A characterization of entropy dimensions of minimal $\mathbb{Z}^3$-SFT

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Abstract

We prove in this text that the possible entropy dimensions of minimal $\mathbb{Z}^3$-SFT are the $\Delta_2$-computable numbers in $[0, 2]$, using Goldbach’s theorem on Fermat numbers.

1 Introduction

Multidimensional subshifts of finite type (SFT) are discrete dynamical systems described as the action of the shift on a compact set of symbol displays on an infinite regular grid. This set is defined by a finite set of local rules on the symbols. It has been established that their dynamicals are related to computability. The most emblematic result in this sense is the characterization of the possible values of the entropy of these systems, by M. Hochman and T. Meyerovitch [HM10], as the set of $\Pi_1$-computable numbers. The $\Pi_1$-computability means that the number can be approximated from above by a sequence of rational numbers produced by an algorithm. This result was followed for instance by the characterization of the entropy dimensions [Mey11] of the possible sets of periods [JV15], with similar recursion-theoretic criteria. These results follow a common outline, which consists in implementing Turing machines in hierarchical structures of particular SFT.

A recent trend is to see the effect of dynamical restrictions on these characterization results. It was already proved in [HM10] that the values of entropy of strongly irreducible bidimensional subshifts are all computable numbers. This means that there is an algorithm which on input $n$ outputs an approximation up to $\frac{1}{n}$. Moreover, the authors proved that this can be done in time $e^{O(n^3)}$. Then R. Pavlov and M. Schraudner provided a construction that allows to realize a subclass of this class.

Other types of restrictions have been studied, for instance minimality. For instance, in [HV17], the authors proved that minimal subshifts can exhibit complex computational behaviors, by exhibiting minimal $\mathbb{Z}^d$-subshifts that have complex Turing spectrum. In [JLP15], the authors provided a construction that allows the realization of all the non-negative real numbers smaller than 2 as entropy dimension of minimal $\mathbb{Z}^2$-subshifts (which are not though subshifts of finite type).

The restriction of minimality acts in a different way on SFT. Notably, the entropy of minimal SFT is zero. As a consequence, this invariant is not pertinent for the study of these systems. Moreover, one could think that this prevents embedding Turing computations in these subshifts, since entropy is often assimilated with complexity.

We prove in this text that this is not the case. First, this is possible to embed Turing computation in minimal SFT. Second, we show that the entropy dimension is a more pertinent invariant for these subshifts.

To our knowledge, the only other constructions of minimal SFT are due to B. Durand and A. Romashchenko [DR17]. They use a fixed-point construction in which are implemented computing machines that check if no forbidden pattern in a recursively enumerable set appear in bi-infinite words on the alphabet $\{0, 1\}$. Since in these constructions the computation areas of the machines are sparse, the degenerated behaviors are simple. Controlling the growth of the computing units,
they could attribute the function of some particular sub-units in order to simulate these behaviors in any computing unit. The minimality of the architecture used for the control on the apparition of the forbidden words is ensured by this simulation.

The idea of simulation is present in both constructions. However, the construction of \cite{DR17} relies on a very rigid architecture. We propose here a more flexible way to ensure the minimality, although complex to formulate. This allows the simulation of a consequent set of patterns.

In this text, we prove the following theorem:

**Theorem 1.** The possible entropy dimension of minimal $\mathbb{Z}^{3}$-SFT are the $\Delta_2$-computable numbers in $[0, 2]$.

Thus, there is a difference with the set of numbers that are the entropy of $\mathbb{Z}^{3}$-SFT: the $\Delta_2$-computable numbers in $[0, 3]$.

The construction used to prove the realization part of this characterization is an adaptation of the construction presented in \cite{Mey11}. This construction is not minimal for the reason that pathological behaviors of the Turing machines can appear, and that it uses a display of random bits that breaks the minimality property.

As in our previous article \cite{GS17a}, there is an attractive analogy between counters used in the construction and DNA. This comes from the separation into coding part and non-coding one. This analogy suggests that the non-coding part is implied in global properties of a living system.

The structure layer used in this construction is a 3d version of the Robinson subshift constructed with three copies of a rigid version of this subshift. It exhibits similar hierarchical structures as the two-dimensional version. Although we don’t prove how here, this structure allows, by adding colors, some structures used in \cite{Hoc99} and \cite{PS15} in order to construct $\mathbb{Z}^{3}$-SFTs to be recovered.

Let us also remark that some of the mechanisms described in the construction presented in this text can be reformulated using substitutions and S-adic systems. However, we don’t use Mozes theorem (proved in \cite{Moz89}) or the result proved in \cite{AS14}. (this theorem states that multidimensional S-adic systems are sofic) for two reasons. The first one is that some of the substitution mechanisms are localized in restricted parts in each configuration. When the mechanism is global, there is still an obstacle for the use of these theorems: the need of more precise properties on the SFT than stated in these theorems. One of these properties is the repetition, in each configuration of the SFT, of patterns whose size some power of two, with period equal as well to some power of two. We instead use directly similar techniques as \cite{Moz89} in the construction.

This text is organized as follows: In section 4, we prove that the entropy dimensions of a minimal SFT are smaller than $d - 1$. Then in section 5 we prove the realization part of the characterization.

\section{Definitions}

In this section, we recall some definitions of subshifts, entropy dimension, $\Delta_2$-computable numbers.

\subsection{Subshifts dynamical systems}

Let $\mathcal{A}$ be some finite set, called alphabet. Let $d \geq 1$ be an integer. The set $\mathcal{A}^{\mathbb{Z}^d}$ is a topological space with the product of the discrete topology on $\mathcal{A}$. Its elements are called configurations. We denote $(e^1, ..., e^d)$ the canonical sequence of generators of $\mathbb{Z}^d$. Let us denote $\sigma$ the action of $\mathbb{Z}^d$ on this space defined by the following equality for all $u \in \mathbb{Z}^d$ and $x$ element of the space:

\[ (\sigma^u.x)_v = x_{v+u}. \]

A compact subset $X$ of this space is called a subshift when this subset is stable under the action of the shift. This means that for all $u \in \mathbb{Z}^d$:

\[ \sigma^u.X \subset X. \]
Consider some finite subset $U$ of $\mathbb{Z}^d$. An element $p$ of $A^U$ is called a pattern on support $U$. This pattern appears in a configuration $x$ when there exists a translate $V$ of $U$ such that $x_V = p$. It appears in a subshift $X$ when it appears in a configuration of $X$. The set of patterns of $X$ that appear in it is called the language of $X$. The number of patterns on support $[1, n]^d$ that appear in $X$ is denoted $N_n(X)$.

We say that a subshift $X$ is minimal when any pattern in its language appears in any of its configurations.

A subshift $X$ defined by forbidding patterns in some finite set $F$ to appear in the configurations, formally:

$$X = \bigcap_{U \subseteq \mathbb{Z}^2} \left\{ x \in A^{\mathbb{Z}^2} : x_U \notin \mathcal{F} \right\}$$

is called a subshift of finite type (SFT).

### 2.2 Computability notions

**Definition 2.** A real number $x$ is said to be $\Delta_2$-computable when there exists a Turing machine which given as input an integer $n$ outputs a rational number $r_n$ such that $x = \lim_{n \to \infty} r_n$.

**Definition 3.** A sequence $(a_n)_n \in \{0, 1\}^N$ is said $\Pi_1$-computable if there exists a Turing machine that, taking as input a couple of integers $(n, i)$, outputs some $\epsilon_{n, i} \in \{0, 1\}$ such that for all $n$, $a_n = \inf_i \epsilon_{n, i}$.

The following lemma establishes a link between $\Delta_2$-computable real numbers and $\Pi_1$-computable sequences.

**Lemma 4.** A real number $z \in [0, 2]$ is $\Delta_2$-computable if and only if there exists some $\Pi_1$-computable sequence $(a_j)_j$ such that

$$z = \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} a_j$$

This lemma is stated as part of Lemma 4.1 in [Mey11].

### 2.3 Entropy dimensions

The upper entropy dimension of $X$ is the number

$$\overline{D}_h(X) = \limsup_n \frac{\log_2 \log_2 (N_n(X))}{\log_2(n)}$$

The lower entropy dimension of $X$ is:

$$\underline{D}_h(X) = \liminf_n \frac{\log_2 \log_2 (N_n(X))}{\log_2(n)}$$

When these numbers are equal, they are referred to as the entropy dimension of $X$, and denoted $D_h(X)$.

**Proposition 5** ([Mey11]). The upper and lower entropy dimensions are topological invariants, as well as the existence and value of the entropy dimension.

**Theorem 6** ([Mey11]). The possible values of the entropy dimension for $\mathbb{Z}^d$-SFT are the $\Delta_2$-computable numbers in $[0, d]$. We state the following proposition in order to show that the entropy dimension is very different from the entropy. This type of properties has an impact on the techniques used in order to realize numbers as topological invariants of subshifts.
Proposition 7. Let \( X \) and \( Z \) be two \( \mathbb{Z}^d \)-subshifts having entropy dimension. The subshift \( X \times Z \) has entropy dimension:
\[
D_h(X \times Z) = \max(D_h(X), D_h(Z)).
\]

Proof. We have the following for all \( n \geq 1 \):
\[
N_n(X \times Z) = N_n(X) \times N_n(Z).
\]

As a consequence,
\[
\log_2(N_n(X \times Z)) = \log_2(N_n(X)) + \log_2(N_n(Z)).
\]

Thus we have
\[
\frac{\max(\log_2 \circ \log_2(N_n(X)), \log_2 \circ \log_2(N_n(Z)))}{\log_2(n)} \leq \frac{\log_2(\log_2(\max(N_n(X), N_n(Z))))}{\log_2(n)}
\]
and
\[
\frac{\log_2 \circ \log_2(N_n(X \times Z))}{\log_2(n)} \leq \frac{\log_2(2 \log_2(\max(N_n(X), N_n(Z))))}{\log_2(n)}
\]

From this follows that
\[
\frac{\log_2 \circ \log_2(N_n(X \times Z))}{\log_2(n)} \leq \log_2(2) + \frac{\max(\log_2 \circ \log_2(N_n(X)), \log_2 \circ \log_2(N_n(Z)))}{\log_2(n)}
\]

As a consequence of the previous inequalities, we have, taking \( n \to +\infty \),
\[
D_h(X \times Z) = \max(D_h(X), D_h(Z)).
\]

\[\square\]

3 Robinson subshift - a rigid version

The Robinson subshift was constructed by R. Robinson \[Rob71\] in order to prove undecidability results. It has been used by in other constructions of subshifts of finite type as a structure layer \[PS15\].

In this section, we present a version of this subshift which is adapted to constructions under the dynamical constraints that we consider. In order to understand this section, it is preferable to read before the description of the Robinson subshift done in \[Rob71\]. Some results are well known and we don’t give a proof. We refer instead to the initial article of R. Robinson.

Let us denote \( X_{adR} \) this subshift, which is constructed as the product of two layers. We present the first layer in Section \[3.1\] and then describe some hierarchical structures appearing in this layer in Section \[3.2\]. In Section \[3.3\] we describe the second layer. This layer allows adding rigidity to the first layer, in order to enforce dynamical properties.

3.1 Robinson layer

The first layer has the following symbols, and their transformation by rotations by \( \frac{\pi}{2} \), \( \pi \) or \( \frac{3\pi}{2} \):

The symbols \( i \) and \( j \) can have value 0, 1 and are attached respectively to vertical and horizontal arrows. In the text, we refer to this value as the value of the 0, 1-counter. In order to simplify the representations, these values will often be omitted on the figures.
In the text we will often designate as **corners** the two last symbols. The other ones are called **arrows symbols** and are specified by the number of arrows in the symbol. For instance a six arrows symbols are the images by rotation of the fifth and sixth symbols.

The **rules** are the following ones:

1. the outgoing arrows and incoming ones correspond for two adjacent symbols. For instance, the pattern

```
  /
/
```

is forbidden, but the pattern

```
  /
/  
```

is allowed.

2. in every $2 \times 2$ square there is a blue symbol and the presence of a blue symbol in position $u \in \mathbb{Z}^2$ forces the presence of a blue symbol in the positions $u + (0, 2), u - (0, 2), u + (2, 0)$ and $u - (2, 0)$.

3. on a position having mark $(i, j)$, the first coordinate is transmitted to the horizontally adjacent positions and the second one is transmitted to the vertically adjacent positions.

4. on a six arrows symbol, like

```
  /  
/  
```

or a five arrow symbol, like

```
  /  
/    
```

the marks $i$ and $j$ are different.

The **Figure** shows some pattern in the language of this layer. The subshift on this alphabet and generated by these rules is denoted $X_R$: this is the Robinson subshift.

The main aspect of this subshift is the following property:

**Definition 8.** A $\mathbb{Z}^d$-subshift $X$ is said aperiodic when for all configuration $x$ in the subshift, and $u \in \mathbb{Z}^d \setminus (0, 0)$,

$$\sigma^u(x) \neq x.$$ 

**Theorem 9 (Rob71).** The subshift $X_R$ is non-empty and aperiodic.

In the following, we state some properties of this subshift. The proofs of these properties can also be found in [Rob71].

### 3.2 Hierarchical structures

In this section we describe some observable hierarchical structures in the elements of the Robinson subshift.

Let us recall that for all $d \geq 1$ and $k \geq 1$, we denote $U_k^{(d)}$ the set $[0, k - 1]^d$. 
3.2.1 Finite supertiles

Let us define by induction the south west (resp. south east, north west, north east) supertile of order $n \in \mathbb{N}$. For $n = 0$, one has

$$
\begin{align*}
S_{stsw}(0) &= \begin{array}{c}
-\end{array}, & S_{stse}(0) &= \begin{array}{c}
\cdot
\end{array}, & S_{stnw}(0) &= \begin{array}{c}
\cdot
\end{array}, & S_{stne}(0) &= \begin{array}{c}
\cdot
\end{array},
\end{align*}
$$

For $n \in \mathbb{N}$, the support of the supertile $S_{stsw}(n+1)$ (resp. $S_{stse}(n+1)$, $S_{stnw}(n+1)$, $S_{stne}(n+1)$) is $\mathbb{U}_{2^n+2}^{(2)}$. On position $u = (2^{n+1} - 1, 2^{n+1} - 1)$ write

$$
\begin{align*}
S_{stsw}(n+1)_u &= \begin{array}{c}
-\end{array}, & S_{stse}(n+1)_u &= \begin{array}{c}
\cdot
\end{array}, & S_{stnw}(n+1)_u &= \begin{array}{c}
\cdot
\end{array}, & S_{stne}(n+1)_u &= \begin{array}{c}
\cdot
\end{array},
\end{align*}
$$

Then complete the supertile such that the restriction to $\mathbb{U}_{2^{n+1}-1}^{(2)}$ (resp. $(2^{n+1}, 0) + \mathbb{U}_{2^{n+1}-1}^{(2)}$, $(0, 2^{n+1}) + \mathbb{U}_{2^{n+1}-1}^{(2)}$, $(2^{n+1}, 2^{n+1}) + \mathbb{U}_{2^{n+1}-1}^{(2)}$) is $S_{stsw}(n)$ (resp. $S_{stse}(n)$, $S_{stnw}(n)$, $S_{stne}(n)$).

Then complete the cross with the symbol

or with the symbol

in the south vertical arm with the first symbol when there is one incoming arrow, and the second when there are two. The other arms are completed in a similar way. For instance, Figure 1 shows the south west supertile of order two.

Proposition 10 ([Rob71]). For all configuration $x$, if an order $n$ supertile appears in this configuration, then there is an order $n+1$ supertile, having this order $n$ supertile as sub-pattern, which appears in the same configuration.

3.2.2 Infinite supertiles

Let $x$ be a configuration in the first layer and consider the equivalence relation $\sim_x$ on $\mathbb{Z}^2$ defined by $i \sim_x j$ if there is a finite supertile in $x$ which contains $i$ and $j$. An infinite order supertile is an infinite pattern over an equivalence class of this relation. Each configuration is amongst the following types (with types corresponding with types numbers on Figure 2):

Figure 1: The south west order 2 supertile denoted $S_{stsw}(2)$ and petals intersecting it.
Figure 2: Correspondence between infinite supertiles and sub-patterns of order $n$ supertiles. The whole picture represents a schema of some finite order supertile.

(i) A unique infinite order supertile which covers $\mathbb{Z}^2$.

(ii) Two infinite order supertiles separated by a line or a column with only three-arrows symbols (1) or only four arrows symbols (2). In such a configuration, the order $n$ finite supertiles appearing in the two infinite supertiles are not necessarily aligned, whereas this is the case in a type (i) or (iii) configuration.

(iii) Four infinite order supertiles, separated by a cross, whose center is superimposed with:
- a red symbol, and arms are filled with arrows symbols induced by the red one. (1)
- a six arrows symbol, and arms are filled with double arrow symbols induced by this one. (2)
- a five arrow symbol, and arms are filled with double arrow symbols and simple arrow symbols induced by this one. (3)

Informally, the types of infinite supertiles correspond to configurations that are limits (for type (ii) infinite supertiles this will be true after alignment [Section 3.3]) of a sequence of configurations centered on particular sub-patterns of finite supertiles of order $n$. This correspondence is illustrated on Figure 2. We notice this fact so that it helps to understand how patterns in configurations having multiple infinite supertiles are sub-patterns of finite supertiles.

We say that a pattern $p$ on support $U$ appears periodically in the horizontal (resp. vertical) direction in a configuration $x$ of a subshift $X$ when there exists some $T > 0$ and $u_0 \in \mathbb{Z}^2$ such that for all $k \in \mathbb{Z}$,

$$x_{u_0 + U + kT(1,0)} = p$$

(resp. $x_{u_0 + U + kT(0,1)} = p$). The number $T$ is called the period of this periodic appearance.

Lemma 11 ([Rob71]). For all $n$ and $m$ integers such that $n \geq m$, any order $m$ supertile appears periodically, horizontally and vertically, in any supertile of order $n \geq m$ with period $2^{m+2}$. This is also true inside any infinite supertile.
3.2.3 Petals

For a configuration $x$ of the Robinson subshift some finite subset of $\mathbb{Z}^2$ which has the following properties is called a petal.

- this set is minimal with respect to the inclusion,
- it contains some symbol with more than three arrows,
- if a position is in the petal, the next position in the direction, or the opposite one, of the double arrows, is also in it,
- and in the case of a six arrows symbol, the previous property is true only for one couple of arrows.

These sets are represented on the figures as squares joining four corners when these corners have the right orientations.

Petals containing blue symbols are called order 0 petals. Each one intersect a unique greater order petal. The other ones intersect four smaller petals and a greater one: if the intermediate petal is of order $n \geq 1$, then the four smaller are of order $n-1$ and the greatest one is of order $n+1$. Hence they form a hierarchy, and we refer to this in the text as the petal hierarchy (or hierarchy).

We usually call the petals valued with 1 support petals, and the other ones are called transmission petals.

Lemma 12 ([Rob71]). For all $n$, an order $n$ petal has size $2^{n+1} + 1$.

We call order $n$ two dimensional cell the part of $\mathbb{Z}^2$ which is enclosed in an order $2n + 1$ petal, for $n \geq 0$. We also sometimes refer to the order $2n + 1$ petals as the cells borders.

In particular, order $n \geq 0$ two-dimensional cells have size $4^{n+1} + 1$ and repeat periodically with period $4^{n+2}$, vertically and horizontally, in every cell or supertile having greater order.

See an illustration on Figure [4].

3.3 Alignment positioning

If a configuration of the first layer has two infinite order supertiles, then the two sides of the column or line which separates them are non dependent. The two infinite order supertiles of this configuration can be shifted vertically (resp. horizontally) one from each other, while the configuration obtained stays an element of the subshift. This is an obstacle to dynamical properties such as minimality or transitivity, since a pattern which crosses the separating line can not appear in the other configurations. In this section, we describe additional layers that allow aligning all the supertiles having the same order and eliminate this phenomenon.

Here is a description of the second layer:

Symbols: nw, ne, sw, se, and a blank symbol.

The rules are the following ones:

- Localization: the symbols nw, ne, sw and se are superimposed only on three arrows and five arrows symbols in the Robinson layer.

- Induction of the orientation: on a position with a three arrows symbol such that the long arrow originate in a corner is superimposed a symbol corresponding to the orientation of the corner.

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• **Transmission rule:** on a three or five arrows symbol position, the symbol in this layer is transmitted to the position in the direction pointed by the long arrow when the Robinson symbol is a three or five arrows symbol with long arrow pointing in the same direction.

• **Synchronization rule:** On the pattern

![Pattern](image1)

or

![Pattern](image2)

in the Robinson layer, if the symbol on the left side is *ne* (resp. *se*), then the symbol on the right side is *nw* (resp. *sw*). On the images by rotation of these patterns, we impose similar rules.

• **Coherence rule:** the other couples of symbols are forbidden on these patterns.

*Global behavior:* the symbols *ne, nw, sw, se* designate orientations: north east, north west, south west and south east. We will re-use this symbolisation in the following. The localization rule implies that these symbols are superimposed on and only on straight paths connecting the corners of adjacent order *n* cells for some integer *n*.

