Discrete-time Flatness and Linearization along Trajectories

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Abstract: The paper studies the relation between a nonlinear time-varying flat discrete-time system and the corresponding linear time-varying systems which are obtained by a linearization along trajectories. It is motivated by the continuous-time case, where it is well-known that the linearization of flat systems along trajectories results in linear time-varying systems which are again flat. Since flatness implies controllability, this constitutes an important verifiable necessary condition for flatness. In the present contribution, it is shown that this is also true in the discrete-time case: We prove that the linearized system is again flat, and that a possible flat output is given by the linearization of a flat output of the nonlinear system. Analogously, the map that describes the parameterization of the system variables of the linear system by this flat output coincides with the linearization of the corresponding map of the nonlinear system. The results are illustrated by two examples.

Keywords: discrete-time systems; flatness; linearization; controllability; time-varying systems

1. INTRODUCTION

The concept of flatness has been introduced in the 1990s by Fliess, Levine, Martin and Rouchon for nonlinear continuous-time systems, see e.g. Fliess et al. (1992), Fliess et al. (1995), or Fliess et al. (1999). Since flatness allows an elegant solution for motion planning problems and a systematic design of tracking controllers, it is of high practical relevance and belongs to the most popular nonlinear control concepts. Nevertheless, checking the flatness of a nonlinear multi-input system is known as a highly nontrivial problem, for which still no complete systematic solution in the form of verifiable necessary and sufficient conditions exists (see e.g. Nicolau and Respondek (2016), Nicolau and Respondek (2017), or Gstöttner et al. (2021) for recent contributions in this field). For this reason, also necessary conditions for flatness are of interest to be able to prove at least that a given system is not flat. One such necessary condition is based on the fact that the linearization of a flat continuous-time system along a trajectory yields a linear time-varying system which is again flat and hence controllable (see e.g. Rudolph (2021)). Since the latter property can be checked easily for linear systems, this connection between a nonlinear system and its linearization constitutes an important necessary condition for the flatness of continuous-time systems.

The purpose of the present contribution is to investigate the relation between a flat system and its linearization along a trajectory in the discrete-time case. Since the linearization of a nonlinear system along a trajectory leads in general to a linear time-varying system but the literature has addressed so far only the time-invariant case, we first need to discuss the concept of discrete-time flatness for time-varying systems. As proposed in Diwold et al. (2022b), we consider discrete-time flatness as the existence of a one-to-one correspondence of the system trajectories to the trajectories of a trivial system. This leads naturally to a formulation which takes into account both forward- and backward-shifts of the system variables as in it is also proposed in Guillot and Millérioux (2020). The point of view adopted e.g. in Sira-Ramirez and Agrawal (2004), Kaldmäe and Kotta (2013), or Kolar et al. (2016), where discrete-time flatness is defined by replacing the time derivatives of the continuous-time case by forward-shifts, is included as a special case and denoted within the present paper as forward-flatness.

As our main result, we prove that the linearization of a flat discrete-time system along a trajectory is again flat, and that a possible flat output is given by the linearization of a flat output of the nonlinear system. Furthermore, we show that the corresponding parameterization of the system variables by the flat output and its shifts coincides with the linearization of the parameterization of the nonlinear system. Like in the continuous-time case, this connection between nonlinear system and linearized system establishes...
an important necessary condition for flatness. Even though for discrete-time systems the property of forward-flatness can be checked efficiently by a generalization of the test for static feedback linearizability (see Kolar et al. (2022)) which is based on a certain decomposition property derived in Kolar et al. (2021), for the more general case including both forward- and backward-shifts of the system variables a computationally feasible test does not yet exist. \(^1\) Hence, as we shall illustrate by our second example, the derived necessary condition is a useful possibility to prove that a given discrete-time system is not flat.

The paper is organized as follows: First, Section 2 deals with the concept of discrete-time flatness for time-varying systems. The core of the paper is then contained in Section 3, which studies the relation between a flat system and the linear time-varying system obtained by a linearization along a trajectory. The presented results are illustrated by two examples in Section 4.

