Finsleroid space with angle and scalar product

G. S. Asanov

Division of Theoretical Physics, Moscow State University
119992 Moscow, Russia
(e-mail: asanov@newmail.ru)
Abstract

A systematic approach has been developed to encompass the Minkowski-type extension of Euclidean geometry such that a one-vector anisotropy is permitted, retaining simultaneously the concept of angle. For the respective geometry, the Euclidean unit ball is to be replaced by the body which is convex and rotund and is found on assuming that its surface (the indicatrix extending the unit sphere) is a space of constant positive curvature. We have called the body the Finsleroid in view of its intrinsic relationship with the metric function of Finsler type. The main point of the present paper is the angle coming from geodesics through the cosine theorem, the underlying idea being to derive the angular measure from the solutions to the geodesic equation which prove to be obtainable in simple explicit forms. The substantive items concern geodesics, angle, scalar product, perpendicularity, and two-vector metric tensor. The Finsleroid-space-associated one-vector Finslerian metric function admits in quite a natural way an attractive two-vector extension.
0. Introduction

The Euclidean geometry is simple and totally spherically symmetric, and corresponds well to our ordinary everyday experience and intuition, while the Finsler or Banach-Minkowski geometries [1-9] are much more extended and sophisticated constructions that may serve to reflect various anisotropic scenarios. When a single vector is distinguished geometrically to be the only isotropic direction in extending the Euclidean geometry, the sphere may not be regarded as an exact carrier of the unit-vector image. So under respective conditions one may expect that some directionally-anisotropic figure should be substituted with the sphere. To this end we shall use the Finsleroid which, being convex and rotund, is not, however, a second-order figure. The constant positive curvature is the fundamental property of the Finsleroid.

The present paper develops and elaborates in much detail the related Finsleroid-geometry (initiated by the author earlier in [10-12]) in the direction of evidencing the concepts of angle and scalar product. No special knowledge of Banach-Minkowski of Finsler geometries is assumed.

It will be recollected that, despite the fact that in geometry one certainly needs to use not only length but also angle and scalar product, various known attempts to introduce the concept of angle in the Minkowski or Finsler spaces were steadily encountered with drawback positions:

"Therefore no particular angular measure can be entirely natural in Minkowski geometry. This is evidenced by the innumerable attempts to define such a measure, none of which found general acceptance“. (Busemann [2], p. 279.)

"Unfortunately, there exists a number of distinct invariants in a Minkowskian space all of which reduce to the same classical euclidean invariant if the Minkowskian space degenerates into a euclidean space. Consequently, distinct definitions of the trigonometric functions and of angles have appeared in the literature concerning Minkowskian and Finsler spaces“. (Rund [3], p. 26)

A short but profound review of the respective attempts can be found in Section 1.7 of the book [4]. The fact that the attempts have never been unambiguous seems to be due to a lack of the proper tools. For the opinion was taken for granted that the angle ought to be defined or constructed in terms of the basic Finslerian metric tensor (and whence ought to be explicated from the initial Finslerian metric function). Let us doubt the opinion from the very beginning. Instead, we would like to raise alternatively the principle that the angle is a concomitant of the geodesics (and not of the metric function proper). The angle is determined by two vectors (instead of one vector in case of the length) and actually implies using a due extension of the Finslerian metric function to a two-vector metric function (to a scalar product). Below, the principle is applying to the Finsleroid space in a systematic way.

The abbreviations FMF and FMT will be used for the Finsleroid metric function and the associated Finslerian metric tensor, respectively. The notation $E_g^{PD}$ will be applied to the Finsleroid space, with the upperscripts “$PD$” meaning “positive-definite”. The characteristic parameter $g$ may take on the values between $-2$ and 2; at $g = 0$ the space is reduced to become an ordinary Euclidean one.
Chapter 1: Synopsis of new conclusions

The angle $\alpha$ obtained in the Finsleroid Geometry under study has the following remarkable property: if the consideration is restricted to the ($N = 2$)-dimensional Finsleroid-Minkowski plane, then $\alpha$ is a factor of the respective Landsberg angle (see Section 1.1). The respective $E_g^{PD}$-Generalized Trigonometric Functions are appeared. Section 1.2 is devoted to reviewing the key and basic concepts determined by the angle. Chapter 1 ends with Section 1.3 in which (on returning the treatment from the auxiliary quasi-Euclidean framework back to the primary Finsleroid space) we are able to display the form of the associated two-vector metric tensor.

1.1. Finsleroid-Minkowski plane. When reducing the consideration to the Minkowski plane, with the dimension $N = 2$ and the orthogonalized form $r_{pq} = \delta_{pq}$ of the input Euclidean metric tensor, the Finsleroid-adapted vector components (cf. the representation (2.83) in Chapter 2) take on the form

\[ R^1 = \frac{K}{hJ} \sin f, \quad R^2 = \frac{K}{J} (\cos f - \frac{1}{2} G \sin f), \]  

(1.1)

from which it follows that

\[ R^2 \frac{\partial R^1}{\partial f} - R^1 \frac{\partial R^2}{\partial f} = \frac{1}{hJ^2} K^2. \]  

(1.2)

Since also $\sqrt{\det(g_{pq})} = J^2$ (cf. Eq. (2.64) in Chapter 2), from the equality (1.2) we conclude

\[ d\alpha_{\text{Finsleroid-Landsberg}} = \frac{1}{h} df \]

(see, e.g., p. 85 of [8] for the definition of the Landsberg angle), where $h$ is the constant (2.13) of Chapter 2.

Therefore, the following theorem is valid.

**Theorem 1.1.** Restricting the Finsleroid geometry to the Minkowski plane, the quantity $f$ in the representation (1.1) is the factor $h$ of the Landsberg angle.

It is also possible to draw

**Theorem 1.2.** The Finsleroid Indicatrix on the Minkowski plane is strongly convex.

**Proof.** Let us verify the relevant criterion formulated on p. 88 of [8]. In terms of our notation, we calculate accordingly:

\[ \frac{\partial^2 R^2}{\partial f^2} \frac{\partial R^1}{\partial f} - \frac{\partial R^2}{\partial f} \frac{\partial^2 R^1}{\partial f^2} - \frac{1}{h^3} = 1 = \frac{1}{h^2}. \]

Since the right-hand side here is always positive, the criterion works fine and, therefore, Theorem 1.2 is valid.

Noting that $ds := \sqrt{g_{pq}(g; R)dR^pdR^q} = \frac{1}{h} df$, we conclude that

\[ ds = \frac{1}{h} df. \]  

(1.3)
In particular, the latter equality entails

**Theorem 1.3.** The length \( L_I := \int ds \) of the Finsleroid Indicatrix is

\[
L_I = \frac{2\pi}{h} \geq 2\pi,
\]

showing the properties

\[
L_I = 2\pi \quad \text{if and only if} \quad g = 0 \quad \text{(the Euclidean case)}
\]

and

\[
L_I \to \infty \quad \text{when} \quad |g| \to 2.
\]

From (1.1) and (1.3) it can readily be explicated that the *Rund equation*

\[
\frac{d^2 R^p}{ds^2} + I \frac{dR^p}{ds} + R^p = 0
\]

holds fine with

\[
I = -g.
\]

If the meaning of the Cartan scalar is acquired to the quantity \( I \) thus appeared in (1.7) (cf. [8]), one may state the following:

**Theorem 1.4.** The Cartan scalar for the Finsleroid-Minkowski plane is the constant which equals the negative of the characteristic Finsleroid parameter \( g \).

Eqs. (1.1) suggest naturally to propose the following \( \mathcal{E}_g^{PD} \)-Generalized Trigonometric Functions:

\[
\cos f := \frac{1}{h} (\cos f - \frac{G}{2} \sin f), \quad \sin f := \frac{1}{hJ} \sin f,
\]

and

\[
\cos^* f := \frac{1}{h^2 J} (\cos f + \frac{G}{2} \sin f).
\]

They reveal the properties

\[
R^1 = K \sin f, \quad R^2 = K \cos f,
\]

and

\[
(\cos f)' = -\frac{1}{h} \sin f, \quad (\sin f)' = \cos^* f,
\]

together with

\[
(\cos^* f)' = -\sin f + G \cos f,
\]

where the prime stands for the derivative with respect to \( f \).

1.2. *Finsleroid angle.* Given two vectors \( R_1 \in V_N \) and \( R_2 \in V_N \). Applying the quasi-Euclidean transformation (see (5.11) in Chapter 2) to Eq. (1.36) of Chapter 3 at any dimension \( N \geq 2 \) results in the following \( \mathcal{E}_g^{PD} \)-scalar product:

\[
<R_1, R_2> = K(g; R_1)K(g; R_2) \cos \left[ \frac{1}{h} \arccos \frac{A(g; R_1)A(g; R_2) + h^2 r_{be} R^b_1 R^e_2}{\sqrt{B(g; R_1)} \sqrt{B(g; R_2)}} \right],
\]

(2.1)
so that the $E^D_{g}$-angle

$$
\alpha(R_1, R_2) = \frac{1}{h} \arccos \frac{A(g; R_1)A(g; R_2) + h^2 r_{bc} R_1^b R_2^c}{\sqrt{B(g; R_1)} \sqrt{B(g; R_2)}}
$$

(2.2)

is appeared between the vectors $R_1$ and $R_2$; the functions $B, K$, as well as the function $A$ can be found in Section 2.2 of Chapter 2.

In the Euclidean limit proper, the angle (2.2) is reduced to read merely

$$
\alpha(R_1, R_2) \bigg|_{g=0} = \arccos \frac{R_1^N R_2^N + r_{bc} R_1^b R_2^c}{\sqrt{(R_1^N)^2 + r_{bc} R_1^b R_1^c} \sqrt{(R_2^N)^2 + r_{be} R_2^e R_2^e}}
$$

(2.2)

For the intermediate angle $\nu$ defined by Eq. (1.44) in Chapter 3, we obtain

$$
\nu = \arctan \frac{s K(g; R_2) \sin \alpha}{K(g; R_1) \Delta s + [K(g; R_2) \cos \alpha - K(g; R_1)] s}. 
$$

(2.3)

The $E^D_{g}$-space general solution to the geodesic equation (presented by Eqs. (2.88)-(2.89) in Chapter 2) reads

$$
R^p(s) = \mu^p(g; t(g; s)), 
$$

(2.4)

where $t(g; s)$ is given by Eq. (1.41) of Chapter 3, and $\mu^p$ are the functions which realize the quasi-Euclidean transformation according to Eqs. (5.14)-(5.15) of Chapter 2; $s$ is the arc-length parameter (see Eq. (2.89) in Chapter 2). The relevant explicit formulas are

$$
R^N(s) = (t^N(s) - \frac{1}{2} G m(s))/k(s), \quad R^a(s) = \frac{1}{h} t^a(s)/k(s)
$$

(2.5)

with

$$
t^N(s) = \frac{K_s}{\sin(h \alpha)} \left[ \frac{A_1}{\sqrt{B_1}} \sin(h(\alpha - \nu)) + \frac{A_2}{\sqrt{B_2}} \sin(h \nu) \right],
$$

(2.6)

where

$$
t^a(s) = h \frac{K_s}{\sin(h \alpha)} \left[ \frac{R_1^a}{\sqrt{B_1}} \sin(h(\alpha - \nu)) + \frac{R_2^a}{\sqrt{B_2}} \sin(h \nu) \right],
$$

(2.7)

and

$$
K_s = \sqrt{K^2(g; R_2) + 2sK(g; R_1)} \sqrt{1 - \left( \frac{K(g; R_2) \sin \alpha}{\Delta s} \right)^2} + s^2,
$$

(2.8)

$$
k(s) = \exp \left( \frac{1}{2} G \arctan \frac{t^N(s)}{m(s)} \right),
$$

(2.9)

$$
m(s) = \sqrt{r_{ab} t^a(s) t^b(s)}.
$$

(2.10)

Along the geodesics, the behaviour law for the squared FMF is quadratic:

$$
K^2(g; R(s)) = a^2 + 2bs + s^2
$$

(2.11)

(cf. Eq. (1.12) in Chapter 3); $a$ and $b$ are integration constants.

Below the picture symbolizes the role which the angles (2.2) and (2.3) are playing in the geodesic line $C$ which joins two points $P_1$ and $P_2.$
On this way the following substantive items can be arrived at.

The $E^P D^g$-Case Cosine Theorem

$$(\Delta s)^2 = (K(g; R_1))^2 + (K(g; R_2))^2 - 2K(g; R_1)K(g; R_2)\cos\alpha.$$ (2.12)

The $E^P D^g$-Case Two-Point Length

$$|R_1 \ominus R_2|^2 = (K(g; R_1))^2 + (K(g; R_2))^2 - 2K(g; R_1)K(g; R_2)\cos\alpha.$$ (2.13)

The $E^P D^g$-Case Scalar Product

$$< R_1, R_2 > = K(g; R_1)K(g; R_2)\cos\alpha.$$ (2.14)

At equal vectors, the reduction

$$< R, R > = K^2(g; R)$$ (2.15)

takes place, that is, the two-vector scalar product (2.1) reduces exactly to the squared FMF.

The $E^P D^g$-Case Perpendicularity

$$< R, R^\perp > = K(g; R)K(g; R^\perp),$$ (2.16)

in which case $\alpha = \pi/2$.

The $E^P D^g$-Case Pythagoras Theorem

$$|R_1 \ominus R_2|^2 = (K(g; R_1))^2 + (K(g; R_2))^2$$ (2.17)

holds fine.

The Finsleroid angle $\alpha$ between two vectors ranges over

$$0 \leq \alpha \leq \alpha_{\text{max}}.$$
where

$$\alpha_{\text{max}} = \frac{1}{h} \pi \geq \pi$$

with equality if and only if $g = 0,$

so that

$$\alpha_{\text{max}} \rightarrow \infty.$$  

The identification

$$|R_2 \ominus R_1|^2 = (\Delta s)^2$$  

yields another lucid representation

$$|R_1 \ominus R_2|^2 = (K(g; R_1))^2 + (K(g; R_2))^2 - 2K(g; R_1)K(g; R_2) \cos \alpha.$$  

We observe the symmetry

$$|R_1 \ominus R_2| = |R_2 \ominus R_1|.$$  

Particularly, from (2.2) it directly ensues that the value of the angle $\alpha$ formed by a vector $R$ with the Finsleroid $R^N$-axis is given by

$$\alpha = \frac{1}{h} \arccos \frac{A(g; R)}{\sqrt{B(g; R)}},$$  

and with $(N - 1)$-dimensional equatorial $\{R\}$-plane of Finsleroid is prescribed as

$$\alpha = \frac{1}{h} \arccos \frac{L(g; R)}{\sqrt{B(g; R)}},$$  

here, $L$ is the function (2.36) of Chapter 2.

1.3. **Two-vector metric tensor.** Let $R \in V_N$ and $S \in V_N$ be two vectors. From (2.1) we obtain

$$\frac{\partial < R, S >}{\partial R^p} = R_p < R, S > + hK(g; S)s_p(g; R, S) \sin \alpha$$

and

$$\frac{\partial < R, S >}{\partial S^q} = S_q < R, S > + hK(g; R)s_p(g; S, R) \sin \alpha.$$  

For the associated $\mathcal{E}_{g}^{PP}$-two-vector metric tensor

$$G_{pq}(g; R, S) := \frac{\partial^2 < R, S >}{\partial S^q \partial R^p}$$

we can find explicitly the representation

$$G_{pq}(g; R, S) = \left( \frac{R_p S_q}{K(g; R)K(g; S)} - h^2 s_p(g; R, S)s_q(g; S, R) \right) \cos \alpha.$$
\[ + \hbar \left[ \left( \frac{R_p}{K(g; R)} s_q(g; S, R) + \frac{S_q}{K(g; S)} s_p(g; R, S) \right) + s_{pq}(g; R, S) \right] \sin \alpha, \]

where

\[ s_p(g; R, S) = \frac{M_p(g; R, S) K(g; R)}{W(g; R, S)^2 B(g; R)} \]

and

\[ s_{pq}(g; R, S) = K(g; S) \frac{\partial s_p(g; R, S)}{\partial S^a} \]

with

\[ W(g; R, S) = \sqrt{B(g; R)B(g; S) - \left[ A(g; R)A(g; S) + h^2 r_{be} R^b S^e \right]^2} \]

and

\[ M_p(g; R, S) = B(g; R) \sqrt{B(g; R)} \sqrt{B(g; S)} \frac{1}{h^2} \frac{\partial}{\partial R^p} A(g; R)A(g; S) + h^2 r_{be} R^b S^e \]

The latter vector has the components

\[ h^2 N_p(g; R, S) = B(g; R)A(g; S) - \left[ A(g; R)A(g; S) + h^2 r_{be} R^b S^e \right] A(g; R) \]

and

\[ h^2 a_p(g; R, S) = B(g; R) \left( \frac{1}{2} g \frac{R^b}{q(R)} A(g; S) + h^2 S^b \right) r_{ab} \]

\[ - \left[ A(g; R)A(g; S) + h^2 r_{be} R^b S^e \right] \left( \frac{1}{2} g R^N + q(R) \right) \frac{R^b}{q(R)} r_{ab}, \]

which can be simplified to get

\[ N_p(g; R, S) = q^2(R)A(g; S) - r_{be} R^b S^e A(g; R) \]

and

\[ a_p(g; R, S) = \left( -R^N R^b A(g; S) + S^b B(g; R) - r_{ec} R^c S^e \left( q(R) + \frac{1}{2} g R^N \right) \frac{R^b}{q(R)} \right) r_{ab}. \]

The identity

\[ M_p(g; R, S) R^p = 0 \]

holds.

