Abstract. Here we develop the theory of the diagrammatics of surface cross sections to prove that there are an infinite number of homology 3-spheres smoothly embeddable in a homology 4-sphere but not in a homotopy 4-sphere. Our primary obstruction comes from work of Daemi.

1. Introduction

It has been a long standing question in low-dimensional topology of which 3-manifolds embed in which 4-manifolds, in particular the 4-sphere.

Problem 1. [8, Problem 3.20] Under what conditions does a closed orientable 3-manifold smoothly imbed in $S^4$?

Budney and Burton tackled this problem by applying a variety of algebraic obstructions to the 11 tetrahedra census of manifolds [1]. One more recent method for obtaining obstructions to 3-manifold embeddings has been to tweak existing gauge theoretic obstructions for studying the homology cobordism group $\mathbb{Z}$. However, all of these obstructions have generally been obstructions to embedding in homology spheres, rather than to embedding into the 4-sphere specifically. Understanding the difference between these two conditions is in some sense the hardest case of Kirby problem 3.20, as most of the existing obstructions to $S^4$ embeddings must vanish.

Question 2. [11, Question 8.3] If a 3-manifold $M$ admits a smooth embedding into a homotopy 4-sphere, does it admit a smooth embedding (into) $S^4$? Are there 3-manifolds that embed in homology 4-spheres which do not embed in $S^4$?

In this paper, we establish a clear difference between these two properties, namely:

Theorem 3. There exist an infinite number of integer homology 3-spheres that are embeddable in an integer homology 4-sphere but not in any homotopy 4-sphere.

For the purposes of this paper, we work entirely in the smooth category. We construct our examples via branched covers of cross sectional slices of knotted spheres. One of the main advantages to this perspective is that there is a natural inclusion of branched covers $\Sigma_n(S^3, K) \subset \Sigma_n(S^4, F)$. This fact has been used to great effect as an obstruction to double slicing of knots, but in the case of our proof, we will use it to describe families of 3-manifolds embeddable in homology 4-spheres.
Figure 1. An example of a knot whose double branched cover embeds in a homology sphere but not a homotopy sphere. Note that the added twists are sufficient because one is added on each bridge in a bridge position diagram of $T_{3,5}$.

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2. The Gauge Theoretic Input

Our primary obstruction will be modeled on a corollary of a result of Taubes, attributed to Akbulut. For the purposes of this statement, a standard intersection form is one that is diagonalizable over the integers.

**Theorem 4.** [10, Proposition 1.7] Let $\Sigma$ denote any homology sphere bounding a simply connected 4-manifold with nonstandard definite intersection form. Then there is no simply connected definite manifold with boundary $\Sigma \# \Sigma$.

Using this theorem, Donald [4] noted in his thesis that one could obtain an obstruction to embedding into $S^4$:

**Corollary 5.** [4, Corollary 3.52] Let $\Sigma$ be as above. If there exists a smooth definite simply connected cobordism $W$ from $\Sigma \# \Sigma$ to a rational homology sphere $Y$, then $Y$ does not embed smoothly in $S^4$.

*Idea of proof.* Assume $Y$ embeds in $S^4$, and let $X$ be one of the components of $S^4 \setminus Y$. Then by Seifert van Kampen, $\pi_1(Y)$ normally generates $\pi_1(X)$. This means that $\pi_1(W) = 1$ normally generates $\pi_1(W \cup_Y X)$ so $W \cup_Y X$ is also simply connected.
Furthermore, $X$ is a rational homology ball, so $W \cup_Y X$ is definite, which by Theorem 4 is a contradiction. □

As Donald notes, it is easy to construct manifolds that satisfy the conditions of this obstruction. For example, if you attach 2-handles to a generating set for the fundamental group of $\Sigma \# \Sigma$, then this describes a simply connected cobordism, and with the proper choice of surgery coefficients, it can be made definite as well. The main challenge with using this obstruction is constructing a manifold that both embeds in a homology sphere and exhibits this definite cobordism.

For the purposes of our proof, however, we will need to slightly modify this construction. The obstruction to embedding in $S^4$ in this case is really just an obstruction to $\pi_1(Y)$ normally generating $\pi_1(X)$ for $X$ some homology ball it bounds. (We note that this makes the above obstruction actually an obstruction to embedding into a homotopy sphere rather than $S^4$ in particular). Therefore, one would not be able to find a manifold obstructable via this method via branched covers of slice knots, as both knots in $S^3$ and knotted surfaces in $B^4$ have fundamental groups normally generated by meridians.

