Abstract

We study an effective Einstein–Finsler theory on tangent Lorentz bundle constructed as a "minimal" extension of general relativity. Black ring and Kerr like ellipsoid exact solutions and soliton configurations are presented. In this endeavor the relevant metric depends not only on four dimensional spacetime coordinates and also on velocity type variables that can be interpreted as additional coordinates in the space of "extra dimensions".

Keywords: Analogous black holes, black rings, tangent Lorentz bundle, modified dispersion relations, modified gravity, relativistic Finsler spaces.

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1 Introduction

Black rings do not exist in the classical general relativity (GR). This provides a strong motivation for conducting studies in dimensions greater than four. The other motivation, equally strong, is string/M-theory. Such a theory is a promising candidate for providing a unified theory of particle interactions and aims to incorporate a consistent theory of quantum gravity.

Originally, a rotating black ring was demonstrated to exist as an exact solution of the vacuum Einstein equations in five dimensional gravity for a black hole with an event horizon topology of $S^1 \times S^2$, which is stationary and asymptotically flat [1, 2, 3]. It is an example of a metric with non–spherical horizon topology and a counterexample to black hole uniqueness theorem [4]. Since then, many new black ring solutions have been discovered, with the inverse scattering method (ISM) playing a central role in this process. The first black ring solution constructed with the ISM was that of Pomeransky and Sen'kov [5], generalizing the Emparan-Reall black ring [3] to include a second angular momentum. Following [6], other (concentric) doubly spinning, asymptotically flat black rings in five dimensions followed. These include black suns [6],
di-rings [7, 8] and bicycling black rings [9]. More recent inverse-scattering constructions include the explicit derivation of the unbalanced Pomeransky-Sen’kov black ring [10, 11], and rings on topologically non-trivial backgrounds [12, 13]. For completeness, we also state that black rings and other extended objects relevant to black ring solutions have also been constructed in supersymmetric theories [14, 15, 16, 17, 18, 19, 20, 21].

Briefly, in general terms, the line element in Finsler geometry is defined as

$$s[x] = \int d\tau F(x(\tau), \dot{x}(\tau)).$$

(1)

The Finsler fundamental generating function \(F\) is a function on the tangent bundle \(TM\) of a manifold \(M\). Given coordinates \((x^a)\) on \(M\), one defines coordinate basis of \(TM\) as \(\partial/\partial y^a\). The coordinates \((x^a, y^a)\) on \(TM\) are called total bundle coordinates. Geometry of spacetimes is generalized from the widely used notion of Finsler geometries with Euclidean signature to the Lorentzian signature [22, 23]. The geodesic equation can be written in arc length parametrization as

$$\ddot{x}^a + N^a_b(x, \dot{x})\dot{x}^b = 0,$$

(2)

where \(N^a_b\) denotes the coefficients of the the Cartan non-linear connection. In Finsler spacetime, \(N^a_b\) is not the usual Levi-Civita connection. The Cartan non-linear connection induces a unique split of the tangent bundle \(TTM\) with basis \((\delta_a, \partial/\partial y^a)\) into horizontal and vertical parts, \(TTM = HTM \oplus VTM\). The horizontal tangent bundle \(HTM\) is spanned by the vector fields

$$\{\delta_a = \frac{\partial}{\partial x^a} - N^b_a \frac{\partial}{\partial y^b}\},$$

(3)

while the vertical tangent bundle \(VTM\) is spanned by \(\partial/\partial y^a\). The dual basis of \(T^*TM\) is given by

$$\{dx^a, \delta y^a = dy^a + N^a_b dx^b\}.$$  

(4)

The tangent bundle \(TM\) of a Finsler spacetime is equipped with a metric \(g\) called the Sasaki metric, which can most conveniently be expressed as

$$g = g^F_{ab} dx^a \otimes dx^b + g^F_{ab} \delta y^a \otimes \delta y^b.$$  

(5)

An added bonus is the fact that the horizontal and vertical tangent spaces are mutually orthogonal with respect to the Sasaki metric. We adapt this formalism to our work in the following.

We study black ring solutions on the tangent bundle \(TMM^3_1\) to a 4D Lorentz manifold \(M^3_1\) with signature \((+, +, −, +)\) using the so–called anholonomic frame deformation method (AFDM). This geometric method admits generic off–diagonal exact solutions in GR and modified gravity theories (MGTs) [24, 25, 26]. Such extensions of GR are in effect Einstein–Finsler gravity (EFG) theories. Here we use the term Einstein–Finsler gravity in a more general context when GR is naturally extended from \(M^3_1\) to \(TMM^3_1\) using the same axiomatic scheme as in the Einstein gravity theory but for certain metric compatible and \(N\)–adapted linear connections defined canonically by the same metric tensor in the total space and arbitrary modified dispersion relations (MDRs). The velocity type coordinates are effectively treated as extra dimension coordinates and can be geometrized on tangent bundles endowed with nonholonomic distributions determined by MDRs.

The paper is organized as follows. In section 2 we show how MDR determine canonical lifts of the geometric/ physical objects and fundamental field equations from the base Lorentzian manifolds to total spaces of respective (co) tangent bundles. We also apply the AFDM for constructing generic off–diagonal stationary solutions and nonholonomic deformations of conventional 4D and 8D locally anisotropic spacetimes. In section 3 two classes of exact solutions for analogous black ring and Kerr black holes on tangent Lorentz bundles are constructed using the AFDM. We study off–diagonal deformations and nonlinear superpositions of the black ring and black hole metrics in section 4. Finally, section 5 is devoted to concluding remarks.
2 Gravity on Tangent Lorentz Bundles

General relativity is conventionally geometrized on a Lorentz manifold $M^3_1$, i.e., on a 4D pseudo–Riemannian spacetime, using the principle of equivalence together with the postulates of special relativity which hold true at any point in $u \in M^3_1$. The metric $g(u)$ and other fundamental fields, geometric or otherwise, depend explicitly on spacetime coordinates $u = \{(x^i, y^a) = (x^1, y^3 = t)\}$.* It is possible to formulate an Einstein like theory with geometric/ physical objects depending additionally on velocity type coordinates. The metric in such an endeavor is taken to be $g(x^i, y^a; y^{a_1}, y^{a_2})$, where $(y^{a_1}, y^{a_2})$ are fiber coordinates. Such constructions were originally elaborated upon in Finsler gravity [27]. The motivation stems from studying small contributions from quantum gravity (QG) and MGTs that can be axiomatized on the tangent bundle $TM^3_1$ in phenomenological models with modified (non–quadratic) dispersion relations (MDRs) see [22].

In order to construct generic off–diagonal solutions, it is convenient to parameterize the local coordinates on $TM^3_1$ in the form $u^a = (x^i, y^a; y^{a_1}, y^{a_2})$, where $(x^i, y^a)$ are considered for a conventional $2+2$ splitting on a base Lorentz manifold $M^3_1$ and $(y^{a_1}, y^{a_2})$ are $2+2$ coordinates for the typical fiber. Indices correspondingly take values $a_1, b_1, c_1, \ldots = 5, 6$ and $a_2, b_2, c_2, \ldots = 7, 8$. Possible dispersion relations of a MGT on $M^3_1$ are computed by perturbing the action of the theory around the Minkowski background and using Fouries transforms of type $\psi(x^i, t) = \int \frac{dp}{(2\pi)^{3/2}} \psi(p) e^{ip\cdot x}$, [27]. For light rays propagation in a modified spacetime, the MBD between the frequency $\omega$ and the wave vector $k_i \rightarrow p_i \sim \gamma^\lambda$ takes the form

$$\omega^2(x^i, y^a; x^p) = c^2 [g_{ij} \gamma^i \gamma^j]^2 + \left[1 + \frac{1}{r} q_{i_1 i_2 \ldots i_r} \gamma^{i_1} \gamma^{i_2} \ldots \gamma^{i_r} (g_{ij} \gamma^i \gamma^j)^{-r} \right],$$

where $c$ is the speed of light in vacuum and the coefficients $q_{i_1 i_2 \ldots i_r}$ are certain functions depending on space coordinates and a set of parameters $x^p$ defining a MGT model. If all $q_{i_1 i_2 \ldots i_r}$ are zero, we obtain the standard quadratic dispersion relation for vacuum waves in special relativity, which holds true at any point $(x^i, y^a) \in M^3_1$.

