ON NONSTANDARD QUANTIZATIONS OF $\mathfrak{osp}(2|1)$ SUPERALGEBRA VIA CONTRACTION AND MAPPING

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Abstract

We develop a generic representation-independent contraction procedure for obtaining, for instance, $R_h$ and $L$ operators of arbitrary dimensions for the quantized $\mathcal{U}_h(\mathfrak{osp}(2|1))$ algebra corresponding to the classical $r_2$ matrix from the pertinent quantities of the standard $q$-deformed $\mathcal{U}_q(\mathfrak{osp}(2|1))$ algebra. Also the quantized $\mathcal{U}_h(\mathfrak{osp}(2|1))$ algebra corresponding to the classical $r_1$ matrix comprising of the generators of the classical $\mathfrak{sl}(2)$ algebra is obtained in terms of a nonlinear basis set.

1 Introduction

Quantum deformations of the Lie superalgebra $\mathfrak{osp}(2|1)$ have been studied [1]-[7] extensively both from the point of view of investigating integrable physical models, and also because of their intrinsic mathematical importance. It has been recently demonstrated [5] that three distinct bialgebra structures exist on the classical $\mathfrak{osp}(2|1)$ superalgebra, and all of them are coboundary. The classical Lie superalgebra $\mathfrak{osp}(2|1)$ has three even ($h, b_{\pm}$) and two odd ($e, f$) generators, obeying the following commutation relations:

$$[h, e] = e, \quad [h, f] = -f, \quad \{e, f\} = -h,$$

$$[h, b_{\pm}] = \pm 2b_{\pm}, \quad [b_+, b_-] = h,$$

$$[b_+, f] = e, \quad [b_-, e] = f, \quad b_+ = e^2, \quad b_- = -f^2. \quad (1.1)$$

The generators ($h, b_{\pm}$) form a subalgebra $\mathfrak{sl}(2) \subset \mathfrak{osp}(2|1)$. The classical universal enveloping algebra $\mathcal{U}(\mathfrak{osp}(2|1))$ is generated by the elements ($h, e, f$). The irreducible representations are parametrized by an integer or a half-integer $j$, and are of dimension $(4j+1)$. For later use we quote here a representation of the algebra [11]:

$$h |j \, m\rangle = 2m |j \, m\rangle, \quad e |j \, m\rangle = |j \, m+1/2\rangle,$$

$$f |j \, m\rangle = -(j+m) |j \, m-1/2\rangle \quad \text{for} \quad j-m \text{ integer,}$$

$$= (j-m+1/2) |j \, m-1/2\rangle \quad \text{for} \quad j-m \text{ half-integer,} \quad (1.2)$$

where $m = j, j-1/2, \ldots, -(j-1/2), -j$. The inequivalent classical $r$-matrices listed in Ref.[5] read as follows:

$$r_1 = h \wedge b_+, \quad r_2 = h \wedge b_+ - e \wedge e,$$

$$r_3 = t(h \wedge b_+ + h \wedge b_- - e \wedge e - f \wedge f). \quad (1.3)$$

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The matrices $r_1$ and $r_2$ satisfy the classical Yang-Baxter equation, whereas the $r_3$ matrix satisfy the modified classical Yang-Baxter equation. The standard $q$-deformation of the $osp(2|1)$ algebra considered in Refs. [1] and [2] corresponds to $r_3$, and it is a quasi-triangular algebra. The parameter $t$ in $r_3$ becomes irrelevant in quantization as it can be absorbed into the deformation parameter. Various applications of the algebra have been studied earlier. The $r_1$ matrix is comprised of the elements of the $sl(2)$ subalgebra. This allows quantization of the $osp(2|1)$ algebra using the inclusion $sl(2) \subset osp(2|1)$. This is done [3] by applying Drinfeld twist for the $sl(2)$ subalgebra to the full $osp(2|1)$ algebra. The Hopf superalgebra $U_h(osp(2|1))$ obtained thereby is triangular. The quantization of the $r_1$ matrix has been obtained in Ref. [3] in terms of the classical basis set with undeformed commutation relations but with coproduct structures deformed in a complicated manner.

The classical matrix $r_2$ has recently been quantized [6] using nonlinear basis elements. A class of invertible maps relating the quantized $U_h(osp(2|1))$ algebra and its classical analog $U(osp(2|1))$ has been found. The twist elements corresponding to these maps and the resultant universal $R$ matrix have been evaluated as series expansions in the deformation parameter. In another work [7] the twist operator has been obtained in a closed-form expression in terms of the elements of the undeformed classical $osp(2|1)$ algebra. The universal $R$ matrix obtained from this twist operator satisfies [7] the quantum triangularity condition. Another important issue observed [4] in this context is that the relevant quantum $R_h$ matrix in the fundamental representation may be obtained via a contraction mechanism from the corresponding $R_q$ matrix of the standard $U_q(osp(2|1))$ algebra in the $q \to 1$ limit. A generalization of this contraction procedure for arbitrary representations, though clearly desirable as it will allow us to systematically obtain various quantities of interest of the $U_h(osp(2|1))$ algebra, has not been achieved so far.

In the present work we complete the two above mentioned tasks. A generic technic developed earlier [8]-[10] allowed us to extract quantum universal $R$ matrix and $T$ matrices for arbitrary representations for Jordanian $U_h(sl(N))$ algebra from the corresponding operators of the standard $q$-deformed $U_q(sl(N))$ algebra. A suitable adaptation of this methodology is used here to obtain the $L$ operator of the Jordanian $U_h(osp(2|1))$ algebra obtained by quantizing the $r_2$ matrix. The standard FRT [11] procedure is then utilized, in conjunction with the $L$ operator obtained here, to derive the Hopf structure of the Borel subalgebra of the Jordanian $U_h(osp(2|1))$ algebra. Our method may be readily used to derive $L$ operators for the higher representations and the corresponding $T$ matrices of the $U_h(osp(2|1))$ algebra.

