Gravitational waves with distinct wavefronts

by

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Abstract

Exact solutions of Einstein’s vacuum equations are considered which describe gravitational waves with distinct wavefronts. A family of such solutions presented recently in which the wavefronts have various geometries and which propagate into a number of physically significant backgrounds is here related to an integral representation which is a generalisation of the Rosen pulse solution for cylindrical waves. A nondiagonal solution is also constructed which is a generalisation of the Rosen pulse, being a cylindrical pulse wave with two states of polarization propagating into a Minkowski background. The solution is given in a complete and explicit form. A further generalisation to include electromagnetic waves with a distinct wavefront of the same type is also discussed.

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1. Introduction

In many physically realistic situations concerning gravitational wave emission and propagation, it seems natural to expect the occurrence of waves with distinct wavefronts. By contrast, most exact solutions which describe gravitational wave pulses have amplitudes which only vanish asymptotically. Many well known soliton wave solutions provide examples of such pulses of infinite duration. The purpose of the present paper is to consider gravitational and electromagnetic waves which have distinct wavefronts and to present some new solutions of this type which have variable polarization. These may include impulsive or shock waves, or waves with much weaker discontinuities on their wavefronts.

In general, a solution of Einstein’s field equations describes a wave with a distinct wavefront if there exists a characteristic null hypersurface (wavefront) which divides the space-time locally into two regions — an unperturbed region where the solution represents some given background field, and a wave region where the solution describes the interaction of some wave with the background field. The appropriate matching conditions for pure gravitational waves are those of O’Brien and Synge [1]. The problem of constructing exact solutions for gravitational waves with distinct wavefronts can be considered as a special case of the characteristic initial value problem. For this case, the boundary data on one of two intersecting characteristic surfaces (i.e. on the wavefront) are determined from the condition of matching the solution through the wavefront with the given background field. The boundary data on the other characteristic remain arbitrary, permitting various wave profiles.

Examples of such waves include the well known class of plane (or pp) gravitational waves (see §21 of [2]) which can be used to describe sandwich waves in a Minkowski background [3]. These will not be considered further here. In the context of cylindrically symmetric gravitational waves, the pulse wave solution of Rosen [4] is also well known (see also [5]). Using standard coordinates $t$ and $\rho$, this is given by $V = 0$ for $t < \rho$ and, for $t \geq \rho$ by

$$V = \int_{0}^{t-\rho} \frac{F(\sigma) \, d\sigma}{\sqrt{(t-\sigma)^2 - \rho^2}}$$

(1.1)

where the function $F(\sigma)$ vanishes outside a finite range $0 < \sigma < T$, and the metric function $V$ is defined below. This solution describes a pulse gravitational wave with a distinct wavefront $t = \rho$ which can be made arbitrarily smooth by taking a sufficient number of derivatives of $F(\sigma)$ to be zero on $\sigma = 0$. However, although this defines a general class of solutions for $V$, the corresponding expressions for all the metric components cannot be obtained in an explicit form.

In recent years, however, further families of exact solutions have been obtained for gravitational waves with distinct, possibly curved, wavefronts which propagate into various backgrounds. For these solutions explicit expressions for all the metric components have been obtained in closed form. Generally, these solutions describe gravitational waves with distinct wavefronts which propagate either into a Minkowski space or into some other
background of cosmological significance. The wavefronts of these waves may be plane when propagating into a vacuum Kasner background, or either plane or cylindrical (or toroidal) when propagating into a Friedmann–Robertson–Walker universe with a stiff perfect fluid [6–9]. Other families have been obtained [10–11] which represent gravitational waves with cylindrical, spherical or toroidal wavefronts propagating into a Minkowski background.

All these space-times admit a two-dimensional abelian group of spacelike isometries. However, so far, solutions have only been given for the case in which the metric is diagonal. This case corresponds to that in which the gravitational waves have constant (linear) polarization. Mathematically, it corresponds to the case in which the main field equation is the linear Euler–Poisson–Darboux equation with non-integer coefficients. (In stationary axisymmetric space-times, this case corresponds to the diagonal Weyl metrics.) An important consequence of the main equation being linear is that we can superpose any wave solution on a variety of backgrounds including a number that are physically significant (e.g. Minkowski in various coordinates, Kasner, FRW stiff fluid etc.).

The first purpose of the present paper is to clarify the relation between these solutions and the integral representation of the Rosen pulse (1.1), or a suitable variation of it. This is achieved in section 3.