Let us associate to a MDR $\omega^2(x^i, y^a; x^p)$ a nonlinear quadratic element

$$ds^2 = F^2(x^i, y^a; y^{a_1}, y^{a_2}) \approx -(cdt)^2 + (g_{ij} \gamma^i \gamma^j)^2 [1 + \frac{1}{r} q_{i_1 i_2 \ldots i_r} \gamma^{i_1} \gamma^{i_2} \ldots \gamma^{i_r} (g_{ij} \gamma^i \gamma^j)^{-r}] + O(q^2). \quad (6)$$

If there are satisfied the homogeneity conditions $F(x^i, y^a; \tau y^{a_1}, \tau y^{a_2}) = \tau F(x^i, y^a; y^{a_1}, y^{a_2})$ for any $\tau > 1$, and nondegenerate positive definite Hessian $g_{ab, mn} = \frac{1}{2} \frac{\partial^2 F}{\partial y^a \partial y^m} \partial y^b$; for $z, m = 1, 2$, the value $F$ is termed the Finsler fundamental function. For a more general class of theories with arbitrary frame and coordinate transforms on $TM^3_1$, the condition of homogeneity of $F$ and positive definiteness of $g_{ab, mn}$ are not usually imposed. We can define as an effective Lagrangian $L = F^2$ on nonzero sections of $TM^3_1$ and express the corresponding Lagrange–Euler equations as semi-spray equations, [28] [29] [30]. This determines a canonical nonlinear connection we call the N–connection, the structure of which is a Whitney sum $\mathbf{N} : TT M^3_1 = h TM^3_1 \oplus v TM^3_1$, for a nonholonomic (equivalently, anholonomic, i.e nonintegrable) splitting into conventional horizontal (h), and vertical (v) subspaces. Fixing a system of local coordinates and frames with $2\cdot 2+2+2$ splitting, a N–connection is determined by its coefficients as

$$\mathbf{N} = \{N^a_i(x^k, y^b), N^{a_1}_i(x^k, y^b, y^{b_1}), N^{a_2}_i(x^k, y^b, y^{b_1}, y^{b_2}); N^a_1(x^k, y^b, y^{b_1}), N^{a_2}_a(x^k, y^b, y^{b_1}, y^{b_2}); N^{a_2}_a(x^k, y^b, y^{b_1}, y^{b_2})\}.$$  

We can associate to $\mathbf{N}$ a so–called N–adapted dual basis

$$e^a = (e^i \rightarrow dx^i, e^a = dy^a + N^a_i dx^i, e^{a_1} = dy^{a_1} + N^{a_1}_i dx^i + N^a_i dy^a, e^{a_2} = dy^{a_2} + N^{a_2}_i dx^i + N^a_i dy^a + N^{a_2}_a dy^{a_1}). \quad (7)$$

Any metric $g = g_{ab}(u^a) e^a \otimes e^b$ can be written as a distinguished metric (d–metric) with respect to a N–adapted coframe [4].

$$g = g_a(x^i) dx^i \otimes dx^i + g_{a_1}(x^i, y^b) e^a \otimes e^a + g_{a_2}(x^i, y^b, y^{b_1}) e^{a_1} \otimes e^{a_1} + g_{a_2}(x^i, y^b, y^{b_2}) e^{a_2} \otimes e^{a_2}. \quad (8)$$

*Indices are labelled in the form $i, j, k, \ldots = 1, 2, 4$ and $i, j, k, \ldots = 1, 2, a, b, c, \ldots = 3, 4$ corresponding to signature $(+, +, -, +)$, and $y^5$ is fixed as the timelike coordinate. Coordinates are also split into either $3 + 1$ or $2 + 2$ splitting as needed.
As alluded to previously, one does not work in Finsler like gravity theories with the Levi–Civita connection \( \nabla \) because it is not adapted to the N–connection splitting. Nevertheless, it is always possible to introduce an "auxiliary" canonical distinguished connection (d–connection) \( \hat{\nabla} \) with a distortion relation \( \hat{\nabla} = \nabla + \hat{\nabla} \) when both connections and the distorting tensor \( \hat{\nabla} \) are uniquely determined by data \((g, N)\) following the conditions that \( \nabla g = 0 \) with zero torsion \( \nabla T \) of \( \nabla \) and, respectively, \( \hat{\nabla} g = 0 \) and the "pure" h- and v–components of torsion \( \hat{T} \) of \( \hat{\nabla} \) are zero. In general, there are nonzero \( h \) and \( v \) components of \( \hat{T} \) but such values are induced by anholonomy coefficients \( W^{\alpha}_{\beta\gamma}[N] \) for

\[
\varepsilon_\alpha \varepsilon_\beta - \varepsilon_\beta \varepsilon_\alpha = W^{\alpha}_{\beta\gamma}[N].
\]

The main idea of the so–called anholonomic frame deformation method (AFDM) \cite{24} is to consider \( \hat{\nabla} \) instead of \( \nabla \) and find solutions of the nonholonomically deformed Einstein equations

\[
\hat{R}_{ij} = \Lambda g_{ij}, \hat{R}_{ab} = \Lambda g_{ab}, \hat{R}_{a1b1} = \Lambda g_{a1b1}, \hat{R}_{a2b2} = 2 \Lambda g_{a2b2},
\]

where \( \hat{R}_{\alpha\beta} \) is the Ricci tensor of \( \hat{\nabla} \) and \( \Lambda, \frac{1}{2} \Lambda, \frac{3}{2} \Lambda \) are effective cosmological constants. The canonical d–connection allows us to decouple the nonlinear system of partial differential equations, PDE, \(10\) for off–diagonal ansatz depending on all spacetime and extra–dimension coordinates with respect to N–adapted frames \(7\). We can integrate the decoupled system of PDE in very general forms with Killing and non–Killing symmetries depending on generating and integration functions and parameters, nontrivial effective sources from extra dimensions and matter field interactions, or QG and MGTs terms. If needed, one can impose additional (Levi–Civita, LC) constraints, \( \hat{T}_{\hat{\nabla} \to 0} \), and extract LC–configurations. Even for the zero torsion constraints, such solutions are, in general, off–diagonal because the anholonomy coefficients \( W^{\alpha}_{\beta\gamma}[N] \) are not zero. The fundamental priority of the AFDM is that it admits exact solutions in GR and MGTs with nontrivial N–connection structure and generic off–diagonal interactions.

### 2.1 Generic off–diagonal ansatz for stationary exact solutions

In this subsection, we study the decoupling property of equations \(10\) for metrics which are stationary in the 4D horizontal part and with three Killing symmetries on \( \partial_3 = \partial_t, \partial_b, \partial_7 \) in the total space. We take the following parameterizations of the N–adapted coefficients in \(8\) and \(7\):

\[
g_1 = \epsilon_1 e^{g(x^3)}, \quad g_2 = h_3(x^k, y^4), \quad g_4 = h_4(x^k, y^4);
\]

\[
g_5 = h_5(x^k, y^4, y^6), \quad g_6 = h_6(x^k, y^4, y^6); \quad g_7 = h_7(x^k, y^4, y^6, y^8), \quad g_8 = h_8(x^k, y^4, y^6, y^8),
\]

\[
N^{\alpha}_{\beta} = n_1(x^k, y^4), \quad N^2 = w_1(x^k, y^4);
\]

\[
N^4 = n_2(x^k, y^4, y^6), \quad N^6 = w_2(x^k, y^4, y^6); \quad N^8 = n_4(x^k, y^4, y^6, y^8);
\]

\[
N^i = n_i(x^k, y^4, y^6, y^8), \quad N^{a_i} = n_i(x^k, y^4, y^6, y^8) \quad \text{for} \quad i = 1, 2; a = 3, 4; a_1 = 5, 6.
\]

