The Goldman-Turaev Lie bialgebra and the Kashiwara-Vergne problem in higher genera

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Abstract

For a compact oriented surface Σ of genus $g$ with $n+1$ boundary components, the space $g(Σ)$ spanned by free homotopy classes of loops in Σ carries the structure of a Lie bialgebra. The Lie bracket was defined by Goldman and it only depends on the orientation of the surface. The Lie cobracket was defined by Turaev and it depends on the framing of Σ. The Lie bialgebra $g(Σ)$ has a natural decreasing filtration such that the Goldman bracket and the Turaev cobracket have degree $(-2)$.

In this paper, we address the following Goldman-Turaev formality problem: construct a Lie bialgebra homomorphism $θ$ from $g(Σ)$ to its associated graded $gr g(Σ)$ such that $gr θ = id$. In order to solve it, we define a family of higher genus Kashiwara-Vergne (KV) problems for an element $F ∈ Aut(L)$, where $L$ is a free Lie algebra.

In the case of $g = 0$ and $n = 2$, the problem for $F$ is the classical KV problem from Lie theory [17, 4]. For $g > 0$, these KV problems are new.

Our main results are as follows. Every solution of the KV problem induces a Goldman-Turaev formality map. Solving the KV problem reduces to two important cases: $g = 0, n = 2$ which admits solutions by [3], and $g = 1, n = 1$ for which we construct solutions in terms of certain elliptic associators following Enriquez [10]. By combining these two results, we obtain a proof of the Goldman-Turaev formality for every $g$ and $n$.

Furthermore, we introduce pro-unipotent groups $KV^{(g,n+1)}$ and $KRV^{(g,n+1)}$ which act on the space of solutions of the KV problem freely and transitively. Groups $KV^{(g,n+1)}$ also act by automorphisms of $g(Σ)$. There are injective group homomorphisms $GT_1 → KV^{(g,n+1)}$, $GRT_1 → KRV^{(g,n+1)}$ from Grothendieck-Teichmüller groups which depend on the pair of pants decomposition of Σ.

Groups $KRV^{(g,n+1)}$ admit graded pro-nilpotent Lie algebras $krv^{(g,n+1)}$. We show that the elliptic Lie algebra $frv^{(1,1)}$ contains a copy of the Grothendieck-Teichmüller Lie algebra $grt_1$ as well as symplectic derivations $δ_{2n}$. 

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1 Introduction

1.1 Goldman-Turaev Lie bialgebras

Let $K$ be a field of characteristic zero and let $\Sigma$ be a compact connected oriented 2-manifold of genus $g$ with $n+1$ boundary components. We label the boundary components by the set $\{0,1,\ldots,n\}$ and choose the basepoint $*$ on the boundary component with label $0$. The space

$$g(\Sigma) := \frac{K\pi}{[K\pi, K\pi]}$$

is isomorphic to the $K$-span of homotopy classes of free loops in $\Sigma$. We denote by $a \mapsto |a|$ the natural projection $K\pi \rightarrow g(\Sigma)$.

In [12], Goldman showed that the space $g(\Sigma)$ carries a natural structure of a Lie algebra defined in terms of intersections of curves representing elements of $\pi$. This construction was originally motivated by the study of the Atiyah-Bott symplectic structures on moduli spaces of flat connections [5] as well as by the Weil-Petersson symplectic geometry by Wolpert [44]. In more detail, for every $n$ there is a Lie homomorphism

$$(g(\Sigma), [\cdot, \cdot]_{\text{Goldman}}) \rightarrow (\text{Fun}(\mathcal{M}(\Sigma, \text{GL}(n))), \{\cdot, \cdot\}_{AB}),$$

where $\mathcal{M}(\Sigma, \text{GL}(n)) = \text{Hom}(\pi, \text{GL}(n))/\text{GL}(n)$ is the representation variety, $\text{Fun}(\mathcal{M}(\Sigma, \text{GL}(n)))$ is the space of algebraic functions on $\mathcal{M}(\Sigma, \text{GL}(n))$ and $\{\cdot, \cdot\}_{AB}$ is the Atiyah-Bott Poisson structure.

Under the Goldman bracket, the homotopy class of a trivial loop $1 \in g(\Sigma)$ is a central element. Hence, the quotient space

$$\tilde{g}(\Sigma) = g(\Sigma)/K1$$

is also a Lie algebra. In [42], Turaev showed that $\tilde{g}(\Sigma)$ carries a canonical structure of Lie bialgebra where the Lie bracket is (induced by) the Goldman bracket and the Lie cobracket is defined in terms of self-intersections of curves on $\Sigma$.

Let $f : T\Sigma \rightarrow \Sigma \times \mathbb{R}^2$ be a framing on $\Sigma$ (a trivialization of the tangent bundle of $\Sigma$). To each free loop $\alpha$ in $\Sigma$, the framing assigns a rotation number,

$$\alpha \mapsto \text{rot}_f(\alpha) \in \mathbb{Z}.$$ 

A choice of framing defines a Lie bialgebra structure on $g(\Sigma)$ with Lie bracket the Goldman bracket and the Lie cobracket $\delta^f$ which depends on $f$. This lift of Turaev’s Lie cobracket comes from Furuta’s observation on a 1-cocycle of the mapping class group [31, §4]. For every $f$, this Lie bialgebra structure descends to the canonical Lie bialgebra structure on $\tilde{g}(\Sigma)$. In what follows, we sometimes drop the index $f$ if the statement applies to all framings.

1.2 Filtrations and associated graded objects

The Goldman-Turaev (GT) Lie bialgebra $(g(\Sigma), [\cdot, \cdot]_{\text{Goldman}}, \delta)$ carries a natural filtration. In the case of $n = 0$, it can be described as follows: the group ring $K\pi$ carries a
natural decreasing filtration by powers of the augmentation ideal which descends to a filtration on $\mathfrak{g}(\Sigma)$ (for the general case, see Section 3). Under this filtration, the Goldman bracket and the Turaev cobracket have filtration degree $(-2)$.

Denote by $\hat{\mathfrak{g}}(\Sigma)$ the completion of $\mathfrak{g}(\Sigma)$ with respect to the filtration described above. For every framing $f$, the Lie bialgebra structure $(\mathfrak{g}(\Sigma), [\cdot, \cdot]_{\text{Goldman}}, \delta^f)$ extends to $\hat{\mathfrak{g}}(\Sigma)$. The associated graded vector space $\text{gr} \hat{\mathfrak{g}}(\Sigma)$ carries the induced Lie bracket $[\cdot, \cdot]_{\text{gr}}$ and the induced Lie cobracket $\delta^f_{\text{gr}}$. Both these operations are of degree $(-2)$.

The vector space $\text{gr} \hat{\mathfrak{g}}(\Sigma)$ admits a nice explicit description. For $n = 0$, it is as follows: let $H := H_1(\Sigma, \mathbb{K})$ be the first homology group of $\Sigma$ and $A = \hat{T}(H)$ the completed free associative algebra spanned by $H$. Then,

$$\text{gr} \hat{\mathfrak{g}}(\Sigma) \cong A/[A, A],$$

where $[A, A] \subset A$ is the (closed) subspace spanned by commutators. One can also identify $\text{gr} \hat{\mathfrak{g}}(\Sigma)$ with the completed graded vector space spanned by cyclic words in $H$.

The following question naturally arises in the framework that we have described above:

**Formality problem for Goldman-Turaev Lie bialgebras:** find a filtration preserving Lie bialgebra isomorphism $\theta: \hat{\mathfrak{g}}(\Sigma) \to \text{gr} \hat{\mathfrak{g}}(\Sigma)$ such that its associated graded map is the identity: $\text{gr} \theta = \text{id}$.

This question can be addressed for any framing $f$ on $\Sigma$. A positive solution of the GT formality problem for some $f$ implies a positive solution for the canonical GT Lie bialgebra (on the completion of $\hat{\mathfrak{g}}(\Sigma)$).

### 1.3 Known results and applications of Goldman-Turaev formality

#### Expansions

Among the maps $\theta: \hat{\mathfrak{g}}(\Sigma) \to \text{gr} \hat{\mathfrak{g}}(\Sigma)$ there is a class which admits an easy description. Recall that $\mathbb{K}\pi$ and its completion $\hat{\mathbb{K}}\pi$ with respect to the augmentation ideal are filtered Hopf algebras and $A = \hat{T}(H)$ is a graded Hopf algebra. Furthermore, the set of primitive elements in $A$ is the completed free Lie algebra $L = \mathbb{L}(H)$ spanned by $H$.

Let $\theta: \hat{\mathbb{K}}\pi \to A$ be a homomorphism of filtered completed Hopf algebras such that

$$\theta(\gamma) = 1 + [\gamma] + \cdots,$$

where $\gamma \in \pi$, $[\gamma] \in H$ is its homology class and $\cdots$ stand for higher degree terms. Then, $\theta$ induces an isomorphism of filtered vector spaces $\hat{\mathfrak{g}}(\Sigma) \to \text{gr} \hat{\mathfrak{g}}(\Sigma)$ (which we denote by the same letter).

In more detail, choose a set of generators $\alpha_i, \beta_i, i = 1, \ldots, g$ and $\gamma_j, j = 0, \ldots, n$ of $\pi$ represented by simple curves which only intersect at the basepoint $*$ such that $\gamma_j$ is a loop around the boundary component with label $j$ and the defining relation in $\pi$ reads

$$\prod_{i=1}^g \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1} \prod_{j=1}^n \gamma_j = \gamma_0.$$
Denote by $[\alpha_i] = x_i, i = 1, \ldots, g$ and $[\beta_j] = y_j, j = 1, \ldots, n$ the corresponding generators of $H$. Then, there is a unique Hopf algebra homomorphism $\theta_{\exp}$ such that $\theta_{\exp}(\alpha_i) = e^{x_i}, \theta_{\exp}(\beta_i) = e^{y_i}, \theta_{\exp}(\gamma_j) = e^{z_j}$. All other maps $\theta$ can be represented as $\theta_F = F^{-1} \circ \theta_{\exp},$

where $F \in \text{Aut}(L)$ is an automorphism of the free Lie algebra $L$ which induces an automorphism of $A$. Our aim will be to encode the properties of the map $\theta_F$ in terms of the properties of the automorphism $F$.

Formality for Goldman bracket

Before studying formality for GT Lie bialgebras, one can address the following easier question:

**Formality problem for Goldman Lie algebras:** find a filtration preserving Lie algebra isomorphism $\theta : \widehat{\mathfrak{g}}(\Sigma) \to \text{gr} \widehat{\mathfrak{g}}(\Sigma)$ such that its associated graded map is the identity: $\text{gr} \theta = \text{id}.$

This problem admits a nice solution:

**Theorem 1.1.** Let $F \in \text{Aut}(L)$ be such that $F : z_j \mapsto f_j^{-1} z_j f_j$, where $f_j \in A$ is a group-like element, and

$$F : \sum_{i=1}^{g} [x_i, y_i] + \sum_{j=1}^{n} z_j \mapsto \log \left( \prod_{i=1}^{g} e^{x_i} e^{y_i} \prod_{j=1}^{n} e^{z_j} \right).$$

Then, the map $\theta_F$ solves the formality problem for Goldman Lie algebras.

This theorem has several proofs available in the literature: Kawazumi and Kuno [20] establish it by viewing the map $\theta_F$ in terms of relative cohomology of Hopf algebras and by interpreting the Goldman bracket using cup products. Recently, the cup product interpretation was also used by Hain [14] to give a proof based on the theory of Hodge structures. In [27], Massuyeau and Turaev give a proof using the theory of non-degenerate Fox pairings. In [33], Naef gives a proof using the moment maps in non-commutative Poisson geometry.

Theorem 1.1 finds the following application in Poisson geometry. Let $G$ be a connected Lie group such that the corresponding Lie algebra $\text{Lie}(G)$ carries a non-degenerate invariant bilinear form (e.g. the Killing form for $G$ semisimple). Then, every map $F$ as in Theorem 1.1 induces a formal Poisson isomorphism

$$\Theta_F : \text{Hom}(H, \text{Lie}(G))/G \to \mathcal{M}(\Sigma, G),$$

where $\text{Hom}(H, \text{Lie}(G))$ is the space of linear maps from $H$ to $\text{Lie}(G)$. It carries the standard Poisson structure with moment map

$$\mu(x, y, z) = \sum_{i=1}^{g} [x_i, y_i] + \sum_{j=1}^{n} z_j.$$
For \( n = 0 \), this is the symplectic structure induced by the intersection pairing on \( H \) and the scalar product on \( \text{Lie}(G) \). If \( F \) is well chosen, the formal map \( \Theta_F \) may have a non-vanishing convergence radius. Then, it defines a Poisson map between the neighborhood of zero in the quotient space \( \text{Hom}(H,\text{Lie}(G))/G \) and the neighborhood of the trivial connection in the representation variety \( \mathcal{M}(\Sigma,G) \). In the case of \( g = 0 \) and \( G \) compact this gives a new proof of the theorem of Jeffrey \([23]\) on the isomorphism of moduli spaces of flat connections and multiplicity spaces; see \([33]\).

**Goldman-Turaev formality in genus zero**

There are several proofs of the GT formality for genus zero surfaces in the literature. In \([26]\), Massuyeau establishes it using the Kontsevich integral. In \([3]\), Alekseev and Naef gave a proof over \( \mathbb{C} \) using the Knizhnik-Zamolodchikov connection. Recall the idea of the proof that we gave in \([2]\).

We denote by \( a \mapsto |a| \) the natural projection \( A \to A/[A,A] \), and recall that the group \( \text{Aut}(L) \) carries a 1-cocycle
\[
j : \text{Aut}(L) \to A/[A,A].
\]
This cocycle is a non-commutative analogue of the log-Jacobian 1-cocycle in differential geometry. The main result of \([2]\) is the following theorem:

**Theorem 1.2.** Suppose \( g = 0 \). Let \( F \in \text{Aut}(L) \) such that \( F(z_j) = f_j^{-1}z_jf_j \), where \( f_j \in A \) is a group-like element, and
\[
F : z_1 + \cdots + z_n \mapsto \log(e^{z_1} \cdots e^{z_n}), \quad j(F) = \sum_{j=1}^{n} h(z_j) - h(\log(e^{z_1} \cdots e^{z_n}))
\]
where \( h \) is a formal power series in one variable (the Duflo function). Then, the map \( \theta_F \) solves the formality problem for Goldman-Turaev Lie bialgebras for a surface of genus zero with \( n + 1 \) boundary components.

For \( n = 2 \), the conditions on \( F \) coincide with the Kashiwara-Vergne (KV) problem in Lie theory \([17] [4]\). By \([4]\), the KV problem admits solutions parametrized by Drinfeld associators. Furthermore, solutions of the problem for \( n > 2 \) are easily constructed from solutions of the \( n = 2 \) problem.

### 1.4 Main results

The main goal of this paper is to establish the formality property for Goldman-Turaev Lie bialgebras for higher genus surfaces. In order to do it, we need to give more information about framings on \( \Sigma \). We will present the case of \( n = 0 \) (for the complete formulation, see Section \([5,4]\)). Choose a basis of \( \pi \) and define the adapted framing with vanishing rotation numbers: \( \text{rot}_{\text{adp}}(\alpha_i) = \text{rot}_{\text{adp}}(\beta_i) = 0 \). The set of framings on \( \Sigma \) is an affine space over \( H^1(\Sigma,\mathbb{Z}) \). For \( p \in H^1(\Sigma,\mathbb{Z}) \), we associate the framing \( f(p) \) with
\[
\text{rot}_{f(p)}(\alpha) = \text{rot}_{\text{adp}}(\alpha) + \langle p, [\alpha] \rangle.
\]
Note that by the Poincaré duality $H^1(\Sigma, \mathbb{Z}) \cong H_1(\Sigma, \mathbb{Z}) \subset H$, we can view $p$ as an element of $H$. Our first main result is the following theorem:

**Theorem 1.3.** For $g > 1$, let $F \in \text{Aut}(L)$ such that

$$F : \sum_{i=1}^{g}[x_i, y_i] \mapsto \log \prod_{i=1}^{g}[e^{x_i}, e^{y_i}], \quad j(F) = r + p - \log \prod_{i=1}^{g}[e^{x_i}, e^{y_i}], \quad (1)$$

where $h$ is a power series in one variable and $r = \sum_{i=1}^{g}(r(x_i) + r(y_i))$ with $r(s) = \log((e^s - 1)/s)$. Then, $\theta_F$ is a solution of the GT formality problem for a surface of genus $g$ with the framing defined by $p$.

For $g = 1$, let $F \in \text{Aut}(L)$ be a solution of equations (1) with $p = 0$. Then, $\theta_F$ is a solution of the GT formality problem for a surface of genus 1 with the adapted framing.

Equations (1) define a new algebraic problem that we call the higher genus Kashiwara-Vergne problem (for the formulation in the case of $n > 0$, see Section 5.4). This problem has interesting new features in comparison to the classical (genus zero) KV problem: these are explicit dependence on the framing of the surface and presence of the new element $r$ which appears in the equation for $j(F)$. Our second main result is as follows:

**Theorem 1.4.** For $g > 1$, the higher genus KV problem admits solutions for any framing. For $g = 1$, it admits solutions for the adapted framing.

The strategy of the proof of this result is as follows: it turns out that one can reduce the higher genus KV problem for arbitrary $g$ and $n$ to two special cases. The first one is the case of $g = 0$ and $n = 2$ (this is the classical KV problem mentioned above) and the second one is the case of $g = 1$ and $n = 1$. This latter case admits solutions in terms of special elliptic associators defined by Enriquez [10].

Furthermore, we study the uniqueness issue for higher genus KV problems. We introduce groups KV$(g,n+1)$ and KRV$(g,n+1)$ which act freely and transitively by left and right multiplication on the space of solutions of the KV problem for a surface of genus $g$ with $n + 1$ boundary components. Every solution of the KV problem establishes an isomorphism between KV$(g,n+1)$ and KRV$(g,n+1)$.

For certain pair of pants decompositions of $\Sigma$, we define injective group homomorphisms from Grothendieck-Teichmüller groups to Kashiwara-Vergne groups:

$$\text{GT}_1 \rightarrow \text{KV}^{(g,n+1)}, \quad \text{GRT}_1 \rightarrow \text{KRV}^{(g,n+1)}$$

An important property of the Kashiwara-Vergne group is given by the following theorem:

**Theorem 1.5.** The group KV$(g,n+1)$ acts by automorphisms of the Goldman-Turaev Lie bialgebra $\tilde{g}(\Sigma)$. The group KRV$(g,n+1)$ acts by automorphisms of the graded Lie bialgebra gr $\tilde{g}(\Sigma)$.
Theorem 1.5 implies that the group $G_T$ acts by automorphisms of Goldman-Turaev Lie bialgebra $\hat{g}(\Sigma)$. This action depends on the pair of pants decomposition of the surface. In the case of a sphere with three boundary components, this action is independent of choices.

The group $KRV^{(g,n+1)}$ is a pro-unipotent group with graded pro-nilpotent Lie algebra $\mathfrak{trv}^{(g,n+1)}$. Recall that there is a canonical injection $\mathfrak{grt}_1 \to \mathfrak{trv}^{(0,2)}$ of the Grothendieck-Teichmüller Lie algebra to the Kashiwara-Vergne Lie algebra for $g = 0$ and $n = 2$; see [4]. In the case of $g = 1$ and $n = 1$, we prove the following result:

**Theorem 1.6.** There is an injective Lie homomorphism $\mathfrak{grt}_1 \to \mathfrak{trv}^{(1,1)}$. Furthermore, $\delta_{2n} \in \mathfrak{trv}^{(1,1)}$, where $\delta_{2n} \in \text{Der}(L)$ is the unique derivation of the free Lie algebra with generators $x$ and $y$ with the property $\delta_{2n}(x) = \text{ad}_{x}^{2n}(y), \delta_{2n}([x,y]) = 0$.

Inspired by considerations of [10], we conjecture that there is an injection $\mathfrak{grt}_{\text{ell}} \to \mathfrak{trv}^{(1,1)}$ of the elliptic Grothendieck-Teichmüller Lie algebra to the Kashiwara-Vergne Lie algebra for genus 1.

One of motivations for studying the Goldman-Turaev formality comes from its application to the Johnson homomorphisms for the mapping class groups. In this paper, by improving previous results [21] we show that the framed Turaev cobracket gives rise to an obstruction for the image of the extended Johnson homomorphism, and that under the choice of GT formality isomorphism this obstruction becomes homogeneous and coincides with the previously known one given by the Enomoto-Satoh trace [9].

**Acknowledgements** We are grateful to V. Turaev for interesting remarks and suggestions. Research of A.A. and F.N. was supported in part by the grant MODFLAT of the European Research Council (ERC), by the grants number 178794 and number 178828 of the Swiss National Science Foundation (SNSF) and by the NCCR SwissMAP of the SNSF. F.N. was supported by the Early Postdoc.Mobility grant of the SNSF. N.K. was supported in part by the grant JSPS KAKENHI 15H03617. Y.K. was supported in part by the grant JSPS KAKENHI 26800044.

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Convention.
Here we collect our convention used in the rest of the paper.

- For complete tensor product, we always use the symbol $\otimes$ instead of $\hat{\otimes}$.
- Let $U$ and $V$ be (complete) $K$-vector spaces and $u \in U, v \in V$. We denote
  $$(u \otimes v) : = v \otimes u \in V \otimes U.$$  
For the next two items, $\mathfrak{A}$ is a (topological) associative $K$-algebra with unit.
- For $a = a' \otimes a''$ and $b = b' \otimes b'' \in \mathfrak{A} \otimes \mathfrak{A}$, we denote
  $$a \cdot b = ab : = a'b' \otimes a''b'', \quad a \circ b : = a'b' \otimes b''a''.$$
Let $[\mathfrak{A}, \mathfrak{A}]$ be the closure of the $K$-span of the elements of the form $ab - ba$ with $a, b \in \mathfrak{A}$, and set
\[ [\mathfrak{A}] := \mathfrak{A}/[\mathfrak{A}, \mathfrak{A}]. \]
The natural projection is denoted by $\cdot : \mathfrak{A} \to [\mathfrak{A}]$.

Adopting this notation, the Goldman-Turaev Lie bialgebra is given by $g(\Sigma) = [K \pi]$ and its linearized version by $A/[A, A] = [A]$.

In our study, the theory of double brackets in the sense of van den Bergh plays a key role. Here we briefly outline basic constructions to fix our convention.

**Definition 1.7.** Let $\mathfrak{A}$ be an associative $K$-algebra with unit. A $K$-linear map $\Pi : \mathfrak{A} \otimes \mathfrak{A} \to \mathfrak{A} \otimes \mathfrak{A}$ is called a **double bracket** on $\mathfrak{A}$ if for any $a, b, c \in \Pi$,
\[
\begin{align*}
\Pi(a, bc) &= \Pi(a, b)(1 \otimes c) + (b \otimes 1)\Pi(a, c), \quad (2) \\
\Pi(ab, c) &= \Pi(a, c)(b \otimes 1) + (1 \otimes a)\Pi(b, c). \quad (3)
\end{align*}
\]

The following notation of Sweedler type is sometimes convenient:
\[ \{a, b\}_\Pi = \Pi(a, b) = \Pi(a, b)' \otimes \Pi(a, b)''. \]
Given a double bracket $\Pi$, we can consider the following auxiliary operations.

(i) The map
\[ (a, b) \mapsto \Pi(a, b)'\Pi(a, b)'' =: \{a, b\}_\Pi \in \mathfrak{A} \]
descends to a map from $[\mathfrak{A}] \otimes \mathfrak{A}$, and is a derivation in the second variable. Namely,
\[ \{[a], b_1 b_2\}_\Pi = \{[a], b_1\}_\Pi b_2 + b_1 \{[a], b_2\}_\Pi, \quad b_1, b_2 \in \mathfrak{A}. \]

(ii) The map
\[ (a, b) \mapsto \Pi(a, b)''\Pi(a, b)' =: \{a, [b]\}_\Pi \in \mathfrak{A} \]
descends to a map from $\mathfrak{A} \otimes [\mathfrak{A}]$, and is a derivation in the first variable. Namely,
\[ \{a_1a_2, [b]\}_\Pi = \{a_1, [b]\}_\Pi a_2 + a_1 \{a_2, [b]\}_\Pi. \]

(iii) The maps (i) and (ii) descend to the same map $[\mathfrak{A}] \otimes [\mathfrak{A}] \to [\mathfrak{A}]$, given by the formula
\[ [a] \otimes [b] \mapsto \{[a], [b]\}_\Pi := \|[a], [b]\|_\Pi = \|[a, [b]\|_\Pi \in [\mathfrak{A}] . \]

All the three constructions above are called the **bracket** associated with $\Pi$. 

9
2 Goldman-Turaev Lie bialgebra

In this section, we describe the structure morphisms of the Goldman-Turaev Lie bialgebra and their upgrades. For proofs, we refer to [12] [42] [21] [29]. When we need to modify the proofs due to the presence of a framing, we explain the necessary changes.

By a surface we mean a connected smooth 2-manifold. Let \( \Sigma \) be a compact oriented surface with non-empty boundary \( \partial \Sigma \). Fix a basepoint \( * \in \partial \Sigma \). We consider the group algebra \( K \pi \) of the fundamental group \( \pi := \pi_1(\Sigma,*) \), as well as its quotient \( K \)-vector space \( |K\pi| = K\pi/[K\pi,K\pi] \).

The space \( |K\pi| \) is denoted by \( \mathfrak{g}(\Sigma) \) in Introduction. Another description of \( |K\pi| \) is as follows. Let \( \hat{\pi} \) be the set of homotopy classes of free loops in \( \Sigma \). Forgetting the basepoint of a loop, we obtain a map \( \pi_1(\Sigma,p) \to \hat{\pi}, \gamma \mapsto |\gamma| \) for any \( p \in \Sigma \). Then the \( K \)-linear extension of the map \( \pi = \pi_1(\Sigma,*) \to \hat{\pi} \) induces a \( K \)-linear isomorphism \( |K\pi| \cong K\hat{\pi} \).

