Towards Efficient Interactive Computation of Dynamic Time Warping Distance

Akihiro Nishi¹, Yuto Nakashima¹, Shunsuke Inenaga¹,², Hideo Bannai³, and Masayuki Takeda¹

¹ Department of Informatics, Kyushu University, Fukuoka, Japan
² PRESTO, Japan Science and Technology Agency, Kawaguchi, Japan
³ M&D Data Science Center, Tokyo Medical and Dental University, Tokyo, Japan

Abstract

The dynamic time warping (DTW) is a widely-used method that allows us to efficiently compare two time series that can vary in speed. Given two strings A and B of respective lengths m and n, there is a fundamental dynamic programming algorithm that computes the DTW distance \( dtw(A, B) \) for A and B together with an optimal alignment in \( \Theta(mn) \) time and space. In this paper, we tackle the problem of interactive computation of the DTW distance for dynamic strings, denoted \( D^2TW \), where character-wise edit operation (insertion, deletion, substitution) can be performed at an arbitrary position of the strings. Let M and N be the sizes of the run-length encoding (RLE) of A and B, respectively. We present an algorithm for \( D^2TW \) that occupies \( \Theta(mN + nM) \) space and uses \( O(m + n + \#_{chg}) \subseteq O(mN + nM) \) time to update a compact differential representation \( DS \) of the DP table per edit operation, where \( \#_{chg} \) denotes the number of cells in \( DS \) whose values change after the edit operation. Our method is at least as efficient as the algorithm recently proposed by Froese et al. running in \( \Theta(mN + nM) \) time, and is faster when \( \#_{chg} \) is smaller than \( O(mN + nM) \) which, as our preliminary experiments suggest, is likely to be the case in the majority of instances. In addition, our result leads to interactive LCS/weighted edit distance computation running in \( O(m + n + \#_{chg}) \subseteq O(mN + nM) \) time per update using \( \Theta(mN + nM) \) space. This improves on Hyyrö et al.’s interactive algorithm that occupies \( \Theta(mn) \) space and uses \( O(mn) \) time per update in the worst case.

1 Introduction

The dynamic time warping (DTW) is a classical and widely-used method that allows us to efficiently compare two temporal sequences or time series that can vary in speed. A fundamental dynamic programming algorithm computes the DTW distance \( dtw(A, B) \) for two strings A and B together with an optimal alignment in \( \Theta(mn) \) time and space [10], where \(|A| = m\) and \(|B| = n\). This algorithm allows one to update the DP table \( D \) for \( dtw(A, B) \) in \( O(m) \) time (resp. \( O(n) \) time) when a new character is appended to B (resp. to A).

In this paper, we introduce the “dynamic” DTW problem, denoted \( D^2TW \), where character-wise edit operation (insertion, deletion, substitution) can be performed at an
arbitrary position of the strings. More formally, we wish to maintain a (space-efficient) representation of $D$ that can dynamically be modified according to a given operation. This representation should be able to quickly answer the value of $D[m, n] = dtw(A, B)$ upon query, together with an optimal alignment achieving $dtw(A, B)$. This kind of interactive computation for (a representation of) $D$ can be of practical merits, e.g. when simulating stock charts, or editing musical sequences. Another example of applications of $D^2$TW is a sliding window version of DTW which computes $dtw(A, B[\ldots, j + d - 1])$ between $A$ and every substring $B[j..j + d - 1]$ of $B$ of arbitrarily fixed length $d$.

Incremental/decremental computation of a DP table is a restricted version of the aforementioned interactive computation, which allows for prepending a new character to $B$, and/or deleting the leftmost character from $B$. A number of incremental/decremental computation algorithms have been proposed for the unit-cost edit distance and weighted edit distance: Kim and Park [8] showed a incremental/decremental algorithm for the unit-cost edit distance that occupies $\Theta(mn)$ space and runs in $O(m + n)$ time per operation. Hyyrö et al. [6] proposed an algorithm for the edit distance with integer weights which uses $\Theta(c(m + n), mn)$ time per operation, where $c$ is the maximum weight in the cost function. This translates into $O(m + n)$ time under constant weights. Schmidt [11] gave an algorithm that uses $\Theta(mn)$ space and runs in $O(n\log m)$ time per operation for a general weighted edit distance. Hyyrö and Inenaga [4] presented a space efficient alternative to incremental/decremental unit-cost edit distance computation which runs in $O(m + n)$ time per operation but uses only $\Theta(mN + nM)$ space, where $M$ and $N$ are the sizes of run-length encoding (RLE) of $A$ and $B$, respectively. Since $M \leq m$ and $N \leq n$ always hold, the $mN + nM$ terms can be much smaller than the $mn$ term for strings that contain many long character runs. Later, Hyyrö and Inenaga [5] presented a space-efficient alternative for edit distance with integer weights, which runs in $O(\min\{c(m + n), mn\})$ time per operation and requires $\Theta(mN + nM)$ space.

Fully-dynamic interactive computation for the (weighted) edit distance was also considered: Let $j^*$ be the position in string $B$ where the modification has been performed. For the unit cost edit distance, Hyyrö et al. [7] presented a representation of the DP table which uses $\Theta(mn)$ space and can be updated in $O(\min\{rc(m + n), mn\})$ time per operation, where $r = \min\{j^*, n - j^* + 1\}$ and $c$ is the maximum weight. They also showed that there exist instances that require $\Omega(\min\{rc(m + n), mn\})$ time to update their data structure per operation.

While computing longest common subsequence (LCS) and weighted edit distance of strings of length $n$ can both be reduced to computing DTW of strings of length $O(n)$ [11][9], a reduction to the other direction is not known. It thus seems difficult to directly apply any of the aforementioned algorithms to our $D^2$TW problem. Also, a conditional lower bound suggests that strongly sub-quadratic DTW algorithms are unlikely to exist [11][2]. Thus, any method that recomputes the naïve DP table $D$ from scratch should take almost quadratic time per update.

This paper takes the first step towards an efficient solution to $D^2$TW. Namely, we present an algorithm for $D^2$TW that occupies $\Theta(mN + nM)$ space and uses $O(m + n + \#_{chg})$ time to update a compact differential representation $DS$ for the DP table $D$ per edit operation, where $\#_{chg}$ denotes the number of cells in $DS$ whose values change after the edit operation. Since $\#_{chg} = O(mN + nM)$ always hold, our method is always at least as efficient as the naïve method that recomputes the full DP table $D$ in $\Theta(mn)$ time, or
the algorithm of Froese et al. [3] that recomputes another sparse representation of \(D\) in \(\Theta(mN+nM)\) time. While there exist worst-case instances that give \(#_{\text{chg}} = \Omega(mN+nM)\), our preliminary experiments suggest that, in many cases, \(#_{\text{chg}}\) can be much smaller than the size of \(DS\) which is \(\Theta(mN+nM)\). Our algorithm also supports efficient batched updates for character-run-wise edit operations.

Since LCS and weighted edit distance can be reduced to DTW, our method implies \(O(m+n+#_{\text{chg}})\)-time or \(O(mN+nM)\)-time algorithms for interactive LCS and weighted edit distance computation, using \(\Theta(mN+nM)\) space. This improves on Hyyrö et al.’s interactive weighted edit distance algorithm [7] that occupies \(\Theta(mn)\) space and takes worst-case \(O(\min\{rc(m+n), mn\}) \subseteq O(mn)\) time per update, when \(r = \Theta(n)\) (e.g. when \(j^* = n/2\) or when \(c = \Theta(\min\{m,n\})\)).

