On the smoothness of Hölder-doubling measures

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1 Introduction

In this paper we consider the question of whether the doubling character of a measure supported on a subset of $\mathbb{R}^m$ determines the regularity of its support (in a classical sense). This problem was studied in [1] for codimension 1 sets under the assumption that the support be flat. Here we study the higher codimension case and remove the flatness hypothesis.

In order to give precise statements we need to introduce some definitions. Fix integer dimensions $0 < n < m$ and a closed set $\Sigma \subset \mathbb{R}^m$. For $x \in \Sigma$ and $r > 0$, set

\begin{equation}
\theta_\Sigma(x, r) = \frac{1}{r} \inf \{D[\Sigma \cap B(x, r), L \cap B(x, r)]; L \text{ is an affine } n\text{-plane through } x\},
\end{equation}

where $B(x, r)$ denotes the open ball of center $x$ and radius $r$ in $\mathbb{R}^m$, and where

\begin{equation}
D[E, F] = \sup \{\text{dist}(y, F); y \in E\} + \sup \{\text{dist}(y, E); y \in F\}
\end{equation}

denotes the usual Hausdorff distance between (nonempty) sets. If there is no ambiguity over the set we are considering we write $\theta(x, r)$ rather than $\theta_\Sigma(x, r)$.

**Definition 1.1** Let $\delta > 0$ be given. We say that the closed set $\Sigma \subset \mathbb{R}^m$ is $\delta$-Reifenberg flat of dimension $n$ if for all compact sets $K \subset \Sigma$ there is a radius $r_K > 0$ such that

\begin{equation}
\theta(x, r) \leq \delta \quad \text{for all} \quad x \in K \quad \text{and} \quad 0 < r \leq r_K.
\end{equation}

Note that it does not make sense to take $\delta$ large (like $\delta \geq 2$), because $\theta(x, r) \leq 2$ anyway.

**Definition 1.2** We say that the closed set $\Sigma \subset \mathbb{R}^m$ is Reifenberg flat with vanishing constant (of dimension $n$) if for every compact subset $K$ of $\Sigma$,

\begin{equation}
\lim_{r \to 0^+} \theta_K(r) = 0,
\end{equation}

where

\begin{equation}
\theta_K(r) = \sup_{x \in K} \theta(x, r).
\end{equation}

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Unless otherwise specified, “measure” here will mean “positive Radon measure”, i.e. “Borel measure which is finite on compact sets.” Let \( \mu \) be a measure on \( \mathbb{R}^m \), set

\[
\text{supp}(\mu) = \{ x \in \mathbb{R}^m; \mu(B(x,r)) > 0 \text{ for all } r > 0 \}.
\]

For a measure \( \mu \) on \( \mathbb{R}^m \), with support \( \Sigma = \text{supp}(\mu) \) we define for \( x \in \Sigma \), \( r > 0 \) and \( t \in (0,1] \) the quantity

\[
R_t(x, r) = \frac{\mu(B(x, tr))}{\mu(B(x, r))} - t^n,
\]

which encodes the doubling properties of \( \mu \).

**Definition 1.3** A measure \( \mu \) supported on \( \Sigma \) is said to be asymptotically optimally doubling if for each compact set \( K \subset \Sigma \), \( x \in K \), and \( t \in \left[ \frac{1}{2}, 1 \right] \)

\[
\lim_{r \to 0^+} \sup_{x \in K} |R_t(x, r)| = 0.
\]

The results in this paper can be summarized as follows: first under the appropriate conditions on \( \theta(x, r) \) (see (1.1)) the asymptotic behavior of \( R_t(x, r) \) as \( r \) tends to 0 fully determines the regularity of \( \Sigma \). Second for asymptotically doubling measures which are Ahlfors regular flatness is an open condition.

We mention the local versions of some of the previous results along these lines.

**Theorem 1.4** ([4], [1]) Let \( \mu \) be an asymptotically doubling measure supported on \( \Sigma \subset \mathbb{R}^m \). If \( n = 1, 2 \), \( \Sigma \) is Reifenberg flat with vanishing constant. If \( n \geq 3 \), there exists a constant \( \delta(n, m) \) depending only on \( n \) and \( m \) such that if \( x_0 \in \Sigma \) and \( \Sigma \cap B(x_0, 2R_0) \) is \( \delta(n, m) \)-Reifenberg flat, then \( \Sigma \cap B(x_0, R_0) \) is Reifenberg flat with vanishing constant.

The converse is also true.

**Theorem 1.5** ([1]) If \( \Sigma \) is a Reifenberg flat set with vanishing constant there exists a measure \( \mu \) supported on \( \Sigma \) which satisfies (1.8).

Precise asymptotic estimates on the quantity \( R_t(x, r) \) yield stronger results about the regularity of \( \Sigma \).

**Theorem 1.6** ([1]) For each constant \( \alpha > 0 \) we can find \( \beta = \beta(\alpha) > 0 \) with the following property. Let \( \mu \) be a measure in \( \mathbb{R}^{n+1} \), set \( \Sigma = \text{supp}(\mu) \), and suppose that for each compact set \( K \subset \Sigma \), there is a constant \( C_K \) such that

\[
\left| \frac{\mu(B(x, tr))}{\mu(B(x, r))} - t^n \right| \leq C_K r^\alpha \quad \text{for } r \in (0,1], \ t \in \left[ \frac{1}{2}, 1 \right] \text{ and } x \in K.
\]

If \( n = 1, 2 \), \( \Sigma \) is a \( C^{1, \beta} \) submanifold of dimension \( n \) in \( \mathbb{R}^{n+1} \). If \( n \geq 3 \), for \( x_0 \in \Sigma \) if \( \Sigma \cap B(x_0, 2R_0) \) is \( \frac{1}{4\sqrt{2}} \)-Reifenberg flat, then \( \Sigma \cap B(x_0, R_0) \) is a \( C^{1, \beta} \) submanifold of dimension \( n \) in \( \mathbb{R}^{n+1} \).
For $n \geq 3$, the preceding theorem fails if one removes the flatness assumption. Indeed, Kowalski and Preiss [3] discovered that the 3-dimensional Hausdorff $\mathcal{H}^3$ measure on the cone $X = \{ x \in \mathbb{R}^4 : x_4^2 = x_1^2 + x_2^2 + x_3^2 \}$ satisfies $\mathcal{H}^3(B(x,r) \cap X) = Cr^3$ for all $x \in X$ and all $r > 0$. Clearly, (1.9) holds in this case and $X$ is non smooth at the origin.

In this paper we extend Theorem 1.6 to general codimensions in $\mathbb{R}^m$, and moreover we prove that, when $n \geq 3$, if one does not assume $\Sigma$ to be Reifenberg flat, one still has that $\Sigma$ is smooth out of a small closed set (like in the case of the cone $X$). The precise statement is the following.

**Theorem 1.7** For each constant $\alpha > 0$ we can find $\beta = \beta(\alpha) > 0$ with the following property. Let $\mu$ be a measure in $\mathbb{R}^m$ supported on $\Sigma$, and suppose that for each compact set $K \subset \Sigma$, there is a constant $C_K$ such that

$$(1.10) \quad \left| \frac{\mu(B(x, tr))}{\mu(B(x, r))} - t^n \right| \leq C_K r^\alpha \quad \text{for } r \in (0,1], \ t \in [\frac{1}{2}, 1] \quad \text{and } x \in K.$$ 

If $n = 1, 2$, $\Sigma$ is a $C^{1,\beta}$ submanifold of dimension $n$ in $\mathbb{R}^m$. If $n \geq 3$, $\Sigma$ is a $C^{1,\beta}$ submanifold of dimension $n$ in $\mathbb{R}^m$ away from a closed set $S$ such that $\mathcal{H}^n(S) = 0$.

We would like to point out that condition (1.10) implies an apparently stronger condition, namely that for each compact set $K \subset \Sigma$, there is a constant $C_K$ depending on $K$, $n$ and $\alpha$ such that

$$(1.11) \quad \left| \frac{\mu(B(x, tr))}{\mu(B(x, r))} - t^n \right| \leq C_K r^\alpha \quad \text{for } r \in (0,1], \ t \in (0, 1] \quad \text{and } x \in K.$$ 

In fact assume that (1.10) holds and let $\tau \in (0,1/2)$. There exits $j \in \mathbb{N}, j \geq 2$ so that $1/2^j \leq \tau < 1/2^{j-1}$ thus $\tau^2 = t \in [1/2, 1/\sqrt{2})$. For $x \in K$, and $r \in (0,1]$, (1.10) yields

$$(1.12) \quad t^{n(j-1)}|\mu(B(x, tr)) - t^n \mu(B(x, r))| \leq C_K r^\alpha t^{n(j-1)} \mu(B(x, r))$$

$$(1.13) \quad t^{n(j-2)}|\mu(B(x, t^2 r)) - t^n \mu(B(x, tr))| \leq C_K r^\alpha t^{n(j-2)} \mu(B(x, tr))$$

$$\vdots$$

$$(1.14) \quad \left| \mu(B(x, t^i r)) - t^n \mu(B(x, t^{i-1} r)) \right| \leq C_K r^\alpha \mu(B(x, t^{i-1} r)).$$

Adding the above inequalities, we obtain that

$$(1.15) \quad |\mu(B(x, \tau r)) - t^n \mu(B(x, r))| \leq C_K r^\alpha \mu(B(x, r)) \sum_{i=0}^{j-1} \left( \frac{1}{(\sqrt{2})^n} \right)^i$$

which implies that for $x \in (0, R), \ x \in K$ and $\tau \in (0, 1/2)$

$$(1.16) \quad \left| \frac{\mu(B(x, \tau r))}{\mu(B(x, r))} - t^n \right| \leq CC_K r^\alpha.$$ 

The constant $C$ depends only on the dimension $n$.  

