ON FC-CENTRAL EXTENSIONS OF GROUPS OF INTERMEDIATE GROWTH

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Abstract. It is shown that FC-central extensions retain sub-exponential volume growth. A large collection of FC-central extensions of the first Grigorchuk group is provided by the constructions in the works of Erschler [Ers06] and Kassabov-Pak [KLP13]. We show that in these examples subgroup separability is preserved. We introduce two new collections of extensions of the Grigorchuk group. One collection gives first examples of intermediate growth groups with centers isomorphic to $\mathbb{Z}^\infty$; and the other provides groups with prescribed oscillating intermediate growth functions.

1. Introduction

The FC-center of a group $G$, denoted by $Z_{FC}(G)$, consists of all elements of $G$ which have finite conjugacy classes. We say $\Gamma$ is an FC-central extension of $G$ if $G$ is a quotient of $\Gamma$ and $\ker(\Gamma \to G)$ is contained in $Z_{FC}(\Gamma)$. A special case is that $\Gamma$ is a central extension of $G$, that is, $\ker(\Gamma \to G)$ is contained in the center of $\Gamma$. In this paper we study FC-central extensions of groups of sub-exponential growth, in particular, of groups related to Grigorchuk groups.

Let $G$ be a group equipped with a finite generating set $S$. Denote by $d_S$ the graph distance on the Cayley graph of $(G,S)$. The growth function $v_{G,S}(n)$ counts the number of elements in the ball of radius $n$, $v_{G,S}(n) = |\{ \gamma \in G : d_S(\gamma, id) \leq n \}|$.

A finitely generated group $G$ is of polynomial growth if there exists $0 < d < \infty$ and a constant $C > 0$ such that $v_{G,S}(n) \leq Cn^d$. It is of exponential growth if $\lim_{n \to \infty} v_{G,S}(n)^{1/n} > 1$. If $v_{G,S}(n)$ is sub-exponential but not bounded by a polynomial, we say $G$ is of intermediate growth. A group of polynomial growth is virtually nilpotent by Gromov’s theorem [Gro81]. The first examples of groups of intermediate growth are constructed by Grigorchuk in [Gri84].

We show that FC-central extensions retain sub-exponential growth:

Lemma 1.1 (= Lemma 3.1). Let $\Gamma$ be a finitely generated FC-central extension of a group $G$. If $G$ is of sub-exponential growth, then $\Gamma$ is of sub-exponential growth as well.

A well-known question of Rosenblatt [Ros81] asks if every Cayley graph with exponential volume growth function admits a Lipschitz embedding of the binary tree. Equivalently, the question asks if no group of exponential growth can be supramenable, see the characterizations of supramenability in [KMR13]. A group

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of sub-exponential growth is supramenable, and results in [Mon17] imply that central extensions of a group of sub-exponential growth are supramenable. Lemma 1.1 shows that one can not find an example that answers Rosenblatt’s question negatively by taking a central extension of an intermediate growth group.

The space $\mathcal{M}_k$ of $k$-marked group the space of marked groups $(G, S)$, where $S = (s_1, \ldots, s_k)$ is an ordered generating $k$-tuple of $G$. Equivalently, $(G, S)$ can be viewed as an epimorphism $F_k \to G$, where $F_k$ is the free group on $k$ generators. The space $\mathcal{M}_k$ is endowed with the Cayley topology, also referred to as the Cayley-Grigorchuk topology. The Cayley topology is induced by the metric $d((G_1, S_1), (G_2, S_2)) = 2^{-k}$, where $k$ is the maximal radius such that the ball around $id_{G_1}$ in the Cayley graph of $(G_1, S_1)$ is identical to the ball of the same radius around $id_{G_2}$ in the Cayley graph of $(G_2, S_2)$. The space $\mathcal{M}_k$ is first used by Grigorchuk in [Gri84] to study the family of Grigorchuk groups $(G_\omega, \omega \in \{0, 1, 2\}^\mathbb{N}$).

Given a collection of $k$-marked groups $((G_i, S_i))_{i \in I}$, one can take the diagonal product $(\Delta, S)$ of this collection, namely the subgroup of $\prod_{i \in I} G_i$ generated by diagonally embedding the generating set of each $G_i$. Taking the diagonal product of a sequence of marked group is a useful tool to construct groups with "exotic" properties. Such a construction has been performed by Erschler in [Ers06] (in the language of piecewise automata), Kassabov and Pak in [KP13], Brieussel and the author in [BZ19]. The goal is often to produce a family of groups with certain desired properties: sub-exponential growth groups with arbitrarily large Følner functions in [Ers06], groups with oscillating volume growth in [KP13], and groups with prescribed random walk characteristics or metric embedding distortion into Banach spaces in [BZ15].

A connection between FC-central extensions and diagonal products is the following. Suppose $((G_i, S_i))_{i = 1}^\infty$ is a sequence of marked groups in $\mathcal{M}_k$ such that each $G_i$ is finite and moreover $(G_i, S_i)$ converges to $(G, S)$ in the Cayley topology, then the diagonal product of $((G_i, S_i))_{i = 1}^\infty$ is an FC-central extension of $G$. For instance, the groups constructed in [Ers06] and [KP13] cited above are FC-central extensions of the first Grigorchuk group arising this way.

In general a central extension does not preserve residual finiteness. By [Ers04], there exist central extensions of the first Grigorchuk group $\mathfrak{G}$ that are not commensurable up to finite kernels with any residually finite groups.

Subgroup separability (also called locally extended residual finiteness, LERF in abbreviation) is a strong form of residual finiteness. A group $G$ is said to be subgroup separable if every finitely generated subgroup is an intersection of subgroups of finite index in $G$. Subgroup separability is a powerful property and is only known to hold for a few special classes of groups, notably, free groups [Bur71], polycyclic groups [Mal83], some 3-manifold groups [LN91], the first Grigorchuk group $\mathfrak{G}$ [GW03], the Gupta-Sidki 3-group [Gar16].

Regarding stability under group constructions, it is known that taking free products [Bur71] and carefully controlled amalgamations preserve subgroup separability, see [Sco78] [Git99]. In Section 6 we show that the type of FC-central extension of the first Grigorchuk group $\mathfrak{G}$ as in [Ers06] and [KP13] preserves subgroup separability. The definition of such extensions is explained at the beginning of Section 5.

Denote by $F$ the free product $(\mathbb{Z}/2\mathbb{Z})*((\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}))$, marked with the generating tuple $S = (a, b, c, d)$, where $a, b, c, d$ are involutions, $F = \langle a \rangle * (\langle b \rangle \times \langle c \rangle)$, $d = bc$. 

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We say that a group $A$ is a marked quotient of $F$ if there is an epimorphism $F \to A$ and $A$ is marked with the generating tuple $(a, b, c, d)$ which is the image of $S$.

**Theorem 1.2** (= Theorem 6.2). Let $\mathcal{L} = (A_n)_{n=1}^{\infty}$ be a sequence of finite marked quotients of $F$. Denote by $\mathfrak{S}$ the first Grigorchuk group and $\Gamma(\mathcal{L})$ the FC-central extension of $\mathfrak{S}$ with $\mathcal{L}$ as input described in Definition 7.1. Then $\Gamma(\mathcal{L})$ is subgroup separable.

One can also determine the commensurability classes of finitely generated subgroups of $\Gamma(\mathcal{L})$, see Theorem 6.1. The proof of Theorem 1.2 relies on the generalized wreath recursion and branching structure in $\Gamma(\mathcal{L})$ (see Section 5) and properties of subgroups of $\mathfrak{S}$ proven in [GW03] (see Subsection 6.2). Groups constructed in [KP13] correspond to taking $\mathcal{L} = (\text{PSL}_2(\mathbb{Z}/n\mathbb{Z}))$. As a consequence of the main theorem of [KP13], the collection of FC-central extensions of $\mathfrak{S}$ which are of the form $\Gamma(\mathcal{L})$ contains uncountably many groups of intermediate growth with pairwise non-comparable growth functions.

The second half of the paper is devoted to constructing more examples of extensions of intermediate growth groups and studying their properties. We start with permutation wreath extensions of the first Grigorchuk group $\mathfrak{S}$, instead of $\mathfrak{S}$ as in previous examples. The reasons for this choice will be clear in the context: in Section 7 we need the Schur multipliers to contain copies of $\mathbb{Z}$ as direct summands; and in Section 10 we rely on the structure of permutation wreath products to provide sharp volume lower bounds.

Let $G$ be a group acting on a countable set $X$ and $L$ be a group. The **permutation wreath product** $W = L \wr_X G$ is the semi-direct product $\oplus_X L \rtimes G$, where $G$ acts on $\oplus_X L$ by permuting coordinates. The permutation wreath extension of $\mathfrak{S}$ of the form $W = A \wr_{\mathfrak{S}} \mathfrak{G}$, where $A$ is a group and $\mathfrak{S}$ is the orbit of the right-most ray $1^\infty$ of the rooted binary tree under the action of $\mathfrak{S}$, is first considered in [BE12].

The goal of Section 7 is to construct groups of intermediate growth with center isomorphic to $\mathbb{Z}^\infty$, where $\mathbb{Z}^\infty$ denotes the direct sum of countably many copies of $\mathbb{Z}$. We take a specific countable $\mathfrak{S}$-set $X$ and construct a sequence of marked groups $\Gamma_n$ which converges to $\mathbb{Z}_1 \wr_X \mathfrak{S}$ in the Cayley topology, such that the center of diagonal product of this sequence is isomorphic to $\mathbb{Z}^\infty$. The question whether there is a group of intermediate growth whose center contains $\mathbb{Z}^\infty$ is asked by Bartholdi and Erschler (personal communication).

**Theorem 1.3.** There exists a finitely generated torsion-free group $\Gamma$ of intermediate growth whose center $Z(\Gamma)$ is isomorphic to $\mathbb{Z}^\infty$.

In Section 8 we take the diagonal products of suitable finite marked groups which result in FC-central extensions of $W = (U \times V) \wr_{\mathfrak{S}} \mathfrak{G}$, where $U, V$ are finite group and $\mathfrak{S}$ is the orbit of $1^\infty$ under the action of $\mathfrak{S}$. We show that the volume growth functions of such diagonal products can be estimated up to good precision, see Theorem 10.3. In particular, such FC-central extensions of $W$ allow to show the following result on prescribed growth function.

For the first Grigorchuk group $\mathfrak{S}$, the upper bound in [Bar98] states that $v_{\mathfrak{S}, \mathfrak{S}}(n) \leq \exp(C n^{\alpha_0})$, where $\alpha_0 = \frac{\log 2}{\log \lambda_0} \approx 0.7674$, and $\lambda_0$ is the positive root of the polynomial $X^3 - X^2 - 2X - 4$. The volume lower bound in [EZ19] shows that $v_{\mathfrak{S}, \mathfrak{S}}(n) \geq \exp(c_0 n^{\alpha_0 - \epsilon})$ for any $\epsilon > 0$. The upper and lower bounds together imply that

$$\lim_{n \to \infty} \frac{\log \log v_{\mathfrak{S}, \mathfrak{S}}(n)}{\log 3} = \alpha_0,$$
that is, the volume exponent of $\mathcal{G}$ exists and is equal to $\alpha_0$.

Recall that a function $f : \mathbb{N} \to \mathbb{N}$ is sub-additive if $f(n + m) \leq f(n) + f(m)$ and it is sublinear if $\lim_{n \to \infty} f(n)/n = 0$.

**Theorem 1.4.** Denote by $\alpha_0$ the volume exponent of the first Grigorchuk group $\mathcal{G}$. There exists an absolute constant $C > 0$ such that the following is true. Let $f : \mathbb{N} \to \mathbb{N}$ be any sublinear, non-decreasing, subadditive function such that $f(1) = 1$ and
\[
 f(n) \geq n^{\alpha_0} \quad \text{for all } n \in \mathbb{N}.
\]

Then there exists a group $\Delta$ equipped with a finite generating set $T$ such that $\Delta$ is an FC-central extension of $W = (\mathbb{Z}/2\mathbb{Z})^3 \wr \mathcal{G}$ and for all $n \in \mathbb{N}$,
\[
 \exp \left( \frac{1}{C} f(n) \right) \leq v_{\Delta,T}(n) \leq \exp (C f(n)).
\]

Note that $\log v_{G,S}$ is necessarily a non-decreasing, subadditive function. The limitation of Theorem 1.4 is that it is based on extensions of $W$, thus the volume functions obtained are always at least the growth function of $W$ (which is equivalent to $e^{n \alpha_0}$ by [BE12]). It is an important open problem whether there exists intermediate growth groups with (lower) volume exponent strictly less than $\alpha_0$. This problem is closely related to the Gap Conjecture of Grigorchuk, see the survey [Gri14].

The groups in Theorem 1.4 are similar to, but not the same kind as those considered in [KP13, Theorem 10.1]. The construction is designed so that volume growth of the diagonal product is controlled by what we call the traverse fields of words (defined in Section 9). The traverse fields are compatible with the word recursion, which allows us to apply the norm contraction inequality to control the volume growth, see Section 9 and 10.

**Relation to prior works.** Theorem 1.4 can be viewed as a further development of earlier works of Bartholdi and Erschler [BE14], Brieussel [Bri14], Kassabov and Pak [KP13], in the sense that it combines some aspects of the constructions in these works. The main theorem of [BE14] states that given a function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that
\[
 (1.1) \quad \phi(2R) \leq 2\phi(R) \leq \phi \left( 2^{\frac{1}{C}} R \right) \quad \text{for all } R \text{ large enough},
\]
then there exists a group $\Gamma$ with finite generating set $S$ and a constant $C > 0$ such that $e^{\phi(n/C)} \leq v_{\Gamma,S}(n) \leq e^{\phi(Cn)}$. The group $\Gamma$ is taken to be $(\mathbb{Z}/2\mathbb{Z})^3 \wr \mathcal{G}_\omega$, where $\omega \in \{0, 1, 2\}^\mathbb{N}$ is chosen carefully according to the given function $\phi$. The groups considered in [Bri14] are the same kind as in [BE14], where estimates are given in terms of upper and lower growth exponents. The use of permutation wreath products in Theorem 1.4 is motivated by the sharp volume estimates in [BE12, BE14]. Note that the collection of admissible functions in Theorem 1.4 is larger than in [BE14] cited above. Indeed, the assumption $2\phi(R) \leq \phi \left( 2^{\frac{1}{C}} R \right)$ in (1.1) implies not only $\phi(R) \geq cR^{\alpha_0}$, but also $\phi(R)/R^{\alpha_0}$ is non-decreasing. Consider a function that oscillates between $n^{\alpha_0}$ and some larger sublinear function $g(n)$, for example, $f$ is a sub-additive function such that
\[
 f(n) \geq n^{\alpha_0}, \quad \lim_{n \to \infty} \inf_{n^{\alpha_0}} \frac{f(n)}{n^{\alpha_0}} < \infty \quad \text{and} \quad \lim_{n \to \infty} \sup_{n \to \infty} \frac{f(n)}{n/\log n} = \infty.
\]
This type of oscillating growth functions is not covered by the results in [BE14], but allowed in Theorem 1.4 The main theorem in [KP13] states that given sub-exponential functions $f_1, f_2, g_1, g_2$ satisfying

$$f_1 \succeq f_2 \succeq g_1 \succeq g_2 = v_{G, S}$$

and some additional conditions, there exists a group $\Gamma$ (which is an FC-central extension of $\mathcal{G}$ as mentioned earlier) with a finite generating set $S$, such that its growth function satisfies $g_2(n) < v_{\Gamma, S}(n) < f_1(n)$ and $v_{\Gamma, S}$ takes values in the intervals $[g_2(n), g_1(n)]$ and $[f_2(n), f_1(n)]$ infinitely often. For an illustration such oscillating growth functions, see [KP13, Figure 1]. The examples in Theorem 1.4 can exhibit oscillating behaviors similar to [KP13] and at the same time the log of the volume functions can be estimated up to an absolute constant. We mention that techniques in Section 10 can be used to improve volume upper bounds of groups considered in [KP13].

Another feature of the family of FC-central extensions of $W$ in Theorem 1.4 is that simple random walks on these groups exhibit rich behavior. The technique in the proof of Theorem 1.4 is relevant for random walks: the random walk is controlled by the traverse field of random words (instead of deterministic words in the volume calculations). In particular, one can show that with appropriate choice of parameters, the entropy of simple random walk on $\Delta$ can be equivalent to any prescribed sublinear function $f$ with $f(n) \geq n^{1/2}$. Estimates of random walk characteristics on $\Delta$ will be discussed elsewhere.

Organization of the paper. In Section 2 we collect some background on permutation wreath products, Grigorchuk groups and the space of marked groups. In Section 3 we show that FC-central extensions preserve sub-exponential volume growth. Section 4 contains a basic lemma on growth of diagonal products and discusses direct sum structure in the diagonal product. The rest of the paper contains three parts, which can be read independently. Part I consists of Section 5 and 6 on the FC-central extensions of the Grigorchuk group $\mathcal{G}$ of the form $\Gamma(\mathcal{L})$. In Section 5 we present the definition of $\Gamma(\mathcal{L})$ and explain the wreath recursions on the formal level. Section 6 is devoted to the study of finitely generated subgroups of $\Gamma(\mathcal{L})$, where we determine the commensurability classes of finitely generated subgroups and show subgroup separability. Part II consists of Section 7 where we construct finitely generated groups of intermediate growth with center isomorphic to $\mathbb{Z}^\infty$. Part III consists of Section 8, 9 and 10 on extensions of $(U \times V) \wr S \mathcal{G}$. In Section 8 we describe the construction and explain the recursive structure in these groups. In Section 9 we introduce the traverse fields of words and show that they are compatible with the wreath recursion. Section 10 contains volume growth estimates for groups described in Section 8. We first use the traverse fields and the norm contraction inequality to estimate growth in each factor group, then combine the results to obtain estimates for the growth of the diagonal product. Finally we present an elementary lemma for approximation of prescribed functions and show explicitly that given a function $f$ as in Theorem 1.4, how to choose appropriate parameters and find a diagonal product whose log volume function is equivalent to $f$.

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2. Preliminaries

2.1. Permutation wreath product. Let $G$ be a group acting on a set $X$ from the right and $L$ be a group. The permutation wreath product $W = L \wr_X G$ is defined as the semi-direct product $\oplus X L \rtimes \Gamma$, where $G$ acts on $\oplus X L$ by permuting coordinates. The support $\text{supp} f$ of $f : X \to L$ consists of points of $x \in X$ such that $f(x) \neq \text{id}_L$. Elements of $\oplus X L$ are viewed as finitely supported functions $X \to L$. The action of $G$ on $\oplus X L$ is it follows that $\text{supp}(g \cdot f) = (\text{supp} f) \cdot g^{-1}$. Elements in $W = L \wr_X G$ are recorded as pairs $(f, g)$ where $f \in \oplus X L$ and $g \in G$. Multiplication in $W$ is given by

$$(f_1, g_1)(f_2, g_2) = (f_1 (g_1 \cdot f_2), g_1 g_2),$$

where $g \cdot f(x) = f(x \cdot g)$.

When $G$ and $L$ are finitely generated, $W = L \wr_X G$ is finitely generated as well. For more information on Cayley graphs of permutation wreath products, see [BE72, Section 2].

We use the notation $\delta^y_x$ for the function $f : X \to L$ such that $f(x) = \gamma$ and $f(y) = \text{id}_L$ for all $y \neq x$. The additive notation $\delta^y_x + \delta^z_y$, $x \neq y$, means the function $f : X \to L$ such that $f(x) = \gamma_1$, $f(y) = \gamma_2$ and $f(z) = \text{id}_L$ for $z \notin \{x, y\}$.

2.2. Grigorchuk groups. The groups $\{G_\omega\}$ were introduced by Grigorchuk in [Gri84] as first examples of groups of intermediate growth. We recall below the definition of these groups. See also the exposition in the book [BH00, Chapter VIII].

The spherically symmetric rooted tree $T_d$ is the tree with vertices $v = v_1 \ldots v_n$ with each $v_j \in \{0, 1, \ldots, d_j - 1\}$. The root is denoted by the empty sequence $\emptyset$.

Edge set of the tree is $\{(v_1 \ldots v_n, v_1 \ldots v_nv_{n+1})\}$. The index $n$ is called the depth or level of $v$, denoted $|v| = n$. Denote by $T^0_d$ the finite subtree of vertices up to depth $n$ and $L_n$ the vertices of level $n$. The boundary $\partial T_d$ of the tree $T_d$ is the set of infinite rays $x = v_1 v_2 \ldots$ with $v_j \in \{0, 1, \ldots, d_j - 1\}$ for each $j \in \mathbb{N}$. Throughout the paper, denote by $\Gamma$ the rooted binary tree, that is, $d$ is the constant sequence with $d_i = 2$.

Let $\{0, 1, 2\}$ be the three non-trivial homomorphisms from the 4-group $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) = \{id, b, c, d\}$ to the 2-group $\mathbb{Z}/2\mathbb{Z} = \{id, a\}$. They are ordered in such a way that $0, 1, 2$ vanish on $d, b, c$ respectively. For example, $0$ maps $id, d$ to $id$ and $b, c$ to $a$. Let $\Omega = \{0, 1, 2\}^\infty$ be the space of infinite sequences over letters $\{0, 1, 2\}$. The space $\Omega$ is endowed with the shift map $s : \Omega \to \Omega, s(\omega_0, \omega_1, \ldots) = \omega_1 \omega_2 \ldots$.

Given an $\omega \in \Omega$, the Grigorchuk group $G_\omega$ acting on the rooted binary tree $\Gamma$ is generated by $\{a, b, c, d, \omega\}$, where $a = (id, id)e$, $e$ transposes 0 and 1, and the automorphisms $b_\omega, c_\omega, d_\omega$ are defined recursively according to $\omega$ as follows. The wreath recursion sends

$$\psi_n : G_{\omega^n} \to G_{\omega^{n+1}} \wr_\{0,1\} \mathcal{S}_2$$

by

$$\psi_n(b_{\omega^n}) = (\omega_n(b), b_{\omega^{n+1}}),$$

$$\psi_n(c_{\omega^n}) = (p_n(c), c_{\omega^{n+1}}),$$

$$\psi_n(d_{\omega^n}) = (p_n(d), d_{\omega^{n+1}}).$$

The string $\omega$ determines the portrait of the automorphisms $b_\omega, c_\omega, d_\omega$ by the recursive definition above. The first Grigorchuk group corresponds to the periodic sequence $\omega = (012)^\infty$ and is often denoted as $G_{012}$. 

