Vacuum non-expanding horizons and shear-free null geodesic congruences

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Abstract
We investigate the geometry of a particular class of null surfaces in spacetime called vacuum non-expanding horizons (NEHs). Using the spin-coefficient equation, we provide a complete description of the horizon geometry, as well as fixing a canonical choice of null tetrad and coordinates on a NEH. By looking for particular classes of null geodesic congruences which live exterior to NEHs but have the special property that their shear vanishes at the intersection with the horizon, a good cut formalism for NEHs is developed which closely mirrors asymptotic theory. In particular, we show that such null geodesic congruences are generated by arbitrary choice of a complex worldline in a complex four-dimensional space, each such choice induces a CR structure on the horizon, and a particular worldline (and hence CR structure) may be chosen by transforming to a privileged tetrad frame.

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1. Introduction
This work is devoted to two related topics with the discussion of the second depending on the results of the first. The associated issues are however distinct from each other. The first topic is the analysis of the geometry of certain special null 3-surfaces embedded in a four-dimensional Lorentzian manifold. These surfaces, \(\mathcal{H}\), referred to as local non-expanding horizons (NEHs), are defined by having the topology of \(S^2 \times \mathbb{R}\), with the additional property that the null generators (the null geodesics of the surface) have both vanishing divergence and vanishing shear. The study of local NEH geometry has a history dating back (to the best of our knowledge) to the early vacuum work of Pajerski et al in 1969 [1, 2], where a special case of a horizon was considered. This was followed recently by the more general and sophisticated approaches of Ashtekar, Lewandowski and their colleagues [3–7]. In a long series of papers, using different gauge conditions than those of Pajerski, they developed a general theory for the intrinsic geometry of horizons for both vacuum and non-vacuum.
In the present work, we return to the earlier approach of Pajerski where all the vacuum equations of the Newman–Penrose or spin-coefficient (SC) formalism are used to give a straightforward, simple derivation with relatively transparent results of the vacuum NEH geometry. Our results also give a complete description of the geometry for both rotating and non-rotating non-expanding horizons in the spin-coefficient formalism.

In addition to our desire to take a second look at horizon geometry, another major reason for our return to the topic of horizons was to investigate certain further geometric structures that live on them which, up to now, have been ignored. The same structures arise naturally and determine a rich structure on the future null infinity ($\mathcal{I}^+$) of asymptotically flat spacetimes. Exploiting the analogous properties between $\mathcal{I}^+$ and a NEH $\mathcal{H}$ (e.g., both $\mathcal{I}^+$ and $\mathcal{H}$ are $S^2 \times \mathbb{R}$ null surfaces with the null generators having vanishing divergence and shear), we can generate these same structures on $\mathcal{H}$.

More precisely, we study null geodesic congruences (NGCs) ‘living’ exterior to $\mathcal{H}$ but that intersect $\mathcal{H}$ with the very special property that their shear vanishes at $\mathcal{H}$. We will refer to such congruences as $\mathcal{H}$-shear-free (this is the analogue of asymptotically shear-free NGCs). In order to do this, the geometry of $\mathcal{H}$ must be locally complexified so that the $\mathcal{H}$ coordinates become complex variables slightly extended away from their real values.

The basic result is that the $\mathcal{H}$-shear-free congruences are determined by solutions to the $\mathcal{H}$-Good cut equation, whose solution space is a four complex dimensional manifold, a space similar to the one that arises while studying asymptotically shear-free NGCs [8]. In the same manner as in the asymptotic case, any arbitrary analytic worldline in this complex space generates an $\mathcal{H}$-shear-free NGC just as in the recently developed physical identification theory on $\mathcal{I}^+$ (cf [9–11]). Among these arbitrary worldlines there is a means of singling out a unique one from which there is hope of developing a physical identification theory on NEHs in further analogy with the more recent work on $\mathcal{I}^+$.

In section 2, we characterize the geometry of vacuum NEHs by first defining a coordinate and null tetrad system, then integrating the spin-coefficient equations on the horizon. This will include the repeated use of gauge freedoms which involve the choice of coordinates and tetrad and allow us to fix these systems completely. Using this rigid structure we study, in section 3, the null geodesic congruences whose shear vanishes on the horizon. Associated with these congruences, we find a good cut equation and show from it that such shear-free NGCs are generated by arbitrary worldlines in a complex four-dimensional space. It is also observed that each choice of such a worldline induces a CR structure on the horizon—the possible choice of a unique worldline would induce a unique CR structure. Section 4 concludes and discusses the results, while an appendix provides a more detailed exposition of the horizon CR structures.

2. Non-expanding vacuum horizon geometry

In the remainder of this paper, we will be working on a generic non-expanding horizon, as defined by Ashtekar et al [5]:

**Definition 1.** A non-expanding vacuum horizon $\mathcal{H}$ is a null 3-submanifold in a spacetime $\mathcal{M}$ which satisfies the following properties:

1. $\mathcal{H}$ is topologically $\mathbb{R} \times S^2$, and there is a projection $\Pi : \mathcal{H} \to S^2$ where the fibres of this projection are null curves in $\mathcal{H}$;
2. The complex divergence and shear of any null tangent vector $l$ to $\mathcal{H}$ vanish;
3. The vacuum Einstein field equations hold on $\mathcal{H}$. 

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A substantial body of research had been dedicated towards understanding the intrinsic geometry induced on NEHs by the global geometry of $\mathcal{M}$; the interested reader may reference [4–7] for a discussion of these findings.

Working with the spin-coefficient (SC) formalism in the following section we re-investigate this issue.

2.1. Coordinates and null tetrads

Consider a region in the spacetime $\mathcal{R} \subset \mathcal{M}$ which is foliated by null surfaces, each of the leaves of this foliation having topology $\mathbb{R} \times S^2$. On $\mathcal{R}$, we choose a coordinate $s$ to label each of these null surfaces (i.e., $s = \text{const}$ determines a null surface with topology $\mathbb{R} \times S^2$). Each constant $s$ slice can then be charted with coordinates $(u, \zeta, \bar{\zeta})$, where $u$ covers the $\mathbb{R}$ portion of the topology and provides a foliation of each null surface, and $(\zeta, \bar{\zeta})$ charts the 2-sphere. We take the complex coordinate $\zeta = e^{\phi \cot(\theta/2)} (\zeta = x + iy)$ to be the usual complex stereographic angle coordinate on $S^2$. Hence, we cover all of $\mathcal{R}$ with the coordinate system $(u, s, \zeta, \bar{\zeta})$.

Initially, we have the freedom $s \rightarrow s^* = G(s)$ and $u \rightarrow u^* = F(u, s, \zeta, \bar{\zeta})$, but we will impose coordinate and null tetrad conditions through the course of this paper which fix the choice of these coordinates entirely.

