DIRAC INDEX AND TWISTED CHARACTERS

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Abstract. Let $G$ be a real reductive Lie group with maximal compact subgroup $K$. We generalize the usual notion of Dirac index to a twisted version, which is nontrivial even in cases when $G$ and $K$ do not have equal rank. We compute ordinary and twisted indices of standard modules. As applications, we study extensions of Harish-Chandra modules and twisted characters.

1. INTRODUCTION

1.1. The Dirac operator has played an important role in representation theory, in particular the realization and properties of the discrete series, work of Parthasarathy, Schmid, and Atiyah-Schmid. On the other hand, the use of the Dirac operator has been an important tool in the determination of the unitary dual and cohomology of discrete groups via the Dirac inequality, as it appears in the work of Borel, Enright, Kumaresan, Parthasarathy, Salamanca-Riba, Vogan, Zuckerman, and many others. This has led to the notion of Dirac cohomology, introduced by Vogan, and developed further by the work of Huang, Pandžić, and others.

In this paper we develop further the connections between the Dirac cohomology of a representation and its distribution character as well as its possible extensions as encoded in the Ext groups. We work in a more general setting than the results alluded to earlier; for example our results are for groups of unequal rank and possibly disconnected. An important motivating example is the case of a complex group viewed as a real group. Most of the results are stated for the case of a real Lie group, which comprises the real points of a linear algebraic connected reductive Lie group, though they might hold for a larger class.

The next sections in the introduction review the known material on Dirac cohomology with particular attention to the modifications necessary to treat the possible disconnectedness of the group. We then give a detailed statement of the results.

1.2. Let $G := G(\mathbb{R})$ be the group of real points of a reductive linear algebraic connected group $G(\mathbb{C})$. In particular, $G$ is in the Harish-Chandra class; i.e., it has only finitely many connected components, the derived group $[G,G]$ has finite center, and the automorphisms $\text{Ad}(g)$, $g \in G$, of the complexified Lie algebra $\mathfrak{g}$ of $G$ are all inner. Furthermore the Cartan subgroups are abelian.
Let $\theta$ be a Cartan involution of $G$ and let $K = G^\theta$ be the corresponding maximal compact subgroup. We do not assume that $K$ is connected. Let $\mathfrak{g}_0 = \mathfrak{t}_0 \oplus \mathfrak{s}_0$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}_0$ of $G$ corresponding to $\theta$. We denote the linear extension of $\theta$ to the complexification $\mathfrak{g}$ of $\mathfrak{g}_0$ again by $\theta$; let $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{s}$ be the corresponding Cartan decomposition (i.e., the decomposition into the $\pm 1$ eigenspaces for $\theta$).

Fix $B$, a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$ which is negative definite on $\mathfrak{t}_0$ and positive definite on $\mathfrak{s}_0$. Let $C(\mathfrak{s})$ be the Clifford algebra of $\mathfrak{s}$ with respect to $B$ and let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$.

The Dirac operator $D$ is defined as

$$D = \sum_i b_i \otimes d_i \in U(\mathfrak{g}) \otimes C(\mathfrak{s}),$$

where $b_i$ is any basis of $\mathfrak{s}$ and $d_i$ is the dual basis with respect to $B$. Then $D$ is independent of the choice of the basis $b_i$. The square of $D$ is given by the following formula due to Parthasarathy [Par72]:

$$D^2 = -(\text{Cas}_\mathfrak{g} \otimes 1 + \|\rho_\mathfrak{g}\|^2) + (\text{Cas}_{\mathfrak{t}_\Delta} + \|\rho_{\mathfrak{t}}\|^2).$$

Here $\text{Cas}_\mathfrak{g}$ is the Casimir element of $U(\mathfrak{g})$ and $\text{Cas}_{\mathfrak{t}_\Delta}$ is the Casimir element of $U(\mathfrak{t}_\Delta)$, where $\mathfrak{t}_\Delta$ is the diagonal copy of $\mathfrak{t}$ in $U(\mathfrak{g}) \otimes C(\mathfrak{s})$, defined using the obvious embedding $\mathfrak{t} \hookrightarrow U(\mathfrak{g})$ and the usual map $\mathfrak{t} \rightarrow \mathfrak{s}_0(\mathfrak{s}) \rightarrow C(\mathfrak{s})$. See [HP06] for details.

**Definition 1.2.** We will denote by $K^\dagger$ the pin double cover of $K$ and by superscript $\dagger$ the various double covers of subgroups of $K$. (We note that it is more usual to denote the pin or spin double covers by superscript $^\dagger$, but in this paper we reserve this notation for a component of the extended group defined below.)

Recall that $K^\dagger$ is obtained from the pullback diagram

$$\begin{array}{ccc}
K^\dagger & \longrightarrow & \text{Pin}(\mathfrak{s}_0) \\
\downarrow & & \downarrow \text{p} \\
K & \longrightarrow & \text{O}(\mathfrak{s}_0),
\end{array}$$

where the bottom arrow is the action map and the right arrow is the canonical double covering map (see e.g. [HP06] for the definition of the Pin group and the covering map).

In other words,

$$K^\dagger = \{(k, g) \in K \times \text{Pin}(\mathfrak{s}_0) \mid \text{Ad}(k)|_{\mathfrak{s}_0} = p(g)\},$$

where $p : \text{Pin}(\mathfrak{s}_0) \rightarrow \text{O}(\mathfrak{s}_0)$ is the covering map.

If $X$ is a $(\mathfrak{g}, K)$-module, and if $S$ is a spin module for $C(\mathfrak{s})$, then $X \otimes S$ is a $(U(\mathfrak{g}) \otimes C(\mathfrak{s}), K^\dagger)$-module, with the action of $u \otimes c \in U(\mathfrak{g}) \otimes C(\mathfrak{s})$ given by

$$(u \otimes c)(x \otimes s) = ux \otimes cs, \quad x \in X, s \in S,$$

and the action of $(k, g) \in K^\dagger$ given by

$$(k, g)(x \otimes s) = kx \otimes gs, \quad x \in X, s \in S.$$

(Recall that $gs$ is defined since $\text{Pin}(\mathfrak{s}_0) \subset C(\mathfrak{s}_0) \subset C(\mathfrak{s})$.) Here we consider $(U(\mathfrak{g}) \otimes C(\mathfrak{s}), K^\dagger)$ as a pair in the usual sense, with $(k, g) \in K^\dagger$ acting on $u \otimes c \in U(\mathfrak{g}) \otimes C(\mathfrak{s})$ by

$$(k, g)(u \otimes c) = \text{Ad}(k)u \otimes gcg^{-1},$$

where $g$ is any basis of $\mathfrak{s}$.
and with the Lie algebra \( \mathfrak{t} \) of \( K^\dagger \) embedded into \( U(\mathfrak{g}) \otimes C(\mathfrak{s}) \) as \( \mathfrak{t}_\Delta \). (Note that our definition of the \( K^\dagger \)-action on \( U(\mathfrak{g}) \otimes C(\mathfrak{s}) \) is different from the one used in [DHI], so the problem noticed in [DHI], pp. 41–42, does not appear here.)

In particular, \( D \) acts on \( X \otimes S \), and one can define the Dirac cohomology of \( X \) as

\[
H_D(X) = \ker D / \ker D \cap \text{Im } D.
\]

\( H_D(X) \) is a \( K^\dagger \)-module, since the elements of \( K^\dagger \) which map to the even part \( \text{Spin}(\mathfrak{s}_0) \) of \( \text{Pin}(\mathfrak{s}_0) \) commute with \( D \), while the elements of \( K^\dagger \) which map to the odd part of \( \text{Pin}(\mathfrak{s}_0) \) anticommute with \( D \).

In the rest of the paper we assume that \( X \) is admissible and has infinitesimal character. In particular, it follows that the Dirac cohomology \( H_D(X) \) is finite-dimensional.

If \( X \) is unitary or finite-dimensional, then

\[
H_D(X) = \ker D = \ker D^2.
\]

Let \( \mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0 \) be a fundamental Cartan subalgebra of \( \mathfrak{g}_0 \), and let \( \mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a} \) be its complexification. We view \( \mathfrak{t}^* \) as a subspace of \( \mathfrak{h}^* \) by extending functionals on \( \mathfrak{t} \) by 0 over \( \mathfrak{a} \). We fix compatible positive root systems \( R^+_\mathfrak{g} \) and \( R^+_\mathfrak{t} \) for \((\mathfrak{g}, \mathfrak{h})\) respectively \((\mathfrak{t}, \mathfrak{t})\). In particular, this determines half sums of roots \( \rho_\mathfrak{g} \) and \( \rho_\mathfrak{t} \) as usual. Write \( W_\mathfrak{g} \) (resp. \( W_\mathfrak{t} \)) for the Weyl group associated with the roots of \((\mathfrak{g}, \mathfrak{h})\) (resp. \((\mathfrak{t}, \mathfrak{t})\)). We identify infinitesimal characters with elements of \( \mathfrak{h}^* \) via the Harish-Chandra isomorphism.

The following result was conjectured by Vogan [V3] and proved in [HP02] for connected \( G \). In this paper we are primarily concerned with disconnected groups. The extension to this case is in [DHI] and follows from [HP02] combined with the remark that Dirac cohomology commutes with restriction to the connected component.

**Theorem 1.3.** Let \( X \) be a \((\mathfrak{g}, K)\)-module with infinitesimal character \( \Lambda \in \mathfrak{h}^* \). Assume that \( H_D(X) \) contains the irreducible \( K^\dagger \)-module \( E_\gamma \) with highest weight \( \gamma \in \mathfrak{t}^* \subset \mathfrak{h}^* \). (When \( K \) and \( K^\dagger \) are disconnected, we follow the conventions of [V2] Section 5.1.) Essentially, \( \gamma \) can be any choice of a highest weight of the restriction of \( E_\gamma \) to the connected component \( K_0^\dagger \).

Then \( \gamma + \rho_\mathfrak{t} = w\Lambda \) for some \( w \in W_\mathfrak{g} \). In other words, the \( \mathfrak{t} \)-infinitesimal character of any \( K^\dagger \)-type contributing to \( H_D(X) \) is conjugate to the \( \mathfrak{g} \)-infinitesimal character of \( X \) by the Weyl group \( W_\mathfrak{g} \).

Further work on Dirac cohomology and unitary modules ([Bar89], [Bs]) includes partial answers to the following problem:

**Determine the Dirac cohomology of any irreducible unitary module.**

There are also various applications, for example relations to other kinds of cohomology. For more details, see [HP02], [HP06], [HPR], [HKP09], [BP11], and [BP15]. Similarly there is a parallel theory for affine graded Hecke algebras, [BCT].

The following notion is a generalization of the usual notion of Dirac index in the equal rank case (we recall the usual notion below). Before defining this generalized Dirac index, we recall the notion of virtual modules.

**Definition 1.4.** For an arbitrary compact group \( K \), let \( \mathcal{C} \) be the category of admissible representations of \( K \). In other words, each object of \( \mathcal{C} \) is a direct sum of irreducible (finite-dimensional) representations of \( K \), with finite (nonnegative) multiplicities.
The Grothendieck group of $\mathcal{C}$ is then
$$G(\mathcal{C}) = \mathbb{Z}^{\hat{K}} = \left\{ \sum_{\gamma \in \hat{K}} n_{\gamma} \gamma \mid n_{\gamma} \in \mathbb{Z} \right\},$$
with the obvious addition, and $\hat{K}$ denotes the set of irreducible representations of $K$. We call the elements of $G(\mathcal{C})$ virtual $K$-modules.

In the context of this paper, $K$ may refer to the maximal compact subgroup $K \subset G$ or one of its covers $\hat{K}^\dagger$ or $(K^+)\dagger$.

In other words, virtual $K$-modules are $\mathbb{Z}$-linear combinations of irreducible $K$-modules. For each object $M$ of $\mathcal{C}$, its image is denoted by $[M]$. When no confusion arises, we will omit the brackets and denote $[M]$ simply by $M$.

The equivalence of the Grothendieck group with $\mathbb{Z}$-linear combinations of characters is standard. In the special case when $K = \{1\}$, virtual $K$-modules are just virtual vector spaces; in this case $\mathcal{C}$ is the category of finite-dimensional vector spaces, and $G(\mathcal{C})$ can be identified with $\mathbb{Z}$ via taking the dimension.

Let $\gamma$ be an automorphism of $(U(\mathfrak{g}) \otimes C(\mathfrak{s}), K^\dagger)$. This means the following:

1. $\gamma$ consists of an automorphism $\gamma^\theta$ of $U(\mathfrak{g}) \otimes C(\mathfrak{s})$ and an automorphism $\gamma^K$ of $K^\dagger$;
2. $\gamma$ is compatible with the action of $K^\dagger$ on $U(\mathfrak{g}) \otimes C(\mathfrak{s})$ in the sense that
   $\gamma^\theta((k,g)(u \otimes c)) = \gamma^K(k,g)\gamma^\theta(u \otimes c)$
   for $(k,g) \in K^\dagger$ and $u \otimes c \in U(\mathfrak{g}) \otimes C(\mathfrak{s})$;
3. the differential of $\gamma^K$ coincides with the restriction of $\gamma^\theta$ to $\mathfrak{t}_{\Delta}$.

Now let $X \otimes S$ be a $(U(\mathfrak{g}) \otimes C(\mathfrak{s}), K^\dagger)$-module as above, with the action denoted by $\pi$. We assume that $X \otimes S$ has a compatible action of $\gamma$; i.e., there is an operator $\pi(\gamma)$ on $X \otimes S$ such that
$$\pi(\gamma) \pi(u \otimes c) \pi(\gamma)^{-1} = \pi(\gamma^\theta(u \otimes c)), \quad u \otimes c \in U(\mathfrak{g}) \otimes C(\mathfrak{s});$$
$$\pi(\gamma) \pi(k) \pi(\gamma)^{-1} = \pi(\gamma^K(k)), \quad k \in K^\dagger.$$

We assume in the following that $\gamma$ is an involution and that
$$\gamma(D) = -D,$$
so $\pi(\gamma)$ and $\pi(D)$ anticommute. Then $\gamma$ preserves $H_D(X)$, and $H_D(X)$ splits into $\pm 1$ eigenspaces for $\gamma$:
$$H_D(X) = H_D(X)^+ \oplus H_D(X)^-.$$
If we denote the fixed points of $\gamma$ in $K^\dagger$ by $K^\dagger_\gamma$, then $K^\dagger_\gamma$ preserves the above decomposition, and we define the $\gamma$-index of $D$ on $X \otimes S$ as the function
$$\chi^X_\gamma(k) = \operatorname{tr}(\gamma k; H_D(X)) = \operatorname{tr}(k; H_D(X)^+) - \operatorname{tr}(k; H_D(X)^-), \quad k \in K^\dagger_\gamma.$$
Equivalently, the $\gamma$-index of $D$ on $X \otimes S$ is the virtual $K^\dagger_\gamma$-module
$$I^\gamma(D) = H_D(X)^+ - H_D(X)^-.$$
We can also decompose all of $X \otimes S$ into the $\pm 1$ eigenspaces for $\gamma$:
$$X \otimes S = (X \otimes S)^+ \oplus (X \otimes S)^-,$$
and this decomposition is invariant under $K^\dagger_\gamma$. Since $D$ anticommutes with $\gamma$, it interchanges these eigenspaces, so we get two operators,
$$D^\pm : (X \otimes S)^\pm \rightarrow (X \otimes S)^\mp.$$
Now we assume that \( X \otimes S \) is admissible for \( K_1^\dagger \); in the examples we are interested in, this will be true. Then we have the following.

**Proposition 1.8.** With notation and assumptions as above,
\[
I_\gamma(X) = (X \otimes S)^+ - (X \otimes S)^-
\]
as \( K_1^\dagger \)-modules. Furthermore, \( I_\gamma(X) \) is the index of the operator \( D^+ \), i.e., \( I_\gamma(X) = \ker D^+ - \coker D^+ \), while the index of the operator \( D^- \) is \(-I_\gamma(X)\).

**Proof.** We can decompose \( X \otimes S \) into eigenspaces of \( \gamma_1 \) since \( \gamma_1 \) is an automorphism. Now we assume that \( X \otimes S \) is the decomposition into eigenspaces of \( \gamma_1 \) as \( K \)-modules. Furthermore, \( I_\gamma(X) = \ker D^+ - \coker D^+ \), while the index of the operator \( D^- \) is \(-I_\gamma(X)\).