Employing the ansatz \(11\) for the d–metric \(8\), the modified Einstein equations \(10\) transform into a system of decoupled PDE,

\[
\epsilon_1 \partial^2_x g + \epsilon_2 \partial^2_y g = 2 \Lambda,
\]

\[
\partial_4 \phi \partial_3 h_3 = 2 h_4 h_3 \Lambda, \quad \partial^2_x n_1 + \gamma \partial_4 n_1 = 0, \quad \beta w_1 - \alpha = 0,
\]

\[
\partial_6 \phi \partial_5 h_5 = 2 h_6 h_5 \Lambda, \quad \partial^2_x n_1 + \gamma \partial_6 n_1 = 0, \quad \beta \partial_4 w_1 \partial_4 - \alpha = 0,
\]

\[
\partial_8 \phi \partial_7 h_7 = 2 h_8 h_7 \Lambda, \quad \partial^2_x n_1 + \gamma \partial_8 n_1 = 0, \quad \beta \partial_4 w_1 \partial_4 - \alpha = 0.
\]

where the second partial derivatives are defined as \( \partial^2_x a = \partial^2_a \partial x^1 \partial x^1 \), and similarly for the other indices \( i_1 = (i, a), i_2 = (i_1, a_1) \). Additional LC–conditions can be imposed for zero torsion, leading to the following
The coefficients in above equations are given by formulas

\[ \phi = \ln \left| \frac{\partial h_3}{h_3 h_4} \right|, \gamma = \partial_4 \ln \left| \frac{h_3^{3/2}}{h_4} \right|, \alpha_i = \frac{\partial_4 h_3}{h_3} \partial_i \phi, \beta = \frac{\partial_4 h_3}{h_3} \partial_4 \phi, \]

\[ 1\phi = \ln \left| \frac{\partial h_5}{h_5 h_6} \right|, 1\gamma = \partial_6 \ln \left| \frac{h_5^{3/2}}{h_6} \right|, 1\alpha_i = \frac{\partial_6 h_5}{h_5} \partial_1 \phi, 1\beta = \frac{\partial_6 h_5}{h_5} \partial_6 \phi, \]

\[ 2\phi = \ln \left| \frac{\partial h_7}{h_7 h_8} \right|, 2\gamma = \partial_8 \ln \left| \frac{h_7^{3/2}}{h_8} \right|, 2\alpha_i = \frac{\partial_8 h_7}{h_7} \partial_2 \phi, 2\beta = \frac{\partial_8 h_7}{h_7} \partial_8 \phi, \]

where

\[ \partial_4 h_3 \neq 0, \partial_6 h_5 \neq 0, \partial_8 h_7 \neq 0 \text{ if (respectively) } \Lambda \neq 0, 1\Lambda \neq 0 \text{ and } 2\Lambda \neq 0. \quad (13) \]

We find exact solutions by integrating "step by step" the system (12) for arbitrary generating functions \( \phi(x^k, y^4), 1\phi(x^k, y^4, y^6), 2\phi(x^k, y^6, y^8) \) (or \( \Phi = e^{\phi}, 1\Phi = e^{1\phi}, 2\Phi = e^{2\phi} \)) and, respectively, nonzero \( \partial_4 \phi, \partial_6 1\phi, \partial_8 2\phi \); arbitrary integration functions \( n_1(x^k), n_2(x^k), n_1(x^k, y^4), n_2(x^k, y^4), n_1(x^k, y^6), n_2(x^k, y^6) \). Such generic off-diagonal solutions are parameterized by quadratic elements

\[ ds^2 = e^{\eta(x^k)}[\epsilon_1(dx^1)^2 + \epsilon_2(dx^2)^2] + \frac{\Phi^2}{4\Lambda}[dt + (1n_1 + 2n_2) \int dy^1(\frac{\partial_1 \Phi}{\Phi})^2 dx^1]^2 + (\partial_4 \Phi)^2 [dy^4 + \frac{\partial_4 \Phi}{\partial_4 \Phi} dx^4]^2 \]

\[ + \frac{(1\Phi)^2}{4\Lambda} \left[ dy^5 + \left( \frac{1}{n_1} + \frac{1}{n_2} \right) \int dy^6(\frac{\partial_1 1\Phi}{1\Phi})^2 dx^1 \right]^2 + (\partial_6 1\Phi)^2 \frac{1\Phi(1\Phi)}{6\Lambda(1\Phi)^2} \left[ dy^6 + \frac{\partial_1 1\Phi}{\partial_6 1\Phi} dx^1 \right]^2 \]

\[ + \frac{(2\Phi)^2}{4\Lambda} \left[ dy^7 + \left( \frac{2}{n_1} + \frac{2}{n_2} \right) \int dy^8(\frac{\partial_2 2\Phi}{2\Phi})^2 dx^2 \right]^2 + (\partial_8 2\Phi)^2 \frac{2\Phi(2\Phi)}{2\Lambda(2\Phi)^2} \left[ dy^8 + \frac{\partial_2 2\Phi}{\partial_8 2\Phi} dx^2 \right]^2, \]

for \( x^{i_2} = (x^{i_1}, y^{a_1}) = (x^i, y^a, y^{a_1}). \) The solutions (14) are, in general, with nonholonomically induced torsion.

We extract pseudo-Riemannian metrics on \( TM^3_1 \) as solutions for Einstein tangent bundles, i.e. for (10) with \( \tilde{h}/z_{\rightarrow} \rightarrow 0 \) if we fix nonholonomic distributions subject to the LC-conditions of zero torsion when the coefficients of the above d-metrics are subjected to additional conditions and generated in the form

\[ \Phi = \Phi, \text{ for } \partial_4 \partial_4 \Phi = \partial_4 \Phi; 1\Phi = 1\Phi, \text{ for } \partial_6 \partial_1 \Phi = \partial_1 \Phi, \partial_6 \Phi = \Phi; 2\Phi = 2\Phi, \text{ for } \partial_2 \partial_2 \Phi = \partial_2 \Phi; \]

\[ 2n_i = 0, \ 2n_{i_2} = 0; \ 1n_i = \partial_1 n(x^k), \ 1n_{i_2} = \partial_{i_2} n(x^k, y^a, y^{a_1}); \]

\[ w_i = \partial_4 \Phi, \partial_6 \Phi = \partial_4 \Phi, \partial_2 \Phi = \partial_2 \Phi; \]

\[ 1w_i = \partial_i 1\Phi, \partial_6 1\Phi = \partial_6 1\Phi, \partial_2 1\Phi = \partial_2 1\Phi; \]

\[ h_3 = \frac{\Phi^2}{4\Lambda}, h_4 = (\partial_4 \Phi)^2/\Lambda, \]

\[ h_5 = \frac{\Phi^2}{4\Lambda}, h_6 = (\partial_6 \Phi)^2/\Lambda, h_7 = 2\Phi^2/4 2\Lambda \]

The generating and integration functions in above formulas define integral varieties and LC-subvarieties of generalized Einstein–Finsler spaces. Such functions can be smooth , or with singular behaviour. For stationary configurations and certain Killing symmetries, they can be determined from asymptotic conditions for well-defined systems of coordinates and possible limits to diagonal configurations with observational effects on the base 4D effective spacetime. We note that the AFDM allows us to construct generic off-diagonal
non–Killing solutions depending on all (higher dimension) spacetime coordinates, \[24\] \[28\] \[29\] \[30\]. In the next section, we elaborate on the conditions when solutions of type \[11\] with possible LC–constraints define black ring configurations with self–consistent imbedding into velocity black ellipsoid/torus backgrounds.