In another problem considered here, we express the Jordanian deformed Hopf algebra $U_h(osp(2|1))$ corresponding to the classical $r_1$ matrix in a nonlinear basis. Here we follow the approach in Ref. [12], where the Jordanian $U_h(sl(2))$ algebra has been introduced in terms of a nonlinear basis set, while retaining the coproduct structure of these basis elements simple. A consequence of our choice of nonlinear basis elements is that, Ohn’s $U_h(sl(2))$ algebra [12] explicitly arises as a Hopf subalgebra in our $U_h(osp(2|1))$ algebra. This feature is not directly evident in the construction given in Ref. [3]. Moreover, while the algebraic relations of our $U_h(osp(2|1))$ algebra are deformed, the coproduct structures are considerably simple. Our approach may be of consequence in building physical models of many-body systems employing coalgebra symmetry [13]. Invertible nonlinear maps, and twist operators pertaining to these maps, exist connecting the deformed and the undeformed basis sets. We will present here a class of invertible maps interrelating the Hopf superalgebra $U_h(osp(2|1))$ based on quantization of the $r_1$ matrix, and its classical analog $U(osp(2|1))$. The twist operators vis-à-vis the above maps will be discussed in the sequel. It is shown that a particular map called ‘minimal twist map’ implements the simplest twist given directly by the factorized form of the universal $R_h$ matrix of the $U_h(osp(2|1))$ algebra. For a ‘non-minimal’ map the twist has an additional factor. We evaluate this twist operator as a series in the deformation parameter.
2 A Contraction Process

A quantization of the classical matrix \( r_2 \) given in \[4\] using a nonlinear basis elements was obtained previously \[6\]. The generating elements \((H, E, F)\) of the \(U_h(osp(2|1))\) algebra go to the classical generators \((h, e, f)\) in the limiting value of the deformation parameter: \( h \to 0 \). Introducing also the elements \((X, Y)\) going to \((b_+, b_-)\) respectively in the said limit, the commutation relations read

\[
[H, E] = \frac{1}{2} (T + T^{-1}) E, \quad [H, F] = -\frac{1}{4} (T + T^{-1}) F - \frac{1}{4} F (T + T^{-1}),
\]

\[
\{E, F\} = -H, \quad [H, T^\pm] = T^{\pm 2} - 1,
\]

\[
[H, Y] = -\frac{1}{2} (T + T^{-1}) Y - \frac{1}{2} Y (T + T^{-1}) - \frac{h}{4} E (T - T^{-1}) F - \frac{h}{4} F (T - T^{-1}) E,
\]

\[
[T^\pm, Y] = \pm \frac{h}{2} (T^\pm H + HT^\pm), \quad E^2 = \frac{T - T^{-1}}{2h}, \quad F^2 = -Y,
\]

\[
[T^\pm, F] = \pm h T^\pm E, \quad [Y, E] = \frac{1}{4} (T + T^{-1}) F + \frac{1}{4} F (T + T^{-1}), \quad (2.1)
\]

where \( T^\pm = \exp(\pm h X) \). The quantum coproduct \((\Delta)\) map for the \(U_h(osp(2|1))\) algebra reads

\[
\Delta(H) = H \otimes T^{-1} + T \otimes H + h E T^{1/2} \otimes E T^{-1/2}; \quad \Delta(E) = E \otimes T^{-1/2} + T^{1/2} \otimes E,
\]

\[
\Delta(F) = F \otimes T^{-1/2} + T^{1/2} \otimes F; \quad \Delta(T^\pm) = T^{\pm 1} \otimes T^\pm,
\]

\[
\Delta(Y) = Y \otimes T^{-1} + T \otimes Y + \frac{h}{2} E T^{1/2} \otimes T^{-1/2} F + \frac{h}{2} T^{1/2} F \otimes E T^{-1/2}. \quad (2.2)
\]

The corresponding counit \((\varepsilon)\) and the antipode \((S)\) maps are given by

\[
\varepsilon(H) = \varepsilon(E) = \varepsilon(F) = \varepsilon(Y) = 0, \quad \varepsilon(T^\pm) = 1,
\]

\[
S(H) = -H - h E^2; \quad S(E) = -E; \quad S(F) = -F + \frac{h}{2} E,
\]

\[
S(T^\pm) = T^\mp; \quad S(Y) = -Y + \frac{h}{2} H + \frac{h^2}{4} E^2. \quad (2.3)
\]

The only primitive element in the above algebra is \(X\). We also note that the Hopf algebra \(U_h(osp(2|1))\) has only one Borel subalgebra generated by the elements \((H, E, X)\). It was observed by Kulish \[4\] that the \(R_h\) matrix in the fundamental representation of the quantized algebra \(U_h(osp(2|1))\) corresponding to the classical \(r_2\) matrix may be obtained via a transformation, singular in the \(q \to 1\) limit, from the corresponding \(R_q\) matrix in the fundamental representation of the standard \(q\)-deformed \(U_q(osp(2|1))\) algebra. The \(R_h^{1/2;1/2}\) matrix, thus obtained in Ref. \[4\], reads

\[
R_h^{1/2;1/2} = \begin{pmatrix}
1 & 0 & h & 0 & 0 & -h & 0 & \frac{h^2}{2T} \\
0 & 1 & 0 & 0 & h & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & h \\
0 & 0 & 0 & 1 & 0 & 0 & -h & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -h \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}. \quad (2.4)
\]
Our task in the present Section is to generalize the above contraction procedure for arbitrary representations; and, in particular, we construct the $L$ operator corresponding to the Borel subalgebra of the Hopf algebra $U_h(osp(2|1))$ from the corresponding $L$ operator of the standard $q$-deformed $U_q(osp(2|1))$ algebra. To this end, we first quote the well-known [1,2] results on the $U_q(osp(2|1))$ algebra. The quasitriangular quantum Hopf superalgebra $U_q(osp(2|1))$, by analogy with the classical $U(osp(2|1))$ algebra, is generated by three elements $(\hbar, \hat{e}, \hat{f})$ obeying the algebraic relations

