EDGECWORTH EXPANSION FOR THE COEFFICIENTS OF RANDOM WALKS ON THE GENERAL LINEAR GROUP

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Abstract. Let \((g_n)_{n \geq 1}\) be a sequence of independent and identically distributed random elements with law \(\mu\) on the general linear group \(\text{GL}(V)\), where \(V = \mathbb{R}^d\). Consider the random walk \(G_n := g_n \ldots g_1, n \geq 1\). Under suitable conditions on \(\mu\), we establish the first-order Edgeworth expansion for the coefficients \(\langle f, G_n v \rangle\) with \(v \in V\) and \(f \in V^*\), in which a new additional term appears compared to the case of vector norm \(\|G_n v\|\).

1. Introduction

Since the pioneering work of Furstenberg and Kesten [13], the study of random walks on linear groups has attracted a great deal of attention, see for instance the work of Le Page [23], Guivarc’h and Raugi [19], Bougerol and Lacroix [5], Goldsheid and Margulis [14], Benoist and Quint [4], and the references therein. Of particular interest is the study of asymptotic properties of the random walk \(G_n := g_n \ldots g_1, n \geq 1\), where \((g_n)_{n \geq 1}\) is a sequence of independent and identically distributed random elements with law \(\mu\) on the general linear group \(\text{GL}(V)\) with \(V = \mathbb{R}^d\). One natural and important way to describe the random walk \((G_n)_{n \geq 1}\) is to investigate the growth rate of the coefficients \(\langle f, G_n v \rangle\), where \(v \in V\), \(f \in V^*\) and \(\langle \cdot, \cdot \rangle\) is the duality bracket: \(\langle f, v \rangle = f(v)\). Bellman [2] conjectured that the classical central limit theorem should hold true for \(\langle f, G_n v \rangle\) in the case when \(g_n\) are positive matrices. This conjecture was proved by Furstenberg and Kesten [13], who established the strong law of large numbers and central limit theorem under the condition that the matrices \(g_n\) are strictly positive and that all the coefficients of \(g_n\) are comparable. For further developments we refer to Kingman [22], Cohn, Nerman and Peligrad [7], Hennion [20].

As noticed by Furstenberg [12], the analysis developed in [13] for positive matrices breaks down for invertible matrices. It turns out that the situation of invertible matrices is much more complicated and delicate. Guivarc’h and Raugi [19] established the strong law of large numbers for the coefficients of products of invertible matrices under an exponential moment condition: for any \(v \in V \setminus \{0\}\) and \(f \in V^* \setminus \{0\}\),

\[
\lim_{n \to \infty} \frac{1}{n} \log |\langle f, G_n v \rangle| = \lambda \quad \text{a.s.,} \tag{1.1}
\]
where $\lambda \in \mathbb{R}$ is a constant independent of $f$ and $v$, called the first Lyapunov exponent of $\mu$. It is worth mentioning that the result (1.1) does not follow from the classical subadditive ergodic theorem of Kingman [22], nor from its recent version by Gouëzel and Karlsson [16]. The central limit theorem for the coefficients has also been established in [19] under the exponential moment condition: if $\int_{GL(V)} N(g) \mu(dg) < \infty$ with $N(g) = \max\{\|g\|, \|g^{-1}\|\}$ for some $\varepsilon > 0$, then for any $t \in \mathbb{R}$,
\[
\lim_{n \to \infty} \mathbb{P}\left( \frac{\log |\langle f, G_n v \rangle| - n\lambda}{\sigma \sqrt{n}} \leq t \right) = \Phi(t),
\]
(1.2)
where $\Phi$ is the standard normal distribution function on $\mathbb{R}$ and $\sigma^2 > 0$ is the asymptotic variance of $\frac{1}{\sqrt{n}} \log |\langle f, G_n v \rangle|$. Recently, using the martingale approximation method, Benoist and Quint [3] have improved (1.2) by relaxing the exponential moment condition to the optimal second moment $\int_{GL(V)} (\log N(g))^2 \mu(dg) < \infty$.

An important and interesting problem is the estimation of the rate of convergence in (1.2). Very recently, under the exponential moment condition, Cuny, Dedecker, Merlevède and Peligrad [9] established a rate of convergence of order $\log n / \sqrt{n}$. Dinh, Kaufmann and Wu [10, 11] improved this result by giving the optimal rate $1 / \sqrt{n}$ under the same exponential moment assumption: there exists a constant $c > 0$ such that for all $n \geq 1$, $t \in \mathbb{R}$, $v \in V$ and $f \in V^*$ with $\|v\| = \|f\| = 1$,
\[
\left| \mathbb{P}\left( \frac{\log |\langle f, G_n v \rangle| - n\lambda}{\sigma \sqrt{n}} \leq t \right) - \Phi(t) \right| \leq \frac{c}{\sqrt{n}}.
\]
(1.3)

The objective of this paper is to further elaborate on the central limit theorem (1.2) by establishing the first-order Edgeworth expansion for the coefficients under the exponential moment condition. We prove that as $n \to \infty$, uniformly in $t \in \mathbb{R}$, $x = \mathbb{R}v \in \mathbb{P}(V)$ and $y = \mathbb{R}f \in \mathbb{P}(V^*)$ with $\|v\| = \|f\| = 1$,
\[
\mathbb{P}\left( \frac{\log |\langle f, G_n v \rangle| - n\lambda}{\sigma \sqrt{n}} \leq t \right) = \Phi(t) + \frac{\Lambda''(0)}{6\sigma^3 \sqrt{n}} (1 - t^2) \phi(t) - \frac{b_1(x) + d_1(y)}{\sigma \sqrt{n}} \phi(t) + o\left( \frac{1}{\sqrt{n}} \right),
\]
(1.4)
where $\phi$ denotes the standard normal density, $\Lambda''(0)$, $b_1(x)$, $d_1(y)$ are defined in Section 2. Notice that the asymptotic bias terms $b_1(x)$ and $d_1(y)$ are new compared with the classical Edgeworth expansion for sums of independent real random variables [24]; $d_1(y)$ is also new compared with the Edgeworth expansion for the vector norm $\|G_n v\|$ [25]. In fact, we will establish a stronger result, that is, the first-order Edgeworth expansion for the couple $(\varphi(G_n \cdot x), \log |\langle f, G_n v \rangle|)$ with a target function $\varphi$ on $\mathbb{P}(V)$, cf. Theorem 2.1. Moreover, we prove a similar result under the changed measure, which can be useful for studying moderate deviations with explicit rates of convergence. Clearly, the expansion (1.4) implies the Berry-Esseen bound (1.3).

The proof of the Edgeworth expansion for the coefficient $\langle f, G_n v \rangle$ turns out to be much more complicated than that for the norm cocycle $\sigma(G_n, x) = \log \frac{\|G_n v\|}{\|v\|}$, $x = \mathbb{R}v \in \mathbb{P}(V)$ recently established in [25]. One of the difficulties is that $\log |\langle f, G_n v \rangle|$ is not a cocycle and cannot be studied with the same approach as $\sigma(G_n, x)$. Our starting point
is the following decomposition which relates the coefficient to the norm cocycle: for any $x = \mathbb{R}v \in \mathbb{P}(V)$ and $y = \mathbb{R}f \in \mathbb{P}(V^*)$ with $\|f\| = 1$, 
\[
\log |\langle f, G_n v \rangle| = \sigma(G_n, x) + \log \delta(G_n \cdot x, y),
\]
where $\delta(x, y) = \frac{|\langle f, v \rangle|}{\|f\| \|v\|}$. For the proof of the Edgeworth expansion (1.4), we first use a partition $(\chi^y_{n,k})_{k \geq 1}$ of the unity to discretize the component $\log \delta(G_n \cdot x, y)$ in (1.5). This allows us to reduce the study of the coefficient to that of the couple formed by norm cocycle $\sigma(G_n, x)$ and the target function $\chi^y_{n,k}(G_n \cdot x)$. It turns out that the Edgeworth expansion for the couple $(\chi^y_{n,k}(G_n \cdot x), \sigma(G_n, x))$ established recently in [25] is not appropriate for our proof because the reminder terms therein are not precise enough. We need to track the dependence of the remainder term on the Hölder norm of the function $\varphi = \chi^y_{n,k}$, see Theorem 3.7. In contrast to the previous work [10], the partition of the unity that we use should become finer and finer as $n \to \infty$, in order to recover the term $d_1(y)$, see Lemma 3.9. Finally, another delicate point is to patch up the expansions for couples $(\chi^y_{n,k}(G_n \cdot x), \sigma(G_n, x))$ by means of the Hölder regularity of the invariant measure $\nu$ and the linearity in $\varphi$ of the asymptotic bias term $b_\varphi(x)$.

