Orbital stability property for coupled nonlinear Schrödinger equations

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Abstract

Orbital stability property for weakly coupled nonlinear Schrödinger equations is investigated. Different families of orbitally stable standing waves solutions will be found, generated by different classes of solutions of the associated elliptic problem. In particular, orbitally stable standing waves can be generated by least action solutions, but also by solutions with one trivial component whether or not they are ground states. Moreover, standing waves with components propagating with the same frequencies are orbitally stable if generated by vector solutions of a suitable single Schrödinger weakly coupled system, even if they are not ground states.

1 Introduction

We consider the following Cauchy problem for two coupled nonlinear Schrödinger equations

\[
\begin{align*}
    i\partial_t \phi_1 + \Delta \phi_1 + \left( |\phi_1|^2 + \beta |\phi_2|^p |\phi_1|^{p-2} \right) \phi_1 &= 0, \\
    i\partial_t \phi_2 + \Delta \phi_2 + \left( |\phi_2|^2 + \beta |\phi_1|^p |\phi_2|^{p-2} \right) \phi_2 &= 0, \\
    \phi_1(0, x) &= \phi_1^0(x), \quad \phi_2(0, x) = \phi_2^0(x),
\end{align*}
\]

where \( \Phi = (\phi_1, \phi_2) \) and \( \phi_i : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}, \phi_i^0 : \mathbb{R}^n \to \mathbb{C}, p > 1 \) and \( \beta \) is a real positive constant.

Coupled nonlinear Schrödinger equations appear in the study of many physical processes. For instance, such equations with cubic nonlinearity model the nonlinear interaction of two wave packets, optical pulse propagation in birefringent fibers or wavelength-division-multiplexed optical systems (see [16, 17], [1, 10] and the references therein).

A soliton or standing wave solution is a solution of the form \( \Phi(x, t) = (u_1(x)e^{i\omega_1 t}, u_2(x)e^{i\omega_2 t}) \) where \( U(x) = (u_1(x), u_2(x)) : \mathbb{R}^n \to \mathbb{C}^2 \) is a solution of the elliptic system

\[
\begin{align*}
    -\Delta u_1 + \omega_1 u_1 &= \left( |u_1|^{2p-2} + \beta |u_2|^p |u_1|^{p-2} \right) u_1, \\
    -\Delta u_2 + \omega_2 u_2 &= \left( |u_2|^{2p-2} + \beta |u_2|^p |u_1|^{p-2} \right) u_2.
\end{align*}
\]

Among all the standing waves we can distinguish between ground and bound states. A ground state corresponds to a least action solution \( U \) of \( \text{ellittico} \), while all the other critical points of the action functional give rise to bound states (or excited states) of \( \text{schr} \). A ground state generates a one-hump soliton of \( \text{schr} \) because it is nonnegative, radially symmetric and decays exponentially at infinity \( \Phi \). On the other hand, vector multi-hump solitons are of much interest in the applications, for

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example they have been observed in photorefractive crystals \[18\].

When investigating stability properties of a given set of solution, it is natural to take into account the rotation invariance of the problem, and this is done by the orbital stability. Roughly speaking, this means that if an initial datum \( \Phi^0 \) is close to a ground state \( U \) then all the orbit generated by \( \Phi^0 \) remains close to the soliton generated by \( U \) up to translations or phase rotations.

For the single Schrödinger equation it has been proved that the orbital stability property is enjoyed by standing waves raised by least action solutions. This result can be deduced from the two following facts (see \[6\] and Section 8 in \[5\]):

(a) every least action solution can be associated, by a bijective correspondence, to a minimum point of the energy constrained to the \( L^2 \) sphere with a suitable choice of the radius.

(b) Conservation laws and compactness properties of this minimization problem imply that the set of minimum points of the energy on this sphere manifold generates stable standing waves.

Moreover, in \[10, 26\] it is proved that every critical point of the action with Morse index larger than one give rise to instability. Taking into consideration the result of \[11\] the stability of the standing wave \( e^{i \omega t} z_\omega \) of the single Schrödinger equation holds if and only if \( z_\omega \) is the minimum point of the associated energy functional constrained to the \( L^2 \) sphere of radius \( \|z_\omega\|_{L^2}^2 \).

This and (a) are the reasons why ground states are the most desirable solution for the single Schrödinger equation.

With this situation in mind, large effort has been done in the last few years to find ground states of \( \ell \). In \[10, 26\] numerical arguments or analytical expansions have been employed to produce different families of solitons. The investigation has been improved by means of variational methods. In \[2, 3, 7, 12, 15, 25\] assumptions on the constant \( \beta \) are stated in order to distinguish between ground states with both nontrivial components (vector ground states) and ground states with one trivial component (scalar ground states). It has been discovered that there exists vector ground states for the constant \( \beta \) sufficiently large in dependence on the frequencies ratio, while if \( \beta \) is small least action solutions have necessarily one trivial component. Moreover, in \[2\] it is clarified the difference between scalar and vector positive solutions in dependence to different geometrical properties of the action functional. For \( \beta \) small the scalar ground states are critical points of the action functional with Morse index equal to one, while for \( \beta \) large these kind of solutions have larger Morse index. Since stable standing waves should be generated only by ground states, these results suggested the idea that stable standing waves should be given by scalar solutions for \( \beta \) small and by vector ground states for \( \beta \) large. This opinion is confirmed also in \[14\] where this topic has been studied for different evolution systems, and the orbital stability property is shown to be enjoyed by standing waves associated to solution of the corresponding elliptic system with Morse index equal to one.

For the cubic NLS systems in the one dimensional case, this subject has been recently studied in some interesting papers using numerical and analytical methods. In \[27\] it is conjectured, based on numerical evidence, that single-hump soliton are stable while multi-hump vector solitons are all linearly unstable and this is proved by numerical and analytical arguments in \[23\] for \( p = 2 \) in \( \ell \) and for any \( p \) for special families of multi-hump vector solitons. In \[22\] a stability criterion is found to study the stability property of some families of single-hump vector solitons. When tackling this matter for weakly coupled Schrödinger equations by means of variational methods, one has to take into account that the \( L^2 \) norms of the components are conserved separately (see \[11\]). So that we can consider different constrains on which minimize the energy. When we choose the sphere with respect of the \( L^2 \times L^2 \) norm, we obtain ground states, however we do not know whether or not they are scalar or vector solutions. Otherwise, we could try to minimize the energy constraining the \( L^2 \) norms separately, this approach will permit us to know in advance if we will find scalar or vector
solutions even if they may be not least action solutions. The first approach consists in solving the minimization problem

\begin{equation}
\mathcal{E}(u) = \inf_{M_p} \mathcal{E} \quad \text{where} \quad M_p = \left\{ U = (u_1, u_2) \in H^1 \times H^1 : \omega_1 \|u_1\|_{L^p}^p + \omega_2 \|u_2\|_{L^p}^p = \gamma \right\}
\end{equation}

and

\[ \mathcal{E}(U) = \mathcal{E}(u_1, u_2) = \frac{1}{2} \|\nabla u_1\|_{L^2}^2 + \frac{1}{2} \|\nabla u_2\|_{L^2}^2 - \frac{1}{2p} \left( \|u_1\|_{L^p}^p + \|u_2\|_{L^p}^p + 2\beta \|u_1 u_2\|_{L^p}^p \right), \]

for \(1 < p < 1 + 2/n\) in order to have global existence of \(u(t)\) (see \([20]\)).

For a suitable choice of \(\gamma\), we will find that this problem has a solution corresponding, in a bijective correspondence, to ground states of \(\text{mainstab}2\). Moreover, the set of the solutions generates stable standing waves (see Theorem \([12]\)). This conclusion is in accordance to the single equation case, since these solutions have Morse index equal to one.

