A result on the bias of sieve profile estimators

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Abstract

This paper complements the paper Andresen and Spokoiny (2014). We show how to control the bias of a sieve type profile estimator under natural conditions on the Hessian of the expected contrast functional.

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1 Introduction

This paper presents a way to control the bias in a sieve profile contrast estimation problem, which we elaborate for simplicity for parameters in $l^2 \equiv \{(x_1, x_2, \ldots) \subset \mathbb{R} : \sum_{k=1}^{\infty} x_k^2 < \infty\}$. More precisely consider a contrast functional $IE_L : \mathbb{R}^p \times l^2 \to \mathbb{R}$. Assume that the goal is to calculate the target parameter $\theta^*$ defined as

$$\theta^* \equiv \arg\max_{\theta} \sup_{\eta \in \mathcal{Y} \subset l^2} IE_L(\theta, \eta) = \Pi_\theta v^* \equiv \Pi_\theta \arg\max_{v \in \mathcal{Y} \subset l^2} IE_L(v),$$

with a set $\mathcal{Y} \subset \mathbb{R}^p \times l^2$. To circumvent the problem of maximizing over an infinite dimensional set $\mathcal{Y} \subset \mathbb{R}^p \times l^2$ define for some $m \in \mathbb{N}$ the following approximation contrast functional

$$IE_{L_m} : \mathbb{R}^{p+m} \to \mathbb{R},$$

$$(\theta, \eta) \mapsto IE_L(\theta, E_l \eta),$$

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where \( E_{l^2} : \mathbb{R}^m \to l^2 \) is the natural embedding operator. Further define the biased target

\[
\theta_m^* = \arg\max_\theta \sup_{\eta \in \mathbb{R}^m} \mathbb{E} \mathcal{L}_m(\theta, \eta) = \Pi_\theta \nu_m^* \overset{\text{def}}{=} \Pi_\theta \arg\max_{\nu = (\theta, \eta) \in \mathbb{R}^{p+m}} \mathbb{E} \mathcal{L}_m(\theta, \eta).
\]

We are interested in a bound for the euclidean distance \( \|\theta^* - \theta_m^*\| \). Further consider the following block representations of the hessian operator \( \mathcal{D}^2(\nu) = -\nabla^2 \mathcal{E} \mathcal{L}(\theta, f) \):

\[
\mathcal{D}^2 = \begin{pmatrix}
\mathcal{D}^2_m & A_m \\
A_m^\top & \mathcal{H}_m^2
\end{pmatrix} \in L(\mathbb{R}^{p+m} \times l^2, \mathbb{R}^{p+m} \times l^2),
\]

where for a vectorspace \( V \) the symbol \( L(V, V) \) denotes the set of linear operators from \( V \) to \( V \), \( p^* = p + m \in \mathbb{N} \) and where

\[
\mathcal{D}^2_m(\nu) \overset{\text{def}}{=} \nabla^2 \mathcal{E} \mathcal{L}(\nu) \in \mathbb{R}^{p \times p},
\]

i.e. the derivatives of \( \mathcal{E} \mathcal{L} \) are only taken with respect to the first \( p + m \in \mathbb{N} \) coordinates of \( \nu = (\theta, \eta) \in \mathbb{R}^p \times l^2 \).

Define the following two matrices

\[
\mathcal{D}_m^2(\nu)^* = \begin{pmatrix}
\Pi_\theta \mathcal{D}^{-2}_m(\nu) \Pi_\theta^\top
\end{pmatrix}^{-1} \in \mathbb{R}^{p \times p}, \quad \text{and} \quad \mathcal{D}_m^2(\nu)^* = \begin{pmatrix}
\Pi_\theta \mathcal{D}^{-2}(\nu) \Pi_\theta^\top
\end{pmatrix}^{-1} \in \mathbb{R}^{p \times p}.
\]

The second result we want to derive is a bound for the difference between \( \mathcal{D}_m(\nu_m^*) \in \mathbb{R}^{p \times p} \) and \( \mathcal{D}(\nu^*) \in \mathbb{R}^{p \times p} \) in spectral norm, where

\[
\nu_m^* \overset{\text{def}}{=} \arg\max_{\nu \in \mathbb{R}^{p+m}} \mathcal{E} \mathcal{L}.
\]

