TIMELIKE SURFACES OF CONSTANT MEAN CURVATURE
\[ \pm 1 \] IN ANTI-DE SITTER 3-SPACE \( \mathbb{H}^3_1(-1) \)

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ABSTRACT. It is shown that timelike surfaces of constant mean curvature \( \pm 1 \) in anti-de Sitter 3-space \( \mathbb{H}^3_1(-1) \) can be constructed from a pair of Lorentz holomorphic and Lorentz antiholomorphic null curves in \( \text{P}SL_2 \mathbb{R} \) via Bryant type representation formulae. These Bryant type representation formulae are used to investigate an explicit one-to-one correspondence, the so-called \textit{Lawson-Guichard correspondence}, between timelike surfaces of constant mean curvature \( \pm 1 \) in \( \mathbb{H}^3_1(-1) \) and timelike minimal surfaces in Minkowski 3-space \( \mathbb{E}^3_1 \). The hyperbolic Gauss map of timelike surfaces in \( \mathbb{H}^3_1(-1) \), which is a close analogue of the classical Gauss map is considered. It is discussed that the hyperbolic Gauss map plays an important role in the study of timelike surfaces of constant mean curvature \( \pm 1 \) in \( \mathbb{H}^3_1(-1) \). In particular, the relationship between the Lorentz holomorphicity of the hyperbolic Gauss map and timelike surface of constant mean curvature \( \pm 1 \) in \( \mathbb{H}^3_1(-1) \) is studied.

1. INTRODUCTION

It is known that surfaces of constant mean curvature \( \pm 1 \) surfaces in hyperbolic 3-space \( \mathbb{H}^3(-1) \) can be constructed from holomorphic null curves in \( \text{P}SL_2 \mathbb{C} = \text{SL}_2 \mathbb{C}/\{ \pm \text{id} \} \) \([4, 23]\), while minimal surfaces in Euclidean 3-space \( \mathbb{E}^3 \) can be constructed from holomorphic null curves in \( \mathbb{C}^3 \) via well-known Weierstraß-Enneper representation formula. It is also known that spacelike surfaces of constant mean curvature \( \pm 1 \) in de-Sitter 3-space \( \mathbb{S}^3_1(1) \) can be constructed from holomorphic null curves in \( \text{P}SL_2 \mathbb{C} \) \([2, 16]\), while spacelike maximal surfaces in Minkowski 3-space \( \mathbb{E}^3_1 \) can be constructed from holomorphic null curves in \( \mathbb{C}^3 \) via an analogue of Weierstraß-Enneper representation formula \([18, 14]\). These are all related by the \textit{Lawson-Guichard correspondence} between minimal surfaces in \( \mathbb{E}^3 \) and surfaces of constant mean curvature \( \pm 1 \) in \( \mathbb{H}^3(-1) \) \([15]\) and the one between spacelike maximal surfaces in \( \mathbb{E}^3_1 \) and spacelike surfaces of constant mean curvature \( \pm 1 \) \([21]\). Note that the
correspondents (they are usually called the cousins) in different space forms satisfy the same Gauß and Mainardi-Codazzi equations.

It is interesting to see that there exists a Lawson-Guichard correspondence between timelike minimal surfaces in \( E^2_3 \) and timelike surfaces of constant mean curvature \( \pm 1 \) in anti-de Sitter 3-space \( H^3_1(-1) \). See sections 5, 9, and 14 (appendix I) for details. In [13], J. Inoguchi and M. Toda show that timelike minimal surfaces can be constructed from a pair of Lorentz holomorphic and Lorentz antiholomorphic null curves in \( \mathbb{R}^3 \) via normalized Weierstraß formula (59). Hence, one might expect a similar construction of timelike surfaces of constant mean curvature \( \pm 1 \) in \( H^3_1(-1) \) in terms of Lorentz holomorphic and Lorentz antiholomorphic null curves. In this paper, we prove that a pair of Lorentz holomorphic and Lorentz antiholomorphic null curves in \( \text{PSL}_2 \mathbb{R} \) gives rise to a timelike surface of constant mean curvature \( \pm 1 \) in \( H^3_1(-1) \). Furthermore, every timelike surface of constant mean curvature \( \pm 1 \) in \( H^3_1(-1) \) can be constructed from a pair of Lorentz holomorphic and Lorentz antiholomorphic null curves in \( \text{PSL}_2 \mathbb{R} \).

An analogue of the hyperbolic Gauß map\(^1\) can be defined for timelike surfaces of constant mean curvature in \( H^3_1(-1) \) and plays an important role in studying timelike surfaces of constant mean curvature \( \pm 1 \) in \( H^3_1(-1) \). It is shown in section 13 that

1. The hyperbolic Gauß map was introduced by C. Epstein in [7] and used by R. L. Bryant to study \( \text{cmc} \, 1 \) surfaces in \( H^3(-1) \) of surfaces in hyperbolic 3-space \( H^3(-1) \) in [4].
The anti-de Sitter (abbreviated: AdS) 3-space $\mathbb{H}^3_1(-1)$ is a Lorentzian 3-manifold of sectional curvature $-1$ that can be realized as the hyperquadric in $E^4_2$:

$$\mathbb{H}^3_1(-1) := \{(x_0, x_1, x_2, x_3) \in E^4_2 : -(x_0)^2 - (x_1)^2 + (x_2)^2 + (x_3)^2 = -1\}.$$

Let $M$ be a connected orientable 2-manifold and $\varphi : M \rightarrow \mathbb{H}^3_1(-1)$ an immersion. The immersion $\varphi$ is said to be *timelike* if the induced metric $I$ on $M$ is Lorentzian. The induced Lorentzian metric $I$ determines a Lorentzian conformal structure $C_I$ on $M$.

Let $(x, y)$ be a Lorentz isothermal coordinate system with respect to the conformal structure $C_I$. Then the first fundamental form $I = \langle d\varphi, d\varphi \rangle$ is given by the matrix

$$I = e^\omega \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The first fundamental form is also written in terms of $(x, y)$ as $I = e^\omega \{-dx^2 + (dy)^2\}$. Let $u := x + y$ and $v := -x + y$. Then $(u, v)$ defines a *null coordinate system* with respect to the conformal structure $C_I$. The first fundamental form $I$ is written in terms of $(u, v)$ as

$$I = e^\omega du dv.$$

In terms of null coordinates $u$ and $v$, the differential operators $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ are computed to be

$$\frac{\partial}{\partial u} = \frac{1}{2} \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial v} = \frac{1}{2} \left( -\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right).$$

The conformality condition is equivalent to

$$\langle \varphi_u, \varphi_u \rangle = \langle \varphi_v, \varphi_v \rangle = 0, \quad \langle \varphi_u, \varphi_v \rangle = \frac{1}{2} e^\omega.$$

Let $N$ be a unit normal vector field of $M$. Then

$$\langle N, N \rangle = 1, \quad \langle \varphi, N \rangle = \langle \varphi_u, N \rangle = \langle \varphi_v, N \rangle = 0.$$

The mean curvature $H$ is given by $H = 2e^{-\omega} < \varphi_{uv}, N >$. Let $Q := < \varphi_{uu}, N >$ and $R := < \varphi_{vv}, N >$. Then the quadratic differentials $Q^2 := Qdu \otimes du$ and $R^2 := Rdv \otimes dv$ are called *Hopf pairs* of $M$. The quadratic differential

$$Q := Qdu^2 + Rdv^2 = Q^2 + R^2$$

is called *Hopf differential*\(^3\). This differential is globally defined on the Lorentz surface $(M, C_I)$. The second fundamental form $II$ of $M$ derived from $N$ is defined by

$$II := -\langle d\varphi, dN \rangle$$

\(^2\)In [6], [11], [13], the quadratic differentials $Q^2$ and $R^2$ are defined as Hopf differentials.\(^3\)The definition of Hopf differential $Q$ was suggested to the author by J. Inoguchi [12].
and it is given by the matrix

\[ II = \begin{pmatrix} Q + R - H e^\omega & Q - R \\ Q - R & Q + R + H e^\omega \end{pmatrix} \]

with respect to Lorentz isothermal coordinate system \((x, y)\). The second fundamental form is related to Hopf differential \(Q\) by

\[ II = Q + HI. \]  

The shape operator \(S\) of \(M\) derived from \(N\) is \(S := -dN\). The shape operator \(S\) is related to \(II\) by

\[ II(X, Y) = \langle SX, Y \rangle \]

for all vector fields \(X, Y\) on \(M\). The shape operator \(S\) is also represented by the matrix \(II \cdot I^{-1}\). The mean curvature \(H\) of \(M\) is

\[ H = \frac{1}{2} \text{tr} S = \frac{1}{2} \text{tr}(II \cdot I^{-1}) \]

and the Gaußian curvature\(^4\) \(K\) of \(M\) is

\[ K := -1 + \det S = -1 + \det(II \cdot I^{-1}). \]

The eigenvalues of \(S\), i.e., the solutions to the characteristic equation

\[ \det(S - \lambda I) = 0, \quad I = \text{identity of } T M \]

are called the principal curvatures. Since the metric \(I\) is indefinite, both principal curvatures may be nonreal complex numbers. The mean curvature \(H\) is the mean of the two principal curvatures and the Gaußian curvature \(K\) is the product of the two principal curvatures minus one.

A point \(p \in M\) is said to be an umbilic point if \(II\) is proportional to \(I\) at \(p\). Equivalently, \(p\) is an umbilic point if and only if the two principal curvatures at \(p\) are the same real number and the corresponding eigenspace is 2-dimensional. A timelike surface is said to be a totally umbilic if all the points are umbilical. The formula \(\text{(1)}\) implies that \(p \in M\) is an umbilic point if and only if \(Q(p) = 0\), i.e., \(p \in M\) is a common zero of Hopf pairs \(Q\) and \(R\).

The Gauß equation which describes a relationship between \(K, H, Q\) and \(R\) takes the following form:

\[ H^2 - K - 1 = 4e^{-2\omega}QR. \]

Note that the condition \(QR = 0\) does not imply the condition \(Q = R = 0\) (See \(\text{(15)}\)).

Let \(M\) be a simply-connected open and orientable 2-manifold and \(\varphi : M \rightarrow \mathbb{H}^2(-1)\) a timelike conformal immersion with unit normal vector field \(N\). Then we can define an orthonormal frame field \(\mathcal{F}\) along \(\varphi\) by

\[ \mathcal{F} = (\varphi, e^{-\frac{\pi}{2}}\varphi_x, e^{-\frac{\pi}{2}}\varphi_y, N) : M \rightarrow O^+(2, 2), \]

\(^4\)This can be easily computed from the Gauß equation \(\text{(15)}\).
where $O^{++}(2,2)$ denotes the identity component of the Lorentz group

$$O(2,2) = \{ A \in \text{GL}_4 \mathbb{R} : < A \mathbf{u}, A \mathbf{v} > = < \mathbf{u}, \mathbf{v} >, \; \mathbf{u}, \mathbf{v} \in \mathbb{E}^4_2 \}.$$  

In terms of null coordinates $(u, v)$, $F$ is defined by

$$F = (\varphi, e^{-\frac{\varphi}{2}}(\varphi_u - \varphi_v), e^{-\frac{\varphi}{2}}(\varphi_u + \varphi_v), N) : M \rightarrow O^{++}(2,2).$$

The semi-Euclidean 4-space $\mathbb{E}^4_2$ is identified with the linear space $M_2 \mathbb{R}$ of all $2 \times 2$ real matrices via the correspondence

$$\mathbf{u} = (x_0, x_1, x_2, x_3) \leftrightarrow \begin{pmatrix} x_0 + x_3 & x_1 + x_2 \\ -x_1 + x_2 & x_0 - x_3 \end{pmatrix}.$$

The scalar product of $\mathbb{E}^4_2$ corresponds to the scalar product

$$< \mathbf{u}, \mathbf{v} > = \frac{1}{2} \{ \text{tr}(\mathbf{u}\mathbf{v}) - \text{tr}(\mathbf{u}) \text{tr}(\mathbf{v}) \}, \; \mathbf{u}, \mathbf{v} \in M_2 \mathbb{R}.$$

Note that $< \mathbf{u}, \mathbf{u} > = -\det \mathbf{u}$. The standard basis $e_0, \; e_1, \; e_2, \; e_3$ for $\mathbb{E}^4_2$ is identified with the matrices

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; k = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

i.e.,

$$x_0 \mathbf{1} + x_1 \mathbf{i} + x_2 \mathbf{j} + x_3 \mathbf{k}' = \begin{pmatrix} x_0 + x_3 & x_1 + x_2 \\ -x_1 + x_2 & x_0 - x_3 \end{pmatrix}.$$

Note that the $2 \times 2$ matrices $x_0 \mathbf{1} + x_1 \mathbf{i} + x_2 \mathbf{j} + x_3 \mathbf{k}'$ form the algebra $\mathbb{H}'$ of split-quaternions. (For more details, see, for example, [13].) Under the identification $\mathbb{E}^4_2$, the group $G$ of timelike unit vectors corresponds to a special linear group

$$\text{SL}_2 \mathbb{R} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2 \mathbb{R} : ad - bc = 1 \right\}.$$

The metric of $G$ induced by the scalar product $< \cdot, \cdot >$ is a bi-invariant Lorentz metric of constant curvature $-1$. Hence, $G$ is identified with $\mathbb{H}'^1(-1)$.  

3. CARTAN’S FORMALISM

Let $\{ e_\alpha : \alpha = 0, 1, 2, 3 \}$ be a frame field of $\mathbb{E}^4_2$, i.e., $\{ e_\alpha(p) : \alpha = 0, 1, 2, 3 \}$ is a basis for the tangent space $T_p \mathbb{E}^4_2$ at each $p \in \mathbb{E}^4_2$. Denote by $\langle \cdot, \cdot \rangle_p$ the scalar product on the tangent space $T_p \mathbb{E}^4_2$, $p \in \mathbb{E}^4_2$. Then

$$< e_\alpha, e_\beta > = \begin{cases} -1 & \text{if } \alpha = \beta = 0 \text{ or } 1, \\ 0 & \text{if } \alpha \neq \beta, \\ 1 & \text{if } \alpha = \beta = 2 \text{ or } 3. \end{cases}$$

There exist unique connection 1-forms $\{ \omega^\alpha_\beta : \alpha, \beta = 0, 1, 2, 3 \}$ such that

$$de_\alpha = \omega^\alpha_\beta e_\beta.$$
We use the index range $1 \leq i, j, k \leq 3$ and denote by $\omega^i$ the connection form $\omega^i_\alpha$. Then the equation (7) can be written

$$\omega^0_i = \omega^i_0 e_0 + \omega^i_1 e_2 + \omega^i_2 e_3,$$

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$$\omega^0_i = \omega^i_0 e_0 + \omega^i_1 e_2 + \omega^i_2 e_3.$$

The connection 1-forms $\{\omega^\beta_\alpha : \alpha, \beta = 0, 1, 2, 3\}$ satisfy:

$$\omega^1_0 = -\omega^0_1, \quad \omega^0_0 = 0, \quad \omega^i_0 = \omega^i_1, \quad \omega^i_0 = \omega^i_1, \quad i = 0, 1, 2, 3,$$

$$\omega^0_i = \omega^i_0, \quad i = 2, 3.$$

Differentiating the equation (8) we get the first structure equation:

$$d\omega^i = -\omega^i_0 \wedge \omega^0_1.$$

Differentiating this first structure equation (11) we get the second structure equation:

$$d\omega^i_0 = -\omega^i_0 \wedge \omega^0_1.$$

For the frame field $\mathcal{F}$ of timelike immersion $\varphi : M \longrightarrow \mathbb{H}_3^1(1)$, we have

$$d\omega^1 = \omega^2 \wedge \omega^3 \quad \text{(The First Structure Equations)}$$

$$d\omega^2 = \omega^1 \wedge \omega^3 \quad \text{(Symmetry Equation)}$$

$$d\omega^3 = -\omega^1 \wedge \omega^2 \quad \text{(Gauß Equation)}$$

$$d\omega^3 = -\omega^1 \wedge \omega^2 \quad \text{(Mainardi-Codazzi Equations)}$$

Proposition 1. Let $\varphi : M \longrightarrow \mathbb{H}_3^1(1)$ be a timelike immersion. If $\{e_1 = e^{-\frac{2}{3}} \varphi_x e_2 = e^{-\frac{2}{3}} \varphi_y e_3 = N\}$ forms an adapted frame field along $\varphi$, then the Gaussian curvature $K$ and mean curvature $H$ of $\varphi$ satisfy the following equations:

$$\omega^1_0 \wedge \omega^2_0 = (K + 1) \omega^1_0 \wedge \omega^2_0,$$

$$\omega^1_0 \wedge \omega^2_0 + \omega^1_0 \wedge \omega^2_0 = -2H \omega^1_0 \wedge \omega^2_0.$$

