Global solutions to the free boundary problem of a three-dimensional chemotaxis-Navier-Stokes system

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ABSTRACT In this paper, we investigate the global solvability of the chemotaxis-Navier-Stokes system on a three-dimensional moving domain of finite depth, bounded below by a rigid flat bottom and bounded above by the free surface. Completing the system with the boundary conditions that match the boundary descriptions in the experiment and numerical simulations, we establish the global existence and uniqueness of solutions near a constant state \((0, \hat{c}, 0)\), where \(\hat{c}\) is the saturation value of the oxygen on the free surface. Our proof is based on the contraction mapping theorem and analysis on a linearized chemotaxis-Stokes problem by using energy methods along with the linear parabolic and elliptic theory. To the best of our knowledge, this is the first analytical work for the well-posedness of chemotaxis-Navier-Stokes system on a time-dependent domain.

MSC: 35A01, 35B40, 35K57, 35Q92, 92C17

KEYWORDS: Free boundary, Chemotaxis, Navier-Stokes, Logarithmic singularity, Global existence

1 Introduction

Background and literature review. When suspension of an oxytactic bacteria denser than water like Bacillus subtilis is placed in a chamber with its upper surface open to the atmosphere, bacterial cells swim up the gradient of the oxygen which diffuses to the suspension through the air-fluid interface and quickly get densely packed below the interface in a relatively thin layer. Subsequently, Rayleigh-Taylor type instabilities occur due to the buoyancy effect and evolve ultimately into the bioconvection patterns observed in the experiments [10, 15, 16]. To describe this chemotaxis-diffusion-convection process the following chemotaxis-Navier-Stokes system has been proposed in [32]:

\[
\begin{align*}
    m_t + \vec{u} \cdot \nabla m + \nabla \cdot (m \chi(c) \nabla c) &= D_m \Delta m \quad \text{for } x \in \Omega_t \text{ and } t > 0, \\
    c_t + \vec{u} \cdot \nabla c + mf(c) &= D_c \Delta c, \\
    \vec{u}_t + \kappa \vec{u} \cdot \nabla \vec{u} + \nabla p + m \nabla \Phi &= D \Delta \vec{u}, \\
    \nabla \cdot \vec{u} &= 0,
\end{align*}
\]

(1.1)

where \(\Omega_t\) is a domain in \(\mathbb{R}^d\) that may evolve with time \(t\). The unknown functions \(m(x,t), c(x,t)\) are bacteria density and oxygen concentration and \(\vec{u}(x,t)\) denotes the fluid velocity with associated pressure \(p(x,t)\). The positive constants \(D_m, D_c\) and \(D\) are diffusion rates of the cells, the oxygen and the velocity respectively. The first two equations in (1.1) describe the chemotactic movement of bacteria towards increasing gradients of the attractive oxygen with chemotactic intensity \(\chi(c) > 0\)
and oxygen consumption rate \( f(c) > 0 \), where both bacteria and oxygen diffuse and are convected with the fluid. In turn, the influence of the bacteria cells on the fluid is through the buoyant forces given by potential \( \Phi(x,t) \).

The striking feature of (1.1) is that it couples the well-known obstacles in theory of hydrodynamics to the typical difficulties in the study of chemotaxis system. Indeed, due to the lack of effective mathematical tools handling the cross-diffusive term \( \nabla \cdot (m\chi(c)\nabla c) \), the answer is still unavailable to the question whether the global weak solutions (constructed in [30]) of the three-dimensional chemotaxis-only subsystem obtained from (1.1) by letting \( \bar{u} = 0 \) may blow up at a finite time or not. On the other hand, the global well-posedness to the three dimensional incompressible Navier-Stokes equations with arbitrary large initial data remains a prominent open problem in hydrodynamics. In spite of these challenges, extensive studies have been conducted numerically and analytically in the last decades and most of the results achieved are focusing on the pattern formation of bacteria cells and global well-posedness for the corresponding initial-boundary problem on fixed spatial domains \( \Omega \) independent of \( t \). Here, we only mention the previous results related to the present paper.

In the case \( \Omega = \mathbb{R}^3 \), for the chemotaxis-Stokes system obtained from (1.1) on neglecting the convective term \( \bar{u} \cdot \nabla \bar{u} \) in the fluid evolution, the global weak solutions have been constructed (cf. [11]) under appropriate smallness assumptions on either the potential function or the initial oxygen concentration along with certain structural conditions on \( \chi \) and \( f \). For the same two-dimensional Cauchy problem, Liu and Lorz (cf. [24]) removed the above smallness assumptions and showed global weak solvability even for the chemotaxis-Navier-Stokes system with under basically the same conditions on \( \chi \) and \( f \) as made in [11]. Uniqueness of such solutions was justified later (c.f. [40]) by taking advantage of a coupling structure of the equations and using the Fourier localization technique. To include the prototypical choices \( \chi = \text{const.} \) and \( f(c) = c \) (cf. [8, 14, 32]), Chae-Kang-Li (cf. [3, 6]) demonstrated the global well-posedness for the chemotaxis-Navier-Stokes system with \( \kappa = 1 \) under smallness assumptions on \( ||c_0||_{L^\infty} \) and relaxed assumptions on \( \chi \) and \( f \). They also obtained some blow-up criteria that allow the weak solutions derived in [24] to become a classical one upon improving the regularity of the initial data. In the case \( \Omega = \mathbb{R}^3 \), the problem of global well-posedness seems to be more delicate: to the best of our knowledge, results available so far are merely confined to local and global small solutions (cf. [3, 6, 11]).

When \( \Omega \) is a fixed bounded domain (independent of \( t \)) in \( \mathbb{R}^d \), \( d = 2,3 \) with smooth boundary, system (1.1) subject to the following boundary conditions:

\[
\nabla m \cdot \bar{n} = 0, \quad \nabla c \cdot \bar{n} = 0, \quad \bar{u} = 0,
\]

with \( \bar{n} \) the outward unit normal to the boundary \( \partial \Omega \), was investigated in [25] and local weak solutions were constructed in the situation \( \chi \) being a constant. Under the structural hypotheses \((f(s)') > 0, (\chi(s)f(s))' \geq 0, Winkler derived the global existence of weak solutions in the 3D case with \( \kappa = 0 \) and of smooth solutions in the 2D case with \( \kappa \in \mathbb{R} \) (cf. [35]). Such smooth solutions in the latter 2D case stabilize to the spatially uniform equilibria \((\bar{m}_0,0,0)\) with \( \bar{m}_0 = \frac{1}{|\Omega|} \int_{\Omega} m(x,0) dx \) in the large time limit (cf. [27]) at an exponential convergence rate (cf. [39]). Their convergence in small-convection limit \( \kappa \rightarrow 0 \) was later justified in [33]. Global existence of weak solutions for the three-dimensional chemotaxis-Navier-Stokes system was established in [36] under certain structural requirements on \( \chi \) and \( f \). Similarly to the 2D case, such weak solutions enjoy eventual smoothness and approach the unique spatially homogeneous steady state \((\bar{m}_0,0,0)\) as \( t \) goes to infinity (cf. [37]). Recently, the global solvability of the chemotaxis-Navier-Stokes system in a three-dimensional unbounded domain \( \Omega \) with infinite extent and finite depth was justified in [29] under appropriate smallness assumptions on initial data.

**Goals and Motivations.** As aforementioned most of the previously analytical studies devoted to the
chemotaxis-Navier-Sotkes system in the literature are confined to the fixed domain setting. However, the domain is normally deformable in natural conditions. For instance, considering a large variety of swimming micro-organisms live in the vast ocean lying above a rigid bottom it is realistic to investigate the dynamics of cell-fluid interactions with upper surface evolving in time. Allowing the motion of the upper surface and completing system (1.1) with appropriate boundary conditions, the linear and nonlinear stability analysis have been carried out in [7] along with supporting numerical simulations in a 2D shallow chamber and the effect of free-surface on bacterial plume patterns and their dynamics in both 2D and 3D cases have been recently explored numerically in [17, 18]. However, the rigorous mathematical analysis for (1.1) in time-dependent domains are lack of investigations even on the natural first question of its well-posedness.

Motivated by the above numerical results and the experiments conducted in [15, 16, 22], we shall investigate the global well-posedness of system (1.1) in the following three-dimensional moving domain above a rigid bottom defined by:

\[ \Omega_t = \{(x_1,x_2,y) \in \mathbb{R}^3 : -1 < y < \eta(x_1,x_2,t)\}, \]

where the surface function \( \eta(x_1,x_2,t) \) depending on the horizontal variable \((x_1,x_2) \in \mathbb{R}^2\) and temporary variable \(t\). For illustration, we denote \( S_F = \{(x_1,x_2,y) \in \mathbb{R}^3 : y = \eta(x_1,x_2,t)\}, S_B = \{(x_1,x_2,y) \in \mathbb{R}^3 : y = -1\} \) and \( \Omega_0 = \{(x_1,x_2,y) \in \mathbb{R}^3 : -1 < y < \eta_0(x_1,x_2)\}. \)

The choice on chemotactic intensity and oxygen consumption rate in the present paper is \( \chi(c) = \frac{1}{c} \) and \( f(c) = c \). Substituting \( \chi(c) = \frac{1}{c} \) into the chemotactic sensitivity function \( \chi(c)\nabla c \) in the first equation of (1.1) leads to the logarithmic sensing \( \nabla \ln c \), which has been experimentally verified in [19]. This logarithm results in a mathematically unfavorable singularity which, however, has been adopt in the literature to describe various chemotaxis process e.g. dynamical interactions between vascular endothelial cells and signaling molecules vascular endothelial growth factor in the initiation of tumor angiogenesis (cf. [22]), the boundary movement of chemotactic bacterial populations (cf. [28]) and the travelling band behavior of bacteria (cf. [20, 26]). With such choice on \( \chi(c) \) and \( f(c) \), the initial-boundary problem studied in the present paper reads

\[
\begin{align*}
\left\{ \begin{array}{ll}
m_t + \bar{u} \cdot \nabla m + \nabla \cdot (m \nabla \ln c) = \Delta m & \text{for } (x_1,x_2,y) \in \Omega_t \text{ and } t > 0, \\
c_t + \bar{u} \cdot \nabla c + mc = \Delta c, \\
\bar{u}_t + \bar{u} \cdot \nabla \bar{u} + \nabla p + m \nabla \Phi = \Delta \bar{u}, \\
\nabla \cdot \bar{u} = 0, \\
m(x_1,x_2,0) = m_0, \quad c(x_1,x_2,0) = c_0, \quad \bar{u}(x_1,x_2,0) = \bar{u}_0 & \text{for } (x_1,x_2,y) \in \Omega_0,
\end{array} \right.
\end{align*}
\]

with the following boundary conditions on \( S_F \):

\[
\begin{align*}
\left\{ \begin{array}{ll}
(\nabla m - m \nabla \ln c) \cdot \bar{n} = 0, \\
c = \hat{c}, \\
\eta_t = u_3 - u_1 \partial_1 \eta - u_2 \partial_2 \eta, \\
pn_i - (\partial_i u_1 + \partial_i u_2)n_j = \left\{ \gamma \eta - \sigma \nabla \cdot \left( \frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}} \right) \right\} n_i
\end{array} \right. \quad \text{for } (x_1,x_2,y) \in \Omega_0,
\end{align*}
\]

and the following boundary conditions on \( S_B \):

\[
m = 0, \quad \partial_3 c = 0, \quad \bar{u} = 0,
\]

where \( \hat{c} \) is a positive constant, \( \bar{n} = (n_1,n_2,n_3) \) is the outward unit normal to \( S_F \) and we sum upon repeated indices following the Einstein convention (this convention will be used in the remaining part of this paper without further clarification). \( D_m, D_c, D \) have been taken to be 1 without loss of generality and initial-boundary conditions have been imposed.

We next briefly introduce the derivation of the boundary conditions in (1.4) on \( S_F \).
• the kinematic condition: denote the free surface by \( d(x_1, x_2, y, t) = y - \eta(x_1, x_2, t) = 0 \). Since fluid particles do not cross the free surface, we have \((\partial_t + \hat{u} \cdot \nabla)(y - \eta(x_1, x_2, t)) = 0\), which results in \( \eta_t = u_3 - u_1 \partial_1 \eta - u_2 \partial_2 \eta \). Further discussion of this condition can be found, e.g., in [34, page 451].

• the normal force balance condition: the last boundary condition in (1.4) states a discontinuity in the normal stress on two sides of the free surface which, is proportional to the mean curvature of the surface and produced by the effect of surface tension, where \((p - \gamma \eta)n_i - (\partial_j u_i + \partial_i u_j)n_j\) is the normal stress tensor, \(\nabla \cdot \left( \frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}} \right) n_i\) is the mean curvature of the surface, \(\gamma > 0\) is the acceleration of gravity and \(\sigma > 0\) denotes the coefficient of surface tension. See [34, page 451-454] for detailed derivation of this condition.

• the zero-flux boundary condition on \( m \) and the Dirichlet boundary condition on \( c \) in (1.4) are physical ones that match the boundary descriptions in the experiment conducted in [32] and the numerical analysis in [7, 17, 18].

As mentioned before, our goal is to establish the global solvability of system (1.3)-(1.5) under appropriate small assumptions on initial data. To this end, we shall first apply the following transformation (cf. [38])

\[
\tilde{c} = -\ln c + \ln \hat{c}
\]

(1.6)
to system (1.3)-(1.5) to resolve the logarithmic singularity in its first equation and study the global well-posedness of the following transformed system

\[
\begin{aligned}
& m_t + \hat{u} \cdot \nabla m - \nabla \cdot (m \nabla \tilde{c}) = \Delta w \quad \text{for } (x_1, x_2, y) \in \Omega, \quad t > 0, \\
& \tilde{c}_t + \hat{u} \cdot \nabla \tilde{c} + |\nabla \tilde{c}|^2 - m = \Delta \tilde{c}, \\
& \tilde{u}_t + \hat{u} \cdot \nabla \tilde{u} + \nabla p + m \nabla \Phi = \Delta \hat{u}, \\
& \nabla \cdot \tilde{u} = 0, \\
& m(x_1, x_2, y, 0) = m_0, \quad \tilde{c}(x_1, x_2, y, 0) = \tilde{c}_0, \quad \tilde{u}(x_1, x_2, y, 0) = \tilde{u}_0 \quad \text{for } (x_1, x_2, y) \in \Omega_0,
\end{aligned}
\]

(1.7)

with

\[
\begin{aligned}
& \left\{ \begin{array}{l}
(\nabla m + m \nabla \tilde{c}) \cdot \tilde{n} = 0, \\
\tilde{c} = 0, \\
\eta_t = u_3 - u_1 \partial_1 \eta - u_2 \partial_2 \eta, \\
pn_i - (\partial_j u_i + \partial_i u_j)n_j = \left\{ \gamma \eta - \sigma \nabla \cdot \left( \frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}} \right) \right\} n_i
\end{array} \right. \\
& \text{on } S_F \times (0, \infty) \quad \text{and} \\
& m = 0, \quad \partial_3 \tilde{c} = 0, \quad \tilde{u} = 0
\end{aligned}
\]

(1.8)

(1.9)

on \( S_B \times (0, \infty) \).

2 Main Results

Notation. For clarity, we specify some notations below.

• \( \Omega = \{ (x_1, x_2, y) \in \mathbb{R}^3 : -1 < y < \eta(x_1, x_2, t) \} \).

• \( S_B = \{ (x_1, x_2, y) \in \mathbb{R}^3 : y = -1 \} \).

• \( S_F = \{ (x_1, x_2, y) \in \mathbb{R}^3 : y = \eta(x_1, x_2, t) \} \).

• \( \Omega_0 = \{ (x_1, x_2, y) \in \mathbb{R}^3 : -1 < y < \eta_0(x_1, x_2) \} \).
\[ S_0 = \{(x_1, x_2, y) \in \mathbb{R}^3 : y = \eta_0(x_1, x_2)\} \]

\[ \Omega = \{(x_1, x_2, y) \in \mathbb{R}^3 : -1 < y < 0\} \]

\[ \Gamma = \{(x_1, x_2, y) \in \mathbb{R}^3 : y = 0\} \]

We denote \( dx = dx_1 dx_2 \) for \( \bar{x} = (x_1, x_2) \in \mathbb{R}^2 \) and denote \( dxdy = dx_1 dx_2 dy \) for \( (x_1, x_2, y) \in \Omega \).

\[ [\vec{v}, \vec{u}] = \frac{1}{2} \int_{\Omega} (\partial_j v_i + \partial_i v_j)(\partial_j u_i + \partial_i u_j) dxdy \] for \( \vec{v} = (v_1, v_2, v_3) \) and \( \vec{u} = (u_1, u_2, u_3) \).

\( \nabla_0 \) denotes tangential gradient along the \( x_1 - x_2 \) plane.

\( H^m (m \geq 1) \) represents \( H^m(\Omega) \) and \( L^p (p \geq 1) \) stands for \( L^p(\Omega) \). For simplicity, we use \( \| \cdot \| \) to denote \( \| \cdot \|_{L^2(\Omega)} \).

\( \mathfrak{D}H^1 \) and \( 0H^1 \) represent the subspace of \( H^1(\Omega) \) consisting of functions which vanish on \( S_B \) and \( \Gamma \), respectively.

For any Banach space \( X \), we use \( X' \) to denote its dual space. In particular, \( H^{-\frac{1}{2}}(\Gamma) \) represents the dual space of \( H^{\frac{1}{2}}(\Gamma) \). We use \( \| \cdot \|_{L_q(\Omega)} \) to denote \( \| \cdot \|_{L_q(0, t; X)} \) for any Banach space \( X \) and \( t > 0 \).

\( \mathcal{H} \) is the harmonic extension operator, also denoted by \( \mathcal{H}(\eta) = \bar{\eta} \), extending functions defined on \( \Gamma \) to harmonic functions on \( \Omega \) with zero Neumann boundary condition on \( S_B \). Specifically, for any \( \eta(x_1, x_2) \) defined on \( \Gamma \), its harmonic extension \( \bar{\eta} \) solves

\[
\begin{align*}
\Delta \bar{\eta} &= 0 \quad \text{in} \ \Omega, \\
\bar{\eta} &= \eta \quad \text{on} \ \Gamma, \\
\partial_3 \bar{\eta} &= 0 \quad \text{on} \ S_B.
\end{align*}
\]

Compatibility conditions. Since the problem (1.7)-(1.9) is posed on a domain with boundary, it is natural to impose on initial data the following compatibility conditions:

\[
\begin{align*}
\left[(\partial_j u_0 i + \partial_i u_0 j)n_0\right]_{\tan} &= 0 \quad \text{on} \ S_0, \\
\nabla \cdot \vec{u}_0 &= 0 \quad \text{in} \ \Omega_0, \\
\left(\nabla m_0 + m_0 \nabla \tilde{c}_0\right) \cdot \bar{n}_0 &= 0, \quad \tilde{c}_0 = 0 \quad \text{on} \ S_0, \\
m_0 &= 0, \quad \partial_3 \tilde{c}_0 = 0 \quad \bar{u}_0 = 0 \quad \text{on} \ S_B, \\
\end{align*}
\]

where “tan” means the tangential component, \( \bar{n}_0 = (n_{01}, n_{02}, n_{03}) \) is the outward unit normal to the initial surface \( S_0 \) and the first condition is obtained by taking the inner product of the last equality in (1.8) with any tangential vector on \( S_0 \).

We are now in a position to state the main results of this paper.

**Theorem 2.1.** Suppose that the initial data \( m_0, \tilde{c}_0, \vec{u}_0 \in H^2(\Omega_0) \) and \( \eta_0 \in H^3(\mathbb{R}^2) \) fulfill the compatibility conditions (2.2). Then there exists a constant \( \varepsilon_0 > 0 \) suitably small such that if \( \| \eta_0 \|_{H^3(\mathbb{R}^2)} + \| m_0 \|_{H^2(\Omega_0)} + \| \tilde{c}_0 \|_{H^2(\Omega_0)} + \| \vec{u}_0 \|_{H^2(\Omega_0)} < \varepsilon_0 \), system (1.7)–(1.9) admits a unique solution \((m, \tilde{c}, \vec{u}, \nabla p, \eta)\) satisfying

\[
\begin{align*}
\sup_{t > 0} \left( \| m(t) \|_{H^2(\Omega_t)} + \| \tilde{c}(t) \|_{H^2(\Omega_t)} + \| \vec{u}(t) \|_{H^2(\Omega_t)} + \| \nabla p(t) \|_{L^2(\Omega_t)} + \| \eta(t) \|_{H^3(\mathbb{R}^2)} \right)^2 \nonumber \\
+ \int_0^t \left( \| m(t) \|_{H^3(\Omega_t)} + \| \tilde{c}(t) \|_{H^3(\Omega_t)} + \| \vec{u}(t) \|_{H^3(\Omega_t)} + \| \nabla p(t) \|_{H^3(\Omega_t)} + \| \nabla_0 \eta(t) \|_{H^{\frac{3}{2}}(\mathbb{R}^2)} \right)^2 dt \\
\leq C \left( \| m_0 \|_{H^2(\Omega_0)} + \| \tilde{c}_0 \|_{H^2(\Omega_0)} + \| \vec{u}_0 \|_{H^2(\Omega_0)} + \| \eta_0 \|_{H^3(\mathbb{R}^2)} \right)^2 \nonumber
\end{align*}
\]
for some positive constant C.

With (1.6) and the results obtained for the transformed system (1.7)-(1.9), we have the following assertions for the initial-boundary value problem (1.3)-(1.5).

**Theorem 2.2.** Let \( \hat{c}, c_0 > 0 \) and \( m_0 \geq 0 \). Suppose the assumptions in Theorem 2.1 hold with \( \tilde{c}_0 = -\ln c_0 + \ln \hat{c} \). Then there exists a unique solution \((m, c, \bar{u}, \nabla p, \eta)\) to system (1.3)-(1.5) satisfying that

\[
(m(x_1, x_2, y, t)) \geq 0 \quad \text{and} \quad c(x_1, x_2, y, t) > 0
\]

for \((x_1, x_2, y) \in \Omega_t \) and \( t > 0 \), and that

\[
\sup_{t \geq 0} \left( \left| \frac{m(t)}{H^2(\Omega_0)} + \left| c(t) - \hat{c} \right| \frac{H^2(\Omega_0)}{H^2(\Omega_0)} + \left| \bar{u}(t) \right| \frac{H^2(\Omega_0)}{H^2(\Omega_0)} + \left| \nabla p(t) \right| \frac{1}{L^2(\Omega_0)} + \left| \frac{\eta(t)}{H^1(\mathbb{R}^2)} \right| \right)^2
\]

\[
+ \int_0^\infty \left( \left| \frac{m(t)}{H^2(\Omega_0)} + \left| c(t) - \hat{c} \right| \frac{H^2(\Omega_0)}{H^2(\Omega_0)} + \left| \bar{u}(t) \right| \frac{H^2(\Omega_0)}{H^2(\Omega_0)} \right)^2 dt
\]

\[
+ \int_0^\infty \left( \left| \nabla p(t) \right| \frac{H^2(\Omega_0)}{H^2(\Omega_0)} + \left| \nabla \theta(t) \right| \frac{H^2(\mathbb{R}^2)}{H^2(\mathbb{R}^2)} \right)^2 dt
\]

\[
\leq C \left( \left| m_0 \frac{H^2(\Omega_0)}{H^2(\Omega_0)} + \left| \ln c_0 - \ln \hat{c} \right| \frac{H^2(\Omega_0)}{H^2(\Omega_0)} + \left| \bar{u}_0 \right| \frac{H^2(\Omega_0)}{H^2(\Omega_0)} + \left| \frac{\eta_0}{H^1(\mathbb{R}^2)} \right| \right)^2
\]

for some positive constant C.

We next briefly outline the main ideas and organisation of this paper. Due to the lack of well-posedness theory for the parabolic systems on a moving domain, we shall transform (1.7)-(1.9) into the nonlinear initial-boundary problem (3.10)-(3.12) on an equilibrium domain in Section 3, whose detailed derivations are postponed to be given in the Appendix. Solvability of the corresponding linear system on the equilibrium domain is demonstrated in Section 4. Having solved the linearized problem, we construct approximation solutions for the nonlinear problem (3.10)-(3.12) by iteration and prove that such approximation solutions are uniformly bounded provided the initial data are sufficient small in Section 5. The proof of Theorem 2.1 and Theorem 2.2 is given in Section 6 by first applying the contraction mapping theorem to the approximation sequence to derive the unique solution of (3.10)-(3.12) on the equilibrium and then reversing the transformation defined in Section 3 to obtain the solution of the original problem.

The overall analysis is based on some a priori estimates along with the well-posedness theory on linear parabolic and elliptic systems. When solving the linear system in Section 4, the main challenge that we are face of is that we can not address the higher energy estimates in the vertical direction for solution \((w, h, \bar{v})\) by directly taking the partial derivatives in the equations (4.1)-(4.2) since the equilibrium domain is not translation-invariant in the vertical spatial variable. By using the translation-invariant property of the domain in the horizontal direction, we start by taking tangential derivatives in the first equation of (4.2) to gain energy bounds for the tangential derivatives of the velocity field \(\bar{v}\). With these tangential bounds for \(\bar{v}\) in hand, one can actually go further to bound the vertical derivatives. Indeed, by taking advantage of the divergence-free condition on \(\bar{v}\), the vertical estimates for its third component \(v_3\) readily follows from these tangential estimates (see Lemma 4.3 and the proof of Proposition 4.3). The vertical bounds for its first and second component, \(v_1\) and \(v_2\) are achieved by employing the elliptic theory with the aid of the first two boundary conditions in (4.2) and the estimation of the time derivative of the velocity field (see Lemma 4.9 and the proof of Proposition 4.3). The other solution components \((w, h)\) are estimated in a similar fashion, i.e. by employing the elliptic theory and trading spatial derivatives with time derivative.

