Hopf cyclic cohomology and Hodge theory for proper actions on complex manifolds

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Abstract We introduce two Hopf algebroids associated to a proper and holomorphic Lie group action on a complex manifold. We prove that the cyclic cohomology of each Hopf algebroid is equal to the Dolbeault cohomology of invariant differential forms. When the action is cocompact, we develop a generalized complex Hodge theory for the Dolbeault cohomology of invariant differential forms. We prove that every cyclic cohomology class of these two Hopf algebroids can be represented by a generalized harmonic form. This implies that the space of cyclic cohomology of each Hopf algebroid is finite dimensional. As an application of the techniques developed in this paper, we generalize the Serre duality and prove a Kodaira type vanishing theorem.

Keywords Cyclic cohomology, complex Hodge theory, proper action, vanishing theorem

MSC 53B35, 58B34

1 Introduction

Hopf algebroid was introduced by Lu [11] in generalizing the notion of Hopf algebra. Connes and Moscovici [3] applied this concept to generalize that of symmetry of ‘noncommutative spaces’. They developed a beautiful theory of cyclic cohomology for a Hopf algebroid, and used it to study the transverse index theory.

If $\Gamma$ is a discrete group acting on a smooth manifold $M$, Kaminker and Tang [7] showed that the graded commutative algebra of differential forms on the action groupoid $M \rtimes \Gamma$ is a topological Hopf algebroid with the coalgebra and antipode structures defined by taking the dual of the groupoid structure.

In the case of a Lie group $G$ action, instead of considering the algebra of differential forms on the groupoid $M \rtimes G$, Tang et al. [16] considered the
algebra $\mathcal{H}(G, M)$ of differential forms valued functions on $G$ and defined a Hopf algebroid structure on this algebra. When the $G$-action is proper, they proved that the cyclic cohomology of this Hopf algebroid is equal to the de Rham cohomology of invariant differential forms. When the $G$-action is cocompact, they developed a generalized Hodge theory for the de Rham cohomology of invariant differential forms and proved that every cyclic cohomology class of this Hopf algebroid is represented by a generalized harmonic form.

Let $G$ be a Lie group acting on a complex manifold $M$. We assume that the $G$-action is proper and holomorphic. Inspired by Tang et al. [16], we introduce two Hopf algebroids $\mathcal{H}(G, M; \overline{\partial})$ and $\mathcal{H}(G, \Omega^0(M); \overline{\partial})$. Similar to [16], using the result of Crainic [4], we are able to compute the cyclic cohomology of each Hopf algebroid, which is equal to the Dolbeault cohomology of $G$-invariant differential forms on $M$.

The main result of this paper is to prove a ‘complex Hodge theorem’ for $G$-invariant $(p, q)$-forms on $M$ when the $G$-action is cocompact. Our approach to this generalized complex Hodge theory is to study a generalized $\overline{\partial}$-Laplacian on the space of $G$-invariant $(p, q)$-forms on $M$ inspired by Tang et al. [16]. With some elliptic estimates, we are able to prove that this operator has essentially the same properties as the standard $\overline{\partial}$-Laplacian on a compact complex manifold. This allows us to prove that every cyclic cohomology class of $\mathcal{H}(G, M; \overline{\partial})$ and $\mathcal{H}(G, \Omega^0(M); \overline{\partial})$ can be uniquely represented by a harmonic form of our generalized $\overline{\partial}$-Laplacian, which implies that the cyclic cohomology of each Hopf algebroid is finite dimensional.

**Theorem 1** Let $G$ be a Lie group acting on a complex manifold $M$, and assume that the $G$-action is holomorphic, proper and cocompact. Then the cyclic cohomology groups of $\mathcal{H}(G, M; \overline{\partial})$ and $\mathcal{H}(G, \Omega^0(M); \overline{\partial})$ are of finite dimension.

Theorem 1 allows us to generalize the classical Serre duality and Kodaira vanishing theorem as follows.

Let $M$ be an $n$-dimensional complex manifold with a holomorphic, proper, and cocompact $G$-action. Let $L$ be a holomorphic line bundle, and let $E$ be a holomorphic vector bundle on $M$. Assume that $L$ admits a holomorphic $G$-action lifted from the action of $G$ on $M$ as well as $E$. Let $H$ be a $G$-invariant Hermitian metric on $M$, and let $H^L$ be a $G$-invariant Hermitian metric on $L$.

1. When the action group $G$ is unimodular, the Serre duality theorem holds for the Dolbeault cohomology groups of $G$-invariant differential forms.
2. (a) When $G$ is unimodular, if $K^* \otimes L$ is semi-positive on $M$ and is positive at a point $x_0 \in M$, then
   \[ H^{0,q}_{\overline{\partial}}(M, L, G) = 0, \quad q \neq 0, \]
   where $K = \Lambda^{0,0}(M)$ denotes the canonical line bundle on $M$.
3. If $L$ is positive, then for $m$ sufficiently large, we have
   \[ H^{0,q}_{\overline{\partial}}(M, L^m \otimes E, G) = 0, \quad q \neq 0. \]
This paper is organized as follows. In Section 2, we introduce two Hopf
algebroids $\mathcal{H}(G, M; \overline{J})$ and $\mathcal{H}(G, \Omega^0\bullet(M); \overline{J})$, and compute their Hopf cyclic
cohomology groups. In Section 3, we study a generalized $\overline{J}$-Laplacian, and prove
that every Hopf cyclic cohomology class of $\mathcal{H}(G, M; \overline{J})$ and $\mathcal{H}(G, \Omega^0\bullet(M); \overline{J})$
can be uniquely represented by a generalized harmonic form. In Section 4, we
prove the Serre duality and a Kodaira type vanishing theorem.

2 Hopf algebroids and cyclic cohomology

2.1 Hopf algebroids

In this paper, we will only work in the category of topological algebras, and
by tensor product $\otimes$ we always mean topological tensor product. We refer
the interested readers to [9] and [10] for the beautiful systematic study of the
general theory of Hopf algebroids.

Let $A$ and $B$ be unital topological algebras. A topological bialgebroid
structure on $A$, over $B$, consists of the following data.

1. A continuous algebra homomorphism $\alpha: B \to A$ called the source map
and a continuous algebra anti-homomorphism $\beta: B \to A$ called the target map,
satisfying
$$\alpha(a)\beta(b) = \beta(b)a(a), \quad \forall a, b \in B.$$

Let $A \otimes_B A$ be the quotient of $A \otimes A$ by the right $A \otimes A$ ideal generated by
$\beta(a) \otimes 1 - 1 \otimes \alpha(a)$ for all $a \in B$.

2. A continuous $B$-$B$ bimodule map $\Delta: A \to A \otimes_B A$, called the coproduct,
satisfying

(a) $\Delta(1) = 1 \otimes 1$;
(b) $(\Delta \otimes B \text{Id})\Delta = (\text{Id} \otimes B \Delta)\Delta: A \to A \otimes_B A \otimes_B A$;
(c) $\Delta(a)(\beta(b) \otimes 1 - 1 \otimes \alpha(b)) = 0$ for $a \in A, b \in B$;
(d) $\Delta(a_1a_2) = \Delta(a_1)\Delta(a_2)$ for $a_1, a_2 \in A$.

3. A continuous $B$-$B$ bimodule map $\epsilon: A \to B$, called the counit, satisfying

(a) $\epsilon(1) = 1$;
(b) $\ker \epsilon$ is a left $A$ ideal;
(c) $(\epsilon \otimes_B \text{Id})\Delta = (\text{Id} \otimes_B \epsilon)\Delta = \text{Id}: A \to A$;
(d) $\epsilon(\alpha(b)\beta(b')a) = \beta(a)b'\epsilon(a)$ and $\epsilon(aa') = \epsilon(a)\epsilon(a') = \epsilon(a)\beta(a')\epsilon(a')$ for any
$a, a' \in A, b, b' \in B$.

A topological Hopf algebroid is a topological bialgebroid $A$, over $B$, which
admits a continuous algebra anti-isomorphism $S: A \to A$ satisfying

1. $S \circ \beta = \alpha$;
2. $m_A(S \otimes \text{Id})\Delta = \beta S: A \to A$, with $m_A: A \otimes A \to A$ the multiplication
on $A$;
3. there is a linear map $\gamma: A \otimes_B A \to A \otimes A$ such that
(a) if \( \pi : A \otimes A \to A \otimes_B A \) is the natural projection, then \( \pi \gamma = \text{Id} : A \otimes_B A \to A \otimes B A \);

(b) \( m_A(\text{Id} \otimes S) \gamma \Delta = \alpha : A \to A \).

A topological \textit{para-Hopf} algebroid is a topological bialgebroid \( A \), over \( B \), which admits a continuous algebra anti-isomorphism \( S : A \to A \) such that

1. \( S^2 = \text{Id} \) and \( S\beta = \alpha \);
2. \( m_A(S \otimes \text{Id}) \Delta = \beta \epsilon S : A \to A \);
3. \( S(a^{(1)})a^{(2)} \otimes_B S(a^{(1)}) = 1 \otimes_B S(a) \).

In the above formula, we have used Sweedler’s notation for the coproduct \( \Delta(a) = a^{(1)} \otimes_B a^{(2)} \).

In this paper, we will only deal with para-Hopf algebroids. As pointed out in [9, Sect. 2.6.13], any para-Hopf algebroid defined above is a Hopf algebroid. Therefore, for simplicity, we will abbreviate ‘para-Hopf algebroid’ to ‘Hopf algebroid’ in the sequel.

