"The Waters I am Entering No One yet Has Crossed": Alexander Friedman and the Origins of Modern Cosmology

Ari Belenkiy
Simon Fraser University, Department of Statistics and Actuarial Science, BC, Canada

Abstract. Ninety years ago, in 1922, Alexander Friedman (1888–1925) demonstrated for the first time that the General Relativity equations admit non-static solutions and thus the Universe may expand, contract, collapse, and even be born. The fundamental equations he derived still provide the basis for the current cosmological theories of the Big Bang and the Accelerating Universe. Later, in 1924, he was the first to realize that General Relativity allows the Universe to be infinite. Friedman’s ideas initially met strong resistance from Einstein, yet from 1931 he became their staunchest supporter. This essay connects Friedman’s cosmological ideas with the 1998–2004 results of the astronomical observations that led to the 2011 Nobel Prize in Physics. It also describes Friedman’s little known topological ideas of how to check General Relativity in practice and compares his contributions to those of Georges Lemaître. Recently discovered corpus of Friedman’s writings in the Ehrenfest Archives at Leiden University sheds some new light on the circumstances surrounding his 1922 work and his relations with Paul Ehrenfest.

"L’acqua ch’io prendo giammai non si corse.”
Dante, Paradiso Canto II

1. Introduction

The 2011 Nobel Prize in Physics was assigned to scientists who independently confirmed that the Universe presently expands in an accelerating manner. Thus one of the scenarios described by Alexander Friedman in 1922 and 1924 was recognized as true. Although the essay “Accelerating Universe” (Nobelprize.org [2011]) composed by the Class for Physics of the Swedish Royal Academy of Science to elucidate the “scientific background of the Nobel Prize in Physics 2011” cites both of Friedman’s works, Friedman (1922, 1924), regrettably, in the essay text, Friedman’s contribution is distorted. It mistakenly ascribes to Friedman (1922) the discovery “that Einstein’s steady state (sic!) solution was really unstable.”¹ Then it erroneously asserts “in 1924, Friedman presented his full equations.” Finally, it wrongly states “in 1927, the Belgian priest and physicist Georges Lemaître working independently from Friedman performed similar calculations based on General Relativity (GR) and arrived at the same results.”

¹A shorter version of this paper appeared in Belenkiy (2012).
²This fact was discovered by Eddington (1930).
This essay remedies these errors. The facts are: already in 1922 Friedman (1922) had set the correct framework for GR, suggesting the most general “line element” for the positively curved space, derived the set of correct equations (now the “Friedman equations”), solved them and discussed all three major scenarios for the expanding Universe. As well, he introduced the expression “Expanding Universe” (in his words: “The Monotone World”). Friedman (1924) further revolutionized the discourse on GR presenting the idea of an infinite Universe, static or non-static, with a constant negative curvature, completing what would be later known as the “FLRW” metric.

Most of the biographical details can be found in Tropp et al. (2006) or Friedman (1966). However, a recent discovery of a number of Friedman’s papers and letters in the Ehrenfest Archives at Leiden University sheds new light on some particular circumstances surrounding the discoveries of the Petrograd physicist. The exact references to the works cited in the text of this essay, as well as the details of GR theory, can be found in Nussbaumer & Bieri (2009). The translation of several excerpts from Friedman (1966) from the Russian is this author’s.

2. Alexander Friedman: A short but very accomplished life

Born in 1888 and raised in St. Petersburg, Friedman studied mathematics at St. Petersburg University under the guidance of Vladimir Steklov, in parallel attending Paul Ehrenfest’s physics seminars. Upon graduation in 1910, he worked primarily in mathematical physics and its applications in meteorology and aerodynamics. From the outbreak of World War I, Friedman served with the Russian air force at the Austrian front as an instructor in ballistics. Taking part in several air reconnaissance flights, he was awarded the military cross for his courage.

When, after the February 1917 Revolution, dozens of new universities were established across Russia, on Steklov’s recommendation Friedman obtained his first professorship in mechanics in Perm near the Ural Mountains. The faculty included several eminent mathematicians. During the Russian Civil War, Perm changed hands twice and the teaching conditions were miserable; the science library was practically missing, as Friedman often complained in the letters to Steklov.

With the Civil War coming to a close, in 1920, Friedman returned to his alma mater, now Petrograd, and started working as a physicist at the Main Geophysical Observatory, rising to director by 1925. He also lectured on mechanics at the Polytechnic Institute and the Railway Institute. Most of his personal research at this time was oriented toward the theories of turbulence and aerodynamics. Additional professional commitments occasioned Friedman’s parallel investigations into Niels Bohr’s quantum theory and Albert Einstein’s general relativity. A month before his untimely death from typhus in September 1925, Friedman made a record breaking 7,400-meter air balloon flight risking his life to conduct health-related scientific experiments. His recollections of this flight were published posthumously (Friedman 1966, pp. 382-5).

Einstein’s special theory of relativity was well-known in Russia from its inception in 1905, but awareness of GR, published in 1915, was delayed due to the First World War. The news of GR, along with the results of the British 1919 astronomical expedition led by Arthur Eddington and confirming GR’s prediction for the gravitational bending of light, caused tremendous excitement in both the scientific milieu and general public throughout revolutionary Russia, where it was considered as another revolution – though in science. Finally, in 1921, shipments of European scientific publications re-
Alexander Friedman and the Origins of Modern Cosmology

Figure 1. Portrait by M.M. Devyatov. Alexander Friedman. Petrograd, 1925. Courtesy of the Voeikov Main Geophysical Observatory (St. Petersburg), http://www.veoikovmgo.ru/ru/istoriya

sumed providing Russian scientists with sufficient access to the contemporary scientific literature. Physicist Vsevolod Frederiks, on his return to Petrograd in 1920, brought insider’s information: interned in Germany during the war, he worked at Göttingen University as a private assistant to David Hilbert, who wrote GR’s equations in covariant form in 1916, about the same time as Einstein.

In collaboration with Frederiks, Friedman organized a seminar dedicated to the study of GR. Together they aimed to write a comprehensive textbook on GR; the first volume, devoted to tensor calculus, appeared in 1924. In parallel, in his own book, The World as Space and Time (Friedman 1923) Friedman developed a philosophical interpretation of GR. But his fame rests on two papers, published in Zeitschrift für Physik in 1922 and 1924 (Friedman 1922, 1924) with new solutions of GR equations. In these papers he introduced the fundamental idea of modern cosmology— that the Universe is dynamic and may evolve in different manners, for example, starting from singularity.

3. Cosmology before Friedman: Rivalry between two static Universes’ models

The 16 (or actually 10 different) equations of GR are:

$$R_{ik} - \frac{1}{2} g_{ik} \bar{R} - \Lambda g_{ik} = -\kappa T_{ik},$$

(1)

2The exact timing remains somewhat controversial, see Corry et al. (1997).
3There also exist English translations of both papers (Friedman 1999a,b).
where indexes \(i\) and \(k\) run from 1 to 4. The first three indexes relate to space while index 4 relates to time, \(g_{ik}\) is the metric tensor, \(R_{ik}\) is the Ricci tensor representing 2-dimensional curvatures, \(\mathbf{R}\) is the scalar space-time curvature, constant \(\kappa = 8\pi G/c^2 = 1.87 \times 10^{-27} \text{ cm} \text{ g}^{-1}\). \(G\) is the gravitational constant, \(c\) is the speed of light in vacuum, and \(T_{ik}\) is the energy-matter tensor representing the “inertia” of the world. The latter was assumed to be \(T_{11} = T_{22} = T_{33} = -p\), where \(p\) is the pressure of radiation, \(T_{44} = c^2 \rho g_{44}\), where \(\rho\) is average density of matter in the Universe, and \(T_{ik} = 0\) for non-diagonal elements. Einstein readily considered a simplification with \(p = 0\). The equality sign in Equation (1) signifies the “equivalence principle” between gravity (on the left) and inertia (on the right).

The left side of Equation (1) is highly non-linear in \(g_{ik}\) and its derivatives of the first and second order. To assure stability of the solution, Einstein introduced a linear term, \(\Lambda g_{ik}\), where the coefficient \(\Lambda\) became known as the “cosmological constant.”

Since finding a solution to this system of equations requires great ingenuity, only two simple solutions were discovered by 1922, one by Einstein, and the other by the Dutch astronomer Willem de Sitter.

The so-called “solution A,” found by Einstein (1917), represented a spatially 3-dimensional spherical, finite Universe with curvature radius \(R\) constant in space and time. In coordinates \((\chi, \theta, \phi)\) the elements of metric tensor are:

\[
g_{11} = -\frac{R^2}{c^2}, \quad g_{22} = -\frac{R^2}{c^2} \sin^2 \chi, \quad g_{33} = -\frac{R^2}{c^2} \sin^2 \chi \sin^2 \theta, \quad g_{44} = 1, \quad (2)
\]

while all other \(g_{ik} = 0\). Einstein’s Universe is a 3-dimensional sphere with a fixed radius, which evolves in time as a 4-dimensional cylinder.

