Bifurcations in the time-delayed Kuramoto model of coupled oscillators: Exact results

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In the context of the Kuramoto model of coupled oscillators with distributed frequencies interacting via a time-delayed mean-field, we derive as a function of the delay exact results for stability boundary between incoherent and synchronized states and the nature in which the latter bifurcates from the former. Our results are based on an unstable manifold expansion in the vicinity of the bifurcation applied to both the kinetic equation for the distribution function for a generic frequency distribution and the Ott-Antonsen(OA)-reduced dynamics for a Lorentzian distribution. Besides elucidating the effects of delay on bifurcation, we show that the Ott-Antonsen-approach, although an ansatz, gives an amplitude dynamics of the unstable modes close to bifurcation that remarkably coincides with the one derived from the kinetic equation. Interestingly, close to the bifurcation, the unstable manifold derived from the kinetic equation has the same form as the OA manifold.

The Kuramoto model enjoys a unique status in nonlinear sciences [1,6], providing arguably the minimal model for spontaneous synchronization commonly observed in nature [7-8], e.g., in fireflies [9], cardiac pacemaker cells [10], electrochemical [11] and electronic [12] oscillators, Josephson junction arrays [13] and power-grids [14]. The model comprises $N$ limit-cycle oscillators of distributed natural frequencies that are globally coupled through the sine of the instantaneous phase differences between them [1,13,7], and has been very successful in explaining synchrony in diverse dynamical setups, showing specifically as $N \rightarrow \infty$ a spontaneous transition from incoherence to synchrony for coupling strength larger than a critical value $\Gamma$. Nevertheless, one encounters situations where due to time delay in signal propagation between the interacting units, the evolution of the dynamical variables depends both on their instantaneous as well as past values. Time delay has important consequences, e.g., for synchronization of biological clocks [14] and digital phase-locked loops [16], and also in information propagation through neural networks [17].

To assess the effects of time delay, a pioneering work within the ambit of the Kuramoto model was its generalization by Yeung and Strogatz [15] to have a time-delayed global coupling between the oscillators: in terms of delay $\tau > 0$, the phase $\theta_j$ of the $j$-th oscillator with natural frequency $\omega_j$ evolves as

$$\frac{d\theta_j(t)}{dt} = \omega_j + \frac{K}{N} \sum_{k=1}^{N} \sin[\theta_k(t-\tau) - \theta_j(t) - \alpha], \quad (1)$$

with $K > 0$ the coupling and $\alpha \in (-\pi/2,\pi/2]$ denoting phase frustration [19]; $\alpha = \tau = 0$ recovers the Kuramoto model. The set $\{\omega_j\}$ denote quenched disordered random variables sampled independently from distribution $G(\omega)$: $\int_{-\infty}^{\infty} d\omega \ G(\omega) = 1$. The phase coherence among the oscillators is measured by $r(t) \equiv (1/N) \sum_{j=1}^{N} e^{i\theta_j(t)}$, with a fully synchronized and an incoherent state corresponding respectively to $|r| = 1$ and $|r| = 0$.

A usual choice for $G(\omega)$ is to have it symmetric about its center $\omega_0$ ($\omega_0$ is the mean when the latter exists). Then, unlike the Kuramoto model, (1) is not invariant under $\theta_j(t) \rightarrow \theta_j(t) - \omega_0 t$, $\omega_j \rightarrow \omega_j - \omega_0 \forall j$, corresponding to viewing the dynamics in a frame rotating uniformly at frequency $\omega_0$ in an inertial frame; viewing in such a frame requires replacing $\alpha$ with $\alpha' \equiv \alpha - \omega_0 \tau$, so that Eq. (1) in terms of $r(t)$ reads

$$\frac{dr_j(t)}{dt} = \omega_j + K \text{Im} \left[ r(t-\tau) e^{-i(\theta_j(t)+\alpha' - \omega_0 \tau)} \right], \quad (2)$$

where the $\omega_j$’s are now distributed according to a distribution $g(\omega)$ that is centered at zero: $g(\omega) \equiv G(\omega - \omega_0)$.

We may anticipate that introducing delay in the Kuramoto model leads to a richer and complex dynamical scenario more challenging to analyze. Indeed, Ref. [18] unraveled a range of new phenomena including bistability between synchronized and incoherent states and unsteady solutions with time-dependent order parameters that do not occur in the original model. For identical $\omega_j$’s and $\alpha = 0$, Yeung and Strogatz derived exact formulas for stability boundaries between the incoherent and synchronized states. For the general case of distributed $\omega_j$’s and $\alpha = 0$, they adduced only numerical results for a Lorentzian frequency distribution to suggest bifurcation of the incoherent state as a function of $K$ that could be either sub- or supercritical depending on $\tau$. Reference [19] obtained the regions of parameter space corresponding to synchronized and incoherent solutions, but for particular frequency distributions. The complex dynamical scenario did not allow a straightforward analytical treatment to answer the following obvious questions for generic frequency distributions: What is the critical value of $K$ at which the incoherent state loses its stability? Can one predict analytically as a function of $\tau$ the nature of bifurcation of the incoherent state? What are the effects of the phase frustration parameter $\alpha’$? Starting from the
the number of oscillators with a given $\omega$, $F(\theta, \omega, t)$ evolves according to a kinetic equation given by the continuity equation [25]:

\[
\frac{\partial F(\theta, \omega, t)}{\partial t} + \omega \frac{\partial F(\theta, \omega, t)}{\partial \theta} + K \frac{\partial}{\partial \theta} \left( \left[ r[F](t - \tau) e^{-i(\theta + \alpha - \omega \tau)} - c.c. \right] F(\theta, \omega, t) \right) = 0.
\]

(3)

Here, c.c. stands for complex conjugation, and we have defined as functions of $F$ the quantity $r[F](t) \equiv \int d\theta d\omega \ e^{i\theta} F(\theta, \omega, t)$. Writing $F(t)$ for $F(\theta, \omega, t)$ and using the formalism of Delay Differential Equation (DDE) [25 27] to define $F_t(\varphi) \equiv F(t + \varphi)$ for $-\tau \leq \varphi \leq 0$, we show in the Supplemental Material (SM) [28] that the kinetic equation is a DDE $\partial F_t(\varphi)/\partial t = (\mathcal{A} F_t)(\varphi)$.

The incoherent state $F_{\infty}(\varphi) = g(\omega)/(2\pi)$ is evidently a stationary solution of the DDE. Perturbations $f_t(\varphi)$, defined as $F_t(\varphi) = F_{\infty}(\varphi) + f_t(\varphi)$ with normalization of $F_t$ implying $\int_0^{2\pi} d\theta f_t(\varphi) = 0$, evolve as

\[
\frac{d}{dt} f_t(\varphi) = (\mathcal{A} f_t)(\varphi) = (\mathcal{D} f_t + \mathcal{F}[f_t])(\varphi); \quad -\tau \leq \varphi \leq 0,
\]

(4)

with $\mathcal{D}$ being the part of the evolution operator that is linear in $f_t$ and $\mathcal{F}$ the part that is nonlinear:

\[
(\mathcal{D} f_t)(\varphi) = \begin{cases} \frac{d}{d\varphi} f_t(\varphi); & -\tau \leq \varphi < 0, \\ \mathcal{L} f_t(\varphi) = L f_t(0) + R f_t(-\tau); & \varphi = 0, \\ \mathcal{F}[f_t](\varphi) = \begin{cases} 0; & -\tau \leq \varphi < 0, \\ \mathcal{N}[f_t]; & \varphi = 0. \end{cases} \end{cases}
\]

(5). (6)

Here, we have

\[
L f = -\omega \frac{\partial}{\partial \theta} f, \quad R f = K g(\omega) \left( r[f] e^{-i(\theta + \alpha - \omega \tau)} + c.c. \right),
\]

(7)

\[
\mathcal{N}[f_t] = \frac{K}{2\pi} \frac{\partial}{\partial \theta} \left( \left[ r[f_t](\tau - \tau) e^{-i(\theta + \alpha - \omega \tau)} - c.c. \right] f_t(0) \right).
\]

Using the scalar product [29] $(h_2(\varphi), h_1(\varphi))_\tau = \int_0^\tau d\xi \langle h_2^*(\xi + \tau), h_1(\xi) \rangle$ adapted for delay problems, where $\langle \cdot, \cdot \rangle$ is the $L_2(\mathbb{T} \times \mathbb{R})$ scalar product, the adjoint operator $\mathcal{D}^\dagger$ satisfying $(h_2(\varphi), \mathcal{D} h_1(\varphi))_\tau = (\mathcal{D}^\dagger h_2(\varphi), h_1(\varphi))_\tau$ may be obtained (cf. SM [28]).

