Relativistic Quantum Transport Theory

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Abstract

Relativistic quantum transport theory has begun to play an important role in the space-time description of matter under extreme conditions of high energy density in out-of-equilibrium situations. The following introductory lectures on some of its basic concepts and methods comprise the sections: 1. Introduction; 2. Aims of transport theory (classical); 3. Quantum mechanical distribution functions - the density matrix and the Wigner function; 4. Transport theory for quantum fields; 5. Particle production by classical fields; 6. Fluid dynamics of relativistic quantum dust.

1 Introduction

The study of the behavior of matter under more and more extreme conditions has a long tradition motivated by the quest for understanding the forces among its constituents on smaller and smaller scales. Not only the attempts to understand such spectacular phenomena as the stellar supernova explosions or theories of even the primordial stages of cosmological evolution, but also the ever increasing collision energy of high-energy particle accelerators and the heavy-ion programs at CERN and RHIC, in particular, witness the most recent stages of this scientific development.

In the latter context, often the complicated space-time dependence of what is really a quantum many-body system or what are highly dynamical interacting quantum fields is described in terms of a perfect fluid model. Since the seminal work by Fermi and Landau this approach has been applied successfully, in order to study global features, such as multiplicity distributions and apparently thermal transverse momentum spectra of produced particles, in high-energy collisions of strongly interacting matter \[1, 2, 3, 4\]. Similarly, the hydrodynamic approximation is often invoked in astrophysical applications and cosmological studies of the early universe \[5\].

The limitations of and likely necessary corrections to the fluid picture, however, have rarely been explored in the microscopic or high energy density domain. Difficulties reside in the derivation of consistent transport equations and in the amount of computational work required to find realistic solutions; see Refs. \[6, 7, 8\], for example, for a review and recent progress concerning selfinteracting scalar particles and the quark-gluon plasma, respectively. More understanding of related hydrodynamic behavior, if any, seems highly desirable.

For example, it has recently been shown that a free scalar field indeed behaves like a perfect fluid in the semiclassical (WKB) regime \[9\]. More generally, the mechanisms of quantum decoherence and thermalization in such systems which can be described hydrodynamically, i.e. the emergence of classical deterministic evolution from an underlying quantum field theory, are of fundamental interest \[10, 11, 12, 13\].
Having said this, it becomes obvious that not only particular applications of relativistic quantum transport theory motivated by experiments or observations – such as a suitable quark-gluon plasma transport theory to be applied to the phenomenology of high-energy heavy-ion collisions, or a transport theory for the electro-weak interactions of the intense neutrino flux from a supernova core with its electron-positron plasma sphere – are of interest, but that many interesting conceptual problems can be found in this field. It is the aim of the present introduction to describe some of its basic concepts and methods.

It seems worth while to emphasize here that transport theory by its very nature aims to describe highly dynamical systems where the time dependence of the phenomena to be studied cannot be neglected. Therefore, one necessarily has to go beyond (thermal) equilibrium field theory, for example. A partial exception consists in linear response theory, where standard field theory methods are employed to calculate the short-time response of the system to necessarily small perturbations.

The plan of these lectures is as follows. In Section 2 we present the motivation for transport theory by taking a cursory look at classical relativistic transport theory and its relation to relativistic hydrodynamics. In Section 3 we introduce quantum mechanical distribution functions. Especially, the need for the density matrix formalism is reviewed in basic terms, and the Wigner function is introduced. In Section 4, as an example, we develop the transport theory for the particular model of interacting quantum fields with a global O(4) symmetry, i.e. the linear sigma model, in the Hartree approximation. In Section 5 particle production by classical fields is described, presumably an important effect during early stages of heavy-ion collisions, solving fermion quantum transport equations perturbatively. Finally, in Section 6, we investigate the fluid dynamical behavior of relativistic quantum dust, solving the free quantum transport equations for arbitrary initial conditions exactly.

2 Aims of transport theory

Concerning the historical development as well as the systems under study, the subject matter of transport theory is most frequently associated with nonequilibrium plasmas of all sorts:

- The quark-gluon plasma with QCD interactions formed shortly after the Big Bang initiating the observable Universe and possibly recreated during high-energy collisions, in particular with heavy nuclear projectiles/targets \((\mathcal{R}+\mathcal{Q})\).

- The \(\nu\bar{\nu}e^+e^-\) plasma with electro-weak interactions created during supernova explosions between the proto-neutron star core and the leftover outer layers of the collapsing star \((\mathcal{R})\).

- \(H, He, \ldots\) fusion plasmas in burning stars.

- Electrodynamic plasma phenomena in the Earth’s ionosphere leading to polar lights and thunderstorms with lightnings.

- Discharge plasmas used in neon lighting and plasma welding.

- The \(e^-\) plasma in metals or semiconductors, the study of which has been advanced in solid state physics with particular attention to quantum effects \((\mathcal{Q})\).

Here we marked by \(\mathcal{R}\) and/or \(\mathcal{Q}\) the systems where relativistic and/or quantum effects are known or expected to play an essential role. – Abstracting from these examples, we notice two common qualitative features among these systems:

- microscopic distance scales (average interparticle distance \(n^{-1/3}\), mean free path \(\lambda_{mfp}\), etc.) \(\approx\) homogeneity scale \(L\);
microscopic time scales (average lifetime $\tau_l$, relaxation time(s) $\tau_r$, etc.)
$\approx$ hydrodynamic time scale $\tau_h = L/c$, where $c_s$ denotes the sound velocity.

Clearly, there will be exceptions to these qualitative statements and more precise characterizations of the plasma state can be given in the individual cases. However, this may suffice here, and we embark on the more formal description in the following. – We refer the reader to the monograph on relativistic kinetic theory of Ref. [14], which presents an excellent detailed exposition of the more traditional material in this context.

2.1 Classical phase space description of many-body dynamics

For a classical many-particle system, consider the probability to find a particle in the 8-dimensional phase space volume element at the (four-vector) position $x$ with (four-vector) momentum $p$,

$$dP(x,p) \equiv f(x,p) d^4x d^4p ,$$

where $f(x,p)$ denotes the corresponding Lorentz scalar phase space density. Generally, in a classical system, the four-momentum is on-shell, such that not all components of $p$ are independent. In particular, we assume that the constraint expressing the energy in terms of the three-momentum and particle mass, $p^0 = +\sqrt{\vec{p}^2 + m^2}$, is incorporated in $f$. – Except when explicitly stated, our units are such that $\hbar = c = k_B = 1$, and we use the Minkowski metric $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$.

A remark is in order here. Clearly, in order to learn not only about the behavior of a typical particle but also about its correlations with others, one principally should study the one-body density $f$ along with a two-body distribution $f(x_1, x_2, p_1, p_2)$, three-body distribution . . . , etc. This is quite complicated in general and we restrict our attention to the one-body density here. Sometimes it is useful to visualize this function alternatively as describing a collection of (test) particles or a single particle with an ensemble of initial conditions.

Now, how does the one-body density $f$ evolve, e.g. from one time-like hypersurface to another? We recall two ingredients of Liouville’s theorem, which will provide the basic tool to answer this question [15]: i) Consider a phase-space volume element which is defined by ‘tracer’ particles forming its surface; then, due the uniqueness of the Newtonian motion, or its relativistic generalization, the number of particles inside is constant (in the absence of scattering interactions). ii) The size of the volume element, being associated with one of Poincaré’s integral invariants (under contact transformations), is also a constant of motion. Then, Liouville’s theorem follows: If there are only conservative forces, then the phase space density is a constant of motion.

Beginning with this theorem, we derive the evolution equation for $f$ in terms of the relativistic proper time $\tau$ as follows:

$$0 = \frac{d}{d\tau} f(x,p) \equiv \frac{d}{d\tau} \sum_i \delta^4 [x_i(\tau) - x] \delta^4 [p_i(\tau) - p] ,$$

$$= \sum_i \left\{ \frac{dx_i^\mu}{d\tau} \frac{\partial}{\partial x^\mu} + \frac{dp_i^\mu}{d\tau} \frac{\partial}{\partial p^\mu} \right\} \delta^4 (x_i(\tau) - x) \delta^4 (p_i(\tau) - p)$$

$$= \left\{ \frac{1}{m} p^\mu \frac{\partial}{\partial x^\mu} + \mathcal{F}^\mu(x) \frac{\partial}{\partial p^\mu} \right\} f(x,p) ,$$

where we used the constancy of the density which is represented in terms of the particle (index $i$) trajectories $\{x_i(\tau), p_i(\tau)\}$ in the first line, carried out the differentiation in the second, and employed the definition of the four-velocity and the equation of motion in the last, respectively; here $\mathcal{F}^\mu(x)$ denotes the external or selfconsistent internal four-force(s). This is the relativistic Vlasov equation.
More generally, allowing for scattering of particles into and out of phase space volume elements, i.e. \((d/d\tau) f(x,p) \neq 0\), the Vlasov equation is replaced by the **Boltzmann equation**:

\[
\left\{ \frac{1}{m} p \cdot \partial_x + \mathbf{F}(x) \cdot \partial_p \right\} f(x,p) = C[f](x,p) ,
\]

where \(C[f]\) denotes the collision term. It can be derived from an analysis of the equations including the two-body densities and generally turns out to be a nonlinear functional of the one-body densities.