In what follows, we often use this identification without mentioning explicitly.

We say that two immersed curves in \( \Sigma \) are generic or in general position if their intersections consist of finitely many transverse double points. Likewise, an immersed curve is said to be generic if its self-intersections consist of a finite number of transverse double points.

For an ordered pair of linearly independent tangent vectors \( (\vec{\alpha},\vec{\beta}) \) at some \( p \in \Sigma \), let \( \varepsilon(\vec{\alpha},\vec{\beta}) := +1 \) if \( (\vec{\alpha},\vec{\beta}) \) is a positive basis for \( T_p\Sigma \), and \( \varepsilon(\vec{\alpha},\vec{\beta}) := -1 \) otherwise.

2.1 Goldman bracket and its upgrade

Let \( \alpha \) and \( \beta \) be free loops on \( \Sigma \) in general position. For each intersection \( p \in \alpha \cap \beta \), let \( \dot{\alpha}_p,\dot{\beta}_p \in T_p\Sigma \) be the tangent vectors of \( \alpha \) and \( \beta \) at \( p \), respectively. Let \( \alpha_p \in \pi_1(\Sigma,p) \) be the loop \( \alpha \) based at \( p \) and define \( \beta_p \in \pi_1(\Sigma,p) \) in the same way. Then the concatenation \( \alpha_p\beta_p \in \pi_1(\Sigma,p) \) and its projection \( |\alpha_p\beta_p| \in \hat{\pi} \) are well defined. The Goldman bracket of \( \alpha \) and \( \beta \) is defined by the formula

\[
[\alpha,\beta]_{\text{Goldman}} := \sum_{p \in \alpha \cap \beta} \varepsilon(\dot{\alpha}_p,\dot{\beta}_p) |\alpha_p\beta_p| \in K\pi \cong |K\pi|.
\]

(4)

As shown in [12] Theorem 5.3], this formula defines a Lie bracket on \( |K\pi| \). That is, \( [\cdot,\cdot] = [\cdot,\cdot]_{\text{Goldman}} \) satisfies

- skew-symmetry condition: \( [v,u] = -[u,v] \) for any \( u,v \in |K\pi| \),
- Jacobi identity: \( [u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0 \) for any \( u,v,w \in |K\pi| \).

In order to introduce an upgrade of the Goldman bracket, we need the following arrangement: take an orientation preserving embedding \( \nu : [0,1] \to \partial \Sigma \) with \( \nu(1) = * \), and let \( \bullet := \nu(0) \). The path \( \nu \) gives an isomorphism of groups

\[
\pi = \pi_1(\Sigma,*) \xrightarrow{\cong} \pi_1(\Sigma,\bullet), \quad \gamma \mapsto \nu\gamma\nu^{-1}.
\]

(5)
Here, $\mathcal{V}$ is the inverse of $\nu$.

Let $\alpha$ and $\beta$ be generic loops based at $\bullet$ and $\ast$, respectively. For each $p \in \alpha \cap \beta$, let $\alpha_{\bullet p}$ be the path from $\bullet$ to $p$ along $\alpha$, and define $\beta_{\ast p}$, $\beta_{sp}$ and $\beta_{ps}$ in a similar way. Then the concatenations $\beta_{sp}\alpha_{\bullet p}\nu$ and $\mathcal{V}\alpha_{\bullet p}\beta_{ps} \in \pi$ are defined. Set

$$\kappa(\alpha, \beta) := \sum_{p \in \alpha \cap \beta} \varepsilon(\dot{\alpha}_{p}, \dot{\beta}_{p}) \beta_{sp}\alpha_{\bullet p}\nu \otimes \mathcal{V}\alpha_{\bullet p}\beta_{ps} \in K\pi \otimes K\pi. \quad (6)$$

Extending by $K$-bilinearly and using the isomorphism (5), we obtain a map $\kappa : K\pi \otimes K\pi \to K\pi \otimes K\pi$.

**Proposition 2.1.** The map $\kappa$ is a double bracket on $K\pi$ in the sense of Definition 1.7. Furthermore, for any $u, v \in K\pi$, we have

$$\kappa(v, u) = -\kappa(u, v) + u \otimes v + v \otimes u - uv \otimes 1 - 1 \otimes vu. \quad (7)$$

**Proof.** For the first statement, see [21, Lemma 4.3.1] and [29]. The second statement is proved in the proof of [29, Lemma 7.2]. □

**Corollary 2.2.** Let $u_{1}, \ldots, u_{l}, v_{1}, \ldots, v_{m} \in K\pi$. Then,

$$\kappa(u_{1} \cdots u_{l}, v_{1} \cdots v_{m}) = \sum_{i,j} (v_{1} \cdots v_{j-1} \otimes u_{1} \cdots u_{i-1}) \kappa(u_{i}, v_{j}) (u_{i+1} \cdots u_{l} \otimes v_{j+1} \cdots v_{m}).$$

**Proof.** This is a direct consequence of (2) and (3). □

Applying the construction explained after Definition 1.7 to the double bracket $\kappa$, we obtain the bracket $\{\cdot, \cdot\} = \{\cdot, \cdot\}_\kappa : |K\pi| \otimes |K\pi| \to |K\pi|$ which equals the Goldman bracket. Another bracket operation $\{\cdot, \cdot\} : |K\pi| \otimes K\pi \to K\pi$ coincides with the action $\sigma$ in [20, §3.2].

**Remark 2.3.** Papakyriakopoulos [34] and Turaev [41] independently introduced essentially the same operations as the map $\kappa$.

**Remark 2.4.** The sign convention in formula (6) is different from [2] and [21].

### 2.2 Turaev cobracket and its upgrade

The original version of the Turaev cobracket [42] is a Lie cobracket on the quotient space $|K\pi|/K1$, where $1 \in \hat{\pi}$ is the class of a constant loop. In this paper, motivated by Furuta’s observation [31, §4], we use a framed version of the Turaev cobracket which is a lift of the original one to the space $|K\pi|$. This lift depends on the choice of framing. Here, a **framing** on $\Sigma$ is (the homotopy class of) a trivialization of the tangent bundle of $\Sigma$, $f : T\Sigma \cong \Sigma \times \mathbb{R}^{2}$. Since by assumptions $\partial\Sigma$ is non-empty, the tangent bundle $T\Sigma$ is trivial and a framing always exists. Let $\hat{\pi}^{+}$ be the set of regular homotopy classes of immersed free loops on $\Sigma$. Given a framing $f$, one can define the rotation number function

$$\text{rot}_f : \hat{\pi}^{+} \to \mathbb{Z}.$$
Remark 2.5. The set of homotopy classes of framings on $\Sigma$ is a torsor under the action of $H^1(\Sigma, \mathbb{Z})$. This action can be seen through the injective assignment $f \mapsto \text{rot}_f$. Namely, if $f$ and $f'$ are framings, then there exists a uniquely determined cohomology class $\chi \in H^1(\Sigma, \mathbb{Z}) \cong \text{Hom}(\pi, \mathbb{Z})$ such that for any $\gamma \in \hat{\pi}^+$,

$$\text{rot}_f(\gamma) = \text{rot}_{f'}(\gamma) + \chi(\gamma).$$ (8)

For a given framing $f$ we now define a framed version of the Turaev cobracket. Let $\alpha \in \hat{\pi}$. By insertion of a suitable number of monogons into $\alpha$, we arrange that $\alpha$ is represented by a generic immersed loop with $\text{rot}_f(\alpha) = 0$. Each double point $p$ of $\alpha$ is traversed by $\alpha$ twice, and there are two linearly independent tangent vectors of $\alpha$ at $p$. Temporarily, we label them by $v_1, v_2 \in T_p \Sigma$ so that the ordered pair $(v_1, v_2)$ is a positive basis for $T_p \Sigma$. Define $\alpha^1_p$ as the loop starting at $p$ in the direction of $v_1$ and going along $\alpha$ until the first return to $p$. Define $\alpha^2_p$ similarly. Then, the framed version of the Turaev cobracket of $\alpha$ is defined by the formula

$$\delta^f(\alpha) := \sum_p \alpha^1_p \wedge \alpha^2_p = \sum_p \alpha^1_p \otimes \alpha^2_p - \alpha^2_p \otimes \alpha^1_p,$$ (9)

where the sum is taken over all self-intersections of $\alpha$.

Proposition 2.6. The map $\delta^f$ is a Lie cobracket on $|\mathbb{K}\pi|$. That is, $\delta^f$ satisfies

- coskew-symmetry condition: $\delta^f(u)^\circ = -\delta^f(u)$ for any $u \in |\mathbb{K}\pi|$
- coJacobi identity: $N(\delta^f \otimes 1)\delta^f = 0 : |\mathbb{K}\pi| \to |\mathbb{K}\pi| \otimes 3$.

Here, $N(u \otimes v \otimes w) = u \otimes v \otimes w + v \otimes w \otimes u + w \otimes u \otimes v$ for $u, v, w \in |\mathbb{K}\pi|$.

Proof. Coskew-symmetry condition follows from the equation (9). To prove the coJacobi identity, let $\alpha$ be a generic immersed loop with $\text{rot}_f(\alpha) = 0$. Then

$$(\delta^f \otimes 1)\delta^f(\alpha) = \sum_p \delta^f(\alpha^1_p) \otimes \alpha^2_p - \delta^f(\alpha^2_p) \otimes \alpha^1_p.$$ (8)

This can be expanded to two terms: one term is a sum of expressions $X(p, q)$ associated with non-ordered pairs $(p, q)$ of self-intersections of $\alpha$ as in the proof of [42, Theorem 8.3], and the other term is the contribution of insertion of monogons which we need to compute $\delta^f(\alpha^1_p)$ and $\delta^f(\alpha^2_p)$. The same argument as in the proof of [42, Theorem 8.3] proves that the first term is in the kernel of $N$.

To compute the second term, notice that for each double point $p$,

$$\text{rot}_f(\alpha^1_p) + \text{rot}_f(\alpha^2_p) = 0.$$ (8)

Therefore, if we insert $m_p$ positive monogons into $\alpha^1_p$ to achieve $\text{rot}_f(\alpha^1_p) = 0$, then we need $m_p$ negative monogons for $\alpha^2_p$. Now, each positive (resp. negative) monogon of $\gamma$ contributes to $\delta^f(\gamma)$ as $\gamma \wedge 1$ (resp. $-\gamma \wedge 1$). Therefore, the second term is computed as

$$\sum_p m_p((\alpha^1_p \wedge 1) \otimes \alpha^2_p + (\alpha^2_p \wedge 1) \otimes \alpha^1_p),$$

and this is in the kernel of $N$. This completes the proof. \qed
There are two refinements of $\delta f$ which are closely related to each other. First of all, we can arrange that the framing $f$ be trivial on a neighborhood of $\nu$. Namely, the velocity vector field of $\nu$ corresponds to $(1, 0) \in \mathbb{R}^2$ through $f$. Let $v_\ast \in T_\ast \Sigma$ and $v_\ast \in T_\ast \Sigma$ be the inward tangent vectors corresponding to $(0, 1) \in \mathbb{R}^2$.

Let $\Pi\Sigma(\ast, \ast)$ be the set of homotopy classes of paths from $\ast$ to $\ast$, and define $\pi\Sigma(\ast, \ast)$ similarly. The path $\nu$ induces natural identifications

$$\pi \cong \Pi\Sigma(\ast, \ast), \quad \gamma \mapsto \nu\gamma,$$

and

$$\pi \cong \Pi\Sigma(\ast, \ast), \quad \gamma \mapsto \gamma\nu.$$

Let $\gamma \in \pi$. Using the bijection $\pi \cong \Pi\Sigma(\ast, \ast)$, we choose a representative of $\gamma$ to be a generic immersed path $\gamma : ([0, 1], 0, 1) \to (\Sigma, \ast, \ast)$ such that $\dot{\gamma}(0) = v_\ast$ and $\dot{\gamma}(1) = -v_\ast$. We further assume that $\gamma$ has the rotation number $-1/2$ with respect to $f$.

For each double point $p$ of $\gamma$, we denote $\gamma^{-1}(p) = \{t_1^p, t_2^p\}$ with $0 < t_1^p < t_2^p < 1$. The velocity vectors $\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p)$ are linearly independent in $T_p \Sigma$. Let $\gamma_{t_1^p, t_2^p}$ be the restriction of $\gamma$ to the interval $[t_1^p, t_2^p]$. This becomes a loop since $\gamma(t_1^p) = \gamma(t_2^p) = p$. Define the paths $\gamma_{t_1^p}$ and $\gamma_{t_2^p}$ similarly. Then the concatenation $\nu\gamma_{t_1^p}\gamma_{t_2^p} \in \pi$ is defined. Set

$$\mu^f(\gamma) := \sum_p \varepsilon(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p)) |\gamma_{t_1^p, t_2^p}| \otimes \nu\gamma_{t_1^p, t_2^p} \in |K\pi| \otimes K\pi. \quad (10)$$

Extending $K$-linearly, we obtain a map

$$\mu^f : K\pi \to |K\pi| \otimes K\pi.$$

By exchanging the roles of $\ast$ and $\ast$, we obtain another map

$$\mu^f_* : K\pi \to K\pi \otimes |K\pi|,$$

In more detail, we now use the bijection $\pi \cong \Pi\Sigma(\ast, \ast)$. Let $\gamma \in \pi$ and represent it as a generic immersed path from $\ast$ to $\ast$ with $\dot{\gamma}(0) = v_\ast$ and $\dot{\gamma}(1) = -v_\ast$. We further assume that $\text{rot}_f(\gamma) = 1/2$. Then, we have

$$\mu^f(\gamma) = -\sum_p \varepsilon(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p)) |\gamma_{t_1^p, t_2^p}| \otimes \nu\gamma_{t_1^p, t_2^p} \in |K\pi| \otimes |K\pi|. \quad (11)$$

Remark 2.7. Our sign convention in formula (10) is different from [2] and [21].

The next proposition describes product formulas for the maps $\mu^f$ and $\mu^f_*$ and their relation to the cobracket $\delta f$.

Proposition 2.8. (i) For any $u, v \in K\pi$,

$$\mu^f(uv) = \mu^f(u)(1 \otimes v) + (1 \otimes u)\mu^f(v) + (| \cdot | \otimes 1)\kappa(u, v), \quad (12)$$

$$\mu^f_*(uv) = \mu^f_*(u)(v \otimes 1) + (u \otimes 1)\mu^f_*(v) + (1 \otimes | \cdot |)\kappa(v, u). \quad (13)$$
(ii) For any $\gamma \in \pi$,

$$\delta^f(|\gamma|) = \text{Alt}(1 \otimes |\cdot|)^f(\gamma) + |\gamma| \wedge 1 \tag{14}$$

$$= (1 \otimes |\cdot|)^f(\gamma) + (|\cdot| \otimes 1)^f(\gamma). \tag{15}$$

Here, $\text{Alt}(a \otimes b) = a \otimes b - b \otimes a$ for $a, b \in |K\pi|$.

Proof. For (i), the same argument as the proof of [21, Lemma 4.3.3] works.

We next prove (ii). Let $\gamma \in \pi$. Using the identification $\pi \cong \Pi(\bullet, \ast)$, we choose its representative by a generic immersion $\gamma : ([0, 1], 0, 1) \to (\Sigma, \bullet, \ast)$ such that $\dot{\gamma}(0) = v_\bullet$, $\dot{\gamma}(1) = -v_\ast$, and $\text{rot}_f(\gamma) = -1/2$. Then inserting a positive monogon into $\gamma \mathcal{V}$, we obtain a generic immersed loop with vanishing rotation number with respect to $f$. This loop has one more self-intersection than $\gamma$, whose contribution to $\delta^f(|\gamma|)$ is $|\gamma| \wedge 1$. We see that $\text{Alt}(1 \otimes |\cdot|)^f(\gamma)$ is equal to the sum of contributions to $\delta^f(|\gamma|)$ from all the other self-intersections. This proves (14). To prove (15), we use (14) and the equality

$$\mu^f(\bullet)(\gamma) = -\mu^f(\ast)(\gamma) + \gamma \wedge 1 - 1 \otimes |\gamma|. \tag{16}$$

The repeated use of the product formulas yields the following more explicit description of the operation $\mu^f$.

Corollary 2.9. Let $u_1, \ldots, u_m \in K\pi$. Then,

$$\mu^f(u_1 \cdots u_m) = \sum_i (1 \otimes u_1 \cdots u_{i-1})^f(u_i)(1 \otimes u_{i+1} \cdots u_m)$$

$$+ \sum_i (|\cdot| \otimes 1)^f(\kappa(u_1 \cdots u_{i-1}, u_i)(1 \otimes u_{i+1} \cdots u_m)).$$

The next proposition describes how the operations $\delta^f$ and $\mu^f$ depend on the choice of framing.

Proposition 2.10. Let $f$ and $f'$ be two framings on $\Sigma$ and let $\chi \in H^1(\Sigma; \mathbb{Z})$ be the cohomology class describing their difference as in Remark 2.5. Then, for any $\gamma \in \pi$,

$$\mu^{f'}(\gamma) = \mu^f(\gamma) + \chi(\gamma) (1 \otimes \gamma),$$

$$\mu^{f'}(\bullet)(\gamma) = \mu^f(\bullet)(\gamma) - \chi(\gamma) (\gamma \otimes 1),$$

$$\delta^{f'}(|\gamma|) = \delta^f(|\gamma|) + \chi(\gamma) (1 \wedge |\gamma|).$$

Proof. Let $\gamma : ([0, 1], 0, 1) \to (\Sigma, \bullet, \ast)$ be a generic immersion such that $\dot{\gamma}(0) = v_\bullet$, $\dot{\gamma}(1) = -v_\ast$, and $\text{rot}_f(\gamma) = -1/2$. Then, by (16), we see that by inserting $\chi(\gamma)$ negative monogons to $\gamma$, we obtain a generic immersion $\gamma'$ with $\text{rot}_{f'}(\gamma') = -1/2$. Since each negative monogon in $\gamma'$ contributes a term $1 \otimes \gamma$ to $\mu^{f'}(\gamma)$, the formula for $\mu^{f'}(\gamma)$ follows.

The other formulas can be proved in a similar way. \qed
2.3 Compatibility conditions

The Goldman bracket and Turaev cobracket define a Lie bialgebra structure on the space $[\mathbb{K}\pi]$.

**Proposition 2.11.** The triple $([\mathbb{K}\pi], [\cdot, \cdot]_{\text{Goldman}}, \delta^f)$ is a Lie bialgebra. That is, $[\cdot, \cdot] = [\cdot, \cdot]_{\text{Goldman}}$ is a Lie bracket on $[\mathbb{K}\pi]$, $\delta^f$ is a Lie cobracket on $[\mathbb{K}\pi]$, and they satisfy the compatibility condition: for any $u, v \in [\mathbb{K}\pi]$,

$$[\delta^f([u, v])] = u.\delta^f(v) - v.\delta^f(u).$$

Here, on the right hand side, $\alpha.\delta^f(\beta)$ denotes the adjoint action of the Lie algebra $[\mathbb{K}\pi]$ on $[\mathbb{K}\pi] \otimes [\mathbb{K}\pi]$. Moreover, this Lie bialgebra structure is involutive in the sense that $[\cdot, \cdot] \circ \delta^f = 0 : [\mathbb{K}\pi] \rightarrow [\mathbb{K}\pi] \otimes [\mathbb{K}\pi] \rightarrow [\mathbb{K}\pi]$.

**Proof.** To prove the compatibility condition, let $\alpha$ and $\beta$ be generic immersed loops such that rot$_f(\alpha) = \text{rot}_f(\beta) = 0$. Observe that for each intersection $p \in \alpha \cap \beta$,

$$\text{rot}_f([\alpha_p, \beta_p]) = \text{rot}_f(\alpha) + \text{rot}_f(\beta).$$

Therefore, to compute $\delta^f([\alpha_p, \beta_p])$, we do not need insertion of monogons. Then, the same argument as in the proof of [42, Theorem 8.3] yields $\delta^f([\alpha, \beta]) = \alpha.\delta^f(\beta) - \beta.\delta^f(\alpha)$.

The involutivity condition can be proved in the same way as [6, Proposition B.1].

There is a refinement of the involutivity condition in the previous proposition.

**Lemma 2.12.** The operations $\{\cdot, \cdot\} : [\mathbb{K}\pi] \otimes [\mathbb{K}\pi] \rightarrow [\mathbb{K}\pi]$ and $\mu^f$ satisfy the involutivity condition

$$\{\cdot, \cdot\} \circ \mu^f = 0 : [\mathbb{K}\pi] \rightarrow [\mathbb{K}\pi] \otimes [\mathbb{K}\pi] \rightarrow [\mathbb{K}\pi].$$

**Proof.** This can be proved in the same way as [21, Proposition 3.2.7]. Note that a negative monogon in $\gamma$ contributes an extra term $1 \otimes \gamma$ to $\mu^f(\gamma)$, but this term is in the kernel of the bracket $\{\cdot, \cdot\} : [\mathbb{K}\pi] \otimes [\mathbb{K}\pi] \rightarrow [\mathbb{K}\pi]$.

2.4 Standard generators and adapted framing

By the classification theorem of surfaces, there are unique integers $g, n \geq 0$ such that $\Sigma$ is diffeomorphic to $\Sigma_{g,n+1}$, a compact oriented surface of genus $g$ with $n + 1$ boundary components. Fix a labeling to the boundary components, $\partial \Sigma = \bigsqcup_{j=0}^n \partial_j \Sigma$, such that $\ast \in \partial_0 \Sigma$. Then there is a generating system for $\pi = \pi_1(\Sigma, \ast)$, $\alpha_i, \beta_i, \gamma_j$, $i = 1, \ldots, g$, $j = 1, \ldots, n$, such that

$$\prod_{i=1}^g \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1} \prod_{j=1}^n \gamma_j = \gamma_0,$$

where $\gamma_0$ is the loop around the 0th boundary $\partial_0 \Sigma$ with negative orientation and $\gamma_j$ is freely homotopic to the $j$th boundary $\partial_j \Sigma$ with positive orientation, see Figure 1.
Remark 2.13. Given a basepoint \(* \in \partial \Sigma\) and a labeling \(\partial \Sigma = \bigsqcup_{j=0}^{n} \partial_j \Sigma\), the choice of a generating system \(\alpha_i, \beta_i, \gamma_j\) as above is unique up to an action of the mapping class group of \(\Sigma\) relative to the boundary \(\partial \Sigma\).

We compute the operation \(\kappa\) on the generators \(\alpha_i, \beta_i, \gamma_j\).

**Proposition 2.14.** For \(i = 1, \ldots, g\),
\[
\begin{align*}
\kappa(\alpha_i, \alpha_i) &= \alpha_i \otimes \alpha_i - 1 \otimes \alpha_i^2, \\
\kappa(\alpha_i, \beta_i) &= \beta_i \otimes \alpha_i, \\
\kappa(\beta_i, \alpha_i) &= \beta_i \otimes \alpha_i - \alpha_i \beta_i \otimes 1 - 1 \otimes \beta_i \alpha_i, \\
\kappa(\beta_i, \beta_i) &= \beta_i \otimes \beta_i - \beta_i^2 \otimes 1.
\end{align*}
\]

If \(x \in \{\alpha_i, \beta_i\}\), \(y \in \{\alpha_j, \beta_j\}\) with \(i < j\), or \(x \in \{\alpha_i, \beta_i\}\), \(y = \gamma_j\) for some \(i\) and \(j\),
\[
\begin{align*}
\kappa(x, y) &= 0, \\
\kappa(y, x) &= x \otimes y + y \otimes x - xy \otimes 1 - 1 \otimes xy.
\end{align*}
\]

For \(j = 1, \ldots, n\),
\[
\kappa(\gamma_j, \gamma_j) = \gamma_j \otimes \gamma_j - 1 \otimes \gamma_j^2.
\]

Let \(1 \leq j < k \leq n\). Then,
\[
\begin{align*}
\kappa(\gamma_j, \gamma_k) &= 0, \\
\kappa(\gamma_k, \gamma_j) &= \gamma_j \otimes \gamma_k + \gamma_k \otimes \gamma_j - \gamma_j \gamma_k \otimes 1 - 1 \otimes \gamma_k \gamma_j.
\end{align*}
\]

**Proof.** Figure 2 computes \(\kappa(\alpha_1, \beta_1)\) and \(\kappa(\beta_1, \alpha_1)\). The other cases are similar. \(\square\)

**Remark 2.15.** Since \(\kappa\) is a double bracket, Proposition 2.14 gives its characterization.

There are simple closed curves freely homotopic to \(|\alpha_i|, |\beta_i|, |\gamma_j|\). We denote them by the same letters \(|\alpha_i|, |\beta_i|, |\gamma_j|\) \(\in \hat{\pi}^+\). Any framing on \(\Sigma\) is determined by the values of its rotation number function on these simple closed curves.
Definition 2.16. Fix a generating system \( \alpha_i, \beta_i, \gamma_j \) as above. The adapted framing is the framing \( f^{adp} \) with its rotation number function \( \text{rot}_{adp} = \text{rot}_{f^{adp}} \) given by

\[
\text{rot}_{adp}(|\alpha_i|) = \text{rot}_{adp}(|\beta_i|) = 0, \quad \text{rot}_{adp}(|\gamma_j|) = -1,
\]

for \( i = 1, \ldots, g \), \( j = 1, \ldots, n \).

Note that the Poincaré-Hopf theorem implies \( \text{rot}_{adp}(|\gamma_0|) = 2g - 1 \).

Let \( \mu^{adp} \) be the operation \( \mu^f \) for the adapted framing.