Technically our algorithm is most related to Hyyrö et al.’s method [6, 7] and Froese et al.’s method [3], but our algorithm is not straightforward from these.

Proofs omitted due to lack of space can be found in Appendix.

## 2 Preliminaries

We consider sequences (strings) of characters from an alphabet \(\Sigma\) of reals. Let \(A = a_1, \ldots, a_m\) be a string consisting of \(m\) characters from \(\Sigma\). The run-length encoding \(\text{rle}(A)\) of string \(A\) is a compact representation of \(A\) such that each maximal run of the same characters in \(A\) is represented by a pair of the character and the length of the run. More formally, let \(\mathbb{N}\) denote the set of positive integers. For any non-empty string \(A\), \(\text{rle}(A) = a_1^{e_1} \cdots a_M^{e_M}\), where \(a_I \in \Sigma\) and \(e_I \in \mathbb{N}\) for any \(1 \leq I \leq M\), and \(a_I \neq a_{I+1}\) for any \(1 \leq I < M\). Each \(a_I^{e_I}\) in \(\text{rle}(A)\) is called a (character) run, and \(e_I\) is called the exponent of this run. The size of \(\text{rle}(A)\) is the number \(M\) of runs in \(\text{rle}(A)\). E.g., for string \(A = \text{aaccddcbbabbbb}\) of length 16, \(\text{rle}(A) = a^2c^3b^2a^1b^4\) and its size is 5.

Dynamic time warping (DTW) is a commonly used method to compare two temporal sequences that may vary in speed. Consider two strings \(A = a_1, \ldots, a_m\) and \(B = b_1, \ldots, b_n\). To formally define the DTW for \(A\) and \(B\), we consider a \(m \times n\) grid graph \(G_{m,n}\) such that each vertex \((i, j)\) has (at most) three directed edges; one to the lower neighbor \((i+1, j)\) (if it exists), one to the right neighbor \((i, j+1)\) (if it exists), and one to the lower-right neighbor \((i+1, j+1)\) (if it exists). A path in \(G_{m,n}\) that starts from vertex \((1,1)\) and ends at vertex \((m,n)\) is called a warping path, and is denoted by a sequence \((1,1), \ldots, (i,j), \ldots, (m,n)\) of adjacent vertices. Let \(P_{m,n}\) be the set of all warping paths in \(G_{m,n}\). Note that each warping path in \(P_{m,n}\) corresponds to an alignment of \(A\) and \(B\). The DTW for strings \(A\) and \(B\), denoted \(\text{dtw}(A, B)\), is defined by \(\text{dtw}(A, B) = \min_{p \in P_{m,n}} \sqrt{\sum_{(i,j) \in p} (a_i - b_j)^2}\).

The fundamental \(\Theta(mn)\)-time and space solution for computing \(\text{dtw}(A, B)\), given in [10], fills an \(m \times n\) dynamic programming table \(D\) such that \(D[i, j] = \text{dtw}(A[1..i], B[1..j])^2\) for \(1 \leq i \leq m\) and \(1 \leq j \leq n\). Therefore, after filling all the cells of \(D\), the desired result \(\text{dtw}(A, B)\) is obtained by \(\sqrt{D[m,n]}\). The value for each cell \(D[i, j]\) is computed by the
following well-known recurrence:
\[
D[1,1] = (a_1 - b_1)^2, \\
D[i,1] = D[i-1,1] + (a_i - b_1)^2 \text{ for } 1 < i \leq m, \\
D[1,j] = D[1,j-1] + (a_1 - b_j)^2 \text{ for } 1 < j \leq n, \\
D[i,j] = \min\{D[i,j-1], D[i-1,j], D[i-1,j-1]\} + (a_i - b_j)^2 \\
\text{for } 1 < i \leq m \text{ and } 1 < j \leq n.
\]

In the rest of this paper, we will consider the problem of maintaining a representation for \( D \), each time one of the strings, \( B \), is dynamically modified by an edit operation (i.e. single character insertion, deletion, or substitution) on an arbitrary position in \( B \). We call this kind of interactive computation of \( dtw(A, B) \) as the dynamic DTW computation, denoted by \( D^2TW \).

Let \( B' \) denote the string after an edit operation is performed on \( B \), and \( D' \) denote the dynamic programming table \( D \) after it has been updated to correspond to \( dtw(A, B') \). In a special case where the edit operation is performed at the right end of \( B \), where we have either \( B' = Bc \) (insertion), \( B' = B[1..n-1] \) (deletion) or \( B' = B[1..n-1]c \) (substitution) with a character \( c \in \Sigma \), then \( D \) can easily be updated to \( D' \) in \( O(m) \) time by simply computing a single column at index \( j = n \) or \( j = n + 1 \) using recurrence (1).

The following lemma states that in our \( D^2TW \) computation, updating \( D \) to \( D' \) leads to a worst-case bound \( \Theta(mn) \), and this cost cannot be amortized.

**Lemma 1** There are strings \( A, B \) and a sequence of \( k \) edits on \( B \) such that \( \Theta(kmn) \) cells in \( D' \) have different values in the corresponding cells in \( D \).

## 3 Our \( D^2TW \) Algorithm based on RLE

We first explain the data structures which are used in our algorithm.

**Differential representation \( DR \) of \( D \).** The first idea of our algorithm is to use a differential representation \( DR \) of \( D \): Each cell of \( DR \) contains two fields that respectively store the horizontal difference and the vertical difference, namely, \( DR[i,j].U = D[i,j] - D[i-1,j] \) and \( DR[i,j].L = D[i,j] - D[i,j-1] \). We let \( DR[i,1].L = 0 \) for any \( 1 \leq i \leq m \) and \( DR[1,j].U = 0 \) for any \( 1 \leq j \leq n \). The diagonal difference \( D[i,j] - D[i-1,j-1] \) can easily be computed from \( DR[i,j].U \) and \( DR[i,j].L \) and thus is not explicitly stored in \( DR[i,j] \).

In our algorithm we make heavy use of the following lemma:

**Lemma 2** For any \( 1 < i \leq m \),
\[
DR[i,j].U = \begin{cases} (a_i - b_1)^2 & \text{if } j = 1, \\ z - DR[i-1,j].L & \text{if } 2 \leq j \leq n, \end{cases}
\]

and for any \( 1 < j \leq n \),
\[
DR[i,j].L = \begin{cases} (a_1 - b_j)^2 & \text{if } i = 1, \\ z - DR[i,j-1].U & \text{if } 2 \leq i \leq m, \end{cases}
\]

where \( z = \min\{DR[i-1,j].L, \ DR[i,j-1].U, \ 0\} + (a_i - b_j)^2 \).
The second key idea of our algorithm is to divide the dynamic programming table $D$ into “boxes” that are defined by intersections of maximal runs of $A$ and $B$. Note that $D$ contains $M \times N$ such boxes. Let \( \text{rle}(A) = A_B^1 \ldots A_B^M \) and \( \text{rle}(B) = B_A^1 \ldots B_A^N \) be the RLEs of $A$ and $B$. Let \( i_T^j = \sum_{i=1}^{j} k_i + 1 \), \( i_B^j = \sum_{i=1}^{j} k_i \), \( j_L^j = \sum_{j=1}^{i} l_j + 1 \), and \( j_R^j = \sum_{j=1}^{i} l_j \). We define a sparse table $DS$ for $DR$ that consists only of the rows and columns on the borders of the maximal runs in $A$ and $B$. Namely, $DS$ is a sparse table that only stores the rows $i_T^j$, $i_B^j$ ($1 \leq I \leq M$) and the columns $j_L^j$, $j_R^j$ ($1 \leq J \leq N$), of $DR$. Each row and column of $DS$ is implemented by a linked list as follows: each cell $DS[i,j]$ has four links to the upper, lower, left, and right neighbors in $DS$ (if these neighbors exist), plus a diagonal link to the right-lower direction. This diagonal link from $DS[i,j]$ points to the first cell $DS[i+h,j+h]$ that is reached by following the right-lower diagonal path from $DS[i,j]$, namely, $h \geq 0$ is the smallest integer such that $i+h = i_T^j$ or $j+h = j_R^j$. Clearly $DS$ occupies $\Theta(mN + nM)$ space. $DS$ can answer $d tw(A,B) = D[m,n]$ in $O(m+n)$ time by tracing $O(m+n)$ cells of $DS$ from $(1,1)$ to $(m,n)$.

