W–Infinity Structure
of the $sl(N)$ Conformal Affine Toda Theories

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Abstract

We reexamine the $W_\infty$ symmetry of the $sl(N)$ Conformal Affine Toda theories. It is shown that it is possible to reduce (nonuniquely) the zero curvature equation to a Lax equation for a first order pseudodifferential operator, whose coefficients are the generators of the $W_\infty$ algebra. This clarifies the known relation between the Conformal Affine Toda theories and the KP hierarchy. A possible correspondence between the matrix models and the Conformal Affine Toda models is discussed.
1 Introduction

The study of the higher–spin extensions of the Virasoro algebra has acquired a central role in the two–dimensional physics. In the conformal field theory \[1\] the \( W_n \) algebras appear as chiral algebras of a huge set of rational conformal field theories. In their classical version the \( W_n \) algebras underlie the integrability structure of the \( n \)th Korteweg–de Vries (KdV)–type hierarchy and related to them matrix Drinfeld–Sokolov (DS) hierarchies \[2,3\]. Both of these two types of hierarchies are integrable hamiltonian systems possessinig a pair of Poisson brackets which are coordinated (i. e. any linear combination of them is again a Poisson bracket). The integrable hierarchies of KdV–type arise in the scaling limit of the matrix models of the two–dimensional gravity coupled to \( c \leq 1 \) conformal matter \[4\] (for a review see \[5\]) and have a deep geometrical interpretation \[6\]. Recently it has been realized that it is not necessary to go to the scaling limit in order to recover integrable hierarchies \[7\].

A natural generalization of the finite classical \( W \) algebras appears when one takes the limit \( n \to \infty \). In this limit the nonlinear part of the second Gelfand–Dickey (GD) bracket disappears and one obtains the so called \( w_\infty \) algebra \[8\]. Another way to construct \( W\)–infinity algebras is to pass directly to the infinite generalization of the KdV–type hierarchies, the Kadomtsev–Petiaushvili (KP) hierarchy. The first GD bracket \[9\] produces the \( W_\infty \) algebra studied in \[10\] . The nonlinear generalizations of \( W\)–infinity algebra are related to the second GD bracket. In \[11\] a large class of \( W \)-infinity algebras are realized in terms of two fields of spin one and two. This construction applies to the Conformal Affine Toda (CAT) theories \[12\]. The \( W\)–symmetry of the CAT models is a strong indication that they are integrable and that their integrability structure is dictated by the KP hierarchy.

In this paper the \( W_\infty \) symmetry of \( sl(N) \) CAT is reconsidered from the point of view of the inverse scattering method. Starting from the spatial component of the Lax connection we show that after a suitable gauge transformation it assumes a very simple form: its matrix part belongs to the Heisenberg subalgebra associated with the principal gradation of the affine Lie algebra \( sl(N) \) plus a term proportional to the derivation of \( sl(N) \). Our approach is similar to the approach used in \[13,14\] to construct a large class of DS–type integrable hierarchies. The main result in these papers is that for each positive graded element of a Heisenberg subalgebra of an untwisted affine Lie algebra there exist a collection of hierarchies related through Miura maps. Therefore, modulo Miura transformations, the inequivalent hierarchies are given by inequivalent Heisenberg subalgebras (the classification of the graded regular elements which belong to certain Heisenberg subalgebra of the loop algebra \( gl(n) \) has been done in \[15\]). The situation in CAT is however different. In this way one can produce only a finite number of conserved quantities. A further reduction in getting the \( W\)–infinity currents consists in restricting the Lax connection on the Fock space associated with grade one element (and its conjugate) of the Heisenberg subalgebra. On this subspace the matrix part of the Lax connection is closely related to the spectral matrix in the one–matrix models.
Therefore it seems reasonable to state that the multiplication by the spectral parameter in the matrix models can be identified with the differentiation along the space direction in CAT. The problem of the reduction to the KP hierarchy is also considered. It turns out that this reduction is not unique. On the other hand, it has been proven recently [16] that both the two– and the four–boson KP hierarchies are gauge equivalent to the standard KP hierarchy.

