Constructing States and Effective Hamiltonians in Lattice QCD

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Abstract

We formulate SU(3) Hamiltonian lattice QCD in terms of the plaquette variables and determine the relevant subspaces of the Hilbert space for the vacuum wave functional and its approximations. We analyze the one- and two-plaquette problems.

In an earlier paper [1] we formulated the fully gauge-fixed SU(N) lattice gauge Hamiltonian and studied the one-plaquette problem for several values of N. This paper is restricted to the case of SU(3) lattice gauge theory. For this case I discuss ways to go beyond the one-plaquette approximation.

The Hamiltonian approach to lattice QCD, although more complicated, has advantages over the Euclidean formulation. [2, 3] In general, one can say that the Hamiltonian approach allows one to split the problem in parts and tackle each part separately, thereby generating more understanding and restricting the eventual computational effort. This should be formulated consistently as restrictions on the full physical Hilbert space to a proper subspace, or likewise, either by formal arguments, or from calculations. One important step in this process is the gauge fixing, such that all the degrees of freedom are physical, and an excitation spectrum, determined from an ab-initio many-body approach, such as RPA, is actually physical, and does not contain oscillations in the gauge parameter. The first and foremost part of the full QCD problem is the determination of the vacuum wave functional.

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The dimensionless QCD lattice Hamiltonian can be split into three parts:

\[ H = 2 \sum_\alpha (E^a_\alpha)^2 - \frac{\lambda}{2} \sum_\alpha h_{aM} + \sum_{<\alpha,\beta>} E^a_\alpha E^a_\beta, \]  

where the sum over \( \alpha \) runs over all plaquettes, and the sum over \(<\alpha,\beta>\) runs over all nearest-neighbour plaquette pairs. The dimensionless energy \( \varepsilon \) is related to the energy \( E \) for a given lattice spacing \( a \) and coupling constant \( g \):

\[ \frac{g^2}{a} \varepsilon + \frac{3}{g^2 a} = E \]  

and the coupling constant \( \lambda \) can be expressed in terms of the gauge-field coupling constant \( g \):

\[ \lambda = \frac{1}{g^2}. \]  

If we ignore the nearest-neighbour interaction, the Hamiltonian is local and the solution is the product wave functional of one-plaquette wave functions.

It is important to realize that although a special unitary \( 3 \times 3 \) matrix \( U \) is parametrized by eight angles, only two are relevant for the one-plaquette problem. Since the potential, or magnetic, term \( h_M \) only depends on the eigenvalues \( x, y \) and \( z \):

\[ h_M = \text{Tr}[U + U^\dagger] = \text{Tr} \begin{bmatrix} x & y & z \\ y & z & x \\ z & x & y \end{bmatrix} + \text{Tr} \begin{bmatrix} x^{-1} & y^{-1} & z^{-1} \\ y^{-1} & z^{-1} & x^{-1} \\ z^{-1} & x^{-1} & y^{-1} \end{bmatrix} \]

\[ = x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \]  

of the plaquette variable \( U \) with the relation \( xyz = 1 \) it is sufficient to know \( x = \exp i\phi_x \) and \( y = \exp i\phi_y \). In the case of several plaquettes, we can simultaneously diagonalize multi-plaquette states with up to five different plaquettes, only for correlations of six and more plaquettes simultaneous diagonalization might not be possible in special cases.

For the ground state at zero temperature there are no excitations in the transverse directions because the Hamiltonian does not couple to these states, so the wave functional is constant for these angles. So for our analysis of

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1. The normalization is not a problem since the group is compact. Correspondingly, the maximal-Abelian gauge is not fully Abelian because of correlations of sixth and higher order spoil the diagonalization in three spatial dimensions.
the vacuum, we can restrict ourselves to the variables \(x\) and \(y\), however, for symmetry reasons, we use also \(z\), keeping in mind that \(z = (xy)^{-1}\).

So in terms of the eigenvalues \(x\), \(y\), and \(z\) of the plaquette variable \(U\) the one-plaquette Hamiltonian is:

\[
h = 2\Delta + 4\Xi - \frac{2}{3}(\Upsilon^2 + 6\Upsilon) - \frac{\lambda}{2}(x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z}) ,
\]

where we define

\[
\Xi = \frac{1}{x-z}(x^2\partial_x - z^2\partial_z) + \frac{1}{x-y}(x^2\partial_x - y^2\partial_y) + \frac{1}{y-z}(y^2\partial_y - z^2\partial_z) ,
\]

\[
\Delta = (x\partial_x)^2 + (y\partial_y)^2 + (z\partial_z)^2 ,
\]

\[
\Upsilon = x\partial_x + y\partial_y + z\partial_z .
\]

The operators \(\Delta\) and \(\Upsilon^2\) are diagonal for each monomial \(x^py^qz^l\), and give the respective eigenvalues, \(p^2 + q^2 + l^2\) and \((p + q + l)^2\). The terms \(\Xi\) and \(\Upsilon\) are due to the curvature of the group manifold given by the Jacobian \((x - y)(y - z)(z - x)\). Only \(\Xi\) mixes the different monomials, however, preserves the order \(p + q + l\). It is easy to see that the electric operator is consistent with the SU(3) group, since for any eigenstate \(\phi(x, y, z)\) one can show that \((xyz)^r\phi(x, y, z)\) is an eigenstate with the same eigenvalue, which then can be mapped on \(\phi(x, y, z)\). In the strong-coupling limit \(\lambda = 0\), the solutions are the eigenstates of the electric operator, which form an orthonormal basis on the group manifold. Generally, we do not have to perform any integration over the group, as the eigenstates of the electric operator guarantees orthogonality of the basis states. Actually, all we need for an eigenvalue equation is a unique labeling of states, orthogonal or not, an invariant subspaces, on which the action of the Hamiltonian is bijective, such that a truncation to a finite subspace is possible. A similarity transform can make the Hamiltonian hermitian if necessary. For the one-plaquette problem these states are orthogonal and characterized by a label \((n, m)\) where \(n > m > 0\):

\[
\langle x, y, z|n, m\rangle = \chi_{(n,m)} = \begin{vmatrix} x^n & x^m & 1 \\ y^n & y^m & 1 \\ z^n & z^m & 1 \\ x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix} ,
\]

with eigenvalues \(\frac{2}{3}(n^2 + m^2 - nm - 3)\). The eigenvalues are degenerate for each pair \((n, m)\) and \((n, n - m)\), which corresponds to the hermitian conjugate
\( (n, m) \rightarrow (n, m) = \sum c_{ijk} m_{ijk}, \quad i + j + k = n + m - 3, \quad i \geq j \geq k \)

| \((n, m)\) | \(\chi_{(n,m)}\) | \(\lambda_{(n,m)}\) |
|---------|----------------|----------------|
| (2,1)   | 1              | 0              |
| (3,1)   | \(m_1\)        | 8/3            |
| (3,2)   | \(m_{1,1}\)    | 8/3            |
| (4,1)   | \(m_{2,1} + m_{1,1}\) | 20/3 |
| (4,2)   | \(m_{2,2} + m_{2,1,1}\) | 6 |
| (4,3)   | \(m_3 + m_{2,1} + m_{1,1}\) | 12 |
| (5,1)   | \(m_{3,1} + m_{2,2} + 2m_{2,1,1}\) | 32/3 |
| (5,2)   | \(m_{3,2} + m_{3,1,1} + 2m_{2,2,1}\) | 32/3 |
| (5,3)   | \(m_{3,3} + m_{3,2,1} + m_{2,2,2}\) | 12 |
| (5,4)   | \(m_4 + m_{3,1} + m_{2,2} + m_{2,1,1}\) | 56/3 |
| (6,1)   | \(m_{4,1} + m_{3,2} + 2m_{3,1,1} + 2m_{2,2,1}\) | 50/3 |
| (6,2)   | \(m_{4,2} + m_{3,3} + m_{4,1,1} + 2m_{3,2,1} + 3m_{2,2,2}\) | 16 |
| (6,3)   | \(m_{4,3} + m_{4,2,1} + 2m_{3,3,1} + 2m_{3,2,2}\) | 50/3 |
| (6,4)   | \(m_{4,4} + m_{4,3,1} + m_{4,2,2} + m_{3,3,2}\) | 56/3 |
| (6,5)   | \(m_{5,1}\) | \(m_{5,1}\) |

Table 1: The eigenstates \((n, m)\) expressed in monomials, for example: 
\[ m_{3,2,2} = x^3y^2z^2 + y^3x^2z^2 + z^3x^2y^2 \] and \[ m_{2,2,2} = x^2y^2z^2. \] These expressions can be reduced, using the relation \(xyz = 1\), however, for most purposes it useful to keep the original form. The last column contains the corresponding eigenvalues.

In principle, for the vacuum wave functional, the basis can be restricted to symmetric combinations only: \(\frac{1}{\sqrt{2}} (|n, m\rangle + |n, n - m\rangle)\). There are several equivalent representations of these states. For example, they can be expressed as polynomials of trace variables: \(t_1 = x + y + z\) and \(t_2 = x^2 + y^2 + z^2\), or in terms of loop variables: \(l_{i,j} = (x^i + y^i + z^i)(x^j + y^j + z^j)\). In the mappings between these representations one makes extensive use of the relation \(xyz = 1\). In order to derive results efficiently we need to go to and fro between representations. We choose the representation in terms of symmetric polynomials of the eigenvalues because of its clarity and versatility. I will return to this representation to check signs and factors in the formulae.

In Table 1 we express the eigenstates up to \(n\) is six in terms of monomials. The, generally integer, coefficients of the mapping between representations can be deduced from group theoretical methods using Young tableaux, but in this paper I will not refer to any group-theoretical result. A general
Figure 1: The occupation numbers of the lowest basis states for the one-plaquette ground state. Due to the symmetry between covariant and contravariant representations, the occupation numbers for the states \((n, m)\) and \((n, n-m)\) are identical.

The formula for an electric eigenstate is given by:

\[
\langle x, y, z | n, m \rangle = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \sum_{k=0}^{i+j+n-m-3} \sigma_{j-i} z^{n+m-2-i-j} x^{i+j+i-j-k} y^{i+j-i-j-k} c_{i,j,k} m_{i,j,k},
\]

which follows from the evaluation of the determinants of Eq. 7. We used the result, which will be useful later for the evaluation of the \(\Xi\) operator:

\[
\frac{x^i y^j - x^j y^i}{x - y} = \sigma_{i-j} (xy)^{\min(i,j)} \sum_{k=0}^{i-j-1} x^{i-j-1-k} y^k.
\]

In all this, \(\sigma\) is the sign function, with \(\sigma_0 = 0\).

If we use this basis up to a certain number \(\mathbb{N}\) \(n\) we can solve the one-plaquette problem for arbitrary \(\lambda = g^{-4}\). Away from the strong coupling limit a large basis is required for convergence, but with 100-200 states a

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\(^2\) The Hamiltonian restricted to this subspace is hermitian.
good convergence is obtained for $\lambda < 50$. In Figure 1 we show the occupation number for the lowest states as a function of the coupling constant. Clearly, a typical expansion around the strong coupling limit with, say, 5-10 states will break down quickly, as many electric eigenstates contribute to the ground state. In Figure 2 we show the lower end of the spectrum.

Therefore, in order to go beyond the one-plaquette approximation, it is not only necessary to construct multiple plaquette states, but also restrict the physical Hilbert space to make any calculation feasible. A straight two-plaquette problem requires already thousands of states for a reasonable convergence, and there will be no easy way to extend that to larger lattices. We will return to this problem later.

From the one-plaquette problem, we are guided to write the Hamiltonian:

$$H = \sum_{\alpha} h_{\alpha} + \sum_{\langle \alpha, \beta \rangle} E_{\alpha}^a E_{\beta}^a,$$  \hspace{1cm} (10)

where $h_{\alpha}$ is the Hamiltonian for the plaquette $\alpha$, which we solved earlier. Although the ground-state and the low-lying states of the one-plaquette problem contain many contributing basis states, it would not be too hard to construct an effective Hamiltonian with only the eigenstates of the lower end of the one-plaquette spectrum. The real question is how to construct the spatial excitations; the important multiple-plaquette correlations, which then in its turn are used to construct the short-range interaction in the effective Hamiltonian. In a way, part of the physics ends up in the coupling-constant and lattice-spacing dependence of the parameters of this effective Hamiltonian.

Since the operator $E_{\alpha}^a E_{\beta}^a = \Omega_{\alpha \beta} + P_{\alpha \beta}$, contains a diagonal part $\Omega_{\alpha \beta}$ and a part $P_{\alpha \beta}$ between plaquette $\alpha$ and plaquette $\beta$, which brings us out of the one-plaquette product wave functional. We can use it as generator for spatial excitations. For a state $| (n_{\alpha}, m_{\alpha}) \rangle | (n_{\beta}, m_{\beta}) \rangle$ the multiple application of the operator $P_{\alpha \beta}$ can generate up to $\min\{n_{\alpha}, n_{\beta}\} - 2$ spatial excitations. In order to see this come about we look at a basis state in terms of its trace variables. One term may be written as:

$$w_{j_1 j_2 \cdots j_l} = \sum_{i_1, i_2, \cdots, i_l} U_{i_1 i_1} U_{i_2 i_2} \cdots U_{i_l i_l},$$  \hspace{1cm} (11)

where $\{j_1 j_2 j_3 \cdots j_l\}$ is a permutation of $\{123 \cdots l\}$. In the case of the identity: $\{j_1 j_2 j_3 \cdots j_l\} = \{123 \cdots l\}$ it is a product of single traces: $w_j = Tr[U]^l$ and for $\{j_1 j_2 j_3 \cdots j_l\} = \{l12 \cdots (l-1)\}$ it yields $w_j = Tr[U']$. So in the end it is not the permutation $\{j_1 j_2 j_3 \cdots j_l\}$ that gives a unique representation, but the cocycles in the permutation; the number of traces and each exponent. This can be labeled by exponents as the partition of $l$. For the special group
SU(N) the representation in terms of trace variables is not unique. Because \(xyz = 1\) every term can be expressed in terms of two traces \(\text{Tr}[U^i] \text{Tr}[U^j]\) since a product of three traces contains terms like \(x^i y^j z^k\) from which a common factor \((xyz)^{\min(i,j,k)}\) can be removed. This has been the greatest headache for people\(^3\) studying SU(3) Hamiltonian lattice gauge theory. It mixes also terms of different order. As soon as one includes more complicated states, in terms of trace variables, the abundance of redundant states is much larger than the number of unique states. There does not seem to be an easy way to determine the unique and complete set in terms of trace variables.

However, we will work from the other way around; we have the electric eigenstates of the SU(3) group on one plaquette, without redundancies. We construct the corresponding trace variables from that, and we can now study the effect of the correlation operator \(P_{\alpha\beta}\). We start best from the SU(3) identity:

\[
\frac{1}{4} \sum_{a=1}^{8} \lambda^a_{ij} \lambda^a_{kl} = \frac{1}{2} \delta_{il} \delta_{kj} - \frac{1}{6} \delta_{ij} \delta_{kl} . \tag{12}
\]

The application of the electric operator corresponds to the insertion of these \(\lambda\)-matrices at the appropriate places. So acting on the right, for example, on the trace above Eq. 11 at places, \(r\) and \(s\), we find that

\[
\{j_1 \cdots j_l\} \rightarrow \frac{1}{2} \{j_1 \cdots j_{r-1} j_s j_{r+1} \cdots j_{s-1} j_r j_{s+1} \cdots j_l\} - \frac{1}{6} \{j_1 \cdots j_l\} . \tag{13}
\]

\(^3\)For SU(2) this problem is much easier.
As simple as that.

We have the electric operator on each plaquette, which is a combination of the permutation operator and the identity operator for each basis state, and we have the inter-plaquette operator $\mathcal{P}_{\alpha\beta}$ that permutes lines between the two plaquettes. So in the two-plaquette problem we distinguish six operators: $P_\alpha$ and $P_\beta$ the permutation operators of lines of plaquette $\alpha$ and plaquette $\beta$ respectively, $\Omega_\alpha$ and $\Omega_\beta$ the diagonal part of the electric operator on each plaquette, $\mathcal{P}_{\alpha\beta}$ the operator that permutes lines of plaquette $\alpha$ and plaquette $\beta$, and $\Omega_{\alpha\beta}$ the diagonal part of the inter-plaquette operator.

Now we can make a number of important observations. Firstly, every eigenstate on a plaquette of the electric operator can be expressed in terms a finite number of normal-ordered permutation operators $P$ without common indices on a given number of lines $l$:

$$
\langle U|n, m\rangle = \sum_{l=0}^{n+m-3} \sum_{k=0}^{\frac{l}{2}} c_{kl} : P^k : w_{1\ldots l} 
,$$

where the normal-ordered permutation operator $: P^k :$ can be expressed in terms of permutations $P(i, j)$ of elements $i$ and $j$, and the inter-plaquette operator $\mathcal{P}$ accordingly:

$$
: P^k : = \sum_{i_1 \neq i_2 \neq \ldots \neq i_k} P(i_1, i_2)P(i_3, i_4)\cdots P(i_{2k-1}, i_{2k})
,$$

$$
: \mathcal{P}_{\alpha\beta}^k : = \sum_{i_1 \neq i_2 \neq \ldots \neq i_k, j_1 \neq j_2 \neq \ldots \neq j_k} P(i_1, j_1)P(i_2, j_2)\cdots P(i_k, j_k)
,$$

where the indices $i_1, \ldots, i_k$ belong to the plaquette variables on the plaquette $\alpha$, and $j_1, \ldots, j_k$ to the plaquette $\beta$. Terms like $P(12)P(23)w_{123} = \text{Tr}[U^3]$ can be reduced to $w_0 + \frac{1}{2} P(12)w_{123} - \frac{1}{2} w_{123} = w_0 + \frac{1}{2} w_{123} - \frac{1}{2} w_{123}$, where $w_0 = \text{Tr}[1]$.

Secondly, the inter-plaquette operator can be normal ordered, without common indices, and truncates at the minimal number of lines on one of the plaquettes. The normal ordering generates additional inter-plaquette permutations:

$$
\mathcal{P}_{\alpha\beta}\mathcal{P}_{\alpha\beta} w_{1\ldots p}^{(\alpha)} w_{1\ldots q}^{(\beta)} = : \mathcal{P}_{\alpha\beta}^2 : w_{1\ldots p}^{(\alpha)} w_{1\ldots q}^{(\beta)}
$$

$$
= \frac{1}{p} \mathcal{P}_{\alpha\beta} \mathcal{P}_{\alpha} w_{1\ldots p}^{(\alpha)} w_{1\ldots q}^{(\beta)} + \frac{1}{q} \mathcal{P}_{\alpha\beta} \mathcal{P}_{\beta} w_{1\ldots p}^{(\alpha)} w_{1\ldots q}^{(\beta)}
$$

$$
+ pq w_{1\ldots p}^{(\alpha)} w_{1\ldots q}^{(\beta)}
.$$
\[ \text{Tr}[U^2] : P^k : w_{1\ldots l} = \frac{(l - 2k)(l - 2k + 1)}{(l + 2)(l + 1)} : P^{k+1} : w_{1\ldots l(l+1)(l+2)} \]  

(19)

So every two-plaquette basis state can be represented by five labels:

\[ \langle \{U_\alpha, U_\beta\} | k, p, l, q, r \rangle = : P^r_{\alpha\beta} : : P^q_{\alpha} : : P^p_{\beta} : w^{(\alpha)}_{1\ldots k} w^{(\beta)}_{1\ldots l}, \]  

(20)

where \( p, q, \) and \( r \) are restricted by \( k \) and \( l \), such that \( p \leq k/2, q \leq l/2, \) and \( r \leq \min\{k, l\} \). Linear combinations of \( : P^p : w_{1\ldots k} \) span the same space as linear combinations of \( (n, m) \) with \( n + m - 3 = k \). An important corollary of this result is that different one-plaquette electric eigenstates do not mix due to the inter-plaquette action, since only one eigenstate with a given eigenvalue lies within the space spanned by \( : P^p : w_{1\ldots k} \).

In terms of electric eigenstates on each plaquette there would be a redundancy, since, for example, \( : P^p_{\alpha\beta} : \langle \{U_\alpha, U_\beta\} | (3, 2)_\alpha (4, 1)_\beta \rangle = 0 \) due to cancellations among the trace variables, however, due to mixing of states with different \( n \) there is no real redundancy, if the commutation relations are worked out. Similarly, a straightforward truncation of the \( l \leq L \) and \( k \leq K \) would violate hermiticity, and it would lead to complex eigenvalues, since the symmetry between the covariant \( (n, m) \) and the contravariant representation \( (n, n - m) \) is broken by this truncation. It is easy to mistake the non-permuting part of Eq. (13) for the diagonal part of inter-plaquette part of the electric operator, however, some care is required here since the basis in trace variables is not orthogonal.

We will illustrate most of the concepts above in a simple example. Let’s consider one-plaquette diagrams with two lines. Starting from \( \text{Tr}[U]^2 = w_{12} \) there are two diagrams: \( w_{12} \) and \( P w_{12} \). The electric operator in this case is:

\[ E^2 = \left( P - \frac{2}{3} \right) + 2 \frac{8}{3}, \]  

(21)

where the last term is the diagonal part from \( \text{Tr}[U] \sum_a \text{Tr}[\lambda^a \lambda^a U] \), and the first part is the specific case of Eq. (13) where \( P = 2P(1, 2) = P(1, 2) + P(2, 1) \) and the eigenvalue equation is given by:

\[ \begin{pmatrix} \frac{14}{3} & 2 \\ 2 & \frac{14}{3} \end{pmatrix} \begin{pmatrix} w_{12} \\ \frac{1}{2} P w_{12} \end{pmatrix} = \lambda \begin{pmatrix} w_{12} \\ \frac{1}{2} P w_{12} \end{pmatrix}, \]  

(22)

which leads to the eigenstates \( (3, 2) \) and \( (4, 1) \) and the eigenvalues: \( \lambda_{(3, 2)} = 8/3 \) and \( \lambda_{(4, 1)} = 20/3 \) as denoted in Table 4.

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4 This equals \((N^2 - 1)/N\) for \( N = 3 \) times the number of matrices \( U \) in the product \( w \).
If we now look at the two-plaquette problem, we see that the one-plaquette electric operator commutes with the inter-plaquette part:

\[ [P_\alpha, P_\beta] = [P_\beta, P_{\alpha\beta}] = [P_\alpha, P_{\alpha\beta}] = 0 \]

therefore we can work independently in the four separate subspaces of \( \{ \phi^{(3,2)}_\alpha, \phi^{(4,1)}_\alpha \} \otimes \{ \phi^{(3,2)}_\beta, \phi^{(4,1)}_\beta \} \). The two-plaquette electric operator has a simple action on each of these states