In addition, we construct a null tetrad system $\{l, n, m, \bar{m}\}$ on $\mathcal{R}$ which obeys the usual inner product relations:

$$l^a n_a = -m^a \bar{m}_a = 1,$$

with all other contractions between the vectors vanishing. We set the null vector $l$ to be the future-directed tangent vector to the constant $s$ null surfaces:

$$l = l^a \frac{\partial}{\partial x^a} = \frac{\partial}{\partial u}, \quad l_a dx^a = ds. \quad (2)$$

By also demanding that $u$ be an affine parameter for null geodesics on these null surfaces, we reduce the remaining coordinate freedom in $u$ to

$$u \rightarrow u^* = Z(s, \zeta, \bar{\zeta})u + A(s, \zeta, \bar{\zeta}), \quad (3)$$

and also fix $l$ as a geodesic tangent vector on the leaves of the $s$-foliation. (Ashtekar and colleagues [12] use an alternative choice of gauge where the $l^a$ is rescaled so that its acceleration is the surface gravity. This makes it rather difficult to compare the results of the present work with their work.)

The most general form for the remaining tetrad vectors under these conditions is

$$l = l^a \frac{\partial}{\partial x^a} = \frac{\partial}{\partial u}, \quad l_a dx^a = ds, \quad (4)$$

$$n = n^a \frac{\partial}{\partial x^a} = U \frac{\partial}{\partial u} + \frac{\partial}{\partial s} + X^A \frac{\partial}{\partial x^A}, \quad (5)$$

$$m = m^a \frac{\partial}{\partial x^a} = \omega \frac{\partial}{\partial u} + \xi^A \frac{\partial}{\partial x^A}, \quad (6)$$

where $A = \{2, 3\}$, $(x^2, x^3) = (\zeta, \bar{\zeta})$ and $U, X^A \in \mathbb{R}, \omega, \xi^A \in \mathbb{C}$ are functions to be determined.

We choose one particular surface, labelled by $s = 0$, in $\mathcal{R}$ as our non-expanding horizon as in definition (1). Our ‘$s$’ freedom is then

$$s \rightarrow s^* = G(s, \zeta, \bar{\zeta}), \quad G(0, \zeta, \bar{\zeta}) = 0. \quad (7)$$
On the horizon this means that both $l$ and $n$ are trivially rescaled. Our coordinate system singles out a NEH, $\delta$, in the region $\mathcal{R}$ as

$$\delta = \{x^a \in \mathcal{R} : s = 0\}. \quad (9)$$

Restricting ourselves to $\delta$, our remaining freedom in the choice of coordinates is

$$u^* = Z(\zeta, \bar{\zeta})u + A(\zeta, \bar{\zeta}), \quad (10)$$

$$s^* = sZ^{-1}, \quad (11)$$

$$\zeta^* = \frac{a\zeta + b}{c\zeta + d}, \quad ad - bc = 1, \quad (12)$$

the four-parameters $\{a, b, c, d\}$ lie in $\text{SL}(2, \mathbb{C})$; $\zeta \rightarrow \zeta^*$ being the fractional linear transformation that maps $S^2$ to itself. The remaining freedom in the $s$ coordinate will simply re-scale the null tetrad by $K$ and can be bundled into $Z$ for the $u$-transformations, which will be fixed later. Under the action of the transformation equation (10), it should be noted that

$$l \rightarrow l^* = Z^{-1}l,$$

so if we demand that

$$l = \frac{\partial}{\partial u}, l^* = \frac{\partial}{\partial u^*},$$

it follows that any transformation of the $u$ coordinate must be accompanied by a re-scaling of the null tetrad as

$$l^* = Z^{-1}l, \quad (13)$$

along with

$$n^* = Zn, \quad (14)$$

to preserve the inner product $l^*n_a = 1$.

The full four-dimensional (contravariant) metric evaluated with this tetrad is given by

$$g^{ab} = l^a n^b + n^a l^b - m^a m^b - \overline{m}^a \overline{m}^b. \quad (15)$$

Without going into details since they are already in the literature [13, 14] we outline another coordinate condition that greatly simplifies the analysis. The metric given by equation (15) induces a 2-metric on the 2-surfaces $u = \text{constant}$ on $\delta$ ($s = 0$) that is determined by the coefficients $\xi^A$ appearing in the tetrad. Since every 2-metric is conformally flat, coordinates on the 2-surface can be introduced so that the metric has the form

$$ds^2 = P^{-2} d\zeta d\overline{\zeta}. \quad (16)$$

This in turn, using the freedom of a spin transformation $m \rightarrow m^* = e^{\omega} m$, allows $\xi^A$ to be chosen, with $P$ real, as

$$\xi^\zeta = -P, \quad \xi^{\overline{\zeta}} = 0,$$

$$\xi^{\overline{\zeta}} = 0, \quad \overline{\xi}^{\zeta} = -P. \quad (17)$$

This form of $\xi^A$ is used throughout this work.

**Note.** The form of equation (6) simplifies to

$$m = \omega \frac{\partial}{\partial u} - P \frac{\partial}{\partial \zeta}. \quad (18)$$
Remark. Note that although the metric, (16), is conformal to a sphere metric, it need not be a sphere metric itself. Hence, we will often write
\[ P = VP_0, \]
where \( V \) is the conformal factor and \( P_0 \) induces a 2-sphere metric. This has important implications for the definition of the \( \tilde{\Omega} \)-operator on \( \mathcal{H} \), which will prove crucial later. For a spin-weight \( s \) function \( f_{(s)} \) defined on \( \mathcal{H} \), we have
\[
\tilde{\Omega} f_{(s)} = P^{1-s} \frac{\partial}{\partial \zeta} (P^s f_{(s)}),
\]
and we will at some points need to compare this to the \( \tilde{\Omega}_0 \)-operator defined on the 2-sphere as
\[
\tilde{\Omega}_0 f_{(s)} = P^{1-s} \frac{\partial}{\partial \zeta} (P^{-s} f_{(s)}),
\]
and
\[
\tilde{\Omega}_0 f_{(s)} = P^{1+s} \frac{\partial}{\partial \zeta} (P^{s} f_{(s)}).
\]
In particular, to work with tensorial spin-\( s \) spherical harmonics in equations that contain \( \tilde{\Omega} \), we must make all expressions in terms of \( \tilde{\Omega}_0 \). This will be particularly important when developing a good cut formalism for non-expanding horizons.

Returning to the discussion of the gauge freedom, we have the choice of null rotations about the \( l \) vector:
\[
\begin{align*}
l \to l^* &= l, \\
m \to m^* &= m + Ll, \\
n \to n^* &= n + Lm + Lm + Ll,
\end{align*}
\]
with \( L \) being an arbitrary spin-weight one function. This preserves the form of equation (4). In the following section, this freedom is used extensively. The analogous null rotation about \( n \) is not of use since it would destroy the tangency conditions placed on \( l \) in the region \( \mathcal{R} \). However the rescaling freedom (boosts), \( l^* = Z^{-1}l, n^* = Zn \) is used later.

In the following subsection, further conditions are placed on both the coordinates and tetrad so that the choice of \( u \) is fixed and the tetrad is made unique.

