Multiplicative operators in the spectral problem of integrable systems

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June 25, 2018

Keywords: Multiplicative operators, hyperelliptic curves, squared eigenfunctions, solitons, Korteweg-de Vries, nonlinear Schrödinger.

Abstract

We consider the spectral problem of the Lax pair associated to periodic integrable partial differential equations. We assume this spectral problem to be a polynomial of degree \(d\) in the spectral parameter \(\lambda\). From this assumption, we find the conservation laws as well as the hyperelliptic curve required to solve the periodic inverse problem. A recursion formula is developed, as well as \(d\) additional conditions which give additional information to integrate the equations under consideration. We also include two examples to show how the techniques developed work. For the Korteweg-de Vries (KdV) equation, the degree of the multiplicative equation is \(d = 1\). Hence, we only have one condition and one recursion formula. The condition gives in each degree of the recursion the conserved densities for KdV equation, recovering the Lax hierarchy. For the Nonlinear Schrödinger (NLS) equation, the degree of the multiplicative operator is \(d = 2\). Hence, we have a couple of conditions that we use to deduce conserved density constants and the Lax hierarchy for such equation. Additionally, we explicitly write down the hyper-elliptic curve associated to the NLS equation. Our approach can be use for other completely integrable differential equations, as long as we have a polynomial multiplicative operator associated to them.
Introduction

John Scott Russell observed for the first time a "soliton" (although he named it a "great wave of translation") in the 1830s in a channel in Edinburgh, Scotland. Russell performed several experiments and reported his results in 1834 (see reference in [32]). But it was not until 1895 that a partial differential equation (PDE) was developed by two Dutch mathematicians, D.J. Korteweg and G. de Vries [20], to describe Russell’s "great wave". In their paper, the authors found the one-soliton solution, a sech²-profile describing the wave observed by Russell, as well as a periodic solution of cnoidal-type. This equation is now known as the Korteweg-de Vries (KdV) equation,

\[
\frac{\partial q}{\partial t} + 6q\frac{\partial q}{\partial x} + \frac{\partial^3 q}{\partial x^3} = 0,
\]

where \(x\) denotes the spatial variable along the channel, \(t\) the temporal variable, and \(q = q(x, t)\) the elevation of the wave at the position \(x\) and at time \(t\).

In 1967, C. Gardner, J. Greene, M. Kruskal and R. Miura [13] found a method to exactly solve the KdV equation, the inverse scattering method (or transform) (IST). A year later, P. Lax generalized this idea stating that all the equations that possess a pair of linear operators to express the equations can be solved by the IST method. Since then, the pair of operators have been called the Lax pair [25]. In 1972, following Lax [25], V.E. Zakharov and A.B. Shabat were able to integrate the nonlinear Schrödinger equation (NLS) [33],

\[
\frac{\partial q}{\partial t} = -\frac{1}{2} \frac{\partial^2 q}{\partial x^2} - \sigma |q|^2 q,
\]

where \(|q|^2\) represents the intensity of an electromagnetic field inside an optical fiber and \(\sigma = \pm 1\) represents the focusing or defocusing nature of the fiber. In 1974, another equation, the sine-Gordon equation, was completely integrated [1] and the method could be further generalized [2]. After that, many other equations, continuous and discrete, were found to be completely integrable (see [32, 5, 31, 3, 4, 29] and references therein).

The IST was originally developed in the real line but earlier in the 1970s, P. Lax worked the KdV equation in a periodic domain [27]. Independently, and about the same years, several authors such as B.A. Dubrovin, A.P. Its, I.M. Krichever, V.B. Matveev and S.P. Novikov were also working the periodic KdV equation using the techniques of algebraic geometry [6, 7, 9, 10, 8, 21, 22, 23, 24, 30, 14]. A good survey of the topic can be found in [6, 21]. In the present paper, we follow the ideas in [6] and [12].

At this point, almost all the Lax pairs found for new integrable systems were matrix operators (as an example, the Lax pair for the KdV equation is scalar, although it also could be written as pair of \(2 \times 2\) matrix operators). However, in the work of Kamchatnov, Kraenkel and Umarov [18] the authors found the corresponding scalar Lax pairs to the usual \(2 \times 2\) Lax matrix representations. In particular, applications can be found in [17, 19, 15, 16]. The authors applied
this technique \cite{17, 19} to the nonlinear Schrödinger equation, whose associated spectral problem is a polynomial of degree 2 in the spectral parameter, to find trains of solitons using the Bohr-Sommerfeld quantization rule, just as it is done in the quantum case for the Schrödinger equation \cite{26}.

In this paper, we consider the spectral problem associated to certain nonlinear evolution equation, \textit{i.e.}, the problem associated to the Lax operator of the spatial evolution. We assume that this problem can be written in scalar form and that the corresponding Lax operator is a \textit{multiplicative operator}. Even more, we assume that this operator is a polynomial of degree \(d\) in the spectral parameter, \(d\) being a positive integer. We consider the work of Dubrovin \textit{et.al.} \cite{6, 8}, but follow Flaschka \cite{12}. Under this approach, we find the corresponding general hyperelliptic curve, a recursion formula to find the coefficients of the \(n\)-soliton solution, the conserved densities of motion, as well as the equations in the Lax hierarchy. The integrals of motion appear as coefficients in the polynomial expansion of the squared eigenfunction solution and the Lax hierarchy appears in extra conditions of solvability for the third order differential equation of the squared eigenfunction. As particular examples, we consider the KdV (1) (\(d = 1\)) and the NLS (2) (\(d = 2\)) equations. As far as the authors know, there is no approach similar to the methodology presented here for \(d\) being an arbitrary positive integer.

To see how this works, consider a solution \(y(x,t)\) of the Schrödinger-type equation,

\[
\frac{\partial^2 y}{\partial x^2} = \hat{L} y, \tag{3}
\]

where \(\hat{L}\) is a multiplicative linear operator, polynomial of degree \(d\) in the spectral parameter \(\lambda\). For the KdV equation (1), \(\hat{L} = \lambda - q\), \textit{i.e.}, \(d = 1\). Since \(\hat{L}\) is a multiplicative operator, multiplying (3) by \(y\) results into

\[
y \frac{\partial^2 y}{\partial x^2} = \hat{L} y^2 \tag{4}
\]

Now, setting \(\phi = y^2\) (this is the \textit{squared eigenfunction} of \(\hat{L}\)), taking the derivative with respect to \(x\), \(\phi' = 2yy'\) and squaring, we obtain: \((y')^2 = (\phi')^2/(4\phi)\). Similarly, taking a second derivative of \(\phi\), we have \(\phi'' = 2(y')^2 + 2yy'' = (\phi')^2/(2\phi) + 2yy''\). Hence, \(yy'' = \frac{1}{2} \phi'' - (\phi')^2/(4\phi)\), and equation (4) becomes

\[
\frac{1}{2} \phi'' - \frac{(\phi')^2}{4\phi} = \hat{L} \phi. \tag{5}
\]

Taking derivatives on both sides, we have

\[
\frac{1}{2} \phi''' - \left( \frac{(\phi')^2}{4\phi} \right)_x = \left[ \hat{L} \phi \right]_x \tag{6}
\]

A straight-forward computation, using equation (5), shows that \(\left( (\phi')^2/(4\phi) \right)_x = (\phi'/\phi) \left( \frac{1}{2} \phi'' - (\phi')^2/(4\phi) \right) = (\phi'/\phi) \hat{L} \phi = \hat{L} \phi', \) since \(\hat{L}\) is a multiplicative operator.
Thus, equation (6) turns to be the following third order differential equation for the squared eigenfunction $\phi$:

$$\phi''' - 2\hat{L}_x \phi - 4\hat{L} \phi' = 0.$$  \hspace{1cm} (7)

To keep computations compact, we define the following bilinear operator:

$$\langle \psi, \phi \rangle := (\psi \phi)_x + \psi \phi_x = \psi_x \phi + 2\psi \phi_x,$$  \hspace{1cm} (8)

so that equation (7) can be written as:

$$\phi''' - 2\langle \hat{L}, \phi \rangle = 0.$$  \hspace{1cm} (9)

Equation (9) is the differential equation to be solved for the squared eigenfunction $\phi$. \textit{Notice that this equation holds no matter what the degree of the operator $\hat{L}$ is.} Here it is where all of our argument is sustained.

