Beyond the Relativistic Point Particle: A Reciprocally Invariant System and its Generalisation

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Abstract

We investigate a reciprocally invariant system proposed by Low and Govaerts et al., whose action contains both the orthogonal and the symplectic forms and is invariant under global $O(2,4) \cap Sp(2,4)$ transformations. We find that the general solution to the classical equations of motion has no linear term in the evolution parameter, $\tau$, but only the oscillatory terms, and therefore cannot represent a particle propagating in spacetime. As a remedy, we consider a generalisation of the action by adopting a procedure similar to that of Bars et al., who introduced the concept of a $\tau$ derivative that is covariant under local $Sp(2)$ transformations between the phase space variables $x^\mu(\tau)$ and $p^\mu(\tau)$. This system, in particular, is similar to a rigid particle whose action contains the extrinsic curvature of the world line, which turns out to be helical in spacetime. Another possible generalisation is the introduction of a symplectic potential proposed by Montesinos. We show how the latter approach is related to Kaluza-Klein theories and to the concept of Clifford space, a manifold whose tangent space at any point is Clifford algebra $Cl(8)$, a promising framework for the unification of particles and forces.

Key words: Born’s reciprocity, phase space, two times, symplectic potential, Kaluza-Klein theory, Clifford space
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1 Introduction

Born’s reciprocity [1] in phase space has undergone a revival in studies of reciprocal relativity [2]. The latter theory was recently formulated in terms of an action [3] that contains a term of the form $\dot{x}^\mu \dot{x}_\mu + \dot{p}^\mu \dot{p}_\mu$ and also a term of the form $p^\mu \dot{x}_\mu - \dot{p}_\mu x^\mu$. The dot denotes the derivative with respect to the parameter $\tau$, which increases monotonically along a particles world line. That action is invariant under global transformations of the group $O(2,6) \cap Sp(2,6) \sim U(1,3)$. We examine the equations of motion for such a system and find that they do not describe a propagating particle in spacetime. This is due to the fact
that the general solution is a closed time-like or space-like loop in spacetime, and thus represents a sort of a classical instanton.

On the other hand, another research direction has focused on generalising the phase space action for a relativistic point particle \[4\]. The system so obtained, besides being Lorentz-invariant, is invariant under local Sp(2) transformations that rotate \(x^\mu\) and \(p^\mu\) into each other, and also under reparametrisations of the world line parameter \(\tau\).

In this paper, we unify the action of Govaerts et al. \[3\] and that of Bars et al. \[4\] in such a way that they become special cases of a single action. We study the corresponding equations of motion and their general solutions. They turn out to describe, in particular, a variant of the so called rigid particle \[5\], i.e., a particle whose action contains the world line’s extrinsic curvature.

Finally, we employ the concept of symplectic potential \[6\] and show that the resulting generalised action is related on one hand to Kaluza-Klein theories, and on the other hand to Clifford algebras and Clifford spaces \[7\] \[8\] \[9\], which have turned out to be promising for the unification of elementary particles and interactions \[9\] \[10\] \[11\].

2 A phase space action that contains the orthogonal and the symplectic form

Let us first consider the action proposed by Govaerts et al. \[3\]:

\[
I = \int d\tau \left[ \dot{x}^\mu \dot{x}_\mu + \dot{p}^\mu \dot{p}_\mu + (\dot{\theta} - \frac{1}{2} \dot{x}^\mu p_\mu + \frac{1}{2} x^\mu \dot{p}_\mu)^2 \right]^{1/2},
\]

(1)

where, besides the variables \(x^\mu(\tau), \mu = 0, 1, 2, 3\), denoting the position of the particle in spacetime, there are also additional variables, \(p^\mu\) and \(\theta\). In order to elucidate their roles, let us consider the equations of motion derived from the action (1):

\[
\delta x^\mu : \frac{d}{d\tau} \left( \frac{x^\mu - \frac{1}{2} \mu p^\mu}{\sqrt{Q}} \right) - \frac{\mu \dot{p}^\mu}{2 \sqrt{Q}} = 0,
\]

(2)

\[
\delta p^\mu : \frac{d}{d\tau} \left( \frac{\dot{p}^\mu + \frac{1}{2} \mu x^\mu}{\sqrt{Q}} \right) + \frac{\mu \dot{x}^\mu}{2 \sqrt{Q}} = 0,
\]

(3)

\[
\delta \theta : \frac{d}{d\tau} \left( \frac{\mu}{\sqrt{Q}} \right) = 0,
\]

(4)

where \(\mu \equiv \dot{\theta} - \frac{1}{2} \dot{x}^\mu p_\mu + \frac{1}{2} x^\mu \dot{p}_\mu\) and \(\sqrt{Q} \equiv (\dot{x}^2 + \dot{p}^2 + \mu^2)^{1/2}\). From Eqs. (2)–(4) we have

\[
\frac{d}{d\tau} \left( \frac{\dot{x}^\mu}{\sqrt{Q}} \right) = C \dot{p}^\mu, \tag{5}
\]

\[
\frac{d}{d\tau} \left( \frac{\dot{p}^\mu}{\sqrt{Q}} \right) = -C \dot{x}^\mu, \tag{6}
\]

\[1\]