2 The Drinfeld-Sokolov Reduction

This section is devoted to a review of the results of [2]. The basic idea is very simple: a system of \( n \) first–order differential equations \( \mathcal{L} \xi = 0 \) (\( \xi \) is a \( n \)-dimensional vector) for a certain class of first–order matrix differential operators can be reduced to a \( n \)-th order scalar differential equation \( \mathcal{L} \xi = 0 \). Drienfeld and Sokolov considered the case when \( \mathcal{L} \) has the following form

\[
\mathcal{L} = \partial_x + Q(x) + \Lambda \quad (1)
\]

where \( Q(x) \) is a lower triangular \( n \times n \) matrix and \( \Lambda = \mathcal{E}_+ + \lambda \mathcal{E}^{-1}, \Lambda^{-1} = \mathcal{E}_- + \lambda^{-1} \mathcal{E}_+^{-1} \). Here and in what follows we shall use the notations \( \mathcal{E}_+ = \sum_{i=1}^{n-1} E_{i+1,i} \), and \( E_{ij} \) is the matrix having one at \((i, j)\)-th site and zero elsewhere. The extra variable \( \lambda \) is usually called the spectral parameter; \( Q(x) + \Lambda \) will be called the matrix part of \( \mathcal{L} \). One easily checks that the identities \( \Lambda^{\pm i} = \mathcal{E}^{\pm i}_+ + \lambda^{\pm 1} \mathcal{E}_+^{\pm i} \) for \( i = 1, \ldots n-1 \) and \( \Lambda^n = \lambda \) are valid. The adjoint action of the element \( \Lambda \) permutes the diagonal matrices \( \Lambda E_{ii} \Lambda^{-1} = E_{i-1,i-1} \).

The evolution of \( \mathcal{L} \) is determined by the Lax equation

\[
\partial_t \mathcal{L} = [\mathcal{A}, \mathcal{L}] \quad (2)
\]

which implies that the spectrum of \( \mathcal{L} \) does not depend on the time parameter \( t \). The operator \( \mathcal{A} \) in the Lax equation cannot be arbitrary. In order to get a consistent time flow one should require that the commutator \([\mathcal{A}, \mathcal{L}]\) has the same form as the matrix \( Q \).

There is a standard procedure (see for example [2], [3]) to build up evolution equations for the operator \( \mathcal{L} \). Given a resolvent \( \mathcal{M} \) of \( \mathcal{L} \), i.e. a formal Laurent expansion \( \mathcal{M} = \sum_{i=0}^{\infty} \mathcal{M}_i \Lambda^i \) with coefficients being diagonal matrices\footnote{It is easy to show that any Laurent polynomial on \( \lambda \) can be uniquely rewritten in powers of \( \Lambda \) with diagonal coefficients} which commutes with \( \mathcal{L} \), one sets \( \mathcal{A} = \mathcal{M}_+ = \sum_{i \geq 0} \mathcal{M}_i \Lambda^i \) (the negative part of \( \mathcal{M} \) is defined as \( \mathcal{M}_- = \sum_{i \leq -1} \mathcal{M}_i \Lambda^i \)). For generic lower diagonal \( Q \) this choice produces a consistent flow since \([\mathcal{M}_+, \mathcal{L}]\) is a polynomial on \( \lambda \) while \([\mathcal{M}_-, \mathcal{L}]\) contains only nonpositive powers of \( \lambda \) and its free term results to be a lower diagonal matrix. In order to find the space of the resolvents \( \mathcal{R}_\mathcal{L} \) of \( \mathcal{L} \) consider the gauge transformation

\[
\mathcal{L}^G = G \mathcal{L} G^{-1} = \partial_x G G^{-1} + G \mathcal{L} G^{-1} \quad (3)
\]
such that
\[ \mathcal{L}^G = \partial_x + \Lambda + \sum_{i=0}^{\infty} h_i(x) \Lambda^{-i} \]  
(4)

where \( h_i \) are functions. The solution is given by the expansion
\[ G(x, \lambda) = 1 + \sum_{i=1}^{\infty} g_i(x) \Lambda^{-i} \]  
(5)

where \( g_i \) should satisfy the recursion relations
\[ h_i(x) + \Lambda g_{i+1}(x) \Lambda^{-1} - g_{i+1}(x) = Q_i(x) + \sum_{p=0}^{i-1} g_{i-p} \Lambda^{p-i} Q_i \Lambda^{i-p} - \]
\[ -\frac{\partial g_i(x)}{\partial x} - \sum_{p=0}^{i-1} h_{i-p}(x) \Lambda^{-p} g_i \Lambda^{p} \]  
(6)

