Generalized Uncertainty Relation Associated with a Monotone or an Anti-Monotone Pair Skew Information

Kenjiro Yanagi* and Satoshi Kajihara†

Abstract. We give a trace inequality related to the uncertainty relation based on the monotone or anti-monotone pair skew information which is one of generalizations of result given by [6]. And it includes the result for generalized Wigner-Yanase-Dyson skew information as a particular case ([14]).

Key Words: Uncertainty relation, Wigner-Yanase-Dyson skew information

1 Introduction

Wigner-Yanase skew information

\[
I_\rho(H) = \frac{1}{2} \text{Tr} \left( (i [\rho^{1/2}, H])^2 \right) = \text{Tr}[\rho H^2] - \text{Tr}[\rho^{1/2} H \rho^{1/2} H]
\]

was defined in [11]. This quantity can be considered as a kind of the degree for non-commutativity between a quantum state \( \rho \) and an observable \( H \). Here we denote the commutator by \([X, Y] = XY - YX\). This quantity was generalized by Dyson

\[
I_{\rho,\alpha}(H) = \frac{1}{2} \text{Tr}[i [\rho^\alpha, H] (i [\rho^{1-\alpha}, H])] = \text{Tr}[\rho H^2] - \text{Tr}[\rho^\alpha H \rho^{1-\alpha} H], \alpha \in [0, 1]
\]
which is known as the Wigner-Yanase-Dyson skew information. It is famous that
the convexity of \( I_{\rho,\alpha}(H) \) with respect to \( \rho \) was successfully proven by E.H.Lieb in \[8\]. And also this quantity was generalized by Cai and Luo

\[
I_{\rho,\alpha,\beta}(H) = \frac{1}{2} \left[ Tr[\rho H^2] + Tr[\rho^{\alpha+\beta} H \rho^{1-\alpha-\beta} H] - Tr[\rho^\alpha H \rho^{1-\alpha} H] - Tr[\rho^\beta H \rho^{1-\beta} H]\right],
\]

where \( \alpha, \beta \geq 0, \alpha + \beta \leq 1 \). The convexity of \( I_{\rho,\alpha,\beta}(H) \) with respect to \( \rho \) was proven by Cai and Luo in \[2\] under some restrictive condition. In this paper we let \( M_n(\mathbb{C}) \) be the set of all \( n \times n \) complex matrices, \( M_{n,sa}(\mathbb{C}) \) be the set of all \( n \times n \) self-adjoint matrices, \( M_{n,+}(\mathbb{C}) \) be the set of strictly positive elements of \( M_n(\mathbb{C}) \) and \( M_{n,+1}(\mathbb{C}) \) be the set of strictly positive density matrices, that is \( M_{n,+1}(\mathbb{C}) = \{ \rho \in M_n(\mathbb{C}) | Tr[\rho] = 1, \rho > 0 \} \). If it is not otherwise specified, from now on we shall treat the case of faithful states, that is \( \rho > 0 \). The relation between the Wigner-Yanase skew information and the uncertainty relation was studied in \[10\]. Moreover the relation between the Wigner-Yanase-Dyson skew information and the uncertainty relation was studied in \[7, 12\]. In our paper \[12\] and \[13\], we defined a generalized skew information and then derived a kind of an uncertainty relations. And also in \[14\] and \[15\] , we gave an uncertainty relation of two parameter generalized Wigner-Yanase-Dyson skew information. In this paper, we consider three parameter generalized Wigner-Yanase-Dyson skew information and give a kind of generalized uncertainty relations which is a generalization of the result of Ko and Yoo \[6\].

## 2 Trace inequality of Wigner-Yanase-Dyson skew information

We review the relation between the Wigner-Yanase skew information and the uncertainty relation. In quantum mechanical system, the expectation value of an observable \( H \) in a quantum state \( \rho \) is expressed by \( Tr[\rho H] \). It is natural that the variance for a quantum state \( \rho \) and an observable \( H \) is defined by \( V_{\rho}(H) = Tr[\rho(H - Tr[\rho H]I)^2] = Tr[\rho H^2] - Tr[\rho H]^2 \). It is famous that we have

\[
V_{\rho}(A)V_{\rho}(B) \geq \frac{1}{4}|Tr[\rho[A, B]]|^2 \tag{2.1}
\]

for a quantum state \( \rho \) and two observables \( A \) and \( B \). The further strong results was given by Schrödinger