Proposition 2.17. For \( i = 1, \ldots, g \),

\[
\mu^{adp}(\alpha_i) = 1 \otimes \alpha_i, \quad \mu^{adp}(\beta_i) = -|\beta_i| \otimes 1.
\]

For \( j = 1, \ldots, n \),

\[
\mu^{adp}(\gamma_j) = 0.
\]

Proof. The values \( \mu^{adp}(\alpha_i) \) and \( \mu^{adp}(\beta_i) \) can be seen from Figure 3, which shows representatives of \( \nu\alpha_i \) and \( \nu\beta_i \) with rotation number \(-1/2\) with respect to \( f^{adp} \). For \( \nu\gamma_j \), there is a representative with no self-intersections. Hence \( \mu^{adp}(\gamma_j) = 0 \).\( \square \)

Remark 2.18. By the product formula (12), Proposition 2.17 uniquely determines the map \( \mu^{adp} \). For other framings, we obtain a similar characterization using Proposition 2.10.
3 Filtrations

In this section, we introduce a filtration on $K\pi$, which we call the weight filtration, and show that it defines the graded version of the Goldman-Turaev Lie bialgebra.

First of all, let us recall the following classical fact (see, e.g., [24]). Let $F$ be a free group of finite rank. The group algebra $KF$ has a structure of a Hopf algebra whose coproduct, augmentation and antipode are given by formulas

$$\Delta(x) = x \otimes x, \quad \varepsilon(x) = 1, \quad \iota(x) = x^{-1}$$

for any $x \in F$. Then the augmentation ideal $IF := \text{Ker}(\varepsilon)$ defines a decreasing filtration $\{IF^m\}_{m \geq 0}$ of two-sided ideals of $KF$, and the (completed) graded quotient

$$\text{gr}(KF) := \prod_{m \geq 0} (IF)^m/(IF)^{m+1}$$

is a graded Hopf algebra generated by the degree one part. Moreover, it is canonically isomorphic to the completed free associative algebra (or tensor algebra) $\hat{T}(H)$ generated by $H = H_1(F, K) = F^{\text{abel}} \otimes_{\mathbb{Z}} K$, and the isomorphism restricted to the degree one part is given by

$$(IF)/(IF)^2 \cong H, \quad (x - 1) \mod (IF)^2 \mapsto [x] := \text{the homology class of } x \in F.$$  

Since the fundamental group $\pi := \pi_1(\Sigma, \ast)$ is a free group of rank $2g + n$, the construction in the preceding paragraph applies to $\pi$. However, when we take into account topological operations such as the Goldman bracket and Turaev cobracket, there is a better way to introduce a filtration on $K\pi$. Roughly speaking, the idea of the weight filtration is to assign weights to generators of $\pi$ such that loops parallel to a boundary component have weight 2, and all the other loops have weight 1. In what follows, we make this idea more precise.

3.1 Two definitions of the weight filtration

We give two definitions of the weight filtration on $K\pi$.

The first method is to use a capping trick. Take $n$ copies of $\Sigma_{1,1}$, a surface of genus 1 with one boundary component, and cap them along $\partial_j \Sigma$, $j = 1, \ldots, n$ to obtain

$$\Sigma = \Sigma \cup \bigcup_{j=1}^n \partial_j \Sigma = \bigcup_{j=1}^n \Sigma_{1,1}.$$  

The fundamental group $\pi := \pi_1(\Sigma, \ast)$ is a free group of rank $2(g + n)$. The inclusion homomorphism $i_* : \pi \to \pi$ and its $K$-linear extension are injective.

**Definition 3.1** (the first definition of $K\pi(m)$). For $m \geq 0$, define the two-sided ideal $K\pi(m)$ of $K\pi$ by the formula

$$K\pi(m) := i_*^{-1}((\pi)^m).$$
Another way to introduce the weight filtration is to use standard generators $\alpha_i, \beta_i, \gamma_j$ for $\pi$. Assign weights to them as follows:

$$\text{wt}(\alpha_i) = \text{wt}(\beta_i) = 1, \quad \text{wt}(\gamma_j) = 2.$$  

**Definition 3.2** (the second definition of $\mathbb{K}\pi(m)$). For $m \geq 0$, define $\mathbb{K}\pi(m)$ to be the two-sided ideal of $\mathbb{K}\pi$ generated by elements of the form

$$(\zeta_1 - 1)(\zeta_2 - 1) \cdots (\zeta_l - 1),$$

where $\zeta_1, \zeta_2, \ldots, \zeta_l \in \{\alpha_i, \beta_i, \gamma_j\}$ and $\sum a \text{wt}(\zeta_a) \geq m$.

Actually, the two methods above define the same filtration on $\mathbb{K}\pi$:

**Proposition 3.3.** The two definitions of $\mathbb{K}\pi(m)$ are equivalent.

The proof of Proposition 3.3 is rather technical, so it is given in §3.2.

**Remark 3.4.** If $n = 0$, the weight filtration is the same as the powers of the augmentation ideal.

For later use, let us collect notations related to Definitions 3.1 and 3.2.

- $i^* : \pi \to T$ is the inclusion homomorphism induced by the inclusion $\Sigma \hookrightarrow \Sigma$.
- Let $H = H_1(\pi, \mathbb{K}) \cong H_1(\Sigma, \mathbb{K})$. Given standard generators $\alpha_i, \beta_i, \gamma_j$ for $\pi$, we denote $x_i = [\alpha_i], y_i = [\beta_i]$ and $z_j = [\gamma_j] \in H$. Let $A = \hat{T}(H) = \prod_{p=0}^{\infty} H \otimes \mathbb{K}$ be the completed free associative algebra generated by $H$, whose multiplication is $\otimes$.
- We can take $2n$ elements $\alpha_{g+1}, \beta_{g+1}, \ldots, \alpha_{g+n}, \beta_{g+n} \in T$ such that

$$i^*(\gamma_j) = [\alpha_{g+j}, \beta_{g+j}] = \alpha_{g+j}\beta_{g+j}\alpha_{g+j}^{-1}\beta_{g+j}^{-1}.$$  

Then $T$ is a free group generated by $\alpha_i, \beta_i, i = 1, \ldots, g + n$.
- Let $\overline{H} := H_1(\pi, \mathbb{K})$ and $\overline{A} = \hat{T}(\overline{H})$. We denote $x_i = [\alpha_i]$ and $y_i = [\beta_i] \in \overline{H}$, for $i = 1, \ldots, g + n$.
- Define the injective $\mathbb{K}$-algebra homomorphism $i^*_+ : A \to \overline{A}$ by

$$i^*_+ : x_i \mapsto x_i, \quad y_i \mapsto y_i, \quad z_j \mapsto [x_{g+j}, y_{g+j}] = x_{g+j}y_{g+j} - y_{g+j}x_{g+j}.$$  

For the moment, we describe immediate consequences of the definitions. (They will not be used in the proof of Proposition 3.3.)

First of all, by Definition 3.1 we see that the filtration $\mathbb{K}\pi(m)$ described in Definition 3.2 does not depend on the choice of standard generators.

Next we show elementary properties of the associated graded of the weight filtration. The weight filtration is multiplicative in the sense that $\mathbb{K}\pi(l) \cdot \mathbb{K}\pi(m) \subset \mathbb{K}\pi(l + m)$ for
any \(l, m \geq 0\). The coproduct \(\Delta : \mathbb{K}\pi \to \mathbb{K}\pi \otimes \mathbb{K}\pi\) is filtration-preserving in the sense that \(\Delta(\mathbb{K}\pi(m)) \subset \sum_{a+b=m} \mathbb{K}\pi(a) \otimes \mathbb{K}\pi(b)\) for any \(m \geq 0\). Then, the completion

\[
\hat{\mathbb{K}\pi} := \lim_m \mathbb{K}\pi / \mathbb{K}\pi(m)
\]

is a filtered Hopf algebra whose filtration is given by \(\hat{\mathbb{K}\pi}(m) := \text{Ker}(\hat{\mathbb{K}\pi} \to \mathbb{K}\pi / \mathbb{K}\pi(m))\), \(m \geq 0\). Then, the completion

\[
\hat{\mathbb{K}\pi} := \lim_m \mathbb{K}\pi / \mathbb{K}\pi(m)
\]

is a filtered Hopf algebra whose filtration is given by \(\hat{\mathbb{K}\pi}(m) := \text{Ker}(\hat{\mathbb{K}\pi} \to \mathbb{K}\pi / \mathbb{K}\pi(m))\), \(m \geq 0\). Then, the completion

\[
\hat{\mathbb{K}\pi} := \lim_m \mathbb{K}\pi / \mathbb{K}\pi(m)
\]

is a graded Hopf algebra. According to Definition 3.1, we see that the inclusion homomorphism \(i_*\) induces an injective graded Hopf algebra homomorphism

\[
\text{gr}^w(\mathbb{K}\pi) \to \text{gr}^w(I\mathbb{K}\pi).
\]

Thirdly, by Definition 3.2, we obtain an explicit generating set for \(\text{gr}^w(\mathbb{K}\pi)\):

**Corollary 3.5.** The graded \(\mathbb{K}\)-algebra \(\text{gr}^w(\mathbb{K}\pi)\) is generated by the following elements of degrees 1 and 2:

\[
\begin{align*}
(\alpha_i - 1) \mod \mathbb{K}\pi(2), \quad (\beta_i - 1) \mod \mathbb{K}\pi(2), \quad (i = 1, \ldots, g), \\
(\gamma_j - 1) \mod \mathbb{K}\pi(3), \quad (j = 1, \ldots, n).
\end{align*}
\]

Finally, the weight filtration on \(|\mathbb{K}\pi|\) is defined by using the projection map \(|\cdot| : \mathbb{K}\pi \to |\mathbb{K}\pi|\). That is, the space \(|\mathbb{K}\pi|\) is filtered by \(|\mathbb{K}\pi(m)|\), \(m \geq 0\), and its completion is a filtered \(\mathbb{K}\)-vector space naturally isomorphic to \(|\hat{\mathbb{K}\pi}| = \hat{\mathbb{K}\pi} / [\hat{\mathbb{K}\pi}, \hat{\mathbb{K}\pi}]|\). The associated graded vector space \(\text{gr}^w(|\mathbb{K}\pi|) \cong \text{gr}^w(|\hat{\mathbb{K}\pi}|)\) is naturally isomorphic to \(\text{gr}^w(\mathbb{K}\pi)|\).

### 3.2 Proof of Proposition 3.3

Let us start with a general situation. For the moment, let \(\pi\) be a free group of finite rank with a fixed basis \(\{x_i\}\). Assume that each generator is given a positive weight \(w_i = \text{wt}(x_i) \in \mathbb{Z}_{>0}\).

For \(m \geq 0\), let \(\mathbb{K}\pi(m)\) be the two-sided ideal of \(\mathbb{K}\pi\) generated by elements of the form

\[
(x_{i_1} - 1) \cdots (x_{i_l} - 1)
\]

where \(\sum_j \text{wt}(x_{i_j}) \geq m\). This defines a decreasing multiplicative filtration on \(\mathbb{K}\pi\).

Let \(I\mathbb{K}\pi\) be the augmentation ideal of \(\mathbb{K}\pi\). Clearly, \((I\mathbb{K}\pi)^m \subset \mathbb{K}\pi(m)\). Conversely, if \(w = \max_i w_i\), then \(\mathbb{K}\pi(m) \subset (I\mathbb{K}\pi)^l\), where \(l = \lceil \frac{m}{w} \rceil\). This shows that we have a canonical isomorphism between the two completions

\[
\hat{\mathbb{K}\pi}^l = \lim_m \mathbb{K}\pi / (I\mathbb{K}\pi)^m \quad \text{and} \quad \hat{\mathbb{K}\pi} = \lim_m \mathbb{K}\pi / \mathbb{K}\pi(m).
\]
Furthermore, \((I\pi)^m\) (resp. \(K\pi(m)\)) induces a filtration \((\hat{I}\pi)^m\) (resp. \(\hat{K}\pi(m)\)) on these isomorphic completions. The natural map from \(K\pi\) to its completion is injective since 
\(\bigcap_m (I\pi)^m = 0\); see [24].

Let \(H = H_1(\pi, K)\) and \(X_i := [x_i] \in H\) the homology class of \(x_i\). The complete associative algebra \(A = \prod_{p=0}^{\infty} H^{\otimes p}\) has the degree filtration defined by \(A^{\text{deg}}_{\geq m} = \prod_{p \geq m} H^{\otimes p}\). Moreover, we introduce the weight filtration on \(A\) as follows. First of all, define the weight of \(X_i\) to be \(w_i\). Then the \(m\)th term \(A^{\text{wt}}_{\geq m}\) of this filtration is defined to be the closure of the \(K\)-span of monomials of the form

\[
X_{i_1} \cdots X_{i_l}
\]

where \(\sum_j \text{wt}(X_{i_j}) \geq m\). Both filtrations on \(A\) described above are multiplicative.

To proceed, let us recall some material on expansions of a free group.

**Definition 3.6** ([18]). An expansion of a free group \(\pi\) is a map \(\theta : \pi \to A\) such that

(i) \(\theta(xy) = \theta(x)\theta(y)\) for any \(x, y \in \pi\),

(ii) \(\theta(x) \equiv 1 + [x] \mod A_{\geq 2}^{\text{deg}}\) for any \(x \in \pi\).

In §3.5, we discuss expansions in more detail.

An example of an expansion is the standard Magnus expansion [24] defined by

\[
\theta_{\text{std}} : \pi \to A, \quad x_i \mapsto 1 + X_i.
\]

We have \(\theta_{\text{std}}((I\pi)^m) \subset A^{\text{deg}}_{\geq m}\) and \(\theta_{\text{std}}(K\pi(m)) \subset A^{\text{wt}}_{\geq m}\) for any \(m \geq 0\), and \(\theta_{\text{std}}\) induces an isomorphism of \(K\)-algebras:

\[
\theta_{\text{std}} : \hat{K}\pi = \hat{K}\pi \xrightarrow{\cong} A.
\]

In fact, the inverse of \(\theta_{\text{std}}\) is given by \(\lambda(X_i) = x_i - 1\). Clearly, \(\lambda\) satisfies \(\lambda(A^{\text{deg}}_{\geq m}) \subset (\hat{I}\pi)^m\), and \(\lambda(A^{\text{wt}}_{\geq m}) \subset \hat{K}\pi(m)\). (Hence these inclusions are actually equalities.) As for the map \(\theta_{\text{std}} : K\pi \to A\), we have

\[
(\theta_{\text{std}})^{-1}(A^{\text{deg}}_{\geq m}) = K\pi \cap (\hat{I}\pi)^m = (I\pi)^m, \quad (\theta_{\text{std}})^{-1}(A^{\text{wt}}_{\geq m}) = K\pi \cap \hat{K}\pi(m) = K\pi(m).
\]

For any expansion \(\theta\), one has

\[
\theta^{-1}(A^{\text{deg}}_{\geq m}) = (I\pi)^m, \quad \text{for any } m \geq 0
\]

([18] Theorem 1.3]). The next lemma asserts that a similar property holds for the weight filtration under a certain assumption on the weights \(w_i\).

**Lemma 3.7.** Assume that the weight \(w_i\) is 1 or 2 for any \(i\). Then, for any expansion \(\theta\), one has \(\theta^{-1}(A^{\text{wt}}_{\geq m}) = K\pi(m)\) for any \(m \geq 0\).
Proof. There exists a $K$-algebra automorphism $U : A \to A$ satisfying $U(A_{\geq m}^{\text{deg}}) \subset A_{\geq m}^{\text{deg}}$, $U = \text{id}$ on $A_{\geq 1}^{\text{deg}}/A_{\geq 2}^{\text{deg}}$, and $\theta = U \circ \theta_{\text{std}}$ ([13, Theorem 1.3]). The assumption on $w_i$ and the second condition on $U$ imply that for any $i$,

$$U(X_i) = X_i + \text{(higher degree terms)} \in A_{\geq w_i}^{\text{wt}}.$$ 

Therefore, $U(A_{\geq m}^{\text{wt}}) \subset A_{\geq m}^{\text{wt}}$ for all $m$. Hence

$$\theta^{-1}(A_{\geq m}^{\text{wt}}) = (\theta_{\text{std}})^{-1}(U^{-1}(A_{\geq m}^{\text{wt}})) = (\theta_{\text{std}})^{-1}(A_{\geq m}^{\text{wt}}) = K\pi(m).$$

Remark 3.8. To see that the assumption of the lemma is necessary, consider the following example: let $\pi = \langle x_1, x_2 \rangle$ with $w_1 = 1$ and $w_2 = 10$, and define $\theta(x_1) = 1 + X_1$ and $\theta(x_2) = 1 + X_2 + X_1^2$. Then $(x_1 - 1)(x_2 - 1) \in K\pi(11)$, while

$$\theta((x_1 - 1)(x_2 - 1)) = X_1(X_2 + X_1^2) = X_1X_2 + X_1^3 \notin A_{\geq 11}^{\text{wt}}.$$

We are now ready to prove equivalence between two definitions of the weight filtration on $K\pi$.

**Proof of Proposition 3.3.** Recall the notation in §3.1. Put weights to the standard generators of $\pi$ as $\text{wt}(\alpha_i) = \text{wt}(\beta_i) = 1$ and $\text{wt}(\gamma_j) = 2$. Applying the construction in this subsection (recall Definition 3.2), we obtain the filtration $K\pi(m)$. Our goal is to prove $K\pi(m) = \tilde{i}^{-1}((I\pi)^m)$ for any $m \geq 0$.

Assign weight 1 to all the generators of $\pi$: we have $\text{wt}(\alpha_i) = \text{wt}(\beta_i) = 1$ for $i = 1, \ldots, g + n$. The injective $K$-algebra homomorphism $\tilde{i}_*: A \to \overline{A}$ satisfies $\tilde{i}^{-1}_*(\overline{A}_{\geq m}^{\text{deg}}) = \tilde{i}^{-1}_*(\overline{A}_{\geq m}^{\text{wt}}) = A_{\geq m}^{\text{wt}}$ for any $m \geq 0$.

**Lemma 3.9.** There exist expansions $\theta : \pi \to A$ and $\overline{\theta} : \pi \to \overline{A}$ such that

$$\tilde{i}_* \circ \theta = \overline{\theta} \circ i_* : K\pi \to \overline{A}.$$ 

**Proof.** This will be proved in [33].

Assuming Lemma 3.9, we continue the proof of Proposition 3.3. Take expansions $\theta$ and $\overline{\theta}$ as in Lemma 3.9. Then

$$\tilde{i}^{-1}_*(\overline{I\pi}^m) = i^{-1}_* \overline{\theta}^{-1}(\overline{A}_{\geq m}^{\text{deg}}) = \theta^{-1}\tilde{i}^{-1}_*(\overline{A}_{\geq m}^{\text{deg}}) = \theta^{-1}(A_{\geq m}^{\text{wt}}) = K\pi(m).$$

Here, the first equality uses ([16]) and the last equality uses Lemma 3.7.

**Remark 3.10.** The same argument proves that for any capping $\Sigma \subset \pi$ such that the inclusion homomorphism $i_* : \pi \to \pi = \pi_1(\overline{\Sigma})$ is injective, one has $i^{-1}_*(K\pi(m)) = K\pi(m)$.
3.3 Structure of the graded quotient

In this section, we describe the graded quotient $\text{gr}^{\text{wt}}(\mathbb{K}\pi)$ in terms of a certain tensor algebra. We keep the notation in §3.1 and §3.2.

There is a 2-step filtration on $H = H_1(\pi, \mathbb{K}) \cong H_1(\Sigma, \mathbb{K})$ defined by

$$H^{(1)} := H, \quad H^{(2)} := \{ x \in H \mid \langle x, y \rangle = 0 \text{ for all } y \in H \},$$

where $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{K}$ is the intersection pairing on $H$.

**Remark 3.11.** Another description of $H^{(2)}$ is $H^{(2)} = \text{Ker}(i_* : H \to \overline{H})$. The classes $[\partial_j \Sigma] = z_j, \ j = 1, \ldots, n$, constitute a $\mathbb{K}$-basis for $H^{(2)}$.

Consider the graded $\mathbb{K}$-vector space $\text{gr} H = H/H^{(2)} \oplus H^{(2)}$. Assign weights to the graded components as follows: $\text{wt}(H/H^{(2)}) = 1, \text{wt}(H^{(2)}) = 2$. Then, this induces a grading on the algebra $\hat{T}(\text{gr} H)$. Notice that choosing standard generators $\alpha_i, \beta_i, \gamma_j$ defines an isomorphism between $H$ and $\text{gr} H$:

$$H \cong \text{gr} H, \quad x_i \mapsto x_i \mod H^{(2)}, \quad y_i \mapsto y_i \mod H^{(2)}, \quad z_j \mapsto z_j.$$

Hence, $\hat{T}(H) \cong \hat{T}(\text{gr} H)$.

**Proposition 3.12.** There is a graded Hopf algebra isomorphism

$$\text{gr}^{\text{wt}}(\mathbb{K}\pi) \cong \hat{T}(\text{gr} H).$$

**Proof.** Since $\pi$ is a free group of finite rank, there is a canonical isomorphism

$$\text{gr}^I(\mathbb{K}\pi) \cong \hat{T}(\mathbb{N})$$

of graded Hopf algebras, as explained in the beginning of this section. Recall from §3.1 that we have an embedding $i_* : \text{gr}^{\text{wt}}(\mathbb{K}\pi) \to \text{gr}^I(\mathbb{K}\pi)$. In what follows,

- we define an embedding $\check{i}_* : \hat{T}(\text{gr} H) \to \hat{T}(\mathbb{N})$ of graded Hopf algebras, and
- we show that the isomorphism (17) restricts to $i_*(\text{gr}^{\text{wt}}(\mathbb{K}\pi)) \cong \check{i}_*(\hat{T}(\text{gr} H))$.

These two steps will complete the proof.

Recall that we cap $n$ copies of $\Sigma_{1,1}$ along $\partial_j \Sigma$, $1 \leq j \leq n$. Denote by $\Sigma_{1,1}^{(j)}$ the surface attached to $\partial_j \Sigma$, and set $M_j := H_1(\Sigma_{1,1}^{(j)}, \mathbb{K})$. The inclusion $\Sigma_{1,1}^{(j)} \to \Sigma$ induces an injection $M_j \to \overline{H}$, and it extends to an embedding $\hat{T}(M_j) \to \hat{T}(\overline{H})$. Let $\omega^{(j)} = [x_{g+j}, y_{g+j}] \in M_j \otimes \hat{T}(M_j)$ be the symplectic element of $\Sigma_{1,1}^{(j)}$.

The inclusion homomorphism $H \to \overline{H}$ induces an injective map $H/H^{(2)} \to \overline{H}$. The primitive elements $\omega^{(j)}$ define the map $H^{(2)} \to \hat{T}(\overline{H}), z_j \mapsto \omega^{(j)}$. Combining these two maps, we obtain the map $\check{i}_* : H \to \hat{T}(\overline{H})$, and it extends uniquely to a Hopf algebra homomorphism $\check{i}_* : \hat{T}(\text{gr} H) \to \hat{T}(\overline{H})$. One can check that this map is injective.

By Corollary 3.3, $\text{gr}^{\text{wt}}(\mathbb{K}\pi)$ is generated as a $\mathbb{K}$-algebra by the elements $(\alpha_i - 1) \mod \mathbb{K}\pi(2), (\beta_i - 1) \mod \mathbb{K}\pi(2)$ and $(\gamma_j - 1) \mod \mathbb{K}\pi(3)$. On the other hand, $\hat{T}(\text{gr} H)$ is generated as a $\mathbb{K}$-algebra by $x_i, y_i, z_j$. Since the images of these elements under $i_*$ and $\check{i}_*$ correspond to each other under isomorphism (17), we obtain the assertion. \qed
3.4 Filtration and topological operations

In this section, we show a property of the weight filtration related to the operations $\kappa$ and $\mu^f$.

**Proposition 3.13.** (i) The map $\kappa$ is of degree $(-2)$ with respect to the weight filtration. Namely, for any $l, m \geq 0$, we have

$$\kappa(\mathbb{K}\pi(l) \otimes \mathbb{K}\pi(m)) \subset \sum_{a+b=l+m-2} \mathbb{K}\pi(a) \otimes \mathbb{K}\pi(b).$$

Here, we understand that if $l + m - 2 < 0$ the right hand side is just $\mathbb{K}\pi \otimes \mathbb{K}\pi$. We adopt the same convention in what follows.

(ii) For any framing $f$ on $\Sigma$, the map $\mu^f$ is of degree $(-2)$.

**Proof.** (i) Consider the operations $\kappa$ on $\Sigma$ and on $\overline{\Sigma}$. The assertion is true for $\Sigma$ by Corollary 2.2. Since the two $\kappa$’s are compatible with the inclusion homomorphism $i^* : \mathbb{K}\pi \rightarrow \mathbb{K}\pi$, we obtain

$$(i_* \otimes i_*) \kappa(\mathbb{K}\pi(l) \otimes \mathbb{K}\pi(m)) \subset \sum_{a+b=l+m-2} (I\pi)^a \otimes (I\pi)^b.$$  

This proves (i).

We next prove (ii). First we assume that $\text{rot}_f(\partial_j \Sigma) = +1$ for $j = 1, \ldots, n$, so that $f$ extends to a framing $\bar{f}$ on $\overline{\Sigma}$. The assertion is true for $\overline{\Sigma}$ by Corollary 2.9 and Corollary 2.2. Since the operations $\mu^f$ and $\mu^{\bar{f}}$ are compatible with $i_*$, we obtain

$$(i_* \otimes i_*) \mu^f(\mathbb{K}\pi(m)) \subset \sum_{a+b=m-2} |(I\pi)^a| \otimes (I\pi)^b.$$  

This proves (ii) under the assumption above.

For any $\chi \in H^1(\Sigma, \mathbb{Z})$ the map $\gamma \mapsto \chi(\gamma) 1 \otimes \gamma$ is a derivation, hence of degree $(-1)$. By the preceding paragraph and Proposition 2.10, $\mu^f$ is of degree $(-2)$ for any $f$.

**Corollary 3.14.** The Goldman bracket and the framed Turaev cobracket are of degree $(-2)$ with respect to the weight filtration.

As a consequence of Proposition 3.13 and Corollary 3.14, the double bracket $\kappa$, the Goldman bracket, the operation $\mu^f$ and the framed Turaev cobracket $\delta^f$ extend to operations on the completions $\hat{\mathbb{K}}\pi$ and $|\hat{\mathbb{K}}\pi|$. In particular, the space $|\hat{\mathbb{K}}\pi|$ has the structure of a Lie bialgebra.

