New explicit examples of fixed points of Poisson shot noise transforms

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Abstract

We show that gamma distributions, generalized positive Linnik distributions, S2 distributions are fixed points of Poisson shot noise transforms. The corresponding response functions are identified via their inverse functions except for some special cases when those can be obtained explicitly. As a by-product, it is proven that log-convexity of the response function is not necessary for selfdecomposability of non-negative Poisson shot noise distribution. Some attention is given to perpetuities of a rather special type which are closely related to our model. In particular, we study the problem of their existence and uniqueness.

Key words: Poisson shot noise transform · shot noise distribution · fixed points · perpetuity · infinite divisibility · selfdecomposability

1 Introduction.

Let $\mathcal{P}^+$ be the set of all probability distributions on the Borel subsets of $\mathbb{R}^+ = [0, \infty)$ and $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a Borel measurable function which in what follows we call response function. Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. It will be assumed throughout the paper that all random variables (r.v.’s) involved are defined there, and this space is rich enough to accumulate independent copies of some r.v.’s. Also from now on notation $\mu = \mathcal{L}(\xi)$ means that $\mu \in \mathcal{P}^+$ is a probability distribution of r.v. $\xi = \xi(\omega)$, $\omega \in \Omega$. The last convention is that we always take the distribution function of measure $\mu$ that is right-continuous. Let $\{\tau_i\}, i = 1, 2, \ldots$ be the points of a Poisson flow with intensity $0 < \lambda < \infty$, and $\xi, \xi_1, \xi_2, \ldots$ be non-negative independent identically distributed (i.i.d.) r.v.’s.

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independent of the Poisson flow. For a fixed function \( h \), let \( P^+_h \) be the subset of \( P^+ \) consisting of probability distributions of r.v. \( \xi \) such that the series

\[
\sum_{i=1}^{\infty} \xi_i h(\tau_i)
\]

is well-defined in the weak convergence sense (and hence in probability and almost surely). Recall that the probability distribution of the latter random series when exists is called *(Poisson) shot noise distribution* (SND, in short).

For a fixed \( \lambda \) let us define a *Poisson shot noise transform* (SNT) \( T_{h,\lambda} : P^+_h \to P^+ \) as follows

\[
T_{h,\lambda}(\mathcal{L}(\xi)) = \mathcal{L}\left( \sum_{i=1}^{\infty} \xi_i h(\tau_i) \right).
\]

At this stage we would like to remark that non-negativity assumption of the model above is not necessary in general. It is imposed here to take into account features of the current presentation. Iksanov, Jurek (2002b) (henceforth to be referred to as IJ(2002)) introduce the SNT for vector-valued response functions and distributions in many dimensions. Furthermore Iksanov, Jurek (2002a) provide conditions on \((\mathcal{L}(\xi), h)\) which ensure the convergence of series (1) for this more general framework.

We will say that a non-degenerate at zero probability distribution \( \mu^* = \mathcal{L}(\xi) \) is a fixed point of the SNT \( T_{h,\lambda} \) and/or the pair \((\lambda, h)\) generates or gives rise to a fixed point \( \mu^* \) if

\[
\mu^* = T_{h,\lambda}(\mu^*).
\]

Formula (3) can be rewritten in terms of the Laplace-Stieltjes transform (LST) \( \varphi^*(s) = \int_0^{\infty} e^{-sx} \mu^*(dx) \) as follows

\[
\varphi^*(s) = \exp\{-\lambda \int_0^{\infty} (1 - \varphi^*(sh(u))) du\}.
\]

Every Poisson SND is infinitely divisible (ID), so is \( \mu^* \). Moreover, \( \mu^* \) has zero drift and Lévy measure \( M^* \) given by its tail as follows:

\[
M^*(x, \infty) = \lambda \int_0^{\infty} \mu^*(x/h(u), \infty) du.
\]

On the other hand by differentiating (4) (it is not hard to verify that this is possible) and by inverting the resulting expression, one gets

\[
\omega^*[0, x] := \int_0^{x} y\mu^*(dy) = \int_0^{x} \mu^*[0, x-y]yM^*(dy)
\]

