Charged Black Holes in Three Dimensional Einstein Theory with Torsion and Chern Simons Terms

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Abstract

We discuss exact solutions to 3D Einstein theory with torsion coupled to gravitational Chern Simons term and charged under Maxwell Chern Simons electrodynamics. Some of the solutions we found represent charged rotating black holes and generalize previous solutions reported in the literature. Also we find that Godel type solution exists in the theory considered. Finally we compute the conserved charges.

1 Introduction

In recent years, there has been a remarkable activity in the study of three dimensional models of gravity. In three dimensions exist a variety of interesting solutions, the well known BTZ black hole [1] is a solution to the Einstein equations with a negative cosmological constant that has very interesting classical and quantum properties and shares several features of the Kerr black hole [2]. Topologically Massive Gravity [3] have been reconsidered recently [4] to construct a chiral theory of gravity and have been extensively analyzed [5], and this gave raise a fruitful discussion that ultimately led to a much better understanding of the model [6], most of these works assume that the connection is torsion free, however it is interesting to consider a more general model of gravity, based on the Riemann Cartan geometry, where the gravitational fields are the vielbein and the Cartan connection, and the curvature and torsion describe the gravitational dynamics. Torsion was added to the BTZ metric in [7], recently in [8], [9] have found solutions in spacetime with torsion and with a nonlinear Born Infeld electrodynamics [10].

In this paper we will be concerned with an interesting toy model of gravity in three dimensions, the topological model of 3D gravity proposed by Mielke and Baekler [11] where the matter contribution will be the Maxwell action increased by a topological term, i.e. we will be concerned with the 3D Einstein gravity with torsion coupled to gravitational topological term and topologically massive electrodynamics, this will allows to study charged solutions of the theory. We will consider this theory formulated in the first order formalism of Poincaré gauge theory of gravity (PGTG) with a geometric structure described by Riemann Cartan spacetime, in PGTG the metric and connection are not independent, but are related to each other by the metricity condition: \( Dg = 0 \), where \( g \) is the metric and \( D \) is the covariant derivative.

Here we will write down exact solutions to this theory representing charged rotating black hole solutions that, in turn, generalize previous solutions reported in the literature. In the case of pure gravity, the black hole content of this models turn out to be a very important question to understand the theory. In particular, this is very important to construct a consistent CFT\(_2\) dual for the model, since the appropriate classification of the bulk geometries that contributes to the partition function is a crucial step to declose quantum gravity.
Of course, the problem of quantizing the theory in presence of matter (e.g. of U(1) matter) is far from being a tractable task to our knowledge; nevertheless, the question about the black hole content of the theory still represents a well motivated classical problem.

We will also comment briefly a solution that generalizes the Godel spacetime obtained in [12] by including the torsion and the gravitational topological term. The Godel like solution is supported by an abelian gauge field, and it is necessary to include an additional Chern Simons term that produces the energy momentum tensor of a pressureless perfect fluid. in [12] it is show that in 2+1 dimensions the Maxwell field minimally coupled to gravity can be the source of such a fluid, when it is increased with a topological mass $\mu_E$.

The structure of the paper is as follows. In section 2 we introduce the model and summarize the procedure followed in [8] to derive the field equations, in section 3 we solve the field equations with an ansatz for the vielbein and the gauge field, in section 4 we calculate the conserved charges, section 5 analize the solution taking some limits and section 6 presents final remarks.

2 Gravitational Equations

Here we follows the procedure presented in [8] to derive the field equations (here we are using a different notation). In the Einstein Cartan geometry the basic gravitational fields are the triad $e^a$ and the Lorentz connection $\omega^{ab}$ and the correspondings field strengths are the torsion $T^a$ and the curvature $R^{ab}$.

To simplify the calculations, in three dimensions we can define:

$$\omega^{ab} = -\epsilon^a_{bc} \omega^c$$

and

$$R^{ab} = -\epsilon^a_{bc} R^c$$

and work with the dual fields $\omega^a$, $R^a$. therefore, we have:

$$T^a = d\omega^a + \frac{1}{2} \epsilon^a_{bc} \omega^b \omega^c$$

$$R^a = d\omega^a + \frac{1}{2} \epsilon^a_{bc} \omega^b \omega^c$$

In local coordinates, we have $e^a = e^a_\mu dx^\mu$, $\omega^a = \omega^a_\mu dx^\mu$, we take our local coordinates to be $x^\mu = t, r, \phi$.

As said in the introduction we will consider the model proposed by Mielke and Baekler charged under Chern Simons electrodynamics, that is, we will consider the model defined by the action:

$$S = \int \left( \frac{1}{\kappa} e_a R^a - \frac{\Lambda}{3} \epsilon_{abc} e^a e^b e^c + \alpha_3 \left( \omega^a d\omega_a + \frac{1}{3} \epsilon_{abc} \omega^a \omega^b \omega^c \right) + \alpha_4 e_a T^a + \mathcal{L}_F \right)$$

$$\mathcal{L}_F = -\frac{1}{2} F^a F - \frac{\mu_E}{2} AF$$

where $\alpha_3$ and $\mu_E$ are the gravitational and electromagnetic Chern Simons coupling constants, respectively, $\alpha_4$ is the torsion coupling, $\Lambda$ is a cosmological constant and $\epsilon_{abc}$ is the completely antisymmetrical symbol ($\epsilon_{012} = +1$).

