A Relation between the Protocol Partition Number and the Quasi-Additive Bound

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Abstract
In this note, we show that the linear programming for computing the quasi-additive bound of the formula size of a Boolean function presented by Ueno [MFCS’10] is equivalent to the dual problem of the linear programming relaxation of an integer programming for computing the protocol partition number. Together with the result of Ueno [MFCS’10], our results imply that there exists no gap between our integer programming for computing the protocol partition number and its linear programming relaxation.

1 Introduction
Proving lower bounds for a concrete computational model is a fundamental problem in the computational complexity theory. In this note, we consider formula size lower bounds for a Boolean function. Karchmer and Wigderson [1] shown that the size of a smallest formula computing a Boolean function $f$ is equal to the protocol partition number of the communication matrix arising from $f$. Karchmer, Kushilevitz and Nisan [2] formulated the problem of computing a lower bound for a protocol partition number as an integer programming problem and introduced a technique, called the rectangle bound, which gives a lower bound by showing a feasible solution of the dual problem of its linear programming relaxation. However, Karchmer, Kushilevitz and Nisan [2] also showed that this technique can not prove a lower bound larger than $4n^2$ for non-monotone formula size in general.

Recently, Ueno [3] introduced a novel technique, called the quasi-additive bound, which is inspired by the notion of subadditive rectangle measures presented by Hrubeš, Jukna, Kulikov and Pudlák [4]. Although the linear programming for computing the quasi-additive bound can be seen as a simple extension of the linear programming for computing the rectangle bound, Ueno [3] showed that the quasi-additive bound can surpass the rectangle bound and it is potentially strong enough to give the matching formula size lower bounds.

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In this note, we show that the linear programming for computing the quasi-additive bound of the formula size of a Boolean function presented by Ueno [3] is equivalent to the dual problem of the linear programming relaxation of an integer programming for computing the protocol partition number. Together with the result of Ueno [3], our results imply that there exists no gap between our integer programming for computing the protocol partition number and its linear programming relaxation. We hope that the results of this note help to understand why the quasi-additive bound is more powerful than the rectangle bound. Furthermore, to the best of our knowledge, no one studied an exact integer programming formulation for computing a protocol partition number. Thus, it may be of independent interests.

2 Preliminaries

Let \( \mathbb{R} \) and \( \mathbb{N} \) be the sets of reals and non-negative integers, respectively. Given a vector \( x \) on a ground set \( U \), we use the notation \( |x| = \sum_{u \in U} x_u \). A relation \( T \) is a non-empty subset of \( X \times Y \times Z \) for some finite sets \( X, Y \) and \( Z \). When we emphasize that a relation \( T \) is a subset of \( X \times Y \times Z \), we say that \( T \) is a relation on \( (X,Y,Z) \). In this note, we assume that for each relation \( T \) on \( (X,Y,Z) \) and \( (x,y) \in X \times Y \) there exists \( z \in Z \) such that \( (x,y,z) \in T \).

A formula is a binary tree with each leaf labeled by a literal and each non-leaf vertex labeled by either of the binary connectives \( \lor \) and \( \land \). A literal is either a variable or its negation. The size of a formula is its number of literals. For a Boolean function \( f \), we define formula size \( L(f) \) as the size of a smallest formula computing \( f \).

Karchmer and Wigderson [1] characterized the size of a smallest formula computing a Boolean function by using the notions of a communication matrix and a protocol partition number. Suppose that we are given a relation \( T \) on \( (X,Y,Z) \). The communication matrix \( M_T \) of \( T \) is defined by a matrix whose rows and columns are indexed by \( X \) and \( Y \) respectively. Furthermore, each cell \( (x,y) \in X \times Y \) of \( M_T \) contains \( z \in Z \) such that \( (x,y,z) \in T \).