(compare to standard representation of positive ID distributions due to Steutel

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Furthermore, (5) reveals that $M_*$ satisfies the relation
\[
\int_0^x yM_*(dy) = \int_0^{h(+0)} \omega_*[0, x/y]\nu(dy),
\]
where $\nu(dx) = -\lambda x h^+(dx)$, and $h^+$ is a generalized inverse of $h$ to be defined in Section 2.
Just from (4)-(7) one can deduce a lot of things about $\mu^*$. See Section 2 for details.
The research of fixed points of the SNT (2) has been initiated in Iksanov (2001). There in fact the following result has been proven: if $h(x) = e^{-x}$, $x \geq 0$ then the condition $\lambda \leq 1$ is necessary and sufficient to guarantee an existence of fixed points $\mu^*$ of SNT $\mathcal{T}_{h, \lambda}$. Furthermore, those fixed points are positive Linnik distributions (exponential for $\lambda = 1$) which are given by the tails of distributions $\mu^*(x, \infty) = \sum_{k=0}^{\infty} (-\beta)^{-k} x^{\lambda k} / \Gamma(1+\lambda k)$, $x \geq 0$, $\beta > 0$, where $\Gamma$ stands for the Euler gamma function, or via the LST's
\[
\int_0^\infty \exp(-zx)\mu^*(dx) = (1 + \beta z^\lambda)^{-1}.
\]
Here it is reasonable to note that 1) Lin (2001) independently proves a closely related result in slightly different settings by using another approach; 2) in Iksanov (2001) the distributions with the LST (8) has been called Mittag-Leffler distributions. However, as explained in Pakes (1995, p. 294) (see also Lin (2001)) this may cause confusion and the name “positive Linnik” is more correct for these distributions.
As it is well-known from Vervaat (1979) or Bondesson (1992), when one studies non-negative SND, there is no loss of generality in assuming that the response function $h$ is right-continuous and non-increasing. Under such assumptions IJ (2002) provide a description of fixed points that correspond to response functions $h$ with $h(+0) \leq 1$, and also directly verify that $h(s) = 1_{[a, a]}(s)$ for some $a > 0$, and $h(s) = s^{-\alpha}$, $\alpha > 1$ give rise to no fixed points for any positive $\lambda > 0$. Also Theorem 1.1(a) from the latter reference implies that a pair $(\lambda, h)$ with $\lambda \int_0^\infty h(u)du > 1$ does not generate fixed points.
Mentioned above are the only known before response functions which permit either to describe fixed points explicitly (that is, to point out its LST or distribution function etc.) or to prove an absence of fixed points. Similarly the problem of not having many explicit examples is often mentioned in the literature on perpetuities. This is not strange. In fact, the reader will observe (see Lemma 3.3 below) that the size-biased distributions which correspond to fixed points of finite mean are perpetuities of a very special kind. Consequently, study of fixed points in our model and that of perpetuities are closely related. Although those have much in common, a certain peculiarity of fixed points requires to work out special methods to treat them. To point out a few features of fixed points under consideration, we only mention their ID and (in most cases) absolute continuity on $(0, \infty)$. This is certainly not a case for general perpetuities.
Somebody may ask why one needs to seek for explicit examples of fixed points? We believe that first it is a quite interesting theoretical problem on its own. Second it is expected that having found the way of construction explicit examples of fixed points, one could say more about some Lebesgue properties of fixed points. For example, which fixed points in addition to just mentioned ID and absolute continuity are selfdecomposable (SD) (certainly provided that the support of $h$ is the whole half-line), or which are unimodal? Those appear to be quite intriguing problems.

2 Main results.

Our first result states that some well-known distributions do appear as fixed points of the SNT (2). Although Proposition 2.1 does not contain an explicit form of the corresponding response functions except for some partial cases (one of them can be found in the proof of Proposition 2.2), no problems occur because the only thing one should know is that those $h$’s are right-continuous and non-increasing with $\int_0^\infty h(u)du = 1$. Let us recall that any right continuous and non-decreasing function $g$ on $(0, \infty)$ allows to define its generalized inverse $g^{-}$ which is right-continuous and non-decreasing as well and given as follows

$$g^{-}(z) = \inf\{u : g(u) < z\}$$

for $z < g(0^{+})$ and 0 otherwise. We also preserve the above notation for "usual" inverse functions which are defined for continuous and strictly monotone $g$ by the relation $g(g^{-}(z)) = g^{-}(g(z)) = z$.

