On the structure of the energy-momentum and the spin currents in Dirac’s electron theory*

Friedrich W. Hehl
Institute for Theoretical Physics, University of Cologne
D-50923 Köln, Germany

Alfredo Macías, Eckehard W. Mielke
Departamento de Física
Universidad Autónoma Metropolitana–Iztapalapa
P.O. Box 55-534, 09340 México D.F., México

Yuri N. Obukhov
Department of Theoretical Physics, Moscow State University
117234 Moscow, Russia

1997-06-03

Abstract
We consider a classical Dirac field in flat Minkowski spacetime. We perform a Gordon decomposition of its canonical energy-momentum and spin currents, respectively. Thereby we find for each of these currents a convective and a polarization piece. The polarization pieces can be expressed as exterior covariant derivatives of the two-forms $\gamma^\alpha$ and $M_{\alpha\beta} = -M_{\beta\alpha}$, respectively. In analogy to the magnetic moment in electrodynamics, we identify these two-forms as gravitational moments connected with the translation group and the Lorentz group, respectively. We point out the relation between the Gordon decomposition of the energy-momentum current and its Belinfante-Rosenfeld symmetrization. In the non-relativistic limit, the translational gravitational moment of the Dirac field is found to be proportional to the spin covector of the electron.

1 Introduction
Fermi systems do not possess classical analogs. Nevertheless, for such systems, we can define a limit for $\hbar \to 0$, even if, in this limit, we do not have ordinary

*To appear in: On Einstein’s Path. Festschrift for E. Schucking on the occasion of his 70th birthday. A. Harvey, ed. (Springer: Berlin 1997/98)
classical theory. We feel that a description of such a limit would be interesting in view of the possibility to develop a better intuition and to build up new models. Conventionally Fermi systems can be quantized by the path integral method. By introducing semi-classical Grassmann variables, for instance, we get a deeper understanding of this method.

In some dual theories with Fermi variables, a detailed knowledge of the classical limit is also interesting, because it allows a reinterpretation in terms of superstrings. This unified treatment of commuting and anticommuting variables is obviously of particular interest for supersymmetric theories, because in this case one is obliged to treat Fermi and Bose fields in a symmetrical way.

Knowing the usefulness of understanding the classical limit, we will investigate, in this article, the structure of the energy-momentum and the spin current of the classical Dirac theory. Of course, the classical Dirac theory can only be made consistent by second quantization. However, for low energy phenomena, when particle creation and annihilation don’t play a role, the classical Dirac theory and its non-relativistic limit convey an appropriate picture of the underlying physics.

An electron carries a spin of \( s_z = \hbar/2 \). It is legitimate to visualize this spin as some sort of intrinsic circular motion. This seems to be clear from the relation of spin to the Lorentz or the three-dimensional rotation group, but it can also be made explicit by studying the quantum mechanics of a Dirac particle in a Coulomb potential. Even in an \( s_1/2 \) state, an electric current is present, i.e., “a polarized Dirac electron is a rotating particle”.

If we take this for granted, it is obvious that the electric charge, which is housed in the electron, induces an Ampère type ring current which, in turn, according to the Oersted-Ampère law, acts as a magnetic moment \( \mu_e \). The specific nature of spin angular momentum yields a gyromagnetic ratio \( \mu_e/(s_z/\hbar) = -2\mu_B \), where \( \mu_B := e\hbar/2m_e \) is Bohr’s magneton (\( e \) is the charge of the electron, \( m_e \) its mass). This gyromagnetic ratio turns out to be twice as large as that of ordinary orbital angular momentum known from Newtonian mechanics.

The electron, besides the electric charge, carries also gravitational charges. In the framework of general relativity, the electron’s mass is the only gravitational charge, and we expect a mass-energy ring current inducing a gravitational moment \( \mu_{GR} \). In the Einstein-Cartan theory, which is the appropriate gravitational theory for a Dirac field in much the same way as general relativity, i.e. Einstein’s theory, is appropriate for classical point particles (without spin), there feature two types of gravitational charges: mass-energy and spin. In such a theoretical framework, we would search, already in the special-relativistic realm, for a translational gravitational moment \( M_T \), linked to the mass, and a Lorentz (or rotational) gravitational moment \( M_L \), linked to the spin aspect. The latter one is to be understood as the spin charge which is carried around by itself.

The expected units of the translational and Lorentz gravitomoments \( \mu_{G_1} = \hbar/2 \) and \( \mu_{G_2} = \hbar^2/4m_e \) are found by substituting in Bohr’s magneton \( \mu_B \) the electric charge by the mass or the spin of the electron, respectively. Then, in analogy to electrodynamics, we should expect gyrogravitational ratios for the
different gravitational moments, namely $\mu_{GR}/(s_z/h) = g_0 \hbar/2$, $M_T/(s_z/h) = g_1 \hbar/2$, and $M_L/(s_z/h) = g_2 \hbar^2/4m_e$, with $g_0$, $g_1$, $g_2$ as the dimensionless gravitational $g$-factors. For the gravitational $g$-factors we wouldn’t be able to predict the values before going into the corresponding computations. But our guess, for reasons of analogy with electrodynamics, would be $g_0 = g_1 = g_2 = 2$.

How could we thoroughly check these ideas? Again appealing to analogies to electrodynamics, the following procedure is near at hand: By using a Gordon type decomposition of the energy-momentum and the spin currents in special relativity, we can provisionally define the gravitational moments. Then we couple the Dirac Lagrangian minimally to the gravitational field, reshuffle the Lagrangian suitably, and identify the factor(s) in front of the gravitational field strength(s) as gravitational moment(s). In general relativity, the gravitational field strength is the Riemannian curvature, in the Einstein-Cartan theory, both the torsion and the (Riemann-Cartan) curvature represent the field strengths, respectively. In this article we make the first step in identifying the gravitational moments of the classical Dirac field within the framework of special relativity.

We find it surprising that, apart from Kobzarev and Okun, nobody seems to care about the gravitational moments of the Dirac field. It is true that the effects will be so small that there is no hope for measuring these moments in the near future. In any case, as a matter of principle, the gravitational moments of a fundamental particle belong to its basic properties and will enter any unification attempt.

We would like to dedicate this article to Engelbert Schücking on the occasion of his 70th birthday.

2 Dirac–Yang-Mills theory

The Dirac Lagrangian is given by the hermitian four-form

$$L_D = L(\vartheta^\alpha, \Psi, D\Psi) = \frac{i}{2} \hbar \left\{ \overline{\Psi} \star \gamma \wedge D\Psi + D\overline{\Psi} \wedge \star \gamma \Psi \right\} + *mc \overline{\Psi} \Psi. \quad (1)$$

The coframe $\vartheta^\alpha$ necessarily occurs in the Dirac Lagrangian, even in special relativity. We are using the formalism of Clifford-valued exterior forms, see the Appendix for the basic definitions. For the mass term, we use the short-hand notation $*m := m \eta = m *1$. The hermiticity of the Lagrangian (1) leads to a charge current which admits the usual probabilistic interpretation.

