Palatini formulation of modified gravity with squared scalar curvature

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Abstract

In this paper we derive the Modified Friedmann equation in the Palatini formulation of $R^2$ gravity. Then we use it to discuss the problem of whether in Palatini formulation a $R^2$ term can drive an inflation. We show that the Palatini formulation of $R^2$ gravity cannot lead to gravity-driven inflation. If considering no zero radiation and matter energy densities, we show that only under rather restrictive assumption about the radiation and matter energy densities there will be a mild power-law inflation $a \sim t^2$. 

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1. Introduction

The expansion of our universe is currently in an accelerating phase now seems well-established [1]. But now the mechanism responsible for this is not very clear. Many authors introduce a mysterious cosmic fluid called dark energy to explain this. See Ref. [2] for a review and Ref. [3] for some recent models. On the other hand, some authors suggested that maybe there does not exist such a mysterious dark energy, but the observed cosmic acceleration is a signal of our first real lack of understanding of gravitational physics [4, 6]. An example is the braneworld theory of Dvali et al. [5].

Recently, some authors proposed to add a $1/R$ term in the Einstein-Hilbert action to modify the General Relativity (GR) [6, 7]. It is interesting that such a term may be predicted by string/M-theory [8]. In the metric formulation, this additional term will give fourth order field equations. It has been shown that this additional term can give accelerating expansion without dark energy [6]. In this framework, Dick [9] considered the problem of weak field approximation. Soussa and Woodard [10] have considered the gravitational response to a diffuse source.

Based on this modified action, Vollick [11] has used Palatini variational principle to derive the field equations. In the Palatini formulation, instead of varying the action only with respect to the metric, one views the metric and connection as independent field variables and vary the action with respect to them independently. This would give second order field equations. For the original Einstein-Hilbert action, this approach gives the same field equations as the metric variation. For a more general action, those two formulations are inequivalent, they will lead to different field equations and thus describe different physics [12]. Flanagan [13] derived the equivalent scalar-tensor description of the Palatini formulation. In Ref. [14], Dolgov and Kawasaki have argued that the fourth order field equations in metric formulation suffer serious instability problem. If this is indeed the case, the Palatini formulation appears even more appealing, because the second order field equations in Palatini formulation are free of this sort of instability [15]. Furthermore, Chiba [16] has argued that the theory derived using metric variation is in conflict to the solar system experiments while in the Palatini formulation, the model seems to be compatible with solar system experiment [17]. Moreover, a convincing motivation to take the Palatini formalism seriously is
that the Modified Friedmann (MF) equation following from it fit the SNe Ia data at an acceptable level [15].

On the other end of cosmic evolution time, the very early stage, it is now generally believed that the universe also undergoes an acceleration phase called inflation. The mechanism driven inflation is also not very clear now. The most popular explanation is that inflation is driven by some inflaton field [18]. Also, some authors suggest that modified gravity could be responsible for inflation [19, 20]. Revealing the mechanisms of inflation is also one of the most important problems in modern cosmology.

As originally proposed by Carroll et al. [6] and later implemented by Nojiri and Odintsov [20], adding correction term $R^m$ with $m > 0$ in addition to the $1/R$ term may explain both the early time inflation and current acceleration without inflaton and dark energy. Furthermore, Nojiri and Odintsov [20] showed that adding a $R^m$ term can avoid the above mentioned instability when considering the theory in metric formulation. In this paper, we will show that in the Palatini formulation, the $R^2$ term cannot lead to a gravity-driven inflation, in opposite to the conclusion when considering the theory in metric formulation [19].

This paper is arranged as follows: in Sec.2 we review the framework of deriving field equations and Modified Friedmann (MF) equations in Palatini formulation; in Sec.3 we discuss the $R^2$ gravity in Palatini formulation and inflation in this model; Sec.4 is devoted to conclusions and discussions.

2. Deriving the Modified Friedmann equation in Palatini formulation

First, we briefly review deriving field equations from a generalized Einstein-Hilbert action of the form $L(R)$ by using Palatini variational principle. See Refs. [15] for details.

