Comment on "Anomalous Edge State in a Non-Hermitian Lattice"

Ye Xiong
Department of Physics and Institute of Theoretical Physics, Nanjing Normal University, Nanjing 210023, P. R. China
National Laboratory of Solid State Microstructures, Nanjing University, Nanjing 210093, P. R. China

Tianxiang Wang and Xiaohui Wang
Department of Physics and Institute of Theoretical Physics, Nanjing Normal University, Nanjing 210023, P. R. China

Peiqing Tong
Department of Physics and Institute of Theoretical Physics, Nanjing Normal University, Nanjing 210023, P. R. China
Jiangsu Key Laboratory for Numerical Simulation of Large Scale Complex Systems, Nanjing Normal University, Nanjing 210023, P. R. China

In this comment, we criticize three main conclusions of the letter [1]. We show that the concept of fractional winding number (FWN) is factitious. Lee’s conclusions on Fig. 3 are finite-size effect and the breakdown of bulk-boundary correspondence (BBBC) cannot be explained by “defective”.

![Figure 1](attachment:image.png)

**FIG. 1.** (color online) (a) The arrows indicates how the eigenvalues in the two bands evolve with wave-vector $k$. The state in the upper band evolves continuously to the lower band after 2π variance of $k$. (b) The winding numbers, $w_+$ and $w_-$, for the upper and the lower bands, respectively. (c) and (d) show the left and the right eigenvectors of a $N = 30$ chain with OBC. The bulk states are changed from extended to localized just because the boundary condition is changed from periodic to open. (e) The energy spectrum for chains with different lengths. The spectrum are changed entirely when $N$ is increased. The parameters are $\gamma = 1$, $r = 0.5$ and $v = 0.52$.

There is no FWN. — For a hermitian Hamiltonian $H_b = x(k)\sigma_x + z(k)\sigma_z$, there are two equivalent ways to calculate the winding number: one is by studying the track of the vector $(x(k), z(k))$ in the plane and the other is by the berry phase: $w_\pm = \frac{1}{\pi} \int_{k=0}^{2\pi} dk A_\pm(k)$, where $A_\pm(k) = -i\langle u_\pm(k)|\partial_k|u_\pm(k)\rangle$ is the Berry connection for the upper + and the lower − bands. But in Lee’s model, the equivalence breaks down and the previous method adopted by Lee gives a wrong answer. As Fig. (a) shows, the states are not periodic in the Brillouin zone (BZ), e.g. the upper band states change continuously to the lower band states after 2π variance of $k$. So the real period for $k$ becomes 4π. It is impossible to define the winding number in one BZ because the evolution of the states is not close in this interval. So FWN for each band is meaningless because it is a gauge variant quantity. The physical quantity should be counted in the real period 4π, which will include the Berry phases of the two bands. So this quality must be specified by a winding number 1.

To confirm our conclusion, we calculate $w_\pm$ by choosing a special gauge: $|u_\pm\rangle = (1, -(r\sin(k) - i/2 \pm E)/(r\cos(k) + v))^T$ and $\langle u_\pm\rangle = \langle \langle u_\pm\rangle \rangle^T$. Here $|u_\pm\rangle$ and $\langle \langle u_\pm\rangle \rangle$ are the left and the right eigenvectors and the Berry connection becomes $A_\pm = -i\langle u_\pm|\partial_k|u_\pm\rangle/\langle \langle u_\pm\rangle |u_\pm\rangle \rangle$ because the model is non-hermitian. The results are presented in Fig. (b). We find $w_\pm \neq 1/2$ but $w_+ + w_- = 1$ as expected. By choosing another gauge, $w_\pm$ are different.

**Finite-size effect.** — In Fig. (e), we present the spectrum for chains with open boundary condition (OBC). The length is changed from $N = 30$ to 800. When $N = 30$, our result is identical to Lee’s. But the spectrum changes entirely when $N = 800$, e.g. the band gap disappears and the spectrum becomes complex. So Lee’s conclusions, such as the PT symmetry is preserved and so on, are caused by the finite size effect.

"Defective" cannot explain BBBC. — “Defective equations” mean equations with less solutions. Lee used it to explain why there is only one $E = 0$ boundary state rather than two. But he further explains BBBC with “defective” without any explanation. In Fig. (c) and (d), we show that the bulk states are changed from ex-
tended to localized just because the boundary condition is changed. We believe that this behavior is the real reason for BBBC. A letter supports us because it shows that the ansatz, that most bulk states are not affected by the boundary condition, are important in deriving the bulk boundary correspondence (BBC)\(^2\). When the ansatz is destroyed, so as to BBC.

**Appendix A: The winding number**

We appreciate the author to express his version of winding number in mathematic form. But we can prove that Eq. (3) in his third reply is wrong because he has misunderstood the relation between the winding number and the Brillouin zone. We will present our reasons from the following three aspects.

Before the beginning of our discussions, we want to first present how we will define the winding number and the difference between ours and Lee’s for the sake of clarity.

