HOMOTOPY INVARIANT PRESHEAVES WITH FRAMED TRANSFERS

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ABSTRACT. The category of framed correspondences $Fr_*(k)$, framed presheaves and framed sheaves were invented by Voevodsky in his unpublished notes [6]. Based on the theory, framed motives are introduced and studied in [3]. The main aim of this paper is to prove the following result, which was first announced in [3].

For any $\mathbb{A}^1$-invariant quasi-stable additive framed presheaf of abelian groups $\mathcal{F}$, the associated Nisnevich sheaf $\mathcal{F}_{\text{nis}}$ is $\mathbb{A}^1$-invariant provided that the base field $k$ is infinite. Moreover, if the base field $k$ is perfect and infinite then every $\mathbb{A}^1$-invariant quasi-stable Nisnevich framed sheaf is strictly $\mathbb{A}^1$-invariant.

This result and the paper are inspired by Voevodsky’s paper [7].

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1. INTRODUCTION

The main result can be restated in terms of $\mathbb{Z}F_*$-presheaves of abelian groups on smooth varieties. Recall that $\mathbb{Z}F_*(k)$ is defined in [3] as an additive category whose objects are those of

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Sm/k and Hom-groups are defined as follows. We set for every $n \geq 0$ and $X,Y \in Sm/k$,
\[
\mathbb{Z}F_n(X,Y) = \mathbb{Z}F_n(X,Y)/(Z_1 \sqcup Z_2 - Z_1 - Z_2),
\]
where $Z_1,Z_2$ are supports of correspondences. In other words, $\mathbb{Z}F_n(X,Y)$ is a free abelian group generated by the framed correspondences of level $n$ with connected supports. We then set
\[
\text{Hom}_{\mathbb{Z}F_0(k)}(X,Y) = \bigoplus_{n \geq 0} \mathbb{Z}F_n(X,Y).
\]

The canonical morphisms $Fr_n(X,Y) \to \text{Hom}_{\mathbb{Z}F_n(k)}(X,Y)$ define a functor $R : Fr_n(k) \to \mathbb{Z}F_n(k)$, which is the identity on objects. For any $\mathbb{A}^1$-invariant quasi-stable $\mathbb{Z}F_1$-presheaf of abelian groups $\mathcal{F}$ the functor $\mathcal{F} \circ R : Fr_n(k) \to \mathbb{A}^1$ is $\mathbb{A}^1$-invariant quasi-stable additive framed presheaf of abelian groups.

By definition, a $Fr_n$-presheaf $\mathcal{F}$ of abelian groups is stable if for any $k$-smooth variety the pull-back map $\sigma_\mathcal{F}^* : \mathcal{F}(X) \to \mathcal{F}(X)$ equals the identity map, where $\sigma_\mathcal{F} = (X \times 0, X \times \mathbb{A}^1, t; pr_X) \in Fr_1(X,X)$. In turn, $\mathcal{F}$ is quasi-stable if for any $k$-smooth variety the pull-back map $\alpha_\mathcal{F}^* : \mathcal{F}(X) \to \mathcal{F}(X)$ is an isomorphism. Also, recall that $\mathcal{F}$ is additive if $\mathcal{F}(\emptyset) = \{0\}$ and $\mathcal{F}(X_1 \sqcup X_2) = \mathcal{F}(X_1) \times \mathcal{F}(X_2)$.

For any $\mathbb{A}^1$-invariant stable additive framed $Fr_n$-presheaf of abelian groups $G$ there is a unique $\mathbb{A}^1$-invariant stable $\mathbb{Z}F_n$-presheaf of abelian groups $\mathcal{F}$ such that $G = \mathcal{F} \circ R$. This follows easily from the additivity theorem [3, Theorem 5.1].

That is the category of $\mathbb{A}^1$-invariant stable additive framed presheaves of abelian groups is equivalent to the category of $\mathbb{A}^1$-invariant stable $\mathbb{Z}F_n$-presheaves of abelian groups.

The latter means that the main result announced in the abstract is equivalent to the following

**Theorem 1.1** (Main). For any $\mathbb{A}^1$-invariant quasi-stable $\mathbb{Z}F_n$-presheaf of abelian groups $\mathcal{F}$, the associated Nisnevich sheaf $\mathcal{F}_{nis}$ is $\mathbb{A}^1$-invariant provided that the base field is infinite. Moreover, if the base field $k$ is perfect and infinite then every $\mathbb{A}^1$-invariant quasi-stable Nisnevich framed sheaf is strictly $\mathbb{A}^1$-invariant and quasi-stable.

In the rest of the paper we suppose that the base field $k$ is infinite.

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2. A few theorems

The main aim of this section is to state a few major theorems on presheaves with framed transfers. As an application, we deduce the following result (which is the first assertion of Theorem [1]).

**Theorem 2.1.** For any $\mathbb{A}^1$-invariant quasi-stable $\mathbb{Z}F_n$-presheaf of abelian groups $\mathcal{F}$, the associated Nisnevich sheaf $\mathcal{F}_{nis}$ is $\mathbb{A}^1$-invariant and quasi-stable.

We need some definitions. We will write $(V,q;g)$ for an element $a$ in $Fr_n(X,Y)$. We also write $Z_\alpha$ to denote the support of $(V,q;g)$. It is a closed subset in $X \times \mathbb{A}^n$ which is finite over $X$ and which coincides with the common vanishing locus of the functions $q_1,\ldots,q_n$ in $V$. Next, by $(V,q;g)$ we denote the image of the element $1\cdot (V,q;g)$ in $\mathbb{Z}F_n(X,Y)$.

**Definition 2.2.** Given any $k$-smooth variety $X$, there is a distinguished morphism $\sigma_X = (X \times \mathbb{A}^1,t; pr_X) \in Fr_1(X,X)$. Each morphism $f : Y \to X$ in $Sm/k$ can be regarded tautologically as a morphism in $Fr_0(Y,X)$. 

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In what follows by $\text{SmOp}/k$ we will mean a category whose objects are pairs $(X, V)$, where $X \in \text{Sm}/k$ and $V$ is an open subset of $X$, with obvious morphisms of pairs.

**Definition 2.3.** Define $\mathbb{Z}F^p_n((W), (X), (V))$ as an additive category whose objects are those of $\text{SmOp}/k$ and Hom-groups are obtained from the $\text{Hom}$-groups of the category $\mathbb{Z}F^p_n(Y, (X), (V))$.

**Notation 2.7.** By definition, the composite $(a, b) \circ (a', b')$ is the pair $((a \circ b), (a' \circ b'))$.

We define $\overline{\mathbb{Z}F}_*(k)$ as an additive category whose objects are those of $\text{SmOp}/k$ and Hom-groups are defined as follows. We set for every $n \geq 0$ and $(X, V), (Y, W) \in \text{SmOp}/k$:

$$\overline{\mathbb{Z}F}_*(k) = \text{Coker}[\mathbb{Z}F_n(\mathbb{A}^1 \times (X, V), (X, V)) \xrightarrow{i_{k-1}} \mathbb{Z}F_n((Y, W), (X, V))]$$

Next, one defines $\overline{\mathbb{Z}F}^p_n(k)$ as an additive category whose objects are those of $\text{SmOp}/k$ and Hom-groups are defined as follows. We set for every $n \geq 0$ and $(X, V), (Y, W) \in \text{SmOp}/k$:

$$\overline{\mathbb{Z}F}^p_n((W), (X), (V)) = \ker[\mathbb{Z}F_n(Y, X) \oplus \mathbb{Z}F_n(W, V) \xrightarrow{i_1} \mathbb{Z}F_n(W, X)]$$

where $i_1 : W \to Y$ is the embedding and $i_2 : V \to X$ is the embedding. In other words, the group $\mathbb{Z}F^p_n((W), (X), (V))$ consists of pairs $(a, b) \in \mathbb{Z}F_n(Y, X) \oplus \mathbb{Z}F_n(Y, W)$ such that $i_1 \circ a = a \circ i_2$.

By definition, we have $\overline{\mathbb{Z}F}_*(k) = \text{Hom}_{\text{SmOp}}((X, V), (X, V))$.

**Notation 2.4.** Given $a \in \mathbb{Z}F_n(Y, X)$, denote by $[a]$ its class in $\overline{\mathbb{Z}F}_*(Y, X)$. Similarly, if $r = (a, b) \in \mathbb{Z}F^p_n((Y, W), (X, V))$, then we will write $[r]$ to denote its class in $\overline{\mathbb{Z}F}^p_n((Y, W), (X, V))$.

If $X^0$ is open in $X$ and $Y^0$ is open in $Y$, then $g(X) \cap Y^0$ will stand for the element $(a, b)$ in the category $\mathbb{Z}F^p_n((W), (X), (V))$.

**Notation 2.7.** If $r = (a, b) \in \mathbb{Z}F^p_n((W), (X), (V))$, then we will write $[r]$ for its class in $\overline{\mathbb{Z}F}^p_n((W), (X), (V))$. For $(X, V)$ in $\text{SmOp}/k$ we write $\langle \sigma_X \rangle$ for the morphism $1 \cdot \sigma_X, 1 \cdot \sigma_Y$ in $\mathbb{Z}F_1((X, V), (X, V))$.

We will denote by $[\langle V, \varphi \rangle]$ the class of the element $[\langle V, \varphi \rangle]$ in $\overline{\mathbb{Z}F}_n((X, V), (Y, W))$.

**Construction 2.8.** Let $\mathcal{F}$ be an $\mathbb{A}^1$-invariant $\mathbb{Z}F_*$-presheaf of abelian groups. Then the assignment $(X, V) \mapsto \mathcal{F}(X, V) := \mathcal{F}(V) / \text{Im}(\mathcal{F}(X))$ and

$$(a, b) \mapsto [(a, b)]_* = b^* : \mathcal{F}(V) / \text{Im}(\mathcal{F}(X)) \to \mathcal{F}(W) / \text{Im}(\mathcal{F}(Y))$$

for any $(a, b) \in \mathbb{Z}F_n((W), (X), (V))$ define a presheaf $\mathcal{F}^{\text{pairs}}$ on the category $\overline{\mathbb{Z}F}_*(k)$.

The neatest aim is to formulate a series of theorems (each of which is of independent interest), which are crucial for the proof of Theorem [1, 1].
Theorem 2.9 (Injectivity on affine line). Let $U \subset \mathbb{A}^1_U$ be an open subset and let $i : V \hookrightarrow U$ be a non-empty open subset. Then there is a morphism $r \in \mathbb{F}_1(U,V)$ such that $[i] \circ [r] = [\sigma_U]$ in $\mathbb{F}_1(U,U)$.

Theorem 2.10 (Excision on affine line). Let $U \subset \mathbb{A}^1_U$ be an embedding. Let $i : V \hookrightarrow U$ be an open inclusion with $V$ non-empty. Let $S \subset V$ be a closed subset. Then there are morphisms $r \in \mathbb{F}_1((U,U-S),(V,V-S))$ and $l \in \mathbb{F}_1((U,U-S),(V,V-S))$ such that $[[r]] \circ [[r]] = [[[\sigma_U]]]$ and $[[l]] \circ [[l]] = [[[\sigma_V]]]$ in $\mathbb{F}_1((U,U-S),(V,V-S))$ and $\mathbb{F}_1((U,V-S),(V,V-S))$ respectively.

Theorem 2.11 (Injectivity for local schemes). Let $X = \text{Spec}(\mathcal{O}_{X,x})$, $i : D \hookrightarrow X$ be a closed subset. Then there exists an integer $N$ and a morphism $r \in \mathbb{F}_N(U,X-D)$ such that $[r] \circ [j] = [\text{can}] \circ [\sigma_U^N]$ in $\mathbb{F}_N(U,X)$ with $j : X - D \hookrightarrow X$ the open inclusion and $\text{can} : U \to X$ the canonical morphism.

Theorem 2.12 (Excision on relative affine line). Let $X = \text{Spec}(\mathcal{O}_{X,x})$, $W = \text{Spec}(\mathcal{O}_{X,x})$. Let $i : V = (\mathbb{A}^1_W)_f \subset \mathbb{A}^1_W$ be an affine open subset, where $f \in \mathcal{O}_{X,x}[r]$ is monic such that $f(0) \in O_{X,x}$. Then there are morphisms $r \in \mathbb{F}_1((\mathbb{A}^1_W, \mathbb{A}^1_W - 0 \times W),(V,V-0 \times W))$ and $l \in \mathbb{F}_1((\mathbb{A}^1_W, \mathbb{A}^1_W - 0 \times W),(V,V-0 \times W))$ such that $[[r]] \circ [[r]] = [[[\sigma_{\mathbb{A}^1_W}]])$ and $[[l]] \circ [[l]] = [[[\sigma_V]]]$ in $\mathbb{F}_1((\mathbb{A}^1_W, \mathbb{A}^1_W - 0 \times W),(V,V-0 \times W))$ and $\mathbb{F}_1((V,V-0 \times W),(V,V-0 \times W))$ respectively.

To formulate further two theorems relating étale excision property, we need some preparations. Let $S \subset X$ and $S' \subset X'$ be closed subsets. Let

$$
\begin{array}{ccc}
V' & \longrightarrow & X' \\
\downarrow & & \downarrow \pi \\
V & \longrightarrow & X
\end{array}
$$

be an elementary distinguished square with $X$ and $X'$ affine $k$-smooth. Let $S = X - V$ and $S' = X' - V'$ be closed subschemes equipped with reduced structures. Let $x \in S$ and $x' \in S'$ be two points such that $\pi(x') = x$. Let $U = \text{Spec}(\mathcal{O}_{X,x})$ and $U' = \text{Spec}(\mathcal{O}_{X',x})$. Let $\pi : U' \to U$ be the morphism induced by $\pi$.

Theorem 2.13 (Injective étale excision). Under the notation above there is an integer $N$ and a morphism $r \in \mathbb{F}_N((U,U-S),(X',X'-S'))$ such that $[[\pi]] \circ [[r]] = [\text{can}] \circ [[[\sigma_V]]]$ in $\mathbb{F}_N((U,U-S),(X,X-S))$, where $\text{can} : U \to X$ is the canonical morphism.

Theorem 2.14 (Surjective étale excision). Under the notation above suppose additionally that $S$ is $k$-smooth. Then there are an integer $N$ and a morphism $l \in \mathbb{F}_N((U,U-S),(X',X'-S'))$ such that $[[l]] \circ [[\pi]] = [\text{can}] \circ [[[\sigma_V]]]$ in $\mathbb{F}_N((U',U'-S'),(X',X'-S'))$ with $\text{can}' : U' \to X'$ the canonical morphism.
We are now in a position to prove the following

**Theorem 2.15.** For any $\mathbb{A}^1$-invariant quasi-stable $\mathbb{Z}_F$-presheaf of abelian groups $\mathcal{F}$ the following statements are true:

1. under the assumptions of Theorem 2.9 the map $i^*: \mathcal{F}(U) \to \mathcal{F}(V)$ is injective;
2. under the assumptions of Theorem 2.10 the map $[[i]]^*: \mathcal{F}(U - S)/\mathcal{F}(U) \to \mathcal{F}(V - S)/\mathcal{F}(V)$ is an isomorphism;
3. under the assumptions of Theorem 2.11 the map $\eta^*: \mathcal{F}(U) \to \mathcal{F}(\mathrm{Spec}(k(X)))$
   is injective, where $\eta: \mathrm{Spec}(k(X)) \to U$ is the canonical morphism;
4. under the assumptions of Theorem 2.11 let $U^h_x$ be the henselization of $U$ at $x$ and let $k(U^h_x)$ be the function field on $U^h_x$. Then the map $\eta^h_+: \mathcal{F}(U^h_x) \to \mathcal{F}(\mathrm{Spec}(k(U^h_x)))$ is injective, where $\eta^h+: \mathrm{Spec}(k(U^h_x)) \to U^h_x$ is the canonical morphism;
5. under the assumptions of Theorems 2.13 and 2.14 the map $[[\Pi]]^*: \mathcal{F}(U - S)/\mathcal{F}(U) \to \mathcal{F}(U' - S')/\mathcal{F}(U')$ is an isomorphism.

**Proof.** Firstly we may assume that $\mathcal{F}$ is stable. Now assertions (1), (3) and (3’) follow from Theorems 2.9 and 2.11. To prove assertions (2), (4) and (5), use Construction 2.8 and apply Theorems 2.10, 2.12, 2.13 and 2.14 respectively (recall that $\mathcal{F}$ is stable).

**Proof of Theorem 2.11.** Firstly, (1) and (2) show that $\mathcal{F}|_{\mathbb{A}^1}$ is a Zariski sheaf. Using (5) applied to $X = \mathbb{A}^1$, one shows that for any open $V$ in $\mathbb{A}^1$ one has $\mathcal{F}_{\mathrm{Nis}}(V) = \mathcal{F}(V)$.

Now consider the following Cartesian square of schemes

$\begin{array}{ccc}
\text{Spec}(k(X)) & \xrightarrow{\eta} & X \\
\downarrow \scriptstyle\iota_{0,k(X)} & & \downarrow \scriptstyle\iota_{0,X} \\
\mathbb{A}^1_{k(X)} & \xrightarrow{\eta \times \text{id}} & X \times \mathbb{A}^1
\end{array}$

Evaluating the Nisnevich sheaf $\mathcal{F}_{\mathrm{Nis}}$ on this square, we get a square of abelian groups

$\begin{array}{ccc}
\mathcal{F}_{\mathrm{Nis}}(\text{Spec}(k(X))) & \xrightarrow{\eta^*} & \mathcal{F}_{\mathrm{Nis}}(X) \\
\downarrow \scriptstyle\iota^*_{0,k(X)} & & \downarrow \scriptstyle\iota^*_{0,X} \\
\mathcal{F}_{\mathrm{Nis}}(\mathbb{A}^1_{k(X)}) & \xrightarrow{(\eta \times \text{id})^*} & \mathcal{F}_{\mathrm{Nis}}(X \times \mathbb{A}^1)
\end{array}$

The map $\iota^*_{0,X}$ is plainly surjective. It remains to check its injectivity. The map $(\eta \times \text{id})^*$ is injective (apply (3’)). As already mentioned in this proof, $\mathcal{F}_{\mathrm{Nis}}(\mathbb{A}^1_{k(X)}) = \mathcal{F}(\mathbb{A}^1_{k(X)})$. Since
We finish the section by proving the following useful statement, which is a consequence of Theorem 2.15 (item 4):

**Corollary 2.16.** Let $X \in Sm/k$, $x \in X$ be a point, $W = \Spec(\mathcal{O}_{X,x})$. Let $\mathcal{V} := \Spec(\mathcal{O}_{W \times \mathbb{A}^1,(x,0)})$ and can : $\mathcal{V} \to W \times \mathbb{A}^1$ be the canonical embedding. Let $\mathcal{F}$ be an $\mathbb{A}^1$-invariant stable $\mathbb{Z}F_\ast$-presheaf of abelian groups. Then the pull-back map

$$[[\text{can}]] : \mathcal{F}(W \times (\mathbb{A}^1 - \{0\})) / \mathcal{F}(W \times \mathbb{A}^1) \to \mathcal{F}(\mathcal{V} - W \times \{0\}) / \mathcal{F}(\mathcal{V})$$

is an isomorphism (both quotients make sense: the second quotient makes sense due to Theorem 2.15 (item 3), the first one makes sense due to homotopy invariance of $\mathcal{F}$).

**Proof.** Consider the $W$-scheme $W \times \mathbb{P}^1$ and effective divisors of the form $H \sqcup D$ on $W \times \mathbb{P}^1$ such that $H$ is a section of the projection $W \times \mathbb{P}^1 \to W$, $D$ is a reduced divisor, $H \cap (W \times 0) = \emptyset$ and $D \cap (W \times 0) = \emptyset$. For such a divisor set $V_{H,D} := W \times \mathbb{P}^1 - (H \sqcup D)$. Note that $(W \times 0) \subset V_{H,D}$.