$$[\hbar, \hat{e}] = \hat{e}, \quad [\hbar, \hat{f}] = -\hat{f}, \quad [\hat{e}, \hat{f}] = -[\hbar]_q, \quad (2.5)$$

where $[x]_q = (q^x - q^{-x})/(q - q^{-1})$. The coalgebraic relations are given by

$$\Delta(\hbar) = \hbar \otimes 1 + 1 \otimes \hbar, \quad \Delta(\hat{e}) = \hat{e} \otimes q^{-\hbar/2} + q^{\hbar/2} \otimes \hat{e}, \quad \Delta(\hat{f}) = \hat{f} \otimes q^{-\hbar/2} + q^{\hbar/2} \otimes \hat{f},$$

$$\varepsilon(\hbar) = \varepsilon(\hat{e}) = \varepsilon(\hat{f}) = 0, \quad S(\hbar) = -\hbar, \quad S(\hat{e}) = -q^{-1/2} \hat{e}, \quad S(\hat{f}) = -q^{1/2} \hat{f}. \quad (2.6)$$

For convenience, we choose the $(4j + 1)$ dimensional irreducible representation of the $U_q(osp(2|1))$ algebra as follows:

$$\hat{h} | j m > = 2m | j m >, \quad \hat{e} | j m >= | j m + 1/2 >, \quad \hat{f} | j m > = -|j + m|_q |j - m + 1/2|_q | j m - 1/2 >, \quad \text{for } j - m \text{ integer},$$

$$= |(j + m)|_q |j - m + 1/2|_q | j m - 1/2 >, \quad \text{for } j - m \text{ half-integer}, \quad (2.7)$$

where $[[x]]_q = (q^x - (-1)^{2x}q^{-x})/(q^{1/2} + q^{-1/2})$. A related numerical quantity used subsequently reads $[n]_+ = (-1)^{n-1} ([n/2])_q$, where $n$ is an integer. The representation (2.7) may be viewed as the $q$-deformation of its classical analog [1,2]. The fundamental representation of the algebra (2.5) reads

$$\hat{h} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \hat{e} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{f} = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \quad (2.8)$$

The universal $R$-matrix of the $U_q(osp(2|1))$ algebra is given by [1]

$$R = q^{\hbar \otimes \hbar} \sum_{n=0}^{\infty} q^{n(n+1)/4} \frac{(1 - q^{-2})^n}{[n]_+!} \left( q^{\hbar/2} \hat{e} \right)^n \otimes \left( q^{-\hbar/2} \hat{f} \right)^n, \quad (2.9)$$

where $[n]_+! = [n]_+ [n - 1]_+ \cdots [1]_+ [0]_+! = 1$.

Following the strategy adopted earlier [8-10] for constructing the Jordanian deformation of the $sl(N)$ algebra, we give here a general recipe for obtaining the quantum $R^{1/2j}_h$ matrix of an arbitrary representation of the $U_h(osp(2|1))$ algebra. Explicit demonstration is given for the $1/2 \otimes j$ representation, as the relevant $R^{1/2j}_h$ matrix may be directly interpreted as the $L$ operator corresponding to the Borel subalgebra of the $U_h(osp(2|1))$ algebra. But our construction may be obviously generalized. The primary ingredient for our method is the $R^{1/2j}_q$ matrix of the $U_q(osp(2|1))$ algebra in the $1/2 \otimes j$ representation. A suitable similarity transformation is performed on this $R^{1/2j}_q$ matrix. The transforming matrix is singular in the $q \to 1$ limit. For the transformed matrix, the singularities, however, systematically cancel yielding a well-defined construction. The transformed matrix, directly furnishes the $R^{1/2j}_h$ for the nonstandard $U_h(osp(2|1))$ algebra. Interpreting, as mentioned above, the $R^{1/2j}_h$ obtained here as the $L$ operator corresponding to the Borel subalgebra of the $U_h(osp(2|1))$ algebra, we use the standard FRT procedure [11] to reconstruct the full Hopf structure of the said Borel subalgebra presented in (2.1)-(2.3). The $R^{1/2j}_q$ matrix of the tensored $1/2 \otimes j$ representation of the
where \( \omega = q - q^{-1} \). We now introduce a transforming matrix \( M \), singular in the \( q \to 1 \) limit, as

\[
M = E_q^2(\eta \hat{e}^2),
\]

where

\[
E_q^2(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}, \quad \eta = \frac{h}{q^2 - 1}.
\]

For any finite value of \( j \) the series (2.12) may be terminated after setting \( \hat{e}^{4j+1} = 0 \); but, we proceed quite generally. The \( R_{q}^{j_1:j_2} \) matrix of the \( U_q(osp(2|1)) \) algebra may now be subjected to a similarity transformation followed by a limiting process

\[
\tilde{R}_{q}^{j_1:j_2} = \lim_{q \to 1} \left[ \left( M_{j_1}^{-1} \otimes M_{j_2}^{-1} \right) R_{q}^{j_1:j_2} \left( M_{j_1} \otimes M_{j_2} \right) \right].
\]

In the followings we will present explicit results for the operator \( \tilde{R}_{q}^{1/2:j} \). In performing the similarity transformation (2.13) we may choose any suitable operator ordering. Specifically, starting from left we maintain the order \( \hat{e} < \hat{h} < \hat{f} \). In our calculation a class of operators

\[
T_{(\alpha)} = (E_q^2(\eta \hat{e}^2))^{-1} E_q^2(q^{2\alpha} \eta \hat{e}^2)
\]

satisfying

\[
(E_q^2(\eta \hat{e}^2))^{-1} q^{\alpha \hat{h}} (E_q^2(\eta \hat{e}^2)) = T_{(\alpha)} q^{\alpha \hat{h}}, \quad T_{(\alpha+\beta)} q^{(\alpha+\beta)\hat{h}} = T_{(\alpha)} q^{\alpha \hat{h}} T_{(\beta)} q^{\beta \hat{h}},
\]

play an important role. To evaluate \( q \to 1 \) limiting value of the operator \( T_{(\alpha)} \), we use the identity

\[
E_q^2(q^2 \eta \hat{e}^2) - E_q^2(q^{-2} \eta \hat{e}^2) = \eta (q^2 - q^{-2}) \hat{e}^2 E_q^2(\eta \hat{e}^2),
\]

which follows from the series (2.12). The identity (2.16) may be rephrased as

\[
T_{(1)} - T_{(-1)} = \eta (q^2 - q^{-2}) \hat{e}^2.
\]

