Shrinking of the fluctuation region in a two-band superconductor

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Received 25 November 2013, revised 26 March 2014
Accepted for publication 16 April 2014
Published 2 July 2014

Abstract
In a two-band superconductor, two qualitatively different fluctuation modes related to the gap modules contribute to the free energy and heat capacity, together with the phase fluctuations. The first mode has divergent temperature behaviour since it accounts for critical fluctuations around the phase transition point, $T_c$, along with pseudo-critical ones associated with the former instability of the weaker superconductivity component. The involvement of these two factors, competing under interband interaction, results in a Ginzburg number that varies with $T_c$ non-monotonically, allowing a reduction of up to 75%. This makes the fluctuations effective in revealing additional superconducting components in the system. The second mode does not diverge, but has a jump at $T_c$, defined uniquely by the strength of the interband interaction. This mode contributes fundamentally beyond the critical domain.

Keywords: two-band superconductor, fluctuation, correlation length, Ginzburg number

1. Introduction
The behaviour of fluctuations around the critical temperature is the most fundamental aspect of the theory of phase transitions. Despite the phenomenological picture given by Landau, there is always a critical region where deviations from the common description appear in the form of universality classes. In particular, the superconductivity can be treated in terms of the Ginzburg–Landau theory in the striking vicinity of the phase transition temperature. However, the scaling predictions for the criticality of superconducting instability seem to follow an $XY$ model [1, 2], similar to the normal–superfluid transition in $^4$He liquid, for example. The quantitative estimation for the temperature interval near the critical point, where fluctuation cannot be considered in the Gaussian–Ginzburg–Landau approximation, is given by the Ginzburg criterion. The latter assesses the corrections stemming from the critical behaviour as unobservable in conventional bulk superconductors.

The extraordinary properties of cuprates [3] started an era of high-$T_c$ superconductors 30 years ago. Nowadays, that promising family also includes magnesium diboride [4], fullerenes [5], and pnictides [6], to name a few. Unlike conventional superconductors, in these compounds the role of critical fluctuations becomes essential. High values of thermal disordering energy, $k_BT_c$, together with small coherence volume due to short correlation lengths, makes it energetically inexpensive to create fluctuating patches of normal material in the superconducting state. As a result, the critical region becomes expanded by several orders of magnitude, reaching about 1 K, or even 10 K, depending on the system dimensionality [7]. Additionally, strong magnetic fields, comparable to $H_{c2}$, suppress the superconductivity by substantially enhancing the role of fluctuations. That enables detailed experimental study of the critical domain.

Numerous experiments indicate the significant impact of superconducting fluctuations on various high-$T_c$ systems, e.g. the reduction of transition temperature [8] and melting of the Abrikosov flux lattice into a vortex liquid [9, 10]. The latter has practical consequences for the applications of high-$T_c$ superconductors, since the lack of vortex pinning results in a loss of zero resistance. On the other hand, preformed Cooper pairs observed above the phase transition point are considered as a possible explanation for various anomalies in the normal state of the underdoped copper oxides, referred to as ‘pseudogap physics’ [11].

0953-2048/14/085006+08$33.00 © 2014 IOP Publishing Ltd Printed in the UK
Although the mechanism behind high-$T_c$ superconductivity is far from clear, the contributions from more than one carrier band can play a decisive role here. This picture is supported by direct experimental evidence of multicomponent scenarios in a variety of relevant materials, see [12–14], for example. The multigap nature opens up possibilities for the fluctuation regimes not peculiar to single-band superconductors. For instance, small interband-phase fluctuations result in collective excitations [15–17] called Leggett mode, which become massless and mix with density fluctuations in a three-gap system at the time-reversal-symmetry-breaking transition [18]. Phase kinks can be generated in non-equilibrium current-carrying states [19]. Recently, the presence of a magnetic flux-carrying metastable topological soliton was predicted in three-band superconductors with broken time-reversal symmetry [20].

The spatial coherency of the gap order parameter plays a central role in the formation of superconducting fluctuations. However, these properties are affected by multiband physics in a serious way. Neither the type-I nor type-II magnetic response, called type-1.5 behaviour [21–23], provides such an example. The phenomenon is based on the existence of several competing coherency channels for the inhomogeneous multiband superconductor. Due to interband pairing, the corresponding correlation lengths describe the joint superconducting condensate as a whole, not de Gious Wazowski [3].

