Ringing in de Sitter spacetime

Alex Buchel

Department of Applied Mathematics
Department of Physics and Astronomy
University of Western Ontario
London, Ontario N6A 5B7, Canada
Perimeter Institute for Theoretical Physics
Waterloo, Ontario N2J 2W9, Canada

Abstract

Hydrodynamics is a universal effective theory describing relaxation of quantum field theories towards equilibrium. Massive QFTs in de Sitter spacetime are never at equilibrium. We use holographic gauge theory/gravity correspondence to describe relaxation of a QFT to its Bunch-Davies vacuum — an attractor of its late-time dynamics. Specifically, we compute the analogue of the quasinormal modes describing the relaxation of a holographic toy model QFT in de Sitter.

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1 Introduction

Isolated strongly interacting systems typically\(^1\) reach a thermal equilibrium state at late times of its dynamical evolution. An approach towards equilibrium is governed by hydrodynamics — a universal effective theory organized as derivative expansion of the local velocity gradients to the temperature of the final equilibrium state. One example is the relativistic hydrodynamics of conformal gauge theories developed in [3,4]. As an effective description, gradient expansion of the gauge theory hydrodynamics has zero radius of convergence due to the existence of the non-hydrodynamic modes in equilibrium plasma [5,6]. Whenever gauge theory allows for a dual holographic description [7,8] in terms of classical supergravity, its thermal equilibrium state is represented by a black hole/black brane in the gravitational dual [9]. Furthermore, linearized hydrodynamic and non-hydrodynamic excitations about the equilibrium state are mapped to the quasinormal modes (QNMs) of the corresponding dual black hole [10]. QNMs encode the information about the relaxation of the near-equilibrium state of a gauge theory plasma [11–14].

Implicit in the above overview was an assumption that QFT dynamics occurs in Minkowski spacetime. Using holographic correspondence\(^2\), it was argued in [21,22] that massive gauge theories in de Sitter spacetime are not in equilibrium at late times: while Bunch-Davies (BD) vacuum is the late-time attractor of a dynamical evolution of a QFT state, the co-moving entropy production rate is nonzero. In this paper we make the first step addressing the question:

\(^1\)There are some exceptions to this lore: condensed matter systems with many-body localization [1]; holographic models with phase-space restricted dynamics [2].

\(^2\)For early work on gauge theories in de Sitter within holographic framework see [15–19].
What is the effective theory of the relaxation towards Bunch-Davies vacuum of a massive QFT?

We restrict our attention to a simple holographic toy model of a $2 + 1$-dimensional massive $QFT_3$ with the effective dual gravitational action:\(^3\)

$$S_4 = \frac{1}{2\kappa^2} \int_{M_4} dx^4 \sqrt{-\gamma} \left[ R + 6 - \frac{1}{2} (\nabla \phi)^2 + \phi^2 \right]. \quad (1.1)$$

The four dimensional gravitational constant $\kappa$ is related to the ultraviolet (UV) conformal fixed point $CFT_3$ central charge $c$ as

$$c = \frac{192}{\kappa^2}. \quad (1.2)$$

$\phi$ is a gravitational bulk scalar with

$$L^2 m_\phi^2 = -2, \quad (1.3)$$

which is dual to a dimension $\Delta_\phi = 2$ operator $O_\phi$ of the boundary theory. $QFT_3$ is a relevant deformation of the UV $CFT_3$ with

$$\mathcal{H}_{CFT} \rightarrow \mathcal{H}_{QFT} = \mathcal{H}_{CFT} + \Lambda O_\phi, \quad (1.4)$$

with $\Lambda$ being the deformation mass scale. We study $QFT_3$ dynamics in de Sitter spacetime with a Hubble constant $H$; thus the metric on $M_4$ boundary, $ds^2_{\partial M_4}$, is taken as

$$ds^2_{\partial M_4} = -dt^2 + e^{2Ht} \left(dx_1^2 + dx_2^2\right). \quad (1.5)$$

Following [21], in the next section we describe gravitational dynamical setup encoding de Sitter evolution of spatially homogeneous and isotropic states of the boundary field theory. We study the late-time attractor of the evolution in section 2.1. In section 2.2 we compute the spectrum of linearized fluctuations of the boundary theory around its BD vacuum. In section 2.3 we use fully nonlinear characteristic formulation of asymptotically $AdS$ dynamics [23] and establish that generic homogeneous and isotropic states of the boundary theory indeed “ring-down” to BD vacuum with frequencies computed in section 2.2. We conclude in section 3.

