A NOTE ON ALMOST CONTACT RIEMANNIAN 3-MANIFOLDS II

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Abstract. We classify Kenmotsu 3-manifolds and cosymplectic 3-manifolds with \( \eta \)-parallel Ricci operator.

Introduction

It is well known that semi-symmetric Sasakian manifolds are of constant curvature 1. On the other hand, semi-symmetric Kenmotsu manifolds are of constant curvature \(-1\). These facts mean that semi symmetry is a strong restriction for Sasakian and Kenmotsu manifolds.

In 3-dimensional geometry, local symmetry, i.e., the parallelism of the Riemannian curvature \( R \) is equivalent to the parallelism of the Ricci operator \( S \).

Cho and Kimura showed that Kenmotsu 3-manifolds whose Ricci operator is parallel along the characteristic flow are of constant curvature \(-1\) [6].

In this paper we study more mild condition on the Ricci operator. More precisely we study Kenmotsu 3-manifolds and cosymplectic 3-manifolds satisfying the following \( \eta \)-parallel condition:

\[
g( (\nabla_X S) Y, Z ) = 0
\]

for all vector fields \( X, Y \) and \( Z \) orthogonal to the structure vector field \( \xi \).

We classify Kenmotsu 3-manifolds satisfying this condition. Moreover we show that there exist Kenmotsu 3-manifolds of non-constant curvature which have \( \eta \)-parallel Ricci operator. In addition we also study cosymplectic 3-manifolds and Sasakian 3-manifolds with \( \eta \)-parallel Ricci operator.

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1. Preliminaries

1.1. Let \((M, g)\) be a Riemannian \(m\)-manifold with its Levi-Civita connection \(\nabla\). Denote by \(R\) the Riemannian curvature of \(M\):
\[
R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \quad X, Y \in \mathfrak{X}(M).
\]
Here \(\mathfrak{X}(M)\) is the Lie algebra of all vector fields on \(M\).

For an endomorphism field \(F\) on \(M\), its divergence \(\text{div} F\) is a vector field defined by
\[
\text{div} F = \text{tr}_g(\nabla F) = \sum_{i=1}^m (\nabla e_i F) e_i.
\]
Here \(\{e_i\}_{i=1}^m\) is a local orthonormal frame field of \((M, g)\).

One can see that the differential \(dr\) of the scalar curvature \(r\) is related to the divergence of the Ricci operator \(S\) by (1.1):
\[
(1.1) \quad dr = 2g(\text{div} S, \cdot).
\]

A Riemannian manifold \((M, g)\) is said to be locally symmetric if \(R\) is parallel, i.e., \(\nabla R = 0\). Clearly every Riemannian manifolds of constant curvature is locally symmetric. More generally \((M, g)\) is said to be semi-symmetric if \(R\) is semi-parallel, i.e., \(R \cdot R = 0\).

1.2. In case \(m = \dim M = 3\), the Riemannian curvature \(R\) is determined by the Ricci tensor \(\rho\). In fact, \(R\) is expressed as
\[
(1.2) \quad R(X, Y)Z = \rho(Y, Z)X - \rho(Z, X)Y
+ g(Y, Z)SX - g(Z, X)SY - \frac{r}{2}(X \wedge Y)Z,
\]
where \((X \wedge Y)Z\) is a curvature-like tensor field defined by
\[
(X \wedge Y)Z = g(Y, Z)X - g(Z, X)Y, \quad X, Y, Z \in \mathfrak{X}(M).
\]
The formula (1.2) implies that a Riemannian 3-manifold \((M, g)\) is locally symmetric if and only if \(R\) is semi-parallel, that is, \(R \cdot S = 0\). More generally \((M, g)\) is semi-symmetric if and only if \(S\) is semi-parallel.

2. Almost contact Riemannian manifolds

2.1. Let \(M\) be a \((2n+1)\)-dimensional manifold. An almost contact structure on \(M\) is a quadruple of tensor fields \((\varphi, \xi, \eta, g)\), where \(\varphi\) is an endomorphism field, \(\xi\) is a vector field, \(\eta\) is a one-form and \(g\) is a Riemannian metric, respectively, such that
\[
(2.1) \quad \varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1,
\]
\[
(2.2) \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad X, Y \in \mathfrak{X}(M).
\]
An $(2n + 1)$-dimensional manifold together with an almost contact structure is called an \emph{almost contact Riemannian manifold} (or \emph{almost contact metric manifold}) \cite{2}. The fundamental 2-form $\Phi$ of $M$ is defined by

$$\Phi(X, Y) = g(X, \varphi Y), \quad X, Y \in \mathfrak{X}(M).$$

If an almost contact Riemannian manifold $(M; \varphi, \xi, \eta, g)$ satisfies the condition:

$$\rho = ag + b\eta \otimes \eta$$

for some functions $a$ and $b$, then $M$ is said to be \emph{$\eta$-Einstein}.

