Fundamental String Solutions in Open String Field Theories

Yoji Michishita *

Center for Theoretical Physics
Massachusetts Institute of Technology
Cambridge MA 02142 USA

November, 2005

Abstract

In Witten’s open cubic bosonic string field theory and Berkovits’ superstring field theory we investigate solutions of the equations of motion with appropriate source terms, which correspond to Callan-Maldacena solution in Born-Infeld theory representing fundamental strings ending on the D-branes. The solutions are given in order by order manner, and we show some full order properties in the sense of $\alpha'$-expansion. In superstring case we show that the solution is 1/2 BPS in full order.

*michishi@lns.mit.edu
1 Introduction and Summary

In Witten’s cubic open string field theory[1] and its extension to superstring such as Berkovits’ superstring field theory[2], it is very difficult to construct solutions with coordinate dependence. This is because string field theory is nonlocal and contains infinitely many derivatives. It prevents us from investigating behavior of higher modes and full order properties. (We consider only classical theory and do not consider string loop correction. Therefore throughout this paper “full order” means exactness in the sense of $\alpha'$-expansion.)

In this paper we investigate an example of such solutions of the equations of motion with appropriate source terms, of which we can derive some full order properties: string field theory counterpart of Callan-Maldacena solution[3]. (For a related topic see [4].) This solution represents configuration of fundamental strings emanating from the D-brane. Since it is also a solution of free U(1) gauge theory, we expect that we can construct the string field theory solutions in order by order manner, starting from the linearized equation and introducing higher order source terms. In section 2 we construct the solution in Witten’s string field theory and see that it has the following properties:

- The coefficient of the massless component $A_\mu$ is equal to the gauge field $\tilde{A}_\mu$ in the effective action with full order correction in $\alpha'$.
- The solution has no tachyon component, and the massless component has no higher order correction.
- Although we have no proof, we give a convincing argument that massive modes have no singularity unlike the massless component.
- Energy-momentum tensor given in [5] has no contribution from massive modes, and is equal to that of free U(1) gauge theory.

In section 4 we construct the solution in Berkovits’ superstring field theory and see that it has almost the same properties as the bosonic one. Moreover we show that it is 1/2 BPS in full order.
2 Solution in Bosonic String Field Theory

Let us consider one single D$p$-brane in the flat space. The bosonic quadratic part of its effective action, in both bosonic and superstring theory, is given by free U(1) gauge theory. Spacetime-filling D-brane action has only gauge field $\tilde{A}_\mu$, and lower dimensional D-brane actions are obtained from it by dropping dependence on coordinates perpendicular to the D-branes. We separate spacetime coordinates $x^\mu$ into $x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1)$, $x^i$ and $x^I$, where $x^0$ and $x^i$ are directions along the D$p$-brane, and $x^1$ and $x^I$ are directions perpendicular to the D$p$-brane. Then $\tilde{A}_1$ and $\tilde{A}_I$ are scalar fields corresponding to $x^1$ and $x^I$ respectively.

Suppose $\tilde{A}_- = 0$, $\tilde{A}_i = 0$, $\tilde{A}_I = 0$, and $\tilde{A}_+ = \tilde{A}_+(x^i)$, then the linearized equation of motion is

$$\sum_i \partial_i \partial_i \tilde{A}_+ = 0.$$  \hspace{1cm} (1)

This is Laplace equation, and “point charge” configurations give solutions:

$$\tilde{A}_+ = \sum_n \frac{c_n}{[\sum_i (x^i - x^i_n)^2]^{p/2}},$$  \hspace{1cm} (2)

where $c_n$ and $x^i_n$ are constants. We assumed $p \geq 3$. For $p = 1$ solutions are sums of segments of linear functions and for $p = 2$ sums of log $\sum_i (x^i - x^i_n)^2$. In these cases momentum expressions (i.e. Fourier transformations) of these solutions require introducing infrared regulators. Since in string field theory we use momentum expression, we do not consider $p = 1$ and 2 in this paper.

For this solution the right hand side of (1) is not actually zero, but a sum of delta function sources. In [3] it has been shown that this configuration represents fundamental strings stretching along $x^1$ direction and ending on the D-brane at $x^i = x^i_n$, and extension of this solution to Born-Infeld theory is again given by (2), without corrections. In this interpretation the presence of the delta function sources is not a problem, because the points $x^i = x^i_n$ are not on the worldvolume of the D-brane (or are regarded to be infinitely far away).

Furthermore in superstring theory this solution is 1/2 supersymmetric, both in linearized U(1) gauge theory [3] and Born-Infeld theory [6].

In fact this solution is an $\alpha'$-exact solution as shown in [7] by computing beta function of the worldsheet sigma model.

Since leading order terms of string field theory action give free U(1) gauge theory, we expect that starting from the solution of (1) we can construct corresponding solutions of string field
equation “order by order”. In this section we investigate the solution in Witten’s cubic bosonic string field theory.

In the bosonic string field theory the equation of motion is

\[ Q\Phi + \Phi^2 = 0. \]  

(3)

Of course the right hand side is not actually zero. To get a right solution we have to put a source term which we will call \( \Delta_n \).

The solution is constructed by expanding \( \Phi \) in some parameter \( g \):

\[ \Phi = g\Phi_0 + g^2\Phi_1 + g^3\Phi_2 + \ldots. \]

(4)

The equation of motion is decomposed into contributions from each order in \( g \):

\[ \Delta_0 = Q\Phi_0, \]

(5)

\[ \Delta_1 = Q\Phi_1 + \Phi_0^2, \]

(6)

\[ \Delta_2 = Q\Phi_2 + \Phi_0\Phi_1 + \Phi_1\Phi_0, \]

(7)

\[ \vdots \]

\[ \Delta_n = Q\Phi_n + \sum_{m=0}^{n-1} \Phi_m\Phi_{n-m-1}, \]

(8)

\[ \vdots \]

Massless part of the lowest order equation (5) is equivalent to that of free \( U(1) \) gauge theory with source terms. So we take the following \( \Phi_0 \) which corresponds to (2):

\[ \Phi_0 = \int \frac{dp|k|}{(2\pi)^d} A_+(k_i)c\partial X^i e^{ik_iX^i}, \]

(9)

where coordinate expression of \( A_+(k_i) \) which is given by \( A_+(x^i) = \int \frac{dp|k|}{(2\pi)^d} A_+(k_i)e^{ik_ix^i} \) satisfies Laplace equation with delta function source terms. Then the string field source term \( \Delta_0 \) is

\[ \Delta_0 = -\alpha' \int \frac{dp|k|}{(2\pi)^d} k^2 A_+(k_i)c\partial c\partial X^i e^{ik_iX^i}. \]

(10)

\( \Phi_0 \) satisfies Siegel gauge condition: \( b_0\Phi_0 = 0 \). We require that at each order this condition is satisfied: \( b_0\Phi_n = 0 \). In addition we require that \( \Delta_n \) with \( n \geq 1 \) also satisfy this condition: \( b_0\Delta_n = 0 \). \( \Delta_0 \) does not satisfy it. This means that \( \Delta_0 \) is the only source for physical components, and \( \Delta_n \) with \( n \geq 1 \) are for unphysical components. This is desirable because, when we eliminate all unphysical massive modes by a gauge fixing condition and solve all equations
for physical massive modes, we have to obtain equation of motion for massless modes with a simple source term to have a solution corresponding to Callan-Maldacena solution.