\(^3\)We set the radius $L$ of an asymptotic $AdS_4$ geometry to unity.
2 Holographic gravitational dynamics

A generic state of the boundary field theory with a gravitational dual \((1.1)\), homogeneous and isotropic in the spatial boundary coordinates \(x = \{x_1, x_2\}\), leads to a bulk gravitational metric ansatz

\[
\text{d}s_4^2 = 2\text{d}t\left(\text{d}r - A\text{d}t\right) + \Sigma^2\text{d}x^2, \tag{2.1}
\]

with the warp factors \(A, \Sigma\) as well as the bulk scalar \(\phi\) depending only on \(\{t, r\}\). From the effective action \((1.1)\) we obtain the following equations of motion:

\[
0 = d'_+\Sigma + d_+\Sigma \left(\ln \Sigma\right)' - \frac{3}{2}\Sigma - \frac{1}{4}\Sigma\phi^2, \\
0 = d'_+\phi + d_+\phi \left(\ln \Sigma\right)' + \frac{d_+\Sigma}{\Sigma}\phi' + \phi, \tag{2.2}
\]

\[
0 = A'' - 2\frac{d_+\Sigma}{\Sigma^2} \Sigma' + \frac{1}{2}d_+\phi \phi', \tag{2.3}
\]

as well as the Hamiltonian constraint equation:

\[
0 = \Sigma'' + \frac{1}{4}\Sigma(\phi')^2, \tag{2.4}
\]

and the momentum constraint equation:

\[
0 = d^2_+\Sigma - 2Ad'_+\Sigma - \frac{d_+\Sigma}{\Sigma^2} \left(A\Sigma^2\right)' + \frac{1}{4}\Sigma \left((d_+\phi)^2 + 2A(6 + \phi^2)\right). \tag{2.5}
\]

In \((2.2)-(2.4)\) we denoted \(' = \frac{\partial}{\partial r}, \cdot = \frac{\partial}{\partial t}, \) and \(d_+ = \frac{\partial}{\partial r} + A\frac{\partial}{\partial r}\). The near-boundary \(r \to \infty\) asymptotic behaviour of the metric functions and the scalar encode the mass parameter \(\Lambda\) and the boundary metric scale factor \(a(t) \equiv e^{Ht}\):

\[
\Sigma = a\left(r + \lambda + \mathcal{O}(r^{-1})\right), \quad A = \frac{r^2}{2} + \left(\lambda - \frac{\dot{a}}{a}\right) r + \mathcal{O}(r^0), \quad \phi = \frac{\Lambda}{r} + \mathcal{O}(r^{-2}). \tag{2.5}
\]

\(\lambda = \lambda(t)\) in \((2.5)\) is the residual radial coordinate diffeomorphism parameter \([23]\). An initial state of the boundary field theory is specified providing the scalar profile \(\phi(0, r)\) and solving the constraint \((2.3)\), subject to the boundary conditions \((2.5)\). Equations \((2.2)\) can then be used to evolve the state.

The subleading terms in the boundary expansion of the metric functions and the scalar encode the evolution of the energy density \(\mathcal{E}(t)\), the pressure \(P(t)\) and the expectation values of the operator \(\mathcal{O}_\phi(t)\) of the prescribed boundary QFT initial state.
Specifically, extending the asymptotic expansion (2.5) for \( \{ \phi, A \} \),

\[
\begin{align*}
\phi &= \frac{\Lambda}{r} + \frac{f_2(t)}{r^2} + O \left( \frac{1}{r^3} \right), \\
A &= \frac{r^2}{2} + \left( \lambda - \frac{\dot{a}}{a} \right) r + \frac{\Lambda^2}{8} - \frac{\dot{a}}{a} \lambda - \dot{\lambda} + \frac{1}{r} \left( \mu(t) - \frac{\Lambda}{4} f_2(t) - \frac{\Lambda^2}{4} \lambda \right) + O \left( \frac{1}{r^2} \right),
\end{align*}
\]

(2.6)

the observables of interest can be computed following the holographic renormalization of the model:

\[
\begin{align*}
2\kappa^2 \mathcal{E}(t) &= - 4\mu + \frac{\dot{a}}{a} \Lambda^2 + \left( \delta_1 \Lambda^3 + 2\delta_2 \Lambda \frac{(\dot{a})^2}{a^2} \right), \\
2\kappa^2 P(t) &= - 2\mu + \frac{1}{2} \Lambda (f_2 + \lambda \Lambda) + \left( - \delta_1 \Lambda^3 - 2\delta_2 \Lambda \frac{\ddot{a}}{a} \right), \\
2\kappa^2 \mathcal{O}_\phi(t) &= - f_2 - \lambda \Lambda + \frac{\dot{a}}{a} \Lambda + \left( 3\delta_1 \Lambda^2 + \delta_2 \left( \frac{\ddot{a}}{a} + 2 \frac{(\dot{a})^2}{a^2} \right) \right),
\end{align*}
\]

(2.7) \hspace{1cm} (2.8) \hspace{1cm} (2.9)

where the terms in brackets, depending on arbitrary constants \( \{ \delta_1, \delta_2 \} \), encode the renormalization scheme ambiguities. Independent of the renormalization scheme, these expectation values satisfy the expected conformal Ward identity

\[
- \mathcal{E} + 2P = - \Lambda \mathcal{O}_\phi.
\]

(2.10)

Furthermore, the conservation of the stress-energy tensor

\[
\frac{d\mathcal{E}}{dt} + 2\frac{\dot{a}}{a} (\mathcal{E} + P) = 0,
\]

(2.11)

is a consequence of the momentum constraint (2.4):

\[
0 = \dot{\mu} + \frac{\dot{a}}{a} \left( 3\mu - \frac{1}{4} \Lambda f_2 \right) - \frac{\Lambda^2}{4} \frac{\dot{a}}{a} \left( \lambda + \frac{\dot{a}}{a} \right) + \frac{\ddot{a}}{a}.
\]

(2.12)

From now on we choose a scheme with \( \delta_i = 0 \).