For the purposes of our proof, we will slightly relax the $\pi_1$ condition on the cobordism. We note that the cobordism does not have to be simply connected, as long as we can guarantee that it becomes simply connected once we cap it off.

Moreover, we will use a strengthening of Taubes’ result due to Daemi [3]. Taubes’ result was generalized to include a wider class of manifolds in the place of $\Sigma$ (namely any manifold with nontrivial Frøyshov $h$ invariant [6]), as well as a generalization on the $\pi_1$ condition. The following theorem follows from Theorems 3 and 5 in [3]:

**Theorem 6.** Let $\Sigma$ denote any homology sphere bounding a simply connected 4-manifold with nonstandard definite intersection form, (or more generally, with $h(\Sigma) \neq 0$). Then any definite manifold $W$ with boundary $\Sigma \# \Sigma$ must have a non-trivial $SU(2)$ representation of its fundamental group that extends a non-trivial representation of $\pi_1(\Sigma \# \Sigma)$.

We note that the existence of an $SU(2)$ representation of the fundamental group will be crucial to the case analysis at the end of our argument.

### 3. Surface slicing and cobordisms

As stated in the introduction, our examples all come from branched covers of knots, and the knots are obtained using cross sectional slices of knotted surfaces.

**Definition 7.** Let $F \subset S^4$ be a knotted surface, the image of an embedding $f : \Sigma_g \hookrightarrow S^4$ of a genus $g$ surface $\Sigma_g$. Then we say $K \subset S^3$ is a slice of $F$ if $K$ can be realized as the intersection of $F$ and some equatorial $S^3$ in $S^4$.

If we fix a knotted surface whose branched cover is a homology 4-sphere, then we can study its slices to create families of manifolds embeddable in the corresponding homology 4-sphere.
3.1. Slices and broken surface diagrams. Here we will explore a new combinatorial technique for constructing surface slices, which will be discussed in more detail in an upcoming work.

The main diagrammatic input into our technique will be the broken surface diagram of a knotted surface, a higher dimensional analogue of a knot diagram \[2\]. In the case of a knot diagram, we project the image of our embedding of \(S^1 \rightarrow S^3\) to \(S^2\), and recover the isotopy type of the knot using information about the points of self intersection for the resulting immersed curve. Similarly, for a broken surface diagram we project the image of our embedding of \(\Sigma_g\) of self intersection for the resulting immersed curve. Similarly, for a broken surface diagram we can also associate a dekker set diagram, a higher dimensional analogue of the chord diagram we would associate to a knot isotopy type.

To a given broken surface diagram we can also associate a chord diagram, a pair \((\alpha, \beta)\) identifying the points on the two curves. This perspective for understanding knotted surfaces can be somewhat unwieldy, as diagrams of immersed surfaces are difficult to visualize. The key advantage in our case, however, is that the diagrams naturally lie in \(S^3\), so curves lying on our immersed surface representative of \(F\) can naturally be associated with knots. Moreover, there is a combinatorial criterion for when these curves actually represent slices of our surface, and the criterion only depends on the dekker set.

**Proposition 8.** Let \(F \subset S^4\) be a knotted surface, let \(F_0 \subset S^3\) be a broken surface diagram representation of \(F\) and let \((\Sigma_g, \{((\alpha_1, \beta_1), f_1), ((\alpha_2, \beta_2), f_2), \ldots (\alpha_n, \beta_n), f_n)\})\) be the corresponding dekker set. Let \(\gamma\) be a separating curve on \(\Sigma_g\), \((\alpha, \beta)\) the components of its complement. If \(f_i(\alpha_i) \subset F_1 \cap \beta_i\) for all \(i\) (or vice versa), then the curve in \(S^3\) represented by \(\gamma\) is a slice of \(F\).

**Proof.** Assume without loss of generality that \(f_i(\alpha_i) \subset F_1 \cap \beta_i\) for every \(i\). To achieve a broken surface diagram projection, we remove two balls from \(S^4\) and then project the resulting \(S^3 \times [-1, 1]\) to \(S^3\). Our goal in this proof is then to reverse this projection process while maintaining \(\gamma\) as a slice. Let \(F_0 \subset S^3 \times \{0\} \subset S^3 \times [-1, 1]\). Then we can push the interior of \(F_1\) down in the projected coordinate and the interior of \(F_2\) up in the projected coordinate.