For each $1 \leq I < M$ and $1 \leq J < N$, we consider the region of $DR$ that is surrounded by the borders of the $I$th and $(I+1)$th runs of $A$, and the $J$th and $(J+1)$th runs of $B$. This region is called a box for $I, J$, and is denoted by $B^{I,J}$. For ease of description, we will sometimes refer to a box $B^{I,J}$ also in $D$ and $DS$.

Proof. $DR[i,1].U = (a_i - b_1)^2$ and $DR[1,j].L = (a_1 - b_j)^2$ are clear from recurrence (1). Now we consider $1 < i \leq m$ and $1 < j \leq n$, and let $d = D[i - 1, j - 1]$, $x = DR[i - 1, j].L$, $y = DR[i, j - 1].U$, and $d+z = D[i, j]$. Then we have $D[i-1, j] = d+x$ and $D[i, j-1] = d+y$ (see Fig. 2). It follows from the definition of $DR$ that $DR[i, j].U = D[i, j] - D[i-1, j] = z-x$ and $DR[i, j].L = D[i, j] - D[i, j-1] = z-y$. Since $D[i, j] = \min\{D[i-1, j-1], D[i-1, j], D[i, j-1]\} + (a_i-b_j)^2$ by recurrence (1), we obtain $d+z = \min\{d, d+x, d+y\} + (a_i-b_j)^2$ which leads to $z = \min\{x, y, 0\} + (a_i-b_j)^2$. \( \square \)

RLE-based sparse differential representation $DS$. The key idea of our algorithm is to divide the dynamic programming table $D$ into “boxes” that are defined by intersections of maximal runs of $A$ and $B$. Note that $D$ contains $M \times N$ such boxes. Let \( \text{rle}(A) = A_B^1 \ldots A_B^M \) and \( \text{rle}(B) = B_A^1 \ldots B_A^N \) be the RLEs of $A$ and $B$. Let \( i_T^j = \sum_{i=1}^{j} k_i + 1 \), \( i_B^j = \sum_{i=1}^{j} k_i \), \( j_L^j = \sum_{j=1}^{i} l_j + 1 \), and \( j_R^j = \sum_{j=1}^{i} l_j \). We define a sparse table $DS$ for $DR$ that consists only of the rows and columns on the borders of the maximal runs in $A$ and $B$. Namely, $DS$ is a sparse table that only stores the rows $i_T^j$, $i_B^j$ ($1 \leq I \leq M$) and the columns $j_L^j$, $j_R^j$ ($1 \leq J \leq N$), of $DR$. Each row and column of $DS$ is implemented by a linked list as follows: each cell $DS[i,j]$ has four links to the upper, lower, left, and right neighbors in $DS$ (if these neighbors exist), plus a diagonal link to the right-lower direction. This diagonal link from $DS[i,j]$ points to the first cell $DS[i+h,j+h]$ that is reached by following the right-lower diagonal path from $DS[i,j]$, namely, $h \geq 0$ is the smallest integer such that $i+h = i_T^j$ or $j+h = j_R^j$. Clearly $DS$ occupies $\Theta(mN + nM)$ space. $DS$ can answer $d tw(A,B) = D[m,n]$ in $O(m+n)$ time by tracing $O(m+n)$ cells of $DS$ from $(1,1)$ to $(m,n)$.

For each $1 \leq I < M$ and $1 \leq J < N$, we consider the region of $DR$ that is surrounded by the borders of the $I$th and $(I+1)$th runs of $A$, and the $J$th and $(J+1)$th runs of $B$. This region is called a box for $I, J$, and is denoted by $B^{I,J}$. For ease of description, we will sometimes refer to a box $B^{I,J}$ also in $D$ and $DS$. 

Figure 1: Illustration for $DS$ that consists only of the cells of $DR$ corresponding to the maximal run boundaries of $A$ and $B$ (white rows and columns).

The gray regions that are surrounded by the box boundaries are not stored in $DS$. 

Figure 2: Illustration for Lemma 2 which depicts the corresponding cells of the dynamic programming table $D$, where $D[i - 1, j - 1] = d$, $D[i - 1, j] = d + x$, $D[i, j - 1] = d + y$, and $D[i, j] = d + z$. 

Proof. $DR[i,1].U = (a_i - b_1)^2$ and $DR[1,j].L = (a_1 - b_j)^2$ are clear from recurrence (1). Now we consider $1 < i \leq m$ and $1 < j \leq n$, and let $d = D[i-1, j-1]$, $x = DR[i-1, j].L$, $y = DR[i, j-1].U$, and $d+z = D[i, j]$. Then we have $D[i-1, j] = d+x$ and $D[i, j-1] = d+y$ (see Fig. 2). It follows from the definition of $DR$ that $DR[i, j].U = D[i, j] - D[i-1, j] = z-x$ and $DR[i, j].L = D[i, j] - D[i, j-1] = z-y$. Since $D[i, j] = \min\{D[i-1, j-1], D[i-1, j], D[i, j-1]\} + (a_i-b_j)^2$ by recurrence (1), we obtain $d+z = \min\{d, d+x, d+y\} + (a_i-b_j)^2$ which leads to $z = \min\{x, y, 0\} + (a_i-b_j)^2$. \( \square \)
3.1 Updating DS

Suppose that an edit operation has been performed at position \(j^*\) of string \(B\) and let \(B'\) denote the edited string. Let \(D'\) denote the dynamic programming table for \(\text{dtw}(A, B')\). Let \(DR'\) denote the difference representation for \(D'\), and \(DS'\) the sparse table for \(DR'\).

Because the prefix \(B[1..j^*-1]\) remains unchanged after the edit operation, for any \(j < j^*\) we have \(DR[i, j] = DR'[i, j]\) by Lemma 2 and recurrence (1). Hence, we can restrict ourselves to the indices \(j \geq j^*\). We define \(\ell\) as a correcting offset of string indices before and after the update: \(\ell = -1\) if a character has been inserted at position \(j^*\) of \(B\), \(\ell = 1\) if a character has been deleted from position \(j^*\) of \(B\), and \(\ell = 0\) otherwise. Now, for any \(j \geq j^*, B'[j] = B[j+\ell]\) and column \(j\) in \(DR'\) corresponds to column \(j+\ell\) in \(DR\).

