PARTITION FUNCTIONS, DUALITY AND THE TUBE METRIC *

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ABSTRACT

The partition function of type IIA and B strings on $R^6 \times K^3$, in the $T^4/Z_2$ orbifold limit, is explicitly computed as a modular invariant sum over spin structures required by perturbative unitarity in order to extend the analysis to include type II strings on $R^6 \times W^4$, where $W^4$ is associated with the tube metric conformal field theory, given by the degrees of freedom transverse to the Neveu-Schwarz fivebrane solution. This generates partition functions and perturbative spectra of string theories in six space-time dimensions, associated with the modular invariants of the level $k$ affine $SU(2)$ Kac-Moody algebra. These theories provide a conformal field theory (i.e. perturbative) probe of non-perturbative (fivebrane) vacua. We contrast them with theories whose $N = (4,4)$ sigma-model action contains $n_H = k + 2$ hypermultiplets as well as vector supermultiplets, and where $k$ is the level just mentioned. In Appendix B we also give a $D = 6$, $N = (1,1)$ ‘free fermion’ string model which has a different moduli space of vacua from the 81 parameter space relevant to the above examples.

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1. Introduction

The non-perturbative formulation of string theory is motivated by dualities, but ultimately it should provide the identity of a set of degrees of freedom on which the theory is consistently defined.

In perturbative string theory, the GSO projections lead to modular invariance of the one-loop amplitudes. We further recall\[1−3\] that in string perturbation theory there are requirements of space-time factorization for vertices and one-loop modular invariance which guarantee higher-loop modular invariance. These requirements are related to the conservation at vertices of quantum numbers associated with the projections, and are satisfied by models that are unitary on a set of states picked out from the whole Fock space by a set of projections defined by number operators. In perturbative string theory, the projections in the various sectors are a primary concept.

Currently, the backbone of non-perturbative string theory is a web of various string perturbative expansions, described by dual pairs of theories\[4,5\] and more generally by matrix theory\[6\]. To construct a complete non-perturbative formulation, the guiding principle which replaces perturbative unitarity is that strong coupling (non-perturbative) extensions of existing perturbative string models appear to be well defined in terms of other perturbative models. In particular, a dual pair of string theories have the same moduli space of vacua, which is already present in the low energy (field theory) description of either theory. In this way the moduli spaces (different ones correspond to different compactifications) become a primary concept.

In this paper, we analyze several superstring vacua in some detail in order to extract from perturbative formulations relevant non-perturbative information which may be useful in shaping the appropriate set of states on which the theory is defined.

In sect.2, we consider the type IIA and IIB superstrings on $R^6 \times K3$ and give the partition function explicitly as sum over spin structures for its orbifold limit $R^6 \times T^4/\mathbb{Z}_2$. We identify the T-duality group for this restricted class of K3 compactifications, and show that it is an invariance of the partition function, although its status as an invariance of the full non-perturbative spectrum of this theory remains conjectural. The generalized GSO projections that define these models are then used in sect.3 to derive the massless spectra, which are seen to correspond to $D = 6$ theories with space-time supersymmetries $N = (1, 1)$ and $N = (2, 0)$ respectively that occur for generic $R^6 \times K3$ compactifications. In sect.4 analogous formulae are given for the type II superstrings on $R^6 \times W4/\mathbb{Z}_2$, where W4 is the ‘tube metric’ conformal field theory (cft) associated with the transverse degrees of freedom of the type II NS fivebrane\[7−10\]. In sect.5, we recall the relationships\[11−16\] among this cft, points in the moduli space corresponding to non-abelian (enhanced) gauge symmetry, the $D = 2, N = (4, 4)$ supersymmetric theories with vector and $n_H = k + 2$ hypermultiplets, level $k \tilde{SU}(2)$ affine algebras, and the theory of $n_H$ coincident NS fivebranes. We discuss the norms of both continuous and discrete states in the ‘tube metric’ cft, and a connection to exact results on Liouville cft\[17\]. Conclusions and comments are found in sect.6 which discusses the presence of enhanced gauge symmetry and its
incorporation in type II variables. We also give a type II free fermion string model with conventional (NS-NS) non-abelian symmetry (for the gauge group $SU(2)^4$) which has a different moduli space of vacua from the above examples. This model shows that all known compactifications of string theory with $N = (1, 1)$ supersymmetry in $D = 6$ are not on the same moduli space of vacua.

2. Partition function

We first compute explicitly type II strings on $\mathbb{R}^6 \times T^4/\mathbb{Z}_2$, where $T^4/\mathbb{Z}_2$ is the $\mathbb{Z}_2$ orbifold limit of $K3$. In the light-cone description, the left and right-moving modes are each taken to be described by 8 bosonic and 8 fermionic worldsheet (primary) fields: $\tilde{A}_s^i, A_s^i, \tilde{\psi}_r^i, \psi_r^i; A_s^i, \tilde{A}_s^i, \tilde{\psi}_r^i, \psi_r^i; 1 \leq i, \leq 4$ and $5 \leq I, \leq 8$, where the superscript $i$ refers to the transverse spatial degrees of freedom, the superscript $I$ to the internal ones, and the subscripts $s,r$ each correspond to either integer or half-integer modding depending on the sector. The partition function or one loop contribution to the vacuum to vacuum amplitude in $D$ space-time dimensions is

$$
\Lambda = -\frac{1}{4\pi(\alpha')^3} \int_F d^2\tau (\text{Im} \tau)^{-2-(D-2)/2} |f(\omega)|^{-2(D-2)} \Lambda_f
$$

(2.1)

where $\Lambda_f$ is the partition function for the fermionic and internal bosonic degrees of freedom,

$$
\Lambda_f = \sum_{\alpha \in \Omega} \delta_\alpha \text{tr}_\alpha \{ \tilde{\omega}^{-1} L_0 - \frac{1}{2} \omega L_0 - \frac{1}{2} \} \prod_{\beta \in \Omega} P_{\alpha,\beta},
$$

(2.2)

i.e. the spectrum of a theory will consist of a set of sectors $\Omega$, characterized by the modding of the internal bosons ($s \in \mathbb{Z}$, untwisted), ($s \in \mathbb{Z} + \frac{1}{2}$, twisted), and of the fermions ($r \in \mathbb{Z}$, untwisted Ramond (R) or twisted Neveu-Schwarz (NS)), and ($r \in \mathbb{Z} + \frac{1}{2}$, untwisted NS or twisted R). The quantities $\delta_\alpha$ and the projection operators $P_{\alpha,\beta}$ are discussed below. The integration region $F = \{ \tau : |\tau| > 1, |\text{Re}\tau| < \frac{1}{2} \}$ is a fundamental region for the modular group that is generated by $\tau \rightarrow \tau + 1, \tau \rightarrow -\frac{1}{\tau}$; and $\tilde{L}_0, L_0$ refer to left, right movers.

To define the orbifold choose a complex basis for the internal fermions, for example for the left-movers: $f_1 = \frac{1}{4\sqrt{2}}(h^5 + ih^6), \tilde{f}_1 = \frac{1}{\sqrt{2}}(h^5 - ih^6), f_2 = \frac{1}{\sqrt{2}}(h^7 + ih^8), \tilde{f}_2 = \frac{1}{4\sqrt{2}}(h^7 - ih^8)$. Then the $\mathbb{Z}_2$ transformation $\theta$ acting on the internal fermions in terms of the number operator $F = \sum_{i=1,2;r} : f_i f_{\bar{r}} :$ (where : $f_0 \tilde{f}_0 := - : \tilde{f}_0 f_0 :$), so that $\theta = (-1)^F$; and similarly $\theta$ acts on the internal bosons by $\theta A_s I \theta^{-1} = -A_s I$. Oscillators with space-time indices are invariant under $\theta$, and $D = 6$.

A sector $\alpha$ is labelled by a twelve-dimensional vector whose components are 0 for NS and 1 for R:

$$
\rho_\alpha = (\bar{\rho}_1, \ldots, \bar{\rho}_4; \rho'_1, \bar{\rho}_2; \rho_1, \ldots, \rho_4; \rho'_1, \rho'_2)
$$

(2.3)

This vector corresponds to boundary conditions of left- and right- modes separately described by 4 real and 2 complex fermions.
The set of states on which the theory is unitary is specified by states that survive
projections defined by number operators which generalize the GSO projection. The pro-
jections are defined by requiring the parity of the number operators, \( N_\beta \) defined in (2.9),
to take on definite values \( \epsilon(\alpha, \beta) \) on any state in the sector \( \alpha \), i.e.
\[
(-1)^{N_\beta} |\alpha = \epsilon(\alpha, \beta),
\]
where each \( \epsilon(\alpha, \beta) \) is either \( \pm 1 \). The (perturbative) spectrum of a model is specified by
a set of sectors \( \Omega \), together with a set \( \{-1)^{N_\beta} : \beta \in \Omega \} \) of parity operators, and their
values \( \epsilon(\alpha, \beta) \) on the sectors \( \alpha \in \Omega \).

