Biased Random Search in Complex Networks

Lasko Basnarkov\textsuperscript{1,2}, Miroslav Mirchev\textsuperscript{1}, and Ljupco Kocarev\textsuperscript{1,2}
\textsuperscript{1}Faculty of Computer Science and Engineering, SS. Cyril and Methodius University, P.O. Box 393, 1000 Skopje, Macedonia
\textsuperscript{2}Macedonian Academy of Sciences and Arts, P.O. Box 428, 1000 Skopje, Macedonia

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We study two types of biased random walk on complex networks, which are based on local information. In the first procedure, the transitions towards neighboring nodes with smaller degrees are favored, while in the second another concept based on a two-hop neighborhood is explored. We have verified by numerical simulations that both procedures reduce the mean searching time of the target. We show theoretically that for well connected networks, where nodes have many neighbors, biasing of the random walk based on inverse of nodes’ degrees leads to nearly optimal search for undirected and directed networks.

I. INTRODUCTION

The pursuit for appropriate models of the nontrivial interconnections between the units of real systems has led to the emergence of the complex networks theory as one of the most fruitful fields in modern science. Instead of being regular, or purely random, the graph of connections between the items rather frequently poses characteristics like the small world property and power law degree distribution. These topological features have strong implications on the dynamics which might be present in the system. A list of such dynamical processes on complex networks of interacting units include synchronization, consensus formation, disease spreading, and so on.

The random walk is one of the most pervasive concepts in natural sciences which is applied in studies of diverse phenomena ranging from simple animal strategies for food location to complex human interactions resulting in stock price variations, or evolution of research interests. A recent paper contains nice review of the topic and long list of references. Large portion of dynamical processes on complex networks like the PageRank algorithm, or various types of searching, are based on or related to the random walk. Random searching process in a complex network is formulated as follows: starting from an arbitrary node, sequentially hop from a node to one randomly chosen neighbor until reaching some previously defined target node. The performance of a searching procedure is measured in terms of the number of steps needed to get from the target and the related quantity is known as first passage time. Due to the stochastic nature of picking the nodes in the sequence, sometimes one can be very lucky and rather quickly find the target, while in most of the trials the number of steps would be larger then the number of nodes in the network, for a typical source-target pair. Therefore, a more informative quantity is the average number needed to complete the task – the Mean First Passage Time (MFPT) – obtained by averaging across all possible realizations of the random choices.

On the other side, there are also deterministic efficient searching algorithms, based on complete information of the underlying graph structure which follow the shortest paths and result in smallest number of steps to reach the target. However, for very large systems, like the World Wide Web, or in dynamical environments like mobile sensor networks, keeping and updating all necessary topological information might be very complicated. Then one could turn towards strategies based on local information only. The classical Generic Random Walk (GRW) needs the smallest amount of information – only the number of neighbors (the degree $k_i$) of each node $i$. Within this approach, the probabilities for choosing among the neighbors of some node $i$ are taken to be identical and equal to the inverse of its degree $p = 1/k_i$. However, this procedure greatly increases the time to completion of the task, which is another type of inconvenience.

There are various attempts for modification of the GRW aimed for speeding up its searching capabilities. Some of these works provided enhancements while others also presented connections with related problems in other fields. For example, as a counterpart of the path integrals, the Maximal Entropy Random Walk was introduced as a modification of GRW which assigns equal probabilities to all paths with equal length starting from a certain node. In another approach, the Lévy random walk which allows for jumps toward more distant nodes besides the (first) neighbors, was proven to decrease the expected time needed to visit all nodes in a network. Combination of the local diffusion and knowledge of the topology has recently been applied for study of routing of neural signals.

