Side-Constrained Dynamic Traffic Equilibria

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We study dynamic traffic assignment with side-constraints. We first give a counter-example to a key result from the literature regarding the existence of dynamic equilibria for volume-constrained traffic models in the classical linear edge-delay model. Our counter-example shows that the feasible flow space need not be convex and it further reveals that classical infinite dimensional variational inequalities are not suited for the definition of general side-constrained dynamic equilibria. We propose a new framework for side-constrained dynamic equilibria based on the concept of feasible $\varepsilon$-deviations of flow particles in space and time. We then show under which assumptions the resulting equilibria can still be characterized by means of quasi-variational and variational inequalities, respectively. Finally, we establish first existence results for side-constrained dynamic equilibria for the non-convex setting of volume-constraints.
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1. Introduction

Traffic assignment problems have been successfully applied in the past decades in order to model, predict and optimize traffic distributions. While in most models, the network infrastructure is equipped with capacities, it is usually assumed that the excess of capacity is possible and leads to congestion, that is, increased travel times, e.g., by increased waiting times in queues. In several realistic scenarios, however, there are also hard capacity constraints that must not be violated by any feasible traffic flow. For instance, hard traffic volume restrictions are imposed by local authorities in order to keep the exhaust gas emissions within urban residential areas below certain threshold values, see Grote et al. [18]. Following Zhong et al. [45], another example includes tunnels, in which the number of vehicles inside the tunnel is limited to maintain sufficient reserve capacity/space for handling any possible incident (e.g., car accidents or disruptions due to disaster).

From a theoretical and computational perspective, the traffic assignment problem with hard side-constraints has been studied extensively for static flows using methods from convex optimization, see [25, 33, 31, 35] and references therein. These works mostly considered edge-capacity constraints and studied the optimization problem minimizing the Beckmann-McGuire-Winsten potential subject to these constraints. The dual variables associated with the capacity constraints are used as additional prices or queueing delays and the optimal solutions are interpreted as unconstrained Wardrop equilibria wrt. generalized travel costs consisting of the actual delay plus the dual prices along a path. This way, using the convexity of both the feasible space and the objective function, such special capacitated equilibria can be completely characterized as solutions to associated variational inequalities. Most works cited above, however, used the solutions of the convex optimization problem or, equivalently, solutions to the associated variational inequalities as the definition of a capacitated user (or Wardrop) equilibrium. Only a few works noted a conceptual gap between introducing a behavioural equilibrium concept in the sense of defining an associated noncooperative game versus using solutions to a variational inequality formulation as its definition – see the discussions in Correa et al. [7, pp.968], Bernstein and Smith [1] and Marcotte et al. [35].

In this article, we revisit the dynamic traffic assignment problem with side-constraints. Based on the path-delay operator model in the $L^2$-function space introduced in seminal works by Friesz et al. [15, 22] (see also Friesz and Han [13] for a recent overview), we consider general side-constraints ranging from edge-volume constraints and path inflow constraints to abstract constraint sets. While the dynamic traffic assignment problem with side-constraints is far less explored compared to the static variant, there are a few works studying fundamental questions related to the existence, structure and computability of constrained dynamic equilibria. One of the central works in this area is the paper by Zhong, Sunalee, Friesz and Lam [45], who were the first to consider side-constraints within the general path-delay operator model. They assumed a fixed flow volume and flexible departure time choice and instantiated the network loading using (linear) volume-delay functions. The side-constraints were given by arc-volume constraints. They defined a side-constrained dynamic user equilibrium via solutions of an associated infinite-dimensional variational inequality (VI) which needs to be solved over the space of capacity-feasible dynamic flows. As one of their main results (Prop. 3.1., pp. 1040), they claimed the existence of side-constrained dynamic user equilibria arguing that the respective VI always admits a solution. The proof of this claim uses that the capacity-feasible dynamic flow space is bounded, closed and convex. Only these properties would allow the invocation of general existence results for VI’s in appropriate function spaces by Browder [3].
1.1. Our Contribution

We study dynamic traffic assignment problems using the general path-delay-operator form as proposed by Friesz et al. [15] and augment this model with side-constraints. In fact, we will consider a more general general walk-delay-operator model, because some interesting side-constraints such as energy-constraints for electric vehicles (see Graf et al. [16]) require cyclic routes. Our contribution consists of four types of results.

1. We first show that the claim of existence of side-constrained dynamic user equilibria defined as solutions to a VI (Zhong, Sumalee, Friesz and Lam [45], (Prop. 3.1., pp. 1040)) is wrong – we give a nontrivial counterexample to this claim. The underlying reason lies in the fact that the side-constrained dynamic flow space need not be convex in general. The consequences of the counter-example are somewhat severe since not only does the assumed existence result break down but, perhaps more seriously, the counterexample reveals that the proposed VI is in fact not a suitable definition of a side-constrained dynamic equilibrium.

2. We introduce – in line with prior works for the static flow model (e.g., Correa et al. [7, pp.968], Bernstein and Smith [1] and Marcotte et al. [35]) – a behavioral equilibrium concept via formally introducing a noncooperative game modeling the space of feasible deviations of users given a dynamic flow. Roughly speaking, a dynamic flow is an equilibrium, if there is no arbitrarily small bundle of users that can switch their strategy in space and time and strictly reduce their travel cost. The precise way a feasible deviation is defined leads to a whole set of equilibrium concepts and we propose several of them including dynamic extensions of Larsson-Patriksson (LP), Bernstein-Smith (BS) and Marcotte-Nguyen-Schoeb (MNS) equilibria, respectively, which were originally proposed for static equilibrium flows.

3. Given an equilibrium concept for side-constrained dynamic equilibrium flows, obvious questions related to their characterization, existence and computability arise. Under mild assumptions on the structure of the side-constraints and the set of feasible deviations, we give necessary and sufficient conditions under which an equilibrium can be described as a solution to an associated quasi-variational inequality or variational inequality, respectively. We further show that equilibrium solutions exist, if the set of side-constrained dynamic flows is convex and the walk-delay operator is sequentially weak-strong continuous.

4. As the counter-example to Zhong et al. [45] suggests, the model with edge-volume constraints can lead to non-convex flow spaces which means that standard existence tools from the infinite dimensional VI theory cannot be used. For modelling volume constraints, we first describe a network loading model and then introduce abstract edge-load functions which include flow volumes as a special case. We show existence of dynamic LP and MNS equilibria under mild continuity assumptions on the edge-load functions. Our existence proof is in some sense constructive as it uses an augmented Lagrangian function approach (see [31] for such an approach for static flows) for violated edge-load constraints and invokes, in a black-box fashion, solutions to the relaxed equilibrium problem. We further show, however, that this augmented Lagrangian approach fails for other, stricter equilibrium concepts such as the dynamic BS equilibrium: we give an example in which the flows for the unconstrained model with penalties do not converge to a (strong) dynamic BS equilibrium.
Finally, it is worth mentioning that our model and the subsequent characterization and existence results require only mild continuity properties of the walk-delay operator and the edge load functions, respectively, and thus apply for several realistic and well-studied network loading models including the Vickrey queueing model with point queues [5, 6, 30, 37, 44], with spillback [41] with departure-time choice [11, 22], the Lighthill-Whitham-Richards (LWR) model [14], the LWR model with spillback [23] and the classical link-delay model of Friesz et al. [12].

1.2. Related Work

Two of the earliest papers in the field of dynamic traffic assignment are papers by Friesz et al. [12, 15] who introduced the formalism of a path-delay operator and investigated variational inequality and optimal control formulations under specific network loading models, see also Boyce, Ran and LeBlanc [2, 39]. For an overview of further relevant works, we refer to the survey article by Friesz and Han [13]. A key development in this field are the identification of certain continuity conditions of the path-delay operator in order to establish equilibrium existence. This has been successfully shown for various network loading models ranging from the link-delay model [46], the Vickrey model with point queues [5, 17, 21, 30, 36] and the Lighthill-Whitham-Richards (LWR) model [14] to the LWR model with spillback [23].

It is worth noting that dynamic traffic models with spillback (cf. [23, 41]), can be interpreted as an alternative way of handling hard capacities limiting the flow volume on particular edges. However, there is an important conceptual difference to side-constrained dynamic traffic assignment models: Spillback models allow the injection of flow into arbitrary paths and whenever the capacity restriction on an edge is reached, additional flow is prohibited from entering this edge and has to wait on the previous edge instead (causing additional congestion there). Thus, the spillback effect influences the agents' behaviour only indirectly via the increased path-delays. In a model with hard side-constraints, on the other hand, the volume constraints directly influence the agents' behaviour by restricting their strategy space: I.e. if entering a certain path at a specific time would lead to a violation of any edge capacity on that path, such a flow is considered to be infeasible. Our model can be seen as a strict generalization of both ideas as we do allow spillback models for the network loading but in addition we can model hard side-constraints.

For dynamic traffic assignment with hard side-constraints not much is known. Zhong et al. [45] considered the path-operator model with a linear volume-delay formulation for the network loading. They defined side-constrained dynamic equilibria as solutions to an associated infinite dimensional variational inequality and claimed existence of such equilibria. Hoang et al. [27] transferred the static BMW equilibrium concept to a dynamic model by discretizing time and then considering a time-expanded network. In a similar way Hamdouch et al. [19] extended the static equilibrium concept from [35] to a dynamic setting.

1.3. Paper Organization

We start the paper by recapping in Section 2 the theory of side-constrained static equilibrium flows. Already for static models, the issue about properly defining side-constrained equilibria arises and we try to sketch the historic development of the key concepts in the field. In Section 3, we introduce the basic dynamic traffic assignment model. In Section 4, we give a counterexample to Zhong, Sumalee, Friesz and Lam [45], (Prop. 3.1., pp.
1040) which illustrates the need of rethinking an appropriate solution concept for traffic assignment models with side-constraints.

In Section 5, we introduce our abstract framework of side-constrained dynamic traffic equilibria. In Section 6, we turn to characterization results of such equilibria in terms of variational or quasi-variational inequalities. Finally, in Section 7, we derive two equilibrium existence results for a class of non-convex volume-constrained traffic models using an augmented Lagrangian penalty function approach.

2. The Static Model

We are given a directed graph $G = (V, E)$ and a set of populations or commodities $I := \{1, \ldots, n\}$, where each commodity $i \in I$ has a demand $d_i > 0$ that has to be routed from a source $s_i \in V$ to a destination $t_i \in V$. The demand interval $[0, d_i]$ represents a continuum of infinitesimally small agents each acting independently choosing a cost minimal path. There are continuous and nondecreasing cost functions $\ell_e : \mathbb{R}^E \to \mathbb{R}_{\geq 0}, i \in I, e \in E$.

A path flow for commodity $i \in I$ is a nonnegative vector $x_i \in \mathbb{R}_{\geq 0}^{\mathcal{P}_i}$ that lives in the path flow polytope:

$$X_i = \left\{x_i \in \mathbb{R}_{\geq 0}^{\mathcal{P}_i} \mid \sum_{p \in \mathcal{P}_i} x_{i,p} = d_i\right\},$$

where $\mathcal{P}_i$ denotes the set of simple $s_i, t_i$-paths in $G$ and $\mathcal{P} := \bigcup_{i \in I} \mathcal{P}_i$. We assume that every $t_i$ is reachable in $G$ from $s_i$ for all $i \in I$, thus, $X_i \neq \emptyset$ for all $i \in I$. Given a path flow vector $x \in X := \times_{i \in I} X_i$, the cost of a path $p \in \mathcal{P}_i$, is defined as $\ell_p(x) := \sum_{e \in p} \ell_e(x)$, where $x_e := \sum_{i \in I} \sum_{p \in \mathcal{P}_i : e \in p} x_{i,p}$ is the aggregated load of edge $e \in E$.

Definition 2.1. A path flow $x^* \in X$ is a Wardrop equilibrium, if for all $i \in I$:

$$\ell_p(x^*) \leq \ell_q(x^*) \text{ for all } p, q \in \mathcal{P}_i \text{ with } x_{i,p}^* > 0.$$

The interpretation here is that all agents are travelling along shortest paths given the overall load vector $(x_e^*)_{e \in E}$. One can characterize Wardrop equilibria by means of variational inequalities as follows (cf. Patriksson [38, Sec. 3.2.1]):

Lemma 2.2. The following statements are equivalent:

1. $x^* \in X$ is a Wardrop equilibrium.
2. $(\ell(x^*), x^* - y) \leq 0 \text{ for all } y \in X$, with $\ell(x^*) := (\ell_p(x^*))_{p \in \mathcal{P}}$

For the case of separable latency functions $\ell$, Dafermos and Sparrow [9] related Wardrop equilibria to Nash equilibria of an associated noncooperative game. Formally, for $\varepsilon > 0$ and $p, q \in \mathcal{P}_i$ with $x_p > 0$, let

$$x_w(\varepsilon, p, q) := \begin{cases} x_w - \varepsilon, & \text{if } w = p \\ x_w + \varepsilon, & \text{if } w = q \\ x_w, & \text{else.} \end{cases}$$

Definition 2.3. A path flow $x^* \in Z$ is a Nash equilibrium, if for all $p, q \in \mathcal{P}_i, x_{i,p}^* > 0, \varepsilon \in (0, x_{i,p}^*], i \in I$, we have

$$\ell_p(x^*) \leq \ell_q(x^*(\varepsilon, p, q)).$$
Dafermos and Sparrow [9] showed that for continuous and separable latency functions, Nash equilibria and Wardrop equilibria coincide.

If $\ell$ is separable and non-decreasing, it is the gradient of the Beckmann-McGuire-Winsten potential function and one can further characterize Wardrop equilibria as optimal solutions to a convex optimization problem:

**Lemma 2.4** (cf. Dafermos [8]). The following statements are equivalent:

1. $x^* \in X$ is a Wardrop equilibrium.
2. $x^* \in \arg \min_{x \in X} \left\{ \sum_{e \in E} \int_{0}^{x_e} \ell_e(z)dz \right\}$.
3. $(\ell(x^*), x^* - y) \leq 0$ for all $y \in X$, with $\ell(x^*) := (\ell_p(x^*))_{p \in P}$.

### 2.1. Side-Constrained Traffic Equilibria

Suppose we have hard edge capacities $c_e \geq 0, e \in E$ which need to be satisfied for a flow $x \in X$ to be capacity-feasible, that is, $x_e \leq c_e, e \in E$. Let $Z := \{ x \in X | x_e \leq c_e, e \in E \}$ denote the set of capacity-feasible path flows. Following Larsson and Patriksson [31] (which we abbreviate henceforth with LP) and Patriksson [38, pp. 73], a side-constrained equilibrium can be defined via the notion of saturated and unsaturated paths. Given a path flow $x \in Z$, a path $p$ is saturated, if there is an edge $e \in p$ with $x_e = c_e$ and, conversely, a path $p$ is unsaturated, if $x_e < c_e$ for all $e \in p$.

**Definition 2.5.** A path flow $x^* \in Z$ is a side-constrained LP-equilibrium, if

\[ \ell_p(x^*) \leq \ell_q(x^*) \text{ for all } p, q \in P_i \text{ with } x_{i,p}^* > 0 \text{ and } q \text{ unsaturated}. \]

For our subsequent discussion it is worth reformulating the definition of an LP-equilibrium in terms of feasible additive $\varepsilon$-deviations in the spirit of Dafermos and Sparrow [9]. For $\varepsilon > 0$ and $q \in P_i$, let

\[ x_w(\varepsilon, q) := \begin{cases} x_w + \varepsilon, & \text{if } w = q \\ x_w, & \text{else.} \end{cases} \]

Define

\[ \ell_e(x) := \begin{cases} \ell_e(x), & \text{if } x_e \leq c_e \\ +\infty, & \text{else.} \end{cases} \]

We obtain the following equivalent definition:

**Lemma 2.6.** A path flow $x^* \in Z$ is a side-constrained LP-equilibrium iff for all $p, q \in P_i, x_{i,p}^* > 0, i \in I$, we have

\[ \ell_p(x^*) \leq \ell_q(x^*(\varepsilon, q)) \text{ for all } \varepsilon \in (0, x_{i,p}^*]. \]

As observed by Marcotte et al. [35], this definition has the drawback of admitting rather artificial equilibrium flows. Consider for instance a graph with one edge followed by two edges in parallel. If the capacity of the first edge is saturated, then *any* feasible flow is an LP equilibrium no matter how the flow is distributed on the two subsequent edges. This leads to unrealistic equilibria in case one of the two edges is longer but carries flow and the other shorter one has free capacity. An alternative equilibrium concept proposed by Smith [43] avoids this problem by allowing for path changes of $\varepsilon > 0$ units of flow provided that *after the change*, the resulting flow is still feasible (this corresponds to considering
deviations of the form $x^*(\varepsilon, p, q)$ in Lemma 2.6). This concept has the drawback that it allows for coordinated deviations of bundles of users, which is unrealistic and, as shown by Smith, leads to non-existence of equilibria for monotonic, continuous and non-separable latency functions.

In response to Smith’s equilibrium concept, Bernstein and Smith (BS) [1] proposed an alternative equilibrium concept addressing the issue of possible coordinated deviations of bundles of users. They added the condition that only deviations need to be considered that involve “small enough” bundles of users.¹

Definition 2.7. (Bernstein and Smith [1, Def. 2,]) A path flow $x^* \in Z$ is a side-constrained BS-equilibrium, if for all $p, q \in P, x_{i,p}^* > 0, i \in I$, we have

$$\tilde{\ell}_p(x^*) \leq \liminf_{\varepsilon \downarrow 0} \tilde{\ell}_q(x^*(\varepsilon, p, q)).$$

Note that the original definition of Bernstein and Smith [1, Def. 2] does not involve capacities but by using latency functions $\tilde{\ell}$ that jump to $+\infty$ as soon as arc capacities are exceeded an equilibrium will be capacity-feasible. Following Correa et al. [7], the definition of BS-equilibrium can be rephrased as “no arbitrarily small bundle of drivers on a common path can strictly decrease its cost by switching to another path”.

2.2. Beckmann-McGuire-Winsten Equilibria and Variational Inequalities

For the case of separable latency functions, a subset of BS-equilibria can be characterized as solutions to an associated convex optimization problem, where the Beckmann-McGuire-Winsten potential function over the convex space of capacity-feasible flows is minimized:

$$\min \sum_{e \in E} \int_0^{x_e} \ell_e(z)dz$$

subject to: $x \in Z$.

(BMW)

We obtain the following well-known characterization (see, e.g. Patriksson [38]).

Lemma 2.8. 1. $x^* \in Z$ is optimal for (BMW) iff $x^*$ solves the following variational inequality

$$\langle \ell(x^*), x^* - y \rangle \leq 0 \text{ for all } y \in Z. \quad (1)$$

2. Every optimal $x^* \in Z$ to (BMW) is a BS-equilibrium but not vice versa.

The second statement implies the existence of BS-equilibria: the space $Z$ is non-empty and compact and by the continuity of the objective in (BMW), the theorem of Weierstraß implies that (BMW) admits an optimal solution (which then is also a BS-equilibrium).

The first and second statement together show that there are BS-equilibria which need not solve the variational inequality stated in (1). Correa et al. [7] termed equilibria coming from optimal solutions to (BMW) as Beckmann-McGuire-Winsten (BMW) equilibria while Marcotte et al. [35] termed them Hearn-Larsson-Patriksson equilibria. A further useful interpretation of solutions to (BMW) is the use of the dual variables associated with the capacity constraints $x \leq c$. It was shown in several works (cf. Hearn [24], Daganzo [10], Larsson and Patriksson [33]) that a BMW-equilibrium can be interpreted as

¹Heydecker [26] introduced yet another equilibrium definition in response to Smith’s definition, where the path costs of $p$ and $q$ are compared after the route switch of $\varepsilon$ units of flow.
an unconstrained equilibrium, if the dual variables are added as additional penalty terms to the user’s cost function. In addition to this natural interpretation and possible implementation of BMW-equilibria via prices, the efficiency properties of BMW-equilibria in terms of induced total travel times are particularly appealing compared to other possible side-constrained equilibria, see Correa et al. [7].

2.3. Discussion

It is very instructive to restate a remark made by Marcotte et al. [35]: “Defining equilibrium meaningfully in side-constrained transportation networks represents a nontrivial task.” Indeed, the above presentation already shows some subtle issues arising when defining a sound notion of side-constrained traffic equilibria. While only a few works formally introduced a behavioral equilibrium concept involving the notion of feasible \( \epsilon \)-deviations (as in Bernstein and Smith [1], Dafermos and Sparrow [9], Smith [43], Heydecker [26]), most of the works in the transportation science literature used directly the optimization formulations of the type (BMW) or the variational inequality formulations as the definition of a side-constrained user equilibrium (cf. Daganzo [10], Hearn and Ramana [25] or Larsson and Patriksson [33, Remark 11]). This observation was also made in Correa et al. [7]: “It is interesting to note that the model [(BMW)] has been used before without the formal introduction of the concept of a capacitated user equilibrium.”

As we will see later in Section 4, within the realm of *dynamic traffic assignments*, defining side-constrained dynamic equilibria via variational inequalities is not only imprecise (as it excludes other “equilibria” being not of this type) but also leads to flawed statements about equilibrium existence and their characterizations. We show that the natural infinite dimensional variational inequality formulation for a class of volume-constrained dynamic traffic assignments need per se not be related to an equilibrium solution. Instead our goal in this work is to transfer behavioral equilibrium concepts in the spirit of Dafermos and Sparrow [9], Bernstein and Smith [1], Smith [43], Heydecker [26] to the domain of *dynamic* side-constrained traffic assignments.

3. Unconstrained Dynamic Flows

We consider the following model based on the walk-delay operator model of Friesz et al. [15]: We are given a finite directed graph \( G = (V, E) \) and some fixed planning horizon \([t_0, t_f] \subseteq \mathbb{R}_{\geq 0}\). Additionally, we have a finite set of commodities \( I \) and for every commodity \( i \in I \) a source node \( s_i \in V \), a sink node \( t_i \in V \) and either a fixed network inflow volume \( Q_i \geq 0 \) (for the model with departure time choice) or a fixed bounded network inflow rate \( r_i \in L^2_+([t_0, t_f]) \) (for the model without departure time choice) where \( L^2_+([t_0, t_f]) \) denotes the set of non-negative \( L^2 \)-integrable functions on \([t_0, t_f]\).