To study the linear stability of $F_{\infty}$, we need the eigen-spectrum of $\mathcal{D}$ and $\mathcal{D}^\dagger$. Besides a continuous spectrum on the imaginary axis (typical of kinetic equations of our type [24 30 31]), $\mathcal{D}$ has (cf. SM [28]) discrete eigenvalues $\lambda$ and $\Lambda^*$ and corresponding eigen-functions $p(\varphi) = \psi_1(\omega) e^{i\theta + \lambda \varphi}$ and $p^\dagger(\varphi)$, with $\psi_1(\omega) = K g(\omega)/(2(\lambda + i\omega)) e^{-\lambda \tau + i(\alpha - \omega \tau)}$ and dispersion relations $\Lambda(\lambda) = \Lambda^*(\lambda*) = 0$. Here, we have

\[
\Lambda(\lambda) = 1 - \frac{K}{2} e^{-\lambda \tau + i(\alpha - \omega \tau)} \int d\omega \ g(\omega) \frac{\lambda + i\omega}{\lambda + i\omega}.
\]

(8)
The eigenfunctions of the adjoint operator \( \mathcal{D}^* \) are \( g(\psi) \propto e^{i\theta - \lambda^* \tau} \) (see SM [28]). When the incoherent stationary state is linearly unstable, the unstable eigenspace is spanned by \( p(\psi) \) and \( p^*(\psi) \). The stationary solution \( F_{st} = g(\omega)/(2\pi) \) will be neutrally stable due to the continuous spectrum generating a dynamics similar to Landau damping [32], provided there are no discrete eigenvalues \( \lambda \). Vanishing of the real part of the eigenvalue with the smallest real part signals criticality above which \( F_{st} \) becomes linearly unstable. Denoting by \( \lambda_i; \lambda \in \mathbb{R} \) the imaginary part of the eigenvalue with the smallest real part, the dispersion relations at criticality give

\[
\cos \delta = \frac{g(-\lambda_i) \pi K_c}{2}; \quad \tan \delta = \frac{\text{PV} \int g(\omega)/(\omega + \lambda_i)}{\pi g(-\lambda_i)},
\]

where PV stands for principal value, and \( \delta \equiv \alpha - (\omega_0 + \lambda_i)\tau \); we require positive \( \cos \delta \) to solve the first equation.

![FIG. 1. Stability region of the incoherent state for \( \alpha = 0 \) for the unimodal Lorentzian (14) with \( \Delta = 0.1, \omega_0 = 3 \) and for the bimodal Lorentzian (15) with \( \Delta = 0.1, \omega_0 = 3, \omega_c = 0.09 \). For \( K > K_c \), the state is unstable. A positive (respectively, a negative) sign of \( \text{Re}(c_3) \) implies a subcritical (respectively, a supercritical) bifurcation as \( K \to K_c^+ \). Consistent simulations for the unimodal Lorentzian were reported in Ref. [18] at \( \tau = 1 \) and \( \tau = 2 \) (vertical dotted lines). Bimodal Lorentzian with \( \Delta/\sqrt{3} < \omega_c < \Delta \) and \( \tau = 0 \) shows subcritical bifurcation [23, 24]; the inset shows (and as verified in Fig. 2) that even a small delay (\( \tau \geq 0.01 \)) makes the bifurcation supercritical.

We want to study the behavior of \( f_1(\psi) \) as \( K \to K_c^+ \), the goal being to uncover the weakly nonlinear dynamics occurring beyond the exponential growth taking place due to the instability as \( K \to K_c^+ \). To this end, we study the behavior of \( f_1(\psi) \) on the unstable manifold, which by definition is tangential to the unstable eigenspace at the equilibrium point \( (K = K_c, \lambda = i\lambda_i) \). The unstable manifold expansion of \( f_1(\psi) \) for \( K > K_c \) reads

\[
f_1(\psi) = A(t)p(\psi) + A^*(t)p^*(\psi) + w[A, A^*](\psi),
\]

with the relations \( (q(\psi), p(\psi))_\tau = 1, (q(\psi), p^*(\psi))_\tau = 0, (q(\psi), w(\psi))_\tau = 0 \) yielding \( A(t) = (q(\psi), f_1(\psi))_\tau \). We require \( w(\psi) \) to be at least quadratic in \( A \). Close to the bifurcation, the order parameter is given by \( r(t) = A^*(t) + O(|A|^2 A^*) \), so that the nature of bifurcation as \( K \to K_c^+ \) is determined by the time evolution of \( A(t) \). We detail in the SM [28] the derivation of this time evolution by developing the unstable manifold \( w[A, A^*] \) to leading order in \( A \). The result is

\[
\dot{A} = \lambda A + c_3|A|^2 A + O(|A|^4 A),
\]

\[
c_3 = -\frac{K^2}{4} \int d\omega \frac{g(\omega)}{(\lambda + i\omega)^3} + i \frac{\tau}{2 K} e^{\lambda \tau - i(\alpha - \omega_0 \tau)} \int d\omega \frac{g''(\omega)}{(\lambda + i\omega) + \omega}
\]

\[
\times \left[ \text{PV} \int d\omega \frac{g'(\omega)}{\lambda_1 + \omega} - \frac{2 \tau}{K_c} \cos \delta - i \left( \pi g''(\lambda_1) - \frac{2 \tau}{K_c} \sin \delta \right) \right]^{-1}
\]

The sign of \( \text{Re}(c_3) \) in the last equation gives the nature of the bifurcation as \( K \to K_c^+ \). Contrary to similar unstable manifold analysis [33–36] \( c_3 \) is not diverging as \( \lambda \to 0^+ + i\lambda_i \), validating formally the asymptotic analysis. Equations (9) and (13) suggest that at bifurcation, the effects of changing \( \tau \) at a fixed \( \alpha \) are the same as those obtained on changing \( \tau \) at a fixed \( \alpha \) such that the combination \( \alpha - \omega_0 \tau \) remains constant.