The integral varieties of solutions determined by \(d\)--metric of type \[14\] (with possible constraints \[15\]) allows us to describe nonholonomic deformations of locally anisotropic 8D spacetimes of "higher symmetry" (for instance, with \(2+2\) Killing vectors) into configurations with "lower symmetry" (for instance, with \(1+1\) Killing vectors). In such cases, certain well defined physical models with higher symmetry can be extended to less symmetric ones with certain conventional "polarization functions" and effective interaction constants. In explicit form, we state these conditions for \(N\)--adapted transforms

\[
[g_\alpha(x^k, y^{a_1}), \tilde{N}(x^k, y^{a_1})] \rightarrow [g_\alpha = \eta_\alpha(x^k, y^{a_1}, y^8)\tilde{g}_\alpha, N = \tilde{N} + \tilde{N}(x^k, y^{a_1}, y^8)],
\]

where \(\eta_\alpha\) and \(\tilde{N}\) are respective "gravitational polarizations" of the coefficients of \(d\)--metric and \(N\)--connection structures.\(^2\) Applying the AFDM, we construct nonholonomic deformations when the "prime" data \([\tilde{g}_\alpha, \tilde{N}]\) may be, or not, a solution of some gravitational field equations but, positively, the "target" data \([g_\alpha = \eta_\alpha\tilde{g}_\alpha, N = \tilde{N} + \tilde{N}]\) are determined by some exact solutions of generalized Einstein–Finsler equations.

For certain physically important cases, we can consider that \([\tilde{g}_\alpha, \tilde{N}]\) are just certain trivial lifts from a 4D stationary vacuum spacetime with two Killing symmetries, when \(\Lambda = 1\Lambda = 2\Lambda = 0\). The target solutions \([g_\alpha, N]\) for an ansatz \[11\] and a \(d\)--metric \[8\] belong, in general, to an integral variety with nonzero effective cosmological constants \(\Lambda, 1\Lambda, 2\Lambda\) (up to certain classes of frame transforms and re–definitions of generating functions and effective sources of gravitational field equations). We can provide similar physical interpretations both for the prime and target metrics in a nonholonomic deformation

\[
[g_\alpha, \tilde{N}] \rightarrow [g_\alpha(\varepsilon) = \eta_\alpha\tilde{g}_\alpha, N(\varepsilon) = \tilde{N} + \tilde{N}(\varepsilon)], \text{ with } \eta_\alpha = 1 + \varepsilon \chi_\alpha \text{ and } \tilde{N}(\varepsilon) = \varepsilon \tilde{N},
\]

for a small parameter \(0 < \varepsilon \ll 1\). Such nonholonomic transforms may not have a smooth limit \(\varepsilon \rightarrow 0\) \(g(\varepsilon) \rightarrow \tilde{g}\) if, for instance, \(\tilde{g}\) is a solution of vacuum Einstein equations (with possible trivial extensions on total space \(TM^3\)) but \(g(\varepsilon)\) is a solution of MGTs with nontrivial sources. In general, \([g_\alpha, \tilde{N}]\) and \([g_\alpha(\varepsilon), N(\varepsilon)]\) may describe nonholonomic configurations with different topology and symmetries and/or very different geometric/physical models. There are necessary additional analysis of the properties of nonholonomic deformations of certain prime geometric data into nonholonomic ones. For small values \(\varepsilon\), it is possible to state such conditions on generating and integration functions and, for instance, for effective de Sitter configurations on \(TM^3\), when two configurations may have analogous behaviour but with certain deformed symmetries and effective nonlinear "polarization" functions.

## 3 Black Ring and Kerr Solutions

The black ring and black hole solutions constructed and reviewed in \[3\] \[4\] occur in extra dimension spacetimes, for instance, in (super) string / gravity and Kaluza–Klein theories. In this section we show how such solutions can be constructed on \(TM^3\). These constructions have a different physical interpretation because higher dimension coordinates are velocity type coordinates and generalize on respective tangent bundles certain 4D gravity theories, in particular, the Einstein gravity.

### 3.1 Neutral black ring solutions on tangent Lorentz bundle

We introduce 5D ring coordinates as in sections 2–3 of \[3\] and extend them trivially to 8D, \(x^1(x), x^2(y), y^3 = t, y^4 = \chi, y^5 = \psi, y^6, y^7, y^8\), where

\[
x^1(x) = \int dx|G(x)|^{-1/2} \quad \text{and} \quad x^2(y) = \int dy|G(y)|^{-1/2}, \text{ for } G(\xi) = (1 - \xi^2)(1 + \nu\xi),
\]

\(^2\)We do not consider summation on indices for values of type \(\eta_\alpha\tilde{g}_\alpha\)
and the constants $\lambda, \nu, \dot{c}$ and function $F(\xi) = 1 + \lambda \xi$ are related via $\dot{c} := (\xi(\lambda - \nu)(1 + \lambda)/(1 - \lambda)$ in order to get black ring configurations for $0 < \nu \leq \lambda < 1$. The heuristic construction of a 5D black ring is $TM^3_1$ can be considered as a boosted black string with velocity like coordinate $\psi$ (on such strings, see [30] and references therein) bent into a circular shape and imbedded into a total space with curved fiber subspace determined by a velocity black hole configuration. The nontrivial coefficients for a "prime" metric $\tilde{g}_{\alpha \beta}$ are chosen

$$
\begin{align*}
\dot{g}_1(x^k) &= e^{\varphi(x^1,x^2)} \dot{g}_2(x^k) = -e^{\varphi(x^1,x^2)} \dot{h}_3(x^k) = A + \frac{(AB)^2}{AB^2 - S}, \\
\dot{h}_5(x^k) &= AB^2 - S, \quad \dot{h}_6 = 1, \dot{h}_7 = \pm 1, \dot{h}_8 = 1, N_3^5 = \dot{w}_3(x^k) = \frac{AB}{AB^2 - S}.
\end{align*}
$$

for $A = -F(x^1)/F(x^2), B = \dot{c}r(1 + y(x^2))/F(x^2), S = -\dot{c}^2 F(x^1)G(x^2)/F(x^2)[x(x^1) - y(x^2)^2]$, where $\dot{r} = \text{const}$ (the $\dot{r}$ is used instead of the radial constant $R$ in [11]; see there the section 3.1 related to the vacuum solution of 5-d Einstein equations by formula (14)). The function $\dot{g}(x^1, x^2)$ is derived from the relation

$$
\begin{align*}
e^{\varphi}[(dx^1)^2 - (dx^2)^2] &= \frac{\dot{r}^2}{(x - y)^2} F(x)[\frac{dx^2}{G(x)} - \frac{dy^2}{F(y)}] \\
\text{and } A(dt - B\psi)^2 - S\psi^2 &= \dot{h}_5(x^k)dt^2 + \dot{h}_3(x^k)[\delta \psi + \dot{w}_3(x^k)dt]^2.
\end{align*}
$$

We obtain an example of d–metric (17) when the coefficients depend only on two coordinates $x^k (x, y)$,

$$
\begin{align*}
ds^2 = e^{\varphi(x^k)}[(dx^1)^2 - (dx^2)^2] + \dot{h}_3(x^k)dt^2 + \dot{h}_4(x^k)d\chi^2 + (dy^5)^2 \pm (dy^6)^2 \pm (dy^7)^2 + \dot{h}_5(x^k)[\psi + \dot{w}_3(x^k)dt]^2,
\end{align*}
$$

where we have changed $y^5$ into $y^8 = \psi$ (this is convenient for further applications of the AFDM). This metric is mathematically equivalent to that for the neutral black ring solution with trivial extension from 5D to 8D in variables on $TM^3_1$. The sign $\pm$ before $dy^6$ and/or $dy^7$ reflects two possibilities to lift geometric objects on $M^3_1$ in the total space. Nevertheless, this is not a model of "two time" physics [31] [32] [33] if the geometric/physical objects on $TM^3$ are considered as certain canonical lifts (for instance, of Sasaki type [34], from $M^3_1$ with one time like coordinate. In our approach, the extra dimensional coordinate $y^8 = \psi$ is of velocity type, i.e. our analogous black ring solution is for a model of "phase" space on tangent Lorentz bundle. We conclude that the class of solutions [19] constitute an example of metrics of type [13] which are degenerate in the sense that the conditions [13] are not satisfied. This is a possibility if $\partial_3 \dot{h}_4 = 0$ and $\Lambda = 0$ for vacuum solutions.

