Quantum states experimentally achieving high-fidelity transmission over a spin chain

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Quantum information processing (QIP) often requires the transfer of known or unknown quantum states from one subspace to another within an information processing device. In recent years, the quantum spin chain has become a prime candidate for quantum communication purposes such as these [1, 2]. In the simplest configuration, where the nearest neighbor couplings are considered to be equal, perfect state transmission is typically not possible between two single spin processors within a linear chain. In other words, there is typically a non-vanishing probability that the initial excitation amplitude can be found outside the receiving spin location [3] at any given time. In principle, however, perfect state transfer (PST) can be realized by properly engineering the couplings between neighboring sites [3]. High fidelity state transmissions can also be obtained using weakly coupled external qubits [4, 5], modifying only one or two couplings [6, 7], or by encoding the states using multiple spins [8, 9]. In [10, 14], a class of states were found to transfer very well across long XY coupled spin chains. The existence of PST has also been established for a variety of interacting media, including, but not limited to, the spin chain model [15]. Recently, exact state swap through a spin ring has been investigated. It was shown that there is a straightforward approach to calculating the probability of the occurrence of an exact state swap [16].

The schemes developed in Ref. [15] prompted the following question. Given an arbitrary spin chain Hamiltonian, can we find initial states which can be used to enable high-fidelity state transmission? In this letter, we answer this question and show that for a uniformly coupled chain, there exists a particular state which reliably transfers quantum information over large distances. We use a multi-spin encoding scheme and find the existence of a three-spin encoding which can provide reliable state transmission. In this case, the encoding and decoding processes can also be realized easily [13]. This report is therefore important from an experimental perspective due to the ease of implementation which is typically favorable.

The method for identifying high-fidelity states. — Consider a spin chain consisting of $N$ sites which evolves according to some Hamiltonian $H$ in a single excitation subspace. Suppose for the moment the initial state of our system is $| \Psi(0) \rangle = |1 \rangle_A \otimes |101...1 \rangle \otimes |0 \rangle_B$, where $A$ and $B$ denote separate processors. After the system evolves, the state at time $t$ will be

$$| \Psi(t) \rangle = U(t) |1 \rangle = \exp(-iHt) |\Psi(0)\rangle,$$

where $\hbar$ is taken to be 1 throughout. Suppose that at some time $\tau$ PST occurs, then

$$| \Psi(\tau) \rangle = U(\tau) |1 \rangle = |N \rangle,$$

where $|N \rangle = |0 \rangle_A \otimes |101...1 \rangle \otimes |1 \rangle_B$. We can use a permutation operator $P_{AB}$ to swap all states in $A$ and $B$, then the quantum information can be transferred from $A$ to $B$. The permutation operator can be expressed as:

$$P_{AB} = \sum_{\alpha, \beta} | \alpha A \rangle \langle \beta A | \otimes | \alpha B \rangle \langle \beta B |,$$

where $\alpha, \beta = 1, 2, ..., 2^k$ represent the standard basis for the $k$ qubits located in processors $A$ and $B$. Clearly $P^1_{AB} = P_{AB}$ and $P^2_{AB} = 1$. $| \alpha(\beta) \rangle_A \langle \beta(A(B)) |$ refers to a state $| \alpha(\beta) \rangle$ in processor $A$ ($B$). From Eq. (2)

$$U(\tau) |1 \rangle = P_{AB} |1 \rangle,$$

Then

$$P_{AB} U(\tau) |1 \rangle = W(\tau) |1 \rangle = |1 \rangle.$$
We introduce the unitary operator \( W(\tau) = P_{AB}U(\tau) \). From Eq. (5), if the state \( |1\rangle \) is an eigenvector of the operator \( W \) at time \( \tau \), PST occurs. The eigenvectors of \( W \) reveal information about the possibilities of a specific state transmission. The problem of solving Schrödinger’s equation now becomes a standard eigen-problem of the operator \( W \).

