Hierarchical Wave Functions and Fractional Statistics
in Fractional Quantum Hall Effect on the Torus

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**ABSTRACT**

One kind of hierarchical wave functions of Fractional Quantum Hall Effect (FQHE) on the torus are constructed. The multi-component nature of anyon wave functions and the degeneracy of FQHE on the torus are very clear reflected in this kind of wave functions. We also calculate the braid statistics of the quasiparticles in FQHE on the torus and show they fit to the picture of anyons interacting with magnetic field on the torus obtained from braid group analysis.
1. Introduction

It is of great interest to see that how FQHE is realized on the surfaces of different topologies. FQHE on the torus is particular interesting because torus provides the simplest example of surface with nontrivial topology. Recently it has been emphasized that a new kind of order called topological order which appears in FQHE (Hall fluid)[1], the chiral spin fluid and the anyon superfluid. FQHE on the torus at filling $\frac{1}{m}$ with m being an integer has $m$-fold center-mass degeneracy [2,3]. This degeneracy actually is a manifestation of the topological order. Topological order describes the global properties of the ground state which depend on the topology of the surface and its low lying excitations. The Field theory of such kind system is controlled by topological field theory, Chern-Simons theory, which is particularly relevant in the application to the condensed matter systems. The relation of Chern-Simons theory to FQHE has been investigated in [4,5,6,7, etc.].

In [8], a kind of hierarchical wave functions (for Laughlin [9] wave function on the torus, see [3]) of FQHE on the torus is constructed by generalizing the one [12] on the plane (the hierarchical construction of FQHE state was first proposed in [10,11]). The hierarchical state is characterized by a generalized Abelian Chern-Simons theory. Furthermore the degeneracy is determined directly from the wave functions, which agrees with the prediction in [13,14].

Another kind of hierarchical wave function of FQHE on the plane have been constructed in [15] and analyzed by plasma analogue (hierarchical wave function in [15] is based the wave function proposed in [11]; for the case on the sphere, see [16]) and this wave function has a very clear physical picture, the hierarchical condensation of the quasiparticles (holes of the parents states). To construct this kind of wave function, we need the wave functions of the condensed quasiparticles in different hierarchy. Those wave functions turn out to be multi-component on the torus. In this paper, we shall construct such kind of wave functions on the torus and see what is the degeneracy of the wave functions. FQHE at hierarchical filling on the torus also has been investigated in [7,17] in the context of Chern-Simons
theory.

In the next section, the notations used in this paper are summarized and some results in [2] are briefly reviewed. In section 3, the hierarchical wave functions on the torus of the type as in [15] are constructed following a simple example. In section 4, the fractional statistics of the quasiparticles in FQHE on the torus is discussed. In section 4, we give some remarks about the large gauge transformations and the modular transformations of the wave functions.

2. Basic Notations and Haldane Wave Functions

Following [2,8], we consider a magnetic field with potential $A = -B y \hat{x}$, the wave function describing an electron in the lowest Landau level has the form

$$\psi(x, y) = e^{-\frac{By}{2} f(z)}, \quad (2.1)$$

where $f(z)$ is the holomorphic function, and the units $e = 1, \hbar = 1$ are used. It is better to use the lagrangian to analyse the symmetry of the theory. The lagrangian of the electron in the magnetic field is

$$L = \sum_{i=1,2} \frac{1}{2} m (v^i)^2 + A^i v^i, \quad (2.2)$$

where $L$ is invariant up to a total time derivative under the translations. The corresponding Noether currents due to the translations are

$$t_x = m \dot{x} - B y, \quad t_y = m \dot{y} + B x. \quad (2.3)$$

The conjugate momenta are

$$p_x = m \dot{x} - B y, \quad p_y = m \dot{y}. \quad (2.4)$$
So

\[ t_x = px, t_y = py + Bx. \] (2.5)

They commute with Hamiltonian

\[ H = \frac{1}{2m} [(px + By)^2 + (py)^2], \] (2.6)

with the commutations \([x, px] = i, [y, py] = i\) when the theory is quantized. We work on a torus by identifying \(z \sim z + m + n\tau\) with \(\tau = \tau_1 + i\tau_2\) and \(\tau_2 \geq 0\). The consistent boundary conditions imposed on the wave function of the electron on this torus are

\[ e^{it_x} \psi = e^{i\phi_1} \psi, e^{it_1 t_x + i\tau_2 t_y} \psi = e^{i\phi_2} \psi, \] (2.7)

with the condition \(\tau_2 B = 2\pi \Phi\), where \(\Phi\) is an integer, which will insure that \(e^{it_x}, e^{it_1 t_x + i\tau_2 t_y}\) commute with each other for the consistence of the equation (2.7).

By using the relation

\[ e^{it_1 t_x + i\tau_2 t_y} = e^{-i\tau_2 \frac{Bx^2}{2\tau_1}} e^{i\tau_1 px + i\tau_2 py} e^{-i\pi \frac{Bx^2}{2\tau_1}}, \]

(2.7) can be written as

\[ f(z + 1) = e^{i\phi_1} f(z), f(z + \tau) = e^{i\phi_2} e^{-i\pi \Phi(2z + \tau)} f(z). \] (2.8)

For the many-particle wave functions, the condition of the equation (2.8) is imposed on every particle.