2. Main results

For any integer $d \geq 1$, denote by $V = \mathbb{R}^d$ the $d$-dimensional Euclidean space. We fix a basis $e_1, \ldots, e_d$ of $V$ and the associated norm on $V$ is defined by $\|v\| = \sum_{i=1}^d |v_i|^2$ for $v = \sum_{i=1}^d v_i e_i \in V$. Let $V^*$ be the dual vector space of $V$ and its dual basis is denoted by $e_1^*, \ldots, e_d^*$, so that $e_i^*(e_j) = 1$ if $i = j$ and $e_i^*(e_j) = 0$ if $i \neq j$. Let $\wedge^2 V$ be the exterior product of $V$ and we use the same symbol $\| \cdot \|$ for the norms induced on $\wedge^2 V$ and $V^*$. We equip $\mathbb{P}(V)$ with the angular distance
\[
d(x, x') = \frac{\|v \land v'\|}{\|v\| \|v'\|} \quad \text{for} \quad x = \mathbb{R}v \in \mathbb{P}(V), \ x' = \mathbb{R}v' \in \mathbb{P}(V). \tag{2.1}
\]
We use the symbol $\langle \cdot, \cdot \rangle$ to denote the dual bracket defined by $\langle f, v \rangle = f(v)$ for any $v \in V$ and $f \in V^*$. Set
\[
\delta(x, y) = \frac{|\langle f, v \rangle|}{\|f\| \|v\|} \quad \text{for} \quad x = \mathbb{R}v \in \mathbb{P}(V), \ y = \mathbb{R}f \in \mathbb{P}(V^*).
\]
Denote by $\mathcal{C}(\mathbb{P}(V))$ the space of complex-valued continuous functions on $\mathbb{P}(V)$, equipped with the norm $\|\varphi\|_\infty := \sup_{x \in \mathbb{P}(V)} |\varphi(x)|$ for $\varphi \in \mathcal{C}(\mathbb{P}(V))$. Let $\gamma > 0$ be a constant and set
\[
\|\varphi\|_\gamma := \|\varphi\|_\infty + [\varphi]_\gamma, \quad \text{where} \quad [\varphi]_\gamma = \sup_{x, x' \in \mathbb{P}(V), x \neq x'} \frac{|\varphi(x) - \varphi(x')|}{d(x, x')^\gamma}.
\]
Consider the Banach space
\[
\mathcal{B}_\gamma := \{ \varphi \in \mathcal{C}(\mathbb{P}(V)) : \|\varphi\|_\gamma < \infty \},
\]
which consists of complex-valued $\gamma$-Hölder continuous functions on $\mathbb{P}(V)$. Denote by $\mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$ the set of all bounded linear operators from $\mathcal{B}_\gamma$ to $\mathcal{B}_\gamma$, equipped with the operator norm $\|\cdot\|_{\mathcal{B}_\gamma \to \mathcal{B}_\gamma}$. The topological dual of $\mathcal{B}_\gamma$ endowed with the induced norm
is denoted by \( B_\gamma \). Let \( B^*_\gamma \) be the Banach space of \( \gamma \)-Hölder continuous functions on \( \mathbb{P}(V^*) \) endowed with the norm

\[
\| \varphi \|_{B^*_\gamma} = \sup_{y \in \mathbb{P}(V^*)} |\varphi(y)| + \sup_{y, y' \in \mathbb{P}(V^*) : y \neq y'} \frac{|\varphi(y) - \varphi(y')|}{d(y, y')^\gamma},
\]

where \( d(y, y') = \frac{|f(y) - f(y')|}{\|f\|_{L^\infty}} \) for \( y = \mathbb{R}f \in \mathbb{P}(V^*) \) and \( y' = \mathbb{R}f' \in \mathbb{P}(V^*) \).

Let \( \text{GL}(V) \) be the general linear group of the vector space \( V \). The action of \( g \in \text{GL}(V) \) on a vector \( v \in V \) is denoted by \( gv \), and the action of \( g \in \text{GL}(V) \) on a projective line \( x = \mathbb{R}v \in \mathbb{P}(V) \) is denoted by \( g \cdot x = \mathbb{R}gv \). For any \( g \in \text{GL}(V) \), let \( \|g\| = \sup_{v \in V \setminus \{0\}} \frac{\|gv\|}{\|v\|} \) and denote \( N(g) = \max\{\|g\|, \|g^{-1}\|\} \). Let \( \mu \) be a Borel probability measure on \( \text{GL}(V) \).

We shall use the following exponential moment condition.

**A1.** There exists a constant \( \varepsilon > 0 \) such that \( \int_{\text{GL}(V)} N(g)^\varepsilon \mu(dg) < \infty \).

Let \( \Gamma_\mu \) be the smallest closed subsemigroup generated by the support of the measure \( \mu \). An endomorphism \( g \) of \( V \) is said to be proximal if it has an eigenvalue \( \lambda \) with multiplicity one and all other eigenvalues of \( g \) have modulus strictly less than \( |\lambda| \). We shall need the following strong irreducibility and proximality condition.

**A2.** (i) (Strong irreducibility) No finite union of proper subspaces of \( V \) is \( \Gamma_\mu \)-invariant. (ii) (Proximality) \( \Gamma_\mu \) contains a proximal endomorphism.

Define the norm cocycle \( \sigma : \text{GL}(V) \times \mathbb{P}(V) \rightarrow \mathbb{R} \) as follows:

\[
\sigma(g, x) = \log \frac{\|gv\|}{\|v\|} \quad \text{for any } g \in \text{GL}(V) \text{ and } x = \mathbb{R}v \in \mathbb{P}(V).
\]

Recall that the first Lyapunov exponent \( \lambda \) is defined by (1.1). By [25, Proposition 3.15], under **A1** and **A2**, the following limit exists and is independent of \( x \in \mathbb{P}(V) \):

\[
\sigma^2 := \lim_{n \to \infty} \frac{1}{n} \mathbb{E} \left[ (\sigma(G_n, x) - n\lambda)^2 \right] \in (0, \infty). \tag{2.2}
\]

For any \( s \in (-s_0, s_0) \) with \( s_0 > 0 \) small enough, we define the transfer operator \( P_s \) as follows: for any bounded measurable function \( \varphi \) on \( \mathbb{P}(V) \),

\[
P_s \varphi(x) = \int_{\text{GL}(V)} e^{s\sigma(g, x)} \varphi(g \cdot x) \mu(dg), \quad x \in \mathbb{P}(V). \tag{2.3}
\]

It will be shown in Lemma 3.1 that there exists a constant \( s_0 > 0 \) such that for any \( s \in (-s_0, s_0) \), the operator \( P_s \in \mathcal{L}(\mathcal{B}, \mathcal{B}) \) has a unique dominant eigenvalue \( \kappa(s) \) with \( \kappa(0) = 1 \) and the mapping \( s \mapsto \kappa(s) \) being analytic. We denote \( \Lambda = \log \kappa \).

Under **A1** and **A2**, the Markov chain \( (G_n \cdot x)_{n \geq 0} \) has a unique invariant probability measure \( \nu \) on \( \mathbb{P}(V) \) such that for any bounded measurable function \( \varphi \) on \( \mathbb{P}(V) \),

\[
\int_{\mathbb{P}(V)} \int_{\text{GL}(V)} \varphi(g \cdot x) \mu(dg) \nu(dx) = \int_{\mathbb{P}(V)} \varphi(x) \nu(dx) =: \nu(\varphi). \tag{2.4}
\]

For any \( \varphi \in \mathcal{B} \), define the functions

\[
b_\varphi(x) := \lim_{n \to \infty} \mathbb{E} [(\sigma(G_n, x) - n\lambda) \varphi(G_n \cdot x)], \quad x \in \mathbb{P}(V). \tag{2.5}
\]
and
\[ d_\varphi(y) := \int_{\mathbb{F}(V)} \varphi(x) \log \delta(x, y) \nu(dx), \quad y \in \mathbb{P}(V^*). \tag{2.6} \]

It will be shown in Lemmas 3.5 and 3.6 that both functions \( b_\varphi \) and \( d_\varphi \) are well-defined and \( \gamma \)-Hölder continuous. Denote \( \phi(u) = \sqrt{2\pi}e^{-u^2/2}, u \in \mathbb{R} \). Let \( \Phi(t) = \int_{-\infty}^{t} \phi(u) du, t \in \mathbb{R} \) be the standard normal distribution function.