Proof. From the symmetry equation (15), we see that there exist smooth functions $h_{ij}$, $i, j = 1, 2$ such that

$$\begin{pmatrix} \omega^1_0 \\ \omega^2_0 \end{pmatrix} = \begin{pmatrix} -h_{11} & h_{12} \\ h_{21} & -h_{22} \end{pmatrix} \begin{pmatrix} \omega^1_0 \\ \omega^2_0 \end{pmatrix} \text{ and } h_{12} = -h_{21}. $$
Note that $\omega^1$ and $\omega^2$ are the dual 1-forms of $e_1$ and $e_2$, resp., and so
\[
\omega_1^1 \wedge \omega_2^2 = (h_{11}h_{22} + h_{12}^2)\omega^1 \wedge \omega^2 = (K + 1)\omega^1 \wedge \omega^2,
\]
where $K$ is the Gaussian curvature of $\varphi$. Thus, the Gaussian equation (10) can be written as
\[
d\omega_2^1 = K\omega^1 \wedge \omega^2.
\]
The mean curvature $H$ of $\varphi$ is $\frac{h_{11} + h_{22}}{2}$.
Hence,
\[
\omega_3^1 \wedge \omega^2 + \omega^1 \wedge \omega_3^2 = -(h_{11} + h_{22})\omega^1 \wedge \omega^2 = -2H\omega^1 \wedge \omega^2.
\]
\[\square\]

4. Lie Group Actions $\mu$ and $\nu$ on $\mathbb{E}_2^4$

The Lie group $\text{SL}_2\mathbb{R} \times \text{SL}_2\mathbb{R}$ acts isometrically on $\mathbb{E}_2^4$ via the group action:
\[
\mu : (\text{SL}_2\mathbb{R} \times \text{SL}_2\mathbb{R}) \times \mathbb{E}_2^4 \to \mathbb{E}_2^4; \quad \mu(g_1, g_2)u = g_1u g_2^t.
\]
This action is transitive on $\mathbb{H}_1^3(-1)$. The isotropy subgroup of $\text{SL}_2\mathbb{R} \times \text{SL}_2\mathbb{R}$ at $1$ is $K = \{(g, (g^{-1})^t) : g \in \text{SL}_2\mathbb{R}\}$ and $\mathbb{H}_1^3(-1)$ is represented as the Lorentzian symmetric space $\text{SL}_2\mathbb{R} \times \text{SL}_2\mathbb{R}/K$. The natural projection $\pi_\mu : \text{SL}_2\mathbb{R} \times \text{SL}_2\mathbb{R} \to \mathbb{H}_1^3(-1)$ is given explicitly by $\pi_\mu(g_1, g_2) = g_1 g_2^t$.

The Lie group $\text{SL}_2\mathbb{R} \times \text{SL}_2\mathbb{R}$ also acts isometrically on $\mathbb{E}_2^4$ via the diagonal action:
\[
\nu : (\text{SL}_2\mathbb{R} \times \text{SL}_2\mathbb{R}) \times \mathbb{E}_2^4 \to \mathbb{E}_2^4; \quad \nu(g_1, g_2)u = g_1u g_2^{-1}.
\]
This action is also transitive on $\mathbb{H}_1^3(-1)$. The isotropy subgroup of $\text{SL}_2\mathbb{R} \times \text{SL}_2\mathbb{R}$ at $1$ is the diagonal subgroup $\Delta$ of $\text{SL}_2\mathbb{R} \times \text{SL}_2\mathbb{R}$, that is, $\Delta = \{(g, g) : g \in \text{SL}_2\mathbb{R}\}$ and $\mathbb{H}_1^3(-1)$ is also represented by $\text{SL}_2\mathbb{R} \times \text{SL}_2\mathbb{R}/\Delta$ as a Lorentzian symmetric space. The natural projection $\pi_\nu : \text{SL}_2\mathbb{R} \times \text{SL}_2\mathbb{R} \to \mathbb{H}_1^3(-1)$ is given explicitly by
\[
\pi_\nu(g_1, g_2) = g_1 g_2^{-1}, \quad (g_1, g_2) \in \text{SL}_2\mathbb{R} \times \text{SL}_2\mathbb{R}.
\]
Moreover, $\text{SL}_2\mathbb{R}$ acts isometrically on $\mathbb{E}_1^3$ via the $\text{Ad}$-action:
\[
\text{Ad} : \text{SL}_2\mathbb{R} \times \mathbb{E}_1^3 \to \mathbb{E}_1^3; \quad \text{Ad}(g)u = gug^{-1}, \quad g \in \text{SL}_2\mathbb{R}, \quad u \in \mathbb{E}_1^3.
\]
The actions $\mu$ and $\nu$ both induces a double covering $\text{SL}_2\mathbb{R} \times \text{SL}_2\mathbb{R} \to O^{++}(2, 2)$ of the Lorentz group $O^{++}(2, 2)$.

Remark 1. In [10], J. Q. Hong used the group action $\mu$ to study a Bryant type representation formula for timelike $\text{cmc}$ 1 surfaces in $\mathbb{H}_1^3(-1)$, In [1], R. Aiyama and K. Akutagawa also used the action $\mu$ to study Kenmotsu-Bryant type representation formula for spacelike $\text{cmc}$ surfaces in $\mathbb{H}_1^3(-1)$. In this paper, we use both actions $\mu$ and $\nu$. 

The frame field \( \{ e_\alpha : \alpha = 0, 1, 2, 3 \} \) can be parametrized by the Lie group \( \text{SL}_2 \mathbb{R} \times \text{SL}_2 \mathbb{R} \) via the Lie group action \( \mu \): for each \( g = (g_1, g_2) \in \text{SL}_2 \mathbb{R} \times \text{SL}_2 \mathbb{R} \),

\[
{e}_0(g) := \mu(g)(1) = g_1 1 g_2,
\]

\[
e_1(g) := \mu(g)(i) = g_1 i g_2,
\]

\[
e_2(g) := \mu(g)(j') = g_1 j' g_2,
\]

\[
e_3(g) := \mu(g)(k') = g_1 k' g_2.
\]

The frame field \( \{ e_\alpha : \alpha = 0, 1, 2, 3 \} \) can also be parametrized by the Lie group \( \text{SL}_2 \mathbb{R} \times \text{SL}_2 \mathbb{R} \) via the Lie group action \( \nu \): for each \( g = (g_1, g_2) \in \text{SL}_2 \mathbb{R} \times \text{SL}_2 \mathbb{R} \),

\[
e_0(g) := \nu(g)(1) = g_1 1 g_2^{-1},
\]

\[
e_1(g) := \nu(g)(i) = g_1 i g_2^{-1},
\]

\[
e_2(g) := \nu(g)(j') = g_1 j' g_2^{-1},
\]

\[
e_3(g) := \nu(g)(k') = g_1 k' g_2^{-1}.
\]

We need the following two equations in order to do some differential geometric computations in Sections 8 and 4.

**Lemma 2.** If the frame field \( \{ e_\alpha : \alpha = 0, 1, 2, 3 \} \) is parametrized by \( \text{SL}_2 \mathbb{R} \times \text{SL}_2 \mathbb{R} \) via the action \( \mu \), then the pull back \( g^{-1}dg \) of Maurer-Cartan form \( \Omega = (\omega^\alpha_\beta) \) can be written as the following equation in the Lie algebra \( \mathfrak{sl}_2 \mathbb{R} \oplus \mathfrak{sl}_2 \mathbb{R} \):

\[
g^{-1}dg = g_1^{-1}dg_1 \oplus g_2^{-1}dg_2,
\]

where

\[
ge_1^{-1}dg_1 = \frac{1}{2} \begin{pmatrix}
\omega^3 + \omega^1_2 \\
-\omega^1 + \omega^2 - \omega^1_3 + \omega^2_3 \\
\omega^1_1 + \omega^2 - \omega^1_3 - \omega^2_3
\end{pmatrix}
\]

and

\[
ge_2^{-1}dg_2 = \frac{1}{2} \begin{pmatrix}
\omega^3 - \omega^1_3 \\
\omega^1 + \omega^2 + \omega^3_3 + \omega^3_1 \\
-\omega^3 + \omega^1 - \omega^3_3
\end{pmatrix}.
\]

**Proof.** For simplicity, let \( \sigma_0 := 1, \sigma_1 := i, \sigma_2 := j', \sigma_3 := k' \). By applying the chain rule,

\[
de_{\sigma_0}(g) = d(g_1 \sigma_0 g_2^t)
\]

\[
= (dg_1) \sigma_0 g_2^t + g_1 \sigma_0 dg_2^t
\]

\[
= g_1 \{ g_1^{-1} (dg_1) \sigma_0 + \sigma_0 (g_2^{-1} dg_2^t) \} g_2^t.
\]

On the other hand,

\[
de_{\sigma_0} = \omega^\beta_\alpha e_\beta = \omega^\beta_\alpha g_1 \sigma_0 g_2.
\]

Hence, we have the equation

\[
(g_1^{-1}dg_1) \sigma_0 + \sigma_0 (g_2^{-1}dg_2^t) = \omega^\beta_\alpha g_2.
\]

The equations (21) and (22) follow from this equation. □
Lemma 3. If the frame field \( \{ e_\alpha : \alpha = 0, 1, 2, 3 \} \) is parametrized by \( SL_2\mathbb{R} \times SL_2\mathbb{R} \) via the action \( \nu \), then the pull back \( g^{-1}dg \) of Maurer-Cartan form \( \Omega = (\omega^a) \) can be written as the following equation in the Lie algebra \( sl_2\mathbb{R} \oplus sl_2\mathbb{R} \):
\[
g^{-1}dg = g_1^{-1}dg_1 \oplus (dg_2^{-1})g_2,
\]
where
\[
g_1^{-1}dg_1 = \frac{1}{2} \begin{pmatrix}
\omega^3 + \omega^1_2 & \omega^1 + \omega^2 - \omega^3_1 - \omega^3_2 \\
-\omega^1 + \omega^2 - \omega^3_1 + \omega^3_2 & -\omega^3 - \omega^2_1 \\
\end{pmatrix}
\]
and
\[
(dg_2^{-1})g_2 = \frac{1}{2} \begin{pmatrix}
\omega^3 - \omega^1_2 & \omega^1 + \omega^2 + \omega^3_1 + \omega^3_2 \\
-\omega^1 + \omega^2 + \omega^3_1 - \omega^3_2 & -\omega^3 + \omega^2_1 \\
\end{pmatrix}.
\]

Proof. Similar to the proof of Lemma 2, we get the equation
\[
(g_1^{-1}dg_1)_{\sigma^1} + \sigma^2(g_2^{-1})g_2 = \omega^a_{\sigma^a}
\]
and the equations (23) and (24) then follow. \( \square \)

5. Timelike \( \text{cmc} \) Surfaces in AdS 3-Space \( \mathbb{H}^3_1(-1) \) and Integrable Systems

Let \( M \) be a simply-connected open and orientable 2-manifold and \( \varphi : M \rightarrow \mathbb{H}^3_1(-1) \) a timelike conformal immersion.

By using a double covering induced by the group action \( \mu \), we can find lift \( \Phi = (\Phi_1, \Phi_2) \) (called a coordinate frame) of \( \mathcal{F} \) to \( SL_2\mathbb{R} \times SL_2\mathbb{R} \):
\[
\mu(\Phi)(1, 1, j', k') = \mathcal{F}.
\]
That is, the lifted framing \( \Phi = (\Phi_1, \Phi_2) : M \rightarrow SL_2\mathbb{R} \times SL_2\mathbb{R} \) satisfies
\[
\mu(\Phi)(1) = \Phi_1 1 \Phi_2^t = \varphi,
\]
\[
\mu(\Phi)(i) = \Phi_1 i \Phi_2^t = e^{-\tilde{\varphi}} \varphi_x,
\]
\[
\mu(\Phi)(j') = \Phi_1 j' \Phi_2^t = e^{-\tilde{\varphi}} \varphi_y,
\]
\[
\mu(\Phi)(k') = \Phi_1 k' \Phi_2^t = N.
\]
Then
\[
\varphi_u = e^\varphi \Phi_1 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Phi_2^t
\]
and
\[
\varphi_v = e^\varphi \Phi_1 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Phi_2^t.
\]

\(^5\)Here, we use the same \( \mu \) for both Lie group action and group representation.
Similarly, by using a double covering induced by the group action $\nu$, we can find lift $\Psi = (\Psi_1, \Psi_2)$ of $\mathcal{F}$ to $\text{SL}_2\mathbb{R} \times \text{SL}_2\mathbb{R}$:

$$\nu(F)(1, i, j', k') = \mathcal{F}.$$ 

The lifted framing $\Psi = (\Psi_1, \Psi_2) : M \rightarrow \text{SL}_2\mathbb{R} \times \text{SL}_2\mathbb{R}$ satisfies

$$\nu(\Psi)(1) = \Psi_1 \Psi_2^{-1} = \varphi,$$

$$\nu(\Psi)(i) = \Psi_1 \Psi_2^{-1} = e^{-\frac{\pi}{2} \varphi_x},$$

$$\nu(\Psi)(j') = \Psi_1 \Psi_2^{-1} = e^{-\frac{\pi}{2} \varphi_y},$$

$$\nu(\Psi)(k') = \Psi_1 \Psi_3^{-1} = N.$$ 

Then

$$\varphi_u = e^{\frac{\pi}{2} \Psi_1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Psi_2^{-1}$$

and

$$\varphi_v = e^{\frac{\pi}{2} \Psi_1} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Psi_2^{-1}.$$ 

Let $s := (\varphi, \varphi_u, \varphi_v, N)$. Then $s$ defines a moving frame on the immersed surface $\varphi$ and satisfy the following Gauß-Weingarten equations:

$$s_u = s \mathcal{U}, \quad s_v = s \mathcal{V},$$

where

$$\mathcal{U} = \begin{pmatrix} 0 & 0 & \frac{1}{2} e^{\omega} & 0 \\ 1 & \omega_u & 0 & -H \\ 0 & 0 & 0 & -2Qe^{-\omega} \\ 0 & Q & \frac{1}{2} e^{\omega} H & 0 \end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix} 0 & \frac{1}{2} e^{\omega} & 0 & 0 \\ 0 & 0 & 0 & -2Re^{-\omega} \\ 1 & 0 & \omega_v & -H \\ 0 & \frac{1}{2} e^{\omega} H & R & 0 \end{pmatrix}.$$ 

The integrability condition of the Gauß-Weingarten equation is the Gauß-Mainardi-Codazzi equation

$$\nu_u = \nu_v + [\mathcal{U}, \mathcal{V}] = 0.$$ 

This Gauß-Mainardi-Codazzi equation is equivalent to

$$\omega_{uv} + \frac{1}{2} e^{\omega}(H^2 - 1) - 2QRe^{-\omega} = 0,$$

$$H_u = 2e^{-\omega}Q_v, \quad H_v = 2e^{-\omega}R_u.$$ 

**Remark 2.** From the equations (32), we see that a timelike surface $\varphi : M \rightarrow \mathbb{H}^1(-1)$ has constant mean curvature if and only if $R_u = Q_v = 0$. In this case, $R$ is said to be *Lorentz antiholomorphic* and $Q$ is said to be *Lorentz holomorphic*, respectively.
Each component framing $\Phi_1$ and $\Phi_2$ of $\Phi$ satisfy the following Lax equations:

$$(\Phi_1)_u = \Phi_1 U_1, \quad (\Phi_1)_v = \Phi_1 V_1;$$

$$(\Phi_2)_u = \Phi_2 U_2, \quad (\Phi_2)_v = \Phi_2 V_2,$$

where

$$U_1 = \begin{pmatrix}
\frac{\omega u}{4} & \frac{1}{2} e^{\frac{\omega}{2}} (H + 1) \\
-e^{-\frac{\omega}{2}} Q & -\frac{\omega u}{4}
\end{pmatrix},$$

$$V_1 = \begin{pmatrix}
\frac{\omega v}{4} & e^{-\frac{\omega}{2}} R \\
-\frac{1}{2} e^{\frac{\omega}{2}} (H - 1) & \frac{\omega u}{4}
\end{pmatrix},$$

$$U_2 = \begin{pmatrix}
\frac{\omega v}{4} & e^{-\frac{\omega}{2}} R \\
-\frac{1}{2} e^{\frac{\omega}{2}} (H - 1) & \frac{\omega u}{4}
\end{pmatrix},$$

$$V_2 = \begin{pmatrix}
\frac{\omega u}{4} & \frac{1}{2} e^{\frac{\omega}{2}} (H + 1) \\
-e^{-\frac{\omega}{2}} Q & -\frac{\omega u}{4}
\end{pmatrix}.$$

The compatibility conditions $(\Phi_1)_uv = (\Phi_1)_vu$ and $(\Phi_2)_uv = (\Phi_2)_vu$ give the Maurer-Cartan equations

$$(U_1)_v - (V_1)_u - [U_1, V_1] = 0$$

and

$$(U_2)_v - (V_2)_u - [U_2, V_2] = 0.$$

Each of these two Maurer-Cartan equations is also equivalent to the Gauß-Mainardi-Codazzi equations \ref{eq:31} and \ref{eq:32}.

