On hypergeometric functions and $k$-Pochhammer symbol

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Abstract

We introduce the $k$-generalized gamma function $\Gamma_k$, beta function $B_k$, and Pochhammer $k$-symbol $(x)_{n,k}$. We prove several identities generalizing those satisfied by the classical gamma function, beta function and Pochhammer symbol. We provided integral representation for the $\Gamma_k$ and $B_k$ functions.

1 Introduction

The main goal of this paper is to introduce the $k$-gamma function $\Gamma_k$ which is a one parameter deformation of the classical gamma function such that $\Gamma_k \to \Gamma$ as $k \to 1$. Our motivation to introduce $\Gamma_k$ comes from the repeated appearance of expressions of the form

$$x(x+k)(x+2k)\ldots(x+(n-1)k)$$

in a variety of contexts, such as, the combinatorics of creation and annihilation operators [4], [5] and the perturbative computation of Feynman integrals, see [3]. The function of variable $x$ given by formula (1) will be denoted by $(x)_{n,k}$, and will be called the Pochhammer $k$-symbol. Setting $k = 1$ one obtains the usual Pochhammer symbol $(x)_n$, also known as the raising factorial [9], [10].

It is in principle possible to study the Pochhammer $k$-symbol using the gamma function, just as it is done for the case $k = 1$, however one of the main purposes of this paper is to show that it is most natural to relate the Pochhammer $k$-symbol to the $k$-gamma function $\Gamma_k$ to be introduce in section 2. $\Gamma_k$ is given by the formula

$$\Gamma_k(x) = \lim_{n \to \infty} \frac{n!k^n(nk)^{x-1}}{(x)_{n,k}}, \quad k > 0, \quad x \in \mathbb{C} \setminus k\mathbb{Z}^-.$$

The function $\Gamma_k$ restricted to $(0, \infty)$ is characterized by the following properties 1) $\Gamma_k(x+k) = x\Gamma_k(x)$, 2) $\Gamma_k(1) = 1$ and 3) $\Gamma_k(x)$ is logarithmically convex. Notice that the characterization

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above is indeed a generalization of the Bohr-Mollerup theorem [2]. Just as for the usual $\Gamma$ the function $\Gamma_k$ admits an infinite product expression given by

$$\frac{1}{\Gamma_k(x)} = x^{k-x} e^{\frac{x}{k}} \prod_{n=1}^{\infty} \left(1 + \frac{x}{nk}\right) e^{-\frac{x}{nk}}. \quad (2)$$

For $\text{Re}(x) > 0$, the function $\Gamma_k$ is given by the integral

$$\Gamma_k(x) = \int_0^{\infty} t^{x-1} e^{-\frac{k}{t}} dt.$$

We deduce from the steepest descent theorem a $k$-generalization of the famous Stirling’s formula

$$\Gamma_k(x + 1) = (2\pi)^{\frac{1}{2}} (kx)^{-\frac{1}{2}} x^{\frac{x}{k}} e^{-\frac{x}{k}} + O\left(\frac{1}{x}\right), \quad \text{for} \quad x \in \mathbb{R}^+.$$

It is an interesting problem to understand how the function $\Gamma_k$ changes as the parameter $k$ varies. Theorem 11 on section 2 shows that the function $\psi(k,x) = \log \Gamma_k(x)$ is a solution of the non-linear partial differential equation

$$-kx^2 \partial_x^2 \psi + k^3 \partial_k^2 \psi + 2k^2 \partial_k \psi = -x(k + 1).$$

