Integral representations of functions and
Addison-type series for mathematical constants

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Abstract

We generalize techniques of Addison to a vastly larger context. We obtain integral representations in terms of the first periodic Bernoulli polynomial for a number of important special functions including the Lerch zeta, polylogarithm, Dirichlet $L$- and Clausen functions. These results then enable a variety of Addison-type series representations of functions. Moreover, we obtain integral and Addison-type series for a variety of mathematical constants.

Key words and phrases
Lerch zeta function, Hurwitz zeta function, polylogarithm function, Dirichlet $L$ functions, Clausen functions, generalized Somos constants, Glaisher-Kinkelin constant, Kinkelin constant, Stieltjes constants, series representation

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Introduction and statement of results

Very recently we have shown how a method of Addison [2] for series representations of the Euler constant \( \gamma \) may be generalized in several directions [6]. In particular, we demonstrated generalizations to the important Stieltjes constants \( \gamma_k(a) \) of analytic number theory. In this paper, we present integral representations for a variety of very useful special functions. These representations are such that Addison-type series for function values and other mathematical constants may be developed and we illustrate this in several instances. The presentation demonstrates the wide applicability of our methods.

We let \( \Phi(z, s, a) \) be the Lerch zeta function and \( \text{Li}_s(z) = z\Phi(z, s, 1) \) be the polylogarithm function [24]. Integral representations, including with contour integrals, are known for \( \Phi \), permitting, among other topics, analytic continuation. We let \( \zeta(s, a) = \Phi(1, s, a) \) be the Hurwitz zeta function, \( \zeta(s) = \zeta(s, 1) \) be the Riemann zeta function [12, 19, 21, 26], \( \psi = \Gamma'/\Gamma \) be the digamma function (e.g., [1]), \( \psi^{(k)} \) be the polygamma functions [1], and \( _pF_q \) the generalized hypergeometric function [3].

We let \( P_1(x) = B_1(x - \lfloor x \rfloor) = x - \lfloor x \rfloor - 1/2 \) be the first periodized Bernoulli polynomial, and \( \{x\} = x - \lfloor x \rfloor \) be the fractional part of \( x \). Being periodic, \( P_1 \) has the Fourier series [1] (p. 805)

\[
P_1(x) = -\sum_{j=1}^{\infty} \frac{\sin 2\pi j x}{\pi j}.
\]

(1.1)

Addison [2] gave an interesting series representation for \( \gamma \) [15],

\[
\gamma = \frac{1}{2} + \frac{1}{2} \sum_{n=1}^{\infty} n \sum_{m=2^{n-1}}^{2^n-1} \frac{1}{2m(m+1)(2m+1)} = 1 - \frac{1}{2} \sum_{n=1}^{\infty} n \sum_{m=2^{n-1}+1}^{2^n} \frac{1}{m(2m-1)}.
\]

(1.2)
His approach uses an integral representation for the Riemann zeta function in terms of $P_1$. Various transformations of Addison’s result are possible, including \cite{15, 18, 22, 27}

$$\gamma = 1 + \sum_{j=3}^{\infty} (-1)^j \frac{\ln(j-1)/\ln 2}{j} = \sum_{j=1}^{\infty} (-1)^j \frac{\ln j/\ln 2}{j} = 1 + \sum_{j=1}^{\infty} \frac{(-1)^j}{j} \left\{ \frac{\ln j}{\ln 2} \right\}. \quad (1.3)$$

Therefore, Addison’s result can be connected to other topics, including the binary expansion of $\gamma$.

In this paper, we present many significant generalizations of Addison’s result. As with the example (1.3), our results can then be connected with the binary or $n$-ary expansion of function values and other fundamental mathematical constants.

**Proposition 1.** We have

$$\Phi(z, s, a) = \frac{1}{a^s} + \frac{z}{2(a+1)^s} + \int_1^{\infty} \frac{z^x}{(x+a)^s} dx + \int_1^{\infty} \left[ \frac{z^x \ln z}{(x+a)^s} - \frac{sz^x}{(x+a)^{s+1}} \right] P_1(x) dx. \quad (1.4)$$

This representation holds for $a \in C/\{-1, -2, \ldots\}$ and $s \in C$ when $|z| < 1$ and for $\Re s > 1$ when $|z| = 1$. From this result follows many others, including for the polylogarithm function:

**Corollary 1.** We have

$$\text{Li}_s(z) = \frac{z}{2} + \int_1^{\infty} \frac{z^x}{x^s} dx + \int_1^{\infty} \left[ \frac{z^x \ln z}{x^s} - s \frac{z^x}{x^{s+1}} \right] P_1(x) dx. \quad (1.5)$$

Proposition 1 and Corollary 1 are applied in the following to develop integral representations for other special functions and series representations for fundamental mathematical constants.

An immediate consequence of Proposition 1 is a connection with a specific generalized hypergeometric function $k+1 F_k$. We have for integers $k \geq 0,$
Corollary 2

\[ \Phi(z, k, a) = \frac{1}{a^k} k F_k(1, a, \ldots, a; a + 1, \ldots, a + 1; z) \]

\[ = \frac{1}{a^k} + \frac{z}{2(a + 1)^k} + \int_1^\infty \frac{z^x}{(x + a)^k} dx + \int_1^\infty \left[ \frac{z^x \ln z}{(x + a)^k} - \frac{\frac{kz^x}{(x + a)^{k+1}}}{(x + a)^{k+1}} \right] P_i(x) dx. \quad (1.6) \]

In turn, Corollary 1 implies other results. As an example, we have the following for the dilogarithm and polylogarithm functions.

Corollary 3. Let \( \text{Re} \alpha > -1 \). Then (a)

\[ \int_0^1 t^{\alpha-1} \text{Li}_2(t) dt = \frac{1}{\alpha} \zeta(2) - \frac{1}{\alpha^2} [\psi(\alpha + 1) + \gamma], \quad (1.7a) \]

recovering a known result (e.g., \([21]\), p. 109). Let \( _2F_1 \) be the Gauss hypergeometric function, \( n \geq 2 \) an integer, and \( h(s) \equiv -\zeta(s)/s + (s + 1)/[2s(s - 1)] \). Then we have (b)

\[ \int_0^1 t^{\alpha-1} \text{Li}_n(t) dt = \frac{1}{2(\alpha + 1)} + \frac{1}{n} _2F_1(1, n; n + 1; -\alpha) + (-1)^{n+1} \sum_{j=1}^n \frac{(-1)^j j}{\alpha^{n-j+1}} h(j) \]

\[ + (-1)^{n+1} \frac{1}{\alpha^n} \left[ \psi(\alpha + 1) - \ln(\alpha + 1) + \frac{1}{2(\alpha + 1)} \right]. \quad (1.7b) \]

We may note the value \( \zeta(3) \) on the right side of (1.7a) when \( \alpha \to 0 \), owing to the expansion \( \psi(\alpha + 1) = -\gamma + \zeta(2)\alpha - \zeta(3)\alpha^2 + \zeta(4)\alpha^3 + O(\alpha^4) \). Indeed, the proof of Corollary 3 shows how other general results for integrals \( \int_0^1 t^{\alpha-1} \text{Li}_n(t) dt \) may be obtained.

We recall definitions of the generalized Clausen function \( \text{Cl}_n \) for nonnegative integers \( n \):

\[ \text{Cl}_n(\theta) = \text{Im} \text{Li}_n(e^{i\theta}), \quad \text{for } n \text{ even}, \]

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\[ = \text{Re Li}_n(e^{i\theta}), \quad \text{for } n \text{ odd}. \quad (1.8) \]

Therefore, from Corollary 1 we obtain the following.

**Corollary 4.** We have (a)

for \( n \) even:

\[ \text{Cl}_n(\theta) = \frac{1}{2} \sin \theta + \int_1^\infty \frac{\sin x\theta}{x^n} dx + \int_1^\infty \frac{1}{x^n} [\theta \cos x\theta - \frac{n}{x} \sin x\theta] P_1(x) dx, \quad (1.9a) \]

for \( n \) odd:

\[ \text{Cl}_n(\theta) = \frac{1}{2} \cos \theta + \int_1^\infty \frac{\cos x\theta}{x^n} dx - \int_1^\infty \frac{1}{x^n} [\theta \sin x\theta - \frac{n}{x} \cos x\theta] P_1(x) dx, \quad (1.9b) \]

(b)

\[ \text{Cl}_2(\theta) = \frac{3}{2} \sin \theta - \theta \text{Ci}(\theta) + \int_1^\infty \frac{1}{x^2} [\theta \cos x\theta - \frac{2}{x} \sin x\theta] P_1(x) dx, \quad (1.10) \]

and (c) for the Catalan constant \( G = \text{Cl}_2(\pi/2) \),

\[ G = \frac{3}{2} - \frac{\pi}{2} \text{Ci} \left( \frac{\pi}{2} \right) + \int_1^\infty \frac{1}{x^2} \left[ \frac{\pi}{2} \cos \left( \frac{\pi x}{2} \right) - \frac{2}{x} \sin \left( \frac{\pi x}{2} \right) \right] P_1(x) dx. \quad (1.11) \]

In parts (b) and (c), \( \text{Ci}(z) \equiv -\int_z^\infty (\cos t)(dt/t) \) is the cosine integral.