By [Gri84], if \( \omega \) is eventually constant, then \( G_\omega \) has polynomial growth; otherwise \( G_\omega \) is of intermediate growth. When \( \omega \) contains infinitely many of each of the three symbols \( \{0,1,2\} \), \( G_\omega \) is an infinite torsion group.

2.3. Marked groups and the Cayley topology. Let \( k \in \mathbb{N} \). We say \((G,S)\) is a \( k\)-marked group if \( S = (s_1, \ldots, s_k) \) is an ordered \( k \)-tuple of elements in \( G \) that generates \( G \). We refer to \( S \) as a \( k\)-marking of \( G \). Equivalently one can think of \( k \)-marking of \( G \) as a quotient map \( \pi : F_k \to G \), where \( F_k \) is the free group on free generators \( \{s_1, \ldots, s_k\} \). We say a map \( \psi : (G_1, S_1) \to (G_2, S_2) \) is a \( k\)-marked group homomorphism, where \( S_i = (s_i^{(1)}, \ldots, s_i^{(k)}) \) for \( i \in \{1,2\} \), if \( \psi \) is a homomorphism \( G_1 \to G_2 \) such that \( \psi(s_j^{(1)}) = s_j^{(2)} \) for all \( j \in \{1,\ldots,k\} \).

Given a collection of \( k\)-marked groups \( \{(G_i, S_i)\}_{i \in I} \), the diagonal product \((\Delta, S)\) of this collection is the subgroup of \( \prod_{i \in I} G_i \) generated by diagonally embedding the generating set of each \( G_i \). Equivalently, let \( F_k \) be the free group on free generators \( S = \{s_1, \ldots, s_k\} \) and identify the \( k\)-marking \( S_i \) of \( G_i \) as the quotient map \( \pi_i : F_k \to G_i \) sending \( S \) to \( S_i \), then the diagonal product \((\Delta, S)\) of the collection \( \{(G_i, S_i)\}_{i \in I} \) is given by

\[ \Delta = F_k / \cap_{i \in I} \ker (\pi_i : F_k \to G_i) , \quad S = \text{image of } S. \]

For some basic properties of diagonal products we refer to [KPR3 Section 4]. In particular, let \((\Delta, S)\) be the diagonal product of a sequence of \( k\)-marked groups that converge to \((G,S)\) in the Cayley topology, then \( G \) is a \( k\)-marked quotient of \( \Delta \).

3. Growth of FC-central extensions

In this section we show that sub-exponential growth is preserved under FC-central extensions. We will need the following classical result of B.H. Neumann [Neu51] which states that if \( N \) is an FC-group (i.e., \( N = Z_{FC}(N) \)), then

- the elements of finite order form a characteristic subgroup \( P \) of \( N \) and \( P \) contains the derived subgroup \( N' \);
- every finite set of finite order elements is contained in a finite normal subgroup of \( N \);
- moreover, if \( N \) is finitely generated then \( P \) is finite.

The key step in the proof of the following lemma is the rearranging of products in Claim 3.2.

**Lemma 3.1.** Let \( G \) be a group of sub-exponential growth. Suppose \( \Gamma \) is a finitely generated FC-central extension of \( \Gamma \), that is,

\[ 1 \to N \to \Gamma \to G \to 1 \]  where \( N \subseteq Z_{FC}(\Gamma) \).

Then \( \Gamma \) has sub-exponential growth as well.

**Proof.** Given two sets \( A, B \subseteq \Gamma \), denote by \( AB \) the product set \( \{gh : g \in A, h \in B\} \).

Take a symmetric finite generating set of \( \Gamma \), adjust the generating set if necessary, we may assume that it is of the form \( T = \{t_1, \ldots, t_\ell\} \cup \{n_1, \ldots, n_p\} \), where the projections \( \pi(t_i) \), \( 1 \leq i \leq \ell \) are pairwise distinct non-trivial elements of \( G \) and \( n_i \in N \) for \( 1 \leq i \leq p \). Write \( T_1 = \{t_1, \ldots, t_\ell\} \) and denote by \( \Gamma_1 \) the subgroup generated by \( T_1 \). Denote by \( Q_0 \) the union of the conjugates of \( \{n_1, \ldots, n_p\} \) in \( \Gamma \). Note that \( B_{\Gamma_1,T}(id,n) \subseteq Q_0^\ast B_{\Gamma_1,T_1}(id,n) \). Since \( Q_0 \) is a finite set in the FC-center
of $\Gamma$, by Neumann’s result, $\langle Q_0 \rangle$ is virtually abelian. Thus to show that $\Gamma$ is of sub-exponential growth, it suffices to show for the subgroup $\Gamma_1$. In what follows we replace $\Gamma$ by $\Gamma_1$ and assume the generating set $T = \{t_1, \ldots, t_s\}$ is symmetric and the projections $\pi(t_i)$, $1 \leq i \leq \ell$ are pairwise distinct non-trivial elements of $G$.

Equip $G$ by the generating set $\hat{T} = \pi(T)$. In what follows balls in $\Gamma$ (resp.) are with respect to word length $|\cdot|_T$ ( $|\cdot|_{\hat{T}}$ resp.). Throughout the proof, fix a choice of section $s : G \to \Gamma$ such that $s(id_G) = id_T$ and on the generating set $\hat{T}$, $s(\pi(t_i)) = t_i$.

For each integer $k \geq 2$, the section $s$ gives rise to a map

$$\beta_k : G^k \to N$$

$$(g_1, \ldots, g_k) \mapsto s(g_1) \cdots s(g_k)s(g_1 \cdots g_k)^{-1}.$$ Denote by $A(k, r)$ the image of $k$ copies of $B_G(id, r)$ under the map $\beta_k$, that is, $A(k, r) := \beta_k(B_G(id, r) \times \cdots \times B_G(id, r))$. By its definition, we have that the size of the set $A_k^r$ is bounded by

$$|A(k, r)| \leq |B_G(id, r)|^k.$$ Since an element $\gamma \in B_T(id, k)$ can be written as

$$\gamma = t_{i_1} \cdots t_{i_{|T|}} = s(\pi(t_{i_1})) \cdots s(\pi(t_{i_{|T|}})) s(\pi(\gamma))^{-1} : s(\pi(\gamma)) \in A(k, 1)s(\pi(\gamma)),$$

it follows that

$$B_T(id, k) \subseteq A(k, 1)s(B_G(id, k)).$$

Denote by $C(k, r)$ the union of the $\Gamma$-conjugates of $A(k, r)$, that is, $C(k, r) := \cup_{g \in \Gamma} g^{-1}A(k, r)g$. Since $A(k, r)$ is a finite subset in $N$ and $N$ is contained in the FC-center of $\Gamma$, the set $C(k, r)$ is a finite subset in $N$ as well. We have the following inclusion relation of sets:

**Claim 3.2.** Let $j$ be an integer such that $j|n$. Then

$$A(n, r) \subseteq C(j, r)^{n/j}A(n/j, r).$$

**Proof of the Claim.** Recall that an element $\gamma$ in $A(n, r)$ can be written in the form $\gamma = s(g_1) \cdots s(g_n)s(g_1 \cdots g_n)^{-1}$, where each $g_j$ is an element in $G$ satisfying $|g_j|_T \leq r$. Divide into blocks of size $j$ and write for each $1 \leq k \leq n/j$,

$$a_k = s(g_{j(k-1)+1}) \cdots s(g_{jk})s(g_{j(k-1)+1} \cdots g_{jk})^{-1},$$

$$x_k = s(g_{j(k-1)+1} \cdots g_{jk}),$$

and let $w_0 = id$,

$$w_k = x_1 \cdots x_k.$$ Then we have

$$\gamma = s(g_1) \cdots s(g_n)s(g_1 \cdots g_n)^{-1}$$

$$= \left( \prod_{k=1}^{n/j} a_kx_k \right) s(g_1 \cdots g_n)^{-1}$$

$$= \left( \prod_{k=1}^{n/j} w_{k-1}a_kw_{k-1}^{-1} \right) w_{n/j}s(g_1 \cdots g_n)^{-1}.$$
By definitions of the sets, we have that $a_k \in A(j, r)$, its conjugate $w_{k-1}a_kw_{k-1}^{-1} \in C(j, r)$, and

$$w_{n/j}s(g_1 \ldots g_n)^{-1} = \left( \prod_{k=1}^{n/j} s(g_j(k-1)+1 \ldots g_{jk}) \right) s(g_1 \ldots g_n)^{-1} \in A(n/j, rj).$$

The claim then follows from \([3.4]\).

Now we return to the proof of the lemma. Since $G$ is assumed to have sub-exponential growth, given any $\epsilon > 0$, there exists a $j_\epsilon \in \mathbb{N}$ such that

$$v_{G, \Gamma}(n) \leq e^{\epsilon n/3} \text{ for all } n \geq j_\epsilon.$$

In Claim \([3.2]\) take $r = 1$ and $j = j_\epsilon$. By Neumann’s result, the finite set $C(j, r)$ generates a virtually abelian group. Since finitely generated virtually abelian groups are either finite or of polynomial growth, there exists constants $K, d > 0$, depending only on $C(j, 1)$, such that for all $m \geq 1$, we have

$$|C(j, 1)|^m \leq Km^d.$$

Recall that by \([3.1]\), we have $|A(m, j)| \leq v_{G, \Gamma}(j)^m$. Combine the two parts, by Claim \([3.2]\) we have that for all $n \in j_\epsilon \mathbb{N}$,

$$|A(n, 1)| \leq \left| C(j, 1)^{n/j_\epsilon} \right| |A(n/j_\epsilon, j_\epsilon)| \leq K \left( \frac{n}{j_\epsilon} \right)^d v_{G, \Gamma}(j)^{n/j_\epsilon} \leq K \left( \frac{n}{j_\epsilon} \right)^d e^{\epsilon n/3}.$$

Combined with \([3.2]\), we have that for all $n \in j_\epsilon \mathbb{N}$,

$$|B_\Gamma(id, n)| \leq |A(n, 1)| \left| B_G(id, n) \right| \leq K \left( \frac{n}{j_\epsilon} \right)^d e^{2\epsilon n/3},$$

where $K$ and $d$ are constants only depending on $C(j, 1)$. In particular, it implies that

$$\liminf_{n \to \infty} \frac{1}{n} \log |B_\Gamma(id, n)| \leq \frac{2}{3} \epsilon.$$

Since this is true for every $\epsilon > 0$, we conclude that $\Gamma$ has sub-exponential growth.

\[\square\]

**Remark 3.3.** In the setting of Lemma \([3.1]\) if $G$ is of polynomial growth, then $\Gamma$ is of polynomial growth too. Indeed, by Gromov’s theorem \([Gro81]\), a polynomial growth group is virtually nilpotent; and any finitely generated FC-central extension of a virtually nilpotent group is virtually nilpotent, see e.g., \([Los01]\).

**Example 3.4.** Let $G$ be a finitely generated group marked with generating $k$-tuple $S$, that is we have an epimorphism $F_k \to G$. Let $R$ be the kernel $R = \ker (F_k \to G)$. Then the group $\Gamma = F_k/[F_k, R]$ is a finitely generated central extension of $G$. By Lemma \([3.1]\) $\Gamma = F_k/[F_k, R]$ has sub-exponential growth if and only if $G = F_k/R$ has sub-exponential growth.
The subgroup $R/[F_k, R]$, which is in the center of $\Gamma$, contains the Schur multiplier of $G$: by the Hopf formula (see e.g., [Rob96 Chapter 11]), we have

$$(R \cap [F_k, F_k]) / [F_k, R] \simeq H^2(G, \mathbb{Z}),$$

where $H^2(G, \mathbb{Z})$ is the Schur multiplier of $G$. The Schur multiplier of the first Grigorchuk group $\mathfrak{S}$ is determined by Grigorchuk in [Gr99]: $H^2(\mathfrak{S}, \mathbb{Z}) \simeq (\mathbb{Z}/2\mathbb{Z})^\infty$. It follows that the central extension $\Gamma = F_4/[F_4, R_{\mathfrak{S}}]$ of $\mathfrak{S} = F_4/R_{\mathfrak{S}}$ contains $(\mathbb{Z}/2\mathbb{Z})^\infty$ in its center.

4. Diagonal product of marked groups

As mentioned in the Introduction, a useful way to produce FC-central extensions is to take the diagonal product of a suitable sequence of marked groups.

**Fact 4.1** (c.f. [EZ19 Remark 9.2]). Suppose $((\Gamma_i, S_i))_{i=1}^\infty$ is a sequence of $k$-marked finite groups that converge to $(G, S)$ in the Cayley topology. Then the diagonal product $(\Delta, S)$ of $((\Gamma_i, S_i))_{i=1}^\infty$ is an FC-central extension of $G$. Moreover, if $G$ is residually finite, then $\Delta$ is residually finite as well.

4.1. Growth. The following lemma is a direct consequence of the definition of convergence in the space of marked groups and sub-multiplicativity of the volume growth function.

**Lemma 4.2.** Suppose $((\Gamma_i, S_i))_{i=1}^\infty$ is a sequence of $k$-marked groups converging to $(G, S)$ in the Cayley topology. The following statements are equivalent.

(i): The diagonal product $(\Delta, S)$ of the sequence $((\Gamma_i, S_i))_{i=1}^\infty$ is of sub-exponential growth.

(ii): For every $i \in \mathbb{N}$, $(\Gamma_i, S_i)$ is of sub-exponential growth and the limit group $(G, S)$ is of sub-exponential growth.

**Proof.** The (i)$\Rightarrow$(ii) direction is obvious because the groups $(\Gamma_i, S_i)$ and $(G, S)$ are quotients of $(\Delta, S)$.

We now prove (ii)$\Rightarrow$(i). Denote by $R_i$ be the largest radius $r$ such that the ball of radius $r$ around identity in $(\Gamma_i, S_i)$ coincide with the ball of same radius around identity in $(G, S)$. Given an index $i_0 \in \mathbb{N}$, consider the diagonal product $(\Delta_{>i_0}, S)$ of the collection $((\Gamma_i, S_i))_{i>i_0}^\infty$. Then by definition the balls of radius $r_{i_0} \equiv \inf_{i>i_0} R_i$ around the identities in the Cayley graphs of $(\Delta_{>i_0}, S)$ and $(G, S)$ coincide. Since the volume function is sub-multiplicative, we have for any $n > r_{i_0}$,

$$v_{\Delta_{>i_0}, S}(n) \leq v_{G,S}(r_{i_0})\left[\frac{n}{r_{i_0}}\right].$$

Regard $(\Delta, S)$ as the diagonal product of $((\Gamma_i, S_i))_{i\leq i_0}$ and $(\Delta_{>i_0}, S)$, we have

$$v_{\Delta, S}(n) \leq \left(\prod_{i=1}^{i_0} v_{\Gamma_i, S_i}(n)\right) v_{G,S}(r_{i_0})\left[\frac{n}{r_{i_0}}\right].$$

Let $\varepsilon > 0$ be any small positive constant. Since $G$ is of subexponential growth, there is a radius $\varepsilon n$ such that $v_{G,S}(n) \leq \exp(\varepsilon n/4)$ for all $n > \varepsilon n$. Since $((\Gamma_i, S_i))_{i=1}^\infty$ converge to $(G, S)$ in the Chabauty topology, there exists an index $i_0$ such that $r_{i_0} > \varepsilon n$. It follows that for any $n > r_{i_0} > \varepsilon n$,

$$v_{\Delta, S}(n) \leq \left(\prod_{i=1}^{i_0} v_{\Gamma_i, S_i}(n)\right) v_{G,S}(r_{i_0})\left[\frac{n}{r_{i_0}}\right] \leq \left(\prod_{i=1}^{i_0} v_{\Gamma_i, S_i}(n)\right) e^{\varepsilon n/2}. $$

\[10\]
Since each group $\Gamma_i$ is assumed to be of sub-exponential growth, there exists a constant $n_{i_0}$ such that for any $i \leq i_0$ and $n \geq n_{i_0}$, we have that $v_{\Gamma_i,S_i}(n) \leq e^{\varepsilon n/2i_0}$. It follows then from (4.1) that for any $n \geq \max \{n_{i_0}, r_{i_0}\}$, $v_{\Delta,S} (n) \leq e^{\varepsilon n}$. Since $\varepsilon$ is arbitrary, we conclude that $(\Delta, S)$ is of subexponential growth.

4.2. The direct sum assumption. In this subsection we consider a splitting condition which makes the structure of the diagonal product more transparent.

**Definition 4.3** (Direct sum assumption (D)). We say a sequence of $k$-marked groups $(\Gamma_i, S_i), i \in \mathbb{N}$ satisfies the direct sum condition (D) over $(G, S)$ if

- $(\Gamma_i, S_i) \to (G, S)$ when $i \to \infty$ in the Cayley topology,
- for each $i$, there is a marked quotient $G_i$ of $\Gamma_i$, such that the diagonal product $\Gamma$ of the sequence $((\Gamma_i, S_i))_{i \in \mathbb{N}}$ satisfies that

$$\ker(\Gamma \to G) = \bigoplus_{i \in \mathbb{N}} \ker(\Gamma_i \to G_i).$$

If satisfied, the direct sum assumption (D) plays an important role in understanding the structure of the diagonal product $\Gamma$. The groups we consider in Section 5 satisfy the direct sum assumption (D). Typically, in situations where (D) can be verified, there is a natural choice of $(G_i)$, e.g., $\Gamma_i$ by construction is an extension of $G_i$, where $(G_i, S_i)$ is a sequence of quotient groups of $(G, S)$ that converges to $(G, S)$ when $i \to \infty$. To verify (D), it then suffices to show:

1. the length of the shortest nontrivial element in $\ker(\Gamma_i \to G_i)$ goes to infinity as $i \to \infty$;
2. for each $i$, one can find words in $F_i$ such that their images in any $\Gamma_j, j > i$ are trivial and the normal closure of their images in $\Gamma_i$ is $\ker(\Gamma_i \to G_i)$.

5. The definition of $\Gamma(\mathcal{L})$ and formal recursions

This section is a preparation for the study of finitely generated subgroups of $\Gamma(\mathcal{L})$ in the next section. Throughout this section, denote by the $F$ the free product $F = \langle \mathbb{Z}/2\mathbb{Z} \ast (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})\rangle$. Mark $F$ with the generating tuple $S = \langle a, b, c, d \rangle$, where $a, b, c, d$ are involutions and $F = \langle a \rangle \ast \langle (b) \times (c) \rangle$, $d = bc$. In Subsection 4.1 we present the type of construction in [Ers06] Section 2 and [KP13] in the language of permutation wreath products, which is algebraically rather transparent. In Subsection 5.2 we set up notations for the formal wreath recursion and explain the branching structure on the formal level. Automorphisms of $\Gamma(\mathcal{L})$ are discussed in Subsection 5.3 and a sufficient condition for $\Gamma(\mathcal{L})$ to satisfy the direct sum assumption (D) is provided in Subsection 5.4.

5.1. The definition of $\Gamma(\mathcal{L}, \omega)$. We use notations introduced in Subsection 2.2. Let $\mathcal{L} = (A_n)_{n=1}^\infty$ be a sequence of marked quotients of $F$ and $\omega$ be a string in $\{0, 1, 2\}^\infty$. For each $n$, fix a bijection

$$(5.1) \quad \psi_n : \{b^{n=\omega}, c^{n=\omega}, d^{n=\omega}\} \to \{b, c, d\}.$$ 

For example, one can choose $\psi_n$ to send $x^{n(\omega)} \mapsto x$, for $x = b, c, d$.

Recall that each letter $0, 1, 2$ denotes a nontrivial homomorphism $\{id, b, c, d\} \to \{id, a\}$. Recall the recursive definition of the tree automorphisms $b_\omega, c_\omega$ and $d_\omega$. 


Denote by \( \pi_n \) the natural projection \( \text{Aut}(T) \to \text{Aut}(T^n) \). Consider the permutation wreath product \( A_n \wr_\pi \pi_n (G_\omega) \) and its subgroup \( \Gamma_n \) defined as

\[
\Gamma_n = \langle S_n \rangle, \quad S_n = (a_n, b_n^\omega, c_n^\omega, d_n^\omega),
\]

where \( a_n = (id, a), \quad x_n^\omega = \left( \delta_{1n}^{\psi_n(x_n^\omega)} + \delta_{1n-1}^{\omega_0(x)}, x \right) \) for \( x \in \{b, c, d\} \).

Note that the definition of \( x_n^\omega \) mimics the sections of the generator \( x_\omega \) on the level \( n \) in \( G_\omega \), \( x \in \{b, c, d\} \), while the difference is that the lamp group is \( A_n \) in \( \Gamma_n \).

**Definition 5.1.** Given a sequence \( \mathcal{L} \) of finite quotients of \( \mathbf{F} \), \( \omega \in \{0, 1, 2\}^\infty \) and bijections \( (\psi_n)_{n=1}^\infty \) as in (5.1). Let the sequence of marked groups \( ((\Gamma_n)^\omega, S_n^\omega)_{n=1}^\infty \) be defined as in (5.2). We call the diagonal product of the sequence \( ((\Gamma_n)^\omega, S_n^\omega)_{n=1}^\infty \) the extension of \( G^\omega \) with \( \mathcal{L} \) as input and denote it as \( \Gamma(\mathcal{L}, \omega) \).

Although most of the statements in this section can be generalized to \( \Gamma(\mathcal{L}, \omega) \), where all three letters 0, 1, 2 appear infinitely often in \( \omega \), in order not to burden the reader with heavier notations, we will focus on the first Grigorchuk group \( \mathfrak{G} = G(012)^\infty \). For the first Grigorchuk group, we suppress the reference to the string \((012)^\infty\) and

\[
\Gamma(\mathcal{L}) = \Gamma(\mathcal{L}, (012)^\infty),
\]

where the bijections are: for \( \omega = (012)^\infty \),

\[
\psi_n(1_\omega) = b, \quad \psi_n(2_\omega) = c, \quad \psi_n(3_\omega) = d \quad \text{for } n \equiv 0 \mod 3,
\]

\[
\psi_n(1_\omega) = c, \quad \psi_n(2_\omega) = d, \quad \psi_n(3_\omega) = b \quad \text{for } n \equiv 1 \mod 3,
\]

\[
\psi_n(1_\omega) = d, \quad \psi_n(2_\omega) = b, \quad \psi_n(3_\omega) = c \quad \text{for } n \equiv 2 \mod 3.
\]

Indexing by \( \mathbb{N} \) in \( \mathcal{L} \) is for convenience. In particular, the shift \( s \), where \( s\mathcal{L} = (A_{n+1})_{n=1}^\infty \), will be useful in the recursion. In our notation, if the diagonal product is taken over a subsequence \( (n_i) \), then for a level \( n \not\in \{n_i : i \in \mathbb{N}\} \), the corresponding \( A_n \) is the trivial group \( \langle id \rangle \).

