HYPERGEOMETRIC FUNCTION AND MODULAR CURVATURE

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Abstract. We first show that hypergeometric functions appear naturally as spectral functions when applying pseudo-differential calculus to decipher heat kernel asymptotic in the situation where the symbol algebra is noncommutative. Such observation leads to a unified (works for arbitrary dimension) method of computing the modular curvature on toric noncommutative manifolds. We show that the spectral functions that define the quantum part of the curvature have closed forms in terms of hypergeometric functions. As a consequence, we are able to obtained explicit expressions (as functions in the dimension parameter) for those spectral functions without using symbolic integration. A surprising geometric consequence is that the functional relations coming from the variation of the associated Einstein-Hilbert action still hold when the dimension parameter takes real values.

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1. Introduction

1.1. Modular geometry. In noncommutative geometry, a notion of intrinsic curvature for certain class of noncommutative manifolds, called modular (scalar) curvature, has only recently begun to be comprehended. The first example on which the computation was carried out in great detail is noncommutative two tori associated to conformal change (with respect to the flat one) of metrics, cf. [CM14, FK13, LM16], for noncommutative four tori, [Fat15, FK15]. Later in [Liu15, Liu17], the author extended such approach to all even dimensional toric noncommutative manifolds (also known as Connes-Landi noncommutative manifolds, cf. [Rie93, CDV08, CDV02, CL01]).

The word “modular”, taken from modular theory for von Neumann algebras, emphasizes the new ingredients (beyond the Riemannian part of the curvature) brought in by the noncommutativity of the metric. In Riemannian geometry, the conformal class of metrics \( g \) can be parametrized by a commutative coordinate, more precisely, a positive smooth function called a Weyl factor \( k = e^h \). Sometimes, it is more convenient to work with its logarithm \( h = \log k \) which is a real-valued coordinate. For the noncommutative manifolds that have been studied, whose topological structure is presented by a \( C^* \)-algebra, the Weyl factor \( k \) becomes a noncommutative coordinate, namely, a positive invertible elements in the ambient \( C^* \)-algebra, which can be written as an exponential \( k = e^h \) with \( h \) self-adjoint. Local geometric invariants such as Riemannian curvature are extracted from coefficients \( V_j(a, \Delta_k) \), viewed as functionals (in \( a \)) on the underlying \( C^* \)-algebra, of the heat kernel expansion:

\[
\text{Tr}(ae^{-t\Delta_k}) \sim_{t \searrow 0} \sum_{j=0}^{\infty} V_j(a, \Delta_k)t^{(j-m)/2},
\]

where \( m \) is the dimension of the manifolds and \( \Delta_k \) is a perturbation of the Laplacian \( \Delta := \Delta_g \) associated to a background metric \( g \), which behaves like a Riemannian one and admits a pseudo-differential calculus, cf. [Liu17]. Intuitively, one can think of \( \Delta_k \) as a quantization\(^1\) of the conformal change of metric: \( g' = kg \). In this paper, we only consider the simplest perturbation: \( \Delta_k := k\Delta \). By analogy with the results in Riemannian geometry (cf. [Gil75]), we define the scalar curvature of the metric \( \Delta_k \) to be the functional density \( R_{\Delta_k} \) of the second term in (1.1):

\[
V_2(a, \Delta_k) = \varphi_0(aR_{\Delta_k})
\]

with respect to \( \varphi_0 \), a tracial functional defined by the volume form of the background metric \( g \). The trace property of \( \varphi_0 \) is a consequence of the Riemannian features of the background metric. In contrast, the volume form of metric \( g' \) or \( \Delta_k \) is associated to the functional \( V_0(a, \Delta_k) \) in (1.1), which is a only state. The modular theory (Tomita-Takesaki theory) asserts that, for such a state, there exists a one parameter group \( \sigma_t \) of automorphisms of the ambient von Neumann algebra measuring to which

\(^1\)The word “quantization” simply means making something into an operator.
extent the state fails to be a trace. A generator of \( \sigma_t \) is called a modular operator. In our case, the modular operator \( y \) is simply the conjugation by the Weyl factor:

\[
y(a) = k^{-1}ak, \quad \sigma_t(a) = y^{it}(a),
\]

where \( a \) belongs to the von Neumann algebra. In terms of the logarithm coordinate \( h = \log k \), we can consider the modular derivation: \( x := \log y = -ad_h = [\cdot, h] \), which is new type of differential generated by the modular automorphisms that one does not see in the commutative setting. For higher \( V_j \)'s, the action of the modular operator or modular derivation is realized by some intriguing spectral functions such as \( K_{\Delta_k} \) and \( H_{\Delta_k} \) appeared in Eq. (5.12). In dimension two \([LM16, CM14]\), the one variable function \( K \) is the Bernoulli generating function, which is related to the Todd class in topology. In dimension four \([Liu17]\), it is the product of the exponential function and the \( j \)-function: \( e^{\pi i}(\sinh(x/2)/(x/2)) \), which shows striking similarity to the Atiyah-Singer local index formula. Seeking for deeper understanding of those spectral functions and related functional relations, both conceptually and computationally, is one of the main motivations that are pushing the project forward.

1.2. Pseudo-differential calculi and hypergeometric functions. The technical tool that being used to decipher the heat kernel expansion is a pseudo-differential calculus which is suitable for studying the spectral geometry of the underlying manifolds, such as Connes’s calculus for noncommutative tori \([Con80]\) and deformation of Widom’s calculus for general toric noncommutative manifolds \([Liu15]\). In general, a pseudo-differential calculus provides a recursive algorithm to construction a sequence of symbols \( \{b_j\}_j \), whose truncated sums \( \sum_0^N b_j \) approximate the resolvent of the elliptic operator in question, as \( N \to \infty \), so that the heat coefficients \( V_j \) can be obtained by certain integration of the corresponding \( b_j \), see (5.4). The first approximation \( b_0 = (p_2 - \lambda)^{-1} \) is the resolvent of \( p_2 \), the leading symbol of the elliptic operator in question. Higher \( b_j \)'s are finite sums of the form:

\[
b_j = \sum b_0^{l_0} \rho_1 b_0^{l_1} \rho_2 \cdots b_0^{l_n} \rho_n b_0^{l_n},
\]

where the exponents \( l_j \)'s are non-negative integers and \( \rho_j \)'s are the derivatives of symbols of the elliptic operator. Notice that \( b_0 \) and \( \rho_j \)'s do not commute in general, even in the commutative case in which the symbols are endomorphism-valued sections acting on some vector bundle. A useful trick to handle the non-commutativity is to rewritten the summands in (1.4) as “contractions”, cf. \([Les17]\):

\[
(b_0^{l_0} \otimes \cdots \otimes b_0^{l_n}) \cdot (\rho_1 \otimes \cdots \otimes \rho_n) := b_0^{l_0} \rho_1 b_0^{l_1} \rho_2 \cdots b_0^{l_n-1} \rho_n b_0^{l_n},
\]

because eventually, \( \rho_j \)'s can be factored out of the integration. The contribution to the action of the modular operator is given by the operator-valued integral:

\[
\int_0^\infty \frac{1}{2\pi i} \int_C e^{-\lambda b_0^{l_0} \otimes \cdots \otimes b_0^{l_n}} d\lambda (\gamma^{m-1} dr)
\]

\[
= \int_0^\infty \frac{1}{2\pi i} \int_C e^{-\lambda (kr^2 - \lambda)^{-l_0} \otimes \cdots \otimes (kr^2 - \lambda)^{-l_n}} d\lambda (\gamma^{m-1} dr),
\]

where \( m \) is the dimension of the manifold and \( k \) is a positive invertible element in some \( C^* \)-algebra. Method used in the previous works involves switching the order of integration, which depends on \( m \) is a subtle way. In this paper, we attack the contour
integral directly by replacing each \((kr^2 - \lambda)^{-l j}\) with its Mellin transformation (2.3). A surprising consequence is that the integral turns out to be (upto a constant factor) a hypergeometric function or its multi-variable generalization. We shall mainly focus on the case in which \(n\) (appeared in (1.5)) equals 1 or 2, since it is sufficient for the studying of the \(V_2\)-term.

By landing the building blocks of the spectral functions into the hypergeometric family, we obtain tons of new functional relations, among which, the differential and contiguous relations are the fundamental ones. Certain patterns of differential relations have already been observed in [Les17, CF16]. The contiguous relations seems like the other side of a coin to the differential relations, which lead us to some number theoretical properties, such as Gauss's continued fraction, see Eq. (2.24). From the computational side, symbolic integration is replaced by differentiation and recurrence relations. In particular, we have remove the restriction in [Liu17] that the manifold has to be even dimensional; we have recurrence relations among all even dimensions and all odd dimensions; we are able to express the spectral functions explicitly in the dimension parameter \(m\).

1.3. **Variation of the EH (Einstein-Hilbert) action.** The geometric application of the local expression of the modular scalar curvature studied in this paper is the variation of the EH-functional/action. In our operator theoretical framework, it is encoded in the second heat coefficient \(V_2(a, \Delta_k)\), which can be viewed as a function in both \(a\) and \(k\). When the metric coordinate \(k\) is fixed, we vary \(a\) to recover the functional density, which is the modular scalar curvature (1.2). On the other hand, we can take \(a = 1\) and view it a functional in \(k\),

\[
V_2(1, \Delta_k) = \varphi_0(R_{\Delta_k}),
\]

where the right hand side mimics the integration of the scalar curvature against the corresponding volume form \(\varphi\) in Eq. (4.1) which is the EH-action in the commutative setting. The modular curvature \(R_{\Delta_k}\) itself involves two spectral functions \(K_{\Delta_k}\) and \(H_{\Delta_k}\) of one and two variables respectively. On the other hand, its integration \(\varphi_0(R_{\Delta_k})\), can be described by only one spectral function \(T_{\Delta_k}\), which is determined, of course, by \(K_{\Delta_k}\) and \(H_{\Delta_k}\), see (4.6). It was shown in [Liu17] that the EH-action determines the scalar curvature functional completely, namely, if we know \(V_2(1, \Delta_k)\) for all positive invertible elements \(k\), then we can recover \(V_2(a, \Delta_k)\) for any “function” \(a\). The proof is carried out by computing the functional gradient (by means of Gâteaux differential, see Definition 4.1) in two different ways: by differentiating the local expression \(\varphi_0(R_{\Delta_k})\) and the trace of the heat operator \(\text{Tr}(e^{-t\Delta_k})\) respectively. The later one is easier to compute, which shows that the gradient is the almost the same as the scalar curvature, in particular, the spectral functions are still \(K_{\Delta_k}\) and \(H_{\Delta_k}\). This is a noncommutative analog of the following fact in Riemannian geometry: for the conformal change of metric \(g' = u^{4/(m-2)}g\), one sees the scalar curvature of \(g'\) in the Euler-Lagrange equation of the Yamabe functional, which is a normalization of the EH-functional, cf. [Yam60, Eq. 1.11]. For the former one, the calculation, which is discussed in Section 3, is much longer. The bottom line is that the spectral functions

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2 Notice that in Eq. (5.12), the volume factor \(k^{jm}\) is already included in \(R_{\Delta_k}\), therefore, one should apply the functional \(\varphi_0\), instead of \(\varphi\), to recover the EH-action.
of the gradient can be completely derived from \( T_{\Delta_k} \), see Eq. (4.13), by two basic transformations. One comes from integration by parts: \( T_{\Delta_k}(u) \mapsto T_{\Delta_k}(u^{-1}) \), see (3.7). The other arises from commutators of the covariant differential (commutative differential) and the modular operator/derivation (noncommutative differential):
\[
T_{\Delta_k}(u) \mapsto D(T_{\Delta_k})(u, v) = \frac{T_{\Delta_k}(uv) - T_{\Delta_k}(u)}{T_{\Delta_k}(v) - 1}.
\]

Such phenomenon fits with the classical picture that the curvature \( R(X, Y) \) is designed to measure the commutator \([\nabla_X, \nabla_Y]\) of different differentials. By equating two approaches, we obtain functional relations stated in Theorem 4.5 which tell us exactly how to reproduce \( K_{\Delta_k} \) and \( H_{\Delta_k} \) from \( T_{\Delta_k} \).

A new input in present paper is that the variation is performed with respect to the Weyl factor \( k \) itself, instead of \( h = \log k \) as in all previous works [LM16, CM14, Fat15, Liu17]. One of the advantages is to get rid of the spectral functions, cf. [CM14, Eq. (6.9), (6.10)], when computing \( \nabla e^h \) and \( \nabla^2 e^h \). Such modification is merely an operation like changing coordinates, which alters nothing of the underlying geometric objects. Nevertheless, picking a good coordinate system could be very helpful when attacking specific problems.

The discussion of EH-action above requires that the dimension of the manifold \( m \) is greater than 2. In fact, in dimension two, the celebrated Gauss-Bonnet theorem asserts that the EH-action is a constant function whose value, upto a normalization, equals the Euler characteristic of the given manifold. The corresponding result for noncommutative two tori was studied in [CT11, FK12]. One can also see this from the factor \((2 - m)/2\) appeared in the right hand side of Eq. (4.9). An interesting functional with non-trivial variation in dimension two is the Ray-Singer log-determinant functional, which has been intensively studied in [CM14].

The last contribution of this paper is the final expressions of \( K_{\Delta_k} \), \( H_{\Delta_k} \) and \( T_{\Delta_k} \) (see Eq. (4.18), (4.19) and (4.23)) as functions of \( m \), the dimension parameter, thanks to the available knowledge of hypergeometric functions. One can performed straightforward verification with the assistant of Mathematica to see that the three functions satisfy the equations given in Theorem 4.5 for all \( m \in (2, \infty) \). For \( m = 2, 3, \ldots \) being integers, such verification gives conceptual confirmation for the validity of the lengthy calculation. While for real values \( m \), functional relations is a surprising but exciting by product, which do not admit a geometric proof so far. Nevertheless, the geometric significance is that it extends the “universality” of those spectral functions in the sense that the pseudo-differential approach of studying local geometry has the potential to be applied onto noncommutative manifolds of non-integer dimensions. In our operator theoretical framework, the (metric) dimension is determined by the Weyl’s law, that is, by the rate of growth of the spectrum of the Laplacian operator, thus it takes real values in a natural way. We refer to [CM08, Sect. 10.2] for detailed explanation and to [Con94] for examples.

1.4. Outline of the paper. In section 2, we show that the Euler type integral representations of hypergeometric functions \( _2F_1 \) and Appell’s \( F_1 \) functions appears naturally in the pseudo-differential approach of heat kernel expansion. Results of this type are called rearrangement lemmas in the literature [Les17, CM14, CT11].
most general form so far is given in Prop. 2.4. Such observation brings in powerful tools for computations. For example, we have elegant differential and contiguous functionals relations to replace symbolic integrations. Among which, Prop. 2.8 is the most important technical result, which reduces double integrals (Appell’s $F_1$ functions) to a divided difference of hypergeometric functions. Section 3 and 4 contains the geometric part: the variation of the EH-action. We try our best to explore the connections between “individual” functionals relations among hypergeometric functions and the “global” relations stated in Theorem 4.5. For completeness, we outline the calculation for the modular curvature on noncommutative tori of arbitrary dimension $m \geq 2$ in the last section.

