Convergence of the mirror to a rational elliptic surface

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Abstract

The construction introduced by Gross, Hacking and Keel in (Several Complex Variables (Springer, New York, NY, 1976)) allows one to construct a formal mirror family to a pair $(S, D)$ where $S$ is a smooth rational projective surface and $D$ a certain type of Weil divisor supporting an ample or anti-ample class. In that paper, they proved two convergence results when the intersection matrix of $D$ is not negative semi-definite and when the matrix is negative definite. In the original version of that paper, they claimed that if the intersection matrix were negative semi-definite, then family extends over an analytic neighbourhood of the origin but gave an incorrect proof. In this paper, we correct this error. We reduce the construction of the mirror to such a surface to calculating certain log Gromov–Witten invariants. We then relate these invariants to the invariants of a new space where we can find explicit formulae for the invariants. From this we deduce analytic convergence of the mirror family, at least when the original surface has an $I_4$ fibre.

1. Introduction

The geometry of surfaces is often a rich playground in which to test new conjectures and constructions. In [15], the authors introduced a construction of a mirror family to certain pairs $(S, D)$, called Looijenga pairs. These are a smooth Fano surface $S$ and an anti-canonical divisor $D$ which is a chain of rational curves. This built upon work of [4] who had made predictions of the mirror families to del Pezzo surfaces. The constructions of both papers make sense when $-K_S$ is not ample, but merely nef, equivalently when $S$ is a rational elliptic surface and $D = F$ is a rational fibre. In this paper, we apply the construction of [15] to such a pair and construct its mirror.

This project is an initial step towards understanding mirror symmetry for type II degenerations of a $K3$ surface $K$, described in [22]. Upto birational modification this is a degeneration of $K$ into two components $S_1$ and $S_2$ meeting along a common smooth anti-canonical divisor $D$. To construct the mirror to $K$, one could attempt to construct the mirrors to the pairs $(S_1, D)$ and $(S_2, D)$, and then attempt to glue the resulting constructions. There is much more work to be done before this can be made precise using the Gross–Siebert program.

Let us begin by giving a short and rough introduction to the Gross–Siebert program. A more detailed explanation can be found in both my thesis [5] and in [6]. Recent work in [20] has given a complete treatment of the theory, but at a very high level.

The Gross–Siebert program proceeds in three steps. Given a log structure on a variety $X$, they construct an integral affine manifold with singularities, $\Sigma(X)$. In this case, it is the dual intersection complex of $D$ as described in [15, Section 1], in the non-toric case it is a cone...
complex with a unique singular point. This carries a canonical scattering diagram defined in [15, section 3.1], controlling how one glues the complex structure on the mirror. Finally one defines a ring of theta functions on the integral points of $\Sigma(X)$ counting pairs of broken lines. These broken lines are introduced in [15, Definition 2.16] and consist of piecewise linear curves, whose non-linearity is controlled by the scattering diagram. This produces a family over Spec $k[P]$ a monoid ring related to effective curve classes in $X$.

In Section 2, we recall the notation and definitions of [15]. The main result is that when $S$ is a rational elliptic surface and $D = D_1 \cup \ldots \cup D_4$ an $I_4$ fibre, the universal cover of the dual intersection complex is the one described in Figure 1.

Section 2 introduces the combinatorics of broken lines and by giving a loose description of the canonical scattering diagram proves that calculations in the ring of theta functions introduced in the previous section reduce to calculate a infinite but tractable list of log Gromov–Witten invariants. In particular, we already see the general form that the mirror must take:

**Lemma 1.1 (Lemma 3.1).** Let $S$ be a rational elliptic surface and $D = D_1 \cup \ldots \cup D_4$ an $I_4$ fibre. Then the product formula of [15] for the ring of $\vartheta$-functions is defined by six algebraic expressions (3.1)–(3.3) whose coefficients are weighted counts of rational curves on $S$ tangent to $D$ to order 1 or 2.

Those tangent to order 1 must decompose as sums of a section plus some number of covers of rational fibres. In Section 4, we study the invariants of such curves by recalling work of Bryan and Leung [8]. By bounding the weights and the number of such curves we are able to show that some of the families of functions appearing above are convergent, which we do later in Lemma 9.3.

The main part of this paper focuses on the threefolds constructed in Section 5. These are families of elliptically fibred K3 surfaces appearing as nef complete intersections in toric varieties. We explain the relevance of these varieties in the start of the section by splitting the families of curves we are interested in into three types: point, curve and bubble components. Roughly these threefolds allow us to count curves on $S$ which ramify over a particular fibre. The threefolds themselves are introduced in Construction 5.5. In Section 6, we apply techniques of Givental to calculate the Gromov–Witten theory of such a space and show that the $I$ and $J$ functions are holomorphic. We interpret the invariants appearing in low degree in terms of the invariants on $S$.

The technical work of the paper concerns relating certain Gromov–Witten invariants between the initial rational elliptic surface $S$ and a related unravelled threefold $X$. This is carried out in Section 7, in particular, we have the following compatibility result:

**Theorem 1.2 (Theorem 7.2).** For a generic choice of unravelling $d : X \to S$, there is an equality between the virtual classes associated to the obstruction theories on $\overline{\mathcal{M}}_{0,0}(X, d^{-1}\beta)$ and $\overline{\mathcal{M}}_{0,0}(S, \beta)$:

$$[\overline{\mathcal{M}}_{0,0}(X, d^{-1}\beta)]^\text{virt} = 2e^! [\overline{\mathcal{M}}_{0,0}(S, \beta)]^\text{virt}.$$
We then finally compare the log invariants of \( S \) to the standard Gromov–Witten invariants of \( S \). This is carried out in Theorems 7.3 and 7.4. All of these are proved by constructing compatible triples between the obstruction theories.

If all the contributions from the components considered above were positive, then we would be able to deduce convergence, since we only sum over a subset of these components. This is not the case and Section 8 is dedicated to showing that the sum of the absolute values of these contributions is also bounded.

From this, we can prove holomorphicity of all the terms in equations (3.1)–(3.3). We do this in Section 9 to prove convergence.

**Theorem 1.3 (9.4).** The equations defining the mirror family to a rational elliptic surface relative to an \( I_4 \) fibre converge in a neighbourhood of the origin.

In Section 10, we interpret the induced mirror family and in particular the fibre at infinity in terms of classical Jacobi \( \Theta \) functions.

**Theorem 1.4 (Theorem 10.1).** Let \( S \) be a rational elliptic surface with \( D \) an \( I_4 \) fibre. There exists a projective closure of the restriction of the mirror family to a locus in the base such that the boundary is a smooth elliptic curve mirror to the generic fibre of \( S \). This duality is induced by an expression for the \( \vartheta \)-functions in terms of Jacobi theta functions.

We should say that these techniques are specific to the given situation. Suppose that we have an elliptically fibred surface \( \rho: S \to \mathbb{P}^1 \) and we wanted to calculate curves tangent to a fibre \( F \) at a single point of maximal order in a class \( \beta \). The discussion relating tangency at \( F \) with the ramification continues as explained later. However, if \( \beta \cdot F > 2 \), then the morphism from the space of log stable maps to the moduli space of stable maps ramifying maximally along \( F \) is not an isomorphism but in an appropriate manner ‘virtually birational’. As far as we can tell, the intersection theory for such morphisms has not been studied. A second problem is that if \( S \) is not a rational elliptic surface, for example, a \( K3 \) surface, then the construction of the unravelled threefold need not be a nef intersection in a toric variety. This means that one cannot apply the techniques of Givental to calculate the Gromov–Witten invariants. Therefore this paper limits any discussion to the described setting.

1.1. Notation and remarks

Throughout this \( S \) will denote a rational elliptic surface, which by [25, Lemma IV.1.2] is isomorphic to the blowup in the nine points lying on the intersection of two cubics in \( \mathbb{P}^2 \). The Chow group \( A_1(S) \) is spanned by \( H \), the pullback of a hyperplane, and \( E_1, \ldots, E_9 \), the exceptional curves of the blowup. The Chow groups \( A_0(S) \) and \( A_2(S) \) are both one dimensional, spanned by a point and a fundamental class, respectively. \( F \) will denote a general fibre, whilst \( F_0 \) will denote an \( I_4 \) fibre, a cycle of four \( -2 \)-curves. We will write \( S^\dagger \) for \( S \) together with the divisorial log structure coming from \( F_0 \). By an inclusion of components \( i: X \to Y \) we mean an isomorphism from \( X \) to a union of connected components of \( Y \).

2. The Gross–Siebert program

The Gross–Siebert program began as an attempt to make rigorous geometric constructions from the SYZ conjecture using logarithmic and tropical geometry. The conjecture as proposed in [30] states that mirror symmetry is approximately a duality of torus fibrations over an integral affine base with singularities \( \Sigma(X) \).
The construction of [15] studies a Looijenga pair, a pair \((S, D)\) where \(S\) is a smooth rational surface and \(D\) an anti-canonical cycle of rational curves. It states that the base for the fibration should be the dual intersection complex of \(D\), defined in [15] at the start of Section 2 which we recall now.

**Definition 2.1.** Starting with \((S, D)\) a Looijenga pair, we define a cone complex which has a unique zero-dimensional strata, a one-dimensional strata \(\langle \mathbb{R}_\geq 0v_i \rangle\) for each component \(D_i\) and a two-dimensional strata \(\langle \mathbb{R}_\geq 0v_i \oplus \mathbb{R}_\geq 0v_{i+1} \rangle\) corresponding to each component of \(D_i \cap D_{i+1}\).

To define an integral affine structure, we glue \(\langle \mathbb{R}_\geq 0v_{i-1} \oplus \mathbb{R}_\geq 0v_i \rangle\) and \(\langle \mathbb{R}_\geq 0v_i \oplus \mathbb{R}_\geq 0v_{i+1} \rangle\) by embedding them into \(\mathbb{R}^2\) via the relation \(v_{i-1} + (D_i)^2v_i + v_{i+1} = 0\). We write this affine manifold \(\Sigma_{(S,D)}\).

Of course this base need not embed into \(\mathbb{R}^2\), if \((S, D)\) is not toric, then there is a unique singularity at the origin. This is precisely as expected from the SYZ picture and is where interesting geometry can enter the picture. In the case at hand \(D = D_1 \cup \ldots \cup D_4\), all the components \(D_i\) satisfy \(D_i^2 = -2\), and the relations in the affine manifold becomes \(v_{i-1} + v_{i+1} = 2v_i\).

In particular, the dual intersection complex drawn in 1 is the universal cover of the dual intersection complex \(\Sigma_{(S,D)}\) with the apparent induced integral structure. Fix a sharp monoid \(P \subset A_1(S)\) which is finitely generated and contains all effective curves. There is a multivalued piecewise linear function \(\phi\) valued in \(P\), defined up to a choice of globally defined piecewise linear function. It changes across \(v_i\) by \(n_i \otimes D_i\).

In the toric case, there is no affine singularity and one can construct the mirror family via the Mumford degeneration. In [15, section 1.3], the authors generalise this construction to the non-toric case by introducing a sheaf of coefficients. See the definition there for the details and notation, but note that if one runs the Gross–Hacking–Keel construction in the toric case, one obtains a family over \(\text{Spec } k[P]\). Pairing an element of \(P\) with the anti-canonical class produces a family over \(\mathbb{A}^1\) recovering the Mumford degeneration.

In general, however, the induced gluing data are not consistent beyond order 0. To correct it, one should correct by using automorphisms formed from a scattering diagram. We have to introduce the definition here since we use slightly different notation from [15], working directly in terms of sheaves.

**Definition 2.2.** Let \(B\) be an affine manifold with a single singularity, with \(B\) homeomorphic to \(\mathbb{R}^2\) with singularity at the origin, so that \(B^* := B \setminus \{0\}\) is an affine manifold. Let \(\mathcal{M}\) be a locally constant sheaf of abelian groups on \(B^*\) with a subsheaf of monoids \(\mathcal{M}^+ \subset \mathcal{M}\) and equipped with a map \(r : \mathcal{M} \to \Lambda_B\) to the sheaf of integral vector fields on \(B\). Let \(\mathcal{J}\) be a sheaf of ideals in \(\mathcal{M}^+\) with stalk \(\mathcal{J}_x\) maximal in \(\mathcal{M}^+_x\) for all \(x \in B^*\). Let \(\mathcal{R}\) denote the sheaf of rings locally given by the completion of \(k[\mathcal{M}^+]\) at \(\mathcal{J}\). A scattering diagram with values in the pair \((\mathcal{M}, \mathcal{J})\) on \(B\) is a function \(f\) which assigns to each rational ray from the origin a section of the restriction of \(\mathcal{R}\) to the ray. We require the following properties of this function.

- For each \(\mathfrak{d}\), one has \(f(\mathfrak{d}) = 1 \mod \mathcal{J}|_{\mathfrak{d}}\).
- For each \(n\), there are only finitely many \(\mathfrak{d}\) for which \(f(\mathfrak{d})\) is not congruent to 1 \mod \(\mathcal{J}|_{\mathfrak{d}}\). These \(\mathfrak{d}\) are called walls.
- For each ray \(\mathfrak{d}\) and for each monomial \(z^p\) appearing in \(f(\mathfrak{d})\), one has \(r(p)\) tangent to \(\mathfrak{d}\). A line for which \(r(p)\) is a positive generator of \(\mathfrak{d}\) for all \(p\) with \(c_p \neq 0\) is called an incoming ray. If instead \(r(p)\) is a negative generator of \(\mathfrak{d}\) for all \(p\) with \(c_p \neq 0\), it is called an outgoing ray.

We denote such an object by the tuple \((B, f, \mathcal{M}, \mathcal{J})\). We say that \((B, f, \mathcal{M}, \mathcal{J})\) is obtained from \((B', f', \mathcal{M}, \mathcal{J})\) by adding outgoing rays if for each ray \(\mathfrak{d}\) one can write \(f(\mathfrak{d}) = f'(\mathfrak{d})(1 + \sum (c_p z^p))\) where for each monoid element \(p\) with \(c_p \neq 0\) the vector \(-r(p)\) is a generator of \(\mathfrak{d}\).
In our case, the sheaf of monoids $M^+$ will be as given in [15, Construction 2.2], with the monoid $M$ being a finitely generated sharp submonoid of $H_2(S,\mathbb{Z})$ containing $NE(S)$, the monoid generated by effective curves on $S$. Being sharp means the only invertible element of $M$ is the identity element, and so the maximal ideal is just the complement of the identity.

The choice of scattering diagram is the canonical scattering diagram, introduced in [15, Section 3.1]. A ray $\mathfrak{d}$ specifies a blow-up of $S$ at the boundary, and let $D_\mathfrak{d}$ be the exceptional curve. The function associated to this ray is a sum of genus zero curves meeting $D_\mathfrak{d}$ at a single point tangent to maximal order according to the sum contained at the top of [15, p. 42].