The symmetry
and the Finslerian limit
\[
\lim_{S^r \to R^r} \left\{ G_{pq}(g; R, S) \right\} = g_{pq}(g; R)
\]
(cf. Eq. (2.6) in Chapter 3) can straightforwardly be verified; the components \( g_{pq}(g; R) \) are presented in Chapter 2 by the list (2.60)-(2.61). The two-vector \( \mathcal{E}_g^{PD} \)-metric tensor can also be obtained as the transform
\[
G_{pq}(g; R, S) = \sigma^r_p(g; R)\sigma^s_q(g; S)n_{rs}(g; t_1, t_2)
\]
(cf. Eq. (5.47) in Chapter 2) of the two-vector quasi-Euclidean tensor (see Eq. (2.2) in Chapter 3), where
\[
t_1^r = \sigma^r(g; R), \quad t_2^s = \sigma^s(g; S)
\]
(cf. Eqs. (5.10) and (5.35) in Chapter 2).

Chapter 2. Centerpiece properties

2.1. Motivation. Below Section 2.2 gives an account of the notation and conventions for the space \( \mathcal{E}_g^{PD} \) and introduces the initial concepts and definitions that are required. The space is constructed by assuming an axial symmetry and, therefore, incorporates a single preferred direction, which we shall often refer as the \( Z \)-axis. After preliminary introducing a characteristic quadratic form \( B \), which is distinct from the Euclidean sum of squares by entrance of a mixed term (see Eq. (2.22)), we define the FMF \( K \) for the space \( \mathcal{E}_g^{PD} \) by the help of the formulae (2.30)-(2.33). A characteristic feature of the formulae is the occurrence of the function “\( \arctan \)”. Next, we calculate basic tensor quantities of the space. There appears a remarkable phenomenon, which essentially simplifies all the constructions, that the associated Cartan tensor occurs being of a simple algebraic structure (see Eqs. (2.66)-(2.67)). In particular, the phenomenon gives rise to a simple structure of the associated curvature tensor (Eq. (2.69)). As well as in the Euclidean geometry the locus of the unit vectors issuing from fixed point of origin is the unit sphere, in the \( \mathcal{E}_g^{PD} \)-geometry under development the locus is the boundary (surface) of the Finsleroid. We call the boundary the Finsleroid Indicatrix. It can rigorously be proved that the Finsleroid Indicatrix is a closed, regular, and strongly convex (hyper)surface. The value of the curvature depends on the parameter \( g \) according to the simple law (2.73). The determinant of the associated FMT is strongly positive in accordance with Eqs. (2.64)-(2.65).

In Section 2.3 we show how the consideration can conveniently be converted into the co-approach. We derive the associated Finsleroid Hamiltonian metric function. The explicit form thereof is entirely similar to the form of the FMF \( K \) up to the substitution of \(-g\) with \( g\). The symmetry between the relevant input representations (2.30)-(2.38) and (3.20)-(3.28) is astonishingly perfect.

Section 2.4 gathers together the lucid facts concerning details of the form of the Finsleroids and co-Finsleroids. The Finsleroid is a generalization of the unit ball and may be visualized as comprising a stretched surface of revolution.

Its form essentially depends on the value of the characteristic parameter, \( g \). Under changing the sign of \( g \), the Finsleroid turns up with respect to its equatorial section. When \(|g| \to 2\), the Finsleroid is extending ultimately tending in its form more and more to the cone. The form and all the properties of the co-Finsleroid are essentially similar to
that of the Finsleroid of the opposite sign of the parameter $g$. Various Maple9-designed figures have been presented to elucidate patterns and details, and to make this Finsleroid-framework a plausible one in methodological as well educational respects, – which also show all the basic features of the Finsleroids.

The $\mathcal{E}^{PD}_g$-space has an auxiliary quasi-Euclidean structure, which is deeply inherent in the development. Section 2.5 introduces for the $\mathcal{E}^{PD}_g$-space the quasi-Euclidean map under which the Finsleroid goes into the unit ball. The quasi-Euclidean space is simple in many aspects, so that relevant transformations make reduce various calculations and may provide one with constructive ideas.

Motivated by these observations, in Section 2.6 we offer the nearest interesting properties of the quasi-Euclidean metric tensor (which is not of a Finslerian type). The tensor is a linear combination of the Euclidean metric tensor and the product of two unit vectors. Therefore, the basic relevant geometric objects, including the Christoffel symbols, the curvature tensor, the orthonormal frames, and the Ricci rotation coefficients, are calculated in simple forms. The quasi-Euclidean space is not flat, but proves to be conformally-flat.

2.2. Bases. Suppose we are given an $N$-dimensional centered vector space $V_N$ with some point “$O$” being the origin. Denote by $R$ the vectors constituting the space, so that $R \in V_N$ and it is assumed that $R$ is issued from the point “$O$”. Any given vector $R$ assigns a particular direction in $V_N$. Let us fix a member $R_{(N)} \in V_N$, introduce the straight line $e_N$ oriented along the vector $R_{(N)}$, and use this $e_N$ to serve as a $R^N$-coordinate axis in $V_N$. In this way we get the topological product

$$V_N = V_{N-1} \times e_N$$  \hspace{1cm} (2.1)

together with the separation

$$R = \{R, R^N\}, \quad R^N \in e_N \quad \text{and} \quad R \in V_{N-1}. \hspace{1cm} (2.2)$$

For convenience, we shall frequently use the notation

$$R^N = Z$$  \hspace{1cm} (2.3)

and

$$R = \{R, Z\}. \hspace{1cm} (2.4)$$

Also, we introduce a Euclidean metric

$$q = q(R)$$  \hspace{1cm} (2.5)

over the $(N - 1)$-dimensional vector space $V_{N-1}$.

With respect to an admissible coordinate basis $\{e_a\}$ in $V_{N-1}$, we obtain the coordinate representations

$$R = \{R^a\} = \{R^1, \ldots, R^{N-1}\}$$  \hspace{1cm} (2.6)

and

$$R = \{R^p\} = \{R^a, R^N\} \equiv \{R^a, Z\},$$  \hspace{1cm} (2.7)

together with

$$q(R) = \sqrt{r_{ab}R^aR^b}; \hspace{1cm} (2.8)$$

the matrix $(r_{ab})$ is assumed to be symmetric and positive-definite. The indices $(a, b, \ldots)$ and $(p, q, \ldots)$ will be specified over the ranges $(1, \ldots, N - 1)$ and $(1, \ldots, N),$
respectively; vector indices are up, co-vector indices are down; repeated up-down indices are automatically summed; \( \delta \) will stand for the Kronecker symbol, such that \( (\delta_{ab}) = \text{diag}(1, 1, \ldots) \). The variables

\[
w^a = R^a / Z, \quad w_a = r_{ab} w^b, \quad w = q / Z, \tag{2.9}
\]

where

\[
w \in (-\infty, \infty), \tag{2.10}
\]

are convenient whenever \( Z \neq 0 \). Sometimes we shall mention the associated metric tensor

\[
r_{pq} = \{ r_{NN} = 1, \ r_{Na} = 0, \ r_{ab} \} \tag{2.11}
\]

meaningful over the whole vector space \( V_N \).

Given a parameter \( g \) subject to ranging

\[-2 < g < 2, \tag{2.12}\]

we introduce the convenient notation

\[
h = \sqrt{1 - \frac{1}{4} g^2}, \tag{2.13}
\]

\[
G = g / h, \tag{2.14}
\]

\[
g_+ = \frac{1}{2} g + h, \quad g_- = \frac{1}{2} g - h, \tag{2.15}
\]

\[
g_+ = -\frac{1}{2} g + h, \quad g_- = -\frac{1}{2} g - h, \tag{2.16}
\]

so that

\[
g_+ + g_- = g, \quad g_+ - g_- = 2h, \tag{2.17}
\]

\[
g_+ + g_- = -g, \quad g^+ - g^- = 2h, \tag{2.18}
\]

\[
(g_+)^2 + (g_-)^2 = 2, \tag{2.19}
\]

\[
(g^+)^2 + (g^-)^2 = 2. \tag{2.20}
\]

The symmetry

\[
g_+ \leftrightarrow -g_-, \quad g^+ \leftrightarrow -g^- \tag{2.21}
\]

holds.

The characteristic quadratic form

\[
B(g; R) := Z^2 + g q Z + q^2 = \frac{1}{2} \left[ (Z + g_+ q)^2 + (Z + g_- q)^2 \right] > 0 \tag{2.22}
\]
is of the negative discriminant, namely
\[ D_{\{B\}} = -4h^2 < 0, \] (2.23)
because of Eqs. (2.12) and (2.13). Whenever \( Z \neq 0 \), it is also convenient to use the quadratic form
\[ Q(g; w) := B/(Z)^2, \] (2.24)
obtaining
\[ Q(g; w) = 1 + gw + w^2 > 0, \] (2.25)
together with the function
\[ E(g; w) := 1 + \frac{1}{2}gw. \] (2.26)
The identity
\[ E^2 + h^2w^2 = Q \] (2.27)
can readily be verified. In the limit \( g \to 0 \), the definition (2.22) degenerates to the quadratic form of the input metric tensor (2.11):
\[ B|_{g=0} = r_{pq}R^p R^q. \] (2.28)
Also
\[ Q|_{g=0} = 1 + w^2. \] (2.29)
In terms of this notation, we propose the FMF
\[ K(g; R) = \sqrt{B(g; R)} J(g; R), \] (2.30)
where
\[ J(g; R) = e^{\frac{1}{2} \Phi(g; R)} \] (2.31)
\[ \Phi(g; R) = \frac{\pi}{2} + \arctan \frac{G}{2} - \arctan \left( \frac{q}{hZ} + \frac{G}{2} \right), \quad \text{if } Z \geq 0, \] (2.32)
\[ \Phi(g; R) = -\frac{\pi}{2} + \arctan \frac{G}{2} - \arctan \left( \frac{q}{hZ} + \frac{G}{2} \right), \quad \text{if } Z \leq 0, \] (2.33)
or in other convenient forms,
\[ \Phi(g; R) = \frac{\pi}{2} + \arctan \frac{G}{2} - \arctan \left( \frac{L(g; R)}{hZ} \right), \quad \text{if } Z \geq 0, \] (2.34)
\[ \Phi(g; R) = -\frac{\pi}{2} + \arctan \frac{G}{2} - \arctan \left( \frac{L(g; R)}{hZ} \right), \quad \text{if } Z \leq 0, \] (2.35)
where
\[ L(g; R) = q + \frac{q}{2}Z, \] (2.36)
and also
\[ \Phi(g; R) = \arctan \frac{A(g; R)}{hq}, \] (2.37)
where
\[ A(g; R) = Z + \frac{1}{2} g q. \]  
(2.38)

This FMF has been normalized to show the properties
\[ -\frac{\pi}{2} \leq \Phi \leq \frac{\pi}{2} \]  
(2.39)

and
\[ \Phi = \frac{\pi}{2}, \text{ if } q = 0 \text{ and } Z > 0; \quad \Phi = -\frac{\pi}{2}, \text{ if } q = 0 \text{ and } Z < 0. \]  
(2.40)

We also have
\[ \tan \Phi = \frac{A}{h q} \]  
(2.41)

and
\[ \Phi|_{Z=0} = \arctan \frac{G}{2}. \]  
(2.42)

It is often convenient to use the sign indicator \( \epsilon_Z \) for the argument \( Z \):
\[ \epsilon_Z = 1, \text{ if } Z > 0; \quad \epsilon_Z = -1, \text{ if } Z < 0; \quad \epsilon_Z = 0, \text{ if } Z = 0. \]  
(2.43)

Under these conditions, we call the considered space the \( E^{PD}_{g} \)-space:
\[ E^{PD}_{g} := \{ V_N = V_{N-1} \times \epsilon_N; R \in V_N; K(g; R); g \}. \]  
(2.44)

The right-hand part of the definition (2.30) can be considered to be a function \( \tilde{K} \) of the arguments \( \{g; q, Z\} \), such that
\[ \tilde{K}(g; q, Z) = K(g; R). \]  
(2.45)

We observe that
\[ \tilde{K}(g; q, -Z) \neq \tilde{K}(g; q, Z), \quad \text{unless } g = 0. \]  
(2.46)

Instead, the function \( \tilde{K} \) shows the property of \( gZ \)-parity
\[ \tilde{K}(-g; q, -Z) = \tilde{K}(g; q, Z). \]  
(2.47)

The \( (N - 1) \)-space reflection invariance holds true
\[ K(g; R) \xrightarrow{R_e} K(g; R) \]  
(2.48)

(such an operation does not influence the quantity \( q \)).

It is frequently convenient to rewrite the representation (2.30) in the form
\[ K(g; R) = |Z| V(g; w), \]  
(2.49)
whenever $Z \neq 0$, with the generating metric function

$$V(g; w) = \sqrt{Q(g; w)} j(g; w). \tag{2.50}$$

We have

$$j(g; w) = J(g; 1, w).$$

Using (2.25) and (2.31)-(2.35), we obtain

$$V' = wV/Q, \quad V'' = V/Q^2, \tag{2.51}$$

$$ (V/Q)' = -gV^2/Q^2, \quad (V^2/Q^2)' = -2(g + w)V^2/Q^3, \tag{2.52}$$

$$ j' = -\frac{1}{2}gj/Q, \tag{2.53}$$

and also

$$ \frac{1}{2}(V^2)' = wV^2/Q, \quad \frac{1}{2}(V^2)'' = (Q - gw)V^2/Q^2, \tag{2.54}$$

$$ \frac{1}{4}(V^2)''' = -gV^2/Q^3, \tag{2.55}$$

together with

$$ \Phi' = -h/Q, \tag{2.56}$$

where the prime ('') denotes the differentiation with respect to $w$.

Also,

$$ (A(g; R))^2 + h^2q^2 = B(g; R) \tag{2.57}$$

and

$$ (L(g; R))^2 + h^2Z^2 = B(g; R). \tag{2.58}$$

The simple results for these derivatives reduce the task of computing the components of the associated FMT to an easy exercise, indeed:

$$ R_p := \frac{1}{2} \frac{\partial K^2(g; R)}{\partial R^p}; $$

$$ R_a = r_{ab}R^b_0K^2/B, \quad R_N = (Z + gq)K^2/B; \tag{2.59} $$

$$ g_{pq}(g; R) := \frac{1}{2} \frac{\partial^2 K^2(g; R)}{\partial R^p \partial R^q} = \frac{\partial R_p(g; R)}{\partial R^q}; $$

$$ g_{NN}(g; R) = [(Z + gq)^2 + q^2]K^2/B^2, \quad g_{Na}(g; R) = gqr_{ab}R^b_0K^2/B^2. \tag{2.60} $$
\[ g_{ab}(g; R) = \frac{K^2}{B} r_{ab} - g \frac{r_{ad}R^d r_{be}R^e Z K^2}{q} B^2. \]  

(2.61)

The reciprocal tensor components are

\[ g^{NN}(g; R) = (Z^2 + q^2) \frac{1}{K^2}, \quad g^{Na}(g; R) = -gqR^a \frac{1}{K^2}, \]

(2.62)

\[ g^{ab}(g; R) = \frac{B}{K^2} r^{ab} + g(Z + gq) \frac{R^e R^b}{q} \frac{1}{K^2}. \]

(2.63)

The determinant of the FMT given by Eqs. (2.60)-(2.61) can readily be found in the form

\[ \det(g_{pq}(g; R)) = [J(g; R)]^{2N} \det(r_{ab}) \]

(2.64)

which shows, on noting (2.31)-(2.33), that

\[ \det(g_{pq}) > 0 \quad \text{over all the space} \quad V_N. \]

(2.65)

The associated angular metric tensor

\[ h_{pq} := g_{pq} - R_p R_q \frac{1}{K^2} \]

proves to be given by the components

\[ h_{NN}(g; R) = q^2 \frac{K^2}{B^2}, \quad h_{Na}(g; R) = -Z r_{ab} R^b \frac{K^2}{B^2}, \]

\[ h_{ab}(g; R) = \frac{K^2}{B} r_{ab} - (Z + q) \frac{r_{ad}R^d r_{be}R^e K^2}{q} B^2, \]

which entails

\[ \det(h_{ab}) = \det(g_{pq}) \frac{1}{V^2}. \]

The use of the components of the Cartan tensor (given explicitly in the end of the present section) leads, after rather tedious straightforward calculations, to the following simple and remarkable result.