Furthermore, all these operations descend to the graded $\mathbb{K}$-algebra $\text{gr}^{\text{wt}}(\mathbb{K}\pi)$ and the graded $\mathbb{K}$-vector space $\text{gr}^{\text{wt}}(|\mathbb{K}\pi|) = |\text{gr}^{\text{wt}}(\mathbb{K}\pi)|$. Since $\text{gr}^{\text{wt}}(\mathbb{K}\pi) \cong \hat{T}(\text{gr} H)$ by Proposition 3.12, we can transfer these operations to $\hat{T}(\text{gr} H)$ and $|\hat{T}(\text{gr} H)|$. They are maps of degree $(-2)$. For example, $\kappa$ induces the double bracket

$$\kappa_{\text{gr}} : \hat{T}(\text{gr} H) \otimes \hat{T}(\text{gr} H) \rightarrow \hat{T}(\text{gr} H) \otimes \hat{T}(\text{gr} H),$$

and the self-intersection map $\mu^f_{\text{gr}}$ induces the map

$$\mu^f_{\text{gr}} : \hat{T}(\text{gr} H) \rightarrow |\hat{T}(\text{gr} H)| \otimes \hat{T}(\text{gr} H).$$
Definition 3.15. The completed Goldman-Turaev Lie bialgebra \( \hat{\mathfrak{g}}(\Sigma) = \hat{\mathfrak{g}}(\Sigma, f) \) with respect to the framing \( f \) is the Lie bialgebra structure on \( |\mathbb{K}\pi| \) described above. The associated graded Lie bialgebra structure on the degree completion of \( \text{gr}^{\text{wt}}(|\mathbb{K}\pi|) \cong |\hat{T}(\text{gr} H)| \) is denoted by \( \text{gr}\hat{\mathfrak{g}}(\Sigma, f) \).

To give a description of \( \kappa_{gr} \) and \( \mu_{gr}^f \), we introduce two bilinear maps on \( \text{gr} H \) as follows. The intersection pairing on \( H \) induces a (non-degenerate) skew-symmetric bilinear form
\[
\langle \cdot, \cdot \rangle : (H/H^{(2)}) \times (H/H^{(2)}) \to \mathbb{K}.
\]
We define the symmetric bilinear operation on \( H^{(2)} = \bigoplus_{j=1}^n \mathbb{K}z_j \) by
\[
\delta : H^{(2)} \times H^{(2)} \to H^{(2)}, \quad \delta(z_j, z_k) := \delta_{jk} z_k.
\]
These maps extend to \( \text{gr} H \times \text{gr} H \) in the following way: \( \langle \cdot, \cdot \rangle \) extends by zero on \( H^{(2)} \) and \( \delta \) extends by zero on \( H/H^{(2)} \). Then,

**Lemma 3.16.** For any \( u, v \in \text{gr} H \),
\[
\kappa_{gr}(u, v) = \langle u, v \rangle (1 \otimes 1) + \delta(u, v) \otimes 1 - 1 \otimes \delta(u, v).
\]

**Proof.** We introduce the notation
\[
\mathbb{K}\pi^{\otimes 2}(m) := \sum_{a+b=m} \mathbb{K}\pi(a) \otimes \mathbb{K}\pi(b), \quad m \geq 0.
\]
To prove the lemma, we may assume that \( u \) and \( v \) are elements in the basis \( x_i, y_i, z_j \in \text{gr} H \), which comes from the standard generators \( \alpha_i, \beta_i, \gamma_j \in \pi \). Then we can check the formula by using Proposition 2.14. For example, we compute
\[
\kappa(\alpha_i - 1, \beta_i - 1) = \beta_i \otimes \alpha_i \equiv 1 \otimes 1 \mod \mathbb{K}\pi^{\otimes 2}(1).
\]
This proves \( \kappa_{gr}(x_i, y_i) = 1 \otimes 1 \). Let us give one more example.
\[
\kappa(\gamma_j - 1, \gamma_j - 1) = \gamma_j \otimes \gamma_j - 1 \otimes \gamma_j^2
\]
\[
= (\gamma_j - 1) \otimes \gamma_j - 1 \otimes (\gamma_j - 1) - 1 \otimes (\gamma_j - 1)^2
\]
\[
\equiv (\gamma_j - 1) \otimes 1 - 1 \otimes (\gamma_j - 1) \mod \mathbb{K}\pi^{\otimes 2}(3).
\]
This proves \( \kappa_{gr}(z_j, z_j) = z_j \otimes 1 - 1 \otimes z_j \). The other cases can be checked similarly. \( \square \)

**Proposition 3.17.** Let \( u = u_1 \cdots u_i, v = v_1 \cdots v_m \in \hat{T}(\text{gr} H) \) with \( u_i, v_j \in \text{gr} H \). Then,
\[
\kappa_{gr}(u, v) = \sum_{i,j} \langle u_i, v_j \rangle v_1 \cdots v_{j-1} u_{i+1} \cdots u_l \otimes u_1 \cdots u_{i-1} u_i v_j + \cdots v_m
\]
\[
+ \sum_{i,j} v_1 \cdots v_{j-1} \delta(v_j, u_i) u_{i+1} \cdots u_l \otimes u_1 \cdots u_{i-1} v_{j+1} + \cdots v_m
\]
\[
- \sum_{i,j} v_1 \cdots v_{j-1} u_{i+1} \cdots u_l \otimes u_1 \cdots u_{i-1} \delta(u_i, v_j) v_{i+1} + \cdots v_m.
\]
Proof. This follows from the previous lemma and the graded version of Corollary 2.2 \qed

We next compute the graded quotient of \( \mu^f \).

**Proposition 3.18.** Let \( f \) be a framing on \( \Sigma \), and set \( q_j := \text{rot}_f(|\gamma_j|) + 1 \in \mathbb{Z} \). Then,

\[
\mu^f_{gr}|_{H/H^{(2)}} = 0, \quad \mu^f_{gr}(z_j) = q_j(1 \otimes 1),
\]

where \( 1 = |1| \in |\hat{T}(\text{gr } H)| \). Furthermore, define the \( \mathbb{K} \)-linear function \( c_f : \text{gr } H \to \mathbb{K} \) by \( H/H^{(2)} \to 0 \) and \( z_j \mapsto q_j \). Then, for any \( u = u_1 \cdots u_m \in |\hat{T}(\text{gr } H)| \) with \( u_i \in \text{gr } H \),

\[
\mu^f_{gr}(u) = \sum_{i=1}^m c_f(u_i) 1 \otimes u_1 \cdots u_{i-1}u_{i+1} \cdots u_m \\
+ \sum_{j<k} \langle u_j, u_k \rangle |u_{j+1} \cdots u_{k-1}| \otimes u_1 \cdots u_{j-1}u_{j+1} \cdots u_k \\
+ \sum_{j<k} |\delta(u_k, u_j) u_{j+1} \cdots u_{k-1}| \otimes u_1 \cdots u_{j-1}u_{j+1} \cdots u_k \\
- \sum_{j<k} |u_{j+1} \cdots u_{k-1}| \otimes u_1 \cdots u_{j-1} \delta(u_j, u_k) u_{k+1} \cdots u_l
\]

**Proof.** Since \( x_i \) and \( y_i \in H/H^{(2)} \) are of degree 1, \( \mu^f_{gr}(x_i) = \mu^f_{gr}(y_i) = 0 \). Therefore, the restriction of \( \mu^f_{gr} \) to \( H/H^{(2)} \) is trivial. Writing \( f = f^{\text{adp}} + \chi \) with \( \chi \in H^1(\Sigma, \mathbb{Z}) \), we have \( q_j = \chi(\gamma_j) \). Then, by Proposition 2.10,

\[
\mu^f(\gamma_j - 1) = \mu^f(\gamma_j) = \mu^{\text{adp}}(\gamma_j) + \chi(\gamma_j) 1 \otimes \gamma_j \equiv q_j(1 \otimes 1).
\]

Here, the last \( \equiv \) is equivalence modulo \( |\mathbb{K}\pi(1)| \otimes |\mathbb{K}\pi(0)| + |\mathbb{K}\pi(0)| \otimes |\mathbb{K}\pi(1)| \). Therefore, \( \mu^f_{gr}(z_j) = q_j(1 \otimes 1) \).

Corollary 2.9 descends to a product formula for \( \mu^f_{gr} \) of the same form, and we have

\[
\mu^f_{gr}(u_1 \cdots u_m) = \sum_i (1 \otimes u_1 \cdots u_{i-1}) \mu^f_{gr}(u_i)(1 \otimes u_{i+1} \cdots u_m) \\
+ \sum_i (1 \cdot |1) \kappa_{gr}(u_1 \cdots u_{i-1}, u_i)(1 \otimes u_{i+1} \cdots u_m).
\]

The first term becomes \( \sum_{i=1}^m c_f(u_i) 1 \otimes u_1 \cdots u_{i-1}u_{i+1} \cdots u_m \). By using Proposition 3.17 we can compute the second term, and we obtain the formula as required. \( \square \)

### 3.5 Expansions

We introduce several definitions about expansions of the fundamental group \( \pi \). In what follows, \( \hat{\text{gr}}^{\text{wt}}(\mathbb{K}\pi) \) is the degree completion of the graded Hopf algebra \( \text{gr}^{\text{wt}}(\mathbb{K}\pi) \).

**Definition 3.19.** A *group-like expansion* is an isomorphism \( \theta : \mathbb{K}\pi \to \hat{\text{gr}}^{\text{wt}}(\mathbb{K}\pi) \) of complete filtered Hopf algebras such that the induced map \( \text{gr} \theta : \hat{\text{gr}}^{\text{wt}}(\mathbb{K}\pi) \to \hat{\text{gr}}^{\text{wt}}(\mathbb{K}\pi) \) is the identity.
Let us make this definition more explicit. We have the isomorphism $\hat{\text{gr}}^{\text{wt}}(\hat{\mathbb{K}}\pi) \cong \hat{T}(\text{gr} H)$ of graded Hopf algebras by Proposition 3.12. Furthermore, let us fix standard generators $\alpha_i, \beta_i, \gamma_j \in \pi$. Then $\text{gr} H$ and $H$ are identified, and

$$\hat{\text{gr}}^{\text{wt}}(\hat{\mathbb{K}}\pi) \cong \hat{T}(\text{gr} H) \cong \hat{T}(H) = A.$$ 

Giving a group-like expansion is equivalent to giving an isomorphism $\theta : \hat{\mathbb{K}}\pi \rightarrow A$ of complete filtered Hopf algebras (the filtration of $A$ being the weight filtration) with the property

$$\theta(\alpha_i) \equiv_2 1 + x_i, \quad \theta(\beta_i) \equiv_2 1 + y_i, \quad \theta(\gamma_j) \equiv_3 1 + z_j,$$

where $\equiv_m$ means equality modulo $A_{\geq m}$.

Remark 3.20. The condition above implies that $\theta$ maps $(\hat{\mathbb{I}}\pi)_m$ to the degree filtration $A^{\text{deg}}_{\geq m}$. However, it is not necessarily the case that the associated graded map $\text{gr}^f \theta : A^{\text{deg}}_{\geq 1}/A^{\text{deg}}_{\geq 2} \rightarrow A^{\text{deg}}_{\geq 1}/A^{\text{deg}}_{\geq 2}$ is the identity. We only know that the matrix presentation of this map with respect to the basis $x_i, y_i, z_j$ is of the form

$$\begin{pmatrix}
I & 0 & 0 \\
0 & I & 0 \\
A & B & I
\end{pmatrix}.$$ 

Fixing the generators $\alpha_i, \beta_i, \gamma_j$, we continue to give several definitions.

Definition 3.21. (i) A group-like expansion $\theta : \hat{\mathbb{K}}\pi \rightarrow A$ is called tangential if for any $j = 1, \ldots, n$, there is a group-like element $g_j \in A$ such that $\theta(\gamma_j) = g_j e^{z_j} g_j^{-1}$.

(ii) A tangential expansion $\theta$ is called special if $\theta(\gamma_0) = \exp(\omega)$, where

$$\omega := \sum [x_i, y_i] + \sum z_j \in A.$$ 

An example of a tangential expansion is the exponential one given by

$$\theta_{\exp} : \alpha_i \mapsto e^{x_i}, \quad \beta_i \mapsto e^{y_i}, \quad \gamma_j \mapsto e^{z_j}.$$ 

Remark 3.22. If $n = 0$, special expansions are called symplectic expansions [25]. If $g = 0$, the terminology is due to [26]. The condition in Definition 3.21 (ii) was given in [22, §7.2]. The existence of special expansions for any $g$ and $n$ is known, and can be deduced from the existence for $n = 0$ and for $g = 0$ by using a gluing argument which is similar to Proposition 3.24 below.

Let $\hat{\text{gr}}^{\text{wt}}(\hat{\mathbb{K}}\pi)$ be the degree completion of the filtered $\mathbb{K}$-vector space $\text{gr}^{\text{wt}}(\mathbb{K}\pi) \cong |\text{gr}^{\text{wt}}(\mathbb{K}\pi)|$. Any group-like expansion $\theta : \mathbb{K}\pi \rightarrow \hat{\text{gr}}^{\text{wt}}(\mathbb{K}\pi)$ induces an isomorphism

$$\theta : |\mathbb{K}\pi| \rightarrow \hat{\text{gr}}^{\text{wt}}(\mathbb{K}\pi)$$

of complete filtered $\mathbb{K}$-vector spaces. Note that the source and the target of this map are the underlying space for the Lie bialgebra $\hat{\mathfrak{g}}(\Sigma)$ and for $\hat{\text{gr}} \hat{\mathfrak{g}}(\Sigma)$, respectively.
Definition 3.23. A homomorphic expansion is a (tangential) group-like expansion which induces an isomorphism $\tilde{\mathfrak{g}}(\Sigma) \to \text{gr} \tilde{\mathfrak{g}}(\Sigma)$ of Lie bialgebras.

By definition, any homomorphic expansion solves the GT formality problem.

To end this section we give a proof of Lemma 3.9 which in fact can be strengthened to the following proposition. The proof is a gluing argument for expansions.

**Proposition 3.24.** Keep the notation in §3.1 and §3.2. Let $\theta : \pi \to A$ be a tangential expansion. Then there is a group-like expansion $\tilde{\theta} : \pi \to \tilde{A}$ such that

$$\tilde{i}_* \circ \theta = \tilde{\theta} \circ i_* : \mathbb{K}\pi \to \tilde{A}.$$  

If $\theta$ is special, then one can choose $\tilde{\theta}$ to be special as well.

**Proof.** Denote standard generators of $\pi_1(\Sigma_{1,1})$ by $\alpha, \beta$, and let $x = [\alpha], y = [\beta] \in H_1(\Sigma_{1,1}, \mathbb{K})$. Let $A_{xy}$ be the complete tensor algebra generated by $H_1(\Sigma_{1,1}, \mathbb{K}) \cong \mathbb{K}x \oplus \mathbb{K}y$. Take a special expansion $\theta' : \pi_1(\Sigma_{1,1}) \to A_{xy}$ for $\Sigma_{1,1}$.

Introducing the map $i^{(j)}_* : A_{xy} \to \tilde{A}$, $x \mapsto x_{g+j}$, $y \mapsto y_{g+j}$, we define the map $\tilde{\theta} : \pi \to \tilde{A}$ by the formula

$$\alpha_i \mapsto \tilde{i}_* \theta(\alpha_i), \quad \beta_i \mapsto \tilde{i}_* \theta(\beta_i), \quad (i = 1, \ldots, g),$$
$$\alpha_{g+j} \mapsto \tilde{i}_* (g_j) i^{(j)}_* (\theta'([\alpha])) \tilde{i}_* (g_j^{-1}), \quad (j = 1, \ldots, n),$$
$$\beta_{g+j} \mapsto \tilde{i}_* (g_j) i^{(j)}_* (\theta'([\beta])) \tilde{i}_* (g_j^{-1}), \quad (j = 1, \ldots, n).$$

To prove $\tilde{i}_* \circ \theta = \tilde{\theta} \circ i_*$, one needs to check $\tilde{\theta}(i_* \gamma_j) = \tilde{i}_* \theta(\gamma_j)$. In fact, since $\theta'([\alpha, \beta]) = \exp([x, y])$, one has $i^{(j)}_* \theta'([\alpha, \beta]) = \tilde{i}_* (e^{g_j})$. Then

$$\tilde{\theta}(i_* \gamma_j) = \tilde{\theta}(i_* \gamma_j) = \tilde{i}_* (g_j) i^{(j)}_* \theta'([\alpha, \beta]) \tilde{i}_* (g_j^{-1}) = \tilde{i}_* (g_j e^{g_j}) \tilde{i}_* (g_j^{-1}) = \tilde{i}_* \theta(\gamma_j).$$

Furthermore, assume that $\theta$ is special, i.e., $\theta(\gamma_0) = \exp(\omega)$. Then,

$$\tilde{\theta}(\prod_{i=1}^{g+n} \alpha_i \beta_i^{-1}) = \tilde{\theta}(i_* \gamma_0) = \tilde{i}_* \theta(\gamma_0) = \tilde{i}_* \exp(\omega) = \exp \left( \sum_{i=1}^{g+n} [x_i, y_i] \right).$$

This shows that $\tilde{\theta}$ is special.

\qed

4 Divergences

In this section, we describe non-commutative versions of the divergence map and their relation to the Turaev cobracket and the maps $\mu^I$ and $\mu^{J\pi}$.

Let us fix standard generators $\alpha_i, \beta_i, \gamma_j$ for $\pi$. This choice induces the choice of generators $x_i, y_i, z_j$ for the completed free associative algebra $A = \hat{T}(H)$.
4.1 Definition and properties of divergence cocycles

Recall the following definitions from [2]. Let $A^T$ denote the algebra obtained from $A$ by adding a new free generator $T$. Let $\text{Der}_T(A^T)$ denote the Lie algebra of derivations on $A^T$ which vanish on $T$ and are of positive $T$-degree, and let $\text{tDer}_T(A^T)$ denote the Lie subalgebra of \textit{tangential derivations}, that is

$$\text{tDer}_T(A^T) := \{ u \in \text{Der}_T(A^T) \mid u(z_j) = [z_j, u_j] \text{ for some } u_j \in A^T \}.$$ 

Lie algebras $\text{Der}_T(A^T)$ and $\text{tDer}_T(A^T)$ carry a grading with respect to $T$. Their parts of $T$-degree equal to one are called \textit{double derivations} and \textit{tangential double derivations}, and denoted by $D_A$ and $\text{tD}_A$, respectively. Those graded components have $A$-bimodule structures where the action of $a \otimes b$ is given by replacing $T$ with $bTa$.

Remark 4.1. By replacement of $T$ with $\otimes$, the space $D_A$ is identified with $\text{Der}(A, A \otimes A)$. For instance, to a derivation of $A^T$ mapping $x \mapsto yTz$ we associate a double derivation of $A$ mapping $x \mapsto y \otimes z$.

Following [2], we define the \textit{divergence map} $\text{Div} : \text{Der}_T(A^T) \to |A^T| \otimes |A^T|$ by formula

$$\text{Div}(u) = \left| \sum_i \frac{\partial u(x_i)}{\partial x_i} + \sum_i \frac{\partial u(y_i)}{\partial y_i} + \sum_j \frac{\partial u(z_j)}{\partial z_j} \right|.$$ 

Here, the operator $\frac{\partial}{\partial x_i} \in \text{Der}(A^T, (A^T)^{\otimes 2})$ is the double derivation which sends $x_i$ to $1 \otimes 1$ and the other generators to 0. The operators $\frac{\partial}{\partial y_i}$ and $\frac{\partial}{\partial z_j}$ are defined in a similar way. Also, $| \cdot |$ denotes the projection $A^T \otimes A^T \to |A^T| \otimes |A^T|$. The divergence map is a 1-cocycle [2 Proposition 3.1]. That is,

$$\text{Div}([u, v]) = u.\text{Div}(v) - v.\text{Div}(u).$$

For $u \in \text{tDer}_T(A^T)$, one can introduce the \textit{tangential divergence map} given by

$$\text{tDiv} : \text{tDer}_T(A^T) \longrightarrow |A^T| \otimes |A^T|,$$

$$u \mapsto \left| \sum_i \frac{\partial u(x_i)}{\partial x_i} + \sum_i \frac{\partial u(y_i)}{\partial y_i} + \sum_j \left[ z_j, \frac{\partial u(z_j)}{\partial z_j} \right] \right|, \quad (18)$$

where $u(z_j) = [z_j, u_j]$. Note that for $a \in A^T$ and $b = b' \otimes b'' \in A^T \otimes A^T$ we have

$$[a, b] := (a \otimes 1)b - b(1 \otimes a) = ab' \otimes a'' - b' \otimes b'' a.$$ 

The tangential divergence is again a 1-cocycle, and the relation between Div and tDiv is given by

$$\text{tDiv}(u) = \text{Div}(u) - \sum_j 1 \wedge |u_j|.$$ 

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With the exponential expansion $\theta_{\exp}$ in mind, set $\alpha_i = e^{x_i}, \beta_i = e^{y_i}, \gamma_j = e^{z_j}$. This gives a set of group-like generators of $A$. We can now define one more version of the divergence map which is convenient for our purposes:

$$
\text{gDiv} : \text{tDer}_T(A^T) \longrightarrow |A^T| \otimes |A^T|,
$$

$$
u \mapsto \sum_i \frac{\partial u(\alpha_i)}{\partial \alpha_i} - 1 \otimes u(\alpha_i)\alpha_i^{-1} + \sum_i \frac{\partial u(\beta_i)}{\partial \beta_i} - 1 \otimes u(\beta_i)\beta_i^{-1} + \sum_j \left[ \gamma_j, \frac{\partial u_j}{\partial \gamma_j} \right].
$$

Here the double derivation $\frac{\partial}{\partial \alpha_i} = \frac{\partial}{\partial e^{x_i}}$, has the property $\frac{\partial}{\partial e^{x_i}} \alpha_i = 1 \otimes 1$, and it can be written as $\frac{\partial}{\partial e^{x_i}} = e^{-x_i} \frac{\text{ad}_{e^{x_i}}}{1-e^{-x_i}} \frac{\partial}{\partial x_i}$. See also Lemma 6.4 in [2]. Similar considerations apply to $\beta_i$ and $\gamma_j$.

**Lemma 4.2.** Let $r(s) = \log((e^s - 1)/s) \in \mathbb{K}[s]$ and $x \in \{x_i, y_i, z_j\}$. For any derivation $u \in \text{Der}_T(A^T)$, the following holds:

$$
\left| \frac{\partial u(e^x)}{\partial e^x} \right| = \left| \frac{\partial u(x)}{\partial x} \right| + u.(1 \otimes |x| + |r(x \otimes 1 - 1 \otimes x)|).
$$

**Proof.** Using the convention in the last part of [11] one computes as follows;

$$
\left| \frac{\partial u(e^x)}{\partial e^x} \right| = \left| \frac{\partial}{\partial e^x}, u \right|(e^x)
= \frac{\partial e^x}{\partial x} \circ \left( \left[ \frac{\partial}{\partial e^x}, u \right](x) \right)
= \frac{\partial e^x}{\partial x} \circ \left( \frac{\partial e^x}{\partial x} \cdot u \left( \frac{\partial x}{\partial e^x} \right) \bigg|_{u(x)} \right)
= \frac{\partial}{\partial e^x} \left( u(x) \right) \cdot \frac{\partial e^x}{\partial x} - \frac{\partial e^x}{\partial x} \cdot u \left( \frac{\partial x}{\partial e^x} \right)
= \frac{\partial u(x)}{\partial x} - u \cdot \left| \log \left( \frac{\partial x}{\partial e^x} \right) \right|
= \left| \frac{\partial u(x)}{\partial x} \right| + u \cdot \left| \log \left( \frac{\partial e^x}{\partial x} \right) \right|.
$$

In the second line, we have used the identity $v(f(x)) = \frac{\partial f(x)}{\partial x} \circ v(x)$ for any derivation $v$ and function in one variable $f(x)$. In the fifth line, we have used the chain rule $\frac{\partial}{\partial f} \circ \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}$, the identity $u.|f(a)| = |f'(a)u(a)|$, and $\frac{\partial e^x}{\partial x} \cdot \frac{\partial x}{\partial e^x} = 1 \otimes 1$. Finally, we compute

$$
\log\left( \frac{\partial e^x}{\partial x} \right) = \log\left( \frac{e^{x \otimes 1} - 1 \otimes x}{2 \otimes 1 - 1 \otimes x} \right) = 1 \otimes x + \log\left( \frac{e^{x \otimes 1 - 1 \otimes x} - 1 \otimes 1}{2 \otimes 1 - 1 \otimes x} \right).
$$

\[\square\]
For \( \rho \in |A^T| \otimes |A^T| \), one defines an exact Lie algebra 1-cocycle \( \partial \rho : u \mapsto u.\rho. \)

Let \( \tilde{\Delta} = (1 \otimes \iota) \Delta : A^T \to A^T \otimes A^T \) with \( \Delta \) the coproduct and \( \iota \) the antipode, and set

\[
\mathbf{r} := |\tilde{\Delta} \sum_i (r(x_i) + r(y_i))| \in |A^T| \otimes |A^T|.
\]

**Proposition 4.3.** The maps \( g\text{Div} \) and \( t\text{Div} \) are related by formula

\[
g\text{Div} = t\text{Div} + \partial \mathbf{r} : \text{tDer}_T(A^T) \to |A^T| \otimes |A^T|.
\]

The map \( g\text{Div} \) is a 1-cocycle on the Lie algebra \( \text{tDer}_T(A^T) \).