2. Hypergeometric functions in heat kernel expansion

In this section, we give a new method to prove and generalize the rearrangement lemmas in the literature [CT11, CM14, Les17]. A surprising discovery is the appearance of hypergeometric function and its multivariable generalizations in the part of the spectral functions. The most general version so far is given in Prop. 2.4.

2.1. Contour integral for the heat operator. We start with a lemma which handles the contour integral that defines the heat operator. In our applications, the elliptic operator $P$ has spectrum contained in $[0, \infty)$. Therefore, we can choose the contour $C$ to be the imaginary axis from $-i\infty$ to $i\infty$, so that

$$e^{-tp} = \frac{1}{2\pi i} \int_C e^{-t\lambda}(P - \lambda)^{-1}d\lambda = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-tx}(P - ix)^{-1}dx.$$  

**Lemma 2.1.** Let $A, B, C$ be positive real numbers and $a, b, c$ be negative integers, we have

$$(2\pi)^{-1} \int_{-\infty}^{\infty} e^{-ix}(A - ix)^a(B - ix)^b dx = \frac{1}{\Gamma(a)\Gamma(b)} \int_0^1 (1 - t)^{a-1}t^{b-1}e^{-(A-B)t}dt$$

and

$$(2\pi)^{-1} \int_{-\infty}^{\infty} e^{-ix}(A - ix)^a(B - ix)^b(C - ix)^c dx$$

$$= \frac{1}{\Gamma(a)\Gamma(b)\Gamma(c)} \int_0^1 \int_0^{1-t} (1 - t - u)^{a-1}u^{b-1}e^{-(A-B)t-(A-C)u}dudt.$$  

**Remark.** Notice that the right hand side of (2.1) is a confluent hypergeometric function:

$$(2\pi)^{-1} \int_{-\infty}^{\infty} e^{-ix}(A - ix)^a(B - ix)^b dx = \frac{e^{-A}}{\Gamma(a + b)} \Gamma(a + b) \int_0^1 (1 - t)^{a-1}t^{b-1}e^{zt}dt.$$  

The identity (2.3) appeared in the heat kernel related work [Gus91] and [AB01]. This was the motivation at the early stage that brought the author’s attention to hypergeometric functions.
Proof. We only prove (2.2) and leave (2.1) to the reader. Observe that Powers like \((A - ix)^a\) can be rewritten in terms of Mellin transform:

\[
(2.5) \quad (A - ix)^a = \frac{1}{\Gamma(a)} \int_0^\infty s^{a-1}e^{-s(A-ix)}ds.
\]

The rest of the computation is straightforward, denote \(\gamma(a, b, c) = [\Gamma(a)\Gamma(b)\Gamma(c)]^{-1}\), then:

\[
(2\pi)^{-1} \int_{-\infty}^{\infty} e^{-ix}(A - ix)^a(B - ix)^b(C - ix)^c dx
\]

\[
= (2\pi)^{-1}\gamma(a, b, c) \int_{-\infty}^{\infty} \int_{[0,\infty)^3} e^{-ix}s^{a-1}t^{b-1}u^{c-1}e^{-s(A-ix)-t(B-ix)-u(C-ix)}dsdtdu dx
\]

\[
= \gamma(a, b, c) \int_{[0,\infty)^3} s^{a-1}t^{b-1}u^{c-1}e^{-sA-Bt-Cu}dsdtdu \left((2\pi)^{-1} \int_{-\infty}^{\infty} e^{ix(s+u+t-1)}dx\right)
\]

\[
= \gamma(a, b, c) \int_0^1 \int_0^{1-t} (1-t-u)^{a-1}t^{b-1}u^{c-1}e^{-(A(B-C)u)}dudt.
\]

For the last equal sign, we use the fact that \((2\pi)^{-1} \int_{-\infty}^{\infty} e^{ixy}dx\) equals the Dirac-delta distribution \(\delta(y)\), therefore the domain of integration is reduced from \([0,\infty)^3\) to \(\{s + t + u = 1, \ s, t, u > 0\}\).

\[\square\]

2.2. Spectral functions in terms of hypergeometric functions. Let \(\mathcal{A}\) be a unital \(C^*\)-algebra. For elementary tensors \(a = (a_0, \ldots, a_n) \in \mathcal{A}^{\otimes n+1}\) and \(\rho = (\rho_1, \ldots, \rho_n) \in \mathcal{A}^{\otimes n}\), we recall the contraction \(a.\rho\) used in [Les17]:

\[a.\rho := a_0.\rho_1.a_1.\rho_2 \cdots a_{n-1}.\rho_n.a_n.\]

For a single element \(k \in \mathcal{A}\) acting on a elementary tensor \(\rho\) above (or a product \(\rho_1 \cdots \rho_n\)), we introduce multiplication in the \(j\)-th slot:

\[
(2.6) \quad k^{(j)} := (1, \ldots, k, 1 \ldots, 1), \ k \text{ is in the } j\text{-th slot, } j = 0, 1, \ldots, n,
\]

and conjugation on the \(j\)-th factor:

\[
(2.7) \quad y_j(k) := (1, \ldots, k^{-1}, 1 \ldots, 1), \ k^{-1} \text{ is in the } (j-1)\text{-th slot, } j = 0, 1, \ldots, n.
\]

Later, we need to replace all the multiplication \(k^{(j)}\), \(j \geq 1\) by conjugation operators:

\[
(2.8) \quad k^{(j)} = k^{(0)}y_1(k) \cdots y_j(k).
\]

In the rest of the paper, one can take \(\mathcal{A}\) to be a noncommutative \(m\)-torus: \(C^\infty(T_m^m)\) (cf. [Ric90]).

When applying pseudo-differential calculus to compute heat kernel asymptotic, one encounter integrals involving the resolvent approximation symbols \(b_j\):

\[
\int_0^\infty \frac{1}{2\pi i} \int_C e^{-\lambda b_j(r, \lambda)}d\lambda r^{(m-1)/2}dr.
\]

In general, \(b_j\) is a finite sum of the form:

\[
(2.9) \quad \sum b^{(j)}_0.\rho_1.b^{(j)}_1.\rho_2 \cdots \rho_i.b^{(j)}_i,
\]
where \( b_0 = (kr^2 - \lambda)^{-1} \) and \( \rho_p \) are of the form \( k^{a_p} \nabla^{b_p} k \) for some integers \( a_p \) and \( b_p \), where \( p = 0, \ldots, l \).

Following the contraction notation in the previous section:

\[
\int_0^\infty \frac{1}{2\pi i} \int_C e^{-\lambda b_j(r, \lambda)} d\lambda r^{(m-1)/2} dr
= \left( \int_0^\infty \frac{1}{2\pi i} \int_C e^{-\lambda (kr^2 - \lambda)^{-j_1} \cdots (k^{(n)}r^2 - \lambda)^{-j_n}} d\lambda r^{(m-1)/2} dr \right) \cdot (\rho_1 \cdots \rho_n).
\]

In this paper, we only need to integrate \( b_2 \) which involves terms with \( l = 1 \) or \( 2 \).

**Proposition 2.2.** Let \( a, b, m \) be positive integers, we abbreviate the conjugation operator defined in (2.7) as: \( y(\rho) := y_1(k)(\rho) = k^{-1} pk \), then

\[
\int_0^\infty \frac{1}{2\pi i} \int_C e^{-\lambda (kr^2 - \lambda)^{-a} \cdot \rho \cdot (k^{(1)}r^2 - \lambda)^{-b}} d\lambda r^{2d_m+1} dr
= (k^{(0)})^{-(d_m+1)} \left( \frac{\Gamma(d_m + 1)}{2\Gamma(a + b)} \right)_2 F_1(d_m + 1; b, b + a; 1 - y) \cdot \rho.
\]

where \( d_m := d_m(a, b) = a + b - 2 + (m - 2)/2 \), and the contour \( C \) can be taken to be the imaginary axis from \(-i\infty \) to \( i\infty \). We denote the result as

\[
K_{a,c}(y; m) = \frac{\Gamma(d_m + 1)}{\Gamma(a + b)} \frac{\Gamma(d_m + 1)}{2\Gamma(a + b + c)} \cdot (\rho_1 \rho_2).
\]

**Remark.** The appearance of \( d_m(a, b) \) is due to the homogeneity, degree \(-4\), of \( b_2 \) in \( r \) (before the substitution \( r \mapsto r^2 \)): \( b_2(\rho r) = c^{-4} b_2(r) \). In general, \( b_j \) is of homogeneity \(-2 - j \) when the corresponding differential operator is of order \( 2 \).

Similarly, we have a two-variable version which gives rise to the Appell hypergeometric functions.

**Proposition 2.3.** Let \( a, b, c, m \) be positive integers, we abbreviate the conjugation operator defined in (2.7) as: \( y_j := y_j(k) \), with \( j = 1, 2 \).

\[
\int_0^\infty \frac{1}{2\pi i} \int_C e^{-\lambda (kr^2 - \lambda)^{-a} \cdot \rho_1 \cdot (k^{(1)}r^2 - \lambda)^{-b} \cdot \rho_2 \cdot (k^{(2)}r^2 - \lambda)^{-c}} d\lambda r^{2d_m+1} dr
= (k^{(0)})^{-(d_m+1)} \left( \frac{\Gamma(d_m + 1)}{2\Gamma(a + b + c)} \cdot (\rho_1 \rho_2) \right)
\]

where \( d_m := d_m(a, b, c) = a + b + c - 2 + (m - 2)/2 \), and the contour \( C \) can be taken to be the imaginary axis from \(-i\infty \) to \( i\infty \). We denote the result as

\[
H_{a,b,c}(y_1, y_2; m) = \frac{\Gamma(d_m + 1)}{\Gamma(a + b + c)} \cdot (\rho_1 \rho_2)
\]

for later discussion.

**Proof.** We first perform a substitution \( r \mapsto r^2 \), thus the volume form \( r^{2d_m+1} dr \) becomes \( r^{d_m} dr/2 \), in which the overall factor \( 1/2 \) will be omitted in the rest of the proof. The contour integral can be computed according to lemma 2.1 in which \( A = k^{(0)}r \),

\[
\cdot \rho_1 \cdot (k^{(1)}r^2 - \lambda)^{-b} \cdot \rho_2 \cdot (k^{(2)}r^2 - \lambda)^{-c} d\lambda r^{2d_m+1} dr
\]
$B = k^{(0)} r$ and $C = k^{(2)} r$ are bounded operators with positive spectrum. A Fubini-type theorem cf. [Les17, Lemma 2.1] has been applied to makes sense of such functional calculus. So far, we have

$$\frac{1}{2\pi i} \int_C e^{-\lambda (k^{(0)} r^2 - \lambda) - a (k^{(1)} r^2 - \lambda) - b (k^{(2)} r^2 - \lambda) - c} d\lambda$$

$$= \gamma(a, b, c) \int_0^1 \int_0^{1-t} (1 - t - u)^a t^{b-1} u^{c-1} e^{-r[k^{(0)} - t(k^{(0)} - k^{(1)}) - u(k^{(0)} - k^{(1))})]dudt}$$

$$= \gamma(a, b, c) \int_0^1 \int_0^{1-t} (1 - t - u)^a t^{b-1} u^{c-1} e^{-r[1-t(1-y_1) - u(1-y_2)]}dudt$$

Next, we apply $\int_0^\infty (\ast) r^{d_m} dr$ to the result above and integrate in $r$ first:

$$\int_0^\infty r^{d_m} e^{-r[k^{(0)} - t(k^{(0)} - k^{(1)}) - u(k^{(0)} - k^{(1))})]}dr$$

$$= (k^{(0)})^{-(d_m+1)} \int_0^\infty r^{d_m} e^{-r[1-t(1-y_1) - u(1-y_2)]}dr$$

$$= (k^{(0)})^{-(d_m+1)} (1 - t(1 - y_1) - u(1 - y_2))^{-(d_m+1)} \left( \int_0^\infty r^{d_m} e^{-r} dr \right)$$

$$= \Gamma(d_m + 1)(k^{(0)})^{-(d_m+1)} (1 - t(1 - y_1) - u(1 - y_2))^{-(d_m+1)} .$$

An operator substitution lemma [Les17, Thm 2.2] has been used twice. Roughly speaking, the lemma allows us to treat mutually commuting positive operators $k^{(j)}$ with $j = 1, 2, 3,$ and $1 - t(1 - y_1) - u(1 - y_2)$ as positive real numbers during the computation. To get the first equal sign, we set $r \mapsto rk^{(0)}$, and use the relations (see Eq. 2.8):

$$k^{(1)} = k^{(0)} y_1, \quad k^{(2)} = k^{(0)} y_1 y_2.$$ 

For the second equal sign, we let $r \mapsto r[1 - t(1 - y_1) - u(1 - y_2)]$, which is a positive operator for $r > 0$ and $0 < t, u < 1$.

After combining the previous two steps, we can finish the proof by observing that the integral in $t$ and $u$ matches the integral representation of Appell’s hypergeometric functions, see Eq. (A.15)

$$\int_0^\infty \frac{1}{2\pi i} \int_C e^{-\lambda (k^{(0)} r^2 - \lambda) - a (k^{(1)} r^2 - \lambda) - b (k^{(2)} r^2 - \lambda) - c} d\lambda(r^{d_m} dr)$$

$$= \Gamma(d_m + 1)(k^{(0)})^{-(d_m+1)} \gamma(a, b, c)$$

$$\int_0^1 \int_0^{1-t} (1 - t - u)^a t^{b-1} u^{c-1} (1 - t(1 - y_1) - u(1 - y_2))^{-(d_m+1)} dudt$$

$$= (k^{(0)})^{-(d_m+1)} \frac{\Gamma(d_m + 1)}{\Gamma(a + b + c)} F_1(d_m + 1; c; b; a + b + c; 1 - y_1 y_2, 1 - y_1) \square$$
2.3. **Multivariable generalization.** In general, the role of the Appell’s $F_1$ in Proposition 2.3 becomes Lauricella Functions of type $F_D^{(n)}$. We start with the series version:

\[(2.14) \quad F_D^{(n)}(a; \alpha_1, \ldots, \alpha_n; c; x_1, \ldots, x_n) = \sum_{\beta_1, \ldots, \beta_n \geq 0} \frac{(a)_{\beta_1 + \cdots + \beta_n} (\alpha_1)_{\beta_1} \cdots (\alpha_n)_{\beta_n} x_1^{\beta_1} \cdots x_n^{\beta_n}}{(c)_{\beta_1 + \cdots + \beta_n} \beta_1! \cdots \beta_n!},\]

which has the following integral representation (cf. [HK74]):

\[(2.15) \quad \int_{R(u_1, \ldots, u_n)} (1 - \sum_{j=1}^{n} u_j)^{-\alpha_1 - \cdots - \alpha_n} u_1^{\alpha_1-1} \cdots u_n^{\alpha_n-1} (1 - \sum_{j=1}^{n} x_j u_j)^{-a} du_1 \cdots du_n = \frac{\Gamma(\alpha_1) \cdots \Gamma(\alpha_n) \Gamma(c - \alpha_1 - \cdots - \alpha_n)}{\Gamma(c)} F_D^{(n)}(a; \alpha_1, \ldots, \alpha_n; c; x_1, \ldots, x_n).\]

The three variable case was first introduced by Lauricella in [Lau93] and more fully by Appell and Kampé de Fériet [AdF26].

