Multilevel finite element discretizations based on local defect correction for nonsymmetric eigenvalue problems

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ABSTRACT

Based on the work of Xu and Zhou (2000), we establish new three-level and multilevel finite element discretizations by local defect correction techniques. Theoretical analysis and numerical experiments show that the discretizations are simple and easy to implement, and can be used to solve nonsymmetric eigenvalue problems with non smooth eigenfunctions efficiently. We also discuss the local error estimates of finite element approximations; it is a new feature here that the estimates apply to the local domains containing corner points.

1. Introduction

Nonsymmetric elliptic eigenvalue problems have an important physical background, such as convection–diffusion in fluid mechanics, environmental problems and so on. Thus finite element methods for solving nonsymmetric eigenvalue problems have become an important topic which has attracted the attention of mathematical and physical fields: [1] discussed a priori error estimates, [2–7] a posteriori error estimates and adaptive algorithms, [8] function value recovery algorithms, [9] two level algorithms, [10,11] extrapolation methods, [2] an adaptive homotopy approach, etc. This paper turns to discuss multilevel finite element discretizations based on local defect correction. The defect correction, also called the residual correction or the iterative improvement, is a technique in numerical linear algebra that can effectively improve the accuracy of solutions of linear algebra equations (e.g., see Section 2.5 in [12]). In 2000, Xu and Zhou [13] combined the defect correction with the finite element method to propose the local defect correction technique for the parallel-computing of elliptic equations. This technique has been developed by He et al. [14,15], Xu and Zhou [16], Dai and Zhou [17], Bi et al. [18] and Yang and Han [19], and successfully applied to Stokes equations, symmetric elliptic eigenvalue problems with non smooth eigenfunctions (including the electronic structure problems) and Stokes eigenvalue problem.

In this paper, we further apply the local defect correction technique to nonsymmetric elliptic eigenvalue problems with non smooth eigenfunctions and (or) convection dominated terms. Our work has the following features. (1) We extend local and parallel three-level finite element discretizations for symmetric eigenvalue problems established by Dai and Zhou [17] to nonsymmetric eigenvalue problems. (2) Based on the transition layer technique in [18], we establish new multilevel finite element discretizations with local refinement, this scheme repeatedly makes defect correction on finer and finer local meshes to make up for abrupt changes of the local mesh size caused by the three level scheme. As expected, theoretical
analysis and numerical experiments show that our schemes are simple and easy to implement, and can be used to solve nonsymmetric eigenvalue problems with non smooth eigenfunctions and (or) convection dominated terms. Numerical experiments show that, compared with the adaptive homotopy approach in [2], our algorithm is also efficient. (3) For the nonsymmetric problems, based on the work of [20,13], we discuss the local error estimates of finite element approximations; it is a new feature here that the estimates apply to the local domains containing corner points (see Theorem 2.1, Lemma 2.3 and Remark 2.1 in this paper).

In this paper, regarding the basic theory of finite elements, we refer to [21–24].

2. Preliminaries

Consider the nonsymmetric elliptic differential operator eigenvalue problem:

$$Lu = -\sum_{i,j=1}^d \partial_j(a_{ij}(x)\partial_i u) + \sum_{i=1}^d b_i(x)\partial_i u + c(x)u = \lambda m(x)u, \quad \text{in } \Omega,$$

$$u = 0, \quad \text{on } \partial \Omega,$$

where $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a polyhedral bounded domain with boundary $\partial \Omega$, $\partial_i u = \frac{\partial u}{\partial n_i}$, $i = 1, 2, \ldots, d$.

Let

$$a(u, v) = \int_{\Omega} \left( \sum_{i,j=1}^d a_{ij}\partial_i u \partial_j v + \sum_{i=1}^d b_i \partial_i u v + c u v \right) \, dx,$$

$$b(u, v) = \int_{\Omega} m u v \, dx.$$

The variational form associated with (2.1)–(2.2) is given by: find $\lambda \in \mathbb{C}$ and $u \in H_0^1(\Omega)$ with $\|u\|_0 = 1$ satisfying

$$a(u, v) = \lambda b(u, v), \quad \forall v \in H_0^1(\Omega).$$

Assume that $a_{ij}, b_i \in W_{1,\infty}(\Omega), c \in L_\infty(\Omega)$ are given real or complex functions in $\Omega$, $m \in L_\infty(\Omega)$ is a given real function which is bounded below by a positive constant in $\Omega$. $L$ is assumed to be uniformly strongly elliptic in $\Omega$, i.e., there is a positive constant $a_0$ such that

$$\text{Re} \sum_{i,j=1}^d a_{ij} \xi_i \xi_j \geq a_0 \sum_{i=1}^d \xi_i^2, \quad \forall \xi \in \Omega, \quad \forall (\xi_1, \xi_2, \ldots, \xi_d) \in \mathbb{R}^d.$$

We assume without loss of generality that $\text{Re} c \geq a_0/2 + \max_{x \in \Omega} |b_i(x)|^2/(2a_0)$ since adding a multiple of $m(x)$ to $c(x)$ only shifts the eigenvalues. Under above assumptions, we have

$$\text{Re} a(u, u) \geq \frac{1}{2} a_0 \|u\|_1^2, \quad \forall u \in H^1(\Omega);$$

and there are constants $M_1$ and $M_2$ such that

$$|a(u, v)| \leq M_1 \|u\|_1 \|v\|_1, \quad \forall u, v \in H^1(\Omega),$$

$$|b(u, v)| \leq M_2 \|u\|_0 \|v\|_0, \quad \forall u, v \in L_2(\Omega).$$

For $D \subset \Omega \subset \Omega_0$, we use $D \subset \subset \Omega_0$ to mean that $\text{dist}(\partial D \setminus \partial \Omega, \partial D \setminus \partial \Omega_0) > 0$.

Assume that $\pi_h(\Omega) = \{\tau\}$ is a mesh of $\Omega$ with the mesh size function $h(x)$ whose value is the diameter $h$, of the element $\tau$ containing $x$, and $h(\Omega) = \max_{x \in \Omega} h(x)$ is the mesh diameter of $\pi_h(\Omega)$. We write $h(\Omega)$ as $h$ for simplicity. Let $V_h(\Omega) \subset C(\overline{\Omega})$, defined on $\pi_h(\Omega)$, be a piecewise polynomial space, and $V_0^h(\Omega) = \{v \in V_h(\Omega), v|_{\partial \Omega} = 0\}$. Given $G \subset \Omega$, we define $\pi_h(G)$ and $V_h(G)$ to be the restriction of $\pi_h(\Omega)$ and $V_h(\Omega)$ to $G$, respectively, and

$$V_0^h(G) = \{v \in V_h^h(G), v|_{\partial G} = 0\}.$$

For any $G \subset \Omega$ mentioned in this paper, we assume that it aligns with $\pi_h(\Omega)$ when necessary.

In this paper, $C$ denotes a positive constant independent of $h$, which may not be the same constant in different places. For simplicity, we use the symbol $\alpha \leq \beta$ to mean that $\alpha \leq C \beta$.

We adopt the following assumptions for meshes and finite element spaces, where (A0)–(A2) can be found in [13] and (A3) is also based on [13].

(A0) There exists $\nu \geq 1$ such that $h(\Omega)^{\nu} \lesssim h(x), \quad \forall x \in \Omega$.

This is apparently a very mild assumption, and most practical meshes including locally refined meshes should satisfy it.
(A1) There exists \( r \geq 1 \) such that for \( w \in H^1_0(\Omega) \cap H^{1+t}(\Omega) \),
\[
\inf_{v \in V^0_0(\Omega)} (\|h^{-1}(w - v)\|_0 + \|w - v\|_1) \lesssim h^t \|w\|_{1+t}, \quad 0 \leq t \leq r.
\]

(A2) *Inverse Estimate.* For any \( v \in V^h(\Omega_0) \), \( \|v\|_{1,\Omega_0} \lesssim \|h^{-1}v\|_{0,\Omega_0} \).

(A3) *Superapproximation.* For \( G \subset \Omega_0 \), let \( \omega \in C^\infty(\Omega) \) with \( \text{supp } \omega \subset G \), then for any \( w \in V^h(G) \), \( w|_{\partial G \cap \partial \Omega} = 0 \), there exists \( v \in V^0_0(G) \) such that
\[
\|h^{-1}(\omega w - v)\|_{1,G} \lesssim \|w\|_{1,G}.
\]

Let \( \pi_h(\Omega) \) consist of shape-regular simplices and (A0) hold, and let \( V^h(\Omega) \subset C(\Omega) \) be a piecewise polynomial space of degree \( \leq r \) defined on \( \pi_h(\Omega) \), then from [13] we know that (A1)–(A3) are valid for this \( V^h(\Omega) \).

The finite element approximation of (2.3) is given by: find \( \lambda_h \in C \) and \( u_h \in V^0_0(\Omega) \) with \( \|u_h\|_0 = 1 \) satisfying
\[
a(u_h, v) = \lambda_h b(u_h, v), \quad \forall v \in V^h_0(\Omega).
\] (2.8)

Thanks to [21], we know the adjoint problem of (2.1)–(2.2) is:
\[
L^*u^* = -\sum_{i,j=1}^d \partial_i(\bar{a}_{ij}\partial_j u^*) - \sum_{i=1}^d \partial_i(\bar{b}_i u^*) + \bar{c} u^* = \lambda^* m u^*, \quad \text{in } \Omega,
\]
\[
u^* = 0, \quad \text{on } \partial \Omega.
\] (2.9)

The corresponding variational form and discrete variational form of (2.9)–(2.10) are given by: find \( \lambda^* \in C \) and \( u^* \in H^1_0(\Omega) \) with \( \|u^*\|_0 = 1 \) satisfying
\[
a(v, u^*) = \overline{\lambda^*} b(v, u^*), \quad \forall v \in H^1_0(\Omega);
\] (2.11)
find \( \lambda^*_h \in C \) and \( u^*_h \in V^0_0 \) with \( \|u^*_h\|_0 = 1 \) satisfying
\[
a(v, u^*_h) = \overline{\lambda^*_h} b(v, u^*_h), \quad \forall v \in V^0_0(\Omega).
\] (2.12)

Note that the primal and dual eigenvalues are connected via \( \lambda = \overline{\lambda^*} \) and \( \lambda_h = \overline{\lambda^*_h} \).