We denote by \( P_i \) a fixed set of \( s_i, t_i \)-walks and assume – wlog. – that these sets are disjoint for different commodities. We then denote by \( P := \bigcup_{i \in I} P_i \) the set of all relevant walks. Note that we allow general walks instead of just simple paths as travelling along cycles

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2We use term “behavioral equilibrium concept” in accordance with the concept of a Nash equilibrium for a noncooperative game involving payoff functions (or preference relations) and strategy spaces which are implicitly defined via feasible deviations as in Dafermos and Sparrow [9]. The term “behavioral” as used in Larsson and Patriksson [32] has the following different meaning: “As such, these models are behavioural, in the sense that the effects of the side constraints are assumed to be immediately transferable to the perception of travel costs among the trip-makers, for example as queueing delays; their solutions are also characterized and interpreted as flows satisfying the Wardrop equilibrium conditions in terms of generalized travel costs that include link queueing delays.”
is necessary in certain applications like electric vehicles (cf. [16]) and can also sometimes be advantageous in networks with hard edge-capacities (e.g. Example C.1). A flow in this network is given by a vector \( h \in L^2_+([t_0, t_f])^\mathcal{P} \) of \( L^2 \)-integrable functions \( h_p : [t_0, t_f] \to \mathbb{R}_{\geq 0} \) denoting the walk inflow rates for all walks of all commodities. We denote by

\[
\Lambda(Q) := \left\{ h \in L^2_+([t_0, t_f])^\mathcal{P} \left| \sum_{p \in \mathcal{P}_i} \int_{t_0}^{t_f} h_p(t) \, dt = Q_i \text{ for all } i \in I, h_p \leq B_p \text{ for all } p \in \mathcal{P} \right. \right\},
\]

and

\[
\Lambda(r) := \left\{ h \in L^2_+([t_0, t_f])^\mathcal{P} \left| \sum_{p \in \mathcal{P}_i} h_p(t) = r_i(t) \text{ for almost all } t \in [t_0, t_f] \text{ and all } i \in I \right. \right\}
\]

the sets of all feasible walk inflows for the model with and without departure time choice, respectively. In the definition of \( \Lambda(Q) \) the values \( B_p \in \mathbb{R}_{\geq 0} \) are some fixed walk-specific bounds on the walk inflow rates. In general, these are needed to ensure existence of equilibria in this model (see Remark D.1). Note, however, that in some models such bounds can be introduced without loss of generality since choosing \( B_p \) large enough only excludes flows which cannot be equilibria anyway (cf. [22, Proposition 5.9]). We also observe that we can always view \( \Lambda(r) \) as a subset of some \( \Lambda(Q) \) by defining \( Q_i := \int_{t_0}^{t_f} r_i(t) \, dt \) and choosing \( B_p \geq \sup_{t \in [t_0, t_f]} r_i(t) \) for every \( p \in \mathcal{P}_i \). For the case of elastic demands we are given a non-increasing inverse demand function \( \Theta_i : [0, Q_i] \to \mathbb{R} \) such that for any possible demand \( Q \leq Q_i \) the value \( \Theta_i(Q) \) is the cost threshold at which a volume of \( Q \) of all particles of commodity \( i \) is still willing to travel while the rest stays at home.

Furthermore, we are given a function

\[
\Psi : L^2_+([t_0, t_f])^\mathcal{P} \to \hat{M}([t_0, t_f])^\mathcal{P}, h \mapsto (\Psi_p(h, \cdot) : [t_0, t_f] \to \mathbb{R}_{\geq 0})_{p \in \mathcal{P}}
\]

mapping walk inflows to effective walk delay, i.e. for any walk inflow \( h \), commodity \( i \), walk \( p \in \mathcal{P}_i \) and time \( t \) the value \( \Psi_p(h, t) \) is to be understood as the total travel cost (e.g. some weighted sum of travel time, penalty for late arrival and cost of energy consumption on the chosen route) of a particle of commodity \( i \) starting at time \( t \) to travel along walk \( p \) under the traffic state induced by the walk inflow \( h \). Here, we denote by \( \hat{M}([t_0, t_f]) \) the set of measurable functions from \([t_0, t_f]\) to \( \mathbb{R} \cup \{ \infty \} \).

We can now define three standard types of dynamic equilibria (cf. e.g. [46, 12, 20]):

**Definition 3.1.**

- \( h^* \in \Lambda(r) \) is a **dynamic equilibrium with fixed inflow rates**, if for all \( i \in I \), the following condition holds:

  \[
  h^*_p(t) > 0 \Rightarrow \Psi_p(h^*, t) \leq \Psi_q(h^*, t) \text{ for almost all } t \in [t_0, t_f], p, q \in \mathcal{P}_i.
  \]  

- \( h^* \in \Lambda(Q) \) is a **dynamic equilibrium with fixed flow volumes and departure time choice**, if for all \( i \in I \), there exists a \( \nu_i \in \mathbb{R} \) such that the following conditions hold:

  \[
  h^*_p(t) > 0 \Rightarrow \Psi_p(h^*, t) \leq \nu_i \text{ for almost all } t \in [t_0, t_f], p \in \mathcal{P}_i
  \]

  \[
  h^*_p(t) < B_p \Rightarrow \Psi_p(h^*, t) \geq \nu_i \text{ for almost all } t \in [t_0, t_f], p \in \mathcal{P}_i.
  \]
• \((h^*, Q^*)\) with \(Q^* \in \mathbb{R}^I_{>0}\) and \(h^* \in \Lambda(Q^*)\) is a dynamic equilibrium with elastic demands and departure time choice, if for all \(i \in I\), there exists a \(\nu_i \in \mathbb{R}\) such that the following conditions hold:

\[
\begin{align*}
    h^*_i(t) > 0 & \Rightarrow \Psi_p(h^*, t) \leq \nu_i \text{ for almost all } t \in [t_0, t_f], p \in \mathcal{P}_i \\
    h^*_i(t) < B_p & \Rightarrow \Psi_p(h^*, t) \geq \nu_i \text{ for almost all } t \in [t_0, t_f], p \in \mathcal{P}_i \\
    Q^*_i < Q_i & \Rightarrow \nu_i = \Theta_i(Q^*_i) \\
    Q^*_i = Q_i & \Rightarrow \nu_i \leq \Theta_i(Q_i).
\end{align*}
\]

(4)

We observe that the model with elastic demands can be seen as a special case of the model with fixed inflow volume. Thus, we will only consider the first two models for the rest of this paper. Note, that this trick is certainly not new and has been applied for static models before, see Patriksson [38, Sec. 2.2.4].

**Lemma 3.2.** Consider a network \(\mathcal{N}\) with elastic demand and departure time choice and construct from this a new network \(\mathcal{N}'\) with fixed flow volumes and departure time choice by adding for every commodity \(i \in I\) an additional new edge \(\tilde{e}_i\) connecting the commodity’s source and sink node and defining \(\mathcal{P}'_i := \mathcal{P}_i \cup \{ \tilde{p}_i \} \) where \(\tilde{p}_i := (\tilde{e}_i)\) is the path consisting of only this new edge. Moreover, for each of these new paths we define the effective walk delay operator by \(\Psi^{\tilde{p}_i}(h, t) := \Theta_i(\sum_{p \in \mathcal{P}_i} \int_{t_0}^{t} h_p(t') dt')\) and choose \(B_{\tilde{p}_i}\) such that \(B_{\tilde{p}_i} \cdot (t_f - t_0) > Q_i\).

Then every dynamic equilibrium in \(\mathcal{N}'\) corresponds (in the natural way) to a dynamic equilibrium in \(\mathcal{N}\) and vice versa.

**Proof.** First, assume that \(h'\) is a dynamic equilibrium with fixed flow volume and departure time choice in \(\mathcal{N}'\). Then we define \(Q^*_i := \sum_{p \in \mathcal{P}_i} \int_{t_0}^{t_f} h'_p(t) dt\) for every commodity \(i \in I\) and \(h^*\) as the restriction of \(h'\) to the subnetwork \(\mathcal{N}\). We clearly have \(h^* \in \Lambda(Q^*)\) and the first two conditions of (4) are satisfied because of (3) for \(h'\) (with the same \(\nu_i\)). Moreover, due to our choice of \(B_{\tilde{p}_i}\) there must be a set of positive measure of times \(t\) with \(h'_p(t) < B_{\tilde{p}_i}^{-1}\) and, therefore, (3) implies \(\nu_i \leq \Psi^{\tilde{p}_i}(h', t) = \Theta_i(Q^*_i)\). Finally, if we have \(Q^*_i < Q_i\), then there must be a set of positive measures of times \(t\) with \(h'_p(t) > 0\) and, thus, (3) also guarantees \(\nu_i \geq \Psi^{\tilde{p}_i}(h', t) = \Theta_i(Q^*_i)\).

Now, assume that \((h^*, Q^*)\) is a dynamic equilibrium with elastic demand and departure time choice. Then, we can extend \(h^*\) to a flow \(h' \in \Lambda(Q)\) by setting \(h'_p(t) := \frac{Q^*_i - Q^*_i}{t_f - t_0}\) for all \(t \in [t_0, t_f]\) and \(i \in I\). Using the same \(\nu_i\) as for \(h^*\) we then immediately have that the conditions in (3) hold for \(h'\) on all paths in \(\mathcal{P}_i\). So, we only have to show that they also hold for the new paths \(\tilde{p}_i\): If we have \(h'_p(t) > 0\) at any time \(t\), we have \(Q^*_i < Q_i\) and, therefore, \(\Psi^{\tilde{p}_i}(h', t) = \Theta_i(Q^*_i) = \nu_i\). Otherwise, we still have \(\Psi^{\tilde{p}_i}(h', t) = \Theta_i(Q^*_i) \geq \nu_i\) for all times \(t\). Hence, in both cases (3) holds for the new paths \(\tilde{p}_i\) as well. \(\square\)

We now want to characterize the first two types of dynamic equilibria using variational inequalities. For this we require that the effective walk delays are bounded in the following sense:

(A1) For any \(p \in \mathcal{P}\) and \(h \in \Lambda(Q)\) the function \(\Psi_p(h, .)\) is bounded.

This then, in particular, implies that all effective walk delays are \(L^2\)-integrable, i.e. \(\Psi_p(h, .) \in L^2([t_0, t_f])\). An example for where this assumption may be violated is a network-loading model involving spillback and an effective walk delay \(\Psi\) that just measures the actual delay. Since spillback can lead to gridlock (cf. [40, pp. 99f]), the delay will be \(\infty\).
for particles caught in such a gridlock. However, even for these cases our model might still be applicable as long as particles always have the option to avoid joining a gridlock and achieving some bounded cost (e.g. some alternative route with infinite capacity or a stay at home-option). This will be formalized in Lemma 5.7.

If assumption (A1) holds, it is well known (cf. e.g. [12, 46]) that, as in the static case, both kinds of dynamic equilibria can be characterized by variational inequalities namely

Find \( h^* \in \Lambda(r) \) such that:
\[
\langle \Psi(h^*), h - h^* \rangle \geq 0 \quad \text{for all } h \in \Lambda(r)
\]
and

Find \( h^* \in \Lambda(Q) \) such that:
\[
\langle \Psi(h^*), h - h^* \rangle \geq 0 \quad \text{for all } h \in \Lambda(Q).
\]

Here, \( \langle, \rangle \) denotes the canonical scalar product on \( L^2([t_0, t_f]) \), i.e.
\[
\langle f, g \rangle := \sum_{\theta \in \Theta} \int_{t_0}^{t_f} f_p(\theta)g_p(\theta) \, d\theta.
\]

**Theorem 3.3.** Assume that (A1) holds. Then, a walk inflow \( h^* \in \Lambda(r) \) (\( h^* \in \Lambda(Q) \)) is a dynamic equilibrium with fixed inflow rates (with fixed flow volume) if and only if \( h^* \) is a solution to \( \langle VI(\Psi, r, [t_0, t_f]) \rangle \) (to \( VI(\Psi, Q, [t_0, t_f]) \)).

Conditions to guarantee the existence of such an element \( h^* \) are given by Lions in [34, Chapitre 2, Théorème 8.1] which, following Cominetti et al. [5], can be restated as follows (see [46, Proof of Theorem 4.2] for how to derive this version from Lions’ result):

**Theorem 3.4.** Let \( C \) be a non-empty, closed, convex and bounded subset of \( L^2([a, b])^d \). Let \( A : C \to L^2([a, b])^d \) be a sequentially weak-strong-continuous mapping. Then, the following variational inequality has a solution \( h^* \in C \):

Find \( h^* \in C \) such that:
\[
\langle A(h^*), h - h^* \rangle \geq 0 \quad \text{for all } h \in C.
\]

Using this theorem together with Theorem 3.3 one can derive existence of dynamic equilibria under suitable additional assumptions:

(A2) The sets \( P_i \) are non-empty and finite.

(A3) The walk inflow bounds \( B_p \) are chosen such that there exists at least one feasible flow \( h \in \Lambda(Q) \).

(A4) For any \( p \in P \) the mapping \( \Lambda(Q) \to L^2([t_0, t_f]), h \mapsto \Psi_p(h,.) \) is sequentially weak-strong continuous.

Here, a mapping \( A : K \to X \) from some subset \( K \subseteq B \) of a reflexive Banach space \( B \) (e.g. \( L^2([t_0, t_f])^P \)) to a topological space \( X \) is sequentially weak-strong continuous if it maps weakly convergent sequences in \( B \) to (strongly) convergent sequences in \( X \). A sequence \( f^n \) converges weakly in \( B \) if for any \( g \in B \) the sequence \( (f^n, g) \) converges in \( \mathbb{R} \).

**Theorem 3.5.** For any network and effective walk delays satisfying (A1) to (A4) there exists a dynamic equilibrium (both with and without departure time choice).

As this theorem can be proven in essentially the same way as analogous existence results for very similar models (cf. e.g. [22, 46]), we omit the proof here. We do, however, provide a proof for the exact setting used here in Appendix D.
4. A Counterexample

We now want to describe the dynamic flow model with hard edge capacities introduced by Zhong, Sumalee, Friesz and Lam [45]. In order to formally define such edge capacities one first needs to be able to translate the individual particles’ strategy choices (i.e. a walk inflow $h$) into the resulting dynamic flow on the edges of the network. This translation process is usually called network loading and it typically uses some given edge delay functions $D_e(h, \theta) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ in order to model the flow dynamics on individual edges. Here, $D_e(h, \theta)$ denotes the travel time along edge $e$ when entering at time $\theta$ under the flow induced by $h$.

Using such edge delays, one can define the edge flow corresponding to a given walk inflow $h$: For this we define the set $\mathcal{R} := \{(p, j) \mid p \in \mathcal{P}, j \in \{\{p\}\} \}$, i.e. $\mathcal{R}$ contains exactly one element for every walk in $\mathcal{P}$ and each of its edges (counted with multiplicity if an edge occurs multiple times on the walk). An edge flow corresponding to a given walk inflow $h$ is then a tuple $(f^+, f^-)$ with $f^+, f^- \in L^2_{+}(\mathbb{R}_{\geq 0})^{|\mathcal{R}|}$ satisfying the following conditions:

- Correspondence to $h$:
  $$f^+_{p,1}(\theta) = h_p(\theta) \text{ for all } p \in \mathcal{P} \text{ and almost all } \theta \in \mathbb{R}_{\geq 0}.$$  
  where we implicitly assume $h_p$ to be zero outside the planning horizon $[t_0, t_f]$.

- Flow conservation at the nodes:
  $$f^+_{p,j}(\theta) = f^-_{p,j-1}(\theta) \text{ for all } (p, j) \in \mathcal{R} \text{ with } j > 1 \text{ and almost all } \theta \in \mathbb{R}_{\geq 0}.$$  

- Flow conservation on the edges:
  $$\int_0^{\theta + D_e(h, \theta)} f^-_{p,j}(\zeta) \, d\zeta = \int_0^{\theta} f^+_{p,j}(\zeta) \, d\zeta \text{ f.a. } (p, j) \in \mathcal{R} \text{ and all } \theta \in \mathbb{R}_{\geq 0}. \quad (5)$$  

For any such edge flow we denote by

$$f^+_e(\theta) := \sum_{(p, j) \in \mathcal{R}: e \text{ is } j\text{-th edge on } p} f^+_{p,j}(\theta) \quad \text{and} \quad f^-_e(\theta) := \sum_{(p, j) \in \mathcal{R}: e \text{ is } j\text{-th edge on } p} f^-_{p,j}(\theta)$$  

the aggregated edge in- and outflow rates.

Two commonly used types of edge delay functions are the linear edge delays and the Vickrey queuing delays. In both models the flow dynamics of an edge $e$ are characterized by two values: The free flow travel time $\tau_e \geq 0$ and the edge’s service rate $\nu_e > 0$. The linear edge delay functions are then defined as

$$D_e(h, \theta) := \tau_e + \frac{x_e(h, \theta)}{\nu_e},$$  

where $x_e(h, \theta) := \int_0^\theta f^+_e(\zeta) \, d\zeta - \int_0^\theta f^-_e(\zeta) \, d\zeta$ is the flow volume on edge $e$ at time $\theta$. The edge delays of the Vickrey point queue model are defined by

$$D_e(h, \theta) := \tau_e + \frac{q_e(h, \theta)}{\nu_e}$$  

where $q_e(h, \theta) := \int_0^\theta f^+_e(\zeta) \, d\zeta - \int_0^{\theta + \tau_e} f^-_e(\zeta) \, d\zeta$ is the queue length on edge $e$ at time $\theta$.  


Given such edge delay functions we can then recursively define walk-delay-functions $D_p$ by setting

$$D_p(h,t) := \begin{cases} D_e(h,t), & \text{if } p \text{ consists of a single edge } e \\ D_e(h,t) + D_p'(h,t + D_e(h,t)), & \text{if } p \text{ starts with edge } e \text{ followed by walk } p'. \end{cases}$$

A typical choice for the effective walk delay is then $\Psi_p(h,t) := D_p(h,t)$ or $\Psi_p(h,t) := D_p(h,t) + P(t + D_p(h,t) - T_A)$ where $T_A$ is the desired arrival time and $P$ some penalty function for early/late arrival.

For both linear edge delays and the edge delays of the Vickrey point queue model it is known (cf. e.g. [5, 22, 46]) that for every walk inflow $h$, there exists a unique corresponding edge flow. Even more, the resulting mapping from $h$ over $(f^+, f^-)$ to $\Psi$ can be shown to be sequentially weak-strong continuous. Thus, unconstrained dynamic equilibria are guaranteed to exist with respect to both these edge delay functions.

Zhong et al. [45] take the dynamic flow model with route and departure time choice and linear edge delays and augment it with additional constraints by associating with every edge $e$ a Lipschitz-continuous and bounded capacity function $c_e : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ and defining the set of all feasible flows as

$$S := \{ h \in \Lambda(Q) \mid x_e(h,\theta) \leq c_e(\theta) \text{ for all } \theta \in \mathbb{R}_{\geq 0}, e \in E \}. \quad (6)$$

They then define side-constrained dynamic user equilibria with route and departure time choice as the solutions $h^*$ to the following variational inequality (cf. [45, eq. (33)]):

$$\text{Find } h^* \in S \text{ such that : } \langle \Psi(h^*), h - h^* \rangle \geq 0 \text{ for all } h \in S \quad (7)$$

Finally, they claim existence of such solutions (and, therefore, of such equilibria) under some mild additional assumptions (cf. [45, Proposition 3.2]). As in the analogous existence results for unconstrained equilibria the proof proposed by Zhong et al. critically relies on the convexity of the set of feasible flows $S$. However, in contrast to the case of unconstrained equilibria, this convexity is not obvious for a constraint set $S$ defined by (6). In fact, it turns out that in general this set need not be convex! This not only invalidates the existence proof but also calls into question whether a variational inequality of the form of (7) is even the right way of defining such equilibria.

To prove our point we will now provide counterexamples both to the convexity of $S$ and the existence of solutions to the variational inequality (7). We first provide a simple example where $S$ is non-convex using infinite service rates and fixed network inflow rates. We then show how this example can be adapted and expanded to the exact model used by Zhong et al. [45] (i.e. using only finite service rates and allowing departure time choice). Finally, we show that in this example the variational inequality (7) does not have a solution.

**Example 4.1.** Consider the network depicted in Figure 1 with a single commodity with source $v$, sink $t$ and a constant network inflow rate of 2 during the planning interval $[0, 2]$. If we use the linear edge delays defined by the values for $\tau_e$ and $\nu_e$ given in the figure, the following two walk inflows are feasible: Either sending flow at a rate of 2 into the path $e_1, e_3$ (during the interval $[0, 2]$) or sending flow at a rate of 2 into the path $e_2, e_3$. In both cases flow arrives at the intermediate node $w$ at a rate of 1 during the interval $[2, 6]$ as one can verify by a straightforward calculation (see Figure 2, left). Thus, the capacity constraint on edge $e_3$ is never violated and both flows are feasible. However, if we send flow at a rate of 1 into each of the two paths during the interval $[0, 2]$, this flow...
will arrive at node \( w \) at a rate of \( \frac{2}{3} \) during the interval \([2, 5]\) from each of the two edges (see Figure 2, right). Thus, it will enter edge \( e_3 \) at a rate of \( \frac{1}{2} \) and therefore violate its capacity constraint after time \( \theta = 2 + \frac{3}{2} \). This shows that the set of feasible flows is not convex in this case.

Note, that a similar effect can be observed when using the Vickrey queuing model for the edge dynamics instead. Then, setting \( \nu_{e} = 1 \) for all edges in the network gives us again an instance where sending all flow at a rate of 2 into one of the two paths is feasible, while any convex combination of these two inflows violates the capacity constraint on edge \( e_3 \).

Now, in order to adjust this example for the model used by Zhong et al. we make the following changes: We add an additional edge \( e_0 \) with time-varying capacity constraint at the beginning forcing flow to arrive at node \( v \) at a rate of 2 even when departure time choice is enabled. Next, we replace the infinite service rate on edge \( e_3 \) by a large but finite service rate of \( \frac{1}{\varepsilon} \) and slightly increase the capacity constraint on edge \( e_3 \) to \( 2 + \varepsilon \). Finally, instead of a fixed inflow rate we now have a fixed flow volume \( Q = 4 \). The resulting network is depicted in Figure 3.

We first show that the capacity constraint on edge \( e_0 \) does indeed reduce the case with departure time choice to the case of fixed inflow rates:

**Claim 1.** Let \( h \in S \) be any feasible flow in the network depicted in Figure 3 and \((f^+, f^-)\) its corresponding edge flow. Then, this flow enters edge \( e_0 \) at a rate of 4 during the interval \([0, 1]\) and leaves it at a rate of 2 during the interval \([1, 3]\) i.e. we have

\[
f_{e_0}^+ = 4 \cdot 1_{[0,1]} \quad \text{and} \quad f_{e_0}^- = 2 \cdot 1_{[1,3]} \quad \text{almost everywhere.}
\]

**Proof.** We first show that in a feasible flow all flow has entered edge \( e_0 \) by time 1. So, let \( \theta := \min \{ \vartheta | \int_{\vartheta}^{0} f_{e_0}^+(\zeta) \, d\zeta = 4 \} \) be the time the last particle enters edge \( e_0 \). We observe
that due to the capacity constraint on edge $e_0$ all flow must have left edge $e_0$ by time 3. Thus, we must have

$$3 \geq \theta + D_{e_0}(h, \theta) = \theta + \frac{1}{4} x_{e_0}(h, \theta)$$

and, therefore,

$$x_{e_0}(h, \theta) \leq 8 - 4\theta \quad (8)$$

as well as $\theta \leq 3 - 1 = 2$. Furthermore, since the total flow volume of 4 must traverse edge $e_0$, at time $\theta$ we must have

$$\int_0^\theta f_{e_0}^-(\zeta) \, d\zeta + x_{e_0}(h, \theta) = 4. \quad (9)$$

Now, let $\theta'$ be the time at which particles have to enter edge $e_0$ in order to leave it at time $\theta$, i.e. $\theta'$ satisfies

$$\theta = \theta' + D_{e_0}(h, \theta') = \theta' + 1 + \frac{1}{4} x_{e_0}(h, \theta'). \quad (10)$$

Since the edge travel time along edge $e_0$ is always at least 1 we must have $\theta' \leq \theta - 1 \leq 2 - 1 = 1$ and, hence, no particle has left edge $e_0$ by time $\theta'$. Thus, we have

$$x_{e_0}(h, \theta') = \int_0^{\theta'} f_{e_0}^+(\zeta) \, d\zeta = \int_0^{\theta'} f_{e_0}^+(h, \theta') \, d\zeta = \int_0^\theta f_{e_0}^-(\zeta) \, d\zeta. \quad (5)$$

Together with (9) this gives us

$$x_{e_0}(h, \theta') = 4 - x_{e_0}(h, \theta). \quad (11)$$

From this, we can now deduce

$$4\theta' = c_{e_0}(\theta') \geq x_{e_0}(h, \theta') \overset{(11)}{=} 4 - x_{e_0}(h, \theta) \overset{(8)}{=} 4 - 8 + 4\theta = 4\theta - 4$$

and, thus,

$$4 \geq 4\theta - 4\theta' \overset{(10)}{=} 4\theta' + 4 + x_{e_0}(h, \theta') - 4\theta' = 4 + x_{e_0}(h, \theta').$$
which gives us \( x_{e_0}(h, \theta') = 0 \). Plugging this back into (11) gives us \( x_{e_0}(h, \theta) = 4 \) and, finally, feasibility of \( h \) together with the the capacity constraint on edge \( e_0 \) then implies \( \theta = 1 \) and, therefore,

\[
\int_0^\theta f_0^+(\zeta) \, d\zeta = 4 \quad \text{for all} \quad \theta \geq 1. \tag{12}
\]

Now, take any time \( \theta \in [0, 1] \) and let \( \bar{\theta} := \theta + D_{e_0}(h, \theta) \) be the time particles entering edge \( e_0 \) at time \( \theta \) arrive at node \( v \). Since \( \bar{\theta} \in [1, 3] \), the feasibility of \( h \) implies

\[
4 - 2\theta - \frac{x_{e_0}(h, \theta)}{2} = 6 - 2\left( \theta + 1 + \frac{x_{e_0}(h, \theta)}{4} \right) = 6 - 2\bar{\theta} = c_{e_0}(\bar{\theta}) \geq x_{e_0}(h, \bar{\theta}) = \int_0^\theta f_0^+(\zeta) \, d\zeta - \int_0^\theta f_0^-(\zeta) \, d\zeta \tag{12} = 4 - \int_0^\theta f_0^-(\zeta) \, d\zeta \tag{5} = 4 - \int_0^\theta f_0^+(\zeta) \, d\zeta = 4 - x_{e_0}(h, \theta).
\]

which implies \( x_{e_0}(h, \theta) \geq 4\theta \). At the same time, feasibility of \( h \) implies \( x_{e_0}(h, \theta) \leq 4\theta \) and, thus, we have \( x_{e_0}(h, \theta) = 4\theta \) for all \( \theta \in [0, 1] \). Since no flow leaves edge \( e_0 \) before time \( \theta = 1 \), this implies \( \int_0^\theta f_0^+(\zeta) \, d\zeta = 4\theta \) and, therefore, \( f_0^-(\zeta) = 4 \) for almost all \( \zeta \in [0, 1] \). A direct computation then shows \( f_0^-(\zeta) = 2 \) for almost all \( \zeta \in [1, 3] \) as well.