### 3.2 Black ring solutions in velocity Kerr backgrounds

We provide an example when the metric (19) is nonholonomically deformed by corresponding MDR (up to frame/coordinate transforms on $TM^3_1$ and/or $M^3_1$) into an analogous black ring interacting with an analogous Kerr black hole determined by velocity type coordinates [35] [36] [37]. We use $\psi$ instead of $\varphi$ for respective fiber’s signature $(+, +, +, +)$. Also, we work with corresponding N–adapted frames and systems of coordinates instead of the "standard" prolate spherical, or the Boyer–Linquist coordinates. The fiber analogs of Boyer–Linquist coordinates $(\tilde{y}^a, \tilde{y}^b)$, for $a_1 = 5, 6$ and $a_2 = 7, 8$, are defined $\tilde{y}^a = \tilde{r}, \tilde{y}^b = \tilde{y}^7(\tilde{r}, \tilde{y}), \tilde{y}^8 = \psi = y^8$, where "tilde" is used in order to emphasis that such coordinates are analogous to the ones on a fiber space. The total black hole analogous mass is $m_0$, and the analogous total angular momentum is $am_0$, for the asymptotically flat, stationary and axisymmetric Kerr in the space of

\[ \text{In this paper, we write } \chi \text{ instead the angle variable } \phi \text{ in R. Emparan, H. S. Real and others' works because in our approach the symbol } \phi \text{ is used for generating functions (we can construct in similar forms alternative classes of solutions with } y^5 = \chi, y^4 = \psi) \text{ and work with inverse functions } x(x^5) \text{ and } y(x^5) \text{ to the respective ones defined above.} \]
relativistic velocities. In such variables, the 8D vacuum metric (19) is generalized to the form

\[
ds^2 = e^\theta[(dx^1)^2 - (dx^2)^2] + h_3 dt^2 + h_4 dx^1 + (\tilde{A} - \tilde{B}/\tilde{C})(dt)^2 + \tilde{\Xi}\Delta^{-1}(d\tilde{r})^2 + \tilde{\Xi}(d\tilde{v})^2
\]

\[+ h_5 \tilde{C}(d\psi + 1\tilde{w}_3(x^k)dt + \tilde{B}^2/\tilde{C}d\tilde{r})^2
\]

\[= \dot{g}_1 dx^1 + \dot{h}_3(dt)^2 + \dot{h}_4(dx^1)^2 + \tilde{h}_5(\tilde{y}^6, \tilde{y}^7)(dt)^2 + \tilde{h}_6(\tilde{y}^6, \tilde{y}^7)(dy^6)^2
\]

\[+ \tilde{h}_7(\tilde{y}^6, \tilde{y}^7)(dy^7)^2 + \tilde{h}_8(x^k, \tilde{y}^6, \tilde{y}^7)(d\psi + \tilde{N}_5^8 dt + \tilde{N}_5^8 d\tilde{r}),
\]

where \(\tilde{\Xi}[\Delta^{-1}(d\tilde{r})^2 + (d\tilde{v})^2] = \tilde{h}_6(dy^6)^2 + \tilde{h}_7(dy^6, dy^7)(dy^7)^2\) for certain coefficients \(\tilde{h}_6(\tilde{y}^6, \tilde{y}^7)\) and \(\tilde{h}_7(\tilde{y}^6, \tilde{y}^7)\), \(\tilde{h}_5(x^k) = AB^2 - S\), and

\[\tilde{A}(\tilde{r}, \tilde{\theta}) = -\tilde{\Xi}^{-1}(\Delta - a^2 \sin^2 \tilde{\theta}), \tilde{B} = \tilde{\Xi}^{-1} a \sin^2 \tilde{\theta} \left[\tilde{A} - (\tilde{r}^2 + a^2)\right],
\]

\[\tilde{C}(\tilde{r}, \tilde{\theta}) = \tilde{\Xi}^{-1} \sin^2 \tilde{\theta} \left[\tilde{r}^2 + a^2 - \Delta a^2 \sin^2 \tilde{\theta}\right], \text{ and } \tilde{A}(\tilde{r}^2) = \tilde{r}^2 - 2m_0 + a^2, \tilde{\Xi}(\tilde{r}, \tilde{\theta}) = \tilde{r}^2 + a^2 \cos^2 \tilde{\theta}.
\]

The nonlinear quadratic line element (20) is determined by data

\[
\dot{g}_1 = e^\theta(x^k), \dot{g}_2 = -e^\theta(x^k), \dot{h}_3 = A(x^k) + \frac{[A(x^k)B(x^k)]^2}{A(x^k)B(x^k) - S(x^k)}\dot{h}_4 = \frac{\dot{r}^2 G(x^1)}{[x^1(x^1) - y(x^2)]^2},
\]

\[
\tilde{h}_5 = \tilde{A} - \tilde{B}^2/\tilde{C}, \tilde{h}_6 = \tilde{h}_6(\tilde{y}^6, \tilde{y}^7), \tilde{h}_7 = \tilde{\Xi}, \tilde{h}_8 = \tilde{h}_5\tilde{C}, \tilde{N}_5^8 = 1\tilde{w}_3(x^k), \tilde{N}_5^8 = \frac{2n_5 = \tilde{B}^2/\tilde{C},
\]

and defines solutions of vacuum Einstein equations on \(TM_3^1\). For any fixed point \((x^k, t, \chi) \in M_3^1\), the fiber coefficients (22) of metric in velocity type fiber variables \((\tilde{t}, \tilde{r}, \tilde{\theta}, \psi)\) determine an analogous Kerr like black hole with effective time like coordinate \(\tilde{y}^i = \tilde{t}\). The black ring configuration is self-consistently embedded in the velocity subspace via terms \(\tilde{h}_8 = \tilde{h}_5\tilde{C}\) and \(\tilde{N}_5^8 = 1\tilde{w}_3(x^k)\).

4 Off–diagonally Deformed Black Rings and Kerr–de Sitter Metrics

For a nonvanishing cosmological constant, the vacuum metrics (20) can be nonholonomically deformed into exact solutions of type (14) for the gravitational field equations (10). There are two varieties of solutions, the static ones exhibiting Killing symmetry in \(\partial/\partial t\) and the ones on solitonic backgrounds that depend explicitly on time. We present representative examples of both in the following. We consider \(N\)-adapted deformations (16) of the prime data (22), \(\tilde{g}_{\alpha\beta} = [\tilde{g}_1, \tilde{h}_3(\tilde{h}_a, \tilde{h}_a, \tilde{h}_a, \tilde{h}_a), \tilde{N}_5^8, \tilde{N}_5^8, \Lambda = 1\Lambda = 2\Lambda = 0\), into target data for an ansatz (11) and d–metric (15),

\[
\tilde{g}_{\alpha\beta} = [\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6, \eta_7, \eta_8, \eta_9, \eta_{10}, \eta_{11}, \eta_{12}], \eta_{13} = \tilde{N}_5^a, \eta_{14} = \tilde{N}_5^a, \eta_{15} = \tilde{N}_5^a, \eta_{16} = \delta_{\tilde{a}}^{\tilde{a}} \delta_{\tilde{a}}^{\tilde{a}} \tilde{N}_5^a + \tilde{N}_5^a;
\]

\[
\eta_2 = \tilde{N}_5^a, \eta_3 = \tilde{N}_5^a, \eta_4 = \tilde{N}_5^a, \eta_5 = \delta_{\tilde{a}}^{\tilde{a}} \delta_{\tilde{a}}^{\tilde{a}} \tilde{N}_5^a + \tilde{N}_5^a, \Lambda \neq 0, \Lambda \neq 0, 2\Lambda \neq 0,
\]

when vacuum 8D solutions in EFG are transformed into off–diagonal de Sitter configurations on \(TM_3^1\).