**Proof.** Using Lemma 4.2, we rewrite the definition of the map \( g\text{Div} \) as follows:

\[
g\text{Div}(u) = t\text{Div}(u) + u.|\tilde{\Delta} \sum_i (r(x_i) + r(y_i))|.
\]

Here we have used that \( \tilde{\Delta} \mathbf{r}(x) = \mathbf{r}(x \otimes 1 - 1 \otimes x) \) and that \( |u(e^x)e^{-x}| = u.|x| \); for the latter equality, cf. [2, Lemma 6.4]. The map \( g\text{Div} \) is a 1-cocycle since it is a sum of the 1-cocycle \( t\text{Div} \) and the coboundary \( \partial \mathbf{r} \).

The cocycles \( g\text{Div} \) and \( t\text{Div} \) are compatible with setting \( T = 1 \). Their restrictions to \( \text{tD}_A \) are again denoted by the same symbols. These maps take values in the \( T \)-linear part of \( |A^T| \otimes |A^T| \), which we shall tacitly identify with \( (|A| \otimes A) \oplus (A \otimes |A|) \). Moreover, they are compatible with the \( A \)-bimodule structure on \( \text{tD}_A \) in the following way:

\[
g\text{Div}(a\phi) = a g\text{Div}(\phi) + (1 \otimes |\cdot|)((\phi'(a) \otimes \phi''(a))),
g\text{Div}(\phi a) = g\text{Div}(\phi) a + (|\cdot| \otimes 1)((\phi'(a) \otimes \phi''(a)) (19)
\]

where \( \phi \in \text{tD}_A, a \in A \) and \( \phi(a) = \phi'(a)T\phi''(a) \).

### 4.2 Relation to cobracket

Let \( \theta_\text{exp} \) be the exponential group-like expansion defined by \( \theta_\text{exp}(\alpha_i) = e^{x_i}, \theta_\text{exp}(\beta_i) = e^{y_i}, \theta_\text{exp}(\gamma_j) = e^{z_j} \). It induces an isomorphism of complete Hopf algebras \( \widehat{K}_\pi \cong \widehat{T}(H) = A \). Using \( \theta_\text{exp} \), we can now transport the topological operations to \( A \). Denote the corresponding double bracket by \( \Pi_\text{exp} \), the Lie cobracket by \( \delta^L_\text{exp} \), and the \( \mu \)-maps by \( (\mu^L)^{\text{exp}}, (\mu^S)^{\text{exp}} \).

Let \( \Pi \) be a double bracket on \( A \). For any \( a \in A \), one has

\( \Pi(a, \cdot) \in \text{Der}(A, A^2) = \text{D}_A \subset \text{Der}_T(A^T); \)

see Remark 4.1. Following [2], we say that \( \Pi \) is **tangential** if \( \Pi(a, \cdot) \in \text{tD}_A \) for any \( a \in A \).

**Lemma 4.4.** The double bracket \( \Pi_\text{exp} \) is tangential.
Proof. It is enough to check that $\Pi_{\text{exp}}(\alpha, \cdot) \in tD_A$ for each group-like generator $\alpha \in \{\alpha_i, \beta_i, \gamma_j\}$. We compute for $a = \gamma_j$, the other cases being similar. Proposition 2.13 shows that $\Pi_{\text{exp}}(\gamma_j, \cdot) \in \text{Der}_T(A^T)$ acts as follows:

\[
\begin{align*}
\gamma_k &\mapsto [\gamma_k, T\gamma_j], \quad \text{for } k = 1, \ldots, j - 1, \\
\gamma_j &\mapsto [\gamma_j, T\gamma_j], \\
\gamma_k &\mapsto 0, \quad \text{for } k = j + 1, \ldots, n.
\end{align*}
\]

Hence, $\Pi_{\text{exp}}(\gamma_j, \cdot)$ is tangential, as required. 

Let $\Pi$ be a tangential double bracket on $A$. Composing $g\text{Div}$ with the map $A \to tD_A, a \mapsto \Pi(a, \cdot)$, one obtains the map $g\text{Div}_\Pi : A \to (|A| \otimes A) \oplus (A \otimes |A|)$. The following identity is a direct consequence of (19) and the fact that $a \mapsto \Pi(a, \cdot)$ is a derivation:

\[
g\text{Div}_\Pi(ab) = a g\text{Div}_\Pi(b) + g\text{Div}_\Pi(a) b + (| \cdot | \otimes 1)\Pi(a, b) + (1 \otimes | \cdot |)\Pi(b, a). \tag{20}
\]

Theorem 4.5. For the adapted framing, one has the following:

\[
g\text{Div}_{\Pi_{\text{exp}}} = (\mu_{\text{adp}})_{\text{exp}} + (\mu_{\text{adp}})_{\text{exp}} : A \to (|A| \otimes A) \oplus (A \otimes |A|).
\]

Proof. By Proposition 2.8 and formula (20), the $\mu$-maps and the map $g\text{Div}_\Pi$ behave in the same way under products. Hence, it is enough to check the equality on generators. We compute for $\alpha_i$, the other cases being similar. Set $u := \Pi_{\text{exp}}(\alpha_i, \cdot) \in t\text{Der}_T(A^T)$. Since $u(\alpha_i) = [\alpha_i, T\alpha_i]$, one has $|u(\alpha_i)\alpha_i^{-1}| = |\alpha_i T - T\alpha_i| = 0$ and

\[
\left| \frac{\partial u(\alpha_i)}{\partial \alpha_i} \right| = \left| [\otimes, T\alpha_i] + [\alpha_i, T\otimes] \right| = 1 \otimes |T\alpha_i| - |T| \otimes |\alpha_i|.
\]

Here, $[\otimes, T\alpha_i] = 1 \otimes T\alpha_i - T\alpha_i \otimes 1$ and $[\alpha_i, T\otimes] = \alpha_i T \otimes 1 - T \otimes \alpha_i$. Since $u(\beta_i) = \beta_i T\alpha_i$,

\[
\left| \frac{\partial u(\beta_i)}{\partial \beta_i} \right| - 1 \otimes u(\beta_i)\beta_i^{-1} = 1 \otimes T\alpha_i - 1 \otimes \beta_i T\alpha_i \beta_i^{-1} = 0.
\]

The contributions from the other generators can be seen to vanish, and

\[
g\text{Div}_{\Pi_{\text{exp}}} (\alpha_i) = 1 \otimes \alpha_i - 1 \otimes |\alpha_i| = (\mu_{\text{adp}})_{\text{exp}}(\alpha_i) + (\mu_{\text{adp}})_{\text{exp}}(\alpha_i),
\]

as required. 

Now, let $f$ be a general framing, differing from the adapted one by $\chi \in H^1(\Sigma, \mathbb{Z})$. Recall that $H = H_1(\Sigma, \mathbb{K})$ is equipped with a pairing $\langle \cdot, \cdot \rangle$. By the choice of standard generators $\alpha_i, \beta_i, \gamma_j$ the kernel of this pairing can be split off. Namely, one can decompose $H = H_{xy} \oplus H_z$, where $\langle \cdot, \cdot \rangle$ is defined a non-degenerate pairing on $H_{xy}$ and $H_z = H^{(2)}$ in the notaion in [35]. With this splitting, the element $\chi$ can be represented as a sum of an element $p \in (H_{xy})^* \cong H_{xy}$ and of a linear form $q \in (H_z)^*$. Here, $p \in H_{xy}$ is identified with the linear form $\langle p, \cdot \rangle$. Furthermore, recall from [2] Lemma 3.2 that the maps

\[
c_j : u \mapsto |u_j| \wedge 1 = |u_j| \otimes 1 - 1 \otimes |u_j|
\]

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Corollary 4.8. One has
c_{\chi} : t\text{Der}_T(A^T) \to |A^T| \otimes |A^T|,
   \quad u \mapsto u.|p| \wedge 1 - c_q(u),
where we view \( p \) as an element of \( H_{cg} \).

Recall that the \( T \)-degree one part of \( t\text{Der}_T(A^T) \) is denoted \( t\text{DA} \) and is an \( A \)-bimodule.
The cocycle \( c_{\chi} \) maps it to the \( T \)-linear part of \( |A^T| \otimes |A^T| \):
\[
c_{\chi} : t\text{DA} \to (|A^T| \otimes |A^T|)^{T-\text{lin}} \cong (|A| \otimes A) \oplus (A \otimes |A|).
\]
This is an \( A \)-bimodule map.

**Proposition 4.6.** For every \( \gamma \in \pi \), we have
\[
c_{\chi}(\{\theta_{\exp}(\gamma), \cdot\}_\Pi_{\exp}) = \chi(\gamma) 1 \land \theta_{\exp}(\gamma).
\]

**Proof.** The map on the left hand side is a composition of a derivation and a bimodule map. Hence, it is a derivation. The map on the right hand side is also a derivation. Therefore, it is sufficient to check the formula on standard generators. We compute the case \( \gamma = \alpha_i \) only, the other cases being similar. Since \( p = \sum_i (p(y_i)x_i - p(x_i)y_i) \), a direct computation using Lemma 6.4 in [2] and Proposition 2.14 shows that
\[
\{\alpha_i, |p|\}_\Pi_{\exp} = -p(x_i) \{\alpha_i, |y_i|\}_\Pi_{\exp} = -p(x_i)|T\alpha_i|.
\]
Hence \( c_{\chi}(\{\theta_{\exp}(\alpha_i), \cdot\}_\Pi_{\exp}) = -p(x_i)|T\alpha_i| \land 1 = \chi(\alpha_i) 1 \land \alpha_i \), as required. \qed

**Corollary 4.7.** Set \( g\text{Div}^f := g\text{Div} + c_{\chi} \). Then,
\[
g\text{Div}^f_{\Pi_{\exp}} = (\mu^f)_{\exp} + (\mu^f_*)_{\exp} : A \to (|A| \otimes A) \oplus (A \otimes |A|).
\]

By setting \( T = 1 \), one obtains the evaluation maps \( A^T \to A \) and \( t\text{Der}_T(A^T) \to t\text{Der}(A) \), where
\[
t\text{Der}(A) := \{(u, u_1, \ldots, u_n) \in \text{Der}(A) \times A^\otimes n | u(z_j) = [z_j, u_j]\}
\]
is the Lie algebra of tangential derivations on \( A \). Using the same formula as \( [18] \), one obtains the tangential divergence cocycle \( t\text{Div} : t\text{Der}(A) \to |A| \otimes |A| \). Setting \( T = 1 \) and the tangential divergence maps are compatible.

By Proposition [2.8] one obtains \( \delta^f \) from \( \mu^f + \mu^f_* \) by applying \(|\cdot|\). Through the exponential group-like expansion \( \theta_{\exp} \), applying \(|\cdot|\) corresponds to setting \( T = 1 \). Therefore, we obtain the following description for the Turaev cobracket under \( \theta_{\exp} \):

**Corollary 4.8.** One has
\[
\delta^f_{\exp} = g\text{Div}^f \circ \Pi_{\exp} : |A| \to |A| \otimes |A|,
\]
where \( \Pi_{\exp} : |A| \to t\text{Der}(A) \), \( g\text{Div}^f = g\text{Div} + c_{\chi} : t\text{Der}(A) \to |A| \otimes |A| \) are the \( T = 1 \) induced maps, denoted by the same letter.
4.3 The group 1-cocycle $j$

Let $L \subset A$ denote the space of primitive elements, that is $L$ is the completed free Lie algebra in generators $x_i, y_i, z_j$. As in [2], one defines the Lie algebra

$$t\text{der} = t\text{der}^{(g,n+1)} := \{(u, u_1, \ldots, u_n) ∈ \text{Der}(L) × L^⊗n \mid u(z_j) = [z_j, u_j]\},$$

and its Lie subalgebra

$$t\text{der}^+ := \{(u, u_1, \ldots, u_n) ∈ \text{Der}^+(L) × L^⊗n \mid u(z_j) = [z_j, u_j]\},$$

together with its integrating group

$$\text{TAut} = \text{TAut}^{(g,n+1)} := \{\text{Aut}^+(L) × L^⊗n \mid F(z_j) = e^{-f_j}z_je^{f_j}\}.$$ Here, $\text{Der}^+(L)$ is the Lie algebra of derivations on $L$ of positive degree with respect to the weight filtration, and it integrates to the group $\text{Aut}^+(L)$.

As proved in [2, Proposition 3.5], the cocycle $t\text{Div}$ takes values in $\tilde{\Delta}(|A|) ⊂ |A| ⊗ |A|$ when restricted to $t\text{der}$. Since $\tilde{\Delta}$ is injective, we occasionally identify $\tilde{\Delta}(|A|)$ with $|A|$. The cocycle $t\text{Div}$ integrates to a group 1-cocycle

$$j : \text{TAut} \to |A| \tilde{\Delta} \subset |A| ⊗ |A|.$$ 

Proposition 4.9. Let $F ∈ \text{TAut}$. Then, for any $u ∈ t\text{Der}_T(A_T)$,

$$F^*t\text{Div}(u) = t\text{Div}(u) + u.j(F^{-1}).$$

Proof. Since $F \mapsto (u \mapsto F^{-1}.t\text{Div}(FuF^{-1}) − t\text{Div}(u))$ and $F \mapsto (u \mapsto u.j(F^{-1}))$ define group 1-cocycles on $\text{TAut}$ with values in $\text{Hom}(t\text{Der}_T(A_T), |A| ⊗ |A|)$, it is enough to check that their induced Lie 1-cocycles agree. In both cases one obtains the same Lie algebra 1-cocycle:

$$\text{tder} \to \text{Hom}(t\text{Der}_T(A_T), |A| ⊗ |A|)$$

$$v \mapsto \left( u \mapsto -v.t\text{Div}(u) + t\text{Div}([v, u]) = -u.t\text{Div}(v) \right).$$

Corollary 4.10. Let $F ∈ \text{TAut}$. Then, for any $u ∈ t\text{Der}_T(A_T)$,

$$F^*g\text{Div}(u) = t\text{Div}(u) + u.(j(F^{-1}) + F^{-1}.r).$$

The cocycles $c_j : u \mapsto |u_j| \wedge 1$ similarly integrate to $C_j : F \mapsto |f_j| \wedge 1$, and $c_q$ to $C_q$ likewise. By the cocycle property, for every $F ∈ \text{TAut}$ we have

$$F^*c_j(u) = c_q(u) + u.(-C_q(F^{-1}) + F^{-1}.|p| \wedge 1)$$
Corollary 4.11. Let $F \in T\text{Aut}$. Then, for any $u \in t\text{Der}_T(A^T)$,

$$F^* g\text{Div}^T(u) = t\text{Div}(u) - c_q(u) + u \cdot (j(F^{-1}) - C_q(F^{-1}) + F^{-1}(r + |p| \wedge 1)),$$

where $t\text{Div} - c_q$ is a degree 0 map, and the exact cocycle term is of positive degree.

It is convenient to introduce a special notation for the Lie algebra 1-cocycle

$$t\text{Div}_q := t\text{Div} - c_q$$

and

$$j_q := j - C_q$$

for the corresponding group 1-cocycle.

5 Homomorphic expansions

The goal of this section is to establish Theorem 1.3 in Introduction in full generality and to introduce higher genus Kashiwara-Vergne problems.

Let us keep standard generators $\alpha_i, \beta_i, \gamma_j \in \pi$ fixed. Given any group-like expansion $\theta : \hat{K} \pi \to A$, one can transfer the topological operations to $A$. The corresponding double bracket, the Lie cobracket, and the $\mu$-map are denoted by $\kappa_\theta$, $\delta f_\theta$, and $\mu f_\theta$.

5.1 Special expansions and formality of Goldman bracket

Consider the double bracket $\Pi_{gr}$ on $A$ such that

$$\Pi_{gr}(x_i, y_j) = -\Pi_{gr}(y_i, x_j) = \delta_{ij}(1 \otimes 1),$$

$$\Pi_{gr}(z_j, z_k) = \delta_{jk}(z_j \otimes 1 - 1 \otimes z_j),$$

and the value of $\Pi_{gr}$ is zero for all the other possible pairs in the generators $x_i, y_i, z_j$. It is skew-symmetric in the sense that $\{b, a\}_{\Pi_{gr}} = -\{a, b\}_{\Pi_{gr}}$. The notation reflects the fact that $\Pi_{gr}$ corresponds to the graded version of $\kappa$; see Lemma 3.16. Using the notation from [2, §4], $\Pi_{gr}$ is given by the formula

$$\Pi_{gr} = \sum_i \left( \frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial y_i} - \frac{\partial}{\partial y_i} \otimes \frac{\partial}{\partial x_i} \right) + \sum_j \frac{\partial}{\partial z_j} \otimes \left[ \frac{\partial}{\partial z_j}, z_j \right].$$

Also recall from [2] Lemma 2.7] the double derivation $\phi_0$ defined by $\phi_0(a) = 1 \otimes a - a \otimes 1$ for all $a \in A$, and following [2, §4.4] put

$$\Pi_s = |\phi_0 \otimes s(\text{ad}_\omega)\phi_0|,$$

where $s$ is a formal power series defined by

$$s(z) = \frac{1}{1 - e^{-z}} - \frac{1}{z} = \frac{1}{2} + \sum_{m=1}^{\infty} \frac{B_{2m}}{(2m)!} z^{2m-1}. \quad (21)$$

Both the double brackets $\Pi_{gr}$ and $\Pi_s$ are tangential.
Theorem 5.1 ([28]). Let $\theta : \hat{K}\pi \rightarrow A$ be a special expansion. Then

$$\kappa_\theta = \Pi_{\text{add}} := \Pi_{\text{gr}} + \Pi_s.$$

Proof. If $n = 0$, the statement is equivalent to [27] Theorem 10.4. The general case follows from a capping argument. For the sake of the reader we provide a proof here.

Consider the capping surface $\Sigma$ as in §3.1 and keep the notation there. For $\Sigma$, one can consider the double brackets $\Pi_{\text{gr}}, \Pi_s$ and $\Pi_{\text{add}} := \Pi_{\text{gr}} + \Pi_s$. By Proposition 3.24, there is a symplectic expansion $\theta : K\pi \rightarrow A$ such that $\tilde{\iota}^* \circ \theta = \theta \circ \iota^*$. Since both maps $\iota^*$ and $\tilde{\iota}^*$ are injective, $\iota^* : \pi \rightarrow \pi$ intertwines the two $\kappa$’s, and $\kappa_\theta = \Pi_{\text{add}}$ by the case $n = 0$ as above. Therefore, it is enough to check that $\tilde{\iota}^* : A ightarrow A$ intertwines $\Pi_{\text{add}}$ and $\Pi_{\text{add}}$.

Since $\tilde{\iota}^*(\omega) = \sum x_i y_i$ and $\tilde{\iota}^*$ intertwines $\Pi_{\text{gr}}$ and $\Pi_{\text{gr}}$, we only need to check on generators. This can be done by a direct computation. \hfill \square

Remark 5.2. The previous theorem admits an alternative proof and interpretation using results from [33]. The key observation used in this proof is the fact that a double bracket is (under some technical assumptions) uniquely defined by its moment map.

Since $\Pi_s$ vanishes on $|A|$ (see [2] Proposition 4.19), one has the following corollary:

Corollary 5.3 ([22, 28]). Any special expansion $\theta$ induces an isomorphism $\theta : \hat{g}(\Sigma) \rightarrow |A| \cong \text{gr } \hat{g}(\Sigma)$ of Lie algebras.

Conversely, in our forthcoming paper, we prove that any group-like expansion inducing the Lie isomorphism is conjugate to a special expansion.

The Lie bracket $\Pi_{\text{gr}}$ on $|A|$ is in fact identified with the Lie bracket for special derivations. Following the notation in [2], we introduce the Lie subalgebra of $\text{tDer}(A)$ defined by

$$\text{sDer}(A) := \{(u, u_1, \ldots, u_n) \in \text{tDer}(A) \mid u(\omega) = 0\}.$$

As will be shown in Lemma 5.9, we have $\{a, \omega\}_{\text{gr}} = 0$ for any $|a| \in |A|$. Hence the $T = 1$ induced map $|A| \rightarrow \text{tDer}(A), |a| \mapsto \{\cdot, |a|\}_{\text{gr}}$ takes values in $\text{sDer}(A)$. Furthermore, similarly to [2] Lemma 8.3, one has the Lie algebra isomorphism $|A|/K1 \cong \text{sDer}(A)$.

We will use the following result on the center of the Lie algebra structure on $|A|$.

Theorem 5.4.

$$Z(|A|, \{\cdot, \cdot\}_{\text{gr}}) = |K[[\omega]]| \oplus \bigoplus_{j=1}^n |K[[z_j]]_{\geq 1}|.$$

This result has been proved in [7] by using the Poisson geometry of quiver varieties. For the sake of completeness, we give an elementary proof in the next subsection.
5.2 Proof of Theorem 5.4

First we need some algebraic preliminaries. For the moment, let $H$ be a finite dimensional $K$-vector space.

**Lemma 5.5** (Euclidean algorithm). Let $l, m$ be non-negative integers with $l + m > 0$. Assume that non-zero vectors $u \in H^\otimes l$ and $v \in H^\otimes m$ are commutative, i.e.,

$uv = vu \in H^\otimes (l+m)$.

Then there exist some $w \in H^\otimes \gcd(l,m)$ and $c, d \in K$ such that $u = cw'$ and $v = dw''$, where $l' = l/\gcd(l, m)$ and $m' = m/\gcd(l, m)$.

**Proof.** The lemma is trivial when $l + m = 1$ or $lm = 0$. In order to use induction on $l+m$, let $k > 1$ and assume that the lemma holds if $l + m < k$.

Suppose $l + m = k$. We may assume that $0 < l \leq m$. Take a $K$-basis $\{u_i\}_{1 \leq i \leq (\dim H)^l}$ of $H^\otimes l$ with $u_1 = u$. Then we have $v = \sum_{i=1}^{(\dim H)^l} u_i v_i'$ for some $v_i' \in H^\otimes (m-l)$, and so

$$\sum_{i=1}^{(\dim H)^l} u_1 u_i v_i' = \sum_{i=1}^{(\dim H)^l} u_i v_i' u_1.$$  

Since $\{u_i\}_{1 \leq i \leq (\dim H)^l}$ are linearly independent, we have $v_i' = 0$ for any $i \neq 1$. This means $v = uv_1'$, $uv_1' = uv_1' u$ and so $uv_1' = v_1' u$. Since $l + (m-l) = m < l + m = k$, we can apply the inductive assumption to $uv_1' = v_1' u$, that is, there exist some $w \in H^\otimes \gcd(l,m)$ and $c, d' \in K$ such that $u = cw'$ and $v_1' = d' w''$. Hence we have $v = uv_1' = cd' w''$. This completes the induction. \[\square\]

A non-zero homogeneous element $u_0 \in H^\otimes l$, $l \geq 1$, is **reduced** if the equation $u_0 = \lambda v_0^d$ with $\lambda \in K$ and $v_0 \in H^\otimes (l/d)$ implies $d = 1$. For example, any non-zero element of $H$ or $\wedge^2 H$ is reduced.

**Proposition 5.6.** Let $u_0 \in H^\otimes l \setminus \{0\}$ be reduced. Then we have

$$\{v \in \hat{T}(H) \mid vu_0 = u_0 v\} = K[[u_0]].$$

**Proof.** Let $v \in \hat{T}(H) \setminus \{0\}$ satisfy $vu_0 = u_0 v$. We may assume that $v$ is homogeneous of degree $m \geq 1$. By Lemma 5.3 there exist some $w \in H^\otimes \gcd(l,m)$ and $c, d \in K$ such that $u_0 = cw'$ and $v = dw''$. Since $u_0$ is reduced, we have $l' = 1$, i.e., $v \in K[[u_0]]$. This completes the proof. \[\square\]

By a similar method we obtain the following strengthened version, which will be used in our forthcoming paper.

**Proposition 5.7.** Let $u_0 \in H^\otimes a \setminus \{0\}$ be reduced. Then we have

$$\{v \in \hat{T}(H) \mid [u_0, v] \in K[[u_0]]\} = K[[u_0]].$$
Proof. Let \( v \) be a homogeneous element satisfying \([u_0, v] \in K[[u_0]]\). If \( \alpha \) does not divide \( \deg v \), then \([u_0, v] = 0\). Hence, by Proposition 5.6, we obtain \( v \in K[[u_0]] \). So it suffices to consider the case \( \deg v = \alpha l, l \geq 0 \). We prove \( v \in K[[u_0]] \) by induction on \( l \geq 0 \). If \( l = 0 \), it is trivial. Assume \( l \geq 1 \). Then we have \( vu_0 = u_0(v - \lambda u_0) \) for some \( \lambda \in K \). By a similar way to the proof of Lemma 5.5, we have some \( v_0 \) such that \( v = u_0v_0 \). Then \( u_0v_0u_0 = u_0(\lambda u_0 - \lambda^2 u_0) \), so \( v_0u_0 = u_0v_0 - \lambda u_0^l \). Applying the inductive assumption to \( v_0 \), we obtain \( v_0 \in K[[u_0]] \), which implies \( v \in K[[u_0]] \). This completes the proof. \( \square \)

Let us go back to the case of \( H = H_1(\Sigma, K) \).

Lemma 5.8. \[
\{a \in A \mid a\omega = \omega a\} = K[[\omega]].
\]

Proof. Recall that in \( \S 3.4 \) we introduced an injective algebra homomorphism \( i_\ast : A = A^{(g, n + 1)} \to A_{\ast} = A^{(g + n, 1)} \) and that it maps \( \omega = \omega^{(g + n + 1)} \) to \( \overline{\omega} = \omega^{(g + n, 1)} \). The lemma follows from Proposition 5.6 for \( \overline{\omega} \). \( \square \)

Now we turn our attention to the bracket associated with \( \Pi_{gr} = \kappa_{gr} \). It can be computed by using results in \( \S 3.4 \). For instance, the following formulas are proved in the same way as Lemma 4.11 in \( \cite{2} \).

Lemma 5.9. Let \( a \in A, j = 1, \ldots, n, h(z) \in K[[z]] \) and \( \hat{h}(z) \) its derivative. Then
\( (i) \ \{[h(z_j)], a\}_{\Pi_{gr}} = \{a, [h(z_j)]\}_{\Pi_{gr}} = 0, \)
\( (ii) \ \{h(\omega), [a]\}_{\Pi_{gr}} = \{[a], h(\omega)\}_{\Pi_{gr}} = 0, \)
\( (iii) \ \{[h(\omega)], a\}_{\Pi_{gr}} = [[a, h(\omega)]\}_{\Pi_{gr}} = \hat{h}(\omega), a] \).