The effect of transmission and synchronization rules is stated by the following lemma:

**Lemma 13.** *In any configuration* \(x\) *of the subshift* \(X_{adR}\), *any order* \(n\) *supertile appears periodically in the whole configuration, with period* \(2^{n+2}\), *horizontally and vertically.*

*Proof.*

• This property is true in an infinite supertile: this is the statement of Lemma [11](#).

Hence the statement is true in a type \((i)\) configuration. This is also true in a type \((iii)\) configuration, since the infinite supertiles are aligned, and that the positions where the order \(n\) supertiles appear are the same in any infinite supertile. This statement uses the property that an order \(n\) supertile forces the presence of an order \(n+1\) one.

• Consider a configuration of the subshift \(X_{adR}\) which is of type \((ii)\). Let us assume that the separating line is vertical, the other case being similar. In order to simplify the exposition we assume that this column intersects \((0,0)\).

1. **Positions of the supertiles along the infinite line:**

   From Lemma [11](#) there exists a sequence of numbers \(0 \leq z_n < 2^{n+2} - 1\) and \(0 \leq z_n' < 2^{n+2} - 1\) such that for all \(k \in \mathbb{Z}\), the orientation symbol on positions \((-1, z_n + k.2^{n+2})\) (in the column on the left of the separating one) is *se* and the orientation symbols on positions \((1, z_n' + k.2^{n+2})\) is *sw*. The symbol on positions \((-1, z_n + k.2^{n+2} + 2^{n+1})\) is then *ne* and is *nw* on positions on positions \((1, z_n' + k.2^{n+2} + 2^{n+1})\): this comes from the fact that an order \(n\) petal has size \(2^{n+1} + 1\).

   Let us prove that for all \(n\), \(z_n = z'_n\). This means that the supertiles of order \(n\) on the two sides of the separating line are aligned.

2. **Periodicity of these positions:**

   Since for all \(n\), there is a space of \(2^n\) columns between the rightmost or leftmost order \(n\) supertile in a greater order supertile and the border of this supertile (by a recurrence argument), this means that the space between the rightmost order \(n\) supertiles of the left infinite supertile and the leftmost order \(n\) supertile of the right infinite supertile is \(2^{n+1} + 1\). Since two adjacent of these supertiles have opposite orientations, this implies that each supertile appears periodically in the horizontal direction (and hence both horizontal and vertical directions) with period \(2^{n+2}\).
3. The orientation symbols force alignment:
Assume that there exists some $n$ such that $z_n \neq z'_n$. Since $\{z_n + k, 2^{n+2}, (n, k) \in \mathbb{N} \times \mathbb{Z}\} = \mathbb{Z}$, this implies that there exist some $m \neq n$ and some $k, k'$ such that

$$z_n + k, 2^{n+2} = z'_m + k', 2^{m+2}.$$ 

One can assume without loss of generality that $m < n$, exchanging $m$ and $n$ if necessary. Then the position $(-1, z_n + k, 2^{n+2} + 2^{n+1})$ has orientation symbol equal to $ne$. As a consequence, the position $(1, z'_m + k', 2^{m+2} + 2^{n-m-1}, 2^{m+2})$ has the same symbol. However, by definition, this position has symbol $se$: there is a contradiction. This situation is illustrated on Figure 3.

\[\square\]

3.4 Completing blocks
Let $\chi : \mathbb{N}^+ \to \mathbb{N}^+$ such that for all $n \geq 1$,

$$\chi(n) = \left\lceil \log_2(n) \right\rceil + 4.$$

Let us also denote $\chi'$ the function such that for all $n \geq 1$,

$$\chi'(n) = \left\lceil \frac{\log_2(n)}{2} \right\rceil + 2.$$

The following lemma will be extensively used in the following of this text, in order to prove dynamical properties of the constructed subshifts:

**Lemma 14.** For all $n \geq 1$, any $n$-block in the language of $X_{adR}$ is sub-pattern of some order $\chi(n)$ supertile, and is sub-pattern of some order $\chi'(n)$ order cell.
Proof. 1. **Completing into an order** $2^{\lceil \log_2(n) \rceil + 1} - 1$ block:

Consider some $n$-block $p$ that appears in some configuration $x$ of the SFT $X_{adR}$. We can complete it into a $2^{\lceil \log_2(n) \rceil + 1} - 1$ block, since $2^{\lceil \log_2(n) \rceil + 1} - 1 \geq 2n - 1 \geq n$ for all $n \geq 1$.

2. **Intersection with four order** $\lfloor \log_2(n) \rfloor$ supertiles:

From the periodic appearance property of the order $\lfloor \log_2(n) \rfloor$ supertiles in each configuration, this last block intersects at most four supertiles having this order. Let us complete $p$ into the block whose support is the union of the supports of the supertiles and the cross separating these.

3. **Possible patterns after this completion according to the center symbol**:

Since this pattern is determined by the symbol at the center of the cross and the orientations of the supertiles, the possibilities for this pattern are listed on Figure 38, Figure 39 and Figure 40. Indeed, when the orientations of the supertiles are like on Figure 38 each of the supertiles forcing the presence of an order $\lfloor \log_2(n) \rfloor + 1$ supertile, the center is a red corner. When the orientations of the supertiles are like on Figure 39 the center of the block can not be superimposed with a red corner since the two west supertiles force an order $\lfloor \log_2(n) \rfloor + 1$ supertile, as well as the two east supertiles. This forces a non-corner symbol on the position considered.

For type 4, 5, 9, 10 patterns, there are two possibilities: the values of the two arms of the central cross are equal or not. Hence the notation $4, 4'$, where $4'$ designates the case where the two values are different.

One completes the alignment layer on $p$ according to the restriction of the configuration $x$.

On these patterns, the value of symbols on the cross is opposed to the value of the symbols on the crosses of the four supertiles composing it.

4. **Localization of these patterns as part of a greater cell**:

The way to complete the obtained pattern is described as follows:

(a) When the pattern is the one on Figure 38 this is an order $\lfloor \log_2(n) \rfloor + 1$ supertile and the statement is proved. Indeed, any order $\lfloor \log_2(n) \rfloor + 1$ supertile is a sub-pattern of any order $\lfloor \log_2(n) \rfloor + 4$ one.

(b) One can see the patterns on Figure 39 and Figure 40 in an order $\lfloor \log_2(n) \rfloor + 1, \lfloor \log_2(n) \rfloor + 2, \lfloor \log_2(n) \rfloor + 3$, or $\lfloor \log_2(n) \rfloor + 4$ supertile, depending on how was completed the initial pattern thus far (this correspondence is shown on Figure 41), hence a sub-pattern of an order $\lfloor \log_2(n) \rfloor + 4$ supertile.

The orientation of the greater order supertiles implied in this completion are chosen according to the symbols of the alignment layer. This layer is then completed.

5. This implies that any $n$-block is the sub-pattern of an order $2\left(\frac{1}{2} \lceil \log_2(n) \rceil \right) + 2$ cell, which is included into an order $\lceil \frac{1}{2} \log_2(n) \rceil + 2$ cell.

**Proposition 15.** The subshift $X_{adR}$ is a minimal SFT.

**Remark 1.** Let use note that the existence of a minimal $\mathbb{Z}^2$-SFT is already known. For instance, see [Bal09] (Theorem 1.35).

**Proof.** Any cell appears in all the configurations of this subshift. Since any pattern is sub-pattern of a cell, it then appears in all the configurations. Hence $X_{adR}$ is a minimal SFT.
4 Obstruction

In this section we prove that the entropy dimensions of a minimal SFT are constrained as follows:

**Proposition 16.** Let $d$ be some positive integer. Let $X$ be a minimal $\mathbb{Z}^d$-SFT. Then $\overline{D_h}(X) \leq d-1$.

**Remark 2.** The proof of this proposition was communicated to us by P. Guillon.

**Proof.** Idea: the idea of the proof is to construct an element of the subshift having low complexity. This construction using only the fact that the subshift is of finite type. Since the subshift is minimal, the complexity of the subshift is equal to the complexity of this element. We deduce the upper bound on the entropy dimension.

Let $X$ be some SFT on alphabet $A$. Let $r > 0$ be the rank of the SFT.

1. **Definition of the annulus and cross supports:**
   Let us denote, for all $n \geq 0$, $r_n = 2^n + 2r$, and define
   \[
   \mathcal{O}_n = [1, r_n]^d \setminus [1 + r, r_n - r + 1]^d,
   \]
   \[
   \mathcal{C}_n = \bigcup_{1 \leq k \leq d} [1, r_n]^{k-1} \times [r_{n-1} - r + 1, r_{n-1} + r] \times [1, r_n]^{d-k},
   \]
   when $n \geq 1$, and $\mathcal{C}_0$ is the set $[r + 1, 2r]^d$. Informally, we call $\mathcal{O}_n$ the outside of set $[1, r_n]^d$, and the inside is the complementary of $\mathcal{O}_n$ in this set.

![Figure 4: Illustration of the definition of the sets $\mathcal{O}_n$ and $\mathcal{C}_n$ (respectively dark gray set on the left and on the right).](image)

2. **An association between patterns on the annuli and patterns on the crosses:**
   Let $\psi_n : \mathcal{L}_{\mathcal{O}_n}(X) \to \mathcal{L}_{\mathcal{C}_n}(X)$ be a function which to some pattern $p$ associates a possible completion of $p$ on $\mathcal{O}_n \cup \mathcal{C}_n$.

   Let us construct recursively a function $\varphi_n : \mathcal{L}_{\mathcal{O}_n}(X) \to \mathcal{L}_{[1, r_n]^d}(X)$ as follows (See an illustration on Figure 5). Consider $p$ a pattern in $\mathcal{L}_{\mathcal{O}_n}(X)$.

   (a) Extend it with the pattern $\psi_n(p)$. 

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1. \( p \)

2. \( \psi_n(p) \)

3. \( \square: \) copies of \( \mathcal{O}_{n-1} \)

Figure 5: Illustration of the construction of the functions \( \varphi_n \).

(b) • if \( n \geq 1 \), consider the restrictions of the obtained pattern on the sets consisting in a product of \( d \) elements of \( \{[1, r_{n-1}], [r_{n-1} + 1, r_n]\} \). These sets are translate of \( \mathcal{O}_{n-1} \). Apply the function \( \varphi_{n-1} \) then the completion on these copies of \( \mathcal{O}_{n-1} \).
• if \( n = 0 \), then the construction is done.

3. **Construction of the configuration:**

Consider the a sequence of patterns \( (p_n)_n \) such that for all \( n \), \( p_n \) has support \( \mathcal{O}_n \), and \( (x_n)_n \) a sequence of configurations whose restriction on

\[
\left[1 - \frac{r_n}{2}, \frac{r_n}{2}\right]^d
\]

is equal to \( \varphi_n(p_n) \). There exists such a sequence, since the restriction of this pattern on \( \mathcal{O}_n \) is globally admissible.

By compactness, we can extract some infinite sub-sequence of \( (x_n)_n \) that converges towards some configuration \( x \) of \( X \).

4. **Complexity of the configuration** \( x \):

Let us fix some \( k \geq 1 \) and let us give an upper bound of the number of \( r_k \)-block that appear in \( x \). Consider \( p \) some \( r_k \)-block that appear in \( x \), in position \( u \). There exists some \( n \geq k \) such that \( \left[1 - \frac{r_n}{2}, \frac{r_n}{2}\right]^d \) contains \( u + \mathcal{C}_k \), and the configuration \( x_n \) coincides with \( x \) on \( u + \mathcal{C}_k \). As we can decompose \( \left[1 - \frac{r_n}{2}, \frac{r_n}{2}\right]^d \) into copies of \( \left[1 - \frac{r_k}{2}, \frac{r_k}{2}\right]^d \) on which the patterns have
their inside determined by their outside, the pattern \( p \) is a sub-pattern of a concatenation of 2\( d \) such blocks (See an illustration of Figure 4).

Hence the number of \( r_k \) blocks in \( x \) is smaller than the number of elements of \( (A^{O_k})^{2^d} \), which is smaller than
\[
|A|^{2^d |O_k|} \leq |A|^{2^{d+2d + r_k d - 1}},
\]
because \( O_k \) is the union of 2\( d \) cuboids having dimensions \( r, r_k, \ldots, r_k \) (as illustrated on Figure 6).

\[\textnormal{Figure 6: The set } O_k \textnormal{ can be decomposed into } 2^d \textnormal{ cuboids. On this picture, } d = 2.\]

Since \( X \) is minimal, any globally admissible pattern appears in \( x \). As a consequence,
\[
N_{r_k}(X) \leq |A|^{2^d 2d + r_k d - 1},
\]
which means that \( D_h(X) \leq d - 1. \)

There is also a computational obstruction:

\textbf{Proposition 17.} Let \( X \) be a \( \mathbb{Z}^3 \)-SFT having entropy dimension. Then \( D_h(X) \) is a \( \Delta_2 \)-computable number.

\textbf{Corollary 18.} Let \( X \) be some minimal \( \mathbb{Z}^3 \)-SFT admitting an entropy dimension. Then its entropy dimension is a \( \Delta_2 \)-computable number in \([0, 2] \).
5 Realization

In this section, we prove the following theorem:

**Theorem 1 (GS17b).** Every $\Delta_2$-computable number in $[0, 2]$ is the entropy dimension of a minimal $\mathbb{Z}^3$-SFT.

5.1 Abstract of the construction

In this section, we provide an abstract of the proof of Theorem 1. We first present the principles of Meyerovitch’s construction for the proof of Theorem 6 since we use these principles in our construction.

5.1.1 Principles of the proof of Theorem 6

Let $z \in [0, 2]$ a $\Delta_2$-computable number ($0$ is easily realized as the entropy dimension of a minimal SFT). Let us describe how to construct a $\mathbb{Z}^2$-SFT whose entropy dimension is $z$. From Lemma 4, there exists some $\Pi_1$-computable sequence $(a_n)_n \in \{0, 1\}^\mathbb{N}$ such that

$$z = 2 \lim_{n \to \infty} \frac{\sum_{k=0}^{n} a_k}{n + 1}.$$

This construction relies on the propagation of a signal through the petal hierarchy, which is transformed through the intersections of petals. The signal has two possible colors ■ and □. In our construction, we call them hierarchy bits. The transformation rule is abstracted on Figure 7. It depends on a bit $f_n$ in $\{0, 1\}$ attached to order $n$ petals. These bits are called frequency bits in our construction.

![Figure 7: Illustration of the signal transformation in Meyerovitch’s construction.](image)

On the corners of the order 0 petals are superimposed random bits in $\{0, 1\}$. The bits $f_n$, $n \geq 0$ are imposed to be $a_n$ by a Turing machine using a similar construction as in [HM10]. This architecture is set up so that it does not contribute to the entropy dimension, by using thin computation areas. Thus only the random bits do contribute to the entropy dimension. The number of corners in an order zero petal which is inside an order $n$ cell whose border is colored ■ is equal to

$$4^{\sum_{k=0}^{n} a_k}.$$
This comes from iterating the transformation rule. This number is 0 when the border of the cell is colored \[
\ block. 
\]

As a consequence, the number of possible sets of random bits over an order \( n \) cell is approximately 

\[ 2^{\sum_{k=0}^{n} a_k}. \]

Moreover, the number of possibilities for the random bits over a block of the structure having the same size as an order \( n \) cell is equal as the number of possibilities for the random bits on such a cell.

Since the entropy dimension is generated by random bits, it follows that the entropy dimension is equal to

\[
\lim_{n} \frac{\log_2(\log_2(2^{\sum_{k=0}^{n} a_k}))}{\log(2^{n+1})} = 2 \lim_{n} \frac{\sum_{k=0}^{n} a_k}{n+1} = z,
\]

since the size of an order \( n \) supertile is approximately \( 2^{n+1} \).

5.1.2 Obstacles to the minimality and their solutions

The obstacles to the minimality property in this construction are due to:

1. the definition of the hierarchy bits, since there is some configuration whose hierarchy bits are all \( \square \) (hence the hierarchy bit \( \blacksquare \) does not appear);
2. the definition of the random bits, which can be all 0 in a configuration, and all 1 in another one;
3. the machines computations: in the space time diagram of a machine in an infinite computation area, there can appear some parts that never appear in a finite space-time diagram.

In this text we propose a construction, abstracted in the following section, that overcomes as follows these difficulties so that the SFT is minimal:

1. The trick used to overcome the difficulties relative to hierarchy bits is to modify the transformation rules defining these bits. This modification is done so that for a sparse set of levels, no matter the color of this level, at least one petal under in the hierarchy is colored \( \square \). The set is chosen sparse enough so that this modification does not contribute to the entropy dimension.
2. We use a counter, called hierarchical counter, in order to alternate the possible sets of random bits. For the incrementation, we group the random bits into sets forming independent counters. That is why we need a third dimension in order to realize the numbers in \([0, 2]\), since the display of random bits is bidimensional. The three-dimensional subshifts that we construct use structures that appear in a three-dimensional version of the Robinson subshift.
3. The difficulties coming from the computations are solved by simulating any finite space-time diagram with any initial tape. In order to have the minimality property, we alternate these space-time diagrams in any configuration using counters called linear counters. They code for the initial tape of the machine. Moreover, a machine detecting an error sends an error signal to its initial tape. This error signal is taken into account if and only if the machine was well initialized.
4. The two types of counters have co-prime periods for different levels, in order to ensure the minimality. Moreover, they are incremented in orthogonal directions.

On Figure \( \text{Figure } 8 \) is some simplified schema of the construction.

The main arguments in the proof of the minimality property of this subshift are the following ones:
1. Any pattern $P$ can be completed into a pattern $P'$ over a three-dimensional cell with controlled size. Hence it is sufficient to prove that any pattern over a three-dimensional cell appears in any configuration. Such a pattern is characterized by the values of the counters of a sequence of cells included in its support, intersecting all the intermediate levels.

2. One can find back any sequence of values for the counters contained in the three-dimensional cell starting from any cell. This is done in two steps. First by jumping multiple times from a cell to the adjacent one having the same order in the direction of incrementation of the linear counter. Then in the direction of incrementation of the hierarchical counter. This is possible since the periods of the counters are co-prime.

This is illustrated on Figure 9. On this figure, $t$ (resp. $t'$) is the function which, taking as input the sequence of values of the counters of cells of intermediate level into a cell, outputs the sequence of the adjacent cell in the incrementation direction of the linear (resp. hierarchical) counter.

Figure 9: Schema of the proof for the minimality property of $X_z$.

5.1.3 Description of the layers

Let $z$ some $\Delta_2$ number in $[0, 2]$ and $p = 2^m - 1$ be some Mersenne number, for some $m$ such that $m/(2^m - 1) < z/2$. Let us construct some minimal $\mathbb{Z}^3$-SFT $X$ which has entropy dimension $z$. Using Lemma 4 there exists some $\Pi_1$-computable sequence $(a_j)_j \in \{0, 1\}^\mathbb{N}$ such that

$$z = 2 \lim_{n \to \infty} \frac{1}{n+1} \sum_{j=0}^{n} a_j.$$ 

We assume, without loss of generality, that for all $k$, $a_{2k} = 0$. Indeed, this does not change the limit

$$\lim_{n \to \infty} \frac{1}{n+1} \sum_{j=0}^{n} a_j.$$
and the computability properties of the sequence \((a_n)_n\). We also assume that for all \(k \in [0, p-1]\), there exist infinitely many \(i \in \mathbb{N}\) such that \(a_{ip+k} = 0\) and infinitely many \(i \in \mathbb{N}\) such that \(a_{ip+k} = 1\).

Let us construct a minimal \(\mathbb{Z}^3\)-SFT \(X_e\) which has entropy dimension \(1/p + z(1 - 1/2p)\). This subshift is presented as the superposition of various layers described as follows:

- **Structure layer [Section 6.1]**: this layer is a three-dimensional version of the Robinson subshift. We use three superimposed copies of the Robinson subshift, each one of these being constant respectively in the directions \(e_1^i, e_2^i, e_3^i\). We analyze in the corresponding section the properties of this layer. These properties are similar to the ones of the two dimensional version, in terms of supertiles (finite and infinite), repetition of the supertiles, cells, and completion of the patterns into supertiles (ensuring the minimality property). Only the order \(qp, q \geq 0\), three-dimensional cells will support functions. The possible functions are the following ones:
  
  - to execute some Turing computations, in order to control the value of the frequency bits and grouping bits, introduced in the next sections.
  - to increment some counter, whose value consists in a sequence of symbols. There are two type of counters:
    1. **Linear counters**, coding for the initial tape of the Turing machine the direction of propagation of its error signal, the states of the machine heads entering on the sides of the area, and the activity of lines and columns of the area. When the column or line is unactive, this column is not used for computation. This is used so that any part of a space-time diagram can be completed into a space-time diagram on a face of a three-dimensional cell.
    2. **Hierarchical counters**, coding for random bits that are superimposed to the frequency bits, which generate the entropy dimension.
  
- to transfer the information contained in the linear counter to the bottom (initial tape) and top line (for the error signal propagation direction) of the machine face, as well as the sides of the face (corresponding the the machine heads entering on the sides of the area).