The gluing data are supposed to be corrected by sums of pairs of pants. A broken line formalises one leg of the pair, pairs of broken lines appear in the next section as pairs of legs.

**Definition 2.3.** A broken line from $v \in B(\mathbb{Z})$ to $P \in B$ on a scattering diagram $(B,f,\mathcal{M},J)$ is a choice of piecewise linear function $l : \mathbb{R}^0 \to B$ and a map $m : \mathbb{R}^0 \to \prod_{t \in \mathbb{R}^0} \mathcal{M}_{l(t)}$ satisfying the conditions of [15, Definition 2.16].

Locally the conditions of [15, Definition 2.16] say that $l$ can only be non-linear along at the intersection with a rational ray, and that at such a point $m$ changes by multiplication by a monomial in $f(\mathfrak{d})^i$ where $i = \langle n, r(m) \rangle$ for $n$ a normal to $\mathfrak{d}$ positive negative in the direction of increasing $t$. In particular, for the canonical scattering diagram the linear term and quadratic terms count curves tangent to $D_\mathfrak{d}$ to order 1 and 2. If a broken line is non-linear, changing by once or twice the primitive generator of $\mathfrak{d}$, then the associated change in the monomial is governed by counts of curves tangent to $D_\mathfrak{d}$ to order 1 or 2.

3. **The structure of the mirror family**

The next insight is that the underlying space of the mirror family should be described in terms of the symplectic cohomology of $S \setminus D$. Rather than proving this directly, Gross Hacking and Keel tropicalise and use this as the definition.

The key data we will need are the count of pairs of pants, which are expressed tropically as pairs of broken lines $(l_P,m_P)$ and $(l_Q,m_Q)$ from $P$ and $Q$, respectively, to an irrational point near to $R$ such that $(\partial l_P/\partial t)|_{t=0} + (\partial l_Q/\partial t)|_{t=0} + R = 0$. Let $T_{P,Q} \to R$ denote the set of such pairs.

Now to construct an algebraic version of the symplectic cohomology, we follow [16] and for each integral point $P$ in $B$ introduce a symbol $\vartheta_P$. As a $k$-vector space, the ring $QH(S,D)$ is freely generated by the $\vartheta_P$. We take the content of [15, Theorem 2.34] as the definition of the product of $\vartheta_P$ and $\vartheta_Q$, so this product is equal to

$$\sum_R \sum_{((l_P,m_P),(l_Q,m_Q)) \in T_{P,Q} \to R} m_P(0)m_Q(0)\vartheta_R.$$ 

This is the approach of [17, 20]. A key result of [15] is that for fixed order $J^n$ and for a consistent scattering diagram in the sense of [15, Definition 2.26], this does not depend on the choice of irrational point near $R$. This consistency property will hold in particular for the canonical scattering diagram by [15, Theorem 3.8]. This sum need not terminate and indeed will in general only produce a power series. However, in the case that $D$ supports an ample divisor, this power series will be a polynomial. Therefore we should search for relations between the $\vartheta$-functions.

Now let us forget everything about the scattering diagram and suppose that it is as bad as possible, so it has rays in every direction. We can use the integral structure to limit the possible terms appearing in the product. Let us explain how to apply this idea in the case of a rational elliptic surface.
**Lemma 3.1.** Let $S$ be a rational elliptic surface and $D = D_1 \cup \ldots \cup D_4$ an $I_4$ fibre. Then the product formula of [15] for the ring of $\vartheta$-functions produces the following equations:

$$\vartheta_{D_1} \vartheta_{D_2} = f_{(2,2)} \vartheta_{D_2} + f_{(6,2)} \vartheta_{D_4} + \sum f_{(i,1)} \vartheta_{D_i} + f_0,$$

$$\vartheta_{D_2} \vartheta_{D_4} = g_{(0,2)} \vartheta_{D_2} + g_{(4,2)} \vartheta_{D_4} + \sum g_{(i,1)} \vartheta_{D_i} + g_0,$$

$$\vartheta_{D_i} = (1 + r_{(i,2)}^i) \vartheta_{D_i} + r_{(i+2,2)}^i \vartheta_{D_{i+2}} + \sum r_{(i,1)}^i \vartheta_{D_i} + r_0^i,$$

where $r_{(i,2)}^i$ and $r_{(i+2,2)}^i$ vanish on the central fibre. The functions $f_{(2,2)}, f_{(6,2)}, g_{(0,2)}, g_{(4,2)}, r_{(i,2)}^i$ and $r_{(i+2,2)}^i$ count certain pairs of broken lines not bending. The terms $f_{(i,1)}, g_{(i,1)}$ and $r_{(i,1)}^i$ count pairs of broken lines where one of the lines bends once off of a ray with primitive $(k,1)$. The remaining terms count pairs of broken lines bending in three different ways, either one broken line bends off of a ray with primitive $(k,2)$, both broken lines bends off of rays with primitives $(k,1)$ and $(k',1)$ or one of the broken lines bends twice, off of rays with primitives $(k,1)$ and $(k',1)$.

**Proof.** We apply the balancing condition to limit the bends which can occur. Consider the $y$-coordinate of a point of $B$, $y(P)$, as pictured in Figure 1. This extends to the tangent bundle of $B$ and we write $y(v)$ for this extension. Importantly for us $y(-)$ is positive on points of $B$, and given vectors $v_i$ with $\sum v_i = 0$ we have $\sum y(v_i) = 0$. Let $(l,m)$ be a broken line, we claim that $y(\partial l/\partial t)$ is an increasing function on the linear components of $l$. At points where $l$ is non-linear by definition the tangent vector changes by a positive generator of the ray $D_i$ and $y(-)$ is positive on such a generator. Similarly if $l$ crosses from one maximal cell to another, then the linearity of $y$ shows that $y(\partial l/\partial t)$ is constant, while convexity of $\varphi$ implies that the value of $\varphi$ has increased.

Now suppose that $(l_1, m_1)$ and $(l_2, m_2)$ are two broken lines from $v_1$ and $v_2$, respectively, combining to form a pair of pants at $P$. By definition we have an equality:

$$y(\partial l_1/\partial t)|_{t=0} + y(\partial l_2/\partial t)|_{t=0} + y(P) = 0.$$

Since $y(P)$ is non-negative and $-y(v_i) \leq y((\partial l_i/\partial t)|_{t=0})$, we see that $y(v_1) + y(v_2)$ determines how many times such broken lines can bend. By restricting to the case where $F_0$ is an $I_4$ fibre, we claim that we need only consider products with $y(v_i) = 1$, as in [6, section 3.2]. This follows since any other monomial can be written as a product of these, plus a correction with either lower $y$-coordinates or the same value of $y(-)$ and coefficients lying in a higher power of $F$. Expanding this term by term in the formal completion, we can reduce to only products of these monomials. Restricting to the product of such elements the only possible pairs of broken lines which can appear are the following.

1. Pairs where neither broken line bends.
2. Pairs where one broken line bends along a monomial $m$ with $y(r(m)) = 1$.
3. Pairs where one broken line bends along a monomial $m$ with $y(r(m)) = 2$.
4. Pairs where both broken line bends along monomials $m_i$ with $y(r(m_i)) = 1$.
5. Pairs where one broken line bends twice along monomials $m_i$ with $y(r(m_i)) = 1$.

Recall from the definition of the canonical scattering diagram that the quantity $m$ corresponds to a curve class in $S$. If $y(r(m)) = 1$, then this is necessarily a primitive generator, corresponding by the formula for the canonical scattering diagram to a curve meeting only one of the $D_i$ at a single point transversely. If $y(r(m)) = 2$, then it is either primitive, corresponding to a curve passing through $D_i \cap D_{i+1}$ transverse to both components, or meeting only one of the $D_i$ at a single point tangent to order 2. In particular, $y(r(m))$ describes the intersection of the corresponding curve with $[F]$. 


From this we see that pairs of type 1 above contribute to \( f(k,2), g(k,2), r^1_{i,2} \) or \( r^1_{i+2,2} \), pairs of type 2 contribute to \( f(i,1), g(i,1) \) or \( r^1_{i,1} \). The remaining three items contribute to \( f_0, g_0 \) or \( r^1_0 \).

Whilst the original paper [15] used relative invariants we here use log Gromov–Witten invariants. The compatibility results of [2] show that these two approaches agree. We turn to [26] for the basic definitions of the subject, and to [19] for the definition of the moduli space and enumerative invariants.

4. Sections of a rational elliptic surface

Bryan and Leung in [8] studied the Gromov–Witten theory of an elliptically fibred K3 surface with a section \( E \) and fibre \( F \). For such a surface, one cannot use the Gromov–Witten theory of the K3, but rather the reduced Gromov–Witten theory, and we will encounter this later in Theorem 8.2.

In particular, in Section 6 they showed that any stable map in this moduli space is a union of a component mapping to \( E \) and some combinatorial data describing a cover of different rational fibres and that infinitesimally there are no deformations smoothing the nodes between the component mapping to \( E \) and those mapping to rational fibres. Then in Lemma 5.7 the authors apply well-known formulae for the Gromov–Witten theory of blow ups of \( \mathbb{P}^2 \) to show that each component of the moduli space contributes either a one or a zero to the total count.

They also performed the same analysis, deducing the same geometric results for the moduli space of stable maps to a rational elliptic surface \( S \). In particular, there is no infinitesimal smoothing of any of the nodes lying on the intersection of the component mapping to the section and those mapping to the fibres. In Theorem 7.2, they provide the following formula for \( I_{0,0,E+nF} \), the number of unmarked rational curves in class \( E + nF \):

\[
\sum_{n=0}^{\infty} I_{0,0,E+nF} z^n = \prod_{m=1}^{\infty} (1 - z^m)^{-12}.
\]

We also need to know that there are not too many classes of sections of a given degree on \( S \). Once we combine these two results we will produce a bound on the total number of sections intersecting an ample divisor to bounded degree. Thankfully bounding the number of curves is an easy exercise in classical algebraic geometry:

**Lemma 4.1 (Bounds on the number of sections).** Let \( d \in \mathbb{N} \), define the set

\[
GZ(S,d) = \{ [C] \in A_1(S) \mid [C] \sim dH - \sum a_iE_i, [C].[F] = 1, p_a([C]) \geq 0 \},
\]

where \( p_a([C]) \) is the arithmetic genus of a generic member of this family. Then \( |GZ(S,d)| \leq (N \sqrt{d} + N)^9 \) for some \( N \) independent of \( d \).

**Proof.** Let \( f : C \rightarrow S \) be a stable map. There is an inequality \( g(C) \leq p_a(f_*[C]) \) and so this counts the number of possible curve classes for stable maps of genus zero. To prove the above bound, we apply the genus formula together with basic intersection theory. Let \([C] = d[H] - \sum a_i[E_i]\) be a curve class with \([C].[K_S] = -1\), so \(3d - \sum a_i = 1\). Then the genus formula states that \(p_a(C) = 1 + 1/2([C].[C] - [C].[F])\) and \(p_g(C) \leq p_a(C)\), so

\[
0 \leq 1 + 1/2([C].[C] - [C].[F]) \\
\leq 1/2 + 1/2[C].[C]
\]
hence
\[ 0 \leq 1 + d^2 - \sum a_i^2 \] (4.2)

By the arithmetic–geometric inequality, \( \sum a_i^2 \) is minimised when all the \( a_i \) are equal. We need to find a bound for \( |a_i - a_j| \) in terms of \( d \) given the inequality above. Thus suppose that \( a_1 + k = 1/8 \sum a_i \), and we will bound \( k \). Again \( \sum a_i^2 \) is minimised when \( a_2 = a_3 = \cdots = a_9 \).

Using \( 3d = \sum a_i + 1 \), we can write out everything explicitly
\[ a_1 = \frac{3d - 1 - 8k}{9}, \]
\[ a_2 = a_3 = \cdots = \frac{3d - 1 + k}{9}. \]

Thus (4.2) implies
\[ 1 + d^2 \geq \sum a_i^2 \geq d^2 - \frac{2}{3}d + \frac{8}{9}k^2 + \frac{1}{9}, \]

and hence
\[ 1 + \frac{3d}{4} \geq k^2. \]

Thus we see that for \( |k| > \sqrt{d} + 2 \) there can be no classes satisfying the above inequality. By symmetry, we may assume that \( a_1 \) is the smallest of the \( a_i \). This then shows that none of the \( a_i \) can be more than \( \sqrt{d} + 2 \) from \((3d - 1)/9\), thus the total possible combinations is bounded by a bound of the above form. \( \square \)

In particular, the number of rational curves of degree at most \( d \) grows slower than \( 2^d \).

5. The unravelled threefold

In this section, we make use of the fibration \( \rho \) to construct a space lifting the Gromov–Witten theory of \( S \). We will take a family of double covers of \( \mathbb{P}^1 \) and construct a threefold \( X \) fibred by K3 surfaces with known singularities. Each fibre will be a double cover of \( S \) and so possesses an elliptic fibration. This construction induces a strong relation between the moduli spaces of stable maps to \( S \) and the moduli space of stable maps to \( X \). Let us begin though by thinking about the geometric properties of the curves we wish to count.

Let \( C \) be an irreducible rational curve on \( S \) with \( C.F = 2 \). Restricting the fibration \( \rho \) to \( C \), we therefore obtain a double cover \( \mathbb{P}^1 \rightarrow \mathbb{P}^1 \). By the Riemann–Hurwitz formula, this map must ramify at two points and up to a choice of involution these two points uniquely specify the double cover. This suggests that there should be a map from the moduli space of stable maps in the class \([C] \) to the Hilbert scheme of two points on \( \mathbb{P}^1 \).

Let \([C] \) be an effective curve class in \( D(S, \mathbb{Z}) \) with \([C].[F] = 2 \). We will denote the moduli space of stable rational unmarked maps to \( S \) with image in class \([C] \) by \( \overline{\mathcal{M}}_{0,0}(S, [C]) \). This moduli space carries a universal curve, \( \pi : \mathcal{C}_{0,0}(S, [C]) \rightarrow \overline{\mathcal{M}}_{0,0}(S, [C]) \), and a universal stable map, \( f : \mathcal{C}_{0,0}(S, [C]) \rightarrow S \). Taking the composition \( \rho \circ f \), we obtain a family of branched double covers of \( \mathbb{P}^1 \). Passing to the stabilisation, we therefore obtain a morphism \( \overline{\mathcal{M}}_{0,0}(S, [C]) \rightarrow \overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, 2) \). The space \( \overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, 2) \) meanwhile is naturally isomorphic to \([\text{Hilb}^2(\mathbb{P}^1)}/C_2 \). Here the \( C_2 \) action on \( \text{Hilb}^2(\mathbb{P}^1) \) is trivial, reflecting the involution on a double cover of \( \mathbb{P}^1 \). Therefore via composition we obtain a morphism \( \text{ram} : \overline{\mathcal{M}}_{0,0}(S, [C]) \rightarrow [\text{Hilb}^2(\mathbb{P}^1)}/C_2 \). A stable map in \( \overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, 2) \) is a double cover of \( \mathbb{P}^1 \).
by a rational curve hence ramified over two points. The map \( \text{ram} \) sends such a curve to the two ramified points.

