Pencils of cubics with eight base points lying in convex position in \( \mathbb{R}P^2 \)

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September 19, 2012

Abstract

To a generic configuration of eight points in convex position in the plane, we associate a list consisting of the following information: for all of the 56 conics determined by five of the points, we specify the position of each of the three remaining points, inside or outside. We prove that the number of possible lists, up to the action of \( D_8 \), is 49, and we give two possible ways of encoding these lists. A generic complex pencil of cubics has twelve singular (nodal) cubics and nine distinct base points, any eight of them determines the ninth one, hence the pencil. If the base points are real, exactly eight of these singular cubics are distinguished, that is to say real with a loop containing some base points. We call combinatorial cubic a topological type (cubic, base points), and combinatorial pencil the sequence of eight successive combinatorial distinguished cubics. We choose representatives of the 49 orbits. In most cases, but four exceptions, a list determines a unique combinatorial pencil. We give a complete classification of the combinatorial pencils of cubics with eight base points in convex position.

1 Pencils of cubics

1.1 Preliminaries

Given nine generic points in \( \mathbb{C}P^2 \), there exists one single cubic passing through them. Given eight generic points in \( \mathbb{C}P^2 \), there exists a one-parameter family of cubics passing through them. We will call such a family a pencil of cubics. Let \( F_0 \) and \( F_1 \) be two cubics of a pencil \( P \). They intersect at a ninth point. As the other cubics of \( P \) are linear combinations of \( F_0 \) and \( F_1 \), they all pass through this ninth point. We call these nine points the base points of \( P \). If eight of the base points are real, the pencil is real, and hence the ninth base point is also real. A pencil of cubics is a line in the space \( \mathbb{C}P^9 \) of complex cubics. Let \( \Delta \) be the discriminantal hypersurface of \( \mathbb{C}P^9 \), formed by the singular cubics. The hypersurface \( \Delta \) is of degree 12. Hence, a generic pencil of cubics intersects \( \Delta \) transversally at 12 regular points. Otherwise stated, a generic pencil \( P \) has exactly 12 singular (nodal) cubics. A non-generic pencil will be called singular.
pencil. Let \( P \) be a real pencil with nine real base points and denote the real part of \( P \) by \( \mathbb{R} P \). Let \( n \leq 12 \) be the number of real singular cubics of \( P \). Let \( C_3 \) be one of these cubics. The double point \( P \) of \( C_3 \) is isolated if the tangents to \( C_3 \) at \( P \) are non-real, otherwise \( P \) is non-isolated. If \( P \) is non-isolated, \( C_3 \setminus P = \mathcal{J} \cup \mathcal{O} \), where \( \mathcal{J} \cup P \) \( \neq 0 \) and \( \mathcal{O} \cup P \) \( = 0 \) in \( H_1(\mathbb{R} P^2) \). We say that \( \mathcal{O} \) is the loop and \( \mathcal{J} \) is the odd component of \( C_3 \). Notice that the loop \( \mathcal{O} \) is convex. The estimation of \( n \) presented hereafter is due to V.Kharlamov, see [2].

One has: \( n = n_1 + n_2 + n_3 \), where \( n_1 \) is the number of cubics with an isolated double point, \( n_2 \) is the number of cubics with a loop containing no base points, and \( n_3 \) is the number of cubics with a loop containing some base points. To evaluate \( n \), one recalculates the Euler characteristic of \( \mathbb{R} P^2 \), fibering \( \mathbb{R} P^2 \) with the cubics of \( P \). Each isolated double point, and each base point contributes by +1; and each non-isolated double point contributes by −1, so that one gets:

\[
1 = \chi(\mathbb{R} P^2) = 9 + n_1 - (n - n_1)
\]

So, \( n - 2n_1 = 8 \). Thus, \( n = 8, 10 \) or 12 and correspondingly, \( n_1 = 0, 1 \) or 2.

Consider a motion in \( \mathbb{R} P \) starting from a cubic with an isolated double point. If we choose the direction of the motion properly, an oval appears, grows, and attaches itself to the odd component, forming a loop that contains no base point. Conversely, starting from a cubic with a loop containing no base point, one can move in the pencil so that there appears a cubic with an oval. As this oval lies inside of the loop, it shrinks when one moves further, and degenerates into an isolated double point. Thus, \( n_2 = n_1 \) and \( n_3 = 8 \) independently of \( n \). Let us call the eight cubics of the third type the distinguished cubics of \( P \). We shall picture \( \mathbb{R} P \) by a circle, divided in eight portions by the eight distinguished cubics. Let us remark that this number \( n_3 = 8 \) is the Welschinger invariant \( W_3 \), see [5]. The number of real rational plane curves of degree \( d \) going through \( 3d - 1 \) generic points of \( \mathbb{R} P^2 \) is always finite. Let \( c_1 \) be the number of such curves with an even number of isolated nodes, and \( c_2 \) be the number of such curves with an odd number of isolated nodes. Welschinger proved that the difference \( W_d = c_1 - c_2 \) does not depend on the choice of the \( 3d - 1 \) points.

Pencils of cubics were applied in [5] to solve an interpolation problem, and in [3], [4] to study the isotopy types realizable by some real algebraic curves in \( \mathbb{R} P^2 \).

### 1.2 Singular pencils

Let \( P = P(1, \ldots, 8) \) be a real pencil of cubics, determined by eight generic points on \( \mathbb{R} P^2 \), say \( 1, \ldots, 8 \). Move these points till \( P \) degenerates into a singular pencil \( P_{\text{sing}} \). The degeneration is generic if and only if \( P_{\text{sing}} \) intersects \( \Delta \) transversally at 10 regular points and

1. \( P_{\text{sing}} \) is tangent to \( \Delta \) at one regular point, or
2. \( P_{\text{sing}} \) crosses transversally a stratum of codimension 1 of \( \Delta \)
In both cases, two singular cubics $C^1_3$ and $C^2_3$ of $\mathcal{P}$ come together to yield one singular cubic $C_3$ of $\mathcal{P}_{\text{sing}}$. The cubic $C_3$ is necessarily real; the cubics $C^1_3$ and $C^2_3$ are either both real (case a), or complex conjugated (case b). Notice that, given a generic cubic $F$ of $\Delta$ with node in some point $p$, one has: $T_F \Delta = \{ F + G | G(p) = 0 \}$. Therefore, $\mathcal{P}_{\text{sing}}$ satisfies the condition 1) if and only if $\mathcal{P}_{\text{sing}}$ has a double base point, at $p$. Otherwise stated, $\mathcal{P}_{\text{sing}}$ is obtained from $\mathcal{P}$ by letting two base points $A$ and $B$ of $\mathcal{P}$ come together. Move $A$ towards $B$ along the line $(AB)$. For simplicity, we assume that $B$ is the point $(0,0)$ of the plane, and the direction is the $x$-axis. The condition that some cubic $H$ passes through $A$ and $B$ becomes at the limit: $H(0) = 0$ and $\frac{\partial H}{\partial x}(0) = 0$. So a cubic of $\mathcal{P}_{\text{sing}}$ is either singular at $A = B$ or tangent to the prescribed direction at $A = B$, depending on whether the other partial derivative $\frac{\partial H}{\partial y}$ vanishes or not at $(0,0)$. In case 2), the cubic $C_3$ must be reducible (product of a line and a conic), or have a cusp. If $C_3$ is reducible, the genericity imposes that three of the base points lie on the line, and the other six lie on the conic. Notice that if $C_3$ has a cusp, the cubics $C^1_3$ and $C^2_3$ are either complex conjugated, or non-distinguished real cubics, one with a loop, the other with an isolated double point.

Consider the generic degenerations of the form:

1. Two base points come together, say $A$ and $B$, or

2. Three base points come onto a line, or equivalently the other six base points come onto a conic.

They split into four subcases 1a), 1b), 2a) and 2b), see Figure[1]. In the right-hand part of the figure, the dotted crosses symbolize the non-real nodes of the cubics $C^1_3$ and $C^2_3$. Case 1a) deserves a supplementary explanation. Let us call elementary arc $AB$ an arc connecting $A$ to $B$ and containing no other base point. As $\mathcal{P}$ can degenerate into $\mathcal{P}_{\text{sing}}$ letting $A$ and $B$ come together, some cubics of $\mathcal{P}$ must have an elementary arc $AB$. Start from such a cubic and move in any direction in $\mathbb{R} \mathcal{P}$. At some moment, the mobile arc $AB$ must glue to another arc, and then disappear. Thus $\mathcal{P}$ has two distinguished cubics corresponding to the openings of the arc $AB$. We call singular elementary arc $AB$ the non-smooth arc $AB$ of either of these cubics. In case 1a), these two cubics are $C^1_3$ and $C^2_3$, they come together to yield the cubic $C_3$, which has a non-isolated double point at $A = B$. All three combinatorial cubics $C_3$, $C^1_3$ and $C^2_3$ are identical outside of a neighbourhood of $A \cup B$.

Let 9 be the ninth base point of $\mathcal{P}$, and $C_3$ be any cubic of $\mathcal{P}$. We call combinatorial cubic $C_3$, the topological type of $(C_3, 1, \ldots , 9)$. We call combinatorial pencil $\mathcal{P}$ the list of the successive eight combinatorial distinguished cubics of $\mathcal{P}$. A combinatorial pencil undergoes a (generic) degeneration only if one of the cases 1a) 2a) takes place. Thus, when considering combinatorial pencils, we forget the strata of $\Delta$ formed by cuspidal cubics.
Figure 1: The degeneration of $\mathcal{P}$ into $\mathcal{P}_{\text{sing}}$
2 Configurations of 8 points lying in convex position in $\mathbb{R}P^2$

2.1 Mutual position of points and conics

We say that $n$ generic points lie in convex position in $\mathbb{R}P^2$ if there exists a line $\Delta \subset \mathbb{R}P^2$ such that the $n$ points lie in convex position in the affine plane $\mathbb{R}P^2 \setminus \Delta$. Notice that any set of $n \leq 5$ points in $\mathbb{R}P^2$ lie in convex position.

Consider $n \geq 5$ generic points lying in strictly convex position in $\mathbb{R}P^2$, say $1, \ldots, n$. Let $L(1, \ldots, n)$ be the list of the $C_5^n$ conics through five of these points, enhanced for each conic, with the position of each of the remaining $n-5$ points (inside or outside).

How many different possibilities can be realized by $L(1, \ldots, n)$ when one lets the points $1, \ldots, n$ move?

For $n = 6$, $L(1, \ldots, 6)$ is determined e.g. by the position of the point 6 with respect to the conic 12345. (Note that the six conics of the list intersect pairwise at four points among $1, \ldots, 6$.) Thus, the number of possibilities realized by the lists $L(1, \ldots, 6)$ is 2, see Figure 2. We write shortly: $\sharp L(1, \ldots, 6) = 2$. Consider now a configuration of seven points $1, \ldots, 7$. There are two possibilities for $L(1, \ldots, 6)$. For either of them, 7 may be placed in seven different ways with respect to the set of conics passing through five points among $1, \ldots, 6$. One checks easily that the data $L(1, \ldots, 6)$, position of 7 determines the list $L(1, \ldots, 7)$. Thus, $\sharp L(1, \ldots, 7) = 14$. We may reprove this another way round: let $C_3$ be a cubic passing through the points $1, \ldots, 7$. By abuse of language we will also call cubic $C_3$ the topological type of $(C_3, 1, \ldots, 7)$. Let $F$ be the equation of $C_3$ and $p$ be a point among $1, \ldots, 7$. The condition $F(p) = 0$ is a linear equation in the coefficients of $F$. If $p$ is a singular point of $C_3$, one gets two supplementary linear equations: $\frac{\partial F}{\partial x}(p) = 0$ and $\frac{\partial F}{\partial y}(p) = 0$. Thus, as $1, \ldots, 7$ are generic, there exists exactly one real nodal cubic passing through $1, \ldots, 7$ and having $p$ as double point. Let $S(1, \ldots, 7)$ be the list of the seven nodal cubics passing through the points $1, \ldots, 7$, one of them being the double point.

Proposition 1 The list $S(1, \ldots, 7)$ can realize fourteen possibilities, denoted by $1\pm, 2\pm, \ldots, 7\pm$.

The lists $1+$ and $1-$ are shown in Figure 3; the other lists $n\pm$ are obtained from $1\pm$ performing on $1, \ldots, 7$ the cyclic permutation that replaces 1 by $n$. (The double point of the last cubic in each list may be an isolated node or a crossing.) Note that any of the five non-extreme cubics of the list $S(1, \ldots, 7)$ determines the whole of this list. For the sake of self-containment, we repeat here the proof from [5].

