Removability of time-dependent singularities in the heat equation

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Abstract

We consider solutions of the linear heat equation with time-dependent singularities. It is shown that if a singularity is weaker than the order of the fundamental solution of the Laplace equation, then it is removable. We also consider the removability of higher dimensional singular sets. An example of a non-removable singularity is given, which implies the optimality of the condition for removability.

1 Introduction

Removability of singularities of solutions is an interesting and important problem in partial differential equations. For the Laplace equation, the removability of a singularity is defined as follows. Let $u$ be a solution of

\[ \Delta u = 0 \quad \text{in } \Omega \setminus \{\xi_0\}, \]

where $\Omega$ is a domain in $\mathbb{R}^N$ and $\xi_0 \in \Omega$. We say that $\xi_0$ is a removable singularity if there exists a classical solution $\tilde{u}$ of the Laplace equation in $\Omega$ such that

\[ \tilde{u} \equiv u \quad \text{in } \Omega \setminus \{\xi_0\}. \]

It is well known \cite{3} that for $N \geq 3$, the singular point $\xi_0$ is removable if and only if

\[ |u(x)| = o(|x - \xi_0|^{2-N}) \quad \text{as} \quad x \to \xi_0. \]

For nonlinear elliptic equations, the removability of a singularity has been studied in many papers and various interesting results have been obtained (see, e.g., Brezis-Veron \cite{1}, Gidas-Spruck \cite{4}, Veron \cite{11}).

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Similarly, for the heat equation
\[ u_t = \Delta u \quad \text{in } \Omega \setminus \{x_0\} \times (0, T) \]
with \( N \geq 3 \) and \( T > 0 \), Hsu [7] proved recently that the singular point \( x_0 \) is removable if and only if
\[ |u(x, t)| = o(|x - x_0|^{2-N}) \quad \text{as } x \to x_0 \]
for every \( t \in (0, T) \). Later, Hui [8] gave a simpler proof for this result. In [6], Hirata extended Hsu and Hui’s result to a semilinear parabolic equation of the form
\[ u_t = \Delta u + |u|^{p-1}u \]
with \( p < N/(N-2) \). See also Sato-Yanagida [9] for non-removable singularities of this equation.

In this paper, we consider the case where a singular point may move in time and study its removability for the heat equation. More precisely, we formulate our problem as follows. For \( T > 0 \) fixed, let \( \xi : [0, T] \to \mathbb{R}^N \) be a continuous function, and \( \Gamma \subset \mathbb{R}^{N+1} \) be a curve given by
\[ \Gamma := \{(x, t) \in \mathbb{R}^{N+1} : x = \xi(t), \ t \in (0, T)\}. \]
We take a domain \( \Omega \subset \mathbb{R}^N \) such that \( \xi(t) \in \Omega \) for \( t \in [0, T] \), and define
\[ D := \{(x, t) \in \mathbb{R}^{N+1} : x \in \Omega \setminus \{\xi(t)\}, \ t \in (0, T)\}. \]
For a solution of
\[ u_t = \Delta u \quad \text{in } D, \quad (1.1) \]
the singularity at \( x = \xi(t) \) is said to be removable if there exists a function \( \tilde{u} \) which satisfies the heat equation in \( \Omega \times (0, T) \) in the classical sense and \( \tilde{u} \equiv u \) on \( D \).

Our first result gives a condition for the removability of such a (moving) singularity.

**Theorem 1.1.** Let \( N \geq 3 \). Suppose that \( \xi \) is Hölder continuous with exponent 1/2 and that \( u \) satisfies \((1.1)\) in the classical sense. Then the singularity of \( u \) at \( x = \xi(t) \) is removable if and only if for any \( 0 < t_1 < t_2 < T \) and \( 0 < \varepsilon < 1 \) there exists \( 0 < r < 1 \) depending on \( t_1, t_2, \varepsilon \) such that
\[ |u(x, t)| \leq \frac{\varepsilon}{|x - \xi(t)|^{N-2}}, \quad 0 < |x - \xi(t)| < r \quad (1.2) \]
for any \( t \in [t_1, t_2] \).

**Theorem 1.2.** Let \( N = 2 \). Suppose that \( \xi \) is Hölder continuous with exponent 1/2 and that \( u \) satisfies \((1.1)\) in the classical sense. Then the singularity of \( u \) at \( x = \xi(t) \) is removable if and only if for any \( 0 < t_1 < t_2 < T \) and \( 0 < \varepsilon < 1 \) the function \( u \) satisfies
\[ |u(x, t)| \leq \varepsilon \log \frac{1}{|x - \xi(t)|}, \quad 0 < |x - \xi(t)| < \varepsilon \quad (1.3) \]
for any \( t \in [t_1, t_2] \).
Here we note that for $N = 1$, if we define $\tilde{u}$ by

$$\tilde{u}(x, t) := \begin{cases} u(x, t) & \text{for } (x, t) \in D, \\ \liminf_{x \uparrow \xi(t)} u(x, t) & \text{for } (x, t) \in \Gamma, \end{cases}$$

then the singularity at $x = \xi(t)$ is removable if and only if $\tilde{u}$ is continuously differentiable at $x = \xi(t)$ for any $t \in (0, T)$.

Next, we consider a higher dimensional singular set whose spatial codimension is greater than or equal to 2. We reformulate our problem as follows. Let $m \geq 1$, $N \geq m + 2$, $T > 0$ and $s = (s_1, s_2, \ldots, s_m) \in \mathbb{R}^m$. We assume that the mapping

$$\xi(s, t) = (\xi_1^1(s, t), \xi_2^1(s, t), \ldots, \xi_N^1(s, t)) : [0, 1]^m \times [0, T] \to \mathbb{R}^N$$

is continuously differentiable with respect to $s$ and Hölder continuous with exponent $1/2$ with respect to $t$. Also, we assume that the Jacobian matrix of $\xi$ with respect to $s$ is non-singular, that is,

$$\text{rank} \begin{pmatrix} \xi_{s_1}^1(s_1, s_2, \ldots, s_m, t) & \ldots & \xi_{s_m}^1(s_1, s_2, \ldots, s_m, t) \\ \vdots & \ddots & \vdots \\ \xi_{s_1}^N(s_1, s_2, \ldots, s_m, t) & \ldots & \xi_{s_m}^N(s_1, s_2, \ldots, s_m, t) \end{pmatrix} = m \quad (1.4)$$

for any $(s_1, s_2, \ldots, s_m) \in [0, 1]^m$ and $t \in [0, T]$. We denote the singular set by

$$\Xi(t) := \{ \xi(s, t) : s \in [0, 1]^m \}$$

and define $\Gamma \subset \mathbb{R}^{N+1}$ by

$$\Gamma := \{(x, t) \in \mathbb{R}^{N+1} : x \in \Xi(t), \ t \in (0, T) \}.$$ 

We also define a distant between $x$ and $\Xi(t)$ by

$$d(x, \Xi(t)) := \min_{s \in [0, 1]^m} |x - \xi(s, t)|.$$ 

Furthermore, let $\Omega \subset \mathbb{R}^N$ be a domain such that

$$\Omega \supset \bigcup_{t \in [0, T]} \Xi(t),$$

and define a domain $D \subset \mathbb{R}^{N+1}$ by

$$D := \{(x, t) \in \mathbb{R}^{N+1} : x \in \Omega \setminus \Xi(t), \ t \in (0, T) \}.$$ 

Now we define removability of a higher dimensional singular set as follows. For a solution of (1.1), the singular set $\Xi(t)$ is said to be removable if there exists a function $\tilde{u}$ which satisfies the heat equation in $\Omega \times (0, T)$ in the classical sense and $u \equiv \tilde{u}$ on $D$.