Here we assume a Minkowski spacetime geometry (no gravity). However, the fermions may carry various internal charges. This is reflected in the gauge covariant derivatives, $D = d + A$, where $A = \xi A^\kappa \tau_\kappa$ is a Lie algebra-valued one-form, the gauge potential, with a set of matrices $\tau_\kappa$ specifying a representation of the generators of the gauge symmetry. The charge $e$ stands for the coupling constant of a fermion–gauge field interaction. In the commutator $[\tau_\kappa, \tau_\lambda] = f^{M}_{\kappa \lambda} \tau_M$, the structure constants $f^{M}_{\kappa \lambda}$ are totally antisymmetric for semisimple groups. The Abelian case is also trivially included, when the internal symmetry
reduces to the one-parameter group with a single generator \( \tau_1 = i \) (then \( f^M_{KL} \equiv 0 \)). This corresponds to electrodynamics with \( e \) as the usual electric charge.

The Lagrangian \( (1) \) yields the Dirac equation
\[
\begin{align*}
i\hbar^* \gamma \wedge D\Psi + e mc \Psi &= 0, \quad (2) \\
i\hbar D\Psi^* \wedge \gamma + e mc \Psi^* &= 0. \quad (3)
\end{align*}
\]
The three-form
\[
J_K := \frac{\delta L_D}{\delta A^K} = -ie \Psi^* \gamma \Psi
\]
is the canonical Noether current, the “isospin” three-form, which turns out to be covariantly conserved, provided the equations of motion \((2)-(3)\) are substituted:
\[
DJ_K = dJ_K + \frac{e}{\hbar} f^M_{KL} A^L \wedge J_M = 0.
\]

Using the Dirac equation, we can, following Gordon [11], decompose the Noether current into two pieces, the convective and the polarization currents. The derivation is rather short: assuming the mass to be nonzero, one can resolve \((2)-(3)\) with respect to the spinor fields:
\[
\Psi = \frac{i\hbar}{mc} (\gamma \wedge D\Psi), \quad \Psi^* = \frac{i\hbar}{mc} (D\Psi^* \wedge \gamma).
\]
Now substitute this into \((4)\) and find
\[
J_K = J^{(c)}_K + J^{(p)}_K,
\]
where the convective three-form current reads
\[
J^{(c)}_K := \frac{e\hbar}{2mc} (\nabla \gamma \nabla K D\Psi - D\nabla K \Psi),
\]
and the polarization current is constructed as a covariant exterior differential of the polarization two–form \( P_K \):
\[
J^{(p)}_K := DP_K, \quad P_K := -i \frac{e\hbar}{2mc} \nabla \gamma \nabla K \Psi.
\]
The covariant exterior differential \( D \) is defined as in \((5)\).

Unlike the total current, its convective and polarization parts are not conserved separately in non-Abelian gauge theory. For example, we can immediately check that
\[
DJ^{(p)}_K = DD P_K = \frac{e}{\hbar} f^N_{KL} F^L \wedge P_N,
\]
where
\[
F^K := dA^K + \frac{e}{2\hbar} f^K_{MN} A^M \wedge A^N
\]
\footnote{For a comprehensive study of the Dirac equation, see Thaller [10].}
is the two-form of the gauge field strength. [We can also write the gauge field strength in the form \( F := dA + A \wedge A = \frac{e}{\hbar} F^K \tau_K \). Note that \( DD\Psi = F\Psi \).] Only in the Abelian case one finds Gordon’s separate conservation of the two currents.

In order to understand the physical meaning of these currents, it is instructive to study the non-relativistic approximation for fermions in a weak external gauge field. In the non-relativistic approximation, a Dirac four-spinor is represented by a pair of two-component spinors, \( \Psi = \left( \begin{array}{c} \psi \\ \chi \end{array} \right) \), with \( \chi = \mathcal{O}(\frac{\hbar}{c}) \psi \) (for positive energy solutions). The polarization two-form is then:

\[
P_K = \frac{e\hbar}{2m} dt \wedge S_K + \mathcal{O} \left( \frac{\psi}{c} \right), \quad S_K := \psi^\dagger (\tau_K) \frac{\sigma}{2} \psi d\mathbf{x}. \tag{12}\]

Consider the interaction term in the Lagrangian, \( A^K \wedge J_K \). It is clear that the convective contribution describes a (non-Abelian generalization of the) usual Schrödinger current,

\[
J_K^{(c)} \approx \frac{e\hbar}{2mc} [\psi^\dagger \tau_K d\psi - (d\psi^\dagger) \tau_K \psi], \tag{13}\]

whereas the polarization contribution reads

\[
A^K \wedge J_K^{(p)} \approx F^K \wedge P_K \approx \frac{e\hbar}{m} dt \wedge S_K \wedge F^K. \tag{14}\]

In the Abelian case (for \( \tau_K = i \)), the standard result of the Dirac electron theory is recovered, \( A \wedge J_K^{(p)} \approx \mu B dt \wedge \eta \). Here \( \eta \) is the volume form of a spatial hypersurface, \( B \) is the magnetic field strength, and \( \mu = \frac{e\hbar}{2m} \psi^\dagger \sigma \psi \).

Technically, we can derive the two currents as follows. Substituting (6) back into the Dirac equation (2)-(3), we find the squared equation:

\[
\left[ D^*D - i* \sigma \wedge F + *(mc/\hbar)^2 \right] \Psi = 0. \tag{15}\]

Eq.(15) can be derived from the Lagrange form \( L_{D^2} = L^{(c)} + L^{(p)} \), with

\[
L^{(c)} := \frac{1}{2} \left( \frac{\hbar^2}{mc} \ast \overline{D\Psi} \wedge D\Psi + *mc \overline{D\Psi} \Psi \right), \tag{16}\]

\[
L^{(p)} := P_K \wedge F^K. \tag{17}\]

Since both pieces of the Lagrangian are separately gauge invariant, we straightforwardly recover the convective and polarization currents, (5) and (6), as the respective Noether currents:

\[
J_K^{(c)} = \frac{\delta L^{(c)}}{\delta A^K}, \quad J_K^{(p)} = \frac{\delta L^{(p)}}{\delta A^K}. \tag{18}\]
3 Gordon decomposition of energy-momentum and spin currents

The *canonical* energy-momentum and spin three-forms are defined in the standard way:

\[
\Sigma_\alpha := e_\alpha [L - \frac{\partial L}{\partial D\Psi} (e_\alpha | D\Psi) - (e_\alpha | D\bar{\Psi})] = -\frac{\partial L}{\partial D\Psi} e_\alpha \Psi, \quad (19)
\]

\[
\tau_{\alpha\beta} := \frac{\partial L}{\partial D\Psi} \ell_{\alpha\beta} \Psi + \bar{\Psi} \ell_{\alpha\beta} \frac{\partial L}{\partial D\Psi}. \quad (20)
\]

Recalling that the spinor generators of the Lorentz group are \(\ell_{\alpha\beta} = \frac{i}{4} \hat{\sigma}_{\alpha\beta}\), we find for the Dirac Lagrangian (1), with \(L = L_D\) inserted into (19)-(20),

\[
\Sigma_\alpha = \frac{i}{\hbar} \left( \bar{\Psi} \gamma D\alpha \Psi - D\alpha \bar{\Psi} \gamma \Psi \right), \quad (21)
\]

\[
\tau_{\alpha\beta} = \frac{\hbar}{4} \theta_\alpha \wedge \theta_\beta \wedge \bar{\Psi} \gamma_{\alpha\beta} \Psi. \quad (22)
\]

Here we denoted \(D\alpha := e_\alpha | D\) and took into account that \(L_D \sim 0\) (upon substitution of the field equations).

