ON THE UNIQUENESS OF AN ERGODIC MEASURE OF FULL DIMENSION FOR NON-CONFORMAL REPELLERS

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Abstract. We give a subclass $\mathcal{L}$ of Non-linear Lalley-Gatzouras carpets and an open set $\mathcal{U}$ in $\mathcal{L}$ such that any carpet in $\mathcal{U}$ has a unique ergodic measure of full dimension. In particular, any Lalley-Gatzouras carpet which is close to a non-trivial general Sierpinski carpet has a unique ergodic measure of full dimension.

1. Introduction. It is well known that a $C^{1+\alpha}$ conformal repeller has a unique ergodic measure of full dimension. This is a consequence of Bowen’s equation together with the classical thermodynamic formalism developed by Sinai-Ruelle-Bowen, see [19], [15], [3], [4] and [17]. Moreover, this measure is a Gibbs state relative to some Hölder-continuous potential. Is this true for non-conformal repellers?

The simplest examples of non-conformal repellers are the general Sierpinski carpets, whose Hausdorff dimension was studied by Bedford [2] and McMullen [14]. They computed the Hausdorff dimension of these sets by establishing the variational principle for the dimension. As a consequence, these repellers have an ergodic measure of full dimension (in fact Bernoulli) and, by [10] (see also [15] and [8]), this measure is unique.

In [9] Lalley and Gatzouras introduced a larger class of non-conformal repellers and computed their Hausdorff dimension also by establishing the variational principle for the dimension, and so these repellers have a Bernoulli measure of full dimension (see also [13] for a random version of this result). In [1] the authors give an example of a Lalley-Gatzouras carpet which has two Bernoulli measures of full dimension. So the answer to the question formulated above is negative.

In this paper, we study this problem – existence and uniqueness – of an ergodic measure of full dimension – for a larger class of non-conformal repellers which we shall call Non-linear Lalley-Gatzouras carpets. As the name suggests, these repellers are the $C^{1+\alpha}$ non-linear versions of the Lalley-Gatzouras carpets. They are defined by an Iterated Function System $\{f_{ij}\}$ where $f_{ij}: [0,1]^2 \to [0,1]^2$, $i = 1,\ldots,m$, $j = 1,\ldots,m_i$ have the skew-product form $f_{ij}(x,y) = (a_{ij}(x,y),b_i(y))$, with the domination condition $0 < |\partial_x a_{ij}(x,y)| < |b'_i(y)| < 1$, and the corresponding attractor $\Lambda$ (see Section 2 for precise definitions). The Hausdorff dimension of these repellers was, essentially, computed in [11] by establishing the variational principle for the dimension. Because of the non-linearity of the transformations $f_{ij}$, the
existence of an ergodic measure of full dimension turns out to be a non-trivial problem. This was proved to be true in \[12\] (in a more general context). Then we have the following.

**Theorem 1.1.** A Non-linear Lalley-Gatzouras carpet has an ergodic measure of full dimension. Moreover, this measure is a Gibbs state for a relativized variational principle.

As we know now (by \[11\]), such a measure is, in general, not unique. The main purpose of this paper is to give sufficient conditions for having a unique ergodic measure of full dimension, based on an idea introduced in Remark 3 of \[12\].

We can introduce a natural topology on the class of Non-linear Lalley-Gatzouras carpets by saying that two of these carpets are close if the corresponding functions of the Iterated Function System are $C^{1+\alpha}$ close (with alphabet $(i,j)$ fixed). We denote by $L$ the subclass of Non-linear Lalley-Gatzouras carpets for which $\partial_{xx}a_{ij} = 0$, i.e. $a_{ij}(x,y) = \tilde{a}_{ij}(y)x + u_{ij}(y)$. Of course, $L$ contains the Lalley-Gatzouras carpets.

In this paper, a general Sierpinski carpet is a Lalley-Gatzouras carpet for which $\partial_{x}a_{ij} = a$ and $b'_i = b$ for some positive constants $a$ and $b$ and every $(i,j)$ (this is a more general definition than usual). We say that such a carpet is non-trivial if $a < b$ and the natural numbers $m_i \geq 2$, $i = 1, ..., m$ are not all equal to each other.

**Theorem 1.2.** There is an open set $U$ in $L$ such that:

(i) $U$ contains all non-trivial general Sierpinski carpets;

(ii) every repeller $K$ in $U$ has a unique ergodic measure of full dimension $\mu_K$;

(iii) the map $U \ni K \mapsto \mu_K$ is continuous.

We believe that Theorem 1.2 also holds in the class of Non-linear Lalley-Gatzouras carpets. The reason for restricting to the subclass $L$ relies on the necessity of considering basic potentials in the relativized variational principle of \[7\], which we use, in order to have additional properties (see Remark 3).

What about repellers defined in $\mathbb{R}^d$, $d \geq 3$? There is a natural way of defining the $d$-dimensional versions of general Sierpinski carpets and Lalley-Gatzouras carpets, which we shall call Sierpinski sponges and self-affine sponges, respectively. Very recently, in \[5\] it was proved that the variational principle for the dimension does not hold, in general, for self-affine sponges (even for $d = 3$). On the other hand, by \[10\], Sierpinski sponges do have a unique ergodic measure of full dimension. So a natural question is:

**Problem.** Does a self-affine sponge close to a Sierpinski sponge have a unique ergodic measure of full dimension?

This paper is organized as follows. In Section 2 we introduce the class of Non-linear Lalley-Gatzouras carpets and say how Theorem 1.1 follows from the works \[11\] and \[12\]. In Section 3 within the more general context of \[12\], we prove some properties of measures of maximal dimension, a relativized version of Ruelle’s formulas for the derivative of the pressure, and a criterium for uniqueness of a measure of maximal dimension (Theorem 3.2). In Section 4 we use this criterium to prove Theorem 1.2. As a byproduct of our results, we obtain a version of Bowen’s equation for a class of non-conformal repellors (Theorems 3.1 and 3.3).

2. Non-linear Lalley-Gatzouras carpets.

2.1. Definition. Let $g_i: [0,1] \rightarrow [0,1]$, $i = 1, ..., m$ be $C^{1+\alpha}$ for some $\alpha > 0$. We say that $\{g_1, ..., g_m\}$ is a Simple Function System (SFS) if:
\begin{itemize}
\item $0 < |g'_i(x)| < 1$ for every $x \in [0, 1]$;
\item the sets $g_i([0, 1])$, $i = 1, \ldots, m$ are pairwise disjoint.
\end{itemize}

Let $f_{ij} : [0, 1]^2 \to [0, 1]^2$, $i = 1, \ldots, m$, $j = 1, \ldots, m_i$ be $C^{1+\alpha}$ such that:

\begin{enumerate}[(H1)]
\item $f_{ij}(x, y) = (a_{ij}(x, y), b_i(y))$;
\item \{$b_1, \ldots, b_m$\} is SFS;
\end{enumerate}

for each $i \in \{1, \ldots, m\}$ and $y \in [0, 1]$.

\begin{enumerate}[(H3)]
\item \{$a_{i1}(\cdot, y), \ldots, a_{im}(\cdot, y)$\} is SFS;
\item $\max_{x \in [0, 1]} |\partial_y a_{ij}(x, y)| < |b'_i(y)|$, $j = 1, \ldots, m_i$.
\end{enumerate}

Then there is a unique nonempty compact set $\Lambda$ of $[0, 1]^2$ such that

\[
\Lambda = \bigcup_{(i,j)} f_{ij}(\Lambda).
\]

We call the pair $\{(f_{ij}, \Lambda)\}$ a Non-linear Lalley-Gatzouras carpet.

When the functions $a_{ij}$ and $b_i$ are linear and $\partial_y a_{ij} = 0$, we get the definition of a Lalley-Gatzouras carpet, see [9] (where equality is allowed in (H4)). When, moreover, $\partial_x a_{ij} = a$ and $b'_i = b$ for some positive constants $a$ and $b$ and every $(i, j)$, we get the definition of a general Sierpinski carpet, see [2] and [14] (in fact, our definition is a little more general).

### 2.2. Hausdorff dimension

The Hausdorff dimension of Non-linear Lalley-Gatzouras carpets was, essentially, computed in [11] by establishing the variational principle for the dimension. In fact, the theorems in [11] are formulated in terms of a Dynamical System $f$ instead of an Iterated Function System $\{f_{ij}\}$, although in its proofs we mainly used the $f_{ij}$ approach. The relation between the two approaches is given by $f_{ij} = (f R_{ij})^{-1}$ where $R_{ij}$ is an element of a Markov partition for $f$. Beside imposing a skew-product structure for $f$ (which translates to (H1)), we considered a $C^2$ perturbation of the 2-torus transformation $f_0(x, y) = (lx, my)$, where $l > m > 1$ are integers. The only reason for doing this is to inherit from the linear system a domination condition (which translates to (H4)) and a simple Markov partition (inducing a full shift) which is smooth. More precisely, the Markov partition is constructed using the invariant foliation by horizontal lines (due to the skew-product structure) and an invariant smooth vertical foliation, which exists because the vertical lines constitute a normally expanding invariant foliation for $f_0$.