\(\textbf{Notation}\) Since we apply differential-geometric concepts, we use index notation and the Einstein summation convention to keep formulas short and readable. However, to highlight the summation range especially for double sums, we also frequently indicate the summation explicitly. For coordinates that represent forward- or backward-shifts of system variables, we use a notation with subscripts in brackets. For instance, the \(\alpha\)-th forward- or backward-shift of a component \(y_j\), \(j \in \{1, \ldots, m\}\) of a flat output \(y\) with \(\alpha \in \mathbb{Z}\) is denoted by \(y_{[\alpha]}\), and \(y_{[\alpha]} = (y_{[\alpha]}^1, \ldots, y_{[\alpha]}^m)\). Furthermore, to facilitate the handling of expressions which depend on different numbers of shifts of different components of a flat output, we use multi-indices. If \(A = (a^1, \ldots, a^m)\) is some multi-index, then \(y_{[A]} = (y_{[a^1]}, \ldots, y_{[a^m]})\).

2. FLATNESS OF TIME-VARYING DISCRETE-TIME SYSTEMS

In this contribution, we consider nonlinear time-varying discrete-time systems

\[
x^{i,+} = f^i(k, x, u), \quad i = 1, \ldots, n
\]

with \(\dim(x) = n\), \(\dim(u) = m\), and smooth functions \(f^i(k, x, u)\). In addition, we assume that the system (1) meets the submersivity condition

\[
\text{rank}(\partial_{x,u} f) = n, \quad (2)
\]

which is quite common in the discrete-time literature, for all time-steps \(k\).

As proposed in Diwold et al. (2022b), where only time-invariant systems are considered, we call a time-varying discrete-time system (1) flat if there exists a one-to-one correspondence between its trajectories \((x(k), u(k))\) and the trajectories \(y(k)\) of a trivial system with \(\dim(y) = \dim(u)\). The trajectories of a trivial system are not restricted by any difference equation and hence completely free. By one-to-one correspondence, we mean that the values of \(x(k)\) and \(u(k)\) at a time-step \(k\) are determined by an arbitrary but finite number of past and values of \(y(k)\), i.e., by the trajectory \(y(k)\) in an arbitrarily large but

\(\text{finite time window. Conversely, the value of } y(k) \text{ at a time-step } k \text{ is determined by an arbitrary but finite number of past and values of } x(k) \text{ and } u(k). \text{ Consequently, the one-to-one correspondence of the trajectories can be expressed by maps of the form}

\[
(x(k), u(k)) = F(k, y(k - r_1), \ldots, y(k + r_2)) \quad (3)
\]

and

\[
y(k) = F(k, x(k - q_1), u(k - q_1), \ldots, x(k + q_2), u(k + q_2)) \quad (4)
\]

with suitable integers \(r_1, r_2, q_1, q_2\) that describe the length of the corresponding finite time windows, cf. Fig. 1. Since the number of forward- and backward-shifts in (3) and (4) can of course be different for the individual components of \(y, x,\) and \(u\), we will later use appropriate multi-indices where it is important.

In the remainder of this section, the framework used in Diwold et al. (2022b) for the analysis of flat time-invariant discrete-time systems is adapted to the time-varying case. First, it is important to note that the representation of a trajectory of the system (1) by both sequences \(x(k)\) and \(u(k)\) contains redundancy, as these sequences are coupled by the system equations (1). By a repeated application of (1), all forward-shifts \(x(k + \alpha)\), \(\alpha \geq 1\) of the state variables are determined by \(x(k)\) and the input trajectory \(u(k + \alpha)\) for \(\alpha \geq 0\):

\[
\begin{align*}
x(k + 1) &= f(k, x(k), u(k)) \\
x(k + 2) &= f(k + 1, x(k + 1), u(k + 1)) \\
&\quad \vdots
\end{align*}
\]

In the case \(\text{rank}(\partial_{x,u} f) = n\), the same is also true for the backward-direction. However, even if the system meets only the weaker submersivity condition (2), there exist \(m\) functions \(g(k, x, u)\) such that the \((n + m) \times (n + m)\) Jacobian matrix

\[
\begin{bmatrix}
\partial_x f & \partial_u f \\
\partial_g & \partial_u g
\end{bmatrix} \quad (5)
\]

is regular for all \(k\). With such functions, the map

\[
x^+ = f(k, x, u) \quad (6)
\]

is locally invertible for all \(k\), and by a repeated application of its inverse

\[
(x, u) = \psi(k, x^+, \zeta) \quad (7)
\]

all backward-shifts \(x(k - \beta), u(k - \beta), \beta \geq 1\) of the state- and input variables are determined by \(x(k)\) and backward-shifts \(\zeta(k - \beta), \beta \geq 1\) of the system variables \(\zeta\) defined by (6):

\[
\begin{align*}
(x(k - 1), u(k - 1)) &= \psi(k - 1, x(k), \zeta(k - 1)) \\
(x(k - 2), u(k - 2)) &= \psi(k - 2, x(k - 1), \zeta(k - 2)) \\
&\quad \vdots
\end{align*}
\]

