SUPERSOLVABLE RESTRICTIONS OF REFLECTION ARRANGEMENTS

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Abstract. Let \( A = (A, V) \) be a complex hyperplane arrangement and let \( L(A) \) denote its intersection lattice. The arrangement \( A \) is called supersolvable, provided its lattice \( L(A) \) is supersolvable. For \( X \) in \( L(A) \), it is known that the restriction \( A^X \) is supersolvable provided \( A \) is.

Suppose that \( W \) is a finite, unitary reflection group acting on the complex vector space \( V \). Let \( A = (A(W), V) \) be its associated hyperplane arrangement. Recently, the last two authors classified all supersolvable reflection arrangements. Extending this earlier work, the aim of this note is to determine all supersolvable restrictions of reflection arrangements. It turns out that apart from the obvious restrictions of supersolvable reflection arrangements there are only a few additional instances. Moreover, in a recent paper we classified all inductively free restrictions \( A(W)^X \) of reflection arrangements \( A(W) \). Since every supersolvable arrangement is inductively free, the supersolvable restrictions \( A(W)^X \) of reflection arrangements \( A(W) \) form a natural subclass of the class of inductively free restrictions \( A(W)^X \).

Finally, we characterize the irreducible supersolvable restrictions of reflection arrangements by the presence of modular elements of dimension 1 in their intersection lattice. This in turn leads to the surprising fact that reflection arrangements as well as their restrictions are supersolvable if and only if they are strictly linearly fibered.

1. Introduction

Let \( A = (A, V) \) be a complex hyperplane arrangement and let \( L(A) \) denote its intersection lattice. We say that \( A \) is supersolvable, provided \( L(A) \) is supersolvable, see Definition 2.3. Thanks to [Sta72, Prop. 3.2] (see Corollary 2.7), for \( X \) in \( L(A) \), the restriction \( A^X \) of a supersolvable arrangement \( A \) is itself again supersolvable.

Now suppose that \( W \) is a finite, unitary reflection group acting on the complex vector space \( V \). Let \( A = (A(W), V) \) be the associated hyperplane arrangement of \( W \). In [HR14, Thm. 1.2], we classified all supersolvable reflection arrangements. Extending this earlier work, the aim of this note is to classify all supersolvable restrictions \( A^X \) for \( A \) a reflection arrangement. Since supersolvability is a rather strong condition, not unexpectedly, there are only very few additional instances apart from the obvious restrictions of supersolvable reflection arrangements.

Moreover, similar to the case of supersolvable reflection arrangements, we are able to characterize the irreducible arrangements in this class merely by the presence of modular elements of dimension 1 in their intersection lattice (see Theorem 1.5). This in turn leads to the

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unexpected, remarkable fact that reflection arrangements as well as their restrictions are supersolvable if and only if they are strictly linearly fibered (see Corollary 1.7).

The classification of the irreducible, finite, complex reflection groups $W$ due to Shephard and Todd [ST54] states that each such group belongs to one of two types. Namely, either $W$ belongs to the infinite three-parameter family $G(r, p, \ell)$ of monomial groups, or else one of an additional 34 exceptional groups, simply named $G_4$ up to $G_{37}$. As a result, proofs of properties of $W$ and its arrangement $A(W)$ frequently also do come in two flavors: conceptional, uniform arguments for the infinite families on the one hand, and adhoc and mere computational techniques for the exceptional instances, on the other, e.g. see [OT92, §6, App. B, App. C] and [OS82]. This dichotomy also prevails the statements and proofs of this paper.

First we recall the main result from [HR14, Thm. 1.2]:

**Theorem 1.1.** For $W$ a finite complex reflection group, $A(W)$ is supersolvable if and only if any irreducible factor of $W$ is of rank at most 2, or is isomorphic either to a Coxeter group of type $A_\ell$ or $B_\ell$ for $\ell \geq 3$, or to a monomial group $G(r, p, \ell)$ for $r, \ell \geq 3$ and $p \neq r$.

It is easy to see that any central arrangement of rank at most 2 is supersolvable, cf. [HR14, Rem. 2.3]. Thus we focus in the sequel on restrictions $A^x$ with $\dim X \geq 3$.

In [AHR13, Thm. 1.2], we classified all inductively free restrictions $A^x$ of reflection arrangements $A$. Since a supersolvable arrangement is inductively free, thanks to work of Jambu and Terao [JTS84, Thm. 4.2] (see Theorem 2.5), the supersolvable restrictions $A^x$ of reflection arrangements $A$ form a natural subclass of the inductively free restrictions.

In order to state [AHR13, Thm. 1.2], we require a bit more notation. For fixed $r, \ell \geq 2$ and $0 \leq k \leq \ell$, we denote by $A^k(r)$ the sequence of intermediate arrangements that interpolate between the two reflection arrangements $A(G(r, r, \ell)) =: A^0(r)$ and $A(G(r, 1, \ell)) =: A^\ell(r)$, defined in [OS82, §2] (see also [OT92, §6.4]). Note that $A^k(r)$ is not a reflection arrangement for $k \neq 0, \ell$. Also note that each of the arrangements does arise as the restriction $A(W)^x$ for suitable $W$ and $X$, e.g. see [OT92, Prop. 6.84]. Here is [AHR13, Thm. 1.2].

**Theorem 1.2.** Let $W$ be a finite, irreducible, complex reflection group with reflection arrangement $A = A(W)$ and let $X \in L(A)$. The restricted arrangement $A^x$ is inductively free if and only if one of the following holds:

(i) $A$ is inductively free;

(ii) $W = G(r, r, \ell)$ and $A^x \cong A^k(r)$, where $p = \dim X$ and $p - 2 \leq k \leq p$;

(iii) $W$ is one of $G_{24}, G_{27}, G_{29}, G_{31}, G_{33}$, or $G_{34}$ and $X \in L(A) \setminus \{V\}$ with $\dim X \leq 3$.

Because the pointwise stabilizer $W_X$ of $X$ in $L(A)$ is itself a complex reflection group, by [Ste64, Thm. 1.5], following [OS82, §3, App.] (cf. [OT92, §6.4, App. C]), we may label the $W$-orbit of $X \in L(A)$ by the type $T$ say, of $W_X$; we thus frequently denote such a restriction $A(W)^x$ simply by the pair $(W, T)$.

Thanks to [HR14, Prop. 2.5] and the factorization property of restrictions (2.2) below, the question of supersolvability of $A^x$ reduces to the case when $A$ is irreducible. Thus, we may
assume that $W$ is irreducible. In light of Theorem 1.2, we can now state the main result of our paper.

**Theorem 1.3.** Let $W$ be a finite, irreducible, complex reflection group with reflection arrangement $A = A(W)$ and let $X \in L(A)$ with $\dim X \geq 3$. Then the restricted arrangement $A^X$ is supersolvable if and only if one of the following holds:

(i) $A$ is supersolvable;

(ii) $W = G(r, r, \ell)$ and $A^X \cong A^p_p(r)$ or $A^{p-1}_p(r)$, where $p = \dim X$;

(iii) $A^X$ is $(E_6, A_3)$, $(E_7, D_4)$, $(E_7, A_2^2)$, or $(E_8, A_3)$.

While Theorem 1.3 gives a complete classification of supersolvable restrictions of reflection arrangements, as far as isomorphism types of such restrictions are concerned, there is quite a bit of redundancy in its statement. For, the last two restrictions in part (iii) of Theorem 1.3 are isomorphic to each other while the first two restrictions are isomorphic to ones in part (ii) (see Lemma 3.3 below) and $A^p_p(r)$ is isomorphic to the reflection arrangement of $G(r, 1, p)$, already covered in part (i). Thus, apart from supersolvable reflection arrangements themselves, there is only one additional family of supersolvable restrictions one for each dimension, and a single exceptional case, up to isomorphism.