Using the metricity condition we get the following useful relation:

$$\omega^a = \tilde{\omega}^a + K^a$$

Latin indices label the components with respect to a local Lorentz frame and Greek indices refers to the coordinate frame.
where $\tilde{\omega}^a$ is the Riemannian connection and $K^a$ is the contortion 1-form.

The contortion is defined through:

$$T^a = \epsilon^a_{bc} K^b e^c$$

(10)

the relation above allow us to express the Cartan curvature in terms of the Riemannian curvature and the contortion as:

$$R^a = \bar{R}^a + \bar{D}K^a + \frac{1}{2} \epsilon^a_{bc} K^b K^c$$

(11)

The spacetime metric is:

$$g = \eta_{ab} e^a \otimes e^b$$

(12)

where we have adopted the convention:

$$\eta_{ab} = \text{diag}(+1, -1, -1)$$

(13)

Next we vary the action with respect to the fields $e^a$ and $\omega^a$ and obtain the gravitational field equations:

$$T^a - \frac{a}{2} \epsilon^a_{bc} e^b e^c = \frac{m}{2} \chi^a$$

(14)

$$R^a - \frac{b}{2} \epsilon^a_{bc} e^b e^c = -\frac{n}{2} \chi^a$$

(15)

where:

$$\chi_a = -\frac{\delta L_F}{\delta e^a}$$

(16)

is the (Maxwell) energy-momentum current, and we have introduced the following constans to simplify the writing of our gravitational equations:

$$a = \frac{\alpha_3 \Lambda + \frac{\alpha_4}{2\kappa}}{\alpha_3 \alpha_4 - \left(\frac{1}{2\kappa}\right)^2}$$

(17)

$$b = -\frac{\alpha_3^2 + \frac{1}{2\kappa} \Lambda}{\alpha_3 \alpha_4 - \left(\frac{1}{2\kappa}\right)^2}$$

(18)

$$m = \frac{\alpha_3}{\alpha_3 \alpha_4 - \left(\frac{1}{2\kappa}\right)^2}$$

(19)

$$n = \frac{\frac{1}{2\kappa}}{\alpha_3 \alpha_4 - \left(\frac{1}{2\kappa}\right)^2}$$

(20)

In this article we find solutions for $\alpha_3 \alpha_4 - \left(\frac{1}{2\kappa}\right)^2 \neq 0$.

The energy momentum tensor is calculated by:

$$S^a_b = \epsilon^a_{abc} F^c_{bc} = -F^{ac} F_{bc} + \frac{1}{4} \delta^a_b (F^{cd} F_{cd})$$

(21)

and we get the energy momentum current as:

$$\chi_a = \epsilon_{abc} s^b e^c$$

(22)

with

$$s^a = -\left( S^a_b - \frac{1}{2} \delta^a_b S \right) e^b$$

(23)
\[ S = S^a_a \quad (24) \]

\( S \) is the trace of the energy momentum tensor.

Replacing the expression for the current, the gravitational equations become:

\[ T^a = \frac{1}{2} \epsilon^a_{bc} (ae^b + ms^b)e^c \quad (25) \]

\[ R^a = \frac{1}{2} \epsilon^a_{bc} (be^b - ns^b)e^c \quad (26) \]

we write the contortion explicitly as:

\[ K^b = \frac{1}{2} (ae^b + ms^b) \quad (27) \]

Now, varying the action with respect to 1-form \( A \) we obtain the modified Maxwell equations (that includes the contribution of the Chern Simons term):

\[ d^a F + \mu_F F = 0 \quad (28) \]

Replacing the expression for the contortion in equation (11), we find the Cartan curvature as:

\[ \bar{R}^a = \bar{R}^a + \frac{m}{2} \bar{D}s^a + \frac{1}{8} \epsilon^a_{bc} (a^2 b^d c^e + 2 am b^d c^e + m^2 b c e) \quad (29) \]

Taken this last equation together with the second gravitational equation (26), we obtain:

\[ \bar{R}^a = -\frac{\gamma}{2} \epsilon^a_{bc} s^b e^c - \frac{m}{2} \bar{D}s^a - \frac{m^2}{8} \epsilon^a_{bc} s^b s^c + \frac{1}{2} \Lambda_{eff} \epsilon^a_{bc} e^b c \quad (30) \]

where we have defined:

\[ \gamma = n + \frac{am}{2} \quad (31) \]

\[ \Lambda_{eff} = b - \frac{a^2}{4} = -\frac{1}{l^2} \quad (32) \]

and:

\[ \bar{D}s^a = ds^a + \epsilon^a_{bc} \bar{\omega}^b s^c \quad (33) \]

is the covariant derivative (with respect to \( \bar{\omega} \)) of \( s^a \).