Applying GR equations (1) the only solution comes when the two (initially independent) parameters, \(\Lambda\) and \(\rho\), become interconnected and expressed via radius \(R\):

\[
\Lambda = \frac{c^2}{R^2} \quad \& \quad \rho = \frac{2}{kR^2}. \quad (3)
\]

Multiplying \(\rho\) by the volume of the 3-dimensional sphere, \(V = 2\pi^2 R^3\), the Universe’s mass could be found as \(M = 4\pi^2 R^3 / \kappa\).

The remarkable consequence of “solution A” was that a good estimate of average density leads to an estimate of the radius and mass of the Universe. With estimate \(\rho = 2 \times 10^{-27} \text{ g cm}^{-3}\), suggested by de Sitter (1917), it seemed that Einstein had achieved his goal and the “final theory” of the spherical Universe was constructed, with constant radius \(R = 750 \times 10^{24} \text{ cm}\), or 800 Mly. De Sitter’s discovery of another solution a month later came as a cold shower for Einstein.

The so-called “solution B,” found by de Sitter (1917), presented a different Universe. Though the rest of \(g_{ik}\) were the same as in Equation (2) importantly \(g_{44} = \cos^2 \chi\),

---

4There is a variant opinion that the equivalence principle means “gravity = space-time” but see Eddington (1920, p. 76).

5Since the metric tensor \(g_{ik}\) has dimension \(s^2\) and the equations (1) are dimensionless, \(\Lambda\) has dimension \(s^{-2}\).

6We use “Mly” for “million light years” and “Gly” for “billion light years.”
i.e., time was curved in the direction of the “radial” spatial coordinate $\chi$. Though spatially spherical, de Sitter’s Universe has a point different from any other point, a center, whereas in Einstein’s solution every point is equivalent to any other. Rays of light don’t move along the space geodesics in de Sitter’s Universe, with the exception of those which pass through the center.

To satisfy GR equations\footnote{i.e., time was curved in the direction of the “radial” spatial coordinate $\chi$. Though spatially spherical, de Sitter’s Universe has a point different from any other point, a center, whereas in Einstein’s solution every point is equivalent to any other. Rays of light don’t move along the space geodesics in de Sitter’s Universe, with the exception of those which pass through the center.} this solution necessitates a non-zero cosmological constant but zero density:

$$\Lambda = \frac{3c^2}{R^2} \quad \& \quad \rho = 0.$$  

(4)

The major feature of this model, absence of matter ($M = \rho V = 0$), violated Ernst Mach’s principle that “inertia cannot exist without matter” and thus made this solution unacceptable for Einstein. Moreover, the function $\cos^2 \chi$ before the time component suggests a singularity at $\chi = \pi/2$. This mystical locus, where time “stops to flow,” Hermann Weyl\footnote{Hermann Weyl\cite{1918}} called the “horizon.” The space of every observer was surrounded by such a “horizon” though the latter was unreachable. However, as de Sitter noticed, the alleged “slowing of time” along the radial component $c$ provided the means to explain the shifts $z = \delta \lambda/\lambda$ of absorption lines in the spectra of nebulae\footnote{The nebulae were not understood as distinct island universes until Hubble’s work on Cepheids in M31 and M33 in 1924 \cite{1925}.} first observed by Vesto Slipher at Lowell Observatory in Flagstaff, Arizona, in 1912 \cite{1913}. Saying that “the observations are still very uncertain, and conclusions drawn from them are liable to be premature,” de Sitter discussed only three nebulae, whose radial velocities have been determined “by more than one astronomer.” The spectrum of the Andromeda nebula showed a blue-shift equivalent to a speed of 311 km s$^{-1}$, but the other two showed more pronounced redshifts of 925 and 1,185 km s$^{-1}$.

Computing the average of three velocities, de Sitter related the “average” redshift of 600 km s$^{-1}$ to his “solution B” via the formula $z = (1/2) \sin^2 \chi$, but deduced from it an absurdly small Universe’s curvature radius $R$ as $5 \times 10^{24}$ cm or 4.5 Mly. Indeed, the 100-inch telescope at the Mount Wilson observatory, established in 1917, could reach as far as 150 Mly\footnote{“The range of the 100-inch Mount Wilson telescope is estimated by Hubble to be $5 \times 10^7$ parsec.”}.

Meanwhile the evidence for the redshifts was mounting mainly due to Slipher’s efforts, and by 1923 reached a score of 36 among 41 spiral nebulae. Eddington popularized this fact in Eddington\cite{1923} (see Fig. 2).

Ready to identify the redshifts with the Doppler effect, most of the workers in this field adopted “solution B” as another means to test GR, looking for a better formula for the redshift and hoping to dismiss the weird “horizon” using various coordinate transformations. The first goal was achieved by Hermann Weyl\cite{1923} and later by Ludwik Silberstein\footnote{According to Lemaître\cite{1927,1931a}: “The range of the 100-inch Mount Wilson telescope is estimated by Hubble to be $5 \times 10^7$ parsec.”}\cite{1924}, who improved the formula for the redshift to $z = \pm \sin \chi + \sin^2 \chi$ and then deduced the value of the radius $R$ ranging from 80 to 120 Mly. The second goal was achieved by Kornel Lanczos\cite{1923}, who reworked “solution B” into a non-static solution with an exponentially growing radius. Later Georges Lemaître\cite{1925} quite elegantly repeated both results. But by then, an absolutely novel and daring idea had been born and developed by an outsider, a physicist from far away “revolutionary” Petrograd.
4. Friedman’s Expanding Universes: Three major scenarios

On June 29, 1922, Zeitschrift für Physik accepted the paper “On the Curvature of Space” by A. Friedman of Petersburg, submitted to the journal by Paul Ehrenfest. Though Friedman (1922) cites there the original works by Einstein (1917) and de Sitter (1917), he certainly learned of these works from Arthur Eddington’s Space, Time and Gravitation (Eddington 1920), available to him in the French edition of 1921, with deliberations on the worth of the two models. But instead of taking sides, the scientist from Petrograd approaches the problem from a wider viewpoint.

The physical demand of homogeneity and isotropy of space does not necessitate a static Universe. Focusing on the most general form of spherical metric, Friedman (1922) finds on top of static solutions A and B, a new class of non-static solutions of GR equations.

Friedman’s dynamical solution is a generalization of Einstein’s 3-dimensional hypersphere Eq. 2 with a constant in space but changing in time curvature radius $R(t)$. In this case, equations yield two ordinary differential equations for $R(t)$ (the “Friedman equations” in modern terminology):

\begin{equation}
\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{c^2}{R^2} - \Lambda = 0 \tag{5}
\end{equation}

and

\begin{equation}
3\frac{\dot{R}^2}{R^2} + 3\frac{c^2}{R^2} - \Lambda = \kappa c^2 \rho. \tag{6}
\end{equation}

---

\footnote{This can be inferred from his letter to Paul Ehrenfest of June 3, 1922 (Fig. 3).}

\footnote{The “dot” and two “dots” above the variable denote, as usual, the first and second time derivatives.}
The second-order Ordinary Differential Equation (Eq. 5) Friedman integrates directly, without using Bianchi identities, arriving at the fundamental equation that governs the dynamics of the Universe:

$$\frac{1}{c^2} \ddot{R}^2 = \frac{A - R + \frac{3}{\kappa} \Lambda R^3}{R}.$$  \hfill (7)

Comparing Eq. 7 with Eq. 6 Friedman finds that the constant of integration, $A$, is related to the density $\rho$ as

$$\rho = \frac{3A}{\kappa R^3}.$$  \hfill (8)

Multiplying Eq. 8 by the volume of hypersphere $V = 2\pi^2 R^3$, Friedman obtains $A = \kappa M / 6\pi^2$. Thus $A$ is proportional to the Universe’s mass $M$ with constant $\kappa$ and represents the gravitational radius of the Universe.
Figure 4. Friedman’s fundamental cubic $C(x)$ that governs the dynamics of the Universe. Its three coefficients are three fundamental constants: cosmological constant, the sign of curvature of space and the gravitational radius of the Universe. Depending on their values, the cubic may have either a. 0, or b. 1 double or c. 2 simple positive roots, which define six scenarios of the Universe’s evolution.

The rest of the 1922 paper is dedicated to analysis of Eq. 7, which after integration becomes

$$t = \frac{1}{c} \int_{R_0}^{R} \sqrt{\frac{x}{A - x + \frac{\Lambda}{3c^2}x^3}} dx + t_0. \quad (9)$$

Taking $R_0$ to be the present radius of the Universe, $t_0$ designates, in Friedman’s words, “the time that passed from Creation,” or the age of the Universe.

