Grover’s search algorithm and the quantum measurement problem

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It is suggested that the individual outcomes of a measurement process can be understood within standard quantum mechanics in terms of the measuring apparatus, treated as a quantum computer, executing Grover’s search algorithm.

1. INTRODUCTION

The measurement problem lies at the root of all interpretational problems of quantum mechanics [1]. Once the wavefunction of a system is assumed to contain the most complete information possible of the system, the measurement problem is inevitable. Let a system be in the pure state

$$|S⟩ = \sum_{i} c_i|S_i⟩,$$

where \(|S_i⟩\) are a complete set of eigenstates of some observable \(A\). If one measures \(A\) on this state, one would get the result \(a_i\) with probability \(|c_i|^2\), and once the measurement is complete, the state is forced into the eigenstate \(|S_i⟩\).

This does not imply that the system is a statistical ensemble of these states \(|S_i⟩\) with probabilities \(|c_i|^2\) in the sense of classical probability theory, and the measurement simply removes the ignorance. The simplest way to see this is to compare the density matrix of a pure state \(|Ψ⟩\),

$$⟨\rho⟩_{ij} = (|Ψ⟩⟨Ψ|)_{ij} = c_i^∗c_j$$

which is non-diagonal with that for a ‘mixture’,

$$⟨\tilde{ρ}⟩_{ij} = |c_i|^2δ_{ij}$$

which is diagonal. Although (2) and (3) give identical results for the probabilities of obtaining the various eigenvalues \(a_i\) of the observable \(A\), they predict quite different results for observables that do not commute with \(A\).

According to standard measurement theory a measurement quenches the off-diagonal interference terms and reduces the density matrix of a pure state like (2) to the diagonal form (3). This loss of coherence or reduction of the state vector cannot result from Schrödinger evolution which is unitary, causal and reversible. A measurement interaction first entangles a system \(S\) with the measuring apparatus \(X\). In general one obtains the state

$$|Ψ⟩ = \sum_{i} c_i|S_i⟩|X_i⟩$$

where the states \(|X_i⟩\) span the pointer basis. This is a unitary Schrödinger process that von Neumann calls ‘process 2’ [2]. It correlates every state \(|S_i⟩\) with a definite apparatus state \(|X_i⟩\). Since, however, this is an entangled state, it has to be reduced to a particular state \(|S_i⟩|X_i⟩\) before the result can be read off. ‘Process 1’ is a non-unitary process that achieves this by projecting the state \(|Ψ⟩\) to this state with the help of the projection operator \(Π_i = |X_i⟩⟨X_i|\). One then obtains the reduced density matrix

$$\rho \rightarrow \tilde{ρ} = \sum_{i} Π_iρΠ_i$$

which is diagonal and represents a heterogeneous mixture with probabilities \(|c_i|^2\). This is the least understood aspect of quantum mechanics and lies at the root of all its interpretations.

There is no universally accepted solution to this problem within standard quantum mechanics. The many-worlds interpretation tries to solve this problem by rejecting the projection postulate and postulating instead that the universe splits into \(n\) orthogonal universes at every measurement, each carrying one possible result of the measurement, and that no communication is possible between these universes [3]. This is considered by some as the only possible solution to the problem, but is rejected by many on aesthetic grounds (it being extravagant in its unverifiable profusion of parallel universes). Environment induced decoherence [4] was also thought to be an alternative solution, but it has become clear now that although an extremely useful concept of great practical importance in its own right, it does not actually solve the measurement problem as it has no explanation for the occurrence of individual events, i.e., why all diagonal elements of the reduced density matrix except one vanish for a single process (like the blackening of a single spot on a photographic plate). It is in this context that the Grover search algorithm [5] offers a plausible solution.
II. THE GROVER SEARCH ALGORITHM

Suppose one wants to search a telephone number in a telephone directory of a large city with \( N \) entries. A classical computer will have to carry out \( O(N) \) operations. Grover’s algorithm can speed up this search and complete it in \( O(\sqrt{N}) \) operations on a quantum computer. We wish to point out that this search algorithm also offers a plausible solution to the measurement problem. The essential point is that the algorithm which is essentially of ‘process 2’ type amplifies the amplitude of an identified target (the amplitude corresponding to a particular eigenstate in this case) at the cost of all other amplitudes to a point where the latter become so small that they cannot be recorded by detectors of finite efficiency. This therefore replaces von Neumann’s ‘process 1’.