When adopting the second approach, we are naturally lead to the minimization problem

\begin{equation}
\mathcal{E}(u) = \inf_{\mathcal{M}(\delta_1, \delta_2)} \mathcal{E} \quad \text{where} \quad \mathcal{M}(\delta_1, \delta_2) = \left\{ U = (u_1, u_2) \in H^1 \times H^1 : \|u_1\|_{L^p}^2 = \delta_1, \|u_2\|_{L^p}^2 = \delta_2 \right\}.
\end{equation}

When \(\delta_2\) (or \(\delta_1\)) is equal to zero we obtain as minimum point the couple \((z_0, 0)\) (or \((0, z_{\omega_2})\)) where \(z_{\omega_1}\) \((z_{\omega_2})\) is the unique positive solution of the first (second) equation in \([20]\). Recall that this solution is a ground state of \(\text{mainstab}3\) only for \(\beta\) small. However, we will show that they still produce orbitally stable standing waves for any \(\beta > 0\) (see Theorem \([20]\)). This result is in accordance with the conjecture in \([27]\). But they are in contrast with the expectation that only ground states should give rise to orbitally stable waves, since they have Morse index greater than one for \(\beta\) large. The case \(\delta_1 = \delta_2\) has been tackled in \([20]\) for \(p = 2, n = 1\) and \(\beta = 1\), and it is proved that the set of solutions of \(\text{mainstab}2\) give rise to orbitally stable solutions of \(\text{mainstab}3\). Here we will extend the result in \([20]\) for higher dimension and for every \(\beta > 0\) see Theorem \([12]\). Moreover, as in \([20]\), we will show that, for a suitable choice of \(\delta\) (and \(\delta_1 = \delta_2 = \delta\)) the set of solutions of \(\text{mainstab}2\) is given by

\[ \mathcal{B} = \left\{ \left( e^{i\theta_1 z_{\omega_1}^\beta}, e^{i\theta_2 z_{\omega_2}^\beta}(-y) \right), \theta_1, \theta_2 \in \mathbb{R}, y \in \mathbb{R}^n \right\} \]

where \(z_{\omega_1}^\beta\) is the unique positive solution of the problem

\[ \begin{cases} -\Delta u + \omega u = (1 + \beta)|u|^{2p-2}u & \text{in } \mathbb{R}^n, \\ u(x) \to 0 & \text{as } |x| \to \infty. \end{cases} \]

Moreover, assuming as in \([20]\) we will demonstrate that \(\mathcal{B}\) also characterizes the set of least action solution of \(\text{mainstab}2\) when \(\omega_1 = \omega_2\), and when one prescribes both of the components to be different from zero (see Theorem \([10]\)). So that our result provide a complete characterization of the set of solutions found in Theorem 1 in \([25]\) and in Theorem 2 in \([23]\) for \(\lambda_j = \omega_1\) for every \(j\). Let us stress again that the set \(\mathcal{B}\) is made of ground states only for \(\beta \geq 1\). Then, for \(\beta\) large we have at least two families of orbitally stable solution of \(\text{mainstab}3\), the ones generated by ground states, which we know have both nontrivial components and the ones produced by the scalar solution. We can reach the same conclusion for any \(\beta\) but for \(\omega_1 = \omega_2\); orbital stability is enjoyed by the standing waves generated by scalar solutions and by vector solution solutions of \(\text{mainstab}3\), the former are ground states for \(\beta\) small, the latter are ground states for \(\beta\) large. Unfortunately we cannot handle the case \(\delta_1 \neq \delta_2\), and to our knowledge the question of whether or not the set of solution of \(\text{mainstab}2\) gives rise of orbitally stable solutions for any \(\delta_1, \delta_2\) is open.
Finally, adapting the arguments in [5], we will also show an instability result in the supercritical case $p = 1 + 2/n$, for ground state solutions, scalar solutions and for the set $B$ (for $\omega_1 = \omega_2 = \omega$), as a consequence of blowing up in finite time. While, for the critical case the instability is produced by every solution of $(1.2)$. 

The paper is organized as follows. In Section 2 we state our main results. The definitions and preliminary results, preparatory to the proofs, are presented in section 3. In section 4 we give the proofs of our main results. A section of conclusion comments the results obtained.

2 Setting of the problem and main results

Our analysis will be carried out in the functional spaces $L^2 = L^2(\mathbb{R}^n, \mathbb{C}) \times L^2(\mathbb{R}^n, \mathbb{C})$ and $H^1 = H^1(\mathbb{R}^n, \mathbb{C}) \times H^1(\mathbb{R}^n, \mathbb{C})$. We recall that the inner product between $u, v \in \mathcal{C}$ is given by $u \cdot v = \Re(u \overline{v}) = 1/2(u^2 + v^2)$. Then for $\omega = (\omega_1, \omega_2), \omega_i \in \mathbb{R}, \omega_i > 0$, we can define an equivalent inner product in $L^2$ given by

$$(\Phi, \Psi) = \Re \int [\omega_1 \phi_1 \overline{\psi}_1 + \omega_2 \phi_2 \overline{\psi}_2], \quad \forall \Phi = (\phi_1, \phi_2), \Psi = (\psi_1, \psi_2)$$

and an equivalent norm

$$||\Phi||^2_{2, \omega} = ||\phi_1||^2_{2, \omega_1} + ||\phi_2||^2_{2, \omega_2}, \quad \text{where} \quad ||\phi_1||^2_{2, \omega_1} = \omega_1 ||\phi_1||^2_1 = \omega_1 \int \phi_1 \overline{\phi}_1$$

for $i = 1, 2$. It is known (see Remark 4.2.13 in [5]) that $(1.1)$ is locally well posed in time, for $p < n/(n-2)$ when $n > 2$ and for any $p$ for $n = 1, 2$, in the space $H^1$ endowed with the norm $||\Phi||^2_{H^1} = ||\nabla \Phi||^2_2 + ||\phi|\Phi||^2_{2, \omega}$ for every $\Phi = (\phi_1, \phi_2) \in H^1$. Moreover we set the $L^p$ norm as $||\Phi||^p_p = ||\phi_1||^p_p + ||\phi_2||^p_p$, for $p \in [1, +\infty)$. It is well known that the masses of the components of a solution and its total energy are preserved in time, that is the following conservation laws hold (see [5]):

mass

$$||\phi_1||^2_2 = ||\phi_1_0||^2_2, \quad ||\phi_2||^2_2 = ||\phi_2_0||^2_2, \quad \text{(2.1)}$$

energy

$$\mathcal{E}(\Phi(t)) = \frac{1}{2} ||\nabla \Phi(t)||^2_2 - F(\Phi(t)) = \frac{1}{2} ||\nabla \Phi_0||^2_2 - F(\Phi_0) = \mathcal{E}(0), \quad \text{(2.2)}$$

where

$$F(\Phi) = \frac{1}{2p} \left(||\Phi||^{2p}_{2p} + 2\beta ||\phi_1 \phi_2||^p_p\right). \quad \text{(2.3)}$$

In [5] it is proved that the solution of this Cauchy problem exists globally in time, under the assumption

$$p < 1 + \frac{2}{n}. \quad \text{(2.4)}$$

In order to study orbital stability properties, we will use the functional energy (see (2.2)) and the action

$$I(U) = \frac{1}{2} ||U||^2_{H^1} - F(U) = \mathcal{E}(U) + \frac{1}{2} ||U||^2_{2, \omega}. \quad \text{(2.5)}$$
Definition 2.1 We will say that a ground state solution \( U \) of \( H \) is a solution of the following minimization problem

\[
I(U) = m_N := \inf_N I(W) \quad \text{where} \quad N := \{W \in H^1 \setminus \{0\} : \langle I'(W), W \rangle = 0 \}.
\]

\( N \) is called in the literature Nehari manifold (see Section 8 in [5, 11]). Moreover, we will denote with \( \mathcal{G} \) the set of the ground state solutions.

Definition 2.2 We will say that a positive bound state solution \( U \) of \( H \) is a solution of the following minimization problem

\[
I(U) = m_2 := \inf_{N_2} I(W) \quad \text{where} \quad N_2 := \{W = (w_1, w_2) \in H^1 : w_1, w_2 \neq 0, \langle \partial_1 I(W), w_1 \rangle = \langle \partial_2 I(W), w_2 \rangle = 0 \},
\]

where \( \partial_1 I(W) (\partial_2 I(W)) \) is the partial derivative with respect to the first (second) component. \( N_2 \) is called in the literature Nehari set (see Section 8 in [5, 11]). Moreover, we will denote with \( \mathcal{B} \) the set of such bound state solutions.