In the present setting, all we need to show is that the sets

\[
R_{(i_1, j_1) (i_2, j_2) \ldots (i_n, j_n)} = f_{i_1,j_1} \circ f_{i_2,j_2} \circ \cdots \circ f_{i_n,j_n}([0, 1]^2)
\]

have vertical boundaries formed by $C^1$ curves with uniformly bounded distortion for all $n \in \mathbb{N}$. But, as we shall see, this is a consequence of the domination condition (H4).

Let

\[
\lambda = \max_{(x, y), (i, j)} \frac{|\partial_x a_{ij}(x, y)|}{|b'_i(y)|}
\]

which is $< 1$ by (H4), and

\[
C = (1 - \lambda)^{-1} \max_{(x, y), (i, j)} \frac{|\partial_y a_{ij}(x, y)|}{|b'_i(y)|}.
\]

We will see that each $f_{ij}$ transforms vertical graphs with distortion $\leq C$ into vertical graphs with distortion $\leq C$. Let $\mathcal{G}_F = \{(F(y), y) : y \in I\}$ with $|F'(y)| \leq C$ for all
Consider the Bernoulli measure $G(y) = a_{ij}((b_i^{-1}(y)),(b_i^{-1}(y)), \ y \in b_i(I)$. We see that (with $z = b_i^{-1}(y)$ and $w = (F(z),z)$

$$G'(y) = \partial_x a_{ij}(w) b'_i(z)^{-1} F'(z) + \partial_y a_{ij}(w) b'_i(z)^{-1},$$

so $|G'(y)| \leq \lambda C + |\partial_y a_{ij}(w) b'_i(z)^{-1}| \leq C$. Then, starting with the vertical graphs $\{0\} \times [0,1]$ and $\{1\} \times [0,1]$ and using induction on $n$, we get the desired property for the sets $R_{(i,j_1),(i,j_2),\ldots,(i,j_n)}$.

Then it follows from the proof of Theorem A in [11] that, there exists $A > 0$ such that, for every $n \in \mathbb{N}$,

$$\dim_H\Lambda = \dim_H\Lambda_n \pm \frac{A}{n},$$

where $\Lambda_n$ is a Lalley-Gatzouras carpet defined using an appropriate linearization of the functions $f_{ij_{1j} \circ F_{ij_{2j}} \circ \cdots \circ F_{ij_{nj}}}$.

More precisely, given $n \in \mathbb{N}$, consider the $n$-tuples $i = (i_1,\ldots,i_n)$ and $j = (j_1,\ldots,j_n)$, where $i_k \in \{1,\ldots,m\}, j_k \in \{1,\ldots,m\}$, $k = 1,\ldots, n$, and write

$$b_i = b_{i_1} \circ b_{i_2} \circ \cdots \circ b_{i_n}, \quad a_{ij} = \pi_1(f_{ij_{1j}} \circ F_{ij_{2j}} \circ \cdots \circ F_{ij_{nj}}),$$

where $\pi_1(x,y) = x$. Note that, because of the skew-product structure,

$$b'_i(y) = \prod_{k=1}^n b'_{i_k}(y_k) \quad \text{and} \quad \partial_x a_{ij}(x,y) = \prod_{k=1}^n \partial_x a_{i_k,j_k}(z_k),$$

where $y_k = b_{i_{k+1}} \circ \cdots \circ b_{i_n}(y), y_n = y$ and $z_k = f_{i_{k+1}j_{k+1}} \circ \cdots \circ f_{i_{nj}j_n}(x,y), z_n = (x,y)$. Consider the numbers

$$\alpha_{ij,n} = \max_{(x,y) \in [0,1]^2} |\partial_x a_{ij}(x,y)| \quad \text{and} \quad \beta_{i,n} = \max_{y \in [0,1]} |b'_i(y)|.$$

Let $p^n = (p^n_i)$ be a probability vector in $\mathbb{R}^{nm}$. We define

$$\lambda_n(p^n) = \frac{\sum_i p^n_i \log p^n_i}{\sum_i p^n_i \log \beta_{i,n}}$$

and $t_n(p^n)$ as being the unique real in $[0,1]$ satisfying

$$\sum_i p^n_i \log \left( \frac{\sum_j \alpha_{ij,n}(p^n)}{\sum_{j'} \alpha_{ij',n}(p^n)} \right) = 0.$$

Consider the Bernoulli measure $\mu_{p^n}$ for the Iterated Function System $\{f_{i_{j1}} \circ \cdots \circ f_{i_{jn}}\}$ that assigns to each $R_{(i_{j1}),\ldots,(i_{jn})}$ the weight

$$\frac{\alpha_{ij,n}(p^n)}{\sum_{j'} \alpha_{ij',n}(p^n)}.$$

**Theorem 2.1** (Proof of Theorem A, [11]). Let $\{\{f_{ij}\}, \Lambda\}$ be a Non-Linear Lalley-Gatzouras carpet. There exist constants $A, B > 0$ such that, for every $n \in \mathbb{N}$,

$$\dim_H\mu_{p^n} = \lambda_n(p^n) + t_n(p^n) \pm \frac{B}{n},$$

and

$$\dim_H\Lambda = \sup_{p^n} \{\lambda_n(p^n) + t_n(p^n)\} \pm \frac{A}{n}.$$
Moreover, \( \{\{f_{ij}\}, \Lambda \} \mapsto \dim_H \Lambda \) is a continuous function in the class of Non-Linear Lalley-Gatzouras carpets.

**Remark 1.** The continuity of \( \{\{f_{ij}\}, \Lambda \} \mapsto \dim_H \Lambda \) follows from the Proof of Corollary \( \Lambda \) in [11]. In fact, there we used the \( C^2 \) topology but it is clear that we can use the \( C^{1+\alpha} \) topology.

As a consequence, the variational principle for the dimension holds, i.e. the Hausdorff dimension of \( \Lambda \) is the supremum of the Hausdorff dimension of ergodic measures (with respect to \( \{f_{ij}\} \)) on \( \Lambda \). In [12] we prove the existence of an ergodic measure of full dimension for \( \Lambda \), which is a Gibbs state for a relativized variational principle. Thus we have Theorem 1.1.

3. Properties of measures of maximal dimension. The results given in this section hold in the more general context of [12]. Let \( (X,T) \) and \( (Y,S) \) be subshifts such that \( (Y,S) \) is a mixing subshift of finite type and a factor of \( (X,T) \), i.e. there exists a continuous and surjective mapping \( \pi : X \to Y \) such that \( \pi \circ T = S \circ \pi \).

We consider the usual metrics on \( X \) and \( Y \). If \( x = (x_1, x_2, \ldots) \), \( x' = (x'_1, x'_2, \ldots) \) are two different points in \( X \), we define \( d_X(x, x') = 2^{-n} \) where \( n \) is the least positive integer such that \( x_n \neq x'_n \). Similarly, we define \( d_Y(\cdot, \cdot) \). In this way, both maps \( T \) and \( S \) are open, Lipschitz-continuous and expanding.

We will use a version of the relativized variational principle developed in the works [6] and [7], so we need to make a few more hypotheses (see e.g. Theorem 2.10 of [6]; there the roles of \( X \) and \( Y \) are interchanged). We assume:

- \( \#\pi^{-1}(y) \geq 2 \) and \( T(\pi^{-1}(y)) = \pi^{-1}(S(y)) \) for every \( y \in Y \);
- \( \pi \) is open and \( d_Y(\pi(x), \pi(x')) \leq d_X(x, x') \) for every \( x, x' \in X \);
- for every \( \varepsilon > 0 \) there exists \( N > 0 \) such that for every \( x \in X \),
  \[ T^N(B(x, \varepsilon) \cap \pi^{-1}(\pi(x))) = \pi^{-1}(\pi(T^N(x))). \]

(Here, \( B(x, \varepsilon) \) is the ball in \( X \) of radius \( \varepsilon \) centered at \( x \).) Since we are assuming this last hypothesis, \( (X,T) \) does not need to be a mixing subshift of finite type as in [12].

3.1. Characterization of measures of maximal dimension. We use the following notation: \( \mathcal{M}(T) \) is the set of all \( T \)-invariant Borel probability measures on \( X \); \( h_\mu(T) \) is the metric entropy of \( T \) with respect to \( \mu \in \mathcal{M}(T) \).