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A first step in the proof of Theorem 1.7 is to prove that if $\mu$ satisfies (1.10), then the restriction $\mu_0$ of $\mathcal{H}^n$ to $\Sigma$ is locally finite, and $d\mu(x) = D(x)d\mu_0(x)$ for some positive density $D(x)$ such that $\log D(x)$ is (locally) Hölder with exponent $\frac{\alpha}{\alpha+1}$. Moreover, $\mu_0$ satisfies the stronger requirement that for all compact sets $K \subset \Sigma$ there is a constant $C_K$ such that

$$\left| \frac{\mu_0(B(x,r))}{\omega_n r^n} - 1 \right| \leq C_K r^{\frac{\alpha}{\alpha+1}} \text{ for all } x \in K \text{ and } 0 < r \leq 1.$$  

We then deduce the conclusion of Theorem 1.7 from (1.15), by a method inspired by [3], [1] and [5]. We first show that if $\mu$ is a Radon measure supported on $\Sigma \subset \mathbb{R}^m$, the local behavior of the quantity $\frac{\mu(B(x,r))}{\omega_n r^n}$ for $x \in \Sigma$ and $r \in (0,1]$ determines the regularity of $\Sigma$ near flat points. The we prove that the set of flat points is open and its complement has $\mathcal{H}^n$ measure 0 (see the two theorems below). Here $\omega_n$ denotes the Lebesgue measure of the unit ball in $\mathbb{R}^n$.

**Theorem 1.8** For each $\alpha > 0$ there exists $\beta = \beta(\alpha) > 0$ with the following property. Suppose $\Sigma = \text{supp}(\mu) \subset \mathbb{R}^m$ for some positive Radon measure $\mu$, and that for each compact set $K \subset \Sigma$ there is a constant $C_K$ such that

$$\left| \frac{\mu(B(x,r))}{\omega_n r^n} - 1 \right| \leq C_K r^{\alpha} \text{ for } x \in K \text{ and } 0 < r < 1.$$  

If $n = 1,2$, $\Sigma$ is a $C^{1,\beta}$ submanifold of dimension $n$ in $\mathbb{R}^m$. If $n \geq 3$, there exists a constant $\delta(n,m)$ depending only on $n$ and $m$ such that if $x_0 \in \Sigma$ and $\Sigma \cap B(x_0,2R_0)$ is $\delta(n,m)$-Reifenberg flat, then $\Sigma \cap B(x_0,R_0)$ is a $C^{1,\beta}$ submanifold of dimension $n$ in $\mathbb{R}^m$.

**Theorem 1.9** For each $\alpha > 0$ there exists $\beta = \beta(\alpha) > 0$ with the following property. Suppose $\Sigma = \text{supp}(\mu) \subset \mathbb{R}^m$ for some positive Radon measure $\mu$, and that for each compact set $K \subset \Sigma$ there is a constant $C_K$ such that

$$\left| \frac{\mu(B(x,r))}{\omega_n r^n} - 1 \right| \leq C_K r^{\alpha} \text{ for } x \in K \text{ and } 0 < r < 1.$$  

If $n = 1,2$, $\Sigma$ is a $C^{1,\beta}$ submanifold of dimension $n$ in $\mathbb{R}^m$. If $n \geq 3$, $\Sigma$ is a $C^{1,\beta}$ submanifold of dimension $n$ in $\mathbb{R}^m$ away from a closed set $S$ such that $\mathcal{H}^n(S) = 0$.

## 2 Preliminaries

In this section we state several results which will be used throughout the paper. The codimension one versions appear in [1]. The reader would realize that the proofs given in there do not depend on the codimension. Thus we do not include proofs.
Proposition 2.1 Let \( \alpha > 0 \) be given. Let \( \mu \) be a measure supported on \( \Sigma \subset \mathbb{R}^m \) and suppose that for all compact sets \( K \subset \Sigma \), there is a constant \( C_K \) such that

\[
|R_t(x, r)| \leq C_K r^\alpha \text{ for } x \in K \text{ and } t, r \in (0, 1].
\]

Then the density

\[
D(x) = \lim_{r \to 0^+} \frac{\mu(B(x, r))}{\omega_n r^n}
\]

(where \( \omega_n \) denotes the \( n \)-dimensional Hausdorff measure of the unit ball in \( \mathbb{R}^n \)) exists for all \( x \in \Sigma \), and

\[
0 < D(x) < +\infty \text{ for } x \in \Sigma.
\]

Moreover, \( \log D(x) \) is locally Hölder; i.e., for all compact sets \( K \subset \Sigma \), we can find \( C'_K \) such that

\[
|\log D(x) - \log D(y)| \leq C'_K |x - y|^{\frac{\alpha}{1+\alpha}} \text{ for } x, y \in K.
\]

Finally, denote by \( \mu_0 \) the restriction of \( \mathcal{H}^n \) to \( \Sigma \), i.e., \( \mu_0 = \mathcal{H}^n \rest \Sigma \). Then \( \mu_0 \) is finite on compact sets,

\[
d\mu(x) = D(x)d\mu_0(x),
\]

and for each compact set \( K \subset \Sigma \) there is a constant \( C''_K \) such that

\[
\left| \frac{\mu_0(B(x, r))}{\omega_n r^n} - 1 \right| \leq C''_K r^{\frac{\alpha}{1+\alpha}} \text{ for } x \in K \text{ and } 0 < r \leq 1.
\]

Remark 2.2 When \( C_K \) in (2.1) is large, (2.1) only gives some information on the doubling properties of \( \mu \) at small scales (i.e., when \( r^{\alpha} < C_K^{-1} \)). Thus, even though we did not say explicitly that \( R_t(x, r) \) is only controlled for \( r \) small enough, this is implicit in (2.1).

Since (2.4) and (2.6) contain some amount of large-scale information, we might be forced in some cases to take huge values of \( C'_K \) and \( C''_K \) that depend on the large-scale behavior of \( \mu \) (and not only on the \( C_K \)). This problem can easily be fixed by restricting the domain of validity of (2.4) to \( |x - y| \leq r_0 \), where \( r_0 \) depends on \( C_K \), and similarly restricting (2.6) to radii \( 0 < r < r_0 \). Then we can get constants \( C'_K \) and \( C''_K \) that depend only on \( C_K \). We could also fix the problem by requiring that \( \mu \) be doubling.

Let \( \mu \) be an \( n \)-Ahlfors regular measure supported on \( \Sigma \subset \mathbb{R}^m \), i.e suppose that for each compact set \( K \subset \Sigma \) there is a constant \( C_K > 1 \) such that

\[
C_K^{-1} < \frac{\mu(B(x, r))}{\omega_n r^n} < C_K
\]
for $x \in K$ and $0 < r < 1$. We follow [3] and introduce some moments for Ahlfors regular measures. Fix a compact set $K$ and for $x_1 \in K$, define the vector $b = b_{x_1, r}$ by

\[(2.8) \quad b = \frac{n + 2}{2\omega_nm^{n+2}} \int_{B(x_1, r)} (r^2 - |y - x_1|^2)(y - x_1)d\mu(y).\]

Also define the quadratic form $Q = Q_{x_1, r}$ on $\mathbb{R}^m$ by

\[(2.9) \quad Q(x) = \frac{n + 2}{\omega_nm^{n+2}} \int_{B(x_1, r)} \langle x, y - x_1 \rangle^2 d\mu(y)\]

for $x \in \mathbb{R}^m$. In all our estimates we use the fact that

\[(2.10) \quad |\mu(B(x, t)) - \omega_nt^n| \leq C_K t^{n+\alpha} \text{ for } x \in \Sigma \cap \overline{B}(x_1, 1) \text{ and } 0 < t < 1,\]

which we get by applying (1.16) with $K^* = \{x \in \Sigma; \text{dist}(x, K) \leq 1\}$.

Roughly speaking the following proposition shows that if the density ratio of $\mu$, $\frac{\mu(B(x, r))}{\omega_nr^n}$ approaches 1 as $r$ tends to 0 in a Hölder fashion then the points in the support of $\mu$ almost satisfy a quadratic equation.

**Proposition 2.3** Let $\mu$ be a measure supported on $\Sigma \subset \mathbb{R}^m$ such that for each compact set $K \subset \Sigma$ there is a constant $C_K$ such that

\[(2.11) \quad \left| \frac{\mu(B(x, r))}{\omega_nr^n} - 1 \right| \leq C_K r^\alpha\]

for $x \in K$ and $0 < r < 1$. For $x_1 \in K$ and $0 < r < 1$, let

\[(2.12) \quad Tr(Q) = \frac{n + 2}{\omega_nm^{n+2}} \int_{B(x_1, r)} |y - x_1|^2 d\mu(y)\]

denote the trace of $Q$. Then

\[(2.13) \quad |Tr(Q) - n| \leq CC_K r^\alpha.\]

Also, if $0 < r < \frac{1}{2}$, for $x \in \Sigma \cap B \left(x_1, \frac{r}{2}\right)$,

\[(2.14) \quad |2\langle b, x - x_1 \rangle + Q(x) - x - x_1|^2| \leq C \frac{|x - x_1|^3}{r} + CC_K r^{2+\alpha}.\]

For a measure $\mu$ supported on $\Sigma$ and satisfying (2.11) we introduce the quantity that allows us to measure the local flatness of $\Sigma$ and prove its regularity. Let $K \subset \Sigma$ be a compact set and let $x_1 \in K$, for small radii $\rho$ consider

\[(2.15) \quad \beta(x_1, \rho) = \inf \left\{ \frac{1}{\rho} \sup \{\text{dist}(y, P); y \in \Sigma \cap B(x_1, \rho) \} \right\}\]

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Here the infimum is taken over all affine $n$-planes $P$ through $x_1$. In particular by (1.1)

$$\beta(x_1, \rho) \leq \theta(x_1, \rho).$$

Note that (1.16) implies that $\mu$ satisfies the hypothesis of Theorem 1.4 which ensures that if $\Sigma \cap B(x_0, 2R_0)$ is Reifenberg flat for some $x_0 \in \Sigma$ then

$$\tag{2.16} \Sigma \cap B(x_0, R_0) \text{ is Reifenberg flat with vanishing constant.}$$

Hence for $x_1 \in \Sigma \cap B(x_0, R_0)$, $\beta(x_1, \rho)$ converges to 0 as $\rho \to 0$ uniformly on compact sets. The key step in the proof of Theorem 1.4 is to show that if $\mu$ satisfies (1.10) then there exists $\gamma > 0$ such that for $\rho$ small $\beta(x_1, \rho) < C_K \rho^\gamma$. This is also the main idea behind the proof of Theorem 1.6. Its implementation in the codimension 1 case is significantly simpler. Once the asymptotic behavior of $\beta$ has been established we simply apply the following theorem which appears in Section 9 in [1]

**Proposition 2.4** Let $0 < \beta \leq 1$ be given. Suppose $\Sigma \cap B(x_0, 2R_0)$ is a Reifenberg flat set with vanishing constant of dimension $n$ in $\mathbb{R}^m$ and that, for each compact set $K \subset \Sigma$, there is a constant $C_K$ such that

$$\tag{2.17} \beta(x, r) \leq C_K r^\beta \text{ for } x \in K \text{ and } r \leq 1.$$ 

Then $\Sigma \cap B(x_0, R_0)$ is a $C^{1,\beta}$ submanifold of dimension $n$ of $\mathbb{R}^m$.

