Approximation Algorithms for Stochastic Boolean Function Evaluation and Stochastic Submodular Set Cover

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1 Introduction

We present approximation algorithms for two problems: Stochastic Boolean Function Evaluation and Stochastic Submodular Set Cover. We also consider a related ranking problem.

Stochastic Boolean Function Evaluation (SBFE) is the problem of determining the value of a given Boolean function $f$ on an unknown input $x$, when each bit $x_i$ of $x$ can only be determined by paying a cost $c_i$. The assumption is that $x$ is drawn from a given product distribution, and the goal is to minimize expected cost. SBFE problems arise in diverse application areas. For example, in medical diagnosis, the $x_i$ might correspond to medical tests, and $f(x) = 1$ if the patient has a particular disease. In database query optimization, $f$ could correspond to a Boolean query on predicates corresponding to $x_1, \ldots, x_n$, that has to be evaluated for every tuple in the database to find tuples that satisfy the query [26, 29, 12, 37]. In Operations Research, the SBFE problem is known as “sequential testing” of Boolean functions. In learning theory, the SBFE problem has been studied in the context of learning with attribute costs.

We focus on developing approximation algorithms for SBFE problems. There have been previous papers on exact algorithms for these problems, but there is very little work on approximation algorithms [38, 28]. Our approach is to reduce the SBFE problems to Stochastic Submodular Set Cover (SSSC). The SSSC problem was introduced by Golovin and Krause, who gave an approximation algorithm for it called Adaptive Greedy. 1 Adaptive Greedy is a generalization of the greedy algorithm for the classical Set Cover problem. We present a new algorithm for the SSSC problem, which we call Adaptive Dual Greedy. It is an extension of the Dual Greedy algorithm for Submodular Set Cover due to Fujito, which is a generalization of Hochbaum’s primal-dual algorithm for the classical Set Cover Problem [15, 16]. We also give a new bound on the approximation achieved by the Adaptive Greedy algorithm of Golovin and Krause.

The following is a summary of our results. We note that our work also suggests many open questions, including approximation algorithms for other classes of Boolean functions, proving hardness results, and determining adaptivity gaps.

The $Q$-value approach: We first show how to solve the SBFE problem using the following basic approach, which we call the $Q$-value approach. We reduce the SBFE problem to an SSSC problem, through the construction of an assignment feasible utility function, with goal value $Q$. Then we apply the Adaptive Greedy algorithm of Golovin and Krause to the SSSC problem, yielding an approximation factor of $(\ln Q + 1)$.

Using this approach, we easily obtain an $O(\log kd)$-approximation algorithm for CDNF formulas (or decision trees), where $k$ is the number of clauses in the CNF and $d$ is the number of terms in the DNF. Previously, Kaplan et al. gave an algorithm also achieving an $O(\log kd)$ approximation, but only for monotone CDNF formulas, unit costs, and the uniform distribution [28].

We also use the $Q$-value approach to develop an $O(\log D)$-approximation algorithm for evaluating linear threshold formulas with integer coefficients. Here $D$ is the sum of the magnitudes of the coefficients. This $O(\log D)$ bound is a weak bound that we improve below, but we adapt the algorithm later to obtain other results.

The $Q$-value approach has inherent limitations. We prove that it will not give an algorithm with a sublinear approximation factor for evaluating read-once DNF (even though there is a poly-time exact algorithm [28, 21]), or for evaluating linear threshold formulas with exponentially large coefficients. In fact, our weak $O(\log D)$ approximation factor for linear threshold formulas cannot be improved to be sublinear in $n$ with the $Q$-value approach. We prove our negative results by introducing a new combinatorial measure of a Boolean function, which we call its $Q$-value.

Adaptive Dual Greedy: We present Adaptive Dual Greedy (ADG), a new algorithm for the SSSC problem.

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1 Golovin and Krause called the problem Stochastic Submodular Coverage, not Stochastic Submodular Set Cover, because the cover is not formed using sets. Our choice of name is for consistency with terminology of Fujito [15].

2 Although our result solves a more general problem than Kaplan et al., they give their $O(\log kd)$ approximation factor in terms of expected certificate cost, which lower bounds the expected cost of the optimal strategy. See Section 2.
We prove that it achieves an approximation factor of of $\alpha$, where $\alpha$ is a ratio that depends on the cover constructed by the algorithm.

3-Approximation for Linear Threshold Formulas: We substitute ADG for Adaptive Greedy in our $O(\log D)$ algorithm for evaluating linear threshold formulas. We show that in this case, $\alpha$ is bounded above by 3, and we get a 3-approximation algorithm.

New bound on Adaptive Greedy: We prove that Adaptive Greedy achieves an $2(\ln P + 1)$-approximation in the binary case (and $k(\ln P + 1)$ in the $k$-ary case) where $P$ is the maximum utility that can be contributed by a single item. The proof of this bound uses the same LP that we use in our analysis of ADG to lower bound the approximation factor, combined with Wolsey’s approach to bounding the analogous algorithm for (non-adaptive) submodular set cover [39]. Our bound generalizes Wolsey’s bound for (non-adaptive) submodular set cover [39], except for an additional factor of 2. Wolsey’s bound generalized the $(\ln s + 1)$ bound for standard set cover, where $s$ is the maximum size of one of the input subsets (cf. [16]).

Simultaneous Evaluation of Linear Threshold Formulas: We apply the above techniques to the problem of simultaneous evaluation of $m$ linear threshold formulas, giving two algorithms with approximation factors of $O(\log mD_{avg})$ and $D_{max}$ respectively. Here $D_{avg}$ and $D_{max}$ are the average and maximum, over the $m$ formulas, of the sum of the magnitude of the coefficients. These results generalize results of Liu et al. for shared filter ordering [30]. We also improve one of Liu’s results for that problem.

Ranking of Linear Functions: We give an $O(\log(mD_{max}))$-approximation algorithm for ranking a set of $m$ linear functions $a_1 x_1 + \ldots + a_n x_n$ (not linear threshold functions), defined over $\{0, 1\}^n$, by their output values, in our stochastic setting. This problem arises in Web search and database query processing. For example, we might need to rank a set of documents or tuples by their “scores”, where the linear functions compute the scores over a set of unknown properties such as user preferences or data source reputations.

2 Stochastic Boolean Function Evaluation and Related Work

Formally, the input to the SBFE problem is a representation of a Boolean function $f(x_1, \ldots, x_n)$ from a fixed class of representations $C$, a probability vector $p = (p_1, \ldots, p_n)$, where $0 < p_i < 1$, and a real-valued cost vector $(c_1, \ldots, c_n)$, where $c_i \geq 0$. An algorithm for this problem must compute and output the value of $f$ on an $x \in \{0, 1\}^n$, drawn randomly from product distribution $D_p$, such that $p_i = \text{Prob}[x_i = 1]$. However, it is not given access to $x$. Instead, it can discover the value of any $x_i$ by “testing” it, at a cost of $c_i$. The algorithm must perform the tests sequentially, each time choosing the next test to perform. The algorithm can be adaptive, so the choice of the next test can depend on the outcomes of the previous tests. The expected cost of the algorithm is the cost it incurs on a random $x$ from $D_p$. (Since each $p_i$ is strictly between 0 and 1, the algorithm must continue doing tests until it has obtained a 0-certificate or 1-certificate for the function.) The algorithm is optimal if it has minimum expected cost with respect to $D_p$. The running time of the algorithm is the (worst-case) time it takes to determine the next variable to be tested, or to compute the value of $f(x)$ after the last test. The algorithm corresponds to a Boolean decision tree (strategy) computing $f$.

If $f$ is given by its truth table, the SBFE Problem can be exactly solved in time polynomial in the size of the truth table, using dynamic programming, as in [22] [34]. The following algorithm solves the SBFE problem with an approximation factor of $n$ for any function $f$, even under arbitrary distributions: Test the variables in increasing cost order (cf. [28]). We thus consider a factor of $n$ approximation to be trivial.

We now review related work. There is a well-known algorithm that exactly solves the SBFE problem for disjunctions: test the $x_i$ in increasing order of ratio $c_i/p_i$ (see, e.g., [17]). A symmetric algorithm works for conjunctions. There is also a poly-time exact algorithm for evaluating a $k$-of-$n$ function (i.e., a function that evaluates to 1 iff at least $k$ of the $x_i$ are equal to 1) [33] [4] [36] [9]. There is a poly-time exact algorithm for evaluating a read-once DNF formula $f$, but the complexity of the problem is open when $f$ is a general read-once formula [7] [21] [20]. The SBFE problem is NP-hard for linear threshold functions [11], but for the special case of unit costs and uniform distribution, testing the variables in decreasing order of the magnitude of their coefficients is optimal [6] [14]. A survey by Ünlüyurt [38] covers other results on exactly solving the
SBFE problem.

There is a sample version of the evaluation problem, where the input is a sample of size $m$ of $f$ (i.e., a set of $m$ pairs $(x, f(x))$), and the problem is to build a decision tree that computes $f$ correctly on the $x$ in the sample that minimizes the average cost of evaluation over the sample. Golovin et al. and Bellala et al. developed $O(\log m)$ approximation algorithms for arbitrary $f$ \cite{19,3}, and there is a 4-approximation algorithm when $f$ is a conjunction \cite{2,13,35}. Moskiov and Chikalov proved a related bound in terms of a combinatorial measure of the sample \cite{32}. Moskiov gave an $O(\log m)$-algorithm for a worst-case cost variant of this problem \cite{31}.

A number of non-adaptive versions of standard and submodular set cover have been studied. For example, Iwata and Nagano \cite{27} studied the “submodular cost set cover” problem, where the cost of the cover is a submodular function that depends on which subsets are in the cover. Beraldi and Ruszczynski addressed a set cover problem where the set of elements covered by each input subset is a random variable, and full coverage must be achieved with a certain probability \cite{5}.

Kaplan et al. gave their $O(\log kd)$ approximation factor for monotone CDNF (and unit costs, uniform distribution) in terms of the expected certificate cost, rather than the expected cost of the optimal strategy. The gap between expected certificate cost and expected cost of the optimal strategy can be large: e.g., for disjunction evaluation, with unit costs, where $\text{Prob}[x_i = 1] = 1/(i + 1)$, the first measure is constant, while the second is $\Omega(\log n)$.