In many applications it is of primary interest to give an estimation of the rate of convergence in the Gaussian approximation (1.2). In this direction we establish the following first-order Edgeworth expansion for the coefficients \( \langle f, G_n v \rangle \).

**Theorem 2.1.** Assume \( A1 \) and \( A2 \). Then, there exists a constant \( \gamma > 0 \) such that for any \( \varepsilon > 0 \), uniformly in \( t \in \mathbb{R}, x = \mathbb{R}v \in \mathbb{P}(V), y = \mathbb{R}f \in \mathbb{P}(V^*) \) with \( \|v\| = \|f\| = 1 \), and \( \varphi \in B_\gamma \), as \( n \to \infty \),

\[
\mathbb{E}\left[ \varphi(G_n x) \mathbb{1}_{\left\{ \frac{\log \|f(G_n x)\|}{\sigma(n^{1-\varepsilon})} \leq t \right\}} \right] = \nu(\varphi) \left[ \Phi(t) + \frac{\Lambda''(0)}{6\sigma^3} (1-t^2) \Phi(t) \right] \\
\quad - \frac{b_\varphi(x) + d_\varphi(y)}{\sigma(n^{1-\varepsilon})} \Phi(t) + \nu(\varphi) \frac{1}{\sqrt{n}} + \|\varphi\|_1 O\left( \frac{1}{n^{1-\varepsilon}} \right). \tag{2.7} \]

When compared with the standard Edgeworth expansion for sums of independent random variables (cf. [24]), we see that two new terms \( b_\varphi(x) \) and \( d_\varphi(y) \) show up, which are explained by the presence of an asymptotic bias for this model. We should also note that the Edgeworth expansion (2.7) for the coefficients is different from that for the norm cocycle \( \sigma(G_n, x) \) obtained in [25], namely, by the presence of the term \( d_\varphi(y) \). The difficulty in proving this precise expansion for coefficient \( \langle f, G_n v \rangle \) consists in obtaining the exact expression of this new asymptotic bias term \( d_\varphi(y) \).

As a consequence of Theorem 2.1 one can get the Berry-Esseen bound (1.3) with the optimal convergence rate, under the exponential moment condition. It is an open problem how to relax the exponential moment condition \( A1 \) for the Edgeworth expansion and for the Berry-Esseen bound. Solving it seems very challenging. Even for the easier case of the norm cocycle, the Berry-Esseen bound \( O(n^{-1/2}) \) is not known under the optimal third moment condition; it is only known under the fourth moment condition, see [8]. For positive matrices, the Edgeworth expansion (2.7) and the Berry-Esseen bound (1.3) have been recently obtained using a different approach in a forthcoming paper [27] under optimal moment conditions.

Finally we would like to mention that all the results of the paper remain valid when \( V \) is \( \mathbb{C}^d \) or \( \mathbb{K}^d \), where \( \mathbb{K} \) is any local field.

3. **Proof of the Edgeworth expansion**

3.1. **Preliminary results.** For any \( z \in \mathbb{C} \), we define the complex transfer operator \( P_z \) as follows: for any bounded measurable function \( \varphi \) on \( \mathbb{P}(V) \),

\[
P_z \varphi(x) = \int_{\text{GL}(V)} e^{z \sigma(g \cdot x)} \varphi(g \cdot x) \mu(dg), \quad x \in \mathbb{P}(V). \tag{3.1} \]

Throughout this paper let \( B_{s_0}(0) := \{ z \in \mathbb{C} : |z| < s_0 \} \) be the open disc with center \( 0 \) and radius \( s_0 > 0 \) in the complex plane \( \mathbb{C} \). The following result shows that the
operator $P_z$ has spectral gap properties when $z \in B_{s_0}(0)$; we refer to \cite{23,21,18,4,25} for the proof based on the perturbation theory of linear operators. Recall that $\mathcal{B}_\gamma'$ is the topological dual space of the Banach space $\mathcal{B}_\gamma$, and that $\mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$ is the set of all bounded linear operators from $\mathcal{B}_\gamma$ to $\mathcal{B}_\gamma$ equipped with the operator norm $\| \cdot \|_{\mathcal{B}_\gamma' \to \mathcal{B}_\gamma}$.

**Lemma 3.1** \cite{4,25}. Assume $A1$ and $A2$. Then, there exists a constant $s_0 > 0$ such that for any $z \in B_{s_0}(0)$ and $n \geq 1$,

$$P^n_z = \kappa^n(z) \nu_z \otimes r_z + L^n_z,$$  \hspace{1cm} (3.2)

where

$$z \mapsto \kappa(z) \in \mathbb{C}, \quad z \mapsto r_z \in \mathcal{B}_\gamma', \quad z \mapsto \nu_z \in \mathcal{B}_\gamma', \quad z \mapsto L_z \in \mathcal{L}(\mathcal{B}_\gamma, \mathcal{B}_\gamma)$$

are analytic mappings which satisfy, for any $z \in B_{s_0}(0)$,

(a) the operator $M_z := \nu_z \otimes r_z$ is a rank one projection on $\mathcal{B}_\gamma$, i.e. $M_z \varphi = \nu_z(\varphi)r_z$ for any $\varphi \in \mathcal{B}_\gamma$;

(b) $M_zL_z = L_zM_z = 0$, $P_zr_z = \kappa(z)r_z$ with $\nu(r_z) = 1$, and $\nu_zP_z = \kappa(z)\nu_z$;

(c) $\kappa(0) = 1$, $r_0 = 1$, $\nu_0 = \nu$ with $\nu$ defined by (2.4), and $\kappa(z)$ and $r_z$ are strictly positive for real-valued $z \in (-s_0, s_0)$.

Using Lemma 3.1, a change of measure can be performed below. For any $s \in (-s_0, s_0)$ with $s_0 > 0$ sufficiently small, any $x \in \mathbb{P}(V)$ and $g \in \text{GL}(V)$, denote

$$q^n_s(x, g) = \frac{e^{s\sigma(g,x)}}{\kappa^n(s)} \frac{r_s(g \cdot x)}{r_s(x)}, \quad n \geq 1.$$ 

Since the eigenvalue $\kappa(s)$ and the eigenfunction $r_s$ are strictly positive for $s \in (-s_0, s_0)$, using $P_s r_s = \kappa(s) r_s$ we get that

$$Q^n_{s,n}(dg_1, \ldots, dg_n) = q^n_s(x, G_n)\mu(dg_1)\ldots\mu(dg_n), \quad n \geq 1,$$

are probability measures and form a projective system on $\text{GL}(V)^N$. By the Kolmogorov extension theorem, there exists a unique probability measure $Q^n_s$ on $\text{GL}(V)^N$ with marginals $Q^n_{s,n}$. We write $\mathbb{E}_{Q^n_s}$ for the corresponding expectation and the change of measure formula holds: for any $s \in (-s_0, s_0), x \in \mathbb{P}(V), n \geq 1$ and bounded measurable function $h$ on $(\mathbb{P}(V) \times \mathbb{R})^n$,

$$\frac{1}{\kappa^n(s)r_s(x)} \mathbb{E} \left[ r_s(G_n \cdot x) e^{s\sigma(G_n,x)} h \left( G_1 \cdot x, \sigma(G_1, x), \ldots, G_n \cdot x, \sigma(G_n, x) \right) \right] = \mathbb{E}_{Q^n_s} \left[ h \left( G_1 \cdot x, \sigma(G_1, x), \ldots, G_n \cdot x, \sigma(G_n, x) \right) \right].$$  \hspace{1cm} (3.3)

Under the changed measure $Q^n_s$, the process $(G_n \cdot x)_{n \geq 0}$ is a Markov chain with the transition operator $Q_s$ given as follows: for any $\varphi \in \mathcal{C}(\mathbb{P}(V))$,

$$Q_s \varphi(x) = \frac{1}{\kappa(s)r_s(x)} P_s(\varphi r_s)(x), \quad x \in \mathbb{P}(V).$$
Under A1 and A2, it was shown in [25] that the Markov operator $Q_s$ has a unique invariant probability measure $\pi_s$ given by
\[ \pi_s(\varphi) = \frac{\nu_s(\varphi r_s)}{\nu_s(r_s)} \quad \text{for any } \varphi \in \mathcal{C}(\mathbb{P}(V)). \quad (3.4) \]
By [25, Proposition 3.13], the following strong law of large numbers for the norm cocycle under the changed measure $Q^x_s$ holds: under A1 and A2, for any $s \in (-s_0, s_0)$ and $x \in \mathbb{P}(V),$
\[ \lim_{n \to \infty} \frac{\sigma(G_n, x)}{n} = \Lambda'(s), \quad Q^x_s\text{-a.s.} \]
where $\Lambda(s) = \log \kappa(s)$.