Any diagonal matrix \( S \) can be represented (nonuniquely) as \( S = \text{Tr}S + \Lambda g \Lambda^{-1} - g \) where \( g \) is diagonal and therefore there exists a (formal) gauge transformation which brings \( \mathcal{L} \) into the form (4). The advantage gained by this gauge transformation is that the resolvent \( R_{\mathcal{L}G} \) has a very simple form: it consists of all the expansions \( \sum_i m_i \Lambda^i \) where the coefficients \( m_i \) are constant scalar matrices. Therefore the operators
\[ \mathcal{A} = \sum_{i \geq 0} m_i \left( G \Lambda^i G^{-1} \right)_+ \]  
(7)

provide consistent equations of motion (2). Moreover, from the gauge transformed Lax equation \( \partial_t \mathcal{L}^G = [\mathcal{A}^G, \mathcal{L}^G] \) where \( \mathcal{A}^G = \partial_t GG^{-1} + GAG^{-1} = \sum_i a^G_i(x) \Lambda^i \) it follows that all the coefficients in the expansion of \( \mathcal{A}^G \) in powers of \( \Lambda \) are scalar matrices. The coefficients \( a^G_i \) are constants on \( x \) for negative \( i \) and \( \partial_t h_i + \partial_x a^G_i = 0 \) for \( i \geq 0 \). Therefore the evolution equation (2) possesses an infinite number of conserved quantities
\[ I_i = \int h_i(x) dx \]  
(8)

Another intriguing property of the flows generated by the operators (7) is that they commute. More precisely, if \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) are resolvents of \( \mathcal{L} \), \( \mathcal{A}_i \) are their positive parts \( \mathcal{A}_i = (\mathcal{M}_i)_+ \), then the evolution equations \( \partial_t \mathcal{L} = \partial_t \mathcal{L} = [(\mathcal{A}_i)_+, \mathcal{L}] \) are in involution \( \partial_t [\partial_1, \partial_2] L = 0 \). The last identity is equivalent to the condition that the curvature \( F_{12} = \partial_1 \mathcal{A}_2 - \partial_2 \mathcal{A}_1 - [\mathcal{A}_1, \mathcal{A}_2] \) is a resolvent of \( \mathcal{L} \). A stronger result is valid, this curvature vanishes identically. In proving this one first observes that \( \partial_t \mathcal{M}_j = [\mathcal{A}_i, \mathcal{M}_j] \) and therefore \( F_{12} = [\mathcal{A}_1, \mathcal{M}_2]_+ + [\mathcal{M}_1, \mathcal{A}_2]_+ + [\mathcal{A}_1, \mathcal{A}_2] \) which vanishes since \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) are commuting.

\[ 2 \]This approach is alternative to the direct diagonalization of the matrix part of \( \mathcal{L} \) considered in [17]
Technically it is quite cumbersome to work with general lower triangular matrices \(Q\) in (1). There exists a gauge transformation \(S\) such that the gauge transformed matrix

\[
U = Q^S = \partial_x SS^{-1} + S(Q + \Lambda)S^{-1} - \Lambda
\]

has only one non-vanishing row: the last one

\[
U = -\sum_{k=0}^{n-1} u_{n-k} E_{n-k}\tag{9}
\]

Till the end of this section we shall skip the index \(S\) for brevity

\[L = \partial_x + \Lambda + U.\]

It is instructive to return to the problem of the construction of consistent evolution equations in this gauge. First we introduce some useful notations. It will be convenient to consider matrices as operators in the space of integral (Volterra) operators spanned by the symbols \(\partial^{-i}\) \((i \geq 1)\) factorized by the subspace of pseudodifferential operators (PDO) of degree less than \(-n\). The multiplication is defined by the generalized Leibniz rules

\[
\partial^{-i-1} f = \sum_{l=0}^{\infty} (-)^l \binom{i + l}{l} f^{(l)} \partial^{-i-l-1} f
\]

for arbitrary function \(f\) on \(x\). Given a matrix with entries \(A_{ij}, i, j = 0, 1, \ldots n - 1\) the vector–columns and vector–rows are introduced as follows

\[
A^i = \sum_{l=0}^{n-1} (-\partial)^{-i-l} A_{li} \quad A_i = \sum_{l=0}^{n-1} A_{il} (-\partial)^l
\tag{10}
\]