Next, we define three path inflows: Flow \( h^1 \) sends flow into the upper path \( e_0, e_1, e_3 \) at a rate of 4 during the interval \([0, 1]\). Flow \( h^2 \) sends flow into the lower path \( e_0, e_2, e_3 \) at a rate of 4 during the same interval. And, finally, \( h^3 \) splits the flow equally between the two paths, i.e. sends flow at a rate of 2 into each of the two paths. We now claim that \( h^1 \) and \( h^2 \) are feasible while \( h^3 \) is not.

**Claim 2.** In the network from Figure 3 the path inflows \( h^1 \) and \( h^2 \) are feasible. The path inflow \( h^3 \) is infeasible for \( \varepsilon < \frac{2}{3} \).

**Proof.** Both inflows \( h^1 \) and \( h^2 \) result in flow arriving at a rate of 1 at node \( w \) (without violating the capacity constraint on edge \( e_0 \) on the way). Since the total flow volume in the network is bounded by 4, the travel time on edge \( e_3 \) is never larger than \( 2 + \frac{4}{3} = 2 + \varepsilon \).

Thus, flow entering this edge at a rate of 1 will never violate its capacity constraint. Therefore, both \( h^1 \) and \( h^2 \) are feasible.

The equal split between the two paths in \( h^3 \) results in flow entering edge \( e_1 \) and \( e_2 \) at a rate of 1 each during \([1, 3]\). This flow exits these edges at a rate of \( \frac{2}{3} \) during the interval \([3, 6]\). Thus, the flow enters edge \( e_3 \) at a combined rate of \( \frac{4}{3} \) during this interval. At time \( \theta = 3 + \frac{3}{4} \cdot (2 + \varepsilon) \) a flow volume of \( 2 + \varepsilon \) has entered edge \( e_3 \). If \( \varepsilon < \frac{2}{3} \), this happens before time \( \theta = 5 \) and, in particular, before any flow has left edge \( e_3 \). Thus, the total flow volume on edge \( e_3 \) is \( 2 + \varepsilon \) at this time, leading to a violation of the capacity constraint immediately after.

Since \( h^3 \) is a convex combination of \( h^1 \) and \( h^2 \), this claim already shows the non-convexity of the set \( S \) of feasible flows. To show that the variational inequality (7) has no solution, we need the following additional property of all feasible flows in the given instance.

**Claim 3.** Let \( h \in S \) be any feasible path inflow for the instance in Figure 3. Then, one of the two paths has a total inflow volume of at most \( 4\varepsilon \).
Proof. Define \( \theta_1 \) as the last time a particle can enter edge \( e_1 \) under the flow induced by \( h \) and still arrive by time 5 at the node \( w \) and define \( \theta_2 \) analogous for edge \( e_2 \). That is, we choose \( \theta_1 \) and \( \theta_2 \) such that we have
\[
\theta_1 + D_{e_1}(h, \theta_1) = 5 \quad \text{and} \quad \theta_2 + D_{e_2}(h, \theta_2) = 5
\]
or, equivalently
\[
\theta_1 + \frac{x_{e_1}(h, \theta_1)}{2} = 3 \quad \text{and} \quad \theta_2 + \frac{x_{e_2}(h, \theta_2)}{2} = 3. \quad (13)
\]
Without loss of generality we assume that \( \theta_1 \leq \theta_2 \). Since \( \theta_1, \theta_2 \leq 3 = \tau_{e_0} + \tau_{e_1} = \tau_{e_0} + \tau_{e_2} \), no flow has left edges \( e_1 \) and \( e_2 \) by time \( \theta_2 \) and, thus
\[
2(\theta_1 - 1) \overset{\text{Claim 1}}{=} \int_0^{\theta_1} f_{e_0}(-\zeta) \, d\zeta \leq x_{e_1}(h, \theta_1) + x_{e_2}(h, \theta_2).
\]
At the same time we also have
\[
x_{e_1}(h, \theta_1) + x_{e_2}(h, \theta_2) \leq x_{e_1}(h, 5) \leq c_{e_2}(5) = 2 + \varepsilon \quad (14)
\]
since all this flow has entered edge \( e_3 \) by time 5 but no flow has left it. Together, this implies \( 2(\theta_1 - 1) \leq 2 + \varepsilon \) and, therefore,
\[
\theta_1 \leq 2 + \frac{\varepsilon}{2}. \quad (15)
\]
Since flow arrives at node \( v \) at a rate of 2 during \([1, 3]\) (cf. Claim 1), we also have
\[
x_{e_1}(h, \theta_1) \leq 2(\theta_1 - 1)
\]
while (13) implies
\[
x_{e_1}(h, \theta_1) = 6 - 2\theta_1. \quad (16)
\]
Together, this gives us \( \theta_1 \geq 2 \) which, combined with (15), results in \( \theta_1 \in [2, 2 + \frac{\varepsilon}{2}] \). Using (16) we then get \( x_{e_1}(h, \theta_1) \in [2 - \varepsilon, 2] \) and, with (14),
\[
x_{e_2}(h, \theta_2) \leq 2 + \varepsilon - x_{e_1}(h, \theta_1) \leq 2 + \varepsilon - (2 - \varepsilon) = 2\varepsilon.
\]
Plugging this back into (13) and solving for \( \theta_2 \) gives
\[
\theta_2 = 3 - \frac{x_{e_2}(h, \theta_2)}{2} \geq 3 - \frac{2\varepsilon}{2} = 3 - \varepsilon.
\]
This, using again the fact that flow arrives at node \( v \) at a rate of 2 after time \( \theta = 1 \), gives us
\[
x_{e_2}(h, 3) \leq x_{e_2}(h, \theta_2) + 2 \cdot (3 - \theta_2) \leq 2\varepsilon + 2\varepsilon = 4\varepsilon.
\]
Since no flow leaves edge \( e_2 \) before time 3 and no flow enters after time 3, this proves our claim. \( \blacksquare \)

Now, let \( h \in S \) be any feasible flow and \( e_2 \) the edge with a total inflow of at most \( 4\varepsilon \). We want to show that \( \langle \Psi(h), h^2 - h \rangle < 0 \). Since the free flow travel times on both paths are equal, we only have to consider the additional flow dependent delays here (as induced by \( h \)). Furthermore, as both \( h \) and \( h^2 \) are feasible flows, they are the same on edge \( e_0 \) (by Claim 1). Thus, the delays experienced on edge \( e_0 \) cancel out. For flow \( h^2 \) edge \( e_2 \)
contributes an additional delay of at most $\frac{4\varepsilon}{2}$ per particle and edge $e_3$ an additional delay of at most $\varepsilon$, leading to a total additional delay of at most $4 \cdot (2\varepsilon + \varepsilon) = 12\varepsilon$. For flow $h$, on the other hand, a flow volume of at least $4 - 4\varepsilon$ traverses edge $e_1$ and is delayed there by all other particles having entered this edge before it (note, that by Claim 1 all particles enter edge $e_1$ during the interval $[1, 3]$ and this edge has a travel time of at least 2). Thus, the total additional delay here is at least
\[\int_0^{4-4\varepsilon} \frac{x}{2} \, dx = 4 - 8\varepsilon + 4\varepsilon^2.\]
This implies
\[\langle \Psi(h), h^2 - h \rangle = \langle \Psi(h), h^2 \rangle - \langle \Psi(h), h \rangle \leq 12\varepsilon - (4 - 8\varepsilon) = 20\varepsilon - 4\]
which is strictly smaller than 0 for $\varepsilon < \frac{1}{5}$. Thus, for such $\varepsilon$ no feasible flow $h$ can be a solution to the variational inequality (7).

Note, that whether or not the instances in Figures 1 and 3 should have an equilibrium ought to depend on the behavioural model, i.e., which alternative strategies are available to individual particles under a given flow. However, a variational inequality of the form of (7) cannot capture such subtleties as it essentially presents every feasible flow as a potential alternative regardless of how different to the current flow it is. In particular, in the first example, the only existing alternative flow can be attained by a collective deviation of all particles and, therefore, should not be relevant for an individual particle’s choice. If, on the other hand, we only consider small “$\varepsilon$-deviations”, the two feasible flows may very well be equilibria as no such small deviation leads to another feasible flow and, thus, individual particles have no potential alternative strategies.

This discussion leads to the following key questions:

- What is a reasonable and useful definition of side-constrained equilibria for dynamic flows?
- Under which conditions can those equilibria be characterized by a variational inequality?
- Under which conditions is existence of such equilibria guaranteed?

We will try to provide answers to all three questions in the following three sections.

5. A General Framework for Side-Constrained Dynamic Equilibria

The general idea of dynamic equilibria is to find a walk inflow $h$ such that (almost) no particle has a better alternative. In the unconstrained case, “alternative” just means any other walk (and departure time) available to that particle’s commodity. In particular, the set of all alternatives of any given particle is independent of the specific flow induced by $h$. This need not be the case anymore for flows with side-constraints as, for example, a certain walk may become unavailable depending on the flow volume already on any particular edge of this walk under $h$. 

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We will now introduce an abstract framework that will allow us to define different types of such side-constrained equilibria. This framework consists of two objects: First, we have a constraint set \( S \subseteq \Lambda(Q) \) defining the set of feasible walk inflows. Second, for every commodity \( i \in I \) and walk \( p \in \mathcal{P}_i \), there is a correspondence

\[
A_p : S \to 2^{\mathcal{P}_i \times \mathcal{M}([t_0, t_f]) \times \mathbb{R}_{\geq 0} \times \mathbb{R}}, \ h \mapsto A_p(h),
\]

mapping every walk inflow \( h \) to the set of admissible \( \varepsilon \)-deviations of commodity \( i \) from walk \( p \). Hereby, \( 2^X \) denotes the power set of \( X \) and \( \mathcal{M}([t_0, t_f]) \) the set of all measurable subsets of \([t_0, t_f] \). An element \((q, J, \varepsilon, \Delta) \in A_p(h) \) then denotes the following admissible \( \varepsilon \)-deviation: Given the walk inflow \( h \), commodity \( i \) is allowed to shift flow at a rate of \( \varepsilon \) in space from walk \( p \) to walk \( q \) and in time from \( J \) to \( J + \Delta \) := \{ t + \Delta \mid t \in J \}. \) For any such \((q, J, \varepsilon, \Delta) \in A_p(h) \) we will denote the walk inflow obtained after this deviation by

\[
H_{p \rightarrow q}(h, J, \varepsilon, \Delta) := (h'_w)_{w \in \mathcal{P}_i} \text{ with } h'_w := \begin{cases} [h_p - \varepsilon \mathbb{1}_J]_+, & \text{for } w = p \\ h_q + \left( h_p^-(\_ - \Delta) - h'_p(\_ - \Delta) \right), & \text{for } w = q \\ h_w, & \text{else} \end{cases}
\]

where

\[
\mathbb{1}_J : [t_0, t_f] \to \mathbb{R}, t \mapsto \begin{cases} 1, & \text{if } t \in J \\ 0, & \text{else} \end{cases}
\]

is the indicator function of the set \( J \) and for any function \( g : [t_0, t_f] \to \mathbb{R} \) the function \([g]_+\) is the non-negative part of \( g \), i.e. the function

\[
[g]_+ : [t_0, t_f] \to \mathbb{R}, t \mapsto \max\{g(t), 0\}.
\]

Furthermore, we implicitly assume that all walk inflows are 0 outside the planning horizon \([t_0, t_f] \). Figure 4 gives an example for how \( H_{p \rightarrow q}(h, J, \varepsilon, \Delta) \) is obtained from \( h \).

![Figure 4: An example for how an admissible \( \varepsilon \)-deviation \((q, J, \varepsilon, \Delta) \in A_p(h) \) changes the path inflow rates on the involved paths \( p \) and \( q \) from the original flow \( h \) (left) to the new flow \( h' := H_{p \rightarrow q}(h, J, \varepsilon, \Delta) \) (right).](image)

From now on we will always require that for any \( h \in S \) and \((q, J, \varepsilon, \Delta) \in A_p(h) \) we have \( H_{p \rightarrow q}(h, J, \varepsilon, \Delta) \in \Lambda(Q) \). This ensures that admissible \( \varepsilon \)-deviations cannot lead to a violation of the walk-inflow bounds \( B_p \) or move inflow outside the planning horizon \([t_0, t_f] \). The latter, in particular, implies that we always have \( H_{p \rightarrow q}(h, J, \varepsilon, \Delta) \in L^2([t_0, t_f])^{\mathcal{P}_i} \) and, thus, \( \Psi \) is well defined on any such flow.

The correspondence

\[
M_i : S \to L^2([t_0, t_f])^{\mathcal{P}_i}, \ h \mapsto \{ h' \in L^2([t_0, t_f])^{\mathcal{P}_i} \mid \exists p \in \mathcal{P}_i, (q, J, \varepsilon, \Delta) \in A_p(h) : h' = H_{p \rightarrow q}(h, J, \varepsilon, \Delta) \}
\]
then returns for any given walk inflow $h$ the set of all possible walk inflows obtained by any of commodity $i$’s admissible $\varepsilon$-deviation. Note that, in general, $M_i$ and $S$ can be completely independent of each other. In particular, a flow obtained by an admissible $\varepsilon$-deviation might be infeasible (i.e. $M_i(h) \not\subseteq S$) and not all feasible flows might be reachable by an admissible $\varepsilon$-deviation even if they only differ by a single commodity’s deviation (i.e. $\{ H_{p\rightarrow q}(h, J, \varepsilon, \Delta) \in S \} \not\subseteq M_i(h)$).

Now, any constraint set $S$ together with a family of deviation correspondences $A_p$ gives rise to the following informal equilibrium concept: A walk inflow $h$ is an equilibrium with respect to $S$ and $A_p$, if it is feasible and no particle can improve by an admissible $\varepsilon$-deviation. To make this mathematically precise, we introduce the concept of admissible deviations to some fixed walk $p$ of commodity $i$ with respect to some $h \in S$ at any fixed time $t \geq 0$ by defining the set

$$U_p(h, t) := \left\{ (q, \Delta) \in \mathcal{P}_i \times \mathbb{R} \mid \forall \delta > 0, \varepsilon > 0 : \exists J \subseteq [t - \delta, t + \delta], \varepsilon' \leq \varepsilon : \int_J \min \{ h_p(t'), \varepsilon' \} \, dt' > 0 \text{ and } (q, J, \varepsilon', \Delta) \in A_p(h) \right\}.$$  

(18)

In words: A walk $q$ and a shift $\Delta$ form an admissible alternative to $p$ at time $t$, if we can shift arbitrarily small amounts of flow in space from $p$ to $q$ and in time from small neighbourhoods of $t$ to small neighbourhoods of $t + \Delta$. This concept can be interpreted as a dynamic extension of the methodology of Bernstein and Smith [1].

Note that, if there is no inflow into a walk $p$ in some neighbourhood of $t$, then $U_p(h, t)$ will always be empty (regardless of the state of the alternative walks). If we assume that the trivial deviation (i.e. shifting flow from $p$ to $p$ and from $J$ to $J$) is always admissible then the converse also holds, i.e. we have: $U_p(h, t)$ is non-empty if and only if there is inflow into $p$ near $t$. We formalize this assumption as

(A5) For any walk inflow $h \in S$, commodity $i \in I$, walk $p \in \mathcal{P}_i$, measurable set $J \in \mathcal{M}([t_0, t_f])$ and $\varepsilon > 0$, we have $(p, J, \varepsilon, 0) \in A_p(h)$.

From now on, we will always assume that the effective walk delay for any given walk inflow $h$ is continuous.

(A6) For any $h \in S$ and $p \in \mathcal{P}$, the function $\Psi_p(h, \cdot) : [t_0, t_f] \to \mathbb{R}$ is continuous.

We can then formally define the concept of a side-constrained dynamic equilibrium in our model as follows:

**Definition 5.1.** Given a graph $G$, a set of commodities $I$, a set of feasible walks $\mathcal{P}$, an effective walk delay operator $\Psi$, a constraint set $S$ and for every walk $p \in \mathcal{P}$ a correspondence $A_p$. Then a feasible flow $h^* \in S$ is a side-constrained dynamic equilibrium (SCDE) wrt. $S$ and $A_p$, if for all $p \in \mathcal{P}$ and all $t \in [t_0, t_f]$ the following condition holds:

$$\Psi_p(h^*, t) \leq \Psi_q(h^*, t + \Delta) \text{ for all } (q, \Delta) \in U_p(h^*, t).$$  

(19)

Note that the above model encompasses both (side-constrained) dynamic equilibria with and without departure time choice. In particular, to disable departure time choice we just choose some $S \subseteq \Lambda(r)$ and define $A_p$ such that it only contains deviations of the form $(q, J, \varepsilon, 0)$.  

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5.1. Some properties of SCDE

We observe that an alternative (negative) definition of side-constrained dynamic equilibria is as follows: A flow is not an equilibrium if there exist two walk-time pairs \((p, t)\) and \((q, t')\) such that the cost of walk \(q\) at time \(t'\) is strictly lower than that of walk \(p\) at time \(t\) and we can shift flow in arbitrarily small neighbourhoods of \(t\) from \(p\) to neighbourhoods of \(t'\) on \(q\). This is formalized in the following lemma:

**Lemma 5.2.** A flow \(h \in S\) is not a SCDE if and only if there exists a commodity \(i \in I\), walks \(p, q \in \mathcal{P}_i\), times \(t, t' \in [t_0, t_f]\), a sequence of measurable sets \(J_n \subseteq [t_0, t_f]\) and a sequence of constants \(\varepsilon_n > 0\) such that

\[
\begin{align*}
\text{a)} & \quad \lim_{n} \varepsilon_n = 0, \\
\text{b)} & \quad \lim_{n} \inf J_n = t = \lim_{n} \sup J_n, \\
\text{c)} & \quad \Psi_p(h, t) > \Psi_q(h, t') \quad \text{and} \\
\text{d)} & \quad \int_{J_n} h_p(\theta) \, d\theta > 0 \quad \text{and} \quad (q, J_n, \varepsilon_n, t' - t) \in A_p(h) \quad \text{for all} \ n \in \mathbb{N}.
\end{align*}
\]

**Proof.** If the conditions from the lemma are satisfied then a), b) and d) together show that \((q, t' - t) \in U_p(h, t)\). Then c) implies that \(h\) is not a SCDE.

If, on the other hand, we know that \(h\) is not a side-constrained dynamic equilibrium, then there must be some \(t \in [t_0, t_f]\), \(i \in I\), \(p, q \in \mathcal{P}_i\) and \(\Delta \in \mathbb{R}\) such that both \(\Psi_p(h, t) > \Psi_q(h, t + \Delta)\) and \((q, \Delta) \in U_p(h, t)\) hold. The latter then implies the existence of \(J_n \in \mathcal{M}([t_0, t_f])\) and \(\varepsilon_n > 0\) satisfying a), b) and d) while the former is exactly c) with \(t' := t + \Delta\).

The next lemma formalizes the intuition that larger sets of admissible \(\varepsilon\)-deviations lead to stronger equilibria.

**Lemma 5.3.** Let \(S\) be a constraint set and \(A_p\) and \(A'_p\) two correspondences defining admissible \(\varepsilon\)-deviations. If \(h^* \in S\) is a SCDE wrt. \(S\) and \(A_p\) and \(A'_p(h^*) \subseteq A_p(h^*)\) holds for every \(p \in \mathcal{P}\) then \(h^*\) is also a SCDE wrt. \(S\) and \(A'_p\).

**Proof.** This follows immediately from the observation that \(A'_p(h^*) \subseteq A_p(h^*)\) implies \(U'_p(h^*, t) \subseteq U_p(h^*, t)\) for all \(t \in [t_0, t_f]\) (where \(U'_p\) denotes the admissible deviations defined by \(A'_p\)).

The following two lemmas show that in many cases, we can wlog. restrict ourselves to smaller sets of feasible walks \(\mathcal{P}_i\) (e.g. in order to make them finite and satisfy (A2)) or assume that the effective walk-delay operators \(\Psi\) are bounded (i.e. satisfy (A1)). For the first lemma we make use of the concept of dominating walks introduced in [16].

**Definition 5.4.** We call a subset \(\mathcal{P}' \subseteq \mathcal{P}\) a dominating walk set, if for every walk \(p \in \mathcal{P}\), \(h \in S\), \(t \in [t_0, t_f]\) and \((q, \Delta) \in U_p(h, t)\), there exists some \((q', \Delta') \in U_p(h, t)\) such that \(q' \in \mathcal{P}'\) and \(\Psi_{q'}(h, t + \Delta') \leq \Psi_q(h, t + \Delta)\).