The field equations follow from the variation in Palatini approach of the generalized Einstein-Hilbert action

$$ S = -\frac{1}{2\kappa^2} \int d^4x \sqrt{-g} L(R) + S_M, $$

where $\kappa^2 = 8\pi G$, $L$ is a function of the scalar curvature $R$ and $S_M$ is the matter action.

Varying with respect to $g_{\mu\nu}$ gives

$$ L'(R) R_{\mu\nu} - \frac{1}{2} L(R) g_{\mu\nu} = \kappa^2 T_{\mu\nu}, $$

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where a prime denotes differentiation with respect to \( R \) and \( T_{\mu\nu} \) is the energy-momentum tensor given by

\[
T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g_{\mu\nu}}. \tag{3}
\]

We assume the universe contains dust and radiation, thus \( T^\mu_{\nu} = \{-\rho_m - \rho_r, p_r, p_r, p_r, p_r\} \) where \( \rho_m \) and \( \rho_r \) are the energy densities for dust and radiation respectively, \( p_r \) is the pressure of the radiation. Note that \( T = g^{\mu\nu} T_{\mu\nu} = -\rho_m \) because of the relation \( p_r = \rho_r / 3 \).

In the Palatini formulation, the connection is not associated with \( g_{\mu\nu} \), but with \( h_{\mu\nu} \equiv L'(R) g_{\mu\nu} \), which is known from varying the action with respect to \( \Gamma^\lambda_{\mu\nu} \). Thus the Christoffel symbol with respect to \( h_{\mu\nu} \) is given by

\[
\Gamma^\lambda_{\mu\nu} = \{_{\lambda}^{\mu}_{\nu}\}_g + \frac{1}{2L'}[2\delta^\lambda_{(\mu} \partial_{\nu)}L' - g_{\mu\nu} g^{\lambda\sigma} \partial_{\sigma}L'], \tag{4}
\]

where the subscript \( g \) signifies that this is the Christoffel symbol with respect to the metric \( g_{\mu\nu} \).

The Ricci curvature tensor is given by

\[
R_{\mu\nu} = R_{\mu\nu}(g) + \frac{3}{2}(L')^{-2} \nabla_{\mu} L' \nabla_{\nu} L' - (L')^{-1} \nabla_{\mu} \nabla_{\nu} L' - \frac{1}{2} (L')^{-1} g_{\mu\nu} \nabla_{\sigma} \nabla_{\sigma} L', \tag{5}
\]

and

\[
R = R(g) - 3(L')^{-1} \nabla_{\mu} L' + \frac{3}{2} (L')^{-2} \nabla_{\mu} L' \nabla^{\mu} L' , \tag{6}
\]

where \( R_{\mu\nu}(g) \) is the Ricci tensor with respect to \( g_{\mu\nu} \), \( R = g^{\mu\nu} R_{\mu\nu} \) and \( \nabla \) is the connection associated to \( g_{\mu\nu} \). Note by contracting \( \{2\} \), we get:

\[
L'(R) R - 2L(R) = \kappa^2 T. \tag{7}
\]

Assume we can solve \( R \) as a function of \( T \) from \( \{7\} \). Thus \( \{5\}, \{9\} \) do define the Ricci tensor with respect to \( h_{\mu\nu} \).

Then let’s derive the MF equation in Palatini formulation. Let us work with the Robertson-Walker metric,

\[
ds^2 = -dt^2 + a(t)^2 (dx^2 + dy^2 + dz^2). \tag{8}
\]

Note that we only consider a flat metric, which is favored by present observations \[1\].
From (8), (5), we can get the non-vanishing components of the Ricci tensor:

\[ R_{00} = -3\ddot{a} + \frac{3}{2}(L')^{-2}(\partial_0 L')^2 - \frac{3}{2}(L')^{-1}\bar{\nabla}_0 \bar{\nabla}_0 L' , \] (9)

\[ R_{ij} = [a\ddot{a} + 2\dot{a}^2 + (L')^{-1}\{0\}_{ij} g \partial_0 L' + \frac{a^2}{2}(L')^{-1}\bar{\nabla}_0 \bar{\nabla}_0 L']\delta_{ij} . \] (10)

