Heuristic Representations of Plasma Momentum Transport

I.H.Hutchinson,
Plasma Science and Fusion Center and
Department of Nuclear Science and Engineering
Massachusetts Institute of Technology,
Cambridge, MA, USA

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Abstract

It is argued that the form commonly adopted for heuristic representation of tokamak momentum diffusion has major shortcomings, and that a non-diffusive momentum-flux term independent of velocity is more appropriate.

1 Introduction

In attempting to understand the self-acceleration of tokamak plasmas[1] and other intriguing momentum transport phenomena in magnetically confined plasmas, a heuristic model of plasma momentum transport has frequently been adopted which separates the momentum transport flux into “diffusive” and “convective” parts[2, 3, 4, 5]. It is shown here that, in this division, which in the absence of fundamental calculations of the transport is ad hoc, the convective part cannot consistently be identified with a term that is equal to the plasma momentum density times a constant coefficient, as is often implied or assumed. Instead, at least a part, and arguably the most interesting and significant part, of the momentum flux must be represented as a term independent of the velocity, rather than proportional to velocity. This velocity-independent part of the momentum flux does not represent, in any meaningful sense, a momentum source. It represents a transport flux of momentum up the velocity gradient that is independent of both the velocity gradient itself (i.e. it is not a form of viscosity) and of the value of the velocity. The latter property is characteristic of momentum that is transported not by mean particle flux but by distinctively momentum transport processes such as by Reynolds stress. Experiments, if they are sufficiently comprehensive, could in principle identify the velocity dependence, if any, of the total “convective” momentum transport, but in practice this is beyond the measurement capability so far.
2 Heuristic flux density representations

2.1 Fick’s Law and its failure

A familiar description of diffusing systems is “Fick’s Law” which assumes that diffusive flux density $\Gamma_Q$ is proportional to the gradient of the property under consideration which we will denote here $Q$.

$$\Gamma_Q = D_Q \frac{dQ}{dx},$$  \hspace{1cm} (1)

where $D_Q$ is called the diffusivity or diffusion coefficient. For convenience we are using a one-dimensional, slab representation in this discussion which is sufficient for immediate purposes.

For treatment of most plasma magnetic confinement problems, this simple approximation is known to be inadequate. Fluxes are present that prove not to be proportional to the gradient of the extensive quantity $Q$. Consequently, while it is still possible to write a Fick’s law and consider it to be the definition of the diffusion coefficient $D_Q$, that diffusion coefficient $D_Q$ is not constant, and not even necessarily non-singular (where $dQ/dx = 0$ for example $D_Q$ becomes formally infinite if the flux there is non-zero). Therefore, for a consistent representation, at a minimum an additional term must be included in the expression for the momentum flux. The extra term that is added to improve the heuristic transport representation ought generally to avoid the singularity in $D_Q$ without introducing singularities of its own. Ideally it would represent an identifiable physical process, but that is not guaranteed by the formal representation.

2.2 Density diffusion

When the quantity under discussion is the density, it is satisfactory to express the additional flux term in the form $V_QQ$, because $n$ never changes sign, and so singularities in $V_Q$, which is called the convection velocity, are not induced. In short, it makes reasonable physical sense to write

$$\Gamma_n = D_n \frac{dn}{dx} + V_n n.$$  \hspace{1cm} (2)

And indeed the “diffusion” (first) and “convection” (second) terms have some mathematical and physical justification in terms of fluid transport. [But notice that the convection velocity, $V_n$, is not in general equal to the fluid velocity $\Gamma_n/n$.]

Let us note, however, that eq (2) (like eq 1) is a purely formal expression. Any flux density could be written this way, as a pure piece of arithmetic, with the only constraint on the $D_n$ and $V_n$ being that the diffusion term and the convection term add up to the correct total flux. The form is suggestive of diagonal and off-diagonal terms in a transport matrix. If such a representation were valid, then one might find that $D_n$ and $V_n$ were independent of the local density and its gradient. Only in the case of approximately constant $D_n$ and $V_n$ does the division into these two terms really have any significance.
2.3 Momentum transport

When, instead, the quantity being transported is the momentum density in a direction (z) transverse to the direction of transport, \( nmv_z \), it is not at all obvious that the second term which seeks to generalize Fick’s law, should be written proportional to the momentum density \( nmv_z \). The first obvious fact is that \( v_z \) can have either sign; and that there are therefore likely places where it goes through zero. Unless the additional momentum flux term is identically zero there, a singularity in the coefficient multiplying \( nmv_z \) will arise. Of course, the arithmetic might be fixed up by adjusting \( D_v \) so that the non-diffusive flux is in fact always zero where \( v_z \) is zero, but such an adjustment cannot be done with constant \( D_v \). Therefore writing the momentum flux, as several recent authors have done, as

\[
\Gamma_v = D_v \frac{d}{dx} (nmv_z) + V_c nmv_z, \quad \text{or} \quad \Gamma_v = nm \left( D_v \frac{d}{dx} v_z + V_c v_z \right)
\]  