![FIG. 2. Order parameter \( r_\infty \) vs. coupling constant \( K \) for the bimodal Lorentzian (15) with \( \Delta = 0.1, \omega_0 = 3, \omega_c = 0.09 \), and for two values of \( \tau \). The data are obtained via numerical integration of (2) with \( N = 64384 \) and timestep \( \delta t = 10^{-2} \). For each \( K \), we run a simulation for time \( t = 2600 \) and compute \( r_\infty \) as the average of \( |r(t)| \) for \( t > 1000 \). The end state of run for a given \( K \) is the initial state of run for the next \( K \). We first increase \( K, K \to K + \delta K \), with \( \delta K = 0.1 \) (or 0.05/0.5 close to/for the bifurcation), and then decreases it, \( K \to K - \delta K \). One observes subcritical bifurcation (hysteresis) for \( \tau = 0 \) and supercritical for \( \tau = 0.1 \) (no hysteresis).
For a fixed $\omega_0$ and by varying $\tau$, one may plot the sign of $c_3$ by computing at criticality $K = K_c(\tau)$ and $\lambda_c(\tau) = 0^+ + i\lambda(\tau)$. The predictions for $\alpha = 0$ are shown in Fig. 1 while Fig. 2 shows simulations for a Lorentzian frequency distribution and a sum of two Lorentzians given respectively by

$$g(\omega) = \frac{\Delta}{\pi} \frac{1}{\omega^2 + \Delta^2},$$

$$g(\omega) = \frac{\Delta}{2\pi} \left( \frac{1}{(\omega - \omega_c)^2 + \Delta^2} + \frac{1}{(\omega + \omega_c)^2 + \Delta^2} \right)$$

where $\Delta > 0$ is the half-width-at-half-maximum and $\pm \omega_c$ in the second case the center frequencies of the two Lorentzians (here, $g$ has two separated maxima for $\omega_c > \Delta/\sqrt{3}$). Fig. 2 shows that as predicted in Fig. 1 (inset) via the sign of $\Re(c_3)(\tau)$, a very small delay (here $\tau = 0.1$) can suppress the subcritical bifurcation present with a bimodal distribution in the absence of delay and turn it into a supercritical bifurcation.

For values $\tau = \tau_n$ satisfying $\omega_0\tau_n = 2n\pi$ for $n \in \mathbb{Z}$, delay in Fig. 2 has no effect. In this case, if the eigenvalue triggering the instability of the incoherent state is real (for (15), it corresponds to $|\omega_c| < \Delta$), it will be of multiplicity two, and so our two-dimensional unstable manifold is still valid. However, if there is a pair of complex eigenvalues (for (15), it corresponds to $|\omega_c| > \Delta$), each one will have a multiplicity two, and one should consider a four-dimensional unstable manifold, as done for $\tau_0 = \tau = 0$ in Ref. [24]. For $\tau \neq \tau_n$, there is a pair of complex eigenvalues of multiplicity one, so that our two-dimensional unstable manifold expansion holds good.

We now apply the formalism of the DDE to the Ott-Antonsen (OA)-reduced dynamics. The OA ansatz for the dynamics [2], discussed in Ref. [21], is recalled in the SM [28]. In this approach, one considers an expansion

$$F(\omega, t) = \frac{g(\omega)}{2\pi} \left[ 1 + \sum_{n=1}^{\infty} \left( F_n(\omega, t) e^{int} + \text{c.c.} \right) \right],$$

where one uses for $F_n(\omega, t)$, the $n$-th Fourier coefficient, the ansatz $F_n(\omega, t) = [z(\omega, t)]^n$. Here, $z(\omega, t)$ is an arbitrary function with certain restrictions [28]. Using (1) and Eq. (3), one derives for the Lorentzian (14) the time evolution of $r(t)$ as the DDE

$$\frac{dr(t)}{dt} + \Delta r(t) + \frac{K}{2} \left[ e^{-i(\alpha - \omega_0)\tau} r(t - \tau) r^2(t) - e^{i(\alpha - \omega_0)\tau} r(t - \tau) \right] = 0. $$

However, this exact expression valid for any $K$ is not in a form that describes in a simple manner the bifurcation of the incoherent stationary state $r_{st} = 0$ to synchrony.

With similar construction and analysis (see SM [28] and [37] for a similar approach) as that for the kinetic equation, perturbations $r_1(\varphi)$ about $r_{st} = 0$ can be studied. A conceptual simplification in the DDE (17) is the absence of continuous spectrum. As discussed in the SM [28], eigenfunctions $p(\varphi)$ of the new linear operator $\mathcal{D}$ (and that of its adjoint) can be constructed. The new dispersion relation is $\Lambda(\lambda) = \lambda + \Delta - (K/2)e^{-\lambda \tau + i(\alpha - \omega_0)\tau} = 0$. The solutions in terms of the Lambert-W function $W_1$ are $\lambda_1 = -\Delta + W_1((K \tau/2)e^{\lambda \tau - i(\alpha - \omega_0)\tau})/\tau$. The stationary solution $r_{st} = 0$ will be linearly stable so long as all the eigenvalues $\lambda$ have a real part that is negative. Vanishing of the real part of the eigenvalue with the smallest real part then signals criticality above which $r_{st} = 0$ is no longer linearly stable. Denoting by $\lambda_1$; $\lambda_i \in \mathbb{R}$ the imaginary part of the eigenvalue with the smallest real part, $\Lambda(\lambda) = 0$ gives at criticality ($K_c/2) \cos \delta = \Delta$, ($K_c/2) \sin \delta = \lambda_1$; these results coincide with those given by Eq. (9) for Lorentzian distribution (14). Similar to $f_1(\varphi)$, unstable manifold expansion of $r_1(\varphi)$ for $K > K_c$ is

$$r_1(\varphi) = A(t)p(\varphi) + w[A](\varphi),$$

where $w[A](\varphi)$, which is at least quadratic in $A$ (in fact, one can prove that it is here cubic in $A$), denotes the component of $r_1(\varphi)$ transverse to the unstable eigenspace, so that $(g(\varphi), w(\varphi)) \tau = 0$. On using the latter equation, together with $(g(\varphi), p(\varphi)) \tau = 1$ in Eq. (18), we get $A(t) = (g(\varphi), r_1(\varphi))\tau$. The unstable manifold may be shown to be an attractor of the dynamics for the type of DDE under consideration [28, 27, 35]. In the SM [28], we obtain the time evolution of $A(t)$ as

$$\dot{A} = \Lambda A + c_3|A|^2 A + O(|A|^2 A),$$

$$c_3 = -\frac{K}{2} - \frac{1}{1 + \tau(K/2)} e^{-\lambda_1(\alpha - \omega_0)\tau} + \lambda_1 \tau,$$

which allows to decide the bifurcation behavior of $r_1(\varphi)$ as $K \to K_c^+$. The relevant parameter to study the type of bifurcation is again the sign of $\Re(c_3)$ as the real part of $\lambda$ approaches zero, so that $\lambda = i\lambda_1$ is purely imaginary. For the Lorentzian distribution, (14), one may check that the normal form obtained from the OA-reduced-dynamics, Eq. (19), and the kinetic equation, Eq. (11), are the same.

We come back to the kinetic equation and consider for generic $g(\omega)$ the decomposition $F_i = F_{iA} + f_i$ with Fourier coefficients $(F_i)_k/(2\pi r) = g(\omega)(\omega_k)/(2\pi)$ and $(\omega_k)$’s the Fourier coefficients on the unstable manifold. Using normalization, we get $(\alpha_i)_{i=1,0}$, and from (10), $(\alpha_1)_{i=1,0} = (A[\varphi] + O(|A|^2)]$. Nonlinear analysis gives $(\alpha_1)_2 = (A[\varphi]^2 + O(|A|^2)|A|)$ (see SM [28]). By induction, we have $k \geq 0$ (see SM [28])

$$(\alpha_i)_k = (\alpha_1)_k^i + O(A^k|A|^2),$$

whose complex conjugate gives the result for $k < 0$. These equations show that close to bifurcation, the unstable manifold has exactly the form of the OA manifold, holding for $\tau = 0$ and $\tau \neq 0$. The OA ansatz fails on adding a second harmonic to Eq. (1) (i.e., interaction $\sim K \sin \theta + J \sin(2\theta)$), and (21) is also not valid.
In this case, with $\tau = 0$, the unstable manifold has singularities yet provides valuable information on bifurcation. These singularities relate to the OA-failure. Studying how [21] changes with this modified interaction could act as genesis for investigating OA-generalization.