The representations of parts (a) and (b) permit the recovery of known properties, including the duplication formula \( \frac{1}{2} \text{Cl}_2(2\theta) = \text{Cl}_2(\theta) - \text{Cl}_2(\pi - \theta) \). The Clausen functions are useful in mathematical physics (e.g., [9]).

In [6] we presented series representations for the digamma function. In particular, since \( H_n = \psi(n + 1) + \gamma, \gamma = -\psi(1) \), for integers \( n \geq 1 \), we effectively obtained series representations for the harmonic numbers \( H_n \equiv \sum_{p=1}^n 1/p \). Therefore, by integration
we obtain results for \( \ln \Gamma \). For instance, we have

**Corollary 5.** We have for \( \Re z > 0 \),

\[
\ln \Gamma(z) = \left( z - \frac{1}{2} \right) - z + 1 - \frac{1}{4} \sum_{n=0}^{\infty} \frac{1}{2^n} \sum_{j=0}^{\infty} \left[ \ln(z+bj) - \ln(1+bj) + \ln(z+b+2bj) - \ln(1+b+2bj) \right] - 2 \ln(2z + b + 2bj) + 2 \ln(2 + b + 2bj)]_{b=2^{-n}},
\]

that follows from Proposition 3(a) of [6]. Thereby, we obtain series for constants such as \( \ln \sqrt{\pi} \). Additionally, if we were to differentiate, for example, (1.8) in [6], we obtain series for polygammic constants such as \( \psi'(1) = \zeta(2) \). More generally, we may find series including for

\[
\psi^{(j)}(1) = (-1)^{j+1}j! \zeta(j + 1).
\]

(1.13)

The numbers of (1.13) are cases of the relation \( \psi^{(n)}(x) = (-1)^{n+1}n! \zeta(n + 1, x) \), and, as is well known, are closely related to generalized harmonic numbers \( H_n^{(r)} \equiv \sum_{j=1}^{n} \frac{1}{j^r}, H_n \equiv H_n^{(1)} \):

\[
H_n^{(r)} = \left( -1 \right)^{r-1} \left( \frac{1}{r-1} \right)! \left[ \psi^{(r-1)}(n + 1) - \psi^{(r-1)}(1) \right].
\]

(1.14)

We have

**Proposition 2.** Let \( \Re s > 0, \Re a > 0 \), and \( k \geq 2 \) be an integer. Then we have the series representation

\[
\zeta'(s,a) + \frac{a^{1-s}}{(s-1)^2} + \frac{\ln a}{2a^s} + \frac{a^{1-s}}{s-1} \ln a
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{k^n} \sum_{j=0}^{\infty} \left\{ \frac{1}{2} \left( \frac{1}{k} - 1 \right) \left[ \frac{\ln(bj + a)}{(bj + a)^s} + \frac{\ln(b(j+1) + a)}{(b(j+1) + a)^s} \right] + \frac{1}{k} \sum_{m=1}^{k-1} \frac{\ln[b(j + m/k) + a]}{(b(j + m/k) + a)^s} \right\}_{b=2^{-n}}.
\]

(1.15)
This Proposition engenders many Corollaries. Taking the limit as $s \rightarrow 1$ in (1.15) gives

**Corollary 4.** We have for $\text{Re } a > 0$

\[
-\gamma_1(a) + \frac{1}{2} \ln a = \sum_{n=0}^{\infty} \frac{1}{k^n} \sum_{j=0}^{\infty} \left\{ \frac{1}{2} \left( \frac{1}{k} - 1 \right) \left[ \frac{\ln(bj + a)}{(bj + a)^s} + \frac{\ln(b(j + 1) + a)}{(b(j + 1) + a)^s} \right] \right. \\
\left. + \frac{1}{k} \sum_{m=1}^{k-1} \frac{\ln[b(j + m/k) + a]}{[b(j + m/k) + a]} \right\}_{b=k-n},
\]

(1.16)

where $\gamma_1(a)$ is the first Stieltjes constant [8, 11, 25, 28]. With (1.15) in hand, one may differentiate with respect to $s$ to find higher order derivatives, and thus series representations for higher order Stieltjes and other constants. As a simple example, we have

**Corollary 7.** For $\text{Re } s > 0$, $\text{Re } a > 0$, and $k \geq 2$ an integer, we have

\[
\zeta''(s, a) - 2 \frac{a^{1-s}}{(s-1)^2} \ln a - \frac{2a^{1-s}}{(s-1)^3} - \frac{\ln^2 a}{2s} - \frac{a^{1-s}}{s-1} \ln^2 a \\
= - \sum_{n=0}^{\infty} \frac{1}{k^n} \sum_{j=0}^{\infty} \left\{ \frac{1}{2} \left( \frac{1}{k} - 1 \right) \left[ \frac{\ln^2(bj + a)}{(bj + a)^s} + \frac{\ln^2(b(j + 1) + a)}{(b(j + 1) + a)^s} \right] + \frac{1}{k} \sum_{m=1}^{k-1} \frac{\ln^2[b(j + m/k) + a]}{[b(j + m/k) + a]^s} \right\}_{b=k-n}.
\]

(1.17)

The following cases for $\zeta'(s)$ could be presented as additional Corollaries. However, due to the celebrated case at $s = 2$, for instance, for the Glaisher-Kinkelin constant $A$ [16, 20] we state the following.

**Proposition 3.** For $\text{Re } s > 0$ we have (a)

\[
\zeta'(s) + \frac{1}{(s-1)^2} = - \frac{1}{4} \sum_{n=0}^{\infty} \frac{1}{2^n} \sum_{j=0}^{\infty} \left[ \frac{2 \ln[b(j + 1/2) + 1]}{[b(j + 1/2) + 1]^s} - \frac{\ln(bj + 1)}{(bj + 1)^s} - \frac{\ln[b(j + 1) + 1]}{[b(j + 1) + 1]^s} \right]_{b=2-n},
\]

(1.18a)
(b) \[
\zeta'(s) + \frac{1}{(s-1)^2} = \frac{1}{3} \sum_{n=0}^{\infty} \frac{1}{3^n} \sum_{j=0}^{\infty} \left[ \frac{\ln[b(j + 1/3) + 1]}{[b(j + 1/3) + 1]^s} - \frac{\ln(bj + 1)}{(bj + 1)^s} - \frac{\ln[b(j + 1) + 1]}{[b(j + 1) + 1]^s} \right]
+ \frac{\ln[b(j + 2/3) + 1]}{[b(j + 2/3) + 1]^s} \bmod 3^n,
\]
and (c) \[
\zeta'(s) + \frac{1}{(s-1)^2} = \frac{1}{8} \sum_{n=0}^{\infty} \frac{1}{4^n} \sum_{j=0}^{\infty} \left[ \frac{2 \ln[b(j + 1/4) + 1]}{[b(j + 1/4) + 1]^s} - \frac{3 \ln(bj + 1)}{(bj + 1)^s} - \frac{3 \ln[b(j + 1) + 1]}{[b(j + 1) + 1]^s} \right]
+ \frac{2 \ln[b(j + 1/2) + 1]}{[b(j + 1/2) + 1]^s} + \frac{2 \ln[b(j + 3/4) + 1]}{[b(j + 3/4) + 1]^s} \bmod 3^n.
\]
We recall that \(\ln A = -\zeta'(2)/\pi^2 + [\ln(2\pi) + \gamma]/12\), or equivalently, due to the functional equation of the Riemann zeta function, \(\zeta'(-1) = 1/12 - \ln A\). Therefore, we have found a family of Addison-type series representations for \(\ln A\). Similarly from our results follow other Corollaries for constants such as \(\zeta'(-2) = -\zeta(3)/4\pi^2\). Other results for the Kinkelin constant \(k = \zeta'(-1)\) are relegated to Appendix A.

The next Proposition describes that a vast array of constants may be written as Addison-type series. We have

**Proposition 4.** Addison-type series may be developed for all of the following quantities:

\[
\int_0^x z^n \psi^{(m)}(a + bz)dz, \quad \int_0^x z^n \psi(a + bz)dz, \quad \int_0^x z^n \ln \Gamma(a + bz)dz. \tag{1.19}
\]

Here, \(n \geq 0\) and \(m \geq 1\) are integers, and \(a, b \in \mathbb{R}\).

**Examples** of this Proposition include the constants

\[
\int_0^{1/2} \ln \Gamma(z)dz = \frac{5}{24} \ln 2 + \ln(A^{3/2} \pi^{1/4}), \tag{1.20a}
\]
\[ \int_0^1 z \ln \Gamma(z) dz = \ln \left( \frac{(2\pi)^{1/4}}{A} \right), \quad (1.20b) \]
\[ \int_0^1 z^2 \ln \Gamma(z) dz = \frac{1}{6} \ln(2\pi) - \ln A + \frac{\zeta(3)}{4\pi^2}, \quad (1.20c) \]
\[ \int_0^{1/2} z \ln \Gamma(z) dz = \frac{1}{96} \ln(16A^{24}\pi^6) - \frac{7}{32} \frac{\zeta(3)}{\pi^2}, \quad (1.20d) \]

and
\[ \int_0^{1/2} z^2 \ln \Gamma(z) dz = \frac{1}{5760}[-55+62 \ln 2+720 \ln A+120 \ln \pi-(540/\pi^2)\zeta(3)-3600\zeta'(-3)]. \quad (1.20e) \]

An explicit example coming from Corollary 5 is:

**Corollary 6.** We have
\[ \int_0^1 \ln \Gamma(z) dz = \ln \sqrt{2\pi} = \frac{3}{4} - \frac{1}{4} \sum_{n=0}^\infty \frac{1}{4^n} \sum_{j=0}^\infty \left\{ j[\ln(bj+1) - \ln(bj)] + (j+1)[\ln(bj+b+1) - \ln(bj+b)] - (2j+1)[\ln(2bj+b+2) - \ln(2bj+b)] \right\}_{b=2^{-n}}. \quad (1.21) \]

We let the hyperfactorial function \[^4\] for \( x \geq 0 \)
\[ \ln K(x) = \frac{1}{2} (x^2 - x) - \frac{x}{2} \ln 2\pi + \int_0^x \ln \Gamma(y) dy, \quad (1.22) \]
satisfying \( K(0) = K(1) = K(2) = 1 \) and for \( x > 0 \), \( K(x+1) = x^x K(x) \). Thus, for nonnegative integers \( n \), \( K(n+1) = 1^{1/2} \cdots n^n \). In addition, we have the value \( K(1/2) = A^{3/2}/(2^{1/4}e^{1/8}) \). As follows from Proposition 4 we have

**Corollary 7.** The hyperfactorial function \( K \) has an integral representation with \( P_1 \), and its values may be written as Addison-type series.