We use units in which all the constants in front of the terms are equal to 1. Indices are raised and lowered by the Minkowsky metric, \(\eta_{\mu\nu}\).
where we have taken into account Eq. (4), which implies
\( \frac{\mu}{\sqrt{Q}} = C. \)  

(7)

From the first order Eqs. (5) and (6), for variables \( x^\mu \) and \( p^\mu \) we obtain the second order equations for variables \( x^\mu \):
\[ \frac{1}{\sqrt{Q}} \frac{d}{d\tau} \left( \frac{1}{\sqrt{Q}} \frac{d}{d\tau} \left( \frac{\dot{x}^\mu}{\sqrt{Q}} \right) \right) + C^2 \frac{\ddot{x}^\mu}{\sqrt{Q}} = 0. \]

(8)

If we choose a parametrisation in which \( \sqrt{Q} = 1 \), then the equations of motion (8) become:
\[ \ddot{x}^\mu + C^2 \dot{x}^\mu = 0, \]

which after the integration gives
\[ \ddot{x}^\mu + C^2 x^\mu = a^\mu. \]

(10)

This is an ordinary equation for oscillatory motion whose general solution is
\[ x^\mu = A^\mu \cos \omega \tau + B^\mu \sin \omega \tau + \frac{a^\mu}{\omega^2}, \]

with \( \omega^2 \equiv C^2. \)

A general solution is thus a closed curve (loop) in spacetime or a sort of instanton. There is no term associated with translational motion.

Let us now consider Eq. (7), which in the gauge \( \sqrt{Q} = 1 \) reads
\[ \dot{\theta} - \frac{1}{2} (p_\mu \dot{x}^\mu - x_\mu \dot{p}^\mu) = C. \]

(12)

This gives
\[ \theta = C \tau + \int d\tau \frac{1}{2} (p_\mu \dot{x}^\mu - x_\mu \dot{p}^\mu) = C \tau + \frac{1}{2} \int (p_\mu dx^\mu - x_\mu dp^\mu), \]

where the integral in the last term runs over the loop in the space of \( \{ x^\mu, p^\mu \} \). We see that variable \( \theta \) is related to the phase, and is equal to the phase if the integration constant, \( C \), is zero.

By inspecting Eq. (4) we find that \( \dot{p}^\mu \) is acceleration. In the gauge \( \sqrt{Q} = 1 \), we have \( \dot{p}^\mu = C^{-1} \ddot{x}^\mu \). However, Eqs. (2) and (3) tell us that \( p_\mu \) is not the conjugate momentum to \( x^\mu \). The conjugate momentum is
\[ \Pi^\mu_x = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{\dot{x}^\mu}{\sqrt{Q}} - \frac{\mu}{2 \sqrt{Q}} p_\mu, \]

(14)

whereas the integration of (5) gives
\[ \frac{\dot{x}^\mu}{\sqrt{Q}} = \frac{\mu}{\sqrt{Q}} p_\mu + C_1. \]

(15)
Thus we have the following relation between the canonical momentum, \( \Pi_\mu \), and the variables, \( p_\mu \):

\[
\Pi_\mu = \frac{1}{2} \frac{\mu}{\sqrt{q}} p_\mu + C_1.
\]

Equation (15) says that \( p_\mu \) up to an integration constant \( C_1 \) equal to the velocity in the subspace spanned by \( \{x_\mu\} \). Analogous relationships hold for the momentum \( \Pi_\mu = \partial L / \partial \dot{p}_\mu \), which is conjugate to \( p_\mu \).

### 3 Including local Sp(2) transformations

The solution of Eq. (11) does not represent a physical propagation of a relativistic particle in spacetime \( \{x_\mu\} \) since a term of the form \( v^\mu \tau \) is missing. In order to include such a term, we must generalise the action in Eq. (1) using the procedure found in ref. [4].

Let us first rewrite the action in Eq. (1) into a more compact notation:

\[
I = \int \mathrm{d}\tau \left[ \dot{z}^a G_{ab} \dot{z}^b + \left( \dot{\theta} - \frac{1}{2} \dot{z}^a J_{ab} \dot{z}^b \right)^2 \right]^{1/2},
\]

where \( z^a \equiv (x^\mu, p^\mu) \), \( \mu = 0, 1, 2, 3 \), and

\[
G_{ab} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & \eta_{\mu\nu} \end{pmatrix} \quad \text{and} \quad J_{ab} = \begin{pmatrix} 0 & \eta_{\mu\nu} \\ -\eta_{\mu\nu} & 0 \end{pmatrix}
\]

are the orthogonal and the symplectic metrics, respectively.

