A NOTE ON POSITIVE ASSOCIATION

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ABSTRACT. We show that if \( A, B, C \) are increasing subsets of \( \Omega := \{0, 1\}^n \) with \( A \neq \emptyset \), then with respect to any product probability measure on \( \Omega \),

if each of the pairs \( \{A \cap B, C\}, \{A \cap C, B\} \) is independent, then \( B \) and \( C \) are independent.

This implies an answer to a motivating question of J. Steif, and is related to a basic, still open variant of that question, and to a well-known conjecture of S. Sahi.

1. INTRODUCTION

This began with a question of Jeff Steif that I first heard in conversation with him and Rob van den Berg many years ago. (The participants have been unable to agree as to the decade in which this conversation took place, or the continent that hosted it, but it was no later than 2002.) The aims of this note are to:

(i) prove a small result (Theorem 2) that implies an answer to Steif’s question;

(ii) point out that a variant (Question 1), which seems as basic a question as one could ask about positive association, remains open; and

(iii) observe a curious connection to a well-known conjecture of Siddhartha Sahi.

Background. Throughout this discussion, \( X_1, \ldots, X_n \) are Bernoulli random variables and \( \mu \) is the law of \( (X_1, \ldots, X_n) \); so \( \mu \) is a probability measure on \( \Omega := \{0, 1\}^n \equiv 2^n \) (with the natural identification of sets with their indicators). Recall that events \( A, B \) (in any probability space) are positively correlated if \( \mathbb{P}(AB) \geq \mathbb{P}(A)\mathbb{P}(B) \). The law of \( X_1, \ldots, X_n \) is positively associated (PA), or has positive association, if

any two events both increasing in the \( X_i \)’s are positively correlated.

The seminal result here is Harris’ Inequality [5], which says product measures are PA. (To be precise, this is what’s given by Harris’ argument; the statement in [5] is less general.) Harris’ Inequality for uniform measure was rediscovered in [6], and in combinatorial circles is sometimes called Kleitman’s Lemma. The most useful extension of Harris (discovered still later but still independently) is the FKG Inequality of Fortuin, Kasteleyn, and Ginibre [3], which says \( \mu \) is PA whenever

\[
\mu(A)\mu(B) \leq \mu(A \cap B)\mu(A \cup B) \quad \forall A, B \in \Omega.
\]

A \( \mu \) satisfying (1) (the “positive lattice condition”) is an FKG measure. See e.g. [4], [8] for some indication of the role of positive association in probability, and [1, Ch. 6] for a quick hint on the combinatorial side.

For the rest of the paper, script capitals \( (A, B, \ldots) \) are nonempty increasing events in \( \Omega \). We use \( AB \) for \( A \cap B \), and denote independence of \( A \) and \( B \) by \( A \perp B \) and dependence by \( A \sim B \).
Underlying independents. One way to prove PA for μ is to realize the Xᵢ’s as increasing functions of independent Bernoullis Y₁, . . . , Yₘ and invoke Harris; more generally, μ is PA if it is a limit of measures obtained in this way. Say μ is FUI (for finitely many underlying independents) in the first case, and UI in the second. This is not as restrictive as it sounds, since FKG measures are FUI,¹ and in fact Steif’s original question was

(2) are all PA measures FUI?

As we will see shortly, the answer is no; but, remarkably, we can’t (as far as I know) rule out a slightly weaker possibility:

Question 1. Are all PA measures UI?

Of course one hopes the answer to this very basic question is again no—that is, positive association is more than Harris’ Inequality—but it seems surprisingly hard to say anything about the law of a UI μ that uses more than positive association. In contrast, the following statement, the technical content of the present note, does manage to distinguish FUI from PA, and to imply the promised negative answer to (2).

Theorem 2. For any FUI μ and (increasing) A, B, C,

(3) if A|B ∩ C and A|C ∩ B (and μ(A) ≠ 0), then B|C.

For the connection to (2), we recall a beautiful result of Doyle, Fishburn and Shepp [2]:

Theorem 3. For a uniform permutation σ of [n], the law, μₙ, of the set of fixed points of σ (that is, of (X₁, . . . , Xₙ), where Xᵢ = 1 {σ(i) = i}) is PA.

Corollary 4. The answer to (2) is negative.