In the last section of this article we study hypergeometric functions from the point of view of the Pochhammer $k$-symbol. We $k$-generalize some well-known identities for hypergeometric functions such as: for any $a \in \mathbb{C}^p$, $k \in (\mathbb{R}^+)^p$, $s \in (\mathbb{R}^+)^q$, $b = (b_1, \ldots, b_q) \in \mathbb{C}^q$ such that $b_i \in \mathbb{C} \setminus s_i \mathbb{Z}^{-}$ the following identity holds

$$F(a, b_s, x)(x) = \frac{1}{\Gamma_k(a)} \int_{(\mathbb{R}^+)^{p+1}} \prod_{j=1}^{p+1} \frac{1}{\Gamma_k(t_j)} \prod_{j=1}^{p+1} e^{-\frac{x}{t_j}} t_j^{a_j-1} \left(\sum_{n=0}^{\infty} \frac{(bx_1 \ldots t_j^{k+1})}{n!} \right) dt, \quad (3)$$

where $(b)_{n,s} = (b_1)_{n_1, s_1} \ldots (b_q)_{n_q, s_q}$, $dt = dt_1 \ldots dt_{p+1}$, $p \leq q$, $\text{Re}(a_j) > 0$ for all $1 \leq j \leq p + 1$, and term-by-term integration is permitted. Our final result Theorem 25 provides combinatorial interpretation in terms of planar forest for the coefficients of hypergeometric functions.

## 2 Pochhammer $k$-symbol and $k$-gamma function

In this section we present the definition of the Pochhammer $k$-symbol and introduce the $k$-analogue of the gamma function. We provided representations for the $\Gamma_k$ function in term of limits, integrals, recursive formulae, and infinite products, as well as a generalization of the Stirling’s formula.

**Definition 1.** Let $x \in \mathbb{C}$, $k \in \mathbb{R}$ and $n \in \mathbb{N}^+$, the Pochhammer $k$-symbol is given by

$$(x)_{n,k} = x(x+k)(x+2k) \ldots (x+(n-1)k).$$
Given \( s, n \in \mathbb{N} \) with \( 0 \leq s \leq n \), the \( s \)-th elementary symmetric function \( \sum_{1 \leq i_1 < \cdots < i_s \leq n} x_{i_1} \cdots x_{i_s} \) on variables \( x_1, \ldots, x_n \) is denoted by \( e^n_s(x_1, \ldots, x_n) \). Part (1) of the next proposition provides a formula for the Pochhammer \( k \)-symbol in terms of the elementary symmetric functions.

**Proposition 2.** The following identities hold

1. \( (x)_{n,k} = \sum_{s=0}^{n-1} e_{s-1}(1, 2, \ldots, n-1)k^s x^{n-s} \).

2. \( \frac{\partial}{\partial k} (x)_{n,k} = \sum_{s=1}^{n-1} s(x)_{s,k}(x + (s + 1)k)_{n-1-s,k} \).

**Proof.** Part (1) follows by induction on \( n \), using the well-known identity for elementary symmetric functions

\[
 e_{s-1}(x_1, \ldots, x_{n-1}) + ne_{s-1}(x_1, \ldots, x_{n-1}) = e^n_s(x_1, \ldots, x_n).
\]

Part (2) follows using the logarithmic derivative. \( \square \)

**Definition 3.** For \( k > 0 \), the \( k \)-gamma function \( \Gamma_k \) is given by

\[
 \Gamma_k(x) = \lim_{n \to \infty} \frac{n!k^n(nk)^{\frac{s-1}{k}}}{(x)_{n,k}}, \quad x \in \mathbb{C} \setminus k\mathbb{Z}^-.
\]

**Proposition 4.** Given \( x \in \mathbb{C} \setminus k\mathbb{Z}^- \), \( k, s > 0 \) and \( n \in \mathbb{N}^+ \), the following identity holds

1. \( (x)_{n,s} = \left( \frac{s}{k} \right)^n \left( \frac{kx}{s} \right)^{\frac{s}{k} - 1} \).

2. \( \Gamma_s(x) = \left( \frac{s}{k} \right)^{\frac{s}{k} - 1} \Gamma_k \left( \frac{kx}{s} \right) \).

**Proposition 5.** For \( x \in \mathbb{C}, \text{Re}(x) > 0 \), we have \( \Gamma_k(x) = \int_0^\infty t^{x-1}e^{-\frac{t}{k}}dt \).