A popular set of assumptions for this derivation is the following: i) Only binary or two-body collisions contribute (in a sufficiently dilute system); ii) Boltzmann’s “Stosszahlansatz” according to which the number of collisions at \(x\) is proportional to \(f(x,p) f(x,p')\); iii) the distribution function varies slowly on the scale of the mean free path, \(\lambda_{\text{mfp}} |\nabla \log f(x,p)| \ll 1\).

We remark that while the Vlasov equation is appropriate for systems with conservative forces only and, thus, describes nondissipative phenomena, the Boltzmann equation incorporates dissipative scattering processes, which lead to entropy production.

In order to illustrate this, we introduce the simple **relaxation time collision term**, which could be derived rigorously as an approximation of a two-body scattering term but can also be seen as a phenomenological ansatz taking the dissipation into account:

\[
C[f] \equiv -\frac{1}{\tau_r} (f - f_0) ,
\]

where \(\tau_r\) denotes the relaxation time parameter and \(f_0\) the equilibrium one-body density, towards which the system will relax.

A particular equilibrium solution of the Boltzmann equation with the above collision term can be obtained from the Jüttner distribution,

\[
f_0(x,p) \equiv \exp[-\beta(U \cdot p + \mu)] ,
\]

where the local parameters \(\beta \equiv 1/T, U^\mu, \mu\) denote the inverse temperature, flow four-velocity, and chemical potential, respectively. Instead of the exponential ‘Boltzmann factor’ one may also use a Fermi-Dirac or Bose-Einstein distribution, depending on the nature of the considered particles.

In the example of a plasma of charged particles, including a static homogeneous neutralizing background, the Lorentz force is \(\mathbf{F}(x) \equiv F^{\mu\nu} j_\nu = (e/m) F^{\mu\nu} p_\nu\), in terms of the field strength tensor \(F^{\mu\nu}\) of external and/or selfconsistently generated internal electromagnetic fields and the particle electric current \(j^\mu\). Inserting this together with the Jüttner distribution, i.e. \(f = f_0\), into the Boltzmann Eq. (4) together with Eq. (5), we obtain the equation:

\[
\frac{\partial}{\partial x_\mu} [\beta(U \cdot p + \mu)] + e\beta F^{\mu\nu} U_\nu = 0 ,
\]

which constrains the parameters of Jüttner distribution. Thus, the simplest solution indeed is the global equilibrium distribution with a constant temperature, a global rest frame, and where the gradient of the chemical potential compensates the electric field \(F^{\mu0}\), \(-\partial_0^\mu = e F^{\mu0}\).

### 2.2 Relation to relativistic hydrodynamics

Turning to the observables to be described by the phase space density \(f\) introduced in Eq. (4), we define the particle (mass, charge) four-current,

\[
N^\mu(x) \equiv \int d^4 p \, p^\mu f(x,p) ,
\]
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and the energy-momentum tensor,

\[ T^{\mu\nu}(x) \equiv \int d^4p \ p^\mu p^\nu f(x,p) \]  

(9)

as the first and second moment of the distribution function, respectively; it is important to recall here that we assume \( f \) to implicitly contain the usual on-shell constraint, i.e. the factors \( \Theta(p^0)\delta(p^2 - m^2) \), as we discussed.

These are the main quantities of interest in the hydrodynamic description of matter, where one integrates out the momentum space information contained in \( f \) [16].

Indeed, it is straightforward to show that \( N^\mu \) and \( T^{\mu\nu} \) obey the appropriate continuity equations related to mass (or charge) and four-momentum conservation.

To begin with, using the Boltzmann Eq. (4) together with Eq. (5), we obtain:

\[ \partial_\mu N^\mu = \int d^4p \ p \cdot \partial_x f = -m \int d^4p \left( \mathcal{F} \cdot \partial_x f + \frac{f - f_0}{\tau_r} \right) = -m \int d^4p \ (f - f_0) \equiv -\partial_\mu \delta N^\mu \]  

(10)

where we find a dissipative contribution \( \delta N^\mu \) on the right-hand side, which vanishes only if the ordinary density equals the equilibrium density determined by \( f_0 \). Similarly, one obtains:

\[ \partial_\mu T^{\mu\nu} = -m \int d^4p \ p^\nu \left( \mathcal{F} \cdot \partial_x f + \frac{f - f_0}{\tau_r} \right) = m \mathcal{F}^\nu \left( \int d^4p \ f \right) - m \frac{\tau_r}{\tau_r} (N^\nu - N_0^\nu) \]  

(11)

where the second term on the right-hand side is related to a dissipative contribution \( -\partial_\mu \delta T^{\mu\nu} \) to the energy-momentum tensor, while the first term presents the external or selfconsistent force density acting on the system; \( N_0^\mu \) is defined like \( N^\mu \), however, with \( f \) replaced by \( f_0 \).

More generally, based on the Chapman-Enskog method, standard forms of the dissipative terms can be constructed incorporating the transport coefficients of shear and bulk viscosity, heat conductivity, particle production/annihilation, and diffusion [16, 17].

Finally, we consider the entropy. It is an important quantity not only in equilibrium thermodynamics, but can be used, for example, to characterize the bulk properties of matter produced in high-energy collisions. In particular, it can be related more or less directly to the observed particle multiplicities [3, 4, 16]. Here we define the entropy four-current:

\[ S^\mu(x) \equiv -\int d^4p \ p^\mu f(x,p) \ln[f(x,p)/f_0(x,p)] \]  

(12)

Calculating as before, we obtain the entropy production formula:

\[ \partial_x \cdot S = \frac{m}{\tau_r} \int d^4p \ \left( \ln[f/f_0] + 1 \right) (f - f_0) \]  

(13)

Since \( \ln(x + 1)(x - 1) \geq 0 \), for all \( x \geq 0 \), we recover Boltzmann’s “H-theorem”:

\[ \partial_x \cdot S \geq 0 \]  

(14)

expressing a positive entropy production which vanishes only in equilibrium, when \( f = f_0 \).

This completes our overview of classical relativistic transport theory and some of its ramifications and we turn to quantum mechanics next.

3 Quantum mechanical distribution functions - the density matrix and the Wigner function

In order to motivate the necessity for a density matrix formulation in quantum mechanics, we recall Feynman’s famous division [18, 19],

\[ \text{Universe} \equiv \text{System} + \text{Rest} \]  

(15)
which says that doing physics and paying attention to the system requires precisely to separate off the rest of the Universe, i.e. the ’environment’ of the system in modern terminology.

Following the presentation in Ref. [18], we introduce generic coordinates \(x\) and \(y\) for the description of \(\mathcal{S}\) and \(\mathcal{R}\), respectively. Assuming the existence of corresponding complete sets of states or wave functions, \(\{\phi_i(x)\}\) and \(\{\psi_j(y)\}\), the most general state vector and normalized wave function of \(\mathcal{U}\) can be expanded, respectively, as:

\[
|\Psi\rangle = \sum_{i,j} c_{ij} |\phi_i\rangle |\psi_j\rangle, \quad \Psi(x,y) = \sum_{i,j} c_{ij} \langle y|\phi_i\rangle \langle x|\psi_j\rangle \equiv \sum_i c_i(y) \phi_i(x),
\]

with complex expansion coefficients \(c_{ij}\) and functions \(c_i(y)\).

Considering an operator \(\hat{A}\) which acts on \(\mathcal{S}\) only, it is given in terms of its matrix elements by:

\[
\hat{A} \equiv \sum_{i'i'} A_{i'i'} |\phi_i\rangle |\psi_{i'}\rangle \langle \phi_{i'}| \langle \psi_i|,
\]

i.e., it acts as a projector on \(\mathcal{R}\). Following the rules, we obtain the expectation value of \(\hat{A}\) in the state \(|\Psi\rangle\) of Eq. (16):

\[
\langle \hat{A} \rangle \equiv \langle \Psi|\hat{A}|\Psi\rangle = \sum_{i'i'} A_{i'i'} c_{i'j} c_{ij}^* = \sum_{i'i'} A_{i'i'} \rho_{i'i'} = \text{Tr} \hat{A} \hat{\rho} = \text{Tr} \hat{\rho} \hat{A}.
\]

It follows from its definition here that the density operator \(\hat{\rho}\), with the density matrix elements identified as \(\rho_{i'i'} = \langle \phi_i|\hat{\rho}|\phi_{i'}\rangle\), is a hermitean operator.

