The moment method in the task of cosmic rays transport with finite speed in the framework of fractal Galaxy model

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ABSTRACT

The task of cosmic rays transport with finite speed in framework of fractal Galaxy model have considered. The moment method have been used for analysis of space characteristics of process. The asymptotic moments (at $t \to \infty$) have been obtained in the task of random walk with finite speed and with power law distributions of free path races $p_\xi(x) = \alpha x^\alpha_0 x^{-\alpha - 1}, x \to \infty, 0 < \alpha < 2$. Accounting a finite speed led to necessity divide the original task into two separate tasks. The former consist in propagation of cosmic rays with finite mathematical expectation of free path races ($1 < \alpha < 2$), and the latter consist in the propagation of cosmic rays with infinite mathematical expectation of free path races ($0 < \alpha < 1$). In the former case the asymptotic distribution is described by stable law and influence of finite speed reduced to decrease of diffusivity. In the latter case a situation cardinally is changing. The asymptotic distribution has U-shape or W-shape form. Distributions of cosmic rays is reconstructed by using the obtained moments.

Subject headings: anomalous diffusion, stable laws, fractal galaxy model, moment method

1. Introduction

Main reason of using of anomalous diffusion model to description of cosmic rays propagation is in turbulent character of magnetic lines. Their structure are extremely complicated and knotty. Magnetic field lines can both propagate over long distances, and be strongly intertwined in small (in comparison to the size of the Galaxy) regions of space, forming a magnetic cloud. Since cosmic rays generally are charged particles then they will be propagated along magnetic field lines. Getting in magnetic clouds cosmic rays can be trapped by them, thus getting in trap or reflected from them.

Let suppose that distribution of distance between the clouds $p_\xi(x)$ has power-law asymptotic

$$p_\xi(x) \propto \alpha x^\alpha_0 x^{-\alpha - 1}, \quad 0 < \alpha \leq 2, \quad x \to \infty,$$

(1)

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then the interstellar magnetic field will form a stochastic fractal. If we further make an assumption of an instantaneous transition of cosmic rays particles between two adjacent magnetic clouds and the power-law distribution of rest times in traps \( q_\tau(t) \propto \beta t_0^{\beta} t^{-\beta-1} \), \( 0 < \beta \leq 1 \), \( t \to \infty \), then the propagation process can be described by Continuous Time Random Walk (CTRW) model \( \text{(Metzler and Klafter 2000)} \). The model describe the processes of anomalous diffusion in which the width of the diffusion packed spreads according to \( \Delta(t) \sim t^{\beta/\alpha} \). Usage of anomalous diffusion model for description of propagation of cosmic rays in the Galaxy lead to conception of fractal distribution of inhomogeneities in the Galaxy or to the Fractal Galaxy model.

Fractal model of the galaxy is a generalization of the homogeneous model proposed by V. L. Ginzburg and S. I. Syrovatskii in \( \text{(Ginzburg and Syrovatskii 1964)} \). If we assume that in the CTRW model the distances between magnetic clouds \( \xi \) and rest time in traps \( \tau \) are distributed according laws with finite variation i.e. \( D\xi < \infty, D\tau < \infty \) (this situation correspond to values \( \alpha = 2, \beta = 1 \)), then in the asymptotic of large time the diffusion packed width will increase according to \( \Delta(t) = \sigma(t) \sim \sqrt{t} \), and the asymptotic distribution is described by Gauss distribution. In this case the fractal model of galaxy pass to homogeneous model. In the case with \( D\xi = \infty \) and \( D\tau = \infty \) (or \( \alpha < 2, \beta < 1 \)) asymptotic distribution of cosmic rays is described by fractional-stable law \( \text{(Uchaikin 2000; Bening et al. 2006)} \) which in effect is Green function of anomalous diffusion equation which is expressed through fractional derivatives.

Knowledge of a Green function allow to us to calculate cosmic rays spectrum. The process of anomalous diffusion for the first time was used for description for cosmic ray propagation in the work \( \text{(Ragot and Kirk 1997)} \). The author is considering propagation of electrons in the framework of CTRW model and on the basis of obtained Green function he is exploring of influence various propagation modes (such as subdiffusion, superdiffusion and normal diffusion) on spectral index of synchrotron radiation. The further advancement of the model of fractal galaxy have obtained in the works of Lagutin A. A. In his works with coauthors are obtained cosmic rays spectra \( \text{(Lagutin et al. 2001a,b)} \), electrons spectra \( \text{(Lagutin et al. 2001b)} \) and mass composition of cosmic rays \( \text{(Lagutin et al. 2001c)} \) on the basis of the anomalous diffusion model. The influence of nearby sources of cosmic rays on their spectra recorded at the Earth \( \text{(Lagutin et al. 2005, 2007, 2008)} \). The work \( \text{(Kermani and Fatemi 2011)} \) is devoted to investigation of a dependence the residence time of cosmic rays in galaxy from parameter \( \alpha \) in the framework of the model. In this work is showing that the residence time is decreased with comparing to homogeneous model. The same conclusion was obtained in our work \( \text{(Uchaikin et al. 2012)} \).

However the CTRW model has one essential disadvantage. The assumption about instantaneous transition of particles between two consecutive collisions lie in the basis of this scheme. This assumption is non-physical. Although the reader may object that in the basis of the homogeneous model of galaxy lies the model of normal diffusion which is also has the same assumption and this model has good agreement with experimental results. And so it is. However it is necessary take into account that in the model of normal diffusion is assumed that \( D\xi < \infty \) and \( D\tau < \infty \) and while the conditions \( \langle \xi \rangle \ll L, \langle \tau \rangle \ll T \) are satisfied, then an influence of finite speed can be neglected.
Here $L$ is a size of diffusion area, $T$ is time of a process, $\langle \xi \rangle$ and $\langle \tau \rangle$ are mean race and mean rest time correspondingly. In the opposite case $\langle \xi \rangle \gtrsim L$, $\langle \tau \rangle \gtrsim T$ it is necessary take into account the finiteness of a speed \cite{Uchaikin and Saenko 2001}. Just that very case is implementing in the anomalous diffusion model at $\alpha < 2$, $\beta < 1$. In this case a values of races can be comparable with the size of the galaxy, and trapping time can be comparable with the residence time of cosmic rays in the galaxy. Then the assumption about instantaneous transition of cosmic rays, to put it mildly, far from true. The same conclusion was obtained in \cite{Uchaikin 2010} where the applicability of the anomalous diffusion model to the description of cosmic rays propagation in the galaxy is analyzed.