Evaluating term by term, the limiting values of \( T_{(\pm 1)}|_{q \to 1} \left( \equiv \tilde{T}_{(\pm 1)} \right) \) are found to be \( \text{finite} \); and, for these finite operators the second equation in (2.15) suggests that

\[
\tilde{T}_{(\pm \alpha)} = (\tilde{T}_{(\pm 1)})^{\alpha},
\]

where \( \tilde{T}_{(\alpha)} = \lim_{q \to 1} T_{(\alpha)} \). Writing \( \tilde{T}_{(\pm 1)} = \tilde{T}^{\pm 1} \) henceforth, we immediately observe that in the \( q \to 1 \) limit, the identity (2.17) assumes to the form

\[
\tilde{T} - \tilde{T}^{-1} = 2 \eta \hat{e}^2,
\]

which may be solved as

\[
\tilde{T}^{\pm 1} = \pm \eta \hat{e}^2 + \sqrt{1 + h^2 \eta^2}.
\]
This is our crucial result. Two other operator identities playing key roles are listed below:

\[
\hat{f} e^{2n} = e^{2n} \hat{f} - \frac{q}{q+1} \{n\}_{q^2} e^{2n-1} \hat{f} - \frac{1}{q+1} \{n\}_{q^{-2}} e^{2n-1} \hat{f}^{-1},
\]

\[
\hat{f}^2 e^{2n} = e^{2n} \hat{f}^2 + \frac{q-1}{q+1} \{n\}_{q^2} e^{2n-1} \hat{f} - q^{-1} \frac{q-1}{q+1} \{n\}_{q^{-2}} e^{2n-1} \hat{f}^{-1} \hat{f}
\]

\[
+ \frac{q}{q+1} \left( \frac{1}{\omega} \{n\}_{q^4} - q^2 \frac{q-1}{q+1} \{n\}_{q^{-2}} \right) e^{2(n-1)} \hat{f}^2
\]

\[
- \frac{1}{q+1} \left( \frac{1}{\omega} \{n\}_{q^{-4}} - q^{-2} \frac{q-1}{q+1} \{n\}_{q^{-2}} \right) e^{2(n-1)} \hat{f}^{-2}
\]

\[
- \frac{q}{(q+1)^3} \{q\{n\}_{q^2} + \{n\}_{q^{-2}}\} e^{2(n-1)},
\]

where \( \{x\}_q = (1 - q^x)/(1 - q) \) and \( \hat{t}^{\pm 1} = q^{\pm h} \). Using the above two identities systematically and passing to the limit \( q \to 1 \), it may be shown that in our construction of the operator \( \tilde{R}_h^{1/2} \) via \( \text{2.13} \), all singularities cancel yielding a well-defined answer

\[
\tilde{R}_h^{1/2} = \begin{pmatrix}
\tilde{T} & \hbar \tilde{T}^{-1} & -\hbar \tilde{T} + \frac{\hbar}{4} (\tilde{T} - \tilde{T}^{-1}) \\
0 & 1 & -\hbar \tilde{T}^{-1} e \\
0 & 0 & \tilde{T}^{-1}
\end{pmatrix},
\]

where \( \tilde{H} = \frac{1}{2} (\tilde{T} + \tilde{T}^{-1}) \) \( h = \sqrt{1 + \hbar^2 e^{-1}} \) \( h \). One way of interpreting \( \text{2.23} \) is to consider it a recipe for obtaining the finite dimensional \( R_h \) matrices of the \( U_h(osp(2\mid 1)) \) algebra. For instance, using the classical \( j = 1 \) representation given in \( \text{1.2} \) we obtain the \( R_h^{1/2;1} (= \tilde{R}_h^{1/2;1}) \) as follows:

\[
\tilde{R}_h^{1;1} = \begin{pmatrix}
1 & 0 & h & 0 & h^2 & 0 & h & 0 & h^2 & 0 & -2h & 0 & h^2 & 0 & h^3 \\
0 & 1 & 0 & h & 0 & 0 & h & 0 & h^2 & 0 & -h & 0 & h^2 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h^2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2h & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -h & 0 & h^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -h & 0 & h^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -h & 0 & h^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & h & 0 & h^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -h & 0 & h^2 & 0 & -h & 0 & h^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -h & 0 & h^2 & 0 & -h & 0 & h^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -h & 0 & h^2 & 0 & -h & 0 & h^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -h & 0 & h^2 & 0 & -h & 0 & h^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -h & 0 & h^2 & 0 & -h & 0 & h^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -h & 0 & h^2 & 0 & -h & 0 & h^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -h & 0 & h^2 & 0 & -h & 0 & h^2 & 0
\end{pmatrix}.
\]
The matrix \((2.23)\) may also be interpreted as the \(L\) operator of the \(U_h(osp(2|1))\) algebra. To this end, we first use the following invertible map of the quantum \(U_h(osp(2|1))\) algebra \((2.1)\) on the classical algebra \((1.1)\):

\[
E = e, \quad H = \hat{H}, \quad F = f + \frac{\hbar}{4} \left( \frac{T - 1}{T + 1} \right) e - \frac{\hbar}{2} \left( \frac{T - 1}{T + 1} \right) e h, \quad T = \hat{T}, \quad Y = F^2. \quad \text{(2.25)}
\]