\[
V_{\rho}(A)V_{\rho}(B) - |Re\{Cov_{\rho}(A, B)\}|^2 \geq \frac{1}{4}|Tr[\rho[A, B]]|^2,
\]
where the covariance is defined by $\text{Cov}_\rho(A, B) = \text{Tr}[\rho(A - \text{Tr}[\rho A])I(B - \text{Tr}[\rho B])]$. However, the uncertainty relation for the Wigner-Yanase skew information failed. (See [10, 7, 12])

$$I_\rho(A)I_\rho(B) \geq \frac{1}{4} |\text{Tr}[\rho[A, B]]|^2.$$ 

Recently, S.Luo introduced the quantity $U_\rho(H)$ representing a quantum uncertainty excluding the classical mixture:

$$U_\rho(H) = \sqrt{V_\rho(H)^2 - (V_\rho(H) - I_\rho(H))^2}, \quad (2.2)$$

then he derived the uncertainty relation on $U_\rho(H)$ in [9]:

$$U_\rho(A)U_\rho(B) \geq \frac{1}{4} |\text{Tr}[\rho[A, B]]|^2. \quad (2.3)$$

Note that we have the following relation

$$0 \leq I_\rho(H) \leq U_\rho(H) \leq V_\rho(H). \quad (2.4)$$

The inequality (2.3) is a refinement of the inequality (2.1) in the sense of (2.4). In [13], we studied one-parameter extended inequality for the inequality (2.3).

**Definition 2.1** For $0 \leq \alpha \leq 1$, a quantum state $\rho$ and an observable $H$, we define the Wigner-Yanase-Dyson skew information

$$I_{\rho, \alpha}(H) = \frac{1}{2} \text{Tr}[(i[\rho^\alpha, H_0])(i[\rho^{1-\alpha}, H_0])] = \text{Tr}[\rho H_0^2] - \text{Tr}[\rho^\alpha H_0 \rho^{1-\alpha} H_0]$$

and we also define

$$J_{\rho, \alpha}(H) = \frac{1}{2} \text{Tr}[[\rho^\alpha, H_0]\{\rho^{1-\alpha}, H_0\}] = \text{Tr}[\rho H_0^2] + \text{Tr}[\rho^\alpha H_0 \rho^{1-\alpha} H_0],$$

where $H_0 = H - \text{Tr}[\rho H]I$ and we denote the anti-commutator by $\{X, Y\} = XY + YX$.

Note that we have

$$\frac{1}{2} \text{Tr}[(i[\rho^\alpha, H_0])(i[\rho^{1-\alpha}, H_0])] = \frac{1}{2} \text{Tr}[(i[\rho^\alpha, H])(i[\rho^{1-\alpha}, H])].$$

but we have

$$\frac{1}{2} \text{Tr}[[\rho^\alpha, H_0]\{\rho^{1-\alpha}, H_0\}] \neq \frac{1}{2} \text{Tr}[[\rho^\alpha, H]\{\rho^{1-\alpha}, H\}].
Then we have the following inequalities:

\[ I_{\rho,\alpha}(H) \leq I_\rho(H) \leq J_\rho(H) \leq J_{\rho,\alpha}(H), \tag{2.5} \]

since we have \( T_r[\rho^{1/2}H\rho^{1/2}] \leq T_r[\rho^\alpha H\rho^{1-\alpha} H]. \) (See [1, 3] for example.) If we define

\[ U_{\rho,\alpha}(H) = \sqrt{V_\rho(H)^2 - (V_\rho(H) - I_{\rho,\alpha}(H))^2}, \tag{2.6} \]

as a direct generalization of Eq. (2.2), then we have

\[ 0 \leq I_{\rho,\alpha}(H) \leq U_{\rho,\alpha}(H) \leq U_\rho(H) \tag{2.7} \]

due to the first inequality of (2.5). We also have

\[ U_{\rho,\alpha}(H) = \sqrt{I_{\rho,\alpha}(H)J_{\rho,\alpha}(H)}. \]

From the inequalities (2.4), (2.6), (2.7), our situation is that we have

\[ 0 \leq I_{\rho,\alpha}(H) \leq I_\rho(H) \leq U_\rho(H) \]

and

\[ 0 \leq I_{\rho,\alpha}(H) \leq U_{\rho,\alpha}(H) \leq U_\rho(H). \]

We gave the following uncertainty relation with respect to \( U_{\rho,\alpha}(H) \) as a direct generalization of the inequality (2.3).

**Theorem 2.1 ([13])** For \( 0 \leq \alpha \leq 1 \), a quantum state \( \rho \) and observables \( A, B \),

\[ U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) \geq \alpha(1 - \alpha)[T_r[\rho[A, B]]]^2. \tag{2.8} \]

Now we define the two parameter extensions of Wigner-Yanase skew information and give an uncertainty relation under some conditions.