We introduce the generalized Somos constants \[^{23}\] for \( t > 1 \)
\[ \sigma_t = \prod_{n=1}^\infty n^{1/t^n}. \quad (1.23) \]
The Somos constant \( \sigma_2 \simeq 1.66169 \) has several representations, including Ramanujan’s nested radical expression. We have

**Proposition 5.** We have (a) the integral representations

\[
\ln \sigma_t = \int_1^\infty \frac{\ln x}{t^x} dx + \int_1^\infty \left( \frac{1}{xt^x} - \frac{(\ln t) \ln x}{t^x} \right) P_1(x) dx, \tag{1.24a}
\]

and (b) the series formulas

\[
\ln \sigma_t = \frac{1}{t-1} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \text{Li}_k \left( \frac{1}{t} \right) = \frac{1}{t-1} \sum_{k=1}^{\infty} \frac{1}{k} \left[ t \text{Li}_k \left( \frac{1}{t} \right) - 1 \right]. \tag{1.25}
\]

On the right side of (1.24a), we may identify \( \int_1^\infty t^{-x} \ln x \ dx = \Gamma(0, \ln t)/\ln t \), with \( \Gamma(x, y) \) the incomplete Gamma function and \( \Gamma(0, x) = -\text{Ei}(-x) \), where \( \text{Ei} \) is the exponential integral (e.g., [1], Ch. 5). On the right side of (1.25), we have \( \text{Li}_1(z) = -\ln(1 - z) \).

**Corollary 10.** We have

\[
\lim_{z \to 0^+} [\ln z + z \ln \sigma_{z+1}] = -\gamma.
\]

Somos’ quadratic recurrence is \( g_n = ng_{n-1}^2 \) with \( g_0 = 1 \). We have

**Corollary 11.** The solution \( \ln g_n \) of Somos’ quadratic recurrence has an integral representation with \( P_1 \).

Our methods directly extend to Dirichlet \( L \) functions, as these may be written as a linear combination of Hurwitz zeta functions. For instance, for \( \chi \) a principal (nonprincipal) character modulo \( m \) and \( \text{Re } s \geq 1 \) (\( \text{Re } s \geq 0 \)) we have

\[
L(s, \chi) = \sum_{k=1}^{\infty} \frac{\chi(k)}{k^s} = \frac{1}{m} \sum_{k=1}^{m} \chi(k) \zeta \left( s, \frac{k}{m} \right). \tag{1.26}
\]
We detail the case of the Dirichlet $L$-function with a character modulo 4 defined by

$$L(s) \equiv \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^s}, \quad \text{Re } s > 1,$$

(1.27)

This function can be easily expressed as

$$L(s) = 4^{-s}[\zeta(s, 1/4) - \zeta(s, 3/4)] = 1 + 4^{-s}[\zeta(s, 5/4) - \zeta(s, 3/4)], \quad \text{Re } s \geq 0,$$

(1.28)

and we have the particular values for nonnegative integers $m$

$$L(2m + 1) = -\frac{(2\pi)^{2m+1}}{2(2m+1)!} B_{2m+1}(1/4),$$

(1.29)

where $B_m$ is the $m$th Bernoulli polynomial. Generally the values of $L$ at odd or even integer argument may be expressed in terms of Euler or Bernoulli polynomials at rational argument and these in turn expressed in terms of the Hurwitz zeta function. Therefore we may in this way obtain many Addison-type series representations for $L(2m)$ and $L(2m + 1)$. These include the special cases of $L(1) = \pi/4$, $L(2) = G \simeq 0.91596559$, Catalan’s constant, and $L(3) = \pi^3/32$.

We have

**Proposition 6.** For $\text{Re } s \geq 0$ we have

$$L(s) = \frac{1}{2} (1 - 3^{-s}) + \frac{1}{4} \frac{1}{(s-1)} (1 - 3^{1-s})$$

$$+ 4^{-s-1} \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{2^n} \left[ \frac{1}{(bj + 1/4)^s} - \frac{2}{[b(j + 1/2) + 1/4]^s} + \frac{1}{[b(j + 1) + 1/4]^s} ight]$$

$$- \frac{1}{(bj + 3/4)^s} + \frac{2}{[b(j + 1/2) + 3/4]^s} - \frac{1}{[b(j + 1) + 3/4]^s} \bigg|_{b=2^{-n}}.$$  

(1.30)
This result itself gives many Corollaries. Besides the above-mentioned examples of \( G = L(2) \) and \( \pi/4 = L(1) \), by differentiation we may obtain an Addison-type series for \( L'(s) = -4^{-s}(\ln 4)[\zeta(s, 1/4) - \zeta(s, 3/4)] + 4^{-s}[\zeta'(s, 1/4) - \zeta'(s, 3/4)] \), giving a series representation for the value

\[
L'(1) = \frac{1}{4} \left\{ (\ln 4) \left[ \psi \left( \frac{3}{4} \right) - \psi \left( \frac{1}{4} \right) \right] + \gamma_1 \left( \frac{3}{4} \right) - \gamma_1 \left( \frac{1}{4} \right) \right\} = \frac{1}{4} \left[ 2\pi \ln 2 + \gamma_1 \left( \frac{3}{4} \right) - \gamma_1 \left( \frac{1}{4} \right) \right],
\]

where we used the reflection formula of the digamma function. In turn, we have the value [10]

\[
\gamma_1 \left( \frac{3}{4} \right) - \gamma_1 \left( \frac{1}{4} \right) = \pi \left\{ \ln 8 + \ln \left( \frac{\Gamma \left( \frac{1}{4} \right)}{\Gamma \left( \frac{3}{4} \right)} \right) \right\}.
\]

We let for \( \Re s > 1 \) and \( a \notin \{-1, -2, \ldots\} \)

\[
H(s, a) \equiv \sum_{n=1}^{\infty} \frac{H_n}{(n + a)^s}.
\]  

(1.31)

There are at least two significant motivations for this function. (i) it can be analytically continued, and for \( a = 0 \) can be expected to share some of the properties of the derivative of the Riemann zeta function. (ii) At positive integer values of \( a \) it yields Euler sums (e.g., [7, 14]), and these have been of interest for some time. Simple examples include

\[
\sum_{n=1}^{\infty} \frac{H_n}{(n + 1)^2} = \zeta(3), \quad \sum_{n=1}^{\infty} \frac{H_n}{(n + 1)^3} = \frac{1}{4} \zeta(4).
\]  

(1.32)

We have
Proposition 7. We have

\[
H(s, a) = \frac{1}{2(a+1)^s} + \int_1^\infty \frac{\psi(x+1) + \gamma}{(x+a)^s} dx + \int_1^\infty \left[ \frac{\psi(x+1)}{(x+a)^s} - s \frac{\psi(x+1) + \gamma}{(x+a)^{s+1}} \right] P_1(x) dx.
\]

(1.33)

We recall some relations useful in the following.

From the representation (cf. e.g., [26], p. 14),

\[
\zeta(s) = \frac{1}{s-1} + \frac{1}{2} - s \int_1^\infty x^{-(s+1)} P_1(x) dx, \quad \text{Re } s > 1,
\]

(1.34)

we obtain

\[
\zeta^{(n)}(s) = \frac{(-1)^n n!}{(s-1)^{n+1}} + (-1)^n n \int_1^\infty \frac{P_1(x)}{x^{s+1}} \ln^{n-1} x \ dx - (-1)^n s \int_1^\infty \frac{P_1(x)}{x^{s+1}} \ln^n x \ dx.
\]

(1.35)

Equation (1.34) extends to

\[
\zeta(s, a) = \frac{a^{-s}}{2} + \frac{a^{1-s}}{s-1} - s \int_0^\infty \frac{P_1(x)}{(x+a)^{s+1}} dx, \quad \text{Re } s > -1.
\]

(1.36)

The defining Laurent expansion for the Stieltjes constants [8, 11, 25, 28] is

\[
\zeta(s, a) = \frac{1}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k \gamma_k(a)}{k!} (s-1)^k, \quad s \neq 1,
\]

(1.37)

wherein \( \gamma_0(a) = -\psi(a) \), and we have the connection to differences of logarithmic sums

\[
\gamma_j(a) - \gamma_j(b) = \sum_{n=0}^{\infty} \left[ \frac{\ln^j(n+a)}{n+a} - \frac{\ln^j(n+b)}{n+b} \right], \quad j \geq 1,
\]

(1.38)

where \( a, b \in C/\{ -1, -2, \ldots \} \).