Hence, every trajectory of the system (1) is uniquely determined both in forward- and backward-direction by

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\(^1\) An interesting approach can be found in Kaldmae (2022) but requires the solution of partial differential equations.

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![Fig. 1. One-to-one correspondence of the trajectories.](image-url)
the values \( \zeta(k - 2), \zeta(k - 1), x(k), u(k), u(k + 1), \ldots \), and the map (4) can actually be written as
\[
y(k) = \varphi(k, \zeta(k - q_1), \ldots, \zeta(k - 1), x(k), u(k), \ldots, u(k+q_2)).
\]

If only a finite time interval is considered, the trajectories of the system (1) can be identified with points of a finite-dimensional manifold \( \Omega_{l_1, \alpha} \times \mathfrak{X} \times \mathfrak{U} \), with coordinates \( \left( \zeta_{l_{-\iota}}, \ldots, \zeta_{l_{-1}}, x, u, u_{[l_1]}, \ldots, u_{[l_2]} \right) \) and sufficiently large integers \( l_{\iota}, l_{\alpha} \).

\[
h(k, \zeta_{l_{-1}}, \ldots, \zeta_{l_{-q_1}}, x, u, \ldots, u_{[q_2]})
\]
denotes a function on \( \mathbb{Z} \times \Omega_{l_{-1}, \alpha} \times \mathfrak{X} \times \mathfrak{U} \) which may depend besides the system trajectory also explicitly on the time-step \( k \in \mathbb{Z} \), then its future values can be determined by a repeated application \( \delta \) of the forward-shift operator \( \delta \) defined by the rule
\[
k \rightarrow k + 1
\]
\[
\zeta_{l_{-1}} \rightarrow \zeta_{l_{-1} + 1} \quad \forall \beta \geq 2
\]
\[
x \rightarrow f(x, k, u)
\]
\[
u^j_{[\alpha]} \rightarrow u^j_{[\alpha] + 1} \quad \forall \alpha \geq 0.
\]
Likewise, its past values can be determined by a repeated application \( \delta \) of the backward-shift operator \( \delta \) defined by the rule
\[
k \rightarrow k - 1
\]
\[
\zeta_{l_{-1}} \rightarrow \zeta_{l_{-1} - 1} \quad \forall \beta \geq 1
\]
\[
x \rightarrow \psi^j_{\iota}(k - 1, x, \zeta_{l_{-1}})
\]
\[
u^j_{[\alpha]} \rightarrow u^j_{[\alpha] - 1} \quad \forall \alpha \geq 1.
\]
where \( \psi^j_{\iota} \) and \( \psi^j_{\alpha} \) are the corresponding components of \( (7) \).

With this framework, flatness for nonlinear time-varying discrete-time systems can be defined as follows. Since flatness is a local concept, in accordance with the discrete-time literature on static and dynamic feedback linearization, only a suitable neighborhood of an equilibrium \( (x_0, u_0) \) (i.e., \( x_0 = f(k, x_0, u_0) \) for all \( k \)) is considered, cf. e.g. Niemeyer and van der Schaft (1990) or Aranda-Bricaire and Moog (2008). However, it is important to emphasize that the concept is still meaningful even in case the conditions do not hold at the equilibrium point itself due to a singularity.

**Definition 1.** The system (1) is said to be flat around an equilibrium \( (x_0, u_0) \), if the \( n + m \) coordinate functions \( x, u \) can be expressed locally by an \( m \)-tuple of functions
\[
y^j(k, \zeta_{l_{-1}}, \ldots, \zeta_{l_{-q_1}}, x, u, \ldots, u_{[q_2]}), \quad j = 1, \ldots, m,
\]
and their forward-shifts
\[
y_{[\alpha]} = \delta^1(y^j(k, \zeta_{l_{-1}}, \ldots, \zeta_{l_{-q_1}}, x, u, \ldots, u_{[q_2]}))
\]
\[
y_{[\beta]} = \delta^2(y^j(k, \zeta_{l_{-1}}, \ldots, \zeta_{l_{-q_1}}, x, u, \ldots, u_{[q_2]}))
\]
up to some finite order. The \( m \)-tuple (12) is called a flat output.

