On Decoding Irregular Tanner Codes with Local-Optimality Guarantees

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Abstract

We consider decoding of binary linear Tanner codes using message-passing iterative decoding and linear programming (LP) decoding in memoryless binary-input output symmetric (MBIOS) channels. We present new certificates that are based on a combinatorial characterization for local-optimality of a codeword in irregular Tanner codes with respect to any MBIOS channel. This characterization is a generalization of [Arora, Daskalakis, Steurer, Proc. ACM Symp. Theory of Computing, 2009] and [Vontobel, Proc. Inf. Theory and Appl. Workshop, 2010] and is based on a conical combination of normalized weighted subtrees in the computation trees of the Tanner graph. These subtrees may have any finite height $h$ (even equal or greater than half of the girth of the Tanner graph). In addition, the degrees of local-code nodes in these subtrees are not restricted to two (i.e., these subtrees are not restricted to skinny trees). We prove that local optimality in this new characterization implies maximum-likelihood (ML) optimality and LP optimality, and show that a certificate can be computed efficiently.

We also present a new message-passing iterative decoding algorithm, called normalized weighted min-sum (NWMS). NWMS decoding is a belief-propagation (BP) type algorithm that applies to any irregular binary Tanner code with single parity-check local codes (e.g., LDPC codes and HDPC codes). We prove that if a locally-optimal codeword with respect to height parameter $h$ exists (whereby notably $h$ is not limited by the girth of the Tanner graph), then NWMS decoding finds this codeword in $h$ iterations. The decoding guarantee of the NWMS decoding algorithm applies whenever there exists a locally optimal codeword. Because local optimality of a codeword implies that it is the unique ML codeword, the decoding guarantee also provides an ML certificate for this codeword.

Finally, we apply the new local optimality characterization to regular Tanner codes, and prove lower bounds on the noise thresholds of LP decoding in MBIOS channels. When the noise is below these lower bounds, the probability that LP decoding fails to decode the transmitted codeword decays doubly exponentially in the girth of the Tanner graph.

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*The material in this paper was presented in part at the 2012 IEEE International Symposium on Information Theory, Cambridge, MA, USA, Jul. 2012, and in part at the 2012 IEEE 27th Convention of Electrical and Electronics Engineers in Israel, Eilat, Israel, Nov. 2012.

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1 Introduction

Modern coding theory deals with finding good error correcting codes that have efficient encoders and decoders [RU08]. Message-passing iterative decoding algorithms based on belief propagation (see, e.g., [Gal63, BGT93, Mac99, LMSS01, RU01]) and linear programming (LP) decoding [Fel03, FWK05] are examples of efficient decoders. These decoders are usually sub-optimal, i.e., they may fail to correct errors that are corrected by a maximum likelihood (ML) decoder.

Many works deal with low-density parity-check (LDPC) codes and generalizations of LDPC codes. LDPC codes were first defined by Gallager [Gal63] who suggested several message-passing iterative decoding algorithms (including an algorithm that is now known as the sum-product decoding algorithm). Tanner [Tan81] introduced graph representations (nowadays known as Tanner graphs) of linear codes based on bipartite graphs over variable nodes and constraint nodes, and viewed iterative decoding as message-passing algorithms over the edges of these bipartite graphs. In the standard setting, constraint nodes enforce a zero-parity among their neighbors. In the generalized setting, constraint nodes enforce a local error-correcting code. One may view a constraint node with a linear local code as a coalescing of multiple single parity-check nodes. Therefore, a code may have a sparser and smaller representation when represented as a Tanner code in the generalized setting. Sipser and Spielman [SS96] studied binary Tanner codes based on expander graphs and analyzed a simple bit-flipping decoding algorithm.

Wiberg et al. [WLK95, Wib96] developed the use of graphical models for systematically describing instances of known decoding algorithms. In particular, the sum-product decoding algorithm and the min-sum decoding algorithm are presented as generic iterative message-passing decoding algorithms that apply to any graph realization of a Tanner code. Wiberg et al. proved that the min-sum decoding algorithm can be viewed as a dynamic programming algorithm that computes the ML codeword if the Tanner graph is a tree. For LDPC codes, Wiberg [Wib96] characterized a necessary condition for decoding failures of the min-sum decoding algorithm by “negative” cost trees, called minimal deviations.

LP decoding was introduced by Feldman, Wainwright, and Karger [Fel03, FWK05] for binary linear codes. LP decoding is based on solving a fractional relaxation of an integer linear program that models the problem of ML decoding. The vertices of the relaxed LP polytope are called pseudocodewords. Every codeword is a vertex of the relaxed LP polytope, however, usually there are additional vertices for which at least one component is non-integral. LP decoding has been applied to several codes, among them: cycle codes, turbo-like and RA codes [FK04, HE05, GB11], LDPC codes [FMS*07, DDKW08, KV06, ADS09, HE11], and expander codes [FS05, Ska11]. Our work is motivated by the problem of finite-length and average-case analysis of successful LP decoding of binary Tanner codes. There are very few works on this problem, and they deal only with specific cases. For example, Feldman and Stein [FS05] analyzed special expander-based codes, and Goldenberg and Burshtein [GB11] dealt with repeat-accumulate codes.

Previous results. Combinatorial characterizations of sufficient conditions for successful decoding of the ML codeword are based on so called “certificates.” That is, given a channel observation $y$ and a codeword $x$, we are interested in a one-sided error test that answers the questions: is $x$ optimal with respect to $y$? is it unique? Note that the test may answer “no” for
a positive instance. A positive answer for such a test is called a certificate for the optimality of a codeword. Upper bounds on the word error probability are obtained by lower bounds on the probability that a certificate exists.

Koetter and Vontobel [KV06] analyzed LP decoding of regular LDPC codes. Their analysis is based on decomposing each codeword (and pseudocodeword) to a finite set of minimal structured trees (i.e., skinny trees) with uniform vertex weights. Arora et al. [ADS09] extended the work in [KV06] by introducing non-uniform weights to the vertices in the skinny trees, and defining local optimality. For a BSC, Arora et al. proved that local optimality implies both ML optimality and LP optimality. They presented an analysis technique that performs finite-length density evolution of a min-sum process to prove bounds on the probability of a decoding error. Arora et al. also pointed out that it is possible to design a re-weighted version of the min-sum decoder for regular codes that finds the locally-optimal codeword if such a codeword exists for trees whose height is at most half of the girth of the Tanner graph. This work was further extended in [HE11] to memoryless binary-input output-symmetric (MBIOS) channels beyond the BSC. The analyses presented in these works [KV06, ADS09, HE11] are limited to skinny trees, the height of which is bounded by a half of the girth of the Tanner graph.

Vontobel [Von10a] extended the decomposition of a codeword (and a pseudocodeword) to skinny trees in graph covers. This enabled Vontobel to mitigate the limitation on the height of the skinny trees by half of the girth of the base Tanner graph. The decomposition is obtained by a random walk, and applies also to irregular Tanner graphs.

Various iterative message-passing decoding algorithms have been derived from the belief propagation algorithm (e.g., max-product decoding algorithm [WLK95], attenuated max-product [FK00], tree-reweighted belief-propagation [WJW05], etc.). The convergence of these belief-propagation (BP) based iterative decoding algorithms to an optimum solution has been studied extensively in various settings (see, e.g., [WLK95, FK00, WF01, CF02, CDE+05, WJW05, RU01, JP11]). However, bounds on the running time required to decode (or on the number of messages that are sent) have not been proven for these algorithms. The analyses of convergence in these works often rely on the existence of a single optimal solution in addition to other conditions such as: single-loop graphs, large girth, large reweighting coefficients, consistency conditions, etc.

Jian and Pfister [JP11] analyzed a special case of the attenuated max-product decoder [FK00] for regular LDPC codes. They considered skinny trees in the computation tree, the height of which is equal or greater than half of the girth of the Tanner graph. Using contraction properties and consistency conditions, they proved sufficient conditions under which the message-passing decoder converges to a locally optimal codeword. This convergence also implies convergence to the LP optimum and therefore to the ML codeword.

While local-optimality characterizations were investigated for the case of finite-length analysis of regular LDPC codes [ADS09, HE11, JP11], no local-optimality characterizations have been stated for the general case of Tanner codes. In this paper we study a generalization of previous local-optimality characterizations, and the guarantees it provides for successful ML decoding by LP decoding and iterative message-passing decoding algorithms. In particular, this paper presents a decoding algorithm for finite-length (regular and irregular) LDPC codes over MBIOS channels with bounded time complexity that combines two properties: (i) it is a message-passing algorithm, and (ii) for every number of iterations (not limited by any function of the girth of the Tanner graph), if the local-optimality characterization is satisfied for some codeword, then the algorithm succeeds to decode the ML codeword and has an ML certificate.
Contributions. We present a new combinatorial characterization for local optimality of a codeword in irregular binary Tanner codes with respect to (w.r.t.) any MBIOS channel (Definition 5). Local optimality is characterized via costs of deviations based on subtrees in computation trees of the Tanner graph. Consider a computation tree with height 2h rooted at some variable node. A deviation is based on a subtree such that (i) the degree of a variable node is equal to its degree in the computation tree, and (ii) the degree of a local-code node equals some constant \(d \geq 2\), provided that \(d\) is at most the minimum distance of the local codes. Furthermore, level weights \(w \in \mathbb{R}^h_+\) are assigned to the levels of the tree. Hence, a deviation is a combinatorial structure that has three main parameters: deviation height \(h\), deviation level weights \(w \in \mathbb{R}^h_+\), and deviation “degree” \(d\). Therefore, the new definition of local optimality is based on three parameters: \(h \in \mathbb{N}, w \in \mathbb{R}^h_+,\) and \(d \geq 2\).

This characterization extends the notion of deviations in local optimality in four ways:

(i) no restrictions are applied to the degrees of the nodes in the Tanner graph,
(ii) arbitrary local linear codes may be associated with constraint nodes,
(iii) deviations are subtrees in the computation tree and no limitation is set on the height of the deviations; in particular, their height may exceed (any function of) the girth of the Tanner graph, and
(iv) deviations may have a degree \(d \geq 2\) in the local-code nodes (as opposed to skinny trees in previous analyses), provided that \(d\) is at most the minimum distance of the local codes.

We prove that local optimality in this new characterization implies ML optimality (Theorem 7). We utilize the equivalence of graph cover decoding and LP decoding for Tanner codes, implied by Vontobel and Koetter [VK05] to prove that local optimality suffices also for LP optimality (Theorem 12). We present an efficient dynamic programming algorithm that computes a local-optimality certificate, and hence an ML certificate \(^2\), for a codeword w.r.t. a given channel output (Algorithm 1), if such certificate exists.

We present a new message-passing iterative decoding algorithm, called normalized weighted min-sum (NWMS) decoding algorithm (Algorithm 2). The NWMS decoding algorithm applies to any irregular Tanner code with single parity-check (SPC) local codes (e.g., LDPC codes and HDPC codes). The input to the NWMS decoding algorithm consists of the channel output and two additional parameters that characterize local optimality for Tanner codes with SPC local codes: (i) a certificate height \(h\), and (ii) a vector of layer weights \(w \in \mathbb{R}^h_+ \setminus \{0^h\}\). (Note that the local codes are SPC codes, and therefore the deviation degree \(d\) equals 2.) We prove that, for any finite \(h\), the NWMS decoding algorithm is guaranteed to compute the ML codeword in \(h\) iterations if an \(h\)-locally-optimal codeword exists (Theorem 13). The decoding guarantee of the NWMS algorithm is not bounded by (any function of) the girth. Namely, the height parameter \(h\) in local optimality and the number of iterations in the decoding is arbitrary and may exceed (any

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\(^1\)We consider computation trees that correspond to a “flooding message update schedule” in the context of iterative message-passing algorithms such as the max-product decoding algorithm.

\(^2\)An ML certificate computed based on local optimality is different from an ML certificate computed by LP decoding [FWK05] in the following sense. In the context of LP decoding, the ML certificate property means that if the LP decoder outputs an integral word, then it must be the ML codeword. Hence, one may compute an ML certificate for a codeword \(x\) and a given channel output by running the LP decoder and compare its result with the codeword \(x\). Local optimality is a combinatorial characterization of a codeword with respect to an LLR vector, which, by Theorem 7 suffices for ML. Hence, one may compute an ML certificate for a codeword \(x\) and a given channel output by verifying that the codeword is locally optimal w.r.t. the channel output. Algorithm 1 is an efficient message-passing algorithm that returns true if the codeword is locally optimal, and therefore provides an ML certificate.
function of) the girth. Because local optimality is a pure combinatorial property, the decoding guarantee of the NWMS decoding algorithm is not asymptotic nor does it rely on convergence. Namely, it applies to finite codes and decoding with a finite number of iterations. Furthermore, the output of the NWMS decoding algorithm can be ML-certified efficiently (Algorithm[1]). The time and message complexity of the NWMS decoding algorithm is $O(|E| \cdot h)$ where $|E|$ is the number of edges in the Tanner graph. Local optimality, as defined in this paper, is a sufficient condition for successfully decoding the unique ML codeword by our BP-based algorithm in loopy graphs.

Previous bounds on the probability that a local-optimality certificate exists [KV06, ADS09, HE11] hold for regular LDPC codes. The same bounds hold also for successful decoding of the transmitted codeword by the NWMS decoding algorithm. These bounds are based on proving that a local-optimality certificate exists with high probability for the transmitted codeword when the noise in the channel is below some noise threshold. The resulting threshold values happen to be relatively close to the BP thresholds. Specifically, noise thresholds of $p^* \geq 0.05$ in the case of a BSC [ADS09], and $\sigma^* \geq 0.735$ ($\frac{E_b}{N_0} \leq 2.67$dB) in the case of a BI-AWGN channel [HE11] are proven for $(3, 6)$-regular LDPC codes whose Tanner graphs have logarithmic girth in the block-length.

Finally, for a fixed height, trees in our new characterization contain more vertices than a skinny tree because the internal degrees are bigger. Hence, over an MBIOS channel, the probability of a locally-optimal certificate with dense deviations (local-code node degrees bigger than two) is greater than the probability of a locally-optimal certificate based on skinny trees (i.e., local-code nodes have degree two). This characterization leads to improved bounds for successful decoding of the transmitted codeword of regular Tanner codes (Theorems 22 and 33).

We extend the probabilistic analysis of the min-sum process by Arora et al. [ADS09] to a sum-min-sum process on regular trees. For regular Tanner codes, we prove bounds on the word error probability of LP decoding under MBIOS channels. These bounds are inverse doubly-exponential in the girth of the Tanner graph. We also prove bounds on the threshold of regular Tanner codes whose Tanner graphs have logarithmic girth. This means that if the noise in the channel is below that threshold, then the decoding error diminishes exponentially as a function of the block length. Note that Tanner graphs with logarithmic girth can be constructed explicitly (see, e.g., [Gal63]).

To summarize, our contribution is threefold.

(i) We present a new combinatorial characterization of local optimality for binary Tanner codes w.r.t. any MBIOS channel. This characterization provides an ML certificate and an LP certificate for a given codeword. The certificate can be efficiently computed by a dynamic programming algorithm. Based on this new characterization, we present two applications of local optimality.

(ii) A new efficient message-passing decoding algorithm, called normalized weighted min-sum (NWMS), for irregular binary Tanner codes with SPC local codes (e.g., LDPC codes and HDPC codes). The NWMS decoding algorithm is guaranteed to find the locally optimal codeword in $h$ iterations, where $h$ determines the height of the local-optimality certificate. Note that $h$ is not bounded and may be larger than (any function of) the girth of the Tanner graph (i.e., decoding with local-optimality guarantee “beyond the girth”).

(iii) New bounds on the word error probability are proved for LP decoding of regular binary Tanner codes.
Organization. The remainder of this paper is organized as follows. Section 2 provides background on ML decoding and LP decoding of binary Tanner codes over MBIOS channels. Section 3 presents a combinatorial certificate that applies to ML decoding for codewords of Tanner codes. In Section 4, we prove that the certificate applies also to LP decoding for codewords of Tanner codes. In Section 5, we present an efficient certification algorithm for local optimality. Section 6 presents the NWMS iterative decoding algorithm for irregular Tanner codes with SPC local codes, followed by a proof that the NWMS decoding algorithm finds the locally-optimal codeword. In Section 7, we use the combinatorial characterization of local optimality to bound the error probability of LP decoding for regular Tanner codes. Finally, conclusions and a discussion are given in Section 8.

2 Preliminaries

2.1 Graph Terminology

Let $G = (V, E)$ denote an undirected graph. Let $\mathcal{N}_G(v)$ denote the set of neighbors of node $v \in V$, and for a set $S \subseteq V$ let $\mathcal{N}_G(S) \triangleq \bigcup_{v \in S} \mathcal{N}_G(v)$. Let $\deg_G(v) \triangleq |\mathcal{N}_G(v)|$ denote the edge degree of node $v$ in graph $G$. A path $p = (v, \ldots, u)$ in $G$ is a sequence of vertices such that there exists an edge between every two consecutive nodes in the sequence $p$. A path $p$ is backtrackless if every two consecutive edges along $p$ are distinct. Let $s(p)$ denote the first vertex (source) of path $p$, and let $t(p)$ denote the last vertex (target) of path $p$. If $s(p) = t(p)$ then the path is closed. A simple path is a path with no repeated vertex. A simple cycle is a closed backtrackless path where the only repeated vertex is the first and last vertex. Let $|p|$ denote the length of a path $p$, i.e., the number of edges in $p$. Let $d_G(r, v)$ denote the distance (i.e., length of a shortest path) between nodes $r$ and $v$ in $G$, and let $\text{girth}(G)$ denote the length of the shortest cycle in $G$. Let $p$ and $q$ denote two paths in a graph $G$ such that $t(p) = s(q)$. The path obtained by concatenating the paths $p$ and $q$ is denoted by $p \circ q$.

An induced subgraph is a subgraph obtained by deleting a set of vertices. In particular, the subgraph of $G$ induced by $S \subseteq V$ consists of $S$ and all edges in $E$, both endpoints of which are contained in $S$. Let $G_S$ denote the subgraph of $G$ induced by $S$.

2.2 Tanner Codes and Tanner Graph Representation

Let $G = (\mathcal{V} \cup \mathcal{J}, E)$ denote an edge-labeled bipartite graph, where $\mathcal{V} = \{v_1, \ldots, v_N\}$ is a set of $N$ vertices called variable nodes, and $\mathcal{J} = \{C_1, \ldots, C_J\}$ is a set of $J$ vertices called local-code nodes. We denote the degree of $C_j$ by $n_j$.