Each of these functions is ensured on a specific face of the cell. The separation of the machine function and the counters functions is a necessity for ensuring the minimality property. This appears clearly in the proof of Proposition 26, which states that any pattern can be completed into a valid pattern over a (finite) three-dimensional cell.

- **Functional areas layer [Section 6.2]**: This layer serves to construct functional areas on the faces of the three-dimensional cells and faces connecting their ridges. This is done in order to localize the possible positions of the counters and machine symbols, and to give a function to these positions (step of computation, vertical or horizontal information transfer), that we call functional positions. This uses a signaling process, similar to a substitution, through hierarchical structures that appear on the faces of the three-dimensional cells and faces connecting these cells.

See Figure 10 for a schema of the functional areas over the faces of a three-dimensional cell. We then select sub-areas of these functional areas that will support computations of the machines and counters. They are constructed sparse enough so that the values of the linear counter and the computations of the Turing machine do not contribute to the entropy dimension of the subshift (they contribute to the complexity function, but not to the entropy dimension because of Proposition 7). Thus the entropy dimension is generated by the values of the hierarchical counter. That is where the number \(p\) is used: the selection process relies on the progressive selection of two dimensional cells on the faces according to a horizontal (resp. vertical) address in \(\{0, 1\}^p\) of these cells relatively to cells higher in the hierarchy. The principle of this address is similar to the coding of a Cantor set: 0 means that the cell is on
the left and 1 on the right (resp. on the bottom and on the top for vertical addresses). As a consequence, this process allows the selection of columns (resp. lines) of the functional areas of the faces, through the selection of order 0 cells.

- **Frequency bits layer** [Section 6.3] The frequency bits are bits in \{0, 1\} and are attached to each order \(qp\) two-dimensional cell on their border, in each of the copies of the Robinson subshift. Two cells having the same order have the same bit. The machine can read these bits on its tape (we can make the selection process of functional areas such that we keep this access to the information), and check that the frequency bit of order \(qp\) cells is \(a_q\) for all \(q \geq 0\).

- **Grouping bits layer** [Section 6.4] The grouping bits are bits in \{0, 1\} that are attached to order \(qp\) two-dimensional cells on their border. The difference with frequency bits is their value. When \(q = 2^k\) for some \(k\) the grouping bit is 1. Else this bit is 0. These bits are imposed by computations of the Turing machines. They are involved in the mechanism that groups the random bits into hierarchical counter values.

- **Linear counter layer** [Section 6.5] This layer supports the value of the linear counter, its incrementation and the transport of the value to the adjacent three-dimensional cells having the same order and to the machine face. The linear counter has values only on the three-dimensional cells that have order \(qp\) for some \(q \geq 0\) (and machines computations are present only on these cells). The incrementation is done on the bottom line of the upper face of the three-dimensional cells.

The value of the counter consists in a word on the alphabet

\[
\mathcal{A} \times \mathcal{Q}^3 \times \{-\rightarrow, \leftarrow\} \times \{\text{on, off}\}^2 \times \mathcal{D},
\]

where \(\mathcal{A}\) is the tape alphabet of the machine, \(\mathcal{Q}\) is the state set. These two sets are chosen to have cardinality \(2^l\) for some integer \(l\). This is possible by completing the machine’s alphabet and state set by elements that do not interact with the others, in order to not alter the work of the machine on well initialized tape. The set \(\mathcal{D}\) a finite set whose cardinality is chosen to be \(2^{4.2^l-2}\), so that the counter alphabet has cardinality which is \(2^{8.2^l}\).

The incrementation mechanism is the one of an adding machine acting on the counter value, except for one step, when the counter value is maximal. When this happen, the action of the adding machine is suspended for one step.
As a consequence, since the number of columns in the selected sub-areas of the functional areas is a power of 2 and that these numbers are different for two $qp$ order cells, the number of values of linear counters for two different of these levels are numbers $2^{2l_1}$ and $2^{2l_2}$, where $l_1 \neq l_2$. Since the counters are suspended for one step, the periods of the two counters are $2^{2l_1} + 1$ and $2^{2l_2} + 1$. These are two different, and thus co-prime, Fermat numbers.

The symbols in the set $\{\rightarrow, \leftarrow\}$ codes for the direction of an error signal propagation. The symbols in $\{\text{on}, \text{off}\}$ tell which ones of the lines and columns are used in the area for computations. The error signal is triggered when the machine ends its computation in an error state. This signal is sent to of the ends of the initial tape (according to the propagation direction), in order to verify that the tape was well initialized. We forbid the coexistence of the error signal with a signal that certifies that the tape was well initialized.

- **Machine layer [Section 6.6]** This layer supports the computations of the machines. These machines check that the $q$th frequency bit, shared by $qp$ cells, is equal to $a_q$. They also check the values of the grouping bits. The initial tape of the machine corresponds to the projection of the linear counter value on $A \times Q$, where $A$ is the tape alphabet and $Q$ is the state alphabet.

The machines and the linear counters are implemented in opposite faces of the three-dimensional cells, in order to ensure the separation of the information. This principle allows the minimality property. The information of the linear counter is connected to the machine through signals that propagate in the other faces of the cells.

- **Hierarchical counter layer [Section 6.8]**: In this layer some bits in $\{0, 1\}$, called hierarchy bits, are superimposed to the two-dimensional cells in the copy of the Robinson subshift parallel to the hierarchical counter face of the three-dimensional cells. These bits are determined by a signaling process through the hierarchical structures of any of the copies of the subshifts $X_{adR}$ in the structure layer. This process relies on the frequency bits. On the border of the order $2^p$ cells, the three hierarchy bits are imposed to be equal.

On the blue corners having hierarchy bit equal to 0 are superimposed some random bits in $\{0, 1\}$. These bits generate the entropy dimension. As a consequence, the machines have control over the entropy dimension through frequency bits.

For all $k$, the $k$th hierarchical counter value is the set of random bits on positions with blue corners that are in an order $2^k p$ order two-dimensional cell and not in an order $2^j p$ cell, with $j < k$.

The value of the $k$th hierarchical counter is incremented on the hierarchical counter face of the three dimensional order $2^k p$ cells. The incrementation mechanism is similar as the one of linear counters, except that it uses discrete curves to represent the value of the counter as a finite sequence.

This signaling process is done in such a way that the number of hierarchy bits on a face of an order $2^k p$ cell, $k \geq 0$, is strictly growing according to $k$. Since this number is also a power of 2, the periods of the hierarchical counters of two different levels are different Fermat numbers.

The direction of incrementation is chosen orthogonal to the incrementation direction of the linear counter. As a consequence, even if a linear counter has the same period as a hierarchical counter, this has no influence on the minimality. On the faces of the other three-dimensional faces, the values of the counters is not changed.

- **Synchronization layer [Section 6.9]**: this layer is used to synchronize the hierarchical counters of three-dimensional cells having the same order which are adjacent in the directions that are orthogonal to their incrementation direction. The linear counter is coded to have directly this synchronization.
After that the $X_z$ is constructed for all $z$, the proof is as follows: take $x$ some $\Delta_2$-computable in $[0,2]$. If $x = 0$, then any subshift having a unique symbol is minimal and has entropy dimension equal to $x$. When $x > 0$, for all $m$ such that $1/(2^m - 1) < x$, there exists some $z_m$ such that $1/p + z_m(1 - 1/2p) = x$. We take $m$ such that
\[
\frac{x - 1/p}{1 - 1/2p} > \frac{m}{2^m - 1}.
\]
For this $m$, $z_m$ is $\Delta_2$-computable, $X_{z_m}$ is minimal and has entropy dimension equal to $x$.

Let us make explicit the local rules that induce these global behaviors.

6 Details of the construction of the subshifts $X_z$:

6.1 Structure layer

In this section, we describe the construction of the structure layer.

For this purpose, we construct a three dimensional equivalent of the subshift $X_{adR}$, and analyze this subshift from the point of view of supertile hierarchical structure, repetition of these supertiles, and infinite supertiles.

This subshift consists in the superposition of three copies of $X_{adR}$, respectively parallel to the vectors $(e_1$ and $e_2)$, $(e_2$ and $e_3)$ and $(e_3$ and $e_1)$.

6.1.1 A minimal three-dimensional version of the Robinson subshift

This subshift has alphabet $A_{adR}^3$.

- **Robinson rules in the two dimensional sections of $\mathbb{Z}^3$:** for $i,j \in \mathbb{Z}^3$ such that $j - i = e_2$, or $e_3$ (resp. $e_1$ or $e_3$, resp. $e_1$ or $e_2$), the first (resp. second, resp. third) coordinates of the triples over these positions verify the rules of the subshift $X_{adR}$. The orientation of the two-dimensional sections of $\mathbb{Z}^3$ where the rules of the subshift $X_{adR}$ are verified is given as follows (the orientation of the Robinson symbols depend on this orientation):

  - for the first coordinate, the horizontal direction is $e_2$ and the vertical one $e_3$. This means that when looking in the direction opposite to the vector $e_3$, and orienting $e_1$ to the right and $e_2$ upwards, we see the usual picture of a configuration of $X_{adR}$.
  - for the second one, the horizontal direction is $e_3$ and the vertical one $e_1$.
  - for the last one, the horizontal direction is $e_1$ and the vertical one $e_2$.

- **Invariance in the orthogonal direction:** For $i,j \in \mathbb{Z}^3$ such that $j - i = e_1$ (resp. $e_2$, resp. $e_3$) second (resp. first) coordinates of the couples over these positions are equal. See Figure 11 for an illustration.

- **Coincidence rules:** When on some position there are at least two corners symbols of the Robinson subshift, then the three symbols are corners having the same color (blue or red).

Moreover, the possible triples of blue corners are the following ones:

1. [Image of possible triple 1]
2. [Image of possible triple 2]
3. [Image of possible triple 3]
4. [Image of possible triple 4]
5. [Image of possible triple 5]
6. [Image of possible triple 6]
7. [Image of possible triple 7]
8. [Image of possible triple 8]
These triples correspond to the corners of a cube as on Figure 12. This cube corresponds to the support of apparition of a pattern whose restriction on each of the coordinates on the corresponding face is a two dimensional order $n$ cell. We call these cubes order $n$ three dimensional cells. Notice that the restriction imposed by these rules is not trivial, since the number of allowed triples of blue corners is 8 which is less than the total number of possibilities, which is $4^3$. There is an equivalent restriction on triples of red corners.

- When on a position there is only one corner, then the couple of other symbols is amongst the following types:
  - (1) Two six arrows symbols or two five arrows symbols, pointing in the same direction, and orthogonal (in $\mathbb{Z}^3$) to the corner: this type of triples corresponds to the center of the faces of the cubes.
  - (2) Two four arrows symbols or two tree arrows symbols, pointing in the same direction and orthogonal (in $\mathbb{Z}^3$) to the corner. This type corresponds to the edges of the cubes and to the edges of connecting their corners.
  - (3) Two six arrows symbols or two five arrows symbols, pointing in the opposite directions of the arms of the corner. This type corresponds to the centers of the ridges.
  - (4) Any couple of four or three arrows symbols that are orthogonal (in $\mathbb{Z}^3$), and parallel to the corner. This type corresponds to the internal faces of the cubes.

The intersection of the cell with the $\mathbb{Z}^2$-section of $\mathbb{Z}^3$ that cut a three dimensional cell in two equal parts is called an internal face of the cell.

- The triples with only arrows symbols which are two by two orthogonal (in $\mathbb{Z}^3$) are forbidden (the other triples of arrows symbols correspond to the ridges of the internal faces, or the inside of these faces).

- On any translate of the subset $U^3_2$ of $\mathbb{Z}^3$, there is an admissible blue triple.

Figure 11: Illustration of the first three rules of the three dimensional Robinson subshift. The symbols $h$ and $v$ indicate respectively the horizontal and vertical directions in each of the copies of the subshift $X_{adR}$. 

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6.1.2 Hierarchical structures

1. Finite supertiles:
In this paragraph, the symbols $sw$, $se$, $nw$, $ne$ mean respectively south west, south east, north west and north east orientations of the corner symbols in the alphabet of the Robinson subshift.

(a) Definition by projection on the faces:
Any block on the language of this layer whose projection over a plane parallel to $e^2$ and $e^3$ (resp. $e^1$ and $e^3$, resp. $e^1$ and $e^2$), considering only the first (resp. second, resp. third) coordinate of the triple, is an order $n$ two-dimensional supertile with orientation $t_1$ (resp. $t_2$, resp. $t_3$) is called a three-dimensional supertile with orientation $t \in \{sw, se, nw, ne\}^3$ and order $n$.

(b) Recursive definition:

| Position of the translate | Orientation of the supertile |
|---------------------------|-----------------------------|
| $(0,0,0)$                 | $(ne,ne,ne)$                |
| $(2^{n+1},0,0)$           | $(ne,se,nw)$                |
| $(0,2^{n+1},0)$           | $(nw,ne,sw)$                |
| $(2^{n+1},2^{n+1},0)$     | $(nw,se,sw)$                |
| $(0,0,2^{n+1})$           | $(se,nw,ne)$                |
| $(2^{n+1},0,2^{n+1})$     | $(ne,sw,nw)$                |
| $(0,2^{n+1},2^{n+1})$     | $(sw,nw,se)$                |
| $(2^{n+1},2^{n+1},2^{n+1})$ | $(sw,sw,sw)$                |

Figure 13: Correspondence table for recursive definition of three-dimensional supertiles. The table gives the orientation of order the $n$ cubic supertile superimposed on the $v + U_{2^{n+2} - 1}$, where $v$ are the entries of the table.

The order $n + 1$ three-dimensional supertile with orientation $t \in \{sw, se, nw, ne\}^3$ can
be constructed from the order $n$ cubic supertiles as follows. The support of the order $n + 1$ cubic supertile is $U_{2n+2-1}^3$. Figure 13 shows positions in the support. On each of the translate of $U_{2n+1-1}^3$ corresponding to these positions, we put an order $n$ cubic supertile whose orientation is given by the table.

In order to complete the construction we put on position $(2^{n+2}, 2^n+2, 2^{n+2})$ a triple of red corners with orientation $t$. The three planes separating the order $n$ supertiles are filled with three arrows symbols or four arrows symbols induced by the corners. On the three lines of these planes intersecting the center position $(2^{n+2}, 2^n+2, 2^{n+2})$ there are triples with a unique corner. Because the corners of the center triple have compatible orientations, these triple are admissible. The reason is that the arrows symbols are orthogonal to the corner.

One can see that in particular, the faces of the cubic supertiles constructed this way verify the recurrence relation used to construct the supertiles of the Robinson subshift.

(c) Admissibility:
One can check that the order 1 cubic supertiles are locally admissible patterns. As a consequence, by a recurrence argument, the three-dimensional supertiles are locally admissible patterns of this subshift. This means that the supertiles do not admit any forbidden pattern as a sub-pattern).

2. Three-dimensional cells:
For all $n \geq 0$, we call order $n$ three-dimensional cell any subset of $\mathbb{Z}^3$ which is the support of a pattern whose projection over one of the coordinates on the corresponding face (the face which is parallel to the copy of the Robinson subshift) is an order $n$ two dimensional cell. The order $n$ three-dimensional cells have size $4^n + 1$: this property comes directly from the properties of the two dimensional cells.

3. Infinite supertiles:
(a) Definition:
Any order $n$ three-dimensional supertile forces the presence of an order $n + 1$ three-dimensional supertile in the direction of its orientation. This comes from the properties of the rigid version of the Robinson subshift $X_{adR}$ listed in Section 3.2. For any configuration $x$ of this structure layer, we denote $\sim_x$ the equivalence relation on $\mathbb{Z}^3$ defined by $i \sim_j x$ if there is a supertile in $x$ which contains $i$ and $j$. This is indeed an equivalence relation, because two supertiles can not intersect but when one of these two supertiles is a sub-pattern of the other. This comes directly from the same fact verified by the Robinson subshift, by projection on the faces of the three-dimensional blocks. An infinite order three-dimensional supertile is an infinite pattern over an equivalence class of this relation.

(b) Types of configurations according to the number of infinite supertiles:
Each configuration is amongst the following types:
(i) A unique infinite order supertile which covers $\mathbb{Z}^3$.
(ii) Two infinite order supertiles separated by a plane. In this case, on the plane, two of the coordinates of the triple are constant and equal to orthogonal three or four arrows symbols (this means that the long arrows of these symbols are orthogonal), whose arrows directions are in the plane. The last coordinates form a configuration of the Robinson subshift. This is a degenerated non-centered face or internal face of a three-dimensional cell. See Figure 14 for an illustration, where the internal faces are colored gray, and the localization of non centered part of one of these faces specified by a red square.
(iii) Four infinite order supertiles separated by two orthogonal planes. On the intersection of the two planes, one coordinate is constant and consists in some (red) corner. The other coordinates can be as follows:
Figure 14: Illustration of the internal faces.

- on the positions of the intersection, there are two arrows symbols with either three or four arrows. This corresponds to a degenerated face superimposed with a configuration of the Robinson with four infinite supertiles.
- on some position of the intersection, there are two six arrows or two five arrows or two three arrows symbols pointing in the opposite direction of the arms of the corner (and the symbols of the other positions are determined). This corresponds to a degenerated centered edge of a three-dimensional cell, illustrated by point 1 on Figure 15
- all the positions of the intersection have two four arrows symbols or two three arrows symbols pointing to the same direction, orthogonal to the corner. This case corresponds to a degenerated non-centered edge, illustrated by point 2 on Figure 15.

Figure 15: Illustration of some locations corresponding to degenerated supertiles.

(iv) Eight infinite order supertiles, separated by three orthogonal planes. The intersection of the three planes contains some corner symbol. Then there are two cases:
- on this intersection there is a triple of red corners with compatible orientations. This corresponds to a degenerated corner of a cube, illustrated by point 3 on Figure 15
- there is only one corner, and there are two six arrows symbols or two five arrows symbols pointing in the opposite direction of the arms of the corner. This case corresponds a degenerated centered face of a cube, illustrated by point 4 on Figure 15.

(c) Proof of the exhaustiveness of this classification:
Let us prove that there is no other possibility.

i. Evaluation of the space separating infinite supertiles:
First, the set of positions that are outside any infinite supertile does not contain any translate of $U_3^2$, because it would imply that it contains some triple of blue corners, and there would be an infinite supertile which does not intersects the others (impossible) or intersects non-trivially another infinite supertile (impossible, because they are equivalence classes).

ii. **Possible supports and combination of them:**
As for the Robinson subshift, the supports of infinite supertiles are some translates of $(\epsilon_1N) \times (\epsilon_2N) \times (\epsilon_3N)$ with $\epsilon_i \in \{-1, 1\}$ (1/8 of $\mathbb{Z}^3$), $(\mathbb{Z}) \times (\epsilon_1N) \times (\epsilon_2N)$, $(\epsilon_1N) \times (\mathbb{Z}) \times (\epsilon_2N)$ with $\epsilon_i \in \{-1, 1\}$ (1/4 of $\mathbb{Z}^3$), some $(\mathbb{Z}) \times (\mathbb{Z}) \times (\epsilonN)$, $(\mathbb{Z}) \times (\epsilonN) \times (\mathbb{Z})$, $(\epsilonN) \times (\mathbb{Z}) \times (\mathbb{Z})$ with $\epsilon \in \{-1, 1\}$ (half $\mathbb{Z}^3$) or $\mathbb{Z}^3$. Because the set of positions that are not in any infinite supertile does not contain any translate of $U_3^2$, the possibilities correspond to the type (i) to (iv) types listed above.

iii. **Reduction of eight infinite supertiles configurations:**
In the case where there are eight supertiles separated by three planes (type (iv)), if there are three corners, then they have compatible orientations. If not, then there is one corner. Indeed, if there were no corner, the intersection of the three planes would be composed by three orthogonal arrows symbols. This is impossible (see the coincidence rules). Then the triple has to be of type (1) of the third coincidence rule. Hence, if it was of type (2) for instance, it would mean that, projecting on the copies of the Robinson that do not correspond to corner, the configurations have two infinite supertiles separated by an infinite line, and the plane generated by translating this line is included in the two orthogonal planes generated by translation of the separating cross of the first copy of the Robinson subshift. That there are only four infinite supertiles.

iv. **Four infinite supertiles:**
In the case of four infinite supertiles, if one position of the intersection has a corner, then this corner is present on the whole line, and the triples on this line are of type (2), (3), (4) types of the coincidence rule, and to the type (iii) of the above description.

v. **Two infinite supertiles:**
When there are two infinite supertiles, by projecting on the copies of the Robinson, we get that two of the copies have two infinite supertiles with a separating line. Moreover, the plane supports a configuration of the Robinson which has a unique infinite supertile, for if it was not the case, there would be two separating planes.

6.1.3 **Properties of this layer**
The structure layer has the following properties:

1. **Non-emptiness:**
A configuration:
   - whose projections on the copies of $X_{adR}$ are all of type (iii) and centered on a red corner
   - and such that the triple of these centers has compatible orientations

is an element of this subshift, hence it is **not empty**.

