Liouvillianity breaking in dissipative interacting Floquet systems under high-frequency drive

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Floquet-Magnus (FM) expansion theory is a powerful tool in periodically driven (Floquet) systems under high-frequency drives. In closed systems, it dictates that their stroboscopic dynamics under a time-periodic Hamiltonian is well captured by the FM expansion, which gives a static effective Hamiltonian. On the other hand, in dissipative systems driven by a time-periodic Liouvillian, it remains an important and nontrivial problem whether the FM expansion gives a static effective Liouvillian which describes a Markovian dynamics. We answer this question for generic systems with local interactions. We find that, while noninteracting systems can either break or preserve Liouvillianity of the FM expansion, generic few-body and many-body interacting systems break it under any finite drive, which is essentially caused by propagation of interactions via higher order terms of the FM expansion. Liouvillianity breaking implies that Markovian dissipative Floquet systems in the high-frequency regimes do not have static (Markovian) counterparts, giving a signature of emergent non-Markovianity. Our theory provides a basic framework for questing unique phenomena in dissipative Floquet systems.

Introduction.—Periodically driven (Floquet) systems have attracted much interest as one of the most important class of nonequilibrium systems, which host unique phases such as Floquet topological phases [4,9] and time crystals [4,9], and enable controls of the phases (Floquet engineering) [10]. In particular, Floquet systems in high-frequency regimes, where their frequency $\omega = 2\pi T$ ($T$: period) is much larger than their energy scale $J$, have been vigorously studied. In closed systems under a time-periodic Hamiltonian $H(t)$, we can analyze their behavior in these regimes in a unified way by the Floquet-Magnus (FM) expansion, which is a perturbation theory in $J/\omega$ [11,13]. Importantly, the FM effective Hamiltonian, which approximately describes the stroboscopic dynamics, is a static Hamiltonian, and hence such systems are understood by conventional techniques in static closed systems, leading to Floquet engineering [10]. Floquet prethermalization [14,17] using eigenstate thermalization hypothesis (ETH) [18,20], and so on. While the hermiticity of the FM effective Hamiltonian provides a powerful tool, it in turn indicates that closed Floquet systems in high-frequency regimes always have counterparts in closed static systems.

Recently, both theoretical and experimental interest has been spreading out over dissipative Floquet systems [21–31], with rapidly developing atomic, molecular, and optical (AMO) platforms [32–34]. Under Markovianity, dissipative Floquet systems obey the Lindblad equation $\partial_t \rho = L(t) \rho$ with $L(t) = L(t + T)$, where a time-periodic Liouvillian $L(t)$ is given by

$$L(t) \rho = -i[H(t), \rho] + \sum_i L_i(t) \rho L_i(t)^\dagger - \frac{1}{2} \{L_i(t)^\dagger L_i(t), \rho\}.$$  \hspace{1cm} (1)

The linearity of $L(t)$ implies a possible extension of the FM expansion to dissipative systems, and a static linear operator $L(t)$ [defined by Eq. (3) below] called the FM effective Lindbladian is obtained. However, in contrast to closed systems, it is an important and nontrivial problem whether the FM effective Lindbladian $L(t)$ is a static Liouvillian, which is a generator of a completely-positive and trace-preserving (CPTP) time-homogeneous dynamical map under Markovianity [35–37]. The Liouvillianity breaks a completely-positive Markovian dynamics [38] and NESS [42], but Liouvillianity itself was not focused on. While Ref. [43] first evaluated Liouvillianity of the FM expansions, these previous studies mainly focused on noninteracting systems. Thus, the knowledge of Liouvillianity of the FM expansions is still lacking in generic systems, especially in interacting systems.

Here, we address the fundamental question whether the FM effective Lindbladian is a Liouvillian in generic systems with particular emphasis on few-body or many-body systems with local interactions. We find out that, while noninteracting systems show model-dependent behaviors, breaking or preservation of Liouvillianity, generic interacting systems experience Liouvillianity breaking of the FM expansions under any finite drive. Importantly, Liouvillianity breaking in interacting systems is essentially attributed to a spreading structure of local interactions in FM expansions. Liouvillianity breaking implies that such Floquet systems cannot be captured as static Markovian systems [44]: namely dissipa-
tive Floquet systems under Markovianity can show emergent non-Markovianity in stroboscopic dynamics. Our theory provides a basic framework for studying unique phenomena in dissipative interacting Floquet systems.

Floquet-Magnus effective Lindbladian.—First, we briefly introduce the FM expansion for dissipative Floquet systems \([11]\). Under a time-periodic Liouvillian \(\mathcal{L}(t)\) [Eq. (4)], we define the effective Lindbladian \(\mathcal{L}_{\text{eff}}\) and the Floquet operator \(U_{\text{eff}}\) by

\[
\mathcal{L}_{\text{eff}} = \frac{1}{T} \log U_{\text{eff}}, \quad U_{\text{eff}} = T \exp \left( \int_0^T \mathcal{L}(t) dt \right). \tag{2}
\]

The FM effective Lindbladian \(\mathcal{L}_{\text{eff}}\) is obtained by the perturbative expansion for \(\mathcal{L}_{\text{eff}}\) up to the \(n\)-th order of \(||\mathcal{L}(t)||_{\text{op}}/\omega\) (\(||||_{\text{op}},\) operator norm), and this results in

\[
\mathcal{L}_{\text{eff}}^n = \sum_{i=0}^n \mathcal{L}_{i}^{(i)}, \quad \mathcal{L}_{i}^{(0)} = \frac{1}{T} \int_0^T \mathcal{L}(t) dt, \tag{3}
\]

\[
\mathcal{L}_{i}^{(1)} = \frac{1}{2T} \int_0^T dt_1 \int_0^{t_1} dt_2 \left[ \mathcal{L}(t_1), \mathcal{L}(t_2) \right]. \tag{4}
\]

This perturbative expansion has the same convergence radius as the one for closed systems, and hence \(\mathcal{L}_{\text{eff}}^n\) well captures the stroboscopic dynamics at \(t = mT\) (\(m \in \mathbb{N}\)) within \(||\mathcal{L}(t)||_{\text{op}}/\omega < O(1)\) [12,15]. Since Liouvillians are closed only with respect to the summation, Liouvillanity of \(\mathcal{L}_{\text{eff}}\) and each \(i\)-th order term \(\mathcal{L}_{i}^{(i)}\) are nontrivial, and to be clarified.

Condition for Liouvillanity.—A super-operator \(\mathcal{L}\) on states \(\rho\) is called a Liouvillan if its time evolution operator \(\exp(\mathcal{L}t)\) becomes a CPTP map for all \(t \geq 0\), and it is equivalent to that \(\mathcal{L}\) is given by the Lindblad form, which is the time-independent version of Eq. (1). Here, we introduce some mathematical tools and describe how to judge Liouvillanity of the FM expansions.

We denote a set of \(d \times d\) matrices by \(\mathbb{M}_d\) and assume a state \(\rho \in \mathbb{M}_d\). We define the Frobenius basis \([F_j])_{j=1}^{d^2}\) as a complete orthonormal set (CONS) for \(\mathbb{M}_d\). Using the Frobenius inner product \(\langle A,B \rangle_F = \text{Tr}(A^\dagger B)\), it satisfies the orthonormality relations \(\langle F_j,F_k \rangle_F = \delta_{jk}\), \(\text{Tr}[F_j] = 0\) if \(j \neq d^2\), and \(F_{d^2} = I/\sqrt{d}\). Next, we introduce the doubled Hilbert space representation [46,47], in which we regard a state \(\rho = (\rho)_{ij} \in \mathbb{M}_d\) as a \(d^2\)-dimensional vector \(\tilde{\rho} = (\rho)_{ij}\). Then, any linear operator on \(\rho\) is represented by a matrix in \(\mathbb{M}_{d^2}\), and a Liouvillian \(\mathcal{L}\) is given by

\[
\mathcal{L} = -i(H \otimes I - I \otimes H^T) + \sum_{j,k=1}^{d^2-1} a_{jk} \left[ F_j \otimes F_k^\dagger - \frac{1}{2} (F_k^\dagger F_j \otimes I + I \otimes F_k^\dagger F_j^T) \right]. \tag{5}
\]

with a hermitian matrix \(H \in \mathbb{M}_d\) (Hamiltonian) and a hermitian positive-semidefinite matrix \([a_{jk}])_{j,k=1}^{d^2-1} \in \mathbb{M}_{d^2-1}\) (dissipator). In this representation, the system size becomes double, and an action \(A_B (A,B \in \mathbb{M}_d)\) is written as \(A \otimes B^T\). We call the system, which \(A \otimes B^T\) acts on, a real (fictitious) system.