### 3 Solving the Gravitational Equations

Now, we consider the following ansatz for the metric (in the local coordinates \( x^\mu = t, r, \phi \)):

\[ ds^2 = \frac{\psi(r)}{f(r)} dt^2 - f(r) \left( d\phi + \frac{Cr}{f(r)} dt \right)^2 - \frac{1}{\psi(r)} dr^2 \quad (34) \]

which (manifestly) corresponds to a stationary circularly symmetric solution\(^2\).

On the other hand, the ansatz for the U(1) field is a spherically symmetric gauge field in the gauge \( A_r = 0 \):

\[ A = A_t dt + A_\phi d\phi \quad (35) \]

\(^2\)Notice that this form for the metric agrees with the one used in [16], see Eq. (39) therein and consider \( r = \bar{r}^2 \).
we consider $A_t$ be a constant and $A_\phi = I + H r$, with $I$ and $H$ constants, this yields:

$$F = dA = H dr d\phi$$

(36)

in the static local coordinates $x^\mu = t, r, \phi$ with $0 \leq \phi < 2\pi$

We will consider $f(r)$ and $\psi(r)$ as being quadratic functions of $r$. That is,

$$f(r) = M r^2 + N r + L$$

(37)

$$\psi(r) = P r^2 + Q r + R$$

(38)

where $M, N, L, P, Q, R$ are integration constants to be determined.

The orthonormal basis is determined up to a local lorentz transformation and we choose the basis to be:

$$e^0 = \sqrt{\frac{\psi(r)}{f(r)}} dt$$

(39)

$$e^1 = \frac{1}{\sqrt{\psi(r)}} dr$$

(40)

$$e^2 = \sqrt{\frac{f(r)}{\psi(r)}} \left( d\phi + \frac{C r}{f(r)} \right)$$

(41)

and for the electric field:

$$F = E(r) e^0 e^1 - B(r) e^1 e^2$$

(42)

"squaring" the electromagnetic tensor we obtain the electromagnetic invariant:

$$F_{ab} F^{ab} = 2(B^2 - E^2)$$

(43)

and the energy momentum tensor results:

$$S_{ab} = \begin{pmatrix}
\frac{1}{2} (E^2 + B^2) & 0 & EB \\
0 & \frac{1}{2} (E^2 - B^2) & 0 \\
-EB & 0 & -\frac{1}{2} (E^2 + B^2)
\end{pmatrix}$$

(44)

the trace of the Maxwell energy momentum tensor reads:

$$S = \frac{1}{2} (E^2 - B^2)$$

(45)

from the Maxwell equations we get the pair of equations:

$$\sqrt{\psi} E' + \frac{1}{2} \sqrt{\psi} \frac{f'}{f} E + \mu E B = 0$$

(46)

$$\sqrt{\psi} B' + \frac{1}{2} \sqrt{\psi} \left( \psi' - \frac{f'}{f} \right) B + \left( C - Cr \frac{f'}{f} + \mu E \right) E = 0$$

(47)

Taking into account that the ansatz for the electric field in the local coordinates is $F = H dr d\phi$, we obtain the electromagnetic field that generate the gravitational field as:

$$E(r) = \frac{CHr}{\sqrt{f}}$$

(48)
Finally, from the gravitational equations we obtain the following algebraic system of equations:

$$B(r) = -H \sqrt{\frac{\psi}{f}}$$

(49)

with:

$$C = \mu_e$$

(50)

from the energy momentum tensor we determine the 1-form $s^a$, we get:

$$s^0 = -\frac{1}{4} (3B^2 + E^2) e^0 - EBe^2$$

(51)

$$s^1 = -\frac{1}{4}(E^2 - B^2)e^1$$

(52)

$$s^2 = \frac{1}{4}(B^2 + 3E^2)e^2 + EBe^0$$

(53)

The null torsion condition yields the Riemannian connection:

$$\tilde{De}^a = de^a + \epsilon_{abc} \bar{\omega}^b e^c = 0$$

(54)

together with the expressions for $e^a$ we have:

$$\bar{\omega}^0 = -\frac{1}{2}C \left(1 - r \frac{f'}{f}\right) e^0 - \frac{1}{2} \sqrt{\psi} f' e^2$$

(55)

$$\bar{\omega}^1 = -\frac{1}{2}C \left(1 - r \frac{f'}{f}\right) e^1$$

(56)

$$\bar{\omega}^2 = -\frac{1}{2 \sqrt{\psi}} \left(\psi - \psi \frac{f''}{f}\right) e^0 + \frac{1}{2} C \left(1 - r \frac{f'}{f}\right) e^2$$