Our results for higher dimensional singular sets are as follows.
Theorem 1.3. Let $N \geq m + 3$. Suppose that $\xi$ satisfies (1.4) and that $u$ satisfies (1.1) in the classical sense. Then the singular set $\Xi(t)$ is removable if and only if for any $0 < t_1 < t_2 < T$ and $0 < \varepsilon < 1$ there exists $0 < r < 1$ depending on $t_1, t_2, \varepsilon$ such that

$$|u(x,t)| \leq \frac{\varepsilon}{d(x,\Xi(t))^{N-m-2}}, \quad 0 < d(x,\Xi(t)) < r$$

for any $t \in [t_1, t_2]$.

Theorem 1.4. Let $N = m + 2$. Suppose that $\xi$ satisfies (1.4) and that $u$ satisfies (1.1) in the classical sense. Then the singular set $\Xi(t)$ is removable if and only if for any $0 < t_1 < t_2 < T$ and $0 < \varepsilon < 1$ the function $u$ satisfies

$$|u(x,t)| \leq \varepsilon \log \frac{1}{d(x,\Xi(t))}, \quad 0 < d(x,\Xi(t)) < \varepsilon$$

for any $t \in [t_1, t_2]$.

By an analogous method to Section 3, we can extend Theorem 1.3 to the case where the singular set consists of $\Xi_1, \Xi_2, \ldots, \Xi_k$, each of which satisfies (1.4) and may intersect with others. By regarding $\Xi_1, \Xi_2, \ldots, \Xi_k$ as local coordinates, the above theorems give a condition for the removability in the case where the singular set is a compact $m$-dimensional $C^1$-manifold in $\mathbb{R}^N$.

Next, we show the existence of a solution of (1.1) whose singularity moves in time and is not removable. Again, let $N \geq 2$, $T > 0$, and $\Gamma \subset \mathbb{R}^{N+1}$ be defined as above.

The next result implies that the conditions in Theorem 1.1 and Theorem 1.2 for the removability are optimal in some sense.

Theorem 1.5. Given any Hölder continuous function $\xi(t) : [0, T] \rightarrow \mathbb{R}^N$ with exponent $\alpha > 1/2$, there exists $u$ defined on a neighborhood of $\Gamma$ such that $u$ satisfies (1.1) in the classical sense but the singularity of $u$ at $x = \xi(t)$ is not removable.

In Section 4, we give an example of a non-removable moving singularity. In fact, this theorem will be proved by solving the following problem:

$$u_t - \Delta u = \delta(x - \xi(t)) \quad \text{in} \ \mathbb{R}^N \times (0, T),$$

where $\delta(\cdot)$ denote the Dirac distribution concentrated at the point $0 \in \mathbb{R}^N$. In this case, we can show that the singularity at $x = \xi(t)$ persists for $t \in (0, T)$ and the solution satisfies

$$u(x,t) = \frac{1}{N(N-2)\omega_N} |x - \xi(t)|^{2-N} + o(|x - \xi(t)|^{2-N}) \quad \text{if} \quad N \geq 3,$$

$$u(x,t) = \frac{1}{2\pi} \log \left( \frac{1}{|x - \xi(t)|} \right) + o\left( \log \frac{1}{|x - \xi(t)|} \right) \quad \text{if} \quad N = 2$$

at $x = \xi(t)$, where we denote by $\omega_N$ volume of unit ball in $\mathbb{R}^N$.

This paper is organized as follows. In Section 2 we prove Theorems 1.1 and 1.2 by cutting a neighborhood of the singularity. In Section 3 we apply this method to a higher dimensional singular set. Section 4 is devoted to the analysis of (1.6).
2 Removability of a moving singularity

In this section, we consider removability of a moving singularity. To show Theorem 1.1, we give the following lemma.

**Lemma 2.1.** Let \( r > 0 \). Suppose that \( \xi(t) : [0, T] \to \mathbb{R}^N \) is H"older continuous with exponent \( \alpha > 0 \). Then there exists a family of cut-off functions \( \{ \eta^r \}_{r > 0} \subset C^\infty(\mathbb{R}^N \times (0, T)) \) such that

\[
\eta^r(x, t) = \begin{cases} 
1 & \text{if } |x - \xi(t)| > r, \\
0 & \text{if } |x - \xi(t)| < r/2,
\end{cases}
\]

and

\[
0 \leq \eta^r \leq 1, \quad |\nabla \eta^r| \leq Cr^{-1}, \quad |\Delta \eta^r| \leq Cr^{-2}, \quad |(\eta^r)_t| \leq Cr^{-1/\alpha},
\]

where \( C > 0 \) is a constant independent of \( x, t \) and \( r \).

**Proof.** Let \( r > 0 \) be fixed. We take standard mollifier \( \rho \in C^\infty(\mathbb{R}) \) by

\[
\rho(t) := \begin{cases} 
Ae^{-1/(1-t^2)} & \text{if } |t| < 1, \\
0 & \text{if } |t| \geq 1,
\end{cases}
\]

where the constant \( A > 0 \) is taken so that \( \int_{\mathbb{R}} \rho(t) \, dt = 1 \). In addition, for each \( \varepsilon > 0 \), we set \( \rho^\varepsilon(t) := (1/\varepsilon)\rho(t/\varepsilon) \). We express \( \xi = (\xi_1, \xi_2, \ldots, \xi_N) \) and define \( \xi^\varepsilon = (\xi_1^\varepsilon, \xi_2^\varepsilon, \ldots, \xi_N^\varepsilon) \) by

\[
\xi_i^\varepsilon(t) := \int_{\mathbb{R}} \rho^\varepsilon(t-s)\xi_i(s) \, ds, \quad i = 1, 2, \ldots, N.
\]

Let \( \varepsilon > 0 \). By H"older continuity of \( \xi \), we obtain

\[
|\xi_i(t) - \xi_i^\varepsilon(t)| \leq L\varepsilon^\alpha \tag{2.1}
\]

for every \( t \in [0, T] \), where \( L > 0 \) is a H"older constant. Moreover, by changing variable \( \tau = (t-s)/\varepsilon \) and simple calculation,

\[
(\xi_i^\varepsilon)_t = \frac{A}{\varepsilon^2} \int_{t-\varepsilon}^{t+\varepsilon} \frac{-2(t-s)/\varepsilon}{(1 - ((t-s)/\varepsilon)^2)^2} \exp \left( -\frac{1}{1 - ((t-s)/\varepsilon)^2} \right) \xi_i(s) \, ds
\]

\[
= \frac{A}{\varepsilon} \int_{-1}^{1} \frac{-2\tau}{(1 - \tau^2)^2} e^{-1/(1-\tau^2)} \xi_i(t - \varepsilon\tau) \, d\tau.
\]

We remark that

\[
\int_{-1}^{1} \frac{-2\tau}{(1 - \tau^2)^2} e^{-1/(1-\tau^2)} \xi_i(t) \, d\tau = 0.
\]
Then, by Hölder continuity, we have
\[
|\langle \xi^\varepsilon \rangle_t| = \frac{A}{\varepsilon} \left| \int_{-1}^{1} \frac{-2\tau}{(1-\tau^2)^2} e^{-1/(1-\tau^2)} (\xi_i(t - \varepsilon \tau) - \xi_i(t)) \, d\tau \right|
\leq A L \varepsilon^{\alpha-1} \int_{-1}^{1} \frac{2|\tau|}{(1-\tau^2)^2} e^{-1/(1-\tau^2)} \, d\tau
= 2AL\varepsilon^{\alpha-1}e^{-1} \leq A\varepsilon^{\alpha-1}.
\] (2.2)

Now we define \( \eta^r \in C^\infty(\mathbb{R} \times (0, T)) \) by
\[
\eta^r(x, t) := \begin{cases} 
\frac{e^{-1/\sigma}}{e^{-1/\sigma} + e^{-1/(1-\sigma)}} & \text{if } \frac{7}{10} r < |x - \xi^\varepsilon(t)| < \frac{4}{5} r, \\
1 & \text{if } |x - \xi^\varepsilon(t)| \geq \frac{4}{5} r, \\
0 & \text{if } |x - \xi^\varepsilon(t)| \leq \frac{7}{10} r,
\end{cases}
\]
where
\[
\sigma = \sigma(x, t; r) = \frac{10}{r} \left( |x - \xi^\varepsilon(t)| - \frac{7}{10} r \right).
\]
It is clear that \( 0 \leq \eta^r(x, t) \leq 1 \).

