HEEGAARD FLOER HOMOLOGY OF MATSUMOTO’S MANIFOLDS

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Abstract. We call a homology sphere presented by two trefoils with the linking number 1 and the framing \((0, n)\) Matsumoto’s manifold. By computing the Heegaard Floer homology of Matsumoto’s manifolds, we give an constraint for contractible bounding of the manifolds. As a corollary, we give a formula of Ozsváth-Szabó’s \(\tau\)-invariant as a total Euler number of the reduced filtration. By using Owens and Strle’s obstruction, we prove that if the \(n\)-twisted Whitehead double of \((2, 2s + 1)\)-torus knot is slice, then \((s, n) = (1, 6), (3, 12)\) only.

1. Introduction and computational results.

Let \(K_1, K_2\) be two knots. We define to be \(M_n(K_1, K_2)\) a 3-manifold presented by \(K_1, K_2\) with the linking number 1 and the framings 0 and \(n\) respectively. See Figure 1. \(M_n(K_1, K_2)\) is a homology sphere. Y. Matsumoto asked in [6] whether \(M_n(T_{2,3}, T_{2,3})\) bounds a contractible 4-manifold or not, where \(T_{r,s}\) is the positive \((r,s)\)-torus knot. Following Y. Matsumoto, we call \(M_n(K_1, K_2)\) Matsumoto’s manifold in this paper. If \(K_1 = T_{2,3}\), then we have

\[
M_n(T_{2,3}, K_2) = S^3_1(D_+(K_2), n),
\]

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where $D_+(K, n)$ means the $n$-twisted positive-Whitehead double of $K$. $S^3_p(K)$ is $p$-surgery of $K$ in $S^3$. If $K_2 = F$ (figure-8 knot), then $M_n(F, K_2) = S^3_{-1}(D_+(K_2, n))$. Thus,

$$K_2 \text{ is slice } \Rightarrow M_n(T_{2,3}, K_2) \text{ and } M_n(F, K_2) \text{ have contractible bounds.}$$

We argue the existence of contractible bounds of $M_n(T_{2,3}, K_2)$ in the present paper. We first review some well-known facts about these manifolds.

If $K_1$ is a slice knot, then $M_n(K_1, K_2)$ bounds a contractible 4-manifold, because $M_n(K_1, K_2)$ is the boundary of 2-handle attachment for the complement of a slice disk of $K$, thus the attachment is $\pi_1 = e$ and $H_2 = H_3 = 0$, hence it is a contractible 4-manifold. Conversely, if $K_1$ is not slice, the contractible boundness is unclear.

If a homology sphere is presented by the $\pm 1$-surgery of a slice knot, then the homology sphere bounds a contractible 4-manifold. Let $F$ be the figure-8 knot. $M_n(T_{2,3}, K_2)$ (or $M_n(F, K_2)$) is the $+1$-surgery (or $-1$-surgery respectively) of $D_+(K_2, n)$, where $D_+(K_2, n)$ is the $n$-twisted Whitehead double of $K_2$. Hence, the contractible boundness problem is related to the slice-ness of $D_+(K_2, n)$. The Alexander polynomial is

$$\Delta_{D_+(K_2, n)}(t) = -nt + 2n + 1 - nt^{-1}.$$ 

Hence, by the classical restriction in [7] of the slice-ness in terms of Alexander polynomial says that what the polynomial is of form $f(t)f(t^{-1})$ is a necessary condition for a knot to be slice. In this case, $-nt + 2n + 1 - nt^{-1}$ is of form $f(t)f(t^{-1})$ if and only if $n = m(m + 1)$ holds for some integer $m$.

We recall from well-known facts to recent results:

**Theorem 1.1** (1976, [15]). $D_+(T_{2,3}, 6)$ is a slice knot.

**Theorem 1.2** (1984, [9]). $M_6(T_{2,3}, T_{2,3})$ bounds a contractible 4-manifold.

**Theorem 1.3** (1997, [1]). $M_0(T_{2,3}, T_{2,3})$ does not bound any contractible 4-manifold.

**Theorem 1.4** (2006, Bar-Natan’s program [2]). $D_+(T_{2,3}, 2)$ is not slice.
Theorem 1.5 (2007, [3]). Let $K$ be a knot in $S^3$.

$$\tau(D_+(K,n)) = \begin{cases} 0 & n \geq 2\tau(K) \\ 1 & n < 2\tau(K) \end{cases}$$

In particular, if $n < 2\tau(K)$, then $D_+(K,n)$ is not slice. Here $\tau(K)$ is Ozsváth-Szabó’s $\tau$-invariant.