Proof. This follows from Theorem 2 and the observation that μ = μ₃ violates (3): μ assigns weight 1/3 to 0 and 1/6 to each of {1}, {2}, {3}, {1, 2, 3} (and 0 to pairs); so, with Aᵢ, j = {σ fixes at least one of i, j}, A = A₁, 2, B = A₁, 3 and C = A₂, 3, we have μ(A) = μ(B) = μ(C) = 1/2, μ(AB) = ··· = 1/3, and μ(ABC) = 1/6, whence the hypotheses of (3) hold but the conclusion does not.

Aside. As its discoverers emphasize, the argument of [2] is a quite painful case analysis. Shouldn’t there be a nicer, more enlightening proof of such an elegant result?

Sahi’s Conjecture. This fascinating (infuriating) conjecture [10] proposes an extension of Harris’ Inequality to k > 2 events; we state just the case k = 3, which has to date proved thoroughly intractable and seems not unlikely to capture the full difficulty of the problem.

Conjecture 5. For a product measure μ and increasing A, B, C ⊆ Ω,

(4) 2μ(ABC) − [μ(AB)μ(C) + μ(AC)μ(B) + μ(BC)μ(A)] + μ(A)μ(B)μ(C) ≥ 0.

¹Since it seems hard to find a reference for this, I include here a sketch of a proof that was shown to me by Rob van den Berg, which he believes is (implicitly) well known in the probability community: (a) Assuming the law, μ, of (X₁, . . . , Xₙ) is FKG, let Z₁, . . . , Zₙ be independent, each uniform from [0, 1], and for i = 1, . . . , n, if Xᵢ = ωᵢ for j < i, let Xᵢ = 1 iff Zᵢ > 1 − μ(Xᵢ = 0|Xⱼ = ωⱼ ∀j < i). This is easily seen to return μ as the law of (X₁, . . . , Xₙ), and it’s not hard to see, using (1), that the Xᵢ’s are nondecreasing in the Zᵢ’s. (b) For each i, the procedure in (a) depends on a finite number of events A(i, j) := {Zᵢ > αᵢ,j}, and it’s easy to realize the indicators 1 A(i, j) (i ∈ [m], j ∈ [m]), say as nondecreasing functions of independent (nonidentical) Bernoullis Yᵢ,j (j ∈ [m]), notice that the law of the Xᵢ’s assigns probability zero to strings of weight 2, so trivially violates (1).
Note $\mu(A) = 1$ recovers Harris. The conjecture is stated in [10] for FKG measures, but this is no more general since FKG measures are FUI. For $k \in \{3, 4, 5\}$, Sahi’s Conjecture was originally stated—as a theorem, but with an incorrect proof—by Richards [9]; he also suggested the possibility of similar inequalities for larger $k$, but without proposed candidates for the coefficients. Progress on the conjecture has been limited (see [7] for the state of the art and [11, 12] for related results), surely a poor reflection of the effort expended on it.

For present purposes the point of all this is that (4), with Harris, implies (3) (since under the hypotheses of (3), (4) becomes $\mu(BC) \leq \mu(B)\mu(C)$); so a positive answer to Question 1, even just for $\mu_3$, would say Sahi’s Conjecture—which of course also implies (4) when $\mu$ is UI—is false. Conversely, Theorem 2 may be considered a tiny step toward Sahi’s Conjecture.

In Section 2 we will give two proofs of Theorem 2. The first of these is very short and rather ad hoc. The second is longer (not long) but feels more systematic, and is included here in the hope that it might be more susceptible to improvement.

2. Proofs

As usual, $\text{min}(A)$ is the set of minimal elements of $A$. One says $i \in [n]$ affects $A$ if $A \cup \{i\} \in A$ for some $A \notin A$ (and $I$ affects $A$ if some $i \in I$ does). We use $Z(A)$ for the set of $i \in [n]$ that affect $A$, noting that

$$Z(A) = \cup\{A : A \in \text{min}(A)\}.$$  

The basis for both our proofs of Theorem 2, an immediate consequence of Harris’ argument, is

$$A|B \iff Z(A) \cap Z(B) = \emptyset. \tag{5}$$

(Equivalently, $A$ and $B$ are independent iff $A \cap B = \emptyset$ whenever $A \in \text{min}(A)$ and $B \in \text{min}(B)$.)

For each of the following arguments we assume $B \sim C$ (and $\mu(A) \neq 0$) and want to show

$$\text{at least one of } AB \sim C, AC \sim B \text{ holds;} \tag{6}$$

so we assume (6) fails and aim for a contradiction.

