Modelling the Temperature State of Bodies with Internal Heat Sources

A V Eremin¹, K V Gubareva¹, A I Popov¹

¹Heat Power Engineering Faculty, Samara State Technical University, Molodogvardeyskaya St., 244, 443100, Samara, Russia

E-mail: r.kristina2017@mail.ru

Abstract. This article presents the results of the development of a numerical - analytical method for solving the problem of thermal conductivity in a plate fuel element. An unsteady temperature field inside a fuel element is investigated for a given spatial distribution of heat sources. The heat release rate is given by the quadratic function of the coordinate. Modeling the temperature state of bodies with internal heat sources allows you to study the operation of equipment in transient modes, control heating/cooling modes of elements, determine temperature stresses, etc. It is shown in the work that regardless of the power of internal sources of heat, the temperature state is stabilized at a temperature level that depends on the Pomerantsev number.

1. Introduction

The study of thermal processes in fuel elements for various purposes is of great interest in heat engineering, in hydraulics, and for power engineering in general. In the general case, heat inside bodies can be released due to the occurrence of exothermic chemical reactions, when exposed to powerful electromagnetic fields, nuclear reactions, etc.

In the study of temperature fields inside fuel rods, various methods of mathematical modeling are used. At present, mainly numerical methods of thermal analysis are used [1 – 5]. Numerical simulation software ANSYS, Comsol, etc. are widely used [6 – 8]. However, when using thermophysical modeling packages, the verification of the results obtained becomes an important issue. It is required to use reliable methods for evaluating the error of results, convergence and stability of solutions.

In this article, it is presented that when setting the boundary conditions of the first kind, temperature stabilization occurs in any case. The maximum temperature in stationary heating mode is determined by the power of the heat source. Solutions for bodies of simple geometric shape are given in [9 – 12]. Such solutions have a number of significant advantages in comparison with numerical ones - the possibility of parametric identification, in-depth analysis of the temperature state (construction of isotherm fields, their velocities, etc.). However, obtaining accurate analytical solutions and their application requires special knowledge and is of little use for engineering use. In this connection, approximate analytical methods are being actively developed [13 – 15].

As a concrete example of using the developed method, we will consider the problem in a fuel element in the form of a plate of a plate δ (see Figure 1).
2. Mathematical statement of the problem

Let us write the equation

\[ \frac{c_p}{\partial t} \frac{\partial T}{\partial t} = \lambda \frac{\partial^2 T}{\partial x^2} + q_0, \]  
(1)

where \( c \) – thermal capacity; \( \rho \) – density of matter, \( T \) – temperature, \( t \) – time, \( x \) – spatial coordinate; \( q_0 = q_0 \left(1 - \frac{x^2}{\delta^2}\right) \) – source power distribution law; \( q_0 \) – maximum heat dissipation power.

Considering that \( \lambda/c = \alpha \) – thermal diffusivity, equation (1) reduced to form

\[ \frac{\partial T}{\partial t} = a \frac{\partial^2 T}{\partial x^2} + \frac{q_0}{c \rho} \left(1 - \frac{x^2}{\delta^2}\right), \]  
(2)

Let’s introduce the following variables and parameters:

\[ \Theta = \frac{T - T_0}{T_{CT} - T_0}; \quad \xi = \frac{x}{\delta}; \quad F_0 = \frac{at}{\delta^2}; \quad P_0 = \frac{q_0 \delta^2}{\lambda (T_{CT} - T_0)}. \]  
(3)

Taking into account (3), equation (2) will be

\[ \frac{\partial \Theta}{\partial F_0} = \frac{\partial^2 \Theta}{\partial \xi^2} + P_0(1 - \xi^2). \]  
(4)

Taking into account the symmetry of the problem, we formulate the boundary conditions for equation (4) (see Figure 1):

\[ \Theta(\xi, 0) = 0; \]  
(5)

\[ \frac{\partial \Theta(0, F_0)}{\partial \xi} = 0; \]  
(6)

\[ \Theta(1, F_0) = 1. \]  
(7)

3. Method description

We introduce into consideration the new required function of time

\[ \varphi(F_0) = \frac{\partial \Theta(1, F_0)}{\partial \xi}. \]  
(8)

The solution to problem (4) - (7) is sought in the form of a polynomial
\[\Theta(\xi, \text{Fo}) = \sum_{i=1}^{n} b_i(\text{Fo}) \xi^{i-1}, \tag{9}\]

where \(n\) – number of members (9); \(b_i(\text{Fo})\) – unknown time-dependent coefficients.

