Bridging the Chiral symmetry and Confinement with Singularity: Bag vs Holography

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**Abstract:** We show that a holographic abelian Higgs model leads us to the Heun’s equation, which is the same one derived for the bag model studied by Lichtenberg et.al. The correspondence between two models resembles the AdS/CFT dictionary. The spectrum follows linear confinement for zero quark mass, while it is highly non-linear for finite quark mass. It can be traced back to the difference in the singularity class of equation of motion made by the quark mass. It suggests that the origin of the chiral symmetry is tied to the dynamics of color confinement.

**Keywords:** Holography, Weyl Semi-Metal, Topological Dipole
1 Introduction

One of the guiding picture of the low energy quantum chromo-dynamics (QCD) is that the vacuum works as a dual superconductor [1, 2] so that confined color flux forms a QCD string whose spectrum is linear in quantum number $n$,

$$\alpha' m^2 = n + \beta,$$

which is called the Regge trajectory or linear confinement. Another guiding symmetry principle for the low energy hadron dynamics is the chiral symmetry, for which the quarks mass should vanish at least approximately. Indeed, the current quark mass contributes less than 1% in counting the proton mass. However, little is understood whether these two are independent or related. While the confinement is certainly consequence of the QCD dynamics, the quark mass is usually considered as a initial condition and the its smallness is considered as a fine tuning problem. If we can relate the two then we would be able to understand the fine tuning problem in terms of the dynamics and two guiding principle would be combined into one.

In a semi-classical bag model of Lichtenberg et.al[4] for the meson, it has been known that the spectrum follows the Regge trajectory if the quark mass vanishes[5]. In a recent paper [3], the spectrum for non-vanishing quark mass was studied and found to be highly
non-linear. Since the model is consistent with the experiment only for the vanishing quark mass and the linear mass spectrum is tied to the dynamics of confinement, one may wonder whether the chiral symmetry is consequence of the confinement dynamics. However, one may also wonder if this is a feature of the specific model or true essence of the QCD dynamics. Therefore it would be nice if one can find such linked feature in other model or other context of reasoning.

In this paper, we consider a holographic fermion in AdS$_4$ interacting with a scalar in a symmetry broken phase. We will find that the Dirac equation in AdS space can be mapped to the Heun’s equation we considered earlier [3] in Bag model in flat space in spite of the difference of the space in which the systems are defined. Furthermore, it turns out that the correspondence between two models looks like a Dictionary of AdS/CFT. For example, the current quark mass in bag model should correspond to the source term of the scalar in holographic model and the string tension corresponds to the scalar condensation, which is precisely the known AdS/CFT dictionary.

We will see that in the presence of the source term, the holographic model has non-linear spectrum also as was the case for the Bag model. Such dynamical difference can be traced back to the difference of the singularity structure of the equation of motion caused by the presence of the quark mass or scalar source. Namely, the equation of motion which was hypergeometric type in the absence of the quark mass, becomes a Heun’s equation in the presence of quark mass. The singularity of the latter is one higher order than that of the former. Such higher order singularity requests higher regularity condition. As a consequence, not only the energy but also some other parameter of the theory should be quantized, which is rather surprising. Such extra quantization is the reason why the hadron spectrum in both models becomes highly non-linear in the presence of the current quark mass. However, such spectrum is not what is observed in the nature and this should be somehow forbidden by the dynamics of the confinement.

2 Holographic models with a scalar interaction

2.1 Holographic Abelian Higgs model with scalar source

For the meson spectrum, we consider the holographic abelian Higgs model in the fixed AdS$_{d+1}$ of radius $L = 1$. The action is given by

$$S = \int d^{d+1}x \sqrt{-g} \left( -\frac{1}{4} F_{\mu\nu}^2 - |D_\mu \Phi|^2 - m_\Phi^2 |\Phi|^2 \right),$$

(2.1)

where $D_\mu = \nabla_\mu - iqA_\mu$ is the covariant derivative. The metric is

$$ds^2 = (dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu)/z^2, \quad \text{with } \eta^{00} = -1.$$

(2.2)