Consider the category, $\mathcal{C}'$, of Zariski neighborhoods of $(W \times 0)$ in $W \times \mathbb{P}^1$ as well as the presheaf $V \mapsto \mathcal{F}(V - W \times 0)/\text{Im}(\mathcal{F}(V))$ on $\mathcal{C}'$. Clearly, the category $\mathcal{C}'$ is co-filtered. By definition, one has

$$\mathcal{F}(\mathcal{V}) = \lim_{\to V} \mathcal{F}(V) \text{ and } \mathcal{F}(\mathcal{V} - W \times 0) = \lim_{\to V} \mathcal{F}(V - W \times 0),$$

where $V$ runs over all Zariski neighborhoods of $(W \times 0)$ in $W \times \mathbb{P}^1$. Thus

$$\mathcal{F}(\mathcal{V} - W \times \{0\}) / \mathcal{F}(\mathcal{V}) = \lim_{\to V} \mathcal{F}(V - W \times 0)/\text{Im}(\mathcal{F}(V)).$$

Let $\mathcal{C}'$ be the full subcategory of $\mathcal{C}$ consisting of objects of the form $V_{H,D}$. Since the base field $k$ is infinite and $W$ is regular local, then the subcategory $\mathcal{C}'$ is cofinal in $\mathcal{C}'$. Thus,

$$\mathcal{F}(\mathcal{V} - W \times \{0\}) / \mathcal{F}(\mathcal{V}) = \lim_{\to V_{H,D}} \mathcal{F}(V_{H,D} - W \times 0)/\text{Im}(\mathcal{F}(V_{H,D})).$$

We claim that for any inclusion $\varepsilon : V_{H_1,D_1} \to V_{H_1,D_1}$, the pull-back map

$$[[\varepsilon]]^* : \mathcal{F}(V_{H_1,D_1} - W \times 0)/\text{Im}(\mathcal{F}(V_{H_1,D_1})) \to \mathcal{F}(V_{H_1,D_1} - W \times 0)/\text{Im}(\mathcal{F}(V_{H_1,D_1}))$$

is an isomorphism. To prove this claim, note that the inclusion above yields an inclusion $H_1 \sqcup D_1 \subset H_2 \sqcup D_2$. We have that either $H_1 = H_2$ or $H_1 \subset D_2$. In the second case one has $H_1 \cap H_2 = \emptyset$, $H_1 \subset D_2$ and $H_2 \cap H_1 \subset H_2 \cap D_2$. If $H_1 = H_2$, then one has inclusions $V_{H_1,D_1} \overset{\alpha}{\to} V_{H_1,D_1} \overset{\beta}{\to} V_{H_1,\emptyset}$. Set $\gamma = \beta \circ \alpha$. By Theorem 2.15 (item 4) the maps $[[\beta]]^*$ and $[[\gamma]]^*$ are isomorphisms. Thus the map $[[\alpha]]^*$ is an isomorphism in this case, too. In the second case consider inclusions $V_{H_1,D_2} \overset{\alpha}{\to} V_{H_1,H_2} \overset{\beta}{\to} V_{H_1,\emptyset}$ and set $\alpha = \beta \circ \gamma$. By Theorem 2.15 (item 4) the maps $[[\beta]]^*$ and $[[\gamma]]^*$ are isomorphisms. Thus the map $[[\alpha]]^*$ is an isomorphism. Now consider inclusions $V_{H_1,H_1} \overset{\delta}{\to} V_{H_1,\emptyset}$ and $V_{H_1,D_1} \overset{\rho}{\to} V_{H_1,\emptyset}$. One has $\delta \circ \gamma = \rho \circ \varepsilon$. We already know that $[[\gamma]]^*$ is an isomorphism. By Theorem 2.15 (item 4) the maps $[[\rho]]^*$ and $[[\delta]]^*$ are isomorphisms. Thus $[[\varepsilon]]^*$ is an isomorphism in the second case, too. The claim is proved. Thus for any $V_{H,D} \in \mathcal{C}'$ the map

$$\mathcal{F}(V_{H,D} - W \times 0)/\text{Im}(\mathcal{F}(V_{H,D})) \to \mathcal{F}(\mathcal{V} - W \times \{0\}) / \mathcal{F}(\mathcal{V})$$

is an isomorphism. In particular, the map

$$\mathcal{F}(W \times \mathbb{A}^1 - W \times 0)/\mathcal{F}(W \times \mathbb{A}^1) = \mathcal{F}(V_{W \times \mathbb{A}^1,D_0} - W \times 0)/\mathcal{F}(V_{W \times \mathbb{A}^1,\emptyset}) \to \mathcal{F}(\mathcal{V} - W \times \{0\}) / \mathcal{F}(\mathcal{V})$$

is an isomorphism, whence the corollary. □
3. Notation and agreements

**Notation 3.1.** Given a morphism \( a \in Fr_n(Y, X) \), we will write \( \langle a \rangle \) for the image of \( 1 \cdot a \) in \( \mathbb{Z}F_n(Y, X) \) and write \([a]\) for the class of \( \langle a \rangle \) in \( \mathbb{Z}F_n(Y, X) \).

Given a morphism \( a \in Fr_n(Y, X) \), we will write \( Z_a \) for the support of \( a \) (it is a closed subset in \( Y \times \mathbb{A}^n \) which finite over \( Y \) and determined by \( a \) uniquely). Also, we will often write

\[
(\mathcal{Y}_a, q_a : \mathcal{Y}_a \to Y \times \mathbb{A}^n; g_a : \mathcal{Y}_a \to X)
\]

for a representative of the morphism \( a \) (here \( \langle \mathcal{Y}_a, \rho : \mathcal{Y}_a \to Y \times \mathbb{A}^n, s : Z_a \to \mathcal{Y}_a \rangle \) is an étale neighborhood of \( Z_a \) in \( Y \times \mathbb{A}^n \)).

**Lemma 3.2.** If the support \( Z_a \) of an element \( a = (\mathcal{Y}, q : \mathcal{Y} \to Y \times \mathbb{A}^n) \in Fr_n(X, Y) \) is a disjoint union of \( Z_1 \) and \( Z_2 \), then the element \( a \) determines two elements \( a_1 \) and \( a_2 \) in \( Fr_n(X, Y) \). Namely,

\[
a_1 = (\mathcal{Y}_1 - Z_2, q|_{\mathcal{Y}_1 - Z_2}; g|_{\mathcal{Y}_1 - Z_2}) \quad \text{and} \quad a_2 = (\mathcal{Y}_2 - Z_1, q|_{\mathcal{Y}_2 - Z_1}; g|_{\mathcal{Y}_2 - Z_1}).
\]

Moreover, by the definition of \( \mathbb{Z}F_n(X, Y) \) one has the equality

\[
\langle a \rangle = \langle a_1 \rangle + \langle a_2 \rangle
\]

in \( \mathbb{Z}F_n(X, Y) \).

**Definition 3.3.** Let \( i_Y : Y' \hookrightarrow Y \) and \( i_X : X' \hookrightarrow X \) be open embeddings. Let \( a \in Fr_n(Y, X) \). We say that the restriction \( a|_{Y'} \) of \( a \) to \( Y' \) runs inside \( X' \), if there is \( a' \in Fr_n(Y', X') \) such that

\[
i_X \circ a' = a \circ i_Y
\]

in \( Fr_n(Y', X) \).

It is easy to see that if there is a morphism \( a' \) satisfying condition (I), then it is unique. In this case the pair \((a, a')\) is an element of \( \mathbb{Z}F_n((Y', Y'), (X', X')) \). For brevity we will write \( \langle \langle a \rangle \rangle \) for \((a, a')\).

**Lemma 3.4.** Let \( i_Y : Y' \hookrightarrow Y \) and \( i_X : X' \hookrightarrow X \) be open embeddings. Let \( a \in Fr_n(Y, X) \). Let \( Z_a \subset Y \times \mathbb{A}^n \) be the support of \( a \). Set \( Z'_a = Z_a \cap Y' \times \mathbb{A}^n \). Then the following are equivalent:

1. \( g_a(Z'_a) \subset X' \);
2. the morphism \( a|_{Y'} \) runs inside \( X' \).

**Proof.** (1) \( \Rightarrow \) (2). Set \( Y' = p_Y^{-1} \cap g^{-1}(X') \), then \( p_Y \circ p_Y \circ p_a = \mathcal{Y}' \to Y \times \mathbb{A}^n \). Then \( a' := \langle Y', q'|_{Y'}, g'|_{Y'} \rangle \in Fr_n(Y', X') \) satisfies condition (I).

(2) \( \Rightarrow \) (1). If \( a|_{Y'} \) runs inside \( X' \), then for some \( a' = \langle Y', q'|_{Y'}, g'|_{Y'} \rangle \in Fr_n(Y', X') \) equality (I) holds. In this case the support \( Z' \) of \( a' \) must coincide with \( Z'_a = Z_a \cap Y' \times \mathbb{A}^n \) and \( g_a|_{Z'} = g'|_{Z'} \).

Since \( g'(Z') \) is a subset of \( X' \), then \( g_a(Z'_a) = g_a(Z') \subset X' \).

**Corollary 3.5.** Let \( i_Y : Y' \hookrightarrow Y \) and \( i_X : X' \hookrightarrow X \) be open embeddings. Let \( h_0 = (\mathcal{Y}_0, q_0; g_0) \in Fr_n(\mathbb{A}^1 \times X, X) \). Suppose \( Z_{h_0} \), the support of \( h_0 \), is such that for \( Z'_0 := Z_0 \cap \mathbb{A}^1 \times Y' \times \mathbb{A}^n \) one has \( g_0(Z'_0) \subset X' \). Then there are morphisms \( \langle h_0 \rangle \in \mathbb{Z}F_n((Y, Y'), (X, X')) \), \( \langle h_1 \rangle \in \mathbb{Z}F_n((Y, Y'), (X, X')) \) and one has an obvious equality

\[
[h_0] = [h_1]
\]

in \( \mathbb{Z}F_n((Y, Y'), (X, X')) \).

**Lemma 3.6** (A disconnected support case). Let \( i_Y : Y' \hookrightarrow Y \) and \( i_X : X' \hookrightarrow X \) be open embeddings. Let \( a \in Fr_n(Y, X) \) and \( Z_a \subset Y \times \mathbb{A}^n \) be the support of \( a \). Set \( Z'_a = Z_a \cap Y' \times \mathbb{A}^n \). Suppose that \( Z_a = Z_{a, 1} \sqcup Z_{a, 2} \). For \( i = 1, 2 \) set \( Y_i = \mathcal{Y}_a - Z_{a, j} \) with \( j \in \{1, 2\} \) and \( j \neq i \). Also set \( q_i = q_a|_{Y_i} \) and \( g_i = g_a|_{Y_i} \). Suppose \( a|_{Y'} \) runs inside \( X' \), then
(1) for each $i = 1, 2$ the morphism $a_i := (\mathcal{Y}_i, q_i; g_i)$ is such that $a_i|_{Y'}$ runs inside $X'$;
(2) $\langle\langle a_1\rangle\rangle = \langle\langle a_1\rangle\rangle + \langle\langle a_2\rangle\rangle$ in $\mathbb{Z}F_0((Y,Y'),(X,X'))$.

4. Some homotopies

Suppose $U, W \subset \mathbb{A}^1_k$ are open and non-empty.

**Lemma 4.1.** Let $a_0 = (\mathcal{Y}, q_0; g_0) \in Fr_1(U, W)$, $a_1 = (\mathcal{Y}, q_1; g_1) \in Fr_1(U, W)$. Suppose that the supports of $a_0$ and $a_1$ coincide. Denote their common support by $Z$. If $g_0|_{Z} = g_1|_{Z}$, then $[a_0] = [a_1]$ in $\mathbb{Z}F_1(U, W)$.

**Proof.** Consider a function $g_\theta = (1 - \theta)g_0 + \theta g_1 : \mathbb{A}^1 \times \mathcal{Y} \to \mathbb{A}^1$ and set $\mathcal{Y}_\theta = g_\theta^{-1}(W)$, $q_\theta = q \circ pr_Y : \mathcal{Y}_\theta \to \mathbb{A}^1$. Next, consider a homotopy

$$h_\theta = (\mathcal{Y}_\theta, q_\theta; g_\theta) \in Fr_1(\mathbb{A}^1 \times U, W).$$

The support of $h_\theta$ equals $\mathbb{A}^1 \times Z \subset \mathbb{A}^1 \times U \times \mathbb{A}^1$. Clearly, $h_0 = a_0$ and $h_1 = a_1$. Whence the lemma.

**Corollary 4.2.** Under the assumptions of Lemma 4.1 let $U' \subset U$ and $W' \subset W$ be open subsets. Suppose that $a_0|_{U'}$ runs inside $W'$. Then $a_1|_{U'}$ runs inside $W'$, the restriction $h_\theta|_{\mathbb{A}^1 \times U'}$ of the homotopy $h_\theta$ runs inside $W'$ and

$$[[a_0]] = [[a_1]]$$

in $\mathbb{Z}F_1((U, U'), (W, W'))$.

**Lemma 4.3.** Let $a_0 = (\mathcal{Y}, q_0u_0; g) \in Fr_1(U, W)$, $a_1 = (\mathcal{Y}, q_1u_1; g) \in Fr_1(U, W)$, where $u_0, u_1 \in k[\mathcal{Y}]$ are units. In this case the supports of $a_0$ and $a_1$ coincide. Denote their common support by $Z$. Suppose $u_0|_{Z} = u_1|_{Z}$, then $[a_0] = [a_1]$ in $\mathbb{Z}F_1(U, W)$.

**Proof.** Set $u_\theta = (1 - \theta)u_0 + \theta u_1 \in k[\mathbb{A}^1 \times \mathcal{Y}]$. Clearly, $u_\theta|_{\mathbb{A}^1 \times Z} = pr_2^*u_0 = pr_2^*(u_1) \in k[\mathbb{A}^1 \times Z]$. Let $\mathcal{Y}_\theta = \{u_\theta \neq 0\} \subset \mathbb{A}^1 \times \mathcal{Y}$. Set,

$$h_\theta = (\mathcal{Y}_\theta, u_\theta q; g \circ pr_Y) \in Fr_1(\mathbb{A}^1 \times U, W).$$

The support of $h_\theta$ equals $\mathbb{A}^1 \times Z \subset \mathbb{A}^1 \times U \times \mathbb{A}^1$. Clearly, $h_0 = a_0$ and $h_1 = a_1$. Whence the lemma.

**Corollary 4.4.** Under the assumptions of Lemma 4.3 let $U' \subset U$ and $W' \subset W$ be open subsets. Suppose $a_0|_{U'}$ runs inside $W'$. Then $a_1|_{U'}$ runs inside $W'$, the restriction $h_\theta|_{\mathbb{A}^1 \times U'}$ of the homotopy $h_\theta$ from the proof of Lemma 4.3 runs inside $W'$ and

$$[[a_0]] = [[a_1]]$$

in $\mathbb{Z}F_1((U, U'), (W, W'))$.

**Lemma 4.5.** Let $U \subset \mathbb{A}^1_k$ be non-empty open as above. Suppose $F_0(Y) = F_1(Y) \in k[U][Y]$. Let $deg_Y(F_0) = deg_Y(F_1) = d > 0$ and let their leading coefficients coincide and are units in $k[U]$. Then,

$$[U \times \mathbb{A}^1, F_0(Y), pr_U] = [U \times \mathbb{A}^1, F_0(Y), pr_U] \in \mathbb{Z}F_1(U, U).$$

**Proof.** Set $F_\theta(Y) = (1 - \theta)F_0(Y) + \theta F_1(Y) \in k[U][\theta, Y]$. Consider a morphism

$$h_\theta = (\mathbb{A}^1 \times U \times \mathbb{A}^1, F_\theta, pr_{\mathbb{A}^1 \times U}) \in Fr_1(\mathbb{A}^1 \times U, U).$$

(4)

Clearly, $h_0 = (U \times \mathbb{A}^1, F_0(Y), pr_U)$ and $h_1 = (U \times \mathbb{A}^1, F_1(Y), pr_U)$. Whence the lemma.
Corollary 4.6. Under the assumptions of Lemma 4.5 let $U' \subset U$ be an open subset. Then $(U \times \mathbb{A}^1, F_0(Y), pr_U)|_{U'}$, $(U \times \mathbb{A}^1, F_1(Y), pr_U)|_{U'}$ runs inside $U'$ and the restriction $h_0|_{\mathbb{A}^1 \times U'}$ of the homotopy $h_0$ from the proof of Lemma 4.5 runs inside $W'$ and

$$[[U \times \mathbb{A}^1, F_0(Y), pr_U]] = [[U \times \mathbb{A}^1, F_0(Y), pr_U]] \in \mathbb{Z}F_1((U, U'), (U, U'))$$

in $\mathbb{Z}F_1((U, U'), (U, U'))$.

Proposition 4.7. Let $U \subset \mathbb{A}_k^1$ and $U' \subset U$ be open subsets. Let $t \in k[\mathbb{A}_k^1]$ be the standard parameter on $\mathbb{A}_k^1$. Set $X := (r \otimes 1)|_{U \times U} \in k[U \times U]$ and $Y := (1 \otimes r)|_{U \times U} \in k[U \times U]$. Then for any integer $n \geq 1$, one has an equality

$$[[U \times U, (Y - X)^{2n+1}, p_2]] = [[U \times U, (Y - X)^{2n}, p_2]] + [[\sigma_U]]$$

in $\mathbb{Z}F_1((U, U'), (U, U'))$.

Proof. Let $m \geq 1$ be an integer. Then

$$[[U \times U, (Y - X)^m, p_2]] = [[U \times U, (Y - X)^m, p_1]] = [[U \times \mathbb{A}_k^1, (Y - X)^m, p_1]] = [[U \times \mathbb{A}_k^1, Y^m, p_1]]$$

in $\mathbb{Z}F_1((U, U'), (U, U'))$. The first equality follows from Corollary 4.2, the third one follows from Corollary 4.6, the middle one is obvious.

There is a chain of equalities in $\mathbb{Z}F_1((U, U'), (U, U'))$:

$$[[U \times \mathbb{A}_k^1, Y^{2n+1}; p_1]] = [[U \times \mathbb{A}_k^1, Y^{2n}(Y + 1); p_1]] =$$

$$= [[U \times (\mathbb{A}_k^1 - \{ -1 \}), Y^{2n}(Y + 1); p_1]] + [[U \times (\mathbb{A}_k^1 - \{ 0 \}), Y^{2n}(Y + 1); p_1]] =$$

$$= [[\mathbb{Y}_0, Y^{2n}; p_1]] + [[\mathbb{Y}_1, (Y + 1); p_1]] =$$

$$= [[U \times \mathbb{A}_k^1, Y^{2n}; p_1]] + [[U \times \mathbb{A}_k^1, (Y + 1); p_1]].$$

Here the first equality holds by Corollary 4.6, the second one holds by Lemma 3.6, the third one holds by Corollary 4.6, the forth one is obvious (replacement of neighborhoods).

Continue the chain of equalities in $\mathbb{Z}F_1((U, U'), (U, U'))$ as follows:

$$[[U \times \mathbb{A}_k^1, Y^{2n}; p_1]] + [[U \times \mathbb{A}_k^1, (Y + 1); p_1]] = [[U \times \mathbb{A}_k^1, (Y - X)^{2n}, p_1]] + [[U \times \mathbb{A}_k^1, Y; p_1]] =$$

$$= [[U \times \mathbb{A}_k^1, (Y - X)^{2n}; p_1]] + [[\sigma_U]] = [[U \times \mathbb{A}_k^1, (Y - X)^{2n}; p_2]] + [[\sigma_U]].$$

Here the first equality holds by Corollary 4.6, the second one holds by the definition of $\sigma_U$ (see Notation 4.7), the third one holds by Corollary 4.2. We proved the equality

$$[[U \times \mathbb{A}_k^1, Y^{2n+1}; p_1]] = [[U \times \mathbb{A}_k^1, (Y - X)^{2n}; p_2]] + [[\sigma_U]].$$

(6)

Combining that with the equality (5) for $m = 2n + 1$ we get the desired equality

$$[[U \times \mathbb{A}_k^1, Y^{2n+1}; p_2]] = [[U \times \mathbb{A}_k^1, (Y - X)^{2n}; p_2]] + [[\sigma_U]]$$

in $\mathbb{Z}F_1((U, U'), (U, U'))$. Whence the proposition. □
5. Injectivity and excision on affine line

The aim of this section is to prove Theorems 2.9 and 2.10.

**Lemma 5.1.** Let $U \subset \mathbb{A}^1$ be open and non-empty. Let $A = \mathbb{A}^1 \setminus U$. Let $G_0(Y), G_1(Y) \in k[U][Y]$ be such that

1. $\deg_Y(G_0) = \deg_Y(G_1)$;
2. both are unitary in $Y$ and the leading coefficients equal one;
3. $G_0|_{U \times A} = G_1|_{U \times A} \in k[U \times A]^\times$.

Then

$$[U \times U, G_0; p_2] = [U \times U, G_1; p_2]$$

in $\mathcal{F}(U, U)$.

**Proof.** One has a homotopy $h_\theta = (\mathbb{A}^1 \times U \times U, G_\theta, p_{\mathbb{A}^1 \times U}) \in \mathcal{F}(\mathbb{A}^1 \times U, U)$, where $G_\theta = (1 - \theta)G_0 + \theta G_1$. Its restriction to $0 \times U$ and to $1 \times U$ coincides with morphisms $(U \times U, G_0; p_2)$ and $(U \times U, G_1; p_2)$ respectively. Whence the lemma.

**Proof of Theorem 2.9.** Under the assumptions of this theorem set $A = \mathbb{A}^1 \setminus U$ and $B = U \setminus V$. For each big enough integer $m \geq 0$ find a polynomial $F_m(Y) \in k[U][Y]$ such that $F_m(Y)$ is of degree $m$ with the leading coefficient equal 1 and such that

1. $F_m(Y)|_{U \times A} = (Y - X)^m|_{U \times A} \in k[U \times A]^\times$;
2. $F_m(Y)|_{U \times B} = 1 \in k[U \times B]^\times$.

Take $n \geq 0$ and set $r = (U \times V, F_{2n+1}; pr_V) - (U \times V, F_{2n}; pr_V) \in \mathcal{F}(U, V)$. Then one has a chain equalities in $\mathcal{F}(U, U)$:

$$[i] \circ [r] = [U \times U, F_{2n+1}; p_2] - [U \times U, F_{2n}; p_2] = [U \times U, (Y - X)^{2n+1}; p_2] - [U \times U, (Y - X)^{2n}; p_2] = [0_U].$$

Here the first equality is obvious, the second one holds by Lemma 5.1, the third one holds by Proposition 4.7. Whence the theorem.

**Corollary 5.2** (of Lemma 5.1). Under the conditions and notation of Lemma 5.1 let $S \subset U$ be a closed subset. Additionally to the conditions (1) – (3) suppose that the following to conditions hold:

4. $G_0(Y)|_{U \times S} = G_1(Y)|_{U \times S}$,
5. $G_0(Y)|_{(U - S) \times S}$ is invertible.

Then one has an equality

$$[[U \times U, G_0; p_2]] = [[U \times U, G_1; p_2]]$$

in $\mathcal{F}(U, U - S, U, U - S)$.