The second purpose is to consider the possible generalisation of these solutions to the nondiagonal case. This problem corresponds physically to the case in which the gravitational wave has variable polarization, or a combination of two states of polarization. Mathematically, it corresponds to the case in which the main field equations can be written as the nonlinear Ernst equation. It is known that this equation admits various symmetry transformations which leave it invariant but transform any one solution into another. Using these transformations it is possible to construct nondiagonal solutions which involve an arbitrary number of parameters. The problem here is therefore twofold. It is to construct solutions that correspond to the physical situation of a wave with a distinct wavefront propagating into a specific background, and also to express those solutions in closed form.

Since the Ernst equation is nonlinear, it will not generally be possible to express solutions in the nondiagonal case as a sum of components. Nor will it be possible to superimpose any particular wave solution on any given background. Nevertheless, a technique for generating exact solutions of this type will be presented in section 4 which is based on a pair of linear equations. Using this, some general classes of wave solutions will be given explicitly in sections 5 and 6. These represent two classes of gravitational waves with cylindrical wavefronts propagating into a Minkowski background. An electromagnetic generalisation of this solution is given in section 7.
2. Space-time geometry and field equations

For vacuum space-times with a two-dimensional abelian group of isometries, with the Killing vectors both spacelike, the line element can be written in the form

$$ds^2 = 2e^{-M}du dv - e^{-U}(\chi dy^2 + \chi^{-1}(dx - \omega dy)^2)$$  \hspace{1cm} (2.1)

in which the functions $U$, $\chi$, $\omega$ and $M$ are functions of the two null coordinates $u$ and $v$ only. The structure of the vacuum field equations for this metric is well known. For the function $U$, these equations imply that

$$e^{-U} = f(u) + g(v),$$  \hspace{1cm} (2.2)

where $f(u)$ and $g(v)$ are arbitrary functions. Two other equations for the function $M$ can be separated from the main system, namely:

$$2U_u M_u = U^2_u - 2U uv + \chi^{-2}(\chi^2_u + \omega^2_u),$$
$$2U_v M_v = U^2_v - 2U uv + \chi^{-2}(\chi^2_v + \omega^2_v).$$  \hspace{1cm} (2.3)

These equations determine $M$ in quadratures provided the main part of the field equations — a closed nonlinear system of two equations for metric functions $\chi$ and $\omega$ — is solved. These last equations can be conveniently expressed in terms of a scalar complex potential

$$Z \equiv \chi + i\omega$$  \hspace{1cm} (2.4)

in the form which is identical to the well known Ernst equation

$$\text{Re} Z(2Z_{uv} - U_u Z_v - U_v Z_u) - 2Z_u Z_v = 0.$$

It may be noted also that, in practice, it is often convenient to treat the arbitrary functions $f$ and $g$ as null coordinates in place of $u$ and $v$. Then, with $e^{-U} = f + g$, the Ernst equation (2.5) takes the form

$$(Z + \bar{Z}) \left( 2Z_{fg} + \frac{1}{f + g} (Z_f + Z_g) \right) = 4Z_f Z_g.$$  \hspace{1cm} (2.6)

For later convenience, we present here also some appropriate expressions for the space-time curvature of the metric (2.1) in terms of the Newman–Penrose null tetrad formalism. For a null tetrad defined for (2.1) in a usual form

$$\ell^\alpha = e^{M/2} \delta_1^\alpha, \quad n^\alpha = e^{M/2} \delta_0^\alpha, \quad m^\alpha = \frac{1}{\sqrt{2\chi}} e^{U/2} \left\{ (\chi + i\omega) \delta_2^\alpha + i\delta_3^\alpha \right\}$$  \hspace{1cm} (2.7)

where $(x^0, x^1, x^2, x^3) \equiv (u, v, x, y)$. The nonzero Newman–Penrose scalars — the projections on this tetrad of the selfdual Weyl tensor — can be expressed as

$$\Psi_0 = \frac{1}{2} e^U \left( \partial_v - i\chi^{-1} \partial_v \omega \right) \left[ \frac{e^{M-U}}{\chi} \partial_v (\chi + i\omega) \right],$$
$$\Psi_2 = \frac{1}{4} e^M \left[ \frac{\partial_u (\chi - i\omega) \partial_v (\chi + i\omega)}{\chi^2} - U_u U_v \right],$$
$$\Psi_4 = \frac{1}{2} e^U \left( \partial_u + i\chi^{-1} \partial_u \omega \right) \left[ \frac{e^{M-U}}{\chi} \partial_u (\chi - i\omega) \right].$$  \hspace{1cm} (2.8)
These components and, in particular, their two combinations — the complex algebraic invariants of the selfdual Weyl tensor $I = 2(\Psi_0\Psi_4 + 3\Psi_2^2)$, $J = 6\Psi_2(-\Psi_0\Psi_4 + \Psi_2^2)$ — describe the properties of the gravitational waves.