(57)

with the expressions obtained above, we get the Riemannian curvature to be:

$$\bar{R}^0 = -\frac{1}{2} C r \frac{f''}{f} \sqrt{\psi} e^0 e^1 - \left(\frac{1}{4} \psi \frac{f''}{f} + \frac{1}{2} \psi \frac{f''}{f} - \frac{1}{4} \psi \frac{f'^2}{f^2} + \frac{1}{4} C^2 \left(1 - r \frac{f'}{f}\right)^2\right) e^1 e^2$$

(58)

$$\bar{R}^1 = -\left(\frac{1}{4} \frac{f''}{f} \left(\psi - \psi \frac{f'}{f}\right) + \frac{1}{4} C \left(1 - r \frac{f'}{f}\right)^2\right) e^0 e^2$$

(59)

$$\bar{R}^2 = \left(-\frac{1}{2} \psi \frac{f''}{f} + \frac{1}{2} \psi'' - \frac{3}{4} \psi \frac{f''}{f} + \frac{3}{4} \psi \frac{f'^2}{f^2} - \frac{3}{4} C^2 \left(1 - r \frac{f'}{f}\right)^2\right) e^0 e^1 - \frac{1}{2} C r \frac{f''}{f} \sqrt{\psi} e^1 e^2$$

(60)

the expressions for the others terms that appears in the gravitational equations are given in appendix A.

Finally, from the gravitational equations we obtain the following algebraic system of equations:

$$Cr \frac{f''}{f} = \gamma \frac{CH^2 r}{f} - \frac{m}{4} \frac{H^2}{4f^2} f' (3\psi + C^2 r^2) + \frac{m}{4} \frac{H^2}{4f} (3\psi' + 2C^2 r) + \frac{m}{2} \frac{H}{f} \left(\frac{\psi' f' - \psi f'}{f^2}\right)$$

$$- \frac{3}{2} \frac{m}{2} \frac{C H^2 r}{f} \left(1 - r \frac{f'}{f}\right) - \frac{m^2}{8} \frac{C H^4 r}{f^2} \left(C^2 r^2 - \psi\right)$$

(61)

$$\frac{3}{4} \psi' \frac{f'}{f} + \frac{1}{4} \psi \frac{f'^2}{f^2} + \frac{3}{4} C^2 \left(1 - r \frac{f'}{f}\right)^2 = \frac{1}{2} \frac{H^2}{f} \left(\psi + C^2 r^2\right) + \frac{m}{2} \frac{H}{f} \left(\frac{\psi f' + r \psi' f - \psi}{f^2}\right)$$

$$- \frac{m}{4} \frac{C^3 H^2 r^2}{f} \left(1 - r \frac{f'}{f}\right) - \frac{m^2}{64} \frac{H^4}{f^2} \left(\psi + 3C^2 r^2\right) \left(C^2 r^2 - \psi\right) - \Lambda_{eff}$$

(62)
allows us to get the constants \( M + m CH \) adding (62) and (63):

Now, we go back to the system of equations and find that only two equations are independent, and this therefore the function \( \psi \):

Adding equations (61) and (65):

\[
-f' \psi + C^2 r^2 f' - 2C^2 r f + f \psi' = 0
\]

from this equation we find a constraint between the coefficients:

\[
P = \frac{R}{L} M + C^2
\]

\[
Q = \frac{R}{L} N
\]

therefore the function \( \psi(r) \) takes the form:

\[
\psi(r) = \frac{R}{L} f(r) + C^2 r^2
\]

Now, we go back to the system of equations and find that only two equations are independent, and this allows us to get the constants \( M \) and \( H \).

adding (62) and (63):

\[
m^2 \frac{R}{L} H^4 + \left( \gamma + \frac{3}{2} m C \right) H^2 - 2M = 0
\]

Multiplying equation (62) by 3 and adding the result to equation (64):

\[
\psi \frac{f''}{f} + \frac{1}{2} \psi'' = \frac{\gamma}{2} f H^2 (\psi + C^2 r^2) + \frac{m C H^2 \psi}{f} + \frac{1}{4} m C H^2 r \psi - \frac{1}{4} m C H^2 r f' \psi \\
- \frac{1}{4} m C H^2 r^2 f' + \frac{1}{4} m C H^2 r^3 f' \\
- \frac{m^2 H^4}{f} \frac{C^2 r^2 f' - 4 \lambda_{eff}}{f^2}
\]

and from this equation we get the other independent relation:

\[
3M \frac{R}{L} + C^2 + 4 \lambda_{eff} = \frac{1}{2} \left( \gamma + \frac{5}{2} m C \right) H^2 \frac{R}{L}
\]
Finally, we are left with only two equations to be solved:

\[
\frac{m^2}{8} R \frac{H^4}{L} + \left( \gamma + \frac{3}{2} m C \right) H^2 - 2 M = 0
\]  

\[
3M \frac{R}{L} + C^2 + 4 \Lambda_{\text{eff}} = \frac{1}{2} \left( \gamma + \frac{5}{2} m C \right) H^2 R \frac{R}{L}
\]  