Here, we analyzed the effects of a time delay in interaction for the Kuramoto model of globally-coupled oscillators with distributed frequencies, for generic choice of the frequency distribution. We derived as a function of the delay exact results for the stability boundary between incoherent and synchronized states and the nature in which the latter bifurcates from the former. We obtained our results in two independent ways, by considering the kinetic equation for the time evolution of the single-oscillator distribution, and by considering for a Lorentzian distribution a reduced equation for the order parameter derived via the Ott-Antonsen ansatz. Fully consistent results derived from the two approaches may have important bearings on their inter-relationship, unraveling which is left for future.

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\[ \text{[1]} \text{ Y. Kuramoto, Chemical oscillations, waves, and turbulence (Springer-Verlag, Berlin, 1984).} \]
\[ \text{[2]} \text{ S. H. Strogatz, From Kuramoto to Crawford: Exploring the onset of synchronization in populations of coupled oscillators, Physica D 143, 1 (2000).} \]
\[ \text{[3]} \text{ J. A. Acebrón, L. L. Bonilla, C. J. Pérez Vicente, F. Ritort and R. Spigler, The Kuramoto model: A simple paradigm for synchronization phenomena, Rev. Mod. Phys. 77, 137 (2005).} \]
\[ \text{[4]} \text{ S. Gupta, A. Campa and S. Ruffo, Kuramoto model of synchronization: Equilibrium and non-equilibrium aspects, J. Stat. Mech.: Theory Exp. R08001 (2014).} \]
\[ \text{[5]} \text{ F. A. Rodrigues, T. K. DM. Peron, P. Ji and J. Kurths, The Kuramoto model in complex networks, Phys. Rep. 610, 1 (2016).} \]
\[ \text{[6]} \text{ S. Gupta, A. Campa and S. Ruffo, Statistical physics of synchronization (Springer-Verlag, Berlin, 2018).} \]
\[ \text{[7]} \text{ A. Pikovsky, M. Rosenblum and J. Kurths, Synchronization: a Universal concept in nonlinear sciences (Cambridge University Press, Cambridge, 2001).} \]
\[ \text{[8]} \text{ S. H. Strogatz, Sync: the emerging science of spontaneous order (Hyperion, New York, 2003).} \]
\[ \text{[9]} \text{ J. Buck, Synchronous rhythmic flashing of fireflies. II., Q. Rev. Biol. 63, 265 (1988).} \]
\[ \text{[10]} \text{ C. S. Peskin, Mathematical aspects of heart physiology (Courant Institute of Mathematical Sciences, New York, 1975).} \]

\[ \text{[11]} \text{ I. Kiss, Y. Zhai and J. Hudson, Emerging coherence in a population of chemical oscillators, Science 296, 1676 (2002).} \]
\[ \text{[12]} \text{ A. A. Temirbayev, Z. Zh. Zhanabaev, S. B. Tarasov, V. I. Ponomarenko and M. Rosenblum, Experiments on oscillator ensembles with global nonlinear coupling, Phys. Rev. E 85, 015204(R) (2012).} \]
\[ \text{[13]} \text{ S. P. Benz and C. J. Burroughs, Coherent emission from twodimensional Josephson junction arrays, Appl. Phys. Lett. 58, 2162 (1991).} \]
\[ \text{[14]} \text{ M. Rohden, A. Sorge, M. Timme and D. Witthaut, Self-Organized synchronization in decentralized power grids, Phys. Rev. Lett. 109, 064101 (2012).} \]
\[ \text{[15]} \text{ L. Herrgen, S. Ares, L. G. Morelli, C. Schröter, F. Jülicher and A. C. Oates, Intercellular coupling regulates the period of the segmentation clock, Curr. Biol. 20, 1244 (2010).} \]
\[ \text{[16]} \text{ L. Wietzel, D. J. Jörg, A. Pollakis, W. Rave, G. Fettweis and F. Jülicher, Self-organized synchronization of digital phase-locked loops with delayed coupling in theory and experiment, PLoS ONE 12, e0171590 (2017).} \]
\[ \text{[17]} \text{ F.-C. Blondeau and G. Chauvet, Stable, oscillatory, and chaotic regimes in the dynamics of small neural networks with delay, Neural Netw. 5, 735 (1992).} \]
\[ \text{[18]} \text{ M. K. S. Yeung and S. H. Strogatz, Time delay in the Kuramoto Model of coupled oscillators, Phys. Rev. Lett. 82, 648 (1999).} \]
\[ \text{[19]} \text{ H. Sakaguchi and Y. Kuramoto, A soluble active rotator model showing phase transitions via mutual entrainment, Prog. Theor. Phys. 76, 576 (1986).} \]
\[ \text{[20]} \text{ E. Montbrió, D. Pazó and J. Schmidt, Time delay in the Kuramoto model with bimodal frequency distribution, Phys. Rev. E 74, 056201 (2006).} \]
\[ \text{[21]} \text{ E. Ott and T. M. Antonsen, Low dimensional behavior of large systems of globally coupled oscillators, Chaos 18, 037113 (2008).} \]
\[ \text{[22]} \text{ E. Ott and T. M. Antonsen, Long time evolution of phase oscillator systems, Chaos 19, 023117 (2009).} \]
\[ \text{[23]} \text{ E. A. Martens, E. Barreto, S. H. Strogatz, E. Ott, P. So and T. M. Antonsen Exact results for the Kuramoto model with a bimodal frequency distribution, Phys. Rev. E 79, 026204 (2009).} \]
\[ \text{[24]} \text{ J. D. Crawford, Amplitude expansions for instabilities in populations of globally-coupled oscillators, J. Stat. Phys. 74, 1047 (1994).} \]
\[ \text{[25]} \text{ T. D. Frank, Kramers-Moyal expansion for stochastic differential equations with single and multiple delays: Applications to financial physics and neurophysics, Phys. Lett. A 360, 552 (2007).} \]
\[ \text{[26]} \text{ J. K. Hale and S. M. V. Lunel, Introduction to functional differential equations (Springer-Verlag, New York, 1993).} \]
\[ \text{[27]} \text{ S. Guo and J. Wu, Bifurcation theory of functional differential equations (Springer-Verlag, New York, 2013).} \]
\[ \text{[28]} \text{ See the Supplemental Material at http://link.aps.org/...} \]
\[ \text{[29]} \text{ J. K. Hale, Linear functional differential equations with constant coefficients, Contributions to Differential Equations 2, 291 (1963).} \]
\[ \text{[30]} \text{ J. D. Crawford and P. D. Hislop, Application of the method of spectral deformation to the Vlasov-poisson system, Annals of Physics 189, 265 (1989).} \]
\[ \text{[31]} \text{ S. H. Strogatz and R. E. Mirollo, Stability of incoherence in a population of coupled oscillators, J. Stat. Phys. 63, 613 (1991).} \]
\[ \text{[32]} \text{ S. H. Strogatz, R. E. Mirollo and P. C. Matthews,} \]
Coupled nonlinear oscillators below the synchronization threshold: relaxation by generalized Landau damping, Phys. Rev. Lett. 68, 2730 (1992).

[33] J. D. Crawford, Scaling and singularities in the entrainment of globally coupled oscillators, Phys. Rev. Lett. 74, 4341 (1995).

[34] J. D. Crawford and K. T. R. Davies, Synchronization of globally coupled phase oscillators: singularities and scaling for general couplings, Physica D 125, 1 (1999).

