Effective Field Theory of small $r$

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The recent observations from CMB have imposed a very stringent upper-limit on the tensor/scalar ratio $r$ of inflation models, which indicates that the primordial gravitational waves (PGW), even though possible to be detected, should be very tiny. However, current experiments on PGW is ambitious to detect such a tiny signal by improving the accuracy to an even higher level. Whatever their results are, it will give us much information about the early universe, not only from the astrophysical side but also from the theoretical side, e.g., model building. In this paper, we are interested in analyzing what kind of inflation models can be favored by future observations, using the facility of the effective field theory (EFT) approach. We show that $r$ can be expressed in EFT language, and more importantly, by plotting the contours in the parameter spaces, we indicate how $r$ can be affected by model parameters in various concrete examples.

I. INTRODUCTION

The detection of gravitational waves (GWs) has become one of the most important tasks in modern astrophysics and cosmology, not only because it is the last proof of the correctness of Einstein’s General Relativity, but also since it can provide an independent probe of our Universe, which is known as the “standard sirens” [1]. Therefore, it is no longer surprising that the first direct detection of GWs from two emerging black holes by the Laser Interferometer Gravitational-Wave Observatory (LIGO) [2] has become the hotspot of nowadays science, and been awarded the Nobel Prize in Physics in 2017. Since then, LIGO and VIRGO announced several events of black hole GWs [3–6] as well as one event of neutron star GWs [7], which indicate the coming of a new “gravitational wave era”.

However, like the electromagnetic waves, the GWs are distributed over a very wide frequency range, from $10^{−16}$ Hz to $10^{12}$ Hz, therefore to get full information of GWs, we need to utilize various probes for different ranges of frequencies. Besides LIGO and VIRGO ($\sim 10^{2}$Hz), the existing and planning programs for GWs detecting includes FAST ($10^{−8}$Hz−nHz with annual modulation, same as PTA) [8], KAGRA (kHz, almost same target-range as those of LIGO and Virgo) [9], LISA/TianQin/Taiji (mHz) [10–12], EPTA (nHz) [13], AliCPT [14] ($\sim 10^{−16}$Hz) and so on, all of which are devoting themselves on building the “Multi-band gravitational waves astronomy”.

Among the various frequency bands, the most difficult to detect might be the one with ultra-low frequency ($\sim 10^{−16}$Hz) and ultra-long wave-lengths (order of the size of the observational universe), as well as very low amplitudes. This kind of gravitational waves are believed to be generated at the very early stages of the universe, probably during the inflationary era [15], and thus dubbed as Primordial Gravitational Waves (PGWs). However, these PGWs, also known as primordial tensor perturbations, can affect the CMB photons before the last scattering, and thus, leave hints on the CMB sky map in the form of $B$ mode polarization [16–18]. Since different evolution of the early universe can give different evolution behavior of the primordial tensor perturbations and also different features of polarization in CMB map, the observations of the polarizations—therefore the primordial gravitational waves—can be used as a probe to test models for the early universe, especially, inflation.

Practically, as done in all-sky surveys from WMAP [19] to PLANCK [20], the detection on polarizations of CMB photons can be transformed into that on parameters of inflation models, such as the spectral amplitude $A_s$, the spectral index $n_s$, as well as the tensor/scalar ratio $r$. The first two corresponds to the primordial scalar perturbations, while the last one involves both scalar and tensor ones. From the newly-released PLANCK 2018 [21], the constraints on these parameters are $ln(10^{10}A_s) = 3.044 \pm 0.0014$ (68% C.L.), $n_s = 0.9649 \pm 0.0042$ (68% C.L.), (TT, TE, EE+lowE+lensing), $r_{0.002} < 0.064$ (95% C.L., TT, TE, EE+lowE+lensing+BK14). As can be seen from these data, we still can have only upper bound for $r$, which is continuously lowered, although $A_s$ and $n_s$ can be constraints both from above and below. It means that the primordial gravitational waves are really very weak and very difficult to

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test, and current constraints to primordial gravitational waves, although having been improved much, still needs much more development.

In 2014, we proposed a ground-based CMB experiment called AliCPT in Ali of Tibet, China, which aims to search for PGWs by detecting such B mode polarization [14]. As an experiment in the northern hemisphere, it can cover up to 65% of the sky map, and thus become very important counterpart to other ground-based experiments, such as that in Chile (Atacama Cosmology Telescope [22], POLARBEAR [23]) and at the South Pole (South Pole Telescope [24], BICEP [25]).

The very ambitious scientific goal of AliCPT is to furtherly improve the sensitivity on r-detection and to put a more stringent limit on r by one order of magnitude [14]. The significance of the detection of PGWs will be at least two-folded: If we succeed in detecting PGW, we will have evidence that the PGWs do exist, giving the tensor/scalar ratio be well within the AliCPT region, namely $r \in (0.064, 0.01)$. On the other hand, if the PGW is still not detected by then, it means that the upper bound of the tensor/scalar ratio will be lowered again, indicating that inflation models with even smaller tensor/scalar ratio ($r < 0.01$) will be favored, examples of which including the Starobinsky model [26], ultra-slow-roll inflation model [27]/constant-roll inflation model [28], among many other models in the literature.

In this paper, we try to explore the general features that give rise to the small tensor/scalar ratio, making use of the effective field theory (EFT) approach. Since the EFT describes everything using geometric operators rather than concrete field operators, it reflects general features of a theory, where the only degree of freedoms matters while the detailed information does not. Therefore, it is proved a powerful tool to study cosmological theories in a general and efficient way. See applications of EFT approach in the study of inflation [29], dark energy [30–32], Horava gravity [33], and the spatial covariant gravity [34], non-singular cosmology [35, 36] and so on. Note that in [35, 36], a more general form of EFT action (based on GLPV action in [31]) has been studied, and was proved to be able to cover both inflation and bouncing cosmology, where the Null Energy Condition is violated. Hereby, following [35, 36], we use the EFT language to re-express $r$, and to find out how $r$ could be within the two regions mentioned above, so as to put constraints on various inflation models.

The rest of the paper is organized as the following: in Sec. II, we set up the EFT framework in light of the work in [35, 36]. In Sec. III, we calculate both scalar and tensor perturbations based on the framework, and re-express the tensor/scalar ratio in the EFT language. In Sec. IV, using concrete examples, we show that the expression of $r$ can be reduced to various relations with model slow-varying parameters, as well as the range of each parameter for $r$ to be within regions of detection/non-detection of PGWs. We also show how $r$ of these models can deviate from the usual consistency relation in inflation models. Sec. V includes our final remarks and discussions.

II. THE EFT FRAMEWORK: BACKGROUND

First of all we briefly review the EFT framework studied in [35, 36] (also in [31]). Based on the metric in the ADM form:

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt),$$

where $N$ and $N^i$ are the lapse function and shift vector, while $h_{ij}$ is the 3-dimentional spatial metric. The most general EFT action is given by [29, 31, 33, 35, 36]:

$$S = \int dt dx \sqrt{-g} \left[ \frac{M_p^2}{2} f(t) R - \Lambda(t) - c(t) g^{00} \right. + \frac{m_4(t)}{2} (\delta g^{00})^2 - \frac{m_3(t)}{2} \delta K \delta g^{00} - m_2(t) \left( \delta K^2 - \delta K_{\mu\nu} \delta K^{\mu\nu} \right) + \frac{\bar{m}_2(t)}{2} R(3) \delta g^{00} \right.$$

$$- \frac{\bar{m}_5(t)}{2} \delta K + \frac{\bar{m}_3(t)}{2} R(3) \delta K + \frac{\bar{\lambda}(t)}{2} (R(3))^2 + ... - \frac{\bar{\lambda}(t)}{M_p^2} \nabla_i R(3) \nabla^i R(3) + ... \right],$$

The first line is background and the rest are for the perturbations up to second order. In the action, we define $\delta K_{\mu\nu} = K_{\mu\nu} - H \Theta_{\mu\nu}$, $\delta K = K - 3H$, where the induced metric $\Theta_{\mu\nu} \equiv g_{\mu\nu} + n_\mu n_\nu$, and the normal vector is defined as $n_\mu \equiv (-N, 0, 0, 0)$. Moreover, since the third and the fourth lines are for higher space (but not time) derivatives, in the following analysis we turned them off by setting $\bar{m}_4 = \bar{m}_5 = \bar{\lambda} = \bar{\lambda} = 0$. 