(73)

(74)

then, we eventually find:

\[
\sigma := \frac{H^2 R}{L} = \frac{4}{m^2} \left( -\frac{2}{3} (\gamma + m \mu) \pm \sqrt{\frac{4}{9} (\gamma + m \mu)^2 - \frac{1}{3} m^2 (\mu^2 + 4 \Lambda_{\text{eff}})} \right)
\]  

(75)

\[
\rho := \frac{M R}{L} = -\frac{1}{3} (\mu_E + 4 \Lambda_{\text{eff}}) + \frac{1}{6} \left( \gamma + \frac{5}{2} m \mu \right) \frac{H^2 R}{L}
\]  

(76)

in turn, equations (75)-(76), along with the ansatz (39)-(42) and the connection (9) represent an exact solution of the theory (5). In the next, we will fix \( N = 2 \).

4 Conserved Charges

In this section we calculate the mass, angular momentum and electric charge of our solution. The asymptotic behavior of the vielbein and the Cartan connection is given in Appendix B.

The canonical generator has the following form in the asymptotic region:

\[
G = -G_1 - G_3
\]  

(77)

with:

\[
G_1 = \xi^a \left[ e^a \mathcal{H}_a + \omega^a \mathcal{K}_a + (\partial_\rho e^a) \pi^t_a + (\partial_\rho \omega^a) \Pi^a_t + (\partial_\rho A_t) \pi^t \right]
\]  

(78)

\[
G_3 = -\lambda \left( \partial_\alpha \pi^a - 2 \mu_E \xi^{a \beta} \partial_a A_\beta \right)
\]  

(79)

\[
\pi^a = -k F^{\alpha a} - \mu_E \xi^{a \beta} A_\beta
\]  

(80)

The expressions for \( \mathcal{H}_a \) and \( \mathcal{K}_a \) can be found in Appendix C of [8] (see also [15]).

Variation of the generator produces:

\[
\delta G_1 = \xi^a \left( -2 \xi^{a \beta} \partial_\alpha \left[ e^a \delta \left( \frac{1}{2 \kappa} \omega_{a \beta} + \alpha_4 e_{a \beta} \right) + \omega^a \delta \left( \frac{1}{2 \kappa} e_{a \beta} + \alpha_3 \omega_{a \beta} \right) \right] + \delta \tau^t_\rho \right) + \text{regular terms}
\]  

(81)

\[
\delta G_3 = -\lambda \left( \partial_\alpha \delta \pi^a - 2 \mu_E \xi^{a \beta} \partial_a \delta A_\beta \right)
\]  

(82)

where \( \delta \tau^t_\rho = \frac{1}{2} \xi^{a \beta} e^a \delta \chi_{a \beta} \) comes from the Maxwell energy-momentum current \( \chi^i = \frac{1}{2} \chi^i_{\mu \nu} dx^\mu dx^\nu \) (see Appendix C of [8]).
Using the asymptotic behavior of the vielbein and the connection (Appendix B) we obtain the conserved charges, however the integrability conditions are satisfied only for particular cases, e.g., for \( L = 0 \) and \( L = \frac{1}{2} \), for this last case we obtain:

\[ Q = 0 \]  

\[ \mathcal{M} = -\left[ \frac{1}{2\kappa}\rho + \frac{1}{\kappa}a\mu E - \frac{1}{4\kappa}m\mu E\sigma + 2\alpha_4\mu E + \alpha_3 \left( \frac{\rho}{2} + \frac{\mu_E^2}{2} + \frac{1}{2}\alpha_4\mu E + \frac{m\sigma\mu E}{8} \right) \left( -\mu_E + a - \frac{3}{4}m\sigma \right) \right] \frac{1}{M} \]

\[ -\frac{1}{2} \sqrt{\frac{2}{\rho}} \frac{A_1 \sqrt{M}}{\mu E} \frac{1}{M} \]  

\[ J = -\left[ \frac{1}{2\kappa} \left( a + \mu E + \frac{1}{4}m\sigma \right) + \frac{\alpha_3}{4} \left( a + \mu E + \frac{1}{4}m\sigma \right)^2 + \alpha_4 + \alpha_3\rho - \frac{1}{2} \sqrt{\frac{2}{\rho} A_1 \sqrt{M}} \right] \frac{1}{M} \]  

and:

\[ A_1 \sqrt{M} = \frac{2}{\sqrt{\rho}} \left[ \frac{1}{\kappa} \left( \frac{\rho}{\sqrt{\rho}} + \frac{2m\sigma}{\rho} \left( \rho + 4\mu_E^2 \right) - \frac{\mu_E^2}{\rho} \right) + \alpha_4 - \alpha_3 \left( \rho + \mu_E^2 \right) \left( -1 + \frac{1}{2}m\mu_E\frac{a}{\rho} \right) \right] \frac{1}{M} \]

\[ A_\phi = I + Hr = \frac{A_t}{\mu E} + Hr \]  