A rectangle of \( M_T \) is a nonempty direct product \( X' \times Y' \subseteq X \times Y \). A rectangle \( X' \times Y' \) is called monochromatic if there exists \( z \in Z \) such that \( (x,y,z) \in T \) for all \( (x,y) \in X' \times Y' \). For a rectangle \( X' \times Y' \), a partition of \( X' \times Y' \) is a pair of rectangles \( X_1' \times Y_1' \) and \( X_2' \times Y_2' \) such that \( X' = X_1' \cup X_2' \) and \( X_1' \cap X_2' = \emptyset \), or a pair of rectangles \( X' \times Y_1' \) and \( X' \times Y_2' \) such that \( Y' = Y_1' \cup Y_2' \) and \( Y_1' \cap Y_2' = \emptyset \).

Suppose that we are given a set \( \mathcal{R} \) of disjoint rectangles. We say that \( \mathcal{R} \) recursively partitions \( M_T \) if \( \bigcup_{R \in \mathcal{R}} R = M_T \) and there exists a rooted binary tree representation of \( \mathcal{R} \) defined as follows. A vertex of this tree corresponds to some rectangle of \( M_T \). Especially, the root vertex corresponds to \( M_T \), and a leaf corresponds to a rectangle in \( \mathcal{R} \). For each non-leaf vertex \( v \), rectangles corresponding to its children consist of a partition of a rectangle corresponding to \( v \). Then, the size of a smallest set of disjoint monochromatic rectangles which recursively partitions \( M_T \) is defied by \( C^P(T) \), called the protocol partition number of \( M_T \).

Given a Boolean function \( f : \{0,1\}^n \rightarrow \{0,1\} \), let \( f^{-1}(1) \) (resp., \( f^{-1}(0) \)) be the set of \( x \in \{0,1\}^n \) such that \( f(x) = 1 \) (resp., \( f(x) = 0 \)). For each Boolean function \( f : \{0,1\}^n \rightarrow \)
{0, 1}, we define the relation $T_f$ by

$$T_f = \{(x, y, i) \in f^{-1}(1) \times f^{-1}(0) \times \{1, \ldots, n\} \mid x_i \neq y_i\}.$$  

(In order to avoid triviality, we assume $f^{-1}(1) \neq \emptyset$ and $f^{-1}(0) \neq \emptyset$.) We are now ready to show the characterization of the size of a smallest formula presented by Karchmer and Wigderson [1].

**Theorem 1** (Karchmer and Wigderson [1]). *For each Boolean function $f$,*

$$C^P(T_f) = L(f).$$

### 2.1 The quasi-additive bound

Here we introduce the *quasi-additive bound* presented by Ueno [3]. Suppose that we are given a relation $T$ on $(X, Y, Z)$. We denote by $C_T$ the set of cells of $M_T$, i.e., $C_T = X \times Y$. Let $\mathcal{R}(T)$ be the set of rectangles of $M_T$, and let $\mathcal{M}(T)$ be the set of monochromatic rectangles of $M_T$. For each $R \in \mathcal{R}(T)$, we denote by $\mathcal{P}(R)$ the set of partitions of $R$. Then, we consider the following linear programming for $\phi \in \mathbb{R}^{C_T}$ and $\psi \in \mathbb{R}^{C_T \times \mathcal{R}(T)}$. The objective is to maximize

$$\sum_{c \in C_T} \phi_c$$

under the constraints that

$$\sum_{c \in R} \phi_c + \sum_{c \in C_T \setminus R} \psi_{c, R} \leq 1$$

for all $R \in \mathcal{M}(T)$, and

$$\sum_{c \in C_T \setminus V} \psi_{c, V} + \sum_{c \in C_T \setminus W} \psi_{c, W} \geq \sum_{c \in C_T \setminus R} \psi_{c, R}$$

for all $R \in \mathcal{R}(T)$ and $\{V, W\} \in \mathcal{P}(R)$. We denote by $\text{LP}(T)$ this linear programming. Let $\text{QA}(T)$ be the optimal objective value of $\text{LP}(T)$, and it is called the quasi-additive bound. Although $\text{LP}(T)$ can be seen as a simple extension of the linear programming for computing the rectangle bound, Ueno [3] showed the following surprising result.