Theorem 1.2 is by slice-ness of $D_+(T_2,3,6)$. Theorem 1.3 was proven by using gauge theory. Theorem 1.4 is by computation of Bar-Natan’s program for computing the reduced Khovanov homology. The paper [3] computes knot Floer homology of $D_+(K,n)$ and the $\tau$-invariant, which gives a homomorphism from the smooth knot concordance group to integers, i.e., $\tau : C_s \to \mathbb{Z}$.

Theorem 1.6 (2013, [17]). Let $n$ be an odd integer. $M_n(T_2,3)$ does not bound any contractible 4-manifold.

Tsuchiya showed the result by computing the Rohlin invariant $\mu$.

We can easily compute the Casson invariant by using the Dehn surgery formula as follows:

$$\lambda(M_n(T_2,3,K)) = \frac{1}{2} \cdot \Delta_{D_+(K,n)}(t)|_{t=1} = -n.$$ 

One of the main purposes of the present paper is to compute of $HF^+$ of $S^3_1(D_+(T_2,3,n))$ and discuss the contractible boundness of $M_n(K_1,K_2)$.

Theorem 1.7. Let $M_n$ be $M_n(T_2,3)$. Then the Heegaard Floer homology of $M_n$ is computed as follows:

$$HF^+(M_n) = \begin{cases} T^+_n \oplus HF_{\text{red}}(M_n) & n \geq 2 \\ T^+_{(-2)} \oplus HF_{\text{red}}(M_n) & n < 2, \end{cases}$$

and further

$$HF_{\text{red}}(M_n) \cong \begin{cases} \mathbb{F}^{n-2}_{(-1)} \oplus \mathbb{F}^2_{(-3)} & n \geq 2 \\ \mathbb{F}^1_{(-2)} \oplus \mathbb{F}^2_{(-3)} & n < 2. \end{cases}$$

This computation is generalized to the case of any knot $K$ as well as $T_2,3$. See Theorem 4.1. On the other hands, one can also compute the Heegaard Floer homology of $M_n(F,K) = S^3_1(D_+(K,n))$. The result is similar to $M_n(T_2,3,K)$ and the values of $d(M_n(F,K))$ is the same as ones of $M_n(T_2,3,K)$.

From Theorem 1.7, the following holds naturally:

Corollary 1.1. When $n < 2$, $M_n$ does not bound any negative-definite 4-manifold.

In particular, if $n < 2$, then $M_n$ does not bound any contractible 4-manifold and $D_+(T_2,3,n)$ is not slice.

Proof. Since in the case of $n < 2$, the correction term $d(M_n)$ is negative integer. $M_n = S^3_1(D_+(T_2,3,n))$ does not bound any negative-definite 4-manifold, because the inequality $c_2^2(s) + b_2(X^4) \leq 4d(Y^3)$ by Ozsváth-Szabó
in [11] does not hold, where $X^4$ is any negative definite bound of homology sphere $Y^3$ and $s$ is any Spin$^c$ structure on $X$.

Hence, clearly, $M_n$ does not have contractible bound. If $D_+(T_{2,3}, n)$ is slice, $S^3_1(D_+(T_{2,3}, n))$ has a contractible bound. This is inconsistent. □

The remaining problem is the following.

**Question 1.1.** Let $n$ be a positive even number and $n \neq 6$. Does Matsumoto’s manifold $M_n$ bound a contractible 4-manifold?

Next, we give a relationship the slice-ness of $K$ and the double branched cover. $\Sigma_2(K)$ is the branched double cover along a knot $K$.

$K$ is slice $\Rightarrow \Sigma_2(K)$ bounds a rational 4-ball

We compute the $\delta$-invariant of Whitehead double of knots.

**Theorem 1.8.** Let $n$ be a non-negative integer and $s$ a positive integer. Then we have

$$
\delta(D_+(T_{2,2s+1}, n)) = \begin{cases} 
0 & n \geq 2s \\
-4(s + \lfloor \frac{n+1}{2} \rfloor) & 0 \leq n < 2s.
\end{cases}
$$

We define $t_s, t_\tau, t_\delta$ as follows:

$$
t_s(K) = \min \{ t \in \mathbb{Z} | s(D_+(K, t)) = 0 \}
$$

$$
t_\tau(K) = \min \{ t \in \mathbb{Z} | \tau(D_+(K, t)) = 0 \}
$$

$$
t_\delta(K) = \min \{ t \in \mathbb{Z} | \delta(D_+(K, t)) = 0 \}
$$

$$
t_{dS^1}(K) = \min \{ t \in \mathbb{Z} | d(S^3_1(D_+(K, t))) = 0 \}.
$$

Hedden in [3] showed that $t_\tau(K) = 2\tau(K)$ and our result says $t_{dS^1}(K) = 2\tau(K)$. Hedden and Ording conjectured $t_{\delta}(T_{2,2n+1}) = 3n - 1$ in [4]. Theorem 1.8 implies that $t_\delta(T_{2,2n+1}) = 2n = t_\tau(T_{2,2n+1})$.