The definition ensures the existence of both maps (3) and (8): The map (8) is given by (12), and the condition that \( x \) and \( u \) can be expressed by (12) and its shifts necessitates the existence of a map
\[
x^i = F^i_{\alpha}(k, y, \ldots, y_{[R-1]}), \quad i = 1, \ldots, n
\]
\[
u^j = F^j_{\beta}(k, y, \ldots, y_{[R-1]}), \quad j = 1, \ldots, m
\]
which corresponds to (3). For notational convenience, we assume like in Diwold et al. (2022b) that the parameterization (13) of \( x \) and \( u \) depends only on forward-shifts of the flat output. This is no restriction, since it can always be achieved by replacing the components of a flat output by their highest backward-shifts that occur in (3). The fact that \( F^i_{\alpha} \) in (13) is independent of the highest forward-shifts \( y_{[R]} \) that are needed to parameterize the control inputs \( u \) follows from an evaluation of the identity
\[
F^i_{\alpha}(k, y, \ldots, y_{[R]}) = F^i_{\alpha}(k, y, \ldots, y_{[R-1]}), \quad F_u(k, y, \ldots, y_{[R]}), \quad i = 1, \ldots, n.
\]
This identity reflects the fact that (3) maps arbitrary trajectories \( y(k) \) of the trivial system to trajectories \((x(k), u(k)) \) of the system (1), which, by definition, must satisfy the difference equation \( x(k+1) = f(k, x(k), u(k)) \). Furthermore, it can be shown in the same way as in the time-invariant case in Diwold et al. (2022b) that the map (13) is unique and that its Jacobian matrix with respect to the variables \( y, y_{[1]}, \ldots, y_{[R]} \) has rank \( n + m \) for all \( k \). As a consequence, the Jacobian matrix of \( F_u \) with respect to \( y, y_{[1]}, \ldots, y_{[R-1]} \) has rank \( n \) for all \( k \). This property is essential for trajectory planning tasks: It ensures that for every initial state \( x_i \) at an arbitrary time-step \( k_i \) and every desired final state \( x_f \) at a time-step \( k_f \geq k_i + r \) with \( r = \max(1^r, \ldots, r^m) \) there exists a trajectory of the flat output such that the set of equations
\[
x_i = F_x(k_i, y(k_i), y(k_i+1), \ldots, y(k_i+R-1))
\]
\[
x_f = F_x(k_f, y(k_f), y(k_f+1), \ldots, y(k_f+R-1))
\]
is satisfied identically. Hence, for a flat system (1) it is possible to reach every desired state regardless of the initial state within \( r \) time steps (locally, where the system is flat). Accordingly, flat systems are locally reachable.

**3. FLATNESS OF THE LINEARIZED SYSTEM**

A linearization of the system (1) along a trajectory \((x(k), u(k))\) yields a linear time-varying system of the form
\[
\Delta x^{i+} = A^i_x(k) \Delta x^i + B^i_y(k) \Delta u^i, \quad i = 1, \ldots, n
\]
with
\[
A^i_x(k) = \partial_x f^i \bigg|_{x=x(k), u=u(k)}
\]
and
\[
B^i_y(k) = \partial_u f^i \bigg|_{x=x(k), u=u(k)}.
\]
For linear time-varying systems (15), the most general linear flat output has the form
\[
\Delta y^i = \sum_{l=1}^{m} \sum_{\beta=1}^{q_1} a^i_{l, \beta} (k) \Delta \zeta_{l_{-\beta}} + \sum_{i=1}^{n} b^i_{l} (k) \Delta x^i + \sum_{l=1}^{m} \sum_{\alpha=0}^{q_2} c^i_{l, \alpha} (k) \Delta u^i, \quad j = 1, \ldots, m.
\]

---

2 Since we use a finite-dimensional framework, it is important to emphasize that an application of (10) or (11) is only meaningful if the integers \( l_{\iota} \) and \( l_{\alpha} \) are chosen large enough such that the considered function (9) does not already depend on \( u_{[l_{\iota}]} \) or \( \zeta_{l_{-\alpha}} \): This is assumed throughout the contribution.