We note that in the above definition, one may allow \( A \) and \( B \) to be differential graded algebras and require all of the above maps to be compatible with the differentials and to be of degree 0. Thus, one would have a differential graded Hopf algebroid (cf. [5]).

2.2 Cyclic cohomology

In this part, we take the simplest approach for the definition of cyclic cohomology of a Hopf algebroid, cf. [8].

We now recall the cyclic module \( A^\natural \) for \( (A, B, \alpha, \beta, \Delta, \epsilon, S) \) introduced by Connes-Moscovici [2].

Define
\[
C^0 = B, \quad C^n = A \otimes_B A \otimes_B \cdots \otimes_B A, \quad n \geq 1.
\]

Faces and degeneracy operators are defined as follows:
\[
\delta_i(a^1 \otimes_B \cdots \otimes_B a^{n-1}) = 1 \otimes_B a^1 \otimes_B \cdots \otimes_B a^{n-1},
\]
\[
\delta_i(a^1 \otimes_B \cdots \otimes_B a^{n-1}) = a^1 \otimes_B \cdots \otimes_B a^{i-1} \Delta a^i \otimes_B \cdots \otimes_B a^{n-1}, \quad 1 \leq i \leq n-1,
\]
\[
\delta_n(a^1 \otimes_B \cdots \otimes_B a^{n-1}) = a^1 \otimes_B \cdots \otimes_B a^{n-1} \otimes_B 1,
\]
\[
\sigma_i(a^1 \otimes_B \cdots \otimes_B a^{n+1}) = a^1 \otimes_B \cdots \otimes_B a^i \otimes_B \epsilon(a^{i+1}) \otimes_B a^{i+2} \otimes_B \cdots \otimes_B a^{n+1}.
\]

The cyclic operators are given by
\[
\tau_n(a^1 \otimes_B \cdots \otimes_B a^n) = (\Delta a^1)(a^2 \otimes \cdots \otimes a^n \otimes 1).
\]

Let
\[
\beta'_n := \sum_{i=0}^{n} (-1)^i \delta_i : C^n \to C^{n+1},
\]
\[ \beta_n := \beta'_n + (-1)^{n+1} \delta_n^{n+1} : C^n \to C^{n+1}, \]
\[ \lambda_n := (-1)^n \tau_n : C^n \to C^n, \]
\[ N_n := \sum_{i=0}^{n} \lambda^i_n : C^n \to C^n. \]

According to [9, Chapter 1],

\[ \begin{array}{c}
0 \rightarrow C^2 \xrightarrow{1-\lambda} C^2 \xrightarrow{N} C^2 \xrightarrow{1-\lambda} C^2 \xrightarrow{N} \ldots \\
0 \rightarrow C^1 \xrightarrow{1-\lambda} C^1 \xrightarrow{N} C^1 \xrightarrow{1-\lambda} C^1 \xrightarrow{N} \ldots \\
0 \rightarrow C^0 \xrightarrow{1-\lambda} C^0 \xrightarrow{N} C^0 \xrightarrow{1-\lambda} C^0 \xrightarrow{N} \ldots \\
\end{array} \]

is a bicomplex. In this complex, the columns are periodic of order 2; for \( p \) even, the \( p \)th column is the Hochschild complex \((C^\bullet, \beta)\); in case \( p \) is odd, the respective column is the acyclic complex \((C^\bullet, \beta')\). The Hochschild cohomology of \( A^\# \) is defined to be the cohomology of \((C^\bullet, \beta)\) and its cyclic cohomology is defined to be the cohomology of the total complex of the bicomplex.

The cyclic (resp. Hochschild) cohomology of \((A, B, \alpha, \beta, \Delta, \epsilon, S)\) is defined to be the cyclic (resp. Hochschild) cohomology of \( A^\# \).

**Remark 1** If \( A, B \) are differential graded algebras, and \((A, B, \alpha, \beta, \Delta, \epsilon, S, d)\) is a differential graded Hopf algebroid with the differential \( d \), then the cyclic cohomology of \((A, B, \alpha, \beta, \Delta, \epsilon, S, d)\) is defined to be the cohomology of the total complex of a tricomplex as in [5].

### 2.3 Hopf algebroids \( \mathcal{H}(G, M; T) \) and \( \mathcal{H}(G, \Omega^0\bullet(M); T) \)

Let \( G \) be a Lie group acting on a complex manifold \( M \). We assume that the \( G \)-action is holomorphic.

Similar to the construction of Hopf algebroid \( \mathcal{H}(G, M) \) in [16], we define \( B \) to be the algebra of complex differential forms on \( M \), and \( A \) to be the algebra of \( B \)-valued functions on \( G \). Then both \( A \) and \( B \) are differential graded algebras with the differential \( \overline{\partial} \).

For a group element \( g \) in \( G \) and a smooth function \( a \) on \( M \), let

\[(g^*(a))(x) := a(x \cdot g).\]

The source and target maps \( \alpha, \beta : B \to A \) are defined as follows:

\[\alpha(b)(g) = b, \quad \beta(b)(g) = g^*(b).\]
When we consider the projective tensor product, the space $A \otimes_B A$ is isomorphic to the space of $B$-valued functions on $G \times G$, i.e.,

$$(\phi \otimes_B \psi)(g_1, g_2) = \phi(g_1)(g_1^*(\psi(g_2))), \quad \phi, \psi \in A.$$  

The coproduct $\Delta: A \to A \otimes_B A$ is defined by

$$\Delta(\phi)(g_1, g_2) = \phi(g_1 g_2),$$  

and the counit map $\epsilon: A \to B$ is defined by $\epsilon(\phi) = \phi(1)$ for $\phi \in A$. Since the $G$-action is holomorphic, it is easy to check that $(A, B, \alpha, \beta, \Delta, \epsilon, \overline{\partial})$ is a differential graded topological bialgebroid.

We define the coproduct $\Delta$ on $A$ by

$$\Delta(\phi)(g_1, g_2) = \phi(g_1 g_2).$$  

One computes (cf. [16])

$$S^2(\phi)(g) = g^*(S(\phi)(g^{-1})) = g^*((g^{-1})^*\phi(g)) = \phi(g),$$

$$(S\beta(b))(g) = g^*(\beta(b)(g^{-1})) = g^*((g^{-1})^*(b)) = b;$$

$$(m_A(S \otimes Id)\Delta)(\phi)(g) = g^*\phi(g^{-1} g) = g^*\phi(1) = g^*(S(\phi)(1)) = (\beta\epsilon S)(\phi)(g);$$

$$(S(a(1)^{(1)} \otimes_B S(a(1)^{(2)}))(a(2) \otimes 1)(g_1, g_2) = (g_1 g_2)^*a((g_1 g_2)^{(-1)} g_1))$$

$$= g^*_1(g^*_2(a(g_2^{-1})))$$

$$= 1 \otimes_B S(a)(g_1, g_2).$$

Therefore, $(A, B, \alpha, \beta, \Delta, \epsilon, S, \overline{\partial})$ is a differential graded Hopf algebroid, we denote it by $\mathcal{H}(G, M; \overline{\partial})$.

If we define $B$ to be the subalgebra $\Omega^0\cdot(M)$ and $A$ to be the algebra of $\Omega^0\cdot(M)$-valued functions on $G$, then we can easily get a new differential graded Hopf algebroid, which we denote by $\mathcal{H}(G, \Omega^0\cdot(M); \overline{\partial})$.

### 2.4 Hopf cyclic cohomology of $\mathcal{H}(G, M; \overline{\partial})$ and $\mathcal{H}(G, \Omega^0\cdot(M); \overline{\partial})$

In this subsection, we compute the Hopf cyclic cohomology of the differential graded Hopf algebroids $\mathcal{H}(G, M; \overline{\partial})$ and $\mathcal{H}(G, \Omega^0\cdot(M); \overline{\partial})$.

First, we rewrite the faces, degeneracy, and cyclic operators of $\mathcal{H}(G, M; \overline{\partial})$. Let $c$ be an $\Omega_c^0(M)$-valued function on $G \times G$. Then

$$\delta_i(c)(g_1, \ldots, g_{n+1}) = \begin{cases} g^*_1(c(g_2, \ldots, g_{n+1})), & i = 0, \\ c(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}), & 1 \leq i \leq n, \\ c(g_1, \ldots, g_n), & i = n + 1, \end{cases}$$

$$\sigma_i(c)(g_1, \ldots, g_{n-1}) = c(g_1, \ldots, g_i, 1, g_{i+1}, \ldots, g_{n-1}), \quad 0 \leq i \leq n - 1,$$

$$\tau_n(c)(g_1, \ldots, g_n) = (g_1 \cdots g_n)^{\epsilon}(g_1 \cdots g_n)^{-1}, g_1, \ldots, g_{n-1}).$$
and
\[ \beta_n(c)(g_1, \ldots, g_{n+1}) = g_1^*(c(g_2, \ldots, g_{n+1})) + \sum_{i=1}^{n} (-1)^i c(g_1, \ldots, g_i, g_{i+1}, \ldots, g_{n+1}) \]
\[ + (-1)^{n+1}c(g_1, \ldots, g_n). \]

Now, recall the definition of differentiable cohomology of a Lie group \( G \). Let \( M \) be a manifold with a left \( G \)-action, and let \( E \) be a \( G \)-equivariant bundle on \( M \). Let
\[ C^p_d(G; E) := C^\infty(G^{\times p}; \Gamma(E)). \]

Then the differential \( d \) on \( C^\bullet_d(G; E) \) is defined by
\[ (dc)(g_1, \ldots, g_{n+1}) = g_1(c(g_2, \ldots, g_{n+1})) + \sum_{i=1}^{n} (-1)^i c(g_1, \ldots, g_i, g_{i+1}, \ldots, g_{n+1}) \]
\[ + (-1)^{n+1}c(g_1, \ldots, g_n), \forall c \in C^p_d(G; E). \]

The differentiable cohomology \( H^\bullet_d(G; E) \) of \( G \) with coefficients in \( E \) is defined to be the cohomology of \( (C^\bullet_d(G; E), d) \).

