A Note On Characterizations of Spherical t-Designs

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Abstract A set \( X_N = \{x_1, \ldots, x_N\} \) of \( N \) points on the unit sphere \( S^d, d \geq 2 \) is a spherical \( t \)-design if the average of any polynomial of degree at most \( t \) over the sphere is equal to the average value of the polynomial over \( X_N \). This paper extends characterizations of spherical \( t \)-designs in \([2]\) from \( S^2 \) to general \( S^d \). We show that for \( N \geq \text{dim}(P_{t+1}) \), \( X_N \) is a stationary point set of a certain non-negative quantity \( A_{N,t} \), and a fundamental system for polynomial space over \( S^d \) with degree at most \( t \), then \( X_N \) is a spherical \( t \)-design. In contrast, we present that with \( N \geq \text{dim}(P_t) \), a fundamental system \( X_N \) is a spherical \( t \)-design if and only if non-negative quantity \( D_{N,t} \) vanishes. In addition, the still unanswered questions about construction of spherical \( t \)-designs are discussed.

Keywords: spherical \( t \)-design; non-negative quantity; nonlinear equation; fundamental system

1 Introduction

Let \( X_N = \{x_1, \ldots, x_N\} \) be a set of \( N \) points on the sphere \( S^d = \{x \mid \|x\|_2 = 1\} \subset \mathbb{R}^{d+1} \), and let \( P_t(S^d) \) be the linear space of restrictions of polynomials of degree at most \( t \) in \( d + 1 \) variables to \( S^d \). The set \( X_N \) is a spherical \( t \)-design if the average of any polynomial of degree at most \( t \) over the sphere is equal to the average value of the polynomial over \( X_N \). That is

\[
\frac{1}{N} \sum_{j=1}^{N} p(x_j) = \frac{1}{\omega_d} \int_{S^d} p(x)d\omega_d(x) \quad \forall p \in P_t(S^d),
\]

where \( d\omega_d(x) \) denotes the surface measure on \( S^d \), and \( \omega_d \) is the surface area of the unit sphere \( S^d \). From (1.1), it can be seen that \( X_N \) is a spherical \( t \)-design if the equal weight cubature rule (with weight \( \omega_d/N \)) is exact for all spherical polynomials \( p \in P_t(S^d) \).

Spherical \( t \)-designs was introduced by Delsarte, Goethals, and Seidel [15] in 1977. And the

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1The work is supported by NSF of China (No.11301222, No.11226305), and NSF of Guangdong Province (No. S2012040007860)
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lower bound on \( N \) was given as the following \([15]\):

\[
N \geq \begin{cases} 
2 \binom{d+s}{d}, & \text{if } t = 2s + 1, \\
\binom{d+s}{d} + \binom{d+s-1}{d}, & \text{if } t = 2s.
\end{cases}
\] (1.2)

In the past three decades, many works have been done on spherical \( t \)-designs, see literature \([2][3][5][7][8][9][17][20][30][31][35]\). Seymour and Zaslavsky \([25]\) showed that a spherical \( t \)-design exists for any \( t \) if \( N \) is sufficiently large. In 2009, there is a comprehensive survey of research on spherical \( t \)-designs in the last three decades provided by Bannai and Bannai \([7]\). As pointed out in \([30]\), \( X_N \) is a spherical \( t \)-design if and only if a certain non-negative quantity \( A_{N,t} \) is zero. Moreover, Sloan and Womersley \([30]\) used a condition based on the mesh norm to help determine if a stationary point of \( A_{N,t} \) is a spherical \( t \)-design. In particular for \( \mathbb{S}^2 \), Chen and Frommer and Lang proved that spherical \( t \)-designs with \( N = (t+1)^2 \) is enough for all \( t \) up to \( t = 100 \). The well conditioned spherical \( t \)-designs \([2]\) have been applied into interpolation, numerical integration and regularized least squares approximation on \( \mathbb{S}^2 \) \([3]\). In very recently, the existence of a spherical \( t \)-design for all \( N \geq ct^d \) for some unknown \( c > 0 \) has been proved by Bondarenko, Radchenko and Viazovska \([11]\). In addition, Bondarenko, Radchenko and Viazovska \([12]\) proved the existence of well separated spherical \( t \)-designs for all \( N \geq ct^d \) for some unknown \( c > 0 \). To our knowledge, there is no constructive proof that spherical \( t \)-designs on general unit sphere \( \mathbb{S}^d \) with \( N \sim \dim(\mathbb{P}_t) \). How to find a spherical \( t \)-design in general \( \mathbb{S}^d \) for large \( t \) is merit to be studied.