Next, we take \( \varepsilon_r = (r/10NL)^{1/\alpha} \). By (2.1), we have
\[
|\xi(t) - \xi^\varepsilon_r(t)| \leq |\xi_i(t) - \xi^\varepsilon_r(t)| + \cdots + |\xi_N(t) - \xi^\varepsilon_r(t)|
\leq L(\varepsilon_r)^\alpha N = r/10.
\]
Here, \( \eta^r(x, t) = 1 \) if \( |x - \xi(t)| > r \), because
\[
|x - \xi^\varepsilon_r(t)| \geq |x - \xi(t)| - |\xi(t) - \xi^\varepsilon_r(t)| > r - (r/10) > 4r/5,
\]
and \( \eta^r(x, t) = 0 \) if \( |x - \xi(t)| < r/2 \), because
\[
|x - \xi^\varepsilon_r(t)| \leq |x - \xi(t)| + |\xi(t) - \xi^\varepsilon_r(t)| < (r/2) + (r/10) < 7r/10.
\]

Finally, we estimate first and second derivatives of \( \eta^r \). It suffices to calculate in the case where \( 7r/10 < |x - \xi^\varepsilon_r(t)| < 4r/5 \). In this case, we have \( 0 < \sigma(x, t; r) < 1 \). By direct calculation, we have
\[
\nabla_x(\eta^r) = \frac{10}{r} X(\sigma) \frac{x - \xi^\varepsilon_r(t)}{|x - \xi^\varepsilon_r(t)|}, \quad (\eta^r)_t = -\frac{10}{r} X(\sigma) \frac{(x - \xi^\varepsilon_r(t)) \cdot \xi^\varepsilon_r(t)}{|x - \xi^\varepsilon_r(t)|},
\]
where
\[
X(\sigma) := \frac{1}{(e^{-1/\sigma} + e^{-1/(1-\sigma)})^2} \left( \frac{1}{\sigma^2} + \frac{1}{(1-\sigma)^2} \right),
\]
and
\[ \Delta_x(\eta^r) = \frac{100}{r^2} Y(\sigma), \]
where
\[ Y(\sigma) := \frac{e^{-1/\sigma}e^{-1/(1-\sigma)}}{(e^{-1/\sigma} + e^{-1/(1-\sigma)})^2} \left[ \left( \frac{N - 1}{\sigma + 7} \right) \left( \frac{1}{\sigma^2} + \frac{1}{(1-\sigma)^2} \right) + \left( \frac{1}{\sigma^4} + \frac{1}{(1-\sigma)^4} \right) (1 - 2\sigma) \right. \]
\[ \left. - \frac{2}{e^{-1/\sigma} + e^{-1/(1-\sigma)}} \left( \frac{e^{-1/\sigma}}{\sigma^4} + \frac{e^{-1/\sigma} - e^{-1/(1-\sigma)}}{\sigma^2(1-\sigma^2)} - \frac{e^{-1/(1-\sigma)}}{(1-\sigma)^4} \right) \right]. \]

Since \( X(\sigma) \) and \( Y(\sigma) \) belong to \( C^\infty(0,1) \) and satisfy
\[ \lim_{\sigma \downarrow 0} |X(\sigma)| = \lim_{\sigma \uparrow 1} |X(\sigma)| = \lim_{\sigma \downarrow 0} |Y(\sigma)| = \lim_{\sigma \uparrow 1} |Y(\sigma)| = 0, \]
we see that \( X(\sigma) \) and \( Y(\sigma) \) are bounded for \( \sigma \in (0,1) \). Moreover, by (2.2), we obtain
\[ |(\eta^r)_t| \leq C_1 r^{-1} A \Lambda (\varepsilon_r)^{\alpha - 1} N = C_2 r^{-1/\alpha}, \]
where \( C_1, C_2 > 0 \) are constants independent of \( x, t, r \). Hence there exists a constant \( C_3 > 0 \) independent of \( x, t, r \) such that
\[ |\nabla \eta^r| \leq C_3 r^{-1}, \quad |\Delta \eta^r| \leq C_3 r^{-2}, \quad |(\eta^r)_t| \leq C_3 r^{-1/\alpha}. \]
The proof is complete. \( \quad \blacksquare \)

**Proof of Theorem 1.1.** Necessity is easily proved by the same argument as in Section 3 of [7]. Indeed, if the singularity of \( u \) at \( x = \xi(t) \) is removable, then \( u \) is bounded near \( x = \xi(t) \).

We prove sufficiency. Let \( 0 < t_1 < t_2 < T \) and \( 0 < \varepsilon < 1 \). By our assumption, there exists \( r = r(t_1, t_2, \varepsilon) > 0 \) such that (1.2) holds. For each \( t \in (0, T) \), we take any sequence \( \{x_i(t)\}_{i=1}^{\infty} \subset \Omega \setminus \{\xi(t)\} \) such that \( |x_i(t) - \xi(t)| \to 0 \) as \( i \to \infty \), and set
\[ \tilde{u}(x, t) := \begin{cases} u(x, t) & \text{for } (x, t) \in D, \\ \liminf_{i \to \infty} u(x_i(t), t) & \text{for } (x, t) \in \Gamma. \end{cases} \]

Our goal is to prove that \( \tilde{u} \) satisfies the heat equation in \( \Omega \times (0, T) \) in the classical sense.

First, we show \( \tilde{u} \in L^1_{\text{loc}}(\Omega \times (0, T)) \). For each \( t \in [t_1, t_2] \), we denote
\[ B(\xi(t), r) := \{ x \in \mathbb{R}^N : |x - \xi(t)| < r \}. \]

By \( N \)-dimensional polar coordinates centered at \( \xi(t) \), we have
\[ \int_{t_1}^{t_2} \int_{B(\xi(t), r)} |x - \xi(t)|^{2-N} dx dt = C_1 (t_2 - t_1) r^2 \quad (2.3) \]

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for some $C_1 = C_1(N) > 0$. Let $K \subset \mathbb{R}^N$ be a compact subset of $\Omega$. Since $\xi(t) \in \Omega$ for $t \in [0, T]$, we can take $r = r(t_1, t_2, \varepsilon) > 0$ so small that $B(\xi(t), r) \subset \Omega$ for every $t \in [t_1, t_2]$. By (1.2) and (2.3), there exists $C_2 > 0$ such that

$$\int_{t_1}^{t_2} \int_K |\tilde{u}(x, t)| \, dx \, dt \leq \int_{t_1}^{t_2} \int_{K \setminus B(\xi(t), r)} |u(x, t)| \, dx \, dt + \varepsilon \int_{t_1}^{t_2} \int_{B(\xi(t), r)} |x - \xi(t)|^{2-N} \, dx \, dt$$

$$\leq C_2 + \varepsilon C_1(t_2 - t_1)r^2 < \infty.$$  

Since $0 < t_1 < t_2 < T$ are arbitrary, we have $\tilde{u} \in L^1_{\text{loc}}(\Omega \times (0, T)).$