Since \( W(\tau) \) is a unitary operator it has a complete set of orthonormal eigenvectors \( \{ |\Psi_m(0)\rangle \}_\tau \) corresponding to eigenvalues \( \{ E_m \}_\tau \),

\[
W(\tau) |\Psi_m(0)\rangle = E_m |\Psi_m(0)\rangle.
\]

This can also be written as

\[
U(\tau) |\Psi_m(0)\rangle = E_m P^\dagger_{AB} |\Psi_m(0)\rangle,
\]

where \( U(\tau) |\Psi_m(0)\rangle \) is the wave function \( |\Psi_m(\tau)\rangle \) of the system which was initially prepared in the eigenstate \( |\Psi_m(0)\rangle \). If \( |\Psi_m(0)\rangle \) is a product state

\[
|\Psi_m(0)\rangle = |A\rangle \otimes |C\rangle,
\]

with \( |A\rangle \) describing the state of processor A and \( |C\rangle \) describing the rest of the system, we can then obtain

\[
|\Psi_m(\tau)\rangle = E_m P^\dagger_{AB} |A\rangle \otimes |C\rangle = E_m |B\rangle \otimes |C'\rangle.
\]

For the single excitation subspace, if one of the eigenvectors \( |\Psi_m(0)\rangle = |1\rangle \) at time \( \tau \), PST occurs. If the eigenvectors are degenerate, an arbitrary linear superposition of these degenerate states is also suitable for PST. Suppose there are \( L \) degenerate eigenvectors \( |\Psi_l(0)\rangle \) \((l = 1, 2, \ldots, L)\), which have common eigenvalues \( E_L \). The state

\[
|\Psi(0)\rangle = \sum_{l=1}^{L} C_l |\Psi_l(0)\rangle.
\]

is an eigenvector of \( W(\tau) \), where \( C_l \) is an arbitrary number. Our analysis describes a method for finding a state which can realize PST. (Note that these states are not all unique.) For a given Hamiltonian, if we initially prepare the state \( |\Psi(0)\rangle \) as an eigenvector of the operator \( W(\tau) \), then after time \( \tau \) PST occurs.

For state transmission from one end spin 1 to another end spin \( N \), the exchange operator is given by \( P_1 = |1\rangle \langle N| + |N\rangle \langle 1| + P_0 \) which is shown in Fig. 1.

where \( P_0 = \sum_j |j\rangle \langle j| \) \((j \neq 1, N)\). We now considered a particular Hamiltonian, but emphasize that our method can be used for any Hamiltonian not only this example.

An experimentally implementable Hamiltonian.— Now consider the recently implemented Hamiltonian called a double quantum (DQ) Hamiltonian [17]:

\[
H = -\sum_{i=1}^{N-1} J_{i,i+1}(X_iX_{i+1} - Y_iY_{i+1}).
\]

where \( J_{i,i+1} \) denotes the coupling between sites \( i \) and \( i+1 \). This nearest neighbor coupled one-dimensional spin chain can be experimentally implemented using solid-state nuclear magnetic resonance [17–19] in \(^{19}\)F spins in a crystal of fluorapatite ((FAp-Ca\(_5\)(PO\(_4\))\(_3\))F) [18, 19]. The system described by Eq. (11) will exhibit free evolution such that the evolution operator at time \( \tau \) will be \( U(\tau) = \exp[-i\tau H] \). We can diagonalize the Hamiltonian \( H \) such that \( H_A = W^\dagger HW \) in the single excitation subspace. The evolution operator can therefore be expressed by \( U(\tau) = W \exp[-i\tau H_W] \) and the N eigenvectors of \( W(\tau) \) can be obtained as a function of \( \tau \). Furthermore, we consider a natural configuration for a DQ Hamiltonian with open ends. The 3-component of the total for the staggered spins is a conserved quantity, \( |\sum_{i\in odd} Z_i - \sum_{i\in even} Z_i\rangle \). For simplicity, we will only consider the single excitation subspace of the full Hilbert space. In this case the total number of flipped spins is one. The basis for this subspace will be denoted as \(|j\rangle \) which indicates that, after flipping, the even (odd) site spins all of the spins reside in the \(|0\rangle \) \(|1\rangle \) state except for the spin at site \( j \) which is in the \(|1\rangle \) \(|0\rangle \) state. For example, in a \( N = 5 \) site chain, the single excitation subspace will be spanned by \(|1\rangle = |10100\rangle, |2\rangle = |00010\rangle, \ldots\). If we flip the even numbered states we find that the total up spin is actually one. We will use this description throughout this paper.