Eq. (2.2) can be expressed as a sum over spin structures\(^1\):
\[
\Lambda_f = \frac{1}{2K+1} \sum_{\alpha \in \Omega} \sum_{\beta \in \Omega} \delta_\alpha \epsilon(\alpha, \beta) \text{tr}_\alpha\{\bar{\omega}^{L_0-\frac{1}{2}} \omega^{L_0-\frac{1}{2}} (-1)^{N_\beta}\}
\]
(2.5)
where \( K \) is the number of basis vectors which generate \( \Omega \). We denote the trace in eq.(2.5)
by \{\alpha, \beta\}, so that, without the factor of \( 2^{-K-1} \), the sum is
\[
\sum_{\alpha, \beta} \delta_\alpha \epsilon(\alpha, \beta) \{\alpha, \beta\},
\]
(2.6)
where
\[
\{\alpha, \beta\} = \text{tr}_\alpha\{\bar{\omega}^{L_0-\frac{1}{2}} \omega^{L_0-\frac{1}{2}} (-1)^{N_\beta}\}
\]
\[
= |\omega|^{-1} |f(\omega)|^{-20} \prod_{i=1}^{4} \left( \bar{\Theta} \left[ \begin{array}{c} \rho^i_\alpha \\ \mu^i_\beta \end{array} \right] (0|\tau) \right) \prod_{i=5}^{6} \left( \bar{\Theta} \left[ \begin{array}{c} \rho^i_\alpha \\ \mu^i_\beta \end{array} \right] (0|\tau) \right)
\]
\[
\times \prod_{j=1}^{4} \left( \Theta \left[ \begin{array}{c} \rho^j_\beta \\ \mu^j_\beta \end{array} \right] (0|\tau) \right) \prod_{j=5}^{6} \left( \Theta \left[ \begin{array}{c} \rho^j_\beta \\ \mu^j_\beta \end{array} \right] (0|\tau) \right) \times \text{internal bosons},
\]
(2.7)
and \( (\bar{\rho}^i_\alpha, \rho^i_\beta) \) and \( (\bar{\mu}^i_\beta, \mu^i_\beta) \) are the the twelve-component vectors describing the sectors \( \alpha \)
and \( \beta \) respectively, i.e. the components are 0 for NS and 1 for R [see for example (2.8)].
\( \Theta \left[ \begin{array}{c} \rho^i \\ \mu^i \end{array} \right] (0|\tau) \) and \( f(\omega) \) are given by (2.14), and \( \delta_\alpha = \delta^{L}_\alpha \delta^{R}_\alpha \) where \( \delta_\alpha = 1 \) if the states of
the sector \( \alpha \) are space-time bosons and \( \delta_\alpha = -1 \) if the states are space-time fermions.
A consistent (perturbative) string theory is such that under modular transformations the
integrand of (2.1) is invariant.

The type II string on \( \mathbb{R}^6 \times \mathbb{T}^4/\mathbb{Z}_2 \) has eight sectors, whose fermion boundary condition
vectors (2.3) are given by
\[
\rho_{b_2} = (1^4, 1^2; 0^4, 0^2) \quad \rho_{b_0b_1b_2} = (1^4, 0^2; 0^4, 1^2)
\]
\[
\rho_{b_0b_2} = (0^4, 0^2; 1^4, 1^2) \quad \rho_{b_1b_2} = (0^4, 1^2; 1^4, 0^2)
\]
\[
\rho_{b_0b_1} = (0^4, 1^2; 0^4, 1^2) \quad \rho_{b_2} = (1^4, 0^2; 1^4, 0^2)
\]
\[
\rho_{b_0} = (1^4, 1^2; 1^4, 1^2) \quad \rho_{\phi} = (0^4, 0^2; 0^4, 0^2),
\]
(2.8)
where \( \{ \phi, b_0, b_0b_2, b_2 \} \) are the sectors that have untwisted bosons, and the \( \mathbb{Z}_2 \) twisted sectors are written as \( \{ b_0b_1, b_1b_2, b_0b_1b_2 \} \). For this theory, the eigenvalues \( \epsilon(\alpha, \beta) \) of the parity operators are given in Table 1, where \( \lambda, \rho, \mu, \nu \) take values \( \pm 1 \), and different choices of \( \lambda, \rho, \mu, \nu \) do not change the theory. Table 1 is derived by requiring modular invariance for the part of \( \Lambda \) given by

\[
\sum_{\alpha, \beta \in \{ \phi, b_0, b_0b_2, b_2 \}} \delta_\alpha \epsilon(\alpha, \beta) \mathrm{tr}_\alpha \{ \tilde{\omega} L_0 - \frac{1}{2} \omega L_0 - \frac{1}{2} (-1)^{N_\beta} \},
\]

and then computing the remaining values of \( \epsilon(\alpha, \beta) \) using \( \epsilon(\alpha, \beta \gamma) = \epsilon(\alpha, \beta) \epsilon(\alpha, \gamma) \), which follows from (2.4).

| \( \alpha \) | \( \epsilon(\alpha, \beta) \) |
|---|---|
| \( \emptyset \) | 0 1 1 -1 1 -1 -1 -1 |
| \( b_0 \) | 1 \( \lambda \) \( \rho \) \( \mu \) \( 1 \) \( 1 \) \( -1 \) \( -1 \) \( -1 \) |
| \( b_1 \) | 1 \( \rho \) \( \rho \) \( \nu \) \( \lambda \rho \) \( 1 \) \( 1 \) \( \rho \nu \) \( \nu \) |
| \( b_2 \) | 1 -\( \mu \) -\( \nu \) \( \mu \) -\( \mu \nu \) -1 -1 \( \mu \nu \) -\( \nu \) |
| \( b_0b_1 \) | 1 \( \lambda \rho \) 1 -\( \mu \nu \) -\( \mu \nu \) -\( \lambda \rho \mu \nu \) -\( \lambda \rho \mu \nu \) -\( \lambda \rho \mu \nu \) -\( \lambda \rho \mu \nu \) |
| \( b_0b_2 \) | 1 -\( \lambda \mu \) \( \rho \nu \) -1 -\( \lambda \rho \mu \nu \) -\( \lambda \mu \) -\( \rho \nu \) -\( \rho \nu \) -\( \lambda \rho \mu \nu \) |
| \( b_1b_2 \) | 1 -\( \rho \mu \) \( \rho \nu \) -\( \mu \nu \) -\( \mu \nu \) -\( \rho \nu \) -\( \mu \rho \) 1 |
| \( b_0b_1b_2 \) | 1 -\( \lambda \rho \mu \) \( \nu \) \( \nu \) -\( \lambda \rho \mu \nu \) -\( \lambda \rho \mu \nu \) 1 -\( \lambda \rho \mu \) |

Table 1

In the fermionic picture, we define the parity of the number operator \( N_\beta \) acting on the sector \( \alpha \) by

\[
(-1)^{N_\beta} \big|_\alpha = (-1)^{\rho_\beta \cdot F} \big|_\alpha ,
\]

where \( F \) is a vector whose components are the operators \( F_j = \sum_r : f^j_r \tilde{f}^j_r : \) for complex fermions and \( \sum_r \psi^j_r \psi^j_r \) for real fermions, and \( r \) is modded according to the boundary condition of the \( j^{th} \) fermion in the sector \( \alpha \).

\[
\rho_\beta \cdot F = \sum_{j=1}^4 \rho_j \tilde{F}_j + \sum_{j=1}^2 \rho'_j \tilde{F}'_j + \sum_{j=1}^4 \rho_j F_j + \sum_{j=1}^2 \rho'_j F'_j 
\]

and the sums \( \tilde{F}_j \) and \( F_j \) distinguish left and right movers, but the pair \( \tilde{f} \) and \( f \) denotes a complex fermion which is either wholly left moving or right moving. In general, the projection operators are defined by

\[
P_{\alpha, \beta} = \frac{1}{2} \{ 1 + \epsilon(\alpha, \beta)(-1)^{N_\beta} \}, \quad \beta \in \Omega.
\]

Since \( (-1)^{N_\beta \gamma} = (-1)^{N_\beta}(-1)^{N_\gamma} \), it follows that

\[
\prod_{\beta \in \Omega} P_{\alpha, \beta} = \prod_{i=0}^{K+1} P_{\alpha, \beta_i}
\]
where, for this theory $K = 2$ and $\beta_0 = b_2, \beta_1 = b_0 b_2, \beta_2 = b_0 b_1$. The conventional GSO projections involve only fermions:

$$P_{\alpha, b_2} = \frac{1}{2} \{ 1 + \epsilon(\alpha, b_2)(-1)^{\rho b_2} \} \tag{2.13a}$$
$$P_{\alpha, b_0 b_2} = \frac{1}{2} \{ 1 + \epsilon(\alpha, b_0 b_2)(-1)^{\rho \rho_0 b_2} \} \tag{2.13b}$$

while the $\mathbb{Z}_2$ projection is incorporated in

$$P_{\alpha, b_0 b_1} = \frac{1}{2} \{ 1 + \epsilon(\alpha, b_0 b_1) \theta \}$$
$$\theta = (-1)^{\rho \rho_0 b_1} [ N = \sum_{r>0} \frac{1}{r} (A_r^I + A_r^I A_r^I) ] \tag{2.13c}$$

The functions in (2.7) have $\omega = e^{2 \pi i \tau}$ and

$$\Theta \left[ \rho \mu \right] (\nu | \tau) = \sum_{n \in \mathbb{Z}} e^{i \pi \tau (n + \frac{\mu}{2})^2} e^{i 2 \pi (n + \frac{\mu}{2}) (\nu + \frac{\mu}{2})} = \omega^{\frac{1}{2} \tau^4} f = \omega^{\frac{1}{2} \tau^4} \prod_{n=1}^{\infty} (1 - \omega^n)$$

The partition function for type II strings on $\mathbb{R}^6 \times \mathbb{T}^4 / \mathbb{Z}_2$ is thus specified by eq.’s (2.8), (2.13) and Table 1. Using this data in (2.1,2,6), we find that many of the spin structures give zero contribution; but also that modular invariance of the integrand of (2.1) can be checked explicitly using the $\epsilon(\alpha, \beta)$ eigenvalues in Table 1 and the modular transformation properties of the functions in (2.14). This assures that one-loop n-point functions as well as the 0-point function are modular invariant.