Bulk mass $m_\Phi^2$ is given in terms of the conformal dimension of the dual operator: $m_\Phi^2 = \Delta(\Delta - d)$. We will fix it such that $\Delta = 2$, so that $m_\Phi^2 = -2$ in $d = 2 + 1$ and $m_\Phi^2 = -4$
for $d = 3 + 1$. For the latter case, $\Delta_{qq} = 2$ is realized in 4 at the lower boundary of conformal window of $N_f/N_c$ [6]. In the rest of this paper, we consider 2+1 case only. The field equation then gives

$$\Phi = M_0 z + M z^2, \text{ in AdS}_4,$$

which is an exact solution of the scalar field equation in the probe limit. In [7], the case $M_0 = 0$ case was considered. In this paper, we consider general $M_0 \neq 0$ case where source is also included. The Maxwell equation then is given by

$$\nabla^\mu F_{\mu \nu} = J_\nu$$

and for the real solution of $\Phi$, the current is simplified to the London equation similarly to the superconductivity,

$$J_\mu = \Phi^2 A_\mu.$$

For the transverse components with $\vec{k} \cdot \vec{A} = 0$, it can be rewritten as Schrödinger equation [8] via $\Psi = \frac{A_x}{\sqrt{d-3/2}}$ :

$$-\Psi''_n + V \Psi_n = E_n \Psi_n,$$

$$V = p^2 - \frac{1}{4} + |\Phi|^2, \text{ where } p = (d-2)/2,$$

and $E_n = \omega^2 - k^2 \equiv m_n^2$. Notice that for $d = 3$,

$$V = (M z + M_0)^2.$$

The $M_0 = 0$ case was analyzed previously in [7, 9] with the result $m_n^2 = M^2(4n + 3)$ for vector mesons.

### 2.2 Fermion with scalar interaction in holography

For the baryon spectrum, we consider following fermion action in AdS space.

$$S_\psi = \int d^{d+1}x \sqrt{-g} \bar{\psi} \left( \Gamma^\mu D_\mu - m - \Phi \right) \psi,$$

where $D_\mu = \partial_\mu + \frac{1}{2} \omega_{ab\mu} \Gamma^{ab}$. In this paper, we consider only $d = 3$. The equation of motion of (2.9) is given by

$$\left( \Gamma^\mu D_\mu - m - \Phi \right) \psi = 0,$$

which can be written as a Schrödinger form Eq.(2.6) with

$$V(z) = \frac{m(m-1) + \Phi^2}{z^2} = \frac{m(m-1)}{z^2} + (M z + M_0)^2.$$

For $M_0 = 0$, the above equations can be shown to have a linear spectrum [7], for example for the fermion case, $E_n = m_n^2 - 2M(m + \frac{1}{2})$. We interpret $m_n^2$ as the constituent quark mass inside a Hadron in confining phase and it was shown that [7]

$$m_n^2 = 4M^2(n + m + 1/2).$$

For $M_0 \neq 0$, we will show in the next section, the above equations of motion will lead to a type of Heun’s equation.
3 Heun’s equation and regularity condition