### 3 Control on the flatness of $\Sigma$

Let $\mu$ be a measure satisfying the hypothesis of Proposition 2.3. Assume $\mu$ is supported on $\Sigma \subset \mathbb{R}^m$, and let $K \subset \Sigma$ be a fixed compact set. Theorem 1.4 ensures that for each small $\delta > 0$, we can find $r_0 \in (0, 10^{-2} R_0)$ depending on $K$ such that

$$\tag{3.1} \theta(x, r) \leq \delta \text{ when } x \in \Sigma \cap B(x_0, R_0), \text{ dist}(x, K) \leq 1, \text{ and } 0 < r \leq 10 r_0.$$ 

As we proceed it might be convenient to make the value of $r_0$ smaller (depending on the constant $C_K$ in (1.16)), to make our estimates simpler. Without loss of generality we may assume that $x_1 = 0 \in \Sigma \cap B(x_0, R_0)$. Let us recall the main properties of $b$ and $Q$ that are used in this section. In particular we do not need to know how $b$ and $Q$ are computed in terms of $\mu$ (see (2.8) and (2.9)). First, $b = b_r \in \mathbb{R}^m$ and

$$\tag{3.2} |b_r| \leq \frac{n + 2}{2 \omega_n r^{n+2}} \int_{B(0,r)} r^2 |y| d\mu(y) \leq \frac{(n + 2) r}{2 \omega_n r^n} \mu(B(0,r)),$$

$$|b_r| \leq \frac{(n + 2) r}{2} \left\{ 1 + \frac{C_K r^\alpha}{\omega_n} \right\} \leq (n + 2) r,$$

by (2.8) and (2.10), provided we assume that $\frac{C_K r^\alpha}{\omega_n} \leq 1$. We do not explicitly need (3.2), but the homogeneity is important to keep in mind.
Next, $Q$ is a quadratic form defined on $\mathbb{R}^m$, (2.9) and (2.10) ensure that for $x \in \mathbb{R}^m$

$$0 \leq Q(x) \leq \frac{n+2}{\omega_n r^{n+2}} \int_{B(0,r)} |x|^2 r^2 d\mu(y) \leq \frac{(n+2)|x|^2}{\omega_n r^n} \mu(B(0,r)) \leq (n+2)|x|^2(1 + \omega_n^{-1} C_K r^\alpha) \leq (2n+4)|x|^2,$$

and

$$|Tr(Q) - n| \leq CC_K r^\alpha$$

by (2.13). It is convenient to set

$$\tilde{Q}(x) = |x|^2 - Q(x).$$

Then (2.14) yields that

$$|2\langle b, x \rangle - \tilde{Q}(x)| \leq Cr^{-1}|x|^3 + CC_K r^{2+\alpha}$$

for $x \in \Sigma \cap B(0, \frac{r}{2})$.

Initially we use (3.4) and (3.6) to derive more information about $Q$ and $b$. We work at scales of the form $\rho = r^{1+\gamma}$ smaller than $r$. Here $\gamma$ is a positive constant that will assume several different values.

It is important to understand how (3.6) is modified by a change of scale. Set

$$\Sigma_\rho = \frac{1}{\rho} \Sigma,$$

and

$$\Sigma'_\rho = \Sigma_\rho \cap B(0, \frac{r}{2\rho}) = \frac{1}{\rho} (\Sigma \cap B(0, \frac{r}{2})).$$

Note that (3.1) guarantees that we can choose an $n$-plane $L$ through the origin such that

$$D[L \cap B(0, \rho), \Sigma \cap B(0, \rho)] \leq \rho \theta(0, \rho) \leq \rho \delta,$$

where $D$ denotes the Hausdorff distance between sets, as in (1.2). (See also (1.1) for the definition of $\theta(0, \rho)$). Moreover for $z \in \Sigma'_\rho$ we can apply (3.6) to $x = \rho z$ and get that

$$|2\langle b, \rho, z \rangle - \tilde{Q}(z)| = \rho^{-2}|2\langle b, x \rangle - \tilde{Q}(x)| \leq C \rho^{-2} r^{-1} |x|^3 + CC_K \rho^{-2} r^{2+\alpha}$$

$$= C \rho r^{-1} |z|^3 + CC_K \rho^{-2} r^{2+\alpha}$$

$$= C r^\gamma |z|^3 + CC_K r^{\alpha-2\gamma}$$

because $\rho = r^{1+\gamma}$. In particular,

$$|\langle 2b, r^{-1-\gamma}, z \rangle - \tilde{Q}(z)| \leq Cr^\gamma + CC_K r^{\alpha-2\gamma} =: \epsilon_0(r, \gamma)$$

for $z \in \Sigma_{r^{1+\gamma}} \cap B(0, \frac{1}{2})$. 8
To motivate the argument in the proof of Theorem 1.7 we briefly recall the main ideas in the proof of Theorem 1.6. (3.11) encodes the information required to estimate the quantity \( \beta(0, \rho_0) \) defined in (2.15). In the codimension 1 case one needs to consider two cases. Either \( b \) in (3.11) is very small, and then one obtains an estimate on the smallest eigenvalue of \( Q \) which allows one to say that at the appropriate scale \( \Sigma \) is very close to the plane normal to the corresponding eigenspace. If \( b \) is “large” then at the appropriate scale \( \Sigma \) is very close to the plane orthogonal to \( b \). In both cases one produces the normal vector which is orthogonal to the plane \( \Sigma \) is close to. In higher codimensions we need to produce an \( m - n \) orthonormal family of vectors whose span is orthogonal to the \( n \)-plane \( \Sigma \) is close to, at a given scale. The difficulty lies on the fact that there is only a single equation at hand, namely (3.11). To overcome this problem we are forced to do a multiscale analysis of (3.11).

Our first intermediate result is an estimate on \( Q \) when \( b_r \) is fairly small. Let us assume that

\[
|b| \leq r^{1+2\theta}
\]

for some \( \theta > 0 \). Then (3.11) and (3.12) ensure that for \( z \in \Sigma_{\rho^{1+\gamma}} \cap B(0, \frac{1}{2}) \) we have

\[
|\tilde{Q}(z)| \leq |\langle 2br^{-1-\gamma}, z \rangle| + Cr^\gamma + CC_K r^{\alpha-2\gamma} \\
\leq r^{2\theta-\gamma} + Cr^\gamma + CC_K r^{\alpha-2\gamma} \\
=: \epsilon_1(r, \theta, \gamma).
\]

Note that (3.13) only provides useful information when \( \gamma \) satisfies

\[
0 < \gamma < 2\theta \quad \text{and} \quad 2\gamma < \alpha.
\]

Choose an orthonormal basis \( (e_1, \ldots, e_m) \) of \( \mathbb{R}^m \) that diagonalizes \( Q \). Thus

\[
Q(z) = \sum_{i=1}^{m} \lambda_i \langle z, e_i \rangle^2
\]

for \( z \in \mathbb{R}^m \). Without loss of generality we may assume that

\[
\lambda_1 \leq \lambda_2 \cdots \leq \lambda_m.
\]

Note that \( \lambda_1 \geq 0 \) because \( Q(z) \geq 0 \) (see (2.9) or (3.3)). Also, by (3.4)

\[
\sum_{i=1}^{m} \lambda_i = Tr(Q) \leq n + CC_K r^\alpha.
\]

In particular, by (3.16) if \( k = m - n \)

\[
m\lambda_1 \leq Tr(Q) \leq n + CC_K r^\alpha < n + \frac{1}{2},
\]

\[
(n + 1)\lambda_k \leq Tr(Q) \leq n + CC_K r^\alpha < n + \frac{1}{2},
\]
provided we take \( r_0 \) small enough. Thus

\[
0 \leq \lambda_1 \leq \frac{2n+1}{2m} \quad \text{and} \quad \lambda_k \leq \frac{2n+1}{2n+2}.
\]

This is just a crude first step. Our next goal is to obtain more precise estimates on \( Q \), when (3.12) holds, i.e., \(|b| \leq r^{1+2\theta}\), under the additional constraint that

\[
0 < \theta < \frac{\alpha}{3}.
\]

**Lemma 3.1** Suppose that (3.12), (3.14) and (3.21) hold. Let \( k = m - n \). For \( r_0 \) small enough and \( \epsilon_2(r, \theta, \gamma) = na^{-2}\epsilon_1(r, \theta, \gamma) \), where \( a \) is a constant that only depends on \( n \) and \( m \), we have

\[
0 \leq \sum_{i=1}^{k} \lambda_i \leq \epsilon_2(r, \theta, \gamma) + CCKr^\alpha,
\]

(3.22)

\[
|\lambda_{k+i} - 1| \leq \epsilon_2(r, \theta, \gamma) + r^{\alpha/2} \quad \text{for} \quad 1 \leq i \leq n,
\]

and

\[
|\tilde{Q}(z) - \sum_{i=1}^{k} \langle z, e_i \rangle^2 | \leq (\epsilon_2(r, \theta, \gamma) + r^{\alpha/2}) |z|^2 \quad \text{for} \quad z \in \mathbb{R}^m.
\]

(3.24)

Note that (3.24) automatically follows from (3.22) and (3.23). In fact if we write \( z = \sum_{i=1}^{m} z_i e_i \), then by (3.5) and (3.15) we have

\[
\tilde{Q}(z) - \sum_{i=1}^{k} \langle z, e_i \rangle^2 = |z|^2 - Q(z) - \sum_{i=1}^{k} z_i^2
\]

\[
= \sum_{i=1}^{n} z_{k+i}^2 - \sum_{i=1}^{m} \lambda_i z_i^2
\]

\[
= - \sum_{i=1}^{k} \lambda_i z_i^2 + \sum_{i=1}^{n} (1 - \lambda_{i+k}) z_i^2.
\]

(3.25)

Note that the choice \( \gamma = \theta \) with \( \theta \) as in (3.13) satisfies (3.14). In this case Lemma 3.1 becomes

**Corollary 3.2** Suppose that (3.12), and (3.21) hold. For \( r_0 \) small enough

\[
0 \leq \sum_{i=1}^{k} \lambda_i \leq Cr^\theta,
\]

(3.26)

\[
|\lambda_{k+i} - 1| \leq Cr^\theta \quad \text{for} \quad 1 \leq i \leq n,
\]

(3.27)
and

\[(3.28) \quad |\tilde{Q}(z) - \sum_{l=1}^{k} \langle z, e_l \rangle^2| \leq Cr_0 |z|^2 \text{ for } z \in \mathbb{R}^m,\]

where \(C\) is a constant that depends on \(K\), \(n\) and \(m\).