Therefore, we may introduce a complete orthonormal basis \(\{|i\rangle\}\) diagonalizing \(\hat{\rho}\): \(\hat{\rho} = \sum_i w_i |i\rangle \langle i|\), with real eigenvalues \(w_i\). Furthermore, choosing \(\hat{A} = 1_\mathcal{S} \otimes 1_\mathcal{R}\), i.e. the identity operator, one finds with the help of Eq. (18) the sum rule: \(1 = \langle \hat{A} \rangle = \text{Tr} \hat{\rho} = \sum_i w_i\). Finally, choosing instead \(\hat{A} = |i'\rangle \langle i'| \otimes 1_\mathcal{R}\), a similar calculation yields: \(w_i = \langle \hat{A} \rangle = \langle \Psi|\hat{A}|\Psi\rangle = \sum_j \langle \psi_j|x\rangle^2 \sum_j \langle \psi_j| |\psi_j\rangle^2 \geq 0\).

Abstracting from the present example, it is postulated that any quantum mechanical system is to be described by a hermitean density operator,

\[
\hat{\rho} = \sum_i w_i |i\rangle \langle i|,
\]

where \(\{|i\rangle\}\) forms a complete orthonormal set, the real expansion coefficients are non-negative, \(w_i \geq 0\), and fulfill a normalization condition, \(\sum_i w_i = 1\). – We remark that the density matrix was first introduced by von Neumann in 1932 in his by now famous book [20]. – Observables in particular and expectation values of operators in general are always to be calculated by:

\[
\langle \hat{A} \rangle = \text{Tr} \hat{\rho} \hat{A} = \sum_i w_i \langle i|\hat{A}|i\rangle.
\]

Consequently, the coefficients \(w_i\) are interpreted as describing the probability to find the system in the state \(|i\rangle\).

As a matter of nomenclature, one distinguishes pure and mixed states of a system, the former being defined by a density operator which is a projector, \(\hat{\rho} = \hat{\rho}^2 \iff (w_{i*} = 1, \text{ all other } w_i = 0)\), and the latter comprising all other cases. – We recall the case of a system in contact with a heat bath, which leads to the mixed state density operator of thermal equilibrium:

\[
\hat{\rho}(\beta) = Z^{-1}(\beta) \sum_n e^{-\beta E_n} |E_n\rangle \langle E_n|, \quad Z(\beta) \equiv \sum_n e^{-\beta E_n},
\]

where \(\{|E_n\rangle\}\) denotes the complete set of normalized energy eigenstates of the system and \(Z\) is its partition function necessary to normalize the exponential ‘Boltzmann factor’. All canonical thermodynamical relations can be easily derived from this density operator, see, for example, [18].
Finally, we may address also in the present context the question how the system, i.e. its density operator, evolves in time. This is easily answered by expanding the eigenstates \{ |i \rangle \} of \( \hat{\rho} \) in terms of the energy eigenstates,

\[ |i(0) \rangle \equiv |i \rangle = \sum_n |E_n \rangle \langle E_n |i(0) \rangle \]  

(22)

which implies according to the Schrödinger equation: \( |i(t) \rangle = \exp(-i \hat{H} t)|i(0) \rangle \), given the Hamiltonian \( \hat{H} \). Then, the evolution of \( \hat{\rho} \) follows:

\[ \hat{\rho}(t) = e^{-i \hat{H} t} \hat{\rho}(0) e^{i \hat{H} t} \]  

(23)

Equivalently, we obtain:

\[ \frac{d}{dt} \hat{\rho}(t) = i [\hat{\rho}(t), \hat{H}] \]  

(24)

which differs from the usual operator evolution equation in quantum mechanics by an additional minus sign on the right-hand side. Both signs, of course, are consistent, since we have:

\[ \langle \hat{A}(t) \rangle \equiv \text{Tr} \, \hat{\rho}(t) \hat{A} = \text{Tr} \, \hat{\rho}(0) e^{i \hat{H} t} \hat{A} e^{-i \hat{H} t} \equiv \langle \hat{A}(t) \rangle \]  

(25)

using the cyclicity of the trace.

The equations (24) and (25) present the problem of quantum transport theory in its most compact form. In particular, Sections 4–6 are devoted to explorations of various more detailed forms of these abstract results and their applications.

In order to connect the density matrix formalism with the classical transport theory, we turn to the density matrix in coordinate and momentum representation.

It helps to visualize the following by imagining a single particle to be described quantum mechanically. Then, in the coordinate representation, using corresponding single-particle wave functions, we have the density matrix elements:

\[ \rho(x', x) = \sum_i w_i \langle x' | i \rangle \langle i | x \rangle = \sum_i w_i \phi_i(x') \phi_i^*(x) \]  

(26)

see Eq. (19). Then, we calculate immediately the probability to find the particle at \( x \):

\[ P(x) \equiv \rho(x, x) = \sum_i w_i |\phi_i(x)|^2 \]  

(27)

in agreement with the standard rules of quantum mechanics. Similarly, in momentum representation,

\[ \rho(p', p) = \sum_i w_i \langle p' | i \rangle \langle i | p \rangle = \sum_i w_i \phi_i(p') \phi_i^*(p) \]  

(28)

which yields:

\[ P(p) \equiv \rho(p, p) = \sum_i w_i |\phi_i(p)|^2 \]  

(29)

i.e. the probability to find the particle with momentum \( p \).

Now, considering a function \( O \) – representing some observable, for example – which is defined over phase space, we obtain its average value by calculating:

\[ \overline{O} \equiv \int dx dp \, O(x, p) f(x, p) \]  

(30)

involving the one-body probability density function \( f \). The question then arises, whether there exists a corresponding quantum mechanical density function which yields the expectation value of operators
in the form of phase space integrals, generalizing Eq. (30). One possible answer is provided by the Wigner function [18, 19, 21]:

$$W(x, p) \equiv \int \mathrm{d}y \, e^{ipy/\hbar} \rho(x + y/2, x - y/2),$$

(31)
i.e. a particular Fourier transform of the density matrix in coordinate space. We remark that the momentum appearing as an argument of $W$ is the conjugate variable to the relative coordinate separating the pair of wave functions which enter $\rho$, cf. Eq. (26); we also indicate here explicitly the $\hbar$-dependence arising in the phase factor.

Furthermore, even though the Wigner function presents a real distribution, due to $W(x, p) = W^*(x, p)$, it may oscillate and indeed does so in most cases. Nevertheless, we easily obtain the following results which match their classical counterparts:

$$\int \frac{dp}{2\pi\hbar} \, W(x, p) = \rho(x, x) = P(x),$$

(32)

$$\int dx \, W(x, p) = \rho(p, p) = P(p).$$

(33)

This implies for the expectation values of functions of operators:

$$\langle O(\hat{x}) \rangle = \text{Tr} \hat{\rho} O(\hat{x}) = \int \frac{dx dp}{2\pi\hbar} \, O(x) W(x, p),$$

(34)

$$\langle O(\hat{p}) \rangle = \text{Tr} \hat{\rho} O(\hat{p}) = \int \frac{dx dp}{2\pi\hbar} \, O(p) W(x, p),$$

(35)

which should be compared to Eq. (30).

However, besides the fact that the Wigner function generally is not positive definite, the expected limitation of the classical/quantum correspondence of various (probability) density functions also shows up, when one considers operators of the form $O(\hat{x}, \hat{p})$. In this case, operator ordering becomes an issue and the above formulae have to be generalized with care. On the other hand, as we shall see shortly in Section 4, the appropriate semiclassical expansion of the Wigner function transport equation does yield the transport equation for the classical probability density function, which we discussed in Section 2.

4 Transport theory for quantum fields

As an example of a field theory we choose the O(4) linear $\sigma$-model, which has a long history of applications in various phenomenological contexts. Recently it has been argued by Wilczek that it represents the QCD chiral order parameter for $n_f = 2$ massless quark flavors [22]. Furthermore, it has been demonstrated by nonperturbative calculation that this model possesses a second order finite temperature phase transition between the spontaneously broken and symmetry restored phases [23]. Most interestingly, the effective mass and coupling go to zero at the phase transition with well-determined critical exponents. This may influence the hydrodynamic behavior of matter described by this model in interesting ways, which we presently study.

Here, our aim is to derive the quantum transport equations in the Hartree approximation. In particular, we will illustrate how an appropriate Wigner function can be introduced and how field theory aspects are related to our earlier considerations of classical transport theory.