Let \( \varphi : X \to \mathbb{R} \) and \( \psi : Y \to \mathbb{R} \) be positive Hölder-continuous functions. We define

\[
D(\mu) = \frac{h_\mu \circ \pi^{-1}(S)}{\int \psi \circ \pi \, d\mu} + \frac{h_\mu(T) - h_\mu \circ \pi^{-1}(S)}{\int \varphi \, d\mu},
\]

and

\[
D = \sup_{\mu \in \mathcal{M}(T)} D(\mu).
\]

We say that \( \mu \) is a measure of maximal dimension if \( D(\mu) = D \).

The expression for \( D(\mu) \) is quite natural in the study of Hausdorff dimension of non-conformal repellers in the plane, since it is the Hausdorff dimension of an ergodic measure \( \mu \) supported on the repeller. More precisely, when we have a good fibration of the space invariant by the dynamics, as in the case of Non-Linear Lalley-Gatzouras carpets, the dimension of \( \mu \) splits into the dimension of the projected measure \( \mu \circ \pi^{-1} \) on the space of fibres plus the dimension of the measure on the fibres. On each of these one-dimensional spaces the usual formula for the dimension
of a measure holds: it is the quotient between the entropy and the Lyapunov exponent of the corresponding measure. The integrals appearing in $D(\mu)$ correspond to Lyapunov exponents when we choose appropriate functions $\psi$ and $\varphi$. When the variational principle for the dimension holds, as in the case of Non-Linear Lalley-Gatzouras carpets, $D$ is the Hausdorff dimension of the repeller.

When $(X, T)$ satisfies weak specification and both $\psi$ and $\varphi$ are constant functions, it was proved in [8] the existence and uniqueness of a measure of maximal dimension. In [12] we prove the existence of an ergodic measure of maximal dimension, and give a characterization of measures of maximal dimension that we shall describe now (for more details see this reference).

We use the following version of the relativized variational principle by [6] and [7]. Given an Hölder-continuous function $\phi: X \to \mathbb{R}$ and $\nu \in \mathcal{M}(S)$, there exists a positive Hölder-continuous function $A_{\phi}: Y \to \mathbb{R}$ (not depending on $\nu$) such that

$$\sup_{\mu \in \mathcal{M}(T)} \left\{ h_{\mu}(T) - h_{\nu}(S) + \int_X \phi \, d\mu \right\} = \int_Y \log A_{\phi} \, d\nu. \tag{1}$$

Moreover, there is a unique measure $\mu$ for which the supremum in (1) is attained which we call the relative equilibrium state with respect to $\phi$ and $\nu$, and $\mu$ is ergodic if $\nu$ is ergodic.

When $\nu = \delta_y$ is the Dirac measure at a fixed point $y$ of $S$, (1) reduces to the classical variational principle (see [18] or [3]) and $\log A_{\phi}(y)$ is the topological pressure on the fibre $\pi^{-1}(y)$ or, equivalently, $A_{\phi}(y)$ is an eigenvalue of a Ruelle’s Perron-Frobenius operator. In general, $A_{\phi}$ is given by means of a fibrewise transfer operator (see equation (6) below) and is called a gauge function.

Given $\nu \in \mathcal{M}(S)$, there is a unique real $t(\nu) \geq 0$ such that

$$\int_Y \log A_{-t(\nu)\phi} \, d\nu = 0.$$  

This can be thought as a random version of Bowen’s equation (see [4]): in applications, as in the case of a Non-Linear Lalley-Gatzouras carpet, $t(\nu)$ is the Hausdorff dimension of a typical fibre with respect to the measure $\nu$. It is easy to see from (1) that

$$t(\nu) = \sup_{\mu \in \mathcal{M}(T)} \frac{h_{\mu}(T) - h_{\mu \circ \pi^{-1}}(S)}{\int \phi \, d\mu},$$

and therefore

$$D = \sup_{\nu \in \mathcal{M}(S)} \left\{ \frac{h_{\nu}(S)}{\int \psi \, d\nu} + t(\nu) \right\}. \tag{2}$$

This new writing of $D$ will allow us to characterize the measures of maximal dimension by using the classical thermodynamical formalism on the space $(Y, S)$, with respect to an appropriate potential.

Let

$$\underline{t} = \inf_{\nu \in \mathcal{M}(S)} t(\nu) \quad \text{and} \quad \overline{t} = \sup_{\nu \in \mathcal{M}(S)} t(\nu)$$

These numbers might be interpreted as the smallest and the biggest, respectively, Hausdorff dimension of fibres, and a parameter $t \in [\underline{t}, \overline{t}]$ as the Hausdorff dimension of typical fibre with respect to some measure in $\mathcal{M}(S)$. We assume the following
technical condition:

the supremum in (2) is not attained at an ergodic measure $\nu$ with $t(\nu) = t$ or $\bar{t}$.

Let $P(\cdot)$ denote the classical Pressure function with respect to $(Y,S)$, and let $\nu_g$ denote the corresponding Gibbs state with respect to the Hölder-continuous potential $g: Y \to \mathbb{R}$. Given $t \in (t, \bar{t})$, let

$$\Phi_t = (t - D)\psi + \beta(t) \log A_{-t^2}$$

(3)

where $\beta(t)$ is the unique real satisfying

$$\int \log A_{-t^2} d\nu_{\Phi_t} = 0$$

(see [12] for details).

Before we proceed, let us give an interpretation for the potential $\Phi_t$. When we are dealing with a $C^{1+\alpha}$ conformal repeller $J$, modeled by the dynamics $(Y,S)$, the Hausdorff dimension of $J$ is given by the unique root $s = s_0$ of celebrated Bowen’s equation $P(-s\psi) = 0$ (see [4] and [17]), and $\nu_{-s_0\psi}$ is its (unique) measure of full dimension. In the present case, we are dealing with a non-conformal repeller $\Lambda$, modeled by the dynamics $(X,T)$, which projects onto $J$. Unless $\Lambda$ is a product, a measure of full dimension of $\Lambda$ does not project, in general, onto $\nu_{-s_0\psi}$ because the fibres, having different Hausdorff dimensions, contribute with different weights. That is why the gauge function $A_{-t^2}$ appears in (3), and the candidates for the projection of a measure of full dimension of $\Lambda$ are $\nu_{\Phi_t}$, ($\beta(t)$ is just a normalizing constant so that $t$ in (3) has the interpretation of Hausdorff dimension of fibres, $t = t(\nu_{\Phi_t})$ for every $t \in (t, \bar{t})$).

Let $\mu_{\Phi_t}$ be the relative equilibrium state with respect to $-t\varphi$ and $\nu_{\Phi_t}$. The following result follows from the proof of Theorem A and Remark 3 in [12], and might be interpreted as a version of Bowen’s equation for non-conformal repellers (see also Theorem 3.3).

**Theorem 3.1** (Proof of Theorem A, [12]). Assume (H).

Then $D(\mu) = D$ if and only if $\mu = \mu_{\Phi_t}$ and $P(\Phi_{t}) = 0$ for some $t \in (t, \bar{t})$.

Also, 0 is the maximum value of $(t, \bar{t}) \ni t \mapsto P(\Phi_{t})$, so there is at least one measure of maximal dimension.

A major difference with respect to Bowen’s equation is that the equation $P(\Phi_{t}) = 0$ might have several solutions $t$, i.e. there might be several measures of maximal dimension.

**3.2. Relativized Ruelle’s formulas.** We begin by recalling some classical Ruelle’s formulas for the derivative of the pressure.

Let $Z = X$ or $Y$. Given $C > 0$ and $0 < \theta \leq 1$, let $\mathcal{H}^{C,\theta}(Z)$ denote the space of Hölder-continuous functions $\phi: Z \to \mathbb{R}$ satisfying

$$|\phi(z_1) - \phi(z_2)| \leq Cd_Z(z_1, z_2)^\theta,$$

for all $z_1, z_2 \in Z$, (4)

and let

$$||\phi||_{\theta} = \inf\{C > 0: (4) \text{ holds}\}.$$

Then $\bigcup_{C > 0} \mathcal{H}^{C,\theta}(Z)$ becomes a Banach space with the norm $||\phi||_{\theta} = \max(||\phi||, ||\phi||_{\theta})$, where $||.||$ is the uniform norm.
Let \( \phi_t : Y \to \mathbb{R} \) be a one-parameter family of continuous functions. We say that \( t \mapsto \phi_t \) is differentiable if its partial derivative in \( t \) exists, let us call it \( \dot{\phi}_t \) or \( \frac{d}{dt} \phi_t \), and it is a one-parameter family of continuous functions.

The following result follows from [18] (see Chapter 5, in particular Exercises 4 and 5).

