)^2} e^{\imath \pi nx/d_x + \imath \pi my/d_y} \times \frac{4}{d_xd_y} \sum_{n,m=1}^{\infty} \sin \left( \frac{\pi nx'}{d_x} \right) \sin \left( \frac{\pi my'}{d_y} \right) \sin \left( \frac{\pi nx}{d_x} \right) \sin \left( \frac{\pi my}{d_y} \right) \times e^{(k^2-(\pi n/d_x)^2-(\pi m/d_y)^2)t_{EW}}.$$ 

Formulæ (A.14), (A.24) and (A.31) give Ewald’s representation of the Green’s function for the rectangular billiard. The integral in (A.14) has to be computed numerically. Now we can recapitulate the advantages of Ewald’s method. First of all, both series $G^{(i)}$ and $G^{(ii)}$ are exponentially convergent. Thus, we can take the analytic continuation and choose real $k$. Secondly, we have separated the part $G^{(i)}$ responsible for the space singularity from the part $G^{(ii)}$ responsible for the pole information. Indeed, $G^{(ii)}$ exponentially converges even when $\mathbf{r} = \mathbf{R}$. Thirdly, only in $G^{(i)}$ there is the term corresponding to $s_1 = s_2 = 0$, $n = m = 0$ that asymptotically tends to infinity when $\mathbf{r} \to \mathbf{R}$. The rest of the series is exponentially convergent. The last observation allows us to compute the renormalized Green’s function.

Using the asymptotic expansion (A.15), we obtain the exact Ewald representation for the renormalized Green’s function,

$$\xi_\beta(\mathbf{R}; k) = \frac{1}{4\pi} \left[ g_0(k^2t_{EW}) + \gamma \right] + \sum_{n,m=-\infty}^{1} \sum_{s_1,s_2=0}^{1} (1 - \delta_{n,0}\delta_{m,0}\delta_{s_1,0}\delta_{s_2,0})(-1)^{s_1+s_2} G^{(i)}(\mathbf{R}, \mathbf{R}_{s_1s_2} + \mathbf{R}_{nm}; k) + \frac{1}{4\pi} \ln \left( \frac{\beta^2}{4t_{EW}} \right) + G^{(ii)}(\mathbf{R}, \mathbf{R}; k),$$

with $G^{(i)}$ to be computed numerically from the integral (29) and $G^{(ii)}$ from the sum (A.31). Now we can compute the perturbed part of the spectrum from the condition $\xi_\beta(\mathbf{R}, k_n) = 0$ (see (25)) using the representation (A.32). This final equation in contrast to (23) lost clarity.
since it depends on the as yet undefined Ewald parameter $t_{Ew}$. To define it, we first consider large values of $k$. Then to avoid exponentially large values of the function $g_0(k^2t_{Ew})$ as well as exponentially large amplitudes of terms with small numbers $n, m$ in the expansion of $G^{(ii)}$, we put $t_{Ew} = 1/k^2$. Obviously, this choice is inappropriate for $k \to 0$, since this would mean computing a huge number of terms in $G^{(ii)}$. So, finally the Ewald parameter can be chosen as

$$
t_{Ew} = \begin{cases} 
1/k^2, & \text{if } k > k_0, \\
1/k_0^2, & \text{if } k \leq k_0,
\end{cases} \quad (A.33)
$$

where $k_0^2 = (\pi/dx)^2 + (\pi/dy)^2$ is the lowest eigenvalue of the unperturbed system. To investigate the spectral statistics, we can assume that $t_{Ew} = 1/k^2$.

Then (A.32) reads

$$
\xi_{\beta}(R; k) = \frac{1}{4\pi}[g_0(1) + \gamma'] + \sum_{n,m=-\infty}^{\infty} \sum_{s_1,s_2=0}^{1} (1 - \delta_{n,0}\delta_{m,0}\delta_{s_1,0}\delta_{s_2,0}) (-1)^{s_1+s_2} G^{(i)}_{t}(R, R_{s_1 s_2} + R_{nm}; k) \\
+ \frac{1}{2\pi} \ln\left(\frac{k\beta}{2}\right) + G^{(ii)}(R, R; k). \quad (A.34)
$$

Although the Poisson resummation is a common tool used to obtain Ewald’s representation of the Green’s function [26–29, 31], one can avoid it and obtain formulae (A.24) and (A.31) more easily.