**Lemma 5.5.** Let \(\mathcal{P}' \subseteq \mathcal{P}\) a dominating walk-set and define

\[
A'_p(h, t) := \{ (q, J, \varepsilon, \Delta) \in A_p(h, t) \mid q \in \mathcal{P}' \}.
\]

Then, any \(h^* \in S' :\{ h \in S \mid h_p \equiv 0 \text{ for all } p \in \mathcal{P} \setminus \mathcal{P}' \}\) is a SCDE wrt. \(S'\) and \(A'_p\) if and only if it is a SCDE wrt. \(S\) and \(A_p\).
Proof. The ‘if’-part follows directly from Lemma 5.3. For the ‘only if’-part, take any walk \( p \in \mathcal{P} \), time \( t \in [t_0, t_f] \) and admissible deviation \((q, \Delta) \in U_p(h^*, t)\). Since \( \mathcal{P}' \) is a dominating walk set, there must be some \((q', \Delta') \in U_p(h^*, t)\) with \( \Psi_q(h^*, t + \Delta') \leq \Psi_q(h^*, t + \Delta) \) and \( q' \in \mathcal{P}' \). Since \( h^* \) is a SCDE, this gives us

\[
\Psi_p(h^*, t) \leq \Psi_q(h^*, t + \Delta') \leq \Psi_q(h^*, t + \Delta).
\]

Therefore, \( h^* \) is a SCDE wrt. \( A_p \).

\[\square\]

**Definition 5.6.** For any \( M \in \mathbb{R} \), we define the truncated effective walk delay operator \( \Psi^M \) of \( \Psi \) by setting

\[
\Psi^M_p(h, t) := \min \{ \Psi_p(h, t), M \} \text{ for all } p \in \mathcal{P}, h \in \Lambda(Q), t \in [t_0, t_f].
\]

**Lemma 5.7.** Given \( M \in \mathbb{R} \) such that for all \( h \in S, t \in [t_0, t_f] \) and \( p \in \mathcal{P} \) with \( U_p(h, t) \neq \emptyset \) there exists some \((q, \Delta) \in U_p(h, t)\) such that \( \Psi_q(h, t + \Delta) < M \), then any walk inflow \( h^* \) is a SCDE with respect to \( \Psi \) if and only if it is a SCDE with respect to \( \Psi^M \).

Proof. First, let \( h^* \) be a SCDE with respect to \( \Psi \). Then, for any \( p \in \mathcal{P}, i \in I \) and \((q, \Delta) \in U_p(h^*, t)\), we have

\[
\Psi^M_p(h^*, t) = \min \{ \Psi_p(h^*, t), M \} \leq \min \{ \Psi_q(h^*, t + \Delta), M \} = \Psi^M_p(h^*, t + \Delta).
\]

Therefore, \( h^* \) is also a SCDE with respect to \( \Psi^M \).

Now, let \( h^* \) be a SCDE with respect to \( \Psi^M \). Again, take any \( p \in \mathcal{P}, i \in I \) and \((q, \Delta) \in U_p(h^*, t)\). Then, by the lemma’s assumption, we have some \((q', \Delta') \in U_p(h^*, t)\) such that \( \Psi_{q'}(h^*, t + \Delta') < M \) implying

\[
\Psi^M_p(h^*, t) \leq \Psi^M_q(h^*, t + \Delta') \leq \Psi_{q'}(h^*, t + \Delta') < M
\]

and, therefore, \( \Psi^M_p(h^*, t) = \Psi_p(h^*, t) \). This, in turn, implies

\[
\Psi_p(h^*, t) = \Psi^M_p(h^*, t) \leq \Psi^M_q(h^*, t + \Delta) \leq \Psi_q(h^*, t + \Delta).
\]

Thus, \( h^* \) is a SCDE with respect to \( \Psi \).

\[\square\]

### 5.2. Some special cases of SCDE

We observe that the standard unconstrained dynamic equilibrium concepts are special cases of our model:

**Lemma 5.8.** Assume that (A6) holds. If we define \( S := \Lambda(Q) \) and

\[
A_p(h) := \left\{ (q, J, \varepsilon, \Delta) \mid q \in \mathcal{P}_i, J \in \mathcal{M}([t_0, t_f]), \varepsilon > 0, J + \Delta \subseteq [t_0, t_f] \text{ and } h_q(t) + \varepsilon \leq B_q \text{ for almost all } t \in J + \Delta \text{ or } q = p \text{ and } \Delta = 0 \right\}
\]

for all \( h \in S, p \in \mathcal{P}_i \) and \( i \in I \), then any flow \( h^* \in S \) is a side-constrained dynamic equilibrium in the sense of (19) if and only if it is a dynamic equilibrium with fixed flow volumes and departure time choice in the sense of (3).
Proof. First, let $h^*$ be an equilibrium in the sense of (3) with corresponding values $v_i$. Now, take any time $t \in [t_0, t_f]$, shift $\Delta$ and walks $\ell, q \in P_i$ such that $\Psi_p(h^*, t) > \Psi_q(h^*, t + \Delta)$. We then have to show that $(q, \Delta)$ is not an admissible deviation, i.e. that $(q, \Delta) \notin U_p(h^*, t)$. For this, we distinguish two cases: If we have $\Psi_q(h^*, t + \Delta) < v_i$, then (A6) guarantees the existence of some neighbourhood $[t - \delta, t + \Delta]$ of $t$ such that $\Psi_q(h^*, t + \Delta) < v_i$ holds for all $t' \in [t - \delta, t + \Delta]$. From (3) we then get that $h^*_p(t') > B_q$ for almost all these $t' \in [t - \delta, t + \Delta]$ which, in turn, implies $(q, \Delta) \notin U_p(h^*, t)$. If, on the other hand, we have $\Psi_q(h^*, t + \Delta) \geq v_i$, then we have $\Psi_p(h^*, t) > v_i$ by our initial assumption. Assumption (A6) then again guarantees the existence of some $\delta > 0$ with $\Psi_p(h^*, t') > v_i$ for all $t' \in [t - \delta, t + \Delta]$. From this (3) allows us to deduce $h^*_p(t') = 0$ for almost all those $t'$ and, hence $U_p(h^*, t) = \emptyset$.

Now, let $h^*$ be an equilibrium in the sense of (19) and assume for contradiction that it is not an equilibrium in the sense of (3). Then, there must exist some commodity $i \in I$ such that there exists no $v_i \in \mathbb{R}$ satisfying (3). In particular, we must have $Q_i > 0$, and therefore

$$v_i := \text{ess sup} \{ \Psi_p(h^*, t) \mid p \in P_i, t \in [t_0, t_f], h^*_p(t) > 0 \} > -\infty$$

where ess sup denotes the essential supremum, i.e.

$$\text{ess sup} \{ g(x) \mid x \in X \} := \inf \{ b \in \mathbb{R} \mid |\{ x \in X \mid g(x) > b \}| = 0 \}$$

where $|\cdot|$ denotes the measure on $X$. Since this $v_i$ cannot satisfy (3), there must be a walk $q \in P_i$, some $\varepsilon, \gamma > 0$ and a set $J_q \subseteq [t_0, t_f]$ of positive measure such that we have

$$h^*_p(t) \leq B_q - \varepsilon \text{ and } \Psi_q(h^*, t) \leq v_i - \gamma \text{ for all } t \in J_q.$$  

At the same time the definition of $v_i$ guarantees the existence of some walk $p \in P_i$ and set $J_p \subseteq [t_0, t_f]$ of positive measure such that

$$h^*_p(t) > 0 \text{ and } \Psi_p(h^*, t) > v_i - \gamma \text{ for all } t \in J_p.$$

Now there must exist some $\Delta \in \mathbb{R}$ such that $(J_p + \Delta) \cap J_q$ has positive measure. Defining $J := J_p \cap (J_q - \Delta)$ gives us a set containing at least one $t \in J$ such that for all $n \in \mathbb{N}$ the set $J_n := [t - \frac{1}{n}, t + \frac{1}{n}] \cap J$ has positive measure. Taking all this together we have

- $\lim_n \inf J_n = t = \lim_n \sup J_n$,
- $\Psi_p(h^*, t) > v_i - \gamma \geq \Psi_q(h^*, t + \Delta)$ since $t \in J \subseteq J_p$ and $t + \Delta \in J + \Delta \subseteq J_q$,
- $\int_{J_n} h^*_p(\theta) d\theta > 0$ for all $n \in \mathbb{N}$ as $J_n \subseteq J_p$ has positive measure and
- $(q, J_n, \frac{\Delta}{n}, \Delta) \in A_p(h^*)$ for all $n \in \mathbb{N}$ since $h^*_q(t) + \frac{\Delta}{n} \leq h^*_q(t) + \varepsilon \leq B_q$ for all $t \in J_n + \Delta \subseteq J_q$.

By Lemma 5.2 this implies that $h^*$ is not a side-constrained dynamic equilibrium – a contradiction to our initial assumption.

Lemma 5.9. Assume that (A6) holds. If we set $S := A(h)$ and

$$A_p(h) := \{ (q, J, \varepsilon, 0) \mid q \in P_i, J \in \mathcal{M}([t_0, t_f]), \varepsilon > 0 \}$$

for all $h \in S$, $p \in P_i$ and $i \in I$, then, any flow $h^* \in S$ is a side-constrained dynamic equilibrium in the sense of (19) if and only if it is a dynamic equilibrium with fixed inflow rate in the sense of (2).
Proof. First, let \( h^* \) be an equilibrium in the sense of (2). Then, for any point \( t \in [t_0,t_f] \) and walks \( p,q \in \mathcal{P}_t \) with \( \Psi_p(h^*,t) > \Psi_q(h^*,t) \) assumption (A6) guarantees the existence of some neighbourhood \( [t - \delta,t + \delta] \) of \( t \) with \( \Psi_p(h^*,t') > \Psi_q(h^*,t') \) for all \( t' \in [t - \delta,t + \delta] \). Equation (2) then implies that \( h^*_p(t') = 0 \) for almost all \( t' \) from this interval and, thus, \( U_p(h^*,t) = \emptyset \). In particular, we have \( q \notin U_p(h^*,t) \) which shows that (19) is satisfied by \( h^* \).

Now, let \( h^* \) be an equilibrium in the sense of (19). Then, for every time \( t \in [t_0,t_f] \) and each pair of walks \( p,q \) with \( \Psi_p(h^*,t) > \Psi_q(h^*,t) \), we have \( q \notin U_p(h^*,t) \). As we are in the unconstrained case, this can only be because there exists some \( \delta_t > 0 \) such that for any \( J \subseteq [t - \delta_t,t + \delta_t] \) and \( \varepsilon > 0 \) we have \( \int f \min \{ h^*_p(t'),\varepsilon \} \, dt' = 0 \) or, equivalently, \( h^*_p(t') = 0 \) for almost all \( t' \in [t - \delta_t,t + \delta_t] \). Since this is true for all such times \( t \), we clearly have

\[
\{ t \in [t_0,t_f] \mid \Psi_p(h^*,t) > \Psi_q(h^*,t) \} \subseteq \bigcup_{t \in [t_0,t_f] : \Psi_p(h^*,t) > \Psi_q(h^*,t)} [t - \delta_t, t + \delta_t].
\]

But then there also exists a countable such covering, i.e. a countable set

\[
K \subseteq \{ t \in [t_0,t_f] \mid \Psi_p(h^*,t) > \Psi_q(h^*,t) \}
\]

such that

\[
\{ t \in [t_0,t_f] \mid \Psi_p(h^*,t) > \Psi_q(h^*,t) \} \subseteq \bigcup_{t \in K} [t - \delta_t, t + \delta_t].
\]

Therefore, for almost all \( t \in \{ t \in [t_0,t_f] \mid \Psi_p(h^*,t) > \Psi_q(h^*,t) \} \) we have \( h^*_p(t') = 0 \) which shows that \( h^* \) satisfies (2). \qed

In general our model requires us to define two objects in order to completely specify a type of side-constraint dynamic equilibrium: The constraint set \( S \) and the admissible \( \varepsilon \)-deviations \( A_p \). We will make use of this flexibility later on (in particular in Section 7) but we can also just take any constraint set \( S \subseteq \Lambda(Q) \) and then derive admissible \( \varepsilon \)-deviations from it: Namely, we can say that a potential \( \varepsilon \)-deviation is admissible if and only if it leads to another feasible flow, i.e.

\[
A_p(h) := \{ (q,J,\varepsilon,\Delta) \mid H_{p \to q}(h,J,\varepsilon,\Delta) \in S \}
\]

or, equivalently,

\[
M_i(h) := \{ h' \in L^2([t_0,t_f])^P \mid \exists p,q,i,J,\varepsilon,\Delta : h' = H_{p \to q}(h,J,\varepsilon,\Delta) \in S \}.
\]

This imposes a global feasibility constraint on the possible deviations: That is, particles are only allowed to deviate if the resulting alternative flow is feasible for all particles (not just the ones deviating). This is a generalization of the capacitated dynamic equilibria defined in [16]. Here, we will call these types of side-constrained dynamic equilibria strict side-constrained dynamic equilibria.

**Definition 5.10.** A strict side-constrained dynamic equilibrium (strict SCDE) with respect to some set \( S \subseteq \Lambda(Q) \) is a side-constrained dynamic equilibrium wrt. \( S \) and admissible \( \varepsilon \)-deviations \( A_p(h) \) defined by (20).

**6. Characterization of SCDE via (Quasi-)Variational Inequalities.**

Similar to the characterizations of unconstrained dynamic equilibria with variational inequalities (cf. Theorem 3.3) we now want to characterize side-constrained dynamic equilibria using quasi-variational inequalities (QVIs). This is not only of theoretical interest.
but there are also algorithms solving them, see e.g. Kanzow and Steck [29], Shehu et al. [42] and references therein. Note, however, that the convergence guarantees given by Shehu et al. require a certain strong monotonicity property for the mapping \( h \mapsto \Psi(h, \cdot) \) which, in general, is not satisfied for dynamic flows whereas Kanzow and Steck only need weak-strong continuity but instead require some convexity assumptions on the sets \( S \) and \( M(h) \).

6.1. Quasi-Variational-Inequality

Let us define the following nonnegative tangent cone with respect to \( S, A_p \) and \( h \).

\[
T(S, A_p, h) := \left\{ v \in L^2([t_0, t_f])^P \left| \exists (h^n)_{n \in \mathbb{N}} \subset M(h), (t_n)_{n \in \mathbb{N}} \subset \mathbb{R}_{>0} : \lim_{n \to \infty} t_n = 0, \lim_{n \to \infty} \frac{h^n - h}{t_n} = v \right. \right\}.
\]

where \( M(h) := \bigcup_{i \in I} M_i(h) \) is the set of all walk inflows reachable by a single admissible \( \varepsilon \)-deviation from \( h \).

We introduce the following assumptions on the correspondences \( A_p \):

**Definition 6.1.** The correspondence \( (A_p)_{p \in P} \) are closed under rate-reduction at \( h \in S \), if for all \( i \in I, q, p \in P_i, \varepsilon, \varepsilon', \Delta \) and \( J \), we have:

\[
(q, J, \varepsilon, \Delta) \in A_p(h), \varepsilon' \leq \varepsilon \implies (q, J, \varepsilon', \Delta) \in A_p(h).
\]

**Definition 6.2.** The correspondences \( (A_p)_{p \in P} \) are closed under time-restriction at \( h \in S \), if for all \( i \in I, q, p \in P_i, \varepsilon, \Delta, J' \) and \( J \), we have:

\[
(q, J, \varepsilon, \Delta) \in A_p(h), J' \subseteq J \implies (q, J', \varepsilon, \Delta) \in A_p(h).
\]

(A7) The admissible \( \varepsilon \)-deviations are closed under rate-reduction at all \( h \in S \).

(A8) The admissible \( \varepsilon \)-deviations are closed under time-restriction at all \( h \in S \).

The first assumption states that whenever flow is allowed to deviate at a certain rate, it is also allowed to deviate at any lower rate. The second assumption states that whenever flow is allowed to deviate during a certain neighbourhood, the same deviation is also allowed during any subset of that neighbourhood. It will turn out that several well-motivated equilibrium concepts fulfil both assumptions (see Observation 7.5 for details).

**Observation 6.3.** Under (A5), assumption (A7) is a weaker assumption than convexity, i.e. if \( M_i(h) \) is convex (at \( h \)), then (A7) holds. This is true because for \( \varepsilon' \leq \varepsilon \), the walk inflow \( H^r_{p \to q}(h, J, \varepsilon', \Delta) \) is a convex combination of \( h = H^r_{p \to q}(h, J, \varepsilon, 0) \) and \( H^r_{p \to q}(h, J, \varepsilon, \Delta) \):

\[
H^r_{p \to q}(h, J, \varepsilon', \Delta) = (1 - \frac{\varepsilon'}{\varepsilon}) \cdot H^r_{p \to q}(h, J, \varepsilon, 0) + \frac{\varepsilon'}{\varepsilon} \cdot H^r_{p \to q}(h, J, \varepsilon, \Delta).
\]

Assumption (A8) is independent of convexity, i.e. there exists a set \( M_i(h) \) which is convex but does not satisfy (A8) and there exists a set \( M_i(h) \) which satisfies (A8) but is not convex. E.g., consider a network with a single commodity with a fixed inflow rate \( r = 1_{[0,2]} \), two nodes \( s \) and \( t \) and two parallel edges \( e_1 \) and \( e_2 \) connecting \( s \) and \( t \) (these are then also the only \( s,t \)-paths \( p \) and \( q \)). We define the following sets \( S \subseteq \Lambda(r) \) with corresponding sets \( A_p(h) := \{ (q, J, \varepsilon, 0) \mid H^r_{p \to q}(h, J, \varepsilon, 0) \in S \} \):

- \( S_1 \) contains all walk inflows \( h \in \Lambda(r) \) which at any point in \( [0,2] \) sent all flow in exactly one of the two paths. Clearly, the resulting admissible \( \varepsilon \)-deviations satisfy (A8). The set \( S \) is, however, not convex as it contains the walk inflow \( h^1 \) which sends all flow into \( p \) as well as the walk inflow \( h^2 \) which sends all flow into \( q \) but not any of their non-trivial convex combinations.
• $S_2 := \text{conv}(h^1, h^2)$ is a convex set, but the corresponding sets $A_p(h)$ do not satisfy 
(A8) since we have $H^*_{p ightarrow q}(h^1, [0, 2], 1, 0) = h^2 \in S_2$ but $H^*_{p ightarrow q}(h^1, [0, 1], 1, 0) \notin S_2$.

We now consider the following quasi-variational inequality:

Find $h^* \in S$ such that:
\[
\langle \Psi(h^*), v \rangle \geq 0 \quad \text{for all } v \in T(S, A_p, h^*). \tag{QVI($\Psi, S, A_p$)}
\]

**Theorem 6.4.** Let $h^* \in S$ be given such that $S$ and $(A_p)_{p \in P}$ satisfy (A7) at $h^*$ and $\Psi$ such that (A6) holds. If $h^*$ is a solution to the quasi-variational inequality (QVI($\Psi, S, A_p$)) then it is a SCDE wrt. $S$ and $(A_p)_{p \in P}$.

**Proof.** Let $h^* \in S$ be a solution to (QVI($\Psi, S, A_p$)) and assume that $h^*$ is not a SCDE. Then there exists a commodity $i$, walks $p, q \in \mathcal{P}$, a shift $\Delta$ and some time $t \in [t_0, t_f]$ such that $\Psi_p(h^*, t) > \Psi_q(h^*, t + \Delta)$ and $(q, \Delta) \in U_p(h^*, t)$. Since $\Psi_q(h^*, \cdot)$ and $\Psi_p(h^*, \cdot)$ are continuous (by (A6)) there must be some constants $\delta, \gamma > 0$ such that $\Psi_p(h^*, t'') - \Psi_q(h^*, t'' + \Delta) \geq \gamma$ holds for all $t'' \in [t - \delta, t + \delta]$. From $(q, \Delta) \in U_p(h^*, t)$ we then get some constant $\varepsilon > 0$ and a measurable set $J \subseteq [t - \delta, t + \delta]$ with $\int_J \min \{h_p^*(t'), \varepsilon\} \cdot dt' > 0$ and \( h := H^*_{p \rightarrow q}(h^*, J, \varepsilon, \Delta) \in M(h^*) \). Because of (A7), we now have $h - h^* \in T(S, A_p, h^*)$. But at the same time we have

\[
\langle \Psi(h^*), h - h^* \rangle = \sum_{w \in P} \int_{t_0}^{t_f} \Psi_w(h^*(t')) \cdot \left(\bar{h}_w(t') - h^*_w(t')\right) \, dt'
\]

\[
= \int \Psi_p(h^*, t') \cdot \left(\bar{h}_p(t') - h^*_p(t')\right) \, dt' + \int_{t + \Delta} \Psi_q(h^*, t') \cdot \left(\bar{h}_q(t') - h^*_q(t')\right) \, dt'
\]

\[
= \int \Psi_p(h^*, t') \cdot \left(\bar{h}_p(t') - h^*_p(t')\right) + \Psi_q(h^*, t' + \Delta) \cdot \left(\bar{h}_q(t' + \Delta) - h^*_q(t' + \Delta)\right) \, dt'
\]

\[
= \int (\Psi_q(h^*, t' + \Delta) - \Psi_p(h^*, t')) \cdot \min \{h_p^*(t'), \varepsilon\} \, dt'
\]

\[
\leq -\gamma \cdot \int \min \{h_p^*(t'), \varepsilon\} \, dt' < 0,
\]

which is a contradiction to $h^*$ being a solution to (QVI($\Psi, S, A_p$)). 

**Remark 6.5.** To see why we need assumption (A7) in the statement of Theorem 6.4 consider again the constraint set $S_1$ from Observation 6.3 and define

\[ A_p(h^1) := \{(q, J, |J|, 0) \mid J \in \mathcal{M}([0, 2])\} \]

Then we have $T(S_1, A_p, h^1) = \{0\}$ and, thus, $h^1$ is a solution to (QVI($\Psi, S, A_p$)) (regardless of the choice of effective walk delay operators) while it is not necessarily a SCDE as we have $U_p(h^1, t) = \{(p, 0), (q, 0)\}$ for all $t \in [0, 2]$ here.

**Theorem 6.6.** Let $h^* \in S$ be given such that $S$ and $(A_p)_{p \in P}$ satisfy (A7) and (A8) at $h^*$. If $h^*$ is a SCDE wrt. $S$ and $(A_p)_{p \in P}$ then it is also a solution to (QVI($\Psi, S, A_p$)).

**Proof.** We show this by contradiction. So, let $h^*$ be a SCDE and assume that there is some $v \in T(S, A_p, h^*)$ with $\langle \Psi(h^*), v \rangle < 0$. By continuity of $\langle., .\rangle$, there exist $h^n \in M(h^*)$ and $t_n > 0$ such that

\[ \langle \Psi(h^*), \frac{h^n - h^*}{t_n} \rangle < 0 \text{ or, equivalently, } \langle \Psi(h^*), h^n - h^* \rangle < 0. \]
Rewriting and using that \( h^a \) is of the form (17), i.e. \( h^a = H_{p \rightarrow q}^r(h^*, J, \varepsilon, \Delta) \) for some \((q, J, \varepsilon, \Delta) \in A_p(h^*)\), yields

\[
0 > \langle \Psi(h^*), h^a - h^* \rangle = \int_J \Psi_p(h^*, t)(h^*_p(t) - h^*_p(t)) \, dt + \int_{J+\Delta} \Psi_q(h^*, t)(h^*_q(t) - h^*_q(t)) \, dt
\]

\[
= \int_J \Psi_p(h^*, t)(h^*_p(t) - h^*_p(t)) + \Psi_q(h^*, t + \Delta)(h^*_q(t + \Delta) - h^*_q(t + \Delta)) \, dt
\]

\[
= \int (\Psi_q(h^*, t + \Delta) - \Psi_p(h^*, t)) \cdot \min \{ h^*_p(t), \varepsilon \} \, dt.
\]

This implies that there is some subset \( J' \subseteq J \) of positive measure with

\[
(\Psi_q(h^*, t + \Delta) - \Psi_p(h^*, t)) \cdot \min \{ h^*_p(t), \varepsilon \} < 0 \quad \text{for all } t \in J'.
\]

Since \( \min \{ h^*_p(t), \varepsilon \} \) is non-negative, this implies \( \Psi_q(h^*, t + \Delta) < \Psi_p(h^*, t) \) for all \( t \in J' \). As \( J' \) has positive measure, it must contain a point \( t \in J' \) such that the intersection of any neighbourhood of \( t \) with \( J' \) also has positive measure. Defining \( J_n := [t - \frac{1}{n}, t + \frac{1}{n}] \cap J' \) then results in a sequence of subsets of \( J' \) of positive measure satisfying \( \lim \sup_n J_n = t \). Since we have both \( H_{p \rightarrow q}(h^*, J, \varepsilon, \Delta) \in M(h^*) \) and \( J_n \subseteq J \), assumptions (A7) and (A8) ensure that \( H_{p \rightarrow q}^r(h^*, J_n, \frac{1}{n}, \Delta) \in M(h^*) \) holds for all \( n \in \mathbb{N}^* \) as well. Furthermore, we have \( \int_{J_n} h^*_p(t') \, dt' > 0 \) for all \( n \) since \( h^*_p(t') > 0 \) for all \( t' \in J' \) and \( J_n \) has positive measure. Altogether, this shows that \( h^* \) is not a SCDE by Lemma 5.2.