We will only repeat the major steps in the previous calculation and left details to interested readers. Let $(A_0, \ldots, A_n) \in (0, \infty)^{n+1}$. Define

\[(2.16) \quad G_{\alpha_0, \ldots, \alpha_n}(A_0, \ldots, A_n) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-ix(A_0 - ix)^{-\alpha_0} \cdots (A_n - ix)^{-\alpha_n}} dx.\]

Similar to Lemma 2.1 we have

\[(2.17) \quad G_{\alpha_0, \ldots, \alpha_n}(A_0, \ldots, A_n) = \gamma(\alpha_0, \ldots, \alpha_n) \int_{R(u_1, \ldots, u_n)} (1 - \sum_{j=1}^{n} u_j)^{\alpha_0-1} u_1^{\alpha_1-1} \cdots u_n^{\alpha_n-1} \]

\[\cdot \exp \left(-A_0(1 - \sum_{j=1}^{n} u_j) - \sum_{j=1}^{0} A_j u_j \right) du_1 \cdots du_n,\]

where $\gamma(\alpha_0, \ldots, \alpha_n) = (\Gamma(\alpha_0) \cdots \Gamma(\alpha_n))^{-1}$ and the domain of integration a standard $n$-simplex:

\[R(u_1, \ldots, u_n) = \left\{ u_1 \geq 0, \ldots, u_n \geq 0, 1 - \sum_{j=1}^{n} u_j \geq 0 \right\}.\]

Next, we set $A_j$ to be positive operators: $A_j = k^{(j)}r$ for $j = 0, 1, \ldots, n$ and $r \geq 0$, where $k^{(j)}$ are the Weyl factors in (2.6). Again, by the operator substitution lemma ([Les17 Lemma 2.1]):

\[\int_{0}^{\infty} \exp \left(-rk^{(0)}(1 - \sum_{j=1}^{n} u_j) - \sum_{j=1}^{0} r k^{(j)} u_j \right) (r^d dr) = \Gamma(d + 1)(k^{(0)})^{-(d+1)} \left(1 - \sum_{j=1}^{m} z_j u_j \right)^{-(d+1)},\]
where \( z_j = 1 - y_1 \cdots y_j \). Notice that we have used the relations (2.8). The final goal is the following integral:

\[
\int_0^\infty G_{\alpha_0, \ldots, \alpha_n}(k^{(0)}r, \ldots, k^{(n)}r)(r^d dr)
\]

\[
= (k^{(0)})^{-(d+1)} \frac{\Gamma(d + 1)}{\Gamma(a_0, \ldots, \alpha_n)} \int_{R(a_1, \ldots, a_n)} (1 - \sum_{j=1}^n u_j)^{a_0 - 1} u_1^{a_1 - 1} \cdots u_n^{a_n - 1}
\]

\[
\cdot (1 - \sum_{j=1}^n z_j u_j)^{-(d+1)} du_1 \cdots du_n.
\]

The integral on the right hand side above is a \( n \)-variable Lauricella function \( F^{(n)}_D \) in (2.15). Sum up, we have obtained the following \( n \)-variable version for the spectral functions.

**Proposition 2.4.** Keep notations.

\[
(2.18)
\]

\[
(k^{(0)})^{d+1} \int_0^\infty G_{\alpha_0, \ldots, \alpha_n}(k^{(0)}r, \ldots, k^{(n)}r)(r^d dr)
\]

\[
= \frac{\Gamma(d + 1)}{\Gamma(a_0 + \cdots + \alpha_n)} F^{(n)}_D(d + 1; \alpha_1, \ldots, \alpha_n; \alpha_0 + \cdots + \alpha_n; z_1, \ldots, z_n),
\]

where \( z_j = 1 - y_1 \cdots y_j \) for \( j = 1, 2, \ldots, n \).

2.4. **Differential and contiguous functional relations.** Let us slightly change the notations in Proposition 2.3. Denote

\[
\tilde{K}_{a,b}(u; m) = \frac{\Gamma(\tilde{d}_m)}{\Gamma(a + b)} 2F_1(\tilde{d}_m; b; a + b; u),
\]

where \( \tilde{d}_m := \tilde{d}_m(a, b) = a + b + m/2 - 2 \). Similarly,

\[
\tilde{H}_{a,b,c}(u, v; m) = \frac{\Gamma(\tilde{d}_m)}{\Gamma(a + b + c)} F_1(\tilde{d}_m; c; a + b + c; u, v)
\]

with \( \tilde{d}_m := \tilde{d}_m(a, b, c) = a + b + c + m/2 - 2 \).

**Proposition 2.5.** For the one variable family defined in (2.19), the following functional relations hold:

\[
(2.21)
\]

\[
K_{a,b}(u; m + 2) = (\tilde{d}_m + ud/du) \tilde{K}_{a,b}(u; m),
\]

\[
(2.22)
\]

\[
\tilde{K}_{a,b+1}(u; m) = (b^{-1}d/du) \tilde{K}_{a,b}(u; m),
\]

\[
(2.23)
\]

\[
\tilde{K}_{a,b+1}(u; m) = (1 + b^{-1}ud/du) \tilde{K}_{a+1,b}(u; m).
\]

Moreover,

\[
(2.24)
\]

\[
\frac{a + b}{\tilde{d}_m} \frac{K_{a+1,b}(u; m)}{K_{a,b}(u; m)} = \frac{2F_1(\tilde{d}_m + 1, b; a + b + 1; u)}{2F_1(\tilde{d}_m, b; a + b; u)},
\]

where the right hand side is a Gauss’s continued fraction.

**Proof.** Follows quickly from the differential relations (A.5) to (A.8).
Proposition 2.6. For the two variable family defined in (2.20),

\[ H_{a,b,c}(u,v;m+2) = (\tilde{d}_m + u\partial_u + v\partial_v)H_{a,b,c}(u,v;m), \]
\[ H_{a,b+1,c}(u,v;m) = b^{-1}\partial_v H_{a,b,c}(u,v;m), \]
\[ H_{a,b,c+1}(u,v;m) = c^{-1}\partial_u H_{a,b,c}(u,v;m). \]

When increasing the parameter \(a\) by one, we encounter a similar notation as in (2.24):
\[
\frac{H_{a+1,b,c}(u,v;m)}{H_{a,b,c}(u,v;m)} = \frac{\tilde{d}_m}{a+b+c} \frac{F_1(\tilde{d}_m + 1; c, b; a + b + c + 1; u, v)}{F_1(d_m; c, b; a + b + c; u, v)}.
\]

**Proof.** Follows quickly from the differential relations (A.17) to (A.21). \(\square\)

Proposition 2.7. Let \(m = \dim M \geq 2\) and \(a, b, c\) be positive integers,

\[ \tilde{K}_{a,b}(z; m+2) = a\tilde{K}_{a+1,b}(z; m) + b\tilde{K}_{a,b+1}(z; m) \]
\[ \tilde{H}_{a,b,c}(u,v;m+2) = a\tilde{H}_{a+1,b,c}(u,v;m) + b\tilde{H}_{a,b+1,c}(u,v;m) \]
\[ + c\tilde{H}_{a,b,c+1}(u,v;m). \]

**Proof.** Notice that (2.28) and (2.29) are equivalent to the following contiguous relations of hypergeometric functions respectively:

\[ (a+b)F_1(\tilde{d}_m + 1; b; a+b; u) \]
\[ = aF_1(\tilde{d}_m + 1; c, b; a+b+c; u, v) \]
\[ + bF_1(\tilde{d}_m + 1; c, b+1; a+b+c+1; u, v) \]
\[ + cF_1(\tilde{d}_m + 1; c+1; b; a+b+c+1; u, v). \]

We prove (2.31) as an example and leave (2.30) to interested readers. Indeed, let \( F = F_1(\alpha; \beta, \beta'; \gamma; u, v) \), from (A.19), (A.20) and (A.21), we can solve for \((u\partial_u + v\partial_v)F\) in two different ways:

\[
(u\partial_u + v\partial_v)F = (\gamma - 1)(F(\gamma) - F)
= \beta(F(\beta) - F) + \beta'(F(\beta') - F),
\]
where \( F(\beta), \ F(\beta') \) and \( F(\gamma) \) stand for rising or lowering the indicated parameter by one. Two sides of (2.31) appear as the two lines on the right hand side above, with \( \alpha = \tilde{d}_m + 1, \ \beta = c, \ \beta' = b \) and \( \gamma = a + b + c + 1. \) \(\square\)

2.5. Comparison to previous results. The main goal of this section is to confirm that the family \( H_{a,b,c}(u,v;m) \) defined in Proposition 2.3 does agree with those cases that have been computed in previous works. First of all, asking a computer algebra system to perform symbolic integration for the double integral (2.13) is tremendously inefficient. Thanks to Proposition A.3, all the \( F_1 \) functions that come from \( H_{a,b,1}(u,v;m) \) can be reduced to hypergeometric functions \( \text{}_2F_1. \) In particular, by taking \( p = 0 \) and \( q = 1 \) in Proposition A.3, we see that \( F_1(a,1,1,b,x,y) \) is a divided difference of \( \text{}_2F_1: \)
Proposition 2.8. For $a \in \mathbb{C}$ and $b \in \mathbb{C}\backslash \mathbb{Z}_{\leq 0}$, we have the following divided difference relation:

\[
F_1(a; 1, 1; b; x, y) = \frac{x_2 F_1(a, 1; b; x) - y_2 F_1(a, 1; b; y)}{x - y} = [x, y](z_2 F_1(a, 1; b; z)).
\]  

(2.32)

and

\[
F_1(a; 1, 2; b; x, y) = b^{-1}(x - y)^{-2}\left[ bx^2 F_1(a, 1; b; x) + by^2 F_1(a, 2; b; y) + x \left(-ay^2 F_1(a + 1, 2; b + 1; y) - 2by F_1(a, 1; b; y)\right)\right]
\]

(2.33)

In dimension $m = \text{dim } M = 2$, we can recover the explicit functions listed at the very end of [CM14]. According to (2.32),

\[
\tilde{H}_{a,1,1}(s,t; 2) = \frac{\Gamma(a+1) F_1(a+1,1;1;a+2;1-t,1-s)}{\Gamma(a+2)}
= \frac{\Gamma(a+1)((s-1)_2 F_1(a+1,1;a+2;1-s)-(t-1)_2 F_1(a+1,1;a+2;1-t))}{\Gamma(a+2)(s-t)}.
\]

For example,

\[
\tilde{H}_{2,1,1}(s,t; 2) = \frac{(t-1)^2 \log(s) + (s-1)((t-1)(s-t) - (s-1) \log(t))}{(s-1)^2(t-1)^2(s-t)},
\]

\[
\tilde{H}_{3,1,1}(s,t; 2) = \frac{(s-1)(t-1)(s-t)((s-3)(t-3s+5) + 2(s-1)^3 \log(t) - 2(t-1)^3 \log(s))}{2(s-1)^3(t-1)^3(s-t)}.
\]

Those functions was computed again in [Les17, Sec. 5.2, 5.3] by applying divided difference repeatedly on to the modified logarithm function $L_0$ in [CT11], which, in particular, is a hypergeometric function:

\[
2 F_1(1,1;2;1-z) = \frac{\ln z}{z - 1}.
\]

One also observes that, in [Les17, Sec. 5.2, 5.3], characteristic differential operators (as in (A.18) to (A.21)) were used in the computation. Such similarity will be investigate in future papers.

In dimension $m = 3$, some explicit functions listed in in [KMS16, Theorem 7.1]. The results are compatible, we give a few examples:

\[
\tilde{K}_{2,1}(1-z; 3) = \frac{3}{8} \sqrt{\pi} \frac{\sqrt{3}}{2} \frac{\pi}{2} F_1\left(\frac{5}{2}, 1; 3; 1 - z\right) = \frac{\sqrt{\pi} \left(\sqrt{3} + 2\right)}{2 \left(\sqrt{3} + 1\right)^2 \sqrt{z}},
\]

\[
\tilde{K}_{2,1}(1-z; 3) = \frac{5 \sqrt{\pi}}{16} \frac{\sqrt{9 \sqrt{3} + 8}}{8} \frac{\pi}{2} F_1\left(\frac{7}{2}, 1; 4; 1 - z\right) = \frac{\sqrt{\pi} \left(3 \sqrt{3} + 9 \sqrt{3} + 8\right)}{8 \left(\sqrt{3} + 1\right)^3 \sqrt{z}}.
\]
and
\[
\tilde{H}_{2,1,1}(1-x, 1-y; 3) = \frac{5\sqrt{\pi}}{16} F_1 \left( \frac{7}{2}; 1, 1; 4; 1 - y, 1 - x \right)
\]
\[
= \frac{\sqrt{\pi} (x\sqrt{y + \sqrt{y} + 4\sqrt{x\sqrt{y} + 2x + 4\sqrt{x + 2y} + 4\sqrt{y} + 2})}{2(\sqrt{x} + 1)^2 \sqrt{y} (\sqrt{y} + 1)}
\]
\[
\tilde{H}_{1,1,1}(1-x, 1-y; 3) = \frac{5\sqrt{\pi}}{16} F_1 \left( \frac{5}{2}; 1, 1; 3; 1 - y, 1 - x \right)
\]
\[
= \frac{\sqrt{\pi} \left( \sqrt{x} + \sqrt{y} + 1 \right)}{\left( \sqrt{x} + 1 \right)^2 \left( \sqrt{y} + 1 \right)} \sqrt{\left( \sqrt{x} + \sqrt{y} \right)}.
\]

It has been shown in the author’s previous work [Liu17, Eq. (3.9)] that for dimension 
\( m = \dim M > 2 \) and even, functions \( \tilde{K}_{a,b}(u; m) \) and \( \tilde{H}_{a,b,c}(u, v; m) \) are jets of the following functions at zero: Eq. (2.34) and (2.35). Using the recurrence relations in 
Proposition (2.7), we can give a short induction proof as below.