Our goal will be to understand the images of different irreducible components of the moduli space \( \overline{M}_{0,0}(S,[C]) \) under this map \( \text{ram} \). To do this we wish to understand how the fibres of \( \pi \) vary as double covers of \( \mathbb{P}^1 \). This requires that we be able to identify which components contribute to the double cover for which we label the different components as follows. Let \( f : C \to S \) be a stable map corresponding to a closed point of \( \overline{M}_{0,0}(S,[C]) \). There are three types of components of \( C \): contracted components, components covering fibres of \( \rho \) and components which cover the base \( \mathbb{P}^1 \) of the fibration. We will assign one of the two labels fib and tr to each component. Components covering fibres will be labelled with fib whilst components covering the base will be labelled tr. Any contracted component which is part of a chain linking two components covering the base will be labelled tr and all the remaining contracted components will be labelled fib. This divides \( C \) into two types of components based on these labels, which we will denote by \( C_{\text{tr}} \) and \( C_{\text{fib}} \). The component \( C_{\text{tr}} \) consists either of a single component mapping as a double cover to the base \( \mathbb{P}^1 \) or as a chain of contracted components joining two components mapping isomorphically to the base or is the union of two disconnected components. Note that \( C_{\text{fib}} \) may very well be disconnected. We write \( C_{\text{contr}} \) for the curve obtained from \( C_{\text{tr}} \) by contracting out the contracted components, noting that if \( C_{\text{tr}} \) is connected, then \( C_{\text{contr}} \) admits a map to \( S \). The following Theorem 5.1 shows that in a fibre of \( \text{ram} \circ \pi \) the component \( C_{\text{tr}} \) is rigid, only the component \( C_{\text{fib}} \) deforms.

**Theorem 5.1** (Fibres of \( \text{ram} \circ \pi \) vary as families of curves). The component \( f(C_{\text{tr}}) \) is locally constant as a subscheme of \( S \) along fibres of \( \text{ram} \circ \pi \). Hence the image under \( \text{ram} \circ \pi \) of a family of curves in \( \overline{M}_{0,0}(S,[C]) \) with \( f(C_{\text{tr}}) \) varying as a subscheme of \( S \) is positive dimensional.

**Proof.** Let \( B \xleftarrow{\pi} C \xrightarrow{f} S \) be a family of genus zero unmarked stable maps contained inside a fibre of \( \text{ram} \). First we treat the case where \( C_{\text{tr}} \) is irreducible.

Fix a choice of section \( E \) for the rational elliptic surface which we take to be the identity for the group law. Choose an étale \( U \to B \) with dense image such that \( (C|U)_{\text{tr}} \) is isomorphic to \( U \times \mathbb{P}^1 \). Thus we may identify fibres over each \( u \in U \) as follows, using the assumption that \( C_{\text{tr}} \) is irreducible. After passing to another étale cover, we may identify \( (C|U)_{\text{tr}} \) with the pull-back of the universal curve over \( \overline{M}_{0,0}(\mathbb{P}^1,2) \). By assumption the image in \( \overline{M}_{0,0}(\mathbb{P}^1,2) \) is a point \( P \). Therefore for each \( u \in U \) we may identify the curve \( C_{u,\text{tr}} \) with the same double cover of \( \mathbb{P}^1 \).

Fix a choice of closed point \( u \in U \) and so a stable map to \( S \) and for each \( v \in U \), we identify the two covers via the above isomorphism as described in Figure 2 and consider \( f(C_{u,v}) - f(C_{u,u}) \) defined inside the fibrewise group law induced by the section of the elliptic fibration. For \( u = v \), this defines a double cover of the section \( E \), while if \( f(C_{u}) \) is varying, then for \( u \neq v \) this is not a double cover of that section. By rigidity of sections, this does not occur.
Now if $C_{tr}$ is reducible, $C_{contr} = C_1 \cup C_2$, then both $C_1$ and $C_2$ are sections and admit no deformations. In particular, this shows that $f$ is constant on these fibres. \hfill \qedsymbol

Let us now study the deformation theory of the different components of $C$. The virtual dimension of the moduli space of genus $g$, $n$-marked stable maps to $S$ in a class $\beta$ is given by

$$(1 - g)(\dim S - 3) + n - K_S.\beta.$$ 

Applying this to our situation, we see that $C$ moves in at least one-dimensional family, and if $C_{tr}$ is connected, then $C_{contr}$ also moves in a one-dimensional family. We will work to relate deformations of these two curves. In any case, there are three types of irreducible components of $\overline{M}_{0,0}(S, \beta)$, point components whose image is dimension zero, curve components whose image is dimension one and bubble components which surject onto $\mathcal{H}ilb^2(\mathbb{P}^1) \cong \mathbb{P}^2$. By Theorem 5.1, the fibres over point components consist only of reparametrisations of components mapping to rational fibres. Schematically the generic types of curves which can occur in each component is contained in Figure 3. We can draw representative curves of each type in the class $E_1 + E_2 + 3F$ for sections $E_1, E_2$ satisfying $E_1 = E_2$, $E_1 \cdot E_2 = 1$ or $E_1 \cdot E_2 = 0$ in the three cases depicted ($E_1 \cdot E_2 = 2$ and higher can occur, and appear as bubble components). The dashed line in the first figure shows that this is a double cover. We will later prove that these are representative of the general case.

Let us begin by proving that if $C$ is in a point component, then $C_{tr}$ is not connected.
Lemma 5.2 (The universal curve over point components). Let $\mathcal{M}$ be a point component. Then the restriction of the universal curve to this component decomposes as two constant components each mapping to potentially distinct sections joined by a tree of curves containing at least one component mapping to a fibre.

Proof. If the transverse component $C_{tr}$ is smooth and connected, then we have seen that we can deform it as a double cover of $\mathbb{P}^1$. This leaves how to glue the component $C_{fib}$ to this deformation. Suppose that $C_{fib}$ meets $C_{tr}$ in $P_1, \ldots, P_n$. If none of these are ramification points of the map $C_{tr} \rightarrow \mathbb{P}^1$, then this deformation gives unique deformations of the $P_i$. If not then take the double cover $\mathbb{A}^1 \rightarrow \mathbb{A}^1$ given by $z \mapsto z^2$. Over this family there is a unique deformation given by choosing a branch to move along. This allows us to lift the glueing data to the entire deformation contradicting Lemma 5.1 and the definition of a point component. Therefore the universal curve must be generically reducible containing two distinct sections. The same argument applies if $C_{tr}$ is the union of two sections meeting at a point.

Now suppose that $C_{tr}$ is connected but not smooth hence of the form two sections joined by a chain of contracted components. Then by stability each contracted component in the chain must have an attached component mapping to a rational fibre. \qed

Now these point components are not relevant to the rest of our work, they cannot carry the correct log structure to be tangent to $F_0$ at a single point, nor are they generic enough to appear in our main construction. However, the bubble components for sure will contribute, and controlling these will be key to our construction. In particular, we can show that curve and bubble components do not meet.

Theorem 5.3 (Curve and bubble components are disjoint). Let $\mathcal{M}$ be a bubble component. Then $\mathcal{M}$ does not meet any curve components of the moduli space.

Proof. We will begin by showing $f(C_{tr})$ does not vary for any stable map in this space, and that there is a section $\sigma$ such that $f : C_{tr} \rightarrow \sigma$ is a double cover. We do this by first studying the restriction of the family to the boundary of the moduli space. Let $\partial \text{Hilb}^2(\mathbb{P}^1)$ denote the conic of degenerate double covers inside $\text{Hilb}^2(\mathbb{P}^1) \cong \mathbb{P}^2$. Take a point $P \in \partial \text{Hilb}^2(\mathbb{P}^1)$ such that the ramification point of the corresponding double cover $C_{tr} \rightarrow \mathbb{P}^1$ is distinct from the images of any of the rational fibres of $S$. In this case, $C_{tr}$ is reducible, the union of two sections meeting at a point which does not lie on one of the singular fibres and admits deformations which remain reducible. Since $C_{tr}$ is reducible, the image is either the union of two sections, or a double cover of a section. The first case is impossible, sections are rigid and so there cannot be a deformation moving them and preserving reducibility. Therefore one has a double cover of a single section, $\sigma$.

Now suppose a curve component intersects $\mathcal{M}$. In this case, one can find the spectrum of a DVR $B = \text{Spec } R$ with closed point $b_0$ and generic point $b_{\text{gen}}$ and a morphism $cl : B \rightarrow \overline{\mathcal{M}}_{0,0}(S, [C])$ such that $cl(b_0) \in \mathcal{M}$, $cl(b_{\text{gen}}) \notin \mathcal{M}$ and $cl(b_{\text{gen}})$ lies in a curve component. Let $\pi : C \rightarrow B$ be the pullback of the universal curve and $f : C \rightarrow S$ the corresponding stable map.

In fact, it is enough to show the following. First, replace $C$ by removing the closures of irreducible components of $C_{\text{gen}}$ mapping into the rational fibres of $S$ and taking the closure inside $C$. Thus we may assume that $(C_{\text{gen}})_{tr} = C_{\text{gen}}$. If we can show that $(C_0)_{tr} = C_0$, then as necessarily $f_*(\{C_0\}) = 2[\sigma]$ and $\sigma^2 = -1$, we see that $f(C_{\text{gen}}) \subseteq \sigma$ also. Thus $f$ yields a double cover of $\sigma$, and hence $cl(b_{\text{gen}})$ in fact lies in a bubble component.

Our goal will be to decorate $C_{\text{gen}}$ with the structure of a log stable map, as described in [19]. Then since the moduli space of log curves is proper, the family $B$ admits a unique completion, and this completion must be a base-change of the classical stable map. But we will now see
that results of [18, proposition 4.3] lead to a contradiction. This states that a maximally tangent curve cannot degenerate to have a component mapping into the boundary. This uses log geometry, but in a subtly different manner from later applications. To this end, we take the divisorial log structure on \( S \) coming from the union \( \cup F_i \) of all singular fibres.

To give a log structure on \( C_{\text{gen}} \), we need to control the location of the singularities. Therefore we first exclude the case that \( C_{\text{gen}} \) has any nodes on the transverse component. Indeed, if \( C_{\text{gen}} \) possesses a node, then it is reducible and the two components map to sections of \( S \to \mathbb{P}^1 \). But then the family is constant since sections admit no deformations. Thus we assume \( f_{\text{gen}}^{-1}(\cup F_i) \) is a set of smooth points of \( C_{\text{gen}} \) and we give \( C_{\text{gen}} \) the divisorial log structure given by this divisor. We then have a log morphism \( C_{\text{gen}} \to S \).

Now we know that the generic fibre has a log structure and hence there is a unique choice of limit of stable log maps, possibly after passing to a branched cover of \( B \), see [19, Theorem 4.1]. We now claim that there are no log stable curves with components mapping into the fibre \( F_0 \). This is a consequence of the proof of [18, Proposition 4.3]. This shows that \( C_{\text{gen}} \) cannot degenerate to one of these curves. By counting the number of components, we show that \( (C_0)_{\text{tr}} = C_0 \). Thus we conclude all curve components are disjoint from \( \mathcal{M} \).

Now if we take a general point of \( \text{Hilb}^2(\mathbb{P}^1) \), there is a canonical way to produce a \( K3 \) surface by taking the fibre product over \( \mathbb{P}^1 \). By the work of Bryan and Leung, the moduli space of stable maps to an elliptically fibred \( K3 \) surface in a class \( E + nF \) is independent of the choice of surface. Let us use this idea to describe the fibres of \( \pi \) over a bubble component.

**Theorem 5.4 (A fibration of bubble components).** Let \( \mathcal{M} \) be a bubble component. The map \( \mathcal{M} \to \text{Hilb}^2(\mathbb{P}^1) \) fibres \( \mathcal{M} \) by moduli spaces of stable maps to (potentially singular) \( K3 \) surfaces. This fibration is trivial away from a codimension one set.

**Proof.** One might think that the fibration should be trivial as different points in a fibre correspond to different parametrisations of the rational fibres. This is only true generically but fortunately we can describe the locus where it does not hold. Let \( R \) denote the locus inside \( \text{Hilb}^2(\mathbb{P}^1) \) of stable maps ramifying over the images of the \( F_i \). Away from \( R \) the fibration will be trivial since the only deformations are deformations of the rational tails. The moduli space on this locus decomposes as a product of covers of the rational tails.

Let \( P \in \text{Hilb}^2(\mathbb{P}^1) \setminus R \cup \partial\text{Hilb}^2(\mathbb{P}^1) \) be a smooth double cover not ramifying over one of the rational fibres. Let \( f : C \to S \) and \( \pi : C \to B \) be a curve in \( \text{ram}^{-1}P \) with \( B \) an infinitesimal extension over \( k[t]/t^2 \). Let \( B_0, C_0 \) denote the reduced structure. We wish to show that this deformation of \( C_0 \) does not smooth any of the nodes between \( C_{0,\text{tr}} \) and the other components. To do this, we construct a moduli space of stable maps to a \( K3 \) surface where none of these smoothings exists and compare the obstruction theories. Take the following diagram

\[
\begin{array}{ccc}
K & \xrightarrow{d} & S \\
\downarrow & & \downarrow \\
C_{0,\text{tr}} & \xrightarrow{\rho \circ f} & \mathbb{P}^1
\end{array}
\]

This defines a smooth elliptically fibred \( K3 \) surface \( K \) (here we are using the assumption that the curve does not ramify over the images of any of the rational fibres). \( C_0 \) maps to both \( C_{0,\text{tr}} \) and \( S \) and hence maps to \( K \) as a section together with a collection of covers of the rational fibres. This shows that the fibres of \( \text{ram} \) are isomorphic to moduli spaces of stable maps to a \( K3 \) surface. In particular, by the beginning of [8, section 5] there are no infinitesimal deformations smoothing the nodes lying on \( C_{0,\text{tr}} \) since there are none on \( K \). In particular this shows that over this set the fibration is trivial.
We now turn our attention to a point on the boundary $P \in \partial \text{Hilb}^2(\mathbb{P}^1)$. We prove that the fibre $\text{ram}^{-1}P$ consists of only reparametrisations of the rational tails, equivalently no family inside this fibre smooths any node lying on the intersection of $C_{tr}$ and $C_{fib}$. The locus where this fibration fails to be locally trivial is then a collection of points inside $\text{Hilb}^2(\mathbb{P}^1)$, otherwise it would have to meet $\partial \text{Hilb}^2(\mathbb{P}^1)$. Take $P \in \partial \text{Hilb}^2(\mathbb{P}^1)$ and $f : C \to S, \pi : C \to B$ an infinitesimal deformation of a point $\pi : C_0 \to B_0$, $f_0 : C_0 \to S$ in $\text{ram}^{-1}P$. Then $C_0$ contains a chain of rational curves connecting two components $D_1$ and $D_2$ which are mapped to sections of $S$. Since the deformation remains within the fibre, it fails to smooth at least one node in this chain. Removing this node disconnects the domain curve $C_0$ into two components $C_a$ and $C_b$ and the deformation restricts to each of these. But $C_a$ and $C_b$ are sections together with rational tails, hence deformations of them fail to smooth any of the nodes connecting the rational tails to the section by the discussion of [8, Section 5]. Pulling back to the curve $C_0$, we obtain the desired statement.