**Corollary 1.4.** Let $W$ be a finite, irreducible, complex reflection group with reflection arrangement $A = A(W)$ and let $X \in L(A)$ with $\dim X \geq 3$. Then $A^X$ is supersolvable if and only if either $A^X$ is isomorphic to a supersolvable reflection arrangement, $A^X \cong A^{p-1}_p(r)$ for some $p \geq 3$, or $A^X \cong (E_7, A_2^2)$.

The definition of supersolvability of $A$ entails the existence of modular elements in $L(A)$ of any possible rank; see §2.3 for the notion of modular elements. In our second main result we show that irreducible, supersolvable restrictions of reflection arrangements are characterized merely by the presence of a modular element of dimension 1.

**Theorem 1.5.** For $W$ a finite, irreducible complex reflection group of rank at least 4, let $A = A(W)$. Let $X \in L(A)$ with $\dim X \geq 3$. Then $A^X$ is supersolvable if and only if there exists a modular element of dimension 1 in its lattice $L(A^X)$.

**Remark 1.6.** (i). Note that Theorem 1.5 also applies to the case $X = V$ and thus gives a new characterization of supersolvable reflection arrangements, cf. [HR14, Thm. 1.3].

(ii). The condition of irreducibility in Theorem 1.5 is necessary, see [HR14, Rem. 2.6].

In view of [OT92, Cor. 5.112, Thm. 5.113], Theorem 1.5 readily implies the following.

**Corollary 1.7.** For $W$ a finite, irreducible complex reflection group let $A = A(W)$ and let $X \in L(A)$. Then $A^X$ is strictly linearly fibered if and only if it is of fiber type.

**Remark 1.8.** See [OT92, Defs. 5.10, 5.11] for the notions of strictly linearly fibered and fiber type arrangements. Corollary 1.7 is rather striking as the latter notion is considerably stronger than the first. In particular, it follows that for $W$ a finite, complex reflection group, the reflection arrangement $A(W)$ is supersolvable if and only if it is strictly linearly fibered. This new fact underlines the very special role played by the class of reflection arrangements and their restrictions among all arrangements.
The paper is organized as follows. In §2 we recall the required notation and facts about supersolvability of arrangements and reflection arrangements from [OT92, §4, §6] and prove some preliminary results. Theorems 1.3 and 1.5 are proved in §3 and §4, respectively. The dichotomy of the classification of the irreducible complex reflection groups \( W \) into the three parameter family \( G(r, p, \ell) \) and an additional 34 exceptional cases (see [OT92, Table B.1]), mentioned above, descends to an analogous dichotomy of the restrictions \( A(W)^X \) into the three-parameter family \( A^X_k(r) \) and a small finite set of exceptional cases. While our proofs for the restrictions that are isomorphic to \( A^X_k(r) \) are conceptual and uniform, the statements of Theorem 1.3 and Corollary 1.4 indicate that there cannot be a uniform argument in general. For \( W \) of exceptional type, our arguments consist of a case-by-case analysis based on some technical general lemmas for 3- and 4-arrangements provided in Section 2.3. Despite the fact that the statement of Theorem 1.5 is uniform, still two instances required some computer calculations.

For general information about arrangements and reflection groups we refer the reader to [Bou68], [OS82] and [OT92]. Throughout, we use the naming scheme of the irreducible finite complex reflection groups due to Shephard and Todd, [ST54].

2. Recollections and Preliminaries

2.1. Hyperplane Arrangements. Let \( V = \mathbb{C}^\ell \) be an \( \ell \)-dimensional complex vector space. A hyperplane arrangement is a pair \((A, V)\), where \( A \) is a finite collection of hyperplanes in \( V \). Usually, we simply write \( A \) in place of \((A, V)\). We only consider central arrangements, i.e. the origin is contained in the center \( T := \bigcap_{H \in A} H \) of \( A \). The empty arrangement in \( V \) is denoted by \( \Phi_\ell \).

The lattice \( L(A) \) of \( A \) is the set of subspaces of \( V \) of the form \( H_1 \cap \ldots \cap H_n \) where \( \{H_1, \ldots, H_n\} \) is a subset of \( A \). For \( X \in L(A) \), we have two associated arrangements, firstly the subarrangement \( A_X := \{H \in A \mid X \subseteq H\} \) of \( A \) and secondly, the restriction of \( A \) to \( X \), defined by \( A^X := \{X \cap H \mid H \in A \setminus A_X\} \).

The lattice \( L(A) \) is a partially ordered set by reverse inclusion: \( X \leq Y \) provided \( Y \subseteq X \) for \( X, Y \in L(A) \). We have a rank function on \( L(A) \) defined by \( r(X) := \text{codim}_V(X) \). The rank \( r(A) \) of \( A \) is the rank of a maximal element in \( L(A) \) with respect to the partial order. With this definition \( L(A) \) is a ranked geometric lattice, [OT92, §2]. The \( \ell \)-arrangement \( A \) is called essential provided \( r(A) = \ell \).

Note that the restriction \( A^X \) is also central, so that \( L(A^X) \) is again a geometric lattice. Let \( A \) be central and let \( X, Y \in L(A) \) with \( X < Y \). We recall the following sublattices of \( L(A) \) from [OT92, Def. 2.10], \( L(A)_Y := \{Z \in L(A) \mid Z \leq Y\} \), \( L(A)^X := \{Z \in L(A) \mid X \leq Z\} \), and the interval \( [X, Y] := L(A)_Y \cap L(A)^X = \{Z \in L(A) \mid X \leq Z \leq Y\} \).

**Lemma 2.1.** Let \( A \) be central and let \( X, Y \in L(A) \) with \( X < Y \). Then \( L(A)^Y \) is a sublattice in \( L(A^X) \). In particular, the former is an interval in the latter.

**Proof.** It follows from [OT92, Lem. 2.11] that \( L(A)^Y = L(A)_Y \subseteq L(A)^X = L(A^X) \) are both sublattices of \( L(A) \) and that \( L(A^Y) = L(A^Y)_Y = [Y, T] \) is an interval in \( L(A^X) \). \( \square \)
Let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ be the product of the two arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$. With the partial order defined on $L(\mathcal{A}_1) \times L(\mathcal{A}_2)$ by $(X_1, X_2) \leq (Y_1, Y_2)$ provided $X_1 \leq Y_1$ and $X_2 \leq Y_2$, there is a lattice isomorphism $L(\mathcal{A}_1) \times L(\mathcal{A}_2) \cong L(\mathcal{A})$ given by $(X_1, X_2) \mapsto X_1 \oplus X_2$, [OT92, §2]. It is easily seen that for $X = X_1 \oplus X_2 \in L(\mathcal{A})$, we have

\begin{equation}
\mathcal{A}^X = \mathcal{A}_1^{X_1} \times \mathcal{A}_2^{X_2}.
\end{equation}

Note that $\mathcal{A} \times \Phi_0 = \mathcal{A}$ for any arrangement $\mathcal{A}$. If $\mathcal{A}$ is of the form $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$, where $\mathcal{A}_i \neq \Phi_0$ for $i = 1, 2$, then $\mathcal{A}$ is called reducible, else $\mathcal{A}$ is irreducible, [OT92, Def. 2.15].

### 2.2. Free Arrangements

Let $S = S(V^*)$ be the symmetric algebra of the dual space $V^*$ of $V$. If $x_1, \ldots, x_\ell$ is a basis of $V^*$, then we identify $S$ with the polynomial ring $\mathbb{C}[x_1, \ldots, x_\ell]$. By denoting the $C$-subspace of $S$ consisting of the homogeneous polynomials of degree $p$ (and 0) by $S_p$, we see that there is a natural $\mathbb{Z}$-grading $S = \bigoplus_{p \in \mathbb{Z}} S_p$, where $S_p = 0$ for $p < 0$.

