RATIONAL LINEAR SUBSPACES OF HYPERSURFACES OVER
FINITE FIELDS

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Abstract. Fix positive integers $n, r, d$. We show that if $n, r, d$ satisfy a suitable inequality, then any smooth hypersurface $X \subset \mathbb{P}^n$ defined over a finite field of characteristic $p$ sufficiently large contains a rational $r$-plane. Under more restrictive hypotheses on $n, r, d$ we show the same result without the assumption that $X$ is smooth or that $p$ is sufficiently large.

1. Introduction

There has been recent interest in studying rational lines and other subvarieties of hypersurfaces over finite fields. For instance, the most direct inspiration for this work, [3], studies the existence of rational lines on cubic hypersurfaces. It is straightforward to establish that there are smooth cubic curves and surfaces over arbitrary finite fields with no rational lines. In [3], Debarre, Laface, and Rouleau establish the existence of rational lines on dimension $n$ cubics over $\mathbb{F}_q$ for all $n \geq 3$ unless $n = 3$ and $q \leq 9$ or $n = 4$ and $q = 3$, and show that there are examples with no rational lines when $n = 3$ and $q \leq 5$.

In this note we consider two questions motivated by [3]. First, we establish the existence of rational linear subspaces of higher degree hypersurfaces, under some assumptions, and give an effective version of our result. Secondly, we explore the existence of rational lines on threefolds over $\mathbb{F}_q$ for $q = 7, 8, 9$. While we were not able to close the existence question, we run some computer experiments that we summarize in the appendix. The statistics in the appendix do not seem to clarify whether all threefolds over $\mathbb{F}_7$ should contain at least one rational line; however, they suggest that all threefolds over $\mathbb{F}_q$ for $q = 8, 9$ probably will.

Our main result guarantees the existence of rational linear subspaces of higher degree on a smooth hypersurface $X$, under some hypotheses:

**Theorem 1.1.** Fix $d$ and $n$ such that $n \geq 2d - 1$ and $n \geq 4$. Let $p$ be a prime outside a finite set depending on $d$ and $n$. Then every smooth hypersurface $X$ of degree $d$ in $\mathbb{P}^n$ over a finite field of characteristic $p$ has a rational line. In addition, for all $r \geq 1$, we
have that $X$ contains a rational $r$-plane if the inequality

$$n \geq 2 \binom{d + r - 1}{r} + r$$

is satisfied.

Without the assumption that $X$ is smooth, on the other hand, we have the following elementary result, generalizing [3, Theorem 6.1].

**Proposition 1.2.** Let $X$ be the zero locus of some degree $d$ polynomial in $\mathbb{P}^n_k$ with $k$ quasi-algebraically closed. Let $r$ be a positive integer. If

$$n \geq r + \binom{d + r}{r + 1},$$

then for any rational $r'$-plane $\Lambda$ contained in $X$ with $r' < r$, there is a rational $r$-plane containing $\Lambda$ and contained in $X$.

Since finite fields are quasi-algebraically closed by [2], this extends Theorem [1.1] to the case of $X$ singular over a finite field, with the downside of having a far more restrictive bound on $d$. As a consequence, we can give an effective bound on the primes $p$ appearing in Theorem [1.1] in some special cases; see Theorem [3.2].

**Remark.** Another recent work that addresses linear subspaces of hypersurfaces is [8]. Therein, Kazhdan and Polischuk study how the existence of linear subspaces defined over the algebraic closure of a perfect field $k$ yield rational linear subspaces of higher codimension over the field itself. Since a smooth hypersurface $X \subset \mathbb{P}^N$ has no linear subspaces of codimension $\frac{N}{2}$ or less, their result applies only to singular varieties, and in that sense is a complement to Theorem [1.1].

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2. **Proofs**

The key geometric input in the proof of Theorem [1.1] will be the following theorem of Beheshti and Riedl.

**Theorem 2.1** ([1], Theorem 1.3). Let $X \subset \mathbb{P}^n_k$ be a smooth hypersurface of degree $d$ and suppose $n \geq 2d - 1$ and $n \geq 4$. Then the Fano variety of lines $F_1(X)$ is irreducible of the expected dimension $2n - d - 3$. 

Likewise, the Fano variety $F_r(X)$ of $r$-planes will be irreducible of the expected dimension $(r + 1)(n - r) - \binom{d + r}{r}$ provided

$$n \geq 2\left(\frac{d + r - 1}{r}\right) + r.$$ 

Given $n, d, r$, we will show this result holds over arbitrary fields of all but finitely many characteristics, allowing us to apply the Lang-Weil bound. Throughout the proof, all schemes will be defined over Spec $\mathbb{Z}$ unless otherwise indicated.

2.1. Proof of Theorem 1.1 Let $\mathbb{P}^N$ be the parameter space of all degree $d$ hypersurfaces in $\mathbb{P}^n$, so $N = \binom{d + n}{d} - 1$. Let $Y \subset \mathbb{P}^N$ be the parameter space of all smooth hypersurfaces, and let $\pi : X \to Y$ be the universal Fano scheme of $r$-planes. This morphism is finitely presented, so by [7 Théorème 9.7.7], the locus of $y \in Y$ such that the Fano scheme $X_y$ is geometrically irreducible is locally constructible. But the previous result says that after changing base to $\mathbb{Q}$, all of $Y$ is in this locus. So, for some open set $U \subset Y$ such that $U_\mathbb{Q} = Y_\mathbb{Q}$, we have that every Fano scheme $X_y$, corresponding to $y \in U$ is geometrically irreducible. The complement $Z := U \setminus Y$ is closed in $Y$, so its image under the final map $Z \to \text{Spec} \mathbb{Z}$ is constructible and does not contain the generic point. So the image is a finite set of primes $p_1, \ldots, p_s$.

Set $R = \text{Spec} \mathbb{Z}\left[\frac{1}{p_1 \cdots p_s}\right]$, so the map $\pi : X_R \to Y_R$ is a finitely presented map with geometrically irreducible fibers. Moreover, applying Theorem 2.1 and upper semicontinuity of fiber dimension [7 Théorème 13.1.3], the fibers are all of dimension at least $(r + 1)(n - r) - \binom{d + r}{r}$. Moreover, we have that $\pi : X_R \to Y_R$ is projective. Then, by the Lang-Weil bound of [10], for $q$ sufficiently large we have that for any $\mathbb{F}_q$ point $y \to Y$ the Fano scheme $X_y$ has a rational point.

2.2. Proof of Proposition 1.2 We may assume $r' = r - 1$, as for more general $r'$ we may simply repeatedly apply this result to produce first an $r' + 1$-plane contained in $X$, then an $r' + 2$-plane, and so on. Say $X$ is cut out by $F$, and let $\Lambda$ be a rational $r - 1$-plane. By applying some automorphism of $\mathbb{P}^n$, we may assume $\Lambda$ is the plane cut out by $x_r = \cdots = x_n = 0$. The space of $r$-planes in $\mathbb{P}^n$ containing $\Lambda$ is naturally identified with the $n - r$-plane $\Lambda^0$ cut out by $x_0 = \cdots = x_{r-1} = 0$; an $r$-plane $\Lambda'$ containing $\Lambda$ is identified with its point of intersection with the plane $\Lambda^0$. Writing $d_* = (d_0, \ldots, d_{r-1})$, we can rewrite $F$ in the form

$$F(x_0, \ldots, x_n) = \sum_{d_*} x_0^{d_0} \cdots x_{r-1}^{d_{r-1}} F_{d_*}(x_r, \ldots, x_n),$$

with each $F_{d_*}$ homogeneous of degree $d - \sum_i d_i$. Because $F$ vanishes on $\Lambda$, $F_{d_*} = 0$ if $d - \sum_i d_i = 0$, and the space of $r$-planes through $\Lambda$ in $X$ is cut out by $\{F_{d_*} : 0 \leq d_0 + \cdots + d_{r-1} < d\}$ on $\Lambda^0$. For $j$ any positive integer, there are $\binom{j+r-1}{r-1}$ ways to write
j as a sum $j = i_0 + \cdots + i_{r-1}$ with each $i_j$ a nonnegative integer. Thus the sum of the degrees of all the $F_{d*}$ is given by

$$\sum_{0 \leq j < d} \binom{j + r - 1}{r - 1} (d - j).$$

Using the hockey-stick identity on this sum gives the sum of degrees,

$$\sum_{0 \leq j < d} \binom{j + r - 1}{r - 1} (d - j) = \binom{d + r}{r + 1}.$$

Since $k$ is quasi-algebraically closed and we have

$$n - r \geq \binom{d + r}{r + 1}$$

by hypothesis, the locus in $\mathbb{P}^{n-r}$ cut out by the $F_I$ includes a rational point, giving a rational $r$-plane through $\Lambda$.

3. Effective results

We would like to investigate how to make the characteristic $p$ in Theorem 1.1 effective. It seems difficult to prove Theorem 2.1 directly in a fixed characteristic $p$: the proof in [1] requires the characteristic zero assumption because [1, Proposition 2.2] (which is a version of [9, Corollary II.3.10.1]) uses generic smoothness. To obtain effective bounds on $p$, we instead use the following effective Lang–Weil bound due to Ghorpade and Lachaud [4]. Below, a variety is a geometrically integral separated scheme of finite type over a field.

**Theorem 3.1** ([4, Theorem 11.1], [5, Theorem 4.1]). Let $X \subset \mathbb{P}^N_{\overline{F}_q}$ be a projective variety of dimension $n$ and degree $d$. Then,

$$|X(\mathbb{F}_q)| - |\mathbb{P}^n(\mathbb{F}_q)| \leq (d - 1)(d - 2)q^{n-(1/2)} + C_+(X)q^{n-1},$$

where $C_+(X)$ only depends on the base change of $X$ to the algebraic closure $\overline{F}_q$, and where

$$C_+(X) \leq 9 \times 2^n \times (m\delta + 3)^{N+1}$$

if $X$ can be defined by the vanishing of $m$ homogeneous polynomials of degrees $d_1, \ldots, d_m$, and $\delta = \max\{d_1, \ldots, d_m\}$.