**Proof of the corollary.** The support $Z_\theta$ of the homotopy $h_\theta$ from the proof of Lemma 5.1 coincides with the vanishing locus of the polynomial $G_\theta$. Since $G_\theta|_{\mathbb{A}^1 \times (U - S) \times S}$ is invertible, then $G_\theta \cap \mathbb{A}^1 \times (U - S) \times S = \emptyset$. By Lemma the homotopy $h_\theta|_{\mathbb{A}^1 \times (U - S) \times U}$ runs inside $U - S$. Hence

$$[[U \times U, G_0; p_2]] = [[h_0]] = [[h_1]] = [[U \times U, G_1; p_2]]$$

in $\mathcal{F}(U, U - S, U, U - S)$. In fact, the second equality here holds by Corollary 5.5. The first and the third equalities hold since for $i = 1, 2$ one has $h_i = (U \times U, G_i; p_2)$ in $\mathcal{F}(U, U)$. □
Proof of Theorem 2.10. Firstly construct a morphism \( r \in \mathbb{Z}F_1((U, U - S), (V, V - S)) \) such that for its class \([r]\) in \( \mathbb{Z}F_1((U, U - S), (V, V - S)) \) one has
\[
[[i]] \circ [[r]] = [[\sigma_U]]
\] (7)
in \( \mathbb{Z}F_1((U, U - S), (U, U - S)) \).
To this end set \( A = \mathbb{A}^1_k - U, B = U - V \). Recall that \( S \subset V \) is a closed subset. Take any big enough integer \( m \geq 1 \) and find a unitary polynomial \( F_m(Y) \) of degree \( m \) satisfying the following properties:
(i) \( F_m(Y)|_{U \times A} = (Y - X)^m|_{U \times A} \in k[U \times A]^\times; \)
(ii) \( F_m(Y)|_{U \times B} = 1 \in k[U \times B]^\times; \)
(iii) \( F_m(Y)|_{U \times S} = (Y - X)^m|_{U \times S} \in k[U \times S]. \)
Note that \( F_m(Y)|_{(U - S) \times S} \in k[(U - S) \times S]^\times. \) Hence by Lemma 3.4, the morphism \( (U \times V, F_m; pr_V) \in Fr_1(U, V) \) being restricted to \( U - S \) runs inside \( V \) s. Thus using Definition 3.3, we get a morphism
\[
\langle (U \times V, F_m; pr_V) \rangle \in \mathbb{Z}F_1((U, U - S), (V, V - S)).
\]
For that morphism one has equalities
\[
[[i]] \circ [[U \times V, F_m; pr_V]] = [[U \times U, F_m; p_2]] = [[U \times U, (Y - X)^m; p_2]]
\]
in \( \mathbb{Z}F_1((U, U - S), (U, U - S)) \). Here the first equality is obvious, the second one follows from Corollary 5.2. Take a big enough integer \( n \). Set
\[
r = \langle (U \times V, F_{2n+1}; pr_V) \rangle - \langle (U \times V, F_{2n}; pr_V) \rangle \in \mathbb{Z}F_1((U, U - S), (V, V - S)).
\]
We claim that
\[
[[i]] \circ [[r]] = [[\sigma_U]]
\]
in \( \mathbb{Z}F_1((U, U - S), (U, U - S)) \). In fact,
\[
[[i]] \circ [[r]] = [[U \times U, (Y - X)^{2n+1}; p_2]] - [[U \times U, (Y - X)^{2n}; p_2]] = [[\sigma_U]].
\]
The first equality proven a few lines above and the second one follow from Proposition 4.7. Whence equality (7) holds.
Now find morphisms \( l \in \mathbb{Z}F_1((U, U - S)), (V, V - S) \) and \( g \in \mathbb{Z}F_1((V, V - S), (V - S, V - S)) \) such that
\[
[[l]] \circ [[i]] - [[j]] \circ [[g]] = [[\sigma_U]]
\] (8)
in \( \mathbb{Z}F_1((V, V - S), (V, V - S)) \). Here \( j : (V - S, V - S) \rightarrow (V, V - S) \) is the inclusion. Clearly, equality (8) yields
\[
[[l]] \circ [[i]] = [[\sigma_U]] \in \mathbb{Z}F_1((V, V - S), (V, V - S)).
\]
Set \( A' = \mathbb{A}^1_k - U, B = U - V \) and recall that \( S \subset V \) is a closed subset. Take an integer \( m \) big enough and find a monic in \( Y \) polynomial \( F_m(Y) \in k[U][Y] \) such that
(i) \( F_m(Y)|_{U \times A} = (Y - X)|_{U \times A} \in k[U \times A]^\times; \)
(ii) \( F_m(Y)|_{U \times B} = 1 \in k[U \times B]^\times; \)
(iii) \( F_m(Y)|_{U \times S} = (Y - X)|_{U \times S} \in k[U \times S]. \)
Note that \( F_m(Y)|_{(U - S) \times S} \in k[(U - S) \times S]^\times. \) Hence by Lemma 3.4, the morphism \( (U \times V, F_m; pr_V) \in Fr_1(U, V) \) being restricted to \( U - S \) runs inside \( V \) s. Thus, using Definition 3.3, we get a morphism
\[
l = \langle (U \times V, F_m; pr_V) \rangle \in \mathbb{Z}F_1((U, U - S), (V, V - S)).
\]
To construct the desired morphism \( g \), find a monic in \( Y \) polynomial \( E_{m-1} \in k[V][Y] \) of degree \( m - 1 \) such that
(i) \( E_{m-1}(Y)|_{V \times A} = 1|_{U \times A} \in k[V \times A]^\times; \)
(ii) \( E_{m-1}(Y)|_{V \times B} = (Y - X)^{-1} \in k[V \times B]^\times; \)
(iii) \( E_{m-1}(Y)|_{V \times S} = 1|_{V \times S} \in k[U \times S]; \)
Claim 5.3. Equality (8) holds for the morphisms \(l\) and \(g\) defined above.

Note firstly that \(l \circ (i') = (V \times V, F_m(Y)|_{V \times Y}; pr_2) \in \mathbb{Z}F_1((V, V - S), (V, V - S))\). Applying Corollary 5.2 to the case \(V \subset \mathbb{A}_k^1, S \subset V\) and \(A := A' \cup B\), we get an equality

\[
[V \times V, F_m(Y)|_{V \times Y}; pr_2] = [V \times V, (Y - \Delta)E_m-1(Y); pr_2]
\]

in \(\mathbb{Z}F_1((V, V - S), (V, V - S))\). By Lemma 5.6 and the fact that \(G \cap \Delta(V) = 0\), one has

\[
[V \times V, (Y - X)E_m-1(Y); pr_2] = [V \times V - G, E_m-1(Y - X); pr_2] + [V \times V - \Delta(V), (Y - X)E_m-1; pr_2] = [V \times V - G, E_m-1(Y - X); pr_2] + [[\mathcal{J}] \circ [[g]]]
\]

in \(\mathbb{Z}F_1((V, V - S), (V - S, V - S))\).

One has a chain of equalities

\[
[[V \times V - G, E_m-1(Y - X); pr_2]] = [[V \times V - G, (Y - X); pr_2]] = [[V \times V, (Y - X); pr_2]] = [[V \times V, Y; pr_1]] = [[V \times \mathbb{A}_k^1, Y; pr_1]] = [[\mathcal{O}_V]].
\]

The first equality holds by condition (iv') and Corollary 4.4. The second one is obvious. The third one is equality (9) for \(m = 1\) from the proof of Proposition 4.7. The forth one is the definition of \(\langle \langle \mathcal{O}_V \rangle \rangle\) (see Definition 2.2 and Notation 2.7). Combining altogether, we get a chain of equalities

\[
[[\mathcal{J}] \circ [[i]] = [[V \times V, F_m(Y)|_{V \times Y}; pr_2]] = [[V \times V, (Y - X)E_m-1(Y); pr_2]] = [[\mathcal{O}_V]] + [[\mathcal{J}] \circ [[g]]],
\]

which proves the claim. Whence the theorem. \qed

6. Excision on relative affine line

Proof of Theorem 2.12. Let \(U = \mathbb{A}_k^1, v \subset U\) be the open \(V\) from Theorem 2.12. Let \(S = 0 \times W\). Note that \(S \subset V\). Set \(A = \mathbb{A}_k^1 - U = 0, B = U - V = \{f = 0\}\). Then \(B\) is finite over \(U\), since \(f\) is monic. Note that \(B \cap (0 \times W) = 0\).

Repeat literally the proof of Theorem 2.10 (see Section 5). \qed

7. Injectivity for local schemes

The main aim of this section is to prove Theorem 2.11. Let \(X \in Sm/k, x \in X\) be a point, \(U = Spec(\mathcal{O}_X,x), i : D \to X\) be a closed subset. Under the notation of Theorem 2.11 we will construct an integer \(N\) and a morphism \(r \in \mathbb{Z}F_N(U, X - D)\) such that

\[
[r] \circ [i] = [can] \circ [\mathcal{O}_U^N]
\]

in \(\mathbb{Z}F_N(U, X)\).
Let \( X' \subset X \) be an open subset containing the point \( x \) and let \( D' = X' \cap D \). Clearly, if we solve a similar problem for the triple \( U, X' \) and \( X' - D' \), then we solve the problem for the original triple \( U, X \) and \( X - D \). So, we may shrink \( X \) appropriately. In particular, we may assume that \( X \) is irreducible and the canonical sheaf \( \omega_{X/k} \) is trivial, i.e. is isomorphic to the structure sheaf \( \mathcal{O}_X \). Let \( d = \dim X \).

Shrinking \( X \) more (and replacing \( D \) with its trace), we can find a commutative diagram of the form

\[
\begin{array}{ccc}
\mathbb{A}^1 \times B & \xrightarrow{\pi} & X \\
\downarrow{pr_B} & & \downarrow{i} \\
B & & D
\end{array}
\]

(10)

where \( p : X \to B \) is an almost elementary fibration in the sense of [4], \( B \) is an affine open subset of the projective space \( \mathbb{P}^{d-1} \), \( \pi \) is a finite surjective morphism, \( p|_D \) is a finite morphism.

The canonical sheaf \( \omega_{X/k} \) remains to be trivial. Since \( p \) is an almost elementary fibration, then it is a smooth morphism such that for each point \( b \in B \) the fibre \( p^{-1}(b) \) is a \( k(b) \)-smooth affine curve. Since \( \pi \) is finite, then the \( B \)-scheme \( X \) is affine.

Set \( U = \text{Spec}(\mathcal{O}_{X, x}) \), \( \mathcal{X} = U \times_B X \), \( \mathcal{D} = U \times_B D \). There is an obvious morphism \( \Delta = (id, \text{can}) : U \to \mathcal{X} \). It is a section of the projection \( p_U : \mathcal{X} \to U \). Let \( p_X : \mathcal{X} \to X \) be the projection to \( X \).

The base change of diagram (30) gives a commutative diagram of the form

\[
\begin{array}{ccc}
\mathbb{A}^1 \times U & \xrightarrow{\Pi} & \mathcal{X} \\
\downarrow{pr_U} & & \downarrow{i} \\
U & & \mathcal{D}
\end{array}
\]

(11)

\[\text{Lemma 7.1 (Ojanguren-Panin). There is a finite surjective morphism } H_0 = (p_U, h_0) : \mathcal{X} \to \mathbb{A}^1 \times U \text{ of } U\text{-schemes such that for the closed subschemes } Z_i := H_0^{-1}(1 \times U) \text{ and } Z_0 := H_0^{-1}(0 \times U) \text{ of } \mathcal{X} \text{ one has}
\]

(i) \( Z_1 \subset \mathcal{X} - \mathcal{D} \);

(ii) \( Z_0 = \Delta(U) \cup Z_0' \text{ (equality of schemes) and } Z_0' \subset \mathcal{X} - \mathcal{D} \).

Now regard \( \mathcal{X} \) as an affine \( \mathbb{A}^1 \times U \)-scheme via the morphism \( \Pi \). And also regard \( \mathcal{X} \) as an \( X \)-scheme via \( p_X \).

\[\text{Remark 7.2. By Lemma 7.1 the class } [\mathcal{O}_\mathcal{X}] \text{ of the structure sheaf of the subscheme } \mathcal{X} \text{ defines a morphism in } \text{Kor}_0(\mathbb{A}^1 \times U, X) \text{ such that for } i = 0, 1 \text{ one has } [\mathcal{O}_\mathcal{X}]|_{(i) \times U} = [\mathcal{O}_Z]. \text{ Moreover, one has } [\mathcal{O}_{Z_0}] = [\text{can}] + [j] \circ [\mathcal{O}_{Z_0}] \text{ and } [\mathcal{O}_{Z_1}] \in \text{Kor}_0(U, X - S) \text{ and } [\mathcal{O}_{Z_1}] \in \text{Kor}_0(U, X - S). \text{ Thus}
\]

\[ [j] \circ ([\mathcal{O}_{Z_0}] - [\mathcal{O}_{Z_1}]) = [\text{can}] \in \text{Kor}_0(U, X).
\]

Below we lift these elements to the category \( \mathbb{Z}F_*(k) \) and equalities to the category \( \overline{\mathbb{Z}}F_*(k) \).

\[\text{Lemma 7.3. There are an integer } N \geq 0, \text{ a closed embedding } \mathcal{X} \hookrightarrow \mathbb{A}^1 \times U \times \mathbb{A}^N \text{ of } \mathbb{A}^1 \times U \text{-schemes, an étale affine neighborhood } (\mathcal{V}, \rho : \mathcal{V} \to \mathbb{A}^1 \times U \times \mathbb{A}^N, s : \mathcal{X} \to \mathcal{V}) \text{ of } \mathcal{X} \text{ in } \mathbb{A}^1 \times U \times \mathbb{A}^N, \text{ functions } q_1, \ldots, q_N \in k[\mathcal{V}] \text{ and a morphism } r : \mathcal{V} \to \mathcal{X} \text{ such that:}
\]

(i) the functions \( q_1, \ldots, q_N \) generate the ideal \( I_s(\mathcal{X}) \) in \( k[\mathcal{V}] \) defining the closed subscheme \( s(\mathcal{X}) \) of \( \mathcal{V} \);

(ii) \( r \circ s = id_s(\mathcal{X}) \);
Claim 7.10. the morphism $r$ is a $U$-scheme morphism if $\mathcal{V}$ is regarded as a $U$-scheme via the morphism $p_U \circ r$ and $\mathcal{X}$ is regarded as a $U$-scheme via the morphism $p_U$.

By Lemma 7.1, $\mathcal{O}_\mathcal{X} = \Delta(U) \cup \mathcal{O}_0$. Set $\mathcal{Y}_0 = \rho^{-1}(0 \times U \times \mathbb{A}^N)$ and let $\mathcal{W}$ be the henselization of $\mathcal{Y}_0$ in $s(\Delta(U))$ (which is the same as the henselization of $0 \times U \times \mathbb{A}^N$ in $\Delta(U)$).

Remark 7.4. By Lemma 7.3 the functions $q_1|_{\mathcal{W}},...,q_N|_{\mathcal{W}}$ generate the ideal $I$ defining the closed subscheme $s(\Delta(U))$ of the scheme $\mathcal{W}$. In particular, the family

$$\langle \rho_1|_{\mathcal{W}},...,\rho_N|_{\mathcal{W}} \rangle \in I/I^2$$

is a free basis of the $k[U]$-module $I/I^2$. Another free basis of the $k[U]$-module $I/I^2$ is the family

$$\langle (t_1 - \Delta^+(t_1))|_{\mathcal{W}},..., (t_1 - \Delta^+(t_1)|_{\mathcal{W}} \rangle \in I/I^2.$$  

Let $A \in \text{GL}_N(k[\mathcal{W}])$ be a unique matrix which converts the second free basis to the first one and let $J := \det(A)$ be its determinant. Replacing $q_1$ by $J^{-1}q_1$, we may and will assume below in this section that $J = 1 \in k[\mathcal{W}]$. This is useful to apply Theorem 7.5 below.

Set $\mathcal{Y}_1 = \rho^{-1}(1 \times U \times \mathbb{A}^N) \cap r^{-1}(\mathcal{X} - \mathcal{D})$. Then $s(\mathcal{D}_1) \subset \mathcal{Y}_1$. In fact, $(r \circ s)(\mathcal{D}_1) = \mathcal{D}_1 \subset \mathcal{X} - \mathcal{D}$ and $\rho(\mathcal{D}_1) \subset 0 \times U \times \mathbb{A}^N$. Thus $\mathcal{Y}_1 \neq 0$.

Construction 7.5 (étale neighborhood of $\mathcal{D}_1$). The morphism $\rho| : \rho^{-1}(1 \times U \times \mathbb{A}^N) \rightarrow 1 \times U \times \mathbb{A}^N$ is étale and the inclusion $i_1 : \mathcal{Y}_1 \rightarrow \rho^{-1}(1 \times U \times \mathbb{A}^N)$ is open. Set $\rho_1 = (\rho|) \circ i_1$. Then the triple

$$\langle \mathcal{Y}_1, \rho_1 : \mathcal{Y}_1 \rightarrow 1 \times U \times \mathbb{A}^N, s_1 = s|_{\mathcal{Y}_1} : \mathcal{Y}_1 \rightarrow \mathcal{Y}_1 \rangle$$

is an étale neighborhood of $\mathcal{D}_1$ in $1 \times U \times \mathbb{A}^N$. Let $r_1 = r|_{\mathcal{Y}_1} : \mathcal{Y}_1 \rightarrow \mathcal{X} - \mathcal{D}$.

Definition 7.6. Set $a_1 = (\mathcal{D}_1, \mathcal{Y}_1, q_1|_{\mathcal{Y}_1},...,q_N|_{\mathcal{Y}_1}; (p_X)|_{\mathcal{X} - \mathcal{D}} \circ r_1) \in F_{\mathcal{Y}_1}(U,X - D)$.

Set $\mathcal{Y}_0 = \rho^{-1}(0 \times U \times \mathbb{A}^N) \cap r^{-1}(\mathcal{X} - \mathcal{D})$. Then $s(\mathcal{D}_0) \subset \mathcal{Y}_0$. In fact, $(r \circ s)(\mathcal{D}_0) = \mathcal{D}_0 \subset \mathcal{X} - \mathcal{D}$ and $\rho(\mathcal{D}_0) \subset 0 \times U \times \mathbb{A}^N$. Thus $\mathcal{Y}_0 \neq 0$.

Construction 7.7. The morphism $\rho| : \rho^{-1}(0 \times U \times \mathbb{A}^N) \rightarrow 0 \times U \times \mathbb{A}^N$ is étale and the inclusion $i_0 : \mathcal{Y}_0 \rightarrow \rho^{-1}(0 \times U \times \mathbb{A}^N)$ is open. Set $\rho_0 = (\rho|) \circ i_0$. Then the triple

$$\langle \mathcal{Y}_0, \rho_0 : \mathcal{Y}_0 \rightarrow 0 \times U \times \mathbb{A}^N, s_0 = s|_{\mathcal{Y}_0} : \mathcal{Y}_0 \rightarrow \mathcal{Y}_1 \rangle$$

is an étale neighborhood of $\mathcal{D}_0$ in $0 \times U \times \mathbb{A}^N$. Let $r_0 = r|_{\mathcal{Y}_0} : \mathcal{Y}_0 \rightarrow \mathcal{X} - \mathcal{D}$.

Definition 7.8. Set $a_0 = (\mathcal{D}_0, \mathcal{Y}_0, q_1|_{\mathcal{Y}_0},...,q_N|_{\mathcal{Y}_0}; (p_X)|_{\mathcal{X} - \mathcal{D}} \circ r_0) \in F_{\mathcal{Y}_1}(U,X - D)$.

Definition 7.9. Set $r = \langle a_1 \rangle - \langle a_0 \rangle \in \mathbb{Z}F_{\mathcal{Y}_1}(U,X - D)$.

Claim 7.10. One has an equality $\langle \rho | r \rangle = \langle \text{can} | \rho| \mathcal{Y}_0 \rangle \in \mathbb{Z}F_{\mathcal{Y}_1}(U,X)$.

In fact, take an element $h_0 = (\mathcal{X}, \mathcal{Y}, q_1, ..., q_N; p_X \circ r) \in F_{\mathcal{Y}_1}(\mathbb{A}^1 \times U,X)$. By Lemma 7.1, the support of $h_0$ is the closed subset $\Delta \cap \mathcal{Y}_0$. Thus by Lemma 7.2, $h_0$ is the sum of two summands. Namely,

$$\langle h_0 \rangle = \rho | a_0 + \langle \Delta(U), \mathcal{Y}, q_1, ..., q_N; p_X \circ (r|_{\Delta(U)}) \rangle$$

in $\mathbb{Z}F_{\mathcal{Y}_1}(U,X)$. By Remark 7.4 and Theorem 12.1 for the second summand one has

$$\langle [\Delta(U), \mathcal{Y}, q_1, ..., q_N; p_X \circ (r|_{\Delta(U)})] \rangle = [p_X \circ (r|_{\Delta(U)}) \circ (s \circ \Delta)] \circ [\mathcal{Y}_0] = \langle \text{can} | \rho| \mathcal{Y}_0 \rangle$$

in $\mathbb{Z}F_{\mathcal{Y}_1}(U,X)$. Clearly, $h_1 = \rho | a_1 \in F_{\mathcal{Y}_1}(U,X)$. Thus one has a chain of equalities

$$\langle \rho | a_1 \rangle = [h_1] = [h_0] = \rho | a_0 + [\text{can} | \rho| \mathcal{Y}_0]$$

in $\mathbb{Z}F_{\mathcal{Y}_1}(U,X)$. Whence the Claim. Whence Theorem 7.11.
8. Preliminaries for the injective part of the étale excision

Let $S \subset X$ and $S' \subset X'$ be closed subsets. Let

\[
\begin{array}{ccc}
V' & \longrightarrow & X' \\
\downarrow & & \downarrow \pi \\
V & \longrightarrow & X
\end{array}
\]

be an elementary distinguished square with affine $k$-smooth $X$ and $X'$. Let $S = X - V$ and $S' = X' - V'$ be closed subschemes equipped with reduced structures. Let $x \in S$ and $x' \in S'$ be two points such that $\Pi(x') = x$. Let $U = \text{Spec}(O_{X,x})$ and $U' = \text{Spec}(O_{X',x'})$. Let $\pi : U' \rightarrow U$ be the morphism induced by $\Pi$.