**Theorem 2.1.** The Cartan tensor associated with the FMF (2.30) is of the following special algebraic form:

\[ C_{pqr} = \frac{1}{N} \left( h_{pq} C_r + h_{pr} C_q + h_{qr} C_p - \frac{1}{C_s C t} C_p C_q C_r \right) \]

(2.66)

with

\[ C_t C^t = \frac{N^2}{4K^2} g^2. \]

(2.67)
Elucidating the structure of the respective curvature tensor
\[ S_{pqrs} := (C_{tqr}C_{p}^{t} - C_{tqs}C_{p}^{t}) \] (2.68)

results in the simple representation
\[ S_{pqrs} = -\frac{C_{r}C^{t}}{N^{2}}(h_{pr}h_{qs} - h_{ps}h_{qr}). \] (2.69)

Inserting here (2.67), we are led to

**Theorem 2.2.** The curvature tensor of the space \( \mathcal{E}^{PD}_{g} \) is of the special type
\[ S_{pqrs} = S^{*}(h_{pr}h_{qs} - h_{ps}h_{qr})/K^{2} \] (2.70)

with
\[ S^{*} = -\frac{1}{4}g^{2}. \] (2.71)

**Definition.** FMF (2.30) generates the Finsleroid
\[ \mathcal{F}^{PD}_{g} := \{ R \in V_{N} : K(g; R) \leq 1 \}. \] (2.72)

**Definition.** The Finsleroid Indicatrix \( \mathcal{I}^{PD}_{g} \) is the boundary of the Finsleroid:
\[ \mathcal{I}^{PD}_{g} := \{ R \in V_{N} : K(g; R) = 1 \}. \] (2.73)

**Note.** Since at \( g = 0 \) the space \( \mathcal{E}^{PD}_{g} \) is Euclidean, then the body \( \mathcal{F}^{PD}_{g=0} \) is a unit ball and \( \mathcal{I}^{PD}_{g=0} \) is a unit sphere.

Recalling the known formula \( R = 1 + S^{*} \) for the indicatrix curvature (see Section 1.2 in [4]), from (2.71) we conclude that
\[ R_{\text{Finsleroid Indicatrix}} = h^{2} = 1 - \frac{1}{4}g^{2}, \] (2.74)

so that
\[ 0 < R_{\text{Finsleroid Indicatrix}} \leq 1 \]

and
\[ R_{\text{Finsleroid Indicatrix}} \xrightarrow{g \to 0} R_{\text{Euclidean Sphere}} = 1. \]

Geometrically, the fact that the quantity (2.74) is independent of vectors \( R \) means that the indicatrix curvature is constant. Therefore, we have arrived at

**Theorem 2.3.** The Finsleroid Indicatrix \( \mathcal{I}^{PD}_{g} \) is a space of constant positive curvature.

Also, on comparing between the result (2.74) and Eqs. (2.22)-(2.23), we obtain

**Theorem 2.4.** The Finsleroid curvature relates to the discriminant of the input characteristic quadratic form (2.22) simply as
\[ R_{\text{Finsleroid Indicatrix}} = -\frac{1}{4}D_{(B)}. \] (2.75)
Points of the indicatrix can be represented by means of the unit vectors $l = \{l^p\}$:

$$l^p = \frac{R^p}{K(g; R)},$$

so that

$$K(g; l) \equiv 1.$$ 

(2.76)

(2.77)

The vectors can conveniently be parameterized as follows:

$$l^a = n^a \sin f \exp \left( \frac{1}{2} G(f - \frac{\pi}{2}) \right), \quad l^N = (\cos f - \frac{1}{2} G \sin f) \exp \left( \frac{1}{2} G(f - \frac{\pi}{2}) \right),$$

(2.78)

where

$$f \in [0, \pi]$$

(2.79)

and $n^a$ are the components that are taken to fulfill

$$r_{ab} n^a n^b = 1;$$

(2.80)

also,

$$J(g; l) = \exp \left( -\frac{1}{2} G(f - \frac{\pi}{2}) \right)$$

(2.81)

(cf. (2.31)). The reader is advised to verify that

$$A(g; l) = \frac{1}{J(g; l)} \cos f$$

and

$$\frac{hq}{A(g; l)} = \tan f.$$  

Therefore, it is appropriate to take

$$f = \arctan \frac{hq}{A},$$

(2.82)

in which case from (2.37) it follows that

$$\Phi(g; l) = \frac{\pi}{2} - f.$$  

At the same time, for the function (2.22) we find

$$B(g; l) = \left( \frac{1}{J(g; l)} \right)^2 = \exp \left( G(f - \frac{\pi}{2}) \right).$$

This method can farther be extended for the whole space by taking the parameterizations

$$R^a = \frac{K}{hJ} n^a \sin f, \quad R^N = \frac{K}{J} (\cos f - \frac{1}{2} G \sin f),$$

(2.83)

which entails

$$\frac{\partial R^p}{\partial K} = \frac{1}{K} R^p,$$

(2.84)
\[
\frac{\partial R^a}{\partial f} = \frac{K}{\hbar J} n^a (\cos f + \frac{1}{2} G \sin f), \quad \frac{\partial R^N}{\partial f} = -\frac{K}{\hbar^2 J} \sin f, \quad (2.85)
\]

\[
\frac{\partial^2 R^a}{\partial f^2} = \frac{K}{\hbar J} n^a \left( G \cos f - (1 - \frac{1}{4} G^2) \sin f \right), \quad (2.86)
\]
and
\[
\frac{\partial^2 R^N}{\partial f^2} = -\frac{K}{\hbar^2 J} (\cos f + \frac{1}{2} G \sin f). \quad (2.87)
\]

Last, we write down the explicit components of the relevant Finsleroid Cartan tensor

\[
C_{pqr} := \frac{1}{2} \frac{\partial g_{pq}}{\partial R^r}:
\]

\[
R^N C_{NNN} = gw^3 V^2 Q^{-3}, \quad R^N C_{aNN} = -gw^a V^2 Q^{-3},
\]

\[
R^N C_{abN} = \frac{1}{2} gwV^2 Q^{-2} r_{ab} + \frac{1}{2} g (1 - gw - w^2) w_a w_b w^{-1} V^2 Q^{-3},
\]

\[
R^N C_{abc} = -\frac{1}{2} g V^2 Q^{-2} w^{-1} (r_{ab} w_c + r_{ac} w_b + r_{bc} w_a) + gw_a w_b w_c w^{-3} \left( \frac{1}{2} Q + gw + w^2 \right) V^2 Q^{-3};
\]

and

\[
R^N C_{N}^{\ N} = gw^3 / Q^2, \quad R^N C_{a}^{\ N} = -gw^a / Q^2,
\]

\[
R^N C_{N}^{\ aN} = -gw (1 + gw) w^a / Q^2,
\]

\[
R^N C_{a}^{\ bN} = \frac{1}{2} gw r_{ab} / Q + \frac{1}{2} g (1 - gw - w^2) w_a w_b / w Q^2,
\]

\[
R^N C_{a}^{\ b} = \frac{1}{2} gw w_a / Q + \frac{1}{2} g (1 + gw - w^2) g^a w_b / w Q^2,
\]

\[
R^N C_{a}^{\ bc} = -\frac{1}{2} g \left( \delta_b^c w_c + \delta_c^b w_a + (1 + gw) r_{ac} w^b \right) / w Q + \frac{1}{2} g (gw Q + Q + 2 w^2) w_a w_b w_c / w^3 Q^2.
\]

The components have been calculated by the help of the formulae (2.51)-(2.54).

The use of the contractions

\[
R^N C_{a}^{\ b} r^{ac} = -\frac{gw^b}{w} \frac{1 + gw}{Q} \left( \frac{N - 2}{2} + \frac{1}{Q} \right)
\]

and
\[ R^NC_{\alpha}^b w^a w^c = -g^{w} Q^2 (1 + gw) w^b \]

is convenient in many calculations.

Also

\[ R^NC_N = \frac{N}{2} gwQ^{-1}, \quad R^NC_a = -\frac{N}{2} g(w_a/w)Q^{-1}, \]
\[ R^NC^N = \frac{N}{2} gw/V^2, \quad R^NC^a = -\frac{N}{2} gw^a(1 + gw)/wV^2, \]
\[ C^N = \frac{N}{2} gwR^NK^{-2}, \quad C^a = -\frac{N}{2} gw^a(1 + gw)w^{-1}R^NK^{-2}, \]
\[ C_p C^p = \frac{N^2}{4K^2} g^2. \]

The respective \( \mathcal{E}^{PD}_g \)-geodesic equation reads

\[ \frac{d^2 R^p}{ds^2} + C_q^p r(g; R) \frac{dR^q}{ds} \frac{dR^r}{ds} = 0, \tag{2.88} \]

where \( s \) is the arc-length parameter defined by

\[ ds = \sqrt{g_{pq}(g; R)dR^pdR^q}. \tag{2.89} \]

2.3. Associated Finsleroid Hamiltonian function. Considering the co-vector space \( \hat{V}_N \) dual to the vector space \( V_N \) used in the preceding Section 2.2, and denoting by \( \hat{R} \) the respective co-vectors, so that \( \hat{R} \in \hat{V}_N \), we may introduce the co-counterparts of the formulas (2.1)-(2.11), obtaining the topological product

\[ \hat{V}_N = \hat{V}_{N-1} \times \hat{e}_N \tag{3.1} \]

and the separation

\[ \hat{R} = \{ \hat{R}, R_N \}, \quad R_N \in \hat{e}_N \quad \text{and} \quad \hat{R} \in \hat{V}_{N-1}. \tag{3.2} \]

Then we put

\[ R_N = \hat{Z}, \tag{3.3} \]
\[ \hat{R} = \{ \hat{R}, \hat{Z} \}, \tag{3.4} \]
and introduce a metric

\[ \hat{q} = \hat{q}(\hat{R}) \tag{3.5} \]

over the \((N - 1)\)-dimensional co-vector space \( \hat{V}_{N-1} \).
With respect to a coordinate basis \( \{ \hat{e}_a \} \) dual to \( \{ e_a \} \), we obtain in \( \hat{V}_{N-1} \) the coordinate representations

\[
\hat{R} = \{ R_a \} = \{ R_1, \ldots, R_{N-1} \}
\]

(3.6)

and

\[
\hat{R} = \{ R_p \} = \{ R_a, R_N \} \equiv \{ R_a, \hat{Z} \}
\]

(3.7)

together with

\[
\hat{q}(\hat{R}) = \sqrt{r^{ab} R_a R_b},
\]

(3.8)

where \( r^{ab} \) are the contravariant components of a symmetric positive-definite tensor defined over \( \hat{V}_{N-1} \); the tensor is determined by the reciprocity \( r_{ab} r^{bc} = \delta_a^c \). The variables

\[
p_a = R_a / \hat{Z}, \quad p^a = r^{ab} p_b, \quad p = \hat{q} / \hat{Z},
\]

(3.9)

where

\[
p \in (-\infty, \infty),
\]

(3.10)

are convenient to apply whenever \( \hat{Z} \neq 0 \). The co-version

\[
r^{pq} = \{ r^{NN} = 1, \quad r^{Na} = 0, \quad r^{ab} \}
\]

(3.11)

of the input metric tensor (2.11) is meaningful over the space \( \hat{V}_N \). The parameter \( g \) introduced in Eqs. (2.12)-(2.13), as well as the explicated formulae (2.14)-(2.21), are applicable in the co-approach, too.

The characteristic quadratic co-form

\[
\hat{B}(g; R) := \hat{Z}^2 - g \hat{q} \hat{Z} + q^2 = \frac{1}{2} \left[ (\hat{Z} + g^+ \hat{q})^2 + (\hat{Z} + g^- \hat{q})^2 \right] > 0
\]

(3.12)

is of the negative discriminant:

\[
D_{\hat{B}} = -4h^2 < 0,
\]

(3.13)

(cf. (2.22) and (2.23)). Whenever \( \hat{Z} \neq 0 \), we can use the quadratic form

\[
\hat{Q}(g; p) := \hat{B} / (\hat{Z})^2,
\]

(3.14)

obtaining

\[
\hat{Q}(g; p) = 1 - gp + p^2 > 0,
\]

(3.15)

and the function

\[
\hat{E}(g; p) := 1 - \frac{1}{2} gp.
\]

(3.16)

Similarly to (2.26)-(2.29), we get

\[
\hat{E}^2 + h^2 p^2 = Q,
\]

(3.17)

\[
\hat{B}|_{g=0} = r^{pq} R_p R_q,
\]

(3.18)

and

\[
\hat{Q}|_{g=0} = 1 + p^2.
\]

(3.19)
This enables us to introduce the Finsleroid Hamiltonian function (FHF for short)

\[ H(g; \hat{R}) = \sqrt{\hat{B}(g; \hat{R}) \hat{J}(g; \hat{R})}, \]  

(3.20)

where

\[ \hat{J}(g; \hat{R}) = e^{-\frac{1}{2}G\Phi(g; \hat{R})} \]  

(3.21)

and

\[ \hat{\Phi}(g; \hat{R}) = \frac{\pi}{2} - \arctan \frac{G}{2} - \arctan \left( \frac{\hat{q}}{h\hat{Z}} - \frac{G}{2} \right), \quad \text{if } \hat{Z} \geq 0, \]  

(3.22)

\[ \hat{\Phi}(g; \hat{R}) = -\frac{\pi}{2} - \arctan \frac{G}{2} - \arctan \left( \frac{\hat{q}}{h\hat{Z}} - \frac{G}{2} \right), \quad \text{if } \hat{Z} \leq 0, \]  

(3.23)

or in other forms,

\[ \hat{\Phi}(g; \hat{R}) = \frac{\pi}{2} - \arctan \frac{G}{2} - \arctan \left( \frac{\hat{L}(g; \hat{R})}{h\hat{Z}} \right), \quad \text{if } \hat{Z} \geq 0, \]  

(3.24)

\[ \hat{\Phi}(g; \hat{R}) = -\frac{\pi}{2} - \arctan \frac{G}{2} - \arctan \left( \frac{\hat{L}(g; \hat{R})}{h\hat{Z}} \right), \quad \text{if } \hat{Z} \leq 0, \]  

(3.25)

where

\[ \hat{L}(g; \hat{R}) = \hat{q} - \frac{g}{2}\hat{Z}; \]  

(3.26)

and

\[ \hat{\Phi}(g; \hat{R}) = \arctan \frac{\hat{A}(g; \hat{R})}{h\hat{q}} \]  

(3.27)

with

\[ \hat{A}(g; \hat{R}) = \hat{Z} - \frac{g}{2}\hat{q}. \]  

(3.28)

The respective range is such that

\[ -\frac{\pi}{2} \leq \hat{\Phi} \leq \frac{\pi}{2} \]  

(3.29)

and

\[ \hat{\Phi} = \frac{\pi}{2}, \quad \text{if } \hat{q} = 0 \text{ and } \hat{Z} > 0; \quad \Phi = -\frac{\pi}{2}, \quad \text{if } \hat{q} = 0 \text{ and } \hat{Z} < 0. \]  

(3.30)

Also,

\[ \hat{\Phi}\big|_{\hat{Z}=0} = -\arctan \frac{G}{2}. \]  

(3.31)

Under these conditions, we arrive at the \( \hat{\mathcal{E}}_{g}^{PD} \)-space:

\[ \hat{\mathcal{E}}_{g}^{PD} := \{ \hat{V}_N = \hat{V}_{N-1} \times \hat{e}_N; \hat{R} \in \hat{V}_N; H(g; \hat{R}); g \}. \]  

(3.32)

The function (3.21) is intimately connected with the function (2.31), namely

\[ \hat{J}(g; \hat{R}) = \frac{J(g; R(g; \hat{R}))}{J(g; R(g; \hat{R}))}. \]  

(3.33)
Treating the right-hand part of the definition (3.20) as a function $\tilde{H}$ of the arguments \{\(g; \hat{q}, \hat{Z}\)}, such that
\[
\tilde{H}(g; \hat{q}, \hat{Z}) = H(g; \hat{R}),
\] (3.34)
the co-counterparts of Eqs. (2.46)-(2.48) hold true:
\[
\tilde{H}(g; \hat{q}, -\hat{Z}) \neq \tilde{H}(g; \hat{q}, \hat{Z}), \quad \text{unless } g = 0;
\] (3.35)
the \(g\hat{Z}\)-parity:
\[
\tilde{H}(-g; \hat{q}, -\hat{Z}) = \tilde{H}(g; \hat{q}, \hat{Z});
\] (3.36)
and
\[
H(g; R) \overset{R_a \leftrightarrow -R_a}{\longrightarrow} H(g; R).
\] (3.37)

We may also represent (3.20) in the form
\[
H(g; \hat{R}) = |\hat{Z}| W(g; p)
\] (3.38)
with the generating co-function
\[
W(g; p) = \sqrt{\hat{Q}(g; p) \hat{j}(g; p)}.
\] (3.39)

We have
\[
\hat{j}(g; p) = \hat{J}(g; 1, p).
\]
Differentiating yields
\[
W' = pW/\hat{Q}, \quad W'' = W/\hat{Q}^2,
\] (3.40)
\[
(W^2/\hat{Q})' = gW^2/\hat{Q}^2, \quad (W^2/\hat{Q}^2)' = 2(g - p)W^2/\hat{Q}^3,
\] (3.41)
\[
\hat{j}' = \frac{1}{2}g\hat{j}/\hat{Q},
\] (3.42)
and
\[
\frac{1}{2}(W^2)' = pW^2/\hat{Q}, \quad \frac{1}{2}(W^2)'' = (\hat{Q} + gp)W^2/\hat{Q}^2,
\] (3.43)
together with
\[
\frac{1}{4}(W^2)''' = gW^2/\hat{Q}^3
\] (3.44)
and
\[
\tilde{\Phi}' = -h/\hat{Q},
\] (3.45)
where the prime (\'\) denotes the differentiation with respect to \(p\).