To prove Lemma 3.1 we need some preliminary results. The first one is the following.

**Lemma 3.3** Let \(L\) denote an \(n\)-plane satisfying (3.9). For \(l = 1, \cdots, k\) let \(v_l\) denote the orthogonal projection of \(e_l\) onto \(L\). If \(\delta\) and \(r_0\) are chosen small enough,

\[(3.29) \quad \left| \sum_{l=1}^{k} x_l v_l - \sum_{l=1}^{k} x_l e_l \right| \geq \tilde{c}^{-1},\]

whenever \(\sum_{l=1}^{k} |x_l|^2 = 1\)

In Lemma 3.3, \(\delta\) and \(r_0\) depend on \(n\), \(\alpha\), \(\theta\) and \(\gamma\). At most 2\(k\) values of \(\theta\) and \(\gamma\), are used depending only on \(\alpha\), and a choice of \(\theta\). Thus one can always choose \(\delta > 0\) and \(r_0 > 0\) to work simultaneously for all our choices. The constant \(C > 1\) depends only on \(n\) and \(m\).

**Proof:** To prove Lemma 3.3 we first estimate \(\tilde{Q}(z)\) for \(z \in L \cap B(0, 1/3)\). Since \(\rho z \in L \cap B(0, \rho)\), where \(\rho = r^{1+\gamma}\), (3.9) guarantees that there is a point \(x \in \Sigma \cap B(0, \rho)\) such that \(|x - \rho z| \leq 2\rho \delta\). If \(\delta\) is small enough, \(|\rho^{-1} x| < \frac{1}{2}\), and so (3.13) ensures that \(|\tilde{Q}(\rho^{-1} x)| \leq \epsilon_1(r, \theta, \gamma)\). Also, \(|\rho^{-1} x - z| = \rho^{-1} |x - \rho z| \leq 2\delta\), and hence (3.3) and (3.5) guarantee that

\[(3.30) \quad |\tilde{Q}(\rho^{-1} x) - \tilde{Q}(z)| \leq C\delta.\]

Altogether,

\[(3.31) \quad |\tilde{Q}(z)| \leq \epsilon_1(r, \theta, \gamma) + C\delta \text{ for } z \in L \cap B(0, \frac{1}{3}).\]

We are now ready to prove (3.29). Let \(u = \sum_{l=1}^{k} x_l e_l\) with \(|u| = 1\). Then \(w = \sum_{l=1}^{k} x_l v_l\) satisfies \(|w| \leq 1\), thus (3.31) guarantees that

\[(3.32) \quad |\tilde{Q}(u)| \leq |\tilde{Q}(w)| + |\tilde{Q}(u) - \tilde{Q}(v)| \leq \epsilon_1(r, \theta, \gamma) + C\delta + |\tilde{Q}(u) - \tilde{Q}(v)|.\]

On the other hand since \(u\) belongs to the span of the first \(k\) eigenvectors of \(Q\) we have that

\[(3.33) \quad \tilde{Q}(u) = 1 - Q(u) \geq 1 - \lambda_k \geq \frac{1}{2n + 2}\]

by (3.5), (3.15), and (3.20). If \(\delta\) and \(r_0\) are small enough, (3.32) and (3.33) imply that

\[(3.34) \quad |\tilde{Q}(u) - \tilde{Q}(w)| \geq \frac{1}{4n + 4}.\]
Thus \( w \) cannot be too close to \( u \) (because of (3.3)), and (3.29) holds.

Now we want to use the fact that \( \Sigma \cap B(x_0, R_0) \) is Reifenberg flat with vanishing constant to get important topological information on \( \Sigma_\rho \cap B(0, \frac{1}{2}) \), \( \rho = r^{1+\gamma} \). Denote by \( P \) the \( n \)-plane through 0 which is orthogonal to \( e_1, \cdots, e_k \). Thus

\[
P = \text{span}^\perp(e_1, \cdots, e_k) = \text{span}(e_{k+1}, \cdots, e_m).
\]

Call \( \pi \) the orthogonal projection onto \( P \). Also denote by \( \pi^* : \mathbb{R}^m \to L \) the projection onto \( L \) parallel to the direction \( \{e_1, \cdots, e_k\} \), i.e.

\[
\pi^*(x) = \pi^*(\sum_{l=1}^m x_l e_l) = \sum_{l=1}^n x_{k+l} e_{k+l} + \sum_{l=1}^k y_l e_l
\]

where the orthogonal projection of \( \sum_{l=1}^k y_l e_l \) into \( L^\perp \) coincides with that of \( \sum_{l=1}^n x_{k+l} e_{k+l} \).

Here \( L \) is as in (3.9), and \( L^\perp \) denotes the \((m-n)\) space orthogonal to \( L \). Denote by \( \pi' \) the orthogonal projection of \( \mathbb{R}^m \) onto \( L^\perp \). Lemma 3.3 ensures that

\[
C^{-1} |\sum_{l=1}^k y_l e_l| \leq |\sum_{l=1}^k y_l e_l - \sum_{l=1}^n y_l v_l| = |\pi'(\sum_{l=1}^k y_l e_l)|
\]

\[
\leq |\pi'(\sum_{l=1}^n x_{k+l} e_{k+l})| \leq |\sum_{l=1}^n x_{k+l} e_{k+l}| \leq |x|.
\]

Thus

\[
|\pi^*(x)| \leq C_0 |x| \quad \text{for } x \in \mathbb{R}^m.
\]

Here \( C_0 = 2C \) where \( C \) is as in (3.29), a constant that depends only on \( n \) and \( m \).

Set \( a = (4C_0)^{-1} \), and recall that \( \rho = r^{1+\gamma} \), where \( \gamma \) satisfies (3.14). The same argument as in [1] guarantees that:

**Lemma 3.4** For every \( \xi \in P \cap B(0, a) \), there is a point \( z \in \Sigma_\rho \cap B(0, \frac{1}{2}) \) such that \( \pi(z) = \xi \).

[See Figure 8.1]

We have gathered all the information needed to prove Lemma 3.1.

**Proof of Lemma 3.1** To prove (3.22) and (3.23), we apply Lemma 3.4 with \( \xi = ae_{k+i} \), \( 1 \leq i \leq n \). We choose \( \gamma \) so that (3.14) holds. We get that for some \( (t^1, \cdots, t^k) \in \mathbb{R}^k \),

\[
z_i = \sum_{l=1}^k t^l e_l + ae_{k+i} \in \Sigma_\rho \cap B(0, \frac{1}{2}).
\]

If we take \( \gamma = \theta \), (3.13) and (3.21) guarantee that

\[
|\tilde{Q}(z_i)| \leq \epsilon_1(r, \theta, \gamma)
\]
Combining (3.5), (3.15) and (3.38) we obtain that

\[ \tilde{Q}(z_i) = |z_i|^2 - Q(z_i) = \sum_{l=1}^{k} (1 - \lambda_l)(t_l^i)^2 + (1 - \lambda_{k+i})a^2. \]

Since \( 1 - \lambda_l \geq (2n + 2)^{-1} \) for \( 1 \leq l \leq k \) (by (3.20)), we get that

\[ (1 - \lambda_{k+i})a^2 \leq \tilde{Q}(z_i) \leq \epsilon_1(r, \theta, \gamma) \]  

(by (3.39)). Thus

\[ \lambda_{k+i} \geq 1 - a^{-2}\epsilon_1(r, \theta, \gamma), \]

for \( 1 \leq i \leq n \), and hence

\[ \sum_{i=1}^{n} \lambda_{k+i} \geq n - \epsilon_2(r, \theta, \gamma). \]

By (3.17) and (3.43) we have that

\[ \sum_{l=1}^{k} \lambda_l = Tr(Q) - \sum_{i=1}^{n} \lambda_i \leq CC_K r^\alpha + a^{-2}\epsilon_1(r, \theta, \gamma) \]

This proves (3.22), because we already know that \( \sum_{l=1}^{k} \lambda_l \geq 0 \). To prove (3.23), we proceed by contradiction and suppose that we can find \( 1 \leq i_0 \leq n \) such that

\[ \lambda_{k+i_0} > 1 + \epsilon_2(r, \theta, \gamma) + r^{\alpha/2}. \]
Then (3.42) and (3.45) yield

\begin{equation}
\sum_{i=1}^{m} \lambda_i \geq \sum_{i=1}^{n} \lambda_{k+i} \geq \lambda_{k+i_0} + (n-1)\left(1 - \frac{\varepsilon_2(r, \theta, \gamma)}{n}\right) > n + r^{\alpha/2}
\end{equation}

This contradicts (3.17), thus (3.45) is impossible and (3.23) holds. We already observed earlier that (3.24) is a consequence of (3.22) and (3.23), and so Lemma 3.1 follows.

Next we use Corollary 3.2 to rewrite (3.11), still under the assumption that (3.12) holds for some \( \theta \in (0, \frac{\alpha}{2}) \). Combining (3.11) and (3.24) we get that for \( z \in \Sigma_r \cap B(0, \frac{1}{2}) \)

\begin{equation}
|\langle 2br^{-1-\gamma}, z \rangle - \sum_{l=1}^{k} \langle z, e_l \rangle |^2 \leq Cr^\gamma + CC_K r^{\alpha-2\gamma} + |\tilde{Q}(z) - \sum_{l=1}^{k} \langle z, e_l \rangle |^2
\end{equation}

\begin{align*}
&\leq Cr^\gamma + CC_K r^{\alpha-2\gamma} + Cr^\theta \\
&= : \varepsilon_3(r, \theta, \gamma).
\end{align*}

Note that here \( \rho = r^{1+\gamma} \) for any \( \gamma > 0 \) (as in (3.11)). Of course (3.47) only provides useful information when \( 0 < \gamma < \frac{\alpha}{2} \).