Kaplan et al. also considered the problem of minimizing the expected cost of evaluating a Boolean function $f$ with respect to a given arbitrary probability distribution, where the distribution is given by a conditional probability oracle \cite{28}. In the work of Kaplan et al., the goal of evaluation differs slightly from ours in that they require the evaluation strategy to output an “explanation” of the function value upon termination. They give as an example the case of evaluating a DNF that is identically true; they require testing of the variables in one term of the DNF in order to output that term as a certificate. In contrast, under our definitions, the optimal strategy for evaluating an identically true DNF formula is a zero-cost one that simply outputs “true” and performs no tests.

Charikar et al. \cite{10} considered the problem of minimizing the worst-case ratio between the cost of evaluating $f$ on an input $x$, and the minimum cost of a certificate contained in $x$. There are also papers on building identification trees of minimum average cost, given $S \subseteq \{0, 1\}^n$, but that problem is fundamentally different than function evaluation because each $x \in S$ must have its own leaf (cf. \cite{1}).

We note that there is a connection between the linear threshold evaluation problem and the Min-Knapsack problem. In Min-Knapsack, you are given a set of items with values $a_i \geq 0$ and weights $c_i \geq 0$, and the goal is to select a subset of the items to put in the knapsack such that the total value of the items is at least $\theta$, and the total weight is minimized. We can therefore solve Min-Knapsack by simulating the above algorithm on the linear threshold formula $\sum_{i=1}^n a_ix_i \geq \theta$, giving the value 1 as the result of each test. It is easy to modify the above analysis to show that in this case the ratio $\alpha$ is at most 2, because $C_0$ is empty. We thus have a combinatorial 2-approximation algorithm for Min-Knapsack, based on ADG. (In fact, the deterministic Dual Greedy algorithm of Fujito would be sufficient here, since the outcomes of the tests are predetermined.) There are several previous combinatorial and non-combinatorial 2-approximation algorithms for Min-Knapsack, and the problem also has a PTAS ( \cite{8}, cf. \cite{24}).

Han and Makino considered an on-line version of the Min-Knapsack where the items are given one-by-one over time \cite{24}. There is also previous work on the “stochastic knapsack” problem, but that work concerns the standard (max) knapsack problem, not Min-Knapsack.

3 Preliminaries

Basic notation and definitions. A table of notation used in this paper is provided in Appendix A.

A partial assignment is a vector $b \in \{0, 1, *\}^n$. We view $b$ as an assignment to variables $x_1, \ldots, x_n$. For partial assignment $b$, we use $\text{dom}(b)$ to denote the set $\{x_i | b_i \neq *\}$. We will use $b \in \{0, 1\}^n$ to represent the outcomes of binary tests, where for $l \in \{0, 1\}$, $b_i = l$ indicates that test $i$ was performed and had outcome
of function \( c \), which we call Classical Set Cover. In Classical Set Cover, the input is a distribution \( g \) whenever we mention extensions to the \( k \)-ary case where \( k > 2 \).

In the (binary) SSSC problem, the input consists of the set \( N \), a cost vector \( (c_1, \ldots, c_n) \), where each \( c_j \geq 0 \), a probability vector \( p = (p_1, \ldots, p_n) \) where \( p \in [0,1] \), an integer \( Q \geq 0 \), and a utility function \( g : (O \cup \{\ast\})^n \rightarrow \mathbb{Z}_{\geq 0} \). Further, \( g(x) = 0 \) if \( x \) is the vector that is all \( * \)’s, and \( g(x) = Q \) if \( x \in O^n \). We call \( Q \) the goal utility. We say that \( b \in (O \cup \{\ast\})^n \) is a cover if \( g(b) = Q \). The cost of cover \( b \) is \( \sum_{j:b_j \neq \ast} c_j \).

Each item \( j \in N \) has a state \( x_j \in O \). We sequentially choose items from \( N \). When we choose item \( j \), we observe its state \( x_j \) (we “test” \( j \)). The states of items chosen so far are represented by a partial assignment \( b \in (O \cup \{\ast\})^n \). When \( g(b) = Q \), we have a cover, and we can output it. The goal is to determine the order in which to choose the items, while minimizing the expected testing cost with respect to distribution \( D_p \). We assume that an algorithm for this problem will be executed in an on-line setting, and that it can be adaptive.

SSS is a generalization of Submodular Set Cover (SSC), which is a generalization of the standard (weighted) Set Cover problem, which we call Classical Set Cover. In Classical Set Cover, the input is a finite ground set \( X \), a set \( F = \{S_1, \ldots, S_m\} \) where each \( S_j \subseteq X \), and a cost vector \( c = (c_1, \ldots, c_m) \) where each \( c_j \geq 0 \). The problem is to find a min-cost “cover” \( F' \subseteq F \) such that \( \bigcup_{S_j \in F'} S_j = X \), and the cost of \( F' \) is \( \sum_{S_j \in F'} c_j \). In SSC, the input is a cost vector \( c = (c_1, \ldots, c_n), \) where each \( c_j \geq 0 \), and a utility function \( g : 2^N \rightarrow \mathbb{Z}_{\geq 0} \) such that \( g \) is monotone and submodular, \( g(\emptyset) = 0 \), and \( g(N) = Q \). The goal is to

\(^3\)To simplify the exposition, we define the SSSS Problem in terms of integer valued utility functions.

\( l \), and \( b_i = * \) indicates that test \( i \) was not performed.

For partial assignments \( a, b \in \{0, 1, *\}^n \), \( a \) is an extension of \( b \), written \( a \sim b \), if \( a_i = b_i \) for all \( b_i \neq * \). We also say that \( b \) is contained in \( a \). Given Boolean function \( f : \{0, 1\}^n \rightarrow \{0, 1\} \), a partial assignment \( b \in \{0, 1, *\}^n \) is a 0-certificate (1-certificate) of \( f \) if \( f(b) = 0 \) (\( f(b) = 1 \)) for all \( a \) such that \( a \sim b \). Given a cost vector \( c = (c_1, \ldots, c_n) \), the cost of a certificate \( b \) is \( \sum_j b_j \neq * c_j \).

Let \( N = \{1, \ldots, n\} \). In what follows, we assume that utility functions are integer-valued. In the context of standard work on submodularity, a utility function \( g : 2^N \rightarrow \mathbb{Z}_{\geq 0} \). Given \( S \subseteq N \) and \( j \in N \), \( g_S(j) \) denotes the quantity \( g(S \cup \{j\}) - g(S) \).

We will also use the term utility function to refer to a function \( g : \{0, 1, *\}^n \rightarrow \mathbb{Z}_{\geq 0} \) defined on partial assignments. Let \( g : \{0, 1, *\}^n \rightarrow \mathbb{Z}_{\geq 0} \), be such a utility function, and let \( b \in \{0, 1, *\}^n \). For \( S \subseteq N \), let \( b^S = b^S_i = b_i \) for \( i \in S \), and \( b^S = * \) otherwise. We define \( g(S, b) = g(b^S) \). For \( j \in N \), we define \( g_{S,b}(j) = g(S \cup \{j\}, b) - g(S, b) \).

For \( l \in \{0, 1, *\} \), the quantity \( b_{x_{l-1}} \) denotes the partial assignment that is identical to \( b \) except that \( b_l = 1 \). We define \( g(b, l) = g(b_{x_{l-1}}) - g(b) \) if \( b_l = * \), and \( g(b, l) = 0 \) if \( l \) is not contained. Given Boolean function \( g \), we will also use the term \( x \) to denote a random \( x \) drawn from product distribution \( D_p \), for fixed \( D_p, b \in \{0, 1, *\}^n \), and \( i \in N \), we use \( E[g(b_i)] \) to denote the expected increase in utility that would be obtained by testing \( i \). In the binary case, \( E[g(b_i)] = p_i g(b, 1) + (1 - p_i) g(b, 0) \). Note that \( E[g(b_i)] = 0 \) if \( b_i \neq * \). For \( b \in \{0, 1, *\}^n \), \( p(b) = \prod_{j \in \text{dom}(b)} p(b, j) \) where \( p(b, j) = p_j \) if \( b_j = 1 \), and \( p(b, j) = 1 - p_j \) otherwise.

Utility function \( g : \{0, 1\}^n \rightarrow \mathbb{Z}_{\geq 0} \) is monotone if for \( b \in \{0, 1, *\}^n \), \( i \in N \) such that \( b_i = * \), and \( b' \in \{0, 1, *\}^n \) and \( l \in \{0, 1\} \), \( g(b_{x_{l-1}}) - g(b) \geq g(b'_{x_{l-1}}) - g(b') \).

The Stochastic Submodular Set Cover (SSSC) problem. The SSSC problem is similar to the SBFE problem, except that the goal is to achieve a cover. Let \( O = \{0, 1, \ldots, k - 1\} \) be a finite set of \( k \) states, where \( k \geq 2 \) and \( * \notin O \). In what follows, we assume the binary case where \( k = 2 \), although we will briefly mention extensions to the \( k \)-ary case where \( k > 2 \).

In the (binary) SSSC problem, the input consists of the set \( N \), a cost vector \( (c_1, \ldots, c_n) \), where each \( c_j \geq 0 \), a probability vector \( p = (p_1, \ldots, p_n) \) where \( p \in [0,1] \), an integer \( Q \geq 0 \), and a utility function \( g : (O \cup \{\ast\})^n \rightarrow \mathbb{Z}_{\geq 0} \). Further, \( g(x) = 0 \) if \( x \) is the vector that is all \( * \)’s, and \( g(x) = Q \) if \( x \in O^n \). We call \( Q \) the goal utility. We say that \( b \in (O \cup \{\ast\})^n \) is a cover if \( g(b) = Q \). The cost of cover \( b \) is \( \sum_{j:b_j \neq \ast} c_j \).
find a subset $S \subseteq N$ such that $g(S) = Q$ and $\sum_{j \in S} c_j$ is minimized. SSC can be viewed as a special case of SSSC in which each $p_j$ is equal to 1.