**Definition 2.2** For \( \alpha, \beta \geq 0 \), a quantum state \( \rho \) and an observable \( H \), we define the generalized Wigner-Yanase-Dyson skew information

\[
I_{\rho,\alpha,\beta}(H) = \frac{1}{2} T_r \left[ (i[\rho^\alpha, H_0])(i[\rho^\beta, H_0])\rho^{1-\alpha-\beta} \right] \\
= \frac{1}{2} \left\{ T_r[\rho H_0^\beta] + T_r[\rho^{\alpha+\beta}H_0\rho^{1-\alpha-\beta}H_0] - T_r[\rho^\alpha H_0\rho^{1-\alpha}H_0] - T_r[\rho^\beta H_0\rho^{1-\beta}H_0] \right\}
\]
and we define
\[ J_{\rho,\alpha,\beta}(H) = \frac{1}{2} \text{Tr} \left[ \{ \rho^\alpha, H_0 \} \{ \rho^\beta, H_0 \} \rho^{1-\alpha-\beta} \right] \]
\[ = \frac{1}{2} \{ \text{Tr}[\rho H_0^2] + \text{Tr}[\rho^{\alpha+\beta} H_0 \rho^{1-\alpha-\beta} H_0] + \text{Tr}[\rho^\alpha H_0 \rho^{1-\alpha} H_0] + \text{Tr}[\rho^\beta H_0 \rho^{1-\beta} H_0] \}, \]
where \( H_0 = H - \text{Tr}[\rho H] I \) and we denote the anti-commutator by \( \{ X, Y \} = XY + YX \). We remark that \( \alpha + \beta = 1 \) implies \( I_{\rho,\alpha}(H) = I_{\rho,\alpha,1-\alpha}(H) \) and \( J_{\rho,\alpha}(H) = J_{\rho,\alpha,1-\alpha}(H) \). We also define
\[ U_{\rho,\alpha,\beta}(H) = \sqrt{I_{\rho,\alpha,\beta}(H) J_{\rho,\alpha,\beta}(H)}. \]

In this paper we assume that \( \alpha, \beta \geq 0 \) do not necessarily satisfy the condition \( \alpha + \beta \leq 1 \). We give the following theorem.

**Theorem 2.2 ([14])** For \( \alpha, \beta \geq 0 \) and \( \alpha + \beta \geq 1 \) or \( \alpha + \beta \leq \frac{1}{2} \) and observables \( A, B \),
\[ U_{\rho,\alpha,\beta}(A) U_{\rho,\alpha,\beta}(B) \geq \alpha \beta |\text{Tr}[\rho[A,B]]|^2. \] (2.9)

And we also define the two parameter extensions of Wigner-Yanase skew information which are different from Definition 2.2.

**Definition 2.3** For \( \alpha, \beta \geq 0 \), a quantum state \( \rho \) and an observable \( H \), we define the generalized Wigner-Yanase-Dyson skew information
\[ \tilde{I}_{\rho,\alpha,\beta}(H) = \frac{1}{2} \text{Tr} \left[ (i[\rho^\alpha, H_0]) (i[\rho^\beta, H_0]) \right] \]
\[ = \text{Tr}[\rho^{\alpha+\beta} H_0^2] - \text{Tr}[\rho^\alpha H_0 \rho^\beta H_0]. \]

and we define
\[ \tilde{J}_{\rho,\alpha,\beta}(H) = \frac{1}{2} \text{Tr} \left[ \{ \rho^\alpha, H_0 \} \{ \rho^\beta, H_0 \} \right] \]
\[ = \text{Tr}[\rho^{\alpha+\beta} H_0^2] + \text{Tr}[\rho^\alpha H_0 \rho^\beta H_0], \]
where \( H_0 = H - \text{Tr}[\rho H] I \) and we denote the anti-commutator by \( \{ X, Y \} = XY + YX \). We remark that \( \alpha + \beta = 1 \) implies \( I_{\rho,\alpha}(H) = \tilde{I}_{\rho,\alpha,1-\alpha}(H) \) and \( J_{\rho,\alpha}(H) = \tilde{J}_{\rho,\alpha,1-\alpha}(H) \). We also define
\[ \tilde{U}_{\rho,\alpha,\beta}(H) = \sqrt{\tilde{I}_{\rho,\alpha,\beta}(H) \tilde{J}_{\rho,\alpha,\beta}(H)}. \]
Then we give the following theorem.