The standard \(\theta\) function is defined as

\[ \theta(z|\tau) = \sum_n \exp(\pi in^2 \tau + 2\pi inz), n \in \text{integer}. \] (2.9)
More generally, the \( \theta \) function* on the lattice \([18]\) is

\[
\theta(z|e, \tau) = \sum_{n_i} \exp(\pi iv^2 \tau + 2\pi iv \cdot z),
\]

(2.10)

where \( v \) is a vector on a \( l \)-dimension lattice, \( v = \sum_{i=1}^{l} n_i e_i \), with \( n_i \) being integers, \( e_i \cdot e_j = A_{ij} \) and \( z = z_i e_i \). The \( \theta \) function in the equation (2.9) is a special case of the \( \theta \) function defined by (2.10) with \( l = 1, e_1 \cdot e_1 = 1 \). Furthermore we define

\[
\theta \left[ \begin{array}{c} a \\ b \end{array} \right] (z|e, \tau) = \sum_{n_i} \exp(\pi i(v + a)^2 \tau + 2\pi i(v + a) \cdot (z + b)),
\]

(2.11)

where \( a, b \) are arbitrary vectors on the lattice. Only the positive matrix \( A \) will be considered, which mean that \( x_i A_{ij} x_j \) always is greater than zero when \( x_i \neq 0 \). This requirement will insure that the \( \theta \) function in the equation (2.10) is well defined.

The dual lattice \( e_i^* \) is defined as

\[
e_i^* \cdot e_j = \delta_{ij},
\]

(2.12)

then we have \( e_i^* \cdot e_j^* = A^{-1}_{i,j} \). It can be verified that

\[
\theta \left[ \begin{array}{c} a \\ b \end{array} \right] (z + e_i|e, \tau) = e^{2\pi i a \cdot e_i} \theta \left[ \begin{array}{c} a \\ b \end{array} \right] (z|e, \tau),
\]

\[
\theta \left[ \begin{array}{c} a \\ b \end{array} \right] (z + \tau e_i|e, \tau) = \exp[-\pi i \tau e_i^2 - 2\pi i e_i \cdot (z + b)] \theta \left[ \begin{array}{c} a \\ b \end{array} \right] (z|e, \tau),
\]

\[
\theta \left[ \begin{array}{c} a \\ b \end{array} \right] (z + e_i^*|e, \tau) = e^{2\pi i a \cdot e_i^*} \theta \left[ \begin{array}{c} a \\ b \end{array} \right] (z|e, \tau),
\]

\[
\theta \left[ \begin{array}{c} a \\ b \end{array} \right] (z + \tau e_i^*|e, \tau) = \exp[-\pi i \tau (e_i^*)^2 - 2\pi i e_i^* \cdot (z + b)] \theta \left[ \begin{array}{c} a + e_i^* \\ b \end{array} \right] (z|e, \tau),
\]

(2.13)

and

\[
\theta \left[ \begin{array}{c} a + e_i \\ b + e_j^* \end{array} \right] (z|e, \tau) = \exp(2\pi i a \cdot e_j^*) \theta \left[ \begin{array}{c} a \\ b \end{array} \right] (z|e, \tau).
\]

(2.14)

In \( 1 \)-dimension lattice with \( e_1 \cdot e_1 = 1 \), the \( \theta \) function is the one defined in the

* In this paper, we only will use \( \theta \) function on one dimensional lattice. This kind of notation is very helpful in the construction of the wave function.
equation (2.9). Moreover

\[ \theta_3(z|\tau) = \theta \left[ \frac{1}{2} \right] (z|\tau), \]  

is an odd function of \( z \). And we have equations

\[ \theta_3(z + 1|\tau) = e^{\pi i} \theta_3(z|\tau), \]
\[ \theta_3(z + \tau|\tau) = \exp \left[ -\pi i \tau - 2\pi i \cdot (z + \frac{1}{2}) \right] \theta_3(z|\tau). \]  

The Laughlin-Jastrow wave functions on the torus at the filling \( \frac{1}{m} \) (\( m \) is an odd positive integer) can be written as

\[ \Psi(z_i) = \exp \left( -\frac{\pi \Phi}{\tau_2} \sum_i y_i^2 \right) F(z_i), \]
\[ F(z_i) = \theta \left[ \frac{a}{b} \right] \left( \sum_i z_i e|e, \tau \right) \prod_{i<j} \left[ \theta_3(z_i - z_j|\tau) \right]^m, \]  

where \( \theta \) function is on 1-dimension lattice, \( e^2 = m, \; i = 1, 2, \ldots, N \) with \( N \) being the number of the electron and \( a = a^*e^*, b = b^*e^* \). Thus

\[ F(z_i + 1) = (-1)^{N-1} e^{2\pi a^*} F(z_i), \]
\[ F(z_i + \tau) = \exp(\pi(N - 1) - 2\pi ib^*) \exp[-i\pi m N(2z_i + \tau)] F(z_i). \]  

Comparing to the equation (2.8), we get

\[ \Phi = mN, \phi_1 = \pi(\Phi + 1) + 2\pi n_1 + 2\pi a^*, \phi_2 = \pi(\Phi + 1) + 2\pi n_2 - 2\pi b^*. \]  

(2.19) has solutions

\[ a_i^* = a_0 + i, \; b^* = b_0, \; i = 0, 1, \ldots, m - 1, \]
\[ a_0 = \frac{\phi_1}{2\pi} + \frac{\Phi + 1}{2}, \; b_0 = -\frac{\phi_2}{2\pi} + \frac{\Phi + 1}{2}, \]  

which will give \( m \) orthogonal Laughlin-Jastrow wave functions (other solutions are not independent on the solutions given in (2.20), which can be seen from the equation (2.14)). So there is \( m \)-fold center-mass degeneracy [2,3] (see also the discussion in section 5).
3. Blok-Wen Hierarchical Wave Function on the Torus