When we come to construct solutions for waves with distinct wavefronts, the appropriate matching conditions must be satisfied across the wavefront. These are the O’Brien–Synge junction conditions [1] which, for metrics of the type (2.1), require the continuity of $U, U_u, \chi, \omega$ and $M$ across any null hypersurface $u =$const,

\[3.\text{ The linear case}\]

Exact solutions for gravitational waves with a distinct wavefront have been considered extensively in the series of papers [6–11] for the diagonal case, in which the wave has linear polarization, and for a variety of backgrounds. In the linear case, for which $\omega = 0$ and the function $Z$ is real, it is convenient to put

\[\chi = e^{-V}.\quad (3.1)\]

With this substitution, the Ernst equation (2.6) reduces to the linear Euler–Poisson–Darboux equation with non-integer coefficients

\[(f + g)V_{fg} + \frac{1}{2}V_f + \frac{1}{2}V_g = 0.\quad (3.2)\]

It is well known that the general solution of the characteristic initial value problem for equation (3.2) admits an explicit integral representation. This can be expressed as the sum of two components whose spectral functions can be related to the corresponding characteristic data by an Abel transform [12]. Each of these components is equivalent to the form of the Rosen pulse solution adapted to a more general context. The purpose of this section is to adapt the corresponding integral representation for the description of gravitational waves with distinct wavefronts on a given background, and to clarify the relation between this general form of the solution and the series of particular exact solutions found explicitly [6–11].

In these cases, space-times describing gravitational waves with a distinct wavefront on the null hypersurface $u = 0$, on which we will normally have $f = 0$, can be constructed by considering solutions of the form

\[V = V_b + \Theta(u)\tilde{V}\quad (3.3)\]

where $V_b(f, g)$ represents the background space-time, $\Theta(u)$ is the Heaviside step function which is zero in the background region and $\tilde{V}(f, g)$ represents the wave component. In order to satisfy the O’Brien–Synge junction conditions at the wavefront, it is necessary that $\tilde{V}(0, g) = 0.$
Assuming that \( f \geq 0 \) and \( f + g > 0 \) behind the wavefront, the general solution for \( \tilde{V} \) can be written in the form

\[
\tilde{V}(f, g) = \int_0^f \frac{F(\sigma) \, d\sigma}{\sqrt{f - \sigma \sqrt{g + \sigma}}}. \tag{3.4}
\]

Putting \( f = t - z \) and \( g = t + z \), this may be expressed as

\[
\tilde{V}(t, z) = \int_0^{t-z} \frac{F(\sigma) \, d\sigma}{\sqrt{t^2 - (z + \sigma)^2}}. \tag{3.5}
\]

However, in some geometrical contexts, (such as for cylindrical waves in which \( f = -(t - \rho) \leq 0, g = t + \rho \) \( f \leq 0, f + g > 0 \) behind the wavefront. In such cases an equivalent integral representations can be given, for which the equivalent of (3.5) is identical to the Rosen pulse solution (1.1).

We concentrate here on the case in which \( f > 0 \) in the wave region. It is also convenient to introduce the new spectral function \( A(\sigma) \equiv F(\sigma)/\sqrt{\sigma} \). Then, \( A(\sigma) \), and hence \( F(\sigma) \), can be determined explicitly in terms of the initial data by considering this integral on the hypersurface \( g = 0 \) and using the Abel transform

\[
\tilde{V}(f, 0) = \int_0^f \frac{A(\sigma) \, d\sigma}{\sqrt{f - \sigma}}, \quad \Leftrightarrow \quad A(\sigma) = \frac{1}{\pi} \int_0^\sigma \frac{\tilde{V}(f, 0) \, df}{\sqrt{\sigma - f}}. \tag{3.6}
\]

which was used by Hauser and Ernst [12] in the initial value problem for colliding plane waves.

In the integral forms of the solution (1.1) and (3.4–5), complete explicit solutions for the metric function \( M \) have not be obtained. However, as shown in [8] and [11], complete solutions with the required properties can be obtained by summing over explicit components each of which have the self-similar form

\[
\tilde{V}_k(f, g) = (f + g)^k H_k \left( \frac{g-f}{f+g} \right) \tag{3.7}
\]

where \( k \) is an arbitrary real parameter, and the functions \( H_k(\zeta) \) satisfy the linear equation

\[
(1 - \zeta^2)H_k'' + (2k - 1)\zeta H_k' - k^2 H_k = 0, \tag{3.8}
\]

with the initial condition \( H_k(1) = 0 \). This satisfies the recursion relations

\[
H_k(\zeta) = \int_1^\zeta H_{k-1}(\zeta') \, d\zeta' \quad \text{so that} \quad H'_k(\zeta) = H_{k-1}(\zeta). \tag{3.9}
\]