To obtain a solution to problem (4) - (7) in the first approximation, we restrict ourselves to three terms in expression (9). Substitute (9) into conditions (6) and (7), as well as into additional condition (8). We obtain the system of equations

\[
\begin{align*}
  b_2 + 2b_1 &= \varphi(\text{Fo}); \\
  b_2 &= 0; \\
  b_1 + b_2 + b_3 &= 1,
\end{align*}
\]

Find

\[
b_1(\text{Fo}) = 1 - \frac{\varphi(\text{Fo})}{2}; \quad b_2(\text{Fo}) = 0; \quad b_3(\text{Fo}) = \frac{\varphi(\text{Fo})}{2}.
\]

Expression (12), taking into account \(b_i(\text{Fo})\) will be written as

\[\Theta(\xi, \text{Fo}) = f_1(\xi)\varphi(\text{Fo}) + 1, \tag{10}\]

where \(f_i(\xi) = \frac{\xi^2}{2} - \frac{1}{2}\) – coordinate function.

For approximate satisfaction (4), we will integrate it, i.e. compose the integral of the heat balance

\[
\int_0^1 \left( \frac{\partial \Theta(\xi, \text{Fo})}{\partial \text{Fo}} \right) d\xi = \int_0^1 \left( \frac{\partial^2 \Theta(\xi, \text{Fo})}{\partial \xi^2} + \text{Po}(1 - \xi^2) \right) d\xi. \tag{11}\]

Calculating the integral, we obtain an equation of the form

\[
\frac{d\varphi(\text{Fo})}{d\text{Fo}} + 3\varphi(\text{Fo}) + 2\text{Po} = 0, \tag{12}\]

find

\[
\varphi(\text{Fo}) = \frac{C_1 e^{-3\text{Fo}}}{3} - \frac{2\text{Po}}{3}, \tag{13}\]

where \(C_1\) – integration constant.

Substituting (13) into (10), we obtain

\[\Theta(\xi, \text{Fo}) = f_1(\xi) \left( \frac{C_1 e^{-3\text{Fo}}}{3} - \frac{2\text{Po}}{3} \right) + 1. \tag{14}\]

To fulfill (5), we compose its residual

\[
\int_0^1 [\Theta(\xi, 0)] f_1(\xi) \, d\xi = 4C_1 - 8\text{Po} - 30 = 0. \tag{15}\]

Из решения уравнения (18) определим \(C_1 = 2\text{Po} + 9\). Expression (14), taking into account \(C_1\), represents the solution of problem (4) - (7) in the first approximation

\[\Theta(\xi, \text{Fo}) = \left( \frac{\xi^2}{2} - \frac{1}{2} \right) \left( \frac{(2\text{Po} + 9)e^{-3\text{Fo}}}{3} - \frac{2\text{Po}}{3} \right) + 1. \tag{16}\]
To obtain a solution to problem (4) - (7) in the second approximation, we will use six terms of the series (9) \((n = 6)\). For the definition \(b_i(Fo)\), we will use three additional conditions.

By writing (4) into \(\xi = 1\) taking into account (6), (7), we obtain the first additional boundary condition

\[
\frac{\partial^2 \Theta(1, Fo)}{\partial \xi^2} = 0.
\]  

(17)

To obtain the second additional condition, we differentiate the original equation with respect to \(\xi\).

\[
\frac{\partial^2 \Theta(0, Fo)}{\partial \xi \partial Fo} - \frac{\partial^3 \Theta(0, Fo)}{\partial \xi^3} = 0.
\]

(18)

Writing down (18) in \(\xi = 0\) taking into account (7), we get

\[
\frac{\partial^3 \Theta(0, Fo)}{\partial \xi^3} = 0.
\]

(19)

The third boundary condition can be obtained by single differentiation (4) with respect to the variable \(\xi = 1\). Taking into account (6), (7) it will be written

\[
2Po + \frac{\partial \varphi(Fo)}{\partial Fo} - \frac{\partial^3 \Theta(1, Fo)}{\partial \xi^3} = 0.
\]

(20)

Substituting (9) into (6), (7), (8) and additional conditions (17), (19), (20). We obtain a system of six algebraic equations, from the solution of which we determine the coefficients \(b_i(Fo)\)

\[
b_1(Fo) = 1 - \frac{7\varphi(Fo)}{10} - \frac{Po}{20} - \frac{1}{40} \frac{\partial \varphi(Fo)}{\partial Fo}; \quad b_2(Fo) = 0;
\]

\[
b_3(Fo) = \frac{1}{12} \frac{\partial \varphi(Fo)}{\partial Fo} + \frac{Po}{6} + \varphi(Fo); \quad b_4(Fo) = 0;
\]

\[
b_5(Fo) = -\frac{1}{8} \frac{d \varphi(Fo)}{d Fo} - \left(\frac{1}{4} Po + \frac{1}{2} \varphi(Fo)\right); \quad b_6(Fo) = \frac{1}{15} \frac{d \varphi(Fo)}{d Fo} + \frac{2Po}{15} + \varphi(Fo).
\]

Calculating integral (11), we obtain the differential equation

\[
\frac{d^2 \varphi(Fo)}{d Fo^2} + 39 \frac{d \varphi(Fo)}{d Fo} + 90 \varphi(Fo) + 60Po = 0.
\]

(21)

His decision

\[
\varphi(Fo) = C_1 \exp(K_1Fo) + C_2 \exp(K_2Fo) - \frac{2Po}{3},
\]

(22)

where \(K_1 = \frac{3\sqrt{129}}{2} - \frac{39}{2}; \quad K_2 = \frac{3\sqrt{129}}{2} + \frac{39}{2}\).

