

BOUNDEDNESS IN A QUASILINEAR PARABOLIC-PARABOLIC KELLER-SEGEL SYSTEM WITH THE SENSITIVITY \( v^{-1}S(u) \)

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**Abstract.** This paper is concerned with global existence and boundedness of classical solutions to the quasilinear fully parabolic Keller-Segel system

\[
\begin{align*}
    \frac{\partial u}{\partial t} &= \nabla \cdot (D(u)\nabla u) - \nabla \cdot \left( \frac{S(u)}{v} \nabla v \right), & x \in \Omega, \ t > 0, \\
    \frac{\partial v}{\partial t} &= \Delta v - v + u, & x \in \Omega, \ t > 0, \\
    \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0, & x \in \partial \Omega, \ t > 0, \\
    u(x,0) &= u_0(x), \ v(x,0) = v_0(x), & x \in \Omega,
\end{align*}
\]

(1.1)

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with smooth boundary, \( N \in \mathbb{N} \) and \( \frac{\partial}{\partial \nu} \) denotes differentiation with respect to the outward normal of \( \partial \Omega \). The initial data \((u_0, v_0)\) is assumed to be a pair of functions fulfilling

\[
u_0 \geq 0, \ u_0 \in C^2(\Omega) \quad \text{and} \quad v_0 > 0, \ v_0 \in C^1(\Omega). \label{initial_conditions}
\]

Moreover we suppose that \( D \) and \( S \) satisfy the following conditions:

\[
\begin{align*}
    D, S &\in C^2([0, \infty)) \quad \text{with} \ S(0) = 0 \quad \text{and} \ S \geq 0, \label{regularity_s}
    \\
    D(u) &\geq K_0(u + 1)^{m-1} \quad \text{with} \ m \in \mathbb{R} \quad \text{and} \ K_0 > 0 \quad \text{for all} \ u \geq 0, \\
    D(u) &\leq K_1(u + 1)^{M-1} \quad \text{with} \ M \in \mathbb{R} \quad \text{and} \ K_1 > 0 \quad \text{for all} \ u \geq 0, \\
    \frac{S(u)}{D(u)} &\leq K(u + 1)^\alpha \quad \text{with} \ \alpha < \frac{2}{N} \quad \text{and} \ K > 0 \quad \text{for all} \ u \geq 0. 
\end{align*}
\]

(1.3) (1.4) (1.5) (1.6)

1. **Introduction.** The Keller-Segel system, which was proposed by Keller and Segel [5] in 1970, describes a motion of cellular slime molds with chemotaxis. The system has been widely studied (see e.g., Hillen and Painter [3]).

In the present paper we consider the following quasilinear fully parabolic Keller-Segel system:

From a mathematical point of view it is important to study whether solutions remain bounded or blow up. As to the problem (1.1) without \( \frac{1}{v} \), i.e., in the case that the chemotaxis term in the first equation in (1.1) is replaced with \( -\nabla \cdot (S(u)\nabla v) \), Tao and Winkler [7] proved...
boundedness of solutions, provided that $D$ and $S$ satisfy (1.3), (1.4), (1.5) and (1.6) and $\Omega$ is convex. Recently this convexity condition of $\Omega$ was removed in [4]. As to blow-up of solutions to the problem (1.1) without $\frac{1}{N}$, Winkler [10, 8] and Cieślak and Stinner [1] established that the solutions blow up in finite time under the conditions that $\frac{S(u)}{D(u)} \geq K u^{\frac{N}{2}} + \eta$ for $u > 1$ with $K > 0$, $\eta > 0$ and that $S(u) \geq cu$ for some $c > 0$. Therefore the optimal exponent is known as $\frac{2}{N}$.

In the last decade, a growing literature has been concerned with signal-dependent sensitivity. The case that the chemotaxis term is $-\chi_0 \nabla \cdot (\frac{u}{v} \nabla v)$ was already proposed in the original model by Keller and Segel from a biological point of view such as the Weber-Fechner law. In [9, 2] it has been shown that (1.1) with $\chi_0 > 0$ small enough has a globally bounded solution, provided that the first equation has the linear diffusion $\Delta u$, i.e., $D(u) \equiv 1$. However, to the best of our knowledge, no results are available for the system with both nonlinear diffusion and signal-dependent sensitivity. As opposed to the case without $\frac{1}{N}$, we find that all solutions of (1.1) are global and bounded in the case $D(u) \equiv 1$ and $S(u) \equiv \chi_0 u$ with sufficiently small $\chi_0 > 0$ in [9, 2]. This means that the case $\alpha = 1$ and sufficiently small $K > 0$ admits global existence and boundedness. As $1 > \frac{2}{N}$ for $N \geq 3$, this fact indicates that the constant $\frac{2}{N}$ is not optimal in the condition (1.6). The question of optimality of (1.6) remains an open problem.