Proof of Propositions
We include here proofs for the Propositions and Corollaries that are not already covered in the previous section.

**Proposition 1.** We apply a summation formula of [26] (p. 14) for functions $\phi(x) \in C^1[\alpha, \beta]$:

$$
\sum_{\alpha<n\leq\beta} \phi(n) = \int_{\alpha}^{\beta} \phi(x)dx + \int_{\alpha}^{\beta} \phi'(x)P_1(x)dx + P_1(\alpha)\phi(\alpha) - P_1(\beta)\phi(\beta). \tag{2.1}
$$

We take $\phi(x) = z^x/(x+a)^s$ and $\alpha$ and $\beta$ to be positive integers, so that $P_1(\alpha) = P_1(\beta) = -1/2$. Then we have

$$
\sum_{n=\alpha+1}^{\beta} \frac{z^n}{(n+a)^s} = \int_{\alpha}^{\beta} \frac{z^x}{(x+a)^s}dx + \int_{\alpha}^{\beta} \left[ \frac{z^x \ln z}{(x+a)^s} - \frac{sz^x}{(x+a)^{s+1}} \right] P_1(x)dx
$$

$$
+ \frac{1}{2} \left[ \frac{z^\beta}{(\beta+a)^s} - \frac{z^\alpha}{(\alpha+a)^s} \right]. \tag{2.2}
$$

In order to obtain the Proposition we take $\alpha = 1$, $\beta \to \infty$, and add to both sides $1/a^s + z/(a+1)^s$, using

$$
\sum_{n=2}^{\infty} \frac{z^n}{(n+a)^s} + \frac{1}{a^s} + \frac{z}{(a+1)^s} = \Phi(z,s,a). \tag{2.3}
$$

**Corollary 1.** For the Corollary we put $a = 1$ in (1.4), make a simple change of variable, and use the 1-periodicity of $P_1$, finding

$$
\Phi(z,s,1) = 1 + \frac{z}{2s+1} + \int_{1}^{\infty} \frac{z^{y-1}}{y^s}dy + \int_{1}^{\infty} \left[ \frac{z^{y-1} \ln z}{y^s} - \frac{sz^{y-1}}{y^{s+1}} \right] P_1(y)dy
$$

$$
= 1 + \frac{z}{2s+1} + \int_{1}^{\infty} \frac{z^{y-1}}{y^s}dy + \int_{1}^{\infty} \left[ \frac{z^{y-1} \ln z}{y^s} - \frac{sz^{y-1}}{y^{s+1}} \right] P_1(y)dy
$$

$$
- \int_{1}^{2} \frac{z^{y-1}}{y^s}dy - \int_{1}^{2} \left[ \frac{z^{y-1} \ln z}{y^s} - \frac{sz^{y-1}}{y^{s+1}} \right] \left( y - \frac{3}{2} \right)dy. \tag{2.4}
$$
The second line of this equation, using integration by parts, evaluates to \(-z^{y-1}y^{-s}(y-3/2)\)|_{y=1} = -1/2 - z/2^{s+1}, giving the Corollary.

**Corollary 3.** By using Corollary 1 and interchanging integrations we have

\[
\int_0^1 t^{\alpha-1} \text{Li}_s(t) \, dt = \frac{1}{2(\alpha + 1)} + \int_1^\infty \frac{1}{x^s(x + \alpha)} \, dx - \int_1^\infty \left[ \frac{1}{x^s(x + \alpha)^2} - \frac{s}{x^{s+1}(x + \alpha)} \right] P_1(x) \, dx
\]

\[
= \frac{1}{2(\alpha + 1)} + \frac{1}{s} \text{$_2$F$_1$(1, s; s+1; -\alpha)} - \int_1^\infty \left[ \frac{1}{x^s(x + \alpha)^2} - \frac{s}{x^{s+1}(x + \alpha)} \right] P_1(x) \, dx. \tag{2.5}
\]

Therefore, recalling that \(\text{$_2$F$_1$(1, 1; 2; -\alpha)} = \ln(1 + \alpha)/\alpha\) and using its derivative with respect to \(\alpha\), at \(s = 2\) we obtain

\[
\int_0^1 t^{\alpha-1} \text{Li}_2(t) \, dt = \frac{1}{2(\alpha + 1)} + \frac{1}{\alpha} - \frac{1}{\alpha^2} \ln(\alpha + 1) - \int_1^\infty \frac{P_1(x) \, dx}{x^2(x + \alpha)^2} - 2 \int_1^\infty \frac{P_1(x) \, dx}{x^3(x + \alpha)}. \tag{2.6}
\]

We then use partial fractions for the integrals on the right side of this equation, together with relation (1.34):

\[
\int_1^\infty \frac{P_1(x) \, dx}{x^2(x + \alpha)^2} + 2 \int_1^\infty \frac{P_1(x) \, dx}{x^3(x + \alpha)} = \int_1^\infty \left[ \frac{2}{\alpha x^3} - \frac{1}{\alpha^2 x^2} + \frac{1}{\alpha^2 (x + \alpha)^2} \right] P_1(x) \, dx
\]

\[
= \frac{2}{\alpha} \left[ \frac{3}{4} - \frac{\zeta(2)}{2} \right] + \frac{1}{\alpha^2} \left( \gamma - \frac{1}{2} \right) + \frac{1}{\alpha^2} \int_1^\infty \frac{P_1(x) \, dx}{(x + \alpha)^2}. \tag{2.7}
\]

The proof of (a) is completed by using the representation (e.g., [6] (2.20))

\[
\psi(a + 1) = \ln(a + 1) - \frac{1}{2(a + 1)} + \int_1^\infty \frac{P_1(y) \, dy}{(y + a)^2}. \tag{2.8}
\]

For part (b), we note that the function \(h(s) = \int_1^\infty x^{-s+1} P_1(x) \, dx\) comes from (1.34). We make use of (2.5) together with the partial fractions decomposition

\[
\frac{1}{x^n(x + \alpha)^2} + \frac{n}{x^{n+1}(x + \alpha)} = (-1)^n \left[ \frac{1}{\alpha^n(x + \alpha)^2} + \sum_{j=1}^n \frac{(-1)^{j+1} \alpha^{n-j+1}}{\alpha^{n-j+1}(x + \alpha)^{j+1}} \right]. \tag{2.9}
\]
Using again the integral representation (2.8), the rest of the Corollary follows.

Remarks. Integrations or differentiations of (2.9) yield families of other relations.

The function of (1.7b) may be written as

\[
\frac{1}{n} \, _2F_1(1, n; n+1; -\alpha) = \frac{(-1)^{n+1}}{\alpha^n} \ln(1 + \alpha) - \sum_{j=1}^{n-1} \frac{(-1)^j}{(n-j)\alpha^j}.
\]  

(2.10)

The \(_2F_1\) function of (2.5) is transformable as

\[
\frac{1}{s} \, _2F_1(1, s; s+1; -\alpha) = \frac{1}{s(1+\alpha)} \, _2F_1\left(1, 1; s+1; \frac{\alpha}{1+\alpha}\right).
\]  

(2.11)

We have the special cases

\[
\int_0^1 t^{\alpha-1} \text{Li}_1(t)\,dt = - \int_0^1 t^{\alpha-1} \ln(1 - t)\,dt = \frac{1}{\alpha} [\psi(\alpha+1) + \gamma], \quad \text{Re } \alpha > 0,
\]  

(2.12)

and

\[
\int_0^1 \frac{1}{t} \text{Li}_s(t)\,dt = \zeta(s+1), \quad \text{Re } s > 0.
\]  

(2.13)

The latter may be easily verified by term-by-term integration of the series form of \(\text{Li}_s\).

**Proposition 2.** We have from (1.36) that

\[
\zeta'(s,a) = -\frac{\ln a}{2a^s} - \frac{a^{1-s}}{(s-1)^2} - \frac{a^{1-s} \ln a}{(s-1)} - \int_0^\infty \frac{P_1(x)\,dx}{(x+a)^{s+1}} + s \int_0^\infty \frac{P_1(x) \ln(x+a)}{(x+a)^{s+1}}\,dx.
\]  

(2.14)

We make use of the functions for \(k \geq 2\) and \(f(x) = -P_1(x)\),

\[
g_k(x) = f(x) - \frac{1}{k} f(kx),
\]  

(2.15)

with \(\sum_{n=0}^\infty \frac{g_n(k^nx)}{k^n} = f(x)\). Then

\[
\zeta'(s,a) + \frac{\ln a}{2a^s} + \frac{a^{1-s}}{(s-1)^2} + \frac{a^{1-s} \ln a}{(s-1)}
\]
\[
\sum_{n=0}^{\infty} \frac{1}{k^n} \int_0^\infty \frac{[1 - s \ln(x + a)]}{(x + a)^{s+1}} g_k(k^n x) \, dx
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{k^{2n}} \int_0^\infty \frac{[1 - s \ln(k^{-n} y + a)]}{(k^{-n} y + a)^{s+1}} g_k(y) \, dy
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{k^{2n}} \sum_{j=0}^{\infty} \int_j^{j+1} \frac{[1 - s \ln(k^{-n} y + a)]}{(k^{-n} y + a)^{s+1}} g_k(y) \, dy
\]

In the last step we have used the values of \( g_k(x) \) on subintervals \([\frac{j-1}{k}, \frac{j}{k}]\) for \( j = 1, 2, \ldots, k \),

\[
g_k(x) = \frac{1}{2} \left( 1 - \frac{1}{k} \right) - \frac{(j-1)}{k}, \quad x \in \left[ \frac{j-1}{k}, \frac{j}{k} \right).
\]

Performing the integrations and collecting the terms gives the Proposition.