Using the hermiticity-preserving property \((\mathcal{L}(t)[\rho])^\dagger = \mathcal{L}(t)[\rho^\dagger]\) \((\forall \rho \in \mathbb{M}_d)\) and the trace-preserving property \(\text{Tr}(\mathcal{L}(t)[\rho]) = 0\) \((\forall \rho \in \mathbb{M}_d)\), the \(n\)-th order FM expansion \(\mathcal{L}_{\text{eff}}^n\) is always written in the same form as Eq. (5) for any \(n\) with hermitian matrices \(H = H^\dagger \in \mathbb{M}_d\) and \([a_{jk}])_{j,k=1}^{d^2-1} \in \mathbb{M}_{d^2-1}\) (See Ref. [43] or Supplemental Materials [48] for the derivation). Note that \([a_{jk}])_{j,k=1}^{d^2-1} \) is not always positive-semidefinite, and hence \(\mathcal{L}_{\text{eff}}^n\) is not always a Liouvillan. Using the orthonormality of the Frobenius basis, the condition for Liouvillanity is summarized as follows:

\[
\mathcal{L}_{\text{eff}}^n \text{ is a Liouvillan} \iff [a_{jk}])_{j,k=1}^{d^2-1} \in \mathbb{M}_{d^2-1} \text{ is positive-semidefinite,} \tag{6}
\]

with

\[
a_{jk} = \text{Tr}[(F_j^\dagger F_k^T) \mathcal{L}_{\text{eff}}^n]. \tag{7}
\]

Each \(i\)-th order term \(\mathcal{L}_{i}^{(i)}\) is also written in the same form as Eq. (5), and thus we can judge its Liouvillanity in the same way. However, we obtain the following result [48]:

(a) \(\mathcal{L}_{i}^{(0)} (= \mathcal{L}_i)\) is always a Liouvillan.

(b) \(\mathcal{L}_{i}^{(i)} (i \geq 1)\) is a Liouvillan if and only if

\[
[a_{ij}] = \left[ \text{Tr}[(F_j^\dagger F_k^T) \mathcal{L}_{i}^{(i)}] \right] = 0. \tag{8}
\]

Thus, except for special cases where \(\mathcal{L}_{i}^{(i)}\) gives no dissipation, any higher order term \(\mathcal{L}_{i}^{(i)} (i \geq 1)\) is not a Liouvillan in general. This brings an essential difference from closed systems. In closed systems, each order term \(H_{i}^{(i)}\) is always a Hamiltonian, and hence the FM effective Hamiltonian \(H_i = \sum_{i=0}^n H_{i}^{(i)}\) is also a Hamiltonian. On the other hand, in dissipative cases, generic higher order terms \(\mathcal{L}_{i}^{(i)} (i \geq 1)\) do harm to the Liouvillanity of \(\mathcal{L}_{\text{eff}}^n\).

Liouvillanity in noninteracting systems.—We first discuss Liouvillanity of the FM effective Lindbladian for noninteracting systems, and show that noninteracting systems host two model-dependent phenomena. We focus on a single spin with \(S = 1/2\), and the Frobenius basis is given by the Pauli matrices \(\sigma^i: F_i = \sigma^i/\sqrt{2} (i = 1,2,3)\), \(F_4 = \sigma^0/\sqrt{2}\). We consider a time-periodic drive (the period \(T = 2\tau > 0\)):

\[
\mathcal{L}(t)\rho = \begin{cases} -i\hbar [\sigma^3,\rho] + \gamma_2 (\tilde{\sigma}^{13}\rho \tilde{\sigma}^{-13} \rho) & (0 \leq t < \tau) \\
\gamma_1 (\sigma^1 \rho \sigma^1 - \rho) & (\tau \leq t < 2\tau), \end{cases} \tag{9}
\]

with \(\tilde{\sigma}^{13} = (\sigma^1 + \sigma^3)/\sqrt{2}\). Here, \(h \in \mathbb{R}\) is the strength of the magnetic field in \(z\)-direction, and the parameters
\(\gamma_1(>0)\) and \(\gamma_2(\geq 0)\) represent dephasing in certain directions. We assume the high-frequency regime where the frequency \(\omega = \pi/\tau\) is much larger than \(h, \gamma_1,\) and \(\gamma_2,\) or equivalently we assume \(h\tau, \gamma_1\tau, \gamma_2\tau \ll 1.\) We discuss two different models \(L_A(t) = \mathcal{L}(t)_{|h=0,\gamma_2=0}\) and \(L_B(t) = \mathcal{L}(t)_{|h=0,\gamma_2>0}\) and evaluate the FM expansion up to the second order. For the first model \(L_A(t),\) we obtain
\[
[a^2_{jk}] = \gamma_1 \begin{pmatrix}
1 - \alpha^{(2)}_A & \alpha^{(1)}_A & 0 \\
\alpha^{(1)}_A & \alpha^{(2)}_A & 0 \\
0 & 0 & 0
\end{pmatrix},
\]
with \(\alpha_A^{(1)} = -h\tau\) and \(\alpha_A^{(2)} = 2(h\tau)^2/3.\) We also obtain the result for the second model \(L_B(t)\) as
\[
[a^2_{jk}] = \begin{pmatrix}
\gamma_2/2 + \gamma_1 + \alpha_B^{(2)} & 0 & \gamma_2/2 + \alpha_B^{(2)} \\
0 & 0 & 0 \\
\gamma_2/2 + \alpha_B^{(2)} & 0 & \gamma_2/2 - \alpha_B^{(2)}
\end{pmatrix},
\]
with \(\alpha_B^{(2)} = (\gamma_1\tau)(\gamma_2\tau)/6 \) and \(\alpha_A^{(1)}\) and \(\alpha_B^{(2)}\) represent the first and the second order terms (the first order term vanishes in the second model). Setting them to zero properly, the zeroth and first order results \([a^0_{jk}]\) and \([a^1_{jk}]\) are reproduced.

By evaluating the positive-semidefinite of \([a^2_{jk}]\) \((n \leq 2)\), we observe two different phenomena: For the first model \(L_A(t),\) the first and the second FM expansions \(L^1_j\) and \(L^2_j\) are not Liouvillians under infinitesimal \(\alpha_A^{(1)}\) and \(\alpha_A^{(2)}\), or equivalently under any finite drive. Similar phenomenon has been observed in Ref. [43], and we discuss the relation in Supplemental Materials [49]. On the other hand, for the second model \(L_B(t),\) the second order FM expansion \(L^1_j\) preserves Liouvillianity within a finite parameter range \(0 < \tau \leq \tau_{\text{max}}.\) These results can be attributed to the structures of \([a^2_{jk}]\). In the first model, higher order terms \([a^{(i)}_{jk}]\) appear in the block different from the zeroth order \([a^0_{jk}]\). As a result, the zero eigenvalue of \([a^0_{jk}]\) in the \((j, k) = (2, 2)\) block is perturbed to a negative value, leading to Liouvillianity breaking. In the second model, \([a^0_{jk}]\) and \([a^{(i)}_{jk}]\) \((i \geq 1)\) are closed in the same block \((j, k) = (1, 1), (1, 3), (3, 1), (3, 3)\), and \([a^0_{jk}]\) is positive-definite in this block. Thus, \([a^0_{jk}]\) does not have negative eigenvalues when the perturbations \([a^{(i)}_{jk}]\) \((i \geq 1)\) are small enough, resulting in Liouvillianity within a finite parameter range.

**Extension to interacting models.**—We now discuss interacting few-body and many-body systems. We consider an \(L\)-site spin chain with \(S = 1/2,\) and the Frobenius basis is a set of the 4\(^2\) matrices in \(\mathbb{M}_{2^L}, \quad F^j = \frac{1}{\sqrt{2^L}} \prod_{i=1}^{L} \sigma^j_i, \quad j = (j_1, \ldots, j_l, \ldots, j_L)\) with \(j_l = 0, 1, 2, 3.\)

By denoting \(\vec{0} = (0, 0, \ldots, 0),\) Liouvillianity of the FM effective Lindbladian \(\mathcal{L}^j\) is confirmed when the matrix \([a^{(n)}_{jk}]_{j,k \neq 0} \in \mathbb{M}_{4^L-1},\) given by \(a^{(n)}_{jk} = \text{Tr}[\left(F^j \otimes F^k\right) \mathcal{L}^j],\) is positive-semidefinite. The difficulty compared to noninteracting systems is that the size of \([a^{(n)}_{jk}]\) is exponentially large in \(L,\) which will be overcome by locality of interactions below.