Collecting the contributions from the different (non-zero) spin structures, we have for type II strings on $\mathbb{R}^6 \times \mathbb{T}^4 / \mathbb{Z}_2$ that

$$\Lambda = - \frac{1}{4 \pi (\alpha')^3} \int \mathcal{D} \tau (\text{Im} \tau)^{-4} |\eta(\tau)|^{-8} \Lambda_f' \tag{2.15a}$$

$$\Lambda_f' = \frac{\theta_4^4 \tau_4}{8} \frac{\Gamma_{4,4}(\tau, \tau)}{|\eta(\tau)|^8} |\eta(\tau)|^{-8} (\bar{\theta}_3^4 - \bar{\theta}_4^4 - \bar{\theta}_2^4) (\theta_3^4 - \theta_4^4 - \theta_2^4)$$

$$+ \frac{2^4 |\eta(\tau)|^4}{\theta_2^2 \theta_4^2} |\eta(\tau)|^{-8} \theta_3^2 \theta_4^2 \theta_3^2 \theta_4^2 (1 - 1 - 1 + 1)$$

$$+ \frac{2^4 |\eta(\tau)|^4}{\theta_2^2 \theta_4^2} |\eta(\tau)|^{-8} \theta_3^2 \theta_4^2 \theta_3^2 \theta_4^2 (2 - 2)$$

$$+ \frac{2^4 |\eta(\tau)|^4}{\theta_3^2 \theta_4^2} |\eta(\tau)|^{-8} \theta_4^2 \theta_3^2 \theta_4^2 \theta_4^2 (2 - 2) \right]. \tag{2.15b}$$
In (2.15b), the factors \((1 - 1 - 1 + 1)\) are all space-time boson contributions, while in lines 3 and 4 each the factors \((2 - 2)\) contribute 2 from space-time bosons and \((-2)\) from space-time fermions. In line 1, the lattice theta function \(\theta_{\Gamma, 4, 4}(\bar{\tau}, \tau) = \sum_{p_L, p_R \in \Gamma_{4, 4}} \omega^{p_L^2} \omega^{2p_R^2}\) is defined for any even, self-dual eight-dimensional Lorentzian lattice \(\Gamma_{4, 4}\). Using the Poisson resummation formula, one can show the quantity \(\frac{\theta_{\Gamma, 4, 4}(\bar{\tau}, \tau)}{|\eta(\tau)|^8}\) is modular invariant. There is a 16-parameter family of such lattices generated by an \(O(4, 4)\) transformation of the \(\oplus 4P_2\) lattice, where \(P_2\) is a two-dimensional lattice with signature \((+ -)\) defined by basis vectors \(\alpha_1^I = \frac{1}{\sqrt{2}}(1, 1)\); \(\alpha_2^I = \frac{1}{\sqrt{2}}(-1, 1)\) with metric \(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\). Since the \(D = 6\) mass squared is invariant under the \(O(4) \times O(4)\) subgroup of the non-compact group \(O(4, 4)\), the number of different parameter families of string models is given by the dimension of the coset \(O(4, 4) / O(4) \times O(4)\) which is 16. The discrete group \(O(4, 4; \mathbb{Z})\) leaves invariant the lattice theta function \(\theta_{\Gamma, 4, 4}(\bar{\tau}, \tau)\) for any given member of the 16-parameter family of lattices. Therefore the partition function given in (2.15) of the type II string on \(R^6 \times T^4 / \mathbb{Z}_2\) has an explicit T-duality symmetry group

\[ G_T = O(4, 4; \mathbb{Z}) \] (2.16)

which is an invariance of the perturbative spectrum. In (2.15) we use \(\theta_3 = \Theta [0, 0] (0|\tau); \theta_4 = \Theta [0, 1] (0|\tau); \theta_2 = \Theta [1, 0] (0|\tau); \Theta [1, 1] (0|\tau) = 0\). The partition function for type II strings on \(R^6 \times T^4\) is given in Appendix A for later comparisons.

3. Massless and lowest lying massive perturbative spectra

In this section we work out the massless and lowest lying massive spectra in these theories, using the projection operators (2.13), rather than directly using facts about the cohomology and moduli space of \(K3\). Our procedure insures that the spectrum at a given mass level agrees with the coefficient of a suitable power of \(\bar{\omega}\omega\) in (2.15). The partition function (2.15) describes a precise set of states on which the theory is known to satisfy (perturbative) unitarity.

String theories that include both perturbative and non-perturbative states (such as type IIA on \(R^6 \times K3\) at values of the \(K3\) moduli corresponding to enhanced non-abelian massless gauge bosons) are not yet sufficiently defined to allow identification of a consistent set of states analogous to the perturbative situation. Nonetheless it is conjectured that such a set of states exists on which unitarity for both perturbative and non-perturbative states, and S-duality could be proved. In this framework, the massless spectrum of the target space theory is frequently computed either via a field theory compactification of type IIA supergravity on \(R^6 \times K3\), using the Betti numbers of \(K3\), or from the moduli space of the 2-dimensional \(N = 4\) superconformal non-linear sigma model with \(K3\) target space, as in [16,18]. Compactifications of the target space field theory in general yield a ‘Kaluza-Klein’ tower of massless and massive states. The massless states derived in this way, together with massless solitons which occur at special moduli values, are assumed to be the massless spectrum of a generalized non-perturbative/perturbative description. Although these techniques do not provide yet a specification of the massive states which
together with this massless spectrum would serve as a consistent set of states on which to define the theory.

On the other hand, computation of the spectrum using the projection operators yields both a ‘Kaluza-Klein’ tower of states from the compactification, and a ‘Regge’ tower of states already present in the $D = 10$ IIA string. The latter is given in the Appendix (A3). This calculation specifies a consistent set of states for the subset of theories parameterized by $\theta \Gamma_{4,4}(\tau, \tau)$. That is to say, given the states described in (2.15), there is no need to add non-perturbative information for the consistency of the theory. Nonetheless, the appearance of BPS soliton solutions of the target space classical field theory, may be a motivation to modify the perturbative description to encompass a bigger theory, a modification which might be essential at larger coupling. We conjecture that a complete definition of the bigger theory will involve an analog of the perturbative GSO projections.

The massless spectrum of the Type IIA superstring on $\mathbb{R}^6 \times T^4 / \mathbb{Z}_2$ is given in terms of representations of the $D = 6$ lightcone little group $\text{Spin}(4) \cong SU(2) \times SU(2)$ which form $D = 6, N = (1, 1)$ spacetime supersymmetry multiplets. The supergravity multiplet is

$$(3, 3) + (3, 1) + (1, 3) + 4(2, 2) + (1, 1) + 2(3, 2) + 2(2, 3) + 2(2, 1) + 2(1, 2). \quad (3.1a)$$

It couples to 20 vector supermultiplets with spin content:

$$(2, 2) + 4(1, 1) + 2(2, 1) + 2(1, 2). \quad (3.1b)$$

We derive (3.1) in the orbifold model of section 2, using the projections (2.13) as follows. The supergravity multiplet and 4 of the vector multiplets come from the untwisted sector, and 16 vector multiplets come from the twisted sector. The physical states satisfy

$$P_{\alpha, b_2} P_{\alpha, b_0 b_2} P_{\alpha, b_0 b_1} |\psi\rangle = |\psi\rangle \quad (3.2)$$

where $P_{\alpha, b_2}, P_{\alpha, b_0 b_2}, P_{\alpha, b_0 b_1}$ are defined in (2.13), via (2.8) and Table 1, and $|\psi\rangle \in \alpha$. Writing the four sums in (2.10) as

$$\rho_\beta \cdot F = \sum_{j=1}^{4} \tilde{\rho}_j \tilde{F}_j + \sum_{j=1}^{2} \tilde{\rho}_j \tilde{F}'_j + \sum_{j=1}^{4} \rho_j F_j + \sum_{j=1}^{2} \rho_j F'_j$$

$$\equiv \tilde{F}_1 + \tilde{F}_2 + \tilde{F}_3 + \tilde{F}_4,$$

we find in the untwisted NS-NS sector ($\alpha = \phi$) that (3.2) corresponds to

$$\frac{1}{8}[1 - (-1)^{\rho_{b_2} \cdot (\tilde{F}_1 + \tilde{F}_2)}][1 - (-1)^{\rho_{b_2} \cdot (\tilde{F}_3 + \tilde{F}_4)}][1 + (-1)^{\rho_{b_2} \cdot (\tilde{F}_2 + \tilde{F}_4)} + N] |\psi\rangle = |\psi\rangle \quad (3.3)$$

So for massless states, either $\tilde{F}_1 = \tilde{F}_3 = 0$; $\tilde{F}_2 = \tilde{F}_4 = 1$ or $\tilde{F}_1 = \tilde{F}_3 = 1$; $\tilde{F}_2 = \tilde{F}_4 = 0$, corresponding to $\tilde{\psi}^J_1 \times \psi^J_1 |0\rangle$ with spin content of 16 scalars and to $\tilde{\psi}^i_1 \times \psi^j_{-\frac{1}{2}} |0\rangle$ with spin content $(3, 3) + (1, 3) + (3, 1) + (1, 1)$. In the untwisted RR sector $\alpha = b_0$, the projections in (3.2) on massless states require $\tilde{F}_1 + \tilde{F}_2 =$even; $\tilde{F}_3 + \tilde{F}_4 =$even, $\tilde{F}_2 + \tilde{F}_4 =$even.
The Ramond ground states in $D = 6$ for the type IIA superstring on $\mathbf{R}^6 \times \mathbf{T}^4$ or $\mathbf{R}^6 \times \mathbf{T}^4 / \mathbb{Z}_2$ corresponds to the spin content:

\[
\begin{align*}
\bar{F}_1 &= \text{even} \quad (2, 1)_{\text{Left}} \\
F_1 &= \text{odd} \quad (1, 2)_{\text{Left}} \\
\bar{F}_3 &= \text{even} \quad (1, 2)_{\text{Right}} \\
F_3 &= \text{odd} \quad (2, 1)_{\text{Right}}.
\end{align*}
\]

Using (3.5), we find the massless states in the untwisted RR sector have spin content $8(2, 2)$. These eight vectors are from $2(2, 1)_{\text{Left}} \times 2(1, 2)_{\text{Right}}$ and $2(1, 2)_{\text{Left}} \times 2(2, 1)_{\text{Right}}$. Similar arguments show the R-NS sector contains massless states $2(2, 3) + 10(2, 1)$ given by $2(2, 1)_{\text{Left}} \times \psi^{-\frac{1}{2}}_{-\frac{1}{2}} |0\rangle$ and $2(1, 2)_{\text{Left}} \times \psi_{\frac{1}{2}}^{-\frac{1}{2}} |0\rangle$; the NS-R sector contains $2(3, 2) + 10(1, 2)$; and the four twisted sectors contain 16 massless vector supermultiplets given by $4|16\rangle$ and $2(2, 1)_{\text{Left}} \times (1, 2)_{\text{Right}} \times |16\rangle$, $2 \times (1, 2)_{\text{Right}} \times |16\rangle$, $|2, 1\rangle_{\text{Left}} \times 2 \times |16\rangle$, where $|16\rangle$ is the degeneracy of the ground state of the twisted bosonic operators $\tilde{A}_s^I, A_s^J$.