In this work we study two simple algorithms built upon the intuition that searching in a network would be faster if the nodes have nearly equal chance to be visited by the random walker. They are based on local information and short memory and result in reduction of the...
search time as compared to the GRW. In the first approach the probability of choosing a node is inversely proportional to its degree, while in the second algorithm the aim is to equalize the probabilities of visiting of the second neighbors. Thus, they could be placed somewhere between the memory and computationally intensive optimal algorithms from one side and less demanding, but slow ones at the opposite. The potential of searching improvement of both algorithms is verified with numerical simulations. We furthermore show theoretically that for the first algorithm, which is degree-based biased RW, the searching is approaching to the optimal one, when each node has many neighbors. This approach is successful even for directed networks when biasing is based on reciprocals of the indegrees. To the best of our knowledge, result of that kind for directed networks does not exist. These strategies could be successfully applied for searching in networks with a dynamic or not completely known structure, such as wireless sensor networks \[18\] and unstructured peer-to-peer networks \[19\], or in some other very large systems where a full knowledge of the network structure can not be obtained to allow a shortest path algorithms application.

The remainder of the text is organized as follows. In Section II we define more precisely the problem of MFPT and related network-wide quantities in the frames of the Markov chain theory. In Section III we study the proposed algorithms, while the numerical results are provided in Section IV. The paper finishes with the conclusions and some ideas about future work.

II. SEARCH IN COMPLEX NETWORKS

The random walk (RW) on complex networks with constant transition probabilities between the nodes is a Markov chain, since the random choices are independent on the past. Thus, one can use the set of tools from Markov chain theory in order to study the properties of the random walk on complex networks. The physicists’ approach to the problem is usually based on generating functions \[20, 21\]. However, we have opted to use the approach with Markov chains, which is favored by mathematicians, because it provides formulas that are more directly related to the network properties (in fact the transition matrix). In this section, we will use Markov chains terminology where it is possible due to its generality, and complex networks terminology where the reasoning is dependent on their features only. First, we will present the required quantities and formulas for the MFPT for Markov chain and then for the purpose of better understanding of the key factors determining the MFPT on complex networks, we will apply some approximations.

A. Exact results for the Mean First Passage Time

Here we provide a very brief overview of the procedure for analytical determination of MFPT for reversible Markov chains, which can be found elsewhere in the literature, for example in \[22\]. Markov chain is a random process where the system with \( N \) discrete states randomly changes one state with another at every iteration. The dynamics of the chain is determined by the transition matrix \( \mathbf{P} \) which consists of the probabilities \( p_{ij} \) for going from any state \( i \) to any other state \( j \), which are time-independent. In the context of the random walk on complex networks, the states are identified with the nodes and the transition elements represent the hopping probabilities of the random walker which in the generic case are inverses of the nodes’ degrees \( p_{ij} = 1/k_i \). It is well known that for irreducible and aperiodic Markov chain, or fully connected network, starting from any initial distribution of states \( \mathbf{w}^{(0)} \), the system goes towards the stationary distribution \( \mathbf{w} \), which represents the frequencies of visiting the states, or nodes in the context of complex networks, by an infinite walk (without stopping). This distribution can be obtained from the transition matrix \( \mathbf{P} \) and is used for determination of the MFPT. To do so, first from the transition matrix \( \mathbf{P} \) one should construct the matrix \( \mathbf{W} \) of identical rows which correspond to the row vector of the stationary distribution \( \mathbf{w} \). This matrix is formally defined as

\[
\mathbf{W} = \lim_{n \to \infty} \mathbf{P}^n, \tag{1}
\]

or otherwise it can be constructed by stacking the rows which are obtained from the stationary condition \( \mathbf{wP} = \mathbf{w} \). At the next step one determines the so called fundamental matrix from

\[
\mathbf{Z} = (\mathbf{I} - \mathbf{P} + \mathbf{W})^{-1}, \tag{2}
\]

where \( \mathbf{I} \) is the \( N \)-dimensional identity matrix. Last, the MFPT matrix elements are calculated from

\[
m_{ij} = \frac{z_{jj} - z_{ij}}{w_j}, \tag{3}
\]