Let $\overrightarrow{\mathcal{C}} \triangleq \{\overrightarrow{\mathcal{C}}^j \mid \overrightarrow{\mathcal{C}}^j \text{ is an } [n_j, k_j, d_j] \text{ code, } 1 \leq j \leq J\}$ denote a set of $J$ linear local codes. The local code $\overrightarrow{\mathcal{C}}^j$ corresponds to the vertex $C_j \in \mathcal{J}$. We say that $v_i$ participates in $\overrightarrow{\mathcal{C}}^j$ if $(v_i, C_j)$ is an edge in $E$. The edges incident to each local-code node $C_j$ are labeled $\{1, \ldots, n_j\}$. This labeling specifies the index of a variable node in the corresponding local code.

A word $x = (x_1, \ldots, x_N) \in \{0, 1\}^N$ is an assignment to variable nodes in $\mathcal{V}$ where $x_i$ is assigned to $v_i$. Let $\mathcal{V}_j$ denote the set $\mathcal{N}_G(C_j)$ ordered according to labels of edges incident to $C_j$. Denote by $x_{\mathcal{V}_j} \in \{0, 1\}^{n_j}$ the projection of the word $x = (x_1, \ldots, x_N)$ onto entries associated with $\mathcal{V}_j$.
The binary Tanner code $C(G, \overline{C}^J)$ based on the labeled Tanner graph $G$ is the set of vectors $x \in \{0, 1\}^N$ such that $x_{V_j}$ is a codeword in $\overline{C}^j$ for every $j \in \{1, \ldots, J\}$. Let us note that all the codes that we consider in this paper are binary linear codes.

Let $d_j$ denote the minimum distance of the local code $\overline{C}^j$. The minimum local distance $d^*$ of a Tanner code $C(G, \overline{C}^J)$ is defined by $d^* = \min_j d_j$. We assume that $d^* \geq 2$.

If the bipartite graph is $(d_L, d_R)$-regular, i.e., the vertices in $V$ have degree $d_L$ and the vertices in $J$ have degree $d_R$, then the resulting code is called a $(d_L, d_R)$-regular Tanner code.

If the Tanner graph is sparse, i.e., $|E| = O(N)$, then it defines a low-density Tanner code. A single parity-check code is a code that contains all binary words with even Hamming weight. Tanner codes that have single parity-check local codes and that are based on sparse Tanner graphs are called low-density parity-check (LDPC) codes.

Consider a Tanner code $C(G, \overline{C}^J)$. We say that a word $x = (x_1, \ldots, x_N)$ satisfies the local code $\overline{C}^j$ if its projection $x_{V_j}$ is in $\overline{C}^j$. The set of words $x$ that satisfy the local code $\overline{C}^j$ is denoted by $C^j$, i.e., $C^j = \{x \in \{0, 1\}^N \mid x_{V_j} \in \overline{C}^j\}$. Namely, the resulting code $C^j$ is the extension of the local code $\overline{C}^j$ from length $n_j$ to length $N$. The Tanner code is simply the intersection of the extensions of the local codes, i.e.,

$$C(G, \overline{C}^J) = \bigcap_{j \in \{1, \ldots, J\}} C^j. \tag{1}$$

2.3 LP Decoding of Tanner Codes over Memoryless Channels

Let $c_i \in \{0, 1\}$ denote the $i$th transmitted binary symbol (channel input), and let $y_i \in \mathbb{R}$ denote the $i$th received symbol (channel output). A memoryless binary-input output-symmetric (MBIOS) channel is defined by a conditional probability density function $f(y_i | c_i = a)$ for $a \in \{0, 1\}$ that satisfies $f(y_i | 0) = f(-y_i | 1)$. The binary erasure channel (BEC), binary symmetric channel (BSC) and binary-input additive white Gaussian noise (BI-AWGN) channel are examples for MBIOS channels. Let $y \in \mathbb{R}^N$ denote the word received from the channel. In MBIOS channels, the log-likelihood ratio (LLR) vector $\lambda = \lambda(y) \in \mathbb{R}^N$ is defined by $\lambda_i(y_i) \triangleq \ln \left( \frac{f(y_i | c_i = 0)}{f(y_i | c_i = 1)} \right)$ for every input bit $i$. For a code $C$, Maximum Likelihood (ML) decoding is equivalent to

$$\hat{x}_{\text{ML}}(y) = \arg \min_{x \in \text{conv}(C)} \langle \lambda(y), x \rangle, \tag{2}$$

where $\text{conv}(C)$ denotes the convex hull of the set $C$ where $\{0, 1\}^N$ is considered to be a subset of $\mathbb{R}^N$.

In general, solving the optimization problem in (2) for linear codes is intractable \cite{BMV17}. Feldman et al. \cite{Feld03, FWK05} introduced a linear programming relaxation for the problem of ML decoding of Tanner codes with single parity-check codes acting as local codes. The resulting relaxation of $\text{conv}(C)$ is nowadays called the fundamental polytope \cite{VK05} of the Tanner graph $G$. We consider an extension of this definition to the case in which the local codes are arbitrary as follows. The generalized fundamental polytope $\mathcal{P} \triangleq \mathcal{P}(G, \overline{C}^J)$ of a

\footnote{Strictly speaking, the operator $\arg \min$ returns a set of vectors because $\langle \lambda(y), x \rangle$ may have multiple minima w.r.t. $\text{conv}(C)$. When $\arg \min$ returns a singleton set, then $\arg \min$ is equal to the vector in that set. Otherwise, it returns a random vector from the set.}
Tanner code $\mathcal{C} = \mathcal{C}(G, \overline{\mathcal{C}}^J)$ is defined by

$$\mathcal{P} \triangleq \bigcap_{C \in \mathcal{C}^J} \text{conv}(C).$$

Note that a code may have multiple representations by a Tanner graph and local codes. Moreover, different representations $(G, C_J)$ of the same code $\mathcal{C}$ may yield different generalized fundamental polytopes $\mathcal{P}(G, C_J)$. If the degree of each local-code node is constant, then the generalized fundamental polytope can be represented by $O(N + |J|)$ variables and $O(|J|)$ constraints. Typically, $|J| = O(N)$, and the generalized fundamental polytope has an efficient representation. Such Tanner codes are often called generalized low-density parity-check codes.

Given an LLR vector $\lambda$ for a received word $y$, LP decoding is defined by the following linear program

$$\hat{x}_{\text{LP}}(y) \triangleq \arg \min_{x \in \mathcal{P}(G, \overline{\mathcal{C}}^J)} \langle \lambda(y), x \rangle.$$  

The difference between ML decoding and LP decoding is that the fundamental polytope $\mathcal{P}(G, \overline{\mathcal{C}}^J)$ may strictly contain the convex hull of $\mathcal{C}$. Vertices of $\mathcal{P}(G, \overline{\mathcal{C}}^J)$ are called pseudocodewords \cite{Fel03, FWK05}. It can be shown that vertices of $\mathcal{P}(G, \overline{\mathcal{C}}^J)$ that are not codewords of $\mathcal{C}$ must have at least one non-integral component.

### 3 A Combinatorial Certificate for an ML Codeword

In this section we present combinatorial certificates for codewords of Tanner codes that apply both to ML decoding and LP decoding. A certificate is a proof that a given codeword is the unique solution of ML decoding and LP decoding. The certificate is based on combinatorial weighted structures in the Tanner graph, referred to as local configurations. These local configurations generalize the minimal configurations (skinny trees) presented by Vontobel \cite{Von10a} as extension to Arora et al. \cite{ADS09}. We note that for Tanner codes, the characteristic function of the support of each weighted local configuration is not necessarily a locally valid configuration. For a given codeword, the certificate is computed by a dynamic-programming algorithm on the Tanner graph of the code (see Section 5).

**Notation:** Let $y \in \mathbb{R}^N$ denote the word received from the channel. Let $\lambda = \lambda(y)$ denote the LLR vector for $y$. Let $G = (V \cup J, E)$ denote a Tanner graph, and let $\mathcal{C}(G)$ denote a Tanner code based on $G$ with minimum local distance $d^*$. Let $x \in \mathcal{C}(G)$ be a candidate for $\hat{x}_{\text{ML}}(y)$ and $\hat{x}_{\text{LP}}(y)$.

**Definition 1 (Path-Prefix Tree).** Consider a graph $G = (V, E)$ and a node $r \in V$. Let $\hat{V}$ denote the set of all backtrackless paths in $G$ with length at most $h$ that start at node $r$, and let

$$\hat{E} \triangleq \{(p_1, p_2) \in \hat{V} \times \hat{V} \mid p_1 \text{ is a prefix of } p_2, \ |p_1| + 1 = |p_2|\}.$$  

We identify the zero-length path in $\hat{V}$ with $(r)$. Denote by $T^h_r(G) \triangleq (\hat{V}, \hat{E})$ the path-prefix tree of $G$ rooted at node $r$ with height $h$. 

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Path-prefix trees of $G$ that are rooted at a variable node or at a local-code node are often called computation trees. We consider also path-prefix trees of subgraphs of $G$ that may be either rooted at a variable node or at a local-code node.

We use the following notation. Vertices in $G$ are denoted by $u,v,r$. Because vertices in $\mathcal{T}^{\ell}(G) = (\hat{V}, \hat{E})$ are paths in $G$, we denote vertices in path-prefix trees by $p$ and $q$. For a path $p \in \hat{V}$, let $s(p)$ denote the first vertex (source) of path $p$, and let $t(p)$ denote the last vertex (target) of path $p$. Denote by $\text{Prefix}^+(p)$ the set of proper prefixes of the path $p$, i.e.,

$$\text{Prefix}^+(p) = \left\{ q \mid q \text{ is a prefix of } p, \ 1 \leq |q| < |p| \right\}.$$ 

Consider a Tanner graph $G = (\hat{V}, \hat{E})$ and let $\mathcal{T}_r^{\ell}(G) = (\hat{V}, \hat{E})$ denote a path-prefix tree of $G$. Let $\hat{V} \triangleq \{ p \mid p \in \hat{V}, \ t(p) \in \hat{V} \}$, and $\hat{J} \triangleq \{ p \mid p \in \hat{V}, \ t(p) \in \hat{J} \}$. Paths in $\hat{V}$ are called variable paths, and paths in $\hat{J}$ are called local-code paths.

The following definitions expand the combinatorial notion of minimal valid deviations [Wib96] and weighted minimal local deviations (skinny trees) [ADS09, Von10a] to the case of Tanner codes.

**Definition 2 (d-tree).** Consider a Tanner graph $G = (\hat{V}, \hat{E})$. Denote by $\mathcal{T}_r^{2h}(G) = (\hat{V}, \hat{J}, \hat{E})$ the path-prefix tree of $G$ rooted at node $r \in \hat{V}$. A subtree $\mathcal{T} \subseteq \mathcal{T}_r^{2h}(G)$ is a $d$-tree if:

(i) $\mathcal{T}$ is rooted at $(r)$,

(ii) for every local-code path $p \in \mathcal{T} \cap \hat{J}$, $\deg_{\mathcal{T}}(p) = d$, and

(iii) for every variable path $p \in \mathcal{T} \cap \hat{V}$, $\deg_{\mathcal{T}}(p) = \deg_{\mathcal{T}_r^{2h}}(p)$.

Note that the leaves of a $d$-tree are variable paths because a $d$-tree is rooted in a variable node and has an even height. Let $\mathcal{T}[r, 2h, d](G)$ denote the set of all $d$-trees rooted at $r$ that are subtrees of $\mathcal{T}_r^{2h}(G)$.

In the following definition we use “level” weights $w = (w_1, \ldots, w_h)$ that are assigned to variable paths in a subtree of a path-prefix tree of height $2h$.

**Definition 3 (w-weighted subtree).** Let $\mathcal{T} = (\hat{V}, \hat{J}, \hat{E})$ denote a subtree of $\mathcal{T}_r^{2h}(G)$, and let $w = (w_1, \ldots, w_h) \in \mathbb{R}^h_+$ denote a non-negative weight vector. Let $w_{\mathcal{T}} : \hat{V} \to \mathbb{R}$ denote a weight function based on the weight vector $w$ for variable paths $p \in \hat{V}$ defined as follows. If $p$ is a zero-length variable path, then $w_{\mathcal{T}}(p) = 0$. Otherwise,

$$w_{\mathcal{T}}(p) \triangleq \frac{w_\ell}{\|w\|_1} \cdot \frac{1}{\deg_G(t(p))} \cdot \prod_{\substack{q \in \text{Prefix}^+(p) \\text{such that } \deg_{\mathcal{T}}(q) \neq 1}} \frac{1}{\deg_{\mathcal{T}}(q) - 1},$$

where $\ell = \lceil \frac{|w|}{2} \rceil$. We refer to $w_{\mathcal{T}}$ as a $w$-weighted subtree.

For any $w$-weighted subtree $w_{\mathcal{T}}$ of $\mathcal{T}_r^{2h}(G)$, let $\pi_{G,\mathcal{T},w} : \hat{V} \to \mathbb{R}$ denote a function whose values correspond to the projection of $w_{\mathcal{T}}$ to the Tanner graph $G$. That is, for every variable node $v$ in $G$,

$$\pi_{G,\mathcal{T},w}(v) \triangleq \sum_{\{p \in \mathcal{T} \mid \ell(p) = v\}} w_{\mathcal{T}}(p).$$

4Vertices in a path-prefix tree of a Tanner graph $G$ correspond to paths in $G$. We therefore refer by variable paths to vertices in a path-prefix tree that correspond to paths in $G$ that end at a variable node.
We remark that: (i) If no variable path in $T$ ends in $v$, then $\pi_{G,T,w}(v) = 0$. (ii) If $h < \text{girth}(G)/4$, then every node $v$ is an endpoint of at most one variable path in $T^{2h}(G)$, and the projection is trivial. However, we deal with arbitrary heights $h$, in which case the projection is many-to-one since many different variable paths may share a common endpoint. Notice that the length of the weight vector $w$ equals the height parameter $h$.

**Definition 4.** Consider a Tanner code $C(G)$, a non-positive weight vector $w \in \mathbb{R}^h$, and $2 \leq d \leq d^\ast$. Let $\mathcal{B}_d^{(w)}$ denote the set of all projections of $w$-weighted $d$-trees to $G$, i.e.,

$$\mathcal{B}_d^{(w)} \triangleq \left\{ \frac{1}{c} \cdot \pi_{G,T,w} \bigg| T \in \bigcup_{r \in V} T[r, 2h, d](G) \right\},$$

where $c \geq 1$ is chosen so that $\mathcal{B}_d^{(w)} \subseteq [0, 1]^N$.

Vectors in $\mathcal{B}_d^{(w)}$ are referred to as projected normalized weighted (PNW) deviations. We use a PNW deviations to alter a codeword in the upcoming definition of local optimality (Definition 5). Our notion of deviations differs from Wiberg’s deviations [Wib96] in three significant ways:

(i) For a $d$-tree $T$, the characteristic function of the support of $w_T$ is not necessarily a valid configuration of the computation tree.

(ii) The entries of $w_T$ are real scaled version of the characteristic function of the support of $w_T$. The scaling obeys a degree normalization along the path from the root of $T$ and a non-negative level weight factor as extension of weighted minimal deviations [ADS09, Von10a].

(iii) We apply a projection operator $\pi$ on $w_T$ to the Tanner graph $G$. The characteristic function of the support of the projection does not induce a tree on the Tanner graph $G$ when $h$ is large.

For two vectors $x \in \{0, 1\}^N$ and $f \in [0, 1]^N$, let $x \oplus f \in [0, 1]^N$ denote the relative point defined by $(x \oplus f)_i \triangleq |x_i - f_i|$ [Fel03]. The following definition is an extension of local optimality [ADS09, Von10a] to Tanner codes on memoryless channels.

**Definition 5** (local optimality). Let $C(G) \subset \{0, 1\}^N$ denote a Tanner code with minimum local distance $d^\ast$. Let $w \in \mathbb{R}^h \setminus \{0^h\}$ denote a non-negative weight vector of length $h$ and let $2 \leq d \leq d^\ast$. A codeword $x \in C(G)$ is $(h, w, d)$-locally optimal w.r.t. $\lambda \in \mathbb{R}^N$ if for all vectors $\beta \in \mathcal{B}_d^{(w)}$,

$$\langle \lambda, x \oplus \beta \rangle > \langle \lambda, x \rangle.$$ 

(6)

Based on random walks on the Tanner graph, the results in [Von10a] imply that $(h, w, d = 2)$-local optimality is sufficient both for ML optimality and LP optimality. The transition probabilities of these random walks are induced by pseudocodewords of the generalized fundamental polytope. We extend the results of Vontobel [Von10a] to “thicker” sub-trees by using probabilistic combinatorial arguments on graphs and the properties of graph cover decoding [VK05]. Specifically, for any $d$ with $2 \leq d \leq d^\ast$ we prove that $(h, w, d)$-local optimality for a codeword $x$ w.r.t. $\lambda$ implies both ML and LP optimality for a codeword $x$ w.r.t. $\lambda$ (Theorems 7 and 12).

The following structural lemma states that every codeword of a Tanner code is a finite conical combination of projections of weighted trees in the computation trees of $G$. 


Lemma 6 (conic decomposition of a codeword). Let $C(G)$ denote a Tanner code with minimum local distance $d^*$, and let $h$ be some positive integer. Consider a codeword $x \neq 0^N$. Then, for every $2 \leq d \leq d^*$, there exists a distribution $\rho$ over $d$-trees of $G$ of height $2h$ such that for every weight vector $w \in \mathbb{R}_+^h \setminus \{0^h\}$, it holds that

$$x = \|x\|_1 \cdot E_\rho[\pi_{G,T,w}].$$

Proof. See Appendix A.

Given Lemma 6, the following theorem is obtained by modification of the proof of [ADS09 Theorem 2] or [HE11, Theorem 6].

Theorem 7 (local optimality is sufficient for ML). Let $C(G)$ denote a Tanner code with minimum local distance $d^*$. Let $h$ be some positive integer and $w = (w_1, \ldots, w_h) \in \mathbb{R}_+^h$ denote a non-negative weight vector. Let $\lambda \in \mathbb{R}^N$ denote the LLR vector received from the channel. If $x$ is an $(h, w, d)$-locally optimal codeword w.r.t. $\lambda$ and some $2 \leq d \leq d^*$, then $x$ is also the unique ML codeword w.r.t. $\lambda$.