## 3 Transformation to an equilibrium domain

Since we do not know how to solve (1.7)-(1.9) locally-in-time on the moving domain \(\Omega_t\), following [4] we shall transform the system to one on the equilibrium domain \(\Omega\). Define \(\theta : \Omega \to \Omega_t = \)
\{(x_1,x_2,y)| (x_1,x_2) \in \mathbb{R}^2, \ -1 < y < \eta(x_1,x_2,t) \} by

\[ \theta(x_1,x_2,y,t) := (\theta_1, \theta_2, \theta_3)(x_1,x_2,y,t) = (x_1,x_2, \bar{\eta} + y(1 + \bar{\eta})) \]  \hspace{1cm} (3.1)

where \( \bar{\eta} \) is the harmonic extension of \( \eta \). Then

\[ d\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & \beta & J \end{pmatrix}, \quad (\xi_{ij})_{3 \times 3} := (d\theta)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -J^{-1}\alpha & -J^{-1}\beta & J^{-1} \end{pmatrix} \]  \hspace{1cm} (3.2)

with

\[ \alpha := (1+y)\partial_1 \bar{\eta}, \quad \beta := (1+y)\partial_2 \bar{\eta}, \quad J := 1 + \bar{\eta} + \partial_3 \bar{\eta}(1+y). \]  \hspace{1cm} (3.3)

For \((x_1,x_2,y,t) \in \Omega \times (0,\infty)\) we define

\[ w(x_1,x_2,y,t) = m(\theta(x_1,x_2,y,t),t), \quad h(x_1,x_2,y,t) = \tilde{c}(\theta(x_1,x_2,y,t),t), \]

\[ q(x_1,x_2,y,t) = p(\theta(x_1,x_2,y,t),t), \quad \phi(x_1,x_2,y,t) = \Phi(\theta(x_1,x_2,y,t),t). \]  \hspace{1cm} (3.4)

The velocity field \( \tilde{v}(x_1,x_2,y,t) = (v_1,v_2,v_3)(x_1,x_2,y,t) \) on the equilibrium domain is defined in the following way

\[ u_i(\theta(x_1,x_2,y,t),t) = J^{-1}(\partial_j \theta_j)(v(x_1,x_2,y,t)), \]  \hspace{1cm} (3.5)

that is,

\[ u_1(\theta(x_1,x_2,y,t),t) = J^{-1}v_1(x_1,x_2,y,t), \quad u_2(\theta(x_1,x_2,y,t),t) = J^{-1}v_2(x_1,x_2,y,t), \]

\[ u_3(\theta(x_1,x_2,y,t),t) = J^{-1}\alpha v_1(x_1,x_2,y,t) + J^{-1}\beta v_2(x_1,x_2,y,t) + v_3(x_1,x_2,y,t) \]  \hspace{1cm} (3.6)

to preserve the divergence-free condition. In particular, when \( t = 0 \)

\[ \theta(x_1,x_2,y,0) = (x_1,x_2, \bar{\eta}_0 + y(1 + \bar{\eta}_0)), \]  \hspace{1cm} (3.7)

where \( \bar{\eta}_0 \) is the harmonic extension of \( \eta_0 \). Corresponding to (3.3)-(3.5) we set

\[ \alpha_0 = (1+y)\partial_1 \bar{\eta}_0, \quad \beta_0 = (1+y)\partial_2 \bar{\eta}_0, \quad J_0 = 1 + \bar{\eta}_0 + \partial_3 \bar{\eta}_0(1+y) \]  \hspace{1cm} (3.8)

and

\[ w_0(x_1,x_2,y) = m_0(\theta(x_1,x_2,y,0)), \quad h_0(x_1,x_2,y) = \tilde{c}_0(\theta(x_1,x_2,y,0)), \]

\[ u_0(\theta(x_1,x_2,y,0)) = J_0^{-1}(\partial_j \theta_j(x_1,x_2,y,0))v_0(x_1,x_2,y). \]  \hspace{1cm} (3.9)

Then by the chain rule and direct computations one can deduce from (1.7)-(1.9) that

\[ \begin{cases} w_t - \Delta w - \nabla \cdot (w \nabla h) = F_4(w,h,\bar{v},\bar{\eta},\nabla q), \\
\eta_t - \Delta h - w = F_5(w,h,\bar{v},\bar{\eta},\nabla q), \\
\bar{v}_t - \Delta \bar{v} + \nabla q + w \nabla \bar{\phi} = \bar{F} (w,h,\bar{v},\bar{\eta},\nabla q), \\
\nabla \cdot \bar{v} = 0,
\end{cases} \]  \hspace{1cm} (3.10)

with the following boundary conditions on \( \Gamma \times (0,\infty): \)

\[ \begin{cases} \partial_3 w + w \partial_3 h = G_4(w,h,\bar{\eta}), \\
\partial_3 v_1 + \partial_1 v_3 = G_1(\bar{v},\bar{\eta}), \\
\partial_3 v_2 + \partial_2 v_3 = G_2(\bar{v},\bar{\eta}), \\
\bar{\eta}_t = v_3, \quad q - 2 \partial_3 v_3 = \gamma \eta - \sigma \Delta_0 \eta - G_3(\bar{v},\bar{\eta}) \end{cases} \]  \hspace{1cm} (3.11)

and the following boundary conditions on \( \Sigma \times (0,\infty): \)

\[ w = 0, \quad \partial_3 h = 0, \quad \bar{v} = 0. \]  \hspace{1cm} (3.12)

The detailed derivation of (3.10)-(3.12) is given in appendix with nonlinear terms \( \bar{F} = (F_1,F_2,F_3), \)

\( \bar{G} = (G_1,G_2,G_3) \) and \( F_4, F_5, G_4 \) defined in (7.10), (7.12), (7.14) and (7.5) respectively.
4 Solvability of the linear problem

This section is devoted to proving the solvability of the linearized version of (3.10)-(3.12). We first split the corresponding linear system into the following two initial-boundary value problems:

\[
\begin{align*}
&\begin{cases}
w_t - \Delta w - \nabla \cdot (a \nabla h) = f_4, & (x_1, x_2, y, t) \in \Omega \times (0, \infty), \\
h_t - \Delta h - w = f_5,
\end{cases} \\
&w(x_1, x_2, y, 0) = w_0, \\n&h(x_1, x_2, y, 0) = h_0,
\end{align*}
\]

and

\[
\begin{align*}
&\begin{cases}
\bar{v}_t - \Delta \bar{v} + \nabla q + \nabla \phi = \bar{f}, & (x_1, x_2, y, t) \in \Omega \times (0, \infty), \\
\nabla \cdot \bar{v} = 0,
\end{cases} \\
&\bar{v}(x_1, x_2, y, 0) = \bar{v}_0,
\end{align*}
\]

\[
\begin{align*}
&\partial_3 \bar{v}_1 + \partial_1 \bar{v}_3 = g_1, \\n&\partial_3 \bar{v}_2 + \partial_2 \bar{v}_3 = g_2 \\
&\text{on } \Gamma \times (0, \infty), \\
&\eta_t = v_3, \\
&\text{on } S_B \times (0, \infty).
\end{align*}
\]

We introduce two notations for later use:

\[
\|f\| := \|f\|_{L^\infty}^2 + \|f_t\|_{L^2} + \|f\|_{L^2} + \|f_t\|_{L^2}.
\]

and

\[
\|\{w, h, \bar{v}, \eta, q\}\| := \|w\| + \|h\| + \|\bar{v}\| + \|\nabla \bar{v}\|_{L^2} + \|\nabla q\|_{L^2} + \|\nabla q_t\|_{L^2} + \|\eta\|_{L^2} + \|\eta_t\|_{L^2} + \|\nabla \phi\|_{L^2}.
\]

To solve system (4.1)-(4.2), the initial and boundary data are required to satisfy the following compatibility conditions:

\[
\begin{align*}
&\begin{cases}
\partial_3 w_0 + a(x_1, x_2, y, 0) \partial_3 h_0 = g_4(x_1, x_2, 0), \\
w_0 = 0 \text{ on } \Gamma, \\n\partial_3 h_0 = 0 \text{ on } S_B,
\end{cases} \\
&w_0 = 0, \\
&h_0 = 0 \text{ on } \Gamma,
\end{align*}
\]

and

\[
\begin{align*}
&\begin{cases}
\partial_3 v_0 + \partial_1 v_3 = g_1(x_1, x_2, 0), \\
\partial_3 v_2 + \partial_2 v_3 = g_2(x_1, x_2, 0) \text{ on } \Gamma, \\
\nabla \cdot \bar{v}_0 = 0 \text{ in } \Omega, \\
\bar{v}_0 = 0 \text{ on } S_B.
\end{cases}
\end{align*}
\]

Then the solvability of the linear system (4.1)-(4.2) is as follows.

**Proposition 4.1.** Let \(w_0, h_0, \bar{v}_0 \in H^2(\Omega), \eta_0 \in H^3(\mathbb{R}^2)\) and

\[
\begin{align*}
&\begin{cases}
\nabla \phi \in L^\infty(0, \infty; H^1), \\
\nabla \phi_t \in L^2(0, \infty; L^2), \\
\bar{g}, \bar{g}_1, \bar{g}_2, \bar{g}_4, \bar{g}_t, \bar{g}_t, \bar{g}_t, \bar{g}_t \in L^2(0, \infty; H^{-\frac{1}{2}}(\Gamma)), \\
\bar{f}, \bar{f}_1, \bar{f}_2, \bar{f}_3, \bar{f}_4, \bar{f}_5 \in L^2(0, \infty; H^{\frac{1}{2}}(\Omega)), \quad \bar{f}_5 \in L^2(0, \infty; H^{-\frac{1}{2}}(\Gamma)),
\end{cases}
\end{align*}
\]

satisfy the compatibility conditions (4.4)-(4.5). Assume further that the function \(a\) fulfills

\[
C_1 (C_2 + 1) \|a\|^2 \leq \frac{1}{2} \quad \text{for all } t \geq 0,
\]
where the constants $C_1$ and $C_2$ are independent of $t$, given in (4.21) and (4.32). Then system (4.1)–(4.2) admits a unique global solution $(w, h, \bar{v}, \eta, \nabla q)$ such that

$$
\| \{ w, h, \bar{v}, \eta, q \} \|^2 \\
\leq C ( \| w_0 \|^2_{H^2(\Omega)} + \| h_0 \|^2_{H^2(\Omega)} + \| \bar{v}_0 \|^2_{H^2(\Omega)} + \| \eta_0 \|^2_{H^2(\mathbb{R}^3)} + 4 \| \bar{v}_0 \|^2_{H^2(\Omega)} + \| \eta_0 \|^2_{H^2(\mathbb{R}^3)}) \\
+ C ( \| f_4 \|^2_{L^2(\Omega)} + \| f_4 u \|^2_{L^2(\partial \Omega)} + \| f_4 \|^2_{L^2(\Omega)} + \| f_4 \|^2_{L^2(\partial \Omega)} + \| f_4 \|^2_{L^2(\Omega)} + \| g_4 \|^2_{L^2(\Omega)}) \\
+ C ( \| g_4 \|^2_{L^2(\Omega)} + \| \bar{f}_4 \|^2_{L^2(\Omega)} + \| \bar{f}_4 \|^2_{L^2(\partial \Omega)} + \| \bar{g}_4 \|^2_{L^2(\Omega)} + \| \bar{g}_4 \|^2_{L^2(\partial \Omega)})
$$

(4.8)

for all $t \geq 0$, where the constant $C$ is independent of $t$.

The remaining part of this section is organized as follows. In next subsection, we shall introduce some preliminaries that will be used later. The solvability of subsystem (4.1) and subsystem (4.2) will be established in subsection 4.2 and subsection 5.2, respectively. Subsection 4.4 is devoted to proving Proposition 4.1 based on the results derived for subsystems (4.1) and (4.2).

4.1 Preliminaries

This subsection is devoted to exhibiting some preliminaries for later use. Noting the divergence-free condition $\nabla \cdot \bar{v} = 0$ and the gradient form $\nabla q$ of the pressure in (4.2), from the following identity

$$
\int_{\Omega} \bar{v} \cdot \nabla q \, dx dy = - \int_{\Omega} (\nabla \cdot \bar{v}) q \, dx dy + \int_{\partial \Omega} (\nabla \cdot \bar{n}) q \, ds
$$

one can see that the vectors $\bar{v}$ and $\nabla q$ are $L^2$-orthogonal if they satisfy the boundary condition $(\bar{v} \cdot \bar{n}) q = 0$ on $\partial \Omega$. This observation has been used in treating equations of incompressible fluids in a fixed domain to remove the pressure as an unknown by projecting the equation onto a subspace of space fields of divergence-free that satisfy the same boundary conditions as the velocity (see e.g. [13], [31]). In the present case, since $\bar{v} \cdot \bar{n} = 0$ on $S_B$ and $\nabla \cdot \bar{v} = 0$ in $\Omega$ it follows from (4.9) that a vector in the gradient form $\nabla \bar{p}$ is $L^2$-orthogonal to $\bar{v}$ if and only if $\bar{p} = 0$ on $\Gamma$. With this in mind, we introduce the projection $P$ on $L^2(\Omega)$ orthogonal to

$$
W := \{ \nabla \bar{p} : \bar{p} \in H^1(\Omega), \bar{p} = 0 \text{ on } \Gamma \}.
$$

(4.10)

For this projection $P$, the following property has been proved in [3].

Lemma 4.1. (\cite[Lemma 2.1]{4}) Let $P$ be the projection on $L^2(\Omega)$ orthogonal to the subspace $W$ given in (4.10). Then $P$ is a bounded operator on $H^k(\Omega)$ for $k \geq 0$. In particular, for $k \geq 1$

$$
P H^k(\Omega) = \{ \bar{v} \in H^k(\Omega) : \nabla \cdot \bar{v} = 0, \bar{v} \cdot \bar{n} = 0 \text{ on } S_B \}.
$$

Moreover, for any $\bar{p}(x_1, x_2, y) \in H^1(\Omega)$ with $\bar{p}(x_1, x_2, 0) = \xi(x_1, x_2)$ on $\Gamma$, the following holds true

$$
P(\nabla \bar{p}) = \nabla \mathcal{H}(\xi).
$$

We shall use the following version of Korn’s inequality, which has been proved in [3].

Lemma 4.2. (\cite[Lemma 2.7]{4}) Let $\bar{v} \in 0 H^1(\Omega)$. Then there exists a constant $C$ such that

$$
\| \bar{v} \|^2_{H^1} \leq C [ \bar{v}, \bar{v} ].
$$

Lemma 4.3. Let $\eta(x_1, x_2, y) = \mathcal{H}(\eta)$ be the harmonic extension of $\eta(x_1, x_2)$. Then for any integer $m \geq 2$

$$
\| \eta \|^2_{H^m} \leq C \| \eta \|^2_{H^{m-\frac{1}{2}}(\mathbb{R}^3)}.
$$

(4.11)
Lemma 3.3(ii) and [4, Lemma 4.2(i)] to system (4.18), one immediately gets

By using the divergence-free condition in elliptic system (4.18), the author in [4] has proved that the vertical derivatives of \( \vec{v} \) can be estimated in terms of its tangential derivatives. Indeed, applying [4, Lemma 3.1] to system (4.13) and using the trace theorem one gets

Applying the standard elliptic theory (see e.g. [3, Lemma 2.8]) to system (4.13) and using the trace theorem one gets

which, along with (4.12) leads to (4.11). The proof is completed.

When solving (4.2), we shall employ the following elliptic system

Applying the standard elliptic theory (see e.g. [3, Lemma 2.8]) to system (4.13) and using the trace theorem one gets

Lemma 4.4. Let \((\vec{v}, \nabla \vec{q})\) be the solution of (4.15). Then for any \(m \geq 2\), the following estimates hold:

for some positive constant \(C\), provided that the right-hand side is finite.

Proof. Following the notations in [4], we define \(P_0\) as the projection in \(L^2(\Omega)\) orthogonal to the subspace consisting of gradients. Then it follows from [4, Lemma 3.1] that \(P_0\) is a bounded operator on \(H^r(\Omega)\) with \(r \geq 0\) satisfying

Applying \(P_0\) to the first equation of (4.15) and using (4.17) we see that \(\vec{v}\) solves

By using the divergence-free condition in elliptic system (4.18), the author in [4] has proved that the vertical derivatives of \(\vec{v}\) can be estimated in terms of its tangential derivatives. Indeed, applying [4, Lemma 3.3(ii)] and [4, Lemma 4.2(i)] to system (4.18), one immediately gets

\[
\|\vec{v}\|_{H^m} \leq C \left( \|\vec{F}\|_{H^{m-2}} + \sum_{k=0}^{m} \|\nabla^k \vec{v}\| \right)
\]
with \( m \geq 2 \). Then it follows from the first equation of (4.15) that
\[
\| \nabla q \|_{\mathcal{H}^{m-2}} \leq C \left( \| \bar{F} \|_{\mathcal{H}^{m-2}} + \| \Delta \bar{v} \|_{\mathcal{H}^{m-2}} \right) \leq C \left( \| \bar{F} \|_{\mathcal{H}^{m-2}} + \sum_{k=0}^{m} \| \nabla_k^2 \bar{v} \| \right)
\]
which, along with (4.19) gives (4.16). The proof is completed.

\[
\square
\]

### 4.2 Solvability of system (4.1)

**Proposition 4.2.** Let \( w_0, h_0 \in \mathcal{H}^2(\Omega) \), \( f_4, f_5, g_4 \) and \( a \) satisfy (4.6), (4.7) and the compatibility conditions (4.4). Then there exists a unique solution of (4.1) such that
\[
\begin{align*}
\| w \|^2 + \| h \|^2 &\leq C_3 \left( \| w_0 \|^2_{\mathcal{H}^2(\Omega)} + \| h_0 \|^2_{\mathcal{H}^2(\Omega)} \right) + C_3 \left( \| f_4 \|^2_{L^2 \mathcal{H}^1(\Gamma)} + \| f_5 \|^2_{L^2 \mathcal{H}^1(\Gamma)} \right) \\
&\quad + C_3 \left( \| f_4 \|^2_{L^2 \mathcal{H}^1(\Gamma)} + \| f_5 \|^2_{L^2 \mathcal{H}^1(\Gamma)} \right) + C_3 \left( \| g_4 \|^2_{L^2 \mathcal{H}^{3/4}(\Gamma)} + \| g_5 \|^2_{L^2 \mathcal{H}^{3/4}(\Gamma)} \right)
\end{align*}
\]
for all \( t \geq 0 \), where the constant \( C_3 \) is independent of \( t \).

The proof of Proposition 4.2 is based on the following two lemmas where the \emph{a priori} estimate on solutions \((w,h)\) is derived.

**Lemma 4.5.** Let the assumptions in Proposition 4.2 hold. Then the solution \((w,h)\) of system (4.1) satisfies
\[
\begin{align*}
\| w \|^2 &\leq \| w_0 \|^2_{\mathcal{H}^2(\Omega)} + C_1 \left( \| a \|^2 \| h \|^2 \right) + C_1 \left( \| g_4 \|^2_{L^2 \mathcal{H}^{3/4}(\Gamma)} + \| g_5 \|^2_{L^2 \mathcal{H}^{3/4}(\Gamma)} \right) \\
&\quad + C_1 \left( \| f_4 \|^2_{L^2 \mathcal{H}^1(\Gamma)} + \| f_5 \|^2_{L^2 \mathcal{H}^1(\Gamma)} \right).
\end{align*}
\]

**Proof.** We first estimate \( \| w_t \|^2_{L^2} + \| \nabla w_t \|^2_{L^2} \). Differentiating the first equation of (4.1) with respect to \( t \) one has
\[
w_t - \Delta w_t - \nabla \cdot (a \nabla h)_t = f_4.
\]
Then multiplying (4.22) by \( 2 w_t \) in \( L^2 \) and using integration by parts, we get
\[
\begin{align*}
\frac{d}{dt} \| w_t \|^2 + 2 \| \nabla w_t \|^2 &= 2 \int_\Omega (\partial_t w + a \partial_t h)_t w_t dx - 2 \int_\Omega (a \nabla h)_t \cdot \nabla w_t dxdy + 2 \int a \nabla h \cdot w_t dxdy \\
&\quad + 2 \int a \nabla h_t \cdot \nabla w_t dxdy + 2 \int f_4 w_t dxdy \\
&= I_1 + I_2 + I_3.
\end{align*}
\]
By the trace theorem we have
\[
I_1 \leq \mathcal{E} \| w_t \|^2_{L^2(\Gamma)} + C(\mathcal{E}) \| g_4 \|^2_{L^2 \mathcal{H}^{3/4}(\Gamma)} \leq C \mathcal{E} \| w_t \|^2_{H^1} + C(\mathcal{E}) \| g_4 \|^2_{H^1(\Gamma)},
\]
where \( C(\mathcal{E}) \) is a constant independent of \( t \), but depending on \( \mathcal{E} \). Then using the Poincaré inequality (Theorem 6.30) thanks to the fact \( w_t = 0 \) on \( S_B \), we can choose \( \mathcal{E} \) small enough such that \( C \mathcal{E} \| w_t \|^2_{H^1} \leq \frac{1}{3} \| \nabla w_t \|^2 \), which inserted into (4.24) gives rise to
\[
I_1 \leq \frac{1}{3} \| \nabla w_t \|^2 + C \| g_4 \|^2_{L^2 \mathcal{H}^{3/4}(\Gamma)}.
\]
It follows from the Sobolev embedding inequality that

\[
I_2 \leq \frac{1}{3} \| \nabla w_t \|^2 + C (\| a \|_{L^3}^2 \| \nabla h_t \|^2 + \| a_t \|_{L^3} \| \nabla h \|^2_{L^3})
\]

\[
\leq \frac{1}{3} \| \nabla w_t \|^2 + C (\| a \|_{L^3}^2 \| \nabla h_t \|^2 + \| a_t \|_{H^1} \| h \|^2_{H^2}).
\]

(4.26)

Noting \( w_t = 0 \) on \( S_B \), one employs the Poincaré inequality again to deduce\[
I_3 \leq \varepsilon \| w_t \|_{H^1}^2 + C(\varepsilon) \| f_{w_t} \|_{(0,H^1)}^2 \leq \frac{1}{3} \| \nabla w_t \|^2 + C \| f_w \|_{(0,H^1)}^2,
\]

where \( \varepsilon \) has been chosen small such that \( \varepsilon \| w_t \|_{H^1}^2 \leq \frac{1}{3} \| \nabla w_t \|^2 \). Substituting (4.25) and (4.27) into (4.23) and integrating the resulting inequality over \((0,t)\) we arrive at

\[
\| w_t \|_{L^2 t}^2 + \| \nabla w_t \|_{L^2 t}^2 \leq \| w_t(0) \|_{H^1}^2 + C(\| g_{w_t} \|_{L^2 t}^2 h_{t, \frac{1}{2}(x)} + \| a \|_{H^1}^2 \| \nabla h_t \|^2 + \| f_{w_t} \|_{L^2 t}^2). 
\]

(4.28)

The estimate of \( \| w_t(0) \|_{H^1}^2 \) follows from the first equation of (4.1) and the compactness theorem (see e.g. [23, Theorem 3.1, Chapter 1]):

\[
\| w_t(0) \|_{H^1}^2 \leq \| w_0 \|_{H^2}^2 + \| a(0) \|_{H^2}^2 \| h(0) \|_{H^1}^2 + \| f_4(0) \|^2
\]

\[
\leq \| w_0 \|_{H^2}^2 + C(\| a \|_{H^1}^2 \| h \|_{L^2 t}^2 h_{t, \frac{1}{2}(x)} + \| f_4 \|_{L^2 t}^2 + \| g_4 \|_{L^2 t}^2))
\]

which, substituted into (4.28) gives rise to

\[
\| w_t \|_{L^2 t}^2 + \| \nabla w_t \|_{L^2 t}^2 \leq \| w_0 \|_{H^2}^2 + \| a(0) \|_{H^2}^2 \| h(0) \|_{H^1}^2 + \| f_4(0) \|^2
\]

\[
C(\| a \|_{H^1}^2 \| h \|_{L^2 t}^2 h_{t, \frac{1}{2}(x)} + \| f_4 \|_{L^2 t}^2 + \| g_4 \|_{L^2 t}^2))
\]

(4.29)

We proceed to estimating \( \| w \|_{L^2 t}^2 \) and \( \| w \|_{L^2 t}^2 \). From (4.1) we know that \( w \) solves the following elliptic system

\[
\begin{aligned}
- \Delta w &= - w_t + \nabla \cdot (a \nabla h) + f_4, & \quad & (x_1, x_2, y, t) \in \Omega \times (0, \infty), \\
\partial_t w &= a \partial_t h + g_4 \quad & & \text{on } \Gamma \times (0, \infty), \\
w &= 0 \quad & & \text{on } S_B \times (0, \infty).
\end{aligned}
\]

(4.30)

Then it follows from the standard elliptic theory (see e.g. [3, Lemma 2.8]) that

\[
\| w \|_{H^r} \leq C (\| w_t \|_{L^2 t} + \| \nabla \cdot (a \nabla h) \|_{L^2 t} + \| f_4 \|_{L^2 t} + \| a \partial_t h \|_{H^r(\Gamma)} + \| g_4 \|_{H^r(\Gamma)})
\]

for fixed \( t > 0 \), with \( r \geq 2 \). Taking \( r = 3 \), one immediately gets

\[
\| w \|_{L^2 t}^2 \leq C (\| w_t \|_{L^2 t}^2 + \| \nabla \cdot (a \nabla h) \|_{L^2 t}^2 + \| f_4 \|_{L^2 t}^2 + \| a \partial_t h \|_{L^2 t}^2 h_{t, \frac{1}{2}(x)} + \| g_4 \|_{L^2 t}^2 h_{t, \frac{1}{2}(x)})
\]

\[
\leq C (\| w_t \|_{L^2 t}^2 + \| a \|_{L^2 t}^2 \| h \|_{L^2 t}^2 h_{t, \frac{1}{2}(x)} + \| f_4 \|_{L^2 t}^2 + \| g_4 \|_{L^2 t}^2 h_{t, \frac{1}{2}(x)}),
\]

(4.31)

where we have used the fact

\[
\| \nabla \cdot (a \nabla h) \|_{L^2 t} + \| a \partial_t h \|_{L^2 t}^2 \leq C (\| a \nabla h \|_{L^2 t} + \| a \|_{L^2 t}^2 h \|_{L^2 t}^2),
\]

thanks to the trace theorem and Sobolev embedding inequality. Combining (4.29) and (4.31) and using the Poincaré inequality \( \| w_t \|_{L^2 t}^2 \leq C \| \nabla w_t \|_{L^2 t}^2 \) (cf. [1, Theorem 6.30]) and the fact \( \| w \|_{L^2 t}^2 \leq C (\| w_t \|_{L^2 t}^2 + \| w \|_{L^2 t}^2) \) thanks to the compactness theorem (cf. [23, Theorem 3.1, Chapter 1]), we obtain (4.22). The proof is completed.

\[\square\]
Lemma 4.6. Suppose that the assumptions in Proposition 4.2 hold. Then the solution \((w, h)\) of (4.1) fulfills
\[
\|h\|^2 \leq \|h_0\|^2_{H^2(\Omega)} + C_2\|w\|^2 + C_2 \left( \|f_s\|^2_{L^2(\Omega)} + \|f_{\tilde{s}}\|^2_{L^2(\partial H^1)} \right). \tag{4.32}
\]

Proof. We first estimate \(\|h_t\|^2_{L^2} + \|\nabla h_t\|^2_{L^2}\). Differentiating the second equation of (4.1) with respect to \(t\), then multiplying the resulting equation with \(2h_t\) in \(L^2\) and using integration by parts we have
\[
\frac{d}{dt}\|h_t\|^2 + 2\|\nabla h_t\|^2 = 2\int_{\Omega} f_s h_t \, dx \, dy + 2\int_{\Omega} w_i h_t \, dx \, dy 
\leq \varepsilon \|h_t\|^2_{H^1} + C(\varepsilon) (\|f_s\|^2_{L^2(\partial H^1)} + \|w_t\|^2), \tag{4.33}
\]
where the constant \(C(\varepsilon)\) depends on \(\varepsilon > 0\). By the Poincaré inequality (see [1, Theorem 6.30]) one can choose \(\varepsilon\) small enough such that
\[
\varepsilon \|h_t\|^2_{H^1} \leq \|\nabla h_t\|^2. \tag{4.34}
\]
Inserting (4.34) into (4.33) and integrating the resulting inequality over \((0, t)\) to derive
\[
\|h_t\|^2_{L^2} + \|h_t\|^2_{H^1} \leq \|h_0\|^2_{H^2} + \|w\|^2 + C(\|f_s\|^2_{L^2(\partial H^1)} + \|f_{\tilde{s}}\|^2_{L^2(\partial H^1)}), \tag{4.35}
\]
where we have used the fact
\[
\|h_t(0)\|^2 \leq \|h_0\|^2_{H^2} + \|w\|^2 + C(\|f_s\|^2_{L^2(\partial H^1)} + \|f_{\tilde{s}}\|^2_{L^2(\partial H^1)}),
\]
thanks to the second equation of (4.1) and the compactness (cf. [23, Theorem 3.1, Chapter 1]). From (4.1) we know that for fixed \(t, h\) solves the following elliptic problem
\[
\begin{aligned}
-\Delta h &= f_s - h_t + w_i \quad (x_1, x_2, y, t) \in \Omega \times (0, \infty), \\
h &= 0 \quad \text{on } \Gamma \times (0, \infty), \\
\partial_t h &= 0 \quad \text{on } S_B \times (0, \infty). 
\end{aligned} \tag{4.36}
\]
Then applying the standard elliptic theory (see e.g. [3, Lemma 2.8]) to (4.36) one deduces that
\[
\|h_t\|^2_{L^2(\partial H^1)} \leq C(\|f_s\|^2_{L^2(\partial H^1)} + \|h_t\|^2_{L^2(\partial H^1)} + \|w\|^2_{L^2(\partial H^1)}),
\]
which, along with (4.35) and the fact \(\|h_t\|^2_{L^2} \leq C(\|h_t\|^2_{L^2(\partial H^1)} + \|h_t\|^2_{L^2(\partial H^1)})\) (cf. [23, Theorem 3.1, Chapter 1]) indicates (4.32). The proof is completed.

We next prove Proposition 4.2 by using Lemma 4.5 and Lemma 4.6.

Proof of Proposition 4.2. The local well-posedness of the initial-boundary problem (4.1) follows from the standard parabolic theory (see e.g. [21, Theorem 9.1]). We omit the details for brevity and proceed to the derivation of (4.20). Multiply (4.21) by \((C_2 + 1)\) then adding the resulting inequality to (4.32) one gets
\[
\|w_t\| + \|h_t\| \leq (C_2 + 1)\|w_0\|_{H^2(\Omega)} + \|h_0\|_{H^2(\Omega)} + C_1(C_2 + 1)\|\alpha\|_{H^2}^2 \|h_t\| + C_1(C_2 + 1)\|f_s\|_{L^2(\partial H^1)} + \|f_{\tilde{s}}\|_{L^2(\partial H^1)}^2 \tag{4.37}
\]
which, in conjunction with (4.7) gives (4.20). Furthermore, from the estimates (4.20) we know that the solution \((w, h)\) persists globally. The proof is finished.
4.3 Solvability of system (4.2)

Proposition 4.3. Suppose \( \tilde{v}_0 \in H^2(\Omega), \eta_0 \in H^3(\mathbb{R}^2), \nabla \phi, \tilde{f} = (f_1, f_2, f_3) \) and \( \tilde{g} = (g_1, g_2, g_3) \) satisfy (4.6) and the compatibility conditions (4.5). Assume further that

\[
 w \in L^\infty(0, \infty; H^2) \cap L^2(0, \infty; H^3), \quad w_i \in L^\infty(0, \infty; L^2) \cap L^2(0, \infty; H^1). 
\]

Then system (4.2) admits a unique global solution \((\tilde{v}, \nabla q, \eta)\) fulfilling

\[
 ||\tilde{v}||_2^2 + ||\nabla \tilde{v}||_{L^2_H}^2 + ||\nabla q||_{L^2_L}^2 + ||\nabla q_1||_{L^2_{H^1}}^2 + ||\eta_0||_{L^2_H}^2 + ||\nabla q_1||_{L^2_{H^1}}^2 + ||\nabla \phi||_{L^2_L}^2 
\]

\[
 \leq C_4 \left( ||\tilde{v}_0||_{H^2(\Omega)}^2 + ||\eta_0||_{H^3(\mathbb{R}^2)}^2 + C_4 ||w||_2^2 \right) \left( ||\nabla \phi||_{L^2_H}^2 + ||\nabla \phi||_{L^2_L}^2 \right) 
\]

\[
 + C_4 \left( ||\tilde{f}||_{L^2_H}^2 + ||\tilde{f}||_{L^2_{(H^1)}}^2 + ||\tilde{g}||_{L^2_H}^2 \right) 
\]

for all \( t \geq 0 \), where \( C_4 \) is a constant independent of \( t \).