where \( w_j \) is the \( j \)-th element of the vector of stationary probabilities, while \( z_{ij} \) are the matrix elements of the fundamental matrix \( \mathbf{Z} \). The element \( m_{ij} \) is the MFPT needed to reach the target \( j \), starting from node \( i \). The last result is exact and can be applied for any irreducible and aperiodic Markov chain, or for any generic or biased random walk on a connected complex network with known transition probabilities, independent on the previously visited nodes (a walk satisfying the Markov property). However, as one can notice the MFPT between any pair of nodes \( i \) and \( j \) depends on the fundamental matrix elements which depend on all elements of the transition matrix \( \mathbf{P} \) due to the inverse of matrix operation in \( \mathbf{Z} \). Thus, from expression of this kind it is not easy to spot which features of the states or of the whole chain influence the MFPT values.
As it is defined the MFPT is a property of the network parameterized by two nodes, or states – the starting one i and the final j. A related property of one state only is obtained by averaging all MFPTs starting from all other states and targeting that state

\[ g_i = \frac{1}{N} \sum_{j=1}^{N} m_{ji}. \]  

(4)

In the literature it was called Global Mean First Passage Time – GMFPT [23]. This property can be also seen as a kind of centrality measure of nodes in a complex network.

B. Approximation of the Mean First Passage Time

To better understand which properties of the nodes or of the whole network determine its searching propensity, one should look more deep at the definition of the MFPT. Again, we will use a wider approach here, one based on Markov chains and obtain results which are applicable for directed and weighted networks as well. As in the previous subsection we denote with w the row vector of probabilities \( w_j \) of RW to be at some state \( j \), or node in the complex networks context. By adding superscripts, \( w^{(0)} \) will denote the initial distribution, \( w^{(n)} \) is the distribution after random walker has made \( n \) steps, and \( w \) will be used for the stationary distribution. When considering the MFPT starting from state \( i \) one should use the initial distribution \( w_i^{(0)} \) determined with \( w_i^{(0)} = 1 \), and \( w_j^{(0)} = 0, \forall j \neq i \). We will not use lower index \( i \) to denote an initial distribution, but only to identify a state or node, since the starting distribution can be determined from the context. By using the transition probability matrix \( P \), the vector of state probabilities of the RW after \( n \) steps is

\[ w^{(n)} = w^{(0)} P^n, \]  

(5)

since the consecutive steps are independent from each other due to the Markov property. The probability that the random walker will reach its target \( j \) for the first time at iteration \( n \) (the First Passage Time) is a product of probabilities

\[ p_{ij}^{(n)} = w_j^{(n)} \prod_{l=1}^{n-1} (1 - w_j^{(l)}), \]  

(6)

where the product term incorporates the probabilities to miss the target \( j \) at all previous steps. Then the MFPT from the initial distribution \( w^{(0)} \) to the target \( j \) is

\[ m_{w^{(0)}j} = \sum_{n=1}^{\infty} np_{ij}^{(n)} = \sum_{n=1}^{\infty} n w_j^{(n)} \prod_{l=1}^{n-1} (1 - w_j^{(l)}), \]  

(7)

where we have assumed that the starting state is appropriately encoded in the initial distribution \( w^{(0)} \).

For irreducible Markov chains, or connected networks, the distribution \( w^{(n)} \) converges to the stationary one with a rate of convergence determined by the eigenvalue with the second largest absolute value \( \lambda^* = \max |\lambda| : \lambda \neq 1 \) of the transition matrix \( P \) [24]. Thus, in the products appearing in the MFPT expression \( (7) \) only the low order multipliers \( 1 - w_j^{(l)} \) (with small \( l \)) will contain probabilities \( w_j^{(l)} \) that are significantly different from the stationary ones. For chains and networks for which \( \lambda^* \) greatly differs from unity, the convergence establishes quickly and one can approximately take that all probabilities are equal to the stationary value \( w_j^{(l)} \approx w_j \) and obtain

\[ m_{w^{(0)}j} \approx \sum_{n=1}^{\infty} n w_j \prod_{l=1}^{n-1} (1 - w_j) = w_j \sum_{n=1}^{\infty} n(1 - w_j)^{n-1}. \]  

(8)

After summing up the series the last expression simplifies to

\[ m_{w^{(0)}j} \approx \frac{1}{w_j}. \]  

(9)