3.1. An Example

Then hierarchical FQHE in [15] describes a hierarchical condensation of holes of the parent states. The wave function can be characterized by matrix $\Lambda$,

$$
\Lambda = \begin{pmatrix}
  p_1 & 1 & 0 & \ldots & 0 & 0 \\
  +1 & -p_2 & -1 & 0 & \ldots & 0 \\
  0 & -1 & p_3 & +1 & 0 & \ldots \\
  \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
  \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
  0 & \ldots & 0 & (-1)^{n-1} & (-1)^n p_{n-1} & (-1)^n \\
  0 & 0 & \ldots & 0 & (-1)^n & (-1)^{n+1} p_n
\end{pmatrix},
$$

(3.1)

where $p_1$ is a positive integer (In the following discussion, we will show that $p_i, i = 2, 3, \ldots, n$ shall be positive even integers if we require that the wave function is well defined on the torus). $\Lambda$ describes a $n$-level hierarchical state. The coordinates of the particles are expressed by $z_{s,i}$. $z_{s,i}$ is the coordinate of the $i^{th}$ particle in level $s$, for example, $z_{s,1,i} = z_i$ is the coordinate of the $i^{th}$ electron. We take a simple example, $n = 2$ hierarchical state, to demonstrate how to construct the hierarchical wave function. In this case, the wave function is supposed to be

$$
\Psi(z_i) = \int \prod_{\alpha} dv_{2,\alpha} \sum_{l=0}^{p_1-1} \Psi_1(z_i, z_{2,\alpha}) l \Psi_2(z_{2,\alpha}) l,
$$

(3.2)

where $dv_{2,\alpha} = dz_{2,\alpha} d\bar{z}_{2,\alpha}$ which are integrated on the torus. $\Psi_1(z_i, z_{2,\alpha}) l$ in (3.2) are the Laughlin wave functions of electrons in the presence of the quasiparticles with the coordinates $z_{2,\alpha}$ and $\Psi_2(z_{2,\alpha}) l$ is the Laughlin type wave function of the quasiparticles. The index $l$ in $\Psi_1$ is the degeneracy index of the Laughlin wave functions with filling at $\frac{1}{p_1}$. However the index $l$ of $\Psi_2$ is the component index of quasiparticle wave function and it reflects the multi-component nature of anyon wave function.
on the torus. In [19], it is found that free anyons have a multicomponent wave function on the torus by using braid group analysis. However even when anyons are exposed to the magnetic field (the quasiparticles in FQHE interact with the magnetic field), by generalizing the results of [19] to the case of anyons interacting with magnetic field, the wave function is still found to be multicomponent (see the discussion in section 4).

The wave functions $\Psi_1$ are a Laughlin wave function with $N_2$ quasiparticles. Now we have the relation $p_1 N_1 + N_2 = \Phi$ and the wave functions are given by

$$
\Psi_1(z_i, z_{2,\alpha})_l = \exp\left(-\frac{\pi \Phi \left(\sum_i y_i^2 + \frac{1}{p_1} \sum_\alpha y_{2,\alpha}^2\right)}{\tau_2}\right) F_1(z_i)_l, \\
F_1(z_i)_l = \theta \left[ \begin{array}{c} a_l/b \\ e^{\sum_i z_i + \sum_\alpha z_{2,\alpha} e^e} \end{array} \right] \theta [\theta_3(z_i - z_j|\tau)]^{p_1} \prod_{i<j} [\theta_3(z_i - z_j|\tau)]^{p_1} \prod_{\alpha<\beta} [\theta_3(z_{2,\alpha} - z_{2,\beta}|\tau)]^{p_1},
$$

where $e^2 = p_1$, $e^* = \frac{1}{e}$ and $a_l, b$ are still given by the equation (2.20), e.g. $a_l^* = a_0 + l$, $b^* = b_0$. As emphasized in [15], $\Psi_1,l$ needs to be a normalized wave functions if we want to construct such kind of hierarchical wave functions. Some may ask how we know $\Psi_1,l$ in (3.3)are the normalized wave functions? The reason is that everything is consistent in the end. Another reason is that $\Psi_1,l$ in (3.3)look like the normalized wave functions in Chern-Simons theory.

Let us consider the wave function $\Psi_2$ now. Firstly $\Psi_2$ is a multicomponent (it has $p_1$ components) wave function. Secondly when two quasiparticles are exchanged anti-clock, the wave function in singular gauge will give a phase $e^{i\theta}$ with $\theta = -\frac{\pi}{p_1}$ (assuming inside the exchanging path, there are no other quasiparticles and we call $\theta$ as the statistical parameter of the quasiparticles). Thirdly it is Laughlin type wave function. Under the magnetic translation of the quasiparticle, the wave function should change up to a unitary transformation (the magnetic translation of the electron is described by equation (2.7); for the case of the quasiparticle, see section 4). The charge of the quasiparticle of the Laughlin state is $\frac{1}{p_1}$. 