Since (2.5)–(2.7) are valid, from Lax–Milgram theorem we know both source problems associated with (2.3) and (2.11) admit a unique solution; and both discrete source problems associated with (2.8) and (2.12) admit a unique solution.

The following operators are used in the proofs later. We define the solution operators \( T : L^2(\Omega) \to H^1_0(\Omega) \) and \( T_h : L^2(\Omega) \to V^0(\Omega) \) as follows:
\[
a(Tg, v) = b(g, v), \quad \forall v \in H^1_0(\Omega),
\] (2.13)
\[
a(T_hg, v) = b(g, v), \quad \forall v \in V^h_0(\Omega).
\] (2.14)

So (2.3) and (2.8) have the equivalent operator forms as below.
\[
Tu = \lambda^{-1}u,
\]
\[
T_hu_h = \lambda_h^{-1}u_h.
\] (2.15)

Define the solution operators \( T^* : L^2(\Omega) \to H^1_0(\Omega) \) and \( T^*_h : L^2(\Omega) \to V^0_0(\Omega) \) satisfying
\[
a(v, T^*f) = b(v, f), \quad \forall v \in H^1_0(\Omega),
\] (2.17)
\[
a(v, T^*_h f) = b(v, f), \quad \forall v \in V^0_0(\Omega).
\] (2.18)

Similarly (2.11) and (2.12) have the equivalent operator forms as below.
\[
T^*u^* = \lambda^*{-1}u^*,
\]
\[
T^*_h u^*_h = \lambda^*_h{-1}u^*_h.
\] (2.19)

It can be proved that \( T \) is completely continuous, and \( T^* \) and \( T^*_h \) are the adjoint operator of \( T \) and \( T_h \) in the sense of the inner product \( b(\cdot, \cdot) \), respectively. In fact,
\[
b(Tu, v) = a(Tu, T^*v) = b(u, T^*v), \quad \forall u, v \in L^2(\Omega),
\]
\[
b(T_hu, v) = a(T_hu, T^*_h v) = b(u, T^*_h v), \quad \forall u, v \in L^2(\Omega).
\]
We need the following regularity assumption. For any \( f \in L^2(\Omega), \) \( Tf \in H^1_0(\Omega) \cap H^{1+\gamma_1}(\Omega) \) and \( T^*f \in H^1_0(\Omega) \cap H^{1+\gamma_2}(\Omega) \) satisfy
\[
\|Tf\|_{1+\gamma_1} \leq C_\beta \|f\|_0, \tag{2.21}
\]
\[
\|T^*f\|_{1+\gamma_2} \leq C_\beta \|f\|_0. \tag{2.22}
\]
According to [25] and the Section 5.5 in [22], the above assumption is reasonable.
Assume \( G \subset \Omega, \) the following local regularity assumption will play a critical role in Theorems 4.2–4.4.
\( R(G). \) For any \( f \in L^2(G), \) there exists a \( \phi \in H^1_0(G) \cap H^{1+\gamma_1}(G) \) satisfying
\[
a(\phi, v) = b(f, v), \quad \forall v \in H^1_0(G),
\]
and
\[
\|\phi\|_{1+\gamma_1, G} \leq C_\gamma \|f\|_{-1+\gamma_1, G}. \tag{2.23}
\]
For any \( g \in L^2(G), \) there exists a \( \varphi \in H^1_0(G) \cap H^{1+\gamma_2}(G) \) satisfying
\[
a(v, \varphi) = b(v, g), \quad \forall v \in H^1_0(G),
\]
and
\[
\|\varphi\|_{1+\gamma_2, G} \leq C_\gamma \|g\|_{-1+\gamma_2, G}. \tag{2.24}
\]
where \( C_\beta, C_\gamma \) are two priori constants, not necessarily the same.

We also need the Ritz projections \( P_h : H^1_0(\Omega) \to V^h_0(\Omega) \) and \( P^*_h : H^1_0(\Omega) \to V^h_0(\Omega) \) defined by
\[
a(u - P_h u, v) = 0, \quad \text{and} \quad a(v, u - P^*_h u) = 0, \quad \forall v \in V^h_0(\Omega). \tag{2.25}
\]
Then \( T_h = P_h T \) and \( T^*_h = P^*_h T^* \) (see [21]). Our error analysis later is based on the error between the approximate eigenfunction and the Ritz projection of the eigenfunction.

Let \( M(\lambda) \) be the space spanned by all generalized eigenfunctions corresponding to \( \lambda \) of \( T, \) \( M_h(\lambda) \) be the space spanned by all generalized eigenfunctions corresponding to all eigenvalues of \( T_h \) that converge to \( \lambda. \) In view of the adjoint problems (2.11) and (2.12), the definitions of \( M^*(\lambda^*) \) and \( M^*_h(\lambda^*) \) are analogous to \( M(\lambda) \) and \( M_h(\lambda). \)

In this paper, we suppose that \( \lambda \) is an eigenvalue of (2.3) with the algebraic multiplicity \( q \) and the ascent \( \alpha = 1, \) then \( \lambda^* = \overline{\lambda} \) is the eigenvalue of (2.11),

Let \( \lambda_h, \lambda_h^* \) be the eigenvalue of (2.8) that converges to \( \lambda \) and \( \lambda_h^* = \overline{\lambda_h}. \)

To build the relationship between the Raleigh quotient and eigenvalues, we also need the following lemma (see [9]):

**Lemma 2.1.** Let \( (\lambda, u) \) be an eigenpair of (2.3), and \( (\lambda^* = \overline{\lambda}, u^*) \) be the associated eigenpair of the adjoint problem (2.11). Then for all \( w, w^* \in H^1_0(\Omega), \) \( b(w, w^*) \neq 0, \)
\[
\frac{a(w, w^*)}{b(w, w^*)} - \lambda = \frac{a(w - u, w^* - u^*)}{b(w, w^*)} - \lambda \frac{b(w - u, w^* - u^*)}{b(w, w^*)}. \tag{2.26}
\]

The following a priori error estimates of the finite element approximations (2.8) and (2.12) can be found in [21,1].

**Lemma 2.2.** Assume (A1) holds with \( t = r + s - 1, M(\lambda) \subset H^{t+s}(\Omega), M^*(\lambda^*) \subset H^{t+s}(\Omega) (0 < s, s_2 < 1). \) Then
\[
|\lambda_h - \lambda| \leq h^{t+s-1+r+s_2-1}, \tag{2.27}
\]
let \( u_h \in M_h(\lambda) \) with \( \|u_h\| = 1, \) then there is \( u \in M(\lambda) \) such that
\[
\|u_h - u\|_1 \leq h^{t+s-1}, \tag{2.28}
\]
\[
\|u_h - u\|_0 \leq h^{t+s-1+r+s_2}; \tag{2.29}
\]
let \( u^*_h \in M^*_h(\lambda^*) \) with \( \|u^*_h\| = 1, \) then there is \( u^* \in M^*(\lambda^*) \) such that
\[
\|u^*_h - u^*\|_1 \leq h^{t+s_2-1}; \tag{2.30}
\]
\[
\|u^*_h - u^*\|_0 \leq h^{t+s_2-1+r+s_1}. \tag{2.31}
\]

**Proof.** See [21]. □

The local a priori error estimate is a fundamental component in the mathematical theory of finite element methods and also plays an important role in adaptive and local parallel algorithms. [20,13], etc. studied the local behavior of finite elements. The following Theorem 2.1 is a simple generalization of Lemma 3.2 in [13]. We can easily prove this theorem by the same argument as Lemma 3.2 in [13].
Theorem 2.1. Suppose that (A0), (A2) and (A3) are valid, \( f \in H^{-1}(\Omega) \) and \( D \subset \subset \Omega_0 \subset \Omega \). If \( w \in V_h^0(\Omega_0) \) and \( w|_{\partial \Omega \cap \partial \Omega_0} = 0 \) satisfy
\[
a(w, v) = f(v), \quad \forall v \in V_h^0(\Omega_0),
\]
then
\[
\|w\|_{1, \Omega} \leq \|w\|_{0, \Omega_0} + \|f\|_{-1, \Omega},
\]
where
\[
\|f\|_{-1, \Omega} = \sup_{\phi \in H_h^1(\Omega_0), \|\phi\|_{1, \Omega_0} = 1} f(\phi).
\]