The IIA superstring on $\mathbf{R}^6 \times K^3$ is conjectured to be S-dual to the heterotic string on $\mathbf{R}^6 \times \mathbf{T}^4$[4,5,19]. This requires both theories to have the same massless spectrum and the same moduli space. At points of enhanced symmetry in the moduli space, the massless spectrum of both theories will be the supergravity multiplet of (3.1a) together with the vector supermultiplet (3.1b) in the adjoint representation of a rank 20 non-abelian group. For enhanced symmetry, the the additional vector supermultiplets imply additional massless scalars, but these have quartic interactions (as required by the gauge and supersymmetry) so their vevs are not new moduli.[5] At generic points in the moduli space, however, the massless spectrum is given in (3.1); and the 81 scalar fields can acquire vacuum expectation values (vevs) which take values in the moduli space $\mathcal{M} \times \mathbf{R}$, where the vev of the dilaton in (3.1a) lives on $\mathbf{R}$ and the vevs of the 80 scalars in the 20 $U(1)$ vector multiplets (3.1b) parameterize

\[
\mathcal{M} = 0(4, 20; \mathbf{Z}) \setminus \frac{0(4, 20)}{0(4) \times 0(20)}.
\]

For the orbifold compactification of IIA on $\mathbf{R}^6 \times \mathbf{T}^4 / \mathbb{Z}_2$, most of these values have been fixed, since from (2.15) the partition function depends only on 16 parameters via $\theta_{\Gamma_{4,4}}(\bar{r}, \tau)$. No choice of these 16 parameters will adjust the $\Gamma_{4,4}$ lattice to enhance the gauge symmetry, i.e. no left or right internal momentum states can appear at the massless level, since for such states the projections would be $\bar{F}_1 = \bar{F}_2 = 0$ (in sector $\phi$ for example), and this does not satisfy (3.4). This reflects the fact[16] that the orbifold which leads to a finite string perturbation theory in (2.15) corresponds to points in the moduli space which do not overlap with points of enhanced symmetry.

All 24 of the $U(1)$ vectors in (3.1), i.e. the (2,2)’s of Spin (4), are from the RR sectors $b_0$ and $b_1$, so from conventional arguments[20] no perturbative states carry any of the 24 $U(1)$ electric charges. Also, the 24 magnetic charges are only carried by non-perturbative states.
In addition to (3.1), we compute the lowest lying massive spectrum in perturbation theory. Expanding (2.15), we see there are 4·(64)² at mass level \( \frac{1}{2}m_L^2 = \frac{1}{2}m_R^2 = \frac{1}{2} \). These occur in the four twisted sectors, via arguments similar to (3.4), and have Spin (4) content equivalent to 64 copies of the \( D = 6, N = (2,2) \) supergravity multiplet given in (3.10). Here the states are massive, with little group Spin (5), and they can be grouped into combinations of \( D = 6 \) massive Spin (5) representations (underlined):

\[
64 \left[ 14 + 5 \begin{array}{c} 10 \\ \end{array} + 10 \begin{array}{c} 5 \\ \end{array} + 14 \begin{array}{c} 1 \\ \end{array} + 4 \begin{array}{c} 16 \\ \end{array} + 16 \begin{array}{c} 4 \\ \end{array} \right].
\]

This is 64 unshortened massive supermultiplets, since they do not have the same spectrum of Spin(4) states as (3.1a). As an example, in sector \( b_0b_1 \), the states surviving the projections are \( \tilde{\psi}^{\dagger} \begin{array}{c} -1/2 \\ \end{array} |2\rangle \times \psi^{\dagger} \begin{array}{c} -1/2 \\ \end{array} |2\rangle \times |16\rangle, \tilde{A}^{\dagger} \begin{array}{c} -1/2 \\ \end{array} |2\rangle \times \tilde{A}^{\dagger} \begin{array}{c} -1/2 \\ \end{array} |2\rangle \times |16\rangle, \tilde{A}^{\dagger} \begin{array}{c} -1/2 \\ \end{array} |2\rangle \times \psi^{\dagger} \begin{array}{c} -1/2 \\ \end{array} |2\rangle \times |16\rangle \) where \(|2\rangle\) refers to the ground state of the internal integer-moded twisted NS fermions. These have Spin(4) content \( 64 \left[ (3,3) + (3,1) + (1,3) + 8(2,2) + 17(1,1) \right] \). As discussed above, the massive states (3.7) are not BPS states (they carry no magnetic charge since they are perturbative, no electric charge since all gauge fields are RR, and as expected they do not fit into ultrashort massive supermultiplets). They do not couple to the \( U(1) \) gauge fields, and thus their mass presumably receives quantum corrections. It is suggested that such states may become unstable from these corrections or non-perturbatively,\(^5\) because they could decay into BPS states and therefore would not appear in an exact S-matrix. It might be possible there are no non-BPS states in the full theory of IIA on \( R^6 \times T^4/Z_2 \), even though they appear in perturbation theory.\(^5\)

By expanding (A4), we see on the dual side (the heterotic string on \( R^6 \times T^4 \)) that there are no (perturbative) states at this mass level. At the next mass level, \( \frac{1}{2}m_L^2 = \frac{1}{2}m_R^2 = 1 \), there are \( 64 \times 3456 \) perturbative states, while on the dual side (heterotic), there are 41,472 perturbative states, again demonstrating the mismatch of perturbative spectra in the S-dual pair.

For the type IIB superstring, the analog of (3.5) is

\[
\begin{align*}
F_1 & = \text{even} \quad |(1,2)\rangle_{\text{Left}} \\
F_1 & = \text{odd} \quad |(2,1)\rangle_{\text{Left}} \\
F_3 & = \text{even} \quad |(1,2)\rangle_{\text{Right}} \\
F_3 & = \text{odd} \quad |(2,1)\rangle_{\text{Right}}.
\end{align*}
\]

So for the Type IIB superstring on \( R^6 \times T^4/Z_2 \) the partition function is the same (2.15) as for IIA on \( R^6 \times T^4/Z_2 \), the number of massless states is the same, but the Spin (4) representations are now form \( D = 6, N = (0,2) \) supermultiplet: the supergravity multiplet

\[
(3,3) + 5(3,1) + 4(3,2)
\]

is coupled to 21 tensor supermultiplets:

\[
(1,3) + 5(1,1) + 4(1,2).
\]
From the projections (2.13), the boundary values (2.8), Table 1, and (3.8), it follows that the massless states in the untwisted sectors are: 

\[(3, 3) + (1, 3) + (3, 1) + (1, 1)\] from NS-NS, 
\[2|(1, 2)_{\text{Left}} \times 2|(2, 1)_{\text{Right}} + 2|(2, 1)_{\text{Left}} \times 2|(2, 1)_{\text{Right}} = 4(1, 3) + 4(3, 1) + 8(1, 1)\]

from RR, and 
\[4(3, 2) + 20(1, 2)\] from R-NS and NS-R. In the twisted sectors, the massless spectrum is 
\[64(1, 1)\] from \(b_0 b_1\), 
\[16(1, 1) + 16(1, 3)\] from \(b_1\), 
\[64(1, 2)\] from \(b_1 b_2\) and \(b_0 b_1 b_2\).

The supergravity multiplet and 5 of the tensor multiplets come from the untwisted sector, and 16 tensor multiplets come from the twisted sector.

From (3.9), one sees there are no vector multiplets (RR nor NS-NS) in Type IIB on \(R^6 \times T^4 / \mathbb{Z}_2\). This holds for generic \(R^6 \times K3\) compactifications, so it seems unlikely that special points in the moduli space would lead to non-abelian gauge symmetry in \(D = 6\), unlike the IIA case.

We include here for comparison, the massless spectrum of type IIA or B on \(R^6 \times T^4\). It is \(D = 6, N = (2, 2)\) with the unique supergravity multiplet:

\[(3, 3) + 5(3, 1) + 5(1, 3) + 16(2, 2) + 25(1, 1) + 4(3, 2) + 4(2, 3) + 20(2, 1) + 20(1, 2)\]

(3.10)

where 8 of the vectors come from NS-NS sector, and 8 from RR sector. The partition function for this theory is given in the Appendix (A1).

The lowest lying spectrum for type II on the tube metric conformal field theory on \(R^6 \times W4 / \mathbb{Z}_2\) described by the partition function given in (4.4) for general \(k\) is comprised of a \(D = 6, N = (1, 1)\) supergravity multiplet and 4 vector supermultiplets, all in the untwisted sector.