The right hand-side of Eq. 9 has physical meaning only when the cubic $C(x)$ in denominator is positive. The cubic $C(x)$ can be positive in three ways (Fig. 4): on a semi-infinite interval that may start either 1) at 0 or 2) at a positive real number, or 3) on a finite segment. This defines three major scenarios of the Universe’s evolution (Fig. 5). If $C(x)$ has a double positive root, then three more scenarios are possible (Fig. 6).

1. The first scenario comes if the cubic has no positive roots and thus is positive on $(0, \infty)$. This happens when $\Lambda > 4c^2/9A^2$, i.e., $\Lambda$ is positive and larger than a certain critical value for a given density $\rho$. The Universe starts from singularity $R = 0$ at $t = 0$; its asymptotic behavior at infinity is $R \approx R_0 \exp[\sqrt{\Lambda/3} (t - t_0)]$. At some inflexion point its expansion changes from deceleration to acceleration.

   According to Eqsns. 7 and 9, the inflexion point occurs where expression $C(x)/x$ reaches its minimum, i.e., when the radius of Universe reaches

   $$R_f = \left(\frac{3c^2A}{2\Lambda}\right)^{1/2} = \left(\frac{\kappa c^2 \rho}{2\Lambda}\right)^{1/2} R_0. \quad (10)$$

   Friedman called this scenario “The Monotone World of the first kind” (“M 1” in Fig. 5).

2. The second situation occurs when $0 < \Lambda < 4c^2/9A^2$. In this case the cubic has two positive roots, $x_1$ and $x_2$ and is positive in intervals $(0, x_1)$ and $(x_2, \infty)$. This presupposes two different scenarios: 2a and 2b. In the 2a scenario, expansion oscillates between $R = 0$ and $R = x_1$. This is the “periodic” solution, viable also for a wider range of $\Lambda$’s, discussed next. In the 2b scenario, expansion starts from a non-zero radius, $R_{\text{min}}$, equal to the greater root, $x_2$, and continues forever in accelerating mode. Its asymptotic behavior at infinity again is $R \approx R_{\text{min}} \exp[\sqrt{\Lambda/3} (t - t_0)]$. Friedman named this scenario “The Monotone World of the second kind” (“M 2” in Fig. 5).
Figure 5. Three possible major scenarios of the Universe’s evolution according to Friedman (1922). The M1 world shows expansion from singularity with a flex point that signifies existence of two stages of evolution: deceleration and acceleration. Bondi (1961, p. 84) calls it “Lemaître’s case.” The M2 world shows expansion from the non-zero radius to infinity. The P world shows periodic evolution, with expansion and contraction phases and point of maximum radius in between. Point $t_0$ is the current stage of the Universe and point $t_f$ is the inflexion point in the M1 world.

Taking in Eq. $9 A = 0$ or, equivalently, $\rho = 0$ (de Sitter’s case), this solution simplifies to $R = \varsigma \cosh(c t / \varsigma)$, where $\varsigma = c \sqrt{3/\Lambda}$ has meaning of the minimal radius, $R_{\text{min}}$, while $\Lambda$ accepts de Sitter’s value $\Lambda = 3c^2/R_{\text{min}}^2$, as in Eq. $4$. Since the function $\cosh(c t / \varsigma)$ smoothly continues to $t < 0$, a symmetric (left) branch may be added to the M2 curve in Fig. 5. This change would modify the M2 scenario: the first phase of the Universe’s evolution becomes infinitely long contraction to $R_{\text{min}} = \varsigma$, followed by an infinitely long expansion back to infinity. Note the intermediate case, $\lambda_{cr} = 4c^2/9A^2$, where the cubic has a positive double root. This case was chosen later by Lemaître (1927). Friedman considered it a “limiting case” – see below.

3. The third scenario results either from scenario 2a, or when $\Lambda \leq 0$. In both cases the cubic has one positive root, $x_1$ (when $\Lambda = 0$ the cubic reduces to the linear function with one positive root) and the interval of positivity is $(0, x_1)$. The Universe starts from singularity $R = 0$ at $t = 0$ and expands in a decelerating manner, then stops at $R = x_1$ and begins contracting back into singularity. The life of the Universe is finite. Friedman called this scenario the “Periodic World” (“P” in Fig. 5) and found its approximate period (for small $\Lambda$) as

$$T_p = \frac{2}{c} \int_0^{x_1} \frac{x}{\sqrt{A - x + \frac{\Lambda}{3c^2} x^3}} dx \approx \frac{\pi A}{c} = \frac{\kappa M}{6\pi c}. \quad (11)$$

Upon assuming the mass of the Universe $M$ is equal to $5 \times 10^{21}$ masses of the Sun, Friedman found $T_p = 10^{10}$ (ten billion) years. Unfortunately we cannot repeat this result. Since the sun’s mass is known as $M_S = 2 \times 10^{33} \text{g}$, we get $M = 10^{55} \text{g}$.
and therefore
\[ T_p \approx \frac{\pi A}{c} = \frac{\kappa M}{6\pi c} = \frac{10^{28} \text{cm}}{3\pi c} \approx \frac{10^{27} \text{cm}}{c}, \tag{12} \]
which is only 1 billion years. One can only guess where the error crawled into Friedman’s reasoning.

Considering in Eq. 9 \( A = 0 \) or, equivalently, \( \rho = 0 \) (de Sitter’s case) Friedman’s formula for \( \Lambda < 0 \) simplifies to \( R = \varsigma \sin(\sqrt{\Lambda} t) \), where \( \varsigma = c \sqrt{3/\Lambda} \) plays a role of maximal radius \( R_{\text{max}} \), while \( \Lambda \) accepts de Sitter’s value \( \Lambda = 3c^2/R_{\text{max}}^2 \), as in Eq. 4.

4. In addition to the three major scenarios, Friedman (1922) mentions two special (“limiting”) cases, which occur when \( \Lambda \) is equal to \( \Lambda_{\text{cr}} = 4c^2/9A^2 \) and the cubic is “degenerate,” i.e., has a double positive root at \( x = 3A/2 \). The double root leads to a logarithmic singularity at the finite radius \( R_E = 3A/2 \). The 2a scenario degenerates into an infinitely long expansion from singularity \( R = 0 \) to the finite radius \( R_E \) at infinity: \( R \approx R_E - \exp[-\sqrt{\Lambda}(t - t_0)] \) (Fig. 6 lower curve). The 2b scenario degenerates into an infinitely long expansion \( R \approx R_E + \exp[\sqrt{\Lambda}(t - t_0)] \) from the finite radius \( R_E \) in the past \( (t < t_0) \) via the exponential growth \( R \approx R_0 \exp[\sqrt{\Lambda}/3(t - t_0)] \) in the future \( (t > t_0) \) (Fig. 6 upper curve). The latter scenario was adopted by Lemaître (1927). Both curves asymptotically converge to radius \( R_E = 3A/2 \), which can be viewed as Einstein’s radius for the static Universe. This static Universe of Einstein is the sixth scenario with \( R_E = 3A/2 = 2GM/\pi c^2 = R_S/\pi \), where \( R_S \) is the Schwarzschild radius.

5. Friedman’s philosophy of the Big Bang and Expanding Universe

In Friedman’s models the cosmological constant \( \Lambda \) is a free parameter to be determined empirically, whereas in Einstein’s and de Sitter’s models \( \Lambda \) was strictly linked to the curvature of the Universe. It is the pair, \( \rho \) and \( \Lambda \), that determines a true scenario for the Universe’s evolution.
In the 1922 paper Friedman remains silent about details, but in his book *World as Space and Time*, sent to print on September 5, 1922, and published the following year, he allows himself to say more. There, in the section “Matter and the Structure of the Universe,” Friedman (1923) describes the Big Bang scenario in the following words:

A non-static Universe represents a variety of cases. For example, it is possible that the radius of curvature constantly increases from a certain initial value; it is also possible that the radius changes periodically. In the latter case the Universe compresses into a point (into nothingness), then increases its radius to a certain value, and then again compresses into a point. Here one may recall the teaching of Indian philosophy about “periods of life.” It also provides an opportunity to speak about the world “created from nothingness.” But all these scenarios must be considered as curiosities which cannot be presently supported by solid astronomical experimental data. So far it is useless, due to the lack of reliable astronomical data, to cite any numbers that describe the life of our Universe. Yet if we compute, for the sake of curiosity, the time when the Universe was created from a point to its present state, i.e., time that has passed from the “creation of the world,” then we get at number equal to tens of billions of usual years.\(^{11}\)

It is interesting that here Friedman mentions only the M2 world and Periodic world but not the M1 world. The reason seems to lie in his low a priori estimate for the cosmological constant, \(\Lambda = 10^{-37} \text{ s}^{-2}\), which is crossed out but still visible in the Russian manuscript of the 1922 paper (see Fig. 3). His estimate \(3A = 10^{27} \text{ cm}\) (see Eq. 12) implies \(\Lambda_{cr} = \frac{4c^2}{9A^2} = 36 \times 10^{-34} \text{ s}^{-2}\), i.e. greater than \(\Lambda\) by four orders, thus making the M1 scenario impossible.