Before passing to examples, we discuss the special case when \( \gamma \) is constructed from an automorphism \( \gamma_1 \) of the pair \((g, K)\) and an automorphism \( \gamma_2 \) of \( C(s) \). As before, to be called an automorphism of the pair \((g, K)\), \( \gamma_1 \) should consist of an automorphism \( \gamma_1^g \) of \( g \) and an automorphism \( \gamma_1^K \) of \( K \), such that the differential of \( \gamma_1^K \) to \( \mathfrak{g} \) and \( \mathfrak{f} \) such that
\[
\gamma_1^g(\text{Ad}(k)Y) = \text{Ad}(\gamma_1^K(k))\gamma_1^g(Y), \quad k \in K, \; Y \in \mathfrak{g}.
\]
Now we set
\[
\gamma_1^g = \gamma_1^g \otimes \gamma_2 : U(g) \otimes C(s) \to U(g) \otimes C(s)
\]
and
\[
\gamma_1^K(k, g) = (\gamma_1^K(k), \gamma_2(g)) \quad (k, g) \in K^\dagger.
\]
It is easy to check that \( \gamma_1^g \) and \( \gamma_1^K \) define an automorphism of the pair \((U(g) \otimes C(s), K^\dagger)\) if \( \gamma_1^K \) is well defined, i.e.,
\[
(k, g) \in K^\dagger \quad \Rightarrow \quad (\gamma_1^K(k), \gamma_2(g)) \in K^\dagger.
\]
In other words, the condition on \( \gamma_1 \) and \( \gamma_2 \) is
\[
(1.9) \quad \text{Ad}(\gamma_1^K(k)) \mid_{s_0} = p(\gamma_2(g)), \quad \text{for all} \; (k, g) \in K^\dagger.
\]
As before, we assume that \( \gamma(D) = -D \).

1.3. **Equal rank case.** The first example is the ordinary Dirac index in the equal rank case. Let \( h_0 = t_0 \) be the compact Cartan subalgebra in \( g_0 \). In this case \( \dim \mathfrak{s} \) is even, so there is only one spin module \( S \), and it is a graded module for \( C(\mathfrak{s}) = C^0(\mathfrak{s}) \oplus C^1(\mathfrak{s}) \), i.e., \( S = S^+ \oplus S^- \), with \( S^\pm \) preserved by \( C^0(\mathfrak{s}) \) and interchanged by \( C^1(\mathfrak{s}) \). (Recall that \( S \) can be constructed as \( \bigwedge \mathfrak{s}^\pm \) with \( \mathfrak{s}^+ \) a maximal isotropic subspace of \( \mathfrak{s} \) and that one can take \( S^+ = \bigwedge_{\text{even}} \mathfrak{s}^+ \) and \( S^- = \bigwedge_{\text{odd}} \mathfrak{s}^+ \).)

Recall that \( \theta \) denotes the Cartan involution of \( g \). It induces \( -\text{Id} \in O(s_0) \) and so gives rise to two elements in \( \text{Pin}(s_0) \). It is easy to see that these elements are
\[
\pm Z_1Z_2 \ldots Z_s \in C(s_0),
\]
where \( Z_1, \ldots, Z_s \) is any orthonormal basis of \( s_0 \). We fix one of these two elements and call it again \( \theta \). In this way \( \theta \) acts on \( S \), and one easily checks that \( S = S^+ \oplus S^- \) is the decomposition into eigenspaces of \( \theta \). Moreover, we can make the choice of \( \theta \) compatible with the choice of \( S^\pm \), so that \( \theta = 1 \) on \( S^+ \) and \(-1 \) on \( S^- \). Furthermore,
we can extend the automorphism $\theta = -\text{Id}$ of $\mathfrak{s}_0$ to an automorphism of $C(\mathfrak{s})$, and this automorphism is exactly the conjugation by the element $\theta \in C(\mathfrak{s})$. (This automorphism is in fact equal to the sign automorphism of $C(\mathfrak{s})$.)

We now consider the automorphism $\gamma$ of $(U(\mathfrak{g}) \otimes C(\mathfrak{s}), K^\perp)$ constructed from the automorphisms $\gamma_1 = \text{Id}$ of $(\mathfrak{g}, K)$ and $\gamma_2 = \theta$ of $C(\mathfrak{s})$. To see that this makes sense, we have to check the condition (1.9). This however immediately follows from $\gamma_1^K = \text{Id}$ and from

$$p(\theta g \theta^{-1}) = p(\theta)p(g)p(\theta^{-1}) = (-\text{Id})p(g)(-\text{Id}) = p(g).$$

It is clear that $\gamma$ is an involution and that $\gamma(D) = -D$. Moreover, since $\theta$ is an inner automorphism of $C(\mathfrak{s})$, it is clear that $\gamma$ automatically acts on $X \otimes S$ for any $(\mathfrak{g}, K)$-module $X$. It is furthermore clear that

$$K^\perp_\gamma = \{(k, g) \in K^\perp \mid g \in \text{Spin}(\mathfrak{s}_0)\}.$$  

In particular, $K^\perp_\gamma$ contains the connected component of $K^\perp$.

We can now consider the $\gamma$-index of $D$ on $X \otimes S$, which we denote simply by $I(X)$ in the present case. It is given by (1.5) or (1.6), with properties described in Proposition 1.8 In particular,

$$I(X) = X \otimes S^+ - X \otimes S^-.$$  

If $K^\perp_\gamma = K^\perp$, i.e., the natural map from $K^\perp$ to $\text{Pin}(\mathfrak{s}_0)$ maps $K^\perp$ into $\text{Spin}(\mathfrak{s}_0)$, then $I(X)$ is the usual index as in [Par72] and the work of Hecht-Schmid and Atiyah-Schmid.

The Dirac index $I(X)$, which can be thought of as the Euler characteristic of Dirac cohomology, is important for several reasons. First, it is directly related to the character of $X$ on the compact Cartan subgroup. This follows from (1.11); see Section 2, especially formula (2.9), for more details. One can likewise compute characters from the Euler characteristic of an appropriate $n$-cohomology, but typically there are cancellations when using $n$-cohomology, and no cancellations when using Dirac cohomology. So one can say that Dirac cohomology is closer to the character. It is also a simpler invariant, which is typically easier to compute than $n$-cohomology.

We remark that one can replace modules with fixed infinitesimal character with arbitrary finite length modules if one modifies the definition of Dirac cohomology and index as in [PS16]. In this paper we only consider modules with infinitesimal character, so we do not need this generalization.

1.4. Unequal rank case. If $\mathfrak{g}$ and $\mathfrak{f}$ do not have equal rank, then the above usual notion of index is trivial. Namely, if dim $\mathfrak{s}$ is even, $S^\pm$ do exist, and we could try to define $I(X)$ as above. It however turns out that $S^+$ and $S^-$ are typically isomorphic as $K^\perp_1$-modules, so $I(X) = 0$ for every $X$; for example, this is always true if $K^\perp$ is connected. If dim $\mathfrak{s}$ is odd, then neither of the two spin modules is graded, so $S^\pm$ cannot be defined as above. One could try to use the two inequivalent spin modules $S_1$ and $S_2$ of $C(\mathfrak{s})$, but they are again isomorphic as $K^\perp$-modules. So the case of real reductive groups is different from the case of graded affine Hecke algebras [CT], [CH], where there are two inequivalent spin modules for $\tilde{W}$ (the analogue of $K^\perp$). Instead, we consider the extended group

$$G^+ = G \rtimes \{1, \theta\},$$
with $\theta$ acting on $G$ by the Cartan involution and with $\theta^2 = 1 \in G$. (The notation is taken from [W2].)

The maximal compact subgroup of $G^+$ is

$$K^+ = K \times \{1, \theta\}.$$  

A $(g, K^+)$-module $(\pi, X)$ can be thought of as a $(g, K)$-module with an additional action of $\theta$ by $\pi(\theta)$, which satisfies

$$
\begin{align*}
\pi(\theta)\pi(k)\pi(\theta) &= \pi(k), & k & \in K; \\
\pi(\theta)\pi(\xi)\pi(\theta) &= \pi(\theta(\xi)), & \xi & \in g.
\end{align*}
$$

(1.12)

We now consider the automorphism $\gamma$ of $(U(g) \otimes C(s), K^\dagger)$ built from the automorphisms $\gamma_1 = \theta$ of $(g, K)$ and $\gamma_2 = \text{Id}$ of $C(s)$. Here $K^\dagger$ still denotes the Pin double cover of $K$, not of $K^+$. The compatibility condition (1.9) is now trivial, and so is the fact that $\gamma$ is an involution satisfying $\gamma(D) = -D$. It is also clear that in this case $K^\dagger_1 = K^\dagger$. Moreover, $\gamma$ acts on $X \otimes S$ whenever $X$ is a $(g, K^+)$-module.

We can now consider the $\gamma$-index of $D$ on $X \otimes S$, which we denote by $I_\theta(X)$ in the present case and call the twisted Dirac index of $X$. It is again given by (1.5) or (1.6), with properties described in Proposition 1.8.

In particular, we have the equality of virtual $K^\dagger$-modules

$$I_\theta(X) = X^+ \oplus S - X^- \otimes S,$$

where $X^\pm$ denote the $\pm 1$ eigenspaces of $\theta$ on $X$.

This setting makes sense in the equal rank case as well. Since $\theta = \text{Ad}k_0$ is inner, any $(g, K)$-module extends naturally to a $G^+ = G \rtimes \{1, \theta\}$-module via $\pi(\theta) = \pi(k_0)$. The resulting twisted index is not substantially different from the usual notion of index. Namely let $\tilde{k}_0 = (k_0, \theta) \in K^\dagger$, where $\theta \in \text{Spin}(s_0)$ is the top degree element acting by $\pm 1$ on $S^\pm$. Then $\tilde{k}_0$ is in $K^\dagger_1$ of (1.10).

Let $\chi_1$ (respectively $\chi_2$) be the function defined by (1.5) for the ordinary (respectively twisted) Dirac index. These functions are both defined for any $k \in K^\dagger_1$. Since $\tilde{k}_0^2$ acts as the identity on $S$, we have

$$
\begin{align*}
\chi_1(\tilde{k}_0k) &= \text{tr}(\tilde{k}_0k; X) \text{tr}(\tilde{k}_0k\theta; S) = \text{tr}(\tilde{k}_0k; X) \text{tr}(\tilde{k}_0^2k; S) \\
&= \text{tr}(\tilde{k}_0k; X) \text{tr}(k; S) = \chi_2(k).
\end{align*}
$$

(1.13)

So we see that the twisted Dirac index $\chi_2$ is the same as the ordinary Dirac index $\chi_1$ with the argument translated by $\tilde{k}_0$.

1.5. **Summary.** In Section 2 we explain the relationship of the ordinary Dirac index to the character on the compact Cartan subgroup and also the relationship of the twisted Dirac index to the twisted character on the fundamental **twisted Cartan subgroup** (Definition 3.3 see also the paragraph below). The main result is Theorem 2.7 which implies that the formulas (1.11) and (1.13) can be interpreted as formulas for the character (respectively twisted character) of $X$.

The material in Section 3 has substantial overlap with the work of Arthur, Kottwitz, Labesse, Langlands, Moeglin, Shelstad, and others on the twisted trace formula. In particular the work of Waldspurger in [Wal10] gives explicit results for various individual cases. We give a treatment of what we need, emphasizing the point of view of group cohomology.
These results are needed for the determination of twisted characters. Analogous to the regular case, twisted characters are determined by their values on the strongly regular semisimple set, and being invariant under conjugation by $G$, determined by the values on the twisted Cartan subgroups.

For the equal rank groups (and $\theta$) twisted Cartan subgroups coincide with the usual Cartan subgroups. In the unequal rank case the main result is Theorem 3.7, which says that for complex groups viewed as real groups, the conjugacy classes of twisted regular semisimple elements are in one-to-one correspondence with conjugacy classes of involutions in the Weyl group. The analogous result for unequal rank real groups is of the same nature but more complicated because there are several conjugacy classes of Cartan subgroups. We discuss examples in Subsection 3.5.

In Section 4 we obtain certain integral formulas for characters of real induced modules. Such results are well known under somewhat more restrictive hypotheses. Our exposition was influenced by the notes of Paul Garrett, available at www.math.umn.edu/~garrett/m/v/characters_ps.pdf. The main result is the character formula (4.2). Our main purpose is to obtain vanishing of characters on certain conjugacy classes of twisted Cartan subgroups, which in turn implies vanishing of the Dirac index.

Section 5 gives a complete discussion of standard modules and their Dirac indices (ordinary and twisted). The main results are Theorem 5.1 which computes twisted indices of standard modules of complex groups; Theorem 5.6 which computes ordinary indices of standard modules in the equal rank case; and Theorem 5.7 which computes twisted indices for standard modules of real groups. Note that Theorem 5.1 is a special case of Theorem 5.7 but we treat it separately because it is easier than the general case.

In principle, knowing the indices of standard modules, one can compute indices of all finite length modules (with a fixed infinitesimal character) by expressing them as $\mathbb{Z}$-linear combinations of standard modules via the Kazhdan-Lusztig algorithm. Conversely, if we know the index by computing the Dirac cohomology explicitly, then we can use the above results to get some of the Kazhdan-Lusztig coefficients. Examples of this are given in Subsection 5.3.

Finally, in Section 6 we study the relationship between indices (ordinary or twisted) and extensions of $(\mathfrak{g}, K)$-modules. The main notion we consider is the Euler-Poincaré pairing defined as the alternating sum of Ext groups between two $(\mathfrak{g}, K)$-modules (see (1.14) below).

Let $\mathcal{K}$ be a compact group and let $(\mu, X)$ and $(\eta, Y)$ be $\mathcal{K}$-modules. Then $\text{Hom}_{\mathbb{C}}[X, Y]$ inherits a $\mathcal{K}$-module structure as well:

$$(k \cdot \lambda) := \eta(k) \circ \lambda \circ \mu(k^{-1}).$$

If $\mathcal{H} \subset \mathcal{K}$ is a normal subgroup, $\text{Hom}_{\mathcal{H}}[X, Y]$ inherits a structure of $\mathcal{K}$-module, which in fact drops down to a $\mathcal{K}/\mathcal{H}$-structure. This extends to the Grothendieck group in the standard way. Precisely if $X = \sum n_i V_i$ and $Y = \sum m_j V_j$ one applies the definition to $\sum n_i m_j \text{Hom}_{\mathcal{H}}[V_i, V_j]$.

In particular, this applies to $\text{Ext}_{(\mathfrak{g}, K)}(X, Y)$, which is the $i$th cohomology of the complex

$$\text{Hom}_{\mathcal{K}}(\bigwedge^i \mathfrak{s} \otimes X, Y)$$
with the usual de Rham-type differential. For any two finite length \((\mathfrak{g}, K)\)-modules \(X\) and \(Y\), the vector spaces \(\text{Ext}^i_{(\mathfrak{g}, K)}(X, Y)\) are finite-dimensional; see for example [BW00], Chapter 1. Furthermore, \(\text{Ext}^i_{(\mathfrak{g}, K)}(X, Y)\) is zero unless \(0 \leq i \leq s\) where \(s = \dim \mathfrak{s}\). Also, if \(X\) and \(Y\) have infinitesimal character, then all \(\text{Ext}^i_{(\mathfrak{g}, K)}(X, Y)\) vanish unless the infinitesimal characters of \(X\) and \(Y\) are the same.

The usual Euler-Poincaré pairing is defined on \(X\) and \(Y\) as the virtual vector space

\[
\text{EP}(X, Y) = \sum_{i=0}^{s} (-1)^i \text{Ext}^i_{(\mathfrak{g}, K)}(X, Y).
\]

(Since the Grothendieck group of finite-dimensional vector spaces is isomorphic to \(\mathbb{Z}\) via taking dimensions, one can also think of \(\text{EP}(X, Y)\) as being an integer.) It is easy to see that \(\text{EP}\) is additive with respect to short exact sequences in each variable, so it makes sense on the level of the Grothendieck group of finite length \((\mathfrak{g}, K)\)-modules; i.e., \(X\) and \(Y\) above can also be virtual \((\mathfrak{g}, K)\)-modules.