**Proposition 2.9.** Assume that the dimension \( m \) is even and greater or equal than 4, 
set \( j_m = (m - 4)/2 \in \mathbb{Z}_{\geq 0} \), we have

\[
(2.34) \quad \tilde{K}_{a,b}(u; m) = \frac{\partial^m}{\partial z^m}_{z=0} (1 - z)^{-a}(1 - u - z)^{-b},
\]
\[
(2.35) \quad \tilde{H}_{a,b,c}(u, v; m) = \frac{\partial^m}{\partial z^m}_{z=0} (1 - z)^{-a}(1 - u - z)^{-b}(1 - v - z)^{-c}.
\]

**Proof.** We use induction on the dimension \( m \). Let us focus on the one variable case 
first. When \( m = 4 \), \( d_m = a + b \), (2.34) follows from the identity:

\[
_2F_1(\alpha, \beta; \gamma, z) = _2F_1(\beta, \alpha; \gamma, z) = (1 - z)^{-\alpha}.
\]

Now assume that (2.34) holds for some even \( m \), it remains to show that

\[
\tilde{K}_{a,b}(u; m + 2) = \frac{\partial^m}{\partial z^m}_{z=0} \frac{d}{dz} \left[ (1 - z)^{-a}(1 - u - z)^{-b} \right]
\]
\[
= a\tilde{K}_{a+1,b}(u; m) + b\tilde{K}_{a,b+1}(u; m),
\]
which is valid due to Proposition (2.7). Therefore (2.34) has been proved by induction. 
Same arguments work for (2.35).

\[\square\]

3. Variational Calculus with respect to a noncommutative variable

**3.1. Notations.** The calculation in this section can be carried out in an abstract setting as in [Les17]. Nevertheless, to smooth the exploration, we take the ambient \( C^* \)-algebra to be a noncommutative two torus \( C(T^2_0) \) for some irrational \( \theta \). Denote by
\( \nabla \) the Levi-Civita connection for the flat metric, and then \( i\nabla_j \) for \( j = 1, 2 \) correspond to the basic derivations, see [CM14, Sect. 1.3].

We start with a fixed self-adjoint element \( h = h^* \) whose exponential \( k = e^h \) defines a Weyl factor. Consider the variation \( \delta_a \) along another self-adjoint operator \( a \):

\[
k_{\varepsilon} := k(\varepsilon) = e^{h + \varepsilon a}, \quad \delta_a := \frac{d}{d\varepsilon} \big|_{\varepsilon=0}.
\]
The modular operator and the modular derivation are denoted by bold letters $\mathbf{y}$ and $\mathbf{x}$ respectively:

\[(3.2) \quad \mathbf{y} = \text{Ad}_k = k^{-1}(\cdot)k, \quad \mathbf{x} = \log \mathbf{y} = -\text{ad}_k = [\cdot, h].\]

They should be compared with the regular font $y$ and $x$, which stand for the positive and real variable, respectively, for spectral functions\(^3\). Also, as explained in section 2.2 when acting on a product of $n$ factors: $\rho_1 \cdots \rho_n$, we shall denote by $\mathbf{y}^{(j)}$, $\mathbf{x}^{(j)}$ or simply $\mathbf{y}_j$, $\mathbf{x}_j$ to indicate that the operator acts only on the $j$-the factor, where $j = 1, \ldots, n$. To make it more explicit, consider a multi-variable function $K(x_1, \ldots, x_n)$ which admits a Fourier transform:

\[(3.3) \quad K(x_1, \ldots, x_n) = \int_{\mathbb{R}^n} \alpha(\xi_1, \ldots, \xi_n) e^{-i(x_1\xi_1 + \cdots + x_n\xi_n)} d\xi_1 \cdots d\xi_n.\]

then the functional calculus $K(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ has the following description

\[(3.4) \quad K(\mathbf{x}_1, \ldots, \mathbf{x}_n) \cdot (\rho_1 \cdots \rho_n) = \int_{\mathbb{R}^n} \beta(\xi_1, \ldots, \xi_n) \mathbf{y}^{-i\xi_1}(\rho_1) \cdots \mathbf{y}^{-i\xi_n}(\rho_n) d\xi_1 \cdots d\xi_n.\]

Let $\varphi_0$ be the canonical trace on $C^\infty(T^2_0)$, later we will need some integration by parts identities.

**Lemma 3.1.** Keep the notations. We have

\[(3.5) \quad \varphi_0 (\rho_1 \cdot K(\mathbf{y}_1)(\rho_2)) = \varphi_0 \left( K(\mathbf{y}_1^{-1})(\rho_1) \cdot \rho_2 \right),\]

\[(3.6) \quad \varphi_0 (K(\mathbf{y}_1, \mathbf{y}_2)(\rho_1 \cdot \rho_2)) = \varphi_0 \left( [K(\mathbf{y}_1, \mathbf{y}_2^{-1})(\rho_1)] \cdot \rho_2 \right),\]

\[(3.7) \quad \varphi_0 (\rho_1 \cdot K(\mathbf{y}_1, \mathbf{y}_2)(\rho_2 \cdot \rho_3)) = \varphi_0 \left( K(\mathbf{y}_2, \mathbf{y}_1^{-1}\mathbf{y}_2^{-1})(\rho_1 \cdot \rho_2) \cdot \rho_3 \right),\]

\[(3.8) \quad \varphi_0 (K(\mathbf{y}_1, \mathbf{y}_2)(\rho_1 \cdot \rho_2) \cdot \rho_3) = \varphi_0 \left( \rho_1 \cdot K(\mathbf{y}_1^{-1}\mathbf{y}_2^{-2}, \mathbf{y}_1)(\rho_2 \cdot \rho_3) \right).\]

\(^3\) In [Lin17], modular operators and modular derivations are denoted by $\hat{\Delta}$ and $\hat{\psi}$ respectively. In this section, they will be treated as arguments of spectral functions in lengthy variational calculus, therefore it should be more natural to denote them as letters and use the bold font whenever we need to emphasize their operator nature.
Proof. We will only check (3.6) and leave the rest to the reader. Let \( \alpha(u,v) \) be the Fourier transform of the function \( H(x_1,x_2) := K(e^{x_1}, e^{x_2}) \) defined as before, then

\[
\varphi_0(K(y_1,y_2)(\rho_1 \cdot \rho_2)) = \varphi_0 \left( \int_{\mathbb{R}^2} \alpha(u,v)y^{-iu}(\varphi_1)y^{-iv}(\rho_2)dudv \right)
\]

\[
= \varphi_0 \left( \int_{\mathbb{R}^2} \alpha(u,v)y^{-i(u-v)}(\rho_1)dudv \cdot \rho_2 \right)
\]

\[
= \varphi_0 \left( [K(y,y^{-1})(\rho_1)] \cdot \rho_2 \right).
\]

Notice that we have used the trace property of \( \varphi_0 \) in order to reach the second step. \( \square \)

3.2. Variational Calculus. In the previous work of modular scalar curvature [LM16, CMI4, Fat15, Liu17, CF16], variation was carried out with respect to \( h = \log k \), which can be think of as a noncommutative coordinate in the tangent space of the moduli space of metrics. In this section, we would like to perform parallel computations with respect to the coordinate \( k \), so that the spectral functions (called local curvature functions in previous works) can be simplified by getting rid of the spectral functions (cf. [CM14, Eq. (6.9), (6.10)]) arising from differentiating \( e^h \), such as \( \nabla e^h \) and \( \nabla^2 e^h \).

It has been pointed out in [Les17] that divided difference plays a crucial role in such variational calculus. Recall

\[
[x,y](T(z)) := \frac{T(x) - T(y)}{x - y}.
\]

(3.9)

For example, \( [x,y](z^m) = (y^m - x^m)/(x - y) \).

The main goal of this section is to derive the variation of the following local expression

\[
\delta_a \varphi_0 \left( k^j T(y)(\nabla k) \cdot \nabla k \right), \quad \forall j \in \mathbb{R}.
\]

(3.10)

where \( \varphi_0 \) is the canonical trace on noncommutative two tori. The final result is given in Theorem 3.10.

**Lemma 3.2.** Let \( k = e^h \) with \( h \) self-adjoint. For \( j \in \mathbb{R} \),

\[
\nabla k^j = k^{j-1} \left( [1,y](z^j) \right) (\nabla k) = k^{j-1} \frac{y^j - 1}{y - 1} (\nabla k).
\]

(3.11)

The same identity holds when the covariant differential \( \nabla \) is replaced by the variation \( \delta_a \) differential.

**Proof.** Recall the notation of the modular derivation \( x \) and modular operator \( y \) in (3.2). Equation (3.11) is another version of the formula (cf. [CM14, Sect. 6] and [Les17])

\[
\nabla e^h = e^h e^x - 1 \frac{1}{x} \nabla h
\]

by changing the variable:

\[
\nabla \log k = k^{-1} \frac{\log y}{y - 1} \nabla k.
\]
Indeed, for $k^j = e^{jh}$,
\[
\nabla k^j = \nabla e^{jh} = e^{jh} \frac{e^{jx} - 1}{jx} \nabla(jh) = e^{jh} \frac{e^{jx} - 1}{jx} (j k^{-1}) \frac{\log y}{y - 1} \nabla k
\]
\[
= k^{-1} j \frac{y^j - 1}{y - 1} \nabla k.
\]

The variation $\delta_a(T(y))$ can be computed by the following Taylor expansion. We refer the proof to [Les17].

**Proposition 3.3** ([Les17], Prop. 3.11). Let $h, b$ be two self-adjoint elements and $T(x)$ be a Schwartz function on $\mathbb{R}$. We have the Taylor expansion for the operator $T(\text{ad}_{h+b})$ up to the first order:
\[
(3.12) \quad T(\text{ad}_{h+b})(\psi) = T(x)(\psi) - ([x_1 + x_2, x_2]T)(b \cdot \psi)
\]
\[
+ ([x_1 + x_2, x_1]T)(\psi \cdot b) + O(b),
\]
as $b \to 0$ and for all $\psi$ in the ambient $C^*$-algebra.

It has an exponential version:

**Proposition 3.4.** Let $T(y)$ be a Schwartz function on $\mathbb{R}^+$, denote
\[
(3.13) \quad D(T)(y_1, y_2) = y_1 [y_1 y_2, y_1] T = \frac{T(y_1 y_2) - T(y_1)}{y_2 - 1}.
\]
Then
\[
(3.14) \quad \delta_a(T(y))(\nabla k) = k^{-1} \left[ y_1^{-1} D(T)(y_1, y_2) \right] \cdot ((\nabla k) \delta_a(k))
\]
\[
- k^{-1} D(T)(y_2, y_1) \cdot (\delta_a(k) \nabla k).
\]
Same identity holds when $\delta_a$ is replaced by the covariant differential $\nabla$.

**Proof.** Denote $\tilde{T}(x) = T(e^x)$, then
\[
\delta_a(T(y)) = \frac{d}{d\varepsilon} \bigg|_{\varepsilon = 0} \tilde{T}(\text{ad}_{h+\varepsilon a}).
\]
Eq. (3.14) follows from (3.12) after some substitutions. □

Now we are ready to compute the commutator of the covariant differential $\nabla$ and the modular action $T(y)$.

**Lemma 3.5.** Keep the notations.
\[
\nabla [T(y)(\nabla k)] - T(y)[\nabla^2 k] = k^{-1} \left[ y_1^{-1} D(T)(y_1, y_2) - D(T)(y_2, y_1) \right] (\nabla k \nabla k),
\]
where the function $D(T)$ is defined in Prop. 3.4.

**Proof.** Use the Leibniz property,
\[
\nabla [T(y)(\nabla k)] = \nabla [T(y)](\nabla k) + T(y)(\nabla^2 k),
\]
where the first term $\nabla [T(y)](\nabla k)$ has been computed in (3.14). □
Now we are fully prepared for computing the variation $\delta_a[k^j T(y)(\nabla k) \cdot (\nabla k)]$, which consists of four parts by the Leibniz property:
\[
\varphi_0 \left( \delta_a[k^j] T(y)(\nabla k) \cdot (\nabla k) \right) + \varphi_0 \left( k^j \delta_a[T(y)](\nabla k) \cdot (\nabla k) \right) \\
+ \varphi_0 \left( k^j T(y)(\delta_a \nabla k) \right) \cdot (\nabla k) + \varphi_0 \left( k^j T(y)(\nabla k) \cdot (\delta_a \nabla k) \right).
\]

We split the calculation into four lemmas and summarize the result at the end.

**Lemma 3.6.** *Keep the notations.*
\[
(3.15) \quad \varphi_0 \left( \delta_a(k^j) T(y)(\nabla k) \cdot (\nabla k) \right) = \varphi_0 \left( \delta_a(k) k^{j-1} (\nabla k) T(y)(\nabla k) \right) \]
\[
=: \varphi_0 \left( \delta_a(k) k^{j-1} \Pi(1)(y_1, y_2)(\nabla k) \cdot (\nabla k) \right),
\]
where
\[
\Pi(1)(T)(y_1, y_2) = [1, y_1^{-1} y_2^{-1}] \delta_a(y_1 y_2)^{j-1} T(y_1).
\]

*Proof.* Apply \((3.14)\) onto $\delta_a(k^j)$, we have
\[
\varphi_0 \left( \delta_a[k^j] T(y)(\nabla k) \cdot (\nabla k) \right) = \varphi_0 \left( y^{j-1} y^{j-1} (\nabla k) T(y)(\nabla k) \right).
\]
Then Eq. \((3.15)\) follows from integration by parts: \((3.8)\). \qed

**Lemma 3.7.** *Keep notations.* We have
\[
\varphi_0 \left( k^j \delta_a [T(y)](\nabla k) \cdot (\nabla k) \right) = \varphi_0 \left( \delta_a(k) k^{j-1} \Pi(2)(T; j)(y_1, y_2)(\nabla k) \cdot (\nabla k) \right),
\]
with
\[
\Pi(2)(T; j)(y_1, y_2) = y_2^{-1} y_1^{-1} D(T)(y_2, y_1^{-1} y_2^{-1}) - (y_1 y_2)^{j-1} D(T)(y_1, y_1^{-1} y_2^{-1}).
\]
where $D(T)$ is defined in Prop. \(3.4\).