We begin with a model driven by Ising interactions and dephasing with the periodic boundary conditions:
\[
\mathcal{L}_\text{C}(t)\rho = \begin{cases}
-iJ_z \sum_{i,l} [\sigma^3_{i+l-1} \rho \sigma^3_i - \rho \sigma^3_i \sigma^3_{i+l-1}] & (0 \leq t < \tau) \\
\gamma \sum_i (\sigma_i^1 \rho \sigma_i^1 - \rho \sigma_i^1 \sigma_i^1) & (\tau \leq t < 2\tau).
\end{cases}
\]

The zeroth order \(L^0_j\) is the time-average \(L^0_j = (\mathcal{L} \mathcal{C}_1 + \mathcal{L} \mathcal{C}_2)/2,\) and the first order is
\[
L^1_j + \frac{J_z \gamma \tau}{2} \sum_l \left\{ \sigma^1_l \otimes \sigma^3_l (\sigma^3_{l-1} + \sigma^3_{l+1}) - \sigma^3_l (\sigma^3_{l-1} + \sigma^3_{l+1}) \otimes \sigma^1_l \right\}.
\]

Up to the second order, though the matrix \([a^2_{jk}]\) possesses \(O(L)\) nonzero components, we can rewrite it in a block-diagonalized form by arranging the order of the basis:
\[
[a^2_{jk}] = \sum_{l=1}^L \begin{pmatrix}
1 - 2\alpha^{(2)} & \alpha^{(1)} & \alpha^{(1)} \otimes -\alpha^{(2)} \\
\alpha^{(1)} & \alpha^{(2)} & 0 \\
-\alpha^{(2)} & 0 & 0
\end{pmatrix} \oplus O_D,
\]
with \(\gamma \equiv 2L^{-1} \gamma, \quad \alpha^{(1)} = -J_z \tau, \quad \alpha^{(2)} = 2(J_z \tau)^2/3,\) and \(D = 4L - 4L - 1.\) The basis of the \(4 \times 4\) matrices \((\cdots)\) is composed of \((\cdots 0, j_l = 1, 0, \ldots, (\cdots 0, j_l = 3, 0, \ldots), (\cdots 0, j_l = 3, 2, 0, \ldots), (\cdots 0, 3, j_l = 1, 3, 0, \ldots).\) In these situations, \([a^{(1)}_{jk}] \) \(([a^{(2)}_{jk}]\) is no longer positive-semidefinite. Therefore, neither the first order FM Lindbladian \(L^1_j\) nor the second order one \(L^2_j\) is a Liouvillian under periodic drives \((\tau \neq 0).\)

We also observe the same behavior in another interacting model (See Supplemental Materials [49]). In both interacting cases, immediate Liouvillianity breaking for any \(\tau > 0\) can be attributed to a spreading structure of \([a^{(i)}_{jk}]\) appearing also in the noninteracting model \(L_B(t).\) However, different from noninteracting cases, higher order terms out of the block where \([a^{(i)}_{jk}]\) lies come from interactions involving a larger number of sites [for example, three-body terms in Eq. (13)], which is proven to be essential in interacting systems.

**Liouvillianity breaking in generic interacting systems.—**—We finally show that immediate Liouvillianity breaking of the FM effective Lindbladian takes place in generic few-body and many-body systems with local interactions. Importantly, the above spreading structure of \([a^{(i)}_{jk}]\) universally appears due to the propagation of local interactions in FM expansions, leading to Liouvillianity breaking. Although we discuss an \(L\)-site spin chain with \(S = 1/2,\) our results are easily generalized to any-dimensional finite systems with finite degrees of freedom.

Here, we assume the \(k\)-locality for interactions, indicating that the Hamiltonian \(H(t)\) and the Lindblad operator
As a result, \( H(t) \) includes at-least two-body interactions. The complex energy per site \( \sim |\mathcal{L}(t)|_{op}/L \) is assumed to be bounded by \( J \), which is physically reasonable. For our interacting model [Eq. (12)], we can take \( k = 2 \) and \( J = 4J + 2\gamma \). Under these assumptions, we obtain the following rigorous bound [48],

\[
|a_{jk}^{(i)}| \leq \frac{(2kJT)^i}{i+1}J \cdot i \cdot 2^L. 
\]  

(15)

As a result, \( a_{jk}^{(i)} \) decays within lower orders in high-frequency expansion up to the order \( n < 1/(2kJT) \). Thus, the problem itself for Liouvillianity does not differ from noninteracting cases where whether \( [a_{jk}^{(0)}] \) can remain positive-semi-definite under perturbations of \( [a_{jk}^{(i)}] \) determines the Liouvillianity. However, the important difference from the noninteracting case is that the locality of interactions restricts the form of \( [a_{jk}^{(n)}] \). In the doubled Hilbert space representation, a \( k \)-local Liouvillian \( \mathcal{L}(t) \) on an \( L \)-site system is interpreted as a non-hermitian Hamiltonian with \( k \)-body interactions on real and fictitious systems which have \( L \)-sites respectively as shown in Fig. 1(a). Under the \( k \)-locality, the commutator \( [\mathcal{L}(t_1), \mathcal{L}(t_2)] \) and thereby \( \mathcal{L}_f^{(j)} \) include at-most \( (2k-1) \)-body interactions, since it is composed of the commutators of local terms in \( \mathcal{L}(t) \) which have overlaps on at-least one-site. Considering that generic \( i \)-th order terms \( \mathcal{L}_f^{(i)} \) are composed of \( i \)-tuple multi-commutators of \( \mathcal{L}(t) \), the sum \( \mathcal{L}_f \) includes at-most \( \{ (n+1)k - n \} \)-body interactions. We denote the number of \( i \) satisfying \( j_i \neq 0 \) in \( \mathcal{L}_f \) by \( n_j \), and then nonzero \( a_{jk}^{(n)} \) ensures the existence of a \( (n_j + n_k) \)-body term \( F_j \otimes F_k^* \), involving both of real and fictitious systems [See Fig. 1(a)]. Thus, the locality constraint gives

\[
a_{jk}^{(n)} = 0 \quad \text{if} \quad (n_j + n_k) > \{ (n+1)k - n \}. 
\]  

(16)

By rearranging the Frobenius basis in ascending order of the locality \( n_j \), the matrix \( [a_{jk}^{(n)}] \) is block-diagonalized as follows:

\[
[a_{jk}^{(n)}] = A_{dn} \oplus O_{4^L - d_n - 1}, \quad A_{dn} \in \mathbb{M}_{d_n}. 
\]  

(17)

The basis of the nontrivial part \( A_{dn} \) is composed of \( \mathcal{L}_f \) with \( 1 \leq n_j \leq (n+1)k - n \), and the size \( d_n \) satisfies

\[
d_n \leq L \mathcal{C}_{(n+1)k-n} \cdot 4^{(n+1)k-n} \sim \frac{(4L)^{(n+1)k-n}}{(n+1)k-n}!. 
\]  

(18)

Furthermore, Eq. (16) also indicates that the elements where both \( n_j \) and \( n_k \) exceed \( \{ (n+1)k/2 - n/2 \} \) vanish \( [\psi]: \) the cell function). Thus, assuming \( d_n < 4^L - 1 \), the nontrivial part \( A_{dn} \) is always written in the form of

\[
A_{dn} = \begin{pmatrix} \hat{A}_{en} & \hat{B}^\dagger \\ \hat{B} & O_{dn-e_n} \end{pmatrix}, \quad \hat{A}_{en} \in \mathbb{M}_{e_n} : \text{hermitian}, 
\]  

(19)

with \( e_n \leq L \mathcal{C}_{(n+1)k/2-n/2} \cdot 4^{(n+1)k/2-n/2} \cdot \mathcal{C}(t) \). This triangular form is attributed to the propagation of interactions via higher order terms [Fig. 1(b)], where the Hamiltonian \( H(t) \) (the dissipation \( L_i(t) \)) causes spread closed within real or fictitious systems (over both systems) [Fig. 1(a)]. If the interactions of \( H(t) \) and \( L_i(t) \) are neighboring (simultaneously acting on at-most the \( k \)-th and \( (k/2) \)-th nearest neighbors respectively), the size \( d_n \) reduces to \( O(4(n+1)k-nL) \).

The increasing dimension \( d_n \) with the order \( n \) means the spreading structure of \([a_{jk}^{(n)}]\) from the zeroth order in common with the models \( \mathcal{L}_B(t) \) [Eq. (9)] and \( \mathcal{L}_C(t) \) [Eq. (12)] showing the immediate Liouvillianity breaking. This perturbs zero eigenvalues in \([a_{jk}^{(n)}]\) and can shift.
them to negative. More rigorously, using the Schur complement [30], the triangular hermitian matrix [Eq. [19]] always has at-least (rank$\mathcal{B}$) negative eigenvalues, and hence $\mathcal{L}_f^n$ for $n \geq 1$ is always a non-Liouvillian as long as $B \neq O$. We conclude that Liouvillianity of the FM effective Lindbladian is always broken in generic interacting systems due to spread of interactions.

We note that our discussion is valid for other types of high-frequency expansions such as the van Vleck expansion [38]. Also, there are exceptional cases for Liouvillianity breaking. For instance, Floquet interacting systems under time-independent dissipation [42] can preserve Liouvillianity of the van Vleck expansion up to the first order which the dissipation does not affect [51].

**Conclusions.**—We have considered dissipative Floquet systems in high-frequency regimes, and have evaluated the Liouvillianity of the FM effective Lindbladian for noninteracting systems and locally interacting systems. While noninteracting systems show model-dependent phenomena, we have provided interacting models rigorously showing the immediate breakdown of Liouvillianity.

We have developed a theoretical framework to judge Liouvillianity breaking in terms of the structure of a hermitian matrix $a^\dagger_n a_n$ determined by the FM effective Lindbladian, and have demonstrated that the spread of interactions via higher order terms in generic interacting systems always causes the Liouvillianity breaking of the FM effective Lindbladian.

Our results show that dissipative Floquet dynamics do not have static counterparts even under high-frequency drive in contrast to closed systems. Thus, we expect some unique phenomena in Floquet systems leading to dissipative Floquet engineering, which we leave for future work. We also note that Liouvillianity breaking shows a sign of emergent non-Markovianity [31], and it should be interesting to seek for what kind of non-Markovian dynamics can describe such Floquet Markovian systems.