The Adaptive Greedy algorithm for Stochastic Submodular Set Cover. The Classical Set Cover problem has a simple greedy approximation algorithm that chooses the subset with the “best bang for the buck” — i.e., the subset covering the most new elements per unit cost. The generalization of this algorithm to SSC, due to Wolsey, chooses the element that adds the maximum additional utility per unit cost [39]. The Adaptive Greedy algorithm of Golovin and Krause, for the SSSC problem, is a further generalization. It chooses the element with the maximum expected increase in utility per unit cost. (Golovin and Krause actually formulated Adaptive Greedy for use in solving a somewhat more general problem than SSSC, but here we describe it only as it applies to SSSC.) We present the pseudocode for Adaptive Greedy in Algorithm 1.

Some of the variables used in the pseudocode are not necessary for the running of the algorithm, but are useful in its analysis. (In Step 5 assume that if $E[g_0(x)] = 0$, the expression evaluates to 0.)

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Algorithm 1: Adaptive Greedy

1. $b \leftarrow (\ast, \ast, \ldots, \ast)$
2. $l \leftarrow 0, F^0 \leftarrow \emptyset$
3. while $b$ is not a solution to SSSC ($f(b) < Q$) do
   4.       $l \leftarrow l + 1$
   5.       $j_l \leftarrow \arg \min_{j \notin F^{l-1}} \frac{c_j}{E[g_0(j)]}$
   6.       $k \leftarrow$ the state of $j_l$ of “test” $j_l$
   7.       $F^l \leftarrow F^{l-1} \cup \{j_l\}$ if $F^l = \text{dom}(b)$
   8.       $b_{j_l} \leftarrow k$
4. return $b$
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Golovin and Krause proved that Adaptive Greedy is a $(\ln Q + 1)$-approximation algorithm, where $Q$ is the goal utility. We will make repeated use of this bound.

4 Function Evaluation and the SSSC Problem

4.1 The $Q$-value approach and CDNF Evaluation.

Definition: Let $f(x_1, \ldots, x_n)$ be a Boolean function. Let $g : \{0, 1, \ast\}^n \rightarrow \mathbb{Z}_{\geq 0}$ be a utility function. We say that $g$ is assignment feasible for $f$, with goal value $Q$, if (1) $g$ is monotone and submodular, (2) $g(\ast, \ast, \ldots, \ast) = 0$, and (3) for $b \in \{0, 1, \ast\}^n$, $g(b) = Q$ if $b$ is a 0-certificate or a 1-certificate of $f$.

We will use the following approach to solving SBFE problems, which we call the $Q$-value approach. To evaluate $f$, we construct an assignment feasible utility function $g$ for $f$ with goal value $Q$. We then run Adaptive Greedy on the resulting SSSC problem. Because $g(b) = Q$ if $b$ is either a 0-certificate or a 1-certificate of $f$, the decision tree that is (implicitly) output by Adaptive Greedy is a solution to the SBFE problem for $f$. By the bound on Adaptive Greedy, this solution is within a factor of $(\ln Q + 1)$ of optimal.

The challenge in using the above approach is in constructing $g$. Not only must $g$ be assignment feasible, but $Q$ should be subexponential, to obtain a good approximation bound. We will use the following lemma, due to Guillory and Bilmes, in our construction of $g$.

Lemma 1. [23] Let $g_0 : \{0, 1, \ast\}^n \rightarrow \mathbb{Z}_{\geq 0}$, $g_1 : \{0, 1, \ast\}^n \rightarrow \mathbb{Z}_{\geq 0}$, and $Q_0, Q_1 \in \mathbb{Z}_{\geq 0}$ be such that $g_0$ and $g_1$ are monotone, submodular utility functions, $g(\ast, \ast, \ldots, \ast) = g_1(\ast, \ast, \ldots, \ast) = 0$, and $g_0(a) \leq Q_0$ and $g_1(a) \leq Q_1$ for all $a \in \{0, 1\}^n$.

Let $Q_\ast = Q_0Q_1$ and let $g_\ast : \{0, 1, \ast\}^n \rightarrow \mathbb{Z}_{\geq 0}$ be such that $g_\ast(b) = Q_\ast - (Q_0 - g_0(b))(Q_1 - g_1(b))$.

Let $Q_\ast = Q_0 + Q_1$ and let $g_\ast : \{0, 1, \ast\}^n \rightarrow \mathbb{Z}_{\geq 0}$ be such that $g_\ast(b) = g_0(b) + g_1(b)$.
Then \( g_v \) and \( g_\land \) are monotone and submodular, and \( g_v(\ast, \ldots, \ast) = g_\land(\ast, \ldots, \ast) = 0 \). For \( b \in \{0, 1, \ast\}^n \), \( g_v(b) = Q_v \) iff \( g_0(b) = Q_0 \) or \( g_1(b) = Q_1 \), or both. Further, \( g_\land(b) = Q_\land \) iff \( g_0(b) = Q_0 \) and \( g_1(b) = Q_1 \).

Using the Q-value approach, it is easy to obtain an algorithm for evaluating CDNF formulas. A CDNF formula for Boolean function \( f \) is a pair \((\phi_0, \phi_1)\) where \( \phi_0 \) and \( \phi_1 \) are CNF and DNF formulas for \( f \), respectively.

**Theorem 1.** There is a polynomial-time \( O(\log kd) \)-approximation algorithm solving the SBFE problem for CDNF formulas, where \( k \) is the number of clauses in the CNF, and \( d \) is the number of terms in the DNF.

**Proof.** Let \( \phi_0 \) be the CNF and \( \phi_1 \) be the DNF. Let \( f \) be the Boolean function defined by these formulas. Let \( k \) and \( d \) be, respectively, the number of clauses and terms of \( \phi_0 \) and \( \phi_1 \). Let \( g_0 : \{0, 1, \ast\}^n \to \mathbb{Z}_\geq 0 \) be such that for \( a \in \{0, 1, \ast\}^n \), \( g_0(a) \) is the number of terms of \( \phi_1 \) set to 0 by \( a \) (i.e. terms with a literal \( x_i \) such that \( a_i = 0 \), or a literal \( \neg x_i \) such that \( a_i = 1 \)). Similarly, let \( g_1(a) \) be the number of clauses of \( \phi_0 \) set to 1 by \( a \). Clearly, \( g_0 \) and \( g_1 \) are monotone and submodular. Partial assignment \( b \) is a 0-certificate of \( f \) iff \( g_0(b) = d \) and a 1-certificate of \( f \) iff \( g_1(b) = k \). Applying the disjunctive construction of Lemma\[\text{[1]}\] to \( g_0 \) and \( g_1 \), yields a utility function \( g \) that is assignment feasible for \( f \) with goal value \( Q = kd \). Applying Adaptive Greedy and its \((\ln Q + 1)\) bound yields the theorem.

Given a decision tree for a Boolean function \( f \), a CNF (DNF) for \( f \) can be easily computed using the paths to the 0-leaves (1-leaves) of the tree. Thus the above theorem gives an \( O(\ln t) \) approximation algorithm for evaluating decision trees, where \( t \) is the number of leaves.

### 4.2 Linear threshold evaluation via the Q-value approach

A linear threshold formula with integer coefficients has the form \( \sum_{i=1}^n a_i x_i \geq \theta \) where the \( a_i \) and \( \theta \) are integers. It represents the function \( f : \{0, 1\}^n \to \{0, 1\} \) such that \( f(x) = 1 \) if \( \sum_{i=1}^n a_i x_i \geq \theta \), and \( f(x) = 0 \) otherwise. We show how to use the Q-value approach to obtain an algorithm solving the SBFE problem for linear threshold formulas with integer coefficients. The algorithm achieves an \( O(\log D) \)-approximation, \( D = \sum_{i=1}^n |a_i| \). This algorithm, like the CDNF algorithm, works by reducing the evaluation problem to an SSC problem. However, the CDNF algorithm reduces the evaluation problem to a stochastic version of Classical Set Cover problem (each \( x_i \) covers one subset of the (term, clause) pairs when it equals 1, and another when it equals 0). Here there is no associated Classical Set Cover problem.

Let \( h(x) = (\sum_{i=1}^n a_i x_i) - \theta \). For \( b \in \{0, 1, \ast\}^n \), let \( \min(b) = \min\{h(b') : b' \in \{0, 1\}^n \text{ and } b' \sim b\} \) and \( \max(b) = \max\{h(b') : b' \in \{0, 1\}^n \text{ and } b' \sim b\} \). Thus \( \min(b) = (\sum_{j: b_j = \ast} a_j b_j) + (\sum_{i: a_i < 0, b_i = \ast} a_i) - \theta \), \( \max(b) = (\sum_{j: b_j \neq \ast} a_j b_j) + (\sum_{i: a_i > 0, b_i = \ast} a_i) - \theta \), and each can be calculated in linear time. Let \( R_{\min} = \min(\ast, \ldots, \ast) \) and \( R_{\max} = \max(\ast, \ldots, \ast) \). If \( R_{\min} \geq 0 \) or \( R_{\max} < 0 \), \( f \) is constant and no testing is needed. Suppose this is not the case.

Let \( Q_1 = -R_{\min} \) and let submodular utility function \( g_1 \) be such that \( g_1(b) = \min\{-R_{\min}, \min(b) - R_{\min}\} \). Intuitively, \( Q_1 - g_1(b) \) is the number of different values of \( h \) that can be induced by extensions \( b' \) of \( b \) such that \( f(b') = 0 \). Similarly, define \( g_0(b) = \min\{R_{\max} + 1, R_{\max} - \max(b)\} \) and \( Q_0 = R_{\max} + 1 \). Thus \( b \) is a 1-certificate of \( f \) iff \( g_1(b) = Q_1 \), and a 0-certificate iff \( g_0(b) = Q_0 \).

We apply the disjunctive construction of Lemma\[\text{[1]}\] to construct \( g(b) = Q - (Q_1 - g_1(b))(Q_0 - g_0(b)) \), which is an assignment feasible utility function for \( f \) with goal value \( Q = Q_1 Q_0 \). Finally, we obtain an \( O(\log D) \) approximation bound by applying the \((\ln Q + 1)\) bound on Adaptive Greedy.