By acting \(b_0\) to the equations of motion and noticing that \(b_0\Phi_n = L_0\Phi_n\), we obtain

\[
\Phi_1 = -\frac{b_0}{L_0}\Phi_0^2, \tag{11}
\]

\[
\Phi_2 = -\frac{b_0}{L_0}(\Phi_0\Phi_1 + \Phi_1\Phi_0)
= \frac{b_0}{L_0}\left(\Phi_0\frac{b_0}{L_0}(\Phi_0^2) + \frac{b_0}{L_0}(\Phi_0^2)\Phi_0\right), \tag{12}
\]

\[
\vdots
\]

\[
\Phi_n = -\frac{b_0}{L_0}\sum_{m=0}^{n-1} \Phi_m\Phi_{n-m-1}, \tag{13}
\]

In this manner \(\Phi_n\) can be expressed by \(\frac{b_0}{L_0}\) and \((n+1)\) copies of \(\Phi_0\). Since \(b_0\) projects out some components of string fields, we have to check if there is more information extracted from the equations of motion by plugging the above solution back into them:

\[
\Delta_n = Q\Phi_n + \sum_{m=0}^{n-1} \Phi_m\Phi_{n-m-1}
= -Q\frac{b_0}{L_0}\sum_{m=0}^{n-1} \Phi_m\Phi_{n-m-1} + \sum_{m=0}^{n-1} \Phi_m\Phi_{n-m-1}
= \frac{b_0}{L_0}Q\sum_{m=0}^{n-1} \Phi_m\Phi_{n-m-1}. \tag{14}
\]

This should be regarded as determining \(\Delta_n\) by lower order solutions. Notice that if lower order \(\Phi_m\) in the right hand side of the above equation satisfy equations of motion without lower order source terms, then \(\Delta_n\) vanishes:

\[
\Delta_n = \frac{b_0}{L_0}\sum_{m=0}^{n-1} ((Q\Phi_m)\Phi_{n-m-1} - \Phi_m(Q\Phi_{n-m-1}))
= -\frac{b_0}{L_0}\left(\sum_{m=1}^{n-1} \sum_{l=0}^{m-1} \Phi_l\Phi_{m-l-1}\Phi_{n-m-1} - \sum_{m=0}^{n-2} \sum_{l=0}^{m-2} \Phi_m\Phi_l\Phi_{n-m-l-2}\right)
= -\frac{b_0}{L_0}\left(\sum_{m=0}^{n-1} \sum_{l=0}^{m-1} \Phi_l\Phi_{m-l-1}\Phi_{n-m-1} - \sum_{l=0}^{m+1} \Phi_l\Phi_{m-l-1}\Phi_{n-m-1}\right)
= 0. \tag{15}
\]
For our solution source terms should not be zero, and we obtain

\[ \Delta_n = \frac{b_0}{L_0} \sum_{m=0}^{n-1} [\Delta_m, \Phi_{n-m-1}], \]

which means that higher order source terms are induced by lower order ones.

Obviously, \( \Phi_n \) has no dependence on momenta along \( x^\pm \). In addition, \( \Phi_n \) has the following property: Let \( \omega \) be any of the vertex operators (or Fock space states) which \( \Phi_n \) consists of. Then \( X^I \) part of \( \omega \) is a Virasoro descendant of the unit operator i.e. a state constructed by acting \( L'_{-n} \) \((n \geq 2)\) on \( |0\rangle \), where \( L'_{-n} \) are Virasoro operators of \( X^I \) part. Moreover, \( n_+(\omega) - n_-(\omega) = n + 1 \), where \( n_+ \) is the number of \( \partial^m X^+ \) (or \( \alpha^+_m \)) in \( \omega \), and \( n_- \) is the number of \( \partial^m X^- \) (or \( \alpha^-_m \)) in \( \omega \).

In summary, matter part of \( \omega \) is in the following form:

\[ \prod_{l=1}^{n_+} \alpha^+_{p_l} \prod_{l=1}^{n_-} \alpha^-_{q_l} \prod_{l} L'_{-u_l} \prod_{l} \alpha^{-i_{u_l}} |k^l\rangle \quad (n_+ - n_- = n + 1, p_l, q_l, u_l \geq 1, t \geq 2). \]

The structure of \( X^I \) part represents symmetry in \( X^I \) directions.

This can be proven by induction as follows. For \( n = 0 \) this is obvious. Suppose \( n \geq 0 \). We take orthonormal basis of the Fock space \( \{|\phi_r\rangle\} \) and its conjugate \( \{\langle \phi_c|\} \). These satisfy \( \langle \phi_c|\phi_r\rangle = \delta_{rs} \). Corresponding vertex operators are denoted by \( \phi_r \) and \( \phi_c \) respectively. Coefficient of \( |\phi_r\rangle \) in the expansion of \( \Phi_n \) by \( \{|\phi_r\rangle\} \) is given by \( \langle \phi_c|\Phi_n\rangle \):

\[ \langle \phi_c|\Phi_n\rangle = \langle \phi_c| - \frac{b_0}{L_0} \sum_{m=0}^{n-1} \Phi_m \Phi_{n-m-1} \rangle \]

\[ = - \left( \frac{b_0}{L_0} \phi_c \sum_{m=0}^{n-1} \Phi_m \Phi_{n-m-1} \right). \]

First we concentrate on \( X^\pm \) sector. Since \( b_0 \) affects only on ghost part and \( L_0 \) gives a numerical factor for each level, we can neglect \( \frac{b_0}{L_0} \). By the assumption of the induction, \( n_+(\Phi_m) - n_-(\Phi_m) \) is \( m + 1 \) and \( n_+(\Phi_{n-m-1}) - n_-(\Phi_{n-m-1}) \) is \( n - m \). There are two processes which change the number of \( X^+ \) and \( X^- \): contraction and conformal transformations in the star product. Since \( X^+ \) has nonzero contraction only with \( X^- \) and vice versa, both processes preserve the difference of these numbers, and the total number of \( X^+ \) and \( X^- \) in the correlator should be equal for nonzero contribution. Therefore \( n_+(\phi_r^c) - n_-(\phi_r^c) \) should be \( -n - 1 \). This means that \( n_+(\phi_r) - n_-(\phi_r) = n + 1 \).

