Infinite Hopf Family of Elliptic Algebras and Bosonization

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Abstract

Elliptic current algebras \(E_{q,p}(\hat{g})\) for arbitrary simply laced finite dimensional Lie algebra \(g\) are defined and their co-algebraic structures are studied. It is shown that under the Drinfeld like comultiplications, the algebra \(E_{q,p}(\hat{g})\) is not co-closed for any \(g\). However putting the algebras \(E_{q,p}(\hat{g})\) with different deformation parameters together, we can establish a structure of infinite Hopf family of algebras. The level 1 bosonic realization for the algebra \(E_{q,p}(\hat{g})\) is also established.

1 Introduction

In this paper we continue our recent study on infinite Hopf family of algebras and obtain new example of such families— infinite Hopf family of elliptic algebras.

The concept of infinite Hopf family of algebras was first introduced our earlier paper \(^6\) in which the algebras \(A_{\bar{h},\eta}(\hat{g})\) are proposed and their co-algebraic structures are specified. In contrast to the standard Hopf structures for the quantum affine algebras and Yangian doubles, the algebras \(A_{\bar{h},\eta}(\hat{g})\), including their most degenerated case \(A_{\bar{h},\eta}(\hat{sl}_2)\) \(^7\), are not evidently co-closed, and their co-algebraic structures are formulated in terms of some generalized Hopf structure, examples are the Hopf family of algebras of \(^7\) and the infinite Hopf family of algebras of our paper \(^6\).

The algebras \(A_{\bar{h},\eta}(\hat{g})\) appeared in \(^6\) are very unusual. For \(g = sl_2\), such algebra was proposed as the scaling limit of the elliptic algebra \(A_{q,p}(\hat{sl}_2)\) and thus inherits two deformation parameters \(\bar{h}\) and \(\eta\). The first parameter \(\bar{h}\) can be viewed as a “quantization parameter”, because in the limit \(\bar{h} \to 0\), the algebra \(A_{\bar{h},\eta}(\hat{g})\) would become a classical algebra. The second parameter \(\eta\) should be viewed as a “family deformation parameter”, because the set of algebras \(A_{\bar{h},\eta}(\hat{g})\) with different \(\eta\) form the first known nontrivial example of infinite Hopf family of algebras, and while \(\eta \to 0\) the family structure become trivial. Another unusual feature of the algebras \(A_{\bar{h},\eta}(\hat{g})\) is that, under the current realization, the generating currents corresponding

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to positive and negative roots are deformed *differently*. Despite their unusual mathematical features, the algebras $A_{\hbar,\eta}(\hat{g})$ are believed to have important applications in integrable quantum field theories such as Sine-Gordon and affine-Toda field theories. Moreover, the study of such kind of algebras would provide better understanding to the $(\hbar,\xi)$-deformed Virasoro and $W$ algebras which are recently under active study.

In this paper we are motivated to study both the elliptic generalizations of the algebras $A_{\hbar,\eta}(\hat{g})$ (or the pre-scaling algebras) and their associated infinite Hopf family of algebras.

The search for elliptic quantum algebras has been lasted for quite some years. Various elliptic deformed algebras have emerged in several different contexts, among them are the Sklyanin algebras of type $s\ell_N$, the algebra $A_{q,p}(\hat{s}\ell_2)$ [4, 5] and its generalization to the $s\ell_N$ case, $A_{q,p}(\hat{s}\ell_N)$, which forms the class of so-called vertex type elliptic algebras, and the “elliptic quantum groups” of Felder et al [1, 2, 3] and the dynamical twisted algebra of Hou et al [6] which form the class of so-called face type elliptic algebras. The above mentioned elliptic algebras are all realized through the (vertex and face type) Yang-Baxter relations. The difference between Sklyanin algebra and the algebras $A_{q,p}(\hat{s}\ell_N)$, as well as that between Felder et al’s algebra and Hou et al’s one lie in that, the modulus for the elliptic entries of $R$-matrices are the same for the former and are different for the latter algebras. Other examples of elliptic algebras are the algebras for the deformed screening currents for the quantum deformed $W$ algebras (defined for any simply-laced underlying Lie algebra $g$) of the first two of the present authors [12] and Konno’s algebra $U_{q,p}(\hat{s}\ell_2)$ [11].

We note that the classification for the elliptic deformed algebras seems far from complete yet. For example, the last two types of algebras are realized as current algebras only and their possible Yang-Baxter type realization are still unknown. Moreover, though the co-structures, or more explicitly, the quasi-Hopf structures, for the vertex and face type algebras realized through Yang-Baxter type relations has recently been clarified due to the work of Fronsdal and Jombo et al, similar structures in the current algebras of [12] are still unknown.

