Observer-based switched-linear system identification

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Abstract

In this paper, we present a methodology to identify discrete-time state-space switched-linear systems (SLSs) from input-output measurements. Continuous-state is not assumed to be measured. The key step is a deadbeat observer based transformation to a switched auto-regressive with exogenous input (SARX) model. This transformation reduces the state-space identification problem to a SARX model estimation problem. Overfitting issues are tackled. The switch and parameter identifiability and the persistence of excitation conditions on the inputs are discussed in detail. The discrete-states are identified in the observer domain by solving a non-convex sparse optimization problem. A clustering algorithm reveals the discrete-states under mild assumptions on the system structure and the dwell times. The switching sequence is estimated from the input-output data by the multi-variable output error state space (MOESP) algorithm and a variant modified from it. A convex relaxation of the sparse optimization problem yields the block basis pursuit denoising (BBPDN) algorithm. Theoretical findings are supported by means of a detailed numerical example. In this example, the proposed methodology is also compared to another identification scheme in hybrid systems literature.

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1 Introduction

Linear time-varying (LTV) systems are frequently used to model systems which have non-stationary properties and undergo small amplitude vibrations. Control design, realization theory, and identification of the LTV systems have received increasing attention in the past years [37, 25, 32]. Linear parameter varying (LPV) systems form a particular type of time-varying systems where the variation depends explicitly on a time-varying signal referred as the scheduling sequence. In state-space realizations, this results in the system matrices changed according to this scheduling sequence.

The state-space models are preferred over the input-output models since in the former multiple inputs and outputs are efficiently handled. Besides, advanced control synthesis methods are readily applied to the LPV state-space descriptions via the linear fractional transformation [34]. Recent studies [4, 16] have shown potential of the LPV system theory for industrial applications in which systems depend on a known scheduling vector. A subspace method to identify multi-input/multi-output (MIMO) LPV state-space systems with affine parameter dependence was proposed in [43]. A major problem is

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large dimensions of data matrices when the scheduling sequence varies arbitrarily. A numerically efficient implementation was presented in [40] using the kernel method [45]. Subspace identification of the MIMO–LPV systems using periodic scheduling sequence was studied in [14].

A special class of the LPV systems is the class of piece-wise affine (PWA) models of discrete-time nonlinear and hybrid systems. A PWA model is obtained by partitioning the state and the input set into a finite number of polyhedral regions. In each region, linear or affine submodels share the same continuous state. The PWA models are hybrid models with dynamical behavior switching among the submodels according to some discrete-event space. They have universal approximation properties, that is, any nonlinear phenomenon can be approximated by a PWA model. Equivalence between the PWA systems and several classes of hybrid systems was established in [19]. Results on analysis, computation, stability, and control of hybrid systems have appeared [29, 30].

Identification of a PWA model is performed in three stages: estimation of the submodel parameters, estimation of the hyperplanes defining partitioning of the state, and estimation of the input set. For models in the regression form, inputs are the regressors. This is a classification problem, that is, each datum is to be associated with a most suitable submodel. It is a very hard problem unless partitioning of the state is fixed a priori. In [15], piece-wise affine auto-regressive with exogenous input (PWARX) models were considered with clustering, linear identification, and pattern recognition techniques to identify both the submodels and the polyhedral partitioning of the regressor set. In [51], an algebro-geometric approach to piecewise-linear (PWL) model identification was proposed. It exploits the connections between the PWL system identification and the polynomial factorization/hyperplane clustering. In [33], a hybrid identification problem was formulated for the hinging hyperplane ARX and the Wiener PWARX models and solved by mixed-integer linear and quadratic programs. When errors are amplitude bounded, a three-stage procedure that uses a modified greedy algorithm for data classification and submodel estimation was proposed in [6].

1.1 Related work

A switched ARX (SARX) model is a hybrid affine model in which a finite number of submodels change only at switches that partition the time interval. The PWA model class is obtained by replacing the regressors with the scheduling sequences. Hence, identification algorithms developed in one model domain may be adapted to another with little effort.

The SARX models were studied in [27, 1, 28, 26]. In all of these works, multiple-input/single-output (MISO) model structures were used. The segmentation problem, that is, the decomposition of a time-varying system into submodels whose parameters are piece-wise constant in time was formulated in [27] as a least-squares estimation problem with sum-of-norms regularization over the state parameter jumps and solved by a standard convex optimization algorithm. A reformulation by a kernel function was introduced in [26]. In [28], this identification problem was cast as a sparsity maximization problem when noise is amplitude bounded and solved by a greedy optimization algorithm. When noise is quadratically bounded, a convex relaxation was also introduced. The algorithm proposed in [1] maximizes sparsity by assigning maximum number of the data points to a hyperplane generated by a submodel. A convex relaxation by the basis pursuit method was also introduced in this work.

The algebro-geometric method proposed in [51] for the PWARX models was extended in [20] to the state-space models by embedding the input-output data in a higher dimensional space. The submodels were extracted by the generalized principal component analysis algorithm. This method is suitable only for small data batches and high signal-to-noise ratio (SNR).

A subspace algorithm was proposed in [44] to identify the SLSs. Although the minimum dwell time requirement is modest, the scheduling sequence is assumed to be known. The state-space identification algorithm proposed in [2] does not restrict the minimum dwell time, yet assumes that continuous-state measurements are available. Without a constraint on the minimum dwell time, a state space identification algorithm for the SLSs was proposed in [3]. This algorithm is based on the observability results
derived in [50] for jump linear systems. Though the minimum dwell time is not constrained, the SLS is assumed pathwise observable. A non-convex optimization based identification algorithm was proposed in [35] assuming that the switching sequence has a bounded variation. In this algorithm, the switching sequence is randomly initialized and the submodels and the initial states are estimated by the Past Outputs Multivariable Output-Error State-Space (PO-MOESP) subspace algorithm [46]. The switching sequence is updated by solving a binary integer programming problem. Next, the submodel parameters are updated. Updating of the submodel clusters and the state-space parameter matrices by a coordinate descent algorithm is continued until a local or the global minimum is attained.

Detection and estimation of jumps in linear systems has been extensively studied in the literature [53, 5, 21, 31, 8]. A deadbeat observer based generalized likelihood ratio (GLR) test was proposed in [21] for the detection and estimation of jumps in the LTI system states. The deadbeat observer controls window size in the GLR test. The GLR test was extended in [31] to the SLSs in the state-space form. In [31], first the number of the local models and the switching sequence were estimated from the GLR test. Then, the Markov parameters of the local models were estimated. In the last step, similar local models were merged.

1.2 Motivation for state-space framework

With few exceptions the contributions surveyed above deal with the PWARX-MISO models. Many existing control analysis and synthesis design methods, on the other hand, rely on the state-space models. The subspace, or more generally, the realization algorithms include some of the very popular methods in system identification. The main reason for their success is that they rely on the numerically robust QR factorization and the singular value decomposition (SVD) for low-rank matrix approximation from input-output data. Models returned by subspace methods are also nearly balanced.

1.3 Contributions

A framework is proposed to identify the discrete states and the switching sequences of the SLSs in the state-space form from the input-output measurements. This framework followed by the basis construction procedure in [7] proposed by the authors of this paper delivers final models suitable for predicting time responses of the SLSs to prescribed inputs. The proposed identification framework is demonstrated to be consistent under some assumptions on the system structure, the dwell times of the discrete states, and noise amplitude in a completely deterministic setting. The switch detection schemes proposed in [21, 31], on the other hand, rely on the stochastic noise descriptions. The proposed framework is exact: any SLS can be recovered in finite time from its noiseless input-output measurements if every discrete state is active in some segment and the minimum dwell time is sufficiently large while in some works [31] output transients may be detrimental due to the state approximations.

1.4 Organization of the paper

The contents of this paper are as follows. In Section 2, we formulate the state space identification problem for single-input/single-output (SISO)-SLS models from input-output data. In Section 3, deadbeat observer-based transformation of the state-space SLS models to the SARX models is studied. This is a key step in reducing the state-space identification problem to an SARX model estimation problem. Back model transformations from the SARX models to the LTV models and from the LTV models to the SLS models are studied. The switches of the SARX model and their identifiability from the input-output data are also studied. This technical section prepares the stage for a non-convex and sparse optimization problem formulation in the next section to estimate the local models. The role played by model transformations is to compress infinite strings of the system Markov parameters into finite sets of the observer Markov parameters at the expense of more complicated discrete state sets and the switching sequences in the transformation domain.
In Section 4 a local model set is retrieved by a clustering algorithm from the solution of a non-convex and sparse optimization problem over long and constant parameter intervals. This set exhausts all discrete states if every discrete state is active in at least one sufficiently long segment. The endpoints of such intervals are the switches. The rest of the switches are estimated from the input-output data by a MOESP type subspace algorithm \cite{47,48} or a discrete optimization algorithm modified from the MOESP algorithm in Section 5.

Convex relaxation of the optimization problem leads to the BBPDN method in Section 6. This is achieved by relaxing the nonconvex mixed \(\ell_0/\ell_1\) norm with the convex mixed \(\ell_2/\ell_1\) norm. Recovery guarantees for the BBPDN \cite{38} and the block orthogonal matching pursuit (BOMP) \cite{39} algorithms have been put forward in the compressive sensing/approximation literature \cite{12,11}. They are replaced in this paper by the switch identifiability and the persistence of excitation conditions. These conditions not only make recovery of the local modes possible, but also guarantee robustness to amplitude-bounded noise if SNR is large. Theoretical findings are supported by means of a detailed numerical example in Section 7. In this example, the proposed method is also compared to a competitive algorithm in hybrid systems literature. Concluding remarks with a brief sketch of future work are presented in Section 8.

2 Problem statement for the SLS identification

In this paper, we consider a special class of the LTV-SISO systems represented by the state-space equations

\[
\begin{align*}
    x(k+1) &= A(k)x(k) + b(k)u(k), \\
    y(k) &= c^T(k)x(k) + d(k)u(k)
\end{align*}
\]  

(1) (2)

where \(u(k) \in \mathbb{R}, y(k) \in \mathbb{R}, x(k) \in \mathbb{R}^n\) are respectively the input, the output, the state sequences, and \(c^T\) denotes the transpose of a given vector (matrix) \(c\). The state dimension \(n\) is assumed to be known and does not change with time.

Let \(\mathbb{N}\) denote the set of positive integers and \(\varphi\) be a switching sequence, that is, a map from \(\mathbb{N}\) onto a finite set \(\mathbb{S} = \{1, \cdots, \sigma\}\) for some fixed \(\sigma \in \mathbb{N}\). Substitute \(l = \varphi(k)\) and suppose that \(A(k) = A_l, b(k) = b_l, c(k) = c_l, d(k) = d_l\). We denote the set of the discrete states (submodels) \(\mathcal{P}_l = (A_l, b_l, c_l^T, d_l)\), \(l = 1, \cdots, \sigma\) by \(\mathcal{P}\). The SISO model (1)–(2) with the state-space matrices changed by \(\varphi\) is an SLS.

A switching sequence \(\varphi(k)\) segments a given interval \([1, N]\) into disjoint intervals \([k_i, k_{i+1}]\) such that

\[
\varphi(k) = \varphi(k_i), \quad k_i \leq k < k_{i+1}
\]

(3)

where \(k_0 = 1\) and \(k_i < k_r \leq N\). Given a segmentation \(\chi\) of \([1, N]\), let \(\delta_i(\chi) = k_{i+1} - k_i, 0 \leq i < i^*\). The minimum dwell time is defined by \(\delta(\chi) = \min_i \delta_i(\chi)\). Thus, \(\delta_i(\chi)\) is the waiting time of the discrete state active in \([k_i, k_{i+1}]\) and the minimum dwell time is the smallest waiting time. We state the requirements on the model structure as follows.

**Assumption 2.1** The SLS model (1)–(3) has \(\sigma\) stable discrete states with MacMillan degree \(n\).

The SLS identification problem for the SISO systems we study in this paper is formulated as follows:

**Problem 2.1** Given input-output data \(u(k), y(k), 1 \leq k \leq N\) of the SLS model (1)–(3) satisfying Assumption 2.7 estimate the discrete states and the switching sequence.

In the course of developing a framework that solves the identification problem posed above, we will impose further conditions on the inputs, \(\mathcal{P}\), and \(\varphi\).
3 Observer-based transformation to SARX model

Let us add and subtract \( g(k)y(k) \) to (1):

\[
x(k + 1) = A(k)x(k) + b(k)u(k) + g(k)y(k) - g(k)y(k), \\
= A_o(k)x(k) + B_o(k)\zeta(k)
\]

(4)

where we used (2), \( g(k) \) is a time-varying gain sequence, and

\[
A_o(k) = A(k) + g(k)c^T(k) \in \mathbb{R}^{n \times n}, \\
B_o(k) = [b(k) + g(k)d(k) - g(k)] \in \mathbb{R}^{n \times 2}, \\
\zeta(k) = [u(k) \ y(k)]^T \in \mathbb{R}^2.
\]

(5)

(6)

(7)

Thus, we arrive at the so-called observer equations

\[
x(k + 1) = A_o(k)x(k) + B_o(k)\zeta(k), \\
y(k) = c^T(k)x(k) + d(k)u(k).
\]

(8)

(9)

The observer response is calculated from (8)–(9)

\[
y(k) = c^T(k)\Phi_o(k,i)x(i) + d(k)u(k) + \sum_{j=i}^{k-1} h_o(k,j)\zeta(j)
\]

(10)

for \( 1 \leq i < k \) by introducing the observer Markov parameters and the observer state transition matrix

\[
h_o(k,i) = c^T(k)\Phi_o(k,i+1)B_o(i), \\
\Phi_o(k,i) = A_o(k-1) \cdots A_o(i)
\]

(11)

(12)

for \( k > i \) and for \( k = i, h_o(k,k) = [d(k) \ 0] \) and \( \Phi_o(k,k) = I_n \). Suppose there are \( \tau, k', k'' \in \mathbb{N} \) such that \( k' < \tau \) and for all \( k' \leq k \leq k'' \), \( \Phi_o(k,k-\tau) = 0 \). Then, (10) simplifies to a linear regression of \( u(k) \) and \( \zeta(k) \)

\[
y(k) = d(k)u(k) + \sum_{j=k-\tau}^{k-1} h_o(k,j)\zeta(j), \quad k' \leq k \leq k''
\]

(13)

which is a time-varying ARX model, and in fact based on (3) an SARX model, described by \( 2\tau + 1 \) parameters. An observer with this property is called deadbeat observer.

**Definition 3.1** An LTV discrete-time observer is said to be a deadbeat observer on the interval \([k' \ k'']\) if there exists a gain sequence \( g(k) \in \mathbb{R}^n \) and a \( \tau < k' \) such that

\[
\Phi_o(k,k-\tau) = 0, \quad \text{for all} \ k' \leq k \leq k''.
\]

(14)

Finding deadbeat observers for arbitrary LTV systems, in particular one with a \( \tau \) as small as possible is not trivial. Suppose for a moment that the system described by (1)–(2) is time-invariant. Thus, we seek a constant gain \( g \in \mathbb{R}^n \). In this case, \( A(k) = A, b(k) = b, c^T(k) = c^T \), \( d(k) = d \) and

\[
h_o(k,i) = c^T(A + gc^T)k-i-1[b + gd - g], \quad k > i.
\]

(15)

For an LTI system, (14) translates to \( (A + gc^T)^\tau = 0 \) for some \( g \in \mathbb{R}^n \). If \( (A,b,c^T,d) \) is minimal, \( (A,c^T) \) is observable and \( \tau = n \). Alternatively, by choosing \( \tau \) large and pushing the eigenvalues of \( A \) to zero, \( \Phi_o(k,k-\tau) = 0 \) may be demanded to hold approximately. Let us illustrate some properties of the deadbeat observers by two numerical examples.
Example 3.1 Let the observability pair \((A, c^T)\) be given by
\[
A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad c^T = [1, 2].
\]
Then, \(A\) has one eigenvalue at 0 and from
\[
\begin{bmatrix} c^T \\ c^TA \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix},
\]
we see that \((A, c^T)\) is observable. With \(g = [g_1, g_2]^T\) the characteristic equation of \(A + gc^T\) is given by
\[
\lambda^2 - \lambda(1 + g_1 + 2g_2) + g_2 = 0.
\]
The observer gain enforcing \(\lambda^2 = 0\) is uniquely calculated as \(g = [-1, 0]^T\). This is expected since \((A, c^T)\) is observable. Since \((A + gc^T)^2 = 0\), \(\min \tau \leq 2\). The pair \((A + gc^T, c^T)\) is also observable. Hence, \((A + gc^T)^k \neq 0\) for all \(k < n\).