**Theorem 2.3 ([15])** For $\alpha, \beta \geq 0$ ($\alpha \beta \neq 0$) and observables $A, B$,

$$
\tilde{U}_{\rho,\alpha,\beta}(A) \tilde{U}_{\rho,\alpha,\beta}(B) \geq \frac{\alpha \beta}{(\alpha + \beta)^2} |\text{Tr}[\rho^{\alpha+\beta}[A, B]]|^2.
$$

**Remark 2.1** We remark that (2.8) is derived by putting $\beta = 1 - \alpha$ in (2.9). Then Theorem 2.2 is a generalization of Theorem 2.1 given in [13].

### 3 Trace inequality of monotone or anti-monotone pair skew information

**Definition 3.1** Let $f(x), g(x)$ be nonnegative continuous functions defined on the interval $[0, 1]$. We call the pair $(f, g)$ a compatible in log-increase, monotone pair (CLI monotone pair, in short) if

(a) $(f(x) - f(y))(g(x) - g(y)) \geq 0$ for all $x, y \in [0, 1]$.

(b) $f(x), g(x)$ are differentiable on $(0, 1)$ and

$$
0 \leq \inf_{0<x<1} \frac{G'(x)}{F'(x)} \leq \sup_{0<x<1} \frac{G'(x)}{F'(x)} < \infty,
$$

where $F(x) = \log f(x), G(x) = \log g(x)$.

**Definition 3.2** Let $f(x), g(x)$ be nonnegative continuous functions defined on the interval $[0, 1]$. We call the pair $(f, g)$ a compatible in log-increase, anti-monotone pair (CLI anti-monotone pair, in short) if

(a) $(f(x) - f(y))(g(x) - g(y)) \leq 0$ for all $x, y \in [0, 1]$.

(b) $f(x), g(x)$ are differentiable on $(0, 1)$ and

$$
-\infty < \inf_{0<x<1} \frac{G'(x)}{F'(x)} \leq \sup_{0<x<1} \frac{G'(x)}{F'(x)} \leq 0,
$$

where $F(x) = \log f(x), G(x) = \log g(x)$.

Let $f(x), g(x), h(x)$ be nonnegative continuous functions defined on $[0, 1]$ and be differentiable on $(0, 1)$. We assume that $(f, g)$ is CLI monotone pair and $(f, h)$ is CLI monotone or anti-monotone pair. We introduce the correlation functions in the following way.
Definition 3.3

\[ I_{\rho,(f,g,h)}(H) = \frac{1}{2} Tr[(i[f(\rho), H_0])(i[g(\rho), H_0])h(\rho)] \]
\[ = -\frac{1}{2} Tr[(f(\rho), H_0)]([g(\rho), H_0])h(\rho)] \]
\[ = -\frac{1}{2} Tr[(f(\rho)H_0 - H_0 f(\rho))(g(\rho)H_0 - H_0 g(\rho))h(\rho)] \]
\[ = -\frac{1}{2} Tr[f(\rho)H_0 g(\rho)H_0 h(\rho) - f(\rho)H_0^2 g(\rho)h(\rho)] \]
\[ + \frac{1}{2} Tr[H_0 f(\rho)g(\rho)H_0 h(\rho) - H_0 f(\rho)H_0 g(\rho)h(\rho)] \]
\[ = -\frac{1}{2} Tr[f(\rho)g(\rho)h(\rho)H_0 - f(\rho)g(\rho)h(\rho)H_0^2] \]
\[ + \frac{1}{2} Tr[f(\rho)g(\rho)H_0 h(\rho)H_0 - g(\rho)h(\rho)H_0 f(\rho)H_0] \]
\[ = \frac{1}{2} \{Tr[f(\rho)g(\rho)h(\rho)H_0^2] + Tr[f(\rho)g(\rho)H_0 h(\rho)H_0]\} \]
\[ - \frac{1}{2} \{Tr[f(\rho)H_0 g(\rho)h(\rho)H_0] + Tr[g(\rho)H_0 f(\rho)h(\rho)H_0]\}. \]

\[ J_{\rho,(f,g,h)}(H) = \frac{1}{2} Tr[\{f(\rho), H_0\}\{g(\rho), H_0\}h(\rho)] \]
\[ = \frac{1}{2} Tr[(f(\rho)H_0 + h_0 f(\rho))(g(\rho)H_0 + H_0 g(\rho))h(\rho)] \]
\[ = \frac{1}{2} Tr[f(\rho)H_0 g(\rho)H_0 h(\rho) + f(\rho)H_0^2 g(\rho)h(\rho)] \]
\[ + \frac{1}{2} Tr[H_0 f(\rho)g(\rho)H_0 h(\rho) + H_0 f(\rho)H_0 g(\rho)h(\rho)] \]
\[ = \frac{1}{2} \{Tr[f(\rho)g(\rho)h(\rho)H_0^2] + Tr[f(\rho)g(\rho)H_0 h(\rho)H_0]\} \]
\[ + \frac{1}{2} \{Tr[f(\rho)H_0 g(\rho)h(\rho)H_0] + Tr[g(\rho)H_0 f(\rho)h(\rho)H_0]\}. \]

\[ U_{\rho,(f,g,h)}(H) = \sqrt{I_{\rho,(f,g,h)}(H)J_{\rho,(f,g,h)}(H)}. \]