Then, the paper is organized as follows. In section 1, it is found a recursion formula to determine the coefficients of the $n$-degree polynomial $\phi_n(x; \lambda)$, the squared eigenfunction, called here $n$-soliton solutions. Additionally, $d$ extra conditions of solvability are found. In section 2, we explicitly construct the hyperelliptic curve associated to the spectral problem, equation (3). Here, a polynomial of degree $2n + d$ is obtained. All the coefficients of the polynomial are computed in terms of the coefficients of the squared eigenfunction $\phi_n(x; \lambda)$ and the operator $\hat{L}$ (considered as polynomials in $\lambda$); therefore, they implicitly depend on the solution $q(x)$. Also, a set of differentials on the hyperelliptic curve is constructed. This set of differentials is essential for the integration and the inverse problem associated to these type of equation, but it will not be done here. The set of differentials is also found in [29, 11, 28]. Section 3 contains the KdV and NLS equations as examples of the theory developed in previous sections. A conclusion section comes at the end of the paper.

Remark.

It is important to mention here that, as in [25, 30], we also work the stationary case of integrable equations only. Therefore, we just consider the spectral problem and the corresponding element of the Lax pair, not the time evolution associated to the second element of the Lax pair.

Notation.

In this work, $f'(x)$ and $f_x(x)$ represent the derivative of $f(x)$ with respect to $x$ (even if $f$ depends on more variables). For the KdV equation (subsection 3.1), we prefer $q'$, $q''$, . . . ; for the derivatives of $q$ of successive order with respect to $x$. But, in the NLS equation (subsection 3.2), we use $q_x$, $q_{xx}$, . . . ; for the derivatives of $q$. In this last case, the derivatives of $E$ and $F$ (see equation 43), are better denoted $E'$, $E''$, $F$, $F''$, etc.
Nomenclature.

A squared eigenfunction, polynomial of degree $n$ in the variable $\lambda$, $\phi_n(x)$, will be called in this paper an $n$-soliton solution. See definition 1.1. (It is usual to call "solitons" to exact solutions of integrable partial differential equations, although we will not name them this way here).

1 Recursion formulæ and conditions of solvability

Assume $\hat{\mathcal{L}}$ is a multiplicative operator, polynomial of degree $d$ in $\lambda$, with coefficients depending on $x$,

$$\hat{\mathcal{L}}(x;\lambda) = \mathcal{L}_0 \lambda^d + \mathcal{L}_1(x) \lambda^{d-1} + \cdots + \mathcal{L}_d(x) = \sum_{j=0}^{d} \mathcal{L}_j(x) \lambda^{d-j},$$

and $\phi_n$ is an squared eigenfunction, and $n$-soliton solution, (associated to $\hat{\mathcal{L}}(x;\lambda)$), also polynomial in $\lambda$, but of degree $n$,

$$\phi_n(x;\lambda) = A_0 \lambda^n + A_1(x) \lambda^{n-1} + \cdots + A_n(x) = \sum_{i=0}^{n} A_i(x) \lambda^{n-i},$$

where $A_0 \neq 0$ and $\mathcal{L}_0 \neq 0$ are $x$-independent, i.e. are constants (that might depend on $t$ in the non-stationary case), $A_i$ and $\mathcal{L}_j$ are functions of $x$ only (and they might also depend on $t$ in the non-stationary case). And $\lambda$ is a constant (in principle, it could be a function of $t$ only in the non-stationary case).

Recursion formulæ for the squared eigenfunctions $\phi_n$ and conditions for the solutions are obtained when solving the equation

$$\phi''''_n - 2\left(\hat{\mathcal{L}}, \phi_n\right) = 0,$$

by setting to zero each coefficient of the polynomial. Notice that the term $2\left(\hat{\mathcal{L}}, \phi_n\right)$ turns to be a polynomial of at most degree $(n + d)$. From here, it is where the recursion formulæ and the $d$ extra conditions arise.

In fact, by bilinearity of the product $\langle \ , \ \rangle$, equation (8) is

$$\phi''''_n - 2\left(\hat{\mathcal{L}}, \phi_n\right) = \sum_{i=0}^{n} A_i'''' \lambda^{n-i} - 2 \left( \sum_{i=0}^{d} \mathcal{L}_i \lambda^{d-i}, \sum_{j=0}^{n} A_j \lambda^{n-j} \right)$$

$$= \sum_{i=0}^{n} A_i'''' \lambda^{n-i} - 2 \sum_{k=0}^{d+n} \left( \sum_{i+j=k, i,j \geq 0} \langle \mathcal{L}_i, A_j \rangle \right) \lambda^{d+n-k}$$

If we set all coefficients of (13) equal to zero, we find relations among the coefficients $A_i$ of the solutions $\phi_n(x;\lambda)$. 
For $k = 0$, the fact that $L_0$ and $A_0$ are constants implies that $\langle L_0, A_0 \rangle = 0$ and, hence, there is no term of degree $(d + n)$. For $k = 1$, it follows $\langle L_0, A_1 \rangle + \langle L_1, A_0 \rangle = 0$, and solving for $A_1$,

$$A_1 = -\frac{1}{2L_0} \int (L_1, A_0) \, dx.$$  

For $k = 2$, we can check that $\langle L_0, A_2 \rangle + \langle L_1, A_1 \rangle + \langle L_2, A_0 \rangle = \langle L_0, A_2 \rangle + \sum_{i=1}^{2} \langle L_i, A_{2-i} \rangle = 0$.

In general, for $k \leq d$ proceeding recursively, we obtain $\langle L_0, A_k \rangle + \sum_{i=1}^{k} \langle L_i, A_{k-i} \rangle = 0$. Solving for $A_k$,

$$A_k = -\frac{1}{2L_0} \int \sum_{i=1}^{k} \langle L_i, A_{k-i} \rangle \, dx.$$  

Starting at $k = d + 1$, the polynomial $\phi'''_n = \sum_{i=1}^{n} A''_i \lambda^{n-i}$ of degree $n - 1$ (remember $A_0 = \text{constant}$) should be now taken into account. For example, for $k = d + 1$, we obtain the formula:

$$A_{d+1} = -\frac{1}{2L_0} \left( \int \sum_{i=1}^{d} \langle L_i, A_{d+1-i} \rangle \, dx - \frac{1}{2} A_1'' \right).$$

Hence, the general recursion formula to compute the coefficients of $\phi_n(x; \lambda)$ is:

$$A_k = -\frac{1}{2L_0} \left( \int \sum_{i=1}^{d} \langle L_i, A_{k-i} \rangle \, dx - \frac{1}{2} A_k'' \right), \quad (14)$$

which holds for $k = 1, 2, \ldots, n$ (considering $A_{-l} = 0$ for $l > 0$).

Now, in order to have a solution to equation (12), a few extra conditions are needed. They come from imposing the coefficients of $\lambda^{d-1}, \lambda^{d-2}, \ldots, \lambda, 1$, in (13), to be zero.