2.2. The spin-coefficients

In this subsection, we write down all of the spin-coefficient (SC) equations on the vacuum NEH \( \mathcal{H} \), integrating many to obtain the full \( u \)-dependence for the SCs. We also simplify these results using the remaining gauge freedom of (10). Since nearly all of our calculations are performed on \( \mathcal{H} \), we will omit notation such as \( f|_0 = f|_{s=0} \), and instead just write \( f \); if the off-horizon variable \( s \) does enter, it will be stated so explicitly.

Before writing down the SC equations, we note that many of the SCs can be restricted or made to vanish \textit{a priori} simply by conditions placed on the null tetrad in the previous section. First of all, the choice of \( u \) as a geodesic parameter, along with the requirement that the vectors \( \{n, m, \overline{m}\} \) be parallely propagated along the \( l \)-congruence results in [14]

\[ \text{Tetrad condition I} \]
\[ \kappa = \varepsilon = \pi = 0, \]
which in fact holds not just on \( \mathcal{H} \) but everywhere in \( \mathcal{R} \). There remains the freedom of the initial choice on the \( n \) and \( m \) vectors before their parallel propagation. This permits the rotation parameter \( L \) in equation (22) to remain a free function of \((\zeta, \zeta')\). This will prove to be useful later.

Next, by (2), we have that \( l \) is a gradient vector field; this follows from \( l_a \, dx^a = ds \) and results in

**Tetrad condition II**

\[
\tau = \bar{\alpha} + \beta, \tag{24}
\]

which also holds everywhere in \( \mathcal{R} \).

Finally, by the definition of \( \mathcal{H} \) as a non-expanding horizon (see definition 1 above), it follows that the (complex) divergence and shear of any null tangent to the horizon must vanish. Hence, we have

**Tetrad condition III**

\[
\rho = \sigma = 0, \tag{25}
\]

holding on \( \mathcal{H} \).

In addition to these three tetrad conditions, it also follows from the Goldberg–Sachs theorem (or the SC equations themselves) that the Weyl tensor components vanish on the horizon as well

\[
\psi_0 = \psi_1 = 0. \tag{26}
\]

We thus have, from the start, that

\[
\kappa = \varepsilon = \pi = \tau - \bar{\alpha} - \beta = \rho = \sigma = 0, \tag{27}
\]

on \( \mathcal{H} \).

To write down the full set of SC equations, we define the differential operators:

\[
D \equiv \frac{\partial}{\partial u},
\]

\[
\delta \equiv \omega \frac{\partial}{\partial u} + \xi^A \frac{\partial}{\partial x^A},
\]

\[
\Delta \equiv \frac{\partial}{\partial s} + U \frac{\partial}{\partial u} + X^A \frac{\partial}{\partial x^A}.
\]

The spin-coefficient equations on \( \mathcal{H} \) are separated into three sets: (1) those containing the operator \( D \) and \( \delta \) or \( \bar{\delta} \), (2) those containing only \( \delta \) and or \( \bar{\delta} \) and (3) those that contain \( \Delta \) with other derivatives. The procedure is to first integrate the \( D \) equations, which determines the \( u \) behaviour, and then substitute those results into the second set which yields relationships between the integration ‘constants’ from the first set. The third set—included for completeness—then would yield the \( s \)-derivatives, the derivatives off \( \mathcal{H} \), of a variety of the variables. They are not of interest to us here. We present each set of equations in three blocks: those for the spin-coefficients themselves, those for the Weyl tensor and those for the metric coefficients. Note that where convenient, we have used the \( \bar{\delta} \)-operator.
The $D$-equations:

\[ D\tau = 0, \]

\[ D\alpha = 0, \]

\[ D\beta = 0, \]

\[ D\gamma = \tau\alpha + \tau\beta + \psi_2, \quad (28) \]

\[ D\lambda = 0, \]

\[ D\mu = \psi_2, \]

\[ D\nu = \tau\mu + \tau\lambda + \psi_3, \]

\[ D\psi_2 = 0, \]

\[ D\psi_3 = -\bar{\delta}\psi_2, \quad (29) \]

\[ D\psi_4 = -\bar{\delta}\psi_3 - 3\lambda\psi_2 - \bar{\omega}\bar{\delta}\psi_2, \]

\[ DU = -(\gamma + \bar{\gamma}) + \bar{\omega}\tau + \omega\tau, \]

\[ DX^A = \tau\xi^A + \bar{\tau}\xi^A, \quad (30) \]

\[ D\omega = -\tau. \]

The $(\delta, \bar{\delta})$-equations

\[ \delta\alpha - \bar{\delta}\beta = \alpha\overline{\alpha} + \beta\overline{\beta} - 2\alpha\beta - \psi_2, \quad (31) \]

\[ \delta\lambda - \bar{\delta}\mu = \mu(\alpha + \beta) + \lambda(\overline{\alpha} - 3\beta) - \psi_3, \]

\[ \delta\omega - \bar{\delta}\overline{\omega} = (\mu - \mu) - \tau\omega + \bar{\tau}\omega, \]

\[ \overline{\delta}\xi^A - \delta\bar{\xi}^A = -\tau\xi^A + \bar{\tau}\xi^A. \quad (32) \]

The $\Delta$-equations (Note. Unless a quantity vanishes on all of $\mathcal{R}$, it is not necessarily true that its $\Delta$-derivative on $\mathcal{F}$ will vanish!):

\[ \Delta\lambda - \bar{\delta}\nu = -(\mu + \overline{\mu})\lambda - (3\gamma + \bar{\gamma})\lambda + 2a\nu - \psi_4, \]

\[ \delta\nu - \Delta\mu = (\mu^2 + \lambda\overline{\lambda}) + (\gamma + \bar{\gamma})\mu - 2\beta\nu, \]

\[ \delta\gamma - \Delta\beta = \mu\tau - \beta(\gamma - \bar{\gamma} - \mu) + \alpha\overline{\alpha}, \quad (33) \]

\[ \delta\tau - \Delta\sigma = (\tau + \beta - \alpha - \tau)\tau, \]

\[ \Delta\rho - \bar{\delta}\tau = (\beta - \alpha - \tau)\tau - \psi_2, \]

\[ \Delta\alpha - \bar{\delta}\gamma = -(\tau + \beta)\lambda + (\overline{\gamma} - \gamma)\alpha + (\beta - \tau)\gamma - \psi_3, \]

\[ \Delta\psi_0 = 0, \]

\[ \Delta\psi_1 = -\bar{\delta}\psi_2, \quad (34) \]

\[ \Delta\psi_2 = \delta\psi_3 - 3\mu\psi_2, \]

\[ \Delta\psi_3 = \delta\psi_4 + 3\nu\psi_2 - 2(\gamma + 2\mu)\psi_3, \]

\[ \delta\psi_4 - \Delta\omega = -v + \lambda\overline{\omega} + (\mu - \gamma + \bar{\gamma})\omega, \]

\[ \delta X^A - \Delta\xi^A = \lambda\xi^A + (\mu - \gamma + \bar{\gamma})\xi^A. \quad (35) \]
The sets of $D$-equations are easily integrated to give the full $u$-dependence of all of the variables on $\mathcal{H}$:

\[
\begin{align*}
\tau &= \tau_0, \\
\alpha &= \alpha_0, \\
\beta &= \beta_0, \\
\gamma &= \gamma_0 + u(\tau_0\alpha_0 + \tau_0\beta_0 + \psi_{2,0}), \\
\lambda &= \lambda_0, \\
\mu &= \mu_0 + u\psi_{2,0}, \\
v &= v_0 + u(\tau_0\mu_0 + \tau_0\lambda_0 + \psi_{3,0}) + \frac{u^2}{2}(\tau_0\psi_{2,0} - \bar{\psi}_2), \\
\psi_2 &= \psi_{2,0}, \\
\psi_3 &= \psi_{3,0} - u\bar{\psi}_2, \\
\psi_4 &= \psi_{4,0} - u(\bar{\psi}_3 + 3\lambda_0\psi_{2,0} + \bar{\omega}_0 + \bar{\psi}_2) + \frac{u^2}{2}(\bar{\psi}_2 + \tau_0\bar{\psi}_2), \\
U &= U_0 + u(\tau_0\psi_{2,0} + \tau_0\psi_{3,0} - \gamma_0 - \bar{\gamma}_0) - \frac{u^2}{2}(4\tau_0\psi_{2,0} + \psi_2 + \bar{\psi}_2), \\
X^A &= X^A_0 - u(\tau_0\xi^A + \tau_0\xi^A), \\
\omega &= \omega_0 - u\tau_0,
\end{align*}
\]

where a subscript $f_0$ indicates that $f$ is a function only on the 2-sphere (i.e., $f_0 = f_0(\zeta, \bar{\zeta})$).

Our procedure is now to feed these relations into the second set of SC equations, i.e., the $(\delta, \bar{\delta})$ equations.

After a straightforward but slightly tedious calculation we obtain from equations (31) and (32) the relations:

\[
\begin{align*}
\alpha_0 &= \frac{\tau_0}{2} - \frac{1}{2} \frac{\partial P}{\partial \bar{\zeta}}, \\
\beta_0 &= \frac{\tau_0}{2} + \frac{1}{2} \frac{\partial P}{\partial \bar{\zeta}}, \\
\psi_{2,0} &= \frac{1}{2}(\bar{\tau}_0 - \bar{\tau}_0) - (P\bar{\tau}_0 - P\bar{\tau}_0 - \bar{\tau}_0 P - \bar{\tau}_0 P), \\
\bar{\omega}_0 &= \mu_0 - \bar{\mu}_0 + \bar{\omega}_0 + u(\psi_{2,0} - \bar{\psi}_{2,0}) + \mu_0 - \bar{\mu}_0, \\
\psi_{3,0} &= \bar{\omega}_0 - \bar{\psi}_{2,0} - \bar{\psi}_{2,0} - \bar{\psi}_{2,0}.
\end{align*}
\]

From equation (40) and the reality of $P$ we have

\[
\begin{align*}
2 \text{Im}(\psi_{2,0}) &= \psi_{2,0} - \bar{\psi}_{2,0} = \bar{\tau}_0 - \bar{\tau}_0.
\end{align*}
\]

This allows us to define a mass aspect on $\mathcal{H}$, analogous to the Bondi mass aspect on $\mathcal{I}^+$,

\[
\Psi \equiv \psi_{2,0} + \bar{\psi}_{2,0} = \bar{\tau}_0.
\]

Additionally, recall that the full spacetime metric $g^{\mu\nu}$, when pushed down to the $u = \text{const}$ 2-surfaces of $\mathcal{H}$, induces a 2-metric with line element:

\[
ds^2 = P^{-2} d\zeta d\bar{\zeta}.
\]
The scalar curvature of this 2-surface, which is topologically $S^2$ for each fixed $u$, is given by
\[ K = 2(P \partial_\tau \tau P - \partial_\tau P \partial_\tau P), \] (45)
so that
\[ \psi_{2,0} = \frac{1}{2} [-K + (\partial_\tau \tau_0 - \partial_\tau \tau_0)]. \] (46)

**Tetrad condition IV**

Using the remaining gauge freedom of the $(\xi, \vec{\tau})$-dependent null rotation about the vector $l$ from (22), and choosing $L$ to have the form
\[ L(\xi, \vec{\tau}) = -\omega_0, \]
the vector $m$ transforms to
\[ m^* = (\omega_0 - u \tau_0) \frac{\partial}{\partial u} - P \frac{\partial}{\partial \xi} - \omega_0 \frac{\partial}{\partial u} \]
\[ = -u \tau_0 \frac{\partial}{\partial u} - P \frac{\partial}{\partial \xi}, \] (47)
so the new $\omega_0^*$ vanishes. Dropping the '*' we have that
\[ \omega_0 = 0. \] (48)
Using these results, i.e., equations (48) and (44), in equation (41), we finally have that
\[ \mu_0 = \bar{\mu}_0. \] (49)

The null rotation freedom is now fixed.

To summarize all of our results thus far, we have
\[ \tau = \tau_0, \] (50)
\[ \alpha = \alpha_0 = \frac{1}{2} \tau_0 - \frac{1}{2} \partial_\tau P, \] (51)
\[ \beta = \beta_0 = \frac{1}{2} \tau_0 + \frac{1}{2} \partial_\xi P, \] (52)
\[ \lambda = \lambda_0, \] (53)
\[ \mu = \mu_0 + u \psi_{2,0}, \quad \mu_0 = \bar{\mu}_0, \] (54)
\[ \nu = \nu_0 + u(\tau_0 \lambda_0 + \tau_0 \mu_0 + \psi_{2,0}) + \frac{u^2}{2}(\tau_0 \psi_{2,0} - \partial_\tau \psi_{2,0}), \] (55)
\[ \gamma = \gamma_0 + u(\tau_0 \alpha_0 + \tau_0 \beta_0 + \psi_{2,0}) \]
\[ = \gamma_0 + u \left( \tau_0 \partial_\tau \tau_0 - \frac{\tau_0}{2} \partial_\tau P + \frac{\tau_0}{2} \partial_\xi P + \psi_{2,0} \right), \] (57)
\[ \omega = -\tau_0 u, \] (58)
\[ \xi^A = \bar{\xi}_0^A = -(P, 0), \xi^A = \bar{\xi}_0^A = -(0, P), \] (59)
\[ U = U_0 - u(\gamma_0 + \bar{\gamma}_0) \]
\[ = U_0 - u \left( 4 \tau_0 \tau_0 + \psi_{2,0} + \bar{\psi}_{2,0} \right), \] (60)
\[ X^A = X_0^A + u(\tau_0 \xi_0^A + \tau_0 \bar{\xi}_0^A), \] (61)
\[ \psi_2 = \psi_{2,0} = \frac{1}{2} [-K + (\partial_\tau \tau_0 - \partial_\tau \tau_0)]. \] (62)
\[ K = 2(\partial_\zeta \partial_\zeta P - \partial_\zeta \partial_\zeta P), \quad (63) \]
\[ \Psi = \psi_2,0 + \overline{\partial} \psi_0 = \overline{\Psi} = -\frac{1}{2} K + \frac{1}{2} \overline{\partial} \tau_0 + \frac{1}{2} \overline{\partial} \tau_0, \quad (64) \]
\[ \psi_3 = \psi_3,0 - u \overline{\partial} \psi_2,0. \quad (65) \]
\[ \psi_3,0 = \overline{\partial} \psi_0 - \tau_0 \psi_0 - \overline{\partial} \mu_0 + \mu_0 \tau_0, \quad (66) \]
\[ \psi_4 = \psi_4,0 - u(\overline{\partial} \psi_3,0 + \overline{\partial} \psi_2,0 + \overline{\partial} \psi_2,0) + \frac{u^2}{2}(\overline{\partial} \psi_2,0 + \tau_0 \overline{\partial} \psi_2,0). \quad (67) \]