We need one more formula about the derivation \( \{[a, \cdot]_{\Pi_{gr}} \). Consider the injective linear map \( N : |A|_{\geq 1} \to A_{\geq 1} \) defined as follows. For \( a = w_1 \cdots w_l \) with \( w_k \in H \), we have
\[
N([a]) = \sum_{k=1}^l w_k \cdots w_l w_{k-1}.
\]
Let \( \{x^*_i, y^*_i, z^*_j\} \subset H^* \) be the dual basis of \( \{x_i, y_i, z_j\} \). For any \( \varphi \in H^* \), we denote \( 1 \otimes \varphi : A_{\geq 1} = A \otimes H \to A, z \otimes w \mapsto \varphi(w)z \). Then for any \( a \in |A|_{\geq 1} \), we have
\[
\begin{align*}
\{[a], x_i\}_{\Pi_{gr}} & = -(1 \otimes y^*_i)N([a]), \\
\{[a], y_i\}_{\Pi_{gr}} & = (1 \otimes x^*_i)N([a]), \\
\{[a], z_j\}_{\Pi_{gr}} & = [z_j, (1 \otimes z^*_j)N([a])].
\end{align*}
\]
(22)

Lemma 5.10. The kernel of the map \( |A|_{\geq 1} \to \text{Der}(A), [a] \mapsto \{[a, \cdot]_{\Pi_{gr}} \) is spanned by the elements \( [z^*_j] \), where \( j = 1, \ldots, n, l \geq 1 \).

Proof. Lemma 5.9(i) shows that for any \( j \) and \( m \geq 1 \), \( [z^*_j]^m \) is in the kernel. In order to prove the other inclusion, suppose that \( [a] \in |A|_{\geq 1} \) satisfies \( \{[a, \cdot]_{\Pi_{gr}} = 0 \). By (22), we have \( (1 \otimes y^*_i)N([a]) = (1 \otimes x^*_i)N([a]) = 0 \). Moreover, by Proposition 5.6, we obtain \( (1 \otimes z^*_j)N([a]) \in K[[z_j]] \). Therefore we have \( N([a]) \in \bigoplus_{j=1}^n K[[z_j]]_{\geq 1} \), i.e., \( [a] \in \bigoplus_{j=1}^n |K[[z_j]]|_{\geq 1} \). This proves the lemma. \( \square \)
Proof of Theorem 5.4. Lemma 5.9 (i)(iii) shows that the elements $|\omega^m|$ and $|z_j^m|$ are in the center of the Lie algebra $\{\cdot,\cdot\}_{\Pi_{\text{gr}}}$. Suppose $|a| \in Z(|A|, \{\cdot,\cdot\}_{\Pi_{\text{gr}}})$. Since $\{|a|, |\cdot|\}_{\Pi_{\text{gr}}} = 0$ for any $b \in A$, Theorem A.1 in [2] implies that there exists some $u_0 \in A$ such that $\{|a|, b\}_{\Pi_{\text{gr}}} = [u_0, b]$ for any $b \in A$. By Lemma 5.9 (ii), $[u_0, \omega] = \{|a|, \omega\}_{\Pi_{\text{gr}}} = 0$. Hence, from Lemma 5.8 we have $u_0 = f(\omega)$ for some $f(z) \in \mathbb{K}[z]$. Set $\hat{f}(z) := \int_0^z f(z)dz$. Then, by Lemma 5.9 (iii), we have $\{|a - \hat{f}(\omega)|, b\}_{\Pi_{\text{gr}}} = \{|a|, b\}_{\Pi_{\text{gr}}} - [u_0, b] = 0$ for any $b \in A$. Applying Lemma 5.10 to $a - \hat{f}(\omega)$, we conclude that $|a|$ has the desired form. \hfill $\square$

5.3 Homomorphic Expansions

Let $\theta : \tilde{K}_\pi \to A$ be any tangential group-like expansion and write it as $\theta = F^{-1} \circ \theta_{\text{exp}}$, where $F \in T\text{Aut}$. Recall that any framing $f$ is determined by the difference with respect to the adapted framing given by $\chi \in H^1(\Sigma, \mathbb{Z})$. If $\theta$ is a special expansion, by Theorem 5.1 it sends $\kappa$ to $\Pi_{\text{add}} = \Pi_{\text{gr}} + \Pi_s$, which is equal to $\Pi_{\text{gr}}$ if one considers the induced map $|A| \to t\text{Der}(A)$.

Proposition 5.11. Let $f$ be a framing defined by $\chi \in H^1(\Sigma, \mathbb{Z})$ with respect to the adapted framing and let $\theta = F^{-1} \circ \theta_{\text{exp}}$ be a special expansion. Then,

$$\delta^f_\theta = (t\text{Div}_q + \partial D) \circ \Pi_{\text{gr}} : |A| \to |A| \otimes |A|,$$

where $D = j_q(F^{-1}) + F^{-1}(r + |p| \wedge 1)$. Moreover, the map $\delta^f_\theta$ is graded if and only if $D \in (|A| \otimes |A|)^{s\text{Der}(A)}$.

Proof. By Corollary 4.8 we have $\delta^f_{\text{exp}} = g\text{Div}^f \circ \Pi_{\text{exp}}$. Applying $F$ yields

$$\delta^f_\theta = F^* \delta^f_{\text{exp}} = (F^* g\text{Div}^f) \circ (F^* \Pi_{\text{exp}}) = (t\text{Div} - c_q + \partial D) \circ (\Pi_{\text{gr}} + \Pi_s) = (t\text{Div}_q + \partial D) \circ \Pi_{\text{gr}},$$

where we have used Corollary 4.11 and the fact that $\Pi_s$ vanishes on $|A|$.

The map $\Pi_{\text{gr}}$ is graded and the maps $t\text{Div}$ and $c_q$ are degree preserving. Since the map $\partial D$ is of positive degree and the image of $\Pi_{\text{gr}}$ is $s\text{Der}(A)$, we conclude that $\delta^f_\theta$ is graded if and only if $D$ is $s\text{Der}(A)$-invariant. \hfill $\square$

Since $\theta$ is a group-like expansion, we have $D \in \tilde{\Delta}(|A|) \subset |A| \otimes |A|$ and hence $D = \tilde{\Delta}(d)$ for some $d \in |A|$. Then, the condition $D \in (|A| \otimes |A|)^{s\text{Der}(A)}$ implies that $u(d) = 0$ for all $u \in s\text{Der}(A)^{\geq 0}$ and thus $\{|d, \cdot|\}_{\Pi_{\text{gr}}} : A \to A$ is a derivation that induces a map $|A| \to |A|$ vanishing on all elements of degree $\geq 2$. By [2, Theorem A.1], such a derivation is inner and hence it vanishes on $|A|$ (in all degrees). Therefore, we conclude that the element $d$ is in the center of the Lie algebra $\{\cdot,\cdot\}_{\Pi_{\text{gr}}}$.

Explicitly, the element $d$ is given by $d = j_q(F^{-1}) + F^{-1}(r + |p|)$, where we regard $j_q(F^{-1})$ and $r$ as elements in $|A|$ through the embedding $\tilde{\Delta}$.

As a summary, we obtain

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Theorem 5.12. Let $f$ and $\theta = F^{-1} \circ \theta_{\text{exp}}$ be as above. Suppose that $\theta$ is a special expansion. Then $\delta_{\theta}^f$ is graded if and only if

$$j_q(F^{-1}) + F^{-1} (\mathbf r + |p|) \in Z(|A|, \{\cdot, \cdot\}_{\Pi_g}).$$

5.4 Kashiwara-Vergne problems in higher genera

As shown in Theorem 5.12, the center $Z(|A|, \{\cdot, \cdot\}_{\Pi_g})$ is spanned by the elements $|\omega^j|, |z^j_l|$, $j = 1, \ldots, n$, $l \geq 0$, where $\omega = \sum_i x_i + \sum_j z_j$. With this in mind, let us turn Theorem 5.12 into a definition.

Definition 5.13 (KV Problem of type $(g, n + 1)$ with framing $(p, q)$). Find an element $F \in T\text{Aut}^{(g, n+1)}$ such that

$$F \left( \sum_{i=1}^g [x_i, y_i] + \sum_{j=1}^n z_j \right) = \log \left( \prod_{i=1}^g (e^{x_i} e^{h_i} e^{-x_i} e^{-y_i}) \prod_{j=1}^n e^{z_j} \right) =: \xi, \quad (\text{KV I}^{(g, n+1)})$$

$$j_q(F) = \mathbf r + |p| + \sum_{j=1}^n |h_j(z_j)| - |h(\xi)| \quad \text{for some } h_j, h \in \mathbb K[[s]]. \quad (\text{KV II}^{(g, n+1)})$$

Here, $j_q(F) = j(F) - C_q(F)$ and $\mathbf r = \sum_i |r(x_i) + r(y_i)|$ with $r(s) = \log((e^s - 1)/s)$.

Definition 5.14. Let $\text{Sol KV}^{(g, n+1)}_{\chi}$ denote the space of solutions of the KV problem of type $(g, n + 1)$ with framing $\chi = (p, q)$. Let moreover $\text{Sol KV}^{(g, n+1)}_{\text{Ad}}$ be the solutions for the adapted framing, i.e., $\text{Sol KV}^{(g, n+1)}_{(0, 0)}$.

Remark 5.15. In the case of $g = 0$, the functions $h_j$ and $h$ agree modulo the linear part \cite[Theorem 8.7]{2}. At present, we do not know whether this holds in the case of positive genus. The functions $h_j$ and $h$ are called the Duflo function.

Remark 5.16. For $g = 0$, one has $p = 0$ and the KV problem is independent of $q$. Indeed, $j_q(F) = j(F) - C_q(F)$ and the cocycle $C_q(F)$ takes values in the linear span of $|z_j|$’s. Hence, it can be absorbed in the expression $\sum_j |h_j(z_j)|$ by changing linear parts of $h_j$’s.

By definition, if $\theta = F^{-1} \circ \theta_{\text{exp}}$ is a special expansion, then the condition $F \in \text{Sol KV}^{(g, n+1)}_{\chi}$ is equivalent to that $\delta_\theta^f$ is graded, i.e., $\theta$ is a homomorphic expansion. In other words, $F \in \text{Sol KV}^{(g, n+1)}_{\chi}$ is equivalent to the fact that $\theta$ solves the GT formality problem.

For the map $\mu_f^l$, we have the following description.

Proposition 5.17. Let $f$ and $\theta = F^{-1} \circ \theta_{\text{exp}}$ be as above, and suppose $F \in \text{Sol KV}^{(g, n+1)}_{\chi}$. Then, for any $a \in A$, we have

$$\mu_f^l(a) = \mu_{\text{gr}}^f(a) + |s''| \otimes a' - |s''| \otimes s' + |g'| \otimes [a, g''].$$

Here, $\mu_{\text{gr}}^f$ is the map in Proposition 5.18.

$g(z) := h'(z)$ is the derivative of $h$, $s$ is given by equation \cite[21]{2} and we write $\tilde{\Delta}(s(\omega)) = s' \otimes s''$ and $\tilde{\Delta}(g(\omega)) = g' \otimes g''$. 40
Proof. Similarly to the proof of Proposition 5.11, \( \theta \) transforms \( \mu^f + \mu_{\bullet}^f \) into
\[
(t\text{Div}_q + \partial D) \circ (\Pi_{\text{gr}} + \Pi_s) : A \to (|A| \otimes A) \oplus (A \otimes |A|),
\]
and taking the \((|A| \otimes A)\)-part gives \( \mu^f_q \). The lowest degree term \( t\text{Div}_q \circ \Pi_{\text{gr}} \) becomes \( \mu^f_{\text{gr}} \).

As in [2, Theorem 7.6], one obtains
\[
\Pi_{\text{gr}}(a, D) = |g'| \otimes [a, g''] + [a, g'] \otimes |g''|
\]
by using Lemma 5.9, and the formula
\[
t\text{Div}(\{a, \cdot\}_s) = |s''| \otimes as' - |s''a| \otimes s' + s''a \otimes |s'| - s'' \otimes |as'|.
\]
All the other terms vanish since \( \{a, \cdot\}_s \) is an inner derivation by \( \sum \lambda_j |z_j| = 0 \).

We describe relations among the KV problems for different framings. In order to do this, let us introduce a Lie subalgebra of \( t\text{der} \):
\[
s\text{der} = s\text{der}^{(g,n+1)} := \{ u \in t\text{der}^{(g,n+1)} | u(\omega) = 0 \}.
\]
Positive degree elements of \( s\text{der} \) integrate to elements in \( T\text{Aut} \) which preserve \( \omega \).

**Proposition 5.18.** If \( g \geq 2 \), then the KV problem of type \((g,n+1)\) for any framing \( \chi = (p,q) \) is equivalent to the KV problem for the adapted framing \( \chi = (0,0) \).

**Proof.** Consider the element of degree one \( u \in s\text{der} \) of the form:
\[
\begin{align*}
x_1 & \mapsto [x_2, x_1], \\
y_1 & \mapsto [x_2, y_1], \\
y_2 & \mapsto [y_1, x_1].
\end{align*}
\]
Note that \( t\text{Div}(u) = 2|x_2| \) and \( c_q(u) = 0 \). By permuting labels, this implies that the image of \( t\text{Div}_q \) contains \( H_{xy} \).

Let \( F \in T\text{Aut} \) and choose \( u \in s\text{der} \) of degree one such that
\[
t\text{Div}_q(u) = [C_q(F)]_1 + |p|,
\]
where \( [C_q(F)]_1 \) is the degree 1 part of \( C_q(F) \). Using that \( F \) fixes \( H_{xy} \), i.e. the degree 1 part of \( |A| \), one computes
\[
\begin{align*}
j_q(F e^u) &= j_q(F) + t\text{Div}_q(u) \\
&= j(F) - C_q(F) + [C_q(F)]_1 + |p| \\
&= j(F) + |p| - [C_q(F)]_2,
\end{align*}
\]
where \( [C_q(F)]_2 = C_q(F) - [C_q(F)]_1 = \sum \lambda_j |z_j| \) is the degree two part of \( C_q(F) \). The equation above implies that \( F \) is a solution of the KV problem with the adapted framing if and only if \( F e^u \) is a solution of the KV problem with framing \((p,q)\) and with the Duflo functions \( h'_j(z_j) = h_j(z_j) - \lambda_j z_j \).
Proposition 5.19. If \( g = 1, \ n \geq 1 \) and \( q \neq 0 \), then the KV problem with framing \( \chi = (p,q) \) is equivalent to the KV problem with the adapted framing \( \chi = (0,0) \).

Proof. Consider the derivation \( u \) of degree one of the form:

\[ y \mapsto z_1, \quad z_1 \mapsto [z_1, x], \]

and note that \( \text{tDiv}(u) = 0 \) and \( c_q(u) = q(z_1)|x| \). By permuting labels, we conclude that the image of the map \( \text{tDiv}_q \) contains \( H_{xy} \). Then the proof of the previous proposition applies verbatim.

\[ \square \]

Proposition 5.20. If \( g = 1 \) and \( q = 0 \), then the KV problem of type \((g,n+1)\) has a solution for at most one value of \( p \).

Proof. Let \( F_i \) be solutions to the KV problem for framings \( p_i \) for \( i = 1, 2 \). One computes the degree 1 term of \( j(F_1^{-1}F_2) \) as follows:

\[ j(F_1^{-1}F_2) = F_1^{-1}(-j(F_1) + j(F_2)) = F_1^{-1}(-|p_1| + |p_2|) + (\text{terms of degree} \geq 2) \]

\[ = |p_2 - p_1| + \ldots \]

This implies that there exists a degree 1 element in \( \text{sder}^{(1,n+1)} \) such that \( \text{tDiv}(u) = |p_2 - p_1| \). However, in degree one the map \( \text{tDiv} \) on \( \text{sder}^{(1,n+1)} \) vanishes (see the proof of the previous proposition).

\[ \square \]

6 Solving KV

In this section, we show that the higher genus KV problems admit solutions.

Theorem 6.1. The KV problem admits solutions in the following cases:

- \( \text{Sol KV}^{(g,n+1)}_{\chi} \neq \emptyset \) for \( g \geq 2 \) and for any framing \( \chi \).
- \( \text{Sol KV}^{(1,n+1)}_{\chi} \neq \emptyset \) if and only if \( q \neq 0 \) or \( q = 0 \) and \( p = 0 \).

In view of Propositions 5.18, 5.19 and 5.20, this is equivalent to showing that the KV problem of type \((g,n+1)\) admits solutions for the adapted framing.

Remark 6.2. The mapping class group \( M(\Sigma) \) acts on the set of homotopy classes of framings on \( \Sigma \) in a natural way. Here the group \( M(\Sigma) \) is the path-component group of the topological group of diffeomorphisms of \( \Sigma \) whose restrictions to the boundary \( \partial \Sigma \) are the identity. In this situation, the datum \( q \) is invariant under the action. As is proved in [19], for a fixed \( q \), the orbit set of the action is finite unless \( g = 1 \) and \( q = 0 \). In the latter case, the orbit set is bijective to the infinite set \( \mathbb{Z}_{\geq 0} \) by the greatest common divisor of the rotation numbers \( \text{rot}_f(\alpha) \) and \( \text{rot}_f(\beta) \). These results match Theorem 6.1 stated above.
We will make use of the following chain rule formula for double derivations. (In fact, this has already been used in the proof of Lemma 4.2.) Let \( \{ v_i \} \) be a set of generators of degree 1 for a completed free associative algebra, and let \( v_i = v_i(w_1, w_2, \ldots) \) be a change of variables. Then

\[
\frac{\partial}{\partial w_i} = \sum_j \frac{\partial}{\partial v_j} \circ \frac{\partial v_j}{\partial w_i}.
\] (23)

The same type of the formula holds for exponential generators as well.

6.1 Gluing

As shown in Figure 4, one can glue \( \Sigma_{g_1,n_1+1}, \Sigma_{g_2,n_2+1} \) and \( \Sigma_0,3 \) to get a surface of genus \( g_1 + g_2 \) with \( n_1 + n_2 + 1 \) boundary components. With this picture in mind, let us denote the variables appearing in the definition of \( \text{tder}^{(g_1+g_2,n_1+n_2+1)} \) by \( x_1^i, y_1^i, z_1^j \) and \( x_2^i, y_2^i, z_2^j \), respectively. Introduce the map

\[
P : \text{tder}^{(0,3)} \to \text{tder}^{(g_1+g_2,n_1+n_2+1)}
\]
as follows: for \( u = (u_1(z_1, z_2), u_2(z_1, z_2)) \in \text{tder}^{(0,3)} \), \( P(u) \) is a derivation which sends

\[
x_i^k \mapsto [x_i^k, u_k(\omega_1, \omega_2)], \quad y_i^k \mapsto [y_i^k, u_k(\omega_1, \omega_2)], \quad z_j^k \mapsto [z_j^k, u_k(\omega_1, \omega_2)],
\]
where

\[
\omega_1 = \sum_i [x_i^1, y_i^1] + \sum_j z_j^1, \quad \omega_2 = \sum_i [x_i^2, y_i^2] + \sum_j z_j^2.
\]

**Proposition 6.3.** The map \( P \) is a Lie algebra homomorphism.

**Proof.** This follows by the direct computation using that for \( k = 1, 2 \) \( P(u)(\omega_k) = [\omega_k, u_k(\omega_1, \omega_2)] \). \( \square \)

Define the linear form \( q = q^P \in (H_2)^* \) by \( q(z_1) = 2g_1, q(z_2) = 2g_2 \).
Proposition 6.4. The map $\mathcal{P}$ is compatible with divergences in the following sense.

$$t\text{Div}(\mathcal{P}(u)) = t\text{Div}_q(u)|_{z_1=\omega_1, z_2=\omega_2}. $$

Proof. Using the chain rule formula (23) and the cyclic property of $|\cdot|$, one computes as follows:

$$t\text{Div}(\mathcal{P}(u))$$

\[
= \sum_{i,k} \left[ \frac{\partial [x_i^k, u_k]}{\partial x_i^k} + \frac{\partial [y_i^k, u_k]}{\partial y_i^k} \right] + \sum_{j,k} \left[ \frac{\partial [z_j^k, u_k]}{\partial z_j^k} \right] 
\]

\[
= \sum_{i,k} \left[ 1 \wedge u_k + [x_i^k, \frac{\partial u_k}{\partial x_i^k}] + 1 \wedge u_k + [y_i^k, \frac{\partial u_k}{\partial y_i^k}] + \sum_{j,k} [z_j^k, \frac{\partial u_k}{\partial z_j^k}] \right] 
\]

\[
= \sum_{i,k} 2g_k 1 \wedge |u_k| + \sum_{i,k} \left[ [x_i^k, \frac{\partial u_k}{\partial x_i^k}] \otimes [y_i^k, \frac{\partial u_k}{\partial y_i^k}] \right] + \sum_{j,k} [z_j^k, \frac{\partial u_k}{\partial z_j^k}] \otimes |\omega_k| 
\]

\[
= \sum_{i,k} 2g_k 1 \wedge |u_k| + \sum_{i,k} \left[ \frac{\partial u_k}{\partial x_i^k} \otimes [x_i^k, \omega_k] \right] + \frac{\partial u_k}{\partial y_i^k} \otimes [y_i^k, \omega_k] \right] + \sum_{j,k} [z_j^k, \frac{\partial u_k}{\partial z_j^k}] \otimes |\omega_k| 
\]

\[
= \sum_{i,k} 2g_k 1 \wedge |u_k| + \frac{\partial u_k}{\partial x_i^k} \otimes |\omega_k| + t\text{Div}(u)|_{z_1=\omega_1, z_2=\omega_2}. 
\]

Remark 6.5. If we consider the adapted framing on $\Sigma_{g_1,n_1+1}$ and $\Sigma_{g_2,n_2+1}$, then $\text{rot}_{\adp}(\gamma_0^1) = 2g_1 - 1$ and $\text{rot}_{\adp}(\gamma_0^2) = 2g_2 - 1$. Thus modifying the adapted framing on $\Sigma_{0,3}$ by using the linear form $\mathcal{Q}$, one obtains a framing on the glued surface $\Sigma_{g_1+g_2,n_1+n_2+1}$.

The map $\mathcal{P}$ integrates to a corresponding map on the group of tangential automorphisms. For $F \in \text{TAut}^{(0,3)}$ one has

$$j(\mathcal{P}(F)) = j_q(F)|_{z_1=\omega_1, z_2=\omega_2}. $$

One straightforwardly defines Lie algebra maps

$$t\text{der}(g_1,n_1+1) \times t\text{der}(g_2,n_2+1) \longrightarrow t\text{der}(g_1+g_2,n_1+n_2+1),$$

that commute with divergences in the obvious way. Again one integrates these maps to corresponding maps on the Lie groups $\text{TAut}$ which now commute with the cocycle $j$.

Proposition 6.6. Suppose $n_1 = 0$ or $g_2 = 0$. Let

$$F_1 \in \text{Sol KV}(g_1,n_1+1), \quad F_2 \in \text{Sol KV}(g_2,n_2+1), \quad F \in \text{Sol KV}^{(0,3)}$$

with the Duflo function for $F_i$ being $h_i^1, \ldots, h_i^{n_i}$, $h^i$ ($i = 1, 2$, respectively) and the Duflo function for $F$ being $h$ modulo the linear part. If $h^1 \equiv h^2 \equiv h$ mod $\mathbb{K}$s, then

$$(F_1 \times F_2) \circ \mathcal{P}(F)$$

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is a solution to $\text{KV}^{(g_1+g_2,n_1+n_2+1)}$ with the Duflot function $h^1_1, \ldots, h^1_{n_1}, h^2_1, \ldots, h^2_{n_2}, h$ modulo the linear part.

**Proof.** One computes

$$(F_1 \times F_2)(\mathcal{P}(F)(\omega_1 + \omega_2)) = (F_1 \times F_2)\log(e^{\omega_1} e^{\omega_2})$$

$$= \log \left( \prod_{i=1}^{g_1} (e^{x_1^i} e^{y_1^i} e^{-x_1^i} e^{-y_1^i}) \prod_{j=1}^{n_1} e^{z_j^1} \prod_{i=1}^{g_2} (e^{x_2^j} e^{y_2^j} e^{-x_2^j} e^{-y_2^j}) \prod_{j=1}^{n_2} e^{z_j^2} \right),$$

which shows (KV I), since one of the middle products is empty by assumption.

For (KV II) one computes modulo the $\mathbb{K}$-span of $z_j^k$'s,

$$j((F_1 \times F_2) \circ \mathcal{P}(F)) = j(F_1 \times F_2) + (F_1 \times F_2) \circ j(\mathcal{P}(F))$$

$$= r_1 + \sum_j h^1_j(z_j^1) - h^1(F_1(\omega_1)) + r_2 + \sum_j h^2_j(z_j^2) - h^2(F_2(\omega_2)) + (F_1 \times F_2)(h(\omega_1) + h(\omega_2) - h(\log(e^{\omega_1} e^{\omega_2})))$$

$$= r_1 + r_2 + \sum_{j,k} h^k_j(z_j^k) - h(\xi).$$

Here, the difference between $j(\mathcal{P}(F)) = j_q(F)|_{z_1=\omega_1, z_2=\omega_2}$ and $j(F)|_{z_1=\omega_1, z_2=\omega_2}$ has no effect since it just contributes to the $\mathbb{K}$-span of $z_j^k$'s. \[\square\]

Using this proposition, the question of existence is reduced to the cases $(g,n+1) = (0,3)$ and $(1,1)$. Recall that by the results of [4] we have $\text{Sol} \text{KV}^{(0,3)} \neq \emptyset$.

### 6.2 Elliptic case

Consider the map $A^{(0,3)} \to A^{(1,1)}$ that sends $z_1 \mapsto \psi_1 = e^x ye^{-x}$ and $z_2 \mapsto \psi_2 = -y$.