Remark. Let us consider a general relation coming from (2.1). Suppose that \( \alpha = 0 \) there, that \( \phi(\beta)P_1(\beta) \to 0 \) as \( \beta \to \infty \), and that \( \int_0^\infty \phi(x) \, dx \) converges. Then employing the \( g_k \) functions of (2.15) we have

\[
\sum_{n=1}^{\infty} \phi(n) = \int_0^\infty \phi(x) \, dx + \int_0^\infty \phi'(x)P_1(x) \, dx - \frac{1}{2} \phi(0)
\]

\[
= \int_0^\infty \phi(x) \, dx - \sum_{n=0}^{\infty} \frac{1}{k^n} \int_0^\infty \phi'(x) g_k(k^n x) \, dx - \frac{1}{2} \phi(0)
\]

\[
= \int_0^\infty \phi(x) \, dx - \sum_{n=0}^{\infty} \frac{1}{k^{2n}} \int_0^\infty \phi'(k^{-n} y) g_k(y) \, dy - \frac{1}{2} \phi(0).
\]
Proposition 4. That the stated integrals may be explicitly evaluated follows from Theorems 4.2, 4.3, and 4.5 of [13]. Then from Corollary 5, the relation below (1.13) for \(\psi^{(n)}(x)\), and Proposition 2, the Proposition follows by a corresponding integration.

Remark. In the Appendix, we discuss the prevalent functions \(A_k(q)\) of [13], writing them in terms of sums of the Stieltjes constants, and delineating some of their properties in that manner.

Proposition 5. We have

\[
\ln \sigma_t = \sum_{n=2}^{\infty} \frac{\ln n}{t^n} = -\left. \frac{\partial \text{Li}_s(1/t)}{\partial s} \right|_{s=0}.
\]

As from Corollary 1,

\[-\frac{\partial \text{Li}_s(z)}{\partial s} = \int_{1}^{\infty} \frac{z^x}{x^s} \ln x \, dx + \int_{1}^{\infty} \left( \frac{z^x}{x^s} (\ln z) \ln x + \frac{z^x}{x^{s+1}} - s \frac{z^x}{x^{s+1}} \ln x \right) P_1(x) \, dx,\]

we obtain part (1.24a). For (1.24b), we use for \(\text{Re } z > 0\)

\[
\ln z = \int_{0}^{\infty} \left( e^{-t} - e^{-tz} \right) \frac{dt}{t},
\]

and then interchange summation and integration to write

\[
\ln \sigma_t = \sum_{n=2}^{\infty} \frac{\ln n}{t^n} = \sum_{n=2}^{\infty} \frac{1}{t^n} \int_{0}^{\infty} \left( e^{-x} - e^{-nx} \right) \frac{dx}{x}.
\]

For part (b) we reorder a double sum as a first demonstration,

\[
\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \text{Li}_k \left( \frac{1}{t} \right) = \sum_{n=1}^{\infty} \frac{1}{t^n} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \frac{1}{n^k} = \sum_{n=1}^{\infty} \frac{1}{t^n} \ln \left( 1 + \frac{1}{n} \right)
\]

\[
= \sum_{n=1}^{\infty} \frac{1}{t^n} [\ln(n+1) - \ln n] = (t-1) \ln \sigma_t.
\]
The other summation in (1.25) is obtained similarly.

For a second proof we appeal to part (a). From Corollary 1 we obtain

\[
\sum_{k=1}^\infty \frac{(-1)^{k-1}}{k} \text{Li}_k \left( \frac{1}{t} \right) = \ln \frac{2}{2t} + \int_1^\infty \frac{1}{tx} \ln \left( \frac{x+1}{x} \right) \, dx + \int_1^\infty \left[ \frac{\ln(1/t)}{tx} \ln \left( \frac{x+1}{x} \right) - \frac{1}{tx} \frac{1}{x(x+1)} \right] P_1(x) \, dx.
\]

(2.24)

We use the property

\[
\int_1^\infty \frac{f(x+1)}{tx} \, dx = \int_2^\infty \frac{f(y)}{ty} \, dy = t \int_1^\infty \frac{f(y)}{ty} \, dy - t \int_1^2 \frac{f(y)}{ty} \, dy,
\]

(2.25)

giving

\[
\sum_{k=1}^\infty \frac{(-1)^{k-1}}{k} \text{Li}_k \left( \frac{1}{t} \right) = \ln \frac{2}{2t} + (t-1) \int_1^\infty \frac{\ln x}{tx} \, dx - t \int_1^2 \frac{\ln x}{tx} \, dx + (t-1) \int_1^\infty \left[ \frac{\ln(1/t)}{tx} \ln x - \frac{1}{xt^x} \right] P_1(x) \, dx - t \int_1^2 \frac{\ln(1/t)}{tx} \ln x P_1(x) \, dx - t \int_1^2 \frac{P_1(x)}{xt^x} \, dx.
\]

(2.26)

By using integration by parts, it is seen that the three integrals on [1, 2), wherein 

\[ P_1(x) = x - 3/2, \]

cancel the \((\ln 2)/(2t)\) term, and the summation formula follows.

**Corollary 10.** We may use (1.24a), (1.24b), or (1.25). Using (1.25) we have

\[
\ln z + z \ln \sigma_{z+1} = \ln z + \sum_{k=1}^\infty \frac{(-1)^{k-1}}{k} \text{Li}_k \left( \frac{1}{z+1} \right).
\]

(2.27)

As \( z \to 0^+ \), the logarithmic singularity is cancelled since \( \text{Li}_1[1/(1+z)] = -\ln z + \ln(1+z) \). Therefore, we obtain

\[
\lim_{z \to 0^+} [\ln z + z \ln \sigma_{z+1}] = \sum_{k=2}^\infty \frac{(-1)^{k-1}}{k} \zeta(k) = \int_0^\infty \left( \frac{e^{-t}}{t} + \frac{1}{1-e^{-t}} \right) \, dt = -\gamma.
\]

(2.28)
Alternatively, from (1.24a), we have

\[
\ln z + z \ln \sigma_{z+1} = \ln z + z \left\{ -\frac{\text{Ei}[-\ln(1+z)]}{\ln(1+z)} + \int_1^\infty \left( \frac{1}{xt^x} - \frac{(\ln t) \ln x}{t^x} \right) P_1(x)dx \right\}
\]

\[
= \ln z + z \left[ -(\gamma + \ln z) \frac{1}{z} + \frac{1}{2}(3 - \gamma - \ln z) + O(z) \right]
\]

\[
= -\gamma + O(z \ln z), 
\]

(2.29)

and the Corollary follows again.

**Remarks.** The series of (2.28) is an example of the expansion [17] (p. 939)

\[
\ln \Gamma(x) = -\ln x - \gamma x + \sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{k} x^k, \quad |x| < 1. 
\]

(2.30)

Similarly using the second equality of (1.25) gives the sum

\[
-1 + \sum_{k=2}^{\infty} \frac{[\zeta(k) - 1]}{k} = -\gamma.
\]

**Corollary 11.** The solution of Somos’s quadratic recurrence is given by [23]

\[
\ln g_n = 2^n \ln \sigma_2 + \frac{1}{2} \frac{\partial \Phi}{\partial s} \left( \frac{1}{2}, 0, n + 1 \right).
\]

(2.31)

From Proposition 1 we have

\[
\frac{\partial \Phi}{\partial s} (z, s, a) = -\frac{\ln a}{a^s} - \frac{z \ln(a+1)}{2(a+1)^s} - \int_1^\infty \frac{z^x \ln(x+a)}{(x+a)^s} \frac{dx}{x^s}
\]

\[
- \int_1^\infty \left[ \frac{z^x (\ln z)}{(x+a)^s} \ln(x+a) + \frac{z^x}{(x+a)^{s+1}} + s \frac{z^x \ln(x+a)}{(x+a)^{s+1}} \right] P_1(x)dx, 
\]

(2.32)

giving

\[
\frac{\partial \Phi}{\partial s} \left( \frac{1}{2}, 0, n + 1 \right) = -\ln(n+1) - \frac{1}{4} \ln(n+2) - \int_1^\infty \frac{\ln(x+n+1)}{2x} dx
\]

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Together with the representation for \( \ln \sigma_2 \) given in Proposition 5, this completes the Corollary.

**Remark.** The result of this Corollary easily extends to the generalized Somos recurrence \( g_n = ng_{n-1}^t \) with \( g_0 = 1 \) and \( n \geq 1 \). For the solution is given by

\[
\ln g_n = t^n \ln \sigma_t + \frac{1}{t} \frac{\partial \Phi}{\partial s} \left( \frac{1}{t}, 0, n+1 \right),
\]

and Proposition 1 again applies.