We shall first prove the solvability of (4.2) in the special case \( g_1 = g_2 = 0 \) in the following Lemma 4.7. The proof of Proposition 4.3 will be given at the end of this subsection. We introduce a space where weak solutions will be defined.

\[
 V = \{ \tilde{v} \in L^2(0, \infty; H^1) | \nabla \cdot \tilde{v} = 0, \quad \int_0^t \nabla \tilde{v}(x_1, x_2, y, s) ds \in L^\infty(0, \infty; L^2(\Gamma)) \}, \quad \tilde{v} = 0 \text{ on } S_B \times (0, \infty). 
\]

For test functions, we introduce the following separable space:

\[
 \mathcal{V} = \{ \phi \in H^1(\Omega) | \nabla \cdot \phi = 0, \quad \nabla_0 \phi \in L^2(\Gamma), \quad \tilde{\phi} = 0 \text{ on } S_B \}. 
\]

Then the existence of weak solutions for (4.2) with \( g_1 = g_2 = 0 \) is as follows.

Lemma 4.7. Let \( \eta_0 \in H^2(\mathbb{R}^2) \) and \( \tilde{v}_0 \in H^1 \) satisfy \( \nabla \cdot \tilde{v}_0 = 0 \). Assume \( g_1 = g_2 = 0, \ g_3 \in L^2(0, \infty; H^{-\frac{1}{2}}(\Gamma)) \) and \( \tilde{f} \in L^2(0, \infty; (0H^1)^\prime) \). Then there exists a weak solution \( \tilde{v} \in V \) of (4.2) with \( \tilde{v}_i \in L^2(0, \infty; (0H^1)^\prime) \) satisfying

\[
 \langle \tilde{v}_i, \phi \rangle + [\tilde{v}_i, \phi] + \gamma \int_\Gamma \left( \eta_0 + \int_0^t v_3 ds \right) \phi_3 dx + \sigma \int_\Gamma \left( \nabla_0 \eta_0 + \int_0^t \nabla_0 v_3 ds \right) \cdot \nabla_0 \phi_3 dx 
\]

\[
 = -\left( w \nabla \phi, \tilde{\phi} \right) + \langle \tilde{f}, \tilde{\phi} \rangle + \langle g_3, \phi_3 \rangle, \quad \forall \phi \in \mathcal{V} \tag{4.38} 
\]

and

\[
 \tilde{v}(x, y, 0) = \tilde{v}_0(x, y) \tag{4.39} 
\]

for a.e. \( t > 0 \), where \( \langle \cdot, \cdot \rangle \) denotes the duality pairing between \( 0H^1 \) and \( (0H^1)^\prime \) and \( \langle \cdot, \cdot \rangle_\Gamma \) denotes the duality pairing between \( H^{\frac{1}{2}}(\Gamma) \) and \( H^{-\frac{1}{2}}(\Gamma) \).

The weak formula (4.38) is formally derived by multiplying the first equation of (4.2) with \( \phi \in \mathcal{V} \) in \( L^2 \) and using the following two facts:

\[
 \int_\Omega (-\Delta \tilde{v} + \nabla q) \cdot \tilde{\phi} dx dy = \int_\Gamma q \phi_3 dx - \int_\Gamma (\partial_q v_3 + \partial_3 v_1) \phi_i dx + [\tilde{v}, \tilde{\phi}] 
\]

and

\[
 \eta(x_1, x_2, t) = \eta_0(x_1, x_2) + \int_0^t v_3(x_1, x_2, 0, s) ds \tag{4.40} 
\]

thanks to \( \eta_i = v_3 \) on \( \Gamma \).
One can prove Lemma 4.7 by first using the Galerkin approximations (see e.g. [12, page 377]) to construct approximating solutions and then passing to limits. Indeed, since $\mathcal{Y}$ is separable (see [9, Lemma 4.1]), there exists a basis $\{\bar{\phi}^k\}_{k \in \mathbb{N}}$ of $\mathcal{Y}$. Define the approximating solutions $\bar{v}^m : [0, \infty) \to \mathcal{Y}$ as follows

$$\bar{v}^m(x_1, x_2, y, t) = \sum_{k=1}^{m} \lambda^k_m(t) \bar{\phi}^k(x_1, x_2, y).$$

By replacing the $\bar{v}$ and $\bar{\phi}$ in (4.38) with $\bar{v}^m$ and $\bar{\phi}^k$ respectively, and using the orthogonality of the basis $\{\bar{\phi}^k\}_{k \in \mathbb{N}}$ in both $\mathcal{Y}$ and $L^2$ one derives a second-order ordinary differential equation on the unknown $\lambda^k_m(t)$. Solving these equations, we obtain the approximating solution $\bar{v}^m \in L^\infty(0, \infty; L^2) \cap L^2(0, \infty; H^1)$. Then passing $m \to \infty$, it follows that the limit function $\bar{v} \in \mathcal{V}$ fulfills the weak formula (4.38) and initial condition (4.39). Since the procedure is standard, we omit the proof of Lemma 4.7 and refer the reader to [9] and [2] for details. We proceed to improving the regularity of the weak solutions under the regularity assumptions on initial data and external forces in Proposition 4.3.

**Lemma 4.8.** Let the assumptions in Proposition 4.3 hold true. Then the weak solutions $\bar{v}$ of (4.2) with $g_1 = g_2 = 0$ fulfill

$$\sum_{0 \leq k \leq 2} \left( \| \nabla_0^k \bar{v} \|^2_{L^2} + \| \nabla_0^k \bar{v} \|^2_{L^2} \right) + \| \eta \|^2_{L^2 H^1(\mathbb{R}^2)} \leq C(\| v_0 \|^2_{H^1(\Omega)} + \| \eta_0 \|^2_{H^1(\mathbb{R}^2)} + C(\| g_3 \|^2_{L^2(\Omega)} + \| \bar{f} \|^2_{L^2 H^1(\Gamma)} + \| \phi \|^2_{L^2 H^1(\Gamma)})$$

(4.41)

for all $t > 0$ with the constant $C$ independent of $t$.

**Proof.** Replacing $\bar{\phi}$ in (4.38) with $\bar{v}$ and using (4.40) one derives

$$\frac{1}{2} \frac{d}{dt} \| \bar{v} \|^2 + \| \bar{v} \|^2 + \frac{1}{2} \frac{d}{dt} \left( \gamma \| \eta \|^2_{L^2(\mathbb{R}^2)} + \sigma \| \nabla_0 \eta \|^2_{L^2(\mathbb{R}^2)} \right)$$

$$= \int_{\Gamma} g_3 v_3 dx + \int_{\Omega} \bar{f} \cdot \bar{v} dxdy - \int_{\Omega} w \nabla \phi \cdot \bar{v} dxdy. \quad \text{(4.42)}$$

The right hand-side of (4.42) can be estimated by using the trace theorem and Sobolev embedding inequality as follows:

$$2 \int_{\Gamma} g_3 v_3 dx + 2 \int_{\Omega} \bar{f} \cdot \bar{v} dxdy - 2 \int_{\Omega} w \nabla \phi \cdot \bar{v} dxdy$$

$$\leq \epsilon (\| \bar{v} \|^2_{L^2(\Gamma)} + \| \bar{v} \|^2 + C(\| g_3 \|^2_{L^2(\Gamma)} + \| \bar{f} \|^2 + \| w \|^2_{L^2_t} \| \nabla \phi \|^2_{L^2_t} ))$$

(4.43)

$$\leq C(\| \bar{v} \|^2_{H^1} + C(\| g_3 \|^2_{L^2(\Gamma)} + \| \bar{f} \|^2 + \| w \|^2_{H^1} \| \nabla \phi \|^2_{H^1}))$$

$$\leq \frac{1}{2} [\bar{v}, \bar{v}] + C(\| g_3 \|^2_{L^2(\Gamma)} + \| \bar{f} \|^2 + \| w \|^2_{H^1} \| \nabla \phi \|^2_{H^1}),$$

where the constant $C(\epsilon)$ depends on $\epsilon$ and in the last inequality $\epsilon$ has been chosen small such that $C(\epsilon) \| \bar{v} \|^2_{H^1} \leq \frac{1}{2} [\bar{v}, \bar{v}]$ thanks to Lemma 4.2. Substituting (4.43) into (4.42) and integrating the resulting inequality with respect to $t$ one deduces that

$$\| \bar{v} \|^2_{L^2} + \int_0^t [\bar{v}, \bar{v}] ds + \| \eta \|^2_{L^2 H^1(\mathbb{R}^2)}$$

$$\leq (\| v_0 \|^2 + \| \eta_0 \|^2_{H^1(\mathbb{R}^2)} + C(\| g_3 \|^2_{L^2(\Omega)} + \| \bar{f} \|^2_{L^2(\Gamma)} + \| w \|^2_{L^2 H^1} \| \nabla \phi \|^2_{L^2 H^1}).$$

(4.44)
We next estimate \(|\nabla_0 \vec{v}|^2_{L^2} + |\nabla_0 \vec{v}|^2_{L^2 H^1(\Omega)}\). Replacing \(\phi\) in (4.38) with \(-\Delta_0 \vec{v}\) and using (4.40) and integration by parts to get

\[
\frac{1}{2} \frac{d}{dt} |\nabla_0 \vec{v}|^2 + [\nabla_0 \vec{v}, \nabla_0 \vec{v}] + \frac{1}{2} \frac{d}{dt} \left( |\gamma| |\nabla_0 \eta||^2_{L^2(\Omega)} + \sigma |\Delta_0 \eta||^2_{L^2(\Omega)} \right) = - \int_\Gamma g_3 \Delta_0 v_3 dx - \int_\Omega \vec{f} \cdot \Delta_0 \vec{v} dxdy + \int_\Omega w \nabla \phi \cdot \Delta_0 \vec{v} dxdy,
\]

(4.45)

where terms on the right-hand side can be estimated by using the trace theorem and Sobolev embedding inequality as follows:

\[
-2 \int_\Gamma g_3 \Delta_0 v_3 dx - 2 \int_\Omega \vec{f} \cdot \Delta_0 \vec{v} dxdy + 2 \int_\Omega w \nabla \phi \cdot \Delta_0 \vec{v} dxdy
\leq \epsilon (|\Delta_0 \vec{v}|^2_{H^1(\Omega)} + |\Delta_0 \vec{v}|^2_{L^2(\Omega)}) + C(\epsilon) (|\nabla g_3|^2_{L^2(\Omega)} + ||\vec{f}||^2_{L^2(\Omega)} + ||\nabla \phi||^2_{L^2(\Omega)})
\leq C \epsilon (|\Delta_0 \vec{v}|^2_{H^1(\Omega)} + |\Delta_0 \vec{v}|^2_{L^2(\Omega)}) + ||\vec{f}||^2_{L^2(\Omega)} + ||\nabla \phi||^2_{L^2(\Omega)}
\leq \frac{1}{2} |\nabla_0 \vec{v}, \nabla_0 \vec{v}| + C(\epsilon) (|\nabla g_3|^2_{L^2(\Omega)} + ||\vec{f}||^2_{L^2(\Omega)} + ||\nabla \phi||^2_{L^2(\Omega)}),
\]

(4.46)

here in the last inequality we have employed Lemma 4.2 and chosen \(\epsilon\) small such that \(C \epsilon |\nabla_0 \vec{v}|^2_{H^1(\Omega)} \leq \frac{1}{2} |\nabla_0 \vec{v}, \nabla_0 \vec{v}|\). Inserting (4.46) into (4.45) and then integrating the resulting inequality with respect to \(t\), it follows that

\[
|\nabla_0 \vec{v}|^2_{L^2} + \int_0^t [\nabla_0 \vec{v}, \nabla_0 \vec{v}] ds + |\nabla_0 \eta||^2_{L^2 H^1(\Omega)}
\leq (|\nabla_0 \vec{v}|^2_{H^1(\Omega)} + ||\eta_0||^2_{H^1(\Omega)}) + C(\epsilon) (|\nabla g_3|^2_{L^2 H^1(\Omega)} + ||\vec{f}||^2_{L^2(\Omega)} + ||\nabla \phi||^2_{L^2(\Omega)}).
\]

(4.47)

Applying \(\Delta_0\) to the first equation of (4.2), then multiplying the resulting equality by \(\Delta_0 \vec{v}\) in \(L^2\) and using integration by parts we obtain

\[
\frac{1}{2} \frac{d}{dt} (|\Delta_0 \vec{v}|^2) + |\Delta_0 \vec{v}, \Delta_0 \vec{v}| + \int_\Gamma (\gamma |\Delta_0 \eta| - \sigma |\Delta_0 \eta|) \Delta_0 v_3 dx
= \int_\Gamma (\Delta_0 g_3) (\Delta_0 v_3) dx + \int_\Omega \Delta_0 \vec{f} \cdot \Delta_0 \vec{v} dxdy - \int_\Omega \Delta_0 (w \nabla \phi) \cdot \Delta_0 \vec{v} dxdy.
\]

(4.48)

It follows from \(\eta_0 = v_3\) on \(\Gamma\) and integration by parts that

\[
\int_\Gamma (\gamma |\Delta_0 \eta| - \sigma |\Delta_0 \eta|) \Delta_0 v_3 dx = \frac{1}{2} \frac{d}{dt} \left( |\nabla_0 \eta||^2_{L^2(\Omega)} + \sigma |\nabla_0 \eta||^2_{L^2(\Omega)} \right).
\]

(4.49)

By a similar argument used in deriving (4.46) one deduces and

\[
\int_\Gamma (\Delta_0 g_3) (\Delta_0 v_3) dx + \int_\Omega \Delta_0 \vec{f} \cdot \Delta_0 \vec{v} dxdy - \int_\Omega \Delta_0 (w \nabla \phi) \cdot \Delta_0 \vec{v} dxdy
\leq \epsilon (|\Delta_0 \vec{v}|^2_{H^1(\Omega)} + |\nabla_0 \Delta_0 \vec{v}|^2_{L^2(\Omega)}) + C(\epsilon) (|\nabla g_3|^2_{L^2 H^1(\Omega)} + ||\vec{f}||^2_{L^2(\Omega)} + ||\nabla_0 \nabla \phi||^2_{L^2(\Omega)}).
\]

(4.50)

where \(\epsilon\) has been chosen small such that \(\epsilon (|\Delta_0 \vec{v}|^2_{H^1(\Omega)} + |\nabla_0 \Delta_0 \vec{v}|^2_{L^2(\Omega)}) \leq \frac{1}{2} |\Delta_0 \vec{v}, \Delta_0 \vec{v}|\) thanks to the trace theorem and Lemma 4.2. Substitute (4.49) and (4.50) into (4.48) then integrating the resulting inequality with respect to \(t\) we arrive at

\[
|\Delta_0 \vec{v}|^2_{L^2} + \int_0^t [\Delta_0 \vec{v}, \Delta_0 \vec{v}] ds + |\Delta_0 \eta||^2_{L^2 H^1(\Omega)}
\leq |\vec{v}_0|^2_{H^2} + ||\eta_0||^2_{H^1(\Omega)} + C(\epsilon) (|\nabla g_3|^2_{L^2 H^1(\Omega)} + ||\vec{f}||^2_{L^2 H^1} + ||\nabla_0 \nabla \phi||^2_{L^2 H^1}).
\]

(4.51)
Collecting (4.44), (4.47) and (4.51) and using the Lemma 4.2 we derive (4.41). The proof is completed.

**Lemma 4.9.** Let the assumptions in Proposition 4.3 hold true. Then the weak solutions \( \tilde{v} \) of (4.2) with \( g_1 = g_2 = 0 \) satisfy

\[
\| \tilde{v} \|_{L^\infty L^2}^2 + \| \tilde{v} \|_{L^2 H^1}^2 \leq C(\| \tilde{v}_0 \|_{H^1}^2 + \| \eta \|_{H^3}^2 + C(\| w \|_{L^2}^2 + \| \nabla \phi \|_{L^2}^2)
\]

\[
+ C(\| \tilde{f} \|_{L^2 H^1}^2 + \| \tilde{f} \|_{L^2 (H^1)^*}^2 + \| g_3 \|_{L^2 H^2(\Gamma)}^2 + \| g_3 \|_{L^2 H^{\frac{3}{2}}(\Gamma)}^2)
\]

(4.52)

for all \( t > 0 \), where the constant \( C \) is independent of \( t \).

**Proof.** Differentiating (4.38) with respect to \( t \) and then replacing \( \bar{\phi} \) with \( \tilde{v} \), leads to

\[
\frac{1}{2} \frac{d}{dt} \| \tilde{v} \|_2^2 + [\tilde{v}, \tilde{v}] + \frac{1}{2} \frac{d}{dt} \left( \gamma \| v_3 \|_{L^2(\Gamma)}^2 + \sigma \| \nabla v_3 \|_{L^2(\Gamma)}^2 \right)
\]

\[
= \int_{\Omega} g_3 \cdot v_3 \, dx + \int_{\Omega} \tilde{f} \cdot \tilde{v} \, dx
\]

(4.53)

where we have used the equality \( -\int_{\Omega} \Delta_0 v_3 v_3 \, dx = \frac{1}{2} \frac{d}{dt} \| v_3 \|_{L^2(\Gamma)}^2 \) thanks to integration by parts. By the trace theorem one has

\[
\int_{\Gamma} g_3 v_3 \, dx + \int_{\Omega} \tilde{f} \cdot \tilde{v} \, dx \leq C \| \tilde{v} \|_{H^{\frac{3}{2}}(\Gamma)}^2 + C \| \tilde{v} \|_{H^1} + C(\| g_3 \|_{H^{\frac{3}{2}}(\Gamma)}^2 + \| \tilde{f} \|_{(H^{1})^*}^2)
\]

\[
\leq C(\| \tilde{v} \|_{H^{\frac{3}{2}}(\Gamma)}^2 + C(\| \nabla \phi \|_{L^2}^2)
\]

\[
\leq C(\| \tilde{v} \|_{H^{\frac{3}{2}}(\Gamma)}^2 + \| \tilde{f} \|_{H^{\frac{1}{2}}(\Gamma)}^2)
\]

(4.54)

where \( \varepsilon \) is small such that \( \varepsilon \| \tilde{v} \|_{H^{\frac{3}{2}}(\Gamma)}^2 \leq \frac{1}{4} \| \tilde{v}, \tilde{v} \| \) by using the Korn’s inequality thanks to the boundary condition \( \tilde{v} = 0 \) on \( S_B \). It follows from the Sobolev embedding inequality that

\[
- \int_{\Omega} (w \nabla \phi) \cdot \tilde{v} \, dx \leq C \| w \|_{L^4} \| \nabla \phi \| \| \tilde{v} \|_{L^4} + C \| w \|_{L^2} \| \nabla \phi \|_{L^2} \| \tilde{v} \|_{L^2}
\]

\[
\leq \varepsilon \| \tilde{v} \|_{H^{\frac{3}{2}}(\Gamma)} + C(\| \nabla \phi \|_{L^2}^2)
\]

\[
\leq \frac{1}{4} \| \tilde{v}, \tilde{v} \| + C(\| \nabla \phi \|_{L^2}^2)
\]

(4.55)

where we have selected \( \varepsilon \) small such that \( \varepsilon \| \tilde{v} \|_{H^{\frac{3}{2}}(\Gamma)} \leq \frac{1}{4} \| \tilde{v}, \tilde{v} \| \) thanks to the Korn’s inequality. Inserting (4.54)-(4.55) into (4.53) and integrating the resulting inequality with respect to \( t \) and using Lemma 4.2 we derive

\[
\| \tilde{v} \|_{L^\infty L^2}^2 + \| \tilde{v} \|_{L^2 H^1}^2 \leq C(\| \tilde{v}_0 \|_{H^1}^2 + \| \tilde{f} \|_{L^2 (H^1)^*}^2 + \| g_3 \|_{L^2 H^2(\Gamma)}^2 + \| g_3 \|_{L^2 H^{\frac{3}{2}}(\Gamma)}^2)
\]

(4.56)

We next estimate the term \( \| \tilde{v}(0) \|_{H^1}^2 \) on the right-hand side of (4.56). Given the regularity assumptions in (4.6), compactness (see e.g. [23, Theorem 3.1, Chapter 1]) implies that

\[
\tilde{f} \in C(0, \infty; L^2), \quad g_3 \in C(0, \infty; H^{\frac{1}{2}}(\Gamma)), \quad w \nabla \phi \in C(0, \infty; L^2).
\]

In particular

\[
\| \tilde{f}(0) \|_L^2 \leq C(\| \tilde{f} \|_{L^2 H^1} + \| \tilde{f} \|_{L^2 (H^1)^*}) \quad \| g_3(0) \|_{H^{\frac{3}{2}}(\Gamma)} \leq C(\| g_3 \|_{L^2 H^{\frac{3}{2}}(\Gamma)} + \| g_3 \|_{L^2 H^{\frac{3}{2}}(\Gamma)})
\]

(4.57)
Applying the standard elliptic theory to (4.60) we get

\[
\|w(0)\nabla\phi(0)\|_{L^2}^2 \leq C(\|w\nabla\phi\|_{L^2_{\mathcal{H}^1}}^2 + \|w(\nabla\phi)\|_{L^2_{\mathcal{L}^2}}) \leq C\|w\|_2^2 (\|\nabla\phi\|_{L^2_{\mathcal{H}^1}}^2 + \|\nabla\phi\|_{L^2_{\mathcal{L}^2}}^2).
\]

(4.58)

The fourth boundary condition in (4.2) implies that

\[
q(0) = 2\partial_3 v_0 + \gamma\eta_0 - \sigma\Delta_0\eta_0 - g_3(0) \quad \text{on } \Gamma.
\]

(4.59)

Applying projection \(P\) to \(\nabla q(0)\) and denoting \(\nabla q_1(0) = P\nabla q(0)\), it follows from Lemma 4.1 and (4.59) that

\[
\begin{cases}
\Delta q_1(0) = 0 & \text{in } \Omega, \\
q_1(0) = 2\partial_3 v_0 + \gamma\eta_0 - \sigma\Delta_0\eta_0 - g_3(0) & \text{on } \Gamma, \\
\nabla q_1(0) \cdot \bar{n} = 0 & \text{on } S_B.
\end{cases}
\]

(4.60)

Applying the standard elliptic theory to (4.60) we get

\[
\|P\nabla q(0)\| = \|\nabla q_1(0)\|_{L^2} \leq C\|2\partial_3 v_0 + \gamma\eta_0 - \sigma\Delta_0\eta_0 - g_3(0)\|_{H^\frac{1}{2}(\Gamma)} \\
\leq C(\|v_0\|_{H^2} + \|\nabla\eta_0\|_{H^1(\mathbb{R}^3)} + \|g_3(0)\|_{H^\frac{1}{2}(\Gamma)}).
\]

(4.61)

On the other hand, applying projection \(P\) to the first equation of (4.2) one deduces that

\[
\|\vec{v}_t(0)\|_{L^2} \leq \|P(\Delta \vec{v}_0 - w(0)\nabla\phi(0) + \vec{f}(0))\|_{L^2} + \|P\nabla q(0)\|_{L^2} \\
\leq C(\|\vec{v}_0\|_{H^2} + \|w(0)\nabla\phi(0)\|_{L^2} + \|\vec{f}(0)\|_{L^2} + \|g_3(0)\|_{H^\frac{1}{2}(\Gamma)}).
\]

(4.62)

Substituting (4.61) into (4.62) one arrives at

\[
\|\vec{v}_t(0)\|_{L^2} \leq C(\|\vec{v}_0\|_{H^2} + \|\nabla\eta_0\|_{H^1(\mathbb{R}^3)} + \|w(0)\nabla\phi(0)\|_{L^2} + \|\vec{f}(0)\|_{L^2} + \|g_3(0)\|_{H^\frac{1}{2}(\Gamma)}),
\]

which, in conjunction with (4.57)-4.58 leads to

\[
\|\vec{v}_t(0)\|_{L^2} \leq C(\|\vec{v}_0\|_{H^2} + \|\nabla\eta_0\|_{H^1(\mathbb{R}^3)} + \|w(0)\nabla\phi(0)\|_{L^2} + \|\vec{f}(0)\|_{L^2} + \|g_3(0)\|_{H^\frac{1}{2}(\Gamma)}) \\
\leq C(\|\vec{v}_0\|_{H^2(\Omega)} + \|\nabla\eta_0\|^2_{H^1(\mathbb{R}^3)} + \|w(0)\nabla\phi(0)\|^2_{L^2} + \|\vec{f}(0)\|^2_{L^2}) \\
+ C(\|\vec{v}_t\|^2_{L^2_{\mathcal{H}^1}} + \|\vec{f}_t\|^2_{L^2_{\mathcal{H}^1}} + \|g_3\|^2_{L^2_{\mathcal{H}^1} - \frac{1}{2}(\Gamma)}).
\]

(4.63)

Plugging (4.63) into (4.56) we obtain the desired estimate. The proof is finished.

\[\square\]

**Lemma 4.10.** Suppose the assumptions in Proposition 3.3 hold. Then (4.2) with \(g_1 = g_2 = 0\) admits a unique solution \((\vec{v}, \nabla q, \eta)\) satisfying

\[
\|\vec{v}\|_{L^2_{\mathcal{H}^1}}^2 + \|\vec{v}_t\|_{L^2_{\mathcal{H}^1}}^2 + \|\nabla\vec{v}_t\|_{L^2_{\mathcal{H}^1}}^2 + \|\nabla q\|_{L^2_{\mathcal{H}^1}}^2 + \|\nabla q_t\|_{L^2_{(\mathcal{H}^1)^\gamma}}^2 + \|\eta\|_{L^2_{\mathcal{H}^1}(\mathbb{R}^3)}^2 \\
\leq C(\|\vec{v}_0\|_{H^2(\Omega)}^2 + \|\nabla\eta_0\|_{H^1(\mathbb{R}^3)}^2 + \|w(0)\nabla\phi(0)\|^2_{L^2} + \|\vec{f}(0)\|^2_{L^2}) \\
+ C(\|\vec{v}_t\|_{L^2_{\mathcal{H}^1}}^2 + \|\vec{f}_t\|_{L^2_{(\mathcal{H}^1)^\gamma}}^2 + \|g_3\|_{L^2_{(\mathcal{H}^1)^\gamma}}^2 + \|g_3\|_{L^2_{(\mathcal{H}^1)^\gamma}}^2).
\]

(4.64)

for all \(t > 0\), where the constant \(C\) is independent of \(t\).