4.1 Stationary solutions

(A.) Off–diagonal deformations of the vacuum solutions (20) are described by target d–metrics

\[
ds^2 = e^\theta[(dx^1)^2 - (dx^2)^2] + \eta_3 \tilde{h}_3(dt + \tilde{n}_1 dx^1)^2 + \eta_4 \tilde{h}_4(dx^1 + \tilde{w}_3 dx^1)^2
\]

\[+ \eta_5(\tilde{A} - \tilde{B}^2/\tilde{C})(dt + \tilde{n}_1 dx^1)^2 + \eta_6 \tilde{\Xi}\Delta^{-1}(d\tilde{r} + \tilde{n}_1 dx^1)^2
\]

\[+ \eta_7(\tilde{d} + 2\tilde{n}_2 dx^2)^2 + \eta_8 \tilde{h}_5(\tilde{C}d\psi + 2\tilde{n}_2 dx^2 + \tilde{n}_8(x^k) dt + (\tilde{B}^2/\tilde{C}) d\tilde{r}),
\]

for \(x^2 = (x^1(x), x^2, y^3, t, y^4 = \chi, \tilde{y}^5 = \tilde{t}, \tilde{y}^6 = \tilde{r}, \tilde{y}^7 = \tilde{\theta}, \tilde{y}^8 = \psi)\). The \(\eta\)–polarizations and \(N\)–coefficients, respectively, are determined in this form:

\[\eta_1 \text{ from } e^\theta(1) = e^\theta(2), \text{ for } q(x^k) \text{ solution of the first equation in (12)};
\]

\[\eta_2 \text{ computed as } \eta_3 = \Phi^2/4A\tilde{h}_3 \text{ and } \eta_4 = (\partial_1 \Phi)^2/\Lambda \Phi^2\tilde{h}_4, \forall \Phi(x^k, y^4), \partial_4 \Phi \neq 0;
\]

\[\eta_5 \text{ as } \eta_5 = (\Phi^2/4\Lambda(\tilde{C}\tilde{A} - \tilde{B}^2)) \text{ and } \eta_6 = (\partial_1 \Phi)^2/\Lambda(\tilde{A})^2 \tilde{\Xi}, \forall \Phi(x^k, y^4, y^6), \partial_4 \Phi \neq 0;
\]

\[\eta_7 \text{ as } \eta_7 = (2\Phi^2/2\Lambda \tilde{\Xi}) \text{ and } \eta_8 = (\partial_8 \Phi^2/2\Lambda(2\Phi^2)\tilde{h}_5\tilde{C}, \forall \Phi(x^k, y^4, y^6, y^8), \partial_8 \Phi \neq 0, \text{ and}
\]

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\[ N_i^3 = \tilde{N}_i^3 = \tilde{n}_i(x^k, y^A) = \tilde{1} \tilde{n}_i(x^k) + \tilde{2} \tilde{n}_i(x^k) \int d\chi (\partial_\Phi)^2/\Phi^5, \quad N_i^4 = \tilde{N}_i^4 = \tilde{u}_i(x^k, y^A) = \partial_\Phi/\partial_4 \Phi; \]

\[ \begin{align*}
N_i^5 &= \tilde{N}_i^5 = \tilde{n}_i(x^k, y^A, y^B) = \frac{1}{2} \tilde{n}_i(x^k, y^A) + \frac{1}{2} \tilde{n}_i(x^k, y^A) \int d\tilde{\Phi}(\partial_\Phi)^2/(1\Phi)^5, \\
N_i^6 &= \tilde{N}_i^6 = \tilde{n}_i(x^k, y^A, y^B) = \partial_i \Phi^4 \Phi^2/(1\Phi)^5, \\
N_i^7 &= \tilde{N}_i^7 = \tilde{n}_i(x^k, y^A, y^B, y^C) = \frac{1}{2} \tilde{n}_i(x^k, y^A, y^B) + \frac{1}{2} \tilde{n}_i(x^k, y^A, y^B) \int d\psi(\partial_\Phi)^2/(2\Phi)^5, \\
N_i^8 &= \tilde{N}_i^8 + \tilde{n}_i(x^k, y^A, y^B, y^C, y^D) = \partial_i \Phi^4 \Phi^2/(2\Phi)^5,
\end{align*} \tag{25} \]

for \( i_1 = (i, a) = 1, 2, 3, 4 \) and \( i_2 = (i, a, a_1) = 1, 2, 3, 4, 5, 6 \), where we fixed the integration functions to get a N-adapted \( 2 + 2 + 2 + 2 \) splitting.

Geometrically, \( d \)-metrics \( (23) \) describe nonholonomic imbedding of analogous black ring and velocity Kerr metrics into more general off-diagonal configurations determined by generation functions (values \( \Phi, \Phi \) and \( \Phi \)) and integration functions of type \( \tilde{1} \tilde{n}_i, \tilde{2} \tilde{n}_i, \tilde{1} \tilde{n}_i, \tilde{2} \tilde{n}_i \), etc. The target metrics are with less symmetries but still contain Killing symmetries on deformations of the prime metrics \( (20) \). For small polarizations constant and possible polarization in the fiber space and velocity like variables generated by nonholonomic physical interpretation for generic off-diagonal solutions is lacking for general nonholonomic deformations.

Considering polarization functions with a small parameter \( \hat{\varepsilon} \), analogous off-diagonal configurations on \( \tilde{\Phi} : \Phi = \Phi, \Phi = \Phi, 2\Phi = \Phi, \Phi, \Phi \) and the integration functions are subject to the conditions \( (15) \). Introducing such values in \( (24) \) and \( (25) \), we generate LC-configurations with effective quadratic element

\[ ds^2 = e^\theta [e_1(dx^1)^2 + e_2(dx^2)^2 + \frac{\phi^2}{4\Lambda} [dt + \partial_1 (1/n)dx^1]^2 + \frac{(\partial_1 \Phi)^2}{\phi^2} [dx + (\partial_1 \tilde{A})dx^1]^2 + \frac{(2\Phi)^2}{4\Lambda} [dr + (\partial_2 \tilde{A})dx^1]^2 + \frac{(2\Phi)^2}{2\Lambda} (2\Phi^2)^2 \psi + (\tilde{A}^2)^2 dx^2]^2. \tag{26} \]

Metrics of this type define generic off-diagonal Einstein manifolds with nonzero effective cosmological constant and possible polarization in the fiber space and velocity like variables generated by nonholonomic deformations of the prime metrics \( (20) \). For small polarizations \( \eta_\alpha = 1 + \varepsilon \eta_\alpha \) and \( \tilde{N}(\varepsilon) = \varepsilon \tilde{N} \) we model analogous black ring and black holes with velocity type variables imbedded into asymptotically flat configurations on \( TM^3 \). In general, the solutions \( (20) \) cannot be diagonalized for coordinate transforms if the anholonomy coefficients \( W_{\beta\gamma} \) are not zero and the N-connection coefficients are not trivial for MDRs.