To begin with, the O(4)-invariant $\sigma$-model action is defined by:

$$S[\phi] \equiv \int \mathrm{d}^4x \left( \frac{1}{2}(\partial \phi)^2 - \frac{1}{2} \mu^2 \phi^2 - \frac{1}{4!} \lambda(\phi^2)^2 \right),$$

(36)
with $\phi \equiv (\phi_1, \phi_2, \phi_3, \phi_4)$ and where the mass parameter is chosen with the ‘wrong’ sign, i.e. $\mu^2 < 0$. The corresponding potential, $V(\phi) \equiv \frac{1}{2} \mu^2 \phi^2 + \frac{1}{4} \lambda (\phi^2)^2$, is of the so-called ‘Mexican hat’ form which leads to spontaneous symmetry breaking at the tree level.

We parametrize the four-component vector $\phi$ in terms of three-component ‘pion’ and one-component ‘sigma’ fields, $\bar{\phi} \equiv (\bar{\pi}, \sigma')$, at each space-time point. Shifting the $\sigma'$-field by the vacuum expectation value $\sigma_0$ of $\bar{\phi}$, which is determined by the minimum of the potential,

$$\frac{dV(\bar{\phi})}{d\bar{\phi}} = (\mu^2 + \frac{1}{3!} \lambda \bar{\phi}^2) \bar{\sigma} = 0 \quad \implies \quad |\sigma_0| = \sqrt{-6\mu^2/\lambda} \ , \quad (37)$$

we define the fluctuation field $\sigma \equiv \sigma' - \sigma_0$.

The effect of this parametrization combined with the shift of one component by the vacuum expectation value is easily seen in the corresponding Heisenberg operator equations of motion,

$$\partial^2 \bar{\pi} + m_\pi^2 \bar{\pi} + \frac{\lambda}{3!} \bar{\pi}^2 \bar{\pi} = 0 \ , \quad (38)$$

$$\partial^2 \sigma + m_\sigma^2 \sigma + \frac{\lambda}{2} \sigma_0 \sigma^2 + \frac{\lambda}{3!} \sigma^3 = 0 \ , \quad (39)$$

which are obtained by varying the action $S[\bar{\phi}]$ with respect to $\bar{\phi}$, introducing parametrization and shift, and considering the fields as quantum field operators. Here we introduced the effective masses:

$$m_\pi^2 \equiv \frac{\lambda}{3!} (\sigma^2 + 2\sigma_0 \sigma) \ , \quad m_\sigma^2 \equiv \frac{\lambda}{3!} (\bar{\pi}^2 + 2\sigma_0^2) \ . \quad (40)$$

We observe that for $\sigma \to 0$ we have $m_\pi^2 \to 0$, while $m_\sigma^2 \to \lambda \sigma_0^2 / 3$, yielding one massive ‘radial’ mode together with three massless ‘Goldstone modes’, in accordance with the Goldstone theorem.

Taking the expectation value Eqs. (38) and (39), we observe that one-point functions, such as the mean field $\bar{\sigma}(x) \equiv \langle \sigma(x) \rangle$, generally are coupled to two-point functions, such as $\lim_{x \to x'} \langle \sigma(x) \sigma(x') \rangle$, and higher $n$-point functions. The coincidence limit of these Wightman functions produces divergences necessitating a renormalization procedure, which we will discuss at the end of this section.

In order to solve the resulting equations, one has to specify the density operator which enters the expectation values, e.g. $\langle \sigma \rangle \equiv \text{Tr } \rho \sigma$, omitting the operator signs used previously, cf. Section 3. This can be done, for example, on a fixed time-like hypersurface, which will be demonstrated for fermions in Section 6. Furthermore, we need to derive equations of motion for the higher $n$-point functions, since taking expectation values of nonlinear operator equations automatically generates an infinite hierarchy of equations, similarly as the Schwinger-Dyson equations for propagators or the BBGKY hierarchy in classical transport theory.

The simplest nonperturbative truncation of the hierarchy of Wightman function equations is produced by the Hartree approximation. It consists in factorizing the $n$-point functions into products of one- or two-point functions, properly taking into account all possible factorizations. For example, we obtain:

$$\langle \sigma_1 \sigma_2 \sigma_3 \rangle = \bar{\sigma}_1 \langle \sigma_2 \sigma_3 \rangle + \bar{\sigma}_2 \langle \sigma_1 \sigma_3 \rangle + \bar{\sigma}_3 \langle \sigma_1 \sigma_2 \rangle + \bar{\sigma}_1 \sigma_2 \sigma_3 \ , \quad (41)$$

where the subscripts refer to different space-time points. Note that the expectation value of any odd power of the proper quantum field vanishes, since we define $\bar{\sigma} \equiv \sigma - \bar{\sigma}$. – This approximation is known to be equivalent to summing all iterated bubbles (‘superdaisies’) in the Feynman diagram calculation of the vacuum effective action in a $\phi^4$-model [24].

Here we will furthermore assume that cross terms vanish, e.g. $\langle \bar{\pi} \sigma \rangle = 0$. It turns out to be consistent with this assumption to set $\langle \bar{\pi} \rangle = 0$, since the classical pion field obtains no source term. In distinction, the Eq. (39) yields the Klein-Gordon type mean field equation:

$$\left( \partial^2 + m_\sigma^2 + \frac{\lambda}{2} \bar{\sigma}^2 + \sigma_0 \bar{\sigma} + \langle \sigma^2 \rangle \right) \bar{\sigma} + \frac{\lambda}{2} \sigma_0 \langle \sigma^2 \rangle \equiv \left( \partial^2 + m_\sigma^2 + \delta m^2 \right) \bar{\sigma} + J = 0 \ , \quad (42)$$
where the two-point function adds to the nonlinear classical force term shifting the effective \( \sigma \)-mass, which contains \( \langle \bar{\pi}^2 \rangle \), and to a source term, \( \delta m^2 \) and \( J \), respectively. For a homogeneous system, for \( \delta m^2 \gg m_\sigma^2 \), and for sufficiently large \( \langle \pi_2^2 \rangle \), the mean field is \( \langle \sigma \rangle = -\sigma_0 \), indicating symmetry restoration with \( \langle \sigma' \rangle = 0 \).

Multiplying Eq. (38) by one more power of the field operator from the left, before applying the Hartree approximation as before, we obtain the two-point function equation:

\[
\left( \partial_2^2 + m_{\sigma,2}^2 + \lambda \sigma_0 \bar{\sigma}_2 + \frac{\lambda}{2} (\bar{\sigma}_2^2 + \langle \pi_2^2 \rangle) \right) \langle \pi_1 \pi_2 \rangle = 0 , \tag{43}
\]

where subscripts “1,2” refer to two different space-time points. Here we also used Eq. (42) at point “2”, multiplied by \( \bar{\sigma}_1 \), in order to simplify Eq. (43) considerably. A similar equation follows from Eq. (38) for the ‘pion’ modes.

The next step consists in defining a suitable Wigner operator,

\[
W_{ab}(x,p) \equiv \int \frac{d^4 y}{(2\pi)^4} e^{-ip\cdot y} \Phi_a(x+\frac{1}{2} y) \Phi_b(x-\frac{1}{2} y) , \tag{44}
\]

where \( \Phi \equiv (\bar{\pi}, \sigma) \) and \( \bar{\Phi} \equiv \Phi - \bar{\Phi} \). This should be compared to the quantum mechanical Wigner function introduced previously, Eq. (B1). We remark that it might be useful for some applications not involving vacuum properties to normal-order the field operators in the definition of \( W \), see, for example, Section 6.

Illustrating the usefulness of \( W \), we write down the energy-momentum tensor for the O(4)-model:

\[
T_{\mu\nu} = \left( \partial_\mu \bar{\Phi} \cdot (\partial_\nu \Phi) - g_{\mu\nu} \left( \frac{1}{2} (\partial \Phi)^2 - \frac{1}{2} \mu^2 \Phi^2 - \frac{1}{4!} \lambda (\Phi^2)^2 \right) \right) \tag{45}
\]

\[
= \int d^4 p \left( p_\mu p_\nu + \frac{1}{4} \partial_{\mu\nu} \partial_x^2 - \frac{1}{2} g_{\mu\nu} (p^2 + \frac{1}{4} \partial_x^2) \right) \langle W_{aa}(x,p) \rangle + g_{\mu\nu} \int d^4 p' \langle W_{aa}(x,p) W_{bb}(x,p') \rangle , \tag{46}
\]

to which must be added the purely classical terms plus mean field dependent interaction terms \( \propto \lambda \) involving \( W \), which can all be further evaluated in Hartree approximation.