**Proposition 1** (Ruelle [18]). If \( \phi_t \in \mathcal{H}^{C,\theta}(Y) \) (with \( C,\theta \) independent of \( t \)), \( t \mapsto \phi_t \) is differentiable and \( \dot{\phi}_t \in \mathcal{H}^{C,\theta}(Y) \) then

\[
\frac{dP(\phi_t)}{dt} = \int \dot{\phi}_t \, d\nu_{\phi_t}.
\]

If, moreover, \( h \in \mathcal{H}^{C,\theta}(Y) \) then

\[
\frac{d}{dt} \int h \, d\nu_{\phi_t} = Q_{\phi_t}(\dot{\phi}_t, h),
\]

where \( Q_{\phi_t}(\cdot, \cdot) : \mathcal{H}^{C,\theta}(Y) \times \mathcal{H}^{C,\theta}(Y) \to \mathbb{R} \) is given by

\[
Q_{\phi_t}(h_1, h_2) = \sum_{n=0}^{\infty} \left( \int_Y (h_1 \circ S^n) \, h_2 \, d\nu_{\phi_t} - \int_Y h_1 \, d\nu_{\phi_t} \int_Y h_2 \, d\nu_{\phi_t} \right).
\]

There exists a constant \( B > 0 \) (depending only on \( C \) and \( \theta \)) such that

\[
|Q_{\phi_t}(h_1, h_2)| \leq B ||h_1||_\theta ||h_2||_\theta.
\]

Also, for each \( h_1, h_2 \in \mathcal{H}^{C,\theta}(Y) \),

\[
\mathcal{H}^{C,\theta}(Y) \ni \phi \mapsto Q_{\phi}(h_1, h_2)
\]

is a continuous function.

**Remark 2.** These formulas are usually presented in the literature for parameter families of the form \( \phi_t = \phi + th \), because this is enough for the study of dimension of conformal repellers (as in Bowen’s equation). In the present paper, we consider parameter families \( \phi_t \) not depending linearly on the parameter \( t \), but Ruelle’s formulas also hold in this case (with \( h \) substituted by \( \dot{\phi}_t \)). See also [16] (Lemma 5.6.4 and Theorems 5.6.5 and 5.7.4).

Now we recall some definitions from [6] and [7] that are used to define \( A_\phi \), for \( \phi \in \mathcal{H}^{C,\theta}(X) \). Given \( y \in Y \), let \( \mathcal{C}_y \) denote the space of bounded continuous functions \( f : \pi^{-1}(y) \to \mathbb{R} \). For each \( y \in Y \) and \( n \in \mathbb{N} \), define the operators \( G^{(n)}_y : \mathcal{C}_y \to \mathcal{C}_y \) by

\[
(G^{(n)}_y f)(x) := \sum_{T^n(x') = T^n(x) \atop \pi(x') = y} \exp \left( \sum_{k=0}^{n-1} \phi(T^k(x')) \right) f(x'),
\]

and

\[
(P^{(n)}_y f)(x) := \frac{(G^{(n)}_y f)(x)}{(G^{(n)}_y 1)(x)}.
\]

Then (see Proposition 2.5 of [7]),

\[
A_\phi(y) := \lim_{n \to \infty} \frac{(G^{(n+1)}_y 1)(x)}{(G^{(n)}_{S(y)} 1)(T(x))},
\]
uniformly in $y \in Y$, $x \in \pi^{-1}(y)$. Moreover (see Corollary 4.14, Remark 4.16 and Proposition 5.5 of [6]), the rate of convergence is exponential depending only in $C$ and $\theta$. Also, for any $y \in Y$, the operators $P_y^{(n)}$ converge to a conditional expectation operator $P_y$ which gives a probability measure $\mu_y$ in $\pi^{-1}(y)$, in the sense that

$$(P_y f)(x) = \int f \, d\mu_y, \quad \text{for any } x \in \pi^{-1}(y).$$

The system $\{\mu_y : y \in Y\}$ is called a Gibbs family for $\phi$.

We will use the following property of $A_\phi$. Given $y \in Y$, consider the operators $V_y : C_y \to C_{S(y)}$ and $U_y : C_{S(y)} \to C_y$ given by

$$(V_y f)(x) := \sum_{x' \in \pi^{-1}(x)} \exp (\phi(x')) f(x'),$$

and

$$(U_y f)(x) := f(T(x)).$$

Then (see Proposition 5.5 of [6])

$$A_\phi(y) P_y = U_y P_{S(y)} V_y.$$

(Note that the operators $C_y^{(n)}$, $P_y^{(n)}$, $P_y$ and $V_y$ depend on the potential $\phi$.)

We say that $\phi \in H^{C,\theta}(X)$ is a basic potential (see Definition 4.1 of [7]), if for $y \in Y$ and $x \in \pi^{-1}(S(y))$ we have

$$A_\phi(y) = (V_y 1)(x),$$

i.e., for each $y \in Y$, the function $V_y 1$ is constant. In this case we have the following.

**Proposition 2** (Proposition 4.4 and Corollary 4.7 of [7]). If $\phi \in H^{C,\theta}(X)$ is a basic potential then:
(a) the Gibbs family for $\phi$ is covariant, i.e.

$$\mu_y \circ T^{-1} = \mu_{S(y)}$$

for each $y \in Y$;
(b) the relative equilibrium state for $\phi$ with respect to $\phi$ and $\nu$ is given by $\mu = \int \mu_y \, d\nu(y)$;

Now we are ready to prove the following.

**Proposition 3.** Let $\varphi \in H^{C,\theta}(X)$ and assume $-t\varphi$ is a basic potential for $t \in (0,1)$, where $0 < \xi < \xi$. Then $t \mapsto A_{-t\varphi}$ is differentiable and

$$\frac{d}{dt} \log A_{-t\varphi} = -\int \varphi \, d\mu_{t,\xi},$$

where $\{\mu_{t,\xi}\}$ is the Gibbs family for $-t\varphi$. Moreover, $\frac{d}{dt} \log A_{-t\varphi} \in H^{D_{\theta} C, \eta(\theta)}(Y)$, for some $D_{\theta} > 0$ and $\eta(\theta) \in (0,1]$, $t \mapsto \frac{d}{dt} \log A_{-t\varphi}$ is differentiable and

$$\frac{d^2}{dt^2} \log A_{-t\varphi} = \int \varphi^2 \, d\mu_{t,\xi} - \left(\int \varphi \, d\mu_{t,\xi}\right)^2.$$

**Proof.** The differentiability of $t \mapsto A_{-t\varphi}$ is an immediate consequence of [7, and

$$\frac{d}{dt} A_{-t\varphi}(y) = -(V_{t,y} \varphi)(x), \quad y \in Y, \, x \in \pi^{-1}(S(y))$$

(10)
Propositions 1 and 3, where \( \nu \) and \( \eta \), of course, we may assume \( \eta \). Since \( \nu \) is \( \pi^{-1}(S(y)) \), then applying [9] to \( \varphi \) we get

\[ A_{-t\varphi}(y)\pi_{t,y} = V_{t,y}\varphi, \]

which together with [10] gives (8). The Hölder-continuity of \( \frac{d}{dt} \log A_{-t\varphi} \) follows from Theorem 2.10 of [6].

In the same way, by [10] we see that \( t \mapsto \frac{d}{dt} \log A_{-t\varphi} \) is differentiable and

\[ \frac{d^2}{dt^2} A_{-t\varphi}(y) = (V_{t,y}\varphi)^2(x), \quad y \in Y, \quad x \in \pi^{-1}(S(y)), \]

and, by [6] applied to \( \varphi^2 \),

\[ A_{-t\varphi}(y)\pi_{t,y}\varphi^2 = V_{t,y}\varphi^2, \]

so that

\[ \frac{d^2}{dt^2} \log A_{-t\varphi}(y) = \frac{1}{A_{-t\varphi}(y)} \frac{d^2}{dt^2} A_{-t\varphi}(y) - \left( \frac{d}{dt} \log A_{-t\varphi}(y) \right)^2, \]

we get [9].

Recall the definition of \( \Phi_t \) from [13].

**Proposition 4.** Assume \( -t\varphi \) is a basic potential for \( t \in (t, \tilde{t}) \), where \( t < \tilde{t} \). Then \( t \mapsto \Phi_t \) is differentiable and

\[ \frac{dP(\Phi_t)}{dt} = \int \psi d\nu_{\Phi_t} - \beta(t) \int \varphi d\mu_{\Phi_t}. \]

Moreover,

\[ \frac{d^2 P(\Phi_t)}{dt^2} = -\beta'(t) \int \varphi d\mu_{\Phi_t} + \beta(t) \int \frac{d^2}{dt^2} \log A_{-t\varphi} d\nu_{\Phi_t} \]

\[ + Q_{\Phi_t}(\psi, \Phi_t) + \beta(t) Q_{\Phi_t} \left( \frac{d}{dt} \log A_{-t\varphi}, \Phi_t \right). \]

**Proof.** Let \( \psi, \varphi \in \mathcal{H}^{C,0}(Z) \), where \( Z = Y \) or \( X \). Fix \( \varepsilon > 0 \) arbitrarily small. It follows from [6] and Theorem 2.10 of [14] that \( A_{-t\varphi} \in \mathcal{H}^{D_1,\eta}(Y) \), for some constants \( D_1 = D_1(C, \theta, \gamma) > 0 \), where \( ||\varphi|| \leq \gamma \), and \( \eta = \eta(\theta) > 0 \), for every \( t \in [t, \tilde{t}] \). Of course, we may assume \( \eta \leq \theta \). It is also proved in [12] that \( \beta(t) \) is continuous for \( t \in [t + \varepsilon, \tilde{t} - \varepsilon] \). So, by [13], we have \( \Phi_t \in \mathcal{H}^{D_2,\eta}(Y) \), for some constant \( D_2 = D_2(C, \theta, \gamma, \varepsilon) \), for every \( t \in [t + \varepsilon, \tilde{t} - \varepsilon] \).

