Laplacian Spectrum of non-commuting graphs of finite groups

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Abstract: In this paper, we compute the Laplacian spectrum of non-commuting graphs of some classes of finite non-abelian groups. Our computations reveal that the non-commuting graphs of all the groups considered in this paper are L-integral. We also obtain some conditions on a group \( G \) so that its non-commuting graph is L-integral.

Key words: non-commuting graph, spectrum, L-integral graph, finite group.

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1 Introduction

Let \( G \) be a finite group with centre \( Z(G) \). The non-commuting graph of a non-abelian group \( G \), denoted by \( A_G \), is a simple undirected graph whose vertex set is \( G \setminus Z(G) \) and two vertices \( x \) and \( y \) are adjacent if and only if \( xy \neq yx \). Various aspects of non-commuting graphs of different finite groups can be found in [1, 4, 8, 12, 23]. In [12], Elvierayani and Abdussakir have computed the Laplacian spectrum of the non-commuting graph of dihedral groups \( D_{2m} \) where \( m \) is odd and suggested to consider the case when \( m \) is even. In this paper, we compute the Laplacian spectrum of the non-commuting graph of \( D_{2m} \) for any \( m \geq 3 \) using a different method. Our method also enables to compute the Laplacian spectrum of the non-commuting graphs of several well-known families finite non-abelian groups such as the quasidihedral groups, generalized quaternion groups, some projective special linear groups, general linear groups etc. In a separate paper [11], we study the Laplacian energy of non-commuting graphs of the groups considered in this paper.

For a graph \( \mathcal{G} \) we write \( \overline{\mathcal{G}} \) and \( V(\mathcal{G}) \) to denote the complement of \( \mathcal{G} \) and the set of vertices of \( \mathcal{G} \) respectively. Let \( A(\mathcal{G}) \) and \( D(\mathcal{G}) \) denote the adjacency matrix and degree matrix of a graph \( \mathcal{G} \) respectively. Then the Laplacian matrix of \( \mathcal{G} \) is given by \( L(\mathcal{G}) = D(\mathcal{G}) - A(\mathcal{G}) \). We write \( L-Spec(\mathcal{G}) \) to denote the Laplacian spectrum of \( \mathcal{G} \) and \( L-Spec(\mathcal{G}) = \{ \alpha_1, \alpha_2, \ldots, \alpha_n \} \) where \( \alpha_1 < \alpha_2 < \cdots < \alpha_n \) are the eigenvalues of

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$L(G)$ with multiplicities $a_1, a_2, \ldots, a_n$ respectively. A graph $G$ is called $L$-integral if $L\text{-Spec}(G)$ contains only integers. As a consequence of our results, it follows that the non-commuting graphs of all the groups considered in this paper are $L$-integral. It is worth mentioning that $L$-integral graphs are studied extensively in \cite{3, 15, 17}.

2 Preliminary results

It is well-known that $L\text{-Spec}(K_n) = \{0^1, n^{n-1}\}$ where $K_n$ denotes the complete graph on $n$ vertices. Further, we have the following results.

**Theorem 2.1.** If $G = l_1K_{m_1} \sqcup l_2K_{m_2} \sqcup \cdots \sqcup l_kK_{m_k}$, where $l_iK_{m_i}$ denotes the disjoint union of $l_i$ copies of $K_{m_i}$ for $1 \leq i \leq k$ and $m_1 < m_2 < \cdots < m_k$, then

$$L\text{-Spec}(G) = \{\alpha \sum_{i=1}^{k} l_i, \alpha l_1(m_1-1), \alpha l_2(m_2-1), \ldots, \alpha l_k(m_k-1)\}.$$ 

**Theorem 2.2.** \cite{18} Theorem 3.6] Let $G$ be a graph such that $L\text{-Spec}(G) = \{\alpha_1 a_1, \alpha_2 a_2, \ldots, \alpha_n a_n\}$ then $L\text{-Spec}(\overline{G})$ is given by

$$\{0, |V(G)| - \alpha_n a_n, |V(G)| - \alpha_{n-1} a_{n-1}, \ldots, |V(G)| - \alpha_1 a_1 - 1\}.$$ 

As a corollary of the above two theorems we have the following result.