We first consider the confluent Heun’s equation\[10–12\] in the context of radial Schrödinger equation
\[
- \left( \frac{1}{r} \frac{d^2}{dr^2} r \right) + V(r) \right] R(r) = E R(r) \tag{3.1}
\]
where \(V\) is the potential given by
\[
V(r) = c^2 r^2 + br - \frac{a}{r} + \frac{L(L+1)}{r^2}. \tag{3.2}
\]
Factoring out the behaviour near \(r = 0\) by \(R(r) = r^L f(r)\), above equation becomes
\[
\frac{d^2 f(r)}{dr^2} + \frac{2(L+1)}{r} \frac{df(r)}{dr} + \left( E - c^2 r^2 - br + \frac{a}{r} \right) f(r) = 0. \tag{3.3}
\]
Factoring out the behaviour near \(\infty\) by \(f(r) = \exp\left(-\frac{c^2 r^2}{2} - \frac{b}{2x} r\right) y(r)\) and redefining \(\rho = \sqrt{cr}\), we get
\[
\rho \frac{d^2 y}{d\rho^2} + \left( -2\rho^2 - b_0 \rho + 2(L+1) \right) \frac{dy}{d\rho} + \left( (\mathcal{E} + b_0^2/4 - (2L+3)) \rho + a_0 - b_0(L+1) \right) y(\rho) = 0 \tag{3.4}
\]
where \(a_0 = a/c^3/2, b_0 = b/c^3/2\) and \(\mathcal{E} = E/c\), which is a bi-confluent Heun (BCH) equation whose canonical form is
\[
\rho \frac{d^2 y}{d\rho^2} + \left( \mu \rho^2 + \varepsilon \rho + \nu \right) \frac{dy}{d\rho} + (\Omega \rho + \varepsilon \omega) y = 0 \tag{3.5}
\]
\(\mu, \varepsilon, \nu, \Omega\) and \(\omega\) are parameters. It has a regular singularity at the origin and an irregular singularity at the infinity of rank 2 [10–12]. Substituting \(y(\rho) = \sum_{n=0}^\infty d_n \rho^n\) into (3.5), we obtain the following recurrence relation:
\[
d_{n+1} = A_n d_n + B_n d_{n-1} \quad \text{for } n \geq 1, \tag{3.6}
\]
where
\[
A_n = -\frac{\varepsilon(n+\omega)}{(n+1)(n+\nu)}, \quad B_n = -\frac{\Omega + \mu(n-1)}{(n+1)(n+\nu)}.
\tag{3.7}
\]
and \(d_1 = A_0 d_0\) for \(n = 0\). Comparing (3.4), (3.5), the former is a special case of the latter with \(\mu = -2, \varepsilon = -b_0, \nu = 2L+2\) and
\[
\omega = L + 1 - \frac{a_0}{b_0}, \quad \Omega = \mathcal{E} + \frac{b_0^2}{4} - (2L+3). \tag{3.8}
\]
Unless \(y(\rho)\) is a polynomial, \(R(r)\) is divergent as \(\rho \to \infty\). Therefore we need to impose regularity conditions by which the solution is normalizable. Through (3.6), we can see that a series expansion becomes a polynomial of degree \(N\) if we impose two conditions
\[
B_{N+1} = d_{N+1} = 0 \quad \text{where } N \in \mathbb{N}_0. \tag{3.9}
\]
Eq. (3.9) is sufficient to give 
\[ d_{N+2} = d_{N+3} = \cdots = 0 \] successively and the solution to eq.(3.4) becomes a polynomial of order \( N \),
\[ y_N(\rho) = \sum_{i=0}^{N} d_i \rho^i. \] (3.10)

Whether imposing both of the equations in eq(3.9) is necessary or not was studied numerically in our previous paper [3].

In general, \( d_{N+1} = 0 \) will define a \( N \)-th order polynomial \( P_{N+1} \) in \( a_0, b_0 \), so that Eq. (3.9) gives
\[ E_{N,L} = 2N + 2L + 3 - b_0^2/4, \]
\[ P_{N+1}(a_0, b_0) = 0. \] (3.11)

where the first comes from \( B_{N+1} = 0 \) or equivalently \( \Omega = -\mu N = 2N \).

Below we give a few lower order polynomial in \( a_0 \) and \( b_0 \) which will be used in next section.

\[
\begin{align*}
P_1(a_0, b_0) &= b_0(L + 1) - a_0, \\
P_2(a_0, b_0) &= (b_0(L + 1) - a_0)(b_0(L + 2) - a_0) - 4(L + 1), \\
P_3(a_0, b_0) &= (L + 1)(L + 2)(L + 3)b_0^3 - (3L(L + 4) + 11)a_0b_0^2 \\
&\quad + \left(3(L + 2)a_0^2 - 4(L + 1)(4L + 9)\right)b_0 - a_0^3 + 4(4L + 5)a_0, \\
P_4(a_0, b_0) &= (L + 1)(L + 2)(L + 3)(L + 4)b_0^4 - 2(2L + 5)(L(L + 5) + 5)a_0b_0^3 \\
&\quad + \left((6L(L + 5) + 35)a_0^3 - (L + 1)(5L(2L + 11)) + 72\right)b_0^2 \\
&\quad - \left(2(2L + 5)a_0^3 + 4(20L(L + 4) + 69)a_0\right)b_0 - 20(2L + 3)a_0^2 \\
&\quad + 144(L + 1)(L + 2), \\
P_5(a_0, b_0) &= (L + 1)(L + 2)(L + 3)(L + 4)(L + 5)b_0^5 \\
&\quad - \left(5L(L + 6)(L(L + 6) + 15) + 274\right)a_0b_0^4 \\
&\quad + \left(5(L + 3)(2L(L + 6) + 15)a_0^2 \\
&\quad - 4(L + 1)(L(5L(4L + 39) + 607) + 600)\right)b_0^3 \\
&\quad - \left(5(2L(L + 6) + 17)a_0^3 - 4(L(15L(4L + 31) + 1096) + 763)a_0\right)b_0^2 \\
&\quad + \left(5(L + 3)a_0^3 - 12(5L(4L + 19) + 98)a_0^2 \\
&\quad + 32(L + 1)(16L(2L + 11) + 225)\right)b_0 \\
&\quad + 20(4L + 7)a_3 - 32(16L(2L + 7) + 89)a_0 - a_5. \\
\end{align*}
\] (3.12)