Furthermore, define the following map inspired by a construction in [10]:

$$\text{TAut}^{(0,3)} \longrightarrow \text{TAut}^{(1,1)},$$

$$F = (f_1, f_2) \longmapsto F^{\text{ell}}, \left\{ \begin{array}{cl}
\alpha \mapsto e^{-f_1(\psi_1, \psi_2)} \alpha e^{f_2(\psi_1, \psi_2)} \\
\beta \mapsto e^{-f_2(\psi_1, \psi_2)} \beta e^{f_2(\psi_1, \psi_2)}
\end{array} \right.$$  

(where $\alpha = e^x$ and $\beta = e^y$ as before) and the corresponding Lie algebra map:

$$\text{tder}^{(0,3)} \longrightarrow \text{tder}^{(1,1)},$$

$$u = (u_1, u_2) \longmapsto u^{\text{ell}}, \left\{ \begin{array}{cl}
\alpha \mapsto [\alpha, u_2(\psi_1, \psi_2)] + (u_2(\psi_1, \psi_2) - u_1(\psi_1, \psi_2))\alpha \\
\beta \mapsto [\beta, u_2(\psi_1, \psi_2)]
\end{array} \right..$$

**Proposition 6.7.** The maps $F \mapsto F^{\text{ell}}$ and $u \mapsto u^{\text{ell}}$ are morphisms of groups and Lie algebras, respectively.
Proof. This follows from $F^{\ell}(\psi_1) = (e^{-f_1}\alpha e^{f_2})(e^{-f_2}y e^{f_2})(e^{-f_2}\alpha^{-1} e^{f_1}) = e^{-f_1}\psi_1 e^{f_1}$ and $F^{\ell}(\psi_2) = e^{-f_2}\psi_2 e^{f_2}$.

Define the linear form $q = q^{\ell} \in (H_\ell)^*$ by $q(z_1) = q(z_2) = 1$.

**Proposition 6.8.** For any $u \in \text{tder}^{(0,3)}$, one has

$$\text{gDiv}(u^{\ell}) = t\text{Div}_q(u)_{|z_1=\psi_1, z_2=\psi_2}.\quad (25)$$

Proof. We use the following chain rule formula: for any $q \in A^{(1,1)}$, one has $[u, \otimes] = \frac{\partial u}{\partial z} [\alpha, \otimes] + \frac{\partial u}{\partial y} [\beta, \otimes]$ and consequently

$$[[u, \otimes]] = [\alpha, \frac{\partial u}{\partial z}] + [\beta, \frac{\partial u}{\partial y}].\quad (25)$$

The first term of the derivation $u^{\ell}$ is an inner derivation by $u_2$ and its $\text{gDiv}$ computes to

$$|[\otimes, u_2] + [\alpha, \frac{\partial u_2}{\partial z}] + [\otimes, u_2] + [\beta, \frac{\partial u_2}{\partial y}]| = |[[\otimes, u_2]]|.$$

Let us identify $z_1 = \psi_1$, $z_2 = \psi_2$ and denote $\partial = \frac{\partial}{\partial z_2}$. The value of $\text{gDiv}$ for the remaining term uses the computation

$$|\left(\frac{\partial}{\partial z_2} (u_2 - u_1)\right)\otimes| = |[\partial (u_2 - u_1), z_1] + [u_2 - u_1, \otimes]|$$

$$= |[\partial_1 u_i, z_1]|$$

to yield

$$|\frac{\partial}{\partial z_2}((u_2 - u_1)\alpha)| - 1 \otimes |u_2 - u_1| = |[\partial_1 (u_2 - u_1), z_1] + [u_2 - u_1, \otimes]|$$

$$= |[\partial_1 u_i, z_1]| + [z_2, \partial_2 u_2] + [\otimes, u_2] + [u_2 - u_1, \otimes]|$$

$$= |[z_1, \partial_1 u_i] + [z_2, \partial_2 u_2] + [\otimes, u_1]|.\quad (25)$$

Here, in the second line we used the same type of the formula as (25) applied for generators $z_1, z_2$.

**Remark 6.9.** As shown in Figure 5, one can construct $\Sigma_{1,1}$ by gluing an annulus to the first and second boundaries of $\Sigma_{0,3}$. For the framing on $\Sigma_{0,3}$ corresponding to $q^{\ell}$, the rotation number of $|\gamma_1|$ and $|\gamma_2|$ are zero. Thus it extends to a framing on $\Sigma_{1,1}$.

By integration, for any $F \in \text{TAut}^{(0,3)}$ one has

$$j(F^{\ell}) = j_q(F)_{|z_1=\psi_1, z_2=\psi_2} - F^{\ell}\cdot r + r.\quad (26)$$

**Lemma 6.10.** Let $F = (f_1, f_2) \in \text{Sol KV}^{(0,3)}$, $h$ the degree $\geq 2$ part of its Duflo function, and $q = q^{\ell}$ as above. Then

$$j_q(F)_{|z_1=\psi_1, z_2=\psi_2} = |r(y) - h(\xi^{\ell})|,$$

where $\xi^{\ell} = \log(e^x e^y e^{-x} e^{-y})$.
Proof. Let us write the linear part of $f_i$ as $c_1^i z_1 + c_2^i z_2$. By inspecting (KV I), we obtain $c_1^2 - c_2^1 = 1/2$, and (KV II) implies that

$$j_q(F) = j(F) - C_q(F) = \left|h(z_1) + h(z_2) - h(\xi)\right| - |c_1^2 z_2 + c_2^1 z_1|,$$

where $\xi = \log(e^{z_1} e^{z_2})$. By putting $z_1 = \psi_1 = e^x ye^{-x}$ and $z_2 = \psi_2 = -y$, we obtain

$$j_q(F)|_{z_1=\psi_1, z_2=\psi_2} = \left|2h_{\text{even}}(y) - h(\xi_{\text{ell}}) + \frac{1}{2}y\right| = |r(y) - h(\xi_{\text{ell}})|,$$

where we used that $2h_{\text{even}}(z) = r(z) - (1/2)z$; see [4, Proposition 6.1].

Define an automorphism of $A^{(1,1)}$ by $\phi : x \mapsto x, y \mapsto e^{ad_x} ye^{-ad_x} - y$. Since $\phi$ is the exponential of the derivation $x \mapsto 0, y \mapsto \log(\frac{e^{ad_x} - 1}{ad_x})y = r(ad_x)y$, one has

$$j(\phi) = |r(x)|. \tag{27}$$

Theorem 6.11. Let $F \in \text{Sol KV}^{(0,3)}$ with the Duflo function $h \in s^2 K[[s]]$. Then $F_{\text{ell}} \circ \phi \in \text{Sol KV}^{(1,1)}$ with the same Duflo function $h$.

Proof. Since $\phi([x, y]) = [x, \phi(y)] = (e^{ad_x} - 1)y = e^x ye^{-x} - y$, one has

$$F_{\text{ell}} \circ \phi([x, y]) = \left(e^{-f_1} e^{x f_2} (e^{-f_2} ye^{f_2})(e^{-f_2} e^{-x f_1}) - e^{-f_2} ye^{f_2}\right)$$

$$= e^{-f_1} \psi_1 e^{f_1} + e^{-f_2} e^{f_2} \psi_2 e^{f_2}$$

$$= F(\psi_1 + \psi_2).$$

From (KV I) for $F$, this is equal to $\log(e^{\psi_1} e^{\psi_2}) = \log(e^x ye^{-x} e^{-y})$, proving (KV I).

Next we prove (KV II). By the previous formulas [26, 27] and Lemma [6,10]

$$j(F_{\text{ell}} \circ \phi) = j(F_{\text{ell}}) + F_{\text{ell}} \cdot j(\phi)$$

$$= |r(y) - h(\xi_{\text{ell}})| - F_{\text{ell}} \cdot r + F_{\text{ell}} \cdot |r(x)|$$

$$= r - |h(\xi_{\text{ell}})|.$$
7 Uniqueness

In this section, we discuss the uniqueness issue for Kashiwara-Vergne problems \( \text{Sol KV}^{(g,n+1)}_\chi \).

7.1 Kashiwara-Vergne groups KRV

The purpose of this section is to introduce and study symmetry groups for higher genus Kashiwara-Vergne problems.

**Definition 7.1.** For \( q \in (H_\mathbb{Z})^* \), let

\[
\text{KRV}^{(g,n+1)}_q = \{ G \in \text{TAut}^{(g,n+1)} | G(\omega) = \omega, j_q(G) = \sum_j |h_j(z_j)| - |h(\omega)| \text{ for some } h_j, h \in \mathbb{K}[[s]] \},
\]

where \( \omega = \sum_j [x_j, y_j] + \sum_i z_i \).

The subset \( \text{KRV}^{(g,n+1)}_q \subset \text{TAut}^{(g,n+1)} \) is a subgroup of \( \text{TAut}^{(g,n+1)} \). Indeed, for \( G, G' \in \text{KRV}^{(g,n+1)}_q \) the product \( GG' \) preserves the element \( \omega \) and

\[
j_q(GG') = j_q(G) + G \cdot j_q(G') = \sum_j |h_j(z_j)| - |h(\omega)| + \sum_j |h'_j(z_j)| - G \cdot |h'(\omega)|
\]

where we have used the cocycle property of \( j_q \) and the facts that \( G \cdot |h'_j(z_j)| = |h'_j(z_j)| \) because \( G \in \text{TAut}^{(g,n+1)} \) and \( G \cdot |h'(\omega)| = |h'(\omega)| \) because \( G \) preserves \( \omega \). We will refer to \( \text{KRV}^{(g,n+1)}_q \) as a Kashiwara-Vergne group.

In genus zero, similar to Remark 5.16, the group \( \text{KRV}^{(0,n+1)}_q \) is independent of \( q \) and can be denoted \( \text{KRV}^{(0,n+1)} \). Equation (28) shows that the Duflo functions

\[
(h_1, \ldots, h_n, h) : \text{KRV}^{(g,n+1)}_q \rightarrow \mathbb{K}[[s]]^{\oplus(n+1)}
\]

define a group homomorphism into \( \mathbb{K}[[s]]^{\oplus(n+1)} \) viewed as an abelian group.

**Theorem 7.2.** Assume that the set \( \text{Sol KV}^{(g,n+1)}_{(p,q)} \) is nonempty. Then, it carries a free and transitive action of the group \( \text{KRV}^{(g,n+1)}_q \) by right multiplications, \( F \mapsto FG \).

**Proof.** Let \( F \in \text{Sol KV}^{(g,n+1)}_{(p,q)} \) and \( G \in \text{KRV}^{(g,n+1)}_q \). Then, \( FG(\omega) = F(\omega) = \xi \) and

\[
j_q(FG) = j_q(F) + F \cdot j_q(G)
\]

\[
= r + |p| + \sum_j |h_j(z_j)| - |h(\xi)| + F \cdot \sum_j |h'_j(z_j)| - F \cdot |h'(\omega)|
\]

Here we have used the facts that \( F \cdot |h'_j(z_j)| = |h'_j(z_j)| \) because \( F \in \text{TAut}^{(g,n+1)} \) and \( F \cdot |h'(\omega)| = |h'(\xi)| \) because \( F(\omega) = \xi \). Hence, \( FG \in \text{Sol KV}^{(g,n+1)}_{(p,q)} \), as required. We
Here we have used the facts that \(h|P\) commute.

Proof. For the first equation in the definition of \(K_{RV}^{(g,n+1)}\) defined by formula \(F\) and \(G\) then, \(Duflo\) functions of the element \(G = F^{-1}F'\) and \(F^{-1}.|h'(z_j)| = |h(z_j)|\) because \(F^{-1}\in \text{TAut}^{(g,n+1)}\) and \(F^{-1}.|h(\omega)| = |h'(\omega)|\) because \(F^{-1}(\xi) = \omega\). Therefore, \(G \in K_{RV}^{(g,n+1)}\), as required.

Note that the Duflon functions of the element \(G = F^{-1}F'\) are given by expressions \(h_j' - h_j\) and \(h' - h\), where \(h_j\) and \(h\) are the Duflo functions of \(F\) and \(h_j'\) and \(h'\) are the Duflo functions of \(F'\).

Observe that for any \(g_1,n_1,g_2,n_2\) there is a subgroup in the product of \(KR\) groups

\[
\Gamma = \{(G_1,G_2,G) \in K_{RV}^{(g_1,n_1+1)} \times K_{RV}^{(g_2,n_2+1)} \times K_{RV}^{(0,3)}; h(G_1) = h(G_2) = h(G)\}.
\]

The following proposition gives a gluing construction for \(KR\) groups:

**Proposition 7.3.** There is a unique group homomorphism \(\Gamma \rightarrow K_{RV}^{(g_1+g_2,n_1+n_2+1)}\) defined by formula

\[
(G_1,G_2,G) \mapsto (G_1 \times G_2)P(G).
\]

**Proof.** For the first equation in the definition of \(K_{RV}^{(g_1+g_2,n_1+n_2+1)}\) observe that \(G_1,G_2\) and \(P(G)\) preserve \(\omega = \omega_1 + \omega_2\). For the second equation, we compute

\[
j_{(q_1,q_2)}((G_1 \times G_2)P(G)) = j_{q_1}(G_1) + j_{q_2}(G_2) + (G_1 \times G_2). j_{(q_1,q_2)}(P(G))
\]

\[
= \sum_j |h_j'(z_j)| + |h_j(z_j)| - |h(\omega_1)| + \sum_j |h_j(z_j)| - |h(\omega_2)|
\]

\[
+ (G_1 \times G_2). |h(\omega_1) + h(\omega_2) - h(\omega)| - C_{(q_1,q_2)}(P(G))
\]

\[
= \sum_j |h_j(z_j)| - |h(\omega)|.
\]

Here we have used the facts that \(h(G_1) = h(G_2) = h(G)\) and that \(C_{(q_1,q_2)}(P(G))\) is a linear combination of \(|\omega_1|\) and \(|\omega_2|\) and, hence, a linear combination of expressions \(|z_j^{(1)}|\) and \(|z_j^{(2)}|\). The Duflo functions \(h_j\) are obtained by adding those linear terms to \(h_j^{(1)}\) and \(h_j^{(2)}\), respectively.

To show that the map \((G_1,G_2,G) \mapsto (G_1 \times G_2)P(G)\) is a group homomorphism recall that \(P\) is a group homomorphism and note that the factors \((G_1 \times G_2)\) and \(P(G)\) commute.

There is an interesting relation between the Kashiwara-Vergne groups in genus one and genus zero.
Proposition 7.4. The map $I : G \mapsto \phi^{-1}G^{\text{ell}}\phi$ defines an injective group homomorphism $\text{KRV}^{(0,3)} \rightarrow \text{KRV}^{(1,1)}$. Furthermore, the Duflo functions of $G$ and $I(G)$ coincide.

Proof. The map $I$ is an injective group homomorphism because so is the map $G \mapsto G^{\text{ell}}$. To show that it maps $\text{KRV}^{(0,3)}$ to $\text{KRV}^{(1,1)}$, let $G \in \text{KRV}^{(0,3)}$ and represent it in the form $G = F_1^{-1}F_2$, where $F_1, F_2 \in \text{SolKV}^{(0,3)}_q$ with $q(z_1) = q(z_2) = 1$. By the proof of Theorem 7.2, the Duflo function of $G$ is given by $h = h_2 - h_1$, where $h_1$ and $h_2$ are the Duflo functions of $F_1$ and $F_2$, respectively.

We can decompose $I(G)$ into a product of two factors in a similar way:

$$I(G) = \phi^{-1}G^{\text{ell}}\phi = (F_1^{\text{ell}}\phi)^{-1}(F_2^{\text{ell}}\phi).$$

By Theorem 6.11, the expressions $F_1^{\text{ell}}\phi, F_2^{\text{ell}}\phi$ are elements of $\text{SolKV}^{(1,1)}$ with Duflo functions $h_1$ and $h_2$. Then, Theorem 7.2 implies that $I(G)$ is an element of $\text{KRV}^{(1,1)}$ with Duflo function $h_2 - h_1 = h$, as required. \qed

Let $T$ be a binary oriented (in the direction of the root) rooted tree with leaves labeled by symbols $[x_i, y_i]$ for $i = 1, \ldots, g$ and $z_j$ for $j = 1, \ldots, n$, the inner vertices labeled by a sum of expressions corresponding to the incoming branches and such that the root is labeled by the element $\omega$. To such a tree, we associate a group homomorphism

$$(\text{KRV}^{(0,3)}_q)^{(2g+n-1)} \rightarrow \text{KRV}^{(g,n+1)}_q.$$

Here $(\text{KRV}^{(0,3)}_q)^{(2g+n-1)}$ is a subgroup in the direct product $(\text{KRV}^{(0,3)}_q)^{(2g+n-1)}$ where all $\text{KRV}^{(0,3)}_q$ elements share the same Duflo function. The group homomorphism above is constructed using the group homomorphism $\text{KRV}^{(0,3)}_q \rightarrow \text{KRV}^{(1,1)}_q$ of Proposition 7.4 for each pair $x_i, y_i$ and then using the group homomorphisms of Proposition 7.3 for each inner vertex of the tree $T$.

The observation above implies that for each tree $T$ there is a diagonal group homomorphism $\text{GRT}_1 \rightarrow \text{KRV}^{(0,3)}_q \rightarrow \text{KRV}^{(g,n+1)}_q$ from the Grothendieck-Teichmüller group $\text{GRT}_1$ to the higher genus Kashiwara-Vergne groups.

The KRV groups naturally act by automorphisms of the graded Lie bialgebra $\text{gr} \hat{\mathfrak{g}}(\Sigma, f)$:

Theorem 7.5. The group $\text{KRV}^{(g,n+1)}_q \subset \text{TAut}^{(g,n+1)}$ acts by automorphisms of the Lie bialgebra structure $\text{gr} \hat{\mathfrak{g}}(\Sigma, f)$ for the framing $f$ defined by $\chi = (p, q)$ for any $p$.

Proof. Elements $G \in \text{KRV}^{(g,n+1)}_q \subset \text{TAut}^{(g,n+1)}$ preserve the element $\omega$. Hence, by Theorem 2.31 in [33] (for the case of $n = 0$, see [27] Theorem 10.4), $G$ preserves the Lie bracket of $\text{gr} \hat{\mathfrak{g}}(\Sigma, f)$. For the cobracket, we compute

$$G^*\delta_f = (G^*t\text{Div}_q) \circ (G^*\Pi_{gr}) = (t\text{Div}_q + \partial j_q(G^{-1})) \circ \Pi_{gr} = t\text{Div}_q \circ \Pi_{gr} = \delta_f.$$ 

Here we have used the facts that $G^*\Pi_{gr} = \Pi_{gr}$ and that $j_q(G^{-1})$ is a central element in $\text{gr} \hat{\mathfrak{g}}(\Sigma, f)$ and, hence, $\partial j_q(G^{-1}) \circ \Pi_{gr} = 0$. \qed

Corollary 7.6. For each tree $T$, there is an action of the group $\text{GRT}_1$ by automorphisms of the graded Lie bialgebra $\text{gr} \hat{\mathfrak{g}}(\Sigma, f)$. 

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7.2 Kashiwara-Vergne Lie algebras

The groups $\text{KRV}_q^{(g,n+1)}$ are pro-unipotent. The corresponding graded Lie algebras are given by

$$\mathfrak{krv}_q^{(g,n+1)} = \{ u \in \text{tder}^{(g,n+1)} | u(\omega) = 0, \text{tDiv}_q(u) = \sum_j |h_j(z_j)| - |h(\omega)| \text{ for some } h_j, h \in \mathbb{K}[[s]] \}.$$ 

Indeed, one can easily show that for $u \in \mathfrak{krv}_q^{(g,n+1)}$ the group element $G = e^u \in \text{KRV}_q^{(g,n+1)}$ and that every $G \in \text{KRV}_q^{(g,n+1)}$ is of that form. Again, in genus zero the Lie algebras $\mathfrak{krv}_q^{(0,n+1)}$ for different $q$'s coincide. In particular, the Lie algebra $\mathfrak{ruv}^{(0,3)}$ (we drop the subscript $q$) was introduced and studied in [4]. It is known that there is an injective Lie homomorphism $\mathfrak{ruv}^{(0,3)} \to \mathfrak{ruv}^{(1,1)}$ (see Theorem 4.6 in [4]).

In the rest of this section, we will focus on the Lie algebra $\mathfrak{ruv}^{(1,1)}$ which corresponds to the second nontrivial case (besides $g = 0, n = 2$) in the existence proof. In this case, the definition above can be written in a more explicit form:

$$\mathfrak{ruv}^{(1,1)} = \{ u \in \text{Der}^{+}(\mathbb{L}(x,y)) | u([x,y]) = 0, \text{tDiv}(u) = -|h([x,y])| \text{ for some } h \in \mathbb{K}[[s]] \}.$$ 

Here $\mathbb{L}(x,y)$ is the free Lie algebra with generators $x$ and $y$, and $\text{Der}^{+}(\mathbb{L}(x,y))$ is the Lie algebra of its derivations of positive degree.

**Proposition 7.7.** The map $I : u \mapsto \text{Ad}_{\phi^{-1}}(u^{\text{ell}})$ defines an injective Lie homomorphism $\mathfrak{ruv}^{(0,3)} \to \mathfrak{ruv}^{(1,1)}$. Furthermore, the Duflo functions of $u$ and $I(u)$ coincide.

**Proof.** This is a direct consequence of Proposition [7.4].

Proposition 7.7 together with Theorem 4.6 in [4] imply that there is an injective Lie algebra homomorphism $\mathfrak{ruv}^{(0,3)} \to \mathfrak{ruv}^{(1,1)}$.

Recall from [10 33] the definition of symplectic derivation $\delta_{2n} \in \text{Der}^{+}(\mathbb{L}(x,y))$: for $n \geq 1$, $\delta_{2n}$ is the unique derivation of $\mathbb{L}(x,y)$ of degree $2n$ such that

$$\delta_{2n}(x) = \text{ad}_{\frac{2n}{2}}(y), \quad \delta_{2n}([x,y]) = 0.$$ 

These conditions uniquely determine the value of $\delta_{2n}(y)$:

$$\delta_{2n}(y) = \sum_{i=0}^{n-1} (-1)^i [\text{ad}_{x}^i(y), \text{ad}_{y}^{2n-1-i}(y)].$$

**Proposition 7.8.** For all $n \geq 1$, we have $\delta_{2n} \in \mathfrak{ruv}^{(1,1)}$.

**Proof.** By definition, we have $\delta_{2n}([x,y]) = 0$ which is the first defining property of elements in $\mathfrak{ruv}^{(1,1)}$. In order to compute $\text{tDiv}(\delta_{2n})$, recall that it takes values in the

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For the use in this section, it is convenient to introduce the following version of the KV problem: 

\[
\text{Proposition 7.9.}
\]

By applying this formula \(2n - 1\) times, we compute

\[
|\langle \partial y \delta_{2n}(y) \rangle_1| = \sum_{i=0}^{n-1} (-1)^i \text{ad}_x^i(y)x^{2n-1-i} - \text{ad}_x^{2n-1-i}(y)x^i = |yx^{2n-1}|.
\]

In conclusion, \(t\text{Div}(\delta_{2n}) = \Delta(yx^{2n-1} - x^{2n-1}y) = 0\) which together with \(\delta_{2n}([x,y]) = 0\) implies that \(\delta_{2n} \in \mathfrak{trv}^{(1,1)}\).

Together, Proposition 7.7 and Proposition 7.8 imply Theorem 1.6 on the structure of the Lie algebra \(\mathfrak{trv}^{(1,1)}\) from the Introduction.

### 7.3 Kashiwara-Vergne groups KV

For the use in this section, it is convenient to introduce the following version of the cocycle \(j\) which depends on both components of the framing \(\chi = (p,q)\):

\[
J_{p,q} = j_0 + \partial(r + |p|), \quad J_{p,q}(F) = j_q(F) + F \cdot (r + |p|) - (r + |p|).
\]

We define Kashiwara-Vergne groups \(KV_{p,q}^{(g,n+1)}\) as follows:

\[
KV_{p,q}^{(g,n+1)} = \{ G \in \text{TAut}^{(g,n+1)}; G(\xi) = \xi, J_{q,p}(G) = \sum_j |h_j(z_j)| - |h(\xi)| \}.
\]

Here \(h_j, h \in K[\![s]\!]\) and \(\xi = \log(\prod_i (e^{x_i^p} e^{y_i^q} e^{-x_i} e^{-y_i} \prod_j e^{z_j})\). Similar to \(KRV_q^{(g,n+1)}\), the groups \(KV_{p,q}^{(g,n+1)}\) are independent of \(q\) for \(g = 0\). Recall that there is a canonical group homomorphism (see Theorem 2.5 in \([\text{I}]\))

\[
\text{GT}_1 \to KV^{(0,3)}.
\]

Similar to the groups \(KRV_q^{(g,n+1)}\), the groups \(KV_{p,q}^{(g,n+1)}\) act freely and transitively on solutions of the KV problem:

\[
\text{Proposition 7.9. The group } KV_{p,q}^{(g,n+1)} \text{ acts freely and transitively on } \text{Sol } KV_{p,q}^{(g,n+1)} \text{ (in case it is nonempty) by left translations } F \mapsto GF.
\]
Proof. The proof is similar to the proof of Theorem 7.2.

The following proposition establishes an isomorphism between groups $KV_{p,q}^{(g,n+1)}$ and $KV_{p,q}^{(g,n+1)}$:

**Proposition 7.10.** Every solution of the KV problem $F \in \text{Sol} KV_{p,q}^{(g,n+1)}$ defines a group isomorphism $KV_{q}^{(g,n+1)} \rightarrow KV_{p,q}^{(g,n+1)}$ given by formula $G \mapsto FGF^{-1}$.