**Corollary 2.3.** If $G = l_1K_{m_1} \sqcup l_2K_{m_2} \sqcup \cdots \sqcup l_kK_{m_k}$, where $l_iK_{m_i}$ denotes the disjoint union of $l_i$ copies of $K_{m_i}$ for $1 \leq i \leq k$ and $m_1 < m_2 < \cdots < m_k$, then

$$L\text{-Spec}(G) = \{0, \left(\sum_{i=1}^{k} l_i m_i - m_k\right) l_k(m_k-1), \ldots, \left(\sum_{i=1}^{k} l_i m_i - m_k\right) l_k(m_k-1), \ldots, \left(\sum_{i=1}^{k} l_i m_i - m_k\right) l_k(m_k-1)\}.$$ 

A group $G$ is called an AC-group if $C_G(x)$ is abelian for all $x \in G \setminus Z(G)$. Various aspects of AC-groups can be found in \cite{1, 10, 21}. The following result gives the Laplacian spectrum of the non-commuting graph of a finite non-abelian AC-group.

**Theorem 2.4.** Let $G$ be a finite non-abelian AC-group. Then

$$L\text{-Spec}(\mathcal{A}_G) = \{0, |G| - |X_1|, |X_2| - |Z(G)|, \ldots, \}$$

$$|G| - |X_n|, |X_1| - |Z(G)| = 1, \ldots, |G| - |Z(G)| = 1, \ldots, ||G| - |Z(G)|\}.$$ 

where $X_1, \ldots, X_n$ are the distinct centralizers of non-central elements of $G$ such that $|X_1| \leq \cdots \leq |X_n|$.

**Proof.** Let $G$ be a finite non-abelian AC-group and $X_i = C_G(x_i)$ where $x_i \in G \setminus Z(G)$ and $1 \leq i \leq n$. Let $x, y \in X_i \setminus Z(G)$ for some $i$ and $x \neq y$ then, since $G$ an AC-group, there is an edge between $x$ and $y$ in $\mathcal{A}_G$. Suppose that $x \in (X_i \cap X_j) \setminus Z(G)$ for some $1 \leq i \neq j \leq n$. Then $[x,x] = 1$ and $[x,x] = 1$. Let $s \in C_G(x)$ then $[s,x_i] = 1$ since $x_i \in C_G(x)$ and $G$ is an AC-group. Therefore, $s \in C_G(x)$ and so $C_G(x) \subseteq C_G(x)$. Again, let $t \in C_G(x)$ then $[t,x] = 1$ since $x \in C_G(x)$ and $G$ is an AC-group. Therefore, $t \in C_G(x)$ and so $C_G(x) \subseteq C_G(x)$. Thus $C_G(x) = C_G(x)$.
Similarly, it can be seen that $C_G(x) = C_G(x_j)$, which is a contradiction. Therefore, $X_i \cap X_j = Z(G)$ for any $1 \leq i \neq j \leq n$. This shows that
\[ A_G = \bigcap_{i=1}^n K_{|X_i| - |Z(G)|}. \] (2.1)

Therefore, by Corollary 2.3 we have
\[ L-\text{Spec}(A_G) = \{0, \left(\sum_{i=1}^n (|X_i| - |Z(G)|) - (|X_n| - |Z(G)|)\right)^{|X_n| - |Z(G)| - 1}, \ldots, \] 
\[ \left(\sum_{i=1}^n (|X_i| - |Z(G)|) - (|X_1| - |Z(G)|)\right)^{|X_1| - |Z(G)| - 1}, \left(\sum_{i=1}^n (|X_i| - |Z(G)|)\right)^{n-1}\}.

Hence, the result follows noting that $\sum_{i=1}^n (|X_i| - |Z(G)|) = |G| - |Z(G)|$. \qed

**Corollary 2.5.** Let $G$ be a finite non-abelian AC-group and $A$ be any finite abelian group. Then
\[ L-\text{Spec}(A_G \times A) = \{0, (|A||G| - |X_n|)|^{A(|X_n| - |Z(G)|) - 1}, \ldots, \] 
\[ (|A||G| - |X_1|)|^{A(|X_1| - |Z(G)|) - 1}, (|A||G| - |Z(G)|)^{n-1}\}.

where $X_1, \ldots, X_n$ are the distinct centralizers of non-central elements of $G$ such that $|X_1| \leq \cdots \leq |X_n|$.

