The smooth transition between the domains with a negative and positive gap. This is a limiting case of the previous model, which is applicable also to more general models of smooth transition. Then the elastic backscattering rate will be found. After that the optical absorption, due to transitions between linear and the Dirac states, will be found. The results will be summarized in the Discussion section. The calculation details are in the Appendix.

\textbf{Volkov-Pankratov Hamiltonian in 2D}

We will base on the VP Hamiltonian for 2D TI. The 2D system is located in plane (x, z). The transition between a TI and an OI is modeled by the one-dimensional dependence of gap \( \Delta(z) = \Delta_0 F(z/l) \), where \( \Delta_0 \) is half of the bulk gap so that dimensionless function \( F(z > 0) > 0 \) and \( F(z < 0) < 0 \). In particular, one can assume that \( F(z) = \tanh(z) \). If parameter \( l \to 0 \), the TI-OI transition is step-like. If \( l \) is large, the transition is smooth.
The two-dimensional VP Hamiltonian reads

$$H_0 = -\Delta(z)\tau_y + v\tau_z (k_x\sigma_x + k_z\sigma_z),$$

where $\tau$ and $\sigma$ are Pauli matrices, which act on orbital and spin subspaces, respectively.

The energy spectrum of Hamiltonian (7) for the TI-OI transition $\Delta(z) = \Delta_0\tanh(z/l)$

$$E_{n,\sigma}^\lambda(k_x) = \lambda\sqrt{v^2k_x^2 + \epsilon_{n,\sigma}^2},$$

$$\epsilon_{n,\sigma}^2 = \Delta_0^2 \left[ 1 - \left( 1 - \frac{n + 1 + \sigma}{2} \frac{v}{\Delta_0} \right)^2 \right],$$

where $n$ are integers, $\lambda = \pm$, and $0 \leq n + (1 + \sigma)/2 \leq \Delta_0/v$.

For $n = 0, \sigma = -1$ we have the Weyl branch with linear spectrum $E_{0,-1}^\lambda = \lambda vk_x$. The other Dirac-like branches with $n, \sigma = 1$ and $n + 1, \sigma = -1$, have gaps. These states are double-degenerate (see Fig. 1).

![Graph of edge states for $\Delta(z) = \Delta_0\tanh(z/l)$](image)

**FIG. 1:** Edge states for $\Delta(z) = \Delta_0\tanh(z/l)$ (green, solid) and for $\Delta(z) = \Delta_0(z/l)$ (red, dashed). The chosen parameters are $\Delta_0 = 22$ meV and $v = 4.2 \times 10^5$ cm/s, and that corresponds to HgTe layer widths 55 Å and 67.6 Å, and $l = 1750.5$ Å. The lowest exact and approximate states coincide. At large numbers the exact states condense when approaching the boundary of 2D states (filled, light blue).

**Elastic backscattering of edge-state-electrons in a two-dimensional topological insulator**

The presence of Dirac edge states inside the characteristic gap of two-dimensional edge states is due to the smooth transition between the positive and negative gaps. A typical transition scale is described by letter $l$.

The two-dimensional gap $2\Delta_0$. The typical distance to the first Dirac state is $\sqrt{\Delta_0v/l}$, where $v$ is the Fermi velocity; the linear branches are of the form $\pm vk_x$, which is less than $\Delta_0$ for $\Delta_0 \gg v/l$. With an abrupt transition, only a linear branch remains inside the two-dimensional gap. When $l$ is very large, the transition can be replaced by a linear dependence $\Delta = \Delta_0z/l$. This dependence leads to an exactly solvable problem. The spectrum consists of a pair of linear branches $\pm vk_x$ and Dirac branches.

In the narrow energy range $|E| < \sqrt{\Delta_0v/l}$, there are only linear topologically protected branches. However, outside the narrow region, these states overlap in energy with Dirac states. At low temperatures, the elastic transition processes between linear and Dirac and between Dirac states are allowed. This gives the backscattering of electrons, including those on linear branches, that is, from a state with a velocity $v$ with momentum $k_z$ to a state with a velocity $-v$ with momentum $-k_z$.

The expected process is two-step: $vk_x \rightarrow \sqrt{v^2k_x^2 + \epsilon_{n,\sigma}^2} \rightarrow (-v)(-k_x)$.

Consider the probability of transition between states $n, k$ and $n', k'$ under the action of $\exp(\imath qz)\exp(\imath q_x x)$. The Dirac-like branches with $n, \sigma = 1$ and $n + 1, \sigma = -1$, have gaps. These states are double-degenerate (see Fig. 1).

**II. LIGHT ABSORPTION IN A TWO-DIMENSIONAL TOPOLOGICAL INSULATOR**

Later on we shall deal with a smooth transition. When the edge state size $\xi = v/\Delta_0$ is less than the transition width $l$, the dependence $\Delta(z)$ may be expanded as $\Delta(z) = \Delta_0z/l$. In such case the edge states can be expressed via oscillator wave functions.