[35] J. D. Crawford, Amplitude equations for electrostatic waves: universal singular behavior in the limit of weak instability, Physics of Plasmas 2, 97 (1995).

[36] J. Barré and D. Métrivet, Bifurcations and singularities for coupled oscillators with inertia and frustration, Phys. Rev. Lett. 117, 214102 (2016).

[37] B. Niu and Y. Guo, Bifurcation analysis on the globally coupled Kuramoto oscillators with distributed time delays, Physica D 266, 23 (2014).

[38] J. Murdock, Normal forms and unfoldings for local dynamical systems (Springer-Verlag, New York, 2006).

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**Supplemental Material for Bifurcations in the time-delayed Kuramoto model of coupled oscillators:** Exact results

**THEORY OF DELAY DIFFERENTIAL EQUATION**

It is evident from the form of equations (3) and (17) of the main text that both may be cast in the general form

$$\frac{\partial}{\partial t} H(t) = \mathcal{M}[H(t), H(\tau - t)].$$

(A1)

To solve for $H(t)$; $t > 0$, one must specify as an initial condition the function $H_0(\varphi)$; $-\tau \leq \varphi \leq 0$. The time evolution of $H_0(\varphi)$ for $-\tau \leq \varphi < 0$ is obtained as $\partial H_0/\partial t = \lim_{\delta \to 0} (H_0(\varphi) - H_0(\varphi + \delta))/\delta = \lim_{\delta \to 0} (H(t + \varphi + \delta) - H(t + \varphi))/\delta = \partial H_0/(\partial \varphi)$. We may then quite generally write

$$\frac{\partial}{\partial t} H_0(\varphi) = (\mathcal{M} H_0)(\varphi); -\tau \leq \varphi \leq 0,$$

(A2)

where we have

$$(\mathcal{M} H_0)(\varphi) = \begin{cases} d\varphi H_0(\varphi); -\tau \leq \varphi < 0, \\ \mathcal{M}[H_0(\varphi), H_{1-\tau}(\varphi)]; \varphi = 0. \end{cases}$$

(A3)

For the kinetic equation (3), with $H_0(\varphi) = F_0(\varphi)$, we have

$$\mathcal{M}[F_0(\varphi), F_{1-\tau}(\varphi)] = -\omega \frac{\partial F_0(\varphi)}{\partial \varphi}$$

$$- \frac{K}{2i} \frac{\partial}{\partial \theta} \left[ (r[F](t - \tau) e^{-i(\theta + \alpha - \omega \tau)} + r^*[F](t - \tau) e^{i(\theta + \alpha - \omega \tau)}) F_0(\varphi) \right] = 0; \varphi = 0.$$

(A4)

Here $r$ denotes complex conjugation, and we have

$$r[F](t) \equiv \int d\theta d\omega e^{i\theta} F_0(\varphi); \varphi = 0.$$  

(A5)

**ADJOINT OF THE LINEAR OPERATOR $\mathcal{D}$ GIVEN IN EQ. (5) OF THE MAIN TEXT**

The adjoint of the linear operator $\mathcal{D}$ given in Eq. (5) of the main text may be obtained by using the equality

$$(h_2(\varphi), \mathcal{D} h_{11}(\varphi)) = (\mathcal{D}^\dagger h_2(\varphi), h_{11}(\varphi)) \tau.$$  

One gets

$$\begin{align}
(\mathcal{D}^\dagger s_1)(\vartheta) & = \begin{cases} -\frac{d}{d\vartheta} s_1(\vartheta); & 0 < \vartheta \leq \tau, \\ L^h s_1(\vartheta) = L^h s_1(0) + R^h s_1(\tau); & \vartheta = 0. \end{cases} \quad (A6)
\end{align}$$

Here, we have

$$L^h h = \omega \frac{\partial}{\partial \theta} h,$$

(A7)

$$R^h h = \frac{K}{4\pi} \left[ r[g(\omega)h] e^{-i(\theta + \alpha + \omega \tau)} + r^*[g(\omega)h] e^{i(\theta + \alpha + \omega \tau)} \right].$$

(A8)

**EIGENFUNCTIONS OF OPERATORS $\mathcal{D}$ AND $\mathcal{D}^\dagger$ GIVEN RESPECTIVELY BY EQ. (5) OF THE MAIN TEXT AND EQ. (A6)**

We first solve the eigenfunction equation

$$(\mathcal{D} P)(\varphi) = \lambda P(\varphi) \quad (A9)$$

for $-\tau \leq \varphi < 0$; we get $P(\varphi) = \Psi e^{i\varphi}$ for arbitrary $\Psi$. Since we would need to expand $f_i(\varphi)$, perturbations about the incoherent stationary state $F_{1\tau}(\varphi)$, in terms of $P(\varphi)$, we choose $\Psi$ as $\Psi(\theta, \omega)$, where $2\pi$-periodicity of $f_i$ implies that so should be $\Psi(\theta, \omega)$. Consequently, we may expand $\Psi(\theta, \omega)$ in a Fourier series in $\theta$ as $\Psi(\theta, \omega) = (2\pi)^{-1} \sum_{k=-\infty}^{\infty} \psi_k(\omega)e^{i k \theta},$ so that $P(\varphi) = (2\pi)^{-1} \sum_{k=-\infty}^{\infty} \psi_k(\omega)e^{i k \theta}$. Using the equation $\mathcal{D} P(\varphi) = \lambda P(\varphi)$ for $\varphi = 0$ and $k = \pm 1$ in the Fourier expansion of $P(\varphi)$, it may be easily seen with the condition $r[\Psi] = r^*[\Psi] = 1$ that $p(\varphi) = \psi_1(\omega)e^{i(\theta + \lambda\varphi)}$ and $p^*(\varphi)$ give two independent eigenfunctions of $\mathcal{D}$ with eigenvalues $\lambda$ and $\lambda^*$, respectively, where the latter satisfy $\Lambda(\lambda) = \Lambda^*(\lambda^*) = 0$ and

$$\psi_1(\omega) = \frac{K}{2} e^{-\lambda \varphi + (\alpha - \omega \tau)} \frac{g(\omega)}{\lambda + i \omega}, \quad (A10)$$

$$\Lambda(\lambda) = 1 - \frac{K}{2} e^{-\lambda \varphi + (\alpha - \omega \tau)} \int d\omega \frac{g(\omega)}{\lambda + i \omega}. \quad (A11)$$
For $k \neq \pm 1$, one has only a continuous spectrum sitting on the imaginary axis.

The eigenfunctions of the adjoint operator $\mathcal{D}^\dagger$ are given by $q(\theta) = \tilde{\psi}_1(e^{i\theta})$ and $q^*(\theta)$ with eigenvalues $\lambda^\ast$ and $\lambda$, respectively, where we fix $\tilde{\psi}_1(0)$ by requiring that $(q(\phi), p(\phi))_\tau = 1 = \int dq \omega \frac{q^*(0)p(0)}{\sqrt{\omega}}$. We thus get

$$\tilde{\psi}_1(\omega) = \frac{1}{2\pi(\Lambda'(\omega)^2)} = \frac{1}{\lambda^\ast - i\omega}, \quad \lambda = \frac{1}{\Lambda'(\lambda)^2} \lambda^\ast - i\omega e^{i\theta - \lambda^\ast \theta}. \quad \text{(A12)}$$

where $\Lambda'(\lambda)$ appears naturally in the normalization.