Assume first that \(\text{rank} \mathfrak{g} = \text{rank} \mathfrak{k}\). If \(X\) and \(Y\) are finite-dimensional, the Euler-Poincaré principle implies that \(\text{EP}(X, Y)\) equals

\[
\sum_i (-1)^i \text{Ext}^i_{(\mathfrak{g}, K)}(X, Y) = \sum_i (-1)^i \text{Hom}_K \big( \wedge^i \mathfrak{s} \otimes X, Y \big)
= \text{Hom}_K \left( \sum_i (-1)^i \wedge^i \mathfrak{s} \otimes X, Y \right) = \text{Hom}_K \left( (S^+ - S^-) \otimes (S^+ - S^-)^* \otimes X, Y \right)
= \text{Hom}_{K^1} \left( X \otimes (S^+ - S^-), Y \otimes (S^+ - S^-) \right) = \text{Hom}_{K^1} (I(X), I(Y)).
\]

In the above computation, we have used the fact that \(\wedge \mathfrak{s} = S \otimes S^*\), which implies that

\[
\sum_i (-1)^i \wedge^i \mathfrak{s} = (S^+ - S^-) \otimes (S^+ - S^-)^*
\]

and also \((1.1)\) for Dirac indices of \(X\) and \(Y\).

Note that \(S \otimes X\) and \(S^* \otimes X\) only admit an action of \(K^1\), but the action on \(S \otimes S^* \otimes X\) factors to \(K\).

For general \(X\) and \(Y\) the above computation does not make sense, as one cannot take an alternating sum of infinite-dimensional vector spaces. However, the end result still holds, as asserted by Theorem 6.5. The proof uses the fact that standard modules generate the Grothendieck group, so it is, at least in principle, enough to understand \(\text{EP}(X, Y)\) in the case when \(X\) is a standard module (see Section 5.2). This special case can be handled by a spectral sequence described in Proposition 6.1. This enables us to pass to finite-dimensional modules where we can use (1.15). This and some additional computations lead to a proof of the result.

An analogous result (still in the equal rank case) has recently been proved by Huang, Milčić, and Sun [HMS] independently. They prove that the Euler-Poincaré pairing of two finite length modules is the same as the elliptic pairing defined in (6.14). Their result combined with the equality between the elliptic pairing and the pairing of Dirac indices ([Hua15], [Ren16]) can be used to derive our Theorem 6.5.

We now drop the equal rank assumption. Let \(X\) and \(Y\) be modules for the extended group \(G^+ = G \rtimes \{1, \theta\}\), where \(\theta\) is the Cartan involution of \(G\). This means we consider \(X\) and \(Y\) as \((\mathfrak{g}, K)\)-modules with a compatible action of \(\theta\), i.e.,
of the group \( \{1, \theta\} \cong \mathbb{Z}_2 \), such that (1.12) holds. Furthermore \( \theta \) acts on \( \bigwedge^i \mathfrak{s} \), by the scalar \((-1)^i\). Thus the complex

\[
\text{Hom}_K(\bigwedge^i \mathbb{F} \otimes X, Y) \cong \text{Hom}_K(\bigwedge^i \mathfrak{s}, \text{Hom}_\mathbb{C}(X, Y))
\]

is a virtual module for \( K^+ / K \cong \{1, \theta\} \), with \( \theta \) acting simultaneously on \( \bigwedge^i \mathfrak{s} \), \( X \), and \( Y \). It is easy to check that this action of \( \theta \) commutes with the differential of the complex, so \( \theta \) also acts on the cohomology of the complex, i.e., on each \( \text{Ext}_i^{(g, K)}(X, Y) \). We can now consider

\[
\text{EP}_\theta(X, Y) = \sum_{i=0}^s (-1)^i \text{Ext}_i^{(g, K)}(X, Y),
\]

not as a virtual vector space, but as a virtual \( \{1, \theta\} \)-module. We want to study the trace of \( \theta \) on \( \text{EP}_\theta(X, Y) \) in the following sense.

**Definition 1.17.** Let \( K \) be a compact group and let \( H \) be a normal subgroup of \( K \). Let \( V = \sum m_j V_j \) be a finite-dimensional (virtual) \( K \)-module, with \( H \) acting trivially. Then the trace of \( k \in K / H \) on \( V \) is the usual

\[
\text{tr}(k, V) = \sum m_j \text{tr}(k, V_j).
\]

Since \( K \) is compact, \( k \) acts semisimply. For each irreducible module,

\[
\text{tr}(k, V_j) = \sum \tau \dim V_\tau, \quad V_\tau = \{ v \in V_j : k \cdot v = \tau v \}.
\]

We will often identify \( \text{tr}(k, V_j) \) with the virtual vector space

\[
\text{tr}(k, V_j) = \sum \tau V_\tau.
\]

This definition will be used in the context of \( K = K^+ \) or \((K^+)\dagger\) and \( H = K \) or \( K^\dagger \). The element \( k \) will simply be \( \theta \) or \( k\theta \) with \( k \in K \).

If \( X \) and \( Y \) are finite-dimensional, we can write the following equalities of virtual vector spaces:

\[
\begin{align*}
\text{tr} \left[ \theta, \text{EP}_\theta(X, Y) \right] &= \text{tr} \left[ \theta \otimes \theta, \text{Hom}_K \left[ \sum (-1)^i \bigwedge^i \mathfrak{s}, \text{Hom}(X, Y) \right] \right] \\
&= \text{tr} \left[ 1 \otimes \theta, \text{Hom}_K \left[ \sum \bigwedge^i \mathfrak{s}, \text{Hom}(X, Y) \right] \right] \\
&= c \text{tr} \left[ 1 \otimes \theta, \text{Hom}_K \left[ S \otimes S^*, \text{Hom}(X, Y) \right] \right] \\
&= c \text{Hom}_{K^\dagger} \left[ (X^+ - X^-) \otimes S, (Y^+ - Y^-) \otimes S \right] \\
&= c \text{Hom}_{K^\dagger} \left[ \text{Id}(X), \text{Id}(Y) \right].
\end{align*}
\]

For this calculation recall that \( \text{Hom} \) for virtual \( K \)-modules is defined by

\[
\text{Hom}_K(\sum \lambda_i V_i, \sum \mu_j V_j) = \sum \lambda_i \mu_j \text{Hom}_K(V_i, V_j),
\]

and likewise for \( K^\dagger \).

The last equality in (1.18) uses the formula (1.13), which gives an equality of virtual \( K^\dagger \)-modules, without the \( \theta \)-action. The constant \( c \) is as follows. If \( \dim \mathfrak{s} \) is even, \( c = 1 \); in this case \( \bigwedge \mathfrak{s} = S \otimes S^* \), where \( S \) is the unique spin module for \( C(\mathfrak{s}) \). If \( \dim \mathfrak{s} \) is odd, \( c = 2 \). There are two spin modules \( S_1 \) and \( S_2 \), and \( \bigwedge \mathfrak{s} = S_1 \otimes S_1^* \oplus S_2 \otimes S_2^* \). Since \( S_1 \) and \( S_2 \) are isomorphic as \( K^\dagger \)-modules, we denote either one of them by \( S \) and write \( \bigwedge \mathfrak{s} = 2 S \otimes S^* \).
Remark 1.19. If we do not bring the $\theta$-action into play and consider just the virtual vector space $EP(X,Y)$, then in the case when $g$ and $k$ do not have equal rank, $EP(X,Y) = 0$ for all $X$ and $Y$; it is enough to check this for $X$ a standard module, and for that case, one can use Lemma 6.7. With the $\theta$-action taken into account, this does not follow. Namely, the conclusion of Lemma 6.7 is no longer valid, since $\sum (-1)^i \wedge^i a$ is nonzero as a virtual $\{1, \theta\}$-module.

As before, the computation (1.18) does not make sense for infinite-dimensional $X$ and $Y$, but the end result still holds. This is the content of Theorem 6.18, which is an analogue of Theorem 6.5 in the twisted setting, and it has a similar proof.

Finally, we stress that Theorem 6.5 and Theorem 6.18 both give equalities of virtual vector spaces and not of virtual $\{1, \theta\}$-modules. Namely, in all Euler characteristic arguments the crucial point is to get cancellations, and these are only possible in the setting of virtual vector spaces, or virtual $K^\dagger$-modules, and not in the setting of virtual $\{1, \theta\}$-modules. Here is a typical example: let $V$ be a finite-dimensional vector space, and let $V^\pm$ denote the $\{1, \theta\}$-module equal to $V$ as a vector space, with $\theta$ acting by $\pm 1$. Then $V^+ - V^-$ is zero as a virtual vector space, but it is nonzero as a virtual $\{1, \theta\}$-module. This is analogous to Remark 1.19.

A strong vanishing result for two discrete series $X$ and $Y$ follows from Schmid’s formula for the $\bar{u}$-cohomology:

\[
\bigoplus_i \Ext^i_{(g,K)}(X,Y) = \Hom_{K^\dagger}(I(X), I(Y)) = \begin{cases} 
0 & \text{if } X \neq Y; \\
\mathbb{C} & \text{if } X = Y.
\end{cases}
\]

Each of the Dirac indices in (1.20) is a single $K^\dagger$-type, and different $X$ and $Y$ have different indices. Conceivably the vanishing of Ext could be established using a calculation similar to (1.18). We do not know of such an argument.

2. $K$-characters as distributions

The results in this section go back to one of Harish-Chandra’s early papers [HIC]. We give some details since we consider groups which are not in the Harish-Chandra class.

2.1. General manifolds. Let $(M, \omega_M)$ and $(N, \omega_N)$ be manifolds with orientation forms. Assume $\Psi : M \rightarrow N$ is a submersion.

Lemma 2.1. For every $n \in N$, $\Psi^{-1}(n)$ is a submanifold. There is a form $\eta_n$ on $\Psi^{-1}(n)$ such that

\[
\omega_M(m) = (\Psi^* \omega_N)(m) \wedge \eta_{\Psi(m)}(m).
\]

Corollary 2.2. There is an onto map $\bar{\Psi} : C_c^\infty(M) \rightarrow C_c^\infty(N)$, denoted $\bar{\Psi}(\phi) := F_\phi$, given by the formula

\[
F_\phi(n) = \int_{\Psi^{-1}(n)} \phi(x) \eta_n(x).
\]

Definition 2.3. If $\Theta$ is a distribution on $N$ let

\[
\Psi^*(\Theta)(\phi) = \Theta(F_\phi).
\]
2.2. Special case. Let $G^+$ be a real reductive group with maximal compact subgroup $K^+$, the fixed points of a Cartan involution $\theta \in G^+$. We do not assume the group is connected, for example $G^+$ could be $G \rtimes \{1, \theta\}$ as described earlier. So $\theta$ may not be in the connected component of the identity. Let $G^+_{\text{reg}}$ be the regular set, $(K^+)' := G^+_{\text{reg}} \cap K^+$. Write $S := G^+ / K^+$, so that $G / K \cong S = \exp \mathfrak{s}$ where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ is the Cartan decomposition.

We choose $M := (S \times (K^+)', dsdk)$ and $N := ((G^+)_{\text{reg}}^\ell, dg)$. Let $\Psi(s, k) := sks^{-1}$. Then $(G^+)_{\text{reg}}^\ell$, the set of regular elliptic elements, is the image of $\Psi$. The results in the previous section imply $\eta(s, k) = \Delta(s) ds$ where $\Delta(s)$ is the Jacobian.

Let $\Theta := \Theta_\pi$ be the distribution character of an admissible $(\mathfrak{g}, K^+)$-module. Bouaziz [Bou87] has extended Harish-Chandra’s results on characters to a larger class of groups so that there exists a function $\Theta_\pi$ analytic on $G^+_{\text{reg}}$ so that

$$\Theta(f) = \text{tr } \pi(f) = \int_K \Theta_\pi(x) f(x) \, dx.$$  

Let $\phi = g(s)f(k)$ for a $g \in C_c^\infty(S)$ and $f \in C_c^\infty((K^+)')$. Then

$$\Theta(F_\phi) = \int_S g(s) \Delta(s) \, ds \int_{K^+} \Theta_\pi(k) f(k) \, dk.$$  

Now let $g$ depend on a parameter $t \in \mathbb{R}$ such that $\text{supp } g_t \to \{1\}$ as $t \to 0$ and $\int_S g_t(s) \Delta(s) \, ds = 1$. We conclude that

$$\lim_{t \to 0} \Theta(F_{\phi_t}) = \int_K \Theta_\pi(k) f(k) \, dk.$$  

**Proposition 2.6.** If $f \in C_c^\infty((K^+)'$, then $\pi(f) := \int_K f(k) \pi(k) \, dk$ is trace class.

**Proof.** The proof is the same as in [Kna01]. \hfill $\square$

**Theorem 2.7.** The distribution $\text{tr } \pi(f)$ for $f \in C_c((K^+)'$) equals $\int_{K^+} \theta_\pi(f) f(k) \, dk$.

**Proof.**

$$\pi(F_{\phi_t}) = \int_S \int_{K^+} g_t(s) \Delta(s) f(k) \pi(s^{-1}ks) \, ds \, dk = \int_S g_t(s) \Delta(s) \pi(s^{-1}) \pi(f) \pi(s) \, ds.$$  

The operators $\pi(s^{-1}) \pi(f) \pi(s)$ are all trace class, and $\text{tr } [\pi(s^{-1}) \pi(f) \pi(s)] = \text{tr } \pi(f)$. Since $\text{supp } g_t \to \{1\}$, we conclude that

$$\lim_{t \to 0} \int_S g_t(s) \Delta(s) \text{tr } [\pi(s^{-1}) \pi(f) \pi(s)] = \text{tr } \pi(f).$$  

The formula follows by comparing (2.5) with (2.8). \hfill $\square$

In the equal rank situation with $\gamma = 1 \otimes \theta$, (1.11) implies that

$$\Theta(X) |_{T_{\text{reg}}} = \frac{\text{ch}(I(X))}{\text{ch}(S^+ - S^-)} |_{T_{\text{reg}}}.$$

Here $\text{ch}(V)$ denotes the usual character of a virtual finite-dimensional $K^1$-module $V$. Since $\text{ch}(S^+ - S^-) |_{T^1}$ is the noncompact part of the Weyl denominator for $\mathfrak{g}$, it does not vanish identically.

In the general case with $\gamma = \theta \otimes 1$, we use the group $G^+ = G \rtimes \{1, \theta\}$. Let $T \subset K$ be a Cartan subgroup of $K$. Then $T\theta$ contains elements in $(K^+)'$; in fact
the elements in \((K^+)^\prime\) in the same connected component as \(\theta\) are all conjugate to \(T_{\text{reg}}\theta\). Then

\[
\Theta(X)(p(t)\theta) = \frac{\text{ch}(I_{\theta}(X))(t)}{\text{ch}(S)(t)} , \quad t \in T_{\text{reg}}^\dagger ,
\]

where \(p : T^\dagger \rightarrow T\) is the covering map. As before, \(\text{ch}(V)\) denotes the usual character of a virtual finite-dimensional \(K^\dagger\)-module \(V\). Note that in this case \(\theta\) acts by the identity on \(S\), so the formula is analogous to (2.9).

3. Twisted conjugacy classes

We review some well-known facts about twisted strongly regular semisimple conjugacy classes and twisted Cartan subgroups; see in particular [Wal10].

3.1. Recall \(G = \mathbb{G}(\mathbb{R})\) are the rational points of a linear algebraic connected group, \(\theta\) is the Cartan involution, \(G^+ = G \times \{1, \theta\}\), and \(\tilde{G} = G\theta\) as before. \(G(\mathbb{R})\) is the fixed points of the conjugation \(\sigma\).

Given an element \(x \in G\), its twisted conjugacy class is the set

\[
\{gx\theta g^{-1} \mid g \in G\} \subset \tilde{G}
\]
or equivalently \(\{gx\theta(g^{-1}) \mid g \in G\} \subset G\).

**Definition 3.1.** An element \(g \in G^+\) is called strongly regular if \(C_g(\text{Ad} g) := \{X \in \mathfrak{g} : \text{Ad} g(X) = X\}\) has minimal dimension.

3.2. Complex groups. We first consider the case of twisted conjugacy classes of strongly regular semisimple elements for a complex group. Write \(G\) for \(G(\mathbb{C})\).

**Proposition 3.2.** For any semisimple strongly regular element \(\tilde{x} = x\theta \in \tilde{G}\) there is a pair \((B, H = TA)\) stabilized by \(\tilde{x}\) and such that \(T\) is the centralizer of \(\tilde{x}\) in \(G\) and \(H\) is the centralizer of \(T\) in \(G\).