Spatial correlation functions.

In this paper, we investigate the spatially inhomogeneous thermal fluctuations of superconductivity in the case when competing coherency scales are present. We analyze the interplay between fluctuation modes related to these length scales. Most attention was previously centered only on the contribution of critical fluctuations to specific heat and conductivity in a multiband scenario [31–33]. However, the competition of superconductivity components was never clarified in this respect. Here we also give special attention to the coherency channel, which is characterized by correlation length with non diverging temperature behaviour. As a result, we find a way to reveal an additional superconducting component in the system by inspecting fluctuations around the phase transition point. The findings may also be of relevance for elementary superconductors, since they become effectively multiband in the nanometre scale [34, 35].

2. Superconductivity fluctuations

We start with a two-component (α = 1, 2) Ginzburg–Landau functional [36] for complex gap-order parameters \( \delta_\alpha = \Re \delta_\alpha + i \Im \delta_\alpha \) in the absence of magnetic field

\[
F = F_s + \sum_\alpha \left( a_\alpha \| \delta_\alpha \|^2 + \frac{b_\alpha}{2} \| \delta_\alpha \|^4 - c_\alpha \delta_\alpha^* \delta_\alpha + K_\alpha \nabla \delta_\alpha \nabla \delta_\alpha^* \right) dV,
\]

where \( F_s \) is free energy without superconductivity. For expansion coefficients we retain full temperature dependence, i.e.

\[
a_\alpha = \frac{W_{12 \alpha \alpha}}{W^{2}} - \rho_\alpha \ln \frac{1.13k_B T}{\hbar \omega_{D}}, \quad b_\alpha = \frac{0.11 \rho_\alpha}{(k_B T)^2}, \quad c = \frac{0.02 \rho_\alpha \hbar v_{F}}{(k_B T)^{3/2}}, \quad W^2 = W_1 W_2 - W_{12}^2.
\]

Here, \( W_{\alpha \alpha} > 0 \) and \( W_{12} = W_{21} \) are matrix elements for the intraband and interband pair-transfer interaction channel, \( \rho_\alpha \) is the density of states at the Fermi level, and \( v_{F} \) is the Fermi velocity in the corresponding band. Electron–electron interactions are assumed to be nonzero and independent of the electron wave vector in the Debye layer \( \pm \hbar \omega_{D} \) around the chemical potential.

In the homogeneous case, the minimization of the Ginzburg–Landau functional leaves us with equations for coupled bulk-order parameters \( \Delta = |\Delta_\alpha| e^{i \phi_\alpha} \), namely,

\[
\Delta_\alpha \left( a_\alpha + b_\alpha |\Delta_\alpha|^2 \right) c \Delta_\alpha = \text{sqn} (c \cos (\phi_1 - \phi_2)) = 1,
\]

where sqn is the sign function. These equations fix the modules of the order parameters and the difference between the phases \( \phi_1 - \phi_2 = \pi n, n \in N \). Since the gap phases remain unspecified, we take real bulk-order parameters \( \Delta_\alpha = \Delta_\alpha^* \).

The condition for the critical point \( a_\alpha a_\beta = c^2 \) (here and elsewhere index ‘c’ implies \( T = T_c \)) has two solutions \( T_{c \alpha} \), which transform into the separate intraband transition temperatures \( T_{c \alpha} \sim \exp \left( \frac{-1}{\rho_\alpha \hbar \omega_{D}} \right) \) in the limit \( W_{12} \to 0 \). The larger solution \( T_{c \alpha} \) corresponds to the superconducting phase transition temperature of the joint condensate, denoted as \( T_c \), and it increases with the interband coupling constant \( |W_{12}| \). The smaller solution \( T_{c \alpha} \) is a monotonically decreasing function of \( |W_{12}| \) which disappears as \( W \to 0 \). These dependencies are explicitly depicted in [37].

2.1. Free energy fluctuations

In the macroscopic system, a homogeneous state with free energy \( F_s \) is very probable and fluctuation effects are bound to the inhomogeneity of the gap-order parameters. To describe statistically small deviations from the homogeneous superconducting (\( \Delta_\alpha \neq 0 \)) or normal (\( \Delta_\alpha = 0 \)) background, we take \( \delta_\alpha \) as \( \Delta_\alpha + \eta_\alpha (r) \) and use the Gaussian approximation.