It is straightforward to get the background equations from action (2), by varying the first line with respect to the lapse function \( N \) and the scale factor \( a \):

\[
3M_p^2[f(t)H^2 + \dot{f}(t)H] = c(t) + \Lambda(t),
\]

\[
-2M_p^2f\dot{H} + 3f(t)H^2 + 2\dot{f}(t)H + \dot{f}(t)] = c(t) - \Lambda(t).
\]

These are actually nothing but the Friedmann equations, and for the minimal coupling theories where \( f(t) = 1 \), one can have \( c(t) = -M_p^2H \) and \( \Lambda(t) = M_p^2(H + 3H^2) \), which are the same as the EFT of inflation firstly proposed in [29]. However, In the nontrivial case where \( f(t) \) is an arbitrary function, the theory is extended to include also cases where gravity part is modified, or there is nonminimally coupling between the field and gravity parts. From Eq.s (3) and (4) one can get:

\[
H(t) = \frac{-\dot{f}}{2f} \pm \frac{3}{12} \left( \frac{\dot{f}}{f} \right)^2 + \frac{4(c + \Lambda)}{M_p^2f},
\]

\[
\dot{H}(t) = \frac{-c}{M_p^2f} - \frac{\dot{f}}{2f} - \left( \frac{\dot{f}}{2f} \right)^2 \pm \frac{3\sqrt{3}}{12} \left[ \frac{\dot{f}^2}{f^2} \right] + \frac{4(c + \Lambda)}{M_p^2f}.
\]

Defining \( \delta_f^{(1)} \equiv \dot{f}/Hf, \delta_f^{(2)} \equiv \ddot{f}/H\dot{f} \), one can also get a neat form of the Hubble parameter squared from Eq. (5) as:

\[
H^2 = \frac{c + \Lambda}{3M_p^2f^2} + \frac{1}{2}H^2(\delta_f^{(1)})^2 \pm \frac{3\sqrt{3}}{6}H^2 \left[ 3(\delta_f^{(1)})^4 + 4(\delta_f^{(1)})^2 \left( \frac{c + \Lambda}{H^2M_p^2f} \right) \right],
\]

which gives the solution

\[
\frac{c + \Lambda}{M_p^2H^2f} = 3(1 + \delta_f^{(1)}).
\]

Moreover, the slow-roll parameter can be written as:

\[
\epsilon \equiv -\frac{\ddot{H}}{H^2} = \frac{3c}{c + \Lambda}(1 + \delta_f^{(1)}) + \frac{1}{2}\delta_f^{(1)}\delta_f^{(2)} - 1.
\]

which will be frequently used in the following analysis.

### III. THE EFT FRAMEWORK: PERTURBATIONS

#### A. scalar perturbation

Using the action (2) and taking the unitary gauge, we find the quadratic action of the scalar perturbation [35]:

\[
S^{(2)}_\zeta = \int d^4x t^3 \left[ c_1\zeta^2 - \left( \frac{c_3}{a} - c_2 \right) \left( \partial\zeta \right)^2 \right],
\]

where \( \zeta \) is the curvature perturbation, and

\[
c_1 = \frac{1}{D} \left( 2m_\zeta^2 + fM_p^2 \right) \left[ 3m_\zeta^2 + 4f^2H^2cM_p^4 + 16m_\zeta^2m_H^2 + M_p^2 \left[ -4f^2 + \dot{f} \left(-6m_\zeta^2 + 4Hm_H^2 + 3fM_p^2 \right) \right] + 2fM_p^2 \left[ 4m_\zeta^2 - \dot{f}M_p^2 + H \left( 4H\zeta H + \dot{f}M_p^2 \right) \right] \right],
\]

\[
c_2 = fM_p^2,
\]

\[
c_3 = \frac{2a}{D} \left( 2m_\zeta^2 + fM_p^2 \right) \left[ 2f^2HM_p^4 + fM_p^2 \left[ -m_\zeta^2 + \dot{f}M_p^2 + 4Hm_\zeta^2 \right] \right],
\]

\[
D = \left[ m_\zeta^2 - 4Hm_H^2 - \left( 2fH + \dot{f} \right) M_p^2 \right]^2 .
\]
According to action (10), one can get the equation of motion:
\[ u'' + c_s^2 k^2 u - \frac{\zeta''}{\zeta} u = 0 , \]
where \( u \equiv z \zeta, \ z \equiv a\sqrt{c_1} \), and prime denotes derivative with respect to the conformal time: \( \eta \equiv \int a^{-1} dt \). The sound speed squared is also defined as:
\[ c_s^2 \equiv \left( \frac{\dot{c}_3}{a} - c_2 \right) / c_1 . \]

For initial condition, we consider the case of the subhorizon region \( c_s^2 k^2 \gg z''/z \), and we assume that the adiabatic condition \( |\omega'/\omega|^2 \ll 1 \) is satisfied, where \( \omega^2 \equiv c_s^2 k^2 - z''/z \), which is true for wide range of parameter choice. Therefore, one can apply the WKB approximation to get:
\[ u_{\text{init}} = \frac{1}{\sqrt{2c_s k}} e^{i \int c_s k d\eta} . \]

On the other hand, for a whole solution, assuming \( z \propto \eta^{4-\nu} \), where the parameter \( \nu \) is assumed to be a constant. Then Eq. (15) becomes
\[ u'' + c_s^2 k^2 u - \frac{4\nu^2 - 1}{4\nu^2} u = 0 , \]
and the solution is the famous Hankel function:
\[ u = C \sqrt{\eta} \left[ H^{(1)}_{\nu} \left( \int c_s k d\eta \right) + H^{(1)}_{-\nu} \left( \int c_s k d\eta \right) \right] , \]
with \( C = \sqrt{\pi}/2 \) comparing to the initial condition (17). In the superhorizon region \( c_s^2 k^2 \ll z''/z \), one has \( H_{\nu}(\int c_s k d\eta) = \sqrt{2/\pi}(\int c_s k d\eta)^{-\nu} \), therefore we have
\[ u = \frac{\eta}{z} \left[ \left( \int c_s k d\eta \right)^{-\nu} + \left( \int c_s k d\eta \right)^{\nu} \right] , \]
\[ \zeta = \frac{u}{z} \propto \frac{1}{\sqrt{2}} \eta^\nu \left( \int c_s k d\eta \right)^{-\nu} \left[ 1 + \left( \int c_s k d\eta \right)^{2\nu} \right] , \]
and the power spectrum is
\[ P_\zeta = \frac{k^3}{2\pi^2} \left| \frac{u}{z} \right|^2 = \frac{k^3}{2\pi^2} \frac{\eta}{2a^2 c_1} \left[ \left( \int c_s k d\eta \right)^{-\nu} + \left( \int c_s k d\eta \right)^{\nu} \right]^2 . \]

We assume slow-varying variable \( \epsilon \equiv -\dot{H}/H^2 \), therefore it is easy to get \( a \sim \eta^{1/(\epsilon - 1)} \), and also \( (aH)^{-1} = (\epsilon - 1)\eta \).

Moreover, for simplicity but without losing generality, we assume the sound speed squared \( c_s = c_s(\eta/\eta_*)^s \) where \( * \) denotes some normalization scale, therefore
\[ P_\zeta = \frac{(\epsilon - 1)^2 H_*^2}{4\pi^2 c_1 s c_s^3} \left( \frac{\eta}{\eta_*} \right)^{-3+2\nu-3s} (c_s k \eta)^{3-2\nu}(s+1)^{2\nu} \left[ 1 + \left( \frac{c_s k \eta}{s+1} \right)^{2\nu} \right]^2 \]
(22)

The current observations indicated that the power spectrum of the scalar perturbation (22) should be (nearly) scale-invariant. In order to be so, one can either have \( \nu \simeq 3/2 \), with \( (c_s k \eta)^{2\nu} \) decreasing:
\[ P_\zeta = \frac{(\epsilon - 1)^2 H_*^2}{4\pi^2 c_1 s c_s^3} \left( \frac{\eta}{\eta_*} \right)^{-3s} (s+1)^{2\nu} , \]
(23)
which is also time-invariant for constant \( c_s (s = 0) \), or have \( \nu \simeq -3/2 \), with \( (c_s k \eta)^{2\nu} \) increasing:
\[ P_\zeta = \frac{(\epsilon - 1)^2 H_*^2}{4\pi^2 c_1 s c_s^3} \left( \frac{\eta}{\eta_*} \right)^{-6-3s} (s+1)^{-2\nu} , \]
(24)
which will be proportional to \( \eta^{-6} \) for constant \( c_s \). Since \( |\eta| \) will be decreasing all the time, both the two cases require \( s + 1 > 0 \).
B. tensor perturbation