Let \(B^{l,j}\) be any box on \(DS'\). For the the top row \(i^T\) of \(B^{l,j}\), we use a linked list \(\Delta^{l,j}_I\) that stores the column indices \(j\) (\(j^L \leq j \leq j^R\)) such that \(DS[i^T, j + \ell] \neq DS'[i^T, j]\), in increasing order. We also compute, in each element of the list, the value for \(D'[i^T, j]\) of the corresponding column index \(j\). We use similar lists \(\Delta^{l,j}_B, \Delta^{l,j}_L, \Delta^{l,j}_R\) for the bottom row, left column, and right column of \(B^{l,j}\), respectively. We compute these lists when an edit operation is performed to string \(B\), and use them to update \(DS\) to \(DS'\) efficiently.

Let \#_{chg} denote the number of cells in our sparse representation such that \(DS[i+\ell,j] \neq DS'[i,j]\). In the next subsections, we prove:

**Theorem 1**  Our \(D^2TW\) algorithm updates \(DS\) to \(DS'\) in \(O(m + n + \#_{chg})\) time.

**Initial step.** Suppose that \(j^*\) is in the \(J\)th run of string \(B\). Let \(B^{l,j}\) be any of the \(M\) boxes of \(DR\) that contain column \(j^*\), where \(j^L \leq j^* \leq j^R\). Due to Lemma 2, (1, \(j^*\)) is the only cell in the first row where we may have \(DS'[1,j^*] \neq DS[1,j^* + \ell]\). \(DS'[1,j^*]\) can be easily computed in \(O(1)\) time by Lemma 2. Then, \(D'[1,j^*]\) can be computed in \(O(j^* \subseteq O(n))\) time by tracing the first row and using \(DS'[1,j].L\) for increasing \(j = 1, \ldots, j^*\). The list \(\Delta^{l,j}_I\) only contains \(j^*\) (coupled with \(D'[1,j^*]\)) if \(DS'[1,j^*] \neq DS[1,j^* + \ell]\), and it is empty otherwise.

Editing string \(B\) at position \(j^*\) incurs some structural changes to \(DS\): (a) \(B^{l,j}\) gets wider by one (insertion of the same character to a run), (b) \(B^{l,j}\) gets narrower by one (deletion of a character), (c) \(B^{l,j}\) is divided into \(2M\) or \(3M\) boxes (insertion of a different character to a run, or character substitution).

In cases (a) and (b), the diagonal links of \(B^{l,j}\) need to be updated. A crucial observation is that the total number of such diagonal links to update is bounded by \(m\) for all the \(M\) boxes \(B^{l,j_1}, \ldots, B^{l,j_M}\), since the destinations of such diagonal links are within the same column of \(DS'\) (\(j^{L}_R + 1\) in case (a), and \(j^{R}_R - 1\) in case (b)). For each box \(B^{l,j}\), if \(j^R - j^L \geq i^T - i^B\) (i.e. \(B^{l,j}\) is a square or a horizontal rectangle), then we scan the top row \(i^T\) from right to left and fix the diagonal links until encountering the first cell in \(i^T\) whose diagonal link needs no updates (see Fig. 8 in Appendix B). The case with \(j^R - j^L < i^T - i^B\) (i.e. \(B^{l,j}\) is a vertical rectangle) can be treated similarly. By the above observation, these costs for all boxes \(B^{l,j}\) that contain the edit position \(j^*\) sum up to \(O(m)\).

In case (a), we shift the right column \(j^R\) of \(DS\) to the right by one position, and reuse it as the right column \(j^R + 1\) of \(DS'\). This incurs two new cells \((i^T, j^R)\) and \((i^B, j^R)\) in \(DS'\) (the gray cells in Fig. 8). We can compute \(DS'[i^T, j^R]\) in \(O(1)\) time using Lemma 2. Now consider to compute \(DS'[i, j^R+1]\) for the new right column. Since this right column initially stores \(DS[i, j^R]\) for the old \(DS\), using Lemma 2, we can compute \(DS'[i, j^R+1]\) in increasing
order of \( i = 1, \ldots, m \), from top to bottom, in \( O(1) \) time each. We can compute \( D'[1, j^*_R + 1] \) in \( O(j^*_R) \) time by simply scanning the first row. Then, we can compute \( D'[i, j^*_R + 1] \) for increasing \( j^*_R = 2, \ldots, m \), using \( DS'[i, j^*_R + 1] \), and construct \( \Delta^*_R \). This takes a total of \( O(m + n) \) time. Finally, \( DS'[i^*_B, j^*_R + 1] \) is computed from \( D'[i^*_B, j^*_R + 1] \) and \( DS'[i^*_B, j^*_R + 1] \) in \( O(1) \) time. Case (b) can be treated similarly.

For case (c), we consider a sub-case where a character substitution was performed completely inside a run of \( B \), at position \( j^* \). This divides an existing box \( B^I,J \) into three boxes \( B^{I,J}, B^{I,J+1}, \) and \( B^{I,J+2} \). Thus, there appear three new columns \( j^* - 1, j^* \), and \( j^* + 1 \) in \( DS' \). Then, the diagonal links for these new columns can easily be computed in \( O(1) \) time each, by scanning row \( I \) from \( j^* + 1 \), from right to left (see Fig. 9 in Appendix). The \( DS' \) values for the cells in these new columns, as well as the \( D' \) values for column \( j^* + 1 \), can also be computed in similar ways to cases (a) and (b) above. The other sub-cases of case (c) can also be treated similarly.

**Updating cells on row \( i^*_T \) and column \( j^*_L \).** In what follows, suppose that we are given a box \( B^I,J \) to the right of the edit position \( j^* \), in which some boundary cell values may have to be updated. For ease of exposition, we will discuss the simplest case with substitution where the column indices do not change between \( DS \) and \( DS' \). The cases with insertion/deletion can be treated similarly by considering the offset value \( \ell \) appropriately.

Now our task is to quickly detect the boundary cells \( (i, j) \) of \( B^I,J \) such that \( DS[i, j] \neq DS'[i, j] \), and to update them. We assume that the boundary cell values of the preceding boxes \( B^{I,J-1} \) and \( B^{I,J-1} \) have already been computed.

In this subsection, we consider how to detect the cells on the top boundary row \( i^*_T \) and the cells on the left boundary column \( j^*_L \) of box \( B^I,J \) that need to be updated, and how to update them. For this sake, we use the following lemma on the values of \( DR \), which is immediate from Lemma 2.

**Lemma 3** Let \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \). Suppose that for any cell \( (i', j') \) with \( i' < i \) or \( j' < j \), the value of \( DR[i', j'] \) has already been computed. If \( DR[i, j - 1] \neq DR'[i, j] \), then \( DR[i, j - 1] \neq DR'[i, j] \).

Intuitively, Lemma 3 states that the cell \( (i, j) \) such that \( DR[i, j] \neq DR'[i, j] \) must be propagated from its left neighbor or its top neighbor. We use this lemma for updating the boundaries of each box \( B^I,J \) stored in \( DS \). Recall that the values on the preceding row \( i^*_T - 1 = i^*_B - 1 \) and on the preceding column \( j^*_L - 1 = j^*_R - 1 \) have already been updated. Then, the cells on \( i^*_T \) and \( j^*_L \) of box \( B^I,J \) with \( DS[i, j] \neq DS'[i, j] \) can be found in constant time each, from the lists \( \Delta^I_{j-1,j} \) and \( \Delta^I_{j-1,j} \) maintained for the preceding row \( i^*_T - 1 = i^*_B - 1 \) and preceding column \( j^*_L - 1 = j^*_R - 1 \), respectively.