Let us see that \( \beta(t) \) is \( C^1 \) for \( t \in (t, \tilde{t}) \). Let

\[ F(t, \beta) = \int \log A_{-t\varphi} d\nu_{(t, \beta)}, \]

where \( \nu_{(t, \beta)} \) is the Gibbs state for the potential \( \phi_{(t, \beta)} = (t - D)\psi + \beta \log A_{-t\varphi} \). By Propositions 1 and 3

\[ \frac{\partial F}{\partial \beta}(t, \beta) = Q_{\phi_{(t, \beta)}}(\log A_{-t\varphi}, \log A_{-t\varphi}) \]

and

\[ \frac{\partial F}{\partial t}(t, \beta) = \int \frac{d}{dt} \log A_{-t\varphi} d\nu_{(t, \beta)} + Q_{\phi_{(t, \beta)}} \left( \log A_{-t\varphi}, \psi + \beta \frac{d}{dt} \log A_{-t\varphi} \right), \]
and so, by 18, \( F \) is \( C^1 \). By 12, \( \frac{dF}{d\beta}(t, \beta) > 0 \) and \( \beta(t) \) is well defined as the unique solution of \( F(t, \beta(t)) = 0 \). Then, it follows by the implicit function theorem that \( \beta(t) \) is \( C^1 \) and

\[
\beta'(t) = -\frac{\partial F}{\partial\beta}(t, \beta(t)) / \frac{\partial F}{\partial\beta}(t, \beta(t)).
\]  

Then \( t \mapsto \Phi_t \) is differentiable,

\[
\Phi_t = \psi + \beta'(t) \log A_{-t\varphi} + \beta(t) \frac{d}{dt} \log A_{-t\varphi} \in \mathcal{H}^{D_{2}\circ}(Y)
\]

for every \( t \in [\ell + \varepsilon, \ell - \varepsilon] \) (after, eventually, increasing \( D_2 \) and decreasing \( \eta \)), and applying Proposition 1 we get

\[
\frac{dP(\Phi_t)}{dt} = \int \psi d\nu_{\Phi_t} + \beta(t) \int \frac{d}{dt} \log A_{-t\varphi} d\nu_{\Phi_t}.
\]

This together with (8) and Proposition 2 gives (11). In the same way, (12) follows by applying Proposition 1 to (17).

Let

\[
\rho(t) = \frac{\int \psi d\nu_{\Phi_t}}{\int \varphi d\mu_{\Phi_t}}.
\]

Proposition 5. Assume (H) and \(-t\varphi\) is a basic potential for \( t \in (\ell, \ell) \). If \( D(\mu) = D \) then \( \mu = \mu_{\Phi_t} \) and \( \beta(t) = \rho(t) \).

Proof. From Theorem 3.1 we have that \( \mu = \mu_{\Phi_t} \) and \( \frac{dP(\Phi_t)}{dt} = 0 \). Then it follows from Proposition 1 that \( \beta(t) = \rho(t) \).

3.3. Criterium for uniqueness of measure of maximal dimension. Now we give sufficient conditions for having \( \frac{dP(\Phi_t)}{dt} < 0 \) which, by Theorem 3.1, implies the existence of a unique measure of maximal dimension (the existence follows from Theorem A in [12], as already noticed in Remark 3 in [12].

We will need a uniform version of Hypothesis (H). Given \( \varepsilon > 0 \) let

if the supremum in (2) is attained at an ergodic measure \( \nu \) then \( t(\nu) \in (\ell + \varepsilon, \ell - \varepsilon) \).

(H\(_{\varepsilon}\))

Theorem 3.2. Let \( \psi \in \mathcal{H}^{C,\beta}(Y) \) and \( \varphi \in \mathcal{H}^{C,\beta}(X) \) be positive. Assume \(-t\varphi\) is a basic potential for \( t \in (\ell, \ell) \). Assume (H\(_{\varepsilon}\)) for some \( \varepsilon > 0 \). Take any \( \gamma > 0 \) such that

(i) \( \psi > \gamma^{-1} \) and \( \gamma^{-1} < \varphi < \gamma; \)
(ii) \( |\beta(t)| < \gamma, \ t \in (\ell + \varepsilon, \ell - \varepsilon); \)
(iii) \( |\beta'(t)| < \gamma, \ t \in (\ell + \varepsilon, \ell - \varepsilon). \)

Then there exists a constant \( \delta = \delta(C, \theta, \varepsilon, \gamma) > 0 \) such that, if \( \|\psi\|_{\theta} < \delta \) and \( \|\varphi\|_{\theta} < \delta \), then there is a unique measure of maximal dimension, say \( \mu_{\psi, \varphi} \), which is ergodic (a Gibbs state for a relativized variational principle).

Moreover, hypotheses (H\(_{\varepsilon}\)) and (i)-(iii) are robust in the following sense: if \( \tilde{\psi} \in \mathcal{H}^{C,\beta}(Y), \tilde{\varphi} \in \mathcal{H}^{C,\beta}(X) \) are positive, \(-t\tilde{\varphi}\) is a basic potential and \( \tilde{\psi}, \tilde{\varphi} \) are \( \|\cdot\|_{\theta}\)-close to, respectively, \( \psi, \varphi \) satisfying these hypotheses, then \( \tilde{\psi}, \tilde{\varphi} \) also satisfy these hypotheses. Then we have that \( (\psi, \varphi) \mapsto \mu_{\psi, \varphi} \) is continuous.
Proof. Unicity of $\mu_{\psi, \varphi}$.

It follows from Theorem 3.1 that measures of maximal dimension are of the form $\mu_{\psi}$, where $P(\Phi_t) = \frac{dP(\Phi_t)}{dt} = 0$ for some $t \in (\ell + \epsilon, \tilde{\ell} - \epsilon)$. Therefore, we only need to prove that $\frac{d^2P(\Phi_t)}{dt^2} < 0$ for $t \in (\ell + \epsilon, \tilde{\ell} - \epsilon)$, and this will be done estimating the 4 terms in (12).

Term 1. First, we observe that it follows from the definitions of $D$ and $t(\nu)$ that both $D$ and $\tilde{\ell}$ are $\leq 2\gamma \log M$, where $M$ is the maximum of the cardinalities of the alphabets of the subshifts $(X, T)$ and $(Y, S)$.

By (13), (14), (15), (8) and Proposition 2 (b), we have

$$\beta'(t) = \int \varphi d\mu_{\psi} - Q_{\Phi, (\log A_{-t\varphi}, \psi + \beta(t) \frac{d}{dt} \log A_{-t\varphi})}. \tag{18}$$

Of course, $\varphi d\mu_{\psi} > 1$. It follows from (12) that $\log A_{-t\varphi}$ is not cohomologous to a constant and so $Q_{\Phi, (\log A_{-t\varphi}, \log A_{-t\varphi})} > 0$. It follows from (6) and Theorem 2.10 of 3 that $\log A_{-t\varphi} \in H^{D_1, \eta}(Y)$, for some constants $D_1 = D_1(C, \theta, \varepsilon, \gamma) > 0$ and $\eta = \eta(\theta) > 0$ (we put the dependence on $\varepsilon$ because $\gamma$ depends on $\varepsilon$). Of course, we may assume $\eta \leq \theta$. In the same way, by (3), we see that $\Phi_t \in H^{D_1, \eta}(Y)$ (after, eventually, increasing $D_1$; we will do this a finite number of times). So we may apply Proposition 1 to obtain

$$Q_{\Phi, (\log A_{-t\varphi}, \log A_{-t\varphi})} \leq D_1.$$

In the same way, see Proposition 3 we have

$$\frac{d}{dt} \log A_{-t\varphi} \in H^{D_1, ||\varphi||_{\psi, \varphi}}(Y), \tag{19}$$

and, applying Proposition 4 again, we get

$$\left| Q_{\Phi, (\log A_{-t\varphi}, \psi + \beta(t) \frac{d}{dt} \log A_{-t\varphi})} \right| \leq D_1(||\psi||_{\theta} + ||\varphi||_{\theta}).$$

Putting all these together in (18), we get

$$-\beta'(t) \int \varphi d\mu_{\psi} \leq -\gamma^{-2} D_1^{-1} + ||\psi||_{\theta} + ||\varphi||_{\theta}, \tag{20}$$

if $||\psi||_{\theta} + ||\varphi||_{\theta} < \gamma^{-2} D_1^{-1}$.