The action is invariant under global transformations,

\[
z \rightarrow z' = \Gamma z,
\]

that preserve both the orthogonal and the symplectic form:

\[
\dot{z}' G' z' = \tilde{z} G z \quad \text{(20)}
\]
\[
\dot{z}' J' z' = \tilde{z} J \tilde{z}. \quad \text{(21)}
\]

This implies:

\[
\tilde{\Gamma} G \Gamma = G \quad \text{(22)}
\]
\[
\tilde{\Gamma} J \Gamma = J, \quad \text{(23)}
\]

where the tilde denotes the transpose. The two conditions (22) and (23) are satisfied by 8 \times 8 matrices, \( \Gamma \in \mathrm{O}(2, 6) \cap \mathrm{Sp}(2, 6) \):

\[
\Gamma = \begin{pmatrix} \Lambda & -M \\ M & \Lambda \end{pmatrix}
\]

\[\text{(24)}\]

\[\footnote{Evolution of the form (11) could correspond to a motion of relativistic particle in AdS space [12]. In that case translations have another form. For the discussion of the nonrelativistic limit of such a system see [13].}\]
such that
\[ \tilde{\Lambda} \eta \Lambda + \tilde{M} \eta M = \eta \] (25)
\[ \tilde{\Lambda} \eta M - \tilde{M} \eta \Lambda = \eta. \] (26)

Matrices \( \Gamma \) can be written in the form:
\[ \Gamma = 1 \otimes \Lambda + i \otimes M, \] (27)
where
\[ 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \] (28)

Here, \( i \), which satisfies \( i^2 = -1 \), is a representation of the imaginary unit, \( i \).

More abstractly, \( \Gamma \) are thus complex \( 4 \times 4 \) matrices such that
\[ \Gamma = \Lambda + iM \in U(1, 3), \] (29)
satisfying
\[ \Gamma^\dagger \eta \Gamma = (\tilde{\Lambda} + i\tilde{M})\eta(\Lambda + iM) = \eta, \] (30)
which implies Eqs. (25) and (26).

The orthogonal and the symplectic metrics have the form
\[ G = 1 \otimes \eta, \quad \text{i.e.,} \quad G_{ab} \equiv G_{(i\mu)(j\nu)} = \delta_{ij} \eta_{\mu\nu}, \] (31)
\[ J = \epsilon \otimes \eta, \quad \text{i.e.,} \quad J_{ab} \equiv J_{(i\mu)(j\nu)} = \epsilon_{ij} \eta_{\mu\nu}, \] (32)
where \( i, j = 1, 2. \)

With \( z^a \equiv z^{i\mu} \equiv X^{i\mu} = (X^{i\mu}, p^{i\mu}) \) the action (17) can be rewritten as
\[ I = \int d\tau \left[ \dot{X}^{i\mu} \delta_{ij} \eta_{\mu\nu} \dot{X}^{j\nu} + (\dot{\theta} - \frac{1}{2} \dot{X}^{i\nu} \epsilon_{ij} \eta_{\mu\nu} X^{j\nu})^2 \right]^{1/2}. \] (33)

According to ref. [4] we will consider \( (X^{i\mu}, X^{2\mu}) \equiv (x^{i\mu}, p^{i\mu}) \) as a doublet under local continuous Sp(2) duality symmetry:
\[ X'^{i\mu} = S^i_k(\tau) X^{k\mu}(\tau), \quad \text{shortly} \quad X'^{i\mu} = S X^{i\mu}. \] (34)

The \( \tau \)-derivative transforms as \( \dot{X}'^{i\mu} = S^i_k(\tau) \dot{X}^{k\mu} + \dot{S}^i_k \dot{X}^{k\mu} \). This can be converted into the covariant \( \tau \)-derivative by introducing the Sp(2) gauge fields:
\[ \frac{DX^{i\mu}}{D\tau} \equiv \dot{X} = \dot{X}^{i\mu} - A_{i\mu}^k X^{k\mu}, \] (35)
that transform according to \( A' = SAS^{-1} + \dot{S}S^{-1} \) so that \( \dot{X}'^{i\mu}(\tau) = S^i_k(\tau) \dot{X}^{k\mu} \).
Under local Sp(2) we have
\[ S^i_k \delta_{ij} \rightarrow S^i_k \delta_{ij} S^j_i, \]  
\[ S^i_k \delta_{ij} S^j_i \neq \delta_{ij}, \]  
we have to replace \( \delta_{ij} \) with a general symmetric metric, \( g_{ij} \), that along the particle’s world line depends on \( \tau \) and transforms according to \( g_{ij} \rightarrow g_{ij}' = S^k_i g_{kj} S^j_i \).

Let us therefore consider the following action:
\[ I = \int \frac{1}{2} \left( X^{\mu} g_{ij} \eta_{\mu\nu} X^{\nu} + \frac{1}{2} \mu X^{\mu} \epsilon_{ij} \eta_{\mu\nu} X^{\nu} \right)^{1/2}, \]
which is invariant under global transformations \( \Gamma \in O(2,6) \cap \text{Sp}(2,6) \), and under local Sp(2) duality transformations that interchange \( x^\mu \) and \( p^\mu \). We now include a constant \( M \) with the dimension of mass.