The map \((2.25)\) satisfies the algebraic relations \((2.1)\). The general structure of maps relating the \(U_h(osp(2|1))\) algebra, obtained by quantizing the classical \(r_2\) matrix, and the classical \(U(osp(2|1))\) algebra has been discussed before \([6]\). Using the map \((2.25)\) the operator \((2.23)\) may be recast as

\[
L \equiv R_\hbar^{\frac{1}{2}j} = \begin{pmatrix}
T & h T^{\frac{1}{2}} E & -h H + \frac{\hbar}{4} (T - T^{-1}) \\
0 & 1 & -h T^{\frac{1}{2}} E \\
0 & 0 & T^{-1}
\end{pmatrix}. \quad \text{(2.26)}
\]

The above \(L\) operator allows immediate construction of the full Hopf structure of the Borel subalgebra of the \(U_h(osp(2|1))\) algebra via the standard FRT formalism \([11]\). The algebraic relations for the generators \((E, T^{\pm 1}, H)\) of the Borel subalgebra is given by

\[
R_\hbar^{\frac{1}{2}j} L_1 L_2 = L_2 L_1 R_\hbar^{\frac{1}{2}j}, \quad \text{(2.27)}
\]

where \(Z_2\) graded tensor product has been used in defining the operators: \(L_1 = L \otimes I, L_2 = I \otimes L\). The coalgebraic properties of the said Borel subalgebra may be succinctly expressed as

\[
\Delta(L) = L \otimes L, \quad \varepsilon(L) = I, \quad S(L) = L^{-1}, \quad \text{(2.28)}
\]

where \(L^{-1}\) is given by

\[
L^{-1} = \begin{pmatrix}
T^{-1} & -h T^{-\frac{1}{2}} E & h H + \frac{\hbar}{4} (T - T^{-1}) \\
0 & 1 & h T^{\frac{1}{2}} E \\
0 & 0 & T
\end{pmatrix}. \quad \text{(2.29)}
\]

This completes our construction of the Hopf structure of the Borel subalgebra of the \(U_h(osp(2|1))\) algebra, obtained by deforming the \(r_2\) matrix, by employing the contraction scheme described earlier. Our recipe \((2.13)\) for obtaining \(R_\hbar^{\frac{1}{2}j}\) matrices for \(j_1 \otimes j_2\) representation of the \(U_h(osp(2|1))\) algebra may be continued arbitrarily. The matrices such as \(R_\hbar^{\frac{1}{2}j}\) may be interpreted as higher dimensional \(L\) operators \([13]\) obeying duality relations with higher representations of \(T\) matrices.

### 3 A nonlinear realization of the quantized \(U_h(osp(2|1))\) algebra corresponding to the \(r_1\) matrix

The classical \(r_1\) matrix has been quantized earlier \([3]\) using the inclusion \(sl(2) \subset osp(2|1)\). These authors have expressed the resultant triangular deformed \(osp(2|1)\) algebra in terms of the classical basis set. On the other hand Ohn \([12]\) employed a nonlinear basis set to formulate the Jordanian deformed \(U_h(sl(2))\) algebra. Consequently, Ohn’s \(U_h(sl(2))\) algebra \([12]\) do not directly appear as a Hopf subalgebra of the deformed \(osp(2|1)\) considered in Ref. \([3]\). Moreover, if the algebraic relations are described in terms of the undeformed classical basis set, the coproduct structures tend to be complicated in nature.

In our work, we present the quantized Hopf structure corresponding to the classical \(r_1\) matrix in terms of nonlinear basis elements. Our algebra explicitly includes Ohn’s \(U_h(sl(2))\) algebra as a Hopf subalgebra.
The coproduct structure we obtain is considerably simple. To distinguish the deformed \(osp(2|1)\) algebra, its generators and the deformation parameter considered in the present Section from the corresponding objects presented in Section 2, we express them in boldfaced notations. The quantized Hopf algebra \(U_h(osp(2|1))\) corresponding to the classical \(r_1\) matrix is generated by the elements \((H, E, F, X, Y)\). Their classical analogs are \((h, e, f, b, \pm)\) respectively. The elements \(T^{\pm 1} = \exp(\pm h X)\) are also introduced. The deformation parameter is denoted by \(h\). The Hopf structure of the \(U_h(osp(2|1))\) algebra is obtained by maintaining the following properties: (i) In the classical limit the quantum coproduct map conforms to the classical cocommutator. (ii) The coproduct map is a homomorphism of the algebra, and it satisfies the coassociativity constraint. (iii) Generator \(X\) is the only primitive element. The commutation relations of the \(U_h(osp(2|1))\) algebra reads

\[
\begin{align*}
[H, E] & = \frac{1}{2}(T + T^{-1})E, \\
[H, F] & = -\frac{1}{4}(T + T^{-1})F - \frac{1}{4}F(T + T^{-1}) - \frac{h}{8}((T - T^{-1})H + H(T - T^{-1}))E \\
& \quad - \frac{h}{8}E((T - T^{-1})H + H(T - T^{-1})), \\
\{E, F\} & = -\frac{1}{4}(T + T^{-1})H - \frac{1}{4}H(T + T^{-1}), \quad [H, T^\pm] = T^{\pm 2} - 1, \\
[H, Y] & = -\frac{1}{2}(T + T^{-1})Y - \frac{1}{2}Y(T + T^{-1}), \quad [T^\pm, Y] = \pm \frac{h}{2}(T^\pm H + HT^\pm), \\
E^2 & = \frac{1}{2h}(T - T^{-1}), \quad [Y, E] = F, \quad [T^\pm, F] = \pm \frac{h}{2}(T^{\pm 2} + 1)E, \\
F^2 & = -Y + \frac{h}{8}(T - T^{-1})H^2 + \frac{h}{4}(T - T^{-1})EF + \frac{3h}{16}(T^2 - T^{-2})H + \frac{h}{4}(T - T^{-1}) \\
& \quad + \frac{9h}{128}(T - T^{-1})^3, \\
\{F, Y\} & = \frac{h}{4}(T - T^{-1})F + \frac{h}{2}(T - T^{-1})EF - \frac{h^2}{4}EH^2 - \frac{3h^2}{8}(T + T^{-1})EH - \frac{h^2}{2}E \\
& \quad - \frac{15h^2}{64}(T - T^{-1})^2E
\end{align*}
\]  