Note that by the condition of the proposition, projecting back onto \(S^3 \times \{0\}\) will return the same immersed surface with the same crossing data. Now \(F_1\) and \(F_2\) do not intersect in their interiors. We can then push the lower sheet of the self intersections of \(F_1\) even lower in the projected coordinate, and the higher sheet of the self intersections of \(F_2\) higher into the projected coordinate, which results in an embedded surface \(F' \subset S^3 \times [-1, 1] \subset S^4\) whose intersection with \(S^3 \times \{0\}\) is \(\gamma\) and
Figure 2. The tangle of the trefoil knot, its corresponding chord diagram (with the arrows pointing to the over crossing), and the dekker set of the corresponding spun knot broken surface diagram projected so that latitudinal circles are mapped to horizontal lines. The dotted lines correspond to the lower sheets, with the solid lines being the upper sheets. The $f_i$ for the pairs of curves match up each point in a given latitudinal line with its corresponding point in the same longitudinal line.

has $\mathcal{F}$ as a broken surface diagram. Because broken surface diagrams capture the isotopy type of a surface, we know that $\gamma$ is a slice of $F$. \hfill \Box

In order to understand how to use this proposition, we first need examples of knotted surfaces and dekker sets for their broken surface diagrams.

3.2. Spun knots.

Example 9. For a knot $K \subset S^3$, we will define a knotted surface in $S^4$ called the spun knot as follows. Let $T_K \subset B^3$ be the two-ended tangle obtained from $K$ by removing a ball from $S^3$ containing a trivial arc of $K$. We can then define the spin of $K$ as $\partial(T_K \times D^2) \subset \partial(B^3 \times D^2) = S^4$. To project this to $S^3$, we can instead project each tangle ball to $D^2$, as $\partial(D^2 \times D^2) \subset \partial(B^3 \times D^2) = S^4$ defines an equatorial sphere in $S^4$. This means that spin of the knot diagram of $T_K$ gives a broken surface diagram for the spin of $K$, and the spin of the chord diagram of $T_K$ gives the dekker set diagram of the spin of $K$. Moreover, we can get a variety of different broken surface diagrams for our spun knot by varying our choice of knot diagram for $T_K$.

By tracing through the chord diagram twice as in Figure 3, we get that $K \# K$ is a slice of the spin of $K$. In fact, the double of the natural ribbon disc for $K \# K$ is the spin of $K$.

Example 10. One convenient feature of Proposition 8 is that it only depends on the way $\mathcal{F}_1$ and $\mathcal{F}_2$ hit the marked curves, so we can arbitrarily modify parts of our slice outside the intersection locus to get new slices from old slices. The dekker set we used in Example 9 has a number of annuli in the complement of the marked curves, so we can apply a Dehn twist along any of these annuli to get new slices. If you
Figure 3. $K \# \overline{K}$ drawn on the decker set for the spun trefoil, and the corresponding knot diagram. The oscillation of the curve to the left and right corresponds exactly to the over/under crossings in the diagram for $K$, as longitudinal lines of the decker set correspond to the immersed curve representation of $T_K$.

Figure 4. An even symmetric union over the trefoil drawn on the decker set for the spun trefoil, and the corresponding knot diagram.

start with the diagram of $K \# \overline{K}$ in Figure 3, these Dehn twists act by applying a full twist if the twist annulus is the spin of an arc in the knot diagram that touches the unbounded region in the diagram (see Figure 4 and Figure 5).

$K \# \overline{K}$ has a canonical ribbon disc given by $T_K \times I \subset B^3 \times I$. We can assign this disc a handle decomposition using a tangle diagram of $T_K$ by assigning a 0-handle (or disc) to each over crossing region and 1-handle (or band) to each arc connecting these regions. If we think of these transformations with this canonical ribbon disc for $K \# \overline{K}$ in mind, we can think of them as adding full twists to the canonical bands.
Figure 5. A local picture demonstrating how the Dehn twist around an annulus acts by applying a full twist, and how such a transformation is equivalent to $-1$ Dehn surgery on a meridian of the band.