(3)

has little to recommend it. It introduces by a pure mathematical ansatz a velocity \( V_c \) that will generally have to become infinite where \( v_z \) changes sign. To avoid such infinities one would have to do fine tuning of \( D_v \), which contradicts the observation made in the previous section that the division into \( D_v \) and \( V_c \) has significance only when specific constraints, such as that \( D_v \) and \( V_c \) are invariant, are applied.

Worse still, the form of eq (3) violates a fundamental property that we should require of non-relativistic physical equations in a translationally-invariant configuration: that they be invariant under Galilean transformation. If we change our frame of reference to one moving with velocity \( v_t \) in the \( \hat{z} \)-direction, then all transformed (primed) velocities are related to the untransformed-frame velocities via \( v_z' = v_z - v_t \). Eq (3) (in its second form for simplicity) becomes

\[
\Gamma'_v = nm \left( D_v \frac{d}{dx} v_z' + V_c v_z' + V_c v_t \right),
\]

(4)

yielding a flux in the \( \hat{x} \)-direction that is not invariant under the transformation, even though, physically, it should be, and indeed yielding a form of the equations in which a new term, \( V_c v_t \), independent of both \( dv'/dx \) and \( v' \) has entered. Expanding on this point in the context of momentum transport, one should realize that if there is a preferred frame of reference, the laboratory frame, for the solution, then that preference is imposed not via the transport equations themselves, but by the boundary condition. In other words, for a translationally invariant confined plasma, there is nothing in the transport equations that specifies the laboratory frame, only in the boundary conditions such as that the velocity \( v_z \) should be zero at the walls in a no-slip situation. [If there were momentum sources these might also select a preferred frame, but the point remains that the flux-density itself should not].

The whole spirit of the heuristic diffusive representation presumes this view. It is therefore a major shortcoming of any proposed heuristic transport expression that it not exhibit Galilean invariance with respect to velocity in a direction of symmetry. Supposing the non-diffusive part of momentum transport to be \( V_c nmv_z \) as, eq (3) does, has precisely this shortcoming.
This invariance failure is partially obscured in a toroidal situation, because transformation to a *rotating* frame of reference itself introduces new terms into the plasma equations: the centrifugal and coriolis forces\[3\]. Since the centrifugal force is an even (and second order) function of the toroidal rotation velocity, it can hardly be a source of rotation. But the coriolis force is not so obviously irrelevant. Nevertheless, it seems unlikely that its effect can explain self-acceleration; not least because the coriolis force does not of itself show a systematic preferred sign of rotation (which the experiments do); it merely amplifies rotation with one or the other sign. It is also certain that constraining the non-diffusive momentum transport term to have a form that is exactly consistent with arising from coriolis forces is, *a priori*, unjustified.

### 2.4 Momentum convection

There is, in fact, a part of the momentum flux-density that is truly convective and ought to be written $nm v_z^2 = \frac{\Gamma_n}{n} v_z^2$. It is the actual momentum convection arising from the plasma flow velocity $v_z = \frac{\Gamma_n}{n}$. This elementary term is of course present in any fluid representation, and is the flux of momentum carried by the (total) particle flux density $\Gamma_n$. This term, while potentially important in transients and essential for maintaining the Galilean invariance in the presence of particle flux, is not of the slightest value as an explanation for tokamak self-acceleration, because that acceleration occurs in plasmas with zero particle source and steady density profiles, for which $\Gamma_n = 0$. Therefore, while one might call the momentum flux-density that arises from a particle pinch such as the Ware pinch a “momentum pinch”, it is completely irrelevant to the explanation of self-acceleration. A particle pinch can plausibly be invoked to explain non-uniform density profiles when there is zero density source and yet $D_n \neq 0$. But in the interesting steady situations, such a particle pinch is always being exactly cancelled by density diffusion, yielding $\Gamma_n = 0$ and hence zero momentum convection.