Example 3.2 Let
\[
A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad c^T = [1, 1].
\]
Both \(A_1\) and \(A_2\) have two eigenvalues at 0 and \((A_1, c^T)\) and \((A_2, c^T)\) are observable. Moreover, \(g = [0, 0]^T\). The matrices \(A_1\) and \(A_2\) generate by multiplication only two other matrices
\[
A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
\]
Hence, \(\{A_1, A_2, A_3, A_4\}\) is a finite non-commutative group without identity.

The deadbeat observers transform the state-space models to the ARX models which are easier to estimate from the input-output data since infinite strings of the system Markov parameters are packed into finite numbers of the regression coefficients. The following result provides an upper bound on \(\tau\) in Definition 3.1. This upper bound does not depend on \(\sigma\).

Lemma 3.1 Let the SLS model \([1]–[3]\) be as in Assumption 2.1. Suppose that \(\delta_\circ(\chi) \geq n\) and \(\delta_0(\chi) \geq 2n\). Then, there exists a deadbeat observer and a gain sequence \(g(k)\) for \([1]–[2]\) satisfying a best possible bound \(\tau \leq 2n - 1\) on \([1, N]\) and \(g(k) = g(k_i), \tau = n\) on \([k_i, k_{i+1})\) for \(1 \leq i < i^*\).

Proof. See Appendix 8.

The minimum dwell time requirement in Lemma 3.1 cannot be dropped. An example using \(A_1\) and \(A_2\) matrices in Example 3.2 is \(\Phi_o(k, i) = A_1A_2A_1 \cdots\) satisfying \(\Phi_o(k, i) \neq 0\) for all \(k\) and \(i\) with \(k > i\). The last statement in the lemma asserts that there is a discrete state set satisfying Assumption 2.1 such that the first conclusion does not hold for a \(\tau < 2n - 1\). From Lemma 3.1 we may write (13) as
\[
y(k) = d(k)u(k) + \sum_{j=k-2n+1}^{k-1} h_o(k, j)\zeta(j), \quad 2n \leq k \leq N.
\]

3.1 Model conversions SARX-to-LTV-to-SLS
In this subsection, we first study recovery of the system Markov parameters of \([1]–[3]\) defined for \(k > i\) by
\[
h(k, i) = c^T(k)\Phi(k, i+1)b(i)
\]
(17)
and for \( k = i \) by \( h(k,k) = d(k) \) where \( \Phi(k,k) = I_n \) and

\[
\Phi(k,i) = A(k-1) \cdots A(i)
\]

with \( I_n \in \mathbb{R}^{n \times n} \) denoting the \( n \) by \( n \) identity matrix and the gain sequence from the observer Markov parameters.

Partition \( h_0(k,i) \) as \( h_0(k,i) = [h_0^{(1)}(k,i) \ h_0^{(2)}(k,i)] \) and let

\[
\gamma(k,i) = h_0^{(1)}(k,i) + h_0^{(2)}(k,i)h_0^{(1)}(i,i), \quad k > i \geq 1.
\]

The following recurrence formula

\[
h(k,i) = \gamma(k,i) + \sum_{j=i+1}^{k-1} h_0^{(2)}(k,j)h(j,i), \quad k > i + 1
\]

initialized with \( h(k,k-1) = \gamma(k,k-1) \) was derived in [23]. For the deadbeat observers, the constraints \( h_o^{(s)}(k,i) = 0 \) for \( s = 1, 2 \) and \( k - i > 2n - 1 \) are invoked in (19) and (20).

Up to a topological equivalence, the quadruples \( A(k), b(k), c^T(k), d(k) \) may be recovered from the system Markov parameters if \( (1) \sim (3) \) is uniform [36]. Two LTV realizations \( (A_1(k), b_1(k), c_1^T(k), d_1(k)) \) and \( (A_2(k), b_2(k), c_2^T(k), d_2(k)) \) have the same Markov parameters if they are topologically equivalent, that is, if there exists a bounded matrix \( T(k) \in \mathbb{R}^{n \times n} \) with a bounded inverse \( T^{-1}(k) \) such that for all \( k \in \mathbb{N} \),

\[
A_2(k) = T(k+1)A_1(k)T^{-1}(k),
\]

\[
b_2(k) = T(k+1)b_1(k)
\]

\[
c_2^T(k) = c_1^T(k)T^{-1}(k),
\]

\[
d_2(k) = d_1(k).
\]

The transformation \( T(k) \) with bounded inverse \( T^{-1}(k) \) is called a Lyapunov transformation. As far as the input-output behavior of an LTV system is concerned, it suffices to estimate \( h(k,i) \) from the input-output measurements.

Next, we estimate \( g(k) \) by a two-step procedure. In the first step, the observer gain Markov parameters defined by

\[
h_m(k,i) = c^T(k)\Phi(k,i+1)g(i), \quad k > i + 1
\]

and \( h_m(i+1,i) = c^T(i+1)g(i) \) are estimated from the observer Markov parameters using the recurrence formula

\[
h_m(k,i) = -h_o^{(2)}(k,i) + \sum_{j=i+1}^{k-1} h_o^{(2)}(k,j)h_m(j,i)
\]

for \( k > i + 1 \) initialized by \( h_m(k,k-1) = -h_o^{(2)}(k,k-1) \) [23]. The second step consists of estimating \( g(k-1) \) from (21). Given \( q \in \mathbb{N} \), which will be fixed later as \( q = 2n \), concatenate the equations in (21) and notice that

\[
\begin{bmatrix}
  h_m(k,k-1) \\
  h_m(k+q-1,k-1) \\
  \vdots \\
  c^T(k) \\
  c^T(k+q-1)\Phi(k+q-1,k)
\end{bmatrix} g(k-1) = \mathcal{O}_q(k)g(k-1)
\]

\[
(23)
\]
where $\mathcal{O}_q(k)$ is the extended observability matrix of (1)–(2) at $k$. Compute $\hat{O}_q(k)$ for $(\hat{A}(k), c^T(k))$ from the realization algorithm outlined in Section 3.1.1 and set

$$\hat{g}(k-1) = \hat{O}_q^1(k) \begin{bmatrix} h_m(k, k-1) \\ \vdots \\ h_m(k+q-1, k-1) \end{bmatrix}. \quad (24)$$

Then, $\hat{g}(k-1) = T(k)g(k-1)$.

The results in this subsection reduce the identification of the state-space models from the input-output data to the estimation of the observer Markov parameters from the input-output data. We outline the above derivations in the form of an algorithm. Execution of Step 7 requires $q$-times applications of Steps 1–6. We will not apply Algorithm 1 to $[2n \ N]$, but to its some specific subsets The above results are summarized in the following.

### Algorithm 1. SARX to LTV model conversion

**Input:** $h_0(k,i), k-2n < i \leq k$

1: Initialize $h(k,k) = h_0^{(1)}(k,k)$
2: Calculate $\gamma(k,i)$ from (19) while $k-2n < i < k$
3: Set $h(k, k-1) = \gamma(k, k-1)$
4: Estimate $h(k,i)$ from (20)
5: Set $h_m(k, k-1) = -\hat{h}_0^{(1)}(k, k-1)$
6: Estimate $h_m(k,i)$ from (21) while $k-2n < i < k-1$
7: Estimate $g(k-1)$ from (24)

**Outputs:** $h(k,i)$ and $g(k-1)$ for $i \leq k$

**Lemma 3.2** Algorithm 1 back transforms the SARX model (13) to the LTV model (1)–(3).

The SARX model is over-parameterized to accommodate the discrete state changes at the switches. The switches are not known a priori. Once they are located, fewer parameters may be used. Zero-padding will only require richer inputs for parameter identifiability as opposed to the parsimonious models. In the rest of this subsection, we show that (1)–(3) subject to Assumption 2.1 and $\delta_c(\gamma) \geq n$ is uniform on the interval $[2n+1 \ N-2n+1]$. This means that (1)–(3) is uniformly bounded, uniformly observable, and uniformly controllable. Recall that (1)–(3) is uniformly controllable if there exist $\kappa_c, \delta_c \in \mathbb{N}$ and $\alpha_0, \alpha_1, \beta_0, \beta_1 > 0$ such that for all $k > \delta_c$,

1. $G_c(k, \kappa_c) > 0$,
2. $\alpha_0 I_n \leq G_c^{-1}(k, \kappa_c) \leq \alpha_1 I_n$,
3. $\beta_0 I_n \leq \Phi^T(k, k-\kappa_c) G_c^{-1}(k, \kappa_c) \Phi(k, k-\kappa_c) \leq \beta_1 I_n$

where the notation $X \geq 0$ ($X > 0$) means that $X$ is a square and positive semi-define (positive definite) matrix and

$$G_c(k, \kappa_c) = \sum_{j=k-\kappa_c}^{k-1} \Phi(k, j+1) b(j) b^T(j) \Phi^T(k, j+1).$$

Likewise, (1)–(3) is uniformly observable if there exist a $\kappa_0 \in \mathbb{N}$ and $\alpha'_0, \alpha'_1, \beta'_0, \beta'_1 > 0$ such that for all $k \leq \delta_0$,

1. $G_0(k, \kappa_0) > 0$,
2. $\alpha'_0 I_n \leq G_0^{-1}(k, \kappa_0) \leq \alpha'_1 I_n$, 

...
3. $\beta'_0 I_n \leq \Phi^T (k + \kappa_o, k)G_o^{-1}(k, \kappa_o)\Phi(k + \kappa_o, k) \leq \beta'_1 I_n$

where

$$G_o(k, \kappa_o) = \sum_{j=k}^{k+\kappa_o-1} \Phi^T(j, k)c(j)c^T(j)\Phi(j, k).$$

**Lemma 3.3** Suppose Assumption [2,7] holds and $\delta_c(\chi) \geq n$. Then, [1–3] is uniform on $(\delta_c \delta_0)$ with $\kappa_0 = \kappa_c = \delta_c = 2n$, and $\delta_0 = N - \kappa_0 + 1$.

**Proof.** See Appendix 8.

We capture the requirements on $\phi$ in the following.

**Assumption 3.1** The switching sequence $\phi$ for the SLS model [1–3] satisfies $\delta_c(\chi) \geq n$, $\delta_0(\chi) \geq \kappa_c$, and $k_r \leq \delta_0$.

The discrete states for $k \leq \delta_c$ and $k > \delta_0$ are minimal from $\delta_0(\chi) \geq \kappa_c$ and $k_r \leq \delta_0$ since they are similar to the discrete states at $k = \delta_c + 1$ and $k = k_r$. From Lemma 3.3, $G_o(k, \kappa_0) > 0$ and if $q \geq \kappa_0$, $\Theta_q(k)$ will have full rank. Hence, $\Theta_q(k)$ is well-defined since $\Theta_q(k) = T^{-1}(k)\Theta_q(k)$.

### 3.1.1 SLS realization from Hankel matrix pairs

For $\delta_c < k \leq \delta_0$, we define a nested sequence of the *Hankel* matrices built from the system Markov parameters

$$H(k) = \begin{bmatrix} h(k, k-1) & \cdots & h(k, k-\kappa_c) \\ \vdots & \ddots & \vdots \\ \cdots & h(k+\kappa_0-1, k-\kappa_c) & \end{bmatrix}$$

and factorize them as $H(k) = \Theta_{\kappa_c}(k)\mathcal{R}_{\kappa_c}(k-1)$ where

$$\mathcal{R}_{\kappa_c}(k-1) = [b(k-1) \cdots \Phi(k, k-\kappa_c+1)b(k-\kappa_c)]$$

is the extended controllability matrix [36]. We have already seen that $\text{rank}(H(k)) = n$ for all $k$. Define the shift matrices of row up and down and column left and right by $J_U = [0 I_{\kappa_c-1}]$ and $J_D = [I_{\kappa_c-1} 0]$ and $J_L = [0 I_{\kappa_c-1}]^T$ and $J_R = [I_{\kappa_c-1} 0]^T$. From the factorization formula with $k$ and $k + 1$ plugged in, we retrieve $A(k)$ from either of the two formulas

$$J_U \Theta_{\kappa_c}(k) = (J_D \Theta_{\kappa_c}(k + 1))A(k),$$

$$A(k)(\mathcal{R}_{\kappa_c}(k-1)J_R) = \mathcal{R}_{\kappa_c}(k)J_L$$

as follows

$$A(k) = (J_D \Theta_{\kappa_c}(k + 1))^T J_U \Theta_{\kappa_c}(k)$$

$$= \mathcal{R}_{\kappa_c}(k)J_L (\mathcal{R}_{\kappa_c}(k-1)J_R)^T$$

where $X^\dagger$ denotes the unique pseudo-inverse of a given full-rank matrix $X$ defined as $(X^TX)^{-1}X^T$. To determine $\Theta_{\kappa_c}(k)$ and $\Theta_{\kappa_c}(k + 1)$. We apply SVD to $H(k)$ and $H(k + 1)$

$$H(k) = U_{\kappa_c}(k)\Sigma(k)V_{\kappa_c}^T(k)$$

$$H(k + 1) = U_{\kappa_c}(k + 1)\Sigma(k + 1)V_{\kappa_c}^T(k + 1)$$

(26)
and let

\[
\begin{align*}
\hat{O}_{\kappa}(k) &= U_{\kappa}(k)\Sigma^{1/2}(k), \\
\hat{\sigma}_{\kappa}(k-1) &= \Sigma^{1/2}(k)V_{\kappa}^T(k), \\
\hat{O}_{\kappa}(k+1) &= U_{\kappa}(k+1)\Sigma^{1/2}(k+1), \\
\hat{\sigma}_{\kappa}(k) &= \Sigma^{1/2}(k+1)V_{\kappa}^T(k+1).
\end{align*}
\] (27)

They provide estimates of \(O_{\kappa}(k)\) and \(\sigma_{\kappa}(k+1)\) for some transformations \(T(k)\) and \(T(k+1)\) as \(O_{\kappa}(k) = \hat{O}_{\kappa}(k)T(k)\) and \(\sigma_{\kappa}(k+1) = \hat{\sigma}_{\kappa}(k+1)T(k+1)\). Let

\[
\hat{A}(k) = (J_D \hat{O}_{\kappa}(k+1))^T J_U \hat{O}_{\kappa}(k)
\] (28)

so that

\[
\hat{A}(k) = T(k+1)A(k)T^{-1}(k).
\] (29)

The estimation of \(b(k)\) and \(c^T(k)\) is in order. Let

\[
J_c = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix},
\]

\[
J_b = J_c^T.
\] (30)

Then,

\[
c^T(k) = J_c O_{\kappa}(k),
\]

\[
b(k) = \sigma_{\kappa}(k)J_b.
\] (31)

As the estimates of \(c^T(k)\) and \(b(k)\), we set

\[
\hat{c}^T(k) = J_c \hat{O}_{\kappa}(k),
\]

\[
\hat{b}(k) = \hat{\sigma}_{\kappa}(k)J_b.
\] (32)

Then, from \(\hat{O}_{\kappa}(k) = T(k+1)\sigma_{\kappa}(k)\)

\[
\hat{c}^T(k) = c^T(k)T^{-1}(k),
\]

\[
\hat{b}(k) = T(k+1)b(k).
\] (33)

Setting \(\hat{d}(k) = h(k,k)\), we see that \((\hat{A}(k), \hat{B}(k), \hat{C}(k), \hat{D}(k))\) is similar to \(\varphi_{(k)}\). The steps above are summarized as a (point-wise in time) realization algorithm.