An almost contact Riemannian manifold $M$ is said to be \emph{normal} if it satisfies $[\varphi, \varphi] + 2d\eta \otimes \xi = 0$, where $[\varphi, \varphi]$ is the Nijenhuis torsion of $\varphi$.

**Definition 2.1.** An almost contact Riemannian manifold $M$ is said to be an \emph{almost Kenmotsu manifold} if it satisfies $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$. A normal almost Kenmotsu manifold is called a \emph{Kenmotsu manifold}.

**Definition 2.2.** An almost contact Riemannian manifold $M$ is said to be an \emph{almost cosymplectic manifold} if it satisfies $d\eta = 0$ and $d\Phi = 0$. A normal almost cosymplectic manifold is called a \emph{cosymplectic manifold}.

**Definition 2.3.** An almost contact Riemannian manifold $M$ is said to be a \emph{contact Riemannian manifold} if it satisfies $d\eta = \Phi$. A normal contact Riemannian manifolds is called a \emph{Sasakian manifold}.

A tangent plane $\Pi_p$ at a point $p$ of an almost contact Riemannian manifold $M$ is said to be \emph{holomorphic} (or \emph{$\varphi$-section}) if it is invariant under $\varphi_p$. It is easy to see that a tangent plane $\Pi_p$ is holomorphic if and only if $\xi_p$ is orthogonal to $\Pi_p$. The sectional curvature $K(\Pi_p)$ of a holomorphic plane $\Pi_p$ is called the \emph{holomorphic sectional curvature} (or \emph{$\varphi$-sectional curvature}) of $M$.

### 2.2. For an arbitrary almost contact Riemannian 3-manifold $M$, we have \cite{14}:

$$\nabla_X \varphi Y = g(\varphi \nabla_X \xi, Y)\xi - \eta(Y)\varphi \nabla_X \xi.$$  

Moreover, we have

$$d\eta = \eta \wedge \nabla_\xi \eta + \alpha \Phi, \quad d\Phi = 2\beta \eta \wedge \Phi,$$

where $\alpha$ and $\beta$ are the functions defined by

$$\alpha = \frac{1}{2}\text{tr}_g(\varphi \nabla \xi), \quad \beta = \frac{1}{2}\text{tr}_g(\nabla \xi) = \frac{1}{2}\text{div} \xi.$$

Olszak \cite{14} showed that an almost contact Riemannian 3-manifold $M$ is normal if and only if $\nabla \xi \circ \varphi = \varphi \circ \nabla \xi$ or, equivalently,

$$\nabla_X \xi = -\alpha \varphi X + \beta (X - \eta(X))\xi, \quad X \in \mathfrak{X}(M).$$

We call the pair $(\alpha, \beta)$ the \emph{type} of a normal almost contact Riemannian 3-manifold $M$.  

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Using (2.4) and (2.6) we note that the covariant derivative $\nabla \varphi$ of $\varphi$ on a 3-dimensional normal almost contact Riemannian manifold is given by
\begin{equation}
(\nabla_X \varphi)Y = \alpha (g(X,Y)\xi - \eta(Y)X) + \beta (g(\varphi X,Y)\xi - \eta(Y)\varphi X).
\end{equation}
Moreover $M$ satisfies (see [3]):
\begin{equation}
2\alpha \beta + \xi(\alpha) = 0.
\end{equation}
Thus if $\alpha$ is a nonzero constant, then $\beta = 0$. In particular a Kenmotsu 3-manifold is a normal almost contact Riemannian 3-manifold of type $(0,1)$. Cosymplectic 3-manifolds are characterised as almost contact Riemannian 3-manifolds of type $(0,0)$. A Sasakian manifold is a normal almost contact Riemannian manifold of type $(1,0)$.

Next, we consider $\eta$-Einstein normal almost contact Riemannian 3-manifolds.