Next, we show that $\tilde{u}$ satisfies the heat equation in $\Omega \times (0, T)$ in the distribution sense. For this purpose, we need a family of cut-off functions $\{\eta^r\}_{r>0} \subset C^\infty(\mathbb{R}^N \times (0, T))$ such that

$$\eta^r(x, t) = \begin{cases} 0 & \text{if } |x - \xi(t)| < r/2, \\ 1 & \text{if } |x - \xi(t)| > r, \end{cases}$$

and

$$0 \leq \eta^r \leq 1, \quad |\nabla \eta^r| \leq C_3r^{-1}, \quad |\Delta \eta^r| \leq C_3r^{-2}, \quad |(\eta^r)_t| \leq C_3r^{-2},$$

where $C_3 > 0$ is a constant independent of $x, t$ and $r$. By Lemma 2.1 and the assumption that $\xi(t)$ is Hölder continuous with exponent $1/2$, we can take such $\{\eta^r\}$. Now, let $\phi \in C^\infty_0(\Omega \times (0, T))$ be a test function. Since $\eta^r \phi \in C^\infty_0(\Omega \times (0, T))$, and $\tilde{u}$ is a classical solution of (1.1), we have

$$\int_\Omega \{\tilde{u}(x, t_2)\phi(x, t_2)\eta^r(x, t_2) - \tilde{u}(x, t_1)\phi(x, t_1)\eta^r(x, t_1)\} \, dx$$

$$= \int_{t_1}^{t_2} \int_\Omega \{(\phi \eta^r)_t + \Delta(\phi \eta^r)\} \, dx \, dt. \tag{2.5}$$

Here, we claim that the following convergence properties hold:

$$\limsup_{\varepsilon \to 0} \left| \int_{t_1}^{t_2} \int_\Omega \tilde{u}\Delta \phi \, dx \, dt - \int_{t_1}^{t_2} \int_\Omega \tilde{u}(\phi \eta^r)_t \, dx \, dt \right| = 0, \tag{2.6}$$

$$\limsup_{\varepsilon \to 0} \left| \int_{t_1}^{t_2} \int_\Omega \tilde{u}\phi_t \, dx \, dt - \int_{t_1}^{t_2} \int_\Omega \tilde{u}(\phi \eta^r)_t \, dx \, dt \right| = 0, \tag{2.7}$$

$$\limsup_{\varepsilon \to 0} \left| \int_\Omega \tilde{u}(x, t_1)\phi(x, t_1) \, dx - \int_\Omega \tilde{u}(x, t_2)\phi(x, t_2) \, dx \right| = 0, \tag{2.8}$$

$$\limsup_{\varepsilon \to 0} \left| \int_\Omega \tilde{u}(x, t_2)\phi(x, t_2) \, dx - \int_\Omega \tilde{u}(x, t_1)\phi(x, t_1) \, dx \right| = 0. \tag{2.9}$$
To show (2.6), we rewrite
\[ \int_{t_1}^{t_2} \int_{\Omega} \tilde{u} \Delta \phi \, dx \, dt - \int_{t_1}^{t_2} \int_{\Omega} \tilde{u} \Delta (\phi \eta^r) \, dx \, dt = \int_{t_1}^{t_2} \int_{\Omega} \tilde{u} (1 - \eta^r) \Delta \phi \, dx \, dt - 2 \int_{t_1}^{t_2} \int_{\Omega} \tilde{u} \nabla \phi \cdot \nabla \eta^r \, dx \, dt - \int_{t_1}^{t_2} \int_{\Omega} \tilde{u} \phi \Delta \eta^r \, dx \, dt =: I_{1,r} - 2I_{2,r} - I_{3,r}. \] 

By (1.2) and (2.4), for sufficiently small \( r = r(t_1, t_2, \varepsilon) > 0 \), we have the inequalities
\[
|I_{1,r}| \leq \| \Delta \phi \|_{L^\infty(\Omega \times (0,T))} \varepsilon \int_{t_1}^{t_2} \int_{B(\xi(t), r)} |x - \xi(t)|^{-N} \, dx \, dt,
\]
\[
|I_{2,r}| \leq \| \nabla \phi \|_{L^\infty(\Omega \times (0,T))} C_3 \frac{\varepsilon}{r} \int_{t_1}^{t_2} \int_{B(\xi(t), r)} |x - \xi(t)|^{-N} \, dx \, dt,
\]
\[
|I_{3,r}| \leq \| \phi \|_{L^\infty(\Omega \times (0,T))} C_4 \frac{\varepsilon}{r^2} \int_{t_1}^{t_2} \int_{B(\xi(t), r)} |x - \xi(t)|^{-N} \, dx \, dt.
\]

Hence, by (2.3) and \( r \in (0, 1) \), we have
\[
|I_{1,r}| \leq \| \Delta \phi \|_{L^\infty(\Omega \times (0,T))} C_1 (t_2 - t_1) \varepsilon r^2 \leq C_4 \varepsilon,
\]
\[
|I_{2,r}| \leq \| \nabla \phi \|_{L^\infty(\Omega \times (0,T))} C_1 C_3 (t_2 - t_1) \varepsilon r \leq C_4 \varepsilon;
\]
\[
|I_{3,r}| \leq \| \phi \|_{L^\infty(\Omega \times (0,T))} C_1 C_3 (t_2 - t_1) \varepsilon \leq C_4 \varepsilon
\]
for some \( C_4 > 0 \). Hence we obtain (2.6). Similarly we obtain (2.7), (2.8) and (2.9) from above estimates.

Thus, the function \( \tilde{u} \) satisfies
\[
\int_{\Omega} \{ \tilde{u}(x, t_2) \phi(x, t_2) - \tilde{u}(x, t_1) \phi(x, t_1) \} \, dx = \int_{t_1}^{t_2} \int_{\Omega} \tilde{u}(\phi_t + \Delta \phi) \, dx \, dt \tag{2.11}
\]
for any \( \phi \in C_0^\infty(\Omega \times (0,T)) \). Since \( 0 < t_1 < t_2 < T \) be arbitrary, the function \( \tilde{u} \in L^1_{loc}(\Omega \times (0,T)) \) satisfies the heat equation in \( \Omega \times (0,T) \) in the distribution sense. By using the Weyl lemma for the heat equation (see, e.g., Section 6 of [5] or [10]), \( \tilde{u} \) satisfies the heat equation in \( \Omega \times (0,T) \) in the classical sense. Since \( \tilde{u} = u \) in \( D \), the singularity of \( u \) at \( x = \xi(t) \) is removable. \( \blacksquare \)

**Proof of Theorem 1.2.** We prove only sufficiency. Let \( 0 < t_1 < t_2 < T \) and \( 0 < \varepsilon < 1 \), and define \( \tilde{u} \) as in the proof of Theorem 1.1. By 2-dimensional polar coordinates, we have
\[
\int_{t_1}^{t_2} \int_{B(\xi(t), \varepsilon)} \log \frac{1}{|x - \xi(t)|} \, dx \, dt \leq C_1 (t_2 - t_1) (1 + \log(1/\varepsilon)) \varepsilon^2 \tag{2.12}
\]
for some \( C_1 > 0 \). This implies \( \tilde{u} \in L^1_{loc}(\Omega \times (0,T)) \).
We show that \( \tilde{u} \) satisfies the heat equation in \( \Omega \times (0, T) \) in the distribution sense. Let \( \phi \in C_0^\infty(\Omega \times (0, T)) \). By Lemma 2.1 and the assumption that \( \xi(t) \) is Hölder continuous with exponent \( 1/2 \), we can take \( \{\eta^\varepsilon\}_{\varepsilon > 0} \subset C^\infty(\mathbb{R}^N \times (0, T)) \) such that