In these notations the multiplication of two matrices can be written as \((AB)^j = \sum_i (\partial)^{-i-1} \text{res}(A_i B^j), (AB)_i = \sum_j \text{res}(A_i B^j) (-\partial)^j\) where the residue of the PDO \(X = \sum X_i (-\partial)^i\) is defined to be \(X_{-1}\). The consistensy of (2) implies \([L, A_i] = 0, i = 0, \ldots, n - 2\) and the equations of motions read \(\partial_t U_{n-1} = -[L, A]_{n-1}\). These equations can be written in the following form

\[
A_{i+1} = -\partial A_i - A_{i-1} L(\lambda) \quad A_n = \partial_t U_{n-1} - \sum_{l=0}^{n-1} u_{n-l}(\lambda) A_l
\]

\[
L(\lambda) = (-\partial)^n + \sum_{i=0}^{n-1} u_{n-i}(\lambda) (-\partial)^i
\tag{11}
\]

where \(u_{n-i}(\lambda) = u_{n-i} - \lambda \delta_{i,0}\). The solution of the upper recursion relation is

\[
A_i = (-\partial)^i A_0 - \left((-\partial)^i A^{n-1}\right)_+ L(\lambda)\tag{12}
\]

and \(X_+\) is the differential part of a PDO \(X; X_- = X - X_+\) is expanded on negative powers of \(\partial\). Setting \(i = n\) in (12) and taking into account that due to the second
\[ \frac{\partial L}{\partial t} = -H(\lambda)(A^{n-1}) \quad H(\lambda)(X) = (L(\lambda)X)_+ L(\lambda) - L(\lambda)(XL(\lambda))_+ \quad (13) \]

\( H(\lambda) \) is the Adler map\(^3\). This completes the reduction of the matrix evolution equation (2) to an evolution equation for a \( n \)th order (scalar) differential operator. Note also that the Adler map splits into two terms \( H(\lambda) = H_2 + \lambda H_1 \) where:

\[
H_1(X) = [X_-, L] \quad H_2(X) = (LX)_+ L - L(XL)_+ \]

(14)

where \( L = L(0) \) and the following recursion relations are obviously valid

\[
H^0(L^{\frac{-n}{n-1}}) + H^\infty(L^\frac{n}{n-1}) = 0 \quad (15)
\]

For \( \lambda \to \infty \) and \( A^{n-1} = L^m \) in (13) the equations of motion coincide with the \( m \)th flow of the \( n \)th KdV–type hierarchy:

\[
\frac{\partial L}{\partial t_m} = -[L^m, L] = [L^m_+, L] \]

(16)

which are hamiltonian with respect to the Gelfand–Dickey (GD) brackets

\[
\{ < L, X >, < L, Y > \}_i^{GD} = < H_i(X)Y > \quad < L, X > = \int \text{res}LXd\lambda \quad (17)
\]

The hamiltonians \( I_{m+n} = -\frac{n}{m+n} \int \text{res}L^\frac{m+n}{n} \) generate the \( m \)th KdV flow with respect to the first GD bracket while the corresponding hamiltonian with respect to the second GD bracket is \( -I_m \). This is seen from the recursion relations (17) written in terms of the GD brackets

\[
\{I_{i+n}, L\}_1 + \{I_i, L\}_2 = 0 \quad (18)
\]

Using the recursion relations it is easy to show that the hamiltonians \( I_i \) are in involution with respect to the both GD brackets. On the other hand the equations (14) allow from a given PDO \( X \) to reconstruct the matrix \( A(X) \) (using the identification between vectors and PDO’s the coefficients of \( X \) are related to the elements of the \( n \)th column

\(^3\)due to the identity \( H(\lambda)(X) = -(L(\lambda)X)_- L(\lambda) + L(\lambda)(XL(\lambda))_- \) it follows that the image of the Adler map belongs to the set of the differential operators of order not greater than \( n - 1 \). The Adler map annihilates the pure differential operators and the PDO of order less than \(-n\).
of $\mathcal{A}$ through the relation $A^{n-1} = X$). Moreover, the Drienfel-Sokolov (DS) brackets coincide with the GD brackets.

$\{< U, A(X)_, < U, A(Y) > \}^{DS} = \int Tr [\mathcal{L}, A(X)] A(Y) dx = \int res H(\lambda) (X) Y dx = \{< L, X >, < L, Y > \}^{GD}$ (19)

We shall leave this section with the following remark. The coefficients $h_i$ in the expansion (4) and the KdV hamiltonian densities $res L$ are not independent. For each $i = 1, 2, \ldots$ there is a linear combination of these two densities which is a total derivative (see [2] for a detailed proof).