**Proof.** It is easy to see that $G \times A$ is an AC-group and $X_1 \times A, X_2 \times A, \ldots, X_n \times A$ are the distinct centralizers of non-central elements of $G \times A$. Hence, the result follows from Theorem 2.4 noting that $Z(G \times A) = Z(G) \times A$. \qed

### 3 Groups with given central factors

In this section, we compute the Laplacian spectrum of the non-commuting graphs of some families of finite non-abelian groups whose central factors are some well-known finite groups. We begin with the following.

**Theorem 3.1.** Let $G$ be a finite group and $\frac{G}{Z(G)} \cong Sz(2)$, where $Sz(2)$ is the Suzuki group presented by $\langle a, b : a^3 = b^5 = 1, b^{-1}ab = a^2 \rangle$. Then
\[ L-\text{Spec}(A_G) = \{0, (15|Z(G)|)^{4|Z(G)| - 1}, (16|Z(G)|)^{|Z(G)| - 5}, (19|Z(G)|)^{5}\}.

**Proof.** We have
\[ \frac{G}{Z(G)} = \langle aZ(G), bZ(G) : a^5Z(G) = b^7Z(G) = Z(G), b^{-1}abZ(G) = a^2Z(G) \rangle.

Observe that
\[ C_G(ab) = Z(G) \cup abZ(G) \cup a^4b^2Z(G) \cup a^5bZ(G), \] 
\[ C_G(a^2b) = Z(G) \cup a^2bZ(G) \cup a^3b^2Z(G) \cup ab^4Z(G), \] 
\[ C_G(a^2b^3) = Z(G) \cup a^2b^3Z(G) \cup ab^2Z(G) \cup a^4bZ(G), \] 
\[ C_G(b) = Z(G) \cup bZ(G) \cup b^2Z(G) \cup b^3Z(G), \] 
\[ C_G(a^3b) = Z(G) \cup a^3bZ(G) \cup a^2b^2Z(G) \cup ab^3Z(G) \] 
and
\[ C_G(a) = Z(G) \cup aZ(G) \cup a^2Z(G) \cup a^3Z(G) \cup a^4Z(G).


are the only centralizers of non-central elements of $G$. Also note that these centralizers are abelian subgroups of $G$. Thus $G$ is an AC-group. We have $|C_G(a)| = |C_G(b)| = 4|Z(G)|$.

Therefore, by Theorem 2.4 the result follows.

**Theorem 3.2.** Let $G$ be a finite group such that $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, where $p$ is a prime integer. Then

$$\text{L-Spec}(A_G) = \{0, ((p^2 - p)|Z(G)|)^{(p^2 - 1)}|Z(G)| - p, ((p^2 - 1)|Z(G)|)^{p^2 - 1}\}.$$

**Proof.** Let $|Z(G)| = n$ then since $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ we have $\frac{G}{Z(G)} = \langle aZ(G), bZ(G) : a^p, b^p, aba^{-1}b^{-1} \in Z(G) \rangle$, where $a, b \in G$ with $ab \neq ba$. Then for any $z \in Z(G)$, we have

$$C_G(a) = C_G(a^i z) = Z(G) \cup aZ(G) \cup \cdots \cup a^{p-1}Z(G) \text{ for } 1 \leq i \leq p - 1,$$

$$C_G(a^i b) = C_G(a^i bz) = Z(G) \cup a^i bZ(G) \cup \cdots \cup a^{(p-1)i}b^{p-1}Z(G) \text{ for } 1 \leq j \leq p.$$ 

These are the only centralizers of non-central elements of $G$. Also note that these centralizers are abelian subgroups of $G$. Therefore, $G$ is an AC-group. We have $|C_G(a)| = |C_G(a^i b)| = pn$ for $1 \leq j \leq p$. Hence, the result follows from Theorem 2.4.

As a corollary we have the following result.