**Proof.** Theorem 7.5 in [St] states that any semisimple automorphism of an algebraic group fixes a pair \((B, H)\) where \(B\) is a Borel subgroup and \(H \subset B\) is a Cartan subgroup. Thus there is a pair \((B, H)\) fixed by \(\theta\). Since any two pairs \((B, H)\) and \((B', H')\) are conjugate by an element in \(G\), there is \(g \in G\) such that \(\text{Ad} \tilde{x}(B, H) = \text{Ad} g(B, H)\). By Lemma 7.3 in [St], there is \(y \in G\) such that \(g^{-1} = y(x\theta)y^{-1}(x\theta)^{-1}\). Thus there is a \(G\)-conjugate of \(x\theta\) which preserves \((B, H)\). Write this conjugate of \(x\theta\) as \(h\theta\) with \(h \in H\); this is possible because the (twisted) conjugate of \(x\) must stabilize \((B, H)\) so must be in \(H\). Write \(H = TA\) where \(\theta\) acts by 1 on \(T\) and \(-1\) on \(A\). Since \(G\) is complex, any element \(a \in A\) can be decomposed as \(a = \theta(b)b^{-1}\), so \(h\theta\) is twisted conjugate to an element \(t\theta\) with \(t \in T\). The fact that \(T\) is the centralizer of \(\tilde{x}\) follows from the assumption of strong regularity.

3.3. Real groups. We now consider the case of a real group. Recall \(G = G(\mathbb{C})\) is the complex group, and \(G = G(\mathbb{R})\) are the fixed points of the conjugation \(\sigma\). Now assume that \(\tilde{x} = x\theta\) is strongly regular and fixed by \(\sigma\), i.e., \(\tilde{x} \in G(\mathbb{R})\). By Proposition 3.2 there is a pair \((B, H)\) which is \(\tilde{x}\)-stable. Write \(H = TA\) where \(T = C_G(\tilde{x})\) and \(H = C_G(T)\). Since \(\sigma\) stabilizes \(\tilde{x}\), it stabilizes \(T\), and therefore also \(H\). Since \(H\) is \(\sigma\)-stable, we can conjugate \(\tilde{x}\) by \(G(\mathbb{R})\) so that \(H\) is also \(\theta\)-stable.

On the other hand, there is \((H_0, H_0)\) so that \(H_0\) is \(\theta\)-stable and \(\sigma\)-stable; \(H_0\) is a fundamental Cartan subgroup with Cartan decomposition \(H_0 = T_0A_0\). In particular
$T_0\theta$ has strongly regular elements, and $C_G(T_0) = \mathbb{H}_0$. By the (proof of) Proposition 3.2, there is $g \in G$ such that $g\bar{x}g^{-1} = t\theta$ with $t \in T_0$. Then
\[ gTg^{-1} = C_G(g\bar{x}g^{-1}) = C_G(t\theta) \supset T_0. \]

Since $\bar{x}$ was assumed strongly regular, the dimension of the centralizer is minimal, so in fact $gTg^{-1} = T_0$. Since $T$ and $T_0$ are both $\sigma$-stable, it follows that $g_T(g^{-1})$ normalizes $T_0$ and therefore also $\mathbb{H}_0$. Thus it is an element of $W(G, \mathbb{H}_0)$ which stabilizes $T_0$. Conversely, given a $T = g^{-1}T_0g$ which is $\sigma$-stable, the associated set of strongly regular elements is in $Tx\theta$ with $x = g^{-1}\theta(g)$. We conclude that in order to classify strongly regular elements in $G(\mathbb{R})\theta$, we need to classify the $\sigma$-stable tori $gT_0g^{-1}$ up to conjugacy by $G(\mathbb{R})$. This is a Galois cohomology problem. If $T_1$ and $T_2$ are conjugate under $G(\mathbb{R})$, then their centralizers $C_G(T_1)$ and $C_G(T_2)$ are conjugate by $G(\mathbb{R})$ as well. We will separate the $gTg^{-1}$ into classes that have the same $\sigma$ and $\theta$-stable centralizer $\mathbb{H}$.

**Definition 3.3.** Representatives of $G(\mathbb{R})$-conjugacy classes of $\sigma$-stable $T\theta$ are called twisted Cartan subgroups.

For the real points $G\theta = G(\mathbb{R})\theta$, the twisted Cartan subgroups will be the fixed points under $\sigma$, denoted $T\theta$.

Recall $G := G(\mathbb{R})$, the real points of a reductive linear algebraic group, the fixed points of a conjugation $\sigma$. Let $g$ be its Lie algebra, $\theta$ the Cartan involution, and $g = \mathfrak{k} \oplus \mathfrak{s}$ the Cartan decomposition. Recall $\mathbb{H}_0 = T_0\mathfrak{h}_0$, the fundamental $\theta$-stable Cartan subgroup with Cartan subalgebra $\mathfrak{h}_0 = t_0 \oplus a_0$. When $G(\mathbb{R})$ is equal rank, $T_0 = \mathbb{H}_0$, and twisted conjugacy classes are just regular conjugacy classes of Cartan subgroups. So we will concentrate on unequal rank groups.

We are looking for $G$-conjugacy classes of $zT_0z^{-1}$ which are $\sigma$-stable. Every $zT_0z^{-1}$ which is $\sigma$-stable must have a centralizer which is a Cartan subgroup which must also be $\sigma$-stable. We sort the $zTz^{-1}$ by the $\sigma$-stable and $\theta$-stable Cartan subgroups which are representatives of $G$-conjugacy classes of real Cartan subgroups. They are of the form $c\mathbb{H}c^{-1}$ with $c$ a Cayley transform. A Cayley transform is a product of Cayley transforms $c_\alpha$ attached to noncompact imaginary roots $\alpha$. If $X_{\pm\alpha}$ are root vectors such that $\theta(X_{\pm\alpha}) = -X_{\pm\alpha}$, and $\sigma(X_{\pm\alpha}) = X_{\mp\alpha}$, then the Cayley transform is $c_\alpha = e^{\pi(X_{\alpha} + X_{-\alpha})/4}$. Then $c_\alpha^2$ represents the Weyl involution $w_\alpha$. We conclude that $T_c := cTc^{-1} \subset \mathbb{H}_c := c\mathbb{H}c^{-1}$ is $\sigma$- and $\theta$-stable. $T_c$ can therefore play the role of $T_0$, and so any $T'$ with centralizer $\mathbb{H}_c$ is then conjugate to $T_c$ by an element representing a Weyl group element in $W_c := W(G, \mathbb{H}_c)$.

We give two examples of conjugacy classes of twisted Cartan subgroups in the unequal rank case.

3.4. Complex groups as real groups. We specialize to the case $G^+ = G \times \{1, \theta\}$ with $G$ a complex group viewed as a real group and consider the problem of classifying twisted regular conjugacy classes. We start with the case of a complex group $G$ viewed as a real group. Its Lie algebra is denoted $\mathfrak{g}$. Let $\theta$ be a Cartan involution, with decomposition $G = KS$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$. Let $-$ be the complex conjugation corresponding to $K$; $(ks) = ks^{-1} = \theta(ks)$, so $-$ equals $\theta$ in this case. Let $B = HN$ be a Borel subgroup with $\theta$-stable Cartan subgroup $H = TA$. Let $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ be the Cartan decomposition. Then $\theta(N) = \overline{N}$, the opposite unipotent radical.
We complexify $G$. Recall that $-$ is conjugation in $G$ with respect to the compact form $K$. Then
\[
\mathbb{G} \cong G \times G, \quad \vartheta_G(g_1, g_2) = (g_2, g_1), \quad G \cong \{(g, \overline{g}) : g \in G\},
\]
\[
\mathbb{H} \cong H \times H, \quad \mathbb{T} \cong \{(h, h) : h \in H\}, \quad \mathbb{A} \cong \{(h, h^{-1}) : h \in H\},
\]
so in particular the conjugation giving $G$ is $\sigma(g_1, g_2) = (\overline{g_2}, \overline{g_1})$. So we are looking for pairs $(a, b) \in \mathbb{G}$ such that $(a, b) \mathbb{T}(a, b)^{-1}$ is $\sigma$-stable, modulo the action of $G$. It follows that for any $t_1 = (h_1, h_1) \in \mathbb{T}$ there must be $t_2 = (h_2, h_2) \in \mathbb{T}$ such that
\[
(3.4) \quad \sigma((a, b)t_1(a, b)^{-1}) = (a, b)t_2(a, b)^{-1}.
\]

We conclude that $\Ad(b^{-1}a), \Ad(b^{-1}a) \in N_G(H)$. In other words if we define $x := b^{-1}a$, then $x, \bar{x} \in N_G(H)$.

Multiplying any $(a, b)$ by $(\bar{c}, c)$ does not change anything; $\Ad(a, b)\mathbb{T}$ is replaced by a $G$-conjugate. So we can take $b = 1$, and then $a$ will satisfy $a, \bar{a} \in N_G(H)$.

Write $a = ks$. Then both $ks$ and $ks^{-1}$ must belong to $N_G(H)$, so it follows that $s^2 \in N_G(H)$. Since $Ad s$ is semisimple with real positive eigenvalues, it follows that $s \in C_G(H) = H$, and therefore $k \in N_G(H)$. So we can use $k$ instead of $a$, and $k$ must satisfy $k^2 \in H$. Thus the conjugacy classes have representatives
\[
(3.5) \quad \{(whw^{-1}, h) : w \in N_K(H)/Z_K(H) := W, \text{ and } w^2 \in H\}.
\]

Suppose that two of these, corresponding to $w_1, w_2 \in W$, give $G$-conjugate sets. Then there is $g \in G$ such that for any $h_1 \in H$ there is $h_2 \in H$ such that
\[
gw_1h_1w_1^{-1}g^{-1} = w_2h_2w_2^{-1}, \quad \overline{gh_1g^{-1}} = h_2.
\]

It follows that $\overline{g} \in N_G(H)$ and therefore also $g \in N_G(H)$. As earlier, if $g = ks$ and $\sigma(g) = ks^{-1}$ stabilize an object, then so does $g^{-1}\sigma(g) = s^{-2}$. Since $s$ is semisimple and has real positive eigenvalues only, $s$ stabilizes the object as well. Thus the sets
\[
\{(sw_1hw_1^{-1}s^{-1}, h)\} = \{(sw_1s^{-1}hsw_1^{-1}s^{-1}, shs^{-1} = h)\}
\]
and $\{(w_1hw_1^{-1}, h)\}$ give the same $G$-conjugacy class. We conclude that $k \in N_G(H)$ and $s \in H$. So we may as well assume $g = k \in N_K(H)$. But then
\[
(3.6) \quad \Ad(gw_1g^{-1})h = \Ad(w_2)h,
\]
for all $h \in H$, so that $w_1$ and $w_2$ are conjugate as elements of the Weyl group $W$. We have proved the following result.

**Theorem 3.7.** The conjugacy classes of twisted Cartan subgroups are in one-to-one correspondence with conjugacy classes of involutions in the Weyl group. More precisely, for any strongly regular semisimple $r \in \overline{G}$, there is a unique conjugacy class of an involution $w$ such that $r$ is conjugate by $G$ to an element of the form $h(w\theta)$ where $h \in H_w$ with
\[
H_w := \{h \in H : \Ad(w\theta)(h) = h\}.
\]

The claim follows from the following additional result; any element of the form $h(w\theta)$ with $h \in H$ is conjugate by $H$ to an element of the form $h_w(w\theta)$ with $h_w \in H_w$. Let $H_w^{\pm} := \{h \in H \mid w\theta(h) = h^{\pm}\}$. In the complex case, any element $h \in H_w^{\mp}$ can be written as $h = w\theta(r)r^{-1}$ for some $r \in H$, because the map
\[
r \mapsto w\theta(r)r^{-1}
\]
is onto $H^-_w$. Therefore any element of the form $thw\theta$ with $t \in H^+_w$ and $h \in H^-_w$ is conjugate to $tw\theta$.

3.5. GL(2n, $\mathbb{R}$). We treat this case in detail; GL(2n + 1, $\mathbb{R}$) is similar.

The fundamental Cartan subalgebra can be realized as

$$\mathfrak{h}_0 = \{\text{diag} \left( \begin{pmatrix} t_1 & \theta_1 \\ -\theta_1 & t_1 \end{pmatrix}, \ldots, \begin{pmatrix} t_n & \theta_n \\ -\theta_n & t_n \end{pmatrix} \right) \}$$

with the usual conjugation $\theta(x) = -x^t$. Its complexification can be written as

$$\mathfrak{h} = \{z := (z_1, \ldots, z_n, z_{n+1}, \ldots, z_{2n}), z_i \in \mathbb{C}\},$$

$$\theta(z_1, \ldots, z_{2n}) = -w_0(z) = -(z_{2n}, \ldots, z_1),$$

$$\sigma(z) = w_0(\overline{z}) = (\overline{z_{2n}}, \ldots, \overline{z_1}),$$

$$t = \{(z_1, \ldots, z_n, -z_n, \ldots, -z_1)\}.$$  

The roots $\epsilon_i - \epsilon_{2n+1-i}$ are noncompact imaginary, and they form a maximal set of strongly orthogonal noncompact roots. The other conjugacy classes of Cartan subalgebras are obtained by applying a set of $n - k$ Cayley transforms about a subset of these roots. A representative is

$$\mathfrak{h}^{n-k} = \{z := (z_1, \ldots, z_k, z_{k+1}, \ldots, z_{2n-k}, z_{2n+1-k}, \ldots, z_{2n})\},$$

$$\theta^{n-k}(z_1, \ldots, z_{2n}) = (-z_2, \ldots, -z_{2n+1-k}, -z_{k+1}, \ldots, -z_{2n-k}, -z_k, \ldots, -z_1),$$

$$\sigma^{n-k}(z) = (\overline{z_2}, \ldots, \overline{z_{2n+1-k}}, \overline{z_{k+1}}, \ldots, \overline{z_{2n-k}}, \overline{z_k}, \ldots, \overline{z_1}).$$

The $t^{n-k}$ obtained from $t$ by applying the Cayley transform is

$$t^{n-k} = \{(z_1, \ldots, z_n, -z_n, \ldots, -z_1)\}.$$  

We consider the case $k = n$, the fundamental Cartan subalgebra. The stabilizer of $t$ is $W^\theta$. The real Weyl group is also $W(G, H) = W^\theta$. This group is formed of changes $z_i \leftrightarrow z_{2n+1-i}$ and permutations $w$ such that the sets $\{1, \ldots, n\}$ and $\{n + 1, \ldots, 2n\}$ are preserved, and if $w(i) = j$, then $w(2n + 1 - i) = 2n + 1 - j$; call this last subgroup $W^C$. It is the diagonal inside $W^c \times W^c$ where the first $W^c \cong S_n$ acts on the first $n$ coordinates and fixes the last $n$, and the second $W^c$ fixes the first $n$ coordinates and acts on the last $n$ coordinates. Representatives of the double cosets $W^\theta \backslash W/W^\theta$ are given by $W^c := (1, W^C)$. This can be seen as follows. Composing $w$ by $r \in W^\theta$ on the left, we can insure that the result takes $(x_1, \ldots, x_n, -x_n, \ldots, -x_1)$ to $(\pm x_1, \ldots, \pm x_n, \ldots)$; if $\pm x_1$ are both beyond place $n$, move $x_1$ to the first $n$ coordinates by reflecting with the appropriate $(i, 2n + 1 - i)$. Continue this way until the first $n$ coordinates are $\pm x_1, \ldots, \pm x_n$. Then use $W^C$ to order the $\pm x_i$ in increasing order. If now $-x_i$ occurs, then compose $w$ with the reflection $r \in W^\theta$ about $(i, 2n + 1 - i)$ on the right to change it so that $x_i$ is in the $i$th coordinate.

Write $w^c$ for $(1, w^C)$. We check that $\sigma(w^ct) = w^ct$ precisely when $w^c$ is an involution. Suppose $w^c(-x_i) = -x_j$ and $w^c(-x_j) = -x_k$. If $k \neq i$, then the fixed points of $\sigma$ on $w^ct$ have strictly smaller dimension than $n$. Finally two involutions give $W^\theta$-conjugate $w^ct$ spaces precisely when the two involutions are conjugate by $W^c$. This follows from observing that conjugating $(1, w^c)$ by a $(w, w) \in W^C \subset W^\theta$ amounts to the same as conjugating $w^c \in W^c$ by $w \in W^c$.

The cases $k < n$ are similar. The centralizer of $t^{n-k}$ is as before. The real Weyl group $W(G, H_0^{n-k})$ is the same as $W^\theta$ on the coordinates 1, \ldots, $k, 2n+1-k, \ldots, 2n$
and the full \(S_{2n-2k}\) on the middle coordinates. Thus the conjugacy classes of \(ut^{n-k}\) are the same as for the fundamental Cartan subgroup for \(GL(2k, \mathbb{R})\), on the coordinates \(1, \ldots, k, 2n+1-k, \ldots, 2n\), and trivial on the middle coordinates.