The choice of small \(\Lambda\) was motivated by his cautious estimate of the age of the Universe as in the last lines of his Russian draft “of the order of 10^{12} years,” which he crossed out in the Russian manuscript at the last moment. Though this figure is reminiscent of James Jeans’ estimate of the age of the solar system, 10^{13} years, Friedman could not have possibly known it. Jeans seems to advocate such a high estimate not earlier than 1928, the second edition of his *Astronomy and Cosmogony*.\(^{12}\) At least Eddington (1920), read by Friedman, is silent on it. Therefore Friedman must have learned it from another source.

And indeed, Eddington (1920, p. 163) gives his own estimate of the radius of the Universe, \(2 \times 10^{11}\) parsecs, which is \(6 \times 10^{11}\) light years and thus of the same order as Friedman’s 10^{12} light years. Eddington’s ad hoc estimate comes from comparison of the radius and “gravitational radius” of the electron.\(^{13}\) This could explain Friedman’s somewhat unclear statement (Friedman 1922) that “hopefully \(\Lambda\) may be found from electrodynamical considerations.”

\(^{11}\)Translated from Friedman (1966, p. 317).

\(^{12}\)Jeans (1928)

\(^{13}\)Though both radiuses used by Eddington, \(7 \times 10^{-35}\) cm and \(2 \times 10^{-13}\) cm, are smaller than the currently accepted values by factors 2 and 1.4, respectively, the net-result for the curvature radius of the Universe is nearly the same.
6. Einstein’s reception of Friedman’s theory in 1922-1923

The main ideas of the Friedman (1922) paper – a dynamic character of the Universe, its metric, his equations, and exhaustive description of possible scenarios for the real Universe have become fundamentals in contemporary cosmology. However, in 1922 they were mostly ignored or rejected.

Friedman’s paper appeared in print in July 1922 in volume 10 of Zeitschrift für Physik and was noticed by Einstein. Most likely, Ehrenfest, Einstein’s close friend, called his attention to it. Einstein’s immediate reaction illustrates how unwelcome the idea of a non-static universe was. In his view, a normal cosmological theory should uphold the static character of the Universe. Accordingly, Einstein initially found Friedman’s solution “suspicious” and in October 1922 published a short note in volume 11 of Zeitschrift für Physik suggesting Friedman’s derivation contained a mathematical error.

In fact, Einstein (1922) mistakenly concluded that Friedman’s equations (with $p = 0$) imply $\frac{d\rho}{dt} = 0$, i.e., constancy of density, and therefore of volume and radius $R$ – in contradiction to the initial assumption. Instead, the correct derivation gives $R^{-3}d(Rp^3)/dt = 0$, which implies constancy of mass and thus adequately represents the “conservation of mass” law.$^{14}$

Learning of Einstein’s note, on December 6, 1922 Friedman wrote a lengthy letter to Einstein, presenting his derivations, but Einstein was already on his world tour, returning to Berlin in March 1923. Only then could he have read Friedman’s letter (Tropp et al. 2006). Later, in May, Yuri Krutkov, Friedman’s colleague, met Einstein twice at Leiden, at Paul Ehrenfest’s, and clarified the confusion. Certainly, the meeting was organized by Ehrenfest, who felt himself personally responsible for presenting Friedman’s paper to Zeitschrift für Physik.

Following this meeting, on May 31, 1923, Einstein (1923) sent another short note to Zeitschrift für Physik accepting the mathematical correctness of Friedman’s results. However, he opined that “the solution has no physical meaning;” wisely he crossed this out from the proofs at the last moment. Einstein was unready to accept the idea of the expanding Universe for 8 more years.$^{15}$

7. In a quest for an infinite Universe: The Universe with negative curvature

Already by 1922 Friedman realized that GR equations$^{14}$ alone fail to provide the final answer for not only the kinematics of the real Universe but also its global structure (the shape and size). The means to choose one solution over another needed to come from elsewhere, for example, astronomy. The Universe in the shape of the 3-dimensional hypersphere $S^3$, for example, admits “ghosts” – double images of the same object in the sky coming from two opposite directions (the second is from passing the “antipodal” point). To avoid such a misleading phenomenon, de Sitter (1917) pioneered the idea that the space of directions on the hypersphere $S^3$ must be considered as the basic

---

$^{14}$Of course, considering zero pressure $p$ deprived the formulation of this law of its most general form found later by Lemaître (1927) as $c^2dM + pdV = 0$, correctly interpreting it as “energy spending for the adiabatic expansion of the Universe.”

$^{15}$In his two notes, Einstein inadvertently introduced the following problem for future historians of cosmology: he christened Friedman with two n’s in his last name whereas Friedman’s 1922 paper was signed with one “n.” It seems Friedman followed the “advice” and submitted his 1924 paper with two n’s.
cosmology space where each pair of “antipodal” points might be viewed as one. A newly arising science of algebraic topology assured that this space is an orientable manifold and its metric element is the same as that of $S^3$. De Sitter’s idea certainly gained some hard currency as Friedman (1924) mentioned it favorably while Lemaître (1927) even adopted it, computing the volume of his Universe as $\pi^2 R^3$, i.e., half of the $S^3$’s volume.

However, Friedman’s major concern lay with the notion of the world’s finiteness, which was firmly entrenched in the minds of scientists, largely, because of Einstein’s staunch adherence to Mach’s philosophy. In all his works, Friedman (1922, 1923, 1924), the physicist from Petrograd insisted that the form of the Riemannian metric does not resolve this problem. His guide at first was algebraic topology. Inspired by Poincare’s theory of the coverings of the Riemannian manifolds, he imagined the possibility of a spherical shaped Universe yet one with infinite diameter and volume. However, the 3-dimensional hypersphere $S^3$ admits only trivial covering and thus this scenario seems impossible. Unabashed, Friedman discussed the “ramified covering” of the sphere, suggesting the “longitude” coordinate $\phi$ may run not from 0 to $2\pi$ but wind over and over again until infinity. However, this idea has an obvious flaw: the two poles would be “covered” only by one point and thus the space would no longer be homogeneous (Fig. 7). Within a year, in his quest for the infinite Universe, Friedman found another argument – this time a geometrical one.

Figure 7. A 2-dimensional sphere covered with an infinite covering. If one folds the cover perpendicular to one axis then its two poles remain “uncovered.”

On advice from his long-term friend and fellow mathematician, Yakov Tamarkin, Friedman checked whether GR allows solutions for a hyperbolic metric with a negative space curvature. Indeed, Friedman (1924) provides a positive answer, with both static and non-static scenarios.

The static scenario, with constant radius $R$, necessitated zero density $\rho$ and cosmological constant $\Lambda$ analogous to de Sitter’s solution for the positive curvature case Eq.

---

This space was called the elliptical space though now it is better known as real projective space $RP^3$. The first constructions of the projective spaces were given by Felix Klein in 1890 and Henri Poincare in 1900 (Klein 1928).
The non-static scenario, with a time-dependent radius of space-curvature \( R = R(t) \), through the same reasoning as in the positive curvature case, yields the fundamental equation,

\[
\frac{1}{c^2} \frac{\dot{R}^2}{R} = \frac{A + R + \frac{\Lambda}{3c^2}R^3}{R},
\]

where the integration constant \( A \) is related to the average density \( \rho \) as in Eq. 8, thus eliminating the possibility to choose the right curvature sign empirically measuring only density. Besides, since the volume of the hyperboloid is infinite, the mass of the matter here is infinite. However, differing from \( \Lambda \) only in the sign before the linear term in the cubic, solution 13 clarified the ontological significance of the remaining three coefficients in the cubic, which are the cosmological constant, the sign of the space curvature and the gravitational radius of the Universe.

Taking in Eq. 13 \( A = 0 \) or, equivalently, \( \rho = 0 \) (de Sitter’s case), this solution simplifies to \( R = \varsigma \sinh(\frac{ct}{\varsigma}) \), where in this case \( \varsigma = c \sqrt{3/\Lambda} \) has the meaning of a scaling parameter.

The 1924 paper was also ignored. Certainly, Einstein paid no attention to it. On meeting Lemaître in 1927, Einstein called the idea of an expanding Universe “abominable.” But growing astronomical evidence, and most notably observations by Hubble (1929) and proof by Eddington (1930) that “solution A” was unstable, changed Einstein’s mind, though in a somewhat unexpected way.

In 1931 Einstein recognized Friedman’s achievement and suggested purging from GR his old “nemesis” – the cosmological constant \( \Lambda \). The next year, in a joint work, Einstein & de Sitter (1932) promoted an idea of the “flat” Universe (i.e., with zero spatial curvature), which is just Eq. 13 with 0 term instead of \( +R \) in the upper right side. The latter idea still exists, together with Friedman’s positive (Friedman 1922) and negative (Friedman 1924) curvature cases, as neither was preferred by the latest empirical data. The former suggestion is not supported by recent astronomical observations: \( \Lambda \) is still necessary for cosmology.