5.2. **Formal wreath recursion.** Denote by \( \mathfrak{S}_2 \) the permutation group of \( \{0, 1\} \) and \( \varepsilon \) the transposition \((0, 1)\). Recall that under the canonical wreath recursion, the generators of the Grigorchuk group \( \mathfrak{G} \) give

\[
a = (id, id) \varepsilon, \quad b = (a, c), \quad c = (a, d), \quad d = (id, b).
\]

We now consider the recursion rules on the formal level. Let \( \varphi \) be the homomorphism

\[
\varphi : \mathbf{F} \to (\{0,1\} \wr \mathfrak{S}_2)
\]

determined by

\[
a \mapsto (id, \varepsilon),
\]

\[
b \mapsto (\delta^c_1 + \delta^a_0, id),
\]

\[
c \mapsto (\delta^d_1 + \delta^a_0, id),
\]

\[
d \mapsto (\delta^b_1, id).
\]

Following the notation of canonical wreath recursion on \( \text{Aut}(T) \), we record \( \varphi(w) = (\delta^w_0 + \delta^w_1, \varepsilon) \) as \((w_0, w_1) \varepsilon \). We refer to the homomorphism \( \varphi \) as the (1-step) **formal wreath recursion.** The word "formal" to indicate that the map is considered
on the free product $F$ instead of rooted tree automorphisms. The homomorphism $\varphi$ can be applied recursively and we have

$$\varphi^k : F \to F \wr G_k.$$  

Mark $\varphi^k(F)$ by the generating tuple of $\langle \varphi^k(a), \varphi^k(b), \varphi^k(c), \varphi^k(d) \rangle$. Denote by $\theta_k$ the projection $F \wr G_k \to G_k$. Because of the recursion rules, the following diagram commute:

$$\begin{array}{ccc}
F & \xrightarrow{\varphi^k} & \varphi^k(F) \\
\pi \downarrow & & \downarrow \theta_k \\
\mathcal{G} & \xrightarrow{\pi_k} & G_k
\end{array}$$

Let $\sigma : \{a, b, c, d\}^* \to \{a, b, c, d\}$ be the substitution

(5.4)  
\[ \sigma : a \mapsto aba, \ b \mapsto d, \ c \mapsto b, \ d \mapsto c. \]

A similar substitution (often referred to as the Lysionok substitution) which sends

\[ a \mapsto aca, \ b \mapsto d, \ c \mapsto b, \ d \mapsto c \]

appears in the recursive presentation [Lys85] of the Grigorchuk group $G$. We may regard the substitution $\sigma$ as a homomorphism $F \to F$.

Denote by $K$ the normal closure of $[a, b]$ in $F$. Let $K$ be the image of $K$ under the projection $\pi : F \to \mathcal{G}$. The group $\mathcal{G}$ is regularly branching over $K$, see e.g., [BGS03] Subsection 1.6.6. Fact 5.2 expresses the branching structure in terms of the formal recursion. Denote by $K_v$ the subgroup of $F \wr L_k G_k$, where $k = |v|$, defined as

$$K_v := \{ (\delta^w_v, id) : w \in K \}.$$

**Fact 5.2.** The substitution $\sigma$ satisfies that $\sigma(K) \subseteq K$ and for any $w \in K$, $\varphi(\sigma(w)) = (id, w)$. Moreover, we have

(5.5)  
$$\bigwedge_{v \in L_k} K_v < \varphi^k(F).$$

**Proof.** It suffices to check on the generator $[a, b]$:

$$\sigma([a, b]) = abadabad = [a, b] [a, b]^d \in K$$

and

$$\varphi(\sigma([a, b])) = \varphi(abadabad) = (id, abab).$$

The calculation above also shows that $\varphi \left( ([b^a, d]^F \right) = K \times K$.

Iterate $k$ times, we have that for $w \in K$, $\varphi^k(\sigma(w)) = (\delta^w_k, id)$. Then (5.5) follows. \qed

The following length reduction property can be seen directly from the rule of multiplication in $F \wr L_k G_k$. For an element $w \in F$, under the formal wreath recursion $\varphi^k : F \to F \wr L_k G_k$, write

$$\varphi^k(w) = ((w_v)_{v \in L_k}, \pi_k(w)).$$

The element $w_v$ is referred to as the *section of $w$ at $v$ under $\varphi^k$*. Recall that $d_S$ denotes the graph distance on the orbital Schreier graph.
Fact 5.3 (Length reduction, c.f. [KP13, Lemma 5.11]). Suppose $w \in F$ is a word length $\ell \leq 2^{k-1} - 1$, then under $\phi^k$, the word length of the section $w_v$ is bounded by 1. More precisely, $w_v \in \{id, b, c, d\}$ for $v \in L_k$ such that $d_S(v, 1^k) \leq 2^{k-1} - 1$; and $w_v \in \{id, a\}$ for $v \in L_k$ such that $d_S(v, 1^{k-1}0) \leq 2^{k-1} - 1$.

Fact 5.3 implies that the sequence of marked groups $((\Gamma_n, S_n))_{n=1}^\infty$ converges to the Grigorchuk group $(G, S)$ in the Cayley topology.

5.3. Lifting automorphisms of $\mathfrak{G}$ to $\Gamma(L)$. The automorphism group of $\mathfrak{G}$ is determined by Grigorchuk and Sidki in [GS04]. For $i \in \mathbb{N}$, let $\theta_i$ be the element in $\text{St}_{\text{Aut}(T)}(i)$ and the sections in level $i$ are given by

$$\theta_i(v) = \text{id}, \quad \text{for all } v \in L_i.$$

By the main theorem of [GS04], $\text{Aut}(\mathfrak{G})$ is isomorphic to the normalizer $N_{\text{Aut}(T)}(\mathfrak{G})$ of $\mathfrak{G}$ in $\text{Aut}(T)$ and

$$(5.6) \quad N_{\text{Aut}(T)}(\mathfrak{G}) = \mathfrak{G}V,$$

where $V$ is the subgroup generated by the collection $\{\theta_i, i \in \mathbb{N}\}$.

In the next section we will need to lift automorphisms of $\mathfrak{G}$. Without further assumptions it is not always guaranteed that an outer automorphism of $\mathfrak{G}$ can be lifted to $\Gamma(L)$, however the following weaker version will be sufficient for our purposes. Denote by $\Delta_{>k} = \Delta_{>k}(L)$ the diagonal product of $((\Gamma_n, S_n))_{n=k+1}^\infty$. Note that $\Delta_{>k}$ can be identified with $\Gamma(L^k)$, where

$$L^k = \left( \{\text{id}\}, \ldots, \{\text{id}\}, A_{k+1}, A_{k+2}, \ldots \right).$$

Denote by $\pi_{>k}$ the marked projection $F \to \Delta_{>k}$. By its definition, we have the embedding

$$(5.7) \quad \varphi_k : \Delta_{>k} \hookrightarrow \Gamma(s^kL) \rtimes_k G_k,$$

where $\varphi_k$ sends

$$a \mapsto (id, a), \quad x \mapsto \left( \delta_{1k}^{\omega_k(x)} + \delta_{1k}^{\omega_{k-1}(x)}, x \right), \quad x \in \{b, c, d\},$$

where $\omega = (012)^\infty$ and $\psi_k$ is specified in [5.3]. Consider the quotient map

$$\varrho_k : F \rtimes_k G_k \to \Gamma(s^kL) \rtimes_k G_k$$

which is induced by the marked projection $F \to \Gamma(s^kL)$. The following diagram commutes and $\varphi_k$ is an isomorphic:

$$\begin{array}{ccc}
F & \xrightarrow{\varphi_k} & \varphi_k(F) \\
\pi_{>k} \downarrow && \downarrow \varrho_k \\
\Delta_{>k} & \xrightarrow{\varphi_k} & \varrho_k \circ \varphi_k(F)
\end{array}$$

In particular, $\Delta_{>k}$ can be viewed as a marked quotient of $\varphi_k(F)$. 
Lemma 5.4. Let \( \tau : \mathfrak{G} \to \mathfrak{G} \) be an automorphism. Then there exists an integer \( k \in \mathbb{N} \) and an automorphism \( \hat{\tau} : \Delta_{>k} \to \Delta_{>k} \) such that \( \pi \circ \hat{\tau} = \tau \circ \pi \):

\[
\begin{align*}
\Delta_{>k} & \xrightarrow{\hat{\tau}} \Delta_{>k} \\
\pi & \downarrow \quad \quad \downarrow \pi \\
\mathfrak{G} & \xrightarrow{\tau} \mathfrak{G}.
\end{align*}
\]

Proof. By [GS04], the automorphism \( \tau \) of \( \mathfrak{G} \) can be written as a finite composition \( \tau = \alpha_1 \ldots \alpha_r \), where each \( \alpha_j \) is either an inner automorphism or conjugation by \( \theta_{ij} \) for some \( i \in \mathbb{N} \). Set \( i_j = 0 \) if \( \alpha_j \) is inner. Let

\[
k = \max_{1 \leq j \leq r} i_j + 1.
\]

To show that \( \tau \) can be lifted to an automorphism of \( \Delta_{>k} \), it suffices to show that for each \( i \in \{1, \ldots, k-1\} \), conjugation by \( \theta_i \) can be lifted to an automorphism of \( \Delta_{>k} \).

Let \( W_i = F \mathfrak{L}_i, G_i \) and \( \theta_i \) be the element of \( W_i \) given by

\[
\theta_i = (f, id_{G_i}), \quad \text{where } f(v) = ad \text{ for all } v \in L_i.
\]

By explicit calculation on the generators, we have that in \( W_1 \),

\[\tag{5.8}
[\theta_1, \varphi(a)] = 1, \quad [\theta_1, \varphi(c)] = (adad, adad), \quad [\theta_1, \varphi(d)] = (1, \{ad, b\}).
\]

Since \( (ad)^4 \in \ker(\varphi) \), under the formal recursion, we have

\[\tag{5.9}
\varphi([\theta_1, \varphi(c)]) = \varphi((dada, adad)) = \varphi^2((ac)^4).
\]

Apply the Lysionok substitution to \( (ac)^4 \), we obtain the word \( (acab)^4 \). Note that \( (acab)^2 \in K \) and \( \varphi((acab)^4) = ((da)^4,(ac)^4) \). Inductively, for \( i \geq 2 \), we have

\[\tag{5.10}
[\theta_i, \varphi^i(a)] = 1, \quad \varphi([\theta_i, \varphi^i(c)]), \quad \varphi([\theta_i, \varphi^i(d)]) \in \varphi^3(K_{i-2}).
\]

Now regard \( \Delta_{>k} \) as a subgroup of \( \Gamma(g^kL)_k G_k \) via the embedding \( \varphi_k \). Define the map \( \hat{\tau}_i \) on \( \Delta_{>k} \) to be

\[\tag{5.11}
\hat{\tau}_i(g) = g^{\hat{\theta}_i} \quad \text{where } \hat{\theta}_i = \varphi_k(\varphi^{k-i}(\theta_i)).
\]

By its definition, it is clear that when projected to \( \mathfrak{G} \), the map is conjugation by \( \theta_i \). It remains to show that for any \( g \in \Delta_{>k} \), \( g^{\hat{\theta}_i} \) is in \( \Delta_{>k} \), equivalently, \( [\hat{\theta}_i, g] \in \Delta_{>k} \). It suffices to check on the generators \( a, c, d \). For the generator \( a \), we have \( ([\hat{\theta}_i, a]) = id \in \Delta_{>k} \). For the other two generators, if \( i = 1 \), then by direct calculations as in (5.8) and (5.9) we have that \( [\hat{\theta}_1, c], [\hat{\theta}_1, d] \in \Delta_{>k} \). For \( i \geq 2 \), by (5.10) we have that \( [\hat{\theta}_1, c] \) and \( [\hat{\theta}_1, d] \) are in the projection of \( \varphi_k(\varphi^{k-i+2}(K_{1-i-2})) \).

By (5.9), \( K_{1-i-2} < \varphi^{i-2}(F) \). It follows then \( [\hat{\theta}_i, c], [\hat{\theta}_i, d] \in \Delta_{>k} \).

\[\square\]

Recall the direct sum assumption (D) as in Definition 4.3. Under (D), Lemma 5.4 can be improved:

Corollary 5.5. Suppose \( \Gamma(L) \) satisfies (D). Let \( \tau : \mathfrak{G} \to \mathfrak{G} \) be an automorphism. Then there exists an automorphism \( \hat{\tau} : \Gamma(L) \to \Gamma(L) \) such that \( \pi \circ \hat{\tau} = \tau \circ \pi \).
\textit{Proof.} It suffices to prove the statement for the automorphism $\tau_i$ of $\mathcal{G}$ which is the conjugation by $\theta_i$, $i \in \mathbb{N}$. By Lemma 5.4, $\tau_i$ can be lifted to an automorphism of $\Delta_{>i+1}$ given by conjugation by $\bar{\theta}_i$ as defined in (5.11) with $k = i + 1$. Denote by $\Delta_{\leq k}$ the diagonal product of the first $k$ factors $((\Gamma_n, S_n))_{n=1}^k$. \textquote{\begin{align*}
\end{align*}\text{Regard } \Gamma = \Gamma(\mathcal{L}) \text{ as the diagonal product of } \Delta_{\leq i+1} \text{ and } \Delta_{>i+1} \text{ and record its elements as } (f, g), \text{ where } f \in \Delta_{\leq i+1} \text{ and } g \in \Delta_{>i+1}. \text{ Define the map } \tilde{\tau}_i \text{ to be}
\end{align*}}\text{(5.12)}
$$
\tilde{\tau}_i ((f, g)) = \left(f, g^{\bar{\theta}_i}\right).
$$

In words, the first factor remains the same, while the second factor, regarded as an element in $\Gamma (\mathcal{L})_{\leq i+1} G_{i+1}$, is conjugated by $\bar{\theta}_i$. To show that map $\tilde{\tau}_i$ is an automorphism of $\Gamma$, we need to show that for $(f, g) \in \Gamma$, the image $\left(f, g^{\bar{\theta}_i}\right)$ is in $\Gamma$, equivalently, $(id, [\bar{\theta}_i, g]) \in \Gamma$. By Lemma 5.4, we have that $[\bar{\theta}_i, g] \in \Delta_{>i+1}$. Under (D), a pair $(f, g) \in \Delta_{\leq i+1} \times \Delta_{>i+1}$ is in the diagonal product $\Delta$ if and only if the projections to $G_{i+1}$ and $\mathcal{G}$ are consistent, that is, $\tilde{\pi}_{i+1}(f) = \pi_{i+1} \circ \pi(g)$, where $\tilde{\pi}_{i+1}$ is the projection $\Delta_{\leq i+1} \to G_{i+1}$, $\pi$ is the projection $\Delta_{>i+1} \to \mathcal{G}$. By direct inspection on the generators, we have that $[\bar{\theta}_i, a], [\bar{\theta}_i, c]$ and $[\bar{\theta}_i, d]$ all project to identity in $G_{i+1}$. It follows that $[\bar{\theta}_i, g]$ projects to identity in $G_{i+1}$ for any $g \in \Delta_{>i+1}$ and $(id, [\bar{\theta}_i, g]) \in \Gamma$.

\textbf{5.4. A sufficient condition for (D).} Although not needed in the next section, we explain a condition on the sequence $\mathcal{L}$ that not only guarantees the direct sum assumption (D), but also allows to identify the FC-center of $\Gamma(\mathcal{L})$ explicitly.

\textbf{Lemma 5.6.} Suppose that $\mathcal{L} = (A_n)_{n=1}^\infty$ is a sequence of marked quotients of $F$ such that the subgroup $K \cap \ker (\varphi)$ projects onto the commutator $[A_n, A_n]$ for all $n \in \mathbb{N}$. Then $((\Gamma_n, S_n))_{n=1}^\infty$ satisfies the direct sum assumption (D) over $\mathcal{G}$ and the FC-center of $\Gamma(\mathcal{L})$ is
$$
Z_{FC} (\Gamma (\mathcal{L})) \cong \bigoplus_{n=1}^\infty \bigoplus_{v \in \mathcal{L}_n} ([A_n, A_n])_v.
$$

\textit{Proof.} By Fact 5.3, the sequence $((\Gamma_n, S_n))_{n=1}^\infty$ converges to the Grigorchuk group $(\mathcal{G}, S)$ in the Cayley topology.

We first verify that for any $n \in \mathbb{N}$ and $g \in [A_n, A_n]$, there exists $\gamma \in \Gamma(\mathcal{L})$ such that the projection of $\gamma$ to $\Gamma_j$, $j \geq n + 1$, is trivial; while the projection of $\gamma$ to $\Gamma_n$ is $(\delta_{1v}, \text{id})$. Let $w \in K \cap \ker (\varphi)$. Apply the substitution $\sigma$ to $w$ for $n$ times, by Fact 5.2, we have $\varphi^n (\sigma^n (w)) = (\delta_1^w, \text{id})$. Since $w \in \ker (\varphi)$, further recursions give trivial image, that is, $\varphi^j (\sigma^n (w)) = \text{id}$ for all $j \geq n + 1$. Thus if $g$ is the image of $w$ in the marked quotient $A_n$, we can take the claimed element $\gamma$ to be the image of $\sigma^n (w)$ in $\Gamma(\mathcal{L})$.

The claim in the previous paragraph implies that if $K \cap \ker (\varphi)$ projects onto the commutator $[A_n, A_n]$ for all $n \in \mathbb{N}$, then
$$
\Gamma (\mathcal{L}) \cong \bigoplus_{n=1}^\infty \bigoplus_{v \in \mathcal{L}_n} ([A_n, A_n])_v.
$$
Denote by $\bar{\Gamma}_n$ the quotient group $\Gamma_n/\bigoplus_{v \in L_n} ([A_n, A_n])_v$. Write $\bar{\mathcal{L}} = (\bar{\Lambda}_n)_{n=1}^{\infty}$, where $\bar{\Lambda}_n$ is the liberalization $A_n/[A_n, A_n]$. By definition of $\Gamma_n$ and Fact 5.3, we have that $(\bar{\mathcal{L}}, \bar{\mathcal{S}})$ converges to $(\mathcal{G}, \mathcal{S})$ and the diagonal product $(\bar{\mathcal{L}}, \bar{\mathcal{S}})$ is isomorphic to $(\mathcal{G}, \mathcal{S})$. It follows that the second item in (D) is satisfied with the sequence $(\bar{\Gamma}_n, \bar{\mathcal{S}}_n)$. By definition it is clear that an element in $(\bar{\mathcal{L}}, \bar{\mathcal{S}})$ is isomorphic to $\mathcal{G}$, in particular, it is ICC, we conclude that the FC-center of $\Gamma(\mathcal{L})$ is isomorphic to $\mathcal{G}$, in particular, it is ICC.

\[ \square \]

**Example 5.7.** Recall that $\mathcal{K} = ([a, b])^F = [B, F]$. By direct calculation, the word $(ad)^4 \in \ker(\varphi)$. Therefore the element $[b, (ad)^4] \in K \cap \ker(\varphi)$. If each $A_n$ is normally generated by the element $[b, (ad)^4]$, then the sequence $\mathcal{L} = (\Lambda_n)$ satisfies the assumption of Lemma 5.6. By [KP13, Lemma 6.1], if $n$ is an integer such that the only prime factors of $n$ are of the form $p \equiv 1 \mod 4$, then there exists a generating set $\{a, b, c, d\}$ of $PSL_2(\mathbb{Z}/n\mathbb{Z})$ such that the element $[c, d, [b, (ad)^4]]$ normally generates $PSL_2(\mathbb{Z}/n\mathbb{Z})$.

## 6. Finitely generated subgroups of $\Gamma(\mathcal{L})$

In this section we study in detail finitely generated subgroups of $\Gamma(\mathcal{L})$. We continue to use notations introduced in Section 4. The main point is that via natural lifting arguments, certain properties of finitely generated subgroups of the Grigorchuk $\mathcal{S}$ are inherited by $\Gamma(\mathcal{L})$. In [GW13], it is proved that $\mathcal{S}$ is subgroup separable and any finitely generated subgroup $H$ of $\mathcal{S}$ is either finite or abstractly commensurable to $\mathcal{S}$. We show similar statements for $\Gamma(\mathcal{L})$.

**Theorem 6.1.** Let $\mathcal{L} = (\Lambda_n)_{n=1}^{\infty}$ be a sequence of finite marked quotients of $F$. Let $H$ be a finitely generated subgroup of $\Gamma(\mathcal{L})$. Then there exists $\ell, p \in \mathbb{N}$ such that $H$ is abstractly commensurable to $\prod_{k=1}^{p} \Gamma(s^k \mathcal{L})$.

**Theorem 6.2.** Let $\mathcal{L} = (\Lambda_n)_{n=1}^{\infty}$ be a sequence of finite marked quotients of $F$. Then $\Gamma(\mathcal{L})$ is subgroup separable.

### 6.1. Generalized sections

Recall the notion of sections of a subgroup $H < Aut(T)$. Given a level $k$ and a vertex $v \in L_k$, consider the level stabilizer $St_H(k) = \{ h \in H : x \cdot h = x \text{ for all } x \in L_k \}$. Under the canonical wreath recursion, $\varphi^k(St_H(k)) < \prod_{u \in L_k} Aut(T_u)$.

Denote by $\phi_v$ the natural projection from the product $\prod_{u \in L_k} Aut(T_u)$ to $Aut(T_v)$. The section of group $H$ at $v$ is defined as $H_v := \phi_v(St_H(k))$, $k = |v|$.

For the diagonal product $\Gamma(\mathcal{L})$, we use the following generalized notion of level stabilizers and sections.

**Notation 6.3 (Generalized stabilizers and sections).** Denote by $\Delta_{>k} = \Delta_{>k}(\mathcal{L})$ the diagonal product of $(\Gamma_n, \mathcal{S}_n)_{n=k+1}^{\infty}$ and $\pi_{>k} : \Gamma(\mathcal{L}) \to \Delta_{>k}$ the marked projection. Let $H$ be a subgroup of $\Gamma(\mathcal{L})$. Given a level $k$, define the generalized level stabilizer as $St_H(k) := \{ h \in H : x \cdot h = x \text{ for all } x \in L_k \}$.
where the action of $\Gamma(\mathcal{L})$ on $L_k$ factors through the projection $\Gamma(\mathcal{L}) \to \mathfrak{S}$. Under the recursion $\varphi_k$ given in (5.7), we have

$$\varphi_k : \text{St}_H(k) \to \prod_{\ell \in L_k} \Gamma(s^k \mathcal{L}).$$

Denote by $\phi_v$ the natural projection from $\prod_{\ell \in L_k} \Gamma(s^k \mathcal{L})$ to the coordinate indexed by $v$. The generalized section of $H$ at $v$ is defined as

$$H_v := \phi_v \circ \varphi_k (\text{St}_H(k)).$$

For an element $\gamma \in \text{St}_H(k)$, write $\gamma_v$ for the section

$$\gamma_v := \phi_v \circ \varphi_k (\gamma).$$

We introduce one more piece of notation.