One of the advantages of the holographic formulation of a QFT dynamics is the natural definition of its far-from-equilibrium entropy density. A gravitational geometry (2.1) has an apparent horizon located at \( r = r_{AH} \), where [23]

\[
d_+ \Sigma \bigg|_{r=r_{AH}} = 0.
\]

(2.13)
Following [24, 25] we associate the non-equilibrium entropy density $s$ of the boundary QFT with the Bekenstein-Hawking entropy density of the apparent horizon

$$a^2 s = \frac{2\pi}{\kappa^2} \Sigma^2 \bigg|_{r=r_{AH}}.$$  \hspace{1cm} (2.14)

Using the holographic background equations of motion (2.2)-(2.4) we find

$$\frac{d(a^2 s)}{dt} = \frac{2\pi}{\kappa^2} (\Sigma^2)' \left( \frac{d\phi}{\phi^2 + 6} \right) \bigg|_{r=r_{AH}}.$$ \hspace{1cm} (2.15)

Following [21] it is easy to prove that the entropy production rate as defined by (2.15) is non-negative, i.e.,

$$\frac{d(a^2 s)}{dt} \geq 0,$$ \hspace{1cm} (2.16)

in holographic dynamics governed by (2.2)-(2.4).

The holographic evolution as explained above is implemented in section 2.3, adopting numerical codes developed in [26, 27].

### 2.1 Bunch-Davies vacua of holographic toy QFT$_3$

Following [21], the equations for the late-time attractor of the evolution (a Bunch-Davies vacuum [21]) can be obtained from (2.2)-(2.4) taking $t \to \infty$ limit with identification

$$\lim_{t \to \infty} \{ \phi, A \}(t, r) = \{ \phi, A \}_v, \quad \lim_{t \to \infty} \frac{\Sigma(t, r)}{a(t)} = \sigma_v(r).$$ \hspace{1cm} (2.17)

Introducing a new radial coordinate

$$x \equiv \frac{H}{r},$$ \hspace{1cm} (2.18)

and denoting

$$\phi_v = p(x), \quad A_v = \frac{H^2}{2x^2} g(x), \quad \sigma_v = \frac{H}{x} f(x),$$ \hspace{1cm} (2.19)

we find

$$0 = f'' + \frac{1}{4}(p')^2 \ f,$$

$$0 = p'' + \left( (fp^2 + 12f't^2 - 12xf + 6f)x^2 \right)^{-1} \left( 2f^2x^4(p')^3 - f^2p(p')^2x^2 + (24x^4(f')^2 + 4fx^2(p^2 - 6x + 6)f')p + 12p((f')^2x^2 - 2fxf' + f^2) \right),$$ \hspace{1cm} (2.20)
along with an algebraic expression for $g$:

$$g = -\frac{2f(f p^2 + 12 f' x^2 - 12 x f + 6 f)}{f^2(p')^2 x^2 - 12(f')^2 x^2 + 24 x f f' - 12 f^2}. \quad (2.21)$$

Vacuum solution has to satisfy the boundary conditions (2.5), and remain nonsingular for $x \in (0, x_{AH}]$, where the location of the apparent horizon $x_{AH}$ is determined from [21]

$$d_+ \Sigma(t, x_{AH}) = 0 \iff \left( f(x) \left( 2x + g(x) \right) - g(x)f'(x) \right) \bigg|_{x=x_{AH}} = 0. \quad (2.22)$$

Without loss of generality we fix the diffeomorphism parameter $\lambda$ so that

$$A_v(x) \bigg|_{x=\frac{1}{3}} = 0. \quad (2.23)$$

We will always have $x_{AH} > \frac{1}{3}$.

It is straightforward to construct an analytic solution to (2.20) as a series expansion in conformal symmetry breaking parameter

$$p_1 \equiv \frac{\Lambda}{H}, \quad (2.24)$$

$$p = p_1 \frac{x}{1 - x} - \frac{x^2 (2x - 1)}{9(x - 1)^3} p_3^3 - \frac{x^2 (875x^3 - 647x^2 + 9x + 51)}{12960(x - 1)^5} p_5^5 + O\left(p_7^7\right), \quad (2.25)$$

$$f = 1 - x + \frac{x(4x - 1)}{24(x - 1)} p_1^2 + \frac{x(4x - 1)(23x^2 - 5x - 5)}{3456(x - 1)^3} p_4^4 + \frac{x(49618x^5 - 46133x^4 + 9055x^3 - 2745x^2 + 3225x - 645)}{6220800(x - 1)^5} p_6^6 + O\left(p_8^8\right), \quad (2.26)$$

which determines following (2.21)

$$g = (1 - 3x) \left( 1 - x + \frac{x(3x - 1)}{12(x - 1)} p_1^2 + \frac{(3x - 1)(19x^2 - 2x - 5)x}{1728(x - 1)^3} p_4^4 \right) + \frac{x(3x - 1)(1937x^4 - 1196x^3 + 54x^2 - 204x + 129)}{622080(x - 1)^5} p_6^6 + O\left(p_8^8\right). \quad (2.27)$$

From (2.22), the apparent horizon is located at

$$x_{AH} = 1 - \frac{1}{6} 6^{2/3} p_1^{2/3} + \frac{1}{12} 6^{1/3} p_1^{4/3} + \frac{1}{9} p_1^2 + \frac{20401}{622080} 6^{2/3} p_1^{8/3} - \frac{685273}{12441600} 6^{1/3} p_1^{10/3} + \frac{40841057}{99532800} p_1^4 + O\left(p_1^{14/3}\right). \quad (2.28)$$
Figure 1: Vacuum energy density $\mathcal{E}_v$ of the holographic toy model as a function of the conformal symmetry breaking deformation $p_1 = \frac{\Lambda}{H}$ (solid black line). Dashed red line indicates perturbative prediction, see (2.29). $c$ is the UV central charge of the model (see (1.2)).