This kind of problem arises for instance when a sieve profile estimator is analyzed. Given a (random) contrast functional \( \mathcal{L} : \mathbb{R}^p \times l^2 \to \mathbb{R} \) one defines \( \mathcal{L}_m \) analogously to \( \mathbb{E} \mathcal{L}_m \) above and the sieve profile estimator

\[
\tilde{\theta}_m = \arg\max_{\theta \in \mathbb{R}^p} \mathcal{L}_m(\theta, \eta) \overset{\text{def}}{=} \Pi_\theta \nu_m^* = \arg\max_{\nu \in \mathbb{R}^{p+m}} \mathcal{E} \mathcal{L}.
\]

The parametric results obtained in Andresen and Spokoiny (2014) claim that the profile estimator \( \tilde{\theta}_m \) estimates well \( \theta_m^* \) if the spread \( \Diamond(x, x) > 0 \) is small. More precisely we have for fixed \( x \) with Theorem 2.1 of Andresen and Spokoiny (2014) applied to \( \tilde{\theta}_m \) from (??) that with probability greater \( 1 - 22.8e^{-x} \)

\[
\|\mathcal{D}_m(\tilde{\theta}_m - \theta_m^*) - \tilde{\xi}_m(\nu_m^*)\| \leq \Diamond(x, x),
\]

where \( \nu_m^* = (\theta_m^*, \eta_m^*) = \arg\max_{\nu} \mathbb{E} \mathcal{L}_m(\nu) \).
This result involves exactly the two kinds of bias from above, i.e. one that concerns the difference \( \theta^*_m - \theta^* \) and the other the difference between \( \bar{D}_m \in \mathbb{R}^{p \times p} \) and \( \bar{D} \in \mathbb{R}^{p \times p} \).

Andresen and Spokoiny (2014) use two assumptions to address this bias. The first one reads:

\[ (\text{bias}) \quad \text{There exists a decreasing function } \alpha : \mathbb{N} \to \mathbb{R}_+ \text{ such that } \| \bar{D}_m (\theta^*_m - \theta^*) \| \leq \tilde{\alpha}(m). \]

The second one:

\[ (\text{bias}') \quad \text{As } m \to \infty \]

\[ \| I - \bar{D}_m (\nu^*)^{-1} \bar{D} (\nu^*)^2 \bar{D}_m (\nu^*)^{-1} \| = o(1), \]

\[ \| I - \bar{D}_m (\nu^*_m)^{-1} \bar{D}_m (\nu^*)^2 \bar{D}_m (\nu^*_m)^{-1} \| = o(1). \]

In this paper we want to present a particular way to obtain such a function \( \tilde{\alpha}(m) \) and to derive \( (\text{bias}') \), which relies on less high level conditions on the smoothness and structure of \( \mathbb{E}\mathcal{L}. \)

2 Main result

Denote by \( \Pi_{p^*} : l^2 \to \mathbb{R}^{p^*} \) the projection to the first \( p^* \in \mathbb{N} \) coordinates of an element of \( l^2 \). To bound the bias \( \| \bar{D}_m (\theta^*_m - \theta^*) \| > 0 \) we present the following condition:

\[ (\nu \Xi) \quad \text{The vector } \nu^\nu \overset{\text{def}}{=} (\text{Id}_{l^2} - \Pi_{p^*}) \nu^* \text{ satisfies } \| \mathcal{H}_m \nu^\nu \|^2 \leq C_{\nu^\nu} m \text{ for some } C_{\nu^\nu} > 0 \]

and with \( \alpha(m) \to 0 \)

\[ \| \mathcal{D}_m^{-1} A_m \nu^\nu \| \leq \alpha(m). \]

Further for any \( \lambda \in [0, 1] \) with some \( \tau(m) \to 0 \)

\[ \| \mathcal{D}_m^{-1} (\nabla_{\nu^\nu} \mathcal{E}\mathcal{L}(\nu^*, \lambda \nu^\nu) - A_m) \nu^\nu \| \leq \tau(m), \]

\[ \| \nu^\nu^\top (\mathcal{D}_{\nu^\nu} - \nabla_{\nu^\nu} \mathcal{E}\mathcal{L}(\nu^*, \lambda \nu^\nu)) \nu^\nu \| \leq C_{\nu^\nu}^2 m. \quad (2.1) \]

To ensure that \( \bar{D}_m \) is close to \( \bar{D} \) we impose the following second condition.