*Proof.* We substitute $\delta_a[T(y)](\nabla k)$ for the right hand side of \((3.14)\):
\[
\varphi_0 \left( k^j \delta_a [T(y)](\nabla k) \cdot (\nabla k) \right) = \varphi_0 \left( k^{j-1} \left[ y_1^{-1} D(T)(y_1, y_2) \right](\nabla k \cdot \delta_a(k)) \cdot (\nabla k) \right) \\
- \varphi_0 \left( k^{j-1} D(T)(y_2, y_1)(\delta_a(k) \nabla k) \cdot (\nabla k) \right).
\]
It remains to bring $\delta_a(k)$ in front and move the modular operators onto $\nabla k$ by the integrations by parts identities \((3.7)\) and \((3.8)\). For the first term:
\[
\varphi_0 \left( k^{j-1} \left[ y_1^{-1} D(T)(y_1, y_2) \right](\nabla k \cdot \delta_a(k)) \cdot (\nabla k) \right) \\
= \varphi_0 \left( \left[ (\nabla k \cdot k^{j-1}) \left[ y_1^{-1} D(T)(y_1, y_2) \right](\nabla k \cdot \delta_a(k)) \right] \right) \\
= \varphi_0 \left( \left[ y_2^{-1} D(T)(y_2, y_1^{-1} y_2^{-1}) \right] \left[ (\nabla k \cdot k^{j-1}) \cdot (\nabla k \cdot \delta_a(k)) \right] \right) \\
= \varphi_0 \left( \delta_a(k) k^{j-1} \left[ y_2^{-1} y_1^{-1} D(T)(y_2, y_1^{-1} y_2^{-1}) \right](\nabla k \cdot (\nabla k)) \right).
\]
For the second term:

\[
\varphi_0 \left( k^{j-1} D(T(y_2, y_1)) (\delta_a(k) \nabla k) \cdot \nabla k \right)
= \varphi_0 \left( \delta_a(k) D(T(y_1, y_1^{-1} y_2^{-1}) \left[ \nabla k \cdot (\nabla k \cdot k^{j-1}) \right] \right)
= \varphi_0 \left( \delta_a(k) k^{j-1} \left( (y_1 y_2)^{j-1} D(T(y_1, y_1^{-1} y_2^{-1})) \right) (\nabla k \cdot \nabla k) \right).
\]

\[\square\]

**Lemma 3.8.** Keep notations,

\[
\varphi_0 \left( k^{j} T(y) (\nabla k) \cdot \delta_a(\nabla k) \right) = \varphi_0 \left( \delta_a(k) k^j I^{(2)}(T)(y)(\nabla^2 k) \right) + \varphi_0 \left( \delta_a(k) k^{j-1} I^{(4)}(T)(y_1, y_2)(\nabla k \cdot \nabla k) \right),
\]

while

\[
I^{(1)}(T)(y) = -T(y),
\]

\[
I^{(3)}(T)(y_1, y_2) = -y_1^{-1} D(T(y_1, y_2) + D(T(y_2, y_1))
- ([1, y_1] id^j) T(y_2).
\]

The proof of Lemma 3.8 and Lemma 3.9 are quite similar. Thus we only prove the more complicated one (Lemma 3.9) as an example.

**Lemma 3.9.** Keep notations. Set \( \mathcal{I}(T, j)(y) = y^j T(y^{-1}) \) with \( j \in \mathbb{R} \).

\[
\varphi_0 \left( k^{j} T(y) (\delta_a(\nabla k)) \cdot \nabla k \right) = \varphi_0 \left( \delta_a(k) k^j I^{(1)}(T)(y)(\nabla^2 k) \right) + \varphi_0 \left( \delta_a(k) k^{j-1} I^{(3)}(T)(y_1, y_2)(\nabla k \cdot \nabla k) \right),
\]

while

\[
I^{(2)}(T)(y) = -\mathcal{I}(T, j)(y),
\]

\[
I^{(4)}(T)(y_1, y_2) = I^{(3)}(\mathcal{I}(T, j))(y_1, y_2)
= -y_1^{-1} D(\mathcal{I}(T, j))(y_1, y_2) + D(\mathcal{I}(T, j))(y_2, y_1)
- ([1, y_1] id^j) \mathcal{I}(T, j)(y_2).
\]

**Proof.** Use the trace property and the fact that \( \delta_a \) commutes with \( \nabla \):

\[
\varphi_0 \left( k^{j} T(y) (\delta_a(\nabla k)) \cdot \nabla k \right)
= - \varphi_0 \left( \delta_a(k) \nabla \left[ T(y^{-1})(\nabla k) k^j \right] \right)
= - \varphi_0 \left( \delta_a(k) \nabla \left[ k^j (y^j T(y^{-1}))(\nabla k) \right] \right),
\]

in which,

\[
\nabla \left[ k^j (y^j T(y^{-1}))(\nabla k) \right] = (\nabla k^j) \cdot (y^j T(y^{-1}))(\nabla k) + k^j \nabla \left[ (y^j T(y^{-1}))(\nabla k) \right].
\]

For the first term,

\[
(\nabla k^j) \cdot (y^j T(y^{-1}))(\nabla k) = k^{j-1}([1, y] z^j)(\nabla k) \cdot (y^j T(y^{-1}))(\nabla k)
= k^{j-1}([1, y_1] z^j)(y_2^j T(y_2^{-1}))(\nabla k \cdot \nabla k)
\]
For the second one, we denote $\tilde{T}(y) = y^j T(y^{-1})$

\[
k^j \nabla \left[ \tilde{T}(y)(\nabla k) \right] = k^j \tilde{T}(y)(\nabla^2 k) + k^{j-1} \left[ y_1^{-1} D(\tilde{T}) \right] (y_1, y_2)(\nabla k \cdot \nabla k) - k^{j-1} D(\tilde{T})(y_2, y_1)(\nabla k \cdot \nabla k).
\]

We group the four lemmas above into our main theorem of this section.

**Theorem 3.10.** For $j \in \mathbb{R}$, let $I^{(\alpha)}(T)$, $\alpha = 1, 2$ and $\Pi^{(\beta)}(T)$, $\beta = 1, 2, 3, 4$, be the operations on the function $T$ defined in the previous lemmas, see also Eq. (4.14), then we have the following variation formula for the local expression:

\[
\delta_a \varphi_0 \left( k^j T(y)(\nabla k) \cdot \nabla k \right) = \varphi_0 \left( \delta_a(k) k^j (I^{(1)} + I^{(2)})(T; j)(y)(\nabla^2 k) \right) + \varphi_0 \left( \delta_a(k) k^{j-1} (\sum_{\beta=1}^4 \Pi^{(\beta)})(T; j)(y_1, y_2)(\nabla k \nabla k) \right).
\]

3.3. **Examples.** The operations $\sum_{\alpha=1}^2 I^{\alpha}(T; j)$ and $\sum_{\beta=1}^4 \Pi^{\beta}(T; j)$ are quite complicated. In fact, $\sum_{\beta=1}^4 \Pi^{\beta}(T; j)$ is expanded to thousands of terms when performing simplification in Mathematica. More conceptual understanding will be investigated in future papers. As we shall see in later computation (such as Eq. (4.22)), typical terms of $T(z)$ are of the form $2 F_1(a, 1; c; 1 - z)$. Let us compute part of $I(T; j)$ and $\Pi(T; j)$ in this case. There two basic operations on $T(z)$ in the variation. One is $\mathcal{I}(T; j)$, which arises from integration by parts. The other, $D(T; j)$ in (3.13), measures the commutator of the classical covariant differential and the modular action.

Let us start with the one variable case, which is related to the Pfaff transformations $A.10$.

**Lemma 3.11.** Let

\[
T(u) = K_{a,b}(u; m) = \frac{\Gamma(\tilde{d}_m)}{\Gamma(a + b)} 2 F_1(\tilde{d}_m, b; a + b; 1 - u),
\]

where $\tilde{d}_m = a + b - 2 + m/2$. Then

\[
\mathcal{I}(T; j)(u) := u^j T(u^{-1}) = u^{j+b} K_{b,a}(u; m).
\]

**Proof.** Let $z = 1 - u^{-1}$ in $A.10$, we obtain

\[
2 F_1(\tilde{d}_m, b; a + b; 1 - u^{-1}) = u^b 2 F_1(\tilde{d}_m, a; a + b; 1 - u),
\]

and (3.16) follows immediately. □

For the two variable situation, we narrow our attention to functions of the form $2 F_1(a, 1; b; 1 - z)$.

**Lemma 3.12.** For $T(z) = 2 F_1(a, 1; b; 1 - z)$ where $a \in \mathbb{C}$ and $b \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$,

\[
[s, t](z T(z)) = [1 - t, 1 - s](z 2 F_1(a, 1; c; z)) + [s, t](z 2 F_1(a, 1; c; 1 - z))
\]

\[
= F_1(a, 1; c; 1 - t, 1 - s) + [s, t](z 2 F_1(a, 1; c; 1 - z))
\]
Proof. We abbreviate $2F_1(a, 1; b; z)$ to $F(z)$ in the calculation:

$$[s, t](zT(z)) = \frac{sF(1 - s) - tF(1 - t)}{s - t} = \frac{(s - 1)F(1 - s) - (t - 1)F(1 - t) + F(1 - s) - F(1 - t)}{s - t} = [1 - t, 1 - s](z_2F_1(a, 1; c; z)) + [s, t](2F_1(a, 1; c; 1 - z)).$$

The first term above gives rise to the Appell $F$ function appeared in (3.17) by Proposition 2.8.

In contrast to Lemma 3.1, we use a different Pfaff transformation in (A.10) for $2F_1(a, 1; c; 1 - z^{-1})$:

$$2F_1(a, 1; c; 1 - z^{-1}) = z_2F_1(c - a, 1; c; 1 - z).$$

Notice that the function $2F_1(c - a, 1; c; 1 - z^{-1})$ does not belong to the family $H_{a,b,c}(u,v;m)$.

Lemma 3.13. Let $T(z) = 2F_1(a, 1; c; 1 - z)$ where $a \in \mathbb{C}$ and $b \in \mathbb{C} \setminus \mathbb{Z}_{<0}$. For fixed $j \in \mathbb{R}$, denote $\tilde{T}(z) := T(T; j) = z^jT(1/z)$, we have

$$[s, t](z\tilde{T}(z)) = t([s, t](z^{j+1}))2F_1(c - a, 1; c; 1 - t)$$

$$s^{j+1}F_1(c - a, 1; 1; c; 1 - t, 1 - s) + s^{j+1}[s, t](2F_1(c - a, 1; c; 1 - z))$$

Proof. According to (3.18), $\tilde{T}$ is a product

$$\tilde{T}(z) = z^j \cdot z_2F_1(c - a, 1; c; 1 - z).$$

Use the Leibniz property for divided differences:

$$[s, t](z\tilde{T}(z)) = ([s, t](z^{j+1}) \cdot t_2F_1(c - a, 1; c; 1 - t)$$

$$+ s^{j+1} [[s, t](z_2F_1(c - a, 1; c; 1 - z))].$$

Lemma 3.14. For $T(z) = 2F_1(a, 1; c; 1 - z)$ where $a \in \mathbb{C}$ and $b \in \mathbb{C} \setminus \mathbb{Z}_{<0},$

$$D(T)(y_1, y_2) = F_1(a, 1; 1, c; 1 - y_1y_2, 1 - y_1)$$

$$+ [y_1, y_1y_2](2F_1(a, 1; c; 1 - z)) - 2F_1(a, 1; c; 1 - y_1y_2).$$

Similarly, for $\tilde{T}(z) = z^jT(1/z) = z^j_2F_1(a, 1; c; 1 - z^{-1}),$

$$D(\tilde{T})(y_1, y_2) = y_1y_2 \left([y_1, y_1y_2](z^{j+1}) \cdot 2F_1(c - a, 1; c; 1 - y_1y_2)$$

$$+ y_1^{j+1}F_1(c - a, 1; 1; c; 1 - y_1y_2, 1 - y_1)$$

$$+ y_1^{j+1}[y_1, y_1y_2](2F_1(c - a, 1; c; 1 - z))$$

$$- (y_1y_2)^{j+1}2F_1(c - a, 1; c; 1 - y_1y_2).$$

□
Proof. Recall that
\[ D(T)(y_1, y_2) = [y_1, y_1y_2](T) = [y_1, y_1y_2](zT(z)) - T(y_1y_2), \]
where the first term \([y_1, y_1y_2](zT(z))\) has been computed in (3.17). \(\square\)

We have another version for Eq. (3.21) as below.

**Lemma 3.15.** For
\[ T(z) = K_{a,1}(z;m) = \frac{\Gamma(a - 1 + m/2)}{a + 1} \cdot {}_2F_1(a - 1 + m/2, 1, a + 1; 1 - z), \]
we have
\[ D(T)(y_1, y_2) = H_{a-1,1}(y_1, y_2) + [y_1, y_1y_2](K_{a,1}(z;m)) - K_{a,1}(y;m). \]

4. Variation of the Einstein-Hilbert action

It has been shown in [Liu15, Liu17] that the spectral functions appeared in the modular curve do not depend on the underlying manifold. Therefore we shall focus only on smooth noncommutative \(m\)-tori: \(C^\infty(\mathbb{T}^m_\theta)\), with the flat Euclidean metric as the background metric. We refer the construction of the algebra \(C^\infty(\mathbb{T}^m_\theta)\) to [Rie90, and [CL01] as a special case of toric noncommutative manifolds.

For the conformal perturbation, we consider the following Laplacian \(\Delta_\varphi = k^{1/2} \Delta k^{1/2}\), which is the even part in the twisted spectral triple studied in [CM14]. The associated modular curvature \(R_{\Delta_\varphi}\) is the functional density of the second heat coefficient (cf. (1.1)):
\[ V_2(a, \Delta_\varphi) = \varphi_0(aR_{\Delta_\varphi}), \quad \forall a \in C^\infty(\mathbb{T}^m_\theta), \]
here \(\varphi\) denotes the state: \(a \in C^\infty(\mathbb{T}^m_\theta) \mapsto \varphi_0(ak)\), which is a twisted trace via the modular operator \(y\):
\begin{align}
(4.1) \quad \varphi(ab) &= \varphi_0(abk) = \varphi_0(bka)\varphi_0(ky(b)a) \\
(4.2) \quad &= \varphi_0(y(b)ak) = \varphi(y(b)a).
\end{align}

We remind the reader again that \(k = e^h\) is a Weyl factor with \(h = h^* \in C^\infty(\mathbb{T}^m_\theta)\) self-adjoint. Bold letters \(x\) and \(y\) stand for the modular derivation and modular operator respectively. Functional calculus, such as \(K(y)(\rho)\) and \(H(y_1, y_2)(\rho_1 \cdot \rho_2)\) (in Eq. (4.3)), was defined in [3.4].

Local expression of \(R_{\Delta_\varphi}\) was first computed in [CM14, FK13]. Later, computation was simplified and generalized to all toric noncommutative manifolds by the author in [Liu15]. For self-containedness, we sketch the computation in section 5 following [Liu15].

**Theorem 4.1.** Upto a constant factor \(\text{Vol}(S^{m-2})/2\), \(R_{\Delta_\varphi}\) is of the form:
\begin{align}
(4.3) \quad R_{\Delta_\varphi} &= k^{j_m} K_{\Delta_\varphi}(y)(\nabla^2k) \cdot g^{-1} + k^{j_m-1} H_{\Delta_\varphi}(y_1, y_2)(\nabla k \nabla k) \cdot g^{-1}, \\
\end{align}
where \(j_m = -m/2\) and
\begin{align}
(4.4) \quad K_{\Delta_\varphi}(y;m) &= \sqrt{y}K_{\Delta_k}(y;m), \\
(4.5) \quad H_{\Delta_\varphi}(y_1, y_2;m) &= \sqrt{y_1y_2}H_{\Delta_k}(y_1, y_2;m).
\end{align}
Proof. It is easier to deal with $R_{\Delta_k}$ first, where $\Delta_k = k\Delta$, whose calculation is put in section 5. Based on Theorem 5.2 the result follows quickly from the relation between $\Delta_k$ and $\Delta_\varphi$: $\Delta_\varphi = k^{1/2}\Delta k^{-1/2} = k^{-1/2}\Delta k^{1/2}$. Same argument as in [Lin17, Eq. (3.19)] shows that

$$R_{\Delta_\varphi} = k^{-1/2}R_{\Delta_k}k^{1/2} = y^{1/2}(R_{\Delta_k}).$$

The functional relations (4.4) and (4.5) follows immediately. \(\square\)

4.1. Functional relations in Higher dimensions. After obtaining the explicit expression of $R_{\Delta_\varphi}$, it is natural to consider related variation problems to the corresponding Einstein-Hilbert action:

$$F_{EH}(k) = V_2(1, \Delta_\varphi) = \varphi_0(R_{\Delta_\varphi}),$$

as a functional in $k$, or equivalently, a functional on the conformal class of metrics defined by the flat metric. Notice that, due to (4.9), $F_{EH}$ is a constant functional in dimension two. Therefore we always assume that the dimension of the underlying manifold is greater or equal than 2.

