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Abstract. In the present paper, we prove that self-approximation of \( \log \zeta(s) \) with \( d = 0 \) is equivalent to the Riemann Hypothesis. Next, we show self-approximation of \( \log \zeta(s) \) with respect to all nonzero real numbers \( d \). Moreover, we partially filled a gap existing in ‘The strong recurrence for non-zero rational parameters’ and prove self-approximation of \( \zeta(s) \) for \( 0 \neq d = a/b \) with \( |a - b| \neq 1 \) and \( \gcd(a, b) = 1 \).

1. Introduction

In 1981 Bagchi [1] discovered that the following almost periodicity holds in the critical strip if and only if the Riemann Hypothesis is true. To state it, let \( \mu \{ A \} \) stand for the Lebesgue measure of a measurable set \( A \), \( D := \{ s \in \mathbb{C} : 1/2 < \Re(s) < 1 \} \) and \( H(K) \) denote the space of non-vanishing continuous functions on a compact set \( K \), which are analytic in the interior, equipped with the supremum norm \( \| \cdot \|_K \). Then Bagchi’s result can be formulated as follows (see also [6, Section 8]).

Theorem A. The Riemann Hypothesis holds if and only if, for every compact set \( K \subset D \) with connected complement and every \( \varepsilon > 0 \), we have

\[
\liminf_{T \to \infty} \frac{1}{T} \mu \{ \tau \in [0, T] : \| \zeta(s + i\tau) - \zeta(s) \|_K < \varepsilon \} > 0. \tag{1}
\]

In 2010 Nakamura [3] showed the following property which might be called self-approximation of the Riemann zeta function.

Theorem B. For every algebraic irrational number \( d \in \mathbb{R} \), every compact set \( K \subset D \) with connected complement and every \( \varepsilon > 0 \), we have

\[
\liminf_{T \to \infty} \frac{1}{T} \mu \{ \tau \in [0, T] : \| \zeta(s + id\tau) - \zeta(s + i\tau) \|_K < \varepsilon \} > 0.
\]

Note that the self-approximation with respect to almost all real numbers \( d \) was also verified in [3]. Afterwards, Pańkowski [5] showed the above result for any irrational number \( d \) whereas Garunkštis [2] and Nakamura [4] investigated the self-approximation for non-zero rational numbers, independently. Unfortunately, the papers [2] and [4] contain a gap in the proof of the main theorem, so actually their methods work only for the logarithm of the Riemann zeta function (see Remark [2.3]).

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In this paper, we prove that self-approximation of \( \text{log } \zeta(s) \) with \( d = 0 \) is equivalent to the Riemann Hypothesis in Theorem 2.1. Next, we show self-approximation of \( \text{log } \zeta(s) \) for all nonzero real numbers \( d \) in Theorem 2.2. Moreover, we partially filled the gap mentioned above and prove self-approximation of \( \zeta(s) \) for \( 0 \neq d = a/b \) with \( |a-b| \neq 1 \) and \( \gcd(a,b) = 1 \) in Theorem 3.1.

2. Self-approximation of \( \text{log } \zeta(s) \)

Firstly, we show the following theorem which is an analogue of Theorem [A].

**Theorem 2.1.** The Riemann Hypothesis holds if and only if, for every compact set \( K \subset D \) with connected complement and for every \( \varepsilon > 0 \), we have

\[
\liminf_{T \to \infty} \frac{1}{T} \mu \{ \tau \in [0,T] : \| \text{log } \zeta(s+i\tau) - \text{log } \zeta(s) \|_K < \varepsilon \} > 0. \tag{2}
\]

**Proof.** If the Riemann hypothesis is true we can apply Voronin’s universality theorem.

Suppose that there exists a zero \( \xi \in D \) of \( \zeta(s) \). Put \( K_{\varepsilon} := \{ s \in \mathbb{C} : |s - \xi| \leq \varepsilon \} \subset D \). Now assume that for a neighborhood \( K_{\varepsilon} \) of \( \xi \) the following relation holds:

\[
\| \text{log } \zeta(s+i\tau) - \text{log } \zeta(s) \|_{K_{\varepsilon}} < \varepsilon. \tag{3}
\]