**Algorithm 2. LTV to SLS model conversion**

**Inputs:** \(h(l+s-1,l-t), 1 \leq t \leq \kappa; 1 \leq s \leq \kappa; l = k, k+1\)

1: Calculate \(\mathcal{H}(k)\) and \(\mathcal{H}(k+1)\) from (25)
2: Apply SVD to \(\mathcal{H}(k)\) and \(\mathcal{H}(k+1)\) as in (26)
3: Estimate \(O_{\kappa}(k)\) and \(\sigma_{\kappa}(k+1)\) from (27)
4: Estimate \(A(k)\) from (28)
5: Estimate \(c^T(k)\) and \(b(k)\) from (32)
6: Set \(\hat{d}(k) = h(k,k)\)

**Outputs:** \(\hat{A}(k), \hat{B}(k), \hat{C}(k), \hat{D}(k)\).
3.2 The switches of the SARX model

Stack the observer Markov parameters and the input-output data into the parameter and the regression vectors defined as

\[
\theta(k) = (d(k) h_o(k,k-1) \cdots h_o(k,k-2n+1))^T, \quad z(k) = (u(k) \xi^T(k-1) \cdots \xi^T(k-2n+1))^T
\]

and write (16) in the form

\[
y(k) = z^T(k) \theta(k), \quad \delta_c < k \leq \delta_o. \quad (35)
\]

Recall from Lemma 3.1 that \( \tau < 2n \) and \( \tau = n \) on the union of intervals \([k_i + n \ k_{i+1})\), \( 1 \leq i < i^* \). The latter means that

\[
\theta(k) = (d(k) h_o(k,k-1) \cdots h_o(k,k-n) \ 0 \cdots 0)^T \quad (36)
\]

on this set. The switches of (35) are the instants \( k \) satisfying

\[
\delta_g(k) = \theta(k) - \theta(k-1) \neq 0. \quad (37)
\]

Let \( \varphi_\theta(k) \) and \( \{s_1, \cdots, s_{j^*}\} \) denote the switching sequence and its range where \( s_0 = 2n \) so that \( \delta_c < s_j \leq \delta_o \) for all \( j \leq j^* \). For a given segmentation of \((\delta_c \ \delta_o)\), i.e., covering by semi-closed disjoint intervals \( \chi_\theta \) as in (3), we allocate a dwell sequence \( \delta_j(\chi_\theta) = s_{j+1} - s_j \), \( 0 \leq j < j^* \) and define the discrete state parameter set as \( \mathcal{P}^\theta = \{ \theta(k) : \delta_c < k \leq \delta_o \} \).

We will present several lemmas to link the segments in the SLS model (1)–(3) to the segments in the observer model (35) and vice versa via Lemma 3.1. The link we provide is partial, that is, it does not carry complete information on \( \chi_\theta \) and \( \mathcal{P}^\theta \); yet, this partial information will be sufficient to retrieve \( \chi \) and \( \mathcal{P} \) from the input-output data.

Lemma 3.4 Consider the SLS model (1)–(3). Let the observer be as in Lemma 3.1. If \( k_i, k_{i+1} \in \chi \) and \( \delta(\chi) > n \), then \([k_i + n \ k_{i+1}) \subseteq [s_j \ s_{j+1})\) for some \( s_j, s_{j+1} \in \chi_\theta \).

Proof. See Appendix 8.

A converse statement is also true. But, first we need an auxiliary result that holds in the more general LTV setting without requiring the observer be as in Lemma 3.1.

Lemma 3.5 Consider the observer (35) for the SLS model (1)–(3). Then, \( h \) and \( h_m \) are locally shift-invariant, that is, \( h(k,k-v) = h(l,l-v) \) and \( h_m(k,k-v) = h_m(l,l-v) \) for all \( k, l \), and \( v \geq 0 \) with \( s_j + v \leq \min\{k,l\} \) and \( \max\{k,l\} < s_{j+1} \).

Proof. See Appendix 8.

Let \( \mathcal{H}_{\alpha \beta}(k) \) denote the the entry of \( \mathcal{H}(k) \) in (25) with the row and the column indices \( \alpha, \beta \). Recalling \( \kappa_c = \kappa_o = 2n \), for all \( 1 \leq \alpha \leq \kappa_o \) and \( 1 \leq \beta \leq \kappa_c \), we will require

\[
\mathcal{H}_{\alpha \beta}(k) = h(k+\alpha-1,k-\beta) = h(k+\alpha,k+1-\beta) = \mathcal{H}_{\alpha \beta}(k+1).
\]

Set \( l = k + \alpha \) and \( l - v = k + 1 - \beta \) from which we derive \( v = \alpha + \beta - 1 \). We get \( s_j + 4n - 1 \leq k \) with \( \alpha = \beta = 2n \) from the first constraint and \( k < s_{j+1} - 2n \) with \( \alpha = 2n \) from the second constraint \( k + \alpha < s_{j+1} \). It follows from Lemma 3.5 that the LTV system is locally shift-invariant on the interval \([\alpha_j \ \beta_j)\) if \( \delta_j(\chi_\theta) \geq 6n \) where

\[
\alpha_j = s_j + 4n - 1 \quad \text{and} \quad \beta_j = s_{j+1} - 2n. \quad (38)
\]
Hence, if \( k \in [\alpha_j, \tilde{\beta}_j] \), then \( H(k) = H(k + 1) \). Moreover, \( d(k) = d(k + 1) \) from Lemma 3.5. Algorithm 2 estimates a realization \((A(k), b(k), c^T(k), d(k))\) of (1)-(3) up to a topological equivalence from the pair \( H(k) \) and \( H(k + 1) \). Since \( T(k) = T(k + 1) \), \( \varphi(k) = \varphi(\alpha_j) \) for all \( k \in [\alpha_j, \tilde{\beta}_j] \). Let
\[
k_i = \max\{k \leq \alpha_j\}, \quad k_f = \min\{k \geq \tilde{\beta}_j\}.
\]
Suppose \( l > i + 1 \). From the definitions of \( k_f \) and \( k_i \), we get \( k_i \leq \alpha_j < k_{i+1} < \tilde{\beta}_j \leq k_f \). Recall that \( \varphi(k) \) is constant on \([\alpha_j, \tilde{\beta}_j]\). Since \( k_{i+1} \) is an interior point of \([\alpha_j, \tilde{\beta}_j]\), \( \varphi(k_{i+1}) = \varphi(\alpha_j) \). But, \( \alpha_j \in [k_i, k_{i+1}] \).

Then, from the equalities \( \varphi(k_i) = \varphi(\alpha_j) \) and \( \varphi(k_i) = \varphi(\alpha_j) \) we reach a contradiction since \( k_{i+1} \in \mathcal{X} \).

Therefore, \( l \leq i + 1 \). Suppose \( l \leq i \). From the chain of the inequalities \( \tilde{\beta}_j \leq k_i \leq k_f \), we get \( \delta_j(\omega) < 6n \) reaching a contradiction. Hence, \( l = i + 1 \). The definitions of \( k_i \) and \( k_f \) yield \( \tilde{\beta}_j \leq k_{i+1} \) and \( k_i \leq \alpha_j \) with \( l = i + 1 \). Reorganize these inequalities as
\[
[\alpha_j, \tilde{\beta}_j) \subset [k_i, k_{i+1}) \quad \text{and} \quad \delta_j(\omega) \geq 6n.
\]

The observer gain \( g(k-1) \) is estimated from (24) using \( \hat{c}(\ell), \hat{A}(\ell), k \leq \ell \leq k + 2n - 2 \) and \( \hat{c}(k + 2n - 1) \). Algorithm 2 estimates \( c(k + 2n - 1) \) from the pair \( H(k + 2n - 1) \) and \( H(k + 2n) \). The estimation of the observer gain Markov parameters \( h_m(k + v, k - 1), 0 \leq v < 2n \) does not require additional data since (22) uses the present and the past samples \( h^{2}(k + v, i), i \leq k + v \). Thus, Lemma 3.5 will be in use if \( k + 4n - 1 < s_j + 1 \). Hence, the left-hand side of (23) does not depend on \( k \) and \( \hat{g}(k - 1) \) is constant if \( k \in [\alpha_j, \tilde{\beta}_j] \) where \( \tilde{\beta}_j = \tilde{\beta}_j - 2n \).

We summarize the results derived above in the following.

**Lemma 3.6** Consider the SLS model (1)-(3) and the observer (35). If \( s_j, s_{j+1} \in \omega \) and \( \delta_j(\omega) \geq 6n \), then for some \( k_i, k_{i+1} \in \mathcal{X} \), \([\alpha_j, \tilde{\beta}_j) \subset [k_i, k_{i+1})\) and if \( \delta_j(\omega) \geq 8n - 1 \), then \( \hat{g}(k - 1) \) is constant for all \( k \in [\alpha_j, \tilde{\beta}_j] \).

Similarly to Lemma 3.5, this lemma holds in the LTV setting, without requiring the observer be as in Lemma 3.1. The proofs of Lemmas 3.5, 3.6 are far more difficult than that of Lemma 3.4 due to the possibility \( \tau \neq n \). We state \( \mathcal{X} \subset \omega \) as a standing assumption and seek conditions which will imply this assumption. The converse statement, i.e., \( \omega \subset \mathcal{X} \) is not true.

**Assumption 3.2** Every switch of \( \mathcal{X} \) is also a switch of \( \omega \).

Using Assumption 3.2, we improve Lemma 3.6 as follows.

**Lemma 3.7** Consider the SLS model (1)-(3) and the observer in Lemma 3.7. Suppose that Assumption 3.2 holds. Let \( s_j, s_{j+1} \in \omega \) be with \( \delta_j(\omega) \geq 5n \). Then, there exist \( k_i, k_{i+1} \in \mathcal{X} \) satisfying \( 0 \leq s_j - k_i < 4n \) and \( k_{i+1} = s_{j+1} \).

**Proof.** See Appendix 8.

Lemma 3.7 guarantees a \( k_i \in \mathcal{X} \) satisfying \( 0 \leq s_j - k_i < 4n \) when Assumption 3.2 holds. Again from Assumption 3.2, \( k_i = s_l \in \omega \) for some \( l \in \mathbb{N} \), but we do not assert \( l = j - 1 \). A result that follows from this observation is if \( \delta_j(\omega) \geq 5n \), then \( \delta_{j-1} < 4n \), that is, a long constant parameter interval is always preceded by a shorter one when Assumption 3.2 holds. The observer assumption in Lemma 3.7 is crucial.

We will state two assumptions implying Assumption 3.2. To this end, write (35) as a sum of the filtered inputs and outputs
\[
y(k) = y^{(1)}(k) + y^{(2)}(k)
\]
\[
= H_o^{(1)}(z;k)u(k) + H_o^{(2)}(z;k)y(k)
\]
(39)
where with \( b_o^{(1)}(k) = b(k) + g(k)d(k) \) and \( b_o^{(2)}(k) = -g(k) \),

\[
H_o^{(1)}(z;k) = c^T(k)(z\mathbf{I} - A_o(k))^{-1}b_o^{(1)}(k) + d(k), \quad (40)
\]

\[
H_o^{(2)}(z;k) = c^T(k)(z\mathbf{I} - A_o(k))^{-1}b_o^{(2)}(k). \quad (41)
\]

From (41), observe that the second term in (39) is a linear combination of the past outputs only. Since \( A_o(k) \) has all eigenvalues at zero for every \( k \), \( H_o^{(1)}(z;k) \) and \( H_o^{(2)}(z;k) \) are time-varying moving-average filters.

**Assumption 3.3** At least one of the following is true:

a. For every \( k \in \mathcal{X} \), \( H_o^{(1)}(z;k) \) is minimal,

b. For every \( k \in \mathcal{X} \), \( H_o^{(2)}(z;k) \) is minimal.

Assumption 3.3 holds for almost all discrete state sets and deadbeat observers. It is a technical assumption required for the consistency of the identification scheme presented in the sequel. As \( k \) changes in \( \mathcal{X} \), a set of transfer functions of MacMillan degree \( n \) is generated by (40)–(41) with a cardinality bounded by \( 2\sigma \). We present three numerical examples to illustrate the minimality properties of the transfer functions defined in (40)–(41).

**Example 3.3** Let \((A, c^T)\) in Example 3.1 be the observability pair of a discrete state \((A,b,c^T,d)\). Since \((A,c^T)\) is observable, \((A_o,c^T)\) is also observable. To check controllability, we calculate the observability matrix of \((A,b)\)

\[
\begin{bmatrix}
 b \\
 Av 
\end{bmatrix} = \begin{bmatrix} b_1 & b_1 + b_2 \\
 b_2 & 0 
\end{bmatrix}.
\]

Thus, \((A,b)\) is controllable if and only if \( b_2 \neq 0 \) and \( b_1 \neq b_2 \). Both conditions are necessary by Assumption 2.1. A simple calculation reveals \( H_o^1(z) = d + (2b_2 + b_1 - d)z^{-1} - b_2z^{-2} \) minimal and \( H_o^2(z) = z^{-2} \) not. Plug \( b_1 = -b_2 \) in \( H_o^1(z) \) to see it is minimal while \((A,b)\) is uncontrollable.

**Example 3.4** Let \((A, c^T)\) in Example 3.2 be the observability pair of \((A,b,c^T,d)\). It is easy to verify that \((A,b)\) is controllable iff \( b_2 \neq 0 \), \( H_o^1(z) = d + (b_1 + b_2)z^{-1} + b_2z^{-2} \), and \( H_o^2(z) = 0 \). Thus, \( H_o^1(z) \) is minimal.

**Example 3.5** Let \( c^T = [1, 1] \), \( d = 1 \), and

\[
A = \begin{bmatrix} 1/2 & 0 \\
1 & -1/2 
\end{bmatrix}, \quad b = \begin{bmatrix} 1 \\
2 
\end{bmatrix}.
\]

It is easy to verify \((A,b,c^T,d)\) minimal, \( g = [-1/8, 1/8]^T \), \( H_o^1(z) = 1 + 3z^{-1} + 0.25z^{-2} \), and \( H_o^2(z) = -0.25z^{-2} \). Both \( H_o^1(z) \) and \( H_o^2(z) \) are minimal.

**Lemma 3.8** Suppose Assumptions 2.1, 3.1 and 3.3 hold. Then, on \((\delta, \delta_0)\) with \( \kappa_c = \kappa_o = 2n \) and \( g(k) \) in Lemma 3.1 \( H_o^{(1)}(q;k) : u \mapsto y^{(1)}(u) \) and \( H_o^{(2)}(q;k) : y \mapsto y^{(2)} \) is uniform.

**Proof**. This lemma is actually a corollary of Lemma 3.3.

**Assumption 3.4** At least one of the following

i. \( \varphi(k) \neq \varphi(l) \iff c(k) \neq c(l) \) and/or \( d(k) \neq d(l) \)

ii. \( \varphi(k) \neq \varphi(l) \iff c(k) \neq c(l) \)

holds for all \( k, l \in (\delta, \delta_0) \).

**Lemma 3.9** Consider the SLS model (1)–(3) with the observer in Lemma 3.1 Suppose \( \delta, (\mathcal{X}) > 2n \). Then, Assumptions 3.3.i.–3.4.i or 3.3.ii.–3.4.ii. imply Assumption 3.2.

**Proof**. See Appendix 8.
3.3 Switch identifiability of the SARX model

The SLS model (1)–(3) and the deadbeat observer in Lemma 3.1 have the same input–output map. We also have $\chi \subset \chi_\theta$ by Assumptions 3.3b.–3.4i. or 3.3b.–3.4ii. In this subsection, we will study conditions rendering a switch in $\chi_\theta$ identifiable from the noiseless input–output data.

**Definition 3.2** [28] If whenever $\phi_\theta(s_{j+1}) \neq \phi_\theta(s_{j+1} - 1)$, it is possible to detect the change in the value of $\phi_\theta$ as soon as $y(s_{j+1})$ is observed, $s_{j+1} \in \chi_\theta$ is causally identifiable.