**Proof.** By Definition 3

\[
 \Gamma_k(x) = \int_0^\infty t^{x-1}e^{-\frac{t}{k}}dt = \lim_{n \to \infty} \int_0^{(nk)^{\frac{1}{k}}} \left( 1 - \frac{t}{nk} \right)^n t^{x-1}dt.
\]

Let \( A_{n,i}(x), \ i = 0, \ldots, n \), be given by \( A_{n,i}(x) = \int_0^{(nk)^{\frac{1}{k}}} \left( 1 - \frac{t}{nk} \right)^i t^{x-1}dt \).

The following recursive formula is proven using integration by parts

\[
 A_{n,i}(x) = \frac{i}{n+k} A_{n,i-1}(x + k).
\]
Also,

\[ A_{n,0}(x) = \int_0^{(nk)^{\frac{1}{k}}} t^{x-1} dt = \frac{(nk)^{\frac{x}{k}}}{x}. \]

Therefore,

\[ A_{n,n}(x) = \frac{n!k^n(nk)^{\frac{x-1}{k}}}{(x)_n,k \left(1 + \frac{x}{nk}\right)}, \]

and

\[ \Gamma_k(x) = \lim_{n \to \infty} A_{n,n}(x) = \lim_{n \to \infty} \frac{n!k^n(nk)^{\frac{x-1}{k}}}{(x)_n,k}. \]

Notice that the case \( k = 2 \) is of particular interest since \( \Gamma_2(x) = \int_0^\infty t^{x-1}e^{-t^2} dt \) is the Gaussian integral.

**Proposition 6.** The \( k \)-gamma function \( \Gamma_k(x) \) satisfies the following properties

1. \( \Gamma_k(x + k) = x\Gamma_k(x) \).

2. \( (x)_n,k = \frac{\Gamma_k(x + nk)}{\Gamma_k(x)} \).

3. \( \Gamma_k(k) = 1 \).

4. \( \Gamma_k(x) \) is logarithmically convex, for \( x \in \mathbb{R} \).

5. \( \Gamma_k(x) = a^{\frac{x}{k}} \int_0^\infty t^{x-1} e^{-\frac{t^2}{k}} dt, \) for \( a \in \mathbb{R} \).

6. \( \frac{1}{\Gamma_k(x)} = xk^{-\frac{x}{k}} e^{\frac{x}{k}} \prod_{n=1}^{\infty} \left(1 + \frac{x}{nk}\right) e^{-n\gamma} \) where \( \gamma = \lim_{n \to \infty} \left(1 + \cdots + \frac{1}{n} - \log(n)\right) \).

7. \( \Gamma_k(x)\Gamma_k(k - x) = \frac{\pi}{\sin \left(\frac{\pi x}{k}\right)} \).

**Proof.** Properties 2), 3) and 5) follow directly from definition. Property 4) is Corollary 12 below. 1), 6) and 7) follows from \( \Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma_k\left(\frac{x}{k}\right) \).

Our next result is a generalization of the Bohr-Mollerup theorem.

**Theorem 7.** Let \( f(x) \) be a positive valued function defined on \((0, \infty)\). Assume that \( f(k) = 1 \), \( f(x + k) = xf(x) \) and \( f \) is logarithmically convex, then \( f(x) = \Gamma_k(x) \), for all \( x \in (0, \infty) \).
Proof. Identity \( f(x) = \Gamma_k(x) \) holds if and only if \( \lim_{n \to \infty} \frac{(x)_{n,k} f(x)}{n! k^n (nk)^{\frac{x}{k}-1}} = 1 \). Equivalently,

\[
\lim_{n \to \infty} \log \left( \frac{(x)_{n,k}}{n! k^n (nk)^{\frac{x}{k}-1}} \right) + \log(f(x)) = 0.
\]