Substituting equations (9) and (10) into the field equations (2), we can get

\[ 6H^2 + 3H(L')^{-1}\partial_0 L' + \frac{3}{2}(L')^{-2}(\partial_0 L')^2 = \kappa^2 (\rho + 3p) + L , \] (11)

where \( H \equiv \dot{a}/a \) is the Hubble parameter, \( \rho \) and \( p \) are the total energy density and total pressure respectively. Assume that we can solve \( R \) in term of \( T \) from Eq.(7), substitute it into the expressions for \( L' \) and \( \partial_0 L' \), we can get the MF equation.

In this paper, we will consider the Palatini formulation of the modified action with a \( R^2 \) term,

\[ L = R + \frac{R^2}{3\beta} , \] (12)

where \( \beta \) is a constant having the dimension of \((mass)^2\). This action has been studied by Starobinsky in metric formulation [19] and it was shown that a gravity-driven inflation can be achieved.

3. Palatini formulation of \( R^2 \) gravity

The field equations follow by substituting Eq.(12) into Eq.(2)

\[ (1 + 2\frac{R}{3\beta})R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R + \frac{R^2}{3\beta}) = \kappa^2 T_{\mu\nu} . \] (13)

Contracting indices gives

\[ R = -\kappa^2 T = \kappa^2 \rho_m . \] (14)

The second equality follows because the radiation has vanishing trace of momentum-energy tensor. This equation is quite remarkable, since it is formally the same as the one given by GR, with only one difference: \( R_{\mu\nu} \) is associated with the conformal transformed metric \( h_{\mu\nu} = L'(R)g_{\mu\nu} \) and \( R = g^{\mu\nu}R_{\mu\nu} \).

From the conservation equation \( \dot{\rho}_m + 3H\rho_m = 0 \) and Eq.(14), we can find that

\[ \partial_0 L' = -2\frac{\kappa^2 \rho_m}{\beta}H . \] (15)
Substituting this into Eq.(11) we can get the Modified Friedmann equation for the $R^2$ gravity:

$$H^2 = \frac{2\kappa^2(\rho_m + \rho_r) + \frac{(\kappa^2\rho_m)^2}{3\beta}}{(1 + 2\frac{\kappa^2\rho_m}{3\beta})(6 + 3F_0(\frac{\kappa^2\rho_m}{\beta})(1 + \frac{1}{2}F_0(\frac{\kappa^2\rho_m}{\beta})))},$$

(16)

where the function $F_0$ is given by

$$F_0(x) = -\frac{2x}{1 + \frac{2}{3}x}.$$  

(17)

It is interesting to see from Eq.(16) that all the effects of the $R^2$ term are determined by $\rho_m$. If $\rho_m = 0$, Eq.(16) simply reduces to the standard Friedmann equation.

Now let’s come to the discussion of inflation. To begin with, note that in the metric formulation of the $R^2$ gravity, inflation is driven by the vacuum gravitational field, i.e. we assume that the radiation and matter energy densities is zero during inflation, thus called ”gravity-driven” inflation. However, in the Palatini formulation, when the radiation and matter energy densities is zero, it can be seen directly from Eq.(16) that the expansion rate will be zero and thus no inflation will happen. Thus, in the Palatini formulation of $R^2$ gravity, we cannot have a gravity-driven inflation. So the only hope that the $R^2$ term can drive an inflation without an inflaton field is that the relationship between the expansion rate and the energy density of radiation and matter will be changed which can lead to inflation (thus what we are talking now is similar to the ”Cardassian” scenario of Freese and Lewis [27]: the current accelerated expansion of the universe is driven by the changed relationship between the expansion rate and matter energy density). We will see that naturally there will be no inflation and a power-law inflation can happen only under specific assumption on $\rho_m$ and $\rho_r$.