We are ready to state our main result. For \( f, g, h \) we let

\[ \beta(f, g, h) \]
\[ = \min \left\{ \frac{m}{(1 + m + n)^2}, \frac{m}{(1 + m + N)^2}, \frac{M}{(1 + M + n)^2}, \frac{M}{(1 + M + N)^2} \right\}, \quad (3.1) \]

where

\[ m = \inf_{0 < x < 1} \frac{G'(x)}{F'(x)}, \quad M = \sup_{0 < x < 1} \frac{G'(x)}{F'(x)} \]
We consider the following two assumptions.

(I) \((f, g), (f, h)\) are CLI monotone pair satisfying

\[
1 + \frac{G(y) - G(x)}{F(y) - F(x)} \leq \frac{H(y) - H(x)}{F(y) - F(x)} \quad \text{for } x < y,
\]

where \(F(x) = \log f(x), G(x) = \log g(x), H(x) = \log h(x)\)

(II) \((f, g)\) is CLI monotone pair and \((f, h)\) is CLI anti-monotone pair satisfying

\[
1 + \frac{G(y) - G(x)}{F(y) - F(x)} + \frac{H(y) - H(x)}{F(y) - F(x)} \geq 0 \quad \text{for } x < y.
\]

**Theorem 3.1** Under the assumption (I) or (II), the following inequality holds:

\[
U_{\rho,(f,g,h)}(A)U_{\rho,(f,g,h)}(B) \geq \beta(f, g, h)\|Tr[f(\rho)g(\rho)h(\rho)[A, B]]\|^2
\]

for \(A, B \in M_{n,sa}(\mathbb{C})\).

**4 Proof of Theorem 3.1**

Let \(\rho = \sum_{i=1}^{n} \lambda_i |\phi_i\rangle \langle \phi_i| \in M_{n+1}(\mathbb{C})\), where \(|\phi_i\rangle\}_{i=1}^{n} is an orthonormal set in \(\mathbb{C}^n\). Let \((f, g)\) be a CLI monotone pair and \((f, h)\) be a CLI monotone or anti-monotone pair. By a simple calculation, we have for any \(H \in M_{n,sa}(\mathbb{C})\)

\[
\begin{align*}
Tr[f(\rho)g(\rho)h(\rho)H_0^2] &= \sum_{i,j} \frac{1}{2} \{f(\lambda_i)g(\lambda_i)h(\lambda_i) + f(\lambda_j)g(\lambda_j)h(\lambda_j)\}|a_{ij}|^2. \quad (4.1) \\
Tr[f(\rho)g(\rho)H_0h(\rho)H_0] &= \sum_{i,j} \frac{1}{2} \{f(\lambda_i)g(\lambda_i)h(\lambda_i) + f(\lambda_j)g(\lambda_j)h(\lambda_i)\}|a_{ij}|^2. \quad (4.2) \\
Tr[f(\rho)H_0g(\rho)h(\rho)H_0] &= \sum_{i,j} \frac{1}{2} \{f(\lambda_i)g(\lambda_j)h(\lambda_i) + f(\lambda_j)g(\lambda_i)h(\lambda_i)\}|a_{ij}|^2. \quad (4.3) \\
Tr[g(\rho)H_0f(\rho)h(\rho)H_0] &= \sum_{i,j} \frac{1}{2} \{g(\lambda_i)f(\lambda_j)h(\lambda_i) + g(\lambda_j)f(\lambda_i)h(\lambda_i)\}|a_{ij}|^2, \quad (4.4)
\end{align*}
\]

where \(a_{ij} = \langle \phi_i|H_0|\phi_j\rangle\) and \(a_{ij} = \overline{a_{ji}}\). From (4.1) - (4.4), we get

\[
I_{\rho,(f,g,h)}(H) = \frac{1}{2} \sum_{i<j} (f(\lambda_i) - f(\lambda_j))(g(\lambda_i) - g(\lambda_j))(h(\lambda_i) + h(\lambda_j))|a_{ij}|^2.
\]
$$J_{ρ,(f,g,h)}(H) \geq \frac{1}{2} \sum_{i<j} (f(λ_i) + f(λ_j))(g(λ_i) + g(λ_j))(h(λ_i) + h(λ_j))|a_{ij}|^2.$$ 