In terms of complex Fourier components \( \Re \Delta_\alpha = \frac{1}{r} \int \Re \delta_\alpha (r) e^{-i \kappa r} dV = \Delta \delta_{k,\alpha} + \Re \eta_{k,\alpha} \) (here \( \delta_{k,\alpha} \) is Kronecker delta) and \( \Im \Delta_\alpha = \frac{1}{r} \int \Im \delta_\alpha (r) e^{-i \kappa r} dV = \Im \eta_{k,\alpha} \), the
Ginzburg–Landau functional reads as

\[ F = F_0 + V \sum \left[ A_\alpha \Re \eta_{\alpha}^2 + B_\alpha \Im \eta_{\alpha}^2 \right. \\
- c \left( \Re \eta_{\alpha}^2 + \Im \eta_{\alpha}^2 \right) + \left. 2 \sum_{\beta \neq \alpha} \left( A_{\alpha \beta} |\eta_{\beta}|^2 + B_{\alpha \beta} |\eta_{\alpha}|^2 \right) \right] \\
- c \left( \Re \eta_{\alpha}^2 + \Im \eta_{\alpha}^2 \right) \right], 
\]

(4)

where \( A_{\alpha} = A_{\alpha} + K_{\alpha} k^2 \), \( A_{\alpha} = a_{\alpha} + 3b_{\alpha} \Delta_{\alpha}^2 > 0 \) and \( B_{\alpha} = B_{\alpha} + K_{\alpha} k^2 \), \( B_{\alpha} = a_{\alpha} + b_{\alpha} \Delta_{\alpha}^2 \geq 0 \). The star sign near the summation denotes half of the \( k \)-space. In this presentation, \( F \) contains only independent degrees of freedom: real \( \Re \eta_{\alpha}^2, \Im \eta_{\alpha}^2 \) and real and imaginary parts of \( \Re \eta_{\alpha}^2, \Im \eta_{\alpha}^2 \) taken in half of the \( k \)-space. To calculate the statistical sum \( Z \), one should integrate \( \exp \left( -\frac{F}{kT} \right) \) over these variables. Note that interband pairing results in the nondiagonal elements in quadratic form \( (4) \), i.e. some degrees of freedom become mixed. As a result, for a macroscopic superconductor, one obtains free energy density \( f = \frac{F}{V} \ln Z \) in the form

\[ f = f_0 + \frac{kT}{V} \sum_{\eta \neq 0} \sin \left[ \left( k^2 + \kappa^2 \right)^{1/2} \right] \\
\left( k^2 + m^2 \right)^{1/2} \right], 
\]

(5)

with \( \kappa = \kappa \eta^2 \) and \( m = \frac{\kappa}{\kappa} \). For the mass factors, one obtains \( m_0^2 = \kappa \eta^2 \) above \( T_c \) and \( m_0^2 = \sum m_0^2 \) below \( T_c \). These formulas generalize single-band (\( W = 0 \)) consideration for which gap-amplitude fluctuations contribute as \( \frac{kT}{V} \sum_{\eta \neq 0} \sin \left( k^2 + \xi^2 \right) \) and gap-phase fluctuations as \( \frac{kT}{V} \sum_{\eta \neq 0} \sin \left( k^2 + m^2 \right) \), see [38]. Here \( \xi \) is the correlation length of the one-gap system diverging at \( T_c \), and \( m \) is the mass of the Goldstone boson disappearing below \( T_c \). In a two-band superconductor four gap channels appear, related to the distinct coherency lengths \( \kappa \eta^{-1} \) and to the mass factors associated with the Goldstone mode \( (m_0) \) and Leggett mode \( (m) \).

Various experimental techniques, e.g. scanning tunneling spectroscopy, muon spin relaxation, and thermal conductivity measurements, point to evidence of distinct coherency scales for two-band superconductivity [39–41]. In our approach, the correlation lengths have non-critical \( (\kappa^{-1}) \) and critical \( (\kappa^{-1}) \) temperature behaviour near the superconducting phase transition point, as follows from the expansions

\[ \kappa^2 \approx \left( \rho^2 + \Delta \rho \right) \frac{T - T_c}{T_c}, \quad \rho^2 = \frac{\rho_{\beta}^2 + \rho_{\alpha}^2}{\rho_{\alpha}^2 + \rho_{\beta}^2}, \]

\[ \kappa^2 \approx \xi_{\alpha}^2 + \xi_{\beta}^2 + \rho^2 \left( \rho_{\alpha}^2 + \Delta \rho \right) \frac{T - T_c}{T_c}, \]

\[ \rho^2 = 2\kappa_{\alpha}^2 + \frac{\rho_{\beta}^2 + \rho_{\alpha}^2}{\rho_{\alpha}^2 + \rho_{\beta}^2}, \]