Two discrete states with parameters $\theta_1, \theta_2 \in \mathcal{B}^\theta$ are one-step indistinguishable from $z(k)$ if $z^T(k)(\theta_1 - \theta_2) = 0$ [28]. Given $s, t \in (\delta, \delta_*]$ with $t \geq s$, we define

$$\mathcal{R}(s, t) = [z(s) \cdots z(t)],$$

$$Y(s, t) = [y(s) \cdots y(t)]^T.$$

Denote the linear space spanned by the columns/rows and the rank of a given matrix $X$ by $\operatorname{range}(X)/\operatorname{span}(X)$ and $\operatorname{rank}(A)$. The following result [28] provides a pair of necessary and sufficient conditions for switch identifiability.

**Lemma 3.10** A switch $s_{j+1}$ is causally identifiable from the input–output data if and only if

$$0 \neq z^T(s_{j+1})(\theta(s_{j+1}) - \theta(s_j)),$$

$$z(s_{j+1}) \in \operatorname{range}(\mathcal{R}(s_j, s_{j+1} - 1)). \quad (42)$$

In the notation of this paper, sufficiency proof in [28] shows that there does not exist a vector $\mu \in \mathbb{R}^{4n-1}$ explaining the data in $[k_i + n \ k_{i+1}]$, that is, $y(k) = z^T(k)\mu$ for all $k_i + n \leq k \leq k_{i+1}$. Here, we disregarded the subinterval $[k_i k_i + n]$ since $\theta(k)$ may not be constant there. In Section 4, $\mu$ will represent the solution of a sparse optimization problem with the feasibility equations $y(k) = z^T(k)\mu$ for all $k_i + n \leq k < k_{i+1}$ satisfied also by the deadbeat observer $y(k) = z^T(k)\theta(n)$, cf. (36). But from Lemma 3.4, $\theta(k)$ is constant for all $k \in [k_i + n k_{i+1}]$. If the inputs are persistently exciting (PE) and $\delta(\chi)$ is large, $\theta(k) = \mu$, see Section 4.1. Then, $\theta(k_i) = \mu = \theta(k_{i+1})$, arriving to a contradiction. We conclude that identifiability of the switches is not changed in the optimization problems (46) and (64).

By replacing $s_{j+1}$ with $s_j$ and decreasing $k$ instead of increasing it Definition 3.2 can be adapted to $s_j$. Likewise, with slight changes, Lemma 3.10 can be adapted to $s_j$. The conditions in the lemma restrict the class of inputs that can be used for identification. If (1)–(3) is strictly proper, then $d(k) = 0$ for all $k$ and the first entry of $\theta(k)$ may be deleted out. When $\delta_\theta(k)$ is calculated, possible pole-zero cancellations in the polynomials $\theta(k)$ and $\theta(k - 1)$ must be taken into account before the zero-padding since they are defined in terms of the deadbeat observer Markov parameters.

Since $\tau = n$ in $[k_i + n k_{i+1})$, $\mathcal{R}(k_i + n, k_{i+1} - 1)$ has no more than $2n + 1$ nonzero rows and by adding the column vectors $z(k_i + n - t), 1 \leq t \leq n$ we cannot make $\mathcal{R}(k_i, k_{i+1} - 1)$ full-rank. Hence, $\mathcal{R}(k_i, k_{i+1} - 1)$ is always rank deficient. If there is no switch of $\chi_\theta$ in $(k_i k_{i+1})$, to check if $k_i \in \chi_\theta$ is identifiable from the input–output data substitute $s_j = k_i$ and $s_{j+1} = k_{i+1}$ in (42) which is equivalent to [28]

$$\operatorname{rank}(Y(k_i, k_{i+1}) \mathcal{R}(k_i, k_{i+1})) > \operatorname{rank}(\mathcal{R}(k_i, k_{i+1})). \quad (43)$$

This inequality is readily checked by counting the number of the nonzero singular values of a given matrix as a measure of its rank. This test tolerates noise with small amplitude. In Section 7, we will illustrate (42) or (43) by means of a numerical example using multi-sine excitations. As we mentioned earlier, $\chi_\theta$ and $\mathcal{B}^\theta$ are very complicated sets. There may even be a continuum of the switches in the interval $(k_i k_i + n)$. Replace $k_i$ with a $k \in (k_i k_i + n)$. The largest $k$ satisfying (43) is the first switch of $\chi_\theta$ to the left of $k_{i+1}$. As noted there is an ambiguity in locating $k_i$ using the rank test (43) by approaching it from $k_{i+1}$. 
4 Estimation of the discrete-states

Evaluating and stacking (35) for $\delta_c < k \leq \delta_d$, we derive the so-called measurement equation $Y = Z\mu$

$$Y = (y(N') \cdots y(N''))^T,$$
$$\mu = (\theta^T(N') \cdots \theta^T(N''))^T,$$
$$Z = \begin{bmatrix} z^T(N') & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & z^T(N'') \end{bmatrix}$$

where $N' = \delta_c + 1 = 2n + 1$ and $N'' = \delta_d = N - 2n + 1$. With $\mu(0) = \theta(2n)$, write the first-order differences in (37) as

$$\mu = [L \ 1_{N''-2n} \otimes I] \begin{bmatrix} \eta \\ \mu(0) \end{bmatrix} \quad (44)$$

where $1_n$ denotes the column vector of ones in $\mathbb{R}^n$ and $A \otimes B$ is the Kronecker product of two matrices $A$ and $B$, and

$$L = \begin{bmatrix} 1 & 0 \\
\vdots & \ddots & \vdots \\
1 & \cdots & 1 \\
\delta_d(N') & \cdots & \delta_d(N'') \end{bmatrix},$$
$$\eta = \begin{bmatrix} \eta \\ \mu(0) \end{bmatrix}.$$

Let $\hat{N} = N'' - 2n$, $M = Z[L \ 1_{\hat{N}} \otimes I]$, and $\zeta = [\eta^T \ \mu^T(0)]^T$. Substituting $M$ and $\zeta$ in $Y = Z\mu$, we derive $Y = M\zeta$.

Now, let $\mathcal{J}(\|\eta(k)\|_2 > 0)$ denote the indicator function defined 1 when $\|\eta(k)\|_2 > 0$ and 0 otherwise. The $\ell_0$ quasi-norm of the sequence $\eta(k)$ is defined by

$$\|\eta\|_{0,3} = \sum_{k=1}^{\hat{N}} \mathcal{J}(\|\eta(k)\|_2 > 0). \quad (45)$$

Since $k$ is a switch if and only if $\delta_d(k) \neq 0$, $\|\eta\|_{0,3}$ is the number of the switches of the deadbeat observer in (35) in the interval $[N' \ N'']$. In the noiseless case, we find all identifiable switches by solving the following non-convex and non-polynomial (NP)-hard sparse optimization problem

$$\min_{\zeta} \|\eta\|_{0,3} \text{ subject to } Y = M\zeta. \quad (46)$$

Let $\hat{\zeta} = [\hat{\eta}^T \ \hat{\mu}^T(0)]^T$ be a solution of (46) and $\tilde{\zeta} = \hat{\zeta} - \zeta$ denote its error. Likewise, $\hat{\mu}$ is the estimate of $\mu$ calculated from (44) with $\hat{\zeta}$. Partition $\zeta$ similarly to $\zeta$. From $M\zeta = 0$, we derive $ZL\tilde{\eta} + Z1_{\hat{N}} \otimes \hat{\mu}(0) = 0$. Thus,

$$\begin{bmatrix} z^T(N') & 0 \\
\vdots & \ddots & \vdots \\
z^T(N'') & \cdots & z^T(N'') \end{bmatrix} \tilde{\eta} = - \begin{bmatrix} z^T(N') \\
\vdots \\
z^T(N'') \end{bmatrix} \hat{\mu}(0). \quad (47)$$

Denote the rows of a given matrix $X$ by $X_k$. From

$$\hat{\eta}(k) = \hat{\eta}(k) - \eta(k) = \hat{\mu}(k) - \mu(k-1) - (\mu(k) - \mu(k-1))$$

$$= \hat{\mu}(k) - \mu(k-1), \quad 1 \leq k \leq \hat{N}.$$
and (47), we derive
\[ z^T (2n+k) \sum_{l=1}^{k} \tilde{\eta}(l) = z^T (2n+k)(\hat{\mu}(k) - \bar{\mu}(0)) = -z^T (2n+k)\bar{\mu}(0). \]

Thus, \( z^T (2n+k)\bar{\mu}(k) = 0 \) for all \( 1 \leq k \leq \hat{N} \).

Consider a segment \([k_i, k_{i+1})\) and suppose Assumptions 3.3a–3.4. hold. Since \( k_i, k_{i+1} \in \chi_\theta \), \( g(k-1) \) is constant on \([k_i + 4n - 1, k_{i+1} - 4n] \) from Lemma 3.6 if \( \delta_t(\chi) \geq 8n - 1 \) and \( \theta(k) \) is constant on \([k_i + n, k_{i+1})\) from Lemma 3.4 if \( \delta_t(\chi) > n \). We assume \( \delta_t(\chi) \geq 8n - 1 \) then. On \([k_i + 4n - 1, k_{i+1} - 4n] \), \( \theta(k) \) is sparse as in (36). Since this interval is a subset of \([k_i + n, k_{i+1})\), this sparse form extends to the entire \([k_i + n, k_{i+1})\). Thus, \( y(k) = z^T(k)\theta_n(k) \) and \( R_n(s,t) = [z_n(s) \cdots z_n(t)] \) where
\[
z_n(k) = (u(k) \zeta(k-1) \cdots \zeta(k-n))^T, \\
\theta_n(k) = (d(k) h_n(k,k-1) \cdots h_n(k,k-n))^T.
\]

We define \( \mu_n(k) \) and \( \hat{\mu}_n(k) \) similarly to \( \theta_n(k) \).

Since \( \hat{\theta}(k) = \theta((k+n), k_i + n \leq k < k_{i+1}) \), \( \eta(k-2n) = 0 \) for all \( k_i + n < k < k_{i+1} \). But \( \tilde{\eta} \) is an optimal solution and \( \zeta \) is a feasible solution. Then, \( \tilde{\eta}(k-2n) = 0 \) and hence \( \tilde{\eta}(k-2n) = 0 \) on \((k_i + n, k_{i+1})\) from the first equality in (48). Plug \( l \) in place of \( k \) in the last equality in (48) and sum over \( l \in (k_i + n k) \) to get \( \hat{\mu}(k-2n) = \hat{\mu}(k-n) \). From the constraints \( z^T(2n+k)\hat{\mu}(k) = 0 \) for all \( k = 1, \ldots, \hat{N} \), then
\[
\mathcal{R}_n(k_i + n + 1, k_{i+1} - 1)\tilde{\mu}_n(k_i - n) = 0.
\]

If
\[
\text{rank}(\mathcal{R}_n(k_i + n + 1, k_{i+1} - 1)) = 2n + 1,
\]
then \( \tilde{\mu}_n(k_i - n) = 0 \) and hence \( \hat{\mu}(k_i - n) = 0 \). Thus, \( \hat{\mu}(k-2n) \) is sparse as in (36) and satisfies \( \hat{\mu}(k-2n) = \mu(k_i - n) \) for all \( k_i + n \leq k < k_{i+1} \). We can now apply Algorithms 1–2 to \( \hat{\theta}_n(k_i + n) \) to extract a state-space realization of \( \varphi(k_i) \). The condition (50) is a PE condition.

### 4.1 Persistence of excitation conditions

Let us rewrite (39) by adding noise \( e(k) \)
\[
(1 - H^{(2)}_o(z;k))y(k) = H^{(1)}_o(z;k)u(k) + e(k). \tag{51}
\]

The one-step-ahead predictors of (55) for all \( k \in (k_i + n, k_{i+1}) \) are derived from (51) as follows
\[
\hat{y}(k|k-1) = H^{(1)}_o(z;k_i)u(k) + H^{(2)}_o(z;k_i)y(k). \tag{52}
\]

Plug \( k = k_i \) and transform (51) into the frequency domain
\[
Y(z) = (1 - H^{(2)}_o(z;k_i))^{-1}H^{(1)}_o(z;k_i)U(z) + V(z) \tag{53}
\]
where \( V(z) = (1 - H^{(2)}_o(z;k_i))^{-1}E(z) \).

It is a well-known fact [49] that when \( u(k) \) and \( y(k) \), with \( u(k) \) independent from \( e(k) \), and with \( e(k) \) persistently exciting of any order, the moving average parameters of \( H^{(2)}_o(z;k_i) \) and \( H^{(2)}_o(z;k_i) \) in (52) are uniquely determined by minimizing the quadratic norm of the prediction errors if \( u(k) \) is PE of order \( 2n + 1 \). In the noiseless case, \( z^*H^{(1)}_o(z;k_i) \) and \( z^*(1 - H^{(2)}_o(z;k_i)) \) must be co-prime and \( u(k) \) be PE of order \( 2n + 1 \). An ergodic or quasi-stationary sequence \( u(k) \) is PE of order \( \rho \) if its power spectrum is nonzero at least at \( \rho \) distinct frequencies. The requirements on the SLS model (1)–(5) and the deadbeat observer in (35) are captured in
Assumption 4.1  For every \( k_i \in \chi \), the polynomials \( z^n H_0^{(1)}(z; k_i) \) and \( z^n(1 - H_0^{(2)}(z; k_i)) \) are co-prime and the latter is Hurwitz, i.e., it has all zeros inside the unit circle.

We illustrate this assumption with a numerical example. Some conditions might be redundant based on the previous assumptions. We defer a detailed study to future work. The co-prime condition guarantees \( \theta_\nu(k) \) is irreducible for all \( k \).

Example 4.1  Let \( (A, b, c, d) \) in Example 3.5 be a discrete state at \( k_i \in \chi \). We calculate \( z^2(1 - H_0^{(2)}(z; k_i)) = z^2 + 0.25 \) verifying it is Hurwitz and \( z^2 H_0^{(1)}(z) = z^2 + 3z + 0.25 \). Thus, \( z^2(1 - H_0^{(2)}(z; k_i)) \) and \( z^2 H_0^{(1)}(z) \) are co-prime as requested.

The requirements on the inputs are stated in

Assumption 4.2  The inputs are \( \text{PE} \) of order at least \( 2n + 1 \).

Proposition 4.1  Consider the SLS model (1)–(3) with the observer in Lemma 3.1. Suppose that Assumptions 4.1–4.2 hold. Then, there exists a \( \gamma > 0 \) such that for every \( k_i \in \chi \) and \( s, t \in [k_i \ k_{i+1}) \) satisfying \( t - s \geq \gamma \),

\[
\text{rank}(\mathcal{R}_n(s, t)) = 2n + 1. \tag{54}
\]

Proof.  See Appendix 8.

From Proposition 4.1, we see that if Assumptions 4.1–4.2 hold and \( \delta_\nu(\chi) \geq \gamma + n + 2 \), then (50) is valid. It is not realistic to expect every segment to meet this requirement; but, there must be a sufficient number of long segments.