\[ (\nu \Xi) \quad \text{Assume that with some } \beta(m) \to 0 \]

\[ \| \mathcal{H}^{-1} A^\top \mathcal{D}_m^{-1} \| \leq \beta(m). \]
Theorem 2.1. Let the conditions $(\kappa)$ and $(\nu \kappa)$ be fulfilled. Further let the condition $(L_{\infty})$ from Section A.1 be satisfied for $E \mathcal{L} : l^2 \to \mathbb{R}$ with $b(r) \equiv b > 0$. Set $r^* = 4C_{\kappa,m}/b$ and let for some $m_0 \in \mathbb{N}$ and all $m \geq m_0$ the condition $(L_0)$ be fulfilled for $D_0 = D_m$ and for any $r \leq r^*$. Then $(\text{bias})$ and $(\text{bias}')$ are satisfied with

$$\hat{\alpha}(m) = \sqrt{1 + \rho^2} \left( \alpha(m) + \tau(m) + 2\delta(2r^*)r^* \right).$$

3 Application to single-index model

We present an example to illustrate how these results can be derived for single-index modeling. Consider the following model

$$Y_i = f(X_i^\top \theta^*) + \varepsilon_i, \quad i = 1, \ldots, n,$$

or some $f : \mathbb{R} \to \mathbb{R}$ and $\theta^* \in S_1^{p,+} = \{ \theta \in \mathbb{R}^p : \|\theta\| = 1, \theta_1 > 0 \} \subset \mathbb{R}^p$. We assume that the support of the $X \in \mathbb{R}^p$ is contained in the ball of radius $s_X > 0$. Further we assume that $f \in \{ f : [-s_X, s_X] \mapsto \mathbb{R} \}$ can be well approximated by an orthonormal $C^2$-Daubechies-wavelet basis, i.e. for a suitable function $e_0 := \psi : [-s_X, s_X] \mapsto \mathbb{R}$ we set for $k = 2^j + j_k$

$$e_k(t) = 2^{j/2} \psi \left( \frac{2^j(t - 2^j s_X)}{s_X} \right), \quad k \in \mathbb{N}.$$

Our aim is to analyze the properties of the profile MLE

$$\hat{\theta} \overset{\text{def}}{=} \arg\max_{\theta} \max_{\eta \in \mathbb{R}^m} \mathcal{L}(\theta, \eta),$$

where

$$\mathcal{L}(\theta, \eta) = \frac{1}{2} \sum_{i=1}^n \left| Y_i - \sum_{k=0}^m \eta_k e_k(X_i^\top \theta) \right|^2.$$

Consider the following assumptions.

(CondX) The measure $P^X$ is absolutely continuous with respect to the Lebesgue measure. The Lebesgue density $d_X$ of $P^X$ is only positive on the ball $B_{s_X}(0) \subset \mathbb{R}^p$ and Lipshitz continuous with Lipshitz constant $L_{d_X} > 0$.

Of course we need some regularity of the link function $f \in \{ f : [-s_X, s_X] \mapsto \mathbb{R} \}$:
(Cond$_{f^*}$) For some $f^* \in \mathbb{R}^\mathbb{N}$

$$f = f_{f^*} = \sum_{k=1}^{\infty} f^*_k e_k,$$

where with some $\alpha > 2$ and a constant $C\|f^*\| > 0$

$$\sum_{l=0}^{\infty} l^{2\alpha} f_{l}^{*2} \leq C_{l}^{2\|f^*\|} < \infty.$$

Lemma 3.1. Assume (Cond$_{f^*}$) and (Cond$_{X}$). Using our orthogonal and sufficiently smooth wavelet basis we get for any $\lambda \in [0,1]$

$$\|\mathcal{D}_{\mathcal{X}}^{1/2} x^*\|^2 < C_1 m^{-2\alpha}, \quad \alpha(m) \leq C_2 m^{-\alpha - 1/2} \sqrt{n},$$

$$\beta(m) \leq C_3 m^{-1/2}, \quad \tau(m) \leq C_4 m^{-2\alpha + 1/2} \sqrt{n},$$

and $|x^*^\top (\mathcal{D}_{\mathcal{X}} - \nabla_{\mathcal{X}} \mathcal{E}(\nu^* , \lambda x^*)) x^*| = 0$.

For details see Andresen (2014).

A Appendix

A.1 The conditions

We adopt the conditions from Section 3 of Spokoiny (2012) with some minor changes. First we present the parametric conditions that apply to parametric models with finite dimensional parameter. Then explain two new conditions that arise in the infinite dimensional setting.