Since \( f \) is logarithmically convex the following inequality holds

\[
\frac{1}{k} \log \left( \frac{f(nk + k)}{f(nk)} \right) \leq \frac{1}{x} \log \left( \frac{f(nk + k + x)}{f(nk + k)} \right) \leq \frac{1}{k} \log \left( \frac{f(nk + 2k)}{f(nk + k)} \right).
\]

As \( f(x + k) = xf(x) \), we have

\[
\frac{x}{k} \log(nk) \leq \log \left( \frac{(x + nk)(x + (n - 1)k) \ldots xf(x)}{n! k^n} \right) \leq \frac{x}{k} \log((n + 1)k)
\]

\[
\log(nk)^{\frac{x}{k}} \leq \log \left( \frac{(x + nk)(x + (n - 1)k) \ldots xf(x)}{n! k^n} \right) \leq \log((n + 1)k)^{\frac{x}{k}}
\]

\[
0 \leq \log \left( \frac{(x + nk)(x + (n - 1)k) \ldots xf(x)}{(nk)^{\frac{x}{k}} n! k^n} \right) \leq \log \left( \frac{(n + 1)k}{nk} \right)^{\frac{x}{k}}
\]

\[
0 \leq \lim_{n \to \infty} \log \left( \frac{(x + nk)(x + (n - 1)k) \ldots xf(x)}{(nk)^{\frac{x}{k}} n! k^n} \right) \leq \lim_{n \to \infty} \log \left( \frac{(n + 1)k}{nk} \right)^{\frac{x}{k}}.
\]

Since

\[
\lim_{n \to \infty} \log \left( \frac{(n + 1)k}{nk} \right)^{\frac{x}{k}} = \frac{x}{k} \log(1) = 0,
\]

we get

\[
0 \leq \lim_{n \to \infty} \log \left( \frac{(x + nk)(x + (n - 1)k) \ldots xf(x)}{(nk)^{\frac{x}{k}} n! k^n} \right) + \log(f(x)) \leq 0.
\]

Therefore, \( f(x) = \Gamma_k(x) \).

A proof of Theorem 8 below may be found in [7].

**Theorem 8.** Assume that \( f : (a, b) \to \mathbb{R} \), with \( a, b \in [0, \infty) \) attains a global minimum at a unique point \( c \in (a, b) \), such that \( f''(c) > 0 \). Then one has

\[
\int_a^b g(x) e^{-\frac{f(x)}{k}} dx = h^\frac{x}{k} e^{-\frac{f(c)}{k}} \sqrt{2\pi} \frac{g(c)}{\sqrt{f''(c)}} + O(h).
\]

As promised in the introduction, we now provide an analogue of the Stirling’s formula for \( \Gamma_k \).

**Theorem 9.** For \( \text{Re}(x) > 0 \), the following identity holds

\[
\Gamma_k(x + 1) = (2\pi)^{\frac{x}{k}} (kx)^{-\frac{1}{2} x - \frac{x}{k}} e^{-\frac{x}{k}} + O\left( \frac{1}{x} \right), \tag{4}
\]

As promised in the introduction, we now provide an analogue of the Stirling’s formula for \( \Gamma_k \).

**Theorem 9.** For \( \text{Re}(x) > 0 \), the following identity holds

\[
\Gamma_k(x + 1) = (2\pi)^{\frac{x}{k}} (kx)^{-\frac{1}{2} x - \frac{x}{k}} e^{-\frac{x}{k}} + O\left( \frac{1}{x} \right), \tag{4}
\]
Proof. Recall that $\Gamma_k(x + 1) = \int_0^\infty t^x e^{-\frac{t}{k}} dt$. Consider the following change of variables $t = x \frac{1}{k} v$, we get

$$\Gamma_k(x + 1) = \int_0^\infty v^x e^{-\frac{(xv)^k}{k}} dv = \int_0^\infty e^{-x\left(\frac{v^k}{k} - \log v\right)} dv.$$ 