Proof. We use the decomposition implied by Lemma 6 to show that for every codeword $x' \neq x$, $\langle \lambda, x' \rangle > \langle \lambda, x \rangle$. Let $z \triangleq x \oplus x'$. By linearity, it holds that $z \in C(G)$. Moreover, $z \neq 0^N$ because $x \neq x'$. Because $d^* \geq 2$, it follows that $\|z\|_1 \geq 2$. By Lemma 6 there exists a distribution $\rho$ over the set $B_d^{(w)}$ of PNW deviations such that $E_\rho[c \cdot \beta] = \frac{1}{\|z\|_1}$, where $c \geq 1$ is the normalizing constant so that $B_d^{(w)} \subseteq [0, 1]^N$ (see Definition 4). Let $\alpha \triangleq \frac{1}{c}$. Let $f : [0, 1]^N \to \mathbb{R}$ be the affine linear function defined by $f(\beta) \triangleq \langle \lambda, x \oplus \beta \rangle = \langle \lambda, x \rangle + \sum_{i=1}^N(-1)^{x_i} \lambda_i \beta_i$. Then,

$$\langle \lambda, x \rangle < E_\rho(\lambda, x \oplus \beta)$$

(by local optimality of $x$)

$$= \langle \lambda, x \oplus E_\rho \beta \rangle$$

(by linearity of $f$ and $E_\beta$)

$$= \langle \lambda, x \oplus \alpha z \rangle$$

(by Lemma 6)

$$= \langle \lambda, (1 - \alpha)x + \alpha(x \oplus z) \rangle$$

$$= \langle \lambda, (1 - \alpha)x + \alpha x' \rangle$$

$$= \langle \lambda, (1 - \alpha)x \rangle + \alpha \langle \lambda, x' \rangle$$

which implies that $\langle \lambda, x' \rangle > \langle \lambda, x \rangle$ as desired.

4 Local Optimality Implies LP Optimality

In order to prove a sufficient condition for LP optimality, we consider graph cover decoding introduced by Vontobel and Koetter [VK05]. We note that the characterization of graph cover decoding and its connection to LP decoding can be extended to the case of Tanner codes in the generalized setting (see, e.g., [Von10b Theorem 25] and [Hal12 Theorem 2.14]).

We use the terms and notation of Vontobel and Koetter [VK05] (see also [HE11 Appendix A]) in the statements of Proposition and Lemma [VK05]. Specifically, let $\tilde{G}$ denote an $M$-cover of $G$. Let $\tilde{x} = x^M \in C(\tilde{G})$ and $\tilde{\lambda} = \lambda^M \in \mathbb{R}^{N \cdot M}$ denote the $M$-lifts of $x$ and $\lambda$, respectively.

In this section we consider the following setting. Let $C(G)$ denote a Tanner code with minimum local distance $d^*$. Let $w \in \mathbb{R}_+^h \setminus \{0^h\}$ for some positive integer $h$ and let $2 \leq d \leq d^*$. 

**Proposition 8** (local optimality of all-zero codeword is preserved by \(M\)-lifts). \(0^N\) is an \((h, w, d)\)-locally optimal codeword w.r.t. \(\lambda \in \mathbb{R}^N\) if and only if \(0^{N \cdot M}\) is an \((h, w, d)\)-locally optimal codeword w.r.t. \(\tilde{\lambda}\).

**Proof.** Consider the surjection \(\varphi\) of \(d\)-trees in the path-prefix tree of \(\tilde{G}\) to \(d\)-trees in the path-prefix tree of \(G\). This surjection is based on the covering map between \(\tilde{G}\) and \(G\). Given a PNW deviation \(\tilde{\beta} \triangleq \pi_{\tilde{G},T,w}\) based on a \(d\)-tree \(T\) in the path-prefix tree of \(\tilde{G}\), let \(\beta \triangleq \pi_{G,\varphi(T),w}\). The proposition follows because \(\langle \lambda, \beta \rangle = \langle \tilde{\lambda}, \tilde{\beta} \rangle\). □

For two vectors \(y, z \in \mathbb{R}^N\), let “*” denote coordinatewise multiplication, i.e., \(y * z \triangleq (y_1 \cdot z_1, \ldots, y_N \cdot z_N\). For a word \(x \in \{0, 1\}^N\), let \((-1)^x \in \{\pm 1\}^N\) denote the vector whose \(i\)th component equals \((-1)^{x_i}\).

**Lemma 9.** For every \(\lambda \in \mathbb{R}^N\) and every \(\beta \in [0, 1]^N\),

\[
\langle (-1)^x \ast \lambda, \beta \rangle = \langle \lambda, x \oplus \beta \rangle - \langle \lambda, x \rangle.
\]

(7)

**Proof.** For \(\beta \in [0, 1]^N\), it holds that \(\langle \lambda, x \oplus \beta \rangle = \langle \lambda, x \rangle + \sum_{i=1}^N (-1)^{x_i} \lambda_i \beta_i\). Hence,

\[
\langle \lambda, x \oplus \beta \rangle - \langle \lambda, x \rangle = \sum_{i=1}^N (-1)^{x_i} \lambda_i \beta_i
\]

\[=
\langle (-1)^x \ast \lambda, \beta \rangle.
\] □

The following proposition states that the mapping \((x, \lambda) \mapsto (0^N, (-1)^x \ast \lambda)\) preserves local optimality.

**Proposition 10** (symmetry of local optimality). For every \(x \in \mathcal{C}\), \(x\) is \((h, w, d)\)-locally optimal w.r.t. \(\lambda\) if and only if \(0^N\) is \((h, w, d)\)-locally optimal w.r.t. \((-1)^x \ast \lambda\).

**Proof.** By Lemma 9, \(\langle \lambda, x \oplus \beta \rangle - \langle \lambda, x \rangle = \langle (-1)^x \ast \lambda, \beta \rangle - \langle (-1)^x \ast \lambda, 0^N \rangle\). □

The following lemma states that local optimality is preserved by lifting to an \(M\)-cover.

**Lemma 11.** \(x\) is \((h, w, d)\)-locally optimal w.r.t. \(\lambda\) if and only if \(\tilde{x}\) is \((h, w, d)\)-locally optimal w.r.t. \(\tilde{\lambda}\).

**Proof.** Assume that \(\tilde{x}\) is a \((h, w, d)\)-locally optimal codeword w.r.t. \(\tilde{\lambda}\). By Proposition 10, \(0^{N \cdot M}\) is \((h, w, d)\)-locally optimal w.r.t. \((-1)^\tilde{x} \ast \lambda\). By Proposition 8, \(0^N\) is \((h, w, d)\)-locally optimal w.r.t. \((-1)^x \ast \lambda\). By Proposition 10, \(x\) is \((h, w, d)\)-locally optimal w.r.t. \(\lambda\). Each of these implications is necessary and sufficient, and the lemma follows. □

The following theorem is obtained as a corollary of Theorem 7 and Lemma 11. The proof is based on a reduction stating that if local optimality is sufficient for ML optimality, then it also suffices for LP optimality. The reduction is based on the equivalence of LP decoding and graph-cover decoding [VK05], and follows the line of the proof of [HE11] Theorem 8.

**Theorem 12** (local optimality is sufficient for LP optimality). If \(x\) is an \((h, w, d)\)-locally optimal codeword w.r.t. \(\lambda\), then \(x\) is also the unique optimal LP solution given \(\lambda\).
5 Verifying Local Optimality

In this section we address the problem of how to verify whether a codeword \( x \) is \((h, w, d)\)-locally optimal w.r.t. \( \lambda \). By Proposition [10], this is equivalent to verifying whether \( 0^N \) is \((h, w, d)\)-locally optimal w.r.t. \((-1)^x \lambda \), where \([-1)^x \lambda \) denotes \((-1)^x \lambda \). Note that the verification algorithm only computes the sign of \( \lambda \). The algorithm returns false if and only if it finds a PNW deviation with non-positive cost. Note that the verification algorithm only computes the sign of \( \min_\beta (\langle \lambda, x \oplus \beta \rangle - \langle \lambda, x \rangle) \). Moreover, the sign of \( \min_\beta (\langle \lambda, x \oplus \beta \rangle - \langle \lambda, x \rangle) \) is invariant under scaling \( \beta \) by any positive constant.

The verification algorithm is listed as Algorithm [11]. It applies dynamic programming to find, for every variable node \( v \), a \( d \)-tree \( T_v \), rooted at \( v \), that minimizes the cost \( \langle (-1)^x \lambda, \pi_{G,T_v,w} \rangle \). The algorithm returns false if and only if it finds a PNW deviation with non-positive cost. Note that the verification algorithm only computes the sign of \( \min_\beta (\langle \lambda, x \oplus \beta \rangle - \langle \lambda, x \rangle) \). Moreover, the sign of \( \min_\beta (\langle \lambda, x \oplus \beta \rangle - \langle \lambda, x \rangle) \) is invariant under scaling \( \beta \) by any positive constant.

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Because \( \|w\|_1 \) contains a “global” information, the division of \( \mu_v \) by \( \|w\|_1 \) does not take place to maintain the property that the verification algorithm is a distributed message passing algorithm.

The algorithm is presented as a message passing algorithm. In every step, a node propagates to its parent the minimum cost of the \( d \)-subtree that hangs from it based on the minimum values received from its children. The message complexity of Algorithm [11] is \( O(|E| \cdot h) \), where \( E \) denotes the edge set of the Tanner graph. Algorithm [11] can be implemented so that the running time of each iteration is: (i) \( O(|E|) \) for the computation of the messages from variable nodes to check nodes, and (ii) \( O(|E| \cdot \log d) \) for the computation of the messages from check nodes to variable nodes.

The following notation is used in Line 8 of the algorithm. For a set \( S \) of real values, let \( \min^{[i]} \{ S \} \) denote the \( i \)th smallest member in \( S \).

Algorithm 1 &lo(x, h, w, d) - An iterative verification algorithm. Let \( G = (\mathcal{V} \cup \mathcal{E}, E) \) denote a Tanner graph. Given an LLR vector \( \lambda \in \mathbb{R}^{|\mathcal{V}|} \), a codeword \( x \in C(G) \), level weights \( w \in \mathbb{R}_+^{|\mathcal{V}|} \), and a degree \( d \in \mathbb{N} \), outputs “true” if \( x \) is \((h, w, d)\)-locally optimal w.r.t. \( \lambda \); otherwise, outputs “false.”

```
1: Initialize: \forall v \in \mathcal{V}: \lambda_v \leftarrow \lambda_v \cdot (-1)^{x_v}
2: \forall C \in \mathcal{E}, \forall v \in \mathcal{N}(C): \mu^{(-1)}_{C \rightarrow v} \leftarrow 0
3: for \ l = 0 to \ h - 1 do
4:  \ \ \ \ for all \ v \in \mathcal{V}, C \in \mathcal{N}(v) \ do
5:  \ \ \ \ \mu^{(l)}_{v \rightarrow C} \leftarrow \frac{w_{h-l}}{\deg_G(v)} \lambda_v + \frac{1}{\deg_G(v) - 1} \sum_{C' \in \mathcal{N}(v) \setminus \{C\}} \mu^{(l-1)}_{C' \rightarrow v}
6: \ \ \ \ end for
7:  \ \ \ \ for all \ C \in \mathcal{E}, v \in \mathcal{N}(C) \ do
8:  \ \ \ \ \mu^{(l)}_{C \rightarrow v} \leftarrow \frac{1}{d-1} \cdot \sum_{i=1}^{d-1} \min^{[i]} \left\{ \mu^{(l)}_{v' \rightarrow C} \mid v' \in \mathcal{N}(C) \setminus \{v\} \right\}
9: \ \ \ \ end for
10: end for
11: \ for all \ v \in \mathcal{V} \ do
12: \ \ \ \ \mu_v \leftarrow \sum_{C \in \mathcal{N}(v)} \mu^{(h-1)}_{C \rightarrow v}
13: \ \ \ \ if \ \mu_v \leq 0 \ then \ \{ \text{min-cost } w \text{-weighted } d \text{-tree rooted at } v \text{ has non-positive value} \}
14: \ \ \ \ \ \ \ \ \ \ return \ \text{false};
15: \ \ \ \ end if
16: \ \ \ \ end for
17: \ return \ \text{true};
```
6 Message-Passing Decoding with ML Guarantee for Irregular LDPC Codes

In this section we present a weighted min-sum decoder (called, NWMS) for irregular Tanner codes with single parity-check local codes over any MBIOS channel. In Section 6.2 we prove that the decoder computes the ML codeword if a locally-optimal codeword exists (Theorem 13). Note that Algorithm NWMS is not presented as a min-sum process. However, in Section 6.2, an equivalent min-sum version is presented.

We deal with Tanner codes based on Tanner graphs \( G = (V \cup J, E) \) with single parity-check local codes. Local-code nodes \( C \in J \) in this case are called check nodes. The graph \( G \) may be either regular or irregular. All the results in this section hold for every Tanner graph, regardless of its girth, degrees, or density.

A huge number of works deal with message-passing decoding algorithms. We point out three works that can be viewed as precursors to our decoding algorithm. Gallager [Gal63] presented the sum-product iterative decoding algorithm for LDPC codes. Tanner [Tan81] viewed iterative decoding algorithms as message-passing iterative algorithms over the edges of the Tanner graph. Wiberg [Wib96] characterized decoding failures of the min-sum iterative decoding algorithm by negative cost trees. Message-passing decoding algorithms proceed by iterations of “ping-pong” messages between the variable nodes and the local-code nodes in the Tanner graph. These messages are sent along the edges.

Algorithm description. Algorithm NWMS\((\lambda, h, w)\), listed as Algorithm 2, is a normalized \(w\)-weighted version of the min-sum decoding algorithm for decoding Tanner codes with single parity-check local codes. The input to algorithm NWMS consists of an LLR vector \( \lambda \in \mathbb{R}^N \), an integer \( h > 0 \) that determines the number of iterations, and a nonnegative weight vector \( w \in \mathbb{R}_+^h \backslash \{0\} \). For each edge \((v, C)\), each iteration consists of one message from the variable node \( v \) to the check node \( C \) (that is, the “ping” message), and one message from \( C \) to \( v \) (that is, the “pong” message). Hence, the message complexity of Algorithm 2 is \( O(|E| \cdot h) \). (It can be implemented so that the running time is also \( O(|E| \cdot h) \)).

Let \( \mu_{v\rightarrow C}^{(l)} \) denote the “ping” message from a variable node \( v \in V \) to an adjacent check node \( C \in J \) in iteration \( l \) of the algorithm. Similarly, let \( \mu_{C\rightarrow v}^{(l)} \) denote the “pong” message from \( C \in J \) to \( v \in V \) in iteration \( l \). Denote by \( \mu_v \) the final value computed by variable node \( v \in V \). Note that the NWMS decoding algorithm does not add \( w_0 \lambda_v \) in the computation of \( \mu_v \) in Line 11 for ease of presentation. The output of the algorithm \( \hat{x} \in \{0,1\}^N \) is computed locally by each variable node in Line 12. In the case where \( \mu_v = 0 \) we chose to assign \( x_v = 1 \) for ease of presentation. However, one can choose to assign \( x_v \) with either a ‘0’ or a ‘1’ with equal probability. Algorithm NWMS may be applied to any MBIOS channel (e.g., BEC, BSC, AWGN, etc.) because the input is the LLR vector.

---

5 Adding \( w_0 \lambda_v \) to \( \mu_v \) in Line 11 requires changing the definition of PNW deviations so that they also include the root of each d-tree.

6 In the case of a BEC, the LLR vector \( \lambda \) is in \( \{+\infty, -\infty, 0\}^N \). In this case, all the messages in Algorithm 2 are in the set \( \{-\infty, 0, +\infty\} \). The arithmetic over this set is the arithmetic of the affinely extended real number system (e.g., for a real \( a \), \( \pm\infty + a = \pm\infty \), etc.). Under such arithmetic, there is no need to assign weights to the LLR value and the incoming messages in the computation of variable-to-check messages in Line 11. Notice that \( +\infty \) is never added to \( -\infty \) since a BEC may only erase bits and can not flip any bit. Therefore, all computed messages in Algorithm 2 are equal to either \( \pm\infty \) or 0.
Algorithm 2 NWMS($\lambda, h, w$) - An iterative normalized weighted min-sum decoding algorithm. Given an LLR vector $\lambda \in \mathbb{R}^N$ and level weights $w \in \mathbb{R}_+^h \setminus \{0^h\}$, outputs a binary string $\hat{x} \in \{0, 1\}^N$.

1: Initialize: $\forall C \in \mathcal{J}, \forall v \in \mathcal{N}(C): \mu_{C \rightarrow v}^{(-1)} \leftarrow 0$
2: for $l = 0$ to $h - 1$ do
3: for all $v \in \mathcal{V}$, $C \in \mathcal{N}(v)$ do \{“PING”\}
4: $\mu_{v \rightarrow C}^{(l)} \leftarrow \frac{\mu_{h \rightarrow v}^{(l-1)} \lambda_v + \frac{1}{\deg_G(v)} \sum_{C' \in \mathcal{N}(v) \setminus \{C\}} C_{C' \rightarrow v}}{\deg_G(v)}$
5: end for
6: for all $C \in \mathcal{J}, v \in \mathcal{N}(C)$ do \{“PONG”\}
7: $\mu_{C \rightarrow v}^{(l)} \leftarrow \left( \prod_{u \in \mathcal{N}(C) \setminus \{v\}} \text{sign}(\mu_{u \rightarrow C}^{(l)}) \right) \cdot \min \left\{ |\mu_{u \rightarrow C}| \mid u \in \mathcal{N}(C) \setminus \{v\} \right\}$
8: end for
9: end for
10: for all $v \in \mathcal{V}$ do \{Decision\}
11: $\mu_v \leftarrow \sum_{C \in \mathcal{N}(v)} \mu_{C \rightarrow v}^{(h-1)}$
12: $\hat{x}_v \leftarrow \begin{cases} 0 & \text{if } \mu_v > 0, \\ 1 & \text{otherwise.} \end{cases}$
13: end for

The upcoming Theorem 13 states that NWMS($\lambda, h, w$) computes an $(h, w, d = 2)$—locally optimal codeword w.r.t. $\lambda$ if such a codeword exists. Hence, Theorem 13 provides a sufficient condition for successful iterative decoding of the ML codeword for any finite number $h$ of iterations. In particular, the number of iterations may exceed (any function of) the girth. Theorem 13 implies an alternative proof of the uniqueness of an $(h, w, d = 2)$—locally optimal codeword that is proved in Theorem 7. The proof appears in Section 6.2.