Finally we claim that the fibres over any point in $\text{Hilb}^2(\mathbb{P}^1) \setminus R$ are isomorphic. This is trivial from the above description. All these spaces are just moduli spaces of covers of the rational tails.

It is worth saying that it is not true that this fibration is globally trivial. We can construct examples where the fibres of $\text{ram}$ over $R \cap \partial \text{Hilb}^2(\mathbb{P}^1)$ are much higher dimension than the surrounding fibres since there could be more moduli in the way we connect the two transverse components together and such examples exist for three sheeted covers of one of the rational tails.

Given an irreducible such curve, $C$, it is an easy calculation that the product $C \times_{\mathbb{P}^1} S$ is a $K3$ surface. But $C$ maps both to itself via the identity and to $S$, compatibly over $\mathbb{P}^1$. Therefore it also maps to this $K3$ surface. The moduli space we are considering provides us with a collection of components mapping to $\text{Hilb}^2(\mathbb{P}^1)$. Of these we are interested in those whose image is at least dimension one so we should search for a curve inside $\text{Hilb}^2(\mathbb{P}^1)$. So let us perform this construction in a family setting.

Construction 5.5. Let $D$ be a generic tri-degree $(1,1,2)$ surface inside $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with projection maps $\pi_L, \pi_M$ and $\pi_R$ to the respective factors. We view this as a family of curves over $\mathbb{P}^1$ via the projection $\pi_L$. Taken this way each fibre is a bi-degree $(1,2)$ hypersurface inside $\mathbb{P}^1 \times \mathbb{P}^1$, so via $\pi_M$ the graph of a double cover of $\mathbb{P}^1$ by $\mathbb{P}^1$. This produces a morphism $\mathbb{P}^1 \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^1,2)$ such that the family $\pi_L : D \to \mathbb{P}^1$ is the pullback of the universal family of double covers. We wish to know that the trivial $C_2$ action on the Hilbert scheme $\text{Hilb}^2(\mathbb{P}^1)$ is intertwined with the involution of $D$ swapping the two sheets of the double cover.

Lemma 5.6. With the above notation, the family $[D/C_2] \to [\mathbb{P}^1/C_2]$ is the pull-back of the universal family of double covers of $\mathbb{P}^1$ over the base.

Proof. By construction, the family $D$ is the pull-back of the universal family of double covers to $\mathbb{P}^1$. Therefore we have a diagram

\[
\begin{array}{ccc}
\mathbb{P}^1 & \xrightarrow{[\mathbb{P}^1/C_2]} & \overline{\mathcal{M}}_{0,0}(\mathbb{P}^1,2) \\
\downarrow & & \downarrow \\
D & \xrightarrow{[D/C_2]} & C
\end{array}
\]
where the front face is a pullback square. Now $\mathbb{P}^1$ is simply connected, hence there is only the trivial $C_2$ torsor over $\mathbb{P}^1$. This shows that $\mathbb{P}^1 \to [\mathbb{P}^1/C_2]$ has a chart given by $\mathbb{P}^1 \amalg \mathbb{P}^1 \to \mathbb{P}^1$. Taking the pull-back of $[D/C_2]$ to this and using that $D$ is the universal curve over $\mathbb{P}^1$, we obtain the desired result. 

We can calculate the multiplicity of the intersection via an example.

**Example 5.7.** Let us study the family with defining equation inside $(\mathbb{P}^1)^3$
\[
x_1x_2x_3^2 + x_1y_2x_3y_3 + 2y_1x_2x_3y_3 + y_1y_2y_3^2,
\]
where the $i$th factor has coordinates $x_i$ and $y_i$. The fibre over $(x_1, y_1)$ is the graph of a double cover of $\mathbb{P}^1$, ramifying where the discriminant 
\[
(x_1y_2 + 2y_1x_2)^2 - 4x_1x_2y_1y_2 = x_1^2y_2^2 + 4y_1^2x_2^2
\]
vanishes. This gives two values for $(x_2 : y_2)$ for each value of $(x_1 : y_1)$. The fibre is singular just when these two values coincide, so when the discriminant of this quadratic vanishes, $-16x_1^2y_1^2$. Therefore there are two singular fibres each with multiplicity two.

In general, this shows that there will be four singular fibres and it is easy to construct examples where these occur as distinct fibres. We have constructed here a collection of double covers of $\mathbb{P}^1$, so the base $\mathbb{P}^1$ should map to $\mathcal{Hilb}^2(\mathbb{P}^1)$. The degenerate fibres inside $\mathcal{Hilb}^2(\mathbb{P}^1)$ form a conic inside $\mathbb{P}^2$, so by counting intersection points we know that the image of this family is also a conic. Therefore to search for curve and bubble components of the moduli space, we are lead naturally to study the following fibre product:

\[
\begin{array}{ccc}
\mathbb{P}^1 & \longrightarrow & S \\
\downarrow & & \downarrow \rho \\
D & \longrightarrow & \mathcal{M}
\end{array}
\]

We call the constructed threefold $X$ an unravelled threefold. Such a threefold is the intersection of a $(3,1,0,0)$ hypersurface and a $(0,1,1,2)$ hypersurface inside $\mathbb{P}^2 \times (\mathbb{P}^1)^3$. The $(0,1,1,2)$ hypersurface may take to be generic whilst the $(3,1,0,0)$ hypersurface depends on the choice of rational elliptic surface $S$. The first hypersurface gives rise to a choice of line $\iota : [\mathbb{P}^1/C_2] \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, 2) \cong [\mathbb{P}^2/C_2]$. We will restrict our range of choices later in the discussion to reflect the surfaces we choose to work with. Let $\mathcal{M}$ be a component of the moduli space $\overline{\mathcal{M}}_{0,0}(S, f_\ast[C])$ and consider the image of $\mathcal{M}$ under the morphism $\text{ram} : \overline{\mathcal{M}}_{0,0}(S, f_\ast[C]) \to \mathcal{Hilb}^2(\mathbb{P}^1) \cong \mathbb{P}^2$. Whenever the image intersects the image of the family of double covers of $\mathbb{P}^1$, one can lift the corresponding stable map to $X$. The existence of a lift suggests that $\overline{\mathcal{M}}_{0,n}(X, f_\ast([C])) \cong \overline{\mathcal{M}}_{0,n}(S, f_\ast[C]) \times \mathcal{Hilb}^2(\mathbb{P}^1)$ along the map $\text{ram}$. Soon we will see that this description induces a relation between the Gromov–Witten invariants of these two moduli spaces.

6. Calculating Gromov–Witten invariants

We can describe this space $X$ as a nef complete intersection inside $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. This allows us to apply techniques of Givental from [13] to calculate the Gromov–Witten theory of $X$. We will begin by giving an explanation of this in the case of $S$ itself, and then explain why this is not enough to calculate the relative invariants we want. Instead we will prove that the $J$ function of $X$ is holomorphic, which provides strong bounds on the growth of the coefficients.

**Example 6.1.** From the description above, the threefold $X$ is the intersection of a $(3,1,0,0)$ hypersurface and a $(0,1,1,2)$ hypersurface in $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. 
Such hypersurfaces were studied in [13] while the $I_X$ function was introduced in [12]. The Gromov–Witten theory of a toric variety is very well understood and by using the defining line bundles we can localise the invariants to the intersection. Givental constructed solutions to two different quantum differential operators, and then proved that the solution spaces to these two were equal. Therefore there is a change of coordinates relating the constructed solutions. Let us recall the theory, starting with a toric variety $T$ with codimension one strata $D_\rho$. Take a basis for the rational Chow ring $A^*(T)$, $\{H_i\}_T$, and a dual basis $\{H^i\}_T$ under cup product.

**Definition 6.2.** The $I_X$ and $J_X$ functions are defined for a smooth complete intersection $X \subset T$. Suppose that $X$ is the common vanishing of sections of line bundles $L_i$ with $-K_T - \sum c_1(L_i)$ nef. Then the $I_X$ function is defined by

$$I_X(t_0, \ldots, t_n) = e^{(t_0+\sum t_i H_i)/\hbar} \text{Eul}(\oplus L_i) \sum_\beta q_\beta \frac{\prod_i (\prod_{m=1}^{c_1(L_i)} (c_1(L_i) + m\hbar) \prod_\rho \prod_{m=-\infty}^{0} (D_\rho + m\hbar))}{\prod_\rho \prod_{m=-\infty}^{0} (D_\rho + m\hbar)},$$

where $q_\beta = e^{\sum t_i H_i \cap \beta}$ and the sum is over $\beta$ effective. This function is valued in $A^*(T)[t_0, \ldots, t_n, \hbar^{-1}]$. The function $J_X$ is a generating function for the gravitational correlators of $X$ and is valued in $A^*(X)[t_0, \ldots, t_n, \hbar^{-1}]$. It is defined by the formula

$$J_X(t_0, \ldots, t_n) = e^{(t_0+\sum t_i H_i)/\hbar} \left(1 + \sum_{\beta, i, k} h^{-(k+1)} q_\beta \langle \tau_i H_0, 1 \rangle_{0, \beta} H^0 \right),$$

where $\langle \tau_i H_0, 1 \rangle_{0, \beta}$ are the gravitational descendants, the sum is over positive $i, k$ and effective curve classes $\beta$. These gravitational descendants include as a subset the Gromov–Witten invariants.

These two functions are not valued in the same ring so cannot be equal. Let $\iota : X \to T$ be the inclusion. The main result of [13] is an equality of generating functions $I_X(t_i) = \iota_* J_X(s_i)$ after some change of basis $t_i \mapsto s_i \in A^*(T)[t_0, \ldots, t_n]$.

**Theorem 6.3 (Mirror symmetry for Givental’s $I$ and $J$ functions).** The functions $I_X$ and $\iota_* J_X$ are equal up to a homogeneous change of variables of the form $t_0 \mapsto t_0 + f_0(z^2) h + h(z^2), t_i \mapsto t_i + f_i(z^2)$, where the $f_i$ and $h$ are homogeneous power series of weights $\deg f_i = 0$ and $\deg h = 1$ with degrees of the variables being given by $c_1(T) - c_1(\sum L_i) = \sum \deg(z^2) H_i$, $\deg h = 1$, $\deg t_0 = 1$ and $\deg t_i = 0$ for $i > 0$.

**Proof.** See [13, Theorem 0.1].

A well-written explanation of how to perform these calculations can be found in [9], along with precise definitions of the descendant invariants appearing in the definition. The rational elliptic surface $S$ is itself a (3,1) hypersurface in $\mathbb{P}^2 \times \mathbb{P}^1$. Unfortunately, this is not powerful enough to reconstruct the relative or open invariants and so it cannot answer our questions by directly calculating with $S$. Instead let us calculate the $J$ function of $X$ and attempt to interpret the answer geometrically. Since we are practising numerology, we will not attempt to prove our claims. Firstly we take a basis for the cohomology $H_1, \ldots, H_4$ given by the duals of pullbacks of hyperplanes from each factor.
The invariants we wish to calculate intersect $H_2$, the class which restricts on each $K3$ fibre to a fibre of the vibration of that $K3$, at a single point. Therefore the number of such curves appear as the coefficient of $H_2$ in the corresponding term in the $J_X$ function. We must double the invariant to account for the tangency condition and double it again since we are really intersecting with a conic inside $\text{Hilb}^3(\mathbb{P}^1)$, this factor of four arises as the four in 5.7. This suggests that each curve on $S$ should be counted with multiplicity four compared to the corresponding curve on $X$.

When we expand out the function $J_X$, the first three coefficients we calculate, the coefficients of $z^{(0,2,0,1)}, z^{(1,2,0,1)}$ and $z^{(2,2,0,1)}$, are $-9, 144$ and $1980$. The first of these we can easily explain, it counts curves which are a double cover of the base $\mathbb{P}^1$ and which are degree 0 on $\mathbb{P}^2$, these are the nine sections appearing as exceptional curves of the blowup. Each admit a bubble component, by the double cover formula the relative invariant should be $-9/4$. The 144 counts curves of the form $H - E_i$, they are degree 1 projected to $\mathbb{P}^2$ and are a double cover of $\mathbb{P}^1$, and for each family there are four lines which are tangent to the boundary to order 2, the coefficient therefore is $9 \times 4 \times 2 \times 2 = 144$.

For the term 1980, we begin to see an interaction between the bubble components and the curve components. Each of the lines $H - E_i - E_j$ admits a bubble component, there are 36 choices of such classes so we expect a contribution of $-36 = -36/4 \times 2 \times 2$. The classes $2H - E_i - E_j - E_k - E_l$ give a curve component of which four elements are tangent to the boundary. This gives a contribution of $126 \times 4 \times 2 \times 2 = 2016$, combining these we obtain the predicted contribution of 1980. Unfortunately, as the degree grows the presumed correspondence with the log invariants becomes intractable. To find the next term, we would have to answer the question ‘given a rational cubic $E$ passing through seven points meeting $E$ in one other point of tangency order 2?’. This is why we must pass to the threefold.

Our main result for this section is that $J_X$ is actually a holomorphic function. We believe that this should be part of the general yoga of hypergeometric equations, but do not know of a reference. For the general theory of holomorphic functions in many variables, we refer to [14]. We will prove that $I_X$ is holomorphic and then deduce the holomorphicity of $J_X$, referring to [5] for an explanation of why the change of basis is holomorphic. Given a vector $v = (v_1, \ldots, v_n) \in \mathbb{N}^n$, we write $x^v$ for the monomial $\prod x_i^{v_i}$.

**Theorem 6.4** (The $I$ function is holomorphic). The function $I_X$ is holomorphic in the $t_i$ for $X$ an unravelled threefold.