For generic $p_1$ we have to resort numerics. Details of the numerical implementation are explained in [21]. Fig. 1 presents the vacuum energy $\mathcal{E}_v$ as a function of $p_1$ in renormalization scheme $\delta_i = 0$. Using perturbative solution (2.25), (2.26) we find

$$2\kappa^2 \frac{\mathcal{E}_v}{H^3} = \frac{1}{3} p_1^2 + \frac{5}{216} p_1^4 - \frac{43}{51840} p_1^6 + \mathcal{O}(p_1^8).$$

(2.29)

Note that in vacuum $P_v = -\mathcal{E}_v$, thus following (2.10),

$$\mathcal{O}_{\phi,v} = 3 \frac{\mathcal{E}_v}{\Lambda}.$$ 

(2.30)

In [22] it was argued that the vacuum of a massive QFT in de Sitter has a constant "entanglement" entropy density $s_{\text{ent}}$, related to the comoving entropy production rate $\mathcal{R}$ at late times. Specifically, parameterizing the comoving entropy production from (2.15) as

$$\lim_{t \to \infty} \frac{1}{H^2 a^2} \frac{d}{dt} (a^2 s) \equiv 2H \times \mathcal{R},$$

(2.31)

the vacuum entropy density $s_{\text{ent}}$ is

$$s_{\text{ent}} \equiv \lim_{t \to \infty} s = H^2 \mathcal{R}.$$ 

(2.32)
Figure 2: Vacuum entropy density $s_{\text{ent}}$ of the holographic toy model (see (2.32)) as a function of the conformal symmetry breaking deformation $p_1 = \Lambda / H$. Dashed red line indicates perturbative prediction, see (2.33). $c$ is the UV central charge of the model (see (1.2)).

Fig. 2 presents the vacuum entropy density as a function of $p_1$ — this result is renormalization scheme independent. Perturbatively,

$$\frac{\kappa^2}{2\pi} \frac{s_{\text{ent}}}{H^2} = \frac{1}{6} 6^{1/3} p_1^{4/3} - \frac{1}{12} p_1^2 - \frac{5}{216} 6^{2/3} p_1^{8/3} - \frac{3359}{311040} 6^{1/3} p_1^{10/3} + \mathcal{O}(p_1^4). \quad (2.33)$$

Following [22], the surface gravity of the apparent horizon equals $(-H)$.

2.2 Spectrum of vacuum linearized fluctuations

For static horizons in holography, quasinormal modes of black holes/black branes represent the physical linearized fluctuations in the dual boundary field theory plasma at equilibrium. In Fefferman-Graham coordinate of the asymptotically AdS bulk geometry the spectrum of QNMs is determined solving Sturm-Liouville problem for the linearized fluctuations with Dirichlet conditions at the asymptotic boundary for the non-normalizable modes of the fluctuating fields, and incoming boundary condition at the horizon [28]. In case of infalling Eddington-Finkelstein coordinates (as in (2.1)), the horizon boundary condition is replaced with the regularity at the trapped surface (the apparent horizon). We stress again that it is the boundary conditions at the horizon and the asymptotic boundary that determine the spectrum of fluctuations.
In analogy to QNMs, we consider linearized fluctuations of the system (2.2)-(2.4) about the late-time attractor solution (2.17). To this end, we define the fluctuations with the harmonic time dependence of frequency $\omega$ as follows:

$$\phi(t, x) = p(x) + \delta H_1(x) e^{-i\omega t}, \quad \Sigma(t, x) = \frac{H}{x} \left(f(x) + \delta H_2(x) e^{-i\omega t}\right),$$

$$A(t, x) = \frac{H}{2x^2} \left(g(x) + \delta H_3(x) e^{-i\omega t}\right),$$

(2.34)

where $\delta$ is the amplitude of the fluctuations. Substituting (2.34) into (2.2)-(2.4) and collecting $O(\delta)$ terms we obtain a consistent set of coupled radial equations of motion for $H_i$:

$$0 = H''_1 + C_{1,1} H'_1 + C_{1,2} H_1 + C_{1,3} H_2 + C_{1,4} H_3,$$

$$0 = H'_2 + C_{2,1} H'_1 + C_{2,2} H_1 + C_{2,3} H_2 + C_{2,4} H_3,$$

$$0 = H'_3 + C_{3,1} H'_1 + C_{3,2} H_1 + C_{3,3} H_2 + C_{3,4} H_3,$$

(2.35)

where the connection coefficients

$$C_{i,j} = C_{i,j} \left[f'(x), p'(x); f(x), p(x); x; \hat{\omega}\right]$$

(2.36)

are functionals of vacuum functions $\{f, p\}$ (see (2.20)) and the reduced frequency

$$\hat{\omega} \equiv \frac{\omega}{H}.$$  

(2.37)