For some finite dimension $p^* \in \mathbb{N}$ the parametric conditions involve a matrix $\mathcal{D}_0^2$ and a central point $\nu^0 \in \mathbb{R}^{p^*}$ that have to be specified before the conditions can be checked.

Remark A.1. For Theorem 2.1 the matrix equals

$$\mathcal{D}_0^2 = -\nabla_{\nu}^2 \mathcal{E}(\nu^*),$$

and $\nu^0 = \nu^*_m$, i.e. the central point does not coincide with the element that defines the matrix $\mathcal{D}_0^2$. It is important to note that condition (L$_0$) thus becomes another constraint on the bias.

The matrix $\mathcal{D}_0^2$ has to satisfy certain regularity conditions. We begin by representing the information matrix in block form:

$$\mathcal{D}_0^2 = \begin{pmatrix} \mathcal{D}_0^2 & A_0 \\ A_0^\top & H_0^2 \end{pmatrix}.$$
Here we restate identifiability conditions:

(I) It holds
\[
\|H_0^{-1} A_0^\top D_0^{-1}\|_\infty^2 =: \rho < 1.
\]

Using the matrix \(D_0 \in \mathbb{R}^{p^* \times p^*}\) and the central point \(v^o \in \mathbb{R}^{p^*}\) we define the local set \(\Theta(x)\) with some \(x \geq 0\)
\[
\Theta(x) \overset{\text{def}}{=} \{ v = (\theta, \eta) \in \Theta, \|D_0(v - v^o)\| \leq x \}.
\]

The local conditions only describe the properties of the process \(L(v)\) for \(v \in \Theta(x)\) with some fixed value \(x > 0\). The global conditions have to be fulfilled on the whole \(\Theta\). We start with the local conditions.

(L_0) For each \(x \leq x_0\), there is a constant \(\delta(x)\) such that it holds on the set \(\Upsilon(x)\) and with spectral norm \(\| \cdot \|\):
\[
\|D_0^{-1} \nabla^2 L(v) D_0^{-1} - I_{p^*}\| \leq \delta(x).
\]

We also need:

(L_x_0) For any \(x > x_0\) there exists a value \(b(x) > 0\), such that
\[
-\frac{E[L(v, v^*)]}{\|D(v - v^*)\|^2} \geq b(\|D(v - v^*)\|).
\]

A.2 Proof of Theorem 2.1

Lemma A.1. Assume that (L_x_0) is satisfied with \(b(x) \equiv b\) and that the condition (x) is satisfied. Then we get \(\|D(v^*_m - v^*)\| \leq x^*\) where \(x^* = 4 C_{\mu^*} m / b\).

Proof. Note that
\[
\|D(v^* - \Pi_{p^*} v^*)\| = \|H_m x^*\|,
\]
such that \(v^* \in \Theta(x^*)\). Further we have \(\nabla E[L(v^*)] = 0\) such that by the Taylor expansion with some \(\lambda \in [0, 1]\)
\[
E[L(\Pi_{p^*} v^*, v^*)] = \|H_m x^*\|^2 + x^* \top (H_m - \nabla_{x^*} E[L(v^*, \lambda x^*)]) x^*.
\]

which gives with (L_x_0) and (x) on \(\Theta(x^*)\) that
\[
|E[L(\Pi_{p^*} v^*, v^*)| \leq \|D(v^* - v^*)\|^2 + C_{\mu^*} m \leq 2 C_{\mu^*} m. \quad (A.1)
\]
Now we show that \( \bm{v}_m^* \) also belongs to \( \Theta(\bm{r}^*) \) for \( \bm{r}^* \geq 4C_\kappa' \cdot m / b \). Suppose for the moment that \( \| \mathcal{D}(\bm{v}_m^* - \bm{v}^*) \| > \bm{r}^* \). By \((\mathcal{L}_{\infty})\), it holds

\[
2 \| \mathcal{E}\mathcal{L}(\bm{v}_m^*, \bm{v}^*) \| \geq b \| \mathcal{D}(\bm{v}_m^* - \bm{v}^*) \|^2 > b \bm{r}^* \cdot 2.
\]

This contradicts \( \| \mathcal{E}\mathcal{L}(\bm{v}_m^*, \bm{v}^*) \| \leq \| \mathcal{E}\mathcal{L}(\Pi_{p^*} \bm{v}^*, \bm{v}^*) \| \) in view of \( \bm{r}^* \geq 4C_\kappa' \cdot m / b \) and (?), so \( \bm{v}_m^* \in \Theta(\bm{r}^*) \).