It is well known (see Section 8 in [5, 11]) that all the solutions of the elliptic problem

\[
\begin{align*}
-\Delta u + \omega u &= (1 + \beta)|u|^{2p-2}u \quad \text{in} \ \mathbb{R}^n, \\
u(x) &\to 0 \quad \text{as} \ |x| \to \infty,
\end{align*}
\]

for \( \omega > 0 \) and \( \beta \geq 0 \), are given by \( \upsilon(x) = e^{ib \omega \theta}(x - y) \) where \( \theta \in \mathbb{R} \) \( y \in \mathbb{R}^n \) and \( \zeta_0^{\omega} \) is the unique least energy solution, where \( \zeta_0^{\omega} \) is the unique positive least energy solution in \( H^1(\mathbb{R}^n, \mathbb{R}) \) of \( -\Delta u + \omega u = (1 + \beta)|u|^{2p-2}u \) in \( \mathbb{R}^n \), \( u(x) \to 0 \) as \( |x| \to \infty \).

Definition 2.3 Problem \( H \) admits also scalar solutions \( U = (u_1, 0) \) (or \( (0, u_2) \)). We will denote with \( \mathcal{S} \) the set of such solutions. The uniqueness result in Section 8 for the single Schrödinger equation, gives us the following characterization for the set \( \mathcal{S} \).

\[
\mathcal{S} = \{e^{ib \omega \theta}(-y), \theta \in \mathbb{R}, y \in \mathbb{R}^n\} \cup \{(0, e^{ib \omega \theta}(-y)), \theta \in \mathbb{R}, y \in \mathbb{R}^n\}
\]

where \( \omega_0 \) is defined in [5, 11] (with \( \omega_1 = \omega \) or \( \omega_2 = \omega \)).

Remark 2.4 The results contained in Sections 8 and 9 show that, depending on the parameters \( \omega_1, \omega_2, \beta \), the set \( \mathcal{G} \) may coincide with either \( \mathcal{B} \) or \( \mathcal{S} \).

For \( \beta \) sufficiently large in dependence on \( \omega_1, \omega_2 \), \( \mathcal{G} = \mathcal{B} \) and the point in \( \mathcal{S} \) are scalar bound states solutions. While, \( \mathcal{G} = \mathcal{S} \) for \( \beta \) small.

In the particular case \( \omega_1 = \omega_2 = 1 \) ground states have both nontrivial components if and only if \( \beta \geq 1 \) (see Section 8 in [5, 11]), so that for \( \beta \geq 1 \) \( \mathcal{G} = \mathcal{B} \), while for \( \beta < 1 \) \( \mathcal{G} = \mathcal{S} \).

Let us recall the orbital stability property for a set of solutions \( \mathcal{F} \), introduced for the single equation case in [11].
Definition 2.5 A set $F \subseteq H^1$ of solutions of Problem (1.2) is orbitally stable if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\inf_{U \in F} \| \Psi^0 - U \|_{H^1} < \delta, \quad \text{then} \quad \sup_{t \geq 0} \inf_{V \in F} \| \Psi(t, \cdot) - V \|_{H^1} < \varepsilon,$$

where $\Psi$ is the global solution of (1.1) with initial datum $\Psi^0$.

Remark 2.6 We call the property in the previous definition orbital stability of $F$ because every element $(u_1, u_2)$ of $F$ generates an orbit given by the standing wave $(e^{i\omega_1 t} u_1, e^{i\omega_2 t} u_2)$.

Roughly speaking, a set $F$ is orbitally stable if any orbit generated from an initial datum $\Psi^0$ close to an element of $F$ remains close to $F$ uniformly with respect to the time.

Up to now the uniqueness of the ground state solution is an open problem for system (1.1), so that in the definition of orbital stability we have to take into account the possibility of a solution $\Psi$ to go from a ground state $U$ to a different ground state solution $V$; with this respect, it would be very interesting to know, at least, if ground states are isolated. In addition, we will show that there exist also other sets of orbitally stable solutions, then our definition has to take into account this aspect.

Our main results are the following ones.

Theorem 2.7 Assume (2.4). For any $\beta, \omega_1, \omega_2 > 0$ the set $G$ is orbitally stable.

Different from the single equations case, we have other families of orbitally stable solutions for the system, as the next results show.

Theorem 2.8 Assume (2.4). For any $\beta, \omega_1, \omega_2 > 0$ the set $S$ is orbitally stable.

Remark 2.9 The preceding results imply that the problem (1.1) possesses at least two family of orbitally stable standing waves for $\beta$ sufficiently large, ground state standing waves and scalar ones. While, for $\beta$ small the stability property of scalar standing waves is a consequence both of Theorems 2.7 and 2.8, since $G = S$ in this case.

If $\omega_1 = \omega_2 = \omega$ the set $B$ is completely characterized in the next result.

Theorem 2.10 Assume $\omega_1 = \omega_2 = \omega > 0$. For any $\beta \geq 0$ it holds

$$B = \left\{ e^{i\theta_1} z_\theta^\omega (\cdot - y), e^{i\theta_2} z_\theta^\omega (\cdot - y), \theta_1, \theta_2 \in \mathbb{R}, y \in \mathbb{R}^n, z_\theta^\omega \text{ defined in (2.8)} \right\}.$$  

In other words the set $B$ is described by the standing wave of the single equation, up to translations and phase shifts of the components.

This characterization of the set $B$ leads us to show that the set $B$ is orbitally stable even for $\beta$ small.

Theorem 2.11 Assume (2.4). For any $\beta \geq 0$ and $\omega_1 = \omega_2 = \omega > 0$ the set $B$ is orbitally stable.

Remark 2.12 1. These results imply that solutions that starts from initial data close to ground states with both nontrivial components remain close to orbits generated by ground states with both nontrivial components. While, solutions that start close to $S$ will stay close to orbits generated by $S$.  

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2. From the preceding results we deduce that, for \( \omega_1 = \omega_2 \), \( \mathcal{B} \) and \( \mathcal{S} \) are always orbitally stable sets independently of \( \beta \). When \( \omega_1 \neq \omega_2 \), we have that \( \mathcal{S} \) is always orbitally stable, while we can prove that \( \mathcal{B} \) is orbitally stable only when it coincides with \( \mathcal{G} \), that is for \( \beta \) large. It is an open problem to study the stability property for \( \mathcal{B} \) for any \( \omega_1 \neq \omega_2 \). Our results cover the following cases:

(a) \( \omega_1 = \omega_2 \) positive; for any \( \beta \geq 0 \).

(b) \( \omega_1 \neq \omega_2 \), \( \beta \) large (in dependence of \( \omega_1/\omega_2 \)) such that \( \mathcal{B} = \mathcal{G} \).

We will also prove an instability result for the sets \( \mathcal{G} \), \( \mathcal{S} \) and \( \mathcal{B} \) (for \( \omega_1 = \omega_2 \)) in dependence of the exponent \( p \). More precisely, we will show the following results.

**Theorem 2.13** Assume \( p < n/(n-2) \). For any \( \omega_1, \omega_2, \beta > 0 \) the following conclusions hold:

a) Let \( p > 1 + 2/n \), then the sets \( \mathcal{G} \), \( \mathcal{S} \) are unstable in the following sense: For any \( U \in \mathcal{G} \) (or \( U \in \mathcal{S} \)) and \( \varepsilon > 0 \) there exists \( U^0 \) with \( \|U^0 - U\|_{H^1} \leq \varepsilon \) such that the solution \( \Phi_\varepsilon \) satisfying \( \Phi_\varepsilon(0) = U^0 \) blows up in a finite time in \( H^1 \).

b) Let \( p = 1 + 2/n \), then every solution of (1.2) is unstable in the sense of the previous conclusion.

**Theorem 2.14** Assume \( 1 + 2/n < p < n/(n-2) \) and \( \omega_1 = \omega_2 = \omega \). For any \( \beta > 0 \), the set \( \mathcal{B} \) is unstable in the following sense:

For any \( U \in \mathcal{B} \) and \( \varepsilon > 0 \) there exists \( U^0 \) with \( \|U^0 - U\|_{H^1} \leq \varepsilon \) such that the solution \( \Phi_\varepsilon \) satisfying \( \Phi_\varepsilon(0) = U^0 \) blows up in a finite time in \( H^1 \).

### 3 Minimization Problems

#### 3.1 Ground states

In this section we will present some general results which will be useful in proving Theorems 2.11 and 2.13. Our orbital stability results will follow by some strict relationship between different minimization problems.