In any system, we define the berry phase as

\[
\gamma_B = i \int \frac{dk}{A} \frac{\langle \langle u \partial_k |u \rangle \rangle}{\langle \langle u |u \rangle \rangle}, \quad (A1)
\]

The closed line integral \(\oint\) stands for the closed path evolution, which means that the wave-functions \(|u\rangle\) and \(\langle \langle u \rangle \rangle\) are on a closed loop, i.e. they must return after a path in the parameter space. We’d like to emphasize that the closed line integral \(\oint\) implicitly indicates that the loop rounds only one time.

The winding number, if exists, should equal to

\[
w = \frac{1}{\pi} \gamma_B, \quad (A2)
\]

irrespective to the length of the closed path. When applying this equation to Lee’s model, as the closed path of \(k\) is \([0, 4\pi]\), the winding number is explicitly written as

\[
w = \frac{i}{\pi} \int_{0}^{4\pi} dk \frac{\langle \langle u \partial_k |u \rangle \rangle}{\langle \langle u |u \rangle \rangle} = 2. \quad (A3)
\]

While Lee thinks that as the closed path goes through the Brillouin zones twice, he divides the above winding number by 2 to count the winding number in one Brillouin zone. So his version of winding number becomes

\[
w = \frac{i}{2\pi} \int_{0}^{4\pi} dk \frac{\langle \langle u \partial_k |u \rangle \rangle}{\langle \langle u |u \rangle \rangle}, \quad (A4)
\]

which is copied from his third reply.

Now we start our discussion from the first aspect.

This is reduction to absurdity so we will adopt Lee’s idea to see what will happen.

As the divergence comes mostly from the effect of the Brillouin zone, we take a much simple model for the sake of clarity. Our demo model is \(H(k) = \cos(k)\sigma_x + \sin(k)\sigma_y\). But we artificially enlarge the Brillouin zone from \([0, 2\pi]\) to \([0, 4\pi]\). We use BZ and BZ’ to denote these two Brillouin zones. So our demo model is the simple Hamiltonian on BZ’.

Now let’s analyze the meaning of each factor in Eq. (A4) so that we can apply it to the demo model. First, the integral is from 0 to 4\(\pi\). This is because the states close one and only one loop in the 4\(\pi\) period in Lee’s model. Second, in Lee’s model the Brillouin zone is \([0, 2\pi]\). Third, there is a factor 1/2 in the front of the integral. This is because the integral goes through the Brillouin zone BZ, \(A = 2\) times. The factor in front of the integral is 1/\(A\). So his equation can be written as

\[
w = \frac{i}{A\pi} \int dk \frac{\langle \langle u \partial_k |u \rangle \rangle}{\langle \langle u |u \rangle \rangle}. \quad (A5)
\]

Now let’s specify the values of these factors in our demo model. First, the closed loop is \([0, 2\pi]\) as the Hamiltonian is a hermitian model. So the integral is from 0 to 2\(\pi\). Second, the Brillouin zone BZ’ is \([0, 4\pi]\) as we artificially chose. Third, \(A = 1/2\) as the integral will only go through half of the Brillouin zone BZ’. Then we apply Lee’s equation and the winding number becomes

\[
w = \frac{2i}{\pi} \int_{0}^{2\pi} dk \frac{\langle \langle u \partial_k |u \rangle \rangle}{\langle \langle u |u \rangle \rangle} = 2, \quad (A6)
\]

as the integral is -i\(\pi\).

We have a few remarks on the above equation. First, as the integral is still from 0 to 2\(\pi\), \(w = 2\) is not caused by the accumulation of the integral in the enlarged Brillouin zone. It is only caused by the factor 1/\(A\) in the front of the integral. Second, \(w = 2\) is wrong because we know the winding number of the demo Hamiltonian is \(w = 1\). Solid state physics tells us that enlarging the Brillouin zone is trivial and should not introduce any modification on the physical quantities. So the winding number must still be 1. Third, if we adopt our expression in Eq. (A2) we can get the correct winding number, \(w = 1\) in this case.

The above demo model illustrates that Lee’s mistake lies in his misunderstanding on the winding number and the Brillouin zone. One may still argue that the integral lies in his misunderstanding on the winding number.

The second aspect.

We want to first clarify a basic question: what is the winding number in 1-dimension? The winding number specifies a mapping from \(S^1\) to \(S^1\). The previous \(S^1\) is a closed path in the parameter space. For hermitian topological insulator, this is the toroidal Brillouin zone, which is usually denoted as \(T^1\). The latter \(S^1\) refers to the closed circle in the Hilbert-space for the wave-functions.
For Lee’s model, there is no such mapping from $T^1 \rightarrow S^1$ because when $k$ changes $2\pi$, $S^1$ is not a closed loop. The mapping restores when $k$ changes $4\pi$ and we denote such mapping as $2 \ast T^1 \rightarrow S^1$. The author keeps trying to fold the mappings $2 \ast T^1 \rightarrow S^1$ to $T^1 \rightarrow S^1$ by dividing the previous one by a factor 2. But the mapping $T^1 \rightarrow S^1$ does not really exist. Referee A has realized this point, although he/she does not support us right now. Let’s quote his/hers words here: “the winding number is only well defined for a case, for which the corresponding eigenvectors are on a closed loop, i.e. they must return to an identical value after closing a path in phase space.”