In this paper, we shall present a new type of elliptic current algebras which we denote as $E_{q,p}(\hat{g})$ (where $g$ can be any classical simply-laced Lie algebra) and study the associated infinite Hopf family of algebras structures. It turns out that the algebras $E_{q,p}(\hat{g})$ are quite similar to the algebras of modified screening currents for the quantum $(q,p)$-deformed $W$-algebras mentioned above in the level 1 bosonic representations. The only difference lies in that, for the algebras $E_{q,p}(\hat{g})$ at level 1, the deformation parameter $q$ is the inverse of the one in the algebras defined in [12] (whilst the parameter $\hat{q}$ is kept unchanged), and we assume here that $|q| < 1$, which corresponds to $|q| > 1$ in [12] (the algebras in [12], however, were defined only for $|q| < 1$ implicitly). This slight difference prevented us from defining the somewhat well-expected structure of infinite Hopf family of algebras in [12].

The organization of this paper is as follows. In section 2, we shall give a definition for the current algebra $E_{q,p}(\hat{g})$. Section 3 is devoted to the study of the structure of associated infinite Hopf family of algebras. In Section 4 we give the bosonic representation for the current algebras $E_{q,p}(\hat{g})$ at level 1. The final section—Section 5—is for some concluding remarks.
2 The elliptic current algebra $\mathcal{E}_{q,p}(\hat{g})$

We first give the definition for the elliptic current algebras $\mathcal{E}_{q,p}(\hat{g})$.

**Definition 2.1** The elliptic current algebra $\mathcal{E}_{q,p}(\hat{g})$ is the associative algebra generated by the currents $E_i(z)$, $F_i(z)$, $H_i^\pm(z)$ with $i = 1, 2, \ldots$, rank($g$), central element $c$ and the unit element $1$ with the following relations,

\begin{align*}
H_i^\pm(z)H_j^\pm(w) & = \frac{\theta_{q}(\frac{c}{w}p^{A_{ij}/2})}{\theta_{\bar{q}}(\frac{c}{w}p^{-A_{ij}/2})} H_j^\pm(w)H_i^\pm(z), \\
H_i^+(z)H_j^-(w) & = \frac{\theta_{q}(\frac{c}{w}p^{A_{ij}+c/2})}{\theta_{\bar{q}}(\frac{1}{w}p^{-A_{ij}-c/2})} H_j^-(w)H_i^+(z), \\
H_i^+(z)E_j(w) & = (-1)^{A_{ij}}(p^{-A_{ij}/2}) \frac{\theta_{q}(\frac{c}{w}p^{A_{ij}/2}p^{c/4})}{\theta_{\bar{q}}(\frac{c}{w}p^{-A_{ij}/2}p^{-c/4})} E_j(w)H_i^+(z), \\
H_i^-(z)E_j(w) & = (-1)^{A_{ij}}(p^{-A_{ij}/2}) \frac{\theta_{q}(\frac{c}{w}p^{A_{ij}/2}p^{-c/4})}{\theta_{\bar{q}}(\frac{c}{w}p^{-A_{ij}/2}p^{c/4})} E_j(w)H_i^-(z), \\
H_i^+(z)F_j(w) & = (-1)^{A_{ij}}(p^{A_{ij}/2}) \frac{\theta_{q}(\frac{c}{w}p^{A_{ij}/2}p^{c/4})}{\theta_{\bar{q}}(\frac{c}{w}p^{-A_{ij}/2}p^{-c/4})} F_j(w)H_i^+(z), \\
H_i^-(z)F_j(w) & = (-1)^{A_{ij}}(p^{A_{ij}/2}) \frac{\theta_{q}(\frac{c}{w}p^{A_{ij}/2}p^{-c/4})}{\theta_{\bar{q}}(\frac{c}{w}p^{-A_{ij}/2}p^{c/4})} F_j(w)H_i^-(z), \\
E_i(z)E_j(w) & = (-1)^{A_{ij}}(p^{-A_{ij}/2}) \frac{\theta_{q}(\frac{c}{w}p^{A_{ij}/2})}{\theta_{\bar{q}}(\frac{c}{w}p^{-A_{ij}/2})} E_j(w)E_i(z), \\
F_i(z)F_j(w) & = (-1)^{A_{ij}}(p^{A_{ij}/2}) \frac{\theta_{q}(\frac{c}{w}p^{A_{ij}/2})}{\theta_{\bar{q}}(\frac{c}{w}p^{-A_{ij}/2})} F_j(w)F_i(z), \\
[E_i(z), F_j(w)] & = \frac{\delta_{ij}}{(p - 1)zw} \left[ \delta \left( \frac{c}{w} \right) H_i^+(zq^{c/2}) - \delta \left( \frac{w}{z}q^{-c} \right) H_i^-(wq^{-c/2}) \right], \\
E_i(z_1)E_i(z_2)E_j(w) - f_{ij}^{(q)}(z_1/w, z_2/w) E_i(z_1)E_j(w) E_i(z_2) + E_j(w)E_i(z_1)E_i(z_2) & + \text{(replacements) } z_1 \leftrightarrow z_2 = 0, \quad A_{ij} = -1, \\
F_i(z_1)F_i(z_2)F_j(w) - f_{ij}^{(q)}(z_1/w, z_2/w) F_i(z_1)F_j(w) F_i(z_2) + F_j(w)F_i(z_1)F_i(z_2) & + \text{(replacements) } z_1 \leftrightarrow z_2 = 0, \quad A_{ij} = -1,
\end{align*}