In this paper we consider the propagation process of a finite speed without traps. When considering walks with finite speed traps can be excluded from consideration. Indeed, a transition from one space point to another occur instantaneous in the CTRW model and at an investigation of non-stationary problems is needed the process that allows to determine a particle position at some time moment. A trapping of particle by a trap is such process in the CTRW model. The introduction of the finite speed eliminates necessity of traps. At any time between two consecutive scattering the particle position can be determined from simple expression $x = vt$. The model with power law distribution of races is considered in this paper.

\section{Equation for moments}

Our task is obtaining of space distribution of cosmic rays and investigation of an influence of finiteness of speed on an asymptotic (at $t \to \infty$) distribution. For solution of the task we will use the moment method which gave good results at the solution of the single-velocity task of transport theory in the case of normal diffusion \cite{Uchaikin and Yarovikov 2000, Uchaikin et al. 2000}. We will use the method proposed in the work \cite{Uchaikin 1998} for finding of moments of distribution.

Let’s consider the process of symmetric walking in the N-dimensional space. Let $R_N(t)$ is N-dimensional random vector which characterises a shift of a particle for the time $t$. The speed of the particle is constant and don’t depend from motion direction and scattering is isotropic. At this assumptions for a random vector of shift we can write the following stochastic relation

$$R_N(t) = \begin{cases} vt\Omega, & \text{with probability } P_\xi(vt)d\Omega_N/S_N, \\ \xi\Omega + R_N(t - \xi/v), & \text{with probability } p_\xi(x)dxd\Omega_N/S_N, \end{cases}$$

where $p_\xi(x)dx$ is the probability that the particle has undergone the scattering at neighbourhood $dx$ of the point $x$, and

$$P_\xi(x) = \int_x^\infty p_\xi(y)dy,$$

$\Omega$ is unit vector, $d\Omega_N$ is element of spherical surface of unit radius at the neighborhood of $\Omega$, $S_N$ is area of sphere $S_N = \int d\Omega_N$. Taking into account the symmetry of the task the odd moments
will equal to zero. Rise the \( m \) to power \( 2n, n = 1, 2, 3, \ldots \) we obtain

\[
R_N^{2n}(t) = \begin{cases} 
(vt)^{2n}, & \text{with probability } P_\xi(vt)d\Omega_N/S_N, \\
(\xi \Omega + R_N(t - \xi/v))^{2n}, & \text{with probability } p_\xi(x)dx d\Omega_N/S_N,
\end{cases}
\]   (3)

Averaging now the expression (3) on random variables and multiplying out, we obtain

\[
m_{2n}^N(t) = (vt)^{2n}P_\xi(vt) + \sum_{n_1+n_2+n_3=n} 2^{n_2} \frac{n!}{n_1!n_2!n_3!} \int_0^vt x^{2n_1+n_2}m_{n_2}^N(t - x/v)m_{n_3}^N(t - x/v)\langle \cos^{n_2} \theta \rangle p_\xi(x)dx,
\]   (4)

where the sum is taken over all solutions of the equation \( n_1 + n_2 + n_3 = n \). Here the notations are introduced: \( m_{2n}^N(t) = \mathbb{E}R_N^{2n}(t) \) is stochastic moment of \( 2n \)-th order of random vector \( \mathbf{R} \) in \( N \)-dimensional space, \( \mathbb{E}X \) is mathematical expectation of random variable \( X \),

\[
\langle \cos^n \theta \rangle = \frac{1}{S_N} \int_{\Omega_N} (\cos \theta)^n d\Omega_N
\]   (5)

is mean cosine, and \( \theta \) is angle between vectors \( \mathbf{R}_N(t - \xi/v) \) and \( \Omega \). An calculation of mean cosine see in Appendix A. How we can see the expression (4) is recurrence relation and we must know the moments of all orders up to \( 2n \)-th for calculation of the moment of \( 2n \)-th order.

Taking into account normalizing condition for probability distribution function (PDF) \( \int p(x, t)dx = 1 \) possible immediately to write \( m_0^N = 1 \). Substituting to the expression (4) value \( n = 1 \) we are obtain the expression for the second moment

\[
m_{2}^N(t) = (vt)^2P_\xi(vt) + \int_0^vt x^2p_\xi(x)dx + \int_0^vt m_{2}^N(t - x/v)p_\xi(x)dx.
\]   (6)

Here is taking into account that \( m_{1}^N(t) = 0, \frac{1}{S_N} \int_{\Omega_N} \cos \theta d\Omega_N = 0, \frac{1}{S_N} \int_{\Omega_N} d\Omega_N = 1 \). Next, you will need the following integral, which is obtained by integrating by parts

\[
\int_0^vt x^k p_\xi(x)dx = -\int_0^vt x^k \frac{dP_\xi(x)}{dx}dx = -(vt)^k P_\xi(vt) + k \int_0^vt x^{k-1}P_\xi(x)dx.
\]   (7)

Using this relation for integrating of the second term in (6), the equation for second moment takes the form

\[
m_{2}^N(t) = 2 \int_0^vt xP_\xi(x)dx + \int_0^vt p_\xi(x)m_{2}^N(t - x/v)dx.
\]   (8)

For the forth moment from the equation (4) we are obtain

\[
m_{4}^N(t) = (vt)^4P_\xi(vt) + \int_0^vt x^4p_\xi(x)dx + \int_0^vt 2x^2m_{2}^N(t - x/v)(1 - 2\langle \cos^2 \theta \rangle)p_\xi(x)dx
\]

\[+ \int_0^vt m_{4}^N(t - x/v)p_\xi(x)dx.
\]   (9)
Taking into account (A2) for finding this integral, and using (7) for calculating of the second term in the equation (9) we obtain

\[ m_N^4(t) = 4 \int_0^{vt} x^3 P_\xi(x) dx + \int_0^{vt} \left( \frac{4+2N}{N} x^2 m_N^2(t-x/v) + m_N^4(t-x/v) \right) p_\xi(x) dx. \]  

(10)

Similarly, we can derive an equation for the moment of any order. Equations for \( m_N^6(t), m_N^8(t) \) \( m_{10}^N(t) \) have form

\[ m_N^6(t) = 6 \int_0^{vt} x^5 P_\xi(x) dx \]

\[ + \int_0^{vt} \left( \frac{3N+12}{N} \left( x^4 m_N^2(t-x/v) + x^2 m_N^4(t-x/v) \right) + m_N^6(t-x/v) \right) p_\xi(x) dx, \]  