We conclude this section by exploiting the remaining coordinate freedom in the choice of the origin of the \( u \) coordinate to further simplify the spin-coefficient results. Recall from (10) that the remaining freedom in \( u \) is
\[ u \rightarrow u^* = Z(\zeta, \overline{\zeta}) u + A(\zeta, \overline{\zeta}), \]
where ‘boosts’ are given with \( Z \) and ‘supertranslations’ with the \( A \). We use the boost freedom to make the spin-coefficient \( \tau \) ‘pure magnetic,’ while the supertranslation freedom will be used to eliminate \( \mu_0 \) altogether.

Recall that under a boost \( Z(\zeta, \overline{\zeta}) \), the tetrad vectors \( l \) and \( n \) must be rescaled as
\[ l \rightarrow l^* = Z^{-1} l, \quad n \rightarrow n^* = Z n, \]
in order to preserve our tetrad conditions. If we write
\[ Z^{-1} \equiv F(\zeta, \overline{\zeta}), \quad (68) \]
then the spin-coefficient \( \tau \) transforms under the boost \( Z \) as
\[ \tau \rightarrow \tau^* = \tau + F^{-1} \overline{\partial} F. \quad (69) \]
Let us assume that the spin-weight one function \( \tau = \tau_0 \) is smooth enough that it can be written as
\[ \tau_0 = \overline{\partial} \tau_0 = \overline{\partial} (\tau_R + i \tau_I), \quad (70) \]
where \( \tau \) is a holomorphic spin-weight zero function on the 2-sphere. By choosing
\[ Z = e^{i \tau}, \quad F = e^{-i \tau}, \quad (71) \]
we see that
\[ \tau_0^* = \tau_0 + F^{-1} \overline{\partial} F = \tau_0 + \overline{\partial} (\log F) \]
\[ = \tau_0 - \overline{\partial} \tau_R = \overline{\partial} (\tau_R + i \tau_I - \tau_R). \quad (72) \]
Hence, with this choice of boost for the \( u \) gauge, we see that \( \tau_0 \) is a pure ‘magnetic’ type function:
\[ \tau_0^* = i \overline{\partial} \tau_I. \quad (73) \]
Note that these results remain true even though \( \overline{\partial} \) is different from \( \partial_0 \).

Finally, we fix the supertranslation freedom by considering the spin-coefficient \( \mu \). Recall that for
\[ \mu = \mu_0 + u \psi_2,0, \]
we have already established that \( \mu_0 \in \mathbb{R} \), so it follows that
\[ \Re(\mu) = \mu_R = \mu_0 + u \Re(\psi_2,0) \]
\[ = \mu_0 - \frac{u}{2} K, \quad (74) \]
where $K$ is the scalar curvature of $u = \text{const}$ cross sections given by equation (45). Under a supertranslation with

$$u^* = u + A, \quad (75)$$

$$A(\zeta, \bar{\zeta}) = \frac{2}{K} \mu_0, \quad (76)$$

the $u$ origin is shifted to the ‘cut’ where $\mu_R$ vanishes. This in turn sets

$$\mu_0^* = 0. \quad (77)$$

To summarize, we have now totally fixed the choice of tetrad and coordinate gauge (except for fractional linear transformations on $\zeta$), resulting in the conditions (dropping the ‘$^*$’ notation) that

$$\omega_0 = 0, \quad \tau_0 = i\partial \tau_I, \quad \mu_0 = 0. \quad (78)$$

These in turn lead to the simplification of several of the expressions in (50)–(67), particularly:

$$\psi_{3,0} = \partial \lambda_0 - \tau_0 \lambda_0.$$

The remaining free functions of $(\zeta, \bar{\zeta})$ are

- complex: $\tau_0, \lambda_0, \gamma_0, \nu_0, \psi_{4,0}$,
- real: $P, U_0, X_A^0$.

A particularly special simple case is when $P = P_0$, i.e., when the 2-metric is that of a sphere.

In the near future, we plan to investigate and hope to be able to assign physical meaning to many of the geometric quantities just defined. For example: where is the interior mass or angular-momentum hiding?

3. $\mathcal{H}$-Shear-free null geodesic congruences

3.1. The good cut equation on $\mathcal{H}$

In the study of $\mathcal{H}$, several fascinating geometric structures (such as $\mathcal{H}$-space and an asymptotic CR geometry) as well as asymptotic physical identifications were discovered by considering null geodesic congruences (NGCs) whose shear was asymptotically vanishing [9–11]. The analogue of such a condition on a non-expanding horizon is to look for those NGCs living exterior to $\mathcal{H}$ whose shear vanishes at the intersection with the horizon. We refer to such NGCs as $\mathcal{H}$-shear-free, and the remainder of this section will be devoted towards their study.

We have at this point a fixed null tetrad $\{l, n, m, \bar{m}\}$ on the NEH, $\mathcal{H}$, where the vector $n$ is the only null vector pointing ‘off’ the horizon (i.e., $n$ is the only vector with a component in the $s$-direction). We now search for other null tetrad systems $\{l^*, n^*, m^*, \bar{m}^*\}$ (at $\mathcal{H}$) that leave $l = l^*$ but force $n^*$ to be shear-free. Now, the shear of the $n$ congruence at $\mathcal{H}$ is given by $-\lambda$ (with $\lambda = \lambda_0(\zeta, \bar{\zeta})$); hence, for an NGC to be $\mathcal{H}$-shear-free, we must be able to transform to a tetrad frame where $\lambda^* \equiv 0$.

Using null rotations of the form (22)

$$l \rightarrow l^* = l, \quad m \rightarrow m^* = m - Ll, \quad n \rightarrow n^* = n - \bar{L}m - \bar{L}\bar{m} + L\bar{L}l, \quad (79)$$

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\(\lambda\) transforms as [15]
\[
\lambda \rightarrow \lambda^* = \lambda - L\pi + 2L^2\varepsilon + L^2\rho - L^3\kappa + \Delta L + L\Delta L. \tag{80}
\]
Simplifying by using our tetrad conditions (23)–(25), the shear-free requirement \(\lambda^* = 0\) yields the \(\mathcal{J}\)-shear-free condition:
\[
\partial L + LL = -\lambda_0(\zeta, \bar{\zeta}), \tag{81}
\]
where \(L = \partial_\mu L\).