Let $\text{Der}(S)$ be the $S$-module of $C$-derivations of $S$ and for $i = 1, \ldots, \ell$ define $D_i := \partial/\partial x_i$ to be the $i$th partial derivation of $S$. Now $D_1, \ldots, D_\ell$ is an $S$-basis of $\text{Der}(S)$ and we call $\theta \in \text{Der}(S)$ homogeneous of polynomial degree $p$ provided $\theta = \sum_{i=1}^\ell f_i D_i$, where $f_i \in S_p$ for each $1 \leq i \leq \ell$. In this case we write $\text{pdeg} \, \theta = p$. By defining $\text{Der}(S)_p$ to be the $\mathbb{C}$-subspace of $\text{Der}(S)$ consisting of all homogeneous derivations of polynomial degree $p$, we see that $\text{Der}(S)$ is a graded $S$-module: $\text{Der}(S) = \bigoplus_{p \in \mathbb{Z}} \text{Der}(S)_p$.

Following [OT92, Def. 4.4], we define the $S$-submodule $D(f)$ of $\text{Der}(S)$ for $f \in S$ by $D(f) := \{ \theta \in \text{Der}(S) \mid \theta(f) \in fS \}$.

If $\mathcal{A}$ is an arrangement in $V$, then for every $H \in \mathcal{A}$ we may fix $\alpha_H \in V^*$ with $H = \text{ker}(\alpha_H)$. We call $Q(\mathcal{A}) := \prod_{H \in \mathcal{A}} \alpha_H \in S$ the defining polynomial of $\mathcal{A}$.

The module of $\mathcal{A}$-derivations is the $S$-submodule of $\text{Der}(S)$ defined by

$$D(\mathcal{A}) := D(Q(\mathcal{A})).$$

The arrangement $\mathcal{A}$ is said to be free if the module of $\mathcal{A}$-derivations $D(\mathcal{A})$ is a free $S$-module.

Note that $D(\mathcal{A})$ is a graded $S$-module $D(\mathcal{A}) = \bigoplus_{p \in \mathbb{Z}} D(\mathcal{A})_p$, where $D(\mathcal{A})_p = D(\mathcal{A}) \cap \text{Der}(S)_p$, see [OT92, Prop. 4.10]. If $\mathcal{A}$ is a free $\ell$-arrangement, then by [OT92, Prop. 4.18] the $S$-module $D(\mathcal{A})$ admits a basis of $\ell$ homogeneous derivations $\theta_1, \ldots, \theta_\ell$. While these derivations are not unique, their polynomial degrees $\text{pdeg} \, \theta_i$ are unique (up to ordering). The set of exponents of the free arrangement $\mathcal{A}$ is the multiset

$$\exp \mathcal{A} := \{ \text{pdeg} \, \theta_1, \ldots, \text{pdeg} \, \theta_\ell \}.$$

For the stronger notion of inductive freeness, see [OT92, Def. 4.53].

### 2.3. Supersolvable Arrangements

Let $\mathcal{A}$ be an arrangement. Following [OT92, §2], we say that $X \in L(\mathcal{A})$ is modular provided $X + Y \in L(\mathcal{A})$ for every $Y \in L(\mathcal{A})$. Let $\mathcal{A}$ be a central (and essential) $\ell$-arrangement. The next notion is due to Stanley [Sta72].

**Definition 2.3.** We say that $\mathcal{A}$ is supersolvable provided there is a maximal chain

$$V = X_0 < X_1 < \cdots < X_{\ell-1} < X_\ell = T$$

of modular elements $X_i$ in $L(\mathcal{A})$, cf. [OT92, Def. 2.32].
It is easy to see that every 2-arrangement is supersolvable, [HR14, Rem. 2.3]. The next result is also straightforward, [HR14, Lem. 2.4].

**Lemma 2.4.** A 3-arrangement \( \mathcal{A} \) is supersolvable if and only if there exists a modular element in \( L(\mathcal{A}) \) of dimension 1.

Thanks to [HR14, Prop. 2.5], supersolvable arrangements behave well with respect to the product construction for arrangements from §2.1.

Supersolvable arrangements are always free, [OT92, Thm. 4.58]:

**Theorem 2.5.** Let \( \mathcal{A} \) be a supersolvable \( \ell \)-arrangement with maximal chain

\[
V = X_0 < X_1 < \cdots < X_{\ell-1} < X_\ell = T
\]

of modular elements \( X_i \) in \( L(\mathcal{A}) \). Define \( b_i = |\mathcal{A}_{X_i} \setminus \mathcal{A}_{X_{i-1}}| \) for \( 1 \leq i \leq \ell \). Then \( \mathcal{A} \) is inductively free with \( \exp \mathcal{A} = \{b_1, \ldots, b_\ell\} \).

Note that it was first proven in [JT84, Thm. 4.2] that every supersolvable arrangement is inductively free. We proceed with some further preliminary results.

**Lemma 2.6.** Let \( X \in L(\mathcal{A}) \) be modular and \( Z \in L(\mathcal{A}) \). Then \( X \cap Z \) is modular in \( L(\mathcal{A}^Z) \).

**Proof.** Let \( Y \in L(\mathcal{A}^Z) = L(\mathcal{A})^Z \subseteq L(\mathcal{A}) \). Since \( X \) is modular in \( L(\mathcal{A}) \), we have \( X + Y \in L(\mathcal{A}) \) and a direct calculation shows \( (Z \cap X) + Y = Z \cap (X + Y) \in L(\mathcal{A}^Z) \). \( \square \)

The following immediate consequence of Lemma 2.6 is due to Stanley, [Sta72, Prop. 3.2].

**Corollary 2.7.** Let \( X < Y \) in \( L(\mathcal{A}) \). If \( \mathcal{A} \) is supersolvable, then so is the interval \([X, Y]\).

Here is a further consequence of Lemma 2.6.

**Corollary 2.8.** Let \( \mathcal{A} \) be a 4-arrangement and let \( Z \in L(\mathcal{A}) \) be of dimension 1. Suppose \( |\mathcal{A}_Z| > |\{H \in \mathcal{A} \mid \mathcal{A}^H \text{ is supersolvable}\}| \). Then \( Z \) is not modular in \( L(\mathcal{A}) \).

**Proof.** By assumption on \( \mathcal{A}_Z \), there is a \( H \in \mathcal{A} \) with \( Z \in L(\mathcal{A}^H) \) such that \( \mathcal{A}^H \) is not supersolvable. Therefore, since \( \mathcal{A}^H \) is a 3-arrangement, \( Z \) is not modular in \( L(\mathcal{A}^H) \), by Lemma 2.4. We conclude that \( Z \) is not modular in \( L(\mathcal{A}) \), thanks to Lemma 2.6. \( \square \)

**Lemma 2.9.** Let \( \mathcal{A} \) be an arrangement. Then for \( X \in L(\mathcal{A}) \), we have \( |\mathcal{A}_X| > |(\mathcal{A}^H)_X| \) for all \( H \in \mathcal{A}_X \).

**Proof.** Note that \( |(\mathcal{A}^H)_X| = |\{H \cap H' \mid H' \in \mathcal{A}_X \setminus \{H\}\}| < |\mathcal{A}_X| \). \( \square \)

The next result gives a useful numerical criterion for the non-supersolvability of a free, irreducible 3-arrangement \( \mathcal{A} \) in terms of the exponents of \( \mathcal{A} \).