One can easily get that the Hochschild cohomology of \( \mathcal{H}(G, M; \overline{\partial}) \) is isomorphic to the differentiable cohomology \( H^\bullet_d(G; \Omega^*(C(M), \overline{\partial})) \).

Let \( \Omega^{p,q}(M)^G \) be the space of \( G \)-invariant \((p, q)\)-forms on \( M \), and let \( H^{p,q}_d(M, G) \) be the \( q \)-th cohomology group of the complex \((\Omega^{p,\bullet})(M, \overline{\partial})\). Then the Hochschild cohomology \( HH^\bullet(\mathcal{H}(G, M; \overline{\partial})) \) and the cyclic cohomology \( HC^\bullet(\mathcal{H}(G, M; \overline{\partial})) \) can be computed as follows.

**Proposition 1** Let \( G \) be a Lie group acting on a complex manifold \( M \). If the \( G \)-action is holomorphic and proper, then
\[ HH^\bullet(\mathcal{H}(G, M; \overline{\partial})) = \bigoplus_{p+q=\bullet} H^{p,q}_d(M, G), \]
\[ HC^\bullet(\mathcal{H}(G, M; \overline{\partial})) = \bigoplus_{p+q=-2k, k \geq 0} H^{p,q}_d(M, G). \] (2.1)

**Proof** Since the \( G \)-action is proper, by [4, Sect. 2.1, Prop. 1] and the theory of homological algebra (cf. [14]), the differentiable cohomology \( H^\bullet_d(G; \Omega^*(C(M), \overline{\partial})) \) is isomorphic to the cohomology of the following complex:
\[ 0 \longrightarrow \Omega^0_C(M)^G \overset{\overline{\partial}}{\longrightarrow} \Omega^1_C(M)^G \overset{\overline{\partial}}{\longrightarrow} \Omega^2_C(M)^G \overset{\overline{\partial}}{\longrightarrow} \ldots \]

Then
\[ HH^\bullet(\mathcal{H}(G, M; \overline{\partial})) = H^\bullet_d(G; \Omega^*(C(M), \overline{\partial})) = \bigoplus_{p+q=\bullet} H^{p,q}_d(M, G). \]
The Hopf cyclic cohomology of $\mathcal{H}(G, M; \overline{\mathcal{D}})$ is isomorphic to the cohomology of the total complex of the following bicomplex:

$$
\begin{array}{cccccccc}
& \Omega^2_C(M)^G & \longrightarrow & 0 & \longrightarrow & \Omega^2_C(M)^G & \longrightarrow & 0 & \longrightarrow \\
\mathcal{D} & \uparrow & \uparrow & \mathcal{D} & \uparrow & \uparrow & \mathcal{D} & \uparrow & \\
0 & \longrightarrow & \Omega^1_C(M)^G & \longrightarrow & 0 & \longrightarrow & \Omega^1_C(M)^G & \longrightarrow & 0 & \longrightarrow \\
\mathcal{D} & \uparrow & \uparrow & \mathcal{D} & \uparrow & \uparrow & \mathcal{D} & \uparrow & \\
0 & \longrightarrow & \Omega^0_C(M)^G & \longrightarrow & 0 & \longrightarrow & \Omega^0_C(M)^G & \longrightarrow & 0 & \longrightarrow \\
\mathcal{D} & \uparrow & \uparrow & \mathcal{D} & \uparrow & \uparrow & \mathcal{D} & \uparrow & \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \\
\end{array}
$$

Then we get (2.1). □

Similarly, for $\mathcal{H}(G, \Omega^0\bullet(M); \overline{\mathcal{D}})$, we have the following result.

**Proposition 2** If the $G$-action is holomorphic and proper, then

$$
HH^\bullet(\mathcal{H}(G, \Omega^0\bullet(M); \overline{\mathcal{D}})) = H^0\mathcal{D}(M, G),
$$

$$
HC^\bullet(\mathcal{H}(G, \Omega^0\bullet(M); \overline{\mathcal{D}})) = \bigoplus_{k \geq 0} H^0(\mathcal{D}; \overline{\mathcal{D}})^{-2k}(M, G).
$$

### 3 Generalized complex Hodge theory and proof of Theorem 1

Throughout this section, let $M$ be an $n$-dimensional complex manifold with a left $G$-action, where $G$ is a locally compact Lie group. Assume that the $G$-action is holomorphic, proper, and cocompact.

Let $H$ be an Hermitian metric on $M$. Without loss of generality, we assume that $H$ is $G$-invariant (cf. [12, (2.3)]). Let $E$ be a holomorphic vector bundle over $M$, and we assume that $E$ admits a holomorphic $G$-action lifted from the action of $G$ on $M$. Let $H^E$ be a $G$-invariant Hermitian metric on $E$, and

$$
E^{p,q} = \Lambda^p(M) \otimes E.
$$

Let $\langle \cdot, \cdot \rangle$ denote the inner product on $E^{p,q}$ induced by $H$ and $H^E$. Then $\Gamma(E^{p,q})$ carries a natural inner product such that for any $s_1, s_2 \in \Gamma(E^{p,q})$ with compact supports,

$$
\langle s_1, s_2 \rangle = \int_M \langle s_1(x), s_2(x) \rangle dx = \int_M s_1(x) \wedge * s_2(x),
$$

(3.1)

where

$$
*: E^{p,q} \rightarrow E^{n-p,n-q}
$$
is the $\mathbb{C}$-antilinear Hodge $*$ operator (cf. [15]).

The compactness of $M/G$ guarantees the existence of a compact subset $Y$ of $M$ such that $G(Y) = M$ (cf. [13, Lemma 2.3]). Let $U, U'$ be two open subsets of $M$ such that $Y \subset U$ and that the closures $\overline{U}$ and $\overline{U'}$ are both compact in $M$, and that $\overline{U} \subset U'$. Following [12], let $f : M \to [0, 1]$ be a smooth function such that $f|_U = 1$ and $\text{supp}(f) \subset U'$.

Let $\| \cdot \|_0$ be the $L^2$-norm associated to the inner product (3.1), and let $H^0(M, E^{p,q})$ be the completion of $\Gamma(E^{p,q})$ under $\| \cdot \|_0$. Let $\Gamma(E^{p,q})^G$ denote the space of $G$-invariant smooth sections of $E^{p,q}$.

Let $\|s\|_{W,0}^2 = \int_W (s(x), s(x))dx.$

For any $s \in \Gamma(E^{p,q})^G$, we have

$$\|s\|_{U,0} \leq \|fs\|_0 \leq \|s\|_{U',0}.$$  

Since $G \cdot U = M$ and $\overline{U'}$ is compact, there are finitely many elements $g_1, \ldots, g_k$ of $G$ such that $g_1U \cup \cdots \cup g_kU$ covers $U'$. It is easy to see that there exists a positive constant $C > 0$ such that

$$\|s\|_{U',0} \leq C\|s\|_{U,0}.$$  

Let

$$f\Gamma(E^{p,q})^G := \{fs : s \in \Gamma(E^{p,q})^G\}.$$  

We define $H^0_f(M, E^{p,q})^G$ to be the completion of the space $f\Gamma(E^{p,q})^G$ under the norm $\| \cdot \|_0$. Let $P_f$ denote the orthogonal projection from $H^0_f(M, E^{p,q})^G$ to $H^0(M, E^{p,q})^G$.

Let $dg := dm(g)$ be the right-invariant Haar measure on $G$, and define the modular function $\chi : G \to \mathbb{R}^+$ by

$$dm(g^{-1}) = \chi(g)dm(g).$$

Then Tang et al. [16] proved that, for any $\mu \in H^0(M, E^{p,q}),$

$$(P_f\mu)(x) = \frac{f(x)}{A_f^2(x)} \int_G \chi(g)f(gx)g^{-1}(\mu(gx))dg,$$  

where

$$A_f(x) := \left(\int_G \chi(g)(f(gx))^2dg\right)^{1/2}$$

is a $G$-equivariant function on $M$, i.e.,

$$A_f^2(gx) = \chi(g)^{-1}A_f^2(x),$$

and is strictly positive.
Define $H^1_0(M, E^{p,q})^G$ to be the completion of the space $f \Gamma(E^{p,q})^G$ under a (fixed) first Sobolev norm associated to the inner product (3.1). And in general, define $H^k_0(M, E^{p,q})^G$ (resp. $H^{-1}_f(M, E^{p,q})^G$) to be the completion of $f \Gamma(E^{p,q})^G$ under the corresponding $H^k$ (resp. $H^{-1}$) norm for $k \geq 2$. (For any open subset $W$ of $M$ and any compactly supported smooth form $s \in \Gamma(E^{p,q})$ with supp$(s) \subset W$,

$$\|s\|_{W,k}^2 = \|(1 + \Delta_{\pi})^{k/2}(s)\|_{W,0}^2, \quad k \geq 1.$$  

Here, $\Delta_{\pi}$ is the usual $\pi$-Laplacian operator on $M$ (cf. [6]).

