A novel hybrid isogeometric element based on two-field Hellinger–Reissner principle to alleviate different types of locking

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Abstract. In the present work, novel hybrid elements are proposed to alleviate the locking anomaly in non-uniform rational B-spline-based isogeometric analysis (IGA) using a two-field Hellinger–Reissner variational principle. The proposed hybrid elements are derived by adopting the independent interpolation schemes for displacement and stress fields. The key highlight of the present study is the choice and evaluation of higher-order terms for the stress interpolation function to provide a locking-free solution. Furthermore, the present study demonstrates the efficacy of the proposed elements with the treatment of several two-dimensional linear-elastic benchmark problems alongside the conventional single-field IGA, Lagrangian-based finite element analysis (FEA), and hybrid FEA formulation. It is shown that the proposed class of hybrid elements performs effectively for analyzing the nearly incompressible problem domains that are severely affected by volumetric locking along with the thin plate and shell problems where the shear locking is dominant. A better coarse mesh accuracy of the proposed method in comparison with the conventional formulation is demonstrated through various numerical examples. Moreover, the formulation is not restricted to the locking-dominated problem domains but can also be implemented to solve the problems of general form without any special treatment. Thus, the proposed method is robust, most efficient, and highly effective against both shear and volumetric locking.

Keywords. Isogeometric analysis; Hellinger–Reissner principle; Hybrid isogeometric analysis; Locking; Finite element analysis; Mixed formulation.

1. Introduction

The finite element analysis (FEA) is a widely practiced numerical procedure to solve the partial differential equations governing a mathematical model for a physical problem. Over a period of time, FEA successfully distinguished itself in various problem domains and found numerous applications in a diverse set of engineering fields. Despite its widespread applications, traditional FEA has certain drawbacks. One of which is the geometry approximation. Most often, the physical domain of the problem is modeled using the computer-aided design (CAD) geometries, which are further treated to create the FE mesh. However, the resultant FE mesh is an approximate version of the actual physical CAD domain. Though refining the mesh considerably improves the approximation but with an additional computing cost and preprocessing efforts. It has been argued that these geometric irregularities can lead to significant errors in analysis. Moreover, the time spent on preprocessing, i.e., to make an analysis-ready FE mesh model, is significantly high as compared to the actual analysis time [1].

To overcome the stated limitations, Hughes et al. introduced the concept of IGA, which provides an efficient integration between CAD and FEA [2]. In IGA, the physical geometry remains invariant regardless of the type or number of elements. Furthermore, once the initial coarse mesh is created, IGA simplifies the mesh refinement by removing further dependency on CAD. The fundamental idea behind IGA was to reinstate the conventional Lagrangian interpolation functions with NURBS basis functions that are extensively utilized in engineering design with the existence of numerous effective and numerically stable algorithms. Furthermore, retention of the essential mathematical properties like non-negativity of basis functions, linear independence, partition of unity, and variation diminishing property [3] of the NURBS assisted the foundation for IGA background.

Since its introduction in 2005, IGA has been widely practiced in different directions and proved to be the powerful method that out-perform FEA in most of the numerical aspects. The ability of the IGA framework in
retaining the exact geometry with higher inter-element continuity, smooth representation of surfaces, and tight coupling between CAD geometries and FEA model lead to effective implementation in the several application domains such as contact formulations [4–6], structural shape optimization [7], fluid and fluid-structure analysis [8–10], structural vibration problem [11], shell and plate problems [12–14], and many more. Furthermore, the effective implementation influenced researchers to make efforts to integrate NURBS-based IGA in existing commercial software like LS-DYNA [15] and Abaqus [16] and to develop several dedicated tools, e.g., GeoPDEs [17], PetIGA [18], and Pegasus [19].

In many aspects, IGA provides a superior and generalized version of conventional FEA. However, it exhibits certain limitations similar to its FE counterpart. For instance, the locking phenomenon that appears during the analysis of thin structural geometries and incompressible material behavior [20]. The term “locking” is often used to label specific situations when the solutions of the nodal variables are underestimated or the near-infinite stiffness promotes the absurd results for the solution. The two most prevalent forms of locking are volumetric and shear locking. The volumetric locking (also known as dilatation or Poisson’s locking) is associated with the Poisson ratio (ν). When the material is incompressible (ν = 0.5) or nearly incompressible (ν ≈ 0.5), the IGA scheme results in unrealistic solutions or low convergence rates for a practical range of discretization [21, 22]. The adverse effects of shear locking can be observed in thin structural geometries where the high slenderness ratios can lead to additional stiffening effects and wildly oscillating shear forces along the length of the element.

To overcome the stated limitation in context of IGA, the explored contributions are limited. The classical shell theories that are extensively used in the analysis of thin structures to handle the shear or membrane locking are successfully incorporated into the IGA framework. Popular theories and their IGA counterpart include the Reissner–Mindlin shell theory [23, 24], Kirchhoff–Love theory [25], and blended shell theory [12]. Furthermore, the contributions are extended to the degenerated shell approach or solid-like shell formulation [23, 26] along with the NURBS-based solid shell elements [27]. Another popular technique to alleviate different types of locking is reduced and selective reduced integration, which has been explored in the framework of IGA [28] with promising results. Furthermore, the methods like $\bar{B}$ and $\bar{F}$ projection techniques are also investigated for handling the volumetric locking in case of incompressible or nearly incompressible problem domain [29, 30]. Along with these, several multi-field variational techniques are found to be effective in alleviating different types of locking in IGA. Popular methods include displacements-pressure formulation [31], assumed natural strains (ANS) [32, 33] and enhanced assumed strains (EAS) [34, 35], which satisfactorily handle the situations where the standard IGA is prone to locking.

It should be pointed out that most of the mixed formulations, modified to work with IGA, are strain-based approaches, and less focus has been given to the stress-based formulations. The advancements in the NURBS-based shell element where the shear locking is addressed using the two-field Hellinger–Reissner principle can be found in the works of Echter et al. [13, 36]. However, the particular focus of these studies was to avoid membrane locking in various shell models using the modification in stress part. Furthermore, the application of Hellinger–Reissner with similar approximating strategies for alleviating locking in shell elements can be seen in [37, 38].

The applicability of the stated stress-based shell models [13, 36–38] is restricted to thin structural geometries, whereas the present formulation is based on a generalized FEA framework which works effectively for thin structures as well as for bulk geometries. Different formulations available in the literature have specific limitations. For instance, the displacement-pressure formulation [31] does not work for shear locking problems, ANS [32, 33] fails in the case of volumetric locking, and EAS [34, 35] has been explored only to alleviate volumetric locking in incompressible problem domains. On the contrary, the proposed stress-based mixed formulation is capable of alleviating both shear and volumetric locking. Also, it can be formulated for lower as well as higher-order elements. Specific modifications of Hellinger–Reissner with similar approximating strategies for composite laminated plates can also be found in [39–41], where a superior accuracy of stress solutions over Reissner mixed formulation [42] has been highlighted.

The present study emphasizes on developing a relatively simple but robust two-field stress-based IGA formulation that is capable of producing locking-free solutions irrespective of the type of locking. The proposed hybrid elements are derived based on a two-field mixed variational principle where displacement and stresses are the independent field variables. The key notion is the choice and evaluation of stress interpolation functions which is inspired from the work of Jog [43, 44] in the context of conventional FEA framework. The efficient FE implementation of the stated approach for large-deformation contact mechanics [45], structural acoustics [46], electromagnetic analysis [47], analysis of electromechanical systems [48], and coupled fluid-structure problem [49] confirms the effectiveness and robustness of the method.

In the present study, the systematic evaluation of stress interpolation functions specific to the NURBS interpolations and its effective implementation in a two-dimensional linear elasticity regime is investigated. The principal concept follows the assessment of normal stress components in relation to derivatives of the respective displacement interpolations and the choice of higher-order terms in
shear components are defined such that they ensure correct stiffness rank and suppress the spurious zero-energy mode, which is necessary to avoid locking. Though the formulation is computationally expensive, it compensates for its efficiency by providing a relatively simple formulation and high coarse mesh accuracy. Elements developed by this theory perform quite well, irrespective of the type of locking. The same element can be used in situations where problems demand incompressibility or near incompressibility of material or to solve plate/shell geometries, and so on. Additionally, the same formulation can be implemented for the standard problems (absence of locking effect) without affecting the solution accuracy, making it easier to implement in coupled problem domains. Another advantage of the hybrid formulation is that no energy or work principle or variational norms are violated, and hence mathematically robust formulation is obtained. The preliminary work of the stated approach has been presented in [50]. However, the present work advances in terms of implementing the proposed two-field hybrid stress formulation in the IGA framework. Furthermore, the section deals with a step-wise implementation procedure followed by Section 4, where several examples with promising results are illustrated. Section 5 summarizes the results and validates the performance of the proposed formulation.

2. Mathematical preliminaries

A \(p\)th degree univariate B-spline interpolation function for a prescribed knot vector \(\Xi = \{\xi_1, \xi_2, \ldots, \xi_{n+p+1}\}\) is defined using the following Cox-de Boor recursion formula [3]

\[
    N_{i,p}^{\xi}(\xi) = \begin{cases} 1 & \text{if } \xi_i \leq \xi < \xi_{i+1} \\ 0 & \text{Otherwise} \end{cases},
\]

for \(p = 0\),

\[
    N_{i,p}^{\xi}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1} + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1},
\]

for \(p \geq 1\).

For the polynomial degree 0 and 1, \(N_{i,p}(\xi)\) will result in standard piecewise constant and Lagrangian interpolation functions, respectively. However, higher degree basis functions \((p \geq 2)\) differ from their FEA counterparts. Further, these interpolations are used as a basis of linear combination to describe a geometric description of a specific curve.

A \(p\)th degree B-spline curve for a given \(\Xi\) and a set of control points is defined as

\[
    C(\xi) = \sum_{i=1}^{n} N_{i,p}(\xi) P_i,
\]

whereas the B-spline surfaces are constructed by considering a bidirectional net of control points, two knot vectors and the tensor product of two univariate B-spline basis functions and given as

\[
    S(\xi, \eta) = \sum_{i=1}^{n} \sum_{j=1}^{m} N_{ij}^{p,q}(\xi, \eta) P_{ij},
\]

Similarly, the B-spline volumes are defined as

\[
    V(\xi, \eta, \zeta) = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{l} N_{ijk}^{p,q,r}(\xi, \eta, \zeta) P_{ijk},
\]

where \(N_{ij}^{p,q}(\xi, \eta)\) and \(N_{ijk}^{p,q,r}(\xi, \eta, \zeta)\) are the bivariate and trivariate B-spline interpolation functions. \(N_{i,p}(\xi), M_{ij}^{pq}(\eta)\) and \(L_{ik}^{qr}(\zeta)\) are \(p, q\) and \(r\)th degree univariate B-spline basis function in \(\xi, \eta\) and \(\zeta\) direction. \(n, m\) and \(l\) are the number of basis functions in corresponding directions. \(P_i, P_{ij},\) and \(P_{ijk}\) are control point coordinates \([x \ y \ z]^T\) for \((i)\)th, \((i,j)\)th, and \((i,j,k)\)th control point.

The evolution of B-spline interpolations to Non-Uniform Rational B-Splines (NURBS) provides an advantage in representing a wide variety of objects, including conic sections such as circles, spheres, cylinders, etc. NURBS are the generalization of B-spline basis functions where ‘Non Uniform’ refers to a non-uniform knot vector whereas ‘Rational B-splines’ describe the rationalization of B-spline basis functions. It inherits all the mathematical properties of B-splines with the added advantage of weights which allows them to offer greater flexibility and accuracy in generation of CAD geometries. A univariate NURBS basis function is defined as

\[
    R_{i,p}(\xi) = \frac{w_i N_{i,p}(\xi)}{W(\xi)}, \quad \text{where } W(\xi) = \sum_{i=1}^{n} w_i N_{i,p}(\xi),
\]

and \(w_i\) is strictly positive \((w_i > 0)\) set of weights. A \(p\)th degree NURBS curve for a given knot vector and a set of control points is defined as
\[ C(\xi) = \sum_{i=1}^{n} R_{i,p}(\xi)P_i = \sum_{i=1}^{n} w_i N_{i,p}(\xi)P_i, \]  
(7)

Similar to B-splines, the NURBS surfaces are constructed by considering a bidirectional net of control points, two knot vectors and the tensor product of two univariate NURBS basis functions and given as

\[ S(\xi, \eta) = \sum_{i=1}^{n} \sum_{j=1}^{m} R_{i,j}(\xi, \eta)P_{i,j} = \sum_{i=1}^{n} \sum_{j=1}^{m} R_{i,p}(\xi)R_{j,q}(\eta)P_{i,j}, \]

\[ = \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{N_{i,p}(\xi)M_{j,q}(\eta)w_{i,j}}{\sum_{i=1}^{n} \sum_{j=1}^{m} N_{i,p}(\xi)M_{j,q}(\eta)w_{i,j}}P_{i,j}, \]
(8)

and the NURBS volumes are defined as

\[ V(\xi, \eta, \zeta) \]
\[ = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{l} R_{i,j,k}(\xi, \eta, \zeta)P_{i,j,k}, \]

\[ = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{l} R_{i,k}(\xi)R_{j,q}(\eta)R_{k,r}(\zeta)P_{i,j,k}, \]

\[ = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{l} \frac{N_{i,p}(\xi)M_{j,q}(\eta)L_{k,r}(\zeta)w_{i,j,k}P_{i,j,k}}{\sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{l} N_{i,p}(\xi)M_{j,q}(\eta)L_{k,r}(\zeta)w_{i,j,k}}, \]
(9)

where \( R_{i,j}(\xi, \eta) \) and \( R_{i,j,k}(\xi, \eta, \zeta) \) are the bivariate and trivariate NURBS interpolation functions, respectively. \( R_{i,p}(\xi), R_{j,q}(\eta) \) and \( R_{k,r}(\zeta) \) are \( p \), \( q \) and \( r \)th degree univariate NURBS basis function in \( \xi, \eta \) and \( \zeta \) direction.

3. Fundamental concepts in two-field hybrid stress formulation in context of IGA

3.1 A classical two-dimensional linear elasticity problem

To understand the underlying concepts in two-field stress formulation and the differentiating features from the conventional single-field IGA formulation [2], it will be wise to appraise the classical two-dimensional linear elasticity problem. Let \( \Omega \) be the open domain with boundary \( \Gamma \), which is composed of two disjoint regions such that \( \Gamma = \Gamma_u \cup \Gamma_t \), where \( \Gamma_u \) is displacement boundary and \( \Gamma_t \) is traction boundary. The governing equations for the linear elasticity problem are given as

\[ \mathbf{V} \cdot \mathbf{\tau} + \mathbf{f} = 0 \text{ on } \Omega, \]
(10a)

\[ \mathbf{n} = t \text{ on } \Gamma, \]
(10b)

\[ \mathbf{u} = \mathbf{u}_0 \text{ on } \Gamma_u, \]
(10c)

\[ t = \tilde{t} \text{ on } \Gamma_t, \]
(10d)

where \( \mathbf{\tau} \) is the Cauchy’s stress tensor, \( \mathbf{n} \) is the unit outward normal to \( \Gamma \), \( t \) is traction defined on the boundary \( \Gamma_t \) and \( \mathbf{f} \) is the body force vector. The small strain tensor \( \mathbf{\varepsilon} \) is defined as

\[ \mathbf{\varepsilon}(\mathbf{u}) = \frac{1}{2} \left[ (\nabla \mathbf{u}) + (\nabla \mathbf{u})^T \right], \]
(11)

and the stress-strain relation is given as \( \mathbf{\tau} = \mathbf{C}\mathbf{\varepsilon} \), where \( \mathbf{C} \) is material constitutive tensor.

3.2 A two-field variational statement

The variational form of the stated governing equations in Section 3.1 is evaluated using the two-field Hellinger–Reissner variational formulation where stress and displacement are the field variables. The involvement of two field variables will necessitate the respective variation for each field. Let \( \delta \mathbf{u} \) and \( \delta \mathbf{\tau} \) be the variation of the displacement field \( \mathbf{u} \) and stress field \( \mathbf{\tau} \) respectively, in such a way that

\[ V_u = \{ \delta \mathbf{u} \in H^1(\Omega) : \delta \mathbf{u} = 0 \text{ on } \Gamma_u \}, \]
(12)

\[ V_t = \{ \delta \mathbf{\tau} \in L^2(\Omega) : \delta \mathbf{\tau} = \delta \mathbf{\tau}^T \}. \]
(13)

If \( (\delta \mathbf{u}, \delta \mathbf{\tau}) \in (V_u \times V_t) \), then the weak form for the given governing equation can be written as

\[ \int_{\Omega} \delta \mathbf{u} \cdot (\nabla \cdot \mathbf{\tau} + \mathbf{f}) \, d\Omega + \int_{\Gamma_t} \delta \mathbf{u} \cdot (\tilde{t} - t) \, d\Gamma \]
\[ + \int_{\Omega} \delta \mathbf{\tau} : [\mathbf{\varepsilon}^\mathbf{\tau} - \mathbf{\varepsilon}^\mathbf{\mathbf{\varepsilon}}] \, d\Omega = 0, \]
(14)

where \( \mathbf{\varepsilon}^\mathbf{\mathbf{\varepsilon}} \) denotes the strain tensor derived with the help of stress-strain relation and \( \mathbf{\varepsilon}^\mathbf{\mathbf{\varepsilon}} \) is the strain tensor derived from the displacements using the Eq. 11.

After satisfying the two conditions, i.e., \( (\delta \mathbf{u}, \delta \mathbf{\tau}) = (\delta \mathbf{u}, 0) \) and \( (\delta \mathbf{u}, \delta \mathbf{\tau}) = (0, \delta \mathbf{\tau}) \), the above equation (Eq. 14) simplifies to the following forms

\[ \int_{\Omega} [\mathbf{\varepsilon} \cdot (\delta \mathbf{u})]^T \mathbf{\tau} \cdot d\Omega = \int_{\Omega} \delta \mathbf{u}^T \mathbf{f} \, d\Omega + \int_{\Gamma_t} \delta \mathbf{u}^T \tilde{t} \, d\Gamma \quad \forall \delta \mathbf{u} \in V_u, \]
(15)

\[ \int_{\Omega} \delta \mathbf{\tau}^T [\mathbf{\varepsilon} \cdot (\mathbf{u}) - \mathbf{C}^{-1} \mathbf{\tau} \cdot d\Omega = 0 \quad \forall \delta \mathbf{\tau}, \]
(16)

where \( \mathbf{\tau}_c \) and \( \mathbf{\tau}_c \) are the vector form of stress and strain tensor, respectively.
3.3 Approximating functions

3.3.1 Interpolation functions for displacement field The current work uses the isoparametric concept where the displacement field \( \mathbf{u} \) is approximated using the NURBS basis functions that are capable of maintaining the exact geometry. The displacement field \( \mathbf{u} \) is interpolated as

\[
\mathbf{u} = \sum_{i=1}^{n_{cp}} R_i \mathbf{u}_i = \mathbf{R} \mathbf{u}, \quad \delta \mathbf{u} = \sum_{i=1}^{n_{cp}} R_i \delta \mathbf{u}_i = \mathbf{R} \delta \mathbf{u},
\]

where \( R_i \) is the NURBS basis function, \( I \) denote the global numbering assigned to the control points, and \( n_{cp} \) is the total number of control points per element.

3.3.2 Stress interpolation functions The next entity of interest in the present formulation is the stress interpolation matrix that benefits in approximating the second independent field, i.e., \( \tau_c \). Let \( \tau_c \) and its variation \( \delta \tau_c \) be interpolated as

\[
\tau_c = P \mathbf{\hat{\tau}}, \quad \delta \tau_c = P \delta \mathbf{\hat{\tau}},
\]

where \( \mathbf{\hat{\tau}} \) is the vector consisting of the stress parameters for the respective element, \( \delta \mathbf{\hat{\tau}} \) is the vector of stress variation parameters, and \( P \) is the stress interpolation matrix. The accuracy of the solution is highly sensitive towards the choice of \( P \). Hence, to ensure the efficient derivation of \( P \), it is mandatory to evaluate the NURBS basis functions in its symbolic design. However, due to the recursive nature of the NURBS basis functions, special treatment needs to be followed to derive the symbolic expressions. At present, the widely recognized NURBS toolbox can only evaluate the basis function values at a given parametric point and does not provide the desired symbolic expressions. However, a mathematical computational software, Mathematica, provides a platform to develop a code to fulfill this provision. Taking advantage of this, a Mathematica code has been developed which is capable of deriving the symbolic expressions for the basis functions. All basis functions and respective derivatives in Section 3.5 are evaluated using Mathematica.

3.4 IGA equilibrium equations for two-field variation principle

Recollecting the expressions for \( \mathbf{u} \) (Eq. 17), \( \tau_c \) (Eq. 18), and if the strains are defined as \( \varepsilon_c(\mathbf{u}) = \mathbf{B} \mathbf{u} \) where \( \mathbf{B} \) is the strain displacement matrix corresponding to the NURBS interpolation functions, then the weak statement for the two-field variational statement (Eqs. 15 and 16) will reduce to

\[
\begin{bmatrix}
-H & G \\
G^T & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{\hat{\beta}} \\
\mathbf{\hat{u}}
\end{bmatrix}
= \begin{bmatrix}
\mathbf{\hat{g}} \\
\mathbf{\hat{f}}
\end{bmatrix},
\]

where \( G = \int_{\Omega} \mathbf{P}^T \mathbf{B} \, d\Omega, \quad H = \int_{\Omega} \mathbf{P}^T \mathbf{S}^p \, d\Omega, \quad \mathbf{\hat{g}} = 0, \quad \mathbf{\hat{f}} = \int_{\Gamma_t} \mathbf{R}^T d\Gamma.
\]

\( S \) denotes the compliance matrix which is evaluated as \( S = \mathbf{C}^{-1} \). Evaluating the expression for \( \mathbf{\hat{\beta}} \) from Eq. 19 will lead to

\[
\frac{\mathbf{G}^T \mathbf{H}^{-1} \mathbf{G}}{k} \mathbf{u} = \mathbf{f} \rightarrow \mathbf{K} \mathbf{u} = \mathbf{f},
\]

where the term \( \mathbf{G}^T \mathbf{H}^{-1} \mathbf{G} \) represents the stiffness matrix for two-field hybrid stress formulation.

3.5 Derivation of proposed interpolations to approximate stresses for two-dimensional elements

Let a single bi-cubic IGA element be modeled using \( \Xi = \mathcal{H} = \{0, 0, 0, 0, 1, 1, 1, 1\} \), and \( p = q = 3 \) where \( \Xi, \mathcal{H} \) and \( p, q \) are the knot vectors and degree of basis function along \( \zeta \) and \( \eta \) direction, respectively. The control points associated with the geometric description of a model are treated as constants, which will only affect the physical domain but not the parametric space.

For the stated bi-cubic element, there will be four univariate basis functions in each \( \zeta \) and \( \eta \) direction. Let \( R_{ij}^p(\zeta) \) and \( R_{ij}^q(\eta) \) be the basis functions in \( \zeta \) and \( \eta \) direction where \( i = 1, 2, 3, 4 \) and \( j = 1, 2, 3, 4 \). Assume a simplified undistorted rectangular geometric description for which the weights associated with the control points are equal to 1; then, the participating basis functions can be evaluated as follows:

\[
R_i = R_{i,j}^p(\zeta, \eta) = R_{ij}^p(\zeta) R_{ij}^q(\eta) = N_{i,p}(\zeta) M_{j,q}(\eta).
\]

The expressions for \( N_{i,p}(\zeta) \) for the range \( 0 \leq \zeta \leq 1 \) and \( M_{j,q}(\eta) \) for \( 0 \leq \eta \leq 1 \) are evaluated as

\[
\begin{align*}
N_{1,3} &= -3 \zeta^3 + 3 \zeta^2 - 3 \zeta + 1, \\
N_{2,3} &= 3 \zeta^3 - 2 \zeta^2 + \zeta, \\
N_{3,3} &= -3 \zeta^3 - 3 \zeta^2, \\
N_{4,3} &= \zeta^3, \\
M_{1,3} &= -3 \eta^3 + 3 \eta^2 - 3 \eta + 1, \\
M_{2,3} &= 3 \eta^3 - 2 \eta^2 + \eta, \\
M_{3,3} &= -3 \eta^3 - 3 \eta^2, \\
M_{4,3} &= \eta^3.
\end{align*}
\]

Hence, the expressions for the resultant two-dimensional participating basis function \( R_i \) can be written as
Finally, the shear stress ($\tau_{ij}$) are obtained by considering the common interpolations of $\tau_{ij}$ and $\tau_{ji}$. Another aspect to consider while deriving the stress interpolation functions is to choose the stress components so that they can vary independently of each other. This means there should not be any shared $\beta$ term in different stress components. This leads to an expression for $\tau_{\xi\eta}$ as a linear combination of the following terms:

$$\{1, \eta, \eta^2, \xi, \eta^2, \xi^2, \eta \xi, \eta\xi^2, \eta^2 \xi, \eta^2 \xi^2, \eta^3 \xi, \eta^3 \xi^2, \eta^2 \xi^2, \eta^3 \xi^2, \eta^3 \xi^3\},$$

whereas $\tau_{\eta\eta}$ will be the linear combination of

$$\{1, \eta, \eta^2, \xi, \eta^2, \xi^2, \eta^2 \xi, \eta^2 \xi^2, \xi^2, \eta \xi^2, \eta^3 \xi^2, \eta^3 \xi^3, \eta^2 \xi^2, \eta^3 \xi^3, \xi \xi^2\}.$$

Finally, the shear stress ($\tau_{\xi\eta}$) components are obtained using the common interpolations of $\tau_{\xi\xi}$ and $\tau_{\eta\eta}$ so that they suppress any spurious zero-energy mode [44]. Therefore, the expression for $\tau_{\xi\eta}$ will be constructed as a linear combination of the terms $\{1, \xi, \eta, \xi^2, \eta \xi, \eta^2 \xi, \xi^2, \eta^2 \xi, \xi^3, \eta \xi^2, \eta^2 \xi^2, \eta^3 \xi, \eta^2 \xi^2, \xi^2, \eta^2 \xi^2, \eta^3 \xi^2, \eta^3 \xi^3\}$. Introduce the stress parameter $\beta_i$ ($i = 1, 2, \ldots, 33$) in such a way that no $\beta_i$ term is shared, which will result in the following expression of stresses:

$$\tau_{\xi\xi} = \beta_1 + \beta_2 \xi + \beta_3 \eta + \beta_4 \xi^2 + \beta_5 \eta \xi + \beta_6 \eta^2 + \beta_7 \eta \xi^2 + \beta_8 \xi \eta^2 + \beta_9 \xi^2 \eta^2 + \beta_{10} \eta^3 \xi + \beta_{11} \eta^2 \xi^2 + \beta_{12} \xi^3,$$

(23)

$$\tau_{\eta\eta} = \beta_{13} + \beta_{14} \xi + \beta_{15} \eta + \beta_{16} \eta^2 \xi + \beta_{17} \eta \xi^2 + \beta_{18} \eta \xi \eta^2 + \beta_{19} \eta^3 + \beta_{20} \eta^2 \xi^2 + \beta_{21} \eta \xi^3 + \beta_{22} \eta \xi^2 \eta^2 + \beta_{23} \xi^3 \eta + \beta_{24} \xi^2 \eta^2 + \beta_{25} \eta^3 \xi^2 + \beta_{26} \eta^2 \xi^3 + \beta_{27} \eta \xi^3 \eta^2 + \beta_{28} \eta \xi^2 \eta^3 + \beta_{29} \xi^3 \eta^2 + \beta_{30} \xi^2 \eta^3 + \beta_{31} \eta^3 \xi^3 + \beta_{32} \eta^2 \xi^3 + \beta_{33} \eta \xi^3 \eta^3,$$

(24)

$$\tau_{\xi\eta} = \beta_{29} + \beta_{30} \xi + \beta_{31} \eta + \beta_{32} \eta^2 \xi + \beta_{33} \eta \xi^2 + \beta_{34} \eta \xi \eta^2 + \beta_{35} \eta^3 + \beta_{36} \eta^2 \xi^2 + \beta_{37} \eta \xi^3 + \beta_{38} \eta \xi^2 \eta^2 + \beta_{39} \xi^3 \eta + \beta_{40} \xi^2 \eta^2 + \beta_{41} \eta^3 \xi^3 + \beta_{42} \eta^2 \xi^3 + \beta_{43} \eta \xi^3 \eta^2 + \beta_{44} \eta \xi^2 \eta^3 + \beta_{45} \xi^3 \eta^2 + \beta_{46} \xi^2 \eta^3 + \beta_{47} \eta^3 \xi^3 + \beta_{48} \eta^2 \xi^3 + \beta_{49} \eta \xi^3 \eta^3.$$

(25)

Considering the fact that, for evaluation of integrals, it is important to define the stress components in the master space. Parametric space and master space are related with a linear mapping. Hence, stress components in a master space will have the same form but with different constants. Therefore, the stress components in a master space ($\xi - \eta$) are given as

$$\begin{bmatrix}
\tau_{\xi\xi} \\
\tau_{\eta\eta} \\
\tau_{\xi\eta}
\end{bmatrix} = 
\begin{bmatrix}
P_1 & P_2 & P_3 \\
P_4 & P_5 & P_6 \\
P_7 & P_8 & P_9
\end{bmatrix}
\begin{bmatrix}
\beta_1 \\
\beta_2 \\
\beta_3
\end{bmatrix} = \mathbf{P}(\xi, \eta)\hat{\mathbf{\beta}},$$

(26)

where $P_2, P_5, P_7, P_8$ are the zero vectors of size $1 \times 12$, $P_3, P_6$ are the zero vectors of size $1 \times 9$, and $P_1, P_5, P_9$ are evaluated as

$$P_1 = \begin{bmatrix}
1, \xi, \eta, \xi^2, \eta \xi, \eta^2, \xi \eta^2, \eta^2 \xi, \eta^3, \xi \eta^3, \eta^2 \xi^2, \xi \eta^2 \xi, \eta^3 \xi, \xi^2 \eta^2, \eta^3 \xi^2, \xi \eta^3 \xi, \eta^2 \xi \eta^2, \eta^3 \xi \eta^2, \xi \eta^2 \xi^2, \eta^3 \xi^2, \xi \eta^3 \xi \eta^2, \eta^3 \xi \eta^3, \xi \eta^3 \xi \eta^3
\end{bmatrix},$$

$$P_5 = \begin{bmatrix}
1, \xi, \eta, \xi^2, \eta \xi, \eta^2, \xi \eta^2, \eta^2 \xi, \eta^3, \xi \eta^3, \eta^2 \xi^2, \xi \eta^2 \xi, \eta^3 \xi, \xi^2 \eta^2, \eta^3 \xi^2, \xi \eta^3 \xi, \eta^2 \xi \eta^2, \eta^3 \xi \eta^2, \xi \eta^2 \xi^2, \eta^3 \xi^2, \xi \eta^3 \xi \eta^2, \eta^3 \xi \eta^3, \xi \eta^3 \xi \eta^3
\end{bmatrix},$$

$$P_7 = \begin{bmatrix}
1, \xi, \eta, \xi^2, \eta \xi, \eta^2, \xi \eta^2, \eta^2 \xi, \eta^3, \xi \eta^3, \eta^2 \xi^2, \xi \eta^2 \xi, \eta^3 \xi, \xi^2 \eta^2, \eta^3 \xi^2, \xi \eta^3 \xi, \eta^2 \xi \eta^2, \eta^3 \xi \eta^2, \xi \eta^2 \xi^2, \eta^3 \xi^2, \xi \eta^3 \xi \eta^2, \eta^3 \xi \eta^3, \xi \eta^3 \xi \eta^3
\end{bmatrix}.$$
energy ($U$) of an element domain which is described using the following expression:

$$U = \frac{1}{2} \mathbf{u}^T \mathbf{K} \mathbf{u} = \frac{1}{2} [\mathbf{G} \mathbf{u}]^T \mathbf{H}^{-1} [\mathbf{G} \mathbf{u}].$$  

(27)

The preceding equation results in zero-strain energy if the admissible displacement ($\mathbf{u}$) coincides with the null space of $\mathbf{G}_e$.

Let $\mathcal{N}$ be the null space of $\mathbf{G}_e$ and $\mathbf{n}_i^0 (i = 1, 2, \text{and } 3)$ are the vector basis of $\mathcal{N}$. $\mathbf{n}_i^0$ are evaluated by substituting the $\mathbf{B}$ matrix (evaluated using the displacement interpolations) and the proposed $\mathbf{P}$ matrix (Eq. 26) into the expression of $\mathbf{G}_e$. Finally, substituting the symbolic expressions of participating NURBS basis functions and $\mathbf{u} = \mathbf{n}_i^0$ in Eq. 17 will result in

$$u = a_1 + a_2 \eta, \quad (28)$$
$$v = a_3 - a_4 \zeta. \quad (29)$$

The above expression of $u$ is nothing but the linear combination of general expressions of translation and rotational rigid body modes for a two-dimensional domain. This demonstrates that the proposed stress interpolations provides a formulation that is free of spurious energy modes and $\mathcal{N}$ which results in zero-energy modes solely consists the rigid body modes for an unconstrained element domain.

Finally, the stress components in master space are related with the physical space with the following transformation:

$$\mathbf{\tau}_e(x, y) = \mathbf{T} \mathbf{\tau}_e(\xi, \eta),$$

(30)

where $\mathbf{T}$ is the transformation matrix derived from a combination of Jacobians relating the corresponding spaces. Jacobian which relates the master space and the physical space is $J = J_2 J_1$, where $J_2$ and $J_1$ are the Jacobians for mapping master space $(\xi, \eta)$ to parametric space $(\zeta, \eta)$ and parametric space to physical space $(x, y)$, respectively [52]. The transformation matrix ($\mathbf{T}$), for two-dimensional problems, is evaluated as follows:

$$\mathbf{T} = \begin{bmatrix} J_{11}^2 & J_{21}^2 & 2J_{11}J_{21} \\ J_{12}^2 & J_{22}^2 & 2J_{12}J_{22} \\ J_{11}J_{12} & J_{21}J_{22} & J_{11}J_{22} + J_{12}J_{21} \end{bmatrix},$$

(31)

where $J_{ij}$ are the components of combined Jacobian ($J$).

A similar approach is followed to identify the stress interpolation functions for bi-quadratic and bi-linear elements. The feasible $\mathbf{P}_e$ for a bi-quadratic element is derived as follows:

$$\begin{bmatrix} \frac{\partial \mathbf{\tau}_e}{\partial \xi} \\ \frac{\partial \mathbf{\tau}_e}{\partial \eta} \end{bmatrix} = \begin{bmatrix} P_1 & P_2 & P_3 \\ P_4 & P_5 & P_6 \\ P_7 & P_8 & P_9 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{p}}_1 \\ \hat{\mathbf{p}}_2 \\ \hat{\mathbf{p}}_3 \end{bmatrix},$$

(32)

where $P_1, P_4, P_7, P_8$ are the zero vectors of size $1 \times 6$, $P_3, P_6$ are the zero vectors of size $1 \times 4$, and $P_1, P_5, P_9$ are given as

$$P_1 = \begin{bmatrix} 1, \xi, \eta, \xi \eta, \eta^2, \xi^2 \end{bmatrix}, \quad P_5 = \begin{bmatrix} 1, \xi, \eta, \xi \eta, \xi^2, \xi \eta^2 \end{bmatrix}, \quad P_9 = \begin{bmatrix} 1, \xi, \eta, \xi \eta \end{bmatrix}.$$

The $\mathbf{P}_e$ for the bi-linear elements is derived as

$$\begin{bmatrix} \frac{\partial \mathbf{\tau}_e}{\partial \xi} \\ \frac{\partial \mathbf{\tau}_e}{\partial \eta} \end{bmatrix} = \begin{bmatrix} 1 & \hat{\mathbf{p}}_1 \\ 0 & 0 & 1 & \hat{\mathbf{p}}_2 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

(33)

It can be observed that the stress-interpolation functions provided in Eq. 33 are identical to the Pian–Sumihara element [53] in FEA context. This is due to the fact that the FEA is a subset of IGA, where for the special case of linear NURBS elements, NURBS-based IGA leads to identical Lagrangian-based FE formulation. However, the stress-interpolation functions for higher-order elements are unique to the specific NURBS element.

Additionally, the spaces obtained for the stress field variable in higher-order NURBS elements are close to the feasible spaces mentioned in [13]. However, the formulation is restricted to plate and shell structures focusing on alleviating the membrane locking in IGA based shell theories. Further, the application of such spaces in approximating the strain field can also be noted in the literature [29, 54]. However, the proposed approach presents a reliable, mathematically simple but robust, and efficient formulation to alleviate locking irrespective of the type of locking. Furthermore, unlike the shell theories, the approach can be directly adapted to solve chunky geometries, making it an attractive alternative to couple the various geometric domains in one framework.

3.6 Mapping associated with IGA

The integrals associated with the evaluation of element stiffness matrices or the force vector are solved using the Gauss–Legendre quadrature rules. However, due to the
involvement of three mapping spaces, IGA necessitates an additional Jacobian. If \( \Omega \) represents the physical space of the problem domain, \( \hat{\Omega} \) denotes the parametric space where the NURBS basis functions are defined, and \( \Omega \) is the master or parent space then, the integrals associated in Eq. \( \text{19} \) are evaluated as follows:

\[
G_e = \int_{\Omega_e} P_e^T B_e \, d\Omega_e = \int_{\hat{\Omega}_e} \left[ TP_e(\hat{\xi}, \hat{\eta}) \right]^T B_e |J_1||J_2| d\hat{\Omega}_e,
\]

\[
H_e = \int_{\Omega_e} P_e^T S P_e \, d\Omega_e,
\]

\[
= \int_{\hat{\Omega}_e} \left[ TP_e(\hat{\xi}, \hat{\eta}) \right]^T S_e \left[ TP_e(\hat{\xi}, \hat{\eta}) \right] |J_1||J_2| d\hat{\Omega}_e.
\]

**Figure 1.** Flowchart showing the systematic implementation procedure and the overview of key steps involved in deriving the stress interpolation function.
3.7 Force vector, boundary conditions, and refinement

The approach for evaluation of the force vector and imposition of boundary conditions is identical to the conventional single-field IGA. Commonly encountered boundary conditions are Dirichlet boundary conditions, Neumann boundary conditions, and Robin boundary conditions. The numerical examples presented hereby deal with the homogeneous Dirichlet boundary condition, i.e., \( u = 0 \) on \( \Gamma_u \), which can be incorporated by regulating the corresponding control variables as zero. The process includes identifying the control points responsible for modeling the boundary of interest and restricting the degrees of freedom corresponding to identified control points. However, special treatment needs to be followed if there is a need to restrict a single point on the domain. In such a case, ensuring the control point lies on the geometry and at a point of interest is mandatory. This can be achieved by reducing the continuity at that location to \( C^{-1} \) and then restricting the point of interest. Furthermore, similar refinement strategies can be used in two-field formulations. The only difference is the choice of \( P \) matrix, which will not change for a particular element during the \( h \)-refinement; however, degree elevation techniques, either \( p \) or \( k \)-refinement, are required to change the \( P \) matrix corresponding to the resulting degree of NURBS basis function.

Finally, the flowchart showing the systematic step-wise implementation procedure and the summary of various steps involved in deriving the stress interpolation function is given in figure 1.

4. Numerical examples

A summary of short-hand notations used in subsequent examples is given as follows: The IGA and H-IGA denote the conventional IGA and the proposed two-field hybrid IGA formulation, respectively. The extensions \( \textbf{d}1, \textbf{d}2, \text{and} \textbf{d}3 \) state the use of linear, quadratic, and cubic basis functions along \( \xi \) and \( \eta \) direction, and \( \text{C}0, \text{C}1, \text{and} \text{C}2 \) are the \( \text{C}0 \), \( \text{C}1 \), and \( \text{C}2 \) inter-element continuity, respectively. Furthermore, the notation FEA and H-FEA represents the conventional FEA and hybrid FEA formulations with extensions \( \text{Q}4 \) and \( \text{Q}9 \) as four and nine node quadrilateral elements, respectively.

4.1 Straight cantilever beam

For the first numerical example, a linear elastic behavior of a two-dimensional cantilever beam of length \( L \) and thickness \( t \) subjected to vertical load \( f_y \) is investigated. The problem is kept simple to test the reliability of the proposed formulation under the influence of the shear locking, see figure 2 [55].

The control point and the respective weights to model the coarsest possible mesh representing the exact geometry is provided in the appendix (table 1). Once the initial mesh is generated, the sequence of meshes is constructed using the \( h \) and \( k \)-refinement. One such mesh of \( 4 \times 1 \) quadratic NURBS elements, for the slenderness ratio \( (L/t) \) 100, is illustrated in figure 3, which highlights the required number of control points, the control point mesh, and the respective discretization of a domain into quadratic NURBS-based elements. In the interest of embracing the proposed formulation, the elaborated results are presented in figure 4, which gives an idea about the deformed configuration of control point mesh and the discretized domain along with the contour plot for vertical displacement for the stated mesh.

The problem is studied by employing the FEA and IGA formulations for three different slenderness ratios (10, 100, and 1000) in order to gradually introduce the shear locking effect into the problem domain. The problem data considered for the three cases is given as follows:

1. \( L/t = 10, L = 100, t = 10, f_y = 4.97018 \times 10^{-3} \)
2. \( L/t = 100, L = 100, t = 1, f_y = 4.9997 \times 10^{-6} \)
3. \( L/t = 1000, L = 100, t = 0.1, f_y = 4.9999 \times 10^{-9} \)

For all three cases, the analytical solution for the vertical displacement (\( u_y \)) at point ‘A’ is 0.02.

The vertical displacement at point ‘A’ is numerically evaluated for all three cases and the corresponding results are presented in figure 5. For the lower value of slenderness ratio \( (L/t) = 10 \) (figure 5a), the locking effect is significantly low whether it is IGA or FEA formulation. However, the proposed hybrid IGA out-performs the conventional formulation with coarse mesh accuracy. For instance, with only two quadratic elements (active dof = 18), the hybrid IGA formulation is capable of providing the results which are in close approximation with the analytical solution. Whereas, with the conventional IGA, further refinement is needed to achieve similar accuracy.

The effect of the shear locking can be distinctly observed in conventional IGA while using the lower degree basis functions with a higher slenderness ratio of the problem domain. As illustrated in figure 5(b and c), conventional IGA with quadratic basis functions locks with a higher value of \( L/t \) and a significant refinement is needed to alleviate the anomaly. On the other hand, the hybrid IGA
performs convincingly well in all the conditions by alleviating the locking and providing superior coarse mesh accuracy. Furthermore, the use of higher degree NURBS, either with conventional or hybrid IGA, significantly reduced the locking. The results for cubic NURBS interpolations are not presented for this particular problem as the formulation, either it is conventional IGA or hybrid IGA, converges to the exact solution with a minimum number of active degrees of freedom. For the point of interest, the results obtained by linear NURBS elements in IGA formulation and Q4 elements in FEA are identical because the NURBS basis functions of degree 1 with weights as 1 will reduce to the conventional Lagrangian basis functions used for Q4 elements.

Furthermore, to have a comprehensive understanding about the performance of the proposed hybrid IGA formulation, the normalized strain energy, $L_2$ error of stress field, and the estimated computational time has been shown in figures 6a, 6c, and 6e, figures 6b, 6d, and 6f, and figure 7, respectively. The strain energy ($W$) for different slenderness ratios is normalized against the analytical solution obtained using the following expression [32]:

$$W = \frac{2f_c^2L^3}{Et^3}.$$  

Whereas the $L_2$ error for stresses are calculated using the following Timoshenko stress solution [55]

$$\sigma_x = f_y(L - x)(y - t/2) \frac{12}{t^3}, \quad \sigma_y = 0,$$

$$\sigma_{xy} = -\frac{6f_y}{t^3} \left[ \frac{r^2}{4} - (y - t/2)^2 \right].$$

Finally, the computational time shown in figure 7 is obtained after compiling the relevant programs on HP Z4-G4 workstation with Intel Xeon W2145 processor (3.7 GHz base frequency, up to 4.5 GHz with Intel turbo boost technology, 11 MB cache, 8 cores) and 96 GB of DDR4 RAM.

4.2 Curved beam

In the present example, a linear elastic behavior of a two-dimensional curved cantilever beam is investigated. The problem is composed of a curved beam subjected to
horizontal load on one end and fixed on the other end. The problem setup and the boundary conditions are illustrated in figure 8a, where $R_{in}$ and $R_{out}$ are the inner and outer radii measured from the origin, $R$ is the mean radius, and $t$ is the thickness of the beam. $f_x$ is the magnitude of the load at the free end such that radial displacement at the tip (Point ‘A’) is evaluated as $0.942$. It is calculated as, $f_x = 0.1 t^3$. \( \nu \) is the Poisson’s ratio and $E$ is the Young’s modulus.

To exactly represent the circular edges of the problem domain, the minimum requirement is to incorporate the quadratic NURBS basis functions along the curvature. The required control point along with the respective weights to model the coarsest possible mesh representing the exact geometry is provided in the appendix (table 2). Once the initial mesh is generated, the sequence of meshes is constructed using the $h$ and $k$-refinement. One such mesh of $8 \times 1$ quadratic NURBS elements, for the slenderness ratio ($R/t$) 10, is illustrated in figures 8b and 8c, which focuses on the number of control points involved, the control point mesh, and the respective element discretization of a domain. To achieve the sense of completeness, the extensive result for the stated mesh is presented in figure 9 which elaborates the deformed configuration of the problem domain (figures 9a and 9b) along with the contour plots for displacement field (figure 9c) obtained by incorporating the hybrid IGA formulation.

The problem is solved using the FEA and IGA formulations for three different slenderness ratios (10, 100, and 1000) in order to gradually introduce the shear locking effect into the problem domain. The problem data considered for the three cases is given as follows:

![Figure 5. Normalized vertical displacement at point ‘A’ for straight rectangular cantilever beam problem for three $L/t$ ratios.](image-url)
Figure 6. Normalized strain energy and $L_2$ error norm of stress for a straight cantilever beam problem for different $L/t$ ratios.
Figure 7. Estimated computational time to achieve the evaluated normalized displacement and $L_2$ error norm of stress for a straight cantilever beam problem for different $L/t$ ratios.
1. \( R/t = 10, \quad R_{in} = 9.5, \quad R_{out} = 10.5, \quad R = 10, \quad t = 1, \quad f_x = 0.1 \)

2. \( R/t = 100, \quad R_{in} = 9.95, \quad R_{out} = 10.05, \quad R = 10, \quad t = 0.1, \quad f_x = 0.1 \times 10^{-3} \)

3. \( R/t = 1000, \quad R_{in} = 9.995, \quad R_{out} = 10.005, \quad R = 10, \quad t = 0.01, \quad f_x = 0.1 \times 10^{-6} \)

For all three cases, the analytical solution for the radial displacement \( (u_r) \) at point ‘A’ is 0.942.

The radial displacement at point ‘A’ is numerically evaluated by employing the different formulations and results are presented in figure 10. For the lower value of slenderness ratio \( (R/t = 10, \text{figure 10a}) \), it can be seen that the locking effect is substantially low whether it is IGA or FEA formulation. However, the proposed hybrid IGA out-performs conventional formulation with coarse mesh accuracy. For instance, the hybrid IGA results for quadratic NURBS basis are in close approximation with the analytical solution even with merely 30 active degrees of freedom, whereas the results for conventional IGA even with the cubic basis are inferior for nearly the same degrees of freedom.

As the slenderness ratio increases, the influence of the shear locking in the conventional IGA formulation can be observed distinctly while using the lower degree basis functions. As illustrated in figures 10b and 10c, conventional IGA with quadratic basis functions locks severely with a higher value of \( R/t \). On the other hand, the hybrid IGA performs convincingly well in all the conditions by alleviating the locking. Furthermore, the use of higher degree NURBS significantly reduces the locking, but hybrid results can be seen marginally better than the conventional formulation.

4.3 Cook’s membrane problem

Next, the Cook’s membrane problem is simulated [29]. The problem setup and the boundary conditions are illustrated in figure 11a, where \( f_y \) is the load per unit length. Setting \( v = 0.4999 \), the problem becomes a typical case of volumetric locking while investigating the nearly

**Figure 8.** (a) The problem definition and (b, c) geometric description of a curved cantilever beam problem \((R/t = 10)\) for \(8 \times 1\) NURBS elements with quadratic basis along \( \xi \) and \( \eta \) direction.

**Figure 9.** (a, b) Deformed geometric description, and (c) contour plot of the magnitude of total displacement for a curved beam problem for \(8 \times 1\) NURBS elements with quadratic basis along \( \xi \) and \( \eta \) direction as illustrated in figures 8b and 8c using hybrid IGA formulation.
incompressible behavior of the domain under combined bending and shear deformation.

The geometric data to construct the coarsest possible mesh, representing the exact geometry, is provided in the appendix (table 3). Once the initial mesh is generated, the sequence of refined meshes are modeled using the refinement techniques. One such mesh of $8 \times 8$ NURBS elements, having the quadratic basis functions along the $\zeta$ and $\eta$ direction, is illustrated in figures 11b and 11c. The elaborated results are presented in the figures 11d and 11f which focuses on the deformed configuration of control point mesh and the discretized domain along with the contour plot for vertical displacement for the stated mesh. The normalized vertical displacement at the point ‘A’ against the reference solution of 7.7 is evaluated as shown in figure 12a. Furthermore, the convergence of the relative $L_2$ error norm of displacement versus the number of active degrees of freedom is shown in figure 12b. As analytical expressions for the displacements are not well established for the stated problem, the reference to evaluate the $L_2$ norm is the well-converged solution of highly-refined mesh of cubic elements.

It can be observed that the conventional FEA for four-node quadrilateral elements and its equivalent IGA formulation locks severely. Even with the significantly high refinement, the results only marginally improve. On the other hand, the hybrid formulation can successfully alleviate locking to produce superior results. Moving to higher degree basis functions, where the IGA basis functions are different from the Lagrangian basis functions, it can be seen that the quadratic NURBS elements lock significantly for lower mesh refinements. However, the proposed hybrid IGA works very well even with a very low number of active degrees of freedom and provides better coarse mesh accuracy. With further elevation in the degree of basis function, the gap between the conventional and hybrid IGA results gets insignificant, yet hybrid IGA is marginally better than the conventional IGA formulation.
4.4 Infinite plate with a hole problem

The problem consists of a two-dimensional infinite plate with a circular hole under constant uni-axial in-plane tension \( T = 1 \) at infinity [29]. Owing to the symmetry of the problem only a quarter portion of the plate is considered, as shown in figure 13a.

The problem setup and the boundary conditions are illustrated in figure 13a, which includes the symmetric boundary condition on edge ED and BC, and the Neumann boundary condition on edge AB and AE. The exact traction, applied on the boundary AB and AE, is evaluated using the analytical expression of stresses (Eqs. 34, 35, and 36) for the stated problem setup.

A plane-strain condition is assumed, and the analytical solution [29] for the stress field is given as follows:

\[
\sigma_{xx} = 1 - \frac{R^2}{r^2} \left( \frac{3}{2} \cos 2\phi + \cos 4\phi \right) + \frac{3R^4}{2r^4} \cos 4\phi, \quad (34)
\]

\[
\sigma_{yy} = -\frac{R^2}{r^2} \left( \frac{1}{2} \cos 2\phi - \cos 4\phi \right) - \frac{3R^4}{2r^4} \cos 4\phi, \quad (35)
\]

\[
\tau_{xy} = -\frac{R^2}{r^2} \left( \frac{1}{2} \sin 2\phi + \sin 4\phi \right) + \frac{3R^4}{2r^4} \sin 4\phi, \quad (36)
\]

where \( r = \sqrt{x^2 + y^2} \) and \( \phi = \tan^{-1}(y/x) \). The analytical expression for the displacement field is given as

\[
u_x(r, \phi) = \frac{R}{8\mu} \left[ \frac{r}{R} (k + 1) \cos \phi \\
+ \frac{2R}{r} \left( (1 + k) \cos \phi + \cos 3\phi \right) - \frac{2R^3}{r^3} \cos 3\phi \right],
\]

\[
u_y(r, \phi) = \frac{R}{8\mu} \left[ \frac{r}{R} (k - 3) \sin \phi + \frac{2R}{r} \left( (1 - k) \sin \phi + \sin 3\phi \right) \\
- \frac{2R^3}{r^3} \sin 3\phi \right],
\]
where \( \mu = \frac{E}{2(1 + v)} \) and \( k = 3 - 4v \) (for plane-strain condition).

The minimal requirement to maintain the exact geometry is to use the quadratic NURBS basis functions along the \( \phi \) direction. Furthermore, the coarsest possible mesh to represent the exact geometry consists of two quadratic elements, one in the radial direction and two in the \( \phi \) direction. The control point coordinates and the respective weights are given in the appendix (tables 4 and 5). Once the initial mesh is generated, the sequence of meshes is constructed using the \( h \)-refinement. One such mesh of \( 10 \times 5 \) elements is illustrated in figures 13b and 13c, which focuses on the number of control points involved, the control point mesh, and the respective element discretization of a domain. For a better understanding, the comprehensive results for the mesh described in figures 13b and 13c are presented in figures 14 and 15, which focuses on the deformed configuration of the problem domain along with the numerical displacement field in comparison with the analytical solution for different Poisson’s ratios.

The problem is solved using the conventional and hybrid IGA alongside their FE counterparts. To test the efficiency and robustness of the method, the problem is studied for the two cases. For the first case, \( v \) is considered as 0.3 so that the solution is unaffected by volumetric locking. Secondly, the \( v \)
Figure 14. (a, b) Deformed geometric description of an infinite plate with a hole problem (a quarter portion) for $10 \times 5$ NURBS elements with quadratic basis along $\zeta$ and $\eta$ direction (magnified by the factor of $1 \times 10^4$ for better visualization), and contour plots for horizontal and vertical displacement for the mesh illustrated in figures 13b and 13c using hybrid IGA formulation (c, d) alongside the analytical solution (e, f) for $\nu = 0.3$. 
Figure 15. (a, b) Deformed geometric description of an infinite plate with a hole problem (a quarter portion) for $10 \times 5$ NURBS elements with quadratic basis along $\xi$ and $\eta$ direction (magnified by the factor of $1 \times 10^4$ for better visualization), and contour plots for horizontal and vertical displacement for the mesh illustrated in figures 13b and 13c using hybrid IGA formulation (c, d) alongside the analytical solution (e, f) for $\gamma = 0.4999$. 
Figure 16. Convergence study of a relative $L_2$ error norm of displacement versus the active degrees of freedom for a plate with hole problem.

Figure 17. (a, b) Normalized strain energy and (c, d) $L_2$ error norm of stress for a plate with hole problem for different Poisson’s ratios.
value is considered as 0.4999 to analyze the nearly incompressible behavior which is highly influenced by locking.

For the first case, the convergence of the relative $L_2$ error norm of displacement versus the number of active degrees of freedom is evaluated as shown in figure 16a. It can be seen that the convergence rates for the conventional and hybrid IGA are closely identical, whether it is for quadratic or cubic basis functions. This authenticates the fact that the hybrid IGA formulation is not restricted to locking-dominated problem domains but can also be effectively used for problems that are not influenced by the locking.

In the second case, the nearly incompressible behavior is investigated by setting the Poisson’s ratio close to 0.5. The convergence curves specific to $\nu = 0.4999$ are presented in figure 16b. It can be seen that the conventional IGA locks while using the quadratic basis functions. Though refinement considerably reduces the error, the proposed hybrid IGA outperforms the conventional IGA formulation in terms of coarse mesh accuracy. Similar results are obtained with cubic basis functions, conventional IGA is less sensitive to locking, but the hybrid IGA provides better accuracy at a relatively low number of elements.

To have thorough insights on the performance of hybrid elements, the problem is further investigated for the normalized strain energy (figures 17a and 17b), $L_2$ error of stress field (figures 17c and 17d), and computational time (figure 18). The reference strain energy ($W_{ref}$) for normalizing the evaluated results is obtained using the well-converged solution of high-refined mesh. The $W_{ref}$ value in case of $\nu = 0.3$ is $7.6937 \times 10^{-5}$ and for $\nu = 0.4999$ is

![Figure 18. Estimated computational time to achieve the evaluated normalized strain energy and $L_2$ error norm of stress for a plate with hole problem for different Poisson’s ratios.](image)
The reference solution for evaluating the $L_2$ error of stress is obtained using Eqs. 34, 35, and 36. Furthermore, the stress concentration factor ($S_f$) has been evaluated using the expression $S_f = \frac{\sigma_{\text{max}}}{T}$ which leads to $S_f = 2.9999$ for $v = 0.3$ and $S_f = 3.6146$ for $v = 0.4999$ for the converged mesh. The programs are compiled on the same workstation with configuration as mentioned in Section 4.1.

5. Conclusions

In the present work, a novel class of hybrid elements is proposed to alleviate the locking in NURBS-based IGA using a two-field Hellinger–Reissner variational principle. The proposed hybrid elements exercise the independent interpolation schemes for displacement and stress field. The displacement field is approximated using the standard NURBS-based interpolations; however, the special treatment is followed in approximating the stress field to ensure locking-free results. The principal concept follows the assessment of normal stress components in relation to derivatives of the respective displacement interpolations and the higher-order terms in shear components are defined such that they ensure correct stiffness rank and suppress the spurious zero-energy mode, which is necessary to avoid locking. To assess the performance, the proposed formulation along with the conventional single-field IGA, Lagrangian-based FEA, and hybrid FEA formulation has been implemented on several two-dimensional linear elastic examples. The results for typical benchmark numerical examples authenticate the potency of the proposed formulation. The two-field hybrid IGA tends to perform well for analyzing nearly incompressible problem domains that are severely affected by volumetric locking as well as for thin plate and shell problems where the shear locking is dominant by alleviating the different types of locking. The same formulation can also be successfully implemented for standard problems (i.e., those without the locking effect) with full reliability as successive mesh refinements converge the solution to the desired results. The effectiveness of the method can be clearly seen while using the lower order NURBS basis functions. However, higher-order basis functions are less affected by locking, but the proposed method outperforms the results in an aspect of coarse mesh accuracy, which eventually leads to lower computational efforts. The results for higher-order NURBS basis are only marginally better for some locking-free problem domains; however, the robustness of the method to perform well in all situations can not be ignored. The extension of the proposed method in a three-dimensional regime and further for non-linear analysis of the locking-dominated problems would be of interest for future study.