**Theorem 2** (Ueno [3]). *For each relation $T$,*

$$\text{QA}(T) = C^P(T),$$

*which implies that $\text{QA}(T_f) = L(f)$ for each Boolean function $f$.*

### 3 Main Results

In this section, we use the same notations for a relation $T$ in Section 2.1. For a relation $T$, let $\Gamma(T)$ be the set of $(R, P)$ such that $R \in \mathcal{R}(T)$ and $P \in \mathcal{P}(R)$, and we define the
integer programming $\text{PN}(T)$ for $x \in \mathbb{N}^{M(T)}$ and $y \in \mathbb{N}^{\Gamma(T)}$ as follows. The objective is to minimize

$$\sum_{R \in M(T)} x_R$$

under the constraints that

$$\sum_{R \in M(T), c \in c} x_R = 1 \quad (1)$$

for all $c \in C_T$ and

$$\sum_{V \in R(T), P \in \mathcal{P}(V), R \in P} \sum_{R \in P} y_{V,P} = \begin{cases} \sum_{P \in \mathcal{P}(R)} y_{R,P} + x_R, & \text{if } R \in \mathcal{M}(T), \\ \sum_{P \in \mathcal{P}(R)} y_{R,P}, & \text{otherwise}, \end{cases} \quad (2)$$

for all $R \in \mathcal{R}^*(T)$, where $\mathcal{R}^*(T) = R(T) \setminus \{C_T\}$. Ueno [5] shown that the dual problem of the linear programming relaxation of $\text{PN}(T)$ is equivalent to $\text{LP}(T)$. Thus, in order to prove the main result, it suffices to show the following theorem.

**Theorem 3.** For each relation $T$, the integer programming $\text{PN}(T)$ computes the protocol partition number of $M_T$.

Theorem 3 clearly follows from the following Lemmas 4 and 5. We say that $x \in \mathbb{N}^{M(T)}$ is feasible to $\text{PN}(T)$ if there exists $y \in \mathbb{N}^{\Gamma(T)}$ such that $(x, y)$ satisfies (1) and (2). Notice that every element of $x \in \mathbb{N}^{M(T)}$ which is feasible to $\text{PN}(T)$ is 0 or 1 by the constraint (1).

**Lemma 4.** Suppose that we are given a relation $T$ and a set $\mathcal{M}'$ of disjoint monochromatic rectangles of $\mathcal{M}(T)$ which recursively partitions $M_T$. Define $x \in \mathbb{N}^{M(T)}$ by

$$x_R = \begin{cases} 1, & R \in \mathcal{M}', \\ 0, & \text{otherwise}, \end{cases}$$

for each $R \in \mathcal{M}(T)$. Then, $x$ is feasible to $\text{PN}(T)$.

**Proof.** Since $\mathcal{M}'$ is a set of disjoint monochromatic rectangles which partitions $M_T$, $x$ clearly satisfies (1). Thus, it suffices to show that there exists $y \in \mathbb{N}^{\Gamma(T)}$ such that $(x, y)$ satisfies (2).

Let $\mathcal{T}$ be a rooted binary tree representation of $\mathcal{M}'$. In the sequel, we do not distinguish between a vertex $v$ of $\mathcal{T}$ and the rectangle to which $v$ corresponds. Define $y \in \mathbb{N}^{\Gamma(T)}$ by

$$y_{R,P} = \begin{cases} 1, & \text{if } R \text{ is a non-leaf vertex of } \mathcal{T} \text{ and the children of } R \text{ consist of a partition } P, \\ 0, & \text{otherwise}, \end{cases}$$

for each $(R, P) \in \Gamma(T)$. Then, we show that $(x, y)$ satisfies (2).

