Stretch IDLA

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May 2, 2014

Abstract

We consider a new IDLA - particle system model, on the upper half planar lattice, resulting in an infinite forest covering the half plane. We prove that almost surely all trees are finite.

1 Introduction

The model of Internal Diffusion Limited Aggregation (IDLA) was introduced by Meakin and Deutch [MD86] as a model for some chemical reactions, particle coalescence and aggregation. IDLA was first studied rigorously in [DF91] and [LBG92]. IDLA is a growth model, starting with a point aggregate \(0 \in \mathbb{Z}^2\), \(A(0) = \{0\}\). At each step a particle exits the origin, preforms a simple random walk (SRW) and stops at the first position outside the aggregate, this position is then added to the aggregate i.e. \(A(n + 1) = A(n) \cup v_n\), where \(v_n\) is the first exit position of a SRW starting at 0 from \(\mathbb{Z}^2 \setminus A(n)\). In [LBG92], Lawler, Bramson and Griffeath prove the asymptotic shape of the IDLA aggregate converges to the Euclidean ball. Assela and Gaudilliere [AG10] and independently Jerison, Levine and Sheffield [JLS12] recently proved the long standing conjecture, that the fluctuations from the Euclidean ball are at most logarithmic.

In this paper we consider an IDLA process in continuous time, introduced to us by Itai Benjamini, which we call Stretch IDLA (SIDLA). This process starts with an infinite line. Every vertex on the line has a Poisson clock, every ring initiates a monotone SRW that can add an edge to the tree rooted at the vertex whose clock rang. We show that even though eventually all vertices are covered, all trees are finite almost surely. See Figures 1.1a and 1.1b for two simulations of the process. The tree rooted at 0 is colored red. In initiating the IDLA in an infinite line, we lose the simplicity of a discrete process, but we gain ergodicity which we use heavily in our analysis.

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1.1 General Notation

We consider the rotated $\mathbb{Z}^2$ lattice in the upper half-plain re-scaled by $\sqrt{2}$. Hereon we abbreviate it $\mathbb{H}$.

$$\mathbb{H} = \{(x, y) : x + y \in 2 \cdot \mathbb{Z}, \ y \geq 0\}.$$  

Viewed as a directed graph, every site $v = (x, y)$ is connected to the cites $v_l = (x - 1, y + 1)$ and $v_r = (x + 1, y + 1)$ we denote $e_l(v) = (v, v_l)$ and $e_r(v) = (v, v_r)$, abbreviate $\mathcal{E}$ the set of edges in $\mathbb{H}$. We denote by $\theta_l = (-1, 1)$ and $\theta_r = (1, 1)$ the vectors spanning the lattice. For a vertex $v = (x, y)$, denote the vertex level by $h(v) = y$. It will also be useful to define the cone of $x$, $C(x) = \{x + i\theta_l + j\theta_r : i, j \in \mathbb{N} \cup \{0\}\}$. This is the set of vertices that can be reached from $x$ using directed edges. Finally we denote $\partial \mathbb{H} = \{(x, 0) : x \in 2 \cdot \mathbb{Z}\}$. See Figure 1.1 for a summary of the notation.

Figure 1.1: The monotone lattice
1.2 Stretch IDLA model description and general remarks

In this section we give a description of the SIDLA, the well-definedness will be proved below. Consider the SIDLA process on $\mathbb{H}$. Let $P_t$ be a measure on $\{0, 1\}^E$, described as follows:

For every site $v$ on the $x$ axis we define the tree $T(v, t)$ to be the tree spanned by time $t$. Thus, $T(v, 0) = \{v\}$, for convenience we denote $T(v) \equiv T(v, \infty)$. Denote by $\partial T(x, t)$ the outer edge boundary of $T(x, t)$, $\partial T(x, t) = \{e = (u, v) : e \notin T(x, t), u \in T(x, t)\}$, and by $\partial^i T(x, t)$ the inner vertex boundary i.e. $\partial^i T(x, t) = \{v \in T(x, t) : \exists u \notin T(x, t), u \in \{v + \theta_r, v + \theta_l\}\}$. At each cite $v$ found on the $x$ axis place an independent Poisson clock of rate 1. Given that a ring occurred at time $t_0$ an edge $e = (u_1, u_2)$ is adjoined to the tree according to the following law:

$$P_{t_0} \{T(v, t_0) = T(v, t_0^-) \cup e\} = \begin{cases} 2^{-h(u_2)} & \text{if } u_1 \in T(v, t_0^-) \text{ and } u_2 \notin \bigcup_{v' \in \partial H} T(v', t_0^-) \\ 0 & \text{Otherwise} \end{cases},$$

for every $e \neq e'$, $P_{t_0} (T(v, t_0) = T(v, t_0^-) \cup e \cup e') = 0$, where

$$t_0^- = t_0^-(v) = \sup\{s > 0 : s < t_0, \text{clock at site } v \text{ rang at time } s\}. $$

This process can be described intuitively in terms of particles: each time $t_0$, the clock at a vertex $u \in \partial \mathbb{H}$ rings, a particle is created, and starts an instantaneous random walk subject to the following law:

1. Being at vertex $v \in T(u, t_0^-)$, the particle chooses one of its neighbours $v_r$ and $v_l$ with probability $\frac{1}{2}$, call the choice $a$.

2. If $a$ is free, the particle marks the edge and occupies the cite.

3. If $a \in \bigcup_{x \neq u} T(x, t_0)$ or $a \in T(u, t_0^-)$ but $(v, a) \notin T(u, t_0^-)$ the particles vanishes.

4. Else it continues as described in (1.) from the newly reached vertex.

Lemma 1.1. The process is well defined.

Proof. Since each vertex can a priori be reached only from a finite number of trees, the well definedness of the process is trivial for every $t \geq 0$. If some edge $e \in \mathcal{E}$ is contained in some tree $T(v, t)$, for every $s > t$, $e \in T(v, s)$ $P_s$-a.s. Thus $\exists \lim_{t \to \infty} P_t$, abbreviate the limiting measure $P$. \hfill $\square$

Definition 1. For a vertex $v \in \partial \mathbb{H}$, let $h(T(v)) = \sup_{x \in T(v)} \{h(x)\}$ denote the hight of the tree rooted at $v$.

Note that in the process described above, every vertex in $\mathbb{H}$ is reached at a finite time a.s. We use this remark in Corollary 3.3 which states that the expected hight of a tree in $P$ is infinity.
1.3 First passage percolation

Definition 2. For an edge \((x, y) = e \in \mathcal{E}\), abbreviate \(h(e) = \max\{h(y), h(x)\}\).

In this section we define a first passage percolation model (FPP). In the next section we will couple the SIDLA with the FPP defined in this section.

Assign for each edge \(e \in \mathcal{E}\) a weight \(\omega(e) \sim \exp(2^{-h(e)})\) independently of all other edges. We denote the measure so constructed by \(\mathbb{P}\). For every monotone path \(\gamma = (e_1, e_2, \ldots, e_n)\) in \(\mathbb{H}\), the length of \(\gamma\) is defined to be \(\lambda(\gamma) = \sum_{i=1}^{n} \omega(e_i)\). For every two points \(x, y \in \mathbb{H}\) such that \(x \in C(y)\) or \(y \in C(x)\), let

\[
d_\omega(x, y) = \min_{\gamma: x \rightarrow y} \lambda(\gamma),
\]

where the minimum is over all monotone paths in \(\mathbb{H}\) connecting \(x\) and \(y\). For a point \(x \in \mathbb{H}\) and a set \(A \subset \mathbb{H}\) connected by a monotone path, let \(d_\omega(x, A) = \inf_{y \in A} d_\omega(x, y)\).

Definition 3. For a vertex \(x \in \partial \mathbb{H}\), let \(\hat{T}(x) = \bigcup_{y \in \mathbb{H}} \{\gamma \mid \gamma\ is\ monotone, \gamma : x \rightarrow y, \lambda(\gamma) = d_\omega(y, \partial \mathbb{H})\}\) i.e. the union of all paths minimizing the distance from points \(y \in \mathbb{H}\) to \(\partial \mathbb{H}\) starting at the vertex \(x\).

Remark 1.2. The uniqueness of the path \(\gamma : x \rightarrow y\), such that \(\lambda(\gamma) = d_\omega(y, \partial \mathbb{H})\), follows from the continuity of the distribution of \(\{\omega(e)\}_{e \in \mathcal{E}}\).

Remark 1.3. Since \(\mathbb{P}\) is a function of i.i.d, \(\mathbb{P}\) is ergodic under the shift \(\theta := \theta_1^{-1} \circ \theta_r\).

2 Coupling SIDLA with FPP

Given a FPP process with distribution \(\mathbb{P}\), we construct a SIDLA process by way of coupling. The construction amounts to associating with each \(x \in \partial \mathbb{H}\) a set of Poisson clock rings and prescribing where each particle ends.

To this end we introduce an auxiliary set of independent Poisson clocks. Given an edge \(e \in \mathcal{E}\) we associate with it a Poisson clock of rate \(2^{-h(e)}\), which we abbreviate \(\text{Poisson}(e)\), such that \(\{\text{Poisson}(e)\}_e\) is an independent set, and independent of the FPP.

We assign a set of rings for \(x\) and particle trajectories as follows: For each finite monotone path \(\gamma \subseteq \{\hat{T}(x) \cup \partial \hat{T}(x)\}\), \(\gamma = (e_1, \ldots, e_{l(\gamma)})\) originating at \(x\) we assign the following rings:

- if \(\gamma \subseteq \hat{T}(x)\) we assign the ring \(\sum_{i=1}^{l(\gamma)} \omega(e_i)\), and the path of the particle will be \(\gamma\).
- if \(\gamma \not\subseteq \hat{T}(x)\) we assign the ring sequence \(\sum_{i=1}^{l(\gamma)} \omega(e_i), \sum_{i=1}^{l(\gamma)} \omega(e_i) + \text{Poisson}(e_{l(\gamma)})\), for each ring in this sequence of rings the particle will be assigned the path \(\gamma\).

Remark 2.1. Note that in the second case, all the particles will vanish, as the vertex at the end of \(\gamma\) will be reached sooner by a particle associated to the FPP tree containing it.

We need to show that this construction results in a Poisson clock at \(v\) for every \(v \in \partial \mathbb{H}\) with the correct rate. We prove this by showing the time differences between every two consecutive rings is distributed exponentially with rate 1.
**Definition 4.** For a finite monotone tree $T$, denote by $\partial S_n(v, t) = \{e \in \partial T(v, t) : h(e) = n\}$.

The next lemma is a combinatorial property of finite monotone trees in $\mathbb{H}$.

**Lemma 2.2.** For every finite monotone tree $T$ in $\mathbb{H}$ with root $x \in \partial \mathbb{H}$ and height $n - 1$ i.e. $n = \max_{e \in \partial T}\{h(e)\}$, then

$$
\sum_{i=1}^{n} \frac{1}{2^i} |\partial S_i| = 1.
$$

**Proof.** We prove by induction on $n$. For $n = 1$, the tree is empty, thus $|\partial S_1| = 2$ and for every $i > 1$, $|\partial S_i| = 0$. We get $\frac{1}{2} = 1$. Now assume the claim is true for $n - 1$, let $T$ be a tree of height $n$. If $|\partial S_1| = 0$, denote by $T_r - \theta_r$ and $T_l - \theta_l$ the two subtrees of $T$ contained in $T \setminus \{x\}$ shifted to $\partial \mathbb{H}$. The subtrees are of height smaller than $n$, and for every $i \leq n$, $|\partial S^r_i| + |\partial S^l_i| = |\partial S_{i+1}|$ thus by the induction hypothesis

$$
\sum_{i=1}^{n-1} \frac{1}{2^i} |\partial S_i| = \frac{1}{2} (|\partial S^r_i| + |\partial S^l_i|) = \frac{1}{2} + \frac{1}{2} = 1. \tag{2.1}
$$

If $|\partial S_1| = 1$, assume wlog assume $T^l = \emptyset$, by the induction hypothesis,

$$
\sum_{i=1}^{n} \frac{1}{2^i} |\partial S_i| = \frac{1}{2} |\partial S_1| + \sum_{i=2}^{n} \frac{1}{2^i} |\partial S_i| = \frac{1}{2} + \frac{1}{2} \sum_{i=2}^{n-1} \frac{1}{2^i} |\partial S^r_i| = 1. \tag{2.2}
$$

□

**Claim 2.3.** The time differences between every two consecutive rings at any vertex $v$ are independent and are distributed exponentially with rate 1.

**Proof.** By induction on the number of rings. The first ring happens at time $\min\{\omega(e_r), \omega(e_l)\}$ which are distributed exponentially $\omega(e_r) \sim \exp(1/2)$, $\omega(e_l) \sim \exp(1/2)$, thus their minimum, is distributed $\min\{\omega(e_r), \omega(e_l)\} \sim \exp(1)$.

Induction step: assuming the $n$ rings have happened we consider the $n+1$st ring time. $T(v, t)$ after the $n$-th ring consists of at most $n$ vertices and edges, in particular $|T(v, t)| < \infty$. Let $w'(e)$ be distributed according to $\mathbb{P}$ independently from $\omega$. By the memoryless property of exponential distribution, the $n+1$st ring time is by definition of the coupling, distributed as $\min_{e \in \partial T(v, t)} w'(e)$.

We prove by induction that

$$
\mu_j = \min_{e \in \bigcup_{k=0}^{j} \partial S_{n-k}(v, t)} \{w'(e)\} \sim \exp \left( \frac{1}{2^{n-j}} \sum_{l=0}^{j} \frac{1}{2^{j-l}} |\partial S_{n-l}(v, t)| \right). \tag{2.3}
$$

The base of induction follows as $\mu_0$ is the minimum of $|\partial S_n(v, t)|$, $\exp \left( \frac{1}{2^n} \right)$ independent random variables. Since

$$
\min_{e \in \partial S_{n-j-1}(v, t)} w'(e) \sim \exp \left( \frac{1}{2^{n-j-1}} |\partial S_{n-j-1}(v, t)| \right),
$$
\[
\mu_{j+1} \sim \min \left\{ \mu_j, \min_{e \in \partial S_{n-j-1}(v,t)} w'(e) \right\} \sim \exp \left( \frac{1}{2n-j-1} \sum_{l=0}^{j+1} \frac{1}{2^{j+1-l}} \left| \partial S_{n-l}(v,t) \right| \right). \tag{2.4}
\]

Thus proving the internal induction. We obtain by Lemma 2.2
\[
\mu_n \sim \exp \left( \sum_{l=0}^{n} \frac{1}{2n-l} \left| \partial S_{n-l}(v,t) \right| \right) \sim \exp(1). \tag{2.5}
\]

3 Finite trees

In this section we will prove the main result of this paper.

**Theorem 3.1.** Given a FPP on \( H \) distributed according to \( \mathbb{P} \), i.e. with weights \( w(e) \sim \exp \left( 2^{-h(e)} \right) \), almost surely all trees are finite, i.e.
\[
\mathbb{P}(|\hat{T}(0)| < \infty) = 1.
\]

**Proof.** Assume for the purpose of contradiction the existence of an infinite tree. Then by shift invariance, \( \beta := \mathbb{P}(|T(0)| = \infty) > 0 \).

Abbreviate \( T^m(x) = \{ v \mid h(v) = m, v \in T(x) \} \). By the ergodic theorem we have
\[
\frac{1}{2n+1} \sum_{x=-n}^{n} |T^m(x)| \mid_{T(x) = \infty} \rightarrow_{n \rightarrow \infty} \mathbb{E} \left[ |T^m(0)| \mid T(0) = \infty \right] \cdot \mathbb{P}(|T(0)| = \infty) = \beta \cdot \mathbb{E} \left[ |T^m(0)| \mid T(0) = \infty \right]. \tag{3.1}
\]

Since all the trees are monotone, each of them resides in a cone, thus
\[
\frac{1}{2n+1} \sum_{x=-n}^{n} |T^m(x)| \mid_{T(x) = \infty} \leq \frac{1}{2n+1} \sum_{x=-n}^{n} |T^m(x)| \leq \frac{2n+2m+1}{2n+1} \rightarrow_{n \rightarrow \infty} 1,
\]
and we get
\[
\mathbb{E} \left[ |T^m(0)| \mid |T(0)| = \infty \right] \leq \frac{1}{\beta}. \tag{3.2}
\]

Fix \( \delta < 1, D = \frac{1}{\beta \delta} \), by Markov's inequality
\[
\mathbb{P} \left( |T^m(0)| > D \mid |T(0)| = \infty \right) \leq \delta.
\]

**Definition 5.** A tree rooted at \( v \) is called slim, if \( 0 < |T^m(v)| < D \) for infinitely many \( n \)'s. We say that a tree is slim at level \( k \) if \( 0 < |T^k(0)| < D \).
Proof. Let $\omega$ be a tree to be slim is 0. For every $\kappa > 0$, let $r_n = \min \{ v = (x, y) \in \mathbb{H} : y = n, x > s, \forall (s, y) \in T^n(0) \}$ be the vertex to the right of $T^n(0)$ and respectively the vertex to the left of $T^n(0)$, which we denote by $l_n$. For every $k \in \mathbb{N}$ denote $\Delta(n) = \mathbb{H} \cap \text{Conv}\{l_n, r_n, l_n + (|T^n(0)| + 1) \theta_r \}$, the triangle based in $T^n(0) \cup l_n \cup r_n$. See Figure 7.1 for clarifications.

**Lemma 3.2.** For every $\kappa > 1$, $\mathbb{P}(d_\omega(l_n, \partial \mathbb{H}) > \kappa 2^{n+1}|\sigma(\{\omega(e) : e \in \cup_{i=1}^n T^i(0)\})) \leq \frac{1}{\kappa} < 1$ a.s.

**Proof.** Let $w_i \sim \text{exp}(2^{-i})$, with law $Q$, be independent of each other and of $\mathbb{P}$. We first prove by induction on $n$ that $d_\omega(l_n, \partial \mathbb{H})$ is dominated by $\sum_{i=1}^n w_i$. For $n = 1$, if $T^1(0)$ consists of $\{\theta_r\}$, then $\omega(\theta_1) > \omega((l_1 - \theta_r, l_1))$ and in particular $d(l_1, \partial \mathbb{H})$ is stochastically dominated by $\exp(1/2)$. If $T^1(0)$ consists of either $\{\theta_t\}$ or $\{\theta_t, \theta_r\}$, $d(l_1, \partial \mathbb{H}) = \min\{\omega(-2, -2 + \theta_t), \omega(-4, -4 + \theta_r)\}$, both are independent of $T^1(0)$, and in particular dominated by $\exp(1/2)$. Assume claim for $l_{n-1}$, if $l_n = l_{n-1} + \theta_t$, since there is no monotone path connecting $T^n(0)$ with the edge $(l_{n-1}, l_n)$, then $\omega(l_{n-1}, l_n)$ is independent of $\cup_{i=1}^{n-1} T^i$, and $d_\omega(l_n, \partial \mathbb{H}) \leq d(l_{n-1}, \partial \mathbb{H}) + \omega(l_{n-1}, l_n)$. If $l_n = l_{n-1} + \theta_r$, then $d_\omega(l_n, \partial \mathbb{H}) < d_\omega(l_{n-1}, \partial \mathbb{H}) + \omega(l_{n-1}, l_n - \theta_t)$. Thus conditioned on the weights of $\cup_{i=1}^{n-1} T^i(0)$, we obtain that

$$0 \leq \omega(l_{n-1}, l_n) \leq \omega(l_{n-1}, l_n) + d(l_{n-1}, \partial \mathbb{H}) - d(l_{n-1}, \partial \mathbb{H}),$$

in particular $\omega(l_{n-1}, l_n)$ is conditionally dominated by $\omega_n$. Thus we got that $d_\omega(l_n, \partial \mathbb{H})$ is conditionally dominated by $\sum_{i=1}^n w_i$. Now

$$\mathbb{P}(d_\omega(l_n, \partial \mathbb{H}) > \kappa 2^{n+1}|\sigma(\{\omega(e) : e \in \cup_{i=1}^n T^i(0)\})) \leq Q\left(\sum_{i=1}^n w_i > \kappa E_Q\left[\sum_{i=1}^n w_i\right]\right) \leq \frac{1}{\kappa} < 1.$$
Let \( M_n = \max\{d_\omega(l_n, \partial \mathbb{H}), d_\omega(r_n, \partial \mathbb{H})\} \). Conditioned on the event that \( T(0) \) was slim in levels \( n_1, \ldots, n_k \) such that \( n_{m+1} - n_m > D \) and \( 2^{n_k+1} > M_{n_k-1} \), \( m = 1, \ldots, k - 1 \), we show that the probability there exists a level \( l \geq n_k + D \) where the tree is slim is bounded away from 1.

Every edge in \( e \in \Delta(n_k) \) has weight distribution \( \omega(e) \sim \exp(2^{-h(e)}) = \exp(2^{-n_k-l}) \) where \( 0 \leq l \leq D + 1 \). Using the exponential distribution properties \( w(e) \sim 2^{n_k} \exp(2^{-l}) \).

The idea that will follow is to show that with positive probability \( \partial \in \Delta(n_k) \setminus T^{n_k}(0) \) belongs to the trees of \( r_{n_k} \) and \( l_{n_k} \) thus killing the tree rooted at 0. To this end let \( w_i \sim \exp(2^{-i}) \), be independent of each other and of \( \mathbb{P} \). We denote the measure so constructed by \( Q \). By Lemma 3.2 we obtain that

\[
\mathbb{P}(d_\omega(l_{n_k}, \partial \mathbb{H}) > M_{n_k-1} + \kappa 2^{n_k+1}) \leq \mathbb{P}(d_\omega(l_{n_k}, \partial \mathbb{H}) > \kappa 2^{n_k+1}) \leq \frac{1}{\kappa} < 1. \tag{3.5}
\]

With positive probability and independent of all the levels lower than \( n_k \), all (finite number) edges \( e \in \Delta(n_k) \) will have weights larger than \( \omega(e) \geq \kappa 2^{D_E}\mathbb{E}[\omega(e)] \), and all edges \( e' = (x, y), \{x, y\} \in \partial \in \Delta(n_k) \setminus T^{n_k}(0) \) will have weights smaller than \( \omega(e') \leq \mathbb{E}[\omega(e')] \). We get that \( \partial \in \Delta(n_k) \setminus T^{n_k}(0) \notin T(0) \).

**Corollary 3.3.** \( \mathbb{E}[h(T(0))] = \infty \)

**Proof.** Assume for the purpose of contradiction that \( \mathbb{E}[h(T(0))] < \infty \), thus

\[
\sum_{i=0}^{\infty} \mathbb{P}(h(T(0)) \geq i) = \sum_{i=0}^{\infty} \mathbb{P}(h(T(i)) \geq i) = \frac{1}{2} \sum_{i=-\infty}^{\infty} \mathbb{P}(h(T(i)) \geq |i|) < \infty. \tag{3.6}
\]

By Borel-Cantelli, for all but a finite number \( i \)'s, \( h(T(i)) < |i| \). Since all trees have finite height, there are infinitely many vertices in \( C(0) \) that are not covered a.s. This is a contradiction to the construction of the SIDLA.

**Remark 3.4.** *Similarly to the SIDLA, one can consider a stretched version of the Eden Model [Ede67] on the upper half plane of the square lattice. Similarly to the coupling argument in Section 2, the stretched Eden can be coupled to FPP with i.i.d. \( \exp(1) \) weights. In the case of a square lattice, Kesten’s result on the speed of convergence of FPP, [Kes93], implies all trees are finite a.s.*
Remark 3.5. To complete the picture one may consider a FPP with exponentially decreasing weights. One can easily see that with probability 1, there exists a $0 > v_1 \in \mathbb{H}$ such that the entire line $\{v_1 + i\theta : i \in \{0\} \cup \mathbb{N}\}$ does not belong to $T(0)$. By symmetry there is a $v_r > 0$ such that the line $\{v_r + i\theta : i \in \{0\} \cup \mathbb{N}\}$ also does not belong to $T(0)$. This shows that the tree $T(0)$ is contained in the triangle made by the two lines and the $x$ axis, and in particular it is finite.

Acknowledgments

The authors wish to thank Itai Benjamini for suggesting this problem. One of the authors would like to thank Ohad Feldheim for a fruitful discussion.

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