Proof. Let \( p \geq 2v - 1 \) be an integer, and
\[
D \subset \subset \Omega_0 \subset \subset \partial \Omega_1 \subset \subset \ldots \subset \subset \Omega_p \subset \subset \Omega.
\]
Choose \( D_1 \subset \Omega \) satisfying \( D \subset \subset D_1 \subset \subset \Omega_p \) and \( \omega \in C^\infty(\tilde{\Omega}) \) such that \( \text{supp} \omega \subset \subset \Omega_p \) and \( \omega \equiv 1 \) on \( D_1 \). Then, from (A3) (with \( G = \Omega_p \)), there exists \( v \in V_h^0(\Omega_p) \) such that
\[
\|\omega^2 w - v\|_{1, \Omega} \leq h_{\Omega_p} \|w\|_{1, \Omega_p},
\]
so we have
\[
|a(w, \omega^2 w - v)| \leq h_{\Omega_p} \|w\|^2_{1, \Omega_p}
\]
and
\[
|f(v)| \leq \|f\|_{-1, \Omega} \|v\|_{1, \Omega_p} \leq \|f\|_{-1, \Omega}(h_{\Omega_p} \|w\|_{1, \Omega_p} + \|\omega w\|_{1, \Omega}).
\]
Since \( v \in V_h^0(\Omega_p) \subset V_h^0(\Omega_0) \), (2.32) implies
\[
a(w, \omega^2 w) = a(w, \omega^2 w - v) + f(v).
\]
Let \( a_0(u, v) = f_1 \sum_{i,j=1}^d a_{ij} \partial_i u \partial_j v \). It can be derived from the proof of Lemma 3.1 in [13] that if \( \Omega_0 \subset \subset \Omega \subset \subset \mathbb{R}^d (d \geq 2) \), \( \omega \in C^\infty(\tilde{\Omega}) \) and \( \text{supp} \omega \subset \subset \Omega_0 \), then
\[
a_0(\omega w, \omega w) \leq |a(w, \omega^2 w) + \|w\|^2_{0, \Omega_0}, \quad \forall w \in H^1_h(\Omega).
\]
It follows from (2.34)-(2.37) that
\[
\|\omega w\|^2_{1, \Omega} \leq a_0(\omega w, \omega w) \leq |a(w, \omega^2 w)| + \|w\|^2_{0, \Omega_0}
\leq |a(w, \omega^2 w - v)| + \|w\|^2_{0, \Omega_0} + |f(v)|
\leq h_{\Omega_p} \|w\|^2_{1, \Omega_p} + \|w\|^2_{0, \Omega_0} + \|f\|_{-1, \Omega}(h_{\Omega_p} \|w\|_{1, \Omega_p} + \|\omega w\|_{1, \Omega}).
\]
thus
\[
\|w\|_{1, \Omega} \leq h_{\Omega_p} \|w\|_{1, \Omega_p} + \|w\|_{0, \Omega_0} + \|f\|_{-1, \Omega_0}.
\]
Similarly, we can get
\[
\|w\|_{1, \Omega_j} \leq h_{\Omega_p} \|w\|_{1, \Omega_{j-1}} + \|w\|_{0, \Omega_0} + \|f\|_{-1, \Omega_0}, \quad j = 1, 2, \ldots, p.
\]
Using (2.38) and (2.39), we get from (A2) and (A0) that
\[
\|w\|_{1, \Omega} \leq h_{\Omega_p}^{p+1/2} \|w\|_{1, \Omega_0} + \|w\|_{0, \Omega_0} + \|f\|_{-1, \Omega_0}
\leq h_{\Omega_p}^{p} h^{-1} \|w\|_{0, \Omega_0} + \|w\|_{0, \Omega_0} + \|f\|_{-1, \Omega_0}
\leq \|w\|_{0, \Omega_0} + \|f\|_{-1, \Omega_0}.
\]
This completes the proof. \( \square \)
Lemma 2.3. Suppose that (A0)–(A3) are valid, and \( D \subseteq \Omega \subseteq \Omega_0 \subseteq \Omega \). Then the following estimates hold:

\[
\begin{align*}
&h^{1/2} \| u - P_h u \|_{1, \Omega} + \| u - P_h u \|_{0, \Omega} \lesssim h^{1/2} \inf_{v \in V_h^0(\Omega)} \| u - v \|_{1, \Omega}, \\
&\| u - P_h u \|_{1, \Omega} \lesssim \inf_{v \in V_h^0(\Omega)} \| u - v \|_{1, \Omega_0} + h^{2/3} \| u - P_h u \|_{1, \Omega}, \\
&\| u_h - u \|_{1, D} = \| u - P_h u \|_{1, D} + \mathcal{O}(\| \lambda_h u_h - \lambda u \|_0).
\end{align*}
\]

(2.40)

(2.41)

(2.42)

Proof. For the proof of (2.40) cf. [22,23]. By virtue of Theorem 2.1, using the proof method of Theorem 3.4 in [13] we can derive (2.41). Referring to Theorem 3.1 in [7] we can deduce the local error estimate (2.42) of the eigenfunction \( u_h \). 

Remark 2.1. In [13], the condition Superapproximation is given as follows.

A.3. Superapproximation. For \( G \subseteq \Omega_0 \), let \( \omega \in C^\infty_0(\Omega) \) with \( \text{supp} \ \omega \subseteq G \). Then for any \( w \in V^h(G) \), there exists \( v \in V_h^0(G) \) such that

\[
\| h^{-1}(\omega w - v) \|_{1,G} \lesssim \| w \|_{1,G}.
\]

In the proof of Lemma 3.2 in [13], the authors choose \( D_1 \subseteq \Omega \) satisfying \( D \subseteq D_1 \subseteq D_2 \subseteq \Omega_0 \) and \( \omega \in C^\infty_0(\Omega) \) such that \( \omega \equiv 1 \) on \( D_1 \) and \( \text{supp} \ \omega \subseteq \Omega_0 \).

Note that the above \( D \) cannot contain the corner points on \( \partial \Omega \). This paper just makes a minor modification, so that the theory of the local error estimates built in [13] applies to the local domains containing the corner points, see Theorem 2.1 and Lemma 2.3. In addition, strictly speaking, according to the above discussions, Condition (A3) in [18] should be modified into the one in this paper.

3. Multilevel discretizations based on local defect correction

Consider the eigenvalue problem (2.3) which has an isolated singular point \( z \in \mathcal{I} \) (e.g., see Fig. 3.1). Let \( \Omega \supseteq \Omega_1 \supseteq \Omega_2 \supseteq \cdots \supseteq \Omega_l \supseteq \cdots \supseteq \Delta \).

Let \( \pi_H(\Omega) \) be a shape-regular grid made up of simplices with size \( H \in (0, 1) \), \( \pi_w(\Omega) \) be a refined mesoscopic shape-regular grid from \( \pi_H(\Omega) \) and \( \pi_h(\Omega_\ell) \) be a locally refined grid from \( \pi_{h_{\ell-1}}(\Omega_\ell-1) \) such that \( h_{\ell-1} = H, h_0 = w, h_i \ll h_{i-1} (i = 0, 1, \ldots, l) \). (See Fig. 3.1 for \( \pi_H(\Omega) \), \( \pi_w(\Omega) \) and \( \pi_h(\Omega_\ell) \) ) Let \( V_H(\Omega), V_w(\Omega) \), and \( \{ V_h^0(\Omega_\ell) \}_1^l \) be finite element spaces of degree less than or equal to \( r \) defined on \( \pi_H(\Omega), \pi_w(\Omega) \) and \( \{ \pi_h(\Omega_\ell) \}_1^l \), respectively, with functions vanishing on \( \partial \Omega \) and continuous in \( \Omega \). Such boundary conditions and continuity conditions guarantee the continuity on \( \mathcal{I} \) of the resulted solutions \( u^{w,h_1}, u^{w,h_1^*}, u^{w,h_i}, u^{w,h_i^*} \) in Schemes 3.1 and 3.2.

Based on Algorithm 3.2 in [17] we establish the following three-level discretization scheme.

Scheme 3.1. Three-level discretizations based on local defect correction.

**Step 1.** Solve (2.3) on a globally coarse grid \( \pi_H(\Omega) \): find \( \lambda_H \in \mathbb{C} \) and \( u_H \in V_H^0(\Omega) \) such that \( \| u_H \|_0 = 1 \) and

\[
a(u_H, v) = \lambda_H b(u_H, v), \quad \forall v \in V_H^0(\Omega).
\]

Let \( \lambda_H = \bar{\lambda}_H \), and find \( u_H^* \in M_H(\lambda^*) \) with \( \| u_H^* \|_0 = 1 \) such that \( |b(u_H^*, u_H^*)| \) has a positive lower bound uniformly with respect to \( H \) (see Remark 3.1).

**Step 2.** Solve two linear boundary value problems on a globally mesoscopic grid \( \pi_w(\Omega) \): find \( u^w \in V_w^0(\Omega) \) such that

\[
a(u^w, v) = \lambda_H b(u^w, v), \quad \forall v \in V_w^0(\Omega);
\]

Fig. 3.1. Coarse grid (left), mesoscopic grid (middle) and locally refined grid (right).
find \( u^{w^*} \in V_0^w(\Omega) \) such that
\[
a(v, u^{w^*}) = \lambda_H b(v, u^{w^*}), \quad \forall v \in V_0^w(\Omega).
\]
Then compute the Rayleigh quotient \( \lambda^w = \frac{a(u^{w^*}, u^{w^*})}{b(u^{w^*}, u^{w^*})} \).

**Step 3.** Solve two linear boundary value problems on a locally fine grid \( \pi_{h_1}(\Omega_1) \): find \( e^{h_1} \in V_0^{h_1}(\Omega_1) \) such that
\[
a(e^{h_1}, v) = \lambda^w b(u^{w^*}, v) - a(u^{w^*}, v), \quad \forall v \in V_0^{h_1}(\Omega_1);
\]
find \( e^{h_1*} \in V_0^{h_1}(\Omega_1) \) such that
\[
a(v, e^{h_1*}) = \lambda^w b(v, u^{w^*}) - a(v, u^{w^*}), \quad \forall v \in V_0^{h_1}(\Omega_1).
\]

**Step 4.** Set
\[
u^{w, h} = \begin{cases} u^{w^*} + e^{h_1} & \text{on } \overline{\Omega}_1, \\ u^{w^*} & \text{in } \Omega \setminus \overline{\Omega}_1, \end{cases}
\]
\[
u^{w, h*} = \begin{cases} u^{w^*} + e^{h_1*} & \text{on } \overline{\Omega}_1, \\ u^{w^*} & \text{in } \Omega \setminus \overline{\Omega}_1, \end{cases}
\]
and compute the Rayleigh quotient
\[
\lambda^{w, h} = \frac{a(u^{w, h_1}, u^{w, h_1*})}{b(u^{w, h_1}, u^{w, h_1*})}, \quad \lambda^{w, h*} = \frac{\lambda^{w, h_1}}{\lambda^{w, h_1}}.
\]

We use \((\lambda^{w, h_1}, u^{w, h_1})\) and \((\lambda^{w, h_1*}, u^{w, h_1*})\) from Scheme 3.1 as the approximate eigenpairs of (2.3) and (2.11), respectively. It is obvious that \((\lambda^w, u^{w^*})\) and \((\lambda^{w^*}, u^{w^*})\) in Scheme 3.1 can be viewed as approximate eigenpairs obtained by the two-grid discretization scheme in [9] from \(\pi_{h_1}(\Omega)\) and \(\pi_w(\Omega)\).

Note that in Scheme 3.1 abrupt changes of the mesh size can appear near \(\partial \Omega_1\). Influenced by the technique on the transition layer proposed by [18], we repeatedly use the local defect correction technique to establish the following multilevel discretization scheme.