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We note that Liouvillianity breaking in stroboscopic dynamics does not mean that the FM expansions are unphysical since Liouvillianity is essential to Markovian dynamics in continuous time. In fact, Refs. [25, 31] also numerically observed the Liouvillianity breaking of the effective Lindbladian [Eq. (2)]. Ref. [41] found that the FM expansions, breaking Liouvillianity which is not discussed in the study but can be confirmed, well describe the stroboscopic dynamics.

Since $||L(t)||_{op}$ scales as the system size $L$ in general, $||L(t)||_{op}/\omega$ usually exceeds the convergence radius in many-body systems. In closed systems, the FM expansion is valid in spite of it in many-body systems with local interactions [14–17]. However, it remains an open question in dissipative cases, while the convergence radius ensures the validity of the FM expansion at-least up to few-body.

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[46] See Supplemental Materials S1. It provides the information about the FM expansion, and the derivation of our result on its generic properties.
[47] See Supplemental Materials S2. It provides detailed analysis on our noninteracting and interacting models.
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[49] See Supplemental Materials S3, in which we discuss some exceptions for Liouvillianity breaking in detail.
Supplemental Materials for
“Liouvillianity breaking in Floquet-Lindblad interacting systems under high-frequency drive”

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S1. FLOQUET-MAGNUS EXPANSIONS AND THEIR PROPERTIES

A. Form of the n-th order Floquet-Magnus effective Lindbladian

In this section, we describe each order term of the Floquet-Magnus (FM) effective Lindbladian. The FM expansion is a perturbative expansion of the effective Lindbladian \( \mathcal{L}_{\text{eff}} \) [Eq. (2) in the main text] in terms of \( ||\mathcal{L}(t)||_{\text{op}}/\omega \). Then, each order term \( \mathcal{L}^{(n)}_{f} \) is given by

\[
\mathcal{L}^{(n)}_{f} = \sum_{\sigma} (-1)^{n-\hat{\theta}(\sigma)} \hat{\theta}(\sigma)! \frac{(n-\hat{\theta}(\sigma))!}{n!(n+1)^2 t^2} \int_{0}^{T} dt_{n+1} \cdots \int_{0}^{t_{2}} dt_{1} [\mathcal{L}(t_{\sigma(n+1)}), [\mathcal{L}(t_{\sigma(n)}), \ldots, [\mathcal{L}(t_{\sigma(2)}), \mathcal{L}(t_{\sigma(1)})] \ldots] , \quad (S1)
\]

\[
\hat{\theta}(\sigma) \equiv \sum_{m=1}^{n} \theta(\sigma(m+1) - \sigma(m)), \quad \theta(x): \text{a step function}, \quad (S2)
\]

where \( \sigma \) represents the permutation of \( \{1, 2, \ldots, n+1\} \) [S1]. We note that each i-th order term is composed of i-tuple multi-commutators of the Liouvillian \( \mathcal{L}(t) \) at different time. The n-th order FM effective Lindbladian \( \mathcal{L}^{(n)}_{f} \) is defined by the summation up to the n-th order, \( \mathcal{L}^{(n)}_{f} = \sum_{i=0}^{n} \mathcal{L}^{(i)}_{f} \).

In the main text, we consider binary drives described by

\[
\mathcal{L}(t) = \begin{cases} \mathcal{L}_1 & (0 \leq t < \tau) \\ \mathcal{L}_2 & (\tau \leq t < 2\tau = T), \end{cases} \quad (S3)
\]

and then the Floquet operator is given by \( \mathcal{U}_{\text{eff}} = \exp(\mathcal{L}_{\text{eff}} \cdot 2\tau) = \exp(\mathcal{L}_2\tau) \exp(\mathcal{L}_1\tau) \). The Baker-Campbell-Hausdorff formula enables the direct calculation of each order term \( \mathcal{L}^{(i)}_{f} \), which results in

\[
\mathcal{L}^{(0)}_{f} = \frac{1}{2}(\mathcal{L}_1 + \mathcal{L}_2), \quad \mathcal{L}^{(1)}_{f} = \frac{\tau}{4}[\mathcal{L}_2, [\mathcal{L}_1, \mathcal{L}_1]], \quad \mathcal{L}^{(2)}_{f} = \frac{\tau^2}{24}[\mathcal{L}_2 - \mathcal{L}_1, [\mathcal{L}_2, [\mathcal{L}_1, \mathcal{L}_1]]], \quad \mathcal{L}^{(3)}_{f} = \frac{\tau^3}{48}[\mathcal{L}_1, [\mathcal{L}_2, [\mathcal{L}_1, \mathcal{L}_2]]], \ldots \quad (S4)
\]

B. Condition for Liouvillianity

Ref. [S2] clarified a way to judge the Liouvillianity of the FM effective Lindbladian. Here, we reformulate this using the Frobenius basis \( \{F_j\} \), while the basis for \( d \)-dimensional square matrices \( M_d \) is not specified in Ref. [S2]. We note that the traceless-property of the Frobenius basis enables us to easily extract an effective Hamiltonian and an effective dissipation from the FM expansions, and to evaluate their Liouvillianity and upper bound, as discussed later. First, we derive the form of the FM expansions, which is the same as Eq. (5) in the main text.

**Theorem 1.** In the doubled Hilbert space representation, the n-th order FM effective Lindbladian \( L^{(n)}_{f} \) is always written in the following form using the Frobenius basis \( \{F_j\} \):

\[
L^{(n)}_{f} = -i(H^n \otimes I - I \otimes (H^n)^T) + \sum_{j,k=1}^{d^2-1} a_{jk}^{(n)} \left[ F_j \otimes F_k^* - \frac{1}{2} (F_k^* F_j^T F_j F_k) \right] , \quad (S5)
\]

where the matrices \( H^n \in M_d \) and \( [a_{jk}^{(n)}] \in M_{d^2-1} \) are hermitian.
Proof.—The Lindbladian $\mathcal{L}(t)$ [Eq. (1) in the main text] satisfies the following conditions at each time $t$:

$$\text{Tr}(\mathcal{L}(t)[\rho]) = 0, \quad (\mathcal{L}(t)[\rho])^\dagger = \mathcal{L}(t)[\rho^\dagger], \quad \forall \rho. \quad (S6)$$

The first condition represents that the time evolution operator $\mathcal{U}(t)$ is trace-preserving, and the second one represents that $\mathcal{L}(t)$ is hermiticity-preserving. Then, the sum, difference, and product of $\mathcal{L}(t)$ satisfy the same properties. For example,

$$(\mathcal{L}(t_1)\mathcal{L}(t_2)[\rho])^\dagger = \mathcal{L}(t_1)[(\mathcal{L}(t_2)[\rho])^\dagger] = \mathcal{L}(t_1)\mathcal{L}(t_2)[\rho^\dagger] \quad (S7)$$

is obtained. Since $\mathcal{L}_r^n$ is composed of the summation of commutators of $\mathcal{L}(t)$, it possesses the same properties:

$$\text{Tr}(\mathcal{L}_r^n[\rho]) = 0, \quad \forall \rho, \quad (\mathcal{L}_r^n[\rho])^\dagger = \mathcal{L}_r^n[\rho^\dagger], \quad \forall \rho. \quad (S8)$$

$$(\mathcal{L}_r^n[\rho])^\dagger = \sum_i x_i X_i \rho X_i^\dagger, \quad x_i \in \mathbb{R}, \quad X_i \in \mathbb{M}_d. \quad (S10)$$

We express $X_i \in \mathbb{M}_d$ as $X_i = \sum_{j=1}^d t^n_{ij} F_j$, then we arrive at

$$\mathcal{L}_r^n[\rho] = \sum_{j,k=1}^d a^n_{jk} F_j \rho F_k^\dagger, \quad a^n_{jk} = \sum_i x_i t^n_{ij} (t^n_{ik})^* = (a^n_{kj})^*. \quad (S11)$$

The latter equation represents the hermiticity of the matrix $[a^n_{jk}]_{j,k=1}^{d^2-1}$. Using the fact $F_d^2 = I_d/\sqrt{d}$ and defining $M^n = (a^n_{d^2,d^2/2d}) I_d + \sum_{j=1}^{d^2-1} a^n_{jd} F_j$ result in

$$\mathcal{L}_r^n[\rho] = M^n \rho + \rho(M^n)^\dagger + \sum_{j,k=1}^{d^2-1} a^n_{jk} F_j \rho F_k^\dagger = i [\text{Im}(M^n), \rho] + \{\text{Re}(M^n), \rho\} + \sum_{j,k=1}^{d^2-1} a^n_{jk} F_j \rho F_k^\dagger. \quad (S12)$$

In the last equality, we define two hermitian matrices $\text{Re}(M) = (M + M^\dagger)/2$ and $\text{Im}(M) = (M - M^\dagger)/2i$. Then,

$$\text{Tr}(\mathcal{L}_r^n[\rho]) = \text{Tr}([\text{Re}(M^n), \rho]) + \sum_{j,k=1}^{d^2-1} a^n_{jk} \text{Tr}[F_j \rho F_k^\dagger] = \text{Tr} \left[ 2\text{Re}(M^n) + \sum_{j,k=1}^{d^2-1} a^n_{jk} F_j^\dagger F_j \right] \rho$$

should be zero regardless of $\rho$ from Eq. (S8). Therefore, $\text{Re}(M^n)$ is given by

$$\text{Re}(M^n) = -\frac{1}{2} \sum_{j,k=1}^{d^2-1} a^n_{jk} F_j^\dagger F_k, \quad (S13)$$

where the hermiticity of $\text{Re}(M^n)$ is ensured by Eq. (S11). Finally, by defining $H^n = -\text{Im}(M^n)$ and using the doubled Hilbert space representation, we obtain Eq. (5).

