Universal super-replication of unitary gates

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Quantum states obey an asymptotic no-cloning theorem, stating that no deterministic machine can reliably replicate generic sequences of identically prepared pure states. In stark contrast, we show that generic sequences of unitary gates can be replicated deterministically at nearly quadratic rates, with an error vanishing on most inputs except for an exponentially small fraction. The result is not in contradiction with the no-cloning theorem, since the impossibility of deterministically transforming pure states into unitary gates prevents the application of the gate replication protocol to states. In addition to gate replication, we show that $N$ parallel uses of a completely unknown unitary gate can be compressed into a single gate acting on $O(\log N)$ qubits, leading to an exponential reduction of the amount of quantum communication needed to implement the gate remotely.

A striking feature of quantum theory is the impossibility of constructing a universal copy machine, which takes as input a quantum system in an arbitrary pure state and produces as output a number of exact replicas \cite{1,2}. Such an impossibility has major implications for quantum error correction \cite{3,4} and cryptographic protocols such as key distribution \cite{5,6}, quantum secret sharing \cite{7,8}, and quantum money \cite{9,10,11,12}. The impossibility of universal copy machines is a hard fact: it equally affects deterministic \cite{13,14,15,16,17,18,19,20,21,22} and probabilistic machines \cite{23,24}, whose performances coincide with those of deterministic machines when it comes to copying completely unknown pure states \cite{15,19}. A similar no-go result holds when the universal machine is presented a large number $N$ of identical copies and is required to produce a larger number $M > N$ of approximate replicas: if the replicas have non-vanishing overlap with the desired $M$-copy state, then the number of extra copies must be negligible compared to $N$ \cite{15}. We refer to this fact as the asymptotic no-cloning theorem, expressing the fact that independent and identically distributed (i.i.d.) sequences of pure states cannot be stretched by any significant amount. The asymptotic no-cloning theorem holds also for non-universal machines designed to copy continuous sets of states, provided that such machines work deterministically \cite{19}.

The impossibility of universal state cloning suggests similar results for quantum gates. Along this line, a no-go theorem for universal gate cloning was proven in Ref. \cite{20}, showing that no quantum network can perfectly simulate two uses of an unknown unitary gate by querying it only once. Optimal networks that approximate universal gate cloning were studied in Refs. \cite{20,21}. Very recently, Dürr and coauthors \cite{22} considered a non-universal setup designed to clone phase gates, i.e. gates generated by time evolution with a known Hamiltonian. In this scenario, they devised a quantum network that approximately simulates up to $N^2$ uses of an unknown phase gate while using it only $N$ times, with vanishing error in the large $N$ limit. Such a result establishes the possibility of super-replication of phase gates—super-replication being the generation of $M \gg N$ high-fidelity replicas from $N \gg 1$ input copies \cite{19}. Remarkably, super-replication of phase gates is achieved deterministically, whereas super-replication of phase states has exponentially small probability of success. The main open question raised by Ref. \cite{22} is whether deterministic super-replication occurs not only for phase gates, but also for arbitrary unitary gates. An affirmative answer would imply that the asymptotic no-cloning theorem only applies to states, whereas it is possible to stretch long i.i.d. sequences of reversible gates by up to a quadratic factor.

In this letter we answer the question in the affirmative, establishing the possibility of universal super-replication of unitary gates, in stark contrast with the asymptotic no-cloning theorem for pure states. Given $N$ uses of a completely unknown unitary gate $U$, we construct a quantum network that simulates up to $N^2$ parallel uses of $U$, providing an output that is close to the ideal target for all possible input states except for an exponentially small fraction. The quadratic replication rate is optimal: every other network producing replicas at a rate higher than quadratic will necessarily spoil their quality, delivering an output that has vanishing overlap with the output of the desired gate. In addition to replication, we consider the task of gate compression, where the goal is to faithfully encode the action of a black box into a gate operating on a smaller quantum system. We show that $N$ uses of a completely unknown gate can be encoded without any loss into a single gate acting only on $(d - 1)(d/2 + 1)\log N$ qubits, thus allowing for an exponential reduction of computational workspace. The number of qubits can be further cut down by a half if one tolerates an error that vanishes on almost all inputs in the large $N$ limit. The compression of i.i.d. gate sequences is the analogue of the compression of i.i.d. state sequences \cite{23}, recently demonstrated experimentally \cite{24}.

Universal super-replication of qubit gates. Let us start from the simple case of qubit gates, represented by unitary matrices in SU(2). A generic gate can be parametrized as $U_{\theta,\mathbf{n}} = \exp[-i\theta \mathbf{n} \cdot \mathbf{j}]$, where $\theta \in [0, 2\pi]$ is a rotation angle, $\mathbf{n} = (n_x, n_y, n_z)$ is a rotation axis, and $\mathbf{j} = (j_x, j_y, j_z)$ is the vector of angular momentum operators $(j_i := \sigma_i/2, i = x, y, z)$. We define $g := (\theta, \mathbf{n})$ and label the unitary gate as $U_g$. Here both $\theta$ and $\mathbf{n}$ are completely unknown, differing from the setting of \cite{22}, where
Theorem 1. The network simulates $M$ parallel uses of an unknown unitary gate $U_g$, while querying it only $N$ times. The simulation is obtained by transforming the input state of $M$ systems into the joint state of $N$ systems plus an ancilla (via quantum channel $C_1$), applying the unknown gate on the $N$ systems, and then recombining them with the ancilla via a quantum channel $C_2$, which finally produces $M$ output systems.

![Diagram](image)

**FIG. 1. Quantum network for gate replication.** The network simulates $M$ parallel uses of an unknown unitary gate $U_g$, while querying it only $N$ times. The simulation is obtained by transforming the input state of $M$ systems into the joint state of $N$ systems plus an ancilla (via quantum channel $C_1$), applying the unknown gate on the $N$ systems, and then recombining them with the ancilla via a quantum channel $C_2$, which finally produces $M$ output systems.

the rotation axis was fixed and only $\theta$ was varying.

In order to replicate gate $U_g$, we consider a network where $N$ parallel uses of $U_g$ are sandwiched between two quantum channels, $C_1$ and $C_2$, as in figure 1. The overall action of the network is described by the channel $C_2 \left( U_g^{\otimes N} \otimes I_A \right) C_1$, with $U_g(\cdot) = U_g \cdot U_g^\dagger$ and $I_A$ denoting the identity on a suitable ancillary system. To construct the channels $C_1$ and $C_2$, we decompose the Hilbert space of $K$ qubits ($K = N, M$) into rotationally invariant subspaces. Choosing $K$ to be even, we have

$$
\mathcal{H}^{\otimes K} \simeq \bigoplus_{j=0}^{K/2} \left( \mathcal{H}_j \otimes \mathcal{M}_j \right),
$$

where $j$ is the quantum number of the total angular momentum, $\mathcal{H}_j$ is a representation space, of dimension $d_j = 2j + 1$, and $\mathcal{M}_j$ is a multiplicity subspace, of dimension $m_j$. The isomorphism in Eq. (1) is called the Schur transform and can be implemented efficiently in a quantum circuit. We now introduce a cutoff on the quantum number $j$ and define the subspace

$$
\mathcal{H}_j^{(K)} = \bigoplus_{j \leq J} \left( \mathcal{H}_j \otimes \mathcal{M}_j \right).
$$