**Lemma A.2.** Assume that \((\mathcal{L}_{\infty})\) is satisfied with \( b(\bm{r}) \equiv b \). Further assume \((\kappa)\) and \((\mathcal{L}_0)\) with central point \( \bm{v}_m^* \in \mathbb{R}^{p^*} \) and operator \( \mathcal{D}_m \). Then we get with \( \bm{r}^* = 4C_\kappa \cdot m / b \)

\[
\| \mathcal{D}_m(\bm{\theta}_m^* - \bm{\theta}^*) \| \vee \| \mathcal{D}_m(\bm{v}_m^* - \bm{v}^*) \| \leq \frac{1 + \rho^2}{1 - \rho^2} (\alpha(m) + \tau(m) + 2\delta(2\bm{r}^*) \bm{r}^*).
\]

**Proof.** Using condition \((\mathcal{L}_0)\) and Taylor expansion we have on \( \Theta_m(\bm{r}) = \{ \| \mathcal{D}_m(\bm{v} - \bm{v}_m^*) \| \leq \bm{r} \} \subset \mathbb{R}^{p+m} \)

\[
\sup_{\bm{v} \in \Theta(\bm{r})} \| \mathcal{D}_m^{-1} \nabla_m \mathcal{E}\mathcal{L}_m(\bm{v}) - \mathcal{D}_m^{-1} \nabla_m \mathcal{E}\mathcal{L}_m(\bm{v}^*) - \mathcal{D}_m(\bm{v} - \bm{v}^*) \|
\leq \sup_{\bm{v} \in \Theta(\bm{r})} \| \mathcal{D}_m^{-1} \nabla_m^2 \mathcal{E}\mathcal{L}_m(\bm{v})^2 \mathcal{D}_m^{-1} - I_{p^*} \|_{\mathcal{D}_m^{-1}(\bm{r}), \mathcal{D}_m^{-1}(\bm{r})^T} \| \mathcal{D}_m \|_{\mathcal{D}_m^{-1}(\bm{r}), \mathcal{D}_m^{-1}(\bm{r})^T} \| \mathcal{D}_m \| \mathcal{D}_m \|
\leq \delta(\bm{r}) \bm{r}.
\]

Because of Lemma A.1 we know that

\[
\| \mathcal{D}_m(\Pi_{p^*} \bm{v}^* - \bm{v}_m^*) \| = \| \mathcal{D}_{E_{i^2}}(\Pi_{p^*} \bm{v}^* - \bm{v}_m^*) \|
\leq \| \mathcal{D}(\Pi_{p^*} \bm{v}^* - \phi^*) \| + \| \mathcal{D}(\bm{v}^* - E_{i^2} \bm{v}_m^*) \| \leq 2\bm{r}^*,
\]

such that \( \Pi_{p^*} \bm{v}_m^* \in \Theta_{0, m}(2\bm{r}^*) \), which gives

\[
\| \mathcal{D}_m(\bm{v}_m^* - \Pi_{p^*} \bm{v}^*) - \mathcal{D}_m^{-1} \nabla_{p+m} \mathcal{E}(\mathcal{L}(\Pi_{p^*} \bm{v}^*) - \mathcal{L}(\bm{v}_m^*)) \| \leq 2\delta(2\bm{r}^*) \bm{r}^*,
\]

from which we derive with the triangle inequality

\[
\| \mathcal{D}_m(\bm{v}_m^* - \bm{v}^*) \| \leq 2\delta(2\bm{r}^*) \bm{r}^* + \| \mathcal{D}_m^{-1} \nabla_{p+m} \mathcal{E}(\mathcal{L}(\Pi_{p^*} \bm{v}^*) - \mathcal{L}(\bm{v}_m^*)) \|.
\]