Proof: Let $1, \ldots, 7$ lie in convex position in $\mathbb{R}P^2$. The points are labeled according to the convex position. We shall perform several cremona transformations $cr : (x_0; x_1; x_2) \to (x_1x_2; x_0x_2; x_0x_1)$ based in triples of points

5
Let us perform a first transformation \( cr \) based in \( 1,2,3 \). After \( cr \), the pencils of lines \( F_i \) based in the points \( i \in \{1,2,3,4\} \) sweep out the remaining points in the following cyclic orderings: \( F_1 : 2,4,5,6,7,3, \) \( F_2 : 1,4,5,6,7,3, \) \( F_3 : 1,4,5,6,7,2, \) \( F_4 : 1,X,Y,Z,3,2, \) where \( \{X,Y,Z\} = \{5,6,7\} \) (hence six possibilities). For all pairs \( i,j \in \{4,5,6,7\} \) with \( i < j \), the image of the line 123ij is a line \( ij \) that meets successively: \( i,j,(23),(13),(12) \). After \( cr \), consider the three conics \( C_2(i),i \in \{1,2,3\} \), determined by \( i,4,5,6,7 \). Assume \( X = 6,Y = 5,Z = 7 \) (see Figure 4).

As any \( F_i,i \in \{1,2,3\} \) meets successively 4,5,6,7 and \( F_4 \) meets successively \( i,6,5,7 \), one has \( C_2(i) = 17456 \). Using the pencils of lines, notice that 6 is separated from 1 by \( (47) \cup (23) \), hence the arc 61 of the conic \( C_2(1) \) cuts the line \( (23) \) once. The point 7 is separated from 4 by \( (16) \cup (ij) \) for any pair \( \{i,j\} \in \{1,2,3\} \), so that the arc 74 of \( C_2(1) \) cuts once each of the base lines \( (12),(13),(23) \). As this arc (oriented from 7 to 4) is contained in the sector \( (46) \cup (47) \) that does not contain 1, it must meet successively the lines \( (23),(13) \) and \( (12) \). This gives us the combinatorial conic \( C_2(1) : 1,7,(23),(13),(12),4,5,6,(23) \). Similarly, one gets the other two combinatorial conics: \( C_2(2) : 2,(13),7,(23),(13),(12),4,5,6, \) and \( C_2(3) : 3,7,(23),(13),(12),4,5,6,(12) \).

Perform the inverse cremona transformation \( cr^{-1} \). One gets the nodal cubics \( C_3(i) = cr^{-1}(C_2(i)),i = 1,2,3 \). Now perform a new cremona transformation \( cr' \) based in \( 1,3,4 \). The cubic \( C_3(3) \) is mapped onto \( C_2'(3) : 3,(14),7,(34),2,(14),(13),5,6 \). Using Bezout’s theorem, one deduces the combinatorial conic \( C_2'(4) : 4,7,(34),2,(14),(13),5,6,(13) \). Perform the inverse cremona transformation \( cr'^{-1} \), one gets the nodal cubic \( C_3(4) = cr'^{-1}(C_2'(4)) \). With other cremona transformations, and similar arguments, one finds out the whole list of seven nodal cubics passing through 1,\ldots,7 with node in one of these points. The case \( X = 6,Y = 5,Z = 7 \) considered here gives rise to the list \( 7+ \) (see Figure 4). If \( X = 5,Y = 6,Z = 7 \), one gets five lists: \( 2-,2+,7-,4+,4- \). If \( X = 7,Y = 6,Z = 5 \) one gets again five lists: \( 1+,1-,3+,3-,5+ \). The other cases \( X = 5,Y = 7,Z = 6; \) \( X = 6,Y = 7,Z = 5 \) and \( X = 7,Y = 5,Z = 6 \) give rise respectively to the lists \( 6+,5- \) and \( 6- \).
| $C_2$ | in | out | in | out |
|-------|----|-----|----|-----|
| 12345 | 6  |     |    | 6   |
| 12346 | 5  | 5   |    |     |
| 12356 | 4  |     |    | 4   |
| 12456 | 3  | 3   |    |     |
| 13456 | 2  |     |    | 2   |
| 23456 | 1  | 1   |    |     |

Figure 2: The two lists $L(1, \ldots 6)$

1+

1−

Figure 3: The lists 1+ and 1−
Figure 4: Construction of the list 7+
In what follows $k < C_2$ means $k$ lies inside of the conic $C_2$. Let us get back to our lists $L(1, \ldots, n)$. For $n = 7$, let $C_2 = ijklm$ be one of the 21 conics in consideration, and let $C_3$ be a cubic from the list $S(1, \ldots, 7)$, having its double point at one of the $i, j, k, l, m$. Applying Bezout’s theorem between $C_2$ and $C_3$, one deduces the position of either of the remaining two points with respect to $C_2$. So, $S(1, \ldots, 7)$ determines $L(1, \ldots, 7)$. It turns out that the 14 lists $S(1, \ldots, 7)$ give rise to distinct lists $L(1, \ldots, 7)$. Thus, the combinatorial data $L(1, \ldots, 7)$ and $S(1, \ldots, 7)$ are equivalent, and $\# L(1, \ldots, 7) = 14$, see Figures 37–38. (For convenience, we have gathered some of the auxiliary tabulars at the end.)

All of these lists are equivalent up to the action of the dihedral group $D_8$.

Consider the group $D_8$ of symmetries of the octagon, generated by the cyclic permutation $a = +1$ and the symmetry with respect to the axis $\sigma = 15$, see Figure 5.

$$D_8 = \{a, \sigma|a^8 = id, \sigma^2 = id, a\sigma = \sigma a^{-1}\}$$

**Proposition 2** Up to the action of $D_8$, $\# L(1, \ldots, 8) = 49$. The total number of generic lists is $784 = 49 \times 16$.

This proposition will be proved in two steps, in sections 2.2 (restriction part) and 4.1 (construction part).

### 2.2 Admissible lists

Let $1, \ldots, 8$ lie in convex position and $\{A, \ldots, G\} = \{1, \ldots, 7\}$. Move 8, leaving the other points fixed and preserving the convex position. Consider an event of the

| 15 | 26 | 37 | 48 | (+1)(15) | (+1)(26) | (+1)(37) | (+1)(48) |
|----|----|----|----|----------|----------|----------|----------|
| 1 ↔ 1 | 1 ↔ 3 | 1 ↔ 5 | 1 ↔ 7 | 1 ↔ 2 | 1 ↔ 4 | 1 ↔ 6 | 1 ↔ 8 |
| 2 ↔ 8 | 2 ↔ 2 | 2 ↔ 4 | 2 ↔ 6 | 3 ↔ 8 | 2 ↔ 3 | 2 ↔ 5 | 2 ↔ 7 |
| 3 ↔ 7 | 4 ↔ 8 | 3 ↔ 3 | 3 ↔ 5 | 4 ↔ 7 | 5 ↔ 8 | 3 ↔ 4 | 3 ↔ 6 |
| 4 ↔ 6 | 5 ↔ 7 | 6 ↔ 8 | 4 ↔ 4 | 5 ↔ 6 | 6 ↔ 7 | 7 ↔ 8 | 4 ↔ 5 |
| 5 ↔ 5 | 6 ↔ 6 | 7 ↔ 7 | 8 ↔ 8 |

Figure 5: Action of $D_8$
form 8 crosses a conic ABCDE, it induces a change of the list \( \hat{G} \). The remaining point \( F \) may lie inside or outside of the conic ABCDE, depending on the list \( L(1, \ldots, 7) \) (see upper and lower part of Figure \( \text{[6]} \)). So there are: 21 choices for the conic ABCDE, two choices of \( G \), and two possible positions of the last point \( F \) with respect to ABCDE. Hence in total 84 possibilities. Figure \( \text{[6]} \) gathers all of these possibilities, showing how the cubic of \( \hat{G} \) with respect to \( F \) outside of ABCDE. For each \( \hat{8} \) we find two possible chains of degenerations of \( \hat{G} \) while moving the point 8. The chain starting with 8± will be denoted by \( \hat{G} \pm (8) \). For each \( \hat{8} \in \{1\pm, \ldots, 7\pm\} \) one watches which of the two possible chains is realized, see Figure \( \text{[7]} \). For each list \( \hat{8} \in \{1\pm, 2\pm, 3\pm, 4\pm\} \), we draw a diagram whose rows are the chains of degenerations of \( 1, \ldots, 7 \). (We drop the other cases \( \hat{8} \in \{7\pm, 6\pm, 5\pm, 4\pm\} \), that can be deduced from the first ones by the action of \( D_8 \)). Let \( C_2 \) and \( C_2' \) be two adjacent conics in a column, we add a vertical arrow from \( C_2 \) to \( C_2' \) if the following holds: 8 outside of \( C_2 \) implies 8 outside of \( C_2' \). See Figures \( \text{[8]-[14]} \).

Chasing in the diagrams, we may find all of the admissible orbits, for the action of \( D_8 \), realizable by the lists \( L(1, \ldots, 8) \). The explicit lists \( L(1, \ldots, 8) \) obtained in this procedure are gathered in Figures \( \text{[9]-[14]} \). We denote these lists by \( L_{1, \ldots, 8} \), according to their appearance order. Note that we cannot completely rule out redundancies: we get sometimes several representants of the same orbit. So, we choose for each orbit one representant that we write with normal fonts, the equivalent lists are written in bold. The lists written in normal fonts will be called for convenience principal lists, even if their choice is not canonical.

Figures \( \text{[8]-[10]} \) show the 64 admissible lists with \( \hat{8} = 1+ \). The first and the last are deduced one from the other by \( \pm 1 \). The 15 lists with \( \hat{5} = 6+ \) are mapped onto the 15 lists with \( \hat{3} = 4+ \) by \( +3 \). The group of 6 lists with \( \hat{2} = 1- \) splits into two subgroups that are mapped one onto the other by the symmetry 15. The group of 20 lists with \( \hat{4} = 3- \) splits into two subgroups that are mapped one onto the other by 26. The group of 6 lists with \( \hat{6} = 5- \) splits into two subgroups that are mapped one onto the other by 37. Up to the action of \( D_8 \), there are 32 admissible lists with \( \hat{8} = 1+ \). Set now \( \hat{8} = 1- \). First thing we rule out the orbit of \( \hat{8} = 1+ \), that is to say any list which is mapped by some element of \( D_8 \) onto a list with \( \hat{8} = 1+ \). In other words, we set: \( \hat{1} \neq 2\pm, 8\pm, \hat{2} \neq 3\pm, 1\pm, 3 \neq 4\pm, 2\pm, 4 \neq 5\pm, 3\pm, \hat{5} \neq 6\pm, 4\pm, 6 \neq 7\pm, 5\pm \) and \( \hat{7} \neq 8\pm, 6\pm \). Figure \( \text{[11]} \) shows the 6 new admissible lists obtained. They split into 2 groups that are mapped one onto the other by 15. For \( \hat{8} = 2+ \) we rule out the orbits of \( \hat{8} = 1\pm \), that is we set: \( \hat{1} \neq 2\pm, 8\pm, \hat{2} \neq 3\pm, 1\pm, 3 \neq 4\pm, 2\pm, 4 \neq 5\pm, 3\pm, \hat{5} \neq 6\pm, 4\pm, 6 \neq 7\pm, 5\pm \) and \( \hat{7} \neq 8\pm, 6\pm \). Figure \( \text{[12]} \) shows the 4 new admissible lists with \( \hat{8} = 2+ \). They split into 2 groups that are mapped one onto the other by 15. Set now \( \hat{8} = 2- \). Once we have ruled out the orbits of \( \hat{8} = 1\pm, 2+, \hat{2} \) we get the 13 new admissible lists shown in Figure \( \text{[13]} \). Some of them are deduced from each other by \( \pm 2, 15 \).
or 26 so that there are only 8 new orbits. Let \( \hat{8} = 3+ \), once we have ruled out the orbits of \( \hat{8} = 1\pm, 2\pm \), we find no new admissible lists. For \( \hat{8} = 3- \), after excluding the orbits of \( \hat{8} = 1\pm, 2\pm, 3+ \), we get the 8 new admissible lists shown in Figure 14. They split into 2 groups that are mapped one onto the other by \( \pm 2 \). At last, let \( \hat{8} = 4+ \), once we have excluded the orbits of \( \hat{8} = 1\pm, 2\pm, 3\pm \), we find no new admissible lists.

The total number of admissible orbits is 49. The 16 lists in each orbit are all distinct, hence there are in total 784 admissible lists.

### 2.3 Extremal lists

Consider a configuration of eight points \( 1, \ldots, 8 \) lying in convex position in the plane. Any piece of information \( \hat{P} = n\pm \), with \( P \in \{1, \ldots, 8\} \) is equivalent to a statement of the form \( F < C_2 \) and \( G > C_2 \), where \( F, G \) are two points among \( 1, \ldots, 8 \), different from \( P \). For example, \( \hat{8} = 1+ \) if and only if \( 7 < \{23456\} \) and \( 1 > 23456 \). The correspondences for \( P = 8 \) are indicated with bold types in the tabulars of Figures 37-38, the other cases are easily deduced from these by the action of \( D_8 \). We say that a list is maximal (or minimal) for some \( F \in 1, \ldots, 8 \) if \( F \) lies outside (or inside) of all the conics determined by five of the other seven points. Say \( F = 8 \), then for each possible \( \hat{8} = n\pm \), there is one maximal and one minimal list. The maximal list \( \max(\hat{8} = n\pm) \) is obtained from the given diagram \( \hat{8} = n\pm \) taking for all \( \hat{m} \) with \( m = 1, \ldots, 7 \) the labels above the first column of horizontal arrows. The minimal list \( \min(\hat{8} = n\pm) \) is obtained from the given diagram \( \hat{8} = n\pm \) taking for all \( \hat{m} \) with \( m = 1, \ldots, 7 \) the labels above the last column of horizontal arrows. These notations are
changes. During the motion A,\,B,\,C,D,E,F,G,H
five points among points fixed and preserving the convex position. When A
then B,\,C,\,D,\,E,\,F,\,G,\,H that separate A
The distance \( \hat{A} \), where \( \hat{A} \) in the cyclic ordering. Let (A,...,H)
L\( \hat{A} \) \( \L_2 \) and \( \L_2 \) \( \min(\hat{1} = 8+), \L_{32} = \max(\hat{1} = 8+), \L_{48} = \max(\hat{1} = 6+), \L_{56} = \max(\hat{1} = 6+), \L_{64} = \max(\hat{1} = 2+), \L_{65} = \max(\hat{1} = 7+), \L_{66} = \max(\hat{1} = 7+), \L_{67} = \max(\hat{1} = 5-), \L_{71} = \min(\hat{1} = 4+), \L_{72} = \min(\hat{1} = 4-).