Solutions with these properties can be expressed in terms of standard hypergeometric functions in the form

\[
(f + g)^k H_k \left( \frac{g-f}{f+g} \right) = c_k \frac{f^{1/2+k}}{\sqrt{f+g}} F \left( \frac{1}{2}, \frac{1}{2} ; \frac{3}{2} + k ; \frac{f}{f+g} \right) \tag{3.10}
\]
where\(^1\), for integer \(k\)
\[
c_k = (-1)^k \frac{2^k \Gamma\left(\frac{3}{2}\right)}{\Gamma\left(k + \frac{3}{2}\right)}.
\]

However, when applying (3.9) for arbitrary values of \(k\), we only require the recursion relation \(c_{k-1} = -\frac{1}{2}(k + \frac{1}{2})c_k\).

These solutions for each \(k\) may be expressed in one of the above integral forms by first evaluating it on the characteristic \(g = 0\) on which
\[
\tilde{V}_k(f, 0) = f^k H_k(-1).
\]

Then, using the Abel transform (3.6), we obtain
\[
A(\sigma) = \frac{k}{\pi} H_k(-1) \int_0^\sigma f^{k-1} \frac{df}{\sqrt{\sigma - f}} = \frac{k}{\pi} H_k(-1) \sigma^{k-1/2} \int_0^1 x^{k-1} \frac{dx}{\sqrt{1 - x}}.
\]

Since the final integral is a constant, it may be concluded that \(A(\sigma)\) is a constant multiple of \(\sigma^{k-1/2}\). It then follows that the particular solution (3.7) corresponds to the Rosen form in which the function \(F(\sigma)\) given in (1.1), (3.4), or (3.5) is a constant multiple of \(\sigma^k\). In addition, it is easy to show directly from (1.1) that replacing \(F(\sigma)\) by some power leads to a self-similar solution. Thus, substituting \(\sigma = t - \rho x\) gives
\[
V = \int_0^{t-\rho} \sigma^k \frac{d\sigma}{\sqrt{(t-\sigma)^2 - \rho^2}} = \rho^k \int_1^{\zeta} (\zeta - x)^k \frac{dx}{\sqrt{x^2 - 1}}.
\]

which is an integral representation of \(H_k(\zeta)\) for \(\zeta \equiv \frac{t}{\rho} \geq 1\).

In the linear case, the general class of solutions which describes waves with the distinct wavefront \(f = 0\) can be expressed as a sum of terms of the form (3.7). This can generally be expressed in the form
\[
\tilde{V} = \int_\alpha^\infty \phi(k) (f + g)^k H_k\left(\frac{4-t}{1+g}\right) \, dk
\]

where \(\phi(k)\) is an arbitrary “spectral amplitude” function, and the lower limit \(k = \alpha\) is chosen to ensure that there are no curvature singularities on the wavefront. This particular representation permits the remaining metric function \(M\) to be determined explicitly (see [8] and [11]), at least for a number of significant backgrounds.

It may be noted that, for a large class of initial data, it is possible to expand the corresponding \(F(\sigma)\) in a Taylor series. In this case it is only necessary to sum terms of the form (3.7) over integer values of \(k > 0\). However, as shown in [9]–[11], in order to include impulses or steps in the Weyl tensor components it is sometimes necessary to take the minimum value of \(k\) in (3.15) as \(\alpha = \frac{1}{2}\). These represent impulsive or shock gravitational

\(^1\) It may be noted that the expression (3.11) for \(c_k\) replaces incorrect expressions in [10] and [11].
waves in some backgrounds. In such situations, it may be appropriate simply to include
other terms with half integer values of $k$.

It has thus been clarified that the above class of solutions (3.3) and (3.15) can be
considered as an alternative and explicit representation of the Rosen pulse. It is in a form
that is applicable to a number of geometrical backgrounds in which a complete solution
for the metric can be obtained explicitly.

4. A simple technique for generating non-diagonal solutions

In the nondiagonal case for which $\omega \neq 0$, $Z$ is necessarily complex and the main equa-
tion (2.6) cannot be reduced to a linear form. However, a number of internal symmetries
of the Ernst equation can be used to generate such solutions.

