SOME STEFFENSEN’S TYPE INEQUALITIES

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Abstract. In this paper, new inequalities connected with the celebrated Steffensen’s integral inequality are proved.

1. Introduction

In 1918 and in order to study certain inequalities between mean values, Steffensen [14] has proved the following inequality (see also [10] & [11]):

Theorem 1. Let \( f \) and \( g \) be two integrable functions defined on \( (a, b) \), \( f \) is decreasing and for each \( t \in (a, b) \), \( 0 \leq g(t) \leq 1 \). Then, the following inequality

\[
\int_{b-\lambda}^{b} f(t) \, dt \leq \int_{a}^{b} f(t) \, g(t) \, dt \leq \int_{a}^{a+\lambda} f(t) \, dt
\]

(1.1)

holds, where, \( \lambda = \frac{b}{A} \int_{a}^{b} g(t) \, dt \).

Some minor generalization of Steffensen’s inequality [14] was considered by T. Hayashi [4], using the substituting \( g(t)/A \) for \( g(t) \), where \( A \) is positive constant. In 1953, another interesting result was proved by Apéry [1], where he extended the Steffensen’s inequality [14] on infinite interval. Namely, we have the following result:

Theorem 2. Let \( f \) be decreasing on \( (0, \infty) \) and let \( g \) be a measurable function on \( (0, \infty) \) such that \( 0 \leq g(t) \leq A \) \( (A \neq 0) \). Then

\[
\int_{0}^{\infty} f(t) \, g(t) \, dt \leq A \int_{0}^{\lambda} f(t) \, dt,
\]

(1.2)

where, \( \lambda = \frac{1}{A} \int_{0}^{\infty} g(t) \, dt \).

For more results concerning new proofs, generalizations, weaker hypothesis or different forms were emerging one after another see [1]–[14], and the references therein.

The aim of this paper is to establish some Steffensen’s type inequalities under various assumptions.

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We may start with the following lemma:

**Lemma 1.** Let $f, g : [a, b] \to \mathbb{R}$ be integrable such that $0 \leq g(t) \leq 1$, for all $t \in [a, b]$ and $\int_a^b g(t) \, dt$ exists. Then we have the following representation

\begin{equation}
\int_a^{a+\lambda} f(t) \, dt - \int_a^b f(t) \, g(t) \, dt = -\int_a^{a+\lambda} \left( \int_a^x (1 - g(t)) \, dt \right) \, df(x) - \int_{a+\lambda}^b \left( \int_x^b g(t) \, dt \right) \, df(x),
\end{equation}

and

\begin{equation}
\int_a^b f(t) \, g(t) \, dt - \int_{b-\lambda}^b f(t) \, dt = -\int_a^{b-\lambda} \left( \int_a^x g(t) \, dt \right) \, df(x) - \int_{b-\lambda}^b \left( \int_x^b (1 - g(x)) \, dt \right) \, df(x),
\end{equation}

where $\lambda := \int_a^b g(t) \, dt$.

**Proof.** Integrating by parts

\[-\int_a^{a+\lambda} \left( \int_a^x (1 - g(t)) \, dt \right) \, df(x) - \int_{a+\lambda}^b \left( \int_x^b g(t) \, dt \right) \, df(x) = -\int_a^{a+\lambda} \left( 1 - g(t) \right) \, df(x) + \int_{a+\lambda}^b \left( \int_x^{a+\lambda} f(x) \, dx \right) \, df(x) \]

\[+ \int_{a+\lambda}^b \left( \int_x^b g(t) \, dt \right) \, df(x) = -\int_a^{a+\lambda} \left( 1 - g(t) \right) \, df(x) + \int_{a+\lambda}^b \left( \int_x^{a+\lambda} f(x) \, dx \right) \, df(x) \]

\[+ \int_{a+\lambda}^b \left( \int_x^b g(t) \, dt \right) \, df(x) = -\lambda \left( f(a) + f(a+\lambda) \int_a^{a+\lambda} g(t) \, dt \right) + \int_{a+\lambda}^b f(x) \, dx \]

\[+ \int_a^{a+\lambda} \left( \int_x^{a+\lambda} f(x) \, dx \right) \, dt + \int_{a+\lambda}^b g(t) \, dt \, dx = \int_a^{a+\lambda} \left( \int_x^{a+\lambda} f(x) \, dx \right) \, dt + \int_{a+\lambda}^b g(t) \, dt \, dx \]