8. On Friedman’s track: Contributions of Georges Lemaître and Edwin Hubble in the 1920s

Between Friedman’s papers of 1922 and 1924 and the astronomical observations of the 1990s that led to the 2011 Nobel Prize in Physics there lie a few groundbreaking achievements: the “Hubble constant” that describes the rate of the Universe’s expansion, and the concept of the “dark matter.” The Hubble (1926) estimate of distances to distant galaxies, led Lemaître (1927) to the discovery of the “Hubble constant.” Lemaître (1934) first gave Friedman’s singularity a physical meaning, that of a “primeval atom” that “blew up” – the idea Fred Hoyle described later as the “Big Bang.” Though Einstein did not see any need of the cosmological constant, Lemaître always valiantly defended its necessity. It is not obvious whose contribution was decisive in shaping modern cosmology.

In a popular exposition of GR, *The Meaning of Relativity*, in three consecutive editions, Einstein (1946; 1950; 1951), that included Appendix 1 “On Cosmological Problem,” discussing the unclear nature of the cosmological constant, Einstein emphasized: “The mathematician Friedman found a way out of this dilemma. His result then found a surprising confirmation in Hubble’s discovery of the expansion of the stellar
system (a redshift of the spectral lines which increases uniformly with distance). The following is essentially nothing but an exposition of Friedman’s idea…” and detailed Friedman’s contribution for the next 15 pages[7]

Later physicists bestowed the title of “father of modern cosmology” upon Georges Lemaître (Peebles 1971, p. 8), while modern astronomers casually credit Edwin Hubble (Perlmutter 2003). In the last decade historians decided to clarify this (Kragh & Smith 2003). The “priority” debate was narrowed to Lemaître and Hubble and concentrated around the passage in Lemaître (1927) which included the derivation of the “Hubble constant” that was omitted in the English translation of his paper (Lemaître 1931a). Some felt behind this omission stood none other than Hubble himself - a view which lacking factual support was recently rejected (Livio 2011). Yet, the consensus was that the “Hubble constant” was solely Lemaître’s idea, thus, supporting Lemaître’s right to be called the “discoverer of the expanding Universe” (Nussbaumer & Bieri 2009, p. 133). However we feel the decision was made too quickly.

Figure 8. Georges Lemaître with Arthur Eddington, Stockholm 1938. Courtesy Georges Lemaître Archives, Catholique University of Louvain.

Indeed, though unaware of Friedman’s 1922 and 1924 groundbreaking works, Lemaître appeared at the junction when the shortcomings of both “Solution A” and “Solution B” became pronounced based on the quickly accumulating astronomical data, coming from Mount Wilson Observatory. Unlike Friedman, who in 1920-25 for the most part studied mostly meteorology, Lemaître in 1924-1925 was working at MIT on his doctorate dedicated to cosmology. As a part of Lemaître’s duties he toured the Mount Wilson and Lowell Observatories and attended the meetings of various astronomical societies. Unlike Friedman, Lemaître had at his desk at least two books with a

---

[7]Somewhat unfortunately, Einstein attributed to Hubble alone what properly belongs to several people, primarily Slipher and de Sitter. Interestingly, Einstein never quoted any of Lemaître’s papers.
systematic exposition of GR and its application to cosmology by Eddington (1923) and Silberstein (1924), as well as distances to the galaxies found by Hubble (1926). Besides, Eddington (1923) brought Lemaître’s attention to the spectral redshift discovery by Slipher, publishing the “radial velocities” of 41 spiral nebulae (Fig. 2).

Noticing that the curvature radius $R(t)$ of his metric is related to the redshift as $z = \dot{R}/R$ and interpreting $dt$ as $r/c$, where $r$ is the distance to a galaxy, on the one side, and using the Doppler effect formula $z \approx v/c$, which holds for small “radial velocity” $v$, on the other, Lemaître was confronted with the equation $\dot{R}/R = v/r$. This makes sense for the exponentially growing radius $R(t)$ he found only if $v/r$ is close to a constant. Seeing a rather weak correlation between $v$ and $r$ for the set of 42 spiral galaxies, cited by several astronomers, and thus weak factual support for this assumption, Lemaître (1927) postulated a linear relation between $v$ and $r$. Moreover, he found the coefficient proportionality $L_1 = 575 \text{ km s}^{-1} \text{ Mpc}^{-1}$ via a “simple” regression method, and $L_2 = 625 \text{ km s}^{-1} \text{ Mpc}^{-1}$, via “weighted” regression, with weight $(1 + r^2)^{-1/2}$ attached to each distance $r$ and corresponding radial velocity $v$.

The “simple” regression method consists of drawing a line from the beginning of the coordinates to the “center of gravity” – the point with two coordinates: “average velocity” $\bar{v}$ and “average distance” $\bar{r}$. Surprisingly, the same method was also employed by Hubble (1929) and de Sitter (1930). The latter two, however, tested the linear relation between $v$ and $r$ from a different perspective – from the relation $v/c \approx r/R$, derived by Silberstein (1924) from de Sitter’s “solution B” and popularized by Lundmark (1924). The constancy of radius $R$ then immediately leads to $v \approx Hr$.

In postulating the linear relation Lemaître was influenced by a similar attempt by Silberstein (1924, p. 551) to fit Harlow Shapley’s data for globular clusters to the asymptotically linear curve (Fig. 9).

Lemaître personally met Silberstein at the meeting of the British Association of the Advancement of Science in Toronto in August 1924 (Flin & Duerbeck 2006). There, on August 13, at the “Cosmical Physics” Sub-section, Silberstein gave a talk “Determination of the Curvature Radius of Space-Time” based on de Sitter’s model, while Lemaître accompanied Eddington who gave a “Citizen’s Lecture” on “Einstein’s Theory of Relativity” on August 9 (BAAS 1925, pp. 19,373). In the paper submitted half a year later, Lemaître (1925) derives Silberstein’s formula, which linearly connects the radius of the Universe with a spectral redshift. Silberstein’s influence can be seen in the fact that Lemaître finally put more trust in $L_2 = 625 \text{ km s}^{-1} \text{ Mpc}^{-1}$ obtained via the “weighted” regression method, where the weight $(1 + r^2)^{-1/2}$ attached to distances and velocities is most detrimental to the more distant galaxies. Silberstein is known to have distrusted the distances to the “extra galactic nebulae.” Pointing out that the different measurements of the distance to Andromeda nebula, made by Knut Lundmark, were discrepant by two orders (Silberstein 1924, p. 522), Silberstein based his computations only on closely lying globular clusters and two Magellanic Clouds (Fig. 9).

The coefficient of proportionality, $L$, could have led to an effective estimate of the age of the Universe ($t_0$ in Eq. 9) had Lemaître embraced a scenario with a finite age of the Universe. However, in his 1927 paper Lemaître entirely ignored the finite

1Lemaître (1927) cites Stromberg (1925), who made some “adaptation” of 41 Slipher’s values and added one extra.

2This method, a precursor of the OLS (ordinary least squares), was first invented by Isaac Newton in 1700 (Belenkiy & Vila-Echague 2003).
Figure 9. The first try to fit the absolute value of the “red shift” (the “Doppler effect”) $C[D]$ vs. the distance $r$ of 11 globular clusters and two Magellanic Clouds to a semi-linear curve. From Silberstein (1924, p. 551) with data taken from Shapley. The dotted circle stands for M33 nebula. All 13 chosen objects are close to the Sun, lying within 100,000 light years.

age scenario and the “Big Bang” solution in particular. Upon rediscovery of the Fried-
man equations (5–6), instead of considering all classes of solutions, Lemaître chose one particular solution, where the cubic (Fig. 4b) has a double positive root, $x_0$. This necessitated accepting a particular (“critical”) value of $\Lambda_{cr} = 4c^2/9A^2$ (in Friedman’s notation). Lemaître identified point $x_0$ with the non-zero initial radius of the Universe, $R_E$. This led Lemaître to Friedman’s first “limiting” case (Fig. 6), where the Universe began expanding from radius $R_E$ to infinity. Thus Lemaître (1927) missed the solution with singularity at the origin, M1 World, now known as the “Big Bang” scenario, the most probable scenario for the expanding Universe according to the astronomical results of 1998-2004 that led to the 2011 Nobel Prize in Physics.

Even in his 1930 letter to Eddington, Lemaître writes: “I consider a Universe of curvature constant in space but increasing with time and I emphasize the existence of a solution in which the motion of the nebulae is always a receding one from time minus infinity to plus infinity” (Nussbaumer & Bieri 2009, p. 122). Thus, as late as 1930 Lemaître still adhered to the “limiting case” scenario. He abandoned it the following year after Eddington pointed out that “such logarithmic infinities have no real physical significance” (Lemaître 1931b), where he discusses the M2 scenario.