Schwarzschild configurations can be extracted from the Kerr ones if \( a = 0 \) in \( (21) \). We denote

\[ \tilde{A}(\tilde{\rho}) = \tilde{A}_{|a=0} = -\tilde{\rho}^2 \Delta, \tilde{B} = \tilde{B}_{|a=0} = 0, \tilde{C}(\tilde{\rho}, \tilde{\vartheta}) = \tilde{C}_{|a=0} = \tilde{\rho}^2 \sin^2 \tilde{\vartheta}, \text{ and } \Delta(\tilde{\rho}^2) = \tilde{\rho}^2 - 2m_0. \]

Considering polarization functions with a small parameter \( \varepsilon \), we construct the off-diagonal solutions

\[ ds^2 = e^\theta [(dx^1)^2 + (dx^2)^2 + (1 + \varepsilon \chi_6) \tilde{h}_3 [dt + \varepsilon \partial_1 (1/n)dx^1]^2 + (1 + \varepsilon \chi_6) \tilde{h}_4 [dx + (\varepsilon \partial_1 \tilde{A})dx^1]^2 + (1 + \varepsilon \chi_6) \tilde{h}_5 \tilde{C}[dx + (\varepsilon \partial_2 \tilde{A})dx^2]^2 + (1 + \varepsilon \chi_6) \tilde{C}[dx + (\varepsilon \partial_2 \tilde{A})dx^2]^2 + (1 + \varepsilon \chi_6) \tilde{h}_6 \tilde{C}[dx + (\varepsilon \partial_2 \tilde{A})dx^2]^2 + (1 + \varepsilon \chi_6) \tilde{h}_7 \tilde{C}[dx + (\varepsilon \partial_2 \tilde{A})dx^2]^2 + (1 + \varepsilon \chi_6) \tilde{h}_8 \tilde{C}[dx + (\varepsilon \partial_2 \tilde{A})dx^2]^2 + (1 + \varepsilon \chi_6) \tilde{h}_9 \tilde{C}[dx + (\varepsilon \partial_2 \tilde{A})dx^2]^2 + (1 + \varepsilon \chi_6) \tilde{h}_10 \tilde{C}[dx + (\varepsilon \partial_2 \tilde{A})dx^2]^2 \tag{27} \]
where the $\varepsilon$–polarsations and $N$–coefficients are respectively parameterized

\begin{align}
1 + \varepsilon_{X4} &= \frac{\Phi^2}{4N_h^2} \text{and} 1 + \varepsilon_{X5} = \frac{(\partial_4 \Phi)^2}{\Lambda \Phi^2} h_4; \\
1 + \varepsilon_{X6} &= (\partial_0 \Phi)^2 1 \Lambda \Phi^2 \text{ and } 1 + \varepsilon_{X7} = (\partial_0 \Phi)^2 \frac{\Delta}{\Lambda}; \\
1 + \varepsilon_{X8} &= (\partial_4 \Phi)^2 \frac{\Lambda}{\Phi^2} 2 \Lambda h_5 C; \text{ and}
\end{align}

\[N^3 = 0, N^4 = N^5 = N^6 = N^7 = N^8 = 0.\]

\[N^1 = \tilde{N}^1 = \tilde{N}^2 = \tilde{N}^3 = \tilde{N}^4 = \tilde{N}^5 = \tilde{N}^6 = \tilde{N}^7 = \tilde{N}^8 = 0.\]

Such parametric solutions are also exact for any fixed value of $\varepsilon$. The generating and integration functions are not arbitrary but restricted to satisfy certain conditions of linear approximation on $\varepsilon$.

Prescribing

\[\chi_5 = 2 \tilde{\zeta} \sin(\omega_0 \chi + \chi_0).\]

We can choose the generating function $2\Phi = \eta(t, y^6, y^8)$ to be a solution of the Kadomtsev–Petviashvili (KP) equation

\[\pm \partial_{y^8}^2 \eta + \partial_0 (\partial_t \eta + \eta \partial_\xi \eta + \varepsilon \partial_{y^8}^3 \eta) = 0\]

where $\partial_{y^8}^2 \eta := \partial^2_\eta \eta / \partial y^8 \partial y^8$. And similarly for the other derivatives. In a similar form, we can consider solutions of any 3-d solitonic equations, for instance, generalized sine–Gordon ones. In the dispersionless limit $\varepsilon \to 0$, we get solutions which do not depend on $y^8$ but preserve locally anisotropic behaviour on $y^6$ and are determined by the Burgers’ equation $\partial_t \eta + \eta \partial_\xi \eta = 0$. Applying the same geometric method outlined in previous section, we construct

\[ds^2 = e^t (\varepsilon_1 (dx^1)^2 + \varepsilon_2 (dx^2)^2) + \frac{\Phi^2}{4 \Lambda} [dt + \partial_i (\varepsilon_1 n) dx^i]^2 + \frac{(\partial_4 \Phi)^2}{\Lambda \Phi^2} [d\chi + (\partial_4 \Phi) dx^4]^2 + \frac{\partial_4 \Phi)^2}{\Lambda \Phi^2} [d\tilde{\eta} + (\partial_4 \Phi) dx^4]^2 + \frac{\partial_4 \Phi)^2}{\Lambda \Phi^2} [d\psi + (\partial_4 \Phi) dx^4]^2;\]
where there are certain differences comparing to \([24]\) and \([25]\). For \([31]\), the polarization functions are

\[
\eta_{i4} \text{ as } \eta_7 = \eta^2/4 \cdot 2 \Lambda \vec{C} \text{ and } \eta_8 = \partial_8 \eta^2 \div 2 \Lambda \partial_8 \vec{C}, \quad \forall \quad \eta(t, y^6, y^8), \partial_8 \forall \neq 0,
\]

and the N–coefficients are

\[
N^7_{i2} = \tilde{N}^7_{i2} = \frac{\tilde{n}_{i2}(t, y^6, y^8)}{\tilde{\eta}} = \frac{\tilde{n}_{i2}(t, y^6) + \tilde{n}_{i2}(t, y^6) \int d\psi(\partial_8 \eta)^2}{\eta^2},
\]

\[
N^8_{i2} = \delta^3_{i2} \tilde{N}^8 + \delta^5_{i2} \tilde{N}^5 + \tilde{N}^6_{i2} = \delta^3_{i2} \tilde{w}_3 + \delta^5_{i2} \tilde{B}^2/\tilde{C} + 2 \tilde{w}_2(t, y^6, y^8) = \partial_{i2} \eta/\partial_8 \eta.
\]

Such solutions are not stationary, i.e., do not possess a Killing symmetry \(\partial_t\) being different from those described by the quadratic element \([26]\). They define a self–consistent imbedding of a black ring solution into velocity 3-d solitonic background determined by \([30]\). There is an off–diagonal interaction between the base and fiber degree of freedoms via term \(1 \tilde{w}_3 \in N^8_{i2}\).

We can positively consider the metrics \([31]\) as solutions for tangent Lorentz bundle black rings interacting with fiber solitons for some small \(\varepsilon\)–deformations. For corresponding parameterizations, the solution describe solitonic propagation of black rings in the total space. In general, the nonlinear superposition of black ring and velocity solitonic degrees of freedom do not have any obvious physical interpretation.

(B.) The solitonic generating function \(2\bar{\Phi} = \bar{\eta}(\bar{t}, y^6, y^8)\) is considered to be a solution of the KP equation \(\pm \partial^2_{\bar{y}\bar{y}} \bar{\eta} + \partial_{\bar{t}}(\partial_{\bar{t}} \bar{\eta}) + \varepsilon \partial^3_{\bar{y}} \bar{\eta} = 0\), with evolution on \(\bar{t}\) instead of \(t\) which is different from integral varieties in the previous example. This class of solitonic solutions is defined by quadratic elements of type

\[
ds^2 = e^{\bar{\eta}}[(dx^1)^2 - (dx^2)^2] + (1 + \varepsilon chatting) \tilde{h}_3 [d\tilde{t} + \varepsilon \partial_t (\bar{t} d\tilde{t})]^2 + (1 + \varepsilon chatting) \tilde{h}_4 [d\tilde{t} + \varepsilon \partial_t (\bar{t} d\tilde{t})]^2
\]

\[
+ (1 + \varepsilon chatting) \tilde{A} [d\tilde{t} + \varepsilon \partial_t (\bar{t} d\tilde{t})]^2 + (1 + \varepsilon chatting) \tilde{h}_5 \tilde{C} [d\psi + \varepsilon \partial_t (\bar{t} d\tilde{t})]^2 + 1 \tilde{w}_3(x^k) dt^2,
\]

which are similar to \([27]\) but with certain modifications of data \([28]\) when

\[
1 + \varepsilon_7 = \tilde{\eta}^2/4 \cdot 2 \Lambda \tilde{C} \text{ and } 1 + \varepsilon_8 = (\partial_8 \tilde{\eta})^2/2 \Lambda \partial_8 \tilde{C}; \quad \text{and}
\]

\[
N^7_{i2} = \delta^3_{i2} \tilde{N}^8 + \delta^5_{i2} \tilde{N}^5 + \tilde{N}^6_{i2} = \delta^3_{i2} \tilde{w}_3 + \varepsilon \tilde{w}_2(\bar{t}, y^6, y^8) = \partial_{i2} \tilde{\eta}/\partial_8 \tilde{\eta}.
\]

Such small solitonic deformations in velocity variables preserve stationarity on 4D base Lorentz manifold. Nevertheless, interactions between the base and fibers is given by the term \(\tilde{w}_3 \in N^8_{i2}\). Such black ring and/or (modified) gravitational solitonic objects are not subject to restrictions of Michelson–Morley type experiments for locally anisotropic aether \([10]\) being constructed as potential relativistic astrophysical objects in compact spacetime regions. Additional constraints are needed to be imposed in order to select LC–configurations.