Appendix I: Geometric data for modeling the base coarse mesh for the presented problems

The section is intended to provide the required control points coordinates and the respective weights to construct the initial mesh for the stated problem domains, see tables 1, 2, 3, 4 and 5. The sequence of the refined meshes are modeled by employing the knot insertion or the degree elevation algorithms in one or both direction. For the reader’s interest, the MATLAB codes based on these algorithms can be found in an open-source code library called NURBS toolbox [56]. The subroutines named nrbkntins and nrbdgelev are of particular interest for the refinement strategies.

Table 1. Control point coordinates ($P_{ij} = [x\ y\ z]$) for modeling the initial coarse mesh (single element) for a rectangular beam of length $L$ and thickness $t$ with linear basis along $\xi$ and $\eta$ direction (weights associated with each control point is 1, knot vector along $\xi$ and $\eta$ direction is $[0, 0, 1, 1]$).

| $j$ | $P_{1j}$ | $P_{2j}$ |
|-----|---------|---------|
| 1   | [0 0 0] | [L 0 0] |
| 2   | [0 t 0] | [L t 0] |

Table 2. Control point coordinates and respective weights ($P_{ij} = [x\ y\ z\ w]$) for modeling a single element representing the problem domain of a curved beam with quadratic basis along $\xi$ and linear basis along $\eta$ direction, the respective knot vectors are $\Xi = [0, 0, 0, 1, 1, 1]$ and $\mathcal{H} = [0, 0, 1, 1]$.

| $j$ | $P_{1j}$ | $P_{2j}$ | $P_{3j}$ |
|-----|---------|---------|---------|
| 1   | [0 $R_{in}$ 0 1] | [$R_{in}$ $R_{in}$ 0 $\frac{1}{\sqrt{2}}$] | [$R_{in}$ 0 0 1] |
| 2   | [0 $R_{out}$ 0 1] | [$R_{out}$ $R_{out}$ 0 $\frac{1}{\sqrt{2}}$] | [$R_{out}$ 0 0 1] |

Table 3. Control point coordinates ($P_{ij} = [x\ y\ z]$) for modeling a single element representing the domain of a Cook’s membrane problem with linear basis along $\xi$ and $\eta$ direction (weights associated with all the control points = 1 and knot vector along $\xi$ and $\eta$ direction is $[0, 0, 1, 1]$).

| $j$ | $P_{1j}$ | $P_{2j}$ |
|-----|---------|---------|
| 1   | [0 0 0] | [48 44 0] |
| 2   | [0 44 0] | [48 60 0] |
Table 4. Control point coordinates ($P_{ij} = [x \ y \ z]$) and respective weights ($w_{ij}$) for modeling a quarter portion of a plate with a hole problem using two quadratic elements (two elements along $\xi$ direction and one element along $\eta$ direction), knot vectors along $\xi$ and $\eta$ are given as; $\Xi = [0, 0, 0, 0.5, 1, 1, 1]$, $\mathcal{H} = [0, 0, 0, 1, 1, 1]$, radius $r = 1$, length $L = 4$, weight $w^p = 0.5(1 + \sqrt{2})$.

| $j$ | $P_{1j}$ | $P_{2j}$ | $P_{3j}$ | $P_{4j}$ | $w_{1j}$ | $w_{2j}$ | $w_{3j}$ | $w_{4j}$ |
|-----|----------|----------|----------|----------|----------|----------|----------|----------|
| 1   | $[-r \ 0 \ 0]$ | $[-r \ 0.414 \ 0]$ | $[-0.414 \ r \ 0]$ | $[0 \ 1 \ 0]$ | $1$ | $w^p$ | $w^p$ | $1$ |
| 2   | $[-2.5 \ 0 \ 0]$ | $[-2.5 \ 0.75 \ 0]$ | $[-0.75 \ 2.5 \ 0]$ | $[0 \ 2.5 \ 0]$ | $1$ | $1$ | $1$ | $1$ |
| 3   | $[-L \ 0 \ 0]$ | $[-L \ L \ 0]$ | $[-L \ L \ 0]$ | $[0 \ L \ 0]$ | $1$ | $1$ | $1$ | $1$ |

Table 5. Control point coordinates ($P_{ij} = [x \ y \ z]$) and respective weights ($w_{ij}$) for modeling a quarter portion of a plate with a hole problem using two cubic elements (two elements along $\xi$ direction and one element along $\eta$ direction), knot vectors along $\xi$ and $\eta$ are given as; $\Xi = [0, 0, 0, 0.5, 1, 1, 1, 1]$, $\mathcal{H} = [0, 0, 0, 1, 1, 1, 1]$.

| $j$ | $P_{1j}$ | $P_{2j}$ | $P_{3j}$ | $P_{4j}$ | $P_{5j}$ |
|-----|----------|----------|----------|----------|----------|
| 1   | $[-1 \ 0 \ 0 \ 1]$ | $[-1.0000 \ 0.2612 \ 0.9024]$ | $[-0.7929 \ 0.7929 \ 0.9047]$ | $[-0.2612 \ 1.0000 \ 0.9024]$ | $[0 \ 1 \ 0 \ 1]$ |
| 2   | $[-2 \ 0 \ 0 \ 1]$ | $[-2.0696 \ 1.5942 \ 0.9349]$ | $[-2.0219 \ 2.0219 \ 0.8698]$ | $[-1.5942 \ 2.0696 \ 0.9349]$ | $[0 \ 2 \ 0 \ 1]$ |
| 3   | $[-3 \ 0 \ 0 \ 1]$ | $[-3.0673 \ 2.8376 \ 0.9675]$ | $[-3.0798 \ 3.0798 \ 0.9349]$ | $[-2.8376 \ 3.0673 \ 0.9675]$ | $[0 \ 3 \ 0 \ 1]$ |
| 4   | $[-4 \ 0 \ 0 \ 1]$ | $[-4 \ 4 \ 0 \ 1]$ | $[-4 \ 4 \ 0 \ 1]$ | $[-4 \ 4 \ 0 \ 1]$ | $[0 \ 4 \ 0 \ 1]$ |

Abbreviations

FEA  Finite element analysis  
CAD  Computer-aided design  
IGA  Isogeometric analysis  
H-IGA  Hybrid isogeometric analysis  
NURBS  Non-uniform rational B-spline  
H-FEA  Hybrid finite element analysis

List of symbols

\( N_i^p(\xi) \) \( p^\text{th} \) degree univariate B-spline basis function
\( N_{ij}^p(\xi, \eta) \) Bivariate B-spline interpolation function
\( N_{ijk}^{pqr}(\xi, \eta, \zeta) \) Trivariate B-spline interpolation function
\( R_p(\xi) \) Univariate NURBS basis function
\( R_{ij}^{pqr}(\xi, \eta) \) Bivariate NURBS interpolation function
\( R_{ijk}^{pqr}(\xi, \eta, \zeta) \) Trivariate NURBS interpolation function
\( R \) Shape function matrix
\( P \) Stress interpolation matrix
\( T \) Transformation matrix
\( B \) Strain-displacement matrix
\( K \) Global stiffness matrix
\( K_e \) Element stiffness matrix
\( u \) Strain energy of an element domain
\( \mathcal{N} \) Null space of \( G_e \)
\( \vec{n}_i^0 \) Vector basis of \( \mathcal{N} \)
\( \mathcal{C} \) Material constitutive tensor
\( \mathcal{U} \) Displacement field
\( \partial u \) Variation of displacement field \( u \)
\( \mathcal{H} \) Knot vector along \( \eta \) direction
\( \eta_p \) Total number of control points per element
\( \bar{t} \) Traction defined on the boundary \( \Gamma \),

Greek symbols

\( \Omega \) Physical space of the problem domain
\( \Omega \) Parametric space of the domain
\( \mathcal{O} \) Master or parent space
\( \Gamma_u \) Displacement boundary
\( \Gamma_t \) Traction boundary
\( \tau \) Cauchy’s stress tensor
\( \epsilon \) Small strain tensor
\( \tau_c \) Engineering form of stress tensor \( \tau \)
\( \delta \tau \) Variation of \( \tau \)
\( \bar{\epsilon} \) Engineering form of strain tensor \( \epsilon \)
(\( \Xi \) Knot vector along \( \xi \) direction
(\( \beta \) Vector consisting of the stress parameters
(\( \delta \beta \) Vector of stress variation parameters

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References

[1] Cottrell J A, Hughes T J R and Bazilevs Y 2009 Isogeometric analysis: Towards integration of CAD and FEA. John Wiley & Sons Ltd, Chichester, United Kingdom
[2] Hughes T J R, Cottrell J A and Bazilevs Y 2005 Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement. Comput. Methods Appl. Mech. Eng. 194: 4135–4195, https://doi.org/10.1016/j.cma.2004.10.008