**Lemma 2.10.** Let \( \mathcal{A} \) be a free, irreducible 3-arrangement. Let \( \exp \mathcal{A} = \{1, b_1, b_2\} \) with \( b_1 \leq b_2 \). If \( |\mathcal{A}_X| \leq b_1 \) for all \( X \in L(\mathcal{A}) \) of dimension 1, then \( \mathcal{A} \) is not supersolvable.
Proof. Note that since \( \mathcal{A} \) is irreducible, \( b_1 > 1 \), [OT92, Thm. 4.29(3)]. Assume \( |A_X| \leq b_1 \) and suppose that \( \mathcal{A} \) is supersolvable with \( V < H < X < 0 \) a maximal chain of modular elements in \( L(\mathcal{A}) \). Observe that \( |A_H \setminus A_V| = |\{H\} \setminus \emptyset| = 1 \). By Theorem 2.5, we have \( |A_X \setminus A_H| \in \{b_1, b_2\} \). However, \( |A_X \setminus A_H| = |A_X| - 1 < b_1 \leq b_2 \), a contradiction. \( \square \)

The following technical lemma is our key tool in order to show that there are no modular elements of dimension 1 in certain non-supersolvable, irreducible 4-arrangements.

**Lemma 2.11.** Let \( \mathcal{A} \) be a free, irreducible 4-arrangement. Assume there is, up to lattice isomorphism, only one supersolvable restriction \( \mathcal{B} := \mathcal{A}^H \) to a hyperplane \( H \in \mathcal{A} \). Let \( 1 < b_1 \leq b_2 \) be the exponents of \( \mathcal{B} \) and let \( c = |\{H' \in \mathcal{A} \mid L(\mathcal{A}'') \cong L(\mathcal{B})\}| \). If \( c \leq b_1 + 1 \), then there are no modular elements of dimension 1 in \( L(\mathcal{A}) \).

**Proof.** Suppose that \( X \in L(\mathcal{A}) \) is modular of dimension 1. Then \( X \) is also modular in \( \mathcal{A}^H \) for all \( H \in A_X \), by Lemma 2.6, and since \( \mathcal{A}^H \) is a 3-arrangement, it is supersolvable, by Lemma 2.4. Lemma 2.10 implies that \( |(\mathcal{A}^H)_X| > b_1 \) and Lemma 2.9 shows that \( |A_X| > b_1 + 1 \geq c \). This is a contradiction to Corollary 2.8. \( \square \)

### 2.4. Reflection Groups and Reflection Arrangements

The irreducible finite complex reflection groups were classified by Shephard and Todd, [ST54]. Let \( W \subseteq GL(V) \) be a finite complex reflection group.

The reflection arrangement \( \mathcal{A} = \mathcal{A}(W) \) of \( W \) in \( V \) is the hyperplane arrangement consisting of the reflecting hyperplanes of the elements in \( W \) acting as reflections on \( V \).

The intermediate arrangements \( A_k(r) \) (with \( 0 \leq k \leq \ell \)) interpolate between the reflection arrangements of \( G(\ell, r, \ell) \) and \( G(\ell, 1, \ell) \). Let \( H_i := \ker(x_i) \) and \( H_{ij}(\zeta) := \ker(x_i - \zeta x_j) \), then \( A_k(r) = \{H_1, \ldots, H_k, H_{ij}(\zeta) \mid 1 \leq i < j \leq \ell, \zeta^r = 1\} \). Note that \( H_{ij}(\zeta) = H_{ji}(\zeta^{-1}) \). We now recall [OS82, Prop. 2.11] (see also [OT92, Prop. 6.82]):

**Proposition 2.12.** Let \( \mathcal{A} = A_k(r) \) and let \( H \in \mathcal{A} \). The type of \( \mathcal{A}^H \) is given in Table 1.

| \( k \)  | \( H \)       | Type of \( \mathcal{A}^H \)          |
|---------|---------------|-------------------------------------|
| 0       | arbitrary     | \( A_{\ell-1}^1(r) \)               |
| 1, \ldots, \ell - 1 | \( H_{ij}(\zeta) \) \quad 1 \leq i < j \leq k \leq \ell | \( A_{\ell-1}^{k-1}(r) \)               |
| 1, \ldots, \ell - 1 | \( H_{ij}(\zeta) \) \quad 1 \leq i \leq k < j \leq \ell | \( A_{\ell-1}^k(r) \)               |
| 1, \ldots, \ell - 1 | \( H_{ij}(\zeta) \) \quad 1 < k < i < j \leq \ell | \( A_{\ell-1}^{k+1}(r) \)               |
| 1, \ldots, \ell - 1 | \( H_i \) \quad 1 \leq i \leq \ell | \( A_{\ell-1}^{\ell-1}(r) \)               |
| \( \ell \) | arbitrary     | \( A_{\ell-1}^{\ell-1}(r) \)               |

Table 1: Restriction types of \( A_k^H(r) \)

Let \( \mathcal{A} = A_k(r) \), fix \( H \in \mathcal{A} \) and let \( (\mathcal{A}, \mathcal{A}' = \mathcal{A} \setminus \{H\}, \mathcal{A}'' = \mathcal{A}^H) \) be the triple with respect to \( H \). In Table 2, we consider the map \( \mathcal{A}' \to \mathcal{A}'' \) given by \( H' \mapsto H' \cap H \), where restricting
to $H_i$ results in the substitution $x_i \to 0$, and restricting to $H_{ij}(\zeta)$ results in the substitution $x_i \to \zeta x_j$.

| $H'$       | restricts to $\tilde{H} \in \mathcal{A}^{H_n}$          | $H'$       | restricts to $\tilde{H} \in \mathcal{A}^{H_{m\alpha}(\eta)}$ |
|------------|----------------------------------------------------------|------------|-------------------------------------------------------------|
| $H_i$      | $i \neq n$ $H_i$                                          | $H_i$      | $i \neq m$ $H_i$                                          |
| $H_{ij}(\zeta)$ | $i, j \neq n$ $H_{ij}(\zeta)$                  | $H_{ij}(\zeta)$ | $i \neq m, j \neq n$ $H_{nij}(\eta^{-1}\zeta)$ |
| $i = n$    | $H_j$                                                   | $i = m$    | $H_n$                                                    |
| $j = n$    | $H_i$                                                   | $\zeta \neq \eta$ | $H_n$                                                   |

Table 2: Restrictions of $\mathcal{A} = \mathcal{A}_k^{\ell}(r)$ to hyperplanes $H_n$ and $H_{mn}(\eta)$

3. PROOF OF THEOREM 1.3

We maintain the notation from the previous sections. Note again that $\mathcal{A}_k^{\ell}(r)$ is not a reflection arrangement for $k \neq 0, \ell$.

**Lemma 3.1.** For $r \geq 2$, $\ell \geq 3$ and $0 \leq k \leq \ell$, the arrangement $\mathcal{A}_k^{\ell}(r)$ is supersolvable if and only if $\ell - 1 \leq k \leq \ell$ or $(k, \ell, r) = (0, 3, 2)$.

**Proof.** First note that $\mathcal{A}_3^0(2)$ is just the Coxeter arrangement of type $D_3 = A_3$. Thus $\mathcal{A}_3^0(2)$ is supersolvable, by Theorem 1.1.

 Else for $k = 0$ and $k = \ell$, the result follows from Theorem 1.1 and for $1 \leq k \leq \ell - 3$, the result is a consequence of Theorems 1.2(ii) and 2.5. For $k = \ell - 1$, this follows from the case for $k = \ell$, the proof of [HR14, Thm. 1.2(iii)] and [OT92, Lem. 2.62]. Finally, for $k = \ell - 2$, we argue by induction on $\ell$ as follows. If $\mathcal{A}$ is any (central) arrangement and $H$ is in $\mathcal{A}$, then $L(\mathcal{A}^H)$ is an intervall in $L(\mathcal{A})$. By Corollary 2.7, if $\mathcal{A}^H$ is not supersolvable, then neither is $\mathcal{A}$. Suppose $\mathcal{A} = \mathcal{A}_k^{\ell-2}(r)$ and $H = \ker(x_i - \zeta x_j) \in \mathcal{A}_k^{\ell-2}(r)$ for some $1 \leq i < j \leq \ell - 2$ and some $r^{th}$ root of unity $\zeta$. By Proposition 2.12, we have $\mathcal{A}^H \cong \mathcal{A}_k^{\ell-3}(r)$, so it suffices to show that $\mathcal{A}_3^1(r)$ is not supersolvable. So, let $\mathcal{A} = \mathcal{A}_3^1(r)$ and let $X \in L(\mathcal{A})$ be of rank 2. Then $|\mathcal{A}_X| \in \{2, 3, r, r + 1\}$. Since $\exp \mathcal{A} = \{1, r + 1, 2r - 1\}$, Lemma 2.10 implies that $\mathcal{A}_3^1(r)$ is not supersolvable.