4 Extra Quantization

We have seen that two parameters \( a_0, b_0 \) should be quantized for polynomial solutions in the modified BCH equation [3]. Here, we consider the case of quantization of \( a_0 \) and \( E \).
We examine a few low order $N$. If we choose $d_0 = 0$ the whole series solution vanishes. So we set $d_0 = 1$ for simplicity.

1. For $N = 0$, Eq. (3.9) gives $B_1 = \frac{-\Omega}{2(2L+3)} = 0$ and $d_1 = A_0 d_0 = 0$. Therefore for $a_0$ and $b_0$ satisfying

\[ P_1(a_0, b_0) = b_0(L + 1) - a_0 = 0, \]  

we have $E = 2L + 3 - \frac{b_0^2}{4}$. The polynomial for its eigenfunction is $y_0(\rho) = 1$.

2. For $N = 1$, $B_2 = \frac{-\Omega + 2}{3(2L+4)}$ and $d_2 = A_1 d_1 + B_1 d_0 = (A_0 A_1 + B_1) d_0$. Requesting $B_2 = d_2 = 0$, we get a relation between $a_0$ and $b_0$, we get

\[ P_2(a_0, b_0) = 0, \]  

which defines a hyperbola in $a_0, b_0$ such that there are always two branches because the discriminant is always positive, $D = b_0^2 + 16(L+1) > 0$: for a given $b_0$, $a_0$ always has real solutions: $2a_0 = b_0(2L + 3) \pm \sqrt{b_0^2 + 16(L+1)}$ and $E = 2L + 5 - \frac{b_0^2}{4}$ as usual. In this case, $y_1(\rho) = 1 + d_1 \rho$ with $d_1 = \left( -b_0 \pm \sqrt{b_0^2 + 16(L+1)} \right)/(4L + 4)$.

3. For $N = 2$, the contour plot of $P_3 = 0$ is given in figure 1. For any given $a_0$, three different $b_0$’s exist. Apart from the region where $a_0, b_0 \sim O(1)$ the curves are approximately linear. Such linearity can be confirmed by drawing the same figure by including asymptotic region as we can see in figure 2, where we used $L = 0$. Notice that the slope of the lines are $b_0/a_0 = 1, 1/2, 1/3$. It is also worthwhile to note that the medium curve with asymptotic slope $1/2$ pass through $a_0 = 0, b_0 = 0$. This happens for all even integer $N$.

4. Similar story holds for $N = 3$ using the equation $P_4 = 0$. See figure 3 and figure 4, where we also used $L = 0$ and the slope of the lines can be read off to give $b_0/a_0 = 1, 1/2, 1/3, 1/4$.

For general $N$, we can show that for large enough $a_0, b_0$, following relation holds.

\[ \frac{a_0}{b_0} = \frac{ac}{b} = L + 1 + K, \]  

for $K = 0, 1, \ldots, N$.  

This means that for any $L$, there are $N+1$ branches of solution satisfying eq. (3.11). This is the quantization of $a_0$ (or $b_0$) for the given value of $b_0$ (or $a_0$). Such extra quantization is an interesting consequence of the Heun’s differential equation. For the hypergeometric type, the differential equation is reduced to two term recurrence relation so that we need to fine tune only one parameter, the energy, to have a normalizable solution. For the Heun’s equation, its higher singularity requests higher regularity: the three term recurrence relation is not reduced to the two term, which in turn leads to an extra quantization of system parameter apart from the energy eigenvalue.
4.1 Quantization of $a_0, b_0$ in the non-linear regime

In the previous section, we analized the asymptotic regime of the potential parameter $a_0, b_0$ and learned that there are extra quantization given by $a_0/b_0 = L + 1 + K$, for integer $K \leq N$. Here we consider the regime where both $|a_0|, |b_0| \leq O(1)$. If we set $a_0 = 0$, the allowed values of $b_0$ are given by the crossing points of $N + 1$ branches of the $P_{N+1} = 0$ with the vertical line $a_0 = 0$. We call such fixing $b$-quantization. See figure 5. Previously we examined the solutions numerically and found that due to the $N, L$ dependence of $b_0$, $\mathcal{E}$ is NOT linear in $N$.