Translational and Lorentz invariance of the Dirac Lagrangian yield the well-known conservation laws for energy-momentum and angular momentum:

\[
D\Sigma_\alpha = 0, \quad D\tau_{\alpha\beta} + \partial_{[\alpha} \Sigma_{\beta]} = 0. \quad (23)
\]

Here \(D\) denotes the Lorentz covariant exterior derivative containing the Levi-Civita connection \(\Gamma_{\alpha \beta}\) of flat Minkowski space. Thus we have, e.g., \(D\Sigma_\alpha = d\Sigma_\alpha - \Gamma_\alpha^\beta \wedge \Sigma_\beta\).

Let us perform the Gordon type decomposition of the energy-momentum current [12, 13, 14, 15, 16]. At first, we substitute (6) into (21) and obtain:

\[
\Sigma_\alpha = \frac{\hbar^2}{2mc} \left[ (D\bar{\Psi}) D\alpha \Psi + D\alpha \bar{\Psi} * \gamma \Psi \right. + i \left( D\bar{\Psi} \wedge * \bar{\sigma} D\alpha \Psi - D\alpha \bar{\Psi} \bar{\sigma} \wedge D\Psi \right). \quad (24)
\]

The Dirac equation (2)-(3) yields:

\[
\begin{align*}
\bar{\Psi} \gamma D\alpha \Psi &= \bar{\Psi} \left( \eta_\alpha \gamma \Psi + mc \right), \\
\bar{\Psi} D\alpha \gamma &\bar{\Psi} &= \bar{\Psi} \left( \eta_\alpha \gamma \Psi - mc \right).
\end{align*} \quad (25, 26)
\]

Substituting this into (21), we find another representation for the energy-momentum, provided we use (6) again:

\[
\Sigma_\alpha = \frac{i\hbar}{2} \left( (e_\alpha | \bar{\Psi} \gamma \Psi \wedge D\Psi - \bar{D}\Psi \wedge (e_\alpha | \gamma \Psi)) + \eta_\alpha mc \bar{\Psi} \right)
\]

\[
= \frac{\hbar^2}{2mc} \left[ (e_\alpha | \bar{D}\Psi) \wedge D\Psi + \bar{D}\Psi \wedge (e_\alpha | \gamma D\Psi) \right. \left. + i \left( D\alpha \bar{\Psi} \bar{\sigma} \wedge D\Psi - \bar{D}\Psi \wedge * \bar{\sigma} D\alpha \Psi \right) \\
- 2i \bar{D}\Psi \wedge (e_\alpha | \gamma D\Psi) + \eta_\alpha mc \bar{\Psi}. \quad (27)
\right]
\]
Then, the sum of (24) and (27) yields
\[ \Sigma_\alpha = \Sigma^{(c)}_\alpha + \Sigma^{(p)}_\alpha, \] (28)
where
\[ \Sigma^{(c)}_\alpha := \frac{mc}{2} \bar{\Psi} \Psi \eta_\alpha + \frac{\hbar^2}{4mc} \left[ \star (D\bar{\Psi}) D_\alpha \Psi + D_\alpha \bar{\Psi} \star D\Psi \right. \]
\[ \left. + (e_\alpha \star D\bar{\Psi}) \wedge D\Psi + D\bar{\Psi} \wedge (e_\alpha \star D\Psi) \right], \] (29)
\[ \Sigma^{(p)}_\alpha := D\hat{M}_\alpha, \] (30)
\[ \hat{M}_\alpha := - \frac{i\hbar^2}{4mc} \left[ \bar{\Psi} (e_\alpha \star \hat{\sigma}) \wedge D\Psi + D\bar{\Psi} \wedge (e_\alpha \star \hat{\sigma}) \Psi \right]. \] (31)

Since we are in flat Minkowski spacetime, the last term in (28), the polarization part of the energy-momentum current, is identically conserved,
\[ D\Sigma^{(p)}_\alpha = D\hat{D}\hat{M}_\alpha = 0. \] (32)
Recalling (23), we immediately obtain a separate conservation of the first term on the right hand side of (28), which we naturally call the convective energy-momentum current:
\[ D\Sigma^{(c)}_\alpha = 0. \] (33)
Moreover, one can immediately check that the convective current is symmetric,
\[ \vartheta_{[\alpha} \wedge \Sigma^{(c)}_{\beta]} = 0. \] (34)

It is remarkable to observe that, similar as the convective current (8) represents a Noether current for the convective Lagrangian (16), the convective energy-momentum (29) precisely turns out to be the canonical Noether current (19) for \( L = L^{(c)} \). The proof is straightforward: substitute (16) into (19) and compare with (29).

As we saw, the ordinary polarization current (9) also emerges as a Noether current for the “Pauli-type” polarization Lagrangian (17) which describes the interaction of the moment two-form with the background gauge field strength \( F^K \). A guess would be that the polarization energy-momentum should arise in a similar way as a canonical current from the respective “Pauli-type” polarization Lagrangian when the gravitational field is “switched on”. The discussion of the precise form of this Lagrangian (as well as of the nature of the gravitational field represented by the coframe) will be considered in a separate publication, see also [17, 18, 19, 20, 21, 22].

Here, however, we have to complete our derivations and to consider the Gordon type decomposition of the spin current. We have good reasons to expect a similar structure of the decomposed three-form \( \tau_{\alpha\beta} \). At first, we note that the canonical spin current (20) for the convective Lagrangian (16) reads:
\[ \tau^{(c)}_{\alpha\beta} = \frac{\partial L^{(c)}}{\partial D\Psi} \ell_{\alpha\beta} \Psi + \bar{\Psi} \ell_{\alpha\beta} \frac{\partial L^{(c)}}{\partial D\Psi} \]
\[ = - \frac{i\hbar^2}{8mc} \left[ \star D\bar{\Psi} \sigma_{\alpha\beta} \Psi - \Psi \sigma_{\alpha\beta} \star D\Psi \right]. \] (35)
Now we use the standard trick by substituting (6) into the spin current (22):

\[
\tau_{\alpha \beta} = \frac{\hbar}{4} \partial_\alpha \wedge \partial_\beta \wedge \bar{\Psi} \gamma_5 \Psi = \frac{\hbar}{8} \bar{\Psi} (\gamma_5 \tilde{\sigma}_{\alpha \beta}^\gamma + \tilde{\sigma}_{\alpha \beta}^\gamma) \Psi
\]

\[
= \frac{i \hbar^2}{8mc} \left( - \bar{D} \Psi \tilde{\sigma}_{\alpha \beta}^\gamma \Psi + \bar{\Psi} \tilde{\sigma}_{\alpha \beta}^\gamma \bar{D} \Psi \\
- i \bar{D} \Psi \wedge \tilde{\sigma}_{\alpha \beta}^\gamma \Psi - i \bar{\Psi} \tilde{\sigma}_{\alpha \beta}^\gamma \bar{D} \Psi \right).
\] (36)