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Thus we find that the wave function $\Psi_2$ shall be written as (we write its complex conjugate),

$$
\bar{\Psi}_2(z_{2,\alpha}) = \exp\left(-\frac{\Phi}{p_1} \sum_{\alpha} y^2_{2,\alpha}\right) F_2(z_{2,\alpha}),
$$

$$
F_2(z_{2,\alpha}) = \theta\left[\frac{a_2,l}{b_2}\right] \left(\sum_{\alpha} z_{2,\alpha}s_2|e_2,\tau\right) \prod_{\alpha<\beta} \left[\theta_3(z_{2,\alpha} - z_{2,\beta}|\tau)\right]^{\frac{1}{p_1}+p_2},
$$

(3.4)

where $e_2^2 = p_1(p_1p_2 + 1)$, $s_2^2 = p_2 + \frac{1}{p_1}$ (the Laughlin type wave function of the quasiparticles on the torus has also been discussed in the context of Chern-Simons theory [7,17] and our construction of the wave function agrees with them, but note that here we work in anyon gauge, which is different from [7,17]). From the form of $\Psi_2$, we can get the relation $|q_2 \cdot \Phi| = \frac{\Phi}{p+1} = N_2(p_2 + \frac{1}{p_1})$ where $q_2$ is the charge of the quasiparticle and equals to $\frac{1}{p_1}$ (assuming electron charge is $-1$). Now we can write relations $p_1N_1 + N_2 = \Phi$, $\frac{\Phi}{p+1} = N_2(p_2 + \frac{1}{p_1})$ as

$$
p_1N_1 + N_2 = \Phi,
$$

$$
N_1 - p_2N_2 = 0.
$$

(3.5)

To fix the parameters $a_{2,l}, b_2$, we impose the condition that

$$
\sum_{l=0}^{p_1-1} \Psi_1(z_i, z_{2,\alpha})_l \Psi_2(z_{2,\alpha})_l
$$

is periodic with the coordinates of the quasiparticles around two nontrivial cycles of the torus in order that the integral in (3.2)is well defined on the torus. So we shall have

$$
\Psi_1(z_{2,\alpha} + 1)_l \Psi_2(z_{2,\alpha} + 1)_l = \Psi_1(z_{2,\alpha})_l \Psi_2(z_{2,\alpha})_l,
$$

$$
\Psi_1(z_{2,\alpha} + \tau)_l \Psi_2(z_{2,\alpha} + \tau)_l = \Psi_1(z_{2,\alpha})_{l+1} \Psi_2(z_{2,\alpha})_{l+1},
$$

(3.6)

with $\Psi_{1,p_1} = \Psi_{1,0}$, $\Psi_{2,p_1} = \Psi_{2,0}$ where $\Psi_{1,p_1} = \Psi_1(z_i, z_{2,\alpha})_l$,$\Psi_{2,p_1} = \Psi_2(z_{2,\alpha})_l$. 


Then we get a set of solution of $a_{2,l}$ and $b_2$

$$a_{2,l} = a_{2,l}^* [p_1(p_1p_2 + 1)]^{-\frac{1}{2}}, b_2 = b_2^* [p_1(p_1p_2 + 1)]^{-\frac{1}{2}},$$

$$a_{2,l}^* = a_0 + l(p_1p_2 + 1) + \lambda p_1, b_2^* = b_0.$$  \hspace{1cm} (3.7)

The solution (3.7) will give $p_1p_2 + 1$ independent wave functions $\Psi$, which means that the degeneracy of the electron ground states is $p_1p_2 + 1$. Now we write the wave functions as

$$\Psi(z_i) = \int \prod_{\alpha} dv_{2,\alpha} \sum_{l=0}^{p_1-1} \Psi_1(z_i, z_{2,\alpha})_l \Psi_2(z_{2,\alpha})_{l,\lambda},$$

where $\lambda$ is the index of the degeneracy of the electron wave functions $\Psi$ and also is the index of the degeneracy of the condensed quasiparticle wave functions $\Psi_2$. We also note that $p_2$ must be positive even integer, otherwise the $\theta$ function in equation (3.4) will not be well defined.

### 3.2. GENERAL HIERARCHICAL WAVE FUNCTIONS

For $n$-level hierarchical wave functions, we define some useful parameters; $d_m = |\det \Lambda (m)|$, where matrix $\Lambda_{i,j}(m) = \Lambda_{i,j}$, $1 \leq i, j \leq m$ is a $m \times m$ matrix with $d_0 = 1$, and

$$e_m = (d_m \cdot d_{m-1})^{\frac{1}{2}},$$

$$s_m = \frac{(d_m)^{\frac{1}{2}}}{(d_{m-1})^{\frac{1}{2}}},$$

$$m = 1, 2, \cdots, n.$$  \hspace{1cm} (3.9)

Now the wave functions are

$$\Psi(z_i)_\lambda_n = \int \prod dv_{2,\alpha} \cdots dv_{n,\alpha} \sum_{\lambda_1, \cdots, \lambda_{n-1}} \Psi_1(z_{1,\alpha}, z_{2,\alpha})_{\lambda_1} \Psi_2(z_{2,\alpha}, z_{3,\alpha})_{\lambda_1, \lambda_2} \cdots \Psi_{n-1}(z_{i,\alpha}, z_{i+1,\alpha})_{\lambda_{i-1, \lambda_i} \cdots} \Psi_n(z_{n,\alpha})_{\lambda_{n-1, \lambda_n}}.$$  \hspace{1cm} (3.10)
where
\[
\lambda_i = 0, 1, \cdots, d_i - 1.
\] (3.11)

Define
\[
\tilde{\Psi}_i(z_i, z_{i+1}, \ldots, z_{d_i-1}) = \Psi_i(z_i, z_{i+1}, \lambda_i, i = \text{odd integers},
\]
\[
\tilde{\Psi}_i(z_i, z_{i+1}, \lambda_i, i = \text{even integers}.
\] (3.12)