\( Q, \mathcal{M} \) and \( J \) denote the electric charge, the mass and the angular momentum respectively.

we also obtain the following equation between the coupling constants:

\[ -\frac{3}{8\kappa}m\sigma\mu E - \frac{1}{2\kappa}\rho + \frac{1}{2\kappa}a\mu E + \alpha_4\mu E + \alpha_3 \left( -\frac{1}{2}\mu_E + \frac{1}{2}a - \frac{3}{8}m\sigma \right) \left( \rho + \frac{1}{2}\mu_E^2 + \frac{1}{2}a\mu E - \frac{3}{8}m\sigma\mu E \right) + \frac{1}{2}\sigma = 0 \]  

(88)

It might be more appropriate to use another approach for the calculation of the conserved charges for this type of spacetime that the used here, e.g., see [12] and references therein.

We briefly comment the relation of our general solution with those of [13], note that our metric can be put in the form of metric (3.16) of that paper making the coordinate transformation \( \rho = -C/r - \frac{RN}{2L} (1 + \frac{1}{2\kappa}E) \).

Therefore we conclude that our solution generalizes the solution (3.16) to spacetime with torsion.

## 5 Particular cases

Now, in order to understand what our solution correspond to, let us consider some particular cases and limits.

Let’s consider the limit \( \alpha_3 \to 0 \) (\( m \to 0 \)), in this case equations (73) and (74) simplify notably:

\[ -\kappa H^2 = M \]

\[ 3M \frac{R}{L} + C^2 + 4\Lambda_{eff} = -\kappa H^2 \frac{R}{L} \]  

(90)
and we obtain:

\[
H^2 \frac{R}{L} = \frac{1}{2\kappa} \left( \mu_E^2 + 4\Lambda_{\text{eff}} \right) \tag{91}
\]

\[
M^2 \frac{R}{L} = -\frac{1}{2} \left( \mu_E^2 + 4\Lambda_{\text{eff}} \right) \tag{92}
\]

the conserved charges simplify to:

\[
Q = 0 \tag{93}
\]

\[
M = \left( \frac{\mu_E^2 + 4\Lambda_{\text{eff}}}{2\kappa} \right) \frac{1}{M} \tag{94}
\]

\[
J = -\frac{\mu_E}{\kappa} \frac{1}{M} \tag{95}
\]

and:

\[
A = \frac{\mu_E}{\sqrt{-\frac{1}{\kappa} M}} \left( \frac{1}{\sqrt{-\frac{1}{\kappa} M}} + Hr \right) \, d\phi \tag{96}
\]

so, we get the following solution for the metric and gauge field:

\[
f(r) = \frac{1}{2\kappa l^2 M} \left( \mu_E^2 l^2 - 4 \right) r^2 + 2r - \frac{\kappa J}{\mu_E} \tag{97}
\]

\[
\psi(r) = \frac{1}{2l^2} \left( \mu_E^2 l^2 + 4 \right) r^2 - 2\kappa Mr + \frac{\kappa^2 M J}{\mu_E} \tag{98}
\]

\[
A = \sqrt{\frac{1}{4\kappa^2 l^2 M} \left( \mu_E^2 l^2 - 4 \right)} \left[ \frac{1}{\sqrt{-\frac{1}{\kappa} M}} dt + \left( r - \frac{1}{\sqrt{-\frac{1}{\kappa} M}} \frac{1}{\mu_E} \left( \mu_E^2 l^2 - 4 \right) \right) d\phi \right] \tag{99}
\]

As pointed out in [13] the gravitational constant can be positive or negative in 2+1 dimensions [14], also we know [3] that in topologically massive gravity the gravitational constant should be taken negative to avoid the appearance of ghosts.

We mention that in this case we have \( M > 0 \) (because of the asymptotic behavior shown in Appendix B) and choose \( \kappa < 0 \) (96) to get a real solution. However, it is possible to make a coordinate transformation: \( \phi \to i\phi, t \to it, r \to -r \) [12] to find a new solution, in this case the metric and the gauge field become (because \( N \) is arbitrary we also make the change \( N \to -N = 2 \)):

\[
ds^2 = \frac{\psi(r)}{-f(r)} dt^2 + (-f(r)) \left( d\phi + \frac{Cr}{-f(r)} dt \right)^2 - \frac{1}{\psi(r)} dr^2 \tag{100}
\]

\[
A = \frac{\mu_E}{\sqrt{-\frac{1}{\kappa} M}} \left( \frac{1}{\sqrt{-\frac{1}{\kappa} M}} + Hr \right) \, d\phi \tag{101}
\]

and:
\[
\mathcal{Q} = 0
\]
\[
\mathcal{M} = \frac{\left(\mu_E^2 + 4\Lambda_{eff}\right)}{2\kappa} \frac{1}{M}
\]
\[
\mathcal{J} = \frac{\mu_E}{\kappa} L
\]

In this case we have a real solution for \( M < 0 \) and \( \kappa > 0 \) and we see that our black hole generalizes the Einstein Maxwell Chern Simons (with negative cosmological constant) solution obtained in [12] (where the gravitational constant is positive) and the solution admit naked closed timelike curves \(^3\). However this solution only is valid for \( L = \frac{1}{M} \) and we emphasize that it might be more appropriate to use the approach used in [12] to obtain more general expressions for the conserved charges.