Heading for the transport theory, we will express Eq. (13) in terms of the Wigner operator. We introduce the abbreviation:

\[
\mathcal{M}^2(x) \equiv m_\sigma^2(x) + \lambda \sigma_0 \bar{\sigma}(x) + \frac{\lambda}{2} \left( \bar{\sigma}_x^2 + \langle \pi_2^2 \rangle \right) . \tag{47}
\]

The two-point functions contained here, \( \langle \bar{\pi}^2 \rangle \) in \( m_\sigma^2 \) and \( \langle \pi_2^2 \rangle \), respectively, can be rewritten using the Wigner operator, e.g.:

\[
\langle \bar{\pi}_x^2(x) \rangle = \int d^4 p \langle W_{\sigma\sigma}(x,p) \rangle . \tag{48}
\]

Then we obtain instead of Eq. (43):

\[
\left( \frac{1}{4} \partial_x^2 - p^2 + i p \cdot \partial_x + \exp(-\frac{i}{2} \partial_x \cdot \partial_p) M^2(x) \right) \langle W_{\sigma\sigma}(x,p) \rangle = 0 , \tag{48}
\]

where the \( x \)-derivative in the exponential acts only on \( M^2 \). In the derivation of this result, and similarly for the ‘pions’, one frequently makes use of suitable partial integrations under the momentum integral from \( W \), as well as expressing the shifted argument of \( M^2 \) by a translation operator giving rise to the exponential [3, 25].

The final step consists in adding to/subtracting from the complex Eq. (48) its adjoint. This yields the transport equation:

\[
\left( p \cdot \partial_x - \frac{1}{\hbar} \sin \left( \frac{\hbar}{2} \partial_x \cdot \partial_p \right) \mathcal{M}^2(x) \right) \langle W_{\sigma\sigma}(x,p) \rangle = 0 , \tag{49}
\]
together with a generalized mass-shell constraint:

\[
\left( \frac{p^2}{4} - \frac{\hbar^2}{4} \partial_x^2 - \cos \left( \frac{\hbar}{2} \partial_x \cdot \partial_p \right) \mathcal{M}^2(x) \right) \langle W_{\sigma \sigma}(x, p) \rangle = 0 ,
\]

where the appropriate powers of \( \hbar \) are reinserted. We stress that the covariant transport equation alone does not suffice to determine the dynamics.

A systematic semiclassical expansion of Eqs. (49)-(50) in powers of \( \hbar \) becomes feasible now. To leading order we obtain:

\[
\left( p \cdot \partial_x - \frac{1}{2} \partial_x \cdot \partial_p \mathcal{M}^2(x) \right) \langle W_{\sigma \sigma}(x, p) \rangle = 0 ,
\]

\[
\left( p^2 - \mathcal{M}^2(x) \right) \langle W_{\sigma \sigma}(x, p) \rangle = 0 ,
\]

which are indeed of the form of a classical Vlasov and mass-shell equation, respectively, discussed in Section 2.1. Note, however, that the effective mass or effective potential, appearing here contains contributions from the mean field, which have to be determined selfconsistently from Eq. (42), and ‘selfenergy’ terms, which we will discuss shortly.

We remark that effects of the higher-order \( \hbar \)-corrections to Eq. (51) and (52) are largely unexplored, since numerically the corresponding higher derivatives lead to strong instabilities. This provides part of the motivation to develop an analytical approach solving the exact quantum transport equations, the first step of which is described in Section 6.

Furthermore, we observe that the equations derived here do not yield any collision terms. This was to be expected, since the Hartree approximation, which is essentially equivalent to the Gaussian approximation in the Schrödinger functional approach, leads to an effective ‘quantum’ Hamiltonian evolution which is nondissipative \[11\]. Only an improved treatment of correlation terms, such as the last term in Eq. (44), will go beyond the collisionless Vlasov dynamics obtained here, see, for example, Refs. \[7, 8\].

Finally, we turn to the ‘selfenergy’ terms mentioned above and to the divergences caused by them, in particular. These terms involve integrals of the kind \( \int d^4p (W_{\sigma \sigma}(x, p)) \). Let us consider the simplest case of a density operator projecting on the vacuum state, \( \rho_{\text{vac}} = |0\rangle \langle 0| \). Then, for a generic free scalar field, the vacuum Wigner function is:

\[
W_{\text{vac}}(p) \equiv \text{Tr} \rho_{\text{vac}} W(x, p) = \langle 0 | W(x, p) | 0 \rangle = (2\pi)^{-3} \delta(p^2 - m^2) ,
\]

where \( m \) denotes its physical mass, and the calculation proceeds, for example, by expanding the field operators appearing in the definition of the Wigner operator in terms of creation and annihilation operators. In this case, we obtain:

\[
\int d^4p W_{\text{vac}}(p) = \int \frac{d^3p}{(2\pi)^3 \omega_p} \equiv I_m ,
\]

with \( \omega_p \equiv (p^2 + m^2)^{1/2} \). This ‘selfenergy’ integral is quadratically divergent.

Since divergences arise from field operators at coinciding points and, as a first approximation, independently of which point it is, we rewrite Eq. (50) accordingly, using Eqs. (40) and (47):

\[
0 = (-p^2 + m_{\text{bare}}^2 + \frac{\lambda}{2} I_m + \frac{\lambda}{2} \int d^4k \{ W(x, k) - W_{\text{vac}}(k) \} W(x, p) - W_{\text{vac}}(p) + W_{\text{vac}}(p))
\]

\[
\equiv \left( -p^2 + m_r^2 + \frac{\lambda}{2} \int d^4k W_r(x, k) \right) (W_r(x, p) + W_{\text{vac}}(p)) ,
\]
with $W(x,p) \equiv \langle W_\sigma(x,p) \rangle$ and the Lagrangian ‘bare’ mass $m^2_{\text{bare}} \equiv \frac{\lambda}{3} \sigma_0^2$, cf. Eq. (41). Here we left out all ‘pion’ and mean field contributions to keep things transparent. These can be added following the same strategy, while the gradient corrections would force us to keep $W(x,p)$ unrenormalized at first sight.

Note that we added and subtracted suitably the divergent term in Eq. (55), thus introducing in Eq. (56) the renormalized mass and Wigner function, $m_r$ and $W_r$, respectively. We complete the renormalization by identifying the physical mass:

$$m^2 = m^2_r + \frac{\lambda}{2} \int \! d^4k \, W_r(x,k) \ .$$

(57)

With this, the Eq. (56) assumes a simple form, $(p^2 - m^2)W_r(x,p) = 0$, indicating that $W_r$ is (in our first approximation) an on-shell distribution:

$$W_r(x,p) = \delta(p^2 - m^2)f(x,p) \ ,$$

(58)

such as a Bose-Einstein distribution with (weakly) $x$-dependent parameters, cf. Section 2.1. Neglecting this dependence altogether and assuming a simple thermal distribution turns Eq. (57) into a mass-gap equation with $m^2_r$ as input parameter and a resulting temperature dependent mass $m^2(T)$.

A mass renormalization as performed here is sufficient to render the mean field Eq. (42) and the operator of two-point function Eq. (43) finite. The gradient corrections in the transport and generalized mass-shell constraint, however, indicate that the renormalized Wigner (or equivalently the two-point) function has to be be renormalized separately, subtracting as above, and thus closing the set of equations. Generally, the situation gets more complicated, when the $x$-dependence of $W_r$ has to be taken serious and, thus, the physical mass becomes space-time dependent, e.g. in the case of a system which evolves from strongly inhomogeneous initial conditions. We will not address these issues here, but turn to a related physical effect in the next section.

5 Particle production by classical fields

The color string, rope, or flux tube models are widely used in phenomenological descriptions of particle production, in high-energy nuclear collisions in particular [4]. The basis here is Schwinger’s nonperturbative calculation of vacuum decay due to charged particle production in constant and homogeneous external electric fields [26, 27]. Considering the inhomogeneous and rapid evolution of the system during a heavy-ion collision, the basic assumptions of this picture are questionable.

Since an appropriate generalization of Schwinger’s result is still not available, one may as well consider the perturbative evaluation of particle production, however, for arbitrarily varying fields. This was originally discussed in Ref. [28] for QCD in abelian dominance approximation. There, also the ensuing modification of semiclassical transport equations, cf. Section 4, and the related vacuum polarization current were obtained. Here we briefly recall part of this calculation which is based on an $O(g^2)$ solution of the quark transport equation in external color fields.