Let us see how this can happen. Let the set \( \{ |S_i⟩|X_i⟩ \} \) (where \( i=1, 2, ..., N \)) be the search elements that a quantum computer inside the apparatus has to deal with. Let these elements be indexed from 0 to \( N - 1 \). This index can be stored in \( n \) bits where \( N = 2^n \). Let the search problem have exactly \( M \) solutions with \( 1 \leq M \leq N \). Let \( f(ξ) \) be a function with \( ξ \) an integer in the range 0 to \( N - 1 \). By definition \( f(ξ) = 1 \) if \( ξ \) is a solution to the search problem and \( f(ξ) = 0 \) if \( ξ \) is not a solution to the search problem. One then needs an oracle that is able to recognize solutions to the search problem [6]. This is signalled by making use of a qubit. The oracle is a unitary operator \( O \) defined by its action on the computational basis:

\[
O: |ξ⟩|q⟩ \rightarrow |ξ⟩|q \oplus f(ξ)⟩
\]  

(6)

where \( |ξ⟩ \) is the index register, \( \oplus \) denotes addition modulo 2, and the oracle qubit \( |q⟩ \) is a single qubit that is flipped if \( f(ξ) = 1 \) and is unchanged otherwise. Thus,

\[
|ξ⟩|0⟩ \rightarrow |ξ⟩|0⟩ \quad \text{if} \quad |ξ⟩ \text{ is not a solution}
\]  

(7)

\[
|ξ⟩|0⟩ \rightarrow |ξ⟩|1⟩ \quad \text{if} \quad |ξ⟩ \text{ is a solution}
\]  

(8)

It is convenient to apply the oracle with the oracle qubit initially in the state \( |q⟩ = (|0⟩ - |1⟩)/\sqrt{2} \) so that

\[
O: |ξ⟩|q⟩ \rightarrow (-1)^{f(ξ)}|ξ⟩|q⟩
\]  

(9)

Then the oracle marks the solutions to the search by shifting the phase of the solution. If there are \( M \) solutions, it turns out that one need only apply the search oracle \( O(\sqrt{N}/M) \) times on a quantum computer.

To start with, the quantum computer, assumed to be an integral part of the final detector, is always in the state \( |0⟩^\otimes n \). The first step in the Grover search algorithm is to apply a Hadamard transform to put the computer in the equal superposition state

\[
|ψ⟩ = \frac{1}{\sqrt{N}} \sum_{ξ=0}^{N-1} |ξ⟩
\]  

(10)

The search algorithm then consists of repeated applications of the Grover iteration or Grover operator \( G \) which can be broken up into the following four operations:

1. The oracle \( O \).
2. The Hadamard transform \( H^\otimes n \).
3. A conditional phase shift on the computer with every computational basis state except \( |0⟩ \) receiving a phase shift of \(-1\), i.e.,

\[
|ξ⟩ \rightarrow (-1)^{f(ξ)}|ξ⟩
\]

4. The Hadamard transform \( H^\otimes n \).

These operations can be carried out on a quantum computer. The combined effect of steps 2, 3 and 4 is

\[
G = H^\otimes n(2|0⟩⟨0| - I)H^\otimes n = 2|ψ⟩⟨ψ| - I
\]  

(11)

where \( |ψ⟩ \) is given by (10).