2.4 Another encoding for the lists and the orbits

For the lists \( L(1,\ldots,8) \), we define the distance between consecutive points A, B in the cyclic ordering. Let (A,...,H) = (1,...,8) up to some cyclic permutation. The distance \( A \to B \) is the number of conics passing through five points among C, D, E, F, G, H that separate A from B, multiplied by \(-1\) if A is the outermost of the two points. If \( \hat{A} = B \), \( \hat{B} = A \) or \( \hat{A} = B = N \epsilon \), with \( N \in \{ C, \ldots, H \} \), then \( A \to B = 0 \). Let now \( A \neq B \). We move A towards B, leaving the other points fixed and preserving the convex position. When A crosses a conic through five points among C, D, E, F, G, H, a pair of lists \( \hat{B}, \tilde{N} \), \( N \in \{ C, D, E, F, G, H \} \) changes. During the motion \( A \to B \), the list \( \hat{B} \) percourses (in one direction or
the other) a piece of one of the complete chains \( \hat{B} \pm (A) \), see Figure 8.

The upper and the lower chain correspond respectively to the case \( H < CDEFG \) and \( H > CDEFG \).

To determine the invariant \( A \rightarrow B \), we proceed as in the following example:

let \( \hat{A} = G^+ \) and \( \hat{B} = A^+ \). One has \( H < CDEFG, CDEFH > B > CDEGH \) and \( A > CDEFG \). Thus, \( A \rightarrow B = -2 \). The tabulars of Figures 15-16 give the values of these invariants for all of the 49 successive principal lists \( L_n \), and their sum \( \sigma \). Note that the 49 octuples \( (1 \rightarrow 2, \ldots, 8 \rightarrow 1) \) listed in Figures 15-16 are all different. Furthermore, their orbits under the action of \( D_8 \) are also all distinct. Observing the tabulars allows to derive the following:

**Proposition 3** Any list \( L(1, \ldots, 8) \) is determined by the octuple \( (1 \rightarrow 2, 2 \rightarrow 3, \ldots, 8 \rightarrow 1) \). Each orbit is determined by an octuple of integer numbers (ranging between \(-6\) and \(+6\)), defined up to cyclic permutation, and reversion with change of all signs. The absolute value \( |\sigma| \) is an invariant of the orbits, that takes all of the even values between 0 and 8. The numbers of orbits realizing \( |\sigma| = 0, 2, 4, 6, 8 \) are respectively: 16, 12, 16, 4 and 1.

We have thus a new encoding for the lists and the orbits.

Denote by \([XY]\) and \([XY']\) the segments \( XY \), the former belonging to the convex hull of the eight points. Note that in the motion \( A \rightarrow B \), the point \( A \) may also cross some conic \( C_2 \) determined by \( B \) and four other points among \( C, D, E, F, G, H \). So, one does not control the changes of the whole list \( L(1, \ldots, 8) \). In particular, let \( C_2 \) be one of the conics \( BDEFG, BDEFH, BDEGH, BDFGH \) or \( BEFGH \). If \( A, C \) lie both outside of \( C_2 \), and the second intersection of \( C_2 \) with the line \( BC \) is a point of \([BC']\), then any path \( A \rightarrow B \) must cross \( C_2 \).

3 **Link between lists and pencils**

3.1 **Isotopies of octuples of points**

The octuples of points \((1, \ldots, 8)\) form a stratified space \( \Sigma \) whose walls are formed by the configurations of points such that six of them lie on a conic.

**Proposition 4** The octuples of points realizing a given list \( L(1, \ldots, 8) \) lie all in the same chamber of \( \Sigma \).
|   | 8+ 8+ 8+ 8+ 8+ 8+ 8+ 8+ |
|---|---------------------------|
| 2 | 8+ 8+ 8+ 8+ 8+ 8+ 8+ 8+ |
| 3 | 8+ 8+ 8+ 8+ 8+ 8+ 8+ 8+ |
| 4 | 8+ 8+ 8+ 8+ 6− 6− 6+ 3− |
| 5 | 8+ 8+ 6− 6+ 6− 6+ 6+ 3− |
| 6 | 8+ 5− 5− 5+ 5− 5+ 3− 3− |
| 7 | 8+ 5− 5+ 5+ 3− 3− 3− 3− |

|   | 8+ 8+ 8+ 8+ 8+ 8+ 8+ 8+ |
|---|---------------------------|
| 2 | 6− 6− 6− 6− 6− 6− 6+ 4− 4+ |
| 3 | 6− 6− 6+ 3− 6+ 3− 3− 3− 3+ |
| 4 | 6− 6+ 6+ 3− 6+ 3− 3+ 3+ 3+ |
| 5 | 6− 5+ 3− 3− 3+ 3+ 3+ 3+ 3+ |
| 7 | 3+ 3+ 3+ 3+ 3+ 3+ 3+ 3+ 3+ |

|   | 8+ 8+ 8+ 8+ 8+ 8+ 8+ 8+ |
|---|---------------------------|
| 2 | 6+ 6+ 6+ 6+ 4− 4− 4− 4+ 1− |
| 3 | 6+ 6+ 4− 4+ 4− 4+ 4+ 1− |
| 4 | 6+ 3− 3− 3+ 3− 3+ 3− 1− 1− |
| 5 | 6+ 3− 3+ 3+ 3− 3− 1− 1− 1− |
| 6 | 1− 1− 1− 1− 1− 1− 1− 1− 1− |
| 7 | 1− 1− 1− 1− 1− 1− 1− 1− 1− |

Figure 9: \( \hat{\delta} = 1+ \), lists \( L_1, \ldots, L_{32} \)
Figure 10: $\hat{8} = 1+$, lists $L_{33}, \ldots L_{64}$

\[
\begin{array}{cccccccccccccccccccc}
\hat{1} & 6- & 6- & 6- & 6- & 6- & 6- & 6- & 6-\\
\hat{2} & 6- & 6- & 6- & 6- & 6- & 6- & 6- & 6-\\
\hat{3} & 6- & 6- & 6- & 6- & 6+ & 6+ & 4- & 4+\\
\hat{4} & 6- & 6- & 6+ & 3- & 6+ & 3- & 3- & 3+\\
\hat{5} & 6- & 6+ & 6+ & 3- & 6+ & 3- & 3+ & 3+\\
\hat{6} & 5- & 5+ & 3- & 3- & 3+ & 3+ & 3+ & 3+\\
\hat{7} & 1+ & 1+ & 1+ & 1+ & 1+ & 1+ & 1+ & 1+
\end{array}
\]

Figure 11: $\hat{8} = 1-$, lists $L_{65}, \ldots L_{70}$

\[
\begin{array}{cccccccccccccccccccc}
\hat{1} & 6+ & 6+ & 6+ & 6+ & 6+ & 6+ & 6+ & 6+\\
\hat{2} & 6+ & 6+ & 6+ & 6+ & 4- & 4- & 4+ & 1-\\
\hat{3} & 6+ & 6+ & 4- & 4+ & 4- & 4+ & 4+ & 1-\\
\hat{4} & 6+ & 3- & 3- & 3+ & 3- & 3+ & 1- & 1-\\
\hat{5} & 6+ & 3- & 3+ & 3+ & 1- & 1- & 1- & 1-\\
\hat{6} & 1+ & 1+ & 1+ & 1+ & 1+ & 1+ & 1+ & 1+\\
\hat{7} & 1+ & 1+ & 1+ & 1+ & 1+ & 1+ & 1+ & 1+
\end{array}
\]

\[
\begin{array}{cccccccccccccccccccc}
\hat{1} & 4- & 4- & 4- & 4- & 4+ & 4+ & 2- & 2+\\
\hat{2} & 4- & 4- & 4+ & 1- & 4+ & 1- & 1- & 1+\\
\hat{3} & 4- & 4+ & 4+ & 1- & 4+ & 1- & 1+ & 1+\\
\hat{4} & 3- & 3+ & 1- & 1- & 1+ & 1+ & 1+ & 1+\\
\hat{5} & 1+ & 1+ & 1+ & 1+ & 1+ & 1+ & 1+ & 1+\\
\hat{6} & 1+ & 1+ & 1+ & 1+ & 1+ & 1+ & 1+ & 1+\\
\hat{7} & 1+ & 1+ & 1+ & 1+ & 1+ & 1+ & 1+ & 1+
\end{array}
\]

\[
\begin{array}{cccccccccccccccccccc}
\hat{1} & 7- & 7+ & 5- & 5+ & 3- & 3+ & 3+\\
\hat{2} & 1+ & 1+ & 1+ & 1+ & 1+ & 1+ & 1+\\
\hat{3} & 1+ & 1+ & 1+ & 1+ & 1+ & 1- & 1-\\
\hat{4} & 1+ & 1+ & 1+ & 1+ & 1- & 1- & 1-\\
\hat{5} & 1+ & 1+ & 1+ & 1- & 1- & 1- & 1-\\
\hat{6} & 1+ & 1+ & 1- & 1- & 1- & 1- & 1-\\
\hat{7} & 1+ & 1- & 1- & 1- & 1- & 1- & 1-
\end{array}
\]

Figure 11: $\hat{8} = 1-$, lists $L_{65}, \ldots, L_{70}$
\[
\hat{1} \, 4+ \ 4- \ 6+ \ 6-
\hat{2} \, 8- \ 8- \ 8- \ 8-
\hat{3} \, 8- \ 8- \ 8- \ 8-
\hat{4} \, 2+ \ 8- \ 8- \ 8-
\hat{5} \, 2+ \ 2+ \ 8- \ 8-
\hat{6} \, 2+ \ 2+ \ 2+ \ 8-
\hat{7} \, 2+ \ 2+ \ 2+ \ 2+
\]

Figure 12: \( \hat{8} = 2+ \), lists \( L_{71}, \ldots, L_{74} \)

\[
\hat{1} \, 4+ \ 4+ \ 6- \ 4+ \ 4+ \ 4- \ 6+
\hat{2} \, 8+ \ 4- \ 6+ \ 6- \ 6+ \ 8+ \ 8+
\hat{3} \, 8- \ 8- \ 8- \ 8- \ 8- \ 8- \ 8-
\hat{4} \, 2+ \ 2+ \ 2+ \ 2+ \ 2+ \ 2+ \ 2+
\hat{5} \, 2+ \ 2- \ 8- \ 2+ \ 2- \ 2+ \ 2+
\hat{6} \, 2+ \ 2- \ 2- \ 2- \ 2- \ 2- \ 2+
\hat{7} \, 2+ \ 2- \ 2- \ 2- \ 2- \ 2- \ 2+
\]

Figure 13: \( \hat{8} = 2- \), lists \( L_{75}, \ldots, L_{87} \)

\[
\hat{1} \, 6+ \ 6+ \ 4- \ 6- \ 4- \ 4-
\hat{2} \, 6- \ 6+ \ 4- \ 6- \ 6- \ 6+
\hat{3} \, 8- \ 8- \ 8- \ 8- \ 8- \ 8-
\hat{4} \, 8- \ 8- \ 8- \ 8- \ 8- \ 8-
\hat{5} \, 8- \ 8- \ 2- \ 8- \ 2+ \ 2+
\hat{6} \, 2+ \ 2- \ 2- \ 2- \ 2- \ 2-
\hat{7} \, 2- \ 2- \ 2- \ 2- \ 2- \ 2-
\]