Note that it was already observed in [JT84, Ex. 5.5] that $\mathcal{A}_3^1(2)$ is not supersolvable.

The following example shows that the supersolvable arrangements from Lemma 3.1 do actually occur as restrictions of the reflection arrangement of $W = G(r, r, \ell)$.

**Example 3.2.** Let $p, r \geq 2$, $\ell = 2p - 1$, and $\zeta$ is an $r$th root of unity. Let $W = G(r, r, \ell)$ and for $1 \leq i \neq j \leq \ell$, let $H_{i, j}(\zeta) = \ker(x_i - \zeta x_j)$ be a hyperplane in $\mathcal{A} = \mathcal{A}(W)$. Then $X = \cap_{i=1}^{p-1} H_{2i-1, 2i}(\zeta)$ belongs to $L(\mathcal{A})$ with $\dim X = \ell - (p - 1) = p$. According to the
If $Y$ is in $L(A)$ with $\dim Y = p$ such that $Y$ is contained in at least one coordinate hyperplane \( \ker(x_i) \), then again by [OS82, Prop. 2.14] ([OT92, Prop. 6.84]), we see that $A^Y \cong A^p_\gamma(r) = A(G(r, 1, p))$ which is supersolvable, by Theorem 1.1.

As indicated in the Introduction, we use the convention to label the $W$-orbit of $X \in L(A)$ by the type $T$ which is the Shephard-Todd label [ST54] of the complex reflection group $W_X$. We then denote the restriction $A^X$ simply by the pair $(W, T)$. The following result is due to Orlik and Terao, [OT92, App. D].

**Lemma 3.3.** We have the following lattice isomorphisms of 3-dimensional restrictions:

1. $(E_6, A_3) \cong A_3^2(2)$;
2. $(E_7, D_4) \cong A_3^3(2)$;
3. $(F_4, A_1) \cong (F_4, A_1) \cong (E_7, A_4^3) \cong (E_7, (A_1A_3)') \cong (E_8, A_1D_4) \cong (E_8, D_5)$;
4. $(E_6, A_1A_2) \cong (E_7, A_4)$;
5. $(E_7, A_2^2) \cong (E_8, A_5)$;
6. $(E_8, A_2^2A_3) \cong (E_8, A_2A_3)$;
7. $(G_{26}, A_0) \cong (G_{32}, C(3)) \cong (G_{34}, G(3, 3, 3))$.

Armed with Lemma 2.10, we can now determine all 3-dimensional supersolvable restrictions for an ambient irreducible, non-supersolvable reflection arrangement.

**Lemma 3.4.** Let $A = A(W)$ be an irreducible, non-supersolvable reflection arrangement of exceptional type. Let $X \in L(A)$ with $\dim X = 3$. Then $A^X$ is supersolvable if and only if $A^X$ is $(E_6, A_3), (E_7, D_4), (E_7, A_2^2),$ or $(E_8, A_5)$.

**Proof.** Using Theorem 1.1, the tables of all orbit types for the irreducible reflection groups of exceptional type in [OS82, App.] (see also [OT92, App. C]) and some explicit computer aided calculations (cf. Remark 4.8), we check which of the 3-dimensional restrictions satisfies the hypothesis of Lemma 2.10. In Table 3 we present in each of the 3-dimensional restrictions $B = A(W)^X$ of the exceptional cases all values $|B_Y|$, where $Y \in L(B)$ is of dimension 1 (disregarding the lattice isomorphisms from Lemma 3.3). Each of the cases labelled “false” satisfies the condition from Lemma 2.10 and is therefore not supersolvable.

| $B : = A^X$ | $\exp B$ | $\{ |B_Y| \mid \rk(Y) = 2 \}$ | supersolvable |
|-------------|----------|------------------|--------------|
| $(F_4, A_1)$ | 1,5,7    | $\{2,3,4\}$     | false        |
| $(F_4, A_1')$ | 1,5,7    | $\{2,3,4\}$     | false        |
| $(G_{29}, A_1)$ | 1,9,11  | $\{2,3,4,5\}$   | false        |
| $(H_{4}, A_1)$ | 1,11,19  | $\{2,3,5,6\}$   | false        |
| $(G_{31}, A_1)$ | 1,13,17  | $\{2,3,6\}$     | false        |
| $(G_{32}, C(3))$ | 1,7,13  | $\{2,4,5\}$     | false        |
| $(G_{33}, A_2^2)$ | 1,7,9   | $\{2,3,4,5\}$   | false        |
Next we argue that the 4 cases labelled “true” in Table 3 are indeed supersolvable:

The restrictions $(E_6, A_3)$ and $(E_7, D_4)$ are supersolvable, thanks to Lemmas 3.1 and 3.3(i), (ii). Moreover, by Lemma 3.3(v), $(E_7, A_7^2) \cong (E_8, A_5)$, so we only have to check the case $(E_7, A_7^2)$.

So, let $B = \{H_0, \ldots, H_{12}\}$ be the arrangement given by $(E_7, A_7^2)$. Since $B$ is a 3-arrangement, we only need to compute the intersections of rank 2. Each $X \in L(B)$ is uniquely given by $B_X$. The $X \in L(B)$ of rank 2 correspond to the following subarrangements $B_X$ of $B$:

| $(G_{34}, A_2)$ | 1,6,7 | $\{2,3,4,5\}$ | false |
| $(G_{34}, A_3^5)$ | 1,13,19 | $\{2,3,4,5,8\}$ | false |
| $(G_{34}, A_1A_2)$ | 1,13,16 | $\{2,3,4,5,7\}$ | false |
| $(G_{34}, A_3)$ | 1,11,13 | $\{2,3,4,6\}$ | false |
| $(G_{34}, G(3, 3, 3))$ | 1,7,13 | $\{2,4,5\}$ | false |
| $(E_6, A_2)$ | 1,4,5 | $\{2,3,4\}$ | false |
| $(E_6, A_1A_2)$ | 1,4,5 | $\{2,3,4\}$ | false |
| $(E_6, A_3)$ | 1,3,4 | $\{2,3,4\}$ | true |
| $(E_7, A_4^2)$ | 1,5,7 | $\{2,3,4\}$ | false |
| $(E_7, A_7^2A_2)$ | 1,5,7 | $\{2,3,4,5\}$ | false |
| $(E_7, A_7^2)$ | 1,5,7 | $\{2,3,4,6\}$ | true |
| $(E_7, (A_1A_3)^{\prime})$ | 1,5,7 | $\{2,3,4\}$ | false |
| $(E_7, (A_1A_3)^{\prime\prime})$ | 1,5,5 | $\{2,3,4\}$ | false |
| $(E_7, A_4)$ | 1,4,5 | $\{2,3,4\}$ | false |
| $(E_7, D_4)$ | 1,3,5 | $\{2,3,4\}$ | true |
| $(E_8, A_1^2A_2)$ | 1,7,11 | $\{2,3,4,5,6\}$ | false |
| $(E_8, A_1^2A_2^2)$ | 1,7,11 | $\{2,3,4,6\}$ | false |
| $(E_8, A_1^2A_3)$ | 1,7,9 | $\{2,3,4,6\}$ | false |
| $(E_8, A_2A_3)$ | 1,7,9 | $\{2,3,4,6\}$ | false |
| $(E_8, A_1A_4)$ | 1,7,8 | $\{2,3,4,5\}$ | false |
| $(E_8, A_5)$ | 1,5,7 | $\{2,3,4,6\}$ | true |
| $(E_8, A_1A_4D_4)$ | 1,5,7 | $\{2,3,4\}$ | false |
| $(E_8, D_5)$ | 1,5,7 | $\{2,3,4\}$ | false |