Setting the coefficient of $\lambda^{d-1}$ equal to zero, we obtain the first condition:

$$A'''_{n-d+1} - \sum_{i=1}^{d} \langle L_i, A_{n+1-i} \rangle = 0.$$  

In general, the $d$ conditions to solve the system are

$$A_{n-s} := A'''_{n-s} - \sum_{i=d-s}^{d} \langle L_i, A_{d+n-s-i} \rangle = 0,$$  

for $0 \leq s \leq d - 1$, where $A_{n-s} = 0$ for values of $s$ such that $n - s < 0$. (This includes the cases $n \leq d$ and $d < n$).
There is an alternative form to express this condition by the use of the recursion formula (14). In fact, writing:

\[ A_{n,s} = A''_{n-s} - 2 \sum_{i=d-s}^{d} \langle \mathcal{L}_i, A_{d+n-s-i} \rangle \]

\[ = A''_{n-s} - 2 \sum_{i=1}^{d} \langle \mathcal{L}_i, A_{d+n-s-i} \rangle + 2 \sum_{i=1}^{d-s-1} \langle \mathcal{L}_i, A_{d+n-s-i} \rangle \]

\[ = 4\mathcal{L}_0 \left\{ \frac{1}{2\mathcal{L}_0} \left( \sum_{i=1}^{d} \langle \mathcal{L}_i, A_{d+n-s-i} \rangle - \frac{1}{2} A'''_{n-s} \right) \right\} + 2 \sum_{i=1}^{d-s-1} \langle \mathcal{L}_i, A_{d+n-s-i} \rangle \]

\[ = 4\mathcal{L}_0 A'_{n+d-s} + 2 \sum_{i=1}^{d-s-1} \langle \mathcal{L}_i, A_{d+n-s-i} \rangle, \quad \text{by the recursion (14)}, \]

therefore obtaining the alternative form for the conditions:

\[ A_{n,s} = 4\mathcal{L}_0 A'_{n+d-s} + 2 \sum_{i=1}^{d-s-1} \langle \mathcal{L}_i, A_{d+n-s-i} \rangle = 0. \quad (16) \]

We will use this alternative form to express Theorems 3.1, 3.3 and 3.4 in the examples for the KdV and the NLS equations.

We must point out that we can construct all the solutions to equation (12) using the general recursion formula subject to the \( d \) conditions.

**Definition 1.1.** Define the normalized \( n \)-soliton solution associated to the linear operator \( \hat{L} \) to be the solution to (9) obtained by the general recursion formula, assuming that \( A_0 = -2\mathcal{L}_0 \), and that all the constants of integration in (14) are zero. Also, name \( \psi_n \) a polynomial solution to (9) of degree \( n \) in \( \lambda \), an \( n \)-soliton solution associated to the linear operator \( \hat{L} \).

Thus, the following theorem holds.

**Theorem 1.1.** A) Each \( n \)-soliton \( \psi_n = \sum_{k=0}^{n} B_k(x)\lambda^{n-k} \) can be written as a linear combination of the normalized \( n \)-solitons: \( \phi_n, \phi_{n-1}, \ldots, \phi_0 \).

B) The linear combination

\[ \psi_n = K_0\phi_n + K_1\phi_{n-1} + \ldots + K_n\phi_0 \]

(with \( K_i \) constant for \( 0 \leq i \leq n \) and \( K_0 \neq 0 \)) is an \( n \)-soliton solution if, and only if, it satisfies the following conditions

\[ \mathcal{B}_{n,s} = \sum_{l=0}^{n} K_l A_{n-l,s} = 0, \quad \text{for} \quad 0 \leq s \leq d-1, \quad (17) \]

where \( A_{n-l,s} \) and \( \mathcal{B}_{n,s} \) are given as in equation (15) for \( \phi_n \) and \( \psi_n \), respectively. Since \( A_{n-1,s} = 0 \) is the \( s \)th condition for the normalized soliton \( \phi_{n-1} \), then \( \mathcal{B}_{n,s} = 0 \) is the \( s \)th condition for the soliton \( \psi_n \), i.e., similar conditions hold for \( \psi_n \).
Proof. A) Let \( A_k \) be the coefficient of the normalized soliton corresponding to the monomial \( \lambda^{n-k} \) for \( \phi_n \), with \( A_0 \neq 0 \). Let

\[
\psi_n(x; \lambda) = \sum_{k=0}^{n} B_k(x) \lambda^{n-k}
\]

be an \( n \)-soliton of degree \( n \), with \( B_0 \neq 0 \). Since \( A_0 \neq 0 \) and \( B_0 \neq 0 \) are constants, then \( K_0 := B_0 A_0^{-1} \) is a constant. Hence, using the recursion formula on the coefficients of \( \psi_n \), we compute:

\[
B_1 = -\frac{1}{2\mathcal{L}_0} \int \langle \mathcal{L}_1, B_0 \rangle \ dx
\]

\[
= K_0 \left( -\frac{1}{2\mathcal{L}_0} \int \langle \mathcal{L}_1, A_0 \rangle \ dx \right) = K_0 A_1 + C_1,
\]

where \( C_1 \) is a constant of integration. If we set \( K_1 = C_1 A_0^{-1} \), a similar computation gives:

\[
B_2 = -\frac{1}{2\mathcal{L}_0} \int \langle \mathcal{L}_1, B_1 \rangle \ dx
\]

\[
= K_0 \left( -\frac{1}{2\mathcal{L}_0} \int \langle \mathcal{L}_1, A_1 \rangle \ dx \right) + K_1 \left( -\frac{1}{2\mathcal{L}_0} \int \langle \mathcal{L}_1, A_0 \rangle \ dx \right)
\]

\[
= K_0 A_2 + K_1 A_1 + C_2 = K_0 A_2 + K_1 A_1 + K_2 A_0
\]

where \( C_2 \) is the constant of integration and \( K_2 := C_2 A_0^{-1} \). We have the following **claim**: there exist constants \( K_0, K_1, \ldots, K_s \) such that

\[
B_j(x) = \sum_{i=0}^{j} K_{j-i} A_i(x)
\]

for \( j \leq s \).

By induction, assume the claim is true for \( s \leq k \). The recursion formula (14) for \( B_{k+1} \) is

\[
B_{k+1} = -\frac{1}{2\mathcal{L}_0} \left( \int \sum_{i=1}^{d} \langle \mathcal{L}_i, B_{k+1-i} \rangle \ dx - \frac{1}{2} B_{k+1-d}' \right).
\]

Hence, using the induction hypothesis of the **claim** for \( s \leq k \),

\[
\sum_{i=1}^{d} \langle \mathcal{L}_i, B_{k+1-i} \rangle = \sum_{i=1}^{d} \sum_{j=0}^{k+1-i} \langle \mathcal{L}_i, \sum_{j=0}^{k+1-i-j} K_{k+1-i-j} A_j \rangle
\]

\[
= \sum_{i=1}^{d} \sum_{j=0}^{k+1-i} K_{k+1-i-j} \langle \mathcal{L}_i, A_j \rangle
\]

8
Descending ordering by the index $k + 1 - i - j$, we obtain that the right-hand-side of (20) can be written as (notice that we are taking the change of variable $l = i + j - 1$):

$$K_k(L_1, A_0) + K_{k-1}(\langle L_1, A_1 \rangle + \langle L_2, A_0 \rangle) + \ldots = \sum_{l=0}^{k} K_{k-l} \left( \sum_{i=0}^{d} \langle L_i, A_{l+1-i} \rangle \right)$$

(21)

But, also

$$B'_{k+1-d} = \left( \sum_{i=0}^{k+1-d} K_{k+1-d-i} A_i \right)'' = \sum_{i=1}^{k+1-d} K_{k+1-d-i} A_i'' = \sum_{l=0}^{k} K_{k-l} A''_{l+1-d}$$

(22)