Also,
\[
(\hat{A}(g; \hat{R}))^2 + h^2 \hat{q}^2 = \hat{B}(g; \hat{R})
\] (3.46)
and
\[
(\hat{L}(g; \hat{R}))^2 + h^2 \hat{Z}^2 = \hat{B}(g; \hat{R}).
\] (3.47)

Similarly to (2.59)-(2.63), the subsequent simple calculations yield the relations
\[
R^p = \frac{1}{2} \frac{\partial H^2(g; \hat{R})}{\partial R_p}.
\]
\[ R^N = (\hat{Z} - g\hat{q})\frac{H^2}{B}, \quad R^a = r^{ab}R_b\frac{H^2}{B}, \quad (3.48) \]

and also

\[ g^{pq}(g; \hat{R}) = \frac{1}{2} \frac{\partial^2 H^2(g; \hat{R})}{\partial R_p \partial R_q} = \frac{\partial R^p(g; \hat{R})}{\partial R_q}; \]

\[ g^{NN}(g; \hat{R}) = [(\hat{Z} - g\hat{q})^2 + \hat{q}^2]\frac{H^2}{B^2}, \quad g^{Na}(g; \hat{R}) = -g\hat{q}r^{ab}R_b\frac{H^2}{B^2}, \quad (3.49) \]

\[ g^{ab}(g; \hat{R}) = \frac{H^2}{B} r^{ab} + g\frac{r^{ad}r^{be}R_e\hat{Z}H^2}{\hat{q}}\frac{H^2}{B^2}. \quad (3.50) \]

The reciprocal tensor components are

\[ g_{NN}(g; \hat{R}) = (\hat{Z}^2 + \hat{q}^2)\frac{1}{H^2}, \quad g_{Na}(g; \hat{R}) = g\hat{q}R_a\frac{1}{H^2}; \quad (3.51) \]

\[ g_{ab}(g; \hat{R}) = \frac{\hat{B}}{H^2} r_{ab} - g(\hat{Z} - g\hat{q})R_aR_b\frac{1}{\hat{q}}\frac{1}{H^2}. \quad (3.52) \]

To arrive from FMF (2.30)-(2.33) at FHF (3.20)-(3.23), it is easy to note that Eq. (2.59) entails the equality

\[ p_a = \frac{w_a}{1 + gw} \]

which inverse is

\[ w_a = \frac{p_a}{1 - gp}. \]

Thus we find

\[ p = \frac{w}{1 + gw}, \quad w = \frac{p}{1 - gp}, \quad 1 + gw = \frac{1}{1 - gp}, \]

which entails the relations

\[ E(g; w) = \frac{\dot{E}(g; p)}{1 - gp}, \]

\[ \frac{E(g; w)}{\sqrt{Q(g; w)}} = \frac{\dot{E}(g; p)}{\sqrt{\dot{Q}(g; p)}}. \]

and

\[ Q(g; w) = \frac{\dot{Q}(g; p)}{(1 - gp)^2} = (1 + gw)^2\dot{Q}(g; p). \quad (3.53) \]
together with
\[
\frac{1 + gw}{Q(g; w)} = \frac{1}{\sqrt{Q(g; w)\hat{Q}(g; p)}}.
\] (3.54)

Starting now with (2.59), we obtain
\[
\hat{Z} = (Z + gq)\frac{K^2}{B} = (Z + gq)\frac{K^2}{Z^2Q} = \frac{1 + gw K^2}{Q} \frac{1}{Z} = \frac{1}{\sqrt{Q\hat{Q}}} \frac{K^2}{Z};
\]
whence
\[
\hat{Z} = \frac{1}{\sqrt{Q\hat{Q}}} \frac{KH}{Z},
\] (3.55)
where the fundamental definition
\[
H(g; \hat{R}) = K(g; R)
\]
for the FHF has been used. Taking into account the representations \( K = ZV \) and \( H = \hat{Z}W \) (see (2.49) and (3.38)), from (3.55) we obtain the identity
\[
V^2W^2 = Q\hat{Q}
\] (3.56)
which, on using \( V = \sqrt{Qj} \) (see (2.50)), just yields the FHF representation (3.20)-(3.23) together with the equality (3.33).

Thus we have proved

**Theorem 2.5.** The representations (3.20)-(3.23) associate the required FHF to the basic FMF given by Eqs. (2.30)-(2.33).

**Definition.** Given the FHF (3.20), the body
\[
\hat{\mathcal{F}}_g^{PD} := \{ \hat{R} \in \hat{V}_N : H(g; \hat{R}) \leq 1 \}
\] (3.57)
is called the co-Finsleroid.

**Definition.** The respective figuratrix defined by the equation
\[
\hat{\mathcal{I}}_g^{PD} := \{ \hat{R} \in \hat{V}_N : H(g; \hat{R}) = 1 \}
\] (3.58)
is called the co-Finsleroid Indicatrix.

We remain it to the reader to verify that Theorems 2.3-2.4 proven in the preceding Section 2.2 can well be re-formulated in the co-approach:

**Theorem 2.6.** The co-Finsleroid Indicatrix \( \hat{\mathcal{I}}_g^{PD} \) is a constant-curvature space with the positive curvature value (2.74):
\[
R_{\text{co-Finsleroid Indicatrix}} = R_{\text{Finsleroid Indicatrix}}
\] (3.59)
and
\[
R_{\text{co-Finsleroid Indicatrix}} = h^2 = 1 - \frac{1}{4}g^2, \quad 0 < R_{\text{co-Finsleroid Indicatrix}} \leq 1.
\] (3.60)
The formula
\[ R_{\text{co-Finsleroid Indicatrix}} = -\frac{1}{4}D_{\{\hat{B}\}} \]  
(3.61) is valid.

We are led also to

Theorem 2.7. The symmetry
\[
\begin{align*}
K(g; R) & \begin{cases} g \leftrightarrow -g, \\ R \leftrightarrow \hat{R} \end{cases} \\
H(g; \hat{R}) & \iff
\end{align*}
\]  
holds fine.

Note. It is useful to verify that the insertion of the relations (2.59) in the right-hand parts of the representations (3.49)-(3.52) leads directly to the FMT components given by the list (2.60)-(2.63).

2.4. Shape of Finsleroid and co-Finsleroid. The Finsleroid is not “uniform” in all directions and, therefore, does not permit general rotations. In terms of the function (2.45), the Finsleroid equation (see the definition (2.72)) reads

\[ \tilde{K}(g; q, Z) = 1. \]  
(4.1)

From (2.30)-(2.35) it follows directly that the value

\[ q^* := q\big|_{Z=0} \]  
(4.2)

of the quantity \( q \) over the Finsleroid is given by

\[ q^*(g) = \exp\left(-\frac{G}{2}\arctan\frac{G}{2}\right); \]  
(4.3)

with the definitions

\[ Z_1(g) = Z\big|_{q=0}, \quad \text{when} \quad Z < 0, \]  
(4.4)

and

\[ Z_2(g) = Z\big|_{q=0}, \quad \text{when} \quad Z > 0, \]  
(4.5)

we obtain

\[ Z_1(g) = -e^{G\pi/4} \quad \text{and} \quad Z_2(g) = e^{-G\pi/4}. \]  
(4.6)

Thus at any given value \( g \) we obtain the simple and explicit value for the altitude of the Finsleroid:

Theorem 2.8. We have

\[ \text{The Altitude of Finsleroid} = Z_2(g) - Z_1(g) = 2\cosh\frac{G\pi}{4}. \]  

The equation (4.1) cannot be resolved to find the function

\[ Z = Z(g; q) \]  
(4.7)
in an explicit form, because of a rather high complexity of the right-hand parts of Eqs. (2.30)-(2.35). Nevertheless, differentiating the identity

$$\tilde{K}(g; q, Z(g; q)) = 1$$ (4.8)

(see (4.1)) yields, on using (2.51), the simple results for the first derivatives:

$$\frac{\partial Z(g; q)}{\partial q} = -\frac{q}{Z + gq}$$ (4.9)

and

$$\frac{\partial^2 Z(g; q)}{\partial q^2} = -\frac{B(g; R)}{(Z + gq)^3}. \quad (4.10)$$

We also get

$$\frac{dZ(g; q)}{dq} \bigg|_{q=0} = 0 \quad \text{and} \quad \frac{dZ(g; q)}{dq} \xrightarrow{z \to +0} -\frac{1}{g}. \quad (4.11)$$

Inversely, for the function

$$q = q(g; Z)$$ (4.12)

obeying (4.1) we obtain

$$\frac{\partial q}{\partial Z} = -g - \frac{Z}{q}$$ (4.13)

and

$$\frac{\partial^2 q}{\partial Z^2} = -\frac{B(g; R)}{q^3} < 0. \quad (4.14)$$

We have

$$\frac{\partial q(g; Z)}{\partial Z} > 0, \quad \text{if} \quad Z < -gq; \quad \frac{\partial q(g; Z)}{\partial Z} < 0, \quad \text{if} \quad Z > -gq. \quad (4.15)$$

Also,

$$\frac{\partial q}{\partial Z} = 0, \quad \text{if} \quad Z = Z^{**} \quad \text{with} \quad Z^{**} = -gq^{**}. \quad (4.16)$$

Inserting this $Z^{**}$ in (2.30)-(2.35) yields

$$\Phi^{**} = \frac{\pi}{2} + \arctan \frac{G}{2} - \arctan \frac{g^2 - 2}{2gh}, \quad \text{if} \quad Z^{**} \geq 0 \sim g < 0, \quad (4.17)$$

$$\Phi^{**} = -\frac{\pi}{2} + \arctan \frac{G}{2} - \arctan \frac{g^2 - 2}{2gh}, \quad \text{if} \quad Z^{**} \leq 0 \sim g > 0, \quad (4.18)$$

and

$$q^{**}(g) = e^{-\frac{1}{2}G\Phi^{**}} \quad (4.19)$$
together with

\[ Z^{**}(g) = -g e^{-\frac{1}{2}G\Phi^{**}}. \tag{4.20} \]

Therefore, the following assertion can be set up for the width of the Finsleroid.

**Theorem 2.9.** With any given \( g \),

The Width of Finsleroid = \( 2q^{**}(g) = 2 e^{-\frac{1}{2}G\Phi^{**}}. \)

The formulas (4.19) and (4.20) may also be interpreted by saying that The Equatorial Section of the Finsleroid is of the radius

\[ r_{\text{Equatorial}} = q^{**} \tag{4.21} \]

and cuts the \( Z \)-axis at

\[ Z_{\text{Equatorial}} = Z^{**}. \tag{4.22} \]

With the parameter value \( |g| \) being increasing, the Finsleroid is stretching in wide and altitude:

\[ q^{**} \xrightarrow{|g| \to 2} \infty \tag{4.23} \]

and

\[ |Z^{**}| \xrightarrow{|g| \to 2} \infty, \tag{4.24} \]

tending in its shape to a cone:

\[ \frac{q^{**}}{|Z^{**}|} \xrightarrow{|g| \to 2} \frac{1}{2}, \tag{4.25} \]

such that the vertex of the Finsleroid tends to approach the origin point “O”. From (4.23) one can infer

**Theorem 2.10.** We have:

The Limiting Vertex Angle = \( 2 \arctan \frac{1}{2} \).

The above formulae, particularly the negative sign of the second derivative (4.10), can be used directly to verify the following

**Theorem 2.11.** The Finsleroid Indicatrix \( I^{PP}_g \) is closed, regular, and strongly convex.

The co-Finsleroid equation

\[ \dot{H}(g; \dot{q}, \dot{Z}) = 1 \tag{4.26} \]

can be studied in a similar way, leading to the relations obtainable from Eqs. (4.3)-(4.20) by means of the formal replacement \( \{g \to -g, R \to \hat{R}\} \), owing to the fundamental symmetry (3.61).

Therefore, we can state the following:

**Theorem 2.12.** The co-Finsleroid Indicatrix \( \hat{I}^{PP}_g \) is closed, regular, and also strongly convex.
Theorem 2.13. At any given parameter $g$, the Finsleroid and the co-Finsleroid mirror one another under the $g$-reflection:

$$\mathcal{F}_g^{PD} \xleftrightarrow{g} \hat{\mathcal{F}}_{-g}^{PD}.$$  \hspace{1cm} (4.27)

All Figures shown below have been prepared by means of a precise use of Maple9.

In Figs.2-7 bold lines serve to draw the Finsleroids, while unit circles simulate the ordinary Euclidean spheres. Fig.2 may be used as a convenient demonstration example (the *trainer*) for the Finsleroid by showing various structure details, including the equatorial section and the characteristic tangents, in a distinct way. We remain it to the reader to evaluate the angles that are depicted in the example and find among them equal cases.

Figs.2-7 clearly support the validity of Theorem 2.11 about regularity and convexity and make an idea of existence a diffeomorphic spherical map (see Eq. (5.1) below) a quite trustworthy one.

If one compares between Fig.2 and Fig.3 between Fig.4 and Fig.5 or between Fig.6 and Fig.7, one observes immediately that the change of sign of the characteristic Finslerian parameter $g$ does *turn up* the figures and, therefore, verifies the fundamental Finslerian $Z$-parity property (as given by Eq. (2.47)) in a due visual way. In a narrow sense, Figs.2-7 show the geometry of the generatrix for the Finsleroids, the latter being (hyper)surfaces of revolution over the $Z$-axis.

It can be traced also how the parameter $g$ effects the shape of Finsleroid. A positive value $g$ (deforms and) shifts the unit sphere in the down-wise manner, respectively a negative value in the up-wise manner.

Fig.8 and Fig.9 model the important functions (4.3) and (4.20), respectively.

There is no need to picture co-Finsleroids, for they mirror Finsleroids with respect to the ($R^N = 0$)-plane (according to our Theorems 2.12 and 2.13 and corresponding Eqs. (3.62) and (4.27)). In particular, at $g = -0.4$, the co-Finsleroid looks like the demonstration example given by Fig.1 at $g = 0.4$ (with interchanging respectively the coordinate axes: $R^N$ with $P_N$ and $R$ with $P$).
Fig 2: $[g=0.4]$

Fig 3: $[g=-0.4]$
Fig 4: [$g=0.2$]

Fig 5: [$g=-0.2$]
Fig 6: $[g=0.6]$

Fig 7: $[g=-0.6]$
Fig 8: [Eq. (4.3)]

Fig 9: [Eq. (4.20)]
2.5. Quasi-Euclidean map of Finsleroid. Theorem 2.11 can be continued farther by indicating the diffeomorphism

\[ \mathcal{F}_g^{PD} \overset{i_g}{\to} \mathcal{B}^{PD} \]

of the Finsleroid \( \mathcal{F}_g^{PD} \subset V_N \) to the unit ball \( \mathcal{B}^{PD} \subset V_N \):

\[ \mathcal{B}^{PD} := \{ R \in \mathcal{B}^{PD} : S(R) \leq 1 \}, \]

where

\[ S(R) = \sqrt{r_{pq} R^p R^q} \equiv \sqrt{(R^N)^2 + r_{ab} R^a R^b} \]

is the input Euclidean metric function (see (2.11)).

The diffeomorphism (5.1) can always be extended to get the diffeomorphic map

\[ V_N \overset{\sigma_g}{\to} V_N \]

of the whole vector space \( V_N \) by means of the homogeneity:

\[ \sigma_g \cdot (bR) = b \sigma_g \cdot R, \quad b > 0. \]

To this end it is sufficient to take merely

\[ \sigma_g \cdot R = \|R\| \cdot i_g \cdot \left( \frac{R}{\|R\|} \right), \]

where

\[ \|R\| = K(g; R). \]

Eqs. (5.1)-(5.7) entail

\[ K(g; R) = S(\sigma_g \cdot R). \]

On the other hand, the identity (2.57) suggests taking the map

\[ \bar{R} = \sigma_g \cdot R \]

by means of the components

\[ \bar{R}^a = \sigma^a(g; R) \]

with

\[ \sigma^a = R^a h J(g; R), \quad \sigma^N = A(g; R) J(g; R), \]

where \( J(g; R) \) and \( A(g; R) \) are the functions (2.31) and (2.38). Indeed, inserting (5.11) in (5.3) and taking into account Eqs. (2.30) and (2.57), we get the identity

\[ S(\bar{R}) = K(g; R) \]

which is tantamount to the implied relation (5.8).