4.2 Clustering of the discrete states

The solution of (46) partitions the interval \([1 \ N]\) where with \( \hat{\zeta}(0) = \hat{\mu}(0) \) we define its switches by the instants \( 1 \leq t \leq N \)

\[
\delta_\zeta(t) = \hat{\zeta}(t) - \hat{\zeta}(t-1) = \hat{\mu}(t) - 2\hat{\mu}(t-1) + \hat{\mu}(t-2) \neq 0.
\]

Let \( \mathcal{T} = \{t_1, \cdots , t_F\} \) denote the switches of \( \hat{\zeta}(t), 0 \leq t \leq N \). We set \( t_0 = 0 \). Suppose \( k_{i+1} \) is identifiable from the input–output data. First, from \( \eta(k - 2n) = 0 \) for all \( k \in [k_i \ n \ k_{i+1}] \), note that \( \eta(k - 2n) = 0 \) on \([k_i + n \ k_{i+1}) \). Thus, there exists a \( J \in \mathcal{T} \) with \([k_i + n \ k_{i+1}) \subset [t_j \ t_{j+1}) \). Hence, \( \hat{\theta}(k) = \hat{\theta}(k_i) \) for all \( k_i + n \leq k < k_{i+1} \). Suppose \( k_{i+1} \) is not a switch of \( \hat{\zeta} \). Then, \( k_{i+1} \) is in \([t_j \ t_{j+1}) \) and \( \hat{\zeta}(k_{i+1} - 2n) = 0 \). Thus, \( \hat{\theta}(k_{i+1}) = \hat{\theta}(k_{i+1} - 1) \neq \hat{\theta}(k_i) \). Since \( \hat{\zeta} \) is a feasible solution of (46), \( k_{i+1} \) is an identifiable switch. A contradiction. We summarize this result as

Lemma 4.1  Consider the SLS model (1)–(3) with the observer in Lemma 3.1. Suppose that \( \delta_\nu(\chi) > 2n \) and Assumptions 3.3a.–3.4i. or 3.3b.–3.4ii. hold. Let \( \hat{\zeta} \) be a solution of (46). Then, every identifiable switch of \( \chi \) is a switch of \( \zeta \).

The conditions in Lemma 4.1 guarantees \( \chi \subset \chi_\theta \). If \( k_i \in \chi \) is identifiable, \( k_i \in \mathcal{T} \) then. Since \( \chi \) is a proper subset of \( \chi_\theta \), the map \( \chi \to \mathcal{T} \) is not necessarily surjective. We state the switch identifiability requirement in the following.

Assumption 4.3  Every switch in \( \chi \) is identifiable from the input–output data.

It is worth to emphasize that the switch identifiability is defined with respect to \( \chi_\theta \) and \( \mathcal{T}_\theta \).
Remark 4.1 If \( k_{i+1} \) is identifiable from the input–output data, a change in \( \theta(k) \) at \( k_{i+1} \) is detectable. However, this does not tell anything about \( \theta(k_i) \) and \( \theta(k_{i+1}) \).

Suppose Assumption 4.3 holds. Let \( k_i, k_{i+1} \in \chi_T \). Since \( \eta(k - 2n) = 0 \) for all \( k \in [k_i + n \ b_k - 1] \), \( \eta(k - 2n) = 0 \) on \([k_i + n \ b_k - 1]\). Then, for some \( J \in \mathcal{T}, \ [k_i + n \ b_k - 1] \subset [t_J t_{J+1}] \). Since \( k_i \in \mathcal{T}, k_i = t_K \) for some \( K \) and \([k_i k_{i+1}] = [t_K t_{J+1}] \). Thus, \( \chi \subset \chi_{\xi} \) with \( \chi_{\xi} \) denoting partitioning of \( [1 N] \) by \( \xi \). Suppose Assumptions 4.1–4.3 hold and \( \delta(\chi) \geq \gamma + n + 2 \). Then, (50) holds and, as we discussed earlier, we can apply Algorithms 1–2 to \( \hat{\theta}_n(t_j + n) \) to extract a realization of \( \varphi(k_i) \). Note that \( k_i \leq t_J \leq k_i + n \). The following assumption ensures that every discrete state in \( \mathcal{P} \) visits a long segment so that its parameters can be identified from the input-output data.

Assumption 4.4 Every discrete state in \( \mathcal{P} \) is active in at least one segment with a dwell time of at least \( \gamma + 2n \).

This assumption lets us recover the discrete states by clustering. In clustering, first a statistics is selected. A statistics based on the realization returned by Algorithm 2 and denoted by the quadruple \( \mathcal{M}(k) = (\hat{A}(k), \hat{b}(k), \bar{c}^T(k), \bar{d}(k)) \) may be chosen as the \( \ell_1 \)-norm of the eigenvalues of \( \hat{A}(k) \), that is, we choose \( \mathcal{M}(\hat{A}(k)) = \sum_{i=1}^n |\lambda_i(\hat{A}(k))| \) or the time-varying \( \mathcal{H}_2 \) norm of \( \mathcal{Y}(k) \). Since \( \mathcal{M}(\hat{A}(\varphi(k))) = \mathcal{M}(\hat{A}(\varphi(l))) \) if \( \varphi(k) = \varphi(l) \), in the noiseless case, \( \mathcal{P} \) may be determined by viewing the graph of \( \mathcal{M}(\hat{A}(\varphi(k))) \). By running a clustering algorithm [13, 42], we recover a collection of the discrete states in \( \mathcal{P} \). This collection exhausts all discrete states in \( \mathcal{P} \) by Assumption 4.4. We summarize these steps as an algorithm.

**Algorithm 3. Estimation of \( \mathcal{P} \)**

**Inputs:** \( u(k) \) and \( y(k) \) for \( 1 \leq k \leq N \)

1: Solve (46) for \( \hat{\xi} \)
2: Choose segments with \( t_{J+1} - t_J \geq \gamma + n + 2 \)
3: Estimate \( h(k, i) \) from Algorithm 1 using \( \hat{\theta}_n(t_j + n) \)
4: Estimate the discrete states from Algorithm 2
5: Choose a statistics either \( \mathcal{M}(\hat{A}(k)) \) or \( \| \mathcal{Y}(k) \|_2 \)
6: Estimate \( \mathcal{P} \) by running [13] over the segments in Step 2

**Output:** \( \mathcal{P} \).

In our numerical study, we will use the density-based clustering algorithm [13] implemented by the `dbscan` command in MATLAB. Another popular clustering method is the \( k \)-means clustering algorithm [42] implemented by the `kmeans` command in MATLAB. Unlike the \( k \)-means clustering algorithm, the density-based clustering algorithm does not need the number of the clusters to be specified \( a \ priori \). We capture the results derived above in the following.

**Theorem 4.1** Consider the SLS model (1)–(3) with noiseless input-output data. If Assumptions 2.1, 3.1, 3.3a., 3.4, or 3.3b., 3.4ii., 4.1, 4.4 hold, Algorithm 3 recovers \( \mathcal{P} \).

## 5 Estimation of the switching sequence

Algorithm 3 delivers not only the discrete state estimates, but also locates the switches at the right endpoints of segments with dwell times at least \( \gamma + n + 2 \) under the stated assumptions in Theorem 4.1. Indeed, let \( \{t_J t_{J+1}\} \) be such a segment. Then, \( k_i \leq t_J \leq k_i + n \) and \( k_{i+1} \leq t_J + 1 \) for some \( i \). But, \( k_{i+1} < t_J + 1 \) is not possible from Assumption 4.3. The switch at \( k_i \) may be determined up to an uncertainty \( n \) by extending \( t_J \) along the negative axis and applying (43) if it has not already been determined by a segment \([t_K t_{K+1}] \) for some \( K \leq J \). More precise estimates may be obtained by using a GLR test with the discrete
state estimates and the input-output data. An option we have yet to explore is to use the switching sequence and the discrete state estimates delivered by the algorithms in this paper for initializing a non-convex optimization algorithm, for example [35]. Switch detection and estimation is a topic of major interest in hybrid systems literature.

If \( \delta_e(\mathcal{X}) \geq \gamma + 2n + 2, \) \( \phi(k) \) is completely determined by Algorithm 3 from Assumption 4.3. It remains to determine the switching sequence on the shorter segments. In this section, given a segment \([t_J, t_{J+1})\) satisfying \( t_{J+1} - t_J < \gamma + n + 2 \) we will discuss how to determine the discrete state that is active in \([t_J, t_{J+1})\), which is unique by Assumption 4.3. This will provide an answer to the value of \( \phi(k) \) on \([t_J, t_{J+1})\).

Let \( l = \phi(t_J), \) pick \( \vartheta \leq t_{J+1} - t_J, \) and form two Hankel matrices from the input–output data

\[
\mathcal{U}_{t_J|\vartheta} = \begin{bmatrix}
u(t_J) & \cdots & u(t_J + \vartheta - 1) \\
\vdots & \ddots & \vdots \\
u(t_J + \vartheta - 1) & \cdots & u(t_J + \vartheta - 1) 
\end{bmatrix},
\]

\[
\mathcal{Y}_{t_J|\vartheta} = \begin{bmatrix}y(t_J) & \cdots & y(t_J + \vartheta - 1) \\
\vdots & \ddots & \vdots \\
y(t_J + \vartheta - 1) & \cdots & y(t_J + \vartheta - 1) \end{bmatrix},
\]

and generate the Toeplitz matrices from \( \mathcal{P}_t = (A_t, b_t, c_t^T, d_t) \)

\[
\Psi_t = \begin{bmatrix}d_t & \cdots & d_t \\
c_t^TA_t^{\vartheta-2}b_t & \cdots & d_t \end{bmatrix}, \quad l \in \mathbb{S}.
\]

The following data equation

\[
\mathcal{Y}_{t_J|\vartheta} = \mathcal{L}_\vartheta \mathcal{X}_{t_J|\vartheta} + \Psi_{\phi(t_J)} \mathcal{U}_{t_J|\vartheta}
\]

can be derived where \( \mathcal{X}_{t_J|\vartheta} \) is the compound state matrix defined by

\[
\mathcal{X}_{t_J|\vartheta} = [x(t_J) \cdots x(t_{J+1} - \vartheta)].
\]

Let the LQ decomposition of \( [\mathcal{Y}_{t_J|\vartheta}^T \mathcal{Y}_{t_J|\vartheta}]^T \) be given by

\[
\begin{bmatrix} \mathcal{U}_{t_J|\vartheta} \\ \mathcal{Y}_{t_J|\vartheta} \end{bmatrix} = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix}
\]

where \( L_{11} \) and \( L_{22} \) are lower triangular matrices and \( Q_1 \) and \( Q_2 \) are orthogonal matrices with dimensions compatible with the row sizes of \( \mathcal{U}_{t_J|\vartheta} \) and \( \mathcal{Y}_{t_J|\vartheta} \). The MOESP algorithm [47, 48] estimates the observability range space by performing first the LQ decomposition (56) and then the SVD

\[
L_{22} = [U_1 U_2] \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} = U_1 \Sigma_1 V_1^T
\]

where \( \Sigma_1 \in \mathbb{R}^{n \times n} \). The extended observability matrix estimate is given by

\[
\hat{O}_\vartheta = U_1 \Sigma_1^{1/2}.
\]

Then, we estimate \( A_{\phi(t_J)} \) from (57)

\[
\hat{A}_{\phi(t_J)} = (J_D \hat{O}_\vartheta) J_U \hat{O}_\vartheta
\]

and calculate \( \mathcal{M}(\hat{A}_{\phi(t_J)}) \). We find the active model index by comparing \( \mathcal{M}(\hat{A}_{\phi(t_J)}) \) with the statistics of the discrete states in \( \hat{\mathcal{X}} \). The MOESP algorithm retrieves \( (A_{\phi(t_J)}, c_{\phi(t_J)}^T) \) up to a similarity transformation if the following three conditions

\[
\text{rank}(\mathcal{X}_{t_J|\vartheta}) = n,
\]
are satisfied. The first condition is the controllability of the discrete state that is active on the interval \([t_j, t_{j+1})\) while the second is satisfied by selecting the PE inputs of order at least \(\vartheta\). The last condition requires data be collected in an open-loop experiment. A pseudo-code implementing this method is outlined below.

**Algorithm 4. Estimation of \(\varphi(t_j)\) by MOESP**

**Inputs:** \(u(k), y(k), 1 \leq k \leq N \text{ and } \mathcal{P}\)

**While** \(t_{j+1} - t_j < \gamma + n + 2\)

1: Compute \(\mathcal{Y}_{t_j|\vartheta}\) and \(\mathcal{Y}_{t_j|\vartheta}\) in (55)
2: Perform the LQ decomposition in (56)
3: Apply SVD to \(L_{22}\)
4: Compute \(\hat{A}_\varphi(t_j)\) from (57)–(58)
5: Compute \(\mathcal{M}(\hat{A}_\varphi(t_j))\)
6: If \(\mathcal{M}(\hat{A}_s) = \mathcal{M}(A_{\varphi(t_j)})\) for some \(s \in \mathcal{S}\), set \(\varphi(t_j) = s\)

**End**

**Output:** \(\varphi(t_j)\).

In Algorithm 4, we substitute \(\vartheta = 2n + 1\). The calculation of \(\hat{\Theta}_\vartheta\) requires \(\text{rank}(V_1) = n\). Therefore, \(L_{22}\) must have at least \(n\) columns. Hence, \(t_{j+1} - t_j \geq \vartheta + n = 3n + 1\). The PE condition on the inputs requires in fact a larger gap. In this section, we directly apply the PE condition to the SLS model (1)–(3) instead of the observer model in Lemma 3.1.

Alternatively, let \(\Xi(l) = \mathcal{Y}_{t_j|\vartheta} - \mathcal{Y}_{t_j|\vartheta}, l \in \mathcal{S}\) and if \(l = \varphi(t_j)\), \(\Xi(l) = \mathcal{O}_{\vartheta} \mathcal{X}_{t_j|\vartheta}\). For each \(l\), perform an SVD as \(\Xi(l) = U_1(l) \Sigma_{1}^{l} V_1^{l} + U_2(l) \Sigma_{2}^{l} V_2^{l}\) and let \(\hat{\Theta}_\vartheta(l) = U_1(l) \Sigma_{1}^{l} [\Sigma_{1}^{l}]^{1/2}\). Define \(\hat{A}_l = (J_0 \hat{\Theta}_\vartheta)^{\dagger} J_0 \hat{\Theta}_\vartheta\), \(l \in \mathcal{S}\). If \(l\) is the right choice, then \(\mathcal{M}(\hat{A}_l) = \mathcal{M}(\hat{A}_{\varphi(t_j)})\). We will call this alternative Algorithm 4'. It is easy to implement and works for \(t_{j+1} - t_j \geq 2n\). For performance guarantees, (59)–(61) are still needed since this algorithm originates from Algorithm 4.

The main result of this section is stated as follows.

**Theorem 5.1** Consider the SLS model (1)–(3) with noiseless input–output data. Assume that \(\chi \subset \mathcal{P}\) and the discrete states in \(\mathcal{P}\) are known up to similarity transformations. Suppose that \(\delta_s(\chi) \geq \gamma + 2n + 2\). Then, \(\varphi\) is determined on \([1, N]\) by Algorithm 3. If \(\delta_s(\chi) < \gamma + 2n + 2\), then Algorithms 4–4' together recovers \(\varphi\) in the segments satisfying (59)–(61).

Since \(\mathcal{M}(\hat{A}_l)\) is invariant to similarity transformations, it suffices to know the discrete states in \(\mathcal{P}\) up to similarity transformations without bringing to a common basis. Algorithm 4–4' may falsely detect the discrete states, in particular over the short segments if the inputs are not PE there. If the SLS model is used to predict outputs, it is necessary to transform the discrete states to a common basis. One can freely select one similarity transformation only. The rest of the transformations are fixed and can be calculated from the input–output data, \(\mathcal{P}\), and \(\chi\). See [7].

### 6 Convex relaxation

Consider the following robust reformulation of (46)

\[
\min_{\zeta} \|\eta\|_{0,3} \quad \text{subject to} \quad \|Y - M\zeta\|_2 \leq \varepsilon
\]

(62)

to model inaccuracies in the measurements. Replace \(Y\) with \(\hat{Y}\) satisfying \(\|\hat{Y} - Y\| \leq \varepsilon\) and set \(\hat{Y} = M\zeta\). We can find an \(\varepsilon_0 > 0\) such that for all \(\varepsilon \leq \varepsilon_0\), every switch in \(\chi\) is identifiable from the input–output
data and (54) holds by virtue of finiteness of $N$. Thus, we see that Algorithm 3 recovers every discrete state in $\mathcal{P}$ within an error that vanishes as $\epsilon$ decreases to 0. Now, replace $\mathcal{P}$ with $\hat{\mathcal{P}}$ in Theorem 5.1 and note that its conclusion is true for all $\epsilon \leq \epsilon_0$, if necessary by reducing $\epsilon_0$. Since $\|\eta\|_{0,2}$ has a fixed optimal value for all $\epsilon \leq \epsilon_0$ and $\hat{\zeta}$ is a feasible solution of (62), we see that any solution of (62) is also a sparse vector. Conversely starting from any solution of (62), we reach the same optimal value. We may now replace Step 1 in Algorithm 3 with (62).