(11)

\[ m_N^8(t) = 8 \int_0^{vt} x^7 P_\xi(x) dx + \int_0^{vt} \left[ \frac{4N+24}{N} \left( x^6 m_N^2(t-x/v) + x^2 m_N^4(t-x/v) \right) \right. \]

\[ + \frac{6N(N+2)+48(N+3)}{N(N+2)} x^4 m_N^4(t-x/v) + m_N^8(t-x/v) \] \]

\[ \left. + \frac{N(N+2)}{N} \left( x^6 m_N^4(t-x/v) + x^2 m_N^6(t-x/v) \right) + m_N^{10}(t-x/v) \right) p_\xi(x) dx, \]  

(12)

\[ m_{10}^N(t) = 10 \int_0^{vt} x^9 P_\xi(x) dx + \int_0^{vt} \left[ \frac{5N+40}{N} \left( x^8 m_N^2(t-x/v) + x^2 m_N^4(t-x/v) \right) \right. \]

\[ + \left( 10 + \frac{120}{N} + \frac{240}{N(N+2)} \right) \left( x^6 m_N^4(t-x/v) + x^4 m_N^6(t-x/v) \right) + m_{10}^N(t-x/v) \] \]

\[ + \left. \frac{N(N+2)}{N} \left( x^6 m_N^4(t-x/v) + x^2 m_N^6(t-x/v) \right) + m_N^{10}(t-x/v) \right) p_\xi(x) dx. \]  

(13)

The solutions of the equations (8), (10), (11), (12), (13) can be found by the method of Laplace transformation. By introducing the notations

\[ \tilde{m}_n(\lambda) = \int_0^{\infty} m_n^N(t) e^{-\lambda t} dt, \quad \tilde{p}_n(\lambda) = \int_0^{\infty} x^n p_\xi(x) e^{-\lambda x} dx, \quad \tilde{P}_n(\lambda) = \int_0^{\infty} x^n P_\xi(x) e^{-\lambda x} dx, \]

in terms of Laplace images the equations for the moments take the form of algebraical equations.
Find the solutions of these equations is easy

\[
\hat{m}_2(\lambda) = \frac{2\hat{P}_1(\lambda/v)}{\lambda(1 - \hat{p}_0(\lambda/v))},
\]

(14)

\[
\hat{m}_4(\lambda) = \frac{4\hat{P}_3(\lambda/v) + \frac{4+2N}{N}\hat{p}_2(\lambda/v)\hat{m}_2(\lambda)}{1 - \hat{p}_0(\lambda/v)},
\]

(15)

\[
\hat{m}_6(\lambda) = \frac{1}{1 - \hat{p}_0(\lambda/v)}\left(\frac{2}{4}\hat{P}_5(\lambda/v) + \frac{3N+12}{N}\hat{p}_4(\lambda/v)\hat{m}_2(\lambda) + \hat{p}_2(\lambda/v)\hat{m}_4(\lambda)\right),
\]

(16)

\[
\hat{m}_8(\lambda) = \frac{1}{1 - \hat{p}_0(\lambda/v)}\left(\frac{10}{6}\hat{P}_7(\lambda/v) + \frac{4N+24}{N}\hat{p}_6(\lambda/v)\hat{m}_2(\lambda) + \hat{p}_2(\lambda/v)\hat{m}_4(\lambda)\right)
+ \frac{6N(N+2)+48(N+3)}{N(N+2)}\hat{p}_4(\lambda/v)\hat{m}_4(\lambda),
\]

(17)

\[
\hat{m}_{10}(\lambda) = \frac{1}{1 - \hat{p}_0(\lambda/v)}\left(\frac{10}{10}\hat{P}_9(\lambda/v) + \frac{5N+40}{N}\hat{p}_8(\lambda/v)\hat{m}_2(\lambda) + \hat{p}_2(\lambda/v)\hat{m}_6(\lambda)\right)
+ \frac{10N(N+2)+120(N+4)}{N(N+2)}\hat{p}_6(\lambda/v)\hat{m}_4(\lambda) + \hat{p}_4(\lambda/v)\hat{m}_6(\lambda)),
\]

(18)

3. Asymptotic for moments

So far we didn’t make any assumption about kind of distributions of races and also we didn’t make any simplifications at the derivation of the system of equations (14). Therefore, the obtained system is exact system of equations for stochastic moments of particle distribution which are performing random walks with finite speed, without traps and at arbitrary kind of race distribution.

In the case exponential distribution of races \( p_\xi(x) = \mu \exp(-\mu x) \) the task is reduced to normal diffusion with finite speed. The exact description of this task is given by Telegraph equation (Uchaikin and Saenko 2001, 2000). In this case the Laplace transformation of (14) have the form

\[
\hat{p}_0(\lambda/v) = \frac{\mu v}{\lambda + \mu v}, \quad \hat{P}_1(\lambda/v) = \frac{v^2}{(\lambda + \mu v)^2}.
\]

Substitute this expressions to (14) we obtain

\[
\hat{m}_2(\lambda) = \frac{2v^2}{\lambda^2(\lambda + \mu v)}.
\]

Now, making the inverse Laplace transformation of this expression we get the expression for stochastic moment of second order

\[
m_2^D(t) = 2(\mu vt - 1 + \exp(-\mu vt))/\mu^2.
\]

This expression has exact coincidence with exact stochastic moment of second order which has been obtained in (Uchaikin and Saenko 2000). Now, if to consider the limit at \( t \to \infty \) we obtain

\[
m_2^D(t) \approx 2Dt, \quad \text{where} \quad D = v/\mu.
\]
This expression represent expression for second moment of normal diffusion with diffusion coefficient $D$.

It is supposed that in the model of fractal galaxy a distance between inhomogeneities of magnetic field $\xi$ are described by distribution with asymptotic form (1). The distributions of this type have the property that the moments of these distributions exist only for orders $\nu < \alpha$. In other words $\mathbb{E} \xi^\nu < \infty$ for $\nu < \alpha$ and $\mathbb{E} \xi^\nu = \infty$ for $\nu \geq \alpha$. It follows therefrom that for values $0 < \alpha \leq 1$ mathematical expectation is infinite and for values $1 < \alpha \leq 2$ mathematical expectation is finite. In connection of this we will consider these two cases separately.