**Notation 6.4** (Saturated subgroup). We say a subgroup $H$ of a product $W_1 \times \ldots \times W_\ell$ is saturated if $\phi_j(H) = W_j$, for every $j \in \{1, \ldots, \ell\}$, where $\phi_j$ is the natural projection $W_1 \times \ldots \times W_\ell \to W_j$. In this case we write $H \leq_s W_1 \times \ldots \times W_\ell$. For example, by the definitions, $\varphi_k (\text{St}_H(k))$ is a saturated subgroup of $\prod_{\ell \in L_k} H_v$.

Suppose $H \leq_s W_1 \times W_2$. Write $L_1 = H \cap (W_1 \times \{id_{W_2}\})$ and $L_2 = H \cap ((\{id_{W_1}\} \times W_2)$. Then there is an isomorphism between $W_1/L_1$ and $W_2/L_2$. In particular, if both $W_1$ and $W_2$ are just-infinite, then either $L_i < f_i$, $W_i$ for $i = 1, 2$ and $H$ is a finite index subgroup of $W_1 \times W_2$; or $L_i$ is trivial and $H$ is isomorphic to $W_1$.

**6.2. Ingredients in the proofs.** The key ingredient in the proof of Theorem 6.1 is the following property of finitely generated subgroups of the Grigorchuk group $\mathfrak{S}$ shown in the work of Grigorchuk and Wilson [GW03].

**Lemma 6.5** (Consequence of [GW03 Theorem 3]). Let $H$ be a finitely generated subgroup of $\mathfrak{S}$. Then there exists a finite level $\ell$ such that for each vertex $v \in T_\ell$, the section $H_v$ of $H$ is either finite or equal to $\mathfrak{S}$.

**Proof.** Denote by $\mathcal{X}$ the set of subgroups of $\mathfrak{S}$ satisfying the statement. To show that $\mathcal{X}$ contains all finitely generated subgroups of $\mathfrak{S}$, we check that the three conditions in [GW03 Theorem 3] are satisfied.

(i) Clearly $\{id\} \in \mathcal{X}$ and $\mathfrak{S} \in \mathcal{X}$.

(ii) Suppose $H \in \mathcal{X}$ is a subgroup such that for each vertex $v \in T_\ell$, the section $H_v$ of $H$ is either finite or equal to $\mathfrak{S}$. Let $L$ be a subgroup of $\mathfrak{S}$ such that $H < L$ and $|L : H| < \infty$. Then $\text{St}_H(\ell) < f_\ell$, $\text{St}_L(\ell)$. It follows that at each vertex $v \in T_\ell$, the section $H_v$ is a finite index subgroup of the section $L_v$. Therefore the sections $L_v$ are either finite or equal to $\mathfrak{S}$, that is, $L \in \mathcal{X}$ as well.

(iii) If $H$ is finitely generated subgroup of $\text{St}_\mathfrak{S}(1)$, and both sections $H_0$ and $H_1$ are in $\mathcal{X}$, then by definition of $\mathcal{X}$ we have that $H \in \mathcal{X}$.

Then $\mathcal{X}$ contains all finitely generated subgroups of $\mathfrak{S}$ by [GW03 Theorem 3].

Another ingredient is the following basic lifting property.

**Lemma 6.6** (Lifting). Suppose $\Gamma$ is a finitely generated FC-central extension of $G$ that fits into $1 \to N \to \Gamma \to \hat{G} \to 1$, where $N$ is torsion. Let $H$ be a subgroup of $\Gamma$.

(i): If $\pi(H)$ is finite and $H$ is finitely generated, then $H$ is finite.
(ii): If $\pi(H)$ is finite index in $G$, then $H$ is finite index in $\Gamma$.

Proof. (i) Suppose $H = \langle T \rangle$ where $T \subseteq \Gamma$ is finite, and $\pi(H)$ is finite. Then $H \cap N$ is a finite index subgroup of $H$. It follows that $H \cap N$ is a finitely generated torsion FC-group, thus finite. It follows that $H$ is finite.

(ii) Consider the subgroup $\bar{H} < \Gamma$ given by $\bar{H} = \{ \gamma \in \Gamma : \pi(\gamma) \in \pi(H) \}$. Then $\bar{H}$ is a finite index subgroup of $\Gamma$. Let $S = \{ s_1, \ldots, s_k \}$ be a finite generating set of $\bar{H}$. Since $\pi(\bar{H}) = \pi(H)$, each generator $s_i = n_is'_i$, where $s'_i \in H$, $n_i \in N$. Since $N < Z_{\text{FC}}(\Gamma)$ and $N$ is torsion, it follows that $\bar{H}$ is contained in the union of finitely many cosets of $\langle s'_1, \ldots, s'_k \rangle$. It follows that $\bar{H}$ is of finite index in $\Gamma$.

The lifting lemma 6.6 implies that the branching structure of $\mathfrak{S}$ is inherited by $\Gamma(\mathcal{L})$. Consider the following subgroup of $\Gamma = \Gamma(\mathcal{L})$ which can be viewed as a generalized vertex rigid stabilizer. Given a vertex $v \in \mathcal{L}_\ell$ and $m \in \mathbb{N}$, define

$$R^\ell_m(T_v) := \{ \gamma \in \text{St}_\ell(\mathcal{L}) : \ell = |v|, \pi_{\leq \ell}(\gamma) = \text{id}, \gamma_u = \text{id} \text{ for all } u \in \mathcal{L}_\ell \setminus \{v\} \},$$

where $\pi_{\leq \ell}$ is the projection $\Gamma(\mathcal{L}) \to \Delta_{\leq \ell}$. We may also regard $R^\ell_m(v)$ as a subgroup of $\Gamma(\mathcal{s^T\mathcal{L}})$. Similar to level rigid stabilizers, we define

$$(6.1) \quad R^\ell_m(T_v) = \prod_{u = uz, |z| = m} R^\ell_m(u).$$

**Lemma 6.7 (Branching in $\Gamma(\mathcal{L})$).** The generalized level rigid stabilizer $R^\ell_m(T_v)$ is of finite index in $\Gamma(\mathcal{s^T\mathcal{L}})$, $\ell = |v|$.

Proof. Let $u$ be a vertex in the level $m$ of the subtree $T_v$. In the group $\mathfrak{S}$, the level rigid stabilizer $R_\mathfrak{S}(u)$ is finitely generated and of finite index in the section $\mathfrak{S}_u$. Let $\{ g_1^u, \ldots, g_k^u \}$ be a generating set of $R_\mathfrak{S}(u)$. For each $g_j^u$, take an element $\bar{g}_j^u \in \Gamma(\mathcal{L})$ such that $\pi(\bar{g}_j^u) = g_j$. Consider the subgroup $F^u$ generated by $\{ \bar{g}_1^u, \ldots, \bar{g}_k^u \}$. Applying Lemma 6.6, we have that the generalized section of $F^u$ at $u$ is of finite index in $\Gamma(\mathcal{s^T\mathcal{L}})$ and at any vertex $v \in \mathcal{L}_{\ell + m} \setminus \{u\}$, the generalized section is of finite index. Moreover, $\Delta_{\leq \ell + m}$ is a finite group. It follows that $F^u \cap R^\ell_m(u)$ is of finite index in $F^u$, thus also of finite index in $\Gamma(\mathcal{s^T\mathcal{L}})$. Take a product over $u$ in the level $m$ of $T_v$, we obtain the statement.

The branching lemma 6.7 will be useful in the proof of subgroup separability. We will also need to lift automorphisms of $\mathfrak{S}$ by Lemma 6.3; see Lemma 6.9 below.

### 6.3. Proofs of Theorem 6.1 and Theorem 6.2

Throughout this subsection, let $H$ be a subgroup of $\Gamma$ generated by a finite set $T$. Denote by $Q = \pi(H)$ its projection to $\mathfrak{S}$.

Applying Lemma 6.5 to the finitely generated subgroup $Q = \langle \pi(T) \rangle$ of $\mathfrak{S}$, there is a finite level $\ell$ such that the sections $Q_v$ for $v \in T_\ell$ are either finite or equal to $\mathfrak{S}$. Fix such a level $\ell$ for $Q$. Denote by $L_\ell'$ the subset of level $\ell$ vertices with $Q_v = \mathfrak{S}$.

Denote by $\pi_{u,v}$ the natural projection $\prod_{x \in T_\ell} Q_x \to Q_u \times Q_v$. Define the following equivalence relation $\sim$ on $L_\ell'$: $u \sim v$ if and only if $\pi_{u,v}(\text{St}_\ell(\mathcal{L})) \cap (Q_u \times \{ \text{id}_{Q_v} \}) = \{ \text{id} \}$. In other words, $u \sim v$ if and only if there exists an isomorphism $\tau_{u,v} : \mathfrak{S} \to \mathfrak{S}$ such that for any $g \in \text{St}_\ell(\mathcal{L})$, $g_v = \tau_{u,v}(g_u)$. It is clear that $\sim$ is an equivalence relation. Denote by $J_1, \ldots, J_p$ the $\sim$ classes in $L_\ell'$. For each equivalence class $J_i$, fix a representative vertex $u_i \in J_i$. 19
For $1 \leq i \leq p$, consider the following subgroup of $Q_{u_i}$:

$$(6.2) \quad \Lambda_i = \left\{ g \in Q_{u_i} : \exists \tilde{g} \in \text{St}_Q(\ell) \text{ such that } (\tilde{g})_v = \begin{cases} \tau_{u_i,v}(g) & \text{for } v \in J_i, \\ \text{id} & \text{otherwise.} \end{cases} \right\}.$$ 

Note that $\Lambda_i$ is a nontrivial normal subgroup in $Q_{u_i} = \mathfrak{G}$. Since $\mathfrak{G}$ is just-infinite, $\Lambda_i$ is of finite index in $Q_{u_i}$.

Now we lift from $Q$ to $H$. By the lifting property in Lemma 6.6, we have:

**Lemma 6.8.** Let $\ell$ be a level such that the sections $Q_v$ for $v \in T_\ell$ are either finite or equal to $\mathfrak{G}$. Then for $v \in L_\ell$, the generalized section $H_v$ is either finite or of finite index in $\Gamma(s^\ell \mathcal{L})$.

**Proof.** Recall that under the recursion $\varphi_\ell$, we have

$$\varphi_\ell (\text{St}_H(\ell)) <_s \prod_{v \in L_\ell} H_v,$$

and the projection to $\mathfrak{G}$ satisfies

$$\pi \left( \text{St}_H(\ell) \right) <_s \prod_{v \in L_\ell} \pi \left( H_v \right), \text{ where each } \pi \left( H_v \right) \text{ is finite or } \mathfrak{G}.$$

First note that $H_v$ is finitely generated: it is a quotient of $\text{St}_H(\ell)$ and $\text{St}_H(\ell)$ is of finite index in $H$. Recall that $H_v$ is regarded as a subgroup of $\Gamma(s^\ell \mathcal{L})$ and $\Gamma(s^\ell \mathcal{L})$ is an FC-central extension of $\mathfrak{G}$. Then by Lemma 6.8, if $\pi(H_v)$ generates a finite subgroup of $\mathfrak{G}$, then $H_v$ is finite; and if $\pi(H_v) = \mathfrak{G}$, then $H_v$ is finite index in $\Gamma(s^\ell \mathcal{L})$.

Recall the definition of the equivalence classes $J_i$, $1 \leq i \leq p$, in $L_\ell$, which depends on the quotient group $Q$. The automorphism lifting lemma 5.4 implies the following.

**Lemma 6.9.** Let $v$ be a vertex in $J_i$ and $u_i$ be the fixed representative of $J_i$. Then the subgroup $L_v$ of $H_v$

$$(6.3) \quad L_v := \{ h \in H_v : \exists \gamma \in \text{St}_H(\ell), \, \gamma_v = h \text{ and } \gamma_{u_i} = \text{id} \}$$

is a finite group contained in $\ker (H_v \rightarrow \mathfrak{G})$.

**Proof.** Recall that an element $g \in \text{St}_Q(\ell)$ satisfies for $v \in J_i$, $g_v = \tau_{u_i,v}(g_{u_i})$, where $\tau_{u_i,v} : \mathfrak{G} \rightarrow \mathfrak{G}$ is an automorphism. It follows that $L_v$ is contained in $\ker (H_v \rightarrow \mathfrak{G})$.

By Lemma 5.4 there exists $k \in \mathbb{N}$ such that the automorphism $\tau_{u_i,v}$ can be lifted to an automorphism $\tilde{\tau}_{u_i,v} : \Delta_{>k}(s^\ell \mathcal{L}) \rightarrow \Delta_{>k}(s^\ell \mathcal{L})$. That is, the following diagram commute:

$$\begin{array}{ccc}
\Delta_{>k}(s^\ell \mathcal{L}) & \xrightarrow{\tilde{\tau}_{u_i,v}} & \Delta_{>k}(s^\ell \mathcal{L}) \\
\pi \downarrow & & \downarrow \pi \\
\mathfrak{G} & \xrightarrow{\tau_{u_i,v}} & \mathfrak{G}.
\end{array}$$

Regard $\Gamma(s^\ell \mathcal{L})$ as the diagonal subgroup of $\Delta_{<k}(s^\ell \mathcal{L}) \times \Delta_{>k}(s^\ell \mathcal{L})$ and write its elements as $\gamma = (\gamma', \gamma'')$. Then for $\gamma \in \text{St}_H(\ell)$, its sections at $v$ and $u_i$ are related...
by

\[(6.4)\quad \gamma_v = (\gamma_v', \gamma_v'') = \left(\gamma_v', n_{\gamma,v} \tilde{\theta}_{u,v} (\gamma_v'') \right),\]

\[(6.5)\quad \text{where } n_{\gamma,v} = \gamma_v'' \tilde{\theta}_{u,v} (\gamma_v'')^{-1} \in \ker \left(\Delta_{>k} (s' \mathcal{L}) \to \mathfrak{S} \right) \].

Recall that \(\text{St}_H(\ell)\) is finitely generated. Take a symmetric finite generating set \(\{\gamma_1, \ldots, \gamma_m\}\) for \(\text{St}_H(\ell)\). Let \(h \in L_v\) and \(\gamma \in \text{St}_H(\ell)\) be such that \(h = \gamma_v\) and \(\gamma_u = id\). Express \(\gamma\) as a product of generators, \(\gamma = \gamma_{j_1} \cdots \gamma_{j_m}\). Then by (6.4), we have that the second coordinate \(\gamma_v''\) can be written as

\[\gamma_v'' = n_{\gamma_{j_1}, \ldots, \gamma_{j_m}} \tilde{\theta}_{u,v} \left(\gamma_{j_1}\right) \cdots n_{\gamma_{j_m}, \gamma} \tilde{\theta}_{u,v} \left(\gamma_{j_m}\right),\]

where \(n_v\) is in the normal closure of \(\{n_{\gamma_{j_1}, \ldots, \gamma_{j_m}}\}\) in \(\Delta_{>k} (s' \mathcal{L})\). Since \(\gamma_u = id\), we have that \(\tilde{\theta}_{u,v} (\gamma_u') = id\) and \(\gamma_v'' = n_{\gamma_v''}\). The kernel \(\ker \left(\Delta_{>k} (s' \mathcal{L}) \to \mathfrak{S} \right)\) is torsion and contained in the FC-center of \(\Delta_{>k} (s' \mathcal{L})\), therefore the normal closure of \(\{n_{\gamma_{j_1}, \ldots, \gamma_{j_m}}\}\) in \(\Delta_{>k} (s' \mathcal{L})\) is finite. Thus \(\gamma_v'' = n_{\gamma_v''}\) is contained in the finite group \(\langle n_{\gamma_{j_1}, \ldots, \gamma_{j_m}} \rangle\). Since \(\Delta_{\leq k} (s' \mathcal{L})\) is finite, we conclude that \(L_v\) is contained in a finite group.

\[\square\]

**Proof of Theorem [6.1]** Let \(H = \langle T \rangle\) be a finitely generated subgroup of \(\Gamma(\mathcal{L})\), \(Q\) be the projection of \(H\) to \(\mathfrak{S}\), and the level \(\ell\) be as in the beginning of this subsection. Since the subgroup \(\pi_{>\ell} (\text{St}_H(\ell))\) is of finite index in \(\pi_{>\ell} (H)\) and the kernel of \(H \to \pi_{>\ell} (H)\) is finite, \(H\) and \(\pi_{>\ell} (\text{St}_H(\ell))\) are commensurable up to finite kernels.

For the product \(\prod_{v \in \mathcal{L}_v} H_v\), take its quotient group \(\prod_{v \in \mathcal{L}_v} \widehat{H}_v\) where

- for \(v \in \{u_1, \ldots, u_p\}\), \(\widehat{H}_v = H_v\),
- for \(v \in J_i \setminus \{u_i\}\), \(1 \leq i \leq p\), \(\widehat{H}_v = H_v / L_v\), where \(L_v\) is defined in (6.3),
- for \(v \notin \mathcal{L}_v, \widehat{H}_v = \{id\}\).

Denote by \(\widetilde{H}_\ell\) the image of \(\pi_{>\ell} (\text{St}_H(\ell))\) in the quotient. Since for \(v \in J_i \setminus \{u_i\}\), \(L_v\) is finite by Lemma [6.9], we have that \(\widetilde{H}_\ell\) and \(H\) are commensurable up to finite kernels. Note that the projection \(\text{St}_H(\ell) \to \text{St}_Q(\ell)\) factors through \(\widetilde{H}_\ell\). With slight abuse of notation we still denote the projection \(\widetilde{H}_\ell \to \text{St}_Q(\ell)\) by \(\pi\).

Next we check that \(\widetilde{H}_\ell\) and \(\prod_{i=1}^{\mathcal{L}_1} H_{u_i}\) are commensurable. Note that for \(v \in J_i\), there is a homomorphism \(\tilde{\theta}_{u,v} : H_{u_i} \to H_v / L_v\) such that for any \(\gamma \in \widetilde{H}_\ell\), we have \(\gamma_v = \tilde{\theta}_{u,v} (\gamma_{u_i})\). It follows that \(\widetilde{H}_\ell\) is isomorphic to its natural projection to \(\prod_{i=1}^{\mathcal{L}_1} H_{u_i}\). Recall the subgroup \(\Lambda_i\) of \(Q_{u_i}\) defined in (6.2) and the property that \(\Lambda_i\) is of finite index in \(Q_{u_i} = \mathfrak{S}\), in particular, it is finitely generated. Take a finite generating set \(\{\gamma_1, \ldots, \gamma_k\}\) of \(\Lambda_i\) and denote by \(g_j\) the element in \(\text{St}_Q(\ell)\) such that \((\gamma_j)_{u_i} = \tau_{u_i,v} (\gamma_j)\) for \(v \in J_i\) and \((\gamma_j)_{u_i} = id\) for \(v \in \mathcal{L}_v \setminus J_i\). Now choose an element \(\tilde{g}_j \in H_{u_i}\) such that \(\pi(\tilde{g}_j) = g_j\). Consider the subgroup of \(H_{u_i}\) generated by \(\{\tilde{g}_1, \ldots, \tilde{g}_k\}\). Let \(R_i\) be the subgroup of \(\prod_{v \in \mathcal{L}_v} \widehat{H}_v\) defined as

\[R_i = \left\{ h : h_{u_i} \in H_{u_i}, h_v = \tilde{\theta}_{u,v} (h_{u_i}) \text{ for } v \in J_i, h_v = id \text{ for } v \notin J_i \right\}\].

Note that \(R_i\) is isomorphic to \(H_{u_i}\). By Lemma [6.6] we have that the group

\[(6.6)\quad \mathcal{R}_i = \langle \tilde{g}_1, \ldots, \tilde{g}_k \rangle \cap R_i\]
is of finite index in both \( \langle \tilde{g}_1, \ldots, \tilde{g}_k \rangle \) and \( R_i \). Since this is true for each \( i \in \{1, \ldots, p\} \), we conclude that \( H_\ell \) and \( \prod_{i=1}^p H_{\ell_i} \) are commensurable.

Recall that by Lemma 6.8, \( H_{\ell_i} \) is of finite index in \( \Gamma (s' \mathcal{L}) \). To summarize, we have shown

\[
H \leftarrow \text{St}_H(\ell) \rightarrow H_\ell \leftarrow \prod_{i=1}^p \mathcal{M}_i \rightarrow \prod_{i=1}^p \Gamma (s' \mathcal{L}) ,
\]

where each arrow indicates a homomorphism of groups with finite kernel and with image of finite index. We conclude that \( H \) and \( \prod_{i=1}^p \Gamma (s' \mathcal{L}) \) are commensurable up to finite kernels. Since they are both residually finite, it follows that \( H \) and \( \prod_{i=1}^p \Gamma (s' \mathcal{L}) \) are abstractly commensurable, see e.g., [H100] IV. 28.

\[\square\]

We now move on to prove subgroup separability. Since \( \text{St}_H(\ell) \) is of finite index in \( H_\ell \), to prove that \( H_\ell \) is separable it suffices to show that \( \text{St}_H(\ell) \) is separable, see e.g., [GW03, Lemma 11]. In the proof we make use of the generalized rigid stabilizers as in the branching lemma 6.7 to find an explicit sequence of subgroups of finite index in \( \Gamma (\mathcal{L}) \) whose intersection is \( \text{St}_H(\ell) \).

**Proof of Theorem 6.2.** We continue to use notations introduced in this section. Recall that for \( 1 \leq i \leq p \), \( u_i \) is a chosen representative of the equivalence class \( J_i \).

Write

\[
A = L_\ell \setminus \{u_1, \ldots, u_p\} .
\]

For a vertex \( v \in A \), either \( H_v \) is finite; or \( H_v \) is infinite and \( v \) is in the equivalence class \( J_i \) for some \( 1 \leq i \leq p \).