On the other hand, if we fix $b_0$ to the value we want, say 1, the allowed values of $a_0$ are given by the crossing points of $N + 1$ branches of the $P_{N+1} = 0$ with the horizontal
Figure 5: Definition of a-(b-) quantization. It depends on whether we fix b or a.

Line $b_0 = 1$. We call such fixing as a-quantization. See figure 5. In this case, $\mathcal{E}$ is linear in $N, L$ and does not depend on a quantized value of $a_0$ as far as it is given by the quantized value that depends on $N, L$ and $b_0$. Table 1 tells us all possible roots of $a_0$’s for each $L$ when $N = 4$ and $b_0 = 1$. Similarly, Table 2 shows us all possible roots of $a_0$’s for each $L$ when $N = 5$ and $b_0 = 1$. As you can see easily from the table, most of the quantized values are in the linear regime where $a_0 \approx (N + L + K)b_0$.

| $a_{00}$ | $a_{01}$ | $a_{02}$ | $a_{03}$ | $a_{04}$ |
|---------|---------|---------|---------|---------|
| $L=0$   | -7.50342| -2.26852| 2.5487  | 7.93985 | 14.2834 |
| $L=1$   | -9.22584| -2.68053| 3.72372 | 10.4374 | 17.7452 |
| $L=2$   | -10.4722| -2.80774| 4.79946 | 12.6207 | 20.8598 |
| $L=3$   | -11.4284| -2.78208| 5.84226 | 14.6311 | 23.7371 |
| $L=4$   | -12.1842| -2.65493| 6.8699  | 16.5287 | 26.4406 |

Table 1: Roots of $a_0$ for $b_0 = 1$, $N = 4$.

From the explicit calculation, we found the following pattern: List N+1 $a_0$ in the increasing order so that let $a_{0K}$ is $K$-th $a_0$, $K = 0, 1, \ldots, N$. Then the polynomial solution for the $a_{0K}$ has $K$ nodes. The number of nodes does not depend on $L$.

Figs. 6 shows us polynomials $y_4$ with $a_{0K}$, $K = 0, 1, 2, 3, 4$ has $K$ nodes in $N = 4$ in $\rho > 0$ regime. We fixed $L = 0$ and $b_0 = 1$. Figs. 7 shows us polynomials $y_5$ with $a_{03}$ has 3 nodes in $N = 5$ in $\rho > 0$ regime independent of the value of $L = 0, 1, 2, 3, 4, 5$. There are two nodes in the unphysical region $\rho < 0$. 
Table 2: Every roots of $a_0$ for $b_0 = 1$, $N = 5$.

| $L$ = 0 | $a_{00}$ | $a_{01}$ | $a_{02}$ | $a_{03}$ | $a_{04}$ | $a_{05}$ |
|---------|----------|----------|----------|----------|----------|----------|
| -10.5701 | -4.75187 | 0.363597 | 5.60184  | 11.6841  | 18.6724  |
| -12.7643 | -5.82539 | 0.801156 | 7.5262   | 14.7189  | 22.5434  |
| -14.4605 | -6.49042 | 1.30825  | 9.19107  | 17.3924  | 26.0593  |
| -15.834  | -6.93228 | 1.86483  | 10.7358  | 19.8467  | 29.319   |
| -16.9777 | -7.22567 | 2.45866  | 12.2089  | 22.1509  | 32.3849  |

Figure 6: Polynomial $y_4$ for each $a_{0K}$, $K$ = 0, · · · , 4. There are $K$ nodes for $a_{0K}$. Here $N = 4$, $L = 0$ and $b_0 = 1$.

Figure 7: Polynomials $y_5$ for various $a_{03}$ with $N = 5$ & $L = 0,1, · · · , 5$. $b_0 = 1$. There are 3 nodes in positive $\rho$ region regardless of $L$.