Spherical \( t \)-designs can be used to realize numerical interpolation (when \( N \) is right) \([2]\), integration \([2][34]\), hyperinterpolation \([26]\), filtered hyperinterpolation \([28]\) and regularized least squares approximation \([3]\). Among most of these approximation schemes, it can be seen that the number of points should be no less than the dimension of polynomial space. Therefore, it is natural that we consider \( N \geq \dim(\mathbb{P}_t) \) in this paper.

Our aim is to explore characterizations of spherical \( t \)-designs on general unit sphere \( \mathbb{S}^d \) with the requirement that \( X_N \) is a fundamental system. We extend the \([2]\) Theorem 3.6] in \( \mathbb{S}^2 \) based on \([2]\) to general sphere \( \mathbb{S}^d, d \geq 2 \). On the other hand, we generalize the nonlinear approach to find a spherical \( t \)-design for \( N \) no less than the dimension of \( \mathbb{P}_t \) in \([2]\). Furthermore, we show that a point set is a spherical \( t \)-design if and only if a nonnegative quantity is zero when \( N \geq \dim(\mathbb{P}_t) \).

In the next section we give some necessary materials of spherical polynomials and known results. In Section 3, we present some results on the unit sphere \( \mathbb{S}^d \). We sketch proofs in Section 4. Finally, in Section 5 we discuss the known properties on spherical \( t \)-designs, and draw attention to some still unknown problems.
2 Preliminaries

In the remainder of this paper, we follow notation and terminology in [2] and [30].

Let \( \{ Y_{\ell,k} : k = 1, \ldots, M(d, \ell), \ell = 0, 1, \ldots, t \} \) be an orthonormal set of (real) spherical harmonics, with \( Y_{\ell,k} \) a spherical harmonic of degree \( \ell \) (see [4] and [21]).

As is known to all, we have

\[
P_t := P_t(S_d) = \text{span}\{ Y_{\ell,k} : k = 1, \ldots, M(d, \ell) \}, \quad \ell = 0, 1, \ldots, t,
\]

where \( a_t \sim b_t \) means that positive constants \( c_1, c_2 \) exist, independently of \( t \), such that \( c_1 a_t \leq b_t \leq c_2 a_t \). The addition theorem [21] for spherical harmonics is

\[
\sum_{k=1}^{M(d,\ell)} Y_{\ell,k}(x) Y_{\ell,k}(y) = \frac{M(d,\ell)}{\omega_d} P_{\ell}^{(d+1)}(x \cdot y), \quad \forall \ x, y \in S_d.
\]

(2.4)

where \( x \cdot y \) is the inner product in \( \mathbb{R}^{d+1} \) and \( P_{\ell}^{(d+1)} \) is the Legendre polynomial in \( \mathbb{R}^{d+1} \) of degree \( \ell \) normalized so that \( P_{\ell}^{(d+1)}(1) = 1 \).

For \( t \geq 1 \), \( N \geq d_t \) let the matrices \( Y_{t0} \) and \( Y_t \) be defined by

\[
Y_{t0} := [Y_{\ell,k}(x_j)], \quad k = 1, \ldots, M(d, \ell), \quad \ell = 1, \ldots, t; \quad j = 1, \ldots, N,
\]

and

\[
Y_t := \begin{bmatrix}
\frac{1}{\sqrt{\omega_d}} e^T \\
Y_{t0}^T
\end{bmatrix} \in \mathbb{R}^{d_t \times N},
\]

(2.6)

where \( e = [1, \ldots, 1]^T \in \mathbb{R}^N \).