**Scheme 3.2.** Multilevel discretizations based on local defect correction.

**Step 1.** The same as that of Step 1 of Scheme 3.1.

**Step 2.** The same as that of Step 2 of Scheme 3.1.

**Step 3.** \( u^{w, h_0} = \lambda^{w, h_0} u^{w^*}, \quad u^{w, h_0*} = \lambda^{w^*} u^{w^*} \) \( \lambda^{w, h_0} \leq \lambda^w, \lambda^{w, h_0*} \leq \lambda^{w^*}. \)

**Step 4.** For \( i = 1, 2, \ldots, l \), execute Step 5 and Step 6.

**Step 5.** Solve linear boundary value problems on locally fine grid \( \pi_{h_i}(\Omega_i) \): find \( e^{h_i} \in V_0^{h_i}(\Omega_i) \) such that
\[
a(e^{h_i}, v) = \lambda^{w, h_i-1} b(u^{w, h_i-1}, v) - a(u^{w, h_i-1}, v), \quad \forall v \in V_0^{h_i}(\Omega_i);
\]
find \( e^{h_i*} \in V_0^{h_i}(\Omega_i) \) such that
\[
a(v, e^{h_i*}) = \lambda^{w, h_i-1} b(v, u^{w, h_i-1*}) - a(v, u^{w, h_i-1*}), \quad \forall v \in V_0^{h_i}(\Omega_i).
\]

**Step 6.** Set
\[
u^{w, h_i} = \begin{cases} u^{w, h_i-1} + e^{h_i} & \text{on } \overline{\Omega}_i, \\ u^{w, h_i-1} & \text{in } \Omega \setminus \overline{\Omega}_i, \end{cases}
\]
\[
u^{w, h_i*} = \begin{cases} u^{w, h_i-1*} + e^{h_i*} & \text{on } \overline{\Omega}_i, \\ u^{w, h_i-1*} & \text{in } \Omega \setminus \overline{\Omega}_i, \end{cases}
\]
and compute
\[
\lambda^{w, h_i} = \frac{a(u^{w, h_i}, u^{w, h_i*})}{b(u^{w, h_i}, u^{w, h_i*})}, \quad \lambda^{w, h_i*} = \frac{\lambda^{w, h_i}}{\lambda^{w^*}}.
\]

We use \((\lambda^{w, h_i}, u^{w, h_i})\) and \((\lambda^{w, h_i*}, u^{w, h_i*})\) from Scheme 3.2 as the approximate eigenpairs of (2.3) and (2.11), respectively.

**Remark 3.1.** We can actually prove the following conclusion:

Let \( u^{h_i}_1 \) be the orthogonal projection of \( u_i \) to \( M_{h_i}^*(\lambda^*) \) in the sense of the inner product \( b(\cdot, \cdot) \), and \( u^{h_i}_1 = \frac{u^{h_i}_1}{\|u^{h_i}_1\|_b} \), then when \( H \) is small enough \( \|b(u_H, u^{h_i}_1)\| \) has a positive lower bound uniformly with respect to \( H \).

Therefore, \( u^{h_i}_1 \) in Step 1 of Schemes 3.1 and 3.2 can be obtained in the way as above.
4. Theoretical analysis

Next we shall discuss the error estimates of Schemes 3.1 and 3.2.

In our analysis, we introduce an auxiliary grid $\pi_b(\Omega)$ which is defined globally, and denote the auxiliary piecewise polynomial space of degree $\leq r$ by $V^h_\Omega(\Omega)$ $(i = 1, 2, \ldots, l)$. We also assume that $\pi_b(\Omega)$ and $V^h_0(\Omega)$ are the restrictions of $\pi_b(\Omega)$ and $V^h_0(\Omega)$ to $\Omega_i$, respectively, and

$$V^h_0(\Omega) \subset V^h(\Omega) \subset V^h_0(\Omega) \subset \cdots \subset V^h_0(\Omega).$$

For $D$ and $\Omega_i$ stated at the beginning of Section 3, let $G_i \subset \Omega$ and $F \subset \Omega$ satisfy $D \subset F \subset G_i \subset \Omega_i$ $(i = 1, 2, \ldots, l)$.

**Theorem 4.1.** Assume that (A0)–(A3) hold, $M(\lambda) \subset H^1(\Omega) \cap H^{r+s}(\Omega) \cap H^{r+1}(\Omega \setminus \bar{D})$ and $(1 < r + s, 0 \leq s < 1)$, $M^*(\lambda^*) \subset H^1(\Omega) \cap H^{r+s}(\Omega) \cap H^{r+1}(\Omega \setminus \bar{D})$ and $(1 < r + s, 0 \leq s < 1)$, and $H$ is properly small. Then there exist $u \in M(\lambda)$ and $u^* \in M^*(\lambda^*)$ such that

$$\|u^w - u\|_1 \leq H^{r+s-1+\gamma_2} + u^{r+s-1}, \quad \|u^w - u\|_0 \leq H^{r+s-1+\gamma_2}, \quad \|u^w - u\|_{1,\Omega \setminus F} \leq H^{r+s-1+\gamma_2} + u^r,$$

$$\|u^{w*} - u^*\|_1 \leq H^{r+s-1+\gamma_1} + u^{r+s-1}, \quad \|u^{w*} - u^*\|_0 \leq H^{r+s-1+\gamma_1}, \quad \|u^{w*} - u^*\|_{1,\Omega \setminus F} \leq H^{r+s-1+\gamma_1} + u^r,$$

$$|\lambda - \lambda| \leq H^{2r+s+s_2+2\gamma_1+\gamma_2} + u^{2r+s+s_2-2}.$$

**Proof.** Let $u \in M(\lambda)$ and $u^* \in M^*(\lambda^*)$ such that $u - u_H$ and $u^* - u^*_H$ both satisfy Lemma 2.2. From (2.13), (2.14), Step 2 of Scheme 3.1, (2.15), Lemmas 2.2 and 2.3, we derive that

$$\|u^w - u\|_1 = \|\lambda T_w u - \lambda Tu\|_1 \leq \|\lambda T_w u - \lambda Tu\|_1 + \|\lambda T_w u - \lambda Tu\|_1 \leq H^{r+s-1+\gamma_2} + u^{r+s-1},$$

then (4.1) follows. By Lemmas 2.2 and 2.3,

$$\|u^w - u\|_{1,\Omega \setminus F} \leq \|\lambda T_w u - \lambda Tu\|_1 + \|P_w T u - Tu\|_{1,\Omega \setminus F} \leq H^{r+s-1+\gamma_2} + u^r,$$

then (4.3) follows. By calculation,

$$\|u^w - u\|_0 \leq \|\lambda T_w u - \lambda Tu\|_0 \leq \|\lambda T_w u - \lambda Tu\|_0 + \|\lambda T_w u - \lambda Tu\|_0 \leq H^{r+s-1+\gamma_2} + u^{r+s-1+\gamma_2},$$

then (4.2) follows.

Similarly we can prove (4.4)–(4.6). From (2.26), we have

$$\lambda - \lambda = \frac{a(u^w - u, u^{w*} - u^*)}{b(u^w, u^{w*})} - \lambda \frac{b(u^w - u, u^{w*} - u^*)}{b(u^w, u^{w*})}.$$

Note that $u_H$ and $u^w$ just approximate the same eigenfunction $u$, $u^*_H$ and $u^{w*}$ approximate the same adjoint eigenfunction $u^*$, and $|b(u_H, u^*_H)|$ has a positive lower bound uniformly with respect to $H$, therefore $b(u^w, u^{w*})$ has a positive lower bound uniformly. Combining (4.1), (4.2), (4.4), (4.5) and (4.8) yields (4.7). \(\square\)

The following Theorem 4.2 is developed from the results of Theorem 3.3 in [17], which characterizes the relationship between the corrected solutions and the Ritz projection on the auxiliary space $V^h_0(\Omega)$. 

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Theorem 4.2. Assume that (A0)–(A3) and \( R(\Omega_i) \) hold (\( i = 1, 2, \ldots, l \)), \( u \in M(\lambda) \) and \( u^* \in M^*(\lambda^*) \). Then

\[
\| u^{w,h_i} - P_h u \|_{1,\Omega} \leq \| u - P_h u \|_{0,\Omega} + h^2_i\| P_h u - u^{w,h_i} \|_{1,\Omega} + \| (\lambda u - \lambda^{w,h_i-1} u^{w,h_i-2}) \|_{0,\Omega} + |\lambda^{w,h_i-1} - \lambda| \| u^{w,h_i-1} - u \|_0 + \| u^{w,h_i-1} - P_h u \|_{1,\Omega} \leq \| u^{w,h_i*} - P_h^* u \|_{1,\Omega} \leq \| u^* - P_h^* u \|_{0,\Omega} + h^2_i\| P_h^* u - u^{w,h_i-1*} \|_{1,\Omega} + \| (\lambda u^* - \lambda^{w,h_i-1*} u^{w,h_i-2*}) \|_{0,\Omega} + |\lambda^{w,h_i-1*} - \lambda^*| \| u^{w,h_i-1*} - u^* \|_0 + \| u^{w,h_i-1*} - P_h^* u \|_{1,\Omega} \|_{1,\Omega} \}, \quad l \geq 1
\]

(4.9)

(4.10)

where \( P_h \) and \( P_h^* \) are both Ritz projections from \( H^1_0(\Omega) \) to \( V_h^0(\Omega) \).

Proof. Due to the inequality

\[
\| u^{w,h_i} - P_h u \|_{1,\Omega} \leq \| u^{w,h_i} - P_h u \|_{1,\Omega} + \| u^{w,h_i} - P_h u \|_{1,\Omega} \leq \| u^{w,h_i} - P_h u \|_{1,\Omega} \leq \| u^{w,h_i} - P_h u \|_{1,\Omega} \]

(4.11)

we shall estimate \( \| u^{w,h_i} - P_h u \|_{1,\Omega} , \| u^{w,h_i} - P_h u \|_{1,\Omega} \) and \( \| u^{w,h_i} - P_h u \|_{1,\Omega} \), respectively.