The Liouvillianity of $\mathcal{L}_r^n$ is determined only by $[a^n_{jk}]$. Using the spectral decomposition of the hermitian matrix, $a^n_{jk} = \sum_{i=1}^{d^2-1} \tilde{x}_i \tilde{t}_{ij} (\tilde{t}_{ik})^*$ ($\tilde{x}_i \in \mathbb{R}$ and $j, k = 1, 2, \ldots, d^2 - 1$), and defining $\tilde{L}_i = \sum_{j=1}^{d^2-1} \sqrt{|\tilde{x}_i|} \tilde{t}_{ij} F_j$, we can rewrite Eq. (S5) as follows:

$$\mathcal{L}_r^n = -i (H^n \otimes I - I \otimes (H^n)^T) + \sum_{i=1}^{d^2-1} \text{sgn}(\tilde{x}_i) \left[ \tilde{L}_i \otimes \tilde{L}_i^* - \frac{1}{2} (\tilde{L}_i^\dagger \tilde{L}_i \otimes I + I \otimes \tilde{L}_i^T \tilde{L}_i^* \right]. \quad (S14)$$

Thus, $\mathcal{L}_r^n$ is a Liouvillian if and only if $[a^n_{jk}]$ is positive-semidefinite ($\tilde{x}_i \geq 0$ for all $i$). Conversely, by expanding $H^n$ by the Frobenius basis as $H^n = \sum_{j=1}^{d^2-1} h^n_j F_j$ (The $j = d^2$ component is irrelevant in the commutator), we obtain

$$\mathcal{L}_r^n = -i \sum_{j=1}^{d^2-1} h^n_j (F_j \otimes I - I \otimes F_j^T) + \sum_{j,k=1}^{d^2-1} a^n_{jk} \left[ F_j \otimes F_k - \frac{1}{2} (F_k^T F_j \otimes I + I \otimes F_j^T F_k^*) \right]. \quad (S15)$$
When we assume that $F_j$ is hermitian (for example, this is satisfied in spin systems in the main text), and then $h_j^n$ is real due to the hermiticity of $H^n$. Multiplying $F_j^\dagger \otimes F_k^T$ $(j, k \neq d^2)$ to Eq. (S5) and taking its trace, we can extract the dissipative components,

$$ a_{jk}^n = \text{Tr}[(F_j^\dagger \otimes F_k^T) \mathcal{L}_f^n] = \langle (F_j \otimes F_k^*), \mathcal{L}_f^n \rangle_F, \quad (S16) $$

where we have used the traceless-property of the Frobenius basis. In a similar way, under the hermiticity of $F_j$, we obtain

$$ \text{Tr}[(F_j \otimes I) \mathcal{L}_f^n] = -i h_j^n \text{Tr}I - \frac{1}{2} \sum_{j'k'} a_{j'k'}^n \text{Tr}[F_j F_{k'} F_{j'}^\dagger] \cdot \text{Tr}I, \quad (S17) $$

and we can extract the effective Hamiltonian terms $h_j^n$ by

$$ h_j^n = \frac{i}{d} \text{Tr}[(F_j \otimes I) \mathcal{L}_f^n] + \frac{i}{2} \sum_{j'k'} \text{Tr}[(F_j^\dagger \otimes F_{k'}^T) \mathcal{L}_f^n] \cdot \text{Tr}[F_j F_{k'} F_{j'}^\dagger]. \quad (S18) $$

We emphasize that Theorem 1 is proven only using the fact that $\mathcal{L}_f^n$ is given by the integrals of polynomial functions of the Liouvillian $\mathcal{L}(t)$. Therefore, each $i$-th order term in the FM effective Lindbladian, $\mathcal{L}_f^{(i)}$, is also written in the same form as Eq. (S5). Other types of high-frequency expansions such as the van Vleck expansion and the Schrieffer-Wolff expansion [S8] also satisfy this theorem, and we can check their Liouvillianity in the same way.

C. Liouvillianity of each $i$-th order term $\mathcal{L}_f^{(i)}$

Here, we derive the propositions (a) and (b) in the main text, which dictate that a higher order term in the FM effective Lindbladian is not a Liouvillian in general.

**Theorem 2.** The zeroth order term of the FM effective Lindbladian, $\mathcal{L}_f^{(0)}$, is always a Liouvillian. On the other hand, for $i \geq 1$, the $i$-th order term $\mathcal{L}_f^{(i)}$ is a Liouvillian if and only if $[a_{jk}^{(i)}] = O$, where the matrix $[a_{jk}^{(i)}]$ is defined by $a_{jk}^{(i)} = \text{Tr}[(F_j^\dagger \otimes F_k^*) \mathcal{L}_f^{(i)}]$. \hfill □

**Proof.**—As discussed in the last subsection, each $i$-th order term $\mathcal{L}_f^{(i)}$ is always written in the same form as Eq. (S5), and $\mathcal{L}_f^{(i)}$ is a Liouvillian if and only if $[a_{jk}^{(i)}] = [\text{Tr}[(F_j^\dagger \otimes F_k^*) \mathcal{L}_f^{(i)}]]$ is positive-semidefinite. Since the zeroth order term is given by the time-average of $\mathcal{L}(t)$, we obtain

$$ a_{jk}^{(0)} = \frac{1}{T} \int_0^T a_{jk}(t) dt. \quad (S19) $$

The hermitian matrix $[a_{jk}^{(0)}]$ becomes positive-semidefinite since $[a_{jk}(t)]$ is positive-semidefinite, and hence $\mathcal{L}_f^{(0)}$ is always a Liouvillian. On the other hand, using the fact that each order term $\mathcal{L}_f^{(i)}$ is composed of $i$-tuple commutators, each $i$-th order term is traceless, $\text{Tr} \left[ \mathcal{L}_f^{(i)} \right] = 0$, for $i \geq 1$. We can calculate the trace in another way using Eq. (S5), and this results in

$$ \text{Tr} \left[ \mathcal{L}_f^{(i)} \right] = -\frac{1}{2} \sum_{j,k=1}^{d^2-1} a_{jk}^{(i)}(\text{Tr}[F_j^\dagger F_k \otimes I] + \text{Tr}[I \otimes F_j^T F_k^*]) = -d \cdot \sum_{j=1}^{d^2-1} a_{jj}^{(n)} = -d \cdot \text{Tr} \left[ (a_{jk}^{(i)})^* \right]. \quad (S20) $$

Therefore, $[a_{jk}^{(i)}]$ is also traceless, and hence the summation of the eigenvalues of $[a_{jk}^{(i)}]$ is zero. Since all of the eigenvalues of hermitian positive-semidefinite matrices cannot be negative, $[a_{jk}^{(i)}]$ is positive-semidefinite if and only if $[a_{jk}^{(i)}] = O$. Using the condition for Liouvillianity, we complete the proof of the theorem. \hfill □

Importantly, this theorem is derived from that the zeroth order $\mathcal{L}_f^{(0)}$ is the time-average of $\mathcal{L}(t)$, and that the higher order terms $\mathcal{L}_f^{(i)}$ are composed of commutators. Thus, this theorem also holds for other types of high-frequency expansion with the same properties, such as the van Vleck expansion and the Schrieffer-Wolff expansion.
D. Upper bound of dissipative terms in the FM effective Lindbladian

We derive the upper bound of the matrix elements $a_{j\bar{k}}^{(i)}$ in few-body or many-body systems with local interactions. Before discussing the result, we rigorously define the locality and the extensiveness of dissipative systems dominated by a Liouvillian $\mathcal{L}$, the time-independent version of Eq. (1) in the main text. We call a Liouvillian $\mathcal{L}$ being $k$-local when its Hamiltonian $H$ and Lindblad operator $L_i$ include at-most $k$-body and $(k/2)$-body interactions respectively. Let us define $\mathcal{L}|_X$ by the terms in $\mathcal{L}$, which nontrivially act on the domain $X$ in the doubled Hilbert space representation. Note that the number of the sites becomes doubled in the doubled Hilbert space representation, and that a domain $X$ is a subset of $\{1, 2, \ldots, 2L\}$. Then, a Liouvillian $\mathcal{L} = \sum_X \mathcal{L}|_X$ is called $J$-extensive when

$$\sum_{X: X \ni i} \|\mathcal{L}|_X\|_{\text{op}} \leq J, \quad \forall i \in \{1, 2, \ldots, 2L\}$$

(S21)

is satisfied. The left hand side means the maximal complex energy at each site $i$, and the extensiveness represents the complex energy per site $\sim ||\mathcal{L}\|_{\text{op}}/2L$ is bounded by $J$. We note that these definitions are the extended versions of those in Ref. [S7, S1] generalized to dissipative cases. With their rigorous definitions, we obtain the following result on the upper bound of $a_{j\bar{k}}^{(i)}$.