\[
\eta^\varepsilon(x, t) = \begin{cases} 
0 & \text{if } |x - \xi(t)| < \varepsilon/2, \\
1 & \text{if } |x - \xi(t)| > \varepsilon,
\end{cases}
\]

and

\[
0 \leq \eta^\varepsilon \leq 1, \quad |\nabla \eta^\varepsilon| \leq C_2 \varepsilon^{-1}, \quad |\Delta \eta^\varepsilon| \leq C_2 \varepsilon^{-2}, \quad |(\eta^\varepsilon)_t| \leq C_2 \varepsilon^{-2}
\]

for some \( C_2 > 0 \). Since \( \tilde{u} \) satisfies (1.1), the equality (2.5) holds for \( r = \varepsilon \). Again, we claim that the convergence properties (2.6), (2.7), (2.8) and (2.9) hold for \( r = \varepsilon \). Let \( I_{1,\varepsilon}, I_{2,\varepsilon} \) and \( I_{3,\varepsilon} \) be defined as in (2.10) with \( r = \varepsilon \). By (1.3) and (2.13), for sufficiently small \( \varepsilon > 0 \), we have

\[
|I_{1,\varepsilon}| \leq \|\Delta \phi\|_{L^\infty(\Omega \times (0, T))} \varepsilon \int_{t_1}^{t_2} \int_{B(\xi(t), \varepsilon)} \log \frac{1}{|x - \xi(t)|} \, dx \, dt,
\]

\[
|I_{2,\varepsilon}| \leq \|\nabla \phi\|_{L^\infty(\Omega \times (0, T))} C_2 \varepsilon \int_{t_1}^{t_2} \int_{B(\xi(t), \varepsilon)} \log \frac{1}{|x - \xi(t)|} \, dx \, dt,
\]

\[
|I_{3,\varepsilon}| \leq \|\phi\|_{L^\infty(\Omega \times (0, T))} C_2 \varepsilon \int_{t_1}^{t_2} \int_{B(\xi(t), \varepsilon)} \log \frac{1}{|x - \xi(t)|} \, dx \, dt.
\]

Hence by (2.12), we have

\[
|I_{1,\varepsilon}| \leq \|\Delta \phi\|_{L^\infty(\Omega \times (0, T))} C_1 (t_2 - t_1) (1 + \log(1/\varepsilon)) \varepsilon^3 \leq C_3 \varepsilon \log(1/\varepsilon),
\]

\[
|I_{2,\varepsilon}| \leq \|\nabla \phi\|_{L^\infty(\Omega \times (0, T))} C_1 C_2 (t_2 - t_1) (1 + \log(1/\varepsilon)) \varepsilon^2 \leq C_3 \varepsilon \log(1/\varepsilon),
\]

\[
|I_{3,\varepsilon}| \leq \|\phi\|_{L^\infty(\Omega \times (0, T))} C_1 C_2 (t_2 - t_1) (1 + \log(1/\varepsilon)) \varepsilon \leq C_3 \varepsilon \log(1/\varepsilon)
\]

for some \( C_3 > 0 \). Hence we obtain (2.6). Similarly we obtain (2.7), (2.8) and (2.9) from above estimates. These imply that \( \tilde{u} \in L^1_{\text{loc}}(\Omega \times (0, T)) \) satisfies the heat equation in \( \Omega \times (0, T) \) in the distribution sense. The remainder is the same as in the proof of Theorem 1.1.

3 Removability of a singular set

Let \( \Xi(t) \subset \mathbb{R}^N, D \subset \mathbb{R}^{N+1}, \Gamma \subset \mathbb{R}^{N+1}, \) and \( \Omega \subset \mathbb{R}^N \) are the sets defined in Section 1. To show Theorems 1.3 and 1.4, we give the following estimates.
Lemma 3.1. There exists $C_1 = C_1(N, m) > 0$ and $C_2 = C_2(m) > 0$ such that for every sufficiently small $r > 0$,

$$
\int_{A_{r,t}} d(x, \Xi(t))^{m+2-N} \, dx \leq C_1 r^2 \quad \text{if} \quad N \geq m + 3,
$$

$$
\int_{A_{r,t}} \log \frac{1}{d(x, \Xi(t))} \, dx \leq C_2 r^2 \left(1 + \log \frac{1}{r}\right) \quad \text{if} \quad N = m + 2 \quad (3.1)
$$

for any $t \in (0, T)$, where $A_{r,t} := \{ x \in \mathbb{R}^N : d(x, \Xi(t)) < r \}$.

**Proof.** We prove the lemma only in the case $N \geq m + 3$. In fact, (3.2) can be proved in the same manner as (3.1). Let $t \in (0, T)$ be fixed. We extend the domain of the function $\xi$ to $[a, b]^m \times [0, T]$ with $a < 0$ and $b > 1$. That is, we take a mapping

$$
\tilde{\xi}(s, t) = (\tilde{\xi}^1(s, t), \tilde{\xi}^2(s, t), \ldots, \tilde{\xi}^N(s, t)) : [a, b]^m \times [0, T] \to \mathbb{R}^N
$$

such that $\tilde{\xi}$ is continuously differentiable in $s$ and continuous in $t$. In addition, we assume that $\tilde{\xi}$ satisfies (1.4) and

$$
\tilde{\xi}|_{[0,1]^m \times [0, T]} = \xi, \quad \tilde{\xi}_{s_i}|_{[0,1]^m \times [0, T]} = \xi_{s_i}, \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, N.
$$

We define

$$
\Xi(t) := \{ \tilde{\xi}(s, t) : s \in [a, b]^m \}.
$$

For each $s \in (a, b)^m$, let $\Pi_{r,t}(s)$ be a subset of a normal plane of $\Xi(t)$ at $\tilde{\xi}(s, t)$ given by

$$
\Pi_{r,t}(s) := \{ x \in A_{r,t} : (x - \tilde{\xi}(s, t)) \cdot \xi_{s_i}(s, t) = 0 \quad \text{for any} \ i = 1, 2, \ldots, m \}.
$$

Since $\tilde{\xi}(\cdot, t)$ is defined on a compact set, there exists a sufficiently small $r > 0$ such that

$$
d(x, \Xi(t)) = |x - \tilde{\xi}(t)|, \quad x \in \Pi_{r,t}(s) \quad (3.3)
$$

for each $s \in (a, b)^m$. Again by compactness, we have

$$
M := \max_{t \in [0, T]} \int_{\Xi(t)} d\sigma^m < \infty, \quad (3.4)
$$

where $d\sigma^m$ is an $m$-dimensional surface element. Since $\tilde{\xi}$ satisfies (1.4), $\Pi_{r,t}(s)$ is an $(N - m)$-dimensional subspace of $\mathbb{R}^N$. Therefore, for each $s \in (a, b)^m$, there exists a congruent transformation $P_s : \mathbb{R}^N \to \mathbb{R}^N$ such that

$$
P_s x = (y_1, y_2, \ldots, y_{N-m}, 0, \ldots, 0), \quad x \in \Pi_{r,t}(s)
$$

for some $y_1, y_2, \ldots, y_{N-m} \in \mathbb{R}$. Now, by using $(N - m)$-dimensional polar coordinates, we obtain

$$
\int_{P_s(\Pi_{r,t}(s))} |y|^{m+2-N} \, dy_1 dy_2 \cdots dy_{N-m} = C_3 \int_0^r \rho^{m+2-N+(N-m-1)} \, d\rho = C_4 r^2, \quad (3.5)
$$
where $C_3, C_4 > 0$ depend on $N, m$ but not on $s, t$.