Each component framing $\Psi_1$ and $\Psi_2$ of $\Psi$ satisfy the following Lax equations:

$$(\Psi_1)_u = \Psi_1 U_1, \quad (\Psi_1)_v = \Psi_1 V_1;$$

$$(\Psi_2)_u = \Psi_2 U_2, \quad (\Psi_2)_v = \Psi_2 V_2,$$

where

$$U_1 = \begin{pmatrix}
\frac{\omega u}{4} & \frac{1}{2} e^{\frac{\omega}{2}} (H + 1) \\
-e^{-\frac{\omega}{2}} Q & -\frac{\omega u}{4}
\end{pmatrix},$$

$$V_1 = \begin{pmatrix}
\frac{\omega v}{4} & e^{-\frac{\omega}{2}} R \\
-\frac{1}{2} e^{\frac{\omega}{2}} (H - 1) & \frac{\omega u}{4}
\end{pmatrix},$$

$$U_2 = \begin{pmatrix}
\frac{\omega v}{4} & e^{-\frac{\omega}{2}} R \\
-\frac{1}{2} e^{\frac{\omega}{2}} (H - 1) & \frac{\omega u}{4}
\end{pmatrix},$$

$$V_2 = \begin{pmatrix}
\frac{\omega u}{4} & \frac{1}{2} e^{\frac{\omega}{2}} (H + 1) \\
-e^{-\frac{\omega}{2}} Q & -\frac{\omega u}{4}
\end{pmatrix}.$$

The compatibility conditions $(\Psi_1)_uv = (\Psi_1)_vu$ and $(\Psi_2)_uv = (\Psi_2)_vu$ give the Maurer-Cartan equations

$$(U_1)_v - (V_1)_u - [U_1, V_1] = 0$$

and

$$(U_2)_v - (V_2)_u - [U_2, V_2] = 0.$$
Again, each of these two Maurer-Cartan equations is equivalent to the Gauß-Mainardi-Codazzi equations \((41)\) and \((42)\).

We now have the following representation formulae for timelike \(\text{cmc}\) surfaces in \(\mathbb{H}^3_1(-1)\).

**Theorem 4.** Let \(M\) be a simply-connected region in Minkowski plane \(\mathbb{E}^2_1 = (\mathbb{R}^2(u, v), dudv)\).

1. Let \(\Phi = (\Phi_1, \Phi_2) : M \rightarrow \text{SL}_2\mathbb{R} \times \text{SL}_2\mathbb{R}\) be a solution to the following Lax equations:

\[
\begin{align*}
(\Phi_1)_u &= \Phi_1 U_1, \quad (\Phi_1)_v = \Phi_1 V_1; \\
(\Phi_2)_u &= \Phi_2 U_2, \quad (\Phi_2)_v = \Phi_2 V_2,
\end{align*}
\]

where

\[
\begin{align*}
U_1 &= \begin{pmatrix}
\frac{\omega_v}{4} & \frac{1}{2} e^{\frac{\omega}{2}} (H + 1) \\
-\frac{1}{2} e^{\frac{\omega}{2}} (H - 1) & -\frac{\omega_v}{4}
\end{pmatrix}, \\
V_1 &= \begin{pmatrix}
-\frac{\omega_v}{4} & e^{-\frac{\omega}{2}} R \\
\frac{1}{2} e^{\frac{\omega}{2}} (H - 1) & \frac{\omega_v}{4}
\end{pmatrix},
\end{align*}
\]

Then \(\varphi := \mu(\Phi)(1) = \Phi_1 \Phi_2^{-1} : M \rightarrow \mathbb{H}^3_1(-1)\) defines a timelike \(\text{cmc}\) \(H\) immersion into \(\mathbb{H}^3_1(-1)\).

2. Let \(\Psi = (\Psi_1, \Psi_2) : M \rightarrow \text{SL}_2\mathbb{R} \times \text{SL}_2\mathbb{R}\) be a solution to the following Lax equations:

\[
\begin{align*}
(\Psi_1)_u &= \Psi_1 U_1, \quad (\Psi_1)_v = \Psi_1 V_1; \\
(\Psi_2)_u &= \Psi_2 U_2, \quad (\Psi_2)_v = \Psi_2 V_2,
\end{align*}
\]

where

\[
\begin{align*}
U_1 &= \begin{pmatrix}
\frac{\omega_v}{4} & \frac{1}{2} e^{\frac{\omega}{2}} (H + 1) \\
-\frac{1}{2} e^{\frac{\omega}{2}} (H - 1) & -\frac{\omega_v}{4}
\end{pmatrix}, \\
V_1 &= \begin{pmatrix}
-\frac{\omega_v}{4} & e^{-\frac{\omega}{2}} R \\
\frac{1}{2} e^{\frac{\omega}{2}} (H - 1) & \frac{\omega_v}{4}
\end{pmatrix},
\end{align*}
\]

Then \(\psi := \nu(\Psi)(1) = \Psi_1 \Psi_2^{-1} : M \rightarrow \mathbb{H}^3_1(-1)\) defines a timelike \(\text{cmc}\) \(H\) immersion into \(\mathbb{H}^3_1(-1)\).

Let \(H_e, H_s, H_h\) be the constant mean curvatures of timelike surfaces in Minkowski 3-space \(\mathbb{E}^3_1\), de Sitter 3-space \(S^3_1(1)\) and anti-de Sitter 3-space \(\mathbb{H}^3_1(-1)\), resp. Then these timelike \(\text{cmc}\) surfaces in each space-form satisfy
the following Gauß-Mainardi-Codazzi equations:

\[ \omega_{uv} + \frac{1}{2} H^2 e^\omega - 2Qe^{-\omega} = 0 \]

Lorentz 3-space \( \mathbb{E}_1^3 \) case,

\[ \omega_{uv} + \frac{1}{2}(H^2_s + 1)e^\omega - 2Qe^{-\omega} = 0 \]

de Sitter 3-space \( \mathbb{S}_1^3(1) \) case,

\[ \omega_{uv} + \frac{1}{2}(H^2_h - 1)e^\omega - 2Qe^{-\omega} = 0 \]

anti-de Sitter 3-space \( \mathbb{H}_1^3(-1) \) case.

By comparing these Gauß-Mainardi-Codazzi equations, we can deduce the Lawson-Guichard correspondence between timelike \( \text{cmc} \) surfaces in \( \mathbb{E}_1^3 \), timelike \( \text{cmc} \) \( H_s \) surfaces in \( \mathbb{S}_1^3(1) \) and timelike \( \text{cmc} \) \( H_h \) surfaces in \( \mathbb{H}_1^3(-1) \), which satisfy the same Gauß-Mainardi-Codazzi equation. Such \( \text{cmc} \) surfaces are called cousins of each other. In particular, we see that there is a bijective correspondence between timelike minimal surfaces \( (H = 0) \) in \( \mathbb{E}_1^3 \) and timelike \( \text{cmc} \) \( \pm 1 \) in \( \mathbb{H}_1^3(-1) \). For this reason, we are mainly interested in timelike \( \text{cmc} \) \( \pm 1 \) surfaces in \( \mathbb{H}_1^3(-1) \). If \( H = \pm 1 \) then the Gauß-Mainardi-Codazzi equations become

\[
\begin{aligned}
\omega_{uv} - 2e^{-\omega}RQ &= 0, \\
R_u = Q_v &= 0.
\end{aligned}
\]

For more details about the Lawson-Guichard correspondence, please see the appendix I (section [I]).

**Remark 3.** Note that one can normalize the Hopf pairs \( Q \) and \( R \) if \( M \) has real distinct principal curvatures or imaginary principal curvatures everywhere. For example, \( Q = \pm R = 1 \) reduces the Gauß-Mainardi-Codazzi equations to the Liouville equation \( \omega_{uv} = \mp 2e^{-\omega} \).

Since the sign of \( H \) depends upon the orientation of a surface (i.e., the orientation of the unit normal vector field \( N \)), hereafter we consider only \( H = 1 \) case.

**Corollary 5.** Let \( M \) be a simply-connected 2-manifold.

(1) Let \( \Phi = (\Phi_1, \Phi_2) : M \to \mathbb{SL}_2\mathbb{R} \times \mathbb{SL}_2\mathbb{R} \) be solutions to the following Lax equations:

\[
\begin{aligned}
(\Phi_1)_u &= \Phi_1 U_1, \quad (\Phi_1)_v = \Phi_1 V_1; \\
(\Phi_2)_u &= \Phi_2 U_2, \quad (\Phi_2)_v = \Phi_2 V_2,
\end{aligned}
\]

where

\[
U_1 = \begin{pmatrix}
\frac{\omega}{4} & e^\frac{\omega}{4} \\
- e^{\frac{\omega}{4}} Q & -\frac{\omega}{4}
\end{pmatrix}, \quad V_1 = \begin{pmatrix}
\frac{\omega}{4} & e^{-\frac{\omega}{4}} R \\
0 & \frac{\omega}{4}
\end{pmatrix},
\]

\[
U_2 = \begin{pmatrix}
\frac{\omega}{4} & e^{-\frac{\omega}{4}} Q \\
0 & \frac{\omega}{4}
\end{pmatrix}, \quad V_2 = \begin{pmatrix}
\frac{\omega}{4} & e^\frac{\omega}{4} R \\
- e^{\frac{\omega}{4}} Q & -\frac{\omega}{4}
\end{pmatrix}.
\]

Then \( \varphi := \Phi_1 \Phi_2 : M \to \mathbb{H}_1^3(-1) \) defines a timelike \( \text{cmc} \) 1 immersion into \( \mathbb{H}_1^3(-1) \).
Let $\Psi = (\Psi_1, \Psi_2) : M \to \text{SL}_2\mathbb{R} \times \text{SL}_2\mathbb{R}$ be solutions to the following Lax equations:

$$(\Psi_1)_u = \Psi_1 U_1, \quad (\Psi_1)_v = \Psi_1 \mathcal{V}_1;$$

$$(\Psi_2)_u = \Psi_2 U_2, \quad (\Psi_2)_v = \Psi_2 \mathcal{V}_2,$$

where

$$U_1 = \left( \begin{array}{cc} e^\frac{v}{4} & -e^{-\frac{u}{4}} \\ -e^{-\frac{v}{4}} & e^\frac{u}{4} \end{array} \right), \quad \mathcal{V}_1 = \left( \begin{array}{cc} -\frac{u}{4} & e^{-\frac{v}{4}}R \\ 0 & \frac{u}{4} \end{array} \right),$$

$$U_2 = \left( \begin{array}{cc} e^\frac{v}{4} & 0 \\ -e^{-\frac{v}{4}} & -\frac{u}{4} \end{array} \right), \quad \mathcal{V}_2 = \left( \begin{array}{cc} -e^{-\frac{v}{4}} & e^{-\frac{v}{4}}R \\ e^{-\frac{v}{4}} & 0 \end{array} \right).$$

Then $\psi := \Psi_1 \Psi_2^{-1} : M \to \mathbb{H}^3_1(-1)$ defines a timelike $\text{cmc} \ 1$ immersion into $\mathbb{H}^3_1(-1)$.

6. Timelike Surfaces of Constant Mean Curvature $\pm 1$ in AdS 3-Space $\mathbb{H}^3_1(-1)$ via the Group Action $\mu$

In [10], J. Q. Hong gave a Bryant type representation formula for timelike surfaces of constant mean curvature 1 in $\mathbb{H}^3_1(-1)$. In this section, we reproduce J. Q. Hong’s Bryant type representation formula in a more general context.

Let $F : M \to \text{SL}_2\mathbb{R} \times \text{SL}_2\mathbb{R}$ be a lift of $\mathcal{F}$ to $\text{SL}_2\mathbb{R} \times \text{SL}_2\mathbb{R}$ via the Lie group action $\mu$, i.e., $\mu(F)(1, i, j, k) = \mathcal{F}$. Let $\Omega := \Omega_1 \oplus \Omega_2 \in \text{sl}_2\mathbb{R} \oplus \text{sl}_2\mathbb{R}$, where $\Omega_i = F^{-1}_i dF_i \in \text{sl}_2\mathbb{R}, \ i = 1, 2$. The Gauß and Mainardi-Codazzi equations are equivalent to Maurer-Cartan equation

$$d\Omega + \Omega \wedge \Omega = 0,$$

which is the null curvature (integrability) condition of the Maurer-Cartan form $\Omega$. Let $F^*$ denote the pull-back map $F^* : (\text{sl}_2\mathbb{R})^* \oplus (\text{sl}_2\mathbb{R})^* \to T^* M$. The Maurer-Cartan forms $\Omega_i = F^{-1}_i dF_i \in \text{sl}_2\mathbb{R}, \ i = 1, 2$ can be written as the following equations:

$$F^{-1}_i dF_i = \frac{1}{2} F^* \left( \begin{array}{ccc} \omega^3 + \omega^1 & \omega^1 + \omega^2 & \omega^1 - \omega^3 \\ -\omega_1 + \omega_2 & -\omega_1 - \omega_2 + \omega_3 & \omega^1 - \omega^3 \\ \omega^3 - \omega^1 & \omega^1 + \omega^2 & -\omega^3 \end{array} \right),$$

$$F^{-1}_2 dF_2 = \frac{1}{2} F^* \left( \begin{array}{ccc} \omega^3 - \omega^2 & \omega^2 + \omega_3 & -\omega^3 \\ -\omega^1 - \omega^2 + \omega_3 & \omega^1 + \omega^2 & -\omega^3 \\ \omega^3 + \omega^2 & \omega^2 + \omega_3 & -\omega^3 \end{array} \right).$$

**Definition 1.** Let $M$ be a 2-manifold. A map $F : M \to \text{SL}_2\mathbb{R}$ is said to be null if $F^*(\phi) = 0$, or equivalently $\det(F^{-1} dF) = 0$, where $\phi$ is the quadratic Cartan-Killing form $\phi = -8 \det(g^{-1} dg)$.

**Theorem 6** (A Bryant type representation formula for timelike $\text{cmc} \pm 1$ Surfaces in $\mathbb{H}^3_1(-1)$). Let $M$ be an open 2-manifold and $F = (F_1, F_2) : M \to \text{SL}_2\mathbb{R} \times \text{SL}_2\mathbb{R}$ an immersion such that

1. $F_1$ is Lorentz holomorphic, i.e., $(F_1)_u = 0$ and $F_2$ is Lorentz antiholomorphic, i.e., $(F_2)_u = 0$,
Then

$$(51) \quad \varphi := \mu(F)(1) = F_1 F_2^t$$

is a smooth conformal timelike immersion into $\mathbb{H}^3_1(-1)$ with cmc $\pm 1$. Conversely, let $M$ be an oriented and simply-connected Lorentzian 2-manifold with globally defined null coordinates.\(^6\) If $\varphi : M \to \mathbb{H}^3_1(-1)$ is a smooth conformal timelike immersion with cmc $\pm 1$, then there exists an immersion $F = (F_1, F_2) : M \to \text{SL}_2\mathbb{R} \times \text{SL}_2\mathbb{R}$ such that $F_1, F_2$ satisfy the conditions (1), (2), and $\varphi = F_1 F_2^t$.