We process column indices \( \Delta^I_{j-1,j} \) in increasing order, and suppose that we are currently processing column index \( \hat{j} \in \Delta^I_{j-1,j} \) in the bottom row \( i^*_B - 1 \) of the preceding box \( B^I_{j-1,j} \). According to the above arguments, this will tell us the cells \( (i^*_T, j) \) in the top row \( i^*_T \) of \( B^I,J \) that need to be updated (i.e., \( DS[i^*_T, j] \neq DS'[i^*_T, j] \)). We assume that, for any \( j' \) with \( j^*_L \leq j' < \hat{j} \), the value of \( DS'[i^*_T, j'] \) has already been computed. Also, we have maintained a partial list for \( \Delta^I_{j-1,j} \) where the last element of this partial list stores the largest \( j'' \) such that \( j^*_L \leq j'' < \hat{j} \) and \( DS[i^*_T, j''] \neq DS'[i^*_T, j''], \) together with the value of \( DS'[i^*_T, j''] \). Now it follows from Lemma 2 that both \( DS'[i^*_T, \hat{j}] \) and \( DS'[i^*_T, \hat{j} - 1] \) can be respectively computed in constant time from \( DS'[i^*_T - 1, \hat{j}] \) and \( DS'[i^*_T, \hat{j} - 1] \).
and thus we can check whether $DS[\hat{i}_T, \hat{j}] \neq DS'[\hat{i}_T, \hat{j}]$ in constant time as well. In case $DS[\hat{i}_T, \hat{j}] \neq DS'[\hat{i}_T, \hat{j}]$, we append $\hat{j}$ to the partial list for $\Delta_{I, J}^{L, J}$. By the definition of $DS$, we have $D'[\hat{i}_T, \hat{j}] = D'[\hat{i}_T - 1, \hat{j}] - DS'[\hat{i}_T, \hat{j}].U$. Since $D'[\hat{i}_T - 1, \hat{j}] = D'[\hat{i}_T - 1, \hat{j}]$ is stored with the current column index $\hat{j}$ in the list $\Delta_B^{I, J}$, $D'[\hat{i}_T, \hat{j}]$ can also be computed in constant time.

Suppose we have processed cell $(\hat{i}_T, \hat{j})$. We perform the same procedure as above for the right-neighbor cells $(\hat{i}_T, \hat{j} + p)$ with $p = 1$ and increasing $p$, until encountering the first cell $(\hat{i}_T, \hat{j} + p)$ such that (1) $DS[\hat{i}_T, \hat{j} + p] = DS'[\hat{i}_T, \hat{j} + p]$, (2) $\hat{j} + p \in \Delta_B^{I, J}$, or (3) $\hat{j} + p = j_R^I + 1$. In cases (1) and (2), we move on to the next element of in $\Delta_B^{I, J}$, and perform the same procedure as above. We are done when we encounter case (3) or $\Delta_B^{I, J}$ becomes empty. The total number of cells $(\hat{i}_T, \hat{j} + p)$ for all boxes in $DS'$ is bounded by $\#_{\text{chg}}$.

In a similar way, we process row indices $\Delta_R^{I, J - 1}$ in increasing order, update the cells on the left column $j_R^I$, and maintain another partial list for $\Delta_L^{I, J}$.

**Updating cells on row $i_B^I$ and column $j_R^I$.** Let us consider how to detect the cells on the bottom row $i_B^I$ and the cells on the right column $j_R^I$ of box $B^{I, J}$ that need to be updated, and how to update them.

The next lemma shows monotonicity on the values of $D$ inside each $B^{I, J}$.

**Lemma 4**\footnote{[3]} \ For any $(i, j)$ with $1 \leq i \leq m$ and $j_L^I < j \leq j_R^I$, $D[i, j] \geq D[i, j - 1]$. For any $(i, j)$ with $i_L^I < i \leq i_B^I$ and $1 \leq j \leq n$, $D[i, j] \geq D[i - 1, j]$.

The next corollary is immediate from Lemma 4.

**Corollary 1** \ For any cell $(i, j)$ with $1 \leq i \leq m$ and $j_L^I < j \leq j_R^I$, $DR[i, j].L \geq 0$. Also, for any cell $(i, j)$ with $i_L^I < i \leq i_B^I$ and $1 \leq j \leq n$, $DR[i, j].U \geq 0$.

Now we obtain the next lemma, which is a key to our algorithm.

**Lemma 5** \ For any cell $(i, j)$ with $i_L^I + 1 < i \leq i_B^I$ and $j_L^I + 1 < j \leq j_R^I$, $DR[i, j] = DR[i - 1, j - 1]$.

**Proof.** By Corollary 1, $DR[i - 1, j].L \geq 0$ and $DR[i, j - 1].U \geq 0$ for $i_L^I + 1 < i \leq i_B^I$ and $j_L^I + 1 < j \leq j_R^I$. Thus clearly $\min\{DR[i - 1, j].L, DR[i, j - 1].U, 0\} = 0$. Therefore, for the value of $z$ in Lemma 2, we have $z = (a_i - b_j)^2$, which leads to

$$
\begin{align*}
DR[i, j].U &= (a_i - b_j)^2 - DR[i - 1, j].L \\
DR[i, j].L &= (a_i - b_j)^2 - DR[i, j - 1].U
\end{align*}
$$

By applying equation \footnote{[3]} to the $DR[i - 1, j].L$ term of equation \footnote{[2]}, we get

$$
DR[i, j].U = (a_i - b_j)^2 - ((a_{i-1} - b_j)^2 - DR[i - 1, j - 1].U).
$$

Recall that $a_i = a_{i-1}$, since we are considering cells in the same box $B^{I, J}$. Thus $DR[i, j].U = DR[i - 1, j - 1].U$. By applying equation \footnote{[2]} to the $DR[i, j - 1].U$ term of equation \footnote{[3]}, we similarly obtain $DR[i, j].L = DR[i - 1, j - 1].L$. \qed
Figure 3: Diagonal propagation of $DR[i, j] \neq DR'[i, j]$ inside box $B^{i, j}$.

For any $i_T + 1 < i \leq i_B$ and $j_L + 1 < j \leq j'_R$, let $\ell$ be the smallest positive integer that satisfies $i - \ell = i_T + 1$ or $j - \ell = j'_L + 1$. By Lemma 5 for any cell $(i, j)$ on the bottom row $i_B$ or on the right column $j'_R$, we have $DS[i, j] = DR[i - \ell, j - \ell]$ and $DS'[i, j] = DR'[i - \ell, j - \ell]$. This means that $DS[i, j] \neq DS'[i, j]$ iff $DR[i - \ell, j - \ell] \neq DR'[i - \ell, j - \ell]$. Thus, finding cells $(i, j)$ with $DS[i, j] \neq DS'[i, j]$ on the bottom row $i_B$ or on the right column $j'_R$ reduces to finding cells $(i', j')$ with $DR[i', j'] \neq DR'[i', j']$ on the row $i_T + 1$ or on the column $j'_L + 1$. See Fig. 3.