**Definition 3.1** Given \( \gamma > 0 \), let us consider the minimization problems

\[
\begin{align*}
\text{min} & \quad I(U) = m_\gamma, \\
\text{min} & \quad E(U) = c_\gamma,
\end{align*}
\]

where \( M_\gamma = \left\{ U \in H^1 : \|U\|_{E,\gamma} = \gamma \right\} \). Moreover, we denote with \( \mathcal{A} \) the set of the solution of problem (1.2).

**Remark 3.2** Notice that, solving problem (1.2) is equivalent to solve problem (1.1), since for every \( V \in M_\gamma \) we have \( I(V) = E(V) + \gamma/2 \).

**Remark 3.3** It results

\[
\mathcal{A} = \{(e^{it_1}u_1, e^{it_2}u_2), t_j \in \mathbb{R}, (u_1, u_2) \in H^1(\mathbb{R}^n, \mathbb{R}^2) \text{ solves } (3.1)\},
\]

where

\[
\sigma_\mathcal{R} = \inf \left\{ E(V) : V \in H^1(\mathbb{R}^n, \mathbb{R}^2), \omega_1\|v_1\|_2^2 + \omega_2\|v_2\|_2^2 = \gamma \right\}.
\]
Indeed, if we consider the minimization problems (3.1) we have that \( c_\gamma = \sigma_{K} \) and if \( U = (u_1, u_2) \) solves (3.2) there exists \( \theta_j \in \mathbb{R} \) such that \( u_j = e^{i\theta_j}|u_j| \) for \( j = 1, 2 \). For more details, see Remark 3.12 of [19],

**Remark 3.4** The conservation laws of the problem suggest that orbital stability has to be studied by using problem (3.2) which can be solved only for \( p < 1 + 2/n \), as for \( p > 1 + 2/n, \mathcal{E} \) (and then \( I \)) is not bounded from below on \( M_\gamma \). We will prove our stability result using problem (3.5) which has a solution for every \( p < n/(n-2) \). At the same time we cannot expect to have a stability result for every \( p < n/(n-2) \), as Theorem 3.1 shows. This aspect is clarified in the following results where we show that problems (3.2) and (3.3) (and then (3.1)) are equivalent for \( p < 1 + 2/n \). Indeed, in this range of exponents we can construct a bijective correspondence between the negative critical values of \( \mathcal{E} \) on \( M_\gamma \) and the critical values of \( I \) on \( N \). While, for \( p > 1 + 2/n \) we cannot derive this map between these critical values.

This suggests that for \( p < 1 + 2/n \) the Nehari manifold and \( M_\gamma \) have the same tangent planes, while when \( p > 1 + 2/n \) the tangent planes are different, so that a minimum point on \( N \) would probably give rise to a different critical point on \( M_\gamma \). This point is crucial in proving orbital stability properties, since the conservation laws show that the dynamical analysis has to be performed on \( M_\gamma \).

In proving many of the results of this section we will make use of the following lemma the proof of which is straightforward.

**Lemma 3.5** For any \( u \in H^1(\mathbb{R}^n, \mathbb{C}) \) and for any positive real numbers \( \lambda, \mu \), we can define the scaling \( u^{\mu, \lambda}(x) = \mu u(\lambda x) \) such that the following equalities hold.

\[
\|u^{\mu, \lambda}\|_2^2 = \lambda^2 \mu^{-n}\|u\|_2^2, \quad \|\nabla u^{\mu, \lambda}\|_2^2 = \lambda^2 \mu^{-n}\|\nabla u\|_2^2, \quad \|u^{\mu, \lambda}\|_{2p}^{2p} = \lambda^2 \mu^{-n}\|u\|_{2p}^{2p}.
\]

**Proof.** The proof is an immediate consequence of a change of variables.

First, we want to show the equivalence between problems (3.3) and (3.2) and (3.4). In order to do this, let us define the sets

\[
K_E = \left\{ c < 0 : \exists U \in M_\gamma : \mathcal{E}(U) = c, \nabla_{M_\gamma}\mathcal{E}(U) = 0 \right\}, \quad K_\mathcal{E} = \left\{ U \in M_\gamma : \nabla_{M_\gamma}\mathcal{E}(U) = 0, \mathcal{E}(U) < 0 \right\},
\]

\[(3.5) \quad K_F = \left\{ m \in \mathbb{R} : \exists U \in N : I(U) = m, I'(U) = 0 \right\}, \quad \tilde{K}_F = \left\{ U \in N : I'(U) = 0 \right\},
\]

where we have denoted with \( \nabla_{M_\gamma}\mathcal{E} \) the tangential derivative of \( \mathcal{E} \) on \( M_\gamma \). The following result holds.

**Theorem 3.6** Assume (3.4). For any \( \beta, \omega_1, \omega_2 > 0 \) the following conclusions hold:

a) there exists a bijective correspondence between the sets \( K_E \) and \( K_F \),

b) there exists a bijective mapping \( T : K_F \to K_E \) given by

\[
T(m) = -\left[ \frac{1}{p-1} - \frac{n}{2} \right] \left[ \frac{\gamma}{2p^* - n} \right] \left[ \frac{1}{m} \right]^{\frac{1}{2p^* - n}}, \quad \text{where } p^* = \frac{p}{p-1} \text{ stands for the conjugate exponent of } p.
\]

**Proof.** In order to prove assertion a), take \( V = (v_1, v_2) \in M_\gamma \) such that \( V \) satisfies

\[
\langle \mathcal{E}'(V), V \rangle = -\n\gamma c, \quad \mathcal{E}(V) = c < 0.
\]

(3.7)
Then, using that \( \langle F'(V), V \rangle = 2pF(V) \), it follows
\[
-\nu^2 - 2c = \langle \mathcal{E}'(V), V \rangle - 2\mathcal{E}(V) = 2F(V)(1 - p) < 0,
\]
showing that \( \nu \) is a positive real number. Therefore it is well defined the map \( T^{\mu,\lambda} : \mathcal{M}_y \to \mathcal{N} \)
\[
T^{\mu,\lambda}(V) = V^{\mu,\lambda},
\]
where \( \mu, \lambda \), are given by
\[
(3.8) \quad \mu = \nu^{-1/2(p-1)}, \quad \lambda = \nu^{-1/2}.
\]
Using this and (3.4) one obtains that \( V^{\mu,\lambda} \) solves (1.2), so that \( T^{\mu,\lambda}(V) \) belongs to \( \tilde{\mathcal{K}}_I \). Vice-versa if \( U \in \tilde{\mathcal{K}}_I \) let us take \( \nu > 0 \) such that
\[
\nu^{1/(p-1)-n/2} = \frac{\gamma}{\|U\|^2_{\mathcal{H}^1}}
\]
and \( \lambda, \mu > 0 \) given by
\[
(3.9) \quad \lambda = \nu^{1/2}, \quad \mu = \nu^{1/2(p-1)},
\]
so that \( U^{\mu,\lambda} \) belongs to \( \mathcal{M}_y, \nabla \mathcal{M}_y, \mathcal{E}(U) = 0 \). This shows that \( (T^{\mu,\lambda})^{-1} = T^{1/\mu,1/\lambda} \).

In order to prove assertion \( b) \), note first that any \( m \in \mathcal{K}_I \) is positive. Indeed, since there exists \( U \in \mathcal{N} \) such that \( \mathcal{I}(U) = m \) and \( \mathcal{I}'(U) = 0 \), it follows
\[
m = \mathcal{I}(U) - \frac{1}{2p} \langle \mathcal{I}'(U), U \rangle = \frac{1}{2} \left( 1 - \frac{1}{p} \right) \|U\|^2_{\mathcal{H}^1} > 0,
\]
so that \( T \) is a well defined and injective map. Let us first show that if \( c \in \mathcal{K}_I \cap \mathbb{R}^- \), then \( c = T(m) \).