Proof. Let $\omega^+ = \omega^1 + \omega^2$, $\omega^- = -\omega^1 + \omega^2$, $\pi^+ = \omega^3_1 + \omega^3_2$, $\pi^- = -\omega^3_1 + \omega^3_2$. Also let

$$F^*(\omega^3 + \omega^1_2) = 2\alpha_1, \quad F^*(\omega^+ - \pi^+) = 2\beta_1, \quad F^*(\omega^- + \pi^-) = 2\gamma_1, \quad F^*(\omega^3 - \omega^1_2) = 2\alpha_2, \quad F^*(\omega^- - \pi^-) = 2\beta_2, \quad F^*(\omega^+ + \pi^+) = 2\gamma_2.$$

Then the Maurer-Cartan equations (21) and (22) become

$$(52) \quad F_1^{-1}dF_1 = \frac{1}{2} F^*(\begin{pmatrix} \omega^3 + \omega^1_2 & \omega^+ - \pi^+ \\ \omega^- + \pi^- & -\omega^3 - \omega^1_2 \end{pmatrix}) = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & -\alpha_1 \end{pmatrix},$$

$$(53) \quad F_2^{-1}dF_2 = \frac{1}{2} F^*(\begin{pmatrix} \omega^3 - \omega^1_2 & \omega^- - \pi^- \\ \omega^+ + \pi^+ & -\omega^3 + \omega^1_2 \end{pmatrix}) = \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & -\alpha_2 \end{pmatrix}.$$  

Note that $F^*(\omega^3) = \alpha_1 + \alpha_2$, $F^*(\omega^+) = \beta_1 + \gamma_2$, $F^*(\omega^-) = -(\beta_2 + \gamma_1)$. Since $\det F_1^{-1}dF_1 = \det F_2^{-1}dF_2 = 0$, $\alpha_1^2 + \beta_1 \gamma_1 = 0$ and $\alpha_2^2 + \beta_2 \gamma_2 = 0$.

Denote by $ds^2$ the metric in $\mathbb{H}^3_1(-1)$ induced by the canonical semi-Riemannian metric in $\mathbb{E}^3$. If we regard $e_0$ as a map $e_0 : \text{SL}_2\mathbb{R} \times \text{SL}_2\mathbb{R} \to \mathbb{H}^3_1(-1)$ given by $e_0(g) = \mu(g)(1)$, $g \in \text{SL}_2\mathbb{R} \times \text{SL}_2\mathbb{R}$, then

$$e_0^*(ds^2) = <de_0, de_0> = \omega^1 e_1 + \omega^2 e_2 + \omega^3 e_3, \omega^1 e_1 + \omega^2 e_2 + \omega^3 e_3> = -(\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2$$

\(^6\)In Lorentzian case, the so-called Riemann Mapping Theorem or Kähler Uniformization Theorem does not hold. So, globally defined null coordinates do not exist, in general, on a simply-connected Lorentzian 2-manifold.
defines an indefinite metric in the oriented orthonormal frame bundle of \( \mathbb{H}_1^3(-1) \). Since \( \varphi = \epsilon_0 \circ F \),

\[
\begin{align*}
\varphi^*(ds^2) & = F^* \circ \epsilon_0^*(ds^2) \\
& = F^*((\omega^3)^2 + \{-(\omega^1)^2 + (\omega^2)^2\}) \\
& = F^*(\omega^3)^2 + F^*(\omega^+)^2 + F^*(\omega^-)^2 \\
& = 2\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2.
\end{align*}
\]

Since \( F \) is an immersion, the last expression defines a metric.

We now show that for the immersion \( \varphi : M \rightarrow \mathbb{H}_1^3(-1) \), \( H \equiv 1 \). Let \( U \subset M \) be a simply-connected open set in which there exists a null coordinate system \((u, v)\) such that \( ds^2 = e^{\omega}dudv \) for some real-valued function \( \omega : U \rightarrow \mathbb{R} \) defined on \( U \). Clearly, \( M \) is covered by such open sets. Let \( \eta = e^{\pi}du \) and \( \xi = e^{\pi}dv \). There exist functions \( A_1, A_2, B_1, B_2, C_1, C_2 \) such that

\[
\begin{align*}
F^*(\omega^3 + \omega_1^1) &= 2A_1\eta, \quad F^*(\omega^3 - \omega_2^1) = 2A_2\xi, \\
F^*(\omega^+ - \pi^+) &= 2B_1\eta, \quad F^*(\omega^- - \pi^-) = 2B_2\xi, \\
F^*(\omega^- + \pi^-) &= 2C_1\eta, \quad F^*(\omega^+ + \pi^+) = 2C_2\xi.
\end{align*}
\]

We, then, have equations

\[
A_1^2 + B_1C_1 = 0, \quad A_2 + B_2C_2 = 0, \quad \text{and} \quad 2A_1A_2 + B_1B_2 + C_1C_2 = 1.
\]

In the open set \( U \),

\[
\begin{align*}
\varphi^*(ds^2) &= (2A_1A_2 + B_1B_2 + C_1C_2)\eta\xi \\
& = \eta\xi.
\end{align*}
\]

Since \( A_1^2 + B_1C_1 = 0 \) and \( A_2 + B_2C_2 = 0 \), there exist smooth functions \( p_1, p_2, q_1, q_2 \) defined in \( U \) (unique up to replacement by \((-p_1, -q_1)\) and \((-p_2, -q_2)\) respectively) such that

\[
\begin{align*}
A_1 &= p_1q_1, \quad A_2 &= p_2q_2, \\
B_1 &= q_1^2, \quad B_2 &= q_2^2, \\
C_1 &= -p_1^2, \quad C_2 &= -p_2^2
\end{align*}
\]

and

\[
2p_1q_1p_2q_2 + q_1^2q_2^2 + p_1^2p_2^2 = (p_1p_2 + q_1q_2)^2 = 1.
\]

By the continuity of \( p_1p_2 + q_1q_2 \), either \( p_1p_2 + q_1q_2 = 1 \) in \( U \) or \( p_1p_2 + q_1q_2 = -1 \) in \( U \). Without loss of generality, we may assume that \( p_1p_2 + q_1q_2 = 1 \). Now define a map \( h : U \rightarrow \text{SL}_2\mathbb{R} \) by \( h = \begin{pmatrix} p_1 & -q_2 \\ q_1 & p_2 \end{pmatrix} \). Then \( \epsilon_0(\pi_1h, F_2(h^{-1})^\dagger) = \ldots \).
such that the associated frame field 
\{e_0, \xi\} of the 1-form \( \xi \) of type \((1,0)\), i.e., a multiple of the 1-form \( \eta \). Similarly, \( (Fh)^*(\omega^-) \) is a 1-form of type \((0,1)\), i.e., a multiple of the 1-form \( \xi \).

Since \( (Fh)^*(\omega^+) = -\eta \) and \( (Fh)^*(\omega^-) = -\xi \), by the equation (20), one can easily see that:

1. If \( \varphi \) satisfies \( H = 1 \) in \( U \), then \( (Fh)^*(\omega^- + \pi^-) \) is a 1-form of type \((1,0)\) and \( (Fh)^*(\omega^+ + \pi^+) \) is a 1-form of type \((0,1)\).
2. If \( (Fh)^*(\omega^- + \pi^-) \) is a 1-form of type \((1,0)\) or \( (Fh)^*(\omega^+ + \pi^+) \) is a 1-form of type \((0,1)\), then \( \varphi \) satisfies \( H = 1 \) in \( U \).

Therefore, we conclude that \( H = 1 \) in \( U \).

Conversely, let \( M \) be an oriented and simply-connected Lorentzian 2-manifold with globally defined null coordinates \((u,v)\). Let \( \varphi : M \rightarrow \mathbb{H}^2_1(-1) \) be a smooth conformal timelike immersion into \( \mathbb{H}^2_1(-1) \) with \( \text{cmc} \ 1 \). Then \( ds^2 = e^{-2\varphi}du^2 \) for some real-valued function \( \omega : M \rightarrow \mathbb{R} \). Let \( \eta = e^{\frac{\varphi}{2}}du \) and \( \xi = e^{\frac{\varphi}{2}}dv \). Then one can choose a lifting \( g = (g_1, g_2) : M \rightarrow \text{SL}_2 \mathbb{R} \times \text{SL}_2 \mathbb{R} \) such that the associated frame field \( \{e_0(g)\} \) is adapted with \( g^*(\omega^+) = -\eta \) and
\[ g^*(\omega^-) = -\xi. \] Since \( g^*(\omega^3) = 0, \)

\[
g_1^{-1} dg_1 = \frac{1}{2} g^* \left( \begin{array}{c} \omega_2^- + \pi^- \\ -\omega_1^- \end{array} \right) = \frac{1}{2} g^* \left( \begin{array}{c} \omega_2^- \\ -\omega_1^- \end{array} \right) + \eta \left( \begin{array}{c} 0 \\ -1 \end{array} \right),
\]

\[
g_2^{-1} dg_2 = \frac{1}{2} g^* \left( \begin{array}{c} -\omega_2^- + \pi^- \\ \omega_1^- \end{array} \right) = \frac{1}{2} g^* \left( \begin{array}{c} -\omega_2^+ \\ \omega_1^- \end{array} \right) + \xi \left( \begin{array}{c} 0 \\ -1 \end{array} \right).
\]

Let \( \zeta = \frac{1}{2} g^* \left( \begin{array}{c} \omega_2^- + \pi^- \\ -\omega_1^- \end{array} \right) \in \mathfrak{sl}_2 \mathbb{R}. \) Then \( d\zeta = -\zeta \wedge \zeta. \) The equation \( d\zeta = -\zeta \wedge \zeta \) satisfies the integrability condition; hence, by the Frobenius Theorem, there exists a smooth map \( h : M \rightarrow \text{SL}_2 \mathbb{R} \) such that \( \zeta = h^{-1} dh. \) Since \( h \in \text{SL}_2 \mathbb{R}, \) it can be written

\[
h = \begin{pmatrix} p_1 & -q_2 \\ q_1 & p_2 \end{pmatrix}, \quad p_1, p_2, q_1, q_2 \in \mathbb{R}.
\]

Set \( F_1 := g_1 h^{-1}. \) Then

\[
F_1^{-1} dF_1 = (g_1 h^{-1})^{-1} d(g_1 h^{-1})
\]

\[
= h(g_1^{-1} dg_1) h^{-1} + h dh^{-1}
\]

\[
= h \left[ \zeta + \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \eta \right] h^{-1} + h dh^{-1}
\]

\[
= h \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} h^{-1} \eta
\]

\[
= \begin{pmatrix} p_1 q_1 & -p_2^2 \\ q_1^2 & -p_1 q_1 \end{pmatrix} \eta.
\]

The differential \( d \) can be written as \( d = \partial' + \partial'', \) where \( \partial' \) is the Lorentz holomorphic part and \( \partial'' \) is the Lorentz antiholomorphic part. Since \( \partial'' F_1 = \frac{\partial F_1}{\partial v} dv = 0, \) \( F_1 \) is Lorentz holomorphic.
Set $F_2 = g_2 h^t$. Then
\[
F_2^{-1} dF_2 = (g_2 h^t)^{-1} d(g_2 h^t)
= (h^t)^{-1} (g_2^{-1} d g_2) h^t + (h^t)^{-1} d h^t
= (h^t)^{-1} \left[ -\xi^t + \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \xi \right] h^t + (h^t)^{-1} d h^t
= (h^t)^{-1} \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} h^t \xi
= \begin{pmatrix} p_2 q_2 & -p_2^2 \\ q_2 & -p_2 q_2 \end{pmatrix} \xi.
\]
Since $\partial F_2 = \partial F_2 / \partial u = 0$, $F_2$ is Lorentz antiholomorphic. Finally,
\[
F_1 F_2 = g_1 h^{-1} (g_2 h^t)^t = g_1 g_2 = \varphi.
\]

Remark 4. Note that, in Theorem 5, $\varphi = F_1 F_2^t$ has cmc 1 (cmc $-1$) if the framing $F$ is orientation preserving (orientation reversing). In order to prove Theorem 5 for orientation reversing framing $F$, one needs to take
\[
F_1^{-1} dF_1 = \begin{pmatrix} p_1 q_2 & p_1^2 \\ -q_2 & -p_1 q_2 \end{pmatrix} \eta, \quad F_2^{-1} dF_2 = \begin{pmatrix} p_2 q_2 & p_2^2 \\ -q_2 & -p_2 q_2 \end{pmatrix} \xi,
\]
and
\[
h = \begin{pmatrix} q_2 & -p_1 \\ p_2 & q_1 \end{pmatrix} \in SL_2 \mathbb{R} \text{ in the proof.}
\]

Remark 5.
\[
ds^2 = \varphi^* (ds^2)
= < d \varphi, d \varphi >
= < d(F_1 F_2^t), d(F_1 F_2)^t >
= - \det \{ F_1^{-1} dF_1 + (F_2^{-1} dF_2)^t \}.
\]
So, $\varphi$ does not assume degenerate points if and only if
\[
\det \{ F_1^{-1} dF_1 + (F_2^{-1} dF_2)^t \} \neq 0.
\]

7. Timelike Surfaces of Constant Mean Curvature ±1 in AdS 3-Space $\mathbb{H}_3^1(-1)$ via the Group Action $\nu$

Let $F : M \rightarrow SL_2 \mathbb{R} \times SL_2 \mathbb{R}$ be a lift of $\mathcal{F}$ to $SL_2 \mathbb{R} \times SL_2 \mathbb{R}$ via the Lie group action $\nu$, i.e., $\nu(F)(\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}) = \mathcal{F}$. Let $\Omega := \Omega_1 \oplus \Omega_2 \in \mathfrak{sl}_2 \mathbb{R} \oplus \mathfrak{sl}_2 \mathbb{R}$, where $\Omega_1 = F_1^{-1} dF_1$, $\Omega_2 = (dF_2^{-1}) F_2 \in \mathfrak{sl}_2 \mathbb{R}$. The Gauß and Mainardi-Codazzi equations are equivalent to Maurer-Cartan equation
\[
d\Omega + \Omega \wedge \Omega = 0,
\]
which is the null curvature (integrability) condition of the Maurer-Cartan form $\Omega$. Let $F^*$ denote the pull-back map $F^* : (\mathfrak{sl}_2\mathbb{R})^* \oplus (\mathfrak{sl}_2\mathbb{R})^* \longrightarrow T^* M$. The Maurer-Cartan forms $\Omega_1 = F_1^{-1}dF_1$, $\Omega_2 = (dF_2^{-1})F_2$ can be written as the following equations:

\begin{align*}
(55) \quad F_1^{-1}dF_1 &= \frac{1}{2} F^* \begin{pmatrix} \omega^1 + \omega^2 - \omega^3 & \omega^1 + \omega^2 - \omega^3 & \omega^1 + \omega^2 - \omega^3 \\
-\omega_1 + \omega_2 - \omega_3 + \omega_4 & -\omega_1 + \omega_2 - \omega_3 + \omega_4 & -\omega_1 + \omega_2 - \omega_3 + \omega_4 \\
-\omega_1 + \omega_2 - \omega_3 + \omega_4 & -\omega_1 + \omega_2 - \omega_3 + \omega_4 & -\omega_1 + \omega_2 - \omega_3 + \omega_4 \\
\end{pmatrix},

(56) \quad (dF_2^{-1})F_2 &= \frac{1}{2} F^* \begin{pmatrix} \omega^1 - \omega^2 + \omega^3 & \omega^1 - \omega^2 + \omega^3 & \omega^1 - \omega^2 + \omega^3 \\
-\omega_1 + \omega_2 - \omega_3 + \omega_4 & -\omega_1 + \omega_2 - \omega_3 + \omega_4 & -\omega_1 + \omega_2 - \omega_3 + \omega_4 \\
-\omega_1 + \omega_2 - \omega_3 + \omega_4 & -\omega_1 + \omega_2 - \omega_3 + \omega_4 & -\omega_1 + \omega_2 - \omega_3 + \omega_4 \\
\end{pmatrix}.
\end{align*}

**Theorem 7** (A Bryant type representation formula for timelike $\text{cmc} \pm 1$ Surfaces in $\mathbb{H}^3_1(-1)$). Let $M$ be an open 2-manifold and $F = (F_1, F_2) : M \longrightarrow \text{SL}_2\mathbb{R} \times \text{SL}_2\mathbb{R}$ an immersion such that

1. $F_1$ is Lorentz holomorphic, i.e., $(F_1)_v = 0$ and $F_2$ is Lorentz antiholomorphic, i.e., $(F_2)_u = 0$,
2. $\det F_1^{-1}dF_1 = \det(dF_2^{-1})F_2 = 0$.

Then

\begin{equation}
(57) \quad \psi := \nu(F)(1) = F_1 F_2^{-1}
\end{equation}

is a smooth conformal timelike immersion into $\mathbb{H}^3_1(-1)$ with $\text{cmc} \pm 1$. Conversely, let $M$ be an oriented and simply-connected Lorentzian 2-manifold with globally defined null coordinates. If $\psi : M \longrightarrow \mathbb{H}^3_1(-1)$ is a smooth conformal timelike immersion with $\text{cmc} \pm 1$, then there exists an immersion $F = (F_1, F_2) : M \longrightarrow \text{SL}_2\mathbb{R} \times \text{SL}_2\mathbb{R}$ such that $F_1$, $F_2$ satisfy the conditions (1), (2), and $\psi = F_1 F_2^{-1}$.