Let us consider the initial problem (A.6) in the rectangular billiard with proper boundary conditions. Then the solution can be written in two equivalent forms: in the form of the image representation $g_i(t; r, R, k)$ and in the form of eigenmode representation $g_e(t; r, R, k)$,

$$
g_i(t; r, R, k) = \sum_{n,m=-\infty}^{\infty} \sum_{s_1,s_2=0}^{1} (-1)^{s_1+s_2} g(t; r, R_{s_1 s_2} + R_{nm}, k), \quad (A.35)
$$

$$
g_e(t; r, R, k) = \frac{4}{d_x d_y} \sum_{n,m=1}^{\infty} e^{(k^2-(\pi n/d_x)^2-(\pi m/d_y)^2)t} \psi_{nm}(x, y) \psi_{nm}(x', y'). \quad (A.36)
$$

Then we write

$$
G(r, R; k) = G^{(i)}(r, R; k) + G^{(ii)}(r, R; k), \quad (A.37)
$$

where

$$
G^{(i)}(r, R; k) = -\int_{t_{Ew}}^{t} g_i(t; r, R, k) dt, \quad (A.38)
$$

$$
G^{(ii)}(r, R; k) = -\int_{C_5} g_e(t; r, R, k) dt. \quad (A.39)
$$

Performing the integration in equations (A.38) and (A.39), we immediately obtain (A.24) and (A.31).
Appendix B. On the scattering length

Let us consider a scattering problem in 2D free space on a point scatterer situated at the point $\mathbf{R}$. Then the solution is

$$\psi_k(\mathbf{r}) = e^{ikr} + f(\mathbf{R}; k) \sqrt{\frac{\pi k}{2}} i H_0^{(1)}(kr), \quad (B.1)$$

where $f(\mathbf{R}; k)$ is the scattering amplitude and $H_0^{(1)}$ is the Hankel function of the first kind. Using (A.11) and (A.20), we obtain from (15)

$$\xi = -\frac{e^{ikR}}{f(\mathbf{R}; k)\sqrt{8\pi k}} + \frac{1}{2\pi} \left( \ln \frac{k\beta}{2} + \gamma \right) - \frac{i}{4}. \quad (B.2)$$

Then (21) gives

$$-\frac{e^{ikR}}{f(\mathbf{R}; k)\sqrt{8\pi k}} + \frac{1}{2\pi} \left( \ln \frac{k\beta}{2} + \gamma \right) - \frac{i}{4} = 0, \quad (B.3)$$

where $\beta$ is a scattering length. Thus

$$f(\mathbf{R}; k) = \sqrt{\frac{\pi}{2k \ln(k\beta/2) + \gamma - i\pi/2}}, \quad (B.4)$$

in agreement with [34] (chapter 132, problem 7). When $|\mathbf{r} - \mathbf{R}| = \beta$, the function $\psi_k(\mathbf{r})$ is close to zero. This illustrates the fact that the scattering length $\beta$ of a given point perturbation does not depend on boundary conditions and can always be defined as the distance from the perturbation point where the wavefunction vanishes.

Appendix C. Renormalization, regularization and scattering length

In this section, we briefly consider the approaches previously used to compute the spectrum of a Šeba billiard.

There are basically two ideas used to compute the spectrum of a Šeba billiard. The first approach [13, 14, 16], considered in detail in section 2 of this paper, is based on the introduction of a point perturbation. It reduces to the boundary condition at the perturbation point (25), which depends on the only parameter, namely the scattering length of the perturbation $\beta$. Within the framework of this approach for computing the spectrum of a Šeba billiard, one has to compute the renormalized Green’s function $\xi_\beta$, which is uniquely defined as

$$\xi_\beta(\mathbf{R}; k) = \lim_{\mathbf{r} \to \mathbf{R}} \left[ G(\mathbf{r}, \mathbf{R}; k) - \frac{1}{2\pi} \ln \left( \frac{|\mathbf{r} - \mathbf{R}|}{\beta} \right) \right]. \quad (C.1)$$