First, we proceed to estimate \( \| u^{w,h_i} - P_h u \|_{1,\Omega} \). From (4.8), (4.6) and (2.25) we derive

\[
a(u^{w,h_i} - P_h u, v) = a(u^{w,h_i}, v) - a(P_h u, v) = a(u^{w,h_i} + e^h, v) - a(u, v)
\]

(4.12)

It is obvious that

\[
\lambda^{w,h_i-1} b(u^{w,h_i-1}, v) - \lambda b(u, v) = (\lambda^{w,h_i-1} - \lambda) b(u, v) + \lambda^{w,h_i-1} b(u^{w,h_i-1} - u, v) \quad \forall v \in V_h^0(\Omega)
\]

(4.13)

which together with (4.12) yields for all \( v \in V_h^0(\Omega) \)

\[
a(u^{w,h_i} - P_h u, v) = (\lambda^{w,h_i-1} - \lambda) b(u, v) + \lambda^{w,h_i-1} b(u^{w,h_i-1} - u, v).
\]

Since \( V_h^0(\Omega) \subset V_h^0(\Omega) \), thus, from the above formula and Theorem 2.1 we deduce that

\[
\| u^{w,h_i} - P_h u \|_{1,\Omega} \leq \| u^{w,h_i} - P_h u \|_{0,\Omega} + \| \lambda^{w,h_i-1} - \lambda \| + \| u^{w,h_i-1} - u \|_{0,\Omega}.
\]

(4.14)

By calculation, we have

\[
\| u^{w,h_i} - P_h u \|_{0,\Omega} \leq \| u^{w,h_i-1} - P_h u \|_{0,\Omega} + \| e^h \|_{0,\Omega} \leq \| u - P_h u \|_{0,\Omega} + \| u - u^{w,h_i-1} \|_{0,\Omega} + \| e^h \|_{0,\Omega},
\]

substituting the above relation into (4.14) we obtain

\[
\| u^{w,h_i} - P_h u \|_{1,\Omega} \leq \| \lambda^{w,h_i-1} - \lambda \| + \| u^{w,h_i-1} - u \|_{0,\Omega} + \| u - P_h u \|_{0,\Omega} + \| e^h \|_{0,\Omega}.
\]

(4.15)

To estimate \( \| e^h \|_{0,\Omega} \), we use the Aubin–Nitsche duality argument. For any given \( f \in L_2(\Omega_i) \), consider the boundary value problem: find \( \psi \in H_0^1(\Omega) \) such that

\[
a(v, \psi) = b(v, f) \quad \forall v \in H_0^1(\Omega).
\]

(4.16)

Let \( \psi \) be the generalized solution of (4.16), \( \varphi_h \) and \( \varphi_{h-1} \) be finite element solutions of (4.16) in \( V_h^0(\Omega) \) and \( V_{h-1}^0(\Omega) \), respectively. Then, from Céa’s lemma, (A1) and \( R(\Omega) \) we have

\[
\| \psi - \varphi_h \|_{1,\Omega} \leq h^2 \| f \|_{0,\Omega}, \quad \| \psi - \varphi_{h-1} \|_{1,\Omega} \leq h^{2} \| f \|_{0,\Omega}.
\]

(4.17)

From (3.6) and (3.8) we get

\[
a(u^{w,h_i}, \varphi_h) = \lambda^{w,h_i-1} b(u^{w,h_i-1}, \varphi_h),
\]

thus by the definitions of \( \varphi, \varphi_h \) and \( e^h \), we deduce that

\[
|b(e^h, f)| = |a(e^h, \varphi)| = |a(e^h, \varphi_h)| = |a(u^{w,h_i} - u^{w,h_i-1}, \varphi_h)| = |a(P_h u - u^{w,h_i-1}, \varphi_h) + a(u^{w,h_i}, \varphi_h) - a(P_h u, \varphi_h)| = |a(P_h u - u^{w,h_i-1}, \varphi_h) + \lambda^{w,h_i-1} b(u^{w,h_i-1}, \varphi_h) - \lambda b(u, \varphi_h)| = |a(P_h u - u^{w,h_i-1}, \varphi - \varphi) + a(P_h u - u^{w,h_i-1}, \varphi - \varphi_{h-1})|
\]
\[ \begin{align*}
&+ a(P_{h_1} u - u^{w,h_{1-1}}, \varphi_{h_{1-1}}) + \lambda^{w,h_{1-1}} b(u^{w,h_{1-1}}, \varphi_{h_{1}}) - \lambda b(u, \varphi_{h_{1}}) \\
&\lesssim h_{1-1}^2 \|P_{h_1} u - u^{w,h_{1-1}}\|_{1,\Omega} \|f\|_{0,\Omega} + \|a(P_{h_1} u - u^{w,h_{1-1}}, \varphi_{h_{1-1}})\|_{\lambda} + |\lambda^{w,h_{1-1}} b(u^{w,h_{1-1}}, \varphi_{h_{1}}) - \lambda b(u, \varphi_{h_{1}})|.
\end{align*} \]

(4.18)

Step 2 of Scheme 3.2 shows that

\[ a(u^{w,h_0}, \varphi_{h_0}) = \lambda^{w,h_0} b(u^{w,h_0}, \varphi_{h_0}), \]

namely, for \( I = 1, \)

\[ a(u^{w,h_{1-1}}, \varphi_{h_{1-1}}) = \lambda^{w,h_{1-1}} b(u^{w,h_{1-1}}, \varphi_{h_{1-1}}), \]

for \( I > 1, \) the above formula follows from (3.6) and (3.8). Therefore,

\[ |a(P_{h_1} u - u^{w,h_{1-1}}, \varphi_{h_{1-1}})| = |a(u - u^{w,h_{1-1}}, \varphi_{h_{1-1}})| \\
= |\lambda b(u, \varphi_{h_{1-1}}) - a(u^{w,h_{1-1}}, \varphi_{h_{1-1}})| \\
= |\lambda b(u, \varphi_{h_{1-1}}) - \lambda^{w,h_{1-2}} b(u^{w,h_{1-2}}, \varphi_{h_{1-2}})| \\
\lesssim \|\lambda u - \lambda^{w,h_{1-2}} u^{w,h_{1-2}}\|_{0,\Omega} \|f\|_{0,\Omega}. \]

It is clear that

\[ |\lambda^{w,h_{1-1}} b(u^{w,h_{1-1}}, \varphi_{h_{1}}) - \lambda b(u, \varphi_{h_{1}})| \lesssim \|\lambda^{w,h_{1-1}} u^{w,h_{1-1}} - \lambda u\|_{0,\Omega} \|f\|_{0,\Omega}. \]

Substituting the above two formulae into (4.18), we derive

\[ |b(e^h, f)| \lesssim (h_{1-1}^2 \|P_{h_1} u - u^{w,h_{1-1}}\|_{1,\Omega} + \|\lambda u - \lambda^{w,h_{1-2}} u^{w,h_{1-2}}\|_{0,\Omega} + \|\lambda^{w,h_{1-1}} u^{w,h_{1-1}} - \lambda u\|_{0,\Omega}) \|f\|_{0,\Omega}. \]

Thus, we get

\[ \|e^h\|_{0,\Omega} \lesssim h_{1-1}^2 \|P_{h_1} u - u^{w,h_{1-1}}\|_{1,\Omega} + \|\lambda u - \lambda^{w,h_{1-2}} u^{w,h_{1-2}}\|_{0,\Omega} + \|\lambda^{w,h_{1-1}} u^{w,h_{1-1}} - \lambda u\|_{0,\Omega}. \]

(4.19)

Substituting (4.19) into (4.15), we obtain

\[ \|u^{w,h_0} - P_{h_0} u\|_{1,D} \lesssim \|u - P_{h_0} u\|_{0,\Omega} + h_{1-1}^2 \|P_{h_1} u - u^{w,h_{1-1}}\|_{1,\Omega} \\
+ \|\lambda u - \lambda^{w,h_{1-2}} u^{w,h_{1-2}}\|_{0,\Omega} + |\lambda^{w,h_{1-1}} - \lambda| + \|u^{w,h_{1-1}} - u\|_{0,\Omega}. \]

(4.20)

Similarly, since \((G \setminus \bar{D}) \subset \subset \Omega,\) we deduce

\[ \|u^{w,h_0} - P_{h_0} u\|_{1,G \setminus \Omega} \lesssim \|u - P_{h_0} u\|_{0,\Omega} + h_{1-1}^2 \|P_{h_1} u - u^{w,h_{1-1}}\|_{1,\Omega} \\
+ \|\lambda u - \lambda^{w,h_{1-2}} u^{w,h_{1-2}}\|_{0,\Omega} + |\lambda^{w,h_{1-1}} - \lambda| + \|u^{w,h_{1-1}} - u\|_{0,\Omega}. \]

(4.21)

The remainder is to analyze \( \|u^{w,h_0} - P_{h_0} u\|_{1,\Omega \setminus \bar{\Omega}}. \) From (3.8), we see that

\[ \|u^{w,h_0} - P_{h_0} u\|_{1,\Omega \setminus \bar{\Omega}} = \|u^{w,h_{1-1}} - P_{h_0} u\|_{1,\Omega \setminus \bar{\Omega}}, \]

which leads to

\[ \|u^{w,h_0} - P_{h_0} u\|_{1,\Omega \setminus \bar{\Omega}} \lesssim \|u^{w,h_{1-1}} - P_{h_0} u\|_{1,\Omega \setminus \bar{\Omega}} + \|u^{w,h_{1-1}} - P_{h_0} u\|_{0,\Omega \setminus \bar{\Omega}} + \|e^h\|_{1,\Omega \setminus \bar{\Omega}} \\
\lesssim \|u^{w,h_{1-1}} - P_{h_0} u\|_{1,\Omega \setminus \bar{\Omega}} + \|e^h\|_{1,\Omega \setminus \bar{\Omega}}. \]

(4.22)

It follows from (3.6), (2.3) and (4.13) that

\[ a(e^h, v) = \lambda^{w,h_{1-1}} b(u^{w,h_{1-1}}, v) - a(u^{w,h_{1-1}}, v) - \lambda b(u, v) + a(u, v) = (\lambda^{w,h_{1-1}} - \lambda) b(u, v) + \lambda^{w,h_{1-1}} b(u^{w,h_{1-1}} - u, v) - a(u^{w,h_{1-1}} - u, v), \quad \forall v \in V_0^h(\Omega), \]

then, thanks to Theorem 2.1, we have

\[ \|e^h\|_{1,\Omega \setminus \bar{\Omega}} \lesssim \|e^h\|_{0,\Omega \setminus \bar{\Omega}} + |\lambda^{w,h_{1-1}} - \lambda| + \|u^{w,h_{1-1}} - u\|_{1,\Omega \setminus \bar{\Omega}}, \]