Let $R$ be a rectangle of $\mathcal{R}^*(T)$ which is not contained in $\mathcal{T}$. In this case, it follows from the definition of $y$ that $y_{R,P} = 0$ for all $P \in \mathcal{P}(R)$ and $y_{V,P} = 0$ for all $(V, P) \in \Gamma(T)$.
such that $R \in P$. Furthermore, even if $R \in \mathcal{M}(T)$, $x_R = 0$ follows from $R \notin \mathcal{M}'$. These imply that [2] satisfies.

Let $R$ be a rectangle of $\mathcal{R}^*(T)$ which is contained in $\mathcal{T}$. Since $R \neq C_T$, $R$ is not the root of $\mathcal{T}$. Hence, there exist the parent $V'$ and the sibling $S$ of $R$ in $\mathcal{T}$. Using the notation $P' = \{R, S\}$, it follows from the definition of $y$ that $y_{V', P'} = 1$ and $y_{V, P} = 0$ for all $(V, P) \in \Gamma(T)$ such that $R \in P$ and $(V, P) \neq (V', P')$. Thus, the left-hand side of (2) is equal to 1, and it suffices to show that the right-hand side of (2) is equal to 1.

If $R$ is a leaf of $\mathcal{T}$ (i.e., $R \in \mathcal{M}'$), $y_{R, P} = 0$ for all $P \in \mathcal{P}(R)$ and $x_R = 1$. Thus, the right-hand side of (2) is equal to 1. In the case where $R$ is a non-leaf vertex of $\mathcal{T}$, $x_R = 0$ follows from $R \notin \mathcal{M}'$. Let $P''$ be a partition of $R$ which consist of the children of $R$ in $\mathcal{T}$. Then, it follows from the definition of $y$ that $y_{R, P''} = 1$ and $y_{R, P} = 0$ for all $P \in \mathcal{P}(R) \setminus \{P''\}$. These facts imply that the right-hand side of (2) is equal to 1. This completes the proof.

**Lemma 5.** Suppose that we are given a relation $T$ and $x \in \mathbb{N}^{\mathcal{M}(T)}$ which is feasible to $\text{PN}(T)$. Define $\mathcal{M}_x$ by

$$
\mathcal{M}_x = \{R \in \mathcal{M}(T) \mid x_R = 1\}.
$$

Then, $\mathcal{M}_x$ is a set of disjoint monochromatic rectangles of $\mathcal{M}(T)$ which recursively partitions $M_T$.

**Proof.** For any relation $T$ and $x \in \mathbb{N}^{\mathcal{M}(T)}$ which is feasible to $\text{PN}(T)$, it follows from (1) that $\mathcal{M}_x$ is a set of disjoint monochromatic rectangles of $\mathcal{M}(T)$ which partitions $M_T$. Thus, what remains is to show that it recursively partitions $M_T$.

For a relation $T$ and $x \in \mathbb{N}^{\mathcal{M}(T)}$, we say that $(T, x)$ is eligible if $x$ is feasible to $\text{PN}(T)$. By induction on $|x|$, we show that the lemma holds for all eligible $(T, x)$. For all eligible $(T, x)$ such that $|x| = 1$, the lemma holds since $\{M_T\}$ is a set of monochromatic rectangles recursively partitions $M_T$.

Assuming that the lemma holds for all eligible $(T, x)$ such that $|x| = k \geq 1$, we consider an eligible $(T, x)$ such that $|x| = k + 1$. Let $y$ be a vector in $\mathbb{N}^{\Gamma(T)}$ such that $(x, y)$ satisfies (1) and (2). For proving the lemma by induction, we first show the following claim.

**Claim 6.** There exists $(R', P') \in \Gamma(T)$ such that

1. every rectangle in $P'$ is monochromatic,
2. $x_{V'} = 1$ for all $V' \in P'$, and
3. $y_{R', P'} > 0$.