**Proof.** From (4.2) we know that \((\vec{v}, \nabla q)\) solves the following elliptic system

\[
\begin{cases}
-\Delta \vec{v} + \nabla q = \vec{f} - (w\nabla\phi) - \vec{v}_t & \text{in } \Omega, \\
\nabla \cdot \vec{v} = 0 & \text{in } \Omega, \\
\partial_3 v_1 + \partial_1 v_3 = 0, \quad \partial_3 v_2 + \partial_2 v_3 = 0 & \text{on } \Gamma.
\end{cases}
\]

(4.65)
for fixed \( t > 0 \). Applying Lemma 4.4 to (4.65) one deduces that

\[
\|\vec{v}\|_{L^2_t H^1} + \|\nabla q\|_{L^2_t H^1} \leq C \left( \|\vec{f}\|_{L^2_t H^1} + \|w\|_{L^2_t H^1} \|\nabla \phi\|_{L^2_t H^1} + \|\vec{v}\|_{L^2_t H^1} \right) + \sum_{k=0}^{2} \|\nabla^k \vec{v}\|_{L^2_t H^1}
\]

which, along with Lemma 4.8-Lemma 4.9 gives rise to

\[
\|\vec{v}\|_{L^2_t H^1}^2 + \|\nabla q\|_{L^2_t H^1}^2 \leq C \left( \|\vec{v}_{0}\|_{H^2(\Omega)}^2 + \|\eta_{0}\|_{H^1(\Omega)}^2 + \|W\|_{L^2}^2 \left( \|\nabla \phi\|_{L^2_t H^1}^2 + \|\nabla \psi\|_{L^2_t L^2}^2 \right) \right) + C \left( \|\vec{f}\|_{L^2_t H^1}^2 + \|\vec{f}\|_{L^2_t H^1}^2 \right) + \|g_3\|_{L^2_t H^1}^2 + \|\sigma\|_{L^2_t H^1}^2 \). \tag{4.66}
\]

Let \( \vec{\phi} \in H^\frac{1}{2}(\Gamma) \). Then for \( i = 1,2 \) it follows from the trace theorem that

\[
\langle \partial_i \vec{v}, \vec{\phi} \rangle_{H^\frac{1}{2}(\Gamma) \times L^2(\Gamma)} = - \int_{\Gamma} \vec{v}_i \cdot \partial_i \vec{\phi} \, dx \leq \|\vec{v}\|_{H^\frac{1}{2}(\Gamma)} \|\partial_i \vec{\phi}\|_{H^\frac{1}{2}(\Gamma)} \leq C \|\vec{v}\|_{H^1} \|\vec{\phi}\|_{H^\frac{1}{2}(\Gamma)}.
\]

Thus

\[
\|\partial_i \vec{v}\|_{L^2_t H^\frac{1}{2}(\Gamma)} \leq C \|\vec{v}\|_{L^2_t H^1} \quad \text{for } i = 1,2. \tag{4.67}
\]

Noting \( \partial_3 v_3 = -\partial_1 v_1 - \partial_2 v_2 \) due to \( \nabla \cdot \vec{v} = 0 \), one gets from (4.67) that

\[
\|\partial_3 v_3\|_{L^2_t H^\frac{1}{2}(\Gamma)} \leq \|\partial_1 v_1\|_{L^2_t H^1} + \|\partial_2 v_2\|_{L^2_t H^1} \leq C \|\vec{v}\|_{L^2_t H^1}. \tag{4.68}
\]

On the other hand, from the boundary conditions in (4.2) we know \( \partial_3 v_3 = -\partial_1 v_1 \) for \( i = 1,2 \). Thus it follows from (4.67) that

\[
\|\partial_3 v_3\|_{L^2_t H^\frac{1}{2}(\Gamma)} \leq \|\partial_i v_i\|_{L^2_t H^1} \leq C \|\vec{v}\|_{L^2_t H^1} \quad \text{for } i = 1,2. \tag{4.69}
\]

Collecting (4.67)-(4.69) and using Lemma 4.9 we arrive at

\[
\|\nabla \vec{v}\|_{L^2_t H^\frac{1}{2}(\Gamma)}^2 \leq C (\|\vec{v}_{0}\|_{H^2}^2 + \|\eta_{0}\|_{H^1(\Omega)}^2 + \|W\|_{L^2}^2 (\|\nabla \phi\|_{L^2_t H^1}^2 + \|\nabla \psi\|_{L^2_t L^2}^2) + C (\|\vec{f}\|_{L^2_t H^1}^2 + \|\vec{f}\|_{L^2_t H^1}^2) + \|g_3\|_{L^2_t H^1}^2 + \|\sigma\|_{L^2_t H^1}^2) \). \tag{4.70}
\]

We proceed to estimating \( \|\nabla q\|_{L^2_t (H^1)} \). Differentiating (4.38) with respect to \( t \) to have

\[
\langle \vec{v}_t, \vec{\phi} \rangle + \vec{v}_t \cdot \vec{\phi} + \gamma \int_{\Gamma} v_3 q_3 \, dx - \sigma \int_{\Gamma} \Delta_0 v_3 q_3 \, dx = - (w_1 \nabla \phi, \vec{\phi}) - (w \nabla \phi, \vec{\phi}) + \langle \vec{f}, \vec{\phi} \rangle + \langle g_3, \phi_3 \rangle_{\Gamma}
\]

for any \( \vec{\phi} \in \mathcal{V} \) and \( a.e. \, t > 0 \), which along with the Sobolev embedding inequality and trace theorem gives

\[
\|\vec{v}_t\|_{L^2_t (H^1)} + \|\nabla q\|_{L^2_t H^1} \leq C (\|\vec{v}_{0}\|_{L^2_t H^1} + \|\nabla q\|_{L^2_t H^1} + \|\nabla \phi\|_{L^2_t H^1} + \|\nabla \psi\|_{L^2_t L^2}^2) + C (\|\vec{f}\|_{L^2_t H^1}^2 + \|\vec{f}\|_{L^2_t H^1}^2) + \|g_3\|_{L^2_t H^1}^2 + \|\sigma\|_{L^2_t H^1}^2) \). \tag{4.71}
\]

For any \( \vec{\phi} \in (0 H^1) \) it follows that

\[
\langle \Delta \vec{v}, \vec{\phi} \rangle = \int_{\Gamma} \partial_3 \vec{v}_t \cdot \vec{\phi} \, dx - \int_{\Omega} \nabla \vec{v}_t \cdot \nabla \vec{\phi} \, dxdy \leq \|\nabla \vec{v}_t\|_{H^\frac{1}{2}(\Gamma)} \|\vec{\phi}\|_{H^\frac{1}{2}(\Gamma)} + \|\vec{v}_t\|_{H^1} \|\vec{\phi}\|_{H^1} \leq C (\|\nabla \vec{v}_t\|_{H^\frac{1}{2}(\Gamma)} + \|\vec{v}_t\|_{H^1}) \|\vec{\phi}\|_{H^1}
\]

for any \( \vec{\phi} \in (0 H^1) \).
which, indicates
\[ \| \Delta \tilde{v} \|_{L^2(H^1)^r} \leq C(\| \nabla \tilde{v} \|_{L^2(H^1)^r} + \| \tilde{v} \|_{L^2(H^1)^r}). \]  
(4.72)

Differentiating the first equation of (4.2) with respect to \( t \) and using (4.71), (4.72) to derive
\[ \| \nabla q_t \|_{L^2((0,\infty)^r)} \leq \| \tilde{v}_t \|_{L^2(H^0)^r} + \| \Delta \tilde{v}_t \|_{L^2(H^0)^r} + \| \nabla \tilde{v} \|_{L^2(H^1)^r} + C(\| \tilde{v} \|_{L^2(H^1)^r} + \| \nabla \tilde{v} \|_{L^2(H^0)^r}) + C(\| \tilde{v} \|_{L^2(H^1)^r} + \| \nabla \tilde{v} \|_{L^2(H^0)^r}) \]
which, along with and (4.66), (4.70) and Lemma 4.9 gives rise to
\[ \| \nabla q_t \|_{L^2((0,\infty)^r)} \leq C(\| \tilde{v}_0 \|_{H^2} + \| \tilde{v}_0 \|_{H^1(\mathbb{R}^2)} + \| \nabla \tilde{v} \|_{L^2(H^1)^r} + C(\| \tilde{v} \|_{L^2(H^1)^r} + \| \nabla \tilde{v} \|_{L^2(H^0)^r}) + C(\| \tilde{v} \|_{L^2(H^1)^r} + \| \nabla \tilde{v} \|_{L^2(H^0)^r}) \]  
(4.73)

Collecting (4.66), (4.70) and (4.73) and using Lemma 4.10, Lemma 4.9 we derived the desired estimates (4.64). Uniqueness follows from (4.64). The proof is finished.

\[ \square \]

Lemma 4.11. Let \((\tilde{v}, \nabla q, \eta)\) be the solution derived in Lemma 4.10 and let \(\mathcal{H}(\eta)\) be the harmonic extension of \(\eta\). Then there exists a constant \(C\) independent of \(t\) such that
\[ \| \nabla \eta \|_{L^2(H^2)^2(\mathbb{R}^2)} + \| \nabla^2 \mathcal{H}(\eta) \|_{L^2(H^1)^2} \leq C(\| \tilde{v} \|_{H^2(\Omega)} + \| \tilde{v}_0 \|_{H^1(\mathbb{R}^2)} + \| \nabla \tilde{v} \|_{L^2(H^1)^r} + C(\| \tilde{v} \|_{L^2(H^1)^r} + \| \nabla \tilde{v} \|_{L^2(H^0)^r}) + C(\| \tilde{v} \|_{L^2(H^1)^r} + \| \nabla \tilde{v} \|_{L^2(H^0)^r}) \]
for all \(t > 0\).

Proof. Noting that the harmonic extension \(\mathcal{H}\) is a linear operator, applying projection \(P\) to \(\nabla q\) and using Lemma 4.11 and the fourth boundary condition in (4.2), we deduce that
\[ P \nabla q = \nabla \mathcal{H}(2\partial_3 v_3 + \gamma \eta - \sigma \Delta_0 \eta - g_3) = 2\nabla \mathcal{H}(\partial_3 v_3) + \nabla \mathcal{H}(\gamma \eta - \sigma \Delta_0 \eta) - \nabla \mathcal{H}(g_3) \]
which, along with Lemma 4.3 and the trace theorem entails that
\[ \| \nabla \mathcal{H}(\gamma \eta - \sigma \Delta_0 \eta) \|_{L^2(H^1)^2} \leq \| P \nabla q \|_{L^2(H^1)^2} + 2 \| \nabla \mathcal{H}(\partial_3 v_3) \|_{L^2(H^1)^2} + \| \nabla \mathcal{H}(g_3) \|_{L^2(H^1)^2} \leq C(\| \nabla q \|_{L^2(H^1)^2} + \| \nabla \tilde{v} \|_{L^2(H^2)^2(\Gamma)} + \| g_3 \|_{L^2(H^2)^2(\Gamma)} \]  
(4.74)

Noting \(\partial_i \mathcal{H}(\gamma \eta - \sigma \Delta_0 \eta) = \mathcal{H}(\partial_i [\gamma \eta - \sigma \Delta_0 \eta])\) for \(i = 1, 2\), we deduce from the trace theorem and (4.74) that
\[ \| \nabla_0 (\gamma \eta - \sigma \Delta_0 \eta) \|_{L^2(H^2)^2(\mathbb{R}^2)} \leq C(\| \nabla q \|_{L^2(H^1)^2} + \| \nabla \tilde{v} \|_{L^2(H^2)^2(\Gamma)} + \| g_3 \|_{L^2(H^2)^2(\Gamma)} \]  
(4.75)

On the other hand, \(\nabla_0 \eta\) solves the following elliptic system
\[ \begin{cases} -\sigma \Delta_0 (\nabla_0 \eta) + \gamma (\nabla_0 \eta) = \nabla_0 (\gamma \eta - \sigma \Delta_0 \eta), \\ \lim_{|\xi| \to \infty} (\nabla_0 \eta) = 0 \end{cases} \]  
(4.76)
We then employ the trace theorem involving time (see e.g. [23, Theorem 2.3, Chapter 4] to select \( \xi \), vanishing near \( S \) satisfying near \( \| \xi \|_{L^2 H^2(\mathbb{R}^2)} \leq C(\| \nabla \eta \|_{L^2 H^1(\mathbb{R}^2)} + \| \nabla \|_{L^2 H^1} + \| g_3 \|_{H^2(\gamma)}) \), \( (4.77) \)

which, in conjunction with the fact \( \partial_i \mathcal{H}(\eta) = \mathcal{H}(\partial_i \eta) \) for \( i = 1, 2 \) and the trace theorem leads to

\[
\| \nabla_0 \mathcal{H}(\eta) \|_{L^2 H^2} \leq \| \nabla_0 \eta \|_{L^2 H^2} \leq C(\| \nabla \eta \|_{L^2 H^1(\mathbb{R}^2)} + \| \nabla \|_{L^2 H^1} + \| g_3 \|_{L^2 H(\gamma)}). \quad (4.78)
\]

Noting \( \Delta \mathcal{H}(\eta) = 0 \) in \( \Omega \), it follows from \( (4.78) \) that

\[
\| \partial_3^2 \mathcal{H}(\eta) \|_{L^2 H^2} = \| \Delta_0 \mathcal{H}(\eta) \|_{L^2 H^2} \leq C(\| \nabla \eta \|_{L^2 H^1(\mathbb{R}^2)} + \| \nabla \|_{L^2 H^1} + \| g_3 \|_{L^2 H(\gamma)}).
\]

which, along with \( (4.78) \) leads to

\[
\| \nabla^2 \mathcal{H}(\eta) \|_{L^2 H^2} \leq C(\| \nabla \eta \|_{L^2 H^1(\mathbb{R}^2)} + \| \nabla \|_{L^2 H^1} + \| g_3 \|_{L^2 H(\gamma)}).
\]

(4.79)

Collecting \( (4.77) \) and \( (4.79) \) and using Lemma 4.10 we derive the desired estimates. The proof is completed.

We are now in the position to prove Proposition 4.3.

**Proof of Proposition 4.3** The compactness theorem (see e.g. see e.g. [23, Theorem 3.1, Chapter 1]) implies that there exists a constant \( C \) independent of \( t \) such that

\[
\| \tilde{g} \|_{C([0, \infty), H^1/2(\gamma))} \leq C(\| \tilde{g} \|_{L^2 H^{1/2}(\gamma)} + \| \tilde{g} \|_{L^2 H^{-1/2}(\gamma)}).
\]

Thus,

\[
\| g_1(x_1, x_2, 0) \|_{H^{1/2}(\gamma)} + \| g_2(x_1, x_2, 0) \|_{H^{1/2}(\gamma)} \leq C(\| \tilde{g} \|_{L^2 H^{1/2}(\gamma)} + \| \tilde{g} \|_{L^2 H^{-1/2}(\gamma)}).
\]

(4.80)

By the trace theorem (see [23, Theorem 7.5, Chapter 1]) one can choose \( \xi_0(x_1, x_2, y) \in H^3(\Omega) \) vanishing near \( S_B \) satisfying

\[
\xi_0(x_1, x_2, 0) = 0, \quad \partial_3 \xi_0(x_1, x_2, 0) = 0, \quad \partial_3^2 \xi_0(x_1, x_2, 0) = (g_2, -g_1, 0)(x_1, x_2, 0)
\]

(4.81)

and

\[
\| \xi_0 \|_{H^1} \leq C(\| g_1(x_1, x_2, 0) \|_{H^{1/2}(\gamma)} + \| g_2(x_1, x_2, 0) \|_{H^{1/2}(\gamma)}),
\]

which, along with \( (4.80) \) gives rise to

\[
\| \xi_0 \|_{H^1} \leq C(\| \tilde{g} \|_{L^2 H^{1/2}(\gamma)} + \| \tilde{g} \|_{L^2 H^{-1/2}(\gamma)}).
\]

(4.82)

We then employ the trace theorem involving time (see e.g. [23, Theorem 2.3, Chapter 4] to select \( \xi(x, t) \) defined in \( \Omega \times (0, \infty) \) vanishing near \( S_B \times (0, \infty) \) and fulfilling

\[
\begin{cases}
\xi(x, y, 0) = \xi_0(x, y), \quad \xi_0(x_1, x_2, y) = 0; \\
\xi_0(x, t) = 0, \quad \partial_3 \xi_0(x_1, 0, t) = 0, \quad \partial_3^2 \xi_0(x_1, 0, t) = (g_2, -g_1, 0)(x_1, t), \quad \partial_3^3 \xi_0(x_1, 0, t) = 0
\end{cases}
\]

(4.83)
with \( \bar{x} = (x_1, x_2) \), since the appropriate compatibility conditions (cf. [23, Theorem 2.3, Chapter 4]) are fulfilled thanks to the requirements on the initial data \( \bar{\xi}_0 \) in (4.81). Moreover, the following estimates hold:

\[
\|\bar{\xi}\|_{L^2_t H^4} + \|\bar{\xi}\|_{L^2_t H^2} + \|\bar{\xi}_t\|_{L^2_t L^2} \leq C(\|\bar{\xi}_0\|_{H^3} + \|g_1\|_{L^2_t H^2(\Gamma)} + \|g_2\|_{L^2_t H^2(\Gamma)}),
\]

(4.84)

where we have used (4.82) in the last inequality. Set

\[
\bar{v}^{(1)}(\bar{x}, y, t) = (v_1^{(1)}, v_2^{(1)}, v_3^{(1)})(\bar{x}, y, t) = \nabla \times \bar{\xi}(\bar{x}, y, t).
\]

Then it follows from (4.83) that

\[
\begin{cases}
\nabla \cdot \bar{v}^{(1)}(\bar{x}, y, t) = 0 & \text{in } \Omega \times (0, \infty), \\
\bar{v}^{(1)}(\bar{x}, y, 0) = \nabla \times \bar{\xi}_0, \\
\partial_1 v_1^{(1)} = \partial_2 v_1^{(1)} = \partial_3 v_1^{(1)} = 0, \quad \partial_3 v_2^{(1)} = g_1(\bar{x}, t), \quad \partial_3 v_3^{(1)} = g_2(\bar{x}, t) & \text{on } \Gamma \times (0, \infty)
\end{cases}
\]

(4.85)

and that \( \bar{v}^{(1)} = 0 \) near \( S_B \times (0, \infty) \). For any \( \bar{\psi} \in H^1 \), integration by parts and direct computation yields

\[
\langle \bar{v}^{(1)}_t, \bar{\psi} \rangle_{(H^1)' \times (H^1)} = \int_{\Omega} (\nabla \times \bar{\xi}_t) \cdot \bar{\psi} dx + \int_{\Gamma} (\nabla \times \bar{\xi}) \cdot \bar{\psi} dl \leq \|\bar{\xi}_t\|_{H^2} \|\bar{\psi}\|_{H^1},
\]

and

\[
\langle \Delta \bar{v}^{(1)}_t, \bar{\psi} \rangle_{(H^1)' \times (H^1)} = \int_{\Omega} \Delta \bar{\xi}_t \cdot (\nabla \times \bar{\psi}) dx + \int_{\Gamma} g_2\psi_2 dx + \int_{\Gamma} g_1\psi_1 dx \\
\leq C(\|\bar{\xi}_t\|_{H^2} + \|\bar{\psi}\|_{H^1}) \|\bar{\psi}\|_{H^1},
\]

which along with (4.84) leads to

\[
\|\bar{v}^{(1)}\|_{L^2_t H^3} + \|\bar{v}^{(1)}_t\|_{L^2_t H^1} + \|\bar{v}^{(1)}_t\|_{L^2_t (H^1)'_y} + \|\Delta \bar{v}^{(1)}_t\|_{L^2_t (H^1)'_y} \leq C(\|\bar{g}\|_{L^2_t H^2(\Gamma)} + \|\bar{\xi}_t\|_{L^2_t H^2(\Gamma)}),
\]

(4.86)

Let \((\bar{v}^{(2)}, \nabla q, \eta)(\bar{x}, y, t)\) be the solution of the following system

\[
\begin{cases}
\bar{v}^{(2)}_t - \Delta \bar{v}^{(2)} + \nabla q + w\nabla \phi = f - \bar{v}^{(1)}_t + \Delta \bar{v}^{(1)}', \quad (x_1, x_2, y, t) \in \Omega \times (0, \infty), \\
\bar{v}^{(2)} = 0, \\
\bar{v}(x_1, x_2, y, 0) = \bar{v}_0(x_1, x_2, y) - \nabla \times \bar{\xi}_0(x_1, x_2, y), \\
\partial_1 v_1^{(2)} + \partial_2 v_2^{(2)} + \partial_3 v_3^{(2)} = 0 \quad \text{on } \Gamma \times (0, \infty), \\
\eta_t = v_3^{(2)}, \quad q - 2\partial_3 v_3^{(2)} = \gamma \eta - \sigma \Delta \eta - g_3 \quad \text{on } \Gamma \times (0, \infty), \\
\bar{v}^{(2)} = 0 \quad \text{on } S_B \times (0, \infty).
\end{cases}
\]

(4.87)

Applying Lemma [4.10] to system (4.87) and using (4.86) and (4.82) one gets

\[
\begin{align*}
&\|\bar{v}^{(2)}\|^2_{L^2_t H^3} + \|\bar{v}^{(2)}_t\|^2_{L^2_t H^1} + \|\nabla \bar{v}^{(2)}_t\|^2_{L^2_t H^1} + \|\nabla q\|^2_{L^2_t H^1} + \|\nabla \phi\|^2_{L^2_t (H^1)'_y} + \|\eta\|^2_{L^2_t H^3(\mathbb{R}^2)} \leq C_4(\|\bar{v}_0\|^2_{H^2} + \|\bar{v}_0\|^2_{H^1(\mathbb{R}^2)} + C_4\|\nabla \phi\|^2_{L^2_t H^1} + \|\nabla \phi\|^2_{L^2_t L^2} + C_4(\|f\|^2_{L^2_t H^2} + \|f\|^2_{L^2_t H^2} + \|\bar{g}\|^2_{L^2_t H^2} + \|\bar{g}\|^2_{L^2_t H^2})),
\end{align*}
\]

(4.88)

where we also used Lemma [4.11]
Let \( \vec{v}(x,y,t) = \vec{v}^{(1)}(x,y,t) + \vec{v}^{(2)}(x,y,t) \). Then from system \((4.87)\) and \((4.85)\) one deduces that \((\vec{v}, \nabla q, \eta)\) solves the initial-boundary value problem \((4.2)\) and it follows from \((4.86)\) and \((4.38)\) that

\[
\| \vec{v} \|_{L^2(H^3)}^2 + \| \nabla \vec{v} \|_{L^2(H^1)}^2 + \| \nabla q \|_{L^2(H^1)}^2 + \| \nabla q_i \|_{L^2(H^1)}^2 + \| \eta \|_{L^2(H^1)}^2 + \| \nabla \eta \|_{L^2(H^1)}^2 + \| \nabla \eta \|_{L^2(H^1)}^2
\]

\[
\leq C_4 \left( \| \vec{v} \|_{H^2}^2 + \| \eta \|_{H^3}^2 \right) + C_4 \left( \| \nabla \phi \|_{L^2(H^1)}^2 + \| \nabla \phi \|_{L^2(L^2)}^2 \right)
\]

\[
+ C_4 \left( \| \vec{f} \|_{L^2(H^1)}^2 + \| \vec{f} \|_{L^2(H^1)}^2 + \| \vec{g} \|_{L^2(H^1)}^2 + \| \vec{g} \|_{L^2(H^1)}^2 \right).
\]

On the other hand, it follows from the compactness theorem (see e.g. [23, Theorem 3.1, Chapter 1]) that

\[
\| \vec{v} \|_{L^2(H^2)}^2 \leq C \left( \| \vec{v} \|_{L^2(H^1)}^2 + \| \vec{v} \|_{L^2(H^1)}^2 \right), \quad \| \nabla q \|_{L^2(H^1)}^2 \leq C \left( \| \nabla q \|_{L^2(H^1)}^2 + \| \nabla q_i \|_{L^2(H^1)}^2 \right),
\]

which, in conjunction with \((4.89)\) gives the desired estimates \((4.37)\). Uniqueness follows from \((4.37)\). The proof is completed.

\[
\Box
\]

4.4 Proof of Proposition 4.1

Proof. Multiplying \((4.20)\) with \(C_4 \left( \| \nabla \phi \|_{L^2(H^1)}^2 + \| \nabla \phi \|_{L^2(L^2)}^2 \right) + 1\) and adding the resulting inequality to \((4.37)\), one derives \((4.8)\). The proof is completed.

\[
\Box
\]

5 Approximation solutions for the nonlinear problem

In this section, we shall first construct approximation solutions for the nonlinear problem \((5.10)-(5.12)\) based on the results obtained on its linearized version \((4.1)-(4.2)\); then proceed to gain a uniform bound for such approximations by estimating the nonlinear terms on the right-hand side of each equation in following approximating system \((5.5)-(5.7)\).

Before constructing the approximation solutions, we exhibit some identities fulfilled by the initial data \(w_0, h_0\) and \(\vec{v}_0\) of the nonlinear problem \((5.10)-(5.12)\). Indeed, by similar arguments used in deriving \((7.11)-(7.6)\) and \((7.12)-(7.13)\) one can deduce from \((2.2)\) that the \(w_0, h_0\) and \(\vec{v}_0\) defined in \((5.9)\) satisfy the following identities

\[
\begin{cases}
\nabla \cdot \vec{v}_0 = 0 & \text{in } \Omega, \\
\partial_3 w_0 = G_4(w_0, h_0, \vec{v}_0) - w_0 \partial_3 h_0 & \text{on } \Gamma, \\
\partial_3 y_0 + \partial_1 y_0 = G_1(\vec{v}_0, \vec{v}_0) + \partial_2 y_0 = G_2(\vec{v}_0, \vec{v}_0) & \text{on } \Gamma, \\
w_0 = 0, \quad \partial_3 h_0 = 0, \quad \vec{v}_0 = 0 & \text{on } S_B,
\end{cases}
\]

where \(G_4(w_0, h_0, \vec{v}_0)\) is derived from \(G_4(w, h, \vec{v})\) by replacing the \(\alpha, \beta, J, \eta, h\) and \(w\) in \((7.6)\) with \(\alpha_0, \beta_0, J_0, \eta_0, h_0\) and \(w_0\), respectively. Similarly, \(G_1(\vec{v}_0, \vec{v}_0)\) and \(G_2(\vec{v}_0, \vec{v}_0)\) are obtained from \(G_1(\vec{v}, \vec{v})\) and \(G_2(\vec{v}, \vec{v})\) by replacing the \(J, \eta\) and \(\vec{v}\) in \((7.12)-(7.13)\) with \(J_0, \eta_0\) and \(\vec{v}_0\), respectively.

We are now in the position to construct the first and second approximations. Let \(\eta(t) \in C([0, \infty))\) be a smooth cut-off function in-time satisfying

\[
\eta(t) = 1 \quad \text{when } t \in [0, 1], \quad \eta(t) = 0 \quad \text{when } t \geq 2
\]

\[
\Box
\]
and let \((w^1, h^1, \bar{v}^1, \nabla q^1, \eta^1) = (w^2, h^2, \bar{v}^2, \nabla q^2, \eta^2)\) be the solutions of the following system:

\[
\begin{cases}
    w_t - \Delta w = 0 \quad \text{in } \Omega \times (0, \infty), \\
    h_t - \Delta h - w = 0, \\
    \bar{v}_t - \Delta \bar{v} + \nabla q + w \nabla \phi = 0, \\
    (w, h, \bar{v}, \eta)(x_1, x_2, y, 0) = (w_0, h_0, \bar{v}_0, \eta_0)(x_1, x_2, y)
\end{cases}
\]  
(5.3)

with the following boundary conditions

\[
\begin{cases}
    \partial_3 w = \zeta(t)(G_4(w_0, h_0, \eta_0) - w_0 \partial_3 h_0), \quad h = 0 \quad \text{on } \Gamma \times (0, \infty), \\
    \partial_3 v_1 + \partial_1 v_3 = \zeta(t)G_1(\bar{v}_0, \eta_0), \quad \partial_3 v_2 + \partial_2 v_3 = \zeta(t)G_2(\bar{v}_0, \eta_0) \quad \text{on } \Gamma \times (0, \infty), \\
    \eta = v_3, \quad q - 2\partial_3 v_3 = \gamma \eta - \sigma \Delta_0 \eta - \zeta(t)G_3(\bar{v}_0, \eta_0) \quad \text{on } \Gamma \times (0, \infty), \\
    w = 0, \quad \partial_3 h = 0, \quad \bar{v} = 0 \quad \text{on } S_B \times (0, \infty).
\end{cases}
\]  
(5.4)

Applying Proposition 4.1 to system (5.3)-(5.4) we obtain the unique solution \((w^1, h^1, \bar{v}^1, \nabla q^1, \eta^1) = (w^2, h^2, \bar{v}^2, \nabla q^2, \eta^2)\) since the required compatibility conditions in Proposition 4.1 follows directly from (5.1)-(5.2).

With the well-defined first and second approximation solutions in hand, we proceed to constructing \((w^{(j+1)}, h^{(j+1)}, \bar{v}^{(j+1)}, \nabla q^{(j+1)}, \eta^{(j+1)})\) with \(j \geq 2\) by solving the following linear system:

\[
\begin{cases}
    w^{(j+1)}_t - \Delta w^{(j+1)} - \nabla \cdot (w^{(j)} \nabla h^{(j+1)}) = F_4(w^{(j-1)}, w^{(j)}, h^{(j)}, \bar{v}^{(j)}, \bar{\eta}^{(j)}) \quad \text{in } \Omega \times (0, \infty), \\
    h^{(j+1)}_t - \Delta h^{(j+1)} - w^{(j+1)} = F_5(h^{(j)}, \bar{v}^{(j)}, \eta^{(j)}), \\
    \bar{v}^{(j+1)}_t - \Delta \bar{v}^{(j+1)} + \nabla q^{(j+1)} + w^{(j+1)} \nabla \phi = \bar{F}(w^{(j)}, \bar{v}^{(j)}, \nabla q^{(j)}, \bar{\eta}^{(j)}), \\
    \bar{v} \cdot v^{(j+1)} = 0, \\
    (w^{(j+1)}, h^{(j+1)}, \bar{v}^{(j+1)}, \eta^{(j+1)})(x_1, x_2, y, 0) = (w_0, h_0, \bar{v}_0, \eta_0)(x_1, x_2, y)
\end{cases}
\]  
(5.5)

with the following boundary conditions on \(\Gamma \times (0, \infty)\) :

\[
\begin{cases}
    \partial_3 w^{(j+1)} + w^{(j)} \partial_3 h^{(j+1)} = G_4(w^{(j)}, h^{(j)}, \bar{\eta}^{(j)}), \quad h^{(j+1)} = 0, \\
    \partial_3 v^{(j+1)} + \partial_1 v^{(j+1)} = G_1(\bar{v}^{(j)}, \bar{\eta}^{(j)}), \quad \partial_3 v_2^{(j+1)} + \partial_2 v_3^{(j+1)} = G_2(\bar{v}^{(j)}, \bar{\eta}^{(j)}), \\
    \eta^{(j+1)} = v_3^{(j+1)}, \quad q^{(j+1)} - 2\partial_3 v_3^{(j+1)} = \gamma \eta^{(j+1)} - \sigma \Delta_0 \eta^{(j+1)} - G_3(\bar{v}^{(j)}, \bar{\eta}^{(j)}), \quad \bar{v}^{(j+1)} = 0, \\
    (w^{(j+1)}, h^{(j+1)}, \bar{v}^{(j+1)}, \eta^{(j+1)})(x_1, x_2, y, 0) = (w_0, h_0, \bar{v}_0, \eta_0)(x_1, x_2, y)
\end{cases}
\]  
(5.6)

and the following boundary conditions on \(S_B \times (0, \infty)\) :

\[
\begin{cases}
    w^{(j+1)}_t = 0, \quad \partial_3 h^{(j+1)} = 0, \quad \bar{v}^{(j+1)} = 0
\end{cases}
\]  
(5.7)

where \(\bar{F}(w^{(j)}, \bar{v}^{(j)}, \nabla q^{(j)}, \bar{\eta}^{(j)}) = (F_1, F_2, F_3)(w^{(j)}, \bar{v}^{(j)}, \nabla q^{(j)}, \bar{\eta}^{(j)}), \quad \bar{G}(\bar{v}^{(j)}, \bar{\eta}^{(j)}) = (G_1, G_2, G_3)(\bar{v}^{(j)}, \bar{\eta}^{(j)})\) and \(F_4(w^{(j)}, h^{(j)}, \bar{\eta}^{(j)}) \) are derived from the \(\bar{F}\), \(\bar{G}\) and \(F_5\) defined in (7.10), (7.12)-(7.14) and (7.5) by replacing all the \(J, \alpha, \beta, w, h, \bar{v}\) and \(\eta\) there with \(J^j, \alpha^j, \beta^j, h, w, \bar{v}^j\) and \(\eta^j\), respectively. Here

\[
\alpha^j = (1 + y) \partial_1 \bar{\eta}^j, \quad \beta^j = (1 + y) \partial_2 \bar{\eta}^j, \quad J^j = 1 + \bar{\eta}^j + \partial_3 \bar{\eta}^j(1 + y),
\]

with \(\bar{\eta}^j = \mathcal{H}(\eta^j)\). \(F_4(w^{(j-1)}, w^{(j)}, h^{(j)}, \bar{v}^{(j)}, \bar{\eta}^{(j)})\) is derived from the \(F_4 \) given in (7.5) by replacing the first line with

\[
\{ (J^j)^{-2}[(\alpha^j)^2 + (\beta^j)^2 + 1] - 1 \} \partial_3 (\partial_3 w^j + w^{(j-1)} \partial_3 h^j) - 2\alpha^j( J^j)^{-1} (\partial_3 \partial_1 w^j + w^{(j-1)} \partial_3 h^j) - 2\beta^j( J^j)^{-1} (\partial_3 \partial_2 w^j + w^{(j-1)} \partial_3 h^j)
\]  
(5.8)

and replacing all the \(J, \alpha, \beta, w, h, \bar{v}\) and \(\eta\) in Line 2 to Line 5 with \(J^j, \alpha^j, \beta^j, h, w^j, \bar{v}^j\) and \(\eta^j\), respectively.
To obtain such \((w^{(j+1)}, h^{(j+1)}, \vec{v}^{(j+1)}, \nabla q^{(j+1)}, \eta^{(j+1)})\) by solving (5.5)-(5.7), from Proposition 4.1, we know that the following compatibility conditions are required to be fulfilled:

\[
\begin{cases}
\nabla \cdot \vec{v}_0 = 0 \quad \text{in} \quad \Omega, \\
\partial_3 w_0 + w_0 \partial_3 h_0 = G_4(w^i(x,y,0), h^i(x,y,0), \eta^i(x,y,0)), \quad h_0 = 0 \quad \text{on} \quad \Gamma, \\
\partial_3 v_{01} + \partial_1 v_{03} = G_1(\vec{v}^i(x,y,0), \eta^i(x,0,0)) \quad \text{on} \quad \Gamma, \\
\partial_3 v_{02} + \partial_2 v_{03} = G_2(\vec{v}^i(x,y,0), \eta^i(x,0,0)) \quad \text{on} \quad \Gamma, \\
w_0 = 0, \quad \partial_3 h_0 = 0, \quad \vec{v}_0 = 0 \quad \text{on} \quad \Sigma_B. 
\end{cases}
\]

(5.9)

With our construction of \((w^i, h^i, \vec{v}^i, \nabla q^i, \eta^i)\), \((w^2, h^2, \vec{v}^2, \nabla q^2, \eta^2)\) and the identities in (5.1), one can easily check that the compatibility conditions in (5.9) hold true for all \(j \geq 2\) by the argument of induction and thus the approximation solutions \((w^{(j+1)}, h^{(j+1)}, \vec{v}^{(j+1)}, \nabla q^{(j+1)}, \eta^{(j+1)})\) with \(j \geq 2\) are well-defined thanks to Proposition 4.1. Actually, by using the fact \(G_1(\vec{v}^2(x,y,0), \eta^2(x,y,0)) = G_1(\vec{v}_0, \eta_0)\), \(G_2(\vec{v}^2(x,y,0), \eta^2(x,y,0)) = G_2(\vec{v}_0, \eta_0)\) and \(G_4(w^2(x,y,0), h^2(x,y,0), \eta^2(x,y,0)) = G_4(w_0, h_0, \eta_0)\) on \(\Gamma\) due to the regularity (4.8) fulfilled by \((w^2, h^2, \vec{v}^2, \nabla q^2, \eta^2)\), one deduces from (5.1) that the compatibility conditions (5.9) holds true for \(j = 2\) and thus derives \((w^3, h^3, \vec{v}^3, \nabla q^3, \eta^3)\) by applying Proposition 4.1 to system (5.5)-(5.7). By similar arguments, one can prove that (5.9) holds true and that (5.5)-(5.7) admits a unique solutions \((w^{(j+1)}, h^{(j+1)}, \vec{v}^{(j+1)}, \nabla q^{(j+1)}, \eta^{(j+1)})\) for all \(j \geq 2\) by the argument of induction and Proposition 4.1. We thus derived the approximation solutions \((w^i, h^i, \vec{v}^i, \nabla q^i, \eta^i)\), \(j \geq 1\) and for such approximations we have the following uniform bounds.