To compress a state inside this subspace, we use the encoding channel defined by

$$
E_j^{(K)}(\rho) := P_j^{(K)} \rho P_j^{(K)} + \text{Tr} \left[ \left( I^{\otimes K} - P_j^{(K)} \right) \rho \right] \rho_0
$$

where $P_j^{(K)}$ is the projector on $\mathcal{H}_j^{(K)}$ and $\rho_0$ is a fixed density matrix with support in $\mathcal{H}_j^{(K)}$. The key observation is that most $K$-qubit states are left unchanged by the channel $E_j^{(K)}$, provided that $K$ is large and $J$ is large compared to $\sqrt{K}$. Denoting by $F_{\Psi}^{(K,J)}$ the fidelity between a generic $K$-partite pure state $|\Psi\rangle$ and its compressed version $E_j^{(K)}(|\Psi\rangle)$, we have the following

**Theorem 1.** If $|\Psi\rangle$ is chosen uniformly at random, then, for every fixed $\epsilon > 0$, the probability that $F_{\Psi}^{(K,J)}$ is smaller than $1 - \epsilon$ satisfies the bound

$$
\text{Prob} \left[ F_{\Psi}^{(K,J)} < 1 - \epsilon \right] < \frac{2(K + 1)}{\epsilon} \exp \left[ -\frac{2J^2}{K} \right].
$$

**Proof.** By Markov's inequality, one has

$$
\text{Prob} \left[ F_{\Psi}^{(K,J)} < 1 - \epsilon \right] < \frac{1}{\epsilon} \mathbb{E} \left[ F_{\Psi}^{(K,J)} \right],
$$

where $\mathbb{E} \left[ F_{\Psi}^{(K,J)} \right]$ is the average of the fidelity over all pure states. In turn, the average fidelity can be lower bounded by the entanglement fidelity [20, 27], given by

$$
F_{E}^{(K,J)} = \langle \Phi_{2K} | \left( E_j^{(K)} \otimes I^{\otimes K} \right) | \Phi_{2K} \rangle,
$$

where $|\Phi_{2K}\rangle$ is a maximally entangled state in $\mathbb{C}^{2K} \otimes \mathbb{C}^{2K}$. The entanglement fidelity satisfies the bound

$$
F_{E}^{(K,J)} \geq \left| \langle \Phi_{2K} | \left( P_j^{(K)} \otimes I^{\otimes K} \right) | \Phi_{2K} \rangle \right|^2 = \frac{\sum d_j m_j K}{2K}^2
$$

where the coefficients $d_j m_j K / 2K$ form a probability distribution, known as the Schur-Weyl measure [28, 29]. For large $K$, the Schur-Weyl measure concentrates around $j = 0$ [30], yielding the bound $F_{E}^{(K,J)} \geq 1 - 2(K + 1) \exp \left[ -\frac{2J^2}{K} \right]$ (Appendix A), which combined with the previous observations implies Eq. (4).

Let us apply Theorem 1 to gate replication. The theorem guarantees that, except for an exponentially small fraction, almost all pure $M$-qubit states can be encoded into a subspace $\mathcal{H}_j^{(M)}$ with $J \gg \sqrt{M}$. To achieve gate replication, we combine this fact with the observation that for $j \leq N/2$, the states in $\mathcal{H}_j^{(M)}$ can be faithfully encoded into $\mathcal{H}_j^{(N)} \otimes \mathcal{H}_j$, where $\mathcal{H}_j$ is a Hilbert space of a suitable ancilla. The encoding is achieved by an isometry $V_j$ that commutes with all rotations, namely

$$
V_j U_g^{\otimes M} = (U_g^{\otimes N} \otimes I_A) V_j \quad \forall U_g \in SU(2).
$$

We are now ready to specify the channels $C_1$ and $C_2$ in the gate replication network of figure 1. For channel $C_1$, we choose $C_1 = V_j E_j^{(M)}$, where $V_j$ is the isometric channel $V_j(\cdot) = V_j \cdot V_j^\dagger$ and $j$ is set to $j = \sqrt{\sqrt{N^2 - \alpha^2}}$ for $M$ growing like $N^{2-\alpha}$, $\alpha > 0$ and to $j = N/2$ for $M$ growing like $N^2$ or faster. For channel $C_2$ we choose the inverse of $V_j$, namely

$$
C_2(\rho) = V_j^\dagger \rho V_j + \text{Tr} \left[ \left( I^{\otimes N} - V_j V_j^\dagger \right) \rho \right] \rho_0.
$$

The action of the network on a generic $M$-qubit state $U_g$ is then given by

$$
C_2(U_g^{\otimes N} \otimes I_A) C_1(|\Psi\rangle) = U_g^{\otimes M} C_2 C_1(|\Psi\rangle).
$$

The first equality coming from Eq. (5) and the second from the fact that $C_2$ is the inverse of $V_j$. Clearly, Eq. (7) implies that the fidelity between the output state and the ideal target $U_g^{\otimes M} |\Psi\rangle$ is equal to $F_{\Psi}^{(M,J)}$ independently of $g$. By Theorem 1, the fidelity is arbitrarily close to one on most input states whenever $M$ grows like $O(N^{2-\alpha})$. In this case, the number of ancillary qubits used by the
protocol scales like $M - N + O\left(N^{1-\alpha/2}\right)$ (Appendix B). For $M$ growing faster than $N^2$, the fidelity tends to zero.

In summary, given $N$ uses of a completely unknown gate, our network simulates up to $N^2$ uses with high fidelity on most input states. The simulation works with probability exponentially close to 1, but can fail on some specific inputs: notably, it fails for inputs of the i.i.d. form $|\psi\rangle^{\otimes M}$, for which the fidelity with the desired output state is zero. The fact that replication works well in the typical case is analogous to other phenomena based on measure concentration, such as quantum equilibration \text{\textcolor{black}{\cite{31}}} and entanglement typicality \text{\textcolor{black}{\cite{32}}}.

\textit{State cloning vs state generation.} Deterministic gate replication is not in contradiction with the asymptotic no-cloning theorem. Indeed, suppose that we wanted to use gate replication to clone a completely unknown pure state $|\psi\rangle = U_\psi |0\rangle$ for some fixed state $|0\rangle$. To this purpose, we would have to retrieve the gate $U_\psi$ from the input state $|\psi\rangle$—a task whose deterministic execution is forbidden by the no-programming theorem \text{\textcolor{black}{\cite{33}}}.

Since the state $|\psi\rangle$ is arbitrary, a symmetry argument precludes the conversion $|\psi\rangle \rightarrow U_\psi |0\rangle$ even probabilistically \text{\textcolor{black}{\cite{19}}}.