The last equality can be written as

$$\xi_\beta(\mathbf{R}; k) = \lim_{\mathbf{r} \to \mathbf{R}} \left[ G(\mathbf{r}, \mathbf{R}; k) - G(\mathbf{r}, \mathbf{R}; q) \right] + \lim_{\mathbf{r} \to \mathbf{R}} \left[ G(\mathbf{r}, \mathbf{R}; q) - \frac{1}{2\pi} \ln \left( \frac{|\mathbf{r} - \mathbf{R}|}{\beta} \right) \right]$$

$$= \lim_{\mathbf{r} \to \mathbf{R}} \left[ G(\mathbf{r}, \mathbf{R}; k) - G(\mathbf{r}, \mathbf{R}; q) \right] + \xi_\beta(\mathbf{R}; q), \quad (C.2)$$

where $q$ is some complex number, $\text{Im}[q^2] > 0$. Substituting (A.1) into (C.2), we obtain

$$\xi_\beta(\mathbf{R}; k) = \sum_{n,m=1}^\infty e^{i k R} \left( \frac{1}{k^2 - E_{nm} - \frac{1}{q^2 - E_{nm}}} + \xi_\beta(\mathbf{R}; q) \right). \quad (C.3)$$
Since series (A.1) is conditionally convergent only, the term-by-term subtraction in (C.3) is questionable. For the rectangular billiard it can be justified by means of the Ewald’s resummation described in appendix A.

Since renormalization means only the removal of space singularity, the renormalizing part

\[-\sum_{n,m} \frac{\psi_{nm}^2(R)}{q^2 - E_{nm}} + \xi_\beta(R; q)\]  

is k-independent.

Another approach [8, 11] does not take care of the boundary condition at the perturbation point but rather considers a formal regularization of the divergent series,

\[\sum_{n,m=1}^\infty \psi_{nm}^2(R) \frac{1}{k^2 - E_{nm}}.\]  

To obtain a spectrum, the regularized series is equated to some constant. The regularization of this kind is not unique; thus the result is, generally speaking, different from (25).

Let us derive a specific form of (C.1). Putting \(q = \sqrt{i\Lambda}, \sqrt{i} = (i + 1)/\sqrt{2}\), where \(\Lambda\) takes some real value, and taking a real part of the right-hand side in (C.1), we obtain

\[\xi_\beta(R; k) = \text{Re}[\xi_\beta(R; \sqrt{i\Lambda})] + \sum_{n,m=1}^\infty \psi_{nm}^2(R) \left( \frac{1}{k^2 - E_{nm}} + \frac{E_{nm}}{E_{nm}^2 + \Lambda^2} \right).\]  

Using the representation (14), we obtain from (C.6)

\[\xi_\beta(R; k) = \text{Re} \left[ \xi_\beta(R; \sqrt{i\Lambda}) \right] + \frac{1}{2\pi} \ln \left( \frac{\beta}{b} \right) + \sum_{n,m=1}^\infty \psi_{nm}^2(R) \left( \frac{1}{k^2 - E_{nm}} + \frac{E_{nm}}{E_{nm}^2 + \Lambda^2} \right).\]  

Substituting (C.7) into (25), we obtain the equation defining a spectrum. This equation was used in previous work [10, 12, 17–20] to compute a spectrum. To relate a spectrum to a scattering length of a given scatterer, one has to compute \(\text{Re}[\xi_\beta(R; \sqrt{i\Lambda})]\). Instead, the authors of [10, 18–20] characterized a scatterer by a given value of a ‘bare coupling constant’ \(v_B\),

\[-v_B^{-1} = \text{Re}[\xi_\beta(R; \sqrt{i\Lambda})] + \frac{1}{2\pi} \ln \left( \frac{\beta}{b} \right).\]  

Depending on the length \(\beta\), the ‘bare coupling constant’ can take any real value. However, without an explicit computation of \(\text{Re}[\xi_\beta(R; \sqrt{i\Lambda})]\), the spectrum cannot be related to a given physical scatterer. The computation of \(\text{Re}[\xi_\beta(R; \sqrt{i\Lambda})]\) cannot be performed within the framework of this approach and requires additional ideas. It can be realized by means of Ewald’s representation (A.32) (see figure C.1). Furthermore, the representation (C.7), despite being exact, has a crucial technical disadvantage: while Ewald’s representation (A.32) gives exponentially convergent series, the convergence in (C.7) is only algebraic. Therefore, a direct computation of large eigenvalues using (C.7) takes an enormous amount of time. In our computation, we found that to compute the renormalized Green’s function below the tenth pole using Ewald’s representation, one needs two orders of magnitude shorter time than using representation (C.7). This shows that representation (C.7) is technically inappropriate for large k values.