Next we want to get a better estimate on the “tangential part” of \( b \). This allows us to estimate \( \beta(0, s) \) as defined in (2.15) for an appropriately chosen \( s \).

**Proposition 3.5** If \( b = \sum_{i=1}^{m} b_i e_i \), then

\begin{equation}
|b_{k+i}| \leq Cr^{1+\eta} \varepsilon_3(r, \theta, \eta) + C_r^{1+4\theta-\eta} \text{ for } 1 \leq i \leq n.
\end{equation}

**Remark 3.6** The goal is to show that given appropriate choices for \( \theta \) and \( \eta \) satisfying (3.21) and (3.14) with \( \eta \) in place of \( \gamma \), (3.48) provides an improvement over (3.12). In the codimension 1 case it was possible to choose \( \gamma = \eta = 3\theta/2 \). The reader will note that this choice does improve estimate (3.12). Unfortunately in the higher codimension setup it is premature to choose \( \eta \) at this stage.

**Proof:** Choose \( \theta \) and \( \gamma = \eta \) such that (3.21) and (3.14) hold. We can then apply Lemma 3.4. Fix \( i \in \{1, 2, \ldots, n\} \) and apply Lemma 3.4 to the two points \( \xi_\pm = \pm ae_{k+i} \). We get \( k \) vectors \( (t_1^\pm, \ldots, t_k^\pm) \) such that

\begin{equation}
z_\pm = \sum_{l=1}^{k} t_l^\pm e_l \pm ae_{k+i} \in \Sigma_{r^{1+\eta}} \cap B(0, \frac{1}{2}).
\end{equation}

Then (3.47) implies that

\begin{equation}
\sum_{l=1}^{k} 2b_l r^{-1-\eta} t_l^\pm \pm 2b_{k+i} r^{-1-\eta} a - \sum_{l=1}^{k} (t_l^\pm)^2 \geq -\varepsilon_3(r, \theta, \eta).
\end{equation}
Set \( f_l(t) = 2b_l r^{-1-\eta} t - t^2 \) for \( 1 \leq l \leq k \). Then
\[
(3.51) \quad f_l(t) = (b_l r^{-1-\eta})^2 - (b_l r^{-1-\eta} - t)^2 \leq (b_l r^{-1-\eta})^2
\]
for all \( t \in \mathbb{R} \). Hence by (3.50) and (3.51) we have that
\[
(3.52) \quad \pm 2b_{k+i} r^{-1-\eta} a \geq -\epsilon_3(r, \theta, \eta) - \sum_{l=1}^{k} f_l(t^\pm_l) \geq -\epsilon_3(r, \theta, \eta) - \sum_{l=1}^{k} (b_l r^{-1-\eta})^2.
\]
Here we have two inequalities, one for each sign \( \pm \). Thus by (3.12)
\[
(3.53) \quad |b_{k+i}| \leq (2a)^{-1} r^{1+\eta} \epsilon_3(r, \theta, \eta) + (2a)^{-1} |b|^2 r^{-1-\eta} \leq C r^{1+\eta} \epsilon_3(r, \theta, \eta) + C r^{1+4\theta-\eta}.
\]
Combining (3.48) and (3.47), we get that for \( z \in \Sigma_\rho \cap B(0, \frac{1}{4}) \), where \( \rho = r^{1+\gamma} \),
\[
(3.54) \quad |\langle 2 \sum_{l=1}^{k} b_l r^{-1-\gamma}, z \rangle - \sum_{l=1}^{k} \langle z, e_l \rangle^2 | \leq \epsilon_3(r, \theta, \gamma) + \sum_{l=1}^{n} 2b_{k+i} r^{-1-\gamma} \langle z, e_{k+i} \rangle |\langle 2 \sum_{l=1}^{k} b_l r^{-1-\gamma}, z \rangle - \sum_{l=1}^{k} \langle z, e_l \rangle^2 |
\]
\[
\leq \epsilon_3(r, \theta, \gamma) + C r^{-1-\gamma} r^{1+\eta} \epsilon_3(r, \theta, \eta) + C r^{4\theta-\eta-\gamma}
\leq C (r^\gamma + r^{\alpha-2\gamma} + r^\theta + r^{2\eta-\gamma} + r^{\alpha-n-\gamma} + r^{\theta+n-\gamma} + r^{4\theta-n-\gamma}).
\]
This holds for \( \theta \) as in (3.21), \( \eta \) satisfying
\[
(3.55) \quad 0 < \eta < 2\theta \quad \text{and} \quad 2\eta < \alpha,
\]
and all \( \gamma > 0 \) as in (3.47). It only provides an interesting estimate for some values of \( \gamma \).
Choose
\[
(3.56) \quad 0 < 4\gamma < \alpha,
\]
and define
\[
(3.57) \quad \epsilon_4(r, \theta, \gamma, \eta) := C (r^\gamma + r^\theta + r^{2\eta-\gamma} + r^{\theta+n-\gamma} + r^{4\theta-n-\gamma}).
\]
Then (3.54) becomes
\[
(3.58) \quad |\langle 2 \sum_{l=1}^{k} b_l r^{-1-\gamma}, z \rangle - \sum_{l=1}^{k} \langle z, e_l \rangle^2 | \leq \epsilon_4(r, \theta, \gamma, \eta).
\]
**Proposition 3.7**  With the notation above we have that
\[
(3.59) \quad |\sum_{l=1}^{k} \langle z, e_l \rangle e_l | \leq 3 \epsilon_4(r, \theta, \gamma, \eta)^{1/2} \text{ for } z \in \Sigma_\rho \cap B(0, \frac{1}{4}).
\]
Here \( \rho = r^{1+\gamma} \). The exponents \( \theta, \gamma \) and \( \eta \) satisfy (3.21), (3.53) and (3.56).
Proof: Set \( z^\perp = \sum_{l=1}^{k}(z, e_l)e_l \) for \( z \in \Sigma_\rho \cap B(0, \frac{1}{2}) \). Then (3.54) can be written

\[
|z^\perp (z^\perp - d)| \leq \epsilon_4(r, \theta, \gamma, \eta),
\]

where \( d = 2b^\perp r^{-1-\gamma} \). This forces

\[
|z^\perp| \leq \epsilon_4(r, \theta, \gamma, \eta)^{\frac{1}{2}}
\]

or

\[
|z^\perp - d| \leq \epsilon_4(r, \theta, \gamma, \eta)^{\frac{1}{2}}.
\]

If \( |d| \leq 2\epsilon_4(r, \theta, \gamma, \eta)^{\frac{1}{2}} \), then (3.59) trivially follows from this. So let us assume that \( |d| > 2\epsilon_4(r, \theta, \gamma, \eta)^{\frac{1}{2}} \). Denote by \( \mathcal{U} \) the connected component of \( \Sigma_\rho \cap B(0, \frac{1}{2}) \) containing the origin, and set

\[
\mathcal{U}_\pm = \{z \in \mathcal{U}; (8.59\pm) \text{ holds}\}.
\]

Obviously \( \mathcal{U}_+ \) and \( \mathcal{U}_- \) are closed in \( \mathcal{U} \), and since \( \mathcal{U} \) is the disjoint union of \( \mathcal{U}_+ \) and \( \mathcal{U}_- \) (because \( |d| > 2\epsilon_4(r, \theta, \gamma, \eta)^{\frac{1}{2}} \)), \( \mathcal{U} \) must be equal to \( \mathcal{U}_+ \). Thus to prove (3.59) it is enough to show that

\[
\Sigma_\rho \cap B(0, \frac{1}{4}) \subset \mathcal{U}.
\]

Since (3.1) holds, \( \Sigma_\rho \) is locally Reifenberg flat and the same argument used in Section 8 of [1] yields (3.62). Proposition 3.7 follows. \( \blacksquare \)

Note that (3.59) says that if \( |b_r| \leq r^{1+2\gamma} \), then

\[
\beta(0, \frac{1}{4}r^{1+\gamma}) \leq 12\epsilon_4(r, \theta, \gamma, \eta)^{\frac{1}{2}} =: \epsilon_5(r, \theta, \gamma, \eta),
\]

where \( \beta(0, s) \) is defined as in (2.15) and

\[
\epsilon_4(r, \theta, \gamma, \eta) = C(r^\gamma + r^\theta + r^{2\gamma - \gamma} + r^{\theta + \eta - \gamma} + r^{4\theta - \eta - \gamma}).
\]

Here \( C \) depends on \( n, m \) and \( K \). The estimate (3.63) holds for all exponents \( \theta, \gamma \) and \( \eta \) satisfying (3.21), (3.55) and (3.56).

Recall that \( b_r = b \). So far we have omitted the dependence of \( r \) to simplify the notation, as there was no room for confusion. From now on we need to keep track of it as it will be made clear shortly.

When (3.12) does not hold, i.e.

\[
|b| > r^{1+2\gamma},
\]

(3.11) and (3.3) tell us that

\[
|\langle z r^{-1-\gamma}, z \rangle| \leq |\bar{Q}(z)| + Cr^\gamma + CCKr^{\alpha - 2\gamma} \leq C
\]

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for \( z \in \Sigma_{r^{1+\gamma}} \cap B(0, \frac{1}{2}) \), provided that we choose \( 0 < \gamma < \frac{\alpha}{2}, r < r_0 \) and \( r_0 \) small enough. Set \( \tau = |b|^{-1}b \). Then

\[
\langle \tau, z \rangle \leq C|b|^{-1}r^{1+\gamma} \leq Cr^{\gamma-2\theta}
\]

for \( z \in \Sigma_{r^{1+\gamma}} \cap B(0, \frac{1}{2}) \). In the codimension 1 case \( |\langle \tau, z \rangle| \) measures the distance from \( z \) to the \( n \)-plane orthogonal to \( \tau \). (3.67) implies that \( \beta(0, \frac{1}{4}r^{1+\gamma}) \leq Cr^{\gamma-2\theta} \). In this case choosing \( \eta, \gamma, \theta \) appropriately one can guarantee that \( \beta(0, \frac{1}{4}r^{1+\gamma}) \) is bounded by a positive power of \( r \). This case is done in [1].