As in case of the QNMs, we insist that the linearized fluctuations $H_i$ do not change boundary QFT data, i.e., we require

$$H_1 = x^2 + O(x^3), \quad H_2 = O(x), \quad H_3 = O(x),$$

(2.38)

as $x \to 0$ (the asymptotic AdS boundary). The $O(x^2)$ term in the $H_1$ asymptotic is simply the definition of the amplitude of the linearized fluctuations. Recall [21] that the vacuum equations of motion have a coordinate singularity when $A_v(x = x_{\text{singularity}}) = 0$. In our case $x_{\text{singularity}} = \frac{1}{3}$, see (2.23). This coordinate singularity occurs always before the apparent horizon: $x_{\text{AH}} > x_{\text{singularity}}$. Turns out that the connection coefficients $C_{i,j}$ are singular at $x_{\text{singularity}}$, and requiring that this is just

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4 Explicit form of (2.35) is provided as a separate file with the arXiv.org submission of this paper.

5 There is no coordinate singularity in the radial coordinate in the characteristic formulation of the dynamical evolution implemented in section 2.3.
a coordinate singularity and the fluctuating fields \( H_i \) are smooth across this point and extend all the way to the apparent horizon \( x_{AH} \), provides the second boundary condition on the spectrum of fluctuations.

To recap: the spectrum of linearized fluctuations about Bunch-Davies vacuum is determined from:

- Dirichlet conditions at the AdS boundary on the non-normalizable modes of the dual gravitational bulk fluctuating fields;

- regularity condition for bulk fluctuating fields at the location \( A_v = 0 \).

It is instructive to solve (2.35) perturbatively in the conformal deformation parameter \( p_1 \), using perturbative expansion for the BD vacuum (2.25)-(2.26). Introducing

\[
H_i(x) = \sum_{k=0}^{\infty} p_1^k H_{i,k}(x), \quad \hat{\omega} = \sum_{k=0}^{\infty} p_1^k \hat{\omega}_k, \quad (2.39)
\]
to leading order \( k = 0 \) we find:

\[
0 = H_{1,0}'' - \frac{2i(3ix^2 - \hat{\omega}_0x - i)}{x(x-1)(3x-1)} H_{1,0}' - \frac{2i(ix - \hat{\omega}_0x - i)}{(3x-1)(x-1)^2x^2} H_{1,0},
\]

\[
0 = H_{2,0}' + \frac{i(2ix - i + \hat{\omega}_0)}{(x-1)(2x-1)} H_{2,0} - \frac{1}{2x(x-1)(2x-1)} H_{3,0}, \quad (2.40)
\]

\[
0 = H_{3,0}' - \frac{2\hat{\omega}_0x(\hat{\omega}_0 + i)}{(x-1)(2x-1)} H_{2,0} + \frac{i(4ix^2 - 5ix - \hat{\omega}_0x + i)}{x(x-1)(2x-1)} H_{3,0}.
\]

Note that to leading order in \( p_1 \) equations for \( H_1 \) and \{\( H_2, H_3 \)\} decouple. The general solution of the first equation in (2.40), subject to (2.38), is

\[
H_{1,0} = \begin{cases} 
-\frac{x}{2(1-x)(1-i\hat{\omega}_0)} \left( 1 - \left( \frac{1-3x}{1-x} \right)^{-1+i\hat{\omega}_0} \right), & \hat{\omega}_0 \neq -i \\
-\frac{x}{2(1-x)} \ln \frac{1-3x}{1-x}, & \hat{\omega}_0 = -i. 
\end{cases} \quad (2.41)
\]

Requiring that \( H_{1,0} \) is analytic at \( x = x_{\text{singularity}} = \frac{1}{3} \) produces the spectrum of fluctuations to leading order in \( p_1 \):

\[
\hat{\omega} \equiv \hat{\omega}^{(n)} = -in + \mathcal{O}(p_1), \quad n = 2, 3, \ldots \quad (2.42)
\]

Note that in a conformal limit \( p_1 \to 0 \) the mode (2.41) disappears from the spectrum — all \( (n) \)-modes are singular at \( x = x_{AH} = 1 + \mathcal{O}(p_1^{2/3}) \). We interpreted this fact as
a statement that the Bunch-Davies vacuum of a CFT does not ring. It is straightforward to check that the remaining two equations in (2.40) do not lead to new spectral branches\(^6\).

The leading order solution (2.42) can be extended to higher orders in \(O(p_1)\). For example, for \(n = 2\) mode we find:

\[
\hat{\omega}^{(2)} = -i \left( 2 + \frac{1}{12} p_1^2 - \frac{1}{54} p_1^4 + \frac{1591}{622080} p_1^6 + O(p_1^8) \right),
\]

\[
H_1^{(2)} = \frac{x^2}{(1-x)^2} + \frac{x^3(19x-6)}{36(x-1)^4} p_1^2 + \frac{x^3(6768x^3 - 3125x^2 - 950x + 535)}{25920(x-1)^6} p_1^4
\]

\[
+ \frac{x^3(17864583x^5 - 14152740x^4 + 1089102x^3 - 157220x^2 + 1292795x - 409080)}{130636800(x-1)^8} p_1^6
\]

\[+ O(p_1^8), \]