(3.1)

and the corresponding coalgebraic structure is given by

\[
\begin{align*}
\Delta(H) & = H \otimes T + T^{-1} \otimes H, \\
\Delta(E) & = E \otimes T^{-1/2} + T^{1/2} \otimes E, \\
\Delta(F) & = F \otimes T^{1/2} + T^{-1/2} \otimes F + \frac{h}{4}T^{-1}E \otimes \left(T^{-1/2}H + HT^{-1/2}\right) - \frac{h}{4}\left(T^{1/2}H + HT^{1/2}\right) \otimes TE, \\
\Delta(T^\pm) & = T^\pm \otimes T^\pm, \\
\Delta(Y) & = Y \otimes T + T^{-1} \otimes Y, \\
\varepsilon(H) & = \varepsilon(E) = \varepsilon(F) = \varepsilon(Y) = 0, \\
\varepsilon(T^\pm) & = 1, \\
S(H) & = -H + 2hE^2, \quad S(E) = -E, \quad S(F) = -F - \frac{h}{2}(T + T^{-1})E, \\
S(T^\pm) & = T^\mp, \quad S(Y) = -Y - hH + h^2E^2.
\end{align*}
\]  

(3.2)

All the Hopf superalgebra axioms can be verified by direct calculation. The universal \(R_h\) matrix of \(U_h(osp(2|1))\) is of the factorized form \([15]\):

\[
R_h = G_{21}^{-1}G, \quad G = \exp(h TH \otimes X),
\]  

(3.3)

which coincides with the universal \(\mathcal{R}_h\) matrix of the \(U_h(sl(2))\) subalgebra \([15]\) involving the highest weight root vector.
Before discussing the general structure of a class of invertible maps of the $U_h(\mathfrak{osp}(2|1))$ algebra on the classical $U(\mathfrak{osp}(2|1))$ algebra, we notice the comultiplication map of a set of three operators

$$T^{-1/2}E, \quad TH, \quad T^{1/2}F + \frac{\hbar}{8}T^{1/2}(T - T^{-1})E - \frac{\hbar}{2}T^{1/2}EH,$$  \hspace{1cm} (3.4)

when acted by the twist operator corresponding to the factorized form of the universal $R_h$ matrix, reduce to the classical cocommutative coproduct:

$$G\Delta(\mathcal{X})G^{-1} = \mathcal{X} \otimes I + I \otimes \mathcal{X},$$ \hspace{1cm} (3.5)

where $\mathcal{X}$ is an element of the set $\mathcal{X}$. From the commutation rules $(3.1)$ it becomes evident that the operators in the set $(3.4)$ satisfy the classical algebra generated by $(e, h, f)$ respectively. A general discussion of the invertible maps between the $U_h(\mathfrak{osp}(2|1))$ algebra based on quantization of the $r_2$ matrix, and the classical $U(\mathfrak{osp}(2|1))$ algebra was given in Ref. [6]. Our present discussion of maps interrelating the $U_h(\mathfrak{osp}(2|1))$ algebra, obtained via quantization of the $r_1$ matrix, and its classical analog $U(\mathfrak{osp}(2|1))$ follow the same pattern. The details of the construction are, however, quite different. A short description of the maps in the present case is given below.

The quantum $U_h(\mathfrak{osp}(2|1))$ algebra may be mapped on the classical $U(\mathfrak{osp}(2|1))$ algebra by using a general ansatz as follows:

$$E = \varphi_1(b_+)e, \quad H = \varphi_2(b_+)h,$$

$$F = \varphi_3(b_+)f + u_1(b_+)e + u_2(b_+)eh, \hspace{1cm} (3.6)$$

where the ‘mapping functions’ $(\varphi_1, \varphi_2, \varphi_3; u_1, u_2)$ depend only on the classical generator $b_+$. In the classical limit $\hbar \to 0$ the above functions satisfy the property: $(\varphi_1, \varphi_2, \varphi_3; u_1, u_2) \to (1, 1, 1; 0, 0)$. The operators $T^{\pm 1}$ may now be expressed as

$$T^{\pm 1} = \pm \hbar b_+ (\varphi_1(b_+))^2 + \sqrt{1 + \hbar^2 b_+^2 (\varphi_1(b_+))^4}. \hspace{1cm} (3.7)$$

Substituting the ansatz $(3.6)$ in the defining relations $(3.1)$ for the $U_h(\mathfrak{osp}(2|1))$ algebra we, for a given function $\varphi_1$, obtain a set of six nonlinear equations for four unknown functions:

$$(\varphi_1(b_+) + 2b_+ \varphi_1'(b_+)) \varphi_2(b_+) - \sqrt{1 + \hbar^2 b_+^2 (\varphi_1(b_+))^4} \varphi_1(b_+) = 0,$$

$$2b_+ \varphi_2(b_+) \varphi_3(b_+) - \varphi_2(b_+) \varphi_3(b_+) + \sqrt{1 + \hbar^2 b_+^2 (\varphi_1(b_+))^4} \varphi_3(b_+) = 0,$$

$$2b_+ \varphi_2(b_+) u_1'(b_+) + \left( \varphi_2(b_+) + \sqrt{1 + \hbar^2 b_+^2 (\varphi_2(b_+))^4} \right) u_1(b_+)$$

$$+ \hbar^2 b_+ \sqrt{1 + \hbar^2 b_+^2 (\varphi_1(b_+))^4} (\varphi_1(b_+))^3 = 0,$$

$$\left( \varphi_2(b_+) - 2b_+ \varphi_2'(b_+) + \sqrt{1 + \hbar^2 b_+^2 (\varphi_1(b_+))^4} \right) u_2(b_+) + \varphi_2(b_+) \varphi_3(b_+)$$

$$+ 2b_+ \varphi_2(b_+) u_2'(b_+) + \hbar^2 b_+ (\varphi_1(b_+))^3 \varphi_2(b_+) = 0,$$

$$\varphi_1(b_+) \left( 2u_1(b_+) + u_2(b_+) - \varphi_3(b_+) \right) + \sqrt{1 + \hbar^2 b_+^2 (\varphi_1(b_+))^4} \varphi_2(b_+) = 0,$$

$$b_+ \varphi_1(b_+) \left( 2u_1(b_+) + u_2(b_+) \right) - b_+ \varphi_1'(b_+) \varphi_3(b_+) - 2b_+ u_2(b_+) + \hbar^2 b_+ (\varphi_1(b_+))^4 = 0. \hspace{1cm} (3.8)$$