Thus we have arrived at

**Theorem 2.14.** The map given explicitly by Eqs. (5.9)-(5.11) assigns the diffeomorphism between the Finsleroid and the unit ball according to Eqs. (5.1)-(5.8).

This motivates the following...
Definition. Under these conditions, the map (5.1) is called the \textit{quasi-Euclidean map of Finsleroid}.

The inverse

\[ R = \mu_g \cdot \bar{R}, \quad \mu_g = (\sigma_g)^{-1} \]  (5.13)

of the transformation (5.9)-(5.11) can be presented by the components

\[ R^p = \mu^p(g; \bar{R}) \]  (5.14)

with

\[ \mu^a = \bar{R}^a / h k(g; \bar{R}), \quad \mu^N = I(g; \bar{R}) / k(g; \bar{R}), \]  (5.15)

where

\[ k(g; \bar{R}) := J(g; \mu(g; \bar{R})) \]  (5.16)

and

\[ I(g; \bar{R}) = \bar{R}^N - \frac{1}{2} G \sqrt{r_{ab} \bar{R}^a \bar{R}^b}. \]  (5.17)

The identity

\[ \mu^p(g; \sigma(g; R)) \equiv R^p \]  (5.18)

can readily be verified. Notice that

\[ \frac{\bar{R}^a}{S(\bar{R})} = \frac{h R^a}{\sqrt{B(g; R)}}, \quad \frac{\bar{R}^N}{S(\bar{R})} = \frac{A(g; \bar{R})}{\sqrt{B(g; R)}}, \]

\[ \sqrt{r_{ab} \bar{R}^a \bar{R}^b} \frac{R^a}{R^N} = \frac{hq}{A(g; \bar{R})}, \quad w^a = \frac{R^a}{R^N} = \frac{\bar{R}^a}{h I(g; \bar{R})}, \]  (5.19)

and

\[ \sqrt{B/R^N} = S/I, \quad \sqrt{Q} = S/I. \]  (5.20)

The \(\sigma_g\)-image

\[ \phi(g; \bar{R}) := \Phi(g; R)|_{R=\mu(g; \bar{R})} \]  (5.21)

of the function \(\Phi\) described by Eqs. (2.32)-(2.42) is of a clear meaning of angle:

\[ \phi(g; \bar{R}) = \arctan \frac{\bar{R}^N}{\sqrt{r_{ab} \bar{R}^a \bar{R}^b}} \]  (5.22)

(Eq. (5.19) has been used) which ranges over
\[-\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}\]  

(5.23)

We have

\[\phi = \frac{\pi}{2}, \quad \text{if} \quad R^a = 0 \quad \text{and} \quad R^N > 0; \quad \phi = -\frac{\pi}{2}, \quad \text{if} \quad R^a = 0 \quad \text{and} \quad R^N < 0,\]

(5.24)

and also

\[\phi \big|_{R^N=0} = 0.\]

(5.25)

Comparing Eqs. (5.16) and (2.31) shows that

\[k = e^{\frac{1}{2}G\phi}.\]

(5.26)

The right-hand parts in (5.11) are homogeneous functions of degree 1:

\[\sigma^p(g; bR) = b\sigma^p(g; R), \quad b > 0.\]

(5.27)

Therefore, the identity

\[\sigma^p_s(g; R)R^s = \bar{R}^p\]

(5.28)

should be valid for the derivatives

\[\sigma^p_s(g; R) := \frac{\partial \sigma^p(g; R)}{\partial R^s}.\]

(5.29)

The simple representations

\[\sigma^N_N(g; R) = \left( B + \frac{1}{2}gqA \right) \frac{J}{B},\]

(5.30)

\[\sigma^a_a(g; R) = -\frac{g(ZA - B)Jr_{ab}R^b}{2q},\]

(5.31)

\[\sigma^N_a(g; R) = \frac{1}{2}gqJ R^a_h,\]

(5.32)

\[\sigma^a_b(g; R) = \left( B \delta^a_b - \frac{gr_{ke}R^eR^aZ}{2q} \right) \frac{Jh}{B},\]

(5.33)

and also the determinant

\[\det(\sigma^p_p) = h^{N-1}J^N\]

(5.34)
are obtained. The relations
\[
\sigma_a^b R^b = J h R^a (AZ + q^2)/B, \quad \tau^{cd} \sigma_c^a \sigma_d^b = J^2 h^2 \left[ \tau^{ab} - g(R^a R^b Z / qB) + \frac{1}{4} g^2 (R^a R^b Z^2 / B^2) \right]
\]
are handy in many calculations involving the coefficients \( \{ \sigma_q^p \} \).

Henceforth, to simplify notation, we shall use the substitution
\[
t^p = \tilde{R}^p.
\] (5.35)

Again, we can note the homogeneity
\[
\mu^p(g; bt) = b \mu^p(g; t), \quad b > 0,
\] (5.36)
for the functions (5.15), which entails the identity
\[
\mu^p_s(g; t^*) = R^p
\] (5.37)
for the derivatives
\[
\mu_q^p(g; t) := \frac{\partial \mu^p(g; t)}{\partial t^q}.
\] (5.38)

We find
\[
\mu_N^N = 1/k(g; t) - \frac{1}{2} g \frac{m(t)}{k(g; t)(S(t))^2}, \quad \mu_a^N = \frac{1}{2} g \frac{r_{ac} t^c I^*(g; t)}{k(g; t)(S(t))^2},
\] (5.39)
\[
\mu_N^a = -\frac{1}{2} g \frac{m(t) t^a}{h k(g; t)(S(t))^2}, \quad \mu_b^a = \frac{1}{h k(g; t)} \delta_b^a + \frac{1}{2} g \frac{t^N t^a r_{bc} t^c}{m(t) h k(g; t)(S(t))^2},
\] (5.40)
where
\[
m(t) = \sqrt{r_{ab} t^a t^b},
\] (5.41)
\[
I^*(g; t) = h m(t) - \frac{1}{2} g t^N,
\] (5.42)
and
\[
S(t) = \sqrt{r_{rs} t^r t^s} = \sqrt{(t^N)^2 + r_{ab} t^a t^b}.
\] (5.43)

The relations
\[
\frac{\partial (1/k(g; t))}{\partial t^N} = -\frac{1}{2} g \frac{m(t)}{h k(g; t)(S(t))^2}, \quad \frac{\partial (1/k(g; t))}{\partial t^a} = \frac{1}{2} g \frac{t^N r_{ab} t^b}{m(t) h k(g; t)(S(t))^2}
\]
are obtained.

Also,
\[
R_p \mu_q^p = t_q, \quad t_p \sigma_q^p = R_q.
\] (5.44)
The unit vectors
\[
L^p := \frac{t^p}{S(t)}, \quad L_p := r_{pq} L^q
\] (5.45)
fulfill the relations
\[ L^q = l^p \sigma^q_p, \quad l^p = \mu^p_q L^q, \quad l_p = \sigma^q_p L_q, \quad L_p = \mu^q_p l_q, \tag{5.46} \]
where \( l^p = R^p/K(g; R) \) and \( l_p = R_p/H(g; R) \) are the initial Finslerian unit vectors.

Now we use the explicit formulae (2.62)-(2.63) and (5.30)-(5.33) to find the transform
\[ n^{rs}(g; t) := \sigma^r_p \sigma^s_q g^{pq} \tag{5.47} \]
of the FMT under the \( F^P_d \)-induced map (5.9)-(5.11), which results in

**Theorem 2.15.** One obtains the simple representation
\[ n^{rs} = h^2 r^{rs} + \frac{1}{4} g^2 L^r L^s. \tag{5.48} \]
The covariant version reads
\[ n_{rs} = \frac{1}{h^2} r_{rs} - \frac{1}{4} G^2 L_r L_s. \tag{5.49} \]
The determinant of this tensor is a constant:
\[ \det(n_{rs}) = h^{2(1-N)} \det(r_{ab}). \tag{5.50} \]

Notice that
\[ L^p L_p = 1, \quad n_{pq} L^q = L_p, \quad n_{pq} L_q = L^p, \quad n_{pq} L^p L^q = 1, \quad n_{pq} t^p t^q = (S(t))^2. \]

Eq. (5.47) obviously entails
\[ g_{pq} = n_{rs}(g; t) \sigma^r_p \sigma^s_q. \tag{5.51} \]
2.6. Quasi-Euclidean metric tensor. Let us introduce

Definition. The metric tensor \( \{ n_{pq}(g; t) \} \) of the form (5.48) is called quasi-Euclidean.

Definition. The quasi-Euclidean space

\[ Q_N := \{ V_N; n_{pq}(g; t); g \} \]  

(6.1)

is an extension of the Euclidean space \( \{ V_N; r_{pq} \} \) to the case \( g \neq 0 \).

The transformation (5.47) can be inverted to read

\[ g_{pq} = \sigma_p^r \sigma_q^s n_{rs}. \]  

(6.2)

For the angular metric tensor (see the formula going below Eq. (2.65) in Chapter 2), from (5.46) and (6.2) we infer

\[ h_{pq} = \sigma_p^r \sigma_q^s H_{rs} \frac{1}{h^2}, \]  

(6.3)

where

\[ H_{rs} := r_{rs} - L_r L_s \]  

(6.4)

is the tensor showing the orthogonality property

\[ L^r H_{rs} = 0. \]  

(6.5)

One can readily verify that

\[ H_{rs} = h^2(n_{rs} - L_r L_s). \]

Theorem 2.16. The quasi-Euclidean metric tensor is conformal to the Euclidean metric tensor.

Indeed, if we consider the map

\[ \vec{R}^p \rightarrow \tilde{R} : \vec{R}^p = \xi(g; \vec{R}) \vec{R}^p / h \]  

(6.6)

with

\[ \xi(g; \vec{R}) = a \left( g; \frac{1}{2} S^2(\vec{R}) \right) \]  

(6.7)

and use the coefficients

\[ k_q^p := \frac{\partial \vec{R}^p}{\partial \vec{R}^q} = (\xi \delta_q^p + a' \vec{R}^p \vec{R}_q) / h \]  

(6.8)

to define the tensor

\[ e^{pq}(g; \vec{R}) := k_q^p k_s^q n^{rs}(g; \vec{R}), \]  

(6.9)

we find that

\[ e^{pq} = \xi^2 r^{pq} \]  

(6.10)

whenever

\[ \xi = \left[ \frac{1}{2} S^2(\vec{R}) \right]^{(h-1)/2}. \]  

(6.11)

The proof of Theorem 2.16 is complete.
The respective determinant representation is found to be merely
\[ \operatorname{det}(c_{pq}) = \xi^{-2N} \operatorname{det}(r_{pq}). \quad (6.12) \]

Let us now use the obtained quasi-Euclidean metric tensor \( n_{pq}(g; t) \) to construct the associated quasi-Euclidean Christoffel symbols \( N^r_{pq}(g; t) \). We find consecutively:

\[ n_{pq,r} := \frac{\partial n_{pq}}{\partial t^r} = -\frac{1}{4}G^2(H_{pr}L_q + H_{qr}L_p)/S, \quad (6.13) \]

and

\[ N^r_{pq} = n^{rs}N_{psq}, \quad N_{pq} = \frac{1}{2}(n_{pr,q} + n_{qr,p} - n_{pq,r}), \quad (6.14) \]

together with

\[ N_{pq}(g; t) = -\frac{1}{4}G^2H_{pq}/S, \quad (6.15) \]

which eventually yields

\[ N^r_{pq}(g; t) = -\frac{1}{4}G^2L^r_{pq}/S. \quad (6.16) \]

Comparing the representation (6.16) with the identity (6.5) shows that

\[ t^p N^r_{pq} = 0, \quad N^r_{pq} = 0, \quad N^s_{t q}, N^t_{p q} = 0. \quad (6.17) \]

Also,

\[ \frac{\partial N^r_{pq}}{\partial t^s} - \frac{\partial N^r_{pq}}{\partial t^q} = -\frac{1}{4}G^2(H_{pq}H^r_{qs} - H_{ps}H^r_{q s})/S^2. \quad (6.18) \]

Using the identities (6.17)-(6.18) in the quasi-Euclidean curvature tensor:

\[ R^r_{pqrs}(g; t) := \frac{\partial N^r_{pq}}{\partial t^s} - \frac{\partial N^r_{pq}}{\partial t^q} + N^w_{pq}N^r_{qs} - N^w_{ps}N^r_{q s}, \quad (6.19) \]

we arrive at the simple result:

\[ R^r_{pqrs}(g; t) = -\frac{1}{4}G^2(H_{pq}H^r_{rs} - H_{ps}H^r_{q r})/S^2. \quad (6.20) \]

This infers the identities

\[ L^pR^p_{pqrs} = L^sR^s_{pqrs} = L^rR^r_{pqrs} = L^sR^s_{pqr} = 0. \quad (6.21) \]

Note. Because of the transformation rules (5.11) and (5.47), the representation (6.20) is tantamount to Eqs. (2.70)-(2.71). Therefore we have got another rigorous proof of Theorem 2.3, and of Eq. (2.74), concerning the Finsleroid curvature.

The quasi-Euclidean orthonormal frames defined by the representations:

\[ n^{pq} = \sum_{p=1}^{N} m^p_p m^q_p \quad (6.22) \]

and

\[ n_{pq} = \sum_{P=1}^{N} f^P_p f^P_q \quad (6.23) \]
prove to be taken as

\[ f^P_q(g; t) = \frac{1}{h} h^P_q + \frac{h - 1}{h} L_q L^P \]  

(6.24)

and

\[ m^q_P(g; t) = h h^P_q + (1 - h) L_P L^q \]  

(6.25)

with

\[ L^P = h^P_q L^q, \quad L_P = h^q_P L_q. \]  

(6.26)

Here, \( h^P_q \) and \( h^q_P \) are the orthonormal frames for the input metric tensor (2.11):

\[ r^{pq} = \sum_{P=1}^{N} h^P_p h^q_q \]  

(6.27)

and

\[ r_{pq} = \sum_{P=1}^{N} h^P_p h^q_q. \]  

(6.28)

We have

\[ f^P_q(g; t) t^q = t^P. \]  

(6.29)

The associated quasi-Euclidean Ricci rotation coefficients:

\[ R^{PQ}_r(g; t) = \left( \partial_p f^Q_q - N^r_{pq} f^Q_r \right) m^q_T \delta^{TP} \]  

(6.30)

are found in the simple explicit form

\[ R^{PQ}_r(g; t) = (h - 1)(L^P f^Q_p - L^Q f^P_p) / S(t). \]  

(6.31)

The structure of the right-hand part of the representation (6.20) is such that the quasi-Euclidean metric tensor \( n_{pq} \) relates to a space of constant curvature only in the Euclidean case \( g = 0 \) proper. At the same time, treating the \( r \)-radius sphere

\[ \mathcal{S}_r := \{ R \in \mathcal{S}_r : S(t) = r \} \]  

(6.32)

(the metric (5.3) has been used) considered as a hypersurface in the quasi-Euclidean space (6.1)

\[ \mathcal{S}_r \subset \mathcal{Q}_N, \]  

(6.33)

we can arrive at

**Theorem 2.17.** The \( \mathcal{Q}_N \)-induced geometry on the sphere (6.32) is a geometry of the constant curvature \( h^2 / r^2 \), where \( h \) is the parameter (2.13).