**Proposition 4.2.** The Einstein-Hilbert action is given by the following local formula:

$$F_{EH}(k) = \varphi_0 \left( k^{j_m - 1}T_{\Delta_\varphi}(y)(\nabla k)(\nabla k) \cdot g^{-1} \right),$$

where the function $T_{\Delta_\varphi}$ is determined by $K_{\Delta_\varphi}$ and $H_{\Delta_\varphi}$,

$$T_{\Delta_\varphi}(y) = -K_{\Delta_\varphi}(1)\frac{y^{j_m} - 1}{y - 1} + H_{\Delta_\varphi}(y, y^{-1}).$$

**Remark.** Observe that:

$$K_{\Delta_\varphi}(1) = K_{\Delta_k}(1), \quad H_{\Delta_\varphi}(y, y^{-1}) = H_{\Delta_k}(y, y^{-1}).$$

**Proof.** Since $\varphi_0$ is a trace, we have in general, $\varphi_0(k^{j_m} K(y)(\rho)) = K(1)\varphi_0(k^{j_m} \rho)$. In particular,

$$\varphi_0 \left( k^{j_m} K_{\Delta_\varphi}(y)(\nabla^2 k) \right) = K_{\Delta_\varphi}(1)\varphi_0 \left( k^{j_m} \nabla^2 k \right)
\quad = K_{\Delta_\varphi}(1)\varphi_0 \left( k^{j_m - 1}\frac{y^{j_m} - 1}{y - 1} (\nabla k) \cdot (\nabla k) \right),$$

here we have used Lemma 3.2 for the second “=” sign. This explains the first term in (4.6). While the second term follows immediately from (3.6):

$$\varphi_0 \left( k^{j_m - 1}H_{\Delta_\varphi}(y_1, y_2)(\nabla k \nabla k) \right) = \varphi_0 \left( [H_{\Delta_\varphi}(y, y^{-1})(\nabla k)] \cdot \nabla k \right).$$

The setting of the variation is identical to the one in the previous section. For $\varepsilon > 0$ and another self-adjoint element $a = a^* \in C^\infty(T^m_\theta)$, we consider the family $k_\varepsilon := k(\varepsilon) = e^{b + \varepsilon a}$ and study the variation $\delta_a := d/d\varepsilon|_{\varepsilon=0}$.  

**Definition 4.1.** Consider the Einstein-Hilbert functional near a fixed Weyl factor $k$. Notice that $a \mapsto \delta_a F_{EH}(k)$ is a linear functional in $a$, we define the gradient of grad $F_{EH}(k)$ (at $k$) to be the unique element in $C^\infty(T^m_\theta)$ with the property that

$$\delta_a F_{EH}(k) = \varphi_0(a \grad F_{EH}).$$
Let us start the computation of \( \delta_a F_{\text{EH}}(k) \) with

\[
\delta_a(\Delta_{\phi}) = \delta_a(k^{1/2} \Delta k^{1/2}) = \delta_a(k^{1/2}) \Delta k^{1/2} + k^{1/2} \Delta \delta_a(k^{1/2}) 
\]

\[(4.8)\]

\[
= (\delta_a(k^{1/2}) k^{-1/2}) \Delta_{\phi} + \Delta_{\phi}(k^{-1/2} \delta_a(k^{1/2})).
\]

According to Duhamel’s formula, for \( t > 0 \),

\[
\delta_a \text{ Tr } \left( e^{-t \Delta_{\phi}} \right) = -t \text{ Tr } \left( \delta_a(\Delta_{\phi}) e^{-t \Delta_{\phi}} \right).
\]

With (4.8) and the trace property, we continue:

\[
\delta_a \text{ Tr } \left( e^{-t \Delta_{\phi}} \right) = -t \text{ Tr } \left( (\delta_a(k^{1/2}) k^{-1/2} + k^{-1/2} \delta_a(k^{1/2})) \Delta_{\phi} e^{-t \Delta_{\phi}} \right)
\]

\[
= t \frac{d}{dt} \text{ Tr } \left( (\delta_a(k^{1/2}) k^{-1/2} + k^{-1/2} \delta_a(k^{1/2})) e^{-t \Delta_{\phi}} \right)
\]

\[
= t \frac{d}{dt} \text{ Tr } \left( f_k e^{-t \Delta_{\phi}} \right).
\]

Use the fact that both \( \delta_a \) and \( \frac{d}{dt} \) can pass through the asymptotic expansion, we see that

\[
\sum_{j=0}^{\infty} \delta_a V_j(1, \Delta_{\phi}) t^{(j-m)/2} = \sum_{j=0}^{\infty} V_j(f_k, \Delta_{\phi}) t \frac{d}{dt} t^{(j-m)/2}.
\]

By equating the coefficients on two sides, we obtained:

**Proposition 4.3.** Let \( f_k = \delta_a(k^{1/2}) k^{-1/2} + k^{-1/2} \delta_a(k^{1/2}) \) defined as before, for \( j = 0, 1, 2, \ldots \),

\[(4.9)\]

\[
\delta_a V_j(1, \Delta_{\phi}) = \frac{j - m}{2} V_j(f_k, \Delta_{\phi}).
\]

We only interested the second coefficient in which \( j = 2 \).

**Theorem 4.4.** Keep the notations. We have the following variation formula for the Einstein-Hilbert action when the dimension \( m > 2 \):

\[
\delta_a F_{\text{EH}} = \varphi_0(\delta_a(k) \text{ grad}_k F_{\text{EH}}),
\]

in which

\[
\frac{2}{2 - m} \text{ grad}_k F_{\text{EH}}
\]

\[(4.10)\]

\[
= k^{j_m - 1} y^{-1/2} K_{\Delta_{\phi}}(y; m)(\nabla^2 k) \cdot g^{-1}
\]

\[
+ k^{j_m - 2} y_1^{-1/2} y_2^{-1/2} H_{\Delta_{\phi}}(y_1, y_2; m)(\nabla k \nabla k) \cdot g^{-1}
\]

\[
= k^{j_m - 1} K_{\Delta_k}(y; m)(\nabla^2 k) \cdot g^{-1} + k^{j_m - 2} H_{\Delta_k}(y_1, y_2; m)(\nabla k \nabla k) \cdot g^{-1},
\]

where \( j_m = -m/2 \).

**Proof.** Notice that

\[
f_k = k^{-1/2} (1 + y^{-1/2}) \delta_a(k^{1/2}) = k^{-1} (1 + y^{-1/2}) \frac{y^{1/2} - 1}{y - 1} (\delta_a(k))
\]

\[
= k^{-1} y^{-1/2}(\delta_a(k)).
\]


Hence

\[ V_2(f_k; \Delta_\varphi) = \varphi_0(f_k R_{\Delta_\varphi}) = \varphi_0(k^{-1}y^{-1/2}(\delta_\alpha(k))R_{\Delta_\varphi}) = \varphi_0(\delta_\alpha(k)y^{1/2}(R_{\Delta_\varphi})k^{-1}) = \varphi_0(\delta_\alpha(k)k^{-1}y^{-1/2}(R_{\Delta_\varphi})). \]

That is

\[ \text{grad}_k F_{EH} = k^{-1}y^{-1/2}(R_{\Delta_\varphi}) = k^{-1}R_{\Delta_\varphi}. \]

To see (4.10), one just needs to substitute \( R_{\Delta_\varphi} \) with its explicit formula stated in Theorem 4.1. \qed

On the other side, we have computed (in Theorem 3.10 with \( j = j_m = j_m - 1 = -m/2 - 1 \)) the variation using the local expression

\[ \delta_\alpha V_2(1, \Delta_\varphi) = \delta_\alpha \varphi_0 \left( k^{j_m-1}[\mathcal{T}_{\Delta_\varphi}(y)(\nabla k)](\nabla k) \right). \]

Namely,

\[ \text{grad}_k F_{EH} = k^{j_m-1}\left[ \sum_{l=1}^{2} I^{(l)}(T)(y; j_m)(\nabla^2 k) \cdot g^{-1} \right. \]

\[ + \left. k^{j_m-2}\sum_{l=1}^{4} \Pi^{(l)}(T)(y_1; y_2; j_m)(\nabla k \nabla k) \cdot g^{-1} \right]. \]

The functional relations follows from (4.11) and (4.10):

**Theorem 4.5.** Let \( T(u) := \mathcal{T}_{\Delta_\varphi}(u; m) \) be the spectral function that defines the Einstein-Hilbert action in Proposition 4.2 and \( K_{\Delta_\varphi}, H_{\Delta_\varphi} \) are the spectral functions for the modular curvature \( R_{\Delta_\varphi} \) in Theorem 5.2.

\[ (4.12) \quad K_{\Delta_\varphi}(u; m) = \left[ \sum_{l=1}^{2} I^{(l)}(T; j_m) \right](u), \quad H_{\Delta_\varphi}(u, v; m) = \left[ \sum_{l=1}^{4} \Pi^{(l)}(T; j_m) \right](u, v), \]

where the right hand sides are defined in Theorem 3.10 with \( j_m = -m/2 - 1 \).

We remind the reader the explicit construction of the right hand sides in Eq. (4.12).

For the one-variable part:

\[ (4.13) \quad (I^{(1)} + I^{(2)})(u; j) = -T(u) - u^jT(u^{-1}), \]

and the two-variable part:

\[ (4.14) \quad \begin{align*}
\Pi^{(1)}(T; j)(u, v) &= T(u) \frac{(uv)^j - 1}{uv - 1}, \\
\Pi^{(2)}(T; j)(u, v) &= \frac{u^{j-1}(T(u^{-1}) - T(v))}{u-1} - \frac{u(uv)^{j-1}(T(v^{-1}) - T(u))}{v-1}, \\
\Pi^{(3)}(T; j)(u, v) &= \frac{(u^j - 1)T(v)}{u-1} + \frac{v(T(uv) - T(v))}{uv-v} - \frac{T(uv) - T(u)}{uv-u}, \\
\Pi^{(4)}(T; j)(u, v) &= \frac{(u^j - 1)v^jT(u^{-1})}{u-1} - \frac{(uv)^jT((uv)^{-1}) - u^jT(u^{-1})}{uv-u} + \frac{v((uv)^jT((uv)^{-1}) - v^jT(v^{-1}))}{uv-v}. 
\end{align*} \]
4.2. Verification of the functional relations. In this section, we give explicit expression of the spectral functions $K_{\Delta_k}$, $H_{\Delta_k}$ and $T_{\Delta_k}$ as functions in $m$, the dimension of the ambient manifold. The upshot is that the functional relations in Theorem (4.5) can be verified by a computer algebra system once for all real value $m$ with $m > 2$. It is indeed a new theorem rather than a double validation, because for general real-value $m$ with $m > 2$, there are no geometric proofs so far.

We recall the main result in the last section,

\begin{align}
K_{\Delta_k}(y; m) &= \frac{4}{m} K_{3,1}(y; m) - K_{2,1}(y; m) \\
&= -\frac{1}{2} \Gamma (m/2 + 1) \ _2F_1 (m/2 + 1, 1, 3, 1 - s) \\
&\quad + \frac{2 \Gamma (m/2 + 2)}{3m} \ _2F_1 (m/2 + 2, 1, 4, 1 - s),
\end{align}

and

\begin{align}
H_{\Delta_k}(s, t; m) &= \left( \frac{4}{m} + 2 \right) H_{2,1,1} (y_1, y_2; m) - \frac{4 y_1 H_{2,2,1} (y_1, y_2; m)}{m} \\
&\quad - \frac{8 H_{3,1,1} (y_1, y_2; m)}{m} \\
&= \frac{1}{6m} 2(m + 2) \Gamma (m/2 + 2) \ _2F_1 (m/2 + 2; 1, 1; 4; 1 - st, 1 - s) \\
&\quad - \frac{1}{6m} \Gamma (m/2 + 3) \ _2F_1 (m/2 + 3; 1, 1; 5; 1 - st, 1 - s) \\
&\quad - \frac{1}{6m} \Gamma (m/2 + 3) \ _2F_1 (m/2 + 3; 1, 2; 5; 1 - st, 1 - s).
\end{align}

The evaluation of $F_1$ family appeared in (4.17) can be reduced to $\ _2F_1$ according to Proposition 2.8. On the other hand, Mathematica is able to provide symbolic evaluation in the parameter $a$ for $\ _2F_1 (a, b; c, z)$ when other two parameters $b, c$ are given numerically.

**Proposition 4.6.** For any $m = \dim M > 2$, the modular curvature $R_{\Delta_k}$ (see Theorem 5.2) has the following explicit modular functions

\begin{align}
K_{\Delta_k}(s; m) &= -\frac{8s^{-\frac{m}{2}} (m(s - 1) - 4s) s^{m/2} + s(m(s - 1) + 4)) \Gamma \left( \frac{m}{2} + 2 \right)}{(m - 2)m^2(m + 2)(s - 1)^3},
\end{align}

and

\begin{align}
H_{\Delta_k}(s, t; m) &= \frac{2}{m} (s - 1)^{-2}(t - 1)^{-2}(st - 1)^{-3} \Gamma (m/2 + 1) \\
&\quad \left[ 2s^{-m/2}(st - 1)^3 + 2(t - 1)^2 \left( \frac{1}{2} m(s - 1)(st - 1) + s(1 - 2s)t + 1 \right) \\
&\quad - 2(s - 1)^2t(st)^{-\frac{m}{2}} \left( \frac{1}{2} m(t - 1)(st - 1) + st^2 + t - 2 \right) \right].
\end{align}

Moreover, the symbolic evaluations still hold when $m \in (2, \infty)$ is a real parameter.
Now, let us move on to $T_{\Delta_k}$ in (4.6). First of all,

\begin{equation}
K_{\Delta_k}(1) = \frac{2\Gamma\left(\frac{m}{2} + 2\right)}{3m} - \frac{\Gamma\left(\frac{m}{2} + 1\right)}{2}.
\end{equation}

To compute $H_{\Delta_k}(y, y^{-1})$, one can, of course, compute the limit of the right hand side of (4.19) as $t \to s^{-1}$. Nevertheless, a better way is to use the reduction formula (A.25) of $F_1$, which implies that

\begin{equation}
H_{a,b,c}(u, u^{-1}; m) = K_{a+c,b}(u; m).
\end{equation}

**Proposition 4.7.** In terms of hypergeometric functions,

\begin{equation}
T_{\Delta_k}(y) = -K_{\Delta_k}(1)\left[\frac{s^{-m/2} - 1}{s - 1} + \frac{1}{3}(1 + \frac{2}{m})\Gamma\left(\frac{m}{2} + 2\right)\right.\nonumber
\begin{align*}
&\quad 2F_1\left(\frac{m}{2} + 2; 1; 4; 1 - s\right) \\
&\quad - \frac{1}{3m}\Gamma\left(\frac{m}{2} + 3\right)2F_1\left(\frac{m}{2} + 3; 1; 5; 1 - s\right) \\
&\quad - \frac{1}{6m}\Gamma\left(\frac{m}{2} + 3\right)s\left[2F_1\left(\frac{m}{2} + 3; 2; 5; 1 - s\right)\right].
\end{align*}
\end{equation}

For $m \in (2, \infty)$, the right hand side above has the following evaluation:

\begin{equation}
T_{\Delta_k}(s; m) = \Gamma(a - 1)6a^{-1}(s - 1)^{-4}s^{-a}\left[-3a^2(-1 + s)^2(-1 + s + s^a(1 + s))\right.\nonumber
\begin{align*}
&\quad + 2a((-1 + s)^3 + (-1 + s)s^a(-2 + s(7 + s))) \\
&\quad a^3(s - 1)^3(s^a + 1) - 12s^2(s^a - 1)\right], \quad \text{where } a = m/2.
\end{align*}
\end{equation}

**Theorem 4.8.** The explicit formulas for $K_{\Delta_k}(u; m)$, $H_{\Delta_k}(u, v; m)$ and $T_{\Delta_k}(u; m)$ defined in (4.18), (4.19) and (4.23) satisfy the functional relations stated in Theorem 4.5, even for real parameter $m$ as long as $m \in (2, \infty)$.