### 4. Partition function for tube metric conformal field theory

The partition function for the type II superstring on \(R^6 \times W4\), for \(W4\) described in sect. 5, is given by

\[
\Lambda = \frac{1}{4\pi (\alpha')^3} \int_\mathcal{F} d^2 \tau (\text{Im} \tau)^{-2} |\eta(\tau)|^{-8} \Lambda'_f
\]

\[
\Lambda'_f = \frac{1}{4} \left( \frac{\text{Im} \tau}{|\eta(\tau)|^2} \right)^{-\frac{1}{2}} Z_k(\bar{\tau}, \tau) \ |\eta(\tau)|^{-8} (\bar{\theta}^4_3 - \bar{\theta}^4_1) (\theta^4_3 - \theta^4_1 - \bar{\theta}^4_2) ,
\]

(4.1)

where the diagonal modular invariant

\[
Z_k(\bar{\tau}, \tau) = \sum_{\lambda=1}^{k+1} \chi_{k,\lambda}(\bar{\tau}) \chi_{k,\lambda}(\tau)
\]

(4.2)

is in correspondence with \(su(2 + k)\), i.e. \(A_{k+1}\) in the ADE classification\([21]\) of modular invariant combinations of level \(k\) affine \(SU(2)\) characters. An irreducible highest weight representation of an affine algebra \(\hat{g}\) is an infinite-dimensional tower of irreducible representations of the finite-dimensional algebra \(g\), and is classified by its highest weight. Allowed
highest weights for level $k$ affine $SU(2)$ are $\lambda = 2\ell + 1$, where $\ell$ is the spin of the $SU(2)$ representation at the top of the tower, and $0 < \ell < \frac{k}{2}$.

The character formula, which counts the states in a given irreducible representation of the level $k$ affine $SU(2)$ is

$$
\chi_{k,\lambda}(\tau) = \frac{1}{\eta^3(\tau)} \sum_{n \in \mathbb{Z}} (n^2(k + 2) + \lambda) \omega^{\frac{(n^2(k+2)+\lambda)^2}{4(k+2)}}. 
$$

The invariant $Z_k(\bar{\tau}, \tau) = \sum_{\lambda=1}^{k+1} \chi_{k,\lambda}(\bar{\tau}) \chi_{k,\lambda}(\tau)$ is defined for $k \geq 0$. For $k = 0$, $Z_0(\bar{\tau}, \tau) = \bar{\chi}_{0,1}(\bar{\tau}) \chi_{0,1}(\tau) = 1$ since $\chi_{0,1}(\tau) = \frac{1}{\eta(\tau)} \omega^{\frac{1}{k}} \sum_{n \in \mathbb{Z}} (4n + 1) \omega^{2n^2 + n} = 1$ using the Jacobi triple product identity for $\eta^3$.

The other ADE modular invariants, for example those corresponding to $D_{\frac{k}{2}+2}$, $k$ even, are discussed in [13].

A modular invariant partition function for the type II superstring on $\mathbb{R}^6 \times W4/\mathbb{Z}_2$ is

$$
\Lambda = -\frac{1}{4\pi (\alpha')^3} \int_{\mathcal{F}} d^2\tau (\text{Im} \tau)^{-4} |\eta(\tau)|^{-8} \Lambda_f'
$$

$$
\Lambda_f' = \frac{1}{8} \frac{(|\text{Im} \tau|^{-\frac{1}{2}})}{|\eta(\tau)|^2} \left[ Z_k \left[ \alpha \right], \tau \right] |\eta(\tau)|^{-8} (\bar{\theta}_3^2 - \bar{\theta}_4^2 - \bar{\theta}_2^2)(\theta_3^2 - \theta_4^2 - \theta_2^2)
$$

$$
+ Z_k \left[ 1 \right], \tau \right] |\eta(\tau)|^{-8} \bar{\theta}_3^2 \bar{\theta}_4^2 \theta_3^2 \theta_4^2 (1 - 1 - 1 + 1)
$$

$$
+ Z_k \left[ 0 \right], \tau \right] |\eta(\tau)|^{-8} \bar{\theta}_3^2 \bar{\theta}_4^2 \theta_3^2 \theta_4^2 (2 - 2)
$$

$$
+ Z_k \left[ 1 \right], \tau \right] |\eta(\tau)|^{-8} \bar{\theta}_3^2 \bar{\theta}_4^2 \theta_3^2 \theta_4^2 (2 - 2) \right] 
$$

where

$$
Z_k \left[ \alpha \right], \tau \right] = \sum_{\lambda=1}^{k+1} e^{i\pi \beta(\lambda-1)} \bar{\chi}_{k,\lambda}(\bar{\tau}) \chi_{k,\lambda+\alpha(k+2-2\lambda)}(\tau) 
$$

transforms under modular transformations as

$$
Z_k \left[ \alpha \right] \rightarrow e^{-i\pi(\frac{k}{2})\alpha^2} Z_k \left[ \alpha \right] \quad \text{for} \quad \tau \rightarrow \tau + 1
$$

$$
Z_k \left[ \alpha \right] \rightarrow e^{i\pi k\alpha^2} Z_k \left[ \alpha \right] \quad \text{for} \quad \tau \rightarrow \frac{-1}{\tau}
$$

(4.6)

and $Z_k \left[ 0 \right], \tau \right] = Z_k(\bar{\tau}, \tau)$ in (4.2). We note that (4.4) is identical to (2.15) when the internal bosons $A^I$ are replaced with Liouville and WZW modes $J^0, J^I$ (see sect.5). The $\mathbb{Z}_2$ twist used in (4.4) is defined in [25].

5. The tube metric fivebrane representation

The solitonic fivebrane solution in sigma-model coordinates has a metric given by

$$
ds^2 = \eta_{MN} dx^M dx^N + (1 + (2\pi T_2)^{-1} \sum_{i=1}^{n_H} \frac{1}{(y_i-y_i')^2}) \delta_{mn} dy^m dy^n
$$

(5.1)
where $0 \leq M, N \leq 5$, and $6 \leq m, n \leq 9$, and magnetic charge $g_6 = \frac{2\pi n_H}{\sqrt{2}kT_2}$. The first term corresponds to a $c = 9$ flat and therefore free cft, and the second term represents a non-trivial $c = 6$ cft describing $n_H$ coincident fivebranes when $y_i = 0$.

For $n_H = k + 2$ and near the semi-wormhole throat $\frac{2\pi n_H}{\sqrt{2}kT_2}\delta_{mn}dy^m dy^n$, the $c = 6$ cft is an $N = 4$ superconformal field theory (scft) whose operator algebra given in Appendix C has generators constructed from four affine Kac-Moody currents $J^A$ of dimension one satisfying an $U(1) \times SU(2)$ KMA (of level $k$) together with a set of four dimension-$\frac{1}{2}$ fields $\psi^A$ satisfying the free fermion algebra:

$$J^0(z)J^0(\zeta) = -(z - \zeta)^{-2} + \ldots$$
$$J^0(z)J^i(\zeta) = O(z - \zeta)^0$$
$$J^i(z)J^j(\zeta) = -\frac{k}{2}(z - \zeta)^{-2} + \epsilon_{ijk}J^k(\zeta)(z - \zeta)^{-1} + \ldots$$
$$\psi^A(z)\psi^B(\zeta) = -(z - \zeta)^{-1}\delta^{AB} + \ldots$$
$$\psi^A(z)J^B(\zeta) = O(z - \zeta)^0.$$  (5.2)

The energy-momentum tensor from the Sugawara and Feigen Fuchs constructions is:

$$L(z) = -\frac{1}{2}J^0 J^0 - \frac{1}{k + 2} J^i J^i - \frac{1}{2} \partial \psi^A \psi^A + \delta \frac{1}{4z^2} + \frac{1}{2} Q \partial J^0(z)$$  (5.3a)

where $\delta = 0, 1$ in the NS or R sector respectively; and $A = (0, i)$. One of the four supercurrents is

$$F(z) = \psi^0 J^0 + \frac{\sqrt{2}}{\sqrt{k + 2}} \psi^i J^i + \frac{\sqrt{2}}{6\sqrt{k + 2}} \epsilon_{ijk} \psi^i \psi^j \psi^k - Q \partial \psi^0(z),$$  (5.3b)

and the level 1 $SU(2)$ current in (C1) is constructed as

$$S^i(z) = \frac{1}{2}(\psi^0 \psi^i + \frac{1}{2} \epsilon_{ijk} \psi^j \psi^k).$$  (5.4)

The superconformal system (5.3) has $c = \frac{3k}{k+2} + 3 + 3Q^2$. We choose the background charge $Q = -\frac{\sqrt{2}}{\sqrt{k+2}}$, so that $c = 6$. From (5.9) we see the Liouville field $J^0(z) = -\partial \phi(z)$ is a space coordinate in this construction, since conventionally $X^M(z)X^N(\zeta) = -\eta^{MN} \ln(z - \zeta) + :X^M(z)X^N(\zeta):$ where the space directions have $\eta^{ii} = 1$, and hermiticity is defined via $a^M(z) \equiv i \partial X^M(z) = \sum_n a^M_n z^{-n-1}$ as $a^M_{-n} = a^M_{n}$. (Since the Liouville field is a space coordinate in this application, the Feigen Fuchs shift in (5.3) is by a real background charge $Q$ which increases the central charge by $3Q^2$.)

For $k = 0$ ($n_H = 2$), the set of fields in (5.2) is reduced to $J^0(z), \psi^A(z)$ since the superconformal system (5.3) becomes

$$L(z) = -\frac{1}{2} J^0 J^0 - \frac{1}{2} \partial \psi^A \psi^A + \delta \frac{1}{4z^2} - \frac{1}{2} \partial J^0$$  (5.5a)

$$F(z) = \psi^0 J^0 + \frac{1}{2} \epsilon_{ijk} \psi^i \psi^j \psi^k + \partial \psi^0,$$  (5.5b)
and the 3 additional supercurrents \( F^i(z) = \psi^i J^0 - \psi^0 J^i - \frac{1}{2} \epsilon_{ijk} \psi^j \psi^k + \partial \psi^i \).