Then when \(1 < i < n\), we have
\[
\tilde{\Psi}_i(z_i, z_{i+1}, \lambda_i, i = \exp(-\pi \Phi(1/d_i-1)\sum_{\alpha} y_{i,\alpha}^2 + \frac{1}{d_i} \sum_{\alpha} y_{i+1,\alpha}^2))
\]
\[
\times F_i(z_i, z_{i+1}, \lambda_i, i = \theta \left[ \sum_{\alpha} z_{i,\alpha} s_i + \sum_{\alpha} z_{i+1,\alpha} s_i^* |e_i, \tau\right] \times
\]
\[
\prod_{\alpha < \beta} [\theta_3(z_{i,\alpha} - z_{i,\beta}|\tau)]^{s_i^*} \prod_{\alpha, \beta} [\theta_3(z_{i,\alpha} - z_{i+1,\beta}|\tau)]^{s_i^*}.
\] (3.13)

And the equation (3.5) is generalized to
\[
\sum_j \Lambda_{ij} N_j = \begin{cases} \Phi, & \text{if } i = 1; \\ 0, & \text{otherwise.} \end{cases}
\] (3.14)

where \(N_j\) is the number of the condensed quasiparticles in level \(i\). Moreover the equation (3.7) becomes
\[
a_{i,\lambda_i-1,\lambda_i} = a_{i,\lambda_i-1,\lambda_i}^* e_i^* , \quad b_i = b_i^* e_i^* ,
\]
\[
e_i^* = \frac{1}{e_i} , \quad s_i^* = \frac{1}{s_i} ,
\]
\[
a_{i,\lambda_i-1,\lambda_i}^* = a_0 + \lambda_{i-1} d_i + \lambda_i d_{i-1} , \quad b_i^* = b_0 .
\] (3.15)

The condition in (3.15) needs to be satisfied in order that
\[
\sum_{\lambda_i-1} \Psi_i(z_{i-2,\alpha}, z_{i-1,\alpha}) \lambda_i-2, \lambda_i \Psi_i(z_{i,\alpha}, z_{i+1,\alpha}) \lambda_i-1, \lambda_i
\]
is periodic with the coordinates \(z_i, \alpha\). \(\Psi_i(z_{i,\alpha}, z_{i+1,\alpha}) \lambda_i-1, \lambda_i\) have \(d_{i-1}\) components
wave functions with \( \lambda_{i-1} \) being the index of the components of the wave functions and have \( d_i \) degeneracy with \( \lambda_i \) being the index of the degeneracy. These wave functions are the wave functions of the condensed quasiparticles of the \( i\text{-level} \) in anyon (singular) gauge. If the electron charge is \(-1\), then the condensed quasiparticle in \( i\text{-level} \) has charge

\[
\frac{(-1)^i}{d_{i-1}},
\]

(3.16)

and the statistics parameter \( \theta \) of the condensed quasiparticle (here we use the anticlock exchange of two quasiparticles and we will get a phase \( e^{i\theta} \). see also section 4) is

\[
(-1)^{i-1} \frac{d_{i-2}}{d_{i-1}}.
\]

(3.17)

\( \Psi_1(z_{1,\alpha}, z_{2,\alpha})_{\lambda_1} \) still are given by the equation (3.3). Finally, we have

\[
\tilde{\Psi}_n(z_{n,\alpha})_{\lambda_{n-1},\lambda_n} = \exp(-\frac{\pi \Phi(\frac{i}{d_{n-1}} \sum_\alpha y_{n,\alpha}^2)}{\tau_2}) F_n(z_{n,\alpha})_{\lambda_{n-1},\lambda_n},
\]

\[
F_n(z_{n,\alpha})_{\lambda_{n-1},\lambda_n} = \theta \left[ \begin{array}{c} a_{n,\lambda_{n-1},\lambda_n} \\ b_n \end{array} \right] \left( \sum_\alpha z_{n,\alpha}s_n|e_{n,\tau} \right) \prod_{\alpha<\beta} [\theta_3(z_{n,\alpha} - z_{n,\beta}|\tau)]^{s_{n,\alpha}^2},
\]

(3.18)

and \( a_{n,\lambda_{n-1},\lambda_n}, b_n \) are still given by the equation (3.15).

Moreover \( p_i, \), \( i = 2, 3, \cdots, n \) shall be positive even integers in order that the \( \theta \) function appeared in the wave function be well defined.

From equation (3.14)we can show that the filling factor equals to \( \nu = \frac{N_1}{\Phi} = \Lambda_{1,1}^{-1} \), where \( N_1 \) is the number of the electrons,

\[
\nu = \frac{1}{p_1 + \frac{1}{p_2 + \frac{1}{p_3 + \cdots + \frac{1}{p_n}}}}.
\]

(3.19)
Finally the degeneracy of the wave functions $\Psi(z_i)_{\lambda_n}$ is

$$d_n = |\det \Lambda|,$$

which actually is the denominator of the filling and agrees with the prediction in the literature [13,14].

4. Fractional Statistics on the torus

The quasiparticles in the FQHE satisfy fractional statistics ([20]; see also references in [21]). In this section we will show that the fractional statistics of the quasiparticles in the FQHE on the torus can be directly calculated from the wave functions. Hence this gives a concrete example how the fractional statistics can be realized on the torus. In [19], it has been proved that the fractional statistics of free anyons on the torus is consistent only with multi-component wave functions (see also [22,23]) and claimed that the fractional quantum hall effect (FQHE) fits to this picture. It is worth to note that the quasiparticles in FQHE interact with the magnetic field, so it needs to modify the results in [19] to the case that anyons interact with the magnetic field if one wants to apply the results from braid group analysis to the quasiparticles of FQHE on the torus. We shall show that even if the anyons are exposed to the magnetic field, the wave functions will still need to be multi-component. We will calculate the braid statistics relation of the quasiparticles in the Laughlin state and compare it with the braid statistics relation of the anyons on the torus from braid group analysis.