**Proof.** To do this, we prove the coefficient of $e^{at_1 + bt_2 + ct_3 + dt_4}$ in $I_X$ grow at most exponentially in $a, b, c$ and $d$. We write out the definition of the $I_X$ function

$$I_X(t_0, \ldots, t_n) = e^{\sum t_i H_i / \hbar} \text{Eul}(\oplus \mathcal{L}_i) \sum_{\beta} q^\beta \frac{\prod_{\rho} (\prod_{m=1}^{\mathcal{L}_i(\beta)} (c_1(\mathcal{L}_i) + m\hbar) \prod_{m=-\infty}^{0} (D_\rho + m\hbar))}{\prod_{\rho} (\prod_{m=-\infty}^{\mathcal{L}_i(\beta)} (D_\rho + m\hbar))}. \quad (6.1)$$

To prove boundedness, with $x_i = e^{t_i}$ and $x^{(a_1, \ldots, a_4)} = \prod x_i^{a_i}$, we only need to prove boundedness of the sum

$$\sum_{x^{(a,b,c,d)}} \frac{\prod_{m=1}^{a+b+c+d} (3H_1 + H_2 + m\hbar) \prod_{m=1}^{b+c+2d} (H_2 + H_3 + 2H_4 + m\hbar)}{\prod_{m=1}^{a+b} (H_1 + m\hbar)^3 \prod_{m=1}^{b} (H_2 + m\hbar)^2 \prod_{m=1}^{c} (H_3 + m\hbar)^2 \prod_{m=1}^{d} (H_4 + m\hbar)^2}. \quad (6.2)$$
Fix a coefficient $H_i = \prod H_i^{n_i}$ and consider the coefficient of $H_{x}^{a(a+b,c,d)} h^{-c+\sum a}$. Expanding out the above product, we obtain the expression

$$\sum x^{(a+b,c,d)} h^{-c} \frac{(3a + b)!(b + c + 2d)!}{a^{3b}b^{2}c^{2}d^{2}l^{2}} \prod_{m=1}^{3a+b} \left( \frac{H_{i}}{m_{i}} + 1 \right)^{b + c + 2d} \prod_{m=1}^{b} \left( \frac{H_{i} - H_{j} + 2H_{k} + 1}{m_{i}} \right)^{2} \prod_{m=1}^{d} \left( \frac{H_{i} - H_{j} + 1}{m_{i}} \right)^{2}.$$ 

Any monomial appearing in this consists of two parts, a term $\frac{(3a + b)!(b + c + 2d)!}{a^{3b}b^{2}c^{2}d^{2}l^{2}}$ with no $h$ term times a correction factor involving a sum of products of terms $1/i$. The correction factor is at most a polynomial in $a, b, c, d$, hence exponentially bounded since there are only finitely many choices of $i$. We now bound the term $\frac{(3a + b)!(b + c + 2d)!}{a^{3b}b^{2}c^{2}d^{2}l^{2}}$ using Stirling’s formula:

$$\frac{(3a + b)!(b + c + 2d)!}{a^{3b}b^{2}c^{2}d^{2}l^{2}} < ke^{c} \cdot \frac{(3a + b)^a}{a^{3a}} \cdot \frac{(b + c + 2d)^b}{b^{b}} \cdot \frac{(b + c + 2d)^c}{c^{c}} \cdot \frac{(b + c + 2d)^d}{d^{d}}.$$ 

Rearranging the first term, we get $(3 + b/a)^{3a}$ and we wish to compare this to $r^{a+b}$. Let $r = 27$, then $27^{a+b} = 3^{3a+3b}$ and we can rewrite the quotient $(3 + b/a)^{3a}/r^{a+b}$ as

$$\frac{(1 + \frac{b}{3a})^{3a}}{3^{b}}.$$ 

Taking log and using that $\log(1 + x) < x$, we find that

$$3a \log \left( 1 + \frac{b}{3a} \right) - b \log(3) < b - b \log(3) < 0.$$ 

Hence for $r > 27$ we see that $(3 + b/a)^{3a} < r^{a+b}$. There are similar choices of $r$ for each other term. Our choice of $r$ is the product of all of these minimal choices. Then since $r^{a+b+c+d}$ dominates all terms of the form $r^{a+b}$, we see that the coefficients of $I_X$ are exponentially bounded. Thus $I_X$ defines a holomorphic function in a neighbourhood of the origin. □ 

**Corollary 6.5.** The function $J_X$ is also holomorphic.

**Proof.** The change of variables from the $I_X$ function to the $J_X$ function is given by holomorphic functions. □ 

### 7. Relating Gromov–Witten invariants

This is the real technical heart of the paper. In this section, we will prove that the Gromov–Witten count of curves on $X$ and on $S$ are related by a Gysin map induced by a regular embedding $\mathbb{P}^{1} \hookrightarrow \mathcal{H}ilb^{2}(\mathbb{P}^{1})$. The invariants we actually wish to count are then related by the composition of two Gysin maps, one induced by an inclusion of components and the other by a regular embedding $\mathbb{P}^{1} \hookrightarrow \mathcal{H}ilb^{2}(\mathbb{P}^{1})$. This inclusion of components causes some problems for us, since we do not know if we have thrown away components with large virtual degree.

Let us begin by proving our philosophy that the moduli spaces of stable maps to $X$ and to $S$ are highly related.
Lemma 7.1 (The unravelled threefold lifts curves). Let $X \subset \mathbb{P}^2 \times (\mathbb{P}^1)^3$ be an unravelled threefold, $\pi_M$ the projection to the third factor, a copy of $\mathbb{P}^1$, and let

$$d^{-1}\beta = \{ \alpha \in A_1(X) \mid \pi_{M,*}\alpha = 0, d_*\alpha = \beta \}$$

be the set of effective classes on $X$ contained inside a fibre of $\pi_M$ and such that $d_*\alpha = \beta$. We write $\overline{M}(X, d^{-1}(\beta))$ for the union of moduli spaces $\bigcup_{\alpha \in d^{-1}(\beta)} \overline{M}_{0,0}(X, \alpha)$. Then consider the following diagram

$$
\begin{array}{ccc}
\overline{M}(X, d^{-1}(\beta)) & \xrightarrow{\psi} & \mathbb{P}^1 \times_{\overline{M}_{0,0}(\mathbb{P}^1, 2)} \overline{M}_{0,0}(S, \beta) \\
\downarrow{pr} & & \downarrow{\iota} \\
\mathbb{P}^1 & \xrightarrow{pr_{\mathbb{P}^1}} & \mathbb{P}^1 \times_{\overline{M}_{0,0}(\mathbb{P}^1, 2)} \overline{M}_{0,0}(S, \beta) \rightarrow \mathbb{P}^1 \times_{\overline{M}_{0,0}(\mathbb{P}^1, 2)} \overline{M}_{0,0}(S, \beta) \\
& & \downarrow{ram} \\
& & \overline{M}_{0,0}(\mathbb{P}^1, 2)
\end{array}
$$

where $pr_X$ associates to a stable map $f : C/B \to X$, the morphism $B \to \mathbb{P}^1$ given by the ramification of the composed map $\rho \circ d \circ f$, where $\rho : S \to \mathbb{P}^1$ is the elliptic fibration. Then the map $\psi$ is an isomorphism, and thus a two to one cover of $[\mathbb{P}^1/C_2] \times_{\overline{M}_{0,0}(\mathbb{P}^1, 2)} \overline{M}_{0,0}(S, \beta) \subset \overline{M}_{0,0}(S, \beta)$.

Proof. By genericity we may assume that the choice of map $\mathbb{P}^1 \to Hilb^2(\mathbb{P}^1)$ avoids all point components and is transverse to the images of all curve components. We use the universal property of products to construct an inverse.

Let $S \xleftarrow{f} C \xrightarrow{\pi} B$ be a family of stable maps inside $\mathbb{P}^1 \times_{\overline{M}_{0,0}(\mathbb{P}^1, 2)} \overline{M}_{0,0}(S, \beta)$. Since $D$ is the restriction of the universal curve on $\overline{M}_{0,0}(\mathbb{P}^1, 2)$ to $\mathbb{P}^1$, there is an induced morphism $C \to D$, and hence by properties of fibre products a unique morphism $C \to X$. This constructs an inverse to $\psi$, which we call it $\phi$.

We now need to prove that the two compositions $\phi \psi$ and $\psi \phi$ are in fact the identity maps. To show that $\psi \cdot \phi$ is the identity we show that $\iota \cdot \psi \cdot \phi = \iota$ and $pr_S \cdot \psi \cdot \phi = pr_S$. These are both automatic from the construction of $\phi$. Indeed, $\iota \cdot \psi \cdot \phi = d_* \cdot \phi = \iota$ and $pr_S \cdot \psi \cdot \phi = pr_X \cdot \phi = pr_S$.

Finally we show the composition $\phi \cdot \psi$ is the identity. Let $f : C \to X$ be a stable map in $\bigcup_{\alpha \in d^{-1}(\beta)} \overline{M}_{0,0}(X, \alpha)$. The image $\psi(f : C \to X)$ is given by pushforward to $S$ and has a unique lift. But $f : C \to X$ is such a lift and so this composition too is the identity map. \qed

Since any degree 2 map from $\mathbb{P}^1$ to $\mathbb{P}^2$ is a local complete intersection, there exists a Gysin map for the inclusion $\iota : \overline{M}(X, d^{-1}(\beta)) \to \overline{M}(S, \beta)$ covering $\mathbb{P}^1 \to Hilb^2(\mathbb{P}^1) \cong [\mathbb{P}^2]$. We follow [23] in constructing a compatibility datum between the two spaces.

Theorem 7.2 (Compatibility of Gromov–Witten invariants between $S$ and $X$). For a generic choice of unravelling $d : X \to S$, there is an equality between the virtual classes associated to the obstruction theories on $\overline{M}_{0,0}(X, d^{-1}(\beta))$ and $\overline{M}_{0,0}(S, \beta)$ under the Gysin map $\iota'$:

$$[\overline{M}_{0,0}(X, d^{-1}(\beta))]^{\text{virt}} = 2\iota'[\overline{M}_{0,0}(S, \beta)]^{\text{virt}}.$$
Proof. We choose a generic unravelling which does not meet any point components and is transverse to any curve components. We have the following commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{d} & S \\
\downarrow f_X & & \downarrow f_S \\
C_{0,0}(X, d^{-1} \beta) & \xrightarrow{\pi_X} & C_{0,0}(S, \beta) \\
\downarrow \pi_X & & \downarrow \pi_S \\
\mathcal{M}_{0,0}(X, d^{-1} \beta) & \xrightarrow{\text{ram}} & \mathcal{M}_{0,0}(S, \beta) \\
\downarrow \text{ram} & & \downarrow \text{ram} \\
\mathbb{P}^1 & \xrightarrow{\gamma} & \mathcal{M}_{0,0}(\mathbb{P}^1, 2)
\end{array}
\]

The desired compatibility datum is a morphism of distinguished triangles

\[
\cdots \to (\iota^* R\pi_{S,*} f^*_S \Theta_{S/k})^\vee \xrightarrow{\phi} (R\pi_X,* f^*_X \Theta_{X/k})^\vee \xrightarrow{\gamma_3} \text{ram}^* L_{\mathbb{P}^1}/\mathbb{P}^2 \to \cdots
\]

\[
\cdots \to \iota^* L_{\mathcal{M}_{0,0}(S, \beta)/\mathfrak{m}} \xrightarrow{\gamma_1} L_{\mathcal{M}_{0,0}(X, d^{-1} \beta)/\mathfrak{m}} \xrightarrow{\gamma_2} L_{\mathcal{M}_{0,0}(X, d^{-1} \beta)/\mathcal{M}_{0,0}(S, \beta)} \to \cdots
\]

To prove this, we must show that the cone over \((\iota^* R\pi_{S,*} f^*_S \Theta_{S/k})^\vee \to (R\pi_X,* f^*_X \Theta_{X/k})^\vee\) is quasi-isomorphic to \(\text{ram}^* L_{\mathbb{P}^1}/\mathbb{P}^2\). First let us show that the map \(\gamma_3 : \text{ram}^* L_{\mathbb{P}^1}/\mathcal{Hilb}^2(\mathbb{P}^1) \to L_{\mathcal{M}_{0,0}(X, d^{-1} \beta)/\mathcal{M}_{0,0}(S, \beta)}\) is an isomorphism. The morphism \(\mathbb{P}^1 \to \mathcal{Hilb}^2(\mathbb{P}^1)\) is a local complete intersection morphism, defined by a single equation \(F\) of degree 2 in \(k[X,Y,Z]\). So in particular, \(F\) forms a length one regular sequence in \(k[X,Y,Z]\). By genericity, we may take the images of \(\mathbb{P}^1\) and the curve components of \(\mathcal{M}_{0,0}(S, \beta)\) to be transverse inside \(\mathcal{Hilb}^2(\mathbb{P}^1)\). Therefore the sequence \(F\) remains regular when pulled back to \(\mathcal{M}_{0,0}(S, \beta)\) and \(\mathcal{M}_{0,0}(X, d^{-1} \beta)\) is just the vanishing locus of the pullback of \(F\). In the case of a regular embedding, there is an explicit formula for the cotangent complex, namely it is \(I/I^2[1]\) where \(I\) is the defining sheaf of ideals. Both source and target of \(\gamma_3\) is \(\text{ram}^* ((F)/(F)^2[1])\) and \(\gamma_3\) is the identity.

Now taking the cone over \(\phi\) there is a morphism of triangles of the following form

\[
\cdots \to (\iota^* R\pi_{S,*} f^*_S \Theta_{S/k})^\vee \to (R\pi_X,* f^*_X \Theta_{X/k})^\vee \xrightarrow{\delta} \text{Cone} \to \cdots
\]

\[
\cdots \to \iota^* L_{\mathcal{M}_{0,0}(S, \beta)/\mathfrak{m}} \to L_{\mathcal{M}_{0,0}(X, d^{-1} \beta)/\mathfrak{m}} \to L_{\mathcal{M}_{0,0}(X, d^{-1} \beta)/\mathcal{M}_{0,0}(S, \beta)} \to \cdots
\]

To begin with, we need a result allowing us to commute pull-back and push-forward. We claim that there is an isomorphism

\[
\iota^* R\pi_{S,*} f^*_S \Theta_{S/k} \cong R\pi_X,* f^*_X \Theta_{X/k} \cong R\pi_X,* f^*_X d^* \Theta_{S/k}.
\]

The second isomorphism is automatic, so we focus on the first. Families of curves are flat and proper, and so the diagram is tor-independent.

