Integrable operators and squares of Hankel matrices

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Abstract

In this note, we find sufficient conditions for an operator with kernel of the form
\[ A(x)B(y) - A(x)B(y)/(x - y) \] (which we call a Tracy–Widom type operator) to be the square of a Hankel operator. We consider two contexts: infinite matrices on \( \ell^2 \), and integral operators on the Hardy space \( H^2(T) \). The results can be applied to the discrete Bessel kernel, which is significant in random matrix theory.

Keywords: discrete-time Lyapunov equation, Tracy–Widom operator, Hankel operator

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1 Introduction

In random matrix theory it is natural (see, e.g. [1]) to consider integrable operators \( T \), where the kernel of \( T \) is
\[ \sum_{j=1}^{n} \frac{A_j(z)B_j(w)}{z - w}, \]
and \( \sum_{j=1}^{n} A_j(z)B_j(z) = 0 \). Here we are concerned with a special class of such operators, namely those with kernel of the form
\[ K(x, y) = \frac{A(x)B(y) - A(y)B(x)}{x - y} \quad (x \neq y), \]
which we shall refer to as Tracy–Widom operators. The variables \( x \) and \( y \) may be non-negative integers, as in the discrete kernels considered in section 2, continuous real parameters, as in e.g. [2], or may live on the circle, as in section 3. We look for conditions under which these operators can be expressed as \( \Gamma^2 \) or \( \Gamma^*\Gamma \), where \( \Gamma \) is a Hankel operator. In particular we recover a result of Borodin et al [3], showing that the discrete Bessel kernel can be written as
\[ \sqrt{\theta} J_x(2\sqrt{\theta})J_{y+1}(2\sqrt{\theta}) - J_y(2\sqrt{\theta})J_{x+1}(2\sqrt{\theta}) = \sum_{k=0}^{\infty} J_{x+k+1}(2\sqrt{\theta})J_{y+k+1}(2\sqrt{\theta}). \]

We can then read off information about \( K \) from knowledge of the Hankel operator \( \Gamma \). For example, a trace formula follows immediately, and the spectrum of \( K \) can be calculated from the spectrum of \( \Gamma \) (which in many cases is easier to calculate). Megretski, Peller and Treil [4] have characterised the self-adjoint bounded linear operators that are unitarily equivalent to Hankel operators: we apply their results to gain spectral information about the operators \( K \).
2 Discrete integrable operators

Define \(\mathbb{N}_0 = \mathbb{N} \cup \{0\}\). We consider infinite matrices with kernel \(K(x,y)\), where \(K(x,y)\) is defined by (2).

Recall that a Hankel matrix \(\Gamma_{\phi} = [\phi(m+n)]_{m,n \geq 0}\) with \((\phi(k)) \in \ell^2\) has square

\[
\Gamma_{\phi}^2 = \left[ \sum_{k=0}^{\infty} \phi(m+k)\phi(n+k) \right]_{m,n=0}^{\infty}.
\]

Nehari’s theorem (see, e.g. [5, p. 3]) states that \(\Gamma_{\phi}\) is a bounded operator on \(\ell^2(\mathbb{N}_0)\) if and only if \((\phi(n))\) are the positive Fourier coefficients of some function in \(L^\infty(T)\). We write the kernel \(K(x,y)\) in matricial form,

\[
K(x,y) = \frac{1}{x-y} \langle Fa(x), a(y) \rangle, \quad (x \neq y)
\]

\[
a(x) = \begin{bmatrix} A(x) \\ B(x) \end{bmatrix}, \quad F = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},
\]

and look for sufficient conditions under which we can construct a function \(\phi : \mathbb{N}_0 \to \mathbb{C}\) with \((\phi(j)) \in \ell^2\), such that

\[
K(x,y) = \sum_{k=0}^{\infty} \phi(x+k)\phi(y+k), \quad (x \neq y).
\]

**Definition 2.1** Let \(S\) be the shift operator on \(\ell^2(\mathbb{N}_0)\), so that \(Sf(x) = f(x-1)\) (where we define \(f(-1) = 0\)), and let \(R\) be the adjoint shift operator \(Rf(x) = f(x+1)\). The forward difference operator \(\Delta\) is defined by \(\Delta f(x) = f(x+1) - f(x)\). Notice that \(\Delta = R_x - I\). Where there are several variables, we write \(R_x, \Delta_y\) and so on.