In codimension \( k = m - n \) we need to produce a \( k \) plane such that \( z^\perp \), the orthogonal projection \( z \in \Sigma_{r^{1+\gamma}} \cap B(0, \frac{1}{2}) \) onto this plane, is bounded by a positive power on \( r \). To accomplish this we need to choose \( 3k \) exponents \( \eta_i, \gamma_i, \theta_i \) and \( k + 1 \) radii \( r_i \) with \( 1 \leq i \leq k \) satisfying

\[
0 < 3\theta_i < \alpha, \quad 0 < 4\gamma_i < \alpha, \quad 0 < \eta_i < 2\theta_i \quad \text{and} \quad 2\eta_i < \alpha,
\]

and
\[
r_1 = r, \quad r_{i+1} = r_i^{1+\gamma_i}.
\]

The difficulty lies on the fact that several additional compatibility conditions arise along the proof, and we need to check that they can be satisfied.

**Case 1** There exists \( i = 1, \ldots, k = m - n \) such that

\[
|b_{r_i}| \leq r_i^{1+2\theta_i},
\]

then (3.63) ensures that

\[
\beta(0, \frac{1}{4}r_{i+1}) \leq \epsilon_5(r_i, \theta_i, \gamma_i, \eta_i).
\]

Thus we have

\[
\beta(0, \frac{r_{k+1}}{4}) \leq \frac{r_{i+1}}{r_{k+1}}\beta(0, \frac{r_{i+1}}{4}) \leq \frac{r_{i+1}}{r_{k+1}}\epsilon_5(r_i, \theta_i, \gamma_i, \eta_i).
\]

**Case 2** For all \( i = 1, \cdots, k = m - n \)

\[
|b_{r_i}| \geq r_i^{1+2\theta_i},
\]

then (3.67) guarantees that

\[
|\langle \tau_i, z \rangle| \leq Cr_i^{\gamma_i-2\theta_i} \quad \text{for} \quad z \in \Sigma_{r_{i+1}} \cap B(0, \frac{1}{2}), \quad \text{where} \quad \tau_i = \frac{b_{r_i}}{|b_{r_i}|}.
\]

Thus

\[
|\langle \tau_i, x \rangle| \leq Cr_i^{1+2\gamma_i-2\theta_i} = Cr_i^{1+2\gamma_i-2\theta_i} \quad \text{for} \quad x \in \Sigma \cap B(0, \frac{r_{i+1}}{2}).
\]
If } j \geq i + 1 \text{ then } r_j \leq r_{i+1}. \text{ Using the definition of } b_{r_j} \text{ which appears in (2.8) we obtain from (3.73) that }
\begin{align}
(3.76) \quad |\langle \tau_i, b_{r_j} \rangle| & \leq Cr_{i+1}r_i^{\gamma_i-2\theta_i}.
\end{align}

The definition of } \tau_j \text{ combined with (3.75) and (3.76) yield }
\begin{align}
(3.77) \quad |\langle \tau_i, \tau_j \rangle| & \leq Cr_j^{-1-2\theta_j}r_{i+1}r_i^{\gamma_i-2\theta_i} = C r_j^{-1-2\theta_j} r_{i+1}^{1+2\gamma_i-2\theta_i} \quad \text{for } j \geq i + 1.
\end{align}

Our goal is to show that in either case there exists } s, \text{ a power of } r, \text{ such that } \beta(0, s) \text{ is bounded above by a power of } r \text{ (i.e. of } s). \text{ In Case 1 it suffices to show that the exponent of } r \text{ in the right hand side of (3.72) is positive. In Case 2 we first need to show that the vectors } \tau_l \text{ for } l = 1, \cdots, k \text{ are linearly independent (in fact almost orthogonal). This is achieved by showing that the exponent of } r \text{ that appears in (3.77) can be made positive. Once we know that the vectors } \tau_l \text{ for } l = 1, \cdots, k \text{ are almost orthogonal (3.73) provides an estimate for } \beta(0, r_{k+1}^{1/4}). \text{ In fact assume that } |\langle \tau_i, \tau_j \rangle| < \frac{1}{2k} \text{ for } i, j = 1, \cdots, k, i \neq j. \text{ Then (3.73) yields that } x^\perp \text{ the orthogonal projection of } x \in \Sigma \cap B(0, r_{k+1}^{1/4}) \text{ satisfies }
\begin{align}
(3.78) \quad |x^\perp| & \leq C \max_{1 \leq i \leq k} r_i^{1+2\gamma_i-2\theta_i}.
\end{align}

Therefore
\begin{align}
(3.79) \quad \beta(0, r_{k+1}^{1/4}) & \leq C r_k^{-1} \max_{1 \leq i \leq k} r_i^{1+2\gamma_i-2\theta_i},
\end{align}

where } C \text{ is a constant that depends on } n, m \text{ and } K. \text{ Our immediate task is to show that by choosing } \theta_i, \eta_i \text{ and } \gamma_i \text{ appropriately and satisfying (3.68) the right hand sides of (3.72), (3.77), and (3.79) can be written as positive powers of } r. \text{ We first focus on the right hand side of (3.77) for } j \geq i + 1. \text{ Recall that }
\begin{align}
(3.80) \quad r_j = r_{j-1}^{1+\gamma_j-1} = r_i^{1+\gamma_j-1},
\end{align}

hence
\begin{align}
(3.81) \quad r_j^{-1-2\theta_j}r_i^{1+2\gamma_i-2\theta_i} & = r_i^{1+2\gamma_i-2\theta_i-(1+2\theta_j)\prod_{l=i}^{j-1}(1+\gamma_l)}.
\end{align}

Thus for each } i = 1, \cdots, k \text{ and } j \geq i + 1 \text{ we need }
\begin{align}
(3.82) \quad 1 + 2\gamma_l - 2\theta_l - (1 + 2\theta_j)\prod_{l=i}^{j-1}(1 + \gamma_l) > 0
\end{align}

Similarly the right hand side of (3.79) yields
\begin{align}
(3.83) \quad r_{k+1}^{-1}r_i^{1+2\gamma_i-2\theta_i} & = r_i^{1+2\gamma_i-2\theta_i-\prod_{l=i}^{k}(1+\gamma_l)},
\end{align}

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which leads to the condition

\[(3.84) \quad 1 + 2\gamma_i - 2\theta_i - \prod_{l=i}^{k}(1 + \gamma_l) > 0, \]

for all \(i = 1, \cdots, k.\)

Note that if \((3.82)\) is satisfied for \(j = k + 1\) then so is \((3.84)\). Moreover \((3.82)\) applied to \(j = i + 1\) requires that for \(i = 1, \cdots, k\)

\[(3.85) \quad \gamma_i > 2\theta_i. \]

The right hand side of \((3.72)\) produces five conditions for each \(i = 1, \cdots, k.\) In fact the term

\[(3.86) \quad r_{k+1}^{-1}r_{i+1}^1 = r_i^{1+\gamma_i-\prod_{l=i}^{k}(1+\gamma_l)} \]

is multiplied by each one of the terms in \(e_5(r_i, \theta_i, \gamma_i, \eta_i).\) We obtain:

\[(3.87) \quad 1 + \frac{3}{2}\gamma_i - \prod_{l=i}^{k}(1 + \gamma_l) > 0, \]

\[(3.88) \quad 1 + \gamma_i + \frac{\theta_i}{2} - \prod_{l=i}^{k}(1 + \gamma_l) > 0, \]

\[(3.89) \quad 1 + \frac{\gamma_i}{2} + \eta_i - \prod_{l=i}^{k}(1 + \gamma_l) > 0, \]

\[(3.90) \quad 1 + \frac{\gamma_i}{2} + \frac{\theta_i}{2} + \frac{\eta_i}{2} - \prod_{l=i}^{k}(1 + \gamma_l) > 0, \]

\[(3.91) \quad 1 + \frac{\gamma_i}{2} + 2\theta_i - \frac{\eta_i}{2} - \prod_{l=i}^{k}(1 + \gamma_l) > 0. \]

Using \((3.84)\) and \((3.68)\) we observe that

\[(3.92) \quad 1 + \frac{3}{2}\gamma_i - \prod_{l=i}^{k}(1 + \gamma_l) \geq 1 + \frac{\gamma_i}{2} + 2\theta_i - \frac{\eta_i}{2} - \prod_{l=i}^{k}(1 + \gamma_l), \]

and

\[(3.93) \quad 1 + \gamma_i + \frac{\theta_i}{2} - \prod_{l=i}^{k}(1 + \gamma_l) \geq 1 + \frac{\gamma_i}{2} + \frac{\theta_i}{2} + \frac{\eta_i}{2} - \prod_{l=i}^{k}(1 + \gamma_l). \]
Thus (3.87) and (3.88) are satisfied whenever (3.68), (3.84), (3.90) and (3.91) hold. At this point we are ready to choose the form of the exponents. Let

\[ (3.94) \quad \gamma_{i+1} = \kappa \gamma_i, \quad \theta_{i+1} = \kappa \theta_i, \quad \eta_{i+1} = \kappa \eta_i, \]

with

\[ (3.95) \quad 0 < \kappa < \frac{1}{16}, \quad 0 < 3\theta_i < \alpha, \quad 0 < 4\gamma_i < \alpha, \quad \text{and} \quad 0 < \eta_i = \frac{3}{2} \theta_i < 2\theta_i. \]

Note that this implies that \(2\eta_i < \alpha\).