Theorem 13 (NWMS decoding algorithm finds the locally optimal codeword). Let $G = (\mathcal{V} \cup \mathcal{J}, E)$ denote a Tanner graph and let $\mathcal{C}(G) \subset \{0, 1\}^N$ denote the corresponding Tanner code with single parity-check local codes. Let $h \in \mathbb{N}_+$ and let $w \in \mathbb{R}_+^h \setminus \{0^h\}$ denote a non-negative weight vector. Let $\lambda \in \mathbb{R}^N$ denote the LLR vector of the channel output. If $x \in \mathcal{C}(G)$ is an $(h, w, d = 2)$—locally optimal codeword w.r.t. $\lambda$, then NWMS($\lambda, h, w$) outputs $x$.

The message-passing algorithm VERIFY-LO (Algorithm 1) described in Section 5 can be used to verify whether NWMS($\lambda, h, w$) outputs the $(h, w, d = 2)$—locally optimal codeword w.r.t. $\lambda$. If there exists $(h, w, d = 2)$—locally optimal codeword w.r.t. $\lambda$, then, by Theorem 7 and Theorem 13, it holds that: (i) the output of NWMS($\lambda, h, w$) is the unique ML codeword, and (ii) algorithm VERIFY-LO returns true for the decoded codeword. If no $(h, w, d = 2)$—locally optimal codeword exists w.r.t. $\lambda$, then algorithm VERIFY-LO returns false for every input code-word. We can therefore obtain a message-passing decoding algorithm with an ML certificate obtained by local optimality by using Algorithms 1 and 2 as follows.

Algorithm ML-CERTIFIED-NWMS($\lambda, h, w$), listed as Algorithm 3, is an ML-certified version of the NWMS decoding algorithm. The input to algorithm ML-CERTIFIED-NWMS consists of an LLR vector $\lambda \in \mathbb{R}^N$, an integer $h > 0$ that determines the number of iterations, and a nonnegative weight vector $w \in \mathbb{R}_+^h \setminus \{0^h\}$. If the ML-CERTIFIED-NWMS decoding algorithm returns a binary word, then it is guaranteed to be the unique ML codeword w.r.t. $\lambda$. Otherwise,
ML-CERTIFIED-NWMS declares a failure to output an ML-certified codeword. The message complexity of Algorithm 2 is $O(|E| \cdot h)$. (It can be implemented so that the running time is also $O(|E| \cdot h)$).

Algorithm 3 ML-CERTIFIED-NWMS($\lambda, h, w$) - An iterative normalized weighted min-sum decoding algorithm with an ML-certified output based on local optimality. Given an LLR vector $\lambda \in \mathbb{R}^N$ and level weights $w \in \mathbb{R}^h_+ \setminus \{0^h\}$, outputs the ML codeword $\hat{x} \in \{0, 1\}^N$ w.r.t $\lambda$ or a “failure”.

1: $x \leftarrow \text{NWMS}(\lambda, h, w)$
2: if $x$ is a codeword then
3: if VERIFY-LO($x, \lambda, h, w, 2$) = true then {$x$ is $(h, w, 2)$-locally optimal w.r.t $\lambda$}
4: return $x$;
5: end if
6: end if
7: return failure;

Remark: Local optimality is a sufficient condition for ML. In case that there is no $(h, w, d = 2)$-locally optimal codeword w.r.t. $\lambda$, then the binary word that the NWMS decoding algorithm outputs may be an ML codeword. Note however, that the ML-CERTIFIED-NWMS decoding algorithm declares a failure to output an ML-certified codeword in this case. In the case where a locally-optimal codeword exists, then both NWMS decoding algorithm and ML-CERTIFIED-NWMS decoding algorithm are guaranteed to output this codeword, which is the unique ML codeword w.r.t $\lambda$.

6.1 Symmetry of NWMS Decoding Algorithm and the All-Zero Codeword Assumption

We define symmetric decoding algorithms (see [RU08, Definition 4.81] for a discussion of symmetry in message passing algorithms).

Definition 14 (symmetry of decoding algorithm). Let $x \in \mathcal{C}$ denote a codeword and let $(-1)^x \in \{\pm 1\}^N$ denote the vector whose $i$th component equals $(-1)^{x_i}$. Let $\lambda$ denote an LLR vector. A decoding algorithm, $\text{DEC}(\lambda)$, is symmetric w.r.t. code $\mathcal{C}$, if

$$\forall x \in \mathcal{C}. \ x \oplus \text{DEC}(\lambda) = \text{DEC}((-1)^x \ast \lambda). \quad (8)$$

The following lemma states that the NWMS decoding algorithm is symmetric. The proof is by induction on the number of iterations.

Lemma 15 (symmetry of NWMS). Fix $h \in \mathbb{N}_+$ and $w \in \mathbb{R}^N_+$. Consider $\lambda \in \mathbb{R}^N$ and a codeword $x \in \mathcal{C}(G)$. Then,

$$x \oplus \text{NWMS}(\lambda, h, w) = \text{NWMS}((-1)^x \ast \lambda, h, w). \quad (9)$$

Proof. See Appendix 14

The following corollary follows from Lemma 15 and the symmetry of an MBIOS channel.
**Corollary 16** (All-zero codeword assumption). Fix $h \in \mathbb{N}_+$ and $w \in \mathbb{R}^N_+$. For MBIOS channels, the probability that the NWMS decoding algorithm fails to decode the transmitted codeword is independent of the transmitted codeword itself. That is,

$$\Pr\{\text{NWMS fails} \} = \Pr\{\text{NWMS}(\lambda, h, w) \neq 0^N \mid c = 0^N\}.$$ 

**Proof.** Following Lemma 15 for every codeword $x$,

$$\Pr\{\text{NWMS}(\lambda, h, w) \neq x \mid c = x\} = \Pr\{\text{NWMS}((-1)^x \ast \lambda, h, w) \neq 0^N \mid c = x\}.$$ 

For MBIOS channels, $f(\lambda_1 | c_2 = 0) = f(-\lambda_1 | c_2 = 1)$. Therefore, the mapping $(x, \lambda) \mapsto (0^N, (-1)^x \ast \lambda)$ preserves the probability measure. We apply this mapping to $(x, (-1)^x \ast \lambda) \mapsto (0^N, (-1)^x \ast (-1)^x \ast \lambda)$ and conclude that

$$\Pr\{\text{NWMS}((-1)^x \ast \lambda, h, w) \neq 0^N \mid c = x\} = \Pr\{\text{NWMS}(\lambda, h, w) \neq 0^N \mid c = 0^N\}.$$ 

Following the contra-positive of Theorem 13 and Corollary 16, provided that the channel is symmetric, for a fixed $h$ and $w \in \mathbb{R}^h_+ \setminus \{0^h\}$, we have

$$\Pr\{\text{NWMS}(\lambda, h, w) \text{ fails}\} \leq \Pr\{\exists \beta \in B_2^{(w)} \text{ s.t. } \langle \lambda, \beta \rangle \leq 0 \mid c = 0^N\}.$$  \hspace{1cm} (10)

Bounds on the existence of a non-positive PNW deviation (i.e., the right-hand side in Equation (10)) are discussed in Section 8.1.

### 6.2 Proof of Theorem 13 – NWMS Decoding Algorithm Finds the Locally Optimal Codeword

**Proof outline.** The proof of Theorem 13 is based on two observations.

(i) We present an equivalent algorithm, called NWMS2 (Section 6.2.1), and prove that Algorithm NWMS2 outputs the all-zero codeword if $0^N$ is locally optimal (Sections 6.2.2, 6.2.3).

(ii) In Lemma 15, we proved that the NWMS decoding algorithm is symmetric. This symmetry is w.r.t. the mapping of a pair $(x, \lambda)$ of a codeword and an LLR vector to a pair $(0^N, \lambda^0)$ of the all-zero codeword and a corresponding LLR vector $\lambda^0 \triangleq (-1)^x \ast \lambda$ (recall that “$\ast$” denotes a coordinate-wise vector multiplication).

To prove Theorem 13, we prove the contrapositive statement, that is, if $x \neq \text{NWMS}(\lambda, h, w)$, then $x$ is not $(h, w, d = 2)$-locally optimal w.r.t. $\lambda$. Let $x$ denote a codeword, and let $(-1)^x$ denote the vector whose $i$th component equals $(-1)^{x_i}$. Define $\lambda^0 \triangleq (-1)^x \ast \lambda$. By definition $\lambda = (-1)^x \ast \lambda^0$.

The proof is obtained by the following derivations. Because $x \neq \text{NWMS}(\lambda, h, w)$, it follows by Lemma 15 (symmetry of NWMS) that $x \neq x \oplus \text{NWMS}(\lambda^0, h, w)$, and hence $0^N \neq \text{NWMS}(\lambda^0, h, w)$. By the upcoming Lemma 21, $0^N$ is not $(h, w, 2)$-locally optimal w.r.t. $\lambda^0$. Because $\lambda = (-1)^x \ast \lambda^0$, it follows by Proposition 10 that $x$ is not $(h, w, 2)$-locally optimal w.r.t. $\lambda$ as required.

We are left to prove Lemma 21 used in the foregoing proof.
6.2.1 NWMS2: An Equivalent Version

The input to Algorithm NWMS includes the LLR vector \( \lambda \). We refer to this algorithm as a min-sum decoding algorithm in light of the general description of Wiberg [Wib96] in the log-domain. In Wiberg’s description, every check node finds a minimum value from a set of functions on the incoming messages, and every variable node computes the sum of the incoming messages and its corresponding channel observation. Hence the name min-sum.

Let \( y \in \mathbb{R}^N \) denote channel observations. For \( a \in \{0, 1\} \), define the log-likelihood of \( y_i \) by \( \lambda_i(a) \triangleq - \log \left( f(y_i|c_i = a) \right) \). Note that the log-likelihood ratio \( \lambda_i \) for \( y_i \) equals \( \lambda_i(1) - \lambda_i(0) \). For \( a \in \{0, 1\} \), let \( \lambda(a) \in \mathbb{R}^N \) denote the log-likelihood vector whose \( i \)th component equals \( \lambda_i(a) \).

Algorithm NWMS2(\( \lambda(0), \lambda(1), h, w \)), listed as Algorithm 4, is a normalized \( w \)-weighted min-sum decoding algorithm. Algorithm NWMS2 computes separate reliabilities for “0” and “1”. Namely, \( \mu^{(l)}_{C \rightarrow v}(a) \) and \( \mu^{(l)}_{C \rightarrow v}(a) \) denote the messages corresponding to the assumption that node \( v \) is assigned the value \( a \) (for \( a \in \{0, 1\} \)). The higher the values of these messages, the lower the likelihood of the event \( x_v = a \).

The main difference between the presentations of Algorithm 2 and Algorithm 4 is in Line 7. Consider a check node \( C \) and valid assignment \( x \in \{0, 1\}^{\operatorname{deg}(C)} \) to variable nodes adjacent to \( C \) with even weight. For every such assignment \( x \) in which \( x_v = a \), the check node \( C \) computes the sum of the incoming messages \( \mu^{(l)}_{u \rightarrow C}(x_u) \) from the neighboring nodes \( u \in N(C) \setminus \{v\} \). The message \( \mu^{(l)}_{C \rightarrow v}(a) \) equals the minimum value over these valid summations.

**Algorithm 4 NWMS2(\( \lambda(0), \lambda(1), h, w \)) - An iterative normalized weighted min-sum decoding algorithm.** Given log-likelihood vectors \( \lambda(a) \in \mathbb{R}^N \) for \( a \in \{0, 1\} \) and level weights \( w \in \mathbb{R}_+^h \setminus \{0^h\} \), outputs a binary string \( \hat{x} \in \{0, 1\}^N \).

1: Initialize: \( \forall C \in \mathcal{J}, \forall v \in N(C), \forall a \in \{0, 1\} : \mu^{(-1)}_{C \rightarrow v}(a) \leftarrow 0 \)
2: for \( l = 0 \) to \( h - 1 \) do
3:     for all \( v \in \mathcal{V}, C \in N(v), a \in \{0, 1\} \) do \{“PING”\}
4:         \( \mu^{(l)}_{C \rightarrow v}(a) \leftarrow \frac{w_{h-l}}{\deg_C(v)} \lambda_v(a) + \frac{1}{\deg_C(v)-1} \sum_{C' \in N(v) \setminus \{C\}} \mu^{(l-1)}_{C' \rightarrow v}(a) \)
5:     end for
6: for all \( C \in \mathcal{J}, v \in N(C), a \in \{0, 1\} \) do \{“PONG”\}
7: \[ \mu^{(l)}_{C \rightarrow v}(a) \leftarrow \min \left\{ \sum_{u \in N(C) \setminus \{v\}} \mu^{(l)}_{u \rightarrow C}(x_u) \middle| x \in \{0, 1\}^{\deg(C)} \quad ||x||_1 \text{ is even} \quad x_v = a \right\} \]
8: end for
9: end for
10: for all \( v \in \mathcal{V} \) do \{Decision\}
11: \( \mu_v(a) \leftarrow \sum_{C \in N(v)} \mu^{(h-1)}_{C \rightarrow v}(a) \)
12: \( \hat{x}_v \left\{ \begin{array}{ll}
0 & \text{if } (\mu_v(1) - \mu_v(0)) > 0, \\
1 & \text{otherwise}.
\end{array} \right. \)
13: end for

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Claim 17. Let $\lambda$, $\lambda(0)$, and $\lambda(1)$ in $\mathbb{R}^N$ denote the LLR vector and the two log-likelihood vectors for a channel output $y \in \mathbb{R}^N$. Then, for every $h \in \mathbb{N}^+$ and $w \in \mathbb{R}_+^h$, the following equalities hold:

1. $\mu_{v\rightarrow C}^{(l)} = \mu_{v\rightarrow C}^{(l)}(1) - \mu_{v\rightarrow C}^{(l)}(0)$ and $\mu_{C\rightarrow v}^{(l)} = \mu_{C\rightarrow v}^{(l)}(1) - \mu_{C\rightarrow v}^{(l)}(0)$ in every iteration $l$.
2. $\mu_v = \mu_v(1) - \mu_v(0)$. Hence $\text{NWMS}(\lambda, h, w)$ and $\text{NWMS}(\lambda(0), \lambda(1), h, w)$ output the same vector $\hat{x}$.

6.2.2 NWMS2 as a Dynamic Programming Algorithm

In Lemma 18 we prove that Algorithm NWMS2 is a dynamic programming algorithm that computes, for every variable node $v$, two min-weight valid configurations. One configuration is 0-rooted and the other configuration is 1-rooted. Algorithm NWMS2 decides $\hat{x}_v = 0$ if the min-weight valid configuration rooted at $v$ is 0-rooted, otherwise decides $\hat{x}_v = 1$. We now elaborate on the definition of valid configurations and their weight.

Valid configurations and their weight. Fix a variable node $r \in V$. We refer to $r$ as the root. Consider the path-prefix tree $\mathcal{T}_2^h(G)$ rooted at $r$ consisting of all the paths of length at most $2h$ starting at $r$. Denote the vertices of $\mathcal{T}_2^h$ by $\mathcal{V} \cup \hat{\mathcal{J}}$, where paths in $\mathcal{V} = \{p \mid p \in \hat{\mathcal{V}}, t(p) \in \mathcal{V}\}$ are variable paths, and paths in $\mathcal{J} = \{p \mid p \in \hat{\mathcal{V}}, t(p) \in \hat{\mathcal{J}}\}$ are parity-check paths. Denote by $(r)$ the zero-length path, i.e., the path consisting of only the root $r$.

A binary word $z \in \{0,1\}^{\mathcal{V}}$ is interpreted as an assignment to variable paths $p \in \hat{\mathcal{V}}$ where $z_p$ is assigned to $p$. We say that $z$ is a valid configuration if it satisfies all parity-check paths in $\mathcal{J}$. Namely, for every check path $q \in \mathcal{J}$, the assignment to its neighbors has an even number of ones. We denote the set of valid configurations of $\mathcal{T}_2^h$ by $\text{vconfig}(\mathcal{T}_2^h)$.

The weight $\mathcal{W}_{\mathcal{T}_2^h}(z)$ of a valid configuration $z$ is defined by weights $\mathcal{W}_{\mathcal{T}_2^h}(p)$ that are assigned to variable paths $p \in \hat{\mathcal{V}}$ as follows. We start with level weights $w = (w_1, \ldots, w_h) \in \mathbb{R}_+^h$ that are assigned to levels of variable paths in $\mathcal{T}_2^h$. Define the weight of a variable path $p \in \hat{\mathcal{V}}$ w.r.t. $w$ by

$$\mathcal{W}_{\mathcal{T}_2^h}(p) \triangleq \frac{w_{|p|/2}}{\deg_G(t(p))} \cdot \prod_{q \in \text{Prefix}^+(p) \cap \hat{\mathcal{V}}} \frac{1}{\deg_G(t(q)) - 1}.$$

There is a difference between Definition 3 and $\mathcal{W}_{\mathcal{T}_2^h}(p)$. A minor difference is that we do not divide by $\|w\|_1$ as in Definition 3. The main difference is that in Definition 3 the product is taken over all paths in $\text{Prefix}^+(p)$. However, in $\mathcal{W}_{\mathcal{T}_2^h}(p)$ the product is taken only over variable paths in $\text{Prefix}^+(p)$.

The weight of a valid configuration $z \in \{0,1\}^{\mathcal{V}}$ is defined by

$$\mathcal{W}_{\mathcal{T}_2^h}(z) \triangleq \sum_{p \in \mathcal{V} \setminus \{r\}} \lambda_{t(p)}(z_p) \cdot \mathcal{W}_{\mathcal{T}_2^h}(p).$$

---

7We use the same notation as in Definition 3.
Given a variable node \( r \in \mathcal{V} \) and a bit \( a \in \{0, 1\} \), our goal is to compute the value of a min-weight valid configuration \( \mathcal{W}^{\text{min}}(r, a) \) defined by

\[
\mathcal{W}^{\text{min}}(T_r^{2h}, a) \triangleq \min \left\{ \mathcal{W}_{T_r^{2h}}(z) \middle| z \in \text{vconfig}(T_r^{2h}), z(r) = a \right\}.
\]

In the following lemma we show that \( \text{NWMS2} \) computes \( \mathcal{W}^{\text{min}}(T_r^{2h}, a) \) for every \( r \in \mathcal{V} \) and \( a \in \{0, 1\} \). The proof is based on interpreting \( \text{NWMS2} \) as dynamic programming. See Appendix C for details.

**Lemma 18.** Consider an execution of \( \text{NWMS2}(\lambda(0), \lambda(1), h, w) \). For every variable node \( r \), \( \mu_r(a) = \mathcal{W}^{\text{min}}(T_r^{2h}, a) \).

From Line 12 in Algorithm \( \text{NWMS2} \) we obtain the following corollary that characterizes \( \text{NWMS2} \) as a computation of min-weight configurations.

