Colombeau Algebra: A pedagogical introduction

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Abstract

A simple pedagogical introduction to the Colombeau algebra of generalised functions is presented, leading the standard definition.

1 Introduction

This is a pedagogical introduction to the Colombeau algebra of generalised functions. I will limit myself to the Colombeau Algebra over $\mathbb{R}$. Rather than $\mathbb{R}^n$. This is mainly for clarity. Once the general idea has been understood the extension to $\mathbb{R}^n$ is not too difficult. In addition I have limited the introduction to $\mathbb{R}$ valued generalised functions. To replace with $\mathbb{C}$ valued generalised functions is also rather trivial.

I hope that this guide is useful in your understanding of Colombeau Algebras. Please feel free to contact me.

There is much general literature on Colombeau Algebras but I found the books by Colombeau himself[1] and the Masters thesis by Tạ Ngọc Trí[2] useful.

2 Test functions and Distributions

The set of infinitely differentiable functions on $\mathbb{R}$ is given by

$$\mathcal{F}(\mathbb{R}) = \{ \phi : \mathbb{R} \rightarrow \mathbb{R} \mid \phi^{(n)} \text{ exists for all } n \}$$ (1)

Test functions are those function which in addition to being smooth are zero outside an interval, i.e.

$$\mathcal{F}_0(\mathbb{R}) = \{ \phi \in \mathcal{F}(\mathbb{R}) \mid \text{there exists } a, b \in \mathbb{R} \text{ such that } f(x) = 0 \text{ for } x < a \text{ and } x > b \}$$ (2)

I will assume the reader is familiar with distributions, either in the notation of integrals or as linear functionals. Thus the most important distributions is the Dirac-$\delta$. This is defined either as a “function” $\delta(x)$ such that

$$\int_{-\infty}^{\infty} \delta(x) \phi(x) dx = \phi(0)$$ (3)

Or as a distribution $\Delta : \mathcal{F}_0(\mathbb{R}) \rightarrow \mathbb{R},$

$$\Delta[\phi] = \phi(0)$$ (4)

We will refer to (3) as the integral notation and (4) as the Schwartz notation. An arbitrary distribution will be written either as $\psi(x)$ for the integral notation or $\Psi$ for the Schwartz notation.

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3 Function valued distributions

The first step in understanding the Colombeau Algebra is to convert distributions into a new object which takes a test functions $\phi$ and gives a functions

$$A : \mathcal{F}_0(\mathbb{R}) \rightarrow \mathcal{F}(\mathbb{R})$$

This is achieved by using translation of the test functions. Given $\phi \in \mathcal{F}_0$ then let

$$\phi^y \in \mathcal{F}_0(\mathbb{R}) \text{, } \phi^y(x) = \phi(x - y) \quad (5)$$

Then in integral notation

$$\overline{\psi}[\phi](y) = \int_{-\infty}^{\infty} \psi(x)\phi(x - y)dx \quad (6)$$

and in Schwartz notation

$$\overline{\Psi}[\phi](y) = \Psi[\phi^y] \quad (7)$$

We will define the Colombeau Algebra in such a way that they include the elements $\overline{\psi}$ and $\overline{\Psi}$. The overline will be used to covert distributions into elements of the Colombeau algebra.

We label the set of all function valued functionals

$$\mathcal{H}(\mathbb{R}) = \{A : \mathcal{F}_0(\mathbb{R}) \rightarrow \mathcal{F}(\mathbb{R})\} \quad (8)$$

We see below that we need to restrict $\mathcal{H}(\mathbb{R})$ further in order to define the Colombeau algebra $\mathcal{G}(\mathbb{R})$.

Observe that we use a slightly non standard notation. Here $A[\phi] : \mathbb{R} \rightarrow \mathbb{R}$ is a function, so that given a point $x \in \mathbb{R}$ then $A[\phi](x) \in \mathbb{R}$. One can equally write $A[\phi](x) = A(\phi, x)$, which is the standard notation in the literature. However I claim that the notation $A[\phi](x)$ does have advantages.