Adding twists to the bands of the canonical ribbon disc for $K \# \overline{K}$ gives a class of knots called symmetric unions of $K$. If every band is twisted for an even number of half twists, then we call this an even symmetric union. With this in mind, our observations gives the following result.

**Proposition 11.** Every even symmetric union of $K$ is a slice of the spin of $K$.

3.3. Cobordisms. The convenient thing about even symmetric unions is that we can realize these full twists to the bands by doing $\pm 1$-surgeries to meridians of the bands, as in Figure 5. Because the homotopy class of each of these surgery curves is a product of two meridians (namely, a meridian of $K$ and its corresponding meridian in $\overline{K}$), the curves lift to two curves in the double branched cover. Therefore, we can attach 2-handles along these curves on one side of $\Sigma_2(S^3, K \# \overline{K}) \times I$ to get us a cobordism between the double branched covers of $K \# \overline{K}$ and our symmetric unions. Moreover, these surgeries preserve the property of being a homology 3-sphere, which will be seen in Lemma 12.

4. The Proof

4.1. Definiteness of cobordisms.

**Lemma 12.** The intersection form of the cobordism $W$ associated to an even symmetric union on $K$ from its double branched cover to the double branched cover of $K \# \overline{K}$ is a diagonal matrix with $\pm 1$’s on the diagonal. In particular, the cobordism is definite if and only if all of the twist regions are of the same sign. Moreover, if double branched cover of $K \# \overline{K}$ is an integer homology sphere, then the branched cover over these even symmetric unions is as well.

**Proof.** Our cobordism $W$ is a branched double cover of a cobordism $M$ from $S^3$ to $S^3$ obtained by attaching $\pm 1$-framed 2-handles along the meridians for bands of a disc for $K \# \overline{K}$. The branch set is the annulus of $K \# \overline{K} \times I \subset S^3 \times I$. To facilitate computation the intersection form of these cobordisms, we can glue on the branched
cover of the disc for $K \# \overline{K}$ to one end of $W$, and use this extra piece to construct an explicit basis for $H_2(M)$ which we will lift to a basis for $H_2(W)$. We note in this case that gluing on a homology ball does not change the intersection form. The meridians of the bands bound disjoint discs in the complement of the $K \# \overline{K}$ disc, as the bands they are meridians of are no longer in the way. We can then cap these discs off with the cores of the 2-handles to get a basis for $H_2(M)$ which we will lift to a basis for $H_2(W)$. This means that in the cover, these disjoint spheres will lift to spheres generating $H_2(W)$. Therefore, the intersection form of $W$ is a diagonal matrix with ones on the diagonal, whose signs are determined by the sign of the twisting. This means that if we twist in a consistent manner, the form will be definite.

This intersection form information also guarantees that the branched covers over these even symmetric unions are also integer homology spheres if the double branched cover of $K \# \overline{K}$ is one. This is because the double branched cover of the symmetric union is obtained by $\pm 1$ framed surgeries on the double branched cover of $K \# \overline{K}$. Moreover, we know from the intersection form that the curves along which we do these surgeries all have pairwise linking number 0, so the property of being an integer homology sphere is preserved.

4.2. The fundamental group criterion.

**Lemma 13.** For an infinite number of the cobordisms associated to even symmetric unions over $K$, the fundamental group of the cobordism is $\pi_1(\Sigma_2(S^3, K))$.

**Proof.** We obtain the cobordism associated to a given even symmetric union by attaching 2-handles to $\Sigma_2(S^3, K \# \overline{K}) \times I$ corresponding to the appropriate twists in the bands. We can instead obtain the cobordism by attaching 2-handles to $(S^3 \setminus K \# \overline{K}) \times I$, taking the appropriate double cover and then reattaching the solid torus in each level set. If we can prove that the cobordism before the cover has fundamental group equal to $\pi_1(S^3 \setminus K)$, then in the cover the cobordism will have fundamental group $\pi_1(\Sigma_2(S^3, K))$.