5 Concluding Remarks

Black ring objects studied in this paper can be real and can be the result of QG effects via nonlinear polarizations in 4D gravity theories even if the space time lacks extra dimensions in the sense of Kaluza-Klein.

 Locally anisobropic black rings may exist as 4D osculatory "shadows" of MGTs with arbitrary MDRs. On tangent Lorentz bundles there are no restrictions on black hole uniqueness theorems. This paves the way to construct various classes of black ellipsoid/hole, cosmological and solitonic solutions of Finsler modified gravitational field equations \([28, 29, 23]\). Such objects may have cosmological implications, for instance, if galitic nucleus may contain analogous black torus solutions with dark matter and dark energy. The AFDM developed in this work can also be applied to construct similar generic off–diagonals solutions with extra dimensions in brane and string gravity.
Physical interpretation of Finsler black ring and black hole solutions depends on the type of nonholonomic constraints we impose, i.e., on the class of generating and integration functions and constants we chose to define our solutions and on resulting off-diagonal deformations and effective polarizations of constants. For small parametric N-connection coefficients and small deformations of standard black ring/hole solutions, we positively get high dimensional black hole solutions with extra "velocity" type coordinates. In such cases, there are typical horizons and singularities similar to the case of black holes in higher dimension theories. It is possible to construct toy 2+2 dimensional Finsler like black hole/ellipsoid models with two "velocity" coordinates with smooth nonholonomic imbedding in a five dimensional spacetime. For such configurations, we can speculate on analogous censorship theorems but these speculations apply only to the specific models constructed. It is also possible to generate solutions for naked singularities imbedded into solitonic like backgrounds \[^3\], with commutative and noncommutative Finsler brane warping/trapping on velocity type coordinates \[^27\] and nontrivial nonlinear connection structure.

Let us discuss the Lorentz invariance violation that is considered to be a general property of Finsler like theories. This is not always correct to assume Lorentz invariance violation is a general property of Finsler like theories because Einstein’s theory of gravity can be described also by Finsler like variables by prescribing a nonholonomic 2+2 splitting on Lorentz manifolds when the local Lorentz invariance is implicit in such constructions\[^4\]. Nevertheless, we always get Lorentz violations if we work with theories on tangent bundles. For compatibility with standard particle physics and gravity theories, we need to consider tangent Lorentz bundles. We can preserve for certain configurations a conventional local Lorentz invariance, which is modified by certain contributions from gravitational interactions depending on spacetime and velocity/momentum type variables. Here we note that even in general relativity, Lorentz transforms are considered only for fixed spacetime points but Lorentz invariance does not apply generally. A similar property exists for the Einstein-Finsler type theories on tangent Lorentz bundles when a corresponding axiomatic can be formulated in a form similar to that for the general relativity theory, see \[^22\]. Violation of the Lorentz symmetry in such Cartan-Finsler type generalized models is a consequence of locally anisotropic gravitational and matter field interactions depending both on base Lorentz manifold coordinates and on fiber like (velocity/momentum) coordinates on (co) tangent bundles.

Various schools on Finsler geometry and generalizations usually work with more general modifications of the Einstein gravity (in many cases, such models do not limit to Einstein’s gravity, e.g., see critical remarks and references in \[^44\]). The first class of such generalized models involve certain modified local Lorentz transforms (broken local Lorentz invariance). Finsler metrics can be related to modified dispersion relations like in Refs. \[^40\] and, for the second class of generalizations, various types of Finsler geometric models can be elaborated upon for different classes of Finsler connections (which may or may not be metric compatible). Such "physical" theories are constructed on different principles than the standard theories of gravity. We could formulate self-consistent commutative and non-commutative Finsler generalizations of Einstein gravity using equivalent geometric variables with the so-called canonical d-connection and the Cartan-Finsler connection. In such cases, it is possible to solve generalized Einstein-Finsler equations in exact forms and derive various classes of generalized black ring/ellipsoid/hole and wormhole solutions as in the present work and in \[^27\] by applying geometric methods summarized in Ref. \[^24\].

Finally, we discuss a recent work on exact solutions in a model of Finsler spacetime \[^45\], which is elaborated upon as an alternative to the models with the Cartan and canonical d-connection. The authors work directly with a Ricci like tensor in Finsler geometry introduced by Akbar-Zadeh \[^46\] which can be defined in symmetric form just from the fundamental Finsler function \(^F\) not involving the concept of linear connection.

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\[^1\]A pseudo-Riemannian manifold can be described in terms of any connection which is defined by a distortion tensor from the standard Levi-Civita connection, for instance, using the Cartan Finsler connection for a fibered structure. Sure, such constructions involve nontrivial torsion and non-metricity geometric objects but the corresponding distortion tensor is determined by a Sasaki type metric structure and conventional N-connection coefficients induced by generic off-diagonal metric terms. These are the so-called Finsler like, or almost Kaehler-Finsler like variables which can be used for the deformation quantization of the Einstein gravity and generating various classes of black hole and cosmological solutions. We can always re-encode all constructions in terms of standard metric and Levi-Civita variables, or introduce generalized tetradic, diadic, spinor variables etc.
Such an approach is based on nice geometric constructions. Nevertheless, it does not provide a self-consistent physical theory without additional motivations for a covariant derivative (linear connection), which should be adapted to the nonholonomic structure. There are also necessary some postulates on nonholonomic frames and physical observers, Lagrangians for anisotropic gravitational and matter field interactions. In our opinion [22, 44] such theories should be with curvature and Ricci tensors for the Cartan–Finsler and/or canonical d-connection. Akbar-Zadeh configurations can be extracted at the end via corresponding classes of nonholonomic deformations of fundamental geometric objects. For anisotropic interactions with matter fields, this is a very difficult theoretical problem.

In general relativity, such issues are solved in a "simplified" way due to the existence of the Levi-Civita connection and the formulation of the general principle of relativity (for arbitrary frames of reference). A physically viable Finsler spacetime geometry can not be derived only from the generalized Finsler metric $F$, or its Hessian, or semi-spray function etc. We need additional assumptions on linear and nonlinear connections (with frame adapted structures) which can be motivated following certain geometric physical principles. In a more general context, it is necessary to formulate some self-consistent generalized gravitational and matter field equations for a model of Finsler gravity which would be compatible with Einstein’s gravity. There are necessary some explicit exact solutions and quantum models for Finsler like gravity theories. In a more advanced theoretical framework such constructions were elaborated upon by using the almost symplectic Cartan-Finsler connection and the canonical d-connection, see details in Refs. [44, 22]. Following such an approach, it was possible to construct physically important black hole like and cosmological exact solutions [24, 27, 28, 23] with various supersymmetric, superstring and noncommutative generalizations [30, 29]. To conclude, Finsler type theories can be important in modern cosmology and astrophysics, and various quantum gravity models, because of global and local anisotropic classical and possible quantum effects. However, classical Lorentz violating terms must be small in order to have compatibility with the data for our Solar System. In our work such an assumption is implicit.

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