**Corollary 3.3.** Let $G$ be a non-abelian group of order $p^3$, for any prime $p$, then

$$\text{L-Spec}(A_G) = \{0, (p^3 - p^2)p^2 - 1, (p^3 - p^2)^p\}.$$ 

**Proof.** Note that $|Z(G)| = p$ and $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Hence the result follows from Theorem 3.2.

**Theorem 3.4.** Let $G$ be a finite group such that $\frac{G}{Z(G)} \cong D_{2m}$, for $m \geq 2$. Then

$$\text{L-Spec}(A_G) = \{0, m|Z(G)|^{(m-1)|Z(G)| - 1}, (2(m-1)|Z(G)|)^{m|Z(G)| - m}, ((2m - 1)|Z(G)|)^m\}.$$ 

**Proof.** Since $\frac{G}{Z(G)} \cong D_{2m}$ we have $\frac{G}{Z(G)} = \langle xZ(G), yZ(G) : x^2, y^m, xyx^{-1}y \in Z(G) \rangle$, where $x, y \in G$ with $xy \neq yx$. It is not difficult to see that for any $z \in Z(G)$,

$$C_G(xy^j) = C_G(xy^j z) = Z(G) \cup xy^j Z(G), 1 \leq j \leq m$$

and

$$C_G(y) = C_G(y^i z) = Z(G) \cup yZ(G) \cup \cdots \cup y^{m-1}Z(G), 1 \leq i \leq m - 1$$

are the only centralizers of non-central elements of $G$. Also note that these centralizers are abelian subgroups of $G$. Therefore, $G$ is an AC-group. We have $|C_G(x^j y)| = 2n$ for $1 \leq j \leq m$ and $|C_G(y)| = mn$, where $|Z(G)| = n$. Hence, the result follows from Theorem 2.4.

Using Theorem 3.4 we now compute the Laplacian spectrum of the non-commuting graphs of the groups $M_{2n,1}, D_{2n}$, and $Q_{4n}$ respectively.
Corollary 3.5. Let \( M_{2m} = \langle a, b : a^m = b^2 = 1, bab^{-1} = a^{-1} \rangle \) be a metacyclic group, where \( m \geq 2 \). Then \( \text{L-Spec}(A_{M_{2m}}) \) is one of the following:

\[
\begin{cases}
\{0, (2m)^{m-1} \}, & \text{if } m \text{ is odd} \\
\{0, (2m)^{m-2} \}, & \text{if } m \text{ is even}.
\end{cases}
\]

Proof. Observe that \( Z(M_{2m}) = \langle b^2 \rangle \) or \( \langle b^2 \rangle \cup a^{\frac{m}{2}} \langle b^2 \rangle \) according as \( m \) is odd or even. Also, it is easy to see that \( M_{2m} \cong D_{2m} \) or \( D_m \) according as \( m \) is odd or even. Hence, the result follows from Theorem 3.3.

As a corollary to the above result we have the following result.

Corollary 3.6. Let \( D_{2m} = \langle a, b : a^m = b^2 = 1, bab^{-1} = a^{-1} \rangle \) be the dihedral group of order \( 2m \), where \( m \geq 2 \). Then

\[
\text{L-Spec}(A_{D_{2m}}) = \begin{cases}
\{0, m^{m-2}, (2m-1)^m \} & \text{if } m \text{ is odd} \\
\{0, m^{m-3}, (2m-4)^m, (2m-2)^m \} & \text{if } m \text{ is even}.
\end{cases}
\]

Corollary 3.7. Let \( Q_{4n} = \langle x, y : y^{2n} = 1, x^2 = y^n, xyx^{-1} = y^{-1} \rangle \), where \( n \geq 2 \), be the generalized quaternion group of order \( 4n \). Then

\[
\text{L-Spec}(A_{Q_{4n}}) = \{0, (2n)^{2n-2}, (4n-4)^n, (4n-2)^n \}.
\]

Proof. The result follows from Theorem 3.3 noting that \( Z(Q_{4n}) = \{1, a^n \} \) and \( Q_{4n} \cong D_{2n} \).