where

\[
f_{ij}^{(a)}(z_1/w, z_2/w) = \left( \psi_{ij}^{(a)} \left( \frac{c}{w} \right) + 1 \right) \left( \psi_{ij}^{(a)} \left( \frac{c}{z} \right) \psi_{ij}^{(a)} \left( \frac{w}{z} \right) + 1 \right), \quad a = q, \ \bar{q},
\]

\[
\psi_{ij}^{(q)}(x) = (-1)^{A_{ij}}(p^{-A_{ij}/2}) \frac{\theta_{q}(x^{-1}p^{A_{ij}/2})}{\theta_{\bar{q}}(x^{-1}p^{-A_{ij}/2})},
\]

\[
\psi_{ij}^{(\bar{q})}(x) = (-1)^{A_{ij}}(p^{A_{ij}/2}) \frac{\theta_{\bar{q}}(x^{-1}p^{-A_{ij}/2})}{\theta_{q}(x^{-1}p^{A_{ij}/2})}.
\]
and $p, q$ are a pair of deformation parameters with norms $|q| < 1$ and $|p| < 1$, $z, w$ are spectral parameters, $\tilde{q}$ and $q$ are connected by the relation

$$\tilde{q}/q = p^\epsilon,$$

and $\theta_q(z)$ is the standard elliptic function given by

$$\theta_q(z) = (z|q)_\infty (q^{-1}|z)\infty (q|q)_\infty,$$

$$(z|q_1, \ldots, q_m)_\infty = \prod_{i_1, i_2, \ldots, i_m=0}^\infty (1 - zq_1^i q_2^i \ldots q_m^i).$$

Quite analogous to the case of $A_{\hbar,\eta}(\hat{g})$, the elliptic current algebra given above enjoys the following features,

- it has two deformation parameters $p, q$ and the “positive” and “negative” currents $E(z)$ and $F(z)$ are deformed differently (each corresponds to one of the two parameters $q$ and $\tilde{q}$ respectively);
- the currents $H_\pm^\pm(z)$ do not commute with themselves in contrast to the $q$-affine and Yangian cases.

These features are also shared by the algebras $A_{q,p}(\hat{sl}_N)$, $A_{q,p,\pi}(\hat{sl}_2)$ and $U_{q,p}(\hat{sl}_2)$.

The second feature has a rather significant consequence. If one consider the subalgebras generated by the currents $H^+(z), E(z)$ or $H^-(z), F(z)$, it would turn out that they do not form nilpotent or even solvable subalgebras. However, in the $q$-affine and Yangian cases similar subalgebras are indeed solvable and with the aid of a properly defined Manin pairing, they give rise to the structure of quantum doubles. The non-solvability of such subalgebras in our case might imply that the algebra $E_{q,p}(\hat{g})$ under consideration does not have a simple quantum double structure.

In order to show the more deep relationship between our algebra and the algebras $A_{\hbar,\eta}(\hat{g})$, we give the following proposition which show that the algebra $E_{q,p}(\hat{g})$ is an elliptic extension of $A_{\hbar,\eta}(\hat{g})$.