Only in late 1931, in a short letter to Nature, responding to Eddington, did Lemaître (1931c) for the first time consider the idea of matter coming from a “discrete number of quanta” – a precursor of the “Big Bang” scenario. Yet by 1931 Lemaître had been
fully aware of Friedman (1922) for at least four years, since his talk with Einstein at the 1927 Solvay conference (Nussbaumer & Bieri 2009, pp. 111-3).

It is rather surprising that, though recognizing that in his 1929 lecture notes Lemaître thanked Einstein for directing his attention to Friedman’s works, which “contained several notions and results later rediscovered by himself,” Nussbaumer & Bieri (2009, p. 111) conclude: “Thus, Lemaître owes nothing to Friedman.” The conclusion is acceptable if taken to mean that Lemaître (1927) is independent of Friedman (1922). But it is remarkably limited in general if we compare the contributions of the two and put it in a 1929 perspective. By 1929 Lemaître had learned from Friedman the idea of “birth from singularity” (the essence of the “Big Bang” scenario) and the idea that the cosmological constant is a fully independent parameter. Oddly enough, Lemaître never discussed the negative curvature or flat cases.

Lemaître had his “15 months of fame” in 1930-1931 (Nussbaumer & Bieri 2009, pp. 126-8), before Einstein was fully “converted” to the idea of an expanding Universe in April 1931. Indeed, already in the first paper written after his trip to the Mount Wilson Observatory in January 1931 (ibid, pp. 146-7), Einstein (1931) immediately emphasized Friedman’s priority: “Several investigators have tried to cope with these new facts by using a spherical space whose radius is variable in time. The first one who attempted this, uninfluenced by observations, was A. Friedmann.” Certainly, Einstein admired Friedman for discovering dynamic solutions purely theoretically, without being driven by the “facts,” as Einstein himself often had been.

Thereafter Lemaître was quoted either only after Friedman or, for example by Einstein, not quoted at all. Lemaître himself recognized the secondary value of his 1927 work, when in the footnote (2) to the 1931 English translation of his 1927 paper Lemaître (1931a) wrote: “Equations of the Universe of variable radius and constant mass have been fully discussed, without reference to the receding velocity of nebulae, by A. Friedmann (1922).” Thus, in 1931, Lemaître credited himself only with finding a link between exponentially increasing radius $R$ of the Universe and redshift phenomena observed for distant galaxies.

Obviously, from a mathematical viewpoint, the Hubble constant $H_o$ is a matter of convenience rather than of fundamental importance, while Friedman’s cubic is fundamental. Deriving three possible scenarios of the Universe’s evolution, Friedman did not need $H_o$. In hindsight he was even fortunate to neglect it. Hubble’s grave mistake in evaluating distances to remote galaxies led to an overestimation of $H_o$ and subsequent underestimation of the age of the Universe by a factor of 8 and thus delayed the acceptance of the “Big Bang” scenario by several decades. Even Einstein in his last years despaired of finding a way out of the dilemma between a short cosmological age of the Universe, of 1.7 billion years, and a longer geological age of the Earth! Only at the time of Einstein’s death in 1955 did Walter Baade (1952) and Allan Sandage (1958) discover Hubble’s mistake in evaluating distances and thus restored confidence in GR and Friedman’s models.

It is interesting how both Friedman and Lemaître thought of their discoveries. Friedman sent both of his papers to the central German physics journal of his time and used to say privately that he managed to “horseshoe Einstein” (Tropp et al. 2006). Lemaître, in contrast, not only published his 1927 paper in a little known journal but
Figure 10. Einstein at the 100-inch telescope at the Mount Wilson Observatory, January 1931, with Edwin Hubble (with a pipe) and Walter Adams watching. Courtesy of the Archives, California Institute of Technology.

seemed to place a low value on his discovery. Quoting Lemaître’s lecture notes of 1929, Nussbaumer & Bieri (2009, p. 111) forgot to mention that in the 23-page long transcript Lemaître discusses the astronomical data only in the context of de Sitter’s model, not mentioning at all his model of 1927. “Lemaître mentions that there is a relation between the velocity of recession and the distance. Yet he does not provide any number to connect the velocity of recession with the distance – the number he himself derived in 1927. In other words, although the paper contains many numbers, there is no estimate of the Hubble constant” (Shaviv 2011). Could Lemaître have been that depressed by Einstein’s cool reception?

9. Friedman confirmed: The Nobel Prize 2011 in Physics

At the end of his 1923 book Friedman concludes:

Einstein’s theory is justified by experience; it explains the old, seemingly inexplicable, phenomena and predicts new remarkable relations. The truest and deepest method to study the world geometry and the structure of the Universe with the help of Einstein’s theory lies in application of this theory to the whole world and use of astronomical observations. So far this method has not given us much since mathematical analysis gives up before the difficulties of the problem and astronomical observations do not provide a reliable basis for experimental study of the Universe. But these
obstacles certainly are of temporary nature and our descendants, without any doubt, will discover the structure of the Universe where we are doomed to live.\footnote{Translated from Friedman (1966, p. 322).}

Interestingly, the model Friedman seemed to be emotionally attached to was the “Periodic World.” Such a world allows multiple “births” and “deaths” of the universe - somewhat in tune with the Pythagoras-Plato-Hindu philosophy of reincarnation. Although already since the late 1980s voices were heard in favor of the positive cosmological constant (Efstathiou et al. 1990), the basic hypothesis with which astronomers attacked the problem in the 1990s was in line with Einstein & de Sitter (1932): that the Universe is flat with zero cosmological constant and thus – despite expanding asymptotically as $R(t) \approx t^{2/3}$ – decelerates.

The result was most surprising! Two groups of astronomers, working independently though using the same technique of observing the so-called “supernovae Ia,” found in 1998-1999 that the galaxies at a distance of 4-5 Gly are farther away than they should be according to a constant speed of expansion given by the currently accepted Hubble constant, $H_0 = 70$ km s$^{-1}$ Mpc$^{-1} = 2 \times 10^{-18}$ s$^{-1}$. Thus, in the last 4-5 billion years, the Universe has been accelerating driven by the strictly positive cosmological term (Riess et al. 1998; Perlmutter et al. 1999).

This result however could not discriminate between M1 and M2 worlds since both models allow for acceleration. From early calculations of CMB anisotropies and their own observations of supernovae, an important relation between two basic parameters, generalized density $\rho$ (including density of the “dark matter”) and “dark energy” (represented by $\Lambda$) the two groups found (ibid; Perlmutter 2003):

$$\frac{\Omega_M}{\Omega_\Lambda} = \frac{\kappa c^2 \rho}{\Lambda} = \frac{0.3}{0.7}. \quad (14)$$

This result has several applications. On one side, for a somewhat arbitrary value of the present day average density $\rho_0 = 3 \times 10^{-30}$ g cm$^{-3}$, it leads to $\Lambda = 12 \times 10^{-36}$ s$^{-2}$ (to have $\sqrt{\Lambda/3} = H_0$).

On the other side, Eq. 8 for the same density $\rho_0$ and the present radius of the Universe $R_0 = 1.4 \times 10^{28}$ cm leads to $3A/2 = 8.23 \times 10^{27}$ cm and gives critical $\Lambda_{cr} = 4c^2/9A^2 = 13 \times 10^{-36}$ s$^{-2}$, which is of the same order of magnitude as $\Lambda$. Thus, Eq. 14 alone could not be decisive when choosing between the M1 and M2 worlds.

The litmus test for the choice between M1 and M2 worlds became the inflexion point. Together with Eq. 14 Friedman’s formula (Eq. 10) leads to a simple relation between the radius $R_f$ at the inflexion point and the present radius $R_0$

$$R_f = \sqrt[3]{\frac{0.3}{20.7}} R_0 = 0.6 R_0. \quad (15)$$

Taking $R_0 = 13.75$ Gly we get $R_f = 8.25$ Gly. Thus, one has to look for galaxies distant from us by more than $r = 5.5$ Gly.

The 1998 data are somewhat contradictory on whether the two most distant supernovae are proportionally fainter than their closer counterparts (cf. Figs 3 and 4 in
Alexander Friedman and the Origins of Modern Cosmology

(Perlmutter 2003). Only in 2004 was the question settled (Riess et al. 2004). According to the formula for the spectral redshift \( z \), given as a result of coupling relativistic Doppler and gravitational effects (Bondi 1961):

\[
1 + z = \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}} \left(1 - \frac{\kappa \rho_0 r^2}{12}\right)}.
\]

(16)

the galaxies observed at \( r = 8.4 - 10.5 \) Gly away (1 < \( z \) < 1.6), are closer to the Milky Way than they should be, given the acceleration rate shown by the galaxies 5 Gly away. Thus, at some time, the Universe switched from deceleration to acceleration. The inflexion point was estimated as being at about \( R_f = 5 \pm 1 \) Gly (\( z = 0.46 \pm 0.13 \)), and the M1 model triumphed.