Indeed take \( V \in \mathcal{M}_y \) corresponding to such \( c \), and take \( T^{\mu,\lambda}(V) = V^{\mu,\lambda} \). Recalling Pohozaev identity (see (5.9) in [18]) and since \( V^{\mu,\lambda} \in \mathcal{N} \) we get
\[
\left( \frac{1}{2} - \frac{1}{2p} \right) \left( \|\nabla V^{\mu,\lambda}\|^2_{L^2} + \|V^{\mu,\lambda}\|^2_{L^2} \right) = m,
\]
\[
(n - 2)\|\nabla V^{\mu,\lambda}\|^2_{L^2} + n\|V^{\mu,\lambda}\|^2_{L^2} = \frac{n}{p} \left( \|\nabla V^{\mu,\lambda}\|^2_{L^2} + \|V^{\mu,\lambda}\|^2_{L^2} \right),
\]
where \( m = \mathcal{I}(V^{\mu,\lambda}) \). We derive
\[
(3.9) \quad \|\nabla V^{\mu,\lambda}\|^2_{L^2} = nm, \quad F(V^{\mu,\lambda}) = \frac{m}{p - 1}, \quad \|V^{\mu,\lambda}\|^2_{L^2} = \left( \frac{2p}{p - 1} - n \right) m.
\]
Using (3.4) we have that
\[
(3.10) \quad \mu^2 = \frac{\|\nabla V\|^2_{L^2}}{nm}, \quad \frac{\mu^{2p}}{\lambda^p} = \frac{F(V)}{m}, \quad \frac{\mu^2}{\lambda^p} \|V\|^2_{L^2} = \left( \frac{2p}{p - 1} - n \right) m.
\]
Since \( V \) is in \( \mathcal{M}_y \), (3.10) yields
\[
\gamma = \|V\|^2_{L^2} = \nu^{1/(p-1)-n/2} \left( \frac{2p}{p - 1} - n \right) m,
\]
this and (5.12) give
\[ \|\nabla V\|_2^2 = n \left( \frac{\gamma}{2p' - n} \right)^{1+\frac{2(p-1)}{2(p'-1)}} \left( \frac{1}{m} \right)^{\frac{2(p-1)}{2(p'-1)}}, \]
\[ F(V) = \frac{1}{p-1} \left( \frac{\gamma}{2p' - n} \right)^{1+\frac{2(p-1)}{2(p'-1)}} \left( \frac{1}{m} \right)^{\frac{2(p-1)}{2(p'-1)}}. \]

All the above calculations imply that, if \( c \) is a negative constrained critical value of \( E \) on \( M_\gamma \) and \( m \) is the corresponding critical value of \( I \), then \( c \) is given by (5.10).

In order to show that \( T^{-1} \) is surjective let us take \( m \) in \( K_I \) and the corresponding \( U \) that satisfies the conditions in (3.6). For any \( \nu > 0 \) we can define
\[ \lambda = \nu^{1/2}, \quad \mu = \nu^{1/2(p-1)} \]
and consider \( U^{\mu,\lambda} \). Using (5.12) and (5.15) and requiring that \( U^{\mu,\lambda} \in M_\gamma \) imply that \( \nu \) is related to \( \gamma \) by the expression
\[ \gamma^{1/(p-1)-n/2} = \frac{\nu}{\mu} \left( \frac{1}{2p' - n} \right). \]

Moreover, since \( U \) is a free critical point of \( I \) we obtain that \( U^{\mu,\lambda} \) is a constrained critical point of \( E \) with Lagrange multipliers equal to \( \nu \). In order to conclude the proof we have to impose that \( E(U^{\mu,\lambda}) = c \). From conditions (3.2), (3.6) and from the definition of \( K_I \) it follows that \( c, m \) and \( \nu \) satisfy
\[ c = m \left[ \frac{n}{2} - \frac{1}{p-1} \right] \nu^{\rho-n/2} \]
and substituting the value of \( \nu \) in dependence of \( \gamma \) implies that \( m = T^{-1}(c) \).

**Corollary 3.7** There exists a bijective correspondence between the sets \( G \) and \( A \).

**Proof.** Let \( V \in A \) and take \( T^{\mu,\lambda}(V) \); Theorem implies that \( T^{\mu,\lambda}(V) \) is a critical point of \( I \), so that we only have to show that
\[ I(T^{\mu,\lambda}(V)) = m = m_N. \]

Indeed, suppose by contradiction that \( m > m_N \). In it is proved that \( m_N \) is achieved by a vector \( U \), then \( U^{1,1}\mu,\lambda \), with \( \mu, \lambda \) as in 4.2, belongs to \( M_\gamma \) and gives a negative critical value \( c \) given by 4.6. Since \( m_N < m \) we get \( c < c_\gamma \) which is a contradiction, so that the claim is true.

Using the preceding result and Theorem 2.1 in we can prove the following statement.

**Theorem 3.8** Assume (2.2). For any \( \beta, \omega_1, \omega_2 > 0 \), there exists a solution of the minimization problem \( \text{arg} \min_{U \in M_\gamma} E(U) \).

**Proof.** As observed in Remark 2.2 problems (3.3), (3.4) are equivalent, so it is enough to show that (3.4) is solved.

By using a Gagliardo-Nirenberg type inequality for systems (see (8) equation (9)), we get that the following inequality holds for any \( U \in M_\gamma 
\[ E(U) \geq \frac{1}{2} \|\nabla U\|_2^{2(p-1)} - \frac{C_{\omega_2}}{p} \gamma^{p-(p-1)n/2}, \]
The following results hold.

Remark 3.9 It is easy to see that every $U$ in $\mathcal{G}$ satisfies

$$||U||^2_{2,\omega} = m_N \left(\frac{2p}{p-1} - n\right),$$

thanks to the regularity properties of $U$ and to Pohozaev identity.

Theorem 3.10 Assume $\gamma_0$ and let $\gamma_0$ be fixed as

$$\gamma_0 = m_N \left(\frac{2p}{p-1} - n\right).$$

Then $m_N = m_\gamma$.

Proof. From the definition of $\gamma_0$ immediately follows that $m_N \geq m_\gamma$. In order to show that the equality is achieved, we only have to observe that $m_\gamma = c_\gamma + \gamma_0/2 = T(m_N) + \gamma_0/2$. Using the definition of $T$ joint with (3.12) yields the conclusion.

Remark 3.11 Consider the minimization problems and

$$m_{N_\gamma} := \inf \{I(W) : W \in H^1(\mathbb{R}^n, \mathbb{R}^2) \setminus \{0\} : \langle I(W), W \rangle = 0\},$$

it results that $m_N = m_{N_\gamma}$. Indeed, $m_N \leq m_{N_\gamma}$. Moreover, if $U \in H^1$, $U = (u_1, u_2)$ is a solution of (3.5), then $||U||^2_{2,\omega} = \gamma_0$, where $\gamma_0$ is defined in (3.11), so that $U \in M_{\gamma_0}$ and Theorem 3.10 implies that $E(U) = c_\gamma$. Then, from Remark 3.11 we deduce that there exist $\theta_1, \theta_2$ such that

$$U = (u_1, u_2) = (e^{i\theta_1}|u_1|, e^{i\theta_2}|u_2|), \quad E(|u_1|, |u_2|) = c_\gamma.$$ 

Finally, Theorem 3.10 proves that $I(|u_1|, |u_2|) = m_N$ and, since $|\nabla u_1| = |\nabla u_2|$ it results that $(|u_1|, |u_2|) \in N$ is a solution of (3.12), that is $m_N$ is achieved on a vector with real valued components. Furthermore, we have shown that

$$\mathcal{G} = \{(e^{i\theta_1}|u_1|, e^{i\theta_2}|u_2|) : \theta_j \in \mathbb{R}, (u_1, u_2) \in H^1(\mathbb{R}^n, \mathbb{R}^2)\}.$$ 

In order to prove the instability result Theorem 3.10 another variational characterization of a ground state solution will be useful. Let us define the functional

$$\mathcal{R}(U) = ||\nabla U||^2_{L^2} - n(p-1)F(U)$$

and the infimum

$$m_\mathcal{P} = \inf_{\mathcal{P}} \mathcal{R}$$

where $\mathcal{P} = \{U \in H^1 : \mathcal{R}(U) = 0\}$.

The following results hold.
Proposition 3.12 Assume that \( p > 1 + 2/n \), then the following conclusions hold:

a) \( P \) is a natural constraint for \( I \);

b) \( m_P = m_N \).

