A note on Gaussian maximal functions

Jonas Teuwen

Delft Institute of Applied Mathematics, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, The Netherlands

Abstract

This note presents a proof that the non-tangential maximal function of the Ornstein-Uhlenbeck semigroup is bounded pointwise by the Gaussian Hardy-Littlewood maximal function. In particular this entails an extension on a result by Pineda and Urbina [1] who proved a similar result for a ‘truncated’ version with fixed parameters of the non-tangential maximal function. We actually obtain boundedness of the maximal function on non-tangential cones of arbitrary aperture.

Keywords: Ornstein-Uhlenbeck semigroup, Mehler kernel, Gaussian maximal function, admissible cones

1. Introduction

Maximal functions are among the most studied objects in harmonic analysis. It is well-known that the classical non-tangential maximal function associated with the heat semigroup is bounded pointwise by the Hardy-Littlewood maximal function, for every \( x \in \mathbb{R}^d \), i.e.,

\[
\sup_{(y,t) \in \mathbb{R}^d, |x-y| < t} |e^{t \Delta} u(y)| \lesssim \sup_{r > 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |u| \, d\lambda, \tag{1}
\]

for all locally integrable functions \( u \) on \( \mathbb{R}^d \) where \( \lambda \) is the Lebesgue measure on \( \mathbb{R}^d \) (cf. [2, Proposition II 2.1.]). Here the action of heat semigroup \( e^{t \Delta} u = \rho_t \ast u \) is given by a convolution of \( u \) with the heat kernel

\[
\rho_t(\xi) := \frac{e^{-|\xi|^2/4t}}{(4\pi t)^d/2}, \text{ with } t > 0 \text{ and } \xi \in \mathbb{R}^d.
\]
In this note we are interested in its Gaussian counterpart. The change from Lebesgue measure to the *Gaussian measure*

\[ d\gamma(x) := \pi^{-\frac{d}{2}} e^{-|x|^2} \, d\lambda(x) \]

introduces quite some intricate technical and conceptual difficulties which are due to its non-doubling nature. Instead of the Laplacian, we will use its Gaussian analogue, the *Ornstein-Uhlenbeck operator* \( L \) which is given by

\[ L := \frac{1}{2}\Delta - \langle x, \nabla \rangle = -\frac{1}{2} \nabla^* \nabla, \]

where \( \nabla^* \) denotes the adjoint of \( \nabla \) with respect to the measure \( d\gamma \). Our main result, to be proved in Theorem 1, is the following Gaussian analogue of (1):

\[ \sup_{(y,t) \in \Gamma_{x}(A,a)} |e^{t^2 L} u(y)| \lesssim \sup_{r>0} \frac{1}{\gamma(B_r(x))} \int_{B_r(x)} |u| \, d\gamma. \]

Here,

\[ \Gamma_{x}(A,a) := \Gamma_{x}(A,a)(\gamma) := \{(y,t) \in \mathbb{R}^{d+1} : |x - y| < At \text{ and } t \leq am(x)\} \]

is the *Gaussian cone* with aperture \( A \) and cut-off parameter \( a \), and

\[ m(x) := \min\left\{ 1, \frac{1}{|x|} \right\}. \]

As shown in [3, Theorem 2.19] the centered Gaussian Hardy-Littlewood maximal function is of weak-type \((1, 1)\) and is \( L^p(\gamma)\)-bounded for \( 1 < p \leq \infty \). In fact, the same result holds when the Gaussian measure \( \gamma \) is replaced by any Radon measure \( \mu \). Furthermore, if \( \mu \) is doubling, then these results even hold for the *uncentered* Hardy-Littlewood maximal function. For the Gaussian measure \( \gamma \) the uncentered weak-type \((1, 1)\) result is known to fail for \( d > 1 \) [4]. Nevertheless, the uncentered Hardy-Littlewood maximal function for \( \gamma \) is \( L^p\)-bounded for \( 1 < p \leq \infty \) [5].

A slightly weaker version of the inequality (4) has been proved by Pineda and Urbina [1] who showed that

\[ \sup_{(y,t) \in \Gamma_{x}} |e^{t^2 L} u(y)| \lesssim \sup_{r>0} \frac{1}{\gamma(B_r(x))} \int_{B_r(x)} |u| \, d\gamma, \]
where
\[ \tilde{\Gamma}_x = \{ (y, t) \in \mathbb{R}_+^d : |x - y| < t \leq \tilde{m}(x) \} \]
is the ‘reduced’ Gaussian cone corresponding to the function
\[ \tilde{m}(x) = \min \left\{ \frac{1}{2}, \frac{1}{|x|} \right\}. \]

Our proof of (4) is shorter than the one presented in [1]. It has the further advantage of allowing the extension to cones with arbitrary aperture \( A > 0 \) and cut-off parameter \( a > 0 \) without any additional technicalities. This additional generality is important and has already been used by Portal (cf. the claim made in [6, discussion preceding Lemma 2.3]) to prove the \( H^1 \)-boundedness of the Riesz transform associated with \( L \).