Let $f(s) = \frac{s^k}{k} - \log(s)$. Clearly $f'(s) = 0$ if and only if $s = 1$. Also $f''(1) = k$. Using Theorem 8, we have

$$\int_0^\infty v^x e^{-\frac{(xv)^k}{k}} dv = \left(2\pi\right)^{\frac{1}{2}} e^{-x^k} + O\left(\frac{1}{x}\right),$$

thus

$$\Gamma_k(x + 1) = \left(2\pi\right)^{\frac{1}{2}} x^\frac{k}{x} e^{-x^k} + O\left(\frac{1}{x}\right).$$

\[\square\]

Proposition 10 and Theorem 11 bellow provide information on the dependence of $\Gamma_k$ on the parameter $k$.

**Proposition 10.** For $\Re(x) > 0$, the following identity holds

$$\partial_k \Gamma_k(x + 1) = \frac{1}{k^2} \Gamma_k(x + k + 1) - \frac{1}{k} \int_0^\infty t^{x+k} \log(t) e^{-\frac{tk}{k}} dt.$$ 

**Proof.** Follows from formula

$$\Gamma_k(x + 1) = \int_0^\infty t^x e^{-\frac{t}{k}} dt.$$

\[\square\]

**Theorem 11.** For $x > 0$, the function $\psi(k, x) = \log \Gamma_k(x)$ is a solution of the non-linear partial differential equation

$$-kx^2 \partial^2 x \psi + k^3 \partial^2 x^2 \psi + 2k^2 \partial x \psi = -x(k + 1).$$

**Proof.** Starting from

$$\frac{1}{\Gamma_k(x)} = xk^{\frac{x}{k}} e^{\frac{x}{k}} \prod_{n=1}^\infty \left(1 + \frac{x}{nk}\right) e^{\frac{x}{nk}}.$$
The following equations can be proven easily.

\[
\psi(k, x) = -\log(x) + \frac{x}{k}\log(k) - \frac{x}{k}\gamma - \sum_{n=1}^{\infty} \left( \log \left( 1 + \frac{x}{nk} \right) - \frac{x}{nk} \right).
\]

\[
\partial_x \psi(k, x) = -\frac{1}{x} + \frac{\log(k) - \gamma}{k} - \sum_{n=1}^{\infty} \left( \frac{1}{x + nk} - \frac{1}{nk} \right).
\]

\[
\partial^2_x \psi(k, x) = \sum_{n=0}^{\infty} \frac{1}{(x + nk)^2}.
\]

\[
\partial_k \psi(k, x) = \frac{x}{k^2} \left( (1 - \log k + \gamma) + \sum_{n=1}^{\infty} \left( \frac{k}{x + nk} - \frac{1}{n} \right) \right).
\]

\[
\partial_k (k^2 \partial_k \psi(k, x)) = -\frac{x}{k} + \sum_{n=1}^{\infty} \frac{x^2}{(x + nk)^2}.
\]

The third equation above shows

**Corollary 12.** The \( k \)-gamma function \( \Gamma_k \) is logarithmically convex on \((0, \infty)\).

We remark that the \( q \)-analogues of the \( k \)-gamma and \( k \)-beta functions has been introduced in \[6\].

## 3 \( k \)-beta and \( k \)-zeta functions

In this section, we introduce the \( k \)-beta function \( B_k \) and the \( k \)-zeta function \( \zeta_k \). We provide explicit formulae that relate the \( k \)-beta \( B_k \) and \( k \)-gamma \( \Gamma_k \), in similar fashion to the classical case.