The general metric, \( G_{ab} \), in the space of \( z^a = X^{i\mu} \) is not “flat,” and one should add a kinetic term for \( G_{ab} \) to the action. By taking a particular solution that gives a certain \( G_{ab}(z) \), we can consider it to be a background metric in which our (test) particle moves. Once \( G_{ab}(z) \) is fixed in such a way, we no longer vary it. Let us now assume that there exists a domain in the space of \( z^a \) in which the metric can be approximated as the product \( G_{ab} = g_{ij} \eta_{\mu\nu} \), where \( i, j = 1, 2 \) and \( \mu, \nu = 0, 1, 2, 3 \). Thus, we arrive at the action (38) for a particle moving in such a background metric. For simplicity reasons we will consider the case in which \( g_{ij} = \delta_{ij} \). The latter equality, of course, no longer holds if we perform a (local) Sp(2) transformation.

Let us now consider the gauge fields, \( A^{ik}(\tau) \) considered in ref. [4]. Indices of those fields are lowered and raised by the Sp(2) invariant metric, \( \epsilon_{ij} \), and its inverse, \( \epsilon^r s \). So we have \( A^{n k} = A_{mk} \epsilon^{mn} \), where \( A_{mk} = A_{km} \). The three independent fields, \( A_{mk} \), are associated with the three independent Sp(2) transformations. The choice of \( A_{mk} \) corresponds to the choice of the local gauge. Variation of the action (38) with respect to \( A_{rs} \) gives
\[ \frac{1}{2} \mu X^{\mu} \epsilon_{ij} \eta_{\mu\nu} X^{\nu} = 0, \]
where
\[ \mu \equiv \dot{\theta} - \frac{1}{2} \epsilon_{ij} X^{\mu} \epsilon_{ij} \eta_{\mu\nu} X^{\nu} \]
and \( \epsilon^r k = \epsilon^{rm} g_{mk} = \epsilon^{rm} \delta_{mk} \), since we take \( g_{mk} = \delta_{mk} \).

The constraints in Eq. (39) are due to the local Sp(2) symmetry of our action (38) and they differ from the constraints obtained in refs. [4] because our action (38) is different. However, equations of motion imply a particular case which approaches that of ref. [4]. Namely, if \( \dot{X}^{ij} = \dot{X}^{ij} - A^{ik} X^{k\mu} = 0 \), then eq. (39) reduces to the constraint of ref. [4].
Variation of (38) with respect to $X^{\ell\rho}$ gives

$$\frac{d}{d\tau} \left( \frac{g_{ik}\eta_{\rho\nu}}{\sqrt{Q}} \dot{X}^{k\nu} \right) + \frac{1}{\sqrt{Q}} A^{i}_{\ell} g_{ik} \eta_{\rho\nu} \dot{X}^{k\nu} + \frac{\mu}{\sqrt{Q}} \dot{X}^{i\nu} \epsilon_{\ell\rho\nu} = 0, \quad (41)$$

where

$$Q \equiv \dot{X}^{i\mu} g_{ij} \eta_{\mu\nu} \dot{X}^{j\nu} + (\dot{\theta} - \frac{1}{2} \dot{X}^{i\mu} \epsilon_{ij} \eta_{\mu\nu} X^{j\nu})^2, \quad (42)$$

whereas variation with respect to $\theta$ gives Eq. (4), in which $Q$ and $\mu$ are now defined according to (42) and (40), respectively.

Multiplying (41) by $\eta^{\rho\sigma} g^{\ell j}$, and taking into account that $A^{i}_{\ell k} = B^{k\ell}$, $B^{k\ell} g^{\ell j} = A^{j}_{k}$, $A^{j}_{k} g^{\ell j} = -A^{i}_{k}$, we have:

$$\frac{d}{d\tau} \left( \frac{\dot{X}^{j\sigma}}{\sqrt{Q}} \right) - A^{j}_{i k} \dot{X}^{k\sigma} \frac{1}{\sqrt{Q}} + \frac{\mu}{\sqrt{Q}} \dot{X}^{i\sigma} \epsilon_{i j} = 0. \quad (43)$$

In the first two terms we recognise the covariant derivative that is analogous to (35), and, therefore, Eq. (43) can be condensed to

$$\frac{D}{D\tau} \left( \frac{\dot{X}^{j\sigma}}{\sqrt{Q}} \right) + \frac{\mu}{\sqrt{Q}} \dot{X}^{i\sigma} \epsilon_{i j} = 0. \quad (44)$$

In the limit $\mu/\sqrt{Q} \gg 1$, Eq. (44) becomes $\dot{X}^{i\sigma} \epsilon_{i j} = 0$, which are the equations of motion considered in ref. [4].