Though gate replication cannot be used for state cloning, it may still provide an advantage in the less demanding task of \textit{state generation}, which consists in producing $M$ copies of the state $|\psi\rangle$ from $N$ uses of the gate $U_\psi$. The advantage does not show up in the universal case, because universal gate replication does not reproduce correctly the action of the gate $U_\psi^{\otimes M}$ on the i.i.d. input state $|0\rangle^{\otimes M}$. However, it does show up in non-universal cases: for example, Ref. \text{\textcolor{black}{\cite{22}}} demonstrated that $N$ uses of a phase gate allow one to generate up to $N^2$ copies of the corresponding phase state. Similarly, we show that universal gate replication allows to generate $M$ maximally entangled states with fidelity

$$F_{\text{gen}}[N \rightarrow M] \geq 1 - 2(M + 1) \exp \left[ -\frac{N^2}{2M} \right]$$

(\text{\textcolor{black}{\cite{23}}}). The protocol for generating entangled states also provides an alternative way to generate phase states: given $N$ uses of a phase gate with phase $\theta$ one can first generate $M \gg N$ approximate copies of the maximally entangled state $((0)|0\rangle + e^{-i\theta}|1\rangle|1\rangle) / \sqrt{2}$ and then apply a CNOT gate to each copy and discard the second qubit of the pair, thus obtaining $M \gg N$ approximate copies of the state $(|0\rangle + e^{-i\theta}|1\rangle) / \sqrt{2}$. We refer to the ability to produce $M \gg N$ states from $N \gg 1$ uses of the corresponding gate as \textit{state super-generation}.

Again, we stress that state super-generation does not challenge the asymptotic no-cloning theorem, because $N$ uses of a gate cannot be obtained deterministically from $N$ copies of the corresponding state. In the concrete examples of phase states and maximally entangled states, the fidelity of the best deterministic $N$-to-$M$ cloner scales as $(N/M)^{1/2}$ and as $(N/M)^{3/2}$, respectively \text{\textcolor{black}{\cite{34}}}, clearly preventing the production of $M \gg N$ high-fidelity clones.

\textit{Probabilistic cloning of maximally entangled states and the optimality of universal gate replication.} The map $U_g \rightarrow |\Phi_g\rangle$ is a one-to-one correspondence between qubit gates and two-qubit maximally entangled states \text{\textcolor{black}{\cite{35,36}}}.

The inverse map $|\Phi_g\rangle \rightarrow U_g$ is implemented by probabilistic teleportation, which succeeds with optimal probability $1/4$ \text{\textcolor{black}{\cite{37,38}}}.

Combined with the super-generation of entangled states, the implementability of the map $|\Phi_g\rangle \rightarrow U_g$ implies that $N$ maximally entangled states can be cloned probabilistically, obtaining up to $N^2$ high fidelity copies, albeit with exponentially small probability $1/4^N$. The result is interesting not only because it provides an explicit protocol achieving super-replication of maximally entangled states, but also because it allows one to prove the optimality of the gate super-replication protocol: if there existed a protocol producing $M \gg N^2$ almost perfect copies of a generic unitary gate, such a protocol could be converted into a probabilistic cloning protocol producing $M \gg N^2$ almost perfect copies of a generic maximally entangled state. Such a protocol is impossible because it would violate the Heisenberg limit for quantum cloning \text{\textcolor{black}{\cite{19}}}.

Even more strongly, since the fidelity of state replication must vanish for $M \gg N^2$ \text{\textcolor{black}{\cite{19}}}, every gate replication protocol simulating $M \gg N^2$ uses must have vanishing entanglement fidelity, and, therefore, vanishing fidelity on most input states. This conclusion applies both to deterministic and probabilistic gate replication protocols.

\textit{Gate compression.} The argument used to prove gate super-replication can also be applied to the task of gate compression, whose goal is to encode the action of a gate $U_x$ into another gate $U'_x$ acting on a smaller physical system. Gate compression protocols are useful in distributed scenarios wherein a server (Alice) is required to apply the gate $U_x$ to an input state provided by a client (Bob). In such a task, it is natural to minimize the total amount of communication between client and server, by compressing the gate $U_x$ into a gate acting on the smallest possible system. Ideally, the compression should be faithful, in the sense that the action of $U_x$ on the original input state is simulated without errors. In practice, Alice and Bob can often tolerate a small error, especially if this allows them to increase the compression rate.

The general form of a gate compression protocol is illustrated in Figure \text{\textcolor{black}{\cite{22}}}. Here we consider the scenario where $U_x$ is an $N$-qubit gate of the i.i.d. form $U_{g}^{\otimes N}$, $U_g$ being an arbitrary rotation of the Bloch sphere. The case of rotations around a fixed axis has been previously considered in \text{\textcolor{black}{\cite{22}}}.

Using the decomposition of Eq. \text{\textcolor{black}{\cite{1}}}, the gate $U_{g}^{\otimes N}$ can be put in the block diagonal form

$$U_{g}^{\otimes N} = \bigoplus_{j=0}^{N/2} U_{g}^{(j)} \otimes I_{\mathcal{M}_j N}$$

where $U_{g}^{(j)}$ is a unitary gate acting on the representation space $\mathcal{A}_j$ and $I_{\mathcal{M}_j N}$ is the identity on the multiplicity space $\mathcal{M}_j N$. The working principle of the compression protocol is to get rid of the multiplicity spaces, on which the gate $U_{g}^{\otimes N}$ acts trivially. Specifically, Bob encodes his $N$-qubit input into the state of a composite system $AB$, where system $A$ has Hilbert space $\mathcal{H}_A = \bigoplus_{j=0}^{N/2} \mathcal{A}_j$ and system $B$ has Hilbert
space $\mathcal{H}_B = \mathcal{M}_{N^2}$, the multiplicity space of largest dimension. Then, he transmits system $A$ to Alice, keeping system $B$ in his laboratory. On her side, Alice encodes the gate $U_g^\otimes N$ into the gate $U_g'' = \bigoplus_{j=0}^{N^2} U_j^{(j)}$, acting only on system $A$. She applies $U_g''$ on the input provided by Bob and returns him the output. Finally, Bob applies to systems $A$ and $B$ a joint decoding operation, thus obtaining an output state of $N$ qubits. All these operations can be devised in such a way that the protocol implements the gate $U_g^\otimes N$ exactly on every $N$-qubit input state (Appendix D). The key feature of this protocol is that Alice and Bob only communicate through the exchange of system $A$, whose dimension grows like $N^2$ instead of $2^N$. With respect to the naive protocol in which Alice and Bob send to each other the input and the output of the gate $U_g^\otimes N$, the protocol allows for an exponential reduction of quantum communication from $2N$ qubits to $4 \log_2 N$ qubits. The amount of quantum communication can be further cut down by nearly a half if a small error is allowed. Indeed, before sending the input to Alice, Bob can apply the encoding operation $E_j^{(N)}$, which compresses the input into a subspace of dimension $O(J^2)$. Setting $J = \lfloor \sqrt{N+\delta} \rfloor$ for some $\delta \in (0, 1)$, this encoding will cause little disturbance, except on an exponentially small fraction of the inputs (cf. theorem [1]). As a result, a completely unknown i.i.d. sequence of $N$ qubit gates on Alice’s side can be approximately reproduced on Bob’s side through the exchange of $2(1+\delta) \log_2 N$ qubits, with arbitrarily small $\delta > 0$ (Appendix D).

