The notion of \((a, b)\)-cores is closely related to rational \((a, b)\) Dyck paths due to Anderson’s bijection, and thus the number of \((a, a + 1)\)-cores is given by the Catalan number \(C_a\). Recent research shows that \((a, a + 1)\) cores with distinct parts are enumerated by another important sequence—Fibonacci numbers \(F_a\). In this paper, we consider the abacus description of \((a, b)\)-cores to introduce the natural grading and generalize this result to \((a, as + 1)\)-cores. We also use the bijection with Dyck paths to count the number of \((2k − 1, 2k + 1)\)-cores with distinct parts. We give a second grading to Fibonacci numbers, induced by bigraded Catalan sequence \(C_{a,b}(q,t)\).

1. Introduction

For two coprime integers \(a\) and \(b\), the rational Catalan number \(C_{a,b}\) and its bigraded generalization \(C_{a,b}(q,t)\) have caught the attention of different researchers due to their connection to algebraic combinatorics and geometry \([5, 4, 7, 9]\). Catalan numbers can be analyzed from the perspective of different combinatorial objects: rational \((a, b)\)-Dyck paths, simultaneous \((a, b)\)-core partitions and abacus diagrams.

In 2015, Amdeberhan \([1]\) conjectured that the number of \((a, a + 1)\)-cores with distinct parts is equal to the Fibonacci number \(F_{a+1}\), and also conjectured the formulas for the largest size and the average size of such partitions. This conjecture has been proven by H.Xiong \([14]\):

**Theorem 1.** (Xiong,15) For \((a, a + 1)\)-core partitions with distinct parts, we have

1. the number of such partitions equals to the Fibonacci number \(F_{a+1}\);
2. the largest size of such partition is \(\left\lfloor \frac{1}{3} \left( \frac{a+1}{2} \right) \right\rfloor\);
3. there are \(\frac{3-(-1)^n}{2}\) such partitions of maximal size;
4. the total number of these partitions and the average sizes are, respectively, given by

\[
\sum_{i+j+k=a+1} F_iF_jF_k \quad \text{and} \quad \sum_{i+j+k=a+1} \frac{F_iF_jF_k}{F_{a+1}}.
\]

Part (1) of the above theorem was independently proved by A.Straub \([13]\).

Another interesting conjecture of Amdeberhan is the number of \((2k − 1, 2k + 1)\)-cores with distinct parts. This conjecture have been proven by Yan, Qin, Jin and Zhou \([15]\):

**Theorem 2.** (YQJZ,16) The number of \((2k − 1, 2k + 1)\)-cores with distinct parts is equal to \(2^{2k-2}\).

The proof uses somewhat complicated arguments about the poset structure of cores. Results by Zaleski and Zeilberger \([17]\) improve the argument using Experimental Mathematics tools in Maple. More recently Baek, Nam and Yu provided simpler bijective proof in \([6]\).
Another set of combinatorial objects that caught the attention of a number of researchers is the set of \((a, as - 1)\)-cores with distinct parts. In particular, there is a Fibonacci-like recursive relation for the number of such cores: \[ N_s(a) = N_s(a - 1) + sN_s(a - 2). \]

**Theorem 3.** (Straub, 16) The number \(N_s(a)\) of \((a, as - 1)\)-core partitions into distinct parts is characterized by \(N_s(1) = 1\), \(N_s(2) = s\) and, for \(a \geq 3\), \[ N_s(a) = N_s(a - 1) + sN_s(a - 2). \]

In this paper, we analyze simultaneous core partitions in the context of Anderson’s bijection and in Section 3 we provide a simple description of the set of \((a, as + 1)\)-cores with distinct parts in terms of abacus diagrams, which also allows us to provide another proof of Theorem 1 parts (1), (2) and (3) in Section 4.

In Section 5 we use the connection between cores and Dyck paths to provide another simple proof of Theorem 2.

In Section 6 we introduce graded Fibonacci numbers \[ F_{a,b}(q) = \sum_{\pi} q^{\text{area}(\pi)}, \] where the sum is taken over all \((a, b)\)-cores \(\pi\) with distinct parts and \(\text{area}\) is some statistic on \((a, b)\)-cores. We show that \(F_{a,a+1}(1) = F_{a+1}\) - the regular Fibonacci sequence, and prove recursive relations for \(F_{a,b}(q) := F_{a,a+1}(q)\). Using properties of \(F_{a,a+1}(q)\) we provide another proof of Theorem 1 part (4).

In Section 7 we introduce bigraded Fibonacci numbers as a summand of bigraded Catalan numbers: \[ F_a^{(s)}(q, t) = \sum_{\pi} q^{\text{area}(\pi)} t^{\text{bounce}(\pi)}, \] where the sum is taken over all \((a, as + 1)\)-Dyck paths corresponding to \((a, as + 1)\)-cores with distinct parts, and statistics \((\text{area, bounce})\) are two standard statistics on Dyck paths (see 11). Using abacus diagrams, we can get a simple formula for \(F_a^{(s)}(q, t)\) and prove a theorem that gives recursive relations similar to the recursive relations for regular Fibonacci numbers. We use the standard notation \((s)_r = 1 + r + \ldots + r^{s-1}\).

**Theorem 4.** Normalized bigraded Fibonacci numbers \(\tilde{F}_a^{(s)}(q, t)\) satisfy the recursive relations
\[ F_{a+1}^{(s)}(q, t) = F_a^{(s)}(q, t) + qt^{s-1} F_{a-1}^{(s)}(q, t), \]
with initial conditions \(\tilde{F}_0^{(s)}(q, t) = \tilde{F}_1^{(s)}(q, t) = 1\).

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2. Background and notation

For two coprime numbers \(a\) and \(b\) consider a rectangle \(R_{a,b}\) on square lattice with bottom-left corner at the origin and top-right corner at \((a, b)\). We call the diagonal from \((0, 0)\) to \((a, b)\) the main diagonal of the rectangle \(R_{a,b}\). An \((a, b)\)-Dyck path is a lattice path from \((0, 0)\) to \((a, b)\) that consists of North and East steps and that lies weakly above the main diagonal. Denote the set of \((a, b)\)-Dyck paths by \(D_{a,b}\).

For a box in \(R_{a,b}\) with bottom-right corner coordinates \((x, y)\), define the rank of the box to be equal to \(ay - bx\) (see Fig. 1 left). Note that a box has positive rank if and only if it lies...
above the main diagonal. For a rational Dyck path \( \pi \), we define the area statistic area(\( \pi \)) to be the number of boxes in \( R_{a,b} \) with positive ranks that are below \( \pi \).

![Figure 1. (9, 11)-Dyck path \( \pi \) and (9, 11)-core \( \kappa \) with \( \text{core}(\pi) = \kappa \).](image)

Denote the set of ranks of all the area boxes of \( \pi \) as \( \alpha(\pi) \). Note that \( \alpha(\pi) \) doesn’t contain any multiples of \( a \) or \( b \) and it has an \((a, b)\)-nested property, i.e. 

\[
(i \in \alpha(\pi), \ i > a) \Rightarrow i - a \in \alpha(\pi), \quad (j \in \alpha(\pi), \ j > b) \Rightarrow j - b \in \alpha(\pi).
\]

Also note that \( \alpha(\pi) \) completely determines the Dyck path \( \pi \).

**Remark 5.** The \((a, b)\)-nested property of \( \alpha(\pi) \) is equivalent to the \((a, b)\)-invariant property of the complement of \( \alpha(\pi) \) (see [9]).

Consider \( i \in \{0, \ldots, a - 1\} \). If we can find a column in \( R_{a,b} \) with a box of rank \( i \), define \( e_i(\pi) \) to be the number of boxes in that column below \( \pi \) and above the main diagonal. If there is no box of rank \( i \) (i.e. if \( b|i) \), define \( e_i(\pi) \) to be zero. Note that 

\[
e_i(\pi) = \left| \{ x \in \alpha(\pi) \mid x \equiv i \pmod{a} \} \right|.
\]

The vector \( e(\pi) = (e_0(\pi), \ldots, e_{a-1}(\pi)) \) is defined to be the area vector of \( \pi \). Note that \( e_0(\pi) \) is always zero.

A partition \( \lambda \) of \( n \) is a finite non-increasing sequence \( (\lambda_1, \lambda_2, \ldots, \lambda_l) \) of positive integers, which sum up to \( n \). Integers \( \lambda_i \) are called parts of the partition \( \lambda \) and \( n \) is called the size of the partition. A partition \( \lambda \) is sometimes represented by its Young diagram (we will use English notation) (see Fig. 1 right). The hook length of a box in the Young diagram of \( \lambda \) is defined to be the number of boxes directly below and directly to the right of the given box, including that box itself.

We say that a partition \( \kappa \) is an \((a, b)\)-core if there are no boxes in the Young diagram of \( \kappa \) with hook length equal to \( a \) or \( b \). Denote the set of all \((a, b)\)-cores as \( K_{a,b} \). For example, a partition \( \kappa = (21, 13, 12, 8, 7, 6, 5, 3, 2, 1) \) belongs to \( K_{9,11} \) (see Fig. 1).

Define the size statistic on cores size(\( \kappa \)) to be the sum of all parts of \( \kappa \). Given an \((a, b)\)-core \( \kappa \), denote the set of hook lengths of the boxes in the first column by \( \beta(\kappa) \). It is not hard to show that \( \beta(\kappa) \) also satisfies \((a, b)\)-nested property (in fact, the nested condition of \( \beta(\kappa) \) and the fact that \( \beta(\kappa) \) doesn’t contain multiples of \( a \) or \( b \) are sufficient for \( \kappa \) to be an \((a, b)\)-core). Note that the set \( \beta(\kappa) \) completely determines the core \( \kappa \).

For two coprime numbers \( a \) and \( b \), there is a bijection path: \( K_{a,b} \rightarrow D_{a,b} \) due to J.Anderson [2]. We can describe the map path by specifying how it acts on sets \( \alpha \) and \( \beta \), namely \( \alpha(\text{path}(\kappa)) = \beta(\kappa) \).
then given by $\partial$ of $\text{SE-step}$ of $\partial$.

The importance of that construction becomes clear when we let partition $\lambda$ with black beads and 0-th position on 0-th runner is always filled with a white bead. The bead otherwise. According to our construction all negative positions on runners are filled with a black bead if the step $i$ the enumeration is consecutively increasing for the steps going from left to right.

NE-steps and SE-steps. We enumerate the steps so that the first NE-step has number 0, and the enumeration is consecutively increasing for the steps going from left to right.

Given a box $x$, there is a correspondence between partitions and abacus diagrams. Let $\lambda$ be the Young diagram of a partition in Russian notation with each row going in North-East direction (see Fig. 2, left). Reading from left to right, the boundary of $\lambda$ (denoted by $\partial(\lambda)$) consists of NE-steps and SE-steps. We enumerate the steps so that the first NE-step has number 0, and the enumeration is consecutively increasing for the steps going from left to right.

A corresponding $a$-abacus diagram is constructed by filling position $k$ on runner $i$ with a black bead if the step $i + ak$ of $\partial(\lambda)$ is an SE-step and filling that position with a white bead otherwise. According to our construction all negative positions on runners are filled with black beads and 0-th position on 0-th runner is always filled with a white bead. The importance of that construction becomes clear when we let partition $\lambda$ to be an $a$-core.

**Proposition 6.** Partition $\lambda$ is an $a$-core if and only if for any SE-step $j$ in $\partial(\lambda)$ step $j - a$ is also an SE-step.

**Proof.** Given a box $x$ of a Young diagram in Russian notation of partition $\lambda$, there is an SE-step of $\partial(\lambda)$ directly in the NE direction of $x$ with number $se(x)$ and there is an NE-step of $\partial(\lambda)$ directly in the SE direction of $x$ with number $ne(x)$. The hook length of box $x$ is then given by $se(x) - ne(x)$.

$(\Leftarrow)$ If for every SE-step $j$ the step $j - a$ is also an SE-step, there is no box $x$ with $se(x) - ne(x) = a$ and $\lambda$ is an $a$-core.

$(\Rightarrow)$ Conversely, for any pair of SE-step $j$ and NE-step $k$ with $j > k$, there exists a box $x$ with $se(x) = j$ and $ne(x) = k$. Thus, if there are no boxes $x$ of hook length $a$, there are no pairs $(j,k)$ with $j - k = a$, and for any SE-step $j$ the step $j - a$ must be also SE.

**Corollary 7.** A partition $\lambda$ is an $a$-core if and only if the set of black beads in the corresponding $a$-abacus diagram is left-justified, i.e. for any runner $i$ there exist an integer $d_i \geq 0$ such all positions $k < d_i$ are filled with black beads and all positions $k \geq d_i$ are filled with white beads.

**Proof.** Remember that a black bead of an abacus diagram of $\lambda$ on a runner $i$ in position $k$ corresponds to an SE-step $j = i + ak$ of $\partial(\lambda)$.
Note that an abacus diagram is left-justified if and only if for any black bead on runner $i$ in position $k$, the bead in position $k - 1$ is also a black one. In turn, that is equivalent to the fact that for any SE-step $j = i + ak$ of $\partial(\lambda)$ the step $j - a = i + a(k - 1)$ is also an SE-step.

Given integers $d_i$ from Corollary \cite{7} denote $d = (d_0, d_1, \ldots, d_{a-1})$. It is useful to think about $d_i$ as a number of black beads in nonnegative positions of a runner $i$. We will call $d$ the abacus vector of an a-core $\lambda$.

Denote the map from a-cores $\lambda$ to integer $a$-dimensional abacus vectors $d$ by $\text{abac}(\lambda)$.

**Proposition 8.** Given an $(a,b)$-core $\kappa$, the a-abacus vector $(d_1, \ldots, d_{a-1}) = \text{abac}(\kappa)$ is equal to the area vector $c(\kappa)$.

**Proof.** We’ll use the notation from Proposition \cite{6} Let $\{x_1, \ldots, x_l\}$ be the set of boxes in the first column of the Young diagram of $\kappa$. Then $\text{ne}(x_i) = 0$ for any $i$ and the set $\{\text{se}(x_1), \ldots, \text{se}(x_l)\}$ covers all positive SE-steps of $\partial(\kappa)$.

The hook length of $x_i$ is thus equal to $\text{se}(x_i) - \text{ne}(x_i) = \text{se}(x_i)$, and the set of hook lengths of $\{x_1, \ldots, x_l\}$ is equal to $\{\text{se}(x_1), \ldots, \text{se}(x_l)\}$. By definition, the set of hook lengths of $\{x_1, \ldots, x_l\}$ is $\beta(\kappa)$. The number of elements in $\beta(\kappa)$ with residue $i$ modulo $a$ is equal to the number of positive SE-steps in $\partial(\kappa)$ with residue $i$ modulo $a$, so $c_i(\kappa) = d_i$ and $c(\kappa) = d$.

**Corollary 9.** Given $\kappa \in K_{a,b}$, the abacus vector $(d_1, \ldots, d_{a-1}) = \text{abac}(\kappa)$ is equal to the area vector $c(\pi)$ of the Dyck path $\pi = \text{path}(\kappa)$.

From our definition $d_0$ is always equal to 0, and thus we often omit that coordinate. For an arbitrary $a$-core $\lambda$ the only condition on the other coordinates $(d_1, \ldots, d_{a-1})$ is that they are all non-negative. It is often useful to think about $(d_1, \ldots, d_{a-1})$ as coordinates of an $a$-core in $\mathbb{Z}^{a-1}$.

**Remark 10.** Embedding of the set of $(a,b)$-cores in $\mathbb{Z}^{a-1}$ has been studied from the point of view of Ehrhart theory in \cite{10}.

To describe the area statistic in terms of vectors $d$, notice that the number of rows in $\lambda$ is equal to the number of positive SE-steps in $\partial(\lambda)$, which correspond to non-negative black beads in the a-abacus diagram. Thus, we define the area statistic of $d$ to be $\text{area}(d) = \sum d_i$.

3. Simultaneous cores with distinct parts.

Let us consider an a-core $\lambda$. We can express the condition that $\lambda$ has distinct parts in terms of the boundary $\partial(\lambda)$.

**Proposition 11.** An a-core $\lambda$ has distinct parts if and only if for each positive SE-step $j$ in $\partial(\lambda)$, the steps $j - 1$ and $j + 1$ are NE-steps.

**Proof.** Suppose there are two consecutive SE-steps $j + 1$ and $j$ in $\partial(\lambda)$. Then there are two rows $\text{row}_i$ and $\text{row}_{i+1}$ in the Young diagram of $\lambda$ that are bordered by steps $j + 1$ and $j$ from NE-side. Then $\lambda_j = \text{length}(\text{row}_i)$ and $\lambda_{i+1} = \text{length}(\text{row}_{i+1})$ are equal, and thus we arrive to a contradiction.

That property can be formulated in terms of vectors $d = \text{abac}(\lambda)$. Given a subset $S$ of $\{0, 1, \ldots, a-1\}$, we call $S$ to be an a-sparse set if for any two elements $n, m \in S$, $\lvert n - m \rvert \neq 1$. The support of the vector $d$ (denoted by $\text{supp}(d)$) is defined to be the set of all indexes $i$ with $d_i > 0$. We call vectors $d = (d_0, d_1, \ldots, d_{a-1})$ with sparse support to be a-sparse vectors. Note that omitting $d_0 = 0$ in $d$ doesn’t change the sparsity of the vector.
Proposition 12. An a-core $\lambda$ has distinct parts if and only if the vector $d = \text{abac}(\lambda)$ is an $a$-sparse vector.

Proof. ($\Rightarrow$) Consider an index $i \in \{1, \ldots, a-1\}$ such that $d_i > 0$. For the corresponding $a$-abacus diagram that means the bead on runner $i$ in position 0 is a black one, and the $i$-th step of $\partial(\lambda)$ is an SE-step. If $\lambda$ has distinct parts, that means steps $i-1$ and $i+1$ of $\partial(\lambda)$ are NE-steps and thus runners $i-1$ and $i+1$ don’t have black beads on nonnegative positions, and $d_{i-1} = d_{i+1} = 0$.

($\Leftarrow$) Conversely, suppose $d = \text{abac}(\lambda)$ is a sparse vector. Any step SE-step $j$ of $\partial(\lambda)$ corresponds to some black bead on runner $i$ in position $k$. Because of the sparsity, beads on runners $i - 1$ and $i + 1$ in position $k$ must be white ones, and thus steps $j - 1$ and $j + 1$ of $\partial(\lambda)$ must be NE-steps. Note that for any SE-step $j$ corresponding to the runner $i = 1$, the step to the left of $j$ is on the 0-th runner and thus is always an NE-step. Similarly, when $i = a - 1$ the step to the right of $j$ is on the 0-th runner in positive position and thus is always an NE-step.

We now consider an additional structure of simultaneous $(a, b)$-cores $\kappa$ in terms of their $a$-abacus diagrams. We use the Proposition 6 one more time, but now looking at $\kappa$ as a $b$-core.

Proposition 13. Let $\kappa$ be an $a$-core with $d = \text{abac}(\kappa)$ and consider positive integer number $b = sa + r$ with $0 \leq r < a$. Then $\kappa$ is a simultaneous $(a, b)$-core if and only if for any index $i$ between 1 and $a - 1$, one of the following is true:

1. $i \geq r$ and $d_i \leq d_{i-r} + s$,
2. $i < r$ and $d_i \leq d_{i+a-r} + s + 1$.

Proof. ($\Rightarrow$) Fix an index $i \in \{1, \ldots, a-1\}$ and consider all black beads on runner $i$ in nonnegative positions $k = 0, \ldots, d_i - 1$. The corresponding SE-steps of $\partial(\kappa)$ enumerated by $j = i, i + a, \ldots, i + a(d_i - 1)$.

From Proposition 6 if $\kappa$ is a $b$-core, then for any positive SE-step $j$ in $\partial(\kappa)$ the step $j - b$ is also an SE-step.

If $i \geq r$, steps $j - b = (i - r) + a(k - s)$ with $k = 0, \ldots, d_i - 1$ correspond to the black beads on the runner $(i-r)$ in positions $k = -s, -s + 1, \ldots, d_i - s - 1$, so the number of nonnegative black beads on the runner $(i-r)$ is greater or equal to $(d_i - s)$, and thus $d_i \leq d_{i-r} - s$.

Similarly, if $i < r$, steps $j - b = (i + a - r) + a(k - s - 1)$ with $k = 0, \ldots, d_i - 1$ correspond to the black beads on the runner $(i + a - r)$ in positions $k = -s - 1, -s + 1, \ldots, d_i - s - 2$, so the number of nonnegative black beads on the runner $(i+a-r)$ is greater or equal to $(d_i - s - 1)$, and thus $d_i \leq d_{i+a-r} + s + 1$.

($\Leftarrow$) Conversely, consider any SE-step $j$ in $\partial(\lambda)$ with the corresponding black bead on runner $i$ and position $k < d_i$.

If $i \geq r$, the step $j - b$ corresponds to a bead on runner $(i-r)$ and position $(k-s)$. Since $k - s < d_i - s \leq d_{i-r}$, that bead must be a black one and the step $(j-b)$ is an SE-step.

If $i < r$, the step $j - b$ corresponds to a bead on runner $(i+a-r)$ and position $(k-s-1)$. Since $k - s - 1 < d_i - s - 1 \leq d_{i+a-r}$, that bead must be a black one and the step $j - b$ is an SE-step.

Together with Proposition 12, Proposition 13 gives a complete description of the simultaneous $(a, b)$-cores with distinct parts.
4. Maximum size of \((a, a + 1)\)-cores with distinct parts.

Combining Proposition 12 and Proposition 13 for the case \(b = a + 1\), we get the following result.

**Theorem 14.** An \((a, a + 1)\)-core \(\kappa\) is an \((a, a + 1)\)-core with distinct parts if and only if \(d = \text{abac}(\kappa)\) has entries \(d_i \in \{0, 1\}\) and the support set \(\text{supp}(d) = \{i : d_i = 1\}\) is an \(a\)-sparse set.

**Proof.** From Proposition 12 the fact that \(\kappa\) has distinct parts is equivalent to the sparsity of the set \(\text{supp}(d)\). Now, taking \(s = 1\) and \(r = 1\) in Proposition 13 \(\kappa\) is an \((a + 1)\)-core if and only if \(d_i \leq d_{i-1} + 1\) for all \(i = 1, \ldots, a - 1\) (condition \(d_0 \leq d_{a-1} + 2\) is always satisfied).

If \(i\) is not in \(\text{supp}(d)\), then \(d_i = 0\) and the equation \(d_i \leq d_{i-1} + 1\) is true. If \(i\) is in \(\text{supp}(d)\), then \(d_{i-1} = 0\) because of the sparsity of \(\text{supp}(d)\) and so \(d_i \leq d_{i-1} + 1\) is equivalent to \(d_i = 1\). \(\square\)

**Theorem 15.** The number of \((a, a + 1)\)-cores with distinct parts is equal to the Fibonacci number \(F_{a+1}\).

**Proof.** By Theorem 14 all \((a, a + 1)\)-cores are in a bijection with \(a\)-sparse sets \(S = \text{supp(abac}(\kappa)) \subseteq \{1, \ldots, a - 1\}\). Let the number of \(a\)-sparse sets be \(G_a\). Then, depending on whether an element \(a - 1\) is in a set, we can divide all \(a\)-sparse sets into two classes, so that \(G_a = G_{a-1} + G_{a-2}\) and \(G(1) = 1\), \(G(2) = 2\). Thus \(G_a = F_{a+1}\). \(\square\)

H. Xiong [14] proved Theorem 15 together with conjectures about the largest size of \((a, a + 1)\)-cores with distinct parts and the number of such cores of maximal size (see Theorem 1). Here we provide another proof, which it is formulated in a different framework and which also will be useful for our future discussion.

**Theorem 16.** The largest size of an \((a, a + 1)\)-core with distinct parts is \(\left\lfloor \frac{a+1}{2} \right\rfloor\). Moreover, the core of maximal size is unique whenever \((a \mod 3) = 0\) or \(2\) and there are two cores of maximal size when \((a \mod 3) = 1\).

**Proof.** Given an \((a, a+1)\)-core \(\kappa\) with distinct parts, take \(d = \text{abac}(\kappa)\), \(\text{supp}(d) = \{i_1, \ldots, i_n\}\), where \(n = n(d) = |\text{supp}(d)| = \text{area}(\kappa)\) and indexes \(0 < i_1 < \ldots < i_n < a\). Denote the gaps between \(i_j\) and \(i_{j+1}\) as \(g_j = i_{j+1} - i_j - 1\) for \(j = 1, \ldots, n - 1\) and \(g_n = a - 1 - i_n\).

Since \(S\) is an \(a\)-sparse set, \(g_j \geq 1\) for \(j = 0, 1, \ldots, n - 1\), and thus we can instead consider nonnegative integer sequence \(g_j = g_j' - 1\) for \(j = 0, \ldots, n - 1\) and \(g_n = g_n'\). Notice that \(\sum_{j=0}^{n} g_j' = a - n\) and \(\sum_{j=0}^{n} s_j = a - 2n\).

**Lemma 17.**

\[
\text{size}(\kappa) = \frac{1}{6}3n(2a + 1 - 3n) - \sum_{j=0}^{n} jg_j.
\]

**Proof.** Following the construction of an abacus diagram (see Fig. 2), each row \(\text{row}_{a+j}\) of the partition \(\kappa\) is bordered by an SE-step \(\text{se}_j \in \partial(\kappa)\), which in turn corresponds to a black bead on the runner \(i_j\) of the abacus diagram.

The length of that row \(\kappa_{a+j}\) is determined by the number of NE-steps in \(\partial(\kappa)\) before the step \(\text{se}_j\). In the abacus diagram, those NE-steps would correspond to the white beads on runners \(i = 0, \ldots, i_j - 1\) in position 0, and the number of those white beads is equal to \(\sum_{k=1}^{j-1} s_k'\).
Summing over all $j$,
\[
\text{size}(\kappa) = \sum_{j=1}^{n} k_{n-j+1} = \sum_{j=1}^{n} \sum_{k=0}^{j-1} g_k' = \sum_{j=0}^{n} (n-j)g_j' = n \sum_{j=0}^{n} g_j' - \sum_{j=0}^{n} jg_j' = n(a-n) - \frac{n(n-1)}{2} - \sum_{j=0}^{n} jg_j = \frac{1}{6} 3n(2a+1-3n) - \sum_{j=0}^{n} jg_j.
\]

Thus, to find a core of largest size, we maximize over $n$ and all nonnegative integer sequences $(g_j)_{j=0}^{n}$ with $\sum g_j = a - 2n$.

\[
\text{max size}(\kappa) = \max_{n \in \mathbb{Z}^+} \max_{g_j \geq 0} \left( \frac{1}{6} 3n(2a+1-3n) - \sum_{j=0}^{n} jg_j \right) = \max_{n \in \mathbb{Z}^+} \left( \frac{1}{6} 3n(2a+1-3n) - \min_{g_j \geq 0} \sum_{j=0}^{n} jg_j \right).
\]

The minimum of $\sum jg_j$ over nonnegative sequences $(g_j)_{j=0}^{n}$ with $\sum g_j = a - 2n$ is equal to $0$ and is uniquely achieved when $g_0 = a - 2n$ and $g_j = 0$ for $j \neq 0$. Thus,

\[
\text{max size}(\kappa) = \max_{n \in \mathbb{Z}^+} \left( \frac{1}{6} 3n(2a+1-3n) \right).
\]

The maximum of the parabola on the right-hand side is achieved at a closest integer point to a number $\frac{2a+1}{6}$.

1. When $(a \mod 3) = 0$, the maximum is achieved at the unique point $n = \frac{a}{3}$ and the value of the maximum is $\frac{1}{3} (\frac{a+1}{2})$.
2. When $(a \mod 3) = 2$, the maximum is achieved at the unique point $n = \frac{a+1}{3}$ and the value of the maximum is $\frac{1}{3} (\frac{a+1}{2})$.
3. When $(a \mod 3) = 1$, there are two integer points equally close to a number $\frac{2a+1}{6}$, which are $n_1 = \frac{a+1}{3}$ and $n_2 = \frac{a+2}{3}$. Both integers give the maximum value equal to $\frac{1}{3} (\frac{a+1}{2})$.

\[
\square
\]

In Section 5, we give a generalization of the Proposition 14 and provide another proof of part (4) of Theorem 1. Before we do that, however, we need to develop a notion of graded Fibonacci numbers in Section 6.

5. Number of $(2k-1, 2k+1)$-cores with distinct parts.

The number of $(2k-1, 2k+1)$-cores with distinct parts was conjectured by A. Straub [13] to be $2^{2k-2}$. That conjecture have been proved by Yan, Qin, Jin and Zhou in [13] using surprisingly deep arguments. Here, we present another perspective on $(2k-1, 2k+1)$-cores using their connection with Dyck paths.

**Theorem 18.** The number of $(2k-1, 2k+1)$-cores with distinct parts is equal to $2^{2k-2}$.

**Proof.** We will use the Dyck path interpretation of cores. For a Dyck path $\pi$, we denoted $\alpha(\pi)$ to be the set of ranks of the area boxes of $\pi$ (see Fig. 1). We’ll also make use of a standard notation $[n] = \{1, 2, \ldots, n\}$

**Lemma 19.** Under the bijection path, the set of $(2k-1, 2k+1)$-cores with distinct parts maps to the set of $(2k-1, 2k+1)$-Dyck paths $\pi$ such that $\alpha(\pi) \cap [2k-1]$ is a sparse set.
Proof of the lemma. Given an \((a, b)\)-core \(\kappa\) with distinct parts and \(\pi = \text{path}(\kappa)\), the sparse set \(\beta(\kappa)\) is equal to \(\alpha(\pi)\). Therefore, we need to prove that the sparsity of \(\alpha(\pi)\) is implied by the sparsity of \(\alpha(\pi) \cap [2k - 1]\).

Suppose, for the sake of contradiction that \(\alpha(\pi) \cap [2k - 1]\) is sparse and there are two elements \(j\) and \(j + 1\) in \(\alpha(\pi)\). Since \(\alpha(\pi)\) is a \(2k - 1\)-nested set, elements \((j \mod 2k - 1)\) are in \(\alpha(\pi)\), they are both in \([2k - 1]\), and they differ by one (since there are no multiples of 2 in \(\alpha(\pi)\)). Therefore, we get a contradiction.

Denote \(T_k\) to be the upper triangle of \(R_{2k-1,2k+1}\), i.e. \(T_k\) consists of all boxes with positive rank in \(R_{2k-1,2k+1}\) (see Fig. 1). Separate \(T_k\) into three parts by a vertical line \(x = k - 1\) and a horizontal line \(y = k + 2\). Below the line \(y = k + 2\) the boxes of \(T_k\) form a staircase-like shape \(A\) that contains, among other boxes, the boxes of odd contents from \([2k - 1]\). To the right of the line \(x = k - 1\) the boxes of \(T_k\) form a staircase shape \(B\) that contains the boxes of even contents from \([2k - 1]\). Above \(y = k + 2\) and to the left of \(x = k - 1\) the boxes of \(T_k\) form a square shape \(C\) of size \(k - 1\).

Now we reflect the shape \(A \cup C\) over the main diagonal \(y = x\) and denote the resulting shape as \(A^T \cup C^T\). Put that shape to the right of \(C \cup B\) to form a rectangular region \(P_k = C \cup B \cup A^T \cup C^T\) (see Fig. 3).

We call a boundary between \(B\) and \(A^T\) to be the main diagonal of \(P_k\). Consider the paths \(\zeta\) from the SW corner of \(P_k\) to the NE corner of \(P_k\) consisting of N and E steps. Denote \(C(\zeta)\) to be the set of contents of boxes below \(\zeta\) in \(C\), denote \(B(\zeta)\) to be the set of contents of boxes below \(\zeta\) in \(B\), denote \(A^T(\zeta)\) to be the set of contents of boxes above \(\zeta\) in \(A^T\) and denote \(C^T(\zeta)\) to be the set of contents of boxes above \(\zeta\) in \(C^T\).

We call \(\zeta\) to be \(C\)-symmetric when \(C(\zeta) = C^T(\zeta)\) (see Fig. 3 and compare with Fig. 1).

Lemma 20. The set of \(C\)-symmetric paths \(\zeta\) in \(P_k\) is in bijection \(\phi\) with the set of \((2k - 1, 2k + 1)\)-Dyck paths with sparse \(\alpha(\pi) \cap [2k - 1]\). Moreover, \(\alpha(\phi(\zeta)) = C(\zeta) \cup B(\zeta) \cup A^T(\zeta)\).

Proof of the lemma. We can define \(\phi\) by the property above: \(\phi(\zeta) = \pi\) if and only if \(\alpha(\pi) = C(\zeta) \cup B(\zeta) \cup A^T(\zeta)\). First, we need to make sure the map \(\phi\) is well-defined, i.e. the set \(\gamma(\zeta) := C(\zeta) \cup B(\zeta) \cup A^T(\zeta)\) is a \((2k - 1, 2k + 1)\)-nested set (i.e. check conditions 4).

Let \(i \in \gamma(\zeta)\). If \(i \in B(\zeta)\) or \(i \in A^T(\zeta)\), conditions 4 are satisfied since \(B(\zeta)\) and \(A^T(\zeta)\) are nested sets by construction. If \(i \in C(\zeta) = C^T(\zeta)\), then \(i - (2k + 1) \in C(\zeta) \cup B(\zeta)\), since \(C(\zeta) \cup B(\zeta)\) is \((2k + 1)\)-nested and \(i - (2k + 1) \in A^T(\zeta) \cup C^T(\zeta)\) since \(A^T(\zeta) \cup C^T(\zeta)\) is \((2k - 1)\)-nested.

Second, we need to check that \(\alpha(\phi(\zeta)) \cap [2k - 1] = \gamma(\zeta) \cap [2k - 1]\) is sparse. Consider \(i \in (B(\zeta) \cup A(\zeta)) \cap [2k - 1]\), and assume without loss of generality that \(i \in B(\zeta)\). Then \(i\) is even and it is bordering odd boxes \(i - 1\) and \(i + 1\) from \(E\) and \(S\) directions. Since \(\zeta\) goes above the box \(i\), it can’t go below boxes \(i - 1\) and \(i + 1\) in \(A^T\), and thus \(i - 1, i + 1 \not\in A^T(\zeta)\). □
Now we want to count the number of $C$-symmetric paths $\zeta$ in $P_k$. If $C(\zeta)$ is non-empty, call $C$-shape of $\zeta$ to be the shape of the diagram under $\zeta$ in $C$, and $C^T$-shape of $\zeta$ is defined correspondingly. Denote $i(\zeta)$ to be the the width of the $C$-shape of $\zeta$ minus 1 (or the height of $C^T$-shape minus 1). Denote $j(\zeta)$ to be the height of the $C$-shape of $\zeta$ minus 1 (or the width of $C^T$-shape minus 1).

The number of $C$-symmetric paths with fixed $i(\zeta) = i$ and fixed $j(\zeta) = j$ is the number of possible paths in $C$ times the number of possible paths in $B \cup A^T$, which is equal to $\binom{i+j}{k+1}(k+1)$. If $C(\zeta)$ is empty, the number of paths is equal to $\binom{i+j}{k+1}(k+1)$. Thus, the total number of paths is

$$\binom{2k}{k+1} + \sum_{i,j \geq 0} \binom{i+j}{k+1}(2k-2-(i+j)) = \binom{2k}{k+1} + \sum_{i,j \geq 0} \binom{i-1}{k+1}(2k-2-j) =$$

$$\binom{2k}{k+1} + \sum_{i,j \geq 0} \binom{2k-1}{k+1} = \binom{2k-1}{k} + \sum_{i,j \geq 0} \binom{2k-1}{k+1} + \sum_{i,j \geq 0} \binom{2k-1}{k+2+i} =$$

$$= \sum_{i \geq 0, 1 \leq j \leq 2k-1} \binom{2k-1}{i} + \frac{1}{2} \sum_{i \geq 0, 1 \leq j \leq 2k-1} \binom{2k-1}{i} = 2^{2k-2}.$$

$\square$

6. Graded Fibonacci numbers and $(a, as + 1)$-cores with distinct parts.

Unfortunately, for general $b$ there is no easy way to combine Proposition 12 and Proposition 13. However it can be achieved for specific values of $b$.

**Theorem 21.** Let $\kappa$ be an $(a, b)$-core for some integer $s$. Then $\kappa$ is an $(a, b)$-core with distinct parts if and only if the abacus vector $d = \text{abac}(\kappa)$ is sparse and $d_i \leq s$ for $i = 1, \ldots, a - 1$.

**Proof.** Similar to the proof of Theorem 14, we use Proposition 12 to get the sparsity of supp($d$). Moreover, using Proposition 13 with $r = 1$ we see that $\kappa$ is an $(a, b)$-core if and only if $d_i \leq d_{i-1} + s$ for all $i = 1, \ldots, a - 1$ (condition $d_0 \leq d_{a-1} + s + 1$ is automatically satisfied since $d_0 = 0$ for any abacus vector $d$).

If $i$ is not in supp($d$), then $d_i = 0$ and the equation $d_i \leq d_{i-1} + s$ is true. If $i$ is in supp($d$), then $d_{i-1} = 0$ because of the sparsity of supp($d$) and so $d_i \leq d_{i-1} + s$ is equivalent to $1 \leq d_i \leq s$.

We use similar argument for the case $b = as - 1$.

**Theorem 22.** Let $\kappa$ be an $(a, b)$-core for some integer $s$. Then $\kappa$ is an $(a, b)$-core with distinct parts if and only if the abacus vector $d = \text{abac}(\kappa)$ is sparse, $d_i \leq s$ for $i \neq a - 1$ and $d_{a-1} \leq s - 1$.

**Proof.** Again, we use Proposition 12 to get the sparsity of supp($d$), and use Proposition 13 with $r = a - 1$ to get inequalities $d_i \leq d_{i+1} + s$ for $i = 0, \ldots, a - 2$ and $d_{a-1} \leq d_0 + (s - 1)$.

If $i$ is not in supp($d$), then $d_i = 0$ and the equation $d_i \leq d_{i+1} + s$ is true. If $i$ is in supp($d$) and $i \neq a - 1$, then $d_{i+1} = 0$ because of the sparsity of supp($d$) and so $d_i \leq d_{i+1} + s$ is equivalent to $1 \leq d_i \leq s$. If $i = a - 1$ and $i$ is in supp($d$), note that $d_0$ is always 0, and thus $d_{a-1} \leq d_0 + (s - 1)$ equivalent to $1 \leq d_{a-1} \leq s - 1$.

For the further analysis we’ll need a generating function of the area statistic of cores $\kappa$. We will call that function to be a graded Fibonacci number.
Definition 23. For two integers $a$ and $b$, the graded Fibonacci number is

\[ F_{a,b}(q) = \sum_{\kappa} q^{\text{area}(\kappa)}, \]

where the sum is taken over all $(a,b)$-cores $\kappa$ with distinct parts.

Remark 24. If the sum above was taken over all $(a,b)$-cores, we would have obtained a graded Catalan number (see [11]).

Remark 25. We don’t require $a$ and $b$ to be coprime. Despite of the fact that the sum would be infinite, the power series would converge for $|q| < 1$. For the further analysis of Catalan numbers with $a$, $b$ not coprime, see [8].

Remark 26. When we set $s \to \infty$, the set of $(a, as + 1)$-cores covers the set of all $a$-cores with distinct parts. Thus we will also be interested in the limit of $F_a^{(s)}$ when $s \to \infty$.

In the light of Theorem 21 from here and until the end of the paper we will only consider the case $b = as + 1$ (although all results that follow are applicable in the case $b = as - 1$ with minor modifications). To shorten the notation, we define $F_a^{(s)} = F_{a,as+1}$. It is helpful to rewrite the sum (3) in terms of vectors $d = \text{abac}(\kappa)$. Denote the set of all $a$-sparse vectors $d = (d_0, d_1, \ldots, d_{a-1})$ with $d_0 = 0$ and $d_i \leq s$ as $\mathcal{A}_a^{(s)}$.

Theorem 27.

\[ F_a^{(s)}(q) = \sum_{d \in \mathcal{A}_a^{(s)}} q^{\Sigma d_i}. \]

Proof. From Proposition 21 the map $\text{abac}: \kappa \to d$ is a bijection from the set of all $(a,b)$-cores $\kappa$ with distinct parts to the set of $a$-sparse vectors $d = (d_0, d_1, \ldots, d_{a-1})$ with $d_i \leq s$ and $d_0 = 0$, i.e. the set $\mathcal{A}_a^{(s)}$.

Also note that bijection $\text{abac}$ sends the area statistic of $\kappa$ to the sum $\sum_{i=0}^{a-1} d_i$, since the number of rows in $\kappa$ is equal to the number of positive SE-steps of $\partial(\kappa)$, which in turn is equal to the number of nonnegative black beads in the abacus diagram of $\kappa$.

Thus,

\[ F_a^{(s)}(q) = \sum_{\kappa} q^{\text{area}(\kappa)} = \sum_{d = \text{abac}(\kappa)} q^{\Sigma d_i} = \sum_{d \in \mathcal{A}_a^{(s)}} q^{\Sigma d_i}. \]

Justification of the term “graded Fibonacci numbers” comes from the proposition below. We will use a standard notation $(s)_q = 1 + q + \ldots + q^{s-1} = \frac{1 - q^s}{1 - q}$.

Theorem 28. Graded Fibonacci numbers $F_a^{(s)}(q)$ satisfy recurrence relation

\[ F_a^{(s)}(q) = F_a^{(s)}(q) + q(s)_q F_a^{(s)}(q) \]

with initial conditions $F_0^{(s)}(q) = F_1^{(s)}(q) = 1$.

Proof. We divide the sum in (4) into two parts: one over vectors $d$ with $a - 1 \not\in \text{supp}(d)$, and the other over vectors $d$ with $a - 1 \in \text{supp}(d)$.

\[ F_a^{(s)}(q) = \sum_{d \in \mathcal{A}_a^{(s)}} q^{\Sigma d_i} = \sum_{d \in \mathcal{A}_a^{(s)}; d_{a-1} = 0} q^{\Sigma d_i} + \sum_{d \in \mathcal{A}_a^{(s)}; d_{a-1} \neq 0} q^{\Sigma d_i} = \sum_{d \in \mathcal{A}_a^{(s)}; d_{a-1} = 0} q^{\Sigma d_i} + \sum_{d \in \mathcal{A}_a^{(s)}; d_{a-1} \neq 0} q^{\Sigma d_i} = F_{a-1}^{(s)}(q) + q(s)_q F_{a-2}^{(s)}(q). \]
For initial conditions, notice that $F_1^{(s)}(q) = 1$, since there is only one $(1, s + 1)$-core, which is empty. The number of $(2, 2s + 1)$-cores with distinct parts is equal to $s + 1$, with corresponding $d_1 = 0, 1, \ldots, s$, and thus $F_2^{(s)}(q) = 1 + q(s)_q$.

Following the recurrence we proved above, we can set $F_0^{(s)}(q) = 1$ for all $s$. □

Remark 29. Evaluating (5) at $q = 1$ would give us a recursive relation for the number of $(a, as + 1)$-cores with distinct parts.

\[
F_a^{(s)}(1) = F_{a-1}^{(s)}(1) + sF_{a-2}^{(s)}(1), \quad F_0^{(s)}(1) = F_1^{(s)}(1) = 1.
\]

In particular, $F_a^{(1)}(1) = F_{a+1}$ is a classical Fibonacci number.

Remark 30. In the limit $s \to \infty$, the relation (5) has the form

\[
F_a^{(\infty)}(q) = F_{a-1}^{(\infty)}(q) + \frac{q}{1 - q} F_{a-2}^{(\infty)}(q).
\]

In light of Remark 26, relation above is the recurrence for the generating function of area $\kappa$ over all $a$-cores with distinct parts.

Theorem 31. (6)

\[
F_a^{(s)}(q) = \sum_{n=0}^{\lfloor a/2 \rfloor} (q(s)_q)^n \binom{a-n}{n}.
\]

Proof. For a fixed $a$-sparse support set $S = \text{supp}(d)$, the sum in (4) is equal to

\[
\sum_{\text{supp}(d)=S} q^{\sum d_i} = \prod_{i \in S} \sum_{d_i=1} q^{d_i} = (q(s)_q)^{|S|}.
\]

For fixed $n = |S|$, the number of possible $a$-sparse support sets $S$ is the number $n$-element subsets of $\{1, 2, \ldots, a-1\}$ such that no two elements are neighbouring each other. The number of such subsets is equal to $\binom{a-n}{n}$.

Summing (7) over all $S$,

\[
F_a^{(s)}(q) = \sum_S (q(s)_q)^{|S|} = \sum_{n=0}^{\lfloor a/2 \rfloor} \sum_{|S|=n} (q(s)_q)^n = \sum_{n=0}^{\lfloor a/2 \rfloor} (q(s)_q)^n \binom{a-n}{n}.
\]

□

Theorem 32. The generating function for $F_a^{(s)}(q)$ with respect to $a$ is

\[
G^{(s)}(x; q) := \sum_{a=0}^{\infty} x^a F_a^{(s)}(q) = \frac{1}{1 - x - q(s)_q x^2}.
\]
Remark 33. In the limit \( s \to \infty \),

\[
F_a^{(\infty)}(q) = \sum_{n=0}^{[a/2]} q^n \binom{a-n}{n}, \quad G^{(\infty)}(x, q) = \frac{1}{1 - x - q x^2}.
\]

Now we consider the case \( s = 1 \) to give a proof of part (4) of Theorem 1.

Remark 34. When \( s = 1 \),

\[
F_a^{(1)}(q) = \sum_{n=0}^{[a/2]} q^n \binom{a-n}{n}, \quad G^{(1)}(x, q) = \frac{1}{1 - x - q x^2},
\]

and

\[
F_a^{(1)}(q) = F_{a-1}^{(1)}(q) + q F_{a-2}^{(1)}(q), \quad F_0^{(1)}(q) = F_1^{(1)}(q) = 1.
\]

Theorem 35. The total sum of the sizes and the average size of \((a, a+1)\)-cores with distinct parts are, respectively, given by

\[
\sum_{i+j+k=a+1} F_i F_j F_k \quad \text{and} \quad \sum_{i+j+k=a+1} \frac{F_i F_j F_k}{F_{a+1}}.
\]

Proof. Denote \( \Phi_d \) to be the total sum of sizes of \((a, a+1)\)-cores with distinct parts. Since the generating function of Fibonacci numbers \( \sum_{i=1}^{\infty} x^i F_i \) is equal to \( \frac{x}{1-x-x^2} \), then in order to prove the theorem it is enough to show that the generating function \( \Gamma(x) := \sum_{a=2}^{\infty} x^{a+1} \Phi_a \) is equal to

\[
\gamma(x) = \sum_{a=2}^{\infty} x^{a+1} \sum_{i+j+k=a+1} F_i F_j F_k = \left( \sum_{i=1}^{\infty} x^i F_i \right)^3 = \left( \frac{x}{1 - x - x^2} \right)^3.
\]

We use the equation (2) to find a formula for \( \Phi_d \).
To evaluate the double sum, we notice that taking \( \lambda \) to be a partition with \( \lambda = (1^{x_1} 2^{x_2} \ldots n^{x_n}) \),

\[
\sum_{g, n, a = 2n} \sum_{i = 0}^{n} ig_i = \sum_{L_i \leq n, L_i \leq 2n} |\lambda| = (\text{number of } \lambda \text{ in a rectangle } n \times (a - 2n)) \cdot (\text{average size of } \lambda \text{ in } n \times (a - 2n)).
\]

Note that the number of partitions that fit rectangle \( n \times (a - 2n) \) is equal to the number of paths from the bottom-right corner of the rectangle to the top-left corner, and thus is equal to \( \binom{n-a}{n} \).

The average size of the partition is equal to half of the area of the rectangle \( n \times (a - 2n) \) because of the symmetry of partitions. Thus, the average size of \( \lambda \) is equal to \( \frac{(a-2n)}{2} \).

Therefore,

\[
\sum_{g, n, a = 2n} \sum_{i = 0}^{n} ig_i = \frac{(a-2n)}{n} \cdot \frac{n(a-2n)}{2}.
\]

Thus, the sum in (13) evaluates to

\[
\Phi_a = \sum_{a} \left( \frac{a-2}{n} \right) \frac{n(a-n)}{2} = \frac{a}{2} \sum_{a} \left( \frac{a-2}{n} \right) n \left( \frac{a-n}{2} \right)n(n-1).
\]

Comparing it with (9), the sum simplifies to

\[
\Phi_a = \frac{a}{2} F_a(1) - \frac{1}{2} F''_a(1),
\]

where the derivative \( F_a(q) \) is a short-hand notation for \( F_a^{(1)}(q) \), and the derivative \( F'' \) is taken with respect to \( q \) (note that \( F_0'(1) = F_1'(1) = 0 \)). Using the expression for a generating function \( G^{(1)}(x; q) \) in (9),

\[
\Gamma(x) = \sum_{a=2}^{\infty} x^{a+1} \Phi_a = \frac{x}{2} \sum_{a=1}^{\infty} x^{a+1} F_a'(1) - \frac{1}{2} \sum_{a=0}^{\infty} x^{a+1} F''_a(1) = \frac{x^2}{2} \sum_{a=1}^{\infty} ax^{a-1} F_a'(1) - \frac{x}{2} \sum_{a=0}^{\infty} x^a F''_a(1) = \frac{x^2}{2} \frac{\partial^2 G^{(1)}(x; 1)}{\partial x \partial q} - \frac{x}{2} \frac{\partial^2 G^{(1)}(x; 1)}{\partial q^2} \quad \square
\]

\[
\frac{x^2}{2} 2x(1-x-x^2) - 2x^2(-1-2x) \quad \frac{x}{2} \frac{2x^4}{(1-x-x^2)^3} = \frac{x^3}{(1-x-x^2)^3}.
\]

\square
7. **Bounce statistic and bigraded Fibonacci numbers.**

In light of Remark 24, we can look at the summand of the bigraded Catalan numbers corresponding to the set of \((a, b)\)-cores with distinct parts. There is a definition of bigraded Catalan numbers in terms of \((a, b)\)-cores directly (see [5]), but the skew length statistic has rather complicated expression in terms of abacus vectors \(d\). Nevertheless, we can use another pair of statistics on the set of \((a, b)\)-Dyck paths in the case \(b = as + 1\), namely \(area\) and \(bounce\).

To define the \(bounce\) statistic of a \((a, as + 1)\)-Dyck path \(\pi\), first we present the construction of a bounce path for a Dyck path \(\pi\) due to N. Loehr [11].

We start at the point \((a, as + 1)\) of the rectangle \(R_{a,as+1}\) and travel in \(W\) (West) direction until we hit an \(N\) (North) step of \(\pi\). Denote \(v_1\) to be the number of \(W\) steps we did in the process, and travel \(w_1 := v_1\) steps in the \(S\) (South) direction.

After that we travel in \(W\) direction until we hit an \(N\) step of \(\pi\) again. Denote \(v_2\) to be the number of \(W\) steps made this time, and travel in \(S\) direction \(w_2 := v_2 + v_1\) steps if \(s > 1\) or \(w_2 := v_2\) steps if \(s = 1\).

In general, on \(k\)-th iteration, after we travel \(v_k \geq 0\) steps in \(W\) direction before hitting an \(N\)-step of \(\pi\), we then travel in \(S\) direction \(w_k := v_k + \ldots + v_{k-s+1}\) steps if \(k \geq s\) or \(w_k := v_k + \ldots + v_1\) steps if \(k < s\). The bounce path always stays above the main diagonal and eventually hits the point \((0, 1)\), where the algorithm terminates (see [11] for details).

To calculate \(bounce\) statistic of \(\pi\), each time our bounce path reaches an \(N\) step of \(\pi\) after traveling \(W\), add up the number of squares to the left of \(\pi\) and in the same row as \(x\). We will call those rows \(bounce\) rows (see Fig. 4).

**Definition 36.** [11] Bigraded rational Catalan number is defined by the equation

\[
C_{a,as+1}(q, t) = \sum_{\pi} q^{area(\pi)} t^{bounce(\pi)},
\]

where the sum is taken over all \((a, as + 1)\)-Dyck paths \(\pi\).

Following that definition, we restrict the sum to define bigraded Fibonacci numbers.
Definition 37. Bigraded rational Fibonacci number is defined by the equation
\begin{equation}
F_a^{(s)}(q, t) = \sum_{\pi} q^{area(\pi)} t^{bounce(\pi)},
\end{equation}
where the sum is taken over all \((a, as + 1)\)-Dyck paths such that \(k = \text{core}(\pi)\) has distinct parts.

Remark 38. After the specialization \(t = 1\) bigraded Fibonacci numbers \(F_a(q, 1)\) are equal to graded \(F_a(q)\) from the previous section.

Under the maps \(\text{core}\) and \(\text{abac}\), there is a correspondence between the Dyck paths \(\pi\) in \(\mathcal{R}_a\) and sparse abacus vectors \(d \in \mathcal{R}_a^{(s)}\). Denote the bounce statistic on \(\mathcal{R}_a^{(s)}\) to be a bounce statistic of the corresponding Dyck path \(\pi\), i.e. \(\text{bounce(abac(core(\pi))) = bounce(\pi)}\).

Theorem 39. Given an abacus vector \(d \in \mathcal{R}_a^{(s)}\),
\begin{equation}
bounce(d) = s^{\left(\frac{a}{2}\right)} - \sum_{i=1}^{a-1} (a - i)d_i.
\end{equation}

Proof. Let \(\pi\) be the corresponding Dyck path, i.e. \(\text{abac(core(\pi))) = d}\). It is easier to consider the statistic \(\text{bounce}'(\pi) = s^{\left(\frac{a}{2}\right)} - \text{bounce}(\pi)\). Here \(s^{\left(\frac{a}{2}\right)}\) counts the total number of boxes in the upper triangle \(T_{a,as+1}\), and thus \(\text{bounce}'(\pi)\) counts the number of boxes in non-bounce rows to the left of \(\pi\) plus the number of area boxes of \(\pi\). Thus, we need to show
\begin{equation}
bounce'(d) = \sum_{i=1}^{a-1} (a - i)d_i.
\end{equation}
We will prove \((18)\) by induction on \(a\). Base case \(a = 1\) is straightforward. Assume now that \((18)\) is true for any \(a \leq k\) and consider the case \(a = k + 1\).

According to Corollary 9 the area vector \(e(\pi) = (d_1, \ldots, d_k)\).

If \(d_1 = e_1(\pi) = 0\), the first \(s\) iterations of the bounce path algorithm yields values for \(W\) steps \(v_1 = 1, v_2 = \ldots = v_s = 0\) and the corresponding steps in \(S\) direction are \(w_1 = w_2 = \ldots = w_s = 1\), ending at the point \((k, ks + 1)\) (and after that point the values of \(v_1, \ldots, v_s\) don’t contribute to the bounce path).

Thus the top \(s\) rows of the upper triangle \(T_{k+1,(k+1)s+1}\) are bounce rows and moreover there are no area boxes of \(\pi\) in those rows. Therefore the top \(s\) rows don’t contribute anything to \(\text{bounce}'(\pi)\), and we can safely erase them, reducing \(a\) by one and reducing all indexes of \(d\) by one. Denoting \(d' = (d'_0, \ldots, d'_{k-1}) = (d_1, \ldots, d_k) = e(\pi)\),
\begin{equation}
bounce'(d) = bounce'(d') = \sum_{i=1}^{k-1} (k - i)d'_i = \sum_{i=2}^{k} (k - (i - 1))d_i = \sum_{i=1}^{k} ((k + 1) - i)d_i.
\end{equation}

If \(d_1 = e_1(\pi) > 0\) (see Fig 3), then \(d_2 = e_2(\pi) = 0\) because of the sparsity of \(d\). The first \(s\) iterations of the bounce path algorithm yields values for \(W\) steps \(v_1 = 1, v_2 = \ldots = v_{s-d_1} = 0, v_{s-d_1+1} = 1, v_{s-d_1+2} = \ldots = v_{2s-d_1} = 0\) with the corresponding steps in the \(S\) direction equal to \(w_1 = \ldots = w_{s-d_1} = 1, w_{s-d_1+1} = \ldots = w_{s} = 2, w_{s+1} = \ldots = w_{2s-d_1} = 1\), ending at the point \((k - 1, (k - 1)s + 1)\) (and after that point the values of \(v_1, \ldots, v_{2s-d_1}\) don’t contribute to the bounce path).

For the top \(2s\) rows of \(T_{k+1,(k+1)s+1}\) the non-bounce rows appear exactly when the bounce path travels 2 steps in the \(S\) direction, i.e. when \(w_i = 2\). Thus, there are \(d_1\) non-bounce rows, each contributing \(k-1\) boxes to statistic \(\text{bounce}'\). Besides that, there are \(d_1\) area boxes...
that also count towards bounce'. Thus, the contribution of the top 2s rows into bounce' is equal to $d_1k$. Denoting $d' = (d'_0, \ldots, d'_{k-2}) = (d_2, \ldots, d_k)$,

$$\text{bounce}'(d) = \text{bounce}'(d') + d_1k = \sum_{i=1}^{k-2} (k - 1 - i)d_{i+2} + d_1k = \sum_{i=1}^{k}((k + 1) - i)d_{i}.$$ 

\[\square\]

**Corollary 40.**

\[r^{-s(\pi)}F^{(s)}_a(q, t) = \sum_{d \in \mathcal{A}^{(s)}} q^\sum d_i t^{-\sum (a-i)d_i} = \sum_{d \in \mathcal{A}^{(s)}} (qt^{-a})^\sum d_i t^{-\sum id_i} = \sum_{d \in \mathcal{A}^{(s)}} q^\sum d_i t^{-\sum id_i}.\]

**Proof.** First two equalities follow directly from (16) and (17). For the last equality we use the symmetry of $\mathcal{A}^{(s)}$ under the reflection of indexes $0 \mapsto 0$, $i \mapsto a - i$. \[\square\]

From (19) it is easier to work with normalized polynomials

\[F^{(s)}_a(q, t) := r^{-s(\pi)}F^{(s)}_a(q, t^{-1}) = \sum_{d \in \mathcal{A}^{(s)}} q^\sum d_i t^{\sum id_i} = \sum_{d \in \mathcal{A}^{(s)}} (qt^a)^\sum d_i t^{-\sum id_i}.\]

**Remark 41.** In terms of Dyck paths, $F^{(s)}_a(q, t)$ is equal to the sum of $q^{a \text{read}(\pi)} \text{bounce}'(\pi)$ over $(a, a+1)$-Dyck paths $\pi$ with core($\pi$) having distinct parts.

Using the simple expression of $F^{(s)}_a(q, t)$ in (20), we prove recursive relations similar to Proposition 28

**Theorem 42.** Normalized bigraded Fibonacci numbers $F^{(s)}_a(q, t)$ satisfy the following relations:

\[F^{(s)}_{a+1}(q, t) = F^{(s)}_a(q, t) + qt^a(s)qt^{a-1}F^{(s)}_{a-1}(q, t);\]

\[F^{(s)}_{a+1}(q, t) = F^{(s)}_a(q, t) + qt(s)qt^{a-1}F^{(s)}_{a-1}(qt^2, t),\]

with initial conditions $F^{(s)}_0(q, t) = F^{(s)}_1(q, t) = 1$.

**Proof.** For equation (21), use the first sum in (20) and divide the set $\mathcal{A}^{(s)}_{a+1}$ into two parts corresponding to vectors $d$ with $d_a = 0$ and with $d_a > 0$.

$$F^{(s)}_{a+1}(q, t) = \sum_{d \in \mathcal{A}^{(s)}_{a+1}} q^\sum d_i t^{\sum id_i} = \sum_{d' \in \mathcal{A}^{(s)}_{a}} q^\sum d'_i t^{\sum id'_i} + \sum_{d \in \mathcal{A}^{(s)}_{a+1}} (qt^a)^d_a \sum_{d' \in \mathcal{A}^{(s)}_{a+1}} q^\sum d'_i t^{\sum id'_i} = F^{(s)}_a(q, t) + qt^a(s)qt^{a-1}F^{(s)}_{a-1}(q, t).$$

For equation (22), use the second sum in (20) and again divide $\mathcal{A}^{(s)}_{a+1}$ into two parts corresponding to vectors $d$ with $d_a = 0$ and with $d_a > 0$.

$$F^{(s)}_{a+1}(q, t) = \sum_{d \in \mathcal{A}^{(s)}_{a+1}} (qt^{a+1})^\sum d_i t^{-\sum id_i} = \sum_{d' \in \mathcal{A}^{(s)}_{a}} (qt^{a+1})^\sum d'_i t^{-\sum id'_i} + \sum_{d \in \mathcal{A}^{(s)}_{a+1}} (qt^{a+1})^d_a \sum_{d' \in \mathcal{A}^{(s)}_{a+1}} (qt^{a+1})^\sum d'_i t^{-\sum id'_i} = F^{(s)}_a(q, t) + qt(s)qt^{a-1}F^{(s)}_{a-1}(qt^2, t).$$

\[\square\]
Remark 43. Setting $s = 1$ and $a \to \infty$, the recurrence (22) gives

\[ \tilde{F}^{(1)}_\infty(1, q, t) = \tilde{F}^{(1)}(q, t) + qt \tilde{F}^{(1)}(qt^2, t), \]

which is an Andrews $q$-difference equation related to Rogers-Ramanujan identities (see [3]).

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