The BBPDN algorithm is a well-known convex relaxation of (62). It recovers any block-sparse vector $\zeta$ from the measurements $Y = M \zeta$ whenever $M$ satisfies a recovery condition. The recovery conditions for the greedy BOMP algorithm happen to coincide with those of the BBDN algorithm. The stability and robustness properties of the BOMP algorithm are however entirely different than those of the BBPDN algorithm. For example, the BOMP algorithm obeys a local stability result, that is, stable recovery is possible for all $\epsilon > \epsilon_0$ for some $\epsilon_0$ [10]. See, [52] for results of the same kind. By comparison, stability is global for the BBPDN algorithm, that is, stable recovery is possible for all $\epsilon > 0$ once a recovery condition is satisfied.

In order a recovery condition to hold, it is necessary that (46) has a unique solution, which is not true in general since $u(k)$ may not be a PE of sufficient order on all segments. In fact, a deadbeat observer generates constant $\theta(k)$ intervals shorter than $4n$, cf. Lemma 3.7 and the comment that follows the lemma. Nevertheless, all one needs to recover $\mathcal{P}$, as expressed in Theorem 5.1, is that Assumption 4.4 holds.

We now formulate an efficient convex optimization problem instead of (46). Similarly to the $\ell_0$ quasi-norm, we define the mixed $\ell_2/\ell_1$ norm over the interval $(\delta_i, \delta_0)$ by

$$
\|\eta\|_{2,1} = \sum_{k=1}^{N} \|\eta(k)\|_2.
$$

(63)

Suppose that the measurements are corrupted by noise and consider the following convex optimization problem

$$
\begin{align*}
\min_{\zeta} & \quad \|\eta\|_{2,1} \quad \text{subject to} \quad \|Y - M \zeta\|_2 \leq \epsilon.
\end{align*}
$$

(64)

The feasible parameter sets $Y = M \zeta$ in (46) ($\epsilon = 0$) and the ellipsoid in (64) ($\epsilon \neq 0$) are not necessarily bounded. If they were, (64) would have a unique and finite solution. Enforcement of the constraint $\|\zeta\|_2 \leq R$ for a large $R$ does not change the optimal value of (64), denoted by $\zeta_{2/1}(k) = \theta_{2/1}(k) - \theta_{2/1}(k-1)$, over the long segments as will be shown shortly, but ensures a unique and bounded solution everywhere.

Let $\epsilon = 0$, adjoin $(\zeta)_{2/1} \leq R$ to (64), and consider $[k_i k_{i+1}]$ with $k_i, k_{i+1} \in \chi$ identifiable from the input-output data and satisfying $\delta_i(\chi) \geq 8n - 1$. If $R$ is chosen very large, any solution $\hat{\zeta}(k)$ of (46) provides a feasible solution to (64). Hence, from $\zeta_{2/1}(k) = 0$ we get $y(k) = z_n^T(k) \theta_{2/1}(k)$ for all $k \in (k_i, k_{i+1})$. Then, $\theta_{2/1}(k_i+n) = \hat{\theta}_n(k+n)$ whenever (50) holds, yet for $k = k_i$ or $k = k_{i+1}$, $\zeta_{2/1}(k) \neq \hat{\zeta}(k)$ is possible. The clustering results in Subsection 4.2 hold with $\zeta_{2/1}(k)$ since $k_i, k_{i+1} \in \chi$ are identifiable from the input-output data.

We can express $\|\zeta\|_2 \leq R$ in a suitable form by letting

$$
\Gamma_s = \begin{bmatrix}
h_0^T(\delta_i, \delta_e - s) \\
\vdots \\
h_0^T(\delta_i, \delta_e - s)
\end{bmatrix}, \quad 0 \leq s < 2n.
$$

(65)

Recall that for all $k \in [k_i+n, k_{i+1})$, $\theta(k)$ is a constant and sparse vector having the structure (36) though $k_i$ and $k_{i+1}$ are not known. As a result, we propose penalizing all Markov parameters $h_0(k, k-l), l > n$. Similarly, we penalize the terms $h_0(k, k-l), l \leq n$, but with a different weight.
We propose the following BBPDN problem:
\[
\begin{align*}
\min_{\zeta} & \quad \|\eta\|_{2,\gamma} \\
\text{subject to} & \quad \|Y - M\zeta\|_2 \leq \varepsilon, \\
& \quad \sum_{s=n+1}^{2n-1} \|\Gamma_s\|_2 \leq \rho_1, \\
& \quad \sum_{s=0}^{n} \|\Gamma_s\|_2 \leq \rho_2
\end{align*}
\] (66)

which can be written in a regularized form by appending the two constraints to the objective function giving rise to
\[
\begin{align*}
\min_{\zeta} & \quad \|Y - M\zeta\|_2 + \lambda \|\eta\|_{2,\gamma} + \sum_{s=0}^{2n-1} \gamma_s \|\Gamma_s\|_2
\end{align*}
\] (67)

where \(\gamma_s = \gamma_1, s > n\) and \(\gamma_s = \gamma_2, s \leq n\). We change the hyper-parameters \(\lambda, \gamma_1, \gamma_2 \geq 0\) by gridding to get a family of Pareto-optimal solutions. The summation in (67) is over the spatial variables; not over the time as in (63). The rationale behind this is to limit the rapid fluctuations of the spatial variables while controlling the computational load. This is especially true when the solution of (67) is iteratively refined.

If the second and the third constraints in (66) were replaced with
\[
\sum_{s=0}^{\delta_0} \sum_{k=\delta_0} \|h_0(k,k-s)\|_2 \leq \rho,
\]
(66) would have a sparse solution, but distributed in time as opposed to the block sparsity induced by the constraints in (66). Block sparsity is most desirable since it does not destroy the switch identifiability while bounding \(\zeta\). An identification procedure extracting the discrete state parameters one after another was presented in [1]. It is based on the \(\ell_1\) relaxation, but has a different measurement model than ours.

Given \(\varepsilon \geq 0\) and \(\rho_1, \rho_2 > 0\), replace \(\hat{\zeta}(k)\) in Step 1 of Algorithm 3 with \(\hat{\zeta}_1(k)\) obtained by solving (66) and save Steps 2–6. The new algorithm will be called Algorithm 5 henceforth. We summarize the results derived in this section.

**Theorem 6.1** Consider the SLS model (1)–(3) with Algorithm 5. Suppose that Assumptions 2.1, 3.1, 3.3a.–3.4i., or 3.3b.–3.4ii., 4.1–4.4 hold. Then, the following are true:

(a) If \(\varepsilon = 0\), there exists a \(\rho_0 > 0\) such that for all \(\rho_1, \rho_2 \geq \rho_0\), Algorithm 5 recovers \(\mathcal{P}\) up to similarity transformations.

(b) There exists an \(\varepsilon_0 > 0\) and \(\rho_0 > 0\) such that for all \(\varepsilon \leq \varepsilon_0\) and \(\rho_1, \rho_2 \geq \rho_0\), Algorithm 5 approximately recovers \(\mathcal{P}\), that is, there exists an absolute constant \(c > 0\) such that
\[
\max_{1 \leq k \leq \sigma} \min_{\hat{\mathcal{P}} \in \mathcal{P}} \|\hat{\mathcal{P}}_k - \mathcal{P}_k\| \leq c\varepsilon.
\]

(c) Assume further that \(\chi \subset \mathcal{T}\). In Algorithms 4–4’, replace \(\mathcal{P}\) with \(\hat{\mathcal{P}}\) estimated by Algorithm 5. Then, there exists an \(\varepsilon_0 > 0\) such that for all \(\varepsilon \leq \varepsilon_0\), \(\varphi\) is determined on \([1 N]\) by Algorithm 3 if \(\delta_*(\chi) \geq \gamma + 2n + 2\). Furthermore, Algorithms 4–4’ together recover \(\varphi\) in the segments satisfying (59)–(61) if \(\delta_*(\chi) < \gamma + 2n + 2\).

### 6.1 Iterative refinement

With \(\varepsilon = 0\) and \(\rho_1 = \rho_2 = \infty\), (66) is the tightest convex relaxation of (46). A better heuristic, the iterative weighted \(\ell_1\) relaxation, was proposed in [2]. The iterative refinement procedure when applied to (67) assumes the following form
Algorithm 6. Iteratively reweighted BBPDN

Given: $\lambda, \gamma_1, \gamma_2, \alpha \geq 0$ and maximum number of iterations $t_{\text{max}}$, initialize weights $w_k^{(0)} = 1, v_s^{(0)} = 1$.

Run the following loop:

while $t \leq t_{\text{max}}$

1. Solve the problem:

$$
\min_\zeta \| Y - M \zeta \|_2 + \lambda \sum_{k=1}^{\hat{N}} w_k^{(t)} \| \eta(k) \|_2 \\
+ \sum_{s=0}^{2n-1} \gamma_s v_s^{(t)} \| \Gamma_s \|_2
$$

2. Update weights for $1 \leq k \leq \hat{N}$ and $0 \leq s < 2n$,

$$
w_k^{(t+1)} = (\alpha + \| \eta(k) \|_2)^{-1}, \\
v_s^{(t+1)} = (\alpha + \| \Gamma_s \|_2)^{-1}.
$$

end while

Outputs: $\hat{\theta}(k), \delta_k < k \leq \delta_0$ and $\hat{\mu}(0)$.

The solution given by Algorithm 6 may be further refined by running Algorithm 6 with the thresholded weights: the weights below a certain value are set to zero and the weights above this value are set to a large number, say $10^5$. This will help improve the fit over segments. Many standard convex optimization solvers could be used to implement Algorithms 5–6. In this paper, we used the CVX package [17] which converts the optimization problem to a second-order cone program (SOCP) and calls a standard interior-point cone solver.

6.2 Summary

In Figure 1, we provide a flowchart to implement the proposed identification scheme. Except Algorithms 5–6, all algorithms in Figure 1 are unsupervised learning algorithms in the terminology of statistical learning theory [41, 18]. Supervised learning algorithms, in contrast, are prediction algorithms estimating the system structure and the parameters by minimizing a suitable norm of the prediction errors constructed from the input-output data. An example is the learning algorithm in [35]. The discrete-state estimates and the switching sequence estimate delivered by the proposed scheme should further be enhanced by a supervised learning algorithm.

7 Numerical example

To illustrate the results derived in this paper, we consider an SLS model with three discrete-states in the state-space form.
Figure 1: Flowchart of the SLS model identification algorithm from the input-output measurements.
Figure 2: The switching sequence in Subsection 7.1.

Figure 3: $\|\Phi_o(k, k - \tau)\|_F$ as a function of $k$ for $\tau = 2, 3, 4$.

$$A_1 = \begin{bmatrix} 0 & -1 \\ 0.9 & 0.6 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 0.4 \\ -1 \end{bmatrix}, \quad c_1 = \begin{bmatrix} -1 \\ -2 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.6 & 1 \\ -1 & -1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \quad c_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -1 & -2 \\ 1 & 1.5 \end{bmatrix}, \quad b_3 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad c_3 = \begin{bmatrix} 0.9 \\ -1 \end{bmatrix},$$

$d_1 = 0.5$, $d_2 = -1.5$, and $d_3 = 2.5$. In this example, we will address several issues in hybrid system identification starting with the observer-based transformation to SARX models.

### 7.1 Observer-based transformation to SARX models

The switching sequence of length $N = 1,000$ plotted in Figure 2 was obtained by sampling from a uniform distribution with a minimum dwell time of $\delta_s(\chi) = 27$ secs. The SLS-to-SARX model conversion was carried out by the pole placement (the `place` command in MATLAB). For $\tau = 2, 3, 4$ and the piecewise constant gain sequence $g(k)$ in Lemma 3.1, $\|\Phi_o(k, k - \tau)\|_F$ is plotted in Figure 3 where $\|X\|_F$ denotes the Frobenius norm of a given matrix $X$. Definition 3.1 holds for all $\tau \geq 3$ and the time-varying ARX model (13) has order 3 on $[1 \ N]$ and 2 in the subintervals $[k_i + 2 \ k_{i+1})$. Since $3 \leq 2n - 1$, $\tau = 3$ is the best possible from Lemma 3.1.
7.2 Switch and submodel identifiability

We first check the switch identifiability conditions in Section 3.3 for $\tau = 2n = 4$ with the gain sequence constructed in Lemma 3.1. We generated a switching sequence of length $N = 1,000$. The minimum dwell time satisfies $\delta_\ast(\chi) = 29$. We excited the SLS model with a multi-sine input consisting of five superposed harmonics with frequencies, phases, and amplitudes randomly selected from uniform distributions. We verified the switch identifiability conditions in a noiseless setup since identifiability of a switch is not lost if perturbations are small enough while we checked the identifiability of the discrete-states by injecting noise with a large SNR. The latter approach, besides being more realistic, has eliminated delicate parametrization issues in designing the PE inputs for noiseless identification experiments. Calculating $z^T(k_i+1)(\theta(k_i+1) - \theta(k_i))$ and plotting it, we see from Figure 4 that the first condition in Lemma 3.10 is satisfied on the switch set $\chi$. Recall from Lemma 3.9 that $\chi \subset \chi_\theta$.

Next, we verify the second condition in Lemma 3.10 on $\chi$ by performing two SVDs: one for $\mathcal{A}(k_i, k_i+1)$ and one for $\mathcal{A}(k_i, k_i+1) = [\mathcal{A}(k_i, k_i+1 \setminus z(k_i+1])]$. The augmentation of $z(k_{i+1})$ into $\mathcal{A}(k_i, k_i+1)$ does not lead to an increase in the number of the significant singular values as shown in Figure 5 where $\|\Sigma_A\|_0$ denotes the number of the nonzero singular values of a given matrix $A \in \mathbb{R}^{n \times m}$. We conclude that by using the multi-sine excitations $\chi$ is identifiable.

Lemma 3.1 tells us that we could have used $\tau = 2n - 1$ instead of $\tau = 2n$. This choice would not change the inner product $z^T(k_{i+1})(\theta(k_{i+1}) - \theta(k_i))$ because zero-padding in $\theta(k)$ has no effect in this
Figure 6: The singular values of $R(k_i + 2n - 1, k_{i+1} - 1)$. 

operation. For the second condition in Lemma 3.10, permute and split $R(s, t)$ as follows

$$
P_z R(s, t) = \begin{bmatrix}
u(s) & \cdots & \nu(t) \\
\vdots & \ddots & \vdots \\
u(s-2n) & \cdots & \nu(t-2n) \\
y(s-1) & \cdots & y(t-1) \\
\vdots & \ddots & \vdots \\
y(s-2n) & \cdots & y(t-2n)
\end{bmatrix} = \begin{bmatrix} R_u(s, t) \\
R_y(s, t) \end{bmatrix}
$$

with $P_z$ denoting an appropriate permutation matrix. If $u(k)$ is PE of a sufficient order, then $R_u(s, t)$ has full rank, that is, rank($R_u(s, t)$) = $2n + 1 = 5$ provided that $t - s$ is large enough. As for the bottom matrix, $n \leq$ rank($R_y(s, t)$) $\leq 2n$. Both limits are attained in Figure 5. As $t - s$ diverges, the lower limit will be more likely to be seen. For intermediate values, $2n - 1$ is possible, and the bar graph would include 8 as a possible value. If we had started with the correct parametrization $\theta \in \mathbb{R}^{2\tau + 1}$ and $\tau = 2n - 1$, then Figure 5 would have 7 only possible value, but the identifiability of $\chi$ would not change.