**Definition 13.** The \( k \)-beta function \( B_k(x, y) \) is given by the formula

\[
B_k(x, y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x + y)}, \quad \text{Re}(x) > 0, \quad \text{Re}(y) > 0.
\]

**Proposition 14.** The \( k \)-beta function satisfies the following identities

1. \( B_k(x, y) = \int_0^{\infty} t^{x-1}(1 + t^k)^{-\frac{x+y}{k}} dt. \)

2. \( B_k(x, y) = \frac{1}{k} \int_0^1 t^{x-1}(1 - t)^{y-1} dt. \)

3. \( B_k(x, y) = \frac{1}{k} B \left( \frac{x}{k}, \frac{y}{k} \right). \)

4. \( B_k(x, y) = \frac{(x + y)}{xy} \prod_{n=0}^{\infty} \frac{nk(nk + x + y)}{(nk + x)(nk + y)}. \)
Definition 15. The $k$-zeta function is given by
\[ \zeta_k(x, s) = \sum_{n=0}^{\infty} \frac{1}{(x + nk)^s}, \text{ for } k, x > 0 \text{ and } s > 1. \]

Theorem 16. The $k$-zeta function satisfies the following identities

1. \( \zeta_k(x, 2) = \partial_x^2 (\log \Gamma_k(x)). \)
2. \( \partial_x^2(\partial_s \zeta_k) \bigg|_{s=0} = -\partial_x^2 (\log \Gamma_k(x)). \)
3. \( \partial_k^m \zeta_k(x, s) = -x(s)_m \sum_{n=0}^{\infty} \frac{n^m}{(x + nk)^{m+s}}. \)

Proof. Follows from equations

\[ \partial_s \zeta_k(x, s) \bigg|_{s=0} = \sum_{n=0}^{\infty} \log(x + nk). \]
\[ \partial_x(\partial_s \zeta_k(x, s)) \bigg|_{s=0} = \sum_{n=0}^{\infty} \frac{1}{(x + nk)}. \]
\[ \partial_x^2(\partial_s \zeta_k(x, s)) \bigg|_{s=0} = -\sum_{n=0}^{\infty} \frac{1}{(x + nk)^2}. \]

4 Hypergeometric Functions

In this section we strongly follow the ideas and notations of [1]. We study hypergeometric functions, see [1] and [8] for an introduction, from the point of view of the Pochhammer $k$-symbol.

Definition 17. Given \( a \in \mathbb{C}^p, k \in (\mathbb{R}^+)^p, s \in (\mathbb{R}^+)^q, b = (b_1, \ldots, b_q) \in \mathbb{C}^q \) such that \( b_i \in \mathbb{C} \setminus s_i \mathbb{Z}^- \). The hypergeometric function \( F(a, k, b, s) \) is given by the formal power series

\[ F(a, k, b, s)(x) = \sum_{n=0}^{\infty} \frac{(a_1)_{n,k_1}(a_2)_{n,k_2} \cdots (a_p)_{n,k_p}}{(b_1)_{n,s_1}(b_2)_{n,s_2} \cdots (b_q)_{n,s_q}} x^n n!. \]

(5)

Given \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \), we set \( \overline{x} = x_1 \ldots x_n \). Using the radio test one can show that the series converges for all \( x \) if \( p \leq q \). If \( p > q + 1 \) the series diverges, and if \( p = q + 1 \), it converges for all \( x \) such that \( |x| < \frac{s_1 \cdots s_q}{k_1 \cdots k_p} \). Also it is easy to check that the hypergeometric function \( y(x) = F(a, k, b, s)(x) \) solves the equation

\[ D(s_1 D + b_1 - s_1) \cdots (s_q D + b_q - s_q)(y) = x(k_1 D + a_1) \cdots (k_p D + a_p)(y), \]

where \( D = x \partial_x \).
Notice that hypergeometric function $F(a, 1, b, 1)$ is given by

$$F(a, 1, b, 1)(x) = \sum_{n=0}^{\infty} \frac{(a)_n \cdots (a_p)_n}{(b)_n \cdots (b_q)_n} \frac{x^n}{n!},$$

and thus agrees with the classical expression for hypergeometric functions. We now show how to transfer from the classical notation for hypergeometric functions to our notation using the Pochhammer $k$-symbol.