4. Kernel computations

Let \(\tau\) be an automorphism of \(G = KS\) of finite order (commuting with \(\theta\)), and write \(G^+ = G \times \{1, \tau\}\). Then \(G^+\) admits a Cartan decomposition \(G^+ = K^+ S\). Let \(P^+ = M^+ N \subset G^+\) be a parabolic subgroup so that \(M^+ = M \times \{1, \tau\}\); in particular, \(\tau(N) = N\). Let \((\rho, V)\) be an admissible representation of \(M^+\), and let \((\pi, \text{Ind}_{P^+} G M^+)\) be the induced representation. The representation space is

\[
\{ f : G^+ \to V_\rho : f(nm^+ g) = \rho(m^+)f(g) \}
\]

with action of \(G^+\) by translation on the right. The space can be identified with \(\{ f : K \to V_\rho \}\). Let \(F \in C_c(\mathcal{G}^+)\). Then \(\pi(F)\) is defined as

\[
\pi(F) := \int_{G^+} \pi(g^+)F(g^+) \, dg^+.
\]

It is well known that \(\pi(F)\) is given by integration against a kernel:

\[
\pi(F)f(x) = \int_{G^+} F(g^+)[\pi(g^+)f](x) \, dg^+ = \int_{G^+} F(x^{-1}g^+)f(g^+) \, dg^+.
\]

Write \(G = PK\), and let \(g^+ = \tau pk_1\), \(x = k_2\). For the case \(F \in C_c(K^+)\), we can rewrite \(\pi(F)f(k_2)\) as

\[
\int_K \int_P \int_P F(k_2^{-1} \tau pk_1) \rho(\tau p) \, dp \, f(k_1) \, dk_1.
\]

So the kernel is

\[
\int_P F(k_2^{-1} \tau pk_1) \rho(\tau p) \, dp.
\]

The distribution character is

\[
\Theta_\pi(F) = \int_K \int_P \int_P F(k_2^{-1} \tau pk_1) \rho(\tau p) \, dp \, dk.
\]

As in the untwisted case, for fixed \(\tau m \in M^+_\text{reg}\), the map

\[
\Psi : N \to N
\]

\[
n \mapsto \text{Ad}(\tau m)(n)n^{-1}
\]

is 1-1 and onto, so we can rewrite the integral as

\[
\Theta_\pi(F) = \int_{M^+_\text{reg}} \Delta(m) \text{tr} \rho(\tau m) \int_K \int_N F(k_2^{-1} n^{-1} \tau mnk) \, dn \, dk \, dm
\]

(4.1)

\[
= \int_{M^+_\text{reg}} \Delta(m) \text{tr} \rho(\tau m) \int_{G(\tau m) \backslash G} F(g \tau mg^{-1}) \, dg \, dm,
\]

where \(G(\tau m)\) is the centralizer of \(\tau m\) in \(G\) and \(\Delta(m)\) is the appropriate Jacobian for the map \(\Psi\).

Recall that there are finitely many twisted Cartan subgroups in \(M\); label them \(\tau H_1, \ldots, \tau H_k\). Then we can rewrite (4.1) as

\[
\Theta_\pi(F) = \sum_{i=1}^k \int_{H_i} D(h_i) \text{tr} \rho(\tau h_i) \int_{H_i \backslash G} F(g \tau h_i g^{-1}) \, dg.
\]
\( \Delta \) and \( D \) are Jacobians; we need not make them explicit. By the generalization of Bouaziz of the results of Harish-Chandra, \( \text{tr} \rho \) and \( \Theta_\pi \) are given by integration against an analytic locally \( L^1 \)-function on the regular set.

**Corollary 4.3.** \( \Theta_\pi \) is zero on any twisted Cartan subgroup which is not conjugate to one in \( M^+ \).

*Proof.* This follows from formula (4.2). \( \square \)

5. **Indices of standard modules**

5.1. **Standard modules for complex groups.** We are using standard notation for complex groups and their representations. See for example [BV85] for a detailed treatment. Note however that there is a difference in that in the reference, \( \lambda_L - \lambda_R \) is a weight of the compact torus \( T \), and \( \lambda_L + \lambda_R \) is a character of \( A \).

Let \( X(\lambda_L, \lambda_R) \) be a standard module with Langlands quotient \( X(\lambda_L, \lambda_R) \). The Cartan subgroup is \( H = TA \), and the parameter corresponds to \( \mu = \lambda_L + \lambda_R \in \hat{T} \), \( \nu = \lambda_L - \lambda_R \in \hat{A} \). Since \( \theta \mu = \mu \) and \( \theta \nu = -\nu \), \( X(\lambda_L, \lambda_R) \) extends to an irreducible module of \( G^+ \) in two distinct ways if there is \( w \) such that \( w\mu = \mu, w\nu = -\nu \). If on the other hand there is no such \( w \in W \), then there is a unique irreducible module \( \overline{X}_G(\lambda_L, \lambda_R) \) which restricts to \( G \) as \( X(\lambda_L, \lambda_R) \oplus \overline{X}(\lambda_R, \lambda_L) \). The characters of these latter modules are 0 on \( \hat{G} \). So we only consider the first kind. In this case \( \lambda_L \) is conjugate to \( \lambda_R \), so we can write the parameter as \( (\lambda, w\lambda) \) for some \( w \in W \). Assume \( 2\lambda \) is regular; it is already integral since it equals \( \mu \in \hat{T} \) for the case \( \overline{X}(\lambda, \lambda) \). Since there must be \( x \in W \) such that

\[
\begin{align*}
x(\lambda + w\lambda) &= \lambda + w\lambda, \\
x(\lambda - w\lambda) &= -\lambda + w\lambda,
\end{align*}
\]

it follows that \( x = w \) is an involution.

Assume that \( \nu \neq 0 \), and let \( P = MN \) be the parabolic subgroup such that

\[
\Delta(M, H) = \{ \alpha : (\alpha, \nu) = 0 \}, \\
\Delta(N, H) = \{ \alpha : (\alpha, \nu) > 0 \}.
\]

Then \( X(\lambda, w\lambda) = \text{Ind}_P^G[X_M(\lambda, w\lambda)] \), and \( X_M(\lambda, w\lambda) = X_M(\mu/2, \mu/2) \otimes \mathbb{C}_\nu \). The module \( X_M(\mu/2, \mu/2) \) is tempered, and it extends in two ways to an irreducible module for \( M^+ \). We will construct the two ways below. Note that, being tempered, \( X_M(\mu/2, \mu/2) = \overline{X}_M(\mu/2, \mu/2) \). Denote by \( (\rho, V_\rho) \) one such \( M^+ \)-module that has \( \overline{X}(\mu/2, \mu/2) \) as its Harish-Chandra module.

The element \( \theta \) does not stabilize \( N \), so it is awkward to define an action on \( X(\lambda, w\lambda) \). However \( \tau := w\theta \) does stabilize \( N \), so it is natural to define its action as follows. Let \( \rho(\theta) : V_\rho \longrightarrow V_\rho \) be the intertwining operator satisfying \( \rho(\theta) \rho(m) = \rho(\theta(m)) \rho(\theta) \). Since \( w\mu = \mu \), we can assume \( w \in M \), so \( \rho(w) \) is well defined. Denote by \( \pi \) the action of \( G \) on \( X(\lambda, w\lambda) \). Then for \( f \in X(\lambda, w\lambda) \), define

\[
[\pi(\tau)f](x) := \rho(\tau^{-1})f(\tau(x)).
\]

We now define the action of \( \theta \) in the case \( X_M(\mu/2, \mu/2) = X_M(\mu/2, \mu/2) \). We suppress the subscript \( M \) since this can be thought of as the case of \( G \) and \( \nu = 0 \). Let \( (\mathfrak{b}, \mathfrak{h}) \) be a \( \theta \)-stable (complex) pair of a Borel subalgebra and a Cartan subalgebra.
The module $X(\lambda, \lambda)$ is tempered irreducible (therefore also unitary) and derived functor induced:

$$\mathcal{L}_i^j(\mathbb{C}_{\tilde{\mu}}) = \begin{cases} X(\lambda, \lambda) & \text{if } i = \dim \mathfrak{n} \cap \mathfrak{k}, \\ 0 & \text{otherwise.} \end{cases}$$

Here $\tilde{\mu} = 2\lambda - 2\rho$, and we are using the unnormalized cohomological induction as in [KV95, Chapter 5].

There are two ways to normalize the action of $\theta$ on $X_\epsilon(\lambda, \lambda)$. The first one is to require that $\theta$ act by $\epsilon$ on the lowest $K$-type $\mu = 2\lambda$. The second one is to denote by $\mathbb{C}_{\tilde{\mu}; \eta}$ the $T^+$-module which is equal to $\mathbb{C}_{\tilde{\mu}}$ as a $T$-module and on which $\theta$ acts by $\eta$. Then $X_\eta(\lambda, \lambda)$ is the $(\mathfrak{g}, K^+)$-module cohomologically induced from $\mathbb{C}_{\tilde{\mu}; \eta}$ viewed as an $(\mathfrak{h}, T^+)$-module. We will use the first normalization, and the relation between the two is $\eta = \epsilon(-1)^{\dim(\mathfrak{w}\cap\mathfrak{s})}$ (see the end of the proof of Theorem 5.1).

**Theorem 5.1.** Normalize the action of $\theta$ on $X_\epsilon(\lambda, \lambda)$ so that it acts by $\epsilon$ on the lowest $K$-type $\mu = 2\lambda$. The twisted index of $X_\epsilon(\lambda, w\lambda)$ is

$$I_\theta[X_\epsilon(\lambda, w\lambda)] = \begin{cases} 0 & \text{if } w \neq 1, \\ r \epsilon E_{2\lambda - \rho} & \text{if } w = 1, \end{cases}$$

where $r = [\text{Spin} : E_\rho]$.

**Proof.** Assume first that $w \neq 1$. There are two distinct actions of $\theta$ on $X_M(\mu, \mu)$; they lift to $X(\lambda, w\lambda)$ and therefore also to $X(\lambda, w\lambda)$. The resulting modules are denoted $X_\epsilon(\lambda, w\lambda)$ and $\overline{X}(\lambda, w\lambda)$ with $\epsilon = \pm 1$. By Frobenius reciprocity, $\theta$ acts by the same scalar on the lowest $K$-type of $X(\lambda, w\lambda)$ and the lowest $M \cap K$-type of $X_M(\mu, \mu)$. This sign is not important for us since the index will be shown to be 0 in both cases.

Let $P = MN$ be the parabolic subgroup determined by $\nu = \lambda - w\lambda$. Since $\nu \neq 0$, $P$ is a proper parabolic subgroup stabilized by $w\theta$. The kernel calculation in Section 4 implies that the distribution character is supported on $\text{Ad} G(Mw\theta)$ which does not intersect $T_0\theta$ (Corollary 4.3). Therefore the character is 0 on $T_0\theta$. Formula 4.10 implies that the index is 0 as claimed.

Assume now $w = 1$. By the usual arguments, one sees that Dirac cohomology of $X = X(\lambda, \lambda)$ is obtained as the PRV component of the tensor product of the lowest $K$-type of $X$ and the spin module $S$. Namely, any $K$-type of $X$ has highest weight of the form $2\lambda + \sum_{\beta} n_{\beta}\beta$, where $\beta$ are positive roots and $n_{\beta}$ are nonnegative integers. On the other hand, any weight of $S$ is of the form $-\rho + \sum_{\beta} m_{\beta}\beta$, with each $m_{\beta}$ being 0 or 1. Putting $k_{\beta} = n_{\beta} + m_{\beta}$, we see that $H_D(X)$ consists of $K^\dagger$-modules $E_\tau$ satisfying

$$\tau + \rho = 2\lambda + \sum_{\beta} k_{\beta}\beta$$

with $\tau + \rho$ conjugate to $2\lambda$, the infinitesimal character of $X$ restricted to $\mathfrak{t}$. In particular, $\|\tau + \rho\|^2 = \|2\lambda\|^2$, so

$$2(2\lambda, \sum_{\beta} k_{\beta}\beta) + \sum_{\beta} k_{\beta}\beta)^2 = 0.$$

Since each of the summands is nonnegative, they all have to be 0, so all $k_{\beta} = 0$, which implies the claim. Thus $H_D(X)$ is a single $K^\dagger$-type $E_{2\lambda - \rho}$, with multiplicity $r = [\text{Spin} : E_\rho]$. 


It remains to consider the action of \( \theta \) on the lowest \( K \)-type of \( X \), which is also the lowest \( K^+ \)-type of \( X_u(\lambda, \lambda) \). This lowest \( K \)-type \( V \) is in the bottom layer, as in section V.6 of [KV95]. Corollary 5.85 of [KV95] gives

\[
\text{Hom}_{K^+}[\mathcal{L}_S(C_{\tilde{\mu}}), V] \cong \text{Hom}_{T^+}[C_{\tilde{\mu}} \otimes \bigwedge^R(n \cap s), V^{n \cap \mathfrak{t}}]
\]

with \( R = \dim(n \cap s) \) and \( S = \dim(n \cap \mathfrak{t}) \). The action of \( \theta \) on \( V \) is by the same scalar \( \eta \) as the action on \( V^{n \cap \mathfrak{t}} \). The action on \( Z \) is by \( \epsilon \), and the action on \( \bigwedge^R n \cap s \) is by \((-1)^R = (-1)^{\dim n \cap s}\). \( \square \)

5.2. Standard modules for real groups. We use the results and some of the notation in [KV95], particularly Chapter XI. The Langlands classification exhibits every irreducible module as a canonical quotient of a standard module. The precise definition and statements of results are summarized in Chapter 11 of [ABV]. We will use only the following description of the standard modules.

**Definition 5.2** (Standard module). A standard module is a (Harish-Chandra induced) module

\[
X(P, \delta, \nu) = \text{Ind}_{P}^{G}[\delta \otimes \mathbb{C}_{\nu}]
\]

where the data \((P, \delta, \nu)\) are as follows:

1. \( P = MAN \) is a real parabolic subgroup of \( G \) with \( M \) equal rank,
2. \( \delta \) is a limit of discrete series representation of \( M \),
3. \( \nu \in \mathfrak{a}^* \) satisfies \( \langle \text{Re} \nu, \alpha \rangle \geq 0 \) and satisfies the “parity condition” (11.10g) in [ABV].

**Theorem 5.3** (Langlands classification). With \((P, \delta, \nu)\) as in Definition 5.2, \( X(P, \delta, \nu) \) has a unique irreducible quotient denoted \( \overline{X}(P, \delta, \nu) \). Two such modules are equivalent if and only if their parameters \((P, \delta, \nu)\) and \((P', \delta', \nu')\) are conjugate under \( G \). Any irreducible admissible module is equivalent to an \( \overline{X}(P, \delta, \nu) \).

Recall that \( G \) comprises the real points of a linear algebraic reductive connected group. We follow [KV95] for the parametrization of the limits of discrete series. The group \( M \) is equal rank but possibly disconnected. Let \( \mathfrak{h} = \mathfrak{t} + \mathfrak{a} \) be a \( \theta \)-stable Cartan subalgebra such that \( \mathfrak{t} \) is a compact Cartan subalgebra of \( \mathfrak{m} \). Let \( H = TA \) be the corresponding Cartan subgroup; by the assumptions on the group it is abelian.

A datum for a limit of discrete series is a \( \theta \)-stable Borel subalgebra \( \mathfrak{b}_M \) containing \( \mathfrak{t} \) and an irreducible representation \( \lambda \) of \( T \), such that

\[
\langle d\lambda + \rho(u), \alpha \rangle \geq 0, \quad \alpha \in \Delta(u),
\]

where \( S = \dim u \cap \mathfrak{t} \). See [KV95] Chapter XI.

**Definition 5.5.** Let \( \mathfrak{b} = \mathfrak{b}_M + \mathfrak{n} \) be a Borel subalgebra containing \( \mathfrak{h} \). We will denote by \( X_\mathfrak{b}(\lambda) \) the derived module \( \mathcal{L}_S(\lambda) \) where we view \( \lambda \) as a 1-dimensional \((\mathfrak{h}, T)\)-module consisting of the datum for \( \delta \) and \( \nu \). In case \( \mathfrak{b} \) is \( \theta \)-stable, this is the familiar module \( A_{\mathfrak{b}}(\lambda) \). Since our \( \mathfrak{b} \) is not necessarily \( \theta \)-stable, we use different notation to avoid confusion.