As an example, let us now choose a gauge in which $A^{1}_{1} = -A^{1}_{2} = -A^{2}_{2} = 0$, $A^{2}_{1} = A_{11} = 0$, whereas $A^{1}_{2} = -A^{2}_{2} \neq 0$, and does not change with $\tau$. Then the equations of motion become

$$\frac{d}{d\tau} \left( \frac{\dot{X}^{1\sigma}}{\sqrt{Q}} \right) + (A^{2}_{2} - \mu) \dot{X}^{2\sigma} \frac{1}{\sqrt{Q}} = 0, \quad (45)$$

$$\frac{d}{d\tau} \left( \frac{\dot{X}^{2\sigma}}{\sqrt{Q}} \right) + \mu \dot{X}^{1\sigma} \frac{1}{\sqrt{Q}} = 0. \quad (46)$$

Using now the notation $X^{1\sigma} \equiv x^{\mu}$, $X^{2\sigma} \equiv p^{\sigma}$, and the relations $\dot{X}^{1\sigma} = \dot{X}^{1\sigma} + A^{2}_{2} X^{2\sigma}$ and $X = \dot{X}^{2\sigma}$, that hold in the above gauge, we have

$$\frac{d}{d\tau} \left( \frac{\dot{x}^{\sigma} + A^{2}_{2} p^{\sigma}}{\sqrt{Q}} \right) + \frac{1}{\sqrt{Q}} (A^{2}_{2} - \mu) \dot{p}^{\sigma} = 0, \quad (47)$$

\footnote{The same symbol $A$ has already been reserved if indices are lowered and raised with $\epsilon_{ij}$ and $\epsilon^{ij}$, respectively, therefore we now use a different symbol.}
\[
\frac{d}{d\tau} \left( \frac{\dot{\mu}}{\sqrt{Q}} \right) + \frac{\mu}{\sqrt{Q}} (\dot{\sigma} + A_{22} \mu) = 0. \tag{48}
\]

Recall that due to the equation of motion (4), \( \mu/\sqrt{Q} = C \), where \( C \) is an integration constant.

Let us choose a reparametrisation of \( \tau \) such that \( \sqrt{Q} = 1 \), so that \( \mu = C \). Equations (47) and (48) then read

\[
\ddot{x}^\mu + (2A_{22} - C) \dot{p}^\mu = 0, \tag{49}
\]

\[
\ddot{p}^\mu + C \dot{x}^\mu + CA_{22} \dot{p}^\mu = 0. \tag{50}
\]

Differentiating (50) and inserting

\[
\dot{p}^\mu = \frac{-\ddot{x}^\mu}{2A_{22} - C} \tag{51}
\]

we obtain

\[
\dddot{x}^\mu + \omega^2 \ddot{x}^\mu = 0, \tag{52}
\]

where \( \omega^2 = C(C - A_{22}) \). A general solution is

\[
x^\mu = -\frac{a^\mu}{\omega^2} \cos \omega \tau - \frac{b^\mu}{\omega^2} \sin \omega \tau + \nu^\mu \tau + x_0^\mu \tag{53}
\]

Integrating (51) gives

\[
p^\mu = \frac{-\ddot{x}^\mu}{2A_{22} - C} + k^\mu. \tag{54}
\]

If we insert this into the second equation of motion (50), then after some straightforward computation we obtain the following relation between \( A_{22} \) and the integration constants \( \nu^\mu, k^\mu, \) and \( C \):

\[
\nu^\mu = -CA_{22} k^\mu. \tag{55}
\]

If \( A_{22} = 0 \), then the 4-velocity \( \nu^\mu \) vanishes and we arrive at the case described in Section 2. Non vanishing \( A_{22} \) is thus necessary in order to have a propagating particle in 4-dimensional spacetime.

If we insert the solution (53), (54) into the constraints (32), we obtain the following conditions on the integration constants

\[
a^\mu a_\mu = b^\mu b_\mu = \nu^\mu \nu_\mu = a^\mu b_\mu = a^\mu v_\mu = b^\mu v_\mu = 0 \tag{56}
\]

This can be satisfied if the spacetime has at least two time-like dimensions.

We have arrived at the solution (53), (54) by choosing \( Q = X_i^\mu \cdot g_{ij} \eta_{\mu\nu} \dot{X}^j + \mu^2 = 1 \) (choice of gauge). But, using (56), we find that the first term in the latter equation vanishes. Therefore, the integration constant \( C = \mu/\sqrt{Q} = \mu \) cannot be arbitrary, but
must be equal to 1. This should not depend on choice of gauge. Indeed, if we choose an arbitrary gauge $Q = i^{ij} g_{ij\mu\nu} \dot{X}^{i\mu} \dot{X}^{j\nu} + \mu^2 = f(X^{i\mu})$, we still have $C = 1$.

A special treatment is necessary, if the quadratic form $Q$, entering the action (38), is light-light, $Q = 0$. Then the solution still has the form (53), (54), together with the condition (56), but with the vanishing integration constant $C = 0$.