**Question 1.2.** When does

$$
\delta(D_+(K, n)) = -4\tau(D_+(K, m))
$$

hold.

This equality is verified in the case of $K = T_{2,3}$ in this equality.

We prove rational bound-ness of double branched cover. This is a kind of generalization of $\delta$-invariant.

**Theorem 1.9.** If $\Sigma_2(D_+(T_{2,2s+1}, n))$ bounds a rational 4-ball, then $n = m(m + 1)$ and $(s, m) = (1, 2), (3, 3)$ holds.

As a corollary, we have the following.

**Corollary 1.2.** Let $n$ be a non-negative integer. If $D_+(T_{2,2s+1}, n)$ is slice, then $(s, n) = (1, 6)$ or $(3, 12)$.

The case of $(s, n) = (1, 6)$ is an actual slice knot due to an example in Rolfsen’s book [15]. See also [9]. The result by Collins [5] denies the slice-ness of $D_+(T_{2,7}, 12)$. However, it is remained whether $\Sigma_2(D_+(T_{2,7}, 12))$ bounds a rational 4-ball.
Question 1.3. Does $\Sigma_2(D_+(T_{2,7}, 12))$ bound a rational ball?

Collins’ obstruction also denies the slice-ness of $D_+(T_{p,q}, m(m+1))$ unless $(p,q) = (m, m+1)$ or $(m+1, m)$. In the case of $D_+(T_{n,n+1}, n(n+1))$, what values do $\delta(D_+(T_{n,n+1}, n))$ become? This computation is easy and may give some interesting observation.

In the end of the paper, we give a formula of $\tau(K)$ as a total Euler number of reduced filtration of $K$. This (Corollary 4.1) might be already known.

$$\tau(K) = \sum_{i=-g}^{g} \chi(\bar{F}(K, i)).$$

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2. Heegaard Floer homology of $M_n$.

Proof of Theorem 1.7. Ozsváth-Szabó’s $\tau$-invariant of $T_{2,3}$ is 1. From the equality $M_n = S^3_1(D_+(T_{2,3}, n))$ and Proposition 7.2 in \cite{3},

$$\hat{HF}(M_n) \cong \begin{cases} \mathbb{F}^{n-4} \oplus \mathbb{F}^{n-3} \oplus V & n \geq 2 \\ \mathbb{F}^{1-n} \oplus \mathbb{F}^{2-n} \oplus \mathbb{F}^{-2} \oplus V & n < 2 \end{cases},$$

where the negative exponents mean the quotient operation in place of the direct sum operation. The summand $V$ is isomorphic to

$$V = \oplus_{i=-1}^1 [H_*(\mathcal{F}(T_{2,3}, i))]^2 \oplus \oplus_{i=-1}^1 [H_{*+1}(\mathcal{F}(T_{2,3}, i))]^2.$$

The chain complex $CFK^\infty(T_{2,3})$ is as follows. Therefore, we have

![Figure 3. $CFK^\infty(T_{2,3})$.](image-url)
The connecting homomorphism \( \delta \) due to the result in [3]. The generator \( C \) is Ozsváth-Szabó's usual action lowering the degree by two. The boundary map vanishes by the map \( \delta \). Therefore, we have

\[
\widetilde{HF}(M_n) = \begin{cases} 
\mathbb{F}^{n-1} \oplus \mathbb{F}^{-1} \oplus \mathbb{F}^{-3} \oplus \mathbb{F}^{-5} & n \geq 2 \\
\mathbb{F}^{-1} \oplus \mathbb{F}^{-1} \oplus \mathbb{F}^{-3} \oplus \mathbb{F}^{-5} & n < 2
\end{cases}
\]

Here we use exact triangle

\[
\cdots \to \widetilde{HF}(M_n) \xrightarrow{\delta} HF^+(M_n) \xrightarrow{\iota} HF^+(M_n) \xrightarrow{\delta} \widetilde{HF}(M_n) \to \cdots.
\]

The map \( \delta \) is the one induced by the natural injection. The multiplication of \( U \) is Ozsváth-Szabó's usual action lowering the degree by two. The connecting homomorphism \( \delta \) shifts the degree by 1. We compute the correction term \( d(M_n) \).

\[
HF^+(S^2(D_+(T_{2,3}, n))) \cong H_*(C\{\max(i, j) \geq 0\}),
\]

where the chain complex \( C \) is \( CFK^\infty(D_+(T_{2,3}, n)) \).