This Theorem states that for the curvature \( \mathcal{C} \) of the hypersurface (6.32) we should get

\[ \mathcal{C}_{\text{quasi-Euclidean}} = h^2 \mathcal{C}_{\text{Euclidean}}, \]  

(6.34)

where

\[ \mathcal{C}_{\text{Euclidean}} = \frac{1}{r^2}. \]  

(6.35)
To verify this, it is sufficient to note that for any admissible parameterizations

\[ t^p = t^p(u^a) \]  

(6.36)

de \( S_r \) the projection factors

\[ T^p_a = \frac{\partial t^p}{\partial u^a}, \]  

(6.37)

where \( a = 1, \ldots, N - 1 \), satisfy the identity

\[ L^p_a T^p_a \equiv 0 \]  

(6.38)

and, therefore, the \( L^p \)-factors disappear in the induced metric tensor

\[ q_{ab} = T^p_a T^q_b n_{pq} \equiv T^p_a T^q_b \left( \frac{1}{\hbar^2} r_{pq} - \frac{1}{4} G^2 L^p L^q \right) \]  

(6.39)

(the formula (5.49) has been applied) leaving us with the representation

\[ q_{ab} = T^p_a T^q_b r_{pq}/\hbar^2. \]  

(6.40)

A particular convenient parameterizations way is

\[ t^a = u^a, \quad t^N = \sqrt{r^2 - u^2} \]

with

\[ u = \sqrt{r_{ab} u^a u^b}, \]

entailing

\[ T^N_a = -\frac{u^a}{\sqrt{r^2 - u^2}}, \quad T^a_b = \delta^a_b \]

together with

\[ \hbar^2 q_{ab}(u) = r_{ab} + \frac{u_a u_b}{r^2 - u^2}, \]

\[ \frac{1}{\hbar^2} q_{ab}(u) = r_{ab} - \frac{1}{r^2} u^a u^b, \]

and

\[ \det(q_{ab}(u)) = \hbar^{2(1-N)} \frac{r^2}{r^2 - u^2} \det(r_{ab}). \]

Constructing the associated Christoffel symbols yields

\[ I_{ab,e} = \frac{r_{ab} u_e}{r^2 - u^2} \frac{1}{\hbar^2} + \frac{u_a u_b u_e}{(r^2 - u^2)^2} \frac{1}{\hbar^2} = \frac{u_e}{r^2 - u^2} q_{ab}, \quad I^e_a b = \frac{\hbar^2}{r^2} u_e q_{ab}, \]

which entails that the associated curvature tensor

\[ R^e_{c ab}(u) := \frac{\partial I^e_c a}{\partial u^b} - \frac{\partial I^e_c b}{\partial u^a} + I^d_a I^c_d b - I^d_b I^c_d a \]

of the sphere (6.33) proves entailing the simple form

\[ R_{ecab}(u) = \hbar^2 \frac{1}{r^2} (q_{ea} q_{cb} - q_{eb} q_{ac}) \]

that directly manifests validity of the equality (6.34).
Chapter 3: Quasi-Euclidean consideration

For the space under study, the geodesics should be obtained as solutions to the equations (2.88)-(2.89) of Chapter 2 through well-known arguments. To avoid complications of calculations involved, it proves convenient to transfer preliminary the consideration into the quasi-Euclidean approach (exposed in Section 2.5). Below, we first consider in Section 3.1 the geodesic equation exploiting attentively the fact that the respective Christoffel symbols $N_{pqr}$ are of sufficiently simple structure. Surprisingly, the equation admits a simple and explicit general solution, as this will be shown in great detail. After that, the angle between two vectors is explicated. The remarkable result is that the angle $\alpha$ found is a factor of the Euclidean one, the angle being normalized such that the Cosine Theorem of ordinary form be rigorously valid if the Euclidean angle is replaced by the $\alpha$. The respective scalar product ensues. In Section 3.2, we introduce the associated two-vector metric tensor and demonstrate that at equality of vectors the tensor reduces exactly to the one-vector FMT of the $\mathcal{E}^{PP}_g$-space. The orthonormal frame thereto is also obtained in a lucid explicit form. After that, in Section 3.3, the possibility of converting the theory into the co-approach is presented, and in Section 3.4 the $\mathcal{E}^{PD}_g$-extension of the parallelogram law of vector addition is derived; it occurs also possible to find the respective sum vector and the difference vector in a nearest approximation with respect to the characteristic parameter $g$.

3.1. Derivation of geodesics and angle in associated quasi-Euclidean space. We start with searching for a general solution to the quasi-Euclidean geodesic equation which, in terms of the coefficients $N_{pqr}$, (given by Eq. (6.16) in Chapter 2), reads

$$\frac{d^2t^p}{ds^2} + N_{q}^{\; p\; r}(g; t) \frac{dt^q}{ds} \frac{dt^r}{ds} = 0. \tag{1.1}$$

Accordingly, we put

$$\sqrt{g_{pq}(g; R)dR^p dR^q} = \sqrt{n_{pq}(g; t)dt^p dt^q} \tag{1.2}$$

and

$$R^p(s) = \mu^p(g; t^r(s)) \tag{1.3}$$

together with

$$\frac{dR^p(s)}{ds} = \mu_q^p(g; t^r(s)) \frac{dt^q(s)}{ds}, \tag{1.4}$$

where $\mu^p(g; t^r)$ and $\mu_q^p(g; t^r)$ are the coefficients that were given in Chapter 2 by Eqs. (5.14)-(5.15) and (5.38)-(5.40), respectively. Let a curve $C$: $t^p = t^p(s)$ be considered, with the arc-length parameter $s$ along the curve being defined by the help of the differential

$$ds = \sqrt{n_{pq}(g; t)dt^p dt^q}, \tag{1.5}$$

where $n_{pq}(g; t)$ is the associated quasi-Euclidean metric tensor (Eq. (5.49) in Chapter 2). Respectively, the tangent vectors

$$w^p = \frac{dt^p}{ds} \tag{1.6}$$

to the curve are unit, in the sense that

$$n_{pq}(g; t)w^p w^q = 1. \tag{1.7}$$
Since \( L_p = \partial S/\partial t^p \), we have
\[
L_p u^p = \frac{dS}{ds}.
\] (1.8)

Here, \( S^2(t) = n_{pq}(g; t)t^pt^q = r_{pq} t^pt^q \) (see Eqs. (5.3) and (5.46) in Chapter 2). Using (1.1) leads to the following equation for geodesics in the quasi-Euclidean space:
\[
\frac{d^2 t}{ds^2} = \frac{1}{4} G_{pq} t^p u^q H_{pq} u^p u^q,
\] (1.9)

where \( H_{pq} = h^2(n_{pq} - L_p L_q) \) (see Eq. (6.4) in Chapter 2) and \( t = \{t^p\} \). We obtain
\[
\frac{d^2 t}{ds^2} = \frac{1}{4} g^2(a^2 - b^2) t^p \quad \text{(1.10)}
\]
and
\[
\frac{d^2 t}{ds^2} = \frac{1}{4} g^2(a^2 - b^2) t^p \quad \text{(1.11)}
\]
with
\[
S^2(s) = a^2 + 2bs + s^2, \quad \text{(1.12)}
\]
where \( a \) and \( b \) are two constants of integration. The formula (1.12) is valid because from
\[
\frac{1}{2} \frac{dS^2}{ds} = r_{pq} t^p u^q \quad \text{(1.13)}
\]
one can deduce
\[
\frac{1}{2} \frac{dS^2}{ds} = r_{pq} t^p u^q + t_q \frac{dS}{ds} = (h^2 n_{pq} + \frac{1}{4} g^2 L_p L_q) u^p u^q + \frac{1}{4} g^2 \left( 1 - \left( \frac{dS}{ds} \right)^2 \right)
\]
\[
= h^2 + \frac{1}{4} g^2 \left( \frac{dS}{ds} \right)^2 + \frac{1}{4} g^2 \left( 1 - \left( \frac{dS}{ds} \right)^2 \right) = 1.
\]

If we put
\[
S(\Delta s) = \sqrt{a^2 + 2b\Delta s + (\Delta s)^2} \quad \text{(1.14)}
\]
and
\[
t_1 = t(0), \quad t_2 = t(\Delta s) \quad \text{(1.15)}
\]
then we get
\[
a = \sqrt{(t_1 t_1)} \quad \text{(1.16)}
\]
and
\[
S(\Delta s) = \sqrt{(t_2 t_2)}. \quad \text{(1.17)}
\]

Here, \( t_1 \) and \( t_2 \) are two vectors emanated from the fixed origin “O”; they point to the beginning of the geodesic and to the end of the geodesic, respectively. The parenthesis couple (..) is used to denote the Euclidean scalar product, so that \( (t_1 t_1) = r_{pq} t^p_1 t^q_1 \), \( (t_2 t_2) = r_{pq} t^p_2 t^q_2 \), \( (t_1 t_2) = r_{pq} t^p_1 t^q_2 \); \( r_{pq} = \delta_{pq} \) in case of orthogonal basis (\( \delta \) stands for the Kronecker symbol).
What value should be prescribed to the scalar product \((t_1 t_2)\)? A special analysis shows that the respective correct choice compatible with the geodesic equation solution (see Eqs. (1.41) and (1.42) below) should read

\[
(t_1 t_2) = aS(\Delta s) \cos \left[ h \arctan \frac{\sqrt{a^2 - b^2} \, \Delta s}{a^2 + b \Delta s} \right].
\] (1.18)

From (1.16)-(1.18) it directly follows that

\[
\frac{\sqrt{a^2 - b^2} \, \Delta s}{a^2 + b \Delta s} = \tan \left[ \frac{1}{h} \arccos \frac{(t_1 t_2)}{\sqrt{(t_1 t_1)(t_2 t_2)}} \right],
\] (1.19)

which entails

\[
\cos \left[ h \arctan \frac{\sqrt{a^2 - b^2} \, \Delta s}{a^2 + b \Delta s} \right] = \frac{(t_1 t_2)}{\sqrt{(t_1 t_1)(t_2 t_2)}},
\]

and

\[
\sin \left[ h \arctan \frac{\sqrt{a^2 - b^2} \, \Delta s}{a^2 + b \Delta s} \right] = \frac{u(t_1, t_2)}{\sqrt{(t_1 t_1)(t_2 t_2)}},
\]

where

\[
u(t_1, t_2) = \sqrt{(t_1 t_1)(t_2 t_2) - (t_1 t_2)^2}.
\] (1.20)

The above equalities suggest the idea to introduce

Definition. The \(E^g_{PD}\)-associated angle is given by

\[
\alpha(t_1, t_2) := \frac{1}{h} \arccos \frac{(t_1 t_2)}{\sqrt{(t_1 t_1)(t_2 t_2)}},
\] (1.21)

so that

\[
\alpha = \frac{1}{h} \alpha_{\text{Euclidean}}.
\] (1.22)

Such an angle is obviously additive:

\[
\alpha(t_1, t_3) = \alpha(t_1, t_2) + \alpha(t_2, t_3)
\] (1.23)

in planar case of the vector triple \(\{t_1, t_2, t_3\}\).

The traditional vanishing at equal vectors holds:

\[
\alpha(t, t) = 0.
\] (1.24)

With any admissible prescribed parameter value \(g\), the range

\[
0 \leq \alpha(t_1, t_2) \leq \frac{1}{h} \pi
\] (1.25)

corresponds to the canonical Euclidean range \((0, \pi)\).

With the angle (1.21), we ought to propose

Definition. Given two vectors \(t\) and \(t^\perp\), the vectors are said to be \(E^g_{PD}\)-perpendicular, if

\[
\cos (\alpha(t, t^\perp)) = 0.
\] (1.26)
Since the vanishing (1.25) implies
\[ \alpha(t, t^\perp) = \frac{\pi}{2}, \] (1.27)
in view of (1.22) we ought to conclude that
\[ \alpha_{Euclidean}(t, t^\perp) = \frac{\pi}{2} h \leq \frac{\pi}{2}. \] (1.28)
Therefore, vectors perpendicular in the quasi-Euclidean sense proper look like acute vectors as observed from associated Euclidean standpoint proper.

With the equality
\[ (\sqrt{a^2 - b^2} \Delta s)^2 + (a^2 + b \Delta s)^2 \equiv a^2 S^2(\Delta s) \] (1.29)
(an implication of Eq. (1.14)) we also establish the relations
\[ \sqrt{a^2 - b^2} \Delta s = a S(\Delta s) \sin \alpha \] (1.30)
and
\[ a^2 + b \Delta s = a S(\Delta s) \cos \alpha, \] (1.31)
where \( \alpha \) is exactly the angle (1.21). They entail the equalities
\[ \frac{b}{\sqrt{a^2 - b^2}} = \frac{S(\Delta s) \cos \alpha - a}{S(\Delta s) \sin \alpha}, \quad \frac{b^2}{a^2} = 1 - \left( \frac{S(\Delta s)}{\Delta s} \right)^2 \sin^2 \alpha \] (1.32)
from which the quantity \( b \) also can be explicated.

Thus each member of the involved set \( \{a, b, \Delta s, S(\Delta s)\} \) can be explicitly expressed through the input vectors \( t_1 \) and \( t_2 \). For various particular cases it is worth rewriting the equality (1.29) as
\[ S^2(\Delta s) = (\Delta s)^2 - a^2 + 2(a^2 + b \Delta s). \] (1.33)
Thus we may naturally set forth the following fundamental items:

The quasi-Euclidean Cosine Theorem
\[ (\Delta s)^2 = S^2(\Delta s) + a^2 - 2a S(\Delta s) \cos \alpha. \] (1.34)

The quasi-Euclidean Two-Point Length
\[ (\Delta s)^2 = (t_1 t_1) + (t_2 t_2) - 2 \sqrt{(t_1 t_1) (t_2 t_2)} \cos \alpha. \] (1.35)

The quasi-Euclidean Scalar Product
\[ < t_1, t_2 > = \sqrt{(t_1 t_1) (t_2 t_2)} \cos \alpha. \] (1.36)

The quasi-Euclidean Perpendicularity
\[ \langle t, t^+ \rangle = \sqrt{(tt)} \sqrt{(t^+ t^+)} \]  

The identification

\[ |t_2 \ominus t_1|^2 = (\Delta s)^2 \]  

yields another lucid representation

\[ |t_2 \ominus t_1|^2 = (t_1 t_1) + (t_2 t_2) - 2\sqrt{(t_1 t_1) \sqrt{(t_2 t_2)}} \cos \alpha \]  

Here the symmetry holds:

\[ |t_2 \ominus t_1| = |t_1 \ominus t_2| \]  

Comparing

\[ (t_1 t_2) = \sqrt{(t_1 t_1)} \sqrt{(t_2 t_2)} \cos h \alpha = \sqrt{(t_1 t_1)} \sqrt{(t_2 t_2)} \cos \alpha_{Euclidean} \]  

(see Eqs. (1.18)-(1.21)) with (1.36) shows that

\[ \langle t_1, t_2 \rangle \neq (t_1 t_2) \quad \text{unless} \quad g = 0. \]  

The consideration can be completed by

**Proposition 3.1.** A general solution to the geodesic equation (1.11) can explicitly be found as follows:

\[
t(s) = \frac{S(s)}{a} \sin \left[ h \arctan \frac{\sqrt{a^2 - b^2} (\Delta s - s)}{a^2 + b\Delta s + (b + \Delta s)s} \right] t_1 + \frac{S(s)}{S(\Delta s)} \sin \left[ h \arctan \frac{\sqrt{a^2 - b^2} s}{a^2 + bs} \right] t_2. \]

(1.41)

The validity of this proposition can be processed by direct insertion of the expression (1.41) in (1.11).

From (1.41) the equality

\[ (t(s) t(s)) = S^2(s) \]  

follows, in agreement with (1.12). Also, it is useful to note that

\[ \arctan \frac{\sqrt{a^2 - b^2} (\Delta s - s)}{a^2 + b\Delta s + (b + \Delta s)s} + \arctan \frac{\sqrt{a^2 - b^2} s}{a^2 + bs} = \arctan \frac{\sqrt{a^2 - b^2} \Delta s}{a^2 + b\Delta s}, \]

where the right-hand part does not involve the variable \(s\). Using (1.19) and (1.21), we conclude

\[ \arctan \frac{\sqrt{a^2 - b^2} (\Delta s - s)}{a^2 + b\Delta s + (b + \Delta s)s} = \alpha - \nu, \]  

(1.43)
\[
\nu = \arctan \frac{\sqrt{a^2 - b^2 s}}{a^2 + bs}.
\] (1.44)

In terms of this angle, the solution (1.41) reads simply
\[
t(s) = \frac{S(s) \sin(h(\alpha - \nu))}{a \sin(h\alpha)} t_1 + \frac{S(s) \sin(h\nu)}{S(\Delta s) \sin(h\alpha)} t_2.
\] (1.45)

The identification
\[
t(s) \bigg|_{s=0} = t_1, \quad t(s) \bigg|_{s=\Delta s} = t_2.
\]
can readily be verified. The Euclidean limit proper for the solution (1.41) is
\[
t(s) \bigg|_{g=0} = \frac{(\Delta s - s) t_1 + s t_2}{\Delta s} = t_1 + \frac{(t_2 - t_1)}{\Delta s} s,
\]
so that the geodesics are got simplified to be straight lines.

Since the general solution (1.41) is such that the right-hand side is spanned by two fixed vectors, \(t_1\) and \(t_2\), we are entitled concluding that

**Proposition 3.2.** Against the quasi-Euclidean treatment, the geodesics under study are plane curves.

Calculating the first derivative
\[
v(s) := \frac{dt(s)}{ds}
\]
of (1.41) yields the formula
\[
v(s) = \frac{b + s}{S^2(s)} t(s) - \frac{\sqrt{a^2 - b^2 h \cos(h(\alpha - \nu))}}{a S(s) \sin(h\alpha)} t_1 + \frac{\sqrt{a^2 - b^2 h \cos(h\nu)}}{S(s) S(\Delta s) \sin(h\alpha)} t_2.
\] (1.46)
The right-hand part here is such that
\[
t(s) \left( v(s) - \frac{b + s}{S^2(s)} t(s) \right) = 0.
\]
From the latter observation we get the equality
\[
t(s) v(s) = b + s
\]
which is tantamount to (1.12).