**DERIVATION OF EQ. (13) OF THE MAIN TEXT**

Let us start with Eq. (10) of the main text that decomposes perturbations $f_t(\phi)$ about the incoherent stationary state $F_{st}$ along the two unstable eigenvectors $p(\phi)$ of $p^\ast(\phi$) and $p^\ast(\phi)$ and the unstable manifold, as

$$f_t(\phi) = A(t)p(\phi) + A^*(t)p^*(\phi) + w[A, A^*](\phi), \quad \text{A14}$$

with the relations $(q(\phi), p(\phi))_\tau = 1$, $(q(\phi), p^*(\phi))_\tau = 0$, $(q(\phi), w(\phi))_\tau = 0$ yielding $A(t) = (q(\phi), f_t(\phi))_\tau$. We require $w(\phi)$ to be at least quadratic in $A$. Let us recall that $f_t(\phi)$ satisfies

$$\frac{df_t(\phi)}{dt} = (\mathcal{D} f_t)(\phi) = (\mathcal{D} f_t + \mathcal{F}(f_t))(\phi); \quad -\tau \leq \phi \leq \tau, \quad \text{A15}$$

with $\mathcal{D}$ being the part of the evolution operator that is linear in $f_t$ and $\mathcal{F}$ the part that is nonlinear:

$$\begin{align*}
(\mathcal{D} f_t)(\phi) &= \begin{cases} 
\frac{d}{d\phi} f_t(\phi); & -\tau \leq \phi < 0, \\
\mathcal{L} f_t(\phi) = L f_t(0) + R f_t(-\tau); & \phi = 0,
\end{cases} \\
(\mathcal{F}(f_t))(\phi) &= \begin{cases} 
0; & -\tau \leq \phi < 0, \\
\mathcal{M}[f_t]; & \phi = 0.
\end{cases} \quad \text{A16}
\end{align*}$$

Here, we have

$$\mathcal{M}[f_t] = -\frac{1}{2K} \left[ \int r[f_t(\tau) e^{-i(\theta + \alpha - \omega \tau)} + c.c.] \right].$$

$$\mathcal{N}[f_t] = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \mathcal{M}[f_k] e^{i\theta}, \quad \mathcal{N}[f_t] = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \mathcal{M}[f_k] e^{i\theta}, \quad \text{A18}$$

We now define the following Fourier expansion needed for further analysis:

$$f_t = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} f_k e^{i\theta},$$

$$\mathcal{N}[f_t] = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \mathcal{N}[f_k] e^{i\theta}, \quad \text{A19}$$

Using Eq. (A18), we then get

$$\mathcal{N}_k[f_t] = -\frac{kK}{2} \left[ \int r[f_t(\tau) e^{-i(\theta + \omega \tau)}(f_t)_{k+1}(0) - r^*[f_t(\tau) e^{i(\alpha - \omega \tau)}(f_t)_{k-1}(0)] \right].$$

$$\quad \text{A22}$$

Note that we have $(f_t)_0 = 0$, so that Eq. (A14) gives $w_0 = 0$; this feature is a major difference with respect to a similar kinetic equation, the Vlasov equation [33]. Note that for $k > 0$ in Eq. (A22), only Fourier modes $(f_k)_k$ with $k > 0$ appear; this does not hold when considering for example a second harmonic in the interaction, so that $(f_{k-1})$ appears in $\mathcal{N}_k$ that leads to singularities in the unstable manifold expansion [33]. By symmetry on the unstable manifold, we have $w_1 = O(A^2 r)$ and for $k > 1$, $w_k = O(A^4)$. From Eq. (A14), we get $(f_1)_1 = A p + w_1$ and for $k \neq \pm 1$, $(f_k)_k = w_k$. The amplitude $A$ may be related to the order parameter close to the bifurcation by $r = A^* + O(A^2)^4$. Using Eq. (A14), we obtain via the projection $(q, (A15))_\tau$ and $(A15) = (q, (A15))_\tau + c.c.$ the time evolution of $A(t)$ and $w$ as

$$\dot{A} = \lambda A + \int d\omega \tilde{\psi}_1^* \mathcal{N}_1[f_t],$$

$$\dot{w} = \mathcal{D} w + \mathcal{F}[f_t] - \left(-p \int d\omega \tilde{\psi}_1^* \mathcal{N}_1[f_t] + c.c.\right).$$

Here, we have used $(q, (A15)) = \int d\omega \theta (q(\omega)) \mathcal{N}[f_t] = \int d\omega \theta \tilde{\psi}_1^*(\omega) e^{-i\theta} \mathcal{N}[f_t]$. From (A22), we get at first order

$$\mathcal{N}_1[f_t] = -A|A|^2 K \left[ \int |r^*[p^*(\tau) e^{-i(\alpha - \omega \tau)} w_{2,0}(0) + O(A|A|^4). \right.$$

$$\quad \text{A25}$$

where we have denoted the leading order of the second harmonic $w_2 = A^2 w_{2,0} + O(A^4|A|)$.

Using the second harmonic of Eq. (A24) and (A2) $2A = 2A^2 \lambda + O(|A^2|^2)$ gives for $\phi \neq 0$, $w_{2,0} = \int d\omega \theta \tilde{\psi}_1^*(\omega) e^{-i\theta} \mathcal{N}_1[f_t]$. The equation for $\phi = 0$ with Eq. (A22) for $k = 2$ gives

$$2A^2 \lambda w_{2,0}(0) = \int d\omega \theta \tilde{\psi}_1^*(\omega) e^{-i\theta} \mathcal{N}_1[f_t] = \int d\omega \theta \tilde{\psi}_1^*(\omega) e^{-i\theta} \mathcal{N}_1[f_t] = \int d\omega \theta \tilde{\psi}_1^*(\omega) e^{-i\theta} \mathcal{N}_1[f_t].$$

We thus get

$$2A^2 \lambda w_{2,0}(0) = 2A^2 \int d\omega \theta \tilde{\psi}_1^*(\omega) e^{-i\theta} \mathcal{N}_1[f_t] = \int d\omega \theta \tilde{\psi}_1^*(\omega) e^{-i\theta} \mathcal{N}_1[f_t] = \int d\omega \theta \tilde{\psi}_1^*(\omega) e^{-i\theta} \mathcal{N}_1[f_t].$$

We then get

$$w_{2,0} = \left( \frac{K}{2} \right)^2 \int d\omega \theta \tilde{\psi}_1^*(\omega) e^{-i\theta} \mathcal{N}_1[f_t] = \int d\omega \theta \tilde{\psi}_1^*(\omega) e^{-i\theta} \mathcal{N}_1[f_t].$$

Plugging Eqs. (A27), (A25) in Eq. (A23), we obtain the desired form for the time evolution of $A(t)$:

$$\dot{A} = \lambda A + c_3 A^2 A + O(|A|^4), \quad \text{A28}$$
where the cubic coefficient $c_3$ is given by
\[
c_3 = -\frac{K^3}{8} e^{-\frac{3\lambda \tau}{\lambda} - i(\alpha - \omega_0 \tau)} \frac{\Delta'(\lambda)}{\Delta(\lambda)} \int d\omega \frac{g(\omega)}{(\lambda + i\omega)^3}.
\]

Here, we have as functionals of $F$ the quantity
\[
r[F](t) = \int d\theta d\omega \ e^{i\theta} F(\theta, \omega, t).
\]  