**Proposition 2.2** In the scaling limit

$$p = e^{\epsilon \hbar}, \quad q = e^{\epsilon \pi}, \quad z = e^{\epsilon u},$$

$$\epsilon \to 0$$

the algebra $E_{q,p}(\hat{g})$ tends to the algebra $A_{\hbar,\eta}(\hat{g})$ defined in [3].
We remark that for the case $g = \mathfrak{sl}_2$, both the algebra $\mathcal{A}_{q,p}(\mathfrak{sl}_2)$ and $U_{q,p}(\hat{\mathfrak{sl}}_2)$ would yield $\mathcal{A}_{\hbar,\eta}(\mathfrak{sl}_2)$ in the scaling limit. Therefore our algebra $\mathcal{E}_{q,p}(\hat{g})$ has the same scaling limit as those two algebras for the special underlying Lie algebra $g = \mathfrak{sl}_2$. However, for general simply-laced $g$, our algebra $\mathcal{E}_{q,p}(\hat{g})$ is the only known algebra which tends to $\mathcal{A}_{\hbar,\eta}(\hat{g})$ in the scaling limit. Actually, the generalization of $\mathcal{A}_{q,p}(\hat{sl}_2)$ to the case of $D, E$ series of Lie algebras are not known to exist. Likewise, the generalization of $U_{q,p}(\hat{sl}_2)$ to any other $g$ is also not known to exist. (We noticed the similarity between our algebra at $g = \mathfrak{sl}_2$ and $U_{q,p}(\hat{sl}_2)$. It is possible that these two algebras are isomorphic, however we do not make this claim because we did not make it out yet. 

Another remark is in order here. The algebra $\mathcal{E}_{q,p}(\hat{g})$, as well as $\mathcal{A}_{\hbar,\eta}(\hat{g})$ defined in [7], should be regarded as *current* algebras only since we do not know the corresponding Yang-Baxter type realizations. Actually, given a Yang-Baxter type relation one can define an associative algebra which is certain deformation of the universal enveloping algebra of some underlying Lie algebra, and, due to the well-known Ding-Frenkel homomorphism, one can find a corresponding current realization which is of important usage for the construction of infinite dimensional representations. However, the inverse to Ding-Frenkel homomorphism is some Riemann problem which often does not possesses a unique solution [9]. Therefore, given the definition of a current algebra such as $\mathcal{E}_{q,p}(\hat{g})$, one actually cannot associate a unique Yang-Baxter type relation without putting in extra constraints. It seems quite possible that both the vertex type and face type elliptic algebras can be obtained from the same current algebra $\mathcal{E}_{q,p}(\hat{g})$ by introducing different sets of constraints which lead to different solutions to the Riemann problem. We hope to consider this problem in later studies.

3 The structure of infinite Hopf family of algebras for $\mathcal{E}_{q,p}(\hat{g})$

The algebra $\mathcal{E}_{q,p}(\hat{g})$ defined in the last section is in fact the representative of an infinite Hopf family of elliptic algebras which we now specify.

Let $\{\mathcal{A}_n, n \in \mathbb{Z}\}$ be a family of associative algebras over $\mathbb{C}$ with unit. Let $\{v_i^{(n)}, i = 1, \ldots, \dim(\mathcal{A}_n)\}$ be a basis of $\mathcal{A}_n$. The maps

$$
\tau_n^\pm : \mathcal{A}_n \to \mathcal{A}_{n \pm 1}
$$

$$
v_i^{(n)} \mapsto v_i^{(n \pm 1)}
$$

are morphisms from $\mathcal{A}_n$ to $\mathcal{A}_{n \pm 1}$. For any two integers $n$, $m$ with $n < m$, we can specify a pair of morphisms

$$
\tau^{(m,n)} = Mor(\mathcal{A}_m, \mathcal{A}_n) \equiv \tau_{m-1}^+ \cdots \tau_{n+1}^+ \tau_n^+: \mathcal{A}_n \to \mathcal{A}_m,
$$

To compare with [11], we one should bare in mind that the following change of notations should be made: $q \rightarrow p$, $p \rightarrow q^2$ and $c \rightarrow -c$. 

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\[ \tau^{(n,m)} = \text{Mor}(A_n, A_m) \equiv \tau_{n+1}^{-} \cdots \tau_{m+1}^{-} \tau_m^{-} : A_m \to A_n \]  

with \( \tau^{(m,n)}\tau^{(n,m)} = \text{id}_m, \ \tau^{(n,m)}\tau^{(m,n)} = \text{id}_n \). Clearly the morphisms \( \tau^{(m,n)} \), \( n, m \in Z \) satisfy the associativity condition \( \tau^{(m,p)}\tau^{(p,n)} = \tau^{(m,n)} \) and thus make the family of algebras \( \{A_n, n \in Z\} \) into a category.