\[
H_2^{(2)} = \frac{-x(13x^2 - 2x + 1)}{72(x-1)^2} p_1 - \frac{x(2867x^4 - 1644x^3 + 646x^2 - 444x + 111)}{34560(x-1)^4} p_1^3
\]

\[
- \frac{x(17822851x^6 - 8946582x^5 - 7415409x^4 + 8763460x^3 - 5161395x^2 + 2036586x - 339431)}{522547200(x-1)^6} p_1^5
\]

\[+ O(p_1^7), \]

\[
H_3^{(2)} = (3x-1) \left( \frac{(4x^2 + x + 1)x}{36(x-1)^2} p_1 + \frac{(1036x^4 + 95x^3 - 447x^2 - 51x + 111)x}{17280(x-1)^4} p_1^3 \right.
\]

\[+ \left. \frac{(8634226x^6 - 6320265x^5 + 1435341x^4 - 731078x^3 - 568056x^2 + 1083183x - 339431)x}{261273600(x-1)^6} p_1^5 \right)
\]

\[+ O(p_1^7), \]

(2.43)

where we fixed the diffeomorphism parameter \(\lambda(t)\) to all orders in \(p_1\) requiring that \(A(t, x = \frac{1}{3}) = 0\). Note that \(n = 2\) mode is purely dissipative. In fact, we find that all modes except for \(n = 3\) are purely dissipative. For example,

\[
\hat{\omega}^{(4)} = -i \left( 4 - \frac{1}{216} p_1^4 + \frac{337}{777600} p_1^6 + O(p_1^8) \right),
\]

\[
\hat{\omega}^{(5)} = -i \left( 5 - \frac{5}{4096} p_1^4 - \frac{589}{15925248} p_1^6 + O(p_1^8) \right),
\]

(2.44)

\(^6\)This is also confirmed comparing with the relaxation to BD vacuum in the full nonlinear dynamics as explained in section 2.3.
Figure 3: Spectra of BD vacuum fluctuations for several low-lying modes $n = 2, 3, 4, 5$. Red dashed lines indicate perturbative predictions perturbative in conformal symmetry breaking parameter $p_1 = \frac{\Lambda}{H}$.

while

$$\hat{\omega}^{(3)} = -3i + \frac{1}{4} \sqrt{2} p_1 - \frac{11i}{192} p_1^2 + \frac{37}{12288} \sqrt{2} p_1^3 - \frac{1855i}{221184} p_1^4 - \frac{2076503}{1132462080} \sqrt{2} p_1^5$$

$$+ \frac{1240993i}{1061683200} p_1^6 + \mathcal{O}(p_1^7).$$

(2.45)

What makes the mode $n = 3$ special is the fact that the connection coefficients $C_{i,j}$ in (2.35) have a simple pole at $\hat{\omega} = -3i$. Unfortunately, we do not understand the physical reason for this.

For general $p_1$ the spectrum of fluctuations can be computed numerically. These results are presented in fig. 3 for $n = \{2, 3, 4, 5\}$ modes. The dashed red lines indicate perturbative approximation (2.43), (2.44) and (2.45). In what follows to refer to the fluctuations in BD vacuum as QNMs.

Although the spectrum is determined solving (2.35) on the radial interval $x \in (0, x_{\text{singularity}})$, we verified that the solution can indeed be smoothly extended to the full interval $x \in (0, x_{AH})$. For example, the radial profile $H_1^{(2)}(x)$ at $p_1 = 1$ is presented in fig. 4.

2.3 Fully nonlinear dynamics and relaxation to BD vacuum

In this section we report results of the fully nonlinear evolution of the toy holographic QFT defined by a dual gravitational action (1.1). Numerical implementation parallel the codes developed in [26,27], and will not be discussed here.
In what follows we focus on the model with $p_1 = 1$. We use the radial coordinate as in (2.18) and evolve in dimensionless time $\tau \equiv H t$. As in [23] we adjust the diffeomorphism parameter $\lambda(t)$ so that the apparent horizon is always at $x_{AH} = 1$. We set the initial condition for the evolution as

$$\phi(0, x) = \phi_{\text{initial}}(x) = p_1 x + A x^2 e^{-x},$$  \hspace{1cm} (2.46)

where $A$ is the amplitude. We also need to supply the initial energy density (see (2.12))

$$\mu(0) \equiv \mu_{\text{initial}}.$$  \hspace{1cm} (2.47)

We verified that BD vacuum is indeed the attractor of long-time dynamics by choosing different initial states for the evolution, i.e., different profiles $\phi_{\text{initial}}$ and/or $\mu_{\text{initial}}$.

Fig. 5 represent a typical dynamical evolution of the boundary QFT state from the initial condition (2.46). As times $\tau = H t \gtrsim 2$ the state relaxes to BD vacuum. The relaxation process is studied in further details as follows.

- We use the last 1000 data points, corresponding to time interval $\tau \in [5.6, 6]$ and fit the observed $O_{\phi}(\tau)$ with a single QNM ansatz:

$$O^{\text{fit}}_{\phi} = \alpha_1 + \alpha_2 e^{-i\alpha_3 \tau},$$  \hspace{1cm} (2.48)

where $\alpha_i$ are constant free parameters. $\alpha_1$ is expected to agree with the BD expectation value and $\alpha_3$ should approximate the frequency of the lowest BD QNM mode, i.e.,

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7 The discussion is generic for the parameter set with stable and convergent evolution of the code.
Figure 5: Fully nonlinear dynamical evolution of the initial boundary QFT state (2.46) (the solid black curve) as a function of $\tau = Ht$. The red line represents the late-time asymptotic value of the operator $O_\phi$ computed in BD vacuum in section 2.1.