The OA ansatz considers in the expansion \ref{OA_ansatz} a restricted class of Fourier coefficients given by \ref{OA_ansatz}
\[
\tilde{F}_n(\omega, t) = [\tilde{z}(\omega, t)]^n
\]
with $\tilde{z}(\omega, t)$ an arbitrary function with the restriction $|\tilde{z}(\omega, t)| < 1$ that makes the infinite series in Eq. \ref{OA_ansatz} convergent one. In implementing the OA ansatz, it is also assumed that $\tilde{z}(\omega, t)$ may be analytically continued to the whole of the complex-$\omega$ plane, that it has no singularities in the lower-half complex-$\omega$ plane, and that $\tilde{z}(\omega, t) \to 0$ as $\text{Im}(\omega) \to -\infty$ \ref{OA_ansatz}. Using Eqs. \ref{OA_ansatz} and \ref{OA_ansatz}, one gets
\[
r(t) \equiv r[F](t) = \int_{-\infty}^{\infty} d\omega \ g(\omega)\tilde{z}(\omega, t).
\]  

On substituting Eqs. \ref{OA_ansatz}, \ref{OA_ansatz}, and \ref{OA_ansatz} in Eq. \ref{OA_ansatz} and on collecting and equating the coefficient of $e^{i\theta}$ to zero, we get
\[
\frac{\partial z(\omega, t)}{\partial t} + i\omega z(\omega, t) + \frac{K}{2} \left[ e^{-i(\alpha - \omega_0 \tau)} r(t - \tau) z^2(\omega, t) - e^{i(\alpha - \omega_0 \tau)} r^*(t - \tau) \right] = 0.
\]  

For the Lorentzian $g(\omega)$, Eq. \ref{Lorentzian}, one may evaluate $r(t)$ by using Eq. \ref{OA_ansatz} to get
\[
r(t) = \frac{1}{2\pi} \int_C d\omega \ z^*(\omega, t) \left[ \frac{1}{\omega - i\Delta} - \frac{1}{\omega + i\Delta} \right]
\]

where the contour $C$ consists of the real-$\omega$ axis closed by a large semicircle in the lower-half complex-$\omega$ plane on which the integral in Eq. \ref{OA_ansatz} gives zero contribution in view of $|\tilde{z}(\omega, t)| \to 0$ as $\text{Im}(\omega) \to -\infty$. The second equality in Eq. \ref{OA_ansatz} is obtained by applying the residue theorem to evaluate the complex integral over the contour $C$. Using Eqs. \ref{OA_ansatz} and \ref{OA_ansatz}, we finally obtain the OA equation for the time evolution of the synchronization order parameter as the DDE \ref{DDE}
\[
\frac{dr(t)}{dt} + \Delta r(t) + \frac{K}{2} \left[ e^{-i(\alpha - \omega_0 \tau)} r(t - \tau) r^*(t - \tau) - e^{i(\alpha - \omega_0 \tau)} r^*(t - \tau) r(t - \tau) \right] = 0.
\]  

whose solution requires as an initial condition the value of $r(t)$ over an entire interval of time $t$, namely, $t \in [-\tau, 0]$. Note that for $\tau = 0$, Eq. \ref{DDE} is a finite-dimensional ODE for $r(t)$ that requires for its solution only the value of $r(t)$ at $t = 0$ as an initial condition. In this case, it has been demonstrated that this single equation contains all the bifurcations and attractors of $r(t)$ as obtained through the evolution of the equations of motion for a Lorentzian $g(\omega)$ and in the limit $N \to \infty$. Equation \ref{DDE} is the Eq. (17) of the main text.
DERIVATION OF EQS. (19, 20) OF THE MAIN TEXT

We start with the time evolution equation for perturbations \( r_l(\varphi) \) about \( r_{st} = 0 \), obtained from Eq. (17) of the main text and rewritten as a DDE, as

\[
\frac{d}{dt} r_l(\varphi) = (\mathcal{D} r_l + \mathcal{F}[r_l](\varphi); \quad -\tau \leq \varphi \leq 0, \tag{A40}
\]

with

\[
(\mathcal{D} r_l)(\varphi) = \begin{cases} \frac{d}{d\varphi} r_l(\varphi); & \tau \leq \varphi < 0, \\ \mathcal{L} r_l(\varphi) = L r_l(0) + R r_l(-\tau); & \varphi = 0, \end{cases} \tag{A41}
\]

and

\[
(\mathcal{F}[r_l])(\varphi) = \begin{cases} 0; & -\tau \leq \varphi < 0, \\ \mathcal{N}[r_l]; & \varphi = 0. \end{cases} \tag{A42}
\]

The adjoint of the linear operator \( \mathcal{D} \) is given by

\[
(\mathcal{D}^\dagger s_l)(\varphi) = \begin{cases} -\frac{d}{d\varphi} s_l(\varphi); & 0 < \varphi \leq \tau, \\ \mathcal{L}^\dagger s_l(\varphi) = L^\dagger s_l(0) + R^\dagger s_l(\tau); & \varphi = 0. \end{cases} \tag{A43}
\]

In these equations, we have

\[
\begin{align*}
L r &= -\Delta r, \tag{A44} \\
R r &= K e^{i(\alpha - \omega_0 \tau)} r, \tag{A45} \\
L^\dagger r &= -\Delta r, \tag{A46} \\
R^\dagger r &= K e^{-i(\alpha - \omega_0 \tau)} r, \tag{A47} \\
\mathcal{N}[r_l] &= -K e^{-i(\alpha - \omega_0 \tau)} r_l(0) - i\alpha r_l(0). \tag{A48}
\end{align*}
\]

Small perturbations \( r_l(\varphi) \) may be expressed as a linear combination of the eigenfunctions of the linear operator \( \mathcal{D} \). To this end, let us then solve the eigenfunction equation

\[
(\mathcal{D} p)(\varphi) = \lambda p(\varphi) \tag{A49}
\]

for \(-\tau \leq \varphi < 0\); we get \( p(\varphi) = p(0)e^{\lambda \varphi} \). The equation \((\mathcal{D} p)(\varphi) = \lambda p(\varphi)\) for \( \varphi = 0 \) gives \( \lambda p(0) = -\Delta p(0) + (K/2)e^{i(\alpha - \omega_0 \tau)} p(0)e^{-\lambda \tau} \). With \( p(0) \neq 0 \), we thus get the dispersion relation

\[
\lambda(\lambda) = \lambda + \Delta - \frac{K}{2} e^{-\lambda \tau + i(\alpha - \omega_0 \tau)} = 0. \tag{A50}
\]

The solution of the above equation gives the discrete eigenvalues \( \lambda_l \) (with \( l \in \mathbb{Z} \)) in terms of the Lambert-W function \( W_l \), as

\[
\lambda_l = \frac{-\Delta \tau + W_l \left( \frac{K \tau}{2} e^{i(\alpha + \Delta \tau - i\omega_0 \tau)} \right)}{\tau}. \tag{A51}
\]

Without loss of generality, we may take \( p(0) = 1 \). We thus conclude that \( p(\varphi) = e^{\lambda \varphi} \) is an eigenfunction of the linear operator \( \mathcal{D} \) for \(-\tau \leq \varphi \leq 0\) with eigenvalue \( \lambda \) provided that \( \lambda \) satisfies \( \Lambda(\lambda) = 0 \). In other words, a discrete set of eigenvalues correspond to \( \mathcal{D} \) for all values of \( \varphi \). Perturbations \( r_l(\varphi) \) may be expressed as a linear combination of the corresponding eigenfunctions. It then follows that the stationary solution \( r_{st} = 0 \) will be linearly stable under the dynamics (A40) so long as all the eigenvalues \( \lambda \) have a real part that is negative. Vanishing of the real part of the eigenvalue with the smallest real part then signals criticality above which \( r_{st} = 0 \) is no longer a linearly-stable stationary solution of Eq. (A40). Denoting by \( \lambda_l ; \lambda_l \in \mathbb{R} \) the imaginary part of the eigenvalue with the smallest real part, we thus have at criticality the following equations obtained from Eq. (A50):

\[
\frac{K_c}{2} \cos(\alpha - (\omega_0 + \lambda_l) \tau) = \Delta, \tag{A52}
\]

\[
\frac{K_c}{2} \sin(\alpha - (\omega_0 + \lambda_l) \tau) = \lambda_l.
\]