We find that when \( \mu_E^2 l^2 = 4 \) the fluid disappears, the energy momentum tensor vanishes and the metric reduces to the BTZ metric in Einstein Cartan spacetime (this is valid also for the general solution (75)-(76)). If \( \mu_E^2 l^2 > 4 \), then \( \mathcal{M} \) has to be positive to get a real solution and if \( \mu_E^2 l^2 < 4 \), \( \mathcal{M} \) is negative (these are the generalized Godel particles and Godel black hole of [12], respectively). When \( \psi(r) \) vanishes we find the horizons \( r_+, r_- \) \( (\psi(r_{\pm}) = 0, f(r_{\pm}) + C^2 r_{\pm}^4 = 0) \).

In the case of not charged solutions, from the gravitational equations (61)-(65) with \( H = 0 \), we recover the BTZ black hole in Riemann Cartan spacetime:

\[
M = 0
\]
\[
C = \frac{2}{l}
\]

6 Final remarks

In conclusion here we have discussed black holes solutions to 3D Einstein theory with torsion coupled to topologically massive gravity and charged under topologically massive electrodynamics. The solution we just found generalizes previous solutions reported in the literature, so that it may contribute to a better understanding of the solutions content of this interesting toy model of gravity.

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8 Appendix A

In this Appendix we show the expressions for some terms that appears in the gravitational equations:

\[
\bar{D}s^a = ds^a + \epsilon_{bc} \bar{\omega}^b s^c
\]

\(^3\)In [12], \( \alpha = -\frac{\mu_E}{l} \).
\[ Ds^0 = -\left( \frac{H^2}{4f^2} f' (3\psi + C^2r^2) \sqrt{\psi} - \frac{H^2}{4f} (3\psi' + 2C^2r) \sqrt{\psi} - \frac{H^2}{8\sqrt{\psi}} (3\psi + C^2r^2) \left( \frac{\psi' f - \psi f'}{f^2} \right) \right) e^0 e^1 \]

\[ + CH^2 \left( \frac{\psi}{f} + \frac{r \psi'}{2f^2} f' \right) e^1 e^2 - C^2 H^2 r \sqrt{\psi} \left( 1 - \frac{r f'}{f} \right) e^0 e^1 - \frac{H^2}{8r \sqrt{\psi}} \left( \psi' - \frac{\psi f'}{f} \right) (C^2r^2 - \psi) e^0 e^1 \]

\[ - \frac{1}{2} C^2 H^2 r \sqrt{\psi} \left( 1 - \frac{r f'}{f} \right) e^1 e^2 - \frac{1}{2} C^2 H^2 r \sqrt{\psi} \left( 1 - \frac{r f'}{f} \right) e^0 e^1 \]

\[ Ds^1 = -\frac{H^2}{f} \left( \frac{1}{2} C \left( 1 - \frac{r f'}{f} \right) (\psi + C^2r^2) - Cr \left( \frac{\psi' - \psi f'}{f} \right) \right) e^0 e^2 \]

\[ Ds^2 = \frac{H^2}{4} \left( -\frac{f'}{2f^2} (\psi + 3C^2r^2) + \frac{1}{f} (\psi' + 6C^2r) \right) \sqrt{\psi} e^1 e^2 - \frac{CH^2}{4f} (\psi + 3C^2r^2) \left( 1 - \frac{r f'}{f} \right) e^0 e^1 \]

\[ + CH^2 \left( \frac{\psi}{f} + \frac{3r \psi'}{2f^2} f' \right) e^1 e^2 - \frac{1}{2} C^2 H^2 r \sqrt{\psi} \left( 1 - \frac{r f'}{f} \right) e^1 e^2 + \frac{1}{8} H^2 \sqrt{\psi} \left( \frac{f'}{f} (C^2r^2 - \psi) \right) e^1 e^2 \]

\[ X^a = e^b e^c s^a \]

\[ X^0 = -\frac{1}{8} (E^2 - B^2) (3E^2 + B^2) e^1 e^2 + \frac{1}{2} EB (E^2 - B^2) e^0 e^1 \]

\[ X^1 = \left( \frac{1}{8} (3E^2 + B^2) (E^2 + 3B^2) - 2E^2 B^2 \right) e^0 e^2 \]

\[ X^2 = \frac{1}{8} (E^2 - B^2) (E^2 + 3B^2) e^0 e^1 + \frac{1}{2} EB (E^2 - B^2) e^1 e^2 \]

\[ Y^a = e^b e^c s^a \]

\[ Y^0 = \frac{1}{2} (E^2 + B^2) e^1 e^2 - EBe^0 e^1 \]

\[ Y^1 = -\frac{1}{2} (B^2 - E^2) e^0 e^2 \]

\[ Y^2 = \frac{1}{2} (E^2 + B^2) e^0 e^1 - EBe^1 e^2 \]