Combining expressions (20, 21), equation (19) becomes

$$B_{k+1} = \sum_{l=0}^{k} K_{k-l} \left( -\frac{1}{2\mathcal{L}_0} \left( \int \sum_{i=1}^{d} \langle L_i, A_{l+1-i} \rangle \ dx \ - \frac{1}{2} A''_{l+1-d} \right) \right)$$

$$= \left( \sum_{l=0}^{k} K_{k-l} A_{l+1} \right) + C_{k+1}$$

But, setting $K_{k+1} := C_{k+1} A_0^{-1}$, we have:

$$B_{k+1} = \sum_{l=0}^{k+1} K_{k+1-l} A_l$$

as needed for proving the claim, equation (18).
Now, using the claim, we can express $\psi_n$ in terms of $\phi_n$, $\phi_{n-1}$, ..., $\phi_0$:

$$
\psi_n = \sum_{k=0}^{n} B_k \lambda^{n-k} = \sum_{k=0}^{n} \left( \sum_{i=0}^{k} K_{k-i} A_i \right) \lambda^{n-k}
$$

$$
= \sum_{k=0}^{n} \left( \sum_{j=0}^{k} K_j A_{k-j} \right) \lambda^{n-k} \quad \text{(setting } j = k - i) \]

$$
= \sum_{j=0}^{n} \sum_{k=j}^{n} K_j A_{k-j} \lambda^{n-k} \quad \text{(exchanging the order of summing)}
$$

$$
= \sum_{j=0}^{n} K_j \left( \sum_{k=j}^{n} A_{k-j} \lambda^{n-k} \right) \quad \text{(setting } l = k - j) \]

$$
= \sum_{j=0}^{n} K_j \phi_{n-j}
$$

(23)

B) From the proof of part A), we can see that $\psi_n = \sum_{j=0}^{n} K_j \phi_{n-j}$ if, and only if, the coefficients $B_l$ of $\psi_n$ relates to the coefficients of $\phi_n$, by the assertion of the equation (18), for $0 \leq j \leq n$.

Now, the definition of $B_{n,s}$ in (15) applied to the coefficients of $\psi_n$ is the equation

$$
B_{n,s} := B_{n-s}'' - 2 \sum_{i=d-s}^{d} \langle \mathcal{L}_i, B_{d+n-s-i} \rangle
$$

But,

$$
\sum_{i=d-s}^{d} \langle \mathcal{L}_i, B_{d+n-s-i} \rangle = \sum_{i=d-s}^{d} \langle \mathcal{L}_i, \sum_{j=0}^{d+n-s-i} K_{d+n-s-i-j} A_j \rangle
$$

$$
= \sum_{i=d-s}^{d} \sum_{j=0}^{d+n-s-i} K_{d+n-s-i-j} \langle \mathcal{L}_i, A_j \rangle \quad \text{(24)}
$$

where the last equation follows by setting $l = d + n - s - i - j$ and ordering by $l$.

Also, $B_{n-s}''' = \sum_{i=0}^{n-s} K_{n-s-i} A_i''' = \sum_{l=0}^{n-s} K_l A_{n-s-l}'$. Now, if $l > n - s$, then $A_{n-s-l} = 0$ (because $n - s - l < 0$). Therefore, we can write

$$
B_{n-s}''' = \sum_{l=0}^{n} K_l A_{n-s-l}''
$$

(25)
Thus, combining equations (24) and (25), we obtain:

\[
B_{n,s} = \sum_{l=0}^{n} K_l A''_{n-s-l} - 2 \sum_{l=0}^{n} K_l \left( \sum_{d-s}^{d} \langle L_i, A_{n+d-s-i-l} \rangle \right)
\]

\[
= \sum_{l=0}^{n} K_l \left( A''_{n-s-l} - 2 \sum_{d-s}^{d} \langle L_i, A_{n+d-s-i-l} \rangle \right)
\]

\[
= \sum_{l=0}^{n} K_l A_{n-l,s}
\]

Then, equation (17) holds, since \(A_{n-l,s} = 0\) is the \(s\)th condition for the normalized soliton \(\phi_{n-i}\). Hence, \(B_{n,s} = 0\) is the \(s\)th condition for the soliton \(\psi_{n-i}\), as we just proved.

\[\square\]

2 Geometry of the \(n\)-soliton.

In this section, we construct the hyperelliptic curve corresponding to the \(n\)-soliton solution related to the eigenvalue problem, equation (3), where \(\hat{L}\) is of degree \(d\) in \(\lambda\). We obtain a hyperelliptic curve,

\[
C_n : \frac{1}{2} Y^2 = -\mathcal{H}_n(X)
\]

where \(\mathcal{H}_n\) is a polynomial of degree \(2n + d\) in the variable \(X\), with constant coefficients. Thus, the genus of the curve \(C_n\) is \(n + (d - 1)/2\) or \(n + (d - 2)/2\), depending on if \(d\) is odd or even, respectively. The factorization of the \(n\)-soliton \(\phi_n = \sum_{i=0}^{n} A_i \lambda^{n-i}\) will give \(n\) different roots \(\lambda_k(x)\) depending on \(x\), such that the points \(P_k = (X_k, Y_k) = (\lambda_k, \phi'_n(x, \lambda_k))\), for \(1 \leq k \leq n\), will represent \(n\) points on the curve.

Hence, if we vary \(x\), a soliton solution defines \(n\) real curves on \(C_n\). These curves on the tangent space define a system of linear differentials. We will also compute these differentials in this section.

2.1 Computation of the function \(\mathcal{H}_n(\lambda)\).

Define

\[
\mathcal{H}(\phi) := \int \phi \left( \phi'' - 2(\hat{L}, \phi) \right) dx = \int \phi \left( \phi''' - 4\hat{L}\phi' - 2\hat{L}'\phi \right) dx
\]

(26)

Thus, integrating:

\[
\int \phi \phi'' dx = \phi\phi'' - \int \phi'\phi'' dx = \phi\phi'' - \frac{1}{2}(\phi')^2
\]

and

\[
- \int (4\phi \hat{L}\phi' + 2\phi \hat{L}'\phi) dx = -2 \int (\hat{L}(2\phi\phi' + \hat{L}'\phi^2)) dx = -2 \int (\hat{L}\phi^2) dx = -2\hat{L}\phi^2,
\]

we obtain

\[
\mathcal{H}(\phi) = \phi\phi'' - \frac{1}{2}(\phi')^2 - 2\hat{L}\phi^2.
\]

(27)

Observing the leading term, we obtain the following result.
Lemma 2.1. If $\phi$ is an $n$-soliton and $\deg(\hat{L}) = d$, then $\mathcal{H}(\phi)$ is a polynomial in $\lambda$, constant with respect to the variable $x$, of degree $2n+d$ with leading coefficient $-2\mathcal{L}_0 A_0^2$, where $A_0$ is the leading coefficient of $\phi$.

Now, if $\phi_n$ is the normalized $n$-soliton of degree $n$, set the function

$$\mathcal{H}_n(\lambda) := \mathcal{H}(\phi_n) = \phi_n \phi_n'' - \frac{1}{2} (\phi_n')^2 - 2\hat{L} \phi_n^2.$$  \hfill (28)

Next, we will give a precise formula to compute $\mathcal{H}_n(\lambda)$ as a polynomial in $\lambda$.

Consider the $n$-soliton solution (11) and the multiplicative operator (10), where $A_0$ and $L_0$ are non-zero constants (i.e., $x$-independent); $A_i$ and $L_j$ are functions of $x$; and $\lambda$ is a constant.

There are two ways to explicitly compute $\mathcal{H}_n(\lambda)$:

1. **Hard way**: Use directly the expression for $\mathcal{H}_n(\lambda) = \phi_n \phi_n'' - \frac{1}{2} (\phi_n')^2 - 2\hat{L} \phi_n^2$. We did it this way with the KdV equation. Details can be found in the proof of Theorem 7.6 in Appendix A of [11].

2. **Easy way**: Use bilinearity in the equation

$$\mathcal{H}_n(\lambda) = \int \phi_n \left( \phi_n''' - 2(\hat{L}, \phi_n) \right) dx + \text{constant of integration.}$$

Notice that when solving $\phi_n''' - 2(\hat{L}, \phi_n) = 0$, we obtain:

(a) the recursive formulæ to compute the coefficients $A_i$ of $\phi_n$;

(b) additional $d$ conditions of solvability.