To prove Theorem 12.3 it suffices to find morphisms $a \in ZF_N((U,U-S)), (X',X'-S'))$ and $b_G \in ZF_N((U,U-S)), (X-S,X-S))$ such that

\[
[[\Pi]] \circ [[a]] - [[j]] \circ [[b_G]] = [[\text{can}]] \circ [[a_G]]
\]  

(12)

in $ZF_N(U,U-S)), (X,X-S))$. Here $j : (X-S,X-S) \rightarrow (X,X-S)$ is the inclusion and $\text{can} : (U,U-S) \rightarrow (X,X-S)$ is the inclusion.

Let $in : X^0 \rightarrow X$ and $in' : (X')^0 \rightarrow X'$ be open such that

1. $x \in X^0$,
2. $x' \in (X')^0$,
3. $\Pi((X')^0) \subset X^0$,
4. the square

\[
\begin{array}{ccc}
V' \cap (X')^0 & \longrightarrow & (X')^0 \\
\downarrow \Pi|_{(X')^0} & & \downarrow \Pi|_{(X')^0} \\
V \cap X^0 & \longrightarrow & X^0
\end{array}
\]

is an elementary distinguished square.

Suppose morphisms $a^o \in ZF_N((U,U-S)), ((X')^0, (X')^0-S'))$, $b_G^o \in ZF_N((U,U-S)), (X^o-S, X^o-S))$ are such that for the inclusions $j^o : (X^o-S, X^o-S) \rightarrow (X^0, X^0-S)$ and $\text{can}_{X^o} : (U,U-S) \rightarrow (X^0, X^0-S)$ one has

\[
[[\Pi|_{(X')^0}]] \circ [[a^o]] - [[j^o]] \circ [[b_G^o]] = [[\text{can}_{X^o}]] \circ [[a^o]].
\]  

(13)

Then the morphisms $a = in' \circ a^o$ and $b_G = in \circ b_G^o$ satisfy property (12). Thus if we shrink $X$ and $X'$ in such a way that properties (1) – (4) are fulfilled and find appropriate morphisms $a^y$ and $b_G^y$, then we find $a$ and $b_G$ subjecting condition (12).

**Remark 8.1.** One way of shrinking $X$ and $X'$ such that properties (1) – (4) are fulfilled is as follows. Replace $X$ by an affine open $X^o$ containing $x$ and then replace $X'$ by $(X')^o = \Pi^{-1}(X^o)$.

Let $X_n'$ be the normalization of $X$ in $\text{Spec}(k(X'))$. Let $\Pi_n : X_n' \rightarrow X$ be the corresponding finite morphism. Since $X'$ is $k$-smooth it is an open subscheme of $X_n'$. Let $Y'' = X_n' - X'$. It is a closed subset in $X_n'$. Since $\Pi_n|_{Y'} : S' \rightarrow S$ is a scheme isomorphism, then $S'$ is closed in $X_n'$. Thus $S' \cap Y'' = 0$. Hence there is a function $f \in k[X_n']$ such that $f|_{Y''} = 0$ and $f|_{S'} = 1$.

**Definition 8.2.** Set $X_{\text{new}}' = (X_n') f$, $Y' = \{ f = 0 \} \subset X$. Note that $X_{\text{new}}'$ is an affine $k$-variety as a principal open subset of the affine $k$-variety $X_n'$. We regard $Y'$ as an effective Cartier divisor of $X_n'$. The subset $Y$ is closed in $X$, because $\Pi_n$ is finite. Set $\Pi_{\text{new}} = \Pi|_{X_{\text{new}}'}$. 

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Remark 8.3. We have that $\Pi_{\text{new}}^{-1}(S) = S'$. Therefore the varieties $X$ and $X'_{\text{new}}$ are subject to properties (1) – (4) of the present section. Below we will work with this $X'_{\text{new}}$. However we will write $X'$ for $X'_{\text{new}}$.

Lemma 8.4. Take $X$ and $X'$ as in Remark 8.3. Shrinking $X$ and $X'$ as described in Remark 8.1, one can find an almost elementary fibration $q : X \to B$ in the sense of [4] ($B$ is affine open in $\mathbb{P}^{p-1}$) such that $q|_{Y \cup S}$ is finite, $\omega_{B/k} \cong \mathcal{O}_B$, $\omega_{X/k} \cong \mathcal{O}_X$.

The shrunk scheme $X'$ will be regarded below as a $B$-scheme via the morphism $q \circ \Pi$.

Remark 8.5. If $q : X \to B$ is the almost elementary fibration from Lemma 8.4, then $\Omega_{X/B}^1 \cong \mathcal{O}_X$. In fact, $\omega_{X/k} \cong q^* (\omega_{B/k}) \otimes \omega_{X/B}$. Thus $\omega_{X/k} \cong \mathcal{O}_X$. Since $X/B$ is a smooth relative curve, then $\Omega_{X/B}^1 = \omega_{X/k} \cong \mathcal{O}_X$.

If, furthermore, $j : X \to B \times \mathbb{A}^N$ is a closed embedding of $B$-schemes, then in $K_0(X)$ one has $\mathcal{A}(j) = (N-1)[\mathcal{O}_X]$, where $\mathcal{A}$ is the normal bundle to $X$ for the imbedding $j$.

Thus increasing the integer $N$, we may assume that the normal bundle $\mathcal{A}$ is isomorphic to the trivial bundle $\mathcal{O}_X^{-1}$.

Proposition 8.6. Let $q : X \to B$ be the almost elementary fibration from Lemma 8.4. Then there are an integer $N \geq 0$, a closed embedding $X \to B \times \mathbb{A}^N$ of $B$-schemes, an étale affine neighborhood $(Y, \rho : Y \to B \times \mathbb{A}^N, s : X \to Y)$ of $X$ in $B \times \mathbb{A}^N$, functions $q_1, \ldots, q_{N-1} \in k[\mathcal{Y}]$ and a morphism $r : Y \to X$ such that:

(i) the functions $q_1, \ldots, q_{N-1}$ generate the ideal $I_{s(Y)}$ in $k[\mathcal{Y}]$ defining the closed subscheme $s(X)$ of $Y$;

(ii) $r \circ s = id_X$;

(iii) the morphism $r$ is a $B$-scheme morphism if $Y$ is regarded as a $B$-scheme via the morphism $p_Y \circ \rho$, and $X$ is regarded as a $B$-scheme via the morphism $q$.

Definition 8.7. Let $x \in S, x' \in S'$ be such that $\Pi(x') = x$. Set $U = \text{Spec}(\mathcal{O}_{X,x})$. There is an obvious morphism $\Delta = (\text{id,can}) : U \to U \times_B X$. It is a section of the projection $p_U : U \times_B X \to U$. Let $\pi : U \to U$ be the restriction of $\Pi$ to $U'$.

Notation 8.8. In what follows we will write $U \times X$ to denote $U \times_B X, U' \times X'$ to denote $U' \times_B X'$, etc. Here $X'$ is regarded as a $B$-scheme via the morphism $q \circ \Pi$.

Proposition 8.9. Under the conditions of Lemma 8.4 and Notation 8.8, there is a function $h_0 \in k[\mathbb{A}^1 \times U \times X] (\theta$ is the parameter on the left factor $\mathbb{A}^1$) such that the following properties hold for the functions $h_0, h_1 := h_0|_{1 \times U \times X}$ and $h_0 := h_0|_{0 \times U \times X}$:

(a) the morphism $(pr, h_0) : \mathbb{A}^1 \times U \times X \to \mathbb{A}^1 \times U \times \mathbb{A}^1$ is finite surjective, and hence the closed subscheme $Z_0 := h_0^{-1}(0) \subset \mathbb{A}^1 \times U \times X$ is finite flat and surjective over $\mathbb{A}^1 \times U$;

(b) for the closed subscheme $Z_0 := h_0^{-1}(0)$ one has $Z_0 = \Delta(U) \cup G$ (an equality of closed subschemes) and $G \subset U \times (X - S)$;

(c) the closed subscheme $(id_U \times \Pi)^* (h_1) = 0$ is a disjoint union of the form $Z_1 \sqcup Z_2$, and $m := (id_U \times \Pi)|_{Z_1}$ identifies $Z_1$ with the closed subscheme $Z_1 := \{h_1 = 0\}$;

(d) $Z_0 \cap \mathbb{A}^1 \times (U - S) \times X = \emptyset$ or, equivalently, $Z_0 \cap \mathbb{A}^1 \times (U - S) \times X \subset \mathbb{A}^1 \times (U - S) \times X$.

Remark 8.10. Item (d) yields the following inclusions: $Z_0 \cap \mathbb{A}^1 \times (U - S) \times X \subset \mathbb{A}^1 \times (U - S) \times (X - S), Z_0 \cap (U - S) \times X \subset (U - S) \times (X - S), and Z_1 \cap (U - S) \times X \subset (U - S) \times (X - S)$.

Applying item (c), we get another inclusion: $Z_1 \cap (U - S) \times X' \subset (U - S) \times (X' - S')$.
Remark 8.11. The class $[\mathcal{O}_{Z_0}]$ of the structure sheaf of the subscheme $Z_0$ defines a morphism in $\text{Kor}_0(\mathbb{A}^1 \times (U, U - S), (X, X - S))$ such that for $i = 0, 1$ one has $[\mathcal{O}_{Z_0}]|_{(i)} = [\mathcal{O}_Z]$. Moreover, by (b) one has $[\mathcal{O}_{Z_0}] = [\text{can}] + [j] \circ [\mathcal{O}_G]$ and $[\mathcal{O}_G] \in \text{Kor}_0((U, U - S), (X, X - S))$.

Thus

$$[\Pi] \circ [\mathcal{O}_{Z_0}] = [\mathcal{O}_Z] = [\text{can}] + [j] \circ [\mathcal{O}_G] \in \text{Kor}_0((U, U - S), (X, X - S)).$$

Below we lift these elements to the category $\mathcal{Z}_F(k)$ and equalities to the category $\mathcal{Z}_F^s(k)$.

9. Reducing Theorem 2.13 to Propositions 8.6 and 8.9

To construct a morphism $b \in Fr_N(U, X)$, we first construct its support in $U \times \mathbb{A}^N$ for an integer $N$, then we construct an étale neighborhood of the support in $U \times \mathbb{A}^N$, then one constructs a framing of the support in the neighborhood, and finally one constructs $b$ itself. In the same manner we construct a morphism $a \in Fr_N(U, X')$ and a homotopy $H \in Fr_N(\mathbb{A}^1 \times U, X)$ between $\Pi \circ a$ and $b$. Using the fact that the support $Z_0$ of $b$ is of the form $\Delta(U) \sqcup G$ with $G \subset U \times (X - S)$, we get an equality

$$\langle b \rangle = \langle b_1 \rangle + \langle b_2 \rangle$$

in $\mathcal{Z} Fr_N(U, X)$. Then we prove that $[b_1] = [\text{can}] \circ [\alpha^0_U]$ and $[b_2]$ factor through $X - S$. Moreover, we are able to work with morphisms of pairs. We will use systematically in this section the data from Proposition 8.6. The details are given below in this section.

Under the assumptions and notation of Proposition 8.6, Lemma 8.6 and Remark 8.3, set $\mathcal{Y}' = X' \times_B \mathcal{V}$. So we have a Cartesian square

$$
\begin{array}{ccc}
\mathcal{Y}' & \xrightarrow{\Pi'} & \mathcal{Y} \\
\downarrow & & \downarrow \\
X' & \xrightarrow{\Pi} & X,
\end{array}
$$

where $\Pi'$ and $\Pi$ are the projections to the first and second factors respectively. The section $s : X \to \mathcal{Y}'$ defines a section $s' = (id, s) : X' \to \mathcal{Y}'$ of $\rho'$. For brevity, we will write below $U \times \mathcal{Y}$ to denote $U \times_B \mathcal{Y}', U \times \mathcal{Y}'$ for $U \times_B \mathcal{Y}'$, and $id \times \rho$ for $id \times_B \rho : U \times_B \mathcal{Y}' \to U \times_B (B \times \mathbb{A}^n) = U \times \mathbb{A}^N$. Let $p_\mathcal{Y} : U \times \mathcal{Y} \to \mathcal{Y}$ be the projection.

Let $X \subset B \times \mathbb{A}^N$ be the closed inclusion from Proposition 8.6. Taking the base change of the latter inclusion by means of the morphism $U \to B$, we get a closed inclusion $U \times X \subset U \times \mathbb{A}^N$.

Under the notation from Proposition 8.6 and Proposition 8.9, construct now a morphism $b \in Fr_N(U, X)$. Let $Z_0 \subset U \times X$ be the closed subset from Proposition 8.9. Then one has the closed inclusions

$$\Delta(U) \sqcup G = Z_0 \subset U \times X \subset U \times \mathbb{A}^N.$$

Let $in_0 : Z_0 \subset U \times X$ be the closed inclusion. Define an étale neighborhood of $Z_0$ in $U \times \mathbb{A}^N$ as follows:

$$(U \times \mathcal{Y}, id \times \rho : U \times \mathcal{Y} \to U \times \mathbb{A}^N, (id \times s) \circ in_0 : Z_0 \to U \times \mathcal{Y}).$$

We will write $\Delta(U) \sqcup G = Z_0 \subset U \times \mathcal{Y}$ for $((id \times s) \circ in_0)(Z_0) \subset U \times \mathcal{Y}$. Let $f \in k[U \times \mathcal{Y}]$ be a function such that $f|_G = 1$ and $f|_{\Delta(U)} = 0$. Then $\Delta(U)$ is a closed subset of the affine scheme $(U \times \mathcal{Y})_f$.

Definition 9.1. Under the notation from Proposition 8.6 and Proposition 8.9 set

$$b' = (Z_0, U \times \mathcal{Y}, p_\mathcal{Y}^* (q_1), \ldots, p_\mathcal{Y}^* (q_{N - 1}), (id \times r)^* (h_0); pr_X \circ (id \times r)) \in Fr_N(U, X).$$
We will sometimes write below \((Z_0, U \times \mathcal{Y}, p^*_\mathcal{Y}(\varphi), (id \times r)^*(h_0); pr_X \circ (id \times r))\) to denote the morphism \(b'\).

To construct the desired morphism \(b \in Fr_N(U, X)\), we need to modify slightly the function \(p^*_\mathcal{Y}(q_1)\) in the framing of \(Z_0\). By Proposition 8.6 and item \((b)\) of Proposition 8.9, the functions

\[
p^*_\mathcal{Y}(q_1), \ldots, p^*_\mathcal{Y}(q_{N-1}), (id \times r)^*(h_0)
\]

generate an ideal \(I_{(id \times s)(\Delta(U))}\) in \(k[(U \times \mathcal{Y})_f]\) defining the closed subscheme \(\Delta(U)\) of the scheme \((U \times \mathcal{Y})_f\). Let \(t_1, t_2, \ldots, t_N \in k[U \times \mathbb{A}^N]\) be the coordinate functions. For any \(i = 1, 2, \ldots, N\), set \(t_i' = t_i - (t_i |_{\Delta(U)}) \in k[U \times \mathbb{A}^N]\). Then the family

\[(t'_1, t'_2, \ldots, t'_N) = (id \times p)^*(t_1), (id \times p)^*(t_2), \ldots, (id \times p)^*(t_N)\]

also generates the ideal \(I = I_{(id \times s)(\Delta(U))}\) in \(k[(U \times \mathcal{Y})_f]\). This holds since (14) is an étale neighborhood of \(Z_0\) in \(U \times \mathbb{A}^N\). By Remark 8.5 the \(k[U] = k[(id \times s)(\Delta(U))]\)-module \(I/I^2\) is free of rank \(N\). Thus the families \((\tilde{t}'_1, \tilde{t}'_2, \ldots, \tilde{t}'_N)\) and \((p_{\mathcal{Y}}(q_1), \ldots, p_{\mathcal{Y}}(q_{N-1}), (id \times r)^*(h_0))\) are two free bases of the \(k([(id \times s) \circ \Delta(U)])\)-module \(I/I^2\). Let \(J \in k[U]^*\) be the Jacobian of a unique matrix \(A \in M_N(k[U])\) which transforms the first free basis to the second one. Set,

\[
q_{1}^{new} = q_{U}^*(J^{-1})q_1 \in k[\mathcal{Y}],
\]

where \(q_U = pr_U \circ (id \times p) : \mathcal{Y} \to U\). Let \(A^{new} \in M_N(k[U])\) be a unique matrix changing the first free basis to the basis

\[(p_{\mathcal{Y}}(q_{1}^{new}), \ldots, p_{\mathcal{Y}}(q_{N-1})), (id \times r)^*(h_0)).\]

Then the Jacobian \(J^{new}\) of \(A^{new}\) is equal to 1:

\[
J^{new} = 1 \in k[U].
\]

We will write

\[(\psi_1, \psi_2, \ldots, \psi_{N-1}) \text{ for } (p_{\mathcal{Y}}(q_{1}^{new}), \ldots, p_{\mathcal{Y}}(q_{N-1})).\]

**Definition 9.2.** Under the notation from Proposition 8.6 and Proposition 8.9 set

\[b = (Z_0, U \times \mathcal{Y}, \psi_1, \ldots, \psi_{N-1}, (id \times r)^*(h_0); pr_X \circ (id \times r)) \in Fr_N(U, X).\]

For brevity, we will sometimes write

\[b = (Z_0, U \times \mathcal{Y}, p^*_\mathcal{Y}(\psi), (id \times r)^*(h_0); pr_X \circ (id \times r)).\]

Under the notation from Proposition 8.6 and Proposition 8.9 construct now a morphism \(a \in Fr_N(U, X)\). Let \(Z_1 \subset U \times X\) be the closed subset from Proposition 8.9. Then one has closed inclusions

\[Z_1 \subset U \times X \subset U \times \mathbb{A}^N.\]

Set \((U \times \mathcal{Y})_o = (U \times X) - Z'_2\) and \((U \times \mathcal{Y})_o = (id \times r)^{-1}((U \times X)_o)\). Let \(in_1 : Z_1 \subset U \times X\) and \(in'_1 : Z'_1 \subset (U \times X)_o\) be closed inclusions. Set,

\[r_o = (id \times r)|_{(U \times \mathcal{Y})_o} : (U \times \mathcal{Y})_o \to (U \times \mathcal{Y})_o.\]

Using the notation of Proposition 8.6 define an étale neighborhood of \(Z_1\) in \(U \times \mathbb{A}^N\) as follows:

\[((U \times \mathcal{Y})_o, (id \times p) \circ (id \times \Pi') : (U \times \mathcal{Y})_o \to U \times \mathbb{A}^N, (id \times s') \circ in'_1 \circ m^{-1} : Z_1 \to (U \times \mathcal{Y})_o)\].

(16)
Definition 9.3. Under the notation of Proposition 8.6 and Proposition 8.9 set,
\[ a = (Z_1, (U \times \mathcal{V})^\circ, (id \times \Pi')^*(\psi_{1}), \ldots, (id \times \Pi')^*(\psi_{N-1}), r_0^*(id \times \Pi)^*(h_1); pr_{\mathcal{X}} \circ r_\circ) \in Fr_N(U, X'). \]
For brevity, we will sometimes write
\[ a = (Z_1, (U \times \mathcal{V})^\circ, (id \times \Pi')^*(\psi), r_0^*(id \times \Pi)^*(h_1); pr_{\mathcal{X}} \circ r_\circ). \]

Under the notation of Proposition 8.6 and Proposition 8.9 let us construct now a morphism \( H_0 \in Fr_N(\mathcal{A} \times U, X) \). Let \( Z_0 \subset \mathcal{A} \times U \times X \) be the closed subset from Proposition 8.9. Then one has closed inclusions
\[ Z_0 \subset A^1 \times U \times X \subset \mathcal{A} \times U \times X^N. \]
Let \( in_\circ : Z_0 \subset A^1 \times U \times X \) be the closed inclusion. Define an étale neighborhood of \( Z_0 \) in \( A^1 \times U \times X^N \) as follows:
\[ (A^1 \times U \times \mathcal{V}, id \times id \times \rho : A^1 \times U \times \mathcal{V} \rightarrow A^1 \times U \times X^N, (id \times id \times \circ) \circ in_\circ : Z_0 \rightarrow A^1 \times U \times \mathcal{V}). \]

Definition 9.4. Under the notation of Propositions 8.6 and 8.9 we set
\[ H_0 = (Z_0, \mathcal{A} \times U \times \mathcal{V}, \psi_1, \ldots, \psi_{N-1}, (id \times id \times \circ)(h_\circ); pr_{\mathcal{X}} \circ (id \times id \times \circ)) \in Fr_N(\mathcal{A} \times U, X). \]
We will sometimes write below \( (Z_0, \mathcal{A} \times U \times \mathcal{V}, \psi, (id \times id \times \circ)^*(h_\circ); pr_{\mathcal{X}} \circ (id \times id \times \circ)) \) to denote the morphism \( H_0 \).

Lemma 9.5. One has equalities \( H_0 = b, H_1 = \Pi \circ a \) in \( Fr_N(U, X) \).

Proof. The first equality is obvious. To check the second one, consider
\[ H_1 = (Z_1, U \times \mathcal{V}, \psi, (id \times \circ)^*(h_1); pr_{\mathcal{X}} \circ (id \times \circ)) \in Fr_N(U, X). \]
Here we use \( (U \times \mathcal{V}, id \times \rho : U \times \mathcal{V} \rightarrow U \times \mathcal{A}^N, (id \times s) \circ in_1 : Z_1 \rightarrow U \times \mathcal{V}) \) as an étale neighborhood of \( Z_1 \) in \( U \times \mathcal{A}^N \). Take another étale neighborhood of \( Z_1 \) in \( U \times \mathcal{A}^N \)
\[ ((U \times \mathcal{V})^\circ, (id \times \rho) \circ (id \times \Pi') : (U \times \mathcal{V})^\circ \rightarrow U \times \mathcal{A}^N, (id \times s') \circ in_1 \circ m^{-1} : Z_1 \rightarrow (U \times \mathcal{V})^\circ \)
and the morphism \( id \times \Pi' : (U \times \mathcal{V})^\circ \rightarrow U \times \mathcal{V} \) regarded as a morphism of étale neighborhoods.
Refining the étale neighborhood of \( Z_1 \) in the definition of \( H_1 \) by means of that morphism, we get a \( N \)-frame \( H'_1 = H_1 \), which has the form
\[ (Z_1, (U \times \mathcal{V})^\circ, (id \times \Pi')^*(\psi), (id \times \Pi')^*(id \times \circ)^*(h_1); pr_{\mathcal{X}} \circ (id \times \circ) \circ (id \times \Pi')). \]
Note that
\[ (id \times \Pi')^*(id \times \circ)^*(h_1) = r_0^*(id \times \Pi)^*(h_1) \] and \( pr_{\mathcal{X}} \circ (id \times \circ) \circ (id \times \Pi') = \Pi \circ pr_{\mathcal{X}} \circ r_\circ. \)
Thus, \( H_1 = H'_1 = \Pi \circ a \) in \( Fr_N(U, X) \). \( \square \)

The following lemma follows from Lemma 3.4 and Remark 8.10.

Lemma 9.6. The morphisms \( a|_{U-S}, b|_{U-S}, H_0|_{\mathcal{A} \times (U-S)} \) and \( \Pi|_{X'-S} \) run inside \( X' - S', X - S, X - S \) and \( X - S \) respectively.

By the preceding lemma the morphisms \( a, b, H_0 \) and \( \Pi \) define morphisms
\[ \langle \langle a \rangle \rangle \in ZF_N((U, U-S), (X', X'-S')), \langle \langle b \rangle \rangle \in ZF_N((U, U-S), (X, X-S)), \langle \langle H_0 \rangle \rangle \in ZF_N(\mathcal{A} \times (U, U-S), (X, X-S)), \langle \langle \Pi \rangle \rangle \in ZF_N((X', X'-S'), (X, X-S)). \]
(see Definition 3.3. Lemma 9.5 and Definition 3.3 yield equalities
\[ \langle \langle \Pi \rangle \rangle \circ \langle \langle a \rangle \rangle = \langle \langle H_1 \rangle \rangle \] and \( \langle \langle H_0 \rangle \rangle = \langle \langle b \rangle \rangle \)
in \(ZF_N((U,U-S),(X,X-S))\).

**Corollary 9.7.** One has an equality \([\Pi] \circ [a] = [b] \) in \(\overline{ZF}_N((U,U-S),(X,X-S))\).

**Proof of Corollary 9.7.** In fact, by Corollary 3.5 one has a chain of equalities
\[
[[\Pi]] \circ [[a]] = [[H_1]] = [[H_0]] = [[b]]
\]
in \(\overline{ZF}_N((U,U-S),(X,X-S))\). \(\square\)

**Reducing Theorem 2.13 to Propositions 8.6 and 8.9.** The support \(Z_0\) of \(b\) is the disjoint union \(\Delta(U) \sqcup G\). Thus, by Lemma 3.6 one has an equality
\[
\langle \langle b \rangle \rangle = \langle \langle b_1 \rangle \rangle + \langle \langle b_2 \rangle \rangle
\]
in \(ZF_N((U,U-S),(X,X-S))\), where
\[
\begin{align*}
b_1 &= (\Delta(U),(U \times \mathcal{F}), \psi_1, ..., \psi_{N-1}, (id \times r)^*(h_0); pr_X \circ (id \times r)), \\
b_2 &= (G,(U \times \mathcal{F} - \Delta(U), \psi_1, ..., \psi_{N-1}, (id \times r)^*(h_0); pr_X \circ (id \times r)).
\end{align*}
\]
By Proposition 8.9 one has \(G \subset U \times (X-S)\). Thus \(b_2 = j \circ b_G\) for an obvious morphism \(b_G \in Fr_N(U,X-S)\). Also,
\[
\langle \langle b_2 \rangle \rangle = \langle \langle j \rangle \rangle \circ \langle \langle b_G \rangle \rangle \in ZF_N((U,U-S),(X,X-S)),
\]
where \(j: (X-S,X-S) \hookrightarrow (X,X-S)\) is a natural inclusion. By the latter comments and Corollary 9.7 one gets an equality
\[
[[\Pi]] \circ [[a]] - [[j]] \circ [[b_G]] = [[b_1]]
\]
in \(\overline{ZF}_N((U,U-S),(X,X-S))\). To prove equality (12), and hence to prove Theorem 2.13, it remains to check that \([[b_1]] = [[can]] \circ [[\sigma_N^0]]\). Recall that one has equality (13). Thus the relation \([[b_1]] = [[can]] \circ [[\sigma_N^0]]\) holds by Theorem 12.1. This finishes the proof of Theorem 2.13. \(\square\)

### 10. Preliminaries for the surjective part of the étale excision

Let \(S \subset X\) and \(S' \subset X'\) be closed subsets. Let
\[
\begin{array}{ccc}
V' & \longrightarrow & X' \\
\downarrow & & \downarrow \pi \\
V & \longrightarrow & X
\end{array}
\]
be an elementary distinguished square with affine \(k\)-smooth \(X\) and \(X'\). Let \(S = X - V\) and \(S' = X' - V'\) be closed subschemes equipped with reduced structures. Let \(x \in S\) and \(x' \in S'\) be two points such that \(\Pi(x') = x\). Let \(U = \text{Spec}(O_{X,x})\) and \(U' = \text{Spec}(O_{X',x'})\). Let \(\pi: U' \to U\) be the morphism induced by \(\Pi\).

To prove Theorem 2.14 it suffices to find morphisms \(a \in \mathbb{Z}F_N((U,U-S),(X',X'-S'))\) and \(b_G \in \mathbb{Z}F_N((U',U'-S'),(X'-S',X'-S'))\) such that
\[
[[a]] \circ [[\pi]] - [[j]] \circ [[b_G]] = [[\text{can}']] \circ [[\sigma_N^0]]
\]
in \(\overline{ZF}_N((U',U'-S'),(X',X'-S'))\). Here \(j: (X'-S',X'-S') \hookrightarrow (X',X'-S')\) and \(\text{can}' : (U',U'-S') \to (X',X'-S')\) are inclusions.

Replace \(X\) by an affine open neighborhood \(in: X^o \hookrightarrow X\) of the point \(x\). Replace \(X'\) by \((X')^o := \Pi^{-1}(X^o)\) and write \(in' : (X')^o \hookrightarrow X'\) for the inclusion. Replace \(V\) by \(V \cap X^o\) and \(V'\) with \(V' \cap \]
(X')^\circ$. Let $\text{can}' : U' \to (X')^\circ$ be the canonical inclusion. Let $j^\circ : ((X')^\circ - S', (X')^\circ - S') \to ((X')^\circ, (X')^\circ - S')$ be an inclusion of pairs. If we find $a^\circ \in \mathcal{Z}F_N((U, U - S)), ((X')^\circ, (X')^\circ - S'))$ and $b_G^\circ \in \mathcal{Z}F_N((U', U' - S'), ((X')^\circ - S', (X')^\circ - S'))$ such that

$$\left[\{a^\circ\} \circ \{\pi\} - \{j^\circ\} \circ \{b_G^\circ\} = \{\text{can}'_\circ\} \circ \{\sigma_N^\circ\}\right],$$

then the morphisms $a = \text{id}' \circ a^\circ$ and $b_G = \text{id}' \circ b_G^\circ$ satisfy condition (18).

Let $X_n'$ be the normalization of $X$ in $\text{Spec}(k(X'))$. Let $\Pi_n : X_n' \to X$ be the corresponding finite morphism. Since $X'$ is $k$-smooth it is an open subscheme of $X_n'$. Let $Y'' = X_n' - X'$. It is a closed subset in $X_n'$. Since $\Pi_n|_S : S' \to S$ is a scheme isomorphism, then $S'$ is closed in $X_n'$. Thus $S' \cap Y'' = 0$. Hence there is a function $f \in k(X_n')$ such that $f|_{Y''} = 0$ and $f|_Y = 1$.

In this section we use agreements and notation from Definition 8.2 and Remark 8.3.

**Proposition 10.1.** Let $q : X \to B$ is the almost elementary fibration from Lemma 8.4 and let $X' = X'_{\text{new}}$ be as in the Remark 8.3. Then there are an integer $N \geq 0$, a closed embedding $j : X' \hookrightarrow B \times \mathbb{A}^N$ of $B$-schemes, an étale affine neighborhood $(\mathcal{V}'', \rho'' : \mathcal{V}'' \to B \times \mathbb{A}^N, \mathcal{S}' : X' \hookrightarrow \mathcal{V}'')$ of $X'$ in $B \times \mathbb{A}^N$, functions $q_1', \ldots, q_{N-1}' \in k[\mathcal{V}'']$ and a morphism $r'' : \mathcal{V}'' \to X'$ such that

(i) the functions $q_1', \ldots, q_{N-1}'$ generate the ideal $I_{X'(\mathcal{V}'')} \subset k[\mathcal{V}'']$ defining the closed subscheme $s'(\mathcal{V}')$ of $X''$;
(ii) $r'' \circ s' = \text{id}_{\mathcal{V}''}$;
(iii) the morphism $r''$ is a $B$-scheme morphism if $\mathcal{V}''$ is regarded as a $B$-scheme via the morphism $\rho'' : \mathcal{V}'' \to X'$ and $X'$ is regarded as a $B$-scheme via the morphism $q \circ \Pi$ from Lemma 8.2.

**Remark 10.2.** If $q : X \to B$ is the almost elementary fibration from Lemma 8.4 then $\Omega^1_{X'/B} \cong \mathcal{O}_X$. In fact, by Remark 8.3, $\Omega^1_{X/B} = \omega_{X/B} \cong \mathcal{O}_X$. The morphism $\Pi : X' \to X$ is étale. Thus $\Omega^1_{X'/B} \cong \mathcal{O}_{X'}$.

If, furthermore, $j' : X' \hookrightarrow B \times \mathbb{A}^N$ is a closed embedding of $B$-schemes, then one has $[\mathcal{N}(j')] = (N - 1)[\mathcal{O}_X]$ in $K_0(X)$, where $\mathcal{N}(j')$ is the normal bundle to $X'$ associated with the imbedding $j$.

Thus increasing the integer $N$, we may assume that the normal bundle $\mathcal{N}(j')$ is isomorphic to the trivial bundle $\mathcal{O}_{\mathcal{X}'}^{\mathcal{N} - 1}$.

**Definition 10.3.** Let $x \in S, x' \in S'$ be such that $\Pi(x') = x$. We put $U = \text{Spec}(\mathcal{O}_{X,x})$. There is an obvious morphism $\Delta' = (\text{id}, \text{can}) : U' \to U' \times_B X'$. It is a section of the projection $p_{X'} : U' \times_B X' \to U'$. Let $p_X : U' \times_B X' \to X'$ be the projection onto $X'$. Let $\pi : U' \to U$ be the restriction of $\Pi$ to $U'$.

**Notation 10.4.** We regard $X$ as a $B$-scheme via the morphism $q$ and regard $X'$ as a $B$-scheme via the morphism $q \circ \Pi$. In what follows we write $U \times X'$ for $U \times_B X', U' \times X'$ for $U' \times_B X'$, etc.

**Proposition 10.5.** Under the conditions of Lemma 8.4 and Notation 10.4 there are functions $F \in k[U \times X']$ and $h_0' \in k[\mathbb{A}^1 \times U' \times X']$ (the parameter is on the left factor $\mathbb{A}^1$) such that the following properties hold for the functions $h_0', h_1' := h_0'|_{U \times X'}$ and $h_0'^\circ := h_0'|_{U \times X'}$:

(a) the morphism $(\text{pr} h_0') : \mathbb{A}^1 \times U' \times X' \to \mathbb{A}^1 \times U' \times \mathbb{A}^1$ is finite and surjective, hence the closed subscheme $Z_0' := (h_0'^\circ)^{-1}(0) \subset \mathbb{A}^1 \times U' \times X'$ is finite flat and surjective over $\mathbb{A}^1 \times U'$;
(b) for the closed subscheme $Z_0' := (h_0'^\circ)^{-1}(0)$ one has $Z_0' = \Delta'(U') \cap G'$ (an equality of closed subschemes) and $G' \subset U' \times (X' - S')$.
(c) \( h'_i = (\pi \times id)^*(F) \) (we write \( Z'_1 \) to denote the closed subscheme \( \{ h'_1 = 0 \} \));

(d) \( Z'_0 \cap \mathbb{A}^1 \times (U' - S') \times S' = \emptyset \) or, equivalently, \( Z'_0 \cap \mathbb{A}^1 \times (U' - S') \times X' \subset \mathbb{A}^1 \times (U' - S') \times (X' - S') \);

(e) the morphism \( (pr_U, F) : U \times X' \to U \times \mathbb{A}^1 \) is finite surjective, and hence the closed subscheme \( Z_1 := F^{-1}(0) \subset U \times X' \) is finite flat and surjective over \( U \);

(f) \( Z_1 \cap (U - S) \times S' = \emptyset \) or, equivalently, \( Z_1 \cap (U - S) \times X' \subset (U - S) \times (X' - S') \).

**Remark 10.6.** Item (d) yields the following inclusions:

\[
\begin{align*}
Z'_0 & \cap (U' - S') \times X' \subset \mathbb{A}^1 \times (U' - S') \times (X' - S'), \\
Z'_0 & \cap (U' - S') \times X' \subset (U' - S') \times (X' - S'), \\
Z'_1 & \cap (U' - S') \times X' \subset (U' - S') \times (X' - S').
\end{align*}
\]

Applying (f) we get another inclusion: \( Z_1 \cap (U - S) \times X' \subset (U - S) \times (X' - S') \).

**Remark 10.7.** The class \([\mathcal{O}_{Z'_0}]\) of the structure sheaf of the subscheme \( Z'_0 \) defines a morphism in \( \text{Kor}_0(\mathbb{A}^1 \times (U', U' - S'), (X', X' - S')) \) such that for \( i = 0, 1 \) one has \([\mathcal{O}_{Z'_0}]|_{(U', U' - S')} = [\mathcal{O}_{Z_1}]\).

Moreover, by (b) one has
\[
[\mathcal{O}_{Z'_0}] = [\text{can}'] + [j] \circ [\mathcal{O}_{G'}]
\]

and \([\mathcal{O}_{G'}] \in \text{Kor}_0((U', U' - S'), (X' - S', X' - S'))\). Thus,
\[
[\mathcal{O}_{Z_1}] \circ \pi = [\mathcal{O}_{Z'_1}] = [\mathcal{O}_{Z'_0}] = [\text{can}'] + [j] \circ [\mathcal{O}_{G'}] \in \text{Kor}_0((U', U' - S'), (X', X' - S')).
\]

Below we will lift these elements to the category \( \mathbb{Z}F_\ast(k) \) and relations to the category \( \mathbb{Z}\mathcal{F}_\ast(k) \).

11. Reducing Theorem 2.14 to Propositions 10.1 and 10.5

We suppose in this section that \( S \subset X \) is \( k \)-smooth. To construct a morphism \( a \in Fr_N(U, X') \), we first construct its support in \( U \times \mathbb{A}^N \) for an integer \( N \), then we construct an étale neighborhood of the support in \( U \times \mathbb{A}^N \), then one constructs a framing of the support in the neighborhood and finally one constructs \( a \) itself. In the same fashion we construct a morphism \( b \in Fr_N(U', X') \) and a homotopy \( H \in Fr_N(\mathbb{A}^1 \times U', X') \) between \( a \circ \pi \) and \( b \). Using the fact that the support \( Z'_0 \) of \( b \) is of the form \( \Delta'(U') \cup G' \) with \( G' \subset U' \times (X' - S') \), we get a relation
\[
\langle b \rangle = \langle b_1 \rangle + \langle b_2 \rangle
\]

in \( \mathbb{Z}F_N(U', X') \). Then we prove that \( b_1 = [\text{can}'] \circ [\sigma_N]^p \) and \( [b_2] \) factors through \( X' - S' \). Moreover, we are able to work with morphisms of pairs. In this section we will use systematically the data from Propositions [10.1] and [10.5] and Notation [10.4]. Details are given below in this section.

Let \( X' \subset B \times \mathbb{A}^N \) be the closed inclusion from Proposition [10.1]. Taking the base change of the latter inclusion by means of the morphism \( U \to B \), we get a closed inclusion \( U \times X' \subset U \times_B (B \times \mathbb{A}^N) = U \times \mathbb{A}^N \).

Under the notation from Proposition [10.1] and Proposition [10.5], construct now a morphism \( b \in Fr_N(U', X') \). Let \( Z'_0 \subset U' \times X' \) be the closed subset from Proposition [10.5]. Then one has closed inclusions
\[
\Delta'(U') \cup G' = Z'_0 \subset U' \times X' \subset U' \times \mathbb{A}^N.
\]

Let \( \text{in}_0 : Z'_0 \subset U' \times X' \) be a closed inclusion. Define an étale neighborhood of \( Z'_0 \) in \( U' \times \mathbb{A}^N \) as follows:
\[
(U \times \mathbb{A}^n, \text{id} \times \rho^n : U' \times \mathbb{A}^n \to U' \times \mathbb{A}^N, (\text{id} \times s) \circ \text{in}_0 : Z'_0 \to U \times \mathbb{A}').
\]

(20)
We will write $\Delta'(U') \cup G' = Z'_0 \subset U' \times \mathcal{Y}''$ for $((id \times s'') \circ im_0)(Z'_0) \subset U' \times \mathcal{Y}''$. Let $f \in k[U \times \mathcal{Y}'']$ be a function such that $f|_{G'} = 1$ and $f|_{\Delta'(U')} = 0$. Then $\Delta'(U')$ is a closed subset of the affine scheme $(U' \times \mathcal{Y}'')_f$.

**Definition 11.1.** Under the notation from Proposition 5.6 and Proposition 5.9, set $b' = (Z'_0, U' \times \mathcal{Y}'', (\pi \times id)^*(p^*_{\mathcal{Y}'}(q'_1), \ldots, p^*_{\mathcal{Y}'}(q'_{N-1})), (id \times r'')^*(h'_0); pr_X \circ (id \times r'')) \in Fr_N(U', X')$.

Here $p^*_{\mathcal{Y}'} : U \times \mathcal{Y}'' \to \mathcal{Y}''$ is the projection. We will sometimes write below $(Z'_0, U' \times \mathcal{Y}'', (\pi \times id)^*(p^*_{\mathcal{Y}'}(q'_1)), (id \times r'')^*(h'_0); pr_X \circ (id \times r''))$ to denote the morphism $b'$.

To construct the desired morphism $b \in Fr_N(U', X')$, we need to modify a bit the function $p^*_{\mathcal{Y}'}(q'_1)$ in the framing of $Z'_0$. By Proposition 10.1 and item (b) of Proposition 10.5, the functions

$$((\pi \times id)^*(p^*_{\mathcal{Y}'}(q'_1)), \ldots, (\pi \times id)^*(p^*_{\mathcal{Y}'}(q'_{N-1})), (id \times r'')^*(h'_0))$$

generate an ideal $I_{(id \times s'')(\Delta'(U'))}$ in $k[(U' \times \mathcal{Y}'')_f]$ defining the closed subscheme $\Delta'(U')$ of the scheme $(U' \times \mathcal{Y}'')_f$. Let $t_1, t_2, \ldots, t_N \in k[U' \times \mathbb{A}^N]$ be the coordinate functions. For any $i = 1, 2, \ldots, N$, set $t'_i = t_i - (t_1 \Delta'(U')) \in k[U' \times \mathbb{A}^N]$. Then the family

$$(t'_1, t'_2, \ldots, t'_N) = (id \times \rho''^*)^*(t_1^*, (id \times \rho''^*)^*(t_2^*), \ldots, (id \times \rho''^*)^*(t_N^*))$$

also generates the ideal $I = I_{(id \times s'')(\Delta'(U'))}$ in $k[(U' \times \mathcal{Y}'')_f]$. This holds since (20) is an étale neighborhood of $Z'_0$ in $U \times \mathbb{A}^N$. By Remark 10.2, the $k[U'] = k[(id \times s'')(\Delta'(U'))]$-module $I/I^2$ is free of rank $N$. Thus the families

$$(\tilde{t}_1', \tilde{t}_2', \ldots, \tilde{t}_N')$$

are two free bases of the $k[((id \times s'')(\Delta'(U'))]-module $I/I^2$. Let $J \in k[U']$ be the Jacobian of a unique matrix $A \in M_N(k[U'])$ changing the first free basis to the second one. There is an element $\lambda \in k[U]$ such that $\lambda |_{S \cap U'} = J |_{S \cap U'}$ (we identify here $S' \cap U'$ with $S \cap U$ via the morphism $\pi |_{S \cap U'}$). Clearly, $\lambda \in k[U']$.

Set

$$(q'_1)^{new} = q^*_U(J^{-1})(q'_1) \in k[\mathcal{Y}''],$$

where $q_U = pr_U \circ (id \times \rho'') : \mathcal{Y}'' \to U$. Let $A^{new} \in M_N(k[U])$ be a unique matrix which transforms the first free basis to the basis

$$(p^*_{\mathcal{Y}'}((q'_1)^{new}), \ldots, p^*_{\mathcal{Y}'}((q'_{N-1})^{new}), (id \times r'')^*(h'_0)).$$

Then the Jacobian $J^{new}$ of $A^{new}$ has the property:

$$J^{new}|_{S \cap U'} = 1 \in k[S' \cap U'].$$

We will write

$$(\psi_1, \psi_2, \ldots, \psi_{N-1}) \text{ for (} p^*_{\mathcal{Y}'}((q'_1)^{new}), \ldots, p^*_{\mathcal{Y}'}((q'_{N-1})^{new}) \text{)}.$$

**Definition 11.2.** Under the notation from Proposition 7.4 and Proposition 10.5, set $b = (Z'_0, U' \times \mathcal{Y}'', (\pi \times id)^*(\psi_1), \ldots, (\pi \times id)^*(\psi_{N-1}), (id \times r'')^*(h'_0); pr_X \circ (id \times r'')) \in Fr_N(U', X')$.

We will often write for brevity

$$b = (Z'_0, U' \times \mathcal{Y}'', (\pi \times id)^*(\psi), (id \times r'')^*(h'_0); pr_X \circ (id \times r'')).$$
Under the notation from Proposition [10.1] and Proposition [10.5] construct now a morphism \( a \in Fr_N(U, X') \). Let \( Z_1 \subset U \times X' \) be the closed subset from Proposition [10.5]. Then one has closed inclusions

\[
Z_1 \subset U \times X' \subset U \times \mathbb{A}^N.
\]

Let \( in_1 : Z_1 \subset U \times X \) be the closed inclusion. Define an étale neighborhood of \( Z_1 \) in \( U \times \mathbb{A}^N \) as follows:

\[
(U \times \mathbb{A}^n, id \times \rho^n : U \times \mathbb{A}^n \to U \times \mathbb{A}^N, (id \times s) \circ in_1 : Z_1 \subset U \times \mathbb{A}^n).
\]  \hfill (22)

**Definition 11.3.** Under the notation from Proposition [10.1] and Proposition [10.5], set

\[
a = (Z_1, U \times \mathbb{A}^n, \psi_1, ..., \psi_{N-1}, (id \times r^n)^*(F); pr_{X'} \circ (id \times r^n)) \in Fr_N(U, X').
\]

We will sometimes write \((Z_1, U \times \mathbb{A}^n, \psi, (id \times r^n)^*(F); pr_{X'} \circ (id \times r^n))\) to denote \(a\).

Under the notation from Proposition [10.1] and Proposition [10.5], construct now a morphism \( H_0 \in Fr_N(\mathbb{A}^1 \times U', X') \). Let \( Z'_0 \subset \mathbb{A}^1 \times U' \times X' \) be the closed subset from Proposition [10.5]. Then one has closed inclusions

\[
Z'_0 \subset \mathbb{A}^1 \times U' \times X' \subset \mathbb{A}^1 \times U' \times \mathbb{A}^N.
\]

Let \( in_0 : Z'_0 \subset \mathbb{A}^1 \times U' \times X' \) be the closed inclusion. Define an étale neighborhood of \( Z'_0 \) in \( \mathbb{A}^1 \times U' \times \mathbb{A}^N \) as follows:

\[
(\mathbb{A}^1 \times U' \times \mathbb{A}^n, id \times id \times \rho^n : \mathbb{A}^1 \times U' \times \mathbb{A}^n \to \mathbb{A}^1 \times U' \times \mathbb{A}^N, (id \times id \times s^n) \circ in_0 : Z'_0 \subset \mathbb{A}^1 \times U' \times \mathbb{A}^n).
\]  \hfill (23)

**Definition 11.4.** Under the notation from Proposition [10.1] and Proposition [10.5], set \( H_0 \) to be equal to

\[
(Z'_0, \mathbb{A}^1 \times U' \times \mathbb{A}^n, pr^*((\pi \times id)^*(\psi)), (id \times id \times r^n)^*(h_0'); pr_{X'} \circ (id \times id \times r^n)) \in Fr_N(\mathbb{A}^1 \times U', X'),
\]

where \( pr : \mathbb{A}^1 \times U' \times \mathbb{A}^n \to U' \times \mathbb{A}^n \) is the projection.

**Lemma 11.5.** One has equalities \( H_0 = b, H_1 = a \circ \pi \in Fr_N(U', X') \).

**Proof.** The first equality is obvious. Let us prove the second one. By Proposition [10.5] one has \( h'_1 = (\pi \times id)^*(F) \). Thus one has a chain of equalities in \( Fr_N(U', X') \):

\[
a \circ \pi = (Z_1, U \times \mathbb{A}^n, (\pi \times id)^*(\psi), (id \times id)^*((id \times r^n)^*(F)); pr_{X'} \circ (id \times id \times \rho^n) \circ (\pi \times id)) = (Z_1, U \times \mathbb{A}^n, (\pi \times id)^*(\psi), (id \times r^n)^*((\pi \times id)^*(F)); pr_{X'} \circ (\pi \times id) \circ (id \times r^n)) = (Z_1, U \times \mathbb{A}^n, (\pi \times id)^*(\psi), (id \times r^n)^*(h'_1)); pr_{X'} \circ (id \times r^n)) = H_1.
\]

\( \square \)

The following lemma follows from Lemma [3.4] and Remark [10.6].

**Lemma 11.6.** The morphisms \( a|_{U-S}, b|_{U'-S'}, H_0|_{\mathbb{A}^1 \times (U'-S')} \) and \( \pi|_{U'-S'} \) run inside \( X' - S', X' - S', X' - S' \) and \( U - S \) respectively.

By the preceding lemma the morphisms \( a, b, H_0 \) and \( \pi \) define morphisms

\[
\langle (a) \rangle \in ZF_N((U, U - S), (X', X' - S')), \langle (b) \rangle \in ZF_N((U', U' - S'), (X', X' - S')),
\]

\[
\langle (H_0) \rangle \in ZF_N(\mathbb{A}^1 \times (U', U' - S'), (X', X' - S')), \langle (\pi) \rangle \in ZF_N((U', U' - S'), (U, U - S)).
\]

(see Definition [3.3]. Lemma [11.5] and Definition [3.3] yield relations

\[
\langle (a) \rangle \circ \langle (\pi) \rangle = \langle (H_1) \rangle \quad \text{and} \quad \langle (H_0) \rangle = \langle (b) \rangle
\]

in \( ZF_N((U', U' - S'), (X', X' - S')) \)).

\[24\]
Corollary 11.7. There is a relation $[[a]] \circ [[\pi]] = [[b]]$ in $\mathcal{ZF}_N((U', U' - S'), (X', X' - S'))$.

Proof of Corollary 11.7. In fact, by Corollary 3.5 one has a chain of equalities

$$[[a]] \circ [[\pi]] = [[H_1]] = [[H_0]] = [[b]]$$

in $\mathcal{ZF}_N((U', U' - S'), (X', X' - S'))$. □

Reducing Theorem 2.14 to Propositions 10.1 and 10.5. The support $Z_0$ of $b$ is the disjoint union $\Delta'(U') \sqcup G'$. Thus, by Lemma 5.6 one has,

$$\langle \langle b \rangle \rangle = \langle \langle \langle b_1 \rangle \rangle + \langle \langle b_2 \rangle \rangle \rangle$$

in $\mathcal{ZF}_N((U', U' - S'), (X', X' - S'))$, where

$$b_1 = (\Delta'(U'), (U' \times \mathbb{V}^n)^{\times}, \psi_1, \ldots, \psi_{N-1}, (id \times r')^*) (h_0^\pi); \quad pr_X \circ (id \times r')$$

$$b_2 = (G', (U' \times \mathbb{V}^n - \Delta'(U'), \psi_1, \ldots, \psi_{N-1}, (id \times r')^*) (h_0^\pi); \quad pr_X \circ (id \times r')$$

By Proposition 10.5 one has $G' \subset U' \times (X' - S')$. Thus $b_2 = j \circ b_{G'}$ for an obvious morphism $b_{G'} \in Fr_N(U', X' - S')$. Also,

$$\langle \langle \langle b_2 \rangle \rangle \rangle = \langle \langle \langle \langle j \rangle \rangle \circ \langle \langle b_{G'} \rangle \rangle \rangle \rangle \in \mathcal{ZF}_N((U', U' - S'), (X', X' - S'))$$

where $j : (X' - S', X' - S') \hookrightarrow (X', X' - S')$ is a natural inclusion. By the latter comments and Corollary 11.7 one gets,

$$[[a]] \circ [[\pi]] - [[j]] \circ [[b_{G'}]] = [[b_1]]$$

in $\mathcal{ZF}_N((U', U' - S'), (X', X' - S'))$. To prove equality (18), and hence to prove Theorem 2.14, it remains to check that $[[b_1]] = [[can']] \circ [[\alpha_{U'}^N]]$.

Let $U'' = (U')^{\prime}_{U \cap U'}$ be the henselization of $U'$ at $S' \cap U'$ and let $\pi' : U'' \rightarrow U'$ be the structure morphism. Recall that $S' \cap U'$ is essentially $k$-smooth. Thus the pair $(U'', S' \cap U')$ is a henselian pair with an essentially $k$-smooth closed subscheme $S' \cap U'$. Recall that one has equality (21). Thus by Theorem 12.2 one has an equality $[[b_1]] \circ [[\pi']] = [[can']] \circ [[\pi]] \circ [[\alpha_{U'}^N]]$ in $\mathcal{ZF}_N((U'', U'' - S'), (X', X' - S'))$. Since $\pi' \circ \alpha_{U'}^N = \alpha_{U'}^N \circ \pi'$ one has,

$$[[b_1]] \circ [[\pi']] = [[can']] \circ [[\alpha_{U'}^N]] \circ [[\pi']] \in \mathcal{ZF}_N((U'', U'' - S'), (X', X' - S'))$$

Applying Theorem 2.13 to the morphism $\pi' : U'' \rightarrow U'$, we see that for an integer $M \geq 0$ one has an equality

$$[[b_1]] \circ [[\alpha_{U'}^M]] = [[can']] \circ [[\alpha_{U'}^{M+N}]] \in \mathcal{ZF}_{M+N}((U', U' - S'), (X', X' - S'))$$

Thus,

$$[[a]] \circ [[\pi]] \circ [[\alpha_{U'}^M]] - [[j]] \circ [[b_{G'}]] \circ [[\alpha_{U'}^M]] = [[can']] \circ [[\alpha_{U'}^{M+N}]] \in \mathcal{ZF}_{M+N}((U', U' - S'), (X', X' - S'))$$

Since $\pi \circ \alpha_{U'}^M = \alpha_{U'}^M \circ \pi$, then we have that

$$[[a]] \circ [[\alpha_{U'}^M]] \circ [[\pi]] - [[j]] \circ [[b_{G'}]] \circ [[\alpha_{U'}^M]] = [[can']] \circ [[\alpha_{U'}^{M+N}]] \in \mathcal{ZF}_{M+N}((U', U' - S'), (X', X' - S'))$$

Set $a_{new} = a \circ \alpha_{U'}^M, h_{new} = b_{G'} \circ \alpha_{U'}^M, N(new) = M + N$. With these in hand the following equality holds:

$$[[a_{new}]] \circ [[\pi]] - [[j]] \circ [[h_{new}^G]] = [[can']] \circ [[\alpha_{U'}^{N(new)}]] \in \mathcal{ZF}_{M+N}((U', U' - S'), (X', X' - S'))$$

The latter equality is of the form (18). Whence Theorem 2.14. □
12. Two useful theorems

**Theorem 12.1.** Let $W$ be an essentially $k$-smooth local $k$-scheme and let $N \geq 1$ be an integer. Let $s : W \to W \times A^N$ be a section of the projection $pr_W : W \times A^N \to W$. Let $W^h_{s(W)} \to W \times A^N$, $s^h : W \to (W \times A^N)_{s(W)}$ be the henselization of $W \times A^N$ at $s(W)$ (particularly, $s = \rho \circ s^h$). Let $X$ be a $k$-smooth scheme. Suppose $\alpha = (s(W), (W \times A^N)_{s(W)}, \varphi, s^h) \in Fr_N(W, X)$, $\alpha = (s(W), (W \times A^N)_{s(W)}, \varphi, s^h) \in Fr_N(W, X)$,

is a $N$-framed correspondence such that the functions $(\varphi_1, \ldots, \varphi_N)$ generate the ideal $I = I_{s(W)}$ of these functions in $k[(W \times A^N)_{s(W)}]$, which vanish on the closed subset $s(W)$. Let $A \in M_N(k[W])$ be a unique matrix transforming the free basis $(t_1, \ldots, t_N)$ of the free $k[W]$-module $1/I^2$ to the free basis $(\tilde{\varphi}_1, \ldots, \tilde{\varphi}_N)$ of the same $k[W]$-module. Suppose that the determinant $J := \det(A) = 1 \in k[W]$. Then,

$$[\alpha] = [g \circ s^h] \circ [\alpha^N] \in ZF_N(W, X).$$

If $W^o \subset W$ is Zariski open and $X^o \subset X$ is Zariski open and $g(s^h(W^o)) \subset X^o$, then

$$[[\alpha]] = [[g \circ s^h]] \circ [[\alpha^N]] \in ZF_N((W, W^o), (X, X^o)).$$

**Theorem 12.2.** Let $W$ be an essentially $k$-smooth local $k$-scheme and $N \geq 1$ be an integer. Let $S \subset W$ be a closed subscheme (essentially $k$-smooth) such that the pair $(W, S)$ is henselian. Let $X$ be a $k$-smooth scheme. Let $s : W \to W \times A^N$, $\alpha \in Fr_N(W, X)$, $A \in M_N(k[W])$, $J := \det(A) \in k[W]$, $s^h$ be the same as in Theorem 12.1. Suppose that $J|S = 1 \in k[S]$. Then,

$$[\alpha] = [g \circ s^h] \circ [\alpha^N] \in ZF_N(W, X).$$

If $W^o \subset W$ is Zariski open and $X^o \subset X$ is Zariski open and $g(s(W^o)) \subset X^o$, then

$$[[\alpha]] = [[g \circ s^h]] \circ [[\alpha^N]] \in ZF_N((W, W^o), (X, X^o)).$$

To prove these two theorems, we need some technical lemmas.

**Lemma 12.3.** Let $W$ be a $k$-smooth affine scheme and let $\mathcal{W} := (W \times A^N)_{h_{W \times 0}}$ be the henselization of $W \times A^N$ at $W \times 0$. Let $\mathcal{W}_0 := (A^1 \times W \times A^N)_{h_{A^1 \times W \times 0}}$ be the henselization of $A^1 \times W \times A^N$ at $A^1 \times W \times 0$. Then there is a morphism $H_0 : \mathcal{W} \to \mathcal{W}$ such that:

(a) $H_1 : \mathcal{W} \to \mathcal{W}$ is the identity morphism;
(b) $H_0 : \mathcal{W} \to \mathcal{W}$ coincides with the composite morphism $\mathcal{W} \to W \times A^N \xrightarrow{pr_W} W \xrightarrow{s_0} \mathcal{W}$.

If $W^o \subset W$ is open, then set $\mathcal{W}^o := p_{W \times A^N}(W^o \times A^N)$ and $\mathcal{W}_0^o := p_{A^1 \times W \times A^N}(A^1 \times W^o \times A^N)$.

In that case $H_0(\mathcal{W}_0^o) \subset \mathcal{W}^o$.

**Corollary 12.4** (of Lemma 12.3). Let $h_0 = (A^1 \times W \times 0, \mathcal{W}_0, \psi; H_0) \in Fr_N(A^1 \times W, \mathcal{W})$. Then one has:

(a) $h_1 = (W \times 0, \mathcal{W}, \psi; id_\mathcal{W}) \in Fr_N(A^1 \times W, \mathcal{W})$;
(b) $h_0 = (W \times 0, \mathcal{W}, \psi; s_0 \circ s^h) \in Fr_N(A^1 \times W, \mathcal{W})$, where $p_W$ is the composite map $\mathcal{W} \to W \times A^N \xrightarrow{pr_W} W$ and $s_0 : W \to \mathcal{W}$ is the canonical inclusion.

Moreover, if $W^o \subset W$ is open, then $h_0 |_{A^1 \times W^o}$ runs inside $\mathcal{W}^o$. 

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Corollary 12.4. Let \( A \in \text{GL}_N(k[W]) \) be a matrix such that \( A_0 = id \). Set \( A := A_1 = \text{GL}_N(k[W]) \).

Take a row \((\psi', \ldots, \psi_N) := (\psi_1, \ldots, \psi_N \cdot P_w(A)) in k[\mathcal{W}]\) and take a \( N \)-framed correspondence

\[
h_0 := (A^1 \times W \times 0, A^1 \times \mathcal{W}, \Psi, \psi, P_w \circ pr_{\mathcal{W}}) \in Fr_N(A^1 \times W, W),
\]

where \( \Psi = (pr_{\mathcal{W}}(\psi_1), \ldots, pr_{\mathcal{W}}(\psi_N)) \cdot P_w(A) \) is a row in \( k[A^1 \times \mathcal{W}] \). Then one has:

(a) \( h_0 = (W \times \mathcal{W}, \psi, P_w) \);

(b) \( h_1 = (W \times \mathcal{W}, \psi, P_w) \).

Moreover, for any open \( W^0 \subset W \) the \( N \)-framed correspondence \( h_0 |_{A^1 \times W^0} \) runs inside \( W^0 \).

Lemma 12.5. Let \( (W \times 0, \mathcal{W}, \psi; p_w) \in Fr_N(W, W) \), where \( p_w : \mathcal{W} \to W \) be the morphism from Corollary [12.4]. Let \( A_0 \in \text{GL}_N(k[W][\theta]) \) be a matrix such that \( A_0 = id \). Set \( A := A_1 \in \text{GL}_N(k[W]) \).

Suppose the functions \( \psi_1, \ldots, \psi_N \) generate the ideal \( I \subset k[\mathcal{W}] \) consisting of all the functions vanishing on the closed subset \( W \times 0 \). Furthermore, suppose that for any \( i = 1, \ldots, N \) one has that \( \psi_i = \theta_i \) in \( I/F \). Set \( \psi_0 := (\psi_0, 1, \ldots, \psi_0, N) \). Set

\[
h_0 := (A^1 \times W \times 0, A^1 \times \mathcal{W}, \psi_0, p_w \circ pr_{\mathcal{W}}) \in Fr_N(A^1 \times W, W).
\]

Then one has:

(a) \( h_0 = (W \times 0, \mathcal{W}, \psi; p_w) \);

(b) \( h_1 = (W \times 0, \mathcal{W}, \psi_1, \ldots, \psi_n; p_w) = (W \times 0, W \times A^1, \ldots, \psi_n; p_w) \equiv \alpha_N \).

Moreover, for any open \( W^0 \subset W \), the \( N \)-framed correspondence \( h_0 |_{A^1 \times W^0} \) runs inside \( W^0 \).

Lemma 12.7. Let \( \alpha = (s(W), (W \times A^N)_{\psi_0}, \psi_1, \ldots, \psi_N; \psi) \in Fr_N(W, X) \) be a \( N \)-framed correspondence, where \( s : W \to W \times A^N, \psi : W \times A^N \to (W \times A^N)_{\psi_0} \) and \( \psi : W \to (W \times A^N)_{\psi_0} \) be as in Theorem [12.7]. Let \( T_i : W \times A^N \to W \times A^N \) be a morphism taking a point \((w, v)\) to the point \((w, v + s(w))\).

Let \( T_i : \mathcal{W} = (W \times A^N)_{\psi_0} \to (W \times A^N)_{\psi_0} \) be the induced morphism. Then one has,

\[
[\alpha] = [W \times 0, \mathcal{W}, \psi_0 \circ T^h_i, \ldots, \psi_n \circ T^h_i, \psi^t T^h_i, \psi^t T^h_i] \in ZF_N(W, X).
\]

Moreover, if \( W^0 \subset W \) is open and if \( X^0 \subset X \) is any open subset such that \( g(s^h(W^0)) \subset X^0 \), then one has,

\[
[[\beta]] = [W \times 0, \mathcal{W}, \psi_0 \circ T^h_i, \ldots, \psi_n \circ T^h_i, \psi^t T^h_i, \psi^t T^h_i] \in ZF_N((W \times W^0), (X \times W^0)).
\]

Proof of Theorem [12.7]. Let \( \alpha \in Fr_N(W, X) \) be the \( N \)-framed correspondence from Theorem [12.7]. By Lemma [12.7] one has an equality in \( ZF_N(W, X) \)

\[
[\alpha] = [W \times 0, \mathcal{W}, \psi_1, \ldots, \psi_n; g \circ T^h_i] = [g \circ T^h_i] \circ [W \times 0, \mathcal{W}, \psi_1, \ldots, \psi_n; \psi^t W],
\]

where \( \psi_i = \psi_1 \circ T^h_i \). By Corollary [12.4], one has an equality in \( ZF_N(W, \mathcal{W}) \):

\[
[W \times 0, \mathcal{W}, \psi_1, \ldots, \psi_n; \psi^t W] = [s_0 \circ W \times 0, \mathcal{W}, \psi; p_w].
\]

Thus, one has

\[
[\alpha] = [g \circ T^h_i \circ s_0] \circ [W \times 0, \mathcal{W}, \psi; p_w] = [g \circ s^h] \circ [W \times 0, \mathcal{W}, \psi; p_w] \in ZF_N(W, X).
\]

Clearly, the functions \( \psi_1, \ldots, \psi_n \) generate the ideal \( I_0 = I_{W^0} \) of these functions in \( k[W] \), which vanish on the closed subset \( W \times 0 \). Let \( A' \in M_N(k[W]) \) be a matrix, which transforms the finite basis \( (t_1, \ldots, t_n) \) of the free \( k[W] \)-module \( I_0/I_0^2 \) to the free basis \( (\psi_1, \ldots, \psi_n) \) of the same \( k[W] \)-module. Clearly, \( \det(A') = \det(A) \). Thus \( \det(A') = 1 \in k[W] \). The ring \( k[W] \) is local. Thus \( A' \) belongs to the group of elementary \( N \times N \) matrices over \( k[W] \). Hence there is a matrix \( A_0 \in M_N(k[W][\theta]) ) \) such that \( A_0 = id \) and \( A_1 = (A')^{-1} \in \text{GL}_N(k[W]) \). By Lemma [12.5] one has an equality

\[
[W \times 0, \mathcal{W}, \psi; p_w] = [W \times 0, \mathcal{W}, \psi, p_w] \in ZF_N(W, W).
\]
with the row \( \psi'_1, \ldots, \psi'_N \) as in Lemma \[12.5\]. By construction, for any \( i = 1, \ldots, N \) the function \( \psi'_i \) has the property: \( \psi'_i \equiv \bar{t}_i \) in \( I_0 / I_0^2 \). By Lemma \[12.6\] one has an equality

\[
[W \times 0, \mathcal{W}, \psi'_i, p_W] = [\alpha^N_W] \in \mathcal{ZF}_N(W, W).
\]

Thus, finally,

\[
[\alpha] = [g \circ s^h] \circ [\alpha^N_W] \in \mathcal{ZF}_N(W, X).
\]

If \( W^o \subset W \) is Zariski open and \( X^o \subset X \) is Zariski open and \( g(\mathcal{S}^h(W^o)) \subset X^o \), then the same arguments prove the relation

\[
[[\alpha]] = [[g \circ s^h]] \circ [[\alpha^N_W]] \in \mathcal{ZF}_N((W, W^o), (X, X^o)).
\]

(28)

Theorem \[12.1\] is proved.

\[\square\]

**Proof of Theorem \[12.2\]**. Repeating literally the proof of Theorem \[12.1\] one gets an equality

\[
[[\alpha]] = [[g \circ s^h]] \circ [[\alpha^N_W]] \circ [[\alpha^N]] \in \mathcal{ZF}_N((W, W^o), (X, X^o)),
\]

(29)

where \([J \cdot t]\) is the class of the element \((J \cdot t)\) corresponding to the 1-framed correspondence

\[
(W \times 0, W \times \mathbb{A}^1, J \cdot t; pr_W) \in Fr_1(W, W)
\]

by means of Definition \[3.3\]. It remains to prove that \([J \cdot t] = [[t]]\) in \( \mathcal{ZF}_N((W, W^o), (W, W^o)) \). Let \( i : S \rightarrow W \) be the inclusion. Since \((W, S)\) is an affine henselian pair with an essentially \( k\)-smooth \( S \), then there is a morphism \( r : W \rightarrow S \) such that \( r \circ i = id_S \). Moreover, there is a morphism \( H_0 : \mathbb{A}^1 \times W \rightarrow W \) such that \( H_1 = id_W \) and \( H_0 \) is the composite map \( W \xrightarrow{r} S \xrightarrow{i} W \). Consider an 1-framed correspondence of the form

\[
h_0 = (\mathbb{A}^1 \times W \times 0, \mathbb{A}^1 \times W \times \mathbb{A}^1, m \circ (J \times id_{\mathbb{A}^1}) \circ (H \times id); pr_{\mathbb{A}^1 \times W}) \in Fr_1(\mathbb{A}^1 \times W, \mathbb{A}^1 \times W),
\]

where \( m : \mathbb{A}^1 \times \mathbb{A}^1 \rightarrow \mathbb{A}^1 \) is the multiplication. Clearly, \( h_1 = (W \times 0, W \times \mathbb{A}^1, J \cdot t; pr_W) \) and

\[
h_0 = (W \times 0, W \times \mathbb{A}^1, r^\ast(i^!(J)) \cdot t; pr_W) = (W \times 0, W \times \mathbb{A}^1, t; pr_W).
\]

Thus \([J \cdot t] = [[t]]\) in \( \mathcal{ZF}_N((W, W^o), (W, W^o)) \). Theorem \[12.2\] is proved.

\[\square\]

13. CONSTRUCTION OF \( h_0, F \) AND \( h_0 \) FROM PROPOSITIONS \[10.5\] AND \[8.9\]

In this section we recall a modification of a result of M. Artin from [1] concerning existence of nice neighborhoods. The following notion (see [4, Defn.2.1]) is a modification of that introduced by Artin in [1, Exp. XI, Déf. 3.1].

**Definition 13.1.** An almost elementary fibration over a scheme \( B \) is a morphism of schemes \( p : X \rightarrow B \) which can be included in a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{j} & \overline{X} \\
\downarrow q & & \downarrow i \\
B & \xleftarrow{q\#} & X_{\neq}
\end{array}
\]

of morphisms satisfying the following conditions:

(i) \( j \) is an open immersion dense at each fibre of \( \overline{q} \), and \( X = \overline{X} \setminus X_{\neq} \);

(ii) \( \overline{q} \) is smooth projective all of whose fibres are geometrically irreducible of dimension one;

(iii) \( q\# \) is a finite flat morphism all of whose fibres are non-empty;
(iv) the morphism $i$ is a closed embedding and the ideal sheaf $I_{X_\infty} \subset \mathcal{O}_{\bar{X}}$ defining the closed subscheme $X_\infty$ in $\bar{X}$ is locally principal.

Let $X$ and $X'$ be as in Lemma 8.4 and let $q : X \to B$ be the almost elementary fibration from Lemma 8.4. The composite morphism $X' \to X \to \bar{X}$ is quasi-finite. Let $\bar{\mathcal{X}}$ be the normalization of $\bar{X}$ in $\text{Spec}(k(X'))$. Let $\bar{\mathcal{X}} : \bar{X} \to \bar{X}$ be the canonical morphism (it is finite and surjective). Then $(\bar{\mathcal{X}})^{-1}(\mathcal{X})$ coincides with the normalization of $\mathcal{X}$ in $\text{Spec}(k(X'))$. Let $f' := f|_{(\bar{\mathcal{X}})^{-1}(\mathcal{X})}$, where $f$ is from Definition 8.4. Let $Y' = \{ f' = 0 \}$ be the closed subscheme of $(\bar{\mathcal{X}})^{-1}(\mathcal{X})$. The morphism $(q \circ (\bar{\mathcal{X}})^{-1}(\mathcal{X})))|_{Y'} : Y' \to B$ is finite, since $q|_{Y} : Y \to B$ is finite and $\bar{\mathcal{X}}$ is finite. Thus $Y'$ is closed in $\bar{X}$. Since $Y'$ is in $(\bar{\mathcal{X}})^{-1}(\mathcal{X})$ it has the empty intersection with $X_\infty$. Hence

$$X' = \bar{X'} - ((\bar{\mathcal{X}})^{-1}(X_\infty) \cap Y').$$

Both $(\bar{\mathcal{X}})^{-1}(X_\infty)$ and $Y'$ are Cartier divisors in $\bar{X}$. The Cartier divisor $(\bar{\mathcal{X}})^{-1}(X_\infty)$ is ample. Thus the Cartier divisor $D' := (\bar{\mathcal{X}})^{-1}(X_\infty) \cap Y'$ is ample as well and $(q \circ \bar{\mathcal{X}})|_{D'} : D' \to B$ is finite.

Set $\Gamma = X'_{\overline{(id)}} \subset X \times_B X'$. Then $\Gamma \subset U \times_B X'$ is a Cartier divisor. The scheme $U'$ contained in $\Gamma$ as open subscheme via the inclusion $(\pi, \text{can'})$, where can$' : U' \to X'$ is the canonical morphism. The composite morphism $pr_{U'} \circ (\pi, \text{can'}) : U' \to U$ coincides with $\pi : U' \to U$. Thus $pr_{X'}|_{\Gamma} : \Gamma \to U$ is étale at the points of $U'$.

Lemma 13.2. Set $\Gamma' = U' \times_U \Gamma \subset U' \times_U X \times_B X' = U' \times_B \bar{X}'$. Then $\Gamma' \subset U' \times_B X'$ is a Cartier divisor. Moreover,

$$\Gamma' = \Delta'(U') \cup G'$$

and $G' \cap U' \times_B S' = \emptyset$.

Remark 13.3. It is easy to check that $\Gamma \cap U \times_B S' = \delta(S')$, where $\delta(s') = (\pi(s'), s')$.

Definition 13.4. Set $\mathcal{D}' = U \times_B D'$ and $\mathcal{D}'' = U' \times_U \mathcal{D}' = U' \times_B D'$. They are Cartier divisors on $U \times_B \bar{X}$ and $U' \times_B \bar{X}'$ respectively. Both $\mathcal{D}'$ and $\mathcal{D}''$ are finite over $B$.

Let $s_0 \in \Gamma(U \times_B \bar{X'}, \mathcal{L}(\mathcal{D}'))$ be the canonical section of the invertible sheaf $\mathcal{L}(\mathcal{D}')$ (its vanishing locus is $\mathcal{D}'$). Let $s_0 \in \Gamma(U \times_B \bar{X'}, \mathcal{L}(\Gamma))$ be the canonical section of the invertible sheaf $\mathcal{L}(\Gamma)$ (its vanishing locus is $\Gamma$). Let $s_{\mathcal{D}'(U')} \in \Gamma(U' \times_B \bar{X'}, \mathcal{L}(\Delta'(U'))$ be the canonical section of the invertible sheaf $\mathcal{L}(\Delta'(U'))$ (its vanishing locus is $\Delta'(U')$). Let $s_{\mathcal{D}''} \in \Gamma(U' \times_B \bar{X'}, \mathcal{L}(\mathcal{D}''))$ be the canonical section of the invertible sheaf $\mathcal{L}(\mathcal{D}''$) (its vanishing locus is $\mathcal{D}''$). Choose an integer $n \gg 0$.

Construction 13.5. Find a section $s_1 \in \Gamma(U \times_B \bar{X'}, \mathcal{L}(n\mathcal{D}'))$ such that:

1. $s_1|_{U \times_B S'} = r_1 \otimes (s_1|_{U \times_B S'})$, where $r_1 \in \Gamma(U \times_B \mathcal{S'}, \mathcal{L}(n\mathcal{D}' - \Gamma))|_{U \times_B S'}$ has no zeros;
2. $s_1|_{\mathcal{D}'}$ has no zeros.

Construction 13.6. Set $t_1 := (\pi \times \text{id})^*(s_1) \in \Gamma(U' \times_B \bar{X'}, \mathcal{L}(n\mathcal{D}''))$. The properties of $t_1$:

1. $t_1|_{U \times_B S'} = r'_1 \otimes (s_1|_{U \times_B S'}) \otimes (s_{\mathcal{D}'(U')}|_{U \times_B S'}) \cdot \lambda$, where $\lambda \in k[U' \times_B \bar{X'}]$ and $r'_1 = (\pi \times \text{id})^*(r_1) \in \Gamma(U' \times_B \bar{X'}, \mathcal{L}(n\mathcal{D}'' - \Gamma));$
2. $t_1|_{\mathcal{D}''}$ has no zeros.

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Construction 13.7. Construct a section \( t_0 \in \Gamma(U' \times_B X', \mathcal{L}(n D'')) \) of the form \( t_0 = t'_0 \otimes s_{\Delta_1(U')} \), where \( t'_0 \in \Gamma(U' \times_B X', \mathcal{L}(n D'' - \Delta(U'))) \) satisfies the following conditions:
\[(1') \quad t'_0|_{D''} = (t_1|_{D''}) \otimes (s_{\Delta_1(U')}|_{D''})^{-1};\]
\[(2') \quad t'_0|_{U' \times_B S'} = r'_1 \otimes (s_{G_1}|_{U' \times_B S'}) \cdot \lambda, \] where \( r'_1 \) and \( \lambda \) are from Construction 13.6.

Lemma 13.8. One has:
\[(1'') \quad t_0|_{D''} = t_1|_{D''} \text{ and both sections have no zeros on } D'';\]
\[(2'') \quad t_0|_{U' \times_B S'} = t_1|_{U' \times_B S'} \text{ and both sections have no zeros on } (U' - S') \times_B S'.\]

Indeed, the first equality is obvious. The second one follows from the chain of equalities
\[t_0|_{U' \times_B S'} = (t'_0|_{U' \times_B S'}) \otimes (s_{\Delta(U')}|_{U' \times_B S'}) = r'_1 \otimes (s_{G_1}|_{U' \times_B S'}) \otimes (s_{\Delta_1(U')}|_{U' \times_B S'}) \cdot \lambda = t_1|_{U' \times_B S'}.

Definition 13.9. Let \( s_0' := (\pi \times \text{id})^*(s_0) \in \Gamma(U' \times_B X', \mathcal{L}(D'')) \). Set
\[h_0'' = \frac{(1 - \theta) t_0 + \theta t_1}{(s_0')^\otimes |_{U' \times_B X'}} \in k[\mathbb{A}^1 \times U' \times_B X']\] and \( F = \frac{s_1|_{U \times X'}}{(s_0)|_{U \times X'}} \in k[U \times_B X'].\)

Proof of Proposition 10.3. The functions \( h_0'' \) and \( F \) satisfy properties (a) – (f) from Proposition 10.3. \( \square \)

In the rest of the section under the hypotheses of Proposition 8.9 we will construct a function \( h_0'' \in k[\mathbb{A}^1 \times U \times X] \).

Let \( X \) and \( X' \) be as in Lemma 8.4 and let \( q : X \to B \) be the almost elementary fibration from Lemma 8.4. The composite morphism \( X' \xrightarrow{\pi} X \xrightarrow{\theta} \bar{X} \) is quasi-finite. Let \( \bar{X}' \) be the normalization of \( \bar{X} \) in \( \text{Spec}(k(X')) \). Let \( \bar{\Pi} : \bar{X}' \to \bar{X} \) be the canonical morphism (it is finite and surjective). Let \( X_\varphi \subset \bar{X} \) be the Cartier divisor from diagram (30). Set \( X'_\varphi := (\bar{\Pi})^{-1}(X_\varphi) \) (scheme-theoretically). Then \( X'_\varphi \) is a Cartier divisor on \( \bar{X}' \). Set
\[E := U \times_B X_\varphi \quad \text{and} \quad E' := U \times_B X'_\varphi.\]

These are Cartier divisors on \( U \times_B \bar{X} \) and \( U \times_B \bar{X}' \) respectively and \((\text{id} \times \bar{\Pi})^*(E) = E'.\)

Choose an integer \( n \gg 0 \). Find a section \( r_1 \in \Gamma(U \times_B S, \mathcal{L}(nE - \Delta(U)|_{U \times_B S}) \) which has no zeros. Let \( s_{\Delta(U)} \in \Gamma(U \times_B \bar{X}, \mathcal{L}(\Delta(U))) \) be the canonical section of the invertible sheaf \( \mathcal{L}(\Delta(U)) \) (its vanishing locus is \( \Delta(U) \)).

Construction 13.10. Find a section \( s'_1 \in \Gamma(U \times_B \bar{X}', \mathcal{L}(nE')) \) such that the following holds:
\[(1) \quad \text{the Cartier divisor } Z'_1 := \{s'_1 = 0\} \text{ has the following properties:}\]
\[(1a) \quad Z'_1 \subset U \times_B X';\]
\[(1b) \quad \text{the morphism } i = (\text{id} \times \bar{\Pi})|_{Z'_1} : Z'_1 \hookrightarrow U \times_B X \text{ is a closed embedding; denote by } Z_1 \text{ the closed subscheme } i(Z'_1) \text{ of the scheme } U \times_B X;\]
\[(1c) \quad (\text{id} \times \bar{\Pi})^{-1}(Z'_1) = Z'_1 \cup Z_1 \text{ (a scheme equality);}\]
\[(2) \quad s'_1|_{U \times_B S'} = (\text{id} \times \bar{\Pi})^*(s_{\Delta(U)}|_{U \times_B S'}) \otimes ((\text{id} \times \bar{\Pi})|_{U \times_B S'})^*(r_1).\]

Remark 13.11. Properties (1a), (1d') and (1b) yield property (1c).

Lemma 13.12. One has \( Z'_1 \cap U \times_B S' = \emptyset.\)

Note that the Cartier divisor \( Z_1 \) in \( U \times_B \bar{X} \) is equivalent to the Cartier divisor \( dE \), where \( d = [k(X') : k(X)] \). Let \( s_1 \in \Gamma(U \times_B \bar{X}, \mathcal{L}(Z_1)) \) be the canonical section (its vanishing locus is \( Z_1 \)). By property (1c) from Construction 13.10 one has an equality
\[(id \times \bar{\Pi})^*(s_1) = (s'_1 \otimes s_2') \cdot \mu, \quad (31)\]
where \( \mu \in k[U]^\times \) and \( s'_2 \in \Gamma(U \times_B \mathcal{X}', \mathcal{L}(Z'_2)) \) is the canonical section of the line bundle \( \mathcal{L}(Z'_2) \).

**Definition 13.13.** Set \( t_1 = s_1 \in \Gamma(U \times_B \mathcal{X}, \mathcal{L}(Z_1)) = \Gamma(U \times_B \mathcal{X}, \mathcal{L}(dnE)) \).

**Construction 13.14.** Construct a section \( t_0 \in \Gamma(U \times_B \mathcal{X}, \mathcal{L}(dnE)) \) of the form \( t_0 = s_{\Delta(U)} \otimes t'_0 \), where \( t'_0 \in \Gamma(U \times_B \mathcal{X}, \mathcal{L}(dnE - \Delta(U))) \), where \( s_{\Delta(U)} \in \Gamma(U \times_B \mathcal{X}, \mathcal{L}(\Delta(U))) \) is the canonical section (its vanishing locus is \( \Delta(U) \) and \( t'_0 \) has the following properties:

1. \( (t_0|_E) = (t_1|_E) \otimes (s_{\Delta(U)}|_E)^{-1} \);
2. \((id \times \bar{\Pi})|_{U \times_B S'}(t'_0|_{U \times_B S'}) = ((id \times \bar{\Pi})|_{U \times_B S'})^*(r_1) \otimes (s'_2|_{U \times_B S'}) \cdot (\mu|_{U \times_B S'}) \), where \( r_1, s'_1 \) are from Construction [13.10] and \( \mu \in k[U]^\times \) is defined just above (since \( U \times_B S' \cong U \times S \) condition (2') on \( t'_0 \) is a condition on \( t'_0|_{U \times B S} \)).

**Lemma 13.15.** The following statements are true:

1. \( t_0|_E = t_1|_E \) and both sections have no zeros on \( E \);
2. \( t_0|_{U \times B S} = t_1|_{U \times B S} \) and both sections have no zeros on \( (U - S) \times_B S \).

Indeed, the first equality is obvious. To prove the second one, it suffices to prove the equality

\[
((id \times \bar{\Pi})|_{U \times B S'})^*(t_0|_{U \times B S}) = ((id \times \bar{\Pi})|_{U \times B S'})^*(t_1|_{U \times B S}).
\]

This equality is a consequence of the following chain of equalities:

\[
((id \times \bar{\Pi})|_{U \times B S'})^*(t_0|_{U \times B S}) = (id \times \bar{\Pi})^* (s_{\Delta(U)})|_{U \times B S'} \otimes ((id \times \bar{\Pi})|_{U \times B S'})^*(r_1) \otimes (s'_2|_{U \times B S'}) \cdot (\mu|_{U \times B S'}) = s'_1|_{U \times B S'} \otimes (s'_2|_{U \times B S'}) \cdot (\mu|_{U \times B S'}) = ((id \times \bar{\Pi})|_{U \times B S'})^*(t_1|_{U \times B S}).
\]

The first equality holds by property (2') from Construction [13.14] the second equality holds by property (2) from Construction [13.10] the third one follows from equality (31) and Definition [13.13].

**Definition 13.16.** Set,

\[
h_{\theta} = \frac{((1 - \theta)t_0 + \theta t_1)|_{\mathbb{A}^1 \times U \times X}}{(s'_E|_{\mathbb{A}^1 \times U \times X})} \in k[\mathbb{A}^1 \times U \times X].
\]

**Proof of Proposition [8.9]**. The function \( h_{\theta} \) satisfies properties (a) – (d) from Proposition [8.9]. \( \Box \)

14. **Homotopy invariance of cohomology presheaves**

In this section we prove Theorems [14.13] and [14.14]. They complete the proof of Theorem [1.1] which is the main result of the paper.

**Definition 14.1.** Let \( \mathcal{F} \) be a homotopy invariant presheaf of abelian groups with \( \mathbb{Z}F_q \)-transfers. Then the presheaf \( X \mapsto \mathcal{F}_-(X) := \mathcal{F}(X \times (\mathbb{A}^1 - 0))/\mathcal{F}(X) \) is also a homotopy invariant presheaf of abelian groups with \( \mathbb{Z}F_q \)-transfers. If the presheaf \( \mathcal{F} \) is a Nisnevich sheaf, then the presheaf \( \mathcal{F}_- \) is also a Nisnevich sheaf. If the presheaf \( \mathcal{F} \) is stable, then the presheaf \( \mathcal{F}_- \) is stable too.

If the presheaf \( \mathcal{F} \) is a Nisnevich sheaf on \( Sm/k \) and \( Y \) is a \( k \)-smooth variety, then denote by \( \mathcal{F}|_Y \) the restriction of \( \mathcal{F} \) to the small Nisnevich site of \( Y \).

**Lemma 14.2.** The category of Nisnevich sheaves with \( \mathbb{Z}F_q \)-transfers is a Grothendieck category.
Lemma 14.3. For any Nisnevich sheaf $\mathcal{F}$ with $\mathbb{Z}F_*$-transfers, any integer $n$ and any $k$-smooth variety $X$, there is a natural isomorphism
\[ H^0_{Nis}(X, \mathcal{F}) = \text{Ext}^n(\mathbb{Z}F_*(X), \mathcal{F}), \]
where the Ext-groups are taken in the Grothendieck category of Nisnevich sheaves with $\mathbb{Z}F_*$-transfers.

Corollary 14.4. For any Nisnevich sheaf $\mathcal{F}$ with $\mathbb{Z}F_*$-transfers and any integer $n$, the presheaf $X \mapsto H^0_{Nis}(X, \mathcal{F})$ has a canonical structure of a $\mathbb{Z}F_*$-presheaf.

Lemma 14.5. For any $k^1$-invariant stable $\mathbb{Z}F_*$-sheaf of abelian groups $\mathcal{F}$, any $k$-smooth variety $Y$ and any $k$-smooth divisor $D$ in $Y$ the canonical morphism
\[ H^1_D(Y, \mathcal{F}) \to H^0_{Nis}(Y, \mathcal{H}^1_D(Y, \mathcal{F})) \]
is an isomorphism.