4 Three special examples.

For the Dirac-$\delta$ we see that

$$\overline{\delta} = \overline{\Delta} = \mathcal{R} \quad (9)$$
where $R \in \mathcal{H}(\mathbb{R})$ is the reflection map

$$R[\phi](y) = \phi(-y)$$

This is because

$$\bar{\delta}[\phi](y) = \int_{-\infty}^{\infty} \delta(x)\phi(x - y)dx = \phi(-y)$$

and is Schwartz notation

$$\bar{\Delta}[\phi](y) = \Delta[\phi^y] = \phi^y(0) = \phi(-y)$$

Regular distribution: Given any function $f \in \mathcal{F}$ then there is a distribution $f^D$ given by

$$f^D[\phi] = \int_{-\infty}^{\infty} f(x)\phi(x)dx$$

Thus we set $\bar{f} = f^D \in \mathcal{H}(\mathbb{R})$ as

$$\bar{f}[\phi](y) = f^D[\phi^y] = \int_{-\infty}^{\infty} f(x)\phi(x - y)dx$$

The other important generalised functions are the regular functions. That is given $f \in \mathcal{F}$ we set

$$\tilde{f} \in \mathcal{H}(\mathbb{R}), \quad \tilde{f}[\phi] = f \quad \text{that is} \quad \tilde{f}[\phi](y) = f(y)$$

The effect of replacing $\bar{\theta}[\phi]$ is to smooth out $\psi$. Examples of $\phi$ are given in figure 1. The action $\bar{\theta}[\phi]$ where $\theta$ is the Heaviside function is given in figure 2.

### 5 Sums and Products

Given two Generalised functions $A, B \in \mathcal{H}(\mathbb{R})$ then we can define there sum and product in the natural way

$$A + B \in \mathcal{H}(\mathbb{R}) \quad \text{via} \quad (A + B)[\phi] = A[\phi] + B[\phi] \quad \text{i.e.} \quad (A + B)[\phi](y) = A[\phi](y) + B[\phi](y)$$

where $R \in \mathcal{H}(\mathbb{R})$ is the reflection map

$$R[\phi](y) = \phi(-y)$$

This is because

$$\bar{\delta}[\phi](y) = \int_{-\infty}^{\infty} \delta(x)\phi(x - y)dx = \phi(-y)$$

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$$\tilde{f} \in \mathcal{H}(\mathbb{R}), \quad \tilde{f}[\phi] = f \quad \text{that is} \quad \tilde{f}[\phi](y) = f(y)$$

The effect of replacing $\bar{\theta}[\phi]$ is to smooth out $\psi$. Examples of $\phi$ are given in figure 1. The action $\bar{\theta}[\phi]$ where $\theta$ is the Heaviside function is given in figure 2.
\[ AB \in \mathcal{H}(\mathbb{R}) \quad \text{via} \quad (AB)[\phi] = A[\phi]B[\phi] \quad \text{i.e.} \quad (AB)[\phi](y) = A[\phi](y)B[\phi](y) \quad (15) \]

We see that the product of delta functions \( \delta^2 \in \mathcal{H}(\mathbb{R}) \) is clearly defined. That is
\[ \delta^2[\phi](y) = (\delta[\phi]\delta[\phi])(y) = \delta[\phi](y)\delta[\phi](y) = (\phi(-y))^2 \]

Although this is a generalised function, it does not correspond to a distribution, via (7). That is there is no distribution \( \Psi \) such that
\[ \Psi = (\delta[\phi]\delta[\phi]). \]

Likewise we can see from figure 2 that \( (\theta)^2[\phi] = (\theta[\phi]\theta[\phi])^2 \neq \theta[\phi]. \)

6 Making \( \bar{f} \) and \( \tilde{f} \) equivalent

Now compare the generalised function \( \bar{f} \) and \( \tilde{f} \). We would like these two generalised functions to be equivalent, so that we can write \( \bar{f} \sim \tilde{f} \). One of the results of making \( \bar{f} \sim \tilde{f} \) is that if \( f, g \in \mathcal{F} \) then
\[ (fg) \sim (\tilde{f} \tilde{g}) = \tilde{f} \tilde{g} \sim f \tilde{g} \]

In the Colombeau algebra, which is a quotient of equivalent generalised functions, we say that \( \bar{f} \) and \( \tilde{f} \) are the same generalised function.