We need the following Hölder regularity of the invariant measure $\pi_s$.

**Lemma 3.2** ([15]). Assume A1 and A2. Then there exist constants $s_0 > 0$ and $\eta > 0$ such that
\[ \sup_{s \in (-s_0, s_0)} \sup_{y \in \mathbb{P}(V^*)} \int_{\mathbb{P}(V)} \frac{1}{\delta(x, y)^\eta} \pi_s(dx) < +\infty. \quad (3.5) \]

We also need the following property:

**Lemma 3.3** ([15]). Assume A1 and A2. Then, for any $\varepsilon > 0$, there exist constants $s_0 > 0$ and $c, C > 0$ such that for all $s \in (-s_0, s_0)$, $n \geq k \geq 1$, $x \in \mathbb{P}(V)$ and $y \in \mathbb{P}(V^*)$,
\[ Q^x_s \left( \log \delta(G_n, x, y) \leq -\varepsilon k \right) \leq Ce^{-ck}. \quad (3.6) \]

Note that (3.6) is stronger than the exponential Hölder regularity of the invariant measure $\pi_s$ stated in Lemma 3.2.

### 3.2. Proof of Theorem 2.1.

In fact we shall prove a more general version of Theorem 2.1 under the changed measure $Q^x_s$. The proof for the case $s = 0$ requires the same effort, so we decide to consider the more general setting. For any $s \in (-s_0, s_0)$ and $\varphi \in \mathcal{B}_\gamma$, define
\[ b_{s,\varphi}(x) := \lim_{n \to \infty} \mathbb{E}_{Q^x_s} \left[ (\sigma(G_n, x) - n\Lambda'(s))\varphi(G_n \cdot x) \right], \quad x \in \mathbb{P}(V) \quad (3.7) \]
and
\[ d_{s,\varphi}(y) = \int_{\mathbb{P}(V)} \varphi(x) \log \delta(x, y) \pi_s(dx), \quad y \in \mathbb{P}(V^*). \quad (3.8) \]

These functions are well-defined and $\gamma$-Hölder continuous, as shown in Lemmas 3.5 and 3.6 below. In particular, we have $b_{0,\varphi} = b_\varphi$ and $d_{0,\varphi} = d_\varphi$, where $b_\varphi$ and $d_\varphi$ are defined in (2.5) and (2.6), respectively.

Our goal of this subsection is to establish the following first-order Edgeworth expansion for the coefficients $\langle f, G_n v \rangle$ under the changed measure $Q^x_s$. Note that $\sigma_s = \sqrt{\Lambda''(s)}$, which is strictly positive under A1 and A2.
Theorem 3.4. Assume A1 and A2. Then, for any \( \varepsilon > 0 \), there exist \( \gamma > 0 \) and \( s_0 > 0 \) such that uniformly in \( s \in (-s_0, s_0) \), \( t \in \mathbb{R} \), \( x = \mathbb{R}v \in \mathbb{P}(V) \), \( y = \mathbb{R}f \in \mathbb{P}(V^*) \) with \( \|v\| = \|f\| = 1 \), and \( \varphi \in \mathcal{B}_\gamma \), as \( n \to \infty \),

\[
\mathbb{E}_{Q_n} \left[ \varphi(G_n \cdot x) \mathbb{I}_{\log ||f,G_n v|| - n \Lambda'(s) \leq t} \right] = \pi_s(\varphi) \left[ \Phi(t) + \frac{\Lambda''(s)}{6\sigma_s^3 \sqrt{n}}(1 - t^2)\phi(t) \right] - \frac{b_{s,\varphi}(x) + d_{s,\varphi}(y)}{\sigma_s \sqrt{n}} \phi(t) + \pi_s(\varphi) \Theta \left( \frac{1}{\sqrt{n}} \right) + \|\varphi\|_\gamma O \left( \frac{1}{n^{1 - \varepsilon}} \right).
\]

Theorem 2.1 follows from Theorem 3.4 by taking \( s = 0 \).

The remaining part of the paper is devoted to establishing Theorem 3.4. We begin with some properties of the function \( b_{s,\varphi} \) (cf. (3.7)).

Lemma 3.5 ([25]). Assume A1 and A2. Then, there exist constants \( s_0 > 0 \), \( \gamma > 0 \) and \( c > 0 \) such that \( b_{s,\varphi} \in \mathcal{B}_\gamma \) and \( \|b_{s,\varphi}\|_\gamma \leq c\|\varphi\|_\gamma \) for any \( s \in (-s_0, s_0) \).

In addition to Lemma 3.5, we shall need the following result on the function \( d_{s,\varphi} \) defined in (3.8).

Lemma 3.6. Assume A1 and A2. Then, there exists \( s_0 > 0 \) such that for any \( s \in (-s_0, s_0) \), the function \( d_{s,\varphi} \) is well-defined. Moreover, there exist constants \( \gamma > 0 \) and \( c > 0 \) such that \( d_{s,\varphi} \in \mathcal{B}_\gamma^* \) and \( \|d_{s,\varphi}\|_\gamma \leq c\|\varphi\|_\infty \) for any \( s \in (-s_0, s_0) \).

Proof. Without loss of generality, we assume that \( \varphi \) is non-negative. Since \( \log a \leq a \) for any \( a \geq 0 \) (with the convention that \( \log 0 = -\infty \)), we have that for any \( \eta \in (0, 1) \),

\[
-\eta \log \delta(x, y) \leq \delta(x, y)^{-\eta},
\]

so that

\[
-d_{s,\varphi}(y) \leq \frac{\|\varphi\|_\infty}{\eta} \int_{\mathbb{P}(V)} \frac{1}{\delta(x, y)^{\eta}} \pi_s(dx).
\]

Choosing \( \eta \) small enough, by Lemma 3.2, the latter integral is bounded by some constant uniformly in \( y \in \mathbb{P}(V^*) \) and \( s \in (-s_0, s_0) \), which proves that \( d_{s,\varphi} \) is well-defined and \( \|d_{s,\varphi}\|_\infty \leq c\|\varphi\|_\infty \) for some constant \( c > 0 \).