For \( k = -1 \), the bosonic fields can correspond to the generators of a non-compact Wolf space and have positive norm.\(^{[22]}\)

For \( L(z) \) hermitian, i.e. \( L_n^+ = L_n^- \), then \( J_n^{0+} = - J_n^{0-} \) for \( n \neq 0 \) since \( Q \) is real, \( J_n^{0+} = -Q - J_0^0 \psi_n^{A+} = -\psi_n^{A-} \), and \( J_n^{1+} = -J_n^{1-} \). Therefore the states \( J_{-n}^0 |\psi\rangle \) for \( n > 0 \) have positive norm:
\[
||J_{-n}^0 |\psi\rangle||^2 = \langle \psi|(-J_n^{0+}) J_{-n}^0 |\psi\rangle = n \langle \psi|\psi\rangle = n |||\psi|||^2 > 0 \tag{5.6}
\]
for \( |||\psi|||^2 > 0 \). A similar argument holds for \( \psi_{-n}^A |\psi\rangle \) and \( J_{-n}^1 |\psi\rangle \). It follows from the above hermiticity conditions on \( J_n^A, \psi_n^A \) that \( F_n^+ = F_{-n} \).

To study vertex operators consider the primary weight one-half superfield whose components are
\[
V_L^i(z) = \psi^i; \quad V_U^i(z) = Q(\frac{1}{2} \epsilon_{ijk} \psi^j \psi^k + J^i) \equiv T^i, \tag{5.7a}
\]
where
\[
F(z) \psi^i(\zeta) = (z - \zeta)^{-1} T^i(\zeta) \]
\[
F(z) T^i(\zeta) = (z - \zeta)^{-2} \psi^i(\zeta) + (z - \zeta)^{-1} \partial \psi^i(\zeta). \tag{5.7b}
\]

We define \( T^i \equiv V_U^i \) and \( 5.7a \) forms a representation of a super Kac-Moody algebra
\[
T^i(z) T^j(\zeta) = -\frac{\delta_{ij}}{(z-\zeta)^2} + \frac{Q \epsilon_{ijk} T^k(\zeta)}{(z-\zeta)} \tag{5.8a}
\]
\[
T^i(z) \psi^j(\zeta) = \frac{Q \epsilon_{ijk} \psi^k(\zeta)}{(z-\zeta)} \tag{5.8b}
\]
\[
\psi^i(z) \psi^j(\zeta) = -\frac{\delta_{ij}}{(z-\zeta)}. \tag{5.8c}
\]

The level of the \( SU(2) \) KMA \( T^i \) is \( k+2 \). Other relevant \( SU(2) \) currents are \( \tilde{S}^i = \frac{1}{2}(-\psi^0 \psi^i + \frac{1}{2} \epsilon_{ijk} \psi^j \psi^k) \) of level 1, and \( \tilde{S}^i + J^i \) of level \( 1+k \).

If we bosonize the \( J^0 \) current by \( J^0(z) = -\partial \phi(z) \), where
\[
\phi(z) \phi(\zeta) = -\ln(z-\zeta) + \ldots \text{ for } |z| > |\zeta|, \tag{5.9}
\]
then the conformal field \( e^{\beta \phi(z)} : \) with \( e^{\beta \phi(0)} : |0\rangle = |\beta\rangle \) is primary with respect to \( L(z) \) with conformal weight \( h_Q = -\frac{1}{2} \beta (\beta + Q) \). We are interested in the case real \( Q \) and real \( h_Q \). The allowed values for \( \beta \) such that \( h_Q \) is real are as follows. For \( h_Q = -\frac{1}{2} \beta (\beta + Q) \) to be real, either \( \beta = -\frac{1}{2} Q + ip \) with real continuous \( p \) so that \( h_Q = \frac{1}{2} Q^2 + \frac{1}{2} p^2 \) is positive; or \( \beta \) is real. If \( \beta \) is real, the highest weight conditions for massless unitary representations require it to take on discrete values, a feature similar to the discrete states in \( c = 1 \) matter coupled to \( 2d \) quantum gravity, i.e. the Liouville field.\(^{[23]}\)
HERMITICITY PROPERTIES, INNER PRODUCTS AND NORMS

We consider the Liouville part of the cft described in (5.3a) given by

\[ L(z) = -\frac{1}{2} J(z) J(z) + \frac{Q}{2} \partial J(z), \]

defining \( J(z) \equiv J^0(z) \) so that for \( n \neq 0, \)

\[ L_n = -J_0 J_n - \frac{Q}{2} (n + 1) J_n - \frac{1}{2} \sum_{m \neq 0, m \neq n} : J_{n-m} J_m : \]

(5.10a)

and

\[ L_0 = -\frac{1}{2} J_0^2 - \frac{Q}{2} J_0 - \sum_{m=1}^{\infty} : J_{-m} J_m : \]

(5.10b)

The current \( J(z) \) is anomalous:

\[ L(z) J(\zeta) = Q(z - \zeta)^{-3} + J(\zeta)(z - \zeta)^{-2} + \partial J(\zeta)(z - \zeta)^{-1} + \ldots \]

so that

\[ [L_n, J_m] = \frac{Q}{2} n(n + 1) \delta_{n,-m} - m J_{n+m} \]

\[ [L_{-1}, J_1] = -J_0 \]

\[ [L_1, J_{-1}] = Q + J_0 \]

(5.11)

The primary field \( e^{\beta \phi(z)} \) satisfies

\[ L(z) e^{\beta \phi(\zeta)} = (z - \zeta)^{-2} \left[ -\frac{1}{2} \beta(\beta + Q) \right] e^{\beta \phi(\zeta)} + (z - \zeta)^{-1} \partial_\zeta e^{\beta \phi(\zeta)} \]

\[ J(z) e^{\beta \phi(\zeta)} = (z - \zeta)^{-1} \beta e^{\beta \phi(\zeta)} \]

\[ e^{\beta \phi(0)} |0\rangle = \psi_\beta \]

\[ J_0 \psi_\beta = \beta \psi_\beta \]

(5.12)

where for \( n > 0, \) \( J_n |0\rangle = 0, L_n |0\rangle = 0. \) From (5.10a) that for \( L_n^* = L_{-n}, \) we must have that for \( n \neq 0, \) \( J_n^* = -J_{-n} \) if \( Q \) is real. From (5.11) it follows that \( J_0^* = -Q - J_0. \) These hermiticity properties correspond to the following definition of inner product:

\[ (\psi_\mu, \psi_q) = \delta_{\mu,-q-Q} \]

(5.13)

since we must have

\[ (\psi_\mu, J_0 \psi_q) = (J_0^* \psi_\mu, \psi_q) \]

\[ = q(\psi_\mu, \psi_q) = ((-J_0 - Q) \psi_\mu, \psi_q) \]

\[ = (-\mu - Q) (\psi_\mu, \psi_q) \]

(5.14)

so that \( (\psi_\mu, \psi_q) = 0 \) unless \( q = -\mu - Q, \) hence (5.13). We can write \( (\psi_\mu, \psi_q) \equiv \langle \mu | q \rangle \) so that (5.13) is \( (\psi_\mu, \psi_q) = \langle \mu | q \rangle = \delta_{\mu,-q-Q}. \) For this definition of hermiticity and corresponding inner product, we have that the adjoint of \( |q\rangle \) is \( \langle q | \) and that

\[ (\psi_q, \psi_q) = \delta_{q,-q-Q} = \delta_{q,-\frac{Q}{2}} \]

(5.15)
i.e. \( \psi_q \equiv |q\rangle \) has zero norm unless \( q = -\frac{Q}{2} \). Note however that \( |q\rangle \) is not null, i.e. it is not orthogonal to every state, since \( (\psi_q - Q, \psi_q) = \delta_{q,q} = 1 \neq 0 \). If the eigenvalues of \( J_0 \) are complex, then the RHS of (5.14) is \((-\mu^* - Q)(\psi_\mu, \psi_q)\), so that (5.15) is

\[
(\psi_q, \psi_q) = \delta_{q,-q^* - Q} = \delta_{\text{Re}q, -\frac{Q}{2}}.
\]

i.e. \( \psi_q \equiv |q\rangle \) has zero norm unless

\[
\text{Re}q = -\frac{Q}{2}.
\]

(Of course we could construct a positive norm state \( \psi = \frac{1}{\sqrt{2}}(|q\rangle + | - q - Q\rangle) \) where \( ||\psi||^2 = 1 \). But then there would also be negative norm state \( \psi' = \frac{1}{\sqrt{2}}(|q\rangle - | - q - Q\rangle) \) where \( ||\psi'||^2 = -1 \), unless this were absent by projection.) In sect. 4, we have assumed the definition of inner product (5.13), so that corresponding to (5.16), positive norm states have \( J_0 \) eigenvalues \( \beta = -\frac{Q}{2} + ip \), with \( \text{h}_Q = \frac{1}{2}Q^2 + \frac{1}{4}p^2 \). Integrating over \( p \) we compute the contribution of the non-compact Liouville mode \( \frac{\text{Im} \tau}{|q(\tau)|^2} \) appearing as the first factor in the partition functions (4.1), (4.4). (Although these theories have this (seventh) non-compact dimension, there is only six-dimensional Lorentz invariance.)