**Proposition 18.** Given $a \in \mathbb{C}^p$, $k \in (\mathbb{R}^+)^p$, $s \in (\mathbb{R}^+)^q$, $b = (b_1, \ldots, b_q) \in \mathbb{C}^q$ such that $b_i \in \mathbb{C} \setminus s_i \mathbb{Z}^-$, the following identity holds

$$F(a, k, b, s)(x) = F\left(\frac{a}{k}, 1, \frac{b}{s}, 1\right) \left(\frac{x^k}{s}\right),$$

where $\frac{a}{k} = \left(\frac{a_1}{k_1}, \ldots, \frac{a_p}{k_p}\right)$, $\frac{b}{s} = \left(\frac{b_1}{s_1}, \ldots, \frac{b_q}{s_q}\right)$ and $1 = (1, \ldots, 1)$.

**Proof.**

$$F(a, k, b, s)(x) = \sum_{n=0}^{\infty} \frac{(a)_n k}{(b)_n s} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{(\frac{a}{k})_n}{(\frac{b}{s})_n} \frac{x^{nk}}{n!} = F\left(\frac{a}{k}, 1, \frac{b}{s}, 1\right) \left(\frac{x^k}{s}\right).$$

**Example 19.** For any $a \in \mathbb{C}$, $k > 0$ and $|x| < \frac{1}{k}$, the following identity holds

$$\sum_{n=0}^{\infty} \frac{(a)_{n,k}}{n!} x^n = (1 - kx)^{-\frac{a}{k}}. \tag{6}$$

We next provide an integral representation for the hypergeometric function $F(a, k, b, s)$. Let us first prove a proposition that we will be needed to obtain the integral representation. Given $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ we denote $x_{\leq i} = (x_1, \ldots, x_i)$.

**Proposition 20.** Let $a, k, b, s$ be as in Definition 17. The following identity holds

$$F(a, k, b, s)(x) = \frac{1}{\Gamma_{k+1}(a_{p+1})} \int_0^{\infty} e^{-\frac{k}{p+1} t^{a_{p+1}} - 1} F(a_{\leq p}, k_{\leq p}, b, s)(x t^{k_{p+1}}) dt \tag{7}$$

when $p \leq q$, Re$(a_{p+1}) > 0$, and term-by-term integration is permitted.

**Proof.**

$$\int_0^{\infty} e^{-\frac{k}{p+1} t^{a_{p+1}} - 1} F(a_{\leq p}, k_{\leq p}, b, s)(x t^{k_{p+1}}) dt = F(a_{\leq p}, k_{\leq p}, b, s)(x) \int_0^{\infty} e^{-\frac{k}{p+1} t^{a_{p+1} + nk_{p+1}} - 1} dt = \Gamma_{k_{p+1}}(a_{p+1}) F(a, k, b, s)(x)$$

\[ \square \]
Theorem 21. For any \(a, k, b, s\) be as in Definition 17. The following formula holds

\[
F(a, k, b, s)(x) = \frac{1}{\Gamma_k(a_j)} \int_{(\mathbb{R}^+)_{p+1}} p \, \prod_{j=1}^{p+1} e^{k_j t_j^{a_j-1}} \left( \sum_{n=0}^{\infty} \frac{1}{(b)_{n,s}} \frac{(xt_1^1 \cdots t_{p+1}^{k_j})^n}{n!} \right) dt,
\]

where \((b)_{n,s} = (b_1)_{n,s_1} \cdots (b_q)_{n,s_q}\), \(dt = dt_1 \cdots dt_{p+1}\), \(p \leq q\), \(\text{Re}(a_j) > 0\) for all \(1 \leq j \leq p+1\), and term-by-term integration is permitted.