The quantity \( D \) can be exponential in \( n \), the number of variables. One might hope to obtain a better approximation factor, still using the Q-value approach, by designing a more clever assignment-feasible utility function with a much lower goal-value \( Q \). However, in the next section we show that this is not possible. Achieving a 3-approximation for this problem, as we do in Section\[\text{[5]}\] requires a different approach.
### 4.3 Limitations of the Q-value approach

The Q-value approach depends on finding an assignment feasible utility function \( g \) for \( f \). We first demonstrate that a generic such \( g \) exists for all Boolean functions \( f \). Let \( Q_0 = \{a \in \{0,1\}^n | f(a) = 0\} \) and \( Q_1 = \{a \in \{0,1\}^n | f(a) = 1\} \). For partial assignment \( b \), let \( g_0(b) = Q_0 - \{a \in \{0,1\}^n | a \sim b, f(a) = 0\} \) with goal value \( Q_0 \), and let \( g_1(b) = Q_1 - \{a \in \{0,1\}^n | a \sim b, f(a) = 1\} \) with goal value \( Q_1 \). Then \( g_0, Q_0, g_1 \) and \( Q_1 \) obey the properties of Lemma 1. Apply the disjunctive construction in that lemma, and let \( g \) be the resulting utility function. Then \( g \) is assignment feasible for \( f \) with goal value \( Q = Q_1 Q_0 \). In fact, this \( g \) is precisely the utility function that would be constructed by the approximation algorithm of Golovin et al. for computing a consistent decision tree of min-expected cost with respect to a sample, if we take the sample to be the set of all \( 2^n \) entries \((x, f(x))\) in the truth table of \( f \). The goal value \( Q \) of this \( g \) is \( 2^\theta(n) \), so in this case the bound for Adaptive Greedy, \((\ln Q + 1)\), is linear in \( n \).

Since we want a sublinear approximation factor, we would instead like to construct an assignment-feasible utility function for \( f \) whose \( Q \) is sub-exponential in \( n \). However, we now show this is impossible even for some simple Boolean functions \( f \). We begin by introducing the following combinatorial measure of a Boolean function, which we call its \( Q \)-value.

**Definition:** The \( Q \)-value of a Boolean function \( f : \{0,1\}^n \rightarrow \{0,1\} \) is the minimum integer \( Q \) such that there exists a assignment feasible utility function \( g \) for \( f \) with goal value \( Q \).

The generic \( g \) above shows that the \( Q \)-value of every \( n \)-variable Boolean function is at most \( 2^{O(n)} \).

**Lemma 2.** Let \( f(x_1, \ldots, x_n) \) be a Boolean function, where \( n \) is even. Further, let \( f \) be such that for all \( n' \leq n/2 \), and for all \( b \in \{0,1,*\}^n \), if \( b_i = b_{n/2+i} = * \) for all \( i \in \{n'+1, \ldots, n/2\} \), the following properties hold: (1) if for all \( i \in \{1, \ldots, n/2\} \), exactly one of \( b_i \) and \( b_{n/2+i} \) is equal to * and the other is equal to 1, then \( b \) is not a 0-certificate or a 1-certificate of \( f \) and (2) if for all \( i \in \{1, \ldots, n' - 1\} \), exactly one of \( b_i \) and \( b_{n/2+i} \) is equal to * and the other is equal to 1, and \( b_{n'} = b_{n/2+n'} = 1 \), then \( b \) contains a 1-certificate of \( f \). Then the \( Q \)-value of \( f \) is at least \( 2^{n/2} \).

**Proof.** Let \( f \) have the properties specified in the lemma. For bitstrings \( r, s \in \{0,1\}^l \), where \( 0 \leq l \leq n/2 \), let \( d_{r,s} \in \{0,1,*\}^n \) be such that \( d_i = r_i \) and \( d_{n/2+i} = s_i \) for \( i \in \{1, \ldots, l\} \), and \( d_i = * \) for all other \( i \). Suppose \( g \) is an assignment feasible utility function for \( f \) with goal value \( Q \). We prove the following claim. Let \( 0 \leq l \leq n/2 \). Then there exists \( r, s \in \{0,1,*\}^l \) such that \( 0 \leq Q - g(d_{r,s}) \leq Q/2^l \), and for all \( i \in \{1, \ldots, l\} \), either \( r_i = 1 \) and \( s_i = * \), or \( r_i = * \) and \( s_i = 1 \).

We prove the claim by induction on \( l \). It clearly holds for \( l = 0 \). For the inductive step, assume it holds for \( l \). We show it holds for \( l + 1 \). Let \( r, s \in \{0,1,*\}^l \) be as guaranteed by the assumption, so \( Q - g(d_{r,s}) \leq n/2^l \).

For \( \sigma \in \{0,1,*\} \), \( r\sigma \) denotes the concatenation of bitstring \( r \) with \( \sigma \), and similarly for \( s\sigma \). By the conditions on \( f \) given in the lemma, \( d_{r,s} \) is not a 0 or 1-certificate of \( f \). However, \( d_{r,1} \) is a 1-certificate of \( f \) and so \( g(d_{r,1}) = Q \). If \( Q - g(d_{r,s}) \leq Q/2^{l+1} \), then the claim holds for \( l + 1 \), because \( r, s \) have the necessary properties. Suppose \( Q - g(d_{r,s}) > Q/2^{l+1} \). Then, because \( g(d_{r,1}) = Q, g(d_{r,s}) \geq Q/2^{l+1} \). Note that \( d_{r,1} \) is the extension of \( d_{r,s} \) produced by setting \( d_{n/2+l+1} \) to 1. Similarly, \( d_{r,s} \) is the extension of \( d_{r,s} \) produced by setting \( d_{n/2+l+1} \) to 1. Therefore, by the submodularity of \( g \), \( g(d_{r,s}) - g(d_{r,s}) \geq g(d_{r,1}) - g(d_{r,s}) \), and thus \( g(d_{r,s}) - g(d_{r,s}) \geq Q/2^{l+1} \).

Let \( A = g(d_{r,s}) - g(d_{r,s}) \) and \( B = Q - g(d_{r,s}) \). Thus \( A \geq Q/2^{l+1} \), and \( A + B = Q - g(d_{r,s}) \) because \( A \geq Q/2^{l+1} \) and the last inequality is from the original assumptions on \( r \) and \( s \). It follows that \( B = Q - g(d_{r,s}) \leq Q/2^{l+1} \) and the claim holds for \( l + 1 \), because \( r, s \) have the necessary properties.

Taking \( l = n/2 \), the claim says there exists \( d_{r,s} \) such that \( Q - g(d_{r,s}) \leq Q/2^{n/2} \). Since \( g \) is integer-valued, \( Q \geq 2^{n/2} \).
Theorem 2. Let \( n \) be even. Let \( f : \{0,1\}^n \to \{0,1\} \) be the Boolean function represented by the read-once DNF formula \( \phi = t_1 \lor t_2 \lor \ldots \lor t_{n/2} \) where each \( t_i = x_i x_{n/2+i} \). The \( Q \)-value of \( f \) is at least \( 2^{n/2} \).

The above theorem shows that the \( Q \)-value approach will not yield a good approximation bound for either read-once DNF formulas or for DNF formulas with terms of length 2.

In the next theorem, we show that there is a particular linear threshold function whose \( Q \)-value is at least \( 2^{n/2} \). It follows that the \( Q \)-value approach will not yield a good approximation bound for linear-threshold formulas either.

We note that the function described in the next theorem has been studied before. As mentioned in [25], there is a lower bound of essentially \( 2^{n/2} \) on the size of the largest integer coefficients in any representation of the function as a linear threshold function with integer coefficients.

Theorem 3. Let \( f(x_1, \ldots, x_n) \) be the function defined for even \( n \), whose value is 1 iff the number represented in binary by bits \( x_1 \ldots x_{n/2} \) is strictly less than the number represented in binary by bits \( x_{n/2+1} \ldots x_n \), and 0 otherwise. The \( Q \)-value of \( f \) is at least \( 2^{n/2} \).

Proof. We define a new function: \( f'(x_1, \ldots, x_n) = f(-x_1, \ldots, -x_{n/2}, x_{n/2+1}, \ldots, x_n) \). That is, \( f'(x_1, \ldots, x_n) \) is computed by negating the assignments to the first \( n/2 \) variables, and then computing the value of \( f \) on the resulting assignment. Function \( f' \) obeys the conditions of Lemma 2 and so has \( Q \)-value at least \( 2^{n/2} \). Then \( f \) also has \( Q \)-value at least \( 2^{n/2} \), because the \( Q \)-value is not changed by the negation of input variables.

Given the limitations of the \( Q \)-value approach we can ask whether there are good alternatives. Our new bound on Adaptive Greedy is \( O(\log P) \), where \( P \) is the maximum amount of utility gained by testing a single variable \( x_j \), so we might hope to use \( P \)-value in place of \( Q \)-value. However, this does not help much: testing all \( n \) variables yields utility \( Q \), so testing one of them alone must yield utility at least \( Q/n \), implying that \( P \geq Q/n \). Another possibility might be to exploit the fact that Golovin and Krause’s bounds on Adaptive Greedy apply to a more general class of utility functions than the assignment feasible utility functions, but we do not pursue that possibility. Instead, we give a new algorithm for the SSSC problem.

5 Adaptive Dual Greedy and a \( 3 \)-approximation for Linear Threshold Evaluation

We now present ADG, our new algorithm for the binary version of the SSSC problem. It easily extends to the \( k \)-ary version, where \( k > 2 \), with no change in the approximation bound. Like Fujito’s Dual Greedy algorithm for the (non-adaptive) SSC problem, it is based on Wolsey’s IP for the (deterministic) Submodular Set Cover Problem. We present Wolsey’s IP in Figure 1.

![Figure 1: Wolsey’s IP for submodular set cover](image)

Wolsey proved that an assignment \( x \in \{0,1\}^n \) to the variables in this IP is feasible iff \( \{j | x_j = 1\} \) is a cover for the associated Submodular Set Cover instance, i.e., iff \( g(\{j | x_j = 1\}) = Q \). We call this Wolsey’s property.