We can perform the same procedure to get the power spectrum for tensor perturbations. According to action (2), one can also obtain the quadratic action of the tensor perturbation [35]:

\[
S_T^{(2)} = \frac{M_p^2}{8} \int d^4 x a^3 D_T \left[ \dot{\gamma}_{ij} - c^2_T \left( \partial_k \gamma_{ij} \right)^2 \right],
\]

(25)

where \( \gamma_{ij} \) is the tensor perturbation, and

\[
D_T = f + 2 \frac{m^2}{M_p^2}, \quad c_T^2 = f \frac{D_T}{c_T}.
\]

(26)

One can also get the equation of motion:

\[
v'' + c_T^2 k^2 v - \frac{z''}{z_T} = 0,
\]

(27)

where \( v \equiv z_T \gamma_{+, \times} \) where \( \gamma_{+, \times} \) are two polarization modes of \( \gamma_{ij} \), \( z_T^2 \equiv a^2 D_T \). Following the same procedure as of the scalar perturbation, and assuming \( z_T \propto \eta^{2-\nu_T} \), one gets the solution:

\[
v = \sqrt{\frac{\eta}{2}} \left[ \left( \int c_T k d\eta \right)^{-\nu_T} + \left( \int c_T k d\eta \right)^{\nu_T} \right],
\]

\[
\gamma = \frac{v}{z_T} \propto \frac{1}{\sqrt{2}} \eta^{\nu_T} \left[ \left( \int c_T k d\eta \right)^{-\nu_T} \left[ 1 + \left( \int c_T k d\eta \right)^{2\nu_T} \right] \right],
\]

(28)

and the power spectrum is:

\[
P_T = \frac{2 k^3}{2\pi^2} \frac{|v|}{z_T^2}
\]

\[
= \frac{(\epsilon - 1)^2 H^2}{2\pi^2 D_T c_T^2} (c_T k \eta)^{3-2\nu_T}(s_T + 1)^{2\nu_T} \left[ 1 + \left( \frac{c_T k \eta}{s_T + 1} \right)^{2\nu_T} \right]^2,
\]

(29)

where we assume \( c_T = c_T_*(\eta/\eta_*)^{\nu_T} \).

The current observations have not provided constraint on the scale variance of primordial tensor power spectrum yet. However, in this work, we restrict ourselves on the case where the tensor spectrum is also scale-invariant, as is for the scalar one. In order to be so, one can either have \( \nu_T \simeq 3/2 \), with \((c_T k \eta)^{2\nu_T}\) decreasing:

\[
P_T = \frac{(\epsilon - 1)^2 H^2}{2\pi^2 D_T c_T^2} \left( \frac{\eta}{\eta_*} \right)^{-3s_T} (s_T + 1)^3
\]

(30)

which is also time-invariant for constant \( c_T \) (\( s_T = 0 \)), or have \( \nu_T \simeq -3/2 \), with \((c_T k \eta)^{2\nu_T}\) increasing:

\[
P_T = \frac{(\epsilon - 1)^2 H^2}{2\pi^2 D_T c_T^2} \left( \frac{\eta}{\eta_*} \right)^{-6-3s_T} (s_T + 1)^{-3}
\]

(31)

which will be proportional to \( \eta^{-6} \) for constant \( c_T \). It is also required that \( s_T + 1 > 0 \). Moreover, it is easy to see that, for \( \nu_T < -3/2 \) or \( \nu_T > 3/2 \), the spectrum would have a red tilt, while for \(-3/2 < \nu_T < 3/2 \), the spectrum would have a blue tilt.

C. tensor/scalar ratio

The tensor/scalar ratio is defined as:

\[
r \equiv \frac{P_T}{P_\zeta},
\]

(32)
where in general, $P_T$ and $P_s$ is given in Eqs. (22) and (29). Hereafter, for simplicity, we stick ourselves only on the cases where tensor spectrum is also scale-invariant. According to the above analysis, one can immediately get the tensor/scalar ratio:

$$r = 2 \frac{D_s c_1^2 (s_T + 1)^3}{D_T c_T^4 (s + 1)^3} .$$

(33)

Moreover, since

$$D_s = c_1, \ c_s = \sqrt{\frac{c_3 / a - c_2}{\sqrt{D_s}}} , \ D_T = \frac{M_p^2}{8} \left[ f + 2 \frac{m_4^2}{M_p^2} \right] , \ c_T = \frac{M_p \sqrt{f}}{\sqrt{8D_T}} ,$$

(34)

one can express $r$ in the EFT language, namely

$$r = 16 \frac{(c_3 / a - c_2)^{3/2} \sqrt{f + 2(m_4 / M_p)^2 (s_T + 1)^3}}{\sqrt{c_1 f^{3/2}} M_p^2 (s + 1)^3} .$$

(35)

Note that the above is our master formula on $r$, which contain EFT functions only. Although not so obvious at this stage, when applied to analysis concrete models, it can provide the convenience of being directly related to the (below-defined) slow-varying parameters of each model through the power of EFT approach, without any need to calculate $r$ model by model.

Before heading to the next section, let us make further comments on Eq. (35): although the relation between $r$ and various parameters seems obscure, it is clear with two parameters, $s_T$ and $s$, which represents the time-dependence of the sound speed with respect to tensor and scalar perturbations, by the previous definition. Marginalizing the effects of other parameters, it is evident that large running of $c_T$ will make $r$ large, while that of $c_s$ will do the opposite. Due to the above, in the following, we will ignore these two parameters for simplicity, by assuming that both $c_T$ and $c_s$ are slow-varying. This is a very common-used assumption in the analysis of inflation models.

IV. CONFRONTING $r$ WITH FUTURE CONSTRAINTS: CONCRETE EXAMPLES

As has been mentioned in the introduction, the detection/non-detection of AliCPT will put $r$ into the ranges of $r \in (0.064, 0.01)/r < 0.01$ separately, while in this section, we will try to discuss how $r$ is related to usually defined slow-roll parameters of concrete models, and what the parameters will be like for $r$ to be within these regions. The models can of course be arbitrary; however, in order to be precise, we choose the models to be consist of the well-known Galileon Lagrangians, namely [37]

$$\mathcal{L}_2 = G_2(\phi, X) ,$$

(36)

$$\mathcal{L}_3 = G_3(\phi, X) \Box \phi ,$$

(37)

$$\mathcal{L}_4 = G_4(\phi, X) R - 2G_{4X}[(\Box \phi)^2 - \phi^{\mu \nu} \phi_{\mu \nu}] ,$$

(38)

$$\mathcal{L}_5 = G_5(\phi, X) G_{\mu \nu \mu} \phi^{\mu \nu} + \frac{1}{3} G_{5X}(\phi, X)[(\Box \phi)^3 - 3 \Box \phi \phi_{\mu \nu} \phi^{\mu \nu} + 2 \phi_{\mu \nu} \phi^{\mu \sigma} \phi^{\sigma \nu}] .$$

(39)

Moreover, according to [35, 36], our EFT action (2) also contains an additional case that has not been included in the Generalized Galileon (a.k.a. Horndeski) Lagrangians, namely:

$$\mathcal{L}_6 = m_4^2 R^{(3)} \delta g^{00} .$$

(40)

In the following, we will choose models which contains two or three of those above Lagrangians, as simple examples.

A. K-essence: $M_p^2 R/2 + \mathcal{L}_2$

We first start with the simplest case of Kessence single field [38]. This case can be written in terms of EFT action (2) with the correspondence:

$$f = 1 , \ \Lambda(2) = \frac{\dot{\phi}^2}{2} G_{2X}(\phi, X) - G_2(\phi, X) , \ c_2 = \frac{\dot{\phi}^2}{2} G_{2X}(\phi, X) , \ m_2 = \frac{\dot{\phi}^4}{4} G_{2XX}(\phi, X) , \ m_3 = m_4 = m_5^2 = 0 .$$

(41)
and according to Eq. (9), the slow-roll parameters can be reduced to

$$\epsilon(2) = \frac{3c_{(2)}(2) H^2 M_p^2}{c_{(2)}(1) + \Lambda_{(2)}} = \frac{3\dot{\phi}^2 G_{2X}}{2(\dot{\phi}^2 G_{2X} - G_2^2)}.$$  

(42)

For canonical scalar field with $G_2(X, \phi) = X - V(\phi)$, the above definition of $\epsilon(2)$ can be connected with the ratio between kinetic and potential terms $\gamma \equiv K/V$: $\epsilon(2) = \gamma/(1 + \gamma)$. Note that it is also coincide with another commonly used definition: $\dot{\phi}^2 G_{2X}/H^2$.