We have shown how to compute $\Delta^{i, j}_T$ for the top row $i_T$ and $\Delta^{i, j}_L$ for the left column $j'_L$. We here explain how to use $\Delta^{i, j}_T$ (we can use $\Delta^{i, j}_L$ in a symmetric manner). We process column indices in $\Delta^{i, j}_T$ in increasing order, and suppose that we are currently processing column index $\hat{j} \in \Delta^{i, j}_T$ in the top row $i_T$ of the current box $B^{i, j}$. We check whether $DR[i_T + 1, \hat{j}] \neq DR'[i_T + 1, \hat{j}]$. For this sake, we need to know the values of $DR[i_T + 1, \hat{j}]$ and $DR'[i_T + 1, \hat{j}]$. Recall that, by Lemma 5, $DR[i_T + 1, \hat{j}]$ is equal to $DR[i_T + 1 + h, \hat{j} + h]$ (= $DS[i_T + 1 + h, \hat{j} + h]$) on the bottom row $i_B$ (if $i_T + 1 + h = i_B$) or on the right column $j'_R$ (if $\hat{j} + h = j'_R$), where $h > 0$. Since the cell $(i_T + 1 + h, \hat{j} + h)$ can be retrieved in constant time by the diagonal link from the cell $(i_T, \hat{j} - 1)$ on the top row $i_T$, we can compute $DR[i_T + 1, \hat{j}]$ in constant time, applying Lemma 5 to the upper-left direction.

Computing $DR'[i_T + 1, \hat{j}]$ is more involved. By Lemma 2, we can compute $DR'[i_T + 1, \hat{j}]$ from $DR'[i_T, \hat{j}], L$ and $DR'[i_T + 1, \hat{j} - 1], U$. Since $(i_T, \hat{j})$ is on the top row $i_T$, $DR'[i_T, \hat{j}], L = DS'[i_T, \hat{j}], L$ has already been computed. Consider to compute $DR'[i_T + 1, \hat{j} - 1], U$. Since $DR'[i_T + 1, \hat{j} - 1], U = D'[i_T + 1, \hat{j} - 1] - D'[i_T, \hat{j} - 1]$, it suffices to compute $D'[i_T, \hat{j} - 1]$ and $D'[i_T + 1, \hat{j} - 1]$. By definition, $D'[i_T, \hat{j} - 1] = D'[i_T, \hat{j}] - DR'[i_T, \hat{j}], L$. Since $\hat{j} \in \Delta^{i, j}_T$, we can retrieve the value of $D'[i_T, \hat{j}]$ from the current element of the list $\Delta^{i, j}_T$, in $O(1)$ time. Since $DR'[i_T, \hat{j}], L = DS'[i_T, \hat{j}], L$, we can compute $D'[i_T, \hat{j} - 1]$ in $O(1)$ time.

What remains is how to compute $D'[i_T + 1, \hat{j} - 1]$. We use the next lemma.

**Lemma 6** For any cell $(i, j)$ with $i_T + 1 < i \leq i_B$ and $j_L + 1 < j \leq j'_R$, let $s = j - j'_L$ and $t = i - i_T$. Then,

$$D[i, j] = D[i_T + \max\{t - s, 0\}, j'_L + \max\{s - t, 0\}] + \min\{s, t\} \cdot (a_i - b_j)^2.$$  

**Proof.** Consider the case where $s > t$. By applying Lemma 4 to recurrence (1), we obtain $D[i, j] = D[i - 1, j - 1] + (a_i - b_j)^2$. Since $a_i = a_i$ and $b_j = b_j'$ for $i_T < i'$ and...
Since $j' < j'$, by repeatedly applying Lemma 4 to the above equation, we get $D[i, j] = D[i, j'] + (s - t) + t \cdot (a_i - b_j)^2$. See also Fig. 4. The case $s \leq t$ is similar and we obtain $D[i, j] = D[i, j] + (t - s) + s \cdot (a_i - b_j)^2$. By merging the two equations for $s > t$ and $s \leq t$, we obtain the desired equation.

Let $k = \hat{j} - j'$. Since $j' + 1 < \hat{j}$, $k \geq 2$. Since $s = \hat{j} - 1 - j' = k - 1$, $t = i_T + 1 - i_T = 1$, and $k \geq 2$, we get $s \geq t$. Thus it follows from Lemma 6 that $D'[i_T + 1, \hat{j} - 2] = D'[i_T + 1, \hat{j} - 1] + (A[i_T] - B[j])^2 = D'[i_T, \hat{j} - 2] + (A[i_T] - B[j])^2$. Since the value $D'[i_T, \hat{j} - 1]$ is already computed and stored in the corresponding element of $D_T^{i_T, \hat{j} - 1}$, we can compute, in $O(1)$ time, $D'[i_T, \hat{j} - 2]$ by $D'[i_T, \hat{j} - 2] = D'[i_T, \hat{j} - 1] = D'[i_T, \hat{j} - 2] + (A[i_T] - B[j])^2$. Thus, we can determine in $O(1)$ time whether $D'[i_T, \hat{j} - 1] \neq D'[i_T, \hat{j} - 1]$, and hence whether $D'[i_T + 1 + h, \hat{j} + h] \neq D'[i_T + 1 + h, \hat{j} + h]$.

Suppose $D'[i_T + 1 + h, \hat{j} + h] \neq D'[i_T + 1 + h, \hat{j} + h]$. This can be computed in constant time using Lemma 6 by $D'[i_T + 1 + h, \hat{j} + h] = D'[i_T + 1 + h, \hat{j} + h] + (A[i_T] - B[j])^2$, where $D'[i_T + 1 + h, \hat{j} + h] = D'[i_T + 1 + h, \hat{j} + h] + (A[i_T] - B[j])^2$. We add the column index $\hat{j} + h$ to list $\Delta^l_{i_T, \hat{j} + h}$ if $\Delta^l_{i_T, \hat{j} + h} + 1 + h = i_T$, and/or add the row index $i_T + 1 + h$ to list $\Delta^l_{i_T, \hat{j} + h}$ if $\Delta^l_{i_T, \hat{j} + h} + 1 + h = i_T$, together with the value of $D'[i_T + 1 + h, \hat{j} + h]$.

The above process of computing $D'[i_T + 1 + h, \hat{j} + h]$ is illustrated in Fig. 4 in Appendix. Suppose we have processed cell $(i_T + 1, \hat{j} + h)$. We perform the same procedure as above for the right-neighbor cells $(i_T + 1, \hat{j} + q)$ with $q = 1$ and increasing $q$, until encountering the first cell $(i_T + 1, \hat{j} + q)$ such that (1) $DR[i_T + 1, \hat{j} + q] = DR'[i_T + 1 + h, \hat{j} + q]$, (2) $\hat{j} + q \in \Delta^l_{i_T, \hat{j} + h}$, or (3) $\hat{j} + q = j'_{R_i} + 1$. In cases (1) and (2), we remove $\hat{j}$ from $\Delta^l_{i_T, \hat{j} + h}$ and move to the next element of $\Delta^l_{i_T, \hat{j} + h}$. We are done when we encounter case (3) or $\Delta^l_{i_T, \hat{j} + h}$ becomes empty. By Lemma 5 the total number of cells $(i_T + 1, \hat{j} + q)$ for all boxes in $DS'$ is $O(#_{chg})$.

**Batched updates.** It immediately follows from the above arguments that our algorithm can efficiently support batched updates for insertion, deletion, substitution of a run of characters. Namely,

**Theorem 2** Let $B'$ be the string after a run-wise edit operation on $B$, and let $m' = |B'|$. $DS$ can be updated to $DS'$ in $O(\max\{m, m'\} + n + #_{chg})$ time where $#_{chg}$ denotes the number of cells where the values differ between $DS$ and $DS'$.