Simplifying the notation, we consider electrically charged Dirac fermions (mass $m$) in arbitrary electromagnetic fields. We define the Fourier transformed vacuum Wigner function:

$$f_{\text{vac}}(q,p) \equiv \int \! d^4x \! d^4y \, e^{i q \cdot x - i p \cdot y} \psi(x + \frac{1}{2}y)\bar{\psi}(x - \frac{1}{2}y)|0\rangle$$

(59)

$$= -(2\pi)^5 \delta^4(q) \delta(p^2 - m^2)\theta(-p^0)(\gamma \cdot p + m) \ ,$$

(60)

which is a 4x4 spinor matrix. The final form here follows similarly as in the scalar case, Eq. (53), using a standard expansion of the field operators [27]. In distinction to the fermion Wigner function introduced in the following Section 6, Eq. (69), where the notation is explained in more detail, we
presently do not normal-order the field operators, since we are interested to study the response of the vacuum, i.e. the modification of $f_{\text{vac}}$, due to external fields.

Our task is to somehow solve the full quantum transport equation which determines the fermionic Wigner function. For the present study the transport equation based on the linear form of the Dirac equation is most useful [3, 23]. – The quadratic form yields transport equations which are closer to the usual form, such as in the scalar field case studied in Section 4. – Presently, following Fourier transformation, we have:

$$
\left( \gamma \cdot (p + \frac{1}{2} q) - m \right) f(q, p) = g \int \frac{d^4q}{(2\pi)^4} \gamma \cdot A(q') f(q' - q, p - \frac{1}{2} q') ,
$$

(61)

where $A^\mu$ denotes the vector potential of the external field. This equation has to be solved together with the constraint:

$$
f^\dagger(q, p) = \gamma^0 f(-q, p) \gamma^0 ,
$$

(62)

which follows from the definition of the fermion Wigner function; equivalently, one may solve Eq. (61) simultaneously with its adjoint.

The perturbative solution in powers of the coupling constant $g$ of Eq. (61) can be found iteratively, starting with the zeroth order solution $f_{\text{vac}}$ as input on the right-hand side.

Thus, at first order one finds the induced vacuum current $J_{(1)}^\mu(q) = \int d^4p \text{Tr} \gamma^\mu f_{(1)}(q, p)$, which can be written explicitly involving the first order (one-loop) QED vacuum polarization tensor and reproduces the known result [27, 28]. Furthermore, since there is a gauge invariant version of Eq. (61) based on a modified definition of the Wigner function [6, 25, 28], this type of calculation might allow to calculate the polarization tensor in a manifestly gauge invariant way.

For our present purposes, the second order solution of the equations is most interesting. The main algebraic complication arises from Eq. (62), which has to be satisfied order by order. We do not give the lengthy expressions here, but turn to the calculation of the spectrum of the produced particles. We have to use reduction formulae, in order to relate the ‘in’ field operators defining the Wigner function to asymptotic on-shell ‘out’ particle states [27]. In this way, we obtain:

$$
2\omega_p \frac{dN}{d^3p} \equiv N_{(2)}(p) = \frac{1}{2(2\pi)^3} \sum_r \bar{u}^r(p)(\gamma \cdot p - m)f_{(2)}(0, p)(\gamma \cdot p - m)u^r(p)
$$

(63)

$$\begin{align*}
= & \frac{g^2}{4\pi^2} \int d^4q \theta(q^0 - p^0)\delta \left( (q - p)^2 - m^2 \right) \\
& \cdot \left( \vec{E}(q) \cdot \vec{E}(-q) - \vec{B}(q) \cdot \vec{B}(-q) - |A(q) \cdot (q - 2p)|^2 \right)
\end{align*}
$$

(64)

where the electromagnetic fields and the vector potential enter; here $\omega_p \equiv \sqrt{p^2 + m^2}$ and $u^r (\bar{u}^r)$ denotes the single-particle (adjoint) spinor wave function, respectively, in the normalization of Ref. [27]. Also the last term of this result is gauge invariant, as can be demonstrated using the constraints of the integral and the fact that the particles are on-shell. Interestingly, with the external fields left completely general, still there is no spatial dependence of this spectrum.

Finally, we calculate the vacuum decay rate $\mathcal{R}$ per unit four-volume, using a more suitable intermediate form of the spectrum result, Eq. (64):

$$
\mathcal{R} = 2 \int \frac{d^3p}{2\omega_p} N_{(2)}(p)
$$

(65)

$$\begin{align*}
= & \frac{g^2}{\pi^2} \int \frac{d^3p}{\omega_p} \frac{d^3p'}{\omega_p} d^4q \delta^4(q - p - p') \left( -\frac{1}{2} \gamma^\mu g_{\mu\nu} + p_\mu p'\nu + p'_\mu p_\nu \right) A^\mu(q)A^\nu(-q) \\
= & \frac{g^2}{12\pi} \int d^4q \theta(q^2 - 4m^2) \left( 1 - \frac{4m^2}{q^2} \right)^{1/2} \left( 1 + \frac{2m^2}{q^2} \right) \left( |\vec{E}(q)|^2 - |\vec{B}(q)|^2 \right)
\end{align*}
$$

(66)

(67)
which confirms the result obtained by other methods in Ref. [27]. Clearly, particle production is an 
electric field effect.

For completeness, let us quote Schwinger’s nonperturbative result [26]:

\[ R = \frac{g^2 E^2}{4\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^2} \exp \left( -\frac{n\pi m^2}{|gE|} \right), \]  

(68)

which forms the basis to date of the phenomenological models mentioned in the beginning of this 
section. For reasons which we discussed, a future calculation bridging the gap between our perturbative 
calculation for arbitrary fields and the nonperturbative result for constant electric fields would be 
extremely useful.

6 Fluid dynamics of relativistic quantum dust

In this section, we will study the relation between relativistic hydrodynamics and the full quantum 
evolution of a free matter field [29]. In particular, we try to answer how a free fermion field and 
its energy-momentum tensor will evolve, given arbitrary initial conditions and especially those of the 
Landau and Bjorken models.

In the absence of interactions, decoherence or thermalization may be present in the initial state, 
corresponding to an impure density matrix, but is followed by unitary evolution. We consider this as a 
“quantum dust” model of the expansion of matter originating from a high energy density preparation 
phase, which the Landau and Bjorken models describe classically [3, 4].

Our approach is independent of the nature of the field, as long as it obeys a standard wave 
equation. To be definite, we choose to work with Dirac fermions and comment about neutrinos 
later. We introduce the spinor Wigner function, i.e., a (4x4)-matrix depending on space-time and 
four-momentum coordinates:

\[ W_{\alpha\beta}(x; p) \equiv \int \frac{d^4 y}{(2\pi)^4} e^{-ip\cdot y} \langle : \bar{\psi}_{\beta}(x + y/2) \psi_{\alpha}(x - y/2) : \rangle, \]  

(69)

where the expectation value refers to the (mixed) state of the system; without interactions, the vacuum 
plays only a passive role and, therefore, is eliminated by normal-ordering the field operators. Note 
that the normalization of the Wigner function is a matter of convention, and we have chosen the most 
convenient one for us here.

All observables can be expressed in terms of the Wigner function here. In particular, the (unsym-
metrized) energy-momentum tensor:

\[ \langle : T_{\mu\nu}(x) : \rangle \equiv i\langle : \bar{\psi}(x) \gamma_{\mu} \overset{\leftrightarrow}{\partial_{\nu}} \psi(x) : \rangle = \text{tr} \gamma_{\mu} \int d^4 p p_{\nu} W(x; p), \]  

(70)

where \( \overset{\leftrightarrow}{\partial} \equiv \frac{1}{2}(\overset{\partial}{\partial} - \overset{\partial}{\partial}) \) and with a trace over spinor indices (conventions as in [23]). Furthermore, the 
dynamics of \( W \) reduces to the usual phase space description, as in Section 2, in the classical limit [23].

The following study is based on the simple fact that the propagation of the free fields entering 
in Eq. (69) from one time-like hypersurface to another is described by the Schwinger function. It is 
the solution of the homogeneous Dirac equation, \( [\gamma \cdot \partial - m] S(x, x') = 0 \), for the initial condition 
\( S(\vec{x}, \vec{x}', x^0 = x'^0) = -i\gamma^0 \delta^3(\vec{x} - \vec{x}') \). Thus, \( \psi(x) = i \int d^3 \vec{x}') S(x, x') \gamma^0 \psi(x') \), and similarly for the adjoint. 
An explicit form is:

\[ iS(x, x') = iS(x - x' = \Delta) = (i\gamma \cdot \partial_{\Delta} + m) \int \frac{d^3 k}{(2\pi)^3 2\omega_k} \left( e^{-ik \cdot \Delta} - e^{-ik \cdot \Delta} \right), \]  

(71)

where \( k_\pm \equiv (\pm \omega_k, \vec{k}) \) and \( \omega_k \equiv (\vec{k}^2 + m^2)^{1/2} \).
Making use of Eqs. (70) and (71), we relate the Wigner function at different times, \( t = x^0, x'^0 \):

\[
W(x; p) = \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot x} \delta^\pm(p_\pm^k) \delta^\pm(p_\mp^k) \int d^3x' e^{ik \cdot x'} \int dp' A_0(p'^k) \gamma^0 W(x'; p') \gamma^0 A(p'^k),
\]

(72)

where \( p_\pm^k \equiv p \pm \frac{\hbar}{2}, \delta^\pm(q) \equiv \pm \delta(q^2 - m^2) \) (for \( q^0/q'0 = \pm 1 \)), \( A(q) \equiv \gamma \cdot q + m \), and \( p'^\mu \equiv (p^0, \vec{p}) \).