If a zero \( \rho \) of \( \zeta(s) \), where \( \rho \in K_{\varepsilon}(\tau) := \{ s \in \mathbb{C} : |s - \xi - i\tau| \leq \varepsilon \} \) does not exist, then the function \( \text{log } \zeta(s+i\tau) \) is analytic in the interior of \( K_{\varepsilon} \) and bounded on \( K_{\varepsilon} \). This contradicts to the above inequality. Hence a zero \( \rho \) of \( \zeta(s) \) exists in \( K_{\varepsilon}(\tau) \). With regard to (3) and the definition of \( K_{\varepsilon}(\tau) \) the zeros \( \xi \) and \( \rho \) are intimately related; more precisely, \( |\rho - \xi - i\tau| < \varepsilon \). Thus two different shifts \( \tau_1 \) and \( \tau_2 \) can lead to the same zero \( \rho \), but their distance is bounded by \( |\tau_1 - \tau_2| < 2\varepsilon \). Therefore we obtain this lemma by modifying the proof of [6] Theorem 8.3 and using the classical Rouché theorem. \( \square \)

The reasoning of [5] Theorem 1.1 and [2] Theorem 1] can be easily applied to prove self-approximation of \( \text{log } \zeta(s) \).

**Theorem 2.2.** For every real number \( d \neq 0 \), every compact set \( K \subset D \) with connected complement and for every \( \varepsilon > 0 \), it holds that

\[
\liminf_{T \to \infty} \frac{1}{T} \mu \{ \tau \in [0,T] : \| \text{log } \zeta(s+i\tau) - \text{log } \zeta(s+id\tau) \|_K < \varepsilon \} > 0. \tag{4}
\]

**Remark 2.3.** In fact, in [2] and [4] it was claimed that the above theorem holds even for the Riemann zeta function instead of the logarithm of \( \zeta(s) \). Unfortunately, the proofs of main results in [2] Theorem 1] and [4] Corollary 1.2] are not sufficient, since it was only shown that \( |\zeta(s+id\tau)/\zeta(s+i\tau) - 1| < \varepsilon \). Obviously we have

\[
|\zeta(s+i\tau) - \zeta(s+id\tau)| = |\zeta(s+i\tau)||\zeta(s+id\tau)/\zeta(s+i\tau) - 1|. \]

So in order to prove self-approximation of \( \zeta(s) \) it should have been proved that \( \zeta(s+i\tau) \) is not too large, namely we need the inequality \( |\zeta(s+i\tau)| < |\zeta(s+id\tau)/\zeta(s+i\tau) - 1|^{-1} \) which is not necessarily satisfied.
3. Self-approximation of $\zeta(s)$

In the following theorem we partially fix the gap existing in [24 Theorem 1] and [4 Corollary 1.2] and prove self-approximation of $\zeta(s)$ in the case $0 \neq d = a/b \in \mathbb{Q}$ with $|a - b| \neq 1$ and $\gcd(a, b) = 1$.

**Theorem 3.1.** For every $0 \neq d = a/b$ with $|a - b| \neq 1$ and $\gcd(a, b) = 1$, every compact set $K \subset D$ and for every $\varepsilon > 0$, we have

$$
\liminf_{T \to \infty} \frac{1}{T} \mu \{ \tau \in [0, T] : \| \zeta(s + i\tau) - \zeta(s + id\tau) \|_K < \varepsilon \} > 0.
$$

(5)

**Proof.** First of all, note that it suffices to consider the case $a \neq b$, or equivalently $d \neq 1$.

Let us take $\omega(p_m) := \exp(2\pi im/(a - b))$, where $p_m$ denotes the $m$-th prime number. Then we have $\omega^a(p) = \omega^b(p)$ since $\omega^{a-b}(p) = 1$. Firstly, we show that

$$
|\zeta(s, \omega^c)| \leq \exp\left(\frac{7(1 - z^{-1-2\sigma})}{2\sigma - 1} + |a - b|(z^{-\sigma} + 2|s|(1 - z^{-\sigma}))\right),
$$