**Proposition 2.1.** Let $M$ be a normal almost contact Riemannian 3-manifold of type $(\alpha, \beta)$. Then $M$ is $\eta$-Einstein if and only if
\begin{equation}
g(\text{grad} \beta - \varphi \text{grad} \alpha, X) = 0
\end{equation}
for all $X \in \mathfrak{X}(M)$ orthogonal to $\xi$. In this case, the Ricci operator $S = aI + b\eta \otimes \xi$ has coefficients:
\begin{equation}
a = \frac{r}{2} + d\beta(\xi) - (\alpha^2 - \beta^2), \quad b = -\frac{r}{2} - 3d\beta(\xi) + 3(\alpha^2 - \beta^2).
\end{equation}
In particular, cosymplectic 3-manifolds, Kenmotsu 3-manifolds and Sasakian 3-manifolds are $\eta$-Einstein.

**2.3. Kenmotsu 3-manifolds.** Let $(M; \varphi, \xi, \eta, g)$ be a Kenmotsu 3-manifold. Then we have
\begin{equation}
(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X,
\end{equation}
\begin{equation}
\nabla_X \xi = X - \eta(X)\xi
\end{equation}
for all $X, Y \in \mathfrak{X}(M)$.

In particular we have $\nabla \xi \xi = 0$. Hence on Kenmotsu 3-manifolds, integral curves (trajectories) of $\xi$ are geodesics.

Every Kenmotsu 3-manifold is $\eta$-Einstein with Ricci operator
\begin{equation}
S = \frac{1}{2}(r + 2)I - \frac{1}{2}(r + 6)\eta \otimes \xi.
\end{equation}
The scalar curvature $r$ is related to the holomorphic sectional curvature function $H$ by $H = r/2 + 2$.

**Corollary 2.1.** The Riemannian curvature of a Kenmotsu 3-manifold is given by
\begin{equation}
R(X,Y)Z = \frac{r + 4}{2}(X \wedge Y)Z + \frac{r + 6}{2} [\xi \wedge ((X \wedge Y)\xi)]Z.
\end{equation}
This curvature formula implies that a Kenmotsu 3-manifold $M$ has constant scalar curvature $r = -6$ if and only if it is of constant curvature $-1$.

More generally we have:

**Proposition 2.2** (cf. [9]). A Kenmotsu 3-manifold $M$ has constant scalar curvature if and only if it is of constant curvature $-1$.

**Proof.** The divergence $\text{div}S$ is computed as

$$\text{div}S = \frac{1}{2}\text{grad}r - \frac{1}{2}dr(\xi)\xi - (r + 6)\xi.$$ 

Thus if $r$ is constant, then $r = -6$ and hence $M$ is of constant curvature $-1$. Conversely if $M$ is of constant curvature $-1$, then $r = -6$. □

From the divergence formula for $S$, we have

$$dr(\xi) = 2g(\text{div}S, \xi) = \xi(r) - \xi(r) - (r + 6) = -(r + 6).$$

Hence we obtain the following result.

**Proposition 2.3.** Let $M$ be a Kenmotsu 3-manifold. Then $M$ satisfies $dr(\xi) = 0$ if and only if $r$ is constant $-6$.

**Corollary 2.2.** A Kenmotsu 3-manifold satisfies the condition

$$\varphi^2(\nabla_{\nabla P} R)(X, Y)Z = 0$$

for all vector fields $X, Y, Z, W \in \mathfrak{X}(M)$ orthogonal to $\xi$ if and only if $M$ is of constant curvature $-1$.

**Proof.** De and Pathak [7, 8] showed that $M$ satisfies (2.10) for all $X, Y, Z, W \in \mathfrak{X}(M)$ orthogonal to $\xi$ if and only if $M$ is of constant scalar curvature. As we have seen above, $M$ is of constant scalar curvature if and only if $M$ is of constant curvature $-1$. □

Note that all the examples of Kenmotsu 3-manifold exhibited in [7, Examples 5.1, 5.2, 5.3] are of constant curvature $-1$.

### 3. $\eta$-parallelism

**3.1.** Kenmotsu [11] showed that locally symmetric Kenmotsu manifolds are of constant curvature $-1$. Thus for Kenmotsu manifolds, local symmetry is a very strong restriction. Instead of local symmetry, we study $\eta$-parallelism for the Ricci operator.

First we recall the notion of $\eta$-parallelism in the sense of Kimura and Maeda.

**Definition 3.1** (cf. [12]). An endomorphism field $P$ of an almost contact Riemannian manifold $M$ is said to be $\eta$-parallel if

$$g((\nabla X P)Y, Z) = 0$$

for all vector fields $X, Y$ and $Z$ orthogonal to $\xi$.