The Eq. (72) implies that the Wigner function obeys a generalized mass-shell constraint and a proper free-streaming transport equation:

\[
[p^2 - m^2 - \frac{\hbar^2}{4} \partial_x^2] W(x; p) = 0,
\]

(73)

\[
p \cdot \partial_x W(x; p) = 0,
\]

(74)

separately for each matrix element. The reinserted \( \hbar \) indicates the important quantum term in the equations, which otherwise have the familiar classical appearance.

Thus Eq. (72) presents an integral solution of the microscopic transport equations for a given initial Wigner function. Furthermore, a semiclassical approximation of the Schwinger function may be used to generate an integral solution of the corresponding classical transport problem.

Next, we decompose the Wigner function with respect to the standard basis of the Clifford algebra, \( W = F + i\gamma^5 P + \gamma^\mu \gamma^5 A_\mu + \frac{1}{2} \sigma^{\mu\nu} S_{\mu\nu} \), i.e., in terms of scalar, pseudoscalar, vector, axial vector, and antisymmetric tensor components [27]. The functions, \( F \equiv \frac{1}{4} tr W, P \equiv -\frac{1}{4} i tr \gamma^5 W, \gamma^\mu \equiv \frac{1}{4} tr \gamma^\mu W, A_\mu \equiv \frac{1}{4} tr \gamma^5 \gamma_\mu W, \) and \( S_{\mu\nu} \equiv \frac{1}{4} tr \sigma_{\mu\nu} W \), which represent physical current densities, are real, due to \( W^\dagger = \gamma^0 W \gamma^0 \) [25]. They individually obey Eqs. (73) and (74).

We assume \( P = 0 = A_\mu \), i.e., we consider a spin saturated system for simplicity. From this point on, a corresponding study of (approximately) massless Standard Model \( \nu_L, \bar{\nu}_R \) neutrinos differs, and is simpler, since \( V_\mu = A_\mu \), while all other densities vanish identically; see, e.g., Ref. [30]. Then, using the ‘transport equation’ which follows directly from the Dirac equation applied to \( W \),

\[
[\gamma \cdot (p + \frac{\hbar}{2} \partial_x) - m] W(x; p) = 0,
\]

and decomposing it accordingly, the following additional relations among the remaining densities are obtained:

\[
\nu^\mu(x; p) = \frac{mp^\mu}{p^2} F(x; p),
\]

(75)

\[
S^{\mu\nu}(x; p) = \frac{1}{2p^2} (p^\nu \partial_x^\mu - p^\mu \partial_x^\nu) F(x; p).
\]

(76)

Note that \( S^{\mu\nu} \) is intrinsically by one order in \( \hbar \) smaller than the other two densities.

We conclude that presently the dynamics of the system is represented completely by the scalar phase space density \( F \), which obeys the same transport equations as the full \( W \) itself. Using Eqs. (70) and (75), we obtain in particular:

\[
\langle : T^{\mu\nu} (x) : \rangle = 4m \int d^4p \frac{p^\mu p^\nu}{p^2} F(x; p),
\]

(77)

which is symmetric and conserved, \( \partial_\mu T^{\mu\nu}(x) = 0 \), on account of Eq. (74). Furthermore, this implies the ‘equation of state’:

\[
\langle : T^{00} (x) : \rangle - \sum_{i=1}^3 \langle : T^{ii} : \rangle = 4m \int d^4p F(x; p),
\]

(78)

which relates energy density and pressure(s). However, applying Eq. (73), we find that this relationship evolves in a wavelike manner, driven by off-shell contributions to the evolving \( F \):

\[
\partial_x^2 \langle : T^{\mu\mu} (x) : \rangle = 16m \int d^4p (p^2 - m^2) F(x; p).
\]

(79)
This differs from classical hydrodynamics with a fixed functional form of the equation of state. Eqs. (77)-(79) hold independently of the initial state, of course, if it evolves without further interaction.

Making use of Eq. (72) in Eq. (71), we now calculate the energy-momentum tensor at any time in terms of the initial scalar density. Employing the decomposition of the Wigner function and commutation and trace relations for the $\gamma$ matrices, as well as Eqs. (73)-(76), we obtain:

$$T^{\mu\nu}(x) = 8m \int \ldots \int \frac{p^\mu p^\nu}{p^2} \left( p^2 + \frac{m^2}{p^2} \left(p^0 p^0 + \vec{p}^2\right) - \frac{1}{4p^2} \left(\left(p^0 k^0\right)^2 - \vec{p}^2 \vec{k}^2\right) + p^0 p^0 k^2 + \frac{p^0}{p^0} \left(k^0 p^0\right)\right) F(x'; \vec{p}, p^0),$$

(80)

where $\int \ldots \int \equiv (2\pi)^{-3} \int d^4p \int d^4k \, e^{-ik\cdot x} \delta^\pm (p^+ - p^-) \int d^3x' e^{ik\cdot x'} \int dp^0$; we also made use of partial integrations and the $\delta$-function constraints. The three terms on the right-hand side stem from the scalar, vector, and antisymmetric tensor components of the initial Wigner function, respectively.

If the initial distribution is an isotropic function of the three-momentum, then $T^{\mu\nu}$ is diagonal at all times, implying that the absence of flow in the initial state will be preserved.

Indeed, we expect the (non-)flow features of the initial distribution to be preserved during the evolution, due to the absence of interactions. Kinetic energy from microscopic particle degrees of freedom will not be converted into collective motion. An interesting question is, how the classical hydrodynamic acceleration of fluid cells due to pressure gradients arises in our present model after coarse graining. This has not been studied yet. Recalling earlier work on the hydrodynamic representation of quantum mechanics, e.g. Refs. [3], [4], and recently deduced classical fluid behavior of quantum fields in WKB approximation [5], however, we study the full quantum evolution here.

We are particularly interested in the exact evolution of $T^{\mu\nu}$, assuming a particle-antiparticle symmetric initial state. This is believed to hold, for example, close to midrapidity in the center-of-mass frame of central high-energy collisions [6]. It implies that the initial $F$ is an even function of the energy variable, $F(x'; \vec{p}, p^0) = F(x'; \vec{p}, -p^0)$. While Eq. (80) allows general initial conditions, we follow the implicit on-shell assumption in classical hydrodynamic models:

$$F(x'; \vec{p}, p^0) = (2\pi)^{-3} m \delta(p^2 - m^2) \left(\Theta(p^0) F(x'; \vec{p}, p^0) + \Theta(-p^0) F(x'; \vec{p}, -p^0)\right).$$

(81)

Fermion blackbody radiation is described by $F(x'; \vec{p}, p^0) \equiv f(p^0/T(x'))$, where $T$ denotes the local temperature, and with $\int f(s) = (e^s + 1)^{-1}$; this is easily illustrated with the help of Eqs. (77) and (81).

Implementing Eq. (81), we obtain the simpler result:

$$\langle \, T^{\mu\nu}(x) \, \rangle = \int \frac{d^3x' d^3k}{(2\pi)^3} \frac{p^\mu p^\nu \cos\left[k\cdot(x' - x)\right]}{\omega_p \omega_+ \omega_-} F(x'; \vec{p}, \omega_p)$$

(82)

$$\cdot\left\{\left((\omega_+ + \omega_-)^2 - \vec{k}^2\right) \cos[(\omega_+ - \omega_-)t] - \left((\omega_+ + \omega_-)^2 - \vec{k}^2\right) \cos[(\omega_+ - \omega_-)t] \right\},$$

where $t \equiv x' - x \equiv (\vec{p}^2 + m^2)^{1/2}$, $\omega_p \equiv ((\vec{p}^2 + \vec{k}^2)^{1/2} + m^2)^{1/2}$; furthermore, $p^0 \equiv \frac{1}{2} \omega_+ \pm \omega_-$. With “+” when multiplying the first and “−” when multiplying the second term of the difference, respectively. Depending on geometry and initial state, further integrations can be done analytically.