2. **Repetition of the supertiles:**
The order $m$ three-dimensional supertile appear periodically in any order $n \geq m$ three-dimensional supertile with period $2^{n+2}$, horizontally and vertically. This comes directly from the similar property of the two-dimensional supertiles. This is also true inside an infinite supertile. Because we use the rigid version of the Robinson subshift, this is also true for the whole configuration, in any configuration of the subshift.
3. Completion result:

**Proposition 19.** Any $n$-block in this layer can be completed into an order $k$ cubic supertile, with $k \geq \lceil \log_2(n) \rceil + 4$. Moreover it can be complete in an order $k$ three-dimensional cell, with

$$k \geq \left\lceil \frac{\lceil \log_2(n) \rceil}{2} \right\rceil + 2.$$

We don’t write the proof of this proposition, since this is similar to the two-dimensional version of the subshift. Moreover, this version is also minimal:

**Corollary 20.** This three-dimensional version of the Robinson subshift is minimal.

6.1.4 Coloration

We use colors in order to simplify the representations of the configurations of this layer. The positions in the edges of the cubes are characterized by having three petal symbols with 0, 1-counter equal to 1 (recall that we call value of the 0, 1-counter the symbols in $\{0, 1\}$ on corner symbols of the Robinson subshift), and we represent this by the symbol $\blacksquare$. The faces positions have exactly two such symbols, and are represented by $\square$. The other faces connecting the edges of the cubes are colored with $\blacksquare$ and are characterized by having a unique petal symbol with 0, 1-counter value equal to 1. See on Figure 16 the representation of the surroundings of a three-dimensional cells with these colorations.

6.2 Functional areas

In this section, we describe how to draw functional areas on these faces. This means that we attribute local functions to the positions of these faces realizing the global functions of the counters and machine computations. These local functions are the execution of one step of computation (including the incrementation step for the counter), and horizontal and vertical transmission of information.

Moreover, we use an addressing mechanism that allows the selection of sparse sub-areas of this functional areas, so that the global functions do not contribute to entropy dimension.
In this section, each sublayer is presented as the superimposition, on the colored faces, of symbols in a finite set $A$. As a consequence, the positions in the intersection of two faces are superimposed with a couple of these symbols, and the other positions in the faces with just one of these symbols. In order to keep the descriptions as simple as possible, each sublayer is presented as having alphabet $A$, while the real alphabet has to include the elements of $A^2$, and the real set of rules corresponds to this alphabet. These rules can be deduced easily from the descriptions above.

Moreover, in each sublayer, the non-blank symbols are superimposed on and only on petals of the copy of the Robinson parallel to this face - we will simply refer to these as petals of this face.

We recall that the petals having 0, 1-counter value equal to 0 are called transmission petals, and the other ones are called support petals.

6.2.1 Orientation in the hierarchy

The purpose of this first sublayer is to give access, to the support petal of each colored face, to the orientation of this petal relatively to the support petal just above in the hierarchy.

Figure 17: Schematic illustration of the orientation rules, showing a support petal and the support petals just under this one in the hierarchy, all of them colored dark gray. The transmission petals connecting them are colored light gray. Transformation positions are colored with a red square. The arrows give the natural interpretation of the propagation direction of the signal transmitting the orientation information.

Symbols:

The symbols are elements of

$$\{\square \, \uparrow \, \downarrow \, \bullet \, \ast \},$$

$$\{\square \, \uparrow \, \downarrow \, \bullet \, \ast \} \times \{\square \, \uparrow \, \downarrow \, \bullet \, \ast \},$$

and a blank symbol.

Local rules:

- Localization: the non blank symbols are superimposed on and only on positions with petal symbols of the gray faces. The symbol is transmitted through the petals, except on transformation positions, defined just below.
• **Transformation positions:** the *transformation positions* are the positions where the transformation rule occurs (meaning that the signal is transformed). These positions depend on the (sub)layer. In this sublayer, these are the positions where a support petal intersects a transmission petal just under in the hierarchy. On these positions is superimposed a couple of symbols, in

\[
\begin{bmatrix}
\begin{array}{c}
\blacksquare \\
\blacksquare \\
\blacksquare \\
\blacksquare \\
\blacksquare \\
\blacksquare \\
\end{array}
\end{bmatrix} \times \begin{bmatrix}
\begin{array}{c}
\blacksquare \\
\blacksquare \\
\blacksquare \\
\blacksquare \\
\blacksquare \\
\blacksquare \\
\end{array}
\end{bmatrix},
\]

while the other petal positions are superimposed with a *simple symbol*, in

\[
\begin{bmatrix}
\begin{array}{c}
\blacksquare \\
\blacksquare \\
\blacksquare \\
\blacksquare \\
\blacksquare \\
\blacksquare \\
\end{array}
\end{bmatrix}.
\]

On the transmission positions, the first symbol corresponds to outgoing arrows (this is the direction from which the signal comes, that is to say the support petal), and the second one to incoming arrows (this is the direction where the signal is transmitted, after transformation).

• **Transformation:** on a transformation position, the second bit of the couple is \( \square \) if the six arrows symbol is \( \square \) or \( \square \).

These positions correspond to the intersection of an order \( n+1 \) support petal with the *north west* order \( n \) transmission petal just under in the hierarchy. *There are similar rules for the other orientations.*

• **Border rule:** on the border of a gray face, the symbol is blank if not on a transmission position. On a transmission position, the first symbol of the couple is blank.

**Global behavior:**

This layer supports a signal that propagates through the petal hierarchy on the colored faces. This signal is transmitted through the petals except on the intersections of a support petal and a transmission petal just above in the hierarchy. On these positions, the symbol transmitted by the signal is transformed into a symbol representing the orientation of the transmission petal with respect to the support petal.

As a consequence, the support petals just under the transmission petal are colored with this orientation symbol. See the schema on Figure 17.

6.2.2 **Functional areas**

**Symbols:**

\[
\begin{bmatrix}
\begin{array}{c}
\blacksquare \\
\blacksquare \\
\blacksquare \\
\blacksquare \\
\blacksquare \\
\blacksquare \\
\end{array}
\end{bmatrix}, \begin{bmatrix}
\begin{array}{c}
\blacksquare \\
\blacksquare \\
\blacksquare \\
\blacksquare \\
\blacksquare \\
\blacksquare \\
\end{array}
\end{bmatrix}, \begin{bmatrix}
\begin{array}{c}
\blacksquare \\
\blacksquare \\
\blacksquare \\
\blacksquare \\
\blacksquare \\
\blacksquare \\
\end{array}
\end{bmatrix}, \begin{bmatrix}
\begin{array}{c}
\blacksquare \\
\blacksquare \\
\blacksquare \\
\blacksquare \\
\blacksquare \\
\blacksquare \\
\end{array}
\end{bmatrix}, \begin{bmatrix}
\begin{array}{c}
\blacksquare \\
\blacksquare \\
\blacksquare \\
\blacksquare \\
\blacksquare \\
\blacksquare \\
\end{array}
\end{bmatrix}, \begin{bmatrix}
\begin{array}{c}
\blacksquare \\
\blacksquare \\
\blacksquare \\
\blacksquare \\
\blacksquare \\
\blacksquare \\
\end{array}
\end{bmatrix}^2.
\]

**Local rules:**

• **Localization:** the non blank symbols are superimposed on and only on petal positions of the colored faces.

• the couples of symbols are superimposed on six arrows symbols positions where the border of a cell intersects the petal just above in the hierarchy (*transformation positions*).

• the symbols are transmitted through through the petals except on transformation positions.
Figure 18: Schematic illustration of the transmission rules of the functional areas sublayer. The transformation positions are marked with red squares on the first schema.

• **Transformation through hierarchy.** On the transformation positions, if the first symbol is \[ \square \] then the second one is equal. For the other cases, if the symbol is

\[ \text{\[ \square \text{ or } \square \text{ or } \square \]} \]

then second color is according to the following rules. The condition on the symbol above corresponds to positions where a support petal intersects the transmission petal just above in the hierarchy and being oriented **north west** with respect to this transmission petal (examples of such positions are represented with large squares on Figure 18).

1. if the orientation symbol is \[ \square \], the second symbol is equal to the first one.
2. if the orientation symbol is \[ \square \], the second symbol is a function of the first one:
   - if the first symbol of the couple is \[ \square \text{ or } \square \text{ or } \square \], then the second symbol is \[ \square \text{ or } \square \text{ or } \square \]
   - if the first symbol is \[ \square \text{ or } \square \text{ or } \square \], then the second one is equal.
   - if the first symbol is \[ \square \text{ or } \square \text{ or } \square \], then the second symbol is \[ \square \text{ or } \square \text{ or } \square \]
3. if the orientation symbol is \[ \square \], the second color is \[ \square \]
4. if the orientation symbol is \[ \square \], the second symbol is a function of the first one:
   - if the first symbol of the couple is \[ \square \text{ or } \square \text{ or } \square \], then the second symbol is \[ \square \text{ or } \square \text{ or } \square \]
   - if the first symbol is \[ \square \text{ or } \square \text{ or } \square \], then the second one is \[ \square \text{ or } \square \text{ or } \square \]
   - if the first symbol is \[ \square \text{ or } \square \text{ or } \square \], then the second symbol is equal.
For the other orientations of the support petal with respect to the transmission petal just above, the rules are obtained by rotation.

See Figure 13 for an illustration of these rules.

- **Border rule:** on the border of a face of a three-dimensional cell, the symbol is blue.

- **Coherence rules:** these rules enable a retroaction of the colors on infinite petals over the order 0 petals that are nearby, ensuring a coherence amongst the hierarchy (they are similar to the property $Q$ in Mozes’ construction).

1. when near a **corner** of a three-dimensional cell, on a pattern

   ![Corner Pattern](image)

   in one of the copies of the Robinson subshift, the corresponding pattern in this sublayer has to be

   ![Sublayer Pattern](image)

   There are similar rules for the other corners. This allows degenerated behaviors which do not happen around finite cells to be avoided. The other rules in this list have the same function. Their analysis is important to understand how it is possible to complete patterns of this subshift into three-dimensional cells.

2. near an **edge** of a three-dimensional cell, the symbols on the corresponding blue corners on the two faces connected by this edge have to match. Moreover, on the middle of an edge, the two blue corners inside each one of the two faces adjacent to the edge that are on the two sides of the middle, are colored blue. For instance, on the pattern

   ![Edge Pattern](image)

   in any of the copies of the Robinson subshift, the pattern in this layer has to be

   ![Layer Pattern](image)

   When on the quarter of the edge, the two symbols have to be all (observer on edge) or all (observer on opposite side), the direction of the arrows corresponding to the orientation of the edge. For instance, on the pattern

   ![Quarter Edge Pattern](image)
the pattern has to be:

![Pattern](image)

3. Inside a colored face, the degenerated behaviors correspond to the degenerated behaviors of the two-dimensional version of the Robinson subshift. When near a corner of a support petal inside the face, the symbols on the blue corners nearby have to be colored according to the color of the corner. For instance, on

![Pattern](image)

in the copy of the Robinson subshift parallel to the face, the pattern in this layer has to be amongst the following ones:

![Patterns](image)

We impose similar restrictions for the other orientations of the corner. On the positions where two petals intersect, the two symbols inside the two-dimensional cell have to be colored with blue, and the two other ones have to be marked with an arrow pointing in the direction parallel to the cell border. For instance, on the pattern

![Pattern](image)

the pattern has to be:

![Pattern](image)

When near the border of a cell not intersecting another petal, then the only restriction is to have arrows marks outside, in the direction of the border.

**Global behavior:**

Like the first sublayer presented in Section 6.2.1, the global behavior of this sublayer consists in coding a substitution. This coding uses a signal that propagates through the petal hierarchy, on any colored face.

This process is similar as the one used by Robinson [Rob71] in order to create areas supporting the computations of Turing machines in a $Z^2$-SFT. However, we localize it here on faces of cubes in a three-dimensional SFT.
The result of this process is that order zero two dimensional cell borders of a colored face are colored with a symbol which represent a function of the blue corners positions - called functional positions - included in the zero order petal just under this petal in the petal hierarchy. These symbols and functions are as follows:

- **blue** if the set of columns and the set of lines in which it is included do not intersect larger order two-dimensional cells. The associated function is to **support a step of computation** (which can be just to transfer the information in the case when the face support a counter), and the corresponding positions are called **computation positions**.

- An **horizontal** (resp. **vertical**) **arrow** directed to the right (resp. to the top) when the set of columns (resp. lines) containing this petal intersects larger order two-dimensional cells but not the set of lines (resp. columns) containing it. The associated function is to **transfer information** in the direction of the arrow (this information can be trivial in the case when the face support a counter. This means that the symbol transmitted is the blank symbol), and the corresponding positions are called **information transfer positions**.

- When the two sets intersect larger order cells, the petal is colored **light gray**. These positions have no function.

See on Figure 19 a schema of the functional positions over the faces of an order 3 three-dimensional cell.

![Figure 19: Schema of the functional positions on the a face of an order 3 three-dimensional cell.](image)

The aim of the following sections, Section 6.2.3, Section 6.2.4, and Section 6.2.5 is to add symbols on the colored faces in order to specify thin sub-areas of the functional areas. These sub-areas will support the computations of the machines and the linear counters. They are thin enough so that the machines and counters do not contribute to the entropy dimension. This entropy dimension is thus generated by the values of the hierarchical counter.

The first sublayer of this set encodes a counter, called \( p \)-counter. It specifies, on each support petal, the arithmetical class of \( n \) modulo \( p \), where \( n \) is the order of the corresponding two-dimensional cell. The second sublayer, based on this \( p \)-counter, gives an address of the columns (resp. lines) amongst the **computation columns** (resp. lines), defined by intersecting computation positions in the face. In the third sublayer, we describe a process which selects a thin subset of these columns (resp. lines) according to the address.

### 6.2.3 The \( p \)-counter

We recall that \( p = 2^m - 1 \) is an integer, fixed at the beginning of the construction.
Symbols:

Elements of $\mathbb{Z}/p\mathbb{Z}$ and of $(\mathbb{Z}/p\mathbb{Z})^2$ and a blank symbol.

Local rules:

- **Localization:**
  - the non blank symbols are superimposed on and only on gray faces, on petals of the copy of the Robinson subshift parallel to this face.
  - The blue petals are superimposed with $\overline{0}$.
  - the couples are superimposed on transformation positions defined as the intersection positions of a support petal and the transmission petal just above in the hierarchy.
  - the other positions have a simple symbol and this symbol is transmitted through these positions.

- **Transformation rule:** on transformation positions, if the first bit is $i$, then the second bit is $i+1$.

- **Coherence rule:** on the edges connecting two faces, the two values of the $p$-counter are equal. On the corners, the three values are equal.

Global behavior:

Each of the support petals on the colored faces is attached with some element of $\mathbb{Z}/p\mathbb{Z}$. This symbol is transmitted through the petals except on transformation positions. These positions are defined to be the intersections of the support petals with the transmission petal just above in the hierarchy, where the transmission petal has value $\overline{k+1}$ and $\overline{k}$ is the value of the support petal. As the blue petals are marked with $\overline{0}$, this imposes that the border petals of the order $n$ two-dimensional cells on the faces are marked with $\overline{1}$.

6.2.4 The $p$-addressing

This sublayer has two subsublayers, one for vertical addresses, and another for horizontal addresses.

Vertical addressing: Symbols:

The symbols of this subsublayer are the following ones.

- Length $k$ words on the alphabet $\{0, 1\}$, for $k = 1 \ldots p$: these symbols are called the vertical address of support petals relatively to the next support petal above in the hierarchy.

- The couples $(w, w')$ such that $w$ and $w'$ are words on alphabet $\{0, 1\}$ with respective lengths $k$ and $k+1$ with $k < p$, and $(w, \square)$ (they represent possible transformations of simple symbols when going through a support petal corresponding to a two-dimensional cell having order different from any $mp$, $m \geq 0$).

- The couples $(w, 0)$ and $(w, 1)$, where $w$ is a length $p$ word on $\{0, 1\}$ (transformations of simple symbols when going through a petal corresponding to order $mp$ cells).

- The symbols $\square$, $(\square, \square)$, $(w, \square)$, $(\square, 0)$ with $w$ a word on $\{0, 1\}$ having length between 1 and $p$, $(\square, 1)$.

- A blank symbol.
Figure 20: Schemata of the transformation rules of the vertical addressing. On the two schemata on the left, the central petal has \( p \)-counter value not equal to \( 0 \). On the two schemata on the right, this value is \( 0 \). The transformation positions are symbolized by red squares. The symbols inside the little petals is the symbol attached to them in this sublayer.

**Local rules:**

- **Localization:**
  - the non-blank symbols are located on the petals of the colored faces.
  - the symbols are transmitted through the petals, except on transformation positions. These positions are defined as the intersection of a support petal and the transmission petal just above in the hierarchy.
  - the transformation positions are written with a couple, and the other petal positions with a simple symbol.
  - the non-transmission positions in counter \( k \) support petals with \( k \leq p - 1 \) are superimposed with a length \( p - k \) word on \( \{0, 1\} \) or the symbol \( \square \).

- **Border rule:** on the border of a gray face, all the positions are colored with \( \square \).

- **Transformation through hierarchy:** on the intersection of a petal support with the transmission petal just above in the hierarchy (transformation positions):
if the symbol is \[ \begin{array}{c}
\text{or } \end{array} \]
corresponding to positions symbolized by a large red square on Figure 20, then second symbol is set according to the following rules:

1. if the orientation symbol is \[ \begin{array}{c}
\text{and the first symbol is some length } k < p \text{ word } w, \text{ then the second symbol is } w0. \text{ If the first symbol is some length } p \text{ word, then the second is 0.} \text{ If the first symbol is } 0, \text{ then the second symbol is equal.}

2. if the orientation symbol is \[ \begin{array}{c}
\text{then the second symbol is } 0.

3. if the orientation symbol is \[ \begin{array}{c}
\text{the second symbol is } if \text{ the counter is not } p − 1, \text{ and 1 else.}

4. if the orientation symbol is \[ \begin{array}{c}
\text{and the first symbol is some length } k \text{ word } w \text{ for } k < p, \text{ then the second symbol is } w0. \text{ If the first symbol is some length } p \text{ word } w, \text{ then the second is 0.} \text{ If the first symbol is } 0, \text{ then is the second.}

There are similar rules when the Robinson symbol corresponds to another orientation of the support petal with respect to the support petal just above in the hierarchy. See Figure 20 for an illustration of these rules.

- Coherence rules: we impose coherence rules in a similar way as in Section 6.2.2. These rules impose that the order 1 petals inside a two-dimensional cell on a gray face are superimposed with \[ \begin{array}{c}
\text{if the value of the } p\text{-counter of the cell border is not } 0 \text{ and is a length } p \text{ word on } \{0, 1\} \text{ if it is } 0.

We don’t describe these rules, in order to keep the exposition as simple as possible, since they are similar as in Section 6.2.2.

Horizontal addressing: We add another subsublayer with the same symbols and similar rules that are abstracted as on the Figure 21.

Global behavior:

In the vertical addressing subsublayer, then on each colored face, the petals support the propagation of a signal which is transformed on the intersections of a support petal with the transmission petal just above in the hierarchy. Except on the transformation positions, the petals are colored with a non-empty word in \{0, 1\} or the symbol \[ \begin{array}{c}
\text{Consider an order } n \text{ colored face, and three integers } k, m \text{ and } j \text{ such that } mp ≤ j < (m + 1)p ≤ kp ≤ n, \text{ the support petals that:}

1. are the border of an order } j \text{ two-dimensional cell,}
2. are included in an order } kp \text{ cell,}
3. such that the columns in the cells intersecting the order } j \text{ cell do not intersect any order } i \text{ cell with } j < i ≤ kp,

are marked with a length \((m + 1)p − j\) word on \{0, 1\} which codes for the column position of this petal in the order \(kp\) cell with respect to the next order \((m + 1)p\) cell in the hierarchy. All the other support petals are colored with \[ \begin{array}{c}
\text{The possible length } l \text{ addresses, with } 1 ≤ l ≤ p \text{ are all the words in } \{0, 1\}^l, \text{ and are arranged in the alphabetic order, } 0^l \text{ being the address of the leftmost petals and } 1^l \text{ the address of the rightmost ones.}

See an example of addresses arrangement on Figure 22.
As a consequence, in an order $kp$ two-dimensional cell, with $k \geq 0$, the columns containing order 0 cells and not intersecting order $i$ cells such that $0 < i \leq kp$, are virtually addressed with a word in $\{0, 1\}^{kp}$, obtained as the concatenation of $k$ length $p$ addresses.

The global behavior of the horizontal addressing sublayer is similar, replacing columns by rows.

The aim of the following section is to select a subset of these columns by selecting a subset of the length $p$ addresses.

### 6.2.5 Active functional areas

In this section, we describe how to select, for all the two dimensional cells on the colored faces, a subset of the functional areas of these cells which is sparse enough to not contribute the entropy dimension. This subset is although large enough so that the machines have access to all the data it has to check (namely, the frequency bits, and grouping bits, presented later).