**Definition 3.1** The category of algebras \( \{A_n, \{\tau^{(n,m)}\}, n, m \in Z\} \) is called an infinite Hopf family of algebras if on each object \( A_n \) of the category one can define the morphisms \( \Delta_n^\pm : A_n \to A_n \otimes A_{n \pm 1}, \epsilon_n : A_n \to C \) and antimorphisms \( S_n^\pm : A_n \to A_{n \pm 1} \) such that the following axioms hold,

\begin{align*}
\bullet & \ (\epsilon_n \otimes \text{id}_{n+1}) \circ \Delta_n^+ = \tau_n^+, \ (\text{id}_{n-1} \otimes \epsilon_n) \circ \Delta_n^- = \tau_n^- \quad (a1) \\
\bullet & \ m_{n+1} \circ (S_n^+ \otimes \text{id}_{n+1}) \circ \Delta_n^+ = \epsilon_{n+1} \circ \tau_n^+, \ m_{n-1} \circ (\text{id}_{n-1} \otimes S_n^-) \circ \Delta_n^- = \epsilon_{n-1} \circ \tau_n^- \quad (a2) \\
\bullet & \ (\Delta_n^- \otimes \text{id}_{n+1}) \circ \Delta_n^+ = (\text{id}_{n-1} \otimes \Delta_n^+) \circ \Delta_n^- \quad (a3)
\end{align*}

in which \( m_n \) is the algebra multiplication for \( A_n \).

**Remark 3.2** We remark here that the presentation of infinite Hopf family of algebras is slightly different from that of \( U \) in the trigonometric case. However, the statement that the algebra \( A_{h,n}(\hat{g}) \) is a representative of an infinite Hopf family of trigonometric algebras still hold true under the present definition of infinite Hopf family of algebras.

Let \( A \) be an associative algebra over \( C \) with unit. A trivial example of infinite Hopf family of algebras is given by the category of algebras \( \{A_n \equiv A, \{\tau^{(n,m)} \equiv \text{id}_A\}, n, m \in Z\} \) with \( \Delta_n^\pm, \epsilon_n \) and \( S_n^\pm \) identified as the standard Hopf algebra structures over \( A \). This trivial example shows that the infinite Hopf family of algebras can be regarded as some deformation of the standard Hopf algebra structure. The maps \( \Delta_n^\pm, \epsilon_n \) and \( S_n^\pm \) in the infinite Hopf family of algebras are called comultiplications, counits and antipodes by this analogy.

Now let us consider the infinite Hopf family of algebras structure of our algebra \( E_{q,p}(\hat{g}) \). For this purpose we introduce some notations. First, we denote the algebra \( E_{q,p}(\hat{g}) \) by \( E_{q,p}(\hat{g})c \), specifying explicitly the central extension \( c \). We see that this algebra is determined uniquely as a current algebra by the defining relations (1) provided the following data are fixed: \( g, \ q, \ p, \ c \). In general, given a series of \( c_n, \ n \in Z \), we can define

\[ q^{(n+1)}/q^{(n)} = pc_n, \]

starting from the data \( q^{(1)} = q, \ c_1 = c \). It is obvious that \( \hat{q} = q^{(2)} \) and hence \( \hat{q}^{(n)} = q^{(n+1)} \). We collect the family of algebras \( \{E_{q^{(n)}, p}(\hat{g})_{c_n}, n \in Z\} \) where \( E_{q^{(n)}, p}(\hat{g})_{c_n} \) is the algebra \( E_{q^{(n)}, p}(\hat{g})c \) with \( q \) replaced by \( q^{(n)} \) and \( c \) by \( c_n \). The generating currents \( H_i^\pm(z), \ E_i(z) \) and \( F_i(z) \) for the algebra \( E_{q^{(n)}, p}(\hat{g})_{c_n} \) are denoted as \( H_i^\pm(z; q^{(n)}), \ E_i(z; q^{(n)}) \) and \( F_i(z; q^{(n)}) \) etc.
The family of algebras \( \{ E_{q(n),p}(\mathcal{G}) \}_{n \in \mathbb{Z}} \) can be easily turned into a category by introducing the morphisms \( \tau^\pm_n \)

\[
\tau^\pm_n : E_{q(n),p}(\mathcal{G})_{c_n} \rightarrow E_{q(n\pm 1),p}(\mathcal{G})_{c_{n \pm 1}}
\]

\[
H^+_i(z; q^{(n)}) \rightarrow H^+_i(z; q^{(n \pm 1)})
\]

\[
E_i(z; q^{(n)}) \rightarrow E_i(z; q^{(n \pm 1)})
\]

\[
F_i(z; q^{(n)}) \rightarrow F_i(z; q^{(n \pm 1)})
\]

\[
c_n \rightarrow c_{n \pm 1}
\]

and defining the compositions \( \tau^{(n,m)} \) as did in [3].