Term 2. Remember from (6),

$$\frac{d^2}{dt^2} \log A_{-t\varphi} = \int \varphi^2 d\mu_{\psi} - \left( \int \varphi d\mu_{\psi} \right)^2 \geq 0,$$

by Cauchy-Schwarz inequality. Clearly,

$$\int \varphi^2 d\mu_{\psi} - \left( \int \varphi d\mu_{\psi} \right)^2 \leq (\sup \varphi)^2 - (\inf \varphi)^2 \leq 2\gamma \left( \sup \varphi - \inf \varphi \right),$$

and $\sup \varphi - \inf \varphi \leq \max \{1, \text{diam}(X)\} ||\varphi||_{\theta}$. So,

$$\left| \beta(t) \int \frac{d^2}{dt^2} \log A_{-t\varphi} d\nu_{\Phi_t} \right| \leq C_0 \gamma^2 ||\varphi||_{\theta},$$

for some constant $C_0$.

Term 3. It follows from (16) and reasoning as in Term 1 that $\Phi_t \in H^{D_1, \eta}(Y)$. So, applying Proposition 4 we get

$$\left| Q_{\Phi_t}(\psi, \Phi_t) \right| \leq D_1 ||\psi||_{\theta}.$$
Term 4. It follows from (19) and Proposition 1 that
\[ \beta(t) Q_{\theta} \left( \frac{d}{dt} \log A_{-t \varphi}, \Phi_t \right) \leq \gamma D_1 \| \varphi \|_{\theta}. \]

Finally, putting all 4 terms together gives
\[ \frac{d^2 P(\Phi_t)}{dt^2} \leq -\gamma^{-2} D_1^{-1} + \| \psi \|_{\theta} + \| \varphi \|_{\theta} + C_0 \gamma^2 \| \varphi \|_{\theta} + D_1 \| \psi \|_{\theta} + \gamma D_1 \| \varphi \|_{\theta} < 0, \]
if we do \( \| \varphi \|_{\theta} < \delta \), \( \| \psi \|_{\theta} < \delta \) and \( \delta = \delta(C, \theta, \varepsilon, \gamma) > 0 \) is chosen sufficiently small.

Robustness of hypotheses

Hyp. (i). It is clear that hypothesis (i) is robust.

Hyp. (ii). We see that \( \varphi \mapsto t_\varphi(\nu) \) is continuous, uniformly in \( \nu \). First, it is clear from (7) that
\[ | \log A_{-t \tilde{\varphi}} - \log A_{-t \psi} | \leq K | \tilde{\varphi} - \varphi |, \]
for some constant \( K = K(\gamma) > 0 \) (and \( t \) varying in a fixed bounded interval). Then, by definition of \( t_\varphi(\nu) \), by (8) and the above, we get
\[ 0 = \int \log A_{-t_\varphi(\nu) \varphi} \, d\nu - \int \log A_{-t_\varphi(\nu) \varphi} \, d\nu \geq \left| \int \log A_{-t_\varphi(\nu) \varphi} - \log A_{-t_\varphi(\nu) \varphi} \, d\nu \right| \]
\[ \geq \gamma^{-1} | t_\varphi(\nu) - t_\varphi(\nu) | - K(\gamma) \| \tilde{\varphi} - \varphi |, \]
which proves the claimed. Since the functions \( \varphi \mapsto t_\varphi(\nu) \) and \( \psi \mapsto \int \psi \, d\nu \) appearing in (12) are continuous, uniformly in \( \nu \), it follows that hypothesis (11) is robust.

Hyp. (ii). The proof that \( (\varphi, \psi) \mapsto \beta_{\varphi, \psi}(t) \) is continuous, uniformly for \( t \) in a compact interval, is essentially contained in (12). In fact, let \( t_0 \in (\xi_\varphi + \varepsilon, t_\varphi - \varepsilon) \) and \( \beta_0 = \beta_{\varphi, \psi}(t_0) \). Then \( F_{\varphi, \psi}(t_0, \beta_0) = 0 \) and, given \( \eta > 0 \) sufficiently small, we have by (13) and continuity (see Proposition 1 of (12)) that there exists \( \delta > 0 \) such that
\[ F_{\varphi, \psi}(t, \beta_0 - \eta) < 0 \quad \text{and} \quad F_{\varphi, \psi}(t, \beta_0 + \eta) > 0 \]
for every \( t \in (t_0 - \delta, t_0 + \delta) \), \( \| \tilde{\varphi} - \varphi \| \leq \delta \) and \( \| \tilde{\psi} - \psi \| \leq \delta \). So, by the intermediate value theorem, there is a unique \( \tilde{\beta}_{\varphi, \psi}(t) \in (\beta_0 - \eta, \beta_0 + \eta) \) such that \( F_{\varphi, \psi}(t, \tilde{\beta}_{\varphi, \psi}(t)) = 0 \). By uniqueness, we have \( \beta_{\varphi, \psi}(t) = \tilde{\beta}_{\varphi, \psi}(t) \) which implies the continuity of \( (\varphi, \psi) \mapsto \beta_{\varphi, \psi}(t) \), uniformly for \( t \) in a compact interval. Then it follows that hypothesis (ii) is robust.

Hyp. (iii). Now we see that \( (\varphi, \psi) \mapsto \beta_{\varphi, \psi}(t) \) is continuous, uniformly for \( t \) in a compact interval. This will follow by (15) if we prove that the functions \( (\varphi, \psi, t, \beta) \mapsto \frac{\partial F_{\varphi, \psi}}{\partial t}(t, \beta) \) and \( \frac{\partial F_{\varphi, \psi}}{\partial \beta}(t, \beta) \) are continuous. From what has been said until now, it is clear that
\[ \phi(\varphi, \psi, t, \beta) = (t - D_{\varphi, \psi}) \psi + \beta \log A_{-t \varphi} \]
is continuous. So the conclusion follows by (13), (14) and Proposition 1. Consequently, hypothesis (iii) is robust.

Therefore, if \( \tilde{\psi}, \tilde{\varphi} \) are as described in statement of Theorem 3.2, there is a unique measure of full dimension \( \mu_{\tilde{\varphi}, \tilde{\psi}} \), and we can infer about its continuity.

Continuity of \( \mu_{\varphi, \psi} \)

Since
\[ (t, \varphi, \psi) \mapsto \Phi_{t, \varphi, \psi} = (t - D_{\varphi, \psi}) \psi + \beta_{\varphi, \psi}(t) \log A_{-t \varphi} \]
is continuous, we get that \((t, \varphi, \psi) \mapsto \mu_{t, \varphi, \psi}\) is also continuous (see Proposition 1 of \[12\]). By Proposition 2 we have that
\[
\mu_{t, \varphi, \psi} = \int \mu_{t, \varphi, y} \, d\nu_{t, \varphi, \psi}(y),
\]
where \(\{\mu_{t, \varphi, y}\}\) is the Gibbs family for \(-t\varphi\). By Theorem 3.1 of \[6\], the Gibbs family \(\{\mu_{t, \varphi, y}\}\) is equal to the family of conditional measures \(\{\mu_y\}\), on the fibers \(\pi^{-1}(y)\), for the measure \(\mu\) which is the classical Gibbs state with respect to the Hölder-continuous potential \(-t\varphi - P(\log A - t\varphi)\). Then it follows that \((t, \varphi, \psi) \mapsto \mu_{t, \varphi, \psi}\) is continuous, uniformly in \(y\), which implies the continuity of \((t, \varphi, \psi) \mapsto \mu_{t, \varphi, \psi}\).

Finally, \(\mu_{\varphi, \psi}\) is the measure \(\mu_{\varphi, \psi}\) where \(t = t(\varphi, \psi)\) is the unique solution of \(\frac{dt}{dt} P(\Phi_{t, \varphi, \psi}) = 0\) (see Theorem 3.1). Since \(\frac{d^2}{dt^2} P(\Phi_{t, \varphi, \psi}) < 0\) and \((\varphi, \psi) \mapsto \frac{dt}{dt} P(\Phi_{t, \varphi, \psi})\) is continuous, uniformly for \(t\) in a compact interval, we get that \(t(\varphi, \psi)\) is continuous, and so is \(\mu_{\varphi, \psi}\).