In Figure 2 we present a new LP, based on Wolsey’s IP, which we call LP1. We use the following notation: \( W = \{w \in \{0,1,\ast\}^n \mid w_j = \ast \ \text{for exactly one value of} \ j\} \). For \( w \in W \), \( j(w) \) denotes the \( j \in N \) where \( w_j = \ast \). Further, \( w^{(0)} \) and \( w^{(1)} \) denote the extensions of \( w \) obtained from \( w \) by setting \( w_{j(w)} = 0 \) and 1, respectively. For \( a \in \{0,1\}^n \), \( j \in N \), \( a^j \) denotes the the partial assignment obtained from \( a \) by setting \( a_j \) to \( \ast \). We will rely on the following observation, which we call the Neighbor Property: Let \( T \) be a decision tree solving the SSSC problem. Given two assignments \( a, a' \in \{0,1\}^n \) differing only in bit \( j \), either \( T \) tests \( j \) on both input \( a \) and input \( a' \), or on neither.
Lemma 3. The optimal value of LP1 lower bounds the expected cost of an optimal decision tree \( T \) for the SSSC instance on \( g, p, \) and \( c. \)

Proof. Let \( X \) be the assignment to the variables \( x_w \) in the LP such that \( x_w = 1 \) if \( T \) tests \( j \) on both assignments extending \( w, \) and \( x_w = 0 \) otherwise. With respect to \( X, \) the expected cost of \( T \) equals \( \sum_{a \in \{0,1\}^n} \sum_j c_j x_{j,a} p(a). \) This equals the value of the objective function, because for \( a, a' \in \{0,1\}^n \) differing only in bit \( j, \) \( P(a) + P(a') = P(a'). \) Finally, for any fixed \( a \in \{0,1\}^n, \) the subset of constraints involving \( a, \) one for each \( S \subseteq N, \) is precisely the set of constraints of Wolsey’s IP, if we take the utility function to be \( g_a(S) = g(S,a). \) Since \( T \) produces a cover for every \( a, \) by Wolsey’s property, the constraints of LP1 involving \( a \) are satisfied. Thus \( X \) is a feasible solution to LP1, and the optimal value of the LP is at most the expected cost of the optimal tree.

We present the pseudocode for ADG in Algorithm 2 (In Step 5 assume that if \( E[g_b(x)] = 0, \) the expression evaluates to 0.) Its main loop is analogous to the main loop in Fujito’s Dual Greedy algorithm, except that ADG uses expected increases in utility, instead of known, deterministic increases in utility and the results of the tests performed on the items already in the cover. The quantity in Step 5 of ADG relies only on the outcomes of completed tests, so ADG can be executed in our stochastic setting.

Algorithm 2: Adaptive Dual Greedy

We now analyze Adaptive Dual Greedy. In the LP in Figure 2, there is a constraint for each \( a, S \) pair. Multiply both sides of such constraints by \( p(a), \) to form an equivalent LP. Take the dual of the result, and

\[
\begin{align*}
\text{Min} & \quad \sum_{w \in W} c_j(w)p(w)x_w \\
\text{s.t.} & \quad \sum_{j \in N} g_{S,a}(j)x_{j,a} \geq Q - g(S,a) \quad \forall a \in \{0,1\}^n, S \subseteq N \\\forall w \in W
\end{align*}
\]

Figure 2: LP1: the Linear Program for Lower Bounding Adaptive Dual Greedy

\[
\begin{align*}
\text{Max} & \quad \sum_{a \in \{0,1\}^n} \sum_{S \subseteq N} p(a) (g(N,a) - g(S,a)) y_{S,a} \\
\text{s.t.} & \quad \sum_{S \subseteq N} (1 - p_j(w)) g_{S,w(0)}(j) y_{S,w(0)} + \sum_{S \subseteq N} p_j(w) g_{S,w(1)}(j) y_{S,w(1)} \leq c_j \quad \forall w \in W \\forall S \subseteq N, a \in \{0,1\}^n
\end{align*}
\]

Figure 3: LP2: the Linear Program for Adaptive Dual Greedy
divide both sides of each constraint by \( p(w) \). (Note that \( p(a)/p(w) = p(a, j(w)) \).) We give the resulting LP, which we call LP2, in Figure 3. The variables in it are \( y_{S,a} \), where \( S \subseteq N \) and \( a \in \{0,1\}^n \).

Consider running ADG on an input \( a \in \{0,1\}^n \). Because \( g(a) = Q \), ADG is guaranteed to terminate with an output \( b \) such that \( g(b) = Q \). Let \( C(a) = \text{dom}(b) \). That is, \( C(a) \) is the set of items that ADG tests and inserts into the cover it constructs for \( a \). We will sometimes treat \( C(a) \) as a sequence of items, ordered by their insertion order. ADG constructs an assignment to the variables \( y_S \) (one for each \( S \subseteq N \)) when it is run on input \( a \). Let \( Y \) be the assignment to the variables \( y_{S,a} \) of LP2, such that \( Y_{S,a} \) is the value of ADG variable \( y_S \) at the end of running ADG on input \( a \).

We now show that \( Y \) is a feasible solution to LP2 and that for each \( a \) and each \( j \in C(a) \), \( Y \) makes the constraint for \( d = a^j \) tight. For \( w \in W \), let \( h_w'(y) \) denote the function of the variables \( y_{S,a} \) computed in the left hand side of the constraint for \( w \) in LP2.

**Lemma 4.** For every \( a \in \{0,1\}^n \), \( j \in N \),
1. \( h_w'(j)(Y) = c_j \) if \( j \in C(a) \), and
2. \( h_w'(j)(Y) \leq c_j \) if \( j \notin C(a) \).

**Proof.** Assignment \( Y \) assigns non-zero values only to variables \( y_{S,a} \) where \( S \) is a prefix of sequence \( C(a) \).

For \( t \in N \), let \( Y^t \) denote the assignment to the \( y_{S,a} \) variables such that \( y_{S,a} \) equals the value of variable \( Y_S \) at the end of iteration \( t \) of the loop in ADG, when ADG is run on input \( a \). (If ADG terminates before iteration \( t \), \( y_{S,a} \) equals the final value of \( y_S \).) Let \( Y^0 \) be the all 0’s assignment. We begin by showing that for all \( t \) and \( a \), \( h_w'(Y^t) = \sum_{S \subseteq N} E[g_{a(S)}(j)] Y^t_{S,a} \). Recall that \( a(S) \in \{0,1,*\}^n \) such that \( \forall i \in S, a(S)_i = a_i \), and \( \forall j \notin S, a(S)_j = * \).

Consider running ADG on \( w(0) \) and \( w(1) \). Since ADG corresponds to a decision tree, the Neighbor Property holds. Then, if \( j \) is never tested on \( w(0) \), it is never tested on \( w(1) \). Let \( Y^t_{S,w(0)} = Y^t_{S,w(1)} \) for all \( S, t \). Thus \( h_w'(Y^t) = \sum_{S \subseteq N} (p_j g_{S,w(1)}(j)) + (1 - p_j) g_{S,w(0)}(j)) Y^t_{S,w(1)} = \sum_{S \subseteq N} E[g_{S,w(j)}] Y^t_{S,w(1)} \) for all \( t \).

Now suppose that \( j \) is tested in iteration \( t \) on input \( w(1) \), and hence on input \( w(0) \). For \( t \leq t \), \( Y^t_{S,w(1)} = Y^t_{S,w(0)} \) for all \( S \). This is not the case for \( t > t \). However, in iterations \( t > t \), \( j \) is already part of the cover, so ADG assigns values only to variables \( Y_S \) where \( j \in S \). For such \( S \), \( g_{S,w(1)}(j) = 0 \). Thus in this case also, \( h_w'(Y^t) = \sum_{S \subseteq N} E[g_{S,w(j)}] Y^t_{S,w(1)} \) for all \( t \).

It is now easy to show by induction on \( t \) that the the two properties of the lemma hold for every \( Y^t \), and hence for \( Y \). They hold for \( Y^0 \). Assume they hold for \( Y^t \). Again consider assignments \( w(1) \) and \( w(0) \). If \( j \) was tested on \( w(1) \) and \( w(0) \) in some iteration \( t_1 < t + 1 \), then \( h_w'(Y^t) = h_w'(Y^{t+1}) \) by the arguments above.

If \( j \) is tested in iteration \( t + 1 \) on both inputs, then the value assigned to \( y_{F(t-1),w(j)} \) by ADG on \( w(1) \) (and \( w(0) \)) equals \( (c_j - h_w'(Y^t))/E[g_{F(t-1),w(j)}] \), and thus \( h_w'(Y^{t+1}) = c_j \). If \( j \) is not tested in iteration \( t + 1 \), and was not tested earlier, the inductive assumption and the greedy choice criterion ensure that \( h_w'(Y^{t+1}) \leq c_j \).

The expected cost of the cover produced by ADG on a random input \( a \) is \( \sum_{a \in \{0,1\}^n} \sum_j \in C(a) p(a) c_j \).

**Lemma 5.** \( \sum_{a \in \{0,1\}^n} \sum_j \in C(a) p(a) c_j = \sum_{a \in \{0,1\}^n} \sum_{S \subseteq N} \sum_j \in C(a) p(a) g_{S,a}(j) Y_{S,a} \).
Proof. For \( j \in N \), let \( W^j = \{ w \in W | w_j = * \} \). Then

\[
\sum_a \sum_{j:S_j \in C(a)} p(a)c_{j}
\]

\( = \sum_j \sum_{a:S_j \in C(a)} p(a)c_{j} \quad \text{switching the order of summation} \)

\( = \sum_j \left( \sum_{w \in W:j \in C(w^{(1)})} p(w^{(1)})c_{j} + \sum_{w \in W:j \in C(w^{(0)})} p(w^{(0)})c_{j} \right) \quad \text{grouping assignments by the value of bit } j \)

\( = \sum_j \left( \sum_{w \in W:j \in C(w^{(1)})} p(w^{(1)})c_{j} + p(w^{(0)})c_{j} \right) \quad \text{because } j \in C(w^{(1)}) \text{ if and only if } j \in C(w^{(0)}) \text{ by the Neighbor Property} \)

\( = \sum_j \sum_{w \in W:j \in C(w^{(1)})} p(w)c_{j} \quad \text{by Lemma 4} \)

\( = \sum_j \sum_{w \in W:j \in C(w^{(1)})} p(w)h_w(Y) \quad \text{by the definition of } h_w \)

\( = \sum_j \sum_{w \in W:j \in C(w^{(1)})} \left( p(w) \sum_{S} (p_i g_{S,w^{(1)}}(j)) Y_{S,w^{(1)}}(1 - p_i g_{S,w^{(0)}}(j)) Y_{S,w^{(0)}}(0) \right) \quad \text{by the definition of } h_w \)

\( = \sum_j \sum_{w \in W:j \in C(w^{(1)})} \sum_{S} p(w^{(1)}) g_{S,w^{(1)}}(j) Y_{S,w^{(1)}} + p(w^{(0)}) g_{S,w^{(0)}}(j) Y_{S,w^{(0)}} \)

\( = \sum_j \left( \sum_{w \in W:j \in C(w^{(1)})} \sum_{S} p(w^{(1)}) g_{S,w^{(1)}}(j) Y_{S,w^{(1)}} + \left( \sum_{w \in W:j \in C(w^{(0)})} \sum_{S} p(w^{(0)}) g_{S,w^{(0)}} Y_{S,w^{(0)}} \right) \right) \quad \text{because } j \in C(w^{(1)}) \text{ if and only if } j \in C(w^{(0)}) \)

\( = \sum_j \sum_{a:S \in C(a)} \sum_S p(a) g_{S,a}(j) Y_{S,a} \)

\( = \sum_a \sum_S \sum_{j:S_j \in C(a)} p(a) g_{S,a}(j) Y_{S,a} \)

We now give our approximation bound for ADG.