**Remark 6.7.** To see why we need assumption (A8) in the statement of Theorem 6.6, consider again the constraint set \( S_2 \) from Observation 6.3 and define the set of admissible \( \varepsilon \)-deviations \( A_p(h^1) := \{ (q, [0, 2], \varepsilon, 0) \mid \varepsilon \geq 0 \} \cup \{ (p, J, \varepsilon, 0) \mid J \in \mathcal{M}([0, 2], \varepsilon \geq 0) \} \) as in the observation. Clearly, this set \( A_p(h^1) \) is not closed under time restriction. Furthermore, we have \( U_p(h^1, t) = \{ (p, 0) \} \) for all \( t \in [0, 2] \) and, therefore, \( h^1 \) is a SCDE. However, it is also easy to see that for certain choices of the effective walk delay operators \( \Psi_p \) and \( \Psi_q \) the flow \( h^1 \) is not a solution to the quasi-variational inequality (QVI(\( \Psi, S, A_p \))) (e.g. choose constant flow independent delays \( \Psi_p \equiv 2 \) and \( \Psi_q \equiv 1 \)).

### 6.2. Variational Inequality

Quasi-variational inequalities may be much harder to solve compared to standard variational inequalities since the feasible search space depends on the solution itself. However, under an additional assumption, we can also use the following variational inequality to characterize SCDE:

\[
\text{Find } h^* \in S \text{ such that :} \\
\langle \Psi(h^*), h - h^* \rangle \geq 0 \quad \text{for all } h \in S. \\
\text{(VI(} \Psi, S)\text{)}
\]

Note, that this variational inequality is of exactly the form of (7) used by Zhong et al. [45] to define their version of side-constrained dynamic equilibria.

For the sufficiency part, we need the following additional assumption stating that small enough admissible \( \varepsilon \)-deviations from a feasible flow lead to another feasible flow:

(A9) For any \( h \in S \) there exists some neighbourhood \( V_h \) of \( h \) such that \( M(h) \cap V_h \subseteq S \).

Note that this assumption is trivially satisfied for strict SCDE as, in this case, by definition any admissible \( \varepsilon \)-deviation leads to a feasible flow.

**Theorem 6.8.** Take any constraint set \( S \) and admissible \( \varepsilon \)-deviations \( (A_p)_{p \in P} \) satisfying (A7) and (A9). Furthermore, assume that \( \Psi \) satisfies (A6). Then, any solution \( h^* \) to the variational inequality (VI(\( \Psi, S)\)) is a SCDE wrt. \( S \) and \( (A_p)_{p \in P} \).
Proof. Let \( h^* \in S \) be a solution to the variational inequality \((\text{VI}(\Psi, S))\). We claim that \( h^* \) is then also a solution to the quasi-variational inequality \((\text{QVI}(\Psi, S, A_p))\). In order to show this, take any \( v \in T(S, A_p, h^*) \). Then there exist sequences of \((h^n)_{n \in \mathbb{N}} \subset M(h^*)\) and \((t_n)_{n \in \mathbb{N}} \subset \mathbb{R}_{>0}\) such that \( \lim_{n \to \infty} \frac{h^n - h^*}{t_n} = v \). Using the continuity of \( \langle \cdot, \cdot \rangle \), we get

\[
\langle \Psi(h^*), v \rangle = \langle \Psi(h^*), \lim_{n \to \infty} \frac{h^n - h^*}{t_n} \rangle = \lim_{n \to \infty} \frac{1}{t_n} \langle \Psi(h^*), h^n - h^* \rangle \geq 0.
\]

The last inequality holds since \( h^n \in M(h) \) implies \( h^n \in S \) for large enough \( n \) due to (A9) and since \( h^* \) is a solution to \((\text{VI}(\Psi, S))\). Thus, we can now apply Theorem 6.4 to conclude that \( h^* \) is indeed a SCDE.

Now, whenever in addition to the assumptions of Theorem 6.8 we know that \((\text{VI}(\Psi, S))\) has a solution, we get a first existence result for SCDE generalizing the existence theorem for unconstrained dynamic equilibria (Theorem 3.5):

**Corollary 6.9.** Let \( S \) be a convex, non-empty, closed and bounded set and \( A_p \) satisfying (A7) and (A9). Furthermore, assume that (A2) holds and \( \Psi \) satisfies (A1), (A4) and (A6). Then there exists a SCDE wrt. \( S \) and \( A_p \).

**Proof.** By Theorem 3.4 the variational inequality \((\text{VI}(\Psi, S))\) has a solution which, by Theorem 6.8, is a SCDE. \(\square\)

**Remark 6.10.** Note that for strict SCDE assumption (A9) holds automatically and (A7) follows from the convexity of \( S \). In particular the existence result for capacitated dynamic equilibria in [16, Theorem 6] is a special case of the above corollary.

On the other hand, the counterexamples from Section 4 also show that assumption (A7) does not necessarily hold for strict SCDE with \( S \) defined by volume-constraints. Thus, even in cases where the variational inequality \((\text{VI}(\Psi, S))\) has a solution, it is not clear whether such a solution is also a SCDE.

For the necessity part we only consider the case of fixed network inflow rates and require the following additional property of the constraint set \( S \) and the admissible \( \varepsilon \)-deviations \( A_p \).

**Definition 6.11.** The set \( S \subseteq \Lambda(r) \) is called **closed with respect to elementary directions**, if for all \( h, h' \in S \) the following holds true: Whenever there exist \( i \in I, p, q \in P_i \) and \( J \subseteq [t_0, t_f] \) with positive measure such that \( h_p(t) - h'_p(t) > 0 \) and \( h_q(t) - h_q(t) > 0 \) for all \( t \in J \), we have \((q, 0) \in U_p(h, t)\) for some \( t \in J \).

This property states that in any walk inflow \( h \in S \) particles are allowed to switch from some walk \( p \) to another walk \( q \) if there exists another feasible walk inflow \( h' \) which has a lower inflow rate into \( p \) and (during the same time) a higher inflow rate into \( q \).

**Theorem 6.12.** Take any constraint set \( S \subseteq \Lambda(r) \) with fixed inflow rates and admissible \( \varepsilon \)-deviations \((A_p)_{p \in P}\) such that \( S \) is closed with respect to elementary directions. Then, every SCDE \( h^* \in S \) wrt. \( S \) and \((A_p)_{p \in P}\) is a solution to the variational inequality \((\text{VI}(\Psi, S))\).

**Proof.** Let \( h^* \in S \) be a SCDE. We have to show that \( h^* \) is a solution to \((\text{VI}(\Psi, S))\), i.e. that for any \( h \in S \) we have \( \langle \Psi(h^*), h - h^* \rangle \geq 0 \). So, assume for contradiction that this is not the case, i.e. there exists some \( h \in S \) with \( \langle \Psi(h^*), h - h^* \rangle < 0 \).

We now define for any pair of walks \( p \) and \( q \) a function \( g_{p \to q} : [t_0, t_f] \to \mathbb{R}_{\geq 0} \) by

\[
g_{p \to q}(t) := \begin{cases} 
(h_p(t) - h_p^*(t)) \cdot \sum_{\substack{q' \in P_i \mid h_q'(t) < h_q^*(t) \land h_q'(t) < h_q^*(t)}} \frac{h_{q'}^*(t) - h_q(t)}{h_{q'}^*(t) - h_q(t)}, & \text{if } h_p(t) > h_p^*(t), h_q(t) < h_q^*(t) \\
0, & \text{else}.
\end{cases}
\]
First, we observe that these functions are non-negative, bounded, well-defined and measurable. Now we define for each of these functions a corresponding function \( h_{p\rightarrow q} \) \( \in L^2([t_0,t_f])^P \) by setting

\[
(h_{p\rightarrow q})_w(t) := \begin{cases} 
0, & \text{if } w \notin \{p,q\} \\
g_{p\rightarrow q}(t) \in \mathbb{R}_+, & \text{if } w = p \\
-g_{p\rightarrow q}(t) \in \mathbb{R}_-, & \text{if } w = q.
\end{cases}
\] (22)

We now claim that these functions add up to precisely the difference between \( h \) and \( h^* \):

**Claim 4.** We have \( h - h^* = \sum_{p,q \in P} h_{p\rightarrow q} \).

**Proof.** Let \( w \in P \) be any walk and \( t \in [t_0, t_f] \) be any time. Then, we distinguish three cases:

**Case 1**  \( (h_w(t) = h^*_w(t)) \): In this case, we have \( g_{p\rightarrow w}(t) = 0 = g_{w\rightarrow q}(0) \) for all \( p,q \in P \) and, thus,

\[
\left( \sum_{p,q \in P} h_{p\rightarrow q} \right)_w = \left( \sum_{q \in P} h_{p\rightarrow w} \right)_w + \left( \sum_{q \in P} h_{w\rightarrow q} \right)_w = \sum_{p \in P} -g_{p\rightarrow w} + \sum_{q \in P} g_{w\rightarrow q} = 0 = (h_w(t) - h^*_w(t)).
\]

**Case 2**  \( (h_w(t) > h^*_w(t)) \): In this case, we have \( g_{p\rightarrow w}(t) = 0 \) for all \( p \in P \) and \( g_{w\rightarrow q}(t) = 0 \) for all \( q \in P \) with \( h_q(t) \geq h^*_q(t) \). For all other \( q \), we have \( g_{w\rightarrow q}(t) = (h_w(t) - h^*_w(t)) \cdot \frac{h^*_q(t) - h_q(t)}{\sum_{q' \in P : h_{q'}(t) < h^*_q(t)} \left(h^*_q(t) - h_{q'}(t)\right)} \) and, thus,

\[
\left( \sum_{p,q \in P} h_{p\rightarrow q} \right)_w = \sum_{p \in P} -g_{p\rightarrow w} + \sum_{q \in P} g_{w\rightarrow q} = 0 + \sum_{q \in P : h_q(t) < h^*_q(t)} (h_w(t) - h^*_w(t)) \cdot \frac{h^*_q(t) - h_q(t)}{\sum_{q' \in P : h_{q'}(t) < h^*_q(t)} \left(h^*_q(t) - h_{q'}(t)\right)} \]

\[
= h_w(t) - h^*_w(t).
\]

Note, that we need fixed inflow rates (i.e. \( S \subseteq \Lambda(r) \)) here to ensure that there exists at least one walk \( q \) with \( h_q(t) - h^*_q(t) \). This is used in the last equality.

**Case 3**  \( (h_w(t) < h^*_w(t)) \): In this case, we have \( g_{w\rightarrow q}(t) = 0 \) for all \( q \in P \) and \( g_{p\rightarrow w}(t) = 0 \) for all \( p \in P \) with \( h_p(t) \leq h^*_p(t) \). For all other \( p \), we have \( g_{p\rightarrow w}(t) = (h_p(t) - h^*_p(t)) \cdot \frac{h^*_w(t) - h_w(t)}{\sum_{q' \in P : h_{q'}(t) < h^*_w(t)} \left(h^*_w(t) - h_{q'}(t)\right)} \) and, thus,

\[
\left( \sum_{p,q \in P} h_{p\rightarrow q} \right)_w = \sum_{p \in P} -g_{p\rightarrow w} + \sum_{q \in P} g_{w\rightarrow q} = -\sum_{p \in P : h_p(t) > h^*_p(t)} (h_p(t) - h^*_p(t)) \cdot \frac{h^*_w(t) - h_w(t)}{\sum_{q' \in P : h_{q'}(t) < h^*_w(t)} \left(h^*_w(t) - h_{q'}(t)\right)} + 0 \]

\[
= -(h^*_w(t) - h_w(t)).
\]
Now, using our initial assumption \( \langle \Psi(h^*), h - h^* \rangle < 0 \), we get
\[
0 > \langle \Psi(h^*), h - h^* \rangle = 4 \sum_{p,q \in P} h_{p \to q}
= \sum_{p,q \in P} \int_{t_0}^{t_f} \Psi_p(h^*, t) \cdot (h_{p \to q})_w(t) dt
= \sum_{p,q \in P} \int_{t_0}^{t_f} \Psi_p(h^*, t) \cdot g_{p \to q}(t) dt + \int_{t_0}^{t_f} \Psi_q(h^*, t) \cdot (-g_{p \to q}(t)) dt
= \sum_{p,q \in P} \int_{t_0}^{t_f} g_{p \to q}(t)(\Psi_p(h^*, t) - \Psi_q(h^*, t)) dt.
\]
Since \( g_{p \to q} \geq 0 \), there must be a subset \( J \subset [t_0, t_f] \) of positive measure with \( g_{p \to q}(t) > 0 \) and \( \Psi_p(h^*, t) - \Psi_q(h^*, t) < 0 \) for all \( t \in J \). As \( g_{p \to q}(t) > 0 \) implies \( h_p^*(t) - h_p(t) < 0 \) and \( h_q^*(t) - h_q(t) > 0 \) and \( S \) is closed with respect to elementary directions, we get \( (p,0) \in U_q(h^*, t) \) for some time \( t \in J \) which, together with \( \Psi_p(h^*, t) - \Psi_q(h^*, t) < 0 \), contradicts that \( h^* \) is a SCDE.

**Observation 6.13.** For any fixed time \( t \in [t_0, t_f] \) the values \( g_{p \to q}(t), p,q \in P \) defined in the proof above solve a transshipment problem defined as follows: We create a complete bipartite graph \( G = (V_1(t) \cup V_2(t), E(t)) \), where nodes in \( V_1(t) \subset P \) are surplus nodes, that is, they fulfill \( b_p(t) := h_p(t) - h_p^*(t) > 0 \), and nodes in \( V_2(t) \subset P \) are deficit nodes fulfilling \( b_q(t) := h_q(t) - h_q^*(t) < 0 \). Note that obviously \( V_1(t) \cap V_2(t) = \emptyset \) for all \( t \in [t_0, t_f] \). For every arc \( (p,q) \in E(t) := V_1(t) \times V_2(t) \) we define capacities \( c_{(p,q)}(t) := \min\{b_p(t), b_q(t)\} \).

**Remark 6.14.** Clearly \( S = \Lambda(r) \) and \( A_p \) defined by (20) also satisfy the assumptions of Theorem 6.12. Thus, Theorems 6.8 and 6.12 together are a generalization of the characterization of unconstrained dynamic equilibria with fixed inflow rates by variational inequalities (i.e. Theorem 3.3).

Another interesting example for which Definition 6.11 applies is the case of monotone box-constraints.

**Example 6.15.** Consider continuous and non-decreasing functions \( z_p : \mathbb{R}_+ \to \mathbb{R}_+, p \in P \) and continuous functions \( v_p : [t_0, t_f] \to \mathbb{R}_+, p \in P \). We get that the set
\[
S := \{ h \in \Lambda(r) \mid z_p(h_p(t)) \leq v_p(t) \text{ for all } p \in P \text{ and almost all } t \in [t_0, t_f] \}
\]
is closed with respect to elementary directions. To see this, let \( h, h' \in S \) with \( h_p(t) - h'^p(t) > 0 \) and \( h_q'(t) - h_q(t) > 0 \) for all \( t \in J, J \) is some set of positive measure.

Then, there must also exist some \( \varepsilon > 0 \) and a subset \( J' \subseteq J \) of positive measure with \( h_p(t) - h'^p(t) \geq \varepsilon \) and \( h_q'(t) - h_q(t) \geq \varepsilon \) for all \( t \in J' \). We can then construct a sequence of sets \( J_n \subseteq J' \) of positive measure satisfying \( J_{n+1} \subseteq J_n \) and \( \lim \inf J_n = \lim \sup J_n = t \) for some \( t \in J' \). Then, for any \( n \in \mathbb{N}^* \) we have
\[
H^p_{p \to q}(h, J_n, \frac{z_p}{n}, 0)_w(t) \leq \max \{ h_w(t), h'_w(t) \}
\]
for all \( t \in J_n \) and \( w \in P \). Since both \( h \) and \( h' \) are from \( S \) this then implies
\[
z_w\left(H^p_{p \to q}(h, J_n, \frac{z_p}{n}, 0)_w(t)\right) \leq z_w\left(\max \{ h_w(t), h'_w(t) \}\right)
= \max \{ z_w(h_w(t)), z_w(h'_w(t)) \} \leq v_w(t)
\]
for almost all \( t \in J_n \) and, thus, \( H^p_{p \to q}(h, J_n, \frac{z_p}{n}, 0) \in S \). Since \( \int_{J_n} \min \{ h_p(t'), \frac{z_p}{n} \} \) dt' > 0 and \( t \in [\inf J_n, \sup J_n] \) hold as well, this is enough to show \( (q,0) \in U_p(h,t) \).
7. Edge Volume Constraints and Network-Loading

In this section, we will now come back to more concrete side-constrained dynamic equilibria, where flows are feasible whenever they obey certain “capacity constraints” on the edges of the network. For the following definitions we will only consider the case of fixed flow volume (and free departure time choice) but note that all definitions can be easily transferred to the case of fixed network inflow rates by choosing $S \cap \Lambda(r)$ instead of the sets $S$ defined here and only allowing admissible $\varepsilon$-deviations with $\Delta = 0$. Furthermore, to simplify the notation we will not explicitly state the condition $H_{p \rightarrow q}(h, \varepsilon, \Delta) \in \Lambda(Q)$ on elements of $A_p(h)$ whenever we define such sets here, i.e. this condition should always be implicitly added when reading a definition of $A_p(h)$ in this section.

In order to formally define SCDE with capacity constraints, we assume that for every edge $e$, we are given a capacity function $c_e : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ and that the flow model is equipped with a weak form of network-loading associating with every walk inflow $h$ two types of functions:

- **Arrival time** functions $\tau^j_p(h,.) : [t_0, t_f] \rightarrow \mathbb{R}_{\geq 0}$ such that for every time $t \in [t_0, t_f]$, walk $p = (v_1, v_2, \ldots, v_{|p|+1}) \in \mathcal{P}$ and $j \in \{1, \ldots, |p| + 1\}$, the value $\tau^j_p(h, t)$ denotes the time at which a particle entering walk $p$ at time $t$ will arrive at the $j$-th node on walk $p$ (or, equivalently leaves the $(j - 1)$-th edge/enters the $j$-th edge of walk $p$). In particular, $\tau^1_p(h, t)$ denotes the time at which a particle starts its journey and $\tau^{|p|+1}_p(h, t)$ denotes the time at which the particle arrives at the end of walk $p$ (i.e. the sink).

- **Edge-load** functions $f_e(h,.) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ such that for every edge $e$ and time $\theta \in \mathbb{R}_{\geq 0}$, the value $f_e(h, \theta)$ denotes some measure of the flow induced by $h$ on edge $e$ at time $\theta$ and which, in a feasible flow, has to be bounded by the edge capacity $c_e(\theta)$.

**Remark 7.1.** Given a model with a full network-loading (as described in Section 4), the arrival time functions would be defined using the corresponding walk-delay functions $D_p$, i.e.

$$\tau^j_p(h, t) := t + D_{p|j-1}(h, t),$$

where $p_{j-1}$ is the prefix of $p$ of length $j - 1$. The edge-load function could then, for example, be the flow volume $x_e(h, \theta)$, the queue length $q_e(h, \theta)$, the cumulative inflow $\int_0^\theta f^+(\zeta) \, d\zeta$ or the current edge inflow rate $f^+_e(\theta)$.

We now define the set of all walk inflows resulting in network flows obeying these edge capacities by

$$S := \{ h \in \Lambda(Q) \mid f_e(h, \theta) \leq c_e(\theta) \text{ f.a. } e \in E \text{ and } \theta \in \mathbb{R}_{\geq 0} \}.$$  \hspace{1cm} (23)

Using this constraint set in order to define a strict SCDE (cf. Definition 5.10) results in an equilibrium that we call **strict capacitated dynamic equilibrium (strict CDE)**. Since, as discussed in Section 4, the constraint set defined by (23) need not be convex, we can, in general, not use the corresponding variational inequality $(\text{VI}(\Psi, S))$ to characterize those types of equilibria. Furthermore, the admissible deviations need not even satisfy (A8) or (A7) (see Example 4.1), so we also cannot apply the quasi-variational inequality. Another problem of strict CDE is that this definition is rather restrictive in what counts as an admissible $\varepsilon$-deviation (and, in turn, leads to a rather weak type of equilibrium). Namely, particles are only allowed to deviate, if their deviation leads to a new flow which is again feasible for all particles – in particular also for the particles which are not themselves
involved in the deviation. In Example 7.4, we provide an instance with a strict SCDE in which particles seem to have a better alternative to their current route choice but are not allowed to deviate because that would lead to infeasibility for other particles not involved in the deviation.

7.1. Dynamic Equilibria of Type LP, BS and MNS

In the following, we want to allow for deviations in which the deviating particles themselves will not violate the capacity constraints while ignoring potential violations by particles not directly involved in the deviation. To formalize this in terms of admissible $\varepsilon$-deviations, we need to answer two questions: At which (time) points during its potential alternative journey must a particle check whether it would violate some capacity constraint and how does a particle check whether it would violate a capacity constraint? For the first question we will consider two potential answers

a) Whenever it enters a new edge, i.e. at all points $\tau^j_q(h,t)$ for $j = 1,\ldots,|q|$ or
b) Whenever a particle travels along an edge, i.e. at all points $\theta \in [\tau^j_q(h,t),\tau^{j+1}_q(h,t)]$ for $j = 1,\ldots,|q|$.

Since a) clearly allows more deviations than b), we will call equilibria using a) strong and equilibria using b) weak. For the second question we will propose three different answers. First, we can follow the approach of Larsson and Patriksson in the static model (cf. Definition 2.5) and require that alternative walks are truly unsaturated at the time of deviation. In other words, at any time at which a deviating particle would arrive at an edge/travel along an edge of the alternative walk (according to the travel times of the current flow), there must be some additional room left on this edge (i.e., $f_e < c_e$ must hold). We can formalize this by

$$A_p(h) := \{ (q,J,\varepsilon,\Delta) \mid \forall t \in J + \Delta, e = (v_j,v_{j+1}) \in q : f_e(h,\tau^j_q(h,t)) < c_e(\tau^j_q(h,t)) \}$$

and

$$A_p(h) := \{ (q,J,\varepsilon,\Delta) \mid \forall e = (v_j,v_{j+1}) \in q, t \in J + \Delta : f_e(h,\theta) < c_e(\theta) \text{ f.a. } \theta \in [\tau^j_q(h,t),\tau^{j+1}_q(h,t)] \}.$$  

We call the resulting equilibrium a strong dynamic Larsson-Patriksson equilibrium (strong LPDE) or a weak dynamic Larsson-Patriksson equilibrium (weak LPDE), respectively.