Table 3: The supersolvable 3-dimensional restrictions of the exceptional groups

\[
\begin{align*}
\{H_0, H_1, H_3, H_6, H_9, H_{12}\}, & \quad \{H_0, H_2, H_{10}\}, & \quad \{H_0, H_4, H_5, H_{11}\}, & \quad \{H_0, H_7, H_8\}, \\
\{H_1, H_2, H_7, H_{11}\}, & \quad \{H_1, H_4, H_{10}\}, & \quad \{H_1, H_5, H_8\}, & \quad \{H_1, H_5, H_8\}, \\
\{H_2, H_3, H_4\}, & \quad \{H_2, H_5, H_6\}, & \quad \{H_2, H_8, H_{12}\}, & \quad \{H_2, H_9\}, \\
\{H_3, H_5, H_7\}, & \quad \{H_3, H_8\}, & \quad \{H_3, H_{10}\}, & \quad \{H_3, H_{11}\}, \\
\{H_4, H_6, H_7\}, & \quad \{H_4, H_8, H_9\}, & \quad \{H_4, H_{12}\}, & \quad \{H_4, H_{12}\}, \\
\{H_5, H_9, H_{10}\}, & \quad \{H_5, H_{12}\}, & \quad \{H_5, H_{12}\}, & \quad \{H_5, H_{12}\}, \\
\{H_6, H_8, H_{10}, H_{11}\}, & \quad \{H_7, H_9\}, & \quad \{H_7, H_{10}, H_{12}\}, & \quad \{H_9, H_{11}\}, \\
\{H_9, H_{11}\}, & \quad \{H_{11}, H_{12}\}. & & \end{align*}
\]
Let $Z := H_0 \cap H_1 \cap H_3 \cap H_6 \cap H_9 \cap H_{12}$. For all $X \in L(B)$ of dimension 1 we have $X + Z \in L(B)$, since $X$ and $Z$ are subsets of a common hyperplane of $B$. Therefore, $Z$ is a modular element of dimension 1 in $L(B)$ and thus $B$ is supersolvable, by Lemma 2.4.

Now, with the aid of Corollary 2.7 and Lemma 3.4 we can show that among irreducible, non-supersolvable reflection arrangement of exceptional type there are no 4-dimensional supersolvable restrictions.

**Lemma 3.5.** Let $A = A(W)$ be an irreducible, non-supersolvable reflection arrangement of exceptional type. Let $X \in L(A)$ with $\dim X = 4$. Then $A^X$ is not supersolvable.

**Proof.** Thanks to Lemma 3.4, for every $W$ as in the statement, other than $W$ of type $E_6$, $E_7$ or $E_8$, already every 3-dimensional restriction is not supersolvable. It thus follows from Corollary 2.7 that in these cases no higher-dimensional restriction is supersolvable either.

So we are left to check the following restrictions $(E_6, A_7^1)$, $(E_6, A_2)$, $(E_7, A_1^3)$, $(E_7, (A_1^3)'')$, $(E_7, A_1^2 A_2)$, $(E_7, A_3)$, $(E_8, A_4)$, $(E_8, A_2^1 A_2)$, $(E_8, A_2^3)$, $(E_8, A_1 A_3)$, $(E_8, A_4)$, $(E_8, D_4)$. Again by Corollary 2.7, it suffices to find a restriction to a hyperplane in these cases which is not supersolvable; in the following table, we exhibit a suitable restriction in each instance:

| $A^X$          | non-supersolvable restriction |
|---------------|-------------------------------|
| $(E_6, A_7^1)$ | $(E_6, A_1 A_2)$              |
| $(E_6, A_2)$  | $(E_6, A_1 A_2)$              |
| $(E_7, A_1^3)$| $(E_7, A_4)$                  |
| $(E_7, (A_1^3)''$ | $(E_7, A_4)$                  |
| $(E_7, A_1 A_2)$| $(E_7, A_4)$                  |
| $(E_7, A_3)$  | $(E_7, A_4)$                  |
| $(E_8, A_4)$  | $(E_8, A_1 D_4)$              |
| $(E_8, A_1^2 A_2)$| $(E_8, A_1 A_4)$              |
| $(E_8, A_2^3)$| $(E_8, A_2 A_3)$              |
| $(E_8, A_1 A_3)$| $(E_8, A_2 A_3)$              |
| $(E_8, A_4)$  | $(E_8, A_1 A_4)$              |
| $(E_8, D_4)$  | $(E_8, A_1 D_4)$              |

This is readily extracted from the information in the tables of [OS82, App.] ([OT92, App. C]).

**Corollary 3.6.** Let $A = A(W)$ be an irreducible, non-supersolvable reflection arrangement of exceptional type. Let $X \in L(A)$ with $\dim X \geq 4$. Then $A^X$ is not supersolvable.

**Proof.** Thanks to Lemma 3.5, every restriction of dimension 4 is non-supersolvable. The result then follows from Corollary 2.7.

**Proof of Theorem 1.3.** Thanks to Corollary 2.7, every interval of a supersolvable lattice is itself supersolvable. Consequently, if $A$ is supersolvable, then so is any restriction $A^X$, by Lemma 2.1. If $A^X$ is as in (ii) or (iii), then $A^X$ is supersolvable, thanks to Lemmas 3.1 and 3.4 and Corollary 3.6. This gives the reverse implication.
If $A^X$ is supersolvable but $A$ is not, then $A^X$ is as in (ii) or (iii), again by Lemmas 3.1 and 3.4 and Corollary 3.6. This gives the forward implication. $\square$

4. Proof of Theorem 1.5

We start by determining all modular elements in the lattice of $A^k_r$. We introduce the following notation: Let $B_k$ be the Boolean arrangement in $C^k$ and define $B^k_\ell := B_k \times \Phi_{\ell-k}$ to be the $\ell$-arrangement consisting of the first $k$ coordinate hyperplanes in $C^\ell$, i.e. $B^k_\ell = \{ \{H_1, \ldots, H_k\}, C^\ell\}$. Then clearly $B^k_\ell \subseteq A^n_m(r)$ for all $m \geq k$.

Lemma 4.1. Let $A = A^k_r$ with $\ell \geq 3$, $0 \leq k \leq \ell$ and $r \geq 2$. Then all elements of $L(B^k_\ell) \subseteq L(A)$ are modular in $L(A)$.

Proof. Let $X \in L(B^k_\ell)$ with $1 \leq \dim X < \ell - 1$ (i.e. $X$ is not trivially modular) and $I = \{1, \ldots, \ell\}$. Then there is a subset $\Delta_X \subseteq \{1, \ldots, k\}$ such that $X = \bigcap_{i \in \Delta_X} H_i$.

Let $Y \in L(A)$ with $\dim Y = p$. Then using the construction in [OS82, §2] (cf. [OT92, §6.4]), we can find a subset $\Delta_Y \subseteq I$ and a partition $\Lambda_Y = (\Lambda_1, \ldots, \Lambda_p)$ of $I \setminus \Delta_Y$ together with $r$th roots of unity $\theta_{ij}$ such that

$$Y = \bigcap_{i \in \Delta_Y} \bigcap_{i,j \in \Lambda_1} H_{ij}(\theta_{ij}) \cap \cdots \cap \bigcap_{i,j \in \Lambda_p} H_{ij}(\theta_{ij}),$$

where we agree that $H_{ii}(\theta) = C^\ell$ for all $i$ and all $\theta$. Consequently, in the following we occasionally omit the intersections corresponding to $\Lambda_i$ with $|\Lambda_i| = 1$.