\[+ \int_a^{a+\lambda} \left( \int_x^{a+\lambda} f(x) \, dx \right) \, dt - \int_{a+\lambda}^b f(x) \, g(x) \, dx = \int_a^{a+\lambda} \left( \int_x^{a+\lambda} f(x) \, dx \right) \, dt - \int_{a+\lambda}^b f(x) \, g(x) \, dx \]

\[= \int_a^{a+\lambda} \left( \int_x^{a+\lambda} f(x) \, dx \right) \, dt - \int_{a+\lambda}^b f(x) \, g(x) \, dx, \]
which gives the desired representation (2.1). The identity (2.2) can be also proved in a similar way, we shall omit the details. □

2.1. Inequalities for bounded variation integrators. Our first result may be stated as follows:

**Theorem 3.** Let $f, g : [a, b] \to \mathbb{R}$ be such that $0 \leq g(t) \leq 1$, for all $t \in [a, b]$ and $\int_a^b g(t) \, df(t)$ exists. If $f$ is of bounded variation on $[a, b]$, then we have

\[
\left| \int_a^{a+\lambda} f(t) \, dt - \int_a^b f(t) \, g(t) \, dt \right| \leq \left[ \int_a^b g(t) \, dt \right] \cdot b \bigvee_a (f),
\]

and

\[
\left| \int_a^b f(t) \, g(t) \, dt - \int_{b-\lambda}^b f(t) \, dt \right| \leq \left[ \int_a^{b-\lambda} g(t) \, dt \right] \cdot b \bigvee_a (f),
\]

where $\lambda := \int_a^b g(t) \, dt$.

**Proof.** We prove the inequality (2.3). Taking the modulus in (2.1) and utilizing the triangle inequality, we get

\[
\left| \int_a^{a+\lambda} f(t) \, dt - \int_a^b f(t) \, g(t) \, dt \right| \\
\leq \left| \int_a^{a+\lambda} \left( \int_a^x (1 - g(t)) \, dt \right) \, df(x) \right| + \left| \int_{a+\lambda}^b \left( \int_x^b g(t) \, dt \right) \, df(x) \right|.
\]

Using the fact that for a continuous function $p : [a, b] \to \mathbb{R}$ and a function $\nu : [a, b] \to \mathbb{R}$ of bounded variation, one has the inequality:

\[
\left| \int_a^b p(t) \, d\nu(t) \right| \leq \sup_{t \in [a, b]} |p(t)| \bigvee_a (\nu),
\]

**[2.5]**
observe that \( \int_a^x (1 - g(t)) \, dt \) is a positive increasing function for \( x \in [a, a + \lambda] \) and \( \int_x^b g(t) \, dt \) is a positive decreasing function for \( x \in [a + \lambda, b] \), it follows that

\[
\left| \int_a^{a+\lambda} f(t) \, dt - \int_a^b f(t) g(t) \, dt \right| \\
\leq \sup_{x \in [a, a+\lambda]} \left[ \int_a^x (1 - g(t)) \, dt \right] \cdot \sup_{x \in [a+\lambda, b]} \left[ \int_x^b g(t) \, dt \right] \cdot \sup_{x \in [a, a+\lambda]} \left[ \int_a^x f(t) \, dt \right] \cdot \sup_{x \in [a+\lambda, b]} \left[ \int_x^b f(t) \, dt \right] \cdot \sup_{x \in [a, a+\lambda]} \left[ \int_a^x f(t) \, dt \right] \cdot \sup_{x \in [a+\lambda, b]} \left[ \int_x^b f(t) \, dt \right] \cdot \sup_{x \in [a, a+\lambda]} \left[ \int_a^x f(t) \, dt \right] \cdot \sup_{x \in [a+\lambda, b]} \left[ \int_x^b f(t) \, dt \right]
\]

since \( \int_a^{a+\lambda} (1 - g(t)) \, dt = \lambda - \int_a^{a+\lambda} g(t) \, dt = \int_a^b g(t) \, dt \) which proves the first inequality in (2.3), the second the inequality follows immediately by assumptions. In similar way and using (2.2) we may deduce the desired inequality (2.4), and we shall omit the details.