(4.23)

where \( F \subset \subset \Omega \) satisfies \( D \subset \subset F \subset \subset G, \) Substituting (4.23) into (4.22) we get

\[ \|u^{w,h_0} - P_{h_0} u\|_{1,\Omega \setminus \bar{\Omega}} \lesssim \|u^{w,h_{1-1}} - P_{h_0} u\|_{1,\Omega \setminus \bar{\Omega}} + \|e^h\|_{0,\Omega \setminus \bar{\Omega}} + |\lambda^{w,h_{1-1}} - \lambda| + \|u^{w,h_{1-1}} - u\|_{1,\Omega \setminus \bar{\Omega}}. \]
It follows from substituting (4.19) into the above inequality that
\[
\|u^{w,h}_1 - P_{h_0} u\|_{1,\Omega;\Gamma} \lesssim \|u^{w,h_1}\|_{1,\Omega;\Gamma} + h_{\Gamma}^{-1}\|P_{h_0} u - u^{w,h_1}\|_{1,\Omega} + \|\lambda u - \lambda^{w,h_1} u^{w,h_1}\|_{0,\Omega} + \|u^{w,h_1} u^{w,h_1} - \lambda u\|_0 + |\lambda^{w,h_1} - \lambda| + \|u^{w,h_1} - u\|_{1,\Omega;\Gamma}. \tag{4.25}
\]
Combining (4.24), (4.20), (4.21) and (4.11), finally, we obtain (4.9).
We can prove (4.10) by using similar arguments. □

**Theorem 4.2** is a critical result in this paper, from which we can readily deduce the following two theorems in relation to the error estimates of the correction solution from Schemes 3.1 and 3.2.

**Theorem 4.3.** Assume that the conditions of Theorems 4.1 and 4.2 hold. Then there exist \( u \in M(\lambda) \) and \( u^* \in M^*(\lambda^*) \) such that
\[
\|u^{w,h}_1 - u\|_{1,\Omega,\Gamma} \lesssim h_1^{-1} + w^r + H^{r+s-1+\gamma_2}, \tag{4.26}
\]
\[
\|u_w, h_1 - u\|_{0,\Omega,\Gamma} \lesssim w^r + H^{r+s-1+\gamma_2}, \tag{4.27}
\]
\[
\|u^{w,h}_1 - u\|_{1,\Omega,\Gamma} \lesssim w^r + H^{r+s-1+\gamma_2}, \tag{4.28}
\]
\[
\|u^{w,h}_1 - u\|_{1,\Omega,\Gamma} \lesssim w^r + H^{r+s-1+\gamma_1}, \tag{4.29}
\]
\[
\|u^{w,h}_1 - u\|_{1,\Omega,\Gamma} \lesssim w^r + H^{r+s-1+\gamma_1}, \tag{4.30}
\]
\[
|\lambda^{w,h_1} - \lambda| \lesssim h_1^{r+s-1+\gamma_2} + w^{2r} + H^{2r+s+2} + \gamma_1 + \gamma_2. \tag{4.31}
\]

**Proof.** Let \( u \in M(\lambda) \) and \( u^* \in M^*(\lambda^*) \) such that both \( u - u_H \) and \( u^* - u^*_H \) satisfy Lemma 2.2. In Theorem 4.2, choose \( l = 1, h_{-1} = H, h_0 = w, u^{w,h}_0 = u, \lambda^{w,h}_0 = \lambda, \) then we get
\[
\|u^{w,h}_1 - P_{h_0} u\|_{1,\Omega,\Gamma} \lesssim \|u - P_{h_0} u\|_{0,\Omega,\Gamma} + w^{2r} \|P_{h_0} u - u^{w}\|_{1,\Omega,\Gamma} + \|\lambda u - \lambda u_H u_H\|_{0,\Omega,\Gamma} + |\lambda^{w,h}_1 - \lambda^{w,h}_1| + \|u^{w,h}_1 - u\|_{0,\Omega,\Gamma} + \|u^{w,h}_1 - u\|_{1,\Omega,\Gamma} \tag{4.32}
\]
Using Lemma 2.3, Theorem 4.1, Lemma 2.2 to estimate the terms at the right hand side of the above formula gives
\[
\|u^{w,h}_1 - P_{h_0} u\|_{1,\Omega,\Gamma} \lesssim h_1^{-1} + w^{2r} H^{r/s-1+1} + H^{r+s-1+2} + w^r H^{r+s-1+2} + w^{r+s-1+2} \notag
\]
\[
+ (w^{r+s-1+2} + w^r) + (w^{r+s-1+2} + w^r) \lesssim H^{r+s-1+2} + w^r. \tag{4.33}
\]
Combining (2.40) and (2.41) yields (4.25), (4.26) and (4.27). By the same argument we can prove (4.28), (4.29) and (4.30). From (2.26), we have
\[
\lambda^{w,h_1} - \lambda = \frac{a(u^{w,h_1} - u, u^{w,h_1} - u^*)}{b(u^{w,h_1}, u^{w,h_1} + u^*)} - \frac{b(u^{w,h_1} - u, u^{w,h_1} - u^*)}{b(u^{w,h_1}, u^{w,h_1} + u^*)}. \tag{4.34}
\]
Note that \( u_H \) and \( u^{w,h_1} \) just approximate the same eigenfunction \( u, u_H \) and \( u^{w,h_1} \) approximate the same adjoint eigenfunction \( u^*, |b(u_H, u_H^*)| \) has a positive lower bound uniformly with respect to \( H \), thus \( b(u^{w,h_1}, u^{w,h_1}) \) has a positive lower bound uniformly. Combining (4.25), (4.26), (4.28), (4.29) and (4.34) yields (4.31). □

For convenient argument, we assume \( s_2 = s, \gamma_1 = \gamma_2 = \gamma \) in the following theorem.

**Theorem 4.4.** Under the conditions of Theorems 4.1 and 4.2, we further assume that
\[
u^r = \Theta(H^{r+s-1+\gamma}), \quad h_1^{r+s-1} \gtrsim H^{r+s-1+\gamma}. \tag{4.35}
\]
Then there exist \( u \in M(\lambda) \) and \( u^* \in M^*(\lambda^*) \) such that
\[
\|u^{w,h}_1 - u\|_{1,\Omega,\Gamma} \lesssim h_1^{-1}, \tag{4.36}
\]
\[
\|u^{w,h}_1 - u\|_{0,\Omega,\Gamma} \lesssim H^{r+s-1+\gamma}, \tag{4.37}
\]
\[
\|u^{w,h}_1 - u\|_{1,\Omega,\Gamma} \lesssim H^{r+s-1+\gamma}, \tag{4.38}
\]
\[
\|u^{w,h}_1 - u\|_{1,\Omega,\Gamma} \lesssim h_1^{r+s-1}, \tag{4.39}
\]
\[
\|u^{w,h}_1 - u\|_{0,\Omega,\Gamma} \lesssim H^{r+s-1+\gamma}. \tag{4.40}
\]
\[ \| u^{w,h_l} - u^* \|_{1,\Omega} \lesssim H^{r+s-1+y}, \] 
(4.41)

\[ |\lambda^{w,h_l} - \lambda^*| \lesssim h_l^{2r+2s-2}. \] 
(4.42)

**Proof.** Let \( u \in M(\lambda) \) and \( u^* \in M^*(\lambda^*) \), such that \( u = u_{hi} \) and \( u^* = u^*_{hi} \) both satisfy Lemma 2.2. The proof of (4.36)–(4.42) is completed by induction.

When \( l = 1 \), Scheme 3.2 is actually Scheme 3.1. Hence, from Theorem 4.1, Theorem 4.3 and (4.35) we know that (4.36)–(4.42) hold for \( l = 0, 1 \).

Suppose (4.36)–(4.42) hold for \( l - 2, l - 1 \), i.e.,

\[ \| u^{w,h_{l-2}} - u \|_{1,\Omega} \lesssim h_{l-2}^{r+s-1}, \] 
(4.43)

\[ \| u^{w,h_{l-2}} - u \|_{0,\Omega} \lesssim H^{r+s-1+y}, \] 
(4.44)

\[ \| u^{w,h_{l-2}} - u \|_{1,\Omega} \lesssim H^{r+s-1+y}, \] 
(4.45)

\[ \| u^{w,h_{l-2}} - u^* \|_{1,\Omega} \lesssim h_{l-2}^{r+s-1}, \] 
(4.46)

\[ \| u^{w,h_{l-2}} - u^* \|_{0,\Omega} \lesssim H^{r+s-1+y}, \] 
(4.47)

\[ \| u^{w,h_{l-2}} - u^* \|_{1,\Omega} \lesssim H^{r+s-1+y}, \] 
(4.48)

\[ |\lambda^{w,h_{l-2}} - \lambda^*| \lesssim h_{l-2}^{2r+2s-2}, \] 
(4.49)

and

\[ \| u^{w,h_{l-1}} - u \|_{1,\Omega} \lesssim h_{l-1}^{r+s-1}, \] 
(4.50)

\[ \| u^{w,h_{l-1}} - u \|_{0,\Omega} \lesssim H^{r+s-1+y}, \] 
(4.51)

\[ \| u^{w,h_{l-1}} - u \|_{1,\Omega} \lesssim H^{r+s-1+y}, \] 
(4.52)

\[ \| u^{w,h_{l-1}} - u^* \|_{1,\Omega} \lesssim h_{l-1}^{r+s-1}, \] 
(4.53)

\[ \| u^{w,h_{l-1}} - u^* \|_{0,\Omega} \lesssim H^{r+s-1+y}, \] 
(4.54)

\[ \| u^{w,h_{l-1}} - u^* \|_{1,\Omega} \lesssim H^{r+s-1+y}, \] 
(4.55)

\[ |\lambda^{w,h_{l-1}} - \lambda^*| \lesssim h_{l-1}^{2r+2s-2}. \] 
(4.56)

Next we shall prove that (4.36)–(4.42) hold for \( l \). Using the above formula and Lemma 2.3 to estimate the terms at the right hand side of (4.9) gives

\[ \| u^{w,h_l} - P_{hi} u \|_{1,\Omega} \lesssim h_l^{r+s-1+y} + h_l^y (h_{l-1}^{r+s-1} + h_{l-1}^{r+s-1}) + H^{r+s-1+y} \]

\[ + H^{r+s-1+y} + (H^{r+s-1+y} + h_l^y) + H^{r+s-1+y} \lesssim H^{r+s-1+y}. \] 
(4.57)

Combining (2.40), (2.41) and (4.57) yields (4.36), (4.37) and (4.38). By the same argument we can prove (4.39)–(4.41), From (2.26), we have

\[ \lambda^{w,h_l} - \lambda = \frac{a(u^{w,h_l} - u, u^{w,h_l} - w^s)}{b(u^{w,h_l}, u^{w,h_l})} - \lambda \frac{b(u^{w,h_l} - u, u^{w,h_l} - w^s)}{b(u^{w,h_l}, u^{w,h_l})}. \] 
(4.58)

Using the similar argument as that of Theorem 4.3 we know that \( |b(u^{w,h_l}, u^{w,h_l})| \) has a positive lower bound uniformly. Combining (4.36), (4.37), (4.39), (4.40) and (4.58) yields (4.42). \( \square \)

**Remark 4.1.** \( \Omega_1 \) in (3.7) and (3.9) can be different from that in (3.6) and (3.8), which also ensures that the corresponding estimates in Theorems 4.2–4.4 still hold.