**Proposition 5.1.** Suppose that the initial data \(w_0, h_0, \vec{v}_0 \in H^2(\Omega)\) and \(\eta_0 \in H^3(\mathbb{R}^2)\) are small enough to fulfill (5.81) and (5.82). Assume further that (5.1) holds. Let \((w^{(j+1)}, h^{(j+1)}, \vec{v}^{(j+1)}, \nabla q^{(j+1)}, \eta^{(j+1)})\) be the solution of (5.5)-(5.7) when \(j \geq 2\). Let \((w^1, h^1, \vec{v}^1, \nabla q^1, \eta^1) = (w^2, h^2, \vec{v}^2, \nabla q^2, \eta^2)\) be the solution of system (5.3)-(5.4). Then there exists a constant \(C\) independent of \(j\) and \(t\) such that

\[
\|\{w^j, h^j, \vec{v}^j, q^j, \eta^j\}\|^2 \leq C(\|w_0\|_{H^2(\Omega)} + \|h_0\|_{H^2(\Omega)} + \|\vec{v}_0\|_{H^2(\Omega)} + \|\eta_0\|_{H^3(\mathbb{R}^2)})^2
\]

for all \(j \geq 1\) and \(t > 0\). Moreover, it holds true for all \(j \geq 1\) and \(t > 0\) that

\[
\frac{1}{2} < J^j < \frac{3}{2}
\]

(5.11)

where \(J^j = 1 + \tilde{\eta}^j + \partial_3 \tilde{\eta}^j(1 + y)\).

In the next subsection, we shall estimate the nonlinear terms \(\vec{F}, \vec{G}, F_4, F_5\) and \(G_4\) in system (5.5)-(5.7) and the proof of Proposition 5.1 will be given in subsection 5.1.

### 5.1 Estimates on nonlinear terms

In estimating the nonlinear terms, we assume that

\[
\frac{1}{2} < J^j < \frac{3}{2} \quad \text{for all} \quad j \geq 1 \text{ and } t > 0,
\]

(5.12)

which, will be verified in the proof of Proposition 5.1 by using the smallness of \(\tilde{\eta}^j\). Assumption (5.12) will be used repeatedly in the proof of the following Lemma 5.1, Lemma 5.8 without further clarification.

We first estimate \(\vec{F}(w^j, \vec{v}^j, \nabla q^j, \eta^j)\) and \(\vec{F}_r(w^j, \vec{v}^j, \nabla q^j, \eta^j)\). It follows from (7.10) that

\[
\vec{F}(w^j, \vec{v}^j, \nabla q^j, \eta^j) = \{J^j\}^{-2} \left[ (\alpha^j)^2 + (\beta^j)^2 + 1 \right] \partial_3^2 \vec{v}^j - 2(J^j)^{-1} \alpha^j \partial_3 \partial_1 \vec{v}^j - 2(J^j)^{-1} \beta^j \partial_3 \partial_2 \vec{v}^j + \vec{Q} + LOT,
\]

(5.13)
where \( \vec{Q} = (Q_1, Q_2, Q_3) \) with
\[
Q_1 = (1 - J^j) \partial_1 q^j + \alpha^j \partial_3 q^j, \quad Q_2 = (1 - J^j) \partial_2 q^j + \beta^j \partial_3 q^j, \\
Q_3 = \{ 1 - (J^j)^{-2} \left[ (\alpha^j)^2 + (\beta^j)^2 + 1 \right] \} \partial_3 q^j + \alpha^j \partial_1 q^j + \beta^j \partial_2 q^j
\]
and the lower order terms (with respect to \( \vec{v}^j \)) LOT are as follows:
\[
LOT \sim (\nabla \bar{\eta}^j)^2 (\nabla^2 \bar{\eta}^j) \nabla \vec{v}^j + (\nabla \bar{\eta}^j)^2 (\nabla^3 \bar{\eta}^j) \cdot \vec{v}^j + \nabla \bar{\eta}^j \cdot \nabla \vec{v}^j + (\nabla^2 \bar{\eta}^j) \vec{v}^j \vec{v}^j + \vec{w} \cdot \nabla \bar{\eta}^j \cdot \nabla \phi.
\]

**Lemma 5.1.** Let the assumptions in Proposition 5.1 and 5.12 hold. Then there exists a constant \( C \) independent of \( j \) and \( t \) such that
\[
\| \vec{F} \|^2_{L^2_t H^4} \leq C \{ \| w^j, h^j, \vec{v}^j, q^j, \eta^j \| \}^4 + C \{ \| w^j, h^j, \vec{v}^j, q^j, \eta^j \| \}^8
\]
for all \( j \geq 2 \) and \( t > 0 \).

**Proof.** By the definition of \( \alpha^j, \beta^j \) and \( J^j \) in (5.3) we know that \( \{ (J^j)^{-2} \left[ (\alpha^j)^2 + (\beta^j)^2 + 1 \right] \} \}
\( (\nabla \bar{\eta}^j)^2 \). Thus it follows from the Sobolev embedding inequality and Lemma 4.3 that
\[
\| \{ (J^j)^{-2} \left[ (\alpha^j)^2 + (\beta^j)^2 + 1 \right] \} \partial_3 \vec{v}^j \|^2_{L^2_t H^4} \\
\leq C \left( \| \nabla \bar{\eta}^j \|^2_{L^2_t L^6} \| \nabla^2 \bar{\eta}^j \|^2_{L^2_t L^4} \| \partial_3 \vec{v}^j \|^2_{L^2_t L^p} + \| \nabla \bar{\eta}^j \|^4_{L^2_t L^2} \| \nabla \partial_3 \vec{v}^j \|^2_{L^2_t L^2} \right) \\
\leq C \| \eta^j \|^4_{L^2_t H^p(\mathbb{R}^2)} \| \vec{v}^j \|^2_{L^2_t H^3}.
\]
Similarly, one gets
\[
\| 2\alpha^j (J^j)^{-1} \partial_3 \partial_1 \vec{v}^j \|^2_{L^2_t H^4} + \| 2\beta^j (J^j)^{-1} \partial_3 \partial_2 \vec{v}^j \|^2_{L^2_t H^4} \leq C \| \eta^j \|^2_{L^2_t H^p(\mathbb{R}^2)} \| \vec{v}^j \|^2_{L^2_t H^3}.
\]
Recalling the definition of \( \vec{Q} \) in (5.13) and using the fact \( (1 - J^j), \alpha^j, \beta^j \sim \nabla \bar{\eta}^j \) we have
\[
\| Q_1 \|^2_{L^2_t H^4} + \| Q_2 \|^2_{L^2_t H^4} + \| Q_3 \|^2_{L^2_t H^4} \leq C ( \| \eta^j \|^2_{L^2_t H^p(\mathbb{R}^2)} + \| \eta^j \|^4_{L^2_t H^p(\mathbb{R}^2)} ) \| \vec{v}^j \|^2_{L^2_t H^4}.
\]
We next estimate LOT. Sobolev embedding inequality and Lemma 4.3 lead to
\[
\| (\nabla \bar{\eta}^j)^2 (\nabla^2 \bar{\eta}^j) \cdot \nabla \vec{v}^j \|^2_{L^2_t H^4} \\
\leq C \| \nabla \bar{\eta}^j \|^2_{L^2_t L^6} \| \nabla^2 \bar{\eta}^j \|^2_{L^2_t L^4} \| \nabla \vec{v}^j \|^2_{L^2_t L^p} + C \| \nabla \bar{\eta}^j \|^4_{L^2_t L^2} \| \nabla^3 \bar{\eta}^j \|^2_{L^2_t L^2} \| \nabla \vec{v}^j \|^2_{L^2_t L^p} + C \| \nabla \bar{\eta}^j \|^4_{L^2_t L^2} \| \nabla^2 \bar{\eta}^j \|^2_{L^2_t L^4} \| \nabla \vec{v}^j \|^2_{L^2_t L^p} \\
\leq C \| \eta^j \|^6_{L^2_t H^p(\mathbb{R}^2)} \| \vec{v}^j \|^2_{L^2_t H^3}
\]
and
\[
\| (\nabla \bar{\eta}^j)^2 (\nabla^3 \bar{\eta}^j) \cdot \vec{v}^j \|^2_{L^2_t H^4} \\
\leq \| \nabla \bar{\eta}^j \|^2_{L^2_t L^6} \| \nabla^2 \bar{\eta}^j \|^2_{L^2_t L^4} \| \nabla \vec{v}^j \|^2_{L^2_t L^p} + C \| \nabla \bar{\eta}^j \|^4_{L^2_t L^2} \| \nabla^4 \bar{\eta}^j \|^2_{L^2_t L^2} \| \vec{v}^j \|^2_{L^2_t L^p} + C \| \nabla \bar{\eta}^j \|^4_{L^2_t L^2} \| \nabla^3 \bar{\eta}^j \|^2_{L^2_t L^4} \| \vec{v}^j \|^2_{L^2_t L^p} \\
\leq C \| \eta^j \|^4_{L^2_t H^p(\mathbb{R}^2)} \| \nabla^2 \bar{\eta}^j \|^2_{L^2_t H^3} \| \vec{v}^j \|^2_{L^2_t H^3} + C \| \eta^j \|^6_{L^2_t H^p(\mathbb{R}^2)} \| \vec{v}^j \|^2_{L^2_t H^3}.
\]
On the other hand, by the Sobolev embedding inequality and (5.22) one deduces that
\[
\| \nabla \bar{\eta}^j \cdot \vec{v}^j \|^2_{L^2_t H^4} + \| \nabla^2 \bar{\eta}^j \cdot \vec{v}^j \|^2_{L^2_t H^4} \leq C ( \| \eta^j \|^2_{L^2_t H^p(\mathbb{R}^2)} + 1 ) \| \vec{v}^j \|^2_{L^2_t H^3} \| \vec{v}^j \|^2_{L^2_t H^3}.
\]
and that
\[ \|w^j\nabla \eta^j \nabla \varphi\|_{L^2_{-H^1}}^2 \leq C\|\eta^j\|_{L^2_{-H^1}(\mathbb{R}^2)}^2 \|w^j\|_{L^2_{-H^1}}^2 \|\nabla \varphi\|_{L^2_{-H^1}}^2. \]  
(5.20)

Combining (5.17)-(5.20) we arrive at
\[ \|LOT\|_{L^2_{-H^1}}^2 \leq C\|\{w^j, h^j, \tilde{v}^j, q^j, \eta^j\}\|_8^4 + C\|\{w^j, h^j, \tilde{v}^j, q^j, \eta^j\}\|_8^8 \]
which, along with (5.14)-(5.16) gives the desired estimate. The proof is completed.

\[ \square \]

**Lemma 5.2.** Let the assumptions in Proposition 5.1 and (5.12) hold true. Then
\[ \|\tilde{R}^j\|_{L^2_{-(\mathcal{H}^1)}}^2 \leq C\|\{w^j, h^j, \tilde{v}^j, q^j, \eta^j\}\|_8^4 + C\|\{w^j, h^j, \tilde{v}^j, q^j, \eta^j\}\|_8^8 \]
for all \( j \geq 2 \) and \( t > 0 \), where the constant \( C \) is independent of \( j \) and \( t \).

**Proof.** By the definition of \( \alpha^j, \beta^j \) and \( J^j \) in (3.3) we know that
\[ \{J^j\}^{-2} [((\alpha^j)^2 + (\beta^j)^2 + 1) - 1, j] \cdot \partial_{3 \tilde{v}^j} \sim \nabla \eta^j \nabla \varphi_{\bar{t}} \partial_{3 \tilde{v}^j} + \nabla \eta^j \nabla \eta_{\bar{t}} \partial_{3 \tilde{v}^j}. \]  
(5.21)

Since \( \eta^j = v^j_3 \) on \( \Gamma \times (0, \infty) \), one has \( \tilde{\eta}^j = \mathcal{H}(\eta^j) = \mathcal{H}(v^j_3(x_1, x_2, 0, t)) \). Then it follows from Lemma 4.3 and the trace theorem that
\[ \|\tilde{\eta}^j\|_{H^m} \leq C\|\eta^j\|_{H^{m-\frac{1}{2}}(\mathbb{R}^2)} = C\|v^j_3\|_{H^{m-\frac{1}{2}}(\Gamma)} \leq C\|\tilde{\varphi}\|_{H^m} \]  
(5.22)

for \( m \geq 2 \). Thus,
\[ \|\nabla \tilde{\eta}^j \nabla \tilde{\eta}^j \partial_{3 \tilde{v}^j} + \nabla \tilde{\eta}^j \nabla \eta_{\bar{t}} \partial_{3 \tilde{v}^j} \|_{L^2_{-L^2}}^2 \leq (\|\nabla \tilde{\eta}^j\|_{L^2_{-L^2}}^2 + \|\nabla \tilde{\eta}^j\|_{L^2_{-L^2}}^4) \|\nabla \tilde{\eta}^j\|_{\tilde{L}^2_{-L^2}} \|\partial_{3 \tilde{v}^j}\|_{\tilde{L}^2_{-L^2}}^2 \]
\[ \leq C(\|\tilde{\eta}^j\|_{L^2_{-H^1}}^2 + \|\eta_{\bar{t}}\|_{L^2_{-H^1}}^4) \|\tilde{\eta}^j\|_{L^2_{-H^1}} \|\tilde{\varphi}\|_{L^2_{-H^1}}^2 \]
\[ \leq C(\|\tilde{\eta}^j\|_{L^2_{-H^1}(\mathbb{R}^2)}^2 + \|\eta_{\bar{t}}\|_{L^2_{-H^1}(\mathbb{R}^2)}^4) \|\tilde{\varphi}\|_{L^2_{-H^1}}^2 \]
which, along with (5.21) entails that
\[ \|\{J^j\}^{-2} [((\alpha^j)^2 + (\beta^j)^2 + 1) - 1, j] \cdot \partial_{3 \tilde{v}^j}\|_{L^2_{-L^2}}^2 \]
\[ \leq C(\|\eta_{\bar{t}}\|_{L^2_{-H^1}(\mathbb{R}^2)}^2 + \|\eta_{\bar{t}}\|_{L^2_{-H^1}(\mathbb{R}^2)}^4) \|\tilde{\varphi}\|_{L^2_{-H^1}}^2. \]  
(5.23)

Using again the definition of \( \alpha^j, \beta^j \) and \( J^j \) in (3.3) to deduce that \( \{J^j\}^{-2} [((\alpha^j)^2 + (\beta^j)^2 + 1) - 1] \sim (\nabla \tilde{\eta}^j)^2 \). Thus for any \( \tilde{\varphi} \in \mathcal{H}^1 \), integration by parts yields
\[ \int_{\Omega} \{J^j\}^{-2} [((\alpha^j)^2 + (\beta^j)^2 + 1) - 1] \cdot (\partial_{3 \tilde{v}^j}) \cdot \tilde{\varphi} \, dx \]
\[ = \int_{\Gamma} \{J^j\}^{-2} [((\alpha^j)^2 + (\beta^j)^2 + 1) - 1] \cdot (\partial_{3 \tilde{v}^j}) \cdot \tilde{\varphi} \, dx \]
\[ - \int_{\Omega} \partial_{3 \tilde{v}^j} \{J^j\}^{-2} [((\alpha^j)^2 + (\beta^j)^2 + 1) - 1] \cdot (\partial_{3 \tilde{v}^j}) \cdot \tilde{\varphi} \, dx \]
\[ - \int_{\Omega} \{J^j\}^{-2} [((\alpha^j)^2 + (\beta^j)^2 + 1) - 1] \cdot (\partial_{3 \tilde{v}^j}) \cdot \partial_{3 \tilde{\varphi}} \, dx \]
\[ \leq C \|\nabla \tilde{\varphi}\|_{L^2_{-H^1}}^2 \|\tilde{\varphi}\|_{L^2_{-H^1}}^2 + C \|\nabla \tilde{\eta}^j\|_{L^2_{-L^2}} \|\nabla \tilde{\eta}^j\|_{L^2_{-L^2}} \|\varphi\|_{L^2_{-L^2}} \|\tilde{\varphi}\|_{L^2_{-L^2}} \]
\[ + C \|\nabla \tilde{\eta}^j\|_{\tilde{L}^2_{-L^2}} \|\partial_{3 \tilde{\varphi}}\|_{\tilde{L}^2_{-L^2}} \]
\[ \leq C \|\nabla \tilde{\varphi}\|_{L^2_{-H^1}}^2 \|\tilde{\varphi}\|_{L^2_{-H^1}}^2 + C \|\nabla \tilde{\varphi}|_{L^2_{-H^1}}^2 \|\tilde{\varphi}\|_{L^2_{-H^1}}^2 \|\tilde{\varphi}\|_{L^2_{-H^1}}^2 \]
where we have used the Sobolev embedding inequality, trace theorem and the following fact

\[
\| (\nabla \tilde{\eta}^j)^2 \phi \|_{H^1_0(\Gamma)} \leq C \| (\nabla \tilde{\eta}^j)^2 \phi \|_{L^2(\Gamma)} + C \| \nabla \tilde{\eta}^j \nabla (\nabla \tilde{\eta}^j)^2 \phi \|_{L^2(\Gamma)} + C \| (\nabla \tilde{\eta}^j)^2 \nabla_\Gamma \phi \|_{L^2(\Gamma)} \\
\leq C \| \nabla \tilde{\eta}^j \|_{L^2(\Gamma)} \| \phi \|_{L^2(\Gamma)} + C \| \nabla \tilde{\eta}^j \|_{L^2(\Gamma)} \| \nabla_\Gamma (\nabla \tilde{\eta}^j)^2 \phi \|_{L^2(\Gamma)} + C \| \nabla \tilde{\eta}^j \|_{L^2(\Gamma)} \| \nabla_\Gamma \phi \|_{L^2(\Gamma)} \\
\leq C \| \eta^j \|_{H^1(\mathbb{R}^2)} \| \phi \|_{H^1},
\]

thanks to Lemma 4.3 and the trace theorem. Then (5.24) entails that

\[
\| \{ (J^j)^{-2} [ (\alpha^j)^2 + (\beta^j)^2 + 1 ] - 1 \} \partial_\Gamma^2 \bar{\nu}^j \|_{L^2(\partial \Omega)} \leq C \| \eta^j \|_{L^2(\mathbb{R}^2)}^4 \left( \| \nabla \bar{\nu}^j \|^2_{L^2(\partial \Omega)} + \| \nabla \bar{\nu}^j \|^2_{L^2(H^1)} \right)
\]

which, along with (5.23) implies

\[
\| \{ (J^j)^{-2} [ (\alpha^j)^2 + (\beta^j)^2 + 1 ] - 1 \} \partial_\Gamma^2 \bar{\nu}^j \|_{L^2(\partial \Omega)} \leq C \| \eta^j \|_{L^2(\mathbb{R}^2)}^4 \left( \| \nabla \bar{\nu}^j \|^2_{L^2(\partial \Omega)} + \| \nabla \bar{\nu}^j \|^2_{L^2(H^1)} \right)
\]

A similar argument used in deriving (5.25) leads to

\[
\| 2 (J^j)^{-1} \alpha \partial_\Gamma \partial_\Gamma \bar{\nu}^j \|_{L^2(\partial \Omega)}^2 + \| 2 (J^j)^{-1} \beta \partial_\Gamma \partial_\Gamma \bar{\nu}^j \|_{L^2(\partial \Omega)}^2 \\
\leq C \| \eta^j \|_{L^2(\mathbb{R}^2)}^2 \left( \| \nabla \bar{\nu}^j \|^2_{L^2(\partial \Omega)} + \| \nabla \bar{\nu}^j \|^2_{L^2(H^1)} \right) + C \left( \| \eta^j \|_{L^2(\mathbb{R}^2)}^2 + 1 \right) \| \bar{\nu}^j \|^2_{L^2(H^1)} \| \bar{\nu}^j \|^2_{L^2(H^1)}.
\]

Noting \((1 - J^j) \sim \nabla \tilde{\eta}^j\), for any \(\phi \in H^1\) we have

\[
\int_{\Omega} [(1 - J^j) \partial_1 q^j, \phi] dxdy \leq \| \nabla \tilde{\eta}^j \|_{L^2} \| \nabla q^j \|_{L^2} \| \phi \|_{L^2} + \| \nabla \tilde{\eta}^j \|_{H^1} \| \nabla q^j \|_{(\partial \Omega)^*} \times \| \phi \|_{H^1} \\
\leq C \| \bar{\nu}^j \|_{H^1} \| \nabla q^j \|_{H^1} \| \phi \|_{H^1} + \| \eta^j \|_{H^1(\mathbb{R}^2)} \| \phi \|_{H^1} \| \nabla q^j \|_{(\partial \Omega)^*},
\]

where we have used the Sobolev embedding inequality, (5.22) and Lemma 4.3 in the last inequality. Thus

\[
\| [(1 - J^j) \partial_1 q^j] \|_{L^2(\partial \Omega)}^2 \leq C \| \bar{\nu}^j \|_{L^2(H^1)} \| \nabla q^j \|_{L^2(H^1)}^2 + C \| \eta^j \|_{L^2(H^1(\mathbb{R}^2))}^2 \| \nabla q^j \|_{L^2(H^1)}^2.
\]

By a similar argument used in deriving (5.27) one deduces that

\[
\| [\alpha \partial_1 q^j] \|_{L^2(\partial \Omega)}^2 \leq C \| \bar{\nu}^j \|_{L^2(H^1)}^2 \| \nabla q^j \|_{L^2(H^1)}^2 + C \| \eta^j \|_{L^2(H^1(\mathbb{R}^2))}^2 \| \nabla q^j \|_{L^2(H^1)}^2
\]

which, along with (5.27) gives rise to

\[
\| Q_1 \|_{L^2(\partial \Omega)}^2 \leq C \| \bar{\nu}^j \|_{L^2(H^1)}^2 \| \nabla q^j \|_{L^2(H^1)}^2 + C \| \eta^j \|_{L^2(H^1(\mathbb{R}^2))}^2 \| \nabla q^j \|_{L^2(H^1)}^2.
\]

Similarly,

\[
\| Q_2 \|_{L^2(\partial \Omega)}^2 \leq C \| \bar{\nu}^j \|_{L^2(H^1)}^2 \| \nabla q^j \|_{L^2(H^1)}^2 + C \| \eta^j \|_{L^2(H^1(\mathbb{R}^2))}^2 \| \nabla q^j \|_{L^2(H^1)}^2
\]

and

\[
\| Q_3 \|_{L^2(\partial \Omega)}^2 \leq C \| \eta^j \|_{L^2(H^1(\mathbb{R}^2))}^2 + \| \eta^j \|_{L^2(H^1(\mathbb{R}^2))}^2 \| \bar{\nu}^j \|_{L^2(H^1)}^2 \| \nabla q^j \|_{L^2(H^1)}^2
\]

+ \[C \| \eta^j \|_{L^2(H^1(\mathbb{R}^2))}^2 \| \nabla q^j \|_{L^2(H^1)}^2.
\]

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We proceed to estimating each term in $(LOT)_t$. For any $\varphi \in H^1$, it follows from Lemma 4.3 and (5.22) that

$$\int_{\Omega} |(\nabla \bar{\eta})^2 \nabla^2 \bar{\eta} \cdot \nabla \bar{v}| \, dx \, dy \leq \|\nabla \bar{\eta}\|_{L^2} \|\nabla^2 \bar{\eta}\|_{L^2} \|\nabla^2 \bar{\eta}\|_{L^2} \|\varphi\|_{L^4} + \|\nabla \bar{\eta}\|_{L^2} \|\nabla^2 \bar{\eta}\|_{L^2} \|\varphi\|_{L^4}$$

Thus

$$\left\langle \left\langle (\nabla \bar{\eta})^2 \nabla^2 \bar{\eta} \cdot \nabla \bar{v}, \delta \right\rangle \right\rangle_{L^2} \leq C(\|\bar{\eta}\|_{H^1(\mathbb{R}^2)}^2 \|\bar{v}\|_{L^2} \|\bar{v}\|_{L^2} + \|\nabla \bar{v}\|_{H^1(\mathbb{R}^2)}^2 \|\bar{v}\|_{L^2}^2). \quad (5.31)$$

For any $\varphi \in H^1$ we get

$$\int_{\Omega} |(\nabla \bar{\eta})^2 \nabla^3 \bar{\eta} \cdot \nabla \bar{v}| \, dx \, dy \leq 2 \|\nabla \bar{\eta}\|_{L^2} \|\nabla \bar{\eta}\|_{L^2} \|\nabla^3 \bar{\eta}\|_{L^2} \|\nabla \bar{v}\|_{L^2} \|\varphi\|_{L^4} + \|\nabla \bar{\eta}\|_{L^2} \|\nabla \bar{\eta}\|_{L^2} \|\nabla^3 \bar{\eta}\|_{L^2} \|\nabla \bar{v}\|_{L^2} \|\varphi\|_{L^4}$$

Thus

$$\left\langle \left\langle (\nabla \bar{\eta})^2 \nabla^3 \bar{\eta} \cdot \nabla \bar{v}, \delta \right\rangle \right\rangle_{L^2} \leq C(\|\bar{\eta}\|_{H^1(\mathbb{R}^2)}^2 \|\bar{v}\|_{L^2} \|\bar{v}\|_{L^2} + \|\nabla \bar{v}\|_{H^1(\mathbb{R}^2)}^2 \|\bar{v}\|_{L^2}^2). \quad (5.32)$$

Employing Lemma 4.3 and (5.22) one can easily deduce that

$$\left\langle \left\langle (\nabla^2 \bar{\eta})^2 \nabla \bar{v}, \delta \right\rangle \right\rangle_{L^2} \leq C(\|\nabla \bar{\eta}\|_{L^2}^2 \|\nabla \bar{v}\|_{L^2}^2 + \|\nabla \bar{\eta}\|_{L^2}^2 \|\nabla \bar{v}\|_{L^2}^2). \quad (5.33)$$

and that

$$\left\langle \left\langle (\nabla^2 \bar{\eta})^2 \nabla \bar{v}, \delta \right\rangle \right\rangle_{L^2} \leq C(\|\nabla \bar{\eta}\|_{L^2}^2 \|\nabla \bar{v}\|_{L^2}^2 + \|\nabla \bar{\eta}\|_{L^2}^2 \|\nabla \bar{v}\|_{L^2}^2). \quad (5.34)$$

Combining (5.31), (5.32), (5.33) and (5.34) arrive at

$$\left\langle \left\langle (LOT)_t, \delta \right\rangle \right\rangle_{L^2} \leq C\left\{w, h^j, \bar{v}^j, q^j, \eta^j \right\}^4 + C\left\{w, h^j, \bar{v}^j, q^j, \eta^j \right\}^8. \quad (5.35)$$

Collecting (5.25)–(5.26), (5.28)–(5.30) and (5.35) we derive the desired estimates. The proof is finished.