This choice of \(\gamma_1, \theta_1, \eta_1\) and \(\kappa\) ensure that (3.68) is satisfied, that \(\eta_i = \frac{3}{2} \theta_i\), and that \(\gamma_i < \frac{1}{4}\). Note that three of the four remaining conditions (3.82), (3.89), (3.90) and (3.91) contain the term \(\prod_{l=i}^k (1 + \gamma_l)\), or a product term which is bounded by it. Using the fact that for \(x \geq 0\)

\[ 1 + x \leq e^x \]

and that for \(x < 1/2\),

\[ e^x \leq 1 + x + x^2 \]

we have

\[ (3.96) \quad \prod_{l=i}^k (1 + \gamma_l) \leq e^{\sum_{l=i}^k \gamma_l} = e^{\sum_{l=i}^{k-1} \kappa^l \gamma_l} \leq e^{\frac{\gamma_i}{1-\kappa}} \leq 1 + \frac{\gamma_i}{1-\kappa} + \left( \frac{\gamma_i}{1-\kappa} \right)^2. \]

Hence (3.82), (3.89), (3.90) and (3.91) become

\[ (3.97) \quad 2\gamma_i - 2\theta_i - 2\kappa \theta_i - (1 + 2\kappa \theta_i) \left( \frac{\gamma_i}{1-\kappa} + \left( \frac{\gamma_i}{1-\kappa} \right)^2 \right) > 0, \]

where we used the fact for \(j \geq i + 1\), \(\theta_j \leq \kappa \theta_i\).

\[ (3.98) \quad \frac{\gamma_i}{2} + \eta_i - \frac{\gamma_i}{1-\kappa} - \left( \frac{\gamma_i}{1-\kappa} \right)^2 > 0, \]

\[ (3.99) \quad \frac{\gamma_i}{2} + \frac{\theta_i}{2} + \frac{\eta_i}{2} - \frac{\gamma_i}{1-\kappa} - \left( \frac{\gamma_i}{1-\kappa} \right)^2 > 0, \]

\[ (3.100) \quad \frac{\gamma_i}{2} + 2\theta_i - \frac{\eta_i}{2} - \frac{\gamma_i}{1-\kappa} - \left( \frac{\gamma_i}{1-\kappa} \right)^2 > 0. \]

Combining (3.94) and (3.95), (3.97), (3.98), (3.99) and (3.100) become

\[ (3.101) \quad 2\gamma_i - 2\theta_i (1 + \kappa) - (1 + 2\kappa \theta_i) \left( \frac{\gamma_i}{1-\kappa} + \left( \frac{\gamma_i}{1-\kappa} \right)^2 \right) > 0, \]

\[ (3.102) \quad \frac{\gamma_i}{2} + \frac{3}{2} \theta_i - \frac{\gamma_i}{1-\kappa} - \left( \frac{\gamma_i}{1-\kappa} \right)^2 > 0, \]

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\[(3.103) \quad \frac{\gamma_1}{2} + \frac{5}{4} \theta_1 - \frac{\gamma_3}{1 - \kappa} - \left( \frac{\gamma_1}{1 - \kappa} \right)^2 > 0.\]

Note that if (3.102) is satisfied so is (3.103). Thus we only have two conditions left to satisfy, namely (3.101) and (3.102). At this point we can choose
\[(3.104) \quad \gamma_1 = \kappa^2 (1 - \kappa) \quad \text{and} \quad \theta_1 = \frac{\kappa^2 (1 - \kappa)}{2 (1 + 4 \kappa)}, \quad \text{provided} \quad 4 \kappa^2 (1 - \kappa) < \alpha.\]

Recalling that \(\kappa < \frac{1}{16}\), a straightforward calculation shows that
\[(3.105) \quad 2 \gamma_1 - 2 \theta_1 (1 + \kappa) - (1 + 2 \kappa \theta_1) \left( \frac{\gamma_1}{1 - \kappa} + \left( \frac{\gamma_1}{1 - \kappa} \right)^2 \right) \geq 2 \kappa^2 (1 - \kappa) - 2 \theta_1 (1 + \kappa) - \kappa^2 (1 + 2 \kappa \theta_1) (1 + \kappa^2) \geq \frac{\kappa^3}{1 + 4 \kappa},\]
and
\[(3.106) \quad \frac{\gamma_1}{2} + \frac{3}{2} \theta_1 - \frac{\gamma_3}{1 - \kappa} - \left( \frac{\gamma_1}{1 - \kappa} \right)^2 \geq \frac{1}{2} \left( \kappa^2 (1 - \kappa) + 3 \theta_1 - 2 \kappa^2 - 2 \kappa^4 \right) \geq \frac{\kappa^2}{4 (1 + 4 \kappa)} \left( 1 - 13 \kappa - 12 \kappa^2 - 16 \kappa^3 \right) \geq \frac{\kappa^2}{32 (1 + 4 \kappa)}.\]

Inequalities (3.105), (3.105) combined with (3.72), (3.63), (3.64), (3.79), (3.82), (3.89), (3.90) and (3.91) show that for \(\kappa\) such that \(4 \kappa^2 (1 - \kappa) < \alpha\)
\[(3.107) \quad \beta(0, \frac{r_{k+1}}{4}) \leq Cr^k \text{ where } r_{k+1} = r \sum_{i=1}^{k} \gamma_i.\]

Note that (3.104) and (3.105) ensure that
\[(3.108) \quad \prod_{t=1}^{k} (1 + \gamma) \leq 1 + \kappa + \kappa^2.\]

Therefore for \(t = \frac{r_{k+1}}{4}\) (3.107) yields
\[(3.109) \quad \beta(0, t) \leq Ct^{\frac{\kappa^3}{(1 + 4 \kappa)(1 + \kappa + \kappa^2)}},\]
where \(C\) is a constant that depends on \(n, m, \alpha\) and our specific choice of \(\kappa\).
4 On the flatness of asymptotically optimally doubling measures

Recall the following result from Preiss [5]. See also [2] [Propositions 6.18 and 6.19] for more details.

**Theorem 4.1** There exists a constant \( \varepsilon_0 > 0 \) depending only on \( n \) and \( d \) such that if \( \nu \) is an \( n \)-uniform measure on \( \mathbb{R}^m \) (normalized so that \( \nu(B(x, r)) = r^n \) for all \( x \in \text{supp}(\nu), \ r > 0 \)) such that its tangent measure \( \lambda \) at \( \infty \) satisfies

\[
\min_{L \in G(n,m)} \int_{B(0,1)} \text{dist}(x, L)^2 d\lambda(x) \leq \varepsilon_0^2,
\]

then \( \nu \) is flat. Here \( G(n,m) \) stands for the collection of all \( n \)-planes in \( \mathbb{R}^m \), \( \lambda \) is normalized so that \( \lambda(B(x, r)) = r^n \) for all \( x \in \text{supp}(\lambda), \ r > 0 \).

We need to define a smooth version of the usual coefficients \( \beta_2 \). To this end, let \( \varphi \) be a \( C_0^\infty \) radial function with \( \chi_{B(0,2)} \leq \varphi \leq \chi_{B(0,3)} \). Let \( B = B(x_0, r) \) be a ball with centered at \( x_0 \in \text{supp}(\mu) \). We denote by

\[
\tilde{\beta}_{2,\mu}(B) = \min_{L \in G(n,m)} \left( \frac{1}{r^{n+2}} \int \varphi \left( \frac{|x - x_0|}{r} \right) \text{dist}(x, L)^2 d\mu(x) \right)^{1/2}.
\]

The following two theorems are the key tools in the proof of Theorem 1.9. We postpone their proofs to the end of the section. We first indicate how they are used to prove Theorem 1.9.

**Theorem 4.2** Let \( \mu \) be an asymptotically optimally doubling measure supported on \( \Sigma \subset \mathbb{R}^m \). Let \( K \subset \mathbb{R}^m \) be compact and suppose that

\[
C_0^{-1}r^n \leq \mu(B(x, r)) \leq C_0 r^n \quad \text{for} \ x \in K \cap \Sigma, \ 0 < r \leq \text{diam}(K).
\]

For any \( \eta > 0 \), there exists \( \delta > 0 \) depending only on \( \eta, n, m, \mu, K \) and \( C_0 \) such that if \( B \) is a ball contained in \( K \) and centered at \( K \cap \Sigma \) with \( \tilde{\beta}_{2,\mu}(B) \leq \delta \), then \( \tilde{\beta}_{2,\mu}(P) \leq \eta \) for any ball \( P \subset B \) centered at \( K \cap \Sigma \).

**Theorem 4.3** Let \( \mu \) be an asymptotically optimally doubling measure supported on \( \Sigma \subset \mathbb{R}^m \). Assume that \( 0 \in \Sigma \). Let \( K \subset \mathbb{R}^m \) be a compact set such that \( B(0,2) \subset K \), and suppose that

\[
C_0^{-1}r^n \leq \mu(B(x, r)) \leq C_0 r^n \quad \text{for} \ x \in K \cap \Sigma, \ 0 < r \leq \text{diam}(K).
\]

Given \( \varepsilon > 0 \), there exists \( \delta \in (0, \varepsilon_0) \) depending only on \( \varepsilon, n, m, \mu, K \) and \( C_0 \) such that if \( \tilde{\beta}_{2,\mu}(B) \leq \delta \), for every ball \( B \subset B(0,2) \) centered at \( K \cap \Sigma \) then there exists \( R > 0 \) such that \( \theta(x, r) < \varepsilon \) for all \( x \in \Sigma \cap B(0,1) \) and \( r < R \).
Corollary 4.4 Let $\mu$ be an asymptotically optimally doubling measure supported on $\Sigma \subset \mathbb{R}^m$. Let $K \subset \mathbb{R}^m$ be compact set and suppose that

$$C_0^{-1} r^n \leq \mu(B(x, r)) \leq C_0 r^n \quad \text{for } x \in K \cap \Sigma, \ 0 < r \leq \text{diam}(K).$$

Given $\epsilon > 0$, there exists $\delta \in (0, \epsilon_0)$ depending only on $\epsilon$, $n$, $\mu$, $K$ and $C_0$ such that if $\bar{\beta}_{2, \mu}(B(x_0, 4R_0)) \leq \delta$, where $x_0 \in \Sigma$ and $B(x_0, 4R_0) \subset K$, then there exists $R > 0$ such that $\theta(x, r) < \epsilon$ for all $x \in \Sigma \cap B(x_0, 2R_0)$ and $r < R$, i.e. $\Sigma \cap B(x_0, 2R_0)$ is $\epsilon$-Reifenberg flat.