**Corollary 19.** Let \( \hat{x} \) denote the output of \( \text{NWMS2}(\lambda(0), \lambda(1), h, w) \). For every variable node \( r \),

\[
\hat{x}_r = \begin{cases} 
0 & \text{if } \mathcal{W}^{\text{min}}(T_r^{2h}, 1) > \mathcal{W}^{\text{min}}(T_r^{2h}, 0), \\
1 & \text{otherwise}.
\end{cases}
\]

Define the \( \mathcal{W}^* \) cost of a configuration \( z \) in \( T_r^{2h} \) to be

\[
\mathcal{W}_{T_r^{2h}}^*(z) \triangleq \sum_{p \in \mathcal{V}} \lambda_{t(p)} \cdot \mathcal{W}_{T_r^{2h}}(p) \cdot z_p.
\]

Note that \( \mathcal{W}_{T_r^{2h}}^*(z) \) uses the LLR vector \( \lambda \) (i.e., \( \lambda_r = \lambda_v(1) - \lambda_v(0) \)).

**Corollary 20.** Let \( \hat{x} \) denote the output of \( \text{NWMS}(\lambda, h, w) \). Let \( z^* \) denote a valid configuration in \( T_r^{2h} \) with minimum \( \mathcal{W}^* \) cost. Then, \( \hat{x}_r = z^*_r \).

**Proof.** The derivation in Equation (11) shows that the valid configuration \( z^* \) that minimizes the \( \mathcal{W}^* \) cost also minimizes the \( \mathcal{W} \) cost.

\[
\arg \min_{z \in \text{vconfig}(T_r^{2h})} \mathcal{W}_{T_r^{2h}}(z) \overset{(a)}{=} \arg \min_{z \in \text{vconfig}(T_r^{2h})} \left\{ \mathcal{W}_{T_r^{2h}}(z) - \mathcal{W}_{T_r^{2h}}(0^{|\mathcal{V}|}) \right\} = \overset{(b)}{=} \arg \min_{z \in \text{vconfig}(T_r^{2h})} \left\{ \sum_{\{p \in \mathcal{V} \mid z_p = 1\}} \lambda_{t(p)}(1) \cdot \mathcal{W}_{T_r^{2h}}(p) - \sum_{\{p \in \mathcal{V} \mid z_p = 1\}} \lambda_{t(p)}(0) \cdot \mathcal{W}_{T_r^{2h}}(p) \right\} = \overset{(c)}{=} \arg \min_{z \in \text{vconfig}(T_r^{2h})} \sum_{p \in \mathcal{V}} \lambda_{t(p)} \cdot \mathcal{W}_{T_r^{2h}}(p) \cdot z_p = \arg \min_{z \in \text{vconfig}(T_r^{2h})} \mathcal{W}_{T_r^{2h}}^*(z).
\]

Equality (a) relies on the fact that \( \mathcal{W}_{T_r^{2h}}(0^{|\mathcal{V}|}) \) is a constant. The summands \( \lambda_{t(p)}(z_p) \cdot \mathcal{W}_{T_r^{2h}}(p) \) in \( \mathcal{W}_{T_r^{2h}}(z) \) with \( z_p = 0 \) are reduced by the substraction of the same summands in \( \mathcal{W}_{T_r^{2h}}(0^{|\mathcal{V}|}) \). This leaves in Equality (b) only summands that correspond to bits \( z_p = 1 \). Equality (c) is obtained by the LLR definition \( \lambda_{t(p)} = \lambda_{t(p)}(1) - \lambda_{t(p)}(0) \).

Let \( \hat{x} = \text{NWMS}(\lambda, h, w) \) and \( \hat{y} = \text{NWMS2}(\lambda(0), \lambda(1), h, w) \). By Corollary 19 and Equation (11), \( \hat{y}_r = z^*_r \). By Claim 17, \( \hat{x}_r = \hat{y}_r \), and the corollary follows. \( \square \)
6.2.3 Connections to Local Optimality

The following lemma states that the NWMS decoding algorithm computes the all-zero codeword if $0^N$ is locally optimal.

**Lemma 21.** Let $\hat{x}$ denote the output of NWMS($\lambda, h, w$). If $0^N$ is $(h, w, d = 2)$-locally optimal w.r.t. $\lambda$, then $\hat{x} = 0^N$.

**Proof.** We prove the contrapositive statement. Assume that $\hat{x} \neq 0^N$. Hence, there exists a variable node $v$ for which $\hat{x}_v = 1$. Consider $T_v^{2h} = (\hat{V} \cup \hat{J}, \hat{E})$. Then, by Corollary 20 there exists a valid configuration $z^* \in \{0, 1\}^{\vert V\vert}$ in $T_v^{2h}$ with $z^*_v = 1$ such that for every valid configuration $y \in T_v^{2h}$ it holds that

$$W_{T_v^{2h}}^*(z^*) \leq W_{T_v^{2h}}^*(y).$$

Let $T(z^*)$ denote the subgraph of $T_v^{2h}$ induced by $\hat{V}(z^*) \cup N(\hat{V}(z^*))$ where $\hat{V}(z^*) = \{ p \in \hat{V} \mid z^*_p = 1 \}$. Note that $T(z^*)$ is a forest. Because $z^*_v = 1$ and $z^*$ is a valid configuration in $T_v^{2h}$, the forest $T(z^*)$ must contain a 2-tree of height $2h$ rooted at the node $v$; denote this tree by $T$. Let $\tau \in \{0, 1\}^{\vert V\vert}$ denote the characteristic vector of the support of $T$, and let $z^0 \in \{0, 1\}^{\vert V\vert}$ denote the characteristic vector of the support of $T(z^*) \setminus \tau$. Then, $z^* = \tau + z^0$, where $z^0$ is also necessarily a valid configuration. By linearity and disjointness of $\tau$ and $z^0$, we have

$$W_{T_v^{2h}}^*(z^*) = W_{T_v^{2h}}^*(\tau + z^0) = W_{T_v^{2h}}^*(\tau) + W_{T_v^{2h}}^*(z^0).$$

Because $z^0$ is a valid configuration, by Equation (12), we have $W_{T_v^{2h}}^*(z^*) \leq W_{T_v^{2h}}^*(z^0)$. By Equation (13), it holds that $W_{T_v^{2h}}^*(\tau) \leq 0$.

Let $w^*_\tau \in R^{\vert V\vert}$ denote the vector whose component indexed by $p \in \hat{V}$ equals $W_{T_v^{2h}}^*(p) \cdot \tau_p$. The vector $w^*_\tau$ is equal to the $w$-weighted 2-tree $w_T$ according to Definition 3. Hence, $\beta = \frac{1}{c} \cdot \pi_{G,T,w} \in B(\mathcal{E}_2(w))$ satisfies $\langle \lambda, \beta \rangle = \frac{1}{c} \cdot W_{T_v^{2h}}^*(\tau) \leq 0$, where $c \geq 1$ is a normalizing constant so that $\frac{1}{c} \cdot \pi_{G,T,w} \in [0,1]^N$ (see Definition 4). We therefore conclude that $0^N$ is not $(h, w, d = 2)$-locally optimal w.r.t. $\lambda$ and the lemma follows.

6.3 Numerical Results for Regular LDPC Codes

We chose a $(3,6)$-regular LDPC code with block length $N = 4896$ for which the girth of the Tanner graph equals 12 [RV00]. We ran up to $h = 400$ iterations of the NWMS decoding algorithm (Algorithm 2) and the ML-CERTIFIED-NWMS decoding algorithm (Algorithm 3) for received words over an AWGN channel. Three choices of level weights $w$ were considered: (1) Unit level weights, $w_\ell \triangleq 1$. This choice reduces local optimality to [ADS09, HET11] (although in these papers $h$ is limited by a quarter of the girth). (2) Geometric level weights $w_\ell \triangleq 3 \cdot (3 - 1)^{\ell-1} = 3 \cdot 2^{\ell-1}$. In this case the NWMS decoding algorithm reduces to the standard min-sum decoding algorithm [Wib96]. (3) Geometric level weights $w_\ell \triangleq 3 \cdot (3 - 1)^{\ell-1} = 3 \cdot (\frac{3}{10})^{\ell-1}$. In this case the NWMS decoding algorithm reduces to normalized BP-based algorithm with $\alpha = 1.25$ [CF02]. The choice of weights in [CF02] was obtained by optimizing density evolution w.r.t. minimum bit error probability.

Figure [II] depicts the word error rate of the NWMS decoding algorithm with respect to these three level weights by solid lines. The word error rate of the ML-CERTIFIED-NWMS decoding
algorithm with respect to these three level weights is depicted by dashed lines, i.e., the dashed lines depict the cases in which the NWMS decoding algorithm failed to return the transmitted codeword certified as locally optimal (and hence ML optimal). The error rate of LP decoding and sum-product decoding algorithm are depicted as well for comparison.

The results show that the choice of unit level weights minimizes the gap between the cases in which the NWMS decoding algorithm fails to decode the transmitted codeword with and without an ML-certificate by local optimality. That is, with unit level weights the main cause for a failure in decoding the transmitted codeword is the lack of a locally optimal codeword. Moreover, there is a tradeoff between maximizing the rate of successful ML-certified decoding by the ML-CERTIFIED-NWMS decoding algorithm, and minimizing the (not necessarily ML-certified) word error rate by the NWMS decoding algorithm. This tradeoff was also observed in [JP11].

Figure 1: Simulations for $(3, 6)$-regular LDPC code of length $N = 4896$ [RV00] over a BI-AWGN channel. Solid lines depict the WER of the NWMS decoding algorithm for three level weights, the sum-product decoding algorithm and LP decoding. Dashed lines depict the word error rate of the ML-CERTIFIED-NWMS decoding algorithm, i.e., the probability that the transmitted codeword is not locally optimal.

7 Bounds on the Error Probability of LP Decoding Using Local Optimality

In this section we analyze the probability that a local optimality certificate for regular Tanner codes exists, and therefore LP decoding succeeds. The analysis is based on the study of a sum-min-sum process that characterizes $d$-trees of a regular Tanner graph. We prove upper
bounds on the error probability of LP decoding of regular Tanner codes in MBIOS channels. The upper bounds on the error probability imply lower bounds on the noise threshold of LP decoding for channels in which the channel parameter increases with noise level (e.g., BSC($\sigma$)). We apply the analysis to a BSC, and compare our results with previous results on expander codes. The analysis presented in this section generalizes the probabilistic analysis of Arora et al. [ADS09] from 2-trees (skinny trees) to $d$-trees for any $d \geq 2$.

In the remainder of this section, we restrict our discussion to $(d_L, d_R)$-regular Tanner codes with minimum local distance $d^*$. Let $d$ denote a parameter such that $2 \leq d \leq d^*$. The upcoming Theorem 22 summarizes the main results presented in this section for a BSC, and generalizes to any MBIOS channel as described in Section 7.3. Concrete bounds are given for a $(2, 16)$-regular Tanner code with code rate at least 0.375 when using [16, 11, 4]-extended Hamming codes as local codes.

**Theorem 22.** Let $G$ denote a $(d_L, d_R)$-regular bipartite graph with girth $g$, and let $C(G) \subset \{0, 1\}^N$ denote a Tanner code based on $G$ with minimum local distance $d^*$. Let $x \in C(G)$ be a codeword. Suppose that $y \in \{0, 1\}^N$ is obtained from $x$ by a BSC with crossover probability $p$. Then,

1. **[finite length bound]** Let $d = d_0$, $p \leq p_0$, $(d_L, d_R) = (2, 16)$, and $d^* = 4$. For the values of $d_0$ and $p_0$ in the rows labeled “finite” in Table 7 it holds that $x$ is the unique optimal solution to the LP decoder with probability at least
   \[
   \Pr \{ x^{LP}(y) = x \} \geq 1 - N \cdot \alpha^{(d-1)|\frac{g}{2}} \tag{14}
   \]
   for some constant $\alpha < 1$.

2. **[asymptotic bound]** Let $d = d_0$, $(d_L, d_R) = (2, 16)$, $d^* = 4$, and $g = \Omega(\log N)$ sufficiently large. For the values of $d_0$ and $p_0$ in the rows labeled “asymptotic” in Table 7 it holds that $x$ is the unique optimal solution to the LP decoder with probability at least $1 - \exp(-N^\delta)$ for some constant $0 < \delta < 1$, provided that $p \leq p_0(d_0)$.

3. Let $d' \triangleq d - 1$, $d'_L \triangleq d_L - 1$, and $d'_R \triangleq d_R - 1$. For any $(d_L, d_R)$ and $2 \leq d \leq d^*$ s.t. $d'_L \cdot d'_R \geq 2$, the codeword $x$ is the unique optimal solution to the LP decoder with probability at least
   \[
   1 - N \cdot \alpha^{(d'_L \cdot d'_R)|\frac{g}{2}} \tag{14}
   \]
   for some constant $\alpha < 1$, provided that
   \[
   \min_{t \geq 0} \left\{ \alpha_1(p, d, d_L, d_R, t) \cdot (\alpha_2(p, d, d_L, d_R, t))^{1/(d'_L \cdot d'_R - 1)} \right\} < 1,
   \]
   where

   \[
   \alpha_1(p, d, d_L, d_R, t) = \sum_{k=0}^{d'_R - 1} \binom{d'_R}{k} p^k (1 - p)^{(d'_R - k)(d'_L - k)} e^{-t(d'_L - 2k)}
   \]

   \[
   + \sum_{k=d'_R}^{d'_L \cdot d'_R} \binom{d'_R}{k} p^k (1 - p)^{(d'_R - k)} e^{t d'_R},
   \]

   \[
   \alpha_2(p, d, d_L, d_R, t) = \binom{d'_R}{d'} ((1 - p)e^{-t} + pe^t)^{d'}.
   \]

\footnote{On the other hand, upper bounds on the error probability imply upper bounds on the channel parameter threshold for channels in which the channel parameter is inverse proportional to the noise level (e.g., BI-AWGN channel described with a channel parameter of signal-to-noise ratio $\frac{E}{N_0}$).}
Table 1: Computed values of $p_0$ for finite $d_0 < d^*$ in Theorem 22 with respect to a BSC.

| finite | | asymptotic | |
|---|---|---|---|
| $d_0$ | $p_0$ | $d_0$ | $p_0$ |
| 3 | 0.0086 | 3 | 0.019 |
| 4 | 0.0218 | 4 | 0.044 |

Values are presented for $(2, 16)$-Tanner code with rate at least 0.375 when using $[16, 11, 4]$-extended Hamming codes as local codes. Values in rows labeled “finite” refer to a finite-length bound: $\forall p \leq p_0$ the probability that the LP decoder succeeds is lower bounded by a function of $d$ and the girth of the Tanner graph (see Equation (14)). Values in rows labeled “asymptotic” refer to an asymptotic bound: For $g = \Omega(\log N)$ sufficiently large, the LP decoder succeeds w.p. at least $1 - \exp(-N^\delta)$ for some constant $0 < \delta < 1$, provided that $p \leq p_0(d_0)$.

**Proof Outline.** Theorem 22 follows from Lemma 25, Lemma 28, Corollary 31, and Corollary 32 as follows. Part 1, that states a finite-length result, follows from Lemma 25 and Corollaries 31 and 32 by taking $s = 0 < h < \frac{1}{2}\text{girth}(G)$ which holds for any Tanner graph $G$. Part 2, that deals with an asymptotic result, follows from Lemma 25 and Corollaries 31 and 32 by fixing $s = 10$ and taking $g = \Omega(\log N)$ sufficiently large such that $s < h = \Theta(\log N) < \frac{1}{2}\text{girth}(G)$. It therefore provides a lower bound on the threshold of LP decoding. Part 3, that states a finite-length result for any $(d_L, d_R)$-regular LDPC code, follows from Lemma 25 and Lemma 28.

We refer the reader to Section 8.2 for a discussion on the results stated in Theorem 22. We now provide more details and prove the lemmas and corollaries used in the proof of Theorem 22.

In order to simplify the probabilistic analysis of algorithms for decoding linear codes over symmetric channels, we apply the assumption that the all-zero codeword is transmitted, i.e., $c = 0^N$. Note that the correctness of the all-zero assumption depends on the employed decoding algorithm. Although this assumption is trivial for ML decoding because of the symmetry of a linear code $C(G)$, it is not immediately clear in the context of LP decoding. Feldman et al. [Fel03, FWK05] noticed that the fundamental polytope $\mathcal{P}(G)$ of Tanner codes with single parity-check local codes is highly symmetric, and proved that for MBIOS channels, the probability that the LP decoder fails to decode the transmitted codeword is independent of the transmitted codeword. The symmetry property of the polytope remains also for the generalized fundamental polytope of Tanner codes based on non-trivial linear local codes. Therefore, one can assume that $c = 0^N$ when analyzing failures of LP decoding to decode the transmitted codeword for linear Tanner codes. The following corollary is the contrapositive statement of Theorem 12 given $c = 0^N$.

**Corollary 23.** For every fixed $h \in \mathbb{N}, w \in \mathbb{R}^h \setminus \{0^h\},$ and $2 \leq d \leq d^*$,

$$\Pr\{\text{LP decoding fails}\} \leq \Pr\{\exists \beta \in B_d^{(w)} \text{ s.t. } \langle \lambda, \beta \rangle \leq 0 \mid c = 0^N\}.$$
7.1 Bounding Processes on Trees

Let $G$ be a $(d_L, d_R)$-regular Tanner graph, and fix $h < \frac{1}{2} \text{girth}(G)$. Let $\mathcal{T}_{v_0}^{2h}(G)$ denote the path-prefix tree rooted at a variable node $v_0$ with height $2h$. Since $h < \frac{1}{2} \text{girth}(G)$, it follows that the projection of $\mathcal{T}_{v_0}^{2h}(G)$ to $G$ is a tree. Usually one regards a path-prefix tree as an out-branching, however, for our analysis it is more convenient to view the path-prefix tree as an in-branching. Namely, we direct the edges of $\mathcal{T}_{v_0}^{2h}$ so that each path in $\mathcal{T}_{v_0}^{2h}$ is directed toward the root $v_0$. For $l \in \{0, \ldots, 2h\}$, denote by $V_l$ the set of vertices of $\mathcal{T}_{v_0}^{2h}$ at height $l$ (the leaves have height $0$ and the root has height $2h$). Let $\tau \subseteq V(\mathcal{T}_{v_0}^{2h})$ denote the vertex set of a $d$-tree rooted at $v_0$.

**Definition 24** $(\lambda, \omega, d)$-Process on a $(d_L, d_R)$-Tree. Let $\omega \in \mathbb{R}_{+}^h$ denote a weight vector. Let $\lambda$ denote an assignment of real values to the variable nodes of $\mathcal{T}_{v_0}$. We define the $\omega$-weighted value of a $d$-tree $\tau$ by

$$\text{val}_\omega(\tau; \lambda) \triangleq \sum_{l=0}^{h-1} \sum_{v \in \tau \cap V_{2l}} \omega_l \cdot \lambda_v.$$ 

Namely, the sum of the values of variable nodes in $\tau$ weighted according to their height.

Given a probability distribution over assignments $\lambda$, we are interested in the probability

$$\Pi_{\lambda, d_L, d_R}(\lambda, \omega) \triangleq \Pr \left\{ \min_{\tau \subseteq V(\mathcal{T}_{v_0}^{2h,d})} \text{val}_\omega(\tau; \lambda) \leq 0 \right\}.$$ 

In other words, $\Pi_{\lambda, d_L, d_R}(\lambda, \omega)$ is the probability that the minimum value over all $d$-trees of height $2h$ rooted in some variable node $v_0$ in a $(d_L, d_R)$-bipartite graph $G$ is non-positive. For every two roots $v_0$ and $v_1$, the trees $\mathcal{T}_{v_0}^{2h}$ and $\mathcal{T}_{v_1}^{2h}$ are isomorphic, hence $\Pi_{\lambda, d_L, d_R}(\lambda, \omega)$ does not depend on the root $v_0$.

With this notation, the following lemma connects between the $(\lambda, \omega, d)$-process on $(d_L, d_R)$-trees and the event where the all-zero codeword is $(\lambda, \omega, d)$-locally optimal. We apply a union bound utilizing Corollary 23, as follows.

**Lemma 25.** Let $G$ be a $(d_L, d_R)$-regular bipartite graph and $\omega \in \mathbb{R}_{+}^h \setminus \{0^h\}$ be a weight vector with $h < \frac{1}{2} \text{girth}(G)$. Assume that the all-zero codeword is transmitted, and let $\lambda \in \mathbb{R}^N$ denote the LLR vector received from the channel. Then, $0^N$ is $(\lambda, \omega, d)$-locally optimal w.r.t. $\lambda$ with probability at least

$$1 - N \cdot \Pi_{\lambda, d_L, d_R}(\lambda, \omega),$$

where $\omega_l = w_{h-l} \cdot d_L^{-1} \cdot (d_L - 1)^{l-h+1} \cdot (d - 1)^{h-l}$, and with at least the same probability, $0^N$ is also the unique optimal LP solution given $\lambda$.

Note the two different weight notations that we use for consistency with [ADS09]: (i) $w$ denotes a weight vector in the context of $(\lambda, \omega, d)$-local optimality certificate, and (ii) $\omega$ denotes a weight vector in the context of $d$-trees in the $(\lambda, \omega, d)$-process. A one-to-one correspondence between these two vectors is given by $\omega_l = w_{h-l} \cdot d_L^{-1} \cdot (d_L - 1)^{l-h+1} \cdot (d - 1)^{h-l}$ for $0 \leq l < h$. From this point on, we will use only $\omega$ in this section.

Following Lemma 25, it is sufficient to estimate the probability $\Pi_{\lambda, d_L, d_R}(\lambda, \omega)$ for a given weight vector $\omega$, a distribution of a random vector $\lambda$, constant $2 \leq d \leq d^*_\omega$, and degrees $(d_L, d_R)$. Arora et al. [ADS09] introduced a recursion for estimating and bounding the probability of the existence of a 2-tree (skinny tree) with non-positive value in a $(\lambda, \omega, 2)$-process. We generalize the recursion and its analysis to $d$-trees with $2 \leq d \leq d^*$.  

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For a set $S$ of real values, let $\min[i] \{ S \}$ denote the $i$th smallest member in $S$. Let $\{ \gamma \}$ denote an ensemble of i.i.d. random variables. Define random variables $X_0, \ldots, X_{h-1}$ and $Y_0, \ldots, Y_{h-1}$ with the following recursion:

\begin{align}
Y_0 &= \omega_0 \gamma \\
X_l &= \sum_{i=1}^{d-1} \min[i] \{ Y_i^{(1)}, \ldots, Y_i^{(d_{R} - 1)} \} \\
Y_l &= \omega_l \gamma + X_{l-1}^{(1)} + \cdots + X_{l-1}^{(d_{L} - 1)} \\
\end{align}

The notation $X^{(1)}, \ldots, X^{(k)}$ and $Y^{(1)}, \ldots, Y^{(k)}$ denotes $k$ mutually independent copies of the random variables $X$ and $Y$, respectively. Each instance of $Y_l, 0 \leq l < h$, uses an independent instance of a random variable $\gamma$. Note that for every $0 \leq l < h$, the $d - 1$ order statistic random variables $\{ \min[i] \{ Y_1^{(1)}, \ldots, Y_1^{(d_{R} - 1)} \} \} = 1 \leq i \leq d - 1$ in Equation (16) are dependent.

Consider a directed tree $T = T_{v_0}$ of height $2h$, rooted at node $v_0$. Associate variable nodes of $T$ at height $2l$ with copies of $Y_l$, and check nodes at height $2l + 1$ with copies of $X_l$, for $0 \leq l < h$. Note that any realization of the random variables $\{ \gamma \}$ to variable nodes in $T$ can be viewed as an assignment $\lambda$. Thus, the minimum value of a $d$-tree of $T$ equals $\sum_{i=1}^{d_{R}} X_{i}^{(i)}$. This implies that the recursion in (15)–(17) defines a dynamic programming algorithm for computing $\min_{\tau \in T_{[v_0, 2h, d]}} \text{val}_{\omega}(\tau; \lambda)$. Now, let the components of the LLR vector $\lambda$ be i.i.d. random variables distributed identically to $\{ \gamma \}$, then

$$\Pi_{\lambda, d, d_{L}, d_{R}}(h, \omega) = \Pr \left\{ \sum_{i=1}^{d_{L}} X_{h-1}^{(i)} \leq 0 \right\}. \quad (18)$$

Given a distribution of $\{ \gamma \}$ and a finite “height” $h$, the challenge is to compute the distribution of $X_l$ and $Y_l$ according to the recursion in (15)–(17). The following two lemmas play a major role in proving bounds on $\Pi_{\lambda, d, d_{L}, d_{R}}(h, \omega)$.

**Lemma 26** ([ADS09]). For every $t \geq 0$,

$$\Pi_{\lambda, d, d_{L}, d_{R}}(h, \omega) \leq \left( \mathbb{E} e^{-tX_{h-1}} \right)^{d_{L}}.$$

Let $d' \triangleq d - 1$, $d'_L \triangleq d_{L} - 1$ and $d'_R \triangleq d_{R} - 1$.

**Lemma 27** (following [ADS09]). For $0 \leq s < l < h$, we have

$$\mathbb{E} e^{-tX_{l}} \leq \left( \mathbb{E} e^{-tX_{s}} \right)^{(d'_{L} \cdot d')^{l-s}} \cdot \prod_{k=0}^{l-s-1} \left( \frac{(d'_{R})}{d'} \right) \left( \mathbb{E} e^{-t\omega_{k}\gamma} \right)^{d_{R}} \left( d'_{L} \cdot d'_{R} \right)^{k}.$$

**Proof.** See Appendix [D]. \qed

In the following subsection we present concrete bounds on $\Pi_{\lambda, d, d_{L}, d_{R}}(h, \omega)$ for a BSC. The bounds are based on Lemmas 26 and 27. The technique used to derive concrete bounds for a BSC may be applied to other MBIOS channels. For example, concrete bounds for a BI-AWGN channel can be derived by a generalization of the analysis presented in [HE11].

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7.2 Analysis for a Binary Symmetric Channel

Consider a binary symmetric channel with crossover probability \( p \) denoted by BSC(\( p \)). In the case that the all-zero codeword is transmitted, the channel input is

\[
\text{scaled vector} = \text{invariant under positive scaling of the vector}.
\]

Therefore, let \( \Pi_{\lambda,d,L,R}(h,\omega) \) is invariant under positive scaling of the vector \( \lambda \), we consider in the following analysis the scaled vector \( \lambda \) in which \( \lambda_i = +1 \) w.p. \( p \), and \(-1 \) w.p. \( (1-p) \).

Following the analysis of Arora et al. [ADS09], we apply a simple analysis in the case of uniform weight vector \( \omega \). Then, we present improved bounds by using a non-uniform weight vector.

7.2.1 Uniform Weights

Consider the case where \( \omega = 1^h \). Let \( \alpha_1 \triangleq \mathbb{E} e^{-tX_0} \) and \( \alpha_2 \triangleq \left( \frac{d_L}{d_R} \right)(\mathbb{E} e^{-t\gamma})^{d'} \) where \( \gamma \sim \lambda_i \), and define \( \alpha \triangleq \min_{\lambda \geq 0} \alpha_1 \cdot \alpha_2^{1/(d_L \cdot d'-1)} \). Note that \( \alpha_1 \leq \alpha_2 \) (see Equation (35) in Appendix D).

We consider the case where \( \alpha < 1 \). By substituting notations of \( \alpha_1 \) and \( \alpha_2 \) in Lemma 27 for \( s = 0 \), we have

\[
\mathbb{E} e^{-tX_1} \leq \left( \mathbb{E} e^{-tX_0} \right)^{(d_L \cdot d')^l} \cdot \prod_{k=0}^{l-1} \left( \left( \frac{d_L}{d_R} \right) \left( \mathbb{E} e^{-t\gamma} \right)^{d'} \right)^{(d_L \cdot d')^k}
\]

\[
= \alpha_1^{(d_L \cdot d')^l} \cdot \prod_{k=0}^{l-1} \alpha_2^{(d_L \cdot d')^k}
\]

\[
= \alpha_1^{(d_L \cdot d')^l} \cdot \alpha_2^{\sum_{k=0}^{l-1} (d_L \cdot d')^k}
\]

\[
= \alpha_1^{(d_L \cdot d')^l} \cdot \alpha_2^{\frac{(d_L \cdot d')^l - 1}{d_L \cdot d' - 1}}
\]

\[
= \left( \alpha_1 \cdot \alpha_2 \right)^{\frac{1}{d_L \cdot d' - 1}} \cdot \alpha_2^{\frac{1}{d_L \cdot d' - 1}}
\]

\[
\leq \alpha_2^{(d_L \cdot d')^l - 1}.
\]

By Lemma 26, we conclude that

\[
\Pi_{\lambda,d,L,R}(h, 1^h) \leq \alpha_2^{d_L \cdot (d_L \cdot d')^h - 1 - d_L}.
\]

To analyze parameters for which \( \Pi_{\lambda,d,L,R}(h, 1^h) \to 0 \), we need to compute \( \alpha_1 \) and \( \alpha_2 \) as functions of \( p, d, d_L, \) and \( d_R \). Note that

\[
X_0 = \begin{cases} 
   d' - 2k & \text{w.p. } \binom{d_R}{k} p^k (1-p)^{d_R-k}, \quad 0 \leq k < d', \\
   -d' & \text{w.p. } \sum_{k=d'}^{d_R} \binom{d_R}{k} p^k (1-p)^{d_R-k}.
\end{cases}
\]

Therefore,

\[
\alpha_1(p, d, d_L, d_R, t) = \sum_{k=0}^{d'-1} \binom{d_R}{k} p^k (1-p)^{(d_R-k)} e^{-(d'-2k)} + \left( \sum_{k=d'}^{d_R} \binom{d_R}{k} p^k (1-p)^{d_R-k} \right) e^{td'},
\]

(20)
and
\[ \alpha_2(p, \lambda, d_L, d_R, t) = \left( \frac{d_R}{d'} \right) ((1 - p)e^{-t} + pe^t)^d'. \] (21)

The above calculations give the following bound on \( \Pi_{\lambda, d_L, d_R}(h, 1^h) \).

**Lemma 28.** Let \( p \in (0, \frac{1}{2}) \) and let \( d, d_L, d_R \geq 2 \) s.t. \( d', \lambda' \geq 2 \). Denote by \( \alpha_1 \) and \( \alpha_2 \) the functions defined in (20)–(21), and let
\[ \alpha = \min_{t \geq 0} \left\{ \alpha_1(p, d, d_L, d_R, t) \cdot \left( \alpha_2(p, d, d_L, d_R, t) \right)^{1/(d'_L, d') - 1} \right\}. \]

Then, for \( h \in \mathbb{N} \) and \( \omega = 1^h \), we have
\[ \Pi_{\lambda, d_L, d_R}(h, \omega) \leq \alpha^{d_L, d_R/2 - 1 - d_L}. \]

Note that if \( \alpha < 1 \), then \( \Pi_{\lambda, d_L, d_R}(h, 1^h) \) decreases doubly exponentially as a function of \( h \).

For \((2, 16)\)-regular graphs and \( d \in \{3, 4\} \), we obtain the following corollary.

**Corollary 29.** Let \( d = 2 \) and \( d_R = 16 \).

1. Let \( d = 3 \) and \( p \leq 0.0067 \). Then, there exists a constant \( \alpha < 1 \) such that for every \( h \in \mathbb{N} \) and \( w = 1^h \),
\[ \Pi_{\lambda, d, d_L, d_R}(h, 1^h) \leq \alpha^{2^h - 1}. \]

2. Let \( d = 4 \) and \( p \leq 0.0165 \). Then, there exists a constant \( \alpha < 1 \) such that for every \( h \in \mathbb{N} \) and \( w = 1^h \),
\[ \Pi_{\lambda, d, d_L, d_R}(h, 1^h) \leq \alpha^{3^h - 1}. \]

The bound on \( p \) for which Corollary 29 applies grows with \( d \). This fact confirms that analysis based on denser trees, i.e., \( d \)-trees with \( d > 2 \) instead of skinny trees, implies better bounds on the error probability and higher lower bounds on the threshold. Also, for \( d > 2 \), we may apply the analysis to \((2, d_R)\)-regular codes; a case that is not applicable by the analysis of Arora et al. [ADS09].

### 7.2.2 Improved Bounds Using Non-Uniform Weights

The following lemma implies an improved bound for \( \Pi_{\lambda, d_L, d_R}(h, \omega) \) using a non-uniform weight vector \( \omega \).

**Lemma 30.** Let \( p \in (0, \frac{1}{2}) \) and let \( d, d_L, d_R \geq 2 \) s.t. \( d'_L, d'_R \geq 2 \). For \( s \in \mathbb{N} \) and a weight vector \( \overline{\omega} \in \mathbb{R}_+^s \), let
\[ \alpha = \min_{t \geq 0} \left\{ \frac{\text{E}e^{-tX_s}}{\left( d'_R \right)^{2\sqrt{p(1 - p)}} d'} \right\}^{1/(d'_L, d') - 1}. \] (22)

Let \( \omega^{(s)} \in \mathbb{R}_+^h \) denote the concatenation of the vector \( \overline{\omega} \in \mathbb{R}_+^s \) and the vector \((\rho, \ldots, \rho) \in \mathbb{R}_+^h\). Then, for every \( h > s \) there exists a constant \( \rho > 0 \) such that
\[ \Pi_{\lambda, d_L, d_R}(h, \omega^{(s)}) \leq \left( \frac{d_R}{d'} \right)^{2\sqrt{p(1 - p)}} d' \cdot \alpha^{d_L, d'_L, d'_R h - s - 1}. \]
Table 2: Computed values of $p_0$ for finite $s$ in Corollary 31 for a BSC. Values are presented for $(d_L, d_R) = (2, 16)$ and $d = 3$.

| $s$ | $p_0$  |
|-----|--------|
| 0   | 0.0086 |
| 1   | 0.011  |
| 2   | 0.0139 |
| 3   | 0.0154 |
| 4   | 0.0164 |
| 5   | 0.0171 |
| 6   | 0.0177 |
| 10  | 0.0192 |

**Proof.** See Appendix E.

Consider a weight vector $\omega$ with components $\omega_l = ((d_L - 1)(d - 1))^l$. This weight vector has the effect that every level in a skinny tree $\tau$ contributes equally to $\text{val}_{\omega}(\tau; |\lambda|)$ (note that $|\lambda| \equiv 1$). For $h > s$, consider a weight vector $\omega^{(\rho)} \in \mathbb{R}^h_+$ defined by

$$
\omega^{(\rho)}_l = \begin{cases} 
\omega_l & \text{if } 0 \leq l < s, \\
\rho & \text{if } s \leq l < h.
\end{cases}
$$

Note that the first $s$ components of $\omega^{(\rho)}$ are geometric while the other components are uniform.

For a given $p$, $d$, $d_L$, and $d_R$, and for a concrete value $s$ we can compute the distribution of $X_s$ using the recursion in (15)–(17). Moreover, we can also compute the value $\min_{t \geq 0} E e^{-tX_s}$. For $(2, 16)$-regular graphs we obtain the following corollaries. Corollary 31 is stated for the case where $d = 3$, and Corollary 32 is stated for the case where $d = 4$.

**Corollary 31.** Let $p \leq p_0$, $d = 3$, $d_L = 2$, and $d_R = 16$. For the following values of $p_0$ and $s$ shown in Table 2 it holds that there exists constants $\rho > 0$ and $\alpha < 1$ such that for every $h > s$,

$$
\Pi_{\lambda, d, d_L, d_R}(h, \omega^{(\rho)}) \leq \frac{1}{420} (p(1 - p))^{-1} \cdot \alpha^{2h-s}.
$$

**Corollary 32.** Let $p \leq p_0$, $d = 4$, $d_L = 2$, and $d_R = 16$. For the following values of $p_0$ and $s$ shown in Table 2 it holds that there exists constants $\rho > 0$ and $\alpha < 1$ such that for every $h > s$,

$$
\Pi_{\lambda, d, d_L, d_R}(h, \omega^{(\rho)}) \leq \frac{1}{60} (p(1 - p))^{-\frac{2}{3}} \cdot \alpha^{3h-s}.
$$

Note that for a fixed $s$, the probability $\Pi_{\lambda, d, d_L, d_R}(h, \omega)$ decreases doubly exponentially as a function of $h$.