It is well known that there exists another complex potential $E$ (usually called the
Ernst potential) which is defined by

$$
E = e^{-U} \chi^{-1} + i\tilde{\omega}, \quad \text{where} \quad \begin{cases}
\tilde{\omega}_u = e^{-U} \chi^{-2} \omega_u, \\
\tilde{\omega}_v = -e^{-U} \chi^{-2} \omega_v
\end{cases}
$$

(4.1)

It is a remarkable property that the function $E$ satisfies the same (Ernst) equation (2.6)
as the function $Z$. The identity of the equations for $Z$ and $E$ manifests the existence of a
discrete symmetry of the vacuum Einstein equations for metrics of the form (2.1). This is
known as the Kramer–Neugebauer mapping (or involution):

$$
Z \leftrightarrow E \quad \text{i.e.} \quad \chi \leftrightarrow e^{-U} \chi^{-1}, \quad \omega \leftrightarrow \tilde{\omega}.
$$

(4.2)

This mapping is not pure gauge and, in general, it generates from any given solution
another which has a different space-time geometry. A second application of this symmetry
simply leads back to the original solution apart from an imaginary constant that has no
physical significance. According to this mapping, any solution of the Ernst equation (2.6)
can be adopted either as $Z$ or $E$. It can then be shown that, for any vacuum space-time
in which $e^{-U} = f + g$, $Z = \chi + i\omega$ and $M = M_0$, there exists another solution given by
(4.2) in which $e^{-U} = f + g$ is unchanged, and $M$ is given by

$$
M = M_0 - \frac{1}{2} \log(f + g) + \log \chi.
$$

(4.3)

Having different curvatures, these two possibilities correspond to different physical prop-
erties.

Another two very useful groups of symmetries of the vacuum Einstein equations in
this case arise from a continuous group of internal symmetries of the Ernst equation. These
are the SL(2, $R$) point transformations of the complex potentials $Z$ and $E$:

$$
Z \rightarrow i \frac{aZ + ib}{cZ + id} \quad \text{or} \quad E \rightarrow i \frac{aE + ib}{cE + id}, \quad \text{both with} \quad ad - bc = 1.
$$

(4.4)
However, there is a very important difference between the corresponding two $\text{SL}(2, \mathbb{R})$ groups of symmetries. The transformations of the $Z$ potential are pure gauge transformations for the metric (2.1) which correspond to a rotation and rescaling of the $x, y$ coordinates relative to its Killing vectors. These leave $M$ invariant for the same $U$ and have no physical significance. By contrast, the similar $\text{SL}(2, \mathbb{R})$ transformations of the $\mathcal{E}$ potential, when expressed in terms of the components of the metric (2.1), can generate changes in the space-time geometry. These latter transformations are known as Ehlers transformations.

Let us now start with an initial real solution $Z_0 = e^{-V}$, where $V$ satisfies (3.2), and take this as an initial solution for $\mathcal{E}$ namely: $\mathcal{E}_0 = e^{-V}$. Then using the Ehlers transformation, as described above, a complex solution of the Ernst equation (2.6) can be obtained by applying a transformation (4.4) for $\mathcal{E}$ to give

$$\mathcal{E} = i \frac{(ae^{-V} + ib)}{(ce^{-V} + id)}, \quad \text{with} \quad ad - bc = 1. \quad (4.5)$$

This has real and imaginary parts given by

$$e^{-U} \chi^{-1} = \frac{1}{d^2 e^V + c^2 e^{-V}}, \quad \tilde{\omega} = \frac{bde^V + ace^{-V}}{d^2 e^V + c^2 e^{-V}}. \quad (4.6)$$

Substituting this into (4.1) yields the equations for $\omega$

$$\omega_f = -2cd(f + g)V_f, \quad \omega_g = 2cd(f + g)V_g. \quad (4.7)$$

It may be noted that, for nondiagonal solutions, both $c$ and $d$ must be nonzero and, without loss of generality, it is always possible to put $c = d = 1/\sqrt{2}$.

The significant and somewhat surprising feature of the equations (4.7) is that they are linear in $V$, which itself satisfies the linear equation (3.2). This feature permits arbitrary solutions for $V$ and the corresponding $\omega$ to be superposed to give a class of solutions of the Ernst equation in the form

$$Z = (f + g) \cosh V + i \omega \quad (4.8)$$

where $V$ satisfies (3.2) and $\omega$ satisfies (4.7) — all linear equations. The arbitrary constant which appears in the integration of (4.7) must be chosen such that the solution can be connected to an appropriate background solution ahead of the wave. In addition, the solution for $M$ using (4.3) is given explicitly by

$$e^{-M} = \sqrt{f + g} \cosh V e^{-M_0} \quad (4.9)$$

where $M_0$ is the corresponding metric function for the initial diagonal solution for which $Z_0 = e^{-V}$.
5. An explicit class of nondiagonal vacuum solutions

At this point it is convenient to introduce the alternative coordinates \( \tau \) and \( \zeta \) defined by

\[
\tau = f + g, \quad \zeta = \frac{g - f}{f + g}.
\]  \hspace{1cm} (5.1)