To estimate \( [d_{s,\varphi}]_\gamma \), we first note that for any \( y' = \mathbb{R}f' \in \mathbb{P}(V^*) \), \( y'' = \mathbb{R}f'' \in \mathbb{P}(V^*) \) and any \( \gamma > 0 \),

\[
|\log \delta(x, y') - \log \delta(x, y'')| = \left| |\log \delta(x, y') - \log \delta(x, y'')| \right| \left\{ \left| \frac{\delta(x, y') - \delta(x, y'')}{\delta(x, y')} \right|^{\gamma'} > \frac{1}{2} \right\}
+ \left| |\log \delta(x, y') - \log \delta(x, y'')| \right| \left\{ \left| \frac{\delta(x, y') - \delta(x, y'')}{\delta(x, y'')} \right|^{\gamma'} \leq \frac{1}{2} \right\}
=: I_1 + I_2.
\]

For \( I_1 \), we easily get

\[
I_1 \leq 2\gamma \left( |\log \delta(x, y')| + |\log \delta(x, y'')| \right) \left| \frac{\delta(x, y') - \delta(x, y'')}{\delta(x, y'')} \right|^{\gamma'}. \]
For $I_2$, since $|\log(1 + a)| \leq 2|a|$ for any $|a| \leq \frac{1}{2}$, we deduce that

$$I_2 = |\log \delta(x, y') - \log \delta(x, y'')|^{1-\gamma} \left| \log \left[ 1 + \frac{\delta(x, y') - \delta(x, y'')}{\delta(x, y'')} \right] \right|^\gamma \left\{ \frac{|\delta(x, y') - \delta(x, y'')|}{\delta(x, y'')} \right\}^{\frac{1}{2}}$$

$$\leq 2\gamma |\log \delta(x, y') - \log \delta(x, y'')|^{1-\gamma} \left| \frac{\delta(x, y') - \delta(x, y'')}{\delta(x, y'')} \right|^\gamma.$$

Therefore,

$$|\log \delta(x, y') - \log \delta(x, y'')| \leq 2\gamma (|\log \delta(x, y')| + |\log \delta(x, y'')|) \left| \frac{\delta(x, y') - \delta(x, y'')}{\delta(x, y'')} \right|^\gamma$$

$$+ 2\gamma |\log \delta(x, y') - \log \delta(x, y'')|^{1-\gamma} \left| \frac{\delta(x, y') - \delta(x, y'')}{\delta(x, y'')} \right|^\gamma.$$

By (3.9), it holds that $-\gamma \log \delta(x, y) \leq \delta(x, y)^{-\gamma}$ for any $\gamma \in (0, 1)$. Hence there exists a constant $c_\gamma > 0$ such that

$$|\log \delta(x, y') - \log \delta(x, y'')|$$

$$\leq c_\gamma \left( \delta(x, y')^{-\gamma} + \frac{\delta(x, y'')}{-\gamma} \right) |\delta(x, y') - \delta(x, y'')|^{\gamma}$$

$$+ c_\gamma \left( \delta(x, y'')^{-\gamma} + \frac{\delta(x, y')}{-\gamma} \right) |\delta(x, y') - \delta(x, y'')|^{\gamma}$$

$$\leq c_\gamma \left( \delta(x, y')^{-\gamma} + \frac{\delta(x, y'')}{-\gamma} \right) |\delta(x, y') - \delta(x, y'')|^{\gamma}$$

$$\leq c_\gamma \left( \delta(x, y')^{-\gamma} + \frac{\delta(x, y'')}{-\gamma} \right) |\delta(x, y') - \delta(x, y'')|^{\gamma}.$$

Since $\|f'\|_\infty - \|f''\|_\infty \leq \sqrt{2}d(y', y'')$ where $d(y', y'')$ is the angular distance on $\mathbb{P}(V^*)$, we have

$$|\delta(x, y') - \delta(x, y'')| = \left| \left\langle f', v \right\rangle \right|_\infty - \left| \left\langle f'', v \right\rangle \right|_\infty \leq \left| \left\langle f' \right\rangle \right|_\infty - \left| \left\langle f'' \right\rangle \right|_\infty \leq \sqrt{2}d(y', y'').$$

By the definition of the function $d_{s, \varphi}$, using the above bounds, we obtain

$$\frac{|d_{s, \varphi}(y') - d_{s, \varphi}(y'')|}{d(y', y'')^{\gamma}} \leq c_\gamma \|\varphi\|_{\infty} \int_{\mathbb{P}(V)} \left( \delta(x, y')^{-\gamma} + \frac{\delta(x, y'')}{-\gamma} \right) \pi_s(dx).$$

By Lemma 3.2, the last integral is bounded by some constant uniformly in $y', y'' \in \mathbb{P}(V^*)$ and $s \in (-s_0, s_0)$, by choosing $\gamma > 0$ sufficiently small. This, together with the fact that $\|d_{s, \varphi}\|_{\infty} \leq c\|\varphi\|_{\infty}$, proves that $d_{s, \varphi} \in \mathbb{B}_{\gamma}$ and $\|d_{s, \varphi}\|_{\gamma} \leq c\|\varphi\|_{\infty}$. □

In the proof of Theorem 3.4 we shall make use of the following Edgeworth expansion for the couple $(G_n \cdot x, \sigma(G_n, x))$ with a target function $\varphi$ on $G_n \cdot x$, which slightly improves [25, Theorem 5.3] by giving more accurate reminder terms. This improvement will be important for establishing Theorem 3.4.
Theorem 3.7. Assume \textbf{A1} and \textbf{A2}. Then, there exist constants $s_0 > 0$ and $\gamma > 0$ such that, as $n \to \infty$, uniformly in $s \in (-s_0, s_0)$, $x \in \mathbb{P}(V)$, $t \in \mathbb{R}$ and $\varphi \in \mathcal{B}_\gamma$,

$$
\mathbb{E}_{Q_n}[\varphi(G_n \cdot x)| \{ s(G_n, x) - n\Lambda(s(x)) \leq t \}] = \pi_s(\varphi) \left[ \Phi(t) + \frac{N''(s)}{6\sigma_n^2 \sqrt{n}}(1 - t^2)\phi(t) \right] - \frac{b_{s,\varphi}(x)}{\sigma_n\sqrt{n}} \phi(t) + \pi_s(\varphi)\phi\left(\frac{1}{\sqrt{n}}\right) + ||\varphi||_\gamma O\left(\frac{1}{n}\right).
$$

Proof. For any $x \in \mathbb{P}(V)$, define

$$
F(t) = \mathbb{E}_{Q_n}[\varphi(G_n \cdot x)| \{ s(G_n, x) - n\Lambda(s(x)) \leq t \}] + \frac{b_{s,\varphi}(x)}{\sigma_n\sqrt{n}} \phi(t), \quad t \in \mathbb{R},
$$

$$
H(t) = \mathbb{E}_{Q_n}[\varphi(G_n \cdot x)] \left[ \Phi(t) + \frac{N''(s)}{6\sigma_n^2 \sqrt{n}}(1 - t^2)\phi(t) \right], \quad t \in \mathbb{R}.
$$

Since $F(-\infty) = H(-\infty) = 0$ and $F(\infty) = H(\infty)$, applying Proposition 4.1 of [25] we get that

$$
\sup_{t \in \mathbb{R}} |F(t) - H(t)| \leq \frac{1}{n} (I_1 + I_2 + I_3 + I_4),
$$

(3.10)

where

$$
I_1 = o\left(\frac{1}{\sqrt{n}}\right) \sup_{t \in \mathbb{R}} |H'(t)|, \quad I_2 \leq C e^{-ca\gamma} ||\varphi||_\gamma, \quad I_3 \leq \frac{c}{n} ||\varphi||_\gamma, \quad I_4 \leq \frac{c}{n} ||\varphi||_\gamma.
$$

Here the bounds for $I_2$, $I_3$ and $I_4$ are obtained in [25]. It is easy to see that

$$
I_1 = o\left(\frac{1}{\sqrt{n}}\right) \mathbb{E}_{Q_n}[\varphi(G_n \cdot x)].
$$

This, together with the fact that

$$
\mathbb{E}_{Q_n}[\varphi(G_n \cdot x)] \leq \pi_s(\varphi) + C e^{-ca\gamma} ||\varphi||_\gamma
$$

(cf. [25]), proves the theorem. \hfill \Box

In the following we shall construct a partition $(\chi_{n,k}^y)_{k \geq 0}$ of the unity on the projective space $\mathbb{P}(V)$, which is similar to the partitions in [26, 15, 10]. In contrast to [26, 15], there is no escape of mass in our partition, which simplifies the proofs. Our partition becomes finer when $n \to \infty$, which allows us to obtain precise expressions for remainder terms in the central limit theorem and thereby to establish the Edgeworth expansion for the coefficients.