The purpose of the present paper is to establish a globally bounded solution of the Keller-Segel system with not only the nonlinear diffusion $\nabla \cdot (D(u) \nabla u)$ but also the singular sensitivity function $\frac{S(u)}{v}$. Our main result reads as follows.

**Theorem 1.1.** Assume that $(u_0, v_0)$ fulfills (1.2). Let $D$ and $S$ satisfy (1.3), (1.4), (1.5) and (1.6) with some $m \in \mathbb{R}$, $M \in \mathbb{R}$, $\alpha < \frac{2}{N}$, $K_0 > 0$, $K_1 > 0$ and $K > 0$. Then there exists a couple $(u, v)$ of nonnegative functions such that

\[
\begin{align*}
    u &\in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)), \\
    v &\in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty))
\end{align*}
\]

which solves (1.1) classically and moreover there exists $C > 0$ such that

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C \quad \text{for all } t > 0.$$

The difficulty in the proof of Theorem 1.1 lies in the singularity of $\frac{1}{N}$. In the present paper, a uniform-in-time lower bound for $v$ ([2]) builds a “bridge” between the regular case ([7, 4]) and the singular case. We will consider approximate problems in Section 2 and prepare some estimates. Section 3 is devoted to discussing convergence of approximate solutions and completing the proof of Theorem 1.1.

2. **Approximate problem.** We consider the following regularization of (1.1):

\[
\begin{align*}
    \frac{\partial u_\varepsilon}{\partial t} &= \nabla \cdot (D(u_\varepsilon) \nabla u_\varepsilon) - \nabla \cdot \left( \frac{S(u_\varepsilon)}{v_\varepsilon + \varepsilon} \nabla v_\varepsilon \right), \quad x \in \Omega, \quad t > 0, \\
    \frac{\partial v_\varepsilon}{\partial t} &= \Delta v_\varepsilon - v_\varepsilon + u_\varepsilon, \quad x \in \Omega, \quad t > 0, \\
    \frac{\partial u_\varepsilon}{\partial \nu} &= \frac{\partial v_\varepsilon}{\partial \nu} = 0, \quad x \in \partial \Omega, \quad t > 0, \\
    (u_\varepsilon(x, 0) &= u_0(x), \quad v_\varepsilon(x, 0) = v_0(x), \quad x \in \Omega,
\end{align*}
\]

where $\varepsilon > 0$. For all $(u_0, v_0)$ satisfying (1.2) we may invoke [7, Lemmas 1.1 and 1.2] to establish local existence of solutions to (2.1) as the following lemma.

**Lemma 2.1.** Let $\varepsilon > 0$. Suppose that $(u_0, v_0)$ fulfills (1.2). Assume that $D$ and $S$ satisfy (1.3), (1.4) and (1.5). Then there exist $T_{\text{max}} \in (0, \infty]$ and a pair $(u_\varepsilon, v_\varepsilon)$ of nonnegative
functions from \(C^0(\Omega \times [0, T_{\text{max}}]) \cap C^{2,1}(\Omega \times (0, T_{\text{max}}))\) solving (2.1) classically in \(\Omega \times (0, T_{\text{max}})\). Moreover,

\[
\text{either } T_{\text{max}} = \infty \text{ or } \lim_{t \to T_{\text{max}}} \| u_{\varepsilon}(t) \|_{L^\infty(\Omega)} + \| v_{\varepsilon}(t) \|_{L^\infty(\Omega)} = \infty;
\]

furthermore, \(u_{\varepsilon}\) has the following mass conservation:

\[
\| u_{\varepsilon}(t) \|_{L^1(\Omega)} = \| u_0 \|_{L^1(\Omega)} \quad \text{for all } t \in (0, T_{\text{max}}).
\]

The following lemma is a cornerstone of this work, which was essentially established in [2, Lemma 2.2]. Mass conservation property plays a key role in the proof of the lemma. In view of the lemma we can ensure a uniform-in-time estimate for \(v_{\varepsilon}\).