It is well known that \( X_N \) a spherical \( t \)-design if and only if Weyl sums vanishes (see for example [30] and [15]), i.e.,

\[
\sum_{j=1}^N Y_{\ell,k}(x_j) = 0, \quad k = 1, \ldots, M(d, \ell), \quad \ell = 1, \ldots, t.
\]

(2.7)

With the aid of (2.5), (2.7) can be written in matrix-vector form as

\[
r(X_N) := Y_{t0}^T e = 0.
\]

(2.8)

where \( r(X_N) \in \mathbb{R}^{d_t-1} \).
Consequently, we can define the nonnegative quantity $A_{N,t}$

$$A_{N,t}(X_N) := \frac{\omega_d}{N^2} r(X_N)^T r(X_N).$$

The distance between any two points $x$ and $y$ on the unit sphere $S^d$ is measured by the geodesic distance $\text{dist}(x, y) := \cos^{-1}(x \cdot y) \in [0, \pi]$.

**Definition 2.1** The mesh norm $h_{X_N}$ of a point set $X_N \subset S^d$ is

$$h_{X_N} := \max_{y \in S^d} \min_{x_i \in X_N} \text{dist}(y, x_i),$$

(2.9)

**Definition 2.2** The set $X_N \subset S^d$ is a fundamental system for $\mathbb{P}_t$ if the zero polynomial is the only element of $\mathbb{P}_t$ that vanishes at each point in $X_N$, that is

$$p \in \mathbb{P}_t, \quad p(x_i) = 0, \quad i = 1, \ldots, N$$

(2.10)

implies $p(x) \equiv 0$ for all $x \in S^d$.

For the case on $S^2$ [2], the definition is also valid when $N$ is large than the dimension of the polynomial space. In this paper, we claim that this statement is also true on $S^d$, see Lemma 3.1.

As shown in [14] [13] and [2], one can find a spherical $t$-design on $S^2$ by solving a system of underdetermined nonlinear equations. In this paper, we define the nonlinear function $C_t : (S^d)^N \rightarrow \mathbb{R}^{N-1}$ as follows:

$$C_t(X_N) := E G_t(X_N) e,$$

(2.11)

where

$$E := [e, -I_{N-1}] \in \mathbb{R}^{(N-1) \times N} \quad \text{and} \quad G_t := Y_t^T Y_t \in \mathbb{R}^{N \times N},$$

(2.12)

### 3 Theorems

In this section we present some results on $S^d$, which are generalized from $S^2$ in [2].

**Lemma 3.1** $X_N$ is a fundamental system for $\mathbb{P}_t$ if and only if $Y_t$ is of full row rank $\dim(\mathbb{P}_t)$.

**Theorem 3.1** If the mesh norm of the point set $X_N$ satisfies $h_{X_N} < \frac{1}{t}$, then $X_N$ is a fundamental system for $\mathbb{P}_t$.

**Remark 1** Lemma [3.7] and Theorem [3.7] can be proved by the same way from [2]. Theorem [3.7] shows the condition of mesh norm is stronger that the condition of fundamental system. The explain for $S^2$ is given by [2].
Theorem 3.2 Let $t \geq 1$ and $N \geq \dim(\mathbb{P}_{t+1})$. Let $X_N \subset S^d$ be a stationary point of $A_{N,t}$. Then $X_N$ is a spherical $t$-design, or there exists a non-zero polynomial $p \in \mathbb{P}_{t+1}$, such that $p(x_j) = 0$ for $j = 1, \ldots, N$.

By the definition of fundamental system (see Definition 2.2), we immediately have the following result.

Corollary 3.1 Let $t \geq 1$ and $N \geq \dim(\mathbb{P}_{t+1})$. Assume $X_N \subset S^d$ is a stationary point of $A_{N,t}$, and $X_N$ is a fundamental system for $\mathbb{P}_{t+1}$. Then $X_N$ is a spherical $t$-design.

Theorem 3.3 Let $N \geq d_t$. Suppose that $X_N = \{x_1, \ldots, x_N\}$ is a fundamental system for $\mathbb{P}_t$. 

Define the nonnegative quantity 

$$D_{N,t}(X_N) := \frac{\omega_d^2}{N^2} C_t(X_N)^T C_t(X_N).$$

We have the following result immediately.