Figure 14: \( \hat{8} = 3- \), lists \( L_{88}, \ldots, L_{95} \)
Figure 15: Distances $A \rightarrow B$ for the principal lists with $\hat{s} = 1+$

| $n$ | $1 \rightarrow 2$ | $2 \rightarrow 3$ | $3 \rightarrow 4$ | $4 \rightarrow 5$ | $5 \rightarrow 6$ | $6 \rightarrow 7$ | $7 \rightarrow 8$ | $8 \rightarrow 1$ | $\sigma$ |
|-----|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|----------|
| 2   | 0                | 0                | 0                | -1               | 0                | 5                | 0                | 4                |          |
| 3   | 0                | 0                | 0                | -1               | 0                | 5                | 0                | 4                |          |
| 4   | 0                | 0                | 0                | 2                | 0                | 4                | 0                | 6                |          |
| 5   | 0                | 0                | -1               | 0                | 0                | -2               | 3                | 0                |          |
| 6   | 0                | 0                | -1               | 1                | 0                | -1               | 3                | 0                |          |
| 7   | 0                | 0                | -2               | 0                | 1                | 0                | 3                | 0                |          |
| 8   | 0                | 0                | -3               | 0                | 0                | 3                | 0                | 0                |          |
| 9   | 0                | 1                | 0                | 0                | 2                | 0                | 2                | 0                |          |
| 10  | 0                | 1                | 1                | 0                | -2               | 0                | 2                | 0                |          |
| 11  | 0                | 1                | -1               | 0                | 1                | -1               | 2                | 0                |          |
| 12  | 0                | 1                | -2               | 0                | 0                | -1               | 2                | 0                |          |
| 13  | 0                | 2                | 0                | 2                | 0                | 2                | 0                | 6                |          |
| 14  | 0                | 2                | -1               | 0                | 1                | 0                | 2                | 0                |          |
| 15  | 0                | 3                | 0                | -1               | 0                | 0                | 2                | 0                |          |
| 16  | -1               | 0                | 0                | 1                | 0                | -3               | 1                | 0                | -2        |
| 17  | -1               | 0                | -1               | 0                | 1                | -2               | 1                | 0                | -2        |
| 18  | -1               | 0                | -2               | 0                | 0                | -2               | 1                | 0                | -4        |
| 19  | -1               | 1                | 0                | 0                | 2                | -1               | 1                | 0                | -2        |
| 20  | -1               | 1                | 1                | 0                | 1                | -1               | 1                | 1                |          |
| 21  | -1               | 2                | 0                | -1               | 0                | -1               | 1                | 0                |          |
| 22  | -2               | 0                | 0                | 3                | 0                | 1                | 0                | 2                |          |
| 23  | -2               | 0                | -1               | 0                | 2                | 0                | 1                | 0                |          |
| 24  | -5               | 0                | 0                | 0                | 0                | 1                | 0                | -4               |          |
| 25  | 0                | 0                | 0                | 1                | 0                | -4               | 0                | 1                | -2        |
| 26  | 0                | 0                | -1               | 0                | 1                | -3               | 0                | 1                | -2        |
| 27  | 0                | 0                | -2               | 0                | 0                | -3               | 0                | 1                | -4        |
| 28  | 0                | 1                | 0                | 2                | -2               | 0                | 1                | 2                |          |
| 29  | 0                | 1                | -1               | 0                | 1                | -2               | 0                | 1                |          |
| 30  | 1                | 0                | 0                | 3                | -1               | 0                | 1                | 2                |          |
| 31  | 4                | 0                | 0                | 0                | -1               | 0                | 1                | -4               |          |
| 32  | 0                | 0                | 0                | 4                | 0                | 0                | 2                | 6                |          |
| 33  | -3               | 0                | 0                | 1                | 0                | 0                | 2                | 0                |          |
| 34  | 0                | 0                | 0                | 0                | 0                | 0                | 6                | 6                |          |

17
Proof: Denote by $h_t$ a homothety centered at any point $P$ different from $1, \ldots, 8$, with rate $t$. The homotheties $h_t, t \in [1, \infty]$ give rise to an isotopy of $1, \ldots, 8$. During this isotopy, the list $L(1, \ldots, 8)$ is preserved, except for the end-point when $1, \ldots, 8$ are all on a line. They are also on some reducible conic. The configuration is thus on the deep stratum formed by the configurations of eight coconic points. Move in this stratum so that the conic becomes non-reducible. With a slight perturbation, we can make the isotopy generic, in other words, it is completely contained in the same chamber except for the end-point. Consider now two configurations realizing the same list. Perform for each of them a generic isotopy till all of the eight points lie on the same conic. Up to an affine transformation mapping one conic onto the other and an isotopy along this conic, we may assume that both isotopies have the same end-point. So we have a path connecting the two configurations and having one single non-generic point. Perturb the path in a neighbourhood of this point. In this neighbourhood, all of the walls intersect pairwise transversally. There are two possible perturbations, one crosses all of the walls twice, the other crosses no wall at all. □

3.2 Nodal lists

All along this section, $1, \ldots, 8$ are eight generic points lying in strictly convex position in $\mathbb{R}P^2$, and $\mathcal{P} = \mathcal{P}(1, \ldots, 8)$ is the pencil of cubics determined by $1, \ldots, 8$. Given two points $X, Y$ among $1, \ldots, 8$, we denote by $(\hat{X}, Y)$ the cubic

| $n$ | $1 \to 2$ | $2 \to 3$ | $3 \to 4$ | $4 \to 5$ | $5 \to 6$ | $6 \to 7$ | $7 \to 8$ | $8 \to 1$ | $\sigma$ |
|-----|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|---------|
| 65  | $-1$      | 0         | 0         | 0         | 0         | $-1$      | 6         | 4         |         |
| 66  | $-2$      | 0         | 0         | 0         | 0         | 1         | 0         | 5         | 4       |
| 67  | $-3$      | 0         | 0         | $-1$      | 0         | 0         | 0         | 4         | 0       |
| 71  | 5         | 0         | 1         | 0         | 0         | 0         | 0         | $-2$      | 4       |
| 72  | 4         | 0         | 0         | $-1$      | 0         | 0         | 0         | $-3$      | 0       |
| 75  | 4         | $-1$      | 1         | 0         | 0         | 0         | 1         | $-1$      | 4       |
| 78  | 3         | $-2$      | 1         | 0         | 0         | $-1$      | 0         | $-1$      | 0       |
| 80  | 3         | $-1$      | 0         | $-1$      | 0         | 0         | 1         | $-2$      | 0       |
| 82  | 1         | $-2$      | 0         | 0         | 1         | $-1$      | 0         | $-3$      | $-4$    |
| 83  | 0         | $-3$      | 0         | 0         | 2         | 0         | 0         | $-3$      | $-4$    |
| 84  | 0         | $-4$      | 0         | $-2$      | 0         | 0         | 0         | $-2$      | $-8$    |
| 86  | 2         | $-2$      | 0         | $-1$      | 0         | $-1$      | 0         | $-2$      | $-4$    |
| 87  | 1         | $-3$      | 0         | $-1$      | 1         | 0         | 0         | $-2$      | $-4$    |
| 88  | $-2$      | 1         | $-1$      | 1         | 0         | 1         | $-1$      | 1         | 0       |
| 90  | $-1$      | 1         | $-1$      | 0         | $-1$      | 1         | $-1$      | 2         | 0       |
| 92  | $-1$      | 0         | $-2$      | 0         | $-1$      | 2         | 0         | 2         | 0       |
| 94  | $-2$      | 0         | $-2$      | 1         | 0         | 2         | 0         | 1         | 0       |

Figure 16: Distances $A \to B$ for the principal lists with $\hat{8} \neq 1+$
of the list $\hat{X}$ with double point at $Y$. Move the eight points, keeping them distinct and strictly convex. The pencil $P$ degenerates into a generic singular pencil $P_{\text{sing}}$ if and only if:

1. the point 9 comes together with one of the points $1, \ldots, 8$, or
2. six of the points $1, \ldots, 8$ come onto a conic.

Note that in case 1, $P_{\text{sing}}$ contains a nodal cubic with node at one of the points $1, \ldots, 8$, this point being also the ninth base point of the pencil.

**Definition 1** The list $L(1, \ldots, 8) = (\hat{1}, \hat{2}, \ldots, \hat{8})$ is nodal if it is realizable by $1, \ldots, 8$ on a nodal cubic with double point at one of these points.

Let $C_3$ be a cubic through $1, \ldots, 8$ with node at one of these points. Up to the action of $D_8$, there are eight possible combinatorial cubics $C_3$, see Figure 17 where the successive cubics are denoted by $(1\pm, 1)_{\text{nod}}, (1-, k)_{\text{nod}}, k = 8, \ldots, 2$. Note that the encoding is consistent with the action of $D_8$.

![Figure 17: The cubics $(1\pm, 1)_{\text{nod}}, (1-, 8)_{\text{nod}}, \ldots (1-, 2)_{\text{nod}}$](image)

**Proposition 5** Up to the action of the group $D_8$, $|L(1, \ldots, 8)_{\text{nodal}}| = 4$. The nodal orbits are the maximal orbits $(8-, 2+), (8-, 2-), (6-, 4+), (6+, 4-)$. Here are representatives of each orbit along with the corresponding nodal cubics.

$L(1, \ldots, 8) = (\hat{1}, \hat{2}, \ldots, \hat{8}) =$

- $\max(\hat{1} = 8-)$, realizable with $(1\pm, 1)_{\text{nod}}, (1-, k)_{\text{nod}}, k = 2, \ldots, 8$,
- $\max(\hat{1} = 8+)$, realizable with $(1\pm, 1)_{\text{nod}}$ and $(1-, 8)_{\text{nod}}$,
- $\max(\hat{1} = 6-)$ and $\max(\hat{1} = 6+)$, both realizable with $(1\pm, 1)_{\text{nod}}$.

**Proof:** Let $1, \ldots, 8$ lie in convex position on some nodal cubic $C_3$, one of these points being the node. Up to the action of $D_8$, we may assume that $C_3$ is one of the cubics $(1\pm, 1)_{\text{nod}}, (1-, k)_{\text{nod}}, k = 2, \ldots, 8$. If $C_3 = (1\pm, 1)_{\text{nod}}$, one has $i \in \{1\pm\}$ for all $i \in \{2, \ldots, 8\}$. Using the two diagrams $8 = 1\pm$ of Figures 39, 40, one finds 14 possibilities for the list $L(1, \ldots, 8)$, namely the maximal lists $\max(\hat{1} = n\pm), n = 2, \ldots, 8$. One has $\hat{1} = 8+ \iff 7 < 23456$ and $8 > 23456$. One can choose the points $2, \ldots, 8$ on the loop of $C_3$ so that this condition is achieved. One has $\hat{1} = 7- \iff 8 < 23456$ and $7 > 23456$. By Bézout’s theorem between 23456 and $(1\pm, 1)_{\text{nod}}$, one cannot choose the points $1, \ldots, 8$ on the loop of this cubic verifying this condition. Finishing this argument with the other possible values of $\hat{1}$, one
finds that \(1, \ldots, 8\) may be chosen on \((1\pm, 1)_{\text{nod}}\) so as to realize the eight lists \(\max(1 = n\pm)\), with \(n = 2, 4, 6, 8\).

Similarly, one proves that \(1, \ldots, 8\) on the cubic \(C_3 = (1-, 8)_{\text{nod}}\) can realize exactly the first two lists \(\max(1 = 8\pm)\); and points \(1, \ldots, 8\) on any cubic \(C_3 = (1-, k)_{\text{nod}}, k \in \{2, \ldots, 7\}\) must realize the first list \(\max(1 = 8-).\)

**Proposition 6** The three conditions hereafter are equivalent:

1. The list \(L(1, \ldots, 8)\) determines the (combinatorial) pencil \(P(1, \ldots, 8)\),
2. The list \(L(1, \ldots, 8)\) is not nodal,
3. \(\forall G \in \{1, \ldots, 8\}, \forall C_3\) cubic of \(\hat{G}\), the position of \(G\) with respect to \(C_3\) is determined by \(L(1, \ldots, 8)\).

**Proof:** \(\neg 2 \Rightarrow \neg 1\): let \(L(1, \ldots, 8)\) be a nodal list. Up to the action of \(D_8\), we may assume it is one of the four lists in Proposition 4. Any one of these lists is realizable with the eight points on a nodal cubic \(C_3 = (1\pm, 1)_{\text{nod}}\). The points \(1, \ldots, 8\) give rise to a singular pencil \(P_{\text{sing}}\) with \(1 = 9\). Perturb the pencil moving 1 away from the node onto the odd component or onto the loop of \(C_3\) (leaving the other seven points fixed). The generic pencil obtained is deduced from \(P_{\text{sing}}\) replacing \((C_3, 1, \ldots, 9)\) by a pair \(C_1^3, C_2^3\). One gets two pairs of distinguished cubics, see Figure 18.

\(2 \Rightarrow 3\): according to Proposition 4, any two configurations of points realizing the same list may be connected by a path inside of their common chamber of \(\Sigma\). As \(L(1, \ldots, 8)\) is not nodal, none of the points \(G\) may cross any cubic of the list \(\hat{G}\).

\(3 \Rightarrow 1\): consider a configuration of points realizing a list \(L(1, \ldots, 8)\) and giving rise to some pencil \(P\). Move the configuration preserving the list, the combinatorial pencil degenerates only if some base point \(A\) among \(1, \ldots, 8\) comes together with 9. But this amounts to say that each point \(G \neq A \in \{1, \ldots, 8\}\) comes onto the cubic \((\hat{G}, A)\).

**Figure 18:** Pairs of distinguished cubics, \((A, B) = (1, 9)\) or \((9, 1)\)
3.3 Pairs of distinguished cubics

Let now 1, . . . , 8 realize any list L(1, . . . , 8). Up to some isotopy replacing the initial configuration 1, . . . , 8 by another one in the same chamber and closer to the deep stratum (configurations of eight coconic points), one may achieve the following condition: for any pair of consecutive points A, B, any one of the 15 conics passing through B but not A, has its second intersection point with the line AB on the segment [AB]. As an immediate consequence, we get the following

**Proposition 7** Let 1, . . . , 8 lie in convex position. Let A, B be two of these eight points, consecutive for the cyclic ordering.