2. The Mehler kernel

The Mehler kernel (see e.g., [7]) is the Schwartz kernel associated to the Ornstein-Uhlenbeck semigroup \((e^{tL})_{t \geq 0}\), that is,
\[ e^{tL}u(x) = \int_{\mathbb{R}^d} M_t(x, \cdot)u \, d\gamma. \tag{7} \]

There is an abundance of literature on the Mehler kernel and its properties. We shall only use the fact, proved e.g. in the survey paper [7], that it is given explicitly by
\[ M_t(x, y) = \frac{\exp \left( -\frac{|e^{-t}x - y|^2}{1 - e^{-2t}} \right) e^{y^2}}{(1 - e^{-2t})^\frac{d}{2}}. \tag{8} \]
Note that the symmetry of the semigroup \( e^{tL} \) allows us to conclude that \( M_t(x, y) \) is symmetric in \( x \) and \( y \) as well. A formula for (8) honoring this observation is:
\[ M_t(x, y) = \frac{\exp \left( -\frac{e^{-2t}|x - y|^2}{1 - e^{-2t}} \right) \exp \left( 2e^{-t}\langle x, y \rangle \right)}{(1 - e^{-t})^\frac{d}{2}} \frac{(1 + e^{-t})^\frac{d}{2}}{(1 - e^{-t})^\frac{d}{2}}. \tag{9} \]
3. Some lemmata

We use $m$ as defined in (6) in our next lemma, which is taken from [8, Lemma 2.3].

1 Lemma. Let $a, A$ be strictly positive real numbers and $t > 0$. We have for $x, y \in \mathbb{R}^d$ that:

1. If $|x - y| < At$ and $t \leq am(x)$, then $t \leq a(1 + aA)m(y)$,
2. If $|x - y| < Am(x)$, then $m(x) \leq (1 + A)m(y)$ and $m(y) \leq 2(1 + A)m(x)$.

The next lemma, taken from [9, Proposition 2.1(i)], will come useful when we want to cancel exponential growth in one variable with exponential decay in the other as long both variables are in a Gaussian cone. For the reader’s convenience, we include a short proof.

2 Lemma. Let $\alpha > 0$ and $|x - y| \leq \alpha m(x)$. Then:

$$e^{-\alpha^2 - 2\alpha|y|^2} \leq e^{\alpha^2 (1 + \alpha)^2 + 2\alpha (1 + \alpha)} e^{2|y|^2}.$$  \hfill (10)

Proof. By the triangle inequality and $m(x)|x| \leq 1$ we get,

$$|y|^2 \leq (\alpha m(x) + |x|)^2 \leq \alpha^2 + 2\alpha + |x|^2.$$  \hfill (11)

This gives the first inequality. For the second we use Lemma 1 to infer $m(x) \leq (1 + \alpha)m(y)$. Proceeding as before we obtain

$$|x|^2 \leq \alpha^2 (1 + \alpha)^2 + 2\alpha (1 + \alpha) + |y|^2,$$

which finishes the proof. \hfill \blacksquare

3.1. An estimate on Gaussian balls

Let $B := B_t(x)$ be the open Euclidean ball with radius $t$ and center $x$ and let $\gamma$ be the Gaussian measure as defined by (2). We shall denote by $S_d$ the surface area of the unit sphere in $\mathbb{R}^d$.

3 Lemma. For all $x \in \mathbb{R}^d$ and $t > 0$ we have the inequality:

$$\gamma(B_t(x)) \leq \frac{S_d \ t^d}{\pi^\frac{d}{2}} e^{2|y|} e^{-|x|^2}. \hfill (10)$$
Proof. Remark that, with $B := B_t(x)$,
\[
\int_B e^{-|\xi|^2} \, d\xi = e^{-|x|^2} \int_B e^{-|\xi-x|^2} \, e^{-2\langle x, \xi-x \rangle} \, d\xi \\
\leq e^{-|x|^2} \int_B e^{-|\xi-x|^2} \, e^{2|x||\xi-x|} \, d\xi \\
\leq e^{-|x|^2} e^{2|x|t} \int_B e^{-|\xi-x|^2} \, d\xi \\
= \pi^d e^{2|x|t} e^{-|x|^2} \gamma(B_t(0)).
\]
So, there holds that
\[
\gamma(B_t(x)) \leq e^{2|x|t} e^{-|x|^2} \gamma(B_t(0)). \tag{11}
\]
We proceed by noting that
\[
\gamma(B_t(0)) \leq \pi^{-\frac{d}{2}} |B_t(0)| \leq \pi^{-\frac{d}{2}} t^{d} S_d, 
\]
and combine this with the previous calculation to obtain
\[
\gamma(B_t(x)) \leq \frac{S_d t^{d}}{\pi^{\frac{d}{2}}} e^{2|x|t} e^{-|x|^2}.
\]
This completes the proof. \hfill \blacksquare