\[\square\]

**Corollary 1.** Let \( f, g : [a, b] \to \mathbb{R} \) be such that \( 0 \leq g(t) \leq 1 \), for all \( t \in [a, b] \) and \( \int_a^b g(t) \, df(t) \) exists. If \( f \) is decreasing on \([a, b]\), then we have

\[
0 \leq \int_a^{a+\lambda} f(t) \, dt - \int_a^b f(t) g(t) \, dt \leq \left| f(a) - f(b) \right| \int_a^{a+\lambda} g(t) \, dt
\]

and

\[
0 \leq \int_a^b f(t) g(t) \, dt - \int_{b-\lambda}^b f(t) \, dt \leq \left| f(a) - f(b) \right| \int_a^{b-\lambda} g(t) \, dt.
\]

**2.2 Inequalities for Lipschitzian integrators.** Inequalities for \( L\)-Lipschitzian integrators may be considered as follows:

**Theorem 4.** Let \( f, g : [a, b] \to \mathbb{R} \) be integrable such that \( 0 \leq g(t) \leq 1 \) for all \( t \in [a, b] \), and \( \int_a^b g(t) \, df(t) \) exists. If \( f \) is \( L\)-Lipschitzian on \([a, b]\), then

\[
\left| \int_a^{a+\lambda} f(t) \, dt - \int_a^b f(t) g(t) \, dt \right| \leq \frac{L}{2} \left[ (b - a - \lambda)^2 + \lambda^2 \right],
\]

and

\[
\left| \int_a^b f(t) g(t) \, dt - \int_{b-\lambda}^b f(t) \, dt \right| \leq \frac{L}{2} \left[ (b - a - \lambda)^2 + \lambda^2 \right],
\]

where \( \lambda := \int_a^b g(t) \, dt \).

**Proof.** Taking the modulus in (2.1) and utilizing the triangle inequality, we get

\[
\left| \int_a^{a+\lambda} f(t) \, dt - \int_a^b f(t) g(t) \, dt \right| \\
\leq \left| \int_a^{a+\lambda} \left( \int_a^x (1 - g(t)) \, dt \right) \, df(x) \right| + \left| \int_{a+\lambda}^b \left( \int_x^b g(t) \, dt \right) \, df(x) \right|.
\]
Using the fact that for a Riemann integrable function \( p : [c, d] \to \mathbb{R} \) and \( L \)-Lipschitzian function \( \nu : [c, d] \to \mathbb{R} \), one has the inequality

\[
\left| \int_c^d p(t) \, d\nu(t) \right| \leq L \int_c^d |p(t)| \, dt.
\]

(2.10)

it follows that

\[
\left| \int_a^{a+\lambda} f(t) \, dt - \int_a^b f(t) \, g(t) \, dt \right|
\]

\[
\leq L \left[ \int_a^{a+\lambda} \left( \int_a^x (1 - g(t)) \, dt \right) \, dx + \int_{a+\lambda}^b \int_x^b g(t) \, dt \, dx \right]
\]

\[
= L \left[ \int_a^{a+\lambda} \left( \int_a^x (1 - g(t)) \, dt \right) \, dx + \int_{a+\lambda}^b \left( \int_x^b |g(t)| \, dt \right) \, dx \right]
\]

\[
\leq L \left[ \int_a^{a+\lambda} \left( \int_a^x (1 - g(t)) \, dt \right) \, dx + \int_{a+\lambda}^b \left( \int_x^b |g(t)| \, dt \right) \, dx \right].
\]

But since \( 0 \leq g(x) \leq 1 \) for all \( x \in [a, b] \), (similarly we have, \( 0 \leq 1 - g(x) \leq 1 \)), then

\[
\left| \int_a^{a+\lambda} f(t) \, dt - \int_a^b f(t) \, g(t) \, dt \right|
\]

\[
\leq L \left[ \int_a^{a+\lambda} \left( \int_a^x |1 - g(t)| \, dt \right) \, dx + \int_{a+\lambda}^b \left( \int_x^b |g(t)| \, dt \right) \, dx \right]
\]

\[
\leq L \left[ \int_a^{a+\lambda} (x - a) \, dx + \int_{a+\lambda}^b (b - x) \, dx \right]
\]

\[
= L \left[ \frac{\lambda^2}{2} + (b - a - \lambda)^2 \right],
\]

which proves (2.8). In a similar way and using (2.2) we may deduce the desired inequality (2.9), and we shall omit the details.