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Figure C.1. The real part (bold line) and the imaginary part (solid line) of the $\xi_b(R; \sqrt{i}\Lambda)$, $b = 1$, $R = (d_x/2, d_y/2)$. The dashed horizontal line has the ordinate $y = 1/8$.

One may think that the advantage of (C.7) is its general form appropriate for any billiard, not only rectangular. To associate the ‘bar coupling constant’ $\upsilon_B$ with the physical properties of the perturbation, there is no way out but an explicit computation of $\text{Re}[\xi_b(R; \sqrt{i}\Lambda)]$. And for an arbitrary billiard one cannot use Ewald’s representation to compute it. We will come back to a generalization for a ‘generic’ billiard in appendix D.

A particular case of the representation (C.7) is $\Lambda = 0$. Then

$$
\xi_\beta(R; k) = \text{Re}[\xi_b(R; 0)] + \frac{1}{2\pi} \ln \left( \frac{\beta}{b} \right) + \sum_{n,m=1}^{\infty} \psi_{nm}^2(R) \left( \frac{1}{k^2 - E_{nm}} + \frac{1}{E_{nm}} \right). 
$$

(C.9)

Using the estimation $E_\nu \simeq (\Delta E)\nu$, where $\nu$ is the sequence number of the eigenvalue $E_\nu$ in the perturbed subspectrum, and (45), we can rewrite (C.9) as

$$
\xi_\beta(R; k) = \text{Re}[\xi_b(R; 0)] + \frac{1}{2\pi} \ln \left( \frac{\beta}{b} \right) + \sum_{\nu=1}^{\infty} \psi_{\nu}^2(R) \left( \frac{1}{E_\nu - \frac{d_x d_y}{4\pi \nu Q}} + \frac{d_x d_y}{4\pi \nu Q} \right),
$$

(C.10)

where $\psi_\nu$ corresponds to $E_\nu$. The representation (C.10) is correct if

$$
\sum_{\nu=1}^{\infty} \psi_{\nu}^2(R) \left( \frac{1}{E_\nu - \frac{d_x d_y}{4\pi \nu Q}} \right)
$$

(C.11)

converges. Comparing (C.10) with (15) of [21], we see that the last representation gives the correct answer (up to a constant) only in the case where $R$ is the center of the rectangular billiard.

Instead of calculating the renormalized Green function, the authors of [8, 11] regularized the divergent eigenfunction representation for $G(R, R; k)$, where $R$ was the central position of a rectangular billiard. In terms of this paper, this means that the renormalized Green’s function
was computed as
\[
\xi_B^\beta (R; k) = \lim_{N \to \infty} \left[ \frac{4}{d_x d_y} \left( \sum_{n=1}^{N} \frac{1}{k^2 - E_n} - \frac{1}{(\Delta E)} \int_0^{E_N} \frac{dz}{k^2 - z} \right) \right] + \text{const}
\]
\[
= \lim_{N \to \infty} \left[ \frac{4}{d_x d_y} \left( \sum_{n=1}^{N} \frac{1}{k^2 - E_n} + \frac{dz}{16\pi \ln \left( \frac{k^2 - E_n}{k^2} \right)} \right) \right] + \text{const}
\]
\[
\simeq \lim_{N \to \infty} \left[ \frac{4}{d_x d_y} \sum_{n=1}^{N} \frac{1}{k^2 - E_n} + \frac{1}{4\pi} \ln \left( \frac{E_N}{k^2} \right) \right] + \text{const}
\]
\[
\simeq \lim_{N \to \infty} \left[ \frac{4}{d_x d_y} \sum_{n=1}^{N} \frac{1}{k^2 - E_n} + \frac{1}{4\pi} \ln \left( \frac{16\pi N}{k^2 d_x d_y} \right) \right] + \text{const}
\]
\[
\simeq \sum_{n=1}^{\infty} \left( \frac{4}{d_x d_y} \frac{1}{k^2 - E_n} + \frac{1}{4\pi n} \right) - \frac{1}{4\pi} \ln \left( \frac{k^2 d_x d_y}{16\pi} \right) - \frac{\gamma}{4\pi} + \text{const},
\]
where \( \bar{f} \) denotes a principal value integral. Comparing \( \xi_B^\beta \) with \( \xi_\beta \), we see that there is an essential discrepancy: whereas \( \xi_\beta (R; k) \) exhibits a logarithmic singularity for \( k \to \infty \), this singularity is canceled for \( \xi_B^\beta (R; k) \) by the logarithmic term in (C.12). We thus conclude that the approach [8, 11], despite properly renormalizing the divergency in the spectral sum, still does not treat the logarithmic \( k \)-dependence of the renormalized Green’s function correctly. The situation will be improved if the regularizing term is chosen as
\[
\frac{1}{\langle \Delta E \rangle} \int_{E_{\text{min}}}^{E_N} \frac{dz}{z}.
\]