From Eqs. (11)-(14), we have

$$D = 4H^2 M_p^2, \quad c_1 = M_p^2 \left( \epsilon(2) + \frac{2m_2^4}{H^2 M_p^2} \right), \quad c_2 = M_p^2, \quad c_3 = \frac{aH}{M_p^2}, \quad c_5 = \frac{H^2 \epsilon(2) M_p^2}{H^2 \epsilon(2) M_p^2 + 2m_2^4},$$

$$D_T = \frac{M_p^2}{8}, \quad c_T^2 = 1.$$  

(43)

so using Eq. (35), one has

$$r = \frac{16\sqrt{2}HM_p^3 \epsilon(2)}{\sqrt{2}H^2 \epsilon(2) M_p^2 + \dot{\phi}^4 G_{2XX}} = \frac{16\epsilon(2)^{3/2}}{\sqrt{\epsilon(2)^{1} + 2\delta_{KXX}}},$$

where we define the parameter $\delta_{KXX} \equiv \dot{\phi}^4 G_{2XX} / (4H^2 f M_p^2)$, with $f$ being unity in the current case.

One can see that for all the parameter choices of $\epsilon(2)$ and $\delta_{KXX}$, the result recovers the standard consistency relation: $r = 16\epsilon c_s$. Moreover, for canonical scalar field $G_{2XX} = 0$, one has $\delta_{KXX} = 0$ and $c_s = 1$, which is so simple that for given potential form, $\epsilon$ and $r$ can be directly related to parameters such as the power-law index of the potential, and the number of e-foldings [39]. Therefore, in the canonical case, the only way of getting small $r$ is to have small $\epsilon$ by requiring the potential to be very flat. Two specific examples are obvious in the literature: the small-field inflation with $\epsilon \sim constant$ (hilltop inflation [40] for example) and the ultra-slow-roll inflation with $\epsilon \sim \eta^{-6}$, which is proposed by [27] (with further studies in e.g., [41]) and extended to constant-roll inflation [28].

For noncanonical scalar field, the sound speed $c_s$ also plays its role. For $G_{2XX} \sim M_2^4 < 0$ which causes $c_s < 1$, $r$ can be suppressed by $c_s$. Examples include ghost condensate inflation [42] as well as DBI inflation [43]. However, as for the single-field inflation, the non-Gaussianities is also related to $c_s$ in terms of $f_{nl} \sim c_s^{-2}$ roughly [44] where $f_{nl}$ describes the amplitude of non-Gaussianities. These models will also meet the danger of making $f_{nl}$ too large to be consistent with the current constraints [45].

The plot of $r$ in parameter space [$\epsilon(2), \delta_{KXX}$] is shown in Fig. 1, where light blue region denotes $0.01 < r < 0.064$, and blue region denotes $r < 0.01$. Note that when $\delta_{KXX} = 0$, according to the consistency relation $r = 16\epsilon$, $r < 0.064$ and $r < 0.001$ take $\epsilon$ into the very narrow region of $\epsilon < 0.004$ and $\epsilon < 0.000625$, respectively. However, when taking $\delta_{KXX}$ into consideration, the allowed region of $\epsilon(2)$ will get much enlarged and larger $\epsilon(2)$ will also be allowed. Moreover, considering the constraints of stability, namely $c_1 > 0$, $c_5^2 > 0$, we must restrict $\epsilon(2)$ and $\delta_{KXX}$ to be within the region of $\epsilon(2) > 0$ and $\epsilon(2) + 2\delta_{KXX} > 0$.

For more complicated case, however, not only the consistency relation $r = 16\epsilon c_s$ will be violated, $\epsilon$ could also be affected and be deviated from $\epsilon(2)$, due to the alterations of both background energy density and pressure, either of which will play a role of suppressing $r$. In order to illustrate, in the following, we write

$$\Lambda = \Lambda(2) + \Delta \Lambda, \quad c = c(2) + \Delta c,$$  

where $\Delta \Lambda$ and $\Delta c$ denotes the derivation of $\Lambda$ and $c$ from those in the case of Kessence. Thus we have

$$\epsilon = \epsilon(2) \left[ 1 + \delta_f^{(1)} - \frac{1}{3} \frac{\Delta c}{M_p^2 H^2} - \frac{1}{3} \frac{\Delta \Lambda}{M_p^2 H^2} \right] + \frac{\Delta c}{M_p^2 H^2} + \frac{1}{2} \delta_f^{(1)} \delta_f^{(2)} - 1,$$  

(46)

so the effects on $\epsilon$ can be corresponded to several parameters such as $\delta_f^{(1)}, \delta_f^{(2)}, \frac{\Delta c}{M_p^2 H^2}$ and $\frac{\Delta \Lambda}{M_p^2 H^2}$.

**B. Galileon: $M_p^2 R/2 + L_2 + L_3$**

The next case to consider is the Galileon case, the proposal of which is inspired by the ghost problems that appeared in DGP models [46]. Although the original proposal introduced the Galileon symmetry, it was later extended to general
case that include also dependence of the field itself, with the addition of the term $G_3(\phi, X)\Box\phi$ [47]. When $G_3$ contains $\phi$ only, this term coincides with the kinetic term only by moduli a total derivative. Here for simplicity we take $G_3$ to be of the form $G_3 = g(\phi)X$, therefore can be written in terms of EFT action (2) with the correspondence:

$$f = 1, \quad \Lambda = \frac{\dot{\phi}^2}{2}G_{2X}(\phi, X) - G_2(\phi, X) - \frac{\dot{\phi}^2}{2}(\ddot{\phi} + 3\dot{\phi})g(\phi), \quad c = \frac{\dot{\phi}^2}{2}G_{2X}(\phi, X) + \frac{\ddot{\phi}^2}{2}(\ddot{\phi} - 3\dot{H})g(\phi) + \frac{\ddot{\phi}^4}{2}g_\phi(\phi),$$

$$m_2^4 = \frac{\dot{\phi}^4}{4}G_{2XX}(\phi, X) - \frac{\dot{\phi}^2}{4}(\ddot{\phi} + 3\dot{H})g(\phi) + \frac{\ddot{\phi}^4}{4}g_\phi(\phi), \quad m_3^3 = -\dot{\phi}^3g(\phi), \quad m_4^2 = \tilde{m}_4^2 = 0,$$

then from Eqs. (11)-(14), we have

$$D = (2HM_p^2 - m_3^3)^2, \quad c_1 = \frac{M_p^2}{(2HM_p^2 - m_3^3)^2}(3m_6^6 + 4H^2\epsilon M_p^4 + 8m_2^4M_p^2), \quad c_2 = M_p^2,$$

$$c_3 = \frac{2aM_p^4}{2HM_p^2 - m_3^3}, \quad c_3^2 = \frac{4H^2\epsilon M_p^4 + 2HM_p^2m_3^3 - m_6^6 + 2(m_3^3)M_p^2}{4H^2\epsilon M_p^4 + 8m_2^4M_p^2 + 3m_6^6},$$

$$D_T = \frac{M_p^2}{8}, \quad c_T^2 = 1.$$

so using Eq. (35), one has

$$r = \frac{16}{(2HM_p^2 - m_3^3)^2}\left[\frac{4H^2\epsilon M_p^4 + 2HM_p^2m_3^3 - m_6^6 + 2(m_3^3)M_p^2}{4H^2\epsilon M_p^4 + 8m_2^4M_p^2 + 3m_6^6}\right]^{3/2}. \quad (49)$$