The results in [KV95], particularly [KV95] Theorem 11.129(c)], imply that also

\[
\mathcal{L}_q(\lambda) = \begin{cases} X(P, \delta, \nu) & \text{if } q = S, \\ 0 & \text{if } q \neq S. \end{cases}
\]
**Theorem 5.6.** Assume $\mathfrak{g}$ and $\mathfrak{k}$ have equal rank. The index of the standard module $X_b(\lambda)$ is

$$I(X_b(\lambda)) = \begin{cases} 
0 & \text{if } b \text{ is not } \theta\text{-stable,} \\
E_{\lambda + \rho(\mathfrak{u} \cap \mathfrak{s})} & \text{if } b \text{ is } \theta\text{-stable.}
\end{cases}$$

**Proof.** If $b$ is $\theta$-stable, then it is well known and easy to see that $H_D(X_b(\lambda))$ is a single $K^\perp$-type $E_{\lambda + \rho(\mathfrak{u} \cap \mathfrak{s})}$. The computation is essentially the same as in the proof of Theorem 5.1. See also [HP02], [HKP09]. This $K^\perp$-type appears in the tensor product of the lowest $K$-type of $X_b(\lambda)$ with the $K^\perp$-type of the spin module $S$ containing the element 1. Hence $H_D(X_b(\lambda)) = H_D^0(X_b(\lambda))$, and the result follows.

Assume now that $b$ is not $\theta$-stable. In view of (1.11), the result will follow if we prove that the character of $X_b(\lambda)$ vanishes on the compact Cartan subalgebra. This can be proved by expressing the standard module $X_b(\lambda)$ using induction in stages, as real induced from a cohomologically induced module; see [KV95, Theorem 11.172 and Corollary 11.173]. To do this, we consider the group $MA = Z_G(a)$ with $M$ equal rank, and we let $P = MAN$ be the associated real parabolic subgroup of $G$. Then $T$ is a compact Cartan subgroup of $M$, and the Borel subalgebra $\mathfrak{b}_m = \mathfrak{b} \cap \mathfrak{m}$ of $\mathfrak{m}$ is $\theta$-stable. It follows that the standard module $X_b(\lambda)$ is Harish-Chandra induced from an $X_{b\cap \mathfrak{m}}(\lambda_M) \otimes \mathbb{C}_\nu$ of $MA \subset P$ to $G$. Now we can apply the kernel computations of Section 4 to conclude that the character of $X_b(\lambda)$ is zero on the compact Cartan subgroup. \hfill \square

We now drop the equal rank assumption, and we consider modules for the extended group $G^\perp$. To understand what the standard modules are in this case, we first note that twisting a standard $(\mathfrak{g},K)$-module $X_b(\lambda)$ by $\theta$, we get $X_{\theta b}(\theta \lambda)$. The reason is as follows. The $(\mathfrak{b},T)$-module $\mathbb{C}_\lambda$ with the action twisted by $\theta$ is isomorphic to the $(\theta \mathfrak{b},T)$-module $\mathbb{C}_{\theta \lambda}$, and $\mathbb{C}_{\theta \lambda}^\#$ with the action twisted by $\theta$ is isomorphic to $\mathbb{C}_{\theta \lambda}$. Thus the $(\mathfrak{g},T)$-module $M(\mathfrak{b},\lambda) := U(\mathfrak{b}) \otimes U(\mathfrak{b}) \mathbb{C}_\lambda$, with the action twisted by $\theta$ is isomorphic to $M(\theta \mathfrak{b},\theta \lambda) := U(\mathfrak{b}) \otimes U(\theta \mathfrak{b}) \mathbb{C}_{\theta \lambda}$. The action of $\mathfrak{g}$ on the corresponding derived module comes from the action of $\mathfrak{g}$ on $\text{Hom}(R(K), M(\mathfrak{b},\lambda))$ given by $(X \cdot F)(k) = (\text{Ad} k(X))(F(k))$. Twisting the action by $\theta$ yields the action of $\mathfrak{g}$ on $\text{Hom}(R(K), M(\theta \mathfrak{b},\theta \lambda))$.

This leads to three cases:

1. $\theta \mathfrak{b} = \mathfrak{b}$ and $\theta \lambda = \lambda$;
2. $X_{\theta \mathfrak{b}}(\theta \lambda)$ is not isomorphic to $X_b(\lambda)$;
3. $X_{\theta \mathfrak{b}}(\theta \lambda)$ is isomorphic to $X_b(\lambda)$, but $\theta \mathfrak{b} \neq \mathfrak{b}$. In this case, we may assume that $\theta \lambda \neq \lambda$; otherwise, we could modify $\theta$ to be in Case 1.

**Case 1.** Now $X_b(\lambda)$ is a module for $G^\perp$, in two ways, distinguished by the sign $\epsilon = \pm 1$ by which $\theta$ acts on the unique lowest $K$-type. We denote the module corresponding to $\epsilon$ by $X_b^\epsilon(\lambda)$.

Another way to obtain these modules is to specify an action of $\theta$ on the $(\mathfrak{h},T)$-module $\mathbb{C}_\lambda$ to make it into an $(\mathfrak{h},T^\perp)$-module and then use cohomological induction to obtain a $(\mathfrak{g},K^\perp)$-module. To link the two constructions, we have to compare the actions of $\theta$ on $\mathbb{C}_\lambda$ and on the lowest $K$-type of $X_b(\lambda)$. Assume that $\theta$ acts by $\epsilon' = \pm 1$ on $\mathbb{C}_\lambda$. Since $\mathbb{C}_\lambda^\# = \Lambda^{\text{top}} u \otimes \mathbb{C}_\lambda$, $\theta$ acts on $\mathbb{C}_\lambda^\#$ by $\epsilon'(-1)^\mathfrak{r}$ where $\mathfrak{r} = \dim u \cap \mathfrak{s}$. Let $V$ be a $K^\perp$-type with $\theta$ acting by $\eta = \pm 1$. By Proposition 5.7 in [KV95],

$$\text{Hom}_{K^\perp}(L_{j}^{K^\perp}(\mathbb{C}_\lambda), V) \cong \text{Hom}_{T^\perp}(\mathbb{C}_\lambda^\#, J^j(\mathfrak{p} \cap \mathfrak{t}, V)),$$
Setting $j = S = \dim (\mathfrak{u} \cap \mathfrak{t})$, we find that this is nonzero precisely when $\eta = (-1)^R \epsilon'$. On the other hand, by the results on page 365 and Theorem 5.80 of [KV95], there is a one-to-one $K^+$-equivariant bottom layer map

$$\mathcal{B} : \mathcal{L}_S^{K^+}(\mathbb{C}_\lambda) \longrightarrow \mathcal{L}_S(\mathbb{C}_\lambda) = X_b(\lambda).$$

Its image is called the bottom layer $K^+$-types. By Proposition 10.24 and Chapter V.6 in [KV95], the lowest $K^+$-type of $X_b(\lambda)$ is the unique $K^+$-type in the bottom layer $K^+$-types. So we see that the module $X_b(\lambda)$ with $\theta$ by $\epsilon$ on the lowest $K$-type is obtained by applying cohomological induction to the module $\mathbb{C}_\lambda$ with $\theta$-action $\epsilon(-1)^R$.

As in the proof of Theorem 5.6, the Dirac cohomology of $X_b(\lambda)$ is equal to the $K^+$-type $E_{\lambda + \rho(\mathfrak{u} \cap \mathfrak{a})}$, but the multiplicity is no longer equal to 1. Rather, the multiplicity is equal to the multiplicity of irreducible summands of the spin module, which is equal to $[2 \epsilon + 1 \dim \mathfrak{a}]$ (recall that $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$). This is the same as $\dim S_{(\mathfrak{h}, \mathcal{T})}$, where $S_{(\mathfrak{h}, \mathcal{T})} = \bigwedge \mathfrak{a}^+$ is the spin module for the pair $(\mathfrak{h}, \mathcal{T})$. The only $K^+$-type of $X_b(\lambda)$ contributing to the Dirac cohomology is the lowest $K$-type, so the action of $\theta$ on the Dirac cohomology is by $\epsilon$. So

$$I_{\theta}(X_b(\lambda)) = \epsilon E_{\lambda + \rho(\mathfrak{u} \cap \mathfrak{a})} \otimes S_{(\mathfrak{h}, \mathcal{T})}.$$ 

Case 2. The corresponding standard module for $G^+$ is $M_{(\mathfrak{b}, \lambda)} = X_b(\lambda) \oplus X_{\theta b}(\theta \lambda)$, with the two summands interchanged by $\theta$. More precisely, fixing an isomorphism $\Phi$ from the $\theta$-twisted $X_b(\lambda)$ onto $X_{\theta b}(\theta \lambda)$, we can define the action of $\theta$ on $v + w \in X_b(\lambda) \oplus X_{\theta b}(\theta \lambda)$ by

$$\theta(v + w) = \Phi^{-1}w + \Phi v \in X_b(\lambda) \oplus X_{\theta b}(\theta \lambda).$$

Choosing a different isomorphism $\Phi$ would lead to an isomorphic module.

Now it follows that $I_{\theta}(M_{(\mathfrak{b}, \lambda)}) = 0$, because $H_G(M_{(\mathfrak{b}, \lambda)})$ decomposes into two pieces interchanged by $\theta$, and hence the trace of $\theta$ on this space is zero.

Case 3. In this case there is $k \in K$ such that $\text{Ad}(k) \theta \mathfrak{b} = \mathfrak{b}$, and also $\text{Ad}(k) \theta \lambda = \lambda$. We can write $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ with $\theta \mathfrak{h} = \mathfrak{h}$ and $\text{Ad}(k) \mathfrak{h} = \mathfrak{h}$. Let $\chi = \lambda + \rho_{\mathfrak{b}}$ denote the infinitesimal character of $X_b(\lambda)$. We assume $\chi$ is regular; for singular $\chi$ the result then follows by the translation principle. Since $\theta \mathfrak{h} = \mathfrak{h}$, we can decompose $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ into eigenspaces for $\theta$. Let $\chi = (\sigma, \nu)$ be the corresponding decomposition of $\chi$.

The above $k$ determines an element $w$ of the Weyl group $W(G, \mathcal{H})$, and

$$w(\sigma, \nu) = (\sigma, -\nu).$$

Moreover, $k$ can be chosen so that $k^2 = (k \theta)^2$ is in the center of $G$, so the above $w$ is an involution.

Similarly to Case 1, we have two options for an action of $k \theta$, which can be distinguished by the sign they produce on the lowest $K$-type. We denote the corresponding standard modules by $X_b^k(\lambda)$, $\epsilon = \pm 1$.

We can compute the twisted character $\Theta$ of $X_b^k(\lambda)$ by using the kernel computation of Section 4 based on $\tau = k \theta$. Let $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{u} \supseteq \mathfrak{b}$ be the parabolic subalgebra of $\mathfrak{g}$ corresponding to $\nu$, so $\mathfrak{m}$ is the centralizer of $\nu$. Then $\tau \mathfrak{m} = \mathfrak{m}$ and $\tau \mathfrak{u} = \mathfrak{u}$. Let $H_0 = T_0 A_0$ be the fundamental Cartan subgroup, and let $T_0^k \theta$ be the set of strongly regular elements. By Corollary 4.3, $\Theta$ is supported on $\text{Ad}(G)(Mk \theta)$. But $T_0^k \theta$ does not meet $\text{Ad}(G)(Mk \theta)$, so it follows that $\Theta = 0$ on $T_0^k \theta$. By equation (2.10) in Section 2, this implies that the twisted Dirac index of $X_b^k(\lambda)$ is 0.
We have proved the following.

**Theorem 5.7.** If $\theta b = b$ and $\theta \lambda = \lambda$, the twisted Dirac indices of the standard $(g, K^+)$-modules $X^r_\theta(\lambda)$ are

$$I_\theta(X^r_\theta(\lambda)) = \epsilon E_{\lambda^\perp} \otimes S_{(b, T)}.$$  

The twisted Dirac indices of all other standard modules are zero. $\square$

5.3. **Applications.** Consider the two special cases of Dirac index we have considered in the introduction. One is the ordinary Dirac index, $I(X) = X \otimes (S^+ - S^-)$ for a $(g, K)$-module $X$, in the equal rank case, while the other is the twisted Dirac index $I_\theta(X) = (X^+ - X^-) \otimes S$ for a $(g, K^+)$-module $X$. In each case, $X$ is a finite length $(g, K)$-module, and there is an involution $\gamma$ acting on $X \otimes S$, equal either to $1 \otimes \theta$ or to $\theta \otimes 1$, and we have

$$H^\pm_D(X) = r \sum m^\pm r E_r,$$

$$I_\gamma(X) = r \left( \sum m^+ r E_r - \sum m^- r E_r \right),$$

where $m^\pm r$ are the multiplicities of $E_r$ in the $\pm$-eigenspaces of $\gamma$, and $r > 0$ is an integer depending on $(g, K)$ only. ($r$ is the multiplicity of the irreducible summands in the spin module.) On the other hand,

$$X = \sum m(X, \chi) X(\chi),$$

where $X(\chi)$ are standard modules. Then

$$I_\gamma(X) = \sum m(X, \chi) I_\gamma(X(\chi)).$$

Specialize $X$ to the cases of irreducible unipotent representations in [BP11]. Write $\overline{X} := X(\lambda, -x \lambda)$ for a unipotent representation. One can verify in all cases that $[\overline{X}]_K$ decomposes with multiplicity 1 and that $E_{2\lambda - \rho}$ occurs as the PRV component of $E_\mu \otimes E_\rho$. Thus one of $H^\pm_D(\overline{X})$ is zero, and $I_\gamma(\overline{X}) = m e(\overline{X}) E_{2\lambda - \rho}$ with $e(\overline{X}) = \pm 1$. Combining this with Theorem 5.1 and equation (5.11) we conclude that

$$e(\overline{X}) = \eta \Phi(\overline{X}, (\lambda, -\lambda)),$$

with $\eta$ as in Theorem 5.1. This can be used to solve for $m(\overline{X}, (\lambda, -\lambda))$.

Similarly, we specialize to the cases of irreducible unipotent representations in [BP15]. One can verify in all cases that one of $m^\pm = 0$ (see below). Thus we can write $I_\gamma(\overline{X}) = \sum e_{\overline{X}}(\tau) m(\tau) E_\tau$. Theorem 5.7 implies that

$$e_{\overline{X}}(\tau) m(\tau) = m(\overline{X}, \chi).$$

In all cases, (2.8) or (2.10) give explicit formulas for the distribution characters restricted to the elliptic strongly regular set, $T_0 \theta$, where $T_0$ is the $\theta$-fixed part of the fundamental Cartan subgroup $H_0 = T_0 A_0$.

We now explain why in the examples of unipotent representations in [BP15], one of $m^\pm = 0$ to be 0 for each $\tau$. Let $(g, K)$ be the pair corresponding to $G = Sp(2n, \mathbb{R})$ or $G = U(p, q)$. Let $\mathfrak{t} \subset \mathfrak{g}$ be a compact Cartan subalgebra, and choose compatible positive root systems $R^+(g, \mathfrak{t}) \supset R^+(\mathfrak{t}, \mathfrak{t})$. Denote $W = W(g, \mathfrak{t})$, $W^+ = W(\mathfrak{t}, \mathfrak{t})$, and let $W^+$ be the subset of $W$ consisting of $w$ which take the fundamental $(g, \mathfrak{t})$-chamber into the fundamental $(\mathfrak{t}, \mathfrak{t})$-chamber.
Recall that for each representation $X$ that we studied, we considered the candidates for $K^{\dagger}$-types that can appear in $H_D(X)$:

$$\tau = w\Lambda - \rho,$$

where $\Lambda \in t^*$ is the infinitesimal character of $X$ and $w \in W^1$. The multiplicity of each such $\tau$ was calculated in [BP15]; it is equal to the number of solutions to the equation

$$w_1\tau = \sigma\rho - \rho_t + \mu^-,$$

with variables $w_1 \in W_k$, $\sigma \in W_1$, and $\mu^-$ the lowest weight of a $K$-type of $X$.

To compute the index $I(X)$, we must determine the multiplicity of each $\tau$ in $H^{\dagger}_D(X)$ and in $H^\dagger_D(X)$. We claim that for each fixed $\tau$ one of these multiplicities is 0. Equivalently, we claim that all $\sigma$ corresponding to a fixed $\tau$ have the same sign.

We identify $\sigma$ with $\sigma\rho$ and recall that each $\sigma\rho$ corresponding to our fixed $\tau$ has a fixed “core” part plus possibly one or two more fixed coordinates. The variable part of $\sigma\rho$ was composed of pairs of neighboring integers that could either be placed on the left of the core, as $(i+1, i)$, or on the right of the core, as $(-i, -i-1)$. Moreover, the number of negative pairs, which lie to the right of the core, is fixed, i.e., does not depend on $\sigma$ (or even on $\tau$). We denote this number by $r$. (The number of positive pairs is thus also fixed.)