**The case of** \( n \geq 2 \). \( C\{i = 0\} \) is filtered chain homotopic to

\[
C\{(0, j)\} \simeq \begin{cases} 
\mathbb{F}^{n-1} \oplus \mathbb{F}^{-1} H_{s-1}(\mathcal{F}(T_{2,3}, i))^2 & j = 1 \\
\mathbb{F}^{n-3} \oplus \mathbb{F}^{-3} H_{s}(\mathcal{F}(T_{2,3}, i))^4 & j = 0 \\
\mathbb{F}^{n-3} \oplus \mathbb{F}^{-3} H_{s+1}(\mathcal{F}(T_{2,3}, i))^2 & j = -1
\end{cases}
\]

due to the result in [3]. The component in \( \widetilde{HF}(M_n) \) attaining the minimal degree in the non-torsion part in \( HF^+(M_n) \) is located at \( (i, j) = (0, 0) \), that is, \( x \in \mathbb{F}(0) \subset C\{(0, 1)\} \). This generator \( x \) vanishes by the boundary map \( d_1^0 : C\{(0, 0)\} \to C\{(0, -1)\} \), because it is the generator of \( \widetilde{HF}(S^3) \). It also vanishes by the map \( d_1^0 : C\{(0, 0)\} \to C\{(-1, 0)\} \). Hence \( x \) is a generator in \( HF^+(M_n) \) and it is clearly \( U \cdot x = 0 \) and \( gr(x) = 0 \). This means \( d(M_n) = 0 \).

**The case of** \( n < 2 \). \( C\{i = 0\} \) is filtered chain homotopic to

\[
C\{(0, j)\} \simeq \begin{cases} 
\mathbb{F}^{2-n} \oplus \mathbb{F}^{1} H_{s-1}(\mathcal{F}(T_{2,3}, i))^2 & j = 1 \\
\mathbb{F}^{3-2n} \oplus \mathbb{F}^{1} H_{s}(\mathcal{F}(T_{2,3}, i))^4 & j = 0 \\
\mathbb{F}^{3-2n} \oplus \mathbb{F}^{1} H_{s+1}(\mathcal{F}(T_{2,3}, i))^2 & j = -1
\end{cases}
\]

due to the result in [3]. The generator \( x \) in \( \widetilde{HF}(S^3) \) lies in \( C\{(0, 1)\} \). That is, \( x \in \mathbb{F}(0) \subset C\{(0, 1)\} \). The boundary map

\[
d_1^0 : C\{(0, 0)\} \to C\{(-1, 0)\}
\]
is surjective due to [3]. The $U$-action $C\{(-1, 0)\} \ni U \cdot x \neq 0$, namely, $U \cdot x$ is the minimal generator in $HF^+(M_n)$. Thus $d(M_n) = gr(U \cdot x) = -2$.

We put $HF^+(M_n) \cong \bigoplus_{i=1}^m W_i$, where $W_i = \mathbb{F}[n_i](d_i) \cong \mathbb{F}[U]/U^{n_i}$. Then we have

$$\widehat{HF}(M_n) = \mathbb{F}(d(M_n)) \oplus_{i=1}^m (\mathbb{F}(d_i) \oplus \mathbb{F}(d_i+2n_i-1)).$$

The component number is

$$m = \begin{cases} n & n \geq 2 \\ 3 - n & n < 2. \end{cases}$$

If some $i$ has $n_i > 1$, then there exists a pair of two components with the degree width $2n_i - 1 \geq 3$. The pair is just the case $n_i = 2$ and the only pair is $\mathbb{F}(0)$ and $\mathbb{F}(-3)$ in the case of $n \geq 2$, due to (2).

**Lemma 2.1.** There does not exist such a pair.

**Proof.** From (2), we may consider the case $n \geq 2$. Suppose that there exist such two pairs in $\widehat{HF}(M_n)$. Then the components $\mathbb{F}[2]_{(-3)}$ and remaining part is

$$\mathcal{T}_{(0)}^+ \oplus \mathbb{F}^{n-4}_{(-1)} \oplus \mathbb{F}^2_{(-2)}.$$ 

The Casson invariant formula implies $\lambda(M_n) = -4 - (n - 4) + 2 = -n + 2$. This is contradiction about (1).

