DETERMINANTS ASSOCIATED TO TRACES ON OPERATOR
BIMODULES

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Abstract. Given a II_1-factor \( M \) with tracial state \( \tau \) and given an \( M \)-bimodule \( \mathcal{E}(M, \tau) \) of operators affiliated to \( M \) we show that traces on \( \mathcal{E}(M, \tau) \) (namely, linear functionals that are invariant under unitary conjugation) are in bijective correspondence with rearrangement-invariant linear functionals on the corresponding symmetric function space \( E \). We also show that, given a positive trace \( \varphi \) on \( \mathcal{E}(M, \tau) \), the map \( \det_\varphi : \mathcal{E}_{\text{log}}(M, \tau) \to [0, \infty) \) defined by
\[
\det_\varphi(T) = \exp(\varphi(\log |T|))
\]
when \( \log |T| \in \mathcal{E}(M, \tau) \) and 0 otherwise, is multiplicative on the \( * \)-algebra \( \mathcal{E}_{\text{log}}(M, \tau) \) that consists of all affiliated operators \( T \) such that \( \log(\mathcal{T}) \in \mathcal{E}(M, \tau) \). Finally, we show that all multiplicative maps on the invertible elements of \( \mathcal{E}_{\text{log}}(M, \tau) \) arise in this fashion.

1. Introduction

Let \( M \) be a von Neumann algebra factor of type II_1, with tracial state \( \tau \). Assume \( M \) has separable predual. The Fuglede–Kadison determinant \( [8] \), is the multiplicative map \( \Delta_\tau : M \to [0, \infty) \) defined by
\[
\Delta_\tau(T) = \lim_{\epsilon \to 0^+} \exp(\tau(\log(|T| + \epsilon))).
\]
In this paper, we prove multiplicativity of analogous determinants corresponding to arbitrary positive traces on arbitrary \( M \)-bimodules of affiliated operators.

Choose any normal representation of \( M \) on a Hilbert space and let \( S(M, \tau) \) be the \( * \)-algebra of (possibly unbounded) operators on the Hilbert space affiliated to \( M \). This algebra, often called the Murray-von Neumann algebra of \( M \), is independent of the representation. See, for example, Section 6 of [11] for an exposition of this theory. Let \( \text{Proj}(M) \) denote the set of projections (i.e., self-adjoint idempotents) in \( M \). For \( A \in S(M, \tau) \) and \( t \in (0, 1) \), \( \mu(t, A) \) denotes the generalized singular number of \( A \), defined by
\[
\mu(t, A) = \inf\{\|A(1 - p)\| \mid p \in \text{Proj}(M), \tau(p) \leq t\},
\]
where \( \| \cdot \| \) is the operator norm. This goes back to Murray and von Neumann; see, for example, Section 2.3 of [14] for some basic theory. We will write simply \( \mu(A) \) for the function \( t \mapsto \mu(t, A) \), which is nonincreasing and right continuous.

Let \( E \) be a complex vector space of measurable functions on \([0, 1]\) with the property that if \( f \) and \( g \) are measurable functions with \( f^* \leq g^* \) and \( g \in E \), then \( f \in E \), where \( f^* \) denotes the decreasing rearrangement of \( |f| \). Following [14], we will call such a space \( E \) a Calkin function space. Note that if \( f \in E \) implies that the dilation \( D_2f \) lies in \( E \), where \( D_2f(t) = f(t/2) \). In particular, every nonzero Calkin function space contains \( L_\infty[0, 1] \).

The corresponding \( M \)-bimodule \( \mathcal{E}(M, \tau) \) is the set of all \( A \in S(M, \tau) \) such that \( \mu(A) \in E \).

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This correspondence, sometimes called the Calkin correspondence in the setting of \((\mathcal{M}, \tau)\), is a bijection from the set of all Calkin function spaces onto the set of all operator \(\mathcal{M}\)-bimodules, by which we mean subspaces of \(S(\mathcal{M}, \tau)\) that are closed under left and right multiplication by elements of \(\mathcal{M}\), and it goes back to Guido and Isola \([9]\). See Theorem 2.4.4 of \([13]\) for the formulation used here. An equivalent version of this is also described in \([4]\).

Note that if \(\mathcal{A} \subseteq \mathcal{M}\) is any unital abelian von Neumann subalgebra that is diffuse (i.e., has no minimal projections), then the \(\ast\)-algebra \(S(\mathcal{A}, \tau |\mathcal{A})\) of affiliated operators is naturally embedded in \(S(\mathcal{M}, \tau)\) and, upon identifying \(\mathcal{A}\) with \(L_\infty(0, 1)\), the elements of \(S(\mathcal{A}, \tau |\mathcal{A})\) are naturally identified with measurable functions on \((0, 1)\). Under these identifications, we have \(E = S(\mathcal{A}, \tau |\mathcal{A}) \cap \mathcal{E}(\mathcal{M}, \tau)\).

By a trace on \(\mathcal{E}(\mathcal{M}, \tau)\), we mean a linear functional \(\varphi \) of \(\mathcal{E}(\mathcal{M}, \tau)\) such that \(\varphi(UAU^*) = \varphi(A)\) for every \(A \in \mathcal{E}(\mathcal{M}, \tau)\) and every unitary \(U \in \mathcal{M}\). A functional \(\varphi_0\) of \(E\) is said to be rearrangement-invariant if \(\varphi_0(f) = \varphi_0(g)\) whenever \(f, g \in E\), \(f, g \geq 0\) and \(f^* = g^*\).

The difficult half of the following result is essentially proved in \([13]\). The proof of the other half is similar to the proof of Lemma 9.4 of \([6]\).

**Theorem 1.1.** Let \(\mathcal{M}\) be a II_1-factor with separable predual. Let \(E\) be a Calkin function space and let \(\mathcal{E}(\mathcal{M}, \tau)\) be the corresponding \(\mathcal{M}\)-bimodule. There is a bijection from the set of all traces of \(\mathcal{E}(\mathcal{M}, \tau)\) onto the set of all rearrangement-invariant functionals of \(E\), whereby a trace \(\varphi\) of \(\mathcal{E}(\mathcal{M}, \tau)\) is mapped to a functional \(\varphi_0\) of \(E\) satisfying

\[
\varphi_0(\mu(A)) = \varphi(A) \text{ whenever } A \in \mathcal{E}(\mathcal{M}, \tau) \text{ and } A \geq 0. \tag{2}
\]

**Proof.** Suppose \(\varphi_0 : E \to \mathbb{C}\) is a rearrangement-invariant linear functional. By the proof of (part of) Theorem 5.2 of \([13]\), there is a trace \(\varphi : \mathcal{E}(\mathcal{M}, \tau) \to \mathbb{C}\) satisfying (2). The statement of that theorem includes additional assumptions about \(E\), namely, that it carries a rearrangement-invariant complete norm. However, the proof found in \([13]\) is valid, verbatim, in the more general situation considered here.

Suppose \(\varphi : \mathcal{E}(\mathcal{M}, \tau) \to \mathbb{C}\) is a trace. We will now show that for any \(A \in \mathcal{E}(\mathcal{M}, \tau)\) that is positive, \(\varphi(A)\) depends only on \(\mu(A)\). Indeed, let \(A_1, A_2 \in \mathcal{E}(\mathcal{M}, \tau)\) be such that \(A_1, A_2 \geq 0\) and \(\mu(A_1) = \mu(A_2)\). Set

\[
B_k = \sum_{n \geq 0} n1_{[n,n+1)}(A_k), \quad C_k = A_k - B_k, \quad k = 1, 2.
\]

Clearly, positive operators \(B_1\) and \(B_2\) have discrete spectrum and \(\mu(B_1) = \mu(B_2)\). Since \(\mathcal{M}\) is a factor, one can choose a unitary element \(U \in \mathcal{M}\) such that \(B_1 = UB_2U^{-1}\). Clearly, \(\varphi(B_1) = \varphi(UB_2U^{-1}) = \varphi(B_2)\). By Theorem 2.3 in \([7]\), we have \(\varphi |\mathcal{M} = c_\varphi \tau |\mathcal{M}\) for a constant \(c_\varphi\). For bounded positive operators \(C_1\) and \(C_2\), we have \(\mu(C_1) = \mu(C_2)\) and also, therefore,

\[
\varphi(C_1) = c_\varphi \tau(C_1) = c_\varphi \tau(C_2) = \varphi(C_2).
\]

Thus, we get

\[
\varphi(A_1) = \varphi(B_1) + \varphi(C_1) = \varphi(B_2) + \varphi(C_2) = \varphi(A_2).
\]

Let \(\mathcal{A}\) be any unital, diffuse, abelian von Neumann subalgebra of \(\mathcal{M}\). As described above, \(E\) is naturally identified with \(S(\mathcal{A}, \tau |\mathcal{A}) \cap \mathcal{E}(\mathcal{M}, \tau)\), and restricting \(\varphi\) to this subalgebra yields a linear functional \(\varphi_0\) on \(E\), which is rearrangement-invariant and satisfies (2), because of the fact that \(\varphi(A)\) depends only on \(\mu(A)\) for all \(A \geq 0\). Using (2), we see that the functional \(\varphi_0\) does not depend on \(\mathcal{A}\), namely, does not depend on which copy of \(E\) we chose in \(\mathcal{E}(\mathcal{M}, \tau)\).

Finally, as \(\varphi\) is uniquely determined by \(\varphi_0\) and the condition (2), we see that the map \(\varphi \mapsto \varphi_0\) is the desired bijection. 

\(\square\)
For convenience, we will use also \( \varphi \), instead of \( \varphi_0 \), to denote the functional on \( E \) corresponding to a trace \( \varphi \) on \( \mathcal{E}(\mathcal{M}, \tau) \).

For example, taking \( E \) to be the function space \( L_1 \) of complex-valued functions on \([0,1]\) that are integrable with respect to Lebesgue measure, the corresponding bimodule is \( \mathcal{L}_1(\mathcal{M}, \tau) \). Moreover, the functional \( f \mapsto \int_0^1 f(t) \, dt \) on \( L_1 \) corresponds to the usual trace \( \tau \) on \( \mathcal{L}_1(\mathcal{M}, \tau) \). Other examples of traces on bimodules are provided by the Dixmier traces on Marcinkiewicz bimodules, which are of interest in noncommutative geometry. See, for example, [3], [2] and [12]; particularly, consider the treatment of functionals supported at zero, but adapted to the case of a \( \hbox{II}_1 \)-factor \( \mathcal{M} \), namely, corresponding to function spaces on \([0,1]\). A specific case (essentially, taken from [3]) is found in Example 3.3.

The Fuglede-Kadison determinant mentioned at the start of this introduction is actually naturally defined on the space, sometimes denoted \( \mathcal{L}_{\log}(\mathcal{M}, \tau) \), of all \( T \in \mathcal{S}(\mathcal{M}, \tau) \) such that \( \log_+(|T|) \in \mathcal{L}_1(\mathcal{M}, \tau) \), where \( \log_+(t) = \max(\log(t), 0) \). See [10] for a development of \( \Delta_\tau \) in this generality, including a proof of multiplicativity.

In the rest of this paper, we will for the most part consider only positive traces \( \varphi \), namely, those satisfying

\[
A \geq 0 \implies \varphi(A) \geq 0
\]

(the exception being Lemma 2.8). Positive traces correspond, under the rubric of Theorem 1.1, to positive rearrangement-invariant linear functionals. In the following, we use the function \( \log_- (t) = - \min(\log(t), 0) \); thus, \( \log = \log_+ - \log_- \).

**Definition 1.2.** Let \( \mathcal{M} \) be a \( \hbox{II}_1 \)-factor and consider a positive trace \( \varphi \) on an \( \mathcal{M} \)-bimodule \( \mathcal{E}(\mathcal{M}, \tau) \). Let \( \mathcal{E}_{\log}(\mathcal{M}, \tau) \) be the set of all \( T \in \mathcal{S}(\mathcal{M}, \tau) \) such that \( \log_+ (|T|) \in \mathcal{E}(\mathcal{M}, \tau) \) and for such \( T \) let

\[
\det_\varphi(T) = \begin{cases} 
\exp(\varphi(\log(|T|))), & \text{ker } T = \{0\} \text{ and } \log_+ (|T|) \in E \\
0, & \text{ker } T = \{0\} \text{ and } \log_+ (|T|) \notin E \\
0, & \text{ker } T \neq \{0\}.
\end{cases}
\]

Thus, in the case \( E = L_1 \) and \( \varphi = \tau \), we have the Fuglede-Kadison determinant: \( \det_\tau = \Delta_\tau \). The natural domain of this determinant by the above rubric should be written \( \mathcal{L}_{\log}(\mathcal{M}, \tau) \), but we will write \( \mathcal{E}_{\log}(\mathcal{M}, \tau) \) for this, in keeping with earlier convention (cf [5], [6]).

The main result of this paper is:

**Theorem 1.3.** For an arbitrary Calkin function space \( E \) on \([0,1]\) and arbitrary positive trace \( \varphi \) on the corresponding bimodule \( \mathcal{E}(\mathcal{M}, \tau) \), the set \( \mathcal{E}_{\log}(\mathcal{M}, \tau) \) is a *-subalgebra of \( \mathcal{S}(\mathcal{M}, \tau) \) and, if \( A, B \in \mathcal{E}_{\log}(\mathcal{M}, \tau) \), then

\[
\det_\varphi(AB) = \det_\varphi(A)\det_\varphi(B).
\]

The proof, presented in the next section, relies on Fuglede and Kadison’s result [3] that \( \Delta_\tau \) is multiplicative on \( \mathcal{M} \) and on the characterization from [4] of sums of \( \mathcal{E}(\mathcal{M}, \tau) \), \( \mathcal{M} \)-commutators. Thus, a special case of this proof yields an alternative proof of Haagerup and Schultz’s result [10] about the extension of the Fuglede–Kadison determinant to \( \mathcal{E}_{\log}(\mathcal{M}, \tau) \).

**Remark 1.4.** It is immediate that \( \det_\varphi(1) = 1 \) and, for \( T \in \mathcal{E}_{\log}(\mathcal{M}, \tau) \), \( \det_\varphi(T) = 0 \) if and only if \( T \) fails to be invertible in \( \mathcal{E}_{\log}(\mathcal{M}, \tau) \).

**Remark 1.5.** In the case that \( \varphi = 0 \), we clearly have, for \( T \in \mathcal{E}_{\log}(\mathcal{M}, \tau) \),

\[
\det_\varphi(T) = \begin{cases} 
1 \text{ if } T \text{ is invertible in } \mathcal{E}_{\log}(\mathcal{M}, \tau) \\
0 \text{ otherwise}
\end{cases}
\]
However, if \( \varphi \neq 0 \), then \( \det \varphi \) is onto \([0, \infty)\).

**Remark 1.6.** It is not difficult to see, in the case \( \varphi = \tau \), that Definition 1.2 agrees with the definition by equation (11), in fact even for all \( T \in \log(M, \tau) \). However, the analogous statement is not true for general traces \( \varphi \). In fact, it obviously fails when \( \varphi = 0 \), (see Remark 1.5, above). See Example 3.3 for specific examples of this failure when \( \varphi \neq 0 \).

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**Proposition 1.7.** For an arbitrary Calkin function space \( E \) on \([0, 1]\) and an arbitrary map

\[
m : \log(M, \tau) \rightarrow [0, \infty)
\]

that is multiplicative, order-preserving and nonzero, there exists a positive trace \( \varphi \) on \( \log(M, \tau) \) such that \( m(X) = \det_{\varphi}(X) \) for every invertible element \( X \) in \( \log(M, \tau) \).

We will show (in Proposition 3.2) that we cannot hope for \( m \) to agree with \( \det_{\varphi} \) on all of \( \log(M, \tau) \).

The proofs of Theorem 1.3 and Proposition 3.2 are contained in the next two sections.

2. **Proof of Theorem 1.3**

Let us begin by describing some further notation and standard conventions.

- \( S(0, 1) \) will denote the set of all complex-valued Borel measurable functions on \([0, 1]\) and \( L_\infty \) will denote the set of all essentially bounded elements of \( S(0, 1) \). As usual, we consider functions that are equal almost everywhere to be the same.
- We will apply the Borel functional calculus to self-adjoint elements \( T \in S(M, \tau) \), and will also use the standard notation \( T_+ = \max(T, 0) \) and \( T_- = -\min(T, 0) \).
- For self-adjoint \( A \in S(M, \tau) \), we consider its eigenvalue function (or spectral scale), defined for \( t \in (0, 1) \) by

\[
\lambda(t, A) = \inf \{ s \in \mathbb{R} \mid \tau(1_{[s, \infty)}(A)) \leq t \},
\]

where, in accordance with notation for the Borel functional calculus, \( 1_{(s, \infty)}(A) \) denotes the spectral projection of \( A \) associated to the interval \((s, \infty)\). This also goes back to Murray and von Neumann. We will write simply \( \lambda(A) \) for the function \( t \mapsto \lambda(t, A) \), which is nonincreasing and right continuous. Note that, if \( A \geq 0 \), then \( \lambda(A) = \mu(A) \).

Moreover, when \( a \leq b \), with \( a \leq \lim_{t \to 0} \lambda(t, A) \) and \( b \geq \lim_{t \to 1} \lambda(t, A) \), we have

\[
\tau(A1_{[a,b]}(A)) = \int_c^d \lambda(t, A) \, dt,
\]

\[
\tau(1_{[a,b]}(A)) = d - c,
\]

where

\[
c = \inf \{ s \mid \lambda(s, A) \leq b \}, \quad d = \sup \{ s \mid \lambda(s, A) \geq a \}.
\]

For any \( T \in S(M, \tau) \), since \( \mu(T) = \mu(|T|) = \lambda(|T|) \), from (5), we get

\[
\tau(1_{[0,\mu(T)]}(|T|)) \geq 1 - t.
\]

- The following inequalities are standard (see, for example, Corollary 2.3.16 of [14]): for all \( A, B \in S(M, \tau) \), if \( s, t > 0 \) and \( s + t < 1 \), then

\[
\mu(s + t, A + B) \leq \mu(s, A) + \mu(t, B),
\]

\[
\mu(s + t, AB) \leq \mu(s, A)\mu(t, B).
\]
Lemma 2.1. Let $T, S \in \mathcal{S}(\mathcal{M}, \tau)$ be self-adjoint. Then for every $t \in (0, \frac{1}{2})$, we have
\[
\left| \int_{2t}^{1-2t} (\log(\mu(u, e^T e^S)) - \lambda(u, T) - \lambda(u, S)) \, du \right| \leq 8t (\mu(t, T) + \mu(t, S)).
\]

Proof. Fix $t \in (0, \frac{1}{2})$ and, using the continuous functional calculus, set
\[
T_0 = \min\{T_+, \mu(t, T)\} - \min\{T_-, \mu(t, T)\},
S_0 = \min\{S_+, \mu(t, S)\} - \min\{S_-, \mu(t, S)\}.
\]
We have
\[
T - T_0 = (T_+ - \mu(t, T))_+ - (T_- - \mu(t, T))_+,
\]
\[
|T - T_0| = (T_+ - \mu(t, T))_+ + (T_- - \mu(t, T))_+ = (|T| - \mu(t, T))_+.
\]
Thus, we have $(T - T_0)1_{[0, \mu(t, T)]}(T) = 0$ and, using (9), we get $\mu(t, T - T_0) = 0$; similarly, we have $\mu(t, S - S_0) = 0$. Using (8), for every $u \in (2t, 1)$ we have
\[
\mu(u, e^T e^S) = \mu(u, e^{T-T_0} \cdot e^{T_0} e^S) \leq \mu(u, e^{T-T_0}) \mu(u - 2t, e^{T_0} e^{-S_0}) \mu(t, e^{S_0-S}),
\]
\[
\mu(u, e^{T_0} e^S) = \mu(u, e^{T_0} - T \cdot e^T e^S) \leq \mu(u, e^{T_0-T}) \mu(u - 2t, e^T e^S) \mu(t, e^{S_0-S}),
\]
Since $\mu(t, e^{T-T_0}) \leq 1$ and $\mu(t, e^{T_0-T}) \leq 1$ and similarly for $S - S_0$, we get
\[
\mu(u, e^T e^S) \leq \mu(u - 2t, e^{T_0} e^S), \quad \mu(u, e^{T_0} e^S) \leq \mu(u - 2t, e^T e^S).
\]
Thus, for $u \in (2t, 1 - 2t)$, we have
\[
\mu(u + 2t, e^{T_0} e^S) \leq \mu(u, e^T e^S) \leq \mu(u - 2t, e^{T_0} e^S).
\]
It follows that
\[
\int_{4t}^{1} \log(\mu(u, e^{T_0} e^S)) \, du \leq \int_{2t}^{1-2t} \log(\mu(u, e^T e^S)) \, du \leq \int_{0}^{1-4t} \log(\mu(u, e^{T_0} e^S)) \, du. \tag{10}
\]
Since $-\mu(t, T) \leq T_0 \leq \mu(t, T)$ and similarly for $S_0$, we also have
\[
e^{-\mu(t,T)-\mu(t,S)} \leq \mu(e^{T_0} e^S) \leq e^{\mu(t,T)+\mu(t,S)}.
\]
Thus,
\[
\| \log(\mu(e^{T_0} e^S)) \|_{\infty} \leq \mu(t, T) + \mu(t, S).
\]
In particular,
\[
\left| \int_{0}^{4t} \log(\mu(u, e^{T_0} e^S)) \, du \right| \leq 4t \| \log(\mu(e^{T_0} e^S)) \|_{\infty} \leq 4t (\mu(t, T) + \mu(t, S)),
\]
\[
\left| \int_{-4t}^{1} \log(\mu(u, e^{T_0} e^S)) \, du \right| \leq 4t \| \log(\mu(e^{T_0} e^S)) \|_{\infty} \leq 4t (\mu(t, T) + \mu(t, S)).
\]
Using (10), we get
\[
\left| \int_{2t}^{1-2t} \log(\mu(u, e^T e^S)) \, du - \int_{0}^{1} \log(\mu(u, e^{T_0} e^S)) \, du \right| \leq 4t (\mu(t, T) + \mu(t, S)).
\]
Since the Fuglede-Kadison determinant $\Delta_\tau$ is multiplicative on $\mathcal{M}$, we have
\[
\int_0^1 \log(\mu(u, e^{T_0}e^{S_0})) \, du = \log(\Delta_\tau(e^{T_0}e^{S_0}))
= \log(\Delta_\tau(e^{T_0})) + \log(\Delta_\tau(e^{S_0})) = \tau(T_0) + \tau(S_0).
\]
But using
\[
\left| \tau(T_0) - \int_{2t}^{1-2t} \lambda(u, T) \, du \right| \leq 4t \mu(t, T),
\]
and the same also for $S$, the assertion follows. \hfill \Box

In the following, we use the notation \cite{10} for the left-continuous versions of monotone functions. (Though, as elements of $E$, $\mu(T)$ and the left-continuous version $\bar{\mu}(T)$ are identified, these functions $\mu(T)$ and similarly $\lambda(T)$ are of interest aside from their membership in $E$, and for correctness at all points of $(0, 1)$ we must use their left-continuous versions in the following inequalities and elsewhere below.)

\textbf{Lemma 2.2.} If $S, T \in \mathcal{S}(\mathcal{M}, \tau)$ are self-adjoint, then for all $u \in (0, 1)$, we have
\[
- \bar{\mu}\left(\frac{1-u}{2}, T\right) - \bar{\mu}\left(\frac{1-u}{2}, S\right) \leq \log(\mu(u, e^T e^S)) \leq \mu\left(\frac{u}{2}, T\right) + \mu\left(\frac{u}{2}, S\right). \tag{11}
\]

\textbf{Proof.} Using \cite{11}, we get
\[
\mu(u, e^T e^S) \leq \mu\left(\frac{u}{2}, e^T\right)\mu\left(\frac{u}{2}, e^S\right) \leq \mu\left(\frac{u}{2}, e^T\right)\mu\left(\frac{u}{2}, e^S\right) = e^{\mu\left(\frac{u}{2}, T\right) + \mu\left(\frac{u}{2}, S\right)} \leq e^{\mu\left(\frac{u}{2}, T\right) + \mu\left(\frac{u}{2}, S\right)}, \tag{12}
\]
which yields the right-most inequality in \eqref{11}. Replacing $S$ with $-T$ and $T$ with $-S$ in \eqref{12}, we get
\[
\mu(u, e^{-S} e^{-T}) \leq e^{\mu\left(\frac{u}{2}, T\right) + \mu\left(\frac{u}{2}, S\right)}, \quad \bar{\mu}(u, e^{-S} e^{-T}) \leq e^{\bar{\mu}\left(\frac{u}{2}, T\right) + \bar{\mu}\left(\frac{u}{2}, S\right)}. \tag{13}
\]
As is well known and easy to show,
\[
\mu(u, e^T e^S) = \frac{1}{\bar{\mu}(1-u, e^{-S} e^{-T})}.
\]
Thus, replacing $u$ with $1-u$ in \eqref{13}, we get
\[
\mu(u, e^T e^S) \geq e^{-\bar{\mu}\left(\frac{1-u}{2}, T\right) - \bar{\mu}\left(\frac{1-u}{2}, S\right)} \geq e^{-\bar{\mu}\left(\frac{1-u}{2}, T\right) - \bar{\mu}\left(\frac{1-u}{2}, S\right)},
\]
which yields the left-most inequality in \eqref{11}. \hfill \Box

The next lemma is a combination of Theorems 3.3.3 and 3.3.4 from \cite{12}.

\textbf{Lemma 2.3.} If $S, T \in \mathcal{M}$ are positive, then
\[
\int_0^t \mu(u, T+S) \, du \leq \int_0^t \left(\mu(u, T) + \mu(u, S)\right) \, du \leq \int_0^{2t} \mu(u, T+S) \, du.
\]

\textbf{Proof.} This follows easily from the fact that, for a positive operator, $T$, we have
\[
\int_0^t \mu(u, T) \, du = \sup\{\tau(pT) \mid p \in \text{Proj}(\mathcal{M}), \tau(p) \leq t\}.
\]
\hfill \Box
For every function $f \in S(0,1)$ that is bounded on compact subsets of $(0,1)$, define
\[
(\Psi f)(t) = \begin{cases} 
\frac{1}{t} \int_t^{1-t} f(s) \, ds, & 0 < t < \frac{1}{2}, \\
0, & \frac{1}{2} \leq t \leq 1.
\end{cases}
\]
Clearly, $\Psi f$ is continuous on $(0,1]$ and $\Psi$ is linear. Note that $\Psi$ is defined on every function arising as $\mu(A)$ or $\lambda(A)$ for $A \in S(M, \tau)$.

**Lemma 2.4.** Let $S, T \in \mathcal{E}(M, \tau)$ be positive. Then
\[
\Psi(\mu(T + S) - \mu(T) - \mu(S)) \in E.
\]

**Proof.** First suppose $S, T \in \mathcal{M}$ are positive. From Lemma 2.3 and the fact that $\tau(T) = \int_0^1 \mu(u, T) \, du$, we have
\[
\int_{2t}^1 \mu(u, T + S) \, du \leq \int_{t}^1 (\mu(u, T) + \mu(u, S)) \, du \leq \int_t^1 \mu(u, T + S) \, du. \tag{14}
\]
For arbitrary positive $S, T \in S(\mathcal{M}, \tau)$, set $T_n = \min\{T, n\}$ and $S_n = \min\{S, n\}$. Since $\mu(T_n) \uparrow \mu(T)$, $\mu(S_n) \uparrow \mu(S)$ and $\mu(T_n + S_n) \uparrow \mu(T + S)$, it follows from the Monotone Convergence Principle that $[14]$ also holds. From (14), we have
\[
\left| \int_t^1 (\mu(u, T + S) - \mu(u, T) - \mu(u, S)) \, du \right| \leq \int_t^{2t} \mu(u, T + S) \, du \leq t\mu(t, T + S).
\]
Thus, for $t \in (0, \frac{1}{2})$, we have
\[
\left| \int_{t}^{1-t} (\mu(u, T + S) - \mu(u, T) - \mu(u, S)) \, du \right|
\leq \left| \int_t^1 (\mu(u, T + S) - \mu(u, T) - \mu(u, S)) \, du \right|
\leq t\mu(t, T + S) + t\mu(1-t, T + S) + t\mu(1-t, T) + t\mu(1-t, S) \leq 4t\mu(t, T + S).
\]
This concludes the proof. \qed

**Lemma 2.5.** Let $T \in S(M, \tau)$ be self-adjoint. Then
\[
\Psi(\lambda(T) - \mu(T^+) + \mu(T^-)) \in L_\infty.
\]

**Proof.** If $T^+ = 0$ or $T^- = 0$, then $\lambda(T) = \mu(T^+) - \mu(T^-)$. Suppose $T^+ \neq 0$ and $T^- \neq 0$. Let $t_0$ be the trace of the support projection of $T^+$. We have
\[
\lambda(u, T) = \begin{cases} 
\mu(u, T^+), & u \in (0, t_0) \\
-\bar{\mu}(1-u, T^-), & u \in [t_0, 1).
\end{cases}
\]
It follows that, for all sufficiently small $t$, we have
\[
t(\Psi\lambda(T))(t) = \int_t^{t_0} \lambda(u, T) \, du + \int_{t_0}^{1-t} \lambda(u, T) \, du
= \int_t^{t_0} \mu(u, T^+) \, du - \int_{t_0}^{1-t} \mu(1-u, T^-) \, du = \int_t^{t_0} \mu(u, T^+) \, du - \int_t^{1-t_0} \mu(u, T^-) \, du
= \int_t^1 (\mu(u, T^+) - \mu(u, T^-)) \, du = t(\Psi(\mu(T^+) - \mu(T^-)))(t),
\]

where the last equality holds because the integrand is zero when \( u \) is sufficiently close to 1. Thus, \( \Psi(\lambda(T) - \mu(T_+) + \mu(T_-))(t) \) vanishes for all \( t \) sufficiently small. Since this function is continuous on \((0,1] \), it is bounded. \( \square \)

**Lemma 2.6.** Let \( S,T \in \mathcal{E}(\mathcal{M},\tau) \) be self-adjoint. Then

\[
\Psi(\lambda(T) + \lambda(S) - \lambda(T+S)) \in E.
\]

**Proof.** We have

\[
(T+S)_+ - (T+S)_- = T_+ - T_- + S_+ - S_-.
\]

Therefore,

\[
(T+S)_+ + T_- + S_+ = (T+S)_- + T_+ + S_+.
\]

Denote the above quantity by \( A \). From Lemma 2.4 we obtain

\[
\Psi(\mu(A) - \mu((T+S)_+) - \mu(T_-) - \mu(S_-)) \in E,
\]

\[
\Psi(\mu(A) - \mu((T+S)_-) - \mu(T_+) - \mu(S_+)) \in E.
\]

Subtracting those formulae, we obtain

\[
\Psi(\mu((T+S)_+) - \mu((T+S)_-) - \mu(T_+) + \mu(T_-) - \mu(S_-) + \mu(S_+)) \in E.
\]

The assertion follows now from Lemma 2.5 as applied to the operators \( T,S \) and \( T+S \), and the fact that \( E \) contains \( L_\infty \). \( \square \)

In the next result, the notation \([\mathcal{E}(\mathcal{M},\tau),\mathcal{M}]\) denotes the space spanned by the set of all commutators of the form \([S,T] = ST - TS\), for \( S \in \mathcal{M} \) and \( T \in \mathcal{E}(\mathcal{M},\tau) \). It amounts to a reformulation of a special case of Theorem 4.6 of [4].

**Theorem 2.7.** Let \( T \in \mathcal{E}(\mathcal{M},\tau) \) be self-adjoint. Then \( T \in [\mathcal{E}(\mathcal{M},\tau),\mathcal{M}] \) if and only if \( \Psi\lambda(T) \in E \).

**Proof.** By Theorem 4.6 of [4], \( T \in [\mathcal{E}(\mathcal{M},\tau),\mathcal{M}] \) if and only if the function

\[
r \mapsto \frac{1}{r} \tau(1_{[0,\mu(r,T)]})(|T|)T
\]

belongs to \( E \). Thus, it will suffice to show that the function

\[
r \mapsto \frac{1}{r} \tau(1_{[0,\mu(r,T)]})(|T|)T - \Psi\lambda(T)(r)
\]

(15)

belongs to \( E \). First suppose \( T_- = 0 \). Then, using \( \lambda(T) = \mu(T) \) and (4), we have

\[
\tau(1_{[0,\mu(r,T)]})(|T|)T = \int_{r'}^1 \mu(t,T) \, dt,
\]

where \( r' = \inf\{s \mid \mu(s,T) \leq \mu(r,T)\} \). Thus \( r' \leq r \) and, for \( 0 < r < \frac{1}{2} \),

\[
\left| \tau(1_{[0,\mu(r,T)]})(|T|)T - \int_r^{1-r} \lambda(t,T) \, dt \right| \leq (r - r')\mu(r,T) + r\mu(1-r,T) \leq 2r\mu(r,T),
\]

which implies that the function (15) belongs to \( E \).

If \( T_+ = 0 \), then we may of course replace \( T \) by \(-T\) and we are done.

Suppose \( T_+ \neq 0 \) and \( T_- \neq 0 \). Letting, \( t_0 = \inf\{t \mid \lambda(t,T_+) \geq 0\} \), we have \( 0 < t_0 < 1 \) and

\[
\lambda(t,T) = \begin{cases} 
\mu(t,T_+), & 0 < t < t_0 \\
\bar{\mu}(1-t,T_+), & t_0 \leq t < 1.
\end{cases}
\]
For $r < t_0$, we have

$$
\tau(1_{[0,\mu(r,T)]}(|T|)T) = \tau(1_{[-\mu(r,T),\mu(r,T)]}(T)T) = \tau(1_{[0,\mu(r,T)]}(T_+)T_+) - \tau(1_{[0,\mu(r,T)]}(T_-)T_-)
$$

$$
= \int_{r'}^{t_0} \lambda(t, T) \, dt + \int_{t_0}^{1-r''} \lambda(t, T) \, dt,
$$

where

$$
r' = \inf\{ s \mid \mu(s, T_+) \leq \mu(r, T) \} \quad \text{(16)}
$$

$$
r'' = \inf\{ s \mid \mu(s, T_-) \leq \mu(r, T) \}. \quad \text{(17)}
$$

Since $\mu(r, T_+) \leq \mu(r, T)$, we have $r', r'' \leq r$. Thus, we have

$$
\left| \tau(1_{[0,\mu(r,T)]}(|T|)T) - \int_r^{1-r} \lambda(t, T) \, dt \right| = \left| \int_{r'}^{r} \lambda(t, T) \, dt + \int_{1-r}^{r''} \lambda(t, T) \, dt \right|
$$

$$
\leq \int_{r'}^{r} \mu(t, T_+) \, dt + \int_{r''}^{r} \mu(t, T_-) \, dt \leq (r-r')\mu(r', T_+) + (r-r'')\mu(r'', T_-) \leq 2r\mu(r, T),
$$

where for the last inequality we used (16)–(17). This shows that the function (15) belongs to $E$ and, thus, completes the proof. □

**Lemma 2.8.** Let $\varphi : \mathcal{E}(\mathcal{M}, \tau) \to \mathbb{C}$ be a trace. If $T \in \mathcal{E}(\mathcal{M}, \tau)$ is self-adjoint and is such that $\Psi(\lambda(T)) \in E$, then $\varphi(T) = 0$.

**Proof.** It follows from Theorem 2.4 that $T \in [\mathcal{E}(\mathcal{M}, \tau), \mathcal{M}]$. Since $\varphi$ is a trace, it follows that $\varphi(T) = 0$. □

**Proof of Theorem 1.6.** For $A \in \mathcal{S}(\mathcal{M}, \tau)$, we have that $A \in \mathcal{E}_{\log}(\mathcal{M}, \tau)$ if and only if $\log_+ \mu(A) \in E$, and this is, in turn, equivalent to $\log(1 + \mu(A)) \in E$. Using the basic equalities (7), (8), we easily see that for $A, B \in \mathcal{E}_{\log}(\mathcal{M}, \tau)$, we have

$$
\log(1 + \mu(A + B)) \leq \log(1 + D_2 \mu(A) + D_2 \mu(B)) \leq \log \left( (1 + D_2 \mu(A))(1 + D_2 \mu(B)) \right)
$$

$$
\log(1 + \mu(AB)) \leq \log(1 + D_2 \mu(A)D_2 \mu(B)) \leq \log \left( (1 + D_2 \mu(A))(1 + D_2 \mu(B)) \right),
$$

where $(D_2 f)(t) = f(t/2)$. But since $\log(1 + D_2 \mu(A)) + \log(1 + D_2 \mu(B)) \in E$, these imply that $A + B$ and $AB$ belong to $\mathcal{E}_{\log}(\mathcal{M}, \tau)$. From this, one easily sees that $\mathcal{E}_{\log}(\mathcal{M}, \tau)$ is a *-subalgebra of $\mathcal{S}(\mathcal{M}, \tau)$.

It remains to show that $\det_\varphi$ is multiplicative. Letting $A, B \in \mathcal{E}_{\log}(\mathcal{M}, \tau)$, we will show (9). We may, without loss of generality, assume $A, B \geq 0$. Indeed, we have $\mu(AB) = \mu(|AB|^*)$. Thus, if the assertion holds for positive operators, then we will have

$$
\det_\varphi(AB) = \det_\varphi(|A||B|^*) = \det_\varphi(|A|)\det_\varphi(|B|^*) = \det_\varphi(A)\det_\varphi(B).
$$

Suppose first that $\log(A), \log(B) \in \mathcal{E}(\mathcal{M}, \tau)$. Denote, for brevity, $T = \log(A)$ and $S = \log(B)$. It follows from Lemma 2.2 that $\log(|AB|) \in E$.

Using Lemma 2.1 and replacing $t$ with $\frac{1}{2}t$, for all $t \in (0, \frac{1}{\tau})$, we get

$$
\left| \int_t^{1-t} \left( \log(\mu(u, e^T e^S)) - \lambda(u, T) - \lambda(u, S) \right) \, du \right| \leq 4t \left( \mu\left( \frac{t}{2}, T \right) + \mu\left( \frac{t}{2}, S \right) \right).
$$

In particular, we have

$$
\Psi\left( \log(\mu(e^T e^S)) - \lambda(T) - \lambda(S) \right) \in E.
$$

It follows from Lemma 2.6 that

$$
\Psi\left( \lambda\left( \log(|e^T e^S|) - T - S \right) \right) \in E.
Using Lemma 2.8, we conclude that
\[ \varphi(\log(|e^T e^S|) - T - S) = 0. \]
This implies (3) for our \( A, B \).

If \( B \) has a nonzero kernel, then so does \( AB \) and (3) holds.

Suppose now that \( \ker B \) is zero but \( \log_-(B) \notin E \). Then, of course, \( \lim_{t \to 1} \mu(t, B) = 0. \)
If \( \ker AB \neq \{0\} \), then (3) holds, so suppose \( \ker AB = \{0\} \). We have, from (3), for all \( t \in (0, \frac{1}{2}) \),
\[ \mu(1-t, AB) \leq \mu(t, A)\mu(1-2t, B) \]
and, thus,
\[ \log(\mu(1-t, AB)) \leq \log(\mu(t, A)) + \log(\mu(1-2t, B)). \]
So, for sufficiently small \( t > 0 \),
\[ \log_- \mu(1-t, AB) + \log_+ \mu(t, A) \geq -\log \mu(1-t, AB) + \log \mu(t, A) \]
\[ \geq -\log \mu(1-2t, B) = \log_- \mu(1-2t, B). \]
Since the function \( t \mapsto \log_- \mu(1-2t, B) \) is not in \( E \), while the function \( t \mapsto \log_+ \mu(t, A) \) does belong to \( E \), we conclude that the function \( t \mapsto \log_- \mu(1-t, AB) \) does not belong to \( E \). Therefore, the function \( \log_- (\mu(AB)) \) does not belong to \( E \) and both left- and right-hand sides of (3) are zero. This concludes the proof of (3) in the degenerate case.

3. PROOF OF PROPOSITION 1.7 AND SOME EXAMPLES

**Lemma 3.1.** Let \( m : \mathcal{E}_{\log}(\mathcal{M}, \tau) \to \mathbb{R} \) be multiplicative and order-preserving. Then for every \( T \in \mathcal{E}_{\log}(\mathcal{M}, \tau) \), \( m(T) \) depends only on \( \mu(T) \).

**Proof.** We may without loss of generality assume \( m \) is not identically zero. Thus, \( m(1) = 1 \).
By Theorem 1 of [1], every unitary element is a product of multiplicative commutators of unitaries (in fact, of symmetries) and it follows that \( m \) sends the entire unitary group of \( \mathcal{M} \) to 1. Thus, by employing the polar decomposition, we have
\[ \forall T \in \mathcal{E}_{\log}(\mathcal{M}, \tau), \quad m(T) = m(|T|). \]
It, therefore, suffices to prove the assertion for positive operators.

Let \( 0 \leq T, S \in \mathcal{E}_{\log}(\mathcal{M}, \tau) \) be such that \( \mu(T) = \mu(S) \). Set
\[ T_\epsilon = \sum_{n \in \mathbb{Z}} (1 + \epsilon)^n 1_{((1+\epsilon)^n, (1+\epsilon)^{n+1})} (T), \quad S_\epsilon = \sum_{n \in \mathbb{Z}} (1 + \epsilon)^n 1_{((1+\epsilon)^n, (1+\epsilon)^{n+1})} (S). \]
For a given \( n \), positive operators \( T_\epsilon \) and \( S_\epsilon \) have discrete spectrum and \( \mu(T_\epsilon) = \mu(S_\epsilon) \). Since \( \mathcal{M} \) is a factor, one can choose a unitary operator \( U_\epsilon \in \mathcal{M} \) such that \( S_\epsilon = U_\epsilon T_\epsilon U_\epsilon^{-1} \). Thus,
\[ m(S_\epsilon) = m(U_\epsilon T_\epsilon U_\epsilon^{-1}) = m(U_\epsilon) m(T_\epsilon) m(U_\epsilon)^{-1} = m(T_\epsilon). \]
Clearly,
\[ S_\epsilon \leq S \leq (1 + \epsilon)S_\epsilon, \quad T_\epsilon \leq T \leq (1 + \epsilon)T_\epsilon. \]
Since \( m \) is order preserving, it follows that
\[ m(S) \leq m(1 + \epsilon) m(S_\epsilon) = m(1 + \epsilon) m(T_\epsilon) \leq m(1 + \epsilon) m(T). \]
Since \( m \) is order preserving, it follows that \( m(1 + \epsilon) \searrow 1 \) as \( \epsilon \searrow 0 \). Passing \( \epsilon \to 0 \), we obtain \( m(S) \leq m(T) \). Similarly, \( m(T) \leq m(S) \). Thus, \( m(S) = m(T) \) and the proof is complete. \( \square \)
Proof of Proposition 3.1. Since the map \( m \) is multiplicative and not identically zero, we must have \( m(1) = 1 \). By Lemma 3.1, \( m(T) \) depends only on \( \mu(T) \) for all \( T \in \mathcal{E}_\log(\mathcal{M}, \tau) \).

Let \( \mathcal{A} \) be any unital, diffuse, abelian von Neumann subalgebra of \( \mathcal{M} \). As in the proof of Theorem 3.3, \( E \) is naturally identified with \( \mathcal{S}(\mathcal{A}, \tau|\mathcal{A}) \cap \mathcal{E}(\mathcal{M}, \tau) \). Given real-valued \( f \in E \), let \( T \in \mathcal{S}(\mathcal{A}, \tau|\mathcal{A}) \cap \mathcal{E}(\mathcal{M}, \tau) \) be the corresponding self-adjoint operator. Note that \( e^T \) is an invertible element of \( \mathcal{E}_\log(\mathcal{M}, \tau) \) and, thus, \( m(e^T) > 0 \). We define

\[
\varphi_0(f) = \log m(e^T). \tag{18}
\]

We will show that \( \varphi_0 \) is \( \mathbb{R} \)-linear. First, given \( f_1, f_2 \in E \) and the corresponding self-adjoint \( T_1, T_2 \in \mathcal{S}(\mathcal{A}, \tau|\mathcal{A}) \cap \mathcal{E}(\mathcal{M}, \tau) \), since \( T_1 \) and \( T_2 \) commute, we have

\[
\varphi_0(f_1 + f_2) = \log m(e^{T_1+T_2}) = \log m(e^{T_1}e^{T_2}) = \log (m(e^{T_1})m(e^{T_2})) = \varphi_0(f_1) + \varphi_0(f_2),
\]

i.e., \( \varphi_0 \) preserves addition. From this, we easily see that \( \varphi_0(rf) = r\varphi_0(f) \) for every rational number \( r \) and real-valued \( f \in E \). This last fact is, of course, equivalent to

\[
m(e^{rT}) = m(e^T)^r \tag{19}
\]

for every self-adjoint \( T \in \mathcal{S}(\mathcal{A}, \tau|\mathcal{A}) \cap \mathcal{E}(\mathcal{M}, \tau) \) and every rational number \( r \). When \( T \geq 0 \), using the order-preserving property of \( m \), we obtain from this that (19) holds for every \( r \in \mathbb{R} \), and similarly when \( T \leq 0 \). For arbitrary self-adjoint \( T \in \mathcal{S}(\mathcal{A}, \tau|\mathcal{A}) \cap \mathcal{E}(\mathcal{M}, \tau) \), writing \( T = T_+ - T_- \) for \( T_+ \) and \( T_- \) positive elements of \( \mathcal{S}(\mathcal{A}, \tau|\mathcal{A}) \cap \mathcal{E}(\mathcal{M}, \tau) \), in the usual way, we get, for all \( r \in \mathbb{R} \),

\[
m(e^{T}) = m(e^{T_+ + (-r)T_-}) = m(e^{T_+})m(e^{-r(T_-)}) = m(e^{T_+})^r m(e^{T_-})^{-r} = (m(e^{T_+}e^{-T_-}))^r = m(e^T)^r.
\]

Thus (19) holds for all self-adjoint \( T \) and all \( r \in \mathbb{R} \), and it follows that \( \varphi_0(rf) = r\varphi_0(f) \) for all real-valued \( f \in E \) and all \( r \in \mathbb{R} \). Thus, we have defined an \( \mathbb{R} \)-linear functional \( \varphi_0 \) on the space of real-valued elements of \( E \). Complexification extends \( \varphi_0 \) to a \( \mathbb{C} \)-linear functional on \( E \).

We now observe that \( \varphi_0 \) is rearrangement-invariant. If \( f \in E \) and \( f \geq 0 \) and if \( T \in \mathcal{S}(\mathcal{A}, \tau|\mathcal{A}) \cap \mathcal{E}(\mathcal{M}, \tau) \) is the corresponding element, then \( \mu(e^T) = f^* \), where \( f^* \) is the nondecreasing rearrangement of \( f \). Since \( m(e^T) \) depends only on \( \mu(e^T) \), we see that \( \varphi_0(f) = \varphi_0(f^*) \) and, thus, \( \varphi_0 \) is rearrangement-invariant.

By Theorem 3.1, there is a unique trace \( \varphi \) on \( \mathcal{E}(\mathcal{M}, \tau) \) such that \( \varphi(T) = \varphi_0(\mu(T)) \) whenever \( T \in \mathcal{E}(\mathcal{M}, \tau) \) is positive. Suppose \( X \) is an invertible element of \( \mathcal{E}_\log(\mathcal{M}, \tau) \) and let us observe that \( m(X) = \det_\varphi(X) \). Since \( m(X) = m(|X|) \) and likewise for \( \det_\varphi \), we may without loss of generality assume \( X \geq 0 \). Thus, there is self-adjoint \( T = \log(X) \in \mathcal{E}(\mathcal{M}, \tau) \) such that \( X = e^T \). Thus, by (18), we have

\[
m(X) = e^{\varphi_0(\log(T))} = e^{\varphi(T)} = \det_\varphi(X),
\]

as required. \( \square \)

The following shows that Proposition 3.1 cannot be improved to obtain \( m = \det_\varphi \) on all of \( \mathcal{E}_\log(\mathcal{M}, \tau) \).

**Proposition 3.2.** Let \( E \) be a symmetric function space. Consider strictly larger symmetric function space \( F \). If \( \psi \) is an arbitrary positive trace on \( F(\mathcal{M}, \tau) \), then

\[
\det_\psi|_{\mathcal{E}_\log(\mathcal{M}, \tau)} \neq \det_\varphi
\]

for each positive trace \( \varphi \) on \( \mathcal{E}(\mathcal{M}, \tau) \).
Let $\psi \in \mathcal{E}(\mathcal{M}, \tau)$ such that $T \notin \mathcal{E}(\mathcal{M}, \tau)$. Take $X = e^{-T}$. Then $X$ is bounded, so belongs to $\mathcal{E}_{\log}(\mathcal{M}, \tau)$. Moreover, $X^{-1} = e^{T}$ belongs to $\mathcal{F}_{\log}(\mathcal{M}, \tau)$, but $X$ is not invertible in $\mathcal{E}_{\log}(\mathcal{M}, \tau)$. Thus, we have

$$\det_{\psi}(X) = e^{-\psi(T)} \neq 0 = \det_{\varphi}(X).$$

See Remark 1.6 for the relevance of the following example.

**Example 3.3.** We give examples of a nonzero trace $\varphi$ on a bimodule $\mathcal{E}(\mathcal{M}, \tau)$ and $T \in \mathcal{E}(\mathcal{M}, \tau)$ such that $\varphi \neq 0$ but

$$\det_{\varphi}(T) \neq \lim_{\epsilon \to 0^+} \det_{\varphi}(|T| + \epsilon). \tag{20}$$

Let $\psi$ be an increasing, continuous, concave function on the interval $[0, 1]$ satisfying

$$\lim_{t \to 0} \frac{\psi(2t)}{\psi(t)} = 1.$$

For example, take $\psi(t) = \frac{1}{2-\log(t)}$. Let $E = M_\psi$ be the Marcinkiewicz space

$$E = \left\{ f \in S(0, 1) \mid \sup_{0 < t < 1} \frac{1}{\psi(t)} \int_0^t f^*(s) \, ds < \infty \right\},$$

where $f^*$ is the decreasing rearrangement of $|f|$. Let $\mathcal{E}(\mathcal{M}, \tau)$ be the corresponding $\mathcal{M}$-bimodule. By Example 2.5(ii) of [3], there is a positive, rearrangement-invariant, linear functional $\varphi$ on $E$ that vanishes on $E \cap L_\infty$, but satisfies $\varphi(\psi^t) = 1$. For $f \in E$ with $f \geq 0$, $\varphi(f)$ is realized as a particular sort of generalized limit as $t \to 0$ of $\frac{1}{\psi(t)} \int_0^t f^*(s) \, ds$. Let $\varphi$ denote also the trace on $\mathcal{E}(\mathcal{M}, \tau)$, according to Theorem 1.1. Thus, we have $\det_{\varphi}(T) = 1$ whenever $T \in \mathcal{M}$ is bounded and has bounded inverse. Consequently, if $T \in \mathcal{M}$ fails to be invertible in $\mathcal{E}_{\log}(\mathcal{M}, \tau)$, for example, because it has a nonzero kernel, then, by Definition 1.2, $\det_{\varphi}(T) = 0$, but the right-hand-side of (20) is equal to 1.

The examples considered hitherto involved non-invertible elements of $\mathcal{E}_{\log}(\mathcal{M}, \tau)$. However, (20) can also fail when $T$ is invertible in $\mathcal{E}_{\log}(\mathcal{M}, \tau)$. For example, take $T \geq 0$ such that $\mu(T)(t) = \exp(-\psi^t(1 - t))$. In particular, $T$ is bounded. Then $\det_{\varphi}(T) = e^{-1}$ but again the right-hand-side of (20) is equal to 1.

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