**Proposition 2.1** a) Let $\alpha, \beta > 0$ and $\gamma \in (0, 1)$. If the function $h$ is defined via its "usual" inverse

$$h^+(x) = \alpha \int_x^1 z^{-1}(1-z)^{\alpha-1}dz, \ x \in (0, 1)$$

then gamma distributions $\mu_{\alpha, \beta}(dx) = \frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-x/\beta}1_{(0, \infty)}(x)dx$ and generalized positive Linnik distributions $\mu_{\alpha, \beta, \gamma}$ given by the LST

$$\int_0^\infty e^{-sx}\mu_{\alpha, \beta, \gamma}(dx) = \frac{1}{(1 + \beta s^\gamma)^\alpha}$$

are fixed points of SNT $T_{h, 1}$ and $T_{h^{-1}/\gamma, 1}$ accordingly.

b) Let $\delta > 0$, $\rho \in (0, 1)$ and $h^+(x) = \ln x + 2x^{-1/2} - 2$, $x \in (0, 1)$. Then S2 distributions $\mu_{\delta}(dx) = d(\sum_{n=-\infty}^\infty (1 - 2\pi^2 x/\delta)e^{-\pi^2 n^2 x/\delta})$ and positive distributions $\mu_{\delta, \rho}$ with the LST

$$\int_0^\infty e^{-sx}\mu_{\delta, \rho}(dx) = \left(\frac{\sqrt{\delta \rho}}{\sinh \sqrt{\delta \rho}}\right)^2$$

are fixed points of SNT’s $T_{h, 1}$ and $T_{h^{-1}/\rho, 1}$ accordingly.
Remark 2.1 Let $\gamma_{\alpha,\beta}$ be a gamma r.v. and $\varepsilon \in (0,1) \cup (2,\infty)$. Unlike the gamma distribution, $L(\gamma_{\alpha,\beta}^\varepsilon)$ cannot be a fixed point of SNT. If $\varepsilon < 1$ this is so, because $L(\gamma_{\alpha,\beta}^\varepsilon)$ is not ID. Whereas for $\varepsilon > 2$ $L(\gamma_{\alpha,\beta}^\varepsilon)$ together with the lognormal distribution are primary examples of laws which are not determined by their moments according to Krein’s criterion. The same is true for $L(\gamma_{\alpha,\beta}^\varepsilon)$ as shown by Pakes, Khattree (1992). Therefore the conclusion follows from Proposition 2.3(b).

Remark 2.2 All distributions of Proposition 2.1 are SD. While the background driving Lévy processes of part a) distributions are compound Poisson (see Ik sanov, Jurek (2002a) for a recent treatment of those and definitions), this is not the case for the others. SD of S2 distributions is easy to verify because as it is shown by Pitman, Yor (2001, Table 1, p.442) their Lévy densities are of the form $k(x)/x = (\delta \sum_{n=-\infty}^{+\infty} e^{-\delta -1\pi^2n^2x})/x$, and hence $k(x)$ is decreasing on $(0,\infty)$. Now distributions given by (11) are SD as these are laws of strictly stable subordinator evaluated at random SD $(S_2)$ time. The observation about SD of such distributions is due to Bondesson (1992, p.19).

Clearly, 1) NOT all fixed points generated by $h$ of unbounded support and 2) NOT all size-biased distributions which correspond to fixed points are SD.

To formulate our second result, recall that Bondesson (1992, p.156) proved that the sufficient condition for SD of SND (1) is log-convexity and strict decreasingness of $h$. The next Proposition states that this is not necessary.

Proposition 2.2 There exist selfdecomposable shot noise distributions which are generated by a response function which is not log-convex.

Suppose that $\mu \in \mathcal{P}^+$ is of finite mean $m := \int_0^\infty x\mu(dx)$. This allows to consider the so-called size-biased distribution $\overline{\mu}(dx) = m^{-1}x\mu(dx)$. Let $\overline{\eta}$, $\eta$ and $A$ be independent r.v.’s with $\overline{\mu} = L(\overline{\eta})$, $\mu = L(\eta)$ and $\nu = L(A)$ which satisfy the distributional equality

$$\overline{\eta} \overset{d}{=} \eta + A\overline{\eta}.$$ (12)

We now cite the problem mentioned by Pitman, Yor (2000, p.35): "given a distribution of $A$...whether there exists such a distribution of $\eta". Recall that in the more recent literature the so defined r.v. $\overline{\eta}$ (as in (12)) is typically called perpetuity.