**Proposition 6.** Using (1.36) we have

\[
L(s) = \frac{1}{4^s} \left\{ \frac{4^s}{2} (1 - 3^{-s}) + \frac{4^{s-1}}{(s-1)} (1 - 3^{1-s}) \right\}
- s \int_0^\infty \left[ \frac{1}{(x+1/4)^{s+1}} - \frac{1}{(x+3/4)^{s+1}} \right] P_1(x) dx \right\},
\]

We then employ the case of the function \( g_2(x) \) of (2.15) in order to evaluate the integrals. We omit further details.

**Remark.** We may extend Proposition 6 with the use of any of the functions \( g_k(x) \).

**Proposition 7.** We apply (2.1) with \( \phi(x) = [\psi(x+1) + \gamma]/(x+a)^s \) and \( \phi(n) = H_n/(n+a)^s \).

**Remark.** This Proposition may be extended by using the generalized harmonic numbers (1.14).

**Summary**

We have obtained integral representations for a wide variety of special functions, including the Lerch zeta, the polylogarithm, Dirichlet \( L \)- and the Clausen functions.
Furthermore, we have shown that the method of Addison may be generalized in several directions to develop series representations for an array of mathematical constants. These constants include, but are not limited to, those that arise from definite integrations of log Gamma, digamma, and polygamma functions, with or without powers, and those that arise from certain derivatives of the Lerch zeta, polylogarithm, and Hurwitz zeta functions. An instance of the latter category is that for the generalized Somos constants $\sigma_t$. By using the stepwise functions $g_k(x)$ in the Addison approach, general series representations with parameter $k$ are obtained. Our methods have broad applicability and permit both the recovery of known results and the development of novel integral and series representations.

**Appendix A: on Kinkelin’s constant $k$ and $\Gamma$ function integrals**

Here we present material on Kinkelin’s constant $k = \zeta'(1) \simeq -0.165421$ together with some particular integrals with the Gamma function. Several expressions for the Kinkelin constant are known [20, 24]. For instance,

$$k = 2 \int_0^\infty \frac{x \ln x}{e^{2\pi x} - 1} dx,$$

and expanding the denominator as a geometric series verifies the relation $k = 1/12 - \ln A$. Another relation is

$$k = \frac{1}{12} - \frac{1}{4} \ln(2\pi) + \int_0^1 x \Gamma(x) dx.$$  \hspace{1cm} (A.2)

Introducing the constants $c_k$ in $\Gamma(z + 1) = \sum_{k=0}^\infty c_k z^k$ for $|z| < 1$, with $c_0 = 1$ and $c_1 = -\gamma$, there is the recurrence $c_{n+1} = \frac{1}{n+1} \sum_{k=0}^n (-1)^{k+1} s_{k+1} c_{n-k}$, where $s_1 = \gamma$ and
s_n = \zeta(n) \text{ for } n \geq 2 \ [17] \ (p. 935). We then have

**Proposition A1.** We have

\[
\int_0^1 x \Gamma(x) \, dx = \frac{1}{2} + \frac{\gamma}{6} + \sum_{k=2}^{\infty} \frac{(-1)^k c_k}{(k+1)(k+2)} = 1 - \frac{\gamma}{2} + \sum_{k=2}^{\infty} \frac{c_k}{k+1}. \quad (A.3)
\]

**Proof.** For the first equality we employ the Taylor expansion of \( \Gamma \) about \( x = 1 \), and then evaluate a simple Beta function integral:

\[
\int_0^1 x \Gamma(x) \, dx = \sum_{k=0}^{\infty} (-1)^k c_k \int_0^\infty x(1-x)^k \, dx = \sum_{k=0}^{\infty} \frac{(-1)^k c_k}{(k+1)(k+2)}. \quad (A.4)
\]

For the second equality we identify the integral with \( \int_0^1 \Gamma(x+1) \, dx \) and proceed similarly.

We have

**Proposition A2.** We have

\[
k = -\frac{\gamma}{12} + \ln 2 - \frac{5}{6} + \frac{1}{2} \sum_{n=2}^{\infty} \frac{(-1)^n [\zeta(n) - 1]}{(n+1)(n+2)}. \quad (A.5)
\]

**Proof.** We reorder the double series

\[
\sum_{n=2}^{\infty} \frac{(-1)^n}{(n+1)(n+2)} \sum_{j=2}^{\infty} \frac{1}{j^n} = \sum_{j=2}^{\infty} \sum_{n=2}^{\infty} \frac{(-1)^n}{(n+1)(n+2)} \frac{1}{j^n} = \sum_{j=2}^{\infty} \sum_{n=2}^{\infty} (-1)^n \left( \frac{1}{n+1} - \frac{1}{n+2} \right) \frac{1}{j^n}
\]

\[
= \sum_{j=2}^{\infty} \left[ \frac{6j^2 + 3j - 1}{6j} + j(j+1) \ln \left( \frac{j+1}{j} \right) \right]
\]

\[
= \frac{11}{6} + \frac{\gamma}{6} - 2 \ln A - 2 \ln 2. \quad (A.6)
\]

The latter expression may be found by using the limit relations \( \gamma = \lim_{N \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \ln n \right) \) and for the Glaisher-Kinkelin constant

\[
\ln A = \lim_{N \to \infty} \left[ \sum_{k=1}^{n} k \ln k - \left( \frac{N^2}{2} + \frac{N}{2} + \frac{1}{12} \right) \ln N + \frac{N^2}{4} \right]. \quad (A.7)
\]
Using the relation \( k = 1/12 - \ln A \) completes the Proposition.

The alternating sum in (A.5) also occurs in [5] (p. 116), where it is calculated with another approach. Corollaries of Proposition A2 follow by using various integral representations of \( \zeta(n) \) in (A.5). For instance, by using (1.26) we have

**Corollary A1.** We have

\[
k = \frac{1}{6} \ln 2 - \frac{5}{18} - \frac{1}{2} \int_1^\infty \left[ 2 - (2x + 1) \ln \left( \frac{x + 1}{x} \right) \right] P_1(x) \, dx. \tag{A.8}
\]

In this expression, the integral term provides a small correction (approximately 2%). This Corollary enables other Addison-type series to be developed for Kinkelin’s constant.

Expecting that the values of \( \int_0^1 x \Gamma(x) \, dx \approx 0.92746 \) and \( \int_0^1 \sin x \, \Gamma(x) \, dx \approx 0.872427 \) are not too different, we have performed a brief investigation of the following integrals.

We have

**Proposition A3.** For \( \lambda > 0 \) and \( |\alpha| \leq \pi/2 \) we have the representation

\[
\int_0^1 \sin(\alpha x) \Gamma(x) \, dx = \int_0^\infty e^{-\lambda t \cos \alpha} \sin(\lambda t \sin \alpha) \frac{(\lambda t - 1)}{t(\ln t + \ln \lambda)} \, dt. \tag{A.9}
\]

**Proof.** This follows by using an integral representation for \( \Gamma(x) \sin \alpha x \) [17] (p. 935) and interchanging integrations.

From A3 we deduce

**Corollary A2.** For \( \lambda > 0 \) we have

\[
\int_0^1 x \Gamma(x) \, dx = \lambda \int_0^\infty e^{-\lambda t} \frac{(\lambda t - 1)}{\ln(\lambda t)} \, dt = \int_0^\infty e^{-u} \frac{(u - 1)}{\ln u} \, du. \tag{A.10}
\]

**Proof.** We let \( \alpha \to 0 \) in (A.9) and use \( e^{-\lambda t \cos \alpha} \sin(\lambda t \sin \alpha) = \alpha \lambda te^{-\lambda t} + O(\alpha^3) \).
We now give a direct proof of this Corollary. We interchange integrations (justified by absolute convergence) and integrate by parts twice:

\[ \int_0^1 x \Gamma(x) dx = \int_0^1 \int_0^\infty e^{-tx} t^{x-1} dt dx = \int_0^\infty e^{-t} \left( \frac{1}{t \ln^2 t} - \frac{1}{\ln t} + \frac{1}{\ln t} \right) dt = -\int_0^\infty \frac{e^{-t}}{\ln t} dt = \int_0^\infty e^{-t \left( \frac{1}{\ln t} \right)} dt. \quad \text{(A.11)} \]

The next result recovers A1 as a special case. Again, \( {}_pF_q \) is the generalized hypergeometric function. We have

**Proposition A4.** We have for \(|\alpha| \leq \pi/2\)

\[ \int_0^1 \sin(\alpha x) \Gamma(x) dx = \alpha \sum_{k=0}^\infty \frac{(-1)^k c_k}{(k+1)(k+2)} \, {}_1F_2 \left( 1; \frac{k+3}{2}, \frac{k+3}{2}; -\frac{\alpha^2}{4} \right) \]

\[ = \frac{1 - \cos \alpha}{\alpha} + \gamma (\alpha - \sin \alpha) + \alpha \sum_{k=2}^\infty \frac{(-1)^k c_k}{(k+1)(k+2)} \, {}_1F_2 \left( 1; \frac{k+3}{2}, \frac{k+3}{2}; -\frac{\alpha^2}{4} \right) \quad \text{(A.12)} \]

\[ = \sum_{k=0}^\infty (-1)^k c_k \left[ \sum_{\ell=0}^{k-1} \frac{\ell!}{\alpha^{\ell+1}} \binom{k}{\ell} \cos \left( \frac{\ell \pi}{2} \right) - \frac{k!}{\alpha^{k+1}} \cos \left( \frac{\alpha - k\pi}{2} \right) \right] \]

\[ = \sum_{k=0}^\infty (-1)^k c_k \left[ \sum_{\ell=0}^{k-1} \frac{\ell!}{\alpha^{\ell+1}} \binom{k}{\ell} \cos \left( \frac{\ell \pi}{2} \right) + 2 \frac{k!}{\alpha^{k+1}} \sin \left( \frac{\alpha}{2} \right) \sin \left( \frac{\alpha - k\pi}{2} \right) \right]. \quad \text{(A.13)} \]