Moreover, in order to take into account of the variation of $\epsilon$, we note that

$$\Delta c = \frac{g(\phi)\dot{\phi}^3}{2M_p^2H^2}\left(\frac{\ddot{\phi}}{H^2} - 3\right) + \frac{g(\phi)\dot{\phi}^4}{2M_p^2H^2},$$

$$\Delta \Lambda = \frac{g(\phi)\dot{\phi}^3}{2M_p^2H^2}\left(\frac{\ddot{\phi}}{H^2} + 3\right),$$

$$\delta_f^{(1)} = \delta_f^{(2)} = 0.$$
It is useful to define the “slow-varying” parameters as $\delta_{\phi X} \equiv \dot{\phi}^3 g(\phi)/(H M^2_p)$, $\delta_{g\phi} \equiv g_\phi(\phi) \dot{\phi}^3/(H^2 M^2_p)$, and $\delta_\phi \equiv \dot{\phi}/H \dot{\phi}$. Therefore the above formula will become
\[
\frac{\Delta^2 \phi}{M^2_{\text{Pl}}} = \frac{1}{2} \delta_{\phi X} (\delta_{g\phi} - 3) + \frac{1}{2} \delta_{g\phi}, \quad \frac{\Delta^2 \phi}{M^2_{\text{Pl}}} = \frac{1}{2} \delta_{\phi X} (\delta_{g\phi} + 3),
\]
thus
\[
\epsilon = \epsilon_2 (1 + \delta_{\phi X} - \frac{1}{6} \delta_{g\phi} + \frac{1}{2} \delta_{\phi X} (\delta_{g\phi} - 3) + \frac{1}{2} \delta_{g\phi}).
\]
Moreover, we have $m^2_3 = -\dot{\phi}^3 g(\phi) = -H M^2_p \delta_{\phi X}$, $(m^3_3) = -g_\phi(\phi) \dot{\phi}^3 - 3g(\phi) \dot{\phi}^2 \ddot{\phi} = -H^2 M^2_p \delta_{g\phi} - 3H^2 M^2_p \delta_{\phi X} \delta_\phi$, and $m^2_4 = H^2 M^2_p \delta_{KXX} - \frac{1}{4} H^2 M^2_p \delta_{g\phi}(\delta_\phi + 3) + \frac{1}{4} H^2 M^2_p \delta_{\phi X}$, where $\delta_{KXX}$ has already been defined previously. In this regard, we get
\[
r = \frac{16}{(2 + \delta_{\phi X})^2} \left[ \frac{4\epsilon_2 + 4\epsilon_2 \delta_{\phi X} - 2\epsilon_2 \delta_{g\phi} (\delta_\phi - 3)}{-8 \delta_{g\phi} + 4 \delta_{g\phi} (\delta_\phi - 3)} \right]^{3/2},
\]
Since one can notice that the dependence of $r$ on $\delta_{g\phi}$ is obvious and is the same as in the Kessence case, we will also ignore it in the following discussions. We now consider a simple case in which $g_\phi(\phi) \approx 0$ ($g \simeq \text{const}$), namely $\delta_{g\phi} \approx 0$, therefore $\epsilon$ and $r$ will become
\[
\epsilon = \epsilon_2 (1 + \delta_{\phi X} - \frac{1}{2} \delta_{\phi X} (\delta_\phi - 3)),
\]
\[
r = \frac{16}{(2 + \delta_{\phi X})^2 (s + 1)^3} \left[ \frac{4\epsilon_2 + 4\epsilon_2 \delta_{\phi X} - 8 \delta_{g\phi} (\delta_\phi - 3)}{-12 \delta_{g\phi} (\delta_\phi - 3)} \right]^{3/2},
\]
Furtherly consider all the parameters are smaller than 1, then the terms of parameters multiplied or squared can be viewed as higher order infinitesimals. Therefore $r$ could be greatly simplified as:
\[
r \simeq 16 \frac{(\epsilon_2 - 2 \delta_{\phi X})^{3/2}}{(\epsilon_2 - 3 \delta_{\phi X})^{3/2}},
\]
For usual case of $\epsilon_2 > 0$, whether $\delta_{g\phi}$ is positive or negative, the inclusion of $\delta_{g\phi}$ will cause the raise of $r$, so we tend to get large $r$ as in G-inflation model [48]. However, for $\epsilon_2 < 0$, which makes phantom-like inflation possible, there is certain room for small $r$ where $\delta_{\phi X}$ is also negative. Note also that due to the requirement of stabilities ($c_1 > 0$, $c_2^2 > 0$), the case of positive $\delta_{g\phi}$ for $\epsilon_2 < 0$ is forbidden.
The plot of $r$ in parameter space $[\epsilon_2, \delta_{\phi X}, \delta_{g\phi}, \delta_\phi]$ is shown in Fig. 2, where light blue region denotes $0.01 < r < 0.064$, and blue region denotes $r < 0.01$. According to the plot, several remarks are as follows: 1) In the first plot, the allowed region extend to where $\epsilon_2 < 0$, making possible for phantom inflation; 2) For other cases, there are also large space for $r$ to be within $(0.064, 0.1)$; 3) Note that for $\delta_{\phi X} = 0$, $r$ can be small for $\delta_{g\phi} \simeq 0$, which is because there is a pole in the numerator of $r$ in Eq. (54). However it seems difficult to make consistency of both $\delta_{\phi X} = 0$ and nonvanishing $\delta_{g\phi}$; 4) From the rightmost column one can see that for $\delta_{\phi X} = 0$, the dependence of $\delta_\phi$ is also decoupled. In the allowed region for $\epsilon_2$ and $\delta_{g\phi}$, the room for small $r$ is also quite large.

The ratio between $r$ and $16 \epsilon_2 s$ can also be straightforwardly calculated as:
\[
\frac{r}{16 \epsilon_2 s} = \frac{[4\epsilon_2 + 4\epsilon_2 \delta_{\phi X} - 2\epsilon_2 (4\epsilon_2 - 3 \delta_{g\phi} (\delta_\phi - 3))]}{(2 + \delta_{\phi X})^2 [\epsilon_2 (1 + \delta_{\phi X} - \frac{1}{2} \delta_{\phi X} (\delta_\phi - 3)) + \frac{1}{2} \delta_{\phi X} (\delta_\phi - 3)]},
\]
which depends on all the parameters, and we also plot this comparison in Fig. 3 (in order to avoid the possible pole in the dominator, we instead plot $r - 16 \epsilon_2 s$, and compare it with 0.), with blue color for $r < 16 \epsilon_2 s$, while yellow color for $r > 16 \epsilon_2 s$. Therefore for Galileon field, the consistency relationship is usually broken.

C. Modified gravity: $L_2 + L_4$

Here we consider the case where $G_4$ terms involves in, which includes nonminimally coupling case as well as $f(R)$ modified gravity case. For simplicity, we take $G_4$ to be purely function of $\phi$, namely $G_4 = M^2_{\text{Pl}} A(\phi)/2$ where $A(\phi)$ is an arbitrary function. Therefore this case can be written in terms of EFT action (2) with the correspondence:
\[
f = A(\phi), \quad \Lambda = \frac{\dot{\phi}^2}{2} G_{2X}(\phi, X) - G_2(\phi, X) , \quad c = \frac{\dot{\phi}^2}{2} G_{2X}(\phi, X), \quad m^4 = \frac{\dot{\phi}^4}{4} G_{2X}(\phi, X), \quad m^4 = m^2 = m^2 = 0.
\]
FIG. 2: The plot of $r$ in parameter space $[\epsilon(2), \delta_{\phi}]$ (setting $\delta_{\phi X} = 0, \delta_\phi = 0$), $[\epsilon(2), \delta_{\phi X}]$ (setting $\delta_{\phi X} = 0, \delta_{\phi} = 0$), $[\epsilon(2), \delta_{\phi}]$ (setting $\delta_{\phi X} = 0, \delta_{\phi} = 0$), $[\delta_{gX}, \delta_{g\phi}]$ (setting $\epsilon(2) = 0.004, \delta_{\phi} = 0$), $[\delta_{gX}, \delta_{g\phi}]$ (setting $\epsilon(2) = 0.004, \delta_{\phi} = 0$), $[\delta_{g\phi}, \delta_{\phi}]$ (setting $\epsilon(2) = 0.004, \delta_{gX} = 0$). Light blue region denotes $0.01 < r < 0.064$, and blue region denotes $r < 0.01$.