**Extension to higher dimensions.** Our gate replication protocol can be easily extended to quantum systems of arbitrary dimension $d < \infty$ (qudits), showing that arbitrary i.i.d. sequences of unitary gates can be replicated at a nearly quadratic rate, with vanishing error on most inputs except for an exponentially small fraction (Appendix E). For $M = O(N^{2-\alpha})$ the number of ancillary qudits used by the protocol is equal to $M = N + O\left(N^{1-\alpha/4}\right)$. As a byproduct of universal gate super-replication, it is possible to achieve super-generation of maximally entangled states, as well as multiphase states of the form $|e_{\theta}\rangle = (|0\rangle + e^{-i\theta_1}|1\rangle + e^{-i\theta_2}|2\rangle + \cdots + e^{-i\theta_{d-1}}|d-1\rangle)/\sqrt{d}$ (Appendix F). Furthermore, $N$ uses of a completely unknown gate can be compressed with zero error into a single gate acting on $(d-1)(d/2+1) \log_2 N$ qudits, a number that can be cut down by nearly a half if one accepts an almost everywhere vanishing error (Appendix G).

In conclusion, we showed that $N$ uses of a completely unknown unitary gate allow one to simulate up to $N^2$ uses of the same gate, with high accuracy on all input states except for an exponentially small fraction. The protocol has optimal rate: any attempt to simulate $N^2$ or more uses is doomed to have vanishing fidelity on most input states. The ability to replicate unitary gates with high fidelity is not in contradiction with the no-cloning theorem, due to the impossibility to deterministically retrieve unitary gates from non-orthogonal pure states. The arguments developed for the gate replication protocol also apply the task of gate compression, useful in distributed scenarios where a server has to apply a gate to an input state provided by a client. In this scenario, an unknown i.i.d. sequence of $N$ unitary gates can be compressed down to a single gate acting only on $O(\log N)$ qudits, thus achieving an exponential reduction of quantum communication between client and server.

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Let us denote the Schur-Weyl measure by \( p_j := \frac{d_j m_j K}{d^K} \). By definition, one has

\[
p_j = \frac{d_j m_j K}{2^K} \leq \frac{(2j + 1)^2}{2^K (K/2 + j + 1)} \binom{K/2}{K/2 + j} \leq \frac{K + 1}{2^K} \binom{K}{K/2 + j},
\]

having used the expressions \( d_j = 2j + 1 \) and \( m_j K = \frac{2j + 1}{K + j + 1} \binom{K}{K/2 + j} \). Then, the probability that the angular momentum number is no larger than \( J \) is lower bounded as

\[
\sum_{j=0}^{J} p_j \geq 1 - (K + 1) \sum_{j>J} \frac{1}{2^K} \binom{K}{K/2 + j} \geq 1 - (K + 1) \exp \left[ -\frac{2J^2}{K} \right],
\]
having used Hoeffding’s inequality. As a consequence, the entanglement fidelity of the channel $\mathcal{E}_j^{(K)}$ is lower bounded as

$$F_E^{(K, J)} = \left( \sum_{j=0}^{J} p_j^{(K)} \right)^2 \geq 1 - 2(K + 1) \exp \left[ -\frac{2J^2}{K} \right]. \quad (A1)$$

### Appendix B: Minimum dimension of the ancilla in the gate replication protocol

In the gate replication protocol, the output of the compression channel $\mathcal{E}_j^{(M)}$ is encoded via an isometry $V_J: \mathcal{H}_j^{(M)} \to \mathcal{H}^\otimes N \otimes \mathcal{H}_A$ satisfying

$$V_J U_g^\otimes M = (U_g^\otimes N \otimes I_A) V_J \quad \forall U_g \in SU(2). \quad (B1)$$

In order for such isometry to exist, the dimension of the ancilla must satisfy the condition $m_{jN} d_A \geq m_{jM}$ for all $j \leq J$. Hence, its minimum value is

$$d_A^{\text{min}} = \left\lceil \min_{j \leq J} \frac{m_{jM}}{m_{jN}} \right\rceil = \left\lceil \frac{(N/2 + J + 1)(M/2 + 1)}{(M/2 + J + 1)(N/2 + J + 1)} \right\rceil.$$ 

For $J/N \ll 1$, Stirling’s approximation gives $\log_2 d_A^{\text{min}} = M - N + O(J^2/N)$. The condition $J/N \ll 1$ is always met in the super-replication regime, when $M$ scales like $N^{2-\alpha}$ for some $\alpha > 0$ and $J$ is set to $\lceil N^{1-\alpha/4} \rceil$. Accordingly, the number of ancillary qubits needed for the implementation of the gate replication protocol is equal to

$$\log_2 d_A^{\text{min}} = M - N + O \left( N^{1-\alpha/2} \right).$$

### Appendix C: Super-generation of two qubit maximally entangled states and single qubit phase states

Given $N$ uses of a generic qubit gate $U_g$, it is easy to obtain up to $N^2$ copies of the maximally entangled state $|\Phi_g\rangle = (U_g \otimes I)|\Phi^+\rangle$, $|\Phi^+\rangle = (|0\rangle|1\rangle + |1\rangle|0\rangle)/\sqrt{2}$, starting from $N$ uses of the qubit gate $U_g$. The protocol is as follows:

1. use the gate replication protocol to simulate $M$ uses of the gate $U_g$
2. apply the simulation of $U_g^\otimes M$ on the first qubit of the each pair in the entangled state $|\Phi^+\rangle^\otimes M$

The protocol produces an approximation of the target state $|\Phi_g\rangle^\otimes M$, given by

$$\rho_g^{(M)} := \left\{ [C_2 (U_g^\otimes N \otimes I_A) C_1] \otimes I_{\text{sec}} \right\} \left( |\Phi^+\rangle \langle \Phi^+ | \right)^\otimes M,$$

where $I_{\text{sec}}$ denotes the identity on the $M$-qubit system consisting of the second qubit of each pair in the entangled state $|\Phi^+\rangle^\otimes M$ and the channel $C_2 (U_g^\otimes N \otimes I_A) C_1$ acts on the $M$-qubit system consisting of the first qubit of each pair (reordering of the Hilbert spaces is assumed where necessary). The fidelity of the approximation is

$$F_{\text{ent}} [N \to M] = \langle \Phi_g |^\otimes M \rho_g^{(M)} | \Phi_g \rangle^\otimes M = \langle \Phi_g |^\otimes M \left\{ [C_2 (U_g^\otimes N \otimes I_A) C_1] \otimes I_{\text{sec}} \right\} \left( |\Phi^+\rangle \langle \Phi^+ | \right)^\otimes M | \Phi_g \rangle^\otimes M$$

$$= \langle \Phi^+ |^\otimes M (C_2 C_1 \otimes I_{\text{sec}}) \left( |\Phi^+\rangle \langle \Phi^+ | \right)^\otimes M | \Phi_g \rangle^\otimes M$$

$$= \langle \Phi^+ |^\otimes M \left( E_j^{(M)} \otimes I_{\text{sec}} \right) \left( |\Phi^+\rangle \langle \Phi^+ | \right)^\otimes M | \Phi_g \rangle^\otimes M$$

$$\equiv F_E^{(M, J)}.$$
Setting $J = N/2$ and $K = M$, Eq. (A1) then implies the bound
\[
F_{\text{gen}}^{\text{ent}}[N \to M] \geq 1 - 2(M + 1) \exp \left[ - \frac{N^2}{2M} \right],
\]
which establishes the possibility to generate up to $N^2$ copies of the state $|\Phi_g\rangle$ from $N$ uses of the gate $U_g$. In other words, maximally entangled states can be super-generated using the corresponding gates.