It is interesting that for positive space curvature, and thus a finite mass Universe, the inflexion point is close or even identical to the Schwarzschild radius of the Universe, \( R_S = 2GM/c^2 \), which (for the finite-mass- and positive-curvature-Universe) for the same density \( \rho_0 \) and radius \( R_0 \) as above can be found as

\[
R_S = \frac{2GM}{c^2} = \frac{\kappa \rho_0 R_0^3}{2} \approx 0.6 R_0.
\]

(17)

If for some reason these two points are identical, the inflexion point marks the point when the Universe turned from being a “black hole” to something visible to the other universes.

10. Friedman’s Legacy: Ninety Years Later

Though the question of the physical scenario of the “Big Bang,” first raised by Lemaître (1934) and expounded by George Gamow (1952) and others, is not yet settled, the mathematical (kinematic) part of the “Big Bang,” described by Friedman (1922, 1924), largely is.

There is a tendency, based on one remark by Vladimir Fock (Friedman 1966, p. 402), to present Friedman as merely a mathematician, unconcerned with the physical implications of his discovery (Kragh & Smith 2003). This remark and its implications, however, completely contradict Friedman’s image as a mathematical physicist with serious achievements in meteorology and hydrodynamics. The wide spectrum of the problems he solved, as seen in Friedman’s “Collected works” (Friedman 1966, pp. 424-447), leaves no doubt that he was interested in verification of his theories. Sadly, Friedman’s premature death in 1925 did not allow him to fulfill this goal for the cosmology problem.

Another argument for why Friedman does not deserve the title of discoverer of the expanding Universe is that he “failed to provide a physical reason for the expanding Universe.” The simple answer to this allegation is that Friedman believed, as many of us still do, that Einstein’s GR equations correctly describe physical reality.

The third argument is that Friedman “did not take part in the debates of 1930s when the expanding Universe concept was widely accepted.” This is untrue in two

---

22R. Smith, private communication of Sept. 15, 2012.

23H. Nussbaumer, private communication of Sept. 15, 2012.
different ways. First, as we have seen, the debates of 1930s came to a dead end because of the wrong, greatly underestimated Hubble constant. As a result, other scenarios, like the “Steady State” Universe were proposed in 1948 (Bondi [1961]), delaying the recognition of the “Big Bang” scenario. On the other hand, the Einstein & de Sitter (1932) paper, which set the standard for the “Big Bang” cosmology for many decades until 1998, was impossible without Friedman (1922, 1924), where the first paper provided the theoretical basis for elimination of the cosmological constant, while the second opened the door for a flat Universe.

As a result of these “arguments,” Friedman’s pioneering role in modern cosmology is often underestimated or misrepresented. For a long time both of Friedman’s papers were quoted by various authors without even mentioning him in the body of the book.24 A more recent example, Friedman’s photo is not present on the cover of the quite comprehensive book Discovering the Expanding Universe by Nussbaumer & Bieri (2009) although all other founding fathers of modern cosmology are there.

It is clear, however, that in shaping the theoretical part of modern cosmology Friedman (1922) went much further than his predecessors and even immediate successors, like Lemaître (1927). According to the memoirs of his wife, Ekaterina Friedman, on this and other occasions her husband used to say citing Dante: “The waters I am entering, no one yet has crossed!” (Friedman 1966, p. 396). And indeed, Friedman’s approach was the first correct application of GR to cosmology, which brought forward the idea of the expanding Universe, possibly born from a singularity. Moreover, realizing that GR may admit different metrics, Friedman (1924) alerted physicists that the Universe could be infinite and negatively curved.

As a philosopher of cosmology, Friedman stands head and shoulders above all the other participants of the great cosmological debate in the 1920s, including Einstein. It is a well-known fact that Einstein later bemoaned the introduction of the cosmological constant since the expansion of the Universe in Friedman’s models could be achieved with a zero cosmological constant. Only in 1930 did Eddington and de Sitter embrace the expanding Universe scenario, with the latter admitting that the “veil fell from his eyes.” Only in 1931 did Lemaître begin thinking of alternatives to his initial model of “logarithmically long awakening” of the Universe from the non-zero radius. In contrast, it is known that Hubble never embraced the expanding Universe model (Nussbaumer & Bieri 2009, p. 120).

Recognizing the scale of his achievements, in the last three editions of the Meaning of Relativity, Einstein acknowledged Friedman’s theoretical groundwork together with Hubble’s astronomical observations, viewing contributions of others as secondary. Sadly, Einstein’s words were ignored. After the 1930s, Hubble alone became credited for the “Hubble constant,” while Lemaître received all the credit for the “Big Bang” theory. However, the persistence of Soviet physicists, who, since the early 1960s, when Soviet communist rulers stopped condemning “Lemaître’s reactionary theory,” and the “Big Bang” theory received a fresh breath, began raising their voices on behalf of Friedman’s achievements, finally paid off and since the 1980s Friedman’s metric and equations began to carry his name.25

24 An outstanding example is given by Bondi (1961).

25 Remarkably, it appears that the first who used the expression “Friedmann’s equation” (with respect to Eq. 7) was Lemaître (1939).
The staunchest defender of Friedman’s legacy, Yakov Zel’dovich, pointed to a singularly touching detail: “Friedman published his works in 1922-1924, in a time of great hardships. Herbert Wells’ impression about Moscow and Petrograd of 1921 was of ‘Russia [lying] in the Shadows.’ In the same issue of the 1922 journal where Friedman’s paper appeared, there was an appeal to German scientists to donate scientific literature to their Soviet colleagues who had been separated from it during the war and the revolution. Under those circumstances, Friedman’s daring discovery was not only a scientific but also a human feat!”

And indeed, Friedman’s letter to Ehrenfest on June 3, 1922 (Fig. 3), announcing the discovery of expanding universes, reveals the level of deprivation of Russian society at that time. In the letter, he asks to be mailed an off-print of de Sitter’s 1917 paper from the Monthly Notices of the Royal Astronomical Society, saying “although the journal might be found in Pulkovo, a suburb of Petrograd, there is no way to fetch it from there and going by foot would be extremely difficult.”

True, in the three subsequent years remaining to him Friedman’s life conditions gradually improved. Alexander Friedman quickly ascended the academic and administrative ladders (Fig. 11) and his early death seems as unfair as in the cases of two other founding fathers of modern cosmology: Hermann Minkowski and Karl Schwarzschild. In Friedman’s obituary, Vladimir Steklov enumerates a variety of papers and books Friedman published during these three years,27 and cites the letter from Professor Hein-

\footnotesize{26Translated from Friedman (1966, p. 404).}

\footnotesize{27Friedman’s bibliography includes 25 titles published between 1922 and 1925. All 25 come after his first paper on cosmology and are exactly half of all of his publications (Friedman 1966, pp. 456-7).}
Belenkiy

rich von Ficker, Director of the Prussian Meteorological Institute, which says in particular that “with Friedman’s death you lost one of the most remarkable disciples, one who will be mourned by every meteorologist. The strongest hope of theoretical meteorology departed with him. This case is especially sad for me since among Russian meteorologists he was the closest to me.” Steklov concludes: “A.A. Friedman died, 37 years old, in the peak of strength and talent.”

One may wonder what Friedman could have accomplished had he be given several more years – at least until 1937. And how much brighter would Friedman’s scientific star shine had Einstein not been initially blind to his revolutionary discoveries.

Acknowledgments. The author acknowledges helpful information about Friedman from Carlo Beenakker (Leiden University), V. M. Kattsov and E. L. Makhotkina (Voeikov Main Geophysical Observatory, St-Petersburg), Sabine Lehr (Springer DE), Liliane Moeurs-Haulotte (Georges Lemaître Centre for Earth and Climate Research), as well as discussions with Alexei Kojevnikov (UBC), Harry Nussbaumer (ETH Zurich), John Peacock (Edinburgh University), Todd Timberlake (Berry College), Michael Way (NASA) and Sarah Olesh (Vancouver).