**Proof.** Let $G \in KV_{q}^{(g,n+1)}$. Then,

$$FGF^{-1}(\xi) = FG(\omega) = F(\omega) = \xi.$$

Furthermore,

$$j_p(FGF^{-1}) = j_q(F) + F.j_q(G) - FGF^{-1}.j_q(F) = r + |p| - FGF^{-1}.(r + |p|) + \sum_j |h_j(z_j)| - |h(\xi)|,$$

where $h_j$ and $h$ are Duflo functions of $G$. The calculation above implies

$$J_{p,q}(FGF^{-1}) = \sum_j |h_j(z_j)| - |h(\xi)|,$$

as required. 

Note that since the groups $KV_{q}^{(g,n+1)}$ are independent of $p$, we conclude that the groups $KV_{p,q}^{(g,n+1)}$ with different $p$ are isomorphic to each other (when the KV problem admits solutions). Similar to the case of KRV groups, there is a relation between the KV groups in genus 0 and genus 1:

**Proposition 7.11.** The map $G \mapsto \text{Gell}$ defines an injective group homomorphism $KV^{(0,3)} \rightarrow KV^{(1,1)}$ preserving the Duflo function.

**Proof.** The proof is similar to the one of Proposition 7.4.

There is also a gluing construction for KV groups. In order to describe it, we need the following pair-of-paints map

$$\tilde{\mathcal{P}} : \text{TAut}^{(0,3)} \rightarrow \text{TAut}^{(g_1+g_2,n_1+n_2+1)}$$

that sends an element $(F, f_1, f_2) \in \text{TAut}^{(0,3)}$ to the automorphism

$$\tilde{\mathcal{P}} : x_i^{k} \mapsto e^{-f_k(\xi_1, \xi_2)} x_i^{k} e^{f_k(\xi_1, \xi_2)}$$

$$y_i^{k} \mapsto e^{-f_k(\xi_1, \xi_2)} y_i^{k} e^{f_k(\xi_1, \xi_2)}$$

$$z_i^{k} \mapsto e^{-f_k(\xi_1, \xi_2)} z_i^{k} e^{f_k(\xi_1, \xi_2)} \quad k = 1, 2,$$

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where

\[ \xi_1 = \log \left( \prod_{i=1}^{g_1} (e^{x_i^1} e^{y_i^1} e^{-x_i^1} e^{-y_i^1}) \prod_{j=1}^{n_1} e^{x_j^1} \right), \quad \xi_2 = \log \left( \prod_{i=1}^{g_2} (e^{x_i^2} e^{y_i^2} e^{-x_i^2} e^{-y_i^2}) \prod_{j=1}^{n_2} e^{x_j^2} \right). \]

Let \( F_1 \) and \( F_2 \) be solutions of equations \( \text{KV I}^{(g_1,n_1+1)} \) and \( \text{KV I}^{(g_1,n_1+1)} \), respectively. Then we have the identity

\[ (F_1 \times F_2) \circ \mathcal{P}(G) = \tilde{\mathcal{P}}(G) \circ (F_1 \times F_2) \]  

for all \( G \in \text{TAut}^{(0,3)} \).

We denote by \( \tilde{\Gamma} \) the following group:

\[ \tilde{\Gamma} = \{(G_1,G_2,G) \in \text{KV}_{(g_1,n_1+1)}^{(0,0)} \times \text{KV}_{(g_2,n_2+1)}^{(0,0)} \times \text{KV}^{(0,3)}; \ h(G_1) = h(G_2) = h(G) \}. \]

**Proposition 7.12.** For \( n_1 = 0 \) or \( g_2 = 0 \), there is a unique group homomorphism \( \tilde{\Gamma} \rightarrow \text{KV}_{(g_1+g_2,n_1+n_2+1)}^{(0,0)} \) given by formula

\[ (G_1,G_2,G) \mapsto (G_1 \times G_2) \tilde{\mathcal{P}}(G). \]

**Proof.** Consider solutions of KV problems in the adapted framings \( F_1,F_1',F_2,F_2',F,F' \) such that \( F_1' = G_1 F_1, F_2' = G_2 F_2, F' = GF \). Then, by Proposition 7.9 the following element is in \( \text{KV}_{(g_1+g_2,n_1+n_2+1)}^{(0,0)} \):

\[ (F_1' \times F_2') \mathcal{P}(F')(F_1 \times F_2) \mathcal{P}(F)^{-1} = (G_1 \times G_2) \tilde{\mathcal{P}}(G). \]

The map \( (G_1,G_2,G) \mapsto (G_1 \times G_2) \tilde{\mathcal{P}}(G) \) is a group homomorphism since \( \tilde{\mathcal{P}} \) is a group homomorphism and the factors \( G_1 \times G_2 \) and \( \tilde{\mathcal{P}}(G) \) commute. \( \square \)

Combining the gluing maps and the relations between KV groups for genus 0 and genus 1, we obtain the following group homomorphisms (parametrized by binary rooted trees):

\[ \text{GT}_1 \rightarrow \text{KV}^{(g,n+1)}_{(0,3)} \rightarrow \left( \text{KV}^{(0,3)}_{(0,3)} \right)^{\times 2g+n-1} \rightarrow \text{KV}^{(g,n+1)}_{(0,0)} \]

**Theorem 7.13.** The group \( \text{KV}^{(g,n+1)}_{(p,q)} \) acts by automorphisms of the Goldman-Turaev Lie bialgebra \( \mathfrak{g}(\Sigma,f) \) for a surface of genus \( g \) with \( n+1 \) boundary components with framing \( f \) defined by \( \chi = (p,q) \).

**Proof.** Let \( G \in \text{KV}_{(p,q)}^{(g,n+1)} \), choose \( F \in \text{Sol} \text{KV}_{(p,q)}^{(g,n+1)} \) and define \( F' = GF \in \text{Sol} \text{KV}_{(p,q)}^{(g,n+1)} \). Then, \( G = F' F^{-1} \). The element \( F^{-1} \) induces an expansion which maps the Goldman bracket to the graded bracket, and the map \( F' \) induces an inverse expansion which maps the graded bracket to the Goldman bracket. Hence, \( F' F^{-1} \) is an automorphism of the Goldman bracket. A similar argument applies to the framed Turaev cobracket. \( \square \)

**Corollary 7.14.** The Goldman-Turaev Lie bialgebra \( \mathfrak{g}(\Sigma,f) \) carries actions of the group \( \text{GT}_1 \) parametrized by certain pair of pants decomposition of \( \Sigma \) (or, equivalently by certain binary rooted trees).
8 Application to Johnson homomorphisms

In this section, we discuss an application of the formality of the Goldman-Turaev Lie bialgebra to the theory of Johnson homomorphisms. We will only consider the case of connected surfaces with one boundary component (that is, $n = 0$).

8.1 Johnson homomorphisms

We briefly review what is known about the Johnson homomorphisms for the mapping class group. For more detail, see, for example, [15, 16, 30, 22].

Recall that the mapping class group of $\Sigma = \Sigma_{g,1}$, $\mathcal{M}(\Sigma_{g,1})$, is the group of isotopy classes of diffeomorphisms of $\Sigma_{g,1}$ fixing the boundary pointwise. The Torelli group $I_{g,1}$ is the subgroup of $\mathcal{M}(\Sigma_{g,1})$ defined to be the kernel of the action on the first homology group $H_1(\Sigma, \mathbb{Z}) = \pi_1(\Sigma)^{\text{abel}}$. The action on the higher order nilpotent quotients of $\pi_1(\Sigma)$ defines a filtration of the Torelli group

$$I_{g,1} = I_{g,1}(1) \supset I_{g,1}(2) \supset I_{g,1}(3) \supset \cdots$$

which is called the Johnson filtration. This is a central filtration, that is, one has $[I_{g,1}(k), I_{g,1}(\ell)] \subset I_{g,1}(k + \ell)$. Then one can consider the graded Lie algebra

$$\text{gr} I_{g,1} := \bigoplus_{k=1}^{\infty} I_{g,1}(k)/I_{g,1}(k + 1)$$

whose Lie bracket is induced by group commutator. The Siegel modular group $Sp_{2g}(\mathbb{Z})$ acts on $\text{gr} I_{g,1}$ in an obvious way.

In the case $n = 0$, the Lie algebra $\text{sd}er^+$ is nothing but the Lie algebra of positive degree symplectic derivations of the free Lie algebra $L$. Following literature, we use the notation $h_{g,1} = \text{sd}er^+$. More precisely, we will use the integral part of $h_{g,1}$. Put $H^Z = H_1(\Sigma, \mathbb{Z})$ and denote by $L^Z = \bigoplus_{k \geq 1} L^Z_k$ the free Lie algebra over $\mathbb{Z}$ generated by $H^Z$. The degree $k$ component $L^Z_k$ is naturally identified with the $k$-th graded quotient of the lower central series of the free group $\pi_1(\Sigma)$. Then the Lie algebra $h^Z_{g,1}$ is defined in the same way as $h_{g,1}$. By restriction to the degree 1 part, we obtain an injection

$$i : h^Z_{g,1} \hookrightarrow \text{Hom}(H^Z, L^Z) = (H^Z)^* \otimes L^Z \cong H^Z \otimes L^Z, \quad D \mapsto D|_{H^Z}. \quad (30)$$

Here, the last isomorphism uses the intersection form, which is non-degenerate in the case $n = 0$. Through this embedding, one has a description for the degree $k$ part of $h^Z_{g,1}$:

$$h^Z_{g,1}(k) = \text{Ker}([\cdot, \cdot] : H^Z \otimes L^Z_{k+1} \to L^Z_k).$$

The $k$th Johnson homomorphism (after refinement by Morita) is a group homomorphism

$$\tau_k : I_{g,1}(k) \to h^Z_{g,1}(k), \quad \varphi \mapsto ([\gamma] \in H^Z \mapsto [\varphi(\gamma)\gamma^{-1}] \in L^Z_{k+1}). \quad (31)$$
The collection \( \{\tau_k\}_k \) induces an injective \( Sp_{2g}(\mathbb{Z}) \)-homomorphism of graded Lie algebras

\[
\tau : \text{gr} I_{g,1} \rightarrow \mathfrak{h}_{g,1}^Z.
\]

(32)

The homomorphism [31] is not surjective in general. This fact was first discovered by Morita [30]. More precisely, Morita defined a homomorphism \( \Theta_k : \mathfrak{h}_{g,1}^Z(k) \rightarrow S^k H^Z \) and showed that \( \Theta_k \circ \tau_k = 0 \) on \( I_{g,1}(k) \) and that \( \Theta_k \otimes \mathbb{Q} \) is surjective for each odd \( k \geq 3 \). Here, \( S^k H^Z \) is the \( k \)th symmetric product of \( H^Z \).

Thus the image of \( \tau \) becomes an object of interest. A deep result of Hain [13] shows that the image of \( \tau \) is generated by the degree 1 part \( \text{Im} \tau_1 \) over \( \mathbb{Q} \). On the other hand, the structure of the cokernel \( \mathfrak{h}_{g,1}^Z / \text{Im} \tau \) is rather mysterious, and its study is an active area of research. Usually one considers this problem over \( \mathbb{Q} \) in order to make use of representation theory, and the main question is formulated as follows.

**Johnson image problem:** For each \( k \geq 1 \), find a \( sp_{2g} \)-module \( M_k \) (with an explicit decomposition into irreducible \( sp_{2g} \)-modules) and a surjective \( sp_{2g} \)-homomorphism \( \mathfrak{h}_{g,1}(k) \rightarrow M_k \) with kernel \( \text{Im} \tau_k \otimes \mathbb{Q} \).

Morita, Sakasai and Suzuki [32] determined the structure of \( M_k \) for \( k = 6 \). For \( k \leq 5 \), see the references in [32].

A refinement of Morita’s map \( \Theta_k \) was introduced by Enomoto and Satoh [9]. For any \( k \geq 2 \), Enomoto and Satoh introduced a homomorphism \( ES_k : \mathfrak{h}_{g,1}^Z(k) \rightarrow |A|_k \), where \( |A|_k \cong H^\otimes k / Z_k \) is the degree \( k \) part of the space of cyclic words in \( H \), and showed that \( ES_k \circ \tau_k = 0 \). The map \( \Theta_k \) factors through \( ES_k \) via a natural map \( |A|_k \rightarrow S^k H \).

The Enomoto-Satoh trace \( ES_k \) is an effective tool for studying the surjectivity of \( \tau_k \); for instance, it plays an important role in the work of Morita, Sakasai and Suzuki mentioned above.

### 8.2 Enomoto-Satoh trace and framed Turaev cobracket

In [21], Morita’s trace map was given a geometric interpretation in terms of the non-framed version of the Turaev cobracket. We now give a similar interpretation of the Enomoto-Satoh trace by using the framed version of the Turaev cobracket \( \delta^f \).

We start with the following observation:

**Lemma 8.1.** The Enomoto-Satoh trace is equal to the divergence map on \( \mathfrak{h}_{g,1} \).

**Proof.** The map \( ES_k \) is given by the following composition:

\[
ES_k : \mathfrak{h}_{g,1}(k) \hookrightarrow H \otimes L_{k+1} \overset{\text{incl.}}{\rightarrow} H \otimes H^\otimes k+1 = H^\otimes k+2 \overset{C_{12}}{\rightarrow} H^\otimes k \overset{\text{proj.}}{\rightarrow} |A|_k.
\]

Here, \( i \) is the map in [30] and \( C_{12} \) is the contraction of the first and second factors by using the intersection form. One can directly verify that this coincides with the divergence map. 

\[\square\]
In [21, 22], the second and third authors gave a non-graded version of the Johnson homomorphisms in a geometric way. They introduced a pro-nilpotent Lie algebra \( L^+(\Sigma) \) as a Lie subalgebra of the completed Goldman Lie algebra \( \hat{\mathfrak{g}}(\Sigma) = |\mathbb{R}_{\pi}| \) and constructed an injective group homomorphism
\[
\tau : I_{g,1} \hookrightarrow L^+(\Sigma).
\] (33)

Here, the group structure of the target is given by the BCH series. The canonical map \( \text{gr} \hat{\mathfrak{g}}(\Sigma) \cong |A| \to \text{sDer}(A) \) induces an isomorphism \( \text{gr} L^+(\Sigma) \cong \mathfrak{h}_{g,1} \), and the associated graded map of (33) coincides with the classical one (32).

Let \( f \) be a framing on \( \Sigma_{g,1} \) and denote by \( I_{f,g,1} \) the subgroup of \( I_{g,1} \) consisting of elements preserving \( f \). Note that \( I_{g,1}(2) \) is contained in \( I_{f,g,1} \) for any \( f \).

**Proposition 8.2.** \( \delta^f \circ \tau = 0 \) on \( I_{f,g,1} \).

**Proof.** Note that the operations \( \delta^f \) and its upgrade \( \mu^f \) are preserved by \( I_{f,g,1} \) because they are defined in terms of self-intersections of loops and the action of \( I_{f,g,1} \) does not change the rotation number function \( \text{rot}_f \). With this in mind, one can apply the proof of [21] Theorem 6.2.1 verbatim.

Once a symplectic expansion \( \theta \) is fixed, one obtains a Lie algebra homomorphism \( \hat{\mathfrak{g}}(\Sigma) \to \text{sDer}(A) \), which restricts to a Lie algebra isomorphism \( \theta : L^+(\Sigma) \cong \text{sder}^+ = \mathfrak{h}_{g,1} \).

By composing the embedding (33) with this isomorphism, we obtain the map
\[
\tau^{\theta} : I_{g,1} \to \mathfrak{h}_{g,1}.
\]

This equals Massuyeau’s improvement [25] of the total Johnson map [18]. By Proposition 8.2 one has \( \delta^f \circ \tau^{\theta} = 0 \) on \( I_{f,g,1} \). Let \( \tau^{\theta}_k \) be the degree \( k \) part of \( \tau^{\theta} \): for any \( \varphi \in I_{g,1} \) write \( \tau^{\theta}(\varphi) = \sum_{k \geq 1} \tau^{\theta}_k(\varphi) \) with \( \tau^{\theta}_k(\varphi) \in \mathfrak{h}_{g,1}(k) \). One has \( \tau^{\theta}|_{I_{g,1}(k)} = \tau_k \) for any \( \theta \).

**Theorem 8.3.** Suppose \( g \geq 2 \) and fix a framing \( f \) on \( \Sigma_{g,1} \). Let \( \theta \) be a symplectic expansion coming from a solution to the Kashiwara-Vergne problem of type \((g,1)\) for the framing \( f \). Then, \( ES_k \circ \tau^{\theta}_k = 0 \) on \( I_{f,g,1}^k \) for any \( k \geq 1 \). In particular, \( ES_k \circ \tau_k = 0 \) on \( I_{g,1}(k) \).

**Proof.** The results in Sections 6 and 7 show that \( \delta^f \) is transformed into \( \text{div} = ES \) through \( \theta \). Hence \( ES \circ \tau^{\theta} = 0 \) on \( I_{f,g,1} \). Since \( ES \) is homogeneous of degree \(-2\), we obtain the assertion.

**Remark 8.4.** This result is a refinement of Corollary 6.3.3 in [21], where the non-framed version of the Turaev cobracket was used. There are explicit differences between the two constraints for the Johnson image; see [8].

**Remark 8.5.** In genus \( 0 \), the following fact is analogous to the theorem above: Let \( \mathfrak{t}_n \) be the Drinfel’d-Kohno Lie algebra of infinitesimal pure braids which sits in \( \text{sder} = \text{sder}^{(0,n+1)} \) in a natural way. Then \( \mathfrak{t}_n \) is in the kernel of the divergence map.
Moreover, Ševera and Willwacher \[36\] gives a decreasing filtration \( s \supset \text{Ker}(\text{div}) \supset \cdots \supset t_n \) by using a certain spectral sequence related to a graph complex. (See also a related work by Felder \[11\].) This can be thought of as a solution to the genus 0 analogue of the Johnson image problem.

**Remark 8.6.** Tsuji \[37\] \[38\] \[39\] \[40\] introduced two skein analogues of the group homomorphism \( \tau \) using the Kauffman bracket skein algebra and the HOMFLY-PT skein algebra of the product \( \Sigma \times [0, 1] \), respectively. Both of them give rise to new constructions of finite type invariants of integral homology 3-spheres.

For the symplectic expansion in Theorem 8.3, the map \( \tau^\theta \) induces an embedding

\[ \mathcal{T}_{g,1}^f \hookrightarrow \text{frv}_{q}^{(g,1)}, \tag{34} \]

where \( f \) corresponds to \( \chi = (p,q) \). In fact, the formula \( \varphi \mapsto \theta \circ \text{DN}(\varphi) \circ \theta^{-1} \), where \( \text{DN}(\varphi) \) means the action of \( \varphi \) on the completed group ring \( \hat{K}\pi \), defines an embedding of \( \mathcal{T}_{g,1}^f \) into the Kashiwara-Vergne group \( \text{KRV}_{q}^{(g,n+1)} \). Composing it with the logarithm we obtain the map in (34).

As we saw in Section 7, for a certain binary rooted tree there is an embedding of the Grothendieck–Teichmüller group \( \text{GRT}_1 \) into the higher genus Kashiwara-Vergne group \( \text{KRV}_{q}^{(g,n+1)} \). In view of the Johnson image problem, it is interesting to clarify how the image of \( \text{GRT}_1 \) and the group \( \mathcal{T}_{g,1}^f \) are related in \( \text{KRV}_{q}^{(g,n+1)} \).

**References**

1. A. Alekseev, B. Enriquez, C. Torossian, Drinfeld associators, braid groups and explicit solutions of the Kashiwara-Vergne equations, Publications mathematiques de l'IHS 112.1, 143–189 (2010)
2. A. Alekseev, N. Kawazumi, Y. Kuno and F. Naef, The Goldman-Turaev Lie bialgebra in genus zero and the Kashiwara-Vergne problem, Adv. Math. 326, 1–53 (2018)
3. A. Alekseev and F. Naef, Goldman-Turaev formality from the Knizhnik-Zamolodchikov connection, preprint, [arXiv:1708.03119] (2017)
4. A. Alekseev and C. Torossian, The Kashiwara-Vergne conjecture and Drinfeld’s associators, Ann. of Math. 175, 415–463 (2012)
5. M. F. Atiyah and R. Bott, The Yang-Mills equations over Riemann surfaces, Philos. Trans. R. Soc. Lond. Ser. A 308 (1505), 523–615 (1983)
6. M. Chas, Combinatorial Lie bialgebras of curves on surfaces, Topology 43, 543–568 (2004)
7. W. Crawley-Boevey, P. Etingof and V. Ginzburg, Noncommutative geometry and quiver algebras, Adv. Math. 209, 274–336 (2007)
[8] N. Enomoto, Y. Kuno and T. Satoh, A comparison of classes in the Johnson cokernels of the mapping class groups of surfaces, preprint, arXiv:1805.02563 (2018)

[9] N. Enomoto and T. Satoh, New series in the Johnson cokernels of the mapping class groups of surfaces, *Algebr. Geom. Topol.* **14**, 627–669 (2014)

[10] B. Enriquez, Elliptic associators, *Sel. Math. New Ser.* **20**, 491–584 (2014)

[11] M. Felder, Internally connected graphs and the Kashiwara-Vergne Lie algebra, *Lett. Math. Phys.* (2018). https://doi.org/10.1007/s11005-018-1052-5

[12] W. Goldman, Invariant functions on Lie groups and Hamiltonian flows of surface group representations, Invent. Math. **85**, 263–302 (1986)

[13] R. Hain, Infinitesimal presentations of the Torelli groups, *J. Amer. Math. Soc.* **10**, 597–651 (1997)

[14] R. Hain, Hodge theory of the Goldman bracket, preprint, arXiv:1710.06053 (2017)

[15] D. Johnson, An abelian quotient of the mapping class group, *Math. Ann.* **249**, 225–242 (1980)

[16] D. Johnson, A survey of the Torelli group, In: *Low Dimensional Topology*, Contemp. Math. 20, AMS Providence 1983, 165–179.

[17] M. Kashiwara and M. Vergne, The Campbell-Hausdorff formula and invariant hyperfunctions, Invent. Math. **47**, 249–272 (1978)

[18] N. Kawazumi, Cohomological aspects of Magnus expansions, preprint, arXiv:math/0505497 (2005)

[19] N. Kawazumi, The mapping class group orbits in the framings of compact surfaces, to appear in: Quarterly J. Math. arXiv:1703.02258

[20] N. Kawazumi and Y. Kuno, The logarithms of Dehn twists, Quantum Topol. **5**, 347–423 (2014)

[21] N. Kawazumi and Y. Kuno, Intersections of curves on surfaces and their applications to mapping class groups, Annales de l’institut Fourier **65**, 2711–2762 (2015)

[22] N. Kawazumi and Y. Kuno, The Goldman-Turaev Lie bialgebra and the Johnson homomorphisms, Handbook of Teichmuller theory, ed. A. Papadopoulos, Volume V, EMS Publishing House, Zurich, 97–165 (2016)

[23] L. Jeffrey, Extended moduli spaces of flat connections on Riemann surfaces, Math. Ann. **298**, 667–692 (1994)

[24] W. Magnus, A. Karass and D. Solitar, *Combinatorial group theory*. Wiley, New York (1966)
[25] G. Massuyeau, Infinitesimal Morita homomorphisms and the tree-level of the LMO invariant, Bull. Soc. Math. France 140, 101–161 (2012)

[26] G. Massuyeau, Formal descriptions of Turaev’s loop operations, Quantum Topol. 9, 39–117 (2018)

[27] G. Massuyeau and V. G. Turaev, Fox pairings and generalized Dehn twists, Annales de l’institut Fourier 63, 2403–2456 (2013)

[28] G. Massuyeau and V. G. Turaev, Tensorial description of double brackets on surface groups and related operations, draft (2012)

[29] G. Massuyeau and V. G. Turaev, Quasi-Poisson structures on representation spaces of surfaces, Int. Math. Res. Not. 2014, 1–64 (2014)

[30] S. Morita, Abelian quotients of subgroups of the mapping class group of surfaces, Duke Math. J. 70, 699-726 (1993)

[31] S. Morita, Casson invariant, signature defect of framed manifolds and the secondary characteristic classes of surface bundles, J. Diff. Geom. 47, 560–599 (1997)

[32] S. Morita, T. Sakasai and M. Suzuki, Structure of symplectic invariant Lie subalgebras of symplectic derivation Lie algebras, Adv. Math. 282, 291–334 (2015)

[33] F. Naef, Poisson brackets in Kontsevich’s “Lie world”, preprint, arXiv:1608.08886 (2016)

[34] C. D. Papakyriakopoulos, Planar regular coverings of orientable closed surfaces, in: Ann. Math. Studies 84, Princeton University Press, Princeton, 1975, 261–292.

[35] A. Pollack, Relations between derivations arising from modular forms, Duke University undergraduate thesis (2009)

[36] P. Ševera and T. Willwacher, Equivalence of formalities of the little discs operad, Duke Math. J. 160, 175–206 (2011)

[37] S. Tsuji, Dehn twists on Kauffman bracket skein algebras, Kodai Math. J. 41, 16–41 (2018)

[38] S. Tsuji, The Torelli group and the Kauffman bracket skein module, Math. Proc. Camb. Soc., doi:10.1017/S030500411700366(2017)

[39] S. Tsuji, Construction of an invariant for integral homology 3-spheres via completed Kauffman bracket skein algebras, preprint: arXiv:1607.01580

[40] S. Tsuji, A formula for the action of Dehn twists on HOMFLY-PT skein modules and its applications, preprint: arXiv:1801.00580

[41] V. G. Turaev, Intersections of loops in two-dimensional manifolds, Mat. Sb. 106(148), 566–588 (1978) English translation: Math. USSR-Sb. 35, 229–250 (1979)
[42] V. G. Turaev, Skein quantization of Poisson algebras of loops on surfaces, Ann. Sci. École Norm. Sup. 24, 635–704 (1991)

[43] M. van den Bergh, Double Poisson Algebras, Trans. Amer. Math. Soc. 360, 5711–5799 (2008)

[44] S. Wolpert, On the symplectic geometry of deformations of a hyperbolic surface, Ann. Math. 117, 207–234 (1983)