**Appendix D. Generic billiard with a smooth boundary**

In this section, we consider the renormalized Green’s function representation allowing one to compute a spectrum of an arbitrary Šeba billiard with a smooth boundary. The approach is basically based on previous work [35, 36] and the boundary element method. The advantage of the present approach as compared to representation (C.10) is the explicit dependence on a scattering length.

Let us first write the equations for the free Green’s function \( G_f \) and for the Green’s function of a billiard,
\[
(\Delta + k^2) G_f (r, R; k) = \delta (r - R),
\]
\[
(\Delta + k^2) G (r, R'; k) = \delta (r - R').
\]

Multiplying the first equation by \( G_f (r, R'; k) \), the second one by \( G_f (r, R; k) \), integrating them over the area of the billiard and taking the difference, we obtain
\[
G (R, R'; k) - G_f (R, R'; k)
\]
\[
= \int_\Omega d^3r \nabla [G (r, R'; k) \nabla G_f (r, R; k) - G_f (r, R; k) \nabla G (r, R'; k)]
\]
\[ = \oint_{\partial \Omega} ds [G(r(s), R'; k) \nabla G_i(r(s), R; k) - G_i(r(s), R; k) \nabla G(r(s), R'; k)] \]

\[ = - \oint_{\partial \Omega} ds G_i(R, r(s); k) \frac{\partial G}{\partial n}(r(s), R'; k), \quad (D.2) \]

where \( \partial \Omega \) is the boundary of the billiard, \( s \) is a coordinate along the boundary \( r = r(s) \), \( ds \) is the outward normal vector of the length \( d \) and \( \partial / \partial n \) is the outward normal derivative with respect to the first argument of a function. Now we compute the the outward normal derivative of (D.2) with respect to its first argument at the point \( R = r(s') \). This gives

\[ \frac{\partial G}{\partial n}(r(s'), R'; k) = \frac{\partial G_i}{\partial n}(r(s'), r(s); k) \frac{\partial G}{\partial n}(r(s), R'; k). \quad (D.3) \]

Thus, we see that \( \partial G/\partial n(r(s'), R'; k) \) obeys the integral Fredholm equation with a kernel

\[ \frac{\partial G_i}{\partial n}(r(s'), r(s); k). \quad (D.4) \]

The theory of Fredholm equations is well developed (see e.g. [37]). After (D.3) has been solved numerically, we can compute the Green’s function as well as the renormalized Green’s function from (D.2). Choosing \( G_i(r, R; k) = Y_0(k| r - R|) \), we obtain

\[ \xi_\beta(R; k) = \frac{1}{2\pi} \left[ \ln \left( \frac{k\beta}{2} \right) + \gamma \right] - \oint_{\partial \Omega} ds Y_0(k|r(s) - R|) \frac{\partial G}{\partial n}(r(s), R; k). \quad (D.5) \]

Equality (D.5) gives a convenient representation for the renormalized Green’s function and explicitly contains the scattering length.

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