1. If A → B = 0, one may move A towards B till A = B, without degeneration of the list L(1, . . . , 8) inbetween.

2. If A → B = ±n, with n > 0, the points A and B are separated from each other by n of the 6 conics determined by C, D, E, F, G, H. One may move A towards B, till A = B, in such a way that the list L(1, . . . , 8) undergoes exactly n degenerations, corresponding to the n conics.

Furthermore, if the initial list L(1, . . . , 8) is non-nodal, then the combinatorial pencil does not degenerate till A = B (case 1) or till A reaches the first of the n conics (case 2).

In what follows, we call cubic C3 a combinatorial cubic (C3,1, . . . , 8) (the position of 9 is not yet specified). We define an encoding for distinguished cubics, that is consistent with the action of D8. Let (1−, N), N = 3, . . . , 8 be the cubic obtained from (1−, N)nod, shifting N away from the node onto the loop, and let (N, 1−), N = 3, . . . , 7 be the cubic obtained from (1−, N)nod, shifting N away from the node onto the odd component, see Figure 19. For (P, Q) = (3, 4), (4, 5), . . . , (8, 1), denote by (1−, PQ), the cubic obtained from (1−, P)nod (or (1−, Q)nod) shifting the node away from P (or Q) into the interior of the arc PQ. Let (1−, E) (E stands for end) be the cubic that could be defined as (1−, N), with N = 2 or as (1−, PQ), with (P, Q) = (2, 3). We chose the specific notation to avoid double notation for one single cubic type. The cubics encoded hereabove are called **cubics of the family 1−**. At last, denote by (81, L) and (81, C) the first and the second cubic in Figure 20. The letters L and C stand respectively for loop and odd component. In section 5 ahead where we classify the pencils, we will consider combinatorial cubics (C3,1, . . . , 9). To encode them, we use the notation defined here for (C3,1, . . . , 8), enhanced with the position of 9. The cubic C3 = (C3,1, . . . , 8) is divided into 9 successive arcs by the points 1, . . . , 8, X (X is the node). If C3 = (81, L), then 9 lies on the arc XX. If C3 is of the family Ne, percourse C3 starting from N in the direction ϵ, and write which oriented arc contains 9 (see Figures 34, 36). Denote by P the combinatorial pencil of cubics determined by 1, . . . , 8. Move 8 towards 1 till the pencil degenerates, see Proposition 7. The only other base point of the pencil that moves is 9. Let Psing be the singular pencil obtained, let C3, C31 and C32
be the three singular cubics involved (see section 1.2). If the list \( L(1, \ldots, 8) \) is non-nodal, we may recover pairs of distinguished cubics of \( \mathcal{P} \) from the list as explained hereafter.

![Figure 19: Cubics (1−, 7), (7, 1−), (1−, 67) and (1−, E)](image)

1. If \( 8 \to 1 = 0 \), the singular cubic \( C_3 \) (with double point at \( 8 = 1 \)) is identical to both \((\hat{8}, 1)\) and \((1, 8)\). If \( \hat{8} = \hat{1} = 2+ \) or \( 7− \), one does not know a priori whether the double points of these two auxiliary cubics are isolated or not. Therefore, the cubics \( C_1^3, C_2^3 \) may be non-real. If \( \hat{8} = \hat{1} \neq 2+ \) or \( 7− \), the cubics \((8, 1)\) and \((1, 8)\) have each a non-isolated double point. All along the motion \( 8 \to 1 \), the position of \( 1 \) with respect to \((\hat{1}, 8)\) and the position of \( 8 \) with respect to \((\hat{8}, 1)\) are preserved. Using Bezout’s theorem with these auxiliary cubics, one finds out the corresponding pair \( C_1^3, C_2^3 \).

In all of the cases, any one of the four data: cubic \((1, 8)\) enhanced with the position of \( 1 \), cubic \((\hat{8}, 1)\) enhanced with the position of \( 8 \), cubic \( C_1^3 \), and cubic \( C_2^3 \) determines the other three. Either case \( \hat{8} = 1+ \), \( \hat{1} = 8+ \) and \( \hat{8} = 1− \), \( \hat{1} = 8− \) gives rise to three pairs \( C_1^3, C_2^3 \) (these three pairs, along with the corresponding auxiliary cubics, are shown in the upper part of Figure 20).

\[
\begin{align*}
(81, L), (81, C) \\
(8+1), (8−, 78) \\
(1+, 12), (1−, 8)
\end{align*}
\]

Each of the other cases gives rise to two pairs \( C_1^3, C_2^3 \), see the tabular hereafter, where \( N \) ranges from \( 3 \) to \( 6 \). Note that all of these cubics \( C_1^3 \), \( C_2^3 \) are distinguished. The case \( \hat{8} = \hat{1} = 7+ \) is shown in the lower part of Figure 20.

\[
\begin{align*}
\hat{8} = \hat{1} = 7+ & \quad \hat{8} = \hat{1} = 2− & \quad \hat{8} = \hat{1} = N+ & \quad \hat{8} = \hat{1} = N− \\
(7+, 8), (1, 7+) & \quad (2−, 8), (12, C) & \quad (N+, 1), (8, N+) & \quad (N−, 8), (1, N−) \\
(78, C), (7+, 1) & \quad (2−, 1), (8, 2−) & \quad (N+, 8), (1, N+) & \quad (N−, 1), (8, N−)
\end{align*}
\]

2. If \( 8 \to 1 \neq 0 \), one has \( C_3 = C_2 \cup (1P), P \in \{2, \ldots, 7\} \). If \( P = 2 \) and both \( 1 \) and \( 2 \) are outside of \( C_2 = 34567 \), then one does not know a priori whether the double points of the reducible cubic \( C_3 \) are real or complex conjugated. Therefore, the cubics \( C_1^3, C_2^3 \) may be non-real. The points
1, 2 lie both outside of 34567 if and only if \( \hat{8} \in \{1^+, 2^-, 4\pm, 6\pm\} \). The supplementary condition \( 8 < 34567 \) is achieved if and only if \( \hat{1} = 2^+ \). So we rule out the case \( \hat{8} \in \{1^+, 2^-, 4\pm, 6\pm\} \) and \( \hat{1} = 2^+ \). For the other cases, find out \( C_2 \) looking up the sequence \( \hat{1} \pm (8) \). Either cubic \( C_1^3, C_2^3 \) may be obtained from the reducible cubic \( C_3 \) by perturbing one of the double points of \( C_3 \). All of the pairs obtained are distinguished cubics. Indeed, let \( s \) be the intersection of the line \((1P)\) with the interior of \( C_2 \), and let \( s_1, s_2 \) be the two arcs of \( C_2 \) on either side of \((1P)\). The loop of each cubic \( C_1^3, C_2^3 \) is obtained perturbing one of the \( s_i \cup s, i \in \{1, 2\} \), and both \( s_i \cup s \) contain some points among \( 1, \ldots, 8 \). We give an example in Figure 21, with \( \hat{1} = 5^+, \hat{5} = 8^- \), the corresponding pair of distinguished cubics is \((5^-, 8^-), (5^+, 1^-)\). Note that \( \hat{1} = 5^+ \) is part of the chain \( \hat{1} \rightarrow (8) \), and \( \hat{5} = 8^- \) is part of the chain \( 5^- \rightarrow \).

4 Constructions

4.1 Elementary changes and inductive constructions

In this section, we finish the proof of Proposition 2. Let us call elementary change the change induced on a list letting one point among \( 1, \ldots, 8 \) cross a conic determined by five others, in some direction. Up to the action of \( D_8 \), there are 19 possible elementary changes, see Figures 22-23. The first one occurs when \( \hat{8} \) goes to the outside of the conic 34567, and \( 1, 2 \) lie both outside of 34567. If the conic 34567 has no real intersection points with the line 12, the combinatorial pencil does not degenerate. Otherwise, one recovers the same set of eight distinguished cubics after the degeneration. The changes of pairs of distinguished cubics induced by the other elementary changes are shown in Figure 24. Recall that a statement of the form \( 6 \) crosses 12345 from the inside to the outside is equivalent to \( 5 \) crosses 12346 from the outside to the inside. All of the 19 elementary changes but one may be thus interpreted as motions of a point \( X \) towards a consecutive point \( Y \) such that \( X \rightarrow Y \neq 0 \), till \( X \) crosses the first conic separating it from \( Y \). The only exception is the third change:

\[
\hat{1} : 8^- \rightarrow 3^+ \quad \hat{2} : 1^- \rightarrow 1^+ 
\]

By Proposition 7, if a list \( L(1, \ldots, 8) \) is realizable, then any list that may be obtained from \( L(1, \ldots, 8) \) by an elementary change (different from the particular one hereabove) is also realizable. We say that the elementary change is always possible. Consider now a list with \( \hat{1} = 8^-, \hat{2} = 1^- \). Note that \( \hat{1} = 8^- \) belongs to the chain \( \hat{1} \rightarrow (8) \), and \( \hat{2} = 1^- \) belongs to the chain \( \hat{2} \rightarrow (8) \). Using Figure 24 we deduce that \( \hat{8} = 1^- \); using Figure 40 we see that \( \hat{N} = 1^- \) for \( N = 3, \ldots, 7 \). The list is \( \text{max}(\hat{1} = 8^-) \). The point \( 8 \) lies inside of all the conics determined by \( 1 \) and four points among \( 2, \ldots, 7 \). Thus one may let \( 8 \) move towards the conic 34567 and cross it without degenerations inbetween. This elementary change is also
Figure 20: Two examples: $\hat{8} = 1\pm, \hat{1} = 8\pm$, and $\hat{8} = \hat{1} = 7+$

Figure 21: $\hat{1} = 5+, \hat{5} = 8-, \hat{8} \in \{5-, 7\pm\}$
always possible. The total number of elementary changes is 224, see Figures 46-47. The 224 pairs \((N = \hat{N}, M = \hat{M})\) in these Figures are called elementary pairs. Start from a list that is already realized, we will say that we perform the change \((\hat{N}, \hat{M})\) without further precision, as there is no ambiguity possible.

Any principal list may be obtained from an extremal list by some sequence of elementary changes and actions of elements of \(D_8\). See Figure 25 where \(n\) stands for a list \(L_n\) (as in Figures 15-16) and each row is a new sequence whose first list was already realized. These sequences are chosen so as to reach all of the principal lists with the least possible number of starting lists (note that some intermediate lists are also extremal).

This finishes the proof of Proposition 2. \(\Box\)

4.2 Lists obtained perturbing four reducible cubics

Among the principal lists, exactly twelve have a set of four disjoint elementary pairs. The lists \(L_n\) with \(n = 11, 12, 13, 14, 19, 20, 21, 22\), have each disjoint elementary pairs \((\hat{1}, \hat{8}), (\hat{2}, \hat{7}), (\hat{3}, \hat{6}), (\hat{4}, \hat{5})\). Each of the four lists \(L_n\) with \(n = 88, 90, 92, 94\) has disjoint elementary pairs \((\hat{1}, \hat{5}), (\hat{2}, \hat{6}), (\hat{3}, \hat{7}), (\hat{4}, \hat{8})\). We explain hereafter how to realize these lists directly.

Consider a pair of ellipses intersecting at four points, \(1, 3, 5, 7\), see Figure 26. Denote by 9 the intersection of the diagonal lines 15 and 37. Draw a vertical and a horizontal line, both passing through 9. Let \(4, 8, 2, 6\) be four supplementary points chosen such as: \(4, 8\) are the intersections of the vertical line with one ellipse; \(2, 6\) are the intersections of the horizontal line with the other ellipse; each pair of points lying on one ellipse is in the interior of the other. By construction, the points \(1, \ldots, 8\) lie in convex position. Moreover, there exist two supplementary conics passing through six points: 234678 and 125678. As a matter of fact, the pencil of cubics \(P\) determined by \(1, \ldots, 8\) has 9 as ninth base point and four distinguished cubics, all of them reducible: 498 ∪ 123567, 296 ∪ 134578, 397 ∪ 124568 and 195 ∪ 234678. Let us perturb the pencil, moving slightly the points 8 and 9. As \(1, 2, 3, 5, 6, 7\) are on a conic, 8 and 4 must stay aligned. The point 8 leaves the three conics 234678, 125678, 134578. Letting 8 cross each conic from the inside to the outside yields the following elementary moves: Conic 23467: \((\hat{1} = 5-, \hat{5} = 1+) \rightarrow (\hat{1} = 5+, \hat{5} = 1-)\). Conic 12456: \((\hat{3} = 7-, \hat{7} = 3+) \rightarrow (\hat{3} = 7+, \hat{7} = 3-)\). Conic 13457: \((\hat{2} = 7+, \hat{6} = 1-) \rightarrow (\hat{2} = 5-, \hat{6} = 3+)\). Let us now move 3 away from the conic 12567. Letting 3 cross 12567 from the inside to the outside yields the following elementary move: \((4 = 7-, \hat{8} = 5+) \rightarrow (4 = 1+, \hat{8} = 3-)\). The first perturbation may be done as to realize six different positions of 8 with respect to the set of conics 23467, 12456, 13457. Then, move 3 to the outside of 12567. We obtain the first five lists, and the last list of Figure 14 four of these lists are: \(L_{88}, L_{90}, L_{92}\) and \((+2) \cdot L_{94}\).