Thus, from the discussion in section 11, we have

$$\phi_n''' - 2(\hat{L}, \phi_n) = \sum_{i=1}^{d} A_{n,d-i} \lambda^{d-i},$$  \hfill (29)

where $A_{n,d-i} = 0$ are the $d$ conditions in equation (15). From the previous equation and the fact that $\phi_n$ is a polynomial of degree $n$,

$$\mathcal{H}_n'(\lambda) = \phi_n (\phi_n''' - 2(\hat{L}, \phi_n))$$

$$= \sum_{k=1}^{d+n} \left( \sum_{\substack{i+j=k \atop d \geq i \geq 1 \atop n \geq j \geq 0}} A_{n,d-i} A_j \right) \lambda^{d+n-k}$$

Integrating (with respect to $x$) and observing that the highest degree in $\mathcal{H}_n(\lambda)$ is $-2\mathcal{L}_0 A_0^2 \lambda^{2n+d}$ (which turns to be the constant of integration by lemma (2.1)), we obtain the following theorem.
Theorem 2.2. The function $H_n(\lambda)$ is a polynomial of degree $2n + d$ with constant coefficients (i.e., they are $x$-independent) and is given by the formula

$$H_n(\lambda) = -2L_0A_0^2\lambda^{2n+d} + \sum_{k=1}^{d+n} \left( \sum_{\substack{i+j=k \\ d \geq i \geq 1 \\ n \geq j \geq 0}} \int A_{n,d-i}A_j \, dx \right) \lambda^{d+n-k}. \quad (30)$$

Remark. Please note that there is a gap (i.e., there is no terms) from $\lambda^{n+d}$ and up to $\lambda^{2n+d-1}$.

2.2 The hyperelliptic curve associated to the solutions.

Consider the normalized $n$-soliton $\phi_n(x) = \sum_{i=0}^{n} A_i(x)\lambda^{n-i}$. We can factorize $\phi_n(x)$ over $\mathbb{C}$ for fixed values of $x$ as

$$\phi_n(x; \lambda) = A_0 \prod_{i=1}^{n} [\lambda - \lambda_i(x)].$$

Here, $\lambda_i(x)$ are the roots of $\phi_n(x)$ which depend on $x$.

Taking the derivative with respect to $x$ of the previous expression, we obtain

$$\phi'_n(x; \lambda) = -A_0 \sum_{j=1}^{n} \lambda_j \prod_{i \neq j} [\lambda - \lambda_i(x)],$$

evaluating at $\lambda = \lambda_k(x)$, we finally get

$$\phi'_n(x, \lambda_k(x)) = -A_0 \lambda_k \prod_{i \neq k} [\lambda_k(x) - \lambda_i(x)]. \quad (31)$$

Now, evaluating $H_n(\lambda)$ in equation (28) at $\lambda = \lambda_k(x)$, and using that $\phi_n|_{\lambda=\lambda_k(x)} = 0$, we obtain

$$H_n(\lambda_k) = -\frac{1}{2} \left( \phi'_n(x, \lambda_k(x)) \right)^2 \quad (32)$$

Now, since $H_n(\lambda)$ is a polynomial of degree $2n + d$ (Theorem 2.2) with constant coefficients with respect to the variable $x$, the equation

$$H_n(X) = -\frac{1}{2} Y^2$$

is the equation of an hyperelliptic curve $\mathcal{H}_n$ of genus $n + (d - 1)/2$ if $d$ is odd, or $n + (d - 2)/2$ if $d$ is even; each $P_k = (X_k, Y_k) = (\lambda_k(x), \phi'_n(x, \lambda_k(x)))$, with $1 \leq k \leq n$, represents a point on the curve. Hence, if we vary $x$, a soliton solution defines $n$ real curves on $\mathbb{C}_n$. The curves on the tangent space define a system of linear differentials. Next, we compute these differentials.
In fact, combining equations (31) and (32), we have that
\[ \sqrt{-2\mathcal{K}_n(\lambda_k)} = \phi'_n(x, \lambda_k(x)) = -A_0\lambda'_k \prod_{i \neq k} [\lambda_k(x) - \lambda_i(x)]. \]

Hence, setting \( R_n = -2\mathcal{K}_n \), we obtain
\[ \lambda'_k \sqrt{R_n(\lambda_k)} = -A^{-1}_0 \prod_{i \neq k} [\lambda_k(x) - \lambda_i(x)]. \]

Now, the differentials
\[ \omega_\mu = \frac{X^{n-1}dX}{\sqrt{R_n(X)}}, \quad 1 \leq \mu \leq n \]
form a basis of the space of differentials of the curve \( \mathcal{K}_n \).

Evaluating those differentials at the points \( P_k \), we obtain
\[ \omega_\mu(P_k) = \omega_\mu(\lambda_k) = \frac{\lambda^{n-1}_k dx}{\sqrt{R_n(\lambda_k)}} = \frac{-A^{-1}_0 \lambda^{n-1}_k}{\prod_{i \neq k} [\lambda_k(x) - \lambda_i(x)]}. \]

Adding up over all points \( P_k \), and using the main Proposition in Appendix B in [11], we get
\[ \sum_{k=1}^{n} \omega_\mu(P_k) = \begin{cases} 0 & \text{if } 1 \leq \mu < n \\ -A^{-1}_0 & \text{if } \mu = n. \end{cases} \] (33)

Other proofs of this fact are found in [29, 28]. These differentials play a fundamental role on integrating the PDEs under consideration.

3 Examples

3.1 The Korteweg-deVries (KdV) equation.

For the the Korteweg-deVries equation (1), we have the associated linear eigenvalue problem (3), where \( \hat{L} = \lambda - q \) is the Schrödinger operator. Equation (7) becomes the linear differential equation:
\[ \phi''' + 4q\phi' + 2q'\phi = 4\lambda\phi', \] (34)

If we set the linear differential operator
\[ \mathcal{M} := \frac{d^3}{dx^3} + 4q\frac{d}{dx} + 2q', \] (35)
the differential equation (34) now reads
\[ \mathcal{M}(\phi) = 4\lambda\phi'. \] (36)
Set
\[ \phi_n(x, t) = F_{-1}(4\lambda)^n + F_0(4\lambda)^{n-1} + \cdots + F_{n-1} \]  
(37)

where \( F_{-1} \) is constant, and \( F_i(x) \) for \( -1 \leq i < n - 1 \) are implicit functions of \( x \). (The factor "4" just helps to normalize the conserved quantities). Notice the shift in the subscript as opposed to equation (11). Now, (36) implies the recursion formula:
\[ F_j = \int M(F_{j-1})dx, \quad \text{for } j = 0, 1, \ldots, n - 1. \]  
(38)

and the condition:
\[ F_n := \int M(F_{n-1})dx \quad \text{is a constant.} \]  
(39)

The function \( F_n(x) \) is the \( n^{th} \) conservation density of the KdV equation.

If we normalize setting \( F_{-1} = \frac{1}{2} \), the first few coefficients of \( \phi_n \) are:
\[ F_0 = q; \quad F_1 = q'' + 3q^2; \quad F_2 = q^{(4)} + 10qq'' + 5(q')^2 + 10q^3, \]  
(40)

which turn to be the conserved densities of the KdV equation.

Using the results of section [1], we can recover the formulae just presented. The recursion formula, equation (14) is
\[ A_k' = -\frac{1}{2}\mathcal{L}_0 \left( \sum_{i=1}^{d} \langle \mathcal{L}_i, A_{k-i} \rangle - \frac{1}{2} A_{k-d}' \right), \]
and the extra condition, equation (16), is \( A_{n,s} = 0 \), with
\[ A_{n,s} = 4\mathcal{L}_0 A_{n+d-s} + 2 \sum_{i=1}^{d-s-1} \langle \mathcal{L}_i, A_{d+n-s-i} \rangle. \]

For the KdV equation, the linear operator is \( \mathcal{L} = \lambda - q(x) \), and the \( n \)-soliton solution is \( \phi_n(x) = A_0\lambda^n + A_1(x)\lambda^{n-1} + \cdots A_n(x) \). Therefore, \( d = 1, \mathcal{L}_0 = 1, \mathcal{L}_1(x) = -q(x) \). Then, the recursion formula becomes:
\[ A_k' = \frac{1}{2} q'(x) A_{k-1} + q(x) A_{k-1}' + \frac{1}{4} A_{k-1}'' , \]
and we recover the recursion formula in equation (38).