3 The multi-index \( R = (r_1, \ldots, r^m) \) contains the number of forward-shifts of each component of the flat output which is needed to express \( x \) and \( u \).

4 Note that if a system can be transformed into Brunovsky normal form then there also exists a flat output which depends only on the state variables.
The corresponding parameterization yields by an application of the chain rule the identity of the nonlinear system (1), which simply states that

\[
\Delta y^j = \sum_{i=1}^{m} \sum_{i'=0}^{r-1} F_{i,j}^g(k) \Delta y^i_{[i]}, \quad j = 1, \ldots, m
\]

\[
\Delta u^j = \sum_{i=1}^{m} \sum_{i'=0}^{r-1} F_{i,j}^u(k) \Delta y^i_{[i]}, \quad j = 1, \ldots, m
\]  

(17)

of the system variables by the flat output and its shifts is also linear. The quantities \( \Delta \zeta \), which allow for the original nonlinear system a minimal parameterization of the past trajectories, can be chosen directly as

\[
\Delta \zeta = \partial_x g^j \bigg|_{x=x(k),u=u(k)} \Delta x^i + \partial_u g^j \bigg|_{x=x(k),u=u(k)} \Delta u^i, \quad j = 1, \ldots, m
\]

(18)

with \( \Delta \zeta \) the corresponding parameterization (13). Computing for both sides of the identity (17) of the state- and input variables of the linearized system (15) by a flat output (16) and its forward-shifts. However, it is important to note that \( \Delta y^j_{[a]} \) denotes here according to (23) only the linearization of the \( \alpha \)-th forward-shift of the flat output (12) of the nonlinear system. Thus, we must prove that (23) coincides with the \( \alpha \)-th forward-shift of the linearized flat output

\[
\Delta y^j = L_{vlin}(\varphi^j), \quad j = 1, \ldots, m
\]  

(24)

along trajectories of the linearized system. Since we work with the linearized equations (19) without a restriction to a particular trajectory of the nonlinear system, the corresponding forward-shift operator, which we denote in the following as \( b_{vlin} \), must shift correctly both the variables \( \ldots, \Delta \zeta_{[-1]}, \Delta x, \Delta u, \Delta \eta_{[1]}, \ldots \) of the linearized system as well as the remaining variables of the nonlinear system. Since the latter serve as placeholders for trajectories of the nonlinear system, they have to be shifted according to the rule (10). Thus, the forward-shift operator \( b_{vlin} \) is defined by the rule

\[
b_{vlin}(k) = k + 1 \quad \forall k \geq 0
\]

(25)

In the following, we show that for an arbitrary function (9) we have

\[
L_{vlin}(\delta(h)) = \delta_{vlin}(L_{vlin}(h)),
\]  

(26)

i.e., shifting along trajectories of the nonlinear system and a subsequent linearization yields the same result as a linearization and a subsequent shift along trajectories of the linearized system. If this property holds for one-fold shifts, then a repeated application immediately yields the desired result

\[
L_{vlin}(\delta^\alpha(\varphi)) = \delta_{vlin}^\alpha(L_{vlin}(\varphi)), \quad \alpha \geq 1.
\]  

(27)

To prove (26), we simply evaluate both sides and show that they are equal. Let us start with the right-hand side: A linearization of the function (9) yields

\[
\Delta x^i = \sum_{i=1}^{m} \sum_{i'=0}^{r-1} \left( \partial_y^i F_{x}^g \circ \varphi_{[0,R-1]} \right) \Delta y^i_{[i]}, \quad i = 1, \ldots, n
\]

\[
\Delta u^j = \sum_{i=1}^{m} \sum_{i'=0}^{r-1} \left( \partial_y^i F_{u}^g \circ \varphi_{[0,R]} \right) \Delta y^i_{[i]}, \quad j = 1, \ldots, m
\]  

(22)

with

\[
\Delta y^j_{[a]} = L_{vlin}(\delta^a(\varphi^j)), \quad \alpha \geq 0, \quad j = 1, \ldots, m.
\]  

(23)

As discussed in Section 2, trajectories \((x(k), u(k))\) can be identified with points on a manifold \((\zeta_{[-l-1]} \times X \times U_{[-l]} \times X \times U_{[-l]} \times \ldots)\), Thus, it makes indeed sense to talk of a neighborhood of a trajectory.