From Eqs. (11)-(14), we have

$$D = (2AH + \dot{A})^2 M_p^4, \quad c_1 = \frac{1}{(2AH + A)^2} A(4A^2 H^2 \epsilon M_p^2 + 3\dot{A}^2 M_p^2 + 8AM_p^4 - 2A\ddot{A}M_p^2 + 2A\dot{A}H M_p^2) ,$$

$$c_2 = Am_p^2, \quad c_3 = \frac{2A^2 M_p^2}{2AH + A}, \quad c_s^2 = \frac{2A\dot{A}H + 3\dot{A}^2 + 4A^2 H^2 \epsilon - 2A\ddot{A}}{2A\dot{A}H + 3\dot{A}^2 + 4A^2 H^2 \epsilon - 2A\ddot{A} + 8Am_p^4},$$

$$D_T = \frac{M_p^2 A(\phi)}{8}, \quad c_T = 1 \ .$$

(60)

Defining $\delta_A^{(1)} \equiv \dot{A}/HA, \delta_A^{(2)} \equiv \ddot{A}/H\dot{A}$, and using Eq. (35), we get

$$r = \frac{16}{(2 + \delta_A^{(1)})^2 (2A^2 + 3\delta_A^{(1)} + 2\epsilon - 2\delta_A^{(2)}) 3/2} \left( \frac{2\delta_A^{(1)} + 3\delta_A^{(2)} + 4\epsilon - 2\delta_A^{(2)} 3/2}{2A^2 + 3\delta_A^{(1)} + 2\epsilon - 2\delta_A^{(2)} 3/2 + 8\delta_{KXX}} \right)^{1/2} \ .$$

(61)

Moreover, in this case we have $\frac{\Delta r}{M_p^2 H^2} = 0$, $\frac{\Delta A}{M_p^2 H^2} = 0$, $\delta_f^{(1)} = \delta_A^{(1)}$, $\delta_f^{(2)} = \delta_A^{(2)}$, therefore

$$\epsilon = \epsilon(2)(1 + \delta_A^{(1)}) + \frac{1}{2}\delta_A^{(1)}(\delta_A^{(2)} - 1) \ .$$

(62)
FIG. 3: The plot of \( r - 16 \epsilon c_s \) in parameter space \([\epsilon(2), \delta_{\phi X}]\) (setting \( \delta_{\phi X} = 0, \delta_{\phi} = 0 \)), \([\epsilon(2), \delta_{\phi}]\) (setting \( \delta_{\phi X} = 0, \delta_{\phi} = 0 \)), \([\epsilon(2), \delta_{\phi}]\) (setting \( \delta_{\phi X} = 0, \delta_{\phi} = 0 \)), \([\epsilon(2), \delta_{\phi}]\) (setting \( \epsilon(2) = 0.004, \delta_{\phi} = 0 \)), \([\delta_{\phi X}, \delta_{\phi}]\) (setting \( \epsilon(2) = 0.004, \delta_{\phi} = 0 \)), \([\delta_{\phi X}, \delta_{\phi}]\) (setting \( \epsilon(2) = 0.004, \delta_{\phi} = 0 \)). Blue region denotes \( r - 16 \epsilon c_s < 0 \), and yellow region denotes \( r - 16 \epsilon c_s > 0 \).

so

\[
\begin{align*}
\frac{r}{c_s^2} &= \frac{16}{(2 + \delta_A^{(1)})^2} \frac{[3\delta_A^{(1)}]^2 + 4\epsilon(2)(1 + \delta_A^{(1)})]^{3/2}}{[3\delta_A^{(1)}]^2 + 4\epsilon(2)(1 + \delta_A^{(1)}) + 8\delta_{KXX}]^{1/2},}
\end{align*}
\]

(63)

For the same reason as above case, we ignore the effects of \( \delta_{KXX} \). In this case, \( c_s^2 \) will exactly be unity, therefore one could get a very neat form of \( r \):

\[
\begin{align*}
\frac{r}{c_s^2} &\approx \frac{16(3\delta_A^{(1)})^2 + 4\epsilon(2) + 4\epsilon(2)\delta_A^{(1)}}{(2 + \delta_A^{(1)})^2}.
\end{align*}
\]

(64)

which only involves two parameters. Since now \( c_s = 1 \), one can see that \( r \) has obviously deviated from the consistency relation: \( r = 16 \epsilon c_s \). Moreover, we also find that \( c_T = 1 \), namely even gravity is modified in the current form, the sound speed of tensor perturbation is unaltered. This indicates that \( c_T \) can only deviate from unity in a more complicated form of modified gravity, e.g. when there is a kinetic coupling to the gravity, as will be demonstrated in the next case. The interest in the deviation of \( c_T \) from unity is spurred by the constraints imposed by the latest GW event from a binary neutron star merger, namely GW170817, and its electromagnetic counterpart GRB170817A, see [50].

Let’s now turn to another interesting case, namely where \( G_2(\phi, X) = G_2(\phi) \) is a pure function of \( \phi \). This, by
conformal transformation, is nothing but $f(R)$ gravity \[51\]. In this case we have $\epsilon = \frac{1}{2}\delta_A^{(1)} - \frac{1}{2}\delta_A^{(1)}$, therefore

$$r = \frac{16(2\delta_A^{(1)} + 3\delta_A^{(1)2} + 4\epsilon - 2\delta_A^{(1)}\delta_A^{(2)})}{(2 + \delta_A^{(1)})^2}$$

$$= \frac{48\delta_A^{(1)2}}{(2 + \delta_A^{(1)})^2}.$$ \hspace{1cm} (65)

namely the number of parameters involved again get reduced, up to only one. Moreover, since $r$ is proportional to $\delta_A^{(1)2}$, it is easier to get small $r$, as long as $\delta_A^{(1)}$ is not too large. For example, for $r < 0.064(0.01)$, one needs $-0.070(-0.028) < \delta_A^{(1)} < 0.076(0.029)$. A concrete example is the famous inflation model proposed by Starobinsky \[26\].

![FIG. 4: The plot of $r$ in parameter space $\epsilon(2), \delta_A^{(1)}$ for nonminimal coupling models (Left panel), and vs. $\delta_A^{(1)}$ for $f(R)$ modified gravity models. Light blue region denotes $0.01 < r < 0.064$, and blue region denotes $r < 0.01$.](image1)

![FIG. 5: The plot of $r - 16\epsilon c_s$ in parameter space $[\epsilon(2), \delta_A^{(1)}]$ (setting $\delta_A^{(2)} = 0$), $[\epsilon(2), \delta_A^{(2)}]$ (setting $\delta_A^{(1)} = 0.01$), $[\delta_A^{(1)}, \delta_A^{(2)}]$ (setting $\epsilon(2) = 0.004$). Blue region denotes $r - 16\epsilon c_s < 0$, and yellow region denotes $r - 16\epsilon c_s > 0$.](image2)

The plot of $r$ in parameter space $[\epsilon(2), \delta_A^{(1)}]$ is shown in Fig. 4, where light blue region denotes $0.01 < r < 0.064$, and blue region denotes $r < 0.01$. Note that the allowed region extend to where $\epsilon(2) < 0$, which means that in presence of nonminimal coupling, $r$ can be within the detectable region even for phantom inflation, but in this case, large $\delta_A^{(1)}$ is needed in order not to cause the instabilities. For $f(R)$ gravity, there is modest space for $\delta_A^{(1)}$ to make $r$ within the detectable region. The comparison of $r$ to the consistency relation $r = 16\epsilon c_s$ is also shown in Fig. 5. One can see
that for the trivial case of $\delta_A^{(1)} = 0$, one can calculate that $r = 16\epsilon_c$, and consistency relation is recovered. However, for positive/negative $\delta_A^{(1)}$, one has $r$ larger/smaller than $16\epsilon_c$.

**D. nonminimal derivative coupling: $M_p^2 R/2 + L_2 + L_5$**

In last case we discussed about nonminimal coupling of field and gravity but missed the possibility of derivative coupling, namely the gravity coupling with derivatives of the inflaton field. This is a very different case from above, which will probably even involve $G_5$. Here we assume $G_5 = h(\phi)$ is a pure function of $\phi$, which is equivalent to the introduction of a term like $G_{\mu\nu} \partial^\mu \phi \partial^\nu \phi$, up to a total derivative [52]. This case can be written in terms of EFT action (2) with the correspondence:

\[
f = 1 - h_\phi(\phi)\dot{\phi}^2,
\]

\[
A = \frac{\dot{\phi}^2}{2} G_{2X}(\phi, X) - G_2(\phi, X) - \frac{1}{2} M_p^2 H \dot{f} - 2 M_p^2 (f - 1) \dot{H} - \frac{1}{2} M_p^2 \ddot{f} + 3M_p^2 H^2 (f - 1),
\]

\[
c = \frac{\dot{\phi}^2}{2} G_{2X}(\phi, X) + \frac{7}{2} M_p^2 H \dot{f} + 2M_p^2 (f - 1) \dot{H} + \frac{1}{2} M_p^2 \ddot{f} + 9M_p^2 H^2 (f - 1),
\]

\[
m_2^4 = \frac{\dot{\phi}^4}{4} G_{2XX}(\phi, X) + \frac{1}{4} M_p^2 H \dot{f} + M_p^2 (f - 1) \dot{H} + \frac{1}{4} M_p^2 \ddot{f},
\]

\[
m_3^3 = M_p^2 \ddot{f} + 4M_p^2 H (f - 1), \quad m_4^2 = -M_p^2 (f - 1), \quad m_5^2 = -M_p^2 (f - 1).
\]