The goal therefore is to restrict the set of possible \( \phi \) so that when they are acted upon by \( (\bar{f} - \tilde{f}) \) the difference is small, where small will be made technically precise. When we think of quantities being small, we need a 1-parameter family of such quantities such that in the limit the difference vanishes. Here we label the parameter \( \epsilon \) and we are interested in the limit \( \epsilon \to 0 \) from above, i.e. with \( \epsilon > 0 \). Given a one parameter set of functions \( g_\epsilon \in \mathcal{F} \) then one meaning to say \( g_\epsilon \) is small is if \( g_\epsilon(y) \to 0 \) for all \( y \). However we would like a whole hierarchy of smallness. That is for any \( q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \) then we can say
\[ g_\epsilon = \mathcal{O}(\epsilon^q) \]

if \( \epsilon^{-q}g_\epsilon(y) \) is bounded as \( \epsilon \to 0 \). Note that we use bounded, rather than tends to zero. However, clearly, if \( g_\epsilon = \mathcal{O}(\epsilon^q) \) then \( \epsilon^{-q+1}g_\epsilon \to 0 \) as \( \epsilon \to 0 \).

We will also need the notion of \( g_\epsilon = \mathcal{O}(\epsilon^q) \) where \( q < 0 \). Thus we wish to consider functions which blow up as \( \epsilon \to 0 \), but not too quickly. Such functions will be called moderate.

Technically we say \( g_\epsilon \) satisfies (16) if for any interval \((a, b)\) there exists \( C > 0 \) and \( \eta > 0 \) such that
\[ \epsilon^{-q}|g_\epsilon(x)| < C \quad \text{for all} \quad a \leq y \leq b \quad \text{and} \quad 0 < \epsilon < \eta \quad (17) \]

We introduce the parameter \( \epsilon \) via the test functions, replacing \( \phi \in \mathcal{F}_0 \) with \( \phi_\epsilon \in \mathcal{F}_0 \) where
\[ \phi_\epsilon(x) = \frac{1}{\epsilon} \phi\left(\frac{x}{\epsilon}\right) \]

Observe as \( \epsilon \to 0 \) then \( \phi_\epsilon \) becomes narrower and taller, in a definite sense more like a \( \delta \)-function. Thus we consider a generalised function \( A \) to be small, if for some appropriate set of test functions \( \phi \in \mathcal{F}_0 \) and for some \( q \in \mathbb{Z}, A[\phi_\epsilon] = \mathcal{O}(\epsilon^q) \).

Let us first restrict \( \phi \in \mathcal{F}_0 \) to be test function which integrate to 1. That is we define \( \mathcal{A}_0 \subset \mathcal{F}_0 \),
\[ \mathcal{A}_0 = \left\{ \phi \in \mathcal{F}_0 \biggm| \int_{-\infty}^{\infty} \phi(x)dx = 1 \right\} \quad (19) \]
Figure 3: Plots of $f(\phi_1|\epsilon=0.2)$ (blue) and $f(\phi_1|\epsilon=0.1)$ (red). $f$ (back), $\bar{f}(\phi_1|\epsilon=0.2)$ (blue) and $\bar{f}(\phi_1|\epsilon=0.1)$ (red).

$(\bar{f} - f)[\phi_1|\epsilon=0.02]$ (blue) and $(\bar{f} - f)[\phi_1|\epsilon=0.01]$ (red).

$(\bar{f} - f)[\phi_3|\epsilon=0.02]$ (blue) and $(\bar{f} - f)[\phi_3|\epsilon=0.01]$ (red).

Figure 3: Plots of $\bar{f}[\phi_1]$ with $f(x) = \tanh(10x)$
Given $\phi \in \mathcal{A}_0$ and setting $z = (x - y)/\epsilon$ so that $x = y + \epsilon z$

$$\mathcal{F}[\phi_\epsilon](y) = f^D[\phi_\epsilon^y] = \int_{-\infty}^{\infty} f(x)\phi_\epsilon(x - y)dx = \frac{1}{\epsilon} \int_{-\infty}^{\infty} f(x)\phi\left(\frac{x - y}{\epsilon}\right)dx = \int_{-\infty}^{\infty} f(y + \epsilon z)\phi(z)dz$$

Thus as $\epsilon \to 0$ then $f(y + \epsilon z) \approx f(y)$ so that, since $\phi \in \mathcal{A}_0$,

$$\mathcal{F}[\phi_\epsilon](y) = \int_{-\infty}^{\infty} f(y + \epsilon z)\phi(z)dz \approx \int_{-\infty}^{\infty} f(y)\phi(z)dz = f(y)\int_{-\infty}^{\infty} \phi(z)dz = f(y) = \mathcal{F}[\phi_\epsilon](y)$$