On the other hand Kon introduced the notion of $\eta$-parallelism as follows:
Definition 3.2 ([13]). The Ricci tensor field $\rho$ of an almost contact Riemannian manifold $M$ is said to be $\eta$-parallel if

$$(\nabla_X \rho)(\varphi Y, \varphi Z) = 0$$

for all vector fields $X$, $Y$ and $Z$ on $M$.

Now we apply these $\eta$-parallelisms on Kenmotsu 3-manifolds. By definition we have

$$(\nabla_X \rho)(\varphi Y, \varphi Z) = g(\nabla_X S)\varphi Y, \varphi Z)$$

for all $X \in \mathfrak{X}(M)$ and $Y$ and $Z$ orthogonal to $\xi$. Hence the $\eta$-parallelism of the Ricci tensor field $\rho$ on an almost contact Riemannian 3-manifold $M$ in the sense of Kon is equivalent to

$$g(\nabla_X S)Y, Z) = 0$$

for all $X \in \mathfrak{X}(M)$ and $Y$ and $Z$ orthogonal to $\xi$. Thus the $\eta$-parallelism of $\rho$ in the sense of Kon is stronger than that of $S$ in the sense of Kimura-Maeda.

To distinguish these two $\eta$-parallelisms, we call the $\eta$-parallelism in the sense of Kon by the name, “strong $\eta$-parallelism”.

4. Kenmotsu 3-manifolds with strongly $\eta$-parallel $S$

4.1. We start our discussions with $\eta$-Einstein almost contact Riemannian 3-manifolds with $\eta$-parallel Ricci operator.

Express the Ricci operator $S$ of an $\eta$-Einstein almost contact Riemannian 3-manifold $M$ as $S = aI + b\eta \otimes \xi$, then we have

$$(\nabla_X S)Y = da(X)Y + db(X)\eta(Y)\xi + b(\nabla_X \eta)Y)\xi$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

4.2. Let us assume that $M$ is a Kenmotsu 3-manifold. Then we have

$$(\nabla_X \eta)Y = g(\nabla_X \xi, Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all $X, Y \in \mathfrak{X}(M)$. Hence from (4.1),

$$(\nabla_X S)Y, Z) = da(X)g(Y, Z) + db(X)\eta(Y)\eta(Z) + b(g(X, Y) - \eta(X)\eta(Y))\eta(Z)$$

for all $X, Y, Z \in \mathfrak{X}(M)$. Next, since

$$a = \frac{1}{2}(r + 2), \quad b = -\frac{1}{2}(r + 6)$$

on Kenmotsu 3-manifolds, we get

$$g((\nabla_X S)Y, Z) = \frac{1}{2}dr(X)g(Y, Z)$$

for all $X, Y$ and $Z \in \mathfrak{X}(M)$ with $\eta(Y) = \eta(Z) = 0$.

Now we take a local orthonormal frame field $\{e_1, e_2, e_3\}$ of $M$ of the form $e_2 = \varphi e_1$, $\eta(e_1) = 0$ and $e_3 = \xi$. 
If we choose $X = Y = Z = e_i$ ($i = 1, 2$), then we get $dr(e_i) = 0$ for $i = 1, 2$. Thus $S$ is $\eta$-parallel if and only if $dr(X) = 0$ for any $X$ orthogonal to $\xi$.

**Proposition 4.1.** A Kenmotsu 3-manifold $M$ has $\eta$-parallel Ricci operator if and only if its scalar curvature satisfies $dr(X) = 0$ for any tangent vector $X$ orthogonal to $\xi$.

Next we assume that $M$ has strongly $\eta$-parallel Ricci operator, then we have

$$0 = g((\nabla_\xi S)e_i, e_i) = \frac{1}{2} dr(\xi), \quad i = 1, 2.$$  

This implies that $r = -6$. Thus we obtain an alternative proof to the following result due to De and Pathak.

**Proposition 4.2** ([8]). A Kenmotsu 3-manifold $M$ has strongly $\eta$-parallel Ricci operator if and only if $M$ is of constant curvature $-1$.