Below we answer the above question for the partial case when $\nu$ is concentrated on $(0,b]$, $b \leq 1$. Denote by $\delta_x$ the delta measure at $x \geq 0$.

Proposition 2.3 a) For any $\nu \neq \delta_1$ concentrated on $(0,b]$, $b \leq 1$ there exist $\mu$’s satisfying (12). For fixed $m > 0$ $\mu$ is a unique solution to (12) such that $m = \int_0^\infty x\mu(dx)$.

b) Those $\mu$’s have finite exponential moments.

c) All $\mu$’s are infinitely divisible with drift 0 and Lévy measure $M$ whose tail is
given as follows \( M(x, \infty) = \int_0^b z^{-1} \mu(xz^{-1}, \infty) \nu(dz) \). Furthermore, \( \mu \)'s are compound Poisson provided \( x^{-1} \nu(dx) \) is integrable at the neighbourhood of zero.

d) If for some \( \varepsilon \in (0, 1] \), \( \int_0^b x^{-\varepsilon} \nu(dx) < \infty \) then
\[
\mu(dx) = q \delta_0 + (1 - q) f_1(0, \infty)(x) dx,
\]
where \( q = 0 \) if \( x^{-1} \nu(dx) \) is not integrable at the neighbourhood of the origin, and \( q \in (0, 1) \) is a unique solution to the equation \( \exp(-b(1 - z)) = z \) if \( \int_0^b x^{-1} \nu(dx) = b \). In words, \( \mu \)'s have an absolutely continuous component on \((0, \infty)\) with density \( f \).

e) All \( \mu \)'s are fixed points of \( T_{h,1} \) with the response function \( h \) given via its generalized inverse \( h^- \) as follows: \( h^-(x) = \int_x^b z^{-1} \nu(dz) \) which implies \( \int_0^{\infty} h(z) dz = 1 \).

**Remark 2.3** It is possible to strengthen the above Proposition in the following way. Let us consider the measure \( \sigma \) such that \( \sigma(dx) = x \mu(dx) \) and rewrite (12) in terms of distributions to obtain the well-known representation of positive infinitely divisible distributions due to Steutel (1970, p. 86):

\[
\sigma[0, x] = \int_0^x \mu[0, x - z] z M(dz),
\]

(13)

\( M \) being the Lévy measure of \( \mu \) which in our case has a feature

\[
\int_0^x z M(dz) = \int_0^b \sigma[0, x/z] \nu(dz).
\]

(14)

As it turned out if \( \int_0^b z^{-\Delta} \nu(dz) < \infty \) for some \( \Delta \in (0, 1) \), we need not presuppose that \( \int_0^{\infty} x \mu(dx) < \infty \). In fact, if a distribution \( \mu \) satisfies (13), (14) then it necessarily has finite first moment. Moreover, given \( m > 0 \) \( \mu \) is the unique distribution of mean \( m \) satisfying (13), (14). This is essentially the content of Theorem 1.1(b) of IJ (2002), but for a special case the proof of that assertion should be taken into account.

### 3 The Proofs.

Four preparatory lemmas are prepared. We begin with a simple observation which can be read from (4) and hence its proof is immediate and omitted. It is singled out as a Lemma only for ease of further references.

**Lemma 3.1** Fixed points of the SNT (2) are invariant under scale transformations, that is, if \( \mathcal{L}(\xi) \) is a fixed point of the SNT so is \( \mathcal{L}(c \xi) \), for any \( c > 0 \).

Throughout the rest of this Section we will assume that for any positive \( \lambda \) response functions \( h \)'s of the SNT \( T_{h,\lambda} \) are subject to **CONDITION A**: they are right-continuous, non-increasing, \( h(+0) \leq 1 \) and \( h \) is not of the form \( h(u) = 1_{[0,a)}(u) \) for some \( a > 0 \).
The next Lemma is a uniqueness result concerning fixed points of the SNT. It is contained in Theorem 1.1(b) of IJ (2002) and has been proven there by using Contraction Principle. We would like to provide an independent, slightly simpler proof.