Expression (9), after substituting the coefficients into it, can be represented as

\[
\Theta(\xi, Fo) = \sum_{j=1}^{2} f_j(\xi) C_j \exp(K_jFo),
\]

(23)
where \( f_j(\xi) = \left( \frac{K_j}{12} + \frac{1}{5} \right) \xi^5 + \left( \frac{-K_j}{8} - \frac{1}{2} \right) \xi^4 + \left( \frac{K_j}{12} + 1 \right) \xi^3 - \left( \frac{1}{40} K_j + \frac{7}{10} \right) \xi^2 \)

We obtain a system of two algebraic equations

\[
\begin{cases}
\int_{0}^{1} \left[ \Theta(\xi,0) \right] f_1(\xi) \, d\xi = 0, \\
\int_{0}^{1} \left[ \Theta(\xi,0) \right] f_2(\xi) \, d\xi = 0.
\end{cases}
\]

from the solution that we find \( C_1 = 0.011\text{Po} + 2.2; C_2 = 0.66\text{Po} + 2.1. \)

For example, for \( \text{Po} = 5 \), constants are equal \( C_1 = 2.25 \) и \( C_2 = 5.4. \)

4. Discussion of the results

The results of calculating the temperature in the second approximation in comparison with the numerical solution are presented in Figure 2, 3. From their analysis it follows that the discrepancy between the results obtained does not exceed 2%.

Expressing \( \xi \) as a function of temperature \( \Theta(\xi,F_0) \) and time \( F_0 \), ratio (14) will be written as

\[
\xi(F_0) = \sqrt{\frac{\varphi(F_0)(2\Theta(\xi,F_0) + \varphi(F_0) - 2)}{\varphi(F_0)}}.
\]

(24)

The first time derivatives of (24) determine the value of the dimensionless velocities of motion of the isotherms \( \nu = \frac{d\xi}{dF_0} \) by coordinate \( \xi \) depending on time \( F_0 \). The ratio will be

\[
\nu(F_0) = \frac{(\Theta(\xi,F_0) - 1) \frac{\partial \varphi(F_0)}{\partial F_0}}{\varphi(F_0)\sqrt{\varphi(F_0)(2\Theta(\xi,F_0) + \varphi(F_0) - 2)}}.
\]

(25)

The graphs of isotherms and velocities obtained using formulas (24), (25) are shown in Figure 4, 5.

![Figure 2. Temperature graph by coordinate (Po = 5).](image)
5. References

[1] Sidambarompoulé X, Notinghera P, Laurentiea J-C, Paillatb T and Leblancb P 2021 *J. of Electrostatics* **112** 103593

[2] Zhoua L, Lianga C, Chen L and Xia Y 2017 *Procedia Engineering* **210** 240-5

[3] Kim C N 2017 *Applied Thermal Engineering* **130** 408-17

[4] Hu D-H, Li M and Li Q 2021 *Applied Thermal Engineering* **188** 116617

[5] Babaieab M and Sayyaadib H 2014 *Energy* **69** 873-90

[6] Choinski D, Wodolażki A, Skupin P, Malcher A and Bernacki K 2021 *Sensors and Actuators A: Physical* **331** 112999

[7] BinRayhan S and Rahman M 2020 *Procedia Structural Integrity* **28** 1892-900

[8] Gallien B, Albaric M, Duffar T, Kakimoto K and M’Hamdid M 2017 *J. of Crystal Growth* **457**
[9] Ryzhkov S V and Kuzenov V V 2019 Int. J. of Heat and Mass Transfer 132 587-92
[10] Mao Q, Li Y, Li G, A Badiei A 2021 Energy 235 121382
[11] Olabi A G, Wilberforce T, Sayed E T, Elsaid K, Atique Rahman S.M. and Abdelkareem M 2021 Int. J. of Thermofluids 10 100072
[12] Guo Z, Song Y, Chen J, Xu Y and Caiqi Zhao 2021 Journal of Building Engineering 43 102582
[13] Pashchenko D and Eremin A 2021 Int. J. of Heat and Mass Transfer 165 120617
[14] Kudinov I V, Kotova E V, Kudinov V A and Eremin A V 2017 J. of Engineering Physics and Thermophysics 90(5) 1317-27
[15] Eremin A V, Kudinov V A and Stefanyuk E V 2018 Fluid Dynamics 53 29-39
[16] Kudinov V A, Eremin A V and Kudinov I V 2017 Thermophysics and Aeromechanics 24(6), 901-907

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