We should note that for some nodes this is a crude approximation since it is clear that it is easier to find the target by random walker which starts from the neighborhood, than from some distant nodes. However, since for majority of the starting nodes the MFPT would be nearly equal to the approximate result \( (9) \), from \( (11) \) one can obtain

\[ g_j \approx 1/w_j. \]  

(10)

For undirected complex networks with \( L \) links, the stationary distribution is \( w_j = k_j/2L \) and thus

\[ g_j \approx m_{w^{(0)}j} \approx \frac{2L}{k_j}. \]  

(11)

where \( k_j \) is the degree of node \( j \) [20]. It is easy to understand such a dependence on the degree \( k_j \), since nodes with more neighbors are easier to be reached than their less connected peers. For directed networks simple result as \( (11) \) does not exist and one should calculate the stationary distribution and MFPTs by other methods, like the one presented in the previous subsection.

When the rate of convergence towards the stationary distribution is slow, one cannot safely apply the above approximations and should use distributions calculated for every iteration, which results in the general expression for the MFPT given by \( (7) \). However, one can draw another conclusion about searching in complex networks on the base of their connectedness. When the network is weakly connected, which means that the distance between pairs of nodes is typically large, then for many start-target pairs, large number of steps \( n_c \) would be needed to obtain a nonzero probability to reach the target from the starting node. Clearly, \( n_c \) equals the length of the shortest path between the pair. In the steps before
the probabilities of missing the target have value 1 in the products in equation (12) and as such they appear in every summand, which makes every term in the sum larger as compared to the better connected networks and consequently results in larger MFPTs.

As the GMFPT is feature of a state, or node taken as target, obtained by averaging over different starting points, one can average across GMFPTs for all states and get a property of the whole chain or network which was introduced as Graph MFPT (GrMFPT) [23]. Thus, the GrMFPT for well connected networks (with a good convergence rate) would be

\[ G = \frac{1}{N} \sum_{i=1}^{N} g_i \approx \frac{1}{N} \sum_{i=1}^{N} \frac{1}{w_i}, \quad (12) \]

or for undirected complex networks

\[ G \approx \frac{2/l}{N} \sum_{i=1}^{N} \frac{1}{k_i}. \quad (13) \]

This expression leads to the conclusion that among networks with the same convergence rate, searching is faster in those with a stationary distribution \( w \) closer to the flat one \( w_i = 1/N \). Accordingly, networks with more evenly distributed degrees have smaller \( G \) and thus searching in them is easier. If one could play with the topology she/he should aim at achieving this by rewiring or adding/removing links without intervening into the transition rules. Otherwise, when the topology is fixed, one could try to improve the searching of the less connected nodes by sacrificing the reachability of the better connected nodes, which is explored in our study.

III. BIASED RANDOM WALK MODELS

A. Inverse degree biasing in undirected networks

In the generic version of the RW on complex networks the hubs have a higher frequency of visits by perpetual walker because having more neighbors implies higher probability to arrive at than their less connected peers. Intuitively, the chances of reaching nodes could be made more fair by decreasing the transition probabilities towards the more connected nodes, while increasing it for the others. Particularly, one could take the probability to move from node \( i \) to its neighbor \( j \) to be inversely proportional to its degree

\[ p_{ij} \sim \frac{1}{k_j} = \frac{1}{\sum_{l \in N_j} A_{jl}}, \quad (14) \]

where \( N_j \) is the set of neighbors of node \( j \). The probability is obtained from the normalization condition

\[ \sum_{j \in N_i} p_{ij} = 1, \quad (15) \]

which leads to transition probabilities

\[ p_{ij} = \frac{1/k_j}{\sum_{l \in N_j} 1/k_l}. \quad (16) \]

This approach allows for an analytical study since one has the transition probabilities – the probability of visiting a node from the previous one. We should stress that this kind of biasing of RW was studied in a more general setting for analysis of transit times between pairs of nodes in [21], where the authors obtained similar conclusions, by using ensemble averaging over different realizations of networks, instead of observing a single network as it is derived here. Also, in a more general setting when the transition probability to the target is taken to be proportional to the power of its degree \( k_j^\alpha \), intensive numerical experiments have shown that best performance for uncorrelated networks is obtained for \( \alpha = -1 \) [22].