**Theorem 4.** Given an instance of SSSC with utility function \( g \) and goal value \( Q \), ADG constructs a cover whose expected cost is no more than a factor of \( \alpha \) larger than the expected cost of the cover produced by the optimal strategy, where \( \alpha = \max \frac{\sum_{a \in \{0,1\}^n} g_{S,a}(j)}{Q - g(S,a)} \), with the max taken over all \( a \in \{0,1\}^n \) and \( S \in \text{Pref}(C(a)) \) such that the denominator is non-zero. Here \( \text{Pref}(C(a)) \) denotes the set of all prefixes of the cover \( C(a) \) that ADG constructs on input \( a \).

**Proof.** By Lemma 5 the expected cost of the cover constructed by ADG is \( \sum_a \sum_S \sum_{j:S_j \in C(a)} p(a) g_{S,a}(j) Y_{S,a} \). The value of the objective function of LP2 on \( Y \) is \( \sum_a \sum_S p(a) (Q - g(S,a)) Y_{S,a} \). For any \( a, Y_{S,a} \) is non-zero if \( S \in \text{Pref}(C(a)) \). Comparing the coefficients of \( Y_{S,a} \) in these two expressions implies that the value of the objective function on \( Y \) is at most \( \max \frac{\sum_{a \in \{0,1\}^n} g_{S,a}(j)}{Q - g(S,a)} \) times the expected cost of the cover. The theorem follows by Lemma 5 and weak duality. \( \square \)

**Theorem 5.** There is a polynomial-time 3-approximation algorithm solving SBFE problem for linear threshold formulas with integer coefficients.

**Proof.** We modify the linear threshold evaluation algorithm from Section 4.2, substituting ADG for Adaptive Greedy. By Theorem 4 the resulting algorithm is within a factor of \( \alpha \) of optimal. We now show that \( \alpha \leq 3 \) in this case.

Fix \( x \) and consider the run of ADG on \( x \). Let \( T \) be the number of loop iterations. So \( C(x) = j_1, \ldots, j_T \) is the sequence of tested items, and \( F^t = \{ j_1, \ldots, j_t \} \). Assume first that \( f(x) = 1 \). Let \( F = F^0 = \emptyset \), and consider the ratio \( \frac{\sum_{j \in C(x)} g_{F,j}(j)}{Q - g(F,x)} \).
We use the definitions and utility functions from the algorithm in Section 4.2. Assume without loss of generality that neither $g_0$ nor $g_1$ is identically 0.

Let $A = -R_{min}$ and let $B = R_{max} + 1$. Thus $Q - g(\emptyset, x) = AB$. Let $C_1$ be the set of items $j_t$ in $C(x)$ such that either $x_{j_t} = 1$ and $a_{j_t} \geq 0$ or $x_{j_t} = 0$ and $a_{j_t} < 0$. Similarly, let $C_0$ be the set of items $j_t$ in $C(x)$, such that either $x_{j_t} = 0$ and $a_{j_t} \geq 0$ or $x_{j_t} = 1$ and $a_{j_t} < 0$.

Testing stops as soon as the goal utility is reached. Since $f(x) = 1$, this means testing on $x$ stops when $b$ satisfies $g_1(b) = Q_1$, or equivalently, $b$ is a 1-certificate of $f$. Thus the last tested item, $j_T$, is in $C_1$. Further, the sum of the $a_{j_t}x_{j_t}$ over all $j_t \in C_1(x)$, excluding $j_T$, is less than $-R_{min}$, while the sum including $j_T$ is greater than or equal to $-R_{min}$. By the definition of utility function $g_1$, $\sum_{j_t \in C_1} g_0(x(j_t)) < AB$. The maximum possible value of $g_0(x(j_T))$ is $AB$. Therefore, $\sum_{j_t \in C_1} g_0(x(j_t)) < 2AB$.

Since $x$ does not contain both a 0-certificate and a 1-certificate of $f$, the sum of the $a_{j_t}x_{j_t}$ over all $j_t \in C_0(x)$ is strictly less than $R_{max}$. Thus by the definition of $g_1$, $\sum_{j_t \in C_0} g_0(x(j_t)) < AB$. Summing over all $j_t \in C(x)$, we get that $\sum_{j_t \in C(x)} g_0(x(j_t)) < 3AB$. Therefore, $\frac{\sum_{j_t \in C(x)} g_0(x(j_t))}{Q - g(F,x)} < 3$, because for $F = \emptyset$, $Q = AB$ and $g(\emptyset, x) = 0$. A symmetric argument holds when $f(x) = 0$.

It remains to show that the same bound holds when $F \neq \emptyset$. We reduce this to the case $F = \emptyset$. Once we have tested the variables in $F^t$, we have an induced linear threshold evaluation problem on the remaining variables (replacing the tested variables by their values). Let $g'$ and $Q'$ be the utility function and goal value for the induced problem, as constructed in the algorithm of Section 4.2. The ratio $\frac{\sum_{j_t \in C(x)} g_0(x(j_t))}{Q - g(F,x)}$ is equal to $\frac{\sum_{j_t \in C(x) - F} g'(x(j_t))}{Q' - g(F,x)}$, where $x'$ is $x$ restricted to the elements not in $F$. By the argument above, this ratio is bounded by 3.

6 A new bound for Adaptive Greedy

We give a new analysis of the Adaptive Greedy algorithm of Golovin and Krause, whose pseudocode we presented in Algorithm 1. Throughout this section, we let $g(j) = \max_{i \in \{0,1\}} g_r(j,i)$ where $r = (\ast, \ldots, \ast)$. Thus $g(j)$ is the maximum increase in utility that can be obtained as a result of testing $j$ (since $g$ is submodular). We show that the expected cost of the solution computed by Adaptive Greedy is within a factor of $2(\ln(\max_{i \in N} g(i) + 1))$ of optimal in the binary case.

In the $k$-ary case, the 2 in the bound is replaced by $k$. Note that $\max_j g(j)$ is clearly upper bounded by $Q$, and in some instances may be much less than $Q$. However, because of the factor of $k$ at the front of our bound, we cannot say that it is strictly better than the $(\ln Q + 1)$ bound of Golovin and Krause. (The bound that is analogous to ours in the non-adaptive case, proved by Wolsey, does not have a factor of 2.)

Adaptive Greedy is a natural extension of the Greedy algorithm for (deterministic) submodular set cover of Wolsey. We will extend Wolsey’s analysis [39], as it was presented by Fujito [16]. In our analysis, we will refer to LP2 defined in Section 5 along with the associated notation for the constraints $h_{a_j}(y) \leq c_j$.

For $x \in \{0,1\}^n$, let $T^x$ be the number of iterations of the Adaptive Greedy while loop on input $x$. Let $b_{\ast j}^t$ denote the value of $b$ at the end of iteration $t$ of the while loop on input $x$, and let $F^t$ denote the value of $F^t$, where $F^t$ is the set of $j$s tested by the end of the $t + 1$st iteration. (The $x$ in the notation may be dropped when it is understood implicitly.) Set $\theta_{\ast j}^t = \min_{j \notin F^{t-1}} \frac{g_{a_j} - g_{a_j+1}}{g_{a_j}}$.

For $j \in N$, let $k_j$ be the value of $t$ that maximizes $(\theta_{\ast j}^t)(g_{a_j^t}(j,1))$. Similarly, let $l_j$ be the value of $t$ that maximizes $(\theta_{\ast j}^t)(g_{a_j^t}(j,0))$, where $x'$ is the assignment obtained from $x$ by complementing $x_j$. Again, let $r$ denote the assignment $\{*, \ldots, *, \}$, and let $H_r^t = H(g_r(j,1))$ and $H_0^t = H(g_r(j,0))$, where $H(n)$ denotes the $n$th harmonic number, which is at most $(\ln n + 1)$. Let $q_j = 1 - p_j$.

To analyze Adaptive Greedy, we define $Y$ to be the assignment to the LP2 variables $y_{S,x}$ setting $y_{F^0,x} = \theta_{\ast j}^1$, $y_{F^t,x} = (\theta_{\ast j}^{t+1} - \theta_{\ast j}^t)$ for $t \in \{1 \ldots T^x - 1\}$ and $y_{S,x} = 0$ for all other $S$. We define $Y^x$ to be the restriction of that assignment to variables $y_{S,x}$ for that $x$. Let $q^x(y) = \sum_{S \subseteq N} y_{S,x}(N - S)y_S^x$. 12
Lemma 6. The expected cost of the cover constructed by Adaptive Greedy is at most \( E[q^*(Y^*)] \), where the expectation is with respect to \( x \sim D_p \).

Proof. By the definition of \( Y^* \), the proof follows directly from the analysis of (non-adaptive) Greedy in Theorem 1 of \cite{16}, by linearity of expectation.

We need to bound the value of \( h'_w(Y) \) for each \( w \in W \). We will use the following lemma from Wolsey’s analysis.