As usual, \(A^T\) is the transpose of a matrix \(A\), while \(B^*\) denotes the adjoint of an operator \(B\).

**Lemma 2.2** (Lyapunov equation) Suppose that \(R\) and \(B\) are bounded linear operators on \(\ell^2\) such that

\[
\sum_{j=0}^{\infty} \langle R^j BB^* (R^*)^j \xi, \xi \rangle < \infty \quad \text{for all } \xi \in \ell^2,
\]

so that the series

\[
K = \sum_{j=0}^{\infty} R^j BB^* (R^*)^j
\]

is convergent in the weak operator topology. Then

\[
K - RKR^* = -BB^*.
\]

**Proof.** Clear from calculation of the left hand side of (7). \(\blacksquare\)

In the following Lemma, we state explicitly the specialisation of the above result to discrete kernels.

**Lemma 2.3** Let \(\Phi(x,y)\) be any function \(\Phi : \mathbb{N}_0^2 \to \mathbb{C}\), and suppose \(\phi : \mathbb{N}_0 \to \mathbb{C}\) is such that \((\phi(j)) \in \ell^2\). Then

\[
\Phi(x,y) = \sum_{k=0}^{\infty} \phi(x+k)\phi(y+k) \quad \text{for all } x,y \in \mathbb{N}_0
\]
if and only if
\[(\Delta_x S_y + \Delta_y)\Phi(x, y) = -\phi(x)\phi(y) \quad \text{for all } x, y \in \mathbb{N}_0 \quad (9)\]

and
\[\Phi(x, y) \to 0 \quad \text{as } x \text{ or } y \to \infty. \quad (10)\]

**Proof.** Suppose (8) holds. Then we have
\begin{align*}
(\Delta_x S_y + \Delta_y) \sum_{k=0}^{\infty} \phi(x+k)\phi(y+k) &= (S_x S_y - I) \sum_{k=0}^{\infty} \phi(x+k)\phi(y+k) \\
&= \sum_{k=0}^{\infty} (\phi(x+k+1)\phi(y+k+1) - \phi(x+k)\phi(y+k)) \\
&= -\phi(x)\phi(y),
\end{align*}
so that (9) holds. By the Cauchy-Schwarz inequality, and since \((\phi(j)) \in \ell^2\), we have
\[\Phi(x, y) = \sum_{k=0}^{\infty} \phi(x+k)\phi(y+k) \leq \left( \sum_{k=0}^{\infty} \phi(x+k)^2 \right)^{1/2} \left( \sum_{k=0}^{\infty} \phi(y+k)^2 \right)^{1/2} \to 0 \quad \text{as } x \text{ or } y \to \infty, \quad (12)\]
which is condition (10). Conversely, suppose that we have (9) and (10), and let
\[G(x, y) = \Phi(x, y) - \sum_{k=0}^{\infty} \phi(x+k)\phi(y+k). \quad (13)\]
By (13), we have
\[(\Delta_x S_y + \Delta_y)G(x, y) = 0 \quad \text{for all } x, y \in \mathbb{N}_0,
\]
so that \(G(x, y) = G(x+1, y+1)\) for all \(x, y \in \mathbb{N}_0\). We then use the hypothesis (10) and the estimate in (12) to show that \(G(x, y) \to 0 \) as \(x \) or \(y \to \infty\), and hence that \(G\) is identically zero for all non-negative integers \(x\) and \(y\), so that (8) holds. \(\blacksquare\)