From Eqs. (11)-(14), we have

\[
D = 4H^2 M_p^4 (3f - 4)^2 = 4M_p^4 (1 + 3h_\phi \dot{\phi}^2)^2,
\]

\[
c_1 = \frac{(2 - f)M_p^2}{4H^2 (3f - 4)^2} \{48H^2 (f - 1)^2 + (2 - f)[2\dot{\phi}^4 G_{2XX}(\phi, X)/M_p^2 + 4H \dot{f} + 4H^2 \epsilon (2 - f)]\},
\]

\[
c_2 = f M_p^2 = (1 - h_\phi \dot{\phi}^2) M_p^2,
\]

\[
c_3 = \frac{-a M_p^2 (2 - f)^2}{H (3f - 4)},
\]

\[
c_4^2 = \frac{4H^2}{(2 - f)} \{\dot{f}(3f - 2)(2 - f)/H - (2 - f)^2 (3f - 4)(1 + \epsilon) - f(3f - 4)^2\} / \{48H^2 (f - 1)^2 + (2 - f)[2\dot{\phi}^4 G_{2XX}(\phi, X)/M_p^2 + 4H \dot{f} + 4H^2 \epsilon (2 - f)]\},
\]

\[
D_T = \frac{M_p^2}{8} (2 - f) = \frac{M_p^2}{8} (1 + h_\phi \dot{\phi}^2),
\]

\[
c_T = \frac{f}{Q_T} = \frac{f}{2 - f} = \frac{1 - h_\phi \dot{\phi}^2}{1 + h_\phi \dot{\phi}^2},
\]

so using Eq. (35), one has

\[
r = \frac{32}{(3f - 4)^2} \{2\dot{f}(2 - f)/H - (2 - f)^2 (3f - 4)(1 + \epsilon) - f(3f - 4)^2\}^{3/2} / \{48(f - 1)(2 - f) + (2 - f)^2 \{2\dot{\phi}^4 G_{2XX}(\phi, X)/(H^2 M_p^2) + 4f / H + 4\epsilon (2 - f)\}\}^{1/2} f^{3/2}.
\]

For ease of our analysis, we define $\delta_{h\phi} \equiv f_{(5)}/f$, where $f_{(5)} = f - 1$. Therefore we have $f_{(5)} = \dot{f}$, $f_{(5)} = \ddot{f}$, and $\delta_{h\phi} = (f_{(5)}/f) = \dot{f}(f - f_{(5)})/f = (\ddot{f}(f - f_{(5)}/f) = H\delta_{h\phi}^{(1)}(1 - h_\phi \dot{\phi}^2)$. Moreover, to avoid ghost and gradient instabilities in this model ($D_T > 0$, $c_T > 0$), we require $-1 < h_\phi \dot{\phi}^2 < 1$, so from the definition one finds $\delta_{h\phi} < 1/2$ needed even for non-inflationary models.
Moreover, in this case we have

$$\frac{\Delta c}{M_p^2 H^2 f} = -\frac{M_p^2 H \dot{f}(5)}{2M_p^2 H^2 f} - \frac{2M_p^2 f(5) \dot{H}}{M_p^2 H^2 f} - \frac{M_p^2 3H \dot{f}(5)}{2M_p^2 H^2 f} + \frac{3M_p^2 H^2 f(5)}{M_p^2 H^2 f}$$

$$= -\frac{1}{2}\delta^{(1)}_f + 2\epsilon\delta_{h\phi} - \frac{1}{2}\delta^{(1)}_f \delta^{(2)}_f + 3\delta_{h\phi},$$

$$\Delta \Lambda \frac{M_p^2 H^2 f(5)}{M_p^2 H^2 f} = \frac{7M_p^2 H f(5)}{2M_p^2 H^2 f} + \frac{2M_p^2 f(5) \dot{H}}{M_p^2 H^2 f} + \frac{M_p^2 3H \dot{f}(5)}{2M_p^2 H^2 f} + \frac{9M_p^2 H^2 f(5)}{M_p^2 H^2 f}$$

$$= \frac{7}{2}\delta^{(1)}_f - 2\epsilon\delta_{h\phi} + \frac{1}{2}\delta^{(1)}_f \delta^{(2)}_f + 9\delta_{h\phi},$$

therefore

$$\epsilon = \frac{\epsilon^{(2)}(1 - 4\delta_{h\phi}) - \delta^{(1)}_f + 3\delta_{h\phi}}{1 - 2\delta_{h\phi}}.$$ 

and

$$r = \frac{16}{(4\delta_{h\phi} - 1)^2} \frac{[6\delta^{(1)}_f \delta_{h\phi}(1 - 2\delta_{h\phi}) + \delta_{h\phi}(3 - 2\delta_{h\phi})(1 - 4\delta_{h\phi}) + (1 - 4\delta_{h\phi})^2(1 - 2\delta_{h\phi})\epsilon^{(2)}]^{3/2}}{[3\delta_{h\phi}(1 + 2\delta_{h\phi}) + (1 - 2\delta_{h\phi})(1 - 4\delta_{h\phi})\epsilon^{(2)} + 2\delta_{h\phi} \frac{\dot{X}}{X}(1 - 2\delta_{h\phi})]^{1/2}}.$$ 

**FIG. 6**: The plot of $r$ in parameter space $[\epsilon^{(2)}, \delta^{(1)}_f]$ (setting $\delta_{h\phi} = 0$), $[\epsilon^{(2)}, \delta_{h\phi}]$ (setting $\delta^{(1)}_f = 0$), $[\delta^{(1)}_f, \delta_{h\phi}]$ (setting $\epsilon^{(2)} = 0.004$). Light blue region denotes $0.01 < r < 0.064$, and blue region denotes $r < 0.01$.

**FIG. 7**: The plot of $r - 16\epsilon c_a$ in parameter space $[\epsilon^{(2)}, \delta^{(1)}_f]$ (setting $\delta_{h\phi} = 0$), $[\epsilon^{(2)}, \delta_{h\phi}]$ (setting $\delta^{(1)}_f = 0$), $[\delta^{(1)}_f, \delta_{h\phi}]$ (setting $\epsilon^{(2)} = 0.004$). Blue region denotes $r - 16\epsilon c_a < 0$, and yellow region denotes $r - 16\epsilon c_a > 0$. 
We now consider a simple case, namely $\delta^{(1)}_f \simeq 0$, which means that $\delta_{h\phi} = \text{const}$. Moreover, we also ignore the effect from $\delta_{KKXX}$. Therefore $\epsilon$ and $r$ will be reduced to:

$$
\epsilon = \frac{\epsilon(1) - 4\epsilon_{h\phi} + 3\delta_{h\phi}}{1 - 2\delta_{h\phi}}, \quad (85)
$$

$$
r = \frac{16(\epsilon(1) - 4\epsilon_{h\phi})(1 - 2\delta_{h\phi}) + 4\delta_{h\phi}^2}{12\delta_{h\phi}(1 - 4\delta_{h\phi})} \approx 16\epsilon(1) + 2\delta_{h\phi} + O(\delta_{h\phi}^2), \quad (86)
$$

where in the last step we expand the expression of $r$ in terms of slow-varying parameters. One could see that the $\epsilon(1)$ and $\delta_{h\phi}$ is anti-proportional to each other in determining $r$. Moreover, for $\delta_{h\phi} \simeq 0$, $\delta^{(1)}_f \simeq 0$ case ($f \simeq 1$), we have

$$
\epsilon \simeq \epsilon(2), \quad r = 16\epsilon(2), \quad (87)
$$

which recovers the canonical field case.

The plot of $r$ in parameter space $[\epsilon(2), \delta^{(1)}_f, \delta_{h\phi}]$ is shown in Fig. 6, where light blue region denotes $0.01 < r < 0.064$, and blue region denotes $r < 0.01$. One can see that (from middle panel) for $\delta^{(1)}_f = 0$, the allowed region lies in where $\epsilon(2)$ and $\delta_{h\phi}$ goes roughly anti-proportional to each other, and large $\epsilon(2)$ is allowable for large negative $\delta_{h\phi}$. Like nonminimal coupling case, negative $\epsilon(2)$ (phantom inflation) is also allowable for large positive $\delta_{h\phi}$. Moreover, when $\delta_{h\phi} = 0$, the dependence of $r$ on $\delta^{(1)}_f$ is also decoupled (left panel). The comparison of $r$ to the consistency relation $r = 16c_s$ is also shown in Fig. 7. From the left panel we can see that, for $\delta_{h\phi} = 0$, the deviation from the consistency relation $r = 16c_s$ (in this case $c_s = 1$) totally depends on $\delta^{(1)}_f$, namely $r > / < 16\epsilon$ when $\delta^{(1)}_f > / < 0$, this is because in this case, $r$ go back to the trivial case $r = 16\epsilon(2) = 16(\epsilon + \delta^{(1)}_f)$.