Here it is convenient to use a variant of the Hamiltonian that was originally proposed by Zhang et al.

$$H_0 = \Delta(z)\tau_z + v(k_z\tau_y - k_z\tau_z\sigma_y).$$
sive VP states. States with linear dispersion, and \( c, \) which satisfy the usual commutation relations 
\[
\hat{c} \hat{c}^{\dagger} = \hat{c}^{\dagger} \hat{c} + 1 \]
where the ladder operators are
\[
\hat{c} = \frac{\lambda}{\sqrt{2}} \left( \frac{z}{\lambda} + ik \right) \quad \text{and} \quad \hat{c}^{\dagger} = \frac{\lambda}{\sqrt{2}} \left( \frac{z}{\lambda} - ik \right),
\]
which satisfy the usual commutation relations \([\hat{c}, \hat{c}^{\dagger}] = 1\), \( \lambda \in \sqrt{2} \). The Hamiltonian \( H \) has the energy spectrum
\[
E_n(k_x) = \lambda v \sqrt{k_x^2 + \frac{2n}{l_0^2}},
\]
where \( \lambda = \pm \). The case \( n = 0 \) corresponds to the edge states with linear dispersion, and \( n > 0 \) represents massive VP states.

The Kubo formula for light polarized along the \( z \)-direction is
\[
\sigma_{zz}(\omega) = i e^2 \sum_{\lambda, \lambda'} \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} \frac{|\langle \psi_{\lambda'}^{\dagger} | \hat{v}_z | \psi_{\lambda} \rangle|^2}{E_n^{\lambda'} - E_n^{\lambda}} \times \frac{f(E_n^{\lambda}) - f(E_n^{\lambda'})}{E_n^{\lambda'} - E_n^{\lambda} - \omega + i\delta}
\]
Here, \( f(E) \) is the Fermi-Dirac distribution function, \( \hat{v}_z = v \tau_y \), \( \delta \to +0 \). For the light polarized along the \( z \)-direction, according to Eq. (15), only the transitions \( n \to n \pm 1 \) are allowed. For \( E_F > 0 \), three transition types can be distinguished. The transitions between the states with \( \lambda = - \) and \( \lambda = + \) are possible at high frequencies, starting from \( \omega > \sqrt{2}v/l_0 \). At frequencies below this value, transitions are possible only between the states with \( \lambda = + \).

\[
\Re[\sigma_{zz}(\omega)] = \frac{e^2}{2\omega} \sum_{m,n \in N} \int_{-\infty}^{\infty} dk_x |\langle \psi_{\lambda}^{\dagger} | \hat{v}_z | \psi_{\lambda'} \rangle|^2 \times \\
\delta \left( \omega - (E_m^{\lambda'} - E_n^{\lambda}) \right) \left( f(E_n^{\lambda}) - f(E_m^{\lambda'}) \right). \tag{12}
\]

In the Fig. 3 is the optical conductivity dependence \( \Re[\sigma_{zz}(\omega)] \) on \( \omega \) for different values of \( E_F \).
III. CONCLUSIONS

We see that, in the absence of scattering, the selection rules allow transitions between the Dirac branches or between the Dirac and linear branches polarization of external in-plane microwave electric field across the edge. Besides, they do not allow the transitions between the linear branches for any external microwave electric field polarization. The transitions are allowed between neighboring transversal states. This statement is valid, however, in the assumption of linear dependence of the gap on the transversal coordinate, and that corresponds to the frequency much less than the gap value apart from edge $2\Delta_0$.

The selection rules determine the lower threshold for light absorption $\sqrt{\Delta_0/\nu l}$. This differs from that for smooth edge from the steep one, where the absorption is limited from below by the quantity $\Delta_0$ due to the processes of transitions between the edge and two-dimensional states.

The light absorption oscillates with the light frequency and the Fermi level, and has the $1/\sqrt{\omega - \omega_p}$ singularities at the thresholds of the pair density of states.

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IV. APPENDIX I. ELECTRONS STATES FOR $\Delta(z) = \Delta_0 \tanh(z/l)$

The eigenfunctions of the Hamiltonian (1) for $\Delta(z) = \Delta_0 \tanh(z/l)$ are

$$\psi_{n,1}^{\lambda} = \begin{pmatrix} \nu k_x, E_{\nu n,\sigma}^{\lambda} \left( \Delta + v \frac{\partial}{\partial z} \right), 1, 0 \end{pmatrix} \Psi_{n,\sigma}, \quad n \geq 1,$$

$$\psi_{n,2}^{\lambda} = \begin{pmatrix} \nu k_x, E_{\nu n,\sigma}^{\lambda} \left( \Delta - v \frac{\partial}{\partial z} \right), 1, 0 \end{pmatrix} \Psi_{n,\sigma}, \quad n \geq 1,$$

$$\psi_{0}^{\lambda} = \begin{pmatrix} \lambda, \nu k_x \left( \Delta + v \frac{\partial}{\partial z} \right), 0, 1 \end{pmatrix} \Psi_{n,\sigma}, \quad n = 0,$$