Consider a pair of ellipses intersecting at four points \(2, 4, 5, 7\), see Figure 27. Denote by 9 the intersection of the lines 27 and 45. Draw a line \(\Delta\) passing through 9 and cutting the ellipses on their arcs 57 and 24. Let 6 and 3 be the intersections of \(\Delta\) with one ellipse, chosen so that these points lie outside of
Figure 22: Elementary changes
\[
\begin{align*}
\hat{1} : 2+ & \rightarrow 2- \\
2 : 1+ & \rightarrow 1- \\
\hat{1} : 3+ & \rightarrow 8- & \hat{1} : 8- & \rightarrow 3+ & \hat{1} : 8- & \rightarrow 3+ \\
2 : 1+ & \rightarrow 1- & \hat{2} : 1- & \rightarrow 1+ & \hat{2} : 8- & \rightarrow 3+ \\
\hat{3} : 1+ & \rightarrow 3+ & \hat{1} : 3+ & \rightarrow 3- & \hat{1} : 2- & \rightarrow 4+ \\
3 : 1+ & \rightarrow 1+ & \hat{3} : 1- & \rightarrow 1+ & \hat{3} : 1+ & \rightarrow 1- \\
\hat{1} : 4+ & \rightarrow 2- & \hat{1} : 4+ & \rightarrow 2- & \hat{1} : 2- & \rightarrow 4+ \\
3 : 1- & \rightarrow 1+ & \hat{3} : 8- & \rightarrow 2+ & \hat{3} : 2+ & \rightarrow 8- \\
\hat{1} : 4+ & \rightarrow 4- & \hat{1} : 4- & \rightarrow 4+ & \hat{1} : 5+ & \rightarrow 3- \\
4 : 1+ & \rightarrow 1- & \hat{4} : 1- & \rightarrow 1+ & \hat{4} : 1+ & \rightarrow 1- \\
\hat{1} : 3- & \rightarrow 5+ & \hat{1} : 3- & \rightarrow 5+ & \hat{1} : 5+ & \rightarrow 3- \\
4 : 1- & \rightarrow 1+ & \hat{4} : 8- & \rightarrow 2+ & \hat{4} : 2+ & \rightarrow 8- \\
\hat{1} : 5- & \rightarrow 5+ & \hat{1} : 4- & \rightarrow 6+ & \hat{1} : 6+ & \rightarrow 4- \\
5 : 1+ & \rightarrow 1- & \hat{5} : 1+ & \rightarrow 1- & \hat{5} : 8- & \rightarrow 2+ \\
\end{align*}
\]

Figure 23: Representants of the 19 orbits of elementary changes

\[
\begin{align*}
(1+, 2) & \rightarrow (1-, E) & (1-, E) & \rightarrow (1+, 2) & (8-, 2) & \rightarrow (23, L) \\
(1, 3+) & \rightarrow (1+, 12) & (1+, 12) & \rightarrow (1, 3+) & (81, L) & \rightarrow (3+, 1) \\
(3, 1+) & \rightarrow (3, 1-) & (3, 1-) & \rightarrow (3, 1+) & (1+, 3) & \rightarrow (1-, 3) \\
(1, 3-) & \rightarrow (1, 3+) & (1, 3+) & \rightarrow (1, 3-) & (12, C) & \rightarrow (1, 4+) \\
(1-, 3) & \rightarrow (1+, 3) & (8-, 3) & \rightarrow (2+, 3) & (2+, 3) & \rightarrow (8-, 3) \\
(1, 4+) & \rightarrow (12, C) & (4+, 1) & \rightarrow (2-, 1) & (2-, 1) & \rightarrow (4+, 1) \\
(4, 1+) & \rightarrow (4, 1-) & (4, 1-) & \rightarrow (4, 1+) & (1+, 4) & \rightarrow (1-, 4) \\
(1, 4+) & \rightarrow (1, 4-) & (1, 4-) & \rightarrow (1, 4+) & (1, 5+) & \rightarrow (1, 3-) \\
(1-, 4) & \rightarrow (1+, 4) & (8-, 4) & \rightarrow (2+, 4) & (2+, 4) & \rightarrow (8-, 4) \\
(1, 3-) & \rightarrow (1, 5+) & (3-, 1) & \rightarrow (5+, 1) & (5+, 1) & \rightarrow (3-, 1) \\
(5, 1+) & \rightarrow (5, 1-) & (1+, 5) & \rightarrow (1-, 5) & (8-, 5) & \rightarrow (2+, 5) \\
(1, 5-) & \rightarrow (1, 5+) & (1, 4-) & \rightarrow (1, 6+) & (6+, 1) & \rightarrow (4-, 1) \\
\end{align*}
\]

Figure 24: Changes of pairs of distinguished cubics
Figure 25: Inductive construction of the principal lists

the second ellipse. Draw a line $\Delta'$ passing through 9 and cutting the second ellipse at two points 8,1 on the arc 72. One may choose $\Delta'$ in such a way that the points 1, ..., 8 lie in convex position. By construction, there exist two supplementary conics passing through six of the points: 123678 and 134568.

The pencil of cubics $P$ determined by 1, ..., 8 has 9 as ninth base point and four distinguished cubics, all of them reducible: $189 \cup 234567$, $369 \cup 124578$, $459 \cup 123678$ and $279 \cup 134568$. Let us perturb the pencil, moving the points 8 and 9. The point 8 leaves the three conics $13456$, $12457$, $12367$. Letting 8 cross each conic from the inside to the outside yields the following elementary moves: Conic $13456$: $(\hat{2} = 6−, \hat{7} = 1−) \rightarrow (\hat{2} = 8+, \hat{7} = 3+)$. Conic $12457$: $(\hat{3} = 6+, \hat{6} = 3+) \rightarrow (3 = 6−, \hat{6} = 3−)$. Conic $12367$: $(\hat{4} = 3−, \hat{5} = 3−) \rightarrow (4 = 6+, \hat{5} = 6+).$ Let us now move 7 away from the conic $23456$. Letting 7 cross each conic from the inside to the outside yields the following elementary move: $(\hat{1} = 8+, \hat{8} = 1+) \rightarrow (\hat{1} = 8−, \hat{8} = 1−).$ The first perturbation may be done so as to realize six different positions of 8 with respect to the set of conics $13456$, $12457$, $12367$. Then, move 7 to the inside of $23456$. We obtain the lists $L_{13}$: move first 7 to the outside of $12458$ and the inside of $12368$; then, move 8 to the outside of $13456$. List $L_{20}$: move first 6 to the outside of $23457$ and the inside of $12378$; then move 8 to the outside of $12457$.

Proposition 8 Let 1, ..., 8 be eight points lying in convex position in the plane and let $k$ be the number of conics passing through exactly 6 of them. One has $k \leq 4$. If $k = 4$, then the points realize, up to the action of $D_8$, one of the two non-generic lists shown in Figures 26, 27. The orbit of the first list has two elements, the orbit of the second list has eight elements.

Proof: Perturbing slightly the configuration 1, ..., 8 must yield a generic list
Figure 26: First configuration of points with 4 reducible cubics

Figure 27: Second configuration of points with 4 reducible cubics
with four distinct elementary pairs, otherwise stated a list that is in the orbit of some of the 12 lists \( L_n \) considered hereabove. Up to the action of \( D_8 \), the original configuration 1, \ldots, 8 must be as shown in Figure 26 or 27. The list \( l \) of Figure 26 is encoded by the data: \( 2, 6 < 134578, 1, 5 > 234678, 3, 7 > 124568 \). This list is invariant by the action of \( \text{id}, 15, 26, 37, 48, +2, -2, +4 \). The list \( l' \) of Figure 27 is encoded by the data: \( 1, 8 > 234567, 5, 4 < 123678, 2, 7 < 134568, 3, 6 > 124578 \). This list is invariant by the action of \((+1)(48)\). \( \square \)

A pencil of cubics with eight base points lying in convex position in the real plane (no 7 of them being coconic) has at most four reducible cubics, the corresponding four lines pass all through the ninth base point. In the next section, we drop the condition of convexity and search for more singular pencils.

4.3 A singular pencil with base points in non-convex position

Let us say that a cubic is \textit{completely reducible} if it is the product of three lines. A complex pencil contains at most four such cubics \([6], [7]\), the corresponding twelve lines and the nine base points are such that: each point lies on four lines and each line passes through three points. Recall that the nine inflection points of a complex cubic \( C_3 \) realize such a configuration. The \textit{Hessian pencil} associated to \( C_3 \) is the pencil generated by \( C_3 \) and its Hessian, based at the inflection points of \( C_3 \). The Hessian pencils realize the upper bound of four completely reducible cubics.

Let us now go back to pencils with only real base points. A pencil with nine real base points cannot have four completely reducible cubics: the Sylvester-Gallai theorem states that given \( n \) points in the real plane, they are either all collinear or there exists a line containing exactly two of them (see e.g.\([1]\)) . Consider the pencil generated by the two completely reducible cubics (one bold, one dotted) in Figure 28, the base points are denoted by \( A, \ldots, I \). We recover an elementary proof of Pascal’s theorem: assume that \( G, H, I \) are on a line, and let \( C_2 \) be a conic passing through five of the other base points. Then, the cubic \((GHI) \cup C_2 \) belongs to the pencil, otherwise stated, \( A, B, C, D, E, F \) are coconic. Conversely, if \( A, \ldots, F \) are coconic, then \( G, H, I \) are aligned. The particular case where \( C_2 \) is the product of two lines is Pappus’ theorem, the pencil here has three completely reducible cubics.

Let us now search for a pencil having the maximal number of six reducible cubics. Each of the corresponding six lines must pass through exactly three base points. If one point lies on four of these lines, then a fifth line intersects these four at four base points, contradiction. If each base point lies on at most two lines, draw five lines, each of them passes through four base points, contradiction again. Thus, one point lies on exactly three lines, six other base points are distributed pairwise on these lines. Draw two further lines, they must intersect at one of these six points, otherwise they would pass each through four base points. We get thus the following distribution of the base points on the
lines and the conics: four points $A, B, C, D$ lie each on three lines and three conics, three points $D, E, F$ lie each on two lines and four conics and the other two points $H, I$ lie each on six conics. Assume that $H, I$ are also real. Up to the action of the symmetric group $S_4$ on the first chosen base points $A, B, C, D$, the sequence of reducible cubics is as shown in Figure 29. Note that the non-singular cubics of the pencil are all disconnected.

Figure 28: Pascal’s and Pappus’ theorems

### 4.4 Symmetrical lists

We proved in section 2.2 that a generic list is preserved by no element of $D_8$, otherwise stated, each orbit contains 16 lists. Let us now search for the non-generic lists that are invariant for some element of $D_8$, we met two of them, $l$ and $l'$, in section 4.2. Let $\lambda_0$ be the list 12345678 consisting of eight conic points. In what follows, $a$ stands for a cyclic permutation, and $\sigma$ for a symmetry 15, 26, 37 or 48. Recall that any two cyclic permutations commute, whereas $a\sigma = \sigma(-a)$. Let $m$ be a cyclic permutation, and $L$ be a list such that $m·L = L$. Any list $L'$ in the orbit of $L$ is also invariant under the action of $m$ and $-m$. Up to the action of $D_8$, we may assume that $L$ contains a conic $C_2 = 123456$, 123457, 123467 or 123567. Moreover, $L$ contains the images of this conic by $m$, $m\circ m \ldots$ and if $P < C_2$, then $m·P < m·C_2$. One finds out easily that $(\pm 1)$, $(\pm 3)$ preserve only $\lambda_0$, whereas $(\pm 2)$, $(\pm 4)$ preserve each three lists: $\lambda_0$, $l$ and $(+1)l$. A list invariant by $(+1)(48)$ must contain the conics 124578, 134568, 123678 and 234567. Thus, $(+1)(48)$ preserves three lists: $\lambda_0$, $l'$ and $(+4)l' = (1)(26)l'$. A list invariant by 15 must contain a conic 234678. Let us first look for the almost generic lists, having no other configuration of six conic points. Let $L$ be such a list, the orbit of $L$ contains eight elements, each symmetry $\sigma$ leaves two of them invariant. If $\sigma·L' = L'$, one may perturb $L'$ in two ways to get generic lists that are deduced from one another by $\sigma$. The 16 generic lists thus obtained form an orbit of $D_8$. We select all of the principal lists having an elementary pair $(N, M)$, with $(N, M) = (1, 5), (2, 6), (3, 7)$ or $(4, 8)$, and such
Figure 29: Pencil with six reducible cubics
that the non-generic list obtained making the six points different from $N, M$\nonumber \coconic is invariant by the symmetry $NM$. Up to cyclic permutations, we get thus 24 almost generic lists invariant by 15, they split in two groups of 12 that are deduced from one another by the action of $+4$ (or 37). These lists may be obtained also as perturbations of more singular lists invariant by 15. The complete set of lists invariant by 15 is displayed hereafter and in Figures 30-32.