**Lemma 3.2** Let \( h \) satisfies Condition A and \( \lambda \int_0^\infty h(z)dz = 1 \). Then \( \mathbb{T}_{h,\lambda} \) has fixed points of finite mean. Given \( m \in (0, \infty) \) there exist a unique fixed point \( \mu^* \) of \( \mathbb{T}_{h,\lambda} \) with \( m = \int_0^\infty x\mu^*(dx) \).

**Proof.** For fixed \( m > 0 \) consider the set of probability measures

\[
\mathcal{P}_{h,m}^+ = \{ \rho \in \mathcal{P}_h^+ : \int_0^\infty x\rho(dx) = m \}.
\]

Starting with \( \mu_0 = \delta_m \), define the sequence

\[
\mu_n := \mathbb{T}_{h,\lambda}\mu_{n-1} := \mathbb{T}_{h,\lambda}^n\mu_0, \quad n = 1, 2, \ldots
\]

which is trivially well-defined on \( \mathcal{P}_{h,m}^+ \) provided \( \int_0^\infty h(z)dz < \infty \). The corresponding LST’s \( \varphi_n(s) = \int_0^\infty e^{-sx}\mu_n(dx), \quad n = 0, 1, \ldots \) satisfy equations

\[
\varphi_0(s) = e^{-ms}, \quad \varphi_n(s) = \exp\{-\lambda\int_0^\infty (1 - \varphi_{n-1}(sh(u)))du\}, \quad n = 1, 2, \ldots \quad (15)
\]

Let us verify that the weak limit of \( \mu_n \), as \( n \to \infty \), exists and has mean \( m \). As it is well-known, this will mean that \( \mathbb{T}_{h,\lambda} \) has a unique fixed point on \( \mathcal{P}_{h,m}^+ \).

In what follows we use some ideas of Durrett, Liggett (1983, the proof of Theorem 2.7). By Jensen’s inequality,

\[
\varphi_1(s) = \mathbb{E}\exp\{-s\sum_{i=1}^\infty \xi_i h(\tau_i)\} \geq \exp\{-s\mathbb{E}\sum_{i=1}^\infty \xi_i h(\tau_i)\} = \varphi_0(s),
\]

that implies \( \varphi_n(s) \geq \varphi_{n-1}(s), \quad n = 1, 2, \ldots, \quad s \geq 0 \). Thus the monotone and bounded sequence \( \{\varphi_n\}, \quad n = 1, 2, \ldots \) has a unique limit \( \varphi(s) \), say, being the LST of a probability measure \( \mu \), say. Since \( \int_0^\infty h(z)dz < \infty \) then by dominated convergence it is easily seen that \( \varphi(s) \) satisfies the fixed point equation (4) or equivalently \( \mu \) is a (possibly degenerate at 0) fixed point of the SNT. It remains to check that \( \mu \in \mathcal{P}_{h,m}^+ \).

Clearly,

\[
\limsup_{s \to +0}(-\varphi'(s)) \leq m. \quad (16)
\]

So we should only study the lower limit.

To this end for \( n = 0, 1, \ldots \) put \( \Phi_n(s) := \log(-\varphi_n'(e^{-s})) \), \( \Psi_n(s) := \log\varphi_n(e^{-s}) \). Note that in view of assumptions \( \pi(dz) := -\lambda z h^{-s}(dz) \) is a probability measure and let \( \theta, \theta_1, \theta_2, \ldots \) be independent rv’s with this distribution. Under these notations one obtains from (15) by change of variable

\[
\Phi_n(s) = \Psi_n(s) + \log \int_0^\infty -\varphi_{n-1}'(e^{-s+t})\pi(dt) = \\
\leq \Psi_n(s) + \mathbb{E}\{\exp\Phi_{n-1}(s - \log \theta)\} \geq \\
\geq \Psi_n(s) + \mathbb{E}\Phi_{n-1}(s - \log \theta) \geq -me^{-s} + \mathbb{E}\Phi_{n-1}(s - \log \theta). \quad (17)
\]
Above the first inequality follows by Jensen’s inequality and the second one follows by monotonicity of \( \{ \Psi_n \} \).