\[5\]
of the nonlinear system (1) into the expressions in brackets
Consequently, after substituting the considered trajectory
flat output (24) along trajectories of the linearized system.
follows then immediately that the quantities $\Delta x$ and
facilitate a comparison, we did not evaluate $\delta$ and $\sum$

$$L_{v_{lin}}(h) = \sum_{j=1}^{m} \sum_{\beta+1}^{\beta} \left( \partial_{\beta} \h \right) \Delta \zeta_{\beta}^{-1} + \sum_{i=1}^{n} \left( \partial_{x_i} h \right) \Delta \alpha_{i}$$

and a subsequent shift operation according to (25) results in

$$\delta_{\text{lin}}(L_{v_{lin}}(h)) = \sum_{j=1}^{m} \sum_{\beta+1}^{\beta} \delta \left( \partial_{\beta} \h \right) \Delta \zeta_{\beta}^{-1} + \sum_{i=1}^{n} \delta \left( \partial_{x_i} h \right) \delta_{\text{lin}}(\Delta \alpha_{i}) +$$

Note that in order to keep the expression short and facilitate a comparison, we did not evaluate $\delta_{\text{lin}}(\Delta \zeta_{-1})$ and $\delta_{\text{lin}}(\Delta x^r)$. Now let us evaluate the left-hand side of (26). The forward-shift of (9) is given by

$$\delta(h) = h(k+1, \zeta_{-1+1}), \ldots, g(k, x, u), f(k, x, u), u_{1}, \ldots, u_{t+1}),$$

and a subsequent linearization yields

$$L_{v_{lin}}(\delta(h)) = \sum_{j=1}^{m} \sum_{\beta+1}^{\beta} \left( \partial_{\beta} \h \right) \Delta \zeta_{\beta}^{-1} + \sum_{i=1}^{n} \left( \partial_{x_i} \delta(h) \right)_{|x=g} L_{v_{lin}}(g^i) +$$

By the definition of the forward-shift operator $\delta$ according to (10), it is straightforward to verify that

$$\partial_{\beta} \h \Delta \zeta_{\beta}^{-1} h \quad \forall \beta \geq 1$$

Together with $L_{v_{lin}}(g^i) = \delta_{\text{lin}}(\Delta \zeta_{\beta}^{-1})$ and $L_{v_{lin}}(f^i) = \delta_{\text{lin}}(\Delta x^r)$ (cf. (18) and (19)), it can thus be observed that (28) and (29) are equal, which proves (26). With (27) it follows then immediately that the quantities $\Delta y^0$ in (22) can also be interpreted as forward-shifts of the linearized flat output (24) along trajectories of the linearized system. Consequently, after substituting the considered trajectory of the nonlinear system (1) into the expressions in brackets of (22) as well as into (24), we have a map (17) which allows to express the state- and input variables $\Delta x$ and $\Delta u$ of the linearized system (15) by a flat output (16) and its forward-shifts. Hence, according to Definition 1, the linearized system (15) is flat.

As already discussed in Section 2, a particularly important property of flat systems is the fact that after substituting the parameterization (13) into the system equations (1) the latter are satisfied identically, cf. (14). This can be written formally as

$$\delta_{y}(F_x(k, y, \ldots, y_{[R-1]})) = f'(k, F_x(k, y, \ldots, y_{[R-1]}), F_u(k, y, \ldots, y_{[R]})) \quad \text{for } i = 1, \ldots, n,$$

with $\delta_{y}$ denoting the forward-shift operator in $y$-coordinates, which is defined by the rule

$$k \rightarrow k + 1$$

in the following, we show that this property holds indeed also for the linearized system. More precisely, we show that the linearized parameterization (22) satisfies the linearized system equations (19) identically (again, it is convenient to perform the calculations without inserting a particular trajectory of the nonlinear system (1)). First, computing the derivative of both sides of (30) with respect to $y^0_{[\alpha]}$ for some $j \in \{1, \ldots, m\}$ and $\alpha^j \in \{0, \ldots, r^j\}$ yields by an application of the chain rule the identity

$$\partial_{y^0_{[\alpha]}} \delta_{y}(F_x) = (\partial_{x} f' \circ F) \partial_{y^0_{[\alpha]}} F_x + (\partial_{u} f' \circ F) \partial_{y^0_{[\alpha]}} F_u$$

Since $\delta_{y}$ only substitutes variables, shifting and subsequently differentiating with respect to $y^0_{[\alpha]}$ is equivalent to differentiating first with respect to $y^0_{[\alpha]}$ and shifting afterwards. Thus, the above identity can be written as

$$\delta_{y}(F_x) = (\partial_{x} f' \circ F) \partial_{y^0_{[\alpha]}} F_x + (\partial_{u} f' \circ F) \partial_{y^0_{[\alpha]}} F_u$$

for $\alpha^j = 1, \ldots, r^j$ and

$$0 = (\partial_{x} f' \circ F) \partial_{y^0_{[\alpha]}} F_x + (\partial_{u} f' \circ F) \partial_{y^0_{[\alpha]}} F_u$$

for $\alpha^j = 0$, and a multiplication with $\Delta y^0_{[\alpha]}$ and subsequent summation yields

$$m \sum_{j=1}^{r^j} \delta_{y}(F_x) \Delta y^0_{[\alpha] + 1} = (\partial_{x} f' \circ F) m \sum_{j=1}^{r^j} \delta_{y}(F_x) \Delta y^0_{[\alpha] + 1} +$$

Substituting the flat output $y$ and its shifts by a trajectory $y(k)$ which corresponds to a considered trajectory of the nonlinear system (1) finally results in an identity of the form

$$\sum_{j=1}^{r^j} \alpha_{j} \Delta y^0_{[\alpha] + 1} = A^j(k) \sum_{j=1}^{r^j} \Delta y^0_{[\alpha] + 1} + B^j(k) \sum_{j=1}^{r^j} \Delta y^0_{[\alpha] + 1},$$
4. EXAMPLES

In this section, the derived results are illustrated by two examples.

4.1 A Flat System

The purpose of the first example is to show that the linearization of a flat system along a trajectory is again flat. Since for practically relevant flat systems like e.g. the gantry crane, the VTOL aircraft, or the induction motor (see Diwold et al. (2022a) or Diwold et al. (2022b)) the corresponding equations would become rather extensive, for demonstrational purposes we use the simple academic example

\[
\begin{align*}
    x_{1+} &= x_1 + u_1 \\
    x_{2+} &= x_2 + u_2 \\
    x_{3+} &= x_3 + u_1 u_2.
\end{align*}
\]  

This system corresponds in fact to an exact discretization of the flat continuous-time system

\[
\begin{align*}
    \dot{x}_1 &= u_1 \\
    \dot{x}_2 &= u_2 \\
    \dot{x}_3 &= u_1 u_2
\end{align*}
\]  

with a sampling time of \( T = 1 \). With the choice \( \zeta^1 = x_1 \), \( \zeta^2 = x_2 \) for the functions \( g(k, x, u) \) of (6), a flat output of (31) is given by

\[
y = (\zeta^1_{[-1]}, x_3 - x_2(x_1 - \zeta^1_{[-1]})),
\]  

and the corresponding parameterization of the state- and input variables (13) reads as

\[
\begin{align*}
    x^1 &= y^1_1 \\
    x^2 &= \frac{y^2 - y^2_1}{y^1 - y^1_1} \\
    x^3 &= \frac{y^1 y^2_1 - y^2 (y^1 - y^1_1) + y^3_1 y^2_1}{y^1 - 2y^1_1 + y^1_2} \\
    u^1 &= y^1_{[-2]} - y^1_1 \\
    u^2 &= \frac{y^1 (y^1_{[-1]} - y^1_2 + y^3_1 (-y^1 + y^1_1 + 2y^1_2))}{y^1_{[-1]} - 2y^1_1 + y^1_2 + y^3_1 (-y^1 + y^1_1) + y^3_2 (-2y^1 + y^1_2) + y^3_3 (-2y^1 + y^1_2) + y^3_4 (-2y^1 + y^1_2)}
\end{align*}
\]  

(34)