**Lemma 2.2.** Let \(\varepsilon > 0\) and \(T > 0\). Suppose that \((u_0, v_0)\) fulfills (1.2). Assume that \(D\) and \(S\) satisfy (1.3), (1.4) and (1.5). Let \((u_{\varepsilon}, v_{\varepsilon})\) be a solution of (2.1) on \([0, T]\). Then there exists \(\delta > 0\) such that

\[
\inf_{x \in \Omega} v_{\varepsilon}(x, t) \geq \delta > 0 \quad \text{for all } t \in (0, T), \quad \varepsilon > 0,
\]

where \(\delta\) does not depend on \(\varepsilon\) and \(T\).

As a preparation for the passage to the limit, we present three lemmas.

**Lemma 2.3.** Let \(\varepsilon > 0\) and \(T > 0\). Suppose that \((u_0, v_0)\) fulfills (1.2). Assume that \(D\) and \(S\) satisfy (1.3), (1.4), (1.5) and (1.6). Let \((u_{\varepsilon}, v_{\varepsilon})\) be a solution of (2.1) on \([0, T]\). Then for all \(p \in [1, \infty)\) and each \(q \in [1, \infty)\) there exist \(C_p > 0\) and \(C_{2q} > 0\) such that

\[
\| u_{\varepsilon}(t) \|_{L^p(\Omega)} \leq C_p \quad \text{for all } t \in (0, T),
\]

\[
\| \nabla v_{\varepsilon}(t) \|_{L^{2q}(\Omega)} \leq C_{2q} \quad \text{for all } t \in (0, T),
\]

where \(C_p\) and \(C_{2q}\) do not depend on \(\varepsilon\) and \(T\).

**Proof.** Proceeding similarly as in [7, Lemma 3.3] and [4, Proposition 3.2], we define \(\phi\) as

\[
\phi(r) := \int_0^r \frac{(\sigma + 1)^{m+p-3}}{D(\sigma)} d\sigma d\rho.
\]

Thus we can calculate

\[
\frac{d}{dt} \int_\Omega \phi(u_{\varepsilon}) = \int_\Omega \phi'(u_{\varepsilon}) \nabla \cdot (D(u_{\varepsilon}) \nabla u_{\varepsilon}) - \int_\Omega \phi(u_{\varepsilon}) \nabla \cdot \left( \frac{S(u_{\varepsilon})}{v_{\varepsilon} + \varepsilon} \nabla v_{\varepsilon} \right)
\]

\[
= - \int_\Omega \phi''(u_{\varepsilon}) D(u_{\varepsilon}) |\nabla u_{\varepsilon}|^2 + \int_\Omega \phi''(u_{\varepsilon}) \frac{S(u_{\varepsilon})}{v_{\varepsilon} + \varepsilon} \nabla u_{\varepsilon} \cdot \nabla v_{\varepsilon}
\]

\[
= - \int_\Omega (u_{\varepsilon} + 1)^{m+p-3} |\nabla u_{\varepsilon}|^2
\]

\[
+ \int_\Omega (u_{\varepsilon} + 1)^{m+p-3} \frac{S(u_{\varepsilon})}{D(u_{\varepsilon}) v_{\varepsilon} + \varepsilon} \nabla u_{\varepsilon} \cdot \nabla v_{\varepsilon}.
\]

Now in virtue of Lemma 2.2 we have the following independent-in-\(\varepsilon\) bound:

\[
\frac{1}{v_{\varepsilon} + \varepsilon} \leq \frac{1}{\delta},
\]

and we are in the same position as [7, (3.10)]. The rest of this proof is the same procedure as in the proofs of [7, Lemma 3.3] and [4, Proposition 3.2].

**Lemma 2.4.** Let \(\varepsilon > 0\) and \(T > 0\). Suppose that \((u_0, v_0)\) fulfills (1.2). Assume that \(D\) and \(S\) satisfy (1.3), (1.4), (1.5) and (1.6). Let \((u_{\varepsilon}, v_{\varepsilon})\) be a solution of (2.1) on \([0, T]\). Then there exist \(C_\infty > 0\) and \(C'_\infty > 0\) such that

\[
\| u_{\varepsilon}(t) \|_{L^\infty(\Omega)} \leq C_\infty \quad \text{for all } t \in (0, T),
\]

\[
\| \nabla v_{\varepsilon}(t) \|_{L^\infty(\Omega)} \leq C'_\infty \quad \text{for all } t \in (0, T),
\]
where \( C_\infty \) and \( C'_\infty \) do not depend on \( \varepsilon \) and \( T \).