Remark. Equation (81) is the same as the asymptotically shear-free condition [16]:
\[
\partial L + LL = \sigma^0(u, \zeta, \bar{\zeta}),
\]
where \(\sigma^0\) is the asymptotic (Bondi) shear at \(\mathcal{J}^+\).

To continue we assume that we are dealing with analytic functions or functions that can be well approximated by analytic functions. We complexify \(\mathcal{J}\) by allowing \(u\) to take on complex values close to the real and \(\bar{\zeta}\) to be independent of, but close to, the complex conjugate of \(\zeta\). Note that this procedure of complexification is covariant since we have assumed the real analyticity of all the relevant functions as well as local coordinate transformations, the analytic continuation into the complex is then independent of the choice of the real coordinates.

To solve equation (81) we transform it—via implicit differentiation—into a simple second-order equation. A complex potential-like function \(T(u, \zeta, \bar{\zeta})\) is introduced, with level-surface values labelled by the complex parameter \(\tau\) (e.g., [9]),
\[
\tau = T(u, \zeta, \bar{\zeta}). \tag{82}
\]
\(T(u, \zeta, \bar{\zeta})\) is defined from \(L(u, \zeta, \bar{\zeta})\) by solutions to the equation
\[
(\partial_{u}T) + LT = 0. \tag{83}
\]
The subscript \((u)\) in \((\partial_{u})\) denotes the application of the \(\partial\) operator while \(u\) is held constant. Later we use \((\partial_{\tau})\), with the analogous meaning. Note that (83) is a CR equation and that both \(T\) and \(\bar{\zeta}\) are CR functions that determine a CR structure from \(L\) on \(\mathcal{J}\). This is in fact the same CR equation one obtains on \(\mathcal{J}^+\) [19]. (See the appendix for details.)

We assume that the relationship \(\tau = T\) can be inverted to give
\[
u = G(\tau, \zeta, \bar{\zeta}), \tag{84}
\]
which for constant \(\tau\) gives a ‘slicing’ of complex \(\mathcal{J}\) and is referred to as a ‘good cut function.’ Using this inversion to change the independent variable from \(u\) to \(\tau\) in equations (81) and (83), where in several steps implicit differentiation is used (cf, [9, 11]), we find, from (83), the relationship between the good cut-function \(G\) and the transformation function \(L\),
\[
L = \partial_{(\tau)} G, \tag{85}
\]
and from (81) the final relation, the ‘good cut equation’ itself [10, 11]:
\[
\partial_{(\tau)}^2 G(\tau, \zeta, \bar{\zeta}) = -\lambda_0(\zeta, \bar{\zeta}). \tag{86}
\]

Before solving equation (86) several remarks are in order.

1. The solution to good cut equation, equation (86), namely \(u = G(\tau, \zeta, \bar{\zeta})\), then gives us parametrically the solution to (81) by
\[
L(u, \zeta, \bar{\zeta}) = \partial_{(\tau)} G(\tau, \zeta, \bar{\zeta}),
\]
\[
u = G(\tau, \zeta, \bar{\zeta}). \tag{87}
\]
(2) There is a very pretty geometric meaning of the null rotation function \( L(u, \zeta, \bar{\zeta}) \). The sphere of null generators of the past light-cone of each point of \( \mathcal{H} \) can be labelled by a stereographic angle with the infinity generator, \( (L = \infty) \) lying on \( \mathcal{H} \). The function \( L(u, \zeta, \bar{\zeta}) \) is the stereographic angle field giving the null directions of the \( \mathcal{H} \)-shear-free vector field \( n^* \) at its intersection with \( \mathcal{H} \).

(3) Ashtekar and colleagues\[5\] introduced their own version of ‘good cuts’. Due to the very different choices of gauge used by them we have not succeeded in finding a relationship between their usage and ours. We suspect that they are unrelated.

3.2. Solving the good cut equation on \( \mathcal{H} \)

Following closely the analogy with the study of asymptotically shear-free NGCs, we show that the solutions to (86) depend only on four complex parameters, i.e. the solution space is a four-complex dimensional manifold. In the case where \( \partial = \partial_0 \) (i.e., where the \( u = \text{const.} \) cross-sections are 2-spheres), this follows easily from the properties of the \( \partial_0 \)-operator on 2-spheres and its operation on spin-weight \( s \) tensorial spherical harmonics. For a general non-expanding horizon however, the situation is slightly more complicated, since \( \partial \) acts on a 2-manifold which is only conformal to a 2-sphere.

We first show that in the homogeneous case (i.e., where \( \lambda_0 = 0 \)), the good cut equation on \( \mathcal{H} \) has a complex four-dimensional solution space. Using this fact, we subsequently prove that this remains true for the general case of (86). We conclude the subsection with the observation that regular \( \mathcal{H} \)-shear-free NGCs are thus generated by complex worldlines in this complex 4-manifold.

To begin, consider the homogeneous good cut equation on \( \mathcal{H} \):

\[
\partial^2 G_0 = 0. \tag{88}
\]

Using (21), (85) and (19), as well as recalling that \( G_0 \) (the general homogeneous solution) is a spin-weight zero function, this can be re-written as

\[
\partial^2 G_0 = \partial \left( P \frac{\partial}{\partial \zeta} G_0 \right) = \partial \left( V \bar{\partial}_0 G_0 \right) = 0
\]

\[
= \frac{\partial}{\partial \zeta} P \left( V \bar{\partial}_0 G_0 \right) = \frac{\partial}{\partial \zeta} P_0 V \left( V \bar{\partial}_0 G_0 \right) = 0
\]

\[
= \bar{\partial}_0 G_0 \cdot \bar{\partial}_0 V^2 + V^2 \partial_0^2 G_0 = 0. \tag{89}
\]

Writing the spin-weight one function \( \partial_0 G_0 \) as

\[
F \equiv \bar{\partial}_0 G_0, \tag{90}
\]

after some algebraic manipulation, we obtain

\[
F^{-1} \bar{\partial}_0 F + V^{-2} \bar{\partial}_0 V^2 = \bar{\partial}_0 [\log (F V^2)] = 0. \tag{91}
\]

In turn this implies that

\[
\bar{\partial}_0 (F V^2) = 0. \tag{92}
\]

Since we have now reduced (88) to a relation involving only the 2-sphere operator, we can easily solve (92), using the properties of \( \bar{\partial}_0 \) and the tensorial spin weight-\( s \) spherical harmonics, as

\[
F V^2 = \zeta' Y^1_{10}(\zeta, \bar{\zeta}), \tag{93}
\]

where \( \zeta' \) are three arbitrary complex parameters. Equation (90) is thus

\[
\bar{\partial}_0 G_0 = V^{-2} \zeta' Y^1_{10}, \tag{94}
\]
with the general solution

\[ G_0(z^a, \zeta, \bar{\zeta}) = \epsilon^0 + \oint_{S^2} K_1(\zeta, \bar{\zeta}; \eta, \bar{\eta}) V^{-2} \epsilon^j Y^1_j \, dS_1. \quad (95) \]