$n = 2$. We find that the BD vacuum expectation value is correct with a relative error of $\sim 3 \times 10^{-6}$ and the relative error in the frequency,

$$\left| 1 - \frac{\alpha_3}{\hat{\omega}(2)} \right| = 2.7 \times 10^{-3},$$

is in excellent agreement with the result in section 2.2.

To check on the spectrum of higher QNMs computed in section 2.2 we restrict to 1000 data points in the intermediate time-range, $\tau \in [4, 4.4]$. We compute residual $\delta$ defined as

$$\delta(\tau) \equiv \left| 1 - \frac{O_{QNM}^\phi(\tau)}{O_\phi(\tau)} \right|,$$

where $O_{QNM}^\phi(\tau)$ is the best QNM approximation to the data in the time subinterval with constant free fit parameters $\alpha_i$ and frequencies $\hat{\omega}^{(n)}$ computed in section 2.2:

$n = 2 : \quad O_{QNM}^\phi = \alpha_1 + \alpha_2 e^{-i\hat{\omega}(2)\tau}$,

$n = 2, 3 : \quad O_{QNM}^\phi = \alpha_1 + \alpha_2 e^{-i\hat{\omega}(2)\tau} + \alpha_3 e^{\text{Im}[\hat{\omega}^{(3)}]_\tau} \cos[\text{Re}[\hat{\omega}^{(3)}]\tau + \alpha_4],$

$n = 2, 3, 4 : \quad O_{QNM}^\phi = \alpha_1 + \alpha_2 e^{-i\hat{\omega}(2)\tau} + \alpha_3 e^{\text{Im}[\hat{\omega}^{(3)}]_\tau} \cos[\text{Re}[\hat{\omega}^{(3)}]\tau + \alpha_4] + \alpha_5 e^{-i\hat{\omega}(4)\tau}.$

The residual $\delta$ is presented in fig. 6. The quality of approximation suggests that the QNMs computed in section 2.2 are all the modes defining the relaxation of the theory to its BD vacuum.
\[ \delta = |1 - O_{QNM}^\phi(\tau)/O_{\phi}(\tau)| \]

Figure 6: Relaxation to BD vacuum via the QNMs computed in section 2.2. The residuals \( \delta \) are computed with the best fit \( O_{QNM}^\phi \) to the evolution data using a constant and: a single \( n = 2 \) mode (blue curve); two modes \( n = 2, 3 \) (red curve); 3 modes \( n = 2, 3, 4 \) (green curve).

3 Conclusion

A surprising fact discovered in [21, 22] is that a vacuum of a massive QFT in de Sitter space-time has a constant entropy density \( s_{\text{ent}} \). We stress that it is important that both the Hubble constant is nonzero, and that the theory is non-conformal. For example, in a simple 2+1 dimensional holographic toy model discussed here

\[ s_{\text{ent}} \sim c \Lambda^{4/3} H^{2/3}, \quad \frac{\Lambda}{H} \ll 1, \quad (3.1) \]

where \( c \) is a UV central charge of the model and \( \Lambda \) is a mass scale of the theory.

Thermal equilibrium states have entropy. (Non)-hydrodynamic modes in equilibrium plasma owe their existence to this entropy — no entropy, nothing to excite. By analogy, the nonvanishing vacuum entropy of a massive QFT in de Sitter suggests that there should be analogous QNM-like excitations about its Bunch-Davies vacuum. In this paper we showed that this is indeed the case.

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References

[1] R. Nandkishore and D. A. Huse, “Many-body localization and thermalization in quantum statistical mechanics,” Annu. Rev. Condens. Matter Phys. 6.1, 15 (2015).

[2] V. Balasubramanian, A. Buchel, S. R. Green, L. Lehner and S. L. Liebling, “Holographic Thermalization, Stability of Antide Sitter Space, and the Fermi-Pasta-Ulam Paradox,” Phys. Rev. Lett. 113, no. 7, 071601 (2014) doi:10.1103/PhysRevLett.113.071601 [arXiv:1403.6471 [hep-th]].

[3] R. Baier, P. Romatschke, D. T. Son, A. O. Starinets and M. A. Stephanov, “Relativistic viscous hydrodynamics, conformal invariance, and holography,” JHEP 0804, 100 (2008) doi:10.1088/1126-6708/2008/04/100 [arXiv:0712.2451 [hep-th]].

[4] S. Bhattacharyya, V. E. Hubeny, S. Minwalla and M. Rangamani, “Nonlinear Fluid Dynamics from Gravity,” JHEP 0802, 045 (2008) doi:10.1088/1126-6708/2008/02/045 [arXiv:0712.2456 [hep-th]].

[5] M. P. Heller, R. A. Janik and P. Witaszczuk, “Hydrodynamic Gradient Expansion in Gauge Theory Plasmas,” Phys. Rev. Lett. 110, no. 21, 211602 (2013) doi:10.1103/PhysRevLett.110.211602 [arXiv:1302.0697 [hep-th]].