While this equilibrium notion seems quite intuitive, it has the same drawback as noted by Marcotte et al. for LP-equilibria in the static model: Namely, one can consider a network consisting of two paths of different length that share their first edge. Then, a flow which only sends flow over the longer path can still be an equilibrium if this flow completely uses the available capacity on the shared first edge. A more lenient definition, thus, would allow $f_e \leq c_e$ to be tight on any common prefix of $p$ and $q$, i.e.

$$A_p(h) := \{ (q,J,\varepsilon,0) \mid \exists \text{ a common prefix } w \text{ of } p \text{ and } q \text{ with } q = w\hat{q} \text{ and } \forall t \in J, e = (v_j,v_{j+1}) \in \hat{q} : f_e(h,\tau^j_q(h,t)) < c_e(\tau^j_q(h,t)) \}$$

and

$$\cup \{ (q,J,\varepsilon,\Delta) \mid \forall t \in J + \Delta, e = (v_j,v_{j+1}) \in q : f_e(h,\tau^j_q(h,t)) < c_e(\tau^j_q(h,t)) \}.$$  

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We call this type of equilibrium a strong dynamic Bernstein-Smith equilibrium (strong BSDE) or a weak dynamic Bernstein-Smith equilibrium (weak BSDE) in the static model also requires that the costs cannot decrease by switching to any other path whenever such a switch would result in another feasible (static) flow (cf. Definition 2.7). Note, however, that in the static model, feasibility for the particles involved in the deviation typically (if the cost functions are separable and nondecreasing) also ensures feasibility for all other particles – thus, the difference between strict CDE and strong BSDE vanishes there, whereas strict CDE are strictly weaker than both strong BSDE and weak BSDE in the dynamic setting.

The following definition summarizes all the different types of equilibria defined in this section:

**Definition 7.2.** Let the constraint set $S$ be defined in (23). Then, a side-constrained dynamic equilibrium is

- a strict capitivated dynamic equilibrium (strict CDE) if the admissible $\varepsilon$-deviations are defined by (20),
- a strong dynamic Larsson-Patriksson equilibrium (strong LPDE) if the admissible $\varepsilon$-deviations are defined by (24),
- a weak dynamic Larsson-Patriksson equilibrium (weak LPDE) if the admissible $\varepsilon$-deviations are defined by (25),
- a strong dynamic Marcotte-Nguyen-Schoeb equilibrium (strong MNSDE) if the admissible $\varepsilon$-deviations are defined by (26) and
- a weak dynamic Marcotte-Nguyen-Schoeb equilibrium (weak MNSDE) if the admissible $\varepsilon$-deviations are defined by (27),
• a strong dynamic Bernstein-Smith equilibrium (strong BSDE) if the admissible $\varepsilon$-deviations are defined by (28),

• a weak dynamic Bernstein-Smith equilibrium (weak BSDE) if the admissible $\varepsilon$-deviations are defined by (29).

The relationships between these different equilibrium concepts are captured in the following proposition:

**Proposition 7.3.** Denoting the sets of equilibria by their respective names we have the following relations between them:

\[
\text{strong LPDE} \subseteq \text{weak LPDE} \\
\text{strong MNSDE} \subseteq \text{weak MNSDE}
\]

and

\[
\text{strong BSDE} \subseteq \text{weak BSDE} \subseteq \text{strict CDE}
\]

If both $f_e$ and $\tau^j_p$ depend continuously on $h$ while $f_e(h,.)$, $\tau^j_p(h,.)$ and $c_e$ are all continuous functions, then we additionally have

\[
\text{strong LPDE} \cup \text{weak LPDE} \\
\text{strong BSDE} \cup \text{weak BSDE}
\]

Furthermore, in general, all these inclusions are proper and the concepts of strict CDE and strong LPDE are independent of each other, i.e. neither includes the other.

**Proof.** The inclusions in the first two diagrams all follow directly from Lemma 5.3 by observing that the respective sets of admissible $\varepsilon$-deviations satisfy the opposite inclusion. E.g. strong MNSDE allow more $\varepsilon$-deviations than strong LPDE and, therefore, strong MNSDE is a stronger equilibrium concept than strong LPDE.

For the inclusion strong BSDE $\subseteq$ strong LPDE, we cannot compare admissible $\varepsilon$-deviations but compare admissible deviations instead. That is, assume that $(q, \Delta) \in U_p(h, t)$ is an admissible deviation according to strong LPDE. Then, due to the continuity of $f_e(h,.)$, $\tau^j_p(h,.)$ and $c_e$, there must be some neighbourhood $V \subseteq [t_0, t_f]$ of $t + \Delta$ such that for all $t' \in V$ and $e = (v_j, v_{j+1}) \in q$, we have $f_e(h, \tau^j_p(h, t')) < c_e(\tau^j_q(h, t'))$. Since all $f_e$ and $\tau^j_q$ also depend continuously on $h$, we then have $f_e(h^\varepsilon, \tau^j_q(h^\varepsilon, t')) < c_e(\tau^j_q(h^\varepsilon, t'))$ for all $t'$ in some smaller neighbourhood $V' \subseteq V$ and $h^\varepsilon := H_{p\rightarrow q}(h, t', \Delta, \varepsilon, \Delta)$ for small enough $\varepsilon$. This then implies $(q, V' - \Delta, \varepsilon, \Delta) \in A_p(h)$ according to strong BSDE and, consequently, $(q, \Delta) \in U_p(h, t)$ according to strong BSDE. Thus, all admissible deviations according to strong LPDE are admissible deviations according to strong BSDE as well and, therefore, the latter is a stronger equilibrium concept than the former. The inclusion weak BSDE $\subseteq$ weak LPDE can be shown in the same way.

For the independence of strict CDE and strong LPDE, we note that the corresponding notion of admissible deviations can be both stricter for strict CDE (a deviation might only be inadmissible because of infeasibility for particles not directly involved in the deviation – see Example 7.4) and stricter for strong LPDE (a deviation might still be possible even if $f_e \leq c_e$ is tight at relevant times because an increased inflow into the alternative walk $q$ does not necessarily increase $f_e$ on all of its edges). The former part also shows that there can exist strict CDE which are not a strong BSDE while the latter part shows that there can exist strong LPDE which are not a strong BSDE. 

\[\square\]
Figure 5: A three commodity network with fixed network inflow rates. All values of $\tau_e$ not explicitly given in the figure are 1 and all $\nu_e$ not given are infinity. Using the Vickrey point queue model for the edge dynamics and the capacity constraint on edge $e$ as volume or inflow rate constraint this network has a unique feasible flow which is a strict CDE but neither a weak BSDE nor a weak LPDE.

Example 7.4. Consider the three commodity network with fixed network inflow rates given in Figure 5. We use the Vickrey point queue model for the edge dynamics and the edge capacity function of edge $e = (s_3,t_{2/3})$ as volume or inflow rate constraint. Then this network has a unique feasible flow: Commodity 1 sends all its flow via the direct edge towards $t_1$ and the other two commodities send all flow along their only available path. This is the only feasible flow (up to changes on a subset of measure zero) since, if commodity 1 were to send any of its flow along the alternative path via $v$ and $s_3$, this would result in a congestion on edge $vs_3$. This, in turn, would lead to some of commodity 2’s particles arriving at $s_3$ after time 2 and, thus, entering edge $e$ at the same time as the particles of commodity 3. This then leads to a violation of the capacity constraint on edge $e$.

Consequently, this unique flow is a strict CDE (since there are no admissible deviations). However it is neither a weak BSDE nor a weak LPDE as particles of commodity 1 could deviate to the shorter path without violating any capacity constraint themselves. In particular, this network does not have any weak BSDE or weak LPDE.

Observation 7.5. Admissible alternatives defined by either (24), (25), (26) or (27) clearly satisfy both (A7) and (A8). Thus, strong LPDE, weak LPDE, strong MNSDE and weak MNSDE with continuous effective walk delays (i.e. satisfying assumption (A6)) are all characterized by their corresponding quasi-variational inequality ($QVI(\Psi,S,A_p)$) (cf. Theorems 6.4 and 6.6).

7.2. Existence Results

As shown in Section 4, feasibility sets defined by capacity constraints need not be convex. Thus, it is not clear whether ($VI(\Psi,S)$) has a solution (at least, we cannot apply Theorem 3.4). Furthermore, in general, it is not guaranteed that small enough admissible $\varepsilon$-deviations from a feasible flow lead to another feasible flow (i.e. assumption (A9))
and, therefore, even if we had a solution to the variational inequality, this would not be guaranteed to also be a SCDE. See Example 7.4 for such an instance.

To still be able to show existence of strong MNSDE and weak MNSDE (under certain additional assumptions), we will therefore employ a different approach using an augmented Lagrangian relaxation of the hard capacity constraints. Before we come to the existence theorem for strong MNSDE, we introduce the additional assumptions we will need there:

**Definition 7.6.** We say that $h$ and $f(h)$ satisfy the principle of causation, if the following condition is satisfied:

For any edge $e \in E$ and interval $[a, b] \subseteq \mathbb{R}_{\geq 0}$ with $0 \leq f_e(h, a) < f_e(h, b)$ there exists some walk $p \in \mathcal{P}$ and subset $J^{-1} \subseteq [t_0, t_f]$ of positive measure such that

\[ e = (v_j, v_{j+1}) \in p, h_p(t') > 0 \text{ f.a. } t' \in J^{-1} \text{ and } \tau_p^j(h, t') \in [a, b] \text{ f.a. } t' \in J^{-1}. \]

The principle of causation states, that if the edge load increases during some interval, then there must exist earlier times at which strictly positive flow is injected into some walk containing $e$ contributing to this increased edge load. Note that some of the assumptions have to be stated slightly differently for the setting with and without departure time choice (DTC). In particular, we will use $\Lambda$ to refer to $\Lambda(Q)$ in the case with DTC and to $\Lambda(r)$ in the case without DTC.

(A10) For any edge $e$ and walk inflow $h$, the function $f_e(h, \cdot)$ is continuous.

(A11) For any edge $e$ and any weakly convergent sequence $h^n$, the sequence $f_e(h^n, \cdot)$ converges uniformly.

(A12) For any $h \in \Lambda$, we have that $h$ and $f(h)$ satisfy the principle of causation.

(A13) For any edge $e$ and walk inflow $h$, we have $f_e(h, 0) \leq c_e(0)$.

(A14) With DTC: For every $h \in \Lambda(Q)$ and $i \in I$, there exists some $p \in \mathcal{P}_i$ and $J \subseteq [t_0, t_f]$ of positive measure such that for all $t \in J$, we have $h_p(t) < B_p$ and $f_e(h, \tau_p^j(h, t)) \leq c_e(\tau_p^j(h, t))$ for all $e = (v_j, v_{j+1}) \in p$.

Without DTC: For every $h \in \Lambda(r)$ and $i \in I$ and $t \in [t_0, t_f]$, there exists some $p \in \mathcal{P}_i$ such that we have $f_e(h, \tau_p^j(h, t)) \leq c_e(\tau_p^j(h, t))$ for all $e = (v_j, v_{j+1}) \in p$.

(A15) For any edge $e$, the capacity function $c_e$ is non-negative.

(A16) For any edge $e$, the capacity function $c_e$ is non-decreasing.

(A17) For any walk $p$ and $j \in \{1, \ldots, |p| + 1\}$ and any weakly convergent sequence $h^n$, the sequence $\tau_p^j(h^n, \cdot)$ converges uniformly.

(A18) For any two walks $p$ and $q$ that share a common prefix $w$, we have $\tau_p^j(h, t) = \tau_q^j(h, t)$ for all $e = (v_j, v_{j+1}) \in w$, $h \in \mathcal{S}$ and $t \in [v_j, t_f]$.

(A19) The mapping $\Lambda \to C([t_0, t_f])^\mathcal{P}, h \mapsto \Psi(h, \cdot)$ is sequentially weak-strong continuous, i.e. for any walk $p \in \mathcal{P}$ and any weakly convergent sequence $h^n$, the sequence $\Psi_p(h^n, \cdot)$ converges uniformly.

(A20) There exists some $M \in \mathbb{R}$ such that for all $h \in \mathcal{S}$, $p \in \mathcal{P}$ and almost all $t \in [t_0, t_f]$, we have $\Psi_p(h, t) \leq M$. 

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Assumption (A14) essentially states that there is always an outside option, i.e. the edge capacities are chosen large enough such that there is always at least one walk which is not oversaturated. Note, that this can be easily accomplished by either having one walk with large enough capacities or by using elastic demands. Example 7.4 shows why this assumption is necessary for the existence of strong LPDE. Assumption (A18) requires that intermediate travel times only depend on the part of the walk already traversed and not on future route choices. Finally, assumption (A19) is a strengthening of assumption (A4) as convergence with respect to the uniform norm implies convergence with respect to the $L^2$-norm.

**Theorem 7.7.** Under assumptions (A2), (A3), (A6) and (A10) to (A20) there always exist strong LPDE and strong MNSDE (both with and without DTC).

**Proof.** As every strong MNSDE is also a strong LPDE (see Proposition 7.3), it suffices to show the existence of the former. We start by defining penalty functions $\xi_e(h, \theta) := \max\{0, f_e(h, \theta) - c_e(\theta)\}, e \in E, \theta \in \mathbb{R}_{\geq 0}$ and write $\xi_p(h, t) := \sum_{e=(v_j, v_{j+1}) \in p} \xi_e(h, \tau^i_p(h, t))$ for all $t \in [t_0, t_f]$. By assumptions (A11), (A17) and (A19), the new $\lambda$-parametrized effective walk delay operator

$$\Psi^\lambda_p(h, t) := \Psi_p(h, t) + \lambda \xi_p(h, t)$$

for all $t \in [t_0, t_f], p \in \mathcal{P}$, satisfies (A4) for any $\lambda > 0$ as well.

**Claim 5.** There exists some $M \in \mathbb{R}_+$ such that we have

$$\min \text{ess inf}_{p \in \mathcal{P}_i} \{ \Psi^\lambda_p(h, t) \mid t \in [t_0, t_f] : h_p(t) < B_p \} \leq M$$

for all $\lambda \geq 0$, $i \in I$ and $h \in \Lambda(Q)$ for the case with DTC and

$$\sup_{t \in [t_0, t_f]} \min_{p \in \mathcal{P}_i} \Psi^\lambda_p(h, t) \leq M$$

for all $\lambda \geq 0$, $i \in I$ and $h \in \Lambda(r)$ for the case without DTC.

**Proof.** We start with the case with DTC: By assumption (A14), for every $h$ and $i \in I$ there exist $p \in \mathcal{P}_i$ and $J \subseteq [t_0, t_f]$ with positive measure such that $h_p(t) < B_p$ and $f_e(h, \tau^i_p(h, t)) \leq c_e(\tau^i_p(h, t))$ and, hence, $\xi_e(h, \tau^i_p(h, t)) = 0$ for all $e = (v_j, v_{j+1}) \in p$ and $t \in J$. Thus, using (A20), we have $\Psi^\lambda_p(h, t) = \Psi_p(h, t) + 0 \leq M$ for all $t \in J$. Since $J$ has positive measure, this implies the claim’s first part.

Now for the case without DTC take any time $t \in [t_0, t_f]$. By assumption (A14) there exists some path $p \in \mathcal{P}_i$ with $f_e(h, \tau^i_p(h, t)) \leq c_e(\tau^i_p(h, t))$ and, hence, $\xi_e(h, \tau^i_p(h, t)) = 0$ for all $e = (v_j, v_{j+1}) \in p$. Thus, using (A20) we have $\Psi^\lambda_p(h, t) = \Psi_p(h, t) + 0 \leq M$ which proves the claim’s second part. \[\square\]

Using this claim we can now, in the setting without side-constraints, apply Lemma 5.7 to replace $\Psi^\lambda$ by the truncated effective walk delay operator $\Psi^{\lambda+1}_M$ which is, in particular, bounded (i.e. satisfies (A1)) while yielding only dynamic equilibria which are also equilibria wrt. $\Psi^\lambda$. As $\Psi^{\lambda+1}_M$ still satisfies (A4) while (A2) and (A3) are part of the theorem’s assumption, we can apply Theorem 3.5 to obtain for any $\lambda \geq 0$ an unconstrained dynamic equilibrium with respect to $\Psi^\lambda$ (with or without DTC).

Let us now take a sequence of strictly positive numbers $(\lambda_n)_{n \in \mathbb{N}}$ with $\lambda_n \to \infty$ and a corresponding sequence $h^n \in \Lambda$ of unconstrained dynamic equilibria with respect to
\[ \Psi^\lambda_n. \] By taking subsequences, \( h^n \) weakly converges to some \( h^* \in \Lambda \) as \( \Lambda \) is bounded and closed (cf. [28, Corollary 7.32]). We will now show that this \( h^* \) is a strong MNSDE by first showing that it is feasible and then that it also satisfies the SCDE-condition (19) for admissible \( \varepsilon \)-deviations defined by (26).

**Claim 6.** \( h^* \) is feasible, i.e., \( h^* \in S. \)

**Proof.** Suppose that \( h^* \notin S \), i.e., there exists some \( e \in E \) and some time \( \theta \in \mathbb{R}_{\geq 0} \) with \( \xi_e(h^*, \theta) > 0 \). Now, by assumption (A11), \( f(h^n) \) converges uniformly to \( f(h^*) \) and, in particular, there exists some \( \delta > 0 \) such that for large enough \( n \in \mathbb{N} \) we have
\[
 f_e(h^n, \theta) - c_e(\theta) = \xi_e(h^n, \theta) \geq \delta.
\]

Using assumptions (A10), (A13), (A15) and (A16), there must be some proper interval \([\theta', \theta] \) such that \( f_e(h^n, \theta') < f_e(h^n, \theta) \) and \( f_e(h^n, \theta) - c_e(\theta) \geq \frac{\delta}{2} \) for all \( \theta \in [\theta', \theta] \). By (A12) this gives us some \( p_n \in \mathcal{P} \) with \( e = (v_{j_n}, v_{j_n+1}) \in p_n \) and some set \( J^{-1}_n \subseteq [t_0, t_f] \) of positive measure, such that for all \( t' \in J^{-1}_n \) we have
\[
 h^n_{p_n}(t') > 0 \quad \text{and} \quad \tau_{j_n}^n(t', h^n) \in [\theta', \theta]
\]
and, thus,
\[
 \Psi_{\lambda_n}(h^n, t') \geq \lambda_n \xi_e(h^n, \tau_{j_n}^n(t', h^n)) \geq \lambda_n \frac{\delta}{2}.
\]

Since \( \lambda_n \to \infty \), there exists some \( n \in \mathbb{N} \) with \( \lambda_n \frac{\delta}{2} > M \) and, hence, for this \( n \) we have a set \( J^{-1}_n \) of positive measure such that for all \( t' \in J^{-1}_n \) we have \( h^n_{p_n}(t') > 0 \) as well as \( \Psi_{\lambda_n}(h^n, t') > M. \) At the same time we have, by Claim 5,
\[
 M \geq \min_{p \in \mathcal{P}_i} \inf \{ \Psi_p(h^n, t) : t \in [t_0, t_f] \} = \nu_i
\]
or
\[
 M \geq \sup_{t \in [t_0, t_f]} \min_{p \in \mathcal{P}_i} \{ \Psi_p(h^n, t) \}
\]
for the case with or without DTC, respectively. But this is now a contradiction to \( h^n \) being an unconstrained dynamic equilibrium. \( \blacksquare \)

**Claim 7.** \( h^* \) is a SCDE wrt. the admissible \( \varepsilon \)-deviations defined in (26).

**Proof.** Suppose that \( h^* \in S \) is not an equilibrium, that is, there is some \( i \in I, p, q \in \mathcal{P}_i \) and \( t \in [t_0, t_f] \) with \( (q, \Delta) \in U_p(h, t) \) and \( \Psi_p(h^*, t) > \Psi_q(h^*, t + \Delta) \). We distinguish two cases: If \( \Delta = 0 \), we define \( w \) as the maximal common prefix of \( p \) and \( q \), i.e., \( p = w\tilde{p} \) and \( q = w\tilde{q} \) for some subwalks \( \tilde{p} \) and \( \tilde{q} \). If \( \Delta \neq 0 \), we define \( w \) as the empty walk and \( \tilde{p} := p \) and \( \tilde{q} := q \).

- With (A6) and (A19), i.e. continuity of \( \Psi_p(h^*, .) \) and uniform convergence of \( \Psi_p(h^n, .) \to \Psi_p(h^*, .) \), we get the existence of some \( \delta > 0 \) such that
\[
 \Psi_p(h^n, t') > \Psi_q(h^n, t' + \Delta) \quad \text{for all} \ t' \in [t - \delta, t + \delta] \quad \text{for} \ n \text{ large enough. (30)}
\]

- From \((q, \Delta) \in U_p(h^*, t)\), we get an \( \varepsilon > 0 \) and some subset \( J \subseteq [t - \delta, t + \delta] \) with \( f_j \min \{ h^n_{\tilde{p}}(t'), \varepsilon \} \) such that \( t' > 0 \) and \((q, J, \varepsilon, \Delta) \in A_p(h^*)\). By the definition of admissible \( \varepsilon \)-deviations (i.e. (26)) this implies
\[
 f_e(h^*, \tau_{\tilde{q}}^n(h^*, t')) < c_e(\tau_{\tilde{q}}^n(h^*, t')) \quad \text{f.a.} \ e = (v_j, v_{j+1}) \in \tilde{q}, t' \in J + \Delta.
\]
- With (A11) and (A17), i.e. the uniform convergence of $f_e(h^n, \cdot)$ to $f_e(h^*, \cdot)$ and $\tau^j_p(h^n, \cdot)$ to $\tau^j_p(h^*, \cdot)$, we have the existence of some $N \in \mathbb{N}$ and $J' \subseteq J$ such that
  \[
  \int_{J'} \min \{ h^*_p(t'), \varepsilon \} \, dt' > 0
  \]
  and
  \[
  f_e(h^n, \tau^j_q(h^n, t')) < c_e(\tau^j_q(h^n, t')) \text{ for all } n \geq N, e = (v_j, v_{j+1}) \in \tilde{q} \text{ and } t' \in J' + \Delta.
  \]

- With weak convergence of $h^n \to h^*$ we now get the existence of some $J_n \subseteq J'$ of positive measure with $h^n_p(t') > 0$ for all $t' \in J_n$ for $n$ large enough:
  To see this, let $\mathbf{1}_{J', p} \in L^2([t_0, t_f])^p$ be the characteristic function of the measurable set $J'$ for walk $p$ and the zero function for all other walks. Then the weak convergence of $h^n$ to $h^*$ implies
  \[
  \lim_n \int_{J'} h^n_p(t') \, dt' = \lim_n \langle h^n, \mathbf{1}_{J', p} \rangle = \langle h^*, \mathbf{1}_{J', p} \rangle = \int_{J'} h^*_p(t') \, dt' > 0.
  \]
  This implies $\int_{J'} h^n_p(t') \, dt' > 0$ for large enough $n$ and, thus, for any such $n$ there exists a subset $J_n \subseteq J'$ of positive measure with $h^n_p(t') > 0$ for all $t' \in J_n$.