Recall that the congruent transformations preserve a distance between any two points and that the function $\tilde{\xi}$ is an extension of $\xi$. Hence by choosing sufficiently small $r > 0$ again if necessary, we have the estimate

$$\int_{A_{r,t}} d(x, \xi(\cdot, t))^{m+2-N} dx \leq MC_4 r^2$$

by using (3.3), (3.4) and (3.5). Thus we obtain (3.1).

**Proof of Theorem 1.3.** We adopt the same approach as in the proof of Theorem 1.1, so we state the outline only.

Let $0 < t_1 < t_2 < T$ and $0 < \varepsilon < 1$. By our assumption, there exists $r = r(t_1, t_2, \varepsilon) > 0$ such that (1.5) holds. For $t \in (0, T)$, we take any sequence $\{x_i(t)\}_{i=1}^\infty \subset \Omega \setminus \Xi(t)$ such that $d(x_i(t), \xi(\cdot, t)) \to 0$ as $i \to \infty$, and set

$$\tilde{u}(x, t) := \begin{cases} u(x, t) & \text{for } (x, t) \in D, \\ \liminf_{i \to \infty} u(x_i(t), t) & \text{for } (x, t) \in \Gamma. \end{cases}$$

By Lemma 3.1, we obtain $\tilde{u} \in L^1_{\text{loc}}(\Omega \times (0, T))$. We show that $\tilde{u}$ satisfies (1.1) in $\Omega \times (0, T)$ in the distribution sense. Let $\phi \in C^\infty(\Omega \times (0, T))$. By an argument similar to Lemma 2.1, we can take $\{\eta_r\}_{r>0} \subset C^\infty(\Omega \times (0, T))$ such that

$$\eta^r(x, t) = \begin{cases} 0 & \text{if } d(x, \xi(\cdot, t)) < r/2, \\ 1 & \text{if } d(x, \xi(\cdot, t)) > r, \end{cases}$$

and $\eta^r$ satisfies the condition (2.4) for some $C > 0$. Since $\tilde{u}$ satisfies (1.1), we have (2.5). By Lemma 3.1 and an argument similar to Section 2, we obtain (2.11). That is, the function $\tilde{u} \in L^1_{\text{loc}}(\Omega \times (0, T))$ satisfies the heat equation in $\Omega \times (0, T)$ in the distribution sense. The remainder is the same as in the proof of Theorem 1.1.

Since (3.2) holds, we can show Theorem 1.4 in the same way. We omit details of the proof.

### 4 Non-removable singularity

In this section, we consider the case where a singularity move in time and is not removable. Without loss of generality, we take $\Omega = \mathbb{R}^N$. Let $N \geq 2$ and $T > 0$. We assume that $\xi : [0, T] \to \mathbb{R}^N$ is arbitrarily given continuous function.

To show Theorem 1.5 we solve the equation (1.6). In this paper, we say that $u$ satisfies (1.6) in the distribution sense if $u$ belongs to $L^1_{\text{loc}}(\mathbb{R}^N \times (0, T))$ and satisfies

$$\int_0^T \int_{\mathbb{R}^N} (\phi_t - \Delta \phi) u \, dx \, dt = \int_0^T \phi(\xi(t), t) \, dt \quad (4.1)$$
for any $\phi \in C^\infty_0(\mathbb{R}^N \times (0, T))$. Now, we denote by
\[
\Phi(x, t) := (4\pi t)^{-N/2} \exp(-|x|^2/4t)
\]
the fundamental solution of the heat equation. Moreover, we define $F$ in $\mathbb{R}^N \times (0, T)$ by
\[
F(x, t) := \int_0^t \Phi(x - \xi(s), t - s) \, ds.
\]
In the following, we will show that $F$ satisfies (1.6) in the distribution sense. In addition, we will give upper and lower estimates of $F$, and we will see that $F$ is an example of Theorem 1.5.

**Proposition 4.1.** The function $F$ satisfies (1.1) in the classical sense.

To show Proposition 4.1, we give the following lemma.

**Lemma 4.1.** The function $F$ satisfies (4.1) in the distribution sense.

**Proof.** First, we show $F \in L^1_{\text{loc}}(\mathbb{R}^N \times (0, T))$. By simple calculation, we have
\[
\int_0^T \int_{\mathbb{R}^N} F(x, t) \, dx \, dt = \int_0^T \int_0^t \left( \int_{\mathbb{R}^N} \Phi(x - \xi(s), t - s) \, dx \right) \, ds \, dt
\]
\[
= \int_0^T \int_0^t ds \, dt = \frac{1}{2} T^2 < \infty,
\]
so that $F \in L^1(\mathbb{R}^N \times (0, T))$. In particular, $F$ belongs to $L^1_{\text{loc}}(\mathbb{R}^N \times (0, T))$.

Next, we show that $F$ satisfies (4.1). For this purpose, let $\phi \in C^\infty_0(\mathbb{R}^N \times (0, T))$ be a test function. For each $t \in (0, \tau)$, we take $\tau \in (0, t)$ and define $F^\tau$ by
\[
F^\tau(x, t) = \int_0^{t-\tau} \Phi(x - \xi(s), t - s) \, ds.
\]
Here $F^\tau$ is bounded for each fixed $\tau$, that is, there exist $C_1(N), C_2(N) > 0$ such that
\[
0 \leq F^\tau(x, t) \leq C_1(N) \int_0^{t-\tau} (t - s)^{-N/2} \, ds \leq C_2(N) \tau^{(2-N)/2}
\]
for each $t \in (0, T)$. Then, integrating by parts yields
\[
\int_0^T \int_{\mathbb{R}^N} (-\phi_t - \Delta \phi) F^\tau \, dx \, dt
\]
\[
= \int_0^T \int_{\mathbb{R}^N} (-\phi_t - \Delta \phi) \left( \int_0^{t-\tau} \Phi(x - \xi(s), t - s) \, ds \right) \, dx \, dt
\]
\[
= \int_0^T \int_{\mathbb{R}^N} \phi(x, t) \Phi(x - \xi(t - \tau), \tau) \, dx \, dt
\]
\[
+ \int_0^T \int_{\mathbb{R}^N} \phi(x, t) \left( \int_0^{t-\tau} \{ \Phi_t(x - \xi(s), t - s) - \Delta \Phi(x - \xi(s), t - s) \} \, ds \right) \, dx \, dt
\]
\[
= \int_0^T \int_{\mathbb{R}^N} \phi(x, t) \Phi(x - \xi(t - \tau), \tau) \, dx \, dt.
\]
Similarly from Section 2.3.1 of [2], we see that
\[
\lim_{\tau \to 0} \int_{\mathbb{R}^N} \phi(x,t)\Phi(x - \xi(t - \tau), \tau) \, dx = \phi(\xi(t), t) \tag{4.2}
\]
for each \( t \in (0, T) \).