Next we consider \( X^I \) sector. By the assumption of the induction, both \( \Phi_m \) and \( \Phi_{n-m-1} \) are Virasoro descendants of the unit operator. If \( \phi_r^c \) is a descendant of a nontrivial primary field
by using the well-known procedure relating a correlator with worldsheet energy-momentum tensors to ones without it, the correlator reduces to one point function of \( \lambda \), which vanishes because of its nonzero conformal dimension. This means that \( \phi_r \) consists of Virasoro descendants of the unit operator.

Ghost part of \( \Phi_n \) can also be restricted further as is explained in [8].

An immediate consequence of the above fact on the number of \( X^\pm \) is that each coefficient of Fock space state in the solution \( \Phi \) receives contribution from only one \( \Phi_n \). (Here we regard states consisting of the same oscillators with different spacetime indices as different states.) In particular, the coefficient of the massless vertex operator \( c\partial X^\mu e^{ik\mu X^\mu} \), which is denoted by \( A_\mu \), is never corrected by higher order contribution, and the coefficient of the lowest state, which represents tachyon, is zero in full order. In addition, we see that the inverses of \( L_0 \) in the expression of \( \Phi_n \) with \( n \geq 1 \) do not cause any problem, because only massless and tachyon components, which is absent in \( \Phi_n \) with \( n \geq 1 \), are problematic.

We can easily see that \( \Delta_n \) also have the same property as \( \Phi_n \) by the same argument: Matter part of \( \Delta_n \) are in the form of (17), there is no more source for massless components than \( \Delta_0 \), and inverses of \( L_0 \) are well defined.

In general, \( A_\mu \) is different from the gauge field \( \tilde{A}_\mu \) in the effective action except at the leading order, because its gauge transformation takes different form from the standard one. They are connected by some field redefinition. In [9] it has been explained how to compute this field redefinition order by order. However, for our solution \( A_\mu \) is equal to \( \tilde{A}_\mu \). This is because higher order terms of the field redefinition contain two or more \( A_\mu \) and possibly derivatives, and since \( \tilde{A}_\mu \) has only one spacetime index, superfluous indices should be contracted with each other. Therefore higher order terms contain \( A_\mu A^\mu \) or \( \partial_\mu A^\mu \), which vanish for our solution. Hence our \( A_\mu \) is also an exact solution of the effective action. This gives another proof of the fact shown in [7].

## 3 Behavior of massive components

In this section we investigate how coefficients of massive states in our solution in the previous section behave by computing those of first and second massive states coming from \( \Phi_1 \) and \( \Phi_2 \), and see more full order properties suggested by it.

First we compute first massive components. It can be easily seen that \( V_1(k) = c\partial X^+ \partial X^+ e^{ik \xi X^i} \).
is the only nonzero component and it is from $\Phi_1$. Since its conjugate operator is $U_1(k) = -\frac{2}{(\alpha')^2}c\partial c\partial X_+ \partial X_+ e^{-ik_1x^1}$, the component is given by the following:

$$\int \frac{dp}{(2\pi)^p} V_1(k) \langle U_1(k) | \Phi_1 \rangle$$

$$= \int \frac{dp k(2)}{(2\pi)^p} \frac{dp k(3)}{(2\pi)^p} V_1(k(2) + k(3))$$

$$\times \left( \frac{4}{3\sqrt{3}} \right)^{2a'(k(2)^2 + k(3)^2 + k(2) \cdot k(3)) + 1} \frac{1}{\alpha'(k(2) + k(3))^2 + 1} A_+(k(2)) A_+(k(3)).$$ (19)

We see that the factor $\left( \frac{4}{3\sqrt{3}} \right)^{2a'(k(2)^2 + k(3)^2 + k(2) \cdot k(3))}$ makes the above integral convergent, since $\frac{4}{3\sqrt{3}} < 1$ and $k(2)^2 + k(3)^2 + k(2) \cdot k(3) = (k(2) + \frac{1}{2}k(3))^2 + \frac{3}{4}k(3)^2$ becomes large as $k(2), k(3) \to \infty$.

In the case of one-center solution $A_+ \propto 1/k^2$, we plot $F_p(r)$, coordinate expression of the above function, defined as follows:

$$F_p(r) = (\alpha')^{p-2} \int \frac{dp k(2)}{(2\pi)^p} \frac{dp k(3)}{(2\pi)^p} e^{i(k(2)+k(3)) \cdot x} \left( \frac{4}{3\sqrt{3}} \right)^{2a'(k(2)^2 + k(3)^2 + k(2) \cdot k(3))}$$

$$\times \frac{1}{\alpha'(k(2) + k(3))^2 + 1} \frac{1}{k(2)^2 k(3)^2},$$ (20)

where $r = \sqrt{(x^i)^2 / \alpha'}$. Figure 1 is the profile of $F_3(r)$. Note that $F_p$ is real, and depends only on $r$ because of the invariance under rotation of $x^i$.

![Figure 1: $F_3(r)$](image-url)
nonlocality of the string field product represented by the factor \( \left( \frac{4}{3\sqrt{3}} \right)^{2\alpha'(k^2_{(2)} + k^2_{(3)} + k^2_{(4)})} \). The nonlocality smears off the singularity. We will see this also happens in the calculation of higher contribution.