Let $U$ be the uniform distribution function on the interval $[0, 1]$: $U(t) = t$ for $t \in [0, 1]$, $U(t) = 0$ for $t < 0$ and $U(t) = 1$ for $t > 1$. Let $a_n = \frac{1}{\log n}$. Here and below we assume that $n \geq 18$ so that $a_n e^{ca\gamma} \leq \frac{1}{2}$. For any integer $k \geq 0$, define

$$
U_{n,k}(t) = U\left(\frac{t - (k - 1)a_n}{a_n}\right), \quad h_{n,k}(t) = U_{n,k}(t) - U_{n,k+1}(t), \quad t \in \mathbb{R}.
$$
It is easy to see that \( U_{n,m} = \sum_{k=m}^{\infty} h_{n,k} \) for any \( m \geq 0 \). Therefore, for any \( t \geq 0 \) and \( m \geq 0 \), we have

\[
\sum_{k=0}^{\infty} h_{n,k}(t) = 1, \quad \sum_{k=0}^{m} h_{n,k}(t) + U_{n,m+1}(t) = 1. \tag{3.11}
\]

Note that for any \( k \geq 0 \),

\[
\sup_{s,t \geq 0, s \neq t} \frac{|h_{n,k}(s) - h_{n,k}(t)|}{|s - t|} \leq \frac{1}{a_n}. \tag{3.12}
\]

For any \( x \in \mathbb{P}(V) \) and \( y \in \mathbb{P}(V^*) \), set

\[
\chi_{n,k}^{y}(x) = h_{n,k}(-\log \delta(x, y)) \quad \text{and} \quad \chi_{n,k}^{y}(x) = U_{n,k}(-\log \delta(x, y)), \tag{3.13}
\]

where we recall that \(-\log \delta(x, y) \geq 0\) for any \( x \in \mathbb{P}(V) \) and \( y \in \mathbb{P}(V^*) \). From (3.11) we have the following partition of the unity on \( \mathbb{P}(V) \): for any \( x \in \mathbb{P}(V), y \in \mathbb{P}(V^*) \) and \( m \geq 0 \),

\[
\sum_{k=0}^{\infty} \chi_{n,k}^{y}(x) = 1, \quad \sum_{k=0}^{m} \chi_{n,k}^{y}(x) + \chi_{n,m+1}(x) = 1. \tag{3.14}
\]

Denote by \( \text{supp}(\chi_{n,k}^{y}) \) the support of the function \( \chi_{n,k}^{y} \). It is easy to see that for any \( k \geq 0 \) and \( y \in \mathbb{P}(V^*) \),

\[
-\log \delta(x, y) \in [a_n(k-1), a_n(k+1)] \quad \text{for any} \ x \in \text{supp}(\chi_{n,k}^{y}). \tag{3.15}
\]

**Lemma 3.8.** There exists a constant \( c > 0 \) such that for any \( \gamma \in (0, 1), k \geq 0 \) and \( y \in \mathbb{P}(V^*) \), it holds \( \chi_{n,k}^{y} \in \mathcal{B}_{\gamma} \) and, moreover,

\[
\|\chi_{n,k}^{y}\|_{\gamma} \leq \frac{ce^{\gamma k a_n}}{a_n}. \tag{3.16}
\]

**Proof.** Since \( \|\chi_{n,k}^{y}\|_{\infty} \leq 1 \), it is enough to give a bound for the modulus of continuity:

\[
[\chi_{n,k}^{y}]_{\gamma} = \sup_{x', x'' \in \mathbb{P}(V): x' \neq x''} \frac{|\chi_{n,k}^{y}(x') - \chi_{n,k}^{y}(x'')|}{d(x', x'')^{\gamma}},
\]

where \( d \) is the angular distance on \( \mathbb{P}(V) \) defined by (2.1). Assume that \( x' = \Re v' \in \mathbb{P}(V) \) and \( x'' = \Re v'' \in \mathbb{P}(V) \) are such that \( \|v'\| = \|v''\| = 1 \). We note that

\[
\|v' - v''\| \leq \sqrt{2}d(x', x''). \tag{3.17}
\]

For short, denote \( B_k = ((k-1)a_n, ka_n] \). Note that the function \( h_{n,k} \) is increasing on \( B_k \) and decreasing on \( B_{k+1} \). Set for brevity \( t' = -\log \delta(x', y) \) and \( t'' = -\log \delta(x'', y) \). First we consider the case when \( t' \) and \( t'' \) are such that \( t', t'' \in B_k \). Using (3.13), (3.12) and the fact that \( |h_{n,k}| \leq 1 \), we have that for any \( \gamma \in (0, 1] \),

\[
|\chi_{n,k}(x') - \chi_{n,k}(x'')| = |h_{n,k}(t') - h_{n,k}(t'')|^{1-\gamma}|h_{n,k}(t') - h_{n,k}(t'')|^{\gamma} \\
\leq 2|h_{n,k}(t') - h_{n,k}(t'')|^{\gamma} \leq 2\frac{|t' - t''|\gamma}{a_n} = \frac{2}{a_n} |\log u' - \log u''|^{\gamma}, \tag{3.18}
\]
where we set for brevity \( u' = \delta (x', y) \) and \( u'' = \delta (x'', y) \). Since \( u' = e^{-t'} \), \( u'' = e^{-t''} \) and \( t, t' \in B_k \), we have \( u'' \geq e^{-ka_n} \) and \( |u' - u''| \leq e^{-(k-1)an} - e^{-ka_n} \). Therefore, for \( n \geq 18 \),

\[
\frac{|u' - u''|}{u''} = \frac{e^{-(k-1)an} - e^{-ka_n}}{e^{-ka_n}} = a_n e^{an} - 1 \leq a_n e^{an} \leq \frac{1}{2},
\]

which, together with the inequality \( |\log (1 + a)| \leq 2|a| \) for any \( |a| \leq \frac{1}{2} \), implies

\[
|\log u' - \log u''| = \left| \log \left( 1 + \frac{u' - u''}{u''} \right) \right| \leq 2 \frac{|u' - u''|}{u''}.
\]  

(3.19)

Since \( u'' \geq e^{-ka_n} \), using the fact that \( \|v'\| = \|v''\| = 1 \) and (3.17), we get

\[
\frac{|u' - u''|}{u''} \leq e^{ka_n} |\delta(x', y) - \delta(x'', y)| = e^{ka_n} \frac{|f(v') - f(v'')|}{\|f\|} \leq e^{ka_n} \|v' - v''\| \leq \sqrt{2} e^{ka_n} d(x', x'').
\]  

(3.20)

Therefore, from (3.18), (3.19) and (3.20), it follows that for \( \gamma \in (0, 1] \),

\[
|\chi_{n,k}^{y}(x') - \chi_{n,k}^{y}(x'')| \leq 6 \frac{e^{\gamma k\alpha_n}}{a_n^\gamma} d(x', x'')^{\gamma}.
\]  

(3.21)

The case \( t', t'' \in B_k \) is treated in the same way.

To conclude the proof we shall consider the case when \( t' = -\log \delta(x', y) \in B_{k-1} \) and \( t'' = -\log \delta(x'', y) \in B_k \); the other cases can be handled in the same way. We shall reduce this case to the previous ones. Let \( x^* \in \mathbb{P}(V) \) be the point on the geodesic line \( [x', x''] \) on \( \mathbb{P}(V) \) such that \( d(x', x^*) = d(x', x'') + d(x^*, x'') \) and \( t^* = -\log \delta(y, x^*) = k\alpha_n \). Then

\[
|\chi_{n,k}^{y}(x') - \chi_{n,k}^{y}(x'')| \leq |\chi_{n,k}^{y}(x') - \chi_{n,k}^{y}(x^*)| + |\chi_{n,k}^{y}(x^*) - \chi_{n,k}^{y}(x'')| \\
\leq 6 \frac{e^{\gamma k\alpha_n}}{a_n^\gamma} d(x', x^*)^{\gamma} + 6 \frac{e^{\gamma k\alpha_n}}{a_n^\gamma} d(x'', x^*)^{\gamma} \\
\leq 12 \frac{e^{\gamma k\alpha_n}}{a_n^\gamma} d(x', x'')^{\gamma}.
\]  

(3.22)

From (3.21) and (3.22) we conclude that \( |\chi_{n,k}^{y}| \leq 12 \frac{e^{\gamma k\alpha_n}}{a_n^\gamma} \), which shows (3.16). 

\[ \square \]

We need the following bounds. Let \( M_n = \lfloor A \log^2 n \rfloor \), where \( A > 0 \) is a constant and \( n \) is large enough. For any measurable function \( \varphi \) on \( \mathbb{P}(V) \), it is convenient to denote

\[
\varphi_{n,k}^{y} = \varphi_{\chi_{n,k}^{y}} \quad \text{for} \quad 0 \leq k \leq M_n - 1, \quad \varphi_{n,M_n}^{y} = \varphi_{\chi_{n,M_n}^{y}}.
\]  

