On the gravitomagnetic clock effect in quantum mechanics

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March 31, 2015

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Abstract

In this article, we discuss how to carryover gravitomagnetic clock effect from classical general relativity to quantum theory and how to calculate this effect in quantum mechanics. Our calculation is valid for semi-classical regime and can be considered as the first step towards a complete gravitomagnetic quantum theory. We also show the analogy between energy levels that corresponds to the clock effect. In fact, it is argued that in quantum mechanics clock effect arises as energy level splitting in gravitomagnetic field.

1 Introduction

General relativity predicts that two freely counter rotating test particles in the exterior field of a central rotating mass take different periods of time to complete the same full orbit; this time difference, first discovered by Cohen and Mashhoon [1] in 1993, is termed as gravitomagnetic clock effect. Since its first formulation, numerous works have been devoted to elucidate its foundation [2-5] and to derive methods to detect it via space borne experiments [6-12]. Gravitomagnetic clock effect (GCE) has been extended by Faruque to include spin of the test particle [13] and the spin-dependence is elucidated further by Faruque [14] and by Bini et al[15]. Mashhoon et al [16] shed more light on the spin dependence of GCE in their work on extended spinning masses. Now, the GCE is one of many gravitomagnetic effects such as, Lense-Thirring precession of gyroscopes, gravitomagnetic time delay, spin-rotation coupling etc., which are awaiting detection in astrophysical systems and also are carrying over to quantum mechanics. Discovering gravitomagnetism in quantum mechanics is a very recent physicist’s drive which was founded by Adler and Chen[17]. Adler and Chen [17] considered Klein-Gordon equation expressed in generally covariant form and coupled to electromagnetic field and found the analog of Lense-Thirring precession in quantum system. Their work endows a spin-zero test mass in orbit in a gravitomagnetic field with a gravitomagnetic moment $\vec{\mu}_{grav}=-\frac{1}{2}\vec{L}$, where $\vec{L}$ is the orbital angular momentum. Finding inspiration from Adler and Chen, we have tried to carryover GCE from classical general relativity to quantum mechanics without actually solving any equation but using the most usual prescription of quantum to classical correspondence as formulated by Bohr. In Section 2, we shall present our main derivation of GCE in quantum system. In Section 3, we shall discuss energy considerations. In Section 4, we will summarize our work.

2 GCE in quantum system

To carry the classical GCE over to quantum mechanics, we shall not try a fully relativistic and fully quantum mechanical formulation of the physical problem that resembles the classical system which shows GCE. Rather, we shall carry the physical problem showing GCE conceptually to a quantum description. We shall use Bohr’s correspondence principle to convert classical angular frequencies to quantum energies. To begin with, we note that the formula for orbital angular frequency of a test particle in circular orbit around a central Kerr source of spin parameter $a = \frac{S}{M}$, $S$ and $M$ being the spin angular momentum and mass of the central rotating body, respectively, and $c$ is the speed of light in vacuum, is, for corotating orbit,

$$\frac{1}{w} = a + \frac{1}{w_k}$$  

(1)

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And that for counter rotating orbit is

\[ \frac{1}{w} = a - \frac{1}{w_k} \]  

(2)

where \( w_k \) is the Keplerian angular frequency of orbital motion. We now divide Eqs.(1) and (2) by \( \hbar \) to find the semiclassical energy of a test particle in states of energies \( E_\pm \) and of z-component of orbital angular momentum \( L_z = \pm \hbar \). These energies are

\[ \frac{1}{E_\pm} = \frac{a}{\hbar} \pm \frac{1}{\hbar w_k} \]  

(3)

In Eq.(3), \( E_+ \) corresponds to a state of z-component of orbital angular momentum \( L_z = +\hbar \) and \( E_- \) corresponds to a state of z-component of orbital angular momentum \( L_z = -\hbar \). This extension of Eqs.(1) and (2), which are classical, to Eq.(3), which is semiclassical, is obvious from the standpoint of Bohr’s correspondence principle. The carrying over of classical motion in corotating (counterrotating) state to quantum physics and finding out the corresponding (gedanken) time period for one complete revolution in orbit is accomplished by demanding that the quantum state undergoes the transformation of the azimuth angle \( \phi \to \phi + 2\pi \) and the time undergoes the transformation \( t \to t + T \), where \( T \) is time period for a full cycle, which in later calculations will be written as \( T_\pm \) to refer to co- and counter rotating orbits which are quantum mechanically \( L_z = \pm \hbar \) orbits. Before carrying out these operations, we note that the only relevant wavefunction of the test particle in the problem under consideration is an eigenstate of z-component of orbital angular momentum and energy. The relevant states can be written as

\[ \psi_1(t, \phi) = A \exp(i\phi) \exp\left(-\frac{i}{\hbar}E_+ t\right) \]  

(4)

and

\[ \psi_2(t, \phi) = A \exp(-i\phi) \exp\left(-\frac{i}{\hbar}E_- t\right) \]  

(5)

In Eq.(4), the state is of energy \( E_+ \) and \( L_z = +\hbar \) and in Eq.(5), the state is of energy \( E_- \) and \( L_z = -\hbar \). The operations mentioned above produces the states \( \psi_1(t + T_+, \phi + 2\pi) \) and \( \psi_2(t + T_-, \phi + 2\pi) \). We appropriately then demand single-valuedness of the quantum wavefunctions and thereby impose

\[ \psi(t, \phi) = \psi(t + T_\pm, \phi + 2\pi) \]  

(6)

One can follow the mathematical steps when implementing the condition (6) on the functions (4) and (5) and those generated from (4) and (5) using the transformations just mentioned. By this way one automatically finds

\[ T_+ = \frac{2\pi \hbar}{E_+} \]  

(7)

and

\[ T_- = -\frac{2\pi \hbar}{E_-} \]  

(8)

Using Eqs.(7) and (8), one finds that

\[ T_+ - T_- = 4\pi a \]  

(9)

which is the established classical GCE. We have thus carried the classical GCE over to quantum mechanical GCE. Both of these are numerically exactly the same. However, one can argue that the states \( E_+ \) and \( E_- \) with \( L_z = \pm \hbar \) are different states and the GCE calculated using them in quantum settings do not mimic the classical picture and so cannot be called the quantum GCE. This is in one sense right, but what is most important in this regard to realize is that what in general relativity is termed as GCE, appears in quantum mechanics as splitting of energy levels through gravitomagnetism. The gravitomagnetic field of the central spinning body (the Kerr source) produces a magnetic field in which the gravitomagnetic moment of the test particle finds itself and the energy of the test particle is different in the two \( L_z = \pm \hbar \) states according gravitomagnetic potential energy \( -\mu_{grav} \cdot \overrightarrow{B} \). Thus a single energy state splits into two (not three, the \( L_z = 0 \) state is missing) states of different total energy. Thus, we get hyperfine splitting. In the next section, we discuss this feature.
3 Energy considerations

As shown in [17], gravitomagnetic moment of a spin-zero test mass in orbit in a gravitomagnetic field is given by

$$\vec{\mu}_{\text{grav}} = -\frac{1}{2} \vec{L}$$  \hspace{1cm} (10)

Now, let us consider a spinning ball of mass $M$ and radius $R$, spinning with angular frequency $\vec{\omega}$. On the equatorial plane the gravitomagnetic field $\vec{B}$ produced by the ball is given by [18]

$$\vec{B} = \frac{4}{5} \frac{GR^2 M \vec{\omega}^2}{c^2 R^3}$$  \hspace{1cm} (11)

This can be rewritten as

$$\vec{B} = \frac{2GI}{c^2 R^3} = \frac{2J}{Mc^2} \omega_k^2$$  \hspace{1cm} (12)

where the moment of the inertia of the ball is $I = \frac{2}{5} MR^2$ and its spin angular momentum $\vec{J} = J \vec{\omega}$. The angular frequency of the test particle in orbit is $\omega_k = \sqrt{\frac{GM}{r^3}}$.

As the system we are considering is that of a test mass either in prograde orbit or in retrograde orbit, the test particle orbital angular momentum along the spin axis is $L_z = +\hbar$ for the prograde orbit and that is $L_z = -\hbar$ for the retrograde orbit. Thus, we find the potential energy associated with the gravitomagnetic moment as

$$E_B = -\vec{\mu}_{\text{grav}} \cdot \vec{B} = \frac{\vec{L} \cdot \vec{J}}{Mc^2} \omega_k^2 = \begin{cases} \frac{\hbar \omega_k^2 J}{Mc^2}, & L_z = +\hbar \\ -\frac{\hbar \omega_k^2 J}{Mc^2}, & L_z = -\hbar \end{cases}$$  \hspace{1cm} (13)

The total energy in $L_z = +\hbar$ and $L_z = -\hbar$ states are found by adding the principle energy of the test mass which is $-\hbar \omega_k$, the Keplerian energy. So, we have

$$E_+ = -\hbar \omega_k + \frac{\hbar \omega_k^2 J}{Mc^2}$$  \hspace{1cm} (14)

$$E_- = -\hbar \omega_k - \frac{\hbar \omega_k^2 J}{Mc^2}$$  \hspace{1cm} (15)

We can rearrange Eqs. (14) and (15) as

$$E_+ = -\hbar \left( \omega_k - \frac{\omega_k^2 J}{Mc^2} \right)$$  \hspace{1cm} (16)

$$E_- = -\hbar \left( \omega_k + \frac{\omega_k^2 J}{Mc^2} \right)$$  \hspace{1cm} (17)