Corollary 3.2 Let $N \geq d_t$. Suppose that $X_N = \{x_1, \ldots, x_N\}$ is a fundamental system for $\mathbb{P}_t$. Then 

$$0 \leq D_{N,t}(X_N) \leq 4(N - 1)M^2(d + 1, t),$$

and $X_N$ is a spherical $t$-design if and only if 

$$D_{N,t}(X_N) = 0.$$

Proof: From the definition of $C_t(X_N)$, see (2.11), and with the aid of addition theorem (2.4), we have 

$$\|C_t(X_N)\| = \|(G_t e)_1 - (G_t e)_{t+1}\|$$

$$\leq \frac{2}{\omega_d} \sum_{j=1}^N \sum_{\ell=0}^t M(d, \ell) \omega_d^2 P_{t+1}^{(d+1)}(x_1 \cdot x_j) - P_{t+1}^{(d+1)}(x_{i+1} \cdot x_j)$$

Then by using the definition of (3.13), we obtain (3.14).
Remark 2 For the special case in $d = 2$, (3.15) was shown in [14 Section 4].

4 Proofs

For the completeness of this paper, we give proofs for Theorem 3.2 and Theorem 3.3

4.1 Proof of Theorem 3.2

This theorem rests on the following Lemma taken from [30].

Lemma 4.2 [30] Let $t \geq 1$, and suppose $X_N$ is a stationary point of $A_{N,t}$. Then either $X_N$ is a spherical $t$-design, or there exists a non-constant polynomial $p \in \mathbb{P}_t$ with a stationary point at each point $x_i \in X_N$, $i = 1, \ldots, N$.

In the following for completeness we give the proof of Theorem 3.2.

Proof: Suppose $X_N$ is not a spherical $t$-design. Then by Lemma 4.2 there exists a non-constant polynomial $q \in \mathbb{P}_t$ with a stationary point at each $x_i \in X_N$, $i = 1, \ldots, N$, i.e.

$$\nabla^* q(x_j) = 0, \quad j = 1, \ldots, N, \quad (4.16)$$

Now define

$$p_i = e_i \cdot \nabla^* q, \quad i = 1, \ldots, d + 1,$$

where $e_1, \ldots, e_{d+1}$ are the unit vectors in the direction of the (fixed) coordinate axes for $\mathbb{R}^{d+1}$, and the dot indicates the inner product in $\mathbb{R}^{d+1}$.

By the stationary property of $q$, each $p_i$ for $i = 1, \ldots, d + 1$ satisfies

$$p_i(x_j) = 0, \quad j = 1, \ldots, N.$$ 

Since $q$ is not a constant polynomial, at least one component of $\nabla^* q$ does not vanish identically, hence at least one of $p_1, \ldots, p_{d+1}$ is not identically zero.

Assume

$$p := p_{i_0} \quad (4.17)$$

is not identically zero. Then because $q$ is a linear combination of spherical harmonics $Y_{\ell,k}$ with $\ell = 1, \ldots, t$ (see [16], Chapter 12), then $p = p_{i_0} = e_{i_0} \cdot \nabla^* q$ is a linear combination of spherical harmonics of degree $\ell - 1$ and $\ell + 1$. Thus for $q \in \mathbb{P}_t$, then $p \in \mathbb{P}_{t+1}$.

Finally (4.16) gives

$$p(x_j) = 0, \quad j = 1, \ldots, N, \quad (4.18)$$

completing the proof.
4.2 Proof of Theorem 3.3

Proof: From (2.12), we have
\[
G_t = \left[ \frac{1}{\sqrt{\omega_d}} e \quad (Y^0_t)^T \right] \left[ \frac{1}{\sqrt{\omega_d}} e^T \quad Y^0_t \right] = \frac{1}{\omega_d} ee^T + (Y^0_t)^T Y^0_t.
\]

Hence, from (2.11) and (2.8) we obtain
\[
C_t(X_N) = \frac{1}{\omega_d} E(Y^0_t)^T Y^0_t e = \frac{1}{\omega_d} E(Y^0_t)^T r(X_N).
\] (4.19)

Let \( X_N = \{x_1, \ldots, x_N\} \) be a fundamental system for \( \mathbb{P}_t \).

Assume \( C_t(X_N) = 0 \), so we have
\[
E(Y^0_t)^T r(X_N) = 0.
\]

Then, all elements of \((Y^0_t)^T r(X_N)\) are equal, i.e. there is a scalar \( \nu \) such that
\[
(Y^0_t)^T r(X_N) = \nu e.
\]

This implies
\[
\left[ \frac{1}{\sqrt{\omega_d}} e \quad (Y^0_t)^T \right] \left[ -\sqrt{\omega_d} \nu \quad r(X_N) \right] = Y^0_t \left[ -\sqrt{\omega_d} \nu \quad r(X_N) \right] = 0.
\]

Since \( Y^0_t^T \) is of full (column) rank, the only solution is
\[
\nu = 0, \quad r(X_N) = 0.
\]

\( X_N \) is a spherical \( t \)-design by following the matrix-vector form of Weyl sums is zero, see (4.19).