**Proof.** We use the same $\omega^+, \omega^-, \pi^+, \pi^-$ and $\alpha_i, \beta_i, \gamma_i$, $i = 1, 2$ as defined in the proof of Theorem 3. Then

\begin{align*}
F_1^{-1}dF_1 &= \begin{pmatrix} \alpha_1 & \beta_1 \\
\gamma_1 & -\alpha_1 \\
\end{pmatrix} \quad \text{and} \quad (dF_2^{-1})F_2 = \begin{pmatrix} \alpha_2 & \gamma_2 \\
\beta_2 & -\alpha_2 \\
\end{pmatrix}.
\end{align*}

Here,

$$ds^2_\psi = F^*(\omega^3)^2 + F^*(\omega^+)^2 + F^*(\omega^-)^2 = 2\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2$$

defines an induced metric of $\psi$ since $F$ is an immersion.

Let $U \subset M$ be a simply-connected open set in which there exists a null coordinate system $(u, v)$ such that $ds^2_\psi = e^{\omega} du dv$, for some real-valued function $\omega : U \longrightarrow \mathbb{R}$. $M$ is covered by such open sets. Let $\eta = e^{\bar{\omega}} du$ and $\xi = e^{\bar{\omega}} dv$. Then, by exactly the same argument in the proof of Theorem 3 there exists a smooth map $h : U \longrightarrow \text{SL}_2\mathbb{R}$ given by $h = \begin{pmatrix} p_1 & -q_2 \\
q_1 & p_2 \\
\end{pmatrix}$ and

$$e_0 \circ Fh = e_0(F_1 h, F_2 h) = F_1 h(F_2 h)^{-1} = F_1 F_2^{-1} = e_0 \circ F.$$
\[(F_1 h)^{-1}d(F_1 h) = h^{-1}(F_1^{-1}dF_1)h + h^{-1}dh \]
\[= \begin{pmatrix}
p_2dp_1 + q_2dq_1 & -p_2dq_2 + q_2dp_2 - \eta \\
-q_1dp_1 + p_1dq_1 & q_1dq_2 + p_1dp_2
\end{pmatrix}
\]

and

\[d(F_2 h)^{-1}F_2 h = (dh^{-1})h + h^{-1}[(dF_2^{-1})F_2]h \]
\[= \begin{pmatrix}
p_1dp_2 + q_1dq_2 & -q_2dp_2 + p_2dq_2 \\
-p_1dq_1 + q_1dp_1 - \xi & q_2dq_1 + p_2dp_1
\end{pmatrix}.\]

It then follows that \((Fh)^*(\omega^3) = 0\), \((Fh)^*(\omega^+) = -\eta\) and \((Fh)^*(\omega^-) = -\xi\).

Hence, \(Fh : U \rightarrow \text{SL}_2\mathbb{R} \times \text{SL}_2\mathbb{R}\) is an oriented framing in \(U\) along the immersion \(\psi = F_1F_2^{-1}\) and \(\psi\) satisfies \(H = 1\) in \(U\).

Conversely, let \(M\) be an oriented and simply-connected Lorentzian 2-manifold with globally defined null coordinates. Let \(\psi : M \rightarrow \mathbb{H}^3_1(-1)\) be a smooth conformal timelike immersion with \(\text{cmc}\) 1. There exist a null coordinate system \((u, v)\) in \(M\) such that \(ds^2_\psi = e^\omega du dv\) for some real-valued function \(\omega : M \rightarrow \mathbb{R}\). Let \(\eta = e^\pi du\) and \(\xi = e^\pi dv\). Then one can choose a lifting \(g : M \rightarrow \text{SL}_2\mathbb{R} \times \text{SL}_2\mathbb{R}\) such that the associated frame field \(\{e_0(g)\}\) is adapted with \(g^*(\omega^+) = -\eta\) and \(g^*(\omega^-) = -\xi\). Since \(g^*(\omega^3) = 0\),

\[g_1^{-1}dg_1 = \frac{1}{2}g^* \begin{pmatrix}
\omega_2^1 & \omega^+ - \pi^+
\omega^+ - \pi^n & -\omega_2^1
\end{pmatrix}
\]
\[= \frac{1}{2}g^* \begin{pmatrix}
\omega_2^1 & -\omega^+ - \pi^+
\omega^+ - \pi^n & -\omega_2^1
\end{pmatrix} + \eta \begin{pmatrix}0 & -1 \\
0 & 0
\end{pmatrix}
\]

and

\[(dg_2^{-1})g_2 = \frac{1}{2}g^* \begin{pmatrix}
-\omega_2^1 & \omega^+ + \pi^+
\omega^+ - \pi^n & \omega_2^1
\end{pmatrix}
\]
\[= \frac{1}{2}g^* \begin{pmatrix}
-\omega_2^1 & \omega^+ + \pi^+
\omega^+ - \pi^n & \omega_2^1
\end{pmatrix} + \xi \begin{pmatrix}0 & 0 \\
-1 & 0
\end{pmatrix}.\]

Let \(\zeta = \frac{1}{2}g^* \begin{pmatrix}
\omega_2^1 & -\omega^+ - \pi^+
\omega^+ - \pi^n & -\omega_2^1
\end{pmatrix} \in \text{sl}_2\mathbb{R}\). Then \(d\zeta = -\zeta \wedge \zeta\) and this equation satisfies the integrability condition; hence, by the Frobenius Theorem, there exists a smooth map \(h : M \rightarrow \text{SL}_2\mathbb{R}\) such that \(\zeta = h^{-1}dh\).

Since \(h \in \text{SL}_2\mathbb{R}\), it can be written \(h = \begin{pmatrix}p_1 & -q_2 \\
q_1 & p_2\end{pmatrix}\), \(p_1, p_2, q_1, q_2 \in \mathbb{R}\). Set \(F_1 = g_1h^{-1}\). Then

\[F_1^{-1}dF_1 = \begin{pmatrix}p_1q_1 & -p_1^2 \\
qu_1^2 & -p_1q_1\end{pmatrix} \eta\]
and $F_1$ is Lorentz holomorphic. Set $F_2 = g_2 h^{-1}$. Then
\[
(dF_2^{-1}) F_2 = (dh) h^{-1} + h[(dg_2^{-1}) g_2] h^{-1} \\
= (dh) h^{-1} + [−ζ + \left( \begin{array}{cc} 0 & 0 \\ -1 & 0 \end{array} \right) \xi] h^{-1} \\
= h \left( \begin{array}{cc} 0 & 0 \\ -1 & 0 \end{array} \right) h^{-1} \xi \\
= (p_2 g_2 \quad q_2^{2} \\
- p_2^{2} \quad - p_2 q_2) \xi
\]
and $F_2$ is Lorentz antiholomorphic. Finally,
\[
F_1 F_2^{-1} = g_1 h^{-1} (g_2 h^{-1})^{-1} = g_1 g_2^{-1} = ψ.
\]

Remark 6. Note that
\[
ds_0^2 = <dψ, dψ> \\
= <d(F_1 F_2^{-1}), d(F_1 F_2^{-1})> \\
= - \det\{F_1^{-1} dF_1 + (dF_2^{-1}) F_2\}.
\]
So, $ψ$ does not assume degenerate points if and only if
\[
\det\{F_1^{-1} dF_1 + (dF_2^{-1}) F_2\} \neq 0.
\]

8. Timelike Minimal Surfaces in $E^3_1$ and the Classical Gauss Map

Recall that the Lie group $G \cong \text{SL}_2 \mathbb{R}$ acts isometrically on Lorentz 3-space $E^3_1$ via the Ad-action. The Ad$(G)$-orbit of $k' = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right)$ is a pseudosphere or de Sitter 2-space:
\[
S^2_1(1) = \{(x_1, x_2, x_3) \in E^3_1 : -x_1^2 + x_2^2 + x_3^2 = 1\}.
\]
The Ad-action of $G$ on $S^2_1(1)$ is transitive as well. The isotropy subgroup of $G$ at $k'$ is the indefinite orthogonal group $\text{SO}_1(2) = \{x_0 \mathbf{1} + x_3 \mathbf{k} : x_0^2 - x_3^2 = 1\}$. Thus, $S^2_1(1)$ can be identified with the symmetric space
\[
\text{SL}_2 \mathbb{R} / \text{SO}_1(2) = \left\{ h \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) h^{-1} : h \in \text{SL}_2 \mathbb{R} \right\}.
\]
The orthogonal subgroup $\text{SO}_1(2)$ can be regarded as the hyperbola $H_0^1$ in a Lorentz plane $E^3_1(x_0, x_3)$. (This is a Lorentz analogue of $S^1 \subset \mathbb{E}^2$.) Note that the group $H_0^1$ is isomorphic to the multiplicative group $\mathbb{R}^\times = (\mathbb{R} \setminus \{0\}, \times)$.

Let $N = (0, 0, 1)$ and $S = (0, 0, -1) \in S^2_1(1)$ be the north and south pole of $S^2_1(1)$. Let $\varphi_+ : S^2_1(1) \setminus \{x_3 = -1\} \rightarrow E^3_1 \setminus H_0^1$ be the stereographic projection.
from the south pole \( S = (0, 0, -1) \), where \( H^1_0 = \{(x_1, x_2) \in \mathbb{E}^2_1 : -x_1^2 + x_2^2 = -1\} \). Then
\[
\varphi_+(x_1, x_2, x_3) = \left( \frac{x_1}{1 + x_3}, \frac{x_2}{1 + x_3} \right) = \left( \frac{x_1 + x_2}{1 + x_3}, \frac{-x_1 + x_2}{1 + x_3} \right) \in \mathbb{E}^2_1(u, v).
\]

Let \( \varphi_- : \mathbb{S}^2_1 \setminus \{x_3 = 1\} \to \mathbb{E}^2_1 \setminus \mathbb{H}^2_0 \) be the stereographic projection from the north pole \( S = (0, 0, 1) \). Then
\[
\varphi_-(x_1, x_2, x_3) = \left( \frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3} \right) = \left( \frac{x_1 + x_2}{1 - x_3}, \frac{-x_1 + x_2}{1 - x_3} \right) \in \mathbb{E}^2_1(u, v).
\]

Note that the classical Gauß map (i.e., the unit normal vector field) \( N \) of timelike surfaces in \( \mathbb{E}^2_1 \) is mapped into de Sitter 2-space \( \mathbb{S}^2_1 \). Thus, the image of classical Gauß map can be represented by the matrices \( h \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) h^{-1}, h \in \text{SL}_2 \mathbb{R} \). If the timelike surface preserves the orientation, then \( h = \left( \begin{array}{cc} p_1 & -q_2 \\ q_1 & p_2 \end{array} \right) \in \text{SL}_2 \mathbb{R} \). So,
\[
\varphi_-(h) \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) h^{-1} = \left( \begin{array}{cc} p_1 p_2 - q_1 q_2 & 2p_1 q_2 \\ 2p_2 q_1 & -p_1 p_2 + q_1 q_2 \end{array} \right) \in \mathbb{S}^2_1(1)
\]
and \( \varphi_-(h) \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) h^{-1} = \left( \begin{array}{cc} p_1 & q_2 \\ q_1 & p_2 \end{array} \right) \in \mathbb{E}^2_1(u, v) \). If the timelike surface reverses the orientation, then \( h = \left( \begin{array}{cc} q_2 & -p_1 \\ p_2 & q_1 \end{array} \right) \in \text{SL}_2 \mathbb{R} \). So,
\[
\varphi_+(h) \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) h^{-1} = \left( \begin{array}{cc} -p_1 p_2 + q_1 q_2 & 2p_1 q_2 \\ 2p_2 q_1 & p_1 p_2 - q_1 q_2 \end{array} \right) \in \mathbb{S}^2_1(1)
\]
and \( \varphi_+(h) \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) h^{-1} = \left( \begin{array}{cc} p_1 & p_2 \\ q_1 & q_2 \end{array} \right) \in \mathbb{E}^2_1(u, v) \).

Here, we recall the following Weierstraß formula for a timelike minimal surface \( \psi : M \to \mathbb{E}^2_1 \) with data \((q, f(q))\) and \((r, g(v))\):
\[
(58) \quad \psi_u = \left( \frac{1}{2} + q^2 \right)(1 + q^2), -q(1 - q^2), -q f(u), \quad \psi_v = \left( -\frac{1}{2}(1 + r^2), -\frac{1}{2}(1 - r^2), -r \right)g(v).
\]

The induced metric of \( \psi \) is
\[
ds^2_\psi = (1 + qr)^2 f(u)g(v)dudv.
\]

Remark 7. The ordered pair \((q, r)\) coincides with the projected Gauß map \( \varphi_- \circ N \) of a timelike minimal surface with data \((q, f(u))\) and \((r, g(v))\).
Remark 8. In [13], J. Inoguchi and M. Toda studied the construction of timelike minimal surfaces via loop group method. Their normalized Wirstraff formula for a timelike minimal surface $\psi : M \rightarrow \mathbb{E}^3_1$ with data $(q, r)$ is

$$\psi_u = \left( \frac{1}{2}(1 + q^2), -\frac{1}{2}(1 - q^2), -q \right), \quad \psi_v = \left( -\frac{1}{2}(1 + r^2), -\frac{1}{2}(1 - r^2), -r \right)$$

and the induced metric of $\psi$ is

$$ds^2 = (1 + qr)^2 du dv.$$

In [13], the signs of coordinate functions in $\psi_u$ and $\psi_v$ are different. The reason is, in [13], $(x_1, x_2, x_3) \in \mathbb{E}^3_1$ is identified with the matrix

$$\begin{pmatrix}
-x_3 & -x_1 + x_2 \\
x_1 + x_2 & x_3 \\
-x_1 + x_2 & -x_3
\end{pmatrix},$$

while in this paper it is identified with

$$\begin{pmatrix}
x_3 & x_1 + x_2 \\
x_1 + x_2 & -x_3 \\
-x_1 + x_2 & -x_3
\end{pmatrix}.$$

Originally, this formula was obtained by M. A. Magid in [17]. However, in [17], the geometric meaning of the data $(q, r)$ is not clarified. In [13], the data $(q, r)$ are retrieved from the normalized potential in their construction. Moreover, $q$ and $r$ are the primitive functions of the coefficients $Q$ and $R$, resp., of Hopf pairs. Note that this is locally true. In general,

$$q_u = \frac{Q}{f(u)}, \quad r_v = \frac{R}{g(v)}.$$ (60) (61)

As is mentioned in Remark 7, $(q, r)$ is the projected Gauß map $\varphi_\perp \circ N$ of a timelike minimal surface given by the Wirstraff formula (59).

9. Lawson-Guichard Correspondence between Timelike $\text{cmc} \pm 1$ Surfaces in $\mathbb{H}^3(-1)$ and Timelike Minimal Surfaces in $\mathbb{E}^3_1$

In Section 5, we discussed the Lawson-Guichard correspondence between timelike cmc surfaces in three different semi-Riemannian space forms $\mathbb{E}^3_1$, $\mathbb{S}^3_1(1)$ and $\mathbb{H}^3(-1)$. In particular, there is a one-to-one correspondence between timelike cmc $\pm 1$ surfaces in $\mathbb{H}^3(-1)$ and timelike minimal surfaces in $\mathbb{E}^3_1$. In this section, we give such bijective correspondence explicitly using the Bryant type representation formulae in Sections 6 and 7.

Let $\varphi : M \rightarrow \mathbb{H}^3(-1)$ be a timelike cmc $-1$ surface. Then, by Theorem 6 (or by Theorem 7), there exists a smooth immersion $F = (F_1, F_2) : M \rightarrow \text{SL}_2 \mathbb{R} \times \text{SL}_2 \mathbb{R}$ such that

1. $F_1$ is Lorentz holomorphic and $F_2$ is Lorentz antiholomorphic.
2. $\det(F_1^{-1} dF_1) = \det(F_2^{-1} dF_2) = 0$
   (or $2' \ \det(F_1^{-1} dF_1) = \det((dF_2^{-1}) F_2) = 0$).
3. $\varphi = F_1 F_2^\perp$ (or $3' \ \varphi = F_1 F_2^{-1}$).
As we have seen in the proof of Theorem 6 (or Theorem 7), locally in an open set $U \subset M$,

$$F_1^{-1} dF_1 = \begin{pmatrix} p_1 q_1 & -p_1^2 \\ q_1^2 & -p_1 q_1 \end{pmatrix} \eta$$

$$= \begin{pmatrix} \frac{p}{q_1^2} & -\frac{p^2}{q_1} \\ 1 & -\frac{p_1}{q_1} \end{pmatrix} q_1^2 \eta$$

and similarly,

$$F_2^{-1} dF_2 = \begin{pmatrix} \frac{p_2}{q_2} & -\frac{p_2^2}{q_2} \\ 1 & -\frac{p_2}{q_2} \end{pmatrix} q_2^2 \xi$$

or

$$(dF_2^{-1}) F_2 = \begin{pmatrix} \frac{p_2}{q_2} & -\frac{p_2^2}{q_2} \\ 1 & -\frac{p_2}{q_2} \end{pmatrix} q_2^2 \xi.$$
10. The Hyperbolic Gauss Map of Timelike Surfaces in $\mathbb{H}^3_1(-1)$

Let $\varphi : M \to \mathbb{H}^3_1(-1)$ be an oriented timelike surface in $\mathbb{H}^3_1(-1)$. At each base point $e_0 = \varphi(m) \in \mathbb{H}^3_1(-1)$, $e_3 \in T_{e_0}\mathbb{H}^3_1(-1)$ is an oriented unit normal vector to the tangent plane $\varphi_*(T_m M)$. The oriented normal geodesic in $\mathbb{H}^3_1(-1)$ emanating from $e_0$, which is tangent to the normal vector $e_3(\varphi(m))$ asymptotically approaches to the null cone $\mathbb{N} = \{ u \in \mathbb{E}^4_2 : \langle u, u \rangle = 0 \}$ at exactly two points $[e_0 + e_3], [e_0 - e_3] \in \mathbb{N}$. The orientation allows us to name $[e_0 + e_3]$ the initial point and $[e_0 - e_3]$ the terminal point.