The following proposition is one of our major results.

**Proposition 3.3** The category of algebras \( \{ E_{q(n),p}(\mathcal{G})_{c_n} \}, \{ \tau^{(n,m)} \} \), \( n, m \in \mathbb{Z} \) form an (elliptic)
infinite Hopf family of algebras with the Hopf family structures given as follows:

- **the comultiplications** \( \Delta^\pm_i \):

  \[
  \Delta^+_n c_n = c_n + c_{n+1},
  \]

  \[
  \Delta^+_n H^+_i(z; q^{(n)}) = H^+_i(zp^{c_{n+1}/4}; q^{(n)}) \otimes H^+_i(zp^{-c_n/4}; q^{(n+1)}),
  \]

  \[
  \Delta^+_n E_i(z; q^{(n)}) = E_i(z; q^{(n)}) \otimes 1 + H^+_i(zp^{c_n/4}; q^{(n)}) \otimes E_i(zp^{c_{n-1}/4}; q^{(n+1)}),
  \]

  \[
  \Delta^+_n F_i(z; q^{(n)}) = 1 \otimes F_i(z; q^{(n+1)}) + F_i(zp^{c_{n-1}/2}; q^{(n)}) \otimes H^+_i(zp^{c_{n+1}/4}; q^{(n+1)}),
  \]

- **the counits** \( \epsilon_n \):

  \[
  \epsilon_n(c_n) = 0,
  \]

  \[
  \epsilon_n(1_n) = 1,
  \]

  \[
  \epsilon_n(H^+_i(z; q^{(n)})) = 1,
  \]

  \[
  \epsilon_n(E_i(z; q^{(n)})) = 0,
  \]

  \[
  \epsilon_n(F_i(z; q^{(n)})) = 0;
  \]
• the antipodes $S_{n}^\pm$:

$$S_{n}^\pm c_n = -c_{n\pm 1},$$

$$S_{n}^\pm H_i^+(z; q^{(n)}) = [H_i^+(z; q^{(n\pm 1)})]^{-1},$$

$$S_{n}^\pm H_i^-(z; q^{(n)}) = [H_i^-(z; q^{(n\pm 1)})]^{-1},$$

$$S_{n}^\pm E_i(z; q^{(n)}) = -H_i^-(z p^{-c_{n\pm 1}/4}; q^{(n\pm 1)})^{-1}E_i(z p^{-c_{n\pm 1}/2}; q^{(n\pm 1)}),$$

$$S_{n}^\pm F_i(z; q^{(n)}) = -F_i(z p^{-c_{n\pm 1}/2}; q^{(n\pm 1)})H_i^+(z p^{-c_{n\pm 1}/4}; q^{(n\pm 1)})^{-1}.$$

The proof for this proposition is by straightforward calculations.

**Remark 3.4** The comultiplications, counits and antipodes given above are analogous to the Drinfeld Hopf structures for q-affine algebras. The difference lies in that, instead of sending elements of the algebra $A_n$ into the tensor product space of the same algebra, the comultiplications $\Delta^\pm$ now send elements of $A_n$ into the tensor product spaces $A_n \otimes A_{n+1}$ and $A_{n-1} \otimes A_n$ respectively of two neighboring algebras in the family. The shift in the suffixes in the notations of target spaces indicate the crucial difference between nontrivial infinite Hopf family of algebras and trivial ones.

In order to understand the meaning of the unusual shift of suffixes mentioned above, we present here another proposition which was first found in [7] in the trigonometric case.

**Proposition 3.5** The comultiplication $\Delta^+_n$ induces an algebra homomorphism

$$\rho : \mathcal{E}_{q^{(n)},p}(\mathcal{g})_{c_n+c_{n+1}} \rightarrow \mathcal{E}_{q^{(n)},p}(\mathcal{g})_{c_n} \otimes \mathcal{E}_{q^{(n+1)},p}(\mathcal{g})_{c_{n+1}},$$

$$X \rightarrow \Delta^+_n X,$$

where $X \in \mathcal{E}_{q^{(n)},p}(\mathcal{g})_{c_n+c_{n+1}}$, $\tilde{X} \in \mathcal{E}_{q^{(n)},p}(\mathcal{g})_{c_n}$ and

$$\tilde{X} = \begin{cases} c_n & \text{if } X = \begin{cases} c_n + c_{n+1} & H_i^+(z; q^{(n)}) \\ E_i(z; q^{(n)}) & F_i(z; q^{(n)}) \end{cases} \\
H_i^+(z; q^{(n)}) & E_i(z; q^{(n)}) \\
E_i(z; q^{(n)}) & F_i(z; q^{(n)}) \end{cases}.$$