**Theorem 3.** We consider an $L$-site system where each site has $f$-degrees of freedom, and suppose that its Liouvillian $\mathcal{L}(t)$ is $k$-local and $J$-extensive at every time $t$. Then, the dissipative terms of each $i$-th order term $\mathcal{L}_{f}^{(i)}$, represented by $|a_{j\bar{k}}^{(i)}|$, has the following upper bound:

$$|a_{j\bar{k}}^{(i)}| \leq \frac{(2kJT)^i}{i+1} J \cdot i! \cdot f^L.$$

(S22)

Proof.—We consider $a_{j\bar{k}}^{(i)}$ for some fixed $j, \bar{k} \neq \bar{0}$, and let $X$ be a domain where $F_{j} \otimes F_{\bar{k}}^* \otimes F_{\bar{r}}^* \otimes F_{\bar{j}}$ nontrivially acts in the $2L$-site system. Let us define $A_{j\bar{k}}^{(i)}$ by

$$A_{j\bar{k}}^{(i)} = \sum_{j', \bar{k}'} \sum_{j' \neq j, \bar{k}' \neq \bar{k}} a_{j'\bar{k}'}^{(i)} \left( F_{j'} \otimes F_{\bar{k}'}^* \otimes F_{\bar{r}}^* \otimes F_{\bar{j}} \right),$$

(S23)

where $\sum_{j', \bar{k}'}$ represents the summation over $j', \bar{k}'$ such that $F_{j'} \otimes F_{\bar{k}'}^* \otimes F_{\bar{r}}^* \otimes F_{\bar{j}}$ nontrivially acts only on the domain $X$ (there exist at-most $(f^2-1)^{|X|}$ terms). Then, $A_{j\bar{k}}^{(i)}$ is the unique term nontrivially acting just on $X$ in $\mathcal{L}_{f}^{(i)}$, and hence we obtain

$$\|A_{j\bar{k}}^{(i)}\|_{\text{op}} \leq J^{(i)},$$

(S24)

where $J^{(i)}$ is the extensiveness of the $i$-th order term $\mathcal{L}_{f}^{(i)}$ from the definition of the extensiveness. When we assume the $k$-locality and the $J$-extensiveness of the Lindbladian $\mathcal{L}(t)$, it is known that $J^{(i)}$ is bounded as follows (See Lemma 5 in Ref. [S1]):

$$J^{(i)} \leq \frac{(2kJT)^i}{i+1} J \cdot i!.$$

(S25)

Let us define the Frobenius norm $|| \cdot ||_F$ for square matrices by $||A||_F = \sqrt{\langle A, A \rangle_F}$, and then, using the Schwartz inequality, we arrive at

$$\left| a_{j\bar{k}}^{(i)} \right| = \left| \left\langle F_j \otimes F_{\bar{k}}, A_{j\bar{k}}^{(i)} \right\rangle \right|_F \leq \left\| \left( F_j \otimes F_{\bar{k}}^* \right) \right\|_F \cdot \left\| A_{j\bar{k}}^{(i)} \right\|_F.$$

(S26)

Using the relation $||A||_F \leq \sqrt{\text{rank} A \cdot ||A||_{\text{op}}}$ and Eqs. (S24) and (S25), we obtain

$$\left| a_{j\bar{k}}^{(i)} \right| \leq \sqrt{\text{rank} \left( A_{j\bar{k}}^{(i)} \right) \cdot ||A_{j\bar{k}}^{(i)}||_{\text{op}}} \leq \frac{(2kJT)^i}{i+1} J \cdot i! \cdot f^L,$$

(S27)

in which $f = 2$ reproduces the result Eq. (17) in the main text. $\square$
S2. DETAILED ANALYSIS OF THE SPECIFIC MODELS

In this section, we provide detailed information about the noninteracting and interacting models described in the main text. We explicitly give their FM effective Lindbladian and their properties.

A. Noninteracting models

The first noninteracting model driven by a z-field and dephasing is described by

$$\mathcal{L}_A(t)\rho = \begin{cases} -i\hbar[\sigma^3, \rho] & (0 \leq t < \tau) \\ \gamma_1(\sigma^1 \rho \sigma^1 - \rho) & (\tau \leq t < 2\tau = T). \end{cases} \quad (S28)$$

The calculation up to the first order using Eqs. (S4) and (S16) results in

$$\mathcal{L}^1_J = -\frac{i\hbar}{2}(\sigma^3 \otimes I - I \otimes \sigma^3) + \frac{\gamma_1}{2}(\sigma^1 \otimes \sigma^1 - 1) + \frac{\gamma_1 h\tau}{2}(\sigma^1 \otimes \sigma^2 - \sigma^2 \otimes \sigma^1), \quad [a_{jk}^1] = \gamma_1 \begin{pmatrix} 1 & -h\tau & 0 \\ -h\tau & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (S29)$$

where we have used the doubled Hilbert space representation for $\mathcal{L}^1_J$. The eigenvalues of the matrix $[a_{jk}^1]$ are 0 and

$$\gamma_1(1 \mp \sqrt{1 + 4(h\tau)^2})/2,$$

and hence it is not positive-semidefinite for any $\tau > 0$, leading to the Liouvillianity breaking of $\mathcal{L}^1_J$ under any finite drive. However, we would like to note that the Liouvillianity can be recovered in this case since the negative eigenvalue of $[a_{jk}^1]$,

$$\gamma_1 \frac{1 - \sqrt{1 + 4(h\tau)^2}}{2} \simeq -\gamma_1(h\tau)^2,$$  

(S30)

is ignorable in the first order approximation. Here, we denote one of the unitary matrices which diagonalize $[a_{jk}^1]$ by $U^1 \in \mathbb{M}_{d^2-1}$, and define the modified positive-semidefinite matrix $[\tilde{a}_{jk}^1]$ by $[\tilde{a}_{jk}^1] = (U^1)\text{diag}(\gamma_1, 0, 0)U^1$. Then, the modified FM effective Lindbladian defined by

$$\tilde{\mathcal{L}}^1_J = -\frac{i\hbar}{2}(\sigma^3 \otimes I - I \otimes \sigma^3) + \sum_{j,k}^d \tilde{a}_{jk}^1 \left( F_j \otimes F_k^* - \frac{1}{2}(F_k^1 F_j \otimes I + I \otimes F_j^T F_k^*) \right) \quad (S31)$$

becomes a Liouvillian, and it correctly describes the dynamics within the same time scale as for $\mathcal{L}^1_J$ because of $\tilde{\mathcal{L}}^1_J \simeq \mathcal{L}^1_J + O((\lambda T)^2)$ with $\lambda = \max(h, \gamma_1)$. On the other hand, the second order FM effective Lindbladian is given by

$$\mathcal{L}^2_J = \mathcal{L}^1_J - \frac{i\hbar(\gamma_1\tau)^2}{6}(\sigma^3 \otimes I - I \otimes \sigma^3) - \frac{\gamma_1(h\tau)^2}{3}((\sigma^1 \otimes \sigma^1 + \sigma^2 \otimes \sigma^2), \quad [a_{jk}^2] = \gamma_1 \begin{pmatrix} 1 - 2(h\tau)^2/3 & -h\tau & 0 \\ -h\tau & 2(h\tau)^2/3 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (S32)$$

The matrix $[a_{jk}^2]$ always has a negative eigenvalue

$$\gamma_1 \frac{1 - \sqrt{1 + 4(h\tau)^2/3 + 16(h\tau)^4/9}}{2} \simeq -\gamma_1 \frac{1}{3}(h\tau)^2,$$  

(S33)

and hence the Liouvillianity breaking of $\mathcal{L}^2_J$ takes place under any finite drive. We note that this negative eigenvalue is no longer ignorable and the Liouvillianity breaking cannot be essentially removed like the first order approximation.

F. Haddadfarshi et al. [S2] have also evaluated the Liouvillianity of the FM effective Lindbladian in a noninteracting single spin and a harmonic oscillator. In their models, they have observed similar phenomena: Liouvillianity breaking under any finite drive, but Liouvillianity can be recovered for any truncation orders in a similar way to the one described here. We would like to note that such procedures are limited to the low order FM expansions in general and that Liouvillianity breaking is essential for the approximate dynamics. In fact, we also calculate the third order and the fourth order matrices $[a_{jk}^n] \ (n = 3, 4)$, and they have non-negligible negative eigenvalues $\sim O((\lambda T)^2)$, and hence Liouvillian breaking is essential also for such higher orders.