**Theorem 3.2.** Let $X \subset \mathbb{P}^N_{\overline{F}_q}$ be a geometrically integral hypersurface of degree $d > 1$ and let $r$ be a positive integer. Assume that

$$n \geq r + \binom{d + r}{r + 1}.$$
and that
\[ \sqrt{q} > \frac{(d - 1)(d - 2) + \sqrt{(d - 1)^2(d - 2)^2 + 4(18(d + 3)^{n+1} - 1)}}{2}. \]

Then \( X \) contains a rational \( r \)-plane.

**Proof.** By Proposition 1.2, for every point \( x \in X(\mathbb{F}_q) \), \( X \) contains a rational \( r \)-plane passing through \( x \). Therefore it suffices to prove that \( X(\mathbb{F}_q) \) is nonempty.

Applying Theorem 3.1 to \( X \), we get that
\[
|X(\mathbb{F}_q)| \geq |\mathbb{P}^{n-1}(\mathbb{F}_q)| - (d - 1)(d - 2)q^{n-3/2} - 18(d + 3)^{n+1}q^{n-2} - 1 + 2q - 2 + q^{-1} + 2q^{-2} + q^{-3} + q^{-4}.
\]

Since \( q \) is positive, the last line of the inequality is greater that zero if and only if
\[
\sqrt{q} > \frac{(d - 1)(d - 2) + \sqrt{(d - 1)^2(d - 2)^2 + 4(18(d + 3)^{n+1} - 1)}}{2}.
\]

Therefore, for \( q \) as in the statement of the theorem, \( X \) contains a rational point, and hence a rational \( r \)-plane. \( \square \)

**Appendix**

In [3] the authors prove that any smooth cubic threefold \( X \subset \mathbb{P}^4_{\mathbb{F}_q} \) contains a rational line whenever \( q \geq 11 \). They also present examples of smooth cubic threefolds with no lines defined over \( \mathbb{F}_q \) for \( q = 2, 3, 4, 5 \). This existence question remains open in the cases \( q = 7, 8, 9 \). We run some computer experiments to collect data about these open cases.

For each \( q \) in \( \{2, 3, 4, 5, 7, 8, 9, 11\} \), we obtained a random sample of \( 10^4 \) cubic threefolds defined over \( \mathbb{F}_q \) and we computed the number of rational lines in each threefold. We repeated this experiment with random samples of \( 10^4 \) smooth cubic threefolds. The sample size was restricted by the exponential growth of the computation time: for instance, running the code on a \( 10^4 \) sample of cubic threefolds took approximately four hours and 42 minutes of CPU time for \( q = 7 \), and three days and 20 hours for \( q = 11 \). The results are summarized in tables 1 and 2, respectively.

We observe in Table 1 that the average number of lines in each sample of cubic threefolds is very close to the theoretical approximation from [3, Formula (24)], given by
\[
q^2 + q + 2 + 2q^{-1} + 2q^{-2} + q^{-3} + q^{-4}.
\]

The smooth cubic threefolds in the second set of samples contain slightly fewer rational lines on average, as recorded in Table 2.
We could not find examples with no rational lines for $q = 7, 8, 9$. When $q = 7$, we found an example of a smooth cubic threefold containing exactly 8 rational lines, given by

\[2x_1^3 + 6x_1^2x_2 + 5x_1x_2^2 + 5x_2^3 + 3x_1^2x_3 + 3x_1x_2x_3 + 4x_1x_3^2 + 4x_2x_3^2 + x_3^3 + x_1^2x_4 + 4x_1x_2x_4 + x_2^2x_4 + 5x_1x_3x_4 + 5x_2x_3x_4 + 5x_3^2x_4 + 2x_1x_4^2 + 6x_2x_4^2 + 3x_3x_4^2 + 5x_4^3 + 2x_1^2x_5 + 2x_1x_5^2 + 2x_2x_3x_5 + 6x_2^2x_5 + 2x_1x_4x_5 + 2x_2x_4x_5 + 4x_3x_4x_5 + 2x_4^2x_5 + 4x_1x_5^2 + 4x_2x_5^2 + 3x_3x_5^2 + x_4x_5^2 + x_5^3 = 0.\]

In Figure 1 we show the sample distributions we obtained for $q = 5$ (for which there exist cubic threefolds with no rational lines), $q = 7$ (for which the existence of rational lines remains open), and $q = 11$ (for which it is known that all smooth cubic threefolds contain a rational line).
Figure 1. Comparing number of lines for $q = 5, 7, 11$. 
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