A general solution, Eq. (53), is a helix. If the integration constants $a^{\mu}$, $b^{\mu}$ vanish, then the oscillating term in (53) disappears, so that $x^{\mu} = v^{\mu} \tau + x^{\mu}_0$, which is a rectilinear motion of the ordinary relativistic particle. This can also be seen by putting $\dot{X}^{i\mu} = 0$ into the equations of motion, (47) and (48), which then reduce to the phase space equations for a relativistic point particle:

$$\dot{p}^{\mu} = 0 \quad \text{and} \quad \dot{x}^{\mu} + A_{22} p^{\mu} = 0$$

Here the index $\mu$ runs not only over the four dimensions of spacetime with signature $(+ - - -)$, but also over one extra time like dimension, and—following ref. [4]—also over one extra space like dimension. Dimensions and signatures occurring in eqs. (17)–(32) should then be modified accordingly. Ghosts can be eliminated by using the Sp(2) gauge symmetry as in ref. [4], which brings us then to an effective system with one time. In ref. [4] it is shown how various gauge choices lead to various 1-time systems with different physical interpretation. It would be interesting to explore this for the case of the generalised action (38) – a project which is beyond the scope of the present letter.

The precursors of 2-time physics for a single particle were models describing two interacting particles (or a particle and string) [14]. Instead of interpreting coordinates $X^{1\mu}$, $X^{2\mu}$ in eq. (38) as describing a single particle in higher dimensions, we could alternatively interpret them as describing two interacting particles in lower dimensions. According to such interpretation, our model would be a variant of the class of models discussed in refs. [14].

The theory considered here goes beyond the relativistic point particle. The fourth order equations of motion (52) can be derived from the second order action:

$$I[x^{\mu}, \dot{x}^{\mu}] = \int d\tau (\ddot{x}^{\mu} \ddot{x}_{\mu} - \omega^2 \dot{x}^{\mu} \dot{x}_{\mu}),$$

which is a functional of variables $x^{\mu}(\tau)$ and velocities $\dot{x}^{\mu}(\tau)$. The second order derivative term in (58) is a gauge fixed 	extit{extrinsic curvature} of the world line. Thus, we have a variant of the “rigid particle” [5], and the variables $(x^{\mu}(\tau), \dot{x}^{\mu}(\tau))$ correspond to the variables $(x^{\mu}(\tau), \dot{p}^{\mu}(\tau))$ of the original action (38). The latter action leads to the relation (54), which says that $p^{\mu}$ are proportional to $\dot{x}^{\mu}$ apart from the integration constants, $k^{\mu}$. Fourth order differential equations of the form similar to eq. (52) appear in the model of the massless particle with rigidity (curvature) [15].
4 Further considerations

4.1 Symplectic potential

Let us now again consider the action (17) and rewrite it in terms of the symplectic potential, \( \theta_a(z) \), as is suggested in ref. [6]:

\[
I = M \int d\tau \left[ \dot{z}^a G_{ab} \dot{z}^b + \left( \dot{\theta} - \theta_a \dot{z}^a \right)^2 \right]^{1/2}.
\] (59)

The symplectic 2-form is defined according to

\[
\omega_{ab} = \partial_a \theta_b - \partial_b \theta_a.
\] (60)

In particular, if

\[
\theta_a = \frac{1}{2} J_{ab} z^b = \frac{1}{2} \left( p_\mu, -x_\mu \right)
\] (61)

then \( \omega_{ab} = J_{ab} \). In general, \( \theta_a \) need not be a field determined by Eq. (61), but can be an arbitrary background field. Introduction of the vector field \( \theta_a \) renders the action (59) invariant under general coordinate transformations, \( z^a \rightarrow z'^a = z'^a(z) \).

The canonical variables are \((z^a, \theta)\) and the conjugate momenta are

\[
\Pi^a = \frac{\partial L}{\partial \dot{z}^a} = M \frac{G_{ac} \dot{z}^c}{\sqrt{Q}} - \Pi^\theta \theta_a,
\] (62)

\[
\Pi^\theta = \frac{\partial L}{\partial \theta} = M \frac{\dot{\theta} - \theta_a \dot{z}^a}{\sqrt{Q}},
\] (63)

where \( \sqrt{Q} \equiv \sqrt{\dot{z}^a G_{ab} \dot{z}^b + (\dot{\theta} - \theta_a z^a)^2} \).

From the identity

\[
\frac{1}{(\sqrt{Q})^2} \left[ \dot{z}^a G_{ab} \dot{z}^b + (\dot{\theta} - \theta_a z^a)^2 \right] - 1 = 0
\] (64)

and using (62) and (63), we find the following constraint amongst the momenta:

\[
\Phi = \left( \Pi^a + \Pi^\theta \theta_a \right) G^{ab} \left( \Pi^b + \Pi^\theta \theta_b \right) + (\Pi^\theta)^2 - M^2 = 0.
\] (65)

Upon quantisation this becomes a Klein-Gordon like equation:

\[
\Phi \Psi = 0,
\] (66)

where momenta are replaced by the operators: \( \Pi^a \rightarrow \hat{\Pi}^a = -i\partial/\partial z^a \) and \( \Pi^\theta \rightarrow \hat{\Pi}^\theta = -i\partial/\partial \theta \).