The second noninteracting model is driven by two different kinds of dephasing, and it is described by

$$\mathcal{L}_B(t)\rho = \begin{cases} \gamma_2(\sigma^{13} \rho \sigma^{13} - \rho) & (0 \leq t < \tau) \\ \gamma_1(\sigma^1 \rho \sigma^1 - \rho) & (\tau \leq t < 2\tau = T). \end{cases} \quad (S34)$$
with $\sigma^{13} = (\sigma^1 + \sigma^3)/\sqrt{2}$. As discussed in the main text, the first order term $L^{(1)}_f = -i(\gamma_1 \gamma_2 \tau/4) (\sigma^2 \otimes I - I \otimes (\sigma^2)^T)$ gives a unitary dynamics, and hence the first order FM Lindbladian $L^1_f$ is always a Liouvillian. The second order calculation results in

$$L^2_f = L^1_f + \frac{\alpha^{(2)}}{2} (-\gamma_3 \sigma^1 \otimes \sigma^3 + \gamma_1 \sigma^1 \otimes \sigma^3 + \gamma_2 \sigma^{1} \otimes \sigma^{3}), \quad [a^{(2)}_{jk}] = \left( \begin{array}{ccc} \frac{\gamma_2}{2} + \gamma_1 + \alpha^{(2)} \gamma_2 & 0 & \frac{\gamma_2}{2} + \alpha^{(2)} \gamma_1 \\ 0 & 0 & 0 \\ \frac{\gamma_2}{2} + \alpha^{(2)} \gamma_1 & 0 & \frac{\gamma_2}{2} - \alpha^{(2)} \gamma_2 \end{array} \right)$$

with $\alpha^{(2)} = (\gamma_1 \tau)(\gamma_2 \tau)/6$. The minimal eigenvalue of the matrix $[a^{(2)}_{jk}]$ is given by

$$\lambda_{\text{min}} = \min \left\{ 0, \frac{\gamma_1 + \gamma_2}{2} - \frac{1}{6} \sqrt{\left( \frac{\gamma_1 \gamma_2 \tau^4}{\gamma_1^2 \gamma_2^2 \tau^4} + 12 \gamma_3^2 \gamma_2^2 \tau^2 + 9(\gamma_1^2 + \gamma_2^2) \right)} \right\}.$$  \hspace{1cm} (S36)

As discussed in the main text, the Liouvillianity of $L^2_f$ is maintained in a finite parameter range $0 \leq \tau \leq \tau_{\text{max}}$, where $\tau_{\text{max}}$ is explicitly given by

$$\tau_{\text{max}} = \left[ \frac{6}{\gamma_1^2 + \gamma_2^2} \right]^{1/2}.$$  \hspace{1cm} (S37)

We also calculate the effective Lindbladian up to the fourth order, and observe the closed structure of $[a^{(i)}_{jk}]$ within the block $j, k = 1, 3$ for $i \leq 4$. Thus, the Liouvillianity is maintained within a certain parameter range of the duration $\tau$, though the threshold value $\tau_{\text{max}}$ changes depending on the order of perturbation.

**B. Interacting models**

We first describe the detailed result for the interacting model in the main text:

$$L_C(t)\rho = \left\{ \begin{array}{ll} -iJ_z \sum_l [\sigma^l \sigma^{3}_{l+1}, \rho] & (0 \leq t < \tau) \\
\gamma \sum_l (\sigma^l \rho \sigma^l - \rho) & (\tau \leq t < 2 \tau = T). \end{array} \right.$$  \hspace{1cm} (S38)

We give the explicit formula for the FM expansions up to the second order as follows.

(The first order)

$$L^1_f = \frac{L_{C1} + L_{C2}}{2} + \frac{J_z \gamma \tau}{2} \sum_l \left\{ \sigma^l \otimes \sigma^l (\sigma^3_{l-1} + \sigma^3_{l+1}) - \sigma^l (\sigma^3_{l-1} + \sigma^3_{l+1}) \otimes \sigma^l \right\},$$

$$[a^{(1)}_{jk}] = \left[ \bigoplus_{l=1}^{L} \gamma \cdot 2^{L-1} \left( \begin{array}{ccc} -1 & -(J_z \tau) & -(J_z \tau) \\ J_z \tau & 0 & 0 \\ -(J_z \tau) & 0 & 0 \end{array} \right) \right] \oplus O_{4^{L-3}L-1}. $$  \hspace{1cm} (S40)

The $3 \times 3$ matrix in $[a^{(1)}_{jk}]$ always possesses a negative eigenvalue $\lambda_{\text{min}} = \gamma \cdot 2^{-2L-2} \left( 1 - \sqrt{1 + 8(J_z \tau)^2} \right)$, and hence the Liouvillianity of the first order FM effective Lindbladian $L^1_f$ is broken under any finite drive $\tau > 0$. Since $\lambda_{\text{min}} \sim \gamma \cdot 2^L O((J_z \tau)^2)$ is small in terms of the truncation order, it seems to be difficult to observe the breakdown. As discussed in the noninteracting models, we can compose the effective Hamiltonian recovering Liouvillianity. However, the Liouvillianity breaking whose degree is characterized by $O((J_z \tau)^2)$ cannot be generally retrieved in the higher order expansions, as discussed below.

(The second order)

$$L^2_f = L^1_f - \frac{\gamma(J_z \tau)^2}{3} \sum_l \left\{ \sigma^l \otimes \sigma^l (1 + \sigma^3_{l-1} \sigma^3_{l+1}) + \sigma^l (1 + \sigma^3_{l-1} \sigma^3_{l+1}) \otimes \sigma^l + \sigma^l (\sigma^3_{l-1} + \sigma^3_{l+1}) \otimes \sigma^l \right\},$$

$$[a^{(2)}_{jk}] = \left[ \bigoplus_{l=1}^{L} \gamma \cdot 2^{L-1} \left( \begin{array}{ccc} 1 - 4(J_z \tau)^2/3 & -(J_z \tau) & -(J_z \tau) \\ (J_z \tau) & 2(J_z \tau)^2/3 & 2(J_z \tau)^2/3 \\ -(J_z \tau) & 2(J_z \tau)^2/3 & 2(J_z \tau)^2/3 \end{array} \right) \right] \oplus O_{4^{L-4}L-1}. $$  \hspace{1cm} (S42)
The enlarged nontrivial parts of \( [a_{jk}^{2}] \) come from the four-body terms in \( L_{f}^{2} \), just caused by the propagation of local interactions discussed in the main text. A matrix in this triangular form always has at least one negative eigenvalue (See Eq. (21) in the main text and Ref. [S8]), and hence the Liouvillianity of \( L_{f}^{2} \) is also broken for any \( \tau > 0 \). We numerically evaluate the minimal eigenvalue of the matrix \( [a_{jk}^{2}] \), denoted by \( \lambda_{\text{min}} \), and it is well fitted to

\[
\lambda_{\text{min}}/[\gamma \cdot 2^{L-1}(J_{z}\tau)^{2}] \simeq \sum_{m=0}^{3} C_{m}(J_{z}\tau)^{m}, \quad C_{0} = -6.67 \times 10^{-1}, \quad C_{1} = 1.97 \times 10^{-2}, \quad C_{2} = -3.08, \quad C_{3} = 2.84
\]

with the root mean square error 3.25 \times 10^{-4} in \( 0 < J_{z}\tau < 0.5 \). The degree of the Liouvillianity breaking is characterized by \( O((J_{z}\tau)^{2}) \), which corresponds to the truncation order \( n = 2 \). Therefore, the second order FM expansion \( L_{f}^{2} \) always breaks Liouvillianity to a non-negligible extent, and it can affect the dynamics.

We also consider interacting models different from the one discussed in the main text. We consider an \( L \)-site Ising spin chain driven by the following time-periodic Liouvillian:

\[
L_{D}(t) = \frac{L_{D1}}{2} - \frac{\gamma}{8} \sum_{i} \left\{ i \sigma_{i}^{x} \otimes \sigma_{i+1}^{y} - i \sigma_{i}^{y} \otimes \sigma_{i+1}^{x} \right\} = \frac{L_{D2}}{2} - \frac{\gamma}{8} \sum_{i} \left\{ i \sigma_{i}^{x} \otimes \sigma_{i+1}^{y} - i \sigma_{i}^{y} \otimes \sigma_{i+1}^{x} \right\}
\]

with \( \sigma_{i}^{x} = (\sigma_{i}^{1} \pm i\sigma_{i}^{2})/2 \). The first step represents the nearest neighbor Ising interactions in \( z \)-direction, and the second one represents a noise which makes the down spin state preferable. The calculation up to the first order results in

\[
L_{f}^{2} = L_{D1} + L_{D2} - \frac{\gamma}{8} \sum_{i} \left\{ i \sigma_{i}^{x} \otimes \sigma_{i+1}^{y} - i \sigma_{i}^{y} \otimes \sigma_{i+1}^{x} \right\} = \frac{L_{D3}}{2} - \frac{\gamma}{8} \sum_{i} \left\{ i \sigma_{i}^{x} \otimes \sigma_{i+1}^{y} - i \sigma_{i}^{y} \otimes \sigma_{i+1}^{x} \right\}
\]

The basis for the nontrivial 4 \times 4 matrices in this equation is composed of \( \{ \ldots, 0, j_{i} = 1, 0, \ldots \}, \ldots, 0, j_{i} = 2, 0, \ldots \}, \ldots, 0, j_{i} = 3, 0, \ldots \}, \ldots, 0, j_{i} = 3, 1, 0, \ldots \}. The first order terms proportional to \( (J_{z}\tau) \) comes from three-body terms of \( L_{f}^{2} \) in the doubled Hilbert space representation, which are just manifestation of the propagating local interactions via the FM expansions. This spreading structure of the matrix \( [a_{jk}^{1}] \) is common in generic interacting systems as discussed in the main text, leading to the results that \( [a_{jk}^{1}] \) always has a negative eigenvalue \( \gamma \cdot 2^{L-2}(1 - \sqrt{1 + (J_{z}\tau)^{2}}) \). Therefore, this interacting model also hosts Liouvillianity breaking of the FM effective Lindbladians under any finite drive, as well as the model in the main text. We note that, since the negative eigenvalue of \( [a_{jk}^{1}] \) is small compared to the truncation order, the Liouvillianity breaking up to the first order can be removed in these models by using the same way for the noninteracting models in the previous section. However, when we consider higher order FM effective Lindbladians, Liouvillianity breaking can be no longer removed in general as well.