Let $\Delta$ be the set of words $w$ in $\{0, 1\}^p$ (amongst vertical or horizontal $p$-addresses) such that there exists $k \in [0, p]$ such that $w = 0^k1^{p-k}$. For instance, when $p = 3$, $\Delta = \{000, 011, 001, 111\}$.

---

Figure 21: Schemata of the transmission rules of the horizontal addressing. The central petals on the two schemata on the left have $p$-counter value not equal to 0. The one on the schemata on the right have $p$-counter value equal to 0.
Figure 22: Schema of the activation signal in a some \((m+1)p\) order two dimensional cell, when \(p = 3\), for the \(mp\) order cells contained in it, for any \(m \geq 0\). The addresses of the active columns are 000, 001, 011 and 111, and are written under these columns. On the right side are written the addresses of all the lines - not only the active ones.

**Symbols:**

The elements of \(\{[\text{ }],[\text{ }]\}^2\), \(\{[\text{ }],[\text{ }]\}^2 \times \{[\text{ }],[\text{ }]\}^2\), and a blank symbol.

**Local rules:**

- **Localization:**
  - the non-blank symbols are located on the petals on gray faces.
  - The transformation positions are the positions where a support petal intersects a transmission petal immediately under in the hierarchy. The transformation positions are marked with a couple of couples, and the other positions with a unique couple.

- **Border rule:** on the border of the faces of the three-dimensional cells, the possible symbols are [ ] and ([ ]).

- the symbols are transmitted through petals and intersections of petals that are not transformation positions.

- **Transformation through hierarchy.** On transformation positions we have the following rules:
  - if the counter is not 0, then the two elements of the couple are the same.
  - if the counter is 0, then the couple is changed from the \(n+1\) order petal to the \(n\) order one as follows:
    1. In the following cases, the couples become ([ ]):
       * the two addresses are in \(\Delta\).
       * the Robinson symbol is \[\text{ }\] or \[\text{ }\] and the orientation symbol is [ ]
**Coherence rules:** we consider, in the following list, patterns that intersect three-dimensional cells having order greater or equal to $p$ (on order $\leq p$ we can impose coherence with a finite set of rules).

1. when near a **corner** of a three-dimensional cell, on a pattern

![Pattern Example](image)

in one of the copies of the Robinson subshift, the corresponding patterns in this sublayer has to be (respectively first and second coordinates):

![Pattern Example](image)

with similar rules for the other corners.

2. near a **edge**, for the corresponding blue corners on the two faces, the symbols have to be equal. On the middle of a edge, the blue corners are colored with $\square$ in the coordinate corresponding to the direction of the edge. The other coordinates are such that when the $p$-counter value on the edge is $0$, the bottom blue corners in the case of a vertical edge, and the left ones in the case of a horizontal edge, are colored with purple, and the other two with gray. When the $p$-counter value on the edge is not $0$, then all the blue corners are colored purple. For instance, on the following pattern with counter $0$ on the edge

![Pattern Example](image)

in any of the copies of the Robinson subshift, the patterns in this layer has to be

![Pattern Example](image)
When on the quarter of the edge, the blue corners are colored with \(\text{red}\) in the coordinate corresponding to the direction of the edge, and the other coordinate is \(\text{gray}\). For instance, on the pattern

\[
\begin{array}{cccc}
  \text{red} & \text{red} & \text{red} & \text{red} \\
  \text{red} & \text{red} & \text{red} & \text{red} \\
  \text{red} & \text{red} & \text{red} & \text{red} \\
  \text{red} & \text{red} & \text{red} & \text{red} \\
\end{array}
\]

the patterns has to be:

\[
\begin{array}{cccc}
  \text{red} & \text{red} & \text{red} & \text{red} \\
  \text{red} & \text{red} & \text{red} & \text{red} \\
  \text{red} & \text{red} & \text{red} & \text{red} \\
  \text{red} & \text{red} & \text{red} & \text{red} \\
\end{array}
\]

3. Inside a face of a three-dimensional cell, the degenerated behaviors correspond to the degenerated behaviors of the 2d version of the Robinson subshift. We consider only two-dimensional cells having order greater or equal to \(p\). When near a corner of a support petal inside the area, the symbols on the blue corners nearby have to be colored according to the color of the corner. On

\[
\begin{array}{cccc}
  \text{red} & \text{red} & \text{red} & \text{red} \\
  \text{red} & \text{red} & \text{red} & \text{red} \\
  \text{red} & \text{red} & \text{red} & \text{red} \\
  \text{red} & \text{red} & \text{red} & \text{red} \\
\end{array}
\]

in the copy of the Robinson subshift parallel to the face, the patterns (respectively first and second coordinate) in this layer have to be

\[
\begin{array}{cccc}
  \text{red} & \text{red} & \text{red} & \text{red} \\
  \text{red} & \text{red} & \text{red} & \text{red} \\
  \text{red} & \text{red} & \text{red} & \text{red} \\
  \text{red} & \text{red} & \text{red} & \text{red} \\
\end{array}
\] or

\[
\begin{array}{cccc}
  \text{red} & \text{red} & \text{red} & \text{red} \\
  \text{red} & \text{red} & \text{red} & \text{red} \\
  \text{red} & \text{red} & \text{red} & \text{red} \\
  \text{red} & \text{red} & \text{red} & \text{red} \\
\end{array}
\]

We have similar restrictions for the other orientations of the corner. On the intersection positions of two petals, the possibilities are the same as on the border of the three-dimensional cell, except is added the possibility that the two patterns of the couple are all gray. On the same example:

\[
\begin{array}{cccc}
  \text{gray} & \text{gray} & \text{gray} & \text{gray} \\
  \text{gray} & \text{gray} & \text{gray} & \text{gray} \\
  \text{gray} & \text{gray} & \text{gray} & \text{gray} \\
  \text{gray} & \text{gray} & \text{gray} & \text{gray} \\
\end{array}
\]

When near the border of a cell not intersecting another petal, or a line with three arrows symbols connecting corners, the symbols corresponding to the direction of the line are purple on the left and gray on the right when it is vertical, and purple on the bottom and gray on the top when it is horizontal. The symbols of the other coordinates on both sides have to match.
4. **Coherence with the functional areas:** the symbol [□□] in this sublayer can be superimposed only on □ in the functional areas sublayer. A symbol with a vertical arrow in the functional areas sublayer can be superimposed only with else (□□) or (□□) in this sublayer. A symbol with a horizontal (resp. vertical) arrow is superimposed with else (□□) or (□□) (resp. (□□)) in this sublayer. When the symbol in the functional areas sublayer is □, then in this layer it is (□□).

**Global behavior:**

In this sublayer, on each of the colored faces, the order 0 cells borders are colored with else □ or □. The ones colored with □ are exactly those that:

1. are contained in an order \( kp, k \geq 0 \), two-dimensional cell,
2. contained in columns of this cell that do not intersect any intermediate order cell,
3. and have virtual address written \( w_1..w_k \), where the words \( w_i \) are length \( p \) words on \( \{0, 1\} \) in the set
   \[ \Delta = \{0^j1^{p-j} : 0 \leq j \leq p\} \]

Such columns are called **active columns** when the order 0 cells border they intersect are colored □.

These petals are colored with a second color, with similar rules, replacing columns by lines. The lines colored □ are called **active lines**.

**Lemma 21.**

1. On any order \( n \) colored face, the number of active columns (resp. active lines) that are not in an order \( jq < n \) two-dimensional cell is \( 2^r(p+1)^q \), where \( n = qp + r \), with \( r < p \).

2. The number of active columns (resp. active lines) in an order \( 2k + 2 \) supertile (centered on an order \( k \) two-dimensional cell) on a gray face is less than \( 2^{p+1}(p+1)^q \), where \( k = qp + r \), with \( r < p \).

**Proof.**

1. (a) The factor \( 2^r \) corresponds to the order \( qp \) cells marked with □ in the addresses sublayer: there are \( 2^r \) columns (resp. lines) of them.

   (b) Then the process of this layer selects \( p + 1 \) addresses of order \( p(q - 1) \) cells relatively to the \( pq \) cells. For these cells, it selects \( p \) addresses of order \( p(q - 2) \) cells relatively to these order \( p(q - 1) \) cells, etc. Hence the factor \( (p + 1)^q \) in the formula.

   (c) The factor 2 comes from the fact that there are two active columns for all order 0 cells.

2. There are two possible cases for the color of the maximal order petal in the supertile: □ or □. In the first case, there are no active columns or lines. In the second one the argument is similar to the first point.

**Corollary 22.** The number of active functional columns in a cell of order \( qp \), for \( q \) integer, is \( 2^{mq} \).

**Lemma 23.** For all \( k < n \), on the left of the first \( k \) order cell in an order \( n \) cell (from left to right), the column is active. Moreover, denote \( p_k \) the number such that this column is the \( p_k \)th column amongst the active ones from left to right. There is an algorithm that computes \( p_k \) given as input \( k \).
Proof. 1. If \( k \) is between \( pq \) and \( pq + r \), all the order \( pq \) cells are selected by the process (colored purple). We pick the one which is just on the right of the leftmost \( k \) order cell. The order \( p(q - 1) \) cell whose address is \( 1...1 \) relatively to this order \( pq \) cell is selected. We repeat this argument, considering a sequence of \( p(q - k) \) order cells, the last one being the order 1 cell just on the right of the \( k \) order cell. The column on the right of the order 1 cell is an active column which is just on the left of the \( k \) order cell.

2. Other values of \( k \):

For the others \( k \), the argument is similar. Consider some \( q' \) such that \( q'p \leq k < (q' + 1)p \). Pick an order \( (q' + 1)p \) order cell which is selected by the process, and then the order \( q'p \) order cell whose address relatively to the first one represents the number \( 2^{k-(q'p+1)-1} \). Then choose successively the corresponding order \( (q'-1)p, ... ,0 \) rightmost cells. The column on the right of the last one is on the left of the picked order \( k \) cell.

Remark 3. The reason of the choice of \( p \) this way is the factor \((p + 1)^{q}\) in the first point of Lemma 21: we need that this number is a power of two, so that the linear counters have as period a Fermat number.

6.2.6 Propagation of the border information inside the colored faces

The point of this section is to give access to all the functional positions in a face of a three-dimensional cell (specified by \( \square \)) to the value of the \( p \)-counter of its border, and a border bit that specifies if a functional position is proper to the face as a two-dimensional cell.

Symbols:

The elements of \( \mathbb{Z}/p\mathbb{Z} \times \{0, 1\} \) and a blank symbol.

Local rules:

- non-blank symbols are superimposed on the faces of the three-dimensional cells. On these faces, they are superimposed on and only on positions having a Robinson symbol (for the copy which is parallel to the face) which is not a two-dimensional cell border symbol.

- a symbol propagates to any position in the neighborhood positions if this position is not in the border of a two-dimensional cell.

- when on a position near the border of a two-dimensional cell from inside (the inside and outside are characterized by the position of the arrows: left or right for vertical arrows, and top or bottom for horizontal ones), then the symbol is equal to the value of the \( p \)-counter on this border. The bit in \( \{0, 1\} \) (called border bit) is 1 when the border is colored with \( \square \) and 1 if colored with \( \square \) or \( \square \).

Global behavior:

All the functional positions of a face are colored with the value of the \( p \)-counter of its border. The ones that are proper to this face as a two-dimensional cell are colored with the border bit 1. The other ones with 0. This is done using the diffusion of a signal on the face. The propagation is stopped by the cells walls and whose. The information transported by these signals is determined on the border of the diffusion areas.
6.3 Frequency bits layer

The frequency bits will help to drive a process that, through signaling in the hierarchy, will generate the entropy dimension of the subshift.

Symbols:

The elements of $\{(0, 1) \cup \Box\}^3$.

Local rules:

- **First coordinate**: non blank symbols for the first coordinate of the couple are superimposed on and only on $np$ cell border positions in the copy of the Robinson subshift parallel to $e^2$ and $e^3$.

- **Second coordinate**: non blank symbols for the second coordinate of the couple are superimposed on and only on order $np$ cell border, $n \geq 0$ petal positions in the copy of the Robinson subshift parallel to $e^3$ and $e^1$.

- **Third coordinate**: non blank symbols for the third coordinate of the couple are superimposed on and only on order $np$ cell border positions in the copy of the Robinson subshift parallel to $e^1$ and $e^2$.

- This means that on a position in $\mathbb{Z}^3$, the number of non-blank coordinates correspond to the color of this position in the structure layer: $\square$ means three, $\Box$ means two, $\Box$ one and $\Box$ zero.

- **Synchronization rule**: on positions colored with $\Box$, the three symbols are equal.

Global behavior:

For all $n \geq 0$, the three dimensional cells having order $np$ are attached with the same bit called **frequency bit** and denoted $f_{bit}(n)$. Part of the work of the machine implemented in the following will be to impose that $f_{bit}(n) = a_n$ for all $n$.

6.4 Grouping bits

This layer supports bits, called **grouping bits**, attached to the two-dimensional cells of the hierarchy in each copies of the Robinson subshift, having order $qp$ for some $q$. They verify similar rules as the frequency bits, except that the Turing machines check that the $2^k$th (this corresponds to some sparse subset of the counter of two-dimensional cells) bit of this sequence is 1 for all $k$, and the other ones are 0.

The grouping bits serve to make group of random bits, described later, into values of the hierarchical counter.

6.5 Linear counter layer

The construction model of M. Hochman and T. Meyerovitch [HM10] implies degenerated behaviors of the Turing machines. For this reason, in order to preserve the minimality property, we use a counter which alternates all the possible behaviors of these machines. We describe this counter here. We attribute to each face a type, as follows. Each type has specific alphabet and rules.

- The top face of three-dimensional cells according to direction $e^3$ (that we call **type 1 face**) supports the following functions:
  
  - incrementation of the counter value in direction $e^2$. This value is a word written on the intersection of active columns and the bottom line of the face (Section 6.5.2).
Figure 23: Localization of the linear counter area (colored yellow). In purple, the localization of non-blank symbols in these areas. The demultiplexing line is represented as a thick dark line. The type of information supported by each face in this layer is listed under the picture.

- Extraction of some information (Section 6.5.3 and Section 6.5.4) from this value.
- the other faces serve for transporting some informations (Section 6.5.5) to the machine face. This is the bottom face of the three-dimensional cells according to direction $\text{e}_3$. I serve also for the synchronization of the counter values on three-dimensional cells having the same order and adjacent in directions $\text{e}_1$ and $\text{e}_3$. The definition of these face is as follows:

- The type 4 (resp. 5) faces are defined as connecting type 1 faces in direction $\text{e}_2$ (resp. $\text{e}_1$).
- The type 2, 3 faces are defined as bottom (resp. top) faces of three-dimensional cells according to direction $\text{e}_2$.
- The type 6, 7 faces are defined as top (resp. bottom) faces of three-dimensional cells according to direction $\text{e}_1$.
- The type $2', 3', 6', 7'$ faces are defined as connecting two type 2, 3, 6, 7 faces in direction $\text{e}_3$.

See Figure 23 for an illustration.

We describe the global behavior at the end of these sections, and the global behavior for the whole layer is written in Section 6.5.6.

In Section 6.5.1 we give some notations used in the construction of this layer.
6.5.1 Alphabet of the linear counter

Let \( l \geq 1 \) be some integer, and \( \mathcal{A}, \mathcal{Q} \) and \( \mathcal{D} \) some finite alphabets such that \(|\mathcal{A}| = |\mathcal{Q}| = 2^l\), and \(|\mathcal{D}| = 2^{4l^2 - 2}\). Denote \( \mathcal{A}_c \) the alphabet \( \mathcal{A} \times \mathcal{Q}^3 \times \mathcal{D} \times \{\leftarrow, \rightarrow\} \times \{\text{on, off}\}^2 \).

The alphabet \( \mathcal{A} \) will correspond to the alphabet of the working tape of the Turing machine after completing it such that it has cardinality equal to \(2^{2l}\) (this is possible by adding letters that interact trivially with the machine heads, and taking \( l \) great enough). The alphabet \( \mathcal{Q} \) will correspond to the set of states of the machine (after similar completion). The arrows will give the direction of the propagation of the error signal. The elements of the set \( \{\text{on, off}\}^2 \) are coefficients telling which ones of the lines and columns are active for computation (which has an influence on how each computation positions work), and the alphabet \( \mathcal{D} \) is an artifact so that the cardinality of \( \mathcal{A}_c \) is \(2^{2l+3}\), in order for the counter to have a period equal to a Fermat number.

Let us fix \( s \) some cyclic permutation of the set \( \mathcal{A}_c \), and \( c_{\max} \) some element of \( \mathcal{A}_c \).

6.5.2 Incrementation

**Symbols:**

The elements of \((\mathcal{A}_c \times \{0, 1\}) \times \{\square, \blacksquare\}\). The elements of \((\mathcal{A}_c \times \{0, 1\})\) are thought as the following tiles. The first symbol represents the south symbol in the tile, and the second one representing the west symbol in the tile:

\[
\begin{array}{c|c}
  s(c) & 1 \\
  \hline 1 & c  \\
\end{array} \quad \begin{array}{c|c}
  c & 0 \\
  \hline 0 & 0  \\
\end{array}
\]

for \( c \neq c_{\max} \), and

\[
\begin{array}{c|c}
  s(c) & 1 \\
  \hline 1 & c  \\
\end{array}
\]

for \( c = c_{\max} \).

The second is called the freezing symbol. The other symbols are the elements of the following sets: \( \mathcal{A}_c \times \{\square, \blacksquare\}, \{\square, \square\}, \mathcal{A}_c, \{0, 1\} \times \{\square, \blacksquare\}\).

**Local rules:**

- **Localization rules:**
  - in this sublayer, the non-blank symbols are superimposed on type 1 faces such that the value of the \( p \)-counter is \( \overline{0} \) and the border bit is 1 (we recall that any position in the face is colored with this information).
  - The elements \( \mathcal{A}_c \times \{\square, \blacksquare\} \), appear on the active columns, except the bottom line (according to direction \( \text{e}^2 \)).
  - The elements of \( \{\square, \square\} \), appear on all the positions of the face except the border and the bottom line of the functional area.
  - The elements of \( \mathcal{A}_c \times \{0, 1\} \times \{\square, \blacksquare\} \) appear on computation positions of the bottom line.
Figure 24: Localization of the linear counter symbols on face 1.

- The elements of \(\{0,1\} \times \{\text{■}, \text{□}\}\) appear on the other positions of the bottom line. See an illustration on Figure 24.

A value of the counter is a possible sequence of symbols of the alphabet \(\mathcal{A}_c\) that can appear on a row, on the positions where this row intersects an active column.

- **Freezing signal:**
  - **On the bottom line:**
    1. on the leftmost bottommost position, the color is ■ if and only if the south symbol in the tile is \(c_{\text{max}}\).
    2. this color propagates towards the right while the south symbol in the tile is \(c_{\text{max}}\). When this is not true, the color becomes white.
    3. the white symbol propagates to the left.
  - **Other positions:**
    1. the color part of the symbol propagates on the gray area on Figure 24.
    2. on the bottom right position of this area (on the top of the position on the bottom right corner of the face), the color is white if the color of the position on bottom is white. When this color is salmon (meaning the value of the counter is maximal), then, if the top right position of the adjacent face 4 is colored salmon, then the considered position is colored white. If not, then it is colored salmon. See Figure 25 for an illustration of possible freezing symbol configuration on face 1.

- **Incrementation of the counter:**
  On the bottom line of the area:
  - on the leftmost position of the bottom line, if the freezing signal of the top left position of the face 4 under is white, then the east symbol in the tile is 1 (meaning that the counter value is incremented in the line). Else it is 0 (meaning this is not incremented).
Figure 25: Possible freezing symbol configurations on face 1.

- on a computation position of this line, the part in \{0,1\} of symbol of the position on the right is the east symbol of the tile. The symbol on the left position is the west symbol of the tile. The symbol on the top position is the north symbol of the tile, and the symbol of the position in type 4 face under is equal to the south symbol of the tile.
- between two computation positions, the symbol in \{0,1\} is transported.

- Transfer of state and letter: in the active columns, while not in the bottom row, the coefficient is transported.

Global behavior:

On type 1 faces having order \(qp\) for some \(q \geq 0\), the counter value is incremented on the bottom line using an adding machine coded with local rules, except when the freezing signal on face 4 under is \(\square\). Then the value is transmitted through the face 4 above to the face 1 of the adjacent three-dimensional cell having the same order in direction \(e^2\). As a consequence, the counter value is incremented cyclically in direction \(e^2\) each time going through a three-dimensional cell. The freezing signal happens each time that the counter reaches its maximal value and stops the incrementation for one step, since during this step, the freezing signal is changed into \(\square\).

6.5.3 Demultiplexing lines

Symbols:

\(\square\) and \(\blacksquare\).

Local rules:

- Localization: the non-blank symbols are superimposed on type 1 faces.
- when not on a position with a corner in the copy of the Robinson subshift parallel to \(e^1\) and \(e^2\), the symbol \(\square\) propagates in directions \(\pm(e^1 + e^2)\).
• when on the north east (resp. south west) corner of the face, the position has symbol \([\square]\). It forces the same symbol in the direction \(-e_1 + e_2\) (resp. \(e_1 + e_2\)).