Suppose that there exists such single pair in $\widehat{HF}(M_n)$. Then the components $\mathbb{F}[2]_{(-3)}$ and remaining part is

$$\mathcal{T}_{(0)}^+ \oplus \mathbb{F}^{n-3}_{(-1)} \oplus \mathbb{F}(-2) \oplus \mathbb{F}(-3).$$

The Casson invariant is $\lambda = -2 - (n - 3) + 1 - 1 = -n + 1$. This is contradiction about (1). □

In the case where there does not exist such pair, namely $n_i = 1$ for any $i$, immediately the following holds:

**Lemma 2.2.** Suppose that there does not exist any pair in $\widehat{HF}(M_n)$. Then in the case of $n \geq 2$, we have

$$HF^+(M_n) = \mathcal{T}_{(0)}^+ \oplus \mathbb{F}^{n-2}_{(-1)} \oplus \mathbb{F}^2_{(-3)}$$

and in the case of $n < 2$, we have

$$HF^+(M_n) = \mathcal{T}_{(-2)}^+ \oplus \mathbb{F}^{1-n}_{(-2)} \oplus \mathbb{F}^2_{(-3)}.$$

□

In the case of $n = 6$, we can also check our computation (Theorem 1.7) by Némethi’s algorithm ([10]) on any plumbed 3-manifold with at most one bad vertex. In fact we can construct the negative definite bound as in Figure 4 for $M_6$. The multiplicity $-1$ vertex is the only bad vertex. Then $HF^+(-M_6)$ can be computed as follows:

$$HF^+(-M_6) = \mathcal{T}_{(0)}^+ \oplus \mathbb{F}^4_{(0)} \oplus \mathbb{F}^2_{(2)}.$$
By reversing the orientation, we get
\[ HF^+(M_6) = T^+_{(0)} \oplus F^4_{(-1)} \oplus F^2_{(-3)}. \]

![Figure 4. The negative definite plumbing of $M_6$.](image)

3. The rational bounding of $\Sigma_2(D_+(T_{2,2s+1}, n))$.

**Question 3.1.** When $n \geq 2$, does $M_n(T_{2,3})$ have a contractible bounding?

At the most cases, it is difficult to find the criterion of whether $K$ is slice or not. Moreover, it is more difficult to find a contractible bounding for a given homology sphere. We consider the slice-ness of $D_+(T_{2,3}, n)$.

**Theorem 3.1.** Let $K = D_+(T_{2,2s+1}, n)$. If $K$ is slice then $(s, n) = (1, 6), (3, 12)$ only. Otherwise, the double branched cover of $K$ does not even have any rational 4-ball bounds.

J. Collins [5] proved similar result in terms of twisted Alexander polynomial and $\omega$-signature. We reprove this fact in terms of Heegaard Floer homology.

First, we compute the $\delta$-invariant (smooth knot concordance invariant) by Manolescu-Owens.

**Definition 3.1 (8).** The smooth knot concordance invariant $\delta : C_8 \to \mathbb{Z}$ is defined to be
\[ \delta(K) = 2d(\Sigma_2(K), c_0), \]
where $\Sigma_2(K)$ is the double branched cover and $c_0$ is the canonical spin structure on $\Sigma_2(K)$.

They proved the following for the untwisted Whitehead double of any knot or alternating knot.

**Theorem 3.2 (8).** For any knot $K$ we have $\delta(D_+(K, 0)) \leq 0$ and inequality is strict, if $\tau(K) > 0$. If $K$ is alternating, then $\delta(D_+(K, 0)) = -4\max\{\tau(K), 0\}$.

Hence, the $\delta$-invariant of the untwisted Whitehead double of $T_{2,2s+1}$ is as follows:
\[ \delta(D_+(T_{2,2s+1}, 0)) = -4s. \]

To prove Theorem 1.8 we use the following:
Proposition 3.1. For integer \( n \) we have
\[
\Sigma_2(D_+(K,n)) = S^3_{2n+1}(K#K^r)
\]
where \( K^r \) is the knot \( K \) with the reverse orientation.

In particular if \( K \) is slice, then \( n = m(m+1) \) holds.

We notice that the condition \( n = m(m+1) \) is also a necessary condition for \( S^3_{2n+1}(K#K^r) \) to bound a rational 4-ball.

Proof. The former assertion is folklore by using the Montesinos trick. By using the Fox-Milnor condition of the Alexander polynomial of \( D_+(K,n) \), it immediately follows.

Before proving Theorem 3.1 and Theorem 1.8, we prepare the rational Dehn surgery formula of the correction terms. Here we give a brief review of the invariants \( V_k, H_k \) in [18]. The maps \( v_k, h_k : A_k^+ \to B^+ \) are defined to be the natural projection
\[ v_k^+ : A_k \to B^+ \]
and the composition of the projection and the identification
\[ h_k : A_k^+ \to C_s \{ j \geq k \} \to B^+ \]
where \( \mathfrak{A}_{k}^T \subset H_{s}(A_k^+) \) is the sub-\( \mathbb{F}[U] \)-module isomorphic to \( U^nH_{s}(A_k) \) for \( n \gg 0 \).