To prove Theorem 3.1 we need to control a lower bound of a functional coming from a CLI monotone or anti-monotone pair. For \( f, g, h \) satisfying (I) or (II), we define a function \( L \) on \([0,1] \times [0,1] \) by

$$L(x, y) = \frac{(f(x)^2 - f(y)^2)(g(x)^2 - g(y)^2)(h(x) + h(y))^2}{(f(x)g(x)h(x) - f(y)g(y)h(y))^2}. \tag{4.5}$$

**Proposition 4.1** Under the assumption (I) or (II)

$$\min_{x,y \in [0,1]} L(x, y) \geq 16\beta(f, g, h),$$

where \( \beta(f, g, h) \) is defined in (3.1).

For the proof of Proposition 4.1 we need the following lemma.

**Lemma 4.1** If \( a, b, c \geq 0 \) satisfy \( 0 < a + b \leq c \) or if \( a, b \geq 0, c \leq 0 \) satisfy \( a + b + c > 0 \), then the inequality

$$\frac{(e^{2ar} - 1)(e^{2br} - 1)(e^{cr} + 1)^2}{(e^{(a+b+c)r} - 1)^2} \geq \frac{16ab}{(a + b + c)^2}$$

holds for any real number \( r \).

**Proof.** We put \( e^r = t \). Then we may prove the following:

$$(t^{2a} - 1)(t^{2b} - 1)(t^c + 1)^2 \geq \frac{16ab}{(a + b + c)^2}(t^{a+b+c} - 1)^2 \tag{4.6}$$

for \( t > 0 \). It is sufficient to prove (4.6) for \( t \geq 1 \) and \( a, b, c \geq 0, 0 < a + b \leq c \) or \( a, b \geq 0, c \leq 0, a + b + c > 0 \).

By Lemma 3.3 in [13] we have for \( 0 \leq p \leq 1 \) and \( s \geq 1 \),

$$(s^{2p} - 1)(s^{2(1-p)} - 1) \geq 4p(1-p)(s - 1)^2.$$ 

We assume that \( a, b \geq 0 \). We put \( p = a/(a+b) \) and \( s^{1/(a+b)} = t \). Then

$$(t^{2a} - 1)(t^{2b} - 1) \geq \frac{4ab}{(a + b)^2}(t^{a+b} - 1)^2.$$ 

Then we have

$$(t^{2a} - 1)(t^{2b} - 1)(t^c + 1)^2 \geq \frac{4ab}{(a + b)^2}(t^{a+b} - 1)^2(t^c + 1)^2.$$
In order to show the aimed inequality, we have to prove that
\[
(t^{a+b} - 1)^2(t^c + 1)^2 \geq \frac{4(a+b)^2}{(a+b+c)^2}(t^{a+b+c} - 1)^2.
\]
Since \(a+b+c > 0\), it is sufficient to prove the following inequality
\[
(t^{a+b} - 1)(t^c + 1) \geq \frac{2(a+b)}{a+b+c}(t^{a+b+c} - 1) \tag{4.7}
\]
for \(t \geq 1\) and \(a,b,c \geq 0, 0 < a + b \leq c\) or \(a,b \geq 0, c \leq 0, a+b+c > 0\). We put
\[
S(t) = (t^{a+b} - 1)(t^c + 1) - \frac{2(a+b)}{a+b+c}(t^{a+b+c} - 1).
\]
Then
\[
S'(t) = t^{c-1}\{(c-a-b)t^{a+b} - c + (a+b)t^{a+b-c}\}.
\]
Here we put
\[
T(t) = (c-a-b)t^{a+b} - c + (a+b)t^{a+b-c}.
\]
Then
\[
T'(t) = (a+b)(c-a-b)t^{a+b-c-1}(t^c - 1).
\]
When \(a+b \leq c\), \(T'(t) \geq 0\). Since \(T(1) = 0, T(t) \geq 0\) for \(t \geq 1\). Then \(S'(t) \geq 0\). Since \(S(1) = 0, S(t) \geq 0\) for \(t \geq 1\). On the other hand when \(c \leq 0, T'(t) \geq 0\). Since \(T(1) = 0, T(t) \geq 0\) for \(t \geq 1\). Then \(S'(t) \geq 0\). Since \(S(1) = 0, S(t) \geq 0\) for \(t \geq 1\). Hence we get (4.7).