The case $0 < \alpha < 1$. For calculation of moments for us will be needed expressions for $\hat{p}_0(\lambda), \hat{p}_n(\lambda), \hat{P}_n(\lambda)$.

After integration by parts for $\hat{p}_0(\lambda)$ we obtain

$$\hat{p}_0(\lambda) = \alpha x_0^\alpha \int_{x_0}^{\infty} x^{-\alpha-1} e^{-\lambda x} dx = e^{-\lambda x_0} - \lambda x_0^\alpha \int_{x_0}^{\infty} x^{-\alpha} e^{-\lambda x} dx.$$  

Next, making the change of integration variable $y = \lambda x$, we obtain

$$\hat{p}_0(\lambda) = e^{-\lambda x_0} - (\lambda x_0)^\alpha \int_{\lambda x_0}^{\infty} y^{-\alpha} e^{-\lambda y} dy = e^{-\lambda x_0} - (\lambda x_0)^\alpha \Gamma(1 - \alpha, \lambda x_0),$$  \hspace{1cm} (19)

where $\Gamma(a, x)$ is incomplete Gamma-function.

According to Tauber’s theorems, asymptotics at $t \to \infty$ corresponds to asymptotics in Laplace space at $\lambda \to 0$. Since in the present case $\mathbb{E} \xi = \infty$, then in the expansion of exponent to series we must retain terms with $\lambda^\nu$, where $\nu < 1$. As a result $\exp(-\lambda x_0) \approx 1$. For incomplete Gamma-function we obtain $\Gamma(1 - \alpha, \lambda x_0) \to \Gamma(1 - \alpha)$ at $\lambda \to 0$. Finally, substituting these expansions to (19) we obtain

$$\hat{p}_0(\lambda) \approx 1 - (\lambda x_0)^\alpha \Gamma(1 - \alpha).$$  \hspace{1cm} (20)

Similarly, we obtain expressions for the remaining transformant

$$\hat{p}_n(\lambda) = \alpha x_0^\alpha \int_{x_0}^{\infty} x^{n-\alpha-1} e^{-\lambda x} dx = \alpha x_0^\alpha \lambda^{\alpha - n} \Gamma(n - \alpha, \lambda x_0) \xrightarrow{\lambda \to 0} \alpha x_0^\alpha \lambda^{\alpha - n} \Gamma(n - \alpha),$$  \hspace{1cm} (21)

$$\hat{P}_n(\lambda) = x_0^\alpha \int_{0}^{\infty} x^{n-\alpha} e^{-\lambda x} dx = x_0^\alpha \lambda^{\alpha - n - 1} \Gamma(n - \alpha + 1).$$  \hspace{1cm} (22)

Substitute (20) and (22) in (14) and canceling the same multipliers for the asymptotics of the moment of the second order we obtain

$$\hat{m}_2(\lambda) = (1 - \alpha) v^2 \Gamma(3) \lambda^{-3}.$$  \hspace{1cm} (23)

Inverse Laplace transform leads to the final result

$$m_2^N(t) = (1 - \alpha) (vt)^2,$$  \hspace{1cm} (23)
Similarly, we obtain the expressions for all the remaining moments

\[ m_4(t) = 1/3 \left( 3 - 2\alpha + \frac{\alpha}{N} - \frac{\alpha^2}{N} \right) (1 - \alpha)(vt)^4, \]

\[ m_6(t) = ((5\alpha^2 - 17\alpha + 15)N^2 + 3\alpha(2\alpha^2 - 5\alpha + 3)N + 2\alpha^2(\alpha^2 - 2\alpha + 1)) \frac{(1 - \alpha)}{15N^2}(vt)^6, \]

\[ m_8(t) = \left( (105 + 79\alpha^2 - 155\alpha - 14\alpha^3)N^4 + (210 - 28\alpha^4 - 12\alpha^2 + 92\alpha^3 - 232\alpha)N^3 \\
- 2\alpha(\alpha - 1)(10\alpha^3 + 7\alpha^2 - 89\alpha + 87)N^2 - \alpha^2(5\alpha^2 + 43\alpha - 72)(\alpha - 1)^2N, \right) \\
- 12\alpha^3(\alpha - 1)^3 \frac{(\alpha - 1)(vt)^8}{105N^3(N + 2)} \]

\[ m_{10}(t) = (48(\alpha - 1)^4\alpha^4 + 2(7\alpha^2 + 113\alpha - 180)(\alpha - 1)^3N\alpha^3 + 5(14\alpha^3 + 53\alpha^2 \\
- 259\alpha + 228)(\alpha - 1)^2N^2\alpha^2 + 5(\alpha - 1)(27\alpha^4 - 46\alpha^3 - 195\alpha^2 + 562\alpha \\
- 390)N^3\alpha + (42\alpha^4 - 344\alpha^3 + 1106\alpha^2 - 1644\alpha + 95)N^5 + (120\alpha^5 \\
- 676\alpha^4 + 1132\alpha^3 + 282\alpha^2 - 2538\alpha + 1890)N^4) \frac{(1 - \alpha)(vt)^{10}}{945N^4(N + 2)}. \]

Let’s introduce concept of width of diffusion packet as \( \Delta(t) = \sqrt{m_2(t)} \). In common case the diffusion packet width widens according to law \( \Delta(t) \propto t^{\gamma} \). There exist three type of diffusion processes in dependence of value exponent \( \gamma \): \( \gamma > 1/2 \) corresponds to superdiffusion, \( \gamma < 1/2 \) corresponds to subdiffusion and \( \gamma = 1/2 \) corresponds to normal diffusion. In turn, the superdiffusion may be divided on the three modes: \( 1/2 < \gamma < 1 \) corresponds to superdiffusion, \( \gamma = 1 \) corresponds to quasiballistic mode and \( \gamma > 1 \) corresponds to superballistic mode. The meaning of the quasiballistic mode consist is that the diffusion packet widens with speed of free motion of particles. The superballistic mode corresponds to widening of the diffusion packet faster than in the case of free particle motion.

Using the concept of diffusion packet width form Eq. (23) can be obtained that \( \Delta(t) = \sqrt{1 - \alpha vt} \). This is corresponds to quasiballistic diffusion. In other words, in the case \( 0 < \alpha < 1 \) when PDF of races don’t has mathematical expectation, the diffusion packet widens with speed of free motion of particles. This is to be expected, as follows from physics considerations in time \( t \) a particle can overcome a distance equal to \( r = vt \). As result, the diffusion packet is remaining clamped in the space area which is defined by the constraint. As we will see further this constraint cardinaly change the shape of diffusion packet.