**Remark 3.** It follows from the proof of Theorem 3.2 that the uniqueness of measure of maximal dimension and the robustness of hypotheses (i.e., everything except, possibly, the continuity of the measure) would also hold without the hypothesis of \(-t\varphi\) being a basic potential, if we could prove that, for some constant \(C > 0\),
\[
\int \frac{dt}{dt} \log A_{-t\varphi} \, d\nu_{t, \varphi, \psi} \geq C^{-1} \min \varphi,
\]
\[
\int \frac{d^2}{dt^2} \log A_{-t\varphi} \, d\nu_{t, \varphi, \psi} \leq C ||\varphi||_{\theta}.
\]
In this case, Theorem 1.2 (except, possibly, the continuity of the measure) would hold in the class of Non-linear Lalley-Gatzouras carpets.

### 3.4. A version of Bowen’s equation for a class of non-conformal repellers.

We have interpreted the result of Theorem 3.1 as a version of Bowen’s equation for non-conformal repellers. To support this statement we should also give \(D\) (defined in the beginning of subsection 3.1) as the zero of some pressure function, at least in some context.

In order to do so, we consider a new parameter \(\Delta\) in place of \(D\) in \[3\], more precisely, for \(t \in (t, \bar{t})\) and \(\Delta \geq 0\), let \[
\Phi_{t, \Delta} = (t - \Delta)\psi + \beta(t, \Delta) \log A_{-t\varphi}
\]
where \(\beta(t, \Delta)\) is the unique real satisfying
\[
\int \log A_{-t\varphi} \, d\nu_{\Phi_{t, \Delta}} = 0 \tag{21}
\]
(see \[12\] for details).

We assume the following technical condition (which will play the role of \(H\)):
\[
t < \bar{t} \text{ and if } \nu \in \mathcal{M}(S) \text{ is ergodic with } t(\nu) = t \text{ or } \bar{t} \text{ then supp } \nu \neq Y. \tag{A}
\]

**Theorem 3.3.** Let \(\psi \in \mathcal{H}^{C, \theta}(Y)\) and \(\varphi \in \mathcal{H}^{C, \theta}(X)\) be positive. Assume \(A\) and \(-t\varphi\) is a basic potential for \(t \in (t, \bar{t})\). There exist \(\varepsilon > 0\) and \(\delta > 0\) such that if \(||\psi||_{\theta} < \delta\) and \(||\varphi||_{\theta} < \delta\), then:

(i) for each \(\Delta \in [t, D + \bar{t}]\), the equation \(\frac{d}{dt} P(\Phi_{t, \Delta}) = 0\) has a unique solution \(t = t(\Delta)\) in \((t, \bar{t} - \varepsilon, \bar{t} + \varepsilon)\);

(ii) the equation \(P(\Phi_{t, \Delta(\Delta)}) = 0\) has a unique solution \(\Delta = \Delta^*\) in \([t, D + \bar{t}]\);

(iii) if \(H\) is satisfied then \(D = \Delta^*\).
Proof. Proof of (i). Given $\Delta \in [t, D + \bar{t}]$, by the proof of Theorem 3.2 we will have that $\frac{d^2}{dt^2} \mathcal{P}(\Phi_{t, \Delta}^\nu) < 0$ for $t \in (t + \epsilon, \bar{t} - \epsilon)$, for some $\epsilon > 0$ to be specified in a moment. The difficulty here is exactly how to get this number $\epsilon > 0$ (independently of $\Delta \in [t, D + \bar{t}]$) because now we are not assuming $(\text{H}_2)$. Assuming this is possible, to prove (i) it is enough to see that $\frac{d^2}{dt^2} \mathcal{P}(\Phi_{t, \Delta}^\nu) < 0$ and $\frac{d^2}{dt^2} \mathcal{P}(\Phi_{t, \Delta}^\nu) > 0$ for some $t_1, t_2 \in (t + \epsilon, \bar{t} - \epsilon)$. By (11) this happens if $\beta(t_1, \Delta) > \bar{\rho}$ and $\beta(t_2, \Delta) < \underline{\rho}$ where

$$\underline{\rho} = \min \frac{\psi}{\max \varphi} \quad \text{and} \quad \bar{\rho} = \max \frac{\psi}{\min \varphi}.$$  

Let $\phi_{t, \Delta, \beta} = (t - \Delta) \psi + \beta \log A_{-t|\varphi} (\beta \in \mathbb{R})$ and $\nu_{t, \Delta, \beta}$ be the corresponding Gibbs state. If $(t_k), t_k \in [t, \bar{t}]$, and $(\beta_k)$ are sequences of real numbers such that $t(\nu_{t_k, \Delta, \beta_k}) \to \bar{t}$, when $k \to \infty$, then we see that $|\beta_k| \to \infty$. Otherwise, by taking subsequences if necessary, we can assume $t_k \to t_*$ and $\beta_k \to \beta_* \in \mathbb{R}$ which would imply $\nu_{t_k, \Delta, \beta_k} \to \nu_{t_*, \Delta, \beta_*}$, and therefore $t(\nu_{t_k, \Delta, \beta_k}) \to t(\nu_{t_*, \Delta, \beta_*})$. The continuity of $\nu \mapsto t(\nu)$ follows from a similar argument used in the proof of Theorem 3.2 when proving robustness of hypothesis $(\text{H}_2)$. So we would have $t(\nu_{t_*, \Delta, \beta_*}) = \bar{t}$ which contradicts (11).

Then if $t \to \bar{t}$, by (22) $t = t(\nu_{t, \Delta}) \to \bar{t}$ and so $|\beta(t, \Delta)| \to \infty$. Since $t \mapsto \beta(t, \Delta)$ is continuous, we must have $\beta(t, \Delta) \to \infty$ or $\beta(t, \Delta) \to -\infty$. We will see that $\beta(t, \Delta) \to \infty$ when $t \to \bar{t}$. Assume, by contradiction, that $\beta(t, \Delta) \to -\infty$ when $t \to \bar{t}$. By an argument used in (12), page 295, we see that

$$\lim_{\beta \to \infty} \int \log A_{-t|\varphi} d\nu_{t, \Delta, \beta} = \inf_{\nu \in \mathcal{M}(S)} \int \log A_{-t|\varphi} d\nu, \tag{22}$$

for every $t \in [t, \bar{t}]$ such that the expression on the right hand side of (22) is non-zero. It is easy to see that this is true for $t \in (t, \bar{t})$ (see page 298 of (12)). But this is also true for $t = \bar{t}$ (in fact, it is negative), otherwise we would have $t(\nu) \geq \bar{t}$ for every $\nu \in \mathcal{M}(S)$, which contradicts $t < \bar{t}$. Then (22) is true for $t$ in the compact interval $[t + \epsilon, \bar{t}]$ (any $\epsilon > 0$). Since, by Proposition 1

$$\frac{\partial}{\partial \beta} \int \log A_{-t|\varphi} d\nu_{t, \Delta, \beta} = Q_{\phi_{t, \Delta, \beta}}(\log A_{-t|\varphi}, \log A_{-t|\varphi}) \geq 0,$$

for every $t \in [t + \epsilon, \bar{t}]$, and the right hand side of (22) is clearly a continuous function of $t$, it follows by Dini’s theorem that the convergence in (22) is uniform with respect to $t \in [t + \epsilon, \bar{t}]$. This implies (assuming $\beta(t, \Delta) \to -\infty$ when $t \to \bar{t}$)

$$0 = \lim_{t \to \bar{t}} \int \log A_{-t|\varphi} d\nu_{t, \Delta} = \inf_{\nu \in \mathcal{M}(S)} \int \log A_{-t|\varphi} d\nu < 0,$$

a contradiction. Then we must have $\beta(t, \Delta) \to \infty$ when $t \to \bar{t}$. Similarly, we see that $\beta(t, \Delta) \to -\infty$ when $t \to \bar{t}$.

Then, for every $\Delta \in [t, D + \bar{t}]$, there exists $\epsilon > 0$ such that $\beta(t_\Delta, \Delta) > \bar{\rho}$ and $\beta(t_\Delta', \Delta) < \underline{\rho}$ for some $t_\Delta, t_\Delta' \in (t + \epsilon, \bar{t} - \epsilon)$. The continuity of $\Delta \mapsto \beta(t, \Delta)$ (actually, $C^\gamma$) follows from the implicit function theorem, using similar arguments as we did in (13), (14), (15). Then, using simple compactness arguments, we see that $\epsilon > 0$, with such a property, can be made independent of $\Delta \in [t, D + \bar{t}]$. As we have seen before (in the proof of Theorem 3.2 when proving robustness of hypothesis (ii)), the dependence of $\beta(t, \Delta)$ on $\psi$ and $\varphi$ is also continuous, so by Arzelà-Ascoli theorem (and its generalization to compact metric spaces by Fréchet) we can make $\epsilon > 0$ only depending on $(\theta, C, \gamma)$ where $||\psi||_\theta, ||\varphi||_\theta \leq C$ and $||\psi||, ||\varphi|| \leq \gamma$. (Also
note that \( t \geq 0 \) and \( D \leq 2\gamma \log K \), where \( K \) is a bound for the cardinality of the alphabets of \((X,T)\) and \((Y,S)\), and \( \psi, \varphi \geq \gamma^{-1} \).