3.2. Off-diagonal kernel estimates on annuli

As is common in harmonic analysis, we often wish to decompose $\mathbb{R}^d$ into sets on which certain phenomena are easier to handle. Here we will decompose the space into disjoint annuli.

Throughout this subsection we fix $x \in \mathbb{R}^d$, constants $A, a \geq 1$, and a pair $(y, t) \in \Gamma_x^{A,a}$. We use the notation $rB$ to mean the ball obtained from the ball $B$ by multiplying its radius by $r$.

The annuli $C_k := C_k(B_t(y))$ are given by:
\[
C_k := \begin{cases} 
2B_t(y), & k = 0, \\
2^{k+1}B_t(y) \setminus 2^kB_t(y), & k \geq 1.
\end{cases} \tag{12}
\]

So, whenever $\xi$ is in $C_k$, we get for $k \geq 1$ that
\[
2^k t \leq |y - \xi| < 2^{k+1} t. \tag{13}
\]
On $C_k$ we have the following bound for $M_{\ell^2}(y, \cdot)$:
Lemma. For all $\xi \in C_k$ for $k \geq 1$ we have:

$$M_{t^2}(y, \xi) \leq \frac{e^{\|y\|^2}}{(1 - e^{-2t^2})^2} \exp\left(2^{k+1}t|y|\right) \exp\left(-\frac{4k}{2e^{2t^2}}\right), \quad (14)$$

Proof. Considering the first exponential which occurs in the Mehler kernel \((9)\) together with \((13)\) gives for $k \geq 1$:

$$\exp\left(-e^{-2t^2} \frac{|y - \xi|^2}{1 - e^{-2t^2}}\right) \leq \exp\left(-\frac{4k}{e^{2t^2} 1 - e^{-2t^2}}\right) \leq \exp\left(-\frac{4k}{2e^{2t^2}}\right), \quad (\dagger)$$

where $(\dagger)$ follows from $1 - e^{-s} \leq s$ for $s \geq 0$. Using the estimate $1 + s \geq 2s$ for $0 \leq s \leq 1$, we find for the second exponential in the Mehler kernel \((9)\), by \((13)\) that

$$\exp\left(2e^{-t^2} \frac{\langle y, \xi \rangle}{1 + e^{-t^2}}\right) \leq \exp(|\langle y, \xi \rangle|) \leq \exp(|\langle y, \xi - y \rangle|) e^{\|y\|^2} \leq \exp\left(2^{k+1}t|y|\right) e^{\|y\|^2}.$$

Combining these estimates we obtain \((14)\), as required. \(\blacksquare\)

The main result

In this section we will prove our main theorem as mentioned in \((4)\) for which the necessary preparations have already been made.

Theorem. Let $A, a > 0$. For all $x \in \mathbb{R}^d$ and all $u \in C_c(\mathbb{R}^d)$ we have

$$\sup_{(y,t) \in \Gamma_x^{(A,a)}} |e^{t^2L}u(y)| \leq \sup_{r>0} \frac{1}{\gamma(B_r(x))} \int_{B_r(x)} |u| \, d\gamma, \quad (15)$$

where the implicit constant only depends on $A, a$ and $d$.

Proof. We fix $x \in \mathbb{R}^d$ and $(y,t) \in \Gamma_x^{(A,a)}$. The proof of \((15)\) is based on splitting the integration domain into the annuli $C_k$ as defined by \((12)\) and estimating on each annulus. More explicit,

$$|e^{t^2L}u(y)| \leq \sum_{k=0}^{\infty} I_k(y), \text{ where } I_k(y) := \int_{C_k} M_{t^2}(y, \cdot)|u(\cdot)| \, d\gamma. \quad (16)$$
We have $t \leq am(x) \leq a$ and, by Lemma 1, $|y| \leq a(1 + aA)$. Together with Lemma 4 we infer, for $\xi \in C_k$ and $k \geq 1$, that

$$M_t^2(y, \xi) \leq \frac{e^{\frac{|y|^2}{2}}}{(1 - e^{-2t^2})^\frac{k}{2}} \exp(2k+1a(1 + aA)) \exp\left(-\frac{4k}{2e^{2a^2}}\right)$$