**Proof of Proposition 4.1.** Let \(x < y\). In the last line of (4.5), dividing both the numerator and the denominator by \((f(x)g(x)h(x))^2\) and by using \(F(x) = \log f(x), G(x) = \log g(x)\) and \(H(x) = \log h(x)\), we get
\[
L(x,y) = \frac{(e^{2(F(y)-F(x))} - 1)(e^{2(G(y)-G(x))} - 1)(e^{H(y)-H(x)} + 1)^2}{(e^{F(y)-F(x)} + G(y)-G(x) + H(y)-H(x) - 1)^2}.
\]
By the generalized mean value theorem, there exist \(z (x < z < y), w (x < w < y)\) such that
\[
\frac{G(y)-G(x)}{F(y)-F(x)} = \frac{G'(z)}{F'(z)} = k(z), \quad \frac{H(y)-H(x)}{F(y)-F(x)} = \frac{H'(w)}{F'(w)} = \ell(w).
\]
Thus we have
\[
L(x,y) = \frac{(e^{2(F(y)-F(x))} - 1)(e^{2k(z)(F(y)-F(x))} - 1)(e^{\ell(w)(F(y)-F(x))} + 1)^2}{(e^{(1+k(z)+\ell(w))(F(y)-F(x))} - 1)^2}.
\]
It follows from Lemma 4.1 that for any \(R > 0\), the function
\[
(k, \ell) \rightarrow A(k, \ell) = \frac{(R^2 - 1)(R^{2k} - 1)(R^\ell + 1)^2}{(R^{1+k+\ell} - 1)^2}
\]
10
defined in $k \in [m, M], \ell \in [n, N]$ is bounded from below by $\min_{m \leq k \leq M, n \leq \ell \leq N} \{A(k, \ell)\}$. It is easy to obtain

$$\min_{m \leq k \leq M, n \leq \ell \leq N} \{A(k, \ell)\} \geq 16\beta(f, g, h).$$

We complete the proof. \hfill \Box

**Proof of Theorem [3.1]** Since

$$Tr[f(\rho)g(\rho)h(\rho)[A, B]] = Tr[f(\rho)g(\rho)h(\rho)[A_0, B_0]]$$

$$= 2i\text{Im}\{Tr[f(\rho)g(\rho)h(\rho)A_0B_0]\}$$

$$= 2i\text{Im} \sum_{\ell < m} (f(\lambda\ell)g(\lambda\ell)h(\lambda\ell) - f(\lambda_m)g(\lambda_m)h(\lambda_m))a_{m\ell}b_{\ell m}$$

$$= 2i \sum_{\ell < m} (f(\lambda\ell)g(\lambda\ell)h(\lambda\ell) - f(\lambda_m)g(\lambda_m)h(\lambda_m)) \text{Im}(a_{m\ell}b_{\ell m})$$

for any $A, B \in M_{n, sa}(\mathbb{C})$, where $a_{m\ell} = \langle \phi_m | A_0 | \phi_\ell \rangle$ and $b_{m\ell} = \langle \phi_m | B_0 | \phi_\ell \rangle$, we have

$$|Tr[f(\rho)g(\rho)h(\rho)[A, B]]| \leq 2 \sum_{\ell < m} |f(\lambda\ell)g(\lambda\ell)h(\lambda\ell) - f(\lambda_m)g(\lambda_m)h(\lambda_m)||Ima_{m\ell}|b_{m\ell}||$$

$$\leq 2 \sum_{\ell < m} |f(\lambda\ell)g(\lambda\ell)h(\lambda\ell) - f(\lambda_m)g(\lambda_m)h(\lambda_m)||a_{m\ell}||b_{m\ell}||$$

By Proposition [4.1] we have

$$\beta(f, g, h)|Tr[f(\rho)g(\rho)h(\rho)[A, B]]|^2$$

$$\leq 4\beta(f, g, h)\sum_{\ell < m} |f(\lambda\ell)g(\lambda\ell)h(\lambda\ell) - f(\lambda_m)g(\lambda_m)h(\lambda_m)||a_{m\ell}||b_{m\ell}||^2$$

$$\leq \frac{1}{4} \sum_{\ell < m} \sqrt{f(\lambda\ell)^2 - f(\lambda_m)^2} |g(\lambda\ell)^2 - g(\lambda_m)^2| h(\lambda\ell) + h(\lambda_m))^2 |a_{m\ell}||b_{m\ell}||^2$$

$$= \frac{1}{4} \sum_{\ell < m} \sqrt{\Delta_f(\ell, m) \Delta_g(\ell, m) \Gamma_h(\ell, m)|a_{m\ell}| \sqrt{\Gamma_f(\ell, m) \Gamma_g(\ell, m) \Gamma_h(\ell, m)|b_{m\ell}|}}$$