It is necessary to know the moment of time at which the process reaches an asymptotic regime. In Fig. 1 are shown the results of calculation of first five even moments obtained by Monte Carlo method and by obtained asymptotic expressions. The wander process with finite speed was modeled for calculation of moments by Monte Carlo method. As can be seen from the figure for these parameter values the wander process reaches the asymptotic regime for times \( T^* \geq 500 \). Let’s study how depend on \( \alpha \) the reaching time of the asymptotics. It is seen from the Fig. 1 the moment of second order reaches the asymptotics slower of all. That is why on Fig. 2 are shown the
calculation results of $m_2^N(t)$ in dependence of time at various values of $\alpha$. As can be seen from the figure, that the bigger a value of parameter $\alpha$, the slower the process reaches the asymptotic regime. However, it can be seen, that for time $T^* \geq 500$ the process has reached the asymptotics for all values of $\alpha$.

**The case** $1 < \alpha < 2$. This case differs from the previous one by existence of expectation of $p_\xi(x)$. Computing the Laplace transform of (3), we obtain

$$\hat{p}_0(\lambda) = \alpha x_0^\alpha \int_{x_0}^{\infty} x^{-\alpha-1} e^{-\lambda x} dx.$$ 

Integrating this expression by part twice we obtain

$$\hat{p}_0(\lambda) = e^{-\lambda x_0} + \frac{\lambda x_0}{1-\alpha} e^{-\lambda x_0} - \frac{(\lambda x_0)^\alpha}{1-\alpha} \Gamma(2-\alpha, \lambda x_0).$$ 

To find the asymptotic behaviour at $t \to \infty$ we again use the Tauberian theorems. Expanding the exponents into series and neglecting terms the terms with power larger than $\alpha$ we obtain

$$\hat{p}_0(\lambda) \approx 1 + \frac{\alpha x_0 \lambda}{1-\alpha} - \frac{(\lambda x_0)^\alpha}{1-\alpha} \Gamma(2-\alpha).$$ 

(28)
Substitute now (28), (22) into (14) we obtain asymptotics of the second moment
\[ \hat{m}_2(\lambda) = \frac{2x_0^2(\alpha - 1)\Gamma(2 - \alpha)v^{2-\alpha}}{\alpha(x_0/v)^{4-\alpha} - (x_0/v)^{2\alpha}\Gamma(2 - \alpha)\lambda^3}. \]

Since \( 1 < \alpha < 2 \), when \( \lambda \to 0 \) in the denominator of the second term can be neglected compared to the first and the asymptotic behavior takes the form
\[ \hat{m}_2(\lambda) \approx \frac{2x_0^{\alpha-1}(\alpha - 1)v^{3-\alpha}\Gamma(4 - \alpha)}{\alpha(3 - \alpha)(2 - \alpha)\lambda^{4-\alpha}}. \]

The inverse Laplace transform of the expression leads to expression of the asymptotics at \( t \to \infty \)
\[ M_N^N(t) \approx \frac{2x_0^{\alpha-1}(\alpha - 1)}{\alpha(3 - \alpha)(2 - \alpha)}(vt)^{3-\alpha}. \] (29)

All other moments are obtained similarly
\[ M_N^4(t) = \frac{4x_0^{\alpha-1}(\alpha - 1)(vt)^{5-\alpha}}{\alpha(5 - \alpha)(4 - \alpha)} + \frac{2(4 + 2N)(\alpha - 1)^2x_0^{2\alpha-2}(\Gamma(2 - \alpha))^2(vt)^{6-2\alpha}}{N\alpha\Gamma(7 - 2\alpha)} \] (30)
\[ M_N^6(t) = \frac{6x_0^{\alpha-1}(\alpha - 1)(vt)^{7-\alpha}}{\alpha(7 - \alpha)(6 - \alpha)} + \frac{6(3N + 12)x_0^{2\alpha-2}\Gamma(4 - \alpha)\Gamma(2 - \alpha)(1 - \alpha)^2(vt)^{8-2\alpha}}{N\alpha\Gamma(9 - 2\alpha)} + \frac{2(3N + 12)(4 + 2N)x_0^{3\alpha-3}\Gamma(2 - \alpha)(\alpha - 1)^3(vt)^{9-3\alpha}}{N^2\alpha\Gamma(10 - 3\alpha)} \] (31)
\[ M_N^8(t) = \frac{8x_0^{\alpha-1}(\alpha - 1)(vt)^{9-\alpha}}{\alpha(9 - \alpha)(8 - \alpha)} + 8\left( \frac{4\Gamma(6 - \alpha)\Gamma(2 - \alpha) + 3(N + 4)(\Gamma(4 - \alpha))^2}{2 + N} \right) \times \frac{(N + 6)x_0^{2\alpha-2}(1 - \alpha)^2(vt)^{10-2\alpha}}{N\alpha\Gamma(11 - 2\alpha)} + \frac{96(N + 6)(N + 4)}{N^2} \times \frac{x_0^{3\alpha-3}\Gamma(4 - \alpha)(\Gamma(2 - \alpha))^2(1 - \alpha)^3(vt)^{11-3\alpha}}{\alpha\Gamma(12 - 3\alpha)} + \frac{48(N + 6)(N + 4)(N + 2)}{N^3} \times \frac{x_0^{4\alpha-4}(\Gamma(2 - \alpha))^4(1 - \alpha)^4(vt)^{12-4\alpha}}{\alpha\Gamma(13 - 4\alpha)} \] (32)
\[ M_{10}^N(t) = \frac{10x_0^{\alpha-1}(\alpha - 1)(vt)^{11-\alpha}}{(11-\alpha)(10-\alpha)\alpha} + \frac{50(N + 8)\Gamma(8-\alpha)\Gamma(2 - \alpha)}{N} \]
\[ + \frac{100(N(14 + N) + 48)\Gamma(6 - \alpha)\Gamma(4 - \alpha)}{N(2 + N)} \frac{x_0^{2\alpha-2}(1 - \alpha)^2(vt)^{12-2\alpha}}{\alpha\Gamma(13 - 2\alpha)} \]
\[ + \frac{40(N + 8)(N + 6)(4\Gamma(6 - \alpha)(2 + N)\Gamma(2 - \alpha) + 3(N + 4)(\Gamma(4 - \alpha))^2)}{N^2(2 + N)} \]
\[ + \frac{2(10N(14 + N) + 480)(2\Gamma(6 - \alpha)(2 + N)\Gamma(2 - \alpha) + 9(N + 4)(\Gamma(4 - \alpha))^2)}{N^2(2 + N)} \]
\[ \times \frac{x_0^{3\alpha-3}(\alpha - 1)^3\Gamma(2 - \alpha)(vt)^{13-3\alpha}}{\alpha\Gamma(14 - 3\alpha)} \]
\[ + \frac{(480(N + 8)(N + 6)(N + 4) + 120(N(14 + N) + 48)(N + 4))}{\alpha\Gamma(15 - 4\alpha)N^3} \}
\[ \times \frac{(1 - \alpha)^4(\Gamma(2 - \alpha))^3\Gamma(4 - \alpha)x_0^{4\alpha-4}(vt)^{14-4\alpha}}{\alpha\Gamma(15 - 4\alpha)N^3} \]
\[ + \frac{240(N + 8)(N + 6)(N + 4)(2 + N)x_0^{5\alpha-5}(\alpha - 1)^5(\Gamma(2 - \alpha))^5(vt)^{15-5\alpha}}{\alpha\Gamma(16 - 5\alpha)} \] \tag{33}