We can identify the orbital frequency in $E_+$ state as

$$\omega_+ = \omega_k - \frac{\omega_k^2 J}{Mc^2}$$  \hspace{1cm} (18)

and in $E_-$ state as

$$\omega_- = \omega_k + \frac{\omega_k^2 J}{Mc^2}$$  \hspace{1cm} (19)

The period of revolution in $E_+$ state is

$$T_+ = \frac{2\pi}{\omega_+} = \frac{2\pi}{\omega_k - \frac{\omega_k^2 J}{Mc^2}}$$

$$= \frac{2\pi}{\omega_k} \left( 1 + \frac{J \omega_k}{Mc^2} \right)$$

$$= \frac{2\pi}{\omega_k} + \frac{2\pi J}{Mc^2}$$  \hspace{1cm} (20)

The period of revolution in $E_-$ state is

$$T_- = \frac{2\pi}{\omega_-} = \frac{2\pi}{\omega_k + \frac{\omega_k^2 J}{Mc^2}}$$

3
\[
\frac{2\pi}{\omega_k} \left( 1 - \frac{J\omega_k}{Mc^2} \right)
\]

Subtracting Eq. (21) from Eq. (20), we obtain

\[
T_+ - T_- = \frac{4\pi J}{Mc^2}
\]

which is the clock effect (Eq. (9) with \(a = \frac{\omega}{Mc^2}\)).

Therefore, we have become able to show that the quantum mechanical energy of a gravitomagnetic dipole of moment \( \vec{\mu}_{grav} = -\frac{1}{2} \vec{L} \) in gravitomagnetic field \( \vec{B} \), produced by a rotating mass, added with the principal Keplerian quantum of energy \(-\hbar \omega_k\), gives two states \(E_+\) and \(E_-\) given in Eqs. (14) and (15). The angular frequencies in these two states are given in Eqs. in (18) and (19). The corresponding periods are \(T_+\) and \(T_-\) which are same as the classical periods giving rise to the classical gravitomagnetic clock effect.

However, the states given in Eq.(3) are not same as those given in Eqs. (14) and (15). There is a fundamental difference in the derivation followed in this section and that in the previous section. Energy considerations followed in this section are more plausible than that followed in the previous section. To adjust the two approaches we note that the energy \(E_+\) given by Eq.(3) is not a bound state, but it can be made to correspond to a bound state by putting a \((-)\)in front. On the other hand the energy \(E_-\) given by Eq. (3) in that of a bound state and it is correct as it is. That is \(E_+\) (\(E_-\))given by Eq. (3) becomes equivalent to those given by Eq. (14) and Eq. (15) only when we perform the adjustment just mentioned.

Finally, we note that the derivation with \(E_\pm\) given by Eq. (3) followed in section 2 is quantum theoretically true in the sense that a positive time should correspond to a positive energy which is true for both \(T_+\) and \(T_-\) given in Eqs. (7) and (8). Thus, keeping in mind these artifacts of a semi-classical theory, we understand that gravitomagnetic clock effect indeed has a counterpart in quantum theory appearing through energy considerations in gravitomagnetic field.

4 Summary

In this article, we have discussed the way to carry the classical gravitomagnetic clock effect in Kerr field and in GEM field over to quantum physics. We have used Bohr’s correspondence principle to convert classical orbital angular frequencies to semiclassical energies. Thus, we have found the corotating and counterrotating orbits in Kerr field to correspond to two different states having different energy and different angular momentum (orbital z-component). When we apply appropriate operations to generate states that help to calculate time periods in a quantum sense and demand single- valuedness of wavefunctions, we automatically find a quantum analog of GCE which is numerically the same as classical GCE. We have thus shown a particular mechanism for discussing GCE in a quantum system.

We also have discussed actual energy considerations involved with the system. A particle with gravitomagnetic moment in a gravitomagnetic field possesses potential energy. When this energy is added with the principal Keplerian energy we automatically find the classical GCE to appear in the quantum system. Note that if the quantum condition of \(L_z = \pm\hbar\) is not applied but \(L\) is left classical then the clock effect would not correspond to what actually is called GCE. Moreover, only micro black holes with quantized spin are candidates of sources of GEM field in which micro-particles with Compton radius below Planck length can show quantum GCE.

Quantum GCE in ordinary quantum systems is so insignificant in magnitude that it might not be measurable in near and far future. However, as long as a particular type of classical GCE depends on only intrinsic properties of orbiter and central rotator, such as spin and mass, the quantum GCE will have the same formula. The numeral insignificance of quantum GCE does not mean that it is not an element of reality. Because reality basically is of quantum nature. Whatever physical phenomena appear in classical physics should have a corresponding quantum phenomena. And quantum phenomena actually are in the root of classical phenomena. Hence, what we have discussed in this article, that is quantum GCE, cannot be denied of existence if classical GCE appears as reality.

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