Consider a $(1+1)$-dimensional system for illustration, assuming that the particles are approximately massless, i.e. $\omega_p \approx |p|$, and that $F$ is even in $p$ (no flow). Specializing to a kind of Landau initial condition, the distribution is prepared on a fixed timelike hypersurface at $t = 0$ [2]. We find the ultrarelativistic equation of state for the only nonvanishing components of $T^{\mu\nu}$, $\epsilon \equiv T^{00} = T^{11} \equiv P$ $(d = 1 + 1)$, which are calculated as a momentum integral following Eq. (82):

$$T^{00}(x, t) = 2 \int \frac{dp}{2\pi} |p| \left(F(x - t; |p|) + F(x + t; |p|)\right)$$

(83)
\[
= \frac{1}{2} \left( T_{00}(x-t, t = 0) + T_{00}(x+t, t = 0) \right), \tag{84}
\]
i.e., a superposition of wavelike propagating momentum contributions in accordance with Eq. (79).

Similarly, a kind of Bjorken initial condition can be specified on a surface of constant proper time \( \tau \). A transformation of Eq. (83) to space-time rapidity and proper time coordinates yields:

\[
T_{00}(y, \tau) = 2 \int \frac{dp}{2\pi} |p| \left( F(-\tau_0 e^{-y/2+\ln \tau/\tau_0}; |p|) + F(\tau_0 e^{y/2+\ln \tau/\tau_0}; |p|) \right), \tag{85}
\]
since \( x \equiv \tau \sinh y/2 \) and \( t \equiv \tau \cosh y/2 \) (\( \tau \geq \tau_0 > 0 \)).

Our results for free fermions show the free-streaming behavior of classical dust, associated with the independent propagation and linear superposition of the momentum contributions to the scalar component \( F \) of the Wigner function here. In particular, the shape function of each mode is preserved and translated lightlike (with dispersion for massive particles). Due to the assumed momentum symmetry, the initial distribution will separate into two components after a finite time, travelling into the forward and backward direction, respectively, with a corresponding dilution at the center. For example, an initial distribution of Gaussian shape will separate into two corresponding humps.

We recall that \( T_{\mu\nu} \) being diagonal implies the absence of ideal hydrodynamic flow, given \( \epsilon = (d-1)P \). This does not depend on whether the initial state is on- or off-shell, see Eq. (80). Therefore, any hydrodynamic behavior must be the effect of a peculiarity of the semiclassical limit \([9]\), of coarse graining \([11, 12, 13]\), or of interactions \([32]\), or a combination of these.

Despite the apparently classical evolution, however, all initial state quantum effects are incorporated and preserved. If the initial dimensionless distribution \( F \) has a dependence on products of momentum and space-time variables, which is characteristic for matter waves, such terms invoke a factor \( 1/\hbar \). Similarly, if it is thermal \( (T) \) but includes the finite size \( (L) \) shell effects or global constraints, then there are quantum corrections involving \( LT/\hbar \) \((k_B = c = 1)\) \([33]\). For typical values of \( LT/\hbar \approx 1 \) these latter corrections are known to lead to corrections on the order of \( 30\% \) in the thermodynamical quantities. They have not been included in semiclassical transport or classical hydrodynamic models of high-energy (nuclear) collisions, but may be large. Here the quantum dust model provides a valuable testing ground to assess the importance of these quantum effects.

Let us summarize briefly the results and perspective of this section:

- Based on the Schwinger function, we obtain the solution of the free quantum transport problem to quadratures for arbitrary on- or off-shell initial conditions.

- In 1+1 dimensions free fermions, i.e. their observables embodied in the energy-momentum tensor, collectively behave like classical dust showing no flow effects, if there is no flow present in the initial condition. Corresponding analytical results in three dimensions can be obtained and will be discussed elsewhere.

- The method presented here may lead to an efficient way of treating interacting particles. Especially when a low-order perturbative expansion is meaningful, interactions could be incorporated in a multiple scattering expansion. With free propagation in between scattering events, a consistent and conceptually simple space-time description of transport phenomena seems feasible.

Thus we conclude our introductory lectures on relativistic quantum transport theory, which should convey some of its basic concepts and hopefully will lead to some of the interesting topics for further study.

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References

[1] E. Fermi, Progr. Theor. Phys. 5, 570 (1950); Phys. Rev. 81, 683 (1951).

[2] L. D. Landau, Izv. Akad. Nauk SSSR, Ser. fiz., 17, 51 (1953); S. Z. Belenkij and L. D. Landau, N. Cim., Suppl., 3, 15 (1956).

[3] J. D. Bjorken, Phys. Rev. D27, 140 (1983).

[4] See the series of proceedings of the International Conferences on Ultra-Relativistic Nucleus-Nucleus Collisions; e.g., Quark Matter ’99, eds. L. Riccati, M. Masera and E. Vercellini (Elsevier, Amsterdam, 1999).

[5] S. Weinberg, Gravitation and Cosmology (Wiley, New York, 1972).

[6] H.-Th. Elze and U. Heinz, Phys. Rep. 183, 81 (1989).

[7] D. Boyanovsky, F. Cooper, H. J. de Vega and P. Sodano, Phys. Rev. D58, 025007 (1998).

[8] J.-P. Blaizot and E. Iancu, Nucl. Phys. B557, 183 (1999); The Quark-Gluon Plasma: Collective Dynamics and Hard Thermal Loops, submitted to Physics Reports, hep-ph/0101103.

[9] G. Domenech and M. L. Levinas, Physica A278, 440 (2000).

[10] W. Zurek (ed.), Complexity, Entropy and the Physics of Information (Addison-Wesley, Reading, Mass., 1990).

[11] H.-Th. Elze, Nucl. Phys. B436, 213 (1995); Phys. Lett. B369, 295 (1996); quant-ph/9710063.

[12] A. M. Lisewski, On the classical hydrodynamic limit of quantum field theories, quant-ph/9905014.

[13] T. A. Brun and J. B. Hartle, Phys. Rev. D60, 123503 (1999).

[14] S. R. de Groot, W. A. van Leeuwen and Ch. G. van Weert, Relativistic Kinetic Theory (North-Holland, Amsterdam, 1980).

[15] H. Goldstein, Classical Mechanics (Addison-Wesley, Cambridge, Mass., 1953), Ch. 8-8.

[16] L. P. Csernai Introduction to Relativistic Heavy Ion Collisions (John Wiley & Sons, New York, 1994); see the lecture notes by L. P. Csernai in these proceedings.

[17] H.-Th. Elze, J. Rafelski and L. Turko, Phys. Lett. B506 (2001) 123.

[18] R. P. Feynman, Statistical Mechanics (Benjamin/Cummings, Reading, Mass., 1972).

[19] D. Han, Y. S. Kim and M. E. Noz, Am. J. Phys. 67 1999 61.

[20] J. von Neumann, Die mathematischen Grundlagen der Quantenmechanik Springer, Berlin, 1932; translated in: Mathematical Foundation of Quantum Mechanics (Princeton University, Princeton, 1955).

[21] P. Carruthers and F. Zachariasen, Rev. Mod. Phys. 55 (1983) 245.

[22] K. Rajagopal and F. Wilczek, Nucl. Phys. B399 (1993) 395.

[23] M. Reuter, N. Tetradis and C. Wetterich, Nucl. Phys. B401 (1993) 567; H.-Th. Elze, Nucl. Phys. A566 (1994) 571c.
[24] J. M. Cornwall, R. Jackiw and E. Tomboulis, Phys. Rev. D10 (1974) 2428.

[25] D. Vasak, M. Gyulassy and H.-Th. Elze, Ann. Phys. (N.Y.) 173, 462 (1987).

[26] J. Schwinger, Phys. Rev. 82 (1951) 664.

[27] C. Itzykson and J.-B. Zuber, Quantum Field Theory (McGraw-Hill, New York, 1980).

[28] H.-Th. Elze, Z. Phys. C38 (1988) 211.

[29] H.-Th. Elze, Fluid dynamics of relativistic quantum dust, J. Phys. G (2002), in press; [hep-ph/0201007].

[30] H.-Th. Elze, T. Kodama and R. Opher, Phys. Rev. D63 (2001) 013008.

[31] E. Madelung, Z. Phys. 40 (1926) 322; G. Holzwarth and D. Schütte, Phys. Lett. B73 (1978) 255; S. K. Ghosh and B. M. Deb, Phys. Rep. 92 (1982) 1.

[32] L. M. A. Bettencourt, F. Cooper and K. Pao, Hydrodynamic scaling from the dynamics of relativistic quantum field theory, [hep-ph/0109108].

[33] H.-Th. Elze and W. Greiner, Phys. Lett. B179, 385 (1986); Phys. Rev. A33, 1879 (1986).