Let's now consider a **new** inner product

\[
\langle \langle \psi_\mu, \psi_q \rangle \rangle = \delta_{\mu,q}.
\]

This corresponds to the hermiticity properties \( J_0^\dagger = J_0 \), and for \( n \neq 0 \), \( J_n^\dagger = -J_{-n} \) if \( Q \) is real, which results in \( L_n^\dagger = L_{-n} + (2J_0 + Q)J_{-n} \) for \( n \neq 0 \), and \( L_0^\dagger = L_0 \). (In a supersymmetric extension \( F_r^\dagger = F_{-r} - (2J_0 + Q)\psi_{0-r}^0\).) To show \( J_0^\dagger = J_0 \), we have from (5.17) that

\[
\langle \langle \psi_\mu, J_0 \psi_q \rangle \rangle = \langle \langle J_0^\dagger \psi_\mu, \psi_q \rangle \rangle = q\langle \langle \psi_\mu, \psi_q \rangle \rangle = \mu \langle \langle \psi_\mu, \psi_q \rangle \rangle = \langle \langle J_0 \psi_\mu, \psi_q \rangle \rangle
\]

where (5.18b) follows from \( \langle \langle \psi_\mu, \psi_q \rangle \rangle = \delta_{\mu,q} \), and \( J_0^\dagger = J_0 \) follows from (5.18a,c). Then

\[
\langle \langle \psi_\mu, \psi_q \rangle \rangle \equiv (\psi_\mu - Q, \psi_q) = \delta_{\mu,q}.
\]

For this definition of hermiticity and corresponding inner product, we have that the adjoint of \( |q\rangle \) is \( <| - q - Q\rangle \) and that

\[
\langle \langle \psi_q, \psi_q \rangle \rangle = \delta_{q,q}, q = 1,
\]

i.e. \( \psi_q \equiv |q\rangle \) has unit norm. If the eigenvalues of \( J_0 \) are complex, then the RHS of (5.18b) is \( \mu^* \langle \langle \psi_\mu, \psi_q \rangle \rangle \), so that (5.20) is \( \mu^* \langle \langle \psi_\mu, \psi_q \rangle \rangle = \delta_{q^*,q} = 1 \), i.e. \( \psi_q \equiv |q\rangle \) has non-zero
norm for any real q. With the new inner product, the discrete states have positive norm. This new definition of norm is seen to agree with that found in recent exact results on Liouville theory\cite{17}. Incorporation of these results as internal degrees of freedom of a string theory would allow us to remove the non-compact contribution $\frac{(\text{Im}\tau)^{-\frac{3}{2}}}{|\eta(\tau)|^2}$, which is modular invariant by itself.

D=2 SUPERSYMMETRIC GAUGE THEORIES

Following recent work on two-dimensional theories, we recall\cite{11–13} that $D = 2$, $N = (4, 4)$ supersymmetric gauge theories can have vector supermultiplets $[2(\pm \frac{1}{2}), 4(0)]$ with gauge group $G$, where the scalars are labelled by $\Phi = \phi_i^a$, $1 \leq i \leq 4; 1 \leq a \leq \text{dim}G$, and hypermultiplets $[2(\pm \frac{1}{2}), 4(0)]$ with scalars $H^{AX}$, $1 \leq A \leq 2; 1 \leq X \leq 2n_H$, where $n_H$ is the number of hypermultiplets. (These 2D multiplets are dimensionally reduced $D = 6$, $N = (1, 0)$ vector $[(2, 2) + 2(1, 2)]$ and hypermultiplets $[2(2, 1) + 4(1, 1)]$, here labelled by their Spin(4) content.) A Coulomb phase has $\langle 0|H|0 \rangle = 0$; $\langle 0|\Phi|0 \rangle \neq 0$ and the surviving massless fields are $r$ massless vector supermultiplets, where $r$ is the rank of $G$. The infrared limit of this $D = 2$ supersymmetric gauge theory is an $N = (4, 4)$ superconformal field theory with central charge $c = 6r$. For $r = 1$ it corresponds to the tube metric cft which has primary fields (4 scalars (bosons) and 4 fermions) given in (5.2).

The level of the $SU(2)$ KMA $J_i$ in (5.2) is $k = n_H - 2$, so a 2D gauge theory with $n_H$ hypermultiplets flows in the infrared to a scft corresponding to the transverse degrees of freedom to $n_H$ coincident fivebranes (5.1). In IIB for example, the 2D gauge theory is the world volume theory of a D1-string containing the $U(1)$ gauge field which can carry away the flux associated with $n_H$ point electric charges, each emerging from the end of a fundamental open string connected to one of $n_H$ nearby D5-branes. (The world volume action of $n_H$ coincident Dirichlet fivebranes is a $D = 6, N=(1,1), U(n_H)$ non-abelian gauge theory). These become Neveu-Schwarz fivebranes in the S-dual picture.

The modular invariant character combinations for level $k$ $\hat{SU}(2)$ were found by Cappelli, Itzykson, Zuber\cite{21} to have an ADE classification $A_{k+1}, k \geq 0$; $D_{\frac{k+2}{2}}, k \geq 4, \text{keven}$; $E_6$, $k = 10$; $E_7$, $k = 16$ ; $E_8$, $k = 28$. The $D = 2$ gauge theory with an $SU(2)$ vector multiplet and $n_H$ hypermultiplets corresponds to the $D_{k+2}$ series\cite{13}. The $A_{k+1}$ modular invariants correspond to one $U(1)$ vector multiplet and $n_H = k + 2$ hypermultiplets; for a given $k$, these invariants are $Z_{k}(\bar{\tau}, \tau) = \sum_{\lambda=1}^{k+1} \chi_{k,\lambda}(\bar{\tau})\chi_{k,\lambda}(\tau)$ described in (4.2), where $\chi_{k,\lambda}(\tau)$ is the character for the highest weight $\lambda$ irreducible representation of the level $k$ affine $SU(2)$ KMA.

The Higgs phase of the 2D theory has $\langle 0|H|0 \rangle \neq 0$; $\langle 0|\Phi|0 \rangle = 0$ and is parameterized by $n_H - n_V$ massless hypermultiplets, with infrared limit of an $N = (4, 4)$ superconformal field theory with central charge $c = 6(n_H - n_V)$.

In the dual pair: heterotic on $R^6 \times T^4$ and type IIA on $R^6 \times K3$, the enhanced symmetry points in the $K3$ moduli space (3.6) have an ADE classification corresponding to simply laced non-abelian symmetries of perturbative states in the heterotic theory. Thus to study a point in this moduli space corresponding to $A_1 \cong SU(2)$ i.e., a theory whose
massless spectrum in spacetime is a $D = 6$, $N = (1,1)$ supergravity multiplet coupled to 19 $U(1)$ and 1 $SU(2)$ vector multiplets, we look on the type II side at the IIA theory which is T-dual to the particular IIB compactification described by the 2D gauge theory $N = (4,4)$ with $U(1)$ vector multiplet ($n_V = 1$) and $n_H = k + 2 = 2$ hypermultiplets; this in turn corresponds to the $k = 0$ tube metric cft generated by (5.5) with primary fields $J^0(z), \psi^0(z)$ and $\psi^i(z)$ (on the left and right-moving sides), to describe the non-spacetime degrees of freedom.

6. Conclusions

The partition functions (4.1) and (4.4) are constructed from the $A_{k+1}$ modular invariant $Z_k(\bar{\tau}, \tau)$ and related twisted expressions (4.5). These correspond to excitations of a type II fundamental string in a background described by degrees of freedom transverse to the NS fivebrane. For general $k$, the lowest lying spectrum is a $D = 6$ supergravity multiplet and 4 $U(1)$ vector multiplets coming from the untwisted sector. (For $k = 0$ the compatibility of this $Z_2$ twist with the global existence of the 2D supercurrent (5.3b), (5.5b) remains problematic[25].) The incorporation of exact results on Liouville cft may modify which states survive in this theory, and hence their gauge symmetry properties. We note the occurrence of the $A_{k+1}$ modular invariants, and contrast this theory with a type II compactification[11−13] described by a 2D supersymmetric gauge theory leading to a $D = 6$, $N = (1,1)$ theory with massless spectrum of 1 SG multiplet, 19 $U(1)$ and 1 $SU(2)$ vector supermultiplets. A deeper understanding of how the conventional type II cft breaks down in this case, due to massless solitons, may guide us to a more economical description of how string theory picks the vacuum. It is believed that the appearance of these massless non-perturbative BPS states may be an important mechanism for the way in which nature incorporates gauge symmetry in string theory.

Of course, some non-abelian symmetry can be incorporated in conventional type II perturbation theory[26]: in Appendix B, we give a model whose internal cft is constructed from free fermions, and which has a 6D, $N = (1,1)$ massless spectrum with 49 scalar fields, and $(SU(2))^4$ non-abelian symmetry. This latter has a 17-dimensional moduli space corresponding to the 16 massless scalars in the $U(1)^4$ Cartan subalgebra vector supermultiplets, and the dilaton, and so is on a different moduli space of vacua.

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Appendix A

ONE-LOOP PARTITION FUNCTIONS FOR $D = 6$ STRING THEORIES:

In addition to expressions for type II on the orbifold and fivebrane given in sections 2, 4, we list here other backgrounds for type II and heterotic strings.