Now let $\hat{\Lambda}_i := \Lambda_i \cap \Delta_X$ for $i = 1, \ldots, p$. If $\hat{\Lambda}_i = \emptyset$ for all $i$, then $Y \subseteq X$ and hence $X + Y = X \in L(A)$. Thus we may assume that $\{i \mid \hat{\Lambda}_i \neq \emptyset\} = \{1, \ldots, q\}$ for some $1 \leq q \leq p$. Then we get

$$X + Y = \bigcap_{i \in \Delta_X \cap \Delta_Y} \bigcap_{i,j \in \hat{\Lambda}_i} H_{ij}(\theta_{ij}) \cap \cdots \cap \bigcap_{i,j \in \hat{\Lambda}_q} H_{ij}(\theta_{ij}) \in L(A),$$

and we are done. $\square$

Lemma 4.2. Let $A = A^k_3(r)$ with $1 \leq k \leq 3$ and $r \geq 2$ and let $X \in L(A)$ of dimension 1. Then $X$ is modular in $L(A)$ if and only if $X = H_i \cap H_j$ for some $1 \leq i < j \leq k$.

Proof. By Lemma 4.1 every such element is modular, so in the following we assume that $X$ is not of this form. If $X = H_i \cap H$ for some $H \in A$ and $k < i \leq 3$, then a simple argument shows that $X$ is not modular. Thus we may assume that $X$ is either $X^n = H_1 \cap H_{23}(\eta)$ or $X_{\theta_2 \theta_3} = H_{12}(\theta_2) \cap H_{13}(\theta_3)$. A simple calculation shows that $X^n + X_{\theta_2 \theta_3} = \ker((\theta_2 \eta + \theta_3)x_1 - \eta x_2 - x_3)$ which is not contained in $L(A)$ whenever $\theta_2 \eta + \theta_3 \neq 0$ and hence proves the lemma. $\square$

Proposition 4.3. Let $\ell \geq 3$, $0 \leq k \leq \ell$ and $r \geq 2$ and let $A = A^k_\ell(r) \not= A^0_3(2)$. Then $X \in L(A)$ is modular in $L(A)$ if and only if $X \in L(B^k_\ell)$ or $X \in A \cup \{0, C^\ell\}$. 

Proof. Let $X \in L(\mathcal{A})$ with $1 \leq \dim X = p \leq \ell - 2$, that is $X$ is none of the obviously modular elements, cf. [OT92, Ex. 2.28]. Using the construction in [OS82, §2] ([OT92, §6.4]), we can find a subset $\Delta_X \subseteq I$ and a partition $\Lambda_X = (\Lambda_1, \ldots, \Lambda_p)$ of $I \setminus \Delta_X$ together with $r$th roots of unity $\theta_j$ such that

$$X = \bigcap_{j \in \Delta_X} H_j \cap \bigcap_{j \in \Lambda_1 \setminus \{m_1\}} H_{m_1j}(\theta_j) \cap \cdots \cap \bigcap_{j \in \Lambda_p \setminus \{m_p\}} H_{mpj}(\theta_j),$$

for $m_i := \min \Lambda_i$ $(1 \leq i \leq p)$. We may assume that $|\Lambda_1| \geq \ldots \geq |\Lambda_p|$. It is easy to see that $X$ is not modular whenever $\Delta_X \not\subseteq \{1, \ldots, k\}$, so we assume $\Delta_X \subseteq \{1, \ldots, k\}$. Then we have $|\Lambda_1| = \ldots = |\Lambda_p| = 1$ if and only if $X \in L(\mathcal{B}_k^\ell)$. By Lemma 4.1, the elements of $L(\mathcal{B}_k^\ell)$ are modular. Thus we assume $|\Lambda_1| > 1$ and show that $X$ is not modular. For $H \in \mathcal{A}$, we know by Proposition 2.12 that $\mathcal{A}^H \cong \mathcal{A}_{k'-1}^\ell(r)$ with $0 < k' \leq \ell - 1$. Using Lemmas 2.6, 4.2 and induction on $\ell$, it suffices to show that for $\ell \geq 4$ we can choose $H \in \mathcal{A}$ such that $X \cap H$ is not modular in $\mathcal{A}_{k'-1}^\ell(r)$. Now we discuss two cases:

**Case 1:** The rank of $X$ is at least 3. Here we have two subcases:

**Case 1a:** If $\Delta_X \neq \emptyset$, then choose $i \in \Delta_X$ and set $H := H_i$. Considering Table 2, we see that in $L(\mathcal{A}^H)$ the element $X \cap H = X$ is of the form

$$X = \bigcap_{j \in \Delta_X \setminus \{i\}} H_j \cap \bigcap_{j \in \Lambda_1 \setminus \{m_1\}} H_{m_1j}(\theta_j) \cap \cdots \cap \bigcap_{j \in \Lambda_p \setminus \{m_p\}} H_{mpj}(\theta_j).$$

We still have $|\Lambda_1| > 1$ and hence $X$ is not contained in the lattice of the Boolean subarrangement $\mathcal{B}_{k-1}^\ell$ of $\mathcal{A}^H$. As $X$ is clearly neither zero nor a hyperplane in $H$ (nor $H$ itself), it is not modular in $L(\mathcal{A}^H)$ by induction hypothesis.

**Case 1b:** If $\Delta_X = \emptyset$, then we may assume $|\Lambda_1| \geq 3$ or $|\Lambda_1| = |\Lambda_2| = 2$, because $X$ is not contained in $\mathcal{A}$. In both cases we set $H := H_{m_1j}(\theta_i)$ for some $i \in \Lambda_1 \setminus \{m_1\}$. Now using Table 2, we see that $X = X \cap H \in L(\mathcal{A}^H)$ is of the form

$$X = \bigcap_{j \in \Lambda_1 \setminus \{i, m_1\}} H_{ij}(\theta_i^{-1}\theta_j) \cap \cdots \cap \bigcap_{j \in \Lambda_p \setminus \{m_p\}} H_{mpj}(\theta_j),$$

where in the case $|\Lambda_1| = 2$ the first intersection is empty. Clearly, we have $|\Lambda_1 \setminus \{i\}| > 1$ or $|\Lambda_2| > 1$ and hence $X$ is not contained in the lattice of the Boolean subarrangement $\mathcal{B}_{k-1}^\ell$ of $\mathcal{A}^H$. As $X$ is clearly neither zero nor a hyperplane in $H$ (nor $H$ itself), it is not modular in $L(\mathcal{A}^H)$ by induction hypothesis.

**Case 2:** Now let $X$ be of rank 2. We may assume that $X$ is either $X_a = H_1 \cap H_{23}(\theta)$, $X_b = H_{12}(\theta_2) \cap H_{13}(\theta_3)$ or $X_c = H_{12}(\theta_2) \cap H_{34}(\theta_4)$. We set $H := H_{23}(\eta) \in \mathcal{A}$ and set $X_\gamma := X_\gamma \cap H$ for $\gamma \in \{a, b, c\}$. Clearly, $X_\gamma \subseteq L(\mathcal{A}^H)$ ($\gamma = a, b, c$) is none of the trivially modular elements. Table 2 shows that in $L(\mathcal{A}^H)$ we get the following cases

$$X_a = H_1 \cap H_{34}(\theta_3^{-1}\eta), \quad X_b = H_{14}(\theta_2\eta) \cap H_{13}(\theta_3) \quad \text{and} \quad X_c = H_{14}(\theta_2\eta) \cap H_{34}(\theta_4).$$

Thus $X_\gamma$ ($\gamma = a, b, c$) is not contained in the lattice of the Boolean subarrangement $\mathcal{B}_{k-1}^\ell$ of $\mathcal{A}^H$ and hence it is not modular by induction hypothesis.