Now, we define the operators

$$\overline{\partial}_f : H^1_0(M, E^{p,q})^G \to H^0_0(M, E^{p,q+1})^G,$$

$$f \alpha \mapsto f \overline{\partial}_f \alpha,$$

$$\overline{\partial}^*_f : H^0_0(M, E^{p,q+1})^G \to H^1_0(M, E^{p,q})^G,$$

$$f \beta \mapsto f \overline{\partial}^*_f \beta - 2P_f(\iota_{\partial_f} \beta),$$

where $\overline{\partial}^* = - * \overline{\partial}$, and $\iota_{\omega}$ is the interior multiplication by the differential form $\omega$ satisfying

$$\langle \iota_{\omega} \varphi, \psi \rangle = \langle \varphi, \overline{\omega} \wedge \psi \rangle, \quad \forall \varphi, \psi \in \Gamma(E^{*,*}).$$

Then

$$\langle \overline{\partial}_f (f \alpha), f \beta \rangle = (f \overline{\partial}_f \alpha, f \beta)$$

$$= (\alpha, \overline{\partial}^* f^2 \beta)$$

$$= (\alpha, f^2 \overline{\partial}^* \beta - 2f \iota_{\partial_f} \beta)$$

$$= (f \alpha, f \overline{\partial}^* \beta) - (f \alpha, 2 \iota_{\partial_f} \beta)$$

$$= (f \alpha, f \overline{\partial}^* \beta) - 2(f \alpha, P_f(\iota_{\partial_f} \beta))$$

$$= (f \alpha, \overline{\partial}^*_f (f \beta)).$$

Let

$$\Delta_{\pi,f} = \overline{\partial}_f \overline{\partial}_f^* + \overline{\partial}^*_f \overline{\partial}_f.$$  

Then we have the following property.

**Proposition 3**  $\Delta_{\pi,f} : H^2_0(M, E^{p,q})^G \to H^0_0(M, E^{p,q})^G$ is Fredholm.

**Proof**  We need to establish a Gårding type inequality.

Let $fs \in H^0_0(M, E^{p,q})^G$. Then

$$\Delta_{\pi,f}(fs) = \overline{\partial}_f(\overline{\partial}_f(fs)) + \overline{\partial}^*_f(\overline{\partial}_f(fs))$$

$$= \overline{\partial}_f(f \overline{\partial}^* s - 2P_f(\iota_{\partial_f} s)) + \overline{\partial}^*_f(f \overline{\partial} s)$$

$$= f \overline{\partial} \cdot \overline{\partial}^* s - 2 \overline{\partial}_f(\iota_{\partial_f s}) + f \overline{\partial} \cdot \overline{\partial}^* s - 2P_f(\iota_{\partial_f \overline{\partial} s})$$

$$= f \Delta_{\pi} s - 2 \overline{\partial}_f(\iota_{\partial_f s}) - 2P_f(\iota_{\partial_f \overline{\partial} s}).$$ (3.3)
Using (3.2), we have
\[
\|\overline{\partial}_J(P_J(\iota_{\partial_f} s))\|_0 = \| f(x) \overline{\partial} \left( \frac{\chi(g)f(gx)g^{-1}(\iota_{\partial_f} s)(gx)}{A_f^2(x)} \right) \|_0
\]
\[
= \| f \int_G \chi(g) \overline{\partial} \left( \frac{g^*f}{A_f^2} \right) (g^{-1}\iota_{\partial_f} s) dg + f \int_G \chi(g) \frac{g^*f}{A_f^2} \overline{\partial}(g^{-1}\iota_{\partial_f} s) dg \|_0
\]
\[
\leq \| f \int_G \chi(g) \overline{\partial} \left( \frac{g^*f}{A_f^2} \right) (g^{-1}\iota_{\partial_f} s) dg \|_0 + \| f \int_G \chi(g) \frac{g^*f}{A_f^2} \overline{\partial}(g^{-1}\iota_{\partial_f} s) dg \|_0, \tag{3.4}
\]
where
\[
\| f \int_G \chi(g) \overline{\partial} \left( \frac{g^*f}{A_f^2} \right) (g^{-1}\iota_{\partial_f} s) dg \|_0 \leq \| f \int_G \chi(g) \overline{\partial} \left( \frac{g^*f}{A_f^2} \right) \| (g^{-1}\iota_{\partial_f} s) dg \|_0
\]
\[
\leq \| f \int_G \chi(g) \overline{\partial} \left( \frac{g^*f}{A_f^2} \right) \| g^*(\partial f) \| g^{-1}s \| dg \|_0
\]
\[
\leq \| f \int_G \chi(g) \overline{\partial} \left( \frac{g^*f}{A_f^2} \right) \| g^*(\partial f) \| dg \|_0.
\]

Since \(\text{supp}(f) \subset U'\),
\[
f \int_G \chi(g) \overline{\partial} \left( \frac{g^*f}{A_f^2} \right) \| g^*(\partial f) \| dg
\]
is finite and continuous everywhere, and therefore, is bounded from above by a constant on \(U'\). Then we have
\[
\| f \int_G \chi(g) \overline{\partial} \left( \frac{g^*f}{A_f^2} \right) (g^{-1}\iota_{\partial_f} s) dg \|_0 \leq C_1 \| s \|_{U',0} \leq C_2 \| f s \|_0. \tag{3.5}
\]

Since the \(G\)-action is holomorphic, the term
\[
\| f \int_G \chi(g) \overline{\partial} \left( \frac{g^*f}{A_f^2} \right) (g^{-1}\iota_{\partial_f} s) dg \|_0 = \| P_J(\overline{\partial}(\iota_{\partial_f} s))\|_0 \leq \| \overline{\partial}(\iota_{\partial_f} s)\|_0. \tag{3.6}
\]

We now choose a cut-off function \(\tilde{f}\) such that \(\tilde{f}|_{\text{supp}(f)} = 1\) and \(\text{supp}(\tilde{f}) \subset U'\). Then we have
\[
\| \overline{\partial}(\iota_{\partial_f} s)\|_0 = \| \overline{\partial}(\iota_{\partial_f}(\tilde{f} s))\|_{U',0}. \tag{3.7}
\]
Since \(\tilde{f}\) is a compactly supported smooth function in \(U'\) and \(\overline{\partial} \circ \iota_{\partial_f}\) is a differential operator of order 1, we have
\[
\| \overline{\partial}(\iota_{\partial_f}(\tilde{f} s))\|_{U',0} \leq C_3 \| \tilde{f} s \|_{U',1}. \tag{3.8}
\]
We compute $\|\tilde{f}s\|_{U',0}$ as

\[
\|\tilde{f}s\|_{U',0}^2 = \|\tilde{f}s\|_{U',0}^2 + (\Delta \eta(\tilde{f}s), \tilde{f}s)_{U'}
= \|\tilde{f}s\|_{U',0}^2 + (\overline{\partial}(\tilde{f}s), \overline{\partial}(\tilde{f}s))_{U'} + (\overline{\partial}'(\tilde{f}s), \overline{\partial}'(\tilde{f}s))_{U'}
= \|\tilde{f}s\|_{U',0}^2 + \|\overline{\partial}(\tilde{f}s)\|_{U',0}^2 + \|\overline{\partial}'(\tilde{f}s)\|_{U',0}^2.
\]

(3.9)

Since $\tilde{f}$ is bounded from above by 1, we have

\[\|\tilde{f}s\|_{U',0} \leq \|s\|_{U',0}.
\]

As $\tilde{f}$ and $\overline{\partial}\tilde{f}$ are both bounded, we have

\[\|\overline{\partial}(\tilde{f}s)\|_{U',0} = \|\overline{\partial}\tilde{f} \wedge s + f\overline{\partial}s\|_{U',0}
\leq \|\overline{\partial}\tilde{f} \wedge s\|_{U',0} + \|f\overline{\partial}s\|_{U',0}
\leq C_4\|s\|_{U',0} + \|\overline{\partial}s\|_{U',0}.
\]

(3.10)

Similarly, as $\tilde{f}$ and $\overline{\partial}\tilde{f}$ are compactly supported, they are both bounded. We have

\[\|\overline{\partial}'(\tilde{f}s)\|_{U',0} = \|\overline{\partial}\tilde{f} \wedge s - \iota_{\overline{\partial}\tilde{f}}s\|_{U',0} \leq C_5\|s\|_{U',0} + \|\overline{\partial}'s\|_{U',0}.
\]

(3.11)

Since $G \cdot U = M$ and $\overline{U'}$ is compact, there are finitely many $g_1, \ldots, g_k$ such that

\[U' \subset g_1U \cup \cdots \cup g_kU.
\]

Since $s$, $\overline{\partial}s$, and $\overline{\partial}'s$ are all $G$-invariant, we have

\[
\|s\|_{U',0} \leq C_6\|s\|_{U,0} \leq C_6\|fs\|_0,
\]

\[
\|\overline{\partial}s\|_{U',0} \leq C_6\|\overline{\partial}s\|_{U,0} \leq C_6\|\overline{\partial}(fs)\|_0.
\]

(3.12)

Summarizing inequalities (3.6)–(3.12), we have

\[
\left\|f \int_G \chi(g) g^{-f} \overline{\partial}(g^{-1}\iota_{\overline{\partial}f}s)dg\right\|_0 \leq A\|fs\|_1.
\]

By (3.4) and (3.5), we have

\[\|\overline{\partial}_f(P_f(\iota_{\overline{\partial}f}s))\|_0 \leq C_2\|fs\|_0 + A\|fs\|_1.
\]

