Dual Space Preconditioning for Gradient Descent

Chris J. Maddison\textsuperscript{1,2,*}, Daniel Paulin\textsuperscript{1,*}, Yee Whye Teh\textsuperscript{1,2}, and Arnaud Doucet\textsuperscript{1}

\textsuperscript{1}Department of Statistics, University of Oxford
\textsuperscript{2}DeepMind, London, UK
\*Both authors contributed equally to this work.

12 February 2019

Abstract

The conditions of relative convexity and smoothness were recently introduced by Bauschke, Bolte, and Teboulle [2] and Lu, Freund, and Nesterov [15] for the analysis of first-order methods optimizing a convex function. Those papers considered conditions over the primal space. We introduce a fully explicit descent scheme with relative smoothness in the dual space between the convex conjugate of the function and a designed dual reference function. For Legendre type convex functions under relative smoothness, our scheme naturally remains in the interior of the domain, despite being fully explicit. We obtain linear convergence under dual relative strong convexity with rates that are invariant under horizontal translations. We show that it is not equivalent to the standard primal scheme by finding a separating class of functions. Our method is a form of non-linear preconditioning of gradient descent that extends the class of convex functions on which explicit first-order methods can be well-conditioned.

1 Introduction

1.1 Setup and Method

We study the minimization of a Legendre (see Definition 2.1) convex function $f: \mathbb{R}^d \to \{\mathbb{R}, \infty\}$,

$$\min_{x \in \text{dom } f} f(x),$$

(1)

where $\text{dom } f = \{x \in \mathbb{R}^d : f(x) < \infty\}$. For such functions, the global minimizer, if it exists, is in $\text{int}(\text{dom } f)$. Our focus is on iterative methods requiring only the oracle computation of the vector of first partial derivatives $\nabla f(x) = (\partial f(x)/\partial x^i)$ of $f$ locally at any point $x \in \text{int}(\text{dom } f)$, known also as first-order methods [20] [23] [21]. These methods produce a sequence of iterates $x_i \in \text{int}(\text{dom } f)$ for $i \geq 0$, and our emphasis is on those that achieve linear convergence in some sense, e.g., the iterates might satisfy $f(x_i) - f(x_*) = O(\lambda^{-i})$ for some rate $\lambda > 1$ and $x_*$ the global minimizer.

The convergence rates of first-order methods on convex functions can be broadly separated by the properties of strong convexity and Lipschitz smoothness. Taken together for twice continuously differentiable $f$, these properties are equivalent to the conditions that eigenvalues of the matrix
of second-order partial derivatives $\nabla^2 f(x) = (\partial^2 f(x)/\partial x^{(m)} \partial x^{(n)}) \in \mathbb{R}^{d \times d}$ are everywhere lower bounded by $\mu > 0$ (strong convexity) and upper bounded by $L > 0$ (smoothness),

$$\mu I \preceq \nabla^2 f(x) \preceq LI \quad \text{for all} \ x \in \text{int(dom } f), \quad (2)$$

where $\preceq$ indicates the partial order of positive semi-definite matrices. Thus, functions whose second derivatives are unbounded or 0 anywhere cannot be both strongly convex and smooth. The first-order conditions equivalent to (2) are more broadly applicable, but the second-order conditions reveals the intuition behind strong convexity and smoothness. By bounding the magnitude of the second derivatives, these conditions control the rate of change of the gradient $\nabla f$. Both bounds play an important role in the performance of first-order methods. On the one hand, for smooth and strongly convex $f$, the iterates of many first-order methods converge linearly. On the other hand, for any first-order method, there exist smooth convex functions and non-smooth strongly convex functions on which its convergence is sub-linear, i.e., $f(x_i) - f(x_\star) \geq \Omega(i^{-2})$ for smooth convex functions. See [20, 23, 21] for these classical results and [12] for other more exotic scenarios.

A central assumption in the worst case analyses of first-order methods is that information about $f$ is restricted to local black-box evaluations of $f$ and $\nabla f$, see [20, 21]. What exactly constitutes a local black-box model is often a subtle matter (see Chapter 4 of [21]), but practically speaking it prevents the analysis of convergence rates from restricting the class of functions based on global knowledge. One way around this is to incorporate global information through additional computation. Recent work by Bauschke, Bolte, and Teboulle [2] and later Lu, Freund, and Nesterov [15] shows how the use of a second convex function $h : \mathbb{R}^d \to \{\mathbb{R}, \infty\}$ with $\text{int(dom } f) \subseteq \text{int(dom } h)$ can open the black-box to condition primal gradient (mirror descent) and dual averaging schemes. These methods generally require the ability to compute $\nabla h$ and solve subproblems of the form

$$\min_{x \in \text{dom } f} \{(c, x) + h(x)\}, \quad (3)$$

for $c \in \mathbb{R}^d$ where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product. For example, the iterates of the primal scheme, due to [19] with a modern formulation by [3], satisfy

$$\nabla h(x_{i+1}) = \nabla h(x_i) - \frac{1}{L_h} \nabla f(x_i), \quad (4)$$

with $L_h > 0$. The question answered by Lu et al. [15] is: how should $h$ and $f$ relate to guarantee the linear convergence of $f(x_i) - f(x_\star)$? They show that the following relative conditions of strong convexity and smoothness are sufficient for linear convergence. Consider $f$ and $h$ that are twice continuously differentiable. Then $f$ is $\mu_h$-strongly convex and $L_h$-smooth relative to $h$, if the following lower (relative strong convexity) and upper (relative smoothness) bounds hold,

$$\mu_h \nabla^2 h(x) \preceq \nabla^2 f(x) \preceq L_h \nabla^2 h(x) \quad \text{for all} \ x \in \text{int(dom } f). \quad (5)$$

Clearly, (5) generalizes (2) while admitting functions $f$ that may be non-smooth under the assumption that the degree of non-smoothness can be encoded in $h$. The optimal choice of $h$ would be $f$, but this is not feasible as this assumes the ability to minimize $f$ itself (the subproblem (3) with $c = 0$). There are two difficulties with condition (5). First, consider horizontal translations of $f$,

$$g(x) = f(x - z) \quad \text{for} \ z \in \mathbb{R}^d. \quad (6)$$
For most $h$, (5) will cease to hold on $q$ and the condition number $\kappa_h = L_h/\mu_h$ will need to change (a point considered by Lu et al. [15]). Even worse, consider $f$ with a singular or unbounded Hessian $\nabla^2 f$ at $x_*$. For (5) to hold on such functions, $\nabla^2 h$ should essentially encode knowledge of the location of $x_*$ by being singular at $x_*$ in the case that $\nabla^2 f(x_*)$ is singular or by growing without bound at $x_*$ in the case that $\nabla^2 f(x_*)$ is unbounded.

In their conclusion, Lu et al. [15] ask an intriguing question about the relative conditions. Consider the following known duality result (e.g. [30, 26, 13, 29]): if $f$ is $\mu$ strongly convex and $L$ smooth on $\text{int}(\text{dom } f)$, then its convex conjugate, $f^*(p) = \sup_{x \in \mathbb{R}^d} \{ \langle x, p \rangle - f(x) \}$, is $1/L$ strongly convex and $1/\mu$ smooth on $\text{int}(\text{dom } f^*)$, i.e., for twice differentiable $f^*$,

$$\mu \nabla^2 f^*(p) \preceq I \preceq L \nabla^2 f^*(p) \text{ for all } p \in \text{int}(\text{dom } f^*).$$

Their question: is there a natural duality theory for relative conditions? Unfortunately, the dual equivalent of (5) does not reveal much additional structure. Here, we consider another way of conditioning the optimization on functions with singular or unbounded $\nabla^2 f(x_*)$ without knowledge of the $x_*$’s location.