In fact since $f(y + \epsilon z) - f(y) = \mathcal{O}(\epsilon)$ we can show using (17) that

$$\text{if } \phi \in \mathcal{A}_0 \text{ then } (\mathcal{F} - \tilde{f})[\phi_\epsilon] = \mathcal{O}(\epsilon) \quad (21)$$

This is good so far, but we want to further restrict the set $\phi$ so that we can satisfy

$$(\mathcal{F} - \tilde{f})[\phi_\epsilon] = \mathcal{O}(\epsilon^q) \quad (22)$$

to any order of $\mathcal{O}(\epsilon^q)$.

Taylor expanding $f(y + \epsilon z)$ to order $\mathcal{O}(\epsilon^{q+1})$ we have

$$f(y + \epsilon z) = \sum_{r=0}^{q} \epsilon^r z^r f^{(r)}(y) + \mathcal{O}(\epsilon^{q+1})$$

Thus

$$(\mathcal{F} - \tilde{f})[\phi_\epsilon](y) = \int_{-\infty}^{\infty} (f(y + \epsilon z) - f(y))\phi(z)dz = \int_{-\infty}^{\infty} \left(\sum_{n=1}^{q} \epsilon^r z^r f^{(r)}(y) + \mathcal{O}(\epsilon^{q+1})\right)\phi(z)dz$$

$$= \sum_{n=1}^{q} \epsilon^r f^{(r)}(y) \int_{-\infty}^{\infty} z^r \phi(z)dz + \mathcal{O}(\epsilon^{q+1})$$

Thus we can satisfy (16) to order $\mathcal{O}(\epsilon^{q+1})$ if the first $q$ moments of $\phi(z)$ vanish:

$$\int_{-\infty}^{\infty} z^r \phi(z)dz = 0 \quad \text{for} \quad 1 \leq r \leq q$$

We now define all the elements with vanishing moments.

$$\mathcal{A}_q = \left\{ \phi \in \mathcal{F}_0(\mathbb{R}) \mid \int_{-\infty}^{\infty} \phi(z)dz = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} z^r \phi(z)dz = 0 \quad \text{for} \quad 1 \leq r \leq q \right\} \quad (24)$$

So clearly $\mathcal{A}_{q+1} \subset \mathcal{A}_q$. We can show that these functions exist. Thus from (23) we have

$$\phi \in \mathcal{A}_q \quad \text{implies} \quad (\mathcal{F} - \tilde{f})[\phi_\epsilon] = \mathcal{O}(\epsilon^{q+1}) \quad (25)$$

Two example test functions $\phi_1 \in \mathcal{A}_1$ and $\phi_3 \in \mathcal{A}_3$ are given in figure 1. The result $\mathcal{F}[\phi_\epsilon]$, (12), (20) is given in fig 3.

The easiest way to construct $\phi \in \mathcal{A}_q$ is to choose a test function $\psi$ and then set

$$\phi(z) = \lambda_0 \psi(z) + \lambda_1 \psi'(z) + \cdots + \lambda_{q-1} \psi^{(q-1)}(z)$$

where $\lambda_0, \ldots, \lambda_{q-1} \in \mathbb{R}$ are constants determined by (24).
7 Null and moderate generalised functions.

As we stated we wanted \( \overline{f} \) and \( \hat{f} \) to be considered equivalent. From (25) we have \( \phi \in \mathbb{A}_q \) then \( (\overline{f} - \hat{f})[\phi] = \mathcal{O}(\epsilon^{q+1}) \). We generalise this notion. We say that \( A, B \in \mathcal{H}(\mathbb{R}) \) are equivalent, \( A \sim B \), if for all \( q \in \mathbb{N} \) there is a \( p \in \mathbb{N} \) such that

\[
\phi \in \mathbb{A}_p \implies A[\phi] - B[\phi] = \mathcal{O}(\epsilon^p)
\]

We label \( \mathcal{N}^{(0)}(\mathbb{R}) \subset \mathcal{H}(\mathbb{R}) \) the set of all elements which are null, that is equivalent to the zero element \( 0 \in \mathcal{H}(\mathbb{R}) \) that is

\[
\mathcal{N}^{(0)}(\mathbb{R}) = \{ A \in \mathcal{H}(\mathbb{R}) \mid A \sim 0 \}
\]