  Taking all this together we can now show a contradiction as follows: Choose $n \in \mathbb{N}$ large enough such that all the previous statements hold. Then, in particular, we have $f_e(h^n, \tau^j_q(h^n, t')) < c_e(\tau^j_q(h^n, t'))$ and, thus,
  \[
  \xi_e(h^n, \tau^j_q(h^n, t')) = 0 \text{ for all } e = (v_j, v_{j+1}) \in \tilde{q} \text{ and } t' \in J_n + \Delta. \tag{31}
  \]
  This then implies
  \[
  \Psi^\lambda_p(h^n, t') = \Psi_p(h^n, t') + \lambda_n \xi_p(h^n, t')
  \]
  \[
  = \Psi_p(h^n, t') + \lambda_n \sum_{e = (v_j, v_{j+1}) \in w_\tilde{q}} \xi_e(h^n, \tau^j_q(h^n, t'))
  \]
  \[
  \geq \Psi_p(h^n, t') + \lambda_n \sum_{e = (v_j, v_{j+1}) \in w} \xi_e(h^n, \tau^j_p(h^n, t'))
  \]
  \[
  > \Psi_q(h^n, t' + \Delta) + \lambda_n \sum_{e = (v_j, v_{j+1}) \in w} \xi_e(h^n, \tau^j_q(h^n, t' + \Delta)) \tag{30}
  \]
  \[
  = \Psi_q(h^n, t' + \Delta) + \lambda_n \sum_{e = (v_j, v_{j+1}) \in w} \xi_e(h^n, \tau^j_q(h^n, t' + \Delta)) \tag{A18}
  \]
  \[
  = \Psi_q(h^n, t' + \Delta) + \lambda_n \sum_{e = (v_j, v_{j+1}) \in w \tilde{q}} \xi_e(h^n, \tau^j_q(h^n, t' + \Delta))
  \]
  for all $t' \in J_n$. For the fourth to last equality, note that $w$ is the empty walk whenever we have $\Delta \neq 0$.

  Now, at the same time, $t' \in J_n$ also implies $h^n_p(t') > 0$. Together with the fact that $J_n$ has positive measure, these two statements are now a contradiction to $h^n$ being an unconstrained dynamic equilibrium. \hfill \blacksquare

  Finally, Claims 6 and 7 together imply that $h^*$ is a strong MNSDE. \hfill \square
Figure 6: An instance with volume constraints, fixed network inflow rates and the Vickrey queueing model for the edge dynamics. All values of $\tau$ not explicitly given are 1 and all missing values of $\nu$ are infinite. In this network there exists a sequence of equilibrium flows for the instance with increasing prices (instead of strict volume constraints) which converges to a feasible flow which is a strong MNSDE but not a strong BSDE.

Note, that the same type of proof cannot be applied to show existence of strong BSDE. While everything up to and including Claim 6 still holds (since the constraint set $S$ remains unchanged), Claim 7 is not guaranteed to hold any more for admissible $\varepsilon$-deviations defined by (28). In fact there exists an instance wherein the sequence $h^n$ from the proof of Theorem 7.7 converges to some $h^*$ which is not a strong BSDE:

**Example 7.8.** The two-commodity network with fixed network inflow rates in Figure 6 shows why existence of strong BSDE cannot be shown by relaxing the constraints and replacing them with fees for violation (as done in the proof of Theorem 7.7 for strong LPDE and strong MNSDE). The given network uses the Vickrey point queue model for the edge dynamics and has a constant volume constraint of 4 on the central edge. For any price $\lambda$ the following is an equilibrium: Commodity 1 uses the central path at a rate of 1 during the interval $[0,5]$ and (additionally) the direct edge towards $t_1$ at a rate of 1 during the interval $[4,5]$. Commodity 2 uses the central path at a rate of $1/\lambda$ and sends everything else over the direct edge towards $t_2$. The flow volume on edge $e$ induced by such a flow split is depicted in Figure 7. Note, that during the interval $[3,4]$ this flow volume is less than 4 and, thus, the particles of commodity 2 starting during $[2,3]$ are indifferent between the central and the right path. During the interval $[5,6]$, on the other hand, the capacity constraint on edge $e$ is violated and particles entering during this interval have to pay a penalty of $\lambda \cdot \frac{1}{\lambda} = 1$. In particular, particles of commodity 1 starting during $[4,5]$ are indifferent between the left and the central path. Therefore, the flow split described above is indeed an unconstrained equilibrium for price $\lambda$. 
Letting $\lambda$ go to $\infty$ these equilibria converge to a flow where commodity 2 sends everything along the direct edge while commodity 1 uses the same split as for every $\lambda$. This is clearly a feasible flow and also a strong MNSDE since the capacity constraint on edge $e$ is tight during $[5, 7]$ and, thus, the central path is not an admissible deviation during $[4, 5]$. However, it is not a strong BSDE as due to the edge dynamics on edge $s_1$ sending more flow into the central path during $[4, 5]$ would not actually increase the flow volume on edge $e$ and, therefore, not violate the volume constraint on this edge. Thus, the central path is an admissible deviation for admissible $\varepsilon$-deviations defined by (28).

The following example shows why it is important for the previous existence proof that the capacity functions $c_e$ are non-decreasing (i.e. why we need assumption (A16)).

Example 7.9. In the network with a decreasing capacity function depicted in Figure 8 the only feasible flow is to send all flow over the edge $e_2$ as sending any flow into edge $e_1$ will lead to a positive flow volume on this edge during the interval $[2, 3]$ (which the edge capacity does not allow for). However, switching from edge $e_2$ to edge $e_1$ is an admissible deviation according to both (24) and (28) during $[0, 1]$ as there is still capacity left on edge $e_1$ at the time these particles would enter edge $e_1$. Thus, this network has no strong BSDE or strong LPDE. At the same time, deviating from edge $e_2$ to edge $e_1$ is not an admissible deviation according to (27) or (29) and, thus, sending all flow via edge $e_2$ is both a weak BSDE and a weak MNSDE.

Indeed, for the existence weak MNSDE, we may remove the assumption that $c_e$ is non-decreasing and allow more general capacity functions as long as we impose some stronger assumptions on the network loading.
(A21) The walk travel times are induced by edge delay functions $D_e(h, t)$, i.e. for any walk $p$ and edge $e = (v_j, v_{j+1}) \in p$, we have $\tau_p^j\!+\!1(h, t) = \tau_p^j(h, t) + D_e(h, \tau_p^j(h, t))$.

(A22) The edge load function satisfies the FIFO-compatible principle of causation, i.e. for any two time $a < b$ with $f_e(h, b) > 0$ and $b > a + D_e(h, a)$, there exists some walk $p \in P$ with $e = (v_j, v_{j+1}) \in p$ and some set $J^{-1} \subseteq [t_0, t_f]$ of positive measure such that $h_p(t) > 0$ and $\tau_p^j(h, t) \in [a, b]$ for all $t \in J^{-1}$.

(A23) For any edge $e$ and walk inflow $h$, the function $f_e(h, \cdot)$ is uniformly continuous.

(A24) For any edge $e$ the capacity function $c_e$ is continuous on $\mathbb{R}_{\geq 0}$.

(A25) For any edge $e$ and walk inflow $h$, we have $f_e(h, \theta) \leq c_e(\theta)$ for all $\theta \in [0, D_e(h, 0)]$.

(A26) The edge delays are non-negative, i.e. $D_e(h, \theta) \geq 0$ for all $h, \theta$ and $e$.

(A27) For any edge $e$ and walk inflow $h$, the edge delay $D_e(h, \cdot)$ is continuous.

(A28) For any edge $e$ and any weakly convergent sequence $h^n$, the sequence $D_e(h^n, \cdot)$ converges uniformly.

(A29) For the case with DTC: For every $h \in \Lambda(Q)$ and $i \in I$, there exists some $p \in P_i$ and $J \subseteq [t_0, t_f]$ of positive measure such that for all $t \in J$, we have $h_p(t) < B_p$ and $f_e(h, \theta) \leq c_e(\theta)$ for all $e = (v_j, v_{j+1}) \in p$ and $\theta \in [\tau_p^j(h, t), \tau_p^{j+1}(h, t)]$.

For the case without DTC: For every $h \in \Lambda(r)$, $t \in [t_0, t_f]$ and $i \in I$, there exists some $p \in P_i$ such that we have $f_e(h, \theta) \leq c_e(\theta)$ for all $e = (v_j, v_{j+1}) \in p$ and $\theta \in [\tau_p^j(h, t), \tau_p^{j+1}(h, t)]$.

The FIFO-compatible principle of causation (A22) states that whenever the edge-load function is positive, there must be particles that have entered but not left the edge before that time assuming that the edge dynamics satisfy the FIFO principle.

**Theorem 7.10.** Under assumptions (A2), (A3), (A6), (A11), (A15) and (A19) to (A29), there always exist weak LPDE and weak MNSDE (both with and without DTC).

**Proof.** By Proposition 7.3 we only need to show the existence of a weak MNSDE. To do that we define a new edge-load function

$$\tilde{f}_e(h, \theta) := \max \left\{ f_e(h, \zeta) - c_e(\zeta) \mid \zeta \in [\theta, \theta + D_e(h, \theta)] \right\}$$

and new constant edge-capacity functions $\tilde{c}_e(\theta) := 0$ for all $\theta \in \mathbb{R}_{\geq 0}$.

**Claim 8.** An admissible $\varepsilon$-deviation with respect to $\tilde{f}_e$ and (26) is an admissible $\varepsilon$-deviation with respect to $f_e$ and (27) and vice versa.

**Proof.** First, let $(q, J, \varepsilon, \Delta)$ be an admissible $\varepsilon$-deviation from $p$ with respect to $\tilde{f}_e$ and (26). Then, for any $t \in J$ and $e = (v_j, v_{j+1}) \in q$, we have

$$0 = \tilde{c}_e(\tau_q^j(h, t)) > \tilde{f}_e(h, \tau_q^j(h, t))$$

$$= \max \left\{ f_e(h, \theta) - c_e(\theta) \mid \theta \in [\tau_q^j(h, t), \tau_q^{j+1}(h, t)] \right\}$$

$$= \max \left\{ f_e(h, \theta) - c_e(\theta) \mid \theta \in [\tau_q^j(h, t), \tau_q^{j+1}(h, t)] \right\}$$

and, therefore $f_e(h, \theta) < c_e(\theta)$ for all $\theta \in [\tau_q^j(h, t), \tau_q^{j+1}(h, t)]$ which implies that $(q, J, \varepsilon, \Delta)$ is an admissible $\varepsilon$-deviation from $p$ with respect to $f_e$ and (27).
Now, if \((q, J, \varepsilon, \Delta)\) is an admissible \(\varepsilon\)-deviation from \(p\) with respect to \(f_e\) and (27), then, we have \(f_e(h, \theta) > c_e(\theta)\) for all \(t \in J, \theta \in [\tau_q^j(h, t), \tau_q^{j+1}(h, t)]\) and \(e = (v_j, v_{j+q}) \in \tilde{q}\) implying – by the same calculation as before – that \(c_e(\tau_q^{j}(h, t)) = 0 > \tilde{f}_e(h, \tau_q^{j}(h, t))\) for all such \(t\) and \(e\). Thus, \((q, J, \varepsilon, \Delta)\) is an admissible \(\varepsilon\)-deviation from \(p\) with respect to \(\tilde{f}_e\) and (26).

Now, using assumption (A26) and the same calculations as in the proof of the previous claim, we observe that the constraint set \(S\) defined using \(\tilde{f}\) and \(\tilde{c}\) is the same as the one defined by \(f\) and \(c\). Thus, Claim 8 implies that any strong MNSDE for \(\tilde{f}\) and \(\tilde{c}\) is also a weak MNSDE for \(f\) and \(c\). Consequently, to show existence of such an equilibrium, it suffices to show that we can apply Theorem 7.7 to the instance with \(\tilde{f}\) and \(\tilde{c}\).

**Claim 9.** The new edge-load functions \(\tilde{f}\) satisfy assumptions (A10) to (A12) and (A14).

**Proof.** Assumptions (A23), (A24) and (A27) together with the definition of \(\tilde{f}\) directly imply that the new edge-load function is again continuous, i.e. satisfies (A10).

To see that (A11) holds for \(\tilde{f}\) take any sequence of walk inflows \(h^n\) converging weakly to some walk inflow \(h^*\). Then for any edge \(e\) we have

\[
\left\| \tilde{f}_e(h^n, \cdot) - \tilde{f}_e(h^*, \cdot) \right\|_\infty = \max_{\theta \in \mathbb{R}_{\geq 0}} \left| \tilde{f}_e(h^n, \theta) - \tilde{f}_e(h^*, \theta) \right|
\]

\[
= \max_{\theta \in \mathbb{R}_{\geq 0}} \left| \tilde{f}_e(h^n, \theta) - f_e(h^n, \theta) \right| + \max_{\theta \in \mathbb{R}_{\geq 0}} \left| f_e(h^n, \theta) - f_e(h^*, \theta) \right|
\]

\[
\leq \max_{\theta \in \mathbb{R}_{\geq 0}} \left| \tilde{f}_e(h^n, \theta) - f_e(h^n, \theta) \right| + \max_{\theta \in \mathbb{R}_{\geq 0}} \left| f_e(h^n, \theta) - f_e(h^*, \theta) \right|
\]

\[
\leq \max_{\theta \in \mathbb{R}_{\geq 0}} \left| \tilde{f}_e(h^n, \theta) - f_e(h^n, \theta) \right| \leq \max_{\theta \in \mathbb{R}_{\geq 0}} \left| \tilde{f}_e(h^n, \theta) - f_e(h^n, \theta) \right|
\]

Now the two terms at the end go to zero as \(n\) goes to infinity by assumptions (A11), (A23), (A24) and (A28) for \(f_e\) and \(c_e\). Thus, (A11) holds for \(\tilde{f}_e\) as well.

Assumption (A29) for \(f\) implies that (A14) holds for \(\tilde{f}\).

Finally, to show that \(\tilde{f}\) satisfies the principle of causation (i.e. (A12)) take any walk inflow \(h\), edge \(e\) and two times \(a < b\) with \(0 \leq \tilde{f}_e(h, a) < \tilde{f}_e(h, b)\). Then, the definition of \(\tilde{f}\) implies that there exists some time \(\theta \in [b, b + D_e(h, b)]\) with

\[
f_e(h, \theta) \geq f_e(h, \theta) - c_e(\theta) > \tilde{f}_e(h, a) \geq 0.
\]

The definition of \(\tilde{f}_e(h, a)\) then implies \(\theta \notin [a, a + D_e(h, a)]\) and, thus, \(\theta > a + D_e(h, a)\). By (A22) this gives us some walk \(p \in \mathcal{P}\) with \(e = (v_j, v_{j+q}) \in p\) and some set \(J^{-1} \subseteq [t_0, t_f]\) of positive measure such that \(h_p(t) > 0\) and \(\tau_q^j(h, t) \in [a, b]\) for all \(t \in J^{-1}\). But this is exactly what we need for (A12) to hold for \(\tilde{f}\).
Since the walk travel times \( \tau^j_p \) are induced by the edge delays (cf. (A21)), assumption (A28) implies that assumptions (A17) and (A18) hold. Furthermore, the new edge-capacities \( \hat{c}_e \) are constant and non negative (i.e. satisfy (A15) and (A16)) and assumption (A25) on \( c \) and \( f \) implies that \( \hat{c} \) and \( \hat{f} \) satisfy (A13). Thus, we can now apply Theorem 7.7 to get the existence of a strong MNSDE \( h^* \) for the edge-load functions \( \hat{f} \) and capacities \( \hat{c} \) which, by Claim 8, is also a weak MNSDE for the original edge-load functions \( f \) and capacities \( c \).

Two important flow models for which we can now apply the above existence results are the Vickrey point queue model and the linear edge delay model described in Section 4. More precisely, let the edge delays be defined as \( D_e(h, \theta) := \tau_e + \frac{q_e(h, \theta)}{\nu_e} \) or \( D_e(h, \theta) := \tau_e + \frac{x_e(h, \theta)}{\nu_e} \) and assume that all free flow travel times and service rates are strictly positive and finite:

(A30) For all edges \( e \in E \) we have \( \tau_e, \nu_e \in \mathbb{R}_{>0} \).

Furthermore, we define the effective walk delays by

\[ \Psi_p(h, t) := C_p(D_p(h, t), t + D_p(h, t)) \]

where \( C_p \) is a continuous function mapping travel and arrival time to total travel cost and \( \tau^j_p \) is derived from the edge delays (i.e. defined as in assumption (A21)). Finally, we use the flow volume on an edge as its edge load, i.e. \( f_e(h, t) := x_e(h, t) \).

As the following lemma will show, both these models then automatically satisfy all the previous assumptions posed on the flow dynamics – thus, we only have to check whether a given network also satisfies the additional assumptions on the network itself to be able to apply the two existence theorems of this section.

**Lemma 7.11.** Under assumptions (A2) and (A30) the two models described above both satisfy assumptions (A6), (A10) to (A12), (A17) to (A23) and (A26) to (A28).

**Proof.** We first note, that both in the Vickrey point queue model and the linear edge delay model any fixed network with finite walk set \( \mathcal{P} \) and fixed flow volume \( Q \) has some constant \( K > 0 \) such that for any walk inflow \( h \in \Lambda(Q) \) and edge \( e \in E \) the support of \( x_e(h,.) \) is in \([0, K]\). This can be shown by the same proof as the one for [16, Lemma 4.4 (full version)]. Together with the continuity of the functions \( C_p \) this already implies that \( \Psi_p \) is bounded, i.e. assumption (A20) holds.

Then, for the Vickrey point queue model, one can show in the same way as in the proofs of [16, Claims 3, 4, 5 (full version)] that the mappings \( h \mapsto \int_0 f^+_e(\theta) \, d\theta \) and \( h \mapsto \int_0 f^-_e(\theta) \, d\theta \) are sequentially weak-strong continuous from \( \Lambda(Q) \subseteq L^2([t_0, t_f])^\mathcal{P} \) to \( C([0, K])^E \). It then immediately follows that the same is true for the mappings \( h \mapsto x_e(h,.) \), \( h \mapsto q_e(h,.) \), \( h \mapsto D_e(h,.) \), \( h \mapsto D_p(h,.) \) and \( h \mapsto \Psi_p(h,.) \). This shows that assumptions (A6), (A10), (A11), (A19), (A23), (A27) and (A28) are satisfied.

For the linear edge delay model the required continuity properties have been shown by Zhu and Marcotte in [46]. Namely, [46, Corollary 5.1] shows that linear edge delays satisfy the strong FIFO condition and, thus, [46, Theorem 3.3] implies that the mapping \( h \mapsto x_e(h,.) \) is sequentially weak-strong continuous from \( \Lambda(Q) \subseteq L^2([t_0, t_f])^\mathcal{P} \) to \( L^2([0, K])^E \). Since both, walk inflow rates and edge outflow rates, are bounded, we can then apply [46, Proposition 3.1] to show that the mapping is even sequentially weak-strong continuous to \( C([0, K])^E \). From this we again immediately get assumptions (A6), (A10), (A11), (A19), (A23), (A27) and (A28).

Assumption (A26) follows directly from the definition of \( D_e \) and the nonnegativity of the queue length \( q_e \) or the flow volume \( x_e \), respectively. Assumptions (A17) and (A18) follow from (A28) in the same way as in the proof of Theorem 7.10.
It remains to show that assumptions (A12) and (A22) (the principles of causation) hold as well. This can be shown in the same way for both models: For (A12) we observe that the flow volume of an edge can only increase during some interval \([a,b]\) if there is positive inflow into this edge for some subset \(J \subseteq [a,b]\) of positive measure. The definition of the flow dynamics then directly implies the existence of some walk \(e = (v_i, v_{i+1}) \in p\) and some set \(J^{-1} \subseteq [t_0, t_f]\) of positive measure such that \(h_p(t) > 0\) and \(\tau^j_p(h, t) \in J \subseteq [a,b]\) for all \(t \in J^{-1}\).

For (A22) we also use flow conservation on edges (5) which implies that any flow on some edge \(e\) at time \(a\) has left this edge by time \(a + D_e(h, a)\). Thus, if there is still positive flow volume on edge \(e\) at some later time \(b > a + D_e(h, a)\), there must have been some additional inflow into this edge between \(a\) and \(b\). This then gives us the desired walk \(p\) and set \(J^{-1}\) in exactly the same way as before.

Corollary 7.12. Using either the Vickrey point queue model or the linear edge delay model together with volume constraints as described above

- any network satisfying assumptions (A2), (A3) and (A13) to (A16) has a strong MNSDE and
- any network satisfying assumptions (A2), (A3), (A15), (A24), (A25) and (A29) has a weak MNSDE.

Proof. This follows directly from Lemma 7.11 together with Theorems 7.7 and 7.10.

We conclude by noting that, in contrast to unconstrained dynamic equilibria, it seems unlikely that an existence result like Corollary 7.12 for the Vickrey point queue model with fixed network inflow rates can be shown using an extension approach instead wherein one “constructs” an equilibrium by iteratively extending the walk inflow over longer and longer time intervals (cf. [30, 40]). This is, because an important ingredient for the success of such a construction process is the observation that, in an unconstrained dynamic equilibrium, later starting particles can never influence earlier starting particles (and, thus, later extensions cannot invalidate earlier ones). For side-constrained dynamic equilibria, however, there exists a simple single-commodity instance wherein this is not true, i.e. even in an equilibrium particles may still overtake each other. Moreover, this instance does indeed have a unique equilibrium which has a prefix which is not an equilibrium on its own:

Example 7.13. Figure 9 depicts a single-commodity network with fixed network inflow rate and a constant volume-constraint on one edge. Under the Vickrey point queue model this instance has a unique weak LPDE: Over the whole interval \([0, 3]\) particles are send into the path \(e_1, vt\) at a rate of 2. Between time 1 and 2 the remaining particles are send over the direct edge towards \(t\) and after time 2 the remaining particles are sent into the path \(e_2, vt\). To see why this is an equilibrium, observe that after time 1 the volume constraint on edge \(e_1\) is always tight and, thus, switching to path \(e_1, vt\) is never an admissible deviation. Furthermore, due to the queue on edge \(vt\) the travel time over path \(e_2, vt\) is strictly larger than 6 during \((0, 2)\).

However, restricting this path inflow to the interval \([0, 2]\) does not yield a weak LPDE as then the queue on edge \(vt\) will never grow beyond a length of 2 and, consequently the travel time over path \(e_2, vt\) will always be strictly shorter then over edge \(st\) (except at time 0).

Note, that in the equilibrium for the whole interval \([0, 3]\) particles travelling along path \(e_2, vt\) will get overtaken by later starting particles travelling along path \(e_1, vt\). This is
something that cannot happen in the unconstrained model as there the former particles could than always improve by copying the latter’s strategy. Here, on the other hand, this is not allowed due to the capacity constraint. This effect is essentially independent of the exact equilibrium concept, i.e. the above observation is equally true for all the equilibrium concepts defined in Definition 7.2.

8. Conclusion

We provided a counterexample to a claimed existence result for dynamic equilibria with side constraints. The implications of this counterexample were shown to be severe since solutions to the canonical infinite dimensional variational inequality are in some sense useless and other approaches seem to be necessary. We provided a general framework for defining side-constrained dynamic equilibria based on two key objects: A constraint set $S$ containing all feasible flows (given as walk inflows) and correspondences $A_p$ providing the flow-dependent set of admissible $\varepsilon$-deviations. We showed that this equilibrium concept not only encompasses the known unconstrained equilibria with and without departure time choice and capacitated dynamic equilibria with convex constraint sets but also allows for a whole range of new dynamic equilibria inspired by static side-constrained equilibria. We provided conditions under which they can be characterized as solutions to a quasi-variational or even a variational inequality. The latter characterization then also provided a first existence result for certain side-constrained dynamic equilibria with convex constraint set. We then turned to equilibria wherein the side-constraints are given by time-varying edge-load constraints. To deal with the non-convexity of the constraint set, we employed an augmented Lagrangian approach by relaxing the hard edge-load-capacities and replacing them by penalty functions. We demonstrated that these existence results apply, in particular, for the widely used Vickrey point queue model as well as the linear edge delay model.

Several important questions remain open. One problem is the multiplicity of equilibria and the issue of selecting a particular type of equilibrium having desirable properties. It is an interesting research direction to characterize equilibrium concepts that admit
equilibrium selection via appropriate optimization or optimal control reformulations whose
optimal solutions provide such desirable properties.

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A. List of Symbols and Notation

| Symbol | Name/Description |
|--------|------------------|
| \( L^2([a, b]) \) | The set of \( L^2 \)-integrable functions from \([a, b]\) to \(\mathbb{R}\) |
| \( L^2_+(([a, b]) \) | The set of \( L^2 \)-integrable functions from \([a, b]\) to \(\mathbb{R}_{\geq 0}\) |
| \((.,.)\) | The scalar product. Specifically, for \(f, g \in L^2([a, b])^d\) the scalar product is defined as \(\langle f, g \rangle := \sum_{j=1}^d f_j(\theta)g_j(\theta)\ d\theta\) |
| \( \mathcal{M}([a, b]) \) | The set of measurable subsets of the interval \([a, b]\) |
| \( G = (V, E) \) | A directed graph with node set \(V\) and edge set \(E\) |
| \([t_0, t_f] \subseteq \mathbb{R}_{\geq 0}\) | The planning horizon, i.e. the time interval during which particles may enter the network |
| \( I \) | The finite set of commodities |
| \( s_i, t_i \) | Source-/sink node of commodity \(i \in I\) |
| \( r_i : [t_0, t_f] \rightarrow \mathbb{R}_{\geq 0}\) | The set of feasible walks of commodity \(i\) |
| \( Q_i \geq 0 \) | The fixed flow volume of commodity \(i\) |
| \( \mathcal{P} = \bigcup_{i \in I} \mathcal{P}_i \) | The set of feasible walks – note that we assume that different commodities have disjoint sets of feasible walks \(\mathcal{P}_i\) |
| \( B_p \geq 0 \) | A given fixed upper bound on the walk inflow rate into walk \(p \in \mathcal{P}\) |
| \( \Lambda(r) \subseteq L^2_+([t_0, t_f])^P \) | The set of feasible walk inflows with fixed network inflow rates \(r\) |
| \( \Lambda(Q) \subseteq L^2_+([t_0, t_f])^P \) | The set of feasible walk inflows with fixed flow volume \(Q\) |
| \( h \in \Lambda(r), h \in \Lambda(Q) \) | A walk inflow vector where \(h_p : [t_0, t_f] \rightarrow \mathbb{R}_{\geq 0}\) describes the rate at which particles of commodity \(i \in I\) enter walk \(p \in \mathcal{P}_i\) |
| \( \mathcal{R} \) | A set containing a tuple \((p, j)\) for every walk \(p \in \mathcal{P}\) and \(j \in |p|\) used to denote the \(j\)-th edge on \(p\) |
| \( f^+ \in L^2_+([t_0, \infty))^{\mathcal{R}} \) | Edge inflow rates: \(f^+_{e,j}(t)\) denotes the rate at which particle on walk \(p\) enters its \(j\)-th edge at time \(t\) |
| \( f^- \in L^2_+([t_0, \infty))^{\mathcal{R}} \) | Edge outflow rates: \(f^-_{e,j}(t)\) denotes the rate at which particle on walk \(p\) leaves its \(j\)-th edge at time \(t\) |
| \( f = (f^+, f^-) \) | Edge flow: a flow described by its edge in- and outflow rates |
| \( x_e(h,.) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \) | (Edge) flow volume: the volume of flow \(x_e(h, \theta) := \int_0^\theta f^+_e(h, \vartheta)\ d\vartheta - \int_0^\theta f^-_e(h, \vartheta)\ d\vartheta\) on edge \(e\) at time \(\theta\) under the flow induced by the walk inflow \(h\) |
| \( q_e(h,.) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \) | Queue length: The length \(q_e(h, \theta) := \int_0^\theta f^+_e(h, \vartheta)\ d\vartheta - \int_0^{\theta + \tau_e} f^-_e(h, \vartheta)\ d\vartheta\) of the queue on edge \(e\) at time \(\theta\) under the flow induced by the walk inflow \(h\) |
| \( D_e(h,.) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \) | Edge delay: the delay \(D_e(h, \theta)\) experience under the flow induced by \(h\) by particles entering edge \(e\) at time \(\theta\) |
Symbol | Name/Description
--- | ---
$D_p(h,.) : [t_0,t_f] \rightarrow \mathbb{R}_{\geq 0}$ | Walk delay: the travel time $D_p(h,t)$ experienced under the flow induced by $h$ by particles entering walk $p$ at time $t$

$\Psi : L^2_+([t_0,t_f])^P \rightarrow \hat{M}([t_0,t_f])^P$ | Effective walk delay: the effective walk delay $\Psi_p(h,t)$ experienced under the flow induced by $h$ by particles entering walk $p$ at time $t$ (comprising e.g. travel time, early/late arrival penalties, energy costs, ...)

$\Psi| : L^2_+([t_0,t_f])^P \rightarrow L^2([t_0,t_f])^P$ | Truncated effective walk delay: the effective walk delay $\Psi_p(h,t)$ experienced under the flow induced by $h$ by particles entering walk $p$ at time $t$ (comprising e.g. travel time, early/late arrival penalties, energy costs, ...)

$\tau^p(h,.) : [t_0,t_f] \rightarrow \mathbb{R}_{\geq 0}$ | Arrival time $\tau^p_j(h,t)$ of particles starting along walk $p$ at time $t$ at the $j$-th node of this walk under the flow induced by $h$

$c_e : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ | Edge capacity: a function denoting the capacity $c_e(\theta)$ of edge $e$ at time $\theta$

$f_e(h,.) : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ | Edge load: $f_e(h,\theta)$ denotes some measure of the flow induced by $h$ on edge $e$ at time $\theta$

$S \subseteq L^2_+([t_0,t_f])^P$ | Constraint set: the set of all feasible walk inflows

$A_p(h) \subseteq P_i \times \mathcal{M}([t_0,t_f]) \times \mathbb{R}_{\geq 0} \times \mathbb{R}$ | Set of admissible $\epsilon$-deviations $(q,J,\epsilon,\Delta)$ under $h$ from walk $p$

$(q,J,\epsilon,\Delta) \in A_p(h)$ | Admissible $\epsilon$-deviation: a tuple denoting an admissible $\epsilon$-deviation of particles in space from walk $p$ to walk $q$ and in time from $J$ to $J + \Delta$

$H_{p\rightarrow q}(h,J,\epsilon,\Delta) \in L^2_+([t_0,t_f])^P$ | The walk inflow obtained from $h$ by an admissible $\epsilon$-deviation $(q,J,\epsilon,\Delta)$

$M_i(h) \subseteq L^2_+([t_0,t_f])^P$ | The set of walk inflows which can be obtained by admissible $\epsilon$-deviations of commodity $i$ from $h$

$U_p(h,t) \subseteq P_i \times \mathbb{R}$ | The set of admissible deviations $(q,\Delta)$ for particles entering walk $p$ at time $t$ under $h$

$(q,\Delta) \in U_p(h,t)$ | Admissible deviation: a tuple denoting an admissible deviation of shifting in space from $p$ to $q$ and in time from $t$ to $t + \Delta$

B. List of Dynamic Equilibrium Concepts

In this paper we consider the following dynamic equilibrium concepts:

- Dynamic equilibrium with fixed inflow rates: A walk inflow is an equilibrium if almost no particle can improve by switching to a different walk – see Definition 3.1.

- Dynamic equilibrium with fixed flow volume and departure choice: A walk inflow is an equilibrium if almost no particle can improve by switching to a different walk and/or departure time – see Definition 3.1.

- Dynamic equilibrium with elastic demands and departure choice: A walk inflow is an equilibrium if almost no particle can improve by switching to a different walk and/or departure time or by staying at home – see Definition 3.1.
• side-constrained dynamic equilibrium (SCDE): Our general equilibrium concept: A
walk inflow is an equilibrium if no particle has an admissible deviation with strictly
better effective walk delay— see Definition 5.1.

• strict side-constrained dynamic equilibrium (strict SCDE): Given any constraint set
$S \subseteq \Lambda(Q)$, the SCDE where admissible $\varepsilon$-deviations are those $\varepsilon$-deviations that lead
to another flow in $S$ – see Definition 5.10.

• strict capacitated dynamic equilibrium (strict CDE): The same as strict SCDE but
specifically for $S$ defined by edge-load constraints (i.e. by (23)) – see Definition 7.2.

• strong dynamic Bernstein-Smith equilibrium (strong BSDE): $S$ defined by edge load
constraints, admissible $\varepsilon$-deviations are those where the resulting flow is feasible for
all deviating particles at times where such particles enter an edge – see Definition 7.2.

• weak dynamic Bernstein-Smith equilibrium (weak BSDE): $S$ defined by edge load
constraints, admissible $\varepsilon$-deviations are those where the resulting flow is feasible for
all deviating particles at all time where such particles travel on an edge – see
Definition 7.2.

• strong dynamic Larsson-Patriksson equilibrium (strong LPDE): $S$ defined by edge load
constraints, admissible $\varepsilon$-deviations are those where deviating particles only
enter unsaturated edges – see Definition 7.2.

• weak dynamic Larsson-Patriksson equilibrium (weak LPDE): $S$ defined by edge load
constraints, admissible $\varepsilon$-deviations are those where deviating particles only travel
on unsaturated edges – see Definition 7.2.

• strong dynamic Marcotte-Nguyen-Schoeb equilibrium (strong MNSDE): $S$ defined
by edge load constraints, admissible $\varepsilon$-deviations are defined as for strong LPDE
except that edge-capacities may be tight on a common prefix of the current and the
alternative walk if $\Delta = 0$ – see Definition 7.2.

• weak dynamic Marcotte-Nguyen-Schoeb equilibrium (weak MNSDE): $S$ defined by
edge load constraints, admissible $\varepsilon$-deviations are defined as for weak LPDE ex-
cept that edge-capacities may be tight on a common prefix of the current and the
alternative walk if $\Delta = 0$ – see Definition 7.2.

C. An Example for the Usefulness of Cycles

The following example shows why particles may prefer to travel along walks containing
cycles in networks with capacity constraints. This means that the equilibria in such a
network can be different depending on whether the strategy space of the particles only
includes paths or also walks containing cycles.

Example C.1. The network in Figure 10 is an example for a single commodity network
with volume-constraints on the edges where it makes a difference for the resulting equi-
librium whether travelling along cycles is allowed or not. If cycles are not allowed, only
half of the flow can use the short edge $e_1$ towards the sink while the rest of the flow has to
take the much longer edge $e_2$. If, on the other hand, cycles are allowed, particles can use
the cycle $s \rightarrow v \rightarrow s$ to essentially wait at the source node until there is again room on
edge $e_1$. In other word, in this example individual particles prefer to travel along a cycle.
D. Existence of Unconstrained Dynamic Equilibria

In Section 3 we restated in Theorems 3.3 and 3.5 a well known characterization and existence result for unconstrained dynamic equilibria in the notation and under the assumptions used throughout this paper. While none of the analogous results from literature known to us (e.g. [5, 12, 22, 46]) exactly match the model used in this paper, the respective theorems can still be proven in essentially the same way. For completeness we provide the adjusted proofs here:

**Proof of Theorem 3.3.** First, consider the case of fixed inflow rates and let $h^*$ be a solution to the variational inequality (VI($\Psi, r, [t_0, t_f]$)). Let $p, q \in P_i$ be any two walks of some commodity $i \in I$ and $J := \{ t \in [t_0, t_f] \mid h_p^*(t) > 0, \Psi_p(t) > \Psi_q(t) \}$ the set of all times with strictly positive inflow of commodity $i$ into walk $p$ where $q$ would be a better alternative. We then define $h$ as the flow obtained by shifting all inflow of commodity $i$ during $J$ from $p$ to $q$, i.e.

$$
\begin{align*}
    h_p(t) &= 0 &\text{for all } t \in J \\
    h_q(t) &= h_p^*(t) + h_p^*(t) &\text{for all } t \in J \\
    h_{p'}(t) &= h_p^*(t) &\text{in all other cases.}
\end{align*}
$$

We then clearly have $h \in \Lambda(r)$ and, as $h^*$ is a solution to (VI($\Psi, r, [t_0, t_f]$)), we get

$$
\begin{align*}
0 &\leq \langle \Psi(h^*), h - h^* \rangle = \int_J \Psi_p(t) \left( 0 - h_p^*(t) \right) \, dt + \int_J \Psi_q(t) \left( h_q^*(t) + h_p^*(t) - h_q^*(t) \right) \, dt \\
&= \int_J (\Psi_q(t) - \Psi_p(t)) \, h_p^*(t) \, dt.
\end{align*}
$$

As $h_p^*$ is strictly positive and $\Psi_q(h, \cdot) - \Psi_p(h, \cdot)$ strictly negative on all of $J$, this implies that $J$ has measure zero. In other words, we have $\Psi_p(h, t) \leq \Psi_q(h, t)$ for almost all $t$ with $h_p^*(t) > 0$. Thus, $h^*$ is indeed a dynamic equilibrium.

For the other direction, let $h^* \in \Lambda(r)$ be a dynamic equilibrium and $h \in \Lambda(r)$ any feasible flow. Defining

$$
\psi_i : [t_0, t_f] \to \mathbb{R}_{\geq 0}, t \mapsto \inf \{ \Psi_p(h^*, t) \mid p \in P_i \}
$$

Figure 10: A single-commodity network with fixed network inflow rate where allowing particles to travel along cycles changes (improves) the equilibrium flow. All edges have a flow independent travel time as given in the figure. Edge $e_1$ is the only edge with a hard volume constraint.
for every commodity $i \in I$ we get
\[ \Psi_p(h^*, t) \left( h_p(t) - h^*_p(t) \right) \geq \psi_i(t) \left( h_p(t) - h^*_p(t) \right) \]
for almost all $t \in [t_0, t_f]$ and every walk $p \in P_i$ (by case-distinction on the second factor being negative or non-negative and using the fact that $h^*$ satisfies (2)). From this we directly get
\[
\langle \Psi(h^*), h - h^* \rangle = \\
\sum_{i \in I} \sum_{p \in P_i} \int_{t_0}^{t_f} \Psi_p(h^*, t) \left( h_p(t) - h^*_p(t) \right) \, dt \\
\geq \sum_{i \in I} \sum_{p \in P_i} \int_{t_0}^{t_f} \psi_i(t) \left( h_p(t) - h^*_p(t) \right) \, dt \\
= \sum_{i \in I} \int_{t_0}^{t_f} \psi_i(t) \left( \sum_{p \in P_i} h_p(t) - \sum_{p \in P_i} h^*_p(t) \right) \, dt \\
= \sum_{i \in I} \int_{t_0}^{t_f} \psi_i(t) (r_i(t) - r_i(t)) \, dt = 0.
\]

Therefore, $h^*$ is indeed a solution to the variational inequality (VI($\Psi, r, [t_0, t_f]$)).

Now, consider the case of fixed flow volumes. Let $h^*$ be a solution to the variational inequality (VI($\Psi, Q, [t_0, t_f]$)) and assume that $h^*$ is not a dynamic equilibrium. Then there must be a commodity $i \in I$, walks $p, q \in P_i$, sets of positive measure $J_p, J_q \subseteq [t_0, t_f]$ and constants $\varepsilon, \nu > 0$ such that
\[
\forall t \in J_p : h^*_p(t) \geq \varepsilon \quad \text{and} \quad \Psi_p(h^*, t) > \nu \\
\forall t \in J_q : h^*_q(t) \leq B_p - \varepsilon \quad \text{and} \quad \Psi_q(h^*, t) < \nu.
\]

We can also assume wlog. that $J_p$ and $J_q$ have the same size (wrt. the Lebesgue measure on $\mathbb{R}$). We now define $h$ as the flow obtained from $h^*$ by shifting flow at a rate of $\varepsilon$ in space from $p$ to $q$ and in time from $J_p$ to $J_q$, i.e.
\[
h_p(t) := h^*_p(t) - \varepsilon \quad \text{for all } t \in J_p \\
h_q(t) := h^*_q(t) + \varepsilon \quad \text{for all } t \in J_q \\
h_{pq}(t) = h^*_p(t) \quad \text{in all other cases.}
\]

Clearly, we have $h \in \Lambda(Q)$ and additionally we get
\[
\langle \Psi(h^*), h - h^* \rangle \\
= \int_{J_p} \Psi_p(h^*, t) \left( h^*_p(t) - \varepsilon - h^*_p(t) \right) \, dt + \int_{J_q} \Psi_q(h^*, t) \left( h^*_q(t) + \varepsilon - h^*_q(t) \right) \, dt \\
= - \int_{J_p} \Psi_p(h^*, t) \varepsilon \, dt + \int_{J_q} \Psi_q(h^*, t) \varepsilon \, dt \\
< - \int_{J_p} \nu \varepsilon \, dt + \int_{J_q} \nu \varepsilon \, dt = 0.
\]

But this is now a contradiction to $h^*$ being a solution to the variational inequality (VI($\Psi, Q, [t_0, t_f]$)). Thus, $h^*$ must have been a dynamic equilibrium.
For the other direction, let \( h^* \in \Lambda(Q) \) be a dynamic equilibrium with corresponding values \( \nu_i \geq 0 \) and \( h \in \Lambda(Q) \) any feasible flow. Then we have
\[
\Psi_p(h^*, t) \left( h_p(t) - h_p^*(t) \right) \geq \nu_i \left( h_p(t) - h_p^*(t) \right)
\]
for every time \( t \in [t_0, t_f] \) and walk \( p \in \mathcal{P}_i \) (by case-distinction on the second factor being negative, positive or zero and the fact that \( h^* \) satisfies \((3))\). From this we directly get
\[
\langle \Psi(h^*), h - h^* \rangle = \\
= \sum_{i \in I} \sum_{p \in \mathcal{P}_i} \int_{t_0}^{t_f} \Psi_p(h^*, t) \left( h_p(t) - h_p^*(t) \right) dt \\
\geq \sum_{i \in I} \sum_{p \in \mathcal{P}_i} \int_{t_0}^{t_f} \nu_i \left( h_p(t) - h_p^*(t) \right) dt \\
= \sum_{i \in I} \nu_i \left( \sum_{p \in \mathcal{P}_i} \int_{t_0}^{t_f} h_p(t) dt - \sum_{p \in \mathcal{P}_i} \int_{t_0}^{t_f} h_p^*(t) dt \right) \\
= \sum_{i \in I} \nu_i (Q - Q) = 0.
\]

Therefore, \( h^* \) is indeed a solution to the variational inequality \((VI(\Psi, Q, [t_0, t_f]))\). \( \square \)

**Proof of Theorem 3.5.** It is easy to see that both \( \Lambda(r) \) and \( \Lambda(Q) \) are non-empty, convex, closed and bounded (with respect to the \( L^2 \)-norm):

- Non-emptyness follows from \((A2)\) and \((A3)\).
- Convexity follows from the fact that the constraints defining \( \Lambda(r) \) and \( \Lambda(Q) \) are all linear.
- For closedness let \( h^n \in L^2_+(\{t_0, t_f\})^{\mathcal{P}} \) be a sequence of functions converging to some \( h \in L^2_+(\{t_0, t_f\})^{\mathcal{P}} \) (with respect to the \( L^2 \)-norm). If all \( h^n \) are from \( \Lambda(r) \) then so is \( h \) as we have
\[
\int_{t_0}^{t_f} \left| r_i(t) - \sum_{p \in \mathcal{P}_i} h_p(t) \right| dt = \int_{t_0}^{t_f} \left| \sum_{p \in \mathcal{P}_i} h^n_p(t) - \sum_{p \in \mathcal{P}_i} h_p(t) \right| dt \\
\leq \sum_{p \in \mathcal{P}_i} \int_{t_0}^{t_f} \left| h^n_p(t) - h_p(t) \right| dt \\
\leq \sum_{p \in \mathcal{P}_i} \left( \int_{t_0}^{t_f} \left( h^n_p(t) - h_p(t) \right)^2 dt \right)^{1/2} \cdot \left( \int_{t_0}^{t_f} 1^2 dt \right)^{1/2} \xrightarrow{n \to \infty} 0
\]
and, therefore, \( \sum_{p \in \mathcal{P}_i} h_p(t) = r_i(t) \) for all \( i \in I \) and almost all \( t \in [t_0, t_f] \). If all \( h^n \) are from \( \Lambda(Q) \) then so is \( h \) as we have
\[
\left| Q_i - \sum_{p \in \mathcal{P}_i} \int_{t_0}^{t_f} h_p(t) dt \right| = \left| \sum_{p \in \mathcal{P}_i} \int_{t_0}^{t_f} h^n_p(t) dt - \sum_{p \in \mathcal{P}_i} \int_{t_0}^{t_f} h_p(t) dt \right| \\
\leq \sum_{p \in \mathcal{P}_i} \int_{t_0}^{t_f} \left| h^n_p(t) - h_p(t) \right| dt \leq \sum_{p \in \mathcal{P}_i} \int_{t_0}^{t_f} \left| h^n_p(t) - h_p(t) \right| dt
\]
\[
\leq \sum_{p \in P} \left( \int_{t_0}^{t_f} \left( h_p^n(t) - h_p(t) \right)^2 \, dt \right)^{1/2} \cdot \left( \int_{t_0}^{t_f} 1^2 \, dt \right)^{1/2} \overset{n \to \infty}{\to} 0.
\]

Furthermore, \( h \) is also bounded by \( B_p \) almost everywhere as otherwise we would have some \( p \in P, \varepsilon > 0 \) and some set \( J \subseteq [t_0, t_f] \) of positive measure with \( h_p(t) \geq B_p + \varepsilon \) for all \( t \in J \). But this would imply the following contradiction:

\[
0 = \lim_{n} \int_{t_0}^{t_f} \left( h_p(t) - h_p^n(t) \right)^2 \, dt \geq \int_{J} \varepsilon^2 \, dt = \varepsilon^2 |J|.
\]

- For boundedness observe that both for \( \Lambda(r) \) and \( \Lambda(Q) \) there exist fixed bounded \( L^2 \)-functions bounding every walk inflow function \( h_p \) of any feasible \( h \) (\( r_i \) and \( B_p \), respectively).

Thus, we can choose \( C = \Lambda(r) \subseteq L^2([t_0, t_f])^P \) or \( C = \Lambda(Q) \subseteq L^2([t_0, t_f])^P \) and \( A = \Psi \) in Theorem 3.4 to obtain a solution to (VI(\( \Psi, r, [t_0, t_f] \))) or (VI(\( \Psi, Q, [t_0, t_f] \))), respectively. By Theorem 3.3 those solution are then also dynamic equilibria.

**Remark D.1.** Here we see why the walk inflow bounds \( B_p \) are needed for the case with departure time choice as these ensure that \( \Lambda(Q) \) is bounded. Without those bounds it is easy to construct instances without an equilibrium: Consider for example a network consisting of just a single edge \( e \) and a flow independent cost function \( \Psi_1, \{e\}(h,t) := t \).

Then for every possible flow all particles entering the network at a time different to \( t = 0 \) can improve by shifting to a time closer to \( 0 \). Thus, there is no equilibrium flow in this instance (and also no solution to the corresponding variational inequality).