We next estimate $\bar{G}(\bar{v}, \bar{\eta})$ and $\bar{G}_t(\bar{v}, \bar{\eta})$. From (7.12)–(7.14) we know that

$$\bar{G}(\bar{v}, \bar{\eta}) \sim (\nabla_0 \eta^j)^4 (\bar{v}) + (\nabla_0 \eta^j)^4 (\bar{v})^2 + (\nabla_0 \eta^j)^2 (\bar{v})^2. \quad (5.36)$$

**Lemma 5.3.** Suppose that the assumptions in Proposition 5.7 and (5.12) hold. Then there exists a constant $C$ independent of $j$ and $t$ such that

$$\|\bar{G}\|_{L^2}^2 + \|\bar{G}_t\|_{L^2}^2 \leq C\left\{w, h^j, \bar{v}^j, q^j, \eta^j \right\}^4 + C\left\{w, h^j, \bar{v}^j, q^j, \eta^j \right\}^8$$

for all $j \geq 2$ and $t > 0$. 

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Proof. It follows from the Sobolev embedding inequality and trace theorem that
\[
\left\| \nabla \eta^j \right\|^2_{L^2 H^2(\Gamma')} \leq C \left\| \nabla \eta^j \right\|^2_{L^2 H^2(\mathbb{R}^2)} \leq C \left\| \nabla \eta^j \right\|^2_{L^2 H^2(\mathbb{R}^2)} \leq C \left\| \nabla \eta^j \right\|^2_{L^2 H^2(\mathbb{R}^2)} \leq C \left\| \nabla \eta^j \right\|^2_{L^2 H^2(\mathbb{R}^2)} \leq C \left\| \nabla \eta^j \right\|^2_{L^2 H^2(\mathbb{R}^2)}.
\] (5.37)

Similarly,
\[
\left\| \nabla \eta^j \right\|^2_{L^2 H^2(\Gamma')} \leq C \left\| \nabla \eta^j \right\|^2_{L^2 H^2(\mathbb{R}^2)} \leq C \left\| \nabla \eta^j \right\|^2_{L^2 H^2(\mathbb{R}^2)} \leq C \left\| \nabla \eta^j \right\|^2_{L^2 H^2(\mathbb{R}^2)} \leq C \left\| \nabla \eta^j \right\|^2_{L^2 H^2(\mathbb{R}^2)} \leq C \left\| \nabla \eta^j \right\|^2_{L^2 H^2(\mathbb{R}^2)}.
\] (5.38)

and
\[
\left\| \nabla \eta^j \right\|^2_{L^2 H^2(\mathbb{R}^2)} \leq C \left\| \nabla \eta^j \right\|^2_{L^2 H^2(\mathbb{R}^2)} \leq C \left\| \nabla \eta^j \right\|^2_{L^2 H^2(\mathbb{R}^2)} \leq C \left\| \nabla \eta^j \right\|^2_{L^2 H^2(\mathbb{R}^2)} \leq C \left\| \nabla \eta^j \right\|^2_{L^2 H^2(\mathbb{R}^2)} \leq C \left\| \nabla \eta^j \right\|^2_{L^2 H^2(\mathbb{R}^2)}.
\] (5.39)

Combining (5.37)- (5.39) and using (5.36) yields
\[
\left\| \tilde{G} \right\|^2_{L^2 H^2(\mathbb{R}^2)} \leq C \left\{ \| w_j \|^2_{H^1(\mathbb{R}^2)} + C \| w_j \|^2_{H^1(\mathbb{R}^2)} \right\} \leq C \left\{ \| w_j \|^2_{H^1(\mathbb{R}^2)} + C \| w_j \|^2_{H^1(\mathbb{R}^2)} \right\}.
\] (5.40)

The Sobolev embedding inequality, the trace theorem and (5.22) lead to
\[
\left\| \nabla \eta^j \right\|^2_{L^2 H^2(\mathbb{R}^2)} \leq C \left\{ \| w_j \|^2_{H^1(\mathbb{R}^2)} + C \| w_j \|^2_{H^1(\mathbb{R}^2)} \right\} \leq C \left\{ \| w_j \|^2_{H^1(\mathbb{R}^2)} + C \| w_j \|^2_{H^1(\mathbb{R}^2)} \right\}.
\] (5.41)

On the other hand, for any $\varphi \in H^2(\mathbb{R}^2)$ one has
\[
\int_{\Gamma'} \left( \nabla \eta^j \right)^4 \nabla \varphi dx \leq \left\| \nabla \eta^j \right\|^2_{L^2 H^2(\mathbb{R}^2)} \left\| \nabla \eta^j \right\|^2_{L^2 H^2(\mathbb{R}^2)} \leq C \left\| \nabla \eta^j \right\|^2_{L^2 H^2(\mathbb{R}^2)} \left\| \nabla \eta^j \right\|^2_{L^2 H^2(\mathbb{R}^2)} \leq C \left\| \nabla \eta^j \right\|^2_{L^2 H^2(\mathbb{R}^2)} \left\| \nabla \eta^j \right\|^2_{L^2 H^2(\mathbb{R}^2)}.
\]

Thus
\[
\left\| \nabla \eta^j \right\|^4_{H^2(\mathbb{R}^2)} \leq C \left\| \nabla \eta^j \right\|^2_{L^2 H^2(\Gamma')} \left\| \nabla \eta^j \right\|^4_{L^2 H^2(\Gamma')} \leq C \left\| \nabla \eta^j \right\|^2_{L^2 H^2(\Gamma')} \left\| \nabla \eta^j \right\|^4_{L^2 H^2(\Gamma')} \leq C \left\| \nabla \eta^j \right\|^2_{L^2 H^2(\Gamma')} \left\| \nabla \eta^j \right\|^4_{L^2 H^2(\Gamma')}.
\]

which, along with (5.41) implies that
\[
\left\| \nabla \eta^j \right\|^4_{L^2 H^2(\Gamma')} \leq C \left\| \nabla \eta^j \right\|^2_{L^2 H^2(\Gamma')} \left\| \nabla \eta^j \right\|^2_{L^2 H^2(\Gamma')} \leq C \left\| \nabla \eta^j \right\|^2_{L^2 H^2(\Gamma')} \left\| \nabla \eta^j \right\|^4_{L^2 H^2(\Gamma')} \leq C \left\| \nabla \eta^j \right\|^2_{L^2 H^2(\Gamma')} \left\| \nabla \eta^j \right\|^4_{L^2 H^2(\Gamma')}.
\] (5.42)

A direct computation yields
\[
\left( \nabla \eta^j \right)^4 \nabla \eta^j \nabla \eta^j = \left( \nabla \eta^j \right)^4 \nabla \eta^j \nabla \eta^j + \left( \nabla \eta^j \right)^4 \nabla \eta^j \nabla \eta^j + \left( \nabla \eta^j \right)^4 \nabla \eta^j \nabla \eta^j + \left( \nabla \eta^j \right)^4 \nabla \eta^j \nabla \eta^j
\]
\[
:= l_1 + l_2 + l_3.
\]

By (5.22), the Sobolev embedding inequality and trace theorem one deduces that
\[
\left\| l_1 \right\|^2_{L^2 H^2(\Gamma')} \leq C \left\| \nabla \eta^j \right\|^2_{L^2 H^2(\Gamma')} \left\| \nabla \eta^j \right\|^2_{L^2 H^2(\Gamma')} \left\| \nabla \eta^j \right\|^2_{L^2 H^2(\Gamma')} \left\| \nabla \eta^j \right\|^2_{L^2 H^2(\Gamma')} \leq C \left\| \nabla \eta^j \right\|^2_{L^2 H^2(\Gamma')} \left\| \nabla \eta^j \right\|^2_{L^2 H^2(\Gamma')} \left\| \nabla \eta^j \right\|^2_{L^2 H^2(\Gamma')} \left\| \nabla \eta^j \right\|^2_{L^2 H^2(\Gamma')}.
\]
and that
\[ ||I_2||_{L^2_t(L^2_\Gamma)}^2 \leq C||\nabla_0 \eta^j||_{L^2_t L^\infty_\Gamma}^8 ||\nabla_0^2 \eta^j||_{L^2_t L^2_\Gamma}^2 ||\bar{\eta}^j||_{L^2_t L^2_\Gamma}^2 \leq C||\eta^j||_{L^2_t H^3_\Gamma}^{10} ||\bar{\eta}^j||_{L^2_t H^3_\Gamma}^2. \]

Sobolev embedding inequality gives
\[ ||I_3||_{L^2_t(L^2_\Gamma)}^2 \leq ||\nabla_0 \eta^j||_{L^2_t L^\infty_\Gamma}^8 ||\nabla_0^2 \eta^j||_{L^2_t L^4_\Gamma}^2 ||\bar{\eta}^j||_{L^2_t L^4_\Gamma}^2 \leq C||\eta^j||_{L^2_t H^5_\Gamma}^{10} ||\bar{\eta}^j||_{L^2_t H^5_\Gamma}^2, \]

which, in conjunction with the above estimates for \( I_1 \) and \( I_2 \) indicates that
\[ ||((\nabla_0 \eta^j)^4 \nabla_0^2 \eta^j)\bar{\eta}^j||_{L^2_t L^2_\Gamma}^2 \leq C||\eta^j||_{L^2_t H^5_\Gamma}^{10} ||\bar{\eta}^j||_{L^2_t H^5_\Gamma}^2 \leq C||\eta^j||_{L^2_t H^5_\Gamma}^{10} ||\bar{\eta}^j||_{L^2_t H^5_\Gamma}^2. \quad (5.43) \]

By a similar argument used in deriving (5.43) one gets
\[ ||((\nabla_0 \eta^j)^4 \nabla_0^2 \eta^j)\bar{\eta}^j||_{L^2_t L^2_\Gamma}^2 \leq C||\eta^j||_{L^2_t H^5_\Gamma}^{10} ||\bar{\eta}^j||_{L^2_t H^5_\Gamma}^2. \quad (5.44) \]

Collecting (5.42)-(5.44) and using (5.36) we have
\[ ||\bar{G}_t||_{L^2_t H^{\frac{1}{2}}_\Gamma}^2 \leq C||\{w^j, h^j, \bar{v}^j, q^j, \eta^j\}||^4 + ||\{w^j, h^j, \bar{v}^j, q^j, \eta^j\}||^{12} \]

which, along with (5.40) gives the desired estimates and completes the proof.

The estimates for \( ||G_4(w^j, h^j, \bar{\eta}^j)||_{L^2_t H^{\frac{1}{2}}_\Gamma}^2 \) and \( ||G_4(w^j, h^j, \bar{\eta}^j)||_{L^2_t H^{\frac{1}{2}}_\Gamma}^2 \) is as follows.

**Lemma 5.4.** Let the assumptions in Proposition 5.1 and (5.12) hold true. Then there exists a constant \( C \) independent of \( j \) and \( t \) such that
\[ ||G_4||_{L^2_t H^{\frac{1}{2}}_\Gamma}^2 + ||G_4||_{L^2_t H^{\frac{1}{2}}_\Gamma}^2 \leq C||\{w^j, h^j, \bar{v}^j, q^j, \eta^j\}||^4 + C||\{w^j, h^j, \bar{v}^j, q^j, \eta^j\}||^{10} \]
for \( j \geq 2 \) and \( t > 0 \).

**Proof.** From (7.5) we know that
\[ G_4(w^j, h^j, \bar{\eta}^j) \sim \nabla_0 \bar{\eta}^j \nabla_0 \eta^j (\nabla_0 w^j + w^j \nabla_0 h^j). \quad (5.45) \]

Then the Sobolev embedding inequality, the trace theorem and Lemma 4.3 entail that
\[ ||G_4||_{L^2_t H^{\frac{1}{2}}_\Gamma}^2 \leq C||\nabla \bar{\eta}^j||_{L^2_t H^{\frac{1}{2}}_\Gamma}^2 ||\nabla \eta^j||_{L^2_t H^{\frac{1}{2}}_\Gamma}^2 (||\nabla w^j||_{L^2_t H^{\frac{1}{2}}_\Gamma}^2 + ||w^j||_{L^2_t H^{\frac{1}{2}}_\Gamma}^2) ||\nabla h^j||_{L^2_t H^{\frac{1}{2}}_\Gamma}^2 \leq C||\eta^j||_{L^2_t H^3_\Gamma}^{10} ||\bar{\eta}^j||_{L^2_t H^3_\Gamma}^2 + ||w^j||_{L^2_t H^3_\Gamma}^2 ||h^j||_{L^2_t H^3_\Gamma}^2. \quad (5.46) \]

We proceed to estimating \( ||G_4||_{L^2_t H^{\frac{1}{2}}_\Gamma}^2 \). First, it follows from the Sobolev embedding inequality, Lemma 4.3 and (5.22) that
\[ ||\nabla_0 \bar{\eta}^j \nabla_0 \eta^j \nabla_0 w^j||_{L^2_t L^2_\Gamma}^2 + ||\nabla_0 \bar{\eta}^j \nabla_0 \eta^j \nabla_0 w^j||_{L^2_t L^2_\Gamma}^2 \leq C||\eta^j||_{L^2_t H^3_\Gamma}^{10} ||\bar{\eta}^j||_{L^2_t H^3_\Gamma}^2 ||\nabla_0 \eta^j||_{L^2_t L^2_\Gamma}^2 ||\nabla_0 w^j||_{L^2_t L^4_\Gamma}^2 \]
\[ \leq C||\eta^j||_{L^2_t H^3_\Gamma}^{10} ||\bar{\eta}^j||_{L^2_t H^3_\Gamma}^2 ||\nabla \eta^j||_{L^2_t H^3_\Gamma}^2 ||\nabla w^j||_{L^2_t H^3_\Gamma}^2 ||h^j||_{L^2_t H^3_\Gamma}^2. \quad (5.47) \]
For any $\varphi \in H^{1}_{\gamma} (\mathbb{R}^{2})$, by using Lemma 4.3 and the trace theorem one gets

$$
\int_{\Gamma} \nabla v \bar{\eta} \nabla v \nabla w_{j} \varphi dx \leq \| \nabla v \bar{\eta} \nabla v \nabla w_{j} \varphi \|_{H^{\gamma}_{\Gamma} (\Gamma)} \leq \| \nabla v \bar{\eta} \nabla v \nabla w_{j} \varphi \|_{H^{\gamma}_{\Gamma} (\Gamma)} \leq C \| \eta \|_{H^{4} (\mathbb{R}^{2})}^{4} \| w_{j} \|_{H^{4} (\mathbb{R}^{2})}^{2} \| H^{4} (\mathbb{R}^{2}) \} \|
$$

Thus,

$$
\| \nabla v \bar{\eta} \nabla v \nabla w_{j} \|_{L^{2}_{H^{\gamma}_{\Gamma} (\Gamma)}}^{2} \leq C \| \eta \|_{L^{2}_{H^{4} (\mathbb{R}^{2})}}^{4} \| w_{j} \|_{L^{2}_{H^{4} (\mathbb{R}^{2})}}^{2} \| H^{4} (\mathbb{R}^{2}) \} \|
$$

which, in conjunction with (5.47) gives rise to

$$
(\nabla v \bar{\eta} \nabla v \nabla w_{j})_{,\frac{1}{2}} \leq C \| \eta \|_{L^{2}_{H^{4} (\mathbb{R}^{2})}}^{4} \| \nabla v \|_{L^{2}_{H^{4} (\mathbb{R}^{2})}}^{2} \| w_{j} \|_{L^{2}_{H^{4} (\mathbb{R}^{2})}}^{2} + \| w_{j} \|_{L^{2}_{H^{4} (\mathbb{R}^{2})}}^{2} \| H^{4} (\mathbb{R}^{2}) \} \|
$$

By a similar argument used in deriving (5.48), one can deduce that

$$
(\nabla v \bar{\eta} \nabla v \nabla w_{j})_{,\frac{1}{2}} \leq C \| \eta \|_{L^{2}_{H^{4} (\mathbb{R}^{2})}}^{4} \| \nabla v \|_{L^{2}_{H^{4} (\mathbb{R}^{2})}}^{2} \| w_{j} \|_{L^{2}_{H^{4} (\mathbb{R}^{2})}}^{2} + \| w_{j} \|_{L^{2}_{H^{4} (\mathbb{R}^{2})}}^{2} \| H^{4} (\mathbb{R}^{2}) \} \|
$$

Combining (5.48) and (5.49) we conclude that

$$
\| G_{\gamma} \|_{L^{2}_{H^{\gamma}_{\Gamma} (\Gamma)}}^{2} \leq C \| \{ w_{j}, h_{j}, \bar{v}_{j}, q_{j}, \eta \} \|^{4} + C \| \{ w_{j}, h_{j}, \bar{v}_{j}, q_{j}, \eta \} \|^{10}
$$

which, along with (5.46) indicates the desired estimates. The proof is finished.

We proceed to estimate $F_{4}(w_{j}(j-1), w_{j}, h_{j}, \bar{v}_{j}, \bar{\eta})$ and $F_{4}(w_{j}(j-1), w_{j}, h_{j}, \bar{v}_{j}, \bar{\eta})$. Recalling the definition of $F_{4}(w_{j}(j-1), w_{j}, h_{j}, \bar{v}_{j}, \bar{\eta})$ in (5.8) we have

$$
F_{4}(w_{j}(j-1), w_{j}, h_{j}, \bar{v}_{j}, \bar{\eta}) = \{ (J) \}^{-2} (\alpha \bar{v})^{2} \bar{\eta}^{2} + (\beta \bar{v})^{2} \bar{\eta}^{2} \bar{\eta} + (\beta \bar{v})^{2} \bar{\eta}^{2} \bar{\eta} + (\beta \bar{v})^{2} \bar{\eta}^{2} \bar{\eta} + \text{LOT}
$$

with

$$
\text{LOT} \sim \nabla \bar{\eta} \nabla w_{j} \bar{\eta} \nabla (\nabla \bar{\eta})^{2} \bar{\eta} \nabla w_{j} \nabla h_{j} + w_{j} \nabla \bar{\eta} \nabla h_{j} \nabla \bar{\eta} \nabla h_{j} + \nabla w_{j} \bar{\eta} \nabla h_{j}.
$$

**Lemma 5.5.** Suppose that the assumptions in Proposition 5.1 and (5.12) hold true. Then there exists a constant $C$ independent of $j$ and $t$ such that

$$
\| F_{4} \|_{L^{2}_{H^{4} (\mathbb{R}^{2})}}^{2} \leq C \| \{ w_{j}, h_{j}, \bar{v}_{j}, q_{j}, \eta \} \|^{4} + C \| \{ w_{j}, h_{j}, \bar{v}_{j}, q_{j}, \eta \} \|^{12}
$$

for $j \geq 2$ and $t > 0$. 

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Proof. First, Recalling the definition of $\alpha$, $\beta$ and $J$ in (3.3) we know $\{(J^j)^{-2}[(\alpha^j)^2+(\beta^j)^2+1]-1\} \sim (\nabla \eta^j)^2$. Thus follows from the Sobolev embedding inequality and Lemma 4.3 that
\begin{align}
\left\|\{(J^j)^{-2}[(\alpha^j)^2+(\beta^j)^2+1]-1\} \partial_3 \left( \partial_3 w^j + w^{(j-1)} \partial_3 h^j \right) \right\|_{L^2_t H^1}^2
\leq C \|\nabla \eta^j\|_{L^2_t H^2}^2 \|\nabla^2 w_j\|_{L^2_t H^4}^2 + C \|\nabla \eta^j\|_{L^2_t H^2(\mathbb{R}^2)}^2 \|\nabla (w^{(j-1)} \nabla h^j)\|_{L^2_t H^1}^2
\leq C \|\eta^j\|_{L^2_t H^4(\mathbb{R}^2)}^4 \|w_j\|_{L^2_t H^1}^4 + \|w^{(j-1)}\|_{L^2_t H^2}^4 \|h^j\|_{L^2_t H^1}^4.
\end{align}
(5.51)

A similar argument leads to
\begin{align}
\| - 2 \alpha(J^j)^{-1} \left( \partial_3 \partial_1 w^j + w^{(j-1)} \partial_3 h^j \right) - 2 \beta(J^j)^{-1} \left( \partial_3 \partial_2 w^j + w^{(j-1)} \partial_3 h^j \right) \|_{L^2_t H^1}^2 &\leq C \|\eta^j\|_{L^2_t H^4(\mathbb{R}^2)}^4 \|w^j\|_{L^2_t H^1}^4 + \|w^{(j-1)}\|_{L^2_t H^2}^4 \|h^j\|_{L^2_t H^1}^4. \tag{5.52}
\end{align}

We next estimate the second term in \textit{LOT} by using the Sobolev embedding inequality and Lemma 4.3
\begin{align}
\left\| (\nabla \eta^j)^2 \nabla^2 \nabla \nabla w_j \right\|_{L^2_t H^2} &\leq \|\nabla \eta^j\|_{L^2_t L^2}^2 \|\nabla^2 \nabla w_j\|_{L^2_t L^2}^2 + \|\nabla \eta^j\|_{L^2_t L^2} \|\nabla^2 \nabla w_j\|_{L^2_t L^2} + \|\nabla \eta^j\|_{L^2_t L^2} \|\nabla^2 \nabla w_j\|_{L^2_t L^2}^2 \\
&\leq C \|\eta^j\|_{L^6_t H^2(\mathbb{R}^2)}^6 \|w_j\|_{L^2_t H^1}^2. \tag{5.53}
\end{align}

By a similar argument in deriving (5.53) one can estimate the other terms in \textit{LOT} to conclude that
\begin{align}
\|\text{LOT}\|_{L^2_t H^1}^2 &\leq C \|\{w^j, h^j, \bar{v}^j, q^j, \eta^j\}\|^4 + C \|\{w^j, h^j, \bar{v}^j, q^j, \eta^j\}\|^4. \tag{5.54}
\end{align}

Collecting (5.51), (5.52) and (5.54) we derive the desired estimate. The proof is completed. \hfill \square

Lemma 5.6. Let the assumptions in Proposition 5.7 and (5.12) hold. Then there exists a constant $C$ independent of $j$ and $t$ such that
\begin{align}
\left\| F_4 \right\|_{L^2_t (\mathcal{H}^1)}^2 &\leq C \|\{w^j, h^j, \bar{v}^j, q^j, \eta^j\}\|^{16} + C \|\{w^j, h^j, \bar{v}^j, q^j, \eta^j\}\|^{16} \\
&+ C \|\{w^{(j-1)}, h^{(j-1)}, \bar{v}^{(j-1)}, q^{(j-1)}, \eta^{(j-1)}\}\|^{20} + C \|\{w^{(j-1)}, h^{(j-1)}, \bar{v}^{(j-1)}, q^{(j-1)}, \eta^{(j-1)}\}\|^{20}
\end{align}
for $j \geq 2$ and $t > 0$. 

Proof. Since $\{\{(J^j)^{-2}[(\alpha^j)^2+(\beta^j)^2+1]-1\}_i \sim \nabla \bar{v}^j \nabla \eta^j + (\nabla \eta^j)^2 \nabla \bar{v}^j \}$ we deduce from the Sobolev embedding inequality, (5.22) and Lemma 4.3 that
\begin{align}
\left\|\{(J^j)^{-2}[(\alpha^j)^2+(\beta^j)^2+1]-1\}_i \partial_3 \left( \partial_3 w^j + w^{(j-1)} \partial_3 h^j \right) \right\|_{L^2_t L^2}^2 &\leq C \|\nabla \bar{v}^j\|_{L^4_t L^4} \|\nabla \eta^j\|_{L^4_t L^4} \|\nabla \bar{v}^j\|_{L^4_t L^4} \|\nabla w^j\|_{L^2_t L^2} \|\nabla (w^{(j-1)} \nabla h^j)\|_{L^2_t L^2} \\
&\leq C \|\eta^j\|_{L^6_t H^2(\mathbb{R}^2)}^6 \|\bar{v}^j\|_{L^2_t H^1}^2 \|\eta^j\|_{L^6_t H^2(\mathbb{R}^2)}^6 \|w^j\|_{L^2_t H^1}^2 \|w^{(j-1)}\|_{L^2_t H^2}^2 \|h^j\|_{L^2_t H^1}^2. \tag{5.55}
\end{align}

On the other hand, for any $\phi \in \mathcal{H}^1$ one gets
\begin{align}
\int_{\Omega} \{\{(J^j)^{-2}[(\alpha^j)^2+(\beta^j)^2+1]-1\}_i \partial_3 \left( \partial_3 w^j + w^{(j-1)} \partial_3 h^j \right) \phi \right\} dx dy &\leq \int_{\Omega} \{\{(J^j)^{-2}[(\alpha^j)^2+(\beta^j)^2+1]-1\}_i \left( \partial_3 w^j + w^{(j-1)} \partial_3 h^j \right) \phi \right\} dx \\
&- \int_{\Omega} \{\{(J^j)^{-2}[(\alpha^j)^2+(\beta^j)^2+1]-1\}_i \left( \partial_3 w^j + w^{(j-1)} \partial_3 h^j \right) \phi \right\} dx \\
&= I_1 + I_2 + I_3. \tag{5.56}
\end{align}
It follows from the boundary condition \( \partial_3 w^j + w^{(j-1)} \partial_3 h^j = G_4(w^{(j-1)}, h^{(j-1)}, \bar{\eta}^{(j-1)}) \) in (5.56) and Lemma 4.3 that

\[
I_1 \leq C \left\| G_4(w^{(j-1)}, h^{(j-1)}, \bar{\eta}^{(j-1)}) \right\|_{H^\frac{1}{2}(\Gamma)} \left\| \left\{ (J^j) - 2 \left( (\alpha^j)^2 + (\beta^j)^2 + 1 \right) \right\} \right\|_{H^\frac{1}{2}(\Gamma)}
\]

\[
\leq C \left\| G_4(w^{(j-1)}, h^{(j-1)}, \bar{\eta}^{(j-1)}) \right\|_{H^\frac{1}{2}(\Gamma)} \left\| \nabla \bar{\eta}^j \right\|_{H^\frac{1}{2}(\Gamma)} \left\| \nabla \phi \right\|_{H^\frac{1}{2}(\Gamma)}
\]

\[
\leq C \left\| G_4(w^{(j-1)}, h^{(j-1)}, \bar{\eta}^{(j-1)}) \right\|_{H^\frac{1}{2}(\Gamma)} \left\| \bar{\eta}^j \right\|_{H^2(\mathbb{R}^2)} \left\| \phi \right\|_{H^1}.
\]

The Sobolev embedding inequality and Lemma 4.3 lead to

\[
I_2 \leq \left\| \nabla \bar{\eta}^j \right\|_{L^2}^2 \left( \left\| \nabla w^j \right\|_{L^2} + \left\| w^{(j-1)} \right\|_{L^4} \left\| \nabla h^j \right\|_{L^4} + \left\| w^{(j-1)} \right\|_{L^2} \left\| \nabla h^j \right\|_{L^2} + \left\| w^{(j-1)} \right\|_{L^2} \left\| \nabla h^j \right\|_{L^2} \right) \left\| \nabla \phi \right\|_{L^2}
\]

\[
\leq C \left\| \bar{\eta}^j \right\|_{H^2(\mathbb{R}^2)}^2 \left( \left\| \nabla w^j \right\|_{H^1} + \left\| w^{(j-1)} \right\|_{H^1} \left\| h^j \right\|_{H^2} + \left\| w^{(j-1)} \right\|_{H^2} \left\| h^j \right\|_{H^1} \right) \left\| \phi \right\|_{H^1}.
\]

Similarly, it follows from the Sobolev embedding inequality and Lemma 4.3 that

\[
I_3 \leq \left( \left\| \nabla \bar{\eta}^j \right\|_{L^2} + \left\| \nabla \bar{\eta}^j \right\|_{L^2} \left\| \nabla \nabla w^j \right\|_{L^2} + \left\| \nabla w^{(j-1)} \right\|_{L^4} \left\| \nabla h^j \right\|_{L^4} + \left\| \nabla w^{(j-1)} \right\|_{L^2} \left\| \nabla h^j \right\|_{L^2} \right) \left\| \nabla \phi \right\|_{L^4}
\]

\[
\leq C \left( \left\| \bar{\eta}^j \right\|_{H^2(\mathbb{R}^2)}^2 + \left\| \nabla \eta^j \right\|_{H^2(\mathbb{R}^2)}^2 \right) \left( \left\| \nabla \phi \right\|_{H^1} + \left\| w^{(j-1)} \right\|_{H^1} \left\| h^j \right\|_{H^2} + \left\| w^{(j-1)} \right\|_{H^2} \left\| h^j \right\|_{H^1} \right) \left\| \phi \right\|_{H^1}.
\]