First, in typical model of $R^2$ inflation, $\beta$ is often taken to be the order of the Planck scale [19]. This is also the most natural value of $\beta$ from an effective field point of view. Thus we naturally have $\kappa^2\rho_m/\beta \ll 1$. Under this condition, it can be seen that from Eq.(17), we have $F_0 \sim 0$, and the MF equation (16) reduces to the standard Friedmann equation:

$$H^2 = \frac{\kappa^2}{3}(\rho_m + \rho_r).$$

(18)

Thus it is obvious that in this case there will be no inflation. Also note that from the BBN constraints on the Friedmann equation [21], $\beta$ should be sufficiently large so that
the condition $\kappa^2 \rho_m / \beta \ll 1$ is satisfied at least in the era of BBN. Thus we conclude that in the most natural case, Palatini formulation of $R^2$ gravity cannot lead to inflation.

Second, let’s assume that in the very early universe, we have $\kappa^2 \rho_m / \beta \gg 1$. In this case, from Eq. (17), the MF equation (16) will reduce to

$$H^2 = \frac{\kappa^2 \rho_m}{21} + \frac{2\beta \rho_r}{7\rho_m} + \frac{2\beta}{7}.$$  (19)

Then we can see that if the $\beta$ term could dominate over the other two terms, it would drive an exponential expansion by the effective cosmological constant $\beta$. But note that this equation is derived under the assumption that $\beta \ll \kappa^2 \rho_m$. Thus inflation cannot be driven by the $\beta$ term. On the other hand, if we assume further that $\rho_r \gg \kappa^2 \rho_m^2 / \beta$, i.e. the second term dominates in the MF equation (19), then from the relation $\rho_r \propto a^{-4}$ and $\rho_m \propto a^{-3}$, the MF equation (19) can be solved to give $a \propto t^2$. Thus, only in this case, we can get a mild power-law inflation. However, current constraint on the rate of power-law inflation reads $p > 21$ where $a \propto t^p$ (see, e.g., Ref. [28]). So this case is not a viable model of inflation.

4. Conclusions and discussions

In summary, in the Palatini formulation, the modified gravity theory with a $R^2$ correction term would not lead to a gravity-driven inflation, in opposite to the conclusion when considering the theory in the metric formulation. And only under the conditions that $\kappa^2 \rho_m / \beta \gg 1$ and $\rho_r \gg \kappa^2 \rho_m^2 / \beta$ we can get a power-law inflation $a \propto t^2$. The difference of those two formulations is now quite obvious. At present, we still can not tell which formulation is physical. But this makes those results more interesting. It is conceivable that quantum effects of the $R^2$ theory in Palatini formulation would also be different from the metric formulation (see Ref. [23] for a review). Such higher derivative terms similar to the $R^2$ term may be induced by quantum effects such as trace anomaly [23, 24]. It has been recently shown [24] that phantom cosmology implemented by trace anomaly induced terms also admits both early time inflation and late time cosmic acceleration. It follows from our consideration that $R^2$ term in Palatini formulation do not support gravity-driven inflation, then we expect that also in phantom cosmology with quantum effects in Palatini formulation, the inflation does not occur.

There are many activities in the study of quantum versions of $R^2$ gravity which seems to be a multiplicatively renormalizable theory (for a review, see Ref. [23]). However,
such a theory has had a serious problem: possible non-unitarity due to the presence of higher derivative terms. It is very promising that in Palatini formulation higher derivative terms do not play such a role as in metric formulation. We expect that the unitarity problem of $R^2$ gravity may be resolved in Palatini formulation. This deserves further investigation.

Finally, it is interesting to explore the $R^2$ correction to the chaotic inflation scenario in Palatini formulation. When written in Einstein frame and in metric formulation, this will correspond to two scalar field inflation; in the Palatini formulation, the model will correspond to a type of k-inflation. More detailed investigations of this idea will be interesting topic of future works.

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