The homomorphisms \( v_k^T, h_k^T \) are the restriction maps on \( \mathfrak{A}_{k}^T \)
\[ v_k^T, h_k^T : \mathfrak{A}^T \cong T^+ \to H_{s}(B^+) \cong T^+ \]
The maps are equivalent to the multiplication by \( U^m \) for some \( m \geq 0 \). We define the exponent \( m \) to be \( V_k \) or \( H_k \) respectively.

The correction term formula by Ni and Wu in [18] is the following:
\[
d(S_{p/q}^3(K), i) = d(L(p, q), i) - 2 \max \{ V_{\lfloor \frac{q}{q} \rfloor}, H_{\lfloor \frac{q}{q} \rfloor} \}.
\]

When \( q \) is an even integer, the canonical Spin\(^c\) structure of \( S_{p/q}^3(K) \) has \( i_0 = \frac{p+q-1}{2} \) (see [19]). Then, since \( V_{\lfloor \frac{q}{q} \rfloor} = H_{\lfloor \frac{q}{q} \rfloor} \), we have
\[
d(S_{p/q}^3(K), i_0) = d(L(p, q), i_0) - 2V_{\lfloor \frac{p+q-1}{q} \rfloor}.
\]

Proof of Theorem 1.8. Let \( T_s \) denote \( T_{2,2s+1} \# T_{2,2s+1}^r \). From Proposition 3.1, we consider the \( d \)-invariant of \( S_{2n+1}^3(T_s) \). Tensoring two copies of the double complex of Figure 3 we get the knot Floer chain complex as in Figure 3. The module \( C_s := C_s[U, U^{-1}] \) is the knot Floer chain complex \( CFK^\infty(T_s) \).

Here we define \( A_k^+ \) and \( B^+ \) as follows:
\[
C_s \{ i \geq 0 \text{ or } j \geq k \} =: A_{s,k}^+

C_s \{ i \geq 0 \} =: B^+
\]

From the chain complex \( C_s \), the invariants \( V_{s,k}, H_{s,k} \) are computed as in the table below.

|   | \( k \) | \( 0 \) | \( 1 \) | \( 2 \) | \( 3 \) | \( \cdots \) | \( 2s-2 \) | \( 2s-1 \) | \( k \geq 2s \) |
|---|---|---|---|---|---|---|---|---|---|
| \( V_{s,k} \) | \( s \) | \( s \) | \( s-1 \) | \( s-1 \) | \( 1 \) | \( 1 \) | \( 0 \) |   |   |
| \( H_{s,k} \) | \( s+1 \) | \( s+1 \) | \( s+2 \) | \( \cdots \) | \( 2s-1 \) | \( 2s \) | \( k \) |   |   |
Figure 5. The chain complex $\bar{C}_s$.

| $k$ | $k \leq -2s - 1$ | $-2s$ | $-2s + 1$ | $\cdots$ | $-1$ | $0$ |
|-----|-----------------|-------|-----------|--------|------|-----|
| $V_{s,k}$ | $|k|$ | $2s$ | $2s$ | $\cdots$ | $s + 1$ | $s$ |
| $H_{s,k}$ | $0$ | $0$ | $1$ | $\cdots$ | $s - 1$ | $s$ |

By using the correction term formula of lens spaces in [16], we have

$$d(L(2r + 1, 2), j) = \begin{cases} \frac{(2k-r-2)^2}{2(2r+1)} j = 2k - 1 \\ \frac{4k^2 - 4kr - 4k + r^2}{2(2r+1)} j = 2k. \end{cases}$$

When $r = 2n$, we have $d(L(4n + 1, 2), i_0) = 0$. We have

$$\delta(D_+(T_{2,2s+1}, n)) = 2d(S^3_{2n+1}(T_s), i_0) = 2d(L(4n + 1, 2), i_0) - 4V_{s,n}$$

$$= -4V_{s,n} = \begin{cases} 0 & n \geq 2s \\ -4(s + \lfloor \frac{n+1}{2} \rfloor) & 0 \leq n < 2s. \end{cases}$$

As a corollary we give a sufficient condition to satisfy $\delta(D_+(K, n)) = 0$ for a knot with non-negative $\tau(K)$.

**Corollary 3.1.** Let $K$ be a knot in $S^3$ with $\tau(K) \geq 0$. If $n \geq 2\tau(K)$, then $\delta(D_+(K, n)) = 0$.

**Proof.** We claim that if $k = 2\tau(K) = \tau(K\#K^r)$, then we have $V_k = 0$. Then, by the decreasing property $V_k \geq V_{k+1} \geq 0$, the assertion required holds.