Substituting the above estimate for \( I_1, I_2 \) and \( I_3 \) into (5.56), we arrive at

\[
\left\| \left\{ (J^j)^{-2} ((\alpha^j)^2 + (\beta^j)^2 + 1) - 1 \right\} \partial_3 (\partial_3 w^j + w^{(j-1)} \partial_3 h^j) \right\|_{L^2_{\mu}(H^1)^{}}^2
\]

\[
\leq C \left( \left\| \eta^j \right\|_{H^2(\mathbb{R}^2)}^4 + \left\| \eta^j \right\|_{L^4_{\mu}(H^1)^{}}^2 \right) \left( \left\| \nabla w^j \right\|_{H^2} + \left\| w^{(j-1)} \right\|_{H^2} \left\| h^j \right\|_{H^2} + \left\| w^{(j-1)} \right\|_{H^2} \left\| h^j \right\|_{H^2} \right)^2
\]

\[
+ C \left( \left\| \nabla \eta^j \right\|_{H^2(\mathbb{R}^2)}^4 \right) \left\| G_4(w^{(j-1)}, h^{(j-1)}, \bar{\eta}^{(j-1)}) \right\|_{L^2_{\mu}(H^1)^{}}^2
\]

which, along with (5.55) and Lemma 5.4, gives rise to

\[
\left\| \left\{ (J^j)^{-2} ((\alpha^j)^2 + (\beta^j)^2 + 1) - 1 \right\} \partial_3 (\partial_3 w^j + w^{(j-1)} \partial_3 h^j) \right\|_{L^2_{\mu}(H^1)^{}}^2
\]

\[
\leq C \left( \left\| \{w^j, h^j, \bar{\eta}^j, q^j, \bar{\eta}^j\} \right\|_{H^1}^4 + C \left( \left\| \{w^j, h^j, \bar{\eta}^j, q^j, \bar{\eta}^j\} \right\|_{H^1}^4 \right)
\]

\[
+ C \left( \left\| \{w^{(j-1)}, h^{(j-1)}, \bar{\eta}^{(j-1)}, q^{(j-1)}, \bar{\eta}^{(j-1)}\} \right\|_{H^1}^4 \right)
\]

\[
+ C \left( \left\| \{w^{(j-1)}, h^{(j-1)}, \bar{\eta}^{(j-1)}, q^{(j-1)}, \bar{\eta}^{(j-1)}\} \right\|_{H^1}^4 \right)
\]

(5.57)

By a similar argument used in deriving (5.57), one deduces that

\[
\left\| 2 \alpha^j (J^j)^{-1} (\partial_3 \partial_3 w^j + w^{(j-1)} \partial_3 \partial_3 h^j) \right\|_{L^2_{\mu}(H^1)^{}}^2
\]

\[
\leq C \left( \left\| \{w^j, h^j, \bar{\eta}^j, q^j, \bar{\eta}^j\} \right\|_{H^1}^4 + C \left( \left\| \{w^j, h^j, \bar{\eta}^j, q^j, \bar{\eta}^j\} \right\|_{H^1}^4 \right)
\]

\[
+ C \left( \left\| \{w^{(j-1)}, h^{(j-1)}, \bar{\eta}^{(j-1)}, q^{(j-1)}, \bar{\eta}^{(j-1)}\} \right\|_{H^1}^4 \right)
\]

\[
+ C \left( \left\| \{w^{(j-1)}, h^{(j-1)}, \bar{\eta}^{(j-1)}, q^{(j-1)}, \bar{\eta}^{(j-1)}\} \right\|_{H^1}^4 \right)
\]

(5.58)

We proceed to estimating \( \left\| (LOT)^j \right\|_{L^2_{\mu}(H^1)^{}}^2 \). For any \( \phi \in \partial H^1 \) it follows from (5.22) and Lemma 4.3 that

\[
\int_{\Omega} \left( \nabla \bar{\eta}^j \right)^2 \nabla \bar{\eta} \nabla w^j \phi \, dx \, dy
\]

\[
\leq \left\| \nabla \bar{\eta}^j \right\|_{L^2} \left\| \nabla \bar{\eta}^j \right\|_{L^2} \left\| \nabla \bar{\eta}^j \right\|_{L^2} \left\| \nabla w^j \right\|_{L^2} \left\| \phi \right\|_{L^2}
\]

\[
+ \left\| \nabla \bar{\eta}^j \right\|_{L^2} \left\| \nabla \bar{\eta}^j \right\|_{L^2} \left\| \nabla w^j \right\|_{L^2} \left\| \phi \right\|_{L^2}
\]

\[
+ C \left( \left\| \bar{\eta}^j \right\|_{H^2(\mathbb{R}^2)}^2 \right) \left\| \phi \right\|_{H^1} + C \left( \left\| \bar{\eta}^j \right\|_{H^2(\mathbb{R}^2)}^2 \right) \left\| \phi \right\|_{H^1}.
\]
Thus
\[\|(\nabla \tilde{\eta})^2 \nabla^2 \tilde{\eta} \nabla w^j \|_{L^2(\mathbb{H})}^2 \leq C\|\eta^j\|^4_{L^r(H^2(\mathbb{R}^2))} \|\tilde{\omega}\|^2_{L^r(H^2)} + C\|\eta^j\|^6_{L^r(H^3(\mathbb{R}^2))} \|w^j\|^2_{L^2(H^1)}. \tag{5.59}\]

By a similar argument used in deriving \((5.59)\) one deduces that
\[\|(\nabla \tilde{\eta}^j) \nabla w^j \tilde{\omega}\|_{L^2(\mathbb{H})}^2 \leq C\|\tilde{\omega}\|^4_{L^r(H^2)} \|w^j\|^2_{L^2(H^2)} + C\|\eta^j\|^2_{L^r(H^1(\mathbb{R}^2))} \|\tilde{\omega}\|^2_{L^2(H^2)} + C\|\eta^j\|^2_{L^r(H^3(\mathbb{R}^2))} \|\tilde{\omega}\|^2_{L^2(H^2)} \tag{5.60}\]
and
\[\|(\nabla \tilde{\eta}^j)^2 \nabla w^j \tilde{\omega}^j\|_{L^2(\mathbb{H})}^2 + \|(\nabla \tilde{\eta}^j)^2 \nabla^2 \tilde{\eta} \nabla w^j \|_{L^2(\mathbb{H})}^2 \leq C\|\tilde{\omega}\|^2_{L^r(H^2)} + C\|\eta^j\|^6_{L^r(H^3(\mathbb{R}^2))} \|\tilde{\omega}\|^2_{L^2(H^2)} \|w^j\|^2_{L^2(H^2)} \tag{5.61}\]
\[+ C\|\eta^j\|^4_{L^r(H^1(\mathbb{R}^2))} \|\tilde{\omega}\|^2_{L^2(H^2)} + \|h\|^2_{L^2(H^1)} \|w^j\|^2_{L^2(H^2)} + \|h\|^2_{L^2(H^1)} \|w^j\|^2_{L^2(H^2)}. \]

By \((5.52)\) one gets
\[\|\|(\nabla \tilde{\omega}^j)\|_{L^2(\mathbb{H})} \leq \|w^j\|^2_{L^2(H^1)} \|\tilde{\omega}\|^2_{L^2(H^2)} + \|w^j\|^2_{L^2(H^1)} \|\tilde{\omega}\|^2_{L^2(H^1)}. \tag{5.62}\]

Collecting \((5.59)-(5.62)\) we conclude that
\[\|\|(LOT)\|_{L^2(\mathbb{H})}^2 \leq C\|\{w^j, h^j, \tilde{\omega}^j, q^j, \eta^j\}\|^4 + C\|\{w^j, h^j, \tilde{\omega}^j, q^j, \eta^j\}\|^6. \tag{5.63}\]

Combining \((5.57), (5.58)\) and \((5.63)\) we derive the desired estimate. The proof is completed.

It remains to estimate \(F_5(h^j, \tilde{\omega}^j, \tilde{\eta}^j)\) and \(F_{5t}(h^j, \tilde{\omega}^j, \tilde{\eta}^j)\). From \((7.5)\) we know that
\[F_5(h^j, \tilde{\omega}^j, \tilde{\eta}^j) = \{(J^j)^{-2}[(\alpha^j)^2 + (\beta^j)^2 + 1] - 1\} \partial_3^4 h^j \]
\[- 2\alpha^j (J^j)^{-1} \partial_3 \partial_1 h^j - 2\beta^j (J^j)^{-1} \partial_3 \partial_2 h^j + LOT \tag{5.64}\]
with
\[LOT \sim \tilde{\omega} \nabla \tilde{\eta} \nabla h^j + (\nabla \tilde{\eta}^j)^2 \nabla^2 \tilde{\eta} \nabla h^j + (\nabla \tilde{\eta}^j)^2 (\nabla h^j)^2 + \nabla h^j \tilde{\eta}^j. \]

**Lemma 5.7.** Let the assumptions in Proposition \(\ref{5.7}\) and \(\ref{5.12}\) hold true. Then there exists a constant \(C\) independent of \(j\) and \(t\) such that
\[\|F_5\|^2_{L^2(H^1)} \leq C\|\{w^j, h^j, \tilde{\omega}^j, q^j, \eta^j\}\|^4 + C\|\{w^j, h^j, \tilde{\omega}^j, q^j, \eta^j\}\|^6 \]
for \(j \geq 2\) and \(t > 0\).

**Proof.** By the definition of \(\alpha^j, \beta^j\) and \(J^j\) in \((3.3)\) we know that \(\{(J^j)^{-2}[(\alpha^j)^2 + (\beta^j)^2 + 1] - 1\} \sim (\nabla \tilde{\eta}^j)^2\). Thus it follows from the Sobolev embedding theorem and Lemma \(\ref{4.3}\) that
\[\|\{(J^j)^{-2}[(\alpha^j)^2 + (\beta^j)^2 + 1] - 1\} \partial_3^4 h^j\|^2_{L^2(H^1)} \leq \|\nabla \tilde{\eta}^j\|^4_{L^2(H^2)} \|\nabla^2 \tilde{\eta}^j\|^2_{L^2(H^2)} + \|\nabla \tilde{\eta}^j\|^2_{L^2(H^1)} \|\nabla^2 h^j\|^2_{L^2(H^1)} \leq C\|\eta^j\|^4_{L^2(H^3(\mathbb{R}^2))} \|h^j\|^2_{L^2(H^1)}. \tag{5.65}\]

Similarly,
\[\|- 2\alpha^j (J^j)^{-1} \partial_3 \partial_1 h^j - 2\beta^j (J^j)^{-1} \partial_3 \partial_2 h^j\|^2_{L^2(H^1)} \leq C\|\eta^j\|^2_{L^2(H^3(\mathbb{R}^2))} \|h^j\|^2_{L^2(H^1)}. \tag{5.66}\]
We next estimate \( \|\text{LOT}\|_{L^2_tH^1} \). By the Sobolev embedding inequality and Lemma 4.3 one gets

\[
\begin{align*}
\|(\nabla \tilde{\eta}^j) \nabla^2 \tilde{\eta}^j \|_{L^2_t H^1}^2 &\leq \|(\nabla \tilde{\eta}^j) \|_{L^4_t L^4}^2 \|\nabla^2 \tilde{\eta}^j\|_{L^2_t L^2}^2 + \|(\nabla \tilde{\eta}^j) \|_{L^4_t L^4}^2 \|\nabla^2 h^j\|_{L^2_t L^2}^2 \\
&\quad + \|(\nabla \tilde{\eta}^j)\|_{L^4_t L^4}^2 \|\nabla^2 \tilde{\eta}^j\|_{L^2_t L^2}^2 \|\nabla^2 h^j\|_{L^2_t L^2}^2 \\
&\leq C \|\tilde{\eta}^j\|_{L^2_t H^3(R^3)}^2 \|h^j\|_{L^2_t H^1}^2 .
\end{align*}
\]

(5.67)

A similar argument used in deriving (5.67) leads to

\[
\|\tilde{\eta}^j \nabla \tilde{\eta}^j \nabla^2 h^j\|_{L^2_t H^1}^2 + \|(\nabla \tilde{\eta}^j) \|_{L^2_t L^2}^2 \|\nabla h^j\|_{L^2_t H^1}^2 \leq C \|\tilde{\eta}^j\|_{L^2_t H^3(R^3)}^2 \|\tilde{\eta}^j\|_{L^2_t H^3(R^3)}^2 \|\nabla h^j\|_{L^2_t H^3}^2 \\
+ C \|\tilde{\eta}^j\|_{L^4_t H^3(R^3)}^4 \|h^j\|_{L^2_t H^1}^2 \|h^j\|_{L^2_t H^1}^2 \|h^j\|_{L^2_t H^1}^2 .
\]

(5.68)

and (5.22) gives

\[
\|\nabla \tilde{h}^j \nabla^2 h^j\|_{L^2_t H^1}^2 \leq C \|h^j\|_{L^2_t H^1}^2 \|\tilde{\eta}^j\|_{L^2_t H^1}^2 .
\]

(5.69)

Substituting (5.65)–(5.69) and using (5.64) we obtain the desired estimate. The proof is finished.

\[ \square \]

**Lemma 5.8.** Suppose that the assumptions in Proposition 5.1 and (5.12) hold true. Then there exists a constant \( C \) independent of \( j \) and \( t \) such that

\[ \|F_{\xi^j}\|_{L^2_t(0;H^1)}^2 \leq C \{\|w^j, h^j, \tilde{\eta}^j, \eta^j\|_{H^1} + \|w^j, h^j, \tilde{\eta}^j, q^j, \eta^j\|_{H^1}\}^8 \]

for \( j \geq 2 \) and \( t > 0 \).

**Proof.** For any \( \varphi \in 0H^1 \), integration by parts leads to

\[
\begin{align*}
\int \Omega \left\{ \left[ (J^j)^{-2}((\alpha_j)^2 + (\beta_j)^2 + 1) - 1 \right] \partial_3^2 h^j \right\} \varphi \, dx \, dy \\
= \int \Omega \left[ (J^j)^{-2}((\alpha_j)^2 + (\beta_j)^2 + 1) - 1 \right] \partial_3^2 h^j \varphi \, dx \, dy \\
- \int \Omega \left[ (J^j)^{-2}((\alpha_j)^2 + (\beta_j)^2 + 1) - 1 \right] \partial_3 h^j_1 \partial_3 \varphi \, dx \, dy \\
- \int \partial_3 \left[ (J^j)^{-2}((\alpha_j)^2 + (\beta_j)^2 + 1) - 1 \right] \partial_3 h^j \varphi \, dx \, dy \\
:= I_1 + I_2 + I_3 .
\end{align*}
\]

Since \( \left[ (J^j)^{-2}((\alpha_j)^2 + (\beta_j)^2 + 1) - 1 \right] \sim \nabla \tilde{\eta}^j \nabla \tilde{\eta}^j + (\nabla \tilde{\eta}^j)^2 \nabla \tilde{\eta}^j \), it follows from the Sobolev embedding inequality, Lemma 4.3 and (5.22) that

\[
\begin{align*}
I_1 &\leq \|\nabla \tilde{\eta}^j\|_{L^4} \|\nabla \tilde{\eta}^j\|_{L^2} \|\nabla^2 h^j\|_{L^2} \|\varphi\|_{L^4} \\
&\leq C \|\tilde{\eta}^j\|_{H^3(R^3)} \|\tilde{\eta}^j\|_{L^2} \|\nabla h^j\|_{L^2} \|\varphi\|_{H^1} .
\end{align*}
\]

Similarly,

\[
\begin{align*}
I_2 + I_3 &\leq \|\nabla \tilde{\eta}^j\|_{L^2} \|\nabla h^j\|_{L^2} \|\varphi\|_{L^2} + \|\nabla \tilde{\eta}^j\|_{L^2} \|\nabla h^j\|_{L^2} \|\tilde{\eta}^j\|_{L^2} \|\nabla \tilde{\eta}^j\|_{L^2} \|\nabla \tilde{\eta}^j\|_{L^2} \|\nabla \varphi\|_{L^4} \\
&\leq C \|\tilde{\eta}^j\|_{L^2} \|\tilde{\eta}^j\|_{L^2} \|\nabla h^j\|_{L^2} \|\varphi\|_{H^1} .
\end{align*}
\]

Collecting the above estimate for \( I_1, I_2 \) and \( I_3 \) we arrive at

\[
\begin{align*}
\left\{ \left[ (J^j)^{-2}((\alpha_j)^2 + (\beta_j)^2 + 1) - 1 \right] \partial_3^2 h^j \right\}_{L^2_t(0;H^1)}^2 \\
\leq C \|\tilde{\eta}^j\|_{L^2_t H^3(R^3)} \|\tilde{\eta}^j\|_{L^2_t H^3(R^3)} \|\tilde{\eta}^j\|_{L^2_t H^3} \|h^j\|_{L^2_t H^1}^2 \\
+ C \|\tilde{\eta}^j\|_{L^4_t H^3(R^3)}^4 \|h^j\|_{L^2_t H^1}^2 \|h^j\|_{L^2_t H^1}^2 .
\end{align*}
\]

(5.70)
By a similar argument used in deriving (5.70) one deduces that
\[
\| 2\alpha^i (J^i)^{-1} \partial_3 \partial_1 h^i \|_{L^2_7(H^3)}^2 + \| 2\beta^i (J^i)^{-1} \partial_3 \partial_2 h^i \|_{L^2_7(H^3)}^2 \\
\leq C(\| \eta_i \|_{L^2_7(H^3)}^2 + 1) \| \bar{\eta}^i \|_{L^2_7(H^3)}^2 + C(\| \eta_i \|_{L^2_7(H^3)}^4 + \| \eta_i \|_{L^2_7(H^3)}^4) \| h^i \|_{L^2_7(H^3)}^2. 
\] (5.71)

We next estimate each term in \((LOT)_i\). By a similar argument used in deriving (5.59) we deduce that
\[
\| [\nabla \bar{\eta}^i]^2 \nabla \bar{\eta}^i \|_{L^2_7(\partial H^1)}^2 \leq C \| \eta_i \|_{L^2_7(H^3)}^2 \| \bar{\eta}^i \|_{L^2_7(H^3)}^2 + C(\| \eta_i \|_{L^2_7(H^3)}^4 + \| \eta_i \|_{L^2_7(H^3)}^4) \| h^i \|_{L^2_7(H^3)}^2.
\] (5.72)

and that
\[
\| [\nabla \bar{\eta}^i \nabla h^i \bar{\eta}^i \|_{L^2_7(\partial H^1)}^2 \leq C \| \bar{\eta}^i \|_{L^2_7(H^3)}^2 \| h^i \|_{L^2_7(H^3)}^2 + C(\| \eta_i \|_{L^2_7(H^3)}^4 + \| \eta_i \|_{L^2_7(H^3)}^4) \| h^i \|_{L^2_7(H^3)}^2.
\] (5.73)

On the other hand, Lemma 4.3 and (5.22) entail that
\[
\| [\nabla \bar{\eta}^i]^2 (\nabla h^i)^2 \|_{L^2_7} \leq C \| \eta_i \|_{L^2_7(H^3)}^2 \| \bar{\eta}^i \|_{L^2_7(H^3)}^2 + C(\| \eta_i \|_{L^2_7(H^3)}^4 + \| \eta_i \|_{L^2_7(H^3)}^4) \| h^i \|_{L^2_7(H^3)}^2.
\] (5.74)

and that
\[
\| [\nabla h^i \eta_i \|_{L^2_7} \leq \| h^i \|_{L^2_7(H^3)}^2 \| \bar{\eta}^i \|_{L^2_7(H^3)}^2 + \| \bar{\eta}^i \|_{L^2_7(H^3)}^2 \| h^i \|_{L^2_7(H^3)}^2.
\] (5.75)

Combining (5.72)–(5.75) we arrive at
\[
\| (LOT)_i \|_{L^2_7(\partial H^1)}^2 \leq C \{ w^{j-1}, h^i, t, q^j, q^j, \eta_i \}^4 + C \{ w^{j-1}, h^i, t, q^j, q^j, \eta_i \}^8.
\] (5.76)

Collecting (5.70), (5.71) and (5.76) we derive the desired estimate. The proof is completed.

\[\square\]

### 5.2 Proof of Proposition 5.1

First, it follows from the Sobolev embedding inequality and Lemma 4.3 that
\[
\| \bar{\eta}^i + \partial_3 \bar{\eta}^i (1 + y) \|_{L^2(L^\infty)} \leq C_5 \| \eta_i \|_{L^2_7(H^3)}\]

which, along with the definition \( J^i = 1 + \bar{\eta}^i + \partial_3 \bar{\eta}^i (1 + y) \) indicates that
\[
\frac{1}{2} < J^i < \frac{3}{2}
\]

provided
\[
C_5 \| \eta_i \|_{L^2_7(H^3)} \leq \frac{1}{2}.
\] (5.77)

Applying Proposition 4.1 to system (5.5)–(5.7) and using Lemma 5.1, Lemma 5.8 and (5.77) we conclude that there exists a constant \( C_6 \) independent of \( j \) and \( t \) such that
\[
\| \{ w^{(j+1)}, h^{(j+1)}, t^{(j+1)}, q^{(j+1)}, \eta^{(j+1)} \} \|^2 \\
\leq C_6 \{ w^j, h^j, t^j, q^j, \eta_j \}^4 + C_6 \{ w^{j-1}, h^{j-1}, t^{j-1}, q^{j-1}, \eta^{j-1} \}^{20} \\
+ C_6 \{ w^{(j-1)}, h^{(j-1)}, t^{(j-1)}, q^{(j-1)}, \eta^{(j-1)} \}^4 \\
+ C_6 \{ w^{(j-1)}, h^{(j-1)}, t^{(j-1)}, q^{(j-1)}, \eta^{(j-1)} \}^{20}
\] (5.78)
for $j \geq 2$ and $t > 0$, provided (5.77) and
\begin{equation}
C_1(C_2 + 1)||w'||^2 < \frac{1}{2},
\end{equation}
where the constant $C_1$ and $C_2$ are defined in Proposition 4.1. On the other hand, applying Proposition 4.1 to system (5.3), (5.4) one deduces that there is a constant $C_7$ independent of $t$ such that
\begin{align}
\|\{w^1,h^1,\bar{v}^1,q^1,\eta^1\}\|^2 &\leq C_7(\|w_0\|_{H^2(\Omega)} + \|\eta_0\|_{H^1(\mathbb{R}^2)}^2), \\
\|\{w^2,h^2,\bar{v}^2,q^2,\eta^2\}\|^2 &\leq C_7(\|w_0\|_{H^2(\Omega)} + \|\eta_0\|_{H^1(\mathbb{R}^2)}^2).
\end{align}
(5.80)
Assume that the initial data satisfy
\begin{equation}
C_5\sqrt{C_7}(\|w_0\|_{H^2(\Omega)} + \|\eta_0\|_{H^1(\mathbb{R}^2)}) < \frac{1}{2},
\end{equation}
(5.81)
and
\begin{align}
C_6C_7(\|w_0\|_{H^2(\Omega)} + \|\eta_0\|_{H^1(\mathbb{R}^2)})^2 &+ C_6C_7(\|w_0\|_{H^2(\Omega)} + \|\eta_0\|_{H^1(\mathbb{R}^2)})^{18} < \frac{1}{2}.
\end{align}
(5.82)
Then we assert that the following holds for all $j \geq 1$:
\begin{equation}
\|\{w^j,h^j,\bar{v}^j,q^j,\eta^j\}\|^2 \leq C_7(\|w_0\|_{H^2(\Omega)} + \|\eta_0\|_{H^1(\mathbb{R}^2)})^2.
\end{equation}
(5.83)
We next prove (5.83) by the argument of induction. By (5.80) we know that (5.83) holds for $j = 1, 2$. Assuming that (5.83) is true for all $1 \leq k \leq j$ with $j \geq 2$, by (5.81) one easily deduces that (5.77) and (5.79) hold for all $1 \leq k \leq j$. Then it follows from (5.78) and (5.82) that
\begin{align}
\|\{w^{(j+1)},h^{(j+1)},\bar{v}^{(j+1)},q^{(j+1)},\eta^{(j+1)}\}\|^2 &< \frac{1}{2}\|\{w^j,h^j,\bar{v}^j,q^j,\eta^j\}\|^2 \\
&\quad + \frac{1}{2}\|\{w^{(j-1)},h^{(j-1)},\bar{v}^{(j-1)},q^{(j-1)},\eta^{(j-1)}\}\|^2 \\
&\leq C_7(\|w_0\|_{H^2(\Omega)} + \|\eta_0\|_{H^1(\mathbb{R}^2)}).
\end{align}
which, implies that (5.83) holds true for $(j + 1)$ and thus it holds for all $j \geq 1$. (5.10) follows from (5.83) and we proceed to proving (5.11). Combining (5.83) and the first inequality in (5.81) one deduces that (5.77) is true for all $j \geq 1$ and thus derives (5.11). The proof is completed.

6 Proof of Theorem 2.1 and Theorem 2.2

Proof of Theorem 2.1 Let
\begin{align}
\delta w^{(j+1)} &= w^{(j+1)} - w^j, \quad \delta h^{(j+1)} = h^{(j+1)} - h^j, \quad \delta q^{(j+1)} = q^{(j+1)} - q^j, \\
\delta \eta^{(j+1)} &= \eta^{(j+1)} - \eta^j.
\end{align}
for $j \geq 3$. Then it follows from (5.5) that
\begin{equation}
\begin{cases}
(\delta w^{(j+1)}), - \Delta \delta w^{(j+1)} - \nabla \cdot (w^{(j+1)}\nabla h^{(j+1)}) = \nabla \cdot (\delta w^{(j+1)}h^j) + \delta F_4 \quad \text{in } \Omega \times (0,\infty), \\
\delta h^{(j+1)} - \Delta \delta h^{(j+1)} = \delta F_5, \\
\delta q^{(j+1)} - \Delta \delta q^{(j+1)} + \nabla \delta q^{(j+1)} + \delta w^{(j+1)}\nabla \phi = \delta F_7, \\
\nabla \cdot \delta \bar{v}^{(j+1)} = 0, \\
(\delta w^{(j+1)}, \delta h^{(j+1)}, \delta \bar{v}^{(j+1)})(x_1, x_2, y, 0) = 0, \quad \delta \eta^{(j+1)}(x_1, x_2, 0) = 0,
\end{cases}
\end{equation}
(6.1)
where
\[ \delta F_4 = F_4(w^{(j-1)}, w^j, h^j, \bar{v}^j, \tilde{\eta}^j) - F_4(w^{(j-2)}, w^{(j-1)}, h^{(j-1)}, \bar{v}^{(j-1)}, \tilde{\eta}^{(j-1)}), \]
\[ \delta F_5 = F_5(h^j, \bar{v}^j, \tilde{\eta}^j) - F_5(h^{(j-1)}, \bar{v}^{(j-1)}, \tilde{\eta}^{(j-1)}), \]
\[ \delta \bar{F} = \bar{F}(w^j, \bar{v}^j, \nabla q^j, \tilde{\eta}^j) - \bar{F}(w^{(j-1)}, \bar{v}^{(j-1)}, \nabla q^{(j-1)}, \tilde{\eta}^{(j-1)}). \]

The boundary conditions on \( \Gamma \times (0, \infty) \) follows from (5.6) as follows:
\[
\begin{align*}
\partial_3 \delta w^{(j+1)} + w_3 \partial_3 \delta h^{(j+1)} &= -\delta w^j \partial_3 h^j + \delta G_4, \quad \delta h^{(j+1)} = 0, \\
\partial_3 \delta v^{(j+1)} + \partial_1 \delta v^{(j+1)} &= \delta G_1, \quad \partial_3 \delta v_2^{(j+1)} + \partial_2 \delta v_3^{(j+1)} = \delta G_2, \\
\delta \eta_t^{(j+1)} &= \delta v_3^{(j+1)}, \quad \delta q^{(j+1)} - 2\partial_3 \delta v_3^{(j+1)} = \gamma \delta \eta^{(j+1)} - \sigma \Delta_0 \delta \eta^{(j+1)} - \delta G_3,
\end{align*}
\] (6.2)
where
\[ \delta G_4 = G_4(w^j, h^j, \bar{v}^j) - G_4(w^{(j-1)}, h^{(j-1)}, \bar{v}^{(j-1)}), \quad \delta G_1 = G_1(\bar{v}^j, \tilde{\eta}^j) - G_1(\bar{v}^{(j-1)}, \tilde{\eta}^{(j-1)}), \]
\[ \delta G_2 = G_2(\bar{v}^j, \tilde{\eta}^j) - G_2(\bar{v}^{(j-1)}, \tilde{\eta}^{(j-1)}), \quad \delta G_3 = G_3(\bar{v}^j, \tilde{\eta}^j) - G_3(\bar{v}^{(j-1)}, \tilde{\eta}^{(j-1)}). \]

The boundary conditions on \( S_B \times (0, \infty) \) follows from (5.7):
\[ \delta w^{(j+1)} = 0, \quad \partial_3 \delta h^{(j+1)} = 0, \quad \delta \bar{v}^{(j+1)} = 0. \] (6.3)