(6)

where $c = a, b$ and $\zeta(s, \omega^c) := \prod_{p \leq z}(1 - \omega(p)^c p^{-s})^{-1}$. In order to prove it let us consider the function $- \sum_{p \leq z} \log(1 - \omega(p)^c p^{-s})$. Then

$$
- \sum_{p \leq z} \log \left(1 - \frac{\omega(p)^c}{p^s}\right) = \sum_{p \leq z} \sum_{k=1}^{\infty} \frac{\omega(p)^c}{kp^k} = \sum_{p \leq z} \frac{\omega(p)^c}{p^s} + \sum_{p \leq z} \sum_{2 \leq k \leq 2} \frac{\omega(p)^c}{kp^k}.
$$

Let us estimate the latter sum on the right hand side. For $\sigma > 1/2$ one has

$$
\left| \sum_{p \leq z} \sum_{2 \leq k \leq 2} \frac{\omega(p)^c}{kp^k} \right| \leq \sum_{p \leq z} \sum_{2 \leq k \leq 2} \frac{1}{p^k} = \sum_{p \leq z} \frac{1}{p^2 - p^\sigma} \leq 7 \sum_{p \leq z} \frac{1}{p^{2\sigma}} \leq 7 \int_1^z t^{-2\sigma} dt = \frac{7(1 - z^{-1-2\sigma})}{2\sigma - 1}.
$$

To consider the former sum, we put

$$
\omega(n) := \begin{cases} 
\omega(p) & \text{if } n = p \leq z, \\
0 & \text{otherwise},
\end{cases}
\quad \Omega_z := \sum_{n=1}^{z} \omega(n)^c.
$$

Then by partial summation, we obtain

$$
\sum_{p \leq z} \frac{\omega(p)^c}{p^s} = \sum_{n=1}^{z} \frac{\omega(n)^c}{n^s} = \Omega_z^{\frac{1}{z^s}} - \sum_{n=1}^{z-1} \Omega_n \left( \frac{1}{(n+1)^s} - \frac{1}{n^s} \right) = \Omega_z^{\frac{1}{z^s}} + s \sum_{n=1}^{z-1} \Omega_n \int_n^{n+1} t^{-s-1} dt.
$$

Let us notice that $\omega(p)^c$ is a nontrivial root of unity, since $\gcd(a, b) = 1$ implies $a - b \nmid a, b$. Hence we have

$$
\left| \sum_{p \leq z} \frac{\omega(p)^c}{p^s} \right| \ll z^{-\sigma} + |s| \int_1^z t^{-\sigma-1} dt = z^{-\sigma} + \frac{|s|}{\sigma} (1 - z^{-\sigma}) \leq \frac{1}{z^\sigma} + 2|s| \left(1 - \frac{1}{z^\sigma}\right),
$$

where the constant in symbol $\ll$ is equal to $|a - b|$. Therefore we obtain (6).
Now it suffices to follow the steps of the proof of [2, Theorem 1] and the following fact. Let $\|x\|$ give the distance from a real number $x$ to the nearest integer. Then the set of positive real numbers $\tau$ satisfying

$$\max_{p_m \leq z} \left\| \frac{\tau \log p_m}{2\pi} - \frac{m}{a - b} \right\| < \delta$$

has a positive density for every positive $\delta$ by the Kronecker approximation theorem. Thus it holds for sufficiently large $z$

$$\| \log \zeta(s + ic\tau) - \log \zeta(s, \omega^c) \|_K < \varepsilon, \quad c = a, b.$$ 

This completes the proof. \qed

References

[1] B. Bagchi, *Recurrence in topological dynamics and the Riemann hypothesis*, Acta Math. Hung. 50 (1987), 227-240.
[2] R. Garunkštis, *Self-approximation of Dirichlet $L$-functions*, J. Number Theory 131 (2011), no. 7, 1286–1295.
[3] T. Nakamura, *The joint universality and the generalized strong recurrence for Dirichlet $L$-functions*, Acta Arith. 138 (2009), no. 4, 357–362.
[4] T. Nakamura, *The generalized strong recurrence for non-zero rational parameters*, Archiv der Mathematik 95 (2010), 549–555.
[5] L. Pańkowski, *Some remarks on the generalized strong recurrence for $L$-functions*, New directions in value-distribution theory of zeta and $L$-functions, 305–315, Ber. Math., Shaker Verlag, Aachen, 2009.
[6] J. Steuding, *Value-Distribution of $L$-functions*, Lecture Notes in Mathematics, 1877, Springer, Berlin (2007).

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