4.1. Fractional Statistics of the Quasiparticles in FQHE on the Torus

The normalized wave functions of the simplest Laughlin state with quasiparticles in anyon (singular) gauge are given by the equation (3.3). These wave functions will give a conjugate representation of the braid statistics for the quasiparticles.
This can be understood as follows; the hierarchical construction of wave functions are $\Psi(z_i) = \int dw_\alpha \sum_l \Psi_1(z_i, w_\alpha)_l \Psi_2(w_\alpha)_l$, where $\Psi_2(w_\alpha)_l$ are the wave functions of the quasiparticles, $\Psi(z_i, w_\alpha)$ are the normalized wave functions of the electrons in the presence of the quasiparticles and both are in singular gauge ($z_i$ are the coordinates of the electrons and $w_\alpha$ are the coordinates of the quasiparticles). From $\Psi_2(w_\alpha)_l$, we can get the braid statistics of the quasiparticles and from $\Psi_1(z_i, w_\alpha)_l$ we can get the complex conjugate representation of the braid statistics of the quasiparticles. Thus $\sum_l \Psi_1(z_i, w_\alpha)_l \Psi_2(w_\alpha)_l$ will give a trivial identity representation of the braid statistics which is needed for the well defined integration with the coordinates of the quasiparticles on the torus. What we discussed in the last section is the case that the wave functions of the quasiparticles are Laughlin type and the hierarchical state is obtained. It is very natural to suggest that even the wave functions of the quasiparticles are not Laughlin type (the quasiparticles are not condensed and may not have Laughlin type wave function), $\sum_l \Psi_1(z_i, w_\alpha)_l \Psi_2(w_\alpha)_l$ still give a trivial identity representation of the braid statistics of the quasiparticles. Thus we can actually read out all braid statistics relation of the quasiparticles from $\Psi(z_i, w_\alpha)_l$ (or the wave functions given by (3.3)) [19] even if we do not know the form of the wave functions $\Psi(w_\alpha)_l$ (if the quasiparticles are condensed and have the Laughlin type wave functions, then $\Psi(w_\alpha)_l$ are given by (3.4)). Now we shall demonstrate how to calculate the braid relation of the fractional statistics of the quasiparticles from $\Psi(z_i, w_\alpha)_l$.

The generators of the braid group are

$$\tau_i, \rho_i, \sigma_k; \quad i = 1, \cdots, n_q; \quad k = 1, \cdots, n_q - 1,$$

(4.1)

where $i$ is the index of anyon (quasiparticle in FQHE) and $n_q$ is the number of anyons. The generators $\sigma_k$ are the anti-clockwise exchanges of anyons $k$ and $k + 1$ (we assume that there are no particles in the region of the exchange path). The generators $\tau_i$ and $\rho_i$ are the magnetic translation operators of the particle $i$ along the fundamental non-contractible loops of the torus (because of the presence of
the magnetic field, we have the magnetic translation symmetry instead of the translation symmetry).

The operators $i, \tau_i, \rho_i$ (acting on the wave function $\Psi_2(w_{\alpha})$) are given by

$$\tau_i = \exp(ip_{w_{i,1}}),$$

$$\rho_i = \exp(i\tau_1 p_{w_{i,1}} + i\tau_2 p_{w_{i,2}} - i2\pi \frac{\Phi}{m} w_{i,1}),$$

(4.2)

because the charge of the quasiparticles is $-\frac{1}{m}$ of the one of the electron, where the filling of the Laughlin state is taken as $\frac{1}{m}$ and $w_{i,1} = Re(w_i), w_{i,2} = Im(w_i)$.

The magnetic translation operators of the quasiparticle $i, \tau_i, \rho_i$ acting on the wave function $\Psi_1(z_i, w_{\alpha})$ are given by

$$\tau_i = \exp(ip_{w_{i,1}}),$$

$$\rho_i = \exp(i\tau_1 p_{w_{i,1}} + i\tau_2 p_{w_{i,2}} + i2\pi \frac{\Phi}{m} w_{i,1}),$$

(4.3)

Actually $\tau_i, \rho_i, \sigma_k$ in (4.3) will give the complex conjugate representation of the braiding operators of the quasiparticles in (4.2) on the torus, since they act on the wave functions $\Psi_1(z_i, w_{\alpha})$. Assume $X$ is a braid operator and let $X$ act on $\Psi_1(z_i, w_{\alpha}) = \Psi_{1,l}$ and $\Psi_2(w_{\alpha}) = \Psi_{2,l}$. Then we will have $X\Psi_{1,l} = \Psi_{1,j}U_1(1)_{jl}$ and $X\Psi_{2,l} = \Psi_{2,j}U_2(2)_{jl}$ where $X$ is an operator defined in (4.3), and $X\Psi_{2,l} = \Psi_{2,j}U_2(1)_{jl}$ where $X$ is an operator defined in (4.2). $U_1(1)$ and $U_2(2)$ are unitary matrices. Moreover, we have relation $U_1(1)_{jl} = U_2(2)_{jl}$, which will insure that $\sum_l (X\Psi_{1,l})(X\Psi_{2,l}) = \sum_l \Psi_{1,l}\Psi_{2,l}$ and this turns out to mean that the function $\sum_l \Psi_{1,l}\Psi_{2,l}$ is periodic with the coordinates of the quasiparticles around two non-contractible loops of the torus. The operators $\tau_i, \rho_i, \sigma_k$ in (4.2) shall commute with the Hamiltonian of the quasiparticles, which is given by

$$H = \sum_l \frac{1}{2M_q} [(p_{w_{i,1}} - \frac{B}{m} w_{i,2})^2 + (p_{w_{i,2}})^2].$$

