Global existence for nonlocal quasilinear diffusion systems in nonisotropic nondivergence form

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Abstract
We consider the quasilinear diffusion problem of \( u = (u^1, \ldots, u^m) \)
\[
\begin{align*}
    u' + \Pi(t, x, u, \Sigma u) A u &= f(t, x, u, \Sigma u) \quad \text{in }]0, T[ \times \Omega, \\
    u &= 0 \quad \text{in }]0, T[ \times \Omega^c, \\
    u(0, \cdot) &= u_0(\cdot) \quad \text{in } \Omega,
\end{align*}
\]
for an open set \( \Omega \subset \mathbb{R}^n \), \( u_0 \in H_0^s(\Omega) := [H_0^{s}(\Omega)]^m \), for \( 0 < s \leq 1 \), and any \( T \in ]0, \infty[. \) Here, \( \Sigma \) denotes an operator which may involve the distributional Riesz fractional gradient \( D_\sigma \) of order \( \sigma \), with \( 0 < \sigma < 2s \), the classical gradient \( D^1 = \partial \) or/and nonlocal derivatives \( D^\sigma \), with \( 0 < \sigma < \min\{2s,1\} \). We show global existence results for various quasilinear diffusion systems in nondivergence form for linear elliptic operators \( A \), including classical elliptic systems, anisotropic fractional equations and systems, and anisotropic local and nonlocal operators of the following type:
\[
(A u)^i = -\sum_{\alpha, \beta, j} \alpha_{i} (A_{\alpha \beta}^j u^\alpha u^\beta), \quad A u = -D^{s}(A(x)D^{s}u), \quad \text{and} \quad (A u)^i = \int_{\mathbb{R}^n} A_i(x, y) u^i(x) - u^i(y) \frac{1}{|x-y|^{n+2s}} dy,
\]
for coercive, invertible matrices \( \Pi \) and suitable vectorial functions \( f \).

Key words
anisotropic fractional derivatives, maximal regularity, nonautonomous evolution equations, nonlocal quasilinear diffusion systems

1 | INTRODUCTION

Suppose that \( \Omega \subset \mathbb{R}^n \) is an open set (not necessarily bounded) and let \( T \in ]0, \infty[. \) Consider the quasilinear diffusion problem for \( u = (u^1, \ldots, u^m) = u(t, x) \)
\[
\begin{align*}
    u' + \Pi(t, x, u, \Sigma u) A u &= f(t, x, u, \Sigma u) \quad \text{in }]0, T[ \times \Omega, \\
    u &= 0 \quad \text{in }]0, T[ \times \Omega^c, \\
    u(0, \cdot) &= u_0(\cdot) \quad \text{in } \Omega,
\end{align*}
\]
with \(u_0 \in H^s_0(\Omega) := [H^s(\Omega)]^m\), for \(0 < s \leq 1\), and where \(\Sigma \in \mathbb{R}^{q \times n}\), for \(0 < q \leq m \times n\), represents a vector field defined by an operator \(A\), which may involve the distributional Riesz fractional gradient \(D^\delta\), of order \(\delta\) or \(\sigma\) or nonlocal derivatives \(D^2\) with \(0 < \sigma < 1\). Here, \(A\) is a time-independent local or nonlocal linear operator defined on the classical Sobolev space \(H^s_0(\Omega)\) with values in its dual space \(H^{-s}(\Omega)\), \(0 < s \leq 1\), bounded, symmetric in \(H^s_0(\Omega)\) and \(L^2(\Omega) = [L^2(\Omega)]^m\)-coercive, that is,

\[
\langle Au, v \rangle = \langle Av, u \rangle \quad \text{for all } u, v \in H^s_0(\Omega),
\]

\[
|\langle Au, v \rangle| \leq a^s \|u\|_{H^s(\Omega)} \|v\|_{H^s(\Omega)} \quad \text{for some } a^s > 0, \forall u, v \in H^s_0(\Omega), \quad \text{and}
\]

\[
\langle Au, u \rangle + \mu \|u\|^2_{L^2(\Omega)} \geq a_s \|u\|^2_{H^s(\Omega)} \quad \text{for some } \mu \geq 0, \quad a_s > 0, \quad \forall u \in H^s_0(\Omega).
\]