Proof of Theorem 1.9: First note that (1.17) ensures that condition (4.5) is satisfied. It also implies that the density of $\mu$ exists and equals 1 everywhere. Therefore Preiss’ work (see [5]) yields that $\Sigma$ is $\mathcal{H}^n$-rectifiable. Furthermore $\mu = \mathcal{H}^n \ll \Sigma$. Thus given $\eta \in (0, \epsilon_0)$ for $\mathcal{H}^n$- a.e $x \in \Sigma$ there exists $\rho > 0$ such that for $r < \rho$, $\theta(x, r) \leq \eta$. Let

$$\mathcal{R} = \{x \in \Sigma : \limsup_{r \to 0} \theta(x, r) = 0\}$$

Note that $\mathcal{H}^n(\mathcal{S}) = 0$ where $\mathcal{S} = \Sigma \setminus \mathcal{R}$. For $x_0 \in \mathcal{R}$ there exists $R_0$ such that $\theta(x_0, r) \leq \eta$ for $r \leq 8R_0$. This implies that $\bar{\beta}_{2, \mu}(B(x_0, 4R_0)) \leq C\eta$, where $C$ only depends on $C_0$. For $\epsilon \in (0, \delta(n, m))$ where $\delta(n, m)$ is as in Theorem 1.8 by Corollary 4.4 we can find $\eta$ so that $C\eta \leq \delta \leq \epsilon_0$, which ensures that $\Sigma \cap B(x_0, 2R_0)$ is $\delta(n, m)$ Reifenberg flat. We use Theorem 1.8 to conclude that $\Sigma \cap B(x_0, R_0)$ is a $C^{1, \beta}$ $n$-dimensional submanifold. In particular this implies that $\mathcal{R}$ is open in $\Sigma$ because, $\Sigma \cap B(x_0, 2R_0) \subset \mathcal{R}$. To prove Theorem 4.2 we need the following result:

Lemma 4.5 Let $\mu$ be an asymptotically optimally doubling measure on $\mathbb{R}^m$. Let $K \subset \mathbb{R}^m$ be compact and let $\delta_0$ be any positive constant. Suppose that

$$C_0^{-1} r^n \leq \mu(B(x, r)) \leq C_0 r^n \quad \text{for } x \in K \cap \Sigma, \ 0 < r \leq \text{diam}(K).$$

There exists some constant $\epsilon_1$ depending on $\epsilon_0$ and $C_0$ (but not on $\delta_0$) and an integer $N > 0$ depending only on $\mu$, $K$, $C_0$, and $\delta_0$, such that if $B$ is a ball centered at $\Sigma$ such that $2^NB \subset K$ and

$$\bar{\beta}_{2, \mu}(2^kB) \leq \epsilon_1 \quad \text{for } 1 \leq k \leq N, \quad \text{then } \bar{\beta}_{2, \mu}(B) \leq \delta_0.$$

Proof: Suppose that the integer $N$ does not exist. Then there exists a sequence of points $\{x_j\} \subset K \cap \Sigma$ and balls $B_j := B(x_j, r_j)$ such that $2^jB_j \subset K$, and

$$\bar{\beta}_{2, \mu}(2^kB_j) \leq \epsilon_1 \quad \text{for } 1 \leq k \leq j,$$

but $\bar{\beta}_{2, \mu}(B_j) > \delta_0$. Clearly, $r_j \to 0$ as $j \to \infty$. For each $j \geq 1$, consider the blow up measure $\mu_j$ defined by

$$\mu_j(A) = \frac{\mu(r_jA + x_j)}{\mu(B_j)}$$
Extracting a subsequence if necessary, we may assume that \( \{\mu_j\} \) converges weakly to another measure \( \nu \), which by [4] [Theorem 2.2] is \( n \)-uniform. We claim that

\[
\tilde{\beta}_{2,\nu}(B(0, 2^k)) \lesssim \varepsilon_1 \quad \text{for all } k \geq 0
\]

and

\[
\tilde{\beta}_{2,\nu}(B(0, 1)) \gtrsim \delta_0.
\]

Assume the claim for the moment. It is easy to check that (4.8) implies that the tangent measure \( \lambda \) of \( \nu \) at \( \infty \) satisfies

\[
\min_{L \in G(n,m)} \int_{B(0,1)} \text{dist}(x, L)^2 d\lambda(x) \leq \varepsilon_0^2,
\]

(assuming \( \varepsilon_1 \leq \varepsilon_0 \) small enough) and so \( \nu \) is flat by Theorem [4.1]. This contradicts (4.9), and the lemma follows.

Let us prove (4.8). Let \( B(0, 2^k) \) be fixed. Extracting a subsequence of \( \{\mu_j\} \), we may assume that the \( n \)-planes \( L_j \) which minimize \( \tilde{\beta}_{2,\mu_j}(B(0, 2^k)) \) converge in the Hausdorff metric to another \( n \)-plane \( \tilde{L} \), and then it easily follows that

\[
\int \varphi\left(\frac{|x|}{2^k}\right) \text{dist}(x, L_j)^2 d\mu_j(x) - \int \varphi\left(\frac{|x|}{2^k}\right) \text{dist}(x, L)^2 d\nu(x) \to 0 \quad \text{as } j \to \infty.
\]

Notice also that

\[
\frac{1}{2^{k(n+2)}} \int \varphi\left(\frac{|x|}{2^k}\right) \text{dist}(x, L_j)^2 d\mu_j(x) = \frac{1}{2^{k(n+2)} \mu(B_j)} \int \varphi\left(\frac{|x-x_j|}{r_j}\right) \text{dist}\left(\frac{x-x_j}{r_j}, L_j\right)^2 d\mu(x)
\]

\[
= \frac{1}{2^{k(n+2)} r_j^2 \mu(B_j)} \int \varphi\left(\frac{|x-x_j|}{2^k r_j}\right) \text{dist}(x, x_j + r_j L_j)^2 d\mu(x)
\]

\[
\approx \frac{1}{(2^k r_j)^{n+2}} \int \varphi\left(\frac{|x-x_j|}{2^k r_j}\right) \text{dist}(x, x_j + r_j L_j)^2 d\mu(x)
\]

\[
\lesssim \varepsilon_1^2,
\]

since \( x_j + r_j L_j \) is the \( n \)-plane that minimizes \( \tilde{\beta}_{2,\mu_j}(B(x_j, 2^k r_j)) \). Inequality (4.8) follows from (4.10) and the preceding estimate.

The proof of (4.9) is analogous. Now let \( L \) be an arbitrary \( n \)-plane. Then we have

\[
\int \varphi(|x|) \text{dist}(x, L)^2 d\nu(x) = \lim_{j \to \infty} \int \varphi(|x|) \text{dist}(x, L)^2 d\mu_j(x)
\]

\[
= \lim_{j \to \infty} \frac{1}{\mu(B_j)} \int \varphi\left(\frac{|x-x_j|}{r_j}\right) \text{dist}\left(\frac{x-x_j}{r_j}, L\right)^2 d\mu(x)
\]

\[
= \lim_{j \to \infty} \frac{1}{r_j^2 \mu(B_j)} \int \varphi\left(\frac{|x-x_j|}{r_j}\right) \text{dist}(x, x_j + r_j L)^2 d\mu(x) \gtrsim \delta_0^2,
\]

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since \( \tilde{\beta}_{2,\mu}(B_j) > \delta_0. \)

**Proof of Theorem 4.2**: Let \( \varepsilon_1 \) be the constant given by Lemma 4.5 and set \( \delta_0 = \min(\varepsilon_1, \eta) \) (recall that \( \varepsilon_1 \) is independent of \( \delta_0 \)). Let \( N \) be the corresponding integer given by the same lemma.

If \( \delta \) is chosen small enough, then we clearly have \( \tilde{\beta}_{2,\mu}(P) \leq \min(\varepsilon_1, \eta) \) for any ball \( P \) centered at any point in \( B \cap \Sigma \) with \( r(P) \geq 2^{-N}r(B) \). By the preceding lemma, by induction on \( j \geq 0 \) we infer that \( \tilde{\beta}_{2,\mu}(P) \leq \min(\varepsilon_1, \eta) \) for any ball \( P \) centered at \( B \cap \Sigma \) with radius \( r(P) \) such that \( 2^{-j-1}r(B) \leq r(P) \leq 2^{-j}r(B) \) (where \( r(B) \) stands for the radius of \( B \)).

**Proof of Theorem 4.3**: We argue by contradiction. Suppose that there exists \( \varepsilon_1 > 0 \) such that for each \( i \geq i_0 \) and each ball \( B \subset B(0,2) \) centered in \( K \cap \Sigma \), \( \tilde{\beta}_{2,\mu}(B) \leq 2^{-i} \leq \varepsilon_0 \) but there are \( x_i \in \Sigma \cap B(0,1) \) and \( r_i > 0 \) with \( \lim_{i \to \infty} r_i = 0 \), so that \( \theta(x_i, r_i) \geq \varepsilon_1 \), i.e \( \theta_{\Sigma_i}(0, 1) \geq \varepsilon_0 \), where \( \Sigma_i = \frac{1}{r_i}(\Sigma - x_i) \). Consider the blow up sequence \( \{\mu_i\} \) defined by

\[
(4.13) \quad \mu_i(E) = \frac{\mu(r_iE + x_i)}{\mu(B(x_i, r_i))}
\]

Modulo passing to a subsequence Theorem 2.2 in [4] ensures that \( \mu_i \) converges weakly to a Radon measure \( \mu_\infty \) which is \( n \)-uniform. Moreover \( \Sigma_i \) converges in the Hausdorff distance sense to \( \Sigma_\infty = \text{supp} \mu_\infty \) uniformly on compact subsets. Therefore \( \theta_{\Sigma_\infty}(0, 1) \geq \varepsilon_0/2 \). Statement (4.10) guarantees that for \( r > 0 \) \( \tilde{\beta}_{2,\mu}(B(0,r)) \) converges to \( \tilde{\beta}_{2,\mu_\infty}(B(0,r)) \). Since for \( r > 0 \) there exists \( i_r \) so that for \( i \geq i_r \) \( \tilde{\beta}_{2,\mu_i}(B(0,r)) \leq 2^{-i} \) then \( \tilde{\beta}_{2,\mu_\infty}(B(0,r)) = 0 \) for every \( r > 0 \). Thus the support of \( \mu_\infty \), \( \Sigma_\infty \) is contained in an \( n \)-plane. Since \( \mu_\infty \) is \( n \)-uniform (and flat at infinity), then \( \Sigma_\infty \) is an \( n \)-plane, which contradicts the fact that \( \theta_{\Sigma_\infty}(0, 1) \geq \varepsilon_0/2 \).

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