Consider the random walk \( S_0 = 0, S_n = -\sum_{i=1}^{n} \log \theta_i, \ n = 1, 2, \ldots \) On iterating (17) one gets

\[
\Phi_n(s) \geq \mathbb{E}\Phi_0(s + S_n) - m e^{-s}(1 + \mathbb{E} \sum_{i=1}^{n-1} \theta_1 \theta_2 \ldots \theta_i).
\]

Since \( \mathbb{E}\log \theta_i = \lambda \int_{0}^{\infty} h(z) \log h(z) dz < 0 \) then by the strong law of large numbers \( S_n \to +\infty \) as \( n \to \infty \) a.s. Consequently by dominated convergence

\[
\lim_{n \to \infty} \mathbb{E}\Phi_0(s + S_n) = \log m.
\] (18)

Since \( \pi \) is concentrated on \( (0, 1) \) then \( \lim_{n \to \infty} (1 + \mathbb{E} \sum_{i=1}^{n-1} \theta_1 \theta_2 \ldots \theta_i) = (1 - \mathbb{E}\theta)^{-1} \).

Therefore by using (18)

\[
\liminf_{s \to +0} (-\varphi'(s)) = \exp\{\liminf_{s \to +\infty} \lim_{n \to \infty} \Phi_n(s)\} \geq m.
\]

This together with (16) show that \( \mu^* := \mu \in P_{h,m}^+ \).

To prove uniqueness let us assume on the contrary that there exists another LST \( \tilde{\varphi}(s) \) with \( \lim_{s \to +0} s^{-1}(1 - \tilde{\varphi}(s)) = m \) that satisfies (4). As in Athreya (1969, Theorem 1), set \( M(s) = \frac{|\tilde{\varphi}(s) - \varphi(s)|}{s} \) for \( s > 0 \) and obtain from (4):

\[
M(s) \leq \int_{0}^{1} M(sz)\pi(dz) \leq \ldots \leq \mathbb{E}M(s\theta_1 \ldots \theta_n).
\] (19)

Further for any \( s > 0 \) \( M(s) \leq |m - s^{-1}(\tilde{\varphi}(s) - 1)| + |s^{-1}(1 - \varphi(s)) - m| \) which gives \( \lim_{s \to +0} M(s) = 0 \).

By the strong law of large numbers and bounded convergence in (19) we conclude that \( M(s) = 0 \) for \( s > 0 \). It remains to recall that \( \tilde{\varphi}(0) = \varphi(0) = 1 \) which yields \( \tilde{\varphi}(s) = \varphi(s) \). This completes the proof.

While our third auxiliary assertion is the key ingredient to the proof of all assertions of Section 2, and in essence makes clear the connection between fixed points of the SNT’s and perpetuities of special kind (12), the fourth one is quite simple and again can be read from (4) with some additional explanations. In Lemma 3.3 all random variables and distributions involved were described just above (12).

**Lemma 3.3** Let for given \( \nu \) as in Proposition 2.3 a r.v. \( \eta \) satisfies (12). Then \( \mu \) is a fixed point of the SNT \( T_{h,1} \) with \( h^-(x) = \int_{x}^{b} z^{-1} \nu(dz) \), \( x \in (0, b) \) and hence \( h \) is subject to Condition A and \( \int_{0}^{\infty} h(z) dz = 1 \).

Conversely, if \( \mu^* \) is a fixed point of the SNT \( T_{h,\lambda} \) with

\[
\lambda \int_{0}^{\infty} h(z) dz = 1 \text{ and } h(+0) = b \in (0, 1]
\] (20)
then the r.v. \( \eta \) with \( \mathcal{L}(\eta) = \mu^* \) satisfies (12) with a r.v. \( A \) whose distribution \( \nu \) is concentrated on \( (0, b] \) and defined as follows:

\[
\nu(dx) = -\lambda x h^{-}(dx).
\]

**Proof.** Let us first note that if \( \nu = \delta_1 \) then \( \mu = \delta_0 \), the case excluded by us. By the same reasoning we remove the indicator function from the class of possible response functions in Condition A.