Maintaining the classical limit the above set of equations may be consistently solved as follows:

$$\varphi_2(b_+) = \frac{\sqrt{1 + \hbar^2 b_+^2 (\varphi_1(b_+))^4} \varphi_1(b_+)}{\varphi_1(b_+) + 2b_+ \varphi_1'(b_+)}$$

$$\varphi_3(b_+) = \frac{1}{\varphi_1(b_+)},$$

$$u_1(b_+) = -\frac{\hbar^2}{4} b_+ (\varphi_1(b_+))^3,$$

$$u_2(b_+) = \frac{1 - \sqrt{1 + \hbar^2 b_+^2 (\varphi_1(b_+))^4} \varphi_2(b_+)}{2b_+ \varphi_1(b_+)}. \hspace{1cm} (3.9)$$
In inverse maps expressing the classical generators in terms of the relevant quantum generators are obtained by assuming the ansatz

\[ e = \psi_1(T), \quad h = \psi_2(T)H, \]
\[ f = \psi_3(T)F + w_1(T)E + w_2(T)EH, \quad (3.10) \]

where \((\psi_1, \psi_2, \psi_3; w_1, w_2)\) are functions of \(T\) obeying the limiting property: \((\psi_1, \psi_2, \psi_3; w_1, w_2) \to (1, 1, 1; 0, 0)\) as \(h \to 0\). The differential equations obeyed by the ‘mapping functions’ introduced in \((3.10)\) read

\[
2(T^2 - 1) \psi_1'(T) \psi_2(T) + (T + T^{-1}) \psi_1(T) \psi_2(T) - 2 \psi_1(T) = 0, \\
2(T^2 - 1) \psi_2(T) \psi_3(T) - (T + T^{-1}) \psi_2(T) \psi_3(T) + 2 \psi_3(T) = 0, \\
\psi_2(T) \left(2(T + T^{-1}) w_1(T) + 4(T^2 - 1) w_1'(T) - h(T^2 - T^{-2}) \psi_3(T)\right) + 4 w_1(T) = 0, \\
(T - T^{-1}) \psi_2(T) (2T w_2'(T) - h \psi_3(T)) + h(T^2 + 1) \psi_2'(T) \psi_3(T) \\
+ \left((T + T^{-1}) \psi_2(T) - 2(T^2 - 1) \psi_2'(T) + 2\right) w_2(T) = 0, \\
2(T - T^{-1}) \psi_1(T) w_2(T) - h(T + T^{-1}) \psi_1(T) \psi_3(T) + 2h \psi_2(T) = 0, \\
(T - T^{-1}) \psi_1(T) \left(4w_1(T) + (T + T^{-1}) w_2(T) - h(T - T^{-1}) \psi_3(T)\right) \\
+ T(T - T^{-1}) \psi_1'(T) \left(2(T - T^{-1}) w_2(T) - h(T + T^{-1}) \psi_3(T)\right) = 0. \quad (3.11)\]

Treating the function \(\psi_1(T)\) as known, and maintaining the limiting properties the remaining functions may be solved uniquely:

\[
\psi_2(T) = \frac{2 \psi_1(T)}{(T + T^{-1}) \psi_1(T) + 2(T^2 - 1) \psi_1'(T)}, \quad \psi_3(T) = \frac{1}{\psi_1(T)}, \\
w_1(T) = \frac{h(T - T^{-1})}{8 \psi_1(T)}, \quad w_2(T) = h \frac{T + T^{-1} - 2 \psi_2(T)}{2(T - T^{-1}) \psi_1(T)}. \quad (3.12)\]

The general structure of the twisting elements corresponding to the given maps may be described as follows. Let \(m\) be a deformation map and \(m^{-1}\) be its inverse:

\[
m : (E, H, F) \to (e, h, f), \quad m^{-1} : (e, h, f) \to (E, H, F). \quad (3.13)\]

The classical \((\Delta_0)\) cocommutative and the quantum \((\Delta)\) non-cocommutative coproducts are related \([16]\) by the twisting element as

\[
\mathcal{G} \Delta \circ m^{-1}(\phi) \mathcal{G}^{-1} = (m^{-1} \otimes m^{-1}) \circ \Delta_0(\phi) \quad \forall \phi \in \mathcal{U}(osp(2|1)), \quad (3.14)\]

where the twisting element \(\mathcal{G} \in \mathcal{U}_h(osp(2|1)) \otimes E\) satisfies the cocycle condition

\[
(\mathcal{G} \otimes \mathcal{I}) ((\Delta \otimes \text{id}) \mathcal{G}) = (\mathcal{I} \otimes \mathcal{G}) ((\text{id} \otimes \Delta) \mathcal{G}). \quad (3.15)\]

Similarly the classical \((S_0)\) and the quantum \((S)\) antipode maps are related as follows:

\[
g S \circ m^{-1}(\phi) g^{-1} = m^{-1} \circ S_0(\phi), \quad g \in \mathcal{U}_h(osp(2|1)). \quad (3.16)\]