(4.4)

We shall remark that, the lagrangian of the quasi-particles are described by vortex (center coordinate) dynamics, and the lagrangian of vortices does not contain any
mass parameters [24]. The Hilbert space of the Hamiltonian which we write above shall be restricted to ground state. The ground state of the above Hamiltonian is the same as the one described by the vortex dynamic. To be rigorous, we shall proceed from the vortex dynamic.

Now we calculate the representation of the operators \( i, \tau_i, \rho_i \) acting on the wave function \( \Psi_1(z_i, w_\alpha)_l \), which are given in (4.3). For simplicity, we now choose \( a_i = ie^{-1}, b = 0, i = 0, 1, \ldots, m - 1 \) in (3.3). Thus we have \( \phi_1 = \phi_2 = \pi(\Phi - 1) \) (any choices \( \phi_1, \phi_2 \) will not affect the braid relations). Now we denote the wave functions \( \Psi_1(z_i, w_\alpha)_l \) as \( \Psi_l \), a multicomponent column. Furthermore we assume

\[
\begin{align*}
  w_{1,1} &< w_{2,1} < \cdots < w_{n_2,1}, \\
  w_{1,2} &< w_{2,2} < \cdots < w_{n_2,2}.
\end{align*}
\]  

(4.5)

Then by applying \( \sigma_i, \tau_i, \rho_i \) on \( \Psi_l \), we have \( \sigma_i = e^{\frac{2\pi i}{m}}I_m \) with \( I_m \) being identity \( m \times m \) matrix and

\[
\begin{align*}
  \tau_1 &= e^{\frac{\pi i (\Phi - 1)}{m}} \begin{pmatrix}
    1 & 0 & \cdots & 0 \\
    0 & c & \cdots & 0 \\
    \vdots & \ddots & \ddots & \vdots \\
    0 & 0 & \cdots & 1
  \end{pmatrix}, \\
  \rho_1 &= e^{\frac{-2\pi i}{m}} \begin{pmatrix}
    0 & \cdots & 0 & 1 \\
    1 & \cdots & 0 & 0 \\
    \vdots & \ddots & \ddots & \vdots \\
    0 & \cdots & 1 & 0
  \end{pmatrix},
\end{align*}
\]

(4.6)

where \( c = e^{\frac{2\pi i}{m}} \). Other \( \tau_i \) and \( \rho_i \) are given by the relation

\[
\tau_{i+1} = e^{\frac{-2\pi i}{m}} \tau_i, \quad \rho_{i+1} = e^{\frac{2\pi i}{m}} \rho_i.
\]

(4.7)
4.2. **Braid Group Analysis of the Fractional Statistics on the Torus**

The main relations of braid statistics of free anyons are [19,25],

\[
\begin{align*}
\sigma_j \sigma_{j+1} \sigma_j &= \sigma_{j+1} \sigma_j \sigma_{j+1}, \\
\tau_{j+1} &= \sigma_j^{-1} \tau_j \sigma_j^{-1}, \rho_{j+1} = \sigma_j \rho_j \sigma_j, \\
\rho_j^{-1} \tau_{j+1} \sigma_j^{-2} \rho_j \sigma_j^{-2} \tau_{j+1}^{-1} &= \sigma_j^2, \\
\sigma_1 \sigma_2 \cdots \sigma_{N-1} \cdots \sigma_2 \sigma_1 &= \rho_1^{-1} \tau_1^{-1} \rho_1 \tau_1. 
\end{align*}
\]

(4.8)

If the anyons are exposed to the magnetic field, we just need to change the last relation in (4.8),

\[
\sigma_1 \sigma_2 \cdots \sigma_{N-1} \cdots \sigma_2 \sigma_1 = \rho_1^{-1} \tau_1^{-1} \rho_1 \tau_1 e^{2\pi iq\Phi},
\]

(4.9)

with \( q \) being the charge of anyons and \( \Phi \) being the magnetic flux out of the torus. The reason for the extra phase is that, when we do the operation of the left equation (4.9), we get a closed curve with zero area which can be deformed to the operation of the right equation [19,25]. However, the curve needs to encompass the whole surface during the deformation. So we get an extra phase (Aharonov and Bohm phase) because anyons now interact with the magnetic field.