This is the \( \varepsilon > 0 \) we use in the proof of Theorem 3.2 to get \( \frac{\partial^2}{\partial \Delta^2} P(\Phi_{t,\Delta}) < 0 \) for \( t \in (t + \varepsilon, t - \varepsilon) \) and \( ||\psi||_\theta < \delta, ||\varphi||_\theta < \delta \), where \( \delta = \delta(\theta, C, \gamma, \varepsilon) > 0 \) is given by Theorem 5. As mentioned in the beginning of this proof, this concludes the proof of (i).

**Proof of (iii).** Since \( \frac{\partial^2}{\partial \Delta^2} P(\Phi_{t,\Delta}) < 0 \) for \( t \in (t + \varepsilon, t - \varepsilon) \), it follows by the implicit function theorem that \((t,D + \bar{t}) \ni \Delta \mapsto t(\Delta) \) is \( C^1 \). Also, by Proposition \[21\],

\[
\frac{\partial}{\partial \Delta} P(\Phi_{t,\Delta}) = -\int \psi \, d\nu_{\Phi_{t,\Delta}}.
\]

Then, by the chain rule,

\[
\frac{d}{d\Delta} P(\Phi_{t(\Delta),\Delta}) = \frac{\partial}{\partial t} P(\Phi_{t,\Delta}) \big|_{t=t(\Delta)} \frac{dt'}{dt} + \frac{\partial}{\partial \Delta} P(\Phi_{t,\Delta}) = -\int \psi \, d\nu_{\Phi_{t,\Delta}} < 0.
\]

So unicity of \( \Delta^* \) follows. To prove existence of \( \Delta^* \), we observe that, from the definition of equilibrium state and \[21\],

\[
P(\Phi_{t(\Delta),\Delta}) = h_{\nu(\Phi_{t(\Delta),\Delta})} (S) + (t(\Delta) - \Delta) \int \psi \, d\nu_{\Phi_{t(\Delta),\Delta}}.
\]

Note that expression above is \( > 0 \) for \( \Delta = t \) (or close to \( t \)), and \( < 0 \) for \( \Delta = D + \bar{t} \) (or close to \( D + \bar{t} \)). Therefore, the existence of \( \Delta^* \in [t, D + \bar{t}] \) follows by the intermediate value theorem.

**Proof of (ii).** By (i), \( \frac{\partial}{\partial \Delta} P(\Phi_{t,D}) = 0 \) has a unique solution \( t = t(D) \) in \((t + \varepsilon, t - \varepsilon)\).

By Theorem \[3.1\] \( t(D) \) must be the unique point of maximum of \((t + \varepsilon, t - \varepsilon) \ni \Delta \mapsto t(\Delta) \) and therefore \( P(\Phi_{t(D),D}) = 0 \). By (ii), we have \( D = \Delta^* \). \( \square \)

4. **Unique ergodic measure of full dimension.** In this section we prove Theorem \[12\]

Consider a non-trivial Sierpinski carpet. More precisely, consider the alphabet \( \mathcal{I} = \{(i,j) : i \in \{1,\ldots,m\} \text{ and } j \in \{1,\ldots,m_i\} \} \) where \( m \geq 1 \) and \( m_i \geq 2 \) are natural numbers such that \( m_i \) are not all equal to each other. For \( (i,j) \in \mathcal{I} \), let

\[
f_{ij}^a(x,y) = (ax + u_{ij}, by + v_i),
\]

where \( 0 < a < b < 1 \) and the positive numbers \( v_i \) and \( u_{ij} \) satisfy \( b + v_i < u_{i+1}, a + u_{ij} < u_{ij+1} \) for all \( (i,j) \in \mathcal{I} \), where \( v_{m+1} = u_{m+1} = 1 \). Let \( \Lambda^o \) be the corresponding attractor, i.e.

\[
\Lambda^o = \bigcup_{(i,j) \in \mathcal{I}} f_{ij}^o(\Lambda^o).
\]

We will consider Non-linear Lalley-Gatzouras carpets \((f_{ij}, \Lambda^o)\),

\[
f_{ij}(x,y) = (a_{ij}(x,y), b_{ij}(y)), \quad (i,j) \in \mathcal{I}
\]

which are close to \((f_{ij}^o, \Lambda^o)\). Note that, since the alphabet \( \mathcal{I} \) is fixed, all of these carpets are topologically modeled by the same Bernoulli shift \( T : X \to X \), where \( X = \mathcal{I}^\mathbb{N} \) and \( T((i_1,j_1),(i_2,j_2),...) = ((i_2,j_2),...) \), via the conjugacy \( h : X \to \Lambda \) given by

\[
h((i_1,j_1),(i_2,j_2),...) = \int_{\mathcal{I}} f_{i_1j_1} \circ f_{i_2j_2} \circ \cdots \circ f_{i_nj_n} ([0,1]^2).\]
Let $\pi: X \to Y$, where $Y = \{1, \ldots, m\}^\mathbb{N}$ and $\pi((i_1, j_1)(i_2, j_2)\ldots) = (i_1i_2\ldots)$. Then $\pi \circ T = S \circ \pi$, where $S: \{1, \ldots, m\}^\mathbb{N} \to \{1, \ldots, m\}^\mathbb{N}$ is the Bernoulli shift given by $S(i_1i_2\ldots) = (i_2\ldots)$.

By (11) and (12), we have that
\[
\dim_H \Lambda = \sup_{\mu \in \mathcal{M}(T)} \left\{ h_{\mu\circ\pi^{-1}}(S) + \frac{h_{\mu}(T) - h_{\mu\circ\pi^{-1}}(S)}{\int \psi \circ \pi d\mu} \right\},
\]
and that the measures of maximal dimension, as defined in previous section, being ergodic coincide with the ergodic measures of full dimension (since the dimension of an ergodic measure is the expression between brackets in equation above). Here $\varphi: X \to \mathbb{R}$ and $\psi: Y \to \mathbb{R}$ are the positive and Hölder-continuous functions given by
\[
\varphi((i_1, j_1)(i_2, j_2)\ldots) = -\log \partial_x a_{i_1j_1}(x, y), \quad \psi(i_1i_2\ldots) = -\log b'_{i_1}(y),
\]
where $(x, y) = h((i_1, j_1)(i_2, j_2)\ldots)$. So we must see that we satisfy Theorem 3.2’s hypotheses.

We note that $-t\varphi$ is a basic potential (remember the definition from (7)) if we restrict to the subclass of carpets $\mathcal{L}$, for then $\partial_x a_{i_1j_1}(x, y)$ does not depend on $x$.

For the general Sierpinski carpet, we have that
\[
t(\nu) = \frac{\sum_{i=1}^{m} p_i \log m_i}{\log a^{-1}},
\]
where $p_i$ is the $\nu$-measure of the cylinder of order 1 $\{(i_1i_2\ldots) \in Y: i_1 = i\}$. Then, (2) reads
\[
D = \frac{1}{\log b^{-1}} \sup_{\nu \in \mathcal{M}(S)} \left\{ h_{\nu}(S) + \sum_{i=1}^{m} p_i \log m_i^\rho \right\},
\]
where $\rho = \frac{\log b}{\log a}$. Since in the variational principle above the potential is constant on cylinders of order 1, it is well known that this supremum is attained at a Bernoulli measure $\nu$ with all $p_i = m_i^\rho / \sum m_i^\rho > 0$ (see Lemma 1.1 and Proposition 2.5 of [3]). Since the numbers $m_i$ are not all equal to each other, it follows that $\tilde{t} < t(\nu) < \overline{t}$. Hence we satisfy hypothesis (H_4), for some $\varepsilon > 0$. Also note that $||\varphi^\alpha||_\alpha = ||\psi^\alpha||_\alpha = 0$.

Then, Theorem 1.2 follows from applying Theorem 3.2 to carpets in $\mathcal{L}$ which are $C^{1+\alpha}$ close to a non-trivial general Sierpinski carpet.

Remark 4. The application of our results goes beyond dynamical systems modeled by a full shift. Whenever the system is modeled by $(X, T, Y, S, \pi)$ as in Section 3 satisfies the variational principle for dimension, and the dimension of an ergodic measure splits into the dimension of the projected measure on the space of fibres plus the dimension of the measure on the fibres (as in formula for $D(\mu)$), we can apply our results on the existence and uniqueness of an ergodic measure of full dimension (Theorems 3.1 and 3.2).

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