**Example 1: nonuniform couplings**— We will consider several different coupling configurations with the potential for high-fidelity state transmission and the best results will be provided at the end of our analysis. First as an example, we consider two pre-engineered couplings: (1) weak couplings at both ends, where \( J_{1,2} = J_{N-1,N} = J_0 \) and \( J_{i,i+1} = J \) elsewhere. (2) couplings termed PST, where \( J_{i,i+1} = \sqrt{J(N-i)} \). It is already known that high fidelity \( (F_{max} \approx 1) \) state transmission for the first configuration [6, 7] and perfect fidelity \( (F_{max} = 1) \) for the second configuration can be gained in a spin system [3]. Here we will use these two kinds couplings to show the applicability of our methods.

For a five-spin system with weak couplings at both ends, we take \( J_{1,2} = J_{4,5} = 0.1J \). J equals -1 elsewhere. The eigenvalues and eigenvectors of the operator \( W(\tau) \) at an arbitrary time \( \tau \) can be obtained numerically. We will consider those which span the single-excitation subspace. In Table 1 we plot the results for \( \tau = 31 \). The first col-
The initial state is \( \Phi(0) \rangle \). We take \( N = 5 \), \( J = 1 \). The state \( |1\rangle \) at site 1 can be transferred exactly to site 4 at time \( \tau = 3.14 \). In Fig. 2(b) we plot the time evolution of the fidelity when transferring a state \( |1\rangle \) from site 1 to 4. We also see that at time \( \tau = 3.14 \) the fidelity is nearly 1. These examples illustrate the validity and practicality of our method while providing a general method to obtain the results.

\[
\begin{align*}
|\Psi_1(0)\rangle &= \sqrt{\frac{1}{2}} (|1\rangle - |4\rangle), \\
|\Psi_3(0)\rangle &= \sqrt{\frac{3}{8}} (|1\rangle + |4\rangle) - \sqrt{\frac{1}{8}} (|2\rangle + |3\rangle), \\
|\Psi_4(0)\rangle &= \sqrt{\frac{1}{8}} (|1\rangle + |4\rangle) + \sqrt{\frac{3}{8}} (|2\rangle + |3\rangle).
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state of the first three spins to the opposite end. The results for a \( N = 6 \) site chain are given in Table IV for time \( \tau = 4.0 \). The eigenvalues of 1 and 2 are roughly degenerate and the approximate relation

\[
|\Psi_1(0)\rangle + |\Psi_2(0)\rangle = -|1\rangle + |3\rangle \quad (14)
\]
can be written in the form of a product state \((-|110\rangle + |011\rangle)_{A} \otimes 10\ldots 1\). The state \((-|110\rangle + |011\rangle)/\sqrt{2}\) is therefore suitable for transmission. We have also checked the case where \( N = 7 \) and find that at time \( \tau = 28.8 \) the above states can be obtained again.

| 1   | 2   | 3   | 4   | 5   | 6   |
|-----|-----|-----|-----|-----|-----|
| 1   | 0.117 -0.993 | -0.493 0.005 0.500 0.500 0.005 -0.500 |
| 2   | -0.117 -0.993 | -0.493 -0.005 0.500 -0.500 0.005 0.500 |
| 3   | 0.252 0.968 | -0.275 0.590 -0.275 -0.275 0.590 -0.275 |
| 4   | -0.252 0.968 | 0.275 0.590 0.275 -0.275 -0.275 0.590 -0.275 |
| 5   | 0.544 0.839 | 0.421 0.379 0.405 0.405 0.379 0.421 |
| 6   | -0.540 0.839 | -0.421 0.379 -0.405 0.405 -0.379 0.421 |