I.e.

\[
\mathcal{N}^{(0)}(\mathbb{R}) = \{ A \in \mathcal{H}(\mathbb{R}) \mid \text{for all } p \in \mathbb{N} \text{ there exists } q \in \mathbb{N} \text{ such that for all } \phi \in \mathbb{A}_q, A[\phi] = \mathcal{O}(\epsilon^p) \}\]

(27)

Examples of null elements are of course \( \overline{f} - \hat{f} \in \mathcal{N}^{(0)}(\mathbb{R}) \), which is true by construction. Another example is \( N \in \mathcal{N}^{(0)}(\mathbb{R}) \) which is given by

\[
N[\phi](y) = \phi(1)
\]

(28)

Since for any \( \phi \in \mathbb{A}_0 \) there exists \( \eta > 0 \) such that \( 1/\eta \) is outside the support of \( \phi \). Thus \( \phi_\epsilon(1) = 0 \) for all \( \epsilon < \eta \) and hence \( N[\phi_\epsilon] = 0 \) so \( N \in \mathcal{N}^{(0)}(\mathbb{R}) \). However, although \( N \in \mathcal{N}^{(0)} \), we can choose \( \phi \) so that \( N[\phi](y) = \phi(1) \) is any value we choose. Thus knowing that a generalised function \( A \) is null says nothing about the value of \( A[\phi] \) but only the limit of \( A[\phi_\epsilon] \) as \( \epsilon \to 0 \).

We would like \( \mathcal{N}^{(0)}(\mathbb{R}) \) to form an ideal in \( \mathcal{H}(\mathbb{R}) \), that is that if \( A, B \in \mathcal{N}^{(0)}(\mathbb{R}) \) and \( C \in \mathcal{H}(\mathbb{R}) \) then

- \( A + B \in \mathcal{N}^{(0)}(\mathbb{R}) \) and
- \( AC \in \mathcal{N}^{(0)}(\mathbb{R}) \).

It is easy to see that the first of these is automatically satisfied. However the second requires one additional requirement. We need

\[
C[\phi_\epsilon] = \mathcal{O}(\epsilon^{-N})
\]

(29)

for some \( N \in \mathbb{Z} \). Thus although \( C[\phi_\epsilon] \to \infty \) as \( \epsilon \to 0 \) we don’t want it to blow up to quickly. Now we have the following:

Given \( A \in \mathcal{N}^{(0)}(\mathbb{R}) \) and \( C \) satisfying (29) and given \( q \in \mathbb{N}_0 \) then there exists \( p \in \mathbb{Z} \) such that \( \phi \in \mathbb{A}_p \) implies \( A[\phi_\epsilon] = \mathcal{O}(\epsilon^{q+N}) \). Hence

\[
(AC)[\phi_\epsilon] = A[\phi_\epsilon]C[\phi_\epsilon] = \mathcal{O}(\epsilon^{q+N})\mathcal{O}(\epsilon^{-N}) = \mathcal{O}(\epsilon^q)
\]

hence \( AC \in \mathcal{N}^{(0)}(\mathbb{R}) \). We call the set of elements \( C \in \mathcal{H}(\mathbb{R}) \) satisfying (29), moderate and set of moderate functions

\[
\mathcal{E}^{(0)}(\mathbb{R}) = \{ A \in \mathcal{H}(\mathbb{R}) \mid \text{there exists } N \in \mathbb{N} \text{ such that for all } \phi \in \mathbb{A}_0, A[\phi_\epsilon] = \mathcal{O}(\epsilon^{-N}) \}
\]

(30)

Examples of moderate functions include

\[
\overline{\Delta}[\phi_\epsilon](y) = \phi_\epsilon(-y) = \frac{1}{\epsilon} \phi\left(-\frac{y}{\epsilon}\right) = \mathcal{O}(\epsilon^{-1}), \quad (\overline{\Delta})''[\phi_\epsilon] = \mathcal{O}(\epsilon^{-n})
\]

and

\[
\hat{f}[\phi_\epsilon](y) = f(y) = \mathcal{O}(\epsilon^0)
\]
8 Derivatives

The last part in the construction of the Colombeau Algebra is to extend all the definitions so that they also apply to the derivatives $\frac{dA[\phi]}{dy}$, $\frac{d^2A[\phi]}{dy^2}$, etc. We require that not only does a moderate function not blow up too quickly, but neither do its derivatives, i.e.