For example, addressing specifically the turbulent transport situation, fluxes of particles and momentum can be considered to arise from the averages of the turbulent transport terms as $\Gamma_n = \langle \tilde{v}_x \tilde{n} \rangle$ and $\Gamma_v = m \langle \tilde{v}_z \tilde{n} \tilde{v}_z \rangle = m \langle v_z \rangle \langle \tilde{v}_x \rangle + m \langle n \rangle \langle \tilde{v}_x \tilde{v}_z \rangle / n = \Gamma_n m v_z + m \langle n \rangle \langle \tilde{v}_x \tilde{v}_z \rangle$, where the last term is directly identifiable as the Reynolds stress, and is manifestly independent of (Galilean) frame of reference.

### 2.5 Self-consistent heuristic momentum flux density

In view of the foregoing considerations, the minimum self-consistent heuristic representation of momentum flux density useful to describe a combination of diffusive and non-diffusive momentum transport is of the form:

$$\Gamma_v = nm \left( D_v \frac{dv_z}{dx} + \frac{\Gamma_n}{n} v_z + V_v v_0 \right),$$  \hspace{1cm} (5)$$

where $v_0$ is merely a conveniently chosen constant characteristic velocity (e.g. the sound speed) that renders the term $V_v v_0$ dimensionally as a product of two velocities, independent
of \( v_z \). Only their product, \( V_v v_0 \), is significant. The middle term, proportional to \( v_z \), must be chosen so as to annihilate the particle density divergence effects arising in Galilean transformations; so its coefficient should not be regarded as a free parameter, but is \( \Gamma_n/n \). This term is not relevant to understanding self-acceleration.

3 Sources, conservation equations and their inversion

3.1 Conservation equation structure

The conservation of plasma \( \hat{\mathbf{z}} \)-momentum in a cylindrically symmetric system is

\[
\frac{\partial}{\partial t}(nm v_z) = \nabla \cdot \nabla v + S_v = \frac{\partial}{\partial r}(r \Gamma_{vr}) + S_v, \tag{6}
\]

where \( S_v \) is a possible local internal momentum source density. The analytic boundary condition on axis is \( \Gamma_{vr}(0) = 0 \); therefore without internal sources \( (S_v = 0) \), in steady state \( (\partial/\partial t = 0) \), the solution is \( \Gamma_{vr} = 0 \). If \( \Gamma_v \) is of the form (5), then trivially

\[
\frac{dv_z}{dr} + \frac{\Gamma_n}{D_v n} v_z + \frac{V_v}{D_v} v_0 = 0. \tag{7}
\]

If the density is also governed by a steady sourceless diffusion equation, then \( \Gamma_n = 0 \) and we find

\[
v_z = v_{za} - \int_a^r \frac{V_v}{D_v} v_0 dr, \tag{8}
\]

where the edge \( (r = a) \) boundary condition is \( v_z = v_{za} \).

Notice that for this source-free situation the term \( V_v v_0 \) is the cause of any non-zero velocity gradient. The role it serves is very similar to that of the source, \( S_v \), in a situation where the flux-density is purely diffusive, \( \Gamma_v = D_v nmv/\partial r \). In that pure-diffusive case, without sources the steady solution for \( v_z \) is uniform, equal to the edge value. But we should not be misled into calling \( nm v_0 V_v \) (or more precisely its divergence) a momentum source. It is not a source; it is a momentum flux density that occurs independent of velocity gradient.

More generally, substituting the form (5) into eq (5), we obtain

\[
\frac{\partial}{\partial t}(nm v_z) - S_v = \frac{\partial}{\partial r} \left[ rnm \left( D_v \frac{\partial v_z}{\partial r} + \frac{\Gamma_n}{n} v_z + V_v v_0 \right) \right]. \tag{9}
\]

For simplicity of discussion of the velocity, let’s assume that density \( n \) is fixed and uniform (general cases have more terms to consider) and that \( \Gamma_n = 0 \), so that this can be written:

\[
L \equiv \frac{\partial}{\partial t}(v_z) - S_v/nm = \frac{\partial}{\partial r} \left[ r \left( D_v \frac{\partial v_z}{\partial r} + V_v v_0 \right) \right] \tag{10}
\]

\[
= \frac{1}{r} \left[ \frac{\partial}{\partial r} (rD_v) \frac{\partial v_z}{\partial r} + (rD_v) \frac{\partial}{\partial r} + v_0 \frac{\partial}{\partial r} (rV_v) \right]. \tag{11}
\]
3.2 Deducing $D$ and $V$ from measurements

If we wish to deduce from measurements the values of $D_v$ and $V_v$, then the left-hand-side terms of eq (10) act in essentially the same way. There are either transients ($\partial v_z/\partial t$) or sources ($S_v$). If neither is present, then the equations are homogeneous, requiring simply $D_v \partial v_z/\partial r + V_v v_0 = 0$, and showing that there is nothing setting the overall size of $D$ or $V$, only their ratio $D_v/V_v$. When non-zero, the LHS determines the divergence of the flux (the total of the RHS), and acts as an inhomogeneous contribution to the radial equation governing the quantities $rD_v$ and $rV_v$.