Here

$$\Psi_{n,\sigma}(\eta) = \frac{A_{n,\sigma} \eta!}{(\varepsilon + 1)n} (1 - \eta^2)^{\varepsilon/2} P_{\nu}^{(\varepsilon,\gamma)}(\eta),$$

where $P_{\nu}^{(\varepsilon,\gamma)}(\eta)$ are Jacobi polynomials, $(\varepsilon + 1)n$ is the Pochhammer symbol, $\eta = \tanh(z/l)$ and $\varepsilon = \Delta_0/v - n - (1 + \sigma)/2$, $A_{n,\sigma}$ are normalization constants.

V. APPENDIX II. ELECTRONS STATES AND MATRIX ELEMENTS FOR $\Delta(z) = \Delta_0 z/l$

The eigenstates of Hamiltonian (8) for $\Delta(z) = \Delta_0 z/l$ are

$$|\psi_{n}^{\lambda} \rangle = (a_{1,n}^{\lambda}|n-1\rangle, a_{2,n}^{\lambda}|n-1\rangle, a_{3,n}^{\lambda}|n\rangle, a_{4,n}^{\lambda}|n\rangle), \quad n \geq 1,$$

$$|\psi_{0}^{\lambda} \rangle = (0, 0, a_{3,0}^{\lambda}|0\rangle, a_{4,0}^{\lambda}|0\rangle), \quad n = 0,$$

where $|n\rangle$ is the eigenstate of harmonic oscillator determined by $\hat{c}_l^\dagger$ and $\hat{c}$. The Hamiltonian written in this basis reads:

$$H_T(n) = v \begin{bmatrix} 0 & ik_x & \sqrt{2n} l_0 \sqrt{1 + \delta} & 0 \\ -ik_x & 0 & 0 & -ik_x \\ \sqrt{2n} l_0 & 0 & 0 & \sqrt{2n} l_0 \\ 0 & \sqrt{2n} l_0 & ik_x & 0 \end{bmatrix}, \quad (13)$$

The corresponding normalized eigenvectors of the Hamiltonian above are:

$$\psi_{n,1}^{\lambda} = \frac{1}{\sqrt{2}} (i \cos \alpha_n, \lambda, 0, \sin \alpha_n), \quad n \geq 1, (14)$$

$$\psi_{n,2}^{\lambda} = \frac{1}{\sqrt{2}} (\lambda, -i \cos \alpha_n, \sin \alpha_n, 0), \quad n \geq 1, (15)$$

$$\psi_{0}^{\lambda} = \frac{1}{\sqrt{2}} (0, 0, i, -\lambda), \quad n = 0. (16)$$

For $n \geq 1$,

$$\cos \alpha_n = \frac{k_x}{\sqrt{k_x^2 + 2n l_0^2}}, \quad \sin \alpha_n = \frac{2n l_0}{\sqrt{k_x^2 + 2n l_0^2}} (17)$$

Matrix elements

$$\langle \psi_{\nu,1}^{\lambda*} | \hat{v}_z | \psi_{\nu,1}^{\lambda} \rangle = iv \times \left[ (a_{3,1,m,d}^\dagger a_{4,1,n,d} - a_{4,1,m,d}^\dagger a_{3,1,n,d}) \delta_{m-1,n} - a_{4,1,m,d}^\dagger a_{3,1,n,d}^\dagger (a_{3,1,n,d} a_{4,1,m,d} \delta_{m,n-1}) \right]. \quad (18)$$

$$\langle \psi_{\nu,1}^{\lambda*} | \hat{v}_z | \psi_{\nu,1}^{\lambda} \rangle = iv \times \left[ \left( a_{3,1,m,d}^\dagger a_{4,1,n,d}^\dagger + a_{4,1,m,d}^\dagger a_{3,1,n,d}^\dagger \right) \delta_{m,n} \right].$$
Here \( \nu_n(k) = \omega - v\left(\sqrt{k^2 + \frac{2(n+1)}{l_0^2}} + \sqrt{k^2 + \frac{2n}{l_0^2}}\right) \), \( \eta_n(k) = \omega - v\left(\sqrt{k^2 + \frac{2(n+1)}{l_0^2}} - \sqrt{k^2 + \frac{2n}{l_0^2}}\right) \), \( k_n = \sqrt{\left(\frac{\omega}{2v}\right)^2 - \frac{2n+1}{l_0^2} + \left(\frac{v}{l_0}\right)^2} \).

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