**S3. SOME EXCEPTIONS FOR LIOUVILLIANITY BREAKING IN INTERACTING SYSTEMS**

In the main text, we claim that local interactions and many-body properties cause Liouvillianity breaking of the FM effective Lindbladian in generic systems. In this section, we would like to pick up some exceptional examples.

The first one is a commutative Liouvillian, which satisfies

\[
[L(t_{1}), L(t_{2})] = 0, \quad \forall t_{1}, \forall t_{2}.
\]

The \( n \)-th order FM effective Lindbladian \( L_{f}^{n} \) is trivially a Liouvillian for any order \( n \) regardless of the presence of interactions due to \( L_{f}^{0} = L_{f}^{n} \). It is known that the time evolution for microscopic dynamics can break Markovianity even in a commutative Liouvillian system, while that for stroboscopic dynamics, \( L_{f}^{0} \), never breaks Markovianity [S3].
Another exceptional example is a dissipative Floquet system under time-independent dissipations, described by

\[ \mathcal{L}(t)\rho = -i[H(t),\rho] + \sum_i \left( L_i\rho L_i^\dagger - \frac{1}{2} \{L_i^\dagger L_i, \rho\} \right), \quad H(t + T) = H(t). \] (S48)

Our discussion on Liouvillian breaking can be applied to other types of high-frequency expansions such as the van Vleck expansion:

\[ \mathcal{L}_n^{V} = \sum_{i=0}^{n} \mathcal{L}_i^{(i)} \mathcal{L}_0^{(0)} = \mathcal{L}_0, \quad \mathcal{L}_n^{(1)} = T \sum_{m=1}^{\infty} \frac{[\mathcal{L}_{m-1}, \mathcal{L}_m]}{4m\pi}, \ldots, \] (S49)

where \( \mathcal{L}_m \) represents the Fourier components \( \mathcal{L}_m = \int_0^T \mathcal{L}(t) e^{-i2\pi mt/T} dt/T \). When the dissipation is time-independent, only the zeroth component \( \mathcal{L}_0 \) includes its effect, and the others \( \mathcal{L}_m \) \( (m \neq 0) \) become equivalent to those of closed Floquet systems driven by the Hamiltonian \( H(t) \). Therefore, the first order term \( \mathcal{L}_n^{(1)} \) always represents a unitary dynamics, and hence the van Vleck effective Lindbladian \( \mathcal{L}_n^{V} \) always becomes a Liouvillian up to the first order \( n \leq 1 \) regardless of interactions as Theorem 2. On the other hand, the higher order effective Lindbladian, in which the dissipation is involved through \( \mathcal{L}_0 \), becomes a non-Liouvillian in the presence of interactions as discussed in the main text. A recent paper by T. N. Ikeda et al., considers this situation with assuming the detailed balance condition [S9]. They obtain generic formula for the nonequilibrium steady state (NESS) up to the first order, in which the existence of NESS is ensured by the Liouvillianity of \( \mathcal{L}_1 \).

We also expect that Liouvillian breaking will be avoided even in the presence of local interactions if we pick up some specific solvable models or those satisfying some special closed algebra of commutators. Even if such systems are found, they will also experience the Liouvillian breaking under perturbations due to the generic structure of \( [a_{jk}^m] \), Eqs. (19) and (21), in the main text.

### S4. EXISTENCE OF NESS AND BREAKDOWN OF TRAJECTORY METHOD

Liouvillian breaking of the FM effective Lindbladian implies that we cannot use conventional theories for static Markovian systems brought by Liouvillianity. We do not know whether individual theories constructed in static systems so far are valid even in the absence of Liouvillianity, but we show that it can possibly break the two important generic notions, the existence of NESS and the validity of the trajectory method.

Let us consider an effectively static system driven by the FM effective Lindbladian \( \mathcal{L}_n^{\text{eff}} \). Nonequilibrium steady states (NESS) exist if and only if \( \mathcal{L}_n^{\text{eff}} \) has at least one zero-eigenvalue and all the eigenvalues of \( \mathcal{L}_n^{\text{eff}} \) have nonpositive real parts. Then, the right eigenstates with zero eigenvalues are called NESS. The existence of NESS is ensured under Liouvillianity. On the other hand, \( \mathcal{L}_n^{\text{eff}} \) is written as

\[ \mathcal{L}_n^{\text{eff}}\rho = -i[H_n,\rho] + \sum_i s_i \left( L_i^n\rho L_i^{n\dagger} - \{L_i^{n\dagger} L_i^n, \rho\} \right), \quad s_i = \pm 1, \] (S50)

and some of \( \{s_i\} \) become negative if Liouvillianity is broken [See Eq. [S14]]. From this representation, \( (\mathcal{L}_n^{\text{eff}})^\dagger I_d = O \) \( (I_d: \text{the } d\text{-dimensional identity matrix}) \) is satisfied, indicating that \( I_d \) is the left eigenstate of \( \mathcal{L}_n^{\text{eff}} \). Thus, \( \mathcal{L}_n^{\text{eff}} \) always has at least one right eigenstate \( \rho_0 \) with zero eigenvalue. However, all the eigenvalues of \( \mathcal{L}_n^{\text{eff}} \) do not necessarily have nonpositive real parts, and hence the state \( \rho_0 \) does not always represent NESS. Therefore, \( \mathcal{L}_n^{\text{eff}} \) which breaks Liouvillianity does not ensure the existence of NESS in general.

In our models \( \mathcal{L}_\alpha(t) \) \( (\alpha = A, B, C, D) \), NESS becomes ill-defined under \( \mathcal{L}_n^{\text{eff}} \) \( (n \geq 1) \) when the frequency is comparable to the energy scale, although such an anomalous effect is not physically accessible. However, we expect that, when the NESS of the Liouvillian \( \mathcal{L}_n^{\text{eff}} \) is degenerated or the Liouvillian gap \( \Delta = \min \{ \text{Re} \lambda \neq 0 \} \lambda : \text{eigenvalue of } \mathcal{L}_n^{\text{eff}} \} \) is small enough, some anomalous behaviors can be observed in physically relevant regimes, which will be left for future work.

Next, we discuss the validity of trajectory method, with which we can efficiently calculate the nonequilibrium dynamics [S10]. In static Liouvillian systems, the Lindblad equation is rewritten as

\[ \partial_t \rho = -i(H_{\text{eff}}\rho - \rho H_{\text{eff}}^\dagger) + \sum_i L_i\rho L_i^\dagger \] (S51)

with a non-hermitian Hamiltonian \( H_{\text{eff}} = H - (i/2) \sum_i L_i^\dagger L_i \). A single trajectory dynamics is a stochastic dynamics composed of non-unitary time evolution under \( H_{\text{eff}} \) and quantum jumps by \( L_i \). Let us assume the initial state...
\[ \rho_0 = |\psi_0\rangle \langle \psi_0 | \] and consider the dynamics of \( |\psi_0\rangle \) for infinitesimal duration \( \delta t \). Up to the first order of \( \delta t \), the state is stochastically updated by \( \exp(-i\text{H}_{\text{eff}} \delta t) |\psi_0\rangle / \sqrt{1 - p} \) with the probability \( 1 - p \) (non-hermitian dynamics) or by \( L_i |\psi_0\rangle / \sqrt{p_i} \) with the probability \( p_i \) (quantum jumps). Here, the probabilities \( p_i \) and \( p \) are given by \( p_i = \langle \psi_0 | L_i^\dagger L_i |\psi_0\rangle \) and \( p = 1 - \sum p_i \) respectively. A series of states \( |\psi(t)\rangle \) obtained by repeating this procedure \( m \) times up to \( t = m\delta t \) is called a trajectory. By taking the statistical ensemble of \( |\psi(t)\rangle \langle \psi(t)| \) over many trajectories with small \( \delta t \), we can reproduce the density operator \( \rho(t) \) obeying the Lindblad equation Eq. (S51). We note that all the eigenvalues of \( \text{H}_{\text{eff}} \) have nonpositive imaginary parts, indicating that the non-unitary time evolution by \( \text{H}_{\text{eff}} \) is always lossy. The lost probability due to this non-unitary dynamics corresponds to the probabilities of quantum jumps by \( L_i \).

On the other hand, if \( L^n_f \) breaks Liouvillianity, the corresponding non-hermitian Hamiltonian becomes \( \text{H}_{\text{eff}} = H - (i/2) \sum s_i L_i^{n_f} L_i^{n_f} \) (\( s_i = \pm 1 \)) where some of \( \{s_i\} \) are \(-1\). Thus, \( \text{H}_{\text{eff}} \) can have eigenvalues with positive imaginary parts, and then stochastic dynamics composed of the non-hermitian Hamiltonian time evolution and quantum jumps becomes ill-defined (some of the probabilities \( p_i \) become negative). This represents the breakdown of the trajectory method in the absence of Liouvillainity for \( L^n_f \).

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