3 Generalization to the $sl(N)$ Conformal Affine Toda Theories

In this section we shall generalize the approach from the previous section to the $sl(N)$ Conformal Affine Toda theories [12]. The equations of motion can be written in a Lax form (2) but the matrix part of $L$ belongs to the affine Lie algebra $\hat{g} = \hat{sl}(N)$. Therefore we shall need some Lie algebraic background. As a basis in $\hat{g}$ we choose the generators $E^{ij}_n (i, j = 1, \ldots N, i \neq j)$, $H^\alpha_i$ ($i = 1, \ldots N-1 = \text{rank} sl(N)$), $\hat{c}$ and $\hat{d}$ with the commutation relations

$[E^{ij}_n, E^{kl}_m] = \delta^{jk} E^{il}_{n+m} - \delta^{il} E^{kj}_{n+m} + n \hat{c} \delta^{jk} \delta^{il} \delta_{n+m,0}$

$[H^{\alpha_i}_n, E^{kl}_m] = \left( \delta^{ik} - \delta^{i+1k} - \delta^{il} + \delta^{i+1l} \right) E^{kl}_{n+m}$

$[\hat{d}, E^{ij}_n] = (Nn + j - i) E^{ij}_n$

$[H^{\alpha_i}_n, H^{\alpha_j}_m] = n \hat{c} \left( 2 \delta^{ij} - \delta^{ij+1} - \delta^{i+1j} \right)$

$[\hat{d}, H^{\alpha_i}_n] = Nn H^{\alpha_i}_n$ (21)

and $\hat{c}$ is the central element. The derivation $\hat{d}$ corresponds to the principal gradation [18]. We choose the invariant inner product as follows

$< E^{ij}_n, E^{kl}_m > = \delta_{n+m,0} \delta^{jk} \delta^{il}$

$< H^{\alpha_i}_n, H^{\alpha_j}_m > = \delta_{n+m,0} \delta_{n+m,0} K_{ij}$

$< \hat{d}, \hat{c} > = N$

$< \hat{d}, H^{\alpha_i}_0 > = 1$

$< \hat{d}, \hat{d} > = \sum_{i,j=1}^{N-1} K^{ij}$ (22)

where $K_{ij}$ and $K^{ij}$ are the entries of the Cartan matrix and of its inverse. The gradation in $\hat{g}$ is introduced naturally through the adjoint action of $\hat{d}$

$\hat{g} = \oplus_{i \in \mathbb{Z}} \hat{g}_i$

$[\hat{d}, \hat{g}_i] = i \hat{g}_i$ (23)
The elements
\[
\mathcal{E}_{nN+i} = \sum_{r=1}^{N-i} E_{n+1}^{r+i} + \sum_{r=1}^{i} E_{n+1}^{N-i+rr} \quad \mathcal{E}_{-nN-i} = \sum_{r=1}^{N-i} E_{-n-1}^{r+i} + \sum_{r=1}^{i} E_{-n-1}^{N-i+rr}
\]
of grade \(\pm(Nn+i)\) \((i = 1, \ldots N - 1)\) and \(\hat{c}\) generate the Heisenberg subalgebra
\[
[\mathcal{E}_n, \mathcal{E}_m] = n\delta_{n+m,0}
\]
(24)

The gradation defined by \(\hat{d}\) is associated to this Heisenberg subalgebra in the sense that \(\mathcal{E}_i \in \hat{g}_i\). The subspaces \(\hat{g}_\pm(Nn+i)\) are spanned by the elements \(E_{n+1}^{r+i}\) and \(E_{n+1}^{N-i+rr}\) for \(i = 1, 2, \ldots, N - 1\) and \(n = 0, 1, \ldots; \hat{g}_\pm Nn\) are spanned by \(H_{\pm n}\) for \(n\) being positive integer number; the zero grade subspace consists of the elements \(H_0^+, \hat{c}\) and \(\hat{d}\). Moreover
\[
\hat{g} = \text{Ker}(ad(\mathcal{E}_1)) \oplus \text{Im}(ad(\mathcal{E}_1)) \oplus \hat{d}
\]
(25)

This property will be important in what follows.