Since the intuition is that biasing the RW to avoid the hubs should lead to optimizing the searching time, it could be also expected that in such a case the invariant density of visits to the states approaches the constant \( 1/N \). We will show that this approaching is really happening under certain circumstances. First, define node-centric, local average of the reciprocal of degrees of the neighbors as

\[ \langle 1/k \rangle_i = \frac{1}{k_i} \sum_{l \in N_i} 1/k_l \quad (17) \]

where the subscript \( i \) in the average denotes that it is calculated only over the set of the neighbors of the node \( i \). Then the normalization sum in Eq. (10) can be expressed through the local average as

\[ \sum_{l \in N_i} 1/k_l = k_i \langle 1/k \rangle_i. \quad (18) \]

Now, consider networks where most of the nodes are well connected, which technically means \( k_i \gg 1 \). Then, for uncorrelated networks, or those where the degree of any node is independent on the degrees of its neighbors, the local average can be approximated with the network average

\[ \langle 1/k \rangle_i \approx \langle 1/k \rangle = \frac{1}{N} \sum_{j=1}^{N} 1/k_j \quad (19) \]

Then, the normalization sum appearing in the denominator in (10) can be conveniently expressed through the network average as

\[ \sum_{l \in N_i} 1/k_l \approx k_i \langle 1/k \rangle. \quad (20) \]

With this approximation the stationary distribution satisfies the set of equations

\[ w_j = \sum_{i \in N_j} \frac{1/k_j}{\langle 1/k \rangle_i} w_i = \frac{1/k_j}{\langle 1/k \rangle_i} \sum_{i \in N_j} w_i/k_i. \quad (21) \]
Now, if one assumes furthermore that the invariant density is the constant one \( w_i = 1/N \), and apply the same approximation of the sum of inverses of degrees the last equation will become identity as follows

\[
\frac{1}{N} = \frac{1}{(1/k)N} \sum_{i \in N_i} \frac{1}{k_i} \approx \frac{1}{(1/k)N} k_j (1/k) = \frac{1}{N}.
\]

(22)

Thus, we have shown that inverse degree biasing of the RW, for networks with good connectivity leads to approximately constant invariant density of visiting frequency. This furthermore implies that GrMFPT would approach to what we think is the optimal one \( G \approx N \). This result is close to the exact one for a fully connected network where \( G = N \). To obtain the last expression one should first observe that the state probabilities are constant \( w_i^{(in)} = 1/N \) and apply equation (19). It is also clear that when the nodes with small degree dominate in the network, the approximation above is cruder and the stationary distribution is much different from the uniform one. Then the searching is slower and it takes more than \( N \) steps on average to find the target.

It seems that the limit of searching propensity of any undirected connected network is \( G \geq N \). However, there is deterministic strategy that is twice faster and which holds for graphs that have a Hamiltonian cycle. It is a path passing through all nodes and visiting each node only once. We emphasize here that determination whether a path passing through all nodes and visiting each node only once. This furthermore implies that GrMFPT would approach an approximately constant invariant density of visiting frequency.

This section, by using biasing with inverse of outdegrees, networks with good connectivity leads to approximately constant invariant density of visiting frequency.

\[
\langle 1/k \rangle^{in} = \frac{1}{N} \sum_{j=1}^{N} 1/k_j^{in}.
\]

(25)

The last approximation is obtained with the same reasoning as the one for the undirected networks (refer to Eqs. (19) and (20)). The stationary distribution satisfies equation similar to that for the undirected networks

\[
w_j = \sum_{i \in N_j^{out}} \frac{1}{k_i^{out}} \langle 1/k \rangle^{in} w_i = \frac{1}{(1/k)^{in}} \sum_{i \in N_j^{out}} \frac{w_i}{k_i^{out}}.
\]

(26)

If one assumes that the invariant density is constant \( w_i = 1/N \), then from Eq. (26) one would have

\[
\frac{1}{N} \approx \frac{1}{N} \langle 1/k \rangle^{in} \sum_{i \in N_j^{out}} \frac{1}{k_i^{out}}.
\]