Next we compute a coefficient of a second massive state \( V_2(k) = c\partial X^+\partial X^+\partial X^+e^{ik_iX^i} \). This is from \( \Phi_2 \) and other nonzero second massive states are in \( \Phi_1 \), which can be computed similarly to \( V_1(k) \). The operator conjugate to \( V_2(k) \) is \( U_2(k) = \frac{4}{3(\alpha')^3}c\partial c\partial X^+\partial X^+e^{-ik_iX^i} \). Therefore the component is

\[
\int \frac{dp_1}{(2\pi)^p} V_2(k) \langle U_2(k) | \Phi_2 \rangle = \int \frac{dp_1}{(2\pi)^p} V_2(k) \left\langle -\frac{b_0}{L_0} U_2(k) | \Phi_1 \Phi_0 + \Phi_0 \Phi_1 \right\rangle = \int \frac{dp_1}{(2\pi)^p} V_2(k) \left( \frac{4}{3(\alpha')^3} \right) \frac{1}{\alpha'k^2 + 1} \langle U_2'(k) | \Phi_0 + \Phi_0 * U_2'(k) | \Phi_1 \rangle
\]

\[
= \int \frac{dp_1}{(2\pi)^p} V_2(k) \left( \frac{1}{\alpha'k^2 + 1} \right) \left\langle U_2'(k) | \Phi_0 + \Phi_0 * U_2'(k) \right\rangle \left( \frac{b_0}{L_0} \right) \Phi_0^2 \right),
\]

where \( U_2'(k) = c\partial X^+\partial X^+e^{-ik_iX^i} \). This can be computed in the same way as 4-point amplitudes by noticing that \( \frac{b_0}{L_0} \) is the string field propagator. Coefficients of higher \( \Phi_n \) are also given by \( (n + 2) \)-point off-shell amplitudes. This fact was pointed out in [10] in a different context.

Technique for computation of off-shell 4-point amplitudes was developed in [11, 12]. By applying it, we obtain

\[
\left\langle U_2'(k) * \Phi_0 + \Phi_0 * U_2'(k) \left| \frac{b_0}{L_0} \right\rangle \Phi_0^2 \right) = \int \frac{dp_1 k_{(2)}}{(2\pi)^p} \frac{dp_2 k_{(3)}}{(2\pi)^p} \frac{dp_3 k_{(4)}}{(2\pi)^p} \left( \frac{1}{\alpha'k^2 + 1} \right) \left( \frac{2\alpha}{1 + \alpha^2} \right)^{2\alpha'(k^2_{(2)} + k^2_{(3)} + k^2_{(4)})} \left( \frac{1 - \alpha^2}{1 + \alpha^2} \right)^{2\alpha'(k^2_{(2)} + k^2_{(3)})^2} \left( \frac{1 - \alpha^2}{1 + \alpha^2} \right)^{2\alpha'(k^2_{(2)} + k^2_{(3)})^2} \left( \frac{1 - \alpha^2}{1 + \alpha^2} \right)^{2\alpha'(k^2_{(2)} + k^2_{(3)})^2} \left( \frac{1 - \alpha^2}{1 + \alpha^2} \right)^{2\alpha'(k^2_{(2)} + k^2_{(3)})^2},
\]

where \( \kappa(\alpha) \) is defined in (A.8) in the appendix.

Let us compare the above integral with on-shell Veneziano amplitude. In the computation of Veneziano amplitude we encounter the following integral:

\[
\int_0^1 dy y^{\alpha'(k^2_{(2)} + k^2_{(3)})^2 - 2}(1 - y)^{\alpha'(k^2_{(3)} + k^2_{(4)})^2 - 2}.
\]
This expression is convergent around $y = 1$ if $\alpha'(k(3)+k(4))^2 > 1$. Divergence at $\alpha'(k(3)+k(4))^2 = 1$ signifies that tachyon mode propagates as an intermediate state. The integral is not well-defined beyond this point, and what we usually do is to replace the integral expression by Beta function which is well-defined except at the poles.

Going back to the expression (22), $1 - y$ corresponds to $\left(\frac{2\alpha}{1+\alpha^2}\right)^2$, and we can see (22) does not have the same problem as (23). This is because $\Phi_1 = -\frac{2\alpha}{L_0} \Phi_0^2$ does not have tachyon and massless components as we have shown earlier and these do not propagate as intermediate states. Therefore we can use the expression of moduli integral in (22) for any values of the momenta.

Then another question is the convergence of the integral of the momenta. Note that $0 \leq \left(\frac{2\alpha}{1+\alpha^2}\right) < 1$ and $0 < \left(\frac{1-\alpha^2}{1+\alpha^2}\right) \leq 1$ in the range of $\alpha$. The equality applies only at the edge of the range. Furthermore in the appendix we show that $0 < \left(\frac{1+\alpha^2}{2}\frac{1}{1-\alpha^2} \kappa(\alpha)\right) \leq 1$. Thus we see that these three factors makes the integral convergent.

The coordinate expression $G_p(r)$ of the above coefficient for one-center case, defined as follows, has the profile shown in Figure 2 for $p = 3$:

$$G_p(r) = (\alpha')^{3p/2-3} \int \frac{dp}{(2\pi)^p} \frac{dp}{(2\pi)^p} \frac{dp}{(2\pi)^p} \frac{1}{\alpha'(k(2)+k(3)+k(4))^2} \frac{1}{k(2)^2} \frac{1}{k(3)^2} \frac{1}{k(4)^2} \left[ e^{i(k(2)+k(3)+k(4)) \cdot r} \frac{1}{\alpha'(k(2)+k(3)+k(4))} \right]$$

$$\times \left[ \frac{8\alpha(1-\alpha^2)}{(1+\alpha^2)^3} \kappa(\alpha)^2 \left( \frac{1 + \alpha^2}{21 - \alpha^2} \kappa(\alpha) \right)^{2 \alpha'(k(2)+k(3)+k(4))^2} \right] \right] (24)$$

Higher $\Phi_n$ have properties similar to $\Phi_1$ and $\Phi_2$ i.e. they are related to $(n+2)$-point off-shell amplitudes and well-defined, and have smooth profiles. The relation to off-shell amplitudes implies that integrals of moduli parameters are well-defined at any values of momenta because $\Phi_n$ do not have tachyon and massless modes, and integrals of momenta are convergent even at $r = 0$ because of the nonlocality. Convergent factors come from the following correlator:

$$\left\langle f_1 \circ (e^{ik(1) \cdot X})(z_1) f_2 \circ (e^{ik(2) \cdot X})(z_2) \cdots f_n \circ (e^{ik(n) \cdot X})(z_n) \right\rangle$$

$$= \prod_i (f_i(z_i))^\alpha k_i^2 \prod_{i \neq j} |f_i(z_i) - f_j(z_j)|^{2 \alpha'(k(i)+k(j))} (2\pi)^p \delta^p \left( \sum_i k(i) \right)$$
\text{where } f_{i}(z) \text{ are conformal transformations appearing in the computation of off-shell amplitudes. Although we have no rigorous proof, we expect that } \sum_{i,j} a_{ij} k_{(i)} \cdot k_{(j)} \text{ is positive for spatial } k_{i} \text{ and works as a convergent factor for integrals of momenta, because any off-shell string amplitude contains this factor and it is highly implausible that this is divergent.}