**Remark 4.2.** After dropping the steps computing \( u_h, u^{h,s}, e^{h,s}, u^{w,h,s}, \lambda^{w,h,s} \) in Scheme 3.2, and replacing \( \lambda^w = \frac{a(u^w, u^{w,s})}{b(u^w, u^{w,s})} \) and \( \lambda^{w,h,s} = \frac{a(u^{w,h_l}, u^{w,h,s})}{a(u^{w,h_l}, u^{w,h,s})} \) with \( \lambda^w, \lambda^{w,h_l} \) and \( \lambda^{w,h,s} = \frac{a(u^{w,h_l}, u^{w,h,s})}{a(u^{w,h_l}, u^{w,h,s})} \), respectively, we are able to establish multilevel discretizations based on local defect correction for symmetric eigenvalue problems. Moreover, the corresponding estimates in Theorems 4.1–4.4 still hold.
In our numericalexperiments, we use Scheme 3.2 to solvetheproblemsuchthat \( \Omega \) and 33.371, respectively. Wewill reportsomenumericalexperiments by linearfiniteelementsonuniformtrianglemeshes.

Concerning this point, the figures of eigenfunction and its adjoint one (see Figs. 5.1–5.2) shows that the values of

\[
\begin{align*}
\text{Remark 4.3. Referring to [17], we can establish the parallel version of Schemes 3.1 and 3.2 and also have the corresponding error estimates in Theorems 4.3–4.4 apparently.}
\end{align*}
\]

5. Numerical experiments

Consider the convection–diffusion equation

\[
- \Delta u + b \cdot \nabla u = \lambda u, \quad \text{in } \Omega, \quad u = 0, \quad \text{on } \partial \Omega,
\]

where \( \Omega = (-1, 1)^2 \setminus \{(0, 1) \times [-1, 0]\} \) or \( \Omega = (-1, 1)^2 \setminus \{(0) \times [-1, 0]\} \). The first eigenfunctions of both problems have singularities at the origin. The exact eigenvalues, which are unknown, are thereby replaced by the approximate eigenvalues of high accuracy. For the problem with \( b = (1, 1)^T \), \( b = (0, 3)^T \) and \( b = (0, 10)^T \) on \( \Omega = (-1, 1)^2 \setminus \{(0, 1) \times [-1, 0]\} \), we take the first eigenvalues approximately as 10.1397, 11.8897 and 34.6397, respectively. For the problem with \( b = (1, 1)^T \), \( b = (0, 3)^T \) and \( b = (0, 10)^T \) on \( \Omega = (-1, 1)^2 \setminus \{(0) \times [0]\} \), we take the first eigenvalues approximately as 10.1397, 11.8897 and 34.6397, respectively. We will report some numerical experiments by linear finite elements on uniform triangle meshes. In our numerical experiments, we use Scheme 3.2 to solve the problem such that \( \Omega_i = (-1, 1)^2 \setminus \{(0, 1) \times [-1, 0]\} \) for the L-shaped domain, \( \Omega_i = (-1, 1)^2 \setminus \{(0, 1) \times [0]\} \) for the slit domain, \( i = 1, 2, \ldots, 6 \), and the locally fine grids have the same degree of freedom as the globally mesoscopic grid (see Tables 1–6).

In our experiments, according to the assumptions of Theorem 4.4, we approximately take \( \gamma_1 = \gamma_2 = 1/2, 2/3 \) and \( s = s_2 = 1/2, 2/3 \) so that (4.35) holds for the slit domain and the L-shaped domain, respectively. We use MATLAB 2011b together with the package of Chen (see [26]) to solve the eigenvalue problems, and the numerical results are shown in Tables 1–6. From Tables 1–4 we can see that without increasing the degree of freedom on locally fine grids, the first local defect correction can largely improve the accuracy of the eigenvalues, and the local defect corrections that follows can gradually improve the accuracy of the eigenvalues by local defect corrections with refined subareas near the singularities at the origin. But Tables 5–6 also indicate that Scheme 3.2 does not perform that well for the problems with \( b = (0, 10)^T \). Concerning this point, the figures of eigenfunction and its adjoint one (see Figs. 5.1–5.2) shows that the values of

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Table 1} & 1st eigenvalue on the L-shaped domain with \( b = (0, 3)^T \). \\
\hline
\text{DOF}_H & \text{DOF}_w & \lambda_H & \lambda_w & \lambda_{w,1} \\
\hline
705 & 2945 & 11.94916 & 11.91247 & 11.89949 \\
2945 & 12033 & 11.91250 & 11.88859 & 11.89343 \\
12033 & 195585 & 11.89859 & 11.89109 & 11.89028 \\
\hline
\lambda_{w,b_2} & \lambda_{w,b_3} & \lambda_{w,b_4} & \lambda_{w,b_5} & \lambda_{w,b_6} \\
\hline
11.89466 & 11.89275 & - & - & - \\
11.89146 & 11.89068 & 11.89037 & - & - \\
11.88996 & 11.88983 & 11.88978 & - & - \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Table 2} & 1st eigenvalue on the L-shaped domain with \( b = (1, 1)^T \). \\
\hline
\text{DOF}_H & \text{DOF}_w & \lambda_H & \lambda_w & \lambda_{w,1} \\
\hline
705 & 2945 & 10.21836 & 10.16730 & 10.15332 \\
2945 & 12033 & 10.16730 & 10.14979 & 10.14438 \\
12033 & 195585 & 10.14979 & 10.14117 & 10.14034 \\
\hline
\lambda_{w,b_2} & \lambda_{w,b_3} & \lambda_{w,b_4} & \lambda_{w,b_5} & \lambda_{w,b_6} \\
\hline
10.14836 & 10.14643 & - & - & - \\
10.14238 & 10.14160 & 10.14129 & - & - \\
10.14002 & 10.13989 & 10.13984 & - & - \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Table 3} & 1st eigenvalue on the slit domain with \( b = (0, 3)^T \). \\
\hline
\text{DOF}_H & \text{DOF}_w & \lambda_H & \lambda_w & \lambda_{w,1} \\
\hline
945 & 3937 & 10.79397 & 10.70640 & 10.66393 \\
3937 & 16065 & 10.70630 & 10.66356 & 10.64247 \\
16065 & 64897 & 10.66353 & 10.64237 & 10.63186 \\
\hline
\lambda_{w,b_2} & \lambda_{w,b_3} & \lambda_{w,b_4} & \lambda_{w,b_5} & \lambda_{w,b_6} \\
\hline
10.64333 & 10.63315 & - & - & - \\
10.63208 & 10.62691 & 10.62434 & - & - \\
10.62664 & 10.62404 & 10.62274 & 10.62209 & - \\
\hline
\end{array}
\]
Table 4
1st eigenvalue on the slit domain with $b = (1, 1)^T$.

| DOF $H$ | DOF $w$ | $\lambda_H$ | $\lambda_w$ | $\lambda_{w,h_1}$ |
|---------|---------|-------------|-------------|------------------|
| 945     | 3937    | 9.06244     | 8.96099     | 8.91741          |
| 3937    | 16065   | 8.96090     | 8.91470     | 8.89333          |
| 16065   | 64897   | 8.91468     | 8.89266     | 8.88207          |
| $\lambda_{w,h_2}$ | $\lambda_{w,h_3}$ | $\lambda_{w,h_4}$ | $\lambda_{w,h_5}$ | $\lambda_{w,h_6}$ |
| 8.98663 | 8.86841 | -           | -           | -                |
| 8.88289 | 8.87772 | 8.87514     | -           | -                |
| 8.87684 | 8.87424 | 8.87294     | 8.87229     | -                |

Table 5
1st eigenvalue on the L-shaped domain with $b = (0, 10)^T$.

| DOF $H$ | DOF $w$ | $\lambda_H$ | $\lambda_w$ | $\lambda_{w,h_1}$ |
|---------|---------|-------------|-------------|------------------|
| 705     | 2945    | 34.58756    | 34.63484    | 34.61999         |
| 2945    | 12033   | 34.63473    | 34.64168    | 34.63605         |
| 12033   | 195585  | 34.64167    | 34.64066    | 34.63982         |
| $\lambda_{w,h_2}$ | $\lambda_{w,h_3}$ | $\lambda_{w,h_4}$ | $\lambda_{w,h_5}$ | $\lambda_{w,h_6}$ |
| 34.61632 | 34.61460 | -           | -           | -                |
| 34.63437 | 34.63363 | 34.63333    | -           | -                |
| 34.63952 | 34.63939 | 34.63934    | -           | -                |

Table 6
1st eigenvalue on the slit domain with $b = (0, 10)^T$.

| DOF $H$ | DOF $w$ | $\lambda_H$ | $\lambda_w$ | $\lambda_{w,h_1}$ |
|---------|---------|-------------|-------------|------------------|
| 945     | 3937    | 33.43287    | 33.42950    | 33.38763         |
| 3937    | 16065   | 33.42885    | 33.40686    | 33.38587         |
| 16065   | 64897   | 33.40671    | 33.39070    | 33.38021         |
| $\lambda_{w,h_2}$ | $\lambda_{w,h_3}$ | $\lambda_{w,h_4}$ | $\lambda_{w,h_5}$ | $\lambda_{w,h_6}$ |
| 33.36894 | 33.35923 | -           | -           | -                |
| 33.37595 | 33.37090 | 33.36836    | -           | -                |
| 33.37510 | 33.37253 | 33.37124    | 33.3706     | -                |

Fig. 5.1. 1st eigenfunction and its adjoint eigenfunction on the L-shaped domain with $b = (0, 10)^T$.

eigenfunctions changes abruptly mainly at the center of the boundary layers, which may lead to the bad performance of Scheme 3.2. Hence for the case $b = (0, 10)^T$, we adopt the parallel version of Scheme 3.2 to make local defect corrections on the boundary layers (see also Figs. 5.1–5.2).