For the extra-conditions, we just have one, with \( s = 0 \) (\( 0 \leq s \leq d - 1 = 0 \)), i.e.,
\[ 4\mathcal{L}_0 A_{n+1}' + 2 \sum_{i=1}^{0} \langle \mathcal{L}_i, A_{1+n-i} \rangle = 0. \]

Therefore, the unique condition is that \( A_{n+1} = 4F_n \) is constant. Same as condition (39).
**Definition 3.1.** A function of the form
\[ \psi_n = A_{-1}(4\lambda)^n + A_0(4\lambda)^{n-1} + \cdots + A_{n-2}(4\lambda) + A_{n-1}, \]
(with \(A_{-1}\) constant and \(A_i(x)\) are functions of \(x\) for \(0 \leq i \leq n - 1\)) solution of \(B(\psi_n) = 4\lambda\psi'_n\), is called a KdV \(n\)-soliton. The normalized KdV \(n\)-soliton, denoted by \(\phi_n\), is the KdV \(n\)-soliton setting: \(A_k = 4^{n-k}F_{k-1}\).

Theorem 1.1 in the KdV case specializes as follows.

**Theorem 3.1.** Each KdV \(n\)-soliton \(\psi_n\) can be written as a linear combination of the normalized solitons: \(\phi_n, \phi_{n-1}, \ldots, \phi_0\). Moreover, a linear combination \(\psi_n = \alpha_n\phi_n + \alpha_{n-1}\phi_{n-1} + \cdots + \alpha_0\phi_0\) (with \(\alpha_i\) constant and \(\alpha_n \neq 0\)) is a KdV \(n\)-soliton if, and only if, \(\alpha_nF_n + \alpha_{n-1}F_{n-1} + \cdots + \alpha_0F_0\) is constant.

Also, Theorem 2.2 has its KdV version.

**Theorem 3.2.** The function \(H_n(\lambda) := \int \phi_n B(\phi_n) dx - 2\lambda \phi_n^2\) is given by the formula:
\[
H_n(\lambda) = -\frac{(4\lambda)^{2n+1}F_{-1}^2}{2} + (4\lambda)^n F_{-1} F_n + (4\lambda)^{n-1} \left[ F_0 F_n - \int F'_0 F_n dx \right] +
+ (4\lambda)^{n-2} \left[ F_1 F_n - \int F'_1 F_n dx \right] + \cdots + \left[ F_{n-1} F_n - \int F'_{n-1} F_n dx \right].
\]

**Proof.** Since \(d = 1\) and \(L_0 = 1\) for the KdV equation, Theorem 2.2 implies that
\[
H_n(\lambda) = -2A_0^2\lambda^{2n+1} + \sum_{k=1}^{n+1} \left( \int A_{n,0} A_{k-1} \, dx \right) \lambda^{n+1-k}.
\]
But, in notation of section 2 we assume that \(\phi_n = \sum_{i=0}^{n} A_i \lambda^{n-i}\), while for the KdV example in this section, we use the classical representation of \(\phi_n\) as the sum
\[ \phi_n = F_{-1}(4\lambda)^n + F_0(4\lambda)^{n-1} + \cdots + F_{n-2}(4\lambda) + F_{n-1}. \]
Equating coefficients, we have the relation among them: \(A_j = 4^{n-j}F_{j-1}\). Hence, the leading term of the polynomial \(H_n(\lambda)\) is
\[
-2A_0^2\lambda^{2n+1} = -2 \left(4^n F_{-1}\right)^2 \lambda^{2n+1} = -\frac{(4\lambda)^{2n+1}F_{-1}^2}{2}.
\]
Now, using that \( A_{n,0} = 4A_{n+1}' \) (by the alternative form in equation (16)), and that \( F_n' = 4A_{n+1} \) and \( A_{k-1} = 4^{n+1-k}F_{k-2} \), we obtain that the sum in (12) becomes

\[
\sum_{k=1}^{n+1} \left( \int F_n' 4^{n+1-k} F_{k-2} \, dx \right) \lambda^{n+1-k} = \sum_{j=0}^{n} \left( \int F_n' F_{j-1} \, dx \right) (4\lambda)^{n-j}.
\]

Now, the result follows after integration by parts. \( \square \)

### 3.1.1 Examples.

Here, we show the normalized \( n \)-soliton solutions for \( n = 0, 1, 2 \) and 3:

\[
\begin{align*}
\phi_0(\lambda) &= \frac{1}{2} \\
\phi_1(\lambda) &= 2\lambda + q \\
\phi_2(\lambda) &= 8\lambda^2 + 4q\lambda + q'' + 3q^2 \\
\phi_3(\lambda) &= 32\lambda^3 + 16q\lambda^2 + 4(q'' + 3q^2)\lambda + q^{(4)} + 10qq'' + 5(q')^2 + 10q^3
\end{align*}
\]

And, for \( n = 0, 1 \) and 2, their corresponding hyperelliptic curves are

\[
\begin{align*}
J_0(\lambda) &= -\frac{1}{2}(\lambda - q) \\
J_1(\lambda) &= -8\lambda^3 + 2\lambda(q'' + 3q^2) + \left[ q(q'' + 3q^2) - \left( \frac{1}{2}(q')^2 + q^3 \right) \right] \\
J_2(\lambda) &= -128\lambda^5 + 8\lambda^2F_2 + 4\lambda K_2 + L_2,
\end{align*}
\]

where

\[
\begin{align*}
F_2 &= q^{(4)} + 10qq'' + 5(q')^2 + 10q^3, \\
K_2 &= qq^{(4)} - q^2 q^{(3)} + 10q^2q^{(2)} + \frac{1}{2}(q^{(2)})^2 + 10q(q')^2 + \frac{25}{2}q^4, \\
L_2 &= q^{(4)}(q^{(2)} + 3q^2) - q^{(3)}\left( \frac{1}{2}q^{(3)} + 6qq' \right) + q^{(2)}(8q^{(2)}q + 6(q')^2 + 30q^3) + 18q^5.
\end{align*}
\]

### 3.2 The Nonlinear Schrödinger (NLS) equation.

The stationary version of the NLS equation (2) is

\[
\frac{1}{2} \frac{\partial^2 q}{\partial x^2} + \sigma|q|^2q = \omega q,
\]

where \( \sigma = \pm 1 \) is the focusing/defocusing parameter and \( \omega \) is a constant. By the work of Kamchatnov, Kraenkel and Umarov [17, 18, 19], we know that the
multiplicative operator $\hat{L}$ of the NLS equation is a polynomial of degree $d = 2$ in $\lambda$:

$$\hat{L} = -\left(\lambda - \frac{i q_x}{2 q}\right)^2 - \sigma \|q\|^2 - \left(\frac{q_x}{2 q}\right)_x,$$

$$\hat{L} = -\lambda^2 + E\lambda + F \quad (43)$$

with

$$E = \frac{i q_x}{q} \quad (44)$$

and

$$F = -\frac{1}{4} E^2 - \sigma \|q\|^2 + \frac{i}{2} E'. \quad (45)$$

Hence, we have the following theorem.

**Theorem 3.3.** The following statements are true.

(i) $H_n(\lambda)$ is a polynomial in $\lambda$ of degree $2n + d = 2n + 2$.

(ii) The recursion formula is

$$A_j = \frac{1}{2} \int \left[ (E, A_{j-1}) + (F, A_{j-2}) \right] dx - \frac{1}{4} A''_{j-2} \quad (46)$$

for $j = 1, \ldots, n$ (assuming $A_{-1} = 0$).