Now let us consider the trajectory

\[
\begin{align*}
    x^1(k) &= \frac{1}{2} k (k - 1) \\
    x^2(k) &= -\frac{1}{2} k (k - 1) \\
    x^3(k) &= -\frac{1}{2} k (k - 1) (2k - 1) \\
    u^1(k) &= k \\
    u^2(k) &= -k
\end{align*}
\]  

and the corresponding trajectory

\[
\begin{align*}
    y^1(k) &= \frac{1}{4} (k - 1) (k - 2) \\
    y^2(k) &= \frac{1}{2} k (k - 1) (k - 2)
\end{align*}
\]  

of the flat output (33). A linearization of the system (31) along (35) yields a linear time-varying system (15) with the matrices

\[
A(k) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -k \end{bmatrix}, 
B(k) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -k \end{bmatrix}, 
\]

and a linearization of the flat output (33) along (35) yields

\[
\Delta y = (\Delta \zeta^1_{[-1]}, \Delta x^3 - (k - 1) \Delta x^2 + \frac{1}{2} k (k - 1) (\Delta x^1 - \Delta \zeta^1_{[-1]}))
\]  

(37)

with \( \Delta \zeta^1 = \Delta x^1 \) and \( \Delta \zeta^2 = \Delta x^2 \) according to (18). With a computer algebra program, it is easy to verify that all state variables \( \Delta x^1, \Delta x^2, \Delta x^3 \) and input variables \( \Delta u^1, \Delta u^2 \) of the linearized system (15) can be expressed by (37) and its forward-shifts. Furthermore, the corresponding map (17) coincides indeed with the linearization

\[
\Delta x^1 = \Delta y^1_1 \\
\Delta x^2 = \frac{1}{2} k (k - 1) (\Delta y^1_1 - 2\Delta y^1_2 + \Delta y^2_1) + \Delta y^2 - \Delta y^2_1 \\
\Delta x^3 = \frac{1}{2} k (k - 1) (k \Delta y^1_1 - (2k - 1) \Delta y^1_2 + (k - 1) \Delta y^1_3) + k \Delta y^2 + (1 - k) \Delta y^2_1 \\
\Delta u^1 = -\Delta y^1_1 + \Delta y^2_1 \\
\Delta u^2 = \frac{1}{2} k (1 - k) \Delta y^1_1 - (3k - 1) \Delta y^1_2 - (3k + 1) \Delta y^1_3 \\
+ \frac{1}{2} k (k - 1) \Delta y^1_3 - \Delta y^2 + 2 \Delta y^2_1 - \Delta y^2_2
\]  

of the map (34) along the trajectory (36).

4.2 A Non-Flat System

As a second example, let us consider the system

\[
\begin{align*}
    x_{1+} &= -\sin(x_1 - x^3) + u_2 \\
    x_{2+} &= (1 - \sin(x_1 - x^3)) u_1 \\
    x_{3+} &= u_2.
\end{align*}
\]  

(38)

A linearization along an arbitrary trajectory \((x(k), u(k))\) results in a linear time-varying system (15) with

\[
A(k) = \begin{bmatrix} -\cos(x_1 - x^3) & 0 & \cos(x_1 - x^3) \\ -\cos(x_1 - x^3) u_1 & 0 & \cos(x_1 - x^3) u_1 \\ 0 & 0 & 0 \end{bmatrix}
\]

and

\[
B(k) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]

Since \( A(k) B(k) = 0 \) for all \( k \) independently of the chosen trajectory, it can be observed immediately that the linearized system (15) is not reachable (conditions for the reachability of linear time-varying discrete-time systems can be found e.g. in Weiss (1972)). Thus, the linearized system cannot be flat, and because of the connection between the flatness of a nonlinear system and its linearization discussed in Section 3, the nonlinear system (38) cannot be flat either. This result can also be obtained in an alternative way by showing e.g. with the method discussed in Aranda-Bricaire et al. (1996) that the considered nonlinear system (38) itself is also not reachable, and hence clearly not flat.

5. CONCLUSION

We have shown that – like in the continuous-time case – the linearization of a flat discrete-time system yields a linear time-varying system which is again flat. Since

\[\text{Like the continuous-time system (32), the system (31) is not flat at equilibrium points. For an equilibrium the flat output is constant, and the parameterization (34) becomes singular.}\]
flatness implies reachability (and consequently also controllability), this property constitutes a useful necessary condition for flatness. Moreover, we have shown that a possible flat output can be obtained by a linearization of a flat output of the nonlinear system, and that the corresponding parameterization of the system variables of the linearized system coincides with the linearization of the parameterization of the system variables of the nonlinear system.

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