For the reader’s convenience, we give a proof of (4.2). Let \( 0 < t < T \) and \( \varepsilon > 0 \) be fixed. We choose \( \delta > 0 \) such that
\[
|\phi(x,t) - \phi(\xi(t), t)| < \varepsilon \tag{4.3}
\]
for any \(|x - \xi(t)| < \delta\). Then, we have
\[
\left| \int_{\mathbb{R}^N} \phi(x,t)\Phi(x - \xi(t - \tau), \tau) \, dx - \phi(\xi(t), t) \right| \\
\leq \int_{\mathbb{R}^N} |\phi(x,t) - \phi(\xi(t), t)|\Phi(x - \xi(t - \tau), \tau) \, dx \\
= \int_{B(\xi(t), \delta)} + \int_{\mathbb{R}^N \setminus B(\xi(t), \delta)} =: I_1 + I_2.
\]

First, by (4.3), we have an estimate of \( I_1 \) as
\[
I_1 \leq \varepsilon \int_{\mathbb{R}^N} \Phi(x - \xi(t - \tau), \tau) \, dx = \varepsilon.
\]

Next, we give an estimate of \( I_2 \). If \(|x - \xi(t)| \geq \delta\) and \(|\xi(t) - \xi(t - \tau)| \leq \delta/2\), then
\[
|x - \xi(t)| \leq |x - \xi(t - \tau)| + |\xi(t - \tau) - \xi(t)| \leq |x - \xi(t - \tau)| + \frac{1}{2}|x - \xi(t)|
\]
Hence \(|x - \xi(t - \tau)| \geq |x - \xi(t)|/2\). By simple calculation,
\[
I_2 \leq 2\|\phi\|_{L^\infty(\mathbb{R}^N \times (0, T))} \int_{\mathbb{R}^N \setminus B(\xi(t), \delta)} (4\pi \tau)^{-N/2} \exp\left(-\frac{|x - \xi(t - \tau)|^2}{4\tau}\right) \, dx \\
\leq C_3 \tau^{-N/2} \int_{\mathbb{R}^N \setminus B(\xi(t), \delta)} \exp\left(-\frac{|x - \xi(t)|^2}{16\tau}\right) \, dx \\
= C_4 \tau^{-N/2} \int_{\delta/4\sqrt{\tau}}^\infty r^{N-1} \exp\left(-\frac{r^2}{16\tau}\right) \, dr \\
= C_5 \int_{\delta/4\sqrt{\tau}}^\infty \sigma^{N-1} e^{-\sigma^2} \, d\sigma \to 0 \quad \text{as} \quad \tau \to 0,
\]
where \( C_3, C_4, C_5 > 0 \) are constants independent of \( \tau \), and \( r = 4\sqrt{\tau}\sigma \). Therefore, if we have \(|\xi(t) - \xi(t - \tau)| \leq \delta/2\) and take \( \tau > 0 \) is sufficiently small, then we obtain \( I_1 + I_2 \leq \varepsilon \). Thus it is shown that (4.2) holds.
From (4.2) and the Lebesgue theorem, we see that $F$ satisfies (4.1), that is,

$$
\int_0^T \int_{\mathbb{R}^N} (-\phi_t - \Delta \phi) F(x, t) \, dx \, dt = \int_0^T \phi(\xi(t), t) \, dt.
$$

(4.4)

Hence the function $F$ satisfies (1.6) in the distribution sense.

**Proof of Proposition 4.1.** Let $\psi \in C_0^\infty(D)$ be a test function, in particular, $\psi \in C_0^\infty(\mathbb{R}^N \times (0, T))$. By (4.4), we have

$$
\int_0^T \int_{\mathbb{R}^N} (-\psi_t - \Delta \psi) F \, dx \, dt = \int_0^T \psi(\xi(t), t) \, dt.
$$

Since $\psi(\xi(t), t) = 0$ for any $t \in (0, T)$, we obtain

$$
\int_0^T \int_{\mathbb{R}^N} (-\psi_t - \Delta \psi) F \, dx \, dt = 0.
$$

Hence $F \in L^1(\mathbb{R}^N \times (0, T))$ satisfies the heat equation in $D$ in the distribution sense. By the Weyl lemma for the heat equation, we conclude that $F$ satisfies (1.1) in the classical sense.

**Proposition 4.2.** Let $N \geq 3$. Suppose that $\xi$ is H"older continuous with exponent $\alpha > 1/2$. Then for each $t \in (0, T)$ the function $F(x, t)$ satisfies

$$
F(x, t) = \frac{1}{N(N-2)\omega_N} |x - \xi(t)|^{2-N} + o(|x - \xi(t)|^{2-N}) \quad \text{as } x \to \xi(t),
$$

where $\omega_N$ is the volume of unit ball in $\mathbb{R}^N$.

**Proof.** We fix $t \in (0, T)$ and set $z := x - \xi(t)$. By changing variable $t - s = |z|^2/(4\sigma)$, we have

$$
F(x, t) = \int_0^t (4\pi(t-s))^{-N/2} \exp \left(-\frac{|z + \xi(t) - \xi(s)|^2}{4(t-s)}\right) \, ds
$$

$$
= 4^{-1-N/2}|z|^{2-N} \int_{|z|^2/4t}^{\infty} \sigma^{(N/2)-2} \exp \left(-\frac{1}{2} \frac{|z|^2}{(t-s)^{1/2}} + \frac{1}{2} \frac{\xi(t) - \xi(s)}{(t-s)^{1/2}} \right) \, d\sigma
$$

(4.5)

$$
= 4^{-1-N/2}|z|^{2-N} I(z, t).
$$

Here, we rewrite $I(z, t)$ as

$$
I(z, t) = \int_{|z|^2/4t}^{\infty} \sigma^{(N/2)-2} e^{-\sigma} \exp \left(-\frac{1}{2} \frac{|z|^2}{(t-s)^{1/2}} \cdot \frac{\xi(t) - \xi(s)}{(t-s)^{1/2}} \right) \exp \left(-\frac{1}{4} \frac{|\xi(t) - \xi(s)|^2}{(t-s)} \right) \chi_{|z|^2/(4t, \infty)}(\sigma) \, d\sigma,
$$

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where \( \chi_A \) is a indicator function of \( A \).

In order to apply the Lebesgue theorem to \( I(z,t) \), we construct a dominating integrable function as follows. By Hölder continuity of \( \xi \), for sufficiently small \(|z| > 0\), we have

\[
\sigma^{(N/2)-2}e^{-\sigma} \exp \left( -\sigma^{1/2} \frac{z}{|z|} \cdot \frac{\xi(t) - \xi(s)}{(t - s)^{1/2}} \right) \exp \left( -\frac{1}{4} \frac{|\xi(t) - \xi(s)|^2}{(t - s)} \right) \chi_{|z|^{2/u,\infty}}(\sigma) \leq \sigma^{(N/2)-2}e^{-\sigma} \exp \left( L\sigma^{1/2} \frac{|z|^2}{4\sigma} \right) \leq \sigma^{(N/2)-2}e^{-\sigma + \sigma^{1-\alpha}},
\]

where \( L > 0 \) is a Hölder constant. Since \( \alpha > 1/2 \), we see that \( \sigma^{(N/2)-2}e^{-\sigma + \sigma^{1-\alpha}} \) becomes a dominating integrable function. On the other hand, by using Hölder continuity of \( \xi \) again, we have

\[
\left| -\sigma^{1/2} \frac{z}{|z|} \cdot \frac{\xi(t) - \xi(s)}{(t - s)^{1/2}} \right| \leq \frac{L}{4^{\alpha-(1/2)}} \sigma^{1-\alpha} |z|^{2\alpha - 1} \to 0 \quad \text{as } |z| \to 0,
\]

\[
\left| -\frac{1}{4} \frac{|\xi(t) - \xi(s)|^2}{(t - s)} \right| \leq \frac{L^2}{4^{2\alpha}} \sigma^{-2\alpha+1} |z|^{4\alpha - 2} \to 0 \quad \text{as } |z| \to 0
\]

for each \( \sigma \in (0, \infty) \). Hence by the Lebesgue theorem, we obtain

\[
\lim_{|z| \to 0} I(z,t) = \int_0^\infty \sigma^{(N/2)-2}e^{-\sigma} \ d\sigma = \Gamma \left( \frac{N}{2} - 1 \right) = \frac{4\pi^{N/2}}{N(N-2)\omega_N},
\]

where \( \Gamma \) denotes the gamma function. Hence by \((4.5)\), we obtain

\[
\lim_{|z| \to 0} \frac{F(x,t)}{|z|^{2-N}} = \frac{1}{N(N-2)\omega_N}.
\]

This completes the proof. \( \blacksquare \)