We immediately recognize the second line as the *convective spin* (35). The last line should thus be related to the polarization spin current. The identities (97)-(98) (see the Appendix) play the crucial role here. Substituting them into (36), we obtain straightforwardly,

\[
\tau_{\alpha \beta} = \tau_{\alpha \beta}^{(c)} + DM_{\alpha \beta} + \partial_\gamma [\tilde{M}^\gamma_{\alpha \beta}],
\] (37)

where we define the *Lorentz gravitational moment* two-form by

\[
M_{\alpha \beta} := \frac{\hbar^2}{16mc} \bar{\Psi} (\gamma_5 \tilde{\sigma}_{\alpha \beta}^\gamma + \tilde{\sigma}_{\alpha \beta}^\gamma) \Psi
\]

\[
= \frac{\hbar^2}{8mc} (\bar{\Psi} \eta_{\alpha \beta} - i \bar{\Psi} \gamma_5 \Psi \partial_\alpha \wedge \partial_\beta),
\] (38)

whereas \( \tilde{M}_\alpha \) is given in (31). The Lorentz moment \( M_{\alpha \beta} \) is very simple in structure. It is additively built up from a scalar and a pseudoscalar piece, i.e., from its 36 components only 2 are independent.

Substituting the Gordon decompositions (28) and (37) into the conservation law of angular momentum (23), we find the *separate* conservation of the convective spin,

\[
D \tau_{\alpha \beta}^{(c)} = 0.
\] (40)

4 Relocalization of energy-momentum and spin

Like for internal symmetries, also in the case of the Poincaré group, the Noether currents are only determined up to an exact two-form. This non-uniqueness has troubled physicists already for quite some time. Within gravitational theory, the question of the “correct” energy-momentum current of matter is as old as general relativity itself [23, 24], see also the review [25]. But only Belinfante [26], in the framework of special relativity, and Rosenfeld [27], within general relativity, gave a general prescription of how one can find the metric or *Hilbert* energy-momentum current from the canonical or *Noether* energy-momentum current of an arbitrary matter field \( \Psi \). The Hilbert current acts as source on the right hand side of the Einstein field equation, whereas the Noether current is of central importance in special-relativistic canonical field theory. We will now turn our attention to this interrelationship between the different energy-momentum currents within the framework of special relativity.
The Noether law for energy-momentum in (23) also holds for an energy-momentum current which is supplemented by a $D$-exact form:

$$\hat{\Sigma}_\alpha(X) := \Sigma_\alpha - DX_\alpha.$$  \hfill (41)

In special relativity, $DD = 0$. This is the only property of $D$ that is needed in this context. The $X_\alpha$ does not interfere with the Noether law:

$$D\Sigma_\alpha = D\hat{\Sigma}_\alpha + DDX_\alpha = D\hat{\Sigma}_\alpha = 0.$$  \hfill (42)

If we insert $\Sigma_\alpha = \hat{\Sigma}_\alpha(X) + DX_\alpha$ into the left hand side of the Noether law for angular momentum in (23), we find

$$D\tau_{\alpha\beta} + \vartheta_{[\alpha \wedge \Sigma_\beta]} = D(\tau_{\alpha\beta} - \vartheta_{[\alpha \wedge X_\beta]} + \vartheta_{[\alpha \wedge \hat{\Sigma}_\beta]}).$$  \hfill (43)

If a relocalized spin $\hat{\tau}_{\alpha\beta}$ is required to fulfill again a law of the type given in (23), i.e. $D\hat{\tau}_{\alpha\beta} + \vartheta_{[\alpha \wedge \hat{\Sigma}_\beta]} = 0$, then

$$\hat{\tau}_{\alpha\beta}(X,Y) := \tau_{\alpha\beta} - \vartheta_{[\alpha \wedge X_\beta]} - DY_{\alpha\beta},$$  \hfill (44)

where $DY_{\alpha\beta}$ is an additional $D$-exact form with $Y_{\alpha\beta} = -Y_{\beta\alpha}$. Thus a relocalization of the energy-momentum is, up to a $D$-exact form, accompanied by an induced transformation of the canonical spin. Therefore we have the following result:

The canonical currents $(\Sigma_\alpha, \tau_{\alpha\beta})$ fulfill the Noether laws (23). Take arbitrary two-forms $X_\alpha$ and $Y_{\alpha\beta} = -Y_{\beta\alpha}$ as superpotentials. Then the relocalized currents

$$\Sigma_\alpha \rightarrow \hat{\Sigma}_\alpha(X) = \Sigma_\alpha - DX_\alpha,$$
$$\tau_{\alpha\beta} \rightarrow \hat{\tau}_{\alpha\beta}(X,Y) = \tau_{\alpha\beta} - \vartheta_{[\alpha \wedge X_\beta]} - DY_{\alpha\beta},$$

satisfy the same relations

$$D\hat{\Sigma}_\alpha = 0, \quad D\hat{\tau}_{\alpha\beta} + \vartheta_{[\alpha \wedge \hat{\Sigma}_\beta]} = 0.$$  \hfill (47)

Accordingly, the Noether identities turn out to be invariant under the relocalization transformation \hfill (47)-(48). As a consequence, the total energy-momentum $P_\alpha$ and the total angular momentum $J_{\alpha\beta}$, up to boundary terms, remain invariant under \hfill (47)-(48),

$$\hat{P}_\alpha = P_\alpha - \int_{\partial H_t} X_\alpha, \quad \hat{J}_{\alpha\beta} = J_{\alpha\beta} - \int_{\partial H_t} (x_{[\alpha \wedge X_\beta]} + Y_{\alpha\beta}),$$  \hfill (48)

where $H_t$ denotes a timelike hypersurface in Minkowski space and $\partial H_t$ its two-dimensional boundary. Provided the superpotentials $X_\alpha$ and $Y_{\alpha\beta}$ approach zero at spacelike asymptotic infinity sufficiently fast, the total quantities are not affected by the relocalization procedure.

Let us put our results of the Gordon decomposition in the last section into this general framework. If we choose as superpotentials the respective gravitational moments,

$$X_\alpha = M_\alpha, \quad Y_{\alpha\beta} = M_{\alpha\beta},$$  \hfill (49)
then the relocalized currents turn out to be the *convective* pieces:

\[
\Sigma^{(c)}_\alpha = \Sigma_\alpha - D \tilde{M}_\alpha = \Sigma_\alpha - \Sigma^{(p)}_\alpha, \tag{50}
\]

\[
\tau^{(c)}_{\alpha \beta} = \tau_{\alpha \beta} - \partial_{[\alpha} \tilde{M}_{\beta]} - D \Sigma^{(p)}_{\alpha \beta} = \tau_{\alpha \beta} - \tau^{(p)}_{\alpha \beta}. \tag{51}
\]

From explicit calculations we know, see (34), that the convective current (50) is symmetric, as one would expect for a Schrödinger type energy-momentum current. Consequently, in special relativity, the decomposed currents have the following properties:

\[
D \Sigma^{(c)}_\alpha = 0, \quad D \Sigma^{(p)}_\alpha = 0, \quad \partial_{[\alpha} \Sigma^{(c)}_{\beta]} = 0, \tag{52}
\]

\[
D \tau^{(c)}_{\alpha \beta} = 0, \quad D \tau^{(p)}_{\alpha \beta} + \partial_{[\alpha} \Sigma^{(p)}_{\beta]} = 0. \tag{53}
\]

Thus a Gordon decomposition in Minkowski spacetime is nothing else but a specific relocalization of the currents. It yields a symmetric energy-momentum current \(\Sigma^{(c)}_\alpha\) with a nonvanishing conserved spin current \(\tau^{(c)}_{\alpha \beta}\). The spin tensor of Hilgevoord et al. \[28, 29\], which was constructed outside of the framework of Lagrangian formalism, coincides with our convective spin current \(\tau^{(c)}_{\alpha \beta}\).