Suppose that the SNT \( T_{h, \lambda} \) has a fixed point \( \mu^* \) and hence \( \varphi^*(s) = \int_0^\infty e^{-sx} \mu^*(dx) \) satisfies (4), that is,

\[
\varphi^*(s) = \exp\left\{-\lambda \int_0^\infty (1 - \varphi^*(sh(u)))du\right\} = \exp\left\{\lambda \int_0^b (1 - \varphi^*(sz))h^{-}(dz)\right\}.
\]

In view of Lemma 3.2 condition (20) implies \( m := \int_0^\infty x \mu^*(dx) < \infty \). Without loss of generality we may and do assume \( m = 1 \) and therefore

\[
\lim_{s \to +0} s^{-1} (1 - \varphi^*(s)) = 1.
\]

Suppose that a r.v. \( \eta \) with \( \mathbb{E}\eta = 1 \) satisfies (12). Then the LT \( \varphi(s) = \mathbb{E}e^{-s\eta} \) solves

\[
\varphi'(s) = \varphi(s) \int_0^\infty \varphi'(sz) \nu(dz).
\]

Note that \( -\varphi'(s) \) is the LST of probability measure \( \mu(dx) = x \mu(dx) \). By using Fubini’s Theorem one has

\[
\ln \varphi(s) = \int_0^s [\ln \varphi(u)]' du = \int_0^s \int_0^\infty \varphi'(uz) \nu(dz) du = \int_0^\infty z^{-1} \nu(dz) (\varphi(sz) - 1)
\]

or equivalently

\[
\varphi(s) = \exp\left\{-\int_0^\infty (1 - \varphi(sz))z^{-1} \nu(dz)\right\}.
\]

Put \( \nu(dz) = -\lambda z h^{-}(dz) \) and note that this implies that the statements "\( \nu \) is a probability measure on \([0, b]\)" and (20) are equivalent. We want to verify that \( \varphi^*(s) = \varphi(s) \). Luckily, the way of doing so mimics that of the proof of the previous Lemma (beginning with "As in Athreya..."); the only difference being

\[
M(s) = \frac{|\varphi(s) - \varphi^*(s)|}{s}.
\]

The proof is completed.

**Lemma 3.4** Assume that \( \lambda \int_0^\infty h(z) dz = 1 \). Then for any \( \alpha \in (0, 1) \) the SNT \( T_{h^{1/\alpha}, \lambda} \) has a fixed point \( \mu^*_\alpha \) whose tail is given by

\[
\mu^*_\alpha(x, \infty) = \int_0^\infty s_\alpha(x t^{-1/\alpha}, \infty) \mu^*(dx)
\]

where \( \mu^* \) is a fixed point of \( T_{h, \lambda} \) with finite mean; \( s_\alpha \) is a strictly positive distribution with index of stability \( \alpha \) or equivalently

\[
\int_0^\infty e^{-sx} \mu^*_\alpha(dx) = \varphi^*(s^{\alpha}),
\]

where \( \varphi^*(s) \) is the LST of \( \mu^* \).
Proof. Set $\varphi^*_\alpha(s) = \varphi^*(s^\alpha)$ and let $m$ be the mean of $\mu^*$. A formal substitution in (4) $s^\alpha$ instead of $s$ gives

$$\varphi^*_\alpha(s) = \exp\{-\lambda \int_0^\infty (1 - \varphi^*_\alpha(sh^{1/\alpha}(u)))du\}$$

(23)

which implies (22) provided the SNT $T_{h^{1/\alpha},\lambda}$ is well-defined or equivalently the integral in (23) converges for small $s$. However the latter is easy since $\lim_{s \to 0^+} s^{-\alpha}(1 - \varphi^*_\alpha(s)) = m$ implies for some $\varepsilon > 0$ and $s_0 = s_0(\varepsilon) > 0 \int_0^{s_0} (1 - \varphi^*_\alpha(sh^{1/\alpha}(u)))du \leq (m + \varepsilon)s^\alpha$ for all $s \in (0, s_0)$. To see that (22) is tantamount to (21), recall that if $\varphi(s) = Ee^{-s\theta}$ then $\varphi(s^\alpha) = Ee^{-\alpha s\theta^{1/\alpha}}$, where $\alpha$ is a positive strictly $\alpha$-stable r.v.