(27)

Now, for networks where the direction of the links is independent on the degree of nodes, the averages of reciprocals of indegrees and outdegrees would be nearly the same

\[
\langle 1/k \rangle^{in} \approx \langle 1/k \rangle^{out}.
\]

(28)

For networks where most of the nodes have many incoming and outgoing links, one can make the following approximation

\[
\sum_{i \in N_j^{out}} \frac{1}{k_i^{out}} \approx k_j^{out} \langle 1/k \rangle^{out} \approx k_j^{in} \langle 1/k \rangle^{in}.
\]

(29)

Plugging the last approximation in the stationary density equation (27), one will see that it is identity. Now, one can use the simplified expression for MFPT (11) and show that \( G \approx N \). We note that, although for directed networks with good connectivity the GrMFPT parallels for the undirected, similar result for less connected networks which depend on the degrees like \( k_i^{out} \) does not exist so far. We remind that it is due to the fact that the relationship between the invariant density \( w_i \) and the degree of a node \( k_i \) for directed networks is unknown. Also, we should mention that although network averages of the reciprocals of in- and outdegrees are nearly equal, the biasing inverse to the outdegrees does not improve the searching. The reason for that is the fact that the sum of inverse of degrees (29) is always proportional to the indegree of the node \( j \) because it accounts for neighbors pointing to the node \( j \). In repeating the analysis in this section, by using biasing with inverse of outdegrees, one can verify that the stationary density condition like (27) is not satisfied.

**B. Inverse degree biasing in directed networks**

The same approximations for the invariant density can be applied for directed networks as well, but with additional condition. First, note that by following the intuition that improvement of GrMFPT can be obtained by equalizing the chances of visiting the nodes, one should try by assigning higher probabilities to nodes with smaller indegree. Thus, the transition probability of the biased random walker would be taken as

\[
p_{ij} = \frac{1/k_i^{in}}{\sum_{l \in N_i^{out}} 1/k_l^{in}},
\]

(23)

where \( N_i^{out} \) denotes the set of neighbors of the node \( i \) where it points to. Furthermore, for networks with good connectivity, one can make similar approximation

\[
\sum_{l \in N_i^{out}} 1/k_l^{in} \approx k_i^{out} \langle 1/k \rangle^{in},
\]

(24)

where \( \langle 1/k \rangle^{in} \) denotes network average of the reciprocal of indegrees

\[
\langle 1/k \rangle^{in} = \frac{1}{N} \sum_{j=1}^{N} 1/k_j^{in}.
\]

C. Biasing with two-hop memory

In the previous subsection, the biasing of RW by taking transition probabilities proportional to the inverse of
node degrees aims at increasing the visiting chances to the less connected nodes. This approach is based only on the nearest neighborhood properties – the first (one hop away) neighbors degrees only. Another strategy could be created by trying to equalize the probabilities to reach the second (two hops away) neighbors, since their number is much larger than the number of first neighbors.

One intuitive way to make such preference is as follows. Assume that at the previous step the walker was at node \( i \), from where it has jumped to the node \( j \), and in the next step it should visit some node \( l \) from the set of neighbors of \( j \). Denote the number of two-hop paths from node \( i \) to \( l \) with \( B(i, l) \). The matrix with elements \( B(i, l) \) is the square of the neighborhood matrix \( B = A^2 \). Then the probability to visit node \( l \) after being at nodes \( i \) and \( j \) in the previous two steps could be conveniently taken as

\[
pr_{i,j}(l) = \frac{1}{\sum_{m \in N_j} B(i,m)} B(j,l) \sum_{m \in N_j} \frac{1}{B(i,m)}.
\]

(30)