Let $C$ denote $CFK^\infty(K\#K^r)$ and $k$ denote $2\tau(K)$. There exist a generator $x \in C\{(i, j) = (0, k)\}$ and some element $\alpha \in C\{\max\{i, j-k\} \geq 0\}$ such that a non-zero class $[x + \alpha] \in H_*(A_k)$, and its image by $v_k^+: H_*(A_k^+) \to H_*(B^+) = T^+$ is the bottom generator. Thus this means that $[x + \alpha] \neq 0$. Clearly, $U \cdot [x + \alpha] \neq 0$, $[x + \alpha]$ is the bottom generator in $A_k^+$. This means $V_k = 0$. 

□
Therefore, for any $n \geq 2\tau(K)$, we have
\[
\delta(D_+(K,n)) = 2d(S^3_{2n+1}(K\#K^4),i_0) = 2d(L(4n+1,2),i_0) - 4V_n = 0 - 0 = 0.
\]

Figure 6. The generator $x$ and some element $\alpha$ in $A_{2\tau(K)}$.

The behavior of $V_n$ when $0 \leq n \leq 2\tau(K)$ deeply depends on the filtered chain complex with respect to $K$.

To show Theorem 3.1, we use the deeper obstruction by Owens and Strle. If $K$ is a slice knot, then the double branched cover $\Sigma_2(K)$ must bound a rational 4-ball.

**Proposition 3.2 ([14]).** Let $Y$ be a rational homology sphere bounding a rational ball $X$. If the order of $H^2(Y)$ is $h = t^2$, then
\[
d(Y, t_0 + \beta) = 0
\]
for any $\beta \in \mathcal{T} \subset H^2(Y)$, where $t_0$ is a Spin$^c$ structure and $|\mathcal{T}| = t$.

Here we discuss whether $\Sigma_2(D_+(T_{2,2s+1}, n))$ bounds rational 4-ball or not. From Proposition 3.1 we prove Theorem 1.9 the half-integer surgery of $T_{2,2s+1}#T_{2,2s+1}$.

**Proof of Theorem 1.9.** Suppose that $X_{s,m} := \Sigma_2(D_+(T_{2,2s+1}, m(m + 1)))$ bounds a rational ball.

Since the canonical Spin$^c$ structure corresponds to $i_0 = 2m(m + 1) + 1$, and Owens and Strle’s subset $t_0 + \mathcal{T}$ is $\{i_0 + \ell(2m + 1) | 0 \leq |\ell| \leq m\}$. By using the formula (3), we have
\[
d(L((2m + 1)^2,2), i_0 + \ell(2m + 1)) = \begin{cases} 2\ell_1(\ell_1 - 1) & \ell = 2\ell_1 - 1 \\ 2\ell_1^2 & \ell = 2\ell_1 \end{cases}
\]

\[
V_{s,|\frac{i_0+\ell(2m+1)}{2}|} = V_{s,m(m+1)+m\ell+\left\lfloor \frac{\ell^2}{2} \right\rfloor} \\
H_{s,|\frac{i_0+\ell(2m+1)-(2m+1)^2}{2}|} = V_{s,m(m+1)-m\ell-\left\lfloor \frac{\ell^2}{2} \right\rfloor} \\
\max\{V_{s,|\frac{i_0+\ell(2m+1)}{2}|}, H_{s,|\frac{i_0+\ell(2m+1)-(2m+1)^2}{2}|}\} = V_{s,m(m+1)-m\ell-\left\lfloor \frac{\ell^2}{2} \right\rfloor}
\]
Thus,
\[
d(X_{s,m}, i_0 + \ell(2m + 1)) = \begin{cases} 2\ell_1(\ell_1 - 1) - 2V_{s,m(m+1-\ell)} - \ell_1 + 1 & \ell = 2\ell_1 - 1 \\ 2\ell_1^2 - 2V_{s,m(m+1-\ell)} & \ell = 2\ell_1 \end{cases}
\]

When \( \ell = 2 \leq m, V_{s,m^2-m-1} = 1 \) holds. Since \( m^2-m-1 \) is odd, \( m^2-m-1 = 2s-1 \), equivalently \( m^2 - m = 2s \). If \( m = 2 \), then \( s = 1 \) holds. When \( m \geq 3 \) and \( \ell = 3 \leq m, V_{s,m^2-2m-1} = 2 \), hence \( m^2 - 2m - 1 = 2s - 3 \), or \( 2s - 4 \) holds. Hence \( (m, s) = (3, 3) \) holds.