(3.23)

**Lemma 3.9.** Assume \( A1 \) and \( A2 \). Then there exist constants \( s_0 > 0 \) and \( c > 0 \) such that for any \( s \in (-s_0, s_0) \), \( y \in \mathbb{P}(V^*) \) and any non-negative bounded measurable function \( \varphi \) on \( \mathbb{P}(V) \),

\[
\sum_{k=0}^{M_n} (k + 1) a_n \pi_s (\varphi_{n,k}^{y}) \leq -d_{s, \varphi} (y) + 2a_n \pi_s (\varphi)
\]
and
\[ \sum_{k=0}^{M_n}(k - 1)a_n\pi_s(\varphi_{n,k}^y) \geq -d_{s,\varphi}(y) - 2a_n\pi_s(\varphi) - c\frac{\|\varphi\|_\infty}{n^2}. \]

**Proof.** Recall that \( d_{s,\varphi}(y) \) is defined in (3.8). Using (3.23) and (3.14) we deduce that
\[
-d_{s,\varphi}(y) = -\sum_{k=0}^{M_n} \int_{\mathbb{P}(V)} \varphi_{n,k}^y(x) \log \delta(x, y)\pi_s(dx)
\geq \sum_{k=0}^{M_n}(k - 1)a_n\pi_s(\varphi_{n,k}^y) = \sum_{k=0}^{M_n}(k + 1)a_n\pi_s(\varphi_{n,k}^y) - 2a_n\pi_s(\varphi),
\]
which proves the first assertion of the lemma.

Using the Markov inequality and the Hölder regularity of the invariant measure \( \pi_s \) (Lemma 3.2), we get that there exists a small constant \( \eta > 0 \) such that
\[
-\int_{\mathbb{P}(V)} \varphi_{n,M_n}^y(x) \log \delta(x, y)\pi_s(dx) \leq c\|\varphi\|_\infty e^{-\eta A \log n} \int_{\mathbb{P}(V)} \delta(x, y)^{-2n\pi_s(dx)} \leq c\frac{\|\varphi\|_\infty}{n^2},
\]
where in the last inequality we choose \( A > 0 \) to be sufficiently large so that \( \eta A \geq 2 \).

Therefore,
\[
-d_{s,\varphi}(y) = -\sum_{k=0}^{M_n} \int_{\mathbb{P}(V)} \varphi_{n,k}^y(x) \log \delta(x, y)\pi_s(dx)
\leq \sum_{k=0}^{M_n}(k + 1)a_n\pi_s(\varphi_{n,k}^y) + c\frac{\|\varphi\|_\infty}{n^2}
\leq \sum_{k=0}^{M_n}(k - 1)a_n\pi_s(\varphi_{n,k}^y) + 2a_n\pi_s(\varphi) + c\frac{\|\varphi\|_\infty}{n^2}
\leq \sum_{k=0}^{M_n}(k - 1)a_n\pi_s(\varphi_{n,k}^y) + 2a_n\pi_s(\varphi) + c\frac{\|\varphi\|_\infty}{n^2}.
\]
This proves the second assertion of the lemma. \( \square \)

**Proof of Theorem 3.4.** Without loss of generality, we assume that the target function \( \varphi \) is non-negative. With the notation in (3.23), we have that for \( t \in \mathbb{R} \),
\[
I_n(t) := \mathbb{E}_{Q^n_x} \left[ \varphi(G_n \cdot x) \mathbb{1}_{\left\{ \frac{\log \|\langle f, G_n v \rangle \| - n\Lambda(s)}{\sigma_s \sqrt{n}} \leq t \right\}} \right]
= \sum_{k=0}^{M_n} \mathbb{E}_{Q^n_x} \left[ \varphi_{n,k}^y(G_n \cdot x) \mathbb{1}_{\left\{ \frac{\log \|\langle f, G_n v \rangle \| - n\Lambda(s)}{\sigma_s \sqrt{n}} \leq t \right\}} \right] =: \sum_{k=0}^{M_n} F_{n,k}(t).
\]
For $0 \leq k \leq M_n - 1$, using (1.5) and the fact that $-\log \delta(x, y) \leq (k + 1)a_n$ when $x \in \text{supp} \varphi_{n,k}^y$, we get

$$F_{n,k}(t) \leq \mathbb{E}_{Q^x} \left[ \varphi_{n,k}^y(G_n \cdot x) \mathbb{1}_{\left\{ \frac{s(G_n,x) - s(G_n,y)}{\sigma_s \sqrt{n}} \leq t + \frac{(k+1)a_n}{\sigma_s \sqrt{n}} \right\}} \right] =: H_{n,k}(t). \quad (3.25)$$

For $k = M_n$, we have

$$F_{n,M_n}(t) \leq \mathbb{E}_{Q^x} \left[ \varphi_{n,M_n}^y(G_n \cdot x) \mathbb{1}_{\left\{ \frac{s(G_n,x) - s(G_n,y)}{\sigma_s \sqrt{n}} \leq t + \frac{(M_n + 1)a_n}{\sigma_s \sqrt{n}} \right\}} \right] + \mathbb{E}_{Q^x} \left[ \varphi_{n,M_n}^y(G_n \cdot x) \mathbb{1}_{\left\{ -\log \delta(G_n \cdot x, y) \geq (M_n + 1)a_n \right\}} \right] =: H_{n,M_n}(t) + W_n. \quad (3.26)$$

By Lemma 3.3 and choosing $A > 0$ large enough, we get

$$W_n \leq \|\varphi\|_\infty Q^x \left( -\log \delta(G_n \cdot x, y) \geq A \log n \right) \leq \frac{c_0}{n^{1/2}} \|\varphi\|_\infty \leq \frac{c_0}{n} \|\varphi\|_\infty. \quad (3.27)$$

Now we deal with $H_{n,k}(t)$ for $0 \leq k \leq M_n$. Denote for short $t_{n,k} = t + \frac{(k+1)a_n}{\sigma_s \sqrt{n}}$. Applying the Edgeworth expansion (Theorem 3.7) we obtain that, uniformly in $s \in (-s_0, s_0)$, $x \in \mathbb{P}(V)$, $t \in \mathbb{R}$, $0 \leq k \leq M_n$ and $\varphi \in \mathcal{B}_r$, as $n \to \infty$,

$$H_{n,k}(t) = \pi_s(\varphi_{n,k}^y) \left[ \Phi(t_{n,k}) + \frac{A''(s)}{6\sigma^2_s \sqrt{n}} (1 - t^2_{n,k}) \phi(t_{n,k}) \right] - \frac{b_{s,\varphi_{n,k}^y}(x)}{\sigma_s \sqrt{n}} \phi(t_{n,k}) + \pi_s(\varphi_{n,k}^y) O \left( \frac{1}{\sqrt{n}} \right) + \|\varphi_{n,k}^y\|_\gamma O \left( \frac{1}{n} \right).$$

Recall that $a_n = \frac{1}{\log n}$ and $M_n = \lfloor A \log^2 n \rfloor$. By the Taylor expansion we have, uniformly in $s \in (-s_0, s_0)$, $x \in \mathbb{P}(V)$, $t \in \mathbb{R}$ and $0 \leq k \leq M_n$,

$$\Phi(t_{n,k}) = \Phi(t) + \frac{(k+1)a_n}{\sigma_s \sqrt{n}} + O \left( \frac{\log^2 n}{n} \right)$$

and

$$(1 - t^2_{n,k}) \phi(t_{n,k}) = (1 - t^2) \phi(t) + O \left( \frac{\log n}{\sqrt{n}} \right).$$

Moreover, using Lemma 3.5, we see that

$$\frac{b_{s,\varphi_{n,k}^y}(x)}{\sigma_s \sqrt{n}} \phi(t_{n,k}) = \frac{b_{s,\varphi_{n,k}^y}(x)}{\sigma_s \sqrt{n}} \phi(t) + \|\varphi_{n,k}^y\|_\gamma O \left( \frac{\log n}{n} \right).$$
Using these expansions and (3.25), (3.26) and (3.27), we get that there exists a sequence 
\((\beta_n)_{n \geq 1}\) of positive numbers satisfying \(\beta_n \to 0\) as \(n \to \infty\), such that for any \(0 \leq k \leq M_n\),
\[
F_{n,k}(t) \leq \pi_s(\varphi^y_{n,k}) \left[ \Phi(t) + \frac{\Lambda''(s)}{6\sigma^3} (1 - t^2) \phi(t) \right] \\
- \frac{b_{s,\varphi^y_{n,k}}(x)}{\sigma_s \sqrt{n}} \phi(t) + \frac{\phi(t)}{\sigma_s \sqrt{n}} \pi_s(\varphi^y_{n,k})(k + 1) a_n \\
+ \pi_s(\varphi^y_{n,k}) \frac{\beta_n}{\sqrt{n}} + \|\varphi^y_{n,k}\|_{\gamma} \frac{c \log n}{n}.
\]
(3.28)