Similarly, we have

\[\|P_f(\iota_{\overline{\partial}f}\overline{\partial}s)\| \leq \|\iota_{\overline{\partial}f}\overline{\partial}s\|_0 \leq C_5\|\overline{\partial}s\|_{U',0} \leq C_8\|\overline{\partial}(fs)\|_0 \leq B\|fs\|_1.
\]
By combining these inequalities, we have
\[ \|\Delta_{\partial f}(fs)\|_0 \geq \|f\Delta_{\partial f}s\|_0 - 2\|\overline{\partial}f(P_f(t_0fs))\|_0 - 2\|P_f(t_0\overline{\partial}s)\|_0 \geq \|f\Delta_{\partial f}s\|_0 - C\|fs\|_1. \] (3.13)

The standard elliptic inequality implies that
\[ \|\Delta f(fs)\|_0 \geq \tilde{A}\|fs\|_2 - \tilde{B}\|fs\|_0. \] (3.14)

We now compute
\[ \Delta_{\partial f}(fs) = (\overline{\partial} \overline{\partial} + \overline{\partial} \overline{\partial}^c)(fs) = f\Delta_{\partial f}s + \overline{\partial}^c(\overline{\partial}f \wedge s) - t_0f\overline{\partial}s - \overline{\partial}(t_0fs) + \overline{\partial}f \wedge \overline{\partial}^c s. \] (3.15)

We notice that \( \overline{\partial}^c(\overline{\partial}f \wedge s), t_0f\overline{\partial}s, \overline{\partial}(t_0fs), \) and \( \overline{\partial}f \wedge \overline{\partial}^c s \) are all differential operators of order less than or equal to 1 for \( s \). So similar estimates as (3.7)–(3.12) show that every piece of them is bounded by a multiple of \( \|fs\|_1 \). Then according to (3.15), we have
\[ \|f\Delta_{\partial f}s\|_0 \geq \|\Delta_{\partial f}(fs)\|_0 - \tilde{C}\|fs\|_1. \] (3.16)

Combining (3.13), (3.14), and (3.16), we have
\[ \|\Delta_{\partial f}(fs)\|_0 \geq \tilde{A}\|fs\|_2 - \tilde{B}\|fs\|_0 - D\|fs\|_1. \]

The so-called Peter-Paul inequality gives us
\[ \|fs\|_1 \leq \frac{1}{2} \frac{\tilde{A}}{\tilde{D}} \|fs\|_2 + \tilde{D}\|fs\|_0. \]

In summary, we have
\[ \|\Delta_{\partial f}(fs)\|_0 \geq D_1\|fs\|_2 - D_2\|fs\|_0. \] (3.17)

Due to the fact that the embedding of \( H^0_G(M, E^{p,q}) \) in \( H^0(M, E^{p,q})^G \) is compact, the above Gårding type inequality implies that \( \Delta_{\partial f} \) is Fredholm. □

**Corollary 1** \( \dim(\ker \Delta_{\partial f}) = \dim(\operatorname{coker} \Delta_{\partial f}) < +\infty. \)

**Lemma 1** \( \ker \Delta_{\partial f} = (\operatorname{Im} \Delta_{\partial f})^\perp \cap H^*_f(M, E^{••})^G. \)

**Proof** We have
\[ f\alpha \in (\operatorname{Im} \Delta_{\partial f})^\perp \cap H^*_f(M, E^{••})^G \iff (f\alpha, \Delta_{\partial f}(f\beta)) = 0, \forall f\beta \in H^*_f(M, E^{••})^G \iff (\Delta_{\partial f}(f\alpha), f\beta) = 0, \forall f\beta \in H^*_f(M, E^{••})^G. \]
Since $H^0_f(M, E^{\bullet, \bullet})^G$ is dense in $H^0_f(M, E^{\bullet, \bullet})^G$, we have $\Delta_{\overline{\nabla}, f}(f\alpha) = 0$, which is equivalent to $f\alpha \in \ker \Delta_{\overline{\nabla}, f}$. □

Lemma 1 together with Corollary 1 implies that

$$\ker \Delta_{\overline{\nabla}, f} = (\text{Im} \Delta_{\overline{\nabla}, f})^\perp.$$ 

So we have the decomposition

$$H^0_f(M, E^{\bullet, \bullet})^G = \ker \Delta_{\overline{\nabla}, f} \oplus \text{Im} \Delta_{\overline{\nabla}, f}.$$ 

Define the projection

$$H_f: H^0_f(M, E^{\bullet, \bullet})^G \to \ker \Delta_{\overline{\nabla}, f}.$$ 

For $f\alpha \in H^0_f(M, E^{\bullet, \bullet})^G$, we have

$$f\alpha - H_f(f\alpha) \in \text{Im} \Delta_{\overline{\nabla}, f}.$$ 

So there is a unique $f\beta \in \text{Im} \Delta_{\overline{\nabla}, f}$ such that

$$\Delta_{\overline{\nabla}, f}(f\beta) = f\alpha - H_f(f\alpha).$$ 

We define in this way the Green operator

$$\mathfrak{G}: f\alpha \mapsto f\beta.$$ 

We will need the following propositions to explore the properties of the Green operator.

**Proposition 4** Let $\{f s_j\}$ be a sequence in $H^0_f(M, E^{p,q})^G$ such that

$$\|f s_j\|_0 \leq C, \quad \|\Delta_{\overline{\nabla}, f}(f s_j)\|_0 \leq C, \quad \forall j,$$

for some constant $C > 0$. Then it has a Cauchy subsequence in $H^0_f(M, E^{p,q})^G$.

**Proof** By (3.17), we have

$$\|f s_j\|_2 \leq A\|\Delta_{\overline{\nabla}, f}(f s_j)\|_0 + B\|f s_j\|_0 \leq (A + B)C.$$ 

So $\{f s_j\}$ is a bounded sequence in $H^2_f(M, E^{p,q})^G$. Then we conclude the result by the fact that $H^2_f(M, E^{p,q})^G$ is compactly embedded in $H^0_f(M, E^{p,q})^G$. □

Now, we prove the regularity for $\Delta_{\overline{\nabla}, f}$.

**Proposition 5** If $f \beta \in H^k_f(M, E^{p,q})^G$ and

$$\Delta_{\overline{\nabla}, f}(f\alpha) = f\beta$$

on $M$, then $f\alpha \in H^{k+2}_f(M, E^{p,q})^G$ for any $k \geq 0$. In particular, if $f\beta$ is a smooth section, then so is $f\alpha$. 
Proof Since $f$ is smooth and compactly supported, it is sufficient to prove the differentiability of $\alpha$. This is a local statement. Since both $\alpha$ and $\beta$ are $G$-invariant and $G \cdot U = M$, we can restrict our analysis to $U$.

According to (3.2) and (3.3), we have

$$\Delta_{\nabla f}(f \alpha) = f \Delta_{\nabla} \alpha - 2 \nabla_{f}(P_{f}(\iota_{\partial f} \alpha)) - 2 P_{f}(\iota_{\partial f} \overline{\partial} \alpha)$$

$$= f \Delta_{\nabla} \alpha - 2 \nabla_{f}(\int_{G} \chi(g) \frac{g^{*} f}{A_{f}^{2}} g^{-1}(\iota_{\partial f} \alpha) dg)$$

$$- 2 \int_{G} \chi(g) \frac{g^{*} f}{A_{f}^{2}} g^{-1}(\iota_{\partial f} \overline{\partial} \alpha) dg.$$ 

Since $\alpha$ and $\overline{\partial} \alpha$ are $G$-invariant, we can find a $G$-invariant smooth $(1,0)$-form $\varpi$ (which depend only on $f$) such that

$$\Delta_{\nabla f}(f \alpha) = f \Delta_{\nabla} \alpha + f \nabla(\iota_{\varpi} \alpha) + f \iota_{\varpi} \overline{\partial} \alpha. \quad (3.18)$$

Notice that on $U$, $f = 1$. Hence, (3.18) implies that

$$\beta = \Delta_{\nabla} \alpha + \nabla(\iota_{\varpi} \alpha) + \iota_{\varpi} \overline{\partial} \alpha$$

on $U$. Since the last two terms are of lower order, the regularity of $\Delta_{\nabla f}$ is a consequence of that of $\Delta_{\nabla}$. 

The Green operator $\mathcal{G}$ has the following properties.

(1) $\mathcal{G}$ is bounded.