## 5 Trivial Lagrangians and relocalization

For the purpose of understanding relocalization from a Lagrangian point of view, let us consider an arbitrary three-form \(U(\Psi, D\Psi)\) constructed from a matter field \(\Psi\) and its derivatives. Here we discard other possible arguments of \(U\) (like the coframe, e.g.). In general, \(\Psi\) can be any matter field, not necessarily the Dirac four-spinor.

If the form \(U\) is invariant under spacetime translations (coordinate transformations) and Lorentz transformations, then, using the standard Lagrange-Noether machinery, cf. \[30\], one derives the first and the second Noether identities:

\[
D X^\alpha \equiv - e_\alpha \cdot dU + (e_\alpha \cdot D\Psi) \frac{\delta U}{\delta \Psi}, \tag{54}
\]

\[
D Y_{\alpha \beta} + \partial_{[\alpha} X_{\beta]} \equiv - \ell_{\alpha \beta} \Psi \frac{\delta U}{\delta \Psi}. \tag{55}
\]

Here the two-forms

\[
\tilde{X}_\alpha := e_\alpha \cdot U - (e_\alpha \cdot D\Psi) \frac{\partial U}{\partial D\Psi}, \tag{56}
\]

\[
\tilde{Y}_{\alpha \beta} := \ell_{\alpha \beta} \Psi \frac{\partial U}{\partial D\Psi}, \tag{57}
\]

are analogs of the three-forms of canonical energy-momentum and spin derived from the “Lagrangian” \(U\). We use the standard notation

\[
\frac{\delta U}{\delta \Psi} := \frac{\partial U}{\partial \Psi} - D \cdot \frac{\partial U}{\partial D\Psi}. \tag{58}
\]

\[2\]... up to a factor of 2 due to different conventions.
We will not assume, however, that the matter field $\Psi$ satisfies the “equation of motion” $\frac{\delta U}{\delta \Psi} = 0$. The relations (54) and (55) are thus strong identities valid for all matter field configurations. The first term on the right-hand side of (54) is usually absent in the first Noether identity due to the fact that the Lagrangian is a form of maximal rank (i.e. four in standard spacetime). Here we have a three-form $U$ in four-dimensional spacetime, and the four-form $dU$ is, in general, nontrivial.

Let us now treat the form $L = dU$ as a specific Lagrangian. It is straightforward to find for the variation:

$$
\delta U = \delta \Psi \frac{\partial U}{\partial \Psi} + \delta D\Psi \wedge \frac{\partial U}{\partial D\Psi} = \delta \Psi \frac{\partial U}{\partial \Psi} + d \left( \delta \Psi \frac{\partial U}{\partial D\Psi} \right).
$$

(59)

Hence, for our specific Lagrangian, we have

$$
\delta L = d(\delta U) = \delta \Psi D \left( \frac{\partial U}{\partial \Psi} \right) + \delta (D\Psi) \wedge \delta U.
$$

(60)

For the partial derivatives, this yields

$$
\frac{\partial L}{\partial \Psi} = D \left( \frac{\partial U}{\partial \Psi} \right),
$$

(61)

$$
\frac{\partial L}{\partial D\Psi} = \delta U.
$$

(62)

Consequently, for the Lagrangian $L = dU$, the equation of motion

$$
\frac{\partial L}{\partial \Psi} = \frac{\partial L}{\partial D\Psi} - D \frac{\partial L}{\partial D\Psi} \equiv 0
$$

(63)

is identically satisfied. Actually, this is no surprise: it is well known that a Lagrangian, which is a total differential, has trivial dynamics.

Nevertheless, although the dynamics is trivial, the conserved currents are not trivial. In particular, the canonical energy-momentum and spin three-forms are defined as usual by

$$
\Sigma_{\alpha} = e_{\alpha} |L - \frac{\partial L}{\partial D\Psi} (e_{\alpha} |D\Psi),
$$

(64)

$$
\tau_{\alpha\beta} = \frac{\partial L}{\partial D\Psi} \ell_{\alpha\beta} \Psi.
$$

(65)

And now we are approaching the crucial point. We recall that $L = dU$. Thus we can substitute (62) into these currents. Then, for the corresponding energy-momentum, we immediately find

$$
\Sigma_{\alpha} = e_{\alpha} |dU - (e_{\alpha} |D\Psi) \frac{\delta U}{\delta \Psi} = -D \tilde{X}_{\alpha},
$$

(66)

where we made use of the first Noether identity (54). Analogously, for the spin current, we have

$$
\tau_{\alpha\beta} = \ell_{\alpha\beta} \Psi \frac{\delta U}{\delta \Psi} = -DY_{\alpha\beta} - \theta_{(\alpha} \wedge \tilde{X}_{\beta)},
$$

(67)
where we used the second Noether identity (55).

This observation underlies the relocalization described above of energy-momentum and spin. Indeed, our derivation shows that if one adds to any matter field Lagrangian $L_{\psi}$ a total divergence $\int dU$, then, for the new Lagrangian $L_{\psi} + dU$, the canonical energy-momentum and spin currents are relocalized according to (45)-(46), with the superpotentials

$$X_{\alpha} = \tilde{X}_{\alpha}, \quad Y_{\alpha\beta} = \tilde{Y}_{\alpha\beta}.$$  

(68)

In this sense, one can say that a relocalization is generated by the three-form $U$ via (56)-(57).

Again looking back to the Gordon decomposition of energy-momentum and spin as a special case of the relocalization procedure, we are now able to generate the corresponding results by means of the simple three-form

$$U = \frac{1}{2} \theta^\alpha \wedge \tilde{M}_{\alpha} = - \frac{i e}{4mc} (\bar{\psi} \sigma \wedge D\psi - \bar{D}\bar{\psi} \wedge \sigma \Psi).$$  

(69)

If we substitute it into (56)-(57), we find, indeed,

$$\tilde{X}_{\alpha} = \tilde{M}_{\alpha}, \quad \text{and} \quad \tilde{Y}_{\alpha\beta} = M_{\alpha\beta},$$  

(70)

compare with (49) and (68). It is remarkable that the translational moment $\tilde{M}_{\alpha}$, via $U$, also generates the corresponding Lorentz moment $M_{\alpha\beta}$.

Moreover, the same three-form $U$ generates the relocalization of the isospin current $J_K$,

$$J_K \to J_K - D\tilde{Z}_K,$$  

(71)

where

$$\tilde{Z}_K = \frac{e}{\hbar} \tau_K \tilde{\Psi} \frac{\partial U}{\partial D\tilde{\Psi}}.$$  

(72)

The proof goes along the same lines as above. We just formulate the relevant Noether identity which arises from the invariance of the three-form $U$ with respect to the gauge transformations under consideration. Substituting (69) into (72), we recover the polarization moment two-form

$$\tilde{Z}_K = P_K = -i \frac{e\hbar}{2mc} \bar{\Psi} \tau_K \sigma \bar{\Psi}.$$  

(73)

6 Belinfante symmetrization of the energy-momentum current

A simple way, within special relativity, to arrive at the “generic” symmetric energy-momentum current, i.e., at the Hilbert current, is to require that the

\footnote{In previous papers, see [31], this Lagrangian prescription was used “on shell” to generate the transition to chiral fermions. However, in the massless limit these fields carry no spin but rather helicity.}
relocalized spin current vanishes. This is what the Belinfante-Rosenfeld symmetrization amounts to. Therefore the Belinfante-Rosenfeld energy-momentum current $t_\alpha$ can be defined as

$$t_\alpha := \hat{\Sigma}_\alpha(X) \quad \text{with} \quad \hat{\tau}_{\alpha\beta}(X, Y) = 0.$$  \hfill (74)

The last equation, together with (46), yields $\tau_{\alpha\beta} = \vartheta_{[\alpha} \wedge X_{\beta]} + DY_{\alpha\beta}$ which can be resolved with respect to the superpotential $X^\beta$ as follows:

$$X^\beta = \mu^\beta - 2e_{\gamma\delta}DY_{\gamma\delta} - \frac{1}{2} \vartheta^{\beta} \wedge (e_{\gamma\delta}DY_{\gamma\delta}).$$  \hfill (75)

Here

$$\mu^\beta := 2e_{\gamma\delta}\tau^{\gamma\delta} + \frac{1}{2} \vartheta^{\beta} \wedge (e_{\gamma\delta}\tau^{\gamma\delta})$$  \hfill (76)

is the spin energy potential. Then the first Noether law in (47) reads alternatively $Dt_\alpha = 0$.

Let us collect the key formulae for our Belinfante-Rosenfeld current with the specific superpotential $X_\alpha$ of (75)-(76):

$$t_\alpha = \Sigma_\alpha - DX_\alpha, \quad \vartheta_{[\alpha} t_{\beta]} = 0, \quad Dt_\alpha = 0.$$  \hfill (77)

For $Y_{\alpha\beta} = 0$, these are the familiar Belinfante-Rosenfeld relations [26, 27]. For particles with spin zero, the improved energy-momentum current can be derived by a suitable choice of the superpotential $Y_{\alpha\beta}$, cf. [19].

It is remarkable that, for a matter field of any spin, we can find a relocalized Belinfante-Rosenfeld energy-momentum current $t_\alpha$, with $Dt_\alpha = 0$. If we consider the motion of a “test” field in a Minkowski spacetime, then our procedure shows that we can always attach to this motion a geodesic line, irrespective of the spin.

As we saw in the previous section, a relocalization of energy-momentum and spin can be generated by a superpotential three-form $U$. However, the finding of an explicit $U$ for the Belinfante relocalization turns out to be a non-trivial problem. Although the general prescription (75) involves $Y_{\alpha\beta}$, a symmetrization of the energy-momentum is already achieved for $Y_{\alpha\beta} = 0$. Moreover, our discussion here was confined to flat Minkowski geometry; but in Riemannian spacetime (which is the arena of general relativity theory) the Belinfante relocalization necessarily demands $Y_{\alpha\beta} = 0$, as was shown in [32].

Accordingly, a puzzling feature of the Belinfante relocalization for the Dirac energy-momentum is the apparent impossibility of constructing a generating form $U$, with $Y_{\alpha\beta} = 0$. Indeed, for the Dirac field, the spin current is given by (22). Hence the spin energy potential (73) reads:

$$\mu_\alpha = \frac{\hbar}{4} \vartheta_\alpha \wedge \nabla_\gamma \gamma_\delta \Psi.$$  \hfill (78)

From (57) it is clear that $Y_{\alpha\beta} = 0$ if and only if $U$ does not depend on the differentials $D\Psi$, i.e. $\partial U/\partial D\Psi = 0$. Then (78) and (75) yield

$$\mu_\alpha = e_\alpha ]U.$$  \hfill (79)
Contracting with the coframe $\vartheta^\alpha$, we find $\vartheta^\alpha \wedge \mu_\alpha = 3U$. However, for the Dirac spin energy potential (78) we get $\vartheta^\alpha \wedge \mu_\alpha \equiv 0$. Therefore we have to conclude that there is no such three-form $U$ which can generate the Belinfante relocalization in the Dirac theory — provided one starts with the canonical currents (21)-(22). Probably the latter requirement has to be given up. One could start with the convective currents (29) and (35) as well. But we will leave that for future consideration.

7 Properties of the gravitational moments and non-relativistic limit

Let us find out the dimensions of the gravitational moments. Recall that the Dirac fields has dimension $[\Psi] = [\overline{\Psi}] = \text{length}^{-3/2}$, whereas $[\vartheta^\alpha] = \text{length}$, and $[e_\alpha] = \text{length}^{-1}$. Thus, we have for the two-forms $[\overline{\vartheta}] = [\vartheta] = \text{length}^2$, and we immediately get

$$[\tilde{M}_\alpha] = [mc], \quad [M_{\alpha\beta}] = [\hbar]. \quad (80)$$

This is consistent with the analogous result for the polarization moment (12) in the Dirac–Yang-Mills theory, where one finds $[P^\kappa] = [e]$ (with $e$ as the non-Abelian charge or the usual electric charge in the Abelian case). Dimensionwise, the “translational charge”, which defines the translational moment, is thus a momentum and the “Lorentz charge” an angular momentum.

In a remarkable way, the gravitational moments are closely related to the spin of a Dirac particle. The relocalization superpotential $U$ and the translational moment $\tilde{M}_\alpha$ both can be expressed in terms of the convective spin alone via the identities:

$$U \equiv -\eta_{\mu\nu} \wedge \ast \tau^{(c)\mu\nu}, \quad \tilde{M}_\alpha \equiv -\eta_{\alpha\mu\nu} \wedge \ast \tau^{(c)\mu\nu}. \quad (81)$$

Since $\tilde{M}_\alpha$ and $\tau^{(c)\mu\nu}$ have the same number of independent components (namely 24), the last algebraic identity may be inverted, giving the convective spin current in terms of the translational moment.

At first sight, it may be unclear that the Lorentz moment (38) is also related to spin. However, let us consider its square invariant:

$$M_{\alpha\beta} \wedge \ast M^{\alpha\beta} = \left(\frac{\hbar^2}{8mc}\right)^2 \left[-(\overline{\Psi}\Psi)^2 + (\overline{\Psi}\gamma_5\Psi)^2\right] \eta_{\alpha\beta} \wedge \vartheta^\alpha \wedge \vartheta^\beta$$

$$= 3 \left(\frac{\hbar^2}{4mc}\right)^2 (\overline{\Psi}\gamma_5\gamma_5\Psi)(\overline{\Psi}\gamma_5\gamma_5\Psi) \eta$$

$$= \frac{1}{2} \tau_{\alpha\beta} \wedge \ast \tau^{\alpha\beta}. \quad (82)$$

Here, in the first line, we used the representation (39). Then we rearranged the products, which are bilinear in the spinor fields, by means of the Fierz identity. As we recognize, in contrast to the translational moment, the Lorentz moment is directly related to the complete Dirac spin $\tau_{\alpha\beta}$. Note that in applying the
Gordon decomposition non-zero mass was assumed. In the limit of massless Dirac particles, the expression in (82) would vanish. Some further insight can be obtained if we calculate the gravitational moments for specific spinor field configurations. The plane waves

$$\Psi = \Psi_0(p)e^{-\frac{i}{\hbar}p_\alpha x^\alpha},$$

as general solution of the free Dirac equation, are very important in this context. Substituting them into (31) and (39), we find:

$$\tilde{M}_\alpha = p_\alpha \frac{\hbar}{2mc} \overline{\Psi}_0^* \tilde{\sigma} \Psi_0,$$

$$M_{\alpha\beta} = \hbar \frac{\hbar}{8mc} \overline{\Psi}_0 \Psi_0 \eta_{\alpha\beta}.$$  

The (reduced) Compton wavelength $\hbar/mc$ in (84)-(85) evidently provides the correct dimension for these two-forms. In deriving (84), we used the identity which holds for any solution of the Dirac equation (i.e. when eqs. (4)- (5) are satisfied):

$$i \left( \overline{\Psi}^* \tilde{\sigma} \wedge D\Psi - D\overline{\Psi}^* \wedge *\tilde{\sigma} \Psi \right) = *D(\overline{\Psi} \Psi).$$

Since for the plane waves (83) the scalar $\overline{\Psi} \Psi = \overline{\Psi}_0 \Psi_0$ is constant (standard normalization is then $\overline{\Psi}_0 \Psi_0 = (mc/\hbar)^3$), we immediately find that the generating three-form (69) vanishes, $U = 0$.

In the non-relativistic approximation, we get for the translational moment (84):

$$\tilde{M}_\alpha = \frac{p_\alpha \hbar}{m} dt \wedge S + O \left( \frac{v}{c} \right), \quad S := \psi_0^\dagger \tilde{\sigma} \psi_0 \, dx.$$  

This result is a complete analog of (12) for the non-relativistic polarization moment in Dirac-Yang-Mills theory.

At the same time, no further clarification of the structure of the Lorentz moment (85) occurs in the non-relativistic limit. In particular, it is not proportional to the spin one-form $S$, unlike the translational moment $\tilde{M}_\alpha$ and the polarization current $F_K$.

8 Discussion

Inherent in the structure of the “inertial currents” $(\Sigma_\alpha, \tau_{\alpha\beta})$ of the Dirac field, namely of energy-momentum and spin, is the existence of convective and polarization pieces, the latter ones being exterior covariant derivatives of the gravitational moments $(\tilde{M}_\alpha, M_{\alpha\beta})$ of translational and Lorentz type, respectively. This discovery is made on the level of special-relativistic field theory, i.e., in (flat) Minkowski spacetime. In other words, we were able to identify the gravitational moments of the Dirac field without any involvement of the gravitational field itself. Rather, we only assumed that the canonical energy-momentum and spin currents are the sources of gravity. With this assumption, which is in accord
with the Einstein-Cartan theory of gravity, we can tell from our results that the field strengths of gravity have to be represented by two two-forms with the following structure: \((T_{\alpha}, R_{\alpha\beta} = -R_{\beta\alpha})\). In a future paper we will show, as a final proof of our conception, that the moments \((M_{\alpha}, M_{\alpha\beta})\) couple, indeed, to the field strengths \((T_{\alpha}, R_{\alpha\beta})\), provided the field strengths are interpreted as torsion and curvature of spacetime.

9 Acknowledgments

This work was partially supported by CONACyT, grant No. 3544–E9311, and by the joint German-Mexican project KFA-Conacyt E130-2924 and DLR-Conacyt 6.B0A.6A. Moreover, EWM acknowledges support by the short-term fellowship 9616160156 of the DAAD (Bonn) and YNO by the project He 528/17-2 of the DFG (Bonn).

10 Appendix

Our general notation is as follows: The spacetime is four–dimensional with a metric \(g\) of signature \((+,-,-,-)\). A local frame is denoted by \(e_\alpha\) \((\alpha = 0, 1, 2, 3; \Xi = 1, 2, 3)\) and the dual coframe by \(\varpi_\alpha\). They fulfill the relation \(e_\alpha \parallel \varpi_\beta = \delta_\beta^\alpha\), with \(\parallel\) denoting the interior product. For a holonomic or coordinate basis, we have \(d\varpi_\alpha = 0\); then there exists a local coordinate system \(\{x^i = (x^0, x^a)\} (i = 0, 1, 2, 3; a = 1, 2, 3)\) such that \(e_\alpha = \delta_i^\alpha \partial_i\) and \(\varpi_\alpha = \delta_i^\alpha dx^i\). Anholonomic indices are always taken from the Greek and holonomic indices from the Latin alphabet. The Hodge star operator is denoted by \(*\). Let \(\eta := *1\) be the volume four-form. The following forms span the exterior algebra at each point of spacetime:

\[
\eta_\alpha := e_\alpha \parallel \eta = *\partial_\alpha, \quad (88)
\]

\[
\eta_{\alpha\beta} := e_\beta \parallel \eta_{\alpha} = *(\partial_\alpha \wedge \partial_\beta), \quad (89)
\]

\[
\eta_{\alpha\beta\gamma} := e_\gamma \parallel \eta_{\alpha\beta} = *(\partial_\alpha \wedge \partial_\beta \wedge \partial_\gamma), \quad (90)
\]

\[
\eta_{\alpha\beta\gamma\delta} := e_\delta \parallel \eta_{\alpha\beta\gamma} = *(\partial_\alpha \wedge \partial_\beta \wedge \partial_\gamma \wedge \partial_\delta). \quad (91)
\]

For the flat metric of Minkowski spacetime, \(o_{\alpha\beta} = \text{diag}(+1, -1, -1, -1)\), we choose the Dirac matrices in the form:

\[
\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^a = \begin{pmatrix} 0 & \sigma^a \\ -\sigma^a & 0 \end{pmatrix}, \quad a = 1, 2, 3. \quad (92)
\]

Here \(\sigma^a\) are the standard \(2 \times 2\) Pauli matrices. Two important elements of the Dirac algebra are:

\[
\tilde{\sigma}_{\alpha\beta} := \frac{i}{2}(\gamma_\alpha \gamma_\beta - \gamma_\beta \gamma_\alpha), \quad (93)
\]

\[
\gamma_5 := -\frac{i}{4!} \eta_{\alpha\beta\gamma\delta} \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \quad (94)
\]
It is convenient to convert the constant $\gamma_{\alpha}$ matrices into Clifford algebra-valued one- or three-forms, respectively:

$$
\gamma := \gamma_{\alpha} \partial^\alpha, \quad \star \gamma = \gamma^\alpha \eta_{\alpha}.
$$

(95)

Correspondingly, we obtain a two-form:

$$
\hat{\sigma} := \frac{1}{2} \hat{\sigma}_{\alpha\beta} \partial^\alpha \wedge \partial^\beta = \frac{i}{2} \gamma \wedge \gamma.
$$

(96)

Two important identities hold for these Clifford algebra-valued objects:

$$
\hat{\sigma}_{\alpha\beta} \star \hat{\sigma} = \eta_{\alpha\beta} - i \gamma_5 \partial_{\alpha} \wedge \partial_{\beta} - 2i \partial_{[\alpha} \wedge \epsilon_{\beta]} \star \hat{\sigma},
$$

(97)

$$
\star \hat{\sigma} \hat{\sigma}_{\alpha\beta} = \eta_{\alpha\beta} - i \gamma_5 \partial_{\alpha} \wedge \partial_{\beta} + 2i \partial_{[\alpha} \wedge \epsilon_{\beta]} \star \hat{\sigma}.
$$

(98)

References

[1] R. Casalbuoni: “On the quantization of systems with anticommuting variables”, Nuovo Cimento A33 (1976) 115-125; “The classical mechanics for Bose-Fermi systems”, Nuovo Cimento A33 (1976) 389-431.

[2] J.L. Martin: “The Feynman principle for a Fermi system”, Proc. Roy. Soc. (London) 251A (1959) 543-549.

[3] P. Ramond: “Dual theory for free fermions”, Phys. Rev. D3 (1971) 2415-2418.

[4] M.B. Green, J.H. Schwarz, and E. Witten: Superstring Theory, 2 volumes (Cambridge University Press: Cambridge, 1987).

[5] L. Corwin, Y. Ne’eman, and S. Sternberg: “Graded Lie algebras in mathematics and physics (Bose-Fermi symmetry)”, Rev. Mod. Phys. 47 (1975) 573-603.

[6] R. Jost: The General Theory of Quantized Fields (Amer. Math. Soc.: Providence, Rhode Island, 1965) p. 39.

[7] E. Wigner: “On unitary representations of the inhomogeneous Lorentz group”, Ann. of Math. 40 (1939) 149-204.

[8] T.T. Chou, C.N. Yang: “Hadronic matter current distribution inside a polarized nucleus and a polarized hadron”, Nucl. Phys. B107 (1976) 1-20.

[9] I.Yu. Kobzarev and L.B. Okun: “Gravitational interaction of fermions”, Sov. Phys. JETP 16 (1963) 1343-1346 [ZhETF 43 (1962) 1904-1909 (in Russian)].

[10] B. Thaller: The Dirac Equation (Springer: Berlin, 1992).
[11] W. Gordon: “Der Strom der Diracschen Elektronentheorie”, Z. Physik **50** (1928) 630-632.

[12] P. von der Heyde: “Is gravitation mediated by the torsion of space-time?”, Z. Naturforsch. **31a** (1976) 1725-1726.

[13] F.W. Hehl: “Four lectures on Poincaré gauge field theory”, in: *Proc. of the 6th Course of Internat. School on Cosmology and Gravitation: Spin, Torsion, Rotation and Supergravity* (Erice, 1979) P.G.Bergmann and V.De Sabbata, eds. (Plenum: New York, 1980) 5-61.

[14] J. Audretsch: “Dirac electron in space-times with torsion: Spinor propagation, spin precession, and nongeodesic orbits”, Phys. Rev. **D24** (1981) 1470-1477; erratum **D25** (1982) 605.

[15] M. Seitz: “The gravitational moment densities of classical matter fields with spin”, Ann. Physik (Leipzig) **41** (1984) 280-290.

[16] M. Seitz: “A quadratic Lagrangian for the Poincaré gauge field theory of gravity”, Class. Quantum Grav. **3** (1986) 175-182.

[17] F.W. Hehl and W.-T. Ni: “Inertial effects of a Dirac particle”, Phys. Rev. **D42** (1990) 2045-2048.

[18] F.W. Hehl, J. Lemke, and E.W. Mielke: “Two lectures on fermions and gravity”, in: *Geometry and Theoretical Physics* (Bad Honnef School, 1990) J. Debrus and A.C.Hirshfeld, eds. (Springer: Berlin, 1992) 56-140.

[19] F.W. Hehl and E.W. Mielke: “Improved expressions for the energy–momentum current of matter”, Festschrift für E. Schmutzer, Wiss. Zeitschr. Friedrich–Schißler–Universität Jena, Naturw. Reihe **39** (1990) 58-65.

[20] J. Audretsch, F.W. Hehl, and C. Lämmerzahl: “Matter wave interferometry and why quantum objects are fundamental for establishing a gravitational theory”, in: *Relativistiv Gravity Research, Proc., Bad Honnaf, Germany 1991*, J. Ehlers and G. Schäfer, eds., Lecture Notes in Physics (Springer) **410** (1992) 368–407.

[21] J. Lemke: “On the gravitational interaction of elementary particles” (in German). Ph.D. thesis, University of Cologne (1992).

[22] A. Macias, E.W. Mielke, and H.A. Morales–Técotl: “Gravitational–geometric phases and translations”, in: *New Frontiers in Gravitation*, G.A. Sardanashvily, ed. (Hadronic Press: Palm Harbor, Florida, 1996) 227-242.

[23] D. Hilbert: “Die Grundlagen der Physik (Erste Mitteilung)”, Königl. Gesellschaft d. Wiss. Göttingen, Nachr. Math.-Phys. Kl. (1915) 395-407.
[24] A. Einstein: “Hamiltonsches Prinzip und allgemeine Relativitätstheorie”, Sitzber. Königl. Preuss. Akad. Wiss. Berlin (1916) 1111-1116.

[25] F.W. Hehl: “On the energy tensor of spinning massive matter in classical field theory and general relativity”, Rep. on Math. Phys. (Toruń) 9 (1976) 55-82.

[26] F. J. Belinfante: “On the spin angular momentum of mesons”, Physica 6 (1939) 887-898; “On the current and the density of the electric charge, the energy, the linear momentum and the angular momentum of arbitrary fields”, Physica 7 (1940) 449-474.

[27] L. Rosenfeld: “Sur le tenseur d’impulsion-energie” Mém. Acad. Roy. Belgique, cl. sc. 18, fasc. 6 (1940).

[28] J. Hilgevoord and S.A. Wouthuysen: “On the spin angular momentum of the Dirac particle”, Nucl. Phys. 40 (1963) 1-12.

[29] J. Hilgevoord and E.A. De Kerf: “The covariant definition of spin in relativistic quantum field theory”, Physica 31 (1965) 1002-1016.

[30] F.W. Hehl, J.D. McCrea, E.W. Mielke, and Y. Ne’eman, “Metric-affine gauge theory of gravity: field equations, Noether identities, world spinors, and breaking of dilaton invariance”, Phys. Rep. 258 (1995) 1-171.

[31] E.W. Mielke, A. Macías, and H.A. Morales–Técotl: “Chiral fermions coupled to chiral gravity”, Phys. Lett. A215 (1996) 14-20.

[32] E.W. Mielke, F.W. Hehl, and J.D. McCrea: “Belinfante type invariance of the Noether identities in a Riemannian and Weitzenböck space-time”, Phys. Lett. A140 (1989) 368-372.