Proof. In order to prove a) let us consider \( U \) a constrained critical point of \( I \) on \( P \), then there exists \( \lambda \in \mathbb{R} \) such that the following identities are satisfied

\[
\begin{align*}
\text{uno} & \quad (1 - 2\lambda)\|\nabla U\|_2^2 + \|U\|_{2,\omega}^2 = 2p[\lambda n(1 - p) + 1]F(U), \\
\text{due} & \quad (1 - 2\lambda)\left(\frac{n}{2} - 1\right)\|\nabla U\|_2^2 + \frac{n}{2}\|U\|_{2,\omega}^2 = n[\lambda n(1 - p) + 1]F(U), \\
\text{tre} & \quad \|\nabla U\|_2^2 = n(p - 1)F(U).
\end{align*}
\]

Hence, using (3.16) in (3.14) we get

\[
\|U\|_{2,\omega}^2 = \left[2\lambda(1 - p) + \frac{2p}{n(p - 1)} - 1\right]\|\nabla U\|_2^2,
\]

and using this and (3.15) in (3.14), and taking into account that \( p > 1 + 2/n \), we obtain that \( \lambda = 0 \), so that \( U \) is a free critical point of \( I \).

In order to prove b) take a minimum point \( U \) of \( I \) in \( P \); from a) it follows that then \( U \) belongs to \( N \) so that \( m_P \geq m_N \); viceversa if \( V \) is a minimum point of \( I \) in \( N \) then \( V \) is a free critical point of \( I \) and Pohozaev identity implies that \( V \in P \) so that \( m_P \leq m_N \), yielding the conclusion.

Lemma 3.13 Assume that \( p > 1 + 2/n \). For any \( U \neq (0,0) \) let us consider \( U^{\kappa_2,\lambda} = (u_1^{\kappa_2,\lambda}, u_2^{\kappa_2,\lambda}) \), for \( u^{\kappa,\lambda} \) defined in Lemma 3.12. The following conclusions hold:

a) there exists a unique \( \lambda_* = \lambda_*(U) \) such that \( U^{\kappa_2,\lambda_*} \) belong to \( P \);

b) the function \( g(\lambda) = I(U^{\kappa_2,\lambda}) \) has its unique maximum point in \( \lambda = \lambda_* \),

c) \( \lambda_* < 1 \) if and only if \( \mathcal{R}(U) < 0 \) and \( \lambda_* = 1 \) if and only if \( \mathcal{R}(U) = 0 \),

d) the function \( g(\lambda) \) is concave on \( (\lambda_*, +\infty) \).

Proof. a): for any \( \lambda > 0 \) it holds

\[
\mathcal{R}\left(U^{\kappa_2,\lambda}\right) = \lambda^2\|\nabla U\|_2^2 - \lambda^{n(p-1)}n(p - 1)F(U),
\]

then there is a unique

\[
\lambda_*(U) = \lambda_* = \left[\frac{\|\nabla U\|_2^2}{n(p - 1)F(U)}\right]^{1/[n(p-1)-2]}
\]

such that \( \mathcal{R}\left(U^{\kappa_2,\lambda_*}\right) = 0 \). Computing the first derivative of \( g(\lambda) \) b) is proved. Since \( p > 1 + 2/n \), c) easily follows. d) immediately follows from writing the second derivative of the function \( g \).

Lemma 3.14 For any \( U \in \mathbb{H}^1 \) with \( \mathcal{R}(U) < 0 \) it results

\[
\mathcal{R}(U) \leq I(U) - m_N.
\]
Proof. Conclusion d) in Lemma 3.13 implies that
\[ g(1) \geq g(\lambda_*) + g'(1)(1 - \lambda_*) . \]

Direct computation yields
\[ I(U) = g(1) \geq g(\lambda_*) + g'(1)(1 - \lambda_*) = g(\lambda_*) + R(U)(1 - \lambda_*) \geq I(U;\gamma^2,\lambda) + R(U) , \]
where in the last inequality we have used that \( R(U) < 0 \). Recalling that \( U;\gamma^2,\lambda \in \mathcal{P} \) and applying conclusion b) of Proposition 3.12 complete the proof.

### 3.2 Bound states

It is well known (see Section 8 in Ca [5]) that \( z^{\omega}_{\beta} \) defined in (2.8) can be characterized as the solution of the following constrained minimization problem
\[
\min_{\mathcal{M}_\delta} E_1(u) = \frac{1}{2} \|
abla u\|_2^2 - \frac{\beta + 1}{2p} \|u\|_p^2 ,
\]
where the functional \( E_1 : H^1(\mathbb{R}^n) \to \mathbb{R} \) is defined by
\[
E_1(u) = \frac{1}{2} \|
abla u\|_2^2 - \frac{\beta + 1}{2p} \|u\|_p^2 .
\]

and when we prescribe
\[
\delta = \delta(\omega) = \frac{\omega^{\frac{n-2}{2} - \frac{2}{p}}}{(\beta + 1)^{\frac{n-2}{2}}} .
\]

Otherwise, \( z^{\omega}_{\beta} \) can be equivalently obtained as the solution of the minimization problem
\[
\min_{\mathcal{N}_1} I_1(u) = m_1 = \min_{\mathcal{N}_1} I_1 u ,
\]
where the functional \( I_1 : H^1(\mathbb{R}^n) \to \mathbb{R} \) is defined by
\[
I_1(u) = \frac{1}{2} \|
abla u\|_2^2 + \frac{\omega}{2} \|u\|_2^2 - \frac{\beta + 1}{2p} \|u\|_p^2 .
\]

The following result is the starting point in proving Theorem 2.10.

**Proposition 3.15** Assume that \( \omega_1 = \omega_2 = \omega \). If \( U \) solves (2.6), it results \( |u_1| = |u_2| \) almost everywhere.

**Proof.** Consider the variational characterization \( z^{\omega}_{\beta} \) as the solution of (3.19), the vector \( Z = (z^{\omega}_{\beta}, z^{\omega}_{\beta}) \) belongs to \( \mathcal{N}_2 \), so that
\[
2m_1 = 2I_1(Z) = I(Z) \geq m_2 .
\]

Let now \( U \) be a solution of (2.6), then, Young inequality yields
\[
m_2 = I(U) \geq I_1(u_1) + I_2(u_2) \geq 2m_1 \geq m_2
\]

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showing that

\[ I(U) = I_1(u_1) + I_1(u_2), \quad \text{and} \quad m_2 = 2m_1. \]

Writing down this equality we get

\[ \frac{\beta}{p} \int |u_1u_2|^p = \frac{\beta}{2p} \int (|u_1|^{2p} + |u_2|^{2p}), \]

that is

\[ \int (|u_1|^p - |u_2|^p)^2 = 0, \]

giving the conclusion.

**Proof of Theorem 2.10.** Let \( U = (u_1, u_2) \in \mathcal{B} \), from Proposition 3.21 we derive that

\[ \|\nabla u_1\|_2^2 + \omega \|u_1\|_2^2 = \|u_1\|_p^{2p} + \beta \|u_1u_2\|_p^p = (\beta + 1)\|u_1\|_p^{2p}, \]

that is \( u_1 \in \mathcal{N}_1 \), so that

\[ I_1(u_1) \geq I_1(z_0^\beta_2). \]

On the other hand, from Proposition 3.21 and (3.22) it follows that \( I(U) = 2I_1(u_1) \) and recalling that \( Z = (z_0^\beta_2, z_0^\beta_2) \in \mathcal{N}_2 \) we derive

\[ 2I_1(u_1) = I(U) \leq I(Z) = 2I_1(z_0^\beta_2). \]

This, (3.21) and (3.17) imply that \( u_1 = e^{\theta_1}z_0^\beta_2(\cdot - y_1) \) for some \( \theta_1 \in \mathbb{R} \) and \( y_1 \in \mathbb{R}^n \). The same argument for \( u_2 \) gives \( u_2 = e^{\theta_2}z_0^\beta_2(\cdot - y_2) \), and Proposition 3.21 yields \( y_1 = y_2 \).

As we did for ground states we want to investigate the connection of Problem (3.16) with a minimization of \( \mathcal{E} \) under suitable constraints. Since we are now considering vectors with both nontrivial components we are naturally lead to study the following problem for \( \mathcal{E} \).

\[ \mathcal{E}(U) := c_{\delta_1, \delta_2} = \inf_{\mathcal{M}_{\delta_1, \delta_2}} \mathcal{E} \quad \text{where} \quad \mathcal{M}_{\delta_1, \delta_2} = \{(u_1, u_2) \in \mathbf{H}^1 : \|u_1\|_2^2 = \delta_1, \|u_2\|_2^2 = \delta_2\} \]

where \( \delta_1, \delta_2 \) are positive real numbers.

If \( \delta_1 \neq \delta_2 \) we do not know how to solve problem (3.22) and, to our knowledge, it is an open problem to prove orbital stability property of solution of Problem (3.22) in this general case.

Therefore, we will focus our attention to the case \( \delta_1 = \delta_2 = \delta \). In this case we have the following minimization problem

\[ \mathcal{E}(U) = c_{\delta, \delta} = \inf_{\mathcal{M}_{\delta, \delta}} \mathcal{E} \quad \text{where} \quad \mathcal{M}_{\delta, \delta} = \{(u_1, u_2) \in \mathbf{H}^1 : \|u_1\|_2^2 = \|u_2\|_2^2 = \delta\}. \]

As we did for the ground state solutions, investigating the relation between Problems (3.16) and (3.23) in the case \( \omega_1 = \omega_2 = \omega \) naturally lead us to choose \( \delta = \delta(\omega) \) such that every solution of Problem (3.23) give rise a solution of Problem (3.22). With this choice we end up with the same characterization of the sets \( \mathcal{B} \) (given in Theorem 2.10) and \( \mathcal{A}_{\delta_1, \delta_2}(\omega) \) set of solutions of (3.22), as the next result shows.

The following result can be proved using the same arguments of (3.23) for the case \( p = 2 \) and \( \beta = 1 \). We include some details for clearness.
Proposition 3.16  It results
\[ \mathcal{A}(\delta(\omega), \delta(\omega)) = \left\{ (e^{\theta_1 + \omega y}, e^{\theta_2 + \omega y}) : \theta_1, \theta_2 \in \mathbb{R}, y \in \mathbb{R}^n \right\}. \]

Proof. Taking \( U = (u_1, u_2) \in \mathcal{A}(\delta(\omega), \delta(\omega)) \), using the variational characterization \( \pi_{\omega}^\ast \) as the solution of (3.14), and arguing as in the proof of Proposition 3.15, yield
\[ \lim \inf_{k \to \infty} \| \Phi_k^0 - U \|_{H^1} = 0 \]
and the corresponding sequence of solution \( \{ \Phi_k \} \) of Problem (4.1) satisfies
\[ \lim \inf_{k \to \infty} \| \Phi_k^0 - (u_1, u_2) \| \geq \epsilon_0. \]

4  Proofs of the Main Results

In this section we will prove the main results concerning the stability (or instability) of the standing waves. In particular in the following subsections we show, in the subcritical case \( 1 < p < 1 + 2/n \), the orbital stability of the sets \( \mathcal{G} \) and \( \mathcal{S} \), and also of the set \( \mathcal{B} \) for \( \omega_1 = \omega_2 \). Finally in subsection 4.3 prove that for \( p > 1 + 2/n \) the sets \( \mathcal{G} \), \( \mathcal{S} \), \( \mathcal{B} \) (for \( \omega_1 = \omega_2 \)) are unstable and for \( p = 1 + 2/n \) the instability holds for every bound state. The proofs of Theorems 3.6, 3.7, 4.1, and 4.2 follow the arguments of [15] for the single equation, while the proof of Theorem 4.3 follows the arguments of [20].

4.1  Stability of the ground state standing waves

Proof of Theorem 4.1  Let us argue by contradiction, and suppose that there exist \( \epsilon_0 > 0 \), \( \{ \tau_k \} \subset \mathbb{R} \) and a sequence of initial data \( \{ \Phi_k^0 \} \subset H^1 \) such that
\[ \lim \inf_{k \to \infty} \| \Phi_k^0 - U \|_{H^1} = 0 \]
and the corresponding sequence of solution \( \{ \Phi_k \} \) of Problem (4.1) satisfies
\[ \lim \inf_{k \to \infty} \| \Phi_k^0 - (u_1, u_2) \| \geq \epsilon_0. \]

Condition definitions (4.1), (4.2), Theorem 3.1, and the continuity properties of the functional \( I \) yield
\[ I(\Phi_k^0) \to m_{\gamma_0}, \quad \| \Phi_k^0 \|^2_{L^2, \omega} = \gamma_0 + o(1). \]

Let us denote \( \Psi_k(x) = \Phi_k(x, t_k) \), then, conservation laws (3.1), (3.2) imply that
\[ I(\Psi_k) \to m_{\gamma_0}, \quad \| \Psi_k \|^2_{L^2, \omega} = \gamma_0 + o(1). \]

Following the arguments of [15], we use the Ekeland variational principle to obtain a new minimizing sequence \( \Psi_k \) which is also a Palais-Smale sequence for \( I \) and we find \( \Psi \) such that
\[ I(\Psi) = m_{\gamma_0}, \quad I'(\Psi) = 0. \]
Then
\[ \frac{1}{2} \left(1 - \frac{1}{p}\right) ||\tilde{\psi}_k||_{H^1}^2 = m_{\rho_0} = I(\tilde{\psi}_k) - \frac{1}{2p} \langle I'(\tilde{\psi}_k), \tilde{\psi}_k \rangle + o(1) = \frac{1}{2} \left(1 - \frac{1}{p}\right) ||\tilde{\psi}_k||_{H^1}^2 + o(1) \]
showing that
\[ ||\tilde{\psi}_k||_{H^1}^2 \rightarrow ||\tilde{\psi}||_{H^1}^2 \]
and, since \( H^1 \) is a Hilbert space, we have the strong convergence in \( H^1 \). By the choice of \( \tilde{\psi}_k \) we derive that also \( \psi_k \rightarrow \tilde{\psi} \) strongly in \( H^1 \), which is an evident contradiction with (4.2).

\[ \tag{4.2} \]

**Remark 4.1** In the proof of the previous result it was crucial to show that the sequence \( \tilde{\psi}_k \) strongly converges in \( H^1 \), to get the desired contradiction. In other words, in proving orbital stability results we made use of conservation laws, minimization property and compactness of the minimizing sequence.

### 4.2 Stability of bound state standing waves

**Proof of Theorem 4.2.** Let us argue for the set of scalar solution with the second component equal to zero, the other case can be handled analogously. The conclusion can be obtained arguing as in the previous Theorem assuming that there exist \( \varepsilon_0 > 0 \), \( \{t_k\} \subset \mathbb{R} \) and a sequence of initial data \( \{\Phi_k^0\} \subset H^1 \) such that
\[ \lim_{k \to \infty} \inf_{t \in \mathbb{R}, y \in \mathbb{R}^d} ||\Phi_k^0 - (e^{i \theta} \zeta_0^k (\cdot - y), 0)||_{H^1} = 0 \]
and the corresponding sequence of solution \( \{\Phi_k\} \) of Problem (1.1) satisfies
\[ \inf_{t \in \mathbb{R}, y \in \mathbb{R}^d} ||\Phi_k (\cdot, t_k) - (e^{i \theta} \zeta_0^k (\cdot - y), 0)|| \geq \varepsilon_0 \tag{4.3} \]
As before, via conservation laws, \( \Psi_k (x) = \Phi_k (x, t_k) \) satisfies
\[ \mathcal{E}(\Psi_k(t)) \rightarrow \mathcal{E}(\zeta_0^0, 0) = c_{\delta(0,0)}, \quad ||\psi_k, 1||_2 = ||\zeta_0^0||_2 + o(1), \quad ||\psi_k, 2||_2 = o(1), \tag{4.4} \]
where
\[ \mathcal{E}(\zeta_0^0, 0) = c_{\delta(0,0)} = \min_{\mathcal{M}(\delta(0,0)) \to \{U \in H^1 : ||u_1||_2 = ||\zeta_0^0||_2 = \delta_0, ||u_2||_2 = 0\}} \mathcal{E} \]
\[ \leq \text{Gagliardo-Nirenberg inequality} \]
\[ \implies \Psi_k \text{ is bounded in } H^1, \text{ then interpolation inequality, joint with (4.4), implies that} \]
\[ \psi_k \rightarrow 0 \quad \text{strongly in } L^2. \tag{4.5} \]
Therefore, Hölder inequality yields
\[ \int ||\psi_k, 1||^p ||\psi_k, 2||^p \leq ||\psi_k, 1||^p_{L^p} ||\psi_k, 2||^p_{L^p} \rightarrow 0. \]
This and (4.5) yield
\[ \mathcal{E}(\Psi_k) = \mathcal{E}_0(\psi_k, 1) + \frac{1}{2} \|
abla \psi_k, 2\|_2^2 + o(1), \tag{4.6} \]
where
where
\[ \mathcal{E}_0(u) = \frac{1}{2} \|\nabla u\|_2^2 - \frac{1}{2p} \|u\|_{L^p}^p. \]
So that we are lead to
\[ \mathcal{E}_0(\psi_{k,1}) \leq \mathcal{E}(\psi_{k}) + o(1) = c_{\delta_0} + o(1). \]
Consider the scalar minimization problem
\[ \mathcal{E}_0(\epsilon_{0}') = c_{\delta_0} = \min_{\mathcal{M}_{\delta_0}} \mathcal{E}_0 \quad \mathcal{M}_{\delta_0} = \{ u \in H^1(\mathbb{R}^n, \mathbb{R}) : \|u\|_2^2 = \|\epsilon_{0}'\|_2^2 \}. \]
It is easy to verify that
\[ c_{\delta_0} = c_{(\delta_0,0)}, \] (4.7)
so that
\[ \mathcal{E}_0(\psi_{k,1}) \leq \mathcal{E}_0(\epsilon_{0}') + o(1) = c_{\delta_0} + o(1), \quad \|\psi_{k,1}\|_2^2 \rightarrow \|\epsilon_{0}'\|_2^2. \] (4.8)
By using Gagliardo-Nirenberg type inequality and arguing as in the proof of Theorem 2.13 we obtain that \( c_{(\delta_0,0)} < 0 \), so that for \( k \) large, \( \mathcal{E}(\psi_{k}) \leq c_{(\delta_0,0)}/2 \). This and (4.8) give the following uniformly a priori lower bound
\[ \|\psi_{k,1}\|_2^2 > \sigma_0, \quad \sigma_0 > 0. \]
This, (4.7) and (4.8) allow us to argue as in (11) (see also (12)), and by means of concentration compactness technique, get the strong convergence (up to a subsequence) of \( \psi_{k,1} \). Then, there exists \( \psi_1 \) such that
\[ \|\psi_1\|_2^2 = \|\epsilon_{0}'\|_2^2, \]
so that \( \mathcal{E}_0(\psi_1) \geq c_{\delta_0} \), but, passing to the limit in (4.6), we have \( \mathcal{E}_0(\psi_{k}) = c_{\delta_0} \). Therefore, there exist \( \theta \in \mathbb{R} \) and \( y \in \mathbb{R}^n \) such that \( \psi_1 = e^{i\theta} e^{i\psi_{n/2}(-y)} \). Moreover, (4.7), (4.8) and (4.9) imply that \( \psi_{k,2} \rightarrow 0 \) in \( H^1 \), giving a contradiction with (10).