Since $m'$ is the length of the string $|B'|$ after modification, $#_{chg}$ in Theorem 2 is bounded by $O(nM + m'N)$. Thus, we can perform a batched run-wise update on our sparse table $DS$ in worst-case $O(\max\{m, m'\} + n + #_{chg}) \subseteq O(m + nM + m'N)$ time. Let $k$ be the total number of characters that are involved in a run-wise batched edit operation from $B$ to $B'$ (namely, a run of $k$ characters is inserted, a run of $k$ characters is deleted, or a run of $k_1$ characters is substituted for a run of $k_2$ characters with $k = k_1 + k_2$). Then a naive $k$-time applications of Theorem 1 to the run-wise batched edit operation requires $O(k(m + n + #_{chg})) \subseteq O(k(mN + nM))$ time. Since $m' \leq m + k$, the batched update of Theorem 2 is faster than the naive method by at most a factor of $k$. We also remark that our batched update algorithm is more efficient than building the sparse DP table of Froese et al.’s algorithm 3 using $\Theta(mN + \max\{m, m'\})$ time and space.
Figure 5: Comparisons of the values of \#chg and the sizes of the sparse table DS on two randomly generated strings A and B. Left: With fixed RLE size \( N = M = 50 \) and varying lengths \( n = m \) from 50 to 500 (horizontal axis). Right: With fixed length \( n = m = 500 \) and varying RLE sizes \( N = M \) from 10 to 500 (horizontal axis).

### 3.2 Evaluation of \#chg

As was proven previously, our D²TW algorithm works in \( O(m + n + \#chg) \) time per edit operation on one of the strings. In this section, we analyze how large the \#chg would be in theory and practice.

Although \#chg = \( \Theta(mN + nM) \) in the worst case for some strings (Theorem 3), our preliminary experiments shown below suggest that \#chg can be much smaller than \( mN + nM \) in many cases.

**Theorem 3** Consider strings \( A = A_{1}^{k} \cdots A_{M}^{k} \) and \( B = B_{1}^{l} \cdots B_{N}^{l} \) of RLE sizes \( M \) and \( N \), respectively, where \( |A| = m = kM \) and \( |B| = n = lN \). We assume lexicographical orders of characters as \( A_{I-1} > A_{I} \) for \( 1 < I \leq M \), \( B_{J-1} < B_{J} \) for \( 1 < J \leq N \), and \( A_{M} > B_{N} \). If we delete \( B[1] \) from \( B \), then \#chg = \( \Omega(mN + nM) \).

We have also conducted preliminary experiments to estimate practical values of \#chg, using randomly generated strings. For simplicity, we set \( m = n \) and \( M = N \) for all experiments. We fixed the alphabet size \( |\Sigma| = 26 \) throughout our experiments. In the first experiment, we fixed the RLE size \( M = N = 50 \), randomly generated two strings \( A \) and \( B \) of varying lengths \( m = n \) from 50 to 500, and compared the values of \#chg and the sizes of DS. For each \( m \), we randomly generated 50 pairs of strings \( A \) and \( B \) of length \( m \) each, and took the average values for \#chg and the sizes of DS when \( B[1] \) was deleted from \( B \). In the second experiment, we fixed the string length \( m = n = 500 \) and randomly generated two strings \( A \) and \( B \) of varying RLE sizes \( M = N \) from 10 to 500. For each \( M \), we randomly generated 50 pairs of strings \( A \) and \( B \) of RLE size \( M \), and took the average values for \#chg and the sizes of DS when \( B[1] \) was deleted from \( B \). The results are shown in Fig. 5. In both experiments, \#chg is much smaller than the size of DS. It is noteworthy that even when the values of \( M (= N) \) and \( m (= n) \) are close, the value of \#chg stayed very small. This suggests that our algorithm can be fast also on strings that are not RLE-compressible.
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A Appendix: Omitted proofs

A.1 Proof for Theorem 3

To prove Theorem 3, we establish the following lemma.

Lemma 7 Let $D$ be the dynamic programming table for the above strings $A$ and $B$. Then,

$$\min\{D[i-1,j], D[i, j-1], D[i-1, j-1]\} = \begin{cases} D[i, j-1] & \text{if } i < j, \\ D[i-1, j] & \text{if } i > j, \\ D[i-1, j-1] & \text{if } i = j. \end{cases}$$

Proof. By recurrence (1), the argument holds for any cells $(1, j)$ and $(i, 1)$, where $1 \leq i \leq m$ and $1 \leq j \leq n$. For any cell $(i, j)$ with $i > 1$ or $j > 1$, suppose that the argument holds for any $(i', j')$ with $i' < i$ and $j' < j$. We consider the five following cases:

1. Case $i < j - 1$: For the cell $(i-1, j)$, it follows from the inductive hypothesis and recurrence (1) that $D[i-1, j] = D[i-1, j-1] + (a_{i-1} - b_j)^2$. Since $(a_{i-1} - b_j)^2 \geq 0$, $D[i-1, j] \geq D[i-1, j-1]$. Similarly, for the cells $(i-1, j-1)$ and $(i, j-1)$, $D[i-1, j-1] = D[i-1, j-2] + (a_{i-1} - b_{j-1})^2$ and $D[i, j-1] = D[i-2, j] + (a_i - b_{j-1})^2$. Since $a_{i-1} \geq a_i$, $a_{i-1} - b_{j-1} \geq a_i - b_{j-1}$. Because $a_i - b_{j-1} > 0$, it holds that $(a_{i-1} - b_{j-1})^2 \geq (a_i - b_{j-1})^2$. By the inductive hypothesis, $D[i-1, j-2] \geq D[i, j-2]$. Thus we have $D[i-1, j-2] + (a_{i-1} - b_{j-1})^2 \geq D[i, j-2] + (a_i - b_{j-1})^2$, which implies $D[i-1, j-1] \geq D[i, j-1]$.

2. Case $i = j - 1$: Analogously to Case 1, we get $D[i-1, j] \geq D[i-1, j-1]$. For the cells $(i-1, j-1)$ and $(i, j-1)$, it follows from the inductive hypothesis and recurrence (1) that $D[i-1, j-1] = D[i-1, j-2] + (a_{i-1} - b_{j-1})^2$ and $D[i, j-1] = D[i-1, j-2] + (a_{i-1} - b_{j-1})^2$. Since $(a_{i-1} - b_{j-1})^2 \geq (a_i - b_{j-1})^2$, $D[i-1, j-2] + (a_{i-1} - b_{j-1})^2 \geq D[i-1, j-2] + (a_i - b_{j-1})^2$, which implies $D[i-1, j-1] \geq D[i, j-1]$.

3. Case $i - 1 > j$: By symmetric arguments to Case 1.

4. Case $i - 1 = j$: By symmetric arguments to Case 2.

5. Case $i = j$: For the cells $(i-1, j)$ and $(i, j-1)$, by the inductive hypothesis $\min\{D[i-2, j], D[i-1, j-1], D[i-1, j]\} = D[i-1, j-1]$ and $\min\{D[i-1, j-1], D[i, j-2], D[i, j-1]\} = D[i-1, j-1]$. Thus $D[i-1, j] \geq D[i-1, j-1]$ and $D[i, j-1] \geq D[i-1, j-1]$.

We are ready to prove Theorem 3.

Proof. For simplicity, we assume that the column indices of $D$ begin with 0. In the grid graph $G_{m,n}$ over $D$, we assign a weight $(a_i - b_j)^2$ to each in-coming edge of cell $(i, j)$. We also consider the grid graph $G_{m-1,n}$ over $D'$ obtained by removing $(1,0), \ldots, (m,0)$ from $G_{m,n}$.