**Proof.** In light of (2.2), we can proceed as in [7, Lemma A.1] and so Lemma 2.3 implies (2.3). As to (2.4), using the representation formula for \( v_\varepsilon \) and standard smoothing estimates, we see that

\[
\| \nabla v_\varepsilon(t) \|_{L^\infty(\Omega)} \leq \| \nabla e^{(\Delta-1)} v_0 \|_{L^\infty(\Omega)} + \int_0^t \| \nabla e^{(t-s)(\Delta-1)} u_\varepsilon(s) \|_{L^\infty(\Omega)} \, ds
\]

\[
\leq c \left( \| v_0 \|_{L^\infty(\Omega)} + \int_0^t (t-s)^{-\frac{1}{2}} - \frac{\varepsilon}{\varepsilon + \theta} c^{\eta(t-s)} \| u_\varepsilon(s) \|_{L^\infty(\Omega)} \, ds \right)
\]

with constants \( c > 0, \eta > 0 \) and \( \theta > 1 \). Now we can choose \( \theta > 1 \) large enough satisfying \( \frac{1}{2} + \frac{N}{2} \cdot \frac{1}{\theta} < 1 \) and (2.3) ensures boundedness of the right-hand side of the above inequality which leads to the conclusion. \( \square \)

**Lemma 2.5.** Let \( \varepsilon > 0 \) and \( T > 0 \). Suppose that \((u_0, v_0)\) fulfills (1.2). Assume that \( D \) and \( S \) satisfy (1.3), (1.4), (1.5) and (1.6). Let \((u_\varepsilon, v_\varepsilon)\) be a solution of (2.1) on \([0, T]\). Then there exists \( C''_\infty > 0 \) such that

\[
\| \nabla u_\varepsilon(t) \|_{L^\infty(\Omega)} \leq C''_\infty \quad \text{for all} \; t \in (0, T),
\]

where \( C''_\infty \) does not depend on \( \varepsilon \) and \( T \).

**Proof.** We can calculate the first equation in (2.1) as

\[
\frac{\partial u_\varepsilon}{\partial t} = \nabla \cdot (D(u_\varepsilon) \nabla u_\varepsilon) - \nabla \left( \frac{S(u_\varepsilon)}{v_\varepsilon + \varepsilon} \right) \cdot \nabla v_\varepsilon - S(u_\varepsilon) v_\varepsilon \Delta v_\varepsilon
\]

\[
= \nabla \cdot (D(u_\varepsilon) \nabla u_\varepsilon) + \frac{S(u_\varepsilon)}{v_\varepsilon + \varepsilon} \nabla v_\varepsilon \cdot \nabla v_\varepsilon - \frac{S'(u_\varepsilon)}{v_\varepsilon + \varepsilon} \nabla u_\varepsilon \cdot \nabla v_\varepsilon - \frac{S(u_\varepsilon)}{v_\varepsilon + \varepsilon} \Delta v_\varepsilon.
\]

From (2.2) we have the following upper estimates:

\[
\left| \frac{S(u_\varepsilon)}{v_\varepsilon + \varepsilon} \nabla v_\varepsilon \right|^2 \leq \frac{S(u_\varepsilon)}{\delta^2} \nabla v_\varepsilon^2,
\]

\[
\left| \frac{S'(u_\varepsilon)}{v_\varepsilon + \varepsilon} \nabla u_\varepsilon \cdot \nabla v_\varepsilon \right| \leq \frac{|S(u_\varepsilon)|}{\delta} \nabla u_\varepsilon \| \nabla v_\varepsilon \|
\]

\[
\left| \frac{S(u_\varepsilon)}{v_\varepsilon + \varepsilon} \Delta v_\varepsilon \right| \leq \frac{S(u_\varepsilon)}{\delta} |\Delta v_\varepsilon|.
\]

By noting that \( u_0 \in C^2(\overline{\Omega}) \), these estimates allow us to apply standard parabolic theory [6, Theorem V.7.2] and to complete the proof. \( \square \)

3. **Proof of the main theorem.** We start by showing that \( \{u_\varepsilon\} \) and \( \{v_\varepsilon\} \) satisfy the Cauchy condition.