The things will be more clear when we throw out the concept of the Brillouin zone and consider $k$ as a parameter. In that case, our equation (A2) still works. But Lee’s equation is ill defined because he cannot define the factor $1/A$ in the front of the integral. One may still argue that Lee counts the winding in one Brillouin zone so we cannot throw out it. This is what we criticize in the comment and the detailed reasons are presented in the next section.

The third aspect.

At first, we want to clarify a few things in Lee’s letter and those in our comment. Lee did not calculate the winding number by a mathematic form in his Letter. His conclusion actually comes from Fig. 2 (c) in the letter. The red solid line is the track for the wave-functions in the upper band, $|u_+(k)\rangle$, and the red dashed line is that for the states in the lower band, $|u_-(k)\rangle$. We want to emphasize that these tracks are for $k \in [0, 2\pi]$, respectively. The upper band states $|u_+\rangle$ change to the lower band states $|u_-\rangle$ when $k$ changes $2\pi$ and a closed loop needs totally $4\pi$ variance. Lee obtains the fractional winding number, $w_+ = 0.5$ for the upper band $|u_+\rangle$ only based on the fact that the red solid line is a half circle around 0. While in our comment, we calculate the winding number $w_+$ for the upper band and find that it is a gauge variant quantity. So it is meaningless. This is easy to be understood because the red solid line is not a closed loop. It is only when $k$ changes $4\pi$, while the state goes through all states in the upper band and the lower band, the loop will close. That is why we have $w_+ + w_- = 1$. This quantity is gauge invariant and is physically meaningful.

We want to point out that Lee’s equations in his third reply contradicts with his conclusions in the Letter. If one insists that the winding number must be defined in the Brillouin zone, he must face up to the truth that there are two bands in this zone. Then he must inevitably specify the two winding numbers for these two bands, respectively. (In traditional topological insulators such as SSH model, the two winding numbers for the two bands are identical so one uses winding number instead of winding numbers for individual bands implicitly.) The things are more clear when we assign the band index to Lee’s equation. We will find e.g. $w_+ = \frac{i}{2\pi} \int_0^{4\pi} dk \frac{\langle \partial_u |u(k)\rangle}{|\langle \partial_u |u(k)\rangle|}$. Let’s start the integral from $|u(k=0)\rangle$. The integral on the right hand side first goes through the upper band in the interval $[0, 2\pi]$. Then it will cover the lower band in the next $[2\pi, 4\pi]$ interval. So the right hand side of the equation includes both the contributions from the upper band and the low band. But Lee is trying to assign it to the upper band independently. This is of course wrong. If Lee insists that his version of winding number is not for the upper band or the lower band individually, but for them all, then he must count the contributions of the upper band and the lower band totally in the Brillouin zone. This will give him $w = 1$. So the mistake is hidden here: This model is $4\pi$ period and the closed loop sweeps both the upper and the lower bands. Lee notices the first property so he divide the winding number by 2. But he forgets the later property which requires him to multiple 2 because there are 2 bands which need to be counted together.

Let’s summary the discussions in the last paragraph because they are very important. There are two kinds of interpretations on Lee’s equation. One is that the winding number is for the individual band. But this will make the winding number for one band include the contribution from the other band. The other is that the winding number is for the two bands. Then from Fig. 2(c) in the letter, the tracks for the two bands (the solid and the dashed red lines) totally form one loop but not a half loop. So the winding number should be 1 instead of 1/2. Our calculation also illustrates that after taking into account the contributions from the two bands, the total winding number $w_+ + w_-$ is 1 but not 1/2.

Remarks: We want to clarify why we use the integral within $[0, 2\pi]$ in our comment. This is because we are inclined to think that Lee is using the first interpretation in his letter. So we defined the corresponding quantities $w_\pm$ for the two bands, respectively. But how are these quantities related to the winding number $w$ defined in this response? In Eq. (A2) $w = \frac{i}{\pi} \int_0^{4\pi} dk \frac{\langle \partial_u |u(k)\rangle}{|\langle \partial_u |u(k)\rangle|}$. Let’s start the integral from $|u(k=0)\rangle$. In the interval $[0, 2\pi]$, the states is in the upper band so the integral in this interval is just $-i\pi w_+$. Then in the interval $[2\pi, 4\pi]$, the wave-functions are in the lower band. So the integral in the later interval is $-i\pi w_-$. Now one will realize that $w = w_+ + w_-$. This is just the argument we presented in the comment. $w_+ + w_- = 1$ is meaningful, while individuals $w_\pm$ are meaningless and the idea of the fractional winding number is artificial.

In summary, we have shown that Lee’s equations are wrong and pointed out where the mistake takes place. J. Phys. A 36, 2125 (2003) is right because the berry phase there includes the contributions from both the upper and the lower band. But one cannot further split it half by half and assign the two parts to the two bands, respectively.
[1] T.E. Lee, Physical Review Letters 116, 133903 (2016).
[2] Abhiject Alase, Emilio Cobanera, Gerardo Ortiz and Lorenza Viola, Physical Review Letters 117, 076804 (2016).