From the obtained expression for \( M_{10}^N(t) \) we see that in the case under consideration the diffusion packet widens according to \( t^{(3-\alpha)/2} \). This corresponds to the subdiffusion. As seen, the diffusion packet widens slower than in the case of the ballistic regime, and the kinematic restriction \( (r = vt) \) will not have influence to the form of the diffusion packet. According to the article \( \text{(Zolotarev et al. 1999)} \) an influence of a finite speed reduces to replacement of the diffusivity \( D \rightarrow D_v, (D_v < D) \) in the superdiffusion equation which is expressed through the Laplacian of fractional order.

Let us analyze the asymptotic behavior. The values of the moments \( 2\sqrt{M_{10}^N(t)} \) are shown on the Fig. 3. It is seen from the figure, the larger of the moment order, the faster it reaches an asymptotic. The second moment reaches an asymptotics slower than others. Although, one can consider, that to the times of order \( T^* > 5 \cdot 10^4 \) already all moments reach asymptotics. Let’s analyze behaviour of the moments in depend of \( \alpha \). Let’s consider a dependence of the \( M_{10}^N(t) \) of \( \alpha \) since it reaches a asymptotics slower than all others. The results of calculation of \( \sqrt{M_{10}^N(t)} \) for \( N = 1 \) at various values of the parameter \( \alpha \) are shown at the Fig. 4. From the calculations one can see that at time \( T^* > 5 \cdot 10^4 \) and at values \( 1 < \alpha \leq 1.8 \) the second moment reaches a asymptotics already, while at the value \( \alpha = 1.95 \) to this time moment asymptotics still not reached. According to this we will be reconstruct the PDF only for values \( \alpha \in (1, 1.8] \) and for time \( t \geq T^* \).
Fig. 3.— The first five even moments for the case $1 < \alpha < 2$ for univariate walking ($N = 1$) in dependence of time $t$. Here $\alpha = 1.8$, $\nu = 0.3$. Black points are results of Monte Carlo method, solid lines are asymptotics (29) - (33). The figure shows the results for $\sqrt{M_{2n}^N(t)}$, $n = 1, 2, 3, 4, 5$. The numeration corresponds to the location curves upward.

Fig. 4.— The second moment $\sqrt{M_2^N(t)}$ for the case $1 < \alpha < 2$ in dependence of time for values of $\alpha$ are shown on the figure. Black points are results of Monte Carlo method, solid lines are asymptotics (29).

4. Reconstruction PDF

As noted in the previous section the kinematic constraint has an essential influence on the shape of the diffusion packet in the case $0 < \alpha < 1$. In the case when the mathematical expectation is finite ($1 < \alpha < 2$), the kinematic constraint has no influence on the shape of the diffusion packet. According to this fact, a reconstruction of the asymptotic distribution will carried out by using different systems of the orthogonal polynomials.

Choice of the polynomial system is carried out according to the information which available about the shape of the distribution. System of polynomials is chosen so that the shape of the distribution is most accurately described by the weight function. According to the a results of the Monte Carlo method it is obtained, that the distribution has U-shape in the case $0 < \alpha < 1$ and it has bell-shape in the case $1 < \alpha < 2$. According to this information in the first case the Tchebychev polynomials of the 1-st kind $T_n(x)$ are used for reconstruction of a distribution and in the second case the Hermite polynomials are used.

In common case the reconstruction procedure consists in follows. Let $P_n(x)$ is the system of orthogonal polynomials on the segment $[a, b]$ with the weight function $w(x)$ with the orthogonality
condition
\[
\int_a^b w(x) P_n(x) P_k(x) dx = \begin{cases} 
0, & n \neq k, \\
h_k, & n = k.
\end{cases}
\]

Then a required probability density function can be developed according to this system of a polynomial
\[
p(x, t) = w(x) \sum_{k=0}^{\infty} c_k(t) P_k(x).
\tag{34}
\]

The expansion coefficients \(c_k(t)\) are found from the expression
\[
c_k(t) = \frac{1}{h_k} \int_a^b p(x, t) P_k(x) dx.
\tag{35}
\]

Take into account the expression \(P_k(x) = \sum_{j=0}^{k} a_j x^j\), it can be obtain the following expression for the coefficients
\[
c_k(t) = \frac{1}{h_k} \sum_{j=0}^{k} a_j m_j(t),
\]
where \(m_j(t)\) are statistical moments of \(j\)-th order of a distribution \(p(x, t)\).

**Case** \(0 < \alpha < 1\). We will be use the Tchebychev polynomials of 1-st kind with weight function
\[
w(\xi) = \frac{1}{\sqrt{1 - \xi^2}}
\]
for a reconstruction of a PDF. The Tchebyshev polynomials possess orthogonal property on the segment \([-1, 1]\) and at the reconstruction of density it is necessary to change coordinates so, that the density was concentrated on this segment. Since we consider the motion with a finite speed, the density distribution is concentrated within the \(-vt \leq x \leq vt\). Outside of the segment the \(p(x, t) = 0\), and transition to the variable \(-1 \leq \xi \leq 1\), where \(\xi = x/vt\), makes it possible to use the Tchebyshev polynomials. At such transition the probability density and moments are transform according to law
\[
p(\xi, t) = \frac{p(x, t)}{vt}, \quad \mu_n^N(t) = \frac{m_n^N(t)}{(vt)^n},
\]
where \(m_n^N(t)\) are moments of the distribution \(p(x, t)\), and \(\mu_n^N(t)\) are moments of the distribution \(p(\xi, t)\).