• the symbol \([\square]\) cannot cross the border of a two-dimensional cell inside the face, except through south west and north east corners.

**Global behavior:**

On the top face, according to direction \(e_3\), of every three-dimensional cells, the diagonal of this face joining the south west corner and the north east one is colored with the symbol \([\square]\) (we recall that each face has specific orientation). See an illustration on Figure 23.

### 6.5.4 Demultiplexing

**Symbols:**

The symbols of this sublayer are the elements of \(A_c\) and a blank symbol.

**Local rules:**

- **Localization rule:** The non-blank symbols are superimposed on the active lines of type 1 faces where that the value of the \(p\)-counter is 0 and the border bit is 1.

- **Transmission rule:** the symbols are transmitted through the columns of the face.

- **Demultiplexing rule:** on symbol \([\square]\) in the demultiplexing lines sublayer, the symbol in the present sublayer is equal to the symbol in the incrementation sublayer, except on the bottom line. In this case, the north symbol of the tile is copied.

**Global behavior:**

As a consequence of the local rules, the value of the counter is copied on the diagonal of type 1 faces having order \(qp\) and transmitted on the active lines of the faces, so that it can be transmitted to type 5, 6, 6', 7, 7' faces.

### 6.5.5 Information transfers

**Symbols:**

The symbols of this sublayer are the elements of the following sets: \(A_c \times \{\square, \square\}\), \(\{\square, \square\}\), \(A_c\), \(A \times Q \times D \times \{\text{on, off}\}\), \(\{\leftarrow, \rightarrow\} \times \{\text{on, off}\}\), \(Q \times \{\text{on, off}\}\), and a blank symbol.

**Local rules:**

- **Localization rules:** The non-blank symbols are superimposed on the positions of faces such that the value of the \(p\)-counter is 0 and the border bit is 1. According to the face type, the possible non-blank symbols and the location on the face are as follows:
  - type 2, 2': elements of \(A \times Q \times D \times \{\text{on, off}\}\), appearing on active lines.
  - type 3, 3': elements of \(\{\leftarrow, \rightarrow\} \times \{\text{on, off}\}\), appearing on active lines.
  - type 5: elements of \(A_c\), appearing on active lines.
  - type 6, 6', 7, 7': elements of \(Q \times \{\text{on, off}\}\), appearing on active columns.
  - type 4: elements of \(A_c \times \{\square, \square\}\), appearing on active columns, and elements of \(\{\square, \square\}\), appearing on the other positions of the face.

- **Information transfer rules:**
– On type 2, 2', 3, 3', 5 faces, the symbols are transmitted through the rows.

– On type 6, 6', 7, 7', the symbols are transmitted through columns.

– On type 4 faces, the symbols in \( A_c \times \{ \text{□□□} \} \) are transmitted through columns, and the ones in \( \{ \text{□□□} \} \) are transmitted through all the face, and the part of the symbols in \( A_c \times \{ \text{□□□} \} \) is equal to the symbols outside the column.

- Connection rules:
  - Across the line connecting face 2, 3, 6, 7 to face 2', 3', 6', 7', the symbols are equal.
  - Across the line connecting face 4 to face 1, on the active columns, the symbols in \( A_c \) on face 4 is equal to the north symbol of the tile on face 1.
  - Across the line connecting faces 2 and 2' (resp. 3 and 3', 6 and 6', 7 and 7') to face 1, if the symbol on the north of the tile on face 1 in the incrementation sublayer (resp. incrementation, demultiplexing, demultiplexing, and demultiplexing layers) is written

\[
w = (a, q^1, q^2, q^3, d, f, o^1, o^2) \in A \times Q^3 \times D \times \{\rightarrow, \leftarrow\} \times \{\text{on, off}\}^2,
\]

then the corresponding symbol on face 2 is equal to \((a, q^1, d, o^1)\) (resp. \((f, o^1), (q^2, o^2), (q^3, o^2), w\)).

Global behavior:

In this sublayer, the information of the counter value supported by face 1 having order \( q \), \( q \geq 0, \) after its incrementation in the first row of this face, is splitted. The parts are transmitted to faces 2, 3, 6, 7 where they will be used by the computing machines. Face 4 transfers information for the incrementation mechanism, and faces 2', 3', 6', 7', 5 allow the synchronization of the counter of adjacent three-dimensional cells having the same order in directions \( e^1 \) and \( e^2 \).

6.5.6 Global behavior

On the top faces of the order \( q \) three-dimensional cells, \( q \geq 0 \), the counter value represents a length \( m \) word on alphabet \( A_c \). It is incremented when going from a three-dimensional cell to the adjacent one in direction \( e^2 \), except once in a cycle, when the counter reaches its maximal value. This time, the value is incremented in two steps. Since the number of active columns is \( 2^{mq} \) (Corollary [27]), the period of the counter (meaning the number of time one has to jump from a three-dimensional cell to the adjacent one in direction \( e^2 \) to see the same value) is equal to

\[
(2^8)^q = 2^{8 \cdot q} + 1.
\]

Since \( m \) and \( l \) are fixed, all these numbers, for \( q \geq 0 \), are two by two different. The important fact about them is that there are all co-prime (Lemma [27]).

Moreover, for two adjacent cells in directions \( e^1 \) or \( e^3 \), their counter values are equal.

6.6 Machines layer

In this section, we present the implementation of Turing machines.

The support of this layer is the bottom face of three-dimensional cells having order \( q \) for some \( q \geq 0 \), according to direction \( e^3 \).

In order to preserve minimality, simulate each possible degenerated behavior of the machines, we use an adaptation of the Turing machine model as follows. The bottom line of the face is initialized with symbols in \( A \times Q \) (we allow multiple heads). The sides of the face are “initialized” with elements of \( Q \) (we allow machine heads to enter at each step on the two sides). As usual in this type of constructions, the tape is not connected. Between two computation positions, the information is transported. In our model, each computation position takes as input up to
four symbols coming from bottom and the sides, and outputs up to two symbols to the top and
sides. Moreover, we add special states to the definition of Turing machine, in order to manage the
presence of multiple machine heads. We describe this model in Section 6.6.2 and then show how
to implement it with local rules in Section 6.6.3.

In Section 6.6.1 we describe signals which activate or deactivate lines and columns of the
computation areas. These lines and columns are used by the machine if and only if they are active.
These signals are determined by the value of the linear counter.

The machine has to take into account only computations starting on well initialized tape and
no machine head entering during computation. For this purpose, we use error signals, described
in Section 6.7.

6.6.1 Computation-active lines and columns

In this section we describe the first sublayer.

*Symbols:*

Elements of \(\{\text{on}, \text{off}\}\)^2, of \(\{\text{on}, \text{off}\}\) and a blank symbol.

*Local rules:*

- **Localization rules:**
  - the non-blank symbols are superimposed on active lines and active columns positions
    on a the bottom face according to direction \(e^3\), with \(p\)-counter equal to \(0\) and border
    bit equal to \(1\).
  - the couples are superimposed on intersections of an active line and an active column,
    the simple symbols are superimposed on the other positions.

- **Transmission rule:** the symbol is transmitted along lines/columns. On the intersections
  the second symbol is equal to the symbol on the column. The first one is equal to the symbol
  on the line.

- **Connection rule:** Across the line connecting type 6,7 (resp. 2,3) face and the machine
  face, and on positions where the bottom line intersects with active columns, the symbol in
  \(\{\text{on}, \text{off}\}\) is equal to the first (resp. second) element of the couple in \(\{\text{on}, \text{off}\}\) in this layer.

*Global behavior:*

On the machine face of any order \(qp\) three-dimensional cell, the active columns and lines are
colored with a symbol in \(\{\text{on}, \text{off}\}\) which is determined by the value of the counter on this cell.
We call columns (resp. lines) colored with on computation-active columns (resp. lines).

6.6.2 Adaptation of computing machines model to minimality property

In this section we present the way computing machines work in our construction. The model we
use is adapted in order to have the minimality property, and is defined as follows:

**Definition 24.** A computing machine \(M\) is some tuple \((Q, A, \delta, q_0, q_e, q_s, \#)\), where \(Q\) is the
state set, \(A\) the alphabet, \(q_0\) the initial state, and \# is the blank symbol, and

\[ \delta : A \times Q \to A \times Q \times \{\leftarrow, \rightarrow, \uparrow\}. \]

The other elements \(q_e, q_s\) are states in \(Q\), such that for all \(q \in \{q_e, q_s\}\), and for all \(a\) in \(A,\)

\[ \delta(a, q) = (a, q, \uparrow). \]

The special states \(q_e, q_s\) in this definition have the following meaning:
• the error state $q_e$: a machine head enters this state when it detects an error, or when it collides with another machine head.

This state is not forbidden in the subshift, but this is replaced by the sending of an error signal, and forbidding the coexistence of the error signal with a well initialized tape. The machine stops moving when it enters this state.

• shadow state $q_s$: because of multiple heads, we need to specify some state which does not act on the tape and does not interact with the other heads (acting thus as a blank symbol). The initial tape will have a head in initial state on the leftmost position and shadow states on the other ones.

Any Turing machine can be transformed in such a machine by adding some state $q_s$ verifying the corresponding properties listed above.

Moreover, we add elements to the alphabet which interact trivially with the machine states. This means that for any added letter $a$ and any state $q$, $\delta(a,q) = (a,q,\uparrow)$, and then machines states which interact trivially with the new alphabet, so that the cardinality of the state set and the alphabet are $2^2$.

When the machine is well initialized, none of these states and letters will be reached. Hence the computations are the ones of the initial machine. As a consequence, one can consider that the machine we used has these properties.

In this section, we use a machine which does the following operations for all $n$:

• write 1 on position $p_n$ if $n = 2^k$ for some $k$ and 0 if not.

• write $a_k^{(n)}$ on positions $p_k$, $k = 1...n$.

The sequence $a$ is the $\Pi_1$-computable sequence defined at the beginning of the construction. The sequence $(a_k^{(n)})$ is a computable sequence such that for all $k$, $a_n = \inf_n a_k^{(n)}$. For all $n$, the position $p_n$ is defined to be the number of the first a active column from left to right which is just on the right of an order $n$ two dimensional cell on a face, amongst active columns.

### 6.6.3 Implementation of the machines

In this section, we describe the second sublayer of this layer.

**Symbols:**

The symbols are elements of the sets $A \times Q$, in $A$, $Q^2$, and a blank symbol.

**Local rules:**

• **Localization:** the non-blank symbols are superimposed on the bottom faces of the three-dimensional cells, according to direction $e^3$. On these faces, they are superimposed on positions of computation-active columns and rows with $p$-counter value equal to 0 and border bit equal to 1. More precisely:
  
  – the possible symbols for computation active columns are elements of the sets $A$, $A \times Q$ and elements of $A \times Q$ are on the intersection with computation-active rows.
  
  – other positions are superimposed with an element of $Q^2$. See an illustration on Figure 26.

• along the rows and columns, the symbol is transmitted while not on intersections of computation-active columns and rows.

• **Connection with the counter:** On active computation positions that are in the bottom line of this area, the symbols are equal to the corresponding subsymbol in $A \times Q$ on face 2 of the linear counter. On the leftmost (resp. rightmost) column of the area that are in a computation-active line, the symbol is equal to the corresponding symbol on face 7 (resp. 6) of the linear counter.
• **Computation positions rules:**

Consider some computation position which is the intersection of a computation-active row and a computation-active column.

For such a position, the **inputs** include:

1. the symbols written on the south position (or on the corresponding position on face 2 when on the bottom line),
2. the first symbol written on the west position (or the symbol on the corresponding position on face 7 when on the west border of the machine face),
3. and the second symbol on the east position (or the symbol on the corresponding position on face 6 when on the west border of the machine face).

The **outputs** include:

1. the symbols written on the north position (when not in the topmost row),
2. the second symbol of the west position (when not in the leftmost column),
3. and the first symbol on the east position (when not on the rightmost column).

Moreover, on the bottom line, the inputs from inside the area are always the shadow state $q_s$.

See Figure 27 for an illustration.

On the first row, all the inputs are determined by the counter and by the above rule. Then, each computation-active row is determined from the adjacent one on the bottom and the value of the linear counter on faces 6 and 7, by the following rules. These rules determine, on each computation position, the outputs from the inputs:

---

Figure 26: Localization of the machine symbols on the bottom faces of the cubes, according to the direction $e_3$. Blue columns (resp. rows) symbolize computation-active columns (resp. rows).
1. **Collision between machine heads:** if there are at least two elements of $Q\setminus\{q_s\}$ in the inputs, then the computation position is superimposed with $(a, q_e)$. The output on the top (when this exists) is $(a, q_e)$, where $a$ is the letter input below. The outputs on the sides are $q_s$. When there is a unique symbol in $Q\setminus\{q_s\}$ in the inputs, this symbol is called the machine head state (the symbol $q_s$ is not considered as representing a machine head).

2. **Standard rule:**
   (a) when the head input comes from a side, then the functional position is superimposed with $(a, q)$. The above output is the couple $(a, q)$, where $a$ is the letter input under, and $q$ the head input. The other outputs are $q_s$. See Figure 28 for an illustration of this rule.
   (b) when the head input comes from under, the above output is:
      - $\delta_1(a, q)$ when $\delta_3(a, q)$ is in $\{\rightarrow, \leftarrow\}$
      - and $(\delta_1(a, q), \delta_2(a, q))$ when $\delta_3(a, q) = \uparrow$.
      The head output is in the direction of $\delta_3(a, q)$ (when this output direction exists) and equal to $\delta_2(a, q)$ when it is in $\{\rightarrow, \leftarrow\}$. The other outputs are $q_s$. See Figure 29 for an illustration.

3. **Collision with border:** When the output direction does not exist, the output is $(a, q_e)$ on the top, and the outputs on the side is $q_s$. The computation position is superimposed with $(a, q)$.
Figure 28: Illustration of the standard rules (1).

\[ \delta_1(a, q) \]
\[ \delta_2(a, q) \]
\[ \delta_3(a, q) = \leftarrow \]

Figure 29: Illustration of the standard rules (2).

\[ \delta_1(a, q) \]
\[ \delta_2(a, q) \]
\[ \delta_3(a, q) = \rightarrow \]

\[ \delta_1(a, q), \delta_2(a, q) \]

4. **No machine head:** when all the inputs in \( Q \) are \( q_s \), and the above output is in \( A \) and equal to its input \( a \).

**Global behavior:**

On the bottom faces according to \( e^3 \) of order \( q_p \) three-dimensional cells, we implemented some computations using our modified Turing machine model. This model allows multiple machine heads on the initial tape and entering in each row. When there is a unique machine head on the leftmost position of the bottom line and only blank letters on the initial tape, and all the lines and columns are computation-active, then the computations are as intended. This means that a the machine write successively the bits \( a_k^{(n)} \) on the \( p_k \)th column of its tape (in order to impose the value of the frequency bits), and moreover writes 1 if \( k \) is a power of two, and 0 if not, in order to impose the value of the grouping bits. It enters in the error state \( q_e \) when it detects an error.

When this is not the case, the computations are determined by the rules giving the outputs on computation positions from the inputs. When there is a collision of a machine head with the border, it enters in state \( q_e \). When heads collide, they fusion into a unique head in state \( q_e \). In Section 6.7 we describe signal errors that helps us to take into account the computations only when the initial tape is empty, all the lines and columns are computation-active, and there is no machine head entering on the sides of the area.

6.7 **Error signals**

In order to simulate any behavior that happens in infinite areas in finite ones, we need error signals. This means that when the machine detects an error (enters a halting state), it sends a signal to
the initialization line to verify it was well initialized: that the tape was empty, that no machine head enters on the sides, and that the machine was initially in the leftmost position of the line, and in initialization state. Moreover, for the reason that we need to compute precisely the number of possible initial tape contents, we allow initialization of multiple heads. The first error signal will detect the first position from left to right in the top row of the area where there is a machine head in error state or the active column is off. This position only will trigger an error signal (described in Section 6.7.3) according to the direction specified just above when it in the top line of the area (the word of arrows specifying the direction is a part of the counter). The empty tape signal detects if the initial tape was empty, and that there was a unique machine head on the leftmost position in initialization state. The empty tape and first error signals are described in Section 6.7.1. The empty sides signal, described in Section 6.7.2, detects if there is no machine head entering on the sides, and that the on/off signals on the sides are all equal to on. The error signal is taken into account (meaning forbidden) when the empty tape, and empty sides signals are detecting an error.

6.7.1 Empty tape, first error signals

Symbols:

The first sublayer has the following symbols:
- symbols in \{ spokeswoman, man, woman\}^2, and a blank symbol.

Local rules:

- Localization: non blank symbols are superimposed on the top line and bottom line of the border of the machine face as a two-dimensional cell.

- First error signal: this signal detects the first error on the top of the functional area, from the left to the right, where an error means a symbol off or qe. The rules are:
  - the topmost leftmost position of the top line of the cell is marked with [woman].
  - the symbol [woman] propagates the the left, and propagates to the right while the position under is not in error state qe and the symbol in \{on, off\} is on.
  - when on position in the top row with an error, the position on the top right is colored [man].
  - the symbol [woman] propagates to the right, and propagates to the left while the positions under is not in error state.

Empty tape signal: this signal detects if the initial tape of the machine is empty. This means that it is filled with the symbol (#, qs) except on the leftmost position where it has to be (#, q0). The signal detects the first symbol which is different from (#, qs) or (#, q0) when on the left, from left to right (first color), and from left to right (second color). Concerning the first color:
  - on the bottom row, the leftmost position is colored with [man].
  - The symbol [woman] propagates to the right unless when on a position under a symbol different from:
    * (#, q0) when on the leftmost functional position,
    * (#, qs) on another functional position.
  - When on these positions, the symbol on the right is [woman].
  - the symbol [woman] propagates to the right.

for the second one:
– on the bottom row, the rightmost position is colored with.
– The symbol propagates to the left except when on a position under a symbol different from:
  * (#,q₀) when on the leftmost functional position,
  * (#,qₘ) on another computation position.
– When on these positions, the symbol on the right is.
– the symbol propagates to the left.

Global behavior:

The top row is separated in two parts: before and after (from left to right) the first error. The left part is colored and the right part . The bottom row is colored with a couple of color. The first one separates the row in two parts. The limit between the two parts is the first occurrence from left to right of a symbol different from (#,qₘ) or (#,q₀) when on the leftmost computation position above. The second color of the couple separates two similar parts from right to left.

See Figure 30 for an illustration.

6.7.2 Empty sides signals

This second sublayer has the same symbols as the first sublayer. The principle of the local rules is similar: the leftmost (resp. rightmost) column is splitted in two parts, the top one colored and the bottom one colored . The limit is the first position from top to bottom where the corresponding symbol across the limit with face 7 (resp. face 6) is (qₘ, on) (resp. qₘ). Moreover, the bottom row of the machine face is colored with a couple of colors. This couple is constant over the row. The first one of the colors is equal to the color at the bottom of the leftmost column. The second one is the color at the bottom of the rightmost one.

See an illustration on Figure 30.

6.7.3 Error signals

Symbols:

(error signal),

Local rules:

• Localization: the non-blank symbols are superimposed on the right, left and top sides of the border of machine faces in three-dimensional cells.

• Propagation: each of the two symbols propagates when inside one of these two areas:
  – the union of the left side of the face and positions colored in the top side of the face.
  – the union of the right side of the face and positions colored in the top side of the face.

• Induction:
  – on a position of the top side of the face which is colored and the position on the right is , if the symbol above in the information transfers layer is → (resp. ←), then there is an error signal on this position and none on the right (resp. there is no error signal on this position and there is one on the right).
  – on the rightmost topmost position of the face, if the first machine signal is , then there is no error signal.
• Forbidding wrong configurations:

there can not be four symbols and an error signal on the same position.

Global behavior:

When there is a machine head in error state in the top row, the first one (from left to right) sends an error signal to the bottom row (see Figure 30) in the direction indicated by the arrow on the corresponding position on face 3. This signal is forbidden if the machine is well initialized. This means that the working tape of the machine is empty in the bottom row. Moreover, there is a unique machine in state $q_0$ in the leftmost position of the bottom row, all the lines and columns are on and there is no machine entering on the sides. This means that the error signal is taken into account only when the computations have the intended behavior.

![Figure 30: Illustration of the propagation of an error signal, where are represented the empty tape, first machine and empty sides signals.](image)

Because for any $n$ and any configuration, there exists some three-dimensional cell in which the machine is well initialized (because of the presence of counters). For any $k$, there exists some $n$ such that the machine has enough time to check the $k$th frequency bit and grouping bit. This means that in any configuration of the subshift, the $k$th frequency bit is equal to $a_k$, and the $2^k$th grouping bit is 1, and the other ones are 0.

6.8 Hierarchical counter layer

This layer has two sublayers.

6.8.1 Grouping border bits

This sublayer supports bits on the top faces of the three-dimensional cells, according to $e^1$, on positions that are not in the border of a two-dimensional cell with grouping bit bit 1. The symbol are 0,1 and propagate to neighbors while these neighbors have not grouping bit equal to 1. When near the border of a cell with grouping bit equal to 1, the bit is 1 if it is the border of the face of the three-dimensional cell. Else, this bit is 0.