1. If 1 or 5 lies on $234678$

(a) One list with eight coconic points: $\lambda_0 = 12345678$

(b) Four lists with seven coconic points: $5 < 1234678, 5 > 1234678, 1 < 2345678, 1 > 2345678$.

2. If $1 > 234678$ and $5 < 234678$

(a) Four almost generic lists $l_1, l_2, l_3, l_4$

(b) Three lists $\lambda_1, \lambda_2, \lambda_3$ having each two more configurations of six coconic points:

\begin{align*}
\lambda_1 : & \quad 123456, 145678; \\
\lambda_2 : & \quad 123457, 135678; \\
\lambda_3 : & \quad 123458, 125678
\end{align*}

3. If $1 < 234678$ and $5 > 234678$

(a) Four almost generic lists $l'_1 = (+4)(l_i)$

(b) Three lists $\lambda'_1 = (+4)(\lambda_i)$ having each two more configurations of six coconic points:

\begin{align*}
\lambda'_1 : & \quad 123458, 125678; \\
\lambda'_2 : & \quad 123457, 135678; \\
\lambda'_3 : & \quad 123456, 145678
\end{align*}

4. If $1 > 234678$ and $5 > 234678$

(a) Eight almost generic lists $l_5, l_6, l_7, l_8, l'_5, l'_6, l'_7, l'_8$

(b) Four lists having each one more configuration of six coconic points:

\begin{align*}
\lambda_4 : & \quad 123578; \\
\lambda_5 : & \quad (+4)(L_4) : 134567; \\
\lambda_6, \lambda_7 : & \quad (+4)(L_6) : 124568
\end{align*}

(c) Two lists having each two more configurations of six coconic points:

$\lambda_8, \lambda_9 = (+4)(\lambda_8): 123567, 134578$

(d) One list with three more configurations of six coconic points:

$\lambda_{10} = l : 124568, 134578, 123567$

5. $1 < 234678$ and $5 < 234678$

(a) Eight almost generic lists $l_9, l_{10}, l_{11}, l_{12}, l'_9, l'_ {10}, l'_ {11}, l'_ {12}$

(b) Four lists having each one more configuration of six coconic points:
\[
\begin{array}{c|cccc}
\hline
 & l_1 & l_2 & l_3 & l_4 \\
\hline
1 & 6+ \leftrightarrow 4- & 6+ \leftrightarrow 4- & 6+ \leftrightarrow 4- & 6+ \leftrightarrow 4- \\
2 & 1- & 4- & 4+ & 4- \\
3 & 1- & 4+ & 4+ & 4- \\
4 & 1- & 3+ & 1- & 3- \\
5 & 1- \leftrightarrow 1+ & 1- \leftrightarrow 1+ & 1- \leftrightarrow 1+ & 1- \leftrightarrow 1+ \\
6 & 1+ & 7- & 1+ & 7+ \\
7 & 1+ & 6- & 6- & 6+ \\
8 & 1+ & 6+ & 6- & 6+ \\
\hline
\end{array}
\]

\[
\begin{array}{c|cccc}
\hline
 & l'_1 & l'_2 & l'_3 & l'_4 \\
\hline
1 & 5- \leftrightarrow 5+ & 5- \leftrightarrow 5+ & 5- \leftrightarrow 5+ & 5- \leftrightarrow 5+ \\
2 & 5+ & 3- & 5+ & 3+ \\
3 & 5+ & 2- & 2- & 2+ \\
4 & 5+ & 2+ & 2- & 2+ \\
5 & 2+ \leftrightarrow 8- & 2+ \leftrightarrow 8- & 2+ \leftrightarrow 8- & 2+ \leftrightarrow 8- \\
6 & 5- & 8- & 8+ & 8- \\
7 & 5- & 8+ & 8+ & 8- \\
8 & 5- & 7+ & 5- & 7- \\
\hline
\end{array}
\]

Figure 30: Almost generic symmetrical lists (first)

\[
\lambda_{11} : 123578; \quad \lambda_{12} = (+4)(\lambda_{11}) : 134567; \\
\lambda_{13}, \lambda_{14} = (+4)(\lambda_{13}) : 124568
\]

(c) Two lists having each two more configurations of six coconic points:

\[
\lambda_{15}, \lambda_{16} = (+4)(\lambda_{15}) : 123567, 134578
\]

(d) One list having three supplementary configurations of six coconic points:

\[
\lambda_{17} = (+1)(\lambda_{10}) = (+1)(l) : 124568, 123567, 134578
\]
### Figure 31: Almost generic symmetrical lists (second)

| $l_5$ | $l_6$ | $l_7$ | $l_8$ |
|-------|-------|-------|-------|
| 1     | 5− ↔ 5+ | 5− ↔ 5+ | 5− ↔ 5+ | 5− ↔ 5+ |
| 2     | 1+    | 7+    | 7+    | 7−    |
| 3     | 1+    | 7−    | 7+    | 1+    |
| 4     | 1+    | 1+    | 1+    | 1+    |
| 5     | 1+ ↔ 1− | 1+ ↔ 1− | 1+ ↔ 1− | 1+ ↔ 1− |
| 6     | 1−    | 1−    | 1−    | 1−    |
| 7     | 1−    | 3+    | 3−    | 1−    |
| 8     | 1−    | 3−    | 3−    | 3+    |

| $l'_5$ | $l'_6$ | $l'_7$ | $l'_8$ |
|--------|--------|--------|--------|
| 1     | 5− ↔ 5+ | 5− ↔ 5+ | 5− ↔ 5+ | 5− ↔ 5+ |
| 2     | 5−    | 5−    | 5−    | 5−    |
| 3     | 5−    | 7+    | 7−    | 5−    |
| 4     | 5−    | 7−    | 7−    | 7+    |
| 5     | 1+ ↔ 1− | 1+ ↔ 1− | 1+ ↔ 1− | 1+ ↔ 1− |
| 6     | 5+    | 3+    | 3+    | 3−    |
| 7     | 5+    | 3−    | 3+    | 5+    |
| 8     | 5+    | 5+    | 5+    | 5+    |

### Figure 32: Almost generic symmetrical lists (third)

| $l'_9$ | $l'_{10}$ | $l'_{11}$ | $l'_{12}$ |
|--------|-----------|-----------|-----------|
| 1     | 6+ ↔ 4−  | 6+ ↔ 4−  | 6+ ↔ 4−  | 6+ ↔ 4−  |
| 2     | 8+    | 6−    | 6+    | 8−    |
| 3     | 8−    | 8+    | 8+    | 8−    |
| 4     | 8−    | 8−    | 8+    | 8−    |
| 5     | 8− ↔ 2+ | 8− ↔ 2+ | 8− ↔ 2+ | 8− ↔ 2+ |
| 6     | 2+    | 2+    | 2−    | 2+    |
| 7     | 2+    | 2−    | 2−    | 2+    |
| 8     | 2−    | 4+    | 4−    | 2+    |

| $l''_9$ | $l''_{10}$ | $l''_{11}$ | $l''_{12}$ |
|---------|------------|------------|------------|
| 1     | 6+ ↔ 4−  | 6+ ↔ 4−  | 6+ ↔ 4−  | 6+ ↔ 4−  |
| 2     | 6+    | 6+    | 6−    | 6+    |
| 3     | 6+    | 6−    | 6−    | 6+    |
| 4     | 6−    | 8+    | 8−    | 6+    |
| 5     | 8− ↔ 2+ | 8− ↔ 2+ | 8− ↔ 2+ | 8− ↔ 2+ |
| 6     | 4+    | 2−    | 2+    | 4−    |
| 7     | 4−    | 4+    | 4−    | 4−    |
| 8     | 4−    | 4−    | 4+    | 4−    |

35
5 Classification of the pencils of cubics

Denote by $\mathcal{P}(1,\ldots,8)$ the number of generic combinatorial pencils of cubics with eight base points in convex position in the plane.

Proposition 9 Up to the action of $D_8$, $\mathcal{P}(1,\ldots,8) = 45$

Proof: Let us call nodal pencil a pencil corresponding to a nodal list. Up to the action of $D_8$, there are four nodal lists, see Proposition 5. It follows from this proposition that the first list gives rise to nine pencils, that are deduced from each others switching 9 with the other base points. The second list gives rise to three pencils obtained from each others by the swaps $9 \leftrightarrow 1$, $9 \leftrightarrow 8$; each of the last two lists gives rise to two pencils obtained one from another by the swap $9 \leftrightarrow 1$.

Let $T$ be the triangle $(12)$, $(67)$, $(17)$ containing 8. The condition that $1,\ldots,8$ realizes the list $\max(\hat{1} = 8 -)$ splits into eight disjoint subconditions. There is an ordering $14567 > (\hat{8},1) > (\hat{8},7) > (\hat{8},6) > \ldots > (\hat{8},2) > 34567$ in $T$: the point 8 lies between two consecutive of these curves. When 8 lies on a cubic $(\hat{8},N)$, otherwise stated, when the eight chosen base points are on a cubic $(1-,N)$ nod, the pencil is singular, with $9 = N$. By Bezout’s theorem with the cubics $(\hat{8},N)$, the degeneration $9 = 8$ may occur only if 8 lies between $(\hat{8},1)$ and $(\hat{8},7)$. Using the method exposed in section 3.3, one gets thus the nine pencils of Figure 33 with $(G,H,I,A,B,C,D,E,F) = (9,2,3,4,5,6,7,8,1), (1,2,3,4,5,6,7,8,9), (1,2,3,4,5,6,7,9,8), (1,2,3,4,5,6,9,7,8), (1,2,3,4,5,9,6,7,8), (1,2,3,4,9,5,6,7,8), (1,2,3,9,4,5,6,7,8), (1,2,9,3,4,5,6,7,8), (1,2,3,4,5,6,7,8)$.

If 8 lies outside of the loop of $(\hat{8},1)$, then one gets the pencil with $G = 9$. If 8 lies between $(\hat{8},1)$ and $(\hat{8},7)$, one gets the next two pencils with $(E,F) = (8,9)$ and $(9,8)$. The other positions of 8 give rise to the other pencils, switching successively 9 with 7, 6, \ldots, 2.

We say that an elementary change is possible for a pencil if one may move the base points till the change occurs, without degeneration of the pencil inbetween. The elementary change $\hat{1} : 8- \rightarrow 8+, \hat{8} : 1- \rightarrow 1+$ (7 enters the conic 23456, 1, 8 outside of 23456) is possible only for the first three nodal pencils corresponding to $\max(\hat{1} = 8-)$, by Proposition 5. This elementary change leaves the sequence of distinguished cubics unchanged. The nodal list with $\hat{1} = 8+$ is thus realizable by the three pencils of Figure 33 with

$(G,H,I,A,B,C,D,E,F) = (9,2,3,4,5,6,7,8,1), (1,2,3,4,5,6,7,8,9), (1,2,3,4,5,6,7,9,8)$

The elementary change $\hat{1} : 8+ \rightarrow 6-, \hat{7} : 1- \rightarrow 1+$ (8 enters the conic 23456, 7 inside and 1 outside of 23456) is possible only for the first
two pencils. For both, perform the change of pairs: $(1-,7), (81, C) \to (1+,7), (1,6-)$, one gets the two pencils corresponding to the nodal list \(\max(\hat{1} = 6-\) \). The elementary change \(1 : 6- \to 6+, 6 : 1- \to 1+ \) (8 enters the conic 23457, 1, 6 outside of 23457) may be performed on both previous pencils replacing the pair $(6,1-), (1,6-)$ by the pair $(6,1+), (1,6+)$, the new pencils obtained realize the nodal list \(\max(\hat{1} = 6+\) \). In Figures 34-36, we classify the pencils of cubics with eight base points lying in convex position. Each pencil is described as a sequence of eight successive combinatorial distinguished cubics \(C_3(1,\ldots,9\), encoded as explained in section 3.3. The upper nine pencils displayed in Figure 34 are the nodal pencils obtained from the list \(\max(\hat{1} = 8-\) \) = \(L_{64}\). The first three of them correspond also to the list \(\max(\hat{1} = 8+\) \) = \(L_{32}\). The next two pencils after the blank line correspond to the list \(\max(\hat{1} = 6-\) \) = \(L_{48}\). The last two pencils after the second blank line correspond to the list \(\max(\hat{1} = 6+\) \) = \(L_{56}\).