Substitute now expression for the Tchebyshev polynomials to the Eq. \((34)\) and Eq. \((35)\) we obtain the expansion for the PDF
\[
p(\xi, t) \approx \frac{1}{\sqrt{1 - \xi^2}} \sum_{k=0}^{5} c_{2k}(t) T_{2k}(\xi), \quad t > T^*.
\tag{36}
where the expansion coefficients have the form

\[ c_l(t) = \frac{l}{2h_l} \sum_{m=0}^{[l/2]} \frac{(-1)^m(l - m - 1)!2^{l-2m}}{m!(l - 2m)!} \mu_{l-2m}(t). \]

Here \([A]\) denote integer part of number \(A\).

The results of the reconstruction are shown on the Figs. 5 and 6. On the figures the dots denote the PDF is obtained by Monte Carlo method, solid curve denote the results of a reconstruction which obtained according to Eq. (36). It is seen from the Fig. 5 that within interval \(0.4 \leq \alpha \leq 0.8\) the PDF is reconstructed very well. However for other values of \(\alpha\) a deficiency appears of the number moments used for the reconstruction. This clear can be see from the Fig. 6 where a typical polynomial behaviour of a function appears. At larger or at smaller values of \(\alpha\) such behaviour becomes stronger and the function can take negative values an therefore can't be used as probability density function.

**Case** \(1 < \alpha < 2\). In this case the Hermite polynomials were used for reconstruction of the PDF. In this case the mathematical expectation exists, and when \(\alpha = 2\) a variance exists also. Moreover, it is known, that when \(\alpha = 2\) the process under consideration reduced to the process of normal diffusion, which has Gauss distribution as asymptotic distribution. Therefore, let’s redefine the

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**Fig. 5.**— Reconstructed PDF for the case \(0 < \alpha < 1\) and values of \(\alpha\) are shown on the figure and \(N = 1, \nu = 0.3, t = 1000\).

**Fig. 6.**— Same as on the Fig. 5
Hermite polynomials such way that the weight function took the form
\[ w(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left( -\frac{x^2}{2\sigma^2} \right), \]
where \( \sigma^2 = M_N^2(t) \). Then, the formula for the Hermite polynomials takes the form
\[ H_n(x) = \frac{n!}{\sigma^{2n}} \sum_{m=0}^{[n/2]} \frac{(-1)^m \sigma^{2m}}{m!2^m(n-2m)!} x^{n-2m}, \]
and the orthogonality condition is \( h_k = k!/\sigma^{2k} \).

Substitute this expressions to (34) and (35) we obtain
\[ p(x, t) \approx \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left( -\frac{x^2}{2\sigma^2} \right) \sum_{k=0}^{5} a_{2k}(t) H_{2k}(x), \]
with the coefficients
\[ a_l(t) = \sum_{m=0}^{[l/2]} \frac{(-1)^m \sigma^{2m}}{m!2^m(k-2m)!} M_N^{l-2m}(t), \]
where the moments \( M_N^{l}(t) \) are defined by expressions (29) - (33).

Nevertheless, despite the fact, that the obtained asymptotics for the moments (29) - (33) coincide with the monte Carlo results (see Fig. 3) and the same approach was used as in the work (Uchaikin et al. 2000) for a reconstruction of the PDF, in that case it don’t succeeds to reconstruct of the PDF. The cause in following. The first, the main asymptotics term was retained at calculation of asymptotics for the moments, all others terms have been neglected. The second, it has been obtained only the first five moments, since the formula (34) is exact only at infinity numbers of summands, then in the case \( 1 < \alpha < 2 \) the number of moments which were obtained not enough for a reconstruction of the PDF.

5. Conclusions and Results

This paper is devoted to investigation the effect of finite speed on the space distribution of the cosmic rays in the framework of random walks without traps and with power-law distribution of the free-path length. This model describes a propagation of the cosmic rays within framework of fractal galaxy model. Accounting the finite speed leads us to necessity to divide the process under consideration by two case: when the expectation of the free-path length exists (1 < \( \alpha < 2 \)), and when expectation free-path length don’t exists (0 < \( \alpha < 1 \)).

In the first case (1 < \( \alpha < 2 \)) from the Eq. (29) follow, that \( \Delta(t) \propto t^{\gamma(\alpha)}, \gamma(\alpha) = (3-\alpha)/2 \) and the diffusion packet widens slower than in the quasiballistic case. This leads to absence of influence of the kinematic constraint to the shape of the diffusion packet, and the regime of propagation...
corresponds to the superdiffusion with $\gamma(\alpha) \in (1/2, 1)$. This case has been considered in the work [Zolotarev et al. 1999], where was shown, that the asymptotic distribution is described by stable-law with characteristic exponent $\alpha$. Accounting the finite speed leads to decreasing of the diffusivity.

Unfortunately, the method of moment doesn’t allow to reconstruct the PDF of the cosmic rays by using first five moments. The reasons of this: both small number of the obtained moments and the fact, that when we were finding the moments, only the main asymptotic term has been retained. The last fact leads to worsening of the results. Nevertheless, the obtained asymptotics for the moments coincide with Monte Carlo results, this allows us to use them for analysis of asymptotic behaviour of the process.

In the case of propagation with infinite free-path length ($0 < \alpha < 1$) the situation cardinally changes. The analysis of the second moment (see Eq. (23)) shows, that the diffusion packet widens according to $\Delta(t) \propto vt$. Such law of widening corresponds to the superdiffusion, more exactly to quasi-ballistic regime of the superdiffusion. It is should be noted, that an influence of finite speed leads to vanishing of the dependence from $\alpha$ in the law of widening of the diffusion packet. Indeed, it is known, that in the framework of the CTRW model with finite of mathematical expectation of rest time the width on the diffusion packet widens according to law $\Delta(t) \propto t^{\gamma(\alpha)}$, where $\gamma(\alpha) = 1/\alpha$. It is known, the CTRW model assumes instantaneous movement of particles between two successive traps. This lead to that in the case $\alpha < 1$ the diffusion packet widens faster than the kinematic constrain $r = vt$. Another words, the diffusion packet widens with bigger speed, than the speed of free motion of particles. Accounting of the finiteness of speed leads to vanishing of the dependence of exponent $\gamma$ of $\alpha$. In the considered case the $\gamma = 1$ at all values within range $0 < \alpha < 1$. It follows from (23) the speed of widening of the diffusion packet is $v_{\Delta(t)} = \sqrt{1 - \alpha v}$ and $v_{\Delta(t)} \to v$ at $\alpha \to 0$ and $v_{\Delta(t)} \to 0$ at $\alpha \to 1$. This leads to the two different shapes of the diffusion packet in those two extreme cases. The reconstruction of the PDF of cosmic rays confirms this conclusion, within range values $0 < \alpha \lesssim 0.6$ the diffusion packet has U-shape form, and within range of values $0.6 < \alpha < 1$ the diffusion packet has W-shape form (see Fig. 5).