## 5. Symbolic computation

The explicit computation for the $b_2$ term in the resolvent approximation (see (1.4)) has been performed several times in different settings and via different methods: [LM16, CM14, FK15, FK13, FK12, CT11, Liu15, Liu17]. In this section, we shall only outline some key steps. As pointed out before, it is sufficient to carried out the computation for noncommutative $m$-tori with the flat background metric. Instead of Connes’s pseudo-differential calculus [Con80], we use the deformed Widom’s calculus with respect to a flat connection so that the full expression of the $b_2$ term can be recorded in a fews lines. Notations used in this section are highly compatible with those in [Liu15].

Let us quickly review the algorithm of constructing resolvent approximation for differential operators via pseudo-differential calculus. We assume the pseudo-differential operators considered in the sections are scalar operators, acting on smooth functions. Symbols of pseudo-differential operators form a subalgebra inside smooth functions on the cotangent bundle which admits a filtration. The associated graded algebra is called the algebra of complete symbols. Let $P$ and $Q$ be two pseudo-differential
operators with symbol $p$ and $q$ respectively. Then the symbol of their composition has a formal expansion

$$\sigma(PQ) = p \ast q \sim \sum_{j=0}^{\infty} a_j(p,q),$$

where each $a_j(\cdot, \cdot)$ is a bi-differential operator such that $a_j(p,q)$ reduce the total degree by $j$.

Consider a second order differential operator $P$ with symbol $\sigma(P) = p_2 + p_1 + p_0$, where $p_j$ homogeneous of order $j$ with $j = 0, 1, 2$. In most of pseudo-differential calculi, $a_0$ has no differential, here we assume that $a_0(p,q) = pq$. With the initial value $b_0 = (p_2 - \lambda)^{-1}$, one can recursively construct $b_j$:

\begin{align}
(5.1) & \quad b_1 = [a_0(b_0, p_1) + a_1(b_0, p_2)](-b_0) \\
(5.2) & \quad b_2 = [a_0(b_0, p_0) + a_0(b_1, p_1) + a_1(b_0, p_1) + a_1(b_1, p_2) + a_2(b_0, p_2)](-b_0).
\end{align}

The construction can be continued while the complexity of the right hand sides increases dramatically.

We now specialize on noncommutative tori (of arbitrary dimension $\geq 2$) from deformation point of view. Let $M = \mathbb{T}^m$, a $m$-dimensional torus with the flat Euclidean metric, and let $\nabla$ be the Levi-Civita connection. For two symbols $p = p(x, \xi)$ and $q = q(x, \xi)$, with $(x, \xi) \in T^*\mathbb{T}^m \cong \mathbb{T}^m \times \mathbb{R}^m$, we have

\begin{equation}
(5.3) \quad a_j(p,q) = \frac{(-i)^j}{j!} (D^j p) \cdot (\nabla^j q), \quad j = 0, 1, 2, \ldots,
\end{equation}

where $D = D_\xi$ is the vertical differential so that $(D^j p)$ is a contravariant $j$-tensor fields. Deformed contracting (what the $\ast$ stands for) with a covariant $j$-tensor field $\nabla^j q$ gives rise to a scalar tensor field which is also a symbol of order $\deg p + \deg q - j$.

For more explanations about the notations, see [Liu15, Liu17].

Consider the perturbed Laplacian $\Delta_k := k\Delta$, with the heat operator as a contour integral,

$$e^{-t\Delta_k} = \int_{C} e^{-t\lambda}(\Delta_k - \lambda)^{-1} d\lambda.$$ 

If we ignore the zero spectrum of $\Delta_j$, the contour $C$ can be chosen to be the imaginary axis from $-i\infty$ to $i\infty$.

Any finite sum $\sum_{j=0}^{N} b_j$ will give an approximation of the resolvent $(\Delta_k - \lambda)^{-1}$ which leads to the asymptotic expansion:

$$\text{Tr}(ae^{-t\lambda}) \sim_{t \searrow 0} \sum_{j=0}^{\infty} t^{(j-m)/2} V_j(a, \Delta_k) = \varphi_0(a\mathcal{R}_j), \quad \forall a \in C^\infty(\mathbb{T}^m_\theta),$$

where $\mathcal{R}_j$ is the functional density which can be explicitly computed by $b_j$:

\begin{equation}
(5.4) \quad \mathcal{R}_j = \int_{T^*_M} \frac{1}{2\pi i} \int_{C} e^{-\lambda b_j(\xi, \lambda)} d\lambda d\xi.
\end{equation}

The symbol of $\sigma(\Delta_k) = \sigma(k\Delta) = p_2 + p_1 + p_0$ is a degree two polynomial in $\xi$:

$$p_2 = k |\xi|^2, \quad p_1 = p_0 = 0.$$
As a function on the cotangent bundle, one can compute the vertical \( D \) and horizontal \( \nabla \) differential of \( p_2 \) as below:

\[
(Dp_2)_j = 2\xi_j, \quad (D^2p_2)_{jl} = 2\mathbf{1}_{jl},
\]

\[
(\nabla p_2)_j = (\nabla k)_j |\xi|^2, \quad (\nabla^2 p_2)_{jl} = (\nabla^2 k)_{jl} |\xi|^2,
\]

where \( \mathbf{1}_{jl} \) stands for the Kronecker-delta symbol. By substituting (5.3) and the derivatives of the symbols (5.5) into general formula (5.2), we obtain the expanded \( b_2 \) term as a function on the cotangent bundle,

\[
b_2(\xi) = 4r^2\xi_j\xi_l b^3_{0,k} k^2(\nabla^2 k)_{l,j} b_0 - r^2\mathbf{1}_{jl} b^3_{0,k} k^2(\nabla^2 k)_{l,j} b_0
\]

\[
+ 4r^4\xi_j\xi_l b^3_{0,k} k(\nabla^2 k)_{l,j} b_0 - 4r^4\xi_j\xi_l b^3_{0,k} k(\nabla^2 k)_{l,j} b_0 - 8r^4\xi_j\xi_l b^3_{0,k} k^2(\nabla^2 k)_{l,j} b_0,
\]

where the summation is taken automatically for repeated indices from 1 to \( m = \dim \mathbb{T}_m \). Let \( r = |\xi|^2 \), where the length is associated to the flat Riemannian metric. We will compute the integration over the fiber \( T^*_x \mathbb{T}_m \) using spherical coordinates:

\[
\int_{T^*_x \mathbb{T}_m} b_2(\xi) d\xi = \int_0^\infty b_2(r) r^{m-1} dr,
\]

with

\[
b_2(r) = \int_{|\xi|^2=1} b_2(\xi) ds,
\]

where \( ds \) the standard volume form for the unit sphere in \( \mathbb{R}^{m-1} \).

**Lemma 5.1.** Let \( ds \) be the standard volume form for the unit sphere in \( \mathbb{R}^{m-1} \), we have

\[
\operatorname{Vol}(S^{m-1}) = \int_{S^{m-1}} ds = \frac{2\pi^{m/2}}{\Gamma(m/2)},
\]

\[
\int_{S^{m-1}} \xi_j\xi_l ds = \frac{\pi^{m/2}}{\Gamma(1 + m/2)} \mathbf{1}_{jl} = \operatorname{Vol}(S^{m-1}) \frac{1}{m} \mathbf{1}_{jl}.
\]

**Proof.** Elementary calculus, left to the reader. \( \square \)

Upto an overall constant factor \( \operatorname{Vol}(S^{m-1}) \), \( b_2(r) = \int_{S^{m-1}} b_2(\xi) ds \) equals

\[
b_2(r) = - r^2 \mathbf{1}_{jl} b^2_{0,k} k(\nabla^2 k)_{l,j} b_0 + \frac{4r^4 \mathbf{1}_{jl} b^3_{0,k} k^2(\nabla^2 k)_{l,j} b_0}{m}
\]

\[
+ 2r^4 \mathbf{1}_{jl} b^3_{0,k} k(\nabla^2 k)_{l,j} b_0 - 4r^4 \mathbf{1}_{jl} b^3_{0,k} k(\nabla^2 k)_{l,j} b_0 + \frac{4r^6 \mathbf{1}_{jl} b^3_{0,k} k^2(\nabla^2 k)_{l,j} b_0}{m}
\]

\[
- \frac{8r^6 \mathbf{1}_{jl} b^3_{0,k} k^2(\nabla^2 k)_{l,j} b_0}{m} - \frac{4r^6 \mathbf{1}_{jl} b^3_{0,k} k^2(\nabla^2 k)_{l,j} b_0}{m}.
\]

The summation over \( i, j \) stands for contraction between contravariant and covariant tensors. To be precise:

\[
(\nabla^2 k) \cdot g^{-1} := \sum I_{ij} (\nabla^2 k)_{i,j} = \text{Tr Hess}(k) = - \Delta k,
\]

\[
(\nabla k \nabla k) \cdot g^{-1} := \sum I_{ij} (\nabla k)_i (\nabla k)_j = \langle \nabla k, \nabla k \rangle_g.
\]
where $g^{-1}$ stands for the metric on the cotangent bundle. We now give some examples on how to apply integration lemma developed at the beginning of the paper. Recall $b_0 = (kr^2 - \lambda)^{-1}$. We first move powers of $k$ in front, for instance,

$$r^2 1_{jl} b_0^2 k. (\nabla^2 k)_{l,j} b_0 = k \left( r^2 1_{jl} b_0^2 (\nabla^2 k)_{l,j} b_0 \right)$$

$$r^6 1_{jl} b_0^2 k. (\nabla k)_{l} b_0^2 k. (\nabla k)_{j} b_0 = k^2 y_1 \left( 1_{jl} b_0^2 (\nabla k)_{l} b_0^2 (\nabla k)_{j} b_0 \right)$$

where in the second line, $y_1$ is the conjugation operator acting on the factor $(\nabla k)_{l}$, which allows us to move the $k$ between $(\nabla k)_{l}$ and $(\nabla k)_{j}$ in front. Then we apply Proposition 2.2

$$\int_0^\infty \frac{1}{2\pi i} \int_C e^{-\lambda} r^2 1_{jl} b_0^2 k. (\nabla^2 k)_{l,j} b_0 d\lambda (r^{m-1} dr)$$

$$= k^{-(m/2+1)} K_{2,1}(y;m)(\nabla^2 k) \cdot g^{-1},$$

and Proposition 2.3

$$\int_0^\infty \frac{1}{2\pi i} \int_C e^{-\lambda} 1_{jl} b_0^2 (\nabla k)_{l} b_0^2 (\nabla k)_{j} b_0 d\lambda (r^{m-1} dr)$$

$$= k^{-(m/2+3)} H_{2,2,1}(y_1, y_2; m)(\nabla k \nabla k) \cdot g^{-1}.$$

The one variable spectral function is obtained by integrating the first two terms in (5.7):

$$K_{\Delta_k}(y;m) = \frac{4}{m} K_{3,1}(y;m) - K_{2,1}(y;m)$$

and for the two variable function, it comes from the last four terms in (5.7):

$$H_{\Delta_k}(y_1, y_2; m) = \left( \frac{4}{m} + 2 \right) H_{2,1,1}(y_1, y_2; m) - \frac{4y_1 H_{2,2,1}(y_1, y_2; m)}{m}$$

$$- \frac{8H_{3,1,1}(y_1, y_2; m)}{m}.$$

Let us sumerize the computation of the whole section as a theorem.

**Theorem 5.2.** For the perturbed Laplacian $\Delta_k = k\Delta$, a closed form of the functional $V_2(\cdot, \Delta_k)$ is given by

$$V_2(a, \Delta_k) = \varphi_0(aR_{\Delta_k}), \ \forall a \in C^\infty(T^n_a),$$

with $R_{\Delta_k} \in C^\infty(T^n_a)$. Upto an overall constant $\text{Vol}(S^{m-1})/2$,

$$R_{\Delta_k} = k^{jm} K_{\Delta_k}(y;m)(\nabla^2 k) \cdot g^{-1} + k^{jm-1} H_{\Delta_k}(y_1, y_2; m)(\nabla k \nabla k) \cdot g^{-1},$$

where $j_m = -m/2$, $y$ and $y_l$ with $l = 1, 2$ are the modular operators, see (2.7). Constructions with the metric $g^{-1}$ are explained in (5.8) and (5.9). Functions $K_{\Delta_k}$ and $H_{\Delta_k}$ are defined in (5.10) and (5.11) respectively.

In terms of hypergeometric functions,

$$K_{\Delta_k}(s;m) = -\frac{1}{2} \Gamma(m/2 + 1) 2F_1(m/2 + 1, 1, 3, 1 - s)$$

$$+ \frac{2 \Gamma(m/2 + 2)}{3m} 2F_1(m/2 + 2, 1, 4, 1 - s).$$
and

\[
H_{\Delta_k}(s, t; m) = \frac{1}{6m} 2(m + 2) \Gamma \left( \frac{m}{2} + 2 \right) F_1 \left( \frac{m}{2} + 2; 1, 1; 4; 1 - st, 1 - s \right)
\]

(5.14)

\[
- \frac{1}{6m} \Gamma \left( \frac{m}{2} + 3 \right) 2F_1 \left( \frac{m}{2} + 3; 1, 1; 5; 1 - st, 1 - s \right)
\]

\[
- \frac{1}{6m} \Gamma \left( \frac{m}{2} + 3 \right) sF_1 \left( \frac{m}{2} + 3; 1, 2; 5; 1 - st, 1 - s \right) .
\]

### Appendix A. Hypergeometric functions

Hypergeometric functions had been studied intensively in the nineteenth century. The pioneers include Gauss (1813), Ernst Kummer (1836) and Riemann (1857), etc. The two variable extension of the hypergeometric functions has four different types known as Appell’s $F_1$ to $F_4$ functions. In this appendix, we only collect some basic knowledge of $2F_1$ and $F_1$ functions that are related to our exploration of modular curvature. Most of the identities quoted in this section can be found in [EMOT53], [OLBC10], [App25] and [AdF26].