**Theorem 2.4** Let \(K(x, y)\) be as defined in (8), with \((a(x))_{x=0}^{\infty} = ([A(x), B(x)]^T)_{x=0}^{\infty} \) a sequence of 2 \(\times\) 1 real vectors such that
\[\sum_{x \geq 0} \|a(x)\|^2 < \infty. \quad (14)\]
Suppose that there exists a sequence of 2 \(\times\) 2 real matrices \(S_x\) such that \(a(x+1) = S_x a(x)\) for all \(x \in \mathbb{N}_0\) and that
\[C = \frac{S_y^T F S_x - F}{x - y} \quad (15)\]
is a constant matrix. Then \(C\) is symmetric. Suppose further that \(C\) has eigenvalues \(\lambda \in \mathbb{R} \setminus \{0\}\) and 0, and let \([\alpha, \beta]^T\) be a real unit eigenvector corresponding to \(\lambda\). Then
\[K(x, y) = -\text{sgn}(\lambda) \sum_{k=0}^{\infty} \phi(x+k)\phi(y+k) \quad \text{for } x, y \in \mathbb{N}_0 \quad (x \neq y), \quad (16)\]
where
\[\phi(x) = |\lambda|^{1/2} (\alpha A(x) + \beta B(x)) \quad (17)\]
and \((\phi(x)) \in \ell^2\).
Proof. We set

$$C = \frac{S^T F S_x - F}{x - y} \tag{18}$$

where $C$ is constant by hypothesis, so that we can exchange the roles of $x$ and $y$, and find that $C^T = C$. We have, for $x \neq y$,

$$\begin{align*}
(\Delta_x S_y + \Delta_y) K(x, y) &= (S_x S_y - I) \frac{1}{x - y} \langle F a(x), a(y) \rangle \\
&= S_x \frac{1}{x - y - 1} \langle F a(x), S_y a(y) \rangle - \frac{1}{x - y} \langle F a(x), a(y) \rangle \\
&= \frac{1}{x - y} \langle S_x a(x), S_y a(y) \rangle - \frac{1}{x - y} \langle F a(x), a(y) \rangle \\
&= \frac{1}{x - y} \langle (S_y F S_x - F) a(x), a(y) \rangle \\
&= \langle C a(x), a(y) \rangle. \quad (19)
\end{align*}$$

Since $C$ is real and symmetric, and by hypothesis has eigenvalues $\lambda \neq 0$ and 0, there exists a real orthogonal matrix $U$ of unit eigenvectors such that

$$U^T C U = \begin{bmatrix}
\lambda & 0 \\
0 & 0
\end{bmatrix}. \quad (20)$$

We have

$$\begin{align*}
(\Delta_x S_y + \Delta_y) K(x, y) &= \langle C a(x), a(y) \rangle \\
&= \left\langle U \begin{bmatrix}
\lambda & 0 \\
0 & 0
\end{bmatrix} U^T a(x), a(y) \right\rangle \\
&= \lambda \left\langle \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix} U^T a(x), U^T a(y) \right\rangle \\
&= \lambda \left\langle \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix} U^T a(x), \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix} U^T a(y) \right\rangle \\
&= \text{sgn}(\lambda) \phi(x) \phi(y), \quad (21)
\end{align*}$$

where

$$\begin{bmatrix}
\phi(x) \\
0
\end{bmatrix} = \begin{bmatrix}
|\lambda|^{1/2} & 0 \\
0 & 0
\end{bmatrix} U^T a(x). \quad (22)$$

Note that $(\phi(x)) \in \ell^2$ by the condition $\sum_{x \geq 0} \|a(x)\|^2 < \infty$, since $U$ is a constant matrix. It is also clear that $K(x, y) \to 0$ as $x$ or $y \to \infty$, by the same condition on $a(x)$. We now let $[\alpha, \beta]^T$ be a real unit eigenvector of $C$ corresponding to $\lambda$, and the result follows by Lemma 2.3.

Corollary 2.5 Let $K(x, y)$ be as defined in (9), with $(a(x))_{x=0}^\infty = ([A(x), B(x)]^T)_{x=0}^\infty$ a sequence of $2 \times 1$ real vectors such that

$$\sum_{x \geq 0} \|a(x)\|^2 < \infty. \quad (23)$$
Suppose that \( a(x+1) = (Lx + M)a(x) \) (for all \( x \in \mathbb{N}_0 \)), where \( L \) and \( M \) are real constant \( 2 \times 2 \) matrices that satisfy
\[
\begin{aligned}
\det L &= 0 \\
\det M &= 1 \\
M^T FL \text{ is symmetric, and has eigenvalues } \lambda \in \mathbb{R} \setminus \{0\} \text{ and } 0.
\end{aligned}
\]
Let \( [\alpha, \beta]^T \) be a real unit eigenvector of \( M^T FL \) corresponding to \( \lambda \). Then
\[
K(x, y) = -\text{sgn}(\lambda) \sum_{k=0}^{\infty} \phi(x+k)\phi(y+k) \quad \text{for all } x, y \in \mathbb{N}_0 \quad (x \neq y),
\]
where \( \phi(x) = |\lambda|^{1/2}(\alpha A(x) + \beta B(x)) \), and \( (\phi(x)) \in l^2 \).