By the definition of $\mathcal{G}$, we only need to prove that there exists a constant $C > 0$ such that for any $f \alpha \in \text{Im} \Delta_{\nabla f} \cap H^{2}_{f}(M, E^{p,q})^{G}$,

$$\|f \alpha\|_{0} \leq C \|\Delta_{\nabla f}(f \alpha)\|_{0}. \quad (3.19)$$

Suppose the contrary. Then there exists a sequence

$$f \alpha_{i} \in \text{Im} \Delta_{\nabla f} \cap H^{2}_{f}(M, E^{p,q})^{G}$$

with

$$\|f \alpha_{i}\|_{0} = 1, \quad \|\Delta_{\nabla f}(f \alpha_{i})\|_{0} \to 0.$$ 

By Proposition 4, a subsequence of $\{f \alpha_{i}\}$, which for convenience we can assume to be $\{f \alpha_{i}\}$ itself, is Cauchy. Thus, $\lim_{i \to \infty}(f \beta, f \alpha_{i})$ exists for each $f \beta \in H^{0}_{f}(M, E^{p,q})^{G}$. We define a linear functional $l$ on $H^{0}_{f}(M, E^{p,q})^{G}$ by setting

$$l(f \beta) = \lim_{i \to \infty}(f \beta, f \alpha_{i}), \quad f \beta \in H^{0}_{f}(M, E^{p,q})^{G}.$$ 

Now, $l$ is clearly bounded, and

$$l(f \beta) = (f \beta, f \alpha)$$
with \( f \alpha_i \to f \alpha \) in \( H^0_f(M, E^{p,q})^G \). Since

\[
l(\Delta f(f \varphi)) = (\Delta f(f \varphi), f \alpha) = \lim_{i \to \infty} (\Delta f(f \varphi), f \alpha_i) = \lim_{i \to \infty} (f \varphi, \Delta f(f \alpha_i)) = 0, \quad \forall f \varphi \in H^2_f(M, E^{p,q})^G.
\]

we know that \( f \alpha \) is a weak solution of \( \Delta f(\xi) = 0 \). It follows from Proposition 5 that \( f \alpha \) is actually smooth and a strong solution of \( \Delta f(\xi) = 0 \).

Hence,

\[
f \alpha \in \ker \Delta f \cap \text{Im} \Delta f = \{0\},
\]

which yields a contradiction.

(2) \( \mathcal{G} \) is self-adjoint.

In fact,

\[
(\mathcal{G}(f \alpha), f \beta) = (\mathcal{G}(f \alpha), f \beta - H_f(f \beta)) = (\mathcal{G}(f \alpha), \Delta f(f \beta)) = (\Delta f(f \alpha), \mathcal{G}(f \beta)) = (f \alpha - H_f(f \alpha), \mathcal{G}(f \beta)) = (f \alpha, \mathcal{G}(f \beta)).
\]

(3) \( \mathcal{G} \) maps a bounded sequence into one with Cauchy subsequences.

Combining (3.17) and (3.19), there exists a constant \( C > 0 \) such that

\[
\|f \alpha\|_2 \leq C\|\Delta f(f \alpha)\|_0, \quad \forall f \alpha \in \text{Im} \Delta f \cap H^2_f(M, E^{p,q})^G.
\]

Then

\[
\|\mathcal{G}(f \beta)\|_2 \leq C\|\Delta f(f \beta)\|_0 \leq C\|f \beta\|_0, \quad \forall f \beta \in H^0_f(M, E^{p,q})^G.
\]

Hence, \( \mathcal{G} \) is a compact operator due to the fact that the embedding of \( H^2_f(M, E^{p,q})^G \) into \( H^0_f(M, E^{p,q})^G \) is compact.

(4) \( \mathcal{G} \) maps \( f \Gamma(E^{p,q})^G \) to \( f \Gamma(E^{p,q})^G \).

It is a corollary directly from Proposition 5.

Let \( H^{p,q}_f(M, E, G) \) denote the \( q \)-th cohomology group of the complex \( (\Gamma(E^{p,\cdot})^G, \bar{\partial}) \), and let \( \mathcal{H}_f(E^{\cdot,\cdot})^G \) denote the kernel of the operator \( \Delta f \). Then we have the following result.

**Proposition 6**  The map \( H_f \) induces an isomorphism

\[
H_f: H^{p,q}_f(M, E, G) \to \mathcal{H}_f(E^{p,q})^G.
\]
Remark 2  For any $f\varphi \in f\Gamma(E^{p,q})^G$, we have

$$ f\varphi = \overline{\partial}_j \overline{\partial}_j \mathfrak{S}(f\varphi) + \overline{\partial}_j \overline{\partial}_j \mathfrak{S}(f\varphi) + H_f(f\varphi), $$

where $\overline{\partial}_j \overline{\partial}_j \mathfrak{S}(f\varphi)$, $\overline{\partial}_j \overline{\partial}_j \mathfrak{S}(f\varphi)$, and $H_f(f\varphi)$ are mutually orthogonal to each other with respect to the inner product on $H^0_p(M, E^{p,q})^G$.

Proof of Proposition 6  Suppose $\alpha \in \Gamma(E^{p,q})^G$ with $\overline{\partial}\alpha = 0$. Then $f\alpha \in f\Gamma(E^{p,q})^G$ satisfies

$$ \overline{\partial}_f(f\alpha) = f\overline{\partial}\alpha = 0. $$

We define $H_f(\alpha)$ to be $H_f(f\alpha)$.

Since

$$ f\alpha = \overline{\partial}_j \overline{\partial}_j \mathfrak{S}(f\alpha) + \overline{\partial}_j \overline{\partial}_j \mathfrak{S}(f\alpha) + H_f(f\alpha), $$

and

$$ (f\alpha, \overline{\partial}_j \overline{\partial}_j \mathfrak{S}(f\alpha)) = (\overline{\partial}_f(f\alpha), \overline{\partial}_j \mathfrak{S}(f\alpha)) = 0, $$

we have

$$ f\alpha = \overline{\partial}_j \overline{\partial}_j \mathfrak{S}(f\alpha) + H_f(f\alpha). \tag{3.20} $$

If $\alpha = \overline{\partial}\beta$ with $\beta \in \Gamma(E^{p,q-1})^G$, then we have

$$ \overline{\partial}_f(f\beta) = f\overline{\partial}\beta = f\alpha. $$

Since

$$ (f\alpha, H_f(f\alpha)) = (\overline{\partial}_f(f\beta), H_f(f\alpha)) = (f\beta, \overline{\partial}_f(H_f(f\alpha))) = 0, $$

we have

$$ H_f(\alpha) = H_f(f\alpha) = 0. $$

This means that $H_f$ is a well-defined map from $H^{p,q}_G(M, E, G)$ to $\mathcal{H}_f(E^{p,q})^G$.

When $H_f(f\alpha) = 0$, by equation (3.20), we have $f\alpha = \overline{\partial}_j \overline{\partial}_j \mathfrak{S}(f\alpha)$. Write $\overline{\partial}_j \mathfrak{S}(f\alpha) = f\beta$. Then

$$ f\beta \in f\Gamma(E^{p,q-1})^G, \quad f\alpha = \overline{\partial}_f(f\beta). $$

Hence, $\alpha = \overline{\partial}\beta$. This implies that $H_f$ is injective.

For any $f\beta \in \mathcal{H}_f(E^{p,q})^G$, by Proposition 6, $f\beta \in f\Gamma(E^{p,q})^G$. Since

$$ \overline{\partial}_f(f\beta) = f\overline{\partial}\beta = 0, $$

we have $\overline{\partial}\beta = 0$, which implies that $\beta$ is a $G$-invariant smooth closed form with $H_f(f\beta) = f\beta$. Then $H_f$ is onto. $\square$

Theorem 1 is a corollary of Propositions 1, 2, 3, and 6.
4 Serre duality and a Kodaira type vanishing theorem

In this section, we will give some applications of the generalized complex Hodge theory in Section 3.

Following Section 3, when $G$ is unimodular, set $h(x) = f(x)/A_f(x)$. Then

- $h$ is a nonnegative smooth function on $M$;
- $h$ is positive on $\overline{U}$, and $\text{supp}(h) \subset U'$;
- according to [12, (2.20)],

$$\int_G h(gx)^2 dg = 1. \quad (4.1)$$

We will use $h$ instead of $f$ in the following discussion.

According to (4.1), for any $\alpha \in \Gamma(E^{\bullet \bullet})^G$, one computes

$$P_h(dh \wedge \alpha)(x) = h(x) \int_G h(gx) g^{-1}((dh \wedge \alpha)(gx)) dg$$

$$= \left( h(x) \int_G h(gx) g^*(dh(gx)) dg \right) \wedge \alpha(x)$$

$$= \frac{1}{2} d \left( \int_G h(gx)^2 dg \right) \wedge h \alpha(x)$$

$$= 0.$$

Since the $G$-action is holomorphic, one can easily get

$$P_h(\partial h \wedge \alpha) = P_h(\overline{\partial} h \wedge \alpha) = 0. \quad (4.2)$$

According to (4.2), one has

$$(P_h(\iota_{\partial h} \alpha), h\beta) = (\iota_{\partial h} \alpha, h\beta) = (h \alpha, \overline{\partial} h \wedge \beta) = (h \alpha, P_h(\overline{\partial} h \wedge \beta)) = 0,$$

$$\forall \beta \in \Gamma(E^{\bullet \bullet})^G,$$

which implies

$$P_h(\iota_{\partial h} \alpha) = 0. \quad (4.3)$$

Similarly, one can easily get

$$P_h(\iota_{\overline{\partial} h} \alpha) = 0. \quad (4.4)$$

According to (4.3), one has

$$\overline{\partial} h(h \alpha) = h \overline{\partial} \alpha - 2 P_h(\iota_{\partial h} \alpha) = h \overline{\partial} \alpha. \quad (4.5)$$

**Theorem 2** When $G$ is unimodular, there are isomorphisms

$$H^{p,q}_{\overline{\partial}}(M, E, G) \xrightarrow{\sim} H^{n-p, n-q}_{\overline{\partial}}(M, E^*, G).$$
Proof. For the standard $\bar{\partial}$-Laplacian $\Delta_{\bar{\partial}} = \bar{\partial}\partial^* + \partial^*\bar{\partial}$, one has

\[ \ast \Delta_{\bar{\partial}} = \Delta_{\bar{\partial}} \ast. \] (4.6)

For any $s \in \Gamma(E^{p,q})^G$, according to (4.5), we have

\[ \bar{\partial} h_s(hs) = h\bar{\partial}s, \quad \bar{\partial}^* h_s(hs) = h\partial^* s. \]

Then

\[ \Delta_{\bar{\partial},h}(hs) = h\Delta_{\bar{\partial}}s. \]

Using (4.6), we have

\[ \Delta_{\bar{\partial},h}(\ast hs) = h\Delta_{\bar{\partial}}(\ast s) = \ast \Delta_{\bar{\partial},h}(hs), \]

so

\[ \ast \Delta_{\bar{\partial},h} = \Delta_{\bar{\partial},h} \ast. \]

It is easy to see that $\ast$ induces isomorphisms

\[ \ast : \mathcal{H}_h(E^{p,q})^G \to \mathcal{H}_h(E^{n-p,n-q})^G. \] (4.7)

By Proposition 6, there are isomorphisms

\[ H_h : \mathcal{H}^{\bullet\bullet}_G(M,E,G) \to \mathcal{H}_h(E^{\bullet\bullet})^G, \] (4.8)

\[ H_h : \mathcal{H}^{\bullet\bullet}_G(M,E^*,G) \to \mathcal{H}_h(E^{\bullet\bullet})^G. \] (4.9)

The result then follows from (4.7)–(4.9).