As we can see from the figure, the asymptotic distribution isn’t the stable distribution, as this could be assumed from the generalized central limit theorem. Indeed, we are considering the process of walking of a particle with finite speed. The realization of the one particle trajectory represents the sum of independent and identical distributed random variables distributed according to (1). Each trajectory represents a individual sum. As result we obtain the series of sums of random variables $\xi_{i,j}$

$$S_{i,N(t)} = \sum_{j=1}^{N(t)} \xi_{i,j}, \quad i = 1 \ldots K, \quad (37)$$

where $i$ is trajectory number. The random variables $\xi_{i,j}$ are identical distributed and independent both between themselves and between series. The distribution of those random variables has power-law asymptotics. One would think, the conditions of generalized limit theorem [Zolotarev 1986]
are fully satisfied here and the limit distribution of such series of the sum have to belong to class of the stable laws. However, as it is seen from the Fig. 5 obtained limit distributions aren’t belong to this class of laws. The reason of this in following. First, we are considering movement with finite speed. This means, that at moment \( t^* \) the particle can’t be found outside of area \( x \in [-v t^*, v t^*] \). The particle is clamped in this area of space. Secondly, we are considering distribution of the sum (37) at some time moment \( t^* \), which is evaluated from the condition

\[
\sum_{i=1}^{N(t^*)} \frac{\xi_{i,j}}{v} \leq t^* < \sum_{i=1}^{N(t^*)+1} \frac{\xi_{i,j}}{v}.
\]  

Hence, the time moment \( t^* \) defines the number of terms in each sum. In the third, the distribution (1) has finite mathematical expectation at \( \alpha > 1 \) and infinite mathematical expectation within region \( 0 < \alpha \leq 1 \). This means, that distribution of this random variable has heavy tail and the smaller the value of the parameter \( \alpha \), the greater part of the distribution concentrates in the tail. Since the position of the particle are constrained by the kinematic limitation \( [-v t, v t] \) the particle can’t leave this area, and since the greater part of probability concentrates in the tail of distribution (1) then, the smaller value of parameter \( \alpha \), the greater part of probability concentrates within the area adjacent to the lines \( x = vt \) and \( x = -vt \). This leads to formation of U-shape and the W-shape forms of the limit distributions.

The stable distributions are obtained if in given scheme we will make the substitution \( N(t) \rightarrow N \), where \( N \) is fixed, but enough large number. In this case the time \( t \) fully is excluded from consideration and we come to the condition of the generalized limit theorem. The question arises, could whether we select a such time that the equality \( N(t) = N \) is satisfied? It appears that it is impossible. Rather, such time exists, but it equals to infinity, since \( E \xi = \infty \). It is worth to note, that in the case \( E \xi < \infty \) it is possible to point such time moment at which the equality \( N(t) = N \) is satisfied. From this follows, that the conditions of generalized limit theorem are fully satisfied and the asymptotic distribution will belong to the class stable laws.

It is worth to note, that the similar task has been considered in the work [Uchaikin and Sibatov 2011]: propagation of cosmic rays within framework of fractal galaxy model with accounting of finiteness of speed. In the work was obtained that in the one-dimensional case the process is described by the telegraph equation which is expressed through derivatives of fractional order. The solution of the equation was obtained in the work which is expressed trough elementary functions. The results of the work fully coincide with the results obtained here.

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A. An calculation of $\langle \cos^n \theta \rangle$

It will be necessary some known formulas for calculation of integral (5). The full solid angle in the spherical coordinates in multivariate space or that the same, the area of surface of unit sphere in multivariate space is defined as

$$S_N = \frac{2\pi^{N/2}}{\Gamma(N/2)}.$$  

The element of solid angle in multivariate space is

$$d\Omega_N = \prod_{k=1}^{N-1} \sin^{k-1}\theta_k d\theta_k.$$  

If we take that angle $\theta$ is angle between the vector $R_N(t - \xi/v)$ and the axis $x_N$ in Cartesian coordinates then we obtain

$$\langle \cos^n \theta \rangle = \frac{1}{S_N} \int \cdots \int_{\Omega} \cos^n \theta_{N-1} \sin^2 \theta_2 \sin^2 \theta_3 \cdots \sin^{N-2} \theta_{N-1} d\theta_1 d\theta_2 d\theta_3 \cdots d\theta_{N-1} = \frac{S_{N-1}}{S_N} \int_0^\pi \cos^n \theta_{N-1} \sin^{N-2} \theta_{N-1} d\theta_{N-1}.$$  

If we place under the sign of the differential $\sin \theta_{N-1}$ and we will make the change of variable $\mu = \cos \theta_{N-1}$, we obtain

$$\langle \cos^n \theta \rangle = \frac{S_{N-1}}{S_N} \int_{-1}^1 \mu^n (1 - \mu^2)^{(N-3)/2} d\mu. \quad (A1)$$

The case when $n$ is even is interesting for us.

Further, it will be necessary following known integral (see integral 2.2.4.9 [Prudnikov et al. 2002])

$$\int_{0}^{a} x^{2m}(a^2 - x^2)^{k-1/2}dx = \frac{a^{2m+2k} \Gamma(m + 1/2)\Gamma(k + 1/2)}{2 \Gamma(m + k + 1)}.$$  

If we substitute this integral and also expressions for $S_N$ and $S_{N-1}$ into Eq. (A1) we finally obtain

$$\langle \cos^{2m} \theta \rangle = \frac{\Gamma(N/2)\Gamma(m + 1/2)}{\sqrt{\pi} \Gamma(N/2 + m)}, \quad (A2)$$

where $m = n/2$.

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