\[\square\]

**Remark 1.** Let \( f, g \) be as in Theorem 4. If \( \int_a^b g(t) \, dt = 0 \), then

\[
\left| \int_a^b f(t) \, g(t) \, dt \right| \leq \frac{1}{2} L (b - a)^2.
\]

(2.11)

2.3. Inequalities for monotonic non-decreasing integrators.

**Theorem 5.** Let \( f, g : [a, b] \to \mathbb{R} \) be integrable such that \( 0 \leq g(t) \leq 1 \) for all \( t \in [a, b] \), and \( \int_a^b g(t) \, df(t) \) exists. If \( f \) is monotone nondecreasing on \([a, b] \), then

\[
\left| \int_a^{a+\lambda} f(t) \, dt - \int_a^b f(t) \, g(t) \, dt \right|
\]

\[
\leq \lambda \left[ f(a + \lambda) - f(a) \right] + (b - a - \lambda) \left[ f(b) - f(a + \lambda) \right],
\]

(2.12)
and

\[
(2.13) \quad \left| \int_a^b f(t) g(t) \, dt - \int_{a-\lambda}^b f(t) \, dt \right| 
\leq \lambda [f(b) - f(b - \lambda)] + (b - a - \lambda) [f(b - \lambda) - f(a)],
\]

where \(\lambda := \int_a^b g(t) \, dt\).

Proof. Taking the modulus in (2.1) and utilizing the triangle inequality, we get

\[
\left| \int_a^{a+\lambda} f(t) \, dt - \int_a^b f(t) g(t) \, dt \right|
\leq \left| \int_a^{a+\lambda} \left( \int_a^x (1 - g(t)) \, dt \right) df(x) \right| + \left| \int_{a+\lambda}^b \left( \int_x^b g(t) \, dt \right) df(x) \right|.
\]

Using the fact that for a monotonic non-decreasing function \(\nu : [a, b] \to \mathbb{R}\) and continuous function \(p : [a, b] \to \mathbb{R}\), one has the inequality

\[
(2.14) \quad \left| \int_a^b p(t) \, d\nu(t) \right| \leq \int_a^b |p(t)| \, d\nu(t).
\]

it follows that

\[
\left| \int_a^{a+\lambda} f(t) \, dt - \int_a^b f(t) g(t) \, dt \right|
\leq \int_a^{a+\lambda} \left| \int_a^x (1 - g(t)) \, dt \right| df(x) + \int_{a+\lambda}^b \left| \int_x^b g(t) \, dt \right| df(x)
\]

\[
= \int_a^{a+\lambda} \left| \int_a^x (1 - g(t)) \, dt \right| df(x) + \int_{a+\lambda}^b \left| \int_x^b g(t) \, dt \right| df(x)
\]

\[
\leq \int_a^{a+\lambda} \left( \int_a^x |1 - g(t)| \, dt \right) df(x) + \int_{a+\lambda}^b \left( \int_x^b |g(t)| \, dt \right) df(x).
\]

But since \(0 \leq g(x) \leq 1\) for all \(x \in [a, b]\), (similarly we have, \(0 \leq 1 - g(x) \leq 1\), then

\[
\left| \int_a^{a+\lambda} f(t) \, dt - \int_a^b f(t) g(t) \, dt \right|
\leq \int_a^{a+\lambda} \left( \int_a^x |1 - g(t)| \, dt \right) df(x) + \int_{a+\lambda}^b \left( \int_x^b |g(t)| \, dt \right) df(x)
\]

\[
(2.15) \quad \leq \int_a^{a+\lambda} (x - a) \, df(x) + \int_{a+\lambda}^b (b - x) \, df(x).
\]