In the last equation the sum in the denominator is used for normalization of the probabilities. This formula assigns a larger weight to nodes \( l \) which have less alternative paths to be reached from node \( i \), i.e. those with smaller \( B(i, l) \). In this way, the probability to visit a node of that kind from \( i \) in two steps will be increased, and closer to that of nodes which are accessible from \( i \) in two steps through more alternative paths. Since for undirected networks every node is a second neighbor to itself, there is a chance to return to the same node \( i \). However, \( B(i, i) = k_i \) and the probability \( pr_{i,j}(i) \) is the lowest within all \( pr_{i,j}(l) \) and thus immediate returning is disfavored. Because the probability to jump to a neighbor as is defined in the last equation depends also on the previously visited node (besides the current) the Markov property is not satisfied, and the formulas given above cannot be applied for determination of MFPT. One can try to make a related Markov model where the state would be identified with the pair of two consecutively visited nodes \((i, j)\) but this approach would be ineffective for large networks since the dimension of such a transition matrix would be of order \( N^2 \times N^2 \). Correspondingly, this strategy should be preferably analyzed by numerical simulations of random paths within the network, which is what we have done here.

IV. NUMERICAL RESULTS

The verification of the searching improvement potential of both proposed random walk algorithms was based on theoretical calculations and numerical experiments. For the classical and inverse degree biased RW the results were obtained analytically from respective transition matrices. The numerical experiments for determination of the MFPT of the two-hop memory based algorithm were performed by simulation of RW. For computational reasons, such experiments were made for 1000 pairs of randomly selected starting and target node for each network. All examined networks have \( N = 1000 \) nodes, but we varied the average node degree to see how it affects the search. Both the analytical and the numerical results were averaged over 100 network instances obtained with the same parameter values.

The results are obtained by averaging of 100 networks with certain \( m \). We have applied such averaging with 100 networks for all other experiments as well.

We studied the most typical types of networks like random graphs, scale-free and small-world networks. For generating such graphs we used algorithms from the NetworkX library in Python which allow construction of the three graph types with given parameter values \([27]\). The random graphs are complex networks created according to the Erdős-Rényi model where every pair of nodes \( i \) and \( j \) is connected with some predefined probability \( p \), which appears as a parameter of the graph together with the number of nodes \( N \) \([1]\). If the probability \( p \) is large enough then the obtained graph would very likely be connected – there will be a path between each pair of nodes. The scale free networks were generated using the Barabási-Albert model which sequentially builds the network by adding nodes one by one \([2]\). As a seed network we use a network of \( m_0 \) nodes without edges and every newly added node forms \( m \) links with the existing network \([28]\). A preferential attachment is employed as the probability to connect to an existing node is taken to be proportional to its degree. The small world networks were built following the Watts-Strogatz model \([2]\). It starts with a regular ring lattice network with \( N \) nodes each connected with \( n \) neighbors, and then randomly rewire the links with some probability \( p \).

As elaborated in the preceding section, when the transition probabilities are given explicitly, the MFPT and GrMFPT can be determined analytically from the transition matrix. In Figure 14 we show the obtained GrMFPTs for the classical random walk (analytically), the biased random walks with a transition probability proportional to the inverse degree (analytically) and the memory based one (numerically) over scale-free networks. The horizontal axis represents the average degree \( \langle k \rangle \) which is approximately \( 2m \), where \( m \) is varied in the range \([2, 15]\). The seed network is composed of \( m_0 = m \) nodes without edges. As we previously mentioned, the results were obtained by averaging of 100 networks with certain \( m \). It can be noticed that when the average node degree is large enough, the two biasing alternatives provide smaller GrMFPT than the GRW, which also approaches the optimal value \( N \). When the majority of the nodes have few neighbors (four or six in this case) the inverse degree biasing worsens the searching in the network. We think that this is probably due to the much higher preference of the weakly connected nodes and consequently decreasing the chances of exploring new areas in the network by avoiding the hubs. The two-hop memory approach is always better than the classical one since it involves memory and biasing which intents to equalize the chances of visiting the second neighbors of a given...
node. Interestingly, for $\langle k \rangle = 14$ and above the simpler inverse degree approach is slightly better than the two-hop memory. One can notice that all curves decrease asymptotically towards the value corresponding to the number of nodes, which arguably is the minimal possible value for the GrMFPT.