Because \( \nabla_{p+m} \mathcal{E}(\bm{v}_m^*) = 0 \) and \( \nabla \mathcal{E}(\bm{v}^*) = 0 \) we find

\[
\| \mathcal{D}_m^{-1} \nabla_{p+m} \mathcal{E}(\mathcal{L}(\Pi_{p^*} \bm{v}^*) - \mathcal{L}(\bm{v}_m^*)) \| = \| \mathcal{D}_m^{-1} \Pi_{p+m} \nabla \mathcal{E}(\mathcal{L}(\Pi_{p^*} \bm{v}^*) - \mathcal{L}(\bm{v}^*)) \|
\]
Using that \( \| \mathcal{D}(\Pi \psi^* - \psi^*) \| \leq r^* \) and condition (\(z\)) we may infer by the Taylor expansion that with some \( \lambda \in [0, 1] \)

\[
\left\| \mathcal{D}_m^{-1} \Pi_{p+m} \nabla \mathbb{E} \left( \mathcal{L}(\Pi_p \psi^*) - \mathcal{L}(\psi^*) \right) \right\|
\leq \left\| \mathcal{D}_m^{-1} A_m \left( E \nabla \mathbb{E} \left[ \mathcal{L}(\Pi_p \psi^*) \right] - A_m \psi^* \right) \right\|
= \left\| \mathcal{D}_m^{-1} A_m \psi^* \right\| + \left\| \mathcal{D}_m^{-1} \left( \nabla \mathbb{E} \left[ \mathcal{L}(\Pi_p \psi^*, \lambda \psi^*) \right] - A_m \right) \right\|.
\]

Due to assumption (\(z\)) the last sum is bounded by \( \alpha(m) + \tau(m) \). Together this gives that

\[
\left\| \mathcal{D}_m (\psi_m^* - \psi^*) \right\| = \alpha(m) + \tau(m) + 2\delta(2r^*)r^*.
\]

Finally we can represent

\[
\mathcal{D}_m^2 = \begin{pmatrix} D^2 & A \\ A^\top & H^2 \end{pmatrix}, \quad \tilde{\mathcal{D}}_m^2 = D^2 - A^\top H^{-2} A.
\]

and due to (\(I\)) this gives

\[
\left\| \tilde{\mathcal{D}}_m (\theta^*_m - \theta^*) \right\|^2 \leq \frac{1 + \rho}{1 - \rho} \left\| \mathcal{D}_m (\psi_m^* - \psi^*) \right\|^2.
\]

\[\square\]

Lemma A.3. Assume (\(vz\)) then

\[
\left\| I - \tilde{\mathcal{D}}_m^{-1} D^2 \mathcal{D}_m^{-1} \right\| \leq \frac{1 + \rho^2 + \beta^2(m)}{1 - \rho^2} \frac{\beta^2(m)}{1 - \beta^2(m)}.
\]

Proof. Take any \( \psi \in \mathbb{R}^p \) with \( \| \psi \| \leq 1 \) and note that with \( \psi = (\theta, \eta, z) \in l^2 \)

\[
\tilde{D}_m^{-1} \tilde{D}_m \psi
= \Pi_{\theta} \arg \max_{\psi \in l^2} \left\{ \theta^\top \tilde{\mathcal{D}}_m \psi - \| \mathcal{D} \psi \|^2 / 2 \right\}
= \Pi_{\theta} \arg \max_{\psi \in \mathbb{R}^{p+m}} \left\{ \theta^\top \tilde{\mathcal{D}}_m \psi - \| \mathcal{D}_m \psi \|^2 / 2 - \inf_{z} (\psi^\top A_m^\top \psi + \| \mathcal{H} \psi \|^2 / 2) \right\}
= \Pi_{\theta} \arg \max_{\psi \in \mathbb{R}^{p+m}} \left\{ \theta^\top \tilde{\mathcal{D}}_m \psi - \| \mathcal{D}_m \psi \|^2 / 2 - \| \mathcal{D}_m^{-1} A_m^\top \psi \|^2 / 2 \right\}
= \Pi_{\theta} \arg \max_{\psi \in \mathbb{R}^{p+m}} g(\psi) \overset{\text{def}}{=} \Pi_{\theta} \psi^*.
\]

Setting the gradient of \( g(\psi) \) equal to zero gives that the maximizer \( \psi^* \in \mathbb{R}^{p+m} \) satisfies

\[
\psi^* = \mathcal{D}_m^{-1} \left( I_p - \mathcal{D}_m^{-1} A_m \mathcal{H}_m^{-2} A_m^\top \mathcal{D}_m^{-1} \right)^{-1} \mathcal{D}_m^{-1} \Pi_{\theta} \tilde{\mathcal{D}}_m \psi,
\]
where $\Pi_\theta^\top : \mathbb{R}^p \to \mathbb{R}^{p+m}$ denotes the canonical embedding of $\mathbb{R}^p$ into $\mathbb{R}^{p+m}$. By assumption we have

$$\| (I_{p^*} - D_m^{-1} A_m H_m^{-2} A_m^\top D_m^{-1})^{-1} - I_{p^*} \| \leq \frac{\beta^2(m)}{1 - \beta^2(m)}.$$ 