where $\Delta_f(\ell, m) = f(\lambda\ell) - f(\lambda_m), \Delta_g(\ell, m) = g(\lambda\ell) - g(\lambda_m)$ and $\Gamma_f(\ell, m) = f(\lambda\ell) + f(\lambda_m), \Gamma_g(\ell, m) = g(\lambda\ell) + g(\lambda_m), \Gamma_h(\ell, m) = h(\lambda\ell) + h(\lambda_m)$. By Schwarz inequality, we have

$$\beta(f, g, h)|Tr[f(\rho)g(\rho)h(\rho)[A, B]]|^2$$

$$\leq \frac{1}{2} \sum_{\ell < m} \Delta_f(\ell, m) \Delta_g(\ell, m) \Gamma_h(\ell, m) |a_{m\ell}|^2$$

$$\times \frac{1}{2} \sum_{\ell < m} \Delta_f(\ell, m) \Delta_g(\ell, m) \Gamma_h(\ell, m) |b_{m\ell}|^2$$

$$\leq I_{\rho, f, g, h}(A) J_{\rho, f, g, h}(B).$$
Similarly we have

\[ \beta(f, g, h) |\text{Tr}[f(\rho)g(\rho)h(\rho)[A, B]]|^2 \leq I_{\rho,(f,g,h)}(B)J_{\rho,(f,g,h)}(A). \]

Hence by multiplying the above two inequalities, we have

\[ \beta(f, g, h) |\text{Tr}[f(\rho)g(\rho)h(\rho)[A, B]]|^2 \leq U_{\rho,(f,g,h)}(A)U_{\rho,(f,g,h)}(B). \]

\[\square\]

When \( h(x) = 1 \), we obtain the result given by Ko and Yoo \[6\].

**Corollary 4.1** ([6]) If \((f, g)\) is CLI monotone pair, then the following inequality holds:

\[ U_{\rho,(f,g)}(A)U_{\rho,(f,g)}(B) \geq \beta(f, g)|\text{Tr}[f(\rho)g(\rho)[A, B]]|^2 \]

for \( A, B \in M_{m,sa}(\mathbb{C}) \), where

\[
\begin{align*}
I_{\rho,(f,g)}(A) &= \frac{1}{2} \text{Tr}[(i[f(\rho), A_0])(i[g(\rho), A_0])], \\
J_{\rho,(f,g)}(A) &= \frac{1}{2} \text{Tr}[\{f(\rho), A_0\}\{g(\rho), A_0\}], \\
U_{\rho,(f,g)}(A) &= \sqrt{I_{\rho,(f,g)}J_{\rho,(f,g)}}, \\
\beta(f, g) &= \min\{\frac{m}{(m+M)^2}, \frac{M}{(m+M)^2}\}.
\end{align*}
\]

We also have the following corollary.

**Corollary 4.2** Let \( f(x) = x^\alpha \) (\( \alpha \geq 0 \)), \( g(x) = x^\beta \) (\( \beta \geq 0 \)), \( h(x) = x^\gamma \) (\( \gamma \geq 0 \) or \( \gamma \leq 0 \)).

(A) If \( \alpha, \beta, \gamma \geq 0 \) satisfy \( 0 < \alpha + \beta \leq \gamma \), then

\[ \beta(f, g, h) = \frac{\alpha\beta}{(\alpha + \beta + \gamma)^2}. \]

(B) If \( \alpha, \beta \geq 0, \gamma \leq 0 \) satisfy \( \alpha + \beta + \gamma > 0 \), then

\[ \beta(f, g, h) = \frac{\alpha\beta}{(\alpha + \beta + \gamma)^2}. \]
Remark 4.1  When $\alpha, \beta \geq 0, \gamma < 0$ satisfy $\alpha + \beta + \gamma > 0$, we remark that $h(x)$ is not continuous function on $[0, 1]$ because

$$\lim_{x \to 0^+} h(x) = +\infty.$$ 

Then in this case by putting $\epsilon > 0$ such that $\epsilon$ is smaller than the minimal eigenvalue of $\rho$, we can assume that $h(x)$ is continuous on $[\epsilon, 1]$. Hence we obtain the same result as Corollary 4.2.

Remark 4.2  When $\gamma = 0$ in (2) of Corollary 4.2, we have the result in [15] (Theorem 2.3). And when $\alpha + \beta + \gamma = 1$ in Corollary 4.2, we have the result in [14] (Theorem 2.2). That is (1) implies $\alpha, \beta \geq 0, \alpha + \beta \leq \frac{1}{2}$ and (2) implies $\alpha, \beta \geq 0, \alpha + \beta \geq 1$.

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