**Lemma 3.1.** Let \( \varepsilon > 0 \) and \( T > 0 \). Suppose that \((u_0, v_0)\) fulfills (1.2). Assume that \( D \) and \( S \) satisfy (1.3), (1.4), (1.5) and (1.6). Let \((u_\varepsilon, v_\varepsilon)\) be a solution of (2.1) on \([0, T]\). Then there exist \( c_1 > 0, c_2 > 0, c_3 > 0, c_4 > 0 \) and \( c_5 > 0 \) such that for all \( \mu > 0, \nu > 0 \) and \( t \in [0, T] \),

\[
|u_\mu(t) - u_\nu(t)|^2_{L^2(\Omega)} + c_1 \| v_\mu(t) - v_\nu(t) \|^2_{L^2(\Omega)}
\]

\[
+ c_2 \int_0^t \| \nabla (u_\mu(s) - u_\nu(s)) \|^2_{L^2(\Omega)} \, ds + c_3 \int_0^t \| \nabla (v_\mu(s) - v_\nu(s)) \|^2_{L^2(\Omega)} \, ds
\]

\[
\leq c_4 |\mu - \nu|^2 e^{c_5 T}.
\]
Proof. Let $\mu > 0$ and $\nu > 0$. Multiplying the difference of the first equations in (2.1) by $(u_\mu - u_\nu)$, we see that
\[
\frac{1}{2} \frac{d}{dt} \|u_\mu - u_\nu\|_{L^2(\Omega)}^2 = \int_\Omega \nabla \cdot (D(u_\mu) \nabla u_\mu - D(u_\nu) \nabla u_\nu)(u_\mu - u_\nu) \\
- \int_\Omega \nabla \cdot \left( \frac{S(u_\mu)}{v_\mu + \mu} \nabla v_\mu - \frac{S(u_\nu)}{v_\nu + \nu} \nabla v_\nu \right)(u_\mu - u_\nu) \\
= - \int_\Omega (D(u_\mu) \nabla u_\mu - D(u_\nu) \nabla u_\nu) \cdot \nabla (u_\mu - u_\nu) \\
+ \int_\Omega \left( \frac{S(u_\mu)}{v_\mu + \mu} \nabla v_\mu - \frac{S(u_\nu)}{v_\nu + \nu} \nabla v_\nu \right) \cdot \nabla (u_\mu - u_\nu) \\
=: I_1 + I_2. \tag{3.2}
\]
As to the first term $I_1$, it follows from (1.3), (1.4), (2.3) and (2.5) that
\[
I_1 = - \int_\Omega (D(u_\mu) \nabla u_\mu - D(u_\nu) \nabla u_\nu) \cdot \nabla (u_\mu - u_\nu) \\
= - \int_\Omega (D(u_\mu) |\nabla (u_\mu - u_\nu)|^2 - \int_\Omega (D(u_\mu) - D(u_\nu)) \nabla u_\nu \cdot \nabla (u_\mu - u_\nu) \\
\leq -\tilde{K}_0 \int_\Omega |\nabla (u_\mu - u_\nu)|^2 + C_{\max} \int_\Omega |\nabla u_\nu| |u_\mu - u_\nu| |\nabla (u_\mu - u_\nu)| \\
\leq -\tilde{K}_0 \|\nabla (u_\mu - u_\nu)\|^2_{L^2(\Omega)} + C_{\max} \int_\Omega |u_\mu - u_\nu||\nabla (u_\mu - u_\nu)|,
\]
where $\tilde{K}_0 := K_0 \min\{1, (C_\infty + 1)^{m-1}\}$ and $C_{\max} := \max_{\sigma \in [0, C_\infty]} D'(\sigma)$. In light of Young’s inequality we deduce that
\[
I_1 \leq -\tilde{K}_0 \|\nabla (u_\mu - u_\nu)\|^2_{L^2(\Omega)} \\
+ \frac{5C_{\max}^2 C_{\infty}^2}{2K_0} \|u_\mu - u_\nu\|^2_{L^2(\Omega)} + \frac{\tilde{K}_0}{10} \|\nabla (u_\mu - u_\nu)\|^2_{L^2(\Omega)}. \tag{3.3}
\]
As to the second term $I_2$ in (3.2), we write it as follows:
\[
I_2 = \int_\Omega \left( \frac{S(u_\mu)}{v_\mu + \mu} \nabla v_\mu - \frac{S(u_\nu)}{v_\nu + \nu} \nabla v_\nu \right) \cdot \nabla (u_\mu - u_\nu) \\
= \int_\Omega \frac{S(u_\mu)}{v_\mu + \mu} \nabla v_\mu \cdot \nabla (u_\mu - u_\nu) \\
+ \int_\Omega \frac{S(u_\nu)}{v_\nu + \nu} \nabla v_\nu \cdot \nabla (u_\mu - u_\nu).
\]
Then (1.3) and (2.3) entail that
\[
I_2 \leq \tilde{C}_{\max} \int_\Omega |u_\mu - u_\nu| |\nabla v_\mu||\nabla (u_\mu - u_\nu)| \\
+ \tilde{C}_{\max} \int_\Omega |H(v_\mu, v_\nu, \mu, \nu)||\nabla (u_\mu - u_\nu)|,
\]
where $\tilde{C}_{\max} := \max_{\sigma \in [0, C_\infty]} S'(\sigma)$, $\tilde{C}_{\max} := \max_{\sigma \in [0, C_\infty]} S(\sigma)$ and
\[
H(v_\mu, v_\nu, \mu, \nu) := \left( \frac{1}{v_\mu + \mu} - \frac{1}{v_\nu + \nu} \right) \nabla v_\mu + \frac{1}{v_\nu + \nu} \nabla (v_\mu - v_\nu). 
\]
From Lemma 2.2 we find that