Summing up our results, we get:

**Theorem 4.1.** Let $M$ be a Kenmotsu 3-manifolds. Then the following properties are mutually equivalent:

- The scalar curvature $r$ is constant along the trajectories of $\xi$, i.e., $\xi(r) = 0$.
- The scalar $r$ is constant.
- The scalar curvature is $-6$.
- The holomorphic sectional curvature function $H$ is constant.
- The Ricci operator is strongly $\eta$-parallel.
- $M$ is locally symmetric.
- $M$ is of constant curvature $-1$.

**Remark 1.** Jun, De and Pathak showed that Kenmotsu manifolds of arbitrary odd dimension with strongly $\eta$-parallel Ricci operator has constant scalar curvature [10, Theorem 5].

In the next section we classify Kenmotsu 3-manifolds with $\eta$-parallel Ricci operator.

5. Kenmotsu 3-manifolds with $\eta$-parallel $S$

5.1. Warped products. We start with the standard examples of Kenmotsu 3-manifold.

Let $(N, h, J)$ be an oriented Riemannian 2-manifold together with the compatible orthogonal complex structure $J$. Take a direct product $M = \mathbb{E}^1(t) \times N$ of real line and $N$. We denote $\pi$ and $\sigma$ the natural projections onto the first and second factors,

$$\pi : M \to \mathbb{E}^1, \quad \sigma : M \to N,$$

respectively. On the direct product $M$, we equip a Riemannian metric $g$ defined by

$$g = dt^2 + f(t)^2 \pi^* h.$$
Here \( f \) is a positive function on \( \mathbb{E}^1(t) \). The resulting Riemannian manifold \((M, g)\) is denoted by \( \mathbb{E}^1 \times f \mathbb{N} \) and called the \textit{warped product} with base \( \mathbb{E}^1 \) and fibre \( \mathbb{N} \). The function \( f \) is called the \textit{warping function}.

On the warped product \( M = \mathbb{E}^1 \times f \mathbb{N} \), we define the vector field \( \xi \) by \( \xi = \partial / \partial t \). Then the Levi-Civita connection \( \nabla \) of \( M \) is given by (cf. [15]):

\[
\nabla_X Y = (\nabla_X Y)_v - \frac{1}{f} g(\nabla_X \xi, Y)f' \xi,
\]

\[
\nabla_\xi X = \nabla_X \xi = f' f X_v,
\]

\[
\nabla_\xi \xi = 0.
\]

Here the superscript \( v \) means the vertical lift operation of vector fields from \( \mathbb{N} \) to \( M \). Define an endomorphism field \( \phi \) on \( M \) by \( \phi X = \{ J(\sigma^* X) \}_v \). Then we get

\[
\nabla_X \xi = \beta(X - \eta(X)\xi),
\]

\[
(\nabla_X \varphi) Y = \beta\{ g(\varphi X, Y) - \eta(Y)\varphi X \}, \quad \beta = f'/f.
\]

Hence \( M = \mathbb{E}^1 \times f \mathbb{N} \) is a normal almost contact Riemannian 3-manifold of type \( (0, \beta) \). In particular, \( \mathbb{E}^1 \times f \mathbb{N} \) is a Kenmotsu manifold if and only if \( f(t) = ce^t \) for some positive constant \( c \). Take a local orthonormal frame field \( \{ \bar{e}_1, \bar{e}_2 \} \) of \( (\mathbb{N}, h) \) such that \( \bar{e}_2 = J\bar{e}_1 \). Then we obtain a local orthonormal frame field \( \{ e_1, e_2, e_3 \} \) by

\[
e_1 = \frac{1}{f} \bar{e}_1, \quad e_2 = \frac{1}{f} \bar{e}_2 = \varphi e_1, \quad e_3 = \xi.
\]

Then sectional curvatures of \( M \) are given by

\[
K(e_1 \wedge e_2) = \frac{1}{f^2} \{ \kappa - (f')^2 \}, \quad K(e_1 \wedge e_3) = K(e_2 \wedge e_3) = -\frac{f''}{f},
\]

where \( \kappa \) is the Gaussian curvature of \( \mathbb{N} \). The components \( \rho_{ij} = \rho(e_i, e_j) \) of the Ricci tensor field are given by

\[
\rho_{11} = \rho_{22} = \frac{\kappa}{f^2} - \frac{f''}{f} - \left( \frac{f'}{f} \right)^2, \quad \rho_{33} = -2 \frac{f''}{f}.
\]

Now we assume that \( M \) is a Kenmotsu manifold, that is, we choose \( f(t) = ce^t \), then we have

\[
\rho_{11} = \rho_{22} = \frac{\kappa}{ce^{2t}} - 2, \quad \rho_{33} = -2.
\]

Thus we have

\[
r = \frac{2\kappa}{c^2e^{2t}} - 6.
\]
5.2. The local structure of Kenmotsu manifolds is described as follows.