The Conformal Affine Toda (CAT) models (12) arise as a conformally invariant extension of the affine Toda theories. The equations of motion are equivalent to the flatness of the connection
\[
\mathcal{L}_+ = \partial_+ + 2\partial_+ \Phi + \mathcal{E}_1 \quad \mathcal{L}_- = \partial_- + e^{-2ad\Phi} \mathcal{E}_{-1}
\]
(26)

where \(\Phi = \frac{1}{2} \sum_{i=1}^{N-1} \phi_i H_0^{\alpha_i} + \eta \hat{d} + \frac{1}{2} \xi \hat{c}\). Here and in what follows \(x^\pm = x \pm t\). In terms of the components of the field \(\Phi\) one gets
\[
\partial_+ \partial_- \Phi_a = e^{\sum_{a=1}^{N-1} K_{ab} \phi_b + 2\eta} - e^{-\sum_{b=1}^{N-1} K_{ab} \phi_b + 2\eta}
\]
\[
\partial_+ \partial_- \eta = 0
\]
(27)
\[
\partial_+ \partial_- \xi = e^{-\sum_{b=1}^{N-1} K_{ab} \phi_b + 2\eta}
\]

where \(K_{\theta \phi} = 2\frac{\theta_\alpha \phi_\beta}{\theta_\beta}, \alpha_\beta\) are the simple roots and \(\theta\) is the highest root (for \(sl(N)\) \(\theta = \alpha_1 + \ldots + \alpha_{N-1}\)).

In the previous section we learned that from a given Lax operator \(\mathcal{L} = \mathcal{L}_x = \mathcal{L}_+ + \mathcal{L}_-\) (1) one can construct infinite set of integrals of motion. The strategy was to perform a gauge transformation such that the gauge transformed connection \(\mathcal{L}^G\) belongs to a maximally commutative subalgebra \(\mathcal{H}\) of the loop algebra \(\hat{gl}(n)\). In the case considered in Sec.2 this algebra is generated by the integer powers of the matrix \(\Lambda\). An important point in this approach is that the highest grade term \(\Lambda\) of \(\mathcal{L}\) is a regular element, i.e. the loop algebra decomposes into a direct sum of \(Ker(ad\Lambda) = \mathcal{H}\) and \(Im(ad\Lambda)\). This allowed to show that there exists a solution of the recursion relations (16). The situation in CAT looks to be quite similar since \(\mathcal{L}_x\) has the same form as (1) but the presence
of the central element and the derivation \( \hat{d} \) have to be taken into account. The case when \( \mathcal{L} \) does not contain \( \hat{d} \) is considered in [14]. One could expect that after adding such a term the situation will drastically change. The reason is that \( \hat{d} \) is not in the image of the adjoint action of the highest grade element in \( \mathcal{L} \) (in our case \( E_1 \)). It is therefore natural to look for a gauge transformation such that the gauge transformed connection has the form

\[
\mathcal{L}^G_x = \mathcal{E}_1 + \partial_x + J(x)\hat{d} + \sum_{i=1}^{\infty} h_i(x)\mathcal{E}_i \tag{28}
\]

where \( G = e^{\sum_{i=1}^{\infty} g_i - i}, g_i - i \in \mathcal{g}_{-i} \) and \( J(x) = 2\partial_x \eta \). For \( g_{-i} \) and \( h_i \) we obtain the following recursion relations

\[
J(x)\hat{d} + [\mathcal{E}_1, g_{-i}] = 2\partial_x \Phi \tag{29}
\]

\[
h_i\mathcal{E}_{i-1} + [\mathcal{E}_1, g_{-i-1}] = P_l \quad l \geq 1 \tag{30}
\]

where \( P_l \) depends on \( g_{-i}, \ldots, g_{-l} \) (for \( l = 0(\text{mod}N) \) the first term in (30) is missing). These recursion relations are solvable due to (25). It is easy to get

\[
g_{-1} = -\sum_{i=1}^{N-1} \partial_+(\phi_i + \zeta)E_0^{i+1} - \partial_+ E^{1N}_{-1}
\]

\[
<\mathcal{E}_1, \mathcal{E}_{-1}> h_1 = \frac{1}{2} <2\partial_+ \Phi + J\hat{d}, 2\partial_+ \Phi - J\hat{d}> - \frac{\partial}{\partial x} <g_{-1}, \mathcal{E}_1> +
\]

\[
+ e^{-2ad}E_{-1}, \mathcal{E}_1>
\]