(6)

where \( \Delta \rho \) is above \( T_c \), however, below \( T_c \) one should take \( \Delta \rho = -3 \rho \) and

\[ \Delta \rho = -3 \rho = \frac{3\rho_{\beta}^2 + \rho_{\alpha}^2}{\rho_{\alpha}^2 + \rho_{\beta}^2}. \]

Note that the same length scales \( \kappa^{-1} \) also follow directly from the Ginzburg–Landau gap equations [42].

For the Leggett mode, we have obtained the same expression as given in [25]. For this phase difference mode, we get

\[ m^2 \approx \xi_{\alpha}^2 + \xi_{\beta}^2 + \left( \rho^2 + \Delta m \right) \frac{T - T_c}{T_c}, \]

(7)

where \( \Delta m = 0 \) above \( T_c \), but below \( T_c \) one should take \( \Delta m = \frac{\rho^2}{\eta \alpha} \). This mass parameter describes the deviations from the equilibrium value of the interband phase.

2.2. Heat capacity fluctuations

The leading contribution to fluctuations of specific heat and conductivity, stemming from the critical behaviour of the correlation length, was previously considered in [31–33]. Next, we calculate specific heat by taking into account all four modes. By using integration in half-space instead of summation \( \sum_{k} \rightarrow \frac{V}{(2\pi)^2} \int \frac{d\mathbf{k}}{V} \), one obtains specific heat capacity \( c = -T \frac{dF}{dT} \) in the form \( c = c_h + \sum_{\alpha} F_{\alpha}(\tau) \), where \( c_h \) is related to the bulk homogeneous state and

\[ F_{\alpha}(\tau) = \left\{ \begin{array}{ll}
\frac{kT}{2\pi^2} \left( \frac{1}{3} k \theta_1(\tau) \tan^{-1} \frac{k_{max}}{\tau} - \theta_2(\tau) k_{max} \right) \\
\frac{kT}{2\pi^2} \left( \frac{2}{3} k \theta_1(\tau) \tan^{-1} \frac{k_{max}}{\tau} - \theta_2(\tau) k_{max} \right) \\
\frac{kT}{2\pi^2} \left( \frac{1}{3} k \theta_1(\tau) \tan^{-1} \frac{k_{max}}{\tau} \right) \\
\frac{kT}{2\pi^2} \left( \frac{2}{3} k \theta_1(\tau) \tan^{-1} \frac{k_{max}}{\tau} \right) 
\end{array} \right. \]

(8)

for different effective dimensionality \( d = 3, 2, 1 \) (\( V = V, S, L \)), respectively. Here \( \tau \) takes values \( \kappa \), \( \kappa, m, m \), prime implies temperature derivative, \( \theta_1(\tau) = 2e^{\tau} + T \tau^2 \) and \( k_{max} \) is the cutoff parameter. Note that the massless Goldstone mode does not contribute to heat capacity fluctuations in the superconducting state, and in the
normal phase one has \( \sum \mathcal{F}_d(r) = 2(\mathcal{F}_d(\kappa_\alpha) + \mathcal{F}_d(\kappa_\beta)) \). Due to the critical behaviour of \( \kappa^{-1} \), the most predominant contribution from superconductivity fluctuations in the normal phase near \( T_c \) has a standard form

\[
\sum \mathcal{F}_d(r) \approx k_b \vartheta^2 \frac{V_d}{V} \left( \frac{T - T_c}{T_c} \right)^{\frac{d - 2}{2}},
\]

(10)

where \( \vartheta = 2\pi^2 T F(2 - \frac{d}{2}) \). Symmetrically below \( T_c \) the fluctuation contribution differs from value (10) by factor \( 2^\frac{d-1}{2} \).

At the same time, \( \mathcal{F}_d(\kappa_\alpha) \) stays finite by approaching the critical point.