One can apply Proposition 5.1 to system (6.1)-(6.3) and following the procedure in subsection 5.1 to estimate the nonlinear terms on the right-hand side of each equation in (6.1)-(6.2) to conclude that
\[
\begin{align*}
\| &\{ \delta w^{(j+1)}, \delta h^{(j+1)}, \delta \bar{v}^{(j+1)}, \delta q^{(j+1)}, \delta \eta^{(j+1)} \} \|^2 \\
\leq & C_8(j) \| \{ \delta w^j, \delta h^j, \delta \bar{v}^j, \delta q^j, \delta \eta^j \} \|^2 \\
+ & C_9(j) \| \{ \delta w^{(j-1)}, \delta h^{(j-1)}, \delta \bar{v}^{(j-1)}, \delta q^{(j-1)}, \delta \eta^{(j-1)} \} \|^2
\end{align*}
\] (6.4)
with
\[ C_8(j) = C \| \{ w^j, h^j, \bar{v}^j, q^j, \eta^j \} \|^{18} + C \| \{ w^{(j-1)}, h^{(j-1)}, \bar{v}^{(j-1)}, q^{(j-1)}, \eta^{(j-1)} \} \|^{18} \]
and
\[ C_9(j) = C \| \{ w^{(j-1)}, h^{(j-1)}, \bar{v}^{(j-1)}, q^{(j-1)}, \eta^{(j-1)} \} \|^{18} + C \| \{ w^{(j-2)}, h^{(j-2)}, \bar{v}^{(j-2)}, q^{(j-2)}, \eta^{(j-2)} \} \|^{18} + C \| \{ w^{(j-3)}, h^{(j-3)}, \bar{v}^{(j-3)}, q^{(j-3)}, \eta^{(j-3)} \} \|^{18} \]
for some constant \( C \) independent of \( j \) and \( t \). By Proposition 5.1 and (6.4) we deduce that there exists a constant \( C_{10} \) independent of \( j \) and \( r \) such that
\[
\begin{align*}
\| &\{ \delta w^{(j+1)}, \delta h^{(j+1)}, \delta \bar{v}^{(j+1)}, \delta q^{(j+1)}, \delta \eta^{(j+1)} \} \|^2 \\
\leq & C_{10} (\| w_0 \|_{H^2} + \| h_0 \|_{H^2} + \| \bar{v}_0 \|_{H^2} + \| \eta_0 \|_{H^3(\mathbb{R}^2)})^2 \times \| \{ \delta w^j, \delta h^j, \delta \bar{v}^j, \delta q^j, \delta \eta^j \} \|^2 \\
+ & C_{10} (\| w_0 \|_{H^2} + \| h_0 \|_{H^2} + \| \bar{v}_0 \|_{H^2} + \| \eta_0 \|_{H^3(\mathbb{R}^2)})^{18} \times \| \{ \delta w^j, \delta h^j, \delta \bar{v}^j, \delta q^j, \delta \eta^j \} \|^2 \\
+ & C_{10} (\| w_0 \|_{H^2} + \| h_0 \|_{H^2} + \| \bar{v}_0 \|_{H^2} + \| \eta_0 \|_{H^3(\mathbb{R}^2)})^2 \times \| \{ \delta w^{(j-1)}, \delta h^{(j-1)}, \delta \bar{v}^{(j-1)}, \delta q^{(j-1)}, \delta \eta^{(j-1)} \} \|^2 \\
+ & C_{10} (\| w_0 \|_{H^2} + \| h_0 \|_{H^2} + \| \bar{v}_0 \|_{H^2} + \| \eta_0 \|_{H^3(\mathbb{R}^2)})^{18} \times \| \{ \delta w^{(j-1)}, \delta h^{(j-1)}, \delta \bar{v}^{(j-1)}, \delta q^{(j-1)}, \delta \eta^{(j-1)} \} \|^2.
\end{align*}
\] (6.5)
Assume that

\[ C_{10} \left( \|w_0\|_{H^2(\Omega)} + \|h_0\|_{H^2(\Omega)} + \|\bar{v}_0\|_{H^2(\Omega)} + \|\eta_0\|_{H^1(\mathbb{R}^2)} \right)^2 + C_{10} \left( \|w_0\|_{H^2(\Omega)} + \|h_0\|_{H^2(\Omega)} + \|\bar{v}_0\|_{H^2(\Omega)} + \|\eta_0\|_{H^1(\mathbb{R}^2)} \right)^{18} \leq \frac{1}{4}. \]  

(6.6)

Then it follows from (6.5) and (6.6) that

\[ \frac{3}{4} \left( \|\delta w^{(j+1)} + \delta h^{(j+1)}, \delta \bar{v}^{(j+1)}, \delta q^{(j+1)}, \delta \eta^{(j+1)} \| \right)^2 \leq \frac{1}{2} \left( \|\delta w^j, \delta h^j, \delta \bar{v}^j, \delta q^j, \delta \eta^j \| \right)^2 + \frac{1}{2} \left( \|\delta w^{(j-1)}, \delta h^{(j-1)}, \delta \bar{v}^{(j-1)}, \delta q^{(j-1)}, \delta \eta^{(j-1)} \| \right)^2 \]

for \( j \geq 3 \) and \( t > 0 \). From (6.7) we know that \( \{w^j, h^j, \bar{v}^j, q^j, \eta^j\} \) is a Cauchy sequence, thus there exists a unique limit \( (w, h, \bar{v}, q, \eta) \) satisfying

\[ \lim_{j \to \infty} \|w^j - w, h^j - h, \bar{v}^j - \bar{v}, q^j - q, \eta^j - \eta\| = 0 \]

and

\[ \|w, h, \bar{v}, q, \eta\|^2 \leq C \left( \|w_0\|_{H^2(\Omega)} + \|h_0\|_{H^2(\Omega)} + \|\bar{v}_0\|_{H^2(\Omega)} + \|\eta_0\|_{H^1(\mathbb{R}^2)} \right)^2. \]  

(6.8)

Passing \( j \to \infty \) in (5.5)–(5.7) we deduce that \( (w, h, \bar{v}, q^j, \eta^j) \) solves (3.10)–(3.12). Moreover, from (5.11) we deduce that \( J \), the Jacobian determinant of \( d\theta \) satisfies

\[ \frac{1}{2} < J < \frac{3}{2}. \]  

(6.9)

which, along with the transformation \( \theta \) given in (5.1), indicates that the \( (m, \tilde{c}, \tilde{u}, \rho) \) defined in (3.4)–(3.6), along with \( \eta \) solves the initial-boundary value problem (1.7)–(1.9). Uniqueness and estimates (2.3) follow directly from (6.8) and (6.9). The proof is finished.

\[ \square \]

We next prove Theorem 2.2 by using Theorem 2.1 and reversing transformation 1.6.

**Proof of Theorem 2.2** First, it follows from the Sobolev embedding inequality and Theorem 2.1 that

\[ \sup_{t > 0} \|\tilde{c}(t)\|_{L^\infty(\Omega_0)} \leq C \sup_{t > 0} \|\tilde{c}(t)\|_{H^2(\Omega)} \leq C \left( \|m_0\|_{H^2(\Omega_0)} + \|\tilde{c}_0\|_{H^2(\Omega_0)} + \|\tilde{u}_0\|_{H^2(\Omega_0)} + \|\eta_0\|_{H^1(\mathbb{R}^2)} \right). \]  

(6.10)

From transformation (1.6) we deduce that

\[ c(x_1, x_2, y, t) - \tilde{c} = \tilde{c} \left( \exp \left\{ -\tilde{c}(x_1, x_2, y, t) \right\} - 1 \right) = \tilde{c} \sum_{k=1}^{\infty} \frac{(-\tilde{c}(x_1, x_2, y, t))^k}{k!}, \]

which along with (6.10) indicates that

\[ \sup_{t > 0} \|c(t) - \tilde{c}\|_{L^2(\Omega_0)} \leq \sup_{t > 0} \|\tilde{c}(t)\|_{L^2(\Omega_0)} \sum_{k=1}^{\infty} \frac{\left( \sup_{t > 0} \|\tilde{c}(t)\|_{L^\infty(\Omega_0)} \right)^{k-1}}{k!} \]

\[ \leq \sup_{t > 0} \|\tilde{c}(t)\|_{L^2(\Omega_0)} \exp \left\{ \sup_{t > 0} \|\tilde{c}(t)\|_{L^\infty(\Omega)} \right\} \]

\[ \leq C \left( \|m_0\|_{H^2(\Omega_0)} + \|\tilde{c}_0\|_{H^2(\Omega_0)} + \|\tilde{u}_0\|_{H^2(\Omega_0)} + \|\eta_0\|_{H^1(\mathbb{R}^2)} \right). \]

(6.11)
Theorem 2.1 and (6.10) further entail that
\[
\sup_{t > 0} \| \nabla c(t) \|_{L^2(\Omega)} \leq \dot{c} \sup_{t > 0} \| \nabla \dot{c}(t) \|_{L^2(\Omega)} \exp \left\{ \sup_{t > 0} \| \ddot{c}(t) \|_{L^\infty(\Omega)} \right\} \leq C \left( \| m_0 \|_{H^2(\Omega_0)} + \| \bar{c}_0 \|_{H^2(\Omega_0)} + \| \bar{u}_0 \|_{H^2(\Omega_0)} + \| \eta_0 \|_{H^4(\mathbb{R}^2)} \right),
\] (6.12)

By a similar reasoning as above one gets
\[
\sup_{t > 0} \| \nabla^2 c(t) \|_{L^2(\Omega)} \leq C \left( \| m_0 \|_{H^2(\Omega_0)} + \| \bar{c}_0 \|_{H^2(\Omega_0)} + \| \bar{u}_0 \|_{H^2(\Omega_0)} + \| \eta_0 \|_{H^4(\mathbb{R}^2)} \right),
\] which, in conjunction with (6.11)-(6.12) leads to
\[
\sup_{t > 0} \| c(t) - \bar{c} \|_{H^2(\Omega_0)} \leq C \left( \| m_0 \|_{H^2(\Omega_0)} + \| \bar{c}_0 \|_{H^2(\Omega_0)} + \| \bar{u}_0 \|_{H^2(\Omega_0)} + \| \eta_0 \|_{H^4(\mathbb{R}^2)} \right).
\] (6.13)

By a similar arguments used in obtaining (6.13) one can easily deduce that
\[
\int_0^\infty \| c(t) - \bar{c} \|_{H^2(\Omega_0)}^2 \leq C \left( \| m_0 \|_{H^2(\Omega_0)} + \| \bar{c}_0 \|_{H^2(\Omega_0)} + \| \bar{u}_0 \|_{H^2(\Omega_0)} + \| \eta_0 \|_{H^4(\mathbb{R}^2)} \right),
\]
which, along with (6.13) and Theorem 2.1 gives the desired estimates (2.4). The nonnegativity of \( m \) follows from the maximum principle and \( c > 0 \) follows from the fact \( c(x_1, x_2, y, t) = \hat{c} \exp\{-\bar{c}(x_1, x_2, y, t)\} \). The proof is completed.

\[ \square \]

7 Appendix

This section is devoted to the derivation of system (3.10)-(3.12). First, it follows from (3.1)-(3.4) that
\[
\partial_t m = \xi_{k_l} \partial_k w; \quad \partial_t \bar{c} = \xi_{k_l} \partial_k h; \quad \partial_t^2 m = \xi_{k_l} \partial_k (\xi_{j_l} \partial_j w), \quad \partial_t^2 \bar{c} = \xi_{k_l} \partial_k (\xi_{j_l} \partial_j h), \quad (7.1)
\]
where the derivatives are with respect to the coordinates in \( \Omega \) and repeated indices are summed. Using (3.2)-(3.3) we deduce from (7.1) that
\[
\nabla m = (\partial_1 w - J^{-1} \alpha \partial_3 w, \partial_2 w - J^{-1} \beta \partial_3 w, \partial_3 w),
\]
\[
\nabla \bar{c} = (\partial_1 h - J^{-1} \alpha \partial_3 h, \partial_2 h - J^{-1} \beta \partial_3 h, \partial_3 h) \quad (7.2)
\]
and that
\[
\Delta m = \Delta w + (J^{-2} - 1) \partial_3^2 w - \partial_1 (J^{-1} \alpha \partial_3 w) - \partial_2 (J^{-1} \beta \partial_3 w),
\]
\[
\Delta \bar{c} = \Delta h + (J^{-2} - 1) \partial_3^2 h - \partial_1 (J^{-1} \alpha \partial_3 h) - \partial_2 (J^{-1} \beta \partial_3 h), \quad (7.3)
\]
Noting that \( \theta_1(x_1, x_2, y, t) = \bar{\eta}(x_1, x_2, y, t) + y[1 + \bar{\eta}(x_1, x_2, y, t)] \) depends on \( t \), one gets from (3.1) and (7.2) that
\[
w_t = m_t + (\partial_3 m) \times (\partial_3 \theta_1) = m_t + (J^{-1} \partial_3 w) \times (1 + y) \bar{\eta},
\]
\[
h_t = \bar{c}_t + (\partial_3 \bar{c}) \times (\partial_3 \theta_2) = \bar{c}_t + (J^{-1} \partial_3 h) \times (1 + y) \bar{\eta}. \quad (7.4)
\]
Substituting (7.2)-(7.4) and (3.6) into (1.7)-(1.9) we get
\[
\left\{ \begin{array}{l}
w_t - \Delta w - \nabla \cdot (w \nabla h) = F_4(w, h, \bar{v}, \bar{\eta}), \quad (x_1, x_2, y, t) \in \Omega \times (0, \infty), \\
h_t - \Delta h - w = F_5(h, \bar{v}, \bar{\eta}), \\
\partial_3 w + w \partial_3 h = G_4(w, h, \bar{\eta}), \quad h = 0 \quad \text{on} \quad \Gamma \times (0, \infty), \\
w = 0, \quad \partial_3 h = 0 \quad \text{on} \quad S_B \times (0, \infty), \end{array} \right. \quad (7.5)
\]
where
\[ \begin{align*}
F_4 &= [J^{-2}(\alpha^2 + \beta^2 + 1) - 1] \partial_3 (\partial_3 w + \omega_3 h) - 2J^{-1}\alpha (\partial_3 \partial_1 w + \omega_3 \partial_1 h) - 2J^{-1}\beta (\partial_3 \partial_2 w + \omega_3 \partial_2 h) \\
&\qquad - J^{-1}v_1 (\partial_3 w - J^{-1}\alpha \partial_3 h) - J^{-1}v_2 (\partial_2 w - J^{-1}\beta \partial_3 h) - J^{-1}(J^{-1}\alpha v_1 + J^{-1}\beta v_1 + v_3) \partial_3 w \\
&\qquad - \partial_1 (J^{-1}\alpha) \partial_3 w - \partial_2 (J^{-1}\beta) \partial_3 w - \alpha \partial_3 (J^{-1}\alpha) \partial_3 w + J^{-1}\beta \partial_3 (J^{-1}\beta) \partial_3 w \\
&\qquad - J^{-1}\alpha (\partial_1 w \partial_3 h + \partial_3 w \partial_1 h) - J^{-1}\beta (\partial_2 w \partial_3 h + \partial_3 w \partial_2 h) - \omega_1 (J^{-1}\alpha) \partial_3 h \\
&\qquad - \omega_2 (J^{-1}\beta) \partial_3 h - wJ^{-1}\alpha \partial_3 (J^{-1}\alpha) \partial_3 h - wJ^{-1}\beta \partial_3 (J^{-1}\beta) \partial_3 h \\
&\qquad + (J^{-1}\partial_3 w) \times (1 + y) \eta_t,
\end{align*} \]

and
\[ \begin{align*}
F_5 &= [J^{-2}(\alpha^2 + \beta^2 + 1) - 1] \partial_3^2 h - 2J^{-1}\alpha \partial_3 \partial_1 h - 2J^{-1}\beta \partial_3 \partial_2 h \\
&\qquad - J^{-1}v_1 (\partial_3 h - J^{-1}\alpha \partial_3 h) - J^{-1}v_2 (\partial_2 h - J^{-1}\beta \partial_3 h) - J^{-1}(J^{-1}\alpha v_1 + J^{-1}\beta v_1 + v_3) \partial_3 h \\
&\qquad - \partial_1 (J^{-1}\alpha) \partial_3 h - \partial_2 (J^{-1}\beta) \partial_3 h + J^{-1}\alpha \partial_3 (J^{-1}\alpha) \partial_3 h + J^{-1}\beta \partial_3 (J^{-1}\beta) \partial_3 h \\
&\qquad - (\partial_1 h - J^{-1}\alpha \partial_3 h)^2 - (\partial_2 h - J^{-1}\beta \partial_3 h)^2 - J^{-2}(\partial_3 h)^2 \\
&\qquad + (J^{-1}\partial_3 h) \times (1 + y) \eta_t,
\end{align*} \]

and
\[ G_4 = J(\alpha^2 + \beta^2 + 1)^{-1} [\partial_1 \eta (\partial_1 w + \omega_1 h) + \partial_2 \eta (\partial_2 w + \omega_2 h)]. \quad (7.6) \]

On the other hand, it follows from (3.4)-(3.5) that
\[ \begin{align*}
\partial_j u_i &= \xi_{kj} \partial_k (J^{-1} v_1 \partial_1 \theta_l) , \\
\partial_j^2 u_i &= \xi_{kj} \partial_k [\xi_{lj} \partial_l (J^{-1} v_m \partial_m \theta_l)] , \\
\partial_i p &= \xi_{ki} \partial_k q . \quad (7.7)
\end{align*} \]

Thus
\[ \tilde{u} \cdot \nabla \tilde{u} + \nabla p + m \nabla \Phi - \Delta \tilde{u} = J^{-1} v_1 \partial_1 \eta \xi_{kj} \partial_k (J^{-1} v_m \partial_m \theta_l) + \xi_{ki} \partial_k q + \omega_1 \xi_{ki} \partial_k \phi \]
\[ - \xi_{kj} \partial_k [\xi_{lj} \partial_l (J^{-1} v_m \partial_m \theta_l)]. \quad (7.8) \]

Differentiating (3.5) with respect to \( t \) one derives
\[ \begin{align*}
u_{it} + (\partial_3 u_j) (\partial_3 \theta_3) &= \partial_t [J^{-1}(\partial_3 \theta_3)] v_j = -J^{-2} F_1 (\partial_3 \theta_3) v_j + J^{-1}(\partial_3 \theta_3) v_j + J^{-1}(\partial_3 \theta_3) v_j \\
\end{align*} \]

which, along with (3.1) and (3.3) leads to
\[ \begin{align*}
\nu_t &= J^{-1}(\partial_3 \theta_3) v_j - \xi_{k3} \partial_k [\xi_{lj} \partial_l (J^{-1} v_m \partial_m \theta_l)] (1 + y) \eta_t \\
&\quad - J^{-2} v_j [\partial_3 \eta] (1 + y)] + J^{-1} v_j \partial_3 \theta_3, \quad (7.9)
\end{align*} \]

with \( \theta_{1t} = 0, \theta_{2t} = 0 \) and \( \theta_{3t} = (1 + y) \eta_t \). Substituting (7.8)-(7.9) into the third equation of (1.7) and multiplying the resulting equality by \( J(\xi_{ij})_{3 \times 3} \) we arrive at
\[ \begin{align*}
\tilde{v}_t - \Delta \tilde{v} + \nabla q + w \nabla \phi &= \tilde{F}(w, \tilde{v}, \nabla q, \eta), \quad (7.10)
\end{align*} \]

where \( \tilde{F} = (F_1, F_2, F_3) \) with
\[ \begin{align*}
F_1 &= [J^{-2}(\alpha^2 + \beta^2 + 1) - 1] \partial_3^2 v_1 - 2J^{-1}\alpha \partial_3 \partial_1 v_1 - 2J^{-1}\beta \partial_3 \partial_2 v_1 \\
&\quad + 2J \partial_1 (J^{-1}) \partial_1 v_1 + J \partial_2 (J^{-1}) \partial_2 v_1 + J \partial_3 (J^{-1}) \partial_3 v_1 + \partial_3 [J^{-1}\partial_3 (J^{-1})] v_1 \\
&\quad + \partial_3 (J^{-2}) - J \partial_1 [\alpha (J^{-1}) \partial_3 v_1 \partial_1 + J \partial_2 (J^{-1}) \partial_3 v_1 - J \partial_3 (J^{-1}) \partial_3 v_1] \\
&\quad - J \partial_2 \beta (J^{-1}) \partial_3 v_1 - \alpha \partial_3 [J^{-1}\partial_1 \partial_3 v_1] - \alpha \partial_3 (J^{-1}) \partial_3 v_1 - \beta \partial_3 [J^{-1}\partial_3 v_1] - \beta \partial_3 (J^{-1}) \partial_3 v_1 \\
&\quad + \alpha \partial_3 [J^{-1}\partial_1 \partial_3 v_1] + \alpha \partial_3 (J^{-2}) \partial_3 v_1 + \beta \partial_3 [J^{-1}\partial_1 \partial_3 v_1] + \beta \partial_3 (J^{-2}) \partial_3 v_1 \\
&\quad + v_1 \partial_1 [J^{-1} v_1] + v_2 \partial_1 \partial_3 v_1 + v_3 \partial_3 (J^{-1} v_1) \\
&\quad + J^{-1} v_1 [\eta_t + \partial_3 \eta_t (1 + y)] + \partial_3 (J^{-1} v_1) (1 + y) \eta_t \\
&\quad + w \alpha \partial_3 \partial_3 \phi + w (1 - J) \partial_1 \phi + \alpha \partial_3 q + (1 - J) \partial_1 q.
\end{align*} \]
One can derive $F_2$ by replacing the terms in line 6 and line 8 of $F_1$ by

$$v_1 \partial_1 (J^{-1} v_2) + v_2 \partial_2 (J^{-1} v_2) + v_3 \partial_3 (J^{-1} v_2) + w \beta \partial_3 \phi + w (1 - J) \partial_2 \phi + \beta \partial_5 q + (1 - J) \partial_2 q$$

and replacing the $v_1$ in other terms of $F_1$ by $v_2$. For brevity, we shall not write out the exact expression of $F_2$. The third component $F_3$ is as follows:

$$F_3 = J^{-2} (\alpha^2 + \beta^2 + 1) - \alpha \partial_2 J^{-1} \alpha \partial_1 v_3 - 2 J^{-1} \alpha \partial_2 J^{-1} \alpha \partial_2 v_3 + J^{-1} \partial_3 (J^{-1}) \partial_3 v_3 - \partial_1 (\alpha J^{-1}) \partial_3 v_3 - \partial_2 (\beta J^{-1}) \partial_3 v_3 + \alpha J^{-1} \partial_3 (\alpha J^{-1}) \partial_3 v_3 + \beta J^{-1} \partial_3 (\beta J^{-1}) \partial_3 v_3 + v_1 J^{-1} \partial_1 v_3 + v_2 J^{-1} \partial_2 v_3 + v_3 J^{-1} \partial_3 v_3 + v_1 v_2 J^{-2} (\partial_1 \alpha + \partial_2 \alpha) + v_2 J^{-2} \partial_2 \beta + \alpha \partial_3 \phi + w \beta \partial_3 \phi + w (1 - J^{-1} (\alpha^2 + \beta^2 + 1)) \partial_3 \phi + \alpha \partial_5 q + \beta \partial_5 q + [1 - J^{-1} (\alpha^2 + \beta^2 + 1)] \partial_3 q.$$

Substitute (7.7) into the last boundary condition in (1.8) we derive on $\Gamma$ that

$$q \mathbf{N}_i - [\xi_{ij} \partial_k (J^{-1} v_1 \partial_i \theta_j) + \xi_{ji} \partial_k (J^{-1} v_1 \partial_i \theta_j)] \mathbf{N}_j = \left\{ \gamma \eta - \sigma \nabla \cdot \left( \frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}} \right) \right\} \mathbf{N}_i,$$

where

$$\tilde{N} := \mathbf{n} \circ \theta = \frac{(-\partial_1 \eta, -\partial_2 \eta, 1)}{\sqrt{1 + (V_0 \eta)^2}}.$$

Taking the inner product of the vector with components given in (7.11) with $T_1 := (1, 0, \partial_1 \eta)$ one deduces that

$$\partial_3 v_1 + \partial_1 v_3 = G_1 (\vec{v}, \vec{\eta}) \quad \text{on} \quad \Gamma \times (0, \infty),$$

with

$$G_1 = 2 \partial_1 (J^{-1} v_1) - J^{-1} \partial_1 \eta \partial_2 (J^{-1} v_1) \partial_1 \eta \partial_2 \eta + [\partial_2 (J^{-1} v_1) - J^{-1} \partial_2 \eta \partial_3 (J^{-1} v_1)] \partial_2 \eta + (1 - J^{-2}) \partial_3 v_1 - [J^{-1} v_1 \partial_3 (J^{-1}) + \partial_1 (J^{-1} v_1 \partial_3 \eta + J^{-1} v_2 \partial_2 \eta)] + [J^{-1} \partial_3 (J^{-1} v_1 \partial_3 \eta + J^{-1} v_2 \partial_2 \eta + v_3)]$$

Taking the inner product of (7.11) with $T_2 := (0, 1, \partial_2 \eta)$ to have

$$\partial_3 v_2 + \partial_2 v_3 = G_2 (\vec{v}, \vec{\eta}) \quad \text{on} \quad \Gamma \times (0, \infty),$$

where $G_2$ has an expression similar to that of $G_1$ and we shall not write out the explicit form of $G_2$ for brevity. Taking the inner product of (7.11) with $\tilde{N}$ one gets

$$q - 2 \partial_3 v_3 = \gamma \eta - \sigma \Delta_0 \eta - G_3 (\vec{v}, \vec{\eta}) \quad \text{on} \quad \Gamma \times (0, \infty),$$

where $G_3$ has an expression similar to that of $G_1$.
where $G_3 = \sigma \nabla_0 \cdot \left( \frac{\nabla_0 \eta}{\sqrt{1 + |\nabla_0 \eta|^2}} \right) - \sigma \Delta_0 \eta + \tilde{G}_3$ with $\tilde{G}_3$ defined in the following way:

$$[1 + (\nabla_0 \eta)^2] \tilde{G}_3$$

$$= -2[\partial_1 (J^{-1} v_1) - J^{-1} \partial_1 \eta \partial_3 (J^{-1} v_3)](\partial_1 \eta)^2 - 2[\partial_2 (J^{-1} v_2) - J^{-1} \partial_2 \eta \partial_3 (J^{-1} v_3)](\partial_2 \eta)^2$$

$$- 2[\partial_2 (J^{-1} v_1) - J^{-1} \partial_2 \eta \partial_3 (J^{-1} v_3)](\partial_2 \partial_3 \eta) - J^{-1} \partial_1 \eta \partial_3 (J^{-1} v_3)](\partial_1 \eta)$$

$$+ [J^{-1} \partial_3 (J^{-1} v_1) + \partial_1 (J^{-1} v_1 \partial_1 \eta + J^{-1} v_2 \partial_2 \eta + v_3)](\partial_1 \eta)$$

$$- [J^{-1} \partial_1 \eta \partial_3 (v_1 J^{-1} \partial_1 \eta + v_2 J^{-1} \partial_2 \eta + v_3)](\partial_2 \eta)$$

$$+ [J^{-1} \partial_3 (J^{-1} v_2) + \partial_2 (J^{-1} v_2 \partial_2 \eta + \partial_2 \eta + v_3)](\partial_2 \partial_3 \eta)$$

$$- [J^{-1} \partial_2 \eta \partial_3 (v_1 J^{-1} \partial_1 \eta + v_2 J^{-1} \partial_2 \eta + v_3)](\partial_2 \partial_3 \eta)$$

$$+ [J^{-1} \partial_3 (J^{-1} v_3) + \partial_3 (J^{-1} v_3 \partial_3 \eta + J^{-1} v_2 \partial_2 \eta + v_3) - J^{-1} \partial_1 \eta \partial_3 (J^{-1} v_1 \partial_1 \eta + J^{-1} v_2 \partial_2 \eta + v_3)](\partial_1 \eta)$$

$$+ [J^{-1} \partial_1 \eta \partial_3 (J^{-1} v_3) + \partial_2 (J^{-1} v_2 \partial_2 \eta + \partial_2 \eta + v_3) - J^{-1} \partial_2 \eta \partial_3 (J^{-1} v_1 \partial_1 \eta + J^{-1} v_2 \partial_2 \eta + v_3)](\partial_2 \eta)$$

$$- 2J^{-1} \partial_3 (J^{-1} v_1 \partial_1 \eta + J^{-1} v_2 \partial_2 \eta + v_3) + 2[1 + (\nabla_0 \eta)^2 - J^{-1}](\partial_3 v_3).$$

Noting that $\alpha = \partial_1 \eta$, $\beta = \partial_2 \eta$ and $J = 1$ on $\Gamma$, we deduce from (3.6) and the third boundary condition in (1.8) that

$$\eta_x = v_3 \quad \text{on} \quad \Gamma \times (0, \infty). \quad (7.15)$$

Moreover, by rewriting (3.6) as

$$v_1(x_1, x_2, y, t) = J u_1(\theta(x_1, x_2, y, t), t), \quad v_2(x_1, x_2, y, t) = J u_2(\theta(x_1, x_2, y, t), t),$$

$$v_3(x_1, x_2, y, t) = u_3(\theta(x_1, x_2, y, t), t) - \alpha u_1(\theta(x_1, x_2, y, t), t) - \beta u_2(\theta(x_1, x_2, y, t), t) \quad (7.16)$$

and applying a direct computation to (7.16) and using the chain rule one can easily deduces that

$$\nabla \cdot \vec{v} = 0 \quad \text{in} \quad \Omega \times (0, \infty). \quad (7.17)$$

Collecting (7.5), (7.10), (7.12)-(7.14), (7.15) and (7.17) we derive (3.10)-(3.12). \hfill \Box

**Acknowledgements**

This work is supported by China Postdoctoral Science Foundation (2019M651269), National Natural Science Foundation of China (11901139).

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