Also, the *initial velocity*
\[
v_1 := \frac{dt}{ds} \bigg|_{s=0}
\]
and the *final velocity*
\[
v_2 := \frac{dt}{ds} \bigg|_{s=\Delta s}
\]
are found from (1.46) to be
\[ v_1 = \frac{b}{a^2} t_1 - \frac{\sqrt{a^2 - b^2}}{a^2} h \cos(h \alpha) t_1 + \frac{\sqrt{a^2 - b^2}}{a S(\Delta s)} \sin(h \alpha) t_2 \]

\[ = \frac{b}{a^2} t_1 - \frac{\sqrt{a^2 - b^2}}{a^2} \left( \frac{t_1 t_2}{u(t_1, t_2)} \right) t_1 + \frac{\sqrt{a^2 - b^2}}{a S(\Delta s)} \sin(h \alpha) t_2 \]

and

\[ v_2 = \frac{b + \Delta s}{S^2(\Delta s)} t_2 - \frac{\sqrt{a^2 - b^2}}{a S(\Delta s)} t_1 + \frac{\sqrt{a^2 - b^2} h \cos(h \alpha)}{S^2(\Delta s)} t_2 \]

(with having used (1.19)), where \( u(t_1, t_2) \) is the function (1.20).

Notice also that

\[ (t_1 v_1) = b, \quad (t_2 v_2) = b + \Delta s, \]

\[ (v_1 v_2) = \left( 1 + \frac{G^2 b^2}{4} \right) h^2 \equiv \frac{(\Delta s)^2 h^2 + (1 - h^2)(t_1 t_2) \sin^2 \alpha}{(\Delta s)^2} \]

and

\[ n_{pq}(g; t_1) v_1^p v_1^q = 1. \]

The difference between the vectors \( v_2 \) and \( v_1 \) can be found to read

\[ v_2 - v_1 = \left[ \frac{b + \Delta s}{S^2(\Delta s)} + \frac{\sqrt{a^2 - b^2} h}{S^2(\Delta s)} \left( \frac{t_1 t_2}{u(t_1, t_2)} \right) - \frac{\sqrt{a^2 - b^2}}{a^2} \left( \frac{t_1 t_2}{u(t_1, t_2)} \right) \right] t_2 \]

\[ - \left[ \frac{b}{a^2} + \frac{\sqrt{a^2 - b^2}}{a^2} \frac{1}{u(t_1, t_2)} - \frac{\sqrt{a^2 - b^2}}{a^2} \left( \frac{t_1 t_2}{u(t_1, t_2)} \right) \right] t_1. \]

It is useful to verify that

\[ v_2 \bigg|_{g=0} = v_1 \bigg|_{g=0} = \frac{t_2 - t_1}{\Delta s} \]

in compliance with the Euclidean rule proper.

Other convenient representations for these vectors are

\[ v_1 \Delta s = (-1 + \frac{1}{(t_1 t_1) C_1}) t_1 + h \frac{\sin \alpha}{\sin \alpha_{\text{Euclidean}}} t_2. \]
\[ \mathbf{v}_2 \Delta s = (1 - \frac{1}{(t_2 t_2)} C_1) t_2 - h \frac{\sin \alpha}{\sin \alpha_{Euclidean}} t_1, \]

and

\[ (\mathbf{v}_2 - \mathbf{v}_1) \Delta s = \left( 1 - h \frac{\sin \alpha}{\sin \alpha_{Euclidean}} \right) (t_2 + t_1) - C_1 \left( \frac{t_2}{t_2 t_2} + \frac{t_1}{t_1 t_1} \right), \]

where

\[ C_1 = \sqrt{(t_1 t_1) \sqrt{(t_2 t_2)} \cos \alpha - h(t_1 t_2) \frac{\sin \alpha}{\sin \alpha_{Euclidean}}}. \]

Using above formulae, we can turn the solution (1.40) into the initial-date form:

Proposition 3.3. The general initial-date solution to the geodesic equation (1.1) reads

\[ t(s) = m(s)t_1 + n(s)v_1 \]

with

\[ m(s) = -\frac{bS(s)}{a \sqrt{a^2 - b^2 h}} \sin(h\alpha) + \frac{S(s)}{a} \cos(h\alpha) + \frac{S(s) \sin(h(\alpha - \nu))}{\sin(h\alpha)} \]

and

\[ n(s) = \frac{a S(s)}{\sqrt{a^2 - b^2 h}} \sin(h\nu). \]

Below the picture symbolizes the character of the solution (1.47).

Fig 10: [The geodesic C and the initial velocity \( \mathbf{v}_1 \)]
For the derivative vectors 
\[ b_1 := \frac{1}{2} \frac{\partial |t_2 \oplus t_1|^2}{\partial t_1}, \quad b_2 := \frac{1}{2} \frac{\partial |t_2 \oplus t_1|^2}{\partial t_2}, \]
we can obtain the simple representations
\[ b_1 = t_1 - \frac{t_1}{\sqrt{(t_1 t_1)}} \sqrt{(t_2 t_2)} \cos \alpha - \frac{\sqrt{(t_2 t_2)}}{h \sqrt{(t_1 t_1)}} d_1 \sin \alpha \]
and
\[ b_2 = t_2 - \frac{t_2}{\sqrt{(t_2 t_2)}} \sqrt{(t_1 t_1)} \cos \alpha - \frac{\sqrt{(t_1 t_1)}}{h \sqrt{(t_2 t_2)}} d_2 \sin \alpha, \]
where the convenient vectors
\[ d_1 = \frac{(t_1 t_1) t_2 - (t_1 t_2) t_1}{u(t_1, t_2)}, \quad d_2 = \frac{(t_2 t_2) t_1 - (t_1 t_2) t_2}{u(t_1, t_2)} \]
have been introduced. It can readily be verified that
\[ (t_1 d_1) = 0, \quad (t_2 d_2) = 0, \]
\[ (d_1 d_2) = -(t_1 t_2), \quad (d_1 d_1) = (t_1 t_1), \quad (d_2 d_2) = (t_2 t_2), \]
\[ (d_1 t_2) = (t_1 d_2) = u(t_1, t_2), \]
and
\[ t_1 b_1 + t_2 b_2 = 2 |t_2 \oplus t_1|^2, \]

For the respective products of vectors we obtain
\[ (b_1 b_1) = (t_1 t_1) + (t_2 t_2) - 2 \sqrt{(t_1 t_1)} \sqrt{(t_2 t_2)} \cos \alpha + \left( \frac{1}{h^2} - 1 \right) (t_2 t_2) \sin^2 \alpha, \]
\[ (b_2 b_2) = (t_2 t_2) + (t_1 t_1) - 2 \sqrt{(t_2 t_2)} \sqrt{(t_1 t_1)} \cos \alpha + \left( \frac{1}{h^2} - 1 \right) (t_1 t_1) \sin^2 \alpha, \]
and
\[ (b_1 b_2) = - \left[ \frac{\sqrt{(t_1 t_1)}}{\sqrt{(t_2 t_2)}} + \frac{\sqrt{(t_2 t_2)}}{\sqrt{(t_1 t_1)}} - 2 \cos \alpha \right] \cos \alpha + \left( \frac{1}{h^2} - 1 \right) \sin^2 \alpha (t_1 t_2) \]
\[-\frac{1}{\hbar} \left( \frac{\sqrt{(t_1 t_1)}}{\sqrt{(t_2 t_2)}} + \frac{\sqrt{(t_2 t_2)}}{\sqrt{(t_1 t_1)}} - 2 \cos \alpha \right) u(t_1, t_2) \sin \alpha.\]

The vectors $b_1$ and $b_2$ extend the difference vectors, for in the Euclidean limit we have simply

$$(b_1|_{g=0} = t_1 - t_2, \quad b_2|_{g=0} = t_2 - t_1).$$

The following limit

$$\lim_{t_2 \rightarrow t_1} \left\{ \frac{(t_1 t_1)(t_2 t_2)}{h \sqrt{(t_1 t_1)} \sqrt{(t_2 t_2)}} \sin \left[ \frac{1}{h} \arccos \frac{(t_1 t_2)}{\sqrt{(t_1 t_1)} \sqrt{(t_2 t_2)}} \right] \right\} = \frac{1}{\hbar^2}$$

is important to note.

3.2. The two-vector metric tensor and frame in quasi-Euclidean space. Now we are able to introduce the quasi-Euclidean two-vector metric tensor $n(g; t_1, t_2)$ by the components

$$n_{pq}(g; t_1, t_2) := \frac{\partial^2 < t_1, t_2 >}{\partial t_2^p \partial t_1^q} = -\frac{1}{2} \frac{\partial^2 |t_2 \otimes t_1|^2}{\partial t_2^p \partial t_1^q}.$$  \hspace{1cm} (2.1)

Straightforward calculations (on the basis of (1.32) and (1.19)) show that

$$n_{pq}(g; t_1, t_2) = \frac{(t_1 t_1)(t_2 t_2)}{h \sqrt{(t_1 t_1)} \sqrt{(t_2 t_2)}} \sin \alpha \frac{r_{pq}}{u(t_1, t_2)} + \frac{1}{\sqrt{(t_1 t_1)} \sqrt{(t_2 t_2)}} A_1 t_1^p t_2^q - \frac{1}{h \sqrt{(t_1 t_1)} \sqrt{(t_2 t_2)}} A_2 d_1^p d_2^q,$$ \hspace{1cm} (2.2)

where

$$A_1 = \cos \alpha - \frac{1}{\hbar} (t_1 t_2) \frac{\sin \alpha}{u(t_1, t_2)} \hspace{1cm} (2.3)$$

and

$$A_2 = \frac{1}{\hbar} \cos \alpha - (t_1 t_2) \frac{\sin \alpha}{u(t_1, t_2)}. \hspace{1cm} (2.4)$$

For the determinant of the tensor (2.2) we find simply

$$\det (n_{pq}(g; t_1, t_2)) = \left( \frac{(t_1 t_1)(t_2 t_2)}{u(t_1, t_2)} \frac{\sin \alpha}{\hbar} \right)^{N-2} h^{-N} \det (r_{ab}). \hspace{1cm} (2.5)$$

Owing to (1.48), we can establish the following fundamental identification:

$$\lim_{t_2 \rightarrow t_1} \left\{ n_{pq}(g; t_1, t_2) \right\} = n_{pq}(g; t), \hspace{1cm} (2.6)$$

where $n_{pq}(g; t)$ is the quasi-Euclidean metric tensor (see (5.49) in Chapter 2).
Differentiating (2.2) results in

\[
\frac{\partial n_{pq}(g; t_1, t_2)}{\partial t_1^s} = -\frac{1}{h} \frac{\sqrt{(t_2 t_2)}}{\sqrt{(t_1 t_1) u(t_1, t_2)}} A_2 d_{1s} r_{pq} + \frac{1}{h} \frac{1}{\sqrt{(t_1 t_1) \sqrt{(t_2 t_2)}} A_1 t_{2q} H_{sp}(t_1) + \frac{1}{h} \frac{1}{\sqrt{(t_1 t_1)}} H_{ps}(t_1) d_{2q} - (t_1 t_2) H_{qs}(t_1) d_{1p}}{u(t_1, t_2)} A_2 d_{1s} t_{1p} t_{2q}
\]

\[
+ \frac{1}{h} \frac{1}{\sqrt{(t_1 t_1) \sqrt{(t_2 t_2)}} A_2 \times \left[ \frac{1}{(t_1 t_1)} ((t_1 t_1) d_{2p} d_{2q} + (t_2 t_2) d_{1p} d_{1q}) d_{1s} + (t_1 t_2) H_{ps}(t_1) d_{2q} - (t_2 t_2) H_{qs}(t_1) d_{1p} \right]
\]

\[
= \frac{1}{(t_1 t_1)} \left[ (1 - \frac{1}{h^2}) \sin \alpha - \frac{(t_1 t_2)}{u(t_1, t_2)} A_2 \right] d_{1s} d_{1p} d_{2q}, \tag{2.7}
\]

where we have used the relations:

\[
\frac{\partial \alpha}{\partial t_1^s} = -\frac{1}{h} \frac{d_1}{(t_1 t_1)}, \quad \frac{\partial \frac{1}{u}}{\partial t_1^s} = -\frac{1}{u^2} d_2,
\]

\[
\frac{\partial A_1}{\partial t_1^s} = \frac{1}{h} \frac{(t_1 t_2)}{(t_1 t_1) u(t_1, t_2)} A_2 d_{1s},
\]

\[
\frac{\partial A_2}{\partial t_1^s} = \frac{1}{h} \frac{(t_1 t_2)}{(t_1 t_1) u(t_1, t_2)} A_1 d_{1s} - (1 - \frac{1}{h^2}) \frac{(t_2 t_2)}{u(t_1, t_2)} \frac{\sin \alpha}{u(t_1, t_2)} d_{1s}
\]

\[
= \frac{d_{1s}}{(t_1 t_1)} \left[ (1 - \frac{1}{h^2}) \sin \alpha + \frac{(t_1 t_2)}{u(t_1, t_2)} A_2 \right].
\]

Since

\[
\lim_{t_2 \to t_1} \left\{ A_1 \right\} = 1 - \frac{1}{h^2}, \quad \lim_{t_2 \to t_1} \left\{ \frac{A_2}{u} \right\} = 0,
\]

\[
\lim_{t_2 \to t_1} \left\{ \frac{\partial n_{pq}(g; t_1, t_2)}{\partial t_1^s} \right\} = (1 - \frac{1}{h^2}) \frac{t_2}{(tt)} H_{sp}(t),
\]

and

\[
\lim_{t_2 \to t_1} \left\{ \frac{\partial n_{pq}(g; t_1, t_2)}{\partial t_2^s} \right\} = (1 - \frac{1}{h^2}) \frac{t_2}{(tt)} H_{sq}(t),
\]

the fundamental consequence

\[
\lim_{t_2 \to t_1} \left\{ \frac{\partial n_{pq}(g; t_1, t_2)}{\partial t_1^s} + \frac{\partial n_{pq}(g; t_1, t_2)}{\partial t_2^s} \right\} = \frac{\partial n_{pq}(g; t)}{\partial t^s}
\]
The expansion with respect to an appropriate orthonormal frame $f_p^R(g; t_1, t_2)$ can be found to read

$$n_{pq}(g; t_1, t_2) = \sum_{R=1}^{N} f_p^R(g; t_1, t_2) f_q^R(g; t_2, t_1)$$

with

$$\sqrt{h} \sqrt{(t_1 t_1)} \sqrt{(t_2 t_2)} f_p^R(g; t_1, t_2) = z h_p^R$$

$$- \frac{1}{(t_1 t_2)} \left[ z - \sqrt{z^2 + (t_1 t_2) \left( h \cos \alpha - (t_1 t_2) \frac{\sin \alpha}{u(t_1, t_2)} \right)} \right] t_2 t_1$$

$$+ \frac{1}{(t_1 t_2)} \left[ z - \sqrt{z^2 + (t_1 t_2) \left( \frac{1}{h} \cos \alpha - (t_1 t_2) \frac{\sin \alpha}{u(t_1, t_2)} \right)} \right] d_2 d_1^R,$$

where

$$z = \sqrt{(t_1 t_1)(t_2 t_2)} \frac{\sin \alpha}{u(t_1, t_2)}.$$
\[-\sqrt{\frac{1}{h}(t_1 t_2) \cos \alpha + u(t_1, t_2) \sin \alpha} t_2^R + \sqrt{\frac{1}{h}(t_1 t_2) \cos \alpha + u(t_1, t_2) \sin \alpha} t_1^R \]

and

\[f_p^R(g; t_1, t_2) t_2^R = \frac{1}{\sqrt{h} \sqrt{(t_1 t_1) (t_2 t_2)}} \sqrt{h(t_1 t_2) \cos \alpha + u(t_1, t_2) \sin \alpha} t_2^R,\]

together with

\[\sum_{R=1}^{N} f_p^R(g; t_1, t_2) t_1^R = \frac{1}{\sqrt{h} \sqrt{(t_1 t_1) (t_2 t_2)}} \sqrt{h(t_1 t_2) \cos \alpha + u(t_1, t_2) \sin \alpha} t_1^R,\]

and

\[\sum_{R=1}^{N} f_p^R(g; t_1, t_2) t_2^R = \frac{1}{\sqrt{h} \sqrt{(t_1 t_1) (t_2 t_2)}} \left( (t_2 t_2) \sqrt{h(t_1 t_2) \cos \alpha + u(t_1, t_2) \sin \alpha} t_1 + \sqrt{\frac{1}{h}(t_1 t_2) \cos \alpha + u(t_1, t_2) \sin \alpha} t_2^R \right)\]

3.3. **Covariant version.** It proves possible to convert the approach into the co-version by introducing the co-vectors

\[T_{1p}(g; t_1, t_2) := n_{pq}(g; t_1, t_2) t_2^p, \quad T_{2q}(g; t_1, t_2) := t_1^p n_{pq}(g; t_1, t_2). \quad (3.1)\]