Combining this with Lemma 2 we obtain

$$M_t^2(y, \xi) \lesssim_{A,a} \frac{e^{\frac{|x|^2}{2}}}{(1 - e^{-2t^2})^\frac{k}{2}} C_k. \tag{17}$$

Also, by (13) we get

$$|x - \xi| \leq |x - y| + |\xi - y| \leq (2k+1 + A)t.$$ 

Let $K$ be the smallest integer such that $2k+1 \geq A$ whenever $k \geq K$. Then it follows that $C_k$ for $k \geq K$ is contained in $B_{2k+2t}(x)$ and for $k < K$ is contained in $B_{2At}(x)$. We set

$$D_k := D_k(x) = \begin{cases} B_{2k+2t}(x) & \text{if } k \geq K, \\ B_{2At}(x) & \text{elsewhere.} \end{cases}$$

Let us denote the supremum on right-hand side of (13) by $M_{\gamma}u(x)$. Using (17), we can bound the integral on the right-hand side of (16) by

$$\int_{C_k} M_t^2(y, \cdot)|u(\cdot)| \, d\gamma \lesssim_{A,a} c_k \frac{e^{\frac{|x|^2}{2}}}{(1 - e^{-2t^2})^\frac{k}{2}} \int_{C_k} |u| \, d\gamma$$

$$\leq c_k \frac{e^{\frac{|x|^2}{2}}}{(1 - e^{-2t^2})^\frac{k}{2}} \int_{D_k} |u| \, d\gamma$$

$$\leq c_k \frac{e^{\frac{|x|^2}{2}}}{(1 - e^{-2t^2})^\frac{k}{2}} \gamma(D_k) M_{\gamma}u(x),$$

where we pause for a moment to compute a suitable bound for $\gamma(D_k)$. As above we have both $t|x| \leq am(x)|x| \leq a$ and $t \leq a$. Together with Lemma 3 applied to $D_k$ for $k \geq K$ we obtain:

$$\gamma(D_k) e^{\frac{|x|^2}{2}} \lesssim_{A} C_{d}^{\frac{S_d}{d}} t^{kd} e^{2k+3t|x|} e^{-|x|^2} e^{\frac{|x|^2}{2}}$$

$$\lesssim_{A,a,d} t^{kd} e^{2k+3a}.$$
Similarly, for $k < K$:
\[
\gamma(D_k)e^{|x|^2} \lesssim_{A,a,d} t^d e^{2Aa}.
\]

Using the bound $t \leq a$, we can infer that
\[
\frac{t^d}{(1 - e^{-2t^2})^{\frac{d}{2}}} \leq \frac{a^d}{(1 - e^{-2a^2})^{\frac{d}{2}}} \lesssim_{a,d} 1.
\]
(note that $s/(1 - e^{-s})$ is increasing). Combining these computations with the ones above for $k \geq K$ we get
\[
\int_{C_k} M_{t^2}(y, \cdot)|u(\cdot)| d\gamma \lesssim_{A,a,d} c_k 2^{kd} e^{2k^2 + 2A} M_{\gamma} u(x),
\]
while for $k < K$ we get
\[
\int_{C_k} M_{t^2}(y, \cdot)|u(\cdot)| d\gamma \lesssim_{A,a,d} c_k M_{\gamma} u(x).
\]

Similarly, for $\xi \in 2B_t(x)$ we obtain:
\[
I_0 := \int_{2B_t} M_{t^2}(y, \cdot)|u(\cdot)| d\gamma \lesssim_{A,a,d} M_{\gamma} u(x).
\]

Inserting the dependency of $c_k$ upon $k$ as coming from (17), we obtain the bound:
\[
|e^{t^2}u(y)| = I_0 + \sum_{k=1}^{K-1} I_k + \sum_{k=K}^{\infty} I_k \lesssim_{A,a,d} \left[ 1 + \sum_{k=1}^{K-1} c_k + \sum_{k=K}^{\infty} c_k 2^{kd} e^{2k^2 + 2a} \right] M_{\gamma} u(x),
\]
\[
\lesssim_{A,a,d} \left[ 1 + \sum_{k=1}^{K-1} e^{-\frac{k^2}{2c^2 a^2}} + \sum_{k=K}^{\infty} 2^{kd} e^{2k+1(1+2a+aA)} e^{-\frac{4k}{2c^2 a^2}} \right] M_{\gamma} u(x),
\]
valid for all $(y,t) \in \Gamma_{x}^{(A,a)}$. As the sum on the right-hand side evidently converges, we see that taking the supremum proves (15).

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