Here \(\Sigma u(t, x) \in \mathbb{R}^q\) represents a vector field defined in \(\mathbb{R}^n\) for \(0 < q \leq m \times n\) and for a.e. \((t, x) \in ]0, T[ \times \Omega\), where \(\Sigma u\) represents a linear combination of fractional or nonlocal derivatives in the form \(D^\delta u\) or \(D^2 u\). We shall assume that \(\Sigma : H^s_0(\Omega) \to [L^2(\mathbb{R}^n)]^q\) is a continuous linear map, so that

\[
\|\Sigma w\|_{L^2(\mathbb{R}^n)} \leq c_\Sigma \|w\|_{H^s(\Omega)} \quad \text{for all } w \in H^s_0(\Omega),
\]

where \(c_\Sigma\) is the constant for the estimate depending only on the structure of \(\Sigma\) and not on \(w \in H^s_0(\Omega)\).

Suppose also that \(f : ]0, T[ \times \Omega \times \mathbb{R}^m \times \mathbb{R}^q \to \mathbb{R}^m\) is measurable, such that it is continuous with respect to the last two variables for almost every \((t, x)\) and satisfies a linear growth condition with respect to the last variable, and \(\Pi : ]0, T[ \times \Omega \times \mathbb{R}^m \times \mathbb{R}^q \to \mathbb{R}^{m \times n}\) is a measurable, coercive, invertible matrix, such that \(\Pi = \Pi(t, x, v, q)\) is continuous with respect to the last two variables \((v, q)\) for almost every \((t, x)\) and

\[
γ|\xi|^2 \leq \Pi \xi : \xi \quad \text{and} \quad \Pi \xi : \xi_\ast \leq \bar{\gamma}|\xi_\ast|^2 \quad \text{for all } \xi, \xi_\ast \in \mathbb{R}^m,
\]

for all \((v, q)\) and almost all \((t, x)\), with \(0 < γ \leq \bar{\gamma}\).

The main purpose of this work is to prove the existence of a solution \(u\) to Problem (1) in the space

\[
H^1(0, T; L^2(\Omega)) \cap L^2(0, T; L^2_\delta) \cap C([0, T]; H^s_0(\Omega)).
\]

Here, \(L^2_\delta = D(A_\delta)\) is the domain of the operator \(A_\delta\), associated with homogeneous Dirichlet boundary condition when \(Au \in L^2(\Omega)\), given by

\[
L^2_\delta = D(A) := \{ u \in H^s_0(\Omega) : A u \in L^2(\Omega)\},
\]

as \(A\) may be regarded as an operator in the classical framework \(H^s_0(\Omega) \subset L^2(\Omega) \subset H^{-s}(\Omega)\). Then, by the second condition of (2), we see that this operator, still denoted by \(A\), is a closed operator in \(L^2_\delta\), and the space \(L^2_\delta\) is a Hilbert subspace of \(H^s_0(\Omega)\) when equipped with the graph norm \(\|u\|^2_{L^2_\delta} = \|u\|^2_{H^s(\Omega)} + \|Au\|^2_{L^2(\Omega)}\). Subsequently, the Bochner space \(L^2(0, T; L^2_\delta)\) is also a Hilbert space.

Note that, for \(u\) in appropriate spaces, \(f_u(t, x) = f(t, x, u(t, x), \Sigma u(t, x))\) and \(\Pi_u(t, x) = \Pi(t, x, u(t, x), \Sigma u(t, x))\) are functions in \(L^2(0, T; L^2(\Omega))\) and \(L^\infty(0, T; L^\infty(\Omega))\), respectively. Problem 1 generalizes the quasilinear equation defined with the classical gradient in [3] to systems of equations with more general derivatives.