\[ E^2 = \lambda^{(\alpha)} + \lambda^{(\beta)} + \frac{1}{2} \left( P_{\alpha\beta} - \frac{4}{3} \right) \]

where the factor \( \frac{1}{2} \) is because the permuting part of the electric operator on the common link is: \( \frac{1}{4} (P_\alpha + P_\beta + 2P_{\alpha\beta}) \). On all the other links there is a single \( \frac{1}{4} P_\beta \) or \( \frac{1}{4} P_\alpha \). For the mixed states we get the eigenvalue equation:

\[
\begin{pmatrix}
9 & 1 & 1 \\
1 & 9 & 1
\end{pmatrix}
\begin{pmatrix}
\phi^{(3,2)}_\alpha \\
\phi^{(3,2)}_\beta
\end{pmatrix}
= \lambda
\begin{pmatrix}
\frac{1}{2} P_{\alpha\beta} \phi^{(3,2)}_\alpha \\
\frac{1}{2} P_{\alpha\beta} \phi^{(3,2)}_\beta
\end{pmatrix},
\]

while for the symmetric states the electric operator yields:

\[
\begin{pmatrix}
\Lambda & 1 & 0 \\
1 & \Lambda + 1 & 1 \\
0 & 1 & \Lambda
\end{pmatrix}
\begin{pmatrix}
\phi^{(\alpha\beta)}_\Lambda \\
\frac{1}{2} P_{\alpha\beta} \phi^{(\alpha\beta)}_\Lambda \\
\frac{1}{4} : P^2_{\alpha\beta} \phi^{(\alpha\beta)}_\Lambda
\end{pmatrix}
= \lambda
\begin{pmatrix}
\phi^{(\alpha\beta)}_\Lambda \\
\frac{1}{2} P_{\alpha\beta} \phi^{(\alpha\beta)}_\Lambda \\
\frac{1}{4} : P^2_{\alpha\beta} \phi^{(\alpha\beta)}_\Lambda
\end{pmatrix},
\]

where \( \Lambda = \frac{38}{3} \) and the positive sign for \((4,1)\) states of each plaquette, i.e., \( \phi^{(\alpha\beta)}_{\Lambda} = \phi^{(\alpha)}_{(4,1)} \phi^{(\beta)}_{(4,1)} \), and \( \Lambda = \frac{14}{3} \) and the negative sign for \((3,2)\). The two largest of the three eigenvalues of the two-plaquette electric eigenstates for \((3,2)\) states are \( 14/3 \) and \( 17/3 \), which correspond with the eigenvalues of coupled \((3,1)\) states. A representation in terms of symmetric polynomials in \( x_\alpha, y_\alpha \) and \( z_\alpha \), and \( x_\beta, y_\beta \) and \( z_\beta \) reveals indeed that these are hermitian conjugate solutions. The lowest eigenvalue \( 8/3 \) of the \((3,2)\) triplet is degenerate with the lowest single-plaquette excitation like \((3,2)\), however, this corresponds to a redundant solution, as the three states of Eq. 26 are linearly dependent for \( \Lambda = (3,2) \), contrary to the case of \( \Lambda = (4,1) \).

This procedure can be carried out to construct any eigenstate of the electric operator without generating many redundant states. Furthermore, it does not rely on the diagonalization of plaquette variable \( U \) in the one-plaquette problem, so it can be applied hierarchically up to any order in a multi-plaquette Hilbert space. However, in the future we hope to map this
procedure completely onto a basis of symmetric polynomials which should remove the last redundancies.

Eventually, as we increase the spatial extent of the states, we can restrict ourselves to the lower energy states of the subsystems from which these states are build. Thereby we restrict the computational effort, which, as mentioned before, is the main problem of an ab-initio Hamiltonian lattice QCD calculation. Finally, I gratefully acknowledge discussions with Niels Walet during the early stages of this work.

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