Let \( R^s_m(T_v) \) be the generalized level rigid stabilizer of \( \Gamma (\mathcal{L}) \) in the subtree \( T_v \) defined in (6.1). Take the subgroup \( U_m \) of \( \Gamma (\mathcal{L}) \) defined as

\[
U_m = \left< \text{St}_H(\ell), \prod_{v \in A} R^s_m(T_v) \right> .
\]

We claim that the sequence \( \{U_m\} \) is the desired approximation for \( \text{St}_H(\ell) \):

**Claim 6.10.** For each \( m \in \mathbb{N} \), the subgroup \( U_m \) is of finite index in \( \Gamma (\mathcal{L}) \), and \( \text{St}_H(\ell) = \cap_{m=1}^\infty U_m \).

**Proof.** By Lemma 6.7, the generalized rigid stabilizer \( R^s_m(T_v) \) is of finite index in \( \nu_v (\Gamma (s' \Gamma)) \). To show that \( U_m \) is of finite index in \( \Gamma \), it suffices to show that for each \( v \in L_\ell \), \( U_m \cap R_\Gamma (v) \) is of finite index in \( R_\Gamma (v) \). If \( v \in A \), then by its definition \( R^s_m(T_v) < U_m \cap R_\Gamma (v) \). If \( v \notin A \) then \( v = u_i \) for some \( 1 \leq i \leq p \). Recall the subgroup \( \Lambda_i \) of \( Q_{u_i} \) defined in (6.2), \( \Lambda_i \) is of finite index in \( Q_{u_i} \). Consider the subgroup of \( \text{St}_H(\ell) \) defined as

\[
\tilde{\Lambda}_i = \{ \gamma \in \text{St}_H(\ell) : \gamma_{u_i} \in \Lambda_i, \gamma_{u_i} = id \} \text{ for } u_i \notin J_i, \gamma_{u_i} \in R^s_m(T_v) \text{ for } v \in J_i \setminus \{u_i\} \} .
\]

By its definition, we have that \( \tilde{\Lambda}_i < U_m \). By the same argument which lifts up a generating set of \( \Lambda_i \) to \( \text{St}_H(\ell) \), we have that \( \phi_{u_i} (\tilde{\Lambda}_i) \) is of finite index in \( H_{u_i} \). It follows that \( U_m \cap R_\Gamma (v) \) is of finite index in \( R_\Gamma (u_i) \).

We now show that \( \cap_{m=1}^\infty U_m \subseteq \text{St}_H(\ell) \). Suppose \( \gamma \in \cap_{m=1}^\infty U_m \), then for any \( m \in \mathbb{N} \), one can write \( \gamma \) as a product \( \gamma = h_m r_m \), where \( h_m \in \text{St}_H(\ell) \) and \( r_m \in \prod_{v \in A} R^s_m(T_v) \). In particular, for any \( m, n \in \mathbb{N} \), \( \gamma = h_m r_m = h_n r_n \). It follows that for \( 1 \leq i \leq p \), \( h_m h_n^{-1} \) has trivial section at \( u_i \): \( (h_m h_n^{-1})_{u_i} = \).
The finite group \( G \) of the diagonal action of \( G \) by \( \{ (1^n, id), (1^n, ab) \} \).

Consider the action of \( G \) on \((v, \gamma) \in X \). Note that the generators \( a, b, c, d \) of \( G \) have distinct images in \( G_3 \). The reason to take \( X = L_n \times G_3 \), where \( L_n \) is the level \( n \) vertices of the rooted tree \( T_n \). Consider the following kind of finite sets where the diagonal action of \( G \) on \( X \) has orbits of trivial sign. Take \( X_n = L_n \times G_3 \), where \( L_n \) is the level \( n \) vertices of the rooted tree \( T_n \). Consider the following action of \( \Phi \) on \( X_n \) which factors through the quotient \( \Phi \to G_n \). The finite group \( G_n \) acts faithfully and level transitively on the finite rooted tree \( T_n \) of \( n \) levels.

With the Schur multiplier in mind, consider the following kind of finite sets where the diagonal action of \( \Phi \) on \( X \) has orbits of trivial sign. Take \( X_n = L_n \times G_3 \), where \( L_n \) is the level \( n \) vertices of the rooted tree \( T_n \). Consider the following action of \( \Phi \) on \( X_n \) which factors through the quotient \( \Phi \to G_n \). The finite group \( G_n \) acts faithfully and level transitively on the finite rooted tree \( T_n \) of \( n \) levels.

The same as in the previous sections, denote by \( G_n \) the finite quotient of the first Grigorchuk group \( \Phi \) under the projection \( Aut(T) \to Aut(T_n) \). We still denote by \( \{ a, b, c, d \} \) the image of the generating set of \( \Phi \) under the projection \( \Phi \to G_n \). The finite group \( G_n \) acts faithfully and level transitively on the finite rooted tree \( T_n \) of \( n \) levels.

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By [Tap81], Lemma 7.1 implies that the Schur multiplier \( H^2 (\mathbb{Z} \langle X_\nu \rangle, G_{n+3}, \mathbb{Z}) \) has a direct summand \( H_1 (\mathbb{Z}) \otimes H_1 (\mathbb{Z}) \) indexed by the orbit of \(((1^n, \text{id}), (1^n, ab))\). Let \( [\beta] \) be a generator of this summand \( H_1 (\mathbb{Z}) \otimes H_1 (\mathbb{Z}) \) (as an abelian group), then we can take the central extension of \( \mathbb{Z} \langle X_\nu \rangle, G_{n+2} \) corresponding to the 2-cocycle \([\beta]\).

More explicitly, we take the following central extension of \( \mathbb{Z} \langle X_\nu \rangle, G_{n+3} \). Denote by \( M_n \), the orbit of \(((1^n, \text{id}), (1^n, ab))\) under the diagonal action of \( G_{n+3} \). Consider the step-2 free nilpotent group \( N_n \) on generators \( \{b_x, x \in X_\nu\} \). In the free nilpotent group \( N_n \), impose the following additional relations:

\[
[b_x, b_y] = 1 \text{ if neither } (x, y) \text{ or } (y, x) \text{ is in } M_n,
\]

(7.1) \[
[b_x, g] = [b_x, b_y] \text{ for any } (x, y) \in M_n, g \in G_{n+3}.
\]

Denote by \( \tilde{N}_n \) the resulting quotient group of \( N_n \). Since \( M_n \) is an orbit of trivial sign, we have that \( \tilde{N}_n \) fits into the exact sequence

\[
1 \to \mathbb{Z} \to \tilde{N}_n \to \mathbb{Z}^{X_\nu} \to 1,
\]

where \( \mathbb{Z} \) is the center of \( \tilde{N}_n \). With slight abuse of notation we still denote by \( b_x \) the image of the generator \( b_x \) in \( \tilde{N}_n \). The group \( G_{n+3} \) acts on \( \tilde{N}_n \) by permuting the generators: \( b_x \cdot g = b_{x,g} \).

By the definitions, we have that \( \tilde{N}_n \) is the center of \( \tilde{N}_n \). Now take the semi-direct product \( \Gamma_n = \tilde{N}_n \rtimes G_{n+3} \). An element of \( \Gamma_n \) is recorded as \((h, g)\), where \( h \in \tilde{N}_n \) and \( g \in G_{n+3} \). Because of the relations (7.1), \( G_{n+3} \) acts trivially on the commutator subgroup of \( \tilde{N}_n \). Therefore \( \Gamma_n \) is a central extension of \( \mathbb{Z} \langle X_\nu \rangle, G_{n+3} \):

**Lemma 7.2.** Let \( \Gamma_n = \tilde{N}_n \rtimes G_{n+3} \) be defined above. Then the center of \( \Gamma_n \) is

\[
Z(\Gamma_n) = \ker (\Gamma_n \to \mathbb{Z} \langle X_\nu \rangle, G_{n+3}) \simeq \mathbb{Z}.
\]

**Proof.** By the definitions, we have that

\[
\ker (\Gamma_n \to \mathbb{Z} \langle X_\nu \rangle, G_{n+3}) = \ker (\tilde{N}_n \to \oplus_{x \in L_n} (b_x)).
\]

Since \( G_{n+3} \) acts trivially on \( [\tilde{N}_n, \tilde{N}_n] \), \( \ker (\Gamma_n \to \mathbb{Z} \langle X_\nu \rangle, G_{n+3}) \subseteq Z(\Gamma_n) \).

Let \( \gamma \in \Gamma_n \) be an element with nontrivial projection to \( \mathbb{Z} \langle X_\nu \rangle, G_{n+3} \), we need to show that it is not in the center of \( \Gamma_n \). Case (i): the projection of \( \gamma \) to \( G_{n+3} \) is nontrivial, then since \( G_{n+3} \) acts faithfully on \( X_\nu \), there exists \( x \in X_\nu \) such that \( x \cdot \pi_{n+3}(\gamma) \neq x \). Take an element \( h \in \Gamma_n \) such that its projection to \( \mathbb{Z} \langle X_\nu \rangle, G_{n+3} \) is \( (b_x, id_{G_{n+3}}) \). Then \( [\gamma, h] \) projects to \( (b_x \pi_{n+3}(\gamma) - b_x, id_{G_{n+3}}) \neq id \), it follows that in this case \( \gamma \notin Z(\Gamma_n) \). Case (ii): the projection of \( \gamma \) to \( \mathbb{Z} \langle X_\nu \rangle, G_{n+3} \) is of the form \( \sum_{x \in X_\nu} \lambda_x b_x, id_{G_{n+3}} \), where there exists at least one \( x \) with \( \lambda_x \neq 0 \). Since \( G_{n+3} \) acts transitively on \( X_\nu \), there exists \( g \in G_{n+3} \) such that \( (1^n, id) \cdot g = x \). Let \( y = (1^n, ab) \cdot g \). Take an element \( h \in \Gamma_n \) such that its projection to \( \mathbb{Z} \langle X_\nu \rangle, G_{n+3} \) is \( (b_y, id_{G_{n+3}}) \). Then we have \([\gamma, h] = [(\lambda_x b_x, b_y), id_{G_{n+3}}] = (\lambda_x [b_x, b_y], b_{(1^n, ab)} \cdot id_{G_{n+3}}) \neq id \). Combining the two cases, we conclude that \( Z(\Gamma_n) \subseteq \ker (\Gamma_n \to \mathbb{Z} \langle X_\nu \rangle, G_{n+3}) \).

Mark the group \( \Gamma_n \) with the generating tuple \( T_n = (a_n, b_n, c_n, d_n, t_n) \), where

\[
x_n = (id_{\tilde{N}_n}, x) \text{ for } x \in \{a, b, c \} \text{ and } t_n = (b_{(1^n, id)}, id_{G_{n+3}}).
\]

We have defined a sequence of marked groups \( ((\Gamma_n, T_n))_{n=1}^{\infty} \). Next we show that the sequence \( (\Gamma_n, T_n) \) converges in the Cayley topology and identify the limit. Let
$S = 1^\infty \cdot \mathfrak{S}$ be the orbit of the right most ray under the action of $\mathfrak{S}$, $S$ consists of infinite strings cofinal with $1^\infty$. Consider the embedding
\begin{equation}
\vartheta : \mathfrak{S} \hookrightarrow (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \wr \mathfrak{S},
\end{equation}
given by $\vartheta(a) = (0, a)$, $\vartheta(b) = (\delta_{1^\infty}, b)$, $\vartheta(c) = (\delta_{1^\infty}, c)$ and $\vartheta(d) = (\delta_{1^\infty}, d)$. Under the embedding $\vartheta$, we write
$$\vartheta(g) = (\Phi_g, g),$$
where the function $\Phi_g : S \rightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ can be viewed as the "germ configuration" of $g$. Consider the action of $\mathfrak{S}$ on the space $S \times (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$ given by
$$(x, \gamma) \cdot g = (x \cdot g, \gamma \Phi_g(x)), \text{ where } x \in S, \gamma \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$ Denote by $X$ the second coordinate is
$$W, T \in S \times (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})$$
are identical. In what follows we denote by $\pi_n$ the marked projection to $G_n$, where $t$ is sent to $id_{G_n}$.

Next we consider the image of the same word $w = x_1 \ldots x_\ell$ in the letters $\{a, b, c, d, t\}$ of length $\ell \leq 2^n - 1$. Consider the image of $w$ in $\Gamma_n$. Consider the inverted orbit of $(1^n, id)$:
$$\mathcal{O}_n(w) = \left\{(1^n, id) \cdot \pi_{n+2} \left((x_1 \ldots x_i)^{-1}\right) : 0 \leq i \leq \ell\right\}.$$
Write $w_i = x_1 \ldots x_i$. By the definition of the action, for $(v, \gamma) = (1^n, id) \cdot \pi_{n+2} (w_i)^{-1}$, the second coordinate is
$$\gamma = \pi_2 \left((w_i)_{1^n \cdot \pi_{n}(w_i)^{-1}}\right).$$
Note that $d\left(1^n, 1^n \cdot \pi_n (w_i)^{-1}\right) \leq |w_i| \leq \ell$. By Lemma 5.3 we have that
$$(w_i)_{1^n \cdot \pi_{n}(w_i)^{-1}} \in \{id, b, c, d\}.$$ It follows that $\gamma \in \{id, b, c, d\}$. Recall that $M_n$ is the orbit of $((1^n, id), (1^n, ab))$ under the diagonal action of $G_{n+3}$. Note that for a pair $((v, \gamma), (u, \gamma'))$ to be in the orbit $M_n$, it is necessarily that $v = u$ and $\pi_{1}(\gamma^{-1}\gamma') = a$. The fact that the second coordinate $\gamma$ is in $\{id, b, c, d\}$ for any $(v, \gamma) \in \mathcal{O}_n(w)$ implies that
$$(\mathcal{O}_n(w) \times \mathcal{O}_n(w)) \cap M_n = \emptyset.$$ Because of the relations imposed in $\tilde{N}_n$, the subset $\{b_{(v, \gamma)}\}_{(v, \gamma) \in \mathcal{O}_n(w)}$ of basis elements generates an abelian subgroup of $\tilde{N}_n$. Therefore the image of $w$ in $\Gamma_n$ can be written as
$$\tilde{\pi}_n(w) = \sum_{(v, \gamma) \in \mathcal{O}_n(w)} z_{(v, \gamma)} b_{(v, \gamma), \pi_{n+3}(w)} \cdot \pi_n(w),$$
where $z_{(v, \gamma)} \in \mathbb{Z}$.

Next we consider the image of the same word $w$ in $W$. By definition of the action on $X$ and the rule of multiplication in $W$, we have that the inverted orbit
\[ \mathcal{O}(w) = \left\{ (1^\infty, id) \cdot \pi (w_1 \ldots w_t)^{-1} : 0 \leq i \leq t \right\} \] satisfies that for any \((v, \gamma) \in \mathcal{O}_n(w)\), the second coordinate \( \gamma \in \\{ id, b, c, d \} \). The image of \( w \) in \( W \) is

\[ \tilde{\pi}(w) = \left( \sum_{(u, \gamma) \in \mathcal{O}(w)} x_{(u, \gamma)} \delta(u, \gamma), \pi(w) \right), \] where \( x_{(u, \gamma)} \in \mathbb{Z} \).

For \((v, \gamma) \in \mathcal{O}_n(w)\), since \( d(v, 1^n) \leq 2^{n-1} - 1 \), we have that the last digit of \( v \) must be 1. Similarly, for \((u, \gamma) \in \mathcal{O}(w)\), \( d(u, 1^n) \leq 2^{n-1} - 1 \) implies \( u \) is of the form \( u = u_1 \ldots u_{n-1} \). Together with the property that the second coordinates must be in \( \{ id, b, c, d \} \) as shown in the previous paragraphs, we have that the following map is a bijection:

\[ \mathcal{O}_n(w) \rightarrow \mathcal{O}(w) \]

\[ (v, \gamma) \rightarrow (v 1^n, \gamma), \]

Moreover, the multiplication rule in \( \Gamma_n \) and \( W \) imply that \( z(v, \gamma) = x(v 1^n, \gamma) \). Finally, by Lemma 5.3 we have that the injective radius of the marked projection \( \mathcal{O} \rightarrow G_{n+3} \) is at least \( 2^{n-1} - 1 \). Thus \( \pi_{n+3}(w) \in G_{n+3} \) can be identifies with \( \pi(w) \in \mathcal{O} \). We conclude that \( \tilde{\pi}_n(w) \) can be identified with \( \tilde{\pi}(w) \).

Next we examine the center of the diagonal product of the sequence \(((\Gamma_n, S_n))_{n=1}^{\infty} \).

**Proposition 7.4.** Let \( \Gamma \) be the diagonal product of the sequence \(((\Gamma_n, S_n))_{n=1}^{\infty} \).

Then the center of \( \Gamma \) is isomorphic to the direct sum

\[ Z(\Gamma) \cong \mathbb{Z}_{\infty} \oplus Z(\Gamma_n). \]

**Proof.** In the factor \( \Gamma_n \), the kernel of \( \Gamma_n \rightarrow \mathbb{Z} \times_\mathbb{Z} G_{n+3} \) is the center \( Z(\Gamma_n) \). By the definition of \( \tilde{N}_n \), its center is generated by the commutator \( [b_{x_n}, b_{y_n}] \), where \( x_n = (1^n, id) \) and \( y_n = (1^n, ab) \). Recall in the marking \( T_n \) of \( \Gamma_n \), the generator \( t_n = (b_{x_n}, id_{G_{n+3}}) \). Take an element \( g \in St_\mathcal{O}(n) \) with the section \( g_{1^n} = ab \). Take a word \( w \) in \( \{ a, b, c, d \} \) which represents \( g \). Then the image of \( w \) in \( \Gamma_n \) is \( \tilde{\pi}_n(w) = (id, \pi_{n+3}(g)) \). Then in \( \Gamma_n \), we have

\[ \tilde{\pi}_n(w)t_n \tilde{\pi}_n(w)^{-1} = (b_{x_n}, \pi_{n+3}(g), id_{G_{n+3}}) = (b_{y_n}, id_{G_{n+3}}). \]

It follows that the center of \( \Gamma_n \) is generated by \( [t_n, \tilde{\pi}_n(w)] \).

We now show that for the word \( w \) chosen in the previous paragraph, the image of \([t, u] \) in \( \Gamma \) is trivial, for all \( j \neq n \).

- For \( j > n \), since the section of \( g \) at \( 1^n = ab \), we have that \( 1^j \cdot g = 1^n(1^j - n \cdot ab) = 1^n0u \), where \( u \) is a string of length \( j-n-1 \). Therefore \( \tilde{\pi}_j(w)t_j \tilde{\pi}_j(w)^{-1} = (b_{1^n0u, \gamma}, id_{G_{n+3}}) \). Note that neither the pair \( (1^j, id), (1^n0u, \gamma) \) nor the pair \((1^n0u, \gamma), (1^j, id) \) can be in the orbit \( M_j \) of \((1^j, id), (1^j, ab) \). Thus the corresponding basis elements commute in \( \tilde{N}_j \). It follows that \( \tilde{\pi}_j([w, t]) = id \).

- For \( j \leq n-3 \), \( \pi_{j+3}(g) = id \) because \( g \in St_\mathcal{O}(n) \). In this case \( x_j, \pi_{j+3}(g) = x_j \), and it follows that \( \tilde{\pi}_j([w, t]) = id \).

- For \( j \in \{ n-2, n-1 \} \), we have that

\[ x_j \cdot \pi_{j+3}(g) = (1^j, \pi_3(g_{1^n})). \]

Since \( g \in St_\mathcal{O}(1^n) \), we have that \( \pi_1(g_{1^n}) = id \). The pair \((x_j, x_j \cdot \pi_{j+3}(g)) \) or \((x_j \cdot \pi_{j+3}(g), x_j) \) being in the orbit of \((1^j, id), (1^j, ab) \) would imply
that \( \pi_1(g_1) = a \), a contradiction. We conclude that \( \tilde{\pi}_j([w,t]) = id \) for \( j = n - 2, n - 1 \).

We have proved that \( \tilde{\pi}_n([t,w]) \) generates the center of \( \Gamma_n \) and \( \tilde{\pi}_j([t,w]) = id \) for \( j \neq n \). It follows that \( Z(\Gamma_n) = \ker(\Gamma_n \to \mathbb{Z} \wr \mathbb{Z} G_{n+3}) \) is a direct summand in the kernel of the projection \( \Gamma \to \Delta \), where \( \Delta \) is the diagonal product of the quotient sequence \( \left( (\mathbb{Z} \wr \mathbb{Z} G_{n+3}, T_n) \right)_{n=1}^{\infty} \). Then we have

\[
\ker(\Gamma \to \Delta) = \bigoplus_{n=1}^{\infty} \ker(\Gamma_n \to \mathbb{Z} \wr \mathbb{Z} G_{n+3}).
\]

The inclusion \( \ker(\Gamma \to \Delta) \subseteq Z(\Gamma) \) is clear by definitions. It remains to verify that \( Z(\Gamma) \subseteq \ker(\Gamma \to \Delta) \). Take an element \( \gamma \notin \ker(\Gamma \to \Delta) \). Then by Lemma 4.2, to show that \( \Gamma \) has torsion. To have an example of a torsion free group as stated, one can further take the diagonal product of \( \Gamma \) and a suitable torsion-free group of intermediate growth.

Instead of \( \Gamma \) we may take the diagonal product of a subsequence \( ((\Gamma_{ni}, S_i))_{i=1}^{\infty} \), thus there are uncountably many examples satisfying the statement of Theorem 1.3.

\[\square\]

**Remark 7.5.** The statement of Proposition 7.4 clearly passes to subsequences: for any increasing subsequence \( (n_i)_{i=1}^{\infty} \), the diagonal product of \( ((\Gamma_{ni}, S_i))_{i=1}^{\infty} \) has center isomorphic to \( \mathbb{Z}^{\infty} \). The proof proceeds by showing the diagonal product \( \Gamma \) in Proposition 7.4 is of sub-exponential growth, then \( \Gamma \) will satisfy the statement of Theorem 1.3 except that \( \Gamma \) has torsion.

We are now ready to prove Theorem 1.3 which states that there exists a torsion free group of intermediate growth whose center is isomorphic to \( \mathbb{Z}^{\infty} \). By construction, each factor group \( \Gamma_n \) is virtually nilpotent, thus of polynomial growth. By Proposition 7.3 \( (\Gamma_n, T_n) \) converges to \( (W, T) \) in the Cayley topology when \( n \to \infty \). Then by Lemma 1.2 to show that \( \Gamma \) is of exponential growth, it suffices to show that \( W \) is of subexponential growth.