In the remainder of this section, we assume that $M$ with the Hermitian metric $H$ is Kähler, and $\omega$ is the corresponding $G$-invariant Kähler form. Let $L$ be a holomorphic line bundle and assume that $L$ admits a holomorphic $G$-action lifted from the action of $G$ on $M$. Let $H^L$ be a $G$-invariant Hermitian metric on $L$.

Following [15, Chapter I], the Hermitian connection $\nabla^L$ defines a unique $\mathbb{C}$-linear map

\[ \tilde{\nabla}^L : C^\infty(M,\Lambda^\bullet T^*_C \otimes L) \to C^\infty(M,\Lambda^{\bullet+1}T^*_C \otimes L), \]

such that

\[ \tilde{\nabla}^L|_{C^\infty(M,L)} = \nabla^L \]

(identifying $L$ with $\Lambda^0 T^*_C \otimes L$) and

\[ \tilde{\nabla}^L(\eta \wedge s) = d\eta \wedge s + (-1)^k \eta \wedge \tilde{\nabla}^L s, \quad \forall \eta \in C^\infty(M,\Lambda^k T^*_C), \forall s \in C^\infty(M,\Lambda^\bullet T^*_C \otimes L). \]

Decompose $\tilde{\nabla}^L$ into

\[ \tilde{\nabla}^L = \nabla^L + \nabla^H, \]
where
\[ \nabla^L; C^\infty (M, L^p,q) \rightarrow C^\infty (M, L^{p+1,q}), \]
\[ \nabla''^L; C^\infty (M, L^p,q) \rightarrow C^\infty (M, L^{p,q+1}). \]

Since \( \nabla^L \) is compatible with the complex structure, we have
\[ \nabla''^L = \overline{\partial}. \]

Let \[ \delta^L = -*\nabla^L * \]
and define the Laplacian
\[ \Delta^L = \delta^L \nabla^L + \nabla^L \delta^L. \]

Let \( \theta^L \) denote the curvature form of \( L \) with respect to the Hermitian connection \( \nabla^L \). Then the famous Kodaira-Nakano Identity [15, (1.58)] states that
\[ \Delta \overline{\partial} = \Delta^L + \sqrt{-1}[\theta^L, i\omega]. \]

(4.10)

Here, we use the Lie bracket notation \( [A, B] = AB - BA. \)

Following [15, Chapter II], let \( x \in M \), and write
\[ \theta^L (x) = \sum_{j,k} a_{jk}(x) \varphi_j \wedge \varphi_k, \]
where \( \{\varphi_1, \ldots, \varphi_n\} \) is an orthonormal basis of the holomorphic cotangent space to \( M \) at \( x \) (we denote this space at \( x \) by \( T^*_x \)). Thus, \( [a_{jk}(x)] \) is an Hermitian matrix. Let \( \lambda_1(x) \leq \lambda_2(x) \leq \cdots \leq \lambda_n(x) \)
denote the eigenvalues of \( [a_{jk}(x)] \), and define
\[ \mu_q(x) = \lambda_{q+1}(x) + \cdots + \lambda_n(x), \quad 0 \leq q \leq n - 1. \]

It is easy to see that \( \mu_q(x) \) are continuous functions on \( M \).

If \( \lambda_n(x) \leq 0 \) for all \( x \in M \), then one says that \( L \) is semi-negative; if \( \lambda_1 \geq 0 \)
for all \( x \in M \), then \( L \) is semi-positive.

The following unique continuation result due to Aronszajn [1] is needed.

**Lemma 2** Let \( A \) be a linear elliptic differential operator of second order defined on a domain \( D \) in \( \mathbb{R}^n \), and let \( u = (u^1, \ldots, u^r) \) be functions on \( D \) satisfying the differential inequalities
\[ |Au| \leq \text{const.} \left\{ \sum_{\alpha} \left| \frac{\partial u^\alpha}{\partial x^i} \right| + \sum_{\beta} |u^\beta| \right\}. \]

(4.11)

If \( u = 0 \) on any open subset of \( D \), then \( u \equiv 0 \) on \( D \).
Now, we have the following result.

**Theorem 3** Assume that $G$ is unimodular. Let $0 \leq q \leq n - 1$. If $\mu_q(x) \leq 0$ for all $x \in M$ and $\mu_q(x_0) < 0$ at a point $x_0$, then $H^{0,q}_\omega(M, L, G) = 0$.

**Remark 3** It is easy to see that $\mu_q(x)$ is a $G$-invariant continuous function on $M$. Since $G \cdot U = M$, we can assume that $x_0 \in U$.

**Proof of Theorem 3** According to Proposition 6, there is an isomorphism

$$H_h: H^{0,q}_\omega(M, L, G) \to \mathbb{H}_h(L^0, q)^G.$$  

So we only need to prove that $\mathbb{H}_h(L^0, q)^G = 0$.

Let $hs \in \mathbb{H}_h(L^0, q)^G$. Then, by (4.10), we have

$$\Delta_h(hs) = \Delta^L(hs) - \sqrt{-1} \iota_\omega(\theta^L \wedge hs).$$

Thus,

$$(\Delta_h(hs), hs) = (\Delta^L(hs), hs) - \sqrt{-1} (\theta^L \wedge hs, \omega \wedge hs). \quad (4.12)$$

Since $hs \in \mathbb{H}_h(L^0, q)^G$, according to (4.5), we have

$$\bar{\partial}_h(hs) = h\bar{\partial}s = 0, \quad \bar{\partial}_h^*(hs) = h\bar{\partial}^*s = 0.$$

With the fact that $s$ is $G$-invariant, we have

$$\bar{\partial}s = \bar{\partial}^*s = 0. \quad (4.13)$$

So

$$(\Delta_h(hs), hs) = (\bar{\partial}(hs), \bar{\partial}(hs)) + (\bar{\partial}^*(hs), \bar{\partial}^*(hs)) = (\bar{\partial}h \wedge s, \bar{\partial}h \wedge s) + (\iota_{\partial h}s, \iota_{\partial h}s) = (|\partial h|^2s, s). \quad (4.14)$$

According to [15, (1.53)], we have

$$(\Delta^L(hs), hs) = (\nabla^L(hs), \nabla^L(hs)) + (\delta^L(hs), \delta^L(hs)).$$

By (4.2), we have

$$(\nabla^L(hs), \nabla^L(hs)) = (\partial h \wedge s + h\nabla^L s, \partial h \wedge s + h\nabla^L s)$$

$$(\partial h \wedge s, \partial h \wedge s) + (h\nabla^L s, h\nabla^L s) + 2 \text{Re}(\partial h \wedge s, h\nabla^L s)$$

$$(\partial h \wedge s, \partial h \wedge s) + (h\nabla^L s, h\nabla^L s) + 2 \text{Re}(P_h(\partial h \wedge s), h\nabla^L s)$$

$$(\partial h \wedge s, \partial h \wedge s) + (h\nabla^L s, h\nabla^L s),$$
while by (4.4), we have
\[(\delta^{\prime \prime \prime}L(hs), \delta^{\prime \prime \prime}L(hs)) = (h\delta^{\prime \prime \prime}L - i\gamma_{hs}, h\delta^{\prime \prime \prime}L - i\gamma_{hs}) = (i\gamma_{hs}, i\gamma_{hs}) + (h\delta^{\prime \prime \prime}L - i\gamma_{hs}, h\delta^{\prime \prime \prime}L - i\gamma_{hs}) - 2 \text{Re}(P_{h}(i\gamma_{hs}, h\delta^{\prime \prime \prime}L - i\gamma_{hs})).\]

Then
\[(\Delta^{\prime \prime \prime}L(hs), hs) = (\|\partial h\|^2 s, s) + (h\nabla^{\prime \prime \prime}L s, h\nabla^{\prime \prime \prime}L s) + (h\delta^{\prime \prime \prime}L s, h\delta^{\prime \prime \prime}L s). \quad (4.15)\]

By (4.12), (4.14), and (4.15), we have
\[\sqrt{-1} (\theta^{\prime \prime \prime}L \wedge hs, \omega \wedge hs) = (h\nabla^{\prime \prime \prime}L s, h\nabla^{\prime \prime \prime}L s) + (h\delta^{\prime \prime \prime}L s, h\delta^{\prime \prime \prime}L s),\]
and
\[\sqrt{-1} (\theta^{\prime \prime \prime}L \wedge hs, \omega \wedge hs) \geq 0. \quad (4.16)\]

We now compute the pointwise inner product \(\langle \theta^{\prime \prime \prime}L \wedge hs, \omega \wedge hs \rangle\).