By [29, tag 081B] and the fact that families of curves are flat, cohomology and base change commute in the derived category. Applying [7, Proposition 5], \(f^*_S \Theta_{S/k}\) admits a resolution by a two term complex of vector bundles \(b : F^{-1} \to F^0\) such that \(\pi_* F^1 = 0\) and \(R^1 \pi_* F^1\) is a vector bundle. Now \(\pi_* f^* \Theta_{S/k}\) is just the cokernel of \(R^1 \pi_* (F^0) \to R^1 \pi_* (F^1)\). But we have seen that cohomology and base change commute, thus we have an isomorphism \(\iota^* R^1 \pi_* (F^i) \cong R^1 \pi_* (\iota^* F^i)\) and \(\iota^* R^0 \pi_* (F^i) = 0\). Now this is enough to show that there is the desired equality.
This allows us to calculate \( \text{Cone} \) from the distinguished tangent sequence triangle on \( X \)

\[
\rightarrow \Theta_{X/k} \rightarrow d^* \Theta_{S/k} \rightarrow \mathbb{T}_{X/S}[1] \rightarrow .
\]  

(7.2)

The complex \( \mathbb{T}_{X/S}[1] \) is the derived dual of the relative cotangent complex, and is a line bundle in degree \(-1\) and supported only on the critical locus of \( X \) over \( S \) in degree 0. The fibres of the map \( d : X \rightarrow S \) are isomorphic to the fibres of \( d : D \rightarrow \mathbb{P}^1 \). By an elementary calculation, this is only singular over two points of \( \mathbb{P}^1 \) where the corresponding double cover degenerates. The choice of \( D \) was arbitrary so we can arrange that these two singular points do not occur over the critical values of the map \( S \rightarrow \mathbb{P}^1 \), so over the images of the singular fibres. This ensures that \( H^i(\mathbb{T}_{X/S}[1]) \) is supported only on two elliptic fibres of the fibration \( \rho : S \rightarrow \mathbb{P}^1 \).

Let us describe the pullback of \( \mathbb{T}_{X/S}[1] \) to fibres of the universal curve of stable maps. Let \( f : C \rightarrow X \) be a stable map. By necessity such a stable map factorises through a (possibly singular) \( K3 \) double cover of \( S \). This shows that \( L^0 f^* \mathbb{T}_{X/S}[1] \) is supported only on a finite set since the universal curve can have no components mapping into an elliptic fibre. In particular, it has support relative dimension zero over the base, and so pushes forward to something supported only in degree zero. By applying the Grothendieck spectral sequence, we therefore see that both \( H^1(\mathbb{R} \pi_* Lf^* \mathbb{T}_{X/S}[1]) \) and \( H^{-1}(\mathbb{R} \pi_* Lf^* \mathbb{T}_{X/S}[1]) \) vanish.

By pushing forward (7.2) and dualising in the derived category, we obtain a long exact sequence

\[
0 \rightarrow H^{-1}(\mathbb{R} \pi_* Lf^* d^* \Theta_{S/k}) \rightarrow H^{-1}(\mathbb{R} \pi_* Lf^* \Theta_{X/k}) \rightarrow H^0(\mathbb{R} \pi_* Lf^* \mathbb{T}_{X/S}[1]) \rightarrow H^0(\mathbb{R} \pi_* Lf^* d^* \Theta_{S/k}) \rightarrow H^0(\mathbb{R} \pi_* Lf^* \Theta_{X/k}) \rightarrow H^1(\mathbb{R} \pi_* Lf^* \mathbb{T}_{X/S}[1]) \rightarrow 0,
\]

we see that \( H^1(\mathbb{R} \pi_* Lf^* \mathbb{T}_{X/S}[1]) \) vanishes by mapping via the canonical isomorphisms to the non-negative part of the cotangent exact sequence:

\[
l^* L\mathcal{N}_{0,0}(S, \beta)/\mathfrak{m} \rightarrow L\mathcal{N}_{0,0}(X, d^{-1} \beta)/\mathfrak{m} \rightarrow L\mathcal{N}_{0,0}(X, d^{-1} \beta)/\mathcal{M}_{0,0}(S, \beta) \rightarrow
\]

and applying the 5-lemma. We now have an exact sequence

\[
0 \rightarrow H^{-1}(\mathbb{R} \pi_* Lf^* d^* \Theta_{S/k}) \rightarrow H^{-1}(\mathbb{R} \pi_* Lf^* \Theta_{X/k}) \rightarrow H^0(\mathbb{R} \pi_* Lf^* \mathbb{T}_{X/S}[1]) \rightarrow H^0(\mathbb{R} \pi_* Lf^* d^* \Theta_{S/k}) \rightarrow H^0(\mathbb{R} \pi_* Lf^* \Theta_{X/k}) \rightarrow H^1(\mathbb{R} \pi_* Lf^* \mathbb{T}_{X/S}[1]) \rightarrow 0
\]

and by the 5-lemma \( H^0(\mathbb{R} \pi_* Lf^* \mathbb{T}_{X/S}[1]) \) maps surjectively to \( \text{rarr}^* L_{\mathcal{M}_{0,1}/\mathcal{M}_{0,2}} \). Taking stalks at a generic point \( \eta = \text{Spec} \ K \) of the moduli space, we may apply the Riemann–Roch theorem to the two tangent sheaf terms and using additivity of Euler characteristics, we find that \( H^0(\mathbb{R} \pi_* Lf^* \mathbb{T}_{X/S}[1]) \) has dimension one as a vector space over \( K \). Now \( (\mathbb{R} \pi_* Lf^* \mathbb{T}_{X/S}[1]) \) has cohomology supported in degrees \(-1\) and 0. Let \( E^* \) be a resolution of \( \mathbb{R} \pi_* Lf^* \mathbb{T}_{X/S}[1] \), which is quasi-isomorphic to the truncation \( \tau_{\leq 0} E^* \) defined in [29, Tag 0118]. Taking a locally free Cartan Eilenberg double resolution of this, we obtain a resolution of \( \mathbb{R} \pi_* Lf^* \mathbb{T}_{X/S}[1] \) by locally free sheaves concentrated in non-positive degree. Therefore after dualising \( \mathbb{R} \pi_* Lf^* \mathbb{T}_{X/S}[1] \) has a resolution concentrated in non-negative degree. This shows that \( H^0(\mathbb{R} \pi_* Lf^* \mathbb{T}_{X/S}[1]) \) embeds into a vector bundle, and hence is torsion free. We prove an analogous result to the statement that a surjective map of line bundles is an isomorphism. Take the morphism

\[
\delta : H^0(\mathbb{R} \pi_* Lf^* \mathbb{T}_{X/S}[1]) \rightarrow L\mathcal{N}_{0,0}(X, d^{-1} \beta)/\mathcal{M}_{0,0}(S, \beta)
\]

of the diagram (7.1). After localising at a point \( x \), we have an isomorphism

\[
L\mathcal{N}_{0,0}(X, d^{-1} \beta)/\mathcal{M}_{0,0}(S, \beta), x \cong \mathcal{O}_{\mathcal{N}_{0,0}(X, d^{-1} \beta), x}
\]

and we know that \( H^0(\mathbb{R} \pi_* Lf^* \mathbb{T}_{X/S}[1]) \) is a rank one torsion free sheaf with \( \delta \) surjective. Therefore we have the following situation: a local ring \( R \), a rank one torsion free \( R \)-module \( M \)
and a surjection $\delta : M \to R$. We claim that this implies that $M$ is isomorphic to $R$. Since $\delta$ is surjective, we can choose $m \in M$ mapping to $1 \in R$. Suppose that there were $n \in M$ not in the submodule of $M$ generated by $m$. The element $n$ maps to $f \in R$ and so at every generic point of $R$, one has $n = fm$. But $M$ is torsion free and so $n = fm$. Therefore $m$ generates $M$ freely, that is, $R \cong M$ as $R$-modules. Therefore $\overline{H}((R\pi_* L f^* T_{X/S}[1])\gamma)$ is isomorphic to $\overline{ram}^* L_{E_1/P^2}$.

The Gysin map $\iota^*$ produces a class on $[\mathbb{P}^1/C_2] \times _{\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1,2)} \overline{\mathcal{M}}_{0,0}(S,\beta)$. However, the moduli space $\overline{\mathcal{M}}_{0,0}(X, d^{-1}\beta)$ is a two to one étale cover of $[\mathbb{P}^1/C_2] \times _{\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1,2)} \overline{\mathcal{M}}_{0,0}(S,\beta)$. Therefore by [23] we obtain the formula

$$2\iota^* [\overline{\mathcal{M}}_{0,0}(S,\beta)]^\text{virt} = [\overline{\mathcal{M}}_{0,0}(X, d^{-1}\beta)]^\text{virt}.$$ 

We call this the comparison formula for bisections on $S$ and sections on $X$. □

This leaves the question of relating the relative invariants on $S$ to the classical invariants. Bryan and Leung studied this problem for the tangent order 1 case. Their motivation was to apply point conditions to stable maps directly by restricting to those stable maps whose marked points map to $P$. The deformation theory then needs to be modified, restricting to those deformations which vanish at the marked point. In their Appendix A, they prove that this produces the correct invariants and extend this to the case of certain divisors. We apply their ideas here, starting with the easiest case. Recall that we write $\overline{\mathcal{M}}(S^1,\beta)$ for the moduli of log stable maps to $S^1$ in a class $\beta$ tangent to the boundary at a single point with maximal tangency.

**Theorem 7.3** (Log invariants of sections are restrictions of Gromov–Witten invariants). Let $E$ be a section of a rational elliptic surface $S$, and let $\beta = E + nF$. Then there is an inclusion of components $\iota : \overline{\mathcal{M}}(S^1,\beta) \to \overline{\mathcal{M}}_{0,0}(S,\beta)$ and this induces an equality of virtual classes $\iota^* [\overline{\mathcal{M}}_{0,0}(S,\beta)]^\text{virt} = [\overline{\mathcal{M}}(S^1,\beta)]^\text{virt}$.

**Proof.** In [8], the authors constructed the moduli space $\overline{\mathcal{M}}_{0,0}(S,\beta)$: any curve is the union of the section $E$ together with different covers of the rational fibres. Since any curve in $\overline{\mathcal{M}}(S^1,\beta)$ has a unique choice of marked point $\sigma$ the forgetful map is injective and since different components correspond only to different covers of the rational fibres, this map is an inclusion of components. Indeed by [18, Proposition 4.3], the components of $\overline{\mathcal{M}}(S^1,\beta)$ are precisely those components of $\overline{\mathcal{M}}_{0,0}(S,\beta)$ where no component of the universal curve maps into the boundary, such curves have a unique log enhancement.

Let us denote by $\mathfrak{M}_{0,0}$ the moduli Artin stack of prestable genus zero curves and by $\mathfrak{M}_{0,1}$ the moduli Artin stack of all pre-stable genus zero one-pointed log curves. Recall that the virtual fundamental class of $\overline{\mathcal{M}}_{0,0}(S,\beta)$ is defined via an obstruction theory relative to $\mathfrak{M}_{0,0}$ and recall from [19] that the virtual fundamental class of the moduli space $\overline{\mathcal{M}}(S^1,\beta)$ is defined by an obstruction theory relative to $\mathfrak{M}_{0,1}$.

We consider the product $\overline{\mathcal{M}}(S,\beta)_{0,1} \times_S F_0$ and restrict to those components of $\overline{\mathcal{M}}(S,\beta)_{0,1}$ where no component of the domain curve maps into $F_0$. We have an inclusion of components $\overline{\mathcal{M}}(S,\beta) \subset \overline{\mathcal{M}}(S,\beta)_{0,1} \times_S F_0$. Furthermore, there is a virtual class on $\overline{\mathcal{M}}(S,\beta)_{0,1} \times_S F_0$ relative to $\mathfrak{M}_{0,1}$ defined by the obstruction bundle induced by the pull-back of the kernel of the natural morphism $\Theta_{S/k} \to N_{F_0/S}$. Note that this kernel is rank two bundle away from the singular points of $F_0$, and no curve in the moduli space passes through these singular points. Sections of the pull-back of this bundle then correspond to infinitesimal deformations that deform the marked point in a direction along $F_0$. It is a folklore result exposited in [8, remark A.5 of the appendix] that the virtual class this defines is equal to the pullback of the
virtual class on $\overline{M}(S, \beta)_{0,0}$. Their result holds so long as $F_0$ and $S$ are both smooth. This is not the case and we instead consider $Spec \ k \in \mathbb{P}^1$ which indeed does have smooth source and target. Then the Gysin map of $F_0$ in $S$ is compatible with the Gysin map of a point in $\mathbb{P}^1$ for which we do have the desired equality.

It remains to compare the virtual classes on $\overline{M}(S, \beta)_{0,1} \times_S F_0$ and on $\overline{M}^\dagger(S, \beta)$. Write $D_i$ for the components of $F_0$. There is a standard exact sequence of sheaves on $S$ given by

$$0 \to \Theta^I_{S^1/k} \to \Theta_{S/k} \to \bigoplus i_* N_{D_i/S} \to 0.$$  

By definition, the curves we wish to count do not map into the intersections $D_i \cap D_j$ and so the pull-backs of $\Theta^I_{S^1/k}$ or $\ker(\Theta_{S/k} \to N_{F_0/S})$ to any stable curve in $\overline{M}^\dagger(S, \beta) \cong \overline{M}(S, \beta)_{0,1} \times_S F_0$ are isomorphic. We consider the two cotangent complexes $L_{\overline{M}^\dagger(S^1, \beta)/\mathfrak{M}^\dagger_{0,1}}$ and $L_{\overline{M}_{0,0}(S, \beta) \times_S F_0/\mathfrak{M}_{0,1}}$. Note the log structure on $\overline{M}^\dagger(S^1, \beta)$ is the pull-back of the basic log structure on $\mathfrak{M}_{0,1}$ under the forgetful map because there are no components or nodes of the domain curve mapping into $F_0$. Thus we have a strict factorisation $\overline{M}^\dagger(S^1, \beta) \to \mathfrak{M}_{0,1} \to \mathfrak{M}^\dagger_{0,1}$, with the second morphism the open embedding of curves with the basic log structure. Thus the two cotangent complexes are isomorphic.

Since we have an isomorphism of obstruction theories and an isomorphism of underlying schemes, the corresponding virtual classes are equal. \hfill \Box

The techniques of Bryan and Leung do not apply directly to the tangency order 2 case. This means we have to work harder.

**Theorem 7.4** (Log invariants of bisections are restrictions of Gromov-Witten invariants). Let $i : \mathbb{P}^1 \to \text{Hilb}^2(\mathbb{P}^1) \cong \mathbb{P}^2$ be the hyperplane of double covers ramifying over $0 \in \mathbb{P}^1$ and $\beta$ be a curve class with $\beta.K_S = -2$. We denote the product $\overline{M}_{0,0}(S, \beta) \times [\mathbb{P}^2/C_2] \mathbb{P}^1$ by $\overline{M}^{ram}(S, \beta)$ with projection maps $\text{pr}$ to $\overline{M}_{0,0}(S, \beta)$ and $\text{ram}$ to $\mathbb{P}^1$. Then the moduli space of log stable maps to $S^1$, $\overline{M}^\dagger(S^1, \beta)$ admits a morphism $j$ to $\overline{M}^{ram}(S, \beta)$ which is an inclusion of components and there is an equality of virtual classes

$$j^* i^! [\overline{M}_{0,0}(S, \beta)]^{virt} = [\overline{M}^\dagger(S^1, \beta)]^{virt} \quad (7.3)$$

**Proof.** The map $j$ is the forgetful map, forgetting the choice of marking. A log curve in this space only meets the boundary in a single point, so maps to the desired space. By construction of the space $\overline{M}^{ram}(S, \beta)$, there is a canonical choice of marked point given by intersecting with the boundary $F_0$ so long as no component of the curve maps into $F_0$. Viewing this point as a logarithmic marked point with contact order 2 with one of the boundary divisors $D_i$ or contact orders 1 with adjacent divisors $D_i, D_j$, we obtain a family of stable log maps. Thus $\overline{M}^\dagger(S^1, \beta)$ is open inside some components of $\overline{M}^{ram}$ but is also proper and hence closed. Thus $j$ is an inclusion of components. To be precise it is those components which do not cover the marked rational fibre and have precisely one point mapping to the boundary, which excludes any point components.