Lemma 7. \cite{22} Given two sequences \( (\alpha(t))_{t=1}^T \) and \( (\beta(t))_{t=0}^{T-1} \), such that both are nonnegative, the former is monotonically non-decreasing and the latter, monotonically non-increasing, and \( \beta(t) \) is a nonnegative integer for any value of \( t \), then

\[
\alpha(1)\beta(0) + (\alpha(2) - \alpha(1))\beta(1) + \ldots + (\alpha(T) - \alpha(T-1))\beta(T-1) \\
\leq \left( \max_{1 \leq t \leq T} \alpha(t) \beta(t-1) H(\beta(t)) \right).
\]

Lemma 8. For every \( x \in \{0,1\}^n \) and \( j \in \{1,\ldots,N\} \), \( h'_{x,j}(Y) \leq c_j 2H(\max_{i \in N} g(i)) \).

Proof. By the submodularity of \( g \), and the greedy choice criterion used by Adaptive Greedy, \( \theta_x^1 \leq \theta_x^2 \leq \ldots \leq \theta_x^T \). By the submodularity of \( g \), \( g_{h_0}(j,0) \geq g_{h_1}(j,0) \geq \ldots \geq g_{h_{T-1}}(j,0) \). Thus Lemma \[7\] applies to the non-decreasing sequence \( \theta_x^1, \theta_x^2, \ldots, \theta_x^T \) and the non-increasing sequence \( g_{h_x}^0(j,0), \ldots, g_{h_x}^1(j,0), \ldots, g_{h_x}^{T-1}(j,0) \). This also holds if we substitute \( (j,1) \) for \( (j,0) \) in the second sequence.

Let \( x' \) be the assignment differing from \( x \) only in bit \( j \). In the following displayed equations, we write \( k \) and \( l \) in place of \( k_j \) and \( l_j \) to simplify the notation.

\[
h'_{x,j}(Y) = \sum_{S \subseteq N} (p_j g_{S,x}(j) + (1 - p_j) g_{S,x'}(j)) Y_{S,x}
\]

by the Neighbor Property, by the same argument as used in the analysis of ADG

\[
= p_j[\theta_x^1 g_{h_x^0}(j,1) + \sum_{i=2}^{T_x} (\theta_x^i - \theta_x^{i-1}) g_{h_x^{i-1}}(j,1)] + q_j[\theta_x^1 g_{h_x^0}(j,0) + \sum_{i=2}^{T_x} (\theta_x^i - \theta_x^{i-1}) g_{h_x^{i-1}}(j,0)] \\
\leq p_j[\theta_x^1 g_{h_x^0}(j,1) H_j^1] + q_j[\theta_x^1 g_{h_x^0}(j,0) H_j^0] \leq 2H(\max_i g(i))
\]

by Lemma \[7\] as indicated above

\[
\leq p_j[\theta_x^1 h_{jk_x^0}(j,1) H_j^1] + q_j[\theta_x^1 h_{jk_x^0}(j,0) H_j^0] + p_j[\theta_x^1 h_{jk_x^0}(j,1) H_j^1] + q_j[\theta_x^1 h_{jk_x^0}(j,0) H_j^0]
\]

since this just adds extra non-negative terms

\[
= \theta_x^1 H_j^1 [p_j g_{h_x^0}(j,1) + q_j g_{h_x^0}(j,0)] + \theta_x^1 H_j^0 [p_j g_{h_x^0}(j,0) + p_j g_{h_x^0}(j,1)]
\]

\[
\leq c_j H_j^1 + c_j H_j^0
due to the greedy choices made by Algorithm \[1\]

\[
\leq c_j 2H(g(j))
\]

\[
\leq c_j 2H(\max_i g(i)) \]

\[\square\]

Theorem 6. Given an instance of SSSC with utility function \( g \), Adaptive Greedy constructs a decision tree whose expected cost is no more than a factor of \( 2(\max_{i \in N} (\ln g(i)) + 1) \) larger than the expected cost of the cover produced by the optimal strategy.
Proof. Let OPT be the expected cost of the cover produced by the optimal strategy. Let AGCOST be the expected cost of the cover produced by Adaptive Greedy, and let \( q(y) \) denote the objective function of LP2. By Lemma 3, the optimal value of LP1 is a lower bound on OPT. By Lemma 4, \( Z = Y/(2H(\max_i g(i))) \) is a feasible solution to LP2. Thus by weak duality, \( q(Z) \leq OPT \). By Lemma 5, \( AGCOST \leq E[q^*(Y^2)] \), and it is easy to see that \( E[q^*(Y^2)] = q(Y) \). Since \( q(Y) = q(Z)(2H(\max_i g(i))) \), \( AGCOST \leq OPT(2H(\max_i g(i))) \).

7 Simultaneous Evaluation and Ranking

Let \( f_1, \ldots, f_m \) be (representations of) Boolean functions from a class \( C \), such that each \( f_i : \{0,1\}^n \rightarrow \{0,1\} \). We consider the generalization of the SBFE problem where instead of determining the value of a single function \( f \) on an input \( x \), we need to determine the value of all \( m \) functions \( f_i \) on the same input \( x \).

The \( Q \)-value approach can be easily extended to this problem by constructing utility functions for each of the \( f_i \), and combining them using the conjunctive construction in Lemma 1. The algorithm of Golovin and Krause for simultaneous evaluation of OR formulas follows this approach (Liu et al. presented a similar algorithm earlier, using a different analysis [30]). We can also modify the approach by calculating a bound based on \( P \)-value, or using ADG instead of Adaptive Greedy. We thus obtain the following theorem, where \( \sum_{i=1}^m \theta_k x_i \leq \theta_k \) is the \( k \)th threshold formula.

**Theorem 7.** There is a polynomial-time algorithm for solving the simultaneous evaluation of linear threshold formulas problem which produces a solution that is within a factor of \( O(\log m D_{\text{avg}}) \) of optimal where \( D_{\text{avg}} \) is the average, over \( k \in \{1, \ldots, m\} \), of \( \sum_{i=1}^n |a_{ki}| \). In the special case of OR formulas, where each variable appears in at most \( r \) of them, the algorithm achieves an approximation factor of \( 2(\ln(\beta_{\text{max}} r) + 1) \), where \( \beta_{\text{max}} \) is the maximum number of variables in any of the OR formulas.

There is also a polynomial-time algorithm for solving the simultaneous evaluation of threshold formulas problem which produces a solution that is within a factor of \( D_{\text{avg}} \) of optimal, where \( D_{\text{max}} = \max_{k=1}^m \sum_{i=1}^n |a_{ki}| \).

Proof. Let \( g^{(1)}, \ldots, g^{(m)} \) be the \( m \) utility functions that would be constructed if we ran the algorithm from Section 4.2 separately on each of the \( m \) threshold formulas that need to be evaluated. Let \( Q^{(1)}, \ldots, Q^{(m)} \) be the associated goal values.

Using the conjunctive construction from Lemma 1, we construct utility function \( g \) such that \( g(b) = \sum_{k=1}^m g^{(k)}(b) \), and \( Q = \sum_{k=1}^m Q^{(k)} \).

To obtain the first algorithm, we evaluate all the threshold formulas by running Adaptive Greedy with \( g \), goal value \( Q \), and the given \( p \), and \( c \), until it outputs a cover \( b \). Given cover \( b \), it is easy to determine for each \( f_k \) whether \( f_k(x) = 1 \) or \( f_k(x) = 0 \).

In the algorithm of Section 4.2 for each \( f_k \), the associated \( Q_k = O(D_k) \), where \( D_k \) is the sum of the absolute values of the coefficients in \( f_k \). Since \( Q = \sum_k Q_k \), the \( O(\log(m D_{\text{avg}})) \) bound follows from the (\( \ln Q + 1 \)) bound for Adaptive Greedy.

Suppose each threshold formula is an OR formula. For \( b \in \{0, 1, *\}^n \), \( \max_{l \in \{0,1\}} g^{(k)}(i, l) = 0 \) if \( x_j \) does not appear in the \( k \)th OR formula, otherwise it is equal to the number of variables in that formula. The \( 2(\ln(\beta_{\text{max}} r) + 1) \) approximation factor then follows by our bound on Adaptive Greedy in Theorem 6.

For the second algorithm, we just use ADG instead of Adaptive Greedy with the same utility function \( g \). By Theorem 3, the approximation factor achieved by ADG is \( \max \sum_j g^{(j)}(S,d) / Q^{(j)}(S,d) \).

We bound this ratio for \( g \). Let \( D_j = \sum_{i=1}^n |a_{ji}|. \) Let \( d \in \{0,1\}^n \) and \( S \in F(x) \). Without loss of generality, assume \( S = \{n'+1, \ldots, n\} \). In the \( k \)th threshold formula, for \( i \geq n' \), replace \( x_i \) with \( d_i \). This induces a new threshold formula on \( n-n' \) variables with threshold \( \theta_{k,d} = \theta_k - D_{k,d} \) whose coefficients sum to \( D_{k,b} = D_k - \sum_{i=n'+1}^n a_id_i \). Let \( b \) be the partial assignment such that \( b_i = d_i \) for \( i \geq n' \), and \( b_i = * \) otherwise. If \( b \) contains either a 0-certificate or a 1-certificate for \( f_k \), then \( Q_k - g^k(S,d) = 0 \).
Otherwise, \(Q_k - g(S, d) = (\theta_{k,b})(D_{k,b} - \theta_{k,b} + 1)\), and \(\sum_j g^k_{S,d}(j) \leq D_{k,b} \max\{\theta, D_{k,b} - \theta_{k,b} + 1\}\.

It follows that \(\frac{\sum_k g^k_{S,d}(j)}{Q_k - g(S, d)} \leq D_{k,b} \leq D_{\max}\).

Since this holds for each \(k\), \(\max\frac{\sum g_{S,d}(j)}{Q - g(S, d)} \leq D_{\max}\) \(\square\)

For the special case of simultaneous evaluation of OR formulas, the theorem implies a \(\beta\)-approximation algorithm, where \(\beta\) is the length of the largest OR formula. This improves the \(2\beta\)-approximation achieved by the randomized algorithm of Liu et al. [30].