**Proof.** Since $|x| \geq 2$, there exists $R \in \mathcal{M}(T)$ such that $x_R = 1$ and $R \neq C_T$. Hence, by (2) there exists $(R, P) \in \Gamma(T)$ such that $y_{R, P} > 0$. Let $(R', P')$ be a pair of $\Gamma(T)$ such that $y_{R', P'} > 0$ and $|R'|$ is minimum. Then, we can show that $(R', P')$ satisfies the above conditions as follows. If $V' \in P'$ is not monochromatic or $x_{V'} = 0$, it follows from (2) that $y_{V', P} > 0$ for some $P \in \mathcal{P}(V')$, which contradicts $|R'|$ is minimum. This completes the proof. 

\[ \Box \]
Let \( P' = \{V', W'\} \) be a pair of \( \Gamma(T) \) satisfying the conditions of Claim 6. Since \( V' \) is monochromatic, there exists some index \( i \) which every cell of \( V' \) contains. Here we consider a new relation \( T' \) obtained from \( T \) by adding an index \( i \) to the entry of every cell of \( W' \). Then, we define \( x' \in \mathbb{N}^{M(T')} \) by

\[
x'_R = \begin{cases} 
1, & \text{if } R = R', \\
0, & \text{if } R \in \{V', W'\}, \\
x_R, & \text{if } R \in M(T) \setminus \{R', V', W'\}, \\
0, & \text{otherwise},
\end{cases}
\]

for each \( R \in M(T') \). Furthermore, we define \( y' \in \mathbb{N}^{\Gamma(T')} \) by

\[
y'_{R,P} = \begin{cases} 
\mathbb{y}_{R,P} - 1, & \text{if } (R, P) = (R', P'), \\
\mathbb{y}_{R,P}, & \text{otherwise},
\end{cases}
\]

for each \( (R, P) \in \Gamma(T') \). Notice that \( y'_{R', P'} \geq 0 \) follows from \( y_{R', P'} > 0 \). Since \( R' \notin M(T) \) or \( x_R = 0 \) by (1) and \( x_{V'} = 1 \), we have \( |x'| = k \). Hence, in order to use the induction hypothesis, we need the following claim.

Claim 7. \((x', y')\) satisfies (1) and (2) for \( T' \).

Proof. Since (1) is satisfied by the definition of \( x' \) and the induction hypothesis, we consider the constraint (2). By the definition of \((x', y')\) and induction hypothesis, it suffices to consider the constraint for \( R', V' \) and \( W' \).

First we consider the constraint for \( R' \). Since \( x'_R - x'_R = 1 \) (if \( R' \) is not contained in \( M(T) \), set \( x'_R = 0 \)) and

\[
\sum_{P \in P(R')} y'_{R', P} - \sum_{P \in \mathbb{P}(R')} y_{R', P} = -1,
\]

the right-hand side of (2) does not change. Hence, since the left-hand side does not change, (2) is satisfied. Next we consider the constraint for \( V' \). The left-hand side of (2) decreases by 1 due to \((R', P')\). Since \( x'_V - x'_V = -1 \), the right-hand side of (2) also decreases by 1. Hence, (2) is satisfies. The same argument is clearly valid for \( V' \). This completes the proof.

By the induction hypothesis, \( M_{x'} \) recursively partitions \( M_{T'} \). It is not difficult to see that we can construct a rooted binary tree representation of \( M_{x'} \) by adding two vertices \( V' \) and \( W' \) under \( R' \) of the rooted binary tree representation of \( M_{x'} \). This completes the proof.

Together with Theorem 2 and the fact that the dual problem of the linear programming relaxation of \( \text{PN}(T) \) is equivalent to \( \text{LP}(T) \), the following main results of this note hold by Theorem 3.

Corollary 8. For each relation \( T \), \( \text{LP}(T) \) is the dual problem of the linear relaxation of the integer programming \( \text{PN}(T) \) for computing the protocol partition number of \( M_T \).

Corollary 9. For each relation \( T \), there exists no gap between the integer programming \( \text{PN}(T) \) for computing the protocol partition number of \( M_T \) and its linear programming relaxation.
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