We can choose the base of the wave functions such that \( \sigma_i = e^{i\theta} I_M \) with \( I_M \) as \( M \times M \) identity matrix [19], then from the second and third equation in (4.8), we get

\[
\tau_i \rho_j = \rho_j \tau_i e^{2i\theta}.
\]

(4.10)

By taking the determinant of the equation (4.10), we have

\[
\exp(2Mi\theta) = 1.
\]

(4.11)

If \( \theta = \pi \frac{r}{s} \) with \( r \) and \( s \) are coprime with each other, so from (4.11), we need
\( M = ns \) with \( n \) being integer. Furthermore from (4.9), we have another equation

\[
e^{2N\theta - 2\pi q\Phi} = 1. \tag{4.12}
\]

which imply that \( \frac{N\theta}{s} - q\Phi \) should be integer. The braid statistics relation of the quasiparticles of FQHE in above example turn out to fit to the picture from general braid group analysis. Now \( \theta = \frac{\pi}{m} \) and the wave functions of the quasiparticles are \( m \) component, so (4.11) is fulfilled. Furthermore in this example, the equation (4.12) becomes \( \frac{n\theta}{m} - \frac{\Phi}{m} = \text{integer} \) with \( n_q \) as the number of the quasiparticles. Because we have relation \( mn_e + n_q = \Phi \) where \( n_e \) is the electron number, so (4.12) is automatically satisfied.

5. Remarks about the Modular Invariance and Large Gauge Transformations

The modular transformations and the large gauge transformations of the wave functions in FQHE have been considered in [7,8,17]. How are about the wave functions constructed here? Let us first discuss the modular transformations of the wave functions. If we require the wave functions \( \Psi(z_i)_{\lambda_n} \) are transformed covariantly under the modular transformation, \( z \rightarrow \frac{z}{\gamma\tau + \delta} \) and \( \tau \rightarrow \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \), then we find that \( a_0, b_0 \) need to equal to \( \frac{1}{2} \). The proving needs the modular transformation of the quasiparticle wave functions \( \Psi_{m,n} \), where \( m \) is the index of the component and \( n \) is the index of the degeneracy. Under the modular transformations \( \tau \rightarrow \tau + 1 \) and \( z \rightarrow z \), the wave functions \( \Psi_{m,n} \) in the case of \( a_0 = \frac{1}{2} \), \( b_0 = \frac{1}{2} \), will be changed up a unitary phase. Under the modular transformations \( \tau \rightarrow \frac{1}{\gamma} \) and \( z \rightarrow \frac{z}{\tau} \), \( \Psi_{m,n} \rightarrow N \sum_{m'} \sum_{n'} f_1(m, m') f_2(n, n') \Psi_{m',n'} \), where \( N \) can depend on the coordinates of the quasiparticles, but it does not depend on \( m', n' \). These results have been discussed in [7,17,27].

Now we come to discuss the large gauge transformations. Due to the nontrivial topology of the torus, we have gauge transformations like \( U_1 = \exp(-\frac{2\pi i y}{\tau_2}) \) and
\[ U_2 = \exp\left[\frac{\pi(\tau z - \tau z_2)}{\tau_2}\right]. \] If we choose the magnetic potential as \( A_1 = (-By + \frac{2\pi c_1}{\tau_2})\hat{x}, \)
\( A_2 = \frac{2\pi c_2}{\tau_2}\hat{y} \) with \( c_1, c_2 \) being constant and \( c = ic_1 - c_2 \), then under the gauge transformations generated by \( U_1 \) and \( U_2 \), we will have \( c \rightarrow c + m + n\tau \). When we take magnetic potential \( A_1 = (-By + \frac{2\pi c_1}{\tau_2})\hat{x}, A_2 = \frac{2\pi c_2}{\tau_2}\hat{y}, \) the wave functions \( \Psi(z_i)_{\lambda_n} \) will depend on \( c \). So we denote the wave functions now as \( \Psi(z_i, c)_{\lambda_n} \). The large gauge transformations on the wave functions are defined as follows; under the transformation \( U_1 \), \( \Psi(z_i, c)_{\lambda_n} \rightarrow \prod_i [U(z_i)_1]^{(-1)}G_1(c)\Psi(z_i, c + 1)_{\lambda_n} \), and under the transformation \( U_2 \), \( \Psi(z_i, c)_{\lambda_n} \rightarrow \prod_i [U(z_i)_2]^{(-1)}G_2(c)\Psi(z_i, c + \tau)_{\lambda_n} \) (Under suitable choices of \( G_1(c), G_2(c) \), those operators actually form an Heisenberg algebra \([7,17]\)). We find that the wave functions \( \Psi(z_i, c)_{\lambda_n} \) change up to a phase under the gauge transformation \( U_1 \) and change from one to another under the gauge transformation \( U_2 \). Thus we can say that the degeneracy will disappear by fixing the gauge of the magnetic field. This maybe offer some reasons why this kind of the center coordinate degeneracy of FQHE on the torus is quite physically irrelevant \([3]\).

The proving of the above statements is quite technical and complicated and also because the discussion about the modular transformations and large gauge transformation of the wave function may not be physically interesting, so we will not pursue it here and just state the results. The discussion of the modular transformations and large gauge transformations for another kind of hierarchical wave functions on the torus can be found in \([8]\).

### 6. Summary and Conclusion

In this paper, we have constructed one kind of hierarchical wave functions on the torus. In order to construct such wave functions on the torus, we need first to have the normalized quasiparticle wave functions on the torus. The parameters of the quasiparticle wave functions can be fixed from the requirement of a well defined integration of the wave functions on the torus by the coordinates of the quasiparticles. Then the degeneracy of the wave function functions is obtained, which shows how the topological order is manifested in this kind of hierarchical
FQHE wave function on the torus and agrees with the prediction in the literature. We also calculate the braid matrix of the fractional statistics of the FQHE quasi-particles on the torus and compare them to the general results from braid group analysis. Finally, the modular transformations and large gauge transformations of the wave functions are briefly discussed, and it is found that the parameters can be fixed if we want the wave functions to be transformed covariantly under the modular transformations, and that the unique ground state appears by fixing the gauge of the magnetic field.

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