**Proposition 4.3.** Let \( N = 2 \). Suppose that \( \xi \) is Hölder continuous with exponent \( \alpha > 1/2 \). Then for each \( t \in (0,T) \) the function \( F(x,t) \) satisfies

\[
F(x,t) = \frac{1}{2\pi} \log \left( \frac{1}{|x - \xi(t)|} \right) + o \left( \log \frac{1}{|x - \xi(t)|} \right) \quad \text{as } x \to \xi(t).
\]

**Proof.** We fix \( t \in (0,T) \) and set \( z := x - \xi(t) \). Setting \( N = 2 \) in \((4.5)\), we have

\[
F(x,t) = (4\pi)^{-1} \int_{|z|^{2/4t}} \sigma^{-1} \exp \left( -\sigma^{1/2} \frac{z}{|z|} \cdot \frac{1}{2} \frac{\xi(t) - \xi(s)}{(t - s)^{1/2}} \right) \ d\sigma \quad \text{(4.6)}
\]

\[
=: (4\pi)^{-1} I(z,t).
\]

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Here, we rewrite \( I(z, t) \) as
\[
I(z, t) = \int_{|z|^2/4t}^{\infty} \sigma^{-1} e^{-\sigma} \exp \left( -\sigma^{1/2} \frac{z}{|z|} \cdot \frac{\xi(t) - \xi(s)}{(t - s)^{1/2}} \right) \exp \left( -\frac{1}{4} \frac{|\xi(t) - \xi(s)|^2}{(t - s)} \right) d\sigma.
\]
First, we claim that the function \( F \) satisfies
\[
\limsup_{|z| \to 0} \frac{F(x, t)}{\log(1/|z|)} \leq \frac{1}{2\pi}. \tag{4.7}
\]
To show this, we give an upper bound of \( I(z, t) \) as
\[
I(z, t) \leq \int_{|z|^2/4t}^{\infty} \sigma^{-1} e^{-\sigma} \exp \left( -\sigma^{1/2} \frac{z}{|z|} \cdot \frac{\xi(t) - \xi(s)}{(t - s)^{1/2}} \right) d\sigma
\]
\[
\leq \int_{|z|^2/4t}^{\infty} \sigma^{-1} e^{-\sigma} \exp \left( \frac{L}{4^{\alpha-(1/2)} |z|^{2\alpha-1}} \right) d\sigma.
\]
For sufficiently small \( |z| > 0 \), we have
\[
I(z, t) \leq \int_{1}^{\infty} \sigma^{-1} e^{-\sigma + \sigma^{1-\alpha}} d\sigma + \exp \left( -\frac{|z|^2}{4t} \right) \exp \left( \frac{L}{4^{\alpha-(1/2)} |z|^{2\alpha-1}} \right) \int_{1}^{\infty} \sigma^{-1} d\sigma
\]
\[
= C(\alpha) + \exp \left( -\frac{|z|^2}{4t} \right) \exp \left( \frac{L}{4^{\alpha-(1/2)} |z|^{2\alpha-1}} \right) \left( 2 \log \frac{1}{|z|} + \log(4t) \right)
\]
for some \( C(\alpha) > 0 \). Hence by (4.6) and the above inequalities, we have
\[
\frac{F(x, t)}{\log(1/|z|)} = \frac{I(z, t)}{4\pi \log(1/|z|)} \leq \frac{1}{2\pi} \exp \left( -\frac{|z|^2}{4t} \right) \exp \left( \frac{L}{4^{\alpha-(1/2)} |z|^{2\alpha-1}} \right) + \frac{C(\alpha) + \exp(-|z|^2/(4t)) \exp(4^{(1/2)-\alpha} L |z|^{2\alpha-1}) \log(4t)}{4\pi \log(1/|z|)}
\]
\[
\to 1/2\pi \quad \text{as } |z| \to 0.
\]
Consequently, we obtain (4.7).

Next, we claim that for any fixed \( \varepsilon \in (0, 1) \) the function \( F \) satisfies
\[
\liminf_{|z| \to 0} \frac{F(x, t)}{\log(1/|z|)} \geq \frac{1}{2\pi} (1 - \varepsilon). \tag{4.8}
\]
To show this, we give a lower bound of \( I(z, t) \). Now \( (|z|^{2-\varepsilon}/4t, |z|^{\varepsilon}/4t) \subset (|z|^2/4t, \infty) \) holds. Then, by using Hölder continuity, we directly calculate
\[
I(z, t) \geq \int_{|z|^2-\varepsilon/4t}^{(1/4t)} \sigma^{-1} e^{-\sigma} \exp \left( -\sigma^{1/2} \frac{z}{|z|} \cdot \frac{\xi(t) - \xi(s)}{(t - s)^{1/2}} \right) \exp \left( -\frac{1}{4} \frac{|\xi(t) - \xi(s)|^2}{(t - s)} \right) d\sigma
\]
\[
\geq \int_{|z|^2-\varepsilon/4t}^{(1/4t)} \sigma^{-1} e^{-\sigma} \exp \left( -\frac{L}{4^{\alpha-(1/2)} |z|^{2\alpha-1}} \sigma^{1-\alpha} \right) \exp \left( -\frac{L^2}{4^{2\alpha} |z|^{4\alpha-2}\sigma^{-2\alpha+1}} \right) d\sigma,
\]
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where $L > 0$ is a Hölder constant. Since we assume $\alpha > 1/2$, we have the following estimate:

$$I(z, t) \geq \exp \left( -\frac{|z|^\varepsilon}{4t} \right) \exp \left( -\frac{L}{4^{\alpha - \frac{1}{2}}} |z|^{2\alpha - 1} \left( \frac{|z|^\varepsilon}{4t} \right)^{1 - \alpha} \right) \times \exp \left( -\frac{L^2}{4^{2\alpha}} |z|^{4\alpha - 2} \left( \frac{|z|^{2 - \varepsilon}}{4t} \right)^{-2\alpha + 1} \right) \int_{|z|^2 - \varepsilon/4t} \sigma^{-1} d\sigma$$

$$= \exp \left( -\frac{|z|^\varepsilon}{4t} \right) \exp \left( -\frac{L}{2} t^{\alpha - 1} |z|^{2\alpha - 1 + \varepsilon(1 - \alpha)} \right) \times \exp \left( -\frac{L^2 t}{4} |z|^{2\alpha - 1} \right) \left( 1 - \varepsilon \right) \log \frac{1}{|z|}.$$

Hence by (4.6) and the above inequalities, we have

$$\frac{F(x, t)}{\log(1/|z|)} = \frac{I(z, t)}{4\pi \log(1/|z|)} \geq \frac{1 - \varepsilon}{2\pi} \exp \left( -\frac{|z|^\varepsilon}{4t} \right) \exp \left( -\frac{L}{2} t^{\alpha - 1} |z|^{2\alpha - 1 + \varepsilon(1 - \alpha)} \right) \times \exp \left( -\frac{L^2 t}{4} |z|^{2\alpha - 1} \right) \rightarrow \left( 1 - \varepsilon \right)/2\pi \quad \text{as } |z| \rightarrow 0,$$

so that (4.8) holds. These two claims imply that for any $\varepsilon \in (0, 1)$ the function $F$ satisfies

$$\frac{1 - \varepsilon}{2\pi} \leq \liminf_{|z| \rightarrow 0} \frac{F(x, t)}{\log(1/|z|)} \leq \limsup_{|z| \rightarrow 0} \frac{F(x, t)}{\log(1/|z|)} \leq \frac{1}{2\pi}.$$

Then

$$\lim_{|z| \rightarrow 0} \frac{F(x, t)}{\log(1/|z|)} = \frac{1}{2\pi}.$$

This completes the proof. □

Now Theorem 1.5 immediately follows from Propositions 4.2, 4.3, and Proposition 4.1.

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