5 Bag model vs Holography

In this section, we will see that two very different physics leads to the same Heun’s equation studied in the previous section. The first one is the bag model studied in [3, 5] and the other is the holographic model.

5.1 Quark-antiquark system with only scalar interaction

A spin-free Hamiltonian with scalar interaction for the meson ($q\bar{q}$) system satisfy the equation [3–5]

$$\left[ \left( m + \frac{1}{2} br \right)^2 + P^2 + \frac{L(L+1)}{r^2} \right] R(r) = \frac{E^2}{4} R(r)$$

(5.1)

where we used $\vec{p}^2 = P^2 + \frac{L(L+1)}{r^2}$ with $P_r = -i \frac{1}{r} \partial_r r$ and $L$ is the angular momentum and $b$ is the string tension. Introducing the reduced radial wave function $u(r) = r R(r)$ and
arrive at
\[-u'' + Vu = \frac{E^2}{4} u, \tag{5.2}\]
\[V = \left( m + \frac{1}{2}br \right)^2 + \frac{L(L+1)}{r^2}. \tag{5.3}\]

For \(m = 0\), the spectrum was obtained in [5] and it is linear in quantum number:
\[E^2 = 4b(N + L + 3/2). \tag{5.4}\]

For \(m \neq 0\), \(b\) cannot be an arbitrary value. It has to be determined by \(b\)-quantization because \(a = 0\) from (3.2) and (5.3). In [3], the value \(b\) for given \(N, L\) was determined numerically, which can be approximately summarized by
\[b \approx 8.72m^2 \left( \frac{4}{7}N + L + \frac{10}{7} \right) \frac{1}{N^2 + \frac{1}{5}N - \frac{1}{30}}, \tag{5.5}\]

which is non-linear in quantum number \(N\) or \(L\).

At the first looking, it is rather surprising that presence of one more parameter \(m\) changes the spectrum so much. As we described earlier, this is because the quark mass is encoded such that its presence changes the singularity type of the equation of motion. Non-vanishing quark mass gives spectrum which is inconsistent with the confinement of color which tied to the Regge trajectory.

### 5.2 Holographic model

Finally we come back to the holographic theory whose equation of motion can be written as
\[\left[ -\frac{d^2}{dz^2} + V(z) \right] u(z) = Eu(z) \tag{5.6}\]

where
\[V(z) = (M_0 + Mz)^2 + \frac{m(m-1)}{z^2}. \tag{5.7}\]

where \(0 \leq z < \infty\) and \(m \in [-1/2, 0]\). If replace \(L \to -m\) and \(rR(r) \to u(z)\) in (3.1), it turns to be (5.6). Now, comparing (5.3) with (5.7), two equations are equivalent to each other with correspondence
\[m \leftrightarrow M_0, \quad b/2 \leftrightarrow M, \quad \text{and} \quad E^2/4 \leftrightarrow E_n. \tag{5.8}\]

It is quite remarkable that two completely different approaches to the Hadron gave almost identical differential equation. Even the spaces in which the differential equations are setup are different. Furthermore above mapping is not just resembling but actually is a dictionary of the AdS/CFT. Indeed, the quark mass corresponds to the source term in the bulk and the condensation corresponds to the string tension.

Notice here also in the presence of the scalar source \(M_0\), the resulting constituent quark masses or Hadron masses are not consistent with the Linear spectrum tied to the color confinement.
6 Conclusion

In this paper, we consider the holographic hadrons in 2+1 dimension as toy models. The spectrum follows linear confinement with zero quark mass, while it is highly non-linear with finite quark mass. The origin of such non-linearity can be traced to the difference in the singularity class of equation of motion that is made by the quark mass. For spinless quarks, 3+1 dimensional bag model of Lichtenberg et.al has the same behavior.

Although it is still too early to say that this is an intrinsic property of light hadrons, the agreement of models of different category suggests that the small quark mass is tied to the confinement dynamics of QCD. It also suggests that the presence of non-zero quark mass is non-trivial from the low energy point of view, because color flux would not allow the quark mass. It could be that the finite quark mass is phenomena of high energy only where neither bag model nor holography is relevant. The real 3+1 dimensional physics is more subtle because the equation of motion involve the logarithmic potential. We want to comeback to this problem in near future.

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