Using these variables, equations (4.7) become

\[
\begin{align*}
\omega_\tau &= \tau \zeta V_\tau - (\zeta^2 - 1)V_\zeta \\
\omega_\zeta &= \tau^2 V_\tau - \tau \zeta V_\zeta.
\end{align*}
\]  \hspace{1cm} (5.2)

Then, putting

\[
V = \tau^k H_k(\zeta)
\]  \hspace{1cm} (5.3)

where \( H_k(\zeta) \) satisfies (3.8), the equations (5.2) can be integrated to yield

\[
\omega = \frac{1}{k+1} \tau^{k+1} [k \zeta H_k + (1 - \zeta^2) H_{k-1}]
\]  \hspace{1cm} (5.4)

in which the constant of integration has been set to zero to put \( \omega = 0 \) on the wavefront on which \( \zeta = 1 \). This solution of the Ernst equation is thus given by

\[
Z(\tau, \zeta) = \tau \cosh V + i\omega
\]  \hspace{1cm} (5.5)

where \( V \) is given by (5.3) and \( \omega \) is given by (5.4).

We may now use the property that equations (4.7) for \( \omega \), and hence (5.2), are linear in \( V \). It follows that separate solutions for \( V \) can be superposed. We thus obtain a general class of complex solutions of the Ernst equation in the form (5.5) where

\[
V = \int_\alpha^\infty \phi(k) \tau^k H_k(\zeta) \, dk.
\]  \hspace{1cm} (5.6)

and

\[
\omega = \int_\alpha^\infty \phi(k) \frac{1}{k+1} \tau^{k+1} [k \zeta H_k + (1 - \zeta^2) H_{k-1}] \, dk.
\]  \hspace{1cm} (5.7)

where \( \phi(k) \) is an arbitrary spectral amplitude function, defined over \([\alpha, \infty)\), subject only to the condition that the above integrals exist. As explained in [11], the lower limit \( \alpha \) for \( k \) must be chosen such that the space-time is nonsingular on the wavefront. This choice also determines the behaviour of the Weyl tensor on the wavefront. It will be shown that, taking \( \alpha = \frac{1}{2} \), introduces an impulsive wave component. Gravitational waves with step wavefronts may alternatively occur if \( \alpha = \frac{3}{2} \).

It may also be observed that the expression for \( \omega \) given by (5.7) is functionally independent of \( V \) as given by (5.6). It follows from this that it is not possible to introduce a rotation that will diagonalise the solution globally. It may thus be concluded that these solutions are genuinely nondiagonal.
We now turn to the question of obtaining an explicit expression for \( M \). It is not necessary to evaluate this by integrating (2.3). Rather, we can use (4.9) and the expressions given in [11] for the diagonal case to put

\[
e^{-M} = |f'g'| \cosh V e^{-S}
\]

where

\[
S = -\frac{1}{2} \int_{\alpha}^{\infty} \int_{\alpha}^{\infty} \phi(k)\phi(k') \frac{\tau^{k+k'}}{k+k'} \left[ kk'H_k H_{k'} + (1 - \zeta^2) H_{k-1} H_{k'-1} \right] dk dk' \\
= -\frac{1}{2} \int_{2\alpha}^{\infty} \frac{d\tau n}{n} \int_{\alpha}^{n-\alpha} \phi(k)\phi(n-k) \left[ k(n-k)H_k H_{n-k} + (1 - \zeta^2) H_{k-1} H_{n-k-1} \right] dk.
\]

(5.9)

This completes the integration of the field equation for our general class of solutions.

6. Possible backgrounds and the character of the wavefronts

In the above discussion it has been assumed that we are considering a wave solution with a wavefront given by \( f = 0 \), or \( \zeta = 1 \). For the linear case considered in [11], it was convenient to take \( V = 0 \) on the wavefront and to superpose such a solution on some given background as in (3.3). In the nonlinear case being considered here, we can similarly superpose a background onto (5.6). This then introduces additional background terms in (5.7).

For the simplest cases, however, in which the background metric is diagonal, \( \omega \) should be zero for \( f \leq 0 \) and the expression for \( V \) in the background must vanish. In this case, no additional terms should appear in (5.6) and, from (5.5), it can be seen that the solution in the background region \( f \leq 0 \) is given by

\[
Z = \tau = f + g,
\]

(6.1)

with line element given by

\[
ds^2 = 2|f'g'|du dv - dx^2 - (f + g)^2 dy^2.
\]

(6.2)

It is now necessary to distinguish two cases according to whether the gradient of \( f + g \) is spacelike or timelike.