\( K_1 \) is a known Green’s function for the \( \partial_0 \)-operator and the measure on \( S^2 \) is \[ dS_1 = -2i \frac{d\eta \wedge d\bar{\eta}}{(1 + \eta\bar{\eta})^2}. \quad (96) \]

We thus see that the homogeneous good cut equation on \( \mathcal{H} \) does indeed depend on four complex parameters, denoted \( z^a \in \mathbb{C} \). The question remains: does this generalize to the fully inhomogeneous case? The answer is in the affirmative. Let us assume that the general solution to the (inhomogeneous) good cut equation can be written as

\[ G = G_0 g, \quad (97) \]

for some undetermined function \( g \). Then we have, using equation (88),

\[ \bar{\partial}^2 (G_0 g) = G_0 \bar{\partial}^2 g + 2 \bar{\partial} G_0 \partial g = -\lambda_0, \quad (98) \]

which, using

\[ f \equiv \partial g, \quad (99) \]

reduces to

\[ \partial (G_0^2 f) = -G_0 \lambda_0. \quad (100) \]

Again recalling the relationship between the \( \bar{\partial} \) and \( \partial_0 \)-operators, this is rewritten as

\[ \partial_0 \left( V G_0^2 f \right) = -G_0 \lambda_0, \quad (101) \]

implying that

\[ f = \bar{\partial} g = -V^{-1} G_0^{-2} \oint_{S^2} K_2(\zeta, \bar{\zeta}; \eta, \bar{\eta}) G_0 \lambda_0 \, dS_2. \quad (102) \]

Noting that \( \partial g = V \partial_0 g \), we obtain

\[ g = -\oint_{S^2} \left( K_1(\zeta, \bar{\zeta}; \eta_2, \bar{\eta}_2) V^{-2}(\eta_2, \bar{\eta}_2) G_0^{-2}(\eta_2, \bar{\eta}_2) \right. \]

\[ \times \left. \oint_{S^2} K_2(\eta_2, \bar{\eta}_2; \eta_1, \bar{\eta}_1) G_0(\eta_1, \bar{\eta}_1) \lambda_0(\eta_1, \bar{\eta}_1) \, dS_1 \right) \, dS_2. \quad (103) \]

Combining this with equation (97), we see that the general solution \( G(\tau, \zeta, \bar{\zeta}) \) depends only on the shear data \( \lambda_0(\zeta, \bar{\zeta}) \) and the four complex parameters of the homogeneous solution, \( z^a \in \mathbb{C} \). It indeed follows that solutions to the good cut equation on \( \mathcal{H} \) live in a complex four-dimensional manifold in the same fashion as solutions to the asymptotic good cut equation!

As pointed out to us by R Penrose, the above construction also allows us to construct twister space on the complexified \( \mathcal{H} \) in a fashion analogous to the construction on future null infinity.

Finally to find the solutions with \( \tau \) dependence all that must be done is to replace \( z^a \) by an arbitrary analytic function of \( \tau \), i.e., by \( z^a(\tau) \). Arbitrary analytic curves in the associated complex four-dimensional space generate solutions to the good cut equation which in turn generate, via the parametric relations, equation (87), \( \mathcal{H} \)-shear-free null geodesic congruences in the neighbourhood of \( \mathcal{H} \). As mentioned earlier, it follows that every choice of such worldline induces a CR structure on the horizon (see the appendix).

It should be noted that if the ambient spacetime is algebraically special, type D, then it—the spacetime itself—possesses a unique shear-free congruence which is also \( \mathcal{H} \)-shear-free.
thereby endowing \( \mathcal{H} \) with a unique CR structure. Since the horizon itself is type II there are no other vacuum spacetimes with a (totally) shear-free NGC. For more general spacetimes, one can ask: is there a way to canonically single out a particular choice of worldline, thereby choosing a unique \( \mathcal{H} \)-shear-free CR structure for any vacuum NEH? This issue is addressed in the following subsection.

### 3.3. Unique choice of worldline

In this final subsection, we describe a method for singling out a unique worldline \( \zeta^a(\tau) \) which in turn generates a unique solution to the good cut equation (86) on \( \mathcal{H} \) and a related unique CR structure. Again, we take our motivation from the extant physical identification theory on \( \mathcal{I}^+ \), which uses similar concerns to single out a unique complex worldline in \( \mathcal{H} \)-space [9–11].

In the asymptotic theory, a unique worldline is singled out by transforming to a tetrad frame where the complex gravitational dipole moment (proportional to the \( l = 1 \) spherical harmonic contribution to \( \psi_0 \)) vanishes. In order to do a similar transformation on a vacuum NEH, we must first identify what will serve as our complex gravitational dipole at \( \mathcal{H} \).

The best choice appears to be the \( l = 1 \) contribution from \( \bar{\psi}_3 \), which has the proper spin-weight (\( s = 1 \)) and transformation behaviour to serve as a base for the description of a dipole. Performing a spherical harmonic expansion

\[
\bar{\psi}_3 = \bar{\psi}_3 Y_{11}^1 + \bar{\psi}_3 Y_{21}^1 + \cdots,
\]

we write

\[
\bar{\psi}_3 = -\frac{6\sqrt{2G}}{c^2} D^l_C = -\frac{6\sqrt{2G}}{c^2} \left( D^l_{(\text{mass})} + ic^{-1} f^l \right),
\]

where \( D^l_C \) is identified as (some form) of a complex ‘gravitational’ dipole (cf, [11]). The process of singling out a unique worldline then becomes the task of transforming to a tetrad frame where \( D^l_C \), or equivalently the new \( \bar{\psi}_3^* \), vanishes. To do this, we perform a null rotation about the null vector \( l \) using the function \( L = \partial G \) from (87). Under such a null rotation, the Weyl tensor component \( \bar{\psi}_3 \) transforms as [15]

\[
\bar{\psi}_3 \rightarrow \bar{\psi}_3^* = \bar{\psi}_3 - 3L\bar{\psi}_2.
\]

The ‘centre-of-mass’ condition, \( \bar{\psi}_3^* = 0 \), thus leads to

\[
0 = \bar{\psi}_3^* - 3(L\bar{\psi}_2)^l, \tag{105}
\]

where \(|l|\) means, ‘extract the \( l = 1 \) part of the product’. Now, both \( \psi_3 \) and \( \psi_2 \) are quantities given in terms of the free data on \( \mathcal{H} \), while the function \( L \) carries the information about the good cut function via the complex worldline \( \zeta^a(\tau) \). Hence, (105) is an algebraic equation for the choice of complex worldline in terms of the free data on the horizon.

This means that we not only single out a particular complex worldline as seen from \( \mathcal{H} \), but also obtain a unique solution to the good cut equation, which induces a unique CR structure on the horizon!

### 4. Discussion and conclusion

In this paper we have set out to do several things. First, we returned to the old topic of non-expanding horizons and gave a complete description of their vacuum geometry using the spin-coefficient formalism. In particular, we exploited the gauge freedom to construct a unique null tetrad and, essentially, a unique choice of coordinates on the horizon. In addition, we reduced the amount of free data on the horizon to five complex and three real quantities.
Besides generalizing the earlier work of Pajerski and extending the more recent results of Ashtekar and others with the spin-coefficient formalism, this also provided us with a platform for investigating the geometric structure induced on NEHs by considering \( \mathfrak{H} \)-shear-free null geodesic congruences.

In section 3, we demonstrated that looking for such \( \mathfrak{H} \)-shear-free congruences results in a good cut equation (86) that closely resembles the well-studied asymptotic good cut equation on \( \mathcal{I}^+ \). Additionally, it was shown that solutions to this equation lie in a complex four-dimensional space, and that arbitrary choices of worldlines in this space generate analytic \( \mathfrak{H} \)-shear-free null geodesic congruences at the horizon. We also showed that by making an identification of a complex dipole term, it is possible to single out a unique such worldline that corresponds—in some sense—to the complex centre of mass. This particular worldline would then induce a unique solution to the good cut equation and thus allow a unique CR structure on the horizon associated with the \( \mathfrak{H} \)-shear-free null geodesic congruence. This unique good cut function is the direct analogue of the universal cut function on \( \mathcal{I}^+ \) of asymptotically flat spacetimes, which is derived in a similar fashion [9, 11].

Through much of this paper, we have motivated many of our calculations using the analogy between a NEH \( \mathfrak{H} \) and future null infinity \( \mathcal{I}^+ \). The next step one would like to take in this analogy would be to construct a means of physical identification on non-expanding horizons using the same tools as in asymptopia. It is at this point where the analogy may break down. It is not at all clear what is the relationship—if any—between the four-dimensional solution space of the good cut equation, when \( P \) does not represent a sphere metric, and the \( \mathcal{H} \)-space arising from the \( P_0 \). An immediate question would be: does this new four-dimensional space define a complex metric analogous to that of \( \mathcal{H} \)-space? This, as well as the issue of giving a physical identification of the worldline, are open questions being investigated.

Despite these difficulties, it should be noted that for the special case where \( u = \text{const} \) cross sections of \( \mathfrak{H} \) have a metric factor \( P = P_0 \) (i.e., spheres), the ambiguities just mentioned largely disappear, and it might be possible to proceed with a physical identification theory.

A final question that is raised by our work here regards the manifold in which the worldlines generating \( \mathfrak{H} \)-shear-free null geodesic congruences live. For \( \mathcal{I}^+ \), a surprising metric construction [18] allowed such worldlines to be interpreted as lying in a complex Minkowski space or \( \mathcal{H} \)-space. We hope that the solution manifold that appears for horizons is isomorphic (or related in some sense) to \( \mathcal{H} \)-space, or at least a complex deformation away from it, but this is currently an open research question.

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Appendix. CR structures on \( \mathfrak{H} \)

A CR structure on a real 3-manifold \( \mathfrak{H} \), with local coordinates \( x^a \), is given intrinsically by equivalence classes of 1-forms: one real, one complex and its complex conjugate [20]. If we denote the real 1-form by \( \mathbb{L} \) and the complex 1-form by \( \mathfrak{M} \), then these are defined up to the transformations:

\[
\begin{align*}
\mathbb{L} &\rightarrow a(x^a) \mathbb{L}, \\
\mathfrak{M} &\rightarrow f(x^a) \mathfrak{M} + g(x^a) \mathbb{L}.
\end{align*}
\]  

(A.1)
(a, f, g) are functions on \( H \): a is non-vanishing and real, f and g are complex function with f non-vanishing. Further it is required that there be a three-fold linear independence relation between these 1-forms [20]:

\[ \Lambda \wedge \mathcal{M} \wedge \overline{\mathcal{M}} \neq 0. \]  
(A.2)

Any 3-manifold with a CR structure is referred to as a three-dimensional CR manifold. There are special classes (referred to as embeddible) of 3D CR manifolds that can be directly embedded into \( C^2 \). For the NEH \( H \), we have the differential equation—the so-called CR equation,

\[ \partial K + L \mathcal{K} = P \frac{\partial K}{\partial \zeta} + L \frac{\partial K}{\partial u} = 0, \]  
(A.3)

where the two linearly independent solutions \( K_{1,2} : H \to \mathbb{C} \) give the embedding of the NEH into \( C^2 \). The first of these solutions is rather obvious:

\[ K_1(u, \zeta, \overline{\zeta}) = \overline{\zeta}, \]  
(A.4)

and the second is given by the complex potential function (see equation (83))

\[ K_2(u, \zeta, \overline{\zeta}) = T(u, \zeta, \overline{\zeta}) = \tau. \]  
(A.5)

To prove that (A.4) and (A.5) indeed describe a CR structure on \( H \), we must show that we can derive the class of 1-forms (A.1).

This is done in the following manner: let \((\overline{\tau}, \zeta)\) be coordinates on \( C^2 \); when these coordinates are restricted to \( H \subset C^2 \), we get

\[ \overline{\tau}|_{H} = \tau, \quad \zeta|_{H} = \zeta = x - iy. \]

Now, taking the exterior derivatives of these quantities (restricted to \( H \)) gives us two (complex) 1-forms on the horizon:

\[ \mathcal{R}_1 = d\overline{\tau}, \]  
(A.6)

\[ \mathcal{R}_2 = d\tau = \frac{\partial T}{\partial x^a} dx^a \\
= \tilde{T} du + \frac{\partial T}{\partial \zeta} d\zeta + \frac{\partial T}{\partial \overline{\zeta}} d\overline{\zeta}, \]  
(A.7)

where \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) are defined up to bi-holomorphic transformations on the coordinates \( \overline{\tau} \) and \( \zeta \) of \( C^2 \) [21]. That is, we retain the freedom:

\[ \mathcal{R}_1 \to f(x^a) \mathcal{R}_1 + g(x^a) \mathcal{R}_2, \quad \mathcal{R}_2 \to h(x^a) \mathcal{R}_1 + j(x^a) \mathcal{R}_2. \]  
(A.8)

We can use (A.8) to make \( \mathcal{R}_2 \) a real 1-form by dividing (A.7) by \( \tilde{T} \) and then using (83) to obtain

\[ \mathcal{R}_2 \to \mathcal{L} = du - \frac{L}{P} d\zeta - \frac{T}{P} d\overline{\zeta}. \]

This reduces the remaining freedom in the forms to (A.1) and gives the required pair of 1-forms (one real, one complex):

\[ \mathcal{L} = du - \frac{L}{P} d\zeta - \frac{T}{P} d\overline{\zeta}, \quad \mathcal{M} = d\overline{\zeta}, \]

which define a CR structure on \( H \), as required.

Note that for each \( L \) (i.e., for each choice of complex worldline \( z^a(\tau) \)), we obtain a different CR structure. In particular, for the worldline corresponding to the complex centre of mass via equation (105), there is a unique CR structure on the horizon generated by the ‘complex centre of mass.’
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