\[
|H(\mu, \nu, \mu, \nu)| \leq \frac{1}{(\mu + \nu)\nu + \nu} |\nu - \mu| |\nabla v_\mu| + \frac{1}{\nu} |\nabla (v_\mu - v_\nu)| \leq \frac{1}{\delta^2} |\nu - \mu| |\nabla v_\mu| + \frac{1}{\delta^2} |\nu - \mu| |\nabla (v_\mu - v_\nu)|.
\]

Thus applying (2.4) and Lemma 2.2, we infer

\[
I_2 \leq \frac{C_{\max}C'}{\delta} \int_\Omega |u_\mu - u_\nu| |\nabla (u_\mu - u_\nu)| + \frac{C_{\max}C'}{\delta^2} \int_\Omega |v_\nu - v_\mu| |\nabla (u_\mu - u_\nu)| + \frac{C_{\max}C'}{\delta^2} \int_\Omega |\nu - \mu| |\nabla (u_\mu - u_\nu)| + \frac{C_{\max}}{\delta} \int_\Omega |\nabla (v_\mu - v_\nu)| |\nabla (u_\mu - u_\nu)|.
\]

Hence Young’s inequality says that

\[
I_2 \leq \frac{5 \hat{C}_{\max} C_{\max} C'}{2\delta^2 K_0} \|u_\mu - u_\nu\|_{L^2(\Omega)}^2 + \frac{\hat{K}_0}{10} \|\nabla (u_\mu - u_\nu)\|_{L^2(\Omega)}^2 + \frac{5 \hat{C}_{\max} C_{\max} C'}{2\delta^2 K_0} \|v_\nu - v_\mu\|_{L^2(\Omega)}^2 + \frac{\hat{K}_0}{10} \|\nabla (u_\mu - u_\nu)\|_{L^2(\Omega)}^2 + |\Omega| \frac{5 \hat{C}_{\max} C_{\max} C'}{2\delta^2 K_0} |\nu - \mu|^2 + \frac{\hat{K}_0}{10} \|\nabla (u_\mu - u_\nu)\|_{L^2(\Omega)}^2 + \frac{5 \hat{C}_{\max} C_{\max} C'}{2\delta^2 K_0} \|\nabla (v_\mu - v_\nu)\|_{L^2(\Omega)}^2 + \frac{\hat{K}_0}{10} \|\nabla (u_\mu - u_\nu)\|_{L^2(\Omega)}^2.
\]

Consequently, combining (3.2) with (3.3) and (3.4), we see that

\[
\frac{1}{2} \frac{d}{dt} \|u_\mu - u_\nu\|_{L^2(\Omega)}^2 + \frac{\hat{K}_0}{2} \|\nabla (u_\mu - u_\nu)\|_{L^2(\Omega)}^2 \leq C_1 \|u_\mu - u_\nu\|_{L^2(\Omega)}^2 + C_2 \|v_\nu - v_\mu\|_{L^2(\Omega)}^2 + C_3 |\nu - \mu|^2 + C_4 \|\nabla (v_\mu - v_\nu)\|_{L^2(\Omega)}^2.
\]

where \(C_1, C_2, C_3\) and \(C_4\) are given by

\[
C_1 := \frac{5 \hat{C}_{\max} C_{\max} C'}{2K_0} + \frac{5 \hat{C}_{\max} C_{\max} C'}{2\delta^2 K_0}, \quad C_2 := \frac{5 \hat{C}_{\max} C_{\max} C'}{2\delta^4 K_0},
\]

\[
C_3 := |\Omega| \frac{5 \hat{C}_{\max} C_{\max} C'}{2\delta^4 K_0}, \quad C_4 := \frac{5 \hat{C}_{\max} C_{\max} C'}{2\delta^2 K_0}.
\]

Similarly, Young’s inequality yields

\[
\frac{1}{2} \frac{d}{dt} \|v_\mu - v_\nu\|_{L^2(\Omega)}^2 \leq -\|\nabla (v_\mu - v_\nu)\|_{L^2(\Omega)}^2 + \frac{1}{4} \|u_\mu - u_\nu\|_{L^2(\Omega)}^2.
\]
Multiplying (3.6) by $2C_4$ and adding (3.5), we have
\[ \frac{1}{2} \frac{d}{dt}(\|u_\mu - u_\nu\|_{L^2(\Omega)}^2 + 2C_4\|v_\mu - v_\nu\|_{L^2(\Omega)}^2) \]
\[ + \frac{K_0}{2} \|\nabla(u_\mu - u_\nu)\|_{L^2(\Omega)}^2 + C_4 \|\nabla(v_\mu - v_\nu)\|_{L^2(\Omega)}^2 \]
\[ \leq \left( C_1 + \frac{C_4}{2} \right) \|u_\mu - u_\nu\|_{L^2(\Omega)}^2 + C_4 \|v_\mu - v_\nu\|_{L^2(\Omega)}^2 + C_3|\mu - \nu|^2 \]
\[ \leq C_5(\|u_\mu - u_\nu\|_{L^2(\Omega)}^2 + 2C_4\|v_\mu - v_\nu\|_{L^2(\Omega)}^2) + C_3|\mu - \nu|^2, \]

where $C_5 := \max\{C_1 + \frac{C_4}{2}, \frac{C_4^2}{2C_5}\}$, and thus Gronwall’s lemma yields
\[ \|u_\mu(t) - u_\nu(t)\|_{L^2(\Omega)}^2 + 2C_4\|v_\mu(t) - v_\nu(t)\|_{L^2(\Omega)}^2 \]
\[ + \int_0^t e^{2C_5(t-s)} \left( K_0 \|\nabla(u_\mu(s) - u_\nu(s))\|_{L^2(\Omega)}^2 + 2C_4 \|\nabla(v_\mu(s) - v_\nu(s))\|_{L^2(\Omega)}^2 \right) \, ds \]
\[ \leq \frac{C_4}{C_5} |\mu - \nu|^2 e^{2C_5t} \]
for all $t \in [0, T]$. Since $e^{2C_5(t-s)} \geq 1 (s \in [0, t])$, we obtain the desired inequality.

We are now in a position to prove Theorem 1.1.

**Proof of Theorem 1.1.** We have $T_{\text{max}} = \infty$ from Lemma 2.4. For all $T > 0$, in view of Lemma 3.1 we find $u$ and $v$ from $L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ such that
\[ u_\varepsilon \to u \quad \text{in } L^\infty(0, T; L^2(\Omega)) \quad \text{as } \varepsilon \to 0, \]
\[ v_\varepsilon \to v \quad \text{in } L^\infty(0, T; L^2(\Omega)) \quad \text{as } \varepsilon \to 0, \]
\[ \nabla u_\varepsilon \to \nabla u \quad \text{in } L^2(0, T; L^2(\Omega)) \quad \text{as } \varepsilon \to 0, \]
\[ \nabla v_\varepsilon \to \nabla v \quad \text{in } L^2(0, T; L^2(\Omega)) \quad \text{as } \varepsilon \to 0. \quad (3.7) \]
\[ (3.8) \]

We will prove that $(u, v)$ is a classical solution of (1.1) and bounded. The proof is divided into two steps.