4 Some well-known groups

In this section, we compute the Laplacian spectrum of the non-commuting graphs of some well-known families of finite groups. We begin with the family of finite groups having order \( pq \) where \( p \) and \( q \) are primes.

Proposition 4.1. Let \( G \) be a non-abelian group of order \( pq \), where \( p \) and \( q \) are primes with \( p \mid (q - 1) \). Then

\[
\text{L-Spec}(A_G) = \{0, (pq - q)^{q-2}, (pq - p)^{pq-2q}, (pq - 1)^q \}.
\]

Proof. It is easy to see that \( |Z(G)| = 1 \) and \( G \) is an AC-group. Also the centralizers of non-central elements of \( G \) are precisely the Sylow subgroups of \( G \). The number of Sylow \( q \)-subgroups and Sylow \( p \)-subgroups of \( G \) are one and \( q \) respectively. Hence, the result follows from Theorem 2.4.

Proposition 4.2. The Laplacian spectrum of the non-commuting graph of the quasidihedral group \( QD_{2^n} = \langle a, b : a^{2^{n-1}} = b^2 = 1, bab^{-1} = a^{2^{n-2} - 1} \rangle \), where \( n \geq 4 \), is given by

\[
\text{L-Spec}(A_{QD_{2^n}}) = \{0, (2^{n-2})^{2^{n-1}-3}, (2^n - 4)^{2^{n-2}}, (2^n - 2)^{2^{n-2}} \}.
\]
Proof. It is well-known that \( Z(QD_{2^n}) = \{1, a^{2^n-2}\} \). Also
\[
C_{QD_{2^n}}(a) = C_{QD_{2^n}}(a^i) = \langle a \rangle \quad \text{for} \quad 1 \leq i \leq 2^{n-1} - 1, i \neq 2^{n-2}
\]
and
\[
C_{QD_{2^n}}(a^i b) = \{1, a^{2^n-2}, a^i b, a^{i+2^n-2} b\} \quad \text{for} \quad 1 \leq j \leq 2^{n-2}
\]
are the only centralizers of non-central elements of \( QD_{2^n} \). Note that these centralizers are abelian subgroups of \( QD_{2^n} \). Therefore, \( QD_{2^n} \) is an AC-group. We have \( |C_{QD_{2^n}}(a)| = 2^{n-1} \) and \( |C_{QD_{2^n}}(a^i b)| = 4 \) for \( 1 \leq j \leq 2^{n-2} \). Hence, the result follows from Theorem \( 2.3 \).

**Proposition 4.3.** The Laplacian spectrum of the non-commuting graph of the projective special linear group \( PSL(2, 2^k) \), where \( k \geq 2 \), is given by
\[
\text{L-Spec}(\mathcal{A}_{PSL(2, 2^k)}) = \{0, (2^{3k} - 2^{k+1} - 1)^2, (2^{3k} - 2^{k+1})^2, (2^{3k} - 2^k - 1)^2\}.
\]

**Proof.** We know that \( PSL(2, 2^k) \) is a non-abelian group of order \( 2^k(2^{2k} - 1) \) with trivial center. By Proposition 3.21 of \([1]\), the set of centralizers of non-trivial elements of \( PSL(2, 2^k) \) is given by
\[
\{xP_{x^{-1}}, xAx^{-1}, xBx^{-1} : x \in PSL(2, 2^k)\}
\]
where \( P \) is an elementary abelian \( 2 \)-subgroup and \( A, B \) are cyclic subgroups of \( PSL(2, 2^k) \) having order \( 2^k, 2^k-1 \) and \( 2^k+1 \) respectively. Also the number of conjugates of \( P, A \) and \( B \) in \( PSL(2, 2^k) \) are \( 2^k, 2^k-1(2^k+1) \) and \( 2^k-1(2^k-1) \) respectively. Note that \( PSL(2, 2^k) \) is an AC-group and so, by \( 2.4 \), we have
\[
\mathcal{A}_{PSL(2, 2^k)} = (2^k + 1)K_{|xP_{x^{-1}}|} \cup 2^{k-1}(2^k + 1)K_{|xAx^{-1}|} \cup 2^{k-1}(2^k - 1)K_{|xBx^{-1}|}.
\]
That is, \( \mathcal{A}_{PSL(2, 2^k)} = (2^k + 1)K_{|xP_{x^{-1}}|} \cup 2^{k-1}(2^k + 1)K_{|xAx^{-1}|} \cup 2^{k-1}(2^k - 1)K_{|xBx^{-1}|} \). Hence, the result follows from Corollary \( 2.3 \).