Taking the square root, we obtain a generalised Dirac equation:

\[
\left[ \gamma^a (\hat{\Pi}^a + \hat{\Pi}^b \theta_a) + \gamma^\theta \hat{\Pi}^\theta + \gamma_M M \right] \Psi = 0.
\] (67)
Here $\gamma^a$ are Clifford numbers, satisfying $\gamma^a \cdot \gamma^b \equiv \frac{1}{2}(\gamma^a \gamma^b + \gamma^b \gamma^a) = G^{ab}$, whereas $\gamma_{\Pi^a}$ and $\gamma_M$ are Clifford numbers that satisfy $\gamma_{\Pi^a}^2 = 1$, $\gamma_M^2 = 1$, $\gamma_{\Pi^a} \cdot \gamma_M = 0$, $\gamma^a \cdot \gamma_{\Pi^a} = 0$, and $\gamma^a \cdot \gamma_M = 0$.

Solutions to the Klein-Gordon like equation (66), for a particular $\theta_a$ given in (61), were considered by Govaerts et al. [3]. They obtained a continuous mass spectrum of bosonic states. By tuning the value of a constant parameter in the action, namely the cosmological constant term $M^2$, it is possible to project out negative norm states, whilst tachyonic states cannot be avoided. It is beyond the scope of the present letter to go into an investigation of the spectrum of fermionic states satisfying the generalised Dirac equation (67).

4.2 Maxwell-like equations for the symplectic potential

A next possible step is to include a kinetic term for the field $\theta_a$. The total classical action is then

$$I[Z^a, \theta^a] = M \int d\tau \left[ \dot{Z}^a G_{ab} \dot{Z}^b + (\dot{\theta} - \theta_a \dot{Z}^a)^2 \right]^{1/2} + \kappa \int d^8z \omega_{ab} \omega^{ab}.$$  (68)

The equations of motion for variables $Z^a$ are just like the Lorentz force law:

$$\frac{1}{\sqrt{Q}} \frac{d}{d\tau} \left( \frac{G_{cb} \dot{Z}^b}{\sqrt{Q}} \right) + \frac{\Pi^a}{M} \omega_{cb} \dot{Z}^b = 0.$$  (69)

If we vary (68) with respect to $\theta_c$ we obtain:

$$\int d\tau \Pi^a \frac{\dot{Z}^c}{\sqrt{Q}} \delta^8(z - Z(\tau)) = \kappa \partial_a \omega^a_c.$$  (70)

These are just like Maxwell equations with a point particle source, where $\theta_a(z)$ corresponds to the electromagnetic potential, $\omega^a_c$ corresponds to the field strength, and $\Pi^a$ corresponds to the electric charge. The difference is that we now have an 8-dimensional space with coordinates $z^a$.

Eight dimensional action (68) and the equations of motion, (69) and (70), contain the ordinary 4-dimensional Maxwell equations as a particular case if $\dot{z}^a = (\dot{x}^\mu, 0)$, i.e., if $\dot{p}^\mu = 0$.

Another particular case is a possible solution to (69) and (70) such that $\theta_a(z)$ is given by Eq. (61). This is expected to be the case if the variables $z^a(\tau)$ satisfy Eqs. (5)–(7), whose

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4. Our $M^2$ corresponds to $\Lambda$ of ref. [3].

5. Tachyons occur already at the classical level since the world lines are loops in spacetime (see solution (11)).

6. We now use a capital symbol in order to distinguish the variables $Z^a(\tau)$, which describe the motion of a particle, from the coordinates $z^a$ of the embedding space.

7. A modification of the spacetime dimensionality and signature is necessary, if we consider the action (68) with the local Sp(2) gauge symmetry. The action (68) does not have such symmetry.
solution (11) is a helix in the 5-dimensional space \( \{x^\mu, \theta\} \). Since \( p^\mu(\tau) \) also satisfies an equation analogous to (13), we have in fact a helix in 9-dimensional space \( \{x^\mu, p^\mu, \theta\} \). Such helical motion of a point particle can in turn be considered as a “source” for the symplectic potential (61), considered as a solution to Eq. (70). And vice versa: in the presence of the potential (61), the particle’s world line is a helix. However, it is important to check whether such a self-consistent helical solution of Eqs. (69), (70) for a single particle indeed takes place. In other words, it remains to be explored whether the symplectic potential (61) can only be due to the presence of other sources, excluding our “test particle” moving in such a background, or perhaps it can also be due to a self-consistent helical motion of a single particle.

However, if we take another possible interpretation, discussed at the end of Sec. 3, namely that the action (68) describes two interacting particles with coordinates \( X^{1\mu}, X^{2\mu} \), which altogether form eight coordinates \( X^{i\mu} \equiv Z^a, i = 1, 2, \mu = 0, 1, 2, 3 \), then it is clear that the trajectories are not straight lines.

### 4.3 Invoking Kaluza-Klein theory

The form

\[
Q = \dot{Z}^a G_{ab} \dot{Z}^b + (\dot{\theta} - \theta \dot{Z}^a)^2
\]

in the action (68) is known in Kaluza-Klein theories. Instead of four dimensions, \( x^\mu \) plus an extra dimension, \( x^5 \), we now have eight dimensions, \( z^a \) plus an extra dimension, \( \theta \). The potential, \( \theta_a \), is proportional to a mixed component of the metric tensor in 9-dimensions, analogous to \( g_{\mu 5} \). Thus, \( Q \) is just a 9-dimensional “quadratic” form:

\[
Q = \dot{Z}^A q_{AB} \dot{Z}^B,
\]

where \( Z^A = (Z^a, \theta) \) and \( q_{AB} \) is a generic metric in 9-dimensions. We arrive at the form (71) by taking the well known Kaluza-Klein ansatz [18] for metric \( q_{AB} \).