**Proof of Theorem 4.3** Instability in the critical or supercritical case

Arguing by contradiction, as in the proofs of the other stability results, and using Theorem 2.10 and Proposition 3.10, we get a positive number \( \epsilon_0 \) and a sequence \( \Psi_k \) such that
\[ \inf_{\psi \in \mathcal{H}, \|\psi\|_2 = \epsilon_0} \mathcal{E}(\psi) \geq \epsilon_0, \quad \|\psi_k\|_2 \rightarrow \|\epsilon_{0}'\|_2, \quad \mathcal{E}(\Psi_k) \rightarrow c_{(\delta_0,0)} \] (4.9)
Following the proof of Lemma 2.3 in (38), it can be proved that the sub-additivity condition holds for Problem (11), then by concentration-compactness arguments (see Section IV in (27)), we obtain that \( \Psi_k \) is compact and the conclusion follows passing to the limit in (4.9).

4.3 Instability in the critical or supercritical case

In this subsection we will prove Theorem 4.3.13 and Proposition 4.3.10.

**Proof of Theorem 4.3.13** In order to prove conclusion \( a \) let us assume \( 1 + 2/n < p < n/(n-2) \) and consider first the set \( \mathcal{G} \). Let \( U \in \mathcal{G} \) so that \( U \notin \mathcal{P} \). When we fix \( U_i = U_{\sigma > s} \), with \( s > 1 \) we get that \( \mathcal{R}(U_{i}) < 1 \) and Conclusion \( c \) of Lemma 4.3.13 implies that \( \mathcal{R}(U_{i}) < 0 \) and Conclusion \( b \) of Lemma 4.3.13 gives
\[ I(U_i) = g(1) < g(\lambda_i) = I(U) = m_N. \] (4.10)
Let $\Phi_\epsilon$ the solution generated by $U_\epsilon$. By stability we have $I(\Phi_\epsilon) = I(U_\epsilon)$ when the solution exists. By continuity $R(\Phi(t)) < 0$ for $t$ small; moreover, Lemma 8.1.4 and (8.10) imply that
\[ R(\Phi(t)) \leq I(\Phi(t)) - m_N = -\sigma < 0, \]
showing that $R(\Phi(t)) < 0$ when the solution exists. Defining the variance function
\[ V(t) = ||x|\Phi(t)||_2^2, \]
it follows that $V'(t) = 8R(\Phi(t)) \leq -8\sigma$. Thus, there exists $T^*$ such that $V(T^*) = 0$ showing, by using Hardy’s inequality, that $\Phi_\epsilon$ blows up in $T^*$ (see (8.1)), which gives conclusion for the set $G$.

Let now $U \in S$ then $U = (u_1, 0)$ (for example) with $u_1 = e^{i\omega_1 z}$, $u_1$ is unstable for the single equation(with $\beta = 0$) then Theorem 8.2.2 implies that there exists $u_{e,1}^\beta$ such that $||u_{e,1} - u_1|| \leq \epsilon$ and the solution generated by $u_{e,1}^\beta, \phi_{e,1}$ blows up in finite time. Now if we choose $U_e = (u_0^{\beta}, 0)$ we get $||U_e - U|| \leq \epsilon$ and the solution generated by $U_e$ is (by the well posedness of the Cauchy problem) $\Phi_e = (\phi_{e,0}, 0)$ blows up in finite time.

Now consider $p = 1 + 2/n$. From Proposition 8.1.2 we get that any $U$ solution of (8.10) satisfies (8.11), then we get $R(U) = 0$ so that $R(\lambda U) < 0$ for any $\lambda > 1$. Let $U_\lambda = \lambda U$ be the initial datum of $\Phi_\lambda$ and $\Phi_0$ the corresponding solution.

\[ 0 > R(U_\lambda) = 2E(U_\lambda) = 2E(\Phi_\lambda) = R(\Phi_\lambda), \]

so that, also in this case, the variance is concave and the solution $\Phi_\lambda$ blows up in finite time.

Proof of Theorem 8.1.4. Let $U \in B$ then $U = (u_1, u_2)$ with $u_1 = e^{i\omega_1 z}$ and $u_2 = e^{i\omega_2 z}$, again $z_0$ is unstable because it is a ground state for the single equation with coefficient $\beta + 1$ in the nonlinear term, so that we can apply, as in the previous result, Theorem 8.2.2 in (8.10) to obtain an initial datum $u_{e,0}$ such that $||u_{e,0} - z_0|| \leq \epsilon$ and the solution that starts from $u_{e,0}$, $\phi_{e,0}$ blows up in finite time. If we choose $U_e = (u_0, u_0)$ we have (by the well posedness of the Cauchy problem) that the solution generating from $U^e$ is $\Phi_e = (\phi_{e,0}, \phi_{e,0})$ and it blows up in finite time.

5 Conclusions

In summary, we have studied the problem of the orbital stability of standing waves in two weakly coupled nonlinear Schrödinger equations. In analogy of what happens for the single equation case we have that least action solutions give rise to orbitally stable standing waves. But, the system admits also other families of orbitally stable standing waves, for example the set of solutions with one trivial component whose elements are not ground states (and have Morse index greater than one) for $\beta$ large. Moreover, for $\omega_1 = \omega_2$ least action solution with both nontrivial components also generate orbitally stable solutions of $I$ and this holds also when they are not ground states of $I$. So that it seems that having Morse index, with respect to $I$, equal to one is not a necessary property to gain orbital stability. In our opinion this is linked to the facts that the $L^2$ norms of the components are conserved separately.

We remark that it remains open the question of the stability for the set of minima of the energy whose components have different $L^2$ norms, at least for $\beta$ small. Moreover, it is an interesting open problem to find conditions, maybe related to the geometrical properties of $I$, on a solution in order to produce instability. More precisely, it would be interesting to understand how to extend the result of (8.10) for this kind of system.
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