It is now easy to see that all $\sigma$ corresponding to the same $\tau$ have equal parity; one can be obtained from another by exchanging some pairs, and moving each pair requires an odd number of steps. One can moreover compute that the sign attached to $\tau$ is $(-1)^r$ times the sign of the fixed part of $\sigma$.

In particular, we see that $I(X)$, and so also the character of $X$ on the compact Cartan subgroup $T$ of $G$, cannot be zero whenever $X$ has nonzero Dirac cohomology. So each such $X$ is elliptic, and using the formulas for $H_D(X)$ in [BP15] together with the above remarks, we can calculate the character on $T$ explicitly.

6. Extensions of modules

Recall from the introduction that the Euler-Poincaré pairing of two (virtual) $(g, K)$-modules $X$ and $Y$ is the virtual vector space

$$\text{EP}(X, Y) = \sum_{i=1}^s (-1)^i \text{Ext}_{(g, K)}^i(X, Y).$$

As explained in the introduction, we should understand $\text{EP}(X, Y)$ in case $X$ is a standard module $X_b(\lambda)$ of Section 5.2. For this we use a special case of the “Frobenius reciprocity spectral sequence” of [V2 Proposition 6.3.2 and Corollary 6.3.4]. (For the $\theta$-stable case, see also [KV95 Corollary 5.121], or [V1 Theorem 6.14], where this spectral sequence is called the Zuckerman spectral sequence.)

**Proposition 6.1.** There is a first quadrant spectral sequence with differential of bidegree $(r, 1-r)$ and $E_2$ term

$$E_2^{pq} = \text{Ext}_{(\mathfrak{h}, T)}^p(C^\#_\lambda, H^q(\bar{u}, Y)),$$

converging to

$$\text{Ext}_{(g, K)}^{p+q-S}(X_b(\lambda), Y).$$
Since \( H^q(\bar{u}, Y) \) are finite-dimensional \((\mathfrak{h}, T)\)-modules by [HS, Theorem 7.22], it makes sense to take the alternating sum of spaces in (6.2). Since (6.3) is obtained from (6.2) by successively taking cohomology with respect to finitely many differentials, we can use the Euler-Poincaré principle to obtain

\[
\sum_{p,q} (-1)^{p+q} \text{Ext}^p_{(\mathfrak{h}, T)}(C^\#_\lambda, H^q(\bar{u}, Y)) = (-1)^S \sum_i (-1)^i \text{Ext}^i_{(\mathfrak{g}, K)}(X_\theta(\lambda), Y).
\]

In other words,

\[
(6.4) \quad \text{EP}(X_\theta(\lambda), Y) = (-1)^S \sum_{p,q} (-1)^{p+q} \text{Ext}^p_{(\mathfrak{h}, T)}(C^\#_\lambda, H^q(\bar{u}, Y)).
\]

6.1. **EP pairing and Dirac index in the equal rank case.** The main result of this subsection is the following.

**Theorem 6.5.** Let \( X \) and \( Y \) be two finite length \((\mathfrak{g}, K)\)-modules with infinitesimal character. Then

\[
(6.6) \quad \text{EP}(X, Y) = \text{Hom}_{K^1}(I(X), I(Y))
\]

in the Grothendieck group of finite-dimensional vector spaces.

**Proof.** As we already remarked, \( \text{EP}(X, Y) = 0 \) unless \( X \) and \( Y \) have the same infinitesimal character, and by Vogan’s conjecture the same is true for the right side. Therefore we may assume that \( X \) and \( Y \) have the same infinitesimal character. Furthermore, it is enough to prove the theorem in case \( X \) is a standard module, since standard modules generate the Grothendieck group of finite length \((\mathfrak{g}, K)\)-modules.

So let \( X = X_\theta(\lambda) \) as in Section 5.2. We first consider the case when \( \mathfrak{b} \) is not \( \theta \)-stable; i.e., \( \mathfrak{h} \) is not contained in \( \mathfrak{t} \), i.e., \( a \neq 0 \). We claim that in this case both sides of (6.6) are equal to zero.

In view of (6.4), \( \text{EP}(X_\theta(\lambda), Y) = 0 \) follows from the following lemma, i.e., its obvious extension to virtual \((\mathfrak{h}, T)\)-modules.

**Lemma 6.7.** Let \( V \) and \( W \) be finite-dimensional \((\mathfrak{h}, T)\)-modules, where \( \mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a} \) is a \( \theta \)-stable Cartan subalgebra of \( \mathfrak{g} \) such that \( a = \dim \mathfrak{a} > 0 \). Then

\[
\sum_{p=0}^a (-1)^p \text{Ext}^p_{(\mathfrak{h}, T)}(V, W) = 0.
\]

**Proof.** The space \( \text{Ext}^p_{(\mathfrak{h}, T)}(V, W) \) is the \( p \)th cohomology of the complex

\[
\text{Hom}_T(\wedge^p \mathfrak{a} \otimes V, W) = \wedge^p \mathfrak{a} \otimes \text{Hom}_T(V, W),
\]

with the usual de Rham differential. Here we used the fact that the adjoint action of \( T \) on \( \mathfrak{a} \) is trivial, so \( \wedge^p \mathfrak{a} \) can be pulled out of the Hom space. Since the complex is finite-dimensional, we can use the Euler-Poincaré principle and conclude that

\[
\sum_{p} (-1)^p \text{Ext}^p_{(\mathfrak{h}, T)}(V, W) = (\wedge^{\text{even}} \mathfrak{a} - \wedge^{\text{odd}} \mathfrak{a}) \otimes \text{Hom}_T(V, W).
\]

The last expression is however zero since \( \wedge^{\text{even}} \mathfrak{a} \) and \( \wedge^{\text{odd}} \mathfrak{a} \) have the same dimension. \( \square \)
The right side of (6.6) is zero as well, by Theorem 5.6.

This leaves us with the case of \( \theta \)-stable \( b \). In this case, since \( M(\mathfrak{h}, T) = M(T) \) is a semisimple category, the higher Ext groups in this category vanish, and so (6.4) becomes

\[
\text{(6.8)} \quad \text{EP}(X_b(\lambda), Y) = (-1)^S \sum_q (-1)^q \text{Hom}_T(C^\#_\lambda, H^q(\bar{u}, Y)) = (-1)^S \text{Hom}_T(C^\#_\lambda, \sum_q (-1)^q H^q(\bar{u}, Y));
\]

the alternating sum can go inside Hom, because all \( H^q(\bar{u}, Y) \) are finite-dimensional. Note that the above arguments actually imply that the spectral sequence given by (6.2) and (6.3) collapses, and hence we in fact get

\[
\text{(6.9)} \quad \text{Ext}_{(g,K)}^{q-S}(X_b(\lambda), Y) = \text{Hom}_T(C^\#_\lambda, H^q(\bar{u}, Y))
\]

for every \( q \). We are however interested primarily in the alternating sum and not in the individual Ext groups.

To compare the right sides of (6.8) and (6.6), we need the following two lemmas. We denote by \( D(\mathfrak{t}, \mathfrak{t}) \) the cubic Dirac operator for the pair \((\mathfrak{t}, \mathfrak{t})\) \cite{Q, Kos99} and by \( I_\mathfrak{t}(E) \) the corresponding Dirac index of a \( \mathfrak{t} \)-module \( E \). The spin module we take to define \( I_\mathfrak{t}(E) \) is \( S_\mathfrak{t} = \bigwedge (u \cap \mathfrak{t}) \). As usual, the difference between the adjoint and the spin action of \( t \) on \( \bigwedge (u \cap \mathfrak{t}) \) is given by a shift by \( \rho(u \cap \mathfrak{t}) \).

**Lemma 6.10.** For any admissible \((g,K)\)-module \( Y \) there is an equality of virtual \( T \)-modules

\[
\sum_q (-1)^q H^q(\bar{u}, Y) = I_\mathfrak{t}(I(Y)) \otimes C_\rho(u).
\]

**Proof.** Writing out the right side we get

\[
I_\mathfrak{t}(I(Y)) \otimes C_\rho(u) = I(Y) \otimes (S^+_\mathfrak{t} - S^-_\mathfrak{t}) \otimes C_\rho(u) = Y \otimes (S^+_\mathfrak{t} - S^-_\mathfrak{t}) \otimes C_\rho(u)
\]

\[
= Y \otimes (\bigwedge \text{even } u \cap s - \bigwedge \text{odd } u \cap s) \otimes (\bigwedge \text{even } u \cap \mathfrak{t} - \bigwedge \text{odd } u \cap \mathfrak{t}) = Y \otimes (\bigwedge \text{even } u - \bigwedge \text{odd } u).
\]

So our lemma is a special case of \cite{V1} Theorem 8.1. It is also a special case of \cite{IIS} Theorem 7.22 (which is stated for homology and not cohomology, but it is easy to pass between homology and cohomology using Poincaré duality).

**Lemma 6.11.** Let \( E_{\mu} \) be the irreducible finite-dimensional \( \mathfrak{t} \)-module with highest weight \( \mu \in \mathfrak{t}^* \). Let \( E \) be any finite-dimensional \( \mathfrak{t} \)-module. Then

\[
\text{(6.12)} \quad \text{Hom}_\mathfrak{t}(E_{\mu}, E) = \text{Hom}_\mathfrak{t}(C_{\mu + \rho(w \cap \mathfrak{t})}, (-1)^S I_\mathfrak{t}(E)).
\]

**Proof.** Let \( \nu \) be a dominant \( \mathfrak{t} \)-weight. By \cite{Kos99}, Dirac cohomology of \( E_{\nu} \) with respect to \( D(\mathfrak{t}, \mathfrak{t}) \) is the \( \mathfrak{t} \)-module

\[
H_D(E_{\nu}) = \bigoplus_{w \in W_\mathfrak{t}} C_w(\nu + \rho(w \cap \mathfrak{t})�.
\]

It follows that

\[
I_\mathfrak{t}(E_{\nu}) = \bigoplus_{w \in W_\mathfrak{t}} (-1)^{S + I(w)} C_w(\nu + \rho(w \cap \mathfrak{t})�.
\]
to check the signs, note that the highest Cartan component of the tensor product $E_{\nu} \otimes S_\nu$ corresponds to $w = 1$ and to $\bigwedge^{\text{top}} u \cap \mathfrak{t}$.

The only dominant weight in the above expression for $I_\mathfrak{t}(E_{\nu})$ corresponds to $w = 1$ and is equal to $(-1)^S \mathbb{C}_{\mu + \rho(u \cap \mathfrak{t})}$. Since $E = \bigoplus_\nu m_\nu E_{\nu}$, we can express the multiplicity $m_\mu$ as the multiplicity of the $t$-module $(-1)^S \mathbb{C}_{\mu + \rho(u \cap \mathfrak{t})}$ in $I_\mathfrak{t}(E) = \bigoplus_\mu m_\mu I_\mathfrak{t}(E_{\nu})$. This implies the lemma.

\begin{remark}
Lemma 6.11 includes the case when $\mu + \rho(u \cap \mathfrak{t})$ is singular, in which case $E_\mu = 0$.
\end{remark}

We now combine (6.8) with Lemma 6.10 remembering that $\mathbb{C}_\lambda^\# = \mathbb{C}_{\lambda + 2\rho(u)}$. It follows that

$$\text{EP}(X_b(\lambda), Y) = \text{Hom}_{T^\dagger}(\mathbb{C}_{\lambda + \rho(u)}, I_\mathfrak{t}(I(Y))),$$

where $T^\dagger$ is the spin double cover of $T$. Using Lemma 6.11 we see that this is further equal to $\text{Hom}_{K^\dagger}(E_{\lambda + \rho(u \cap \mathfrak{t})}, I(Y))$. The claim now follows from Theorem 5.6. This finishes the proof of Theorem 6.5. \hfill \Box

6.2. Elliptic pairing, equal rank case. As mentioned in the introduction, Theorem 6.5 is also related to the elliptic pairing of (finite length) $(\mathfrak{g}, K)$-modules. The elliptic pairing was defined by Arthur [Art93] (see also [Hua15] and [Ren16]). We assume for simplicity that $G$ and $K$ are connected and that $G$ contains a Cartan subgroup equal to a compact maximal torus $T \subseteq K$ (consisting of elliptic elements). Let $X, Y$ be irreducible $(\mathfrak{g}, K)$-modules, and let $\Theta_X, \Theta_Y$ denote the characters of the corresponding group representations, viewed as functions on $G$. The elliptic pairing of $X$ and $Y$ is

$$\langle X, Y \rangle_{\text{ell}} = \frac{1}{|W(G, T)|} \int_T |\det[(1 - \text{Ad}(t))]_{\mathfrak{g}/\mathfrak{t}}| \Theta_X(t) \Theta_Y(t) dt,$$

where $W(G, T)$ denotes the Weyl group of $G$ with respect to $T$ and $dt$ denotes the normalized Haar measure on $T$.

The elliptic pairing is extended linearly to the Grothendieck group of finite length $(\mathfrak{g}, K)$-modules; as in Definition 1.4 we call the elements of this Grothendieck group virtual $(\mathfrak{g}, K)$-modules. By [Hua15] and [Ren16], the elliptic pairing of two virtual $(\mathfrak{g}, K)$-modules is equal to their “Dirac index pairing”, i.e., to

$$\langle X, Y \rangle_{\text{DI}} = \int_{K^\dagger} \text{ch}(I(X))(k) \text{ch}(I(Y))(k) dk = \text{Hom}_{K^\dagger}(I(X), I(Y)).$$

(Here we identified the virtual vector space $\text{Hom}_{K^\dagger}(I(X), I(Y))$ with its dimension; see Definition 1.4. The second equality then follows from the standard orthogonality relations for $K^\dagger$.)

So we conclude the following.

\begin{corollary}
For any two finite length $(\mathfrak{g}, K)$-modules $X$ and $Y$, \[ \langle X, Y \rangle_{\text{ell}} = \text{EP}(X, Y) = \text{Hom}_{K^\dagger}(I(X), I(Y)). \]
\end{corollary}

We note that the equality

$$\langle X, Y \rangle_{\text{ell}} = \langle X, Y \rangle_{\text{DI}}$$

can also be derived from (2.39). Namely, by the Weyl integral formula we can write

$$\langle X, Y \rangle_{\text{DI}} = \frac{1}{|W(K, T)|} \int_{T^\dagger} |\det[(1 - \text{Ad}(t))]_{\mathfrak{t}/\mathfrak{t}}| \text{ch}(I(X))(t) \text{ch}(I(Y))(t) dt.$$
On the other hand we can substitute (2.9) into (6.14). Since \( W(G, T) = W(K, T) \), and since \( g/t = t/t \oplus s \) implies that
\[
\det[(1 - \text{Ad}(t))|_{g/t}] = \det[(1 - \text{Ad}(t))|_{t/t}] \det[(1 - \text{Ad}(t))|_{s}],
\]
the equality (6.16) will follow once we check that
\[
\det[(1 - \text{Ad}(t))|_{s}] = |\text{ch}(S^+ - S^-)(t)|^2.
\]
This follows from
\[
\Lambda_{\text{even}}^s - \Lambda_{\text{odd}}^s = (S^+ - S^-) \otimes (S^+ - S^-)^*
\]
and from the following simple lemma applied to \( A = -\text{Ad}(t) \).

**Lemma 6.17.** Let \( V \) be an \( n \)-dimensional complex vector space and let \( A: V \to V \) be a diagonalizable linear operator. Let \( \Lambda^i(A): \Lambda^i(V) \to \Lambda^i(V) \) be defined by
\[
\Lambda^i(A)(v_1 \wedge \cdots \wedge v_i) = Av_1 \wedge \cdots \wedge Av_i.
\]
Then
\[
\det(1 + A) = \sum_{i=0}^{n} \text{tr}(\Lambda^i(A)).
\]

**Proof.** If \( \lambda_1, \ldots, \lambda_n \) are the eigenvalues of \( A \), then the equality becomes
\[
\prod_{j=1}^{n} (1 + \lambda_j) = \sum_{i=0}^{n} \sum_{1 \leq j_1 < \cdots < j_i \leq n} \lambda_{j_1} \cdots \lambda_{j_i},
\]
which is obviously true. \( \square \)

### 6.3. The general case

We now drop the equal rank assumption. Let \( X \) and \( Y \) be modules for the extended group \( G^+ = G \rtimes \{1, \theta\} \), and consider the twisted Euler-Poincaré pairing \( \text{EP}_\theta(X, Y) \) defined in the introduction.