As in the single section case, we relate potentially distinct obstruction theories on $\overline{M}^\dagger(S^1, \beta)$ and $\overline{M}_{0,0}(S, \beta)$. Recall that the obstruction theories on $\overline{M}^\dagger(S^1, \beta)$ and $\overline{M}_{0,0}(S, \beta)$ are given by $R\pi_\text{S, ram} \mathcal{F}^\text{S} \Theta^\dagger_{S^1/k}$ and $R\pi_\text{S, ram} \mathcal{F}^\text{S} \Theta^\dagger_{S/k}$, respectively. There is an exact sequence

$$0 \to \Theta^I_{S^1/k} \to \Theta_{S/k} \to \bigoplus \mathcal{O}_{D_i}(D_i) \to 0,$$
where $E_0 = \bigcup D_i$. Pulling back to the universal curve over $\overline{M}(S^1, \beta)$, pushing forward along $\pi$ and dualising, we therefore obtain a distinguished triangle
\[
(R\pi_*f^* \bigoplus \mathcal{O}_{D_i}(D_i))^\vee \to (R\pi_*f^* \Theta_{S/k})^\vee \to (R\pi_*f^* \Theta_{S^1/k})^\vee \to .
\]

Now $f^* \bigoplus \mathcal{O}_{D_i}(D_i)$ is supported only at a single point of each fibre of $\pi$, hence it pushes forwards to a line bundle on $\overline{M}(S^1, \beta)$. By [27], consider the log cotangent triangle associated to the maps $\overline{M}(S^1, \beta) \to \mathfrak{M}_{0,1} \to \mathfrak{M}_{0,0}$. The map $\overline{M}(S^1, \beta) \to \mathfrak{M}_{0,0}$ factors through the basic log stable maps, and so the cotangent complex of this is isomorphic to the restriction of $\overline{M}(S, \beta) \to \mathfrak{M}_{0,0}$. Thus we obtain an exact sequence together with a morphism of distinguished triangles from the one above:
\[
\cdots \longrightarrow (R\pi_*f^* \bigoplus \mathcal{O}_{D_i}(D_i))^\vee \phi \longrightarrow (R\pi_*f^* \Theta_{S/k})^\vee \longrightarrow (R\pi_*f^* \Theta_{S^1/k})^\vee \longrightarrow \cdots
\]
\[
\cdots \longrightarrow i^*L^1_{\overline{M}_{0,1}/\overline{M}_{0,0}} \longrightarrow j^*L^1_{\overline{M}(S, \beta)/\mathfrak{M}_{0,0}} \longrightarrow L^1_{\overline{M}(S^1, \beta)/\mathfrak{M}_{0,1}} \longrightarrow \cdots
\]

Since $(R\pi_*f^* \Theta_{S/k})^\vee$ and $(R\pi_*f^* \Theta_{S^1/k})^\vee$ are obstruction theories, we obtain by the 4-lemma that $(R\pi_*f^* \bigoplus \mathcal{O}_{D_i}(D_i))^\vee$ maps surjectively in degree 0 to $i^*L^1_{\overline{M}_{0,1}/\overline{M}_{0,0}}$. The complex $i^*L^1_{\overline{M}_{0,1}/\overline{M}_{0,0}}$ is a line bundle supported in degree 0. But $(R\pi_*f^* \bigoplus \mathcal{O}_{D_i}(D_i))^\vee$ is also a line bundle supported in degree 0 and maps surjectively and hence isomorphically to $i^*L^1_{\overline{M}_{0,1}/\overline{M}_{0,0}}$.

This produces an equality of virtual classes on $\overline{M}(S^1, \beta)$
\[
j^*(i^!([\mathfrak{M}_{0,0}(S, \beta)]^\text{virt})) = [\overline{M}(S^1, \beta)]^\text{virt}
\]
as desired.

We can combine this with our relation between the invariants of $X$ and $S$, producing the following theorem.

**Corollary 7.5 (A formula for bisections on $S$).** There are equalities
\[
\deg (\text{ram}^*(2H) \cdot [\mathfrak{M}_{0,0}(S, \beta)]^\text{virt}) = \deg ([\mathfrak{M}_{0,0}(X, d^{-1}\beta)]^\text{virt})
\]
and
\[
\deg (j^*(\text{ram}^*(H) \cdot [\mathfrak{M}_{0,0}(S, \beta)]^\text{virt})) = \deg ([\overline{M}(S^1, \beta)]^\text{virt}).
\]

### 8. Bounding Gromov–Witten invariants

As we said at the start of the previous section, there is no reason why when we pass to the log moduli space we do not remove components with high degree, potentially ruining our bound of the $J$-function. This section is dedicated to proving that restriction of the virtual class to curve components is positive, whilst the restriction to bubble components is not too negative. We begin by proving that the point components make no contribution to either of our invariants.

**Lemma 8.1 (Point components are trivial).** The restriction of $\text{ram}^*H \cdot [\overline{M}_{0,0}(S, \beta)]^\text{virt}$ to point components is zero.

**Proof.** This follows from conservation of number and choosing a different hyperplane to intersect with. \qed
By relating the invariants on $S$ to those on a $K3$ fibre of $X$, we can apply techniques from [8]. The authors there prove positivity of the virtual class restricted to various components of the moduli space.

**Theorem 8.2 (Curve components are positive).** The restriction of $\text{ram}^* H \cdot [\overline{\mathcal{M}}_{0,0}(S, \beta)]^{\text{virt}}$ to the pre-image of a curve component $\mathcal{M}$ is positive.

**Proof.** By our compatibility result relating Gromov–Witten invariants on $S$ and $X$, it is enough to prove this positivity on the threefold $X$. This problem was studied by Maulik and Pandharipande in [24] but we can prove the equality here via a direct calculation. Recall that the reduced invariants on a $K3$ surface $K$ defined in [24, section 2] are given by an obstruction theory $T$ fitting into an exact triangle

$$
\tau_{\geq -1} \mathcal{R} \pi_* \omega_\pi \otimes H^0(K, \omega_K) \to Rf_* \Theta_{K/k}^\vee \to T^o \to,
$$

where $\tau_{\geq -1} \mathcal{R} \pi_* \omega_\pi \otimes H^0(K, \omega_K)$ is a line bundle supported in degree $-1$. Suppose now that $i : K \hookrightarrow X$ is a fibre of the $K3$ fibration of $X$ into which all the stable curves in $\mathcal{M}$ map. There is an exact sequence on $K$

$$
0 \to R\Theta_K/k \to i^* \Theta_X/k \to \mathcal{N}_{K/X} \to 0,
$$

where the normal bundle $\mathcal{N}_{K/X}$ is trivial since $K$ is a fibre. Pulling this back to the universal curve, pushing forwards to the moduli space and dualising, we obtain a distinguished triangle on $\overline{\mathcal{M}}_{0,0}(X, \beta)$

$$
\mathcal{L} \to (Rf_* i^* \Theta_X/k)^\vee \to (Rf_* \Theta_{K/k})^\vee \to$

where $\mathcal{L}$ is a line bundle supported in degree 0. By composition we find a morphism $(Rf_* i^* \Theta_X/k)^\vee \to T^o$ and let $\text{Cone}$ be the cone over this. Applying the octahedral axiom, we obtain a diagram

$$
\begin{array}{ccc}
\mathcal{L}[1] & \to & \text{Cone}[1] \\
Rf_* i^* \Theta_X/k & \to & T^o \\
Rf_* \Theta_{K/k}^\vee & \to & \tau_{\geq -1} \mathcal{R} \pi_* \omega_\pi \otimes H^0(K, \omega_K)[1]
\end{array}
$$

showing that the following triangle is distinguished

$$
\mathcal{L} \to \text{Cone} \to \tau_{\geq -1} \mathcal{R} \pi_* \omega_\pi \otimes H^0(K, \omega_K) \to .
$$

Now a generic choice of unravelling the universal curve over a curve component have the property that each component is the moduli space of stable maps to $K$ for a finite number of fibres $K$. In particular, the degree 0 cohomology of $(Rf_* i^* \Theta_X/k)^\vee$ and $T^o$ are isomorphic since both are isomorphic to the tangent space of $\mathcal{M}$. This is enough to show that in fact $\text{Cone}$ vanishes since $\text{Cone}$ has a two term resolution by line bundles, the differential is a surjection of line bundles since the degree 0 cohomology vanishes and hence this surjection is an isomorphism.
Now $\mathcal{M}$ is a component of the moduli space of stable maps both to $K$ and $X$ and the two induced obstruction theories are isomorphic. The result is known from [8] that each component of the moduli space contributes either one or zero to the total count. 

Now on the bubble component, we prove that the virtual class cannot become too negative.

**Theorem 8.3 (Bubble components are bounded).** Let $X$ be a generic unravelling of $S$, $\overline{\mathcal{M}}_{0,0}^{\text{bub}} \subset \overline{\mathcal{M}}_{0,0}(X, d^{-1} \beta)$ the pre-image of a bubble component, a family over $\iota : \mathbb{P}^1 \to \text{Hilb}^2(\mathbb{P}^1)$. Choosing a point $P : \text{Spec } k/C_2 \to \mathbb{P}^2/C_2$ in the image of $\iota$ defines a K3 surface $\text{cov} : K \to S$. This induces a map $P : \overline{\mathcal{M}}_{0,0}(K, \text{cov}^\ast \beta) \to \overline{\mathcal{M}}_{0,0}^{\text{bub}}$ and generically fibres $\overline{\mathcal{M}}_{0,0}^{\text{bub}}$ by moduli spaces of stable maps to K3 surfaces, $K_1 : \overline{\mathcal{M}}_{0,0}^{\text{bub}} \to \mathbb{P}^1$. We claim that there is a constant $q \in \mathbb{Q}$ with $P_\ast [\overline{\mathcal{M}}_{0,0}(K, \text{cov}^\ast \beta)]^{\text{virt}} = q [\overline{\mathcal{M}}_{0,0}(X, d^{-1} \beta)]^{\text{virt}}$ and that $q$ is dependent only on the choice of $X$.

**Proof.** This question is related to the comparison theory of Maulik and Pandharipande in [24]. In [24], the authors consider a family of non-singular K3 surfaces over a non-singular curve $C$, $p : X \to C$. The Gromov–Witten invariants of $X$ are related to those of a fibre $K$ by multiplication by the degree of the Hodge bundle of $X \to C$. The $q$ appearing here is the degree of the equivalent Hodge bundle term, as can be seen by comparing the relative theory we define to the one appearing in [24].

In our situation, however, there are some singular fibres of the fibration $p : X \to \mathbb{P}^1$. There are 24 ordinary double points, occurring where the double cover $\mathbb{P}^1 \to \mathbb{P}^1$ ramifies over the image of a rational fibre. One can take a resolution of this family as described in [24, Example 5.1] to obtain a new family of non-singular K3 surfaces with the same Gromov–Witten invariants. In their paper, Maulik and Pandharipande move to an analytic space to construct the small resolution. This possibly produces an algebraic space rather than a scheme. To construct an algebraic space from the small resolution, we proceed as follows. We appeal to [3, Theorem 7.3] which says that any Moishezon manifold is the analytification of an algebraic space. This resolution is indeed a Moishezon manifold since it is a small resolution of another Moishezon manifold. Therefore we consider a small resolution $\pi : \tilde{X} \to X$ with $\tilde{X}$ a smooth algebraic space. The moduli stack of stable maps to a Deligne Mumford stack was constructed in [1]. In that paper, the authors also proved that the usual technology for constructing virtual classes can be applied to such objects and so the techniques of [11] continue to apply. However, there are also two singular fibres where the K3 surface degenerates to the union of two rational elliptic surfaces meeting along a smooth elliptic fibre. We augment this singular family to a smooth log family. Let $K_1$ and $K_2$ denote these two singular fibres and we write $X^\dagger$ for $X$ with the divisorial log structure induced by the $K_i$. Since we have not imposed any tangency conditions, the classical and log Gromov–Witten invariants coincide. Now by construction the map $p : X^\dagger \to \mathbb{P}^1$ is log smooth and we can define a relative theory by taking the cone over the natural morphism

$$\tau_{\geq -1} R\pi_\ast (\omega^\dagger_X)^\vee \otimes H^0(X, \omega_{X/\mathbb{P}^1}) \to R\pi_\ast f'^\ast (\Theta_{X/\mathbb{P}^1}).$$

This gives a relative obstruction theory which may play the role of the relative theory defined in [24, section 2.2]. In particular, their equation

$$[\overline{\mathcal{M}}_{0,0}(\pi, \epsilon)]^{\text{virt}} = c_1(K^\ast) \cap [\overline{\mathcal{M}}_{0,0}(\pi, \epsilon)]^{\text{red}}$$

from page 22 continues to hold, but replacing $K$ by the relative log canonical line bundle with fibre $H^0(X_\xi, K_{X_\xi}^\ast)$ over a point $\xi \in \overline{\mathcal{M}}_{0,0}(X, d^{-1} \beta)$, the class $[\overline{\mathcal{M}}_{0,0}(\pi, \epsilon)]^{\text{virt}}$ by the class $[\overline{\mathcal{M}}_{0,0}(X, d^{-1} \beta)]^{\text{virt}}$ and $\overline{\mathcal{M}}_{0,0}(\pi, \epsilon)]^{\text{red}}$ by a class which is fibrewise the class $[\overline{\mathcal{M}}_{0,0}(K, \text{cov}^\ast \beta)]^{\text{virt}}$. 


This proves the desired result since we may take the Chern class to be supported away from the singular fibres, and here this relative Hodge bundle does not depend on the rational tails. □

9. Assembling the family

We now have a description of the product formula on the mirror of the terms appearing in the scattering diagram and of the piecewise linear function \( \phi \) on the universal cover. Therefore we can assemble all of these together to prove convergence of the mirror. This pits the growth of the function \( \phi \) against the growth of the number of curves, so we begin with bounding the growth of \( \phi \).