A.1. Gauss Hypergeometric functions. For $|z| < 1$, the (Gauss) hypergeometric function $2F_1(a, b; c; z)$ is represented by the hypergeometric series

\[
2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n z^n}{(c)_n n!}, \tag{A.1}
\]

where the coefficients are given by Pochhammer symbols:

\[
(q)_n = \frac{\Gamma(q + n)}{\Gamma(q)}, \tag{A.2}
\]

What we need in the paper is the following Euler type integral representation:

\[
2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 (1 - t)^{c-b-1} t^{b-1} (1 - zt)^{-a} dt. \tag{A.3}
\]

It is a solution of Euler’s hypergeometric differential equation

\[
z(1 - z) \frac{d^2 w}{dz^2} + (c - (a + b + 1)z) \frac{dw}{dz} - abw = 0 \tag{A.4}
\]

For $F := 2F_1(a, b; c; z)$, there are six associated contiguous functions obtained by applying $\pm 1$ on only one of the parameters $a, b$ and $c$. We abbreviate them as $F(a+), F(b+), F(c-)$, etc. Gauss showed that $F$ can be written as a linear combination of any two of its contiguous functions, which leads to 15 (6 choose 2) relations. They can be derived from the differential relations among the family

\[
\frac{d}{dz} (2F_1(a, b; c; z)) = \frac{ab}{c} 2F_1(a + 1, b + 1; c + 1; z), \tag{A.5}
\]
also

\begin{align}
F(a+) &= F + \frac{1}{a} z \frac{d}{dz} F, \\
F(b+) &= F + \frac{1}{b} z \frac{d}{dz} F, \\
F(c-) &= F + \frac{1}{c} z \frac{d}{dz} F. 
\end{align}

(A.6)

(A.7)

(A.8)

Combine with the second order ODE (A.4), we have

**Proposition A.1.** One can read all 15 relations among the contiguous functions by equating any two lines of the right hand side of Eq. (A.9):

\[
z \frac{dF}{dz} = \frac{a b}{c} F(a+, b+, c+) = a(F(a+) - F) = b(F(b+) - F) = (c - 1)(F(c-) - F) = \frac{(c - a)F(a-) + (a - c + bz)F}{1 - z} = \frac{(c - b)F(b-) + (b - c + az)F}{1 - z} = z \frac{(c - a)(c - b)F(c+) + c(a + b - c)F}{c(1 - z)}. 
\]

(A.9)

There are other type of symmetries among the hypergeometric family. For example, under fractional linear transformation

\[
\begin{align}
2F_1(a, b; c; z) &= (1 - z)^{-b} 2F_1(c - a, b; c; \frac{z}{z - 1}) \\
2F_1(a, b; c; z) &= (1 - z)^{-a} 2F_1(a, c - b; c; \frac{z}{z - 1}). 
\end{align}
\]

(A.10)

They are known as Pfaff transformations. Then the Euler transformation

\[
2F_1(a, b; c; z) = (1 - z)^{c-a-b} 2F_1(c - a, c - b; c; z) 
\]

(A.11)

follows quickly.

**Proposition A.2.** For \( p \in \mathbb{Z}_{>0} \),

\[
2F_1(a, 1; c; z) = \frac{(1 - c)_p}{(a - c + 1)_p} 2F_1(a, 1; c - p; z) + \frac{1}{z} \sum_{k=1}^{p} \frac{(1 - c)_k}{(a - c + 1)_k} \left( \frac{z - 1}{z} \right)^{k-1}.
\]

(A.12)

\[
\text{Since } 2F_1(a, 1; 1; z) = (1 - z)^{-a}, \text{ when } a \text{ and } c \text{ belong to the natural domain of the right hand side of (A.12), it provides a symbol evaluation for } 2F_1(a, 1; c; z).
\]
A.2. **Appell Hypergeometric functions.** The hypergeometric series (A.1) has a variety of generalizations to multi-variable cases. For the two-variable case, Appell introduced four types of series $F_1$ to $F_4$. So far, $F_1$ is directly related to the modular curvature functions. To deal with some symbolic computation, we need $F_2$ as a bridge.

\[
F_1(a; b, b'; c; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{m!n!(c)_{m+n}} x^m y^n \\
F_2(a; b, b'; c, c'; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{m!n!(c)_{m+n}(c')_{n+m}} x^m y^n,
\]

where Pochhammer symbols (A.2) is used in the coefficients. They have a double integral representation:

\[
\frac{\Gamma(b)\Gamma(b')\Gamma(c - b - b')}{\Gamma(c)} F_1(a; b, b', c; x, y) = \int_0^1 \int_0^{1-t} u^{b-1}v^{b'-1}(1 - u - v)^{c-b-b'-1}(1 - xu - yv)^{-a} dudv.
\]

\[
\frac{\Gamma(b)\Gamma(b')\Gamma(c - b)\Gamma(c' - b')}{\Gamma(c)\Gamma(c')} F_2(a; b, b'; c, c'; x, y) = \int_0^1 \int_0^1 u^{b-1}v^{b'-1}(1 - u)^{c-b-1}(1 - v)^{c'-b'-1}(1 - xu - yv)^{-a} dudv.
\]

Parallel to the differential system for Gaussian hypergeometric functions, we have a similar relations for rising the parameters via differential operators: $x\partial_x$ and $y\partial_y$. Denote by $F_1 := F_1(a; b, b'; c, x, y)$ and $F_1(a+) := F_1(a + 1; b, b'; c, x, y)$, same pattern applies to $F_1(b+), F_1(b'+)$ and $F_1(c+)$, then

\[
\partial_x F_1 = F_1(a+, b+, c+) \quad \text{and} \quad \partial_y F_1 = F_1(a+, b'+, c+),
\]

also

\[
F_1(a+) = a^{-1}(a + x\partial_x + y\partial_y)F_1 \\
F_1(b+) = b^{-1}(b + x\partial_x)F_1 \\
F_1(b') = b'^{-1}(b' + y\partial_y)F_1 \\
F_1 = c^{-1}(c + x\partial_x + y\partial_y)F_1(c+).
\]

For $F_1$ itself, it is a solution of the PDE system:

\[
[x(1-x)\partial_x^2 + y(1-x)\partial_x\partial_y + [c - (a + b + 1)]\partial_x - by\partial_y - ab] F_1 = 0
\]

\[
[y(1-y)\partial_y^2 + x(1-y)\partial_x\partial_y + [c - (a + b' + 1)]\partial_x - b'x\partial_y - ab'] F_1 = 0
\]

The $F_1$ family reduces to the hypergeometric functions in the situations:

\[
F_1(a; b, b'; c, 0, y) = _2F_1(a, b'; c; y), \\
F_1(a; b, b'; c, x, 0) = _2F_1(a, b; c; x).
\]
In addition,
\[ F_1(a; b, b'; c; x, x) = (1 - x)^{c-a-b-b'}_2F_1(c-a; c-b-b'; x) = 2F_1(a, b; b'; x), \]
(A.25)
\[ F_1(a; b, b'; b+b'; x, y) = (1 - y)^{-a}_2F_1(a, b; b+b'; x-y). \]

The $F_1$ family can be computed via the reduction formula, cf. [OLBC10, Sec. 16.16] or [EMOT53, Sec. 5.10, 5.11],
\[ F_1(a, b, b', c; x, y) = \left(\frac{x}{y}\right)^{b'}_2F_2(b+b'; a, b'; c, b+b'; x, y) \]
(A.26)
\[ = \left(\frac{y}{x}\right)^{b}_2F_2(b+b'; a, b'; c, b+b', y, 1-y/x). \]

**Proposition A.3** ([OSS05], Theorem 2). For $a \in \mathbb{C}$, $b \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, $p, q \in \mathbb{Z}_{\geq 0}$, $p < q$ and $|x| + |y| < 1$,
\[ F_2(q+1, a, p+1; b, p+2; x, y) \]
\[ = - \frac{p!}{q!(1-q)_p} \frac{p+1}{y^{p+1}}_2F_1(a-q-p; b; x) \]
(A.27)
\[ + \frac{p+1}{y^{p+1}} \sum_{k=0}^{p} \frac{(-1)^k}{(q-k)(1-y)^{q-k}} \binom{p}{k} \sum_{m=0}^{p-k} (-x)^m \binom{p-k}{m} \frac{(a)_m}{(b)_m} \]
\[ \cdot _2F_1(a+m, q-k; b+m; \frac{x}{1-y}). \]

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**References**

[AB01] Ivan G. Avramidi and Thomas Branson. Heat kernel asymptotics of operators with non-Laplace principal part. *Rev. Math. Phys.*, 13(7):847–890, 2001.

[AdF26] P. Appell and J.K. de Fériet. *Fonctions hypergéométriques et hypersphériques: polynomes d’Hermite*. Gauthier-Villars, 1926.

[App25] Paul Appell. Sur les fonctions hypergéométriques de plusieurs variables, les polynomes d’hermite et autres fonctions sphériques dans l’hyperespace. 1925.

[CDV02] Alain Connes and Michel Dubois-Violette. Noncommutative finite-dimensional manifolds. I. Spherical manifolds and related examples. *Comm. Math. Phys.*, 230(3):539–579, 2002.

[CDV08] Alain Connes and Michel Dubois-Violette. Noncommutative finite dimensional manifolds. II. Moduli space and structure of noncommutative 3-spheres. *Comm. Math. Phys.*, 281(1):23–127, 2008.

[CF16] A. Connes and F. Fathizadeh. The term $a_4$ in the heat kernel expansion of noncommutative tori. *ArXiv e-prints*, November 2016, 1611.09815.

[CL01] Alain Connes and Giovanni Landi. Noncommutative manifolds, the instanton algebra and isospectral deformations. *Comm. Math. Phys.*, 221(1):141–159, 2001.

[CM08] A. Connes and M. Marcolli. *Noncommutative geometry, quantum fields and motives*, volume 55. Amer Mathematical Society, 2008.

[CM14] Alain Connes and Henri Moscovici. Modular curvature for noncommutative two-tori. *J. Amer. Math. Soc.*, 27(3):639–684, 2014.
[Con80] A. Connes. C*-algèbres et géométrie différentielle. CR Acad. Sci. Paris Sér. AB, 290(13):A599–A604, 1980.

[Con94] Alain Connes. Noncommutative geometry. Academic Press, Inc., San Diego, CA, 1994.

[CT11] Alain Connes and Paula Tretkoff. The Gauss-Bonnet theorem for the noncommutative two torus. In Noncommutative geometry, arithmetic, and related topics, pages 141–158. Johns Hopkins Univ. Press, Baltimore, MD, 2011.

[EMOT53] Arthur Erdélyi, Wilhelm Magnus, Fritz Oberhettinger, and Francesco G. Tricomi. Higher transcendental functions. Vols. I, II. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953. Based, in part, on notes left by Harry Bateman.

[Fat15] Farzad Fathizadeh. On the scalar curvature for the noncommutative four torus. J. Math. Phys., 56(6):062303, 14, 2015.

[FK12] Farzad Fathizadeh and Masoud Khalkhali. The Gauss-Bonnet theorem for noncommutative two tori with a general conformal structure. J. Noncommut. Geom., 6(3):457–480, 2012.

[FK13] Farzad Fathizadeh and Masoud Khalkhali. Scalar curvature for the noncommutative two torus. J. Noncommut. Geom., 7(4):1145–1183, 2013.

[FK15] Farzad Fathizadeh and Masoud Khalkhali. Scalar curvature for noncommutative four-tori. J. Noncommut. Geom., 9(2):473–503, 2015.

[Gil75] Peter B. Gilkey. The spectral geometry of a Riemannian manifold. J. Differential Geometry, 10(4):601–618, 1975.

[Gus91] V. P. Gusynin. Asymptotics of the heat kernel for nonminimal differential operators. Ukraîn. Mat. Zh., 43(11):1541–1551, 1991.

[HK74] Akio Hattori and Toshihusa Kinura. On the Euler integral representations of hypergeometric functions in several variables. J. Math. Soc. Japan, 26:1–16, 1974.

[KMS16] M. Khalkhali, A. Moatadelro, and S. Sadeghi. A Scalar Curvature Formula For the Noncommutative 3-Torus. ArXiv e-prints, October 2016, 1610.04740.

[Lau93] G. Lauricella. Sulle funzioni ipergeometriche a piu variabili. Rendiconti del Circolo Matematico di Palermo, 7(1):111–158, Dec 1893.

[Les17] Matthias Lesch. Divided differences in noncommutative geometry: rearrangement lemma, functional calculus and expansational formula. J. Noncommut. Geom., 11(1):193–223, 2017.

[Liu15] Yang Liu. Modular curvature for toric noncommutative manifolds. to appear in J. Noncommut. Geom., 10 2015, 1510.04608.

[Liu17] Yang Liu. Scalar curvature in conformal geometry of connes–landi noncommutative manifolds. Journal of Geometry and Physics, 121:138 – 165, 2017.

[LM16] Matthias Lesch and Henri Moscovici. Modular Curve and Morita Equivalence. Geom. Funct. Anal., 26(3):818–873, 2016.

[OLBC10] Frank W. J. Olver, Daniel W. Lozier, Ronald F. Boisvert, and Charles W. Clark, editors. NIST handbook of mathematical functions. U.S. Department of Commerce, National Institute of Standards and Technology, Washington, DC; Cambridge University Press, Cambridge, 2010. With 1 CD-ROM (Windows, Macintosh and UNIX).

[OSS05] Sheldon B. Opps, Nasser Saad, and H. M. Srivastava. Some reduction and transformation formulas for the Appell hypergeometric function $F_2$. J. Math. Anal. Appl., 302(1):180–195, 2005.

[Rie90] Marc A. Rieffel. Noncommutative tori—a case study of noncommutative differentiable manifolds. In Geometric and topological invariants of elliptic operators (Brunswick, ME, 1988), volume 105 of Contemp. Math., pages 191–211. Amer. Math. Soc., Providence, RI, 1990.

[Rie93] Marc A. Rieffel. Deformation quantization for actions of $\mathbb{R}^d$. Mem. Amer. Math. Soc., 106(506):x+93, 1993.

[Yam60] Hidehiko Yamabe. On a deformation of riemannian structures on compact manifolds. Osaka Math. J., 12(1):21–37, 1960.