Note that $\tilde{D}_m \Pi_\theta D_m^{-2} \Pi_\theta^\top \tilde{D}_m v = v$ which gives

$$\| (I - \tilde{D}_m \tilde{D}_m^{-2} \tilde{D}_m) v \| = \| v - \tilde{D}_m \Pi_\theta v^\circ \| = \| \tilde{D}_m \Pi_\theta D_m^{-2} \Pi_\theta^\top \tilde{D}_m v - \tilde{D}_m \Pi_\theta v^\circ \|$$

$$= \| \tilde{D}_m \Pi_\theta D_m^{-1} \left( (I_{p^*} - D_m^{-1} A_m H_m^{-2} A_m^\top D_m^{-1})^{-1} - I_{p^*} \right) D_m^{-2} \Pi_\theta^\top \tilde{D}_m v \|$$

$$\leq \frac{\beta^2(m)}{1 - \beta^2(m)} \| \tilde{D}_m \Pi_\theta D_m^{-2} \Pi_\theta^\top \tilde{D}_m v \|$$

$$= \frac{\beta^2(m)}{1 - \beta^2(m)} \| v \| = \frac{\beta^2(m)}{1 - \beta^2(m)}.$$

This implies

$$\| I - \tilde{D}_m^{-1} \tilde{D}_m^{-2} \tilde{D}_m \| \leq \| I - \tilde{D}_m \tilde{D}_m^{-2} \tilde{D}_m \| \| \tilde{D}_m^{-1} \tilde{D}_m^{-2} \tilde{D}_m \| \leq \frac{1 + \rho^2 + \beta^2(m)}{1 - \rho^2} \frac{\beta^2(m)}{1 - \beta^2(m)}.$$ 

\[\square\]

**Lemma A.4.** Assume $v \in \mathcal{K}$ then we get with $r^* = 4\mathcal{C}_{r^*} m / \beta$

$$\| I - \tilde{D}_m (v^*_m)^{-1} \tilde{D}_m (v^*_m)^2 \tilde{D}_m (v^*_m)^{-1} \| \leq \delta(r^*) \frac{1}{1 - 2\delta(r^*)}.$$ 

**Proof.** Denote $\tilde{D}_{mm} \overset{\text{def}}{=} \tilde{D}_m (v^*_m)$, $D_{mm} \overset{\text{def}}{=} D_m (v^*_m)$ and $\tilde{D}_m \overset{\text{def}}{=} \tilde{D}_m (v^*)$, $D_m \overset{\text{def}}{=} D_m (v^*)$. We simply calculate

$$\| I - \tilde{D}_{mm}^{-1} \tilde{D}_m^{-2} \tilde{D}_{mm} \| \leq \| \tilde{D}_{mm}^{-1} \tilde{D}_m^{-2} \tilde{D}_{mm} - I_p \| \| \tilde{D}_m^{-1} \tilde{D}_m^{-2} \tilde{D}_m^{-1} \|$.

Now we get with condition (L_0) and Lemma A.1

$$\| \tilde{D}_{mm}^{-1} \tilde{D}_m^{-2} \tilde{D}_{mm} - I_p \| = \| \tilde{D}_{mm} \left( \Pi_\theta D_m^{-2} \Pi_\theta^\top - \Pi_\theta D_m^{-2} \Pi_\theta^\top \right) \tilde{D}_{mm} \|$$

$$\leq \| I - \tilde{D}_{mm}^{-1} \tilde{D}_m^{-2} \tilde{D}_{mm} \| \leq \| I - \tilde{D}_{mm}^{-1} \tilde{D}_m^{-2} \tilde{D}_{mm} \| \| D_{mm}^{-2} D_{mm} \|$$

$$\leq \delta(r^*) \| D_{mm}^{-2} D_{mm} \|.$$ 

This implies

$$\| D_{mm}^{-2} D_{mm} \| \leq 1/(1 - \delta(r^*)),$$ 

$$\| \tilde{D}_{mm}^{-1} \tilde{D}_m^{-2} \tilde{D}_{mm}^{-1} \| \leq (1 - \delta(r^*)) / (1 - 2\delta(r^*)).$$
Together this gives the claim.

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