The biasing brings search improvement for the purely random Erdős-Rényi graphs also, as it is shown in Figure [11]. We generated per 100 network instances with $N = 1000$ nodes for different average node degree $\langle k \rangle$ by varying the link existence probability $p \in [0.005, 0.015]$. We note that it is very difficult to generate connected Erdős-Rényi graphs of that size for lower values of $p$. As it can be seen the inverse degree biasing always gives lower average GrMFPT than the classical random walk, while the two-hop memory outperforms them both.

In Figure [12], we show how the biasing affects the random walk in Watts-Strogatz networks, where the rewiring probability is 0.2. Unlike for the other network types under study, the inverse degree biasing does not improve the GrMFPT. This is probably due to the smaller degree variability in this kind of networks. On the other hand, the two-hop memory approach still reduces the GrMFPT, as it was the case for the other network types.

We have also made numerical experiments, in order to see whether a mechanism behind the search improvement in complex networks is the flattening of the stationary distribution of the visiting frequency by perpetual random walker. A convenient quantity for estimating the deviation of one distribution from another is the Kullback-Leibler (KL) divergence [29]. In the case when one has two discrete distributions $P(i)$ and $Q(i)$, it is defined as

$$D_{KL}(P||Q) = \sum_i P(i) \log \frac{P(i)}{Q(i)}. \quad (31)$$

One can notice from the definition that this is asymmetric quantity, $D_{KL}(P||Q) \neq D_{KL}(Q||P)$, and within the definition provided above, $P$ has the role of the prior, or the distribution with which we compare. In our case it is the constant $P(i) = 1/N$. This divergence vanishes when the two distributions coincide. In Figure [2a] is shown the KL divergence between the uniform density and those for the classical and inverse degree biased RW, in BA networks. As can be noticed, the biasing produces invariant density that is closer to the flat one, than the classical approach does. Also, the larger the average degree is, the approximation of the invariant density with the uniform one is more correct, as the theoretical analysis in previous sections suggests. However, even though for networks with smaller average degree the biasing makes the distribution closer to the uniform, searching is slower than for the GRW. One of the reasons for such a result might be the fact that approximation of the MFPT which expresses it through the stationary density (10) is not valid.

Similarly, Figure [2b] shows the KL divergence between the uniform density and the classical and inverse degree approaches in ER networks. Once again, the inverse degree biasing yields a density that is closer to the uniform one than the classical random walk, which is probably the reason for the lower GrMFPT obtained in Figure [3].

FIG. 1: GrMFPT in (a) BA, (b) ER, and (c) WS networks with different average node degree $\langle k \rangle$ for the three cases: GRW (red circles), inverse degree (blue squares) and two-hop memory (green triangles).
We also tested the searching improvement in directed networks. In Figure 3a are shown the GrMFPT for directed ER network by GRW and inverse indegree biasing. It even outperforms the two-hop memory based searching which appeared best in undirected networks. As expected biasing based on inverse of outdegree performs slower than the GRW (results are not shown).

The flattening of the invariant density is the reason for search improvement in directed networks as well. We have verified that, as expected, for well connected networks when biasing of random walk is based on inverse of indegrees, the invariant density is closer to the constant, than that of the GRW. In Figure 3b are shown the KL divergence of the GRW on directed ER networks with two biasing alternatives: one based on inverse of indegrees, and another on the oudegrees. The results are in concordance with the theoretical analysis presented in section III B.

V. CONCLUSIONS

In this work we studied the potential for improvement of searching in complex networks by applying biasing of the classical random walk. We have examined two approaches that avoid the hubs by making the transition probabilities proportional to the inverse of the degree of the nodes, or by accounting for the two-hop-paths between the nodes. It was obtained that for very large and well connected networks, the inverse-degree biasing procedure is approaching the optimal searching which is characterized by the average number of steps needed to find the target node being slightly larger than the number of nodes in the network. This optimal achievement is due to the flattening towards the constant of the invariant density of visits of the nodes by an infinite ran-
dom walker. The two-hop memory approach has shown searching improvement as well, and although being more complex, it is promising since it shows better performance than the classical random walk even for networks where the majority of nodes have a small degree. Finally, we have shown that the inverse degree biasing based on indegree, leads to nearly optimal random search in directed networks as well.

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