Conversely, suppose \( X_N \) is a spherical \( t \)-design. By using (2.8), \( r(X_N) = 0 \). From (4.19) we have
\[
C_t(X_N) = 0.
\]

\( \blacksquare \)

5 Discussion

The geometry of a configuration on the unit sphere is a very important issue when one considers numerical interpolation \[2\], potential theory \[18\], and numerical integration \[27\][19]. It is known that a spherical \( t \)-design with a fixed number of points can have arbitrarily small minimum distance between points (see [18]). Thus, a spherical \( t \)-design can be with bad geometry \[2\][20].
**Definition 5.3** A point set \( X_N \subset S^d \) is well separated, if the separation distance

\[
\delta_{X_N} := \min_{x_i, x_j \in X_N, i \neq j} \text{dist}(x_i, x_j) \geq \frac{cd}{N^d}.
\]  

(5.20)

The well conditioned spherical designs is a well separated spherical designs investigated by [2] for \( N = (t + 1)^2 \) on \( S^2 \). The existence of well separated spherical designs has been proved in recently [12].

It is known that mesh norm is the covering radius for covering the sphere with spherical caps of the smallest possible equal radius centered at the points in \( X_N \), while the separation distance \( \delta_{X_N} \) is twice the packing radius, so \( h_{X_N} \geq \delta_{X_N} / 2 \). As mentioned by [19], when the mesh ratio \( \rho_{X_N} \):

\[
\rho_{X_N} := \frac{2h_{X_N}}{\delta_{X_N}} \geq 1
\]

is smaller, the more uniformly are the points distributed on \( S^d \). That is to say, mesh ratio can be regarded as a good measure for the quality of the geometric distribution of \( X_N \). For more information about point sets on sphere and their applications, we refer to [24] and [16].

For choosing the point set \( X_N \), if the points may be freely chosen, then we shall see that there is merit in employing spherical \( t \)-design to be nodes with some appropriate value of \( t \) in practical problems. Spherical \( t \)-designs have many applications: interpolation, hyperinterpolation, numerical integrations, filtered hyperinterpolations and regularized least squares approximations and so on. Among these approximation schemes, especially on constructive approximations, spherical \( t \)-design plays an irreplaceable role. Thus, the study on how to construct the point set is really necessary.

As is shown in [5][7], the power of analytical constructions for spherical \( t \)-designs is limited. It is merit to study how to obtain (approximated) spherical \( t \)-designs by numerical methods. For example, with the aid of nonlinear optimization techniques, there are strong numerical results that there exist spherical \( t \)-designs with close to \((t + 1)^2 / 2\) points [30][36]. Moreover, “symmetric spherical \( t \)-designs” [36] enjoy nice geometrical distribution. This paper provides two ways to determinant spherical \( t \)-designs on \( S^d \) in fundamental systems for given \( t \) with \( N \geq \text{dim}(P_t) \) points. It is evident that high dimensional optimization problems have to be taken into account. Consequently, how to find a spherical \( t \)-design on \( S^d \) for a given large \( t \) by reliable numerical methods? This is a challenge problem even on \( S^2 \). Interval method [1] provided a useful way to guarantee there is a very small neighborhood which contained a true spherical \( t \)-design and computed spherical \( t \)-design [14][13]. Can we extend this tool to general sphere \( S^d \)? This may be the thing to be aimed at.
Clearly, a lot of work is need before we can claim to really understand spherical \( t \)-design and its properties and applications.

6 Acknowledgements

The author would like to acknowledge Eiichi Bannai and Yaokun Wu for their encouragement on this work. He is indebted to Ian. H. Sloan and Xiaojun Chen for discussions on spherical \( t \)-designs and to Rob S. Womersley, Shuogang Gao, Takayuki Okuda, Makoto Tagami, Wei-Hsuan Yu for stimulating suggestions.

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