By Lemma 3.8, it holds that for any \(\gamma \in (0, 1]\) and \(0 \leq k \leq M_n\),
\[
\|\varphi^y_{n,k}\|_{\gamma} \leq c \|\varphi\|_{\infty} n^{\gamma A} \log n + \|\varphi\|_{\gamma}.
\]
(3.29)

From (3.7), it follows that \(b_{s,\varphi}(x) = \sum_{k=0}^{M_n} b_{s,\varphi^y_{n,k}}(x)\). Therefore, summing up over \(k\) in (3.28), using (3.29) and taking \(\gamma > 0\) to be sufficiently small such that \(\gamma A < \varepsilon/2\), we obtain
\[
I_n(t) = \sum_{k=0}^{M_n} F_{n,k}(t) \leq \pi_s(\varphi) \left[ \Phi(t) + \frac{\Lambda''(s)}{6\sigma^3} (1 - t^2) \phi(t) \right] \\
- \frac{b_{s,\varphi}(x)}{\sigma_s \sqrt{n}} \phi(t) + \frac{\phi(t)}{\sigma_s \sqrt{n}} \sum_{k=0}^{M_n} \pi_s(\varphi^y_{n,k})(k + 1) a_n \\
+ \pi_s(\varphi) \frac{\beta_n}{\sqrt{n}} + \|\varphi\|_{\gamma} \frac{c \log n}{n^{1-\varepsilon}}.
\]

Using Lemma 3.9 and the fact that \(a_n \to 0\) as \(n \to \infty\), we obtain the desired upper bound.

The lower bound is established in the same way. Instead of (3.25) we use the following lower bound, which is obtained using (1.5) and the fact that \(- \log \delta(x, y) \geq (k - 1) a_n\) for \(x \in \text{supp} \varphi^y_{n,k}\) and \(0 \leq k \leq M_n\),
\[
F_{n,k}(t) \geq \mathbb{E}_{\mathcal{F}_t^G} \left[ \varphi^y_{n,k}(G_n \cdot x) \mathbf{1}_{\left\{ \frac{\Lambda''(s)}{\sigma_s \sqrt{n}} - \frac{\Lambda''(s)}{\sigma_s \sqrt{n}} \leq t + \frac{(k-1) a_n}{\sigma_s \sqrt{n}} \right\}} \right].
\]
(3.30)

Proceeding in the same way as in the proof of the upper bound, using (3.30) instead of (3.25) and (3.26), we get
\[
I_n(t) = \sum_{k=0}^{M_n} F_{n,k}(t) \geq \pi_s(\varphi) \left[ \Phi(t) + \frac{\Lambda''(s)}{6\sigma^3} (1 - t^2) \phi(t) \right] \\
- \frac{b_{s,\varphi}(x)}{\sigma_s \sqrt{n}} \phi(t) + \frac{\phi(t)}{\sigma_s \sqrt{n}} \sum_{k=0}^{M_n} \pi_s(\varphi^y_{n,k})(k + 1) a_n \\
+ \pi_s(\varphi) \frac{O\left(\frac{1}{\sqrt{n}}\right)}{\sqrt{n}} + \|\varphi\|_{\gamma} O\left(\frac{1}{n^{1-\varepsilon}}\right).
\]

The lower bound is obtained using again Lemma 3.9 and the fact that \(a_n \to 0\) as \(n \to \infty\).
References

[1] Aoun, R., Sert, C.: Law of large numbers for the spectral radius of random matrix products. *American Journal of Mathematics*, 143(3): 995-1010, 2021.

[2] Bellman R.: Limit theorems for non-commutative operations. I. *Duke Math. J.*, 21(3): 491-500, 1954.

[3] Benoist Y., Quint J. F., Central limit theorem for linear groups. *The Annals of Probability*, 44(2): 1308-1340, 2016.

[4] Benoist Y., Quint J. F.: Random walks on reductive groups. *Springer International Publishing*, 2016.

[5] Bougerol P., Lacroix J.: Products of random matrices with applications to Schrödinger operators. *Birkhäuser Boston*, 1985.

[6] Bourgain J., Furman A., Lindenstrauss E., Mozes S., Stationary measures and equidistribution for orbits of nonabelian semigroups on the torus. *Journal of the American Mathematical Society*, 24(1): 231-280, 2011.

[7] Cohn H., Nerman O., Peligrad M.: Weak ergodicity and products of random matrices. *Journal of Theoretical Probability*, 6(2): 389-405, 1993.

[8] Cuny C., Dedecker J., Merlevêde F., Peligrad M.: Berry-Esseen type bounds for the left random walk on $GL_d(\mathbb{R})$ under polynomial moment conditions. To appear in *The Annals of Probability*, hal-03329189, 2021.

[9] Cuny C., Dedecker J., Merlevêde F., Peligrad M.: Berry-Esseen type bounds for the matrix coefficients and the spectral radius of the left random walk on $GL_d(\mathbb{R})$. *Comptes Rendus Mathématique*, 360: 475-482, 2022.

[10] Dinh T. C., Kaufmann L., Wu H.: Berry-Esseen bound and local limit theorem for the coefficients of products of random matrices. arXiv:2110.09032, 2021.

[11] Dinh T. C., Kaufmann L., Wu H.: Berry-Esseen bounds with targets and Local Limit Theorems for products of random matrices. arXiv preprint arXiv:2111.14109v2, 2021.

[12] Furstenberg H.: Noncommuting random products. *Transactions of the American Mathematical Society*, 108(3): 377-428, 1963.

[13] Furstenberg H., Kesten H.: Products of random matrices. *The Annals of Mathematical Statistics*, 31(2): 457-469, 1960.

[14] Goldsheid I. Y., Margulis G. A.: Lyapunov indices of a product of random matrices. *Uspekhi Matematicheskikh Nauk*, 44(5): 11-71, 1989.

[15] Guivarc’h Y.: Exposants de Liapunoff des marches aléatoires à pas markovien. *Publications mathématiques et informatique de Rennes 1*: 1-16, 1980.

[16] Guivarc’h Y., Le Page É.: Spectral gap properties for linear random walks and Pareto’s asymptotics for affine stochastic recursions. *Ann. Inst. Henri Poincaré Probab. Stat.*, 52(2): 503-574, 2016.

[17] Guivarc’h Y., Raugi A.: Frontiere de Furstenberg, propriétés de contraction et théorèmes de convergence. *Probab. Theory Related Fields*, 69(2): 187-242, 1985.

[18] Hennion H.: Limit theorems for products of positive random matrices. *The Annals of Probability*, 25(4): 1545-1587, 1997.

[19] Hennion H., Hervé L.: Limit theorems for Markov chains and stochastic properties of dynamical systems by quasi-compactness. Vol. 1766. *Springer-Verlag*, Berlin, 2001.

[20] Kingman J. F. C.: Subadditive ergodic theory. *Ann. Probab.*, 883-899, 1973.

[21] Le Page É.: Théorèmes limites pour les produits de matrices aléatoires. *In Probability measures on groups*. Springer Berlin Heidelberg, 258-303, 1982.
[24] Petrov V. V: Sums of independent random variables. Springer, 1975.
[25] Xiao H., Grama I., Liu Q.: Berry-Esseen bound and precise moderate deviations for products of random matrices. Journal of the European Mathematical Society, 24(8): 2691-2750, 2022.
[26] Xiao H., Grama I., Liu Q.: Large deviation expansions for the coefficients of random walks on the general linear group. arXiv:2010.00553, 2020.
[27] Xiao H., Grama I., Liu Q.: Edgeworth expansion and large deviations for the coefficients of products of positive random matrices. arXiv:2209.03158, 2022.

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