Consider a cell $(i, j)$ with $1 < i < j$ that is on the top row of some box $B^{i,j}$, namely $i = i_T = Ik + 1$ for some $1 \leq I \leq M - 1$. By Lemma 7, $p_1 = (1,0), \ldots, (i, i-1), \ldots, (i, j)$
is the minimum weight path from \((1,0)\) to \((i,j)\). Similarly, \(p_1' = (1,1), \ldots, (i,i), \ldots, (i,j)\) is the minimum weight path from \((1,1)\) to \((i,j)\).

For the cell \((i-1,j)\) that is the upper neighbor of \((i,j)\) and is on the bottom row of box \(B^{i-1,j}\), by analogous arguments to the above, \(p_2 = (1,0), \ldots, (i-1,i-2), \ldots, (i-1,j)\) is the minimum weight path from \((1,0)\) to \((i-1,j)\), and \(p_2' = (1,1), \ldots, (i-1,i-1), \ldots, (i-1,j)\) is the minimum weight path from \((1,1)\) to \((i-1,j)\).

Let \(p_3\) be the sub-path of \(p_1\) and \(p_2\) ending at \((i-1,i-2)\), \(p_4\) the sub-path of \(p_1'\) and \(p_2'\) ending at \((i-1,i-1)\), \(p_5\) the sub-path of \(p_1\) and \(p_1'\) from \((i,i)\) to \((i,j)\), and \(p_6\) the sub-path of \(p_2\) and \(p_2'\) from \((i-1,i-1)\) to \((i-1,j)\). Let \(e_1\) be the edge from \((i-1,i-2)\) to \((i,i-1)\), \(e_2\) be the edge from \((i,i-1)\) to \((i,i)\), \(e_3\) be the edge from \((i-1,i-2)\) to \((i-1,i-1)\), and \(e_4\) be the edge from \((i-1,i-1)\) to \((i,i)\). See Figure 6 that depicts these paths and edges.

![Figure 6: Minimum weight paths to \((i,j)\) and \((i-1,j)\) over \(D\) and \(D'\)](image)

For any path \(p\) in \(G_{m,n}\), let \(w(p)\) denote the total weights of edges in \(p\). Now we have \(w(p_1) = w(p_3) + w(e_1) + w(e_2) + w(p_5)\) and \(w(p_1') = w(p_4) + w(e_4) + w(p_5)\). By the definition of DTW, \(D[i,j]\) stores the cost of the minimum weight path from \((1,0)\) to \((i,j)\), and \(D'[i,j]\) stores the cost of the minimum weight path from \((1,1)\) to \((i,j)\). Thus \(D[i,j] = w(p_3) + w(e_1) + w(e_2) + w(p_5)\) and \(D'[i,j] = w(p_4) + w(e_4) + w(p_5)\). Similarly, \(D[i-1,j] = w(p_3) + w(e_3) + w(p_6)\) and \(D'[i-1,j] = w(p_4) + w(p_6)\).

Now \(DS[i,j], U = D[i-1,j] - D[i,j] = w(e_3) + w(p_6) - (w(e_1) + w(e_2) + w(p_5))\), and \(DS'[i,j], U = D'[i-1,j] - D'[i,j] = w(p_6) - (w(e_4) + w(p_5))\). This leads to \(DS[i,j], U = DS'[i,j], U = w(e_3) + w(e_4) - (w(e_1) + w(e_2))\). Recall that \(w(e_2) = w(e_4) = (A[i] - B[j])^2\). Thus \(DS[i,j], U = DS'[i,j], U = w(e_3) - w(e_1)\). Also, because \(A[i] \neq A[i-1], (A[i] - B[i-1])^2 = w(e_1) \neq w(e_3) = (A[i] - B[i-1])^2\). Consequently, \(DS[i,j], U = DS'[i,j], U \neq 0\).

Therefore, for any cell \((i,j)\) with \(1 < i < j\) that lies on the top row of any character in \(A\), \(DS[i,j] \neq DS'[i,j]\). Since each top row \(i^T = Ik + 1\) \((1 \leq I \leq M - 1)\) contains \(n - (Ik + 1)\) such cells, and since \(n = kM\), there are \(\sum_{I=1}^{M-1} (n - (Ik + 1)) = n(M - 1) - k((M - 1)M)/2 - (M - 1) = (n - 2)(M - 1)/2\) such cells for top rows \(J = 1, \ldots, M - 1\). Symmetric arguments show that there are \(\sum_{J=1}^{N-1} (m - (Jl + 1)) = (m - 2)(N - 1)/2\) cells with \(DS[i,j] \neq DS'[i,j]\) for left rows \(J = 1, \ldots, N - 1\). Thus, \(\#_{\text{adj}}^{\text{c}} \geq ((n - 2)(M - 1) + (m - 2)(N - 1))/2 \in \Omega(mN + nM)\). \(\square\)

A.2 Proof for Lemma 1

**Proof.** The lemma can be shown in a similar manner to the proof for Theorem 3 above. In so doing, we set \(M = n\) and \(N = n\) in the strings \(A\) and \(B\) of Section 3.2, and consider strings \(A\) and \(B\) such that \(A[i-1] > A[i]\) for \(1 < i \leq m\) and \(B[j-1] < B[j]\) for \(1 < j \leq n\), and
$A[m] > B[n]$. Since deleting the leftmost character $B[1]$ of $B$ is symmetric to appending a new character $b_1$ to $B$ such that $b_1 < B[1]$, we get $\Omega(mn)$ lower bound for the number of cells where $D[i, j] \neq D'[i, j]$ per appended character. If we repeat this by recursively appending $k$ new characters $b_i$ such that $b_i < b_{i-1} < \cdots < B[1]$ for $i = 2, \ldots, k$, we get $\Omega(mn)$ lower bound for the number of cells where $D[i, j] \neq D'[i, j]$ for each $b_i$. Hence there are a total of $\Theta(kmn)$ cells in $D'$ that differ from the corresponding cells in $D$, for $k$ edit operations on $B$. \qed
B Appendix: Supplemental figures

Figure 7: In the worst case, the values of $\Theta(mn)$ cells of the DP table for $\text{dtw}(A, B)$ can change after editing $B$.

Figure 8: Case (a) of the initial step. The dashed arcs are the old diagonal links in $DS$, and the sold arcs are the modified diagonal links in $DS'$. The gray cells depict cells $(i_I^T, j_J^R)$ and $(i_B^T, j_R^R)$.

Figure 9: Case (c) of the initial step, where character substitution has been performed at position $j^*$. The dashed arcs are the old diagonal links in $DS$ from row $i_T^I$ up to $j^*$, and the sold arcs are the modified diagonal links from new column $j^*$ in $DS'$. 
Figure 10: Illustration of the process for computing $DR'[i_{T}^{l} + 1, \hat{j}]$, which is initially unknown. The gray cells show those for which both values of $D'$ and $DR'$ are unknown, the vertically striped cells show those for which only the value of $D'$ is known, the horizontally striped cells show those for which only the value of $DR'$ is known, and the white cells show those for which both values of $D'$ and $DR'$ are known. At the final step (lower-right), the cell $(i_{T}^{l} + 1, \hat{j})$ is horizontally striped, meaning that we have computed the desired value $DR'[i_{T}^{l} + 1, \hat{j}]$. 