*Proof.* The Taylor expansion of \( \Gamma \) about \( x = 1 \) is first used to write

\[ \int_0^1 \sin(\alpha x) \Gamma(x) dx = \sum_{k=0}^\infty c_k \int_0^1 \sin \alpha x (x - 1)^k dx \]

\[ = \sum_{k=0}^\infty (-1)^k c_k \sum_{j=0}^\infty \frac{(-1)^j}{(2j+1)!} \alpha^{2j+1} \int_0^1 x^{2j+1} (1 - x)^k dx \]

\[ = \sum_{k=0}^\infty (-1)^k c_k \sum_{j=0}^\infty (-1)^j \alpha^{2j+1} \frac{\Gamma(k+1)}{\Gamma(2j+k+3)} \frac{(1)^j}{j!}. \quad \text{(A.14)} \]
where the integral evaluated in terms of the Beta function $B$ as $B(2j + 2, k + 1)$, and $(z)_j = \Gamma(z + j)/\Gamma(z)$ is the Pochhammer symbol. We use Legendre’s duplication formula for the Gamma function to write $\Gamma(2j + k + 3) = \frac{1}{\sqrt{\pi}} 2^{2j+k+2} \Gamma \left( j + \frac{k+3}{2} \right) \Gamma \left( j + \frac{k}{2} + 2 \right)$ and $\Gamma(k + 3) = \frac{1}{\sqrt{\pi}} 2^{k+2} \Gamma \left( \frac{k+3}{2} \right) \Gamma \left( \frac{k}{2} + 2 \right)$. These relations yield

$$
\int_0^1 \sin(\alpha x) \Gamma(x) dx = \sum_{k=0}^{\infty} c_k \frac{\Gamma(k + 1)}{\Gamma(k + 3)} \sum_{j=0}^{\infty} \frac{(-1)^j \alpha^{2j+1}}{2^{2j}} \left( \frac{k+3}{2} \right)_j \left( \frac{k}{2} + 2 \right)_j j!
$$

$$= \alpha \sum_{k=0}^{\infty} \frac{(-1)^k c_k}{(k+1)(k+2)} \, _1F_2 \left( 1; \frac{k+3}{2}, 2 + \frac{k}{2}; -\frac{\alpha^2}{4} \right). \quad (A.15)$$

For the alternative form (A.13) we write the first integral on the right side of (A.14) as

$$
\int_0^1 \sin \alpha x (1 - x)^k dx = \int_0^1 \sin[\alpha(1 - v)] v^k dv
$$

$$= \int_0^1 \left[ \sin \alpha \cos \alpha v - \sin \alpha v \cos \alpha \right] v^k dv
$$

$$= \sin \alpha \sum_{\ell=0}^{k} \ell! \left( \frac{k}{\ell} \right) v^{k-\ell} \sin \left( \alpha v + \frac{\ell \pi}{2} \right) \bigg|_{v=0}^{v=1} + \cos \alpha \sum_{\ell=0}^{k} \ell! \left( \frac{k}{\ell} \right) v^{k-\ell} \cos \left( \alpha v + \frac{\ell \pi}{2} \right) \bigg|_{v=0}^{v=1}, \quad (A.16)
$$

where [17] (pp. 183-184) were used. We then find

$$
\int_0^1 \sin \alpha x (1 - x)^k dx = \sum_{\ell=0}^{k} \ell! \left( \frac{k}{\ell} \right) \left[ \sin \alpha \sin \left( \alpha + \frac{\ell \pi}{2} \right) + \cos \alpha \cos \left( \alpha + \frac{\ell \pi}{2} \right) \right] \frac{1}{\alpha^{\ell+1}}
$$

$$- \frac{k!}{\alpha^k} \left[ \sin \alpha \sin \left( \frac{k \pi}{2} \right) + \cos \alpha \cos \left( \frac{k \pi}{2} \right) \right], \quad (A.17)
$$

leading to (A.13).

Remark. When $\alpha \to 0$, $\, _1F_2 \to 1$ in (A.12) and (A.3) results.
Appendix B: The functions $A_k(q)$ of [13]

The functions for integers $k$

$$A_k(q) \equiv k \frac{\partial}{\partial z} \zeta(z, q) \bigg|_{z=1-k}, \quad (B.1)$$

are very useful in evaluating integrals over the Hurwitz zeta function [13]. We present a study of these functions, first writing a sum representation for them in terms of the Stieltjes constants. Since much other information is known about these functions, including their connection with negapolygamma functions, we effectively unify many descriptions of these sets of functions. We show that it is possible to recover known properties [13] of the $A_k(q)$ functions via their representation in terms of the Stieltjes constants. Then we write an integral representation for these functions when $k < 2$, and mention two summation relations. Here we take $0 \leq q \leq 1$.

**Lemma B1.** We have

$$A_k(q) = -\frac{1}{k} - k \sum_{n=0}^{\infty} \frac{\gamma_{n+1}(q)}{n!} k^n, \quad (B.2)$$

and

$$A_1(q) = \ln \Gamma(q) - \frac{1}{2} \ln 2\pi = -1 - \sum_{n=0}^{\infty} \frac{\gamma_{n+1}(q)}{n!}. \quad (B.3)$$

**Lemma B2.** The relations $\int_0^1 A_k(q) dq = 0$ and

$$\sum_{n=0}^{\infty} \frac{k^n}{n!} \int_0^1 \gamma_{n+1}(q) dq = -\frac{1}{k^2} \quad (B.4)$$

are equivalent.

**Lemma B3.** We have $A_k(0) = A_k(1) = k \zeta'(1 - k)$ for $k \geq 2$. 

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Let $B_k(q) = -k\zeta(1-k,q)$ be the Bernoulli polynomials. Then we have

**Lemma B4.** We have

$$B_k(q) = 1 - k \sum_{n=0}^{\infty} \frac{\gamma_n(q)}{n!} k^n. \quad (B.5)$$

**Corollary B1.** We have for the Bernoulli numbers $B_k = B_k(0)$

$$(-1)^k B_k = B_k(1) = 1 - k\gamma - k \sum_{n=1}^{\infty} \frac{\gamma_n}{n!} k^n, \quad (B.6)$$

and in particular at $k = 1$ we recover the special case

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n!} = \frac{1}{2} - \gamma. \quad (B.7)$$

**Lemma B5.** We have $A_{k+1}(q) = A_k(q) + k q^{k-1} \ln q$ and thus for integers $n \geq 0$

$$A_{k+n}(q) = A_k(q) + \ln q \sum_{j=0}^{n-1} (k+j)q^{k+j-1}. \quad (B.8)$$

**Lemma B6.** We have

$$A'_{k+1}(q) = (k+1) \left[ A_k(q) + \frac{1}{k} B_k(q) \right]. \quad (B.9)$$

Proof of Lemmas. **Lemma B1.** We use the expansion (1.37) and the definition (B.1). The relation (B.3) corresponds to Corollary 5 of [11].

**Lemma B2.** We use Proposition 4 of [8], differentiating there with respect to $s$, giving

$$\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k!} (s-1)^k \int_0^1 \gamma_{k+1}(a) da = -\frac{1}{(1-s)^2}. \quad (B.10)$$

Relabeling variables, putting $k \to n$ and then putting $s = 1 - n$ gives the result.

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Lemma B3. We introduce the quantity $C_n(a) = \gamma_n(a) - \frac{1}{a} \ln^a a$, that for $n \geq 1$ has the integral representation

$$C_n(a) = \int_1^\infty P_1(x-a) \frac{\ln^{n-1} x}{x^2} (n - \ln x) dx,$$

(B.11)

where again $P_1(x)$ is the first periodic Bernoulli polynomial. Then from Lemma 1 we find

$$A_k(q) = -\frac{1}{k} - k \sum_{n=0}^\infty \frac{C_{n+1}(q)}{n!} k^n - k q^{k-1} \ln q.$$  

(B.12)

Using the 1-periodicity of $P_1(x)$ yields the relation $A_k(0) = A_k(1)$.

Lemma B4. This follows directly from the expansion (1.37).

Supplements to Corollary B1. The Bernoulli numbers $B_k$ are well known to vanish for $k \geq 3$ an odd integer. That this condition holds in (B.6) is evident upon applying the fact of the trivial zeros of the Riemann zeta function at $z = -2n$, $n \geq 1$.

Upon the use of $a = 1$ in (B.11) inserted in (B.7), we easily recover the known result $\int_1^\infty P_1(x) dx / x^2 = 1/2 - \gamma$.