Our method is based on the following insight. On the one hand, in the primal space the iterates $x_i$ of an optimizer approach the point $x_*$ (for $f$ with a minimum), which depends on $f$. On the other hand, in the dual space the iterates $\nabla f(x_i)$ approach the same point 0 for any $f$. This basic regularity can be exploited. In more detail, we require a second convex function $k : \mathbb{R}^d \to [\mathbb{R}, \infty]$ of Legendre type, differentiable on $\text{int}(\text{dom } f^*)$, and with a minimum at $0 \in \text{dom } k$. Informally, we make the identification $p_i = \nabla f(x_i)$, and run the classical iterative scheme (4) in the dual space “searching” for the $p$ that minimizes $k$ (i.e., $p = 0$) with $f^*$ as the “reference function”,

$$\nabla f^*(p_{i+1}) = \nabla f^*(p_i) - \frac{1}{L_k} \nabla k(p_i),$$

for some $L_k > 0$. Rewriting iteration (8) in the primal space, we obtain the method below.

| Gradient descent with dual space preconditioning. | Given $f : \mathbb{R}^d \to [\mathbb{R}, \infty]$ Legendre convex, $k : \mathbb{R}^d \to [\mathbb{R}, \infty]$ Legendre convex with a minimum at 0, and $x_0 \in \text{int}(\text{dom } f)$, such that $k$ is $L_k$-smooth relative to $f^*$ on $\text{int}(\text{dom } f^*)$. For all $i \geq 0$,

$$x_{i+1} = x_i - \frac{1}{L_k} \nabla k(\nabla f(x_i)).$$ |

We refer to $k$ as a dual reference function, despite it playing the role of an objective in the dual space, because our ultimate goal is to minimize $f$. The convergence of $k(\nabla f(x_i)) - k(0)$ essentially follows from the work of [2] [15] with some important modifications to justify its fully explicit nature. In particular, for convergence we require that $k$ is $L_k$-smooth relative to $f^*$. For linear convergence we require that $k$ is $\mu_k$-strongly convex relative to $f^*$ for $\mu_k > 0$. In other words,

$$\mu_k \nabla^2 f^*(p) \preceq \nabla^2 k(p) \preceq L_k \nabla^2 f^*(p) \text{ for all } p \in \text{int}(\text{dom } f^*).$$

Note that, with the exception of quadratic $k$ or $h$, (5) and (10) are generally not equivalent conditions and $\mu_h \neq \mu_k$, $L_h \neq L_k$. 

3
This method has the following properties. First, we are able to show under dual relative smoothness that the explicit iteration (9) remains in \( \text{int}(\text{dom } f) \). Second, the convergence rates are invariant under horizontal translations of \( f \). Horizontal translations in the primal space become affine translations in the dual space, \( g^*(p) = f^*(p) + \langle z, p \rangle \) for \( g \) in (10), and \( \nabla^2 g^*(p) = \nabla^2 f^*(p) \), while generally \( \nabla^2 g(x) \neq \nabla^2 f(x) \). Thus, the set of functions \( f \) satisfying (10) is closed under translations of \( f \), and thus the dual condition number \( \kappa_d = L_k/\mu_k \) is now invariant under translations of \( f \). Finally, if the behavior of \( \nabla^2 k(p) \) for \( p \) in the neighborhood of 0 matches that of \( \nabla^2 f^*(\nabla f(x)) = (\nabla^2 f(x))^{-1} \) for \( x \) about \( x_* \), then our method will exhibit local linear convergence. Taken together, \( k \) centered at 0 is capable of encoding the kind of behavior in \( f \) centered at \( x_* \), without knowing the where \( x_* \) is.

The primal gradient (11) and dual preconditioned scheme (9) are not equivalent methods in the following sense. We find a simple separating class of functions with singular behavior of \( \nabla^2 f(x) \) satisfies (10), but for which a single, cheap-to-compute, \( \nabla k \) satisfies (11). Thus, \( k \) expresses global assumptions about \( f \) in a fashion distinct to \( h \).

Some caution is warranted. First, the relationship between the convergence of \( k(\nabla f(x_i)) − k(0) \) and other quantities of interest, like \( f(x_i) − f(x_*) \) or \( x_i − x_* \), will likely be problem specific. Nonetheless, under dual relative strong convexity and the symmetry conditions considered in (2), it will directly imply linear convergence of \( f(x_i) − f(x_*) \) with \( \kappa_d \) conditioning. Second, the conditions guiding the design of \( k \) are relative to the convex conjugate \( f^* \). So, there is no free lunch and the central difficulty of this method is in the design of \( k \). Still, to our knowledge (9) has not yet been described in this generality. We believe conditions such as (10) are also of independent interest and may provide new avenues for incorporating hard-won domain-specific knowledge into improving the conditioning of optimizers or other algorithms.

The remainder of the paper is mostly spent justifying this informal discussion. It is organized as follows. In the rest of this section we review the related literature. In Section 2 we review the basic results of convex analysis and the conditions of relative strong convexity and relative smoothness. In Section 3 we show the linear convergence of (9) under the assumption that \( k \) is strongly convex relative to \( f^* \). Finally, in Section 4 we illustrate our method with a power function example that separates it from the primal scheme.

### 1.2 Related literature

Nemirovski and Yudin [20] derived lower bounds for first-order methods under the standard oracle model of local black-box evaluations of \( f \) or \( \nabla f \). Nesterov obtained upper bounds of matching order for first-order methods on smooth convex functions and smooth strongly convex functions [22, 21]. Often, the assumption of strong convexity can be relaxed, and under a weaker quadratic growth condition, linear rates can be obtained by several well known optimization methods [17].

Several other authors obtained linear rates under a variety of assumptions for non-strongly convex or non-uniformly smooth functions, see, e.g., [18, 12, 14, 8, 28, 25].

Bauschke et al. were the first to propose studying the condition of relative smoothness for first-order schemes [2]. Lu et al. [15] first provided the proof of linear convergence of the primal gradient and dual averaging schemes under both relative strong convexity and smoothness. In particular, the analysis of Lu et al. is central to our result. Bolte et al. [4] propose an elegant extension of the relative conditions to capture the behavior of some non-convex problems. Accelerated versions of the primal schemes have been proposed in [11] and [10]. Relative smoothness has been used in the analysis of stochastic composite least-squares problems [9]. Recently, relative smoothness was applied to symmetric non-negative matrix factorization [7].

The primal gradient (11) and dual preconditioned scheme (9) are not equivalent methods in the following sense. We find a simple separating class of functions with singular behavior of \( \nabla^2 f(x) \) satisfies (10), but for which a single, cheap-to-compute, \( \nabla k \) satisfies (11). Thus, \( k \) expresses global assumptions about \( f \) in a fashion distinct to \( h \).

Some caution is warranted. First, the relationship between the convergence of \( k(\nabla f(x_i)) − k(0) \) and other quantities of interest, like \( f(x_i) − f(x_*) \) or \( x_i − x_* \), will likely be problem specific. Nonetheless, under dual relative strong convexity and the symmetry conditions considered in (2), it will directly imply linear convergence of \( f(x_i) − f(x_*) \) with \( \kappa_d \) conditioning. Second, the conditions guiding the design of \( k \) are relative to the convex conjugate \( f^* \). So, there is no free lunch and the central difficulty of this method is in the design of \( k \). Still, to our knowledge (9) has not yet been described in this generality. We believe conditions such as (10) are also of independent interest and may provide new avenues for incorporating hard-won domain-specific knowledge into improving the conditioning of optimizers or other algorithms.