\textbf{First proof.} (A Venn diagram may be helpful here.) Notice to begin that, for any increasing $D$ and $E$,

$$\text{min}(DE)$$

is the set of minimal elements of \{$D \cup E : D \in \text{min}(D), E \in \text{min}(E)$\}.

We now use $A$ (possibly subscripted) for members of $\text{min}(A)$ and so on.

Recalling that we assume $B \sim C$, choose $B$ and $C$ with $B \cap C \neq \emptyset$ and $B \cup C$ minimal subject to this, and observe (with justification below) that

$$B \cup C \in \text{min}(BC).$$

\textbf{Proof.} Suppose instead that $B \cup C \supseteq B_0 \cup C_0$, say with $B_0 \neq B$. Then $B_0 \cap (C \setminus B) \neq \emptyset$ (since otherwise $B_0 \subseteq B$). On the other hand, $C_0 \cap B \neq \emptyset$ (else $C \neq C_0 \subseteq C$) implies $C_0 \supseteq C \setminus B$ (else $B \cup C_0 \subseteq B \cup C$ contradicts our choice of $(B, C)$). But then $B_0 \cap C_0 \neq \emptyset$, which is again a contradiction. \hfill \Box

Choose $A$ with $A \setminus (B \cup C)$ minimal. Since (we assume) $AB|C$, there must be some $A_1 \cup B_1 \in \text{min}(AB)$ with $A_1 \cup B_1 \subseteq (A \cup B) \setminus C$; in particular $B_1 \subseteq (A \cup B) \setminus C$, implying $\emptyset \neq B_1 \setminus B \subseteq A \setminus (B \cup C)$. But then, since $AC|B$, there is $A_2 \cup C_2 \subseteq (A \cup C) \setminus B_1$; which contradicts our choice of $A$ since $A_2 \setminus (B \cup C) \subseteq A \setminus (B \cup C \cup B_1) \subseteq A \setminus (B \cup C)$.

\hfill \Box
Second proof. Let $I = Z(B) \cap Z(C) (\neq \emptyset)$, $J = Z(B) \setminus Z(C)$, $K = Z(C) \setminus Z(B)$, and $L = [n] \setminus (I \cup J \cup K)$.

We are (again) assuming (6) fails, so in particular,

(7) $I$ doesn’t affect either of $AB$, $AC$.

Of course we may also assume

(8) $A \subseteq B \cup C$,

since replacing $A$ by $A \cap (B \cup C)$ has no effect on $AB$, $AC$.

Observation 1. $I \cap Z(A) = \emptyset$.

Proof. Suppose $i \in I$, $A \notin A$, $A' := A \cup \{i\} \in A$, and (w.l.o.g.; see (8)) $A' \in B$. Then $AB \cap \{A, A'\} = \{A'\}$, implying $i \in Z(AB)$ and contradicting (7). $\square$

Observation 2. If $X \subseteq J$, $X \notin B$ and $X \cup I \in B$, then $X \cup K \cup L \notin A$ (so $X \cup Y \notin A \ \forall Y \subseteq K \cup L$).

(And similarly with $B$ replaced by $C$ and the roles of $J$ and $K$ interchanged.)

Proof. Otherwise $X \cup K \cup L \cup I \in AB$ and $X \cup K \cup L \notin AB$ (since $X \notin B$ and $(K \cup L) \cap Z(B) = \emptyset$), contradicting (7). $\square$

Notice that $I \subseteq Z(B) = I \cup J$ implies that there is some $X \subseteq J$ with $B \cap \{X, X \cup I\} = \{X \cup I\}$. Let $X$ be of this type and, similarly, let $Y \subseteq K$ satisfy $Y \notin C$ and $Y \cup I \in C$.

Let $A$ be minimal in $A$ with $A \supseteq X \cup Y$. Then $A \cap I = \emptyset$ (by Observation 1), so by Observation 2,

(9) $A \cap J \supseteq X$ and $A \cap K \supseteq Y$.

(We just need one of these.) Now

$$A \cup I \in ABC$$

(since $X \cup I \in B$ and $Y \cup I \in C$; we just need $A \cup I \in AC$), while minimality of $A$ and Observation 1 give

$$(A \cup I) \setminus \{j\} \notin A$$

for any $j \in (A \cap J) \setminus X$ (and (9) says there is such a $j$). Thus any such $j$ is in $Z(AC) \cap Z(B)$, so that, contrary to assumption, (6) does hold.

$\blacksquare$

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