Type II superstring on $\mathbb{R}^6 \times T^4$:

$$\Lambda = -\frac{1}{4\pi(\alpha')^3} \int d^2 \tau (\text{Im} \, \tau)^{-4} |\eta(\tau)|^{-8} \Lambda_f'$$

$$\Lambda_f' = \frac{1}{4} \frac{\theta \gamma_{4,4}(\bar{\tau}, \tau)}{|\eta(\tau)|^8} |\eta(\tau)|^{-8} (\bar{\theta}^4 - \bar{\theta}_4^4 - \bar{\theta}_2^4)(\theta^4 - \theta_4^4 - \theta_2^4). \quad (A1)$$

Type II superstring on $\mathbb{R}^6 \times \mathbb{R}^4/\mathbb{Z}_2$:

$$\Lambda = -\frac{1}{4\pi(\alpha')^3} \int d^2 \tau (\text{Im} \, \tau)^{-4} |\eta(\tau)|^{-8} \Lambda_f'$$

$$\Lambda_f' = \frac{1}{8} \left[ \frac{(\text{Im} \, \tau)^{-2}}{|\eta(\tau)|^8} |\eta(\tau)|^{-8} (\bar{\theta}^4 - \bar{\theta}_4^4 - \bar{\theta}_2^4)(\theta^4 - \theta_4^4 - \theta_2^4) ight.$$

$$+ \frac{2^4 |\eta(\tau)|^4}{\theta^2_2 \theta^2_4} |\eta(\tau)|^{-8} \bar{\theta}_2^2 \bar{\theta}_4^2 \theta^2_2 \theta^2_4 (1 - 1 - 1 + 1)$$

$$+ \frac{2^4 |\eta(\tau)|^4}{\theta^2_4 \theta^2_3} |\eta(\tau)|^{-8} \bar{\theta}_3^2 \bar{\theta}_4^2 \theta^2_3 \theta^2_4 (2 - 2)$$

$$+ \frac{2^4 |\eta(\tau)|^4}{\theta^2_3 \theta^2_4} |\eta(\tau)|^{-8} \bar{\theta}_4^2 \bar{\theta}_3^2 \theta^2_4 \theta^2_3 (2 - 2) \left]. \quad (A2)$$

Type II superstring on $\mathbb{R}^{10}$:

$$\Lambda = -\frac{1}{4\pi(\alpha')^3} \int d^2 \tau (\text{Im} \, \tau)^{-4} |\eta(\tau)|^{-8} \Lambda_f'$$

$$\Lambda_f' = \frac{1}{4} \frac{(\text{Im} \, \tau)^{-2}}{|\eta(\tau)|^8} |\eta(\tau)|^{-8} (\bar{\theta}^4 - \bar{\theta}_4^4 - \bar{\theta}_2^4)(\theta^4 - \theta_4^4 - \theta_2^4). \quad (A3)$$
Heterotic string on $\mathbb{R}^6 \times \mathbb{T}^4$:

\[
\Lambda = - \frac{1}{4\pi(\alpha')^3} \int_{\mathcal{F}} d^2 \tau (\text{Im} \tau)^{-4} |\eta(\tau)|^{-8} \Lambda'_f
\]

\[
\Lambda'_f = \frac{1}{2} \frac{\theta_{\Gamma_{20,4}}(\bar{\tau}, \tau)}{\eta(\bar{\tau})^{-20}} |\eta(\tau)|^{-8}(\theta^4_3 - \theta^4_4 - \theta^4_2).
\]

Heterotic string on $\mathbb{R}^{10}$:

\[
\Lambda = - \frac{1}{4\pi(\alpha')^3} \int_{\mathcal{F}} d^2 \tau (\text{Im} \tau)^{-4} |\eta(\tau)|^{-8} \Lambda'_f
\]

\[
\Lambda'_f = \frac{1}{2} \frac{(\text{Im} \tau)^{-2} \theta_{\Gamma_{16}}(\bar{\tau})}{|\eta(\tau)|^8} \eta(\tau)^{-4}(\theta^4_3 - \theta^4_4 - \theta^4_2).
\]

where the lattice theta function

\[
\theta_{\Gamma_{16}}(\bar{\tau}) = \frac{1}{2}(\theta^1_{16} + \theta^4_{16} + \theta^2_{16})
\]

is identical for the two 16-dimensional even self-dual Euclidean lattices.
Appendix B

$D = 6, \, N = (1, 1)$ FREE FERMION MODEL

A type II string free fermion model with a $D = 6, \, N = (1, 1)$ massless spectrum of one supergravity multiplet from $\tilde{\psi}^i \times \psi^j |0\rangle, \psi^i \times \psi^j |0\rangle, \tilde{\psi}^i \times 2(2, 1) + (1, 2)_{\text{Right}}$; and 12 vector supermultiplets in the adjoint of $SU(2)^4$ from $\tilde{\psi}^a \times \{\psi^j |0\rangle, \psi^j |0\rangle, 2(2, 1) + (1, 2)_{\text{Right}}\}$, has left and right-moving modes each described by 4 bosonic and 12 fermionic worldsheet (primary) fields: $\tilde{A}^i_n, \tilde{\psi}^i_r, \psi^a_r; A^i_n, \psi^i_r, \psi^j_r, \psi^j_r; 1 \leq i \leq 4, 1 \leq a \leq 12, 1 \leq J \leq 4,$ and $1 \leq j \leq 8$ where the superscript $i$ refers to the transverse spatial degrees of freedom, the superscripts $a, J, j$ to the internal ones, and all bosons and fermions are real. This string has four sectors whose fermion boundary condition vectors are given by

$$\rho_{b_1} = (0^{16}; 1^4 0^{12}) \quad \rho_{b_0} = (1^{16}; 1^{16})$$
$$\rho_{b_0 b_1} = (1^{16}; 0^4 1^{12}) \quad \rho_{\phi} = (0^{16}; 0^{16}).$$

The projections replacing (2.13) are

$$P_{\alpha, b_1} = \frac{1}{2} \{1 + \epsilon(\alpha, b_1)(-1)^{\rho_{b_1} \cdot F}\} \quad (B2)$$
$$P_{\alpha, b_0 b_1} = \frac{1}{2} \{1 + \epsilon(\alpha, b_0 b_1)(-1)^{\rho_{b_0 b_1} \cdot F}\} \quad (B3)$$

where $\rho_{b} \cdot F = \sum_{j=1}^{16} \tilde{\rho}_j \tilde{F}_j + \sum_{j=1}^{16} \rho_j F_j$, and the values of $\epsilon(\alpha, \beta)$ are found in Table 2.

| $\alpha$ | $\beta$ | $\epsilon(\alpha, \beta)$ | $\rho_0$ | $\rho_1$ | $\rho_{01}$ |
|---------|---------|--------------------------|---------|---------|-----------|
| $\emptyset$ | $\emptyset$ | 1 | 1 | -1 | -1 |
| $b_0$ | $\lambda \mu$ | $\mu$ | $\lambda$ |
| $b_1$ | $-\mu$ | $\mu$ | -1 |
| $b_0 b_1$ | $-\lambda$ | -1 | $\lambda$ |

Table 2

The partition function is

$$\Lambda = -\frac{1}{4\pi(\alpha')^3} \int d^2 \tau (\text{Im} \tau)^{-4} |\eta(\tau)|^{-8} \Lambda' \quad (B4)$$

$$\Lambda' = \frac{1}{4} \left[ |\eta(\tau)|^{-16} (|\theta_3|^8 \bar{\theta}_3^4 - |\theta_4|^8 \bar{\theta}_4^4 - |\theta_2|^8 \bar{\theta}_2^4) (\theta_3^4 - \theta_4^4 - \theta_2^4) \right].$$

(B5)
Appendix C

\( N = 4, c = 6 \) SUPERCONEFORMAL ALGEBRA

The currents of (2.5) close on the \( N = 4 \) superconformal algebra with \( c = 6 \) which is given by

\[
\begin{align*}
L(z)L(\zeta) &= \frac{c}{2} \frac{1}{(z-\zeta)^4} + \frac{2L(\zeta)}{(z-\zeta)^2} + \frac{\partial L(\zeta)}{z-\zeta} \\
L(z)F(\zeta) &= \frac{3}{2} \frac{F(\zeta)}{(z-\zeta)^2} + \frac{\partial F(\zeta)}{z-\zeta} \\
L(z)F^i(\zeta) &= \frac{3}{2} \frac{F^i(\zeta)}{(z-\zeta)^2} + \frac{\partial F^i(\zeta)}{z-\zeta} \\
L(z)S^i(\zeta) &= \frac{S^i(\zeta)}{(z-\zeta)^2} + \frac{\partial S^i(\zeta)}{z-\zeta} \\
F(z)F(\zeta) &= \frac{2c}{3} \frac{1}{(z-\zeta)^3} + \frac{2L(\zeta)}{(z-\zeta)} \\
F(z)F^i(\zeta) &= \frac{4S^i(\zeta)}{(z-\zeta)^2} - \frac{2\partial S^i(\zeta)}{z-\zeta} \\
F^i(z)F^j(\zeta) &= \frac{\delta^{ij}2c}{3} \frac{1}{(z-\zeta)^3} - \frac{4\epsilon_{ij\ell}S^\ell(\zeta)}{(z-\zeta)^2} - \frac{2\epsilon_{ij\ell}\partial S^\ell(\zeta)}{z-\zeta} + \frac{2\delta^{ij}L(\zeta)}{z-\zeta} \\
S^i(z)S^j(\zeta) &= \frac{-n\delta^{ij}}{2(z-\zeta)^2} + \frac{\epsilon_{ij\ell}S^\ell(\zeta)}{z-\zeta} \\
S^i(z)F(\zeta) &= \frac{F^i(\zeta)}{2(z-\zeta)} \\
S^i(z)F^j(\zeta) &= \frac{1}{(z-\zeta)} \left[ -\delta^{ij}F(\zeta) + \epsilon_{ij\ell}F^\ell(\zeta) \right].
\end{align*}
\]

(C1)

The central charge \( c \) and the level \( n \) of the \( SU(2)_n \) currents \( S^i \) are related by \( c = 6n \). The condition \( c = 6 \) sets \( S^i \) at level one.

We define complex supercurrents as

\[
\begin{align*}
\mathcal{G}^1 &\equiv \frac{F - iF^3}{\sqrt{2}}, & \mathcal{G}^2 &\equiv \frac{F^2 - iF^1}{\sqrt{2}} \\
\bar{\mathcal{G}}^1 &\equiv \frac{F + iF^3}{\sqrt{2}}, & \bar{\mathcal{G}}^2 &\equiv \frac{F^2 + iF^1}{\sqrt{2}}
\end{align*}
\]

and note that \( \mathcal{G}^1, \mathcal{G}^2 \) transform as a doublet under \( S^i \) etc.
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