We want to study the behavior of \( r_l(\varphi) \) as \( K \to K^e \), the goal being to uncover the weakly nonlinear dynamics occurring beyond the exponential growth taking place due to the instability as \( K \to K^u \). To this end, we want to study the behavior of \( r_l(\varphi) \) on the unstable manifold, which by definition is tangential to the unstable eigenspace spanned by the eigenfunctions \( p(\varphi) \) at the equilibrium point \((K = K_c, \lambda = \lambda_l)\). This manifold may be shown to be an attractor of the dynamics for the type of DDE under consideration and is therefore of interest to study. To proceed, we need the eigenfunctions of the adjoint operator \( \mathcal{D}^\dagger \), which will be useful in discussing the unstable manifold expansion. It is easily checked that \( \mathcal{D}^\dagger \) has the eigenfunction \( q(\varphi) = q(0)e^{-\lambda' \varphi} \) associated with the eigenvalue \( \lambda^* \) satisfying \( \Lambda'(\lambda^*) = 0 \), that is, we get the same dispersion relation as for \( \mathcal{D} \). We may choose \( q(0) \) such that \( (q(\varphi), p(\varphi))_\tau = 1 \). Using \( (h_{2L}(\varphi), h_{1L}(\varphi))_\tau = (h_{2L}(\varphi), h_{1L}(\varphi)) + \int_{-\tau}^0 d\xi (h_{2L}(\xi + \tau), R h_{1L}(\xi)) \), and noting that in the present case, \((q(0), p(0)) = q^*(0)p(0), \) we get

\[
q^*(0) + \int_{-\tau}^0 d\xi q^*(0)e^{-\lambda(\xi + \tau)} K_2 e^{i(\alpha - \omega_0 \tau)} e^{i\xi} = 1, \tag{A53}
\]

yielding

\[
q^*(0) = \frac{1}{1 + \tau K_2 e^{-\lambda(\xi + \tau)} e^{i\xi}} = \frac{1}{\Lambda'(\lambda)}. \tag{A54}
\]

The unstable manifold expansion of \( r_l(\varphi) \) for \( K > K_c \) reads

\[
r_l(\varphi) = A(t)p(\varphi) + w[A](\varphi), \tag{A55}
\]
where \( w[A](\varphi) \), which is at least quadratic in \( A \) (in fact, one can prove that it is cubic in \( A \) in the present case), denotes the component of \( r_{l}(\varphi) \) transverse to the unstable eigenspace, so that \( (q(\varphi), w(\varphi))_{\tau} = 0 \). On using the latter equation, together with \( (q(\varphi), p(\varphi))_{\tau} = 1 \) in Eq. (A55), we get \( A(t) = (q(\varphi), r_{l}(\varphi))_{\tau} \). The time evolution of \( A(t) \) is then obtained as

\[
\dot{A} = (q(\varphi), \dot{r}_{l}(\varphi))_{\tau} = (q(\varphi), (\mathcal{D} r_{l} + \mathcal{F}[r_{l}]) (\varphi))_{\tau} = (q(\varphi), A(t) \lambda p(\varphi) + \mathcal{D} w(\varphi) + \mathcal{F}[r_{l}]) (\varphi)_{\tau} = \lambda A + q^{*} (0) \mathcal{N}[r_{l}],
\]

where the dot denotes derivative with respect to time. Here, in arriving at the second and the third equality, we have used Eqs. (A40) and (A55), while in obtaining the last equality, we have used in the third step \( (q(\varphi), \mathcal{D} w(\varphi))_{\tau} = (\mathcal{D}^\dagger q(\varphi), w(\varphi))_{\tau} = \lambda^* (q(\varphi), w(\varphi))_{\tau} = 0 \). Since we can prove that \( w \) is \( O(|A|^2 A) \), while we see that \( \mathcal{N}[r_{l}] \) is of order three in \( r_{l} \), the leading-order contribution to the nonlinear term on the right hand side of Eq. (A56) is obtained as \( \mathcal{N}[r_{l}] = \mathcal{N}[A(t)] p(\varphi) + O(|A|^2 A) = -(K/2) e^{-i(\omega - \omega_{0} \tau)} p_{+} (-\tau) p^{2} (0) |A(t)|^2 A(t) + O(|A|^2 A) \). Using this result and Eq. (A54) in Eq. (A56), we get eventually the so-called normal form for the time evolution of \( A \) as

\[
\dot{A} = \lambda A + \frac{K}{2} e^{i(\omega_{0} - \lambda) \tau - i\alpha} |A|^2 A + O(|A|^2 A).
\]

The above (yielding Eqs. (19) and (20) of the main text) is the desired finite-dimensional ordinary differential equation corresponding to the infinite-dimensional equation (A40), which allows to decide the bifurcation behavior of \( r_{l}(\varphi) \) as \( K \rightarrow K_{+}^{q} \). The relevant parameter to study the type of bifurcation is given by the sign of the second term on the right hand side. Denoting this term by \( c_{3} \), we then need to study the sign of the real part of \( c_{3} \) as the real part of \( \lambda \) approaches zero, so that \( \lambda = i\lambda_{t} \) is purely imaginary:

\[
\text{Re}(c_{3}) = -\frac{K_{c}}{2} \text{Re} \left( \frac{e^{i(\omega_{0} + \lambda_{t}) \tau - i\alpha}}{1 + \tau/2 e^{-i(\omega_{0} + \lambda_{t}) \tau + i\alpha}} \right).
\]

**DERIVATION OF EQ. (21) OF THE MAIN TEXT**

For generic \( g(\omega) \), we start the decomposition \( F_{i} = F_{st} + f_{i} \), with Fourier coefficients \( (F_{i})_{k}/(2\pi) = g(\omega)/(2\pi)(\alpha_{k})_{k} \), where the \( (\alpha_{k})_{k} \)'s are the Fourier coefficients on the unstable manifold. Using \( \int_{0}^{2\pi} d\theta f_{i}(\varphi) = 0 \), we get \( (\alpha_{k})_{0} = 1 \). From Eq. (A14), we get

\[
(\alpha_{k})_{1}(\varphi) = Ap(\varphi) + O(A|A|^2).
\]

Similarly, Eqs. (A27) and (A14) give

\[
(\alpha_{k})_{2} = (Ap(\varphi))^2 + O(A^2|A|^2).
\]

As in Eq. (A26), we may write equations for the \( k \geq 1 \) Fourier modes with \( w_{k,0} = A^{k} w_{k,0} + O(A^k|A|) \), obtaining

\[
w_{k,0} = w_{k,0}(0) e^{i\omega_{0} \tau}.
\]

\[
 k(\lambda + i\omega) w_{k,0}(0) = \frac{k K}{2} r^{*}[\alpha] (-\tau) e^{i(\alpha - \omega_{0} \tau)} w_{k-1,0}(0) + O(|A|^2),
\]

where we have used

\[
(\dot{A})^{k} = k\lambda A^{k} + O(A^k|A|^2),
\]

and the fact that the dominant contribution in Eq. (A22) always involves for \( k > 1 \) the \( f_{k-1} \) term and not \( f_{k+1} \). By induction, we can deduce for \( k \geq 0 \) that

\[
(\alpha_{k})_{k} = (\alpha_{k})_{k} + O(A^{k}|A|^2),
\]

and by taking the complex conjugate of the last equation, we obtain the corresponding equation for \( k < 0 \). Equation (A64) is Eq. (21) of the main text.