**Proposition 4.4.** The Laplacian spectrum of the non-commuting graph of the general linear group \( GL(2, q) \), where \( q = p^n > 2 \) and \( p \) is a prime integer, is given by
\[
\text{L-Spec}(\mathcal{A}_{GL(2, q)}) = \{0, (q^4 - q^3 - 2q^2 + q + 1)^2, (q^4 - q^3 - 2q^2 + 1)^2, (q^4 - q^3 - 2q^2 + 2)^2\}
\]
\[
(q^4 - q^3 - 2q^2 + 3q - 1)^2, (q^4 - q^3 - q^2 + 1)^2, (q^4 - q^3 - q^2 + 2q)^2\}.
\]

**Proof.** We have \( |GL(2, q)| = (q^2 - 1)(q^2 - q) \) and \( |Z(GL(2, q))| = q - 1 \). By Proposition 3.26 of \([1]\), the set of centralizers of non-central elements of \( GL(2, q) \) is given by
\[
\{xDx^{-1}, xIx^{-1}, xP_{Z(GL(2, q))}x^{-1} : x \in GL(2, q)\}
\]
where \( D \) is the subgroup of \( GL(2, q) \) consisting of all diagonal matrices, \( I \) is a cyclic subgroup of \( GL(2, q) \) having order \( q^2 - 1 \) and \( P \) is the Sylow \( p \)-subgroup of \( GL(2, q) \) consisting of all upper triangular matrices with \( 1 \) in the diagonal. The orders of \( D \) and \( P_{Z(GL(2, q))} \) are \( (q - 1)^2 \) and \( q(q - 1) \) respectively. Also the number of conjugates
Hence, it can be seen that all the centralizers of non-central elements of $GL(2,q)$ are constructed by Hanaki (see [14]), and so, by (2.1), we have $A_{GL(2,q)} = \frac{q(q+1)}{2}K_{|\mu|D_{2q-1}|-q+1} \cup \frac{q(q-1)}{2}K_{|\mu|D_{2q-1}|-q+1} \cup (q+1)K_{|\mu|P_{Z(GL(2,q))}|x^{-1}|q+1}$. That is, $A_{GL(2,q)} = \frac{q(q+1)}{2}K_{2-3q+2} \cup \frac{q(q-1)}{2}K_{-q-2} \cup (q+1)K_{2-2q+1}$. Hence, the result follows from Corollary 2.3.

**Proposition 4.6.** Let $F = GF(2^n)$, $n \geq 2$ and $\vartheta$ be the Frobenius automorphism of $F$, that is, $\vartheta(x) = x^2$ for all $x \in F$. Then the Laplacian spectrum of the non-commuting graph of the group

$$A(n, \vartheta) = \left\{ U(a, b) = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & \vartheta(a) & 1 \end{bmatrix} : a, b \in F \right\}$$

under matrix multiplication given by $U(a, b)U(a', b') = U(a + a', b + b' + a'\vartheta(a))$ is

$$\text{L-Spec}(A(n, \vartheta)) = \{0, (2^{2n} - 2^{n+1})^{(2^n - 1)^2}, (2^{2n} - 2^n)^{2^n - 2}\}.$$  

**Proof.** Note that $Z(A(n, \vartheta)) = \{ U(0, b) : b \in F \}$ and so $|Z(A(n, \vartheta))| = 2^n$. Let $U(a, b)$ be a non-central element of $A(n, \vartheta)$. It can be seen that the centralizer of $U(a, b)$ in $A(n, \vartheta)$ is $Z(A(n, \vartheta)) \cup U(a, 0)Z(A(n, \vartheta))$. Clearly, $A(n, \vartheta)$ is an AC-group and so, by (2.1), we have $A_{A(n, \vartheta)} = (2^n - 1)K_{2^n}$. Hence the result follows from Corollary 2.3.