The same analysis can be applied to } \Delta_{n}: \text{ Although } \Delta_{0} \text{ is a sum of delta functions, } \Delta_{n} \text{ with } n \geq 1 \text{ are not localized to points and have smooth profiles. This is not surprising, because the equation of motion is covariant under gauge transformation, and therefore the source term should also be covariant. So even if the source term is localized to points in some gauge, its gauge transformation is not localized due to the nonlocality of the string star product.}

In [3], in the free U(1) gauge theory it was shown that the coefficient in the gauge field } \tilde{A}_{\mu} \text{ is determined by charge quantization and the energy around the singularity } r = 0 \text{ is equal to the length times string tension.}

In our case the same charge quantization is also applied to } A_{\mu}. \text{ So we expect that massive modes do not contribute to the energy. The fact that massive modes are smooth at } r = 0 \text{ also suggests this. Therefore let us see energy-momentum tensor for our solution. For definiteness we use the energy-momentum tensor } T_{\mu\nu} \text{ given in [5] as Noether current of translation symmetry. Although this tensor itself is not gauge invariant, total energy and momentum computed}
from it are expected to be gauge invariant. * This tensor consists of coefficient fields in the string field and derivatives. Since this has only two spacetime indices \( \mu \) and \( \nu \), superfluous indices should be contracted with each other. We have shown that nonzero component fields have one or more + indices. If they are contracted with – indices in the derivatives, we have vanishing contribution because our solution has no \( x^\pm \) dependence. If they are contracted with – indices of other fields, then the + indices and – indices are paired, and the excess of + indices should be \( \mu \) and \( \nu \). Therefore difference of the number of + and – index in any nonzero term in \( T_{\mu\nu} \) is equal to or less than two. The only term which satisfies this requirement is \( \partial^i A_+ \partial_i A_+ \), and \( T_{++} \) is the only nonvanishing component of \( T_{\mu\nu} \).

Thus we see that not only the massless modes do not contribute to \( T_{\mu\nu} \), but \( T_{\mu\nu} \) is exactly equal to the energy-momentum tensor of free U(1) gauge theory. Note that the above argument can be applied to any definition of energy-momentum tensor consisting of two or more coefficient fields in the string field and derivatives.

One may wonder if the expansion (4) is meaningful. By the charge quantization \( g \) is proportional to the string coupling \( g_s \). In addition, massive modes have no divergent point, and each coefficient in \( \Phi \) receives contribution from only one \( \Phi_n \). We have seen that our solution shares some full order properties with that of [3]. These facts strongly suggest that the expansion (4) is meaningful at least in small \( g_s \) region.

### 4 Solution in Superstring Field Theory

In this section we investigate supersymmetric version of the solution in the previous sections. We use Berkovits’ superstring field theory. The equation of motion is

\[
0 = \eta_0 (e^{-\Phi} Q e^\Phi) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} \eta_0 \Phi_n \Phi_n \ldots .
\]

As in the previous section, we expand \( \Phi \) around the solution of the linearized equation \( \Phi_0 \):

\[
\Phi = g \Phi_0 + g^2 \Phi_1 + g^3 \Phi_2 + \ldots ,
\]

\[
\Phi_0 = \int \frac{d^p k}{(2\pi)^p} A_+ (k_i) \xi c \psi^i e^{-\phi} e^{ik_i} X^i .
\]

*I would like to thank A. Sen for clarifying this point.
\( \Phi_n \) satisfy the following equations:

\[
\begin{align*}
\Delta_0 &= \eta_0 Q \Phi_0, \\
\Delta_1 &= \eta_0 \left( Q \Phi_1 - \frac{1}{2} [\Phi_0, Q \Phi_0] \right), \\
\Delta_2 &= \eta_0 \left( Q \Phi_2 + \frac{1}{6} [\Phi_0, [\Phi_0, Q \Phi_0]] - \frac{1}{2} [\Phi_0, Q \Phi_1] - \frac{1}{2} [\Phi_1, Q \Phi_0] \right), \\
&\vdots \\
\Delta_n &= \eta_0 \left( Q \Phi_n \\
&+ \sum_{m=1}^{n} \sum_{n_1, n_2, \ldots, n_{m+1}} \frac{(-1)^m}{(m+1)!} [\Phi_{n_1}, [\Phi_{n_2}, \ldots, [\Phi_{n_m}, Q \Phi_{n_{m+1}}]]] \right),
\end{align*}
\]

where

\[
\Delta_0 = \alpha' \int \frac{d^p k}{(2\pi)^p} k^2 A_+(k_i) c \partial c \psi^+ e^{-\phi} e^{ik_i X^i}.
\]

We impose the gauge fixing conditions \( b_0 \Phi_n = \tilde{G}_0^- \Phi_n = 0 \), and for \( n \geq 1 \) \( b_0 \Delta_n = 0 \). This condition, with \( \tilde{G}_0^- \) defined as follows[13], is slightly different from the familiar one \( \xi_0 \Phi_n = 0 \).

\[
\tilde{G}_0^- = \left[ Q, \oint \frac{dz}{2\pi i} z b_0 \xi(z) \right] = \oint \frac{dz}{2\pi i} z (\xi T - \partial \xi bc - bc \phi G_m - \eta e^{2\phi} b \partial b),
\]

where \( T \) is the total worldsheet energy momentum tensor, and \( G_m \) is matter part of the worldsheet supercurrent. This operator is more useful than \( \xi_0 \) because of the following relations:

\[
\{ \eta_0, \tilde{G}_0^- \} = L_0, \quad \{ Q, \tilde{G}_0^- \} = \{ b_0, \tilde{G}_0^- \} = 0,
\]

and therefore \( \frac{\tilde{G}_0^-}{L_0} \) is the inverse of \( \eta_0 \) on string fields annihilated by \( \tilde{G}_0^- \). Note that \( \Phi_0 \) obeys \( b_0 \Phi_0 = \tilde{G}_0^- \Phi_0 = 0 \).