**Proof.** We have \( M^T FM = F \det M \) (indeed, this is true for any \( 2 \times 2 \) matrix) and hence \( M^T FM = F \). Likewise \( L^T FL = 0 \). Setting \( S_x = Lx + M \) as in Theorem 2.4, we now have
\[
\frac{S_x^T FS_x - F}{x - y} = \frac{(L_y + M)^T F(L_x + M) - F}{x - y} = \frac{M^T FLx - (M^T FL)^T y}{x - y} \quad \text{(since } F^T = -F) \]
\[
= \frac{M^T FL(x - y)}{x - y} \quad \text{(since } M^T FL \text{ is symmetric by hypothesis)}
\]
\[
= M^T FL.
\]
Hence \( C = (S_x^T FS_x - F)/(x - y) \) is a constant matrix. Thus, together with the summability criterion on the sequence \( (a(x)) \), the hypotheses of Theorem 2.4 are all satisfied, so we have the result.

**Example 2.6**

Let \( J_{\nu}(z) \) be the Bessel functions of the first kind of order \( \nu \), and write \( J_x = J_x(2\sqrt{\theta}) \), where \( \theta \) is a positive real parameter. The discrete Bessel kernel
\[
J(x, y; \theta) = \sqrt{\theta} \frac{J_x J_{y+1} - J_y J_{x+1}}{x - y}
\]
arises in the study of various discrete-variable random matrix models, as in [6] and [3]. Note that \( J_x \) is an entire function of \( x \), so that \( J(x, x; \theta) \) is well-defined via L’Hopital’s rule. In the notation of Corollary 2.5 we take
\[
a(x) = \begin{bmatrix} \sqrt{\theta} J_x \\ J_{x+1} \end{bmatrix}.
\]
The standard formula (see [3, p. 379])
\[
e^{2t \sin \theta} = J_0(2t) + 2 \sum_{m=1}^{\infty} J_{2m}(2t) \cos 2m\theta + 2i \sum_{m=1}^{\infty} J_{2m-1}(2t) \sin(2m - 1)\theta
\]
and Parseval’s identity can be used to show that \( J_0(2t)^2 + 2 \sum_{m=1}^{\infty} J_m(2t)^2 = 1 \) for all real \( t \), and hence that the sequence \( (J_x)_{x=0}^{\infty} \) is square summable. Thus the condition \( \sum_{x \geq 0} \|a(x)\|^2 < \infty \) is satisfied.

The 3-term recurrence relation for the Bessel functions
\[
J_{x+2}(2z) - \frac{x+1}{z} J_{x+1}(2z) + J_x(2z) = 0
\]
becomes
\[
a(x + 1) = \begin{bmatrix} \sqrt{\theta} J_{x+1} \\ J_{x+2} \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{\theta} \\ -\frac{1}{\sqrt{\theta}} & \frac{x+1}{\sqrt{\theta}} \end{bmatrix} \begin{bmatrix} \sqrt{\theta} J_x \\ J_{x+1} \end{bmatrix},
\] (30)
and so we have \( a(x + 1) = (Lx + M)a(x) \), where
\[
L = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{\sqrt{\theta}} \end{bmatrix}
\]
and
\[
M = \begin{bmatrix} 0 & \sqrt{\theta} \\ -\frac{1}{\sqrt{\theta}} & \frac{1}{\sqrt{\theta}} \end{bmatrix}.
\]

It is clear that these matrices satisfy \( \det L = 0 \) and \( \det M = 1 \), and we have
\[
M^TFL = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix},
\]
so we pick the unit eigenvector \([\alpha, \beta]^T = [0, 1]^T\). Thus, the function \( \phi(x) \) in Corollary 2.5 is \( J_{x+1}(2\sqrt{\theta}) \), and we recover a result of Borodin et al in [3]
\[
J(x, y; \theta) = \sum_{k=0}^{\infty} J_{x+k+1}(2\sqrt{\theta}) J_{y+k+1}(2\sqrt{\theta}), \quad x, y \in \mathbb{N}_0,
\] (31)
without their use of asymptotic formulae for the Bessel functions.