Recall the embedding \( \vartheta \) of \( \mathcal{G} \) into \((\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \wr \mathcal{S} \) in (7.2), where we write \( \vartheta(g) = (\Phi_g, g) \). The group \( W \) can be viewed as a subgroup of \( A \wr \mathcal{S} \), where \( A = \mathbb{Z} \wr (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \), via the embedding

\[
W \to A \wr \mathcal{S} \quad (f,g) \mapsto (\psi_f, g) \text{ where } \psi_f(v) = \left((f(v, \gamma))_{\gamma \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}}, \Phi_f(v)\right), \; v \in \mathcal{S}.
\]

Since \( A \) contains \( \mathbb{Z}^4 \) as a finite index subgroup, it is of polynomial growth. Then by [BE12] Lemma 5.1, the growth function of the permutation wreath product \( A \wr \mathcal{S} \) is equivalent to \( \exp(n^{\alpha_0} \log n) \). It follows that the subgroup \( W \) also has growth function bounded by \( \exp(n^{\alpha_0} \log n) \).

Combined with Proposition 7.4, we have that for any subsequence \( (n_i) \), the diagonal product \( \Delta = \Delta(n_i) \) is a group of subexponential growth whose center is isomorphic to \( \mathbb{Z}^{\infty} \).
The first example of a torsion-free group of intermediate growth is constructed by Grigorchuk in [Gri85]. More precisely, in [Gri85] a torsion-free group \( \hat{G} \) is constructed, which is an extension of \( \mathfrak{G} \) and of intermediate growth. The group \( \hat{G} \) can be described as follows. Take the subgroup \( \Lambda \) of \( \mathbb{Z}^3 \) generated by \[ \hat{a} = (0, a), \quad \hat{b} = (\delta_{11}^1, b), \quad \hat{c} = (\delta_{12}^1, c), \quad \hat{d} = (\delta_{13}^1, d), \] where \( \{e_1, e_2, e_3\} \) is the standard basis of \( \mathbb{Z}^3 \). Note that \( \vartheta(\mathfrak{G}) \cong \mathfrak{G} \) is a marked quotient of \( \Lambda \) and \[ \ker (\Lambda \to \mathfrak{G}) = \bigoplus_{\varepsilon \in S} \langle e_1^\varepsilon, e_2^\varepsilon, e_3^\varepsilon \rangle \cong \bigoplus_{\varepsilon \in S} \mathbb{Z}^3. \]

Take \( Z = \langle \hat{a} \rangle \) and mark it with the generating tuple \( \left( \hat{a}, \hat{b}, \hat{c}, \hat{d} \right) \), where \( \hat{b} = \hat{c} = \hat{d} = 0 \). Then take \( \hat{G} \) to be the diagonal product of \( (\Lambda, \langle \hat{a}, \hat{b}, \hat{c}, \hat{d} \rangle) \) and \( (\mathbb{Z}, \langle \hat{a}, \hat{b}, \hat{c}, \hat{d} \rangle) \). Add a trivial generator \( t = \hat{v} \) and mark \( \hat{G} \) with \( \hat{T} = \{a, b, c, d, t\} \). By [Gri85], \( \hat{G} \) is a torsion-free group of intermediate growth.

Finally, take the diagonal product \( \hat{\Delta} \) of \( (\Delta, T) \) and \( (\hat{G}, \hat{T}) \). Since both \( (\Delta, T) \) and \( (\hat{G}, \hat{T}) \) are of sub-exponential growth, \( \hat{\Delta} \) is of subexponential growth as well.

We now show \( \hat{\Delta} \) is torsion-free. If the image of a word \( \omega \) in \( \{a, b, c, d, t\} \) in \( \hat{\Delta} \) is non-trivial, then its projection to \( \hat{G} \) must be trivial since \( \hat{G} \) is torsion-free. It follows that the projection of \( \omega \) to \( \Delta \) is a non-trivial finite order element in \( \ker (\Delta \to \mathfrak{G}) \). However, by its construction, \( \ker (\Delta \to \mathfrak{G}) \) is torsion-free, a contradiction.

The center of \( \hat{G} \) is \( \langle a^2 \rangle \) and the center of \( \Delta \) is contained in the normal closure of \( t \). Since \( a^2 \) evaluates to identity in \( \Delta \) and \( t \) evaluates to identity in \( \hat{G} \), we conclude that \[ Z (\hat{\Delta}) = Z (\Delta) \times Z (\hat{G}) \cong \bigoplus_{\varepsilon = 1}^{\infty} \ker (\Gamma_n \to \mathbb{Z} \times_k G_{n+3}) \oplus \langle a^2 \rangle \cong \mathbb{Z}^\infty. \]

The group \( \hat{\Delta} \) satisfies all the requirements.

\[ \Box \]

8. Extensions of \( W_\omega \)

8.1. The construction of diagonal products. Let \( U, V \) be two nontrivial finite groups such that one of them has at least 3 elements. In this section we consider the extensions of \( W_\omega = (U \times V) \wr S \mathbb{G}_\omega \), where \( S \) is the orbit of \( 1^\infty \) under the action of the Grigorchuk \( G_\omega \). Recall that for the sequence \( \omega = \omega_0 \omega_1 \ldots \in \{0, 1, 2\}^{\infty} \), each symbol \( \omega_k \) corresponds to a homomorphism \( \{id, b, c, d\} \to \{id, a\} \).

Similar to the extensions of \( \mathfrak{G} \) in Section 5, the input to the construction is a sequence of finite marked quotients of \( U \ast V \). Enumerate the non-identity elements of \( U \) and \( V \) as \( \{u_1, \ldots, u_p\} \) and \( \{v_1, \ldots, v_q\} \), \( p = |U| - 1 \) and \( q = |V| - 1 \). Let \( L = (F_n)_{n=1}^{\infty} \) be a sequence of quotients of \( U \ast V \), where each \( L_n \) is marked with the image of \( (u_1, \ldots, u_p, v_1, \ldots, v_q) \) under the projection \( U \ast V \to F_n \).

Consider the permutation wreath product \( \Delta_n = F_n \wr_{L_n} G_\omega \), marked with the following specific generating tuple
\[ T_n = (\langle id, a \rangle, \langle id, b \rangle, \langle id, c \rangle, \langle id, d \rangle, u_{1, n}, \ldots, u_{p, n}, v_{1, n}, \ldots, v_{q, n}, \rangle, \] where \( u_{i, n} = (\delta_{1i}^0, id) \) and \( v_{j, n} = (\delta_{1j}^0, id) \), \( 1 \leq i \leq p, 1 \leq j \leq q \).
With slight abuse of notation we still write $a$ for the generator $(id, a)$, similarly for $b, c, d$. We now explain the choice of generators $u_{i,n}$ and $v_{j,n}$, which mimics the wreath recursion in $G_\omega$. Recall that at each level of the rooted binary tree $T$, the orbital Schreier graph of $1^n$ is a finite line segment of $2^n$ vertices, connected with self-loops and multiple edges. The vertex $1^n$ is at one end of the Schreier graph and $1^{n-1}0$ at the other end. Thus on the Schreier graph the generators $\{u_1, \ldots, u_p\}$ are placed at one end $1^n$; the generators $\{v_1, \ldots, v_q\}$ are placed at the other end $1^{n-1}0$. It will be clear later that the choice of these two locations is essential: they are at the opposite ends of the finite Schreier graph (thus distance $2^n - 1$ apart) and at the same time they are siblings on the tree $T$.

Let $M$ be the free product

$$M = (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) * U * V,$$

marked with generating tuple $T = \{a, b, c, d, u_1 \ldots u_p, v_1 \ldots v_q\}$. Then by the definitions, each $\Delta_n$ is a marked quotient of $M$ with the generating set $T$ projected onto $T_n$.

**Definition 8.1.** Given a sequence of marked quotients $L = (F_n)_{n=1}^\infty$ of $U * V$, the group $\Delta = \Delta (L, \omega)$ is defined as the diagonal product of marked groups $(\Delta_n, T_n)_{n=1}^\infty$, where $\Delta_n = F_n \wr \omega$ and the marking $T_n$ is given in (8.1). Denote the marking of $\Delta$ as $T = \{a, b, c, d, u_1 \ldots u_p, v_1 \ldots v_q\}$.

**8.2. The FC-center of $\Delta(L, \omega)$.** Mark $W_\omega = (U \times V) \wr \omega G_\omega$ by the generating tuple

$$T_0 = \{a, b, c, d, (\delta_1^u, id), \ldots (\delta_1^v, id), (\delta_2^u, id), \ldots (\delta_1^v, id)\}.$$

Given a vertex $v = v_1 \ldots v_n \in L_n$, denote by $\hat{v}$ its sibling, that is the other child of $v_1 \ldots v_{n-1}$. Note the following.

**Fact 8.2.** Let $L = (F_n)_{n=1}^\infty$ be a sequence of marked quotients of $U * V$. Then the sequence $(\Delta_n, T_n)_{n=1}^\infty$ converges to $(W_\omega, T_0)$ in the Cayley topology.

**Proof.** Because the $U$-generators and the $V$-generators are place $2^n - 1$ apart on the level $n$ Schreier graph in $T_n$, the ball of radius $2^{n-1} - 1$ around identity in $(\Delta_n, T_n)$ coincide with the ball of same radius around identity in $(W_\omega, T_0)$. An explicit identification is given by for $(f, g) \in \Delta_n$, where the function $f$ satisfies that for $d(1^n, v) \leq 2^{n-1} - 1$, $f(v) \in U$ and for $d(1^n, v) > 2^{n-1} - 1$, $f(v) \in V$, it is identified with $(\phi, g) \in W_\omega$, where

$$\phi(v_1^\infty) = (f(v), f(\hat{v})) \in U \times V, \text{ for } d(1^n, v) \leq 2^{n-1} - 1; \phi(x) = id \text{ otherwise.}$$

Given an element $\gamma = (\gamma_n) \in \Delta$, denote by $\gamma_\infty$ its projection to $W_\omega$ as explained Fact 8.2. Fact 8.2 implies the following.

**Fact 8.3.** Suppose each group $F_n$ in $L$ is an FC-group, then the FC-center of $\Delta = \Delta (L, \omega)$ is

$$Z_{FC}(\Delta) = \ker (\Delta \to W_\omega).$$

**Proof.** Since the group $W_\omega$ is ICC, we have that $Z_{FC}(\Delta) \subseteq \ker (\Delta \to W_\omega)$. To show containment in the other direction, for any element $\gamma \in \ker (\Delta \to W_\omega)$, since $(\Delta_n, T_n) \to (W_\omega, T_0)$ when $n \to \infty$, there exists an index $n_0$ such that the projection $\gamma_n$ of $\gamma$ to $\Delta_n$ is trivial for all $n \geq n_0$. It follows also that the projection of $\gamma$ to $G_\omega$ is trivial. Then the conjugacy class of $\gamma$ in $\Delta$ is contained in the finite product
Lemma 8.4. Suppose \( \text{closure of } \gamma \in \bigoplus_{x \in \mathbb{L}_n} F_n, n < n_0 \). We conclude that \( \gamma \) is in the FC-center of \( \Delta \).

\[
\text{The proof is similar to Lemma 5.6 or Lemma 7.4. In the factor group }
\]

\( \text{Proof.} \) The proof is similar to Lemma 5.6 or Lemma 7.4. In the factor group \( \Gamma_n \), take \( g \) to be a shortest element in \( G_\omega \) such that \( 1^{n-1}0 = 1^n \cdot g \). The word length of \( g \) is \( 2^n - 1 \). Then

\[
g_{v_j,n}g^{-1} = \left( \delta_{1^{n-1}0,g^{-1}}, id \right) = \left( \delta_{1^n}, id \right),
\]

\[
[u_{i,n},g_{v_j,n}g^{-1}] = \left( \delta_{[u_i,v_j]}, id \right).
\]

For the choice of \( g \) above, for any \( k < n \), \( 1^k \cdot g = 1^k \). For \( k > n \), since \( d(1^k,1^{n-1}0) = 2^k - 1 \), we have that \( 1^k \cdot g \neq 1^{k-10} \). Therefore if \( k \neq n \), then \( 1^{k-10} \cdot g^{-1} \neq 1^k \) and

\[
g_{v_j,k}g^{-1} = \left( \delta_{1^{k-1}0,g^{-1}}, id \right),
\]

\[
[u_{i,k},g_{v_j,k}g^{-1}] = \left( \delta_{[u_i,v_j]}^{1^n}, \delta_{1^{k-1}0,g^{-1}} \right), id = id.
\]

The calculation above implies that for each \( n \), \( \langle [U,V] \rangle^{F_n} \) is a direct summand in \( \ker(\Gamma \to W_\omega) \). Taking the normal closure of these summands, we obtain the statement.

\[
\text{Write } \bar{F}_n = F_n/\langle [U,V] \rangle^{F_n} \text{ and } \bar{\pi}_n : F_n \to \bar{F}_n. \text{ With slight abuse of notation we also write } \bar{\pi}_n \text{ for the projection } U \times V \to \bar{F}_n. \text{ Write } \pi_U : U \times V \to V, \pi_V : U \times V \to V. \text{ Recall that } \bar{v} \text{ is the sibling of } v. \text{ Suppose now } U \text{ and } V \text{ are abelian. Consistent with the markings } T_0 \text{ and } T_n, \text{ there is a projection map }
\]

\[
\bar{\varrho}_n : (U \times V) \wr G_\omega \to \bar{F}_n \wr \mathbb{L}_n G_\omega
\]

\[
(f,g) \mapsto (\psi_f,g),
\]

where

\[
\psi_f(v) = \sum_{z \in S} \bar{\pi}_n(\pi_U(f(vz)), \pi_V(f(\bar{v}z))).
\]

This map is a homomorphism because for any tree automorphism \( g \), if \( v = u \cdot g \) then \( \bar{v} = \bar{u} \cdot g \). By induction on word length, one can verify the following diagram commutes:

\[
\begin{array}{ccc}
\Delta & \xrightarrow{\pi_n} & \Delta_n \\
\downarrow & & \downarrow \\
W_\omega & \xrightarrow{\bar{\varrho}_n} & \bar{F}_n \wr \mathbb{L}_n G_\omega.
\end{array}
\]

In other word, for any element \( \gamma = (\gamma_n)_{n=1}^{\infty} \in \Delta \), \( \gamma_n \) and \( \gamma_\infty \) have consistent projections to \( \bar{F}_n \wr \mathbb{L}_n G_\omega \).
Corollary 8.5. Suppose $\mathcal{L} = (F_n)_{n=1}^\infty$ is a sequence of marked quotients of $U \ast V$, where $U, V$ are nontrivial finite abelian groups. Suppose each group $F_n$ in $\mathcal{L}$ is an FC-group. Then the FC-center of $\Delta = \Delta(\mathcal{L}, \omega)$ is

\[ Z_{\text{FC}}(\Delta) = \bigoplus_{n=1}^\infty \bigoplus_{x \in L_n} \langle [U, V] \rangle^{F_n}. \]

Proof. By Fact 8.3 and Lemma 8.4, it remains to show that $\ker(\Delta \to W_\omega) \subseteq \bigoplus_{n=1}^\infty \bigoplus_{x \in L_n} \langle [U, V] \rangle^{F_n}$. If $\gamma \in \ker(\Delta(\mathcal{L}, \omega) \to W_\omega)$, then as in the proof of Fact 8.3, there exists an index $n_0$ such that the projection $\gamma_n$ of $\gamma$ to $\Delta_n$ is trivial for all $n \geq n_0$. For $n < n_0$, $\gamma_n \in \bigoplus_{x \in L_n} F_n$. Suppose there is an index $n$ and $x \in L_n$ such that $(\gamma_n)_x \notin \langle [U, V] \rangle^{F_n}$, in other words the projection of $(\gamma_n)_x$ to $F_n/\langle [U, V] \rangle^{F_n}$ is nontrivial. We have then the projection $\gamma_\infty$ of $\gamma$ to $W_\omega$ is nontrivial, contradicting with the choice of $\gamma$. It follows that $\gamma \in \bigoplus_{n<n_0} \bigoplus_{x \in L_n} \langle [U, V] \rangle^{F_n}$. \qed

8.3. Recursions for $\Delta(\mathcal{L}, \omega)$. Let a sequence $\mathcal{L} = (F_n)_{n=1}^\infty$ of marked quotients of $U \ast V$ and a string $\omega = \omega_0\omega_1\ldots \in \{0,1,2\}^\infty$ be given. We continue to use notations introduced in the previous subsection. Denote by $\Delta_{>k}$ the diagonal product of factors with index $n > k$, $((\Delta_n, T_n))_{n=k+1}^\infty$. We will also consider the group $\Delta(\mathbf{s}^k\mathcal{L}, \mathbf{s}^k\omega)$ with shifted parameters, where $\mathbf{s}^k\mathcal{L} = (F_{k+1}, F_{k+2}, \ldots)$ and $\mathbf{s}^k\omega = \omega_k\omega_{k+1}\ldots$.

Similar to Section 5, we first consider a formal recursion on the level of the free product $\mathbf{M}$, then project down to its quotients. Recall that $\mathbf{M}$ is the free product $\mathbf{Z}/2\mathbf{Z} \ast (\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}) \ast U \ast V$, marked with the generating tuple $\mathbf{T}$. Denote by $\mathbf{S}_2$ the symmetric group of $\{0,1\}$, generated by the involution $\varepsilon = (0,1)$. Consider the homomorphism

$\theta : \mathbf{M} \to \mathbf{M}_{\{0,1\}} \mathbf{S}_2$

$\begin{array}{ll}
\theta(a) & = (id, \varepsilon), \\
\theta(b) & = (\delta_1^b + \delta_0^{\omega_0(b)}, id), \\
\theta(c) & = (\delta_1^c + \delta_0^{\omega_0(c)}, id), \\
\theta(d) & = (\delta_1^d + \delta_0^{\omega_0(d)}, id).
\end{array}$

Denote by $\pi_1$ the projection $\mathbf{M}_{\{0,1\}} \mathbf{S}_2 \to \Delta(\mathbf{s}\mathcal{L}, \mathbf{s}\omega) \mathbf{M}_{\{0,1\}} \mathbf{S}_2$ induced by the marked projection $\mathbf{M} \to \Delta(\mathbf{s}\mathcal{L}, \mathbf{s}\omega)$.

Lemma 8.6. The homomorphism $\theta$ induces an embedding $\theta : \Delta_{>1} \to \Delta(\mathbf{s}\mathcal{L}, \mathbf{s}\omega)_{\{0,1\}} \mathbf{S}_2$ and the following diagram commute:

$\begin{array}{ccc}
\mathbf{M} & \xrightarrow{\theta} & \theta(\mathbf{M}) \\
\downarrow{\pi} & & \downarrow{\pi_1} \\
\Delta_{>1} & \xrightarrow{\theta} & \theta(\Delta_{>1}).
\end{array}$

Proof. We first describe $\theta$. The group $\Delta_{>1}$ is defined as the diagonal product of $((\Delta_n, T_n))_{n=2}^\infty$. Componentwise, for each $n \geq 2$, we have the embedding

$\theta_n : F_n \ast L_n G_\omega \to (F_n \ast L_{n-1} G_{\omega})_{\{0,1\}} \mathbf{S}_2$,

$(f, g) \mapsto (\psi_{f,g}, \pi_1(g))$,

where $\psi_{f,g} : \{0,1\} \to F_n \ast L_{n-1} G_{\omega}$ is

$\psi_{f,g}(i) = \left((f(i), u_i) \in L_{n-1}, g_i \right), \quad i \in \{0,1\}$,
and \((g_0, g_1) \pi_1(g)\) is the image of \(g\) under the canonical wreath recursion in \(\text{Aut}(T)\).

Explicitly, for the generators in \(T_n\), we have
\[
\begin{align*}
\theta_n(x) &= (id, \varepsilon); \\
\theta_n(x) &= (\psi_x, id), \text{ where } \psi_x(0) = (i_0, \omega_0(x)), \quad \psi_x(1) = (id, x), \quad x \in \{b, c, d\}; \\
\theta_n(u_{i,n}) &= (\delta^q_{1,i}, id), \quad \theta_n(v_{j,n}) = (\delta^q_{1,j}, id), \quad 1 \leq i \leq p, \quad 1 \leq j \leq q.
\end{align*}
\]

The map \(\theta\) on \(\Delta_{>1}\) is defined as for \(\gamma = (\gamma_n)_{n=2}^\infty\), where \(\gamma_n \in \Delta_n, \theta((\gamma_n)_{n=2}^\infty) = (\theta_n(\gamma_n))_{n=2}^\infty\). By inspecting the generators, we have that \(\theta\) maps \(\Delta_{>1}\) into the permutation wreath product \(\Delta(s^\omega L, s^\omega \omega) \wr_{\{0, 1\}} \mathfrak{S}_2\), where \(a \mapsto (id, \varepsilon), x \mapsto (\delta^x_1 + \delta^x_0, id)\) for \(x \in \{b, d, d\}, \tilde{u}_j \mapsto (\delta^q_{1,j}, id)\) and \(\tilde{v}_j \mapsto (\delta^q_{1,j}, id)\).

The map \(\theta\) is an embedding because componentwise, each \(\theta_n\) is an embedding.

The commutative diagram can be checked by induction on the word length of \(w \in M\).

\[
\begin{align*}
\theta^n : \Delta_{>n} &\to \Delta(s^n L, s^n \omega) \wr_{\{0, 1\}} \pi_n(G) \\
&\text{a} \mapsto (id, a), \\
x &\mapsto (\delta^{\omega_{n-1}(x)}_{1,n-1} + \delta^x_1, s), \text{ for generator } x \in \{b, c, d\}, \\
\gamma &\mapsto (\delta^\gamma_{1,n}, id), \text{ for generator } \gamma \in \{\tilde{u}_1, \ldots, \tilde{u}_p, \tilde{v}_1, \ldots, \tilde{v}_1\}.
\end{align*}
\]

9. The Traverse Fields and Contraction Properties

In this section, we consider the traverse field associated with a word \(w\) in the letters \(\{a, b, c, d\}\), under the action of the first Grigorchuk group \(\mathfrak{G}\) on \(T\). We show that the traverse fields are compatible with the formal recursion on words. In what follows, \(a, b, c, d\) acts on \(T\) as the generators of \(\mathfrak{G}\).

Given a word \(w = z_1 \ldots z_m\) in the alphabet \(\{a, b, c, d\}\) and a level \(n\), consider the inverted orbits of the pair \((1^n, 1^{n-1}0)\) under the action of \(\mathfrak{G}\):
\[
I_n(w) := \left((1^n, 1^{n-1}0) \cdot (1^n, 1^{n-1}0) \cdot z_1^{-1}, \ldots, (1^n, 1^{n-1}0) \cdot (z_1 \ldots z_m)^{-1}\right).
\]

Note that \(I_n(w)\) is an ordered sequence of points rather than a set.