**Lemma 9.1 (Bounding the function \( \phi \)).** Suppose that \( m > n > 0 \) are integers, then the value \( \phi((m, 1)) - \phi((n, 1)) \) is bounded by \( (2 \lfloor m/4 \rfloor - 1)D_m - 1 \) where the index \( m \) is taken mod 4.

**Proof.** This follows from the explicit formula for \( \phi \) given in [5, Lemma 13]. □

Recall the structure of the equations (3.1) through (3.3). We will work through each term appearing and prove that they all converge in a neighbourhood of the origin.

**Proposition 9.2.** The coefficients \( f_{(k, 2)}, g_{(k, 2)} \) and \( r_{i(k, 2)} \) of these equations converge in a neighbourhood of the large complex structure limit point.

**Proof.** By symmetry it is enough to prove convergence only of the first equation and of one of the products \( \vartheta D_i \).

The coefficient \( f_{2, 2} \) is the sum over pairs which do not bend anywhere and end close to \((2, 2)\). The pairs of pants are pairs of broken lines from \((4n, 1)\) and \((-4n + 2, 1)\). Near \((2, 2)\) these lines carry the monomials \( z^{\phi(4n, 1)} \) and \( z^{\phi(-4n + 2, 1)} \). Therefore, we want to study the convergence of

\[
\sum |z^{\phi(4n, 1)} + \varphi((-4n + 2, 1))| < \sum |z^{n^2F}|,
\]

which converges since it decays at least exponentially. Exactly the same argument applies to show the convergence of \( f_{(4, 2)} \), \( r_{i} \) and \( r_{i+2} \). In all these cases, none of the broken lines can bend, there is a \( \mathbb{Z} \) indexed family of pairs of pants and the monomials are controlled entirely by the function \( \phi \). For the product \( \vartheta D_i \), there is precisely one pair of broken lines which lie in only one maximal cell of \( B \). This explains the leading coefficient of 1 appearing in the formulae. □

We now study the contribution to the product of points with \( y(P) = 1 \), that is, the theta functions \( \vartheta D_i \), restricting our analysis by symmetry to \( \vartheta D_1, \vartheta D_3 \) and \( \vartheta D_4 \). Let us begin with \( \vartheta D_1 \) where we describe all pairs of broken lines which contribute to \( \vartheta D_4 \). All such pairs contain one bend which for the purpose of proving convergence we can assume occurs on the first broken line. We may also assume that the first broken line is from \((4n, 1)\) with \( m, n \) positive and scatters off of \((4n, 1)\) with a monomial \( z^{E+kF} \) for \( E \) a section meeting \( D_1 \) and \( k \) non-negative. As a consequence, the second line is from \((-4m, 1)\). This pair of lines contributes

\[
f_{0, 0, E+kF} z^{E+\phi((4m+4n, 1)) - \phi((4n, 1)) + \phi((-4m, 1)) + kF}
\]

to the coefficient. This sum is taken over all \( m, n > 0, k \geq 0 \) and sections \( S \) meeting \( D_1 \) in one point.

**Lemma 9.3 (Single bends converge).** The following sum converges

\[
\sum_{m, n, k, S} f_{0, 0, S+kF} z^{S+\phi((4m+4n, 1)) + \phi((-4n, 1)) - \phi((4m, 1)) + kF}.
\]
Proof. We rewrite this sum as
\[
\left( \sum_{S,k} I_{0,0,S+kF}^k z^{S+kF} \right) \left( \sum_{m,n} z^{\phi((4m+4n,1)) + \phi((-4n,1)) - \phi((4m,1))} \right).
\] (9.1)

We know that the term \( I_{0,0,S+kF}^k \) has modulus at most \( 2^k \), therefore for \( |z^F| < 1/2 \) and \( |z^S| < 1 \) and after applying Lemma 4.1 one has convergence of the left-hand sum.

To prove convergence of the right-hand sum we expand out the sum using the explicit formula for \( \phi \) found above and bound it
\[
\sum_{m,n} |z^{\phi((4m+4n,1)) - \phi((4m,1)) + \phi((-4n,1))}| \leq \sum_{m,n} |z^{\phi((-4n,1)) + ((2m+2n-1)D_3)}|.
\]

And this converges since \( \phi((-4n,1)) \) is positive and grows quadratically in \( n \). □

The same argument shows that the pairs of broken lines corresponding to items 4 and 5 in the classification in Lemma 3.1 converge. Pairs corresponding to broken lines corresponding to item 3 come in two types, depending on whether the unique bend is along a ray with primitive generator \((k,1)\) or with primitive generator \((2k+1,2)\). The second case corresponds to sections passing through the intersection \( D_i \cap D_{i+1} \) for appropriate \( i \). These again have already been shown to converge by the above arguments. The first case corresponds to counts of curves tangent to \( F_0 \) at a single point to order two. They converge by combining our results Corollary 6.5, Theorem 7.2, Theorem 7.3, Theorem 7.4, Theorem 8.2 and Theorem 8.3, the combinatorics including the function \( \phi \) is explicitly calculated in [5, Lemmas 15].

Therefore we have shown that all the functions appearing in the defining equations of the mirror family are in fact holomorphic in a neighbourhood of the origin.

Theorem 9.4. The equations defining the mirror family to a rational elliptic surface relative to an \( I_4 \) fibre converge in a neighbourhood of the origin.

Proof. This follows from combining the equations (3.1)–(3.3), the holomorphicity of the \( J \) function of \( X \) in Theorem 6.5, the relations between the Gromov–Witten invariants of \( X \) and the log Gromov–Witten invariants of \((S,D)\) in Sections 7 and 8 and the preceding two lemmata of this section. □

10. Recognising the family
The above convergence result does not tell us very much about the family that one obtains. Fortunately the geometry of the singular locus allows us to say much more. In particular, we will be able to relate the divisor at infinity to the generic fibre of the surface \( S \).

Theorem 10.1. Let \( S \) be a rational elliptic surface with \( D \) an \( I_4 \) fibre. There exists a projective closure of the restriction of the mirror family to a locus in the base such that the boundary is a smooth elliptic curve mirror to the generic fibre of \( S \). This duality is induced by an expression for the \( \vartheta \)-functions in terms of Jacobi theta functions.

Consider equations (3.1)–(3.3), only looking at the terms of order two, these simplify to:
\[
\vartheta_{D_1} \vartheta_{D_3} = f_{(2,2)} \vartheta_{2D_2} + f_{(6,2)} \vartheta_{2D_4},
\] (10.1)
\[
\vartheta_{D_2} \vartheta_{D_4} = f_{(0,2)} \vartheta_{2D_1} + f_{(4,2)} \vartheta_{2D_3},
\] (10.2)
\[ \vartheta_{2D_1} = \frac{\vartheta_{D_1}^2 (1 + r_{1+2}^2) - r_{1+2}^2 \vartheta_{D_1+2}^2}{(1 + r_{1+2}^2)(1 + r_{1+2}^2) - r_{1+2}^2 r_{2+2}^2}. \] (10.3)

These equations correspond to taking the projective closure of the above family and looking at the fibre at infinity. Substituting in the final equation into the first two, we obtain the following pair of quadrics:

\[ \vartheta_{D_1} \vartheta_{D_3} = \frac{f(2,2)(1 + r_{4,2}^2) - f(6,2) r_{4,2}^2}{(1 + r_{4,2}^2)(1 + r_{4,2}^2) - r_{4,2}^2 r_{4,2}^2} \vartheta_{D_2}^2 + \frac{f(6,2)(1 + r_{2,2}^2) - f(2,2) r_{2,2}^2}{(1 + r_{2,2}^2)(1 + r_{2,2}^2) - r_{2,2}^2 r_{2,2}^2} \vartheta_{D_4}^2, \] (10.4)

\[ \vartheta_{D_2} \vartheta_{D_4} = \frac{f(0,2)(1 + r_{3,2}^2) - f(4,2) r_{3,2}^2}{(1 + r_{3,2}^2)(1 + r_{3,2}^2) - r_{3,2}^2 r_{3,2}^2} \vartheta_{D_3}^2 + \frac{f(4,2)(1 + r_{1,2}^2) - f(4,2) r_{1,2}^2}{(1 + r_{1,2}^2)(1 + r_{1,2}^2) - r_{1,2}^2 r_{1,2}^2} \vartheta_{D_4}^2. \] (10.5)

There are some relations between the functions involved in this description. To simplify this description, let us restrict to the locus where the areas of each of the pair of quadrics:

\[ X_1X_3 = tX_2^2 + tX_3^2, \quad X_2X_4 = tX_1^2 + tX_3^2. \]

Putting these equations into Sage, we can arrange them into Weierstrass form (using the WeierstrassForm function). This produces the equation

\[ v^2 = u^3 - u(-t^8/3 - 7t/24 - 1/768) - (2t^{12}/27 - 11t^8/72 - 11t^4/1152 + 1/55296). \]

A generically smooth elliptic curve with \( j \)-invariant

\[ \frac{16777216t^{24} + 44040192t^{20} + 38731776t^{16} + 11583488t^{12} + 151296t^8 + 672t^4 + 1}{65536t^{20} - 16384t^{16} + 1536t^{12} - 64t^8 + t^4}. \] (10.6)

Making the substitution \( s = 4t^2 \), we obtain the following expression.

\[ \frac{16 (s^4 + 14s^2 + 1)^3}{s^2(s - 1)^4(s + 1)^4}. \]

We compare this to the formula for the \( j \)-invariant in terms of the Jacobi modulus \( k \). By definition, the \( j \)-invariant is given by

\[ 256 \frac{(k^4 - k^2 + 1)^3}{k^4(k^2 - 1)^2}. \]

After a change of coordinates \( k = \frac{u^{+1/4}}{\sqrt{u}} \), we find that this is equal to

\[ 256 \frac{(u^4 + 7u^2/8 + 1/256)^3}{(u^4 + 1/4u^6 + 3/128u^4 + (-1/1024)u^2 + 1/65536)u^2} \]

and so lining this up with equation (10.6), we find that \( u = s/4 \). Substituting backwards, we find \( k = \frac{t^{+1/4}}{t} \).

Now let us evaluate the function \( t \) for the locus where the \( z^{[D_i]} \) are all equal to \( v \) and we express the functions in terms of the area of a general smooth fibre, say \( e^{i \alpha} = \prod_i z^{[D_i]} = v^4 \) (the first \( i \) being the imaginary number).

\[ f(k,2)(v) = \sum_n v^{(4n+2)^2} = \Theta_2(0, \rho) \]

\[ 1 + r_{i,2}^2(v) = \sum_{n \text{ even}} v^{4n^2} = (\Theta_3(0, \rho) + \Theta_4(0, \rho))/2 \]
and
\[ r^i_{j+2,2}(v) = \sum_{n \text{ odd}} v^{4n^2} = \frac{(\Theta_3(0, \rho) - \Theta_4(0, \rho))}{2}, \]
where \( \Theta_i(z, \rho) \) are the Jacobi theta functions:
\[ \Theta_1(z, \rho) = -i \sum_n (-1)^n e^{(2n+1)i\pi z} e^{(n+\frac{1}{2})^2i\pi \rho} \]
\[ \Theta_2(z, \rho) = \sum_n e^{(2n+1)i\pi z} e^{(n+\frac{1}{2})^2i\pi \rho} \]
\[ \Theta_3(z, \rho) = \sum_n e^{2ni\pi z} e^{n^2i\pi \rho} \]
and
\[ \Theta_4(z, \rho) = \sum_n (-1)^n e^{2ni\pi z} e^{n^2i\pi \rho}. \]

Substituting these into equations (10.4) and (10.5), we find that \( t \) is equal to the ratio \( \frac{\Theta_3(0, \rho)}{2\Theta_3(0, \rho)} \).
In particular, this is not constant in \( \rho \) and so for generic choices is smooth. Furthermore, the Jacobi modulus \( k \) of this curve is equal to \( \frac{\Theta_2(0, \rho)^2 + \Theta_3(0, \rho)^2}{2\Theta_2(0, \rho)\Theta_3(0, \rho)} \). But by definition we have that \( k \) is also \( \Theta_2(0, \tau)^2/\Theta_3(0, \tau)^2 \) where \( \tau \) is the fundamental period of the elliptic curve. Of course there should be a relation between these two, using \([21, 3.1c and 3.1d]\):
\[ \frac{\Theta_2(0, \rho)^2 + \Theta_3(0, \rho)^2}{2\Theta_2(0, \rho)\Theta_3(0, \rho)} = \frac{\Theta_3(0, \rho/2)^2}{\Theta_2(0, \rho/2)^2}. \]

The modular group acts on the Jacobi theta functions, with \( \tau \mapsto \tau + 1 \) swapping the pairs \( \Theta_3 \) and \( \Theta_4 \) and \( \Theta_1 \) and \( \Theta_2 \). The map \( \tau \mapsto -1/\tau \) swaps \( \Theta_2 \) and \( \Theta_4 \) but preserves \( \Theta_1 \) and \( \Theta_3 \).
Conjugating the first of these by the second, we see that
\[ \frac{\Theta_3(0, \rho/2)^2}{\Theta_2(0, \rho/2)^2} = \frac{\Theta_3(0, \frac{\rho}{2-r})^2}{\Theta_3(0, \frac{\rho}{2+r})^2}. \]

But this is also equal to \( \frac{\Theta_3(\tau)^2}{\Theta_3(\tau)^2} \) and so we can deduce that \( \tau \) and \( \frac{\rho}{2-r} \) are equal up to conjugacy under \( SL(2, \mathbb{Z}) \). But we have just seen that \( \rho \) and \( \frac{\rho}{2-r} \) are also conjugate. Therefore the two associated elliptic curves are isomorphic. The limit as the area of a curve approaches zero corresponds to the limit as the imaginary part of \( \rho \) goes to positive infinity. In this limit, the value of \( \tau \) also approaches the cusp point, and so the corresponding elliptic curve degenerates to the Tate curve. Of course we have already seen this, in the limit that we turn off all corrections this elliptic curve degenerates to a cycle of four rational curves.

This construction connects very pleasantly to the story of mirror symmetry for elliptic curves as described by Dijkgraaf in \([10]\) or Polischuk and Zaslow in \([28]\). These describe the mirror to an elliptic curve with complexified Kähler class \( \alpha \) as the elliptic curve with periods \( (1, \alpha) \). We do not need to specify a complex structure on the first curve since the Fukaya category does not depend on this choice. The choice of volume \( e^{\pi \tau} \) corresponds to a choice of complexified Kähler class on the general fibre, and thus we see a philosophical explanation of the above equality.

If we restrict now to the locus where \( z^{[C]} \) vanishes for \([C], [F] > 0\), the above surface degenerates further, only the above terms appear in the defining equation. Since now the equation is homogeneous in affine space, we recover the family as being a deformation of a cone over the mirror elliptic curve.
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