Following [15] and [8] (see also [43]), for \(0 < s < 1\), the Riesz fractional gradient \(D^s u\) may be defined component-wise in the integral form for vectors \(u = (u^1, u^2, \ldots, u^m) \in H^s_0(\Omega)\), respectively, by

\[
D^s u^j(x) := c_{n,s} \int_{\mathbb{R}^n} \frac{u^j(x) - u^j(y)}{|x-y|^{n+s}} \frac{x_i - y_i}{|x-y|} \, dy, \quad i = 1, \ldots, n, \quad j = 1, \ldots, m, \quad 0 < s < 1,
\]

where \(c_{n,s} = 2^s \pi^{-\frac{n+1}{2}} \Gamma\left(\frac{1+s}{2}\right)\) is given in terms of the Gamma-function, and \(u\) is extended by 0 outside \(\Omega\). It is well known that \(D^s\) applies continuously from \(H^s_0(\Omega)\) into \([L^2(\mathbb{R}^n)]^{m \times n}\), see, for instance, [15]. For \(s > 1/2\), we suppose \(\Omega\) satisfies the extension property, so that we can assume that the extension of \(u\) is in \(H^s(\mathbb{R}^n)\) whenever \(u \in H^s_0(\Omega)\). Similarly, the scalar
nonlocal derivative $D^s u$ may be defined component-wise, following [22], for a suitable kernel $\alpha_s(x, y)$ in the form

$$D^s u^j(x) := \int_{\mathbb{R}^n} (u^j(x) - u^j(y))\alpha_s(x, y) \, dy, \quad j = 1, \ldots, m, \quad s > 0,$$

(6)

such that $D^s$ is a linear continuous operator from $H^s_0(\Omega)$ to $L^2(\mathbb{R}^n)$. There are different possibilities to choose the singular or nonsingular kernel $\alpha_s$. In particular, it includes the case when

$$\alpha_s(x, y) = c_{n,s}|x - y|^{-n-s},$$

which corresponds to the fractional Laplacian $(-\Delta)^{s/2}$ of order $s/2$.

For $1 \leq s < 2$, we define the Riesz fractional gradients by

$$D^s u^j = D_1^{s-1}(\partial_{x_i} u^j), \quad 1 \leq s < 2,$$

(7)

where $\partial_{x_i} = \partial/\partial x_i$ denotes the classical partial derivative, so $D_1 u^j = \partial_{x_i} u^j$.

The linear operator $A$ may involve the classical gradient $\partial$, the Riesz fractional gradient $D^s$, or the nonlocal derivative $\nabla^s$, as long as it is bounded and $L^2(\Omega)$-coercive, as in the following examples. When $s = 1$, this includes the local operator given by

$$\langle A u, v \rangle = \langle L u, v \rangle = \sum_{\alpha, \beta, i, j} A^{\alpha \beta}_{ij} \partial_{x_i} u^j \partial_{x_\beta} v^i$$

(8)

with a bounded, coercive tensor $A = (A^{\alpha \beta}_{ij}(x))$, symmetric in $\alpha$ and $\beta$, where $\langle \cdot, \cdot \rangle$ is understood as the duality between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ such that (2) holds. Here, $\alpha$ and $\beta$ run from 1 to $n$, and $i$ and $j$ run from 1 to $m$. A typical example is when $A$ satisfies the Legendre–Hadamard condition defined later in (21), which is well known to satisfy (2) (see, for instance, Theorem 3.42 of [27]).