Super-generation of entangled states also implies super-generation of phase states. Suppose that one is given $N$ uses of a phase gate $U_{n,\theta}$, where the rotation axis $n$ is known. For definiteness, let us fix $n$ to be the $z$-axis, so that $U_{n,\theta}$ is diagonal in the computational basis $\{|0\rangle, |1\rangle\}$. Then, using the entanglement generation protocol, one can produce $M$ approximate copies of the state $|\Phi_g\rangle = (|0\rangle|0\rangle + e^{-i\theta}|1\rangle|1\rangle)/\sqrt{2}$. The latter can be transformed into $M$ approximate copies of the state $|\psi_g\rangle = (|0\rangle + e^{-i\theta}|1\rangle)/\sqrt{2}$, by applying CNOT gates and discarding the second qubit of each pair. Denoting by $\rho_g^{(M)}$ the approximation of the $M$ entangled states, the fidelity of the protocol is given by
\[
F_{\text{gen}}^{\text{phase}}[N \to M] = (e^{-\theta})^\otimes M \text{Tr}_{\text{sec}} \left( [\text{CNOT}^{\otimes M} \rho_g^{(M)} \text{CNOT}^{\otimes M}] |\psi_g\rangle^\otimes M \right) \geq (|\langle e_\theta | 0 \rangle\rangle^\otimes M \text{CNOT}^{\otimes M} \rho_g^{(M)} \text{CNOT}^{\otimes M} (|\psi_\theta\rangle|0\rangle)^\otimes M
= (|\Phi_g\rangle^\otimes M \rho_g^{(M)} |\Phi_g\rangle^\otimes M
\equiv F_{\text{gen}}^{\text{ent}}[N \to M]
\geq 1 - 2(M + 1) \exp \left[ - \frac{N^2}{2M} \right].
\]

Note that the protocol used to super-generate the phase state $|\psi_g\rangle$ is non-universal due to the presence of the CNOT gate, which is defined in the computational basis $\{|0\rangle, |1\rangle\}$.

**Appendix D: Alice’s and Bob’s operations in the gate compression protocol**

Let us consider first the exact gate compression protocol, which faithfully encodes the gate $U_g^{\otimes N}$, acting on the Hilbert space $\mathcal{H}^{\otimes N} = \bigoplus_{j=0}^{N/2} (\mathcal{A}_j \otimes \mathcal{M}_j)$, into the gate $U_g' = \bigoplus_{j=0}^{N/2} U_g^{(j)}$, acting on the Hilbert space $\mathcal{A} = \bigoplus_{j=0}^{N/2} \mathcal{A}_j$. In this case, Alice’s operations are given by
\[
A_1(\cdot) := V \cdot V^\dagger
A_2(\cdot) := V^\dagger \cdot V + \text{Tr}[(I - VV^\dagger) \cdot \rho_0],
\]
where $\rho_0$ is a fixed density matrix on $\mathcal{H}$ and $V : \mathcal{H} \to \mathcal{H}^{\otimes N}$ is the isometry $V := \bigoplus_{j=0}^{N/2} I_{\mathcal{A}_j} \otimes |\mu_j\rangle$, where $I_{\mathcal{A}_j}$ is the identity on $\mathcal{A}_j$ and $|\mu_j\rangle$ is a fixed pure state in $\mathcal{M}_j$. By construction, one has
\[
A_2 U_g^{\otimes N} A_1 = U_g' \quad \forall U_g \in \text{SU}(2),
\]
where $U_g$ and $U_g'$ are the channels associated to $U_g$ and $U_g'$, respectively. Eq. (D2) expresses the fact that Alice’s operations encode the action of the gate $U_g^{\otimes N}$ into the gate $U_g'$.

In order to show that the encoding is faithful, one has to show that Bob can simulate the gate $U_g^{\otimes N}$ using $U_g'$. To this purpose, consider the operations
\[
B_1(\cdot) := W \cdot W^\dagger
B_2(\cdot) := W^\dagger \cdot W + \text{Tr}[(I - WW^\dagger) \cdot \beta_0],
\]
where $\beta_0$ is a fixed density matrix on $\mathcal{H}^{\otimes N}$ and $W$ is the isometry
\[
W : \mathcal{H}^{\otimes N} \to \mathcal{H}_A \otimes \mathcal{H}_B, \quad W := \bigoplus_{j=0}^{N/2} (I_{\mathcal{A}_j} \otimes W_j),
\]
where $\mathcal{H}_B := \mathcal{M}_{0N}$ is the multiplicity space of largest dimension and $W_j$ is a fixed isometry from $\mathcal{M}_j$ to $\mathcal{H}_B$. With this definition, one has
\[
B_2 (U_g' \otimes I_B) B_1 = U_g^{\otimes N} \quad \forall U_g \in \text{SU}(2),
\]
expressing the fact that Bob can retrieve the action of the gate $U_g^\otimes N$ from the use of $U_g'$. According to Eqs. (D2) and (D4), the operations $A_1, A_2, B_1, B_2$ define an exact gate compression protocol.

The approximate protocol is a straightforward variation of the above: the gate $U_g^\otimes N$ is encoded into the gate $U_g' = \sum_{j=0}^J U_g^{(j)}$, with $J = \lceil \sqrt{N^{1+\delta}} \rceil$ for some $\delta \in (0, 1)$. Alice’s operations are of the same form as those in Eq. (D1), with the only difference that now the domain of the isometry $V$ is not the whole space $\mathcal{H}_A$, but rather the subspace $\mathcal{H}_{A,J} := \sum_{j=0}^J \mathcal{R}_j$. Bob’s operations are of the same form as those in Eq. (D3), with the difference that the operation $B_1$ is replaced by $B_{1,J} := B_1 \mathcal{E}_j^{(N)}$, which outputs states in $\mathcal{H}_{A,J} \otimes \mathcal{H}_B$.

Appendix E: Universal gate replication in dimension $d \geq 2$

The gate replication protocol for qudits is the immediate generalization of the protocol for qubits. The main steps in the construction of the protocol are as follows: First, we define a suitable subspace $\mathcal{H}_j^{(M)} \subseteq \mathcal{H} \otimes M$ and construct the encoding operation

$$\mathcal{E}_j^{(M)}(\cdot) = P_j^{(M)} \cdot P_j^{(M)} + \text{Tr}\left[\left(\mathbb{I} \otimes P_j^{(M)}\right)\rho_0\right], \quad (E1)$$

where $P_j^{(M)}$ is the projector on $\mathcal{H}_j^{(M)}$ and $\rho_0$ is some fixed state with support in $\mathcal{H}_j^{(M)}$. We then show that, provided that $J$ is large enough, the encoding channel induces little disturbance on most input states, with the exception of an exponentially small fraction. Furthermore, we show that the output of the encoding channel can be faithfully encoded into a composite system consisting of $N$ identical qudits and an ancilla $A$, via an isometry $V_J : \mathcal{H}_j^{(M)} \rightarrow \mathcal{H} \otimes N \otimes \mathcal{H}_A$ satisfying the condition

$$V_J U_g^\otimes M = (U_g^\otimes N \otimes I_A) V_J, \quad \forall U_g \in \text{SU}(d). \quad (E2)$$

The action of the isometry $V_J$ can be inverted by the decoding channel

$$\mathcal{D}_J(\rho) := V_J^\dagger \rho V_J + \text{Tr}\left[\left(\mathbb{I} \otimes V_J V_J^\dagger\right)\rho\right] \rho_0. \quad (E3)$$

Given these ingredients, the gate replication protocol follows the same steps as the gate replication protocol for qubits:

1. Send the $M$ input systems to the input of the encoding channel $\mathcal{E}_j^{(M)}$ with $J = \lceil N^{1-\alpha/4} \rceil$ for $M$ growing as $N^{2-\alpha}$, $\alpha > 0$ and $J = N/d$ for $M$ growing as $N^2$ or faster
2. Apply the isometry $V_J$ to the output of $\mathcal{E}_j^{(M)}$, thus encoding it into the Hilbert space $\mathcal{H} \otimes N \otimes \mathcal{H}_A$
3. Apply the gate $U_g^\otimes N$ on the $N$ systems
4. Send the $N$ systems and the ancilla to the input of the decoding channel $\mathcal{D}_J$.

In the following we show how to construct the encoding map $\mathcal{E}_j^{(M)}$ and the isometry $V_J$, proving that the above protocol can simulate $N^2$ parallel uses of the gate $U_g$ with high fidelity on most inputs.

1. Decomposition of the Hilbert space

Let $U_g$ be a generic element of the group $\text{SU}(d)$, parametrized by a suitable vector $g \in \mathbb{R}^{d^2-1}$. For a given integer $K \geq 0$, the irreducible representations in the decomposition of $U_g^\otimes K$ are labelled by Young diagrams with $K$ boxes arranged into $d$ rows. A Young diagram is completely specified by the lengths of its rows, which can be put into a vector $\lambda = (\lambda_1, \ldots, \lambda_d)$ of non-negative integers satisfying the conditions

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq 0, \quad \sum_{i=1}^d \lambda_i = K. \quad (E4)$$

We denote by $Y_{K,d}$ the set of all such vectors and, from now on, we identify Young diagrams with the corresponding vectors, referring to $\lambda$ as a “Young diagram”. Note that one has

$$Y_{K,d} \subset \mathcal{T}_{K,d}, \quad (E5)$$
where $T_{K,d}$ is the set of all partitions of $K$ into $d$ non-negative integers.

With the above notation, the Hilbert space of $K$ identical systems can be decomposed as

$$\mathcal{H}^\otimes K \simeq \bigoplus_{\lambda \in Y_{K,d}} (\mathcal{R}_\lambda \otimes \mathcal{M}_\lambda), \quad (E6)$$

where $\mathcal{R}_\lambda$ is a representation space, of dimension $d_\lambda$, and $\mathcal{M}_\lambda$ is the corresponding multiplicity space, of dimension $m_\lambda$. The dimensions and multiplicities satisfy the following bounds [30, 40]:

$$d_\lambda \leq (K + 1)^{\frac{d(d-1)}{2}}, \quad (E7)$$

and

$$\binom{K}{\lambda} (K + 1)^{\frac{d(d-1)}{2}} \leq m_\lambda \leq \binom{K}{\lambda}, \quad (E8)$$

where $\binom{K}{\lambda} := \frac{K!}{\lambda_1! \cdots \lambda_d!}$ is the multinomial coefficient.

Relative to the decomposition $(E6)$, the gate $U^\otimes K_g$ can be written in the block diagonal form

$$U^\otimes K_g \simeq \bigoplus_{\lambda \in Y_{K,d}} \left[ U^{(\lambda)}_g \otimes I_{\mathcal{M}_\lambda} \right],$$

where $U^{(\lambda)}_g$ is an irreducible representation (irrep) of $SU(d)$ and $I_{\mathcal{M}_\lambda}$ is the identity matrix on $\mathcal{M}_\lambda$.

Different Young diagrams in $Y_{K,d}$ correspond to different irreps. However, for $K \neq K'$ two Young diagrams $\lambda \in Y_{K,d}$ and $\lambda' \in Y_{K',d}$ can correspond to the same irrep, provided that one has

$$\lambda = \lambda' + \lambda_0 \mathbf{1}, \quad (E9)$$

where $\lambda_0 \in \mathbb{Z}$ is some fixed integer and $\mathbf{1} \in \mathbb{R}^d$ is the vector with all entries equal to 1.

### 2. Concentration of the Schur-Weyl measure

The Schur-Weyl measure is the probability distribution over the Young diagrams in $Y_{K,d}$ defined as

$$p_\lambda := \frac{d_\lambda m_\lambda}{d^K}.$$

For large $K$, the Schur-Weyl measure is concentrated on the Young diagrams with rows of length approximately equal to $K/d$ [30]. Specifically, let us denote by $Y^{(K)}_J$ the set of Young diagrams with the last row no shorter than $K/d - J$, i.e.

$$Y^{(K)}_J := \left\{ \lambda \in Y_{K,d} \mid \lambda_d \geq \frac{K}{d} - J \right\}, \quad (E10)$$

and by $\overline{Y}^{(K)}_J := Y_{K,d} \setminus Y^{(K)}_J$ its complement. Then, we have the following

**Lemma 1.** The Schur-Weyl measure of $\overline{Y}^{(K)}_J$ is upper bounded as

$$\text{Prob} \left[ \lambda \in \overline{Y}^{(K)}_J \right] \leq (K + 1)^{\frac{d(d-1)}{2}} \exp \left[ -\frac{2J^2}{K} \right]. \quad (E11)$$

**Proof.** Using Eqs. $(E7)$ and $(E8)$, we obtain the bound

$$\text{Prob} \left[ \lambda \in \overline{Y}^{(K)}_J \right] \leq (K + 1)^{\frac{d(d-1)}{2}} \sum_{\lambda \in \overline{Y}^{(K)}_J} q_\lambda, \quad (E12)$$
where \( q_{\lambda} := \frac{1}{d^K} \binom{K}{\lambda} \) is the multinomial distribution. In turn, the summation in the r.h.s. of Eq. (E12) can be upper bounded by extending the range from Young diagrams to general partitions of \( K \), as follows

\[
\sum_{\lambda \in \mathcal{Y}(J)} q_{\lambda} \leq \sum_{\lambda \in \mathcal{Y}(J), \lambda_d < \frac{K}{d} - J} q_{\lambda}
\]

\[
= \sum_{\lambda_d < \frac{K}{d} - J} \left( \frac{1}{d} \right)^{\lambda_d} \left( 1 - \frac{1}{d} \right)^{K - \lambda_d} \binom{K}{\lambda_d}
\]

\[
\leq \exp \left[ -\frac{2J^2}{K} \right],
\]

having used Hoeffding’s inequality. Combining the above bound with Eq. (E12) one obtains the desired result. \( \square \)

3. The encoding operation

Define the subspace \( \mathcal{H}_J^{(K)} \subseteq \mathcal{H}^{\otimes K} \) as

\[
\mathcal{H}_J^{(K)} := \bigoplus_{\lambda \in \mathcal{Y}(J)} (\mathcal{H}_\lambda \otimes \mathcal{M}_\lambda)
\]

and the encoding operation \( \mathcal{E}_J^{(K)} \) as in Eq. (E1). Denote by \( F_{E}^{(K,J)} \) be the entanglement fidelity of \( \mathcal{E}_J^{(K)} \), given by

\[
F_{E}^{(K,J)} = \langle \Phi_{dK} | \left( \mathcal{E}_J^{(K)} \otimes I^{\otimes K} \right) (|\Phi_{dK}\rangle \langle \Phi_{dK}|) |\Phi_{dK}\rangle,
\]

\(|\Phi_{dK}\rangle\) being a maximally entangled state in \( \mathbb{C}d^K \otimes \mathbb{C}d^K \). Then, we have the following

**Lemma 2.** The entanglement fidelity of channel \( \mathcal{E}_J^{(K)} \) is lower bounded as

\[
F_{E}^{(K,J)} \geq 1 - 2 \left( K + 1 \right) \frac{d(d-1)}{2} \exp \left[ -\frac{2J^2}{K} \right].
\]

**Proof.** By definition of the encoding channel, the entanglement fidelity satisfies the bound

\[
F_{E}^{(K,J)} \geq \left( \Phi_{dK} | \left( \mathcal{E}_J^{(K)} \otimes I^{\otimes K} \right) (|\Phi_{dK}\rangle \langle \Phi_{dK}|) |\Phi_{dK}\rangle \right)^2
\]

\[
= \left( \sum_{\lambda \in \mathcal{Y}(J)} \frac{d_{\lambda} m_{\lambda}}{2^K} \right)^2
\]

\[
\equiv \left\{ \text{Prob } [Y_J^{(K)}] \right\}^2
\]

\[
\geq 1 - 2 \text{Prob } [Y_J^{(K)}].
\]

Inserting Eq. (E11) in the bound one obtains the desired result. \( \square \)

Using the bound on the entanglement fidelity, it is immediate to obtain a bound on the probability that a random \( K \)-partite state \( |\Psi\rangle \) has high fidelity with the state \( \mathcal{E}_J^{(K)} (|\Psi\rangle \langle \Psi|) \). The result is a generalization of theorem 1 in the main text, which now reads

**Theorem 2.** Let \( F_\Psi^{(K,J)} \) be the fidelity between the pure state \( |\Psi\rangle \in \mathcal{H}^{\otimes K} \) and the state \( \mathcal{E}_J^{(K)} (|\Psi\rangle \langle \Psi|) \). If \( |\Psi\rangle \) is chosen uniformly at random, then one has

\[
\text{Prob } [F_\Psi^{(K,J)} < 1 - \epsilon] < 2 \left( K + 1 \right) \frac{d(d-1)}{2} \exp \left[ -\frac{2J^2}{K} \right] \frac{\epsilon}{\epsilon}
\]

for every fixed \( \epsilon > 0 \).
Proof. Recall that the average of fidelity over all pure states can be is lower bounded by the entanglement fidelity. Combining this fact with Markov’s inequality one obtains
\[
\text{Prob}\left[F_\psi^{(K,J)} < 1 - \epsilon\right] < \frac{1 - F_E^{(K,J)}}{\epsilon}.
\]
Inserting Eq. (E14) in the above bound one obtains the desired result.

The construction of the gate replication protocol uses theorem 2 with \( K = M \). The theorem will be used also in the approximate gate compression protocol, in that case by setting \( K = N \).

4. Embedding into the space of \( N \) systems

We now construct the isometry \( V_J \), which embeds the subspace \( \mathcal{H}_J^{(M)} \) into the Hilbert space of \( N \) identical copies and a suitable ancilla. Here we assume that \( M - N \) is a multiple of \( d \). Let us decompose the target Hilbert space as
\[
\mathcal{H}^{\otimes N} \otimes \mathcal{A} \simeq \bigoplus_{\lambda \in Y_{N,d}} (\mathcal{R}_\lambda \otimes \mathcal{M}_\lambda \otimes \mathcal{H}_A).
\]
By Eq. (E2), the isometry \( V_J \) must be of the form
\[
V_J := \bigoplus_{\lambda \in Y_J^{(M)}} (I_\lambda \otimes V_\lambda),
\]
where \( I_\lambda \) is a unitary isomorphism between the representation space \( \mathcal{R}_\lambda \) and the representation space \( \mathcal{R}_{\lambda'} \), with
\[
\lambda' := \lambda - \frac{M - N}{d} 1,
\]
and \( V_\lambda \) is an isometry from \( \mathcal{M}_\lambda \) to \( \mathcal{M}_{\lambda'} \otimes \mathcal{H}_A \). In order for Eq. (E17) to hold, two conditions must be met:

1. all the irreducible representations corresponding to Young diagrams in \( Y_J^{(M)} \) should be contained in the decomposition of \( U_g^{\otimes N} \)

2. the dimension of \( \mathcal{M}_{\lambda'} \otimes \mathcal{H}_A \) should be larger than the dimension of \( \mathcal{M}_\lambda \) for every \( \lambda \in Y_J^{(M)} \).

Condition 1 is equivalent to the requirement that every Young diagram \( \lambda \in Y_J^{(M)} \) be of the form
\[
\lambda = \frac{M - N}{d} 1 + \lambda' \quad \lambda' \in Y_J^{(N)}.
\]
The minimum ancilla dimension compatible with Condition 2 is then given by
\[
d_{\text{A}}^{\text{min}} = \max_{\lambda' \in Y_J^{(N)}} \left[ \frac{m(M,d)}{m(\lambda',N,d)} \left( \frac{M - N}{d} 1 + \lambda' \right) \right].
\]
Using Eq. (E19) we can estimate how many ancillary qudits are required. In the super-replication regime (i.e. when \( M \) grows as \( N^{2-\alpha} \) for some \( 0 < \alpha < 1 \)), the minimum number is asymptotically equal to \( M - N \), up to terms that are negligible compared to \( N \). Indeed, we can use the bound
\[
(M + 1)^{-d(d+1)/2} \max_{\lambda' \in Y_J^{(N)}} \left( \frac{M - N}{d} 1 + \lambda' \right) \leq d_{\text{A}}^{\text{min}} \leq (N + 1)^{-d(d+1)/2} \max_{\lambda' \in Y_J^{(N)}} \left( \frac{M - N}{d} 1 + \lambda' \right),
\]
following from Eq. (E8). Setting \( J = \lceil N^{1-\alpha/4} \rceil \), we can use Stirling’s approximation, which for \( J/K \ll 1 \), \( K = M, N \), yields
\[
\log_d \left( \frac{K}{\lambda} \right) = K - O(J^2/K), \quad \forall \lambda \in Y_J^{(K)}.
\]
Inserting the above approximation in the bounds of Eq. (E20), we finally obtain the equality \( \log_d d_{\text{A}}^{\text{min}} = M - N + O(J^2/N) \). which, recalling that \( J \) was set to \( \lceil N^{1-\alpha/4} \rceil \), yields
\[
\log_d d_{\text{A}}^{\text{min}} = M - N + O\left( N^{1-\alpha/2} \right).
\]
The above expression quantifies the number of ancillary qudits needed to achieve super-replication.
5. The fidelity of gate super-replication

We are now ready to evaluate the fidelity of the universal gate replication network. Let us set $C_1 := V_J E_J^{(M)}$ and $C_2 := D_J$, where $E_J^{(M)}$ is the encoding channel of Eq. (E1) and $D_J$ is the decoding map of Eq. (E3), which inverts the isometric channel $V_J$. Applying the gate replication protocol to a generic pure state $|\Psi\rangle \in D^{\otimes M}$, one obtains the output state

$$C_2 (U_0^{\otimes N} \otimes I_A) C_1 (|\Psi\rangle \langle \Psi|) = U_0^{\otimes M} C_2 C_1 (|\Psi\rangle \langle \Psi|) = U_0^{\otimes M} E_J^{(M)} (|\Psi\rangle \langle \Psi|)$$

for every gate $U_0 \in SU(d)$. Like in the qubit case, the fidelity between the output state and the ideal target $U_0^{\otimes M} |\Psi\rangle$ is equal to the fidelity between $E_J^{(M)} (|\Psi\rangle \langle \Psi|)$ and $|\Psi\rangle$, equal to $F^{(M,J)}_\Psi$. For $M = O(N^{2-\alpha})$, $\alpha > 0$, the choice $J = \lceil N^{1-\alpha/4} \rceil$, guarantees that the fidelity is arbitrarily close to 1 on all states except a low probability subset: for every fixed $\epsilon > 0$, one has

$$\text{Prob} \left[ F^{(M,J)}_{\Psi} < 1 - \epsilon \right] < 2 (M + 1) \frac{2(d+1)}{d^2} \exp \left[ -\frac{2\sqrt{N\alpha}}{\epsilon} \right],$$

having used Eq. (E22).

### Appendix F: Super-generation of maximally entangled states and multiphase states

Like in the qubit case, gate replication can be used to generate up to $N^2$ copies of a generic maximally entangled state starting from $N$ uses of the corresponding gate. Setting $J = N/d$ in the gate replication protocol and following the same steps that led to the derivation of Eq. (C1) we obtain

$$F_{\text{gen}}^{\text{ent}} [N \rightarrow M] = F_{\text{E}}^{(M,N/d)}$$

$$\geq 1 - 2 (M + 1) \frac{2(d+1)}{d^2} \exp \left[ -\frac{2N^2}{d^2 M} \right].$$

In addition, the protocol can be easily adapted in order to achieve super-generation of a generic multiphase state

$$|e_\theta\rangle = \frac{1}{\sqrt{d}} \left[ |1\rangle + e^{-i\theta_1} |2\rangle + \cdots + e^{-i\theta_{d-1}} |d-1\rangle \right] \quad \theta \in [0, 2\pi)^{<d-1>},$$

starting from $N$ uses of the multiphase gate $U_\theta = |0\rangle \langle 0| + e^{-i\theta_1} |1\rangle \langle 1| + e^{-i\theta_2} |2\rangle \langle 2| + \cdots + e^{-i\theta_{d-1}} |d-1\rangle \langle d-1|$. Indeed, it is enough to

1. generate $M$ copies of the entangled state $|\Phi_\theta\rangle = |0\rangle \langle 0| + e^{-i\theta_1} |1\rangle \langle 1| + e^{-i\theta_2} |2\rangle \langle 2| + \cdots + e^{-i\theta_{d-1}} |d-1\rangle \langle d-1|)/\sqrt{d}$,
2. apply the inverse of the control-shift gate $C - \text{SHIFT}$ on each pair of qubits, where $C - \text{SHIFT} := \sum_{n=0}^{d-1} |n\rangle \langle n| \otimes S^n$, $S$ being the cyclic shift on the computational basis, and
3. discard the second qubit of each pair.

The same steps that led to Eq. (C2) show that the fidelity of the above protocol is lower bounded as

$$F_{\text{gen}}^{\text{multiphase}} [N \rightarrow M] \geq 1 - 2 (M + 1) \frac{2(d+1)}{d^2} \exp \left[ -\frac{2N^2}{d^2 M} \right],$$

thus guaranteeing super-generation of multiphase states.

### Appendix G: Universal gate compression in dimension $d \geq 2$

The gate compression protocols for qudits are of the same form of the protocols for qubits, with the only difference that the angular momentum number $j$ is replaced by the vector $\lambda$ that parametrizes the Young diagrams. Here we quantify the compression rates achieved for qudits.
Let us start from zero-error compression. The protocol encodes the gate

\[ U_g^{\otimes N} \simeq \bigoplus_{\lambda \in \mathcal{Y}_{N,d}} \left[ U_g^{(\lambda)} \otimes I_{\mathcal{H}_\lambda} \right], \]

into the gate \( U'_g := \bigoplus_{\lambda \in \mathcal{Y}_{N,d}} U_g^{(\lambda)} \), acting on the smaller Hilbert space \( \mathcal{H}_A = \bigoplus_{\lambda \in \mathcal{Y}_{N,d}} \mathcal{R}_\lambda \). The dimension of \( \mathcal{H}_A \) can be upper bounded as

\[
d_A = \sum_{\lambda \in \mathcal{Y}_{N,d}} d_\lambda < \left( \max_{\lambda \in \mathcal{Y}_{N,d}} d_\lambda \right) |\mathcal{Y}_{N,d}| \leq (N + 1) \frac{d(d-1)}{2} |\mathcal{Y}_{N,d}|, \tag{G1}\]

having used Eq. (E7) in the last inequality. Since the total number of Young diagrams is smaller than the number of partitions in \( T_{N,d} \), one has

\[
|\mathcal{Y}_{N,d}| < \binom{N + d - 1}{d - 1} < (N + 1)^{d-1},
\]

which, inserted in Eq. (G1), yields

\[
d_A < (N + 1)^{(d-1)(d/2+1)}. \tag{G2}\]

Hence, \( N \) uses of a generic qudit gate can be compressed without errors into a single gate acting on \((d-1)(d/2+1)\log_2 N\) qubits in the large \( N \) limit.

Like in the qubit case, the approximate protocol is a straightforward variation of the zero-error protocol. In the approximate version, the gate \( U_g^{\otimes N} \) is encoded into the gate \( U'_g = \bigoplus_{\lambda \in \mathcal{Y}'(N)} U_g^{(\lambda)} \), with \( J = \lfloor \sqrt{N^{1+\delta}} \rfloor \) for some \( \delta \in (0,1) \). Now, all the elements of \( \mathcal{Y}'^{(N)} \) are of the form

\[
\lambda = \frac{N - N'}{d} + \lambda' \quad N' := dJ, \quad \lambda' \in \mathcal{Y}_{N',d}. \]

Replacing \( N \) with \( N' \) in the steps leading to Eq. (G2) we obtain \( d_A < \left[ d(\sqrt{N^{1+\delta}} + 1) \right]^{(d-1)(d/2+1)} \), meaning that the gate \( U_g^{\otimes N} \) can be compressed to a gate acting on \((d-1)(d/2+1)(1+\delta)/2 \log_2 N\) qubits in the asymptotic limit.