Define a map $G : M \to \mathbb{N}$ by $G(m) = [e_0 + e_3](m)$ for each $m \in M$. This map is an analogue of the hyperbolic Gauß map $H$ of surfaces in hyperbolic 3-space $\mathbb{H}^3_1(-1)$. The map will still be called the hyperbolic Gauß map here.

Let $\varphi : M \to \mathbb{H}^3_1(-1)$ be a timelike surface in $\mathbb{H}^3_1(-1)$ and $d\sigma^2$ denote the induced metric on $\mathbb{N}^3$. Then

$$d\sigma^2_\varphi := (e_0 + e_3)^*(d\sigma^2) = <d(e_0 + e_3), d(e_0 + e_3)> = -(\omega^1 + \omega_3^1)^2 + (\omega^2 + \omega_3^2)^2.$$

**Proposition 8.** The hyperbolic Gauß map $[e_0 + e_3] : M \to \mathbb{N} ([e_0 - e_3] : M \to \mathbb{N})$ of a timelike surface $\varphi : M \to \mathbb{H}^3_1(-1)$ is conformal if and only if $\varphi$ satisfies $H = 1$ ($H = -1$) or $\varphi$ is totally umbilic.

**Proof.** We will assume the same settings in the proof of Proposition 1. Then

$$d\sigma^2_\varphi = -(\omega^1 + \omega_3^1)^2 + (\omega^2 + \omega_3^2)^2$$
$$= -((1 - h_{11})\omega^1 + h_{12}\omega^2)^2 + (-h_{12}\omega^1 + (1 - h_{22})\omega^2)^2$$
$$= (2H(H - 1) - K)ds^2_\varphi - (H - 1)((h_{11} - h_{22})(\omega^1)^2 - 4h_{12}\omega^1 \otimes \omega^2 + (h_{11} - h_{22})(\omega^2)^2).$$

Thus, $[e_0 + e_3]$ is conformal, i.e., $d\sigma^2_\varphi$ is a multiple of $ds^2_\varphi$ if and only if $(H - 1)(h_{11} - h_{22}) = (H - 1)h_{12} = 0$.

If $H = 1$, then $d\sigma^2_\varphi = -Kds^2_\varphi$. Suppose $H \neq 1$. Let $U = \{ m \in M : H(m) \neq 1 \}$. Then $U$ is clearly open in $M$. Since $H \neq 1$ on $U$, $h_{11} = h_{22}$ and $h_{12} = 0$ on $U$. The second fundamental form $H$ is then

$$II = -h_{11}(\omega^1)^2 + 2h_{12}\omega^1 \otimes \omega^2 + h_{22}(\omega^2)^2$$
$$= h_{11}(-\omega^1)^2 + (\omega^2)^2$$
$$= HH \text{ on } U.$$

By comparing with the equation (1), we see that the Hopf differential $Q = 0$ on $U$, i.e., $\varphi(U)$ is totally umbilic. Note that $H$ must be constant on $U$, since $Q = R = 0$. Let $V$ be a connected component of $U$. Since $H$ is constant on $V$ and $H$ is continuous on $M$, $H$ is constant on $V$. This implies that $H \neq 1$ on $V$ and so $\overline{V} \subset U$. Since $V$ is connected, so is $\bar{V}$. However, $V$ is a connected

---

7The hyperbolic Gauß map was introduced by C. Epstein ([7]) and was used by R. L. Bryant in his study of cmc 1 surfaces in hyperbolic 3-space $\mathbb{H}^3(-1)$ ([11]).
component; thus, $V = \bar{V}$. It then follows from the connectedness of $M$ that $M = V$. Therefore, $\varphi$ is totally umbilic on $M$. The converse is trivial. □

Remark 9. Note that $[\epsilon_0 + \epsilon_3] : M \rightarrow \mathbb{N}$ ([$\epsilon_0 - \epsilon_3] : M \rightarrow \mathbb{N}$) is the hyperbolic Gauss map of a timelike surface $\varphi : M \rightarrow H^3_1(1)$ if $\varphi$ preserves (reverses) the orientation.

Remark 10. If $\varphi$ satisfies $\pm$ and is totally umbilic, then the hyperbolic Gauss map is constant, since $d\sigma^2 \varphi = 0$. It is also shown in Section 13 that if $\varphi$ satisfies $\pm$ and is totally umbilic, then the (projected) hyperbolic Gauss map is both Lorentz holomorphic and antiholomorphic; hence it is constant. By the equation (2), the Gaussian curvature $K = H^2 - 1 = 0$. Thus, $\varphi$ may be regarded as a horosphere type surface in $H^3_1(1)$.

The null cone $\mathbb{N}$ satisfies the quadric equation

$$-(x_0)^2 - (x_1)^2 + (x_2)^2 + (x_3)^2 = 0.$$ 

If $x_0 \neq 0$, then the above equation can be written

$$-\frac{(x_1)^2}{x_0} + \frac{(x_2)^2}{x_0} + \frac{(x_3)^2}{x_0} = 1,$$

i.e., $\mathbb{N}$ can be locally identified with de Sitter 2-space $S^2_1(1)$. With this identification, the hyperbolic Gauss map can be mapped into de Sitter 2-space $S^2_1(1)$.

Thus, we may be able to relate the hyperbolic Gauss map of a timelike cmc $\pm 1$ surface in $H^3_1(1)$ and the Gauss map of corresponding timelike minimal surface in $E^3_1$. This relationship is discussed in the next section.

Let $(x_0, x_1, x_2, x_3) \in \mathbb{N}$. Denote by $[x_0, x_1, x_2, x_3]$ the null line generated by the null vector $(x_0, x_1, x_2, x_3)$. By using nonhomogeneous coordinates,

$$[x_0, x_1, x_2, x_3] = \left[1, \frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}\right] \text{ provided } x_0 \neq 0$$

$$\cong \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}\right) \in S^2_1(1)$$

$$\cong \left(\frac{x_1}{x_0 + x_3}, \frac{x_2}{x_0 + x_3}\right) \in E^2_1 \text{ via the projection } \varphi_+$$

or

$$[x_0, x_1, x_2, x_3] \cong \left(\frac{x_1}{x_0 - x_3}, \frac{x_2}{x_0 - x_3}\right) \in E^2_1 \text{ via the projection } \varphi_-.$$ 

Finally, we have the identification:

(64) \hspace{1cm} [x_0, x_1, x_2, x_3] \cong \left(\frac{x_1 + x_2}{x_0 + x_3}, \frac{-x_1 + x_2}{x_0 + x_3}\right) \in E^2_1(u, v)

or

(65) \hspace{1cm} [x_0, x_1, x_2, x_3] \cong \left(\frac{x_1 + x_2}{x_0 - x_3}, \frac{-x_1 + x_2}{x_0 - x_3}\right) \in E^2_1(u, v).
11. THE HYPERBOLIC GAUSS MAP AND THE SECONDARY GAUSS MAP

Let \( \varphi : M \rightarrow \mathbb{H}^3(1) \) be a timelike \( \text{cmc} -1 \) surface in \( \mathbb{H}^3(1) \). Then by Theorem 6 (or by Theorem 7), there exists a smooth immersion \( F = (F_1, F_2) : M \rightarrow \text{SL}_2 \mathbb{R} \times \text{SL}_2 \mathbb{R} \) such that

1. \( F_1 \) is Lorentz holomorphic and \( F_2 \) is Lorentz antiholomorphic.
2. \( \det (F_1^* dF_1) = \det (F_2^* dF_2) = 0 \)
   (or (2) \( \det (F_1^* dF_1) = \det ((dF_2^{-1}) F_2) = 0 \)).
3. \( \varphi = F_1 F_2^* \) (or (3) \( \varphi = F_1 F_2^{-1} \)).

Let \( F_1 = \begin{pmatrix} F_{11} & F_{12} \\ F_{13} & F_{14} \end{pmatrix} \) and \( F_2 = \begin{pmatrix} F_{21} & F_{22} \\ F_{23} & F_{24} \end{pmatrix} \).

Then

\[
(e_0 + e_3)(F) = F_1(1 + k') F_2^*
= 2 \begin{pmatrix} F_{11} & F_{12} \\ F_{13} & F_{14} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} F_{21} & F_{23} \\ F_{22} & F_{24} \end{pmatrix}
= 2 \begin{pmatrix} F_{11} F_{21} & F_{11} F_{23} \\ F_{13} F_{21} & F_{13} F_{23} \end{pmatrix}.
\]

By the identification (65),

\[
[e_0 + e_3](F) = \begin{pmatrix} F_{11} F_{21} & F_{11} F_{23} \\ F_{13} F_{21} & F_{13} F_{23} \end{pmatrix} \in \mathbb{E}_2^2(u, v).
\]

Similarly,

\[
[e_0 - e_3](F) = [F_1(1 - k') F_2]^* = \begin{pmatrix} F_{12} & F_{22} \\ F_{14} & F_{24} \end{pmatrix} \in \mathbb{E}_2^2(u, v).
\]

If \( (e_0 \pm e_3)(F) = F_1(1 \pm k') F_2^{-1} \), then

\[
[e_0 + e_3](F) \cong \begin{pmatrix} F_{11} & F_{24} \\ F_{13} & F_{22} \end{pmatrix} \in \mathbb{E}_1^2(u, v),
\]

\[
[e_0 - e_3](F) \cong \begin{pmatrix} F_{12} & F_{23} \\ F_{14} & F_{21} \end{pmatrix} \in \mathbb{E}_1^2(u, v).
\]
Locally,
\[
(e_0 + e_3)(Fh) = F_1 h(1 + k') (F_2 (h^{-1})^t)
\]
\[
= 2F_1 h \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} h^{-1} F_2^t
\]
\[
= 2F_1 \begin{pmatrix} p_1 & -q_2 \\ q_1 & p_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -q_1 \\ p_1 \end{pmatrix} \begin{pmatrix} F_{21} \\ F_{22} \\ F_{23} \\ F_{24} \end{pmatrix}.
\]

The last expression is simplified to the matrix:
\[
2 \begin{pmatrix} (F_{11} p_1 + F_{12} q_1)(F_{21} p_2 + F_{22} q_2) & (F_{11} p_1 + F_{12} q_1)(F_{23} p_2 + F_{24} q_2) \\ (F_{13} p_1 + F_{14} q_1)(F_{21} p_2 + F_{22} q_2) & (F_{13} p_1 + F_{14} q_1)(F_{23} p_2 + F_{24} q_2) \end{pmatrix}.
\]

Thus, by the identification (65),
\[
[(e_0 + e_3)(Fh)] = [F_1 h(1 + k') (F_2 (h^{-1})^t)]
\]
\[
\cong \begin{pmatrix} F_{11} p_1 + F_{12} q_1 \\ F_{13} p_1 + F_{14} q_1 \end{pmatrix} \begin{pmatrix} F_{21} p_2 + F_{22} q_2 \\ F_{23} p_2 + F_{24} q_2 \end{pmatrix} \in \mathbb{E}^2_1(u, v).
\]

Note that the hyperbolic Gauß map \(e_0 + e_3\) is orientation preserving while \(e_0 - e_3\) is orientation reversing. So,
\[
[(e_0 - e_3)(Fh)] = [F_1 h(1 - k') (F_2 (h^{-1})^t)]
\]
\[
\cong \begin{pmatrix} F_{11} p_1 - F_{12} q_1 \\ F_{13} p_1 - F_{14} q_1 \end{pmatrix} \begin{pmatrix} F_{21} p_2 - F_{22} q_2 \\ F_{23} p_2 - F_{24} q_2 \end{pmatrix} \in \mathbb{E}^2_1(u, v),
\]

where \(h = \begin{pmatrix} q_2 & -p_1 \\ p_2 & q_1 \end{pmatrix} \). (See Remark 4.)

Similarly, if \((e_0 \pm e_3)(Fh) = F_1 h(1 \pm k')(F_2 h)^{-1}\), then
\[
[(e_0 + e_3)(Fh)] \cong \begin{pmatrix} F_{11} p_1 + F_{12} q_1 \\ F_{13} p_1 + F_{14} q_1 \end{pmatrix} \begin{pmatrix} F_{23} p_2 - F_{24} q_2 \\ F_{22} p_2 - F_{21} q_2 \end{pmatrix} \in \mathbb{E}^2_1(u, v),
\]
\[
[(e_0 - e_3)(Fh)] \cong \begin{pmatrix} F_{11} p_1 - F_{12} q_1 \\ F_{13} p_1 - F_{14} q_1 \end{pmatrix} \begin{pmatrix} F_{24} p_2 + F_{23} q_2 \\ F_{22} p_2 + F_{21} q_2 \end{pmatrix} \in \mathbb{E}^2_1(u, v).
\]

Let \(q := \frac{p_1}{q_1}\) and \(r := \frac{p_2}{q_2}\). The ordered pair \((q, r)\) is called the secondary Gauß map\(^8\). In terms of the secondary Gauß map \((q, r)\), locally in an open

\(^8\)As seen in section 3, this secondary Gauß map coincides with the projected Gauß map of a corresponding timelike minimal surface in \(E^3_1\).
set $U \subset M$,

$$F_1^{-1} dF_1 = \begin{pmatrix} q & -q^2 \\ 1 & -q \end{pmatrix} f(u) du = \begin{pmatrix} q & -q^2 \\ 1 & -q \end{pmatrix} \eta,$$

$$F_2^{-1} dF_2 = \begin{pmatrix} r & -r^2 \\ 1 & -r \end{pmatrix} g(v) dv = \begin{pmatrix} r & -r^2 \\ 1 & -r \end{pmatrix} \xi,$$

$$(dF_2^{-1}) dF_2 = \begin{pmatrix} r & 1 \\ -r^2 & -r \end{pmatrix} g(v) dv = \begin{pmatrix} r & 1 \\ -r^2 & -r \end{pmatrix} \xi.$$ 