Then the equations of motion can be solved order by order:

\[
\begin{align*}
\Phi_1 &= \frac{1}{2} \tilde{G}_0^- \eta_0 \frac{b_0}{L_0} [\Phi_0, Q \Phi_0], \\
\Phi_2 &= \frac{\tilde{G}_0^-}{L_0} \eta_0 \frac{b_0}{L_0} \left( -\frac{1}{6} [\Phi_0, [\Phi_0, Q \Phi_0]] + \frac{1}{2} [\Phi_0, Q \Phi_1] + \frac{1}{2} [\Phi_1, Q \Phi_0] \right),
\end{align*}
\]

\[
\vdots
\]
\[ \Phi_n = -\frac{G_0}{L_0} \eta_0 \frac{b_0}{L_0} \sum_{m=1}^{n} \sum_{n_1, n_2, \ldots, n_{m+1}} \frac{(-1)^m}{(m+1)!}[\Phi_{n_1}, [\Phi_{n_2}, [\ldots, [\Phi_{n_m}, Q\Phi_{n_{m+1}}]]\ldots]]. \] (38)

Therefore, \( \Delta_n = \eta_0 \frac{b_0}{L_0} Q \sum_{m=1}^{n} \sum_{n_1, n_2, \ldots, n_{m+1}} \frac{(-1)^m}{(m+1)!}[\Phi_{n_1}, [\Phi_{n_2}, [\ldots, [\Phi_{n_m}, Q\Phi_{n_{m+1}}]]\ldots]]. \) (39)

As in the bosonic case, if \( \Phi_m \) satisfy equations of motion with \( \Delta_m = 0 \) for \( m < n \), then \( \Delta_n = 0 \). To prove this, notice the following identity:

\[ Q(e^{-\Phi}Qe^{\Phi}) + (e^{-\Phi}Qe^{\Phi})^2 = 0. \] (40)

Therefore

\[ Q\eta_0(e^{-\Phi}Qe^{\Phi}) = [\eta_0(e^{-\Phi}Qe^{\Phi}), (e^{-\Phi}Qe^{\Phi})]. \] (41)

We expand \( \Phi \) in \( g \) and extract order \( g^{n+1} \) contribution of this equation. From the left hand side,

\[ Q\eta_0(e^{-\Phi}Qe^{\Phi})\big|_{g^{n+1}} = Q\eta_0 \sum_{m=1}^{n} \sum_{n_1, n_2, \ldots, n_{m+1}} \frac{(-1)^m}{(m+1)!}[\Phi_{n_1}, [\Phi_{n_2}, [\ldots, [\Phi_{n_m}, Q\Phi_{n_{m+1}}]]\ldots]]. \] (42)

Using equations of motion for lower order than \( g^{n+1} \), the right hand side gives

\[ [\eta_0(e^{-\Phi}Qe^{\Phi}), (e^{-\Phi}Qe^{\Phi})]\big|_{g^{n+1}} = \sum_{l=0}^{n-1} \sum_{m=0}^{n-l-1} \sum_{n_1, n_2, \ldots, n_{m+1}} \frac{(-1)^m}{(m+1)!}[\Phi_{n_1}, [\Phi_{n_2}, [\ldots, [\Phi_{n_m}, Q\Phi_{n_{m+1}}]]\ldots]]. \] (43)

Therefore

\[ \Delta_n = \frac{b_0}{L_0} \sum_{l=0}^{n-1} \sum_{m=0}^{n-l-1} \sum_{n_1, n_2, \ldots, n_{m+1}} \frac{(-1)^m}{(m+1)!}[\Phi_{n_1}, [\Phi_{n_2}, [\ldots, [\Phi_{n_m}, Q\Phi_{n_{m+1}}]]\ldots]]. \] (44)

This shows that if \( \Delta_m = 0 \) for \( m < n \), then \( \Delta_n = 0 \).

Analogously to the bosonic case, \( \Phi_n \) has no dependence on momenta along \( x^\pm \), and has the following property: \( n_+(\omega) - n_-(\omega) = n + 1 \), where \( \omega \) is any of the vertex operators (or Fock
space states) of which \( \Phi_n \) consists, \( n_+ (\omega) \) is the number of \( \partial^m X^+ s \) and \( \partial^r \psi^+ \) (or \( \alpha^+_m \) and \( \psi^+_r \)) in \( \omega \), and \( n_- (\omega) \) is the number of \( \partial^m X^- s \) and \( \partial^r \psi^- \) (or \( \alpha^-_m \) and \( \psi^-_r \)) in \( \omega \). In addition, \((X^I, \psi^I)\) part of \( \omega \) is a super-Virasoro descendant of the unit operator. In other words, the matter part of \( \omega \) is in the following form:

\[
\prod_{l=1}^{N_+} \alpha^+_p l \psi^+_q l \prod_{l=1}^{M_+} \psi^-_r l \prod_{l=1}^{N_-} \alpha^-_s l \prod_{l=1}^{M_-} L'_l - u_l \prod_{l=1}^{G'_r} \psi^j_l - v_l \prod_{l=1}^{k_l} \right| k_l \rangle (45)
\]

where \( L'_n \) and \( G'_r \) are \((X^I, \psi^I)\) parts of Virasoro operator and worldsheet supercharge respectively.

This can be proven by almost the same argument as in the bosonic case. Here we have new ingredients: \( \eta_0 \), \( Q \) and \( \tilde{G}^-_0 \). \( \eta_0 \) does not affect the matter sector. \( Q \) and \( \tilde{G}^-_0 \) can replace \( X^\pm \) by \( \psi^\pm \) and vice versa, but preserve \( n_\pm \). They map a super-Virasoro descendant of the unit operator to other descendants of it. \( \Delta_n \) also satisfy these properties as can be seen from almost the same argument.

Therefore this solution has the same properties as in the bosonic case: each coefficient of Fock space state in the solution \( \Phi \) receives contribution from only one \( \Phi_n \). In particular, the coefficient \( A_\mu \) of the massless mode \( \xi^\psi^+ e^{-\Phi} e^{ik_i X^i} \) is never corrected by higher order contribution. The inverses of \( L_0 \) in the expression of \( \Phi_n \) with \( n \geq 1 \) do not cause any problem. \( A_\mu \) is equal to the gauge field in the effective action. This gives another proof of the fact shown in [7]. Massive modes are convergent even at the singular points of the massless mode. Energy-momentum tensor as Noether current of translation symmetry is equal to that of free U(1) gauge theory.