Lastly, we check the discrete-state identifiability for the observer order $\tau = 2n - 1$ on the intervals $(k_i + \tau, k_{i+1})$ and the same gain sequence. We corrupted the output measurements by an additive Gaussian noise to achieve an SNR of 40dB. The PE condition is satisfied in both the noiseless and the noisy cases, see Figure 6. On the top subfigure, notice that $R(k_i + 2n - 1, k_{i+1} - 1)$ has 6 significant singular values exceeding by one the theoretical prediction $2n + 1$. Past discrete states in $(\infty, k_i)$ set a nonzero initial state.

7.3 Monte Carlo simulations

We consider the same SLS and start it at $x(0) = [1, 0]^T$. We sampled the switching function from a uniform distribution by selecting a minimum dwell time $\delta_{\min}(\chi) = 26$. The SLS model was excited with a multi-sine input consisting of five superposed harmonics generated as in Section 7.2. In the identification experiment, the length of the input-output data was $N = 2,000$. The output measurements were corrupted by additive Gaussian noise to achieve the SNRs of 40dB, 30dB, and 20dB.

We used the CVX package and selected the hyper-parameters of the optimization problem by griding: at 30dB $\lambda = 2$, $\gamma_1 = 10^3$ to enforce sparsity on the over-parameterized part, but not too large to avoid numerical ill conditioning and $\gamma_2 = 10^{-5}$ to control nonzero components of the parameter vector, cf. (36), but not too small to avoid again numerical ill conditioning. The weights $\gamma_1$ and $\gamma_2$ may be selected more or less the same for the three SNRs. However, for best results $\lambda$ should be adjusted according to the SNR. The segments satisfying $\delta_{\min}(\chi_{\theta}) \geq 150$ were chosen to extract discrete-state estimates. Points in these segments accounted about 70% of the time interval. Applying Algorithm 5, we estimated the
discrete-states. The eigenvalue-based statistics \( M(\hat{A}_\phi(k)) \) was applied to them. The \texttt{dbscan} clustering algorithm retrieved all discrete states. In Figure 7, the histogram of the clustering results for a single run of the algorithm at 30dB SNR is plotted for the three largest clusters. The number of the discrete states is then correctly estimated as \( \hat{\sigma} = 3 \).

Figure 7: The histogram of the \texttt{dbscan} clustering results from Algorithm 5 at 30dB SNR.

Figure 8 shows the true eigenvalues and the eigenvalue estimates of the discrete states at three different SNRs. The estimates are very accurate even for the smallest SNR.

Figure 8: The true and the estimated discrete state eigenvalues: the true ‘+’, the estimated ‘○’ at 40dB, 30dB, and 20dB SNRs, the top to the bottom respectively for different noise realizations.

Now, we verify that the optimal solution of (46) is indeed a deadbeat observer. Recall that this condition is implicitly enforced through the feasible parameter set. From (24), we estimated \( g(k) \) and calculated the eigenvalues of \( A_\phi(k) \) for the 3 submodels. In Figure 9, the Monte Carlo simulation results are plotted at 30dB SNR. The submodel eigenvalues require two steps only to the origin, that is, \( \tau = n = 2 \).

Using Algorithm 4 for \( 2n < k < N - 2n \) and excluding points in the long segments, we estimated \( \phi(k) \) at 30dB SNR. The estimation results plotted in Figure 10 show perfect match to the true switching sequence except few points zoomed in a separate figure for clarity, see Figure 11. In Figure 12, the estimate of \( \chi_\theta \) is plotted. Recall that \( \chi_\theta \) is derived from \( \chi \) under an observer transformation. Three groups of points are visible in the figure: points lying in the long segments, points lying in the short segments, and points lying in the very short segments. Recall that points in the third group belong to intervals with lengths less than \( 4n \). Points in the first group are determined by Algorithm 5.

In Figure 13 at 30dB SNR the components of the deadbeat observer gains are plotted for a single run.
Figure 9: The eigenvalues of $A_o(k)$ for the 3 submodels at 30dB SNR for several runs with different noise realizations.

Figure 10: The switching sequence estimate $\hat{\chi}$ (dotted) and the true switching sequence $\chi$ (solid) for the SLS model.

Figure 11: The switching sequence estimate (dotted) and the true switching sequence (solid) obtained by zooming from Figure 10.

of the algorithm over the interval $(2n, N - 2n)$. To show that over-parameterization does not influence estimation accuracy, with the ordo notation $X = O(\varepsilon)$ meaning $\|X\|_2 \leq \varepsilon$, we list $\hat{\theta}(k)$ delivered by Algorithm 5 for the first three long segments:

$$
\begin{align*}
\theta_1 &= (0.47, 1.33, 0.57, -0.97, -0.86, X_1)^T \\
\theta_2 &= (-1.5, 0.88, -0.4, -4.29, -0.40, X_2)^T \\
\theta_3 &= (2.49, 0.50, 0.49, -8.61, -0.49, X_3)^T
\end{align*}
$$
where $X_k^T = O(10^{-17}) \in \mathbb{R}^4$, $k = 1, 2, 3$. Hence, $X_k \approx 0$. Recall that $\tau = n = 2$ can be chosen on $[k_i + n, k_{i+1})$ for all $k_i \in \chi$ by Lemma 3.1.

7.3.1 Average case performance

We end this numerical example by studying the average case performance of the proposed scheme and comparing it with another scheme [2] from the literature on hybrid systems. The scheme presented in [2] is similar to the scheme put forward in this paper in that it utilizes a sparse optimization stage. But, it assumes availability of the state measurements $x(k)$ which is a rather unrealistic assumption unless $C(k) = I_n$ and $D(k) = 0$ for all $k$ in [2]. In this work, computational complexity of the state estimation was not reported.

We will perform Monte Carlo simulations by producing a new noise realization for the SLS in the beginning of this section. The average case performance will first be assessed by computing the relative error $\|M_\ell - \hat{M}_\ell\|_F / \|M_\ell\|_F$ for the quadruples $\hat{\mathcal{P}}_\ell = (\hat{A}_\ell, \hat{b}_\ell, \hat{c}_\ell^T, \hat{d}_\ell)$, $\ell \in \mathcal{S}$, where

$$
\hat{M}_\ell = \begin{bmatrix}
\hat{A}_\ell & \hat{b}_\ell \\
\hat{c}_\ell^T & \hat{d}_\ell
\end{bmatrix}
$$

is the linear map $(x^T(k) \ u^T(k))^T \mapsto (x^T(k+1) \ u^T(k))^T$ induced by $\hat{\mathcal{P}}_\ell$. This criterion was used in [2]. We sum the relative errors over the $\sigma$ discrete states and average them over 100 noise realizations. The result will be denoted by $\delta_r$. In Table 1, the mean value for $\delta_r$ is displayed for 20, 30, and 40dB SNRs for the two schemes. The proposed scheme outperforms the scheme in [2]. A second criterion for the
performance assessment will be defined on validation data sets as follows. On a fresh data set for each noise realization, we compute a variance-accounted-for (VAF) value

\[ VAF = \max \left\{ 1 - \frac{\text{var}(y(k) - \hat{y}(k))}{\text{var}(y(k))}, 0 \right\} \times 100\% \quad (68) \]

In Table 1, the mean of VAF for a range of the SNRs are shown. An inspection of the table reveals better performance for the presented scheme when subjected to output predictions. The average run time of the proposed scheme per noise realization denoted by \( t_{CPU} \) equals 55 secs. For the scheme proposed in \([2]\), \( t_{CPU} = 10.5 \) secs. The latter scheme is computationally less expensive since it does not include the cost of state estimation. Figure 14 reinforces our conclusion drawn earlier. It shows the distribution of VAF over 100 noise realizations. The VAF concentrates at 100 for the proposed scheme whereas they are scattered for the scheme in \([2]\).
estimate the discrete states up to arbitrary similarity transformations. This stage was complemented by the retrieval of switching sequences via a MOESP subspace algorithm. Convex relaxation of the non-convex optimization problem by the BBPDN method was demonstrated to be effective both theoretically and numerically.

Identifiability of discrete states of an SLS was exhibited as the PE conditions over segments. When a PE condition fails over a segment, the parameters of the discrete state that is active in this segment cannot be determined uniquely. In fact, this failure was shown to be an intrinsic feature of all deadbeat observers. As a result, recovery and robustness guarantees for the sparse optimization stage cannot be derived for the entire observation interval. In the literature on hybrid systems, a system identification procedure typically starts with the estimation of switching sequence from input-output data. Then, local models are parametrically estimated from input-output data over segments. In the current paper, we carried out system identification in reverse order. This approach facilitated local analysis, which was resorted very often in the paper.

In this paper, we considered SISO-SLSs to avoid delicate parameterization issues. Extension to MIMO-SLSs is not difficult by recognizing that the derivations in Sections 3–6 except for 3.3 and 4.1 apply to MIMO-SLSs verbatim. Section 3.3 requires only notational changes. The PE condition in 4.1 can be generalized by utilizing the matrix-fraction descriptions and the co-prime factorizations, see [22]. Replacement of the BBPDN algorithm in Figure 1 with a greedy algorithm, for example the BOMP algorithm, is an open problem we have not tackled in this paper.

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Appendix A

Proof of Lemma 3.7 For each $1 \leq k \leq N$, there is a gain $g(k)$ satisfying

$$\left( A_{\varphi(k)} + g(k)c_{\varphi(k)}^T \right)^n = 0. \quad (69)$$

Extend $g(k)$ such that $g(k) = g(l)$ if $\varphi(k) = \varphi(l)$. Thus, $g(k)$ is a sequence of at most $\sigma$ vectors $g_{\varphi(k)}$, $i = 0, \cdots, i^*$. 

Figure 15: The true and the estimated outputs on the validation data set. The SNR is 30dB and VAF is 99.39%
Let $\epsilon, \ell \in \mathbb{N}$ be such that $0 < k - \epsilon < \ell < k$. Write

$$\Phi_o(k, k - \epsilon) = \Phi_o(k, \ell)\Phi_o(\ell, k - \epsilon). \tag{70}$$

First, let us assume that $\epsilon \geq 2n - 1$. At least one of the inequalities $\ell - (k - \epsilon) \geq n$ or $k - \ell \geq n$ must be true then. If not, $\ell - (k - \epsilon) < n$ and $k - \ell < n$ or $\ell - (k - \epsilon) \leq n - 1$ and $k - \ell \leq n - 1$ and summing the last two inequalities, we get $\epsilon \leq 2n - 2$. We reach a contradiction.

Suppose $\ell \in \chi$. Then, $\ell = k_i$ for some $0 < i < i^*$. Assume $\delta_{i-1}(\chi) \geq n$ and $l - (k - \epsilon) \geq n$. Then, from (69)

$$\Phi_o(l, k - \epsilon) = A_o(l - 1) \cdots A_o(l - n)\Phi_o(l - n, k - \epsilon) = A_o^\ell(k_{i-1})\Phi_o(l - n, k - \epsilon) = 0.$$ 

If $k - \ell \geq n$ and $\delta_i(\chi) \geq n - 1$, again from (69)

$$\Phi_o(k, \ell) = \Phi_o(k, \ell + n)A_o(l + n - 1) \cdots A_o(l) = \Phi_o(k, \ell + n)A_o^\ell(k_i) = 0.$$ 

Hence, if $\delta_i(\chi) \geq n$, $\epsilon \geq 2n - 1$, and there is a switch in the interval $(k - \epsilon, k)$, from (70) we get $\Phi_o(k, k - \epsilon) = 0.$ If there is no switch in $(k - \epsilon, k)$, $(k - \epsilon, k) \subset [k_{i-1}, k_i)$ for some $i$ and

$$\Phi_o(k, k - \epsilon) = A_o(k - 1) \cdots A_o(k - \epsilon) = A_o^\epsilon(k_{i-1}) = 0.$$ 

Since $k$ was arbitrary, $\Phi_o(k, k - \epsilon) = 0$ for all $k$ if $\delta_i(\chi) \geq n$ and $\epsilon \geq 2n - 1$. Then, $\tau^* \leq 2n - 1$ if $\delta_i(\chi) \geq n$.

If $\delta_i(\chi) \geq n$, then $\tau^* < 2n$ is a tight bound. In fact, suppose that there exist three switches $k_i, k_{i-1}, k_{i+1}$ satisfying the equalities $k_i - k_{i-1} = k_{i+1} - k_i = n - 1$. Set $k = k_{i+1}$, $\ell = k_i$, and $\epsilon = 2n - 2$. Then, $\Phi_o(l, k - \epsilon) \neq 0$ if $(A_j + g_j(c^T_j, c^T_j))$ with $j = \delta_i(k_{i-1})$ is observable since $l - k + \epsilon = n - 1$. Next, $\Phi_o(k, l) \neq 0$ if $(A_j + g_j(c^T_j, c^T_j))$ with $j = \delta_i(k_i)$ is observable since $k - l = n - 1$. Thus, $\Phi_o(k, k - \epsilon) \neq 0$ and $\tau^* > 2n - 2$.

It follows that $\Phi_o(k, k - 2n + 1) = 0$ if $k \geq 2n$ and $\delta_i(\chi) \geq n$. The requirements $k \geq 2n$ and $\tau^* = 2n - 1$ are automatically satisfied by selecting a gain $g_{\varphi(k_i)}$ from $\delta_i(\chi) \geq 2n$. If for $i > 0$ and $k \in [k_i \cup n, k_{i+1})$, with $g(k) = g_{\varphi(k_i)}$ we have

$$\Phi_o(k, k - n) = (A_{\varphi(k_i)} + g_{\varphi(k_i)}c^T_{\varphi(k_i)})^n = 0.$$ 

Appendix B

**Proof Lemma 3.3.** The proof of this lemma is similar to the proof of Lemma 3.1. Consider the set of the inequalities $k - \kappa_c < \ell < k - 1$, $\ell \in \mathbb{N}$ where $\kappa_c \geq 2n$. Then, at least one of the inequalities $\ell - k + \kappa_c \geq n$ or $k - \ell - 1 \geq n$ must be true.

Suppose $k - \ell - 1 \geq n$ and $\ell \in \chi$. Then, for some $0 < i < i^*$, $\ell = k_i$. Assume $\delta_i(\chi) \geq n$. For $\ell \leq j \leq \ell + n - 1,$

$$\Phi(k, j + 1)b(j) = \Phi(k, \ell + n)A(\ell + n - 1) \cdots A(j + 1)b(j) = \Phi(k, k_i + n)A^{k_i + n - j - 1}(k_i)b(k_i)$$
Since $\delta_0(\chi) \geq n$ and $k - \ell > n$. Thus,

$$G_c(k, \kappa_{c}) = \sum_{j=k-\kappa_{c}}^{k-1} \Phi(k, j+1)b(j)b^T(j)\Phi^T(k, j+1)$$

$$\geq \sum_{j=k_i}^{k_i+n-1} \Phi(k, k_i + n)\sum_{j=k_i}^{k_i+n-1} A_{\Phi(k_i)}^{k_i+n-j-1}b_{\Phi(k_i)}b_{\Phi(k_i)}^T$$

$$\cdot [A_{\Phi(k_i)}^{k_i+n-j-1}]^T \Phi^T(k, k_i + n)$$

$$= \Phi(k, k_i + n)\sum_{v=0}^{n-1} A_{\Phi(k_i)}^v b_{\Phi(k_i)}b_{\Phi(k_i)}^T$$

$$\cdot [A_{\Phi(k_i)}^v]^T \Phi^T(k, k_i + n).$$

Since $\mathcal{P}_{\Phi(k_i)}$ is stable, $\Phi(k, k_i + n)$ is nonsingular. The middle term is the controllability Grammian of the minimal discrete state $\mathcal{P}_{\Phi(k_i)}$. Hence, it is positive definite.