For any \(x \in M\), let \(\{\varphi_1, \ldots, \varphi_n\}\) be an orthonormal basis for \(T^*_x\) such that
\[\theta^{\prime \prime \prime}L(x) = \sum_{j=1}^{n} \lambda_j(x) \varphi_j \wedge \overline{\varphi}_j.\]

Let
\[s(x) = \sum_{I} s_I(x) \varphi_{i_1} \wedge \cdots \wedge \overline{\varphi}_{i_q} \otimes e,\]
where
\[I = (i_1, \ldots, i_q), \quad 1 \leq i_1 < \cdots < i_q \leq n,\]
and \(e\) is a unit vector in \(L_x\). Recalling that (cf. [6])
\[\omega(x) = \sqrt{-1} \sum_{j=1}^{n} \varphi_j \wedge \overline{\varphi}_j,\]
since \(s \in \Gamma(L^{0,q})^G\), we obtain
\[\sqrt{-1} \langle \theta^{\prime \prime \prime}L \wedge hs, \omega \wedge hs \rangle_x = \sum_{j=1}^{n} \langle \lambda_j(x) \varphi_j \wedge \overline{\varphi}_j \wedge hs(x), \varphi_j \wedge \overline{\varphi}_j \wedge hs(x) \rangle = \sum_{I} \sum_{j \notin I} \lambda_j(x) |hs_I(x)|^2.\]

Since
\[\sum_{j \notin I} \lambda_j(x) \leq \lambda_{q+1}(x) + \cdots + \lambda_n(x) \leq \mu_q(x)\]
for all multi-indices $I$, we conclude that
\[ \sqrt{-1} \langle \theta^L \wedge hs, \omega \wedge hs \rangle \leq \mu_q \langle hs, hs \rangle \] (4.17)
at all points of $M$. Integrating (4.17), according to (4.16), we obtain
\[ \int_M \mu_q \langle hs, hs \rangle dx \geq \sqrt{-1} \int_M \langle \theta^L \wedge hs, \omega \wedge hs \rangle dx \geq 0. \]
Since $\mu_q(x) \leq 0$ for all $x \in M$, we obtain
\[ \int_M \mu_q \langle hs, hs \rangle dx = 0. \]
By Remark 3, since $\mu_q(x_0) < 0$ and $x_0 \in U$, $s$ vanishes on a neighborhood of $x_0$.

Using (4.13), we have
\[ \Delta \sigma s = 0. \]
Then it is easy to see that $s$ satisfies the differential inequalities (4.11) locally.
Hence, by Lemma 2, $s \equiv 0$. \hfill \square

Let
\[ K = \Lambda^{n,0}(M) \]
do not the canonical line bundle on $M$. We equip $K$ with the metric induced by the metric of $TM$, and with the corresponding Hermitian connection. Then we have the following result.

**Corollary 2** When $G$ is unimodular, the following statements hold true.

1. If $L$ is semi-negative on $M$ and negative at a point $x_0 \in M$, then
\[ H_{\sigma}^{0,q}(M, L, G) = 0, \quad q \neq n. \]

2. If $K^* \otimes L$ is semi-positive on $M$ and positive at a point $x_0 \in M$, then
\[ H_{\sigma}^{0,q}(M, L, G) = 0, \quad q \neq 0. \]

**Proof** (1) follows immediately by Theorem 3.

As for (2), [15, (1.32)] shows that
\[ \theta^L = -\partial \bar{\partial} \log H^L. \]
Then it is easy to see that the dual of $K^* \otimes L$ is semi-negative on $M$ and negative at $x_0$. According to (1), we have
\[ H_{\sigma}^{0,q}(M, K \otimes L^*, G) = H_{\sigma}^{0,q}(M, L^*, G) = 0, \quad q \neq n. \]
(2) then follows immediately by Theorem 2. \hfill \square
Let $E$ be a holomorphic vector bundle on $M$ with a $G$-invariant Hermitian metric $H^E$ as in Section 3. Let $\nabla^E$ be the corresponding Hermitian connection on $E$, and let $\theta^E$ be its curvature. Then we have the following result.

**Theorem 4** If $L$ is positive, then, for $m$ sufficiently large, we have

$$H^{0,q}_\partial(M, L^m \otimes E, G) = 0, \quad q \neq 0.$$  

**Proof** Since $K, K^*$ are holomorphic Hermitian line bundles which are such that $K \otimes K^*$ is canonically trivial, by comparing the corresponding $\bar{\partial}$ operators, one can identify $(\Gamma((L^m \otimes E)^{0,q})^G, \bar{\partial})$ with $(\Gamma((L^m \otimes E \otimes K^*)^{n,q})^G, \bar{\partial})$. According to Proposition 6, we only need to prove

$$\mathcal{H}_f((L^m \otimes E \otimes K^*)^{n,q})^G = 0, \quad q \neq 0.$$  

Set $E' = E \otimes K^*$. Let $\theta^{E'}$ be the curvature of the Hermitian connection on $E'$.

Using the Kodaira-Nakano Identity [15, (1.58)], we have

$$\Delta_{\bar{\partial}} = \Delta^L \otimes E' + \sqrt{-1} [m\theta^L + \theta^{E'}, \iota] \omega]. \quad (4.18)$$  

Let $f s \in \mathcal{H}_f((L^m \otimes E')^{n,q})$. Then, according to (4.18), we get

$$(\Delta_{\bar{\partial}}(fs), fs) = (\Delta^L \otimes E'(fs), fs) + \sqrt{-1} [\theta^{E'}, \iota](fs), fs)$$

$$+ m(\sqrt{-1} \theta^L \wedge \iota \omega(fs), fs). \quad (4.19)$$  

Since $fs \in \mathcal{H}_f((L^m \otimes E')^{n,q})$, we have

$$\bar{\partial}_f(fs) = f \bar{\partial}s = 0$$  

and

$$\bar{\partial}'_f(fs) = f \bar{\partial}' s - 2P_f(\iota_\partial fs) = 0.$$  

Then

$$(\Delta_{\bar{\partial}}(fs), fs) = (\bar{\partial}(fs), \bar{\partial}(fs)) + (\bar{\partial}'(fs), \bar{\partial}'(fs))$$

$$= (\bar{\partial} f \wedge s, \bar{\partial} f \wedge s) + (2P_f(\iota_\partial fs) - \iota_\partial fs, 2P_f(\iota_\partial fs) - \iota_\partial fs).$$

Since

$$(P_f(\iota_\partial fs), \iota_\partial fs) = (P_f(\iota_\partial fs), P_f(\iota_\partial fs)),$$

we get

$$(\Delta_{\bar{\partial}} (fs), fs) = (\bar{\partial} f \wedge s, \bar{\partial} f \wedge s) + (\iota_\partial fs, \iota_\partial fs).$$

Then there exist $C_1, C_2 > 0$ such that

$$(\Delta_{\bar{\partial}} (fs), fs) \leq C_1 \|s\|^2_{L^2, 0} \leq C_2 \|fs\|^2_0, \quad \forall f s \in \mathcal{H}_f((L^m \otimes E')^{n,q})^G. \quad (4.20)$$

Clearly,

$$(\Delta^L \otimes E'(fs), fs) \geq 0, \quad (4.21)$$

\[\square\]
and there exists $C_3 > 0$ such that
\begin{equation}
|\langle \sqrt{-1} \theta^E, \omega \rangle(f) - \langle f, f \rangle| \leq C_3 \|f\|_0^2.
\end{equation}

Take $x \in M$, and let $\{\varphi_1, \ldots, \varphi_n\}$ be an orthonormal basis for $T_x^*$ such that
\begin{align*}
\theta^L(x) &= \sum_{j=1}^n \lambda_j(x) \varphi_j \wedge \varphi_j, \\
\omega(x) &= \sqrt{-1} \sum_{j=1}^n \varphi_j \wedge \overline{\varphi}_j.
\end{align*}

Since $L$ is positive, and the $G$-action is cocompact, there exists $c > 0$ such that for any $x \in M$, $\lambda_j(x) \geq c$ (for all $1 \leq j \leq n$). We compute
\begin{align*}
\langle \sqrt{-1} \theta^L \wedge \omega(f), f \rangle &= \langle \omega(f), \omega(f) \rangle \\
&= \sum_{j=1}^n \lambda_j \langle \omega(f) \wedge \overline{\omega}(f), \omega(f) \rangle \\
&\geq cq \langle f, f \rangle, \quad \forall f \in \mathcal{H}_f((L^m \otimes E')^{n,q})^G.
\end{align*}

Then
\begin{equation}
(\sqrt{-1} \theta^L \wedge \omega(f), f) \geq cq \|f\|_0^2.
\end{equation}

Summarizing (4.19)–(4.23), we have
\begin{equation}
(mcq - C_2 - C_3) \|f\|_0^2 \leq 0, \quad \forall f \in \mathcal{H}_f((L^m \otimes E')^{n,q})^G.
\end{equation}

From (4.24), we can easily get that, if
\begin{equation*}
q > 0, \quad m > \frac{C_2 + C_3}{cq},
\end{equation*}
then
\begin{equation*}
\mathcal{H}_f((L^m \otimes E')^{n,q})^G = 0.
\end{equation*}

The proof of Theorem 4 is completed. \hfill \Box

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