Likewise, the comultiplication $\Delta^-_n$ induces an algebra homomorphism

$$\tilde{\rho} : \mathcal{E}_{q^{(n-1)},p}(\mathcal{g})_{c_{n-1}+c_n} \rightarrow \mathcal{E}_{q^{(n-1)},p}(\mathcal{g})_{c_{n-1}} \otimes \mathcal{E}_{q^{(n)},p}(\mathcal{g})_{c_n},$$

$$X \rightarrow \Delta^-_n \tilde{X},$$

where $X \in \mathcal{E}_{q^{(n-1)},p}(\mathcal{g})_{c_{n-1}+c_n}$, $\tilde{X} \in \mathcal{E}_{q^{(n)},p}(\mathcal{g})_{c_n}$ and
\[ \hat{X} = \begin{cases} c_n \ \frac{H^+}{1} (z; q^n) \\ E_i (z; q^n) \\ F_i (z; q^n) \end{cases} \text{ if } X = \begin{cases} c_{n-1} + c_n \\ \frac{H^+}{1} (z; q^{n-1}) \\ E_i (z; q^{n-1}) \\ F_i (z; q^{n-1}) \end{cases} \].

**Corollary 3.6** Let \( m \) be a positive integer. The iterated comultiplication \( \Delta^{(m)+}_n = (id_n \otimes id_{n+1} \otimes ... \otimes id_{n+m-2} \otimes \Delta^{+}_{n+m-1}) \circ (id_n \otimes id_{n+1} \otimes ... \otimes id_{n+m-3} \otimes \Delta^{+}_{n+m-2}) ... \circ (id_n \otimes \Delta^{+}_{n+1}) \circ \Delta^{+}_n \) induces an algebra homomorphism \( \rho^{(m)} \)

\[ \rho^{(m)} : \mathcal{E}_{q(n), p} (\hat{g})_{c_n + c_{n+1} + ... + c_{n+m}} \to \mathcal{E}_{q(n), p} (\hat{g})_{c_n} \otimes \mathcal{E}_{q(n+1), p} (\hat{g})_{c_{n+1}} \otimes ... \otimes \mathcal{E}_{q(n+m), p} (\hat{g})_{c_{n+m}} \]

in the spirit of Proposition 3.5.

**Remark 3.7** We stress here that the maps \( \rho, \ \bar{\rho} \) and \( \rho^{(m)} \) are algebra homomorphisms, whilst \( \tau^{\pm}, \ \tau^{(n, m)} \) and \( \Delta^{\pm}_{n} \) etc are only algebra morphisms. The difference between algebra morphisms and algebra homomorphisms lies in that the latter preserves the structure functions whilst the former does not.

The above proposition and its corollary shows that although the algebras \( \mathcal{E}_{q(n), p} (\hat{g})_{c_n} \) are not co-closed, the tensor product representations can still be defined using the comultiplications \( \Delta^{\pm}_{n} \). In particular, if the \( c_n = 1 \) representation for \( \mathcal{E}_{q(n), p} (\hat{g})_{c_n} \) is available, then the representations at any positive integer \( c_n \) are all available, left aside the reducibility problems of such representations. This statement is of particular importance when one need to realize that the algebra \( \mathcal{E}_{q, p} (\hat{g})_{c} \) is non-empty for \( c \in \mathbb{Z}_{+} \setminus \{1\} \).

## 4 Free boson realization of the algebra \( \mathcal{E}_{q, p} (\hat{g}) \) at \( c = 1 \)

Having established the infinite Hopf family of algebras structure of the elliptic current algebra \( \mathcal{E}_{q, p} (\hat{g}) \), we now turn to consider its simplest infinite dimensional representation, i.e. the free boson realization at \( c = 1 \).