We obtain the following immediate consequence of Proposition 4.3.
Corollary 4.4. Let $\mathcal{A} = \mathcal{A}(G(r, p, \ell))$ and $X \in L(\mathcal{A})$. Then $\mathcal{A}^X$ is supersolvable if and only if $L(\mathcal{A}^X)$ contains a modular element of dimension 1.

Proof. It follows from Lemma 3.1 and Proposition 4.3 that $\mathcal{A}_k^\ell(r)$ is supersolvable if and only if there is a modular element of dimension 1 in $L(\mathcal{A}_k^\ell(r))$. Now the proofs of [OS82, Prop. 2.5] and [OS82, Prop. 2.14] (see also [OT92, Prop. 6.77 and 6.84]) imply the assertion. □

We record another consequence of Proposition 4.3 which together with [OS82, Prop. 2.14] ([OT92, Prop. 6.84]) gives an alternative proof of Theorem 1.3(ii).

Corollary 4.5. $\mathcal{A}_k^\ell(r)$ is supersolvable if and only if $k \in \{ \ell - 1, \ell \}$ or $(k, \ell, r) = (0, 3, 2)$.

In the remainder of this section we consider the case when $W$ is of exceptional type. We start with a further consequence of Lemma 3.4.

Corollary 4.6. Let $\mathcal{A} = \mathcal{A}(W)$ be an irreducible, non-supersolvable reflection arrangement of exceptional type. Let $X \in L(\mathcal{A})$ with $\dim X = 4$. Then $\mathcal{A}^X$ is supersolvable if and only if $L(\mathcal{A}^X)$ contains a modular element of dimension 1.

Proof. Thanks to Lemma 3.4, for every $W$ as in the statement, other than $W$ of type $E_6$, $E_7$ or $E_8$, every 3-dimensional restriction already fails to be supersolvable. Consequently, using Lemma 2.4, there are no modular elements of dimension 1 in $L(\mathcal{A})$.

We are left to check the 4-dimensional restrictions for $W$ of type $E_6$, $E_7$ and $E_8$.

Using [OS82, App.] (cf. [OT92, App. C]), it is easily seen that each of $(E_7, (A_3^2))$, $(E_8, A_1^4)$, $(E_8, A_2^2A_2)$, and $(E_8, D_4)$ has no supersolvable restriction to a hyperplane and therefore none of them admits a modular element of dimension 1, by Lemmas 2.4 and 2.7.

The next set of instances we consider are ones which have only one type of restriction to a hyperplane which is supersolvable. We can then use the condition of Lemma 2.11 to show that none of these cases admits a modular element of dimension 1. We use the tables of [OS82, App.] ([OT92, App. C]) to determine the number of restrictions which are supersolvable and denote this number again by $c$, as in Lemma 2.11:

$(E_6, A_2^2)$: the supersolvable restriction has exponents $\{1, 3, 4\}$ and $c = 3$.
$(E_7, (A_3^2))$: the supersolvable restriction has exponents $\{1, 3, 5\}$ and $c = 1$.
$(E_7, (A_1A_2))$: the supersolvable restriction has exponents $\{1, 5, 7\}$ and $c = 4$.
$(E_7, A_3)$: the supersolvable restriction has exponents $\{1, 3, 5\}$ and $c = 3$.
$(E_8, A_2^2)$: the supersolvable restriction has exponents $\{1, 5, 7\}$ and $c = 6$.
$(E_8, A_1A_3)$: the supersolvable restriction has exponents $\{1, 5, 7\}$ and $c = 4$.

Unfortunately, Lemma 2.11 does not apply for $(E_6, A_2)$ and $(E_8, A_4)$, since $c$ is too big in these instances. Using a computer, we checked directly that there are no modular elements of dimension 1 in these two cases (cf. Remark 4.8). □

Corollary 4.7. Let $\mathcal{A} = \mathcal{A}(W)$ be an irreducible, non-supersolvable reflection arrangement of exceptional type. Let $X \in L(\mathcal{A})$ with $\dim X \geq 3$. Then $\mathcal{A}^X$ is supersolvable if and only if $L(\mathcal{A}^X)$ contains a modular element of dimension 1.
Proof.  For \( \dim X = 3 \), the assertion follows from Lemma 2.4 and Corollary 4.6 shows that this also holds for \( \dim X = 4 \). Since all restrictions \( \mathcal{A}^X \) with \( \dim X \geq 4 \) are not supersolvable for the exceptional types, the result follows for \( \dim X > 4 \) by Lemma 2.6. \( \square \)

Proof of Theorem 1.5.  The forward implication is clear from Definition 2.3.

The reverse implication follows for general central arrangements of rank up to 3 by [HR14, Rem. 2.3] and Lemma 2.4. For \( W = G(r, p, \ell) \), the result is Corollary 4.4. Finally, for \( W \) of exceptional type and \( X \in L(A) \) with \( \dim X \geq 4 \) such that \( \mathcal{A}^X \) is not supersolvable, the result follows from Corollary 4.7.

Note that Corollary 4.4 and Corollary 4.7 also apply in the case \( X = V \). This gives the equivalence in the statement for the underlying reflection arrangement \( \mathcal{A}(W) \) itself (cf. Remark 1.6(i)). \( \square \)

Remark 4.8.  In order to establish several of our results we first use the functionality for complex reflection groups provided by the CHEVIE package in GAP (and some GAP code by J. Michel) (see [S+97] and [GHL+96]) in order to obtain explicit linear functionals \( \alpha \) defining the hyperplanes \( H = \ker \alpha \) of the reflection arrangement \( \mathcal{A}(W) \) and the relevant restrictions \( \mathcal{A}(W)^X \).

We then use the functionality of SAGE ([S+09]) to determine the data in Table 3 and in the proof of Lemma 3.4. Moreover, SAGE was used to establish the fact that the two restrictions \((E_6, A_2)\) and \((E_8, A_4)\) do not admit modular elements of dimension 1 in the proof of Corollary 4.6.

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References

[AHR13] N. Amend, T. Hoge and G. Röhrle, Inductively free restrictions of reflection arrangements, http://arxiv.org/abs/1310.1708.

[Bou68] N. Bourbaki, Éléments de mathématique. Groupes et algèbres de Lie. Chapitre IV-VI, Actualités Scientifiques et Industrielles, No. 1337, Hermann, Paris, 1968.

[GHL+96] M. Geck, G. Hiß, F. Lübeck, G. Malle, and G. Pfeiffer, CHEVIE — A system for computing and processing generic character tables, Appl. Algebra Engrg. Comm. Comput. 7 (1996), 175–210.

[HR14] T. Hoge and G. Röhrle, On supersolvable reflection arrangements, Proc. AMS, to appear.

[JT84] M. Jambu and H. Terao, Free arrangements of hyperplanes and supersolvable lattices, Adv. in Math. 52 (1984), no. 3, 248–258.

[OS82] P. Orlik and L. Solomon, Arrangements defined by unitary reflection groups, Math. Ann. 261, (1982), 339–357.

[OT92] P. Orlik and H. Terao, Arrangements of hyperplanes, Springer-Verlag, 1992.

[S+97] M. Schönert et al., GAP – Groups, Algorithms, and Programming – version 3 release 4, 1997.

[ST54] G.C. Shephard and J.A. Todd, Finite unitary reflection groups. Canadian J. Math. 6, (1954), 274–304.

[Sta72] R. P. Stanley, Supersolvable lattices, Algebra Universalis 2 (1972), 197–217.

[S+09] W. A. Stein et al., Sage Mathematics Software, The Sage Development Team, 2009, http://www.sagemath.org.
[Ste64] R. Steinberg, *Differential equations invariant under finite reflection groups*, Trans. Amer. Math. Soc. **112**, (1964), 392–400.

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