**Proposition 4.6.** Let $F = GF(p^n)$, $p$ be a prime. Then the Laplacian spectrum of the non-commuting graph of the group

$$A(n, p) = \left\{ V(a, b, c) = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} : a, b, c \in F \right\}$$

under matrix multiplication $V(a, b, c)V(a', b', c') = V(a + a', b + b' + ca', c + c')$ is

$$\text{L-Spec}(A(n, p)) = \{0, (p^{3n} - p^{2n})^{p^{3n-2p^n-1}}, (p^{3n} - p^n)^{p^n}\}.$$  

**Proof.** We have $Z(A(n, p)) = \{ V(0, 0, 0) : b \in F \}$ and so $|Z(A(n, p))| = p^n$. The centralizers of non-central elements of $A(n, p)$ are given by

(i) If $b, c \in F$ and $c \neq 0$ then the centralizer of $V(0, b, c)$ in $A(n, p)$ is $\{ V(0, b', c') : b', c' \in F \}$ having order $p^{2n}$.

(ii) If $a, b \in F$ and $a \neq 0$ then the centralizer of $V(a, b, 0)$ in $A(n, p)$ is $\{ V(a', b', 0) : a', b' \in F \}$ having order $p^{2n}$.

(iii) If $a, b, c \in F$ and $a \neq 0, c \neq 0$ then the centralizer of $V(a, b, c)$ in $A(n, p)$ is $\{ V(a', b', caa^{-1}) : a', b' \in F \}$ having order $p^{2n}$.

It can be seen that all the centralizers of non-central elements of $A(n, p)$ are abelian. Hence $A(n, p)$ is an AC-group and so, by (2.1), we have

$$A_{A(n, p)} = K_{p^{2n}-p^n} \cup K_{p^{2n}-p^n} \cup (p^n - 1)K_{p^{2n}-p^n} = (p^n + 1)K_{p^{2n}-p^n}.$$  

Hence the result follows from Corollary 2.3.

We would like to mention here that the groups considered in Proposition 4.5, 4.6 are constructed by Hanaki (see [14]). These groups are also considered in [10], in order to compute their numbers of distinct centralizers.
5 Some consequences

Note that the non-commuting graphs of all the groups considered in Section 3 and 4 are L-integral. In this section, we determine some conditions on $G$ so that its non-commuting graph becomes L-integral.

A finite group is called an $n$-centralizer group if it has $n$ numbers of distinct element centralizers. It clear that 1-centralizer groups are precisely the abelian groups. There are no 2, 3-centralizer finite groups. The study of these groups was initiated by Belcastro and Sherman [6] in the year 1994. We have the following results regarding $n$-centralizer groups.

**Proposition 5.1.** If $G$ is a finite 4-centralizer group then $\mathcal{A}_G$ is L-integral.

**Proof.** Let $G$ be a finite 4-centralizer group. Then, by [6, Theorem 2], we have $\frac{G}{Z(G)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Therefore, by Theorem 3.2 we have

$$L\text{-Spec}(\mathcal{A}_G) = \{0, (2|Z(G)|)^{3|Z(G)|-3}, (3|Z(G)|)^{2}\}. $$

Hence, $\mathcal{A}_G$ is L-integral. \qed

Further, we have the following result.

**Proposition 5.2.** If $G$ is a finite $(p+2)$-centralizer $p$-group for any prime $p$, then $\mathcal{A}_G$ is L-integral.

**Proof.** Let $G$ be a finite $(p+2)$-centralizer $p$-group. Then, by [5, Lemma 2.7], we have $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Therefore, by Theorem 3.2 we have

$$L\text{-Spec}(\mathcal{A}_G) = \{0, ((p^2 - p)|Z(G)|^{p^2-1}|Z(G)|^{p-1}, (p^2 - 1)|Z(G)|^p\}. $$

Hence, $\mathcal{A}_G$ is L-integral. \qed

**Proposition 5.3.** If $G$ is a finite 5-centralizer group then $\mathcal{A}_G$ is L-integral.

**Proof.** Let $G$ be a finite 5-centralizer group. Then by [6, Theorem 4] we have $\frac{G}{Z(G)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ or $D_6$. Now, if $\frac{G}{Z(G)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ then by Theorem 5.2 we have $L\text{-Spec}(\mathcal{A}_G) = \{0, (6|Z(G)|)^{8|Z(G)|-4}, (8|Z(G)|)^3\}$ and hence $\mathcal{A}_G$ is L-integral. If $\frac{G}{Z(G)} \cong D_6$ then, by Theorem 5.4 we have

$$L\text{-Spec}(\mathcal{A}_G) = \{0, (3|Z(G)|)^{2|Z(G)|-1}, (4|Z(G)|)^{3|Z(G)|-1}, (5|Z(G)|)^3\} $$

and hence $\mathcal{A}_G$ is L-integral. Therefore, the result follows. \qed

We also have the following corollary.

**Corollary 5.4.** Let $G$ be a finite non-abelian group and \{${x}_1, x_2, \ldots, x_r$\} be a set of pairwise non-commuting elements of $G$ having maximal size. Then $\mathcal{A}_G$ is L-integral if $r = 3, 4$.

**Proof.** By Lemma 2.4 in [2], we have that $G$ is a 4-centralizer or a 5-centralizer group according as $r = 3$ or 4. Hence the result follows from Proposition 5.1 and Proposition 5.3. \qed
The commuting probability of a finite group $G$ denoted by $\Pr(G)$ is the probability that any two randomly chosen elements of $G$ commute. Clearly, $\Pr(G) = 1$ if and only if $G$ is abelian. The study of $\Pr(G)$ is originated from a paper of Erdős and Turán [13]. Various results on $\Pr(G)$ can be found in [7, 9, 19]. The following results show that $A_G$ is L-integral if $\Pr(G)$ has some particular values.

**Proposition 5.5.** If $\Pr(G) \in \{\frac{5}{14}, \frac{2}{5}, \frac{11}{27}, \frac{1}{2}, \frac{5}{8}\}$ then $A_G$ is L-integral.

*Proof.* If $\Pr(G) \in \{\frac{5}{14}, \frac{2}{5}, \frac{11}{27}, \frac{1}{2}, \frac{5}{8}\}$ then as shown in [22, pp. 246] and [20, pp. 451], we have $\frac{G}{Z(G)}$ is isomorphic to one of the groups in $\{D_{14}, D_{10}, D_8, D_6, \mathbb{Z}_2 \times \mathbb{Z}_2\}$. If $\frac{G}{Z(G)}$ is isomorphic to $D_{14}, D_{10}, D_8$ or $D_6$ then, by Theorem 3.4 it follows that $A_G$ is L-integral. If $\frac{G}{Z(G)}$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ then, by Theorem 3.2 it follows that $A_G$ is L-integral. Hence, the result follows.

**Proposition 5.6.** Let $G$ be a finite group and $p$ the smallest prime divisor of $|G|$. If $\Pr(G) = \frac{2^2p + 1}{p}$ then $A_G$ is L-integral.

*Proof.* If $\Pr(G) = \frac{2^2p + 1}{p}$ then by [16, Theorem 3] we have $\frac{G}{Z(G)}$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. Now, by Theorem 3.2 it follows that $A_G$ is L-integral.

**Proposition 5.7.** If $G$ is a non-solvable group with $\Pr(G) = \frac{1}{2}$ then $A_G$ is L-integral.

*Proof.* By [7, Proposition 3.3.7], we have that $G$ is isomorphic to $A_5 \times B$ for some abelian group $B$. Since $A_5$ is an AC-group, by Corollary 3.3 it follows that $A_G$ is L-integral.

A graph is called planar if it can be embedded in the plane so that no two edges intersect geometrically except at a vertex to which both are adjacent. We conclude this paper with the following result.

**Proposition 5.8.** Let $G$ be a finite group then $A_G$ is L-integral if $A_G$ is planar.

*Proof.* It was shown in Proposition 2.3 of [11] that $A_G$ is planar if and only if $G$ is isomorphic to $D_6, D_8$ or $Q_8$. Therefore, by Corollary 5.6 and Corollary 5.7 the result follows.

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