The main result of this subsection is Theorem 6.18.

**Theorem 6.18.** Let \( X \) and \( Y \) be two finite length modules for \( G^+ \) (i.e., \((g, K^+)\)-modules), with infinitesimal character. Then

\[
\text{EP}_\theta(X, Y) = c \text{Hom}_{K^+}(I_\theta(X), I_\theta(Y))
\]

in the Grothendieck group of finite-dimensional vector spaces. The constant \( c \) is as in the discussion following Definition 1.17, particularly formula (1.18).

The idea is as before to prove the theorem in case \( X \) is a standard module when we can express \( \text{EP}_\theta(X, Y) \) by (6.4).

We treat separately each of the three cases from Section 5.2.

**Case 1.** Now \( \theta b = b \) and \( \theta \lambda = \lambda \), and we have two standard \((g, K^+)\)-modules \( X^\epsilon_\lambda \), \( \epsilon = \pm 1 \), with \( \theta \)-action given by \( \epsilon \) on the lowest \( K \)-type.

Since \( u \) and \( \bar{u} \) are \( \theta \)-stable, \( \theta \) acts on the complex \( \text{Hom}_{C}(\Lambda \bar{u}, Y) = \Lambda u \otimes Y \) used to compute the \( \bar{u} \)-cohomology of \( Y \) by acting on both \( \Lambda u \) and \( Y \). This action commutes with the differential and therefore descends to cohomology. Thus the right side of the identity (6.19) has a natural action of \( \theta \). As defined at the beginning of Section 6.3, so does the left side, and we claim that the two actions are compatible; i.e., (6.19) is an equality of virtual \( \{1, \theta\}\)-modules. To see this, we invoke the Fundamental Spectral Sequence in Chapter V, Section 8 of [KV95]. Let
X, Y be \((\mathfrak{g}, \hat{K})\)-modules. The action of \(\theta\) on \(\text{Ext}^{i}_{(\mathfrak{g}, K)}(X, Y)\) is defined via the complex \(\text{Hom}_{K}[\Lambda^{i} s \otimes X, Y]\). Standard cohomological properties of \(\text{Ext}\) imply that we get the same action on \(\text{Ext}^{i}_{(\mathfrak{g}, K)}(X, Y)\) by using a complex \(\text{Hom}_{(\mathfrak{g}, K)}[P^{i}, Y]\) with \(P^{i} \to X \to 0\) any projective \((\mathfrak{g}, \hat{K})\)-resolution. Using this definition of the action of \(\theta\) we can trace the action of \(\theta\) through the spectral sequence in Proposition 5.113 and the particular cases in Theorem 5.120 of [KV95]. The key fact needed is that the isomorphism in Proposition 2.34 of [KV95], in our notation
\[
\text{Hom}_{\mathfrak{g}, K}[P^{0, K}_{\xi, T}(\mathbb{C}^{\#}), Y] \cong \text{Hom}_{\mathfrak{b}, T}[\mathbb{C}^{\#}, \mathcal{F}^{\vee} Y],
\]
is compatible with the \(\theta\)-action. This is straightforward.

So taking the trace of \(\theta\) on both sides of \((6.4)\), we get
\[
(6.20) \quad \text{tr}[\theta, \text{EP}_{\theta}(X^{\xi}_{\theta}(\lambda), Y)] = (-1)^{S} \sum_{p} (-1)^{p} \text{tr}[\theta, \text{Ext}^{p}_{(\mathfrak{b}, T)}(\mathbb{C}^{\#}, \sum_{q} (-1)^{q} H_{q}(\bar{u}, Y))].
\]

Denoting the finite-dimensional virtual \((\mathfrak{h}, T)\)-module \(\sum_{q} (-1)^{q} H_{q}(\bar{u}, Y)\) by \(Z\), we can use the \((\mathfrak{h}, T)\)-version of \((6.18)\) and write the right side of \((6.20)\) as
\[
(6.21) \quad (-1)^{S} c \text{Hom}_{T^{+}}(\text{tr}[\theta, \mathbb{C}^{\#}_{\lambda}] \otimes S_{(\mathfrak{h}, T)}, \text{tr}[\theta, Z] \otimes S_{(\mathfrak{h}, T)}).
\]

Here \(S_{(\mathfrak{h}, T)}\) is a spin module for the pair \((\mathfrak{h}, T)\), constructed as \(\Lambda a^{+}\) for some maximal isotropic subspace \(a^{+}\) of \(a\).

Since \(\theta\) acts by \(\epsilon\) on \(\mathbb{C}^{\#}_{\lambda}\), \(\text{tr}[\theta, \mathbb{C}^{\#}_{\lambda}] = \epsilon \mathbb{C}^{\#}_{\lambda}\) as a virtual \(T^{+}\)-module.

For the other term in \((6.21)\), we use the following analogue of Lemma 6.10.

**Lemma 6.22.** Let \(b = \mathfrak{h} \oplus u\) be a \(\theta\)-stable Borel subalgebra of \(\mathfrak{g}\). Let \(Y\) be an admissible module for \(G^{+}\). Then
\[
\text{tr}[\theta, \sum_{q} (-1)^{q} H_{q}(\bar{u}, Y)] \otimes S_{(\mathfrak{h}, T)} = I_{\mathfrak{t}}(I_{\theta}(Y)) \otimes \mathbb{C}_{\rho}(u)
\]
as virtual vector spaces.

**Proof.** As in the proof of Lemma 6.10, we can use [VI] Theorem 8.1, or [HS] Theorem 7.22, to identify
\[
\sum_{q} (-1)^{q} H_{q}(\bar{u}, Y) = Y \otimes (\Lambda^{\text{even}} u - \Lambda^{\text{odd}} u)
\]
as virtual \((\mathfrak{h}, T)\)-modules. This identification is compatible with \(\theta\)-actions.

On the other hand,
\[
\text{tr}[\theta, Y \otimes (\Lambda^{\text{even}} u - \Lambda^{\text{odd}} u)] \otimes S_{(\mathfrak{h}, T)} = \text{tr}[\theta, Y \otimes (\Lambda^{\text{even}} u \cap s - \Lambda^{\text{odd}} u \cap s) \otimes (\Lambda^{\text{even}} u \cap \mathfrak{k} - \Lambda^{\text{odd}} u \cap \mathfrak{k})] \otimes S_{(\mathfrak{h}, T)}
\]
\[
= (Y^{+} - Y^{-}) \otimes \Lambda u \cap s \otimes S_{(\mathfrak{h}, T)} \otimes (\Lambda^{\text{even}} u \cap \mathfrak{k} - \Lambda^{\text{odd}} u \cap \mathfrak{k}).
\]

Since the spin module \(S\) for \((\mathfrak{g}, K)\) can be identified with \(\Lambda u \cap s \otimes S_{(\mathfrak{h}, T)} \otimes \mathbb{C}_{-\rho(u \cap s)}\), and the spin module \(S_{\mathfrak{t}}\) for \((\xi, T)\) can be identified with \(\Lambda u \cap \xi \otimes \mathbb{C}_{-\rho(u \cap \xi)}\), the above expression is equal to
\[
(Y^{+} - Y^{-}) \otimes S \otimes (S_{\mathfrak{t}^{+}} - S_{\mathfrak{t}^{-}}) \otimes \mathbb{C}_{\rho}(u) = I_{\mathfrak{t}}(I_{\theta}(Y)) \otimes \mathbb{C}_{\rho}(u).
\]

\[\square\]
5.7 the right side of the equality claimed by Theorem 6.18 is 0 for

\[ \langle \rangle \]

defined by

Elliptic pairing, twisted case.

Thus \( \text{tr}[\theta, \text{EP}_\theta(X^e_b(\lambda), Y)] = c \text{Hom}_K^1(I_\theta(X^e_b(\lambda)), I_\theta(Y)), \)

and this proves Theorem 6.18 in this case.

Case 2. For the standard module \( M_{b,\lambda}, \text{EP}_\theta(M_{b,\lambda}, Y) \) decomposes into two pieces interchanged by \( \theta \), so

\[ \text{tr}[\theta, \text{EP}_\theta(M_{b,\lambda}, Y)] = 0 \]
as a virtual vector space. (Note that it is not equal to 0 as a virtual \( \{1, \theta\} \)-module.) Since also \( I_\theta(M_{b,\lambda}) = 0 \) by Theorem 5.7, the statement of Theorem 6.18 holds trivially in the present case.

Case 3. Recall that there is \( k \in K \) such that \( k\theta b = b \) and \( k\theta \lambda = \lambda \). By Theorem 5.7 the right side of the equality claimed by Theorem 6.18 is 0 for \( X = X^e_b(\lambda) \). To see that the left side of that equality is also 0, we first note that \( K \) acts trivially on the complex Hom\( K(\wedge \mathfrak{s} \otimes X, Y) \) which computes the \( \text{Ext}^{[g,K]}(X, Y) \). Hence \( K \) also acts trivially on \( \text{EP}(X, Y) \), and so the action of \( \theta \) on \( \text{EP}(X, Y) \) is the same as the action of \( k\theta \). On the other hand, similarly as in Case 1, \( k\theta \) acts on both sides of (6.24), in a compatible way, and thus it is enough to show that the trace of \( k\theta \) on the right side of (6.24) is 0. For this, we need an analogue of Lemma 6.7. Note that \( \mathfrak{b} = \mathfrak{t} + \mathfrak{a} \) with \( \mathfrak{a} \neq 0 \). Since \( \theta \lambda \neq \lambda \), \( \lambda \) is not identically 0 on \( \mathfrak{a} \). So \( k\theta \lambda = \lambda \) implies that \( k\theta \) has a nonzero +1-eigenspace in \( \mathfrak{a} \). Denoting this eigenspace by \( \mathfrak{a}_1 \), we can write \( \mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \) with \( \mathfrak{a}_2 \) invariant for \( k\theta \).

Let \( \tilde{\tau} \) be the automorphism of the pair \( (\mathfrak{b}, T) \) acting by \( k\theta \). Let \( T^\circ = T \rtimes \{1, \tilde{\tau}\} \). Let \( V \) and \( W \) be virtual \( (\mathfrak{b}, T^\circ) \)-modules. As in the proof of Lemma 6.7 we can write

\[
\sum_p (-1)^p \text{Ext}^p_{(\mathfrak{b}, T)}(V, W) = (\bigwedge \text{even} \mathfrak{a} - \bigwedge \text{odd} \mathfrak{a}) \otimes \text{Hom}_T(V, W)
\]

\[
= (\bigwedge \text{even} \mathfrak{a}_1 - \bigwedge \text{odd} \mathfrak{a}_1) \otimes (\bigwedge \text{even} \mathfrak{a}_2 - \bigwedge \text{odd} \mathfrak{a}_2) \otimes \text{Hom}_T(V, W).
\]

Thus \( \text{tr[} \tilde{\tau}, \sum_p (-1)^p \text{Ext}^p_{(\mathfrak{b}, T)}(V, W) \text{]} = 0 \) as a virtual vector space, since the action of \( \tilde{\tau} \) on \( (\bigwedge \text{even} \mathfrak{a}_1 - \bigwedge \text{odd} \mathfrak{a}_1) \) is trivial, so the trace of \( \tilde{\tau} \) on it is equal to the dimension, which is 0.

This completes the proof of Theorem 6.18.

6.4 Elliptic pairing, twisted case. As in the untwisted case, the two sides of the equality in Theorem 6.18 are equal to the twisted elliptic pairing of \( X \) and \( Y \), defined by

\[
\langle X, Y \rangle_{\text{ell}, \theta} = \frac{1}{|W|} \int_{T^+} |\det[(1 - \text{Ad}(t^+))]|_{\mathfrak{g}/\mathfrak{t}} |\Theta_X(t^+)\Theta_Y(t^+)| dt^+,
\]
where as before \( W = W(G, T) = W(K, T) \) is the Weyl group, and \( dt^+ \) denotes the normalized Haar measure on \( T^+ \). Here \( X \) and \( Y \) are finite length \((g, K^+)-\)modules or virtual \((g, K^+)-\)modules. The definition of \( \langle \cdot, \cdot \rangle_{\ell, \theta} \) is included in the considerations of Arthur \cite{Art93}; it can also be found in \cite[Section 7.3]{W2}. We assume for simplicity that \( G \) and \( K \) are connected and that \( H = TA \) is a Cartan subgroup of \( G \) such that \( T \subseteq K \) is a maximal torus (consisting of elliptic elements). The case of most interest is when \( H \neq T \); i.e., \( G \) and \( K \) do not have equal rank. So we will assume this.

Since \( T^+ = T \cup T = T \cup T\theta \), the integral in (6.26) is the sum of two pieces, and we claim that the integral over \( T \) is in fact 0. This will follow if we prove

\[
\det((1 - \Ad(t)))_{g/t} = 0, \quad t \in T.
\]

Since \( g/t = t/t \oplus s \) as \( T \)-modules, it is enough to prove that

(6.27) \[
\det((1 - \Ad(t)))_{s} = 0, \quad t \in T.
\]

To see this, we use Lemma 6.17 to rewrite the left hand side as

\[
\tr[\Ad(t); \Lambda^{\text{even}} s - \Lambda^{\text{odd}} s].
\]

We can decompose \( s = a \oplus s' \) as \( T \)-modules; recall that we are assuming \( a \neq 0 \). Then

\[
\Lambda^{\text{even}} s - \Lambda^{\text{odd}} s = (\Lambda^{\text{even}} a - \Lambda^{\text{odd}} a) \otimes (\Lambda^{\text{even}} s' - \Lambda^{\text{odd}} s'),
\]

and since \( \Ad(t) \) is the identity on \( a \), its trace on \( \Lambda^{\text{even}} a - \Lambda^{\text{odd}} a \) is 0. This proves (6.27), and we see that the integration in (6.26) is only over \( T = T\theta \), i.e.,

(6.28) \[
\langle X, Y \rangle_{\ell, \theta} = \frac{1}{|W|} \int_{T^+} \left| \det((1 - \Ad(t\theta)))_{g/t} \right| t \Theta_X(t\theta) \Theta_Y(t\theta) dt.
\]

On the other hand, we define the twisted Dirac index pairing of \( X \) and \( Y \) as

\[
\langle X, Y \rangle_{\text{DL}, \theta} = \text{Hom}_{K^1} (I_\theta(X), I_\theta(Y)) = \int_{K^1} \text{ch}(I_\theta(X)) (k) \text{ch}(I_\theta(Y)) (k) dk
\]

\[
= \frac{1}{|W|} \int_{T^+} \left| \det((1 - \Ad(t)))_{t/t} \right| \text{ch}(I_\theta(X)) (t) \text{ch}(I_\theta(Y)) (t) dt.
\]

(The last equality uses the Weyl integral formula.)

We claim that the elliptic pairing of \( X \) and \( Y \) is equal to \( c \) times their twisted Dirac index pairing, with \( c \) as in Theorem 6.18. To see this, we follow the discussion below Corollary 6.15 we compare the above formula for \( \langle X, Y \rangle_{\text{DL}, \theta} \) with the expression for \( \langle X, Y \rangle_{\ell, \theta} \) obtained by substituting (2.10) into (6.28), i.e., with

\[
\langle X, Y \rangle_{\ell, \theta} = \frac{1}{|W|} \int_{T^+} \left| \det((1 - \Ad(t\theta)))_{g/t} \right| \frac{\text{ch}(I_\theta(X)) (t) \text{ch}(I_\theta(Y)) (t)}{\text{ch}(S) (t)^2} dt.
\]

Since \( \theta \) is 1 on \( t/t \) and \( -1 \) on \( s \), writing \( g/t = t/t \oplus s \) implies that \( \det((1 - \Ad(t\theta)))_{g/t} = \det((1 - \Ad(t)))_{t/t} \det((1 + \Ad(t)))_{s} \). By Lemma 6.17

\[
\det((1 + \Ad(t)))_{s} = \text{ch}(\Lambda(s))(t) = c \text{ch}(S \otimes S^*)(t) = c |\text{ch}(S)(t)|^2.
\]

So we see that indeed \( \langle X, Y \rangle_{\ell, \theta} = c \langle X, Y \rangle_{\text{DL}, \theta} \).

The corollary summarizes the discussion.
Corollary 6.29. For any two finite length \((\mathfrak{g}, K^+)\)-modules \(X\) and \(Y\),
\[
\langle X, Y \rangle_{\text{ell}, \theta} = \text{EP}_\theta(X, Y) = \epsilon \text{Hom}_{K^+}(I_\theta(X), I_\theta(Y)).
\]

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