First we introduce the Heisenberg algebra \( \mathcal{H}_{q, p}(g) \) with generators \( a_i[n], \ P_i, \ Q_i, \ i = 1, ..., \) rank(\( g \)), \( n \in \mathbb{Z} \setminus \{0\} \) and generating relations

\[ [a_i[n], a_j[m]] = \frac{1}{n} \frac{(1 - q^{-n})(p^{nA_{ij} / 2} - p^{-nA_{ij} / 2})(1 - (pq)^n)}{1 - p^n} \delta_{n,m}, \]
\[ [P_i, Q_j] = A_{ij}, \]

where \( A_{ij} \) is the Cartan matrix for the Lie algebra \( g \). Let
\[ s^+_i[n] = \frac{a_i[n]}{q^n - 1}, \quad s^-_i[n] = -\frac{a_i[n]}{(pq)^{-n} - 1} \]

and define the (deformed) free boson fields

\[ \varphi_i(z) = \sum_{n \neq 0} s^+_i[n]z^{-n}, \quad \psi_i(z) = \sum_{n \neq 0} s^-_i[n]z^{-n}. \]

**Proposition 4.1** The following bosonic expressions give a level \( c = 1 \) realization for the algebra \( E_{q,p}(\hat{\mathfrak{g}}) \) on the Fock space of the Heisenberg algebra \( \mathcal{H}_{q,p}(\mathfrak{g}) \).

\[
\begin{align*}
E_i(z) &= e^{Q_i z P_i} : \exp[\varphi_i(z(pq)^{1/2})] : , \\
F_i(z) &= e^{-Q_i z P_i} : \exp[-\psi_i(zq^{1/2})] : , \\
H^+_i(z) &= : E_i(zp^{1/4})F_i(zp^{-1/4}) :, \\
H^-_i(z) &= : E_i(zp^{-1/4})F_i(zp^{1/4}) :, 
\end{align*}
\]

where \( : : \) means taking all subexpressions consisted of \( a_i[n] \) with \( n > 0 \) and \( P_i \) to the right of expressions consisted of \( a_i[n] \) with \( n < 0 \) and \( Q_i \).

The proof for this proposition is also by straightforward but tedious calculations.

**5 Concluding Remarks**

In this paper we obtained the new elliptic current algebras \( E_{q,p}(\hat{\mathfrak{g}}) \) and showed that these algebras have a structure of infinite Hopf family of algebras. So far we have obtained two kinds of nontrivial infinite Hopf family of algebras: trigonometric (for the algebras \( \mathcal{A}_{\hbar,\eta}(\hat{\mathfrak{g}}) \) ) and elliptic (for the algebras \( E_{q,p}(\hat{\mathfrak{g}}) \) ). It is thus an interesting question to ask whether there exists any rational algebras which has the same co-algebraic structure.

It is interesting to mention that the comultiplications appearing in such co-structures are all of the Drinfeld type, which closes over the currents themselves and does not require the resolution to the inverse problem (Riemann problem) of the Ding-Frenkel homomorphism. Recall that two kinds of comultiplications (and thus two kinds of Hopf algebra structures) are known for the standard \( q \)-affine algebras. The algebras \( \mathcal{A}_{\hbar,\eta}(\hat{\mathfrak{g}}) \) and \( E_{q,p}(\hat{\mathfrak{g}}) \) should be considered as some deformation of \( q \)-affine algebras and under such deformations the difference between the two Hopf algebra structures for \( q \)-affine algebras become clear: the standard Hopf structure for \( q \)-affine algebras is inherited into the Yang-Baxter type realizations for the algebras \( \mathcal{A}_{q,p}(\hat{\mathfrak{sl}}_2) \) and \( \mathcal{B}_{q,\lambda}(\hat{\mathfrak{g}}) \) [8] and define the quasi-triangular quasi-Hopf structures in those algebras, and the Drinfeld type Hopf structure is inherited into the current realizations for the algebras \( \mathcal{A}_{\hbar,\eta}(\hat{\mathfrak{g}}) \) and
\( \mathcal{E}_{q,p}(\hat{g}) \) and gives rise to the structure of infinite Hopf family of algebras. The relation between the quasi-triangular quasi-Hopf structure and infinite Hopf family of algebras is an interesting open problem to be answered in later studies.

We should emphasize that this work is only a preliminary study for the algebras \( \mathcal{E}_{q,p}(\hat{g}) \) themselves. Besides the definition and level 1 bosonic realization, we know very little about these algebras, especially their detailed representation theory, vertex operators, Yang-Baxter type realizations etc. The physical applications should also be considered.

Finally, the structures of infinite Hopf family of algebras is still poorly understood yet. We do not know whether there exists a quantum double construction over the infinite Hopf family of algebras and, if not, what kind of new structure will take the place of the standard quantum doubles. Also, the classical counterpart of the infinite Hopf family of algebras is unknown and it seems that all these problems deserve further investigations.

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