From the equations of motion (27), (29) and (30) it follows that

\[
\mathcal{L}_i^G = G\mathcal{L}_i G^{-1} = \mathcal{E}_1 + \partial_i + J(x)\hat{d} + h_1 \mathcal{E}_{-1} + \sum_{i \geq 2} a_i \quad a_i \in \mathcal{g}_{-i} \tag{32}
\]

Substituting in the commutator \([\mathcal{L}_x^G, \mathcal{L}_i^G] = 0\) the expansions (28) and (32) we see that it vanishes if \( \partial_+ J = \partial_+ h_1 = 0 \) and all \( a_i \) in (32) belong to the Heisenberg subalgebra but \( h_i \) for \( i \geq 2 \) are not conserved.

In order to build up the rest of the conserved currents we have to use a different approach. Let \( v \) be a lowest weight vector of \( \hat{g} \), i. e. \( v \) is annihilated by all the negative degree elements of \( \hat{g} \) (in particular, by \( \mathcal{E}_{-i}, \quad i \geq 1 \)). Denote by \( \mathcal{F} \) the Fock space generated by the action of \( \mathcal{E}_1 \) on \( v \). It is obvious from (28) and our conclusion that \( a_i \) in (32) belong to the Heisenberg subalgebra that \( \mathcal{F} \) is invariant under the action of \( \mathcal{L}_x^G \) and \( \mathcal{L}_i^G \) and that on \( \mathcal{F} \)

\[
\mathcal{L}_x^G = \mathcal{L}_i^G \tag{33}
\]
Moreover, any element of $\mathcal{F}$ can be uniquely expressed as a polynomial on $L_x$ acting on $v$

$$\mathcal{E}_1 v = (L_x^G - d_0)v \quad \hat{d}v = d_0v$$

$$\mathcal{E}_1^n v = \left( (L_x^G)^n + \sum_{k=1}^{n} u_k(n)(L_x^G)^{n-k} \right) v$$

(34)

where the functions $u_k(n)$ satisfy the following recursion relations

$$u_k(n+1) = u_k(n) + (\partial_x - (n + d_0)J(x)) u_{k-1}(n) + c\text{nh}(x)u_{k-2}(n - 1)$$

where $\hat{c}\mathcal{F} = c\mathcal{F}$; $h = h_1$. From the Lax equation on the Fock space $\mathcal{F}$ it follows that $
abla_t u_k(n) = \partial_x u_k(n)$ and therefore $u_k(n)$ are densities of conserved currents.

We first present a heuristic derivation of $W_{\infty}$ currents. We shall further set $c = 1$. Denote by $V$ a matrix solution of the auxiliary linear problem

$$\partial_x V = (E_1 + J(x)\hat{d} + E_{-1}) V$$

From (28) one obtains the equations

$$\left( \frac{\partial}{\partial x} - d_0J(x) \right) V(x)v = V(x)\mathcal{E}_1 v$$

$$\left( \frac{\partial}{\partial x} - (d_0 + 1)J(x) \right) \mathcal{E}_1 V(x) + h(x)V(x)v = V(x)\mathcal{E}_1^2 v$$

(35)

and therefore

$$LV(x)\mathcal{E}_1 v = V(x)\mathcal{E}_1^2 v \quad L = \partial_x - (d_0 + 1)J(x) + h(x) \frac{1}{\partial_x - d_0J(x)}$$

(36)

Using the generalized Leibniz rule we expand the first order PDO $L$ in powers of $\partial_x$

$$L = \partial_x - (d_0 + 1)J(x) - \sum_{s=1}^{\infty} u_{s-1}(x)(-\partial_x)^{-s}$$

(37)

The coefficients $u_s$ are expressed in terms of the Faà di Bruno polynomials $P_i(J) = (\partial_x + J(x))^i(1)$

$$u_i(x) = h(x)P_i(-d_0J) \quad i = 0, 1, \ldots$$

(38)

In this form the generators of the $W_{1+\infty}$ algebra appeared in [11]. It is worthwhile to note that nevertheless that the KP $L$-operator appears in the left hand side of (36), this equation is not a scalar pseudodifferential equation. Therefore it seems that the KP structure in CAT is encoded in a different manner.