Our next concern is to count the pencils of cubics up to the action of \(D_8\) and construct representants of each equivalence class. Let us call non-essential changes the elementary changes that leave the pencils unchanged (the orbit of the first change in Figures 22-23). We have already established that the nodal lists give rise to 13 orbits of pencils, see Figure 34. Two lists obtained from one another by a non-essential change must be both nodal or both non-nodal. For any non-nodal list \(L_n\), denote by \(\mathcal{P}_n\) the corresponding pencil of cubics. Hereafter, \((5,6)\) stand always for the same elementary change \((\hat{5} : 6- \to 6+, 6 : 5- \to 5+)\); and \((1,8)\) stands for \((\hat{1} : 8+ \to 8-, 8 : 1+ \to 1-\)\). One has:

\[
\begin{align*}
(5,6) \cdot L_3 &= L_4, \\
26 \circ (+1)(48) \circ (1,8) \cdot L_4 &= (+3) \circ (1,8) \cdot L_4 = L_{15}, \\
(5,6) \circ 37 \cdot L_3 &= (5,6) \cdot L_{17} = L_{18}.
\end{align*}
\]

Thus \(\mathcal{P}_4 = \mathcal{P}_3; \mathcal{P}_{15} = (+3) \cdot \mathcal{P}_4; \mathcal{P}_{18} = 37 \cdot \mathcal{P}_3\). One has:
\((\hat{5}, \hat{6}) \cdot L_5 = L_6,\)
\((\hat{5}, \hat{6}) \circ 37 \cdot L_5 = (\hat{5}, \hat{6}) \cdot L_9 = L_{10},\)
\(26 \circ (+1)(48) \circ (1, \hat{8}) \cdot L_6 = (+3) \circ (1, \hat{8}) \cdot L_6 = L_{23}.\)

Thus, \(P_6 = P_5; \ P_{10} = 37 \cdot P_5; \ P_{23} = (+3) \cdot P_6.\) One has \((\hat{5}, \hat{6}) \circ 37 \cdot L_2 = L_{34}.\)
Thus, \(P_{34} = 37 \cdot P_2.\) Finally, \((+1)(48) \circ (1, \hat{8}) \cdot L_6 = L_m\) for \((n, m) = (7, 26), (8, 25), (11, 22), (12, 21), (13, 20), (14, 19).\) Thus \(P_m = (+1)(48) \cdot P_n.\) The non-nodal principal lists split into two subsets: 28 lists with \(\hat{8} = 1+\) and 17 lists with \(\hat{8} \neq 1+.\) The first set gives rise to 15 orbits of pencils. In the second set, there is a one-to-one correspondence between the lists and the equivalence classes of pencils, see Figures 35-36. There are in total \(13 + 15 + 17 = 45\) equivalence classes of pencils.

To construct the non-nodal pencils in the easiest way, we may follow the sequences of elementary changes from Figure 25 with four starting lists: \(L_2, L_{65}, L_{71}, L_{88}.\) Construct directly the starting pencils \(L_{65}, L_{71}, L_{88}\) using the method exposed in section 3.3. To get the starting pencil \(L_2\) in the shortest way, observe that the list \(L_2\) is obtained from the (non-principal and nodal) list \(L_1\) by an elementary move \((\hat{6}, \hat{7}).\) The list \(L_1 = \max(\hat{8} = 1+) = (+1)(48) \max(\hat{1} = 8-)\) is realizable by nine pencils. However, the elementary change \((\hat{6}, \hat{7})\) is possible only for the first of them, shown in the first row of Figure 35. \(\Box\)
Figure 34: Pencils \( \max(\hat{1} = 8-) \), \( \max(\hat{1} = 8+) \), \( \max(\hat{1} = 6-) \), \( \max(\hat{1} = 6+) \)
Figure 35: Pencils with $\hat{8} = 1+$
| (1+, 2) | (3, 1+) | (1+, 4) | (5, 1+) | (1+, 6) | (7, 1+) | (1, 7−) | (1−, 8) |
|-------|-------|-------|-------|-------|-------|-------|-------|
| 82    | 8X    | 84    | 8X    | 86    | 8X    | 8X    | 28    |
| (1+, 2) | (3, 1+) | (1+, 4) | (5, 1+) | (1+, 6) | (7, 1+) | (1, 7−) | (1−, 8) |
| 82    | 8X    | 84    | 8X    | 86    | 6X    | 2X    |
| (1+, 2) | (3, 1+) | (1+, 4) | (5, 1+) | (1, 5−) | (1−, 6) | (7, 1−) | (1−, 8) |
| 82    | 8X    | 84    | 8X    | 6X    | 26    | 2X    |
| (4, 2+) | (2+, 5) | (6, 2+) | (2+, 7) | (8, 2+) | (2, 8−) | (8−, 3) | (4+, 1) |
| 1X    | 15    | 1X    | 17    | 1X    | 1X    | 13    |
| (4−, 1) | (2+, 5) | (6, 2+) | (2+, 7) | (8, 2+) | (2, 8−) | (8−, 3) | (4, 8−) |
| 51    | 15    | 1X    | 17    | 1X    | 1X    | 13    |
| (4, 2+) | (2+, 5) | (6, 2+) | (2+, 7) | (8, 2+) | (8−, 3) | (4+, 1) |
| 1X    | 15    | 1X    | 17    | 7X    | 3X    |
| (4−, 1) | (2+, 5) | (6, 2+) | (2+, 7) | (8−, 3) | (4, 8−) |
| 51    | 15    | 1X    | 17    | 7X    | 3X    |
| (4−, 1) | (2+, 5) | (6, 2+) | (2+, 7) | (8−, 3) | (4, 8−) |
| 51    | 15    | 5X    | 3X    |
| (4−, 1) | (2+, 5) | (6, 2+) | (2+, 7) | (8−, 3) | (4, 8−) |
| 51    | 15    | 5X    | 3X    |
| (8−, 5) | (6+, 1) | (2+, 6−) | (2−, 7) | (8−, 3) | (4, 8−) |
| 51    | 5X    | 35    | 3X    | 37    |
| (8−, 5) | (6+, 1) | (2+, 6−) | (2−, 7) | (8−, 3) | (4, 8−) |
| 15    | 51    | 5X    | 3X    | 37    |
| (5, 1−) | (1, 5+) | (1+, 4) | (3−, 8) | (7−, 3+) | (7+2) | (1−, 6) |
| 2X    | 4X    | 84    | 48    |
| (1, 5−) | (5, 1+) | (1+, 4) | (3−, 8) | (7−, 3+) | (7+2) | (1−, 6) |
| 6X    | 8X    | 84    | 48    |
| (1, 5−) | (5, 1+) | (1+, 4) | (3−, 8) | (7−, 3+) | (7+2) | (1−, 6) |
| 6X    | 8X    | 84    | 48    |
| (5, 1−) | (1, 5+) | (1+, 4) | (3−, 8) | (7−, 3+) | (7+2) | (1−, 6) |
| 2X    | 4X    | 84    | 48    |

Figure 36: Pencils with \( \hat{8} \neq 1+ \)
Figure 37: The lists $\hat{8} = L(1\ldots7)$

6 Tabulars
| \( C_2 \) | \( 8 \) | \( 4^- \) | \( 5+ \) | \( 5^- \) | \( 6+ \) | \( 6^- \) | \( 7+ \) | \( 7^- \) |
|---|---|---|---|---|---|---|---|---|
| 23456 | 1,7 | 1,7 | 1,7 | 1,7 | 1,7 | 1,7 | 1,7 | 1 |
| 23457 | 1 | 6 | 6 | 1 | 1 | 6 | 1,6 | 1,6 |
| 23467 | 1,5 | 1,5 | 1 | 5 | 5 | 1 | 5 | 1,5 |
| 23567 | 1,4 | 1,4 | 1,4 | 1,4 | 1,4 | 1,4 | 1,4 | 1,4 |
| 24567 | 3 | 1 | 1 | 3 | 3 | 1 | 3 | 1 |
| 34567 | 1,2 | 1,2 | 1,2 | 1,2 | 1,2 | 1,2 | 1,2 | 1,2 |
| 13456 | 7 | 2 | 2 | 7 | 2 | 7 | 2 | 7 |
| 13457 | 2,6 | 2,6 | 2,6 | 2,6 | 2,6 | 2,6 | 2,6 | 2,6 |
| 13467 | 5 | 2 | 2 | 5 | 2,5 | 2,5 | 2,5 | 2,5 |
| 13567 | 2 | 4 | 4 | 2 | 2 | 2 | 2 | 2 |
| 14567 | 2,3 | 2,3 | 2,3 | 2,3 | 2,3 | 2,3 | 2,3 | 2,3 |
| 12456 | 3,7 | 3,7 | 3,7 | 3,7 | 3,7 | 3,7 | 3,7 | 3,7 |
| 12457 | 3 | 6 | 6 | 3 | 6 | 3 | 6 | 3,6 |
| 12467 | 3,5 | 3,5 | 3 | 5 | 5 | 5 | 5 | 3 |
| 12567 | 3,4 | 3,4 | 3,4 | 3,4 | 3,4 | 3,4 | 3,4 | 3,4 |
| 12356 | 7 | 4 | 4 | 7 | 4 | 7 | 4 | 7 |
| 12357 | 4,6 | 4,6 | 4,6 | 4,6 | 4,6 | 4,6 | 4,6 | 4,6 |
| 12367 | 5 | 4 | 4 | 5 | 4,5 | 4,5 | 4,5 | 4,5 |
| 12346 | 5 | 7 | 7 | 5 | 5,7 | 5,7 | 5,7 | 5,7 |
| 12347 | 5,6 | 5,6 | 6 | 5 | 5 | 6 | 5,6 | 5,6 |
| 12345 | 6,7 | 6,7 | 6,7 | 6,7 | 6,7 | 6,7 | 6,7 | 6,7 |

Figure 38: The lists \( \hat{8} = L(1, \ldots 7) \), continued
Figure 39: \( \hat{8} = 1^+ \)
Figure 40: $\hat{8} = 1$
Figure 41: $\hat{8} = 2^+$
Figure 42: $\hat{8} = 2$
Figure 43: $\hat{8} = 3+$
Figure 44: \( \hat{8} = 3^- \)
Figure 45: $\hat{8} = 4+$
Figure 46: Elementary changes
|  3 |  6+ ↔ 4- | 6+ ↔ 4- | 5+ ↔ 5- | 5+ ↔ 5- |
|  5 |  2- ↔ 4+ | 3- ↔ 3+ | 2- ↔ 4+ | 3- ↔ 3+ |
|  8 |  2- ↔ 4+ | 3- ↔ 3+ | 2- ↔ 4+ | 3- ↔ 3+ |
|  3 |  8+ ↔ 6- | 8+ ↔ 6- | 7+ ↔ 7- | 7+ ↔ 7- |
|  7 |  2- ↔ 4+ | 3- ↔ 3+ | 2- ↔ 4+ | 3- ↔ 3+ |
|  3 |  8- ↔ 8+ | 8- ↔ 8+ | 7- ↔ 1+ | 7- ↔ 1+ |
|  8 |  2- ↔ 4+ | 3- ↔ 3+ | 2- ↔ 4+ | 3- ↔ 3+ |
|  4 |  5- ↔ 5+ | 5- ↔ 5+ | 3- ↔ 6+ | 3- ↔ 6+ |
|  5 |  3- ↔ 6+ | 4- ↔ 4+ | 3- ↔ 6+ | 4- ↔ 4+ |
|  4 |  7+ ↔ 5- | 7+ ↔ 5- | 6+ ↔ 6- | 6+ ↔ 6- |
|  6 |  3- ↔ 5+ | 4- ↔ 4+ | 3- ↔ 5+ | 4- ↔ 4+ |
|  4 |  7- ↔ 7+ | 7- ↔ 7+ | 6- ↔ 8+ | 6- ↔ 8+ |
|  7 |  3- ↔ 5+ | 4- ↔ 4+ | 3- ↔ 5+ | 4- ↔ 4+ |
|  4 |  1+ ↔ 7- | 1+ ↔ 7- | 8+ ↔ 8- | 8+ ↔ 8- |
|  8 |  3- ↔ 5+ | 4- ↔ 4+ | 3- ↔ 5+ | 4- ↔ 4+ |
|  5 |  6- ↔ 6+ | 6- ↔ 6+ | 4- ↔ 7+ | 4- ↔ 7+ |
|  6 |  4- ↔ 7+ | 5- ↔ 5+ | 4- ↔ 7+ | 5- ↔ 5+ |
|  5 |  8+ ↔ 6- | 8+ ↔ 6- | 7+ ↔ 7- | 7+ ↔ 7- |
|  7 |  4- ↔ 6+ | 5- ↔ 5+ | 4- ↔ 6+ | 5- ↔ 5+ |
|  5 |  8- ↔ 8+ | 8- ↔ 8+ | 7- ↔ 1+ | 7- ↔ 1+ |
|  8 |  4- ↔ 6+ | 5- ↔ 5+ | 4- ↔ 6+ | 5- ↔ 5+ |
|  6 |  7- ↔ 7+ | 7- ↔ 7+ | 5- ↔ 8+ | 5- ↔ 8+ |
|  7 |  5- ↔ 8+ | 6- ↔ 6+ | 5- ↔ 8+ | 6- ↔ 6+ |
|  6 |  1+ ↔ 7- | 1+ ↔ 7- | 8+ ↔ 8- | 8+ ↔ 8- |
|  8 |  5- ↔ 7+ | 6- ↔ 6+ | 5- ↔ 7+ | 6- ↔ 6+ |
|  7 |  8- ↔ 8+ | 8- ↔ 8+ | 6- ↔ 1+ | 6- ↔ 1+ |
|  8 |  6- ↔ 1+ | 7- ↔ 7+ | 6- ↔ 1+ | 7- ↔ 7+ |

Figure 47: Elementary changes, continued
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