### 7.3 Analysis for MBIOS Channels

Theorem 22 generalizes to MBIOS channels as follows.
Table 3: Computed values of \( p_0 \) for finite \( s \) in Corollary \( \text{[32]} \) for a BSC. Values are presented for \((d_L, d_R) = (2, 16)\) and \( d = 4 \).

| \( s \) | \( p_0 \) |
|---|---|
| 0 | 0.0218 |
| 1 | 0.0305 |
| 2 | 0.0351 |
| 3 | 0.0375 |

**Theorem 33.** Let \( G \) denote a \((d_L, d_R)\)-regular bipartite graph with girth \( \Omega(\log N) \), and let \( \mathcal{C}(G) \subset \{0, 1\}^N \) denote a Tanner code based on \( G \) with minimum local distance \( d^* \). Consider an MBIOS channel, and let \( \lambda \in \mathbb{R}^N \) denote the LLR vector received from the channel given \( c = 0^N \). Let \( \gamma \in \mathbb{R} \) denote a random variable independent and identically distributed to components of \( \lambda \). Then, for any \((d_L, d_R)\) and \( 2 \leq d \leq d^* \) s.t. \((d_L - 1)(d - 1) \geq 2 \), LP decoding succeeds with probability at least \( 1 - \exp(-N^\delta) \) for some constant \( 0 < \delta < 1 \), provided that

\[
\min_{t \geq 0} \left\{ \mathbb{E} e^{-tX_0} \cdot \left( \frac{(d_R - 1)}{(d - 1)} \left( \mathbb{E} e^{-t\gamma} \right)^{(d-1)} \right)^{(d_L-1)(d-1)-1} \right\} < 1.
\]

where \( X_0 = \sum_{i=1}^{d-1} \min[i] \{ \gamma^{(1)}, \ldots, \gamma^{(d-1)} \} \) and the random variables \( \gamma^{(i)} \) are independent and distributed identically to \( \gamma \).

8 Conclusions and Discussion

We have presented a new combinatorial characterization of local optimality for irregular Tanner codes w.r.t. any MBIOS channel. This characterization provides an ML certificate and an LP certificate for a given codeword. Moreover, the certificate can be efficiently computed by a dynamic programming algorithm. Two applications of local optimality are presented based on this new characterization. (i) A new message-passing decoding algorithm for irregular LDPC codes, called NWMS. The NWMS decoding algorithm is guaranteed to find the locally-optimal codeword if one exists. (ii) Bounds for LP decoding failure to decode the transmitted codeword are proved in the case of regular Tanner codes. We discuss these two applications of local optimality in the following subsections.

8.1 Applying NWMS Decoding Algorithm to Regular LDPC Codes

The NWMS decoding algorithm is a generalization of the min-sum decoding algorithm (a.k.a. max-product algorithm in the probability-domain) and other BP-based decoding algorithms in the following sense. When restricted to regular Tanner graphs and exponential level weights (to cancel the normalization in the variable node degrees), the NWMS decoding algorithm reduces to the standard min-sum decoding algorithm \([\text{WLK95, Wib96}]\). Reductions of the NWMS decoding algorithm to other BP-based decoding algorithms (see, e.g., attenuated max-product \([\text{FK00}]\) and normalized BP-based \([\text{CF02, CDE+05}]\) can be obtained by other weight level functions.
Many works on the BP-based decoding algorithms study the convergence of message passing algorithms to an optimum solution on various settings (e.g., [WF01, WJW05, RU01, JP11]). However, bounds on the running time required to decode have not been proven for these algorithms. The analyses of convergence in these works often rely on the existence of a single optimal solution in addition to other conditions such as: a single cycle, large girth, large reweighing coefficient, consistency conditions, etc. On the other hand, the NWMS decoding algorithm is guaranteed to compute the ML codeword within $h$ iterations if a locally optimal certificate with height parameter $h$ exists for some codeword. Moreover, the certificate can be computed efficiently (see Algorithm 1).

In previous works [ADS09, HE11], the probability that a locally optimal certificate with height parameter $h$ exists for some codeword was investigated for regular LDPC codes with $h < \frac{1}{4}\text{girth}(G)$. Consider a $(d_L, d_R)$-regular LDPC code whose Tanner graph $G$ has logarithmic girth, let $h < \frac{1}{4}\text{girth}(G)$ and define a constant weight vector $w \triangleq \frac{1}{h}$. In that case, the message normalization by variable node degrees has the effect that each level of variable nodes in a 2-tree contributes equally to the cost of the $w$-weighted value of the 2-tree. Hence, the set $B_2^{(w)}$ of PNW deviations is equal to the set of $(d_L - 1)$-exponentially weighted skinny trees [ADS09, HE11]. Following Equation (10), we conclude that the previous bounds on the probability that a locally optimal certificate exists [ADS09, HE11] apply also to the probability that the NWMS and ML-CERTIFIED-NWMS decoding algorithms successfully decode the transmitted codeword.

Consider $(3, 6)$-regular LDPC codes whose Tanner graphs $G$ have logarithmic girth, and let $h = \frac{1}{4}\text{girth}(G)$ and $w = \frac{1}{h}$. Then, NWMS($\lambda, h, w$) and ML-CERTIFIED-NWMS($\lambda, h, w$) succeed in recovering the transmitted codeword with probability at least $1 - \exp(-N^\delta)$ for some constant $0 < \delta < 1$ in the following cases:

1. In a BSC with crossover probability $p < 0.05$ (implied by [ADS09, Theorem 5]).
2. In a BI-AWGN channel with $\frac{E_b}{N_0} \geq 2.67\text{dB}$ (implied by [HE11, Theorem 1]).

It remains to explore good weighting schemes (choice of vectors $w$) for specific families of irregular LDPC codes, and prove that a locally optimal codeword exists with high probability provided that the noise is bounded. Such a result would imply that the NWMS decoding algorithm is a good, efficient replacement for LP decoding.

8.2 Bounds on the Word Error Probability for LP Decoding of Tanner Codes

In Section 7, we proved bounds on the word error probability of LP decoding of regular Tanner codes. In particular, we considered a concrete example of $(2, 16)$-regular Tanner codes with [16, 11, 4]-Hamming codes as local codes and Tanner graphs with logarithmic girth. The rate of such codes is at least 0.375. For the case of a BSC with crossover probability $p$, we prove a lower bound of $p^* = 0.044$ on the noise threshold. Below that threshold the word error probability decreases doubly exponential in the girth of the Tanner graph.

Most of the research on the error correction of Tanner codes deals with families of expander Tanner codes. How do the bounds presented in Section 7 compare with results on expander Tanner codes? The error correction capability of expander codes depends on the expansion, thus a fairly large degree and huge block lengths are required to achieve good error correction. Our example for which results are stated in Theorem 22(1) and 22(2) relies only on a 16-regular graph with logarithmic girth. Sipser and Spielman [SS96] studied Tanner codes based on expander
graphs and analyzed a simple bit-flipping iterative decoding algorithm. Their novel scheme was later improved, and it was shown that expander Tanner codes can even asymptotically achieve capacity in a BSC with an iterative decoding bit-flipping scheme \cite{Zem01, BZ02, BZ04}. In these works, a worst-case analysis (for an adversarial channel) was performed as well.

The best result for iterative decoding of such expander codes, reported by Skachek and Roth \cite{SR03}, implies a lower bound of $p^* = 0.0016$ on the threshold of a certain iterative decoder for rate $0.375$ codes. Feldman and Stein \cite{FS05} proved that LP decoding can asymptotically achieve capacity with a special family of expander Tanner codes. They also presented a worst-case analysis, which in the case of a code rate of $0.375$, proves that LP decoding can recover any pattern of at most $0.0008N$ bit flips. This implies a lower bound of $p^* = 0.0008$ on the noise threshold. These analyses yield overly pessimistic predictions for the average case (i.e., a BSC). Theorem 22 deals with average case analysis and implies that LP decoding can correct up to $0.044N$ bit flips with high probability. Furthermore, previous iterative decoding algorithms for expander Tanner codes deal only with bit-flipping channels. Our analysis for LP decoding applies to any MBIOS channel, in particular, it can be applied to the BI-AWGN channel.

However, the lower bounds on the noise threshold proved for Tanner codes do not improve the best previous bounds for regular LDPC codes with the same rate. An open question is whether using deviations denser than skinny trees for Tanner codes can beat the best previous bounds for regular LDPC codes \cite{ADS09, HE11}. In particular, for a concrete family of Tanner codes with rate $\frac{1}{2}$, it would be interesting to prove lower bounds on the threshold of LP decoding that are larger than $p^* = 0.05$ in the case of a BSC, and $\sigma^* = 0.735$ in the case of a BI-AWGN channel (upper bound smaller than $\frac{Eb}{No} = 2.67$dB).

**Acknowledgement**

The authors would like to thank Pascal O. Vontobel for many valuable comments and suggestions that improved the paper. We appreciate the helpful comments that were made by the reviewers.

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A Constructing Codewords from Projection of Weighted Trees – Proof of Lemma 6

In this section we prove Lemma 6, the key structural lemma in the proof of Theorem 7. This lemma states that every codeword of a Tanner code is a finite sum of projections of weighted trees in the computation trees of \( G \).

Throughout this section, let \( C(G) \) denote a Tanner code with minimum local distance \( d^* \), let \( x \) denote a nonzero codeword, let \( h \) denote some positive integer, and let \( w \in \mathbb{R}^h_+ \) denote level weights.

The proof of Lemma 6 is based on Lemmas 34–35 and Corollary 36. Lemma 34 states that every codeword \( x \in C(G) \) can be decomposed into a set of weighted path-prefix trees. The number of trees in the decomposition equals \( \|x\|_1 \). Lemma 35 states that every weighted path-prefix tree is a convex combination of weighted \( d \)-trees. This lemma implies that the projection of a weighted path-prefix tree is equal to the expectation of projections of weighted \( d \)-trees.

For a codeword \( x \in C(G) \subset \{0,1\}^N \), let \( V_x = \{v \in V | x_v = 1\} \). Let \( G_x \) denote the subgraph of the Tanner graph \( G \) induced by \( V_x \cup \mathcal{N}(V_x) \). Note that the degree of every local-code node in \( G_x \) is at least \( d \).

Lemma 34. For every codeword \( x \neq 0^N \), for every weight vector \( w \in \mathbb{R}^h_+ \), and for every variable node \( v \in V \), it holds that

\[
x_v = \sum_{r \in V_x} \pi_{\mathcal{G}(G_x),w}(v).
\]
\[
v = t(p)
\]

Figure 2: Set of all backtrackless paths \( P_\ell(v) \) as augmentation of the set \( P_{\ell-1}(v) \) as viewed by the path-suffix tree of height \( \ell \) rooted at \( v \), in the proof of Lemma 6. Note that if \( \ell \) is odd, then every path that ends at variable node \( v \) starts at a local-code node. If \( \ell \) is even, then every path that ends at variable node \( v \) starts at a variable node.

**Proof.** If \( x_v = 0 \), then \( \pi_{G,T^{2h}(G_x)}(v) = 0 \). It remains to show that equality holds for variable nodes \( v \in V_x \).

Consider an all-one weight vector \( \eta = 1^h \). Construct a path-suffix tree rooted at \( v \). The set of nodes of a path-suffix tree rooted at \( v \) contains paths that end at node \( v \) (in contrast to path-prefix trees where the set of nodes contains paths that start at the root). Level \( \ell \) of this path-suffix tree consists of all backtrackless paths in \( G_x \) of length \( \ell \) that end at node \( v \) (see Figure 2). We denote this level by \( P_\ell(v) \).

We use the same notational convention for \( \eta \) as for \( w \) in Definition 3, i.e., \( \eta_T \) denotes a weight function based on weight vector \( \eta \) for variable paths in \( T \). We claim that for every \( v \in V_x \) and \( 1 \leq \ell \leq 2h \),

\[
\sum_{p \in P_\ell(v)} \eta_{T_{s(p)}}(p) = \frac{1}{h}.
\]  
(23)

The proof is by induction on \( \ell \). The induction basis, for \( \ell = 1 \), holds because \( |P_1(v)| = \deg_{G}(v) \) and \( \eta_{T_{s(p)}}(p) = \frac{1}{||\eta||_1} \cdot \frac{1}{\deg_{G}(v)} = \frac{1}{h} \cdot \frac{1}{\deg_{G}(v)} \) for every \( p \in P_1(v) \). The induction step is proven as follows. For each \( p \in P_\ell(v) \), let \( \text{aug}(p) \triangleq \{ q \in P_{\ell+1}(v) \mid p \text{ is a suffix of } q \} \). Note that \( |\text{aug}(p)| = \deg_{G_x}(s(p)) - 1 \). Moreover, for each \( q \in \text{aug}(p) \),

\[
\frac{\eta_{T_{s(q)}}(q)}{\eta_{T_{s(p)}}(p)} = \frac{1}{\deg_{G_x}(s(p)) - 1}.
\]  
(24)

Hence,

\[
\sum_{q \in \text{aug}(p)} \eta_{T_{s(q)}}(q) = \eta_{T_{s(p)}}(p).
\]

Finally, \( P_{\ell+1}(v) \) is the disjoint union of \( \bigcup_{p \in P_\ell(v)} \text{aug}(p) \). It follows that

\[
\sum_{q \in P_{\ell+1}(v)} \eta_{T_{s(q)}}(q) = \sum_{p \in P_\ell(v)} \eta_{T_{s(p)}}(p).
\]  
(25)
By the induction hypothesis we conclude that \( \sum_{q \in P_{x+1}(v)} \eta_{T_{2h}^q}(q) = 1/h \), as required. Note that the sum of weights induced by \( \eta \) on each level is \( 1/h \), both for levels of paths beginning in variable nodes and in local-code nodes. In the rest of the proof we focus only on even levels that start at variable nodes. We now claim that

\[
\sum_{p \in P_{2h}(v)} w_{T_{2h}^p}(p) = \frac{w}{\|w\|_1}. \tag{26}
\]

Indeed, by Definition \[3\] it holds that \( w_{T_{2h}^p}(p) = \eta_{T_{2h}^p}(p) \cdot \frac{w}{\|w\|_1} \cdot h \) for every \( p \in P_{2h}(v) \). Therefore, Equation (26) follows from Equation (23).

The lemma follows because for every \( v \in \mathcal{V}_x \),

\[
\sum_{r \in \mathcal{V}_x} \pi_{G,T_{2h}^r(G_x),w}(v) = \sum_{\ell=1}^h \sum_{p \in P_{2\ell}(v)} w_{T_{2\ell}^p}(p) = \sum_{\ell=1}^h \frac{w_{\ell}}{\|w\|_1} = 1.
\]

Lemma 35. Consider a subgraph \( G_x \) of a Tanner graph \( G \), where \( x \in C(G) \setminus \{0^N\} \). Then, for every variable node \( r \in G_x \), every positive integer \( h \), every \( 2 \leq d \leq d^* \), and every weight vector \( w \in \mathbb{R}^h_+ \), it holds that

\[
w_{T_{2h}^r(G_x)} = E_{\rho_r}[w_T]
\]

where \( \rho_r \) is the uniform distribution over \( T[r, 2h, d](G_x) \).

Proof. Let \( G_x = (\mathcal{V}_x \cup \mathcal{J}_x, E_x) \) and let \( w_{T_{2h}^r(G_x)} \) denote a \( w \)-weighted path-prefix tree rooted at node \( r \) with height \( 2h \). We claim that the expectation of \( w \)-weighted \( d \)-trees \( w_T \in T[r, 2h, d](G_x) \) equals \( w_{T_{2h}^r(G_x)} \) if \( w_T \) is chosen uniformly at random.

Let \( \rho_r \) denote the uniform distribution over \( T[r, 2h, d](G_x) \). A random \( d \)-tree in \( T[r, 2h, d](G_x) \) can be sampled according to \( \rho_r \), as follows. Start from the root \( r \). For each variable path, take all its augmentations, and for each local-code path choose \( d-1 \) distinct augmentations uniformly at random. Let \( T \in T[r, 2h, d](G_x) \) denote such a random \( d \)-tree, and consider a variable path \( p \in T_{2h}^r(G_x) \). Then,

\[
Pr_{\rho_r}\{p \in T\} = \prod_{\{q \in \text{Prefix}(p) \mid t(q) \in \mathcal{J}_x\}} \frac{d-1}{\deg_{G_x}(t(q))-1}. \tag{27}
\]

Note the following two observations: (i) if \( p \notin T \), then \( w_T(p) = 0 \), and (ii) if \( p \in T \), then the value of \( w_T(p) \) is constant, i.e., \( w_T(p) = w_T(p) \) for all \( T' \) such that \( p \in T' \). Let \( \alpha(p) \) denote this constant, i.e., \( \alpha(p) \triangleq w_T(p) \) for some \( T \in T[r, 2h, d](G_x) \) such that \( p \in T \). From the two observations above we have

\[
E_{\rho_r}[w_T(p)] = \alpha(p) \cdot Pr_{\rho_r}\{p \in T\}. \tag{28}
\]

\(^9\)Note the difference between an augmentation of a variable path in a path-prefix tree and a path-suffix tree. In a path-prefix tree, an augmentation appends a node to the end of the path. In a path-suffix tree, an augmentation adds a node before the beginning of the path.
Note that for a variable path \( p \in \mathcal{T} \), \(|p|\) is even because \( \mathcal{T} \) is rooted at a variable node \( r \). By Definition 3, for a variable path \( p \in \mathcal{T} \) we have

\[
\alpha(p) = w_{\mathcal{T}}(p) = \frac{|p|/2}{\|w\|_1} \cdot \frac{1}{\deg_G(t(p))} \cdot \frac{1}{(d-1)^{|p|/2}} \cdot \prod_{q \in \text{Prefix}^+(p) \mid t(q) \in \mathcal{V}_x} \frac{1}{\deg_G(t(q)) - 1}.
\]  

(29)

By substituting (27) and (29) in (28), we conclude that

\[
\mathbb{E}_{\rho_r}[w_{\mathcal{T}}(p)] = \frac{|p|/2}{\|w\|_1} \cdot \frac{1}{\deg_G(t(p))} \cdot \prod_{q \in \text{Prefix}^+(p)} \frac{1}{\deg_G(t(q)) - 1} = w_{\mathcal{T}^{2^h}(G_x)}(p).
\]

\[\square\]

**Corollary 36.** For every positive integer \( h \), every \( 2 \leq d \leq d^* \), and every weight vector \( w \in \mathbb{R}^h_+ \), it holds that

\[
\pi_{G, \mathcal{T}^{2^h}(G_x), w} = \mathbb{E}_{\rho_r}[\pi_{G, \mathcal{T}, w}]
\]

where \( \rho_r \) is the uniform distribution over \( \mathcal{T}^h[V, 2^h, d](G_x) \).