We use a similar approach to solve the Linear Function Ranking problem. In this problem, you are given a system of linear functions \(f_1, \ldots, f_m\), where for \(j \in \{1, \ldots, m\}\), \(f_j = a_{ij}x_1 + a_{j2}x_2 + \ldots + a_{jn}x_n\), and the coefficients \(a_{ij}\) are integers. You would like to determine the sorted order of the values \(f_1(x), \ldots, f_m(x)\), for an initially unknown \(x \in \{0,1\}^n\). (Note that the values of the \(f_j(x)\) are not Boolean.) We consider the problem of finding an optimal testing strategy for this problem, where as usual, \(x \sim D_p\), for some probability vector \(p\), and there is a cost vector \(c\) specifying the cost of testing each variable \(x_i\).

Note that there may be more than one correct output for this problem if there are ties. So, strictly speaking, this is not a function evaluation problem. Nevertheless, we can still exploit our previous techniques. For each system of linear equations \(f_1, \ldots, f_m\) over \(x_1, \ldots, x_n\), and each \(x \in \{0,1\}^n\), let \(f(x)\) denote the set of permutations \(\{f_{j_1}, f_{j_2}, \ldots, f_{j_m}\}\) of \(f_1, \ldots, f_m\) such that \(f_{j_1}(x) \leq f_{j_2}(x) \leq \ldots \leq f_{j_m}(x)\). The goal of sorting the \(f_j\) is to output some permutation that we know definitively to be in \(f(x)\). Note that in particular, if e.g., \(f_1(x) < f_2(x)\), it may be enough for us to determine that \(f_i(x) < f_j(x)\).

**Theorem 8.** There is an algorithm that solves the Linear Function Ranking problem that runs in time polynomial in \(m, n, D_{\max}\), and achieves an approximation factor that is within \(O(\log(mD_{\max}))\) of optimal, where \(D_{\max}\) is the maximum value of \(\sum_{i=1}^n |a_{ij}|\) over all the functions \(f_j\).

**Proof.** For each pair of linear equations \(f_i, f_j\) in the system, where \(i < j\), let \(f_{ij}\) denote the linear function \(f_i - f_j\). We construct a utility function \(g^{(ij)}\) with goal value \(Q^{(ij)}\). Intuitively, the goal value of \(g^{(ij)}\) is reached when there is enough information to determine that \(f_{ij}(x) \geq 0\), or when there is enough information to determine that \(f_{ij}(x) < 0\).

The construction of \(g^{(ij)}\) is very similar to the construction of the utility function in our first threshold evaluation algorithm. For each \(i, j\) pair, let \(\min_{ij}(b)\) be the minimum value of \(f_{ij}(b')\) on any assignment \(b' \in \{0,1\}^n\) such that \(b' \sim b\), and let \(\max_{ij}(b)\) be the maximum value. Let \(R_{\max(ij)} = \max_{ij}(\ast, \ldots, \ast)\) and let \(R_{\min(ij)} = \min_{ij}(\ast, \ldots, \ast)\).

Let \(g^{(ij)}_{<}: \{0,1, \ast\}^n \rightarrow \mathbb{Z}_{\geq 0}\), be defined as follows. If \(R_{\max(ij)} < 0\), then \(g^{(ij)}_{<}(b) = 0\) for all \(b \in \{0,1, \ast\}^n\) and \(Q^{(ij)}_{<} = 0\). Otherwise, let \(g^{(ij)}_{<}(b) = \min\{R_{\max(ij)}, R_{\max(ij)} - \max_{ij}(b)\}\) and \(Q^{(ij)}_{<} = R_{\max(ij)}\). It follows that for \(b \in \{0,1, \ast\}^n\), \(f_{ij}(b') \leq f_{j}(b')\) for all extensions \(b' \sim b\) iff \(g^{(ij)}_{<}(b) = Q^{(ij)}_{<}\).

We define \(g^{(ij)}_{>}\) and \(Q^{(ij)}_{<}\) symmetrically, so that \(f_{i}(b') \geq f_{j}(b')\) for all extensions \(b' \sim b\) iff \(g^{(ij)}_{>}(b) = Q^{(ij)}_{>}\).

We apply the disjunctive construction of Lemma 1 to combine \(g^{(ij)}_{>}\) and \(g^{(ij)}_{<}\) and their associated goal values. Let the resulting new utility function be \(g^{(ij)}\) and let its goal value be \(Q^{(ij)}\). As in the analysis of the algorithm in Section 4.2, we can show that \(Q^{(ij)}\) is \(O(D^2)\), where \(D\) is the sum of the magnitudes of the coefficients in \(f_{ij}\).

Using the AND construction of Lemma 1, we get our final utility function \(g = \sum_{i < j} g^{(ij)}\) with goal value \(Q = \sum_{i < j} Q^{(ij)}\).

We now show that achieving the goal utility \(Q\) is equivalent to having enough information to do the ranking. Until the goal value is reached, there is still a pair \(i, j\) such that it remains possible that \(f_i(x) >
\(f_j(x)\) (under one setting of the untested variables), and it remains possible that \(f_j(x) < f_i(x)\) (under another setting). In this situation, we do not have enough information to output a ranking we know to be valid.

Once \(g(b) = Q\), the situation changes. For each \(i, j\) such that \(f_i(x) < f_j(x)\), we know that \(f_i(x) \leq f_j(x)\). Similarly, if \(f_i(x) > f_j(x)\), then at goal utility \(Q\), we know that \(f_i(x) \geq f_j(x)\). If \(f_i(x) = f_j(x)\) at goal utility \(Q\), we may only know that \(f_i(x) \geq f_j(x)\) or that \(f_i(x) \leq f_j(x)\). We build a valid ranking from this knowledge as follows. If there exists an “directed cycle,” i.e. a sequence \(i_1, \ldots, i_m, m \geq 2\), such that we know that \(f_{i_1}(x) \leq f_{i_2}(x) \leq \ldots \leq f_{i_m}(x)\) and \(f_{i_m}(x) \leq f_{i_1}(x)\). It follows that \(f_{i_1}(x) = \ldots = f_{i_m}(x)\). In this case, we can delete \(f_{i_2}, \ldots, f_{i_m}\), recursively rank \(f_{i_1}\) and the remaining \(f_i\), and then insert \(f_{i_2}, \ldots, f_{i_m}\) into the ranking next to \(f_{i_1}\).

Applying Adaptive Greedy to solve the SSSC problem for \(g\), the theorem follows from the \((\ln Q + 1)\) approximation bound for Adaptive Greedy, and the fact that \(Q = O(D_{\max}^2 m^2)\).

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A Table of notation

- $x_i$: the $i$th variable
- $p_i$: probability that variable $x_i$ is 1
- $c_i$: cost of testing $x_i$
- $p$: the probability product vector $(p_1, p_2, \ldots, p_n)$
- $c$: the cost vector $(c_1, c_2, \ldots, c_n)$
- $b$: a partial assignment, an element of $\{0, 1, *\}^n$
- $\text{dom}(b)$: $\{b_i | b_i \neq *\}$, the set of variables of $b$ that have already been tested
- $a \sim b$: $a$ extends $b$ (is identical to $b$ for all variables $i$ such that $b_i \neq *$)
- $D_p$: product distribution, defined by $p_i$ the probability that variable $x_i$ is 1
- $\mathbf{Q}$: the cost vector $(c_1, c_2, \ldots, c_n)$
- $b$: a partial assignment, an element of $\{0, 1, *\}^n$
- $\text{dom}(b)$: $\{b_i | b_i \neq *\}$, the set of variables of $b$ that have already been tested
- $\mathbf{Q}$: goal utility
- $P$: maximum utility that testing a single variable $x_i$ can contribute
- $g$: utility function defined on partial assignments with a value in $\{0, \ldots, Q\}$
- $N$: the set $\{1, \ldots, n\}$
- $S$: a subset of $N$
- $g(S, b)$: utility of testing only the items in $S$, with outcomes specified by $b$
- $g_{\mathbf{S}, b}(j)$: $g(S \cup \{j\}, b) - g(S, b)$
- $b_{x_i \leftarrow l}$: $b$ extended by testing variable $i$ with outcome $l$
- $k$: number of clauses in a CNF
- $d$: number of terms in a DNF
- $m$: the number of linear threshold formulas in the simultaneous evaluation problem
- $\min(b)$: the minimum possible value of the linear threshold function for any extension of $b$
- $\max(b)$: symmetric to $\min(b)$, but maximum
- $R_{\min}$: $\min(\ast, \ldots, \ast)$
- $R_{\max}$: $\max(\ast, \ldots, \ast)$
- $W$: the set of partial assignments that contain exactly one $*$
- $w^{(0)}$, $w^{(1)}$: for $w \in W$, the extensions obtained from $w$ by setting the $*$ to 0 and 1, respectively
- $j(w)$: for $w \in W$, the $j$ for which $w_j = *$
- $a^j$: the partial assignment produced from $a$ by setting the $j$th bit to $*$ for assignment $a$
- $a'$: the assignment produced from $a$ by complementing the $j$th bit
- $g_a(S)$: $g(S, a)$
- $y_{S,a}$: the variable in LP2 for SSC associated with subset $S$ and assignment $a$
- $C(a)$: the sequence of items tested by ADG on assignment $a$, in order of testing
- $Y_{S,a}$: the value of ADG variable $y_S$ after running ADG on input $a$
- $h_{\mathbf{w}}(y)$: the left hand side of the constraint in LP2 for $w$ (a function of the $y_{S,a}$ variables)
- $Y^t$: assignment to the $y_{S,a}$ variables s.t. $y_{S,a}$ is the value of ADG variable $y_S$ at the end of iteration $t$ of its while loop, when ADG is run on input $a$
- $T^x$: the number of iterations of the Adaptive Greedy (AG) while loop on input $x$
- $b^t_x$: the value of $b$ on input $x$ after the $t$th iteration of the loop of AG on $x$
- $Y^x$: the assignment to the LP2 variables used in the analysis of the new bound for AG
- $q^r(y)$: $\sum_{S \subseteq N} g_{S,x}(N - S) y_S^r$
- $F^t$: variable of ADG, the set containing the first $t$ variables it tests
- $g(j)$: equals $\max_{t \in \{0, 1\}} g_r(j, l)$ where $r = (\ast, \ldots, \ast)$, in analysis of Adaptive Greedy