(iii) The following two conditions hold:

**Condition A**, corresponding to $s = 1$:

$$A'_{n+1} = 0 \quad (47)$$

**Condition B**, corresponding to $s = 0$:

$$-4A'_{n+2} + 2A_{n+1}E' = 0 \quad (48)$$

*Proof.* The conditions stated in the Theorem are the alternative conditions $B_{n,s} = 0$ in equation (16) for the NLS equation. In this case, we have $d = 2$, $\mathcal{L}_0 = -1$, $\mathcal{L}_1 = E$, and thus, the conditions are:

$$B_{n,1} = -4A'_{n+1} = 0$$

and

$$B_{n,0} = -4A'_{n+2} + 2(E, A_{n+1}) = -4A'_{n+2} + 2A_{n+1}E' = 0;$$

hence $A_{n+1}$ is constant, by condition $B_{n,1} = 0$.

Theorem 1.1 in the NLS example can be stated as follows.
Theorem 3.4. Each NLS \( n \)-soliton solution \( \psi_n \) can be written as a linear combination of the normalized solitons: \( \phi_n, \phi_{n-1}, \ldots, \phi_0 \). Moreover, consider a linear combination

\[
\psi_n = \alpha_n \phi_n + \alpha_{n-1} \phi_{n-1} + \ldots + \alpha_0 \phi_0,
\]

where \( \phi_i \) is the normalized NLS \( i \)-th soliton, \( \alpha_i \) is constant, and \( \alpha_n \neq 0 \); then \( \psi_n \) is a NLS \( n \)-soliton if, and only if,

1. \( \alpha_n A_{n+1} + \alpha_{n-1} A_n + \ldots + \alpha_0 A_1 \) is constant.
2. \( \sum_{j=0}^{n} \alpha_j (-4A_{j+2} + 2A_{j+1}E) \) is constant.

Proof. The result follows after integration of conditions in theorem 3.3.

Theorem 3.5. The function \( \mathcal{H}_n(\lambda) \) for the NLS equation. For the NLS equation, the function \( \mathcal{H}_n(\lambda) \), which defines the hyperelliptic curve for the \( n \)-soliton solution, is given by the following formula:

\[
\mathcal{H}_n(\lambda) = 8\lambda^{2n+2} - 8A_{n+1}\lambda^{n+1} + \sum_{i=1}^{n} \left\{ \int [-4A_i A'_{n+1} + A_{i-1} B_n] \, dx \right\} \lambda^{n+1-i} + \int A_n B_n \, dx,
\]

where \( B_n = A'''_n - 2\langle \mathcal{L}_2, A_n \rangle \).

Proof. Extending the recursive formula (46) to \( j = n+1 \), Condition A becomes:

\[
0 = A''_{n+1} - 2 \left[ \langle E, A_n \rangle + \langle F, A_{n-1} \rangle \right] = -4 \left\{ \frac{1}{2} \left[ \langle E, A_n \rangle + \langle F, A_{n-1} \rangle \right] - \frac{1}{4} A'''_{n-1} \right\} = -4A'_{n+1},
\]

i.e., \( A_{n+1} \) is constant. In this case \( A_{n+1} \) is the new coefficient for the \((n+1)\)-soliton \( \phi_{n+1}(x; \lambda) \). It also represents the \((n+1)\)th conservation density of the NLS equation.

Hence, by equation (29),

\[
\phi'''_n - 2\langle \mathcal{L}, \phi_n \rangle = -4A'_{n+1}\lambda + B_n.
\]

and

\[
\mathcal{H}'_n(\lambda) = \phi_n (\phi'''_n - 2\langle \mathcal{L}, \phi_n \rangle) = \left( 2\lambda^n + \sum_{i=1}^{n} A_i \lambda^{i-1} \right) \left( -4A'_{n+1}\lambda + B_n \right)
\]

\[
= -8A_{n+1}\lambda^{n+1} + \sum_{i=1}^{n} (-aA_i A'_{n+1} + A_{i-1} B_n) \lambda^{n+1-i} + A_n B_n.
\]

Notice that the leading coefficient of \( \phi_n \) is \(-2\mathcal{L}_0 = 2\), by the definition of the normalized solitons. We obtain the result integrating. We just have to notice that the highest degree term in \( \mathcal{H}_n(\lambda) \) is \( 8\lambda^{2n+2} \), which follows from the highest term in \( \lambda \) in the summand \(-2\mathcal{L}\phi^n_n\) in equation (28). This term is the constant of integration.
3.2.1 Computations of some normalized $n$-soliton solutions

As in the KdV case, we compute some normalized $n$-soliton solutions for the NLS equation, using the recursion formula (46).

The normalized 0-soliton is

$$\phi_0 := A_0 = 2.$$  

Thus, $A_1 = \frac{\int (E, 2) dx}{2} = \frac{1}{2} \int 2E_x dx = E + C$ with $C$ a constant. If we set $C = 0$, we obtain the normalized 1-soliton

$$\phi_1 := 2\lambda + E,$$

with

$$A_1 = E = \frac{q_x}{q}$$  

being a constant for the 1-soliton solution.

Continuing with the recursion, we obtain $A_2 = \frac{1}{2} \int [(E, E) + (F, 2)] = \frac{3}{4} E^2 + F + C.$ Taking again $C = 0$, we get

$$\phi_2 := 2\lambda^2 + E\lambda + \left(\frac{3}{4} E^2 + F\right).$$

We can check that

$$A_2 = \frac{3}{4} E^2 + F = -\frac{1}{2} \frac{q_{xx}}{q} - \sigma |q|^2$$

is a constant for the 2-soliton solution. See section 3.2.4.

Using the recursion formula (46), and considering all constants of integration equal to zero, we compute the normalized 3-soliton solution:

$$\phi_3 := 2\lambda^3 + E\lambda^2 + \left(\frac{3}{4} E^2 + F\right) \lambda + \left(\frac{5}{8} E^3 + \frac{3}{2} F E - \frac{1}{4} E''\right)$$

with

$$A_3 = \frac{q_{xxx}}{q} + 6\sigma |q|^2 \frac{q_x}{q}$$

constant for the 3-soliton solution. See section 3.2.5.

Similarly, the 4-soliton solution is:

$$\phi_4 := 2\lambda^4 + E\lambda^3 + \left(\frac{3}{4} E^2 + F\right) \lambda^2 + \left(\frac{5}{8} E^3 + \frac{3}{2} F E - \frac{1}{4} E''\right) \lambda$$

$$+ \left(\frac{35}{64} E^4 + \frac{15}{8} E^2 F + \frac{3}{4} F^2 - \frac{5}{16} (E')^2 - \frac{5}{8} E E'' - \frac{1}{4} F''\right).$$
3.2.2 Successive derivatives of $E = \frac{iq_x}{q}$.

The first derivative of $E$ is $E' = \frac{iq_{xx}}{q} - \frac{iq^2}{q^2}$. Setting

$$E_{(2)} = \frac{iq_{xx}}{q},$$

we can write the first derivative as:

$$E' = E_{(2)} + iE^2$$

Now, define:

$$E_{(n)} := \frac{iq^{(n)}}{q},$$

Hence, we easily compute:

$$E'_{(n)} = E_{(n+1)} + iE_{(n)}E.$$ Using this notation, we can easily compute higher order derivatives of $E$. For example, $E'' = E'_{(2)} + 2iEE' = E_{(3)} + iE_{(2)}E + 2iE (E_{(2)} + iE^2)$, i.e.,

$$E'' = E_{(3)} + 3iE_{(2)}E - 2E^3$$

and $E''' = E'_{(3)} + 3i(E_{(2)}E)' - 6E^2E' = E''_{(4)} + 4iE_{(3)}E - 12E^2E_{(2)} - 6iE^4 + 3iE_{(2)}E^2$.