**Lemma 5.1** ([11]). A Kenmotsu 3-manifold $M$ is locally isomorphic to a warped product $I \times f N$ whose base $I \subset \mathbb{R}^1(t)$ is an open interval, $N$ is a surface and warping function $f(t) = ce^t$, $c > 0$. The structure vector field is $\xi = \partial/\partial t$.

Now let $M$ be a Kenmotsu 3-manifold and take a local warped product representation $I \times f N$.

Take a local isothermal coordinates $(x, y)$ on $N$ and represent $h$ as $h = e^{\omega}(dx^2 + dy^2)$. Then

$$\bar{e}_1 = e^{-\omega/2} \frac{\partial}{\partial x}, \quad \bar{e}_2 = e^{-\omega/2} \frac{\partial}{\partial y}.$$  

Thus we have that $S$ is $\eta$-parallel if and only if $\kappa_x = \kappa_y = 0$, that is, $\kappa$ is constant. Under the constancy of $\kappa$, $d\tau(\xi) = 0$ holds if and only if $\kappa = 0$. In this case $M$ is of constant curvature $-1$.

**Theorem 5.1.** A Kenmotsu 3-manifold has $\eta$-parallel Ricci operator if and only if it is locally isomorphic to the warped product $\mathbb{E}^1 \times ce^t N$, where $N$ is of constant curvature.

Thus the global warped products

$$\mathbb{E}^1 \times ce^t S^2(\kappa), \quad \mathbb{E}^1 \times ce^t H^2(\kappa)$$

are Kenmotsu 3-manifolds whose Ricci operator is $\eta$-parallel but not strongly $\eta$-parallel.

**Remark 2.** In [6], Cho and Kimura showed that a Kenmotsu 3-manifold $M$ satisfies $\ell_{\xi}S = 0$ if and only if $M$ is of constant curvature $-1$. They also showed that Kenmotsu 3-manifolds whose Ricci operator is parallel along the characteristic flow (i.e., $\nabla_{\xi}S = 0$) are of constant curvature $-1$. Recently Cho classified locally symmetric almost Kenmotsu 3-manifolds [5].

**Problem 5.1.**

1. Classify almost Kenmotsu 3-manifolds with $\eta$-parallel Ricci operator.
2. Classify Kenmotsu 3-manifolds with semi $\eta$-parallel Ricci operator, i.e.,

$$g((R(X, Y)S)Z, W) = 0$$

for all vector fields $X$, $Y$, $Z$ and $W$ orthogonal to $\xi$.

6. Cosymplectic 3-manifolds

In this section we study cosymplectic 3-manifolds with $\eta$-parallel Ricci operator. On a cosymplectic 3-manifold $M$, we have

$$\nabla\varphi = 0, \quad \nabla\xi = 0.$$  

In particular we have $\nabla_{\xi}\xi = 0$. Hence on cosymplectic 3-manifolds, integral curves (trajectories) of $\xi$ are geodesics.
Example 6.1. Let \((N, h, J)\) be an oriented Riemannian 2-manifold with the compatible complex structure \(J\). On the direct product manifold \(M = N \times E^1\), we equip the product metric \(g = \pi^* h + dt^2\). Here \(\pi : M \to N\) is the natural projection. Define the endomorphism field \(\varphi\) on \(M\) by
\[
\varphi X = \{J\pi_X\}^h,
\]
where \(h\) is the horizontal lift operation. Define the vector field \(\xi\) and the 1-form \(\eta\) by \(\xi = \partial/\partial t\) and \(\eta = dt\). Then the resulting almost contact Riemannian 3-manifold \((M, \varphi, \xi, \eta, g)\) is cosymplectic.

The local structure of cosymplectic 3-manifolds is described as follows.

**Lemma 6.1 ([4, Lemma 2]).** A cosymplectic 3-manifold \(M\) is locally isomorphic to the Riemannian product \(N \times I\) whose base \(N = (N, h)\) is a Riemannian 2-manifold. The standard fibre \(I\) is an open interval with coordinate \(t\). The metric is \(g = \pi^* h + dt^2\), where \(\pi : N \times I \to N\) is the natural projection. The structure vector field is \(\xi = \partial/\partial t\).