The transforming operator \(g\) for the antipode map may be expressed in terms of the twist operator \(\mathcal{G}\) as

\[
g = \mu \circ (\text{id} \otimes S) \mathcal{G}, \quad (3.17)\]

where \(\mu\) is the multiplication map.
The first map considered here plays a key role in the present construction. With the choice
\[ \varphi_1(b_+) = (1 - 2\hbar b_+)^{-1/4}, \quad \psi_1(T) = T^{-1/2}, \quad (3.18) \]
we obtain the following direct map
\[
E = (1 - 2\hbar b_+)^{-1/4} e, \quad H = \sqrt{(1 - 2\hbar b_+) h}, \quad T^{\pm 1} = (1 - 2\hbar b_+)^{\mp 1/2}, \\
F = (1 - 2\hbar b_+)^{1/4} f - \frac{\hbar^2}{4} b_+ (1 - 2\hbar b_+)^{-3/4} e + \frac{\hbar}{2} (1 - 2\hbar b_+)^{1/4} e h 
\]
and its inverse
\[
e = T^{-1/2} E, \quad h = TH, \quad f = T^{1/2} F + \frac{\hbar}{8} T^{1/2} (T - T^{-1}) E - \frac{\hbar}{2} T^{1/2} EH. \quad (3.19)\]

It may be observed from (3.5) that the operator \( G \) corresponding to the factorized form of the universal \( R_h \) matrix given in (3.3) plays the role of the twist operator \( G \) for the map (3.19) and its inverse. In this sense we refer to it as the 'minimal twist map'. The operator \( g \) transforming the antipode map may be, à la (3.16), explicitly evaluated in a closed form:
\[
g = \exp \left( -\frac{1}{2} TH(1 - T^{-2}) \right). \quad (3.20)\]

Combining (3.21) with the property (3.17) we now immediately obtain a disentanglement relation, which, if expressed in terms of classical generators, reads as follows:
\[
\mu \left[ \exp \left( \frac{1}{2} \hbar \otimes \ln(1 - 2\hbar b_+) \right) \right] = \exp(-\hbar b_+). \quad (3.22)\]

In the above relation \( \hbar \) may be treated as an arbitrary parameter. To our knowledge the above disentanglement formula involving the classical \( sl(2) \) generators was not observed before.

Another map, where the Cartan element \( H \) of the deformed \( U_h(osp(2|1)) \) algebra remains diagonal, is given by 'mapping functions'

\[
\varphi_1(b_+) = \frac{1}{\sqrt{1 - \frac{\hbar^2 b_+^2}{4}}}, \quad \psi_1(T) = \text{sech} \left( \frac{\hbar X}{2} \right). \quad (3.23)\]

The \( U_h(osp(2|1)) \) algebra may now be mapped on the classical \( U(osp(2|1)) \) algebra as
\[
E = \frac{1}{\sqrt{1 - \frac{\hbar^2 b_+^2}{4}}} e, \quad H = h, \quad T^{\pm 1} = \frac{1 \mp \frac{\hbar b_+}{2}}{1 \mp \frac{\hbar b_+}{2}}, \\
F = \sqrt{1 - \frac{\hbar^2 b_+^2}{4}} f - \frac{\hbar^2 b_+}{4} \left( 1 - \frac{\hbar^2 b_+^2}{4} \right)^{3/4} e - \frac{\hbar^2 b_+}{4} \sqrt{1 - \frac{\hbar^2 b_+^2}{4}} eh. \quad (3.24)\]

The inverse map now reads
\[
e = \text{sech} \left( \frac{\hbar X}{2} \right) E, \quad h = H, \quad \]
\[
f = \cosh \left( \frac{\hbar X}{2} \right) F + \frac{\hbar}{4} \sinh(hX) \cosh \left( \frac{\hbar X}{2} \right) E + \frac{\hbar}{2} \sinh \left( \frac{\hbar X}{2} \right) EH. \quad (3.25)\]
The twist operator $G$ for the map (3.24), unlike the previous example of closed-form expression for the ‘minimal twist map’ given in (3.5), may be determined only in a series:

$$G = I \otimes I + \frac{\hbar}{2} r + \frac{\hbar^2}{8} (r^2 + H \otimes X^2 + X^2 \otimes H) + O(\hbar^3),$$

where $r = H \otimes X - X \otimes H$. The corresponding transforming operator for the antipode map may be easily determined from (3.16):

$$g = 1 - \hbar X + \frac{1}{2} \hbar^2 X^2 + O(\hbar^3).$$

(3.27)

4 Conclusion

In the present work we have studied two aspects of the deformations of the $osp(2|1)$ superalgebra. Generalizing the approach in Ref. [4] we, in Section 2, have found a generic representation-independent way of extracting various structures like arbitrary finite dimensional $R_\hbar$ matrices, $L$ operators and $T$ matrices for the $U_\hbar(osp(2|1))$ algebra - obtained via quantization of the $r_2$ matrix - from the corresponding quantities of the standard $q$-deformed $U_q(osp(2|1))$ algebra. This approach may be used, for instance, to construct the higher dimensional $T$ matrix elements and the corresponding non-commutative spaces invariant under the coaction of these $T$ matrix elements. This will be elaborated elsewhere.

In Section 3 of the present work we have recast the algebra $U_\hbar(osp(2|1))$ algebra - generated by the quantization of the classical $r_1$ matrix - in terms of the nonlinear basis elements. There are several benefits of doing this. Ohn’s Jordanian deformation [12] of the $sl(2)$ algebra explicitly appear as a Hopf subalgebra of our $U_\hbar(osp(2|1))$ algebra. Moreover, our coproduct rules are considerably simple. Our approach is expected to be useful in constructing, as advocated in [13] physical many-body models of deformed integrable systems obeying the coalgebra symmetry. Explicit construction of such models based on the coproduct structures of the $osp(2|1)$ algebras will be presented elsewhere. Of particular interest are the possible physical effects arising out of the distinct coproduct structures of the two quantized algebras $U_\hbar(osp(2|1))$ and $U_\hbar(osp(2|1))$ discussed in Sections 2 and 3 respectively.

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