The remainder of the paper is mostly spent justifying this informal discussion. It is organized as follows. In the rest of this section we review the related literature. In Section 2 we review the basic results of convex analysis and the conditions of relative strong convexity and relative smoothness. In Section 3 we show the linear convergence of (9) under the assumption that \( k \) is strongly convex relative to \( f^* \). Finally, in Section 4 we illustrate our method with a power function example that separates it from the primal scheme.

1.2 Related literature

Nemirovski and Yudin [20] derived lower bounds for first-order methods under the standard oracle model of local black-box evaluations of \( f \) or \( \nabla f \). Nesterov obtained upper bounds of matching order for first-order methods on smooth convex functions and smooth strongly convex functions [22, 21]. Often, the assumption of strong convexity can be relaxed, and under a weaker quadratic growth condition, linear rates can be obtained by several well known optimization methods [17]. Several other authors obtained linear rates under a variety of assumptions for non-strongly convex or non-uniformly smooth functions, see, e.g., [18, 12, 14, 8, 28, 25].

Bauschke et al. were the first to propose studying the condition of relative smoothness for first-order schemes [2]. Lu et al. [15] first provided the proof of linear convergence of the primal gradient and dual averaging schemes under both relative strong convexity and smoothness. In particular, the analysis of Lu et al. is central to our result. Bolte et al. [4] propose an elegant extension of the relative conditions to capture the behavior of some non-convex problems. Accelerated versions of the primal schemes have been proposed in [11] and [10]. Relative smoothness has been used in the analysis of stochastic composite least-squares problems [9]. Recently, relative smoothness was applied to symmetric non-negative matrix factorization [7].
We recently introduced a family of Hamiltonian descent methods in [16]. These methods use a Legendre convex function, called the kinetic energy, to condition the optimization. In a forthcoming update, we will show that the central condition exploited by those methods is relative smoothness and strong convexity in the dual space, between \( f^* \) and the kinetic energy \( k \). To our knowledge, (9) has not been described in these terms, and it along with Hamiltonian descent are the first to exploit the first-order equivalents of (10) in a somewhat generic fashion.

2 Convex analysis

2.1 Convex conjugate and Legendre functions

In this section we review some basic facts of convex analysis that will be used throughout. Let \( h : \mathbb{R}^d \to \{ \mathbb{R}, \infty \} \) be a convex function with domain \( \text{dom} h = \{ x : \mathbb{R}^d : h(x) < \infty \} \). \( h \) is proper if \( \text{dom} h \neq \emptyset \) and \( h(x) > -\infty \) for all \( x \in \mathbb{R}^d \) (Section 4 of [24]). A proper convex \( h \) is closed if it is lower semi-continuous (Section 7 of [24]). To indicate \( \text{dom} h = \mathbb{R}^d \), we simply define \( h : \mathbb{R}^d \to \mathbb{R} \) as ranging only over the reals. Let \( \| \cdot \| \) indicate the Euclidean norm.

The convex conjugate \( h^* : \mathbb{R}^d \to \{ \mathbb{R}, \infty \} \) of \( h \) is defined as

\[
    h^*(p) = \sup\{ \langle x, p \rangle - h(x) : x \in \text{dom} h \}
\]

and is a proper closed convex function. For such \( h \), we have that \( (h^*)^* = h \) (Corollary 12.2.1 of [24]). It is easy to show from the definition that if \( g : \mathbb{R}^d \to \{ \mathbb{R}, \infty \} \) is another closed proper convex function such that \( g(x) \leq h(x) \) for all \( x \in \mathbb{R}^d \), then \( h^*(p) \leq g^*(p) \) for all \( p \in \mathbb{R}^d \). For \( h \) differentiable on \( \text{int}(\text{dom} h) \), we have for \( x \in \text{int}(\text{dom} h) \) by Theorem 26.4 of [24],

\[
    \langle x, \nabla h(x) \rangle = h(x) + h^*(\nabla h(x)).
\]

For more on \( h^* \), we refer readers to [24, 6, 5].

We make heavy use in this work of a special type of convex functions, the Legendre type. Intuitively, these functions can be thought of as generalizations of positive definite quadratics and their gradient maps as generalizations of positive definite linear maps.

**Definition 2.1** (Legendre type convex function, Chapter 25 of [24]). Let \( h : \mathbb{R}^d \to \{ \mathbb{R}, \infty \} \) be a closed proper convex function. \( h \) is Legendre type, if

1. \( \text{int}(\text{dom} f) \neq \emptyset \).
2. \( h \) is differentiable on \( \text{int}(\text{dom} h) \), with \( \| \nabla h(x_i) \| \to \infty \) for every \( \{ x_i : i \in \mathbb{N} \} \subset \text{int}(\text{dom} h) \) converging to a boundary point of \( \text{dom} h \) as \( i \to \infty \).
3. \( h \) is strictly convex on \( \text{int}(\text{dom} h) \).

One of the key properties of Legendre functions is that the gradient map of \( h^* \) is the inverse map of the gradient of \( h \). Moreover, this gives us a simple characterization of the inverse of \( \nabla^2 h(x) \), which we summarize in the well-known elementary lemma below.

**Lemma 2.1.** Let \( h : \mathbb{R}^d \to \{ \mathbb{R}, \infty \} \) be a Legendre convex function, then \( h^* \) is also Legendre. The gradient map \( \nabla h \) is one-to-one and onto \( \text{int}(\text{dom} h^*) \), continuous in both directions, and for all \( x \in \text{int}(\text{dom} h) \)

\[
    \nabla h^*(\nabla h(x)) = x.
\]
If $h$ is twice continuously differentiable on an open set containing $x$, then
\[
\nabla^2 h^*(\nabla h(x)) \nabla^2 h(x) = \nabla^2 h(x) \nabla^2 h^*(\nabla h(x)) = I.
\] (14)

Proof. The first part is due to Rockafellar; see Theorem 26.5 in [24]. For (14), note that, by the inverse function theorem, $\nabla h^*$ is continuously differentiable at $\nabla h(x)$ under the assumption that $\nabla h$ is continuously differentiable on an open set containing $x$. The remainder follows by the chain rule applied to (13).

This lemma confirms that Legendre functions can only be minimized in their interior.

Lemma 2.2. Let $h : \mathbb{R}^d \to \{\mathbb{R}, \infty\}$ be a Legendre convex function with a minimum at $x_\star \in \text{dom} h$. $x_\star$ is unique and furthermore $x_\star \in \text{int}(\text{dom} h)$.

Proof. First, we argue that $x_\star$ cannot be found on the boundary by contradiction. Suppose that $x_\star$ is a boundary point. Since $\text{int}(\text{dom} h) \neq \emptyset$, by convexity there exists a line segment connecting the boundary point $x_\star$ and any other interior point $a$. However, by Lemma 26.2 of [24], we know that the directional derivative converges to $-\infty$ as we tend towards the boundary point on this line segment, hence $x_\star$ could not be a minimum of $h$. Thus we conclude that $x_\star \in \text{int}(\text{dom} h)$. By property 3, Legendre functions are strictly convex on their interior, and thus $x_\star$ is unique.

The final lemma will be used in our analysis to show that $k$ is radially unbounded.