In the case in which the gradient of \( f + g \) is spacelike, it is convenient to put

\[
f = -\frac{1}{2}(t - \rho), \quad g = \frac{1}{2}(t + \rho).
\]

(6.3)

Then, relabelling the other coordinates by putting \( x = z \) and \( y = \phi \in [0, 2\pi) \), the background is seen to be Minkowski space in cylindrical coordinates

\[
ds^2 = dt^2 - d\rho^2 - \rho^2 d\phi^2 - dz^2
\]

(6.4)
and the wavefront \( f = 0 \) is the expanding cylinder \( \rho = t \). The complete solution, as constructed above, thus describes a cylindrical gravitational wave with a distinct wavefront. This can be seen to be an explicit nondiagonal generalisation of the Rosen pulse in which the complete integral of all the metric functions has been obtained.

In the alternative case in which the gradient of \( f + g \) is timelike, it is convenient to put

\[
    f = \frac{1}{2}(t - z), \quad g = \frac{1}{2}(t + z).
\]

In this case the background space-time is given by the line element

\[
    ds^2 = dt^2 - dz^2 - dx^2 - t^2 dy^2. \tag{6.6}
\]

This can be seen to be one of the particular forms of Minkowski space that was considered in [11]. Using the results of sections 3 and 4 of [11], the complete solutions in this case can be seen to describe nondiagonal generalisations of the solutions given there which represent gravitational waves propagating into a Minkowski background with half-cylindrical wavefronts driven by two separating singular null half-planes.

The behaviour of the gravitational wave near the wavefront in each case will be determined by the component with the lowest value of \( k \). We may thus consider \( V \) to be given by

\[
    V \approx \phi(\alpha)\tau^\alpha H_\alpha(\zeta) \Theta(u) \tag{6.7}
\]

near the wavefront \( u = 0, f = 0 \) or \( \zeta = 1 \), and we assume that \( \phi(\alpha) \neq 0 \). Just behind the wavefront \( V \) will be small and using (5.5), (5.4) and (3.10), to first order, we obtain

\[
    Z \approx \tau + i\omega \\
    \approx \tau + i\frac{\phi(\alpha)}{\alpha + 1} \tau^{\alpha+1}[\alpha\zeta H_\alpha + (1 - \zeta^2)H_{\alpha-1}] \Theta(u) \\
    \approx g - i\phi(\alpha)c_\alpha g^{-1/2}f^{\alpha+1/2} \Theta(u). \tag{6.8}
\]

in which, for integer \( \alpha \), \( c_\alpha \) is given by (3.11) — otherwise \( c_\alpha \) may be chosen arbitrarily. Substituting these expressions into (2.8) we find that, near the wavefront, the non-zero components of the Weyl tensor are given by

\[
    \Psi_4 \approx \frac{i}{2} \frac{f'}{g}(\alpha + \frac{1}{2})\phi(\alpha)c_\alpha g^{-1/2}f^{\alpha-1/2} \delta(u) + \frac{i}{2} \frac{f'}{g}(\alpha^2 - \frac{1}{4})\phi(\alpha)c_\alpha g^{-1/2}f^{\alpha-3/2} \Theta(u) \\
    \Psi_2 \approx \frac{i}{4} (\alpha + \frac{1}{2})\phi(\alpha)c_\alpha g^{-3/2}f^{\alpha-1/2} \Theta(u) \\
    \Psi_0 \approx \frac{3i}{8} \frac{g'}{f}\phi(\alpha)c_\alpha g^{-5/2}f^{\alpha+1/2} \Theta(u). \tag{6.9}
\]

It can thus be seen that this class of solutions includes an impulsive gravitational wave if \( \alpha = \frac{1}{2} \), the gravitational wave includes a step (or shock) if \( \alpha = \frac{3}{2} \), the Weyl tensor is \( C^n \) across the wavefront if \( \alpha > n + \frac{3}{2} \).
7. Electromagnetic waves with distinct wavefronts

For an electrovac space-time with two Killing vectors, it is known that the main field equations can be expressed as the Ernst equations

\[
(\text{Re} \, \mathcal{E} - H\bar{H})\nabla^2 \mathcal{E} = (\nabla \mathcal{E})^2 - 2\bar{H}(\nabla H).(\nabla \mathcal{E})
\]

\[
(\text{Re} \, \mathcal{E} - H\bar{H})\nabla^2 H = (\nabla \mathcal{E}).(\nabla H) - 2\bar{H}(\nabla H)^2
\]

where \( \mathcal{E} \) is given by

\[
\mathcal{E} = e^{-U}\chi^{-1} + i\tilde{\omega} + H\bar{H}.
\]

In this case, \( H \) is the electromagnetic potential and \( \mathcal{E} \) is the generalisation of the vacuum \( \mathcal{E} \)-potential which involves the functions \( e^{-U}\chi^{-1} \) and \( \tilde{\omega} \), given by (4.1), rather than the function \( Z \) which contains the explicit metric functions \( \chi \) and \( \omega \) as in (2.4).