A new property which is not in bosonic theory is supersymmetry. Therefore let us investigate supersymmetry of this solution. Supersymmetry transformation of R-sector string field \( \Psi \) is given by [14]

\[
\delta(\eta_0 \Psi) = -\eta_0 s (e^{-\Phi}(Qe^\Phi)) \tag{46}
\]

where

\[
s = \oint \frac{dz}{2\pi i} e^{i\pi/4} \bar{\epsilon}_A \xi(z) e^{-\frac{i}{2} \phi(z)} \Sigma^A (z), \tag{47}
\]

\( \bar{\epsilon}_A \) is a constant ten-dimensional Majorana-Weyl spinor, and \( \Sigma^A (z) \) is a spin operator. \( e^{-\frac{i}{2} \phi(z)} \Sigma^A (z) \) is regarded as Grassmann odd. The action of \( s \) on a string field is defined as the contour integral of (47) around it.
It is easy to see that the linearized solution $\Phi_0$ is 1/2 supersymmetric at the linearized level, since on-shell linearized transformation for massless fields is the same as that of the U(1) gauge theory. Because of $A_- = A_i = A_I = 0$ and $A_+ = A_+(k_i)$, the transformation of gaugino $\psi^A(k)$ is
\[
\delta \psi^A(k) = ik_iA_+(k_i)(\Gamma^i + \epsilon)^A.
\]
(48)

We see that the unbroken supersymmetry parameter is given by $\Gamma^i + \epsilon = 0$.

In fact, the full solution is also 1/2 supersymmetric with the same unbroken parameter. This can be shown as follows. First, notice that when $\Gamma^i + \epsilon = 0$, $\Phi_0$ satisfies
\[
s\Phi_0 = s\eta_0\Phi_0 = sQ\Phi_0 = s\eta_0Q\Phi_0 = 0,
\]
(49)
and $s$ commutes with $\frac{\bar{G}_0}{L_0}$ and $\frac{b_0}{L_0}$. Then by plugging our solution, $e^{-\Phi}(Qe^{\Phi})$ is expressed by $\Phi_0$, $\frac{\bar{G}_0}{L_0}\eta_0$, $Q$ and $\frac{b_0}{L_0}$. Using Leibniz rule for $Q$ and $\eta_0$, and $\{Q, \frac{b_0}{L_0}\} = \{\eta_0, \frac{\bar{G}_0}{L_0}\} = 1$, we can rewrite $e^{-\Phi}(Qe^{\Phi})$ in such a form that any $Q$ and $\eta_0$ act directly on one of $\Phi_0$. Since $s$ also satisfies Leibniz rule when it acts on products of string fields, we can again rewrite $se^{-\Phi}(Qe^{\Phi})$ in such a form that $s$ acts directly on one of $\Phi_0$, $\eta_0\Phi_0$, $Q\Phi_0$ or $\eta_0Q\Phi_0$. Thus we can see $se^{-\Phi}(Qe^{\Phi}) = 0$ and therefore $\delta(\eta_0\Psi) = 0$.

5 Discussion

We have shown that our solutions have various full order properties in the sense of $\alpha'$-expansion. Among them, the fact that massive modes have no singularity lacks a rigorous proof for third and higher massive states coming from $\Phi_n$ with $n \geq 3$. It is desirable to give a proof of it, because this fact is important for not only our solutions, but also general structure of off-shell amplitudes.

We have constructed higher order source terms for unphysical modes along with higher order contributions to the solutions, and have seen that those are not localized to points. This is natural in a sense, because full order string theory is a nonlocal theory unlike its low energy effective theory. Although this is expected not to affect the equation of motion for massless modes obtained after integrating out all the massive modes, it is better to give other evidences that our source terms really correspond to endpoints of fundamental strings.

 Readers might wonder why massive modes do not contribute to the energy-momentum tensor, in spite of the fact that they satisfy Siegel gauge condition and therefore they are
physical excitations. It may be useful to consider if this fact has any deep meaning for physical properties of massive modes.

The order by order method employed here can be applied to other systems e.g. closed string field theory. It is interesting to construct solutions corresponding to, for example, macroscopic fundamental string solution or pp-wave solution, which are also known as $\alpha'$-exact solutions in supergravity. We can expect to derive some full order properties of those solutions by the same method as in this paper.

**Acknowledgments**

The author wishes to thank S. Iso, Y. Okawa and A. Sen for useful discussions, and especially B. Zwiebach for reading the manuscript and giving helpful comments. This work is supported in part by funds provided by the U.S. Department of Energy (D.O.E.) under cooperative research agreement DF-FC02-94ER40818, and by the Nishina Memorial Foundation.

**Appendix**

**A**

In this appendix we show that the momentum integral of (22) is convergent, by seeing that the factor $\frac{1}{2\pi} \frac{1+\alpha^2}{1-\alpha^2} \kappa(\alpha)$ is less than or equal to 1. First we give the definition of $\kappa(\alpha)$.

4-point amplitudes can be computed by mapping four vertex operators on four upper half planes by $w = h_i(Z_i)$, defined as follows,

$$h_1(Z) = h_2(Z) = \ln Z - \frac{\tau}{2}, \quad (A.1)$$

$$h_3(Z) = h_4(Z) = -\ln Z + \pi i + \frac{\tau}{2}, \quad (A.2)$$

and the Giddings map $z = z(w)$ [11], defined implicitly as follows,

$$w = \frac{\tau}{2} + N \int_{+0}^{\zeta} d\zeta \frac{\sqrt{\zeta^2 + \gamma^2} \sqrt{\zeta^2 + \gamma^{-2}}}{(\zeta^2 - \alpha^2)(\zeta^2 - \alpha^{-2})}, \quad (A.3)$$

$$N = \frac{2\alpha(\alpha^{-2} - \alpha^2)}{\sqrt{\alpha^2 + \gamma^2} \sqrt{\alpha^2 + \gamma^{-2}}}, \quad (A.4)$$

to one single upper half plane, on which the four vertex operators are at $z = \pm \alpha$ and $z = \pm \alpha^{-1}$. 16
and $\gamma$ are functions of $\alpha$, and implicitly determined by the following equations.

$$\frac{\pi}{2} = N \int^{\gamma}_0 d\zeta \frac{\sqrt{\gamma^2 - \zeta^2} \sqrt{\gamma^2 - \zeta^2}}{(\zeta^2 + \alpha^2)(\zeta^2 + \alpha^{-2})},$$

(A.5)

$$\tau = N \int^{\gamma^{-1}}_{\gamma} d\zeta \frac{\sqrt{\zeta^2 - \gamma^2} \sqrt{\gamma^2 - \zeta^2}}{(\zeta^2 + \alpha^2)(\zeta^2 + \alpha^{-2})}.$$  

(A.6)

$\gamma$ is a monotonously increasing function, and $0 \leq \gamma \leq 1$ as can be seen from the fact that $z = i\gamma$ and $z = i\gamma^{-1}$ are where two of the four strings meet, and therefore $z = i\gamma$ is always below $z = i\gamma^{-1}$ on the imaginary axis. $\tau$ is a modulus to be integrated over $0 \leq \tau \leq \infty$ which corresponds to $\alpha_0 \equiv \sqrt{2} - 1 \geq \alpha \geq 0$. Near $\alpha = 0$, $\gamma \sim \sqrt{3}\alpha$, and $\gamma = 1$ only at $\alpha = \alpha_0$. Figure 3 is the profile of $\gamma$.