Lemma 2.3. Let $h : \mathbb{R}^d \to \{\mathbb{R}, \infty\}$ be a Legendre convex function with a minimum at 0. Then, $h$ is radially unbounded, i.e., if $x_i \in \mathbb{R}^d$ is a sequence such that $\|x_i\| \to \infty$, then $h(x_i) \to \infty$.

Proof. First, by Lemma 2.2 it follows that 0 $\in \text{int} \text{dom} h$ and it is the unique minimum of $h$. Thus, we can define the sphere $S = \{x \in \mathbb{R}^d : \|x\| = r\}$ for some $r > 0$ such that $S \in \text{int}(\text{dom} h)$. By continuity of $h$ in the interior of its domain, and the uniqueness of the minimum at zero, we have $\inf_{x \in S} h(x) > h(0)$. Now, assume without loss of generality that $\|x_i\| > r$. By strict convexity property 3 of Legendre functions we have
\[
h(0) + \frac{\|x_i\|}{r} \left( h \left( \frac{rx_i}{\|x_i\|} \right) - h(0) \right) < h(0) + (h(x_i) - h(0))
\] (15)
and thus
\[
h(x_i) > h(0) + \frac{\|x_i\|}{r} \left( \inf_{x \in S} h(x) - h(0) \right).
\] (16)
Our result follows by taking $i \to \infty$.

2.2 Relative smoothness and relative strong convexity

Relative conditions [2, 15] are defined via a fundamental measure of divergence. Given a convex Legendre function $h : \mathbb{R}^d \to \{\mathbb{R}, \infty\}$ the Bregman divergence (see [1] for a review) is defined $\forall x \in \text{dom} h, \forall y \in \text{int}(\text{dom} h)$ as,
\[
D_h(x, y) = h(x) - h(y) - \langle \nabla h(y), x - y \rangle.
\] (17)
An important special case of (17) is $h(x) = \|x\|^2_2/2$ whose Bregman divergence is the classical Euclidean distance squared $D_{\|\cdot\|^2_2}(x, y) = \|x - y\|^2_2/2$. Relative strong convexity and relative smoothness relate two Legendre convex functions via their respective Bregman divergences.
Definition 2.2 (From [2] [15]). Let \( g, h : \mathbb{R}^d \to \{\mathbb{R}, \infty\} \) be convex Legendre functions. \( g \) is \( \mu \)-strongly convex relative to \( h \) on a convex set \( Q \) if \( Q \subseteq \text{int}(\text{dom } g) \cap \text{int}(\text{dom } h) \), and for every \( x, y \in Q \),
\[
D_g(x, y) \geq \mu D_h(x, y).
\]
\( g \) is \( L \)-smooth relative to \( h \) on a convex set \( Q \) if \( Q \subseteq \text{int}(\text{dom } h) \cap \text{int}(\text{dom } g) \), and for every \( x, y \in Q \),
\[
D_g(x, y) \leq LD_h(x, y).
\]

Here, again, the special cases of relative strong convexity and smoothness with respect to \( h(x) = \|x\|^2/2 \) are exactly the classical conditions of strong convexity and smoothness, the first-order equivalents of (2). The following lemma, taken from [15], describes a variety of equivalent definitions for relative strong convexity and smoothness.

Lemma 2.4 (From [15]). Let \( g, h : \mathbb{R}^d \to \{\mathbb{R}, \infty\} \) be convex Legendre functions. The following are equivalent

1. \( g \) is \( \mu \)-strongly convex relative to \( h \) on \( Q \).
2. \( g(x) - \mu h(x) \) is convex on \( Q \).
3. \( \mu \langle \nabla h(x) - \nabla h(y), x - y \rangle \leq \langle \nabla g(x) - \nabla g(y), x - y \rangle \) for all \( x, y \in Q \).

The following are equivalent

1. \( g \) is \( L \)-smooth relative to \( h \) on \( Q \).
2. \( Lh(x) - g(x) \) is convex on \( Q \).
3. \( \langle \nabla g(x) - \nabla g(y), x - y \rangle \leq L \langle \nabla h(x) - \nabla h(y), x - y \rangle \) for all \( x, y \in Q \).

For completeness, we present some elementary lemmas, which are minor modifications to known results. The first lemma relates relative first-order conditions to their second-order equivalents, which were discussed in Section 1.

Lemma 2.5. Let \( g, h : \mathbb{R}^d \to \{\mathbb{R}, \infty\} \) be convex Legendre functions. Let \( g, h \) be continuously differentiable for all \( x \in Q \) (a convex set) and twice continuously differentiable at all \( x \in Q \setminus \{x^*\} \) where \( x^* \in Q \).

1. \( g \) is \( \mu \)-strongly convex relative to \( h \) on \( Q \) iff \( \forall x \in Q \setminus \{x^*\} \)
\[
\mu \nabla^2 h(x) \preceq \nabla^2 g(x).
\]
2. \( g \) is \( L \)-smooth relative to \( h \) on \( Q \) iff \( \forall x \in Q \setminus \{x^*\} \)
\[
\nabla^2 g(x) \preceq L \nabla^2 h(x).
\]

Proof. For relative strong convexity, \( (\Rightarrow) \) follows directly from part one of Theorem 2.1.4 of [21] applied to \( f(x) = g(x) - \mu h(x) \). For \( (\Leftarrow) \), we have \( f(x) = g(x) - \mu h(x) \) convex. Now, let \( x, y \in Q \).
and $t \in [0, 1]$ and define $x_t = y + t(x - y)$. There can be at most one time $a \in [0, 1]$ such that $x_a = x_*$. Take $a$ to be that time, if it exists, or some arbitrary $a \in [0, 1]$, otherwise. Now,

$$\langle \nabla g(x) - \mu \nabla h(x) - \nabla g(y) + \mu \nabla h(y), x - y \rangle$$

$$= \langle \nabla f(x) - \nabla f(y), x - y \rangle$$

$$(1) \lim_{\tau \downarrow a} \langle \nabla f(x) - \nabla f(x_\tau), x - y \rangle + \lim_{\tau \uparrow a} \langle \nabla f(x_\tau) - \nabla f(y), x - y \rangle$$

$$(2) \lim_{\tau \downarrow a} \int^1_\tau \langle x - y, \nabla^2 f(x_t)(x - y) \rangle dt + \lim_{\tau \uparrow a} \int^\tau_0 \langle x - y, \nabla^2 f(x_t)(x - y) \rangle dt \geq 0$$

(1) follows by the continuity of $\nabla f$ and (2) by the fundamental theorem of calculus. The smoothness result follows analogously.

The second lemma will give us tools for controlling the growth of gradients under relative conditions.

**Lemma 2.6.** Let $g, h : \mathbb{R}^d \to \{\mathbb{R}, \infty\}$ be convex Legendre functions. If $g$ is $\mu$-strongly convex relative to $h$ on $Q$, then $\forall p, q \in \nabla h(Q)$

$$\mu D_h\star (p, q) \leq D_g(\nabla h\star(q), \nabla h\star(p)).$$

If $g$ is $L$-smooth relative to $h$ on $Q$, then $\forall p, q \in \nabla h(Q)$

$$D_g(\nabla h\star(q), \nabla h\star(p)) \leq LD_h\star(p, q).$$

**Proof.** The following identity can be easily verified from (12), $\forall x, y \in Q$,

$$D_h(x, y) = D_h\star(\nabla h(y), \nabla h(x)).$$

For strong convexity, by Lemma 2.3 we have $\nabla h\star(p), \nabla h\star(q) \in Q$ and

$$\mu D_h\star(p, q) = \mu D_h(\nabla h\star(q), \nabla h\star(p)) \leq D_g(\nabla h\star(q), \nabla h\star(p)).$$

The relative smoothness result follows similarly.

The third lemma is a generalized Pythagorean theorem for Bregman divergences [27, 15].

**Lemma 2.7** (From [27, 15]). Let $h : \mathbb{R}^d \to \{\mathbb{R}, \infty\}$ be a Legendre convex function, and $\phi : \mathbb{R}^d \to \{\mathbb{R}, \infty\}$ any proper closed convex function with $\text{int}(\text{dom} h) \subseteq \text{int}(\text{dom} \phi)$. If for $y \in \text{int}(\text{dom} h)$,

$$z_* = \arg \min_{z \in \text{int}(\text{dom} h)} \{\phi(z) + D_h(z, y)\}$$

then for all $x \in \text{int}(\text{dom} h)$

$$\phi(x) + D_h(x, y) \geq \phi(z_*) + D_h(z_*, y) + D_h(x, z_*).$$
3 Analysis of the dual preconditioned scheme

The theorem guaranteeing the convergence of the dual preconditioned scheme uses the analysis of Lu et al. [15], but must be adapted to ensure that the iterates do not escape the interior of the domain of \( f \). Moving the scheme back into the primal space, we find that for this fully explicit method \( k(\nabla f(x_i)) - k(0) \) converges with rate \( \mathcal{O}(i^{-1}) \) when \( k \) is \( L_k \)-smooth relative to \( f^* \) and with rate \( \mathcal{O}((1 - \frac{\mu_k}{L_k})^i) \) when \( k \) is also \( \mu_k \)-strongly convex relative to \( f^* \) with \( \mu_k > 0 \). For this case, define the dual condition number \( \kappa_k = L_k/\mu_k \). First, we consider the descent lemma that guarantees \( k(\nabla f(x_i)) \) is well-defined and monotonically decreasing for all \( i \geq 0 \).

**Lemma 3.1 (Descent lemma).** Given \( f : \mathbb{R}^d \to \{\mathbb{R}, \infty\} \) Legendre convex, \( k : \mathbb{R}^d \to \{\mathbb{R}, \infty\} \) Legendre convex with a minimum at 0, and \( x_0 \in \text{int}(\text{dom } f) \). If \( k \) is \( \mu_k \)-strongly convex and \( L_k \)-smooth relative to \( f^* \) on \( \text{int}(\text{dom } f^*) \), then for all \( i \geq 0 \) we have

1. \( x_i \in \text{int}(\text{dom } f) \).
2. For \( i > 0 \) and all \( p \in \text{int}(\text{dom } f^*) \),
   \[
   k(\nabla f(x_i)) \leq k(p) + (L_k - \mu_k)D_{f^*}(p, \nabla f(x_{i-1})) - L_kD_{f^*}(p, \nabla f(x_i)).
   \]  
   \[
   (18)
   \]
3. For \( i > 0 \), we have \( k(\nabla f(x_i)) \leq k(\nabla f(x_{i-1})) \).

**Proof.** This proof is based on the proof of Lu et al. [15], but crucially we must ensure that the iterates do not escape the interior of the domain of \( f \). First, note \( \text{int}(\text{dom } f^*) \subseteq \text{int}(\text{dom } k) \) (by relative smoothness). Thus, by Lemma 2.1 we have \( \nabla f(x) \in \text{int}(\text{dom } k) \) for all \( x \in \text{int}(\text{dom } f) \).

We proceed by induction. For \( i = 0 \) we have \( x_0 \in \text{int}(\text{dom } f) \) by assumption. Now, for \( i > 0 \), assume the induction hypothesis for \( x_{i-1} \). First, define

\[
x_L = x_{i-1} - \frac{1}{L} \nabla k(\nabla f(x_{i-1}))
\]

for \( L > 0 \). Because \( x_{i-1} \in \text{int}(\text{dom } f) \), the following set is not empty,

\[
S = \{ L \geq L_k : x_L \in \text{int}(\text{dom } f) \}
\]

(20)

For all \( L \in S \) we can identify \( p_{i-1} = \nabla f(x_{i-1}) \) and \( p_L = \nabla f(x_L) \). By Lemma 2.1 we have

\[
\nabla f^*(p_L) = \nabla f^*(p_{i-1}) - \frac{1}{L} \nabla k(p_{i-1})
\]

(21)

and therefore \( p_L \) satisfies the stationary condition of the following subproblem,

\[
\min_{p \in \text{int}(\text{dom } f^*)} \left\{ \frac{1}{L} \langle \nabla k(p_{i-1}), p - p_{i-1} \rangle + D_{f^*}(p, p_{i-1}) \right\}.
\]

(22)

Note that from the three-point Lemma 2.1 applied with \( h = f^* \), \( \phi(p) = \frac{1}{L} \langle \nabla k(p_{i-1}), p - p_{i-1} \rangle \), \( x = p \), \( y = p_{i-1} \) and \( z_* = p_L \), we have

\[
\frac{1}{L} \langle \nabla k(p_{i-1}), p - p_{i-1} \rangle + D_{f^*}(p, p_{i-1}) \geq \frac{1}{L} \langle \nabla k(p_{i-1}), p_L - p_{i-1} \rangle + D_{f^*}(p_L, p_{i-1}) + D_{f^*}(p, p_L).
\]

(23)
Now, putting everything together we have

\[ k(p_L) \leq (a) k(p_{i-1}) + (\nabla k(p_{i-1}), p_L - p_{i-1}) + L_k D_f^* (p_L, p_{i-1}) \]
\[ \leq (b) k(p_{i-1}) + (\nabla k(p_{i-1}), p_L - p_{i-1}) + LD_{f^*} (p_L, p_{i-1}) \]
\[ \leq (c) k(p_{i-1}) + (\nabla k(p_{i-1}), p - p_{i-1}) + LD_{f^*} (p, p_{i-1}) - LD_{f^*} (p, p_L) \]
\[ \leq (d) k(p) + (L - \mu_k) D_{f^*} (p, p_{i-1}) - LD_{f^*} (p, p_L). \]  

(a) follows from $L_k$-smoothness relative to $f^*$, (b) from $L_k \leq L$ and the non-negativity of the Bregman divergence, (c) from (23), and (d) from $\mu_k$-strong convexity of $k$ relative to $f^*$. Taking $p = p_{i-1}$ reveals that

\[ k(p_L) + LD_{f^*} (p_L, p_L) \leq k(p_{i-1}). \]  

Now, our goal is to show that $x_i \in \text{int}(\text{dom } f)$ by showing that $L_k \in S$. We proceed by contradiction, so suppose $L_k \notin S$. Then $x_L \in \mathbb{R}^d \setminus \text{int}(\text{dom } f)$. Then we can find $M \geq L_k$ such that $x_M \in \partial(\text{dom } f)$. Now take a sequence $L_j \to M$ such that $L_j > M$. By the above discussion for all $j \geq 0$ we have $k(\nabla f(x_L)) \leq k(\nabla f(x_i))$. $k$ being minimized at 0 means it satisfies Lemma 2.3 and thus is radially unbounded. This implies that $\|\nabla f(x_L)\| \leq C$ for some $C > 0$ and all $j \geq 0$. But this contradicts the requirement from property 2 of Legendre functions that $\|\nabla f(x_L)\| \to \infty$ since $x_L \to x_M \in \partial(\text{dom } f)$ by assumption. This completes the proof that $x_i = x_{L_k} \in \text{int}(\text{dom } f)$. Since $L_k \in S$, (24) ensures that $\mathbb{2}$ holds and (25) ensures that $\mathbb{3}$ holds.

Now we are ready to analyze the convergence of the dual preconditioned gradient descent.

**Theorem 3.2.** Given $f : \mathbb{R}^d \to \{\mathbb{R}, \infty\}$ Legendre convex, $k : \mathbb{R}^d \to \{\mathbb{R}, \infty\}$ Legendre convex with a minimum at 0, and $x_0 \in \text{int}(\text{dom } f)$. If $k$ is $\mu_k$-strongly convex and $L_k$-smooth relative to $f^*$ on $\text{int}(\text{dom } f^*)$, then for all $i \geq 1$ and $p \in \text{int}(\text{dom } f^*)$ the iterates of $\mathbb{9}$ satisfy,

\[ k(\nabla f(x_i)) - k(p) \leq \frac{\mu_k D_f^* (p, \nabla f(x_0))}{\left(1 + \frac{\mu_k}{L_k - \mu_k}\right)^i} \leq \frac{L_k - \mu_k}{i} D_f^* (p, \nabla f(x_0)) \]  

where, in the case that $\mu_k = 0$, the middle expression is defined in the limit as $\mu_k \downarrow 0$. In particular, if $f$ is minimized at $x_\star \in \text{dom } f$, then $\nabla f(x_i) \to 0$ and therefore $x_i \to x_\star$.

**Remark 1.** Requiring that $k$ be minimized at 0 is not so onerous. Let $l$ satisfy the requirements on $k$ in Theorem 3.2 and $0 \in \text{int}(\text{dom } l)$, but with a minimum other than 0. Then $k(p) = l(p) - (\nabla l(0), p)$ will suffice for Theorem 3.2.

**Remark 2.** As mentioned in (15), the first inequality of (24) implies linear convergence as,

\[ \frac{\mu_k D_f^* (0, \nabla f(x_0))}{\left(1 + \frac{\mu_k}{L_k - \mu_k}\right)^i} \leq L_k \left(1 - \frac{\mu_k}{L_k}\right)^i D_f^* (0, \nabla f(x_0)). \]  

Whether the linear convergence of $k(\nabla f(x_i))$ implies the specific rates of convergence of other quantities of interest, such as $x_i - x_\star$ or $f(x_i) - f(x_\star)$ will likely depend on the problem itself.
Now, the proof of the second inequality follows from (1 + \( \mu_k \))

\[
D_f(x_*, x_i) \leq \frac{L_k}{\nu_k} \left( 1 - \frac{\mu_k}{L_k} \right)^i D_f(x_0, x_*).
\]

Furthermore, consider the following symmetry measure for \( D_f \) defined by Bauske et al. 2,

\[
\alpha(f) = \inf \left\{ \frac{D_f(x, y)}{D_f(y, x)} : (x, y) \in \text{int}(\text{dom } f) \times \text{int}(\text{dom } f), x \neq y \right\}
\]

If \( \alpha(f) > 0 \), we immediately get the linear convergence of \( f(x_i) - f(x_*) \) with condition number \( \kappa_k \).

Indeed, a weaker measure of symmetry from the minimum suffices, \( \inf_{x \in \text{int}(\text{dom } f)} D_f(x_*, x)/D_f(x, x_*) \).

### Proof of Theorem 3.2
Again this proof is based on the proof of Lu et al. 13, but adapted to our setting. Note that

\[
\frac{1}{\sum_{j=1}^{i} \left( \frac{L_k}{L_k - \mu_k} \right)^j} = \frac{\mu_k}{L_k \left( 1 + \frac{\mu_k}{L_k - \mu_k} \right)^i - 1}
\]

which follows from geometric series analysis and only holds when \( \mu_k > 0 \). Now, note that by Lemma 3.1 we have \( x_i \in \text{int}(\text{dom } f) \) for all \( i \geq 0 \) and can thus make the identification \( p_i = \nabla f(x_i) \). Now, it follows by (18) of Lemma 3.1 then induction, that for all \( i > 0 \)

\[
\sum_{j=1}^{i} \left( \frac{L_k}{L_k - \mu_k} \right)^j k(p_j) \leq \sum_{j=1}^{i} \left( \frac{L_k}{L_k - \mu_k} \right)^j k(p) + L_k D_{f^*}(p, p_0) - \left( \frac{L_k}{L_k - \mu_k} \right)^i L_k D_{f^*}(p, p_i).
\]

Because \( k(p_i) \) is monotonically decreasing by Lemma 3.1 and \( D_{f^*} \) is non-negative, we have

\[
\left( \sum_{j=1}^{i} \left( \frac{L_k}{L_k - \mu_k} \right)^j \right) (k(p_i) - k(p)) \leq L_k D_{f^*}(p, p_0) - \left( \frac{L_k}{L_k - \mu_k} \right)^i L_k D_{f^*}(p, p_i) \leq L_k D_{f^*}(p, p_0).
\]

By using (29) and rearranging, we obtain

\[
k(p_i) - k(p) \leq \frac{L_k D_{f^*}(p, p_0)}{\sum_{j=1}^{i} \left( \frac{L_k}{L_k - \mu_k} \right)^j} \left( \frac{\mu_k}{L_k \left( 1 + \frac{\mu_k}{L_k - \mu_k} \right)^i - 1} \right)
\]

Now, the proof of the second inequality follows from \( 1 + \frac{\mu_k}{L_k - \mu_k} \geq 1 + \frac{\mu_k}{L_k - \mu_k} \). Finally, if \( f \) is minimized at \( x_* \), then it is unique and \( x_* \in \text{int}(\text{dom } f) \) by Lemma 2.2. Substituting \( p = 0 \) into (20), we necessarily have \( \nabla f(x_i) \to 0 \) and thus \( x_i \to x_* \).

### Remark 3
For a given \( k \), the set of functions satisfying Theorem 3.2 is closed under horizontal translation. To see why, let \( f \) satisfy Theorem 3.2. Define \( g(x) = f(x - z) \) for \( z \in \mathbb{R}^d \). Then, by Theorem 12.3 of [24],

\[
g^*(p) = f^*(p) + \langle z, p \rangle.
\]

First, note that \( \text{dom } g^* = \text{dom } f^* \). Bregman divergences of functions that differ only in affine terms are identical [1], so we have for all \( p, q \in \text{int}(\text{dom } f^*) \)

\[
D_{g^*}(p, q) = D_{f^*}(p, q).
\]

Thus \( g \) satisfies Theorem 3.2.
4 A power function example

4.1 The class of functions

Given \( a > 2 \) and \( A \in \mathbb{R}^{d \times d} \) such that \( A > 0 \), consider the class of functions \( f : \mathbb{R}^d \to \mathbb{R} \),

\[
f(x) = \frac{1}{a} \|Ax - c\|^a
\]

(33)

for \( c \in \mathbb{R}^d \). This class of functions separates the primal scheme from the dual one in the following precise sense. On the one hand, there exists a \( k : \mathbb{R}^d \to \mathbb{R} \) (cheap to compute) such that the sufficient conditions (10) of the dual scheme are satisfied for all \( f \) in this class. On the other hand, for all twice differentiable \( h : \mathbb{R}^d \to \mathbb{R} \), there exists an \( f \) from this class such that the sufficient conditions (5) for linear convergence of the primal scheme are violated.

4.2 Design of \( k \) for the dual scheme

Let \( b = a/(a - 1) \), and

\[
k(p) = \frac{\|p\|^b}{b}.
\]

Notice that this \( k \) is “cheap” relative to the problem at hand, because it does not require \( A \) or \( A^{-1}c \). In this case, the iterates of (9) are given by,

\[
x_{i+1} = x_i - \frac{1}{L_k} \left( \frac{\|A^T(Ax_i - c)\|}{\|Ax_i - c\|} \right)^{b-2} A^T(Ax_i - c). \tag{34}
\]

We show that the dual relative convexity and smoothness conditions hold for this \( f \) and \( k \), allowing us to find a suitable \( L_k \).

In this case, \( f^*(p) \) can be computed exactly. First, note that for \( g(x) = \|x\|^a/a \), its convex conjugate is given by \( g^*(p) = \|p\|^b/b \). This follows from the fact that the supremum at

\[
g^*(p) = \sup_x \langle x, p \rangle - g(x)
\]

is reached when \( p = \nabla g(x) = \nabla \left( \langle x, x \rangle^{a/2}/a \right) = \|x\|^{a-2}x \), and hence \( x = \|p\|^{b-2}p \). Now using the fact that \( f(x) = g(Ax - c) \), we have

\[
f^*(p) = \sup_x \langle x, p \rangle - g(Ax - c)
\]

\[
= \sup_y \langle A^{-1}(y + c), p \rangle - g(y)
\]

\[
= \sup_y \langle y, A^{-T}p \rangle - g(y) + \langle A^{-1}c, p \rangle
\]

\[
= g^*(A^{-T}p) + \langle A^{-1}c, p \rangle = \frac{1}{b} \|A^{-T}p\|^b + \langle A^{-1}c, p \rangle.
\]

It is straightforward to show that the gradient and the Hessian of \( f^* \) are

\[
\nabla f^*(p) = \|A^{-T}p\|^{b-2}A^{-1}A^{-T}p + A^{-1}c,
\]

\[
\nabla^2 f^*(p) = \|A^{-T}p\|^{b-2}A^{-1}A^{-T} + (b-2)\|A^{-T}p\|^{b-4}A^{-1}A^{-T}pp'A^{-1}A^{-T},
\]

12
the latter one is only defined for \( p \neq 0 \). Suppose that \( p \neq 0 \) in the rest of the proof. Using the fact that \( A^{-T}pp'A^{-1} = (A^{-T}p)(A^{-T}p)' \leq \|A^{-T}p\|^2 I \), it follows that

\[
\|A^{-T}p\|^2 A^{-1} A^{-T} - A^{-1} A^{-T} pp'A^{-1} A^{-T} = A^{-1}(\|A^{-T}p\|^2 I - A^{-T}pp'A^{-1}) A^{-T} \geq 0,
\]

thus \( \|A^{-T}p\|^2 A^{-1} A^{-T} pp'A^{-1} A^{-T} \leq \|A^{-T}p\|^2 A^{-1} A^{-T} \). Hence for any \( p \neq 0 \),

\[
(b - 1)\|A^{-T}p\|^2 A^{-1} A^{-T} \leq \nabla^2 f^*(p) \leq \|A^{-T}p\|^2 A^{-1} A^{-T}
\]

(35)

using the fact that \(-1 < b - 2 \leq 0\). As a special case, for \( k(p) = \|p\|^b/b \), we have

\[
(b - 1)\|p\|^{b-2} I \leq \nabla^2 k(p) \leq \|p\|^{b-2} I,
\]

(36)

for any \( p \neq 0 \).

Let \( \sigma_{\min}(A) \) and \( \sigma_{\max}(A) \) denote the smallest and largest singular values of \( A \), then it is easy to show that \( \sigma_{\min}(A) = \min(A^T) \), \( \sigma_{\max}(A) = \max(A^T) \), \( \sigma_{\min}(A^{-1}) = (\sigma_{\max}(A))^{-1} \), \( \sigma_{\max}(A^{-1}) = (\sigma_{\min}(A))^{-1} \). Based on these, and the fact that \( b \leq 2 \), we have

\[
(\sigma_{\max}(A))^{-2} (\sigma_{\min}(A))^{-2} \|p\|^{b-2} I \leq \|A^{-T}p\|^2 A^{-1} A^{-T} \leq (\sigma_{\min}(A))^{-2} (\sigma_{\max}(A))^{-2} \|p\|^{b-2} I.
\]

Using (35), (36), and (37), for any \( p \neq 0 \), we have \( \mu_k \nabla^2 f^*(p) \leq \nabla^2 k(p) \leq L_k \nabla^2 f^*(p) \) for

\[
\mu_k = (b - 1)(\sigma_{\min}(A))^{-2} (\sigma_{\max}(A))^{-2},
L_k = (b - 1)^{-1} (\sigma_{\max}(A))^{-2} (\sigma_{\min}(A))^{-2},
\kappa_k = \frac{L_k}{\mu_k} = (b - 1)^{-2} \left( \frac{\sigma_{\max}(A)}{\sigma_{\min}(A)} \right)^{4-b}.
\]

Using Lemma 2.35, we can see that the conditions of relative smoothness and convexity hold, and hence the dual gradient scheme is applicable. Here \( \nabla f(x) = \|Ax - c\|^{a-2} A^T (Ax - c) \) and

\[
k(\nabla f(x)) = \frac{1}{b} \|Ax - c\|^{a-2} A^T (Ax - c) \geq \frac{1}{b} \sigma_{\min}(A) \|Ax - c\|^{(a-1)b}
\]

\[
= \frac{1}{b} \sigma_{\min}(A) \|Ax - c\|^{a} = \frac{a}{b} \sigma_{\min}(A) f(x),
\]

hence by (27), we have

\[
f(x_i) - f(x_*) \leq (b - 1) \frac{L_k}{\sigma_{\min}(A)} \left( 1 - \frac{\mu_k}{L_k} \right)^i D_{f^*}(0, \nabla f(x_0)).
\]

4.3 Lack of \( h \) for the primal scheme

The argument that every fixed \( h \) fails to satisfy the sufficient conditions on at least one \( f \) proceeds by contradiction. Assume there existed such a twice differentiable \( h \). Because \( h \) is twice differentiable the conditions of [15] require that there exists \( \mu_h, L_h > 0 \) such that the following holds,

\[
\mu_h \nabla^2 h(x) \leq \nabla^2 \left( \frac{1}{a} \|Ax - c\|^a \right) \leq L_h \nabla^2 h(x),
\]

(38)
for some positive constants $\mu_h, L_h$. Now, notice $\nabla (\|Ax - c\|^a/a) = \nabla (\|Ax - c\|^{a-2} A^T (Ax - c))$, and

$$\nabla^2 \left( \frac{1}{a} \|Ax - c\|^a \right) = \|Ax - c\|^{a-2} A^T A (a-2) \|Ax - c\|^{a-4} A^T (Ax - c) (Ax - c)^T A,$$

for $Ax - c \neq 0$, and 0 for $Ax - c = 0$. Hence, we must have $\nabla^2 h(A^{-1}c) = 0$. But, since $A^{-1}c$ ranges over all of $\mathbb{R}^d$ for certain choices of $c$, we are forced to take $h(x) \equiv 0$, if we wish to satisfy the strong convexity requirement of [15] for all $f$ in this class. However, this $h$ does not satisfy the smoothness requirement of [2, 15].

Moreover, even if we gave ourselves access to $h(x) = \|x - A^{-1}c\|^a/a$, one can show with a similar argument as we have done for $k$ that $f$ is $\mu_h$-relatively convex and $L_h$-relatively smooth with respect to $h$ for

$$\mu_h = a^{-1} (\sigma_{\min}(A))^a,$$

$$L_h = a (\sigma_{\max}(A))^a,$$

$$\kappa_h = \frac{L_h}{\mu_h} = a^2 \left( \frac{\sigma_{\max}(A)}{\sigma_{\min}(A)} \right)^a$$

so the condition number is larger than for the dual scheme (since $(b-1)^{-1} = a - 1 < a$ and $4 - b = 3 - (a - 1)^{-1} < a$ as long as $a > 2$).

Similarly to this, the example $f(x) = \|Ax-b\|_4^4/4 + \|Cx-d\|_2^2/2$ from page 339 of [15] can be also shown to have better conditioning under the dual scheme than under the primal mirror descent scheme ($\kappa_k$ is smaller than $\kappa_h$) when $\sigma_{\max}(A)/\sigma_{\min}(A)$ is large, for reference function $h(x) = \|x\|_4^4/4 + \|x\|_2^2/2$ versus dual reference function $k(p) = \left( \|p\|_2^{-2} + \|p\|_2^{-4/3} \right)^{-1}$. The calculations are messy so we omit them here.

Acknowledgements

We thank David Balduzzi for his comments and helpful discussions. This material is based upon work supported in part by the U.S. Army Research Laboratory and the U.S. Army Research Office, and by the U.K. Ministry of Defence (MoD) and the U.K. Engineering and Physical Research Council (EPSRC) under grant number EP/R013616/1. CJM acknowledges the support of the Natural Sciences and Engineering Research Council of Canada (NSERC) under reference number PGSD3-460176-2014. YWT’s research leading to these results has received funding from the European Research Council under the European Union’s Seventh Framework Programme (FP7/2007-2013) ERC grant agreement no. 617071.

References

[1] Arindam Banerjee, Srujana Merugu, Inderjit S Dhillon, and Joydeep Ghosh. Clustering with Bregman divergences. *Journal of Machine Learning Research*, 6:1705–1749, 2005.

[2] Heinz H Bauschke, Jérôme Bolte, and Marc Teboulle. A descent lemma beyond Lipschitz gradient continuity: first-order methods revisited and applications. *Mathematics of Operations Research*, 42(2):330–348, 2016.
[3] Amir Beck and Marc Teboulle. Mirror descent and nonlinear projected subgradient methods for convex optimization. *Operations Research Letters*, 31(3):167–175, 2003.

[4] Jérôme Bolte, Shoham Sabach, Marc Teboulle, and Yakov Vaisbourd. First order methods beyond convexity and Lipschitz gradient continuity with applications to quadratic inverse problems. *SIAM Journal on Optimization*, 28(3):2131–2151, 2018.

[5] Jonathan Borwein and Adrian S Lewis. *Convex Analysis and Nonlinear Optimization: Theory and Examples*. Springer Science & Business Media, 2010.

[6] Stephen Boyd and Lieven Vandenberghe. *Convex Optimization*. Cambridge university press, 2004.

[7] Radu-Alexandru Dragomir, Jérôme Bolte, and Alexandre d’Aspremont. Fast Gradient Methods for Symmetric Nonnegative Matrix Factorization. *arXiv e-prints*, page arXiv:1901.10791, January 2019.

[8] Dmitriy Drusvyatskiy and Adrian S Lewis. Error bounds, quadratic growth, and linear convergence of proximal methods. *Mathematics of Operations Research*, 2018.

[9] Nicolas Flammarion and Francis Bach. Stochastic composite least-squares regression with convergence rate $O(1/n)$. In *Proceedings of The 30th Conference on Learning Theory, (COLT)*, 2017.

[10] David H Gutman and Javier F Peña. A unified framework for Bregman proximal methods: subgradient, gradient, and accelerated gradient schemes. *arXiv preprint arXiv:1812.10198*, 2018.

[11] Filip Hanzely, Peter Richtarik, and Lin Xiao. Accelerated Bregman proximal gradient methods for relatively smooth convex optimization. *arXiv preprint arXiv:1808.03045*, 2018.

[12] Anatoli Juditsky and Yuri Nesterov. Deterministic and stochastic primal-dual subgradient algorithms for uniformly convex minimization. *Stochastic Systems*, 4(1):44–80, 2014.

[13] Sham Kakade, Shai Shalev-Shwartz, and Ambuj Tewari. On the duality of strong convexity and strong smoothness: Learning applications and matrix regularization. *Unpublished Manuscript*, [http://ttic.uchicago.edu/~shai/papers/KakadeShalevTewari09.pdf](http://ttic.uchicago.edu/~shai/papers/KakadeShalevTewari09.pdf), 2009.

[14] Hamed Karimi, Julie Nutini, and Mark Schmidt. Linear convergence of gradient and proximal-gradient methods under the Polyak-Lojasiewicz condition. In *Joint European Conference on Machine Learning and Knowledge Discovery in Databases*, pages 795–811. Springer, 2016.

[15] Haihao Lu, Robert M Freund, and Yurii Nesterov. Relatively smooth convex optimization by first-order methods, and applications. *SIAM Journal on Optimization*, 28(1):333–354, 2018.

[16] Chris J. Maddison, Daniel Paulin, Yee Whye Teh, Brendan O’Donoghue, and Arnaud Doucet. Hamiltonian descent methods. *arXiv e-prints*, page arXiv:1809.05042, September 2018.

[17] Ion Necoara, Yu Nesterov, and Francois Glineur. Linear convergence of first order methods for non-strongly convex optimization. *Mathematical Programming*, pages 1–39, 2018.
[18] Arkadii S Nemirovskii and Yurii E Nesterov. Optimal methods of smooth convex minimization. *USSR Computational Mathematics and Mathematical Physics*, 25(2):21–30, 1985.

[19] AS Nemirovskii and DB Yudin. Effective methods for the solution of convex programming problems of large dimensions. *Ekonom. i Mat. Metody*, 15(1):135–152, 1979.

[20] Arkadii S Nemirovsky and David B Yudin. *Problem Complexity and Method Efficiency in Optimization*. Wiley Interscience, 1983.

[21] Yurii Nesterov. *Introductory Lectures on Convex Optimization: A Basic Course*, volume 87. Springer Science & Business Media, 2013.

[22] Yurii E Nesterov. A method for solving the convex programming problem with convergence rate $O(1/k^2)$. *Dokl. Akad. Nauk SSSR*, 269:543–547, 1983.

[23] Boris T Polyak. *Introduction to Optimization*. 1987.

[24] R. Tyrrell Rockafellar. *Convex Analysis*. Princeton University Press, 1970.

[25] Vincent Roulet and Alexandre d’Aspremont. Sharpness, restart and acceleration. In *Advances in Neural Information Processing Systems*, pages 1119–1129, 2017.

[26] Shai Shalev-Shwartz and Yoram Singer. Online learning: Theory, algorithms, and applications. 2007.

[27] Paul Tseng. On accelerated proximal gradient methods for convex-concave optimization. Technical report, MIT, 2008.

[28] Tianbao Yang and Qihang Lin. Rsg: Beating subgradient method without smoothness and strong convexity. *Journal of Machine Learning Research*, 19(1):1–33, 2018.

[29] Xingyu Zhou. On the Fenchel duality between strong convexity and Lipschitz continuous gradient. *arXiv e-prints*, page arXiv:1803.06573, March 2018.

[30] Constantin Zălinescu. *Convex Analysis in General Vector Spaces*. World Scientific, 2002.