Thus, we have the following equations:

$$dF_1 = \begin{pmatrix} dF_{11} & dF_{12} \\ dF_{13} & dF_{14} \end{pmatrix} = \begin{pmatrix} F_{11} & F_{12} \\ F_{13} & F_{14} \end{pmatrix} \begin{pmatrix} q & -q^2 \\ 1 & -q \end{pmatrix} \eta,$$

$$= \begin{pmatrix} F_{11}q + F_{12} & -(F_{11}q + F_{12}q) \\ F_{13}q + F_{14} & -(F_{13}q + F_{14}q) \end{pmatrix} \eta,$$

$$dF_2 = \begin{pmatrix} dF_{21} & dF_{22} \\ dF_{23} & dF_{24} \end{pmatrix} = \begin{pmatrix} F_{21} & F_{22} \\ F_{23} & F_{24} \end{pmatrix} \begin{pmatrix} r & -r^2 \\ 1 & -r \end{pmatrix} \xi,$$

$$= \begin{pmatrix} F_{21}r + F_{22} & -(F_{21}r + F_{22})r \\ F_{23}r + F_{24} & -(F_{23}r + F_{24})r \end{pmatrix} \xi,$$

and

$$dF_2^{-1} = \begin{pmatrix} dF_{24} & -dF_{22} \\ -dF_{23} & dF_{21} \end{pmatrix} = \begin{pmatrix} r & 1 \\ -r^2 & -r \end{pmatrix} \begin{pmatrix} F_{24} & -F_{22} \\ -F_{23} & F_{21} \end{pmatrix} \xi,$$

$$= \begin{pmatrix} F_{24}r - F_{23} & -(F_{24}r - F_{23})r \\ -(F_{24}r - F_{23})r & (F_{24}r - F_{23})r \end{pmatrix} \xi,$$

i.e., $dF_2$ is also given by

$$dF_2 = \begin{pmatrix} (F_{22}r - F_{21})r & F_{22}r - F_{21} \\ (F_{24}r - F_{23})r & F_{24}r - F_{23} \end{pmatrix} \xi.$$
Hence, the hyperbolic Gauß map can be written:

\[(e_0 + e_3)(Fh) = [F_1 h \mathbf{1} + \mathbf{k}'(F_2(h^{-1})^t)] \]

\[\cong \begin{pmatrix} F_{11}q + F_{12} & F_{21}r + F_{22} \\ F_{13}q + F_{14} & F_{23}r + F_{24} \end{pmatrix} \]

\[= \begin{pmatrix} \frac{dF_{11}}{dF_{13}} \cdot \frac{dF_{21}}{dF_{23}} \\ \frac{dF_{12}}{dF_{14}} \cdot \frac{dF_{22}}{dF_{24}} \end{pmatrix} \in \mathbb{E}_2^1(u, v),\]

and

\[(e_0 - e_3)(Fh) = [F_1 h \mathbf{1} - \mathbf{k}'(F_2(h^{-1})^t)] \]

\[\cong \begin{pmatrix} F_{11}q - F_{12} & F_{21}r - F_{22} \\ F_{13}q - F_{14} & F_{23}r - F_{24} \end{pmatrix} \]

\[= \begin{pmatrix} \frac{dF_{11}}{dF_{13}} \cdot \frac{dF_{21}}{dF_{23}} \\ \frac{dF_{12}}{dF_{14}} \cdot \frac{dF_{22}}{dF_{24}} \end{pmatrix} \in \mathbb{E}_1^2(u, v),\]

12. The Generalized Gauß Map and the Hyperbolic Gauß Map

Let \(G(2, \mathbb{E}_2^1)\) be the Grassmannian manifold of oriented timelike 2-planes in \(\mathbb{E}_2^2\). Let \(\varphi : M \rightarrow \mathbb{H}_3^1(-1)\) be an oriented timelike surface in \(\mathbb{H}_3^1(-1)\). At each point \(p \in M\), there is a (timelike) tangent plane to the surface \(\varphi:\)

\[\varphi_*(T_pM) = [\varphi_x \wedge \varphi_y]_p = [(\varphi_x + \varphi_y) \wedge (-\varphi_x + \varphi_y)]_p\]
spanned by a timelike vector $\varphi_x$ and a spacelike vector $\varphi_y$ or equivalently, by two null vectors $\varphi_x + \varphi_y$ and $-\varphi_x + \varphi_y$. Define a map

$$G : M \longrightarrow G(2, E^4_2); p \in M \mapsto [\varphi_x \wedge \varphi_y]_p = [(\varphi_x + \varphi_y) \wedge (\varphi_x + \varphi_y)]_p.$$  

This map is called the generalized Gauss map of a timelike surface $\varphi : M \longrightarrow \mathbb{H}^3_1(-1)$.

Let $u$ be a null vector, i.e., $(u, u) = 0$ and $[u]$ denote the null line generated by $u$. Let $Q^2_0 := \{[u] \in \mathbb{R}P^3_2 : (u, u) = 0 \}$. Then there is an embedding

$$\Gamma : G(2, E^4_2) \longrightarrow Q^2_0 \times Q^2_0; [v \wedge w] \mapsto ([v + w], [-v + w]),$$

where $v$ is a timelike vector and $w$ is a spacelike vector. That is, $G(2, E^4_2) \cong \text{Im} \Gamma \subset Q^2_0 \times Q^2_0$. The Lie group $\text{SL}_2 \mathbb{R} \times \text{SL}_2 \mathbb{R}$ acts on $G(2, E^4_2)$ transitively via the actions:

$$\mu : (\text{SL}_2 \mathbb{R} \times \text{SL}_2 \mathbb{R}) \times G(2, E^4_2) \longrightarrow G(2, E^4_2);$$  

$$\mu(g, [v \wedge w]) := [\mu(g, v) \wedge \mu(g, w)] = [(g_1v g_2^{-1}) \wedge (g_1w g_2^{-1})]$$

and

$$\nu : (\text{SL}_2 \mathbb{R} \times \text{SL}_2 \mathbb{R}) \times G(2, E^4_2) \longrightarrow G(2, E^4_2);$$  

$$\nu(g, [v \wedge w]) := [\nu(g, v) \wedge \nu(g, w)] = [(g_1v g_2^{-1}) \wedge (g_1w g_2^{-1})]$$

for $g = (g_1, g_2) \in \text{SL}_2 \mathbb{R} \times \text{SL}_2 \mathbb{R}$ and $[v, w] \in G(2, E^4_2)$. Note that

$$\mu(g, [v \wedge w]) \cong ([g_1(v + w) g_2^{-1}], [g_1(-v + w) g_2^{-1}]) \in Q^2_0 \times Q^2_0;$$  

$$\nu(g, [v \wedge w]) \cong ([g_1(v + w) g_2^{-1}], [g_1(-v + w) g_2^{-1}]) \in Q^2_0 \times Q^2_0.$$

The isotropy subgroup of $\text{SL}_2 \mathbb{R} \times \text{SL}_2 \mathbb{R}$ with the actions $\mu$ and $\nu$ at

$$[e_1 \wedge e_2] = [(e_1 + e_2) \wedge (-e_1 + e_2)]$$

is $\mathbb{R}_+ \times \mathbb{R}_+$, where $\mathbb{R}_+ := \left\{ \begin{pmatrix} r & 0 \\ 0 & \frac{1}{r} \end{pmatrix} : r \in \mathbb{R} \setminus \{0\} \right\}$. Thus, the Grassmannian manifold $G(2, E^4_2)$ can be represented as a symmetric space

$$\frac{\text{SL}_2 \mathbb{R} \times \text{SL}_2 \mathbb{R}}{\mathbb{R}_+ \times \mathbb{R}_+} \cong \left\{ \left( \begin{pmatrix} g_1 & 0 \\ 0 & 0 \end{pmatrix} \right) g_2^{-1}, \left( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} g_2 \right) \right\} \in Q^2_0 \times Q^2_0 : g_1, g_2 \in \text{SL}_2 \mathbb{R} \right\}.$$
Denote by $G(2, \mathbb{E}^4)_-^-$ the Grassmannian manifold of \textit{negatively} oriented time-like 2-planes in $\mathbb{E}^4_2$. Then

\[
G(2, \mathbb{E}^4)_-^-= \frac{\text{SL}_2 \mathbb{R} \times \text{SL}_2 \mathbb{R}}{\mathbb{R}^+ \times \mathbb{R}^+} \cong \left\{ \left( \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix}, \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} \right) : g_1, g_2 \in \text{SL}_2 \mathbb{R} \right\}.
\]

Let us write $g_1 = \left( \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \right)$ and $g_2 = \left( \begin{bmatrix} g_{21} & g_{22} \\ g_{23} & g_{24} \end{bmatrix} \right)$. Define a projection map

\[
\phi = (\phi_1, \phi_2) : G(2, \mathbb{E}^4)_-^-= \mathbb{E}^2_1(u, v) \times \mathbb{E}^2_1(u, v)
\]

by

\[
\left( \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix}, \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} \right) \overset{(\phi_1, \phi_2)}{\mapsto} \left( \begin{bmatrix} g_{11} & g_{21} \\ g_{13} & g_{23} \end{bmatrix} \right)
\]

or

\[
\left( \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix}, \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} \right) \overset{(\phi_1, \phi_2)}{\mapsto} \left( \begin{bmatrix} g_{11} & g_{21} \\ g_{13} & g_{23} \end{bmatrix} \right).
\]

Similarly, we also define a projection map

\[
\phi^- = (\phi_1^-, \phi_2^-) : G(2, \mathbb{E}^4)_-^-= \mathbb{E}^2_1(u, v) \times \mathbb{E}^2_1(u, v)
\]

by

\[
\left( \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix}, \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} \right) \overset{(\phi_1^-, \phi_2^-)}{\mapsto} \left( \begin{bmatrix} g_{12} & g_{22} \\ g_{14} & g_{24} \end{bmatrix} \right)
\]

or

\[
\left( \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix}, \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} \right) \overset{(\phi_1^-, \phi_2^-)}{\mapsto} \left( \begin{bmatrix} g_{12} & g_{22} \\ g_{14} & g_{24} \end{bmatrix} \right).
\]

Let $\varphi : M \longrightarrow \mathbb{H}_1^3(-1)$ be a timelike surface from an oriented and simply-connected open 2-manifold $M$ into $\mathbb{H}_1^3(-1)$ with $ds^2_\varphi = e^\omega(-dx^2 + dy^2) = e^\omega du dv$. Then there exists an adapted framing $F : M \longrightarrow \text{SL}_2 \mathbb{R} \times \text{SL}_2 \mathbb{R}$ of $\varphi$.
such that $e_1 \circ F = e^{-\omega} \phi_x$ and $e_2 \circ F = e^{-\omega} \phi_y$. The generalized Gauß map $G$ of $\phi$ can be written

$$G = [(e_1 \circ F) \wedge (e_2 \circ F)]$$

$$= [(e_1 + e_2)(F) \wedge (-e_1 + e_2)(F)]$$

$$\cong \left( \begin{bmatrix} F_1 & F_1^t \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} F_1 & F_1^t \end{bmatrix} \right) : M \rightarrow G(2, \mathbb{E}^2_2).$$

Let $G_1 = \phi_1 \circ G$ and $G_2 = \phi_2 \circ G$. Then $G_1 = \frac{F_{11}}{F_{13}}, G_2 = \frac{F_{21}}{F_{23}}.$ Note that the ordered pair $(G_1, G_2) = (\frac{F_{11}}{F_{13}}, \frac{F_{21}}{F_{23}})$ is the same as the hyperbolic Gauß map $[(e_0 + e_3)(F)] = [F_1(1 + k)F_2].$

If $G \cong \left( \begin{bmatrix} F_1 & F_1^t \end{bmatrix}, \begin{bmatrix} F_1 & F_1^t \end{bmatrix} \right)$, then

$$(G_1, G_2) = \left( \frac{F_{11}}{F_{13}}, \frac{F_{24}}{F_{22}} \right) = [F_1(1 + k')F_2^{-1}] = [(e_0 + e_3)(F)].$$

Let us define $G^- : M \rightarrow G(2, \mathbb{E}^2_2)$ by

$$G^- = [(-e_1 + e_2)(F) \wedge (e_1 + e_2)(F)]$$

$$\cong \left( \begin{bmatrix} F_1 & F_1^t \end{bmatrix}, \begin{bmatrix} F_1 & F_1^t \end{bmatrix} \right).$$

Let $G^- = \phi_1 \circ G^- , G^- = \phi_2 \circ G^-$. Then

$$(G^-_1, G^-_2) = \left( \frac{F_{12}}{F_{14}}, \frac{F_{22}}{F_{24}} \right) = [F_1(1 - k')F_2^{-1}] = [(e_0 - e_3)(F)].$$

If $G^- \cong \left( \begin{bmatrix} F_1 & F_1^t \end{bmatrix}, \begin{bmatrix} F_1 & F_1^t \end{bmatrix} \right)$, then

$$(G^-_1, G^-_2) = \left( \frac{F_{12}}{F_{14}}, \frac{F_{23}}{F_{21}} \right) = [F_1(1 - k')F_2^{-1}] = [(e_0 - e_3)(F)].$$

13. The Lorentz Holomorphicity of Hyperbolic Gauß Map and Timelike cmc ±1 Surfaces in $\mathbb{H}^3(1)$

In this section, we study the relationship between Lorentz holomorphicity of (projected) hyperbolic Gauß map and timelike cmc ±1 surfaces in $\mathbb{H}^3(1).$ Their relationship is summarized as the following theorem. Here, we assume that $\omega := \Phi_1 \Phi_2 : M \rightarrow \mathbb{H}^3(1)$ and $\psi := \Psi_1 \Psi_2 : M \rightarrow \mathbb{H}^3(1)$ are timelike surfaces in $\mathbb{H}^3(1),$ where $\Phi := (\Phi_1, \Phi_2) : M \rightarrow SL_2 \mathbb{R} \times SL_2 \mathbb{R}$ and $\Psi := (\Psi_1, \Psi_2) : M \rightarrow SL_2 \mathbb{R} \times SL_2 \mathbb{R}$ are solutions of Lax equations \cite{15} and \cite{6}, resp., in Theorem \cite{15}. Let us regard $[(e_0 + e_3)(\Phi)] = [\Phi_1(1 + k)\Phi_2]$ and $[(e_0 + e_3)(\Psi)] = [\Psi_1(1 + k)\Psi_2^{-1}]$ as the projected hyperbolic Gauß map.
We prove only part (1). The rest can be proved similarly. Since

\((\Phi_{11} \Phi_{21}) \in E^2_1(u, v)\) (\((\Phi_{11} \Phi_{13} \Phi_{23}) \in E^2_1(u, v)\)). Also, regard \([(e_0 - e_3)(\Phi)] = [\Phi(1 - k')\Phi^2_2] \([(e_0 - e_3)(\Psi)] = [\Psi(1 - k')\Psi^2_2] \) as the projected hyperbolic Gauss map \((\Phi_{12} \Phi_{22} \Phi_{24}) \in E^2_1(u, v)\) (\((\Phi_{12} \Phi_{14} \Phi_{24}) \in E^2_1(u, v)\)).

Then we have the following theorem holds.

**Theorem 9.**

1. \([(e_0 + e_3)(\Phi)] \([(e_0 + e_3)(\Psi)] \) is Lorentz antiholomorphic if and only if \(\varphi \) satisfies \(H = 1\) and \(Q = 0\).

2. \([(e_0 + e_3)(\Phi)] \ [(e_0 + e_3)(\Psi)] \) is Lorentz holomorphic if and only if \(\varphi \) satisfies \(H = 1\) and \(R = 0\).

3. \([(e_0 - e_3)(\Phi)] \ [(e_0 - e_3)(\Psi)] \) is Lorentz antiholomorphic if and only if \(\varphi \) satisfies \(H = -1\) and \(Q = 0\).

4. \([(e_0 - e_3)(\Phi)] \ [(e_0 - e_3)(\Psi)] \) is Lorentz holomorphic if and only if \(\varphi \) satisfies \(H = -1\) and \(R = 0\).

**Proof.** We prove only part (1). The rest can be proved similarly. Since \n
\([(e_0 + e_3)(\Phi)] = \left(\frac{\Phi_{11}}{\Phi_{13}}, \frac{\Phi_{21}}{\Phi_{23}}\right),\]

\([(e_0 + e_3)(\Phi)] \) is Lorentz antiholomorphic, i.e., \([(e_0 + e_3)(\Phi)]_u = 0\) if and only if \((\Phi_{11})_u \Phi_{13} - \Phi_{11} \Phi_{13})_u = 0\) and \((\Phi_{21})_u \Phi_{23} - \Phi_{21} \Phi_{23})_u = 0\).