For a vertex \(v \in L_n\), keep a record of the pattern of visits from the inverted orbit \(I_n(w)\). Formally, for \(x \in L_n\), let \(\hat{P}(x, w)\) be a string in \(\{0, 1\}\) which is defined recursively as:
\[
\hat{P}(x, id) = \begin{cases} 
1 & \text{if } x = 1^n, \\
0 & \text{if } x = 1^{n-1}0, \\
\emptyset & \text{otherwise};
\end{cases}
\]

and for \(s \in \{a, b, c, d\}\),
\[
\hat{P}(x, ws) = \begin{cases} 
\hat{P}(x, w) & \text{if } x \notin \{1^n \cdot (ws)^{-1}, 1^{n-1}0 \cdot (ws)^{-1}\}, \\
\hat{P}(x, w)1 & \text{if } x = 1^n \cdot (ws)^{-1}, \\
\hat{P}(x, w)0 & \text{if } x = 1^{n-1}0 \cdot (ws)^{-1}.
\end{cases}
\]
After collapse consecutive 1’s into a single 1, consecutive 0’s into a single 0, we obtain from \( \tilde{P}(x, w) \) a string \( P(x, w) \) alternating in 0 and 1. For example, if \( \tilde{P}(x, w) = 000111100 \) then \( P(x, w) = 010 \).

**Definition 9.1** (Traverse field). The collection \( \{P(x, w)\}_{x \in L_n} \) defined above is called the traverse field of the word \( w \) on level \( n \) under the action of \( G \).

The following example illustrates the definitions.

**Example 9.2.** Let \( n = 2 \) and \( w = abaca \). Then
\[
I_2(abaca) = \{(11, 10), (01, 00), (01, 00), (10, 11), (10, 11), (00, 01)\}.
\]
For the vertex \( v = 01 \), we have \( \tilde{P}(01, abaca) = 110 \) and \( P(01, abaca) = 10 \).

Consider the formal recursion \( \varphi \) on the level of words where
\[
a \mapsto (0, 0) \varepsilon, \ b \mapsto (a, c), \ c \mapsto (a, d), \ d \mapsto (\emptyset, b).
\]
Write \( \varphi(w) = (w_0, w_1) \varepsilon^s \). Note that no reduction is performed on the words \( w_0, w_1 \).

Under the formal recursion we have the following. Given two strings \( u = u_1 \ldots u_k \) and \( v = v_1 \ldots v_l \) in \( \{0, 1\} \), we write \( u \subset v \) if there is an increasing, injective map \( \tau : \{1, \ldots, k\} \to \{1, \ldots, l\} \) such that \( v_{\tau(i)} = u_i \) for all \( 1 \leq i \leq k \). In other words, \( u \subset v \) if the string \( u \) can be embedded into \( v \) in an order preserving way.

**Lemma 9.3.** Let \( w \) be a word in \( \{a, b, c, d\} \) and \( \varphi(w) = (w_0, w_1) \varepsilon^s \). Then for \( x \in \{0, 1\}^n, n \geq 1, \) we have that
\[
P(0x, w) \subset P(x, w_0) \quad \text{and} \quad P(1x, w) \subset P(x, w_1).
\]

**Proof.** We prove the claim by induction on the length of \( w \). It is trivially true for the empty word. Suppose the claim is true for all words of length at most \( m \).

Let \( w \) be a word of length \( m + 1 \). Write \( w = w'x, x \in \{a, b, c, d\} \) and \( \varphi(w') = (w'_0, w'_1) \varepsilon^s \).

**Case 1:** \( w \) ends in \( a \). For \( s = 0 \), the new point in the inverted orbit \( I_n(w) \) is
\[
(1^n, 1^{n-1} \cdot 0) \cdot w^{-1} = (0 \cdot 1^{n-1} \cdot (w'_0)^{-1}, 0 \cdot 1^{n-2} \cdot (w'_0)^{-1}) \cdot w^{-1}.
\]
In the left subtree, note that \( (u, v) = (1^{n-1} \cdot (w'_0)^{-1}, 1^{n-2} \cdot (w'_0)^{-1}) \) is the same as the last point in the inverted orbit of \( (1^{n-1}, 1^{n-2} \cdot 0) \) under \( w'_0 \). By the induction hypothesis, we have \( P(0x, w') \subset P(x, w'_0) \). Since the new point \( (u, v) \) is a repetition of the last point in the inverted orbit under \( w'_0 \), we have that \( P(0x, w) \subset P(x, w'_0) \) as well. On the right subtree we have \( P(1x, w) = P(1x, w') \), which is contained by \( P(x, w'_1) \) by the induction hypothesis. The argument for \( s = 1 \) is the same with left and right subtrees swapped.

**Case 2:** \( w \) ends in \( b, c, d \). In this case, the last two points in the inverted orbit \( I_n(w) \) are the same: \( (1^n, 1^{n-1} \cdot 0) \cdot w^{-1} = (1^n, 1^{n-1} \cdot 0) \cdot w' \). Therefore \( P(ix, w) = P(ix, w') \) for \( i \in \{0, 1\}, x \in \{0, 1\}^n \). Since there is no reduction on the words, \( w'_i \) is a prefix of \( w_i \), thus \( P(x, w'_i) \subset P(x, w'_i) \). It follows by the induction hypothesis that \( P(ix, w) = P(ix, w') \subset P(x, w'_i) \subset P(x, w_i) \).

Next we consider admissible word reductions. Following [BE12], we say a word \( w \) is pre-reduced, if it does not contain consecutive occurrences of \( b, c, d \) (while consecutive occurrences of \( a \) is allowed). Given a word \( w \), denote by \( \check{w} \) its pre-reduction. Since \( (1^n, 1^{n-1} \cdot 0) \) is fixed by \( b, c, d \), by the definitions we have:
Fact 9.4. Let \( n \geq 1 \). Then for \( x \in L_n \), \( P(x, w) = P(x, \dot{w}) \), that is, on level \( n \), the traverse fields of \( w \) and its pre-reduction \( \dot{w} \) are the same.

Since \( (1^n, 1^{n-1}0) \) is not fixed by \( a \), inserting or deleting \( a^2 \) may change the traverse field. This is the reason that we consider pre-reduced words, instead of reduced words in the free product \( F \).

In [Bar98, Proposition 4.2] the following length contraction bound is established, see also [BE12, Lemma 4.2]: there is a norm \( \| \cdot \| \) such that for any pre-reduced word \( w \) in \( \{a, b, c, d\} \), we have

\[
\| w_0 \| + \| w_1 \| \leq \eta \| w \| + C,
\]

where \( \varphi(w) = (w_0, w_1)\varepsilon^s \), \( \eta \) is the real root of \( X^3 + X^2 + X - 1 \), \( C = \eta \| a \| \).

Lemma 9.5. There exists a constant \( C > 0 \) such that for any pre-reduced word \( w \) in \( \{a, b, c, d\} \) and \( n \geq 1 \), we have

\[
A(n, w) \leq C (\eta^k |w| + 2^k) \quad \text{for any} \quad 1 \leq k \leq n - 2.
\]

Proof. Lemma 9.3 implies that for a word \( w \) such that \( \varphi(w) = (w_0, w_1)\varepsilon^s \), \( s \in \{0, 1\} \), we have

\[
A(n, w) \leq A(n - 1, w_0) + A(n - 1, w_1).
\]

By Fact 9.4 \( A(n - 1, w_i) = A(n - 1, \dot{w}_i) \). Iterating \( k \) times, we have that

\[
A(n, w) \leq \sum_{v \in L_k} A(n - k, w_v),
\]

where each step of the recursion involves first applying \( \varphi \) to the word \( w \), then pre-reducing \( w_0 \) and \( w_1 \).

Iterate the contraction inequality (9.1), we have that

\[
\sum_{v \in L_k} \| w \|_v \leq \eta^k \| w \| + C 2^k \frac{\eta}{1 - \eta/2},
\]

where \( \| \cdot \| \) is the norm in the contraction inequality (9.1).

By the definition of the traverse field, we have the obvious bound

\[
A(k, w) = \sum_{x \in L_k} |P(x, w)| \leq 2|w| + 2 \leq 2C' \| w \| + 2.
\]

Putting these bounds together, we have

\[
A(n, w) \leq \sum_{v \in L_{n-2}} (2C' \| w \|_v + 2) \leq 2C' \eta^k \| w \| + C'' 2^k.
\]

The statement follows by changing \( \| w \| \) back to \( |w| \). \( \square \)
10. Volume growth estimates on $\Delta$ and proof of Theorem [1.3]

In this section we estimate volume growth of the diagonal product $\Delta(\mathcal{L}) = \Delta(\mathcal{L}, (012)\infty)$ defined in the Section 8. The main point is that volume growth of $\Delta$ is controlled by the traverse field (described in Section 9) and growth in the groups $\mathcal{L} = (F_n)_{n=1}^\infty$. Our estimates on $\Delta$ are summarized in Theorem 10.9 which is stated and proved in Subsection 10.2.

Theorem 10.9 and the flexibility of choices of $\mathcal{L}$ in the construction allow us to establish Theorem 1.4. Similar estimates can be shown for $\Delta(\mathcal{L}, \omega)$, where $\omega$ is a string such that all three letters 0, 1, 2 appear infinitely often. The change needed is to apply results in [BEL] and replace the contraction rate $\eta$ in $\mathcal{G}$ with a sequence of contraction rates $\eta_1, \eta_2, \ldots$ associated with $\omega$. Since Theorem 10.9 is sufficient for our purposes, we do not write the estimates for more general $\omega$ which would involve heavier notations.

10.1. Growth estimates in each factor group. In this subsection we focus on one factor group in the diagonal product $\Delta(\mathcal{L})$. Let $F = \langle U, V \rangle$ be a group generated by two finite subgroups $U, V$. Denote by $v_F$ the volume growth function of $F$ with respect to the generating set $U \cup V$.

Let $n \in \mathbb{N}$, consider the permutation wreath product $\Delta_n = F \wr_n \mathcal{G}$ marked with the generating set $T_n$ [31]. Recall that $T_n = (a, b, c, d, u_{1,n}, \ldots, u_{p,n}, v_{1,n}, \ldots, v_{q,n})$. Note that in $\Delta_n$, $u_{i,n}$ and $v_{j,n}$ commute, for $1 \leq i \leq p, q \leq j \leq q$.

Consider a reduced word $w$ in the free product $\mathcal{M}$ as in Subsection 8.1. Denote by $\hat{w}$ the word obtained from $w$ by deleting letters in $\{u_1, \ldots, u_p, v_1, \ldots, v_q\}$. That is, the word $w$ is of the form $w = u_1 z_1 u_2 z_2 \ldots u_k z_k$, where each $u_i$ is a word in $\{a, b, c, d\}$ and $z_i \in U \ast V$. And the corresponding $\hat{w} = w_1 w_2 \ldots w_k$.

Now evaluate the word $w = w_1 z_1 w_2 z_2 \ldots w_k z_k$ in $(\Delta_n, T_n)$. Denote by $\pi_U$ the projection $U \ast V \to U = \langle id, u_{1,n}, \ldots, u_{p,n} \rangle$ and $\pi_V$ the projection $U \ast V \to V = \langle id, v_{1,n}, \ldots, v_{q,n} \rangle$. Denote the image of $w$ in $\Delta_n$ by $((f_w(x))_{x \in \mathbb{L}_n}, \pi(w))$, where $\pi$ is the marked projection $\mathcal{M} \to \mathcal{G}$. By the multiplication rule in the permutation wreath product, at each point $x \in \mathbb{L}_n$, the lamp configuration is the ordered product

$$f_w(x) = \prod_{i=1}^k \left( \pi_U(z_i) 1_{\{1^n \cdot (u_1 \ldots u_i)_{-1} = x\}} + \pi_V(z_i) 1_{\{1^n \cdot 10 \cdot (u_1 \ldots u_i)_{-1} = x\}} \right),$$

with the convention that if $x \notin \{1^n \cdot (u_1 \ldots u_i)_{-1}, 1^n \cdot 10 \cdot (u_1 \ldots u_i)_{-1}\}$, then the $i$-th factor is identity. From (10.1), we have that the length of the word $f_w(x)$ in the letters $U \cup V$ is dominated by the number of alternating visits of the inverted orbit of $1^m$ and $1^{m-1} \cdot 10$ under $w_1 \ldots w_k$, which is recorded in the traverse field $P(x, w)$. In particular, we have

$$|f_w(x)|_{U \cup V} \leq |P(x, \hat{w})|.$$

We now proceed to show an upper bound of volume growth of $(\Delta_n, T_n)$ in terms of the quantity $A(k, w) = \sum_{x \in \mathbb{L}_k} P(x, w)$ defined in (9.2). Denote $A(n, r)$ the maximum

$$A(n, r) := \max \{A(n, w) : w \text{ is a word in } \{a, b, c, d\} \text{ of length } \leq r\}.$$

Lemma 10.1 (Growth upper bound). The volume function of $(\Delta_n, T_n)$ satisfies:

- for $r \leq (2/\eta)^n$, $\log v_{\Delta_n, T_n}(r) \leq C n^{\alpha_0}$, where $C$ only depends on $|U| + |V|$;
Next we explain that the inequality is nearly optimal. Consider the substitution $w = \text{total volume of particular, for large } r$ in the inverted orbit of $x$ under a word $w$. By the contraction inequality in Lemma 9.5, we have

$$\exp \left( \Phi_F \left( \frac{A(n,r)}{2^n} \right) \right) \leq \exp \left( \frac{A(n,r)}{2^n} + 1 \right).$$

**Proof.** For $r < (2/\eta)^n$, write $k = \left\lfloor \log_{2/\eta} r \right\rfloor$. Regard $\Delta_n$ as the permutation wreath product on level $k$,

$$\Delta_n \simeq (F \wr L_{n-k} G_{\pi_k(G_\omega)}) \wr L_k \pi_k(G_\omega),$$

where $\omega = (012)^\infty$. Then by [BE12, Lemma 5.1], bounding the growth of $F \wr L_{n-k} G_{\pi_k(G_\omega)}$ by the exponential function, we have $v_{\Delta_n,T_n}(r) \leq \exp(C_{\eta^2} r) \cdot |\pi_k(G_\omega)| \leq \exp(C_{\eta^2} r)$, where $\alpha_0 = \log 2 / \log(2/\eta)$.

By [BE12] if $|w| \leq r$, then

$$\sum_{x \in L_n} |f_\omega(x)|_{U \cup V} \leq \sum_{x \in L_n} |P(x, \hat{w})| = A(n, \hat{w}) \leq A(n, r).$$

It follows that the set $B_{\Delta_n}(id, r) \cap \ker(\Delta_n \to \emptyset)$ in the set $B'(r)$ of elements $\{(f(x))_{x \in L_n}, id\}$, where $g \in B_{\emptyset}(id, r)$ and $\sum_{x \in L_n} |f(x)|_{U \cup V} \leq A(n, r)$. The size of $B'(r)$ is bounded by

$$|B'(r)| \leq \left( A(n, r) + 2^n \right) \cdot \exp \left( 2^n \Phi_F \left( \frac{A(n,r)}{2^n} \right) \right).$$

The second item in the statement follows then by plugging in the bound $\binom{n}{k} \leq (ne/k)^k$.

**Remark 10.2.** When $F$ is finite, we may replace $\Phi_F$ by $\max \{ \Phi_F, \log |F| \}$. In particular, for large $r$ such that $A(n,r) \geq 2^n \text{Diam}(F, U \cup V)$, we simply take the total volume of $\bigoplus_{x \in L_n} F$ and have

$$\log v_{\Delta_n,T_n}(r) \leq 2^n |F| + \log v_{\emptyset,S}(r).$$

By the contraction inequality in Lemma 9.5 we have

$$A(n,r) \leq C (\eta^n r + 2^n).$$

Next we explain that the inequality is nearly optimal. Consider the substitution

$$\zeta : ab \mapsto abadac, \; ac \mapsto abab, \; ad \mapsto acac.$$ 

Take $w_n = \zeta^n(a^d)$. It is shown in [BE12, Proposition 4.7] that the sequence $w_j = \zeta^j(a^d)$ has asymptotically maximal inverted orbit growth. Recall that the inverted orbit of $x$ under a word $w = z_1 \ldots z_k$ is defined as the set $O(x,w) = \{ x, x \cdot z_1^{-1}, \ldots, x \cdot (z_1 \ldots z_k)^{-1} \}$.

**Fact 10.3.** The inverted orbit of $1^n$ under $w_n$ contains $L_n$.

**Proof.** Let $w$ be a word in $\{ a, b, c, d \}$ and write $\varphi(w) = (w_0, w_1) \in \varphi^*$. Then similar to the proof of Lemma 9.3 we have

$$O(1^n, w) = O(1^{n-1}, w_0) \cup 1O(1^{n-1}, w_1).$$
Under the formal recursion we have \( \varphi(w_n) = (w_{n-1}^{-1}, w_{n-1}) \). Then the statement follows by induction on \( n \).

Since \( \mathcal{G} \) acts by tree automorphisms, Fact \textbf{10.13} implies that the inverted orbit of \( 1^n-0 \) under \( w_n \) covers \( \mathbb{L}_n \) as well. Note also that the image of \( w_n \) in \( \mathcal{G} \) is in the level-\( n \) stabilizer. It follows that for the word \( w_n w_n^{-1} \ldots w_n w_n^{-1} \), that is, the concatenation of \( w_n \) and its reverse word \( w_n^{-1} \) for \( r \) times, its traverse field at each point \( x \in \mathbb{L}_n \) is of length at least \( 2r \). The length of \( w_n \) is comparable to \( (2/\eta)^n \), see the proof of \textbf{BE12, Proposition 4.7}. The existence of such words implies that there is a constant \( C > 0 \) which doesn’t depend on \( n, r \), such that

\[
A(n, C (2/\eta)^n r) \geq 2^n r.
\]

Using the word \( w_n \) we can find explicitly distinct words in \( \Delta_n \) within a distance to the identity.

**Lemma 10.4 (Growth lower bound).** There exists a constant \( C > 0 \) such that

- for \( r \leq (2/\eta)^n \), \( \log v_{\Delta_n, T_n}(r) \geq \frac{1}{C_0} n^{\alpha_0} \);
- for \( r > (2/\eta)^n \),

\[
\log v_{\Delta_n, T_n}(r) \geq 2^n \log v_{F, U \cup V} \left( \frac{\eta^n r}{C^{2^n}} \right).
\]

**Proof.** Let \( C \) be a constant such that \( |w_j| \leq C (2/\eta)^j \) for all \( n \in \mathbb{N} \).

For \( r > (2/\eta)^n \), in \( \Delta_n \) consider configurations \( (f_x)_{x \in \mathbb{L}_n} \in \oplus_{x \in \mathbb{L}_n} F \) such that \( |f_x|_{U \cup V} \leq \ell \). For each \( f_x \), fix a representing word \( r_x \) in \( U \) and \( V \) of shortest length for \( f_x \). We explicitly find a word in \( M \) that writes an element \( ((f_x), id) \).

Write \( w_n = z_1 z_2 \ldots z_k \). Recall that the inverted orbit of \( w_n \) visits all points on \( \mathbb{L}_n \). For \( x \in \mathbb{L}_n \), denote by \( j_x \) the smallest index \( j \) such that \( 1^n \cdot (z_1 \ldots z_j)^{-1} = x \). For \( \ell = 1 \), for each \( x \) such that \( f_x \in \{u_1, \ldots, u_p\} \), insert the word \( f_x \) after \( x_{j_x} \); after the first round of insertion, for each \( x \) such that \( f_x \in \{v_1, \ldots, v_q\} \), insert the word \( f_x \) after \( z_{j_x} \), where \( x \) is the sibling of \( x \). Then by the multiplication formula \textbf{10.11}, we have that after the insertions the resulting word evaluate to \((f_x, \pi(w_n)) \) in \( \Delta_n \). For general \( \ell \), one can repeat this procedure \( \ell \) times, each time reduce \( \max_{x \in \mathbb{L}_n} |f_x|_{U \cup V} \) by 1. The total length of the word is bounded by \( \ell (|w_n| + 2^n) \). Therefore for \( \ell \in \mathbb{N} \), we have

\[
v_{\Delta_n, T_n} (\ell (|w_n| + 2^n)) \geq (v_{F, U \cup V}(\ell))^{2^n}.
\]

For \( r < (2/\eta)^n \), write \( j = \left\lfloor \log_{2/\eta} r \right\rfloor \). Then by argument in the previous paragraph for \( \ell = 1 \) with \( w_n \) replaced by \( w_j \) shows that

\[
v_{\Delta_n, T_n} (|w_j| + 2^j) \geq 2^j.
\]

The statement follows.

The estimates on one factor group \( \Delta_n \) are summarized in the following. We use the notation that for two positive numbers \( x, y \), \( x \asymp_C y \) if \( y/C \leq x \leq Cy \).
Proposition 10.5 (Piecwise description of the growth function of \((\Delta_n, T_n)\)). Suppose the growth function \(v_{F,U \cup V}\) satisfies that there exists a constant \(C_0 > 1\) and a non-decreasing concave function \(\Phi_F\) such that for \(1 \leq \ell \leq \text{Diam}(F,U \cup V)\),

\[
\log v_{F,U \cup V}(\ell) \asymp_{C_0} \Phi_F(\ell) \quad \text{and} \quad \Phi_F(\ell) \geq \frac{1}{C_0} \log(1 + \ell), \quad \Phi_F(1) = 1.
\]

There exists a constant \(C > 1\), which only depends on \(C_0\), such that the volume growth function of \((\Delta_n, T_n)\) satisfies:

(i): For \(r \leq (2/\eta)^n\),

\[
\log |v_{\Delta_n, \ell}(n)| \asymp C n^\alpha.
\]

(ii): For \((2/\eta)^n \leq r < (2/\eta)^n \text{Diam}(F,U \cup V)\),

\[
\log |v_{\Delta_n, \ell}(n)| \asymp C (2^n)^{\Phi(1)} n^\alpha \eta^n r + \log v_{\Phi, F,U \cup V}(r).
\]

(iii): In the case that \(F\) is a non-trivial finite group, for \(r \geq (2/\eta)^n \text{Diam}(L,U \cup V)\),

\[
\log |v_{\Delta_n, \ell}(n)| \asymp C (2^n)^{\Phi(1)} \log |F| + \log v_{\Phi, F,U \cup V}(r).
\]

Proof. The statement (i) follows from the first items in Lemma 10.1 and Lemma 10.3. Statement (ii) follows from Lemma 10.3, the second items in Lemma 10.1 and Lemma 10.4 and the assumption on \(v_{F,U \cup V}\). Statement (iii) follows from Remark 10.6.

\[\square\]

Remark 10.6. Recall that the balls of radius \(2^{n-1} - 1\) around identities in \((\Delta_n, T_n)\) and \((W,T)\) are identical. Proposition 10.5(i) shows that the growth functions of \((\Delta_n, T_n)\) and \((W,T)\) remain equivalent up to the radius \((2/\eta)^n\), although balls are no longer identical. The estimates show that the growth in the lamp groups becomes more visible only after the radius \((2/\eta)^n\), in the way specified in (ii).