In trying to construct the KP hierarchy within CAT we first note that after a suitable $GL(\infty)$ gauge transformation the connection (28) assumes the form

$$\mathcal{L}_U = \partial_x + \mathcal{E}_1 + U \quad U = -\sum_{i=1}^{\infty} u_{i-1}E_{0i}$$

(39)

we define $GL(\infty)$ as the group of all semi–infinite invertible matrices
which is a natural generalization of (39). The generators $\mathcal{E}_{\pm 1}$ and $\hat{d}$ are expressed in terms of the elementary semi–infinite matrices as follows

$$
\mathcal{E}_1 = \sum_{i=0}^{\infty} E_{i+1} \quad \mathcal{E}_{-1} = -\sum_{i=0}^{\infty} (i+1) E_{ii+1} \quad \hat{d} = \sum_{i=0}^{\infty} iE_{ii} + d_0
$$

The difference with respect to the finite dimensional case is that neither the gauge transformation which brings (28) into the upper form nor the functions $u_i$ are uniquely fixed. Instead of (38) one could take $\tilde{u}_i = u_i + P_i$, where $P_i$ are spin $i$ differential polynomials on $u_{i-1}, \ldots u_0, J$. This problem has been discussed in [19] from the point of view of the hamiltonian reduction of the two–loop WZNW models. Similarly to the previous section we introduce the vector–columns and vector–rows

$$
A_i = \sum_{l=0}^{\infty} (-\partial)^{1-l} A_{il} \quad A_i = \sum_{l=0}^{\infty} A_{il} (-\partial)^{-l}
$$

Looking for a consistent time evolution equations $\partial_t U = [A, \mathcal{L}]$ with given $A^0$ together with the initial condition $A_0 = A^0 L - (A^0 L)_-$ where $L$ is the PDO

$$
L = - (\partial) + \sum_{i=0}^{\infty} u_{i-1} (-\partial)^{-i}
$$

we get the solution

$$
A_i = (-\partial)^{-i} \left( A^0 L \right)_+ - \left( (-\partial)^{-i} A^0 \right)_+ L
$$

and the equations of motion can be written in the form (43)

$$
\frac{\partial L}{\partial t} = -H_2(A^0) \quad H_2 = (LX)_+ L - (LX)_+ L
$$

Setting in the upper equation $A^0 = L^{k-1}$ one recognises (up to a sign) the $k$th flow of the KP hierarchy. The same flow can be also reproduced from the Adler map $H_1(X) = [L, X_+] - [L, X]_+$. This follows from the recursion relations $H_1(L^k) + H_2(L^{k-1}) = 0$. $H_1$ and $H_2$ define the first and the second GD brackets

$$
\{ < L, X >, < L, Y > \}_i^{GD} = < H_i(X), Y > \quad < X, Y > = \int resXY dx
$$

The KP equations are hamiltonian with respect to these Poisson brackets. The corresponding hamiltonians are proportional to $\int resL^k dx$. Their involution follows from the recursion relations for the two Adler maps $H_1$ and $H_2$. The first KP–flow implies that the coefficient of the KP operator are chiral $\partial_t u_i = \partial_x u_i$. Similarly to (19) one could pass from the GD to the DS brackets.

Note also that one can rewrite the auxiliary linear problem as

$$
\frac{\partial \xi}{\partial x}(x) = Q(x)\xi(x)
$$
where $\xi$ is the transposed of an arbitrary row of the matrix $V$ on $\mathcal{F}$ and

$$
Q(x) = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots \\
-h & J & 1 & 0 & \cdots \\
0 & -2h & 2J & 1 & \cdots \\
0 & 0 & -3h & 3J & \cdots \\
& & & & \ddots \\
& & & & & \ddots \\
\end{pmatrix} + d_0 J(x) 
$$

(47)

We thus conclude that $Q$ has the same form as the spectral matrix (the operator of multiplication by the spectral parameter ) in the one–matrix models [7]. In contrast to the CAT models, the first (”space”) flow in the one–matrix models is generated by the upper diagonal part of $Q$, $Q_+$. This simple observation looks to be a promising starting point in relating the matrix models and the CAT.

We shall finish with the following remark. Since CAT are conformally invariant, they separate into two sectors – chiral and antichiral. In this section we considered the conserved quantities in the chiral sector. The antichiral currents are obtained through the same procedure if one starts from the Lax connection

$$
\mathcal{L}_+ = \partial_+ + e^{2ad\Phi} \mathcal{E}_1 \quad \mathcal{L}_- = \partial_- + 2\partial_- \Phi + \mathcal{E}_- 
$$

(48)

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