Fix a basis for $\pi_1(S^3 \setminus K)$. The group $\pi_1(S^3 \setminus (K \# \overline{K}))$ is an amalgamated product of two copies of $\pi_1(S^3 \setminus K)$, which is a free product with an added relation of identifying a fixed meridian in one copy with the corresponding meridian in the other copy. The 2-handles added in the cobordism from $K \# \overline{K}$’s complement to the symmetric union complement add a relation of identifying a meridian in one copy of $\pi_1(S^3 \setminus K)$ with the corresponding meridian in the other $\pi_1(S^3 \setminus K)$. Because knot groups are generated by meridians, we can add enough twists to identify a generating set for $\pi_1(S^3 \setminus K)$. In this case, $\pi_1$ of the cobordism is $\pi_1(S^3 \setminus K)$, meaning that in the cover the cobordism will have fundamental group equal to $\pi_1(\Sigma_2(S^3, K))$. □

**Proof of Theorem 3.** If we set $K = T_{3,5}$, then $\Sigma_2(S^3, K) = P$ the Poincare homology sphere, which satisfies the necessary conditions for $\Sigma$ in Theorem 6. Moreover, the branched double cover of the spin of $K$ will be the spin of $P$ (i.e. $\tilde{\partial}(\partial(P, B^3 \times D^2))$, which is a homology 4-sphere. Therefore, it suffices for the proof of Theorem 3 to
prove that for some family of these even symmetric unions the associated cobordisms are definite, and that we can guarantee the resulting manifold cannot admit an appropriate SU(2) representation after capping it off.

Let $J$ be an even symmetric union on $K$ such that the associated cobordism $W$ from $\Sigma_p(S^3, K \# K)$ to $\Sigma_p(S^3, J)$ is definite and $\pi_1(W) = \pi_1(P)$ (See Figure 1). Let $Y = \Sigma_p(S^3, J)$. Assume that $Y$ embeds smoothly into a homotopy 4-sphere $S$, and let $X_1, X_2$ be the two integer homology balls it bounds which together compose $S$.

Because $X_1$ and $X_2$ are integer homology balls, they don’t affect the definiteness of the intersection form, so it suffices to show for the proof that gluing one of the two $X_i$ to $W$ will result in a manifold with no appropriate SU(2) representation. To do this, let $W_i$ be the space obtained by gluing $X_i$ to $W$ along $Y$, and let $T$ be the singular space obtained by gluing both $X_i$ to $W$ along $Y$ (see Figure 6). Let $G_i$ be the image of the inclusion induced map $\iota : \pi_1(W) \to \pi_1(W_i)$.

Because $W$ is built from $Y \times I$ by attaching 2-handles, we know that $\pi_1(W)$ is a quotient of $\pi_1(Y)$. Because the $X_i$ are submanifolds of $S$, we also know that $\pi_1(W_i)$ is normally generated by $G_i$, which is a quotient group of $\pi_1(P)$. Moreover, because the $X_i$ glue together to form $S$, we know that $\pi_1(T) = 1$, as it is constructed by gluing on thickened 2 cells to $S$. By Seifert-van Kampen, we then know that $\pi_1(W_i), \pi_1(P)$, and $1$ fit into a pushout diagram as in Figure 7. Moreover, we know that $\pi_1(P) = \text{SL}_2(\mathbb{F}_5)$, the group of 2 by 2 matrices over the field of 5 elements with determinant 1.

$\text{SL}_2(\mathbb{F}_5)$ has only $A_5$ as a nontrivial quotient, so we can conclude our proof with a case analysis on $G_i$:

**Case 1:** $G_i = 1$

If $G_i = 1$, then $\pi_1(W_i) = 1$, and $W_i$ is the definite cobordism required for our obstruction, as $\pi_1(W_i)$ admits no nontrivial representations.

**Case 2:** $G_i = A_5$
Figure 7. The pushout diagram containing $\pi_1(W_1)$ and $\pi_1(W_2)$.

If $G_i = A_5$, then $W_i$ does not admit a nontrivial SU(2) representation that extends nontrivially to the boundary. This is because $A_5$ does not admit a nontrivial SU(2) representation (as it is a simple group and not the fundamental group of a spherical 3-manifold), and $G_i$ is the image of $\pi_1(\Sigma \# \Sigma)$ in $W_i$.

**Case 3: $G_i = \text{SL}_2(\mathbb{F}_5)$**

The last possible case is that $G_1 = G_2 = \text{SL}_2(\mathbb{F}_5)$. In this case, both of our initial maps ($f$ and $g$) in our pushout are injective. In this case, the associated free amalgamated product at the end of the commutative diagram cannot be the trivial group, as $\alpha$ and $\beta$ must be injective as well by the normal form theorem for free products with amalgamation [9, p.187]. The resulting contradiction completes the case analysis. □

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