3.2.3 0-soliton solution for the NLS equation

We have that $\phi = A_0$ is a constant. Since $\hat{L} = -\lambda^2 + E\lambda + F$ thus, Condition A (equation (47)) gives $\langle E, A_0 \rangle = A_0E' = 0$. Hence, $E = iq_x/q = i(\ln q)' = k$ is constant. Then, $q$ satisfies the linear equation

$$q_x = -ikq,$$

which is the 0th equation in the NLS Lax hierarchy. Thus, it follows that $q = Ce^{-ikx}$, with $C$ and $k$ constants.

Condition B in (48) is $\langle F, A_0 \rangle = A_0F' = 0$. Hence, $F$ is constant and

$$F = \frac{1}{4}E^2 - \sigma\|q\|^2 + \frac{i}{2}E' = -\frac{1}{4}k^2 - \sigma\|q\|^2.$$

Hence, $\|q\|^2$ is constant, because $F$, $E$ and $\sigma$ are constants. This is the first conserved density of the NLS equation. Thus, we can conclude that $q = Ce^{-ikx}$ with $k \in \mathbb{R}$.

3.2.4 1-soliton solution for NLS equation

Condition A in (47) implies that $A_2 = \frac{3}{4}E^2 + F$ is constant. Now, using the expression of $F$ in (45), we conclude that

$$A_2 = \frac{1}{2}E^2 - \sigma\|q\|^2 + \frac{i}{2}E' = \text{constant}.$$
But, from (49) and (50) and from defining the constant \( \omega = -A_2 \), we obtain the stationary nonlinear Schrödinger equation:

\[
\sigma \|q\|^2 q + \frac{1}{2} q_{xx} = \omega q. \tag{53}
\]

Condition B in (48) implies that \(-4A_3 + 2A_2 E = \Omega\) is constant. After substitution of the values of \(A_2\) and \(A_3\), it becomes: \(-E^3 - 4FE + E'' = \Omega\). Now using (45) and (52), we simplify to:

\[
4\sigma \|q\|^2 q_x q + q_{xxx} q - q_{xx} q_x = \Omega q^2. \tag{54}
\]

From (53), we obtain:

\[
q_{xx} = 2\omega q - 2\sigma \bar{q} q^2,
\]

and taking its derivative,

\[
q_{xxx} = 2\omega q_x - 2\sigma \bar{q}_x q^2 - 4\sigma \|q\|^2 q_x.
\]

Substituting these expressions for \(q_{xx}\) and \(q_{xxx}\) to reduce the order of the derivatives in (54), we finally obtain that

\[
\bar{q}_q - \bar{q} q = \frac{\sigma \Omega}{2}
\]

where \(\Omega\) a constant. This is the second conserved density for the NLS equation.

### 3.2.5 2-soliton for NLS

Condition A for the 2-soliton becomes \(A_3 = \frac{5}{8} E^3 + \frac{3}{2} FE - \frac{1}{4} E''\) equals a constant. Multiplying by \(-4\) and substituting the values of \(E''\), \(E'\) and \(F\), and using (50), (52) and (49), we obtain \(6\sigma \|q\|^2 E + \|E(3)\| = i\omega_2\), where \(\omega_2\) is constant. In terms of \(q\), we have the following condition:

\[
q_{xxx} + 6\sigma \|q\|^2 q_x = \omega_2 q, \tag{56}
\]

which is is the complex modified Korteweg-deVries (mKdV) equation, which is the second equation in the NLS Lax hierarchy.

Now, Condition B for the 2-soliton is \(-4A_4 + 2A_3 E = \text{constant}\), which in terms of \(E, F\) and their derivatives, is

\[
\frac{-15}{16} E^4 - \frac{9}{2} FE^2 - 3F^2 + \frac{5}{4} (E')^2 + 2EE'' + E'' = \text{constant}.
\]

Using (45) to substitute \(F\) and \(E''\) and express only in terms of \(E\) and its derivatives, we obtain that the following expression is a constant:

\[
\frac{3}{2} (E')^2 + \frac{3}{2} E^2 + \frac{i}{2} E'' + 3\sigma \|q\|^2 E^2 + 3i\|q\|^2 E' - 3\|q\|^4 - \sigma \|q\|^2 = \text{constant}.
\]

From the expressions of \(E', E''\) and \(E'''\) in subsection 3.2.2 we simplify this second condition to:

\[
-\frac{1}{2} EE(3) + \frac{i}{2} E(4) + 3i\sigma \|q\|^2 E(2) - 3\|q\|^4 - \sigma \|q\|^2 = \Omega_2,
\]

where \(\Omega_2\) is a constant. In terms of \(q\) and its derivatives, we get

\[
\frac{1}{2} q_x q_{xxx} - \frac{1}{2} q_{xxx} q - 3\sigma \|q\|^2 q_x q - 3\|q\|^4 q^2 - \sigma \|q\|^2 q x q^2 = \Omega_2 q^2 \tag{57}
\]
Using condition (56) and its derivative (in order to reduce the order of the derivatives in (57)), we finally obtain,

\[ \|q_x\|^2 - \sigma \|q\|^4 - \frac{1}{3} \|q\|_{xx}^2 = \frac{\sigma \Omega_2}{3}, \]  

(58)

which is the third conserved density of the NLS equation.

4 Conclusions

In this paper, we have considered the spectral problem associated to completely integrable partial differential equations (PDEs) in the sense of Lax pairs theory. The spectral operator is assumed to be scalar, linear, multiplicative, and of polynomial form of degree \(d\) in the spectral parameter \(\lambda\). We assume that the solution to the PDE, \(q(x)\), is stationary and periodic in \(x\). We translated the spectral problem into a linear third order differential equation for the associated squared eigenfunctions (this is a standard linearization of the problem). We called \(n\)-solitons to the polynomial solutions of degree \(n\) in \(\lambda\) of the squared eigenfunction equations.

Then, we rewrite the linearized problem by introducing a bilinear form. Hence, using linear algebra and matching coefficients of same degree, we obtain recursion formulae to compute the basic \(n\)-solitons, which we proved generate the solutions to the considered differential equation. We called this formula the recursion formula of the \(n\)-solitons.

We also discovered \(d\) extra conditions which are necessary to be satisfied in order to have solutions to the considered equation. This \(d\) conditions give also important information of the system. We also show that the \(n\)-solitons solutions can be parametrized by points on a hyperelliptic curve with genus \(n + (d - 1)/2\) or \(n + (d - 2)/2\), depending on if \(d\) is odd or even, respectively. We find a formula for the equation of the hyperelliptic curve in terms of the coefficients of the \(n\)-solitons and the \(d\) conditions of the system, which define constants of motion of the system.

As examples, we consider two classical equations, the KdV and the NLS equations, which have multiplicative scalar operators of degree \(d = 1\) and \(d = 2\), respectively. We found (the recursion formula for) the coefficients of the \(n\)-soliton solutions (the squared eigenfuntions), the hyperelliptic curve and \(d\) extra conditions for each case. In the KdV case, the extra condition \((d = 1)\) states that the coefficients of the \(n\)-soliton solutions are constants and they turn to be the conserved densities of the KdV equation. For the NLS case, we have two extra conditions \((d = 2)\). One of them provides us again the coefficients of the \(n\)-soliton which are the conserved densities of NLS. The second condition represents the corresponding equations of the Lax hierarchy of the NLS equation.

Finally, these approach can be used in other integrable PDEs, if the pair Lax associated to the PDE is a scalar problem.
Acknowledgment.

The authors are very thankful with E. Martínez-Ojeda (UACM) for the comments, suggestions and useful discussions regarding the present work.

Espínola-Rocha dedicates this work to the memory of his teacher and mentor, Prof. A.A. Minzoni, who passed away on July 1st, 2017. The first beauties of integrable systems that he learned, were taught by Minzoni.

Portillo-Bobadilla dedicates this work to his wife and son, his supportive family. And also to the beautiful city of Guanajuato!

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