Now, suppose $\ell - k + \kappa_{c} \geq n$ and $\ell \in \chi$ again. Assume $\delta_{\ell-1}(\chi) \geq n$. For $\ell - n \leq j \leq \ell - 2$,

$$\Phi(k, j+1)b(j) = \Phi(k, \ell)A(\ell - 1) \cdots A(j+1)b(j)$$

$$= \Phi(k, k_i)A^{\ell-j-1}(k_{i-1})b(k_{i-1})$$

since $\ell - n \geq \kappa_{c}$ and $\delta_{\ell-1}(\chi) \geq n$. Thus,

$$G_c(k, \kappa_{c}) \geq \Phi(k, k_i)\sum_{j=k_{i-1}}^{\ell-2} A_{\Phi(k_{i-1})}^{k_{i-1}-j-1}b_{\Phi(k_{i-1})}b_{\Phi(k_{i-1})}^T$$

$$\cdot [A_{\Phi(k_{i-1})}^{k_{i-1}-j-1}]^T \Phi^T(k, k_i)$$

$$= \Phi(k, k_i)\sum_{v=0}^{n-1} A_{\Phi(k_{i-1})}^v b_{\Phi(k_{i-1})}b_{\Phi(k_{i-1})}^T$$

$$\cdot [A_{\Phi(k_{i-1})}^v]^T \Phi^T(k, k_i) > 0$$

since $\mathcal{P}_{\Phi(k_i)}$ is stable and hence $\Phi(k, k_i)$ is nonsingular and the middle term being the controllability Grammian of the minimal discrete state $\mathcal{P}_{\Phi(k_{i-1})}$. Thus, if $\delta_i(\chi) \geq n$, $\kappa_{c} \geq 2n$, and there is a switch in $(k-\kappa_{c} k-1)$, then $G_c(k, \kappa_{c}) > 0$. If there is no switch in $(k-\kappa_{c} k-1) \subset (k_{i-1} k_i)$ for some $i$ and from the $\ell - k + \kappa_{c} \geq n$ case, we again have $G_c(k, \kappa_{c}) > 0$. Fix $\kappa_{c}$ as $\kappa_{c} = 2n$. Since $k$ lies in the compact set $[1 N]$ and $\sigma < \infty$, one can easily determine some constants $\alpha_i, \alpha_0, \beta_0, \beta_1$, and $\delta_i$ in the definition of uniform observability. The boundedness of $\{1-3\}$ is obvious.

Although duality arguments may be used to show that $G_o(k, \kappa_{o})$ is positive definite, we will prove it directly. This time, we will consider the inequalities $k < \ell < k + \kappa_{o} - 1$ with $\delta_i(\chi) \geq n$ and fix $\kappa_{o} = 2n$. We first examine the $\ell - k \geq n$ case with $\ell = k_i$ for some $0 < i < i^*$. Then,

$$G_o(k, \kappa_{o}) = \sum_{j=k}^{k+\kappa_{o}-1} \Phi^T(j, k)c(j)c^T(j)\Phi(j, k)$$

$$\geq \sum_{j=k}^{k+n-1} \Phi^T(j, k)c(j)c^T(j)\Phi(j, k)$$

$$= \sum_{v=0}^{n-1} [A_{\Phi(k_{i-1})}^v]^T c_{\Phi(k_{i-1})}c_{\Phi(k_{i-1})}^T A_{\Phi(k_{i-1})}^v > 0$$
Since \( \delta_{i-1}(\chi) \geq n \). If \( k + \kappa_0 - 1 - \ell \geq n \), from \( \delta_i(\chi) \geq n \)

\[
G_o(k, \kappa_0) \geq \sum_{j=\ell}^{\ell+n-1} \Phi^T(\ell, k) \Phi(j, \ell) c^T(j) \Phi(j, \ell) \Phi(\ell, k)
\]

\[
= \Phi^T(\ell, k) \sum_{v=0}^{n-1} [A^V_{\phi(k)}]^T c_{\phi(k)} c_{\phi(k)} A^V_{\phi(k)} \Phi(\ell, k) > 0.
\]

If there is no switch in \( (k + \kappa_0 - 1) \), we then have for some \( i \), \( (k + \kappa_0 - 1) \in [k_{i-1}, k_i) \) and from the last case above we get \( G_o(k, \kappa_0) > 0 \). Similar comments to the uniform controllability case apply for the constants \( \alpha_0', \alpha'_i, \beta_0', \beta'_i \), and \( \delta_1 \) in the definition of uniform observability. From the definitions of \( G_c(k, \kappa_0) \) and \( G_c(k, \kappa_c) \), we see that \((1) - (3)\) may be demanded uniform in \( \kappa_c + 1 \ - \kappa_0 + 1 \).

**Appendix C**

**Proof of Lemma 3.4** Suppose \( k_i + n \leq k < k_{i+1} \). The first two components of the observer Markov parameters are

\[
h_o(k, k-v) = \begin{cases} [d(k_i) \ 0], & v = 0; \\ c^T(k_i) B_o(k_i), & v = 1 \end{cases}
\]

since \( d(k) = d(k_i) \), \( c(k) = c(k_i) \), and \( B_o(k-v) = B_o(k_i) \) for all \( 0 \leq v \leq n \). Recall that \( g(k) = g(k_i) \) for all \( k \in [k_i \ k_{i+1}) \) from Lemma 3.1. Hence, for all \( k \in [k_i \ k_{i+1}] \)

\[
A_o(k) = A(k) + g(k) c^T(k) = A(k_i) + g(k_i) = A_o(k_i)
\]

and therefore for \( v \geq 2 \)

\[
h_o(k, k-v) = c^T(k) A_o(k-1) \cdots A_o(k-v+1) B_o(k-v) = c^T(k_i) A_o^{v-1}(k_i) B_o(k_i - v).
\]

Thus, \( h_o(k, k-v) = h_o(k_i + n, k_i + n - v) \), \( v \in [0 \ n] \). Then,

\[
\theta(k) = \theta(k_i + n), \quad k_i + n \leq k_{i+1}.
\]

Pick \( s_j \) as the greatest lower bound on \( k_i + n \) and \( s_{j+1} \) as the smallest upper bound on \( k_{i+1} \) in \( \chi_\theta \).

**Appendix D**

**Proof of Lemma 3.5** Let \( l, j \in [s_j + 2n - 1, s_{j+1}) \). Then for \( 0 \leq v < 2n \), \( h_o(k-v, k-v) = h_o(l-v, l-v) = d(l-v) \) and \( h_o(k, k-v) = h_o(l, l-v) \). In \( (19) \) in place of \( i \) in \( \gamma(k, i) \), substitute \( k-v \)

\[
\gamma(k, k-v) = h_o^\text{(1)}(k, k-v) + h_o^\text{(2)}(k, k-v) h_o^\text{(1)}(k-v, k-v) = h_o^\text{(1)}(l, l-v) + h_o^\text{(2)}(l, l-v) h_o^\text{(1)}(l-v, l-v) = \gamma(l, l-v), \quad 0 \leq v < 2n.
\]

Since \( l \geq s_j + 2n - 1 \), notice that \( d(l-v) = d(s_j) \). Hence, as long as \( k-v \geq s_j \), we let \( v \geq 2n \). Then, \( h_o^\text{(1)}(k-v, k-v) \) is well-defined and equals to \( d(s_j) \). If \( v \geq 2n \), \( h_o^\text{(1)}(k, k-v) = 0 \) and \( h_o^\text{(2)}(k, k-v) = 0 \). Thus, \( \gamma(k, k-v) = 0 \). Combining both cases if \( \max\{k, l\} \leq s_{j+1} \), we derive

\[
\gamma(k, k-v) = \gamma(l, l-v), \quad s_j + v \leq \min\{k, l\}.
\]
Now, for $v = 1$ we have

$$h(k, k - 1) = \gamma(k, k - 1) = \gamma(l, l - 1) = h(l, l - 1).$$

For $v > 1$, we proceed by induction. Consider the second term on the right-hand side of (20). Denote it by $J(v)$. Substitute $k - v$ in place of $i$ in $J(v)$ and change the variable $j$ inside the summand to $\mu$. Then, change $\mu$ to $\xi = \mu + v - k$:

$$J(v) = \sum_{\mu = k-v+1}^{k-1} h^{(2)}_k(k, \mu) h(\mu, k-v), \quad v \geq 2$$

$$= \sum_{\xi = 1}^{v-1} h^{(2)}_k(k, k-v + \xi) h(k-v + \xi, k-v)$$

$$= \sum_{\xi = 1}^{v-1} h^{(2)}_k(\ell, \ell - v + \xi) h(k-v + \xi, k-v).$$

Thus, $J(v)$ is a linear combination of the system Markov parameters $h(k-1, k-v), \ldots, h(k-v+1, k-v)$. For $v = 2$, $h(k-1, k-2) = \gamma(k-1, k-2) = \gamma(l, l-1)$ and $h(k-2, k-3)$ becomes a linear combination of $\gamma(l, l-2)$ and $\gamma(l, l-1)$. Put $v = 3$. Then, $h(k-1, k-3)$ and $h(k-2, k-3)$ are the only terms in the linear combination. Hence, $h(k, k-3)$ is a linear combination of $\gamma(l, l-3), \gamma(l, l-2), \gamma(l, l-1)$.

Assume $h(k, k-v)$ a linear combination of $\gamma(l, l-v), \ldots, \gamma(l, l-1)$. Then, $J(v+1)$ becomes a linear combination of $h(k-1, k-v-1), \ldots, h(k-v, k-v-1) or \gamma(l, l-v), \ldots, \gamma(l, l-1)$ and so does $h(k, k-v)$ a linear combination of $\gamma(l, l-v)$ for $\mu = 1, \ldots, v+1$. This completes the induction and $h(k, k-v) = h(l, l-v)$. The lower bound $s_j + v \leq \min(k, l)$ is not violated throughout the iterations. The iterations stops when $\max(k, l) < s_{j+1}$ is reached.

For the last part, observe that (22) is driven by the shift-invariant terms $h^{(2)}_k(k, \xi), 0 < k - \xi < 2n$.

**Appendix E**

*Proof of Lemma 3.7* From Lemma 3.6

$$[s_j + 4n - 1, s_{j+1} - 2n) \subset [k_i, k_{i+1}).$$

Suppose $s_j - k_i \geq 4n$. Recall from Lemma 3.4 that for some $s_{t\ell}$ and $s_{t+1\ell}$ in $\chi_\theta$, $[k_i+n, k_{i+1}) \subset [s_{t\ell}, s_{t+1\ell})$. Then, from the inequalities $s_{t\ell} \leq k_i + n \leq s_j - 3n$ we must have $j > l$. Next, from the chain of inequalities $s_{j+1} - 2n \leq k_{i+1} \leq s_{i+1} \leq s_j$, we obtain $\delta_j(\chi_\theta) \leq 2n$. A contradiction. Thus, $s_j - k_i < 4n$. The case $s_j < k_i$ is not possible. If it were possible, $s_{j\ell} = s_j$ for some $\ell > j$ since $k_i \in \chi_\theta$. But, $k_i \leq s_j + 4n - 1 or s_{\ell j} < 4n$. Hence, $\delta_j(\chi_\theta) < 4n$. Contradiction. Thus, $s_j - k_i \geq 0$.

Suppose $s_{j+1} > k_{i+1}$. From Assumption 3.2, $k_{i+1} \in \chi_\theta$. Then, $s_{t\ell} = k_{i+1}$ for some $\ell \leq j$. Since $s_{j+1} - 2n \leq k_{i+1}$, we then have $\delta_j(\chi_\theta) \leq 2n$, a contradiction. Hence, $s_{j+1} \leq k_{i+1}$. From $[k_i+n, k_{i+1}) \subset [s_{\ell j}, s_{\ell+1})$, we then have $s_{j+1} \leq k_{i+1} \leq s_{\ell + 1}$. Hence $j \leq \ell$. The case $\ell > j$ is not possible because otherwise the inequalities $s_{j+1} \leq s_{\ell} \leq k_i + n \leq s_j + 5n - 1$ would yield the contradiction $\delta_j(\chi_\theta) < 5n$. Thus, $j = \ell$ and $s_{j+1} = k_{i+1}$.

**Appendix F**

*Proof of Lemma 3.9* Similarly to (25), we define the triangular Hankel matrices

$$\mathcal{H}_0^{(1)}(k) = \begin{bmatrix} h^{(1)}_0(k, k-1) & h^{(1)}_0(k, k-\kappa_c) & \cdots & h^{(1)}_0(k, k-\kappa_c) \\ \vdots & \ddots & \ddots & \vdots \\ \cdots & \cdots & h^{(1)}_0(k + \kappa_0 - 1, k-\kappa_c) \end{bmatrix}$$
and introduce the extended observability and controllability matrices

\[
\mathcal{O}_{\kappa_0}^{(1)}(k) = [c(k) \cdots \Phi_0^T(k + \kappa_0, k)c(k + \kappa_0 - 1)]^T,
\]

\[
\mathcal{P}_{\kappa_0}^{(1)}(k - 1) = [b_0^{(1)}(k - 1) \cdots \Phi_0(k, k - \kappa_0 + 1)b_0^{(1)}(k - \kappa_0)]
\]

for \( \delta_\kappa < k \leq \delta_0 \) and factorize \( \mathcal{H}_0^{(1)}(k) \) as

\[
\mathcal{H}_0^{(1)}(k) = \mathcal{O}_{\kappa_0}^{(1)}(k)\mathcal{P}_{\kappa_0}^{(1)}(k - 1).
\]

From Lemma 3.8, rank(\( \mathcal{H}_0^{(1)}(k) \)) = 2 for all \( k \). Write

\[
\theta^{(1)}(k_i) = [d(k_i) \ c^T(k_i)\mathcal{P}_{\kappa_0}^{(1)}(k_i - 1)],
\]

\[
\theta^{(1)}(k_i - 1) = [d(k_i - 1) \ c^T(k_i - 1)\mathcal{P}_{\kappa_0}^{(1)}(k_i - 2)].
\]

Since \( \delta(\chi) > 2n \), \( \mathcal{P}_{\kappa_0}^{(1)}(k_i - 1) = \mathcal{P}_{\kappa_0}^{(1)}(k_i - 2) \) and from Lemma 3.8, \( \mathcal{P}_{\kappa_0}^{(1)}(k_i - 1) \) has full rank. Suppose \( k_i \in \chi \). Then, \( \varphi(k_i - 1) \neq \varphi(k_i) \) and \( [ c^T(k_i) \ d(k_i) ] \neq [ c^T(k_i - 1) \ d(k_i - 1) ] \) from Assumption 3.4. Therefore, \( \theta^{(1)}(k_i - 1) \neq \theta^{(1)}(k_i) \) if \( d(k_i) \neq d(k_i - 1) \). If \( c(k_i) \neq c^T(k_i - 1) \), the same conclusion is drawn since \( \mathcal{P}_{\kappa_0}^{(1)}(k_i - 1) \) and \( \mathcal{P}_{\kappa_0}^{(1)}(k_i - 2) \) are equal and have full rank. It follows that \( \theta(k_i - 1) \neq \theta(k_i) \) and \( k_i \in \chi_\delta \).

Suppose \( ii. \) holds in Assumption 3.4, construct \( \mathcal{H}_0^{(2)}(k) \) similarly to \( \mathcal{H}_0^{(1)}(k) \) from the Markov parameters \( h_0^{(2)}(k, i) \). Factorization of \( \mathcal{H}_0^{(2)}(k) \) shows that \( \theta^{(2)}(k_i - 1) \neq \theta^{(2)}(k_i) \) if \( c^T(k_i) \neq c^T(k_i - 1) \). Then, \( \theta(k_i - 1) \neq \theta(k_i) \) and \( k_i \in \chi_\delta \).

**Appendix G**

**Proof of Proposition 4.1** Fix \( i \) first and hence \( \varphi(k_i) \). Assumptions 4.1–4.2 are sufficient for the existence of PE inputs for the transfer function

\[
G(z;k_i) = (1 - H_0^{(2)}(z; k_i))^{-1}H_0^{(1)}(z; k_i).
\]

Thus, (54) is true for some \( \gamma \geq 0 \). Let \( \gamma = \sup \gamma \) which is finite for there are no more than \( \sigma \) discrete states as \( i \) changes. Thus, (54) holds with \( \gamma \) and all \( k_i \in \chi \).

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