We see that the point-particle action, based on the simple quadratic form \( Q \) in 9-dimensions, contains the action (11) of Govaerts et al. [3] as a particular case.

### 4.4 Invoking Clifford space

Instead of considering \( \theta \) as an extra dimension, we can consider it as a scalar coordinate of the Clifford space. The concept of the Clifford space, \( C \), has been investigated in refs. [7, 8] (See also [17]). This is a manifold whose tangent space at any point of \( C \) is Clifford algebra. Now, instead of starting from the 4-dimensional Minkowski space, we can start from the 8-dimensional space whose points are described by coordinates \( z^a = (x^\mu, p^\mu) \), and build

\footnote{It is a loop in Minkowski space \( \{x^\mu\} \), but there is another dimension corresponding to \( \theta(\tau) \) that satisfies Eq. (13).}
up the corresponding 256-dimensional Clifford space \(2^8 = 256\). The basis of its tangent space is given by
\[
\gamma^A = \{1, \gamma^a, \gamma^{a1} \wedge \gamma^{a2}, ..., \gamma^{a1} \wedge \gamma^{a2} \wedge ... \wedge \gamma^{a8}\}. \tag{73}
\]
While \(z^a\) are coordinates associated with vectors \(\gamma^a\), the extra quantity, \(\theta\), can be considered as a coordinate associated with the scalar unit \(\mathbb{1}\). We now take \(z^A\) as coordinates, \(q_{AB}\) as the metric tensor, and \(Q\) of Eq. (72) as the quadratic form of the full 256-dimensional space, \(C\). A generic quadratic form is given by expression (72). As a special case, for a suitable choice of metric \(q_{AB}\), we obtain the Kaluza-Klein like quadratic form (71). The latter form, in turn, for a particular choice of \(\theta_a\) given in Eq. (61), becomes the form entering the action (17) of Govaerts et al. [3].

## 5 Conclusion

Born’s reciprocity principle between coordinates and momenta has led investigators to the theory of Govaerts et al. [3] on the one hand, and to the theory of Bars et al. [4] on the other hand. The former theory is based on an action that contains an orthogonal and a symplectic form, whilst the latter theory contains a symplectic form only, the action being invariant under local \(\text{Sp}(2)\) transformations, besides being Lorentz invariant. We have shown how these two theories can be unified by means of a single action (38). The constraints (39) that arise from varying the action with respect to the \(\text{Sp}(2)\) gauge fields, \(A_{ij}\), contain, besides the term considered by Bars et al. [4], an addition term. We have found that the generalised constraints also require at least two time like dimensions, just as in refs. [4]. Their important result that in 2-time physics ghosts can be eliminated by using the \(\text{Sp}(2)\) gauge symmetry holds for this generalised case as well. What remains to be explored is how various gauge choices—in analogy to the results of refs. [4]—lead to various 1-time systems with different physical interpretations.

One possible generalisation of the action by Govaerts et al. [3] is to put it into a form invariant under local \(\text{Sp}(2)\) transformations that transform \(x^\mu\) and \(p^\mu\) into each other, as discussed above. Another possible generalisation is in introducing the symplectic potential, \(\theta_a\). By doing so, we obtain an action that is invariant under general coordinate transformations of coordinates \(z^a = (x^\mu, p^\mu)\). Such action can be considered, à la Kaluza-Klein, as coming from an action that is just a line element in 9-dimensions, \(\theta\) being the ninth dimension—besides the eight dimensions, \(z^a\); the symplectic potential is then related to certain components of the 9-dimensional metric. Moreover, we can go even further by invoking the concept of Clifford space, \(C\), and consider \(\theta\) as one of the dimensions of \(C\). Instead of a 16-dimensional Clifford space whose tangent space at any point is Clifford algebra \(\text{Cl}(4)\), we have now a 256-dimensional Clifford space with \(\text{Cl}(8)\) as a tangent space. Unification of fundamental interaction within the framework of Clifford algebras has been
investigated by many authors [10, 11, 9], and there exist strong arguments that Cl(8) is more suitable for such purpose than Cl(4). However, if starting from 8-dimensional vector space, a question arises as to what is a physical meaning of the extra four dimensions. In this paper we have pointed out that those ”extra dimensions” come from phase space: a particle is described not only by its position, \( x^\mu \), in 4-dimensional spacetime, but also by its momentum, \( p^\mu \). That physics has to be formulated in phase space has been proposed in a number of works [2, 16, 19]. In this letter we have looked at such concepts from a broader perspective, found connections amongst various directions of research known from the literature, and pointed out how they are related to the unification of particles and forces within the framework of Cl(8).

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