The preceding results are identities of kernels for \( x \neq y \). Evidently, the sum in the right-hand side of (16) makes sense for \( x = y \), and hence gives one possible extension of the left-hand side to the case \( x = y \). We use the extension to define an operator \( K \) with matrix given by \( K(x, y) \).

**Proposition 2.7** Suppose that the vector \([A(x), B(x)]^T \) satisfies the conditions of Theorem 2.4 so that \( K(x, y) = \sum_{k=0}^{\infty} \phi(x+k)\phi(y+k) \). Suppose also that \( \sum_{n=0}^{\infty} n\phi(n)^2 < \infty \). Then the operator \( K \) represented by the matrix \([K(x, y)]_{x, y=0}^{\infty} \) is trace class and has trace:
\[
\text{trace } K = \sum_{x=0}^{\infty} (x + 1)\phi(x)^2.
\] (32)

**Proof.** The summability condition on \( \phi \) ensures that \( \Gamma_\phi \) is Hilbert-Schmidt, which implies that \( K = \Gamma_\phi^2 \) is trace-class. We have
\[
\text{trace } K = \sum_{x=0}^{\infty} K(x, x) = \sum_{x=0}^{\infty} \sum_{k=0}^{\infty} \phi(x+k)^2
\] (33)
from which the result follows immediately.

**Definition 2.8** For a compact and self-adjoint operator \( W \) on a Hilbert space \( H \), the spectral multiplicity function \( \nu_W(\lambda) : \mathbb{R} \to \{0, 1, \ldots\} \cup \{\infty\} \) is given by
\[
\nu_W(\lambda) = \dim \{ x \in H : Wx = \lambda x \} \quad (\lambda \in \mathbb{R}).
\] (34)
We now give the consequences of a result of Peller, Megretski and Treil in [4] in the case of discrete integrable operators.

**Proposition 2.9** Suppose that $\Gamma_\phi$ and $K$ are as in Proposition 2.7. Then $\Gamma_\phi$ and $K$ are compact and self-adjoint, and

- (i) $\nu_K(0) = 0$ or $\nu_K(0) = \infty$;
- (ii) for $\lambda > 0$, $\nu_K(\lambda) < \infty$ and $\nu_K(\lambda) = \nu_{\Gamma_\phi}(\sqrt{\lambda}) + \nu_{\Gamma_\phi}(-\sqrt{\lambda})$;
- (iii) if $\nu_K(\lambda)$ is even, then $\nu_{\Gamma_\phi}(\sqrt{\lambda}) = \nu_{\Gamma_\phi}(-\sqrt{\lambda})$;
- (iv) if $\nu_K(\lambda)$ is odd, then $\left|\nu_{\Gamma_\phi}(\sqrt{\lambda}) - \nu_{\Gamma_\phi}(-\sqrt{\lambda})\right| = 1$.

**Proof.** (i) follows from Beurling’s theorem (see [5], page 15), while (ii) is elementary. Peller, Megretski and Treil show in [4] that for any compact and self-adjoint Hankel operator $\Gamma$, the spectral multiplicity function satisfies $|\nu_{\Gamma}(\lambda) - \nu_{\Gamma}(-\lambda)| \leq 1$. Using this, and (ii), statements (iii) and (iv) follow immediately. 

**Remark 2.10** The Carleman operator $\Gamma : L^2(0, \infty) \to L^2(0, \infty)$ is given by

$$\Gamma f(x) = \int_0^\infty \frac{1}{x+t} f(t) \, dt,$$

so $\Gamma^2$ has kernel of Tracy-Widom type

$$\Gamma^2 f(u) = \int_0^\infty \frac{\log u - \log t}{u-t} f(t) \, dt.$$  

Carleman showed that $\Gamma$ is a positive self-adjoint Hankel operator with continuous spectrum $[0, \pi]$ of multiplicity two (see [5, p. 442]), so the Tracy–Widom type operator $\Gamma^2$ has spectrum $[0, \pi^2]$, also of multiplicity two. This contrasts with (iii) and (iv) of Proposition 2.9.

### 3 Integrable operators on $H^2$

Let $H^2$ be the usual Hardy space on the unit circle $T$, with orthonormal basis $\{1, z, z^2, \ldots\}$, and let $R_+ : L^2 \to H^2$ and $R_- : L^2 \to L^2 \ominus H^2$ be the Riesz orthogonal projection operators. We let $M_\phi$ denote multiplication by $\phi$, and define the Toeplitz operator on $H^2$ with symbol $\phi$ to be $T_\phi = R_+ M_\phi R_+$. Let $J : L^2 \to L^2$ be a flip operator, whose operation on a function $f \in H^2$ is $J f(z) = \bar{f}(\bar{z})$. Note that $J$ maps $H^2$ onto $L^2 \ominus H^2$ (and vice versa) and that $J^2 = I$. The Hankel operator $\Gamma_\phi$ on $H^2$ with symbol $\phi \in L^\infty$ is then

$$\Gamma_\phi = JR_- M_\phi.$$  

We let the integral operator $W$ on $L^2(T)$ have kernel

$$W(e^{i\theta}, e^{i\phi}) = \frac{f(e^{i\theta})g(e^{i\phi}) - f(e^{i\phi})g(e^{i\theta})}{1 - e^{i(\theta - \phi)}},$$

where $W$ operates on a function $f \in L^2(T)$ in the usual way:

$$Wf(e^{i\theta}) = \frac{1}{2\pi} \int_T W(e^{i\theta}, e^{i\phi}) f(e^{i\phi}) \, d\phi.$$
Lemma 3.1 Suppose that $f, g \in L^\infty$ have $\bar{f} = g$. Then $W$ defines a bounded and self-adjoint operator on $L^2$. Further, $R_+ W R_+ : H^2 \to H^2$ satisfies

$$R_+ W R_+ = \Gamma_f^* \Gamma_f - \Gamma_g^* \Gamma_g.$$ \hspace{1cm} (40)

Moreover, when $f$ is continuous, $R_+ W R_+$ is compact.

Proof. The condition $\bar{f} = g$ gives immediately $W(e^{i\theta}, e^{i\phi}) = W(e^{i\phi}, e^{i\theta})$, and so $W$ is self-adjoint. It can easily be seen that the Riesz projection $R_+$ has distributional kernel $1/(1 - e^{i(\theta - \phi)})$, and so $W$ decomposes as

$$W = M_g [M_f, R_+] - M_f [M_g, R_+],$$ \hspace{1cm} (41)

where all the operators are bounded. A simple calculation now shows that

$$R_+ W R_+ = (T_{gf} - T_g T_f) - (T_{fg} - T_f T_g),$$ \hspace{1cm} (42)

and we apply the standard formulae $T_{hk} - T_h T_k = \Gamma_{h(z)} \Gamma_k(z)$ and $\Gamma_h^* = \Gamma_{\bar{h}(\bar{z})}$ (see [9, p. 253]) to get equation (40). The last statement follows by Hartman’s theorem: the Hankel operators on the right-hand side of (40) are compact when $f$ is continuous.

Remark 3.2 We continue functions $f \in L^2$ to harmonic functions on $\mathbb{D}$ by means of the Poisson kernel, as in [5, p. 718].

Proposition 3.3 Suppose $f = \bar{g} \in L^\infty$, where $g$ is holomorphic inside $\mathbb{D}$. Then

$$R_+ W R_+ = \Gamma_f^* \Gamma_f.$$ \hspace{1cm} (43)

Further, if $R_+ W R_+$ has finite rank, then $f$ is rational.

Proof. Take $f = \bar{g}$ in Lemma 3.1 to obtain the first part of the result. For the second part, note that

$$\text{Range}(R_+ W R_+) = \text{Ker}(\Gamma_f^* \Gamma_f) = \text{Ker}(\Gamma_f) = \text{Range}(\Gamma_f^*),$$ \hspace{1cm} (44)

and apply Kronecker’s theorem: $\Gamma_k$ has finite rank if and only if $k$ is rational, so $\Gamma_{\bar{f}(\bar{z})}$ has finite rank if and only if $\bar{f}(\bar{z})$ is rational, which implies that $f$ is rational.

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