References

Baade, W. 1952, A Revision of the Extra-Galactic Distance Scale, Transactions of the International Astronomical Union, 8, 397
BAAS 1925, Report of the British Association for the Advancement of Science., vol. 92nd Meeting (1924) (London.). Http://www.biodiversitylibrary.org/bibliography/2276, URL http://www.biodiversitylibrary.org/item/96033
Beenakker, C. 2012, Friedmann Papers (Lorentz-Institute, Leiden University). URL http://www.lorentz.leidenuniv.nl/Friedmann
Belenkiy, A. 2012, Alexander Friedmann and the Origins of Modern Cosmology, Physics Today, 65, 38. URL http://link.aip.org/link/?PTO/65/38/1
Belenkiy, A., & Vila-Echagüe, E. 2005, History of One Defeat: Reform of the Julian Calendar as Envisioned by Isaac Newton, Notes and Records of the Royal Society, 59, 223. URL http://rsnr.royalsocietypublishing.org/content/59/3/223.abstract
Bondi, H. 1961, Cosmology (Cambridge University Press)
Corry, L., Renn, J., & Stachel, J. 1997, Belated decision in the hilbert-einstein priority dispute, Science, 278, 1270. URL http://www.sciencemag.org/content/278/5341/1270.full.pdf
de Sitter, W. 1917, Einstein’s Theory of Gravitation and its Astronomical Consequences. Third paper, MNRAS, 78, 3
— 1930, On the Magnitudes, Diameters and Distances of the Extragalactic Nebulae and their Apparent Radial Velocities (Errata: 5 V, 230), Bulletin of the Astronomical Institutes of the Netherlands, 5, 157
Eddington, A. S. 1920, Space, Time and Gravitation. An Outline of the General Relativity Theory (Cambridge University Press)
— 1923, The Mathematical Theory of Relativity (Cambridge University Press)
— 1930, On the Instability of Einstein’s Spherical World, MNRAS, 90, 668
Efstathiou, G., Sutherland, W. J., & Maddox, S. J. 1990, The Cosmological Constant and Cold Dark Matter, Nat, 348, 705
Einstein, A. 1917, Kosmologische Betrachtungen zur Allgemeinen Relativitätstheorie, Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften (Berlin), Seite 142-152., 142
— 1922, Bemerkung zu der Arbeit von A. Friedmann Über die Krümmung des Raumes, Zeitschrift für Physik, 11, 326
— 1923, Notiz zu der Arbeit von A. Friedmann Über die Krümmung des Raumes, Zeitschrift für Physik, 16, 228
— 1931, Zum kosmologischen Problem der Allgemeinen Relativitätstheorie, Sitzungsberichte der Preussischen Akademie der Wissenschaften, Physikalisch-mathematische Klasse, 235
— 1946, The Meaning of Relativity: Third Edition with an Appendix (Princeton University Press)
— 1950, The Meaning of Relativity: with Further Appendix (Princeton University Press)
— 1951, The Meaning of Relativity (Princeton University Press)
Einstein, A., & de Sitter, W. 1932, On the Relation between the Expansion and the Mean Density of the Universe, Proceedings of the National Academy of Science, 18, 213
Flin, P., & Duerbeck, H. W. 2006, in Albert Einstein Century International Conference, edited by J.-M. Alimi, & A. Füzfa, vol. 861 of American Institute of Physics Conference Series, 1087
Friedman, A. 1922, Über die Krümmung des Raumes, Zeitschrift für Physik, 10, 377
— 1923, Mir Kak Prostranstvo i Vremya [The World as Space and Time] (Petrograd: Academia)
— 1924, Über die Möglichkeit einer Welt mit Konstanter Negativer Krümmung des Raumes, Zeitschrift für Physik, 21, 326
— 1966, Collected Works (Nauka, Moscow)
— 1999a, On the Curvature of Space, General Relativity and Gravitation, 31, 1991
— 1999b, On the Possibility of a World with Constant Negative Curvature of Space, General Relativity and Gravitation, 31, 2001
Gamow, G. 1952, The Creation of the Universe (Viking Press). URL http://books.google.com/books?id=orO7AAAAIAAJ
Hubble, E. P. 1925, Cepheids in Spiral Nebulae, Popular Astronomy, 33, 252
— 1926, Extragalactic Nebulae., ApJ, 64, 321
— 1929, A Relation between Distance and Radial Velocity among Extra-Galactic Nebulae, Proceedings of the National Academy of Science, 15, 168
Jeans, J. H. 1928, Astronomy and Cosmogony (Cambridge University Press). URL http://books.google.se/books?id=Sf8QcgAACAAJ
Klein, F. 1928, Vorlesungen über Nichteuklidische Geometrie, Die Grundlehren der mathematischen Wissenschaften (Springer-Verlag). URL http://books.google.ca/books?id=lPFdtwAACAAJ
Kragh, H., & Smith, R. W. 2003, Who Discovered the Expanding Universe?, History of Science, 41, 141
Lanczos, K. 1923, Über die Rotverschiebung in der de Sitter’s Welt, Zeitschrift für Physik, 17, 168
Lemaître, G. 1925, Note on de Sitter’s universe, Journal of Mathematics and Physics, 4, 188
— 1927, Un Univers Homogène de Masse Constante et de Rayon Croissant Rendant Compte de la Vitesse Radiale des Nébuleuses Extra-Galactiques, Annales de la Societe Scientifique de Bruxelles, 47, 49
— 1931a, Expansion of the Universe, A Homogeneous Universe of Constant Mass and Increasing Radius Accounting for the Radial Velocity of Extra-Galactic Nebulae, MNRAS, 91, 483
— 1931b, The Expanding Universe, MNRAS, 91, 490
— 1931c, The Beginning of the World from the Point of View of Quantum Theory., Nat, 127, 706
— 1934, Evolution of the Expanding Universe, Proceedings of the National Academy of Science, 20, 12
— 1949, Cosmological Application of Relativity, Reviews of Modern Physics, 21, 357
Livio, M. 2011, Lost in Translation: Mystery of the Missing Text Solved, Nat, 479, 171
Lundmark, K. 1924, The Determination of the Curvature of Space-Time in de Sitter’s world, MNRAS, 84, 747
Belenkiy

Nobelprize.org 2011, The Accelerating Universe (Swedish Academy of Sciences). Scientific Background on the Nobel Prize in Physics 2011: Compiled by the Class for Physics of the Royal Swedish Academy of Sciences. URL http://www.nobelprize.org/nobel_prizes/physics/laureates/2011/

Nussbaumer, H., & Bieri, L. 2009, Discovering the Expanding Universe (Cambridge University Press)

Peebles, P. J. E. 1971, Physical Cosmology (Princeton University Press)

Perlmutter, S. 2003, Supernovae, Dark Energy, and the Accelerating Universe, Physics Today, 56, 040000

Perlmutter, S., Aldering, G., Goldhaber, G., Knop, R. A., Nugent, P., Castro, P. G., Deustua, S., Fabbro, S., Goobar, A., Groom, D. E., Hook, I. M., Kim, A. G., Kim, M. Y., Lee, J. C., Nunes, N. J., Pain, R., Pennypacker, C. R., Quimby, R., Lidman, C., Ellis, R. S., Irwin, M., McMahon, R. G., Ruiz-Lapuente, P., Walton, N., Schaefer, B., Boyle, B. J., Filippenko, A. V., Matheson, T., Fruchter, A. S., Panagia, N., Newberg, H. J. M., Couch, W. J., & Supernova Cosmology Project 1999, Measurements of Omega and Lambda from 42 High-Redshift Supernovae. ApJ, 517, 565. arXiv:astro-ph/9812133

Riess, A. G., Filippenko, A. V., Challis, P., Clocchiatti, A., Diercks, A., Garnavich, P. M., Gilliland, R. L., Hogan, C. J., Jha, S., Kirshner, R. P., Leibundgut, B., Phillips, M. M., Reiss, D. J., Schmidt, B. P., Schommer, R. A., Smith, R. C., Spyromilio, J., Stubbs, C., Suntzeff, N. B., & Tonry, J. 1998, Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant, AJ, 116, 1009. arXiv:astro-ph/9805201

Riess, A. G., Strolger, L. G., Tonry, J., Casertano, S., Ferguson, H. C., Mobasher, B., Challis, P., Filippenko, A. V., Jha, S., Li, W., Chornock, R., Kirshner, R. P., Leibundgut, B., Dickinson, M., Livio, M., Giavalisco, M., Steidel, C. C., Benitez, T., & Tsvetanov, Z. 2004, Type Ia Supernova Discoveries at z > 1 from the Hubble Space Telescope: Evidence for Past Deceleration and Constraints on Dark Energy Evolution. ApJ, 607, 665. arXiv:astro-ph/0402512

Sandage, A. 1958, Current Problems in the Extragalactic Distance Scale. ApJ, 127, 513

Shaviv, G. 2011, Did Edwin Hubble plagiarize?, ArXiv e-prints.[1107.0442]

Silberstein, L. 1924, The Theory of Relativity (Macmillan). URL http://books.google.ca/books?id=X91FAQAAIAJ

Slipher, V. M. 1913, The Radial Velocity of the Andromeda Nebula, Lowell Observatory Bulletin, 2, 56

Stromberg, G. 1925, Analysis of Radial Velocities of Globular Clusters and Non-Galactic Nebulae. ApJ, 61, 353

Tropp, E. A., Frenkel, V. Y., Chernin, A. D., Dron, A., & Burov, M. 2006, Alexander A Friedmann (Cambridge University Press)

Weyl, H. 1918, Raum. Zeit. Mater. Vorlesungen über Allgemeine Relativitätstheorie (J. Springer). URL http://books.google.ca/books?id=eIVMAAAAMAAJ

— 1923, Zur Allgemeinen Relativitätstheorie, Physikalische Zeitschrift, 24, 230