### 2.3. Ginzburg number

The way in which the Ginzburg number can be introduced is not unique [38]. The estimate of the critical region can be found by comparing the fluctuation energy with superconducting condensation energy, or by comparing the Aslamazov–Larkin correction to conductivity with its normal value. Here we use an approach based on the corrections to heat capacity. Note that corresponding de

The latter expression is analogous to the single-band counterpart. However, here \( \rho \) and \( \Delta c \) are generally no longer power-law functions of the critical temperature as they are for ordinary superconductivity. As a result, one-band power-law scaling \( G_{ij} \sim T^{\frac{d-2}{2}} \) does not hold, and \( G_{ij}(T_c) \) becomes a more general function in the two-gap case.

### 3. Discussion

#### 3.1. Inverse correlation lengths and masses

The interaction between superconductivity components changes the properties of the joint condensate in a remarkable manner. Without interband pairing, the condensate splits into two non-interacting superconducting subsystems with corresponding phase transition points at \( T_{c\alpha} \) and \( T_{c\beta} \). The latter temperatures determine the critical behaviour for the quantities related to these subsystems, e.g. divergences of correlation length and relaxation time, or jump of heat capacity. Since weak interband coupling acts on the weaker-superconductivity constituent as an external field [43], "applied" interband interaction smears and eliminates the criticality at lower \( T_{c\alpha} \) and mixes both superconductivity components. This results, in particular, in the appearance of the correlation lengths \( \kappa^{-1} \) with qualitatively different peculiarities [24–28] (see figure 1). The noncritical length \( \kappa^{-1} \) changes weakly with temperature. However, the behaviour of the critical one \( \kappa^{-1} \), points clearly to the phase transition temperature of the two-gap system by diverging at \( T_c \). With that, the former autonomous phase transition of the weaker-superconductivity component, smeared by the interband coupling, becomes visible, in the case of tiny interband pairings as noticeable nonmonotonicity below \( T_c \). The latter is accompanied by the presence of an avoided crossing point between \( T_{c\alpha} \) and \( T_{c\beta} \).

In the regime of very weak interband pairings (roughly, \( W_{ij}^2 \leq 10^{-4} W_{ij}^2 W_{ij}^2 \)), the presence of an avoided crossing point and the memory of weaker-superconductivity criticality play decisive roles for the differential properties of correlation lengths. The first feature manifests itself in the step-like peculiarity between \( T_{c\alpha} \) and \( T_{c\beta} \), which is seen in figure 1 for both characteristics \( \kappa^{-1} \) simultaneously. The second feature forces \( \kappa^{-1} \) to increase rapidly near lower \( T_{c\alpha} \). This behaviour reflects a jump at the phase transition point of the weaker-superconductivity constituent smeared by the 'external field' with intensity \( W_{ij}^2 \). The increase of interband interaction suppresses the effects of both features.

The temperature derivative of \( \kappa^{-1} \) has a step at \( T_c \) which follows ‘the law of 2’, i.e. \( \left| \frac{\kappa^{-1}}{\kappa^{-1}} \right| \approx 2 \) (see figure 1). Due to
a non-critical character, that ‘law’ is violated for $\kappa^2$. However, the mixing of the superconductivity components results in the jump of the $\kappa^2$ derivative at $T_c$ as well. The height of the jump is determined by $\Delta \rho$, and for small values of $W_{12}$ one has $\Delta \rho \sim W_{12}$. The jump becomes more pronounced in the superconductors with stronger interband pairings, since $\Delta \rho$ increases monotonically by raising $|W_{12}|$ and quickly approaches $|\Delta \rho|$. Note that there is a limiting value $W \approx 0$ at which real noncritical correlation length disappears from both the Ginzburg–Landau approach and microscopic theory [24–26].

The mass parameter $m_\alpha$ which reflects the excitation of the Leggett mode, also behaves noncritically. Similarly to $\kappa^2$, its temperature derivative is characterized by the finite jump $\Delta \rho$ at the phase transition point. However, as interband coupling decreases, mass $m_\alpha$ softens near the former critical point of the weaker-superconductivity component. In the limit $W_{12} \to 0$ the relevant mode becomes massless at lower $T_{c1}$ due to the Goldstone theorem.

The memory of the former criticality for the weaker-superconductivity, avoided crossing point and jump $\Delta \rho$, play a crucial role for heat-capacity fluctuations, since the latter are defined by derivatives of $\rho_+$. The contributions from the peculiarities indicated will alternate by increasing interband interaction. Although the coupling between gap order parameters is not an easily tunable parameter in experiments, it is still possible to vary it by changing the proximation between distinct electron subsystems [44–46], e.g. through doping, pressure, or direct adjustment.