Proof. The local-global spectral sequence yields an exact sequence of the form
\[ H^0_{Nis}(Y, \mathcal{H}^0_D(Y, \mathcal{F})) \to H^1_D(Y, \mathcal{F}) \to H^0_{Nis}(Y, \mathcal{H}^1_D(Y, \mathcal{F})) \to H^1_{Nis}(Y, \mathcal{H}^1_D(Y, \mathcal{F})). \]
By Theorem [2.15] item (3') the sheaf $\mathcal{H}^1_D(Y, \mathcal{F})$ vanishes.

Lemma 14.6. Let $X$ be an essentially $k$-smooth local henselian scheme and let $D \subset X$ be a $k$-smooth divisor. Let $i : D \hookrightarrow X$ be the closed embedding. Then for any $\mathbb{A}^1$-invariant stable $\mathbb{Z}F_*$-sheaf of abelian groups $\mathcal{F}$ the Nisnevich sheaves $i_* (\mathcal{F}_{-1|D})$ and $\mathcal{H}^1_D(X, \mathcal{F})$ on the small Nisnevich site of $X$ are isomorphic.

Proof. The group $H^1_D(X, \mathcal{F})$ is isomorphic to $\mathcal{F}(X - D)/\text{Im}(\mathcal{F}(X)) = \mathcal{F}(X - D)/\mathcal{F}(X)$. The latter equality makes sense by Theorem [2.15] item (3'). Since $X$ is essentially $k$-smooth and henselian, and $D$ is essentially $k$-smooth, then there is a morphism $r : X \to D$ such that the composite map $D \overset{\delta}{\to} X \overset{\gamma}{\to} D$ is the identity. Let $x \in X$ be the closed point. Clearly, $x \in D$. Set $V := \text{Spec}(\mathcal{O}_D[\mathbb{A}^1,-(r)])$. Let $f \in k[X]$ be a function defining the smooth divisor $D$. Then the morphism
\[ (r, f) : X \to D \times \mathbb{A}^1 \]
takes values in $V$. We keep the same notation for the corresponding morphism $(r, f) : X \to V$.

Note that $(r, f)^{-1}(D \times 0) = D$. Thus the morphism $(r, f)$ induces a homomorphism
\[ [[(r, f)]]_* : \mathcal{F}(V-D\times 0)/\mathcal{F}(V) \to \mathcal{F}(X-D)/\mathcal{F}(X). \]

We claim that it is an isomorphism. To prove this claim note that the morphism $(r, f)$ induces a scheme isomorphism $X^h_D \to V^h_D$, where $X^h_D$ is the henselization of $X$ at $D$ and $V^h_D$ is the henselization of $V$ at $D \times 0$. Now Theorem [2.15] item (5) implies the claim.

By Corollary [2.16] the pull-back map
\[ \mathcal{F}_{-1}(D) = \mathcal{F}(D \times (\mathbb{A}^1 - 0))/\mathcal{F}(D \times \mathbb{A}^1) \to \mathcal{F}(V-D\times 0)/\mathcal{F}(V) \]
is an isomorphism, too. Thus there is a natural isomorphism
\[ \mathcal{F}_{-1}(D) = \mathcal{F}(D \times (\mathbb{A}^1 - 0))/\mathcal{F}(D \times \mathbb{A}^1) \overset{[(\text{can})^*]}{\to} \mathcal{F}(V-D)/\mathcal{F}(V) \overset{[[r,f]]_*}{\to} \mathcal{F}(X-D)/\mathcal{F}(X) = \mathcal{F}(X-D)/\text{Im}(\mathcal{F}(X)) \overset{\delta}{\to} H^1_D(X, \mathcal{F}), \]
leading to an isomorphism of Nisnevich sheaves $i_* (\mathcal{F}_{-1|D}) \cong \mathcal{H}^1_D(X, \mathcal{F})$ on the small Nisnevich site of $X$. □
Lemma 14.7. Let \( X \) be an essentially k-smooth local henselian scheme and let \( D \subset X \) be a k-smooth divisor. Let \( I : D \times \mathbb{A}^1 \rightarrow X \times \mathbb{A}^1 \) be the closed embedding. Then for any \( \mathbb{A}^1 \)-invariant stable \( \mathbb{Z} \)-sheaf of abelian groups \( \mathcal{F} \) the two Nisnevich sheaves \( I_* (\mathcal{F}_{-1}|_{D \times \mathbb{A}^1}) \) and \( \mathcal{H}^1_{D \times \mathbb{A}^1}(X \times \mathbb{A}^1, \mathcal{F}) \) on small Nisnevich site of \( X \times \mathbb{A}^1 \) are isomorphic.

Proof. The proof is similar to that of Lemma 14.6.

Corollary 14.8. The pull-back map \( p_X^*: H^1_D(X, \mathcal{F}) \rightarrow H^1_{D \times \mathbb{A}^1}(X \times \mathbb{A}^1, \mathcal{F}) \) is an isomorphism, where \( p_X : X \times \mathbb{A}^1 \rightarrow X \) is the projection.

Proof. One can check that the following diagram commutes

\[
\begin{array}{ccc}
\mathcal{F}_{-1}(D \times \mathbb{A}^1) & \xrightarrow{\beta} & H^0_{Nis}(X \times \mathbb{A}^1, \mathcal{H}^1_{D \times \mathbb{A}^1}(X \times \mathbb{A}^1, \mathcal{F})) \\
\mathcal{F}_{-1}(D) & \xrightarrow{\alpha} & H^0_{Nis}(X, \mathcal{H}^1_D(X, \mathcal{F})) \\
\end{array}
\]

where the maps \( \alpha \) and \( \beta \) are isomorphisms of Lemmas 14.6 and 14.7 respectively, the maps \( \varphi \) and \( \psi \) are isomorphisms of Lemma 14.5, the vertical maps are the pull-back maps. The sheaf \( \mathcal{F}_{-1} \) is homotopy invariant, because so is the sheaf \( \mathcal{F} \). It follows that the map \( p_D^* \) is an isomorphism, whence the corollary.

Corollary 14.9. Under the hypotheses of Lemma 14.7 the boundary map

\[ \partial : \mathcal{F}((X - D) \times \mathbb{A}^1) \rightarrow H^1_{D \times \mathbb{A}^1}(X \times \mathbb{A}^1, \mathcal{F}) \]

is surjective.

Proof. By Corollary 14.8 the pull-back map \( p_X^*: H^1_D(X, \mathcal{F}) \rightarrow H^1_{D \times \mathbb{A}^1}(X \times \mathbb{A}^1, \mathcal{F}) \) is an isomorphism. The boundary map \( \partial : \mathcal{F}(X - D) \rightarrow H^1_D(X, \mathcal{F}) \) is surjective since \( X \) is local henselian, whence the corollary.

Proposition 14.10. Under the hypotheses of Lemma 14.7 the map

\[ H^1_{Nis}(X \times \mathbb{A}^1, \mathcal{F}) \rightarrow H^1_{Nis}((X - D) \times \mathbb{A}^1, \mathcal{F}) \]

is injective.

Proof. This follows from Corollary 14.9.

Proposition 14.11. Let \( \mathcal{F} \) be an \( \mathbb{A}^1 \)-invariant stable \( \mathbb{Z} \)-sheaf of abelian groups. Then

\[ H^1_{Nis}(\mathbb{A}^1_K, \mathcal{F}) = 0. \]

Proof. Let \( a \in H^1_{Nis}(\mathbb{A}^1_K, \mathcal{F}). \) We want to prove that \( a = 0 \). The Nisnevich topology is trivial at the generic point of the affine line \( \mathbb{A}^1_K \). Therefore there is a Zariski open subset \( U \) in \( \mathbb{A}^1_K \) such that the restriction of \( a \) to \( U \) vanishes. Let \( Z \) be the complement of \( U \) in \( \mathbb{A}^1_K \) regarded as a closed subscheme with the reduced structure (it consists of finitely many closed points). Let \( V := \bigsqcup_{z \in Z} (\mathbb{A}^1_K)_z \), where each summand is the henselization of the affine line at \( z \in Z \). Then the cartesian square
Proof.\ The exact sequence
\[
0 \to H^1_{\text{Nis}}(Y, p_* (\mathcal{F})) \to H^1_{\text{Nis}}(Y \times \mathbb{A}^1, \mathcal{F}) \to H^0_{\text{Nis}}(Y, R^1 p_* (\mathcal{F}))
\]
and the fact that the sheaf \(\mathcal{F}\) is homotopy invariant show that for
\[
A := \text{Ker}[i_0^*: H^1_{\text{Nis}}(Y \times \mathbb{A}^1, \mathcal{F}) \to H^1_{\text{Nis}}(Y, \mathcal{F})]
\]
the map \(\beta|_A : A \to H^0_{\text{Nis}}(Y, R^1 p_* (\mathcal{F}))\) is injective. The stalk of the sheaf \(R^1 p_* (\mathcal{F})\) at a point \(y \in Y\) is \(H^1_{\text{Nis}}(Y^h \times \mathbb{A}^1, \mathcal{F})\), where \(Y^h = \text{Spec}(\mathcal{O}^h_{Y,y})\) is the henselization of the local scheme \(\text{Spec}(\mathcal{O}_{Y,y})\). By Proposition \[\ref{prop:local-coh}\] there is a closed subset \(Z\) in \(Y\) such that \(\beta(a)|_{Y-Z} = 0\). Since the field \(k\) is perfect, there is a proper closed subset \(Z_1 \subset Z\) such that \(Z-Z_1\) is \(k\)-smooth. Then \(Z-Z_1\) is a \(k\)-smooth closed subvariety in \(Y-Z_1\).

We claim that \(a_1 := a|_{(Y-Z_1)\times \mathbb{A}^1} = 0\). In fact, \(a_1|_{(Y-Z_1)\times 0} = 0\). Thus it suffices to check that all stalks of the element \(\beta(a_1)\) vanish. Let \(y \in Y-Z_1\) be a point. If \(y \in Y-Z_1\) then \(\beta(a_1)_y = 0\), because \(\beta(a_1)|_{Y-Z_1} = 0\). If \(y \in Z-Z_1\) then shrinking \(Y-Z_1\) around \(y\) we may assume that there is a \(k\)-smooth divisor \(D\) in \(Y-Z_1\) containing \(Z-Z_1\). In this case \(a_1|_{Y-D} = 0\). Now Proposition \[\ref{prop:local-coh}\] shows that \(\beta(a_1)_y = 0\). We have proved that \(a_1 = 0\).

Now there is a proper closed subset \(Z_2 \subset Z_1\) such that \(Z_1-Z_2\) is \(k\)-smooth. Then \(Z_1-Z_2\) is a \(k\)-smooth closed subvariety in \(Y-Z_2\). Arguing just as above, we conclude that \(a_2 := a|_{(Y-Z_2)\times \mathbb{A}^1} = 0\). Continuing this process finitely many times, we conclude that \(a = 0\).

Theorem 14.13. Suppose the base field \(k\) is infinite and perfect. If \(\mathcal{F}\) is an \(\mathbb{A}^1\)-invariant stable \(\mathcal{Z}_{\mathcal{F}}\)-sheaf of abelian groups, then the \(\mathcal{Z}_{\mathcal{F}_{*}}\)-presheaf of abelian groups \(X \mapsto H^1_{\text{Nis}}(X, \mathcal{F})\) is \(\mathbb{A}^1\)-invariant and stable.

Proof. By Corollary \[\ref{cor:local-coh}\] the presheaf \(X \mapsto H^1_{\text{Nis}}(X, \mathcal{F})\) has a canonical structure of a \(\mathcal{Z}_{\mathcal{F}_{*}}\)-presheaf. Let \(X\) be a \(k\)-smooth variety. The assignment \(X \mapsto (\sigma_X : \mathcal{F}(X) \to \mathcal{F}(X))\) is an endomorphism of the Nisnevich sheaf \(\mathcal{F}|_{\text{Sm/k}}\). Thus for each \(n\) it induces an endomorphism of the cohomology presheaf \(\sigma^*: H^n(\mathcal{F}) \to H^n(\mathcal{F})\). Since \(\sigma^*\) acts on \(\mathcal{F}\) as the identity, it acts as the identity on the presheaf \(H^n(\mathcal{F})\). We see that the \(\mathcal{Z}_{\mathcal{F}_{*}}\)-presheaf \(H^n(\mathcal{F})\) is stable.

To show that the presheaf \(X \mapsto H^1_{\text{Nis}}(X, \mathcal{F})\) is \(\mathbb{A}^1\)-invariant, note that the pull-back map \(i_0^*: H^1_{\text{Nis}}(X \times \mathbb{A}^1, \mathcal{F}) \to H^1_{\text{Nis}}(X, \mathcal{F})\) is surjective. It is also injective by Proposition \[\ref{prop:local-coh}\]. Our theorem now follows.

Theorem 14.14. Suppose the base field \(k\) is infinite and perfect. Let \(\mathcal{F}\) be an \(\mathbb{A}^1\)-invariant stable \(\mathcal{Z}_{\mathcal{F}_{*}}\)-sheaf of abelian groups. Then for any integer \(n \geq 2\), the presheaf \(X \mapsto H^n_{\text{Nis}}(X, \mathcal{F})\) is an \(\mathbb{A}^1\)-invariant and stable \(\mathcal{Z}_{\mathcal{F}_{*}}\)-presheaf of abelian groups.
Proof. We can apply the same arguments as in the proof of Theorem \[14.13\] to show that the presheaf \( X \mapsto H^a_{\text{Nis}}(X, \mathcal{F}) \) is a \( \mathbb{Z}F_s \)-presheaf of abelian groups, which is, moreover, stable.

It remains to check that the presheaf is homotopy invariant. We may assume till the end of the proof that each presheaf \( X \mapsto H^a_{\text{Nis}}(X, \mathcal{F}) \) with \( j < n \) is homotopy invariant.

In order to complete the proof of the theorem, we shall need a couple of lemmas. The first lemma is as follows.

Lemma 14.15. For any \( A^1 \)-invariant stable \( \mathbb{Z}F_s \)-sheaf of abelian groups \( \mathcal{F} \), any \( k \)-smooth variety \( Y \), any \( k \)-smooth divisor \( D \) in \( Y \) and any integer \( n \geq 2 \), one has the equality \( H^n_{\text{Nis}}(Y, \mathcal{F}) = 0 \).

Proof. Applying the local-global spectral sequence, it is sufficient to check that the groups \( H^i_{\text{Nis}}(X \times A^1, \mathcal{H}^{-j}_{D \times A^1}(X \times A^1, \mathcal{F})) \) vanish for \( i + j = n \) (here \( i, j \geq 0 \)). Let \( j = 1 \). By Lemma \[14.7\] one has \( \mathcal{H}^{-1}_{D \times A^1}(X \times A^1, \mathcal{F})) = I_s(\mathcal{F}_{-1}) \). Thus,

\[
H^{n-1}_{\text{Nis}}(X \times A^1, \mathcal{H}^{-1}_{D \times A^1}(X \times A^1, \mathcal{F})) = H^{n-1}_{\text{Nis}}(X \times A^1, I_s(\mathcal{F}_{-1})) = H^{n-1}_{\text{Nis}}(D \times A^1, \mathcal{F}_{-1}) = 0.
\]

The latter equality holds since \( D \) is local henselian. If \( j = 0 \), then the sheaf \( \mathcal{H}^{-1}_{D \times A^1}(X \times A^1, \mathcal{F}) \) vanishes. Obviously, the stalk of this sheaf vanishes at every point \( z \in (X - D) \times A^1 \). If \( z \in D \times A^1 \) then the stalk vanishes by Theorem \[2.15\] (item (3')). Let \( 2 \leq j \leq n \). Then the sheaf \( \mathcal{H}^{-j}_{D \times A^1}(X \times A^1, \mathcal{F}) \) vanishes. Obviously, the stalk of this sheaf vanishes at every point \( z \in (X - D) \times A^1 \). If \( z \in D \times A^1 \) then the stalk of the sheaf at the point \( z \) is equal to the group \( H^{-j}_{\text{Nis}}(\mathcal{F}, \mathcal{D}) \), where \( \mathcal{D} \) is the henselization of \( X \times A^1 \) at \( z \) and \( \mathcal{F} \) is the henselization of \( D \times A^1 \) at \( z \). Since \( H^{-j}_{\text{Nis}}(\mathcal{D}, \mathcal{F}) = 0 \) one has equalities

\[
H^{-j}_{\text{Nis}}(\mathcal{F}, \mathcal{D}) = H^{-j}_{\text{Nis}}(\mathcal{D} - \mathcal{D}, \mathcal{F})/\text{Im}[H^{-j}_{\text{Nis}}(\mathcal{F}, \mathcal{D})] = H^{-j}_{\text{Nis}}(\mathcal{D} - \mathcal{D}, \mathcal{F})/H^{-j}_{\text{Nis}}(\mathcal{F}, \mathcal{D}).
\]

The latter equality makes sense by Theorem \[2.15\] (item (3')).

Now, applying Theorem \[2.15\] (item (5)) and Corollary \[2.16\] to the presheaf \( H^{-j}_{\text{Nis}}(\mathcal{F}, \mathcal{D}) \) and arguing as in the proof of Lemma \[14.6\] we get an isomorphism

\[
H^{-j}_{\text{Nis}}(\mathcal{F}, \mathcal{D})/H^{-j}_{\text{Nis}}(\mathcal{F}, \mathcal{D}) \cong H^{-j}_{\text{Nis}}(\mathcal{D} \times \mathbb{G}_m, \mathcal{F})/H^{-j}_{\text{Nis}}(\mathcal{D} \times \mathbb{G}_m, \mathcal{F}).
\]

It suffices to check that \( H^{-j}_{\text{Nis}}(\mathcal{D} \times \mathbb{G}_m, \mathcal{F}) = 0 \). The presheaf \( H^{-j}_{\text{Nis}}(\mathcal{D} \times \mathbb{G}_m, \mathcal{F}) \) is a homotopy invariant \( \mathbb{Z}F_s \)-presheaf, which is also stable. By Theorem \[2.15\] (item (3')) the map

\[
H^{-j}_{\text{Nis}}(\mathcal{D} \times \mathbb{G}_m, \mathcal{F}) \to H^{-j}_{\text{Nis}}(\text{Spec}(k(D)) \times \mathbb{G}_m, \mathcal{F}) = H^{-j}_{\text{Nis}}(\mathcal{G}_{m,k(D)}, \mathcal{F})
\]

is injective. By Theorem \[2.9\] the map \( H^{-j}_{\text{Nis}}(\mathcal{G}_{m,k(D)}, \mathcal{F}) \to H^{-j}_{\text{Nis}}(\text{Spec}(k(D)(t)), \mathcal{F}) \) is injective. The latter group vanishes, because the Nisnevich topology on \( \text{Spec}(K) \) is trivial, where \( K \) is a finitely generated field over \( k \). Thus \( H^{-j}_{\text{Nis}}(\mathcal{D} \times \mathbb{G}_m, \mathcal{F}) = 0 \) and \( H^{-j}_{\text{Nis}}(\mathcal{F}, \mathcal{D}) = 0 \), too. The lemma follows.

Let us return to the proof of Theorem \[14.14\]. Under the hypotheses of Lemma \[14.7\], the preceding lemma implies the map

\[
H^0_{\text{Nis}}(X \times A^1, \mathcal{F}) \to H^0_{\text{Nis}}((X - D) \times A^1, \mathcal{F})
\]

is injective.

Next, we claim that for a \( k \)-smooth variety \( Y \) and the projection \( p : Y \times A^1 \to Y \) the Nisnevich sheaves \( R^jp_*(\mathcal{F}) \) vanish for \( j = 1, \ldots, n - 1 \). In fact, such a sheaf is associated with
the presheaf \( U \mapsto H^j_{\text{Nis}}(U \times \mathbb{A}^1, \mathcal{F}) \). The presheaf \( H^j_{\text{Nis}}(U, \mathcal{F}) \) is homotopy invariant. Thus \( H^j_{\text{Nis}}(U \times \mathbb{A}^1, \mathcal{F}) = H^j_{\text{Nis}}(U, \mathcal{F}) \). Since \( j \geq 1 \) the associated Nisnevich sheaf vanishes.

Since the Nisnevich sheaves \( R^j p_* (\mathcal{F}) \) vanish for \( j = 1, \ldots, n-1 \), one has an exact sequence

\[
0 \to H^n_{\text{Nis}}(Y, p_*(\mathcal{F})) \xrightarrow{\alpha} H^n_{\text{Nis}}(Y \times \mathbb{A}^1) \xrightarrow{\beta} H^0_{\text{Nis}}(Y, R^n p_*(\mathcal{F})).
\]

Since the sheaf \( \mathcal{F} \) is homotopy invariant, then for

\[
A := \text{Ker}\{H^0_{\text{Nis}}(Y \times \mathbb{A}^1, \mathcal{F}) \to H^1_{\text{Nis}}(Y, \mathcal{F})\}
\]

the map \( \beta|_A : A \to H^0_{\text{Nis}}(Y, R^n p_*(\mathcal{F})) \) is injective. Arguing as in the proof of Proposition 14.12 and using the fact that map (32) is injective, we get the following

**Lemma 14.16.** Suppose the base field \( k \) is infinite and perfect. Let \( \mathcal{F} \) be an \( \mathbb{A}^1 \)-invariant stable \( \mathbb{Z}_F \)-sheaf of abelian groups. Let \( Y \) be a \( k \)-smooth variety and let \( a \in H^n_{\text{Nis}}(Y \times \mathbb{A}^1, \mathcal{F}) \) be an element such that its restriction to \( Y \times \{0\} \) vanishes. Then \( a = 0 \).

The pull-back map \( i^0_\ast : H^n_{\text{Nis}}(Y \times \mathbb{A}^1, \mathcal{F}) \to H^n_{\text{Nis}}(Y, \mathcal{F}) \) is surjective by functoriality. By Lemma (14.16) it is also injective. This completes the proof of Theorem 14.14. \( \square \)

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