Using Riemann–Stieltjes integral we may observe

\[
\int_a^{a+\lambda} (x - a) \, df(x) = \lambda f(a + \lambda) - \int_a^{a+\lambda} f(x) \, dx,
\]

and

\[
\int_{a+\lambda}^b (b - x) \, df(x) = - (b - a - \lambda) f(a + \lambda) + \int_{a+\lambda}^b f(x) \, dx.
\]
Utilizing the monotonicity of \( f \) on \([a, b]\), we get
\[
\int_{a}^{a+\lambda} f(x) \, dx \geq \lambda f(a), \quad \int_{a}^{b} f(x) \, dx \leq (b - a - \lambda) f(b)
\]
and therefore, by (2.15) we get
\[
\left| \int_{a}^{a+\lambda} f(t) \, dt - \int_{a}^{b} f(t) g(t) \, dt \right| 
\leq \int_{a}^{a+\lambda} (x - a) \, df(x) + \int_{a+\lambda}^{b} (b - x) \, df(x) 
\leq \lambda \left[ f(a + \lambda) - f(a) \right] + (b - a - \lambda) \left[ f(b) - f(a + \lambda) \right],
\]
which proves (2.12). In a similar way and using (2.2) we may deduce the desired inequality (2.13), and we shall omit the details. \( \square \)

**Remark 2.** Let \( f, g \) be as in Theorem 7. If \( \int_{a}^{b} g(t) \, dt = 0 \), then
\[
(2.16) \quad \left| \int_{a}^{b} f(t) g(t) \, dt \right| \leq (b - a) \left[ f(b) - f(a) \right].
\]

2.4. **Inequalities for absolutely continuous integrators.** Another result for absolutely continuous integrators is incorporated in the following theorem:

**Theorem 6.** Let \( f, g : [a, b] \to \mathbb{R} \) be integrable such that \( 0 \leq g(t) \leq 1 \), for all \( t \in [a, b] \) such that \( \int_{a}^{b} g(t) \, df(t) \) exists. If \( f \) is absolutely continuous on \([a, b]\) with \( f' \in L_{p}[a, b] \), \( 1 \leq p \leq \infty \), then we have

\[
(2.17) \quad \left| \int_{a}^{a+\lambda} f(t) \, dt - \int_{a}^{b} f(t) g(t) \, dt \right| 
\leq \begin{cases} 
\frac{1}{2} \left[ \lambda^2 + (b - a - \lambda)^2 \right] \| f' \|_{\infty, [a, b]}, & \text{if } f' \in L_{\infty}[a, b]; \\
\frac{\| f' \|_{p, [a, b]}}{(q+1)^{1/q}} \left[ \lambda^{(q+1)/q} + (b - a - \lambda)^{(q+1)/q} \right], & \text{if } f' \in L_{p}[a, b], p > 1;
\end{cases}
\]

and

\[
(2.18) \quad \left| \int_{a}^{b} f(t) g(t) \, dt - \int_{b-\lambda}^{b} f(t) \, dt \right| 
\leq \begin{cases} 
\frac{1}{2} \left[ \lambda^2 + (b - a - \lambda)^2 \right] \| f' \|_{\infty, [a, b]}, & \text{if } f' \in L_{\infty}[a, b]; \\
\frac{\| f' \|_{p, [a, b]}}{(q+1)^{1/q}} \left[ \lambda^{(q+1)/q} + (b - a - \lambda)^{(q+1)/q} \right], & \text{if } f' \in L_{p}[a, b], p > 1;
\end{cases}
\]

\[
\left[ \int_{a+\lambda}^{b} g(t) \, dt \right] \| f' \|_{1}, & \text{if } f' \in L_{1}[a, b]
\]
Proof. Assume that \( f' \in L_\infty[a, b] \) and utilizing the triangle inequality, we get

\[
\left| \int_a^{a+\lambda} f(t) \, dt - \int_a^b f(t) \, g(t) \, dt \right| \\
\leq \left| \int_a^{a+\lambda} \left( \int_a^x (1 - g(t)) \, dt \right) f'(x) \, dx \right| + \left| \int_a^b \left( \int_a^b g(t) \, dt \right) f'(x) \, dx \right|
\]

\[
\leq \int_a^{a+\lambda} \left| \int_a^x (1 - g(t)) \, dt \right| |f'(x)| \, dx + \int_a^b \left| \int_a^b g(t) \, dt \right| |f'(x)| \, dx
\]

\[
\leq \|f'\|_{\infty,[a,a+\lambda]} \int_a^{a+\lambda} \left( \int_a^x |1 - g(t)| \, dt \right) \, dx + \|f'\|_{\infty,[a+\lambda,b]} \int_{a+\lambda}^b (x - a) \, dx + \|f'\|_{\infty,[a+\lambda,b]} \int_{a+\lambda}^b (b - x) \, dx
\]

\[
\leq \frac{1}{2} \left[ \lambda^2 + (b - a - \lambda)^2 \right] \|f'\|_{\infty,[a,b]},
\]

and the first inequality in (2.17) is proved.