Every cosymplectic 3-manifold is \(\eta\)-Einstein with Ricci operator
\[
S = \frac{r}{2}I - \frac{r}{2}\eta \otimes \xi.
\]
The holomorphic sectional curvature function \(H\) is given by \(H = r/2\).

**Corollary 6.1.** The Riemannian curvature of a cosymplectic 3-manifold is given by
\[
R(X, Y)Z = \frac{r}{2}\{\xi \wedge \{(X \wedge Y)\}\}Z.
\]

Using this formula, the covariant derivative of \(S\) is computed as
\[
(\nabla_X S)Y = \frac{1}{2}dr(X)(Y - \eta(Y)\xi).
\]
The divergence \(\text{div} S\) is computed as
\[
\text{div} S = \frac{1}{2}(\text{grad} r - \eta(\text{grad} r)\xi).
\]
This implies the formula
\[
dr(X) = g(\text{grad} r, X) - \eta(\text{grad} r)\eta(X).
\]
Equivalently,
\[
dr = dr - \eta(\text{grad} r)\eta
\]
From this formula we have \(\xi(r) = 0\). This implies that \(\text{div} S = \text{grad} r/2\) and \(\nabla\xi S = 0\).

Now let us consider cosymplectic 3-manifolds with \(\eta\)-parallel Ricci operator.

If we assume that \(\eta(Y) = \eta(Z) = 0\), in (6.1), we obtain
\[
g((\nabla_X S)Y, Z) = \frac{1}{2}dr(X)g(Y, Z) = 0
\]
for all \(X \in \mathfrak{X}(M)\).
Now we take a local orthonormal frame field \( \{ e_1, e_2, e_3 \} \) of the form \( e_2 = \varphi e_1 \), \( \eta(e_1) = 0 \) and \( e_3 = \xi \). If we choose \( X = Y = Z = e_i \) in (6.1) for \( i = 1, 2 \), then we get \( dr(e_i) = 0 \) for \( i = 1, 2 \). Thus \( S \) is \( \eta \)-parallel if and only if \( dr(X) = 0 \) for any vector field \( X \) orthogonal to \( \xi \). Since we know that \( dr(\xi) = 0 \), \( S \) is \( \eta \)-parallel if and only if \( S \) is strongly \( \eta \)-parallel.

**Proposition 6.1.** Let \( M \) be a cosymplectic 3-manifold with Ricci operator \( S \). Then \( S \) is \( \eta \)-parallel if and only if \( r \) is constant. In such a case \( S \) is strongly \( \eta \)-parallel.

Since the holomorphic sectional curvature function \( H \) is related to \( r \) by \( H = r/2 \), the \( \eta \)-parallelism of \( S \) is equivalent to the constancy of \( H \).

**Corollary 6.2.** Let \( M \) be a cosymplectic 3-manifold with Ricci operator \( S \). Then \( S \) is \( \eta \)-parallel if and only if \( M \) has constant holomorphic sectional curvature.

Let \( (N, h) \) be an oriented Riemannian 2-manifold and \( M = N \times E^1 \) the direct product with product metric. We equip the natural cosymplectic structure on \( M \). Then the scalar curvature \( r \) of \( M \) is \( r = 2\kappa \). Here \( \kappa \) is the Gaussian curvature. Hence \( S \) is \( \eta \)-parallel if and only if \( \kappa \) is constant.

Thus \( S^2(\kappa) \times E^1 \) and \( \mathbb{H}^2(\kappa) \times E^1 \) are non-constant curvature cosymplectic manifolds with \( \eta \)-parallel Ricci operator.

**Theorem 6.1.** Let \( M \) be a cosymplectic 3-manifold. Then the following properties are mutually equivalent:

- The scalar curvature is constant.
- The holomorphic sectional curvature function \( H \) is constant.
- The Ricci operator is \( \eta \)-parallel.
- The Ricci operator is strongly \( \eta \)-parallel.
- \( M \) is locally symmetric.

### 7. Sasakian 3-manifolds

For a Sasakian 3-manifold \( M \), we have

\[
S = aI + b\eta \otimes \xi, \quad a = \frac{1}{2}(r - 2), \quad b = \frac{1}{2}(6 - r).
\]

The holomorphic sectional curvature is \( H = r/2 - 2 \). Hence we get

\[
\text{div} S = \frac{1}{2}(\text{grad} r - dr(\xi)\xi).
\]

This formula implies \( dr = dr - dr(\xi)\eta \). Hence we have \( dr(\xi) = 0 \).