On the other hand, from the Lax equations \([\text{Eq}].\)

\[\begin{pmatrix}
(\Phi_{11})_u & (\Phi_{12})_u \\
(\Phi_{13})_u & (\Phi_{14})_u
\end{pmatrix}
= \begin{pmatrix}
\frac{\Phi_{11} \Phi_{12}}{\Phi_{13} \Phi_{14}} & -\frac{\Phi_{12}}{\Phi_{14}} \frac{e^{-\frac{\omega u}{4}} Q}{\Phi_{13} \Phi_{14}} \\
\frac{\Phi_{11}}{\Phi_{13}} \Phi_{14} & -\frac{\Phi_{12}}{\Phi_{14}} \frac{e^{-\frac{\omega u}{4}} Q}{\Phi_{13} \Phi_{14}}
\end{pmatrix}
\]

\[\begin{pmatrix}
\frac{1}{2} \Phi_{11} \omega u - \Phi_{12} e^{-\frac{\omega u}{4}} Q - \frac{1}{2} \Phi_{12} \omega u \\
\frac{1}{2} \Phi_{13} \omega u - \Phi_{14} e^{-\frac{\omega u}{4}} Q - \frac{1}{2} \Phi_{14} \omega u
\end{pmatrix}
\]

and

\[\begin{pmatrix}
(\Phi_{21})_u & (\Phi_{22})_u \\
(\Phi_{23})_u & (\Phi_{24})_u
\end{pmatrix}
= \begin{pmatrix}
\frac{\Phi_{21} \Phi_{22}}{\Phi_{23} \Phi_{24}} & -\frac{\Phi_{22}}{\Phi_{24}} \frac{e^{-\frac{\omega u}{4}} Q}{\Phi_{23} \Phi_{24}} \\
\frac{1}{2} \Phi_{23} \omega u - \Phi_{24} e^{-\frac{\omega u}{4}} Q - \frac{1}{2} \Phi_{24} \omega u
\end{pmatrix}
\]

Thus,

\[\begin{pmatrix}
(\Phi_{11})_u & (\Phi_{13})_u & (\Phi_{11})_u (\Phi_{13})_u = e^{-\frac{\omega u}{4}} Q, \\
(\Phi_{21})_u & (\Phi_{23})_u & (\Phi_{21})_u (\Phi_{23})_u = e^{-\frac{\omega u}{4}} (H - 1).
\end{pmatrix}\]

It then follows immediately that \([(e_0 + e_3)(\Phi)] \) is Lorentz antiholomorphic if and only if \(\varphi \) satisfies \(H = 1\) and \(Q = 0\). \(\square\)

**Corollary 10.**

1. \([(e_0 + e_3)(\Phi)] \ [(e_0 + e_3)(\Psi)] \) is constant if and only if \(\varphi \) satisfies \(H = 1\) and is totally umbilic \((Q = 0, i.e., Q = R = 0)\).
\[(e_0 - e_3)(\Phi) \text{ is constant if and only if } \varphi(\psi) \text{ satisfies } H = -1 \text{ and is totally umbilic.}\]

14. **Appendix I: The Lawson-Guichard Correspondence between Timelike cmc Surfaces in Different Semi-Riemannian Space Forms**

In section 5, we discussed the *Lawson-Guichard correspondence* or simply *Lawson correspondence* between timelike cmc surfaces in semi-Riemannian space forms $E_3^1$, $S^3_1(1)$, and $H^3_1(-1)$. In fact, this Lawson correspondence was already known to A. Fujioka and J. Inoguchi (8 and 9). In this appendix, we study the Lawson correspondence in a more general setting.

Let $\bar{M}$ be a semi-Riemannian manifold and $M \subset \bar{M}$ a hypersurface with the sectional curvatures $\bar{K}$ and $K$, resp. Let $S$ be the shape operator derived from the unit normal vector field $N$ on the hypersurface $M$. If $X, Y$ span a nondegenerate tangent plane on $M$, then the Gauß equation is given by

$$K(X, Y) = \bar{K}(X, Y) + \epsilon \frac{\langle S(X), X \rangle \langle S(Y), Y \rangle - \langle S(X), Y \rangle^2}{\langle X, X \rangle \langle Y, Y \rangle - \langle X, Y \rangle^2},$$

where $\epsilon = \langle N, N \rangle$. (See B. O'Neill [20] on p. 107.) We begin with the following theorem which can be found, for example, in T. Weinstein [24] on p. 158.

**Theorem 11** (Fundamental Theorem of Surface Theory: Lorentzian Version). *Given a simply-connected Lorentz surface with global null coordinate system, there exists a timelike immersion with the first and the second fundamental forms $I$ and $II$ if and only if $I$ and $II$ satisfy the Gauß and Mainardi-Codazzi equations.*

Let $\mathcal{M}^3(\bar{K})$ be the semi-Riemannian 3-manifold with constant sectional curvature $\bar{K}$. For example, $\mathcal{M}^3(-1) = H^3_1(-1)$, $\mathcal{M}^3(0) = E^3_1$, and $\mathcal{M}^3(1) = S^3_1(1)$. For a conformal timelike immersion $\varphi : M \rightarrow \mathcal{M}^3(\bar{K})$ with induce metric $\langle \cdot, \cdot \rangle$, Levi-Civita connection $\nabla$, Gaußian curvature $K$ and shape operator $S$, the Gauß and Mainardi-Codazzi equations are satisfied:

- **Gauß Equation**
  $$K - \bar{K} = \det S$$

- **Mainardi-Codazzi Equation**
  $$\nabla_X S(Y) = \nabla_Y S(X)$$

for all smooth vector fields $X, Y, Z \in T(M)$.

Assume that $\varphi$ has a constant mean curvature $H = \frac{1}{2} \text{tr}(S)$. For any $c \in \mathbb{R}$, define

$$\tilde{S} = S + cI, \quad \tilde{K} = K - c \text{tr}(S) - c^2,$$

9Since $\mathcal{M}^3(\bar{K})$ has a constant sectional curvature and $M$ is a hypersurface immersed into $\mathcal{M}^3(\bar{K})$, the Mainardi-Codazzi equation becomes (68). See B. O'Neill [20] on p. 115 for more details.
where $I$ is the identity transformation. Then the Gauß equation and the Mainardi-Codazzi equation still hold when $S$ and $K$ are replaced by $\tilde{S}$ and $\tilde{K}$, resp.:

$$K - \tilde{K} = K - (\bar{K} - c \text{tr}(S) - c^2)$$
$$= K - \tilde{K} + c \text{tr}(S) + c^2$$
$$= \det(S) + c \text{tr}(S) + c^2$$
$$= \det(S + cI)$$
$$= \det(\tilde{S}).$$

Note that the Gaußian curvature $K$ of $M$ is intrinsic and does not change.

$$\langle \nabla_X S \rangle(Y) = \nabla_X S(Y) - S(\nabla_X Y),$$
$$\langle \nabla_Y S \rangle(X) = \nabla_Y S(X) - S(\nabla_Y X).$$

Since $\langle \nabla_X S \rangle(Y) = \langle \nabla_Y S \rangle(X)$,

$$S([X,Y]) = S(\nabla_X Y - \nabla_Y X)$$
$$= S(\nabla_X Y) - S(\nabla_Y X)$$
$$= \nabla_X S(Y) - \nabla_Y S(X).$$

Now,

$$\tilde{S} = S([X,Y]) + c[X,Y]$$
$$= \nabla_X Y S(Y) - \nabla_Y S(X) + c[X,Y]$$
$$= \nabla_X (S + cI)(Y) - \nabla_Y (S + cI)(X)$$
$$= \nabla_X \tilde{S}(Y) - \nabla_Y \tilde{S}(X).$$

Therefore, there exists an immersion $\tilde{\varphi} : M \to M^3(\bar{K})$ with induced metric $\langle \cdot, \cdot \rangle$ and shape operator $\tilde{S}$, and $\tilde{\varphi}(M)$ is isometric to $\varphi(M)$. The mean curvature $\tilde{H}$ of $\tilde{\varphi}(M)$ is

$$\tilde{H} = \frac{1}{2} \text{tr}(\tilde{S}) = \frac{1}{2} \text{tr}(S) + c = H + c$$

and this shows the Lawson-Guichard correspondence between timelike \textit{cmc} $H$ surfaces in $M^3(\bar{K})$ and timelike \textit{cmc} $(H + c)$ surfaces in $M^3(\bar{K}) = M^3(\bar{K} - 2cH - c^2)$. In particular, when $H = \bar{K} = 0$ and $c = 1$, we have the Lawson-Guichard correspondence between timelike minimal surfaces in $\mathbb{E}^3_1$ and timelike \textit{cmc} 1 surfaces in $\mathbb{E}^3_1(-1)$. If $H = 0$ and $\bar{K} = c = 1$, then we have the Lawson-Guichard correspondence between timelike minimal surfaces in de Sitter 3-space $\mathbb{S}^3_1(1)$ and timelike \textit{cmc} 1 surfaces in $\mathbb{E}^3_1$. 
15. Appendix II: Some Examples of timelike cmc ±1 Surfaces in $\mathbb{H}_1^2(-1)$

In this appendix, we present some examples of timelike cmc 1 surfaces in $\mathbb{H}_1^2(-1)$.

Let us consider the following stereographic projections in order to view the isometric images of timelike cmc ±1 surfaces in $\mathbb{H}_1^2(-1)$ into the interior $\text{Int} S^2_t(1) = \{(x_1, x_2, x_3) \in \mathbb{E}^3 : -(x_1)^2 + (x_2)^2 + (x_3)^2 < 1\}$ of de Sitter 2-space $S^2_t(1)$.

Let $\varphi_+: \mathbb{H}_1^2(-1) \setminus \{x_0 = -1\} \to E_1^3 \setminus S^2_t(1)$ be the stereographic projection from $-e_0 = (-1, 0, 0, 0)$. Then

$$\varphi_+(x_0, x_1, x_2, x_3) = \left( \frac{x_1}{1 + x_0}, \frac{x_2}{1 + x_0}, \frac{x_3}{1 + x_3} \right).$$

Let $\varphi_- : \mathbb{H}_1^2(-1) \setminus \{x_0 = 1\} \to E_1^3 \setminus S^2_t(1)$ be the stereographic projection from $e_0 = (1, 0, 0, 0)$. Then

$$\varphi_-(x_0, x_1, x_2, x_3) = \left( \frac{x_1}{1 - x_0}, \frac{x_2}{1 - x_0}, \frac{x_3}{1 - x_3} \right).$$

Cut $\mathbb{H}_1^2(-1)$ into two halves by the hyperplane $x_0 = 0$. Denote by $\mathbb{H}_1^2(-1)_+$ ($\mathbb{H}_1^2(-1)_-$) the half containing $e_0 = (1, 0, 0, 0)$ ($-e_0 = (-1, 0, 0, 0)$). Then $\varphi_+ : \mathbb{H}_1^2(-1)_+ \to \text{Int} S^2_t(1)$ and $\varphi_- : \mathbb{H}_1^2(-1)_- \to \text{Int} S^2_t(1)$.

Example 1 (Timelike Enneper Cousin in $\mathbb{H}_1^2(-1)$ (Isothermic Type)). Let $(q, r) = (u, v)$. Then using the Bryant-Umehara-Yamada type representation, we set up the following initial value problem:

$$F_1^{-1} dF_1 = \begin{pmatrix} \frac{u}{u^2} \\ \frac{1}{-u} \end{pmatrix} du, \quad F_2^{-1} dF_2 = \begin{pmatrix} \frac{v}{v^2} \\ \frac{1}{-v} \end{pmatrix} dv$$

with the initial condition $F_1(0) = F_2(0) = 1$. This initial value problem has a unique solution

$$F_1(u, v) = \begin{pmatrix} \cosh u & \sinh u - u \cosh u \\ \sinh u & \cosh u - u \sinh u \end{pmatrix},$$

$$F_2(u, v) = \begin{pmatrix} \cosh v & \sinh v - v \cosh v \\ \sinh v & \cosh v - v \sinh v \end{pmatrix}$$

which are Lorentz holomorphic and Lorentz antiholomorphic null immersions into $\text{SL}_2 \mathbb{R}$. The Bryant type representation formula then yields a timelike cmc 1 surface in $\mathbb{H}_1^2(-1)$. The resulting surface is a correspondent of isothermic type timelike Enneper surface in $\mathbb{E}^3_1$ under the Lawson-Guichard correspondence. For this reason, the resulting surface is called isothermic type timelike Enneper cousin in $\mathbb{H}_1^2(-1)$.

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\textsuperscript{10}For details about isothermic and anti-isothermic timelike surfaces, please see [9] or [13].
Figure 1 shows different views of isothermic type timelike Enneper cousin in $\mathbb{H}_1^3(-1)$ projected via $\varphi_+$ into the interior of the boundary $S_1^2(1)$.

**Figure 1.** Isothermic type timelike Enneper cousin projected into $\text{Int}S_1^2(1)$ via $\varphi_+$ with light cone and the boundary $S_1^2(1)$ in $E_1^3$. 
Figure 2 shows different views of isothermic type timelike Enneper surface in $\mathbb{E}_1^3$.

\begin{figure}[h]
\centering
\begin{subfigure}{0.4\textwidth}
  \centering
  \includegraphics[width=\textwidth]{a.png}
  \caption{(a)}
\end{subfigure}
\hfill
\begin{subfigure}{0.4\textwidth}
  \centering
  \includegraphics[width=\textwidth]{b.png}
  \caption{(b)}
\end{subfigure}
\hfill
\begin{subfigure}{0.4\textwidth}
  \centering
  \includegraphics[width=\textwidth]{c.png}
  \caption{(c)}
\end{subfigure}
\hfill
\begin{subfigure}{0.4\textwidth}
  \centering
  \includegraphics[width=\textwidth]{d.png}
  \caption{(d)}
\end{subfigure}
\caption{ Isothermic type timelike Enneper surface in $\mathbb{E}_1^3$ with light cone}
\end{figure}
Example 2 (Timelike Enneper Cousin in $\mathbb{H}^3_1(-1)$ (Anti-isothermic Type)). Let $(q,r) = (-u, v)$. Then using the Bryant-Umehara-Yamada type representation \(^5\), we set up the following initial value problem:

$$
F^{-1}_1 dF_1 = \begin{pmatrix} -u & -u^2 \\ 1 & u \end{pmatrix} du, \quad F^{-1}_2 dF_2 = \begin{pmatrix} v & -v^2 \\ 1 & -v \end{pmatrix} dv
$$

with the initial condition $F_1(0) = F_2(0) = 1$. This initial value problem has a unique solution

$$
F_1(u,v) = \begin{pmatrix} \cos u & -\sin u + u \cos u \\ \sin u & \cos u + u \sin u \end{pmatrix},
$$

$$
F_2(u,v) = \begin{pmatrix} \cosh v & \sinh v - v \cosh v \\ \sinh v & \cosh v - v \sinh v \end{pmatrix}
$$

which are Lorentz holomorphic and Lorentz antiholomorphic null immersions into $SL_2 \mathbb{R}$. The Bryant type representation formula \(^6\) then yields a timelike cmc 1 surface in $\mathbb{H}^3_1(-1)$. The resulting surface is a correspondent of anti-isothermic type timelike Enneper surface in $\mathbb{E}^3_1$ under the Lawson-Guichard correspondence. For this reason, the resulting surface is called anti-isothermic type timelike Enneper cousin in $\mathbb{H}^3_1(-1)$.

Figure 3 shows different views of anti-isothermic type timelike Enneper cousin in $\mathbb{H}^3_1(-1)$ projected via $\wp_+^*$ into the interior of the boundary $S^2_1(1)$.

Figures 4 and 5 show different views of anti-isothermic type timelike Enneper surface in $\mathbb{E}^3_1$.

Example 3 (cmc 1 B-scroll in $\mathbb{H}^3_1(-1)$). Let $(q,r) = (u, 0)$. Then using the Bryant-Umehara-Yamada type representation \(^5\), we set up the following initial value problem:

$$
F^{-1}_1 dF_1 = \begin{pmatrix} u & -u^2 \\ 1 & -u \end{pmatrix} du, \quad F^{-1}_2 dF_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} dv
$$

with the initial condition $F_1(0) = F_2(0) = 1$. This initial value problem has a unique solution

$$
F_1(u,v) = \begin{pmatrix} \cos u & \sinh u - u \cosh u \\ \sinh u & \cosh u - u \sinh u \end{pmatrix},
$$

$$
F_2(u,v) = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}
$$

which are Lorentz holomorphic and Lorentz antiholomorphic null immersions into $SL_2 \mathbb{R}$. The Bryant type representation formula \(^6\) yields a timelike cmc 1 surface

$$
\varphi = F_1 F_2^* = \begin{pmatrix} \cosh u & -(u-v) \cosh u - \sinh u \\ \sinh u & -(u-v) \sinh u + \cosh u \end{pmatrix}
$$

in $\mathbb{H}^3_1(-1)$. The resulting surface is a correspondent of minimal B-scroll in $\mathbb{E}^3_1$ under the Lawson-Guichard correspondence.
Figure 3. Anti-isothermic type timelike Enneper cousin projected into Int$S^2_1(1)$ via $\varphi_+$ with light cone and the boundary $S^2_1(1)$ in $E^3_1$.

Figure 6 shows different views of cmc 1 B-scroll projected in $\mathbb{H}_3^1(-1)$ via $\varphi_+$ into the interior of the boundary $S^2_1(1)$.

Figure 7 shows different views of minimal B-scroll in $E^3_1$. 
Figure 4. Anti-isothermic type timelike Enneper surface in $E^3_1$ with light cone

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Figure 5. Anti-isothermic type timelike Enneper surface in $\mathbb{E}^3$ with self-intersection

of discussions with Dr. Jun-ichi Inoguchi who was also visiting Texas Tech University during that time.
Figure 6. $\text{cmc} \pm 1$ B-scroll projected into $\text{Int} S^2_1(1)$ via $\varphi_+$ with light cone and the boundary $S^2_1(1)$ in $E^3_1$.

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Figure 7. Minimal B-scroll in $\mathbb{E}^3_1$ with light cone

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