Assume \( f' \in L_p[a, b] \), using Hölder integral inequality for \( p > 1, \frac{1}{p} + \frac{1}{q} = 1 \), we also have

\[
\left| \int_a^{a+\lambda} f(t) \, dt - \int_a^b f(t) \, g(t) \, dt \right| \\
\leq \int_a^{a+\lambda} \left| \int_a^x (1 - g(t)) \, dt \right| |f'(x)| \, dx + \int_a^b \left| \int_a^b g(t) \, dt \right| |f'(x)| \, dx
\]

\[
\leq \left( \int_a^{a+\lambda} \left( \int_a^x |1 - g(t)| \, dt \right)^q \, dx \right)^{1/q} \left( \int_a^{a+\lambda} |f'(x)|^p \, dx \right)^{1/p}
\]

\[
+ \left( \int_{a+\lambda}^b \left( \int_a^b |g(t)| \, dt \right)^q \, dx \right)^{1/q} \left( \int_{a+\lambda}^b |f'(x)|^p \, dx \right)^{1/p}
\]

\[
\leq \|f'\|_{p,[a,a+\lambda]} \left( \int_a^{a+\lambda} (x - a)^q \, dx \right)^{1/q} + \|f'\|_{p,[a+\lambda,b]} \left( \int_{a+\lambda}^b (b - x)^q \, dx \right)^{1/q}
\]

\[
\leq \|f'\|_{p,[a,b]} \left[ \lambda^{(q+1)/q} + (b - a - \lambda)^{(q+1)/q} \right],
\]

giving the second inequality in (2.17).
Finally, we also observe that
\[
\left| \int_a^{a+\lambda} f(t) \, dt - \int_a^b f(t) \, g(t) \, dt \right| \\
\leq \sup_{x \in [a,a+\lambda]} \left[ \int_a^x (1 - g(t)) \, dt \right] \cdot \int_a^{a+\lambda} |f'(x)| \, dx + \sup_{x \in [a+\lambda,b]} \left[ \int_x^b g(t) \, dt \right] \cdot \int_{a+\lambda}^b |f'(x)| \, dx \\
= \int_a^{a+\lambda} (1 - g(t)) \, dt \cdot \int_a^{a+\lambda} |f'(x)| \, dx + \int_{a+\lambda}^b g(t) \, dt \cdot \int_{a+\lambda}^b |f'(x)| \, dx \\
= \left[ \int_{a+\lambda}^b g(t) \, dt \right] \|f'\|_1,
\]
which proves the last inequality in (2.17). The inequalities in (2.18) may be proved in the same way using the identity (2.2), we shall omit the details. □

**Remark 3.** One may deduce new inequalities of Hayashi’s type by using the substituting \(g(t)/A\) for \(g(t)\), where \(A\) is nonzero positive constant, and then using the identities
\[
(2.19) \quad \int_a^{a+\lambda} f(t) \, dt - \frac{1}{A} \int_a^b f(t) \, g(t) \, dt \\
= -\frac{1}{A} \int_a^{a+\lambda} \left( \int_t^x (A - g(t)) \, dt \right) \, df(x) - \frac{1}{A} \int_{a+\lambda}^b \left( \int_x^b g(t) \, dt \right) \, df(x),
\]
and
\[
(2.20) \quad \frac{1}{A} \int_a^b f(t) \, g(t) \, dt - \int_{b-\lambda}^b f(t) \, dt \\
= -\frac{1}{A} \int_a^{b-\lambda} \left( \int_t^x g(t) \, dt \right) \, df(x) - \frac{1}{A} \int_{b-\lambda}^b \left( \int_x^b (A - g(x)) \, dt \right) \, df(x),
\]
where \(\lambda := \frac{1}{A} \int_a^b g(t) \, dt\).

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