Moreover we have

\[
g((\nabla_X S)Y, Z) = \frac{1}{2}dr(X)g(Y, Z)
\]

for all \( X \in \mathfrak{X}(M) \) and \( Y, Z \) orthogonal to \( \xi \). The Ricci operator \( S \) on a Sasakian 3-manifold \( M \) is \( \eta \)-parallel if and only if \( dr(X) = 0 \) for all \( X \) orthogonal to \( \xi \). Since \( \xi(r) = 0 \) holds on every Sasakian 3-manifold, we have the following result.
Proposition 7.1. The following properties are mutually equivalent for Sasakian 3-manifolds.

• The scalar curvature $r$ is constant.
• The holomorphic sectional curvature $H$ is constant.
• The Ricci operator is $\eta$-parallel.
• The Ricci operator is strongly $\eta$-parallel.

Thus 3-dimensional Sasakian space forms are examples of Sasakian 3-manifolds with strongly $\eta$-parallel Ricci operator. As is well known Sasakian 3-manifold is locally symmetric if and only if it is of constant curvature 1.

Among the three classes (cosymplectic, Kenmotsu, Sasakian), only for the class of Kenmotsu 3-manifolds, $\eta$-parallelism of $S$ is weaker than the strong $\eta$-parallelism of $S$.

Remark 3. Sasakian manifolds $\text{Nil}_3 = \mathbb{R}^3(-3)$ and $\widetilde{\text{SL}}_2\mathbb{R}$, cosymplectic 3-manifolds $S^2(\kappa) \times E^1$ and $H^2(\kappa) \times E^1$ are model spaces of Thurston geometries. Moreover these space are included in the 2-parameter family of homogeneous Riemannian spaces referred as to the Bianchi-Cartan-Vranceanu family, see [1].

References

[1] M. Belkhelfa, F. Dillen, and J. Inoguchi, Surfaces with parallel second fundamental form in Bianchi-Cartan-Vranceanu spaces, in: PDE’s, Submanifolds and Affine Differential Geometry (Warsaw, 2000), pp. 67–87, Banach Center Publ. 57, Polish Acad. Sci., Warsaw, 2002.
[2] D. E. Blair, Riemannian Geometry of Contact and Symplectic Manifolds, Progress in Mathematics, 203, Birkhäuser Boston, Inc., Boston, 2002.
[3] D. E. Blair and J. A. Oubiña, Conformal and related changes of metric on the product of two almost contact metric manifolds, Publ. Math. 34 (1990), no. 1, 199–207.
[4] D. E. Blair and L. Vanhecke, Symmetries and $\varphi$-symmetric spaces, Tôhoku Math. J. 39 (1987), no. 3, 373–383.
[5] J. T. Cho, Local symmetry on almost Kenmotsu three-manifolds, Hokkaido Math. J. 45 (2016), no. 3, 435–442.
[6] J. T. Cho and M. Kimura, Reeb flow symmetry on almost contact three-manifolds, Differential Geom. Appl. 35 (2014), 266–273.
[7] U. C. De, On $\Phi$-symmetric Kenmotsu manifolds, Int. Electron. J. Geom. 1 (2008), no. 1, 33–38.
[8] U. C. De and G. Pathak, On 3-dimensional Kenmotsu manifolds, Indian J. Pure Appl. Math. 35 (2004), no. 2, 159–165.
[9] J. Inoguchi, A note on almost contact Riemannian 3-manifolds, Bull. Yamagata Univ. Natur. Sci. 17 (2010), no. 1, 1–6.
[10] J.-B. Jun, U. C. De, and G. Pathak, On Kenmotsu manifolds, J. Korean Math. Soc. 42 (2005), no. 3, 435–445.
[11] K. Kenmotsu, A class of almost contact Riemannian manifolds, Tôhoku Math. J. 24 (1972), 93–103.
[12] M. Kimura and S. Maeda, On real hypersurfaces of a complex projective space, Math. Z. 202 (1989), no. 3, 299–311.
[13] M. Kon, Invariant submanifolds in Sasakian manifolds, Math. Ann. 219 (1976), no. 3, 277–290.
[14] Z. Olszak, Normal almost contact metric manifolds of dimension three, Ann. Polon. Math. 47 (1986), 42–50.
[15] B. O’Neill, Semi-Riemannian Geometry with Application to Relativity, Academic Press, Orlando, 1983.

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