9 Appendix B

The asymptotic behavior of the vielbein and the cartan connection are determined by the following expansion:

\[ e^1_i \sim \sqrt{\frac{P}{M}} \left( 1 + \frac{1}{2} \frac{Q}{P} \frac{N}{M} \frac{1}{r} + O(r^2) \right) \]

\[ e^1_r \sim O(r) \]

\[ e^2_\phi \sim \sqrt{M} \left( r + \frac{1}{2} \frac{N}{M} + \frac{1}{2} \frac{L}{M} - \frac{1}{4} \frac{N^2}{M^2} \frac{1}{r} + O(r^2) \right) \]

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\[ \omega_i^2 \sim \frac{C}{\sqrt{M}} \left( 1 - \frac{1}{2} \frac{N}{M} r + \mathcal{O}(r^2) \right) \]

\[ \omega_0^0 \sim -\frac{1}{2} C \sqrt{\frac{P}{M}} \left( 1 + \frac{1}{2} \left( \frac{Q}{P} - \frac{N}{M} \right) \frac{1}{r} + \mathcal{O}(r^2) \right) \]

\[ \omega_0^\phi \sim -\sqrt{PM} \left( r + \frac{1}{2} \frac{Q}{P} + \frac{1}{2} \left( -\frac{L}{M} + \frac{R}{P} + \frac{1}{8} \frac{Q^2}{M^2} - \frac{1}{4} \frac{Q}{P^2} \right) \frac{1}{r} + \mathcal{O}(r^2) \right) \]

\[ \bar{\omega}_t \sim \mathcal{O}(r) \]

\[ \bar{\omega}_r \sim \frac{1}{2} \frac{C^2}{\sqrt{M}} \left( -1 + \frac{1}{2} \frac{N}{M} r + \mathcal{O}(r^2) \right) \]

\[ \omega^\phi \sim \frac{1}{2} C \sqrt{M} \left( -r + \frac{1}{2} \frac{N}{M} + \frac{3}{2} \left( \frac{L}{M} - \frac{1}{4} \frac{N^2}{M^2} \right) \frac{1}{r} + \mathcal{O}(r^2) \right) \]

\[ K_t^0 \sim \frac{1}{2} \left( a - \frac{3}{4} mH^2 \frac{R}{L} \right) \sqrt{\frac{P}{M}} \left( 1 + \frac{1}{2} \left( \frac{Q}{P} - \frac{N}{M} \right) \frac{1}{r} + \mathcal{O}(r^2) \right) \]

\[ K^0_\phi \sim \frac{1}{2} mCH^2 \sqrt{\frac{P}{M}} \left( r + \frac{1}{2} \left( \frac{Q}{P} - \frac{N}{M} \right) + \frac{1}{2} \left( \frac{R}{P} - \frac{L}{M} - \frac{1}{4} \frac{Q^2}{M^2} - \frac{1}{2} \frac{QN}{P} + \frac{3}{4} \frac{N^2}{M^2} \right) \frac{1}{r} + \mathcal{O}(r^2) \right) \]

\[ K_t^2 \sim \frac{1}{2} \left( a \frac{C}{\sqrt{M}} - \frac{3}{4} mH^2 \frac{R}{L} \frac{C}{\sqrt{M}} \right) \left( 1 - \frac{N}{2} \frac{1}{M} r + \mathcal{O}(r^2) \right) \]

\[ K^2_\phi \sim \frac{1}{2} a \sqrt{M} \left( r + \frac{1}{2} \frac{N}{M} + \frac{1}{2} \left( \frac{L}{M} - \frac{1}{4} \frac{N^2}{M^2} \right) \frac{1}{r} \right) + \frac{1}{8} mH^2 \sqrt{\frac{P}{M}} \left( (P + 3C^2) r + \left( Q - \frac{1}{2} \frac{N}{M} (P + 3C^2) \right) \right) \]

\[ + \frac{1}{8} mH^2 \sqrt{\frac{P}{M}} \left( R - \frac{1}{2} \frac{NQ}{M} + \left( \frac{1}{2} \frac{L}{M} + \frac{3}{8} \frac{N^2}{M^2} \right) (P + 3C^2) \right) \frac{1}{r} + \mathcal{O}(r^2) \]

\[ \omega^a_\mu = \bar{\omega}^a_\mu + K^a_\mu \]

where \( \mathcal{O}(r^n) \sim \frac{1}{r^n} \).

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