![Figure 3: $\gamma(\alpha)$](image)

The above conformal mappings for the vertex operators give the following factor, which appears in (22):

$$(\alpha^{-1}\kappa(\alpha))^{\alpha'k^2}(\alpha^{-1}\kappa(\alpha))^{\alpha'k^2}(\alpha\kappa(\alpha))^{\alpha'k^2}(\alpha\kappa(\alpha))^{\alpha'k^2},$$

(A.7)

where

$$\kappa(\alpha) = \exp(I(\alpha)), 

(A.8)$$

$$I(\alpha) = \int^{\alpha}_0 d\zeta \left[ \frac{N}{(\zeta^2 - \alpha^2)(\zeta^2 - \alpha^{-2})} + \frac{1}{\zeta - \alpha} \right] = \int^{1}_{\alpha} d\zeta \left[ \frac{N}{(1 - \zeta^2)(1 - \alpha^4\zeta^2)} + \frac{1}{\zeta - 1} \right].$$

(A.9)

$^\dagger$Although this looks different from eq.(3.13) in [12], this is equal to it as can be seen by partial integration and replacing $\ln(1 - w)$ by $\int^w d\zeta \frac{1}{1 - \zeta}$. 
The two terms of the integrand are divergent at ζ = α, but their sum is not. Though it is difficult to perform this integral at generic α, it is possible at the edges of the range of α:

\[
I(0) = \frac{8}{3\sqrt{3}},
\]

\[
I(\alpha_0) = \ln \sqrt{2}.
\]

To show \(0 < \frac{1 + \alpha^2}{2(1 - \alpha^2)} \kappa(\alpha) \leq 1\), we add some extra terms to the integrand which sum up to zero:

\[
I(\alpha) = \int_0^1 d\zeta \left[ N\alpha \frac{\sqrt{\alpha^2 \zeta^2 + \gamma^2 \sqrt{\alpha^2 \zeta^2 + \gamma^2}}}{(1 - \zeta^2)(1 - \alpha^4 \zeta^2)} - 2(1 - \alpha^2) \frac{1 + \alpha^2 \zeta^2}{(1 - \zeta^2)(1 - \alpha^4 \zeta^2)} + \int_0^1 d\zeta \left[ 2(1 - \alpha^2) \frac{1 + \alpha^2 \zeta^2}{(1 - \zeta^2)(1 - \alpha^4 \zeta^2)} - 2(1 - \alpha_0^2) \frac{1 + \alpha_0^2 \zeta^2}{(1 - \zeta^2)(1 - \alpha_0^4 \zeta^2)} \right] \right] - \frac{2\alpha^2(1 - \alpha^2)(1 - \gamma^2)}{(\alpha^2 + \gamma^2)(1 + \alpha^2 \gamma^2)} \times \int_0^1 d\zeta \left[ (1 + \alpha^2) \sqrt{\frac{(\alpha^2 \zeta^2 + \gamma^2)(1 + \alpha^2 \gamma^2 \zeta^2)}{(\alpha^2 + \gamma^2)(1 + \alpha^2 \gamma^2)}} + 1 + \alpha^2 \zeta^2 \right]^{-1}.
\]

The third integral is equal to \(I(\alpha_0)\), and the second integral can be explicitly done because the integrand is a rational function:

\[
\int_0^1 d\zeta \left[ 2(1 - \alpha^2) \frac{1 + \alpha^2 \zeta^2}{(1 - \zeta^2)(1 - \alpha^4 \zeta^2)} - 2(1 - \alpha_0^2) \frac{1 + \alpha_0^2 \zeta^2}{(1 - \zeta^2)(1 - \alpha_0^4 \zeta^2)} \right] = \ln \left( \sqrt{2} \frac{1 - \alpha^2}{1 + \alpha^2} \right).
\]

The sum of two terms of the integrand in the first integral is not singular at ζ = 1. The final result of this manipulation is

\[
I(\alpha) = \ln \left( \frac{2 - \alpha^2}{1 + \alpha^2} \right) - \frac{2\alpha^2(1 - \alpha^2)(1 - \gamma^2)}{(\alpha^2 + \gamma^2)(1 + \alpha^2 \gamma^2)} \times \int_0^1 d\zeta \left[ (1 + \alpha^2) \sqrt{\frac{(\alpha^2 \zeta^2 + \gamma^2)(1 + \alpha^2 \gamma^2 \zeta^2)}{(\alpha^2 + \gamma^2)(1 + \alpha^2 \gamma^2)}} + 1 + \alpha^2 \zeta^2 \right]^{-1}.
\]

Then we obtain the following expression of \(\frac{1 + \alpha^2}{2(1 - \alpha^2)} \kappa(\alpha)\):

\[
\frac{1 + \alpha^2}{2(1 - \alpha^2)} \kappa(\alpha) = \exp \left( - \frac{2\alpha^2(1 - \alpha^2)(1 - \gamma^2)}{(\alpha^2 + \gamma^2)(1 + \alpha^2 \gamma^2)} \times \int_0^1 d\zeta \left[ (1 + \alpha^2) \sqrt{\frac{(\alpha^2 \zeta^2 + \gamma^2)(1 + \alpha^2 \gamma^2 \zeta^2)}{(\alpha^2 + \gamma^2)(1 + \alpha^2 \gamma^2)}} + 1 + \alpha^2 \zeta^2 \right]^{-1} \right).
\]

It is easy to see that the exponent of the right hand side is always negative, and zero only at α = α_0 (where γ = 1). Thus \(0 < \frac{1 + \alpha^2}{2(1 - \alpha^2)} \kappa(\alpha) \leq 1\), and the momentum integral of (22) is convergent. Figure 4 is the profile of \(\frac{1 + \alpha^2}{2(1 - \alpha^2)} \kappa(\alpha)\).
Figure 4: $\frac{14+\alpha^2}{2(1-\alpha^2)}\kappa(\alpha)$

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