$$ \left( A[\phi] \right)^{(n)} = \frac{d^n}{dy^n} (A[\phi]) \in \mathcal{E}^{(0)}(\mathbb{R}) $$

(31)

Thus we define the set of moderate function as

$$ \mathcal{E}(\mathbb{R}) = \left\{ A \in \mathcal{H}(\mathbb{R}) \mid \left( A[\phi] \right)^{(n)} \in \mathcal{E}^{(0)}(\mathbb{R}) \text{ for all } n \in \mathbb{N}, \phi \in A_0 \right\} $$

(32)

That is

$$ \mathcal{E}(\mathbb{R}) = \left\{ A \in \mathcal{H}(\mathbb{R}) \mid \text{for all } n \in \mathbb{N}_0 \text{ there exists } N \in \mathbb{N} \text{ such that for all } \phi \in A_0, \left( A[\phi] \right)^{(n)} = O(\epsilon^{-N}) \right\} $$

(33)

Likewise we require that for two generalised functions to be equivalent then we require that all the derivatives are small

$$ \mathcal{N}(\mathbb{R}) = \left\{ A \in \mathcal{H}(\mathbb{R}) \mid \left( A[\phi] \right)^{(n)} \in \mathcal{N}^{(0)}(\mathbb{R}) \text{ for all } n \in \mathbb{N} \right\} $$

(34)

That is

$$ \mathcal{N}(\mathbb{R}) = \left\{ A \in \mathcal{H}(\mathbb{R}) \mid \text{for all } n \in \mathbb{N}_0 \text{ and } q \in \mathbb{N} \text{ there exists } p \in \mathbb{N} \text{ such that for all } \phi \in A_p, \left( A[\phi] \right)^{(n)} = O(\epsilon^q) \right\} $$

(35)

9 Quotient Algebra

We write the Colombeau Algebra as a quotient algebra,

$$ \mathcal{G}(\mathbb{R}) = \mathcal{E}(\mathbb{R})/\mathcal{N}(\mathbb{R}) $$

(36)

This means that, with regard to elements $A, B \in \mathcal{E}(\mathbb{R})$ we say $A \sim B$ if $A - B \in \mathcal{N}(\mathbb{R})$. For elements in $A, B \in \mathcal{G}(\mathbb{R})$ we simply write $A = B$.

Given $A \in \mathcal{G}(\mathbb{R})$, then in order to get an actual number we must first choose a representative $B \in \mathcal{E}(\mathbb{R})$ of $A \in \mathcal{G}(\mathbb{R})$, then we must choose $\phi \in A_0$ and $y \in \mathbb{R}$ then the quantity $B[\phi](y) \in \mathbb{R}$.

10 Summary

We can summarise the steps needed to go from distributions to Colombeau functions:

- Convert distributions which give a number $\Psi[\phi]$ as an answer to functionals $A[\phi]$ which give a function as an answer.
- Construct the sets of test functions $A_q$, so that $\bar{f} \sim \tilde{f}$, i.e. $\bar{f} - \tilde{f} \in \mathcal{N}^{(0)}(\mathbb{R})$
- Limit the generalised functions to elements of $\mathcal{E}^{(0)}(\mathbb{R})$ so that the set $\mathcal{N}^{(0)}(\mathbb{R}) \subset \mathcal{E}^{(0)}(\mathbb{R})$ is an ideal.
- Extend the definitions of $\mathcal{E}^{(0)}(\mathbb{R})$ and $\mathcal{N}^{(0)}(\mathbb{R})$ to $\mathcal{E}(\mathbb{R})$ and $\mathcal{N}(\mathbb{R})$ so that they also apply to derivatives.
- Define the Colombeau Algebra as the quotient $\mathcal{G}(\mathbb{R}) = \mathcal{E}(\mathbb{R})/\mathcal{N}(\mathbb{R})$.

The formal definition, we define $\mathcal{E}(\mathbb{R})$ via (33), then $\mathcal{N}(\mathbb{R})$ via (35) and (24). Then define the Colombeau Algebra $\mathcal{G}(\mathbb{R})$ as the quotient (36).
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