The Grover operator \( G \) can be regarded as a rotation in the two dimensional space spanned by the vector \( |ψ⟩ \) which is a uniform superposition of the solutions to the search problem. To see this, define the normalized states
\[|\alpha\rangle = \frac{1}{\sqrt{N-M}} \sum_\xi |\xi\rangle \quad (12)\]
\[|\beta\rangle = \frac{1}{\sqrt{M}} \sum_\xi |\xi\rangle \quad (13)\]

where \(\sum_\xi\) indicates a sum over all \(\xi\) that are solutions to the search problem and \(\sum_\xi\) a sum over all \(\xi\) that are not solutions to the search problem. The initial state can be written as

\[|\psi\rangle = \sqrt{\frac{N-M}{N}} |\alpha\rangle + \sqrt{\frac{M}{N}} |\beta\rangle \quad (14)\]

so that the apparatus (with the built-in quantum computer) is in the space spanned by \(|\alpha\rangle\) and \(|\beta\rangle\) to start with. Now notice that the oracle operator performs a rotation about the vector \(|\alpha\rangle\) in the plane defined by \(|\alpha\rangle\) and \(|\beta\rangle\), i.e.,

\[O(a|\alpha\rangle + b|\beta\rangle) = a|\alpha\rangle - b|\beta\rangle \quad (15)\]

Similarly, \(G\) also performs a reflection in the same plane about the vector \(|\psi\rangle\), and the effect of these two reflections is a rotation. Therefore, the state \(G^k|\psi\rangle\) remains in the plane spanned by \(|\alpha\rangle\) and \(|\beta\rangle\) for all \(k\). The rotation angle can be found as follows. Let \(\cos \theta/2 = \sqrt{N-M}/N\) so that \(|\psi\rangle = \cos(\theta/2)|\alpha\rangle + \sin(\theta/2)|\beta\rangle\). Then one can show that

\[G|\psi\rangle = \cos \frac{3\theta}{2} |\alpha\rangle + \sin \frac{3\theta}{2} |\beta\rangle \quad (16)\]

so that \(\theta\) is indeed the rotation angle, and so

\[G^k|\psi\rangle = \cos \left(\frac{2k+1}{2} \theta\right) |\alpha\rangle + \sin \left(\frac{2k+1}{2} \theta\right) |\beta\rangle \quad (17)\]

Thus, repeated applications of the Grover operator rotates the vector \(|\psi\rangle\) close to \(|\beta\rangle\). When this happens, an observation in the computational basis produces one of the outcomes superposed in \(|\beta\rangle\) with high probability. Grover has shown later that the Hadamard transforms can be replaced by more general transforms [7]. Finally, we would like to point out that in a quantum measurement only one outcome must occur and hence, the number \(M\) of simultaneous solutions that the Grover algorithm searches is always unity.

Let us see how it works out in the typical case of the Stern-Gerlach experiment. The index register in this case consists of the two states \(|X_1\rangle \uparrow\) and \(|X_1\rangle \downarrow\) where \(|X_1\rangle = |(x_1, y_1)\rangle\) and \(|X_1\rangle = |(x_2, y_2)\rangle\) correspond to the two spots on a two-dimensional screen in the \((x, y)\) plane. Since \(M = 1\) and \(N = 2\), the initial state of the detector is a 50 - 50 superposition of \(|\alpha\rangle\) and \(|\beta\rangle\). What happens in an individual event (the appearance of a single spot at certain times either at \((x_1, y_1)\) or \((x_2, y_2)\) is that the Grover search \(G\) rotates the initial state of the apparatus to one of the two solutions, each solution occurring with 50% probability.

### III. CONCLUSIONS

What we have shown is that the Grover search is a plausible explanation of how individual events occur in standard quantum mechanics. The entire process is unitary, i.e., of type 2 but yet it is possible to get information from these events because the amplitudes that are made arbitrarily small by the Grover search cannot be detected by measuring devices of finite efficiency. The projection postulate accounts for individual events but at the expense of unitarity. The decoherence approach, on the other hand, preserves unitarity but fails to account for individual events. In this sense the explanation in terms of the Grover search is different from both the von Neumann projection postulate and the environmentally induced decoherence approach. This proposal is also entirely different from the many-worlds interpretation as it works in a single universe. The mystery of the von Neumann projection or the Everett splitting of the universe is replaced by the riddle of how precisely a measuring apparatus executes the Grover algorithm.
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