This means that the only functional positions which have bit 1 are the functional position on a face of an order $2^k p$ three-dimensional cell which are not in an order $2^j p$ two-dimensional cell with $j < k$. 

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We use a similar mechanism as the one presented in Section 6.2.1 so that the order $n$ two-dimensional cells have access to the grouping bit and the frequency bit of the two-dimensional order $n + 1$ cell just above in the hierarchy, as well as its $p$-addresses (for the addresses there is not border rule or $\epsilon$ symbol).

6.8.2 Hierarchy bits

*Symbols:*

- elements of $\{\square, \blacksquare\}^2$ and a blank symbol.

*Local rules:*

- **Localization:** the non-blank symbols of this layer are localized on the petal positions of the copy of the Robinson subshift which is parallel to $e^2$ and $e^3$.
- the couples are superimposed on and only on transformation positions.
- the symbol is transmitted through arrows and intersection of order $n$ and $n + 1$ petals such that the order $n$ one has counter 0 or the value of the $p$-counter is not $p - 1$. The other intersection positions are transformation positions.
- **Transformation:** on a transformation position, we have the following rules:
  - if the grouping bit of the above cell in the hierarchy is 1, then we have the following:
    * in the following cases, the second color is $\blacksquare$:
      1. the Robinson symbol is $\blacksquare$, or $\square$ and the orientation symbol is $\square$
      2. the Robinson symbol is $\blacksquare$, or $\square$ and the orientation symbol is $\square$
      3. the Robinson symbol is $\blacksquare$, or $\square$ and the orientation symbol is $\square$
      4. the Robinson symbol is $\blacksquare$, or $\square$ and the orientation symbol is $\square$
    * in the following cases, the second color is equal to the first one:
      1. the Robinson symbol is $\blacksquare$, or $\square$ and the orientation symbol is $\square$
      2. the Robinson symbol is $\blacksquare$, or $\square$ and the orientation symbol is $\square$
      3. the Robinson symbol is $\blacksquare$, or $\square$ and the orientation symbol is $\square$
      4. the Robinson symbol is $\blacksquare$, or $\square$ and the orientation symbol is $\square$
    * in the other cases, the second color is $\square$.
  - when:
    * grouping bit of the cell above in the hierarchy is 0,
    * the value of the $p$-counter is $0$,
    * and its frequency bit is 0,
  then the transformation rule relies on the addresses. If the couple of addresses is in
  \[\{(1^p, 0^p), (0^p, 1^p), (0^p, 0^p), (1^p, 1^p)\},\]
  then the color is transmitted. In the other cases, it becomes $\square$.
  - when:
    * the grouping bit of the cell above in the hierarchy is 0,
Figure 31: Illustration of the transformation rules of the hierarchy bits when the grouping bit is 1.

* the value of the $p$-counter is $\overline{0}$ and the frequency bit is 1,
* or the value of the $p$-counter is not $\overline{0}$,
then the second color is equal to the first one.

− **Coherence rules**: since the hierarchy bits are superimposed on the copy of the Robinson subshift parallel to $e^2$ and $e^3$, the degenerated behaviors of this layer correspond to the ones of the Robinson subshift. In this list we consider patterns intersecting two-dimensional cells having order greater or equal to $p$.

1. when near a **corner**, on a pattern

   ![Pattern Example](image)

   the pattern in this layer has to be amongst the following ones:

   ![Pattern Options](image)

   respectively when

   (a) the central corner is colored gray,
   (b) this corner is purple and the grouping bit is 1.
   (c) this corner is purple, the grouping bit is 0 and else the $p$-counter has not value $\overline{0}$, or this value is $\overline{0}$ and the frequency bit is 1, or the addresses are in
   $$\{(0^p, 0^p), (0^p, 1^p), (1^p, 0^p), (1^p, 1^p)\}.$$  
   (d) this corner is purple, the grouping bit is 0, the $p$-counter has value $\overline{0}$, the frequency bit is 0, and the addresses are not in
   $$\{(0^p, 0^p), (0^p, 1^p), (1^p, 0^p), (1^p, 1^p)\}.$$  

   There are similar rules for the other corners. This allows degenerated behaviors which do not happen around finite cells to be avoided. The other rules in this list have the same function.

2. near a the middle or the quarter of an edge of a cell or near a corner of a petal with 0, 1-counter value equal to 0, the symbols on the blue corners are gray if
the border of the cell is gray. If this is purple, the grouping bit is 0, the p-counter has value 0 and the frequency bit is 0, and the addresses are not in \{(0\text{p}, 0\text{p}), (0\text{p}, 1\text{p}), (1\text{p}, 0\text{p}), (1\text{p}, 1\text{p})\}, the blue corners are colored gray. In the other cases, they are colored purple. For instance, on the pattern

the pattern has to be amongst the following ones:

\[ \begin{array}{c|c|c} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \quad \begin{array}{c|c|c} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \quad \begin{array}{c|c|c} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \quad \begin{array}{c|c|c} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \]

\textit{Global behavior:}

Over the copy of the Robinson subshift which is parallel to \(e^2\) and \(e^3\), we make a signal propagate and transform on intersections of support petals and the transmission petal just above in the hierarchy. With this signal, we color the blue corners with a hierarchy bit in \{0, 0\}. This process relies on the frequency bits in a similar way as in the proof of Theorem 6 except that each time the signal enters inside an order \(2^k\) two-dimensional cell, not matter the color of the cell, the support petals just under in the hierarchy are colored \(x\). This leads to the preservation of the minimality property. After synchronization, Lemma 25 count the cardinality of these sets.

6.8.3 Random bits

\textit{Symbols:}

Elements of \(\{0, 1\}\) (these are the random bits, which generate entropy dimension) and a blank symbol.

\textit{Local rules:}

- **Localization:** the non blank symbols are superimposed on and only on positions with north west blue corner having a hierarchy bit equal to \(x\).

- **Transformation:** for any position \(u \in \mathbb{Z}^3\) which is not on the top (according to \(e^1\)) face of a three-dimensional cell, the random bit on position \(u\) is equal to the random bit on position \(u + e^1\). The transformation rule on these faces will be stated in Section 6.8.5.

The next sections describe the set of random bits as a value (similarly to the linear counter) incremented by 1 going through the top face of a three-dimensional cell according to direction \(e^1\).

6.8.4 Convolutions

In this sublayer is generated a path on the top faces of the three-dimensional cells according to direction \(e^1\).

\textit{Symbols:}
Local rules:

- **Localization:** the non-blank symbols are superimposed only on the top faces of the three-dimensional cells according to direction $e_1$.

- **Transmission:** the arrows and blank symbols propagate through five or four arrows symbols of the Robinson copy parallel to $e_2$ and $e_3$.

- **Triggering positions:**
  The position on the top or bottom line of the border of the face, with six arrows symbol in the Robinson copy parallel to this face is called **triggering position**. These positions are superimposed with $\square$, and this symbol can be only on this position. The presence of this symbol implies a right arrow symbol on the position on the right, a left arrow on the left and a blank symbol above if on the top line. When on the bottom line, the direction of the arrows is reversed, and the blank symbol is forced upwards.

- **Correspondence for other types of positions:**
  The following tables give a correspondence between other types of positions and the symbol superimposed on them in this layer. Table 1 is related to positions on the border of a face. In this table, induction refers to the fact that the position type imposes the presence of a symbol on a position nearby. The symbol and the position are specified in the table. Table 2 refers to positions inside a face.

- **Determination of symbols on the border:**
  A cross symbol cannot be present on the border.

| Type of position                  | Symbol | Inducted symbols | Inducted positions |
|----------------------------------|--------|------------------|-------------------|
| Corners of the area              |        |                  |                   |
| On the top line, superimposed with $\square$ | $\square$ |                  | right, resp. left position |
| Top line                         |        |                  | right, resp. left position |
| Leftmost column                  |        |                  | bottom, resp. top position |
| Leftmost column                  |        |                  | bottom, resp. top position |
| Rightmost column                 |        |                  | bottom, resp. top position |
| Rightmost column                 |        |                  | bottom, resp. top position |
| Bottommost line                  |        |                  | right, resp. left position |
| Bottommost line                  |        |                  | right, resp. left position |

Table 1: Correspondence table for positions on the border of a face.

Global behavior:

The global behavior is the drawing of a curve starting and ending at some position at the center of the top or bottom row of the border (depending on the orientation of the three-dimensional cell) of the top face according to $e_1$ of three-dimensional cells. This position is specified by purple color, and the path is going through every position having a random bit on it. This serves as a circuit for the incrementation signal of the hierarchical counter. See Figure 37.
6.8.5 Incrementation signal

Symbols:

This sublayer has symbols in \{\[\] \} or \{\[\][\[\]]\}^2 and a blank symbol.

Local rules:

- **Localization**: the non blank symbols are superimposed only on non blank convolution symbols when on the top face according to e^1 of a three-dimensional cell.

- when outside these faces, the non-blank symbols are superimposed on the positions that don’t have a petal symbol, and the symbol on a position u ∈ Z^3 is transmitted to next positions u ± e^2 and u ± e^3 when these positions are not in the border of a two-dimensional cell with grouping bit equal to 1. These positions have a simple symbol.

- the two bits symbols can be superimposed only on positions with a cross symbol in the convolution sub-sublayer, and on top faces of three-dimensional cells according to e^1.

- **Initialization**: The position with purple symbol in the convolution sublayer is superimposed with \[\].

- **Transmission**: the color \[\] is transmitted to the next position in the direction of the arrow in the convolution sublayer if:
  
  - this next position has random bit equal to 1 or the grouping border bit is 0 (we use this rule so that the counter in a 2^kp order two-dimensional cell does not interact with the random bits that are in a 2^jp order cell, with j < k),
  
  - or the next position has the purple symbol in the convolution sublayer.

  The symbol is changed only in these cases. In other cases, the next position is marked with \[\].

- the symbol \[\] is transmitted through the arrow unless the next position has the purple symbol.

- **Freezing signal**: on the position u with purple symbol in the convolution sublayer, if the color on the next position with arrow pointing on this position is \[\], then the symbols on positions u − e^1 and u + e^1 are different. Else, they are equal. When a position on the face have symbol \[\], then the symbol on positions u − e^1 and u + e^1 is \[\].

---

Table 2: Correspondence table for positions inside a face.
• **Incrementation:** on position $u$ of a top face according to $e^1$, if the grouping bit is 1 and the grouping border bit is 1, the symbol is $\square$. The symbol on position $u + e^1$ is $\blacksquare$. The random bit on position $u$ is 0. Else, it is equal to the random bit on position $u$.

• **Coherence rule:** we use coherence rules so that on degenerated behaviors of the structure layer, the sense of propagation of the incrementation signal is respected. For instance, on a pattern inside the face being

![Pattern Diagram]

the possible colorings are:

![Possible Colorings]

or

![Possible Colorings]

**Global behavior:**

When going through an order $n \neq 2^k p$ three-dimensional cell for any $k$, the random bits are not changed. When going through the top face according to $e^1$ of an order $2^k p$ order three-dimensional cell, the random bits that are in the corresponding two-dimensional cell and are not in a $2^j p$ cell with $j < k$ are grouped (using the grouping bits and the grouping border bits). The sequence of these bits is incremented by 1 - in a similar way as is incremented the linear counter, using the convolutions - when the freezing signal is $\square$. It is not incremented when it is $\blacksquare$. This last case happens only once when this set of bits is in maximal position. Because of this, and the formula proved in the first point of Lemma 25, the period of this counter is $2^{2k} + 1$, where

$$l_k = 4[(k+1)(p-1)+1] + 2(2^k - 1 - c_k) + 4pc_k$$

The functions $k \mapsto 2^k - 1 - c_k$ and $k \mapsto c_k$ are non-decreasing, and the function $k \mapsto (k+1)(p-1)$ is strictly increasing. Hence all the numbers $l_k$ are different. As a consequence, all these numbers are different Fermat numbers.

### 6.9 Synchronization layer

We use this layer to synchronize the hierarchical counters in the orthogonal directions of their incrementation direction $e^1$ (we recall that from the way we coded the linear counters, they are already synchronized in the orthogonal directions of their incrementation direction $e^2$) for three-dimensional cells having the same order in case of the linear counter. We synchronize only $2^k p$ cells for the same $k \geq 0$ (this is the purpose of grouping bits).

This layer has two sub-layers.

#### 6.9.1 Synchronization areas

The aim of this first sublayer is to localize places where synchronization of hierarchical counters occurs.

**Symbols:**

The symbols are $\big((\square, \blacksquare)\big)^2$.

**Local rules:**
Figure 32: Illustration of the synchronization areas rules. The plane generated by the first symbol of the couple is on the left, and the other one on the right.

- the first symbol of the couple is transmitted in the directions $\pm e^2$, and the second one in the directions $\pm e^3$.

- on a position $u \in \mathbb{Z}^3$ which is not superimposed with a corner in the copy of the Robinson subshift parallel to $e^3$ and $e^1$, the first symbol of the couple is transmitted to positions $u + e^3 + e^1$ and $u - e^3 - e^1$.

- on a position $u \in \mathbb{Z}^3$ which is not superimposed with a corner in the copy of the Robinson subshift parallel to $e^1$ and $e^2$, the first symbol of the couple is transmitted to positions $u + e^1 + e^2$ and $u - e^1 - e^2$.

- the symbol $\square$ as first symbol of the couple can not be superimposed on a position in the border of a two-dimensional cell in the Robinson copy parallel to $e^3$ and $e^1$ except a corner.

- the symbol $\square$ as second symbol of the couple can not be superimposed on a position in the border of a two-dimensional cell in the Robinson copy parallel to $e^1$ and $e^2$ except a corner.

- on a position $u$ with a north east or south west corner in the Robinson copy parallel to $e^3$ and $e^1$, the first symbol of the couple is $\square$. According to the orientation of the corner, it forces the presence of another symbol $\square$ on another position as follows:

  1. if a north east corner, the symbol is forced on position $u - e^3 - e^1$
  2. if a south west corner, it is forced on $u + e^3 + e^1$.

Moreover, when the grouping bit is 0, it forces the symbol $\square$ on positions:

  1. if a north east corner, the symbol is forced on position $u + e^3 + e^1$
  2. if a south west corner, it is forced on $u - e^3 - e^1$.

- on a position $u$ with a north east or south west corner in the Robinson copy parallel to $e^3$ and $e^1$, the second symbol of the couple is $\square$. According to the orientation of the corner, it forces the presence of another symbol $\square$ on another position as follows:

  1. if a north east corner, the symbol is forced on position $u - e^1 - e^2$
  2. if a south west corner, it is forced on $u + e^1 + e^2$.

Moreover, when the grouping bit is 0, it forces the symbol $\square$ on positions:

  1. if a north east corner, the symbol is forced on position $u + e^1 + e^2$
  2. if a south west corner, it is forced on $u - e^1 - e^2$. 

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Global behavior:

The global behavior induced by these rules is the drawing, for any three-dimensional cell having order $2^kp$ for some $k \geq 0$, of two planes crossing it. One of them links the north east edge parallel to $e^3$ to the south west one. The other one links the north east edge parallel to $e^4$ to the south west one, as on Figure 32. The first plane is drawn by the first symbol and generated by the vectors $e^1 + e^4$ and $e^2$. The other one is drawn by the second symbol and is generated by the vectors $e^1 + e^2$ and $e^3$.

For all $k$, these planes cross order $j$ cells for all $j < 2^kp$ that are inside the $qp$ order three-dimensional cell. However, an order $j \neq 2^kp$ cell for any $k$, one of these planes can cross this cell if and only if it is on the diagonal plane of an order $2^kp$ cell.

6.9.2 Synchronization of the hierarchical counters

This sub-sublayer consists of two copies of the random bit sublayer, the hierarchy bit sublayer and the incrementation signal sublayer, except that the first set of copies is parallel to $e^1$ and $e^2$ and the second one to $e^1$ and $e^3$ (instead of $e^2$ and $e^3$ for the hierarchical counter). Moreover, in these copies, there is no incrementation signal. This means that the rules on the faces of three-dimensional cells are the same as outside. Furthermore, when the first symbol of the synchronization areas sublayer is $\square$ then the random bit in the first copy is equal to the random bit of the hierarchical counter. When the second symbol of the synchronization areas sublayer is $\square$ then the random bit in the second copy is equal to the random bit of the hierarchical counter.

The global behavior induced is that the values of the hierarchical counter in two adjacent, in else direction $e^2$ or $e^3$, order $2^kp$ three-dimensional cells, are equal.

As a consequence, one can state the following lemma:

Lemma 25. For all $q \neq 2^k$ for any $k$, the number $r_q$ of possible colorings by hierarchy bits and random bits on a $2qp + 2$ order supertile (centered on an order $qp$ cell) verifies the following inequalities:

$$\left[\alpha(p).q^{\lambda(p)}\right]4^{c_q}.16^{p q^3} \geq \log_2(r_q) \geq 4^{c_q}.16^{p q^3}$$

where $c_{q,1}$ is the number of $i \leq q − 1$ such that $a_i = 1$ and $c_{q,0}$ is the number of $i \leq q − 1$ such that $a_i = 0$, and $\alpha(p), \lambda(p) > 0$ depend only on $p$.

Proof. A formula on the number of proper blue corner positions in a cell having hierarchy bit $\square$:

For all $k \geq 0$, in an order $2^kp$ cell, the number of blue corners having hierarchy bit $\square$ that are not in an order $2^p/p$ cell, $j < k$, is

$$d_k = 4.4^{k}.16^{k(p−1)}.4^{2k−1−c_k−k}.4.16^{p−1}.16^{pc_k},$$

where $c_k$ is the number of $i \leq 2^k − 1$ such that the $i$th frequency bit $a_i$ is 1.

Indeed:

1. the process that rules the coloring with hierarchy bits of this cell starts on 4 order $2^k p − 1$ cells, hence the factor $4.16^{p−1}$.
2. The factor $4^k.16^{k(p−1)}$ comes from the transitions occurring when the grouping bit is 1 (there are $k$ such transitions).
3. The factor $4^{2k−1−c_k−k}$ comes from the transitions occurring when the grouping bit is 0 and the frequency bit is 0.
4. The factor $4.16^{pc_k}$ comes from the transitions when the grouping bit is 0 and the frequency bit is 1.
5. The factor 4 comes from the fact that there are four blue corners for each order 0 cell.
• On the hierarchy bits of a supertile:

An order $2qp + 2$ supertile is centered on an order $qp$ cell. There are two possibilities for the hierarchy bits, as follows.

1. The border of this cell is colored □ This means that all the blue corners in the supertiles colored with □ except the ones that are in an order $2^i p$ cell with $2^i \leq q$. This is equivalent to $j \leq \lfloor \log_2(q) \rfloor$. Since the $j$th hierarchical counter for all $j \leq \lfloor \log_2(q) \rfloor$ in this supertile are synchronized, the number of possible colorings of this supertile by random bits is given by the product for $j \leq \lfloor \log_2(q) \rfloor$ of the numbers of possible colorings by random bits of the set of blue corners that are in an order $2^i p$ cell and not in an order $2^i p$ cell with $i < j$. These numbers are $2^{d_j}$, so the total number of random bits displays is $2^{\lambda_1(q)}$, where

$$\lambda_1(q) = \sum_{j=0}^{\lfloor \log_2(q) \rfloor} d_j.$$

2. The border is colored □ In this case, we also have to count the number of hierarchy bits that are in the supertile but not in an order $2^i p$ cell with $j \leq \lfloor \log_2(q) \rfloor$. This case is similar to the first point of this proof, and the number of random bits displays in this case is $2^{\lambda_2(q)}$, where:

$$\lambda_2(q) = \sum_{j=0}^{\lfloor \log_2(q) \rfloor} d_j + 4.4^{\lfloor \log_2(q) \rfloor + 1} 16^{\lfloor \log_2(q) \rfloor (p-1)} 4^{c_2 - \lfloor \log_2(q) \rfloor} 16^{pc_4}.$$

As a consequence, the total number of possibilities for the hierarchy bits and random bits is

$$2^{\lambda_1(q)} + 2^{\lambda_2(q)} \leq 2 \cdot 2^{\lambda_2(q)},$$

since $\lambda_1(q) \leq \lambda_2(q)$.

• Inequalities:

The lower bound is clear. For the upper bound, we have, following the last point:

$$\log_2(r_q) \leq \left(4.4^{\lfloor \log_2(q) \rfloor + 1} 16^{\lfloor \log_2(q) \rfloor (p-1)} 4^{c_2 - \lfloor \log_2(q) \rfloor} 16^{pc_4} + \sum_{j=0}^{\lfloor \log_2(q) \rfloor} d_j + 1 \right) + 1.$$

Hence from the fact that the functions $k \mapsto c_k$ and $k \mapsto 2^k - 1 - c_k$ are non-decreasing and the inequality $\log_2(q) \geq \lfloor \log_2(q) \rfloor$:

$$\log_2(r_q) \leq 2(\log_2(q) + 1) 4^{\log_2(q) + 1} 16^{\log_2(q)(p-1)} 4^{c_2 - \log_2(q) + 1} 16^{pc_4} = 2(\log_2(q) + 1) 4^{2} 16^{\log_2(q)(p-1)} 4^{c_2} 16^{pc_4}.$$

Since $2(\log_2(q) + 1) 4^{2} 16^{\log_2(q)(p-1)}$ can be bounded by $\alpha(p)q^{\lambda(p)}$ for some functions $\alpha$ and $\lambda$, this provides the upper bound.

\[\square\]

7 Properties of the subshifts $X_z$:

7.1 Pattern completion

In this section, we prove the following proposition, which will serve to prove that $X_z$ is minimal. We assume the reader has some familiarity with the properties of the Robinson subshift.
Proposition 26. Let $P$ some $n$-block in the language of $X_z$. Then $p$ can be completed into an admissible pattern over a three-dimensional cell.