If one has measurements of $\partial v_z/\partial r$ for all relevant radii, then instantaneously eq (10) is simple linear functional equation for $(rD_v),(rV_v)$. But its solution is not unique. For example one possible solution is $D_v = 0$ and $\partial rV_v/\partial r = rL/v_0$, another is to take $V_v = 0$. On the basis of only one instantaneous (or steady) case nothing can be done to narrow down the possible solution space. In other words, $D$ and $V$ cannot be separated. However, if a range of different profiles of $L(r)$ is available, for example because a transient can be time-resolved to give a variety of different $L$, or because the sources’ spatial profile can be varied, and if $\partial v_z/\partial r$ is measured for each case, then some best-fit for $rD_v$ and $rV_v$ can be determined (as a function of $r$), provided they are the same for all cases. Without this last stipulation (or some other about how the $D$ and $V$ for different cases are related), then one is no better off than with one instantaneous case.

From a practical viewpoint, the process of solving for $rD_v$ and $rV_v$ will presumably be to (1) discretize them into a finite representation in terms of a set of coefficients times appropriate functions, (2) obtain the linear (matrix) relationships between the coefficients of the discretized representation and the values of $L(r,t)$, (3) invert the matrix equation, probably in some regularized least-squares sense, to obtain the coefficients. If, as is usually the case, the amount of linearly independent information is strongly limited by constraints on the experiments and on their uncertainty, then a very judicious choice of representation is important, and only very few coefficients can be deduced. For example, perhaps the simplest low order representation is to take $D_v = \text{const}$ (independent of $r$), and $V_v = rV_{va}/a$, just two independent coefficients ($D$ and $V_{va}/a$). The $V_v$ representation has been chosen recognizing that analyticity on axis requires $V_v = 0$ there. Then in principle, two independent profiles of $L$ provide sufficient rank to solve for the coefficients. Of course this is no guarantee that the profiles chosen are correct, and preferably a much bigger range of profiles and more comprehensive functional representation should be used.

3.3 Identifying flow-drive

In particular, in flow-drive experiments (e.g. [6]), it would be completely unjustified to assume a highly simplified representation of the form $D_v = \text{const}$ and $V_v = rV_{va}/a$ and then try to invert the process and deduce from the profiles of $v_z$ the spatial dependence of $S_v$. An identification of unknown source profiles would instead have to rely for example upon the RHS of eq (10), the transport terms, being negligible compared with inertia for sufficiently rapid accelerations, in a manner analogous to the “break of slope” analysis of heating steps.
Present experiments don’t normally have such a clear separation of timescales for momentum source transients. The result is that there is no basis on which to distinguish between changes in the term \( v_o \partial (r V_v) / \partial r \), and changes in the term \( S_v / n m \). In other words, one can’t tell from velocity measurements whether there is actually a momentum source \( (S_v) \) or whether the flow-drive works by changing the momentum transport \( (V_v) \). In the latter case, if the effect on \( V_v \) is localized in radius, the radial integral of the \( v_o \partial (r V_v) / \partial r \) term across the whole perturbed region is zero: an embodiment of the fact that the total equivalent force arising from this term is zero. Sometimes this is summarized by referring to the transport-alteration influence as “dipolar”. If the transport alteration is zero outside some radius \( r_o \), and there are no changes to \( S_v \), then the solution of the conservation equation should give unperturbed velocity profile outside \( r_o \). But if the transport alteration extends all the way to the boundary, so that \( V_v \) is altered at the boundary, this conservation no longer applies, the influence need not be “dipolar”, and the velocity perturbation can extend to the edge.

It should be obvious from the above that knowing how the \( V_v v_o \) term varies with \( v_z \) is crucial. Here, we’ve taken it to be independent of \( v_z \) because of the arguments about Galilean invariance. In principle we could include two different terms \( V_v v_o \) and \( V_c v_z \), and then solve for each, if we have sufficient linearly independent \( L \) values, thus identifying experimentally the balance between the two types of term. But it is already a stretch for most experiments to separate \( D \) from \( V \) at all, let alone bringing in other possible terms. Tokamak experiments so far have not had the precision or comprehensive resolution to do so.

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