[6] A. Buchel, M. P. Heller and J. Noronha, “Entropy Production, Hydrodynamics, and Resurgence in the Primordial Quark-Gluon Plasma from Holography,” Phys. Rev. D 94, no. 10, 106011 (2016) doi:10.1103/PhysRevD.94.106011 [arXiv:1603.05344 [hep-th]].

[7] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2, 231 (1998) [Int. J. Theor. Phys. 38, 1113 (1999)] [arXiv:hep-th/9711200].

[8] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, “Large N field theories, string theory and gravity,” Phys. Rept. 323, 183 (2000) [hep-th/9905111].

[9] E. Witten, “Anti-de Sitter space, thermal phase transition, and confinement in gauge theories,” Adv. Theor. Math. Phys. 2, 505 (1998) [hep-th/9803131].
[10] E. Berti, V. Cardoso and A. O. Starinets, “Quasinormal modes of black holes and black branes,” Class. Quant. Grav. 26, 163001 (2009) doi:10.1088/0264-9381/26/16/163001 [arXiv:0905.2975 [gr-qc]].

[11] A. Buchel, M. P. Heller and R. C. Myers, “Equilibration rates in a strongly coupled nonconformal quark-gluon plasma,” Phys. Rev. Lett. 114, no. 25, 251601 (2015) doi:10.1103/PhysRevLett.114.251601 [arXiv:1503.07114 [hep-th]].

[12] J. F. Fuini and L. G. Yaffe, “Far-from-equilibrium dynamics of a strongly coupled non-Abelian plasma with non-zero charge density or external magnetic field,” JHEP 1507, 116 (2015) doi:10.1007/JHEP07(2015)116 [arXiv:1503.07148 [hep-th]].

[13] R. A. Janik, G. Plewa, H. Soltanpanahi and M. Spalinski, “Linearized nonequilibrium dynamics in nonconformal plasma,” Phys. Rev. D 91, no. 12, 126013 (2015) doi:10.1103/PhysRevD.91.126013 [arXiv:1503.07149 [hep-th]].

[14] A. Buchel and A. Day, “Universal relaxation in quark-gluon plasma at strong coupling,” Phys. Rev. D 92, no. 2, 026009 (2015) doi:10.1103/PhysRevD.92.026009 [arXiv:1505.05012 [hep-th]].

[15] A. Buchel, “Gauge / gravity correspondence in accelerating universe,” Phys. Rev. D 65, 125015 (2002) doi:10.1103/PhysRevD.65.125015 [hep-th/0203041].

[16] A. Buchel, P. Langfelder and J. Walcher, “On time dependent backgrounds in supergravity and string theory,” Phys. Rev. D 67, 024011 (2003) doi:10.1103/PhysRevD.67.024011 [hep-th/0207214].

[17] A. Buchel, “Compactifications of the N = 2* flow,” Phys. Lett. B 570, 89 (2003) doi:10.1016/j.physletb.2003.07.030 [hep-th/0302107].

[18] A. Buchel and A. Ghodsi, “Braneworld inflation,” Phys. Rev. D 70, 126008 (2004) doi:10.1103/PhysRevD.70.126008 [hep-th/0404151].

[19] A. Buchel, “Inflation on the resolved warped deformed conifold,” Phys. Rev. D 74, 046009 (2006) doi:10.1103/PhysRevD.74.046009 [hep-th/0601013].

[20] A. Buchel and D. A. Galante, “Cascading gauge theory on dS4 and String Theory landscape,” Nucl. Phys. B 883, 107 (2014) doi:10.1016/j.nuclphysb.2014.03.022 [arXiv:1310.1372 [hep-th]].
[21] A. Buchel and A. Karapetyan, “de Sitter Vacua of Strongly Interacting QFT,” JHEP 1703, 114 (2017) doi:10.1007/JHEP03(2017)114 [arXiv:1702.01320 [hep-th]].

[22] A. Buchel, “Verlinde Gravity and AdS/CFT,” arXiv:1702.08590 [hep-th].

[23] P. M. Chesler and L. G. Yaffe, “Numerical solution of gravitational dynamics in asymptotically anti-de Sitter spacetimes,” JHEP 1407, 086 (2014) doi:10.1007/JHEP07(2014)086 [arXiv:1309.1439 [hep-th]].

[24] I. Booth, “Black hole boundaries,” Can. J. Phys. 83, 1073 (2005) doi:10.1139/p05-063 [gr-qc/0508107].

[25] P. Figueras, V. E. Hubeny, M. Rangamani and S. F. Ross, “Dynamical black holes and expanding plasmas,” JHEP 0904, 137 (2009) doi:10.1088/1126-6708/2009/04/137 [arXiv:0902.4696 [hep-th]].

[26] P. Bosch, A. Buchel and L. Lehner, “Unstable horizons and singularity development in holography,” arXiv:1704.05454 [hep-th].

[27] A. Buchel, “Singularity development and supersymmetry in holography,” arXiv:1705.08560 [hep-th].

[28] P. K. Kovtun and A. O. Starinets, “Quasinormal modes and holography,” Phys. Rev. D 72, 086009 (2005) doi:10.1103/PhysRevD.72.086009 [hep-th/0506184].