### 3.2. Heat capacity fluctuations

In what follows, we discuss the superconductivity fluctuations in a two-band approach. The presence of two correlation lengths and two masses allows one to identify relevant channels of fluctuation, $F_\alpha(\tau)$, that contribute to heat capacity. Their temperature dependencies are essentially different. See figure 2 for $d = 3$ (the dependencies remain the same for $d = 2, 1$ with $y$-scale factor about $10^2$, 1, correspondingly). The contribution $F_\alpha(\kappa_+)$ strongly dominates at the phase transition point with expected divergent behaviour (10). Moreover, it reflects the memory of the weaker-superconductivity criticality by demonstrating a maximum with the well in the middle. This structure appears in the vicinity of $\tau \approx 0.83$ in the left panel of figure 2. The well stems from the differentiability of $\kappa_+^{-1}$ near $T_c$ instead of singularity. By decreasing interband interaction, the depth of the well and the height of its borders will gradually increase together with a decrease in the width of the well. For the vanishing value of $W_{12}$, the well near $T_c$ disappears, leaving us with well-known divergent behaviour at the lower phase transition point.

In the systems with tiny interband pairings, the avoided crossing point in the behaviours of $\kappa_+$ manifests itself as the extremes of $F_\alpha(\kappa_+)$, which compensate for each other (see the left panel of figure 2 at $\tau \approx 0.93$). Therefore, as the net effect from the avoided crossing point is zero in a superconductor with weakly interacting gaps, both contributions $F_\alpha(\kappa_+)$ are equally important. By decreasing interband interaction, the extrema related to the avoided crossing point rise, tighten, and finally disappear for $W_{12} = 0$. Note that in this limiting case, $F_\alpha(\kappa_+)$ will be replaced by band contributions for $\alpha = 1, 2$ with autonomous divergent behaviours at $T_{c1}$ or $T_{c2}$, correspondingly.

Interband phase fluctuations are also affected by the memory of the weaker-superconductivity criticality. Due to softening of the Leggett mode in the vicinity of the lower $T_{c1}$, fluctuation channel $F_\alpha(m_\tau)$ contributes considerably to the heat capacity in this temperature region when interband pairing is very weak.

Whereas $F_\alpha(\kappa_+)$ and $F_\alpha(m_\tau)$ behave noncritically, these fluctuation modes are always overshadowed by $F_\alpha(\kappa_-)$ near $T_c$. However, for sufficiently strong interband couplings, the considerable jump $\Delta \rho$ leads to the observable step-like temperature behaviour of $F_\alpha(\kappa_-)$ and $F_\alpha(m_\tau)$. As a result, a discrepancy appears between the two-band consideration, $F_\alpha(\kappa_-)$, and the single-mode approach, where only the unique critical mode $F_\alpha(\kappa_-)$ is taken into account. The effect is observable in the right panel of figure 2, and it is sensitive to the value $k_{\text{max}}$ chosen.

To summarize, beyond the critical region, the manifestation of fluctuations in heat capacity is qualitatively distinguishable in a two-band superconductivity model and single-gap/single-mode scenarios as follows.

(i) The single-gap approach fails to reproduce the memory of the weaker-superconductivity criticality. Intuitively, the latter effect can be incorporated by taking into consideration an additional single-band subsystem, resulting in the enhancement of the fluctuations in the relevant temperature region. However, this attempt fails because a self-consistent two-gap model predicts a maximum for the fluctuations near lower $T_{c\alpha}$, disguised by a deep well in the middle. The reduction of the heat capacity value near the former critical point of the
weaker-superconductivity component due to fluctuations is an essential feature of a two-band scenario.

(ii) In the single-mode approach, there appears to be an enhancement of the fluctuations below the phase transition point, if the interband pairing is tiny (at \( t \approx 0.93 \) in the left panel of figure 2). This behaviour has no physical meaning. By increasing the interband interaction constant, the enhancement can be suppressed. Nonetheless, the peak of heat-capacity fluctuations remains significantly overestimated in width. These shortcomings can be removed by including the fluctuation mode associated with noncritical correlation length.

Note also that by restricting ourselves to considering uniform spatial mode only (i.e. a homogeneous system), thermal fluctuations in a finite-size two-band superconductor reveal the memory of the former autonomous phase transition in the band with weaker superconductivity [47].

3.3. Width of the fluctuation region

The estimations of the critical region require accurate measurements of conductivity, thermal expansion, specific heat or magnetization. Usually, the width of the critical domain is investigated in various high-\( T \) materials only under applied magnetic fields [48–52]. Next we discuss the qualitative behaviour of \( G_{i \delta} \) as a function of the critical temperature in a two-band model. Figure 3 demonstrates that the dependence can be very dissimilar to the single-band analog for which the Ginzburg number is always a monotone function, \( G_{i \delta} \sim T_c^{-1/2} \). Surprisingly, the presence of another superconducting component can result in the non-monotone behaviour with \( T_c \).

To analyze the functional relation \( G_{i \delta}(T_c) \), where critical temperature changes under interband interaction, we notice that this function always qualitatively follows the dependence \( G_{i \delta}([W_1]) \). With that, the Ginzburg number always increases with interband coupling, if \([W_1]\) exceeds some finite value. However, there are two ways that the Ginzburg number changes near \( W_1 = 0 \). It can have global minimum or alternatively local maximum at \( W_1 = 0 \) (see insets in the right panel in figure 3). These two regimes are separated by the condition

\[
1 + \frac{d}{4 - d} \frac{v_{1s}^2}{v_{1s}^2} \geq \frac{2(d - 1) \rho_1 W_{1s} - \rho_2 W_{2s}}{4 - d} \frac{\rho_1^2 W_{1s}^2}{\rho_2^2 W_{1s}^2},
\]

(13)

where the upper (lower) sign corresponds to the maximum (minimum) of \( G_{i \delta} \) near \( W_1 = 0 \). For \( d = 3, 2 \) the separation lines in the parameter space are depicted in right panel in figure 3. Formally, there should always be a local maximum at \( W_1 = 0 \) for \( G_{i \delta} \). Thus, by moving from single-band to two-gap description (i.e. by turning interband interaction on), the fluctuation effects may be enhanced or suppressed as one can observe in the insets of figure 3.

The small discrepancy between intraband critical temperatures \( T_{ci} \) and the strong memory of the former instability

![Figure 3](image.png)

**Figure 3.** Left: the plot of \( G_{i \delta}(W_1) \) for the same model parameters as in figure 1, except \( W_2 = 0.31 \) (red), 0.29 (blue), 0.22 (green) and 0.167 eV cell (black curve). For these values, \( T_{c1}^0 = 0.95, 0.72, 0.24, \) and 0.05, correspondingly. The lower dashed curve represents the limiting case \( W_2 = 0.32 \) eV cell \( (T_{c2} = T_{c1}) \). The upper dashed curve corresponds to the single-band \( W_2 \rightarrow 0 \) dependence on the critical temperature normalized to \( T_{c1} \). Although the latter temperature in our model calculations equals 23.5 K, resulting in a very small \( G_{i \delta} \) value, the Ginzburg number becomes magnified by several orders of magnitude by taking the parameters of a real high-\( T \) material. Right: two regions in the parameter space where \( G_{i \delta} \) behaves monotonically or non-monotonically with \([W_1]\) increase. The solid line is the separation border for \( d = 3 \), and the dashed line for \( d = 2. \) Red/blue/green/black points represent the curves with the same colour as in the left panel. Note that the maximum of \( G_{i \delta}(W_1) \) tightens as \( \rho_2 \) decreases for \( \rho_1 W_{1s} \) approaches and disappears in the limit \( \rho_1 W_{1s} = \rho_1 W_{1f} \).

of the weaker-superconductivity component support the shrinking of the fluctuation region. Its reduction can be enhanced by bringing together intraband critical points, \( \rho_2 W_{2s} \rightarrow \rho_1 W_{1s} (T_{c2} \rightarrow T_{c1}) \) (see figure 3). In this process the heat-capacity jump \( \Delta C \), appearing in the denominator of \( G_{i \delta} \), grows for any \( \rho_2 \) finite \( W_{2s} \). That jump increases abruptly for \( W_{1s} = 0 \), by becoming a superposition of two jumps related to the superconducting subsystems involved. As a result, one obtains effective single-band dependence \( G_{i \delta} \sim T_{c1}^{-1/2} \) when \( T_{c1} \) and \( T_{c2} \) coincide. The maximal drop of \( G_{i \delta}(T_c) \) can be found from the ratio

\[
\frac{G_{i \delta}(W_{1s}) \rightarrow 0; \rho_2 W_{2s} \rightarrow \rho_1 W_{1s}}{G_{i \delta}(W_{1f} = 0, \rho_2 W_{2s} < \rho_1 W_{1s})} \approx \left( 1 + \frac{\rho_1}{\rho_2} \right)^{-1/2},
\]