Proof of Proposition 2.1 (a). Let $\gamma(\alpha, \beta)$ be a r.v. with gamma distribution with parameters $a, b > 0$, that is, its probability density function (p.d.f) is $p_{a,b}(x) = \frac{b^a}{\Gamma(a)}x^{a-1}e^{-bx}, x > 0$, and $\beta(c, d)$ be a r.v. with beta distribution of the first kind with parameters $c, d > 0$, that is with p.d.f. $q_{c,d}(x) = \frac{\Gamma(c+d)}{\Gamma(c)\Gamma(d)}x^{c-1}(1-x)^{d-1}, x \in (0, 1)$. The well-known result due to Stuart (1962) asserts that for any positive $\alpha_1, \alpha_2 \gamma(1, \alpha_2) \overset{d}{=} \beta(1, \alpha_1) \gamma(1 + \alpha_1, \alpha_2)$. This together with the obvious equality $\gamma(1 + \alpha_1, \alpha_2) \overset{d}{=} \gamma(1, \alpha_2) + \gamma(\alpha_1, \alpha_2)$ imply for $\alpha_1 = \alpha_2 = \alpha$

$$\gamma(1 + \alpha, \alpha) \overset{d}{=} \gamma(\alpha, \alpha) + \beta(1, \alpha)\gamma(1 + \alpha, \alpha).$$

(24)

It remains to note that $E\gamma(\alpha, \alpha) = 1$ and hence (12) is nothing more than (24) with $\eta \overset{d}{=} \gamma(1 + \alpha, \alpha)$, $\eta \overset{d}{=} \gamma(\alpha, \alpha)$ and $A \overset{d}{=} \beta(1, \alpha)$. By Lemma 3.3 $\gamma(\alpha, \alpha)$ is a fixed point of the SNT $T_{h,1}$ with $h$ defined by its inverse $h^\tau(x) = \int_x^1 z^{-1}q_{1,\alpha}(z)dz$. To complete the study of gamma distributions it suffices to note that fixed points of the SNT (2) are scale invariant by Lemma 3.1.

Since $\int_0^\infty e^{-z^2} \mu_{\alpha,\beta}(dx) = \int_0^\infty e^{-z^2} \mu_{\alpha,\beta}(dx)$, an appeal to Lemma 3.4 finishes the proof.

b) Pitman, Yor (2000, Proposition 12(i,iii)) proved that $S$ distribution $\mu_2$ given via the LST $\varphi_2(s) = \left(\frac{\sqrt{2s}}{\sinh \sqrt{2s}}\right)^2$ satisfies (12) with $\nu = L(A)$ such that $\nu(dx) = (x^{-1/2} - 1)dx, x \in (0, 1)$. Hence, by Lemma 3.3 $\mu_2$ is a fixed point of $T_{h,1}$ with $h$ being defined via its inverse $h^\tau(x) = \int_x^1 z^{-1} \nu(dz) = \ln x + 2x^{-1/2} - 2, x \in (0, 1)$. An appeal to Lemma 3.1 proves that the same is true for $\mu_3$.

The conclusion regarding distributions given by (11) comes from Lemma 3.4. This finishes the proof.

Remark 3.1 Formula (24) is well-known and especially often mentioned in the literature on perpetuities. There are some its extensions which can be found in
Proof of Proposition 2.2. We provide an explicit example of such a possibility. In fact, we intend to show that 1) the response function \( h(u) = \frac{1}{(\cosh u)^2} \) generate SD fixed points of the SNT \( T_{h,1} \) being \( \gamma(1/2, 1/2) \) distributions; 2) so defined \( h \) is log-concave.

To this end let us turn to Proposition 2.1 to obtain that \( \gamma(1/2, 1/2) \) distribution is a fixed point of \( T_{h,1} \) where \( h \) is given via its inverse \( h^{-1}(u) = 2^{-1} \int_u^1 z^{-1} (1 - z)^{-1/2} dz = \frac{1}{2} \ln \frac{1 - (1 - u)^{1/2}}{1 + (1 + u)^{1/2}}, \) \( u > 0. \) Now it is easily seen that the corresponding \( h \) is of the form stated above by appealing at final stage to the well-known relation

\[ 1 - (\tanh u)^2 = \frac{1}{(\cosh u)^2}. \]

Log-concavity of \( h \) follows from the relation \((\ln h(u))'' = \frac{-2}{(\cosh u)^2} < 0. \) This completes the proof.

Proof of Proposition 2.3. e) is our Lemma 3.3. All the other parts of the Proposition can be obtained by appealing to e) as follows: a) is a consequence of Lemma 3.2; b) finiteness of some exponential moments is a part of Theorem 1.1(b) of IJ(2002); c) is quite trivial and can be read from (5); d) is a part of Theorem 1.2 of IJ(2002).

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