**Proof.** By definition of \( \pi_{G, \mathcal{T}^{2^h}(G_x), w} \), we have

\[
\pi_{G, \mathcal{T}^{2^h}(G_x), w}(v) = \sum_{\{p \in \mathcal{T}^{2^h}(G_x) \mid t(p) = v\}} w_{\mathcal{T}^{2^h}(G_x)}(p).
\]

(30)

By Lemma 35 and linearity of expectation we have

\[
\sum_{\{p \in \mathcal{T}^{2^h}(G_x) \mid t(p) = v\}} w_{\mathcal{T}^{2^h}(G_x)}(p) = \sum_{\{p \in \mathcal{T}^{2^h}(G_x) \mid t(p) = v\}} \mathbb{E}_{\rho_r}[w_{\mathcal{T}}(p)] = \mathbb{E}_{\rho_r} \left[ \sum_{\{p \in \mathcal{T}^{2^h}(G_x) \mid t(p) = v\}} w_{\mathcal{T}}(p) \right].
\]

(31)

Now, for variable paths \( p \) that are not in a \( d \)-tree \( \mathcal{T} \), \( w_{\mathcal{T}}(p) = 0 \). Hence, if a \( d \)-tree \( \mathcal{T} \) is a subtree of \( \mathcal{T}^{2^h}(G_x) \), then

\[
\sum_{\{p \in \mathcal{T}^{2^h}(G_x) \mid t(p) = v\}} w_{\mathcal{T}}(p) = \sum_{\{p \in \mathcal{T} \mid t(p) = v\}} w_{\mathcal{T}}(p) = \pi_{G, \mathcal{T}, w}(v).
\]

(32)

From Equations (30)–(32) we conclude that for every \( v \in \mathcal{V} \),

\[
\pi_{G, \mathcal{T}^{2^h}(G_x), w}(v) = \mathbb{E}_{\rho_r}[\pi_{G, \mathcal{T}, w}(v)].
\]

\[\square\]

Before proving Lemma 6, we state a lemma from probability theory.
Lemma 37. Let \( \{ \rho_r \}_{r=1}^{K} \) denote \( K \) probability distributions. Let \( \rho \equiv \frac{1}{K} \sum_{r=1}^{K} \rho_r \). Then,

\[
\sum_{r=1}^{K} \mathbb{E}_{\rho_r}[x] = K \cdot \mathbb{E}_{\rho}[x].
\]

Proof of Lemma \( \Box \) By Lemma 34 and Corollary 36, we have for every \( v \in \mathcal{V}_x \)

\[
x_v = \sum_{r \in \mathcal{V}_x} \pi_{G,T} p_h(G_x,w)(v) \\
= \sum_{r \in \mathcal{V}_x} \mathbb{E}_{\rho_r}[\pi_{G,T,w}] .
\]

Let \( \rho \) denote the distribution defined by \( \rho \triangleq \frac{1}{\|x\|_1} \cdot \sum_{r \in \mathcal{V}_x} \rho_r \). By Lemma 37 and Equation (33),

\[
x_v = \|x\|_1 \cdot \mathbb{E}_{\rho}[\pi_{G,T,w}],
\]

and the lemma follows. \( \Box \)

### B Symmetry of NWMS – Proof of Lemma 15

Proof. Let \( \mu_{v \rightarrow C}^{(l)}[\lambda] \) denote the message sent from \( v \) to \( C \) in iteration \( l \) given an input \( \lambda \). Let \( \mu_{C \rightarrow v}^{(l)}[\lambda] \) denote the corresponding message from \( C \) to \( v \). From the decision of NWMS in Line 12, it’s sufficient to prove that \( \mu_{v \rightarrow C}^{(l)}[\lambda] = (-1)^{x_v} \cdot \mu_{C \rightarrow v}^{(l)}[(-1)^x \ast \lambda] \) and \( \mu_{C \rightarrow v}^{(l)}[\lambda] = (-1)^{x_v} \cdot \mu_{C \rightarrow v}^{(l-1)}[(-1)^x \ast \lambda] \) for every \( 0 \leq l \leq h - 1 \).

The proof is by induction on \( l \). The induction basis, for \( l = -1 \), holds because \( \mu_{C \rightarrow v}^{(-1)}[\lambda] = (-1)^{x_v} \cdot \mu_{C \rightarrow v}^{(-1)}[(-1)^x \ast \lambda] = 0 \) for every codeword \( x \).

The induction step is proven as follows. By induction hypothesis we have

\[
\mu_{v \rightarrow C}^{(l)}[\lambda] = \frac{w_{h-l}}{\deg_G(v)} \lambda_v + \frac{1}{\deg_G(v) - 1} \sum_{C \in \mathcal{N}(v) \setminus \{C\}} \mu_{C \rightarrow v}^{(l-1)}[\lambda]
\]

\[
= (-1)^{x_v} \cdot \left( \frac{w_{h-l}}{\deg_G(v)}(-1)^{x_v} \lambda_v + \frac{1}{\deg_G(v) - 1} \sum_{C \in \mathcal{N}(v) \setminus \{C\}} \mu_{C \rightarrow v}^{(l-1)}[(-1)^x \ast \lambda] \right)
\]

\[
= (-1)^{x_v} \cdot \mu_{C \rightarrow v}^{(l)}[(-1)^x \ast \lambda].
\]

For check to variable messages we have by the induction hypothesis,

\[
\mu_{C \rightarrow v}^{(l)}[\lambda] = \left( \prod_{u \in \mathcal{N}(C) \setminus \{v\}} \text{sign}(\mu_{u \rightarrow C}^{(l)}[\lambda]) \right) \cdot \min_{u \in \mathcal{N}(C) \setminus \{v\}} \{ |\mu_{u \rightarrow C}^{(l)}[\lambda]| \}
\]

\[
= \left( \prod_{u \in \mathcal{N}(C) \setminus \{v\}} \text{sign}((-1)^{x_u} \cdot \mu_{u \rightarrow C}^{(l)}[(-1)^x \ast \lambda]) \right)
\]

\[
\cdot \min_{u \in \mathcal{N}(C) \setminus \{v\}} \{ |(-1)^{x_u} \cdot \mu_{u \rightarrow C}^{(l)}[(-1)^x \ast \lambda]| \}
\]

\[
= \left( \prod_{u \in \mathcal{N}(C) \setminus \{v\}} (-1)^{x_u} \right) \cdot \mu_{C \rightarrow v}^{(l)}[(-1)^x \ast \lambda].
\]
Therefore, paths must start with edge \( \text{rooted at node } v \) (see Figure 4). Consider a path-prefix tree \( T \). In such a case, backtrackless path because \( \text{optimal configurations in } T \) of optimal valid subconfigurations and prove invariants for the messages of algorithm computation of optimal valid configurations and subconfigurations. In this appendix we define Figure 3: Substructures of a path-prefix tree \( T_r^{2h}(G) \) in a dynamic programming that computes optimal configurations in \( T_r^{2h}(G) \).

Because \( x \) is codeword, for every single parity-check \( C \) we have \( \prod_{u \in A(C) \setminus \{v\}} (-1)^{x_u} = (-1)^{x_v} \).

Therefore, \( \mu_{C \rightarrow v}^{(l)}[\lambda] = (-1)^{x_v} \cdot \mu_{C \rightarrow v}^{(l)}[(-1)^{\lambda} \ast \lambda] \) and the claim follows.

\[ \square \]

\section{C Optimal Valid Subconfigurations in the Execution of NWMS2}

The description of algorithm NWMS2 as a dynamic programming algorithm deals with the computation of optimal valid configurations and subconfigurations. In this appendix we define optimal valid subconfigurations and prove invariants for the messages of algorithm NWMS2.

Denote by \( T_{C \rightarrow v}^{2l+2} \) a path prefix tree of \( G \) rooted at node \( v \) with height \( 2l + 2 \) such that all paths must start with edge \( (v, C) \) (see Figure 3(a)). Denote by \( T_{v \rightarrow C}^{2l+1} \) a path prefix tree of \( G \) rooted at node \( C \) with height \( 2l + 1 \) such that all paths start with edge \( (C, v) \) (see Figure 3(b)).

Consider the message \( \mu_{C \rightarrow v}^{(2l+2)} \). It is determined by the messages sent along the edges of \( T_{v \rightarrow C}^{2l+1}(G) \) that hang from the edge \( (v, C) \). We introduce the following notation of this subtree (see Figure 4). Consider a path-prefix tree \( T_r^{2h}(G) \) and a variable path \( p \) such that

(i) \( p \) is a path from root \( r \) to a variable node \( v \),
(ii) the last edge in \( p \) is \( (C', v) \) for \( C' \neq C \), and
(iii) the length of \( p \) is \( 2(h - l - 1) \).

In such a case, \( T_{C \rightarrow v}^{2l+2} \) is isomorphic to the subtree of \( T_r^{2h} \) hanging from \( p \) along the edge \( (p, p \circ (v, C)) \). Hence, we say that \( T_{C \rightarrow v}^{2l+2} \) is a substructure of \( T_r^{2h}(G) \). Similarly, if there exists a backtrackless path \( q \) in \( G \) from \( r \) to \( C \) with length \( 2(h - l - 1) \) that does not end with edge \( (v, C) \), we say that \( T_{v \rightarrow C}^{2l+1} \) is a substructure of \( T_r^{2h}(G) \).

Let \( T_{\text{sub}} \) denote a substructure \( T_{C \rightarrow v}^{2l+2} \) or \( T_{v \rightarrow C}^{2l+1} \). A binary assignment \( z \in \{0, 1\}^{|V(T_{\text{sub}})|} \) to variable paths \( \hat{V}(T_{\text{sub}}) \) is a valid subconfiguration if it satisfies every parity-check path \( q \in T_{\text{sub}} \) with \( |q| \geq 1 \). We denote the set of valid subconfigurations of \( T_{\text{sub}} \) by \( v\text{config}(T_{\text{sub}}) \).

Define the weight of a variable path \( q \in \hat{V}(T_{\text{sub}}) \) w.r.t. level weights \( w = (w_1, \ldots, w_h) \in \mathbb{N}^h \).
\[ R^h_+ \text{ by} \]

\[ \mathcal{W}_{\text{sub}}(T_{\text{sub}},q) \triangleq \frac{w_{h-l-1+|[q]/2]}{\deg_G(t(q))} \cdot \prod_{q' \in \text{Prefix}^+(q) \cap \hat{V}(T_{\text{sub}})} \frac{1}{\deg_G(t(q'))-1}. \]

The weight of a valid subconfiguration \( z \) for a substructure \( T_{\text{sub}} \) is defined by

\[ \mathcal{W}_{\text{sub}}(T_{\text{sub}},z) \triangleq \sum_{\{q \in \hat{V}(T_{\text{sub}}) \mid |q| \geq 1\}} \lambda(t(q))(z_q) \cdot \mathcal{W}_{\text{sub}}(T_{\text{sub}},q). \]

Define the minimum weight of substructures \( T_{2l+1}^{2l+1} \) and \( T_{C \rightarrow v}^{2l+2} \) for \( a \in \{0,1\} \) as follows.

\[ \mathcal{W}_{\text{sub}}^{\text{min}}(T_{v \rightarrow C}^{2l+1},a) \triangleq \min \left\{ \mathcal{W}_{\text{sub}}(T_{v \rightarrow C}^{2l+1},z) \mid z \in \text{vconfig}(T_{v \rightarrow C}^{2l+1}), z_{(C,v)} = a \right\}. \]

and

\[ \mathcal{W}_{\text{sub}}^{\text{min}}(T_{C \rightarrow v}^{2l+2},a) \triangleq \min \left\{ \mathcal{W}_{\text{sub}}(T_{C \rightarrow v}^{2l+2},z) \mid z \in \text{vconfig}(T_{C \rightarrow v}^{2l+2}), z_{(v)} = a \right\}. \]

The minimum weight substructures satisfy the following recurrences.

**Proposition 38.** Let \( a \in \{0,1\} \), then

1) for every \( 1 \leq l \leq h - 1 \),

\[ \mathcal{W}_{\text{sub}}^{\text{min}}(T_{v \rightarrow C}^{2l+1},a) = \frac{w_{h-l}}{\deg_G(v)} \cdot \lambda_v(a) + \frac{1}{\deg_G(v) - 1} \cdot \sum_{C' \in \mathcal{N}(v) \setminus \{C\}} \mathcal{W}_{\text{sub}}^{\text{min}}(T_{C' \rightarrow v}^{2(l-1)+2},a). \]
2) For every $0 \leq l \leq h - 1$,
\[
W_{\text{min}}^{\text{sub}}(T_{C \rightarrow v}^{2l+2}, a) = \min \left\{ \sum_{u \in N(C) \setminus \{v\}} W_{\text{sub}}(T_{u \rightarrow C}^{2l+1}, x_u) \bigg| x \in \{0, 1\}^{\deg_G(C)}, \|x\|_1 \text{ even, } x_v = a \right\}.
\]

The following claim states an invariant over the messages $\mu_{v \rightarrow C}^{(l)}(a)$ and $\mu_{C \rightarrow v}^{(l)}(a)$ that holds during the execution of NWMS2.

**Claim 39.** Consider an execution of NWMS2($\lambda(0), \lambda(1), h, w$). Then, for every $0 \leq l \leq h - 1$,
\[
\mu_{v \rightarrow C}^{(l)}(a) = W_{\text{min}}^{\text{sub}}(T_{2l+1}^{2l+2}, v \rightarrow C, a),
\]
and
\[
\mu_{C \rightarrow v}^{(l)}(a) = W_{\text{min}}^{\text{sub}}(T_{2l+2}^{2l+1}, C \rightarrow v, a).
\]

**Proof.** The proof is by induction on $l$. The induction basis, for $l = 0$, holds because $\mu_{v \rightarrow C}^{(0)}(a) = W_{\text{sub}}(T_{v \rightarrow C}^1, v \rightarrow C, a) = w_{\deg_G(v)}(a)$ for every edge $(v, C)$ of $G$. The induction step follows directly from the induction hypothesis and Proposition 38.

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**D Proof of Lemma 27**

**Proof.** We prove the lemma by induction on the difference $l - s$. We first derive an equality for $E e^{-tY_l}$ and a bound for $E e^{-tX_l}$. Since $Y_l$ is the sum of mutually independent variables,
\[
E e^{-tY_l} = (E e^{-t\omega_l}) (E e^{-tX_{l-1}})^{d'_R}. \tag{34}
\]

By definition of $X_l$ we have the following bound,
\[
e^{-tX_l} = e^{-t \sum_{j=1}^{d'} \min_{i \leq d'_R} \{Y_l^{(i)} | 1 \leq i \leq d'_R\}}
= \prod_{j=1}^{d'} e^{-t \min_{i \leq d'_R} \{Y_l^{(i)} | 1 \leq i \leq d'_R\}}
\leq \sum_{S \subseteq [d'_R]} \prod_{i \in S} e^{-tY_l^{(i)}}.
\]

By linearity of expectation and since $\{Y_l^{(i)}\}_{i=1}^{d'_R}$ are mutually independent variables, we have
\[
E e^{-tX_l} \leq \binom{d'_R}{d'} (E e^{-tY_l})^{d'} \tag{35}
\]

By substituting (34) in (35), we get
\[
E e^{-tX_l} \leq \left( E e^{-tX_{l-1}} \right)^{\binom{d'_R}{d'}} \left( E e^{-t\omega_l} \right)^{d'} \tag{36}
\]

which proves the induction basis where $s = l - 1$. Suppose, therefore, that the lemma holds for $l - s = i$, we now prove it for $l - (s + 1) = i + 1$. Then by substituting (36) in the induction
hypothesis, we have

\[ E e^{-tX_t} \leq \left( E e^{-tX_s} \right)^{(d'_L,d')^{l-s}} \prod_{k=0}^{l-s-1} \left( \frac{d'_R}{d'} \right) \left( E e^{-t\omega_{l-k} \gamma} \right)^{d'} \left( d'_L,d' \right)^{k} \]

\[ \leq \left[ \left( E e^{-tX_{s-1}} \right)^{(d'_L,d')^{l-s}} \prod_{k=0}^{l-s-1} \left( \frac{d'_R}{d'} \right) \left( E e^{-t\omega_{l-k} \gamma} \right)^{d'} \left( d'_L,d' \right)^{k} \right] \cdot \prod_{k=0}^{l-s-1} \left( \frac{d'_R}{d'} \right) \left( E e^{-t\omega_{l-k} \gamma} \right)^{d'} \left( d'_L,d' \right)^{k} \]

\[ = \left( E e^{-tX_{s-1}} \right)^{(d'_L,d')^{l-s+1}} \prod_{k=0}^{l-s} \left( \frac{d'_R}{d'} \right) \left( E e^{-t\omega_{l-k} \gamma} \right)^{d'} \left( d'_L,d' \right)^{k} \]

which concludes the correctness of the induction step for a difference of \( l - s + 1 \). \( \square \)

**E Proof of Lemma 30**

**Proof.** By Lemma 27, we have

\[ E e^{-tX_{l-1}} \leq \left( E e^{-tX_s} \right)^{(d'_L,d')^{h-s-1}} \cdot \left( \frac{d'_R}{d'} \right) \left( E e^{-t\rho_n} \right)^{d'} \left( d'_L,d' \right)^{h-s-1} \]

Note that \( E e^{-t\rho_n} \) is minimized for \( e^{t\rho} = \sqrt{p(1-p)} \). Hence,

\[ E e^{-tX_{l-1}} \leq \left( E e^{-tX_s} \right)^{(d'_L,d')^{h-s-1}} \cdot \left( \frac{d'_R}{d'} \right) \left( 2 \sqrt{p(1-p)} \right)^{d'} \left( d'_L,d' \right)^{h-s-1} \]

Let \( \alpha \triangleq \min_{t \geq 0} \left\{ E e^{-tX_s} \left( \frac{d'_R}{d'} \right) \left( 2 \sqrt{p(1-p)} \right)^{d'} \left( d'_L,d' \right)^{-\frac{1}{d'_L,d'} - 1} \right\} \). Let \( t^* = \arg \min_{t \geq 0} E e^{-tX_s} \), then

\[ E e^{-t^*X_{l-1}} \leq \alpha \left( d'_L,d'-1 \right)^{h-s-1} \cdot \left( \frac{d'_R}{d'} \right) \left( 2 \sqrt{p(1-p)} \right)^{d'} \left( d'_L,d' \right)^{-\frac{1}{d'_L,d'} - 1} \]

Using Lemma 26, we conclude that

\[ \Pi_{\lambda,d,\lambda,d} \left( h, \omega^{(\rho)} \right) \leq \alpha \left( d'_L,d'-1 \right)^{h-s-1} \cdot \left( \frac{d'_R}{d'} \right) \left( 2 \sqrt{p(1-p)} \right)^{d'} \left( d'_L,d' \right)^{-\frac{1}{d'_L,d'} - 1} \]

and the lemma follows. \( \square \)