10.2. Growth estimates for \(\Delta\). We have proved in the previous subsection that if the growth functions of the lamp groups in \(L\) satisfies the assumption of Proposition 10.5, then the growth function of \((\Delta_n, T_n)\) can be estimated with good precision. We now proceed to estimate the growth function of \((\Delta, T)\). To this end we impose the following uniform assumption on the sequence \(L\).

Assumption 10.7. Suppose \(L = (F_n)_{n=1}^\infty\) is a sequence of marked quotients of \(U \times V\) which satisfies the following conditions. There exists a non-decreasing concave function \(\Phi : [0, \infty) \to [0, \infty), \Phi(1) = 1\), and a constant \(C_0 > 1\), such that for any \(n \in \mathbb{N}\) and \(0 \leq \ell \leq \text{Diam}(F_n, U \cup V)\), we have \(\Phi(\ell) \geq \frac{1}{C_0} \log (1 + \ell), \) and

\[
\log v_{F_n, U \cup V}(\ell) \asymp_{C_0} \Phi(\ell).
\]

Note that in Assumption 10.7, the groups \(F_n\) can be finite or infinite. Also the sequence \((F_n)_{n=1}^\infty\) does not need to have monotone properties, for example, it could be that on an infinite set of indices, \(F_n = \{\text{id}\}\).

Example 10.8 (Linear \(\Phi\)). Suppose \(L = (F_n)_{n=1}^\infty\) is a sequence of expanders, that is, there exists a constant \(\lambda_0 > 0\), such that for any \(n \in \mathbb{N}\), and \(A \subseteq F_n\) with \(|A| \leq \frac{|F_n|}{2}\), we have

\[
|\partial_{U \cup V} A| \geq \lambda_0 |A|.
\]
where \( \partial_{U \cup V} A = \{ x \in A : \exists s \in U \cup V, \, xs \notin A \} \). Then expansion in (10.3) implies that

\[
v_{F_n, U \cup V}(r) \geq (1 + \lambda_0)^r \quad \text{for all } r \text{ s.t. } |B_{F_n}(id, r - 1)| \leq \frac{|F_n|}{2}.
\]

It follows that if \( L = (F_n)_{n=1}^\infty \) is a sequence of expanders, where each \( F_n \) is a finite marked quotient of \( U \ast V \), then \( L \) satisfies Assumption 10.7 with \( \Phi(x) = x \) and a constant \( C_0 \) that only depends on \( \lambda_0 \) and \( |U| + |V| \).

We now put combine the bounds in the individual factors to give estimates on the growth function of the diagonal product \( \Delta = \Delta(L, (012)^\infty) \). Given \( r \in \mathbb{N} \), denote by \( J_r \) the index set

\[
J_r(L) := \left\{ j \in \mathbb{N} : r \geq \left( \frac{2}{\eta} \right)^j \text{ and } F_j \neq \{id\} \right\}.
\]

Write \( k_r = \left\lceil \log_{2/\eta} r \right\rceil \). The key observation in the upper bound direction is to regard \( \Delta \) as the diagonal product of \( (\Delta_n, T_n)_{n=1}^{k_r} \) and \( (\Delta > k_n, T) \), and then apply Proposition 10.5 to each of these factors. The lower bound simply comes from taking the maximum of the growth functions of the factors.

**Theorem 10.9** (Volume growth of \( \Delta \)). Suppose \( L = (F_n)_{n=1}^\infty \) is a sequence of marked quotients of \( U \ast V \) that satisfies Assumption 10.7 with function \( \Phi \) and constant \( C_0 > 0 \). Let \( \Delta = \Delta(L, (012)^\infty) \) be the diagonal product defined in the Section 8. Then there is a constant \( C > 0 \) which only depends on \( C_0 \) such that

\[
\log v_{\Delta, r}(r) \leq C \sum_{j \in J_r(L)} 2^j \Phi \left( \min \left\{ \left( \frac{\eta}{2} \right)^j r, \mathrm{Diam} (F_j, U \cup V) \right\} \right) + C r^{a_0}.
\]

and

\[
\log v_{\Delta, r}(r) \geq \frac{1}{C} \max_{j \in J_r(L)} 2^j \Phi \left( \min \left\{ \left( \frac{\eta}{2} \right)^j r, \mathrm{Diam} (F_j, U \cup V) \right\} \right) + \frac{1}{C} r^{a_0}.
\]

**Proof.** We first prove the volume upper bound. Denote by \( N = \ker (\Delta \rightarrow W) \). Since \( v_{\Delta, r}(r) \leq |B_{\Delta}(id, r) \cap N| \cdot v_{W, T_n}(r) \), and \( \log v_{W, T_n}(r) \asymp n^{a_0} \) by [BE12], it suffices to show the upper bound for \( |B_{\Delta}(id, r) \cap N| \). Regard \( \Delta \) as the diagonal product of \( (\Delta_n, T_n)_{n=1}^{k_r} \) and \( (\Delta > k_n, T) \). Note that if \( \gamma \in N \) and \( F_j = \{id\}, \) then the projection of \( \gamma \) to \( \Delta_j = \emptyset \) is trivial. Thus we only need to count in factors indexed by \( J_r(L) \) and \( \Delta > k_r \). For \( j \in J_r(L) \), we invoke Assumption 10.7 and apply the second item in Lemma 10.1 and Lemma 9.3 then

\[
\log \left| B_{\Delta_j}(id, r) \cap \ker (\Delta_j \rightarrow \emptyset) \right| \leq C' 2^j \Phi \left( \min \left\{ \left( \frac{\eta}{2} \right)^j r, \mathrm{Diam} (F_j, U \cup V) \right\} \right),
\]

where \( C' \) only depends on \( C_0 \). For the factor \( \Delta > k_r \), recall that it is isomorphic to a subgroup of the permutation wreath product \( \Delta_s^{k_r}L \ast (012)^\infty \circ \pi_{k_r}(\emptyset) \). Therefore by Proposition 10.7 (i), we have that

\[
v_{\Delta > k_r, r}(r) \leq C'' r^{a_0},
\]
where $C''$ is an absolute constant. Therefore
\[
\log |B_\Delta (id, r) \cap N| \leq \sum_{j \in J_r} \log |B_{\Delta_j} (id, r) \cap \ker (\Delta_j \rightarrow \mathcal{G})| + v_{\Delta \rightarrow x_r, T}(r)
\]
\[
\leq C' \sum_{j \in J_r} 2^j \Phi \left( \min \left\{ \left( \frac{\eta}{2} \right)^j r, \text{Diam} (F_j, U \cup V) \right\} \right) + C'' v^{\alpha_0}.
\]
We have proved the upper bound.

To show the lower bound, first note that since $W$ is a marked quotient of $\Delta$, we have $v_{\Delta,T}(n) \geq v_{W,T_0}(n)$. By the second item in Lemma 10.4 we have that for each $j \in J_r,$
\[
\log v_{\Delta_j, T_j}(r) \geq 2^j \log v_{F_j, U \cup V} \left( \left( \frac{\eta}{2} \right)^j \frac{r}{C} \right)
\]
\[
\geq \frac{1}{C_0} 2^j \Phi \left( \min \left\{ \left( \frac{\eta}{2} \right)^j r, \text{Diam} (F_j, U \cup V) \right\} \right)
\]
\[
\geq \frac{1}{C_0} 2^j \Phi \left( \min \left\{ \left( \frac{\eta}{2} \right)^j r, \text{Diam} (F_j, U \cup V) \right\} \right).
\]
Since $v_{\Delta,T} \geq \max \{v_{\Delta_j, T_j}, j \in J_r, v_{W,T_0}\}$, the statement follows.

\[
\square
\]

10.3. Approximations of prescribed functions. In this subsection, let $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a continuous function that is non-decreasing, subadditive and $f(x)/x \rightarrow 0$ when $x \rightarrow \infty$. Suppose in addition that there is a constant $\lambda > 2$ such that $f(x) \geq x^\alpha$ for all $x > 0$, where $\alpha = \frac{1}{\log_2 \lambda}$. We describe a procedure to approximate $f$ by quantities that appear in the bounds of Theorem 10.9.

Given such a function $f$, define recursively the following two sequences. Let $\theta_0 = 0, \alpha_0 = 0$. For $j \in \mathbb{N}$, define
\[
m_j' = \min \left\{ m > m_{j-1} : \frac{f(\lambda^m)}{\lambda^m} \leq (2/\lambda)^{\theta_{j-1}+1} \right\}
\]
\[
m_j'' = \min \left\{ m > m_{j-1} : f(\lambda^m) \geq 2f(\lambda^{m_{j-1}}) \right\},
\]
\[
m_j = \max \{m_j', m_j''\};
\]
and
\[
\theta_j := \log_{2/\lambda} \left( \frac{f(\lambda^{m_j})}{\lambda^{m_j}} \right).
\]
Although suppressed in the notation, the sequences $(m_j)$ and $(\theta_j)$ are determined by $f$ and $\lambda$. Define the function $\phi : \{\theta_1, \theta_2, \ldots\} \rightarrow \mathbb{R}_+$ to be
\[
\phi(\theta_j) = \lambda^{m_j - \theta_j}.
\]
The choice of these sequences is to guarantee the following approximation.

**Lemma 10.10.** There is a constant $c_\lambda > 0$ which only depends on $\lambda$, such that for $j \in \mathbb{N}$, and any $x \in [\lambda^{m_{j-1}}, \lambda^{m_j})$,
\[
c_\lambda \sum_{i : \lambda^{m_i} \leq x} 2^i \min \left\{ \lambda^{-\theta_i} x, \phi(\theta_i) \right\} \leq f(x) \leq \lambda \max_{i : \lambda^{m_i} \leq x} 2^{\theta_i} \min \left\{ \lambda^{-\theta_i} x, \phi(\theta_i) \right\}.
\]
Proof. For $i, m$ such that $\theta_i \leq m$, denote by $A_{i,m}$ the quantity
\[ A_{i,m} = 2^{\theta_i} \min \{ \lambda^{m - \theta_i}, \phi(\theta_i) \} , \]
for $i$ such that $\theta_i > m$, set $A_{i,m} = 0$. Let $m \in [m_{j-1}, m_j)$. Note that by definitions, we have that $\theta_{j-1} < m$; moreover, for any $i \leq j - 1$, $\phi(\theta_i) = \lambda^{m_i - \theta_i} \leq \lambda^{m - \theta_i}$; for $i \geq j$, $\phi(\theta_i) = \lambda^{m_i - \theta_i} > \lambda^{m - \theta_i}$. Therefore for $m \in [m_{j-1}, m_j)$,
\[ A_{i,m} = \begin{cases} 2^{\theta_j} \phi(\theta_i) = 2^{\theta_j} \lambda^{m_i - \theta_i} & \text{if } i \leq j - 1 \\ (2/\lambda)^{\theta_j} \lambda^m & \text{if } i \geq j, \theta_i \leq m. \end{cases} \]

Claim 10.11. The sequence $(\phi(\theta_i))_{i=1}^\infty$ is non-decreasing.

Proof of the Claim. Note that by the definitions, we have
\[ \phi(\theta_i) = \lambda^{m_i - \theta_i} = f(\lambda^{m_i}) / 2^{\theta_i}. \]
Consider the following two cases.
Case 1: $m_i' > m_i''$. In this case $m_i = m_i'$ and $f(\lambda^{m_i}) / \lambda^{m_i} = (2/\lambda)^{\theta_i-1+1}$, $\theta_i = \theta_{i-1} + 1$. Therefore
\[ \phi(\theta_i) = f(\lambda^{m_i}) / 2^{\theta_i} = f(\lambda^{m_i}) / 2^{\theta_i-1+1} \geq f(\lambda^{m_i-1}) / 2^{\theta_i-1} = \phi(\theta_{i-1}). \]
Case 2: $m_i' \leq m_i''$. In this case $m_i = m_i''$ and $f(\lambda^{m_i}) = 2 f(\lambda^{m_i-1}), \theta_i \geq \theta_{i-1} + 1$.
Therefore
\[ \phi(\theta_i) = f(\lambda^{m_i}) / 2^{\theta_i} = 2 f(\lambda^{m_i-1}) / 2^{\theta_i} \geq 2 f(\lambda^{m_i-1}) / 2^{\theta_i-1+1} = \phi(\theta_{i-1}). \]

We now return to the proof of the lemma. Let $m = \log_\lambda x$. The summation on the left-hand side, over $i$ such that $\theta_i \leq m$, splits into two parts, $i \geq j - 1$ and $i \geq j$ (the second part might be empty). Since by the definitions $\theta_i \geq \theta_{i-1} + 1$ and by the Claim $\phi(\theta_i)$ is non-decreasing, we have that $\max_{i \leq j-1} A_{i,m} = A_{j-1,m}$,
\[ \sum_{i \leq j-1} A_{i,m} = \sum_{i \leq j-1} 2^{\theta_i} \phi(\theta_i) \leq 2^{\theta_j-1+1} \phi(\theta_j-1) = 2(2/\lambda)^{\theta_j-1} \lambda^{m_j-1} = 2 A_{j-1,m}. \]
If the set $\{ i : \theta_i \leq m, i \geq j \}$ is non-empty (equivalently, $\theta_j \leq m$), then we have a decreasing geometric sum
\[ \sum_{i \geq j, \theta_i \leq m} A_{i,m} = \sum_{i \geq j, \theta_i \leq m} (2/\lambda)^{\theta_j} \lambda^m \leq \frac{1}{1 - 2/\lambda} \frac{1}{2/\lambda} \frac{\lambda^m}{2/\lambda} A_{j,m}. \]
Similar to the discussion in the proof of the Claim, we consider two cases. Case 1 $m_i' > m''$. In this case, $\theta_j = \theta_{j-1} + 1$ and
\[ (2/\lambda)^{\theta_j-1} \geq f(\lambda^{m}) / \lambda^{m} > (2/\lambda)^{\theta_j}. \]
Then we have the upper bound
\[ f(\lambda^{m}) \leq \lambda^m (2/\lambda)^{\theta_j-1} \leq \lambda \max_{i \theta_i \leq m} A_{i,m}; \]
and the lower bound
\[ f(\lambda^{m}) \geq \lambda^m (2/\lambda)^{\theta_j} \geq \min \left\{ \left( \frac{1}{2} - \frac{1}{\lambda} \right) \frac{1}{2 \lambda} \right\} \sum_{i \theta_i \leq m} A_{i,m}. \]
Case 2: $m_i' > m_i''$. In this case
\[ f(\lambda^{m_j-1}) \leq f(\lambda^{m}) \leq 2 f(\lambda^{m_j-1}), \]
and the terms for $i \geq j$ are non-negative, so
\[ \frac{1}{1 - 2/\lambda} \frac{1}{2/\lambda} \frac{\lambda^m}{2/\lambda} A_{j,m} = \frac{1}{1 - 2/\lambda} \frac{1}{2/\lambda} \frac{\lambda^m}{2/\lambda} A_{j,m} = \frac{1}{1 - 2/\lambda} \frac{1}{2/\lambda} \frac{\lambda^m}{2/\lambda} A_{j,m}. \]

□
and
\[ \frac{A_{j,m}}{A_{j-1,m}} \leq \frac{\lambda^{m_j - \delta_j} 2^\delta_j}{\lambda^{m_{j-1} - \delta_{j-1}} 2^\delta_{j-1}} = \frac{\lambda^{-\delta_j} f(\lambda^{m_j})}{\lambda^{-\delta_{j-1}} f(\lambda^{m_{j-1}})} = 2\lambda^{-\delta_j + \delta_{j-1}} \leq \frac{2}{\lambda}. \]

Then we have the upper bound
\[ f(\lambda^m) \leq 2 f(\lambda^{m_{j-1}}) = 2(2/\lambda)^{\delta_{j-1}} \lambda^{m_{j-1}} \leq 2 \max_{i: \theta_i \leq m} A_{i,m}, \]
and the lower bound
\[ f(\lambda^m) \geq f(\lambda^{m_{j-1}}) = A_{j-1,m} \geq \frac{1}{2} \min \left\{ \frac{1}{2}, \frac{\lambda}{2} - 1 \right\} \sum_{i: \theta_i \leq m} A_{i,m}. \]

The statement of the lemma follows by combing these two cases, where \( c_\lambda = \min \left\{ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} - 1 \right\} \).

\( \square \)

### 10.4. Proof of Theorem 1.4

The goal of this subsection is to apply Theorem 10.9 to prove the prescribed growth function theorem 1.4 stated in the Introduction.

First we fix a choice of an expander sequence \((\Gamma_i)_{i=1}^\infty\) such that \(\Gamma_n\) is a marked quotient of \(U \ast V\), both \(U\) and \(V\) are abelian and there is a constant \(K_0\) such that
\[ \text{Diam}(\Gamma_n, U \cup V) \approx_{K_0} n. \]

Denote by \(\delta_0 > 0\) the lower bound for Cheeger constants of \((\Gamma_i)_{i=1}^\infty\). Such expander sequences exist: for example, by the Margulis construction, one can start with a group with Kazhdan’s property (T) which is generated by two finite abelian subgroups and take a sequence of its finite quotients. For an explicit sequence of such expanders, see e.g., [BZ15 Example 2.3]. Alternatively, one can take \(\Gamma_i = \text{PSL}_2(\mathbb{Z}/5^i\mathbb{Z})\). There exists an explicit marking of \(\Gamma_i\) with \(U = \mathbb{Z}/2\mathbb{Z}\) and \(V = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\); and \(((\Gamma_i, U \cup V))_{i=1}^\infty\) forms a sequence of expanders, see [KPI23 Lemma 6.1].

Given a function \(f\) satisfying the assumptions of Theorem 1.4, the strategy is to find a sequence \(L = (F_n)\) such that the log-volume bounds for \(\Delta(L)\) in Theorem 10.9 approximate the function \(f\) up to a fixed constant. The choice of parameters is provided by the approximation lemma 10.10. Namely, we choose \(F_n\) to be nontrivial groups only along the sequence \((\theta_j)\), and at the level \([\theta_j]\), we take the corresponding lamp group to have diameter comparable to \(\phi(\theta_j)\).

**Proof of Theorem 1.4** Take \(\Gamma_i = \text{PSL}_2(\mathbb{Z}/5^i\mathbb{Z})\) and choose the marking on \(\Gamma_i\) by \(U = \mathbb{Z}/2\mathbb{Z}\) and \(V = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\) as in [KPI23 Lemma 6.1]. Then there is a constant \(K_0\) such that \(\text{Diam}(\Gamma_i, U \cup V) \approx_{K_0} i\) for all \(i\) and \(((\Gamma_i, U \cup V))_{i=1}^\infty\) forms a sequence of expanders. Denote by \(\delta_0\) the infimum of the Cheeger constants of \((\Gamma_i, U \cup V)\).

Given a function \(f: \mathbb{N} \to \mathbb{N}\) satisfying the assumptions of the statement, we may continue it to a sub-additive continuous function \(\tilde{f}: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}\) such that \(f(x) \geq x^{\alpha_0}\) for all \(x \geq 0\). Recall the contraction ratio \(\eta\) as in Theorem 10.9 and the fact that \(\alpha_0 = 1/\log_2(2/\eta)\). Let \(\lambda = 2/\eta\). Recall the sequences \((m_j), (\theta_j)\) and \((\phi(\theta_j))\) associated with \(f\) and \(\lambda\) defined in Subsection 10.3. Note that the assumption \(f(x) \geq x^{\alpha_0}\) implies that \(\phi(\theta_j) \geq 1\) for all \(j \in \mathbb{N}\).
Now we choose the sequence $\mathcal{L} = (F_n)_{n=1}^\infty$ by
\[
F[\theta_j] = \Gamma[\phi(\theta_j)] \text{ for } j \in \mathbb{N};
F_n = \{id\} \text{ for } n \notin \{\lfloor \theta_1 \rfloor, \lfloor \theta_2 \rfloor, \ldots \}.
\]

Take the diagonal product $\Delta = \Delta (\mathcal{L}, (012)\infty)$ as defined in Section 8.1.

As explained in Example 10.8, the sequence $\mathcal{L}$ defined above satisfies Assumption 10.7 with $\Phi(x) = x$ and constant $C_0$ which only depends on the constant $K_0$ and $\delta_0$ associated with the expander sequence $(\Gamma_i)$. Therefore by Theorem 10.9, there is a constant $C$, which only depends on $K_0$ and $\delta_0$, such that
\[
\log v_{\Delta,T}(r) \leq C \sum_{j: \lfloor \theta_j \rfloor \leq \log \lambda r} 2^j \min \left\{ \left( \frac{\eta}{2} \right)^j r, \phi(\theta_j) \right\} + Cr^{\epsilon_\alpha_0}.
\]

and
\[
\log v_{\Delta,T}(r) \geq \frac{1}{C} \max_{j: \lfloor \theta_j \rfloor \leq \log \lambda r} 2^j \min \left\{ \left( \frac{\eta}{2} \right)^j r, \phi(\theta_j) \right\} + \frac{1}{C} r^{\epsilon_\alpha_0}.
\]

Then by the approximation lemma 10.10 we have that for all $r \geq 1$,
\[
\log v_{\Delta,T}(r) \leq \frac{C}{C_{\lambda}} f(r) + Cr^{\epsilon_\alpha_0} \leq \left( \frac{C}{C_{\lambda}} + C \right) f(r);
\]
\[
\log v_{\Delta,T}(r) \geq \frac{1}{C} f(r) + \frac{1}{C} r^{\epsilon_\alpha_0} \geq \frac{1}{C} f(r).
\]

By Fact 8.3 and Corollary 8.5 we have that $\Delta$ is an FC-central extension of $W = (U \times V) \wr \mathfrak{G} = (\mathbb{Z}/2\mathbb{Z})^3 \wr \mathfrak{G}$. We have finished the proof of the statement. 

\[\square\]

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