TABLE IV: The eigenvalues and corresponding eigenvectors of the operator \( W(\tau) \) at \( \tau = 4.0 \) using a 3 spin encoding. Here \( N = 6 \).

| 1   | 2   | 3   | 4   | 5   |
|-----|-----|-----|-----|-----|
| 1   | 0.999 -0.025 | -0.263 -0.263 0.850 -0.263 -0.263 |
| 2   | -0.999 0.041 | 0.500 -0.500 0.000 -0.500 0.500 |
| 3   | -0.997 0.076 | 0.500 -0.500 0.000 0.500 -0.500 |
| 4   | 0.997 0.076 | 0.500 0.500 0.000 -0.500 -0.500 |
| 5   | 0.998 0.067 | 0.425 0.425 0.526 0.425 0.425 |

TABLE V: The eigenvalues and corresponding eigenvectors of the operator \( W(\tau) \) at \( \tau = 47.2 \) using a 2-spin encoding. Here \( N = 5 \).

![Fig. 3](image-url)

FIG. 3: (Color online.) Length dependence of the maximum fidelity achievable \( F_{\text{max}} \) and the associated arrival times \( T_{\text{max}} \) for the state (a) \((-|110\rangle + |011\rangle)/\sqrt{2}\) and (b) \((|11\rangle - |00\rangle)/\sqrt{2}\). The time is searched within the interval \([0, 50]\).

The state \((-|11\rangle + |00\rangle)/\sqrt{2}\) and \((-|110\rangle + |011\rangle)/\sqrt{2}\) can be transferred with high fidelity across chains of arbitrary length \( N \). In Fig. 3 we plot the maximum fidelity \( F_{\text{max}} \) and the associated arrival time \( T_{\text{max}} \) as a function of chain length \( N \). The analytic expression with eigenvalues \( E_m = -2J \cos(\pi m/(N + 1)) \) and eigenvectors \( |\Psi_m(0)\rangle = \sqrt{2/(N + 1)} \sum_j \sin(q_{jm}) |j\rangle \) are used. For practical implementation of our protocol, the maximum fidelity is found in the time \([0, 50]\). For the two-spin encoding, the high fidelity associated with short chain lengths cannot be achieved with increasing chain length. \( F_{\text{max}} \) quickly decreases with increasing \( N \). However, this robustness can be observed even for long chains using the three-spin encoding. The fidelity is exceptionally large for a relatively long chain. Therefore, using this state, a high-fidelity state transfer can be gained. \( F_{\text{max}} = 0.96 \) for \( N = 6 \) at \( \tau = 4.0 \), \( F_{\text{max}} = 1.00 \) for \( N = 7 \) at \( \tau = 28.8 \) which agrees with our previous analysis \[13, 14\]. Note that we only consider two and three-spin encodings here. For encodings using more than three spins, we conjecture that for odd spin encodings some states can be found to possess high-fidelity transmission even over long chains. From Fig. 3(b) we find that the arrival time \( T_{\text{max}} \) typically increases with increasing chain length \( N \) except for some deviation with small values of \( N \). We also find that the \( T_{\text{max}} \) associated with the three-spin encoding is a little longer than in the two-spin encoding case for \( N > 24 \). This suggests that encodings using larger Hilbert spaces require longer waiting times for the maximum fidelity.

**Conclusions.** In conclusion, we have introduced a method to find states which can be transmitted through spin channels with high fidelity. The method can be easily implemented numerically and can be applied to \( N \) site encodings, with \( N \) arbitrary. Using our method we have provided examples for the DQ Hamiltonian which exhibit uniform and nonuniform exchange couplings. For the uniform chain, a 3-spin encoding \((-|110\rangle + |011\rangle)/\sqrt{2}\) was found to exhibit high fidelity state transmission. Using a simple similarity transformation \[17\], our results can be extended to the standard Heisenberg XY model. In this case we have provided an explanation for the appearance of the class of initial states which were previously discovered \[13, 14\]. These states are exceptional due to the fact that they use simple encodings and transfer extremely well. Our work therefore provides a new method, new results, and an explanation of previously known important results.

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