### E. Beyond Horndeski theory: $M_p^2 R/2 + L_2 + L_6$

The last case we’re considering is the beyond Horndeski model. In [35, 36], we considered the beyond Horndeski model, which can realize a non-singular universe without either ghost or gradient instabilities, by introducing a $\tilde{m}_4^2$ term different than $m_4^2$ in the action (2). Here we assume that $\tilde{m}_4^2 = q(\phi)M_p^2$ being a pure function of $\phi$ and $q(\phi)$ is a dimensionless function, then this case can be written in terms of EFT action (2) with the correspondence:

$$
f = 1, \quad \Lambda = \frac{\phi^2}{2} G_{2X}(\phi, X) - G_{2}(\phi, X), \quad c = \frac{\phi^2}{2} G_{2X}(\phi, X),
$$

$$
m_4^2 = \frac{\phi^4}{4} G_{2XX}(\phi, X), \quad m_3^2 = m_2^2 = 0, \quad \tilde{m}_4^2 = q(\phi)M_p^2. \quad (88)
$$

From Eqs. (11)-(14), we have

$$
D = 4H^2 M_p^4, \quad c_1 = \frac{1}{H^2} (H^2 \epsilon M_p^2 + 2m_2^2), \quad c_2 = M_p^2, \quad (89)
$$

$$
c_3 = \frac{aM_p^2}{H} \left( 1 + 2 \tilde{m}_4^2 M_p^2 \right), \quad c_4 = \frac{H^2 \epsilon M_p^2 + 2H^2 \tilde{m}_4^2 (1 + \epsilon) - 2H \tilde{m}_4^2}{H^2 \epsilon M_p^2 + 2m_4^2}, \quad (90)
$$

$$
D_T = \frac{M_p^2}{8}, \quad c_T = 1. \quad (91)
$$

In this case, we have

$$
\frac{\Delta c}{M_p^2 H^2} = \frac{\Delta c}{M_p^2 H^2} = \delta^{(1)}_f = \delta^{(2)}_f = 0, \quad \text{therefore} \quad \epsilon = \epsilon(2), \quad \text{namely there is no derivation of} \ \epsilon \ \text{from the effect of} \ L_6. \ \text{Define} \ \delta_q = \dot{q}/(Hq), \ \text{we have}
$$

$$
r = \frac{16(\epsilon(2) + 2\dot{q}(1 + \epsilon(2) - \delta_q))^{3/2}}{\epsilon(2) + 8\delta_{KKXX}^{1/2}}, \quad (92)
$$

From the expression we can see that, for the case of $\delta_q \simeq 0$ ($q(\phi)$ is nearly constant), to have smaller $r$ than that is given by consistency relation, one must have $q < 0$ ($\delta_{KKXX}$ also ignored). However, the constraint from positivity of $c_1$ and $c_2^2$ requires that $\epsilon(2) + 8\delta_{KKXX} > 0$ as well as $\epsilon(2) + 2q(1 + \epsilon(2)) - \delta_q > 0$. 


FIG. 8: The plot of $r$ in parameter space $[\epsilon_{(2)}, q]$ (setting $\delta_q = 0$), $[\epsilon_{(2)}, \delta_q]$ (setting $q = 0$), $[q, \delta_q]$ (setting $\epsilon_{(2)} = 0.004$). Light blue region denotes $0.01 < r < 0.064$, and blue region denotes $r < 0.01$.

FIG. 9: The plot of $r - 16\epsilon c_s$ in parameter space $[\epsilon_{(2)}, q]$ (setting $\delta_q = 0$), $[\epsilon_{(2)}, \delta_q]$ (setting $q = 0$), $[q, \delta_q]$ (setting $\epsilon_{(2)} = 0.004$). Blue region denotes $r - 16\epsilon c_s < 0$, and yellow region denotes $r - 16\epsilon c_s > 0$.

The plot of $r$ in parameter space $[\epsilon_{(2)}, q, \delta_q]$ is shown in Fig. 8, where light blue region denotes $0.01 < r < 0.064$, and blue region denotes $r < 0.01$. For $q < 0$, small $r$ can still be obtained (left panel). For $q = 0$, $r$ go back to the trivial case $r = 16\epsilon c_s$ and the dependence of $r$ on $\delta_q$ is also decoupled (middle panel). For $q$ around 0 and $\delta_q$ around 1, there still exists room for small $r$ even when $\epsilon_{(2)} = 0.004$, which is at the edge of $r = 0.064$ in the absence of $L_6$ (right panel). The comparison of $r$ to the consistency relation $r = 16\epsilon c_s$ is also shown in Fig. 9.

V. CONCLUSION

The next decade will be a Gravitational Wave decade, with many more experiments of GWs getting down to work, and many more signals of GWs will be discovered. Especially, the ambitious ground-based experiment AliCPT in Tibet, China, is aiming to search for signals of PGWs with improved accuracy in the coming few years [14]. The current and future constraints has divided the amplitude of $r$ into three parts, namely $r > 0.064$ (disfavored by current data), $r \in (0.064, 0.01)$ (within the observable window of next experiments like AliCPT) as well as $r < 0.01$ (still waiting for further detections). In this paper, with the facility of the EFT approach, we have formulated the tensor/scalar ratio in the generic setup including the various scalar-tensor theories and theoretically studied which kind of inflation models let $r$ fall into the last two regions. We have analyzed the relation between $r$ and other slow-varying parameters and obtained the corresponding regions in parameter spaces. Furthermore, we have also discussed the deviation of $r$ from the consistency relation in each model. We summarize our conclusive remarks in the following:

1. Making use of EFT approach, the tensor/scalar ratio $r$ for the given action (2) can be expressed as in Eq. (35). Note that this expression is applicable for both cases where the power spectrum is constant or growing for scalar and
tensor perturbations.

2. From the expression, one can see the running of sound speed affects $r$ in an obvious way. When $s > 0$ or $-1 < s_T < 0$, $r$ will get suppressed, or vice versa, where we simply assumed $c_s \sim \eta^s$, $c_T \sim \eta^{sT}$. Note that the conclusion might not be applicable to more complicated cases.

3. For Kessence model where only $M^2_R/2$ and $L_2$ are involved in the Lagrangian, $r$ is in accord with the consistency relation, $r = 16c_c$. In this case, various ways can be done to suppress $r$, e.g., to have small $\epsilon$, or small sound speed, by non-canonicity of the scalar field. However, the small sound speed will meet the danger of large non-Gaussianities.

4. For Lagrangians beyond $L_2$, $r$ will deviate from the consistency relation, which will cause another mechanism for small $r$. Moreover, this allows for phantom inflation to give rise to small $r$.

5. For nonminimal coupling theory, namely $L_4$ with $G_4 = G_4(\phi)$, the sound speed of tensor perturbations is still unity, $c_p = 1$. However, it will change when kinetic coupling to gravity also involves. Phantom inflation is allowed to have small $r$ in these cases.

6. Small $r$ can also be obtained by taking into account the beyond-Horndeski part, namely $L_6$, even in the absence of $L_3, L_4$, and $L_5$.

These remarks may not be brand-new; however, we have them confirmed at a more general level, namely the framework of effective field theory. Moreover, by showing more detailed and more precise relationships between $r$ and those parameters, we hope our analysis and numerical plots will be useful for concrete model-buildings.

Before ending, let us remind that for the current discussions. We have focused only on $r$ and have not taken into account constraints from other variables on the early universe models, such as spectral index and non-Gaussianities. Although we assumed that the power spectra of these models are scale-invariant, it deserves to consider how such a scale-invariance would impose constraints on those model parameters. Moreover, the non-Gaussianities is also an interesting probe of the early universe. In Ref. [53] authors discussed that in matter bounce scenario driven by Horndeski theory one could not get small $r$ while keeping $f_{nl}$ small enough to be within the current constraints (a.k.a. no-go theorem), while it is also interesting to consider such constraints for other early universe scenarios/models (examples has been given in [54]). We will address the above discussion in the future works.

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