Hence, we obtain \( (m, s) = (2, 1), (3, 3) \).

Therefore Theorem 3.1 follows immediately from Theorem 3.1.

4. Generalization and reduced knot filtration.

Let \( M_n(K) \) denote \( M_n(T_{2,3}, K) \). We define \( \tilde{CF}(K) = \bigcup_i \mathcal{F}(K, i) \), i.e., it is chain homotopy equivalent to \( \tilde{CF}(K) \simeq \tilde{CF}(S^3) \). Here we define to be \( \epsilon \) the composition
\[
\epsilon_i : \mathcal{F}(K, i) \hookrightarrow \tilde{CF}(K) \to \mathbb{F}(0)
\]
for any \( i \), where the last map is what the homological generator map to 1 and other elements to 0. Furthermore, the map \( \tilde{CF}(K) \to \mathbb{F}(0) \) is splittable. We put the kernel of \( \varphi \)
\[
\tilde{F}(K, i) := \ker(\epsilon_i).
\]

Then \( \tilde{F}(K, i) \) is a filter on \( \tilde{CF}(K) := \bigcup_i \tilde{F}(K, i) \). The chain complex \( \tilde{CF}(K) \) is acyclic, because \( \tilde{CF}(K) \to \mathbb{F}(0) \) induces an isomorphism on the homology.
We call \( \tilde{F}(K, i) \) reduced knot filtration.

**Theorem 4.1.** The Heegaard Floer homology of \( M_n(K) \) is computed as follows:
\[
HF^+(M_n(K)) \cong \begin{cases} \mathcal{T}^+_{(0)} \oplus HF_{\text{red}}(M_n(K)) & n \geq 2\tau(K) \\ \mathcal{T}^+_{(-2)} \oplus HF_{\text{red}}(M_n(K)) & n \geq 2\tau(K) \end{cases}
\]

and further,
\[
HF_{\text{red}}(M_n(K)) \cong \begin{cases} \mathbb{F}^{n-2\tau(K)}_{(-1)} \oplus_{i=-g} H_{s+1}(\tilde{F}(K, i))^2 & n \geq 2\tau(K) \\ \mathbb{F}^{2\tau(K)-n-1}_{(-2)} \oplus_{i=-g} H_{s+1}(\tilde{F}(K, i))^2 & n < 2\tau(K) \end{cases}
\]

**Proof.** In the same way that \( K \) is the right-handed trefoil,
\[
d(M_n(K)) = \begin{cases} 0 & n \geq 2\tau(K) \\ -2 & n < 2\tau(K) \end{cases}
\]

\[
HF_{\text{red}}(M_n(K)) \cong \begin{cases} \mathbb{F}^{n-2g-2}_{(-1)} \oplus_{i=-g} H_{s+1}(\mathcal{F}(K, i))^{(1)} & n \geq 2\tau(K) \\ \mathbb{F}^{2\tau(K)-n-1}_{(-2)} \oplus \mathbb{F}^{2\tau(K)-2g-2}_{(-1)} \oplus_{i=-g} H_{s+1}(\mathcal{F}(K, i))^{(2)} & n < 2\tau(K) \end{cases}
\]
For each \( i \) with \( i \geq \tau(K) \), \( H_{*+1}(\mathcal{F}(K, i)) \) contains at least one summand \( F_{\tau(K)}(K, i) \). Thus, we have the following:

\[
\mathbb{F}^{-2g-2}_{\tau(K)} \cong \bigoplus_{i=-g}^{g} H_{*+1}(\mathcal{F}(K, i))^2 \cong \bigoplus_{i=-g}^{g} \left[H_{*+1}(\mathcal{F}(K, i))/\mathbb{F}(-1)\right]^2 \\
\cong \bigoplus_{i=-g}^{g} \left[H_{*+1}(\mathcal{F}(K, i))/\mathbb{F}(-1)\right]^2
\]

As a corollary we have:

**Corollary 4.1.** The total Euler number of reduced knot filtration is \( \tau(K) \).

\[
\sum_{i=-g}^{g} \chi(\mathcal{F}(K, i)) = \tau(K).
\]

**Proof.** If \( n \geq 2\tau(K) \), then we have

\[
\lambda(M_n(K)) = -n = -(n - 2\tau(K)) + \sum_{i=-g}^{g} \chi(H_{*+1}(\mathcal{F}(K, i))^2).
\]

By using this the sum of Euler numbers of the chain complex \( \mathcal{F}(K, i) \) is \( \tau(K) \).

In the case of \( n < 2\tau(K) \), by the same argument, we get the same result.

\[\square\]

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