It is well known that a Bonnor transformation [13] can be used to relate a class of diagonal electromagnetic solutions to a class of nondiagonal vacuum solutions. The precise result may be stated in the form that, if \( Z_0 \) is any complex solution of the vacuum Ernst equation (2.5) or (2.6) using (2.4) with the corresponding \( M_0 \), then a solution (7.2) of the Ernst equations for an electromagnetic field is given by

\[
\mathcal{E} = Z_0\bar{Z}_0, \quad H = \frac{1}{2}(\bar{Z}_0 - Z_0)
\]

and the remaining metric function is given by

\[
e^{-M} = \frac{4(f + g)^2}{|f'g'|^3(Z_0 + \bar{Z}_0)^2}e^{-4M_0}.
\]

The inverse transformation also holds.

We may now apply this to the initial complex solution \( Z_0 = (f + g)\cosh V_o + i\omega_o \) as in (4.8) with the corresponding \( M_0 \). This then gives an electrovac solution in which

\[
\mathcal{E} = (f + g)^2\cosh^2 V_o + \omega_o^2, \quad H = -i\omega_o,
\]

where \( V_o \) and \( \omega_o \) are solutions of the linear equations (3.2) and (4.7). It follows that the new metric functions are given by

\[
\chi = (f + g)^{-1}\text{sech}^2 V_o, \quad \omega = 0, \quad e^{-M} = |f'g'|^{-3}\text{sech}^2 V_o e^{-4M_0}
\]

Taking \( V_o \) and \( \omega_o \) in the forms (5.6) and (5.7), then \( e^{-M} \) takes the form

\[
e^{-M} = |f'g'|\cosh^2 V_o e^{-4S}
\]

where \( S \) is given by (5.9). This solution describes an electromagnetic wave with wavefront given by \( f = 0 \) propagating into a vacuum background region in which \( \chi = \tau^{-1} = (f + g)^{-1} \). This background is exactly equivalent to that considered in section 6, and so these solutions describe a combination of gravitational and electromagnetic waves with cylindrical or half-cylindrical wavefronts propagating into a Minkowski background.
8. Concluding remarks

In the above sections we have effectively given an alternative and explicit representation of the Rosen pulse solution. This new representation is significant for a number of reasons. To start with, it enables a complete integration of the subsidiary field equations, giving an explicit expression for $M$ in terms of a spectral parameter associated with the pulse. It also enables the solution to be used in alternative contexts in addition to cylindrical waves. This then describes a variety of exact solutions which represent gravitational waves with distinct wavefronts and which propagate into a number of different backgrounds.

We have also described a method of constructing a particular class of complex solutions of the Ernst equation which are obtained from a set of linear equations. This has enabled us to explicitly construct a non-diagonal generalisation of the Rosen pulse. These describes a family of cylindrical waves in vacuum or electrovacuum with a distinct wavefront and with two states of polarization that propagate into a Minkowski background.

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References

[1] O’Brien, S. and Synge, J. L. (1952) Commun. Dublin Inst. Adv. Stud. A, no 9.
[2] Kramer, D., Stephani, H., MacCallum, M. A. H., and Herlt, E. (1980) Exact solutions of Einstein’s field equations. (Cambridge University Press).
[3] Bondi, H., Pirani, F. A. E. and Robinson, I. (1959) Proc. Roy. Soc. A, 251, 519–33.
[4] Rosen, N. (1954) Bull. Res. Council Israel., 3, 328–32.
[5] Marder, L. (1958) Proc. Roy. Soc. A, 244, 524–37.
[6] Griffiths, J. B. (1993) Class. Quantum Grav. 10, 975–83.
[7] Bičák, J. and Griffiths, J. B. (1994) Phys. Rev. D, 49, 900–6.
[8] Alekseev, G. A. and Griffiths, J. B. (1995) Phys. Rev. D, 52, 4497–502.
[9] Bičák, J. and Griffiths, J. B. (1996) Ann. Phys., 252, 180–210.
[10] Alekseev, G. A. and Griffiths, J. B. (1996) Class. Quantum Grav., 13, L13–8.
[11] Alekseev, G. A. and Griffiths, J. B. (1996) Class. Quantum Grav., 13, 2191–209.
[12] Hauser, I. and Ernst, F. J. (1989) J. Math. Phys., 30, 872–887.
[13] Bonnor, W. B. (1961) Z. Phys., 161, 439–44.