**Step 1.** In this step we prove that $(u, v)$ is a weak solution of (1.1). Let $\varphi \in C^\infty(\Omega \times [0, \infty))$. We can fix $T > 0$ such that $\text{supp } \varphi \subset \Omega \times [0, T)$. Multiplying the first equation in (2.1) by $\varphi$ and integrating it over $\Omega \times (0, T)$, we can see
\[ -\int_0^T \int_\Omega u_\varepsilon \frac{d\varphi}{dt} \, dx = -\int_0^T \int_\Omega D(u_\varepsilon) \nabla u_\varepsilon \cdot \nabla \varphi \]
\[ + \int_0^T \int_\Omega \frac{S(u_\varepsilon)}{v_\varepsilon + \varepsilon} \nabla v_\varepsilon \cdot \nabla \varphi + \int_\Omega u_0 \varphi(\cdot, 0). \quad (3.9) \]

To accomplish the passage to the limit of approximate solutions we will confirm convergence of each term. Firstly from the convergence $u_\varepsilon \to u$ in $L^2(0, T; L^2(\Omega))$ as $\varepsilon \to 0$ due to (3.1), we easily check
\[ -\int_0^T \int_\Omega u_\varepsilon \frac{d\varphi}{dt} \to -\int_0^T \int_\Omega u \frac{d\varphi}{dt} \quad \text{as } \varepsilon \to 0. \]

Next we consider convergence of the first term on the right-hand side of (3.9). We observe that
\[ |D(u_\varepsilon) \nabla \varphi| \leq \max_{\sigma \in [0, C_\infty]} D(\sigma) \cdot |\nabla \varphi| \in L^2(0, T; L^2(\Omega)) \]
due to (1.3) and $D(u_\varepsilon) \to D(u)$ pointwisely as $\varepsilon \to 0$. Thus it follows that
\[ D(u_\varepsilon) \nabla \varphi \to D(u) \nabla \varphi \quad \text{in } L^2(0, T; L^2(\Omega)) \quad \text{as } \varepsilon \to 0. \quad (3.10) \]
Therefore invoking (3.7) and (3.10), we can show the following convergence:

\[-\int_0^T \int_\Omega D(u_\varepsilon) \nabla u_\varepsilon \cdot \nabla \varphi \, dt \rightarrow -\int_0^T \int_\Omega D(u) \nabla u \cdot \nabla \varphi \quad \text{as } \varepsilon \to 0.\]

As to the second term, (1.3) and Lemma 2.2 yield

\[|\frac{S(u_\varepsilon)}{v_\varepsilon + \varepsilon} \nabla \varphi| \leq \frac{\tilde{C}_{\text{max}}}{\delta} |\nabla \varphi| \in L^2(0, T; L^2(\Omega))\]

and

\[\frac{S(u_\varepsilon)}{v_\varepsilon + \varepsilon} \nabla \varphi \rightarrow \frac{S(u)}{v} \nabla \varphi \quad \text{pointwisely as } \varepsilon \to 0, \quad \text{and hence we can establish}\]

\[\frac{S(u_\varepsilon)}{v_\varepsilon + \varepsilon} \nabla \varphi \rightarrow \frac{S(u)}{v} \nabla \varphi \quad \text{in } L^2(0, T; L^2(\Omega)) \quad \text{as } \varepsilon \to 0.\]

In the same fashion as before (3.8) implies

\[\int_0^T \int_\Omega \frac{S(u_\varepsilon)}{v_\varepsilon + \varepsilon} \nabla v_\varepsilon \cdot \nabla \varphi \rightarrow \int_0^T \int_\Omega \frac{S(u)}{v} \nabla v \cdot \nabla \varphi \quad \text{as } \varepsilon \to 0.\]

Therefore we can accomplish the passage of the limit and hence

\[-\int_0^T \int_\Omega \frac{d}{dt} \varphi \, \frac{d}{dt} + \int_0^T \int_\Omega D(u) \nabla u \cdot \nabla \varphi + \int_0^T \int_\Omega \frac{S(u)}{v} \nabla v \cdot \nabla \varphi + \int_\Omega u_0 \varphi(\cdot, 0).\]

As to the second equation in (1.1), we can similarly deduce the following identity:

\[-\int_0^T \int_\Omega \frac{d}{dt} \varphi \, \frac{d}{dt} + \int_0^T \int_\Omega \nabla v \cdot \nabla \varphi - \int_0^T \int_\Omega v \varphi - \int_\Omega v_0 \varphi(\cdot, 0).\]

Thus we conclude that \((u, v)\) is a weak solution of (1.1).

**Step 2.** Using standard semigroup techniques and parabolic Schauder estimates, we deduce from straightforward regularity arguments that \((u, v)\) is a global classical solution of (1.1). Consequently, we have a globally bounded classical solution \((u, v)\) of (1.1) such that \(u\) belongs to \(L^\infty(\overline{\Omega} \times [0, \infty))\) in light of boundedness of \(\{u_\varepsilon\}_{\varepsilon > 0}\) in \(L^\infty(\overline{\Omega} \times [0, \infty))\) (Lemma 2.4).

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