Lemma B5. This follows from the functional equation

$$\gamma_k(q+1) = \gamma_k(a) - \frac{1}{q} \ln q.$$  

(B.13)

Lemma B6. We use Lemma B1 and Proposition 7 of [8] so that

$$A'_{k+1}(q) = -(k+1) \sum_{n=0}^\infty \frac{\gamma'_{n+1}(q)}{n!} (k+1)^n$$

$$= (k+1) \sum_{n=0}^\infty \frac{(k+1)^n}{n!} (n+1)! (-1)^n+1 n+1 \sum_{\ell=n}^\infty \frac{(-1)^\ell}{\ell!} \left( \frac{\ell + 1}{n+1} \right) \gamma_\ell(q)$$

$$= (k+1) \sum_{n=0}^\infty (k+1)^n (-1)^{n+1} \sum_{\ell=n}^\infty \frac{(-1)^\ell}{\ell!} \left( \frac{\ell}{n} \right) (\ell + 1) \gamma_\ell(q)$$

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\[-(k+1) \sum_{\ell=0}^{\infty} \frac{(\ell+1)}{\ell!} \gamma_{\ell+1}(q) k^\ell \]
\[-(k+1) \sum_{\ell=0}^{\infty} \frac{1}{\ell!} [k \gamma_{\ell+1}(q) + \gamma_{\ell}(q)] k^\ell. \quad (B.14)\]

Above, we reordered the double sum and used the binomial theorem. Lemma B6 then follows in light of Lemma B4.

Remark. Various expressions for values of \( A_k \) in [13] together with Lemma 1 give Corollaries for certain sums of the Stieltjes constants. For instance from Lemma 3.5 of [13] we have

\[ A_k(1/2) = (-1)^{k-1} B_k 2^{1-k} \ln 2 - (1 - 2^{1-k}) k \zeta'(1 - k) = -\frac{1}{k} - k \sum_{n=0}^{\infty} \frac{\gamma_{n+1}(1/2)}{n!} k^n, \]

and from Lemma 3.6 there we have

\[ A_2(q) = (1 - \gamma - \ln 2\pi)(q^2 - q + 1/6) - \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{\ln n}{n^2} \cos(2\pi nq) + \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{\sin(2\pi nq)}{n^2} \]
\[ = -\frac{1}{2} - 2 \sum_{j=0}^{\infty} \frac{2^j}{j!} \gamma_{j+1}(q). \quad (B.15)\]

On the right side of the first line of this equation we recognize the values \( C_{2q} \).

From Lemma B1 we also find the generating function

\[ \sum_{k=1}^{\infty} \frac{A_k(q)}{k!} z^k = \gamma - \text{Ei}(z) + \ln z - ke^z \sum_{j=0}^{\infty} \frac{\gamma_{j+1}(q)}{j!} B_j(z), \quad (B.17)\]

where \( \text{Ei} \) is the exponential integral and \( B_j \) the Bell polynomial.

From using the relation \( \zeta'(0, s) = \ln \Gamma(s) - (1/2) \ln 2\pi \) and various integral representations, we have
**Corollary B2.** We have for $\text{Re } s > 0$

$$- \int_0^1 \frac{P_1(x)dx}{x + s} = \int_0^\infty \left( \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right) \frac{e^{-ts}}{t} dt = 2 \int_0^\infty \frac{\tan^{-1}(t/s)}{e^{2\pi t} - 1} dt. \quad (B.18)$$

Further Corollaries follow by repeated differentiation with respect to $s$ and the use of other operations.

We conclude by giving an integral representation for $A_k(q)$ and two summations relations for it. We have

**Proposition B1.** For $k < 2$ we have

$$A_k(q) = k \left[ - \frac{\ln q}{2q^{1-k}} - \frac{q^k}{k^2} \ln q - \int_0^\infty \frac{P_1(x)dx}{(x + q)^2-k} + (1-k) \int_0^\infty \frac{P_1(x)}{(x + q)^{2-k}} \ln(x + q)dx \right]. \quad (B.19)$$

Thus we have

**Corollary B3.** We have

$$A_1(q) = \left( q - \frac{1}{2} \right) \ln q - q - \int_0^\infty \frac{P_1(x)dx}{x + q}. \quad (B.20)$$

For the proof of this Proposition, we apply (2.14) and the definition (B.1). As far as Corollary B3, it is easy to see that we recover the equivalent of the relation (e.g., [8], (A.2) or [12], p. 107)

$$\ln \Gamma(s + 1) = \left( s + \frac{1}{2} \right) \ln s - s + \frac{1}{2} \ln 2\pi - \int_0^\infty \frac{P_1(x)dx}{x + s}. \quad (B.21)$$

We announce the following summation relations. The proofs will appear elsewhere.

**Proposition B2.** Let $p \geq 1$ and $q \geq 1$ be integers, $b \geq 0$, and $\min(p/q, q/p) > b$. 
Then we have
\[
\sum_{r=1}^{q} A_k \left( \frac{pr}{q} - b \right) = -\ln \left( \frac{q}{p} \right) \sum_{r=1}^{q} B_k \left( \frac{pr}{q} - b \right) + \left( \frac{q}{p} \right)^{1-k} \frac{1}{p^k} \sum_{\ell=0}^{1-k} A_k \left( 1 + \frac{(\ell - b)q}{p} \right).
\]  
(B.22)

**Proposition B3.** Let \( \mathcal{P} \) denote the prime numbers. Then for \( p \in \mathcal{P} \) and integer \( N \geq 0 \) we have
\[
(1 - p^{-k}) A_k(1) - (-1)^k p^{-k-1} \ln p B_k = (-1)^k (N + 1) \ln p (1 - p^{-k-1}) B_k
\]
\[
+ p^{(N+1)(k-1)} \sum_{1 \leq j < p^{N+1} \atop (j,p)=1} A_k \left( \frac{j}{p^{N+1}} \right).
\]  
(B.23)

It is possible to derive other such summation relations.
References

[1] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, Washington, National Bureau of Standards (1964).

[2] A. W. Addison, A series representation for Euler’s constant, Amer. Math. Monthly 74, 823-824 (1967).

[3] G. E. Andrews, R. Askey, and R. Roy, Special Functions, Cambridge University Press (1999).

[4] L. Bendersky, Sur la fonction gamma généralisée, Acta Math. 61, 263-322 (1933).

[5] J. Choi and H. M. Srivastava, Sums associated with the Zeta function, J. Math. Anal. Appl. 206, 103-120 (1997).

[6] M. W. Coffey Addison-type series representation for the Stieltjes constants, J. Number Th. 130, 2049-2064 (2010); arXiv:0912.2391 (2009).

[7] M. W. Coffey, On some log-cosine integrals related to $\zeta(2)$, $\zeta(3)$, and $\zeta(6)$, J. Comput. Appl. Math. 159, 205-214 (2003).

[8] M. W. Coffey, New results on the Stieltjes constants: Asymptotic and exact evaluation, J. Math. Anal. Appl. 317, 603-612 (2006); arXiv:math-ph/0506061.

[9] M. W. Coffey, On a three-dimensional symmetric Ising tetrahedron, and contributions to the theory of the dilogarithm and Clausen functions, J. Math. Phys. 49, 043510-1-043510-32, (2008); arxiv/math.ph/0801.0273v2 (2008).
[10] M. W. Coffey, On representations and differences of Stieltjes coefficients, and
other relations, to appear in Rocky Mtn. J. Math.; arXiv/math-ph/0809.3277v2
(2008).

[11] M. W. Coffey, Series representations for the Stieltjes constants, arXiv:0905.1111
(2009).

[12] H. M. Edwards, Riemann’s Zeta Function, Academic Press, New York (1974).

[13] O. Espinosa and V. Moll, On some integrals involving the Hurwitz zeta function:
part 2, Ramanujan J. 6, 449-468 (2002).

[14] P. Flajolet and B. Salvy, Euler sums and contour integral representations, Exptl.
Math. 7, 15-35 (1998).

[15] I. Gerst, Some series for Euler’s constant, Amer. Math. Monthly, 76, 273-275
(1969).

[16] J. W. L. Glaisher, On the product $1^1 2^2 \cdots n^n$, Mess. Math. 7, 43-47 (1877).

[17] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products,
Academic Press, New York (1980).

[18] G. H. Hardy, Note on Dr. Vacca’s series for $\gamma$, Quart. J. Pure Appl. Math. 43,
215-216 (1912).

[19] A. Ivić, The Riemann Zeta-Function, Wiley New York (1985).
[20] J. Kinkelin, "Über eine mit der Gammafunktion verwandte Transcendente und deren Anwendung auf die Integralrechnung," J. Reine und Angew. Math. 57, 122-158 (1860).

[21] B. Riemann, "Über die Anzahl der Primzahlen unter einer gegebenen Grösse," Monats. Preuss. Akad. Wiss., 671 (1859-1860).

[22] H. F. Sandham and D. F. Barrow, Advanced Problems and Solutions, Amer. Math. Monthly 58, 116-117 (1951).

[23] M. Somos, Several constants related to quadratic recurrences, unpublished note (1999); E. W. Weisstein, "Somos's Quadratic Recurrence Constant." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/SomossQuadraticRecurrenceConstant.html

[24] H. M. Srivastava and J. Choi, Series associated with the zeta and related functions, Kluwer (2001).

[25] T. J. Stieltjes, Correspondance d’Hermite et de Stieltjes, Volumes 1 and 2, Gauthier-Villars, Paris (1905).

[26] E. C. Titchmarsh, The Theory of the Riemann Zeta-Function, 2nd ed., Oxford University Press, Oxford (1986).

[27] G. Vacca, A new series for the Eulerian constant \( \gamma = 0.577\ldots \), Quart. J. Pure Appl. Math. 41, 363-366 (1909-10).
[28] J. R. Wilton, A note on the coefficients in the expansion of $\zeta(s, x)$ in powers of $s - 1$, Quart. J. Pure Appl. Math. 50, 329-332 (1927).