Let $P$ some $n$-block in the language of $X_z$ which appears in some configuration $x$. Let us prove that it can be completed into a three-dimensional supertile. We follow some order in the layers for the completion. First we complete the pattern in the structure layer, then in the functional areas layer, the linear counter and machine layers and then the hierarchy bits and the hierarchical counter. After this we prove how this supertile can be completed into a three-dimensional cell.

7.1.1 Completion of the structure

When the pattern $P$ is a sub-pattern of an infinite supertile of $x$, this is clear that the projection of $P$ into the structure layer can be completed into a finite supertile in the configuration $x$.

When the support of $P$ crosses the separating area between the supports of the infinite super-tiles, as in the proof of Proposition 14, we can still complete the projection of $P$ over the structure layer into an order $k$ three-dimensional supertile, with $k$ great enough.

7.1.2 Completion of the functional areas

The completion in the functional areas layer is more subtle and depends on how the support of the pattern crosses the area between the supports of infinite supertiles. We will choose $k$ great enough in each of the cases. We tell how to complete the border of the greatest three-dimensional cell included in the supertile, since the smaller cells can be completed according to the configuration $x$. Here is a list of all the possible cases:

- When the pattern intersects the corner of an infinite three-dimensional cell, we complete the projection of $P$ on the structure layer on the inside of each of the faces into the inside of a two-dimensional cell (without the border). The size of the two-dimensional cell is chosen according to the value of the $p$-counter and the grouping bit of the three-dimensional cell corner. The coherence rules allow the coloring of the petals in this two-dimensional cell to be determined. Indeed, since this is the corner of a three-dimensional cell, the border rules imply that the borders of the faces are colored with $\square$ in the functional areas sublayer, and $\blacksquare$ in the active areas sublayer. The coherence rules allow the colors of the blue corners around the corner of the three-dimensional infinite cell to be determined. The transformation rules imply that there is a unique possibility for the coloring of the border of the order 0 two-dimensional cells around, then the order 1 two-dimensional cells, etc. This thus determines the color of the other petals in the inside of the two dimensional cells. See an illustration on Figure 33 for the functional areas sublayer.

![Figure 33: Illustration of the sequence of implications of the coherence rule of the functional areas sublayer.](image)

The insides of these two-dimensional cells can be considered as the quarter of a greater cell. This allows completing the structure of the pattern $P$ into a finite three-dimensional cell.
which is colored in the functional areas layer as any three-dimensional cell that appears in the subshift.

- When the pattern intersects an infinite gray face, one can complete the pattern into the structure layer such that the projection over the copy of $X_{adR}$ parallel to the face is a supertile. We use Proposition 14 in case that the patterns intersect the area between infinite two-dimensional supertiles over the face. We chose the order of the cell according to the value of the $p$-counter on the border of this cell. As we have coherence rules for the inside of the faces, this supertile can be colored as any finite supertile on the faces in the functional areas layer. There are the following possibilities (in all these cases, the symbols on this supertile are determined):
  - the central corner of the supertile is colored in the functional areas sublayer, and as a consequence is colored with the \((\square, \square)\) in the active areas sublayer. The color of the petal intersecting this one, and then the other petals in the supertile, is thus determined according to the orientation symbol written on the central corner.
  - this corner is colored \((\square, \rightarrow)\) in the functional areas sublayer, and else \((\square, \square)\) or \((\square, \square)\) in the active areas sublayer.
  - it is colored or \((\square, \rightarrow)\) in the functional areas sublayer, and else \((\square, \square)\) or \((\square, \square)\) in the active areas sublayer.
  - this corner is colored with \(\square\), and there is no restriction in the active areas sublayer.

Concerning the functional areas sublayer, all these cases can be encountered in any arbitrary large colored faces with arbitrary value of the counter. Indeed, they are encountered on any supertile whose orientations with respect to its $n+1$, $n+2$ and $n+3$ supertiles lie in a particular set. This set corresponds to the rules of the functional areas sublayer, and to the condition that the $n+3$ order supertile is colored blue. This happens for all the values of the $p$-counter in any two-dimensional cell. Because some columns or lines are active and others not, all the cases in the active areas sublayer are possible, with restrictions corresponding to the coherence rules.

- The pattern intersects an edge between gray faces. This case is similar to the previous one, since the coherence rules impose that the symbols have to match on the sides of the edges.

### 7.2 Completion of the linear counter and machines computations

In this section, we tell how to complete the pattern over this supertile in the linear counter and machines layers. Since there are rules connecting the symbols of the two layers on the edges, we only have to tell how to complete separately these two layers when knowing only one part (near a corner, a edge, or inside) of the face corresponding to else the linear counter or the machine.

**The linear counter:**

1. when knowing a part of face 1, 2, 3, 4 or 5 which does not intersect the bottom line of face 1, top line of faces 4, 5 and right column for faces 2 (meaning places where there is incrementation of the counter), the only difficulty for completing comes from the freezing signal, which is \(\square\) only when all the letters are equal to $c_{\text{max}}$. Thus when the supertile is colored with this signal, there is nothing to do but to complete the face adding only letters equal to $c_{\text{max}}$. When the freezing signal is $\square$, then we have to add letters that are different from $c_{\text{max}}$.

2. when knowing some part of the incrementation positions, the completion depends on if we known the right or left part of the line (looking in the direction of face 1):
   - when knowing the right part, the freezing signal on the top face and bottom face are determined. We complete the left part of the bottom line of face 1 according to the
freezing signal on the right part. If at some point the freezing signal becomes \(\square\) after being \(\square\) (from left to right), this means that we can complete the letters on the right equal to \(c_{\text{max}}\). When the freezing signal is all blank, we can complete by adding at least some letter different from \(c_{\text{max}}\), and when it is all \(\square\) we can complete by adding only letters equal to \(c_{\text{max}}\).

**The machines.** When completing the machine face, there are two types of difficulties. The first one is managing the various signals: machine signals, first error, empty tape, empty side signals, and error signals. The second one is managing the space-time diagram of the machine. When the machine face is all known, there is no completion to make. Hence we describe the completion according to the parts of the face that are known (meaning that they appear in the initial pattern). Since there is strictly less difficulty to complete knowing only a part inside the face than on the border (since the difficulties come from completing the space-time diagram of the machine, in a similar way than for the border), we describe the completion only when a part of the border is known.

- When knowing the top right corner of the machine face:

  1. if the signal detects the first machine in error state from left to right (becomes \(\square\) after being \(\square\)), then we already know the propagation direction of the error signal. Then we complete the first machine signal and the error signal according to what is known. For completing the space-time diagram of the machine, the difficulty comes from the fact that this is possible that when completing the trajectory of two machine heads according to the local rules, they have to collide reversely in time. This is not possible in our model. This is where we use the on/off signals. We complete first the non already determined signals using only the symbol off, as illustrated on Figure 34.

        Then the space-time diagram is completed by only transporting the information.

        In the end, we completed with any symbols in \(Q\) or \(A \times Q\) where the symbols are not determined. We can do this since they do not interact with the known part of the space-time diagram. Then we complete empty tape and empty side signals according to the determined symbols.

  2. if the signal is all \(\square\) then one can complete the face without wondering about the error signal.

![Diagram](image-url)
3. if the signal is all \( \text{on} \), we complete in the same way as for the first point, and the \( \text{off} \)
signals contribute to the first machine signal being all \( \text{on} \). If there is no error signal on
the right side, one can complete so that the first machine in error state has above the
arrow \( \leftarrow \) indicating that the error signal has to propagate to the left. If there is an error
signal on the right, then we set this arrow as \( \rightarrow \). This is illustrated on Figure 35.

Figure 35: Illustration of the completion of the arrows according to the error signal in the known
part of the area, designated by a dashed rectangle.

- when knowing the top left corner, the difficulty comes from the direction of propagation of
  the error signal. This is ruled in a similar way as point 3 of the last case.

- when knowing the bottom right corner or bottom left corner: the completion is similar as in
  the last points, except that we have to manage the empty tape and sides signals. The difficult
  point comes from the signal, whose propagation direction is towards the known corner. If
  this signal detects an error before entering in the known area, we complete so that the added
  symbols in \{on, off\} are all off: this induces the error. When this signal enters without
  detecting an error, we complete all the symbols so that they do not introduce any error. A
  particular difficulty comes from the case when the bottom right corner is known. Indeed,
  when the signal enters without having detecting an error, this means we have to complete
  the pattern so that a machine head in initial state is initialized in the leftmost position of
  the bottom row. Since the pattern is completed in the structure layer into a great enough
  cell, this head can not enter in the known pattern.

- when the pattern crosses only a edge or the center of the face, the completion is similar (but
  easier since these parts have less information, then we need to add less to the pattern).

What is left to describe for the completion into a cubic supertile is the completion of the
hierarchical counter layer.

7.2.1 Completion of the hierarchical counter layer

We can complete the three copies of the hierarchy bits and random bits on the completed supertile
in a coherent way with the hierarchy bits and random bits on the pattern \( P \). The colors in this layer
are determined by a triple of colors in \{., .\}, corresponding to petal having maximal order in
the three supertiles defining the cubic supertile. We have to prove how to view a three-dimensional
cell having order \( n + 1 \) with these colors in a greater three-dimensional cell, since as a consequence
we can see the supertile around this order \( n + 1 \) cell in the orientation corresponding to orientation
marked on the supertile. It is sufficient to view this order \( n + 1 \) three-dimensional cell as a part
of an order \( 2^k p - 1 \) cell with \( k \) great enough that is inside a greater cell and nearest to one of
the corner (the colors of its borders are ., ., .). There are three cases:
1. When the order \( n + 1 \) cell has its three colors equal to \( \square \), then it can be found near the extremal corners of the \( 2^k p - 1 \) one (thanks to the transformation rules of the hierarchy bits layer).

2. When there is only one color equal to \( \square \), then consider the set of cells near a edge, having its direction orthogonal to the directions of the face of the three-dimensional cell having border colored \( \square \). This corresponding two-dimensional cell is colored \( \square \) for all these three-dimensional cells. We pick one of the cells in this set such that the other faces have border colored with \( \square \). This is possible because the other two two-dimensional cells have the same color for the cells in this set (because the \( 2^k p - 1 \) order cell has its borders all colored with \( \square \)).

3. When there are two colors equal to \( \square \), then we pick some order \( n + 1 \) two-dimensional cell on the diagonal of the face of the \( 2^k p - 1 \) order three-dimensional cell parallel to the face of the \( n + 1 \) order cell which border is colored \( \square \) such that the line and columns of order \( n + 1 \) two-dimensional cells on the face contains two-dimensional cells colored with \( \square \) (for instance the middle lines/columns of the face). Hence we can pick a cell with colors (\( \square \square \square \)) in the column of order \( n + 1 \) three-dimensional cell under the picked two-dimensional cell.

7.3 Computation of the entropy dimension

Let us give upper and lower bounds on the limsup and liminf of

\[
\frac{\log_2(\log_2(N_n(X_z)))}{\log_2(n)}.
\]

Let \( P \) some \( n \)-block in the language of \( X_z \).

7.3.1 Upper bound

One can complete \( P \) into an an order \( 2pq_n + 2 \) supertile, where

\[
g_n = \lceil \log_2(n)/2p \rceil + 1.
\]

The number of patterns over some \( k \) order supertile is less than \( K \) times the number of random bits layout over a \( k \) supertile times the number of possible values of the linear counter inside a supertile, where \( K > 0 \) is a constant (indeed, all the other layers are determined by a symbol in a finite set).

As a consequence of Lemma 25 and Lemma 21 the number of patterns over a \( 2pq_n + 2 \) order supertile is smaller than

\[
K |\mathcal{A}_c| 2^{p+1} (p+1)^{q_n} 2^{\alpha(p)} q_n^{\lambda(p)} 4^{\alpha_n} b_{16 b_{16}}^{c_{16}^n}.
\]

This implies that

\[
\log_2(N_n(X_z)) \leq \log_2(K) + 2.2^p (p + 1)^{q_n} \log_2(|\mathcal{A}_c|) + \alpha(p) q_n^{\lambda(p)} 4^{\alpha_n} b_{16 b_{16}}^{c_{16}^n},
\]
and then
\[
\frac{\log_2 \circ \log_2(N_n(X_z))}{\log_2(n)} \leq \frac{\log_2(\alpha(p), q_n^{\lambda(p)})}{\log_2(n)} + 2 \frac{\epsilon_0}{\log_2(n)} + \frac{4 \epsilon_1^p}{\log_2(n)} \left( 1 + \frac{2}{\log_2(K) + 2^{(p+1)q_n} \log_2(|A_n|)} \right)
\]

The first term of this sum tends towards zero, since
\[
\frac{\log_2(\alpha(p), q_n^{\lambda(p)})}{\log_2(n)} = O\left( \frac{\log_2 \circ \log_2(n)}{\log_2(n)} \right)
\]

The fact that the third term tends towards zero comes from the choice of \( p \). Indeed, we have
\[
\log_2(K) + 2^{2^p(p+1)q_n} \log_2(|A_n|) = O(2^{\log_2(n) \log_2(p+1)}) = O(n^{m/2(2^m-1)}).
\]

Moreover, by definition
\[
\frac{\epsilon_1}{q_n} \to z/2
\]

and as a consequence
\[
2^{\epsilon_1^p q_n} \geq n^{z/2}
\]

Since \( m/p < z/2 \), this means the third term tends towards zero. Thus we have
\[
\limsup_n \frac{\log_2 \circ \log_2(N_n(X_z))}{\log_2(n)} \leq \limsup_n \left( 2 \frac{\epsilon_0}{\log_2(n)} + \frac{4 \epsilon_1^p}{\log_2(n)} \right) \leq 2 \frac{1-z/2}{2p} + 4p/2p = 1/p + z(1-1/2p)
\]

### 7.3.2 Lower bound

For the lower bound, we do the reverse inclusion: if \( m \) is chosen great, any \( n \)-block \( P \) in the language of the subshift \( X_z \) contains some order \( 2pq_n + 2 \) super tile,
\[
q_n' = \lfloor \log_2(n)/2p \rfloor - 2,
\]

since these super tiles are repeated with period \( 2^{2^p q_n' + 4} \leq \frac{n}{2^{2^p q_n'}} \), and have size \( 2^{2^p q_n' + 3} \leq \frac{p}{2} \).

Then the number of \( n \)-blocks in the language of \( X_z \) is greater than
\[
2^{4 \epsilon_0 q_n' + 16 \epsilon_1^p q_n'}.
\]

Similar computation as done for the upper bound leads to
\[
\liminf_n \frac{\log_2 \circ \log_2(N_n(X_z))}{\log_2(n)} \geq 1/p + z(1-1/2p),
\]

which means \( X_z \) has an entropy dimension and this dimension is equal to
\[
D_h(X_z) = 1/p + z(1-1/2p).
\]

### 7.4 Minimality

In this section we prove that the subshift \( X_z \) is minimal. See [36] for a schema of the proof. This proof relies on the following lemma:

**Lemma 27** (Globach’s theorem). The numbers \( F_n = 2^{2^n} + 1, n \geq 0 \) are coprime.
Proof. Let \( m > n \geq 0 \). Then
\[
F_m = 2^{2m} + 1 = (2^{2m} + 1 - 1)2^{m-n} + 1 = \sum_{k=0}^{2m-n} \binom{2m-n}{k} (-1)^{-k} F_n^k + 1
\]
\[
= F_n \sum_{k=1}^{2m-n} \binom{2m-n}{k} (-1)^{k} F_n^{k-1} + 2
\]
This means that a common divisor of \( F_n \) and \( F_m \) divide 2, but 2 does not divide \( F_n \), so \( F_n \) and \( F_m \) are coprime. \( \square \)

Consider some block \( P \) in the language of \( X \), and complete it into a pattern \( P' \) over an order \( 2^k p \) three-dimensional cell. Pick some configuration \( x \in X \), and consider the restriction of \( x \) on the hierarchy bits layer (over the copy of the Robinson subshift parallel to \( e^2 \) and \( e^3 \)). On can find some \( u \) such that the projection of the three dimensional cell over the hierarchy bits layer appear in position \( u \) (using the same arguments as in Section 7.2).

For all \( i \), a three-dimensional cell appears on position \((2i4^{2^mp})e^1 + u\). Comparing the cell in position \((v + 2i4^{2^mp})e^1 + u\) and \((v + 2(i+1)4^{2^mp})e^1 + u\), the second one has the same lineary counters as the first one, and the \( j \)th hierarchical counters values are incremented \( 4(2^j-2^i)p \) times for all \( j \) in the the second one, with respect to the first.

Let \( t \) be the following application:
\[
t : \mathbb{Z}/p_1\mathbb{Z} \times \ldots \times \mathbb{Z}/p_k\mathbb{Z} \to \mathbb{Z}/p_1\mathbb{Z} \times \ldots \times \mathbb{Z}/p_k\mathbb{Z}
\]
\[
(i_1, \ldots, i_k) \mapsto (i_1 + 4(2^{j_1}-1)p, \ldots, i_k + 1)
\]
where \( p_1, \ldots, p_k \) are the periods of \( k \) first hierarchical counters. This is a minimal application, meaning that for all \( i, j \in \mathbb{Z}/p_1\mathbb{Z} \times \ldots \times \mathbb{Z}/p_k\mathbb{Z} \), there exists some \( n \) such that \( t^n(i) = j \). Indeed, considering some \( i \), denote \( n_0 \) the smallest positive integer such that \( t^{n_0}(i) = 1 \). This means that \( p_j \) divides \( n_04(2^{j_1}-2^i)p \) for all \( j \), and because \( p_j \) is a Fermat number, it is odd, and this implies that \( p_j \) divides \( n_0 \). For the numbers \( p_j \) are coprime (Lemma 27), this implies that \( p_1 \times \ldots \times p_j \) divides \( n_0 \). Because this number is a period of the application \( t \), this means that \( p_1 \times \ldots \times p_j \) is the smallest period of every element of \( \mathbb{Z}/p_1\mathbb{Z} \times \ldots \times \mathbb{Z}/p_k\mathbb{Z} \) under the application \( t \).

As a consequence, for all \( i \), the finite sequence \( (t^n(i)) \), for \( n \) going from 0 to \( p_1 \times \ldots \times p_j - 1 \), takes all the possible values in \( \mathbb{Z}/p_1\mathbb{Z} \times \ldots \times \mathbb{Z}/p_k\mathbb{Z} \).

![Figure 36: Schema of the proof for the minimality property of \( X_2 \).](image)

Hence one can find in \( x \) an order \( 2^k p \) three-dimensional cell which supports values of the hierarchical counter equal to the ones that are in \( P' \), iterating \( n_0 \) times this shift.

Now consider \( u' \) the position where this cell appears in \( x \), and the cells appearing in the same configuration on positions \( 2i4^{2^mp}e^2 + u' \). Using a similar argument as above, on the function
\[
t' : \mathbb{Z}/p_1'\mathbb{Z} \times \ldots \times \mathbb{Z}/p_k'\mathbb{Z} \to \mathbb{Z}/p_1'\mathbb{Z} \times \ldots \times \mathbb{Z}/p_k'\mathbb{Z}
\]
\[
(i_1, \ldots, i_k) \mapsto (i_1 + 4(2^{j_1}-1)p, \ldots, i_k + 1)
\]
where $p'_1, \ldots, p'_k$ are the periods of $k$ first linear counters, one can find some $i$ such that on position $2i + 2e^2 + u'$, the cell supports the same values of linear counters than $P'$, iterating $n_1$ times this shift. Moreover, since the hierarchical counters are synchronized in the direction $e^2$, this cells has also the same hierarchical counter values as $P'$.

For the pattern on the cell is determined by the values of the counters, this cell supports the pattern $P'$.

Hence $X_z$ is a minimal SFT.

**Remark 4.** It is trivial to find some minimal $\mathbb{Z}^3$-SFT having entropy dimension equal to zero. However, it is not to find one having entropy dimension equal to two, and we don’t know a simpler proof than the construction presented in this text.

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Figure 37: Illustration of the convolutions rules.
Figure 38: Possible orientations of four neighbor supertiles having the same order (1).

Figure 39: Possible orientations of four neighbor supertiles having the same order (2).
Figure 40: Possible orientations of four neighbor supertiles having the same order (3).
Figure 41: Illustration of the correspondence between patterns of Figure 39 and parts of a supertile.