Applying (2.2), we get

\[T_1 = \frac{\sqrt{(t_2 t_2)} t_1 \cos \alpha + \sqrt{(t_2 t_2)} d_1 \sin \alpha}{h \sqrt{(t_1 t_1)}} \quad (3.2)\]

and

\[T_2 = \frac{\sqrt{(t_1 t_1)} t_2 \cos \alpha + \sqrt{(t_1 t_1)} d_2 \sin \alpha}{h \sqrt{(t_2 t_2)}}. \quad (3.3)\]

The equality

\[(t_1 T_1) + (t_2 T_2) = 2 < t_1, t_2 > = 2 \sqrt{(t_1 t_1) (t_2 t_2)} \cos \alpha \quad (3.4)\]

holds. Also,

\[\lim_{t_2 \to t_1 = t} \{ T_1 \} = \lim_{t_2 \to t_1 = t} \{ T_2 \} = t. \quad (3.5)\]
The metric tensor (2.1)-(2.2) is obtainable from these vectors as follows:

\[ n_{pq}(g; t_1, t_2) = \frac{\partial T_{1p}}{\partial t^q_2} = \frac{\partial T_{2q}}{\partial t^p_1} \]  

(3.6)

The respective products are found to be

\( (T_1 T_1) = (t_2 t_2) (\cos^2 \alpha + \frac{1}{h^2} \sin^2 \alpha), \quad (T_2 T_2) = (t_1 t_1) (\cos^2 \alpha + \frac{1}{h^2} \sin^2 \alpha) \)  

(3.7)

and

\( (T_1 T_2) = (\cos^2 \alpha - \frac{1}{h^2} \sin^2 \alpha) (t_1 t_2) + 2 \frac{1}{h} u(t_1, t_2) \cos \alpha \sin \alpha, \)  

(3.8)

together with

\[ u(T_1, T_2) = \sqrt{(T_1 T_1)(T_2 T_2) - (T_1 T_2)^2}. \]  

(3.10)

From (3.7)-(3.8) it follows that

\[ (\cos^2 \alpha + \frac{1}{h^2} \sin^2 \alpha)^2 u(t_1, t_2) = \frac{2}{h} (T_1 T_2) \sin \alpha \cos \alpha - (\cos^2 \alpha - \frac{1}{h^2} \sin^2 \alpha) u(t_1, t_2), \]  

(3.9)

where

Using these formulas in calculating the co-representation

\[ \alpha = \hat{\alpha}(T_1, T_2) \]  

(3.11)

for the angle (1.21) yields the following implicit equation:

\[ \cos(h \alpha) = \frac{(\cos^2 \alpha - \frac{1}{h^2} \sin^2 \alpha)(T_1 T_2) + \frac{2}{h} \sin \alpha \cos \alpha u(T_1, T_2)}{(\cos^2 \alpha + \frac{1}{h^2} \sin^2 \alpha) \sqrt{(T_1 T_1)(T_2 T_2)}}. \]  

(3.12)

The respective co-version of the scalar product (1.36) reads

\[ < T_1, T_2 >= \sqrt{(T_1 T_1)(T_2 T_2)} \cos \alpha. \]  

(3.13)

On this way the set (3.2)-(3.3) can be inverted, yielding
\[ \mathbf{t}_1(g; \mathbf{T}_1, \mathbf{T}_2) = \frac{1}{\xi} \left[ \frac{\sqrt{(t_1 t_2)}}{\sqrt{(t_1 t_3)}} \left( \cos \alpha - \frac{1}{h} \frac{(t_1 t_2)}{u(t_1, t_2)} \sin \alpha \right) \mathbf{T}_1 \right. \\
\left. - \frac{1}{h} \sqrt{(t_1 t_1)} \sqrt{(t_2 t_2)} \sin \alpha \mathbf{T}_2 \right] \] (3.14)

and

\[ \mathbf{t}_2(g; \mathbf{T}_1, \mathbf{T}_2) = \frac{1}{\xi} \left[ \frac{\sqrt{(t_2 t_2)}}{\sqrt{(t_1 t_1)}} \left( \cos \alpha - \frac{1}{h} \frac{(t_1 t_2)}{u(t_1, t_2)} \sin \alpha \right) \mathbf{T}_2 \right. \\
\left. - \frac{1}{h} \sqrt{(t_1 t_1)} \sqrt{(t_2 t_2)} \sin \alpha \mathbf{T}_1 \right] \] (3.15)

where

\[ \xi = \left( \cos \alpha - \frac{1}{h} \frac{(t_1 t_2)}{u(t_1, t_2)} \sin \alpha \right)^2 - \frac{1}{h^2} \left( \frac{\sin \alpha}{u(t_1, t_2)} \right)^2 (t_1 t_1)(t_2 t_2), \]

or

\[ \xi = \cos^2 \alpha - \frac{1}{h^2} \sin^2 \alpha - \frac{2 \sin \alpha \cos \alpha}{h u(t_1, t_2)} (t_1 t_2). \] (3.16)

On taking into account (3.9), this function can be written merely as

\[ \xi = -\frac{u(T_1, T_2)}{u(t_1, t_2)}. \] (3.17)

Thus we find

\[ \mathbf{t}_1 = \frac{\sqrt{(T_2 T_2)}}{\sqrt{(T_1 T_1)}} T_1 \cos \alpha + \frac{1}{h^2} \sin \alpha D_1 \left( \cos^2 \alpha + \frac{1}{h^2} \sin^2 \alpha \right) \] (3.18)

and

\[ \mathbf{t}_2 = \frac{\sqrt{(T_1 T_1)}}{\sqrt{(T_2 T_2)}} T_2 \cos \alpha + \frac{1}{h^2} \sin \alpha D_2 \left( \cos^2 \alpha + \frac{1}{h^2} \sin^2 \alpha \right), \] (3.19)

where

\[ D_1 = \frac{(T_1 T_1)T_2 - (T_1 T_2)T_1}{u(T_1, T_2)}, \quad D_2 = \frac{(T_2 T_2)T_1 - (T_1 T_2)T_2}{u(T_1, T_2)}. \] (3.20)

The identities

\[ (T_1 D_1) = 0, \quad (T_2 D_2) = 0, \]
\[(D_1 D_2) = -(T_1 T_2), \quad (D_1 D_1) = (T_1 T_1), \quad (D_2 D_2) = (T_2 T_2),\]

and

\[(D_1 T_2) = (T_1 D_2) = u(T_1, T_2)\]

hold.

By the help of (3.20)-(3.21), and in close similarity to (3.6), the co-variation

\[N^{pq}(g; T_1, T_2) := \frac{\partial t_1^p}{\partial T_{2q}} = \frac{\partial t_2^q}{\partial T_{1p}}\]

for the two-vector metric tensor (2.2) can be arrived at.

3.4. \(E_g^{PD}\)-parallelogram law. Let \(t_1, t_2,\) and \(t_3\) be three vectors issued from the same origin “O”, subject to the conditions that the angle between \(t_1\) and \(t_2\) is acute and the vector \(t_3\) is positioned between the vectors \(t_1\) and \(t_2\). Let us denote the end points of the vectors \(t_1, t_2,\) and \(t_3\) as \(X_1, X_2,\) and \(X_3,\) respectively. On joining the points \(X_1\) and \(X_3,\) and also \(X_2\) and \(X_3,\) by means of \(E_g^{PD}\)-geodesics, we get a tetragonal figure, to be denoted as \(P_4.\)

Using Eqs. (1.16), (1.17), and (1.35), we set forth the following couple equations:

\[(t_2 t_2) = (t_1 t_1) + (t_3 t_3) - 2\sqrt{(t_1 t_1) \sqrt{(t_3 t_3)}} \cos \left[ \frac{1}{h} \arccos \frac{(t_1 t_3)}{\sqrt{(t_1 t_1) \sqrt{(t_3 t_3)}}} \right] \quad (4.1)\]

and

\[(t_1 t_1) = (t_3 t_3) + (t_2 t_2) - 2\sqrt{(t_3 t_3) \sqrt{(t_2 t_2)}} \cos \left[ \frac{1}{h} \arccos \frac{(t_2 t_3)}{\sqrt{(t_2 t_2) \sqrt{(t_3 t_3)}}} \right], \quad (4.2)\]

which can also be rewritten in the convenient form

\[\sqrt{(t_3 t_3)} - \frac{(t_2 t_2) - (t_1 t_1)}{\sqrt{(t_3 t_3)}} = 2\sqrt{(t_1 t_1)} \cos \left[ \frac{1}{h} \arccos \frac{(t_1 t_3)}{\sqrt{(t_1 t_1) \sqrt{(t_3 t_3)}}} \right] \quad (4.3)\]

and

\[\sqrt{(t_3 t_3)} - \frac{(t_1 t_1) - (t_2 t_2)}{\sqrt{(t_3 t_3)}} = 2\sqrt{(t_2 t_2)} \cos \left[ \frac{1}{h} \arccos \frac{(t_2 t_3)}{\sqrt{(t_2 t_2) \sqrt{(t_3 t_3)}}} \right]. \quad (4.4)\]

In (4.1), the left-hand part is the squared length of the straight side \(OX_2\) and the right-hand side is the squared length of the geodesic side \(X_1X_3.\) According to (4.2), the lengths of \(OX_1\) and \(X_2X_3\) are equal. Under these conditions, the figure \(P_4\) does attribute the general property of the Euclidean parallelogram that the lengths of opposite sides are equal. In this vein, we introduce the following

**Definition.** Subject to the equations (4.1) and (4.2), the tetragonal figure \(P_4\) is called the \(E_g^{PD}\)-parallelogram, and the vector \(t_3\) is called the \(E_g^{PD}\)-sum vector:

\[t_3 = t_1 \oplus t_2. \quad (4.5)\]
Fig 11: [The Finsleroid-parallelogram law is applied]

Note. The qualitative distinction here from Euclidean patterns is that the sides $X_1X_3$ and $X_2X_2$ of the $\mathcal{P}_4$ are curved lines in general, namely geodesic arcs, which generally cease to be straight under the $\mathcal{E}_g^{PP}$-extension.

Finding the sum vector (4.5) implies solving the set of the equations (4.3) and (4.4). We shall proceed approximately, namely taking

$$\frac{1}{h} = 1 + k$$

(4.6)

and

$$t_3 = t_1 + t_2 + kc(t_1, t_2), \quad k \ll 1.$$  (4.7)

Under these conditions, on inserting (4.6) and (4.7) in (4.3), we find

$$\sqrt{(t_2 + t_1)^2 + k \frac{(t_2 + t_1)c}{\sqrt{(t_2 + t_1)^2}}} = \frac{(t_2 t_2) - (t_1 t_1)}{\sqrt{(t_2 + t_1)^2}} \left(1 - k \frac{(t_2 + t_1)c}{(t_2 + t_1)^2}\right)$$

$$= 2\sqrt{(t_1 t_1)} \cos \left[(1 + k) \arccos \frac{(t_1 t_3)}{\sqrt{(t_1 t_1)} \sqrt{(t_3 t_3)}}\right]$$

$$= 2 \frac{(t_1 t_3)}{\sqrt{(t_3 t_3)}} - 2k \sqrt{(t_1 t_1)} \sqrt{1 - \left(\frac{t_1 (t_2 + t_1)}{\sqrt{(t_1 t_1)} \sqrt{(t_2 + t_1)^2}}\right)^2 \arccos \frac{t_1 (t_2 + t_1)}{\sqrt{(t_1 t_1)} \sqrt{(t_2 + t_1)^2}}}$$

$$= 2k \frac{(t_1 c)}{\sqrt{(t_2 + t_1)^2}} + \frac{t_1 (t_2 + t_1)}{\sqrt{(t_2 + t_1)^2}} \left(1 - k \frac{(t_2 + t_1)c}{(t_2 + t_1)^2}\right)$$

$$- 2k \sqrt{(t_1 t_1)} \sqrt{1 - \left(\frac{t_1 (t_2 + t_1)}{\sqrt{(t_1 t_1)} \sqrt{(t_2 + t_1)^2}}\right)^2 \arccos \frac{t_1 (t_2 + t_1)}{\sqrt{(t_1 t_1)} \sqrt{(t_2 + t_1)^2}}}.$$
which entails
\[
\frac{(t_2 + t_1)c}{\sqrt{(t_2 + t_1)^2}} + \frac{(t_2t_2) - (t_1t_1)}{(t_2 + t_1)^2} \frac{(t_2 + t_1)c}{\sqrt{(t_2 + t_1)^2}}
\]
\[
= 2 \frac{t_1c}{\sqrt{(t_2 + t_1)^2}} - 2 \frac{t_1(t_2 + t_1)}{(t_2 + t_1)^2} \frac{(t_2 + t_1)c}{\sqrt{(t_2 + t_1)^2}}
\]
\[
-2\sqrt{(t_1t_1)} \left[ 1 - \left( \frac{t_1(t_2 + t_1)}{\sqrt{(t_1t_1)\sqrt{(t_2 + t_1)^2}}} \right)^2 \arccos \frac{t_1(t_2 + t_1)}{\sqrt{(t_1t_1)\sqrt{(t_2 + t_1)^2}}} \right].
\]
Therefore we obtain
\[
t_2c = -u(t_1, t_2) \arccos \frac{t_1(t_2 + t_1)}{\sqrt{(t_1t_1)\sqrt{(t_2 + t_1)^2}}},
\]
where \(u(t_1, t_2)\) is the function (1.20). Similarly, from (4.4) it follows that
\[
t_1c = -u(t_1, t_2) \arccos \frac{t_2(t_2 + t_1)}{\sqrt{(t_2t_2)\sqrt{(t_2 + t_1)^2}}},
\]
If we use now the symmetrized expansion
\[
c = m(t_1, t_2)t_1 + n(t_1, t_2)t_2,
\]
then we find
\[
m(t_1, t_2) =
\]
\[
\frac{1}{u(t_1, t_2)} \left( (t_1t_2) \arccos \frac{t_1(t_2 + t_1)}{\sqrt{(t_1t_1)\sqrt{(t_2 + t_1)^2}}} - (t_2t_2) \arccos \frac{t_2(t_2 + t_1)}{\sqrt{(t_2t_2)\sqrt{(t_2 + t_1)^2}}} \right)
\]
and
\[
n(t_1, t_2) =
\]
\[
\frac{1}{u(t_1, t_2)} \left( (t_1t_2) \arccos \frac{t_2(t_2 + t_1)}{\sqrt{(t_2t_2)\sqrt{(t_2 + t_1)^2}}} - (t_1t_1) \arccos \frac{t_1(t_2 + t_1)}{\sqrt{(t_1t_1)\sqrt{(t_2 + t_1)^2}}} \right).
\]
Since
\[
m(t_1, t_2) = n(t_2, t_1),
\]
we just deduce the approximate solution
\[
t_1 \oplus t_2 \approx t_1 + t_2 + \left( \frac{1}{h} - 1 \right) \left( m(t_1, t_2)t_1 + m(t_2, t_1)t_2 \right), \quad \frac{1}{h} - 1 \ll 1.
\]
Alternatively, the solution $t_2 = t_2(t_1, t_3)$ to the set of equations (4.1)-(4.2) can naturally be treated as the $\mathcal{E}^{PD}$-difference of vectors $t_3$ and $t_1$:

$$t_2 = t_3 \ominus t_1.$$  

Again, restricting ourselves to the approximation, from (4.1)-(4.2) we obtain

$$t_3 \ominus t_1 \approx t_3 - t_1 + \left(\frac{1}{h} - 1\right)s(t_1, t_3), \quad \frac{1}{h} - 1 \ll 1,$$

with

$$s(t_1, t_3) = \frac{1}{u(t_1, t_3)}\left\{ \begin{align*}
(t_1 t_1) \arccos \frac{(t_1 t_3)}{\sqrt{(t_1 t_1)} \sqrt{(t_3 t_3)}} \\
-(t_3 - t_1, t_1) \arccos \frac{(t_3 - t_1, t_3)}{\sqrt{(t_3 - t_1, t_3 - t_1)} \sqrt{(t_3 t_3)}} (t_3 - t_1) \\
+ \left[(t_3 - t_1, t_3 - t_1) \arccos \frac{(t_3 - t_1, t_3)}{\sqrt{(t_3 - t_1, t_3 - t_1)} \sqrt{(t_3 t_3)}} \right] t_1 \right\}.$$

Here it is useful to note that

$$(t_3 - t_1, s) = u(t_1, t_3) \arccos \frac{(t_1 t_3)}{\sqrt{(t_1 t_1)} \sqrt{(t_3 t_3)}},$$

$$(t_1, s) = u(t_1, t_3) \arccos \frac{(t_3 - t_1, t_3)}{\sqrt{(t_3 - t_1, t_3 - t_1)} \sqrt{(t_3 t_3)}},$$

and

$$u(t_3 - t_1, t_3) = u(t_1, t_3).$$

The problem of finding the relevant sum vector $t_1 \oplus t_2$ and difference vector $t_3 \ominus t_2$ in general exact forms is open and seems to be difficult.

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