Specifically speaking, for the L-shaped domain, we find that it is better to make local defect corrections near the origin on slightly small area $\Omega^1_i = \left( \frac{-1}{2^{i+1}}, \frac{1}{2^{i+1}} \right)^2 \setminus [[0, \frac{1}{2^{i+1}}] \times [\frac{-1}{2^{i+1}}, 0]]$ for both eigenfunction and its adjoint eigenfunction; as for the other local defect correction areas, we set as $\Omega^2_i = \left( \frac{-1}{2^{i+1}}, \frac{1}{2^{i+1}} \right) \times (1 - \frac{1}{2^i}, 1)$ for the eigenfunction, and $\Omega^3_i = \left( \frac{-1}{2} - \frac{1}{2^i}, -\frac{1}{2} + \frac{1}{2^i} \right) \times (-1, -1 + \frac{1}{2^i})$ for the adjoint eigenfunction, respectively; the related numerical results
Here we set
\[ \text{DOF}_w \approx 4 \times \text{DOF}_{\Omega^1_i} \approx \frac{3}{4} \times \text{DOF}_{\Omega^2_i} \approx \frac{3}{2} \times \text{DOF}_{\Omega^3_i} \quad (i = 1, 2, \ldots). \]

For the slit domain, we set as the local defect correction area \( \Omega^1_i = \left( \frac{1}{2}, \frac{1}{2} \right)^2 \setminus \{0\} \times [\frac{1}{2}, \frac{1}{2}] \) for both eigenfunction and its adjoint eigenfunction, \( \Omega^2_i = \left( \frac{1}{2}, \frac{1}{2} \right) \times (1 - \frac{1}{2}, 1) \) for the eigenfunction, \( \Omega^3_i = \left( \frac{1}{2} - \frac{1}{2}, \frac{1}{2} + \frac{1}{2} \right) \times (-1, -1 + \frac{1}{2}) \) and \( \Omega^4_i = (-\frac{1}{2} - \frac{1}{2}, -\frac{1}{2} + \frac{1}{2}) \times (-1, -1 + \frac{1}{2}) \) for the adjoint eigenfunction, respectively; the related numerical results are given in Table 8. Here we set
\[ \text{DOF}_w = \text{DOF}_{\Omega^1_i} = \text{DOF}_{\Omega^2_i} \approx 2 \times \text{DOF}_{\Omega^3_i} = 2 \times \text{DOF}_{\Omega^4_i} \quad (i = 1, 2, \ldots). \]

Table 7.12 and Table 7.16 in [2] show that, using the adaptive homotopy method to solve the L-shaped domain problem with \( b = (10, 0)^T \), the approximate eigenvalue can have 4–5 significant digits with \( \text{DOF} = 154,994 \) and 124,469, thus the adaptive homotopy method is efficient. However, by our algorithm, the approximate eigenvalue can have 6 significant digits with \( \text{DOF}_H = 12,033 \) (see Table 7), which also indicates our algorithm is efficient.

Moreover, as an example with multiple eigenvalue, we compute the 8th eigenvalue (\( \approx 74,348022 \)) with the multiplicity \( q = 2 \) on the L-shaped domain with \( b = (0, 10)^T \). Its corresponding eigenfunction and adjoint eigenfunction are shown in Fig. 5.3. Accordingly, the local defect correction areas are set as follows: \( \Omega^1_i = \left( \frac{1}{2} - \frac{1}{2}, -\frac{1}{2} + \frac{1}{2} \right) \times (-1, -1 + \frac{1}{2}) \) and \( \Omega^2_i = \left( \frac{1}{2} - \frac{1}{2}, -\frac{1}{2} + \frac{1}{2} \right) \times (1 - \frac{1}{2}, 1) \) for the eigenfunction, \( \Omega^3_i = (-\frac{1}{2} - \frac{1}{2}, -\frac{1}{2} + \frac{1}{2}) \times (-1, -1 + \frac{1}{2}) \) and \( \Omega^4_i = (-\frac{1}{2} - \frac{1}{2}, -\frac{1}{2} + \frac{1}{2}) \times (-1, -1 + \frac{1}{2}) \) for the adjoint eigenfunction; the related numerical results are given in Table 9. Here we set
\[ \text{DOF}_w \approx \frac{3}{2} \times \text{DOF}_{\Omega^1_i} \approx \frac{3}{2} \times \text{DOF}_{\Omega^2_i} \approx \frac{3}{2} \times \text{DOF}_{\Omega^3_i} \quad (i = 1, 2, \ldots). \]

### Table 7

| DOF_H | DOF_w | \( \lambda_H \) | \( \lambda^m \) | \( \lambda^{m, \lambda_1} \) |
|-------|-------|-----------------|----------------|----------------------|
| 705   | 2945  | 34.58756        | 34.63484       | 34.64245             |
| 2945  | 12033 | 34.63474        | 34.64169       | 34.64162             |
| 12033 | 19558 | 34.64166        | 34.64067       | 34.64017             |

### Table 8

| DOF_H | DOF_w | \( \lambda_H \) | \( \lambda^m \) | \( \lambda^{m, \lambda_1} \) |
|-------|-------|-----------------|----------------|----------------------|
| 945   | 3937  | 33.42827        | 33.42950       | 33.40900             |
| 3937  | 16065 | 33.42885        | 33.40686       | 33.38916             |
| 16065 | 64897 | 33.40671        | 33.39070       | 33.38103             |

\[
\begin{aligned}
\lambda^{m, \lambda_2} &= \lambda^{m, \lambda_3} = \lambda^{m, \lambda_4} = \lambda^{m, \lambda_5} = \lambda^{m, \lambda_6} \\
&= 33.38429, 33.37474, 33.37474, 33.37221, 33.37221, 33.37155
\end{aligned}
\]
Table 9
8th eigenvalue on the L-shaped domain with $b = (0, 10)^T$.

| $\text{DOF}_H$ | $\text{DOF}_w$ | $\lambda_H$ | $\lambda_w$ | $\lambda_{w,h_1}$ |
|----------------|----------------|-------------|-------------|------------------|
| 705           | 2945           | 73.42218    | 74.08564    | 74.17571         |
| 2945          | 12033          | 74.10236    | 74.28455    | 74.30779         |
| 12033         | 195585         | 74.28569    | 74.34399    | 74.34548         |

$\lambda_{w,h_2}$ $\lambda_{w,h_3}$ $\lambda_{w,h_4}$ $\lambda_{w,h_5}$ $\lambda_{w,h_6}$

|                |                |             |             |             |
|----------------|----------------|-------------|-------------|-------------|
| 74.18664       | 74.18802       | 74.18812    | –           | –           |
| 74.31049       | 74.31084       | 74.31086    | –           | –           |
| 74.34565       | 74.34567       | 74.34568    | –           | –           |

Table 10
8th eigenvalue on the L-shaped domain with $b = (10, 10x_1)^T$.

| $\text{DOF}_H$ | $\text{DOF}_w$ | $\lambda_H$ | $\lambda_w$ | $\lambda_{w,h_1}$ | $\lambda_{w,h_2}$ | $\lambda_{w,h_3}$ |
|----------------|----------------|-------------|-------------|------------------|------------------|------------------|
| 705            | 2945           | 82.3472     | 82.8171     | 82.8484          | 82.8525          | 82.8516          |
|                |                | –4.4951i    | –4.6349i    | –4.6474i         | –4.6431i         | –4.6436i         |
| 2945           | 12033          | 82.8439     | 82.9742     | 82.9882          | 82.9910          | 82.9906          |
|                |                | –4.6157i    | –4.6509i    | –4.6515i         | –4.6540i         | –4.6525i         |
| 12033          | 195585         | 82.9755     | 83.0168     | 83.0174          | 83.0175          | 83.0175          |
|                |                | –4.6493i    | –4.6664i    | –4.6665i         | –4.6664i         | –4.6664i         |

Fig. 5.3. 8th eigenfunction and its adjoint eigenfunction on the L-shaped domain with $b = (0, 10)^T$.

In the end, as an example with complex eigenvalue, we also compute the 8th eigenvalue ($\approx 83.01891 – 4.66823i$, an approximation of higher accuracy) on the L-shaped domain with $b = (10, 10x_1)^T$. After plotting the eigenfunction and its adjoint eigenfunction, the local defect correction areas are set as follows: $\Omega_i^1 = (1 - \frac{1}{2^i}, 1)^2$ for the eigenfunction, $\Omega_i^2 = (-1, -1 + \frac{1}{2^i}) \times (1 - \frac{1}{2^i}, 1)$ for the adjoint eigenfunction; the related numerical results are given in Table 10. Here we set

$$\text{DOF}_w \approx 3 \times \text{DOF} \Omega_i^1 \approx \frac{3}{2} \times \text{DOF} \Omega_i^2 \quad (i = 1, 2, \ldots).$$

Remark 5.1. In this paper, we employ the geometric intuition to seek the locally refined areas: First we solve the primal and adjoint eigenfunctions by finite element methods on a coarser mesh and then plot their figures (see Figs. 5.1–5.4) to determine the refinement strategy. This method has certain values in application. However, the a posteriori choice of determining the defect correction areas is a more significant and challenging task.

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Fig. 5.4. 8th eigenfunction (real part at the left top, imaginary part at the right top) and its adjoint eigenfunction (real part at the left bottom, imaginary part at the right bottom) on the L-shaped domain with $b = (10, 10x_1)^T$.

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