Proof. Use equation (7) and induction on \(p\). \(\square\)

Example 22. For \(k = (2, 2, \ldots, 2)\), the hypergeometric function \(F(a, 2, b, s)(x)\) is given by

\[
F(a, 2, b, s) = \prod_{j=1}^{p+1} \frac{1}{\Gamma_2(a_j)} \int_{(\mathbb{R}^+)_{p+1}} p \, \prod_{j=1}^{p+1} e^{2 t_j^{a_j-1}} \left( \sum_{n=0}^{\infty} \frac{1}{(b)_{n,s}} \frac{(xt_1^1 \cdots t_{p+1}^{k_j})^n}{n!} \right) dt,
\]

where \(dt = dt_1 \cdots dt_n\), \((b)_{n,s} = (b_1)_{n,s_1} \cdots (b_q)_{n,s_q}\), \(\text{Re}(a_j) > 0\) for all \(1 \leq j \leq p+1\) and term-by-term integration is permitted.

We now proceed to study the combinatorial interpretation of the coefficient of hypergeometric functions.

Definition 23. A planar forest \(F\) consist of the following data:

1. A finite totally order set \(V_r(F) = \{r_1 < \cdots < r_m\}\) whose elements are called roots.
2. A finite totally order set \(V_i(F) = \{v_1 < \cdots < v_n\}\) whose elements are called internal vertices.
3. A finite set \(V_t(F)\) whose elements are called tail vertices.
4. A map \(N : V(T) \to V(T)\).
5. Total order on \(N^{-1}(v)\) for each \(v \in V(F) := V_r(F) \cup V_i(F) \cup V_t(F)\).

These data satisfies the following properties:

- \(N(r_j) = r_j\), for all \(j = 1, \ldots, m\) and \(N^k(v) = r_j\) for some \(j = 1, \ldots, m\) and any \(k \gg 1\).
- \(N(V(F)) \cap V_t(F) = \emptyset\).
- For any \(r_j \in V_r(F)\), there is an unique \(v \in V(F)\), \(v \neq r_j\) such that \(N(v) = r_j\).

Definition 24. a) For any \(a, k \in \mathbb{N}^+\), \(G_{n,k}^a\) denotes the set of isomorphisms classes of planar forest \(F\) such that

1. \(V_r(F) = \{r_1 < \cdots < r_a\}\).
2. \(V_t(F) = \{v_1 < \cdots < v_n\}\).
3. $|N^{-1}(v_i)| = k + 1$ for all $v_i \in V_i(F)$.

4. If $N(v_i) = v_j$, then $i < j$.

b) For any $a, k \in (\mathbb{N}^+)^p$, we set $G_{n,k}^a = G_{n,k_1}^{a_1} \times \cdots \times G_{n,k_p}^{a_p}$.

Figure 1 provides an example of an element of $G_{9,2}^3$.

![Figure 1: Example of a forest in $G_{9,2}^3$.](image)

**Theorem 25.** Given $a, k \in (\mathbb{N}^+)^p$, $b, s \in (\mathbb{N}^+)^q$ and $n \in \mathbb{N}^+$, we have

$$\frac{\partial^n}{\partial x^n} F(a, k; b, s)(x)\big|_{x=0} = \frac{|G_{n,k}^a|}{|G_{b,s}^a|}.$$

**Proof.** It enough to show that $(a)_{n,k} = |G_{n,k}^a|$, for any $a, k, n \in \mathbb{N}^+$. We use induction on $n$. Since $(a)_{1,k} = a$ and $(a)_{n+1,k} = (a)_{n,k}(a + nk)$, we have to check that $|G_{1,k}^a| = a$, which is obvious from Figure 2 and $|G_{n+1,k}^a| = |G_{n,k}^a|(a + nk)$. It should be clear that the any forest in $G_{n+1,k}^a$ is obtained from a forest $F$ in $G_{n,k}^a$, by attaching a new vertex $v_{n+1}$ to a tail of $F$, see Figure 3. One can prove easily that $|V_i(F)| = a + nk$, for all $F \in G_{n,k}^a$. Therefore $|G_{n+1,k}^a| = |G_{n,k}^a|(a + nk)$.

![Figure 2: Example of a forest in $G_{1,4}^a$.](image)
Figure 3: Attaching vertex $v_{n+1}$ to a forest in $G_{n,k}^a$.

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