(14)

For the parabolic system, the latter function has a global minimum at \( \frac{\rho_1}{\rho_2} = 1 \) (\( d = 3 \)) with the value 0.25. For \( d = 2,1 \) that minimum is higher. Thus, \( G_{i \delta} \) can be reduced up to 75% by increasing \( T_{c1} \).

We notice that when \( T_{c2} = T_{c1} \), the Ginzburg number actually behaves non-monotonically in the vicinity of \( W_{1s} = 0 \). This is caused by the non-critical correlation length, which becomes divergent \( \kappa_{cs}^2 = 0 \) for \( W_{1s} = 0 \). As a result, the corresponding fluctuation channel \( \mathcal{F}_d(\kappa_0) \) contributes to the critical region for \( W_{1s} = 0 \), increasing its width. Consequently, the non-monotonicity of \( G_{i \delta}(T_c) \) is related to the interplay
between criticalities of superconductivity components driven by interband pairing.

The interpretation of the $G_i(T_c)$ behaviour in a two-band system can be given as follows: let us fix intraband critical temperatures $T_{c\alpha}$ by fixing parameters $\rho_0$ and $\omega_{\alpha\alpha}, \alpha = 1, 2$. By turning on the interband interaction, these two points become replaced by the phase transition temperature of the joint condensate, $T_c = T_{c\alpha}$, and by $T_{s\alpha}$ which represents the memory of the lower $T_{c\alpha}$. It is important to note that $T_{c\alpha}$ always increases, but $T_{s\alpha}$ decreases with $W_{i\alpha}$. By using single-band analog, $G_i$ should raise with interband coupling, since the latter suppresses $T_{s\alpha}$. Thus, there are two opposite tendencies associated with the temperatures $T_{c\alpha}$ which drive the behaviour of $G_i$. Their interplay becomes essential when the former instability of the weaker-superconductivity component is located close to the phase transition point of the joint condensate, and the memory of it is not completely erased by the interband interaction.

In the vicinity of the phase transition point, deviations from the mean-field predictions for the critical temperature, superfluid density, Josephson current, tunneling conductance (due to fluctuation-induced suppression of density of states), etc., are defined by the value of $G_i$. It would be interesting to analyze these observables for different high-$T_c$ materials, keeping in mind that corrections to the Bardeen–Cooper–Schrieffer scenario from fluctuations can shed light on the presence of additional superconducting components and point to the proportion between the intrinsic superconductivities of the subsystems involved.

4. Conclusions

We have demonstrated that qualitatively new types of fluctuations must be taken into account in the thermodynamics of superconductivity if one involves several electron bands in the pairing mechanism. Together with gap-phase fluctuations (Goldstone and Leggett modes), two distinct channels appear for gap-amplitude fluctuations, in a two-band scenario. The distinctness stems from the corresponding correlation lengths, which have remarkably different properties. The first one diverges at the phase transition point, $T_c$, and refers also to the former criticality of the weaker-superconductivity constituent, smeared by interband interaction. The corresponding mode, representing both the actual and the former superconducting instabilities, dominates in the vicinity of the phase transition point of the joint condensate. That effectively involves two critical temperatures in the evolution of the Ginzburg number. By manipulating the proximization between distinct electron subsystems, non-monotonic behaviour of the Ginzburg number with $T_c$ can be produced, unlike its single-band counterpart. The critical region can shrink up to 25% of the value corresponding to the single-band limit. In this way, fluctuations reflect the two-band nature of superconductivity. The second correlation length is always finite, but its temperature derivative has a jump at $T_c$, defined uniquely by the interband coupling. The fluctuations related to this length scale should be taken into account as one explores two-band superconductivity outside the critical domain.

Acknowledgments

This study was supported by the European Union through the European Regional Development Fund (Centre of Excellence ‘Mesosystems: Theory and Applications’, TK114), the Estonian Science Foundation grant no. 8991; the Estonian Ministry of Education and Research through the Institutional Research Funding IUT2-27 and the COST MP1201 NanoSC Action.

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