$A$ may also be the anisotropic fractional operator

$$\langle A_A u, v \rangle = \langle L_A u, v \rangle = \sum_{\alpha, \beta, i, j} A^{\alpha \beta}_{ij} D_\alpha^s u^j D_\beta^s v^i$$

(9)

for $s \leq 1$, where $D_\alpha^s$ coincides with $\partial_{x_\alpha}$ in the classical case of $s = 1$, where $\langle \cdot, \cdot \rangle$ is now understood as the duality between $H^{-s}(\Omega)$ and $H^s_0(\Omega)$. A sufficient condition on $A$ to satisfy (2) is when $A$ has constant coefficients and satisfies the Legendre–Hadamard condition in (21) below with a constant $a > 0$. Indeed, following the proof of Step 1 of Theorem 3.42 of [27], we have, for a smooth function $u$ with compact support, extending it by zero outside $\Omega$,
Here, we denote the Fourier transform of \( f \) by \( \hat{f} \). By density, this coercivity inequality holds for all \( u \in H^s_0(\Omega) \). Recall also that, with \( 0 < s \leq 1 \),

\[
\widehat{D^s_i u^j} = -(2\pi)^i \xi_i |\xi|^{-1 + s} \hat{u}^j
\]

for all \( u^j \in H^n(\mathbb{R}^n) \), where \( i = \sqrt{-1} \), as given in Theorem 1.4 of [43, part I]. We have also made use of Parseval’s identity \( \langle f, g \rangle_{L^2} = \langle \hat{f}, \hat{g} \rangle_{L^2} \).

We can also consider the anisotropic nonlocal operator defined by \( \mathbb{A} : H^s_0(\Omega) \to H^{-s}(\Omega) \)

\[
\mathbb{A} u = L_A^s u = P.V. \int_{\mathbb{R}^n} A(x, y) \frac{u(x) - u(y)}{|x-y|^{n+2s}} \, dy
\]

with a symmetric, bounded, coercive matrix kernel \( A = (A_{ij}(x, y)) \), that is, for almost all \((x, y)\) in \( \mathbb{R}^n \times \mathbb{R}^n \),

\[
a_+ |\xi|^2 \leq A \xi \cdot \xi \leq a^* |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^m,
\]

for \( s < 1 \) and for given constants \( 0 < a_+ \leq a^* \). As a consequence, for all \( u, v \in H^s_0(\Omega) \),

\[
\langle L_A^s u, v \rangle = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A(x, y) \frac{u(x) - u(y)}{|x-y|^{n+2s}} \cdot \frac{v(x) - v(y)}{|x-y|^{n+2s}} \, dy \, dx \leq a^* \| u \|_{H^s_0(\Omega)} \| v \|_{H^s_0(\Omega)}
\]

and

\[
\langle L_A^s u, u \rangle \geq a_+ \| u \|^2_{H^s_0(\Omega)}.
\]

The fractional Sobolev spaces \( H^s(\mathbb{R}^n) \) for all real \( s \) are defined by

\[
H^s(\mathbb{R}^n) = \{ u \in L^2(\mathbb{R}^n) : |\xi| \mapsto (1 + |\xi|^2)^{s/2} \hat{u}(\xi) \in L^2(\mathbb{R}^n) \},
\]

with norm

\[
\| u \|_{H^s(\mathbb{R}^n)} = \| (1 + |\xi|^2)^{s/2} \hat{u} \|_{L^2(\mathbb{R}^n)},
\]

where \( \hat{u} \) is the Fourier transform of \( u \). For \( 0 < s < 1 \), this norm is well known to be equivalent to

\[
\| u \|^2_{H^s(\mathbb{R}^n)} = \| u \|^2_{L^2(\mathbb{R}^n)} + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dx \, dy =: \| u \|^2_{L^2(\mathbb{R}^n)} + [u]^2_{H^s(\mathbb{R}^n)}.
\]

On the other hand, as it was shown in [43] and [38], the \( H^s(\mathbb{R}^n) \)-norm given by (11) is in fact equal to

\[
\| u \|^2_{H^s(\mathbb{R}^n)} = \| u \|^2_{L^2(\mathbb{R}^n)} + \frac{2}{c_{n, s}^2} \int_{\mathbb{R}^n} |D^s u|^2 = \| u \|^2_{L^2(\mathbb{R}^n)} + \frac{2}{c_{n, s}^2} \| D^s u \|^2_{L^2(\mathbb{R}^n)}.
\]

If \( \Omega \) has a Lipschitz boundary, hence it satisfies the extension property, \( H^s(\Omega) \) coincides with the space of restrictions to \( \Omega \) of the elements of \( H^s(\mathbb{R}^n) \) as in [37] and [19], with norm

\[
\| u \|_{H^s(\Omega)} = \inf_{U = u \text{ a.e. } \Omega} \| U \|_{H^s(\mathbb{R}^n)}.
\]

The subspace \( H^s_0(\Omega) \) is the usual Sobolev space, for \( 0 < s \leq 1 \), given by the closure of \( C_c^\infty(\Omega) \) in \( H^s(\Omega) \) for general open sets \( \Omega \subset \mathbb{R}^n \), as in [37], and \( H^{-s}(\Omega) \) its dual. Since \( C_c^\infty(\Omega) \) is dense in \( H^s(\mathbb{R}^n) \) if and only if \( s \leq \frac{1}{2} \), in this case, \( H^1_0(\Omega) = H^0(\Omega) \). Otherwise, if \( s > \frac{1}{2} \), \( H^s_0(\Omega) \) is strictly contained in \( H^s(\Omega) \). On the other hand, as in [19], for bounded sets with Lipschitz
boundary, $\mathcal{O} \subset \mathbb{R}^n$, $C_0^\infty(\mathcal{O})$ is dense in $H^s(\mathcal{O})$ for all $s \geq 0$. These density properties can be further extended for $s > 1$, with an abuse of notation, by defining $H_0^s(\Omega)$ to be the space

$$H_0^s(\Omega) := \{ u \in H^s(\mathbb{R}^n) : \text{supp } u \subset \bar{\Omega} \}.$$

Consider the maximal regularity space

$$MR := H^1(0, T; L^2(\Omega)) \cap L^2(0, T; L^2_\mathcal{A}),$$

equipped with the norm for $0 < s \leq 1$

$$\|u\|_{MR}^2 := \int_0^T \|u'(t)\|_{L^2(\Omega)}^2 + \int_0^T \|u(t)\|_{H^s(\Omega)}^2 + \int_0^T \|\mathcal{A}u(t)\|_{L^2(\mathbb{R}^n)}^2,$$

(14)

so that the linear inhomogeneous problem

$$u'(t) + \mathcal{A}u(t) = f(t) \quad \text{for a.e. } t \in ]0, T[, \quad u(0) = 0,$$

with a source term $f \in L^2(0, T; L^2(\Omega))$ is well-posed in $MR$.

Classically, parabolic quasilinear systems in nondivergence form have frequently been considered (see [2, 5, 23, 26, 30, 31, 34, 41] and their references). These systems of equations have multiple physical, chemical, and biological applications such as in reaction–diffusion systems (see, e.g., [39]), phase-field models (see, e.g., [40]), and population models (see, for instance, [35] and [9]). Parabolic equations have also been extended to the case of nonlocal reaction–diffusion (see, for instance, [7] and [13]).

In this work, we extend the method introduced in [3], which is based on a combination of Schaefer's fixed point theorem with maximal regularity of solutions to parabolic problems in the Hilbertian framework [6], to local and nonlocal quasilinear systems in nondivergence form. In Section 2, we shall first consider the linear problem, extending the approach of [3] to systems by introducing a suitable time-dependent matrix $Y$. We shall obtain the solution to the general nonautonomous linear problem in $MR$. This result is exemplified with three linear systems corresponding to the types described above. Furthermore, in some of these examples, the space-variable regularity can be improved by assuming that $\Omega$ is a bounded Lipschitz domain.

Next, we shall use the existence and regularity results for the linear problem to obtain the existence of a solution to the quasilinear problem by a fixed point argument. Using the currently known regularity of the solutions to the Dirichlet problems associated with the operator $\mathcal{A}$, in Section 3, we obtain the existence of a solution for the global quasilinear nondivergent systems for the general operators $\mathcal{A}$ satisfying (2), and for the operator $\Sigma$ with fractional gradients $D^\sigma$ or nonlocal derivatives $D^\sigma$ of order $\sigma$, first for $\sigma < s$ and afterwards also for particular operators satisfying additional regularity properties, up to and including $\sigma = s \leq 1$. Our results extend the case of the nonlocal vectorial problem with no source function considered in [33], as well as the vectorial semilinear case in [1, 39] and [4]. Our results also generalize [3] to systems of the form (1) defined in a bounded or unbounded open set $\Omega \subset \mathbb{R}^n$, for more general derivatives, which can take any positive order less than $s$. These results are new even in the classical case of $s = 1$.

In Section 4, the existence results are then generalized to a larger range of $\sigma$, namely, $s < \sigma < 2s$, in the case of $\Omega$ bounded with Lipschitz boundary, by making use of known regularity results for vectorial local and nonlocal operators. In particular, we generalize from the scalar case of [3] with $s = 1$, to nonlocal quasilinear diffusion systems. As a result, we can also consider quasilinear diffusion equations and systems with gradients $D^\sigma$ of order $\sigma > s$, which may be greater than 1, generalizing the results of [3, 4] and [33]. These results may provide useful applications, particularly in population models and advection–diffusion systems, as we try to exemplify in model problems.

## 2 A NONAUTONOMOUS LINEAR PROBLEM

In this section, we first consider the linear problem for a system with a symmetric operator $\mathcal{A}$ satisfying (2), for $0 < s \leq 1$. This includes the classical case of $s = 1$ considered in [4] and we shall extend the approach of [3] to systems, by introducing a suitable matrix $Y$. 
We first observe that for \( u \in MR \), \( \mathbb{A}u(t) \in L^2(\Omega) \) for a.e. \( t \in [0, T] \). Furthermore, by the definition of \( \mathbb{A} \) as a symmetric time-independent operator, we have the following well-known result (see, for instance, [17, p. 480]), which we include here for completeness.

Lemma 1. Let \( u \in MR \). Then, \( \int_{\Omega} \mathbb{A}u(\cdot) \cdot u(\cdot) \in W^{1,1}(0, T) \) and
\[
\frac{d}{dt} \int_{\Omega} \mathbb{A}u(t) \cdot u(t) = 2 \int_{\Omega} \mathbb{A}u(t) \cdot u'(t) \quad \text{for a.e. } t \in [0, T].
\]
Furthermore, the continuous embedding holds
\[
MR \hookrightarrow C([0, T]; H^s_0(\Omega)).
\]

Proof. We first take \( u \in C^1([0, T]; L^{2,\mathbb{A}}) \). Then, since \( \mathbb{A} \) is symmetric and time-independent, we have
\[
\int_{\Omega} \mathbb{A}u(t) \cdot u'(t) = \frac{1}{2} \left( \int_{\Omega} \mathbb{A}u(t) \cdot u(t) + \int_{\Omega} u(t) \cdot \mathbb{A}u'(t) \right) = \frac{1}{2} \frac{d}{dt} \int_{\Omega} \mathbb{A}u(t) \cdot u(t).
\]

Since \( C^1([0, T]; L^{2,\mathbb{A}}) \) is dense in \( MR \), the result holds for arbitrary \( u \in MR \) by approximation.

The second part of the lemma may be proved as in Proposition 3.6 of [20]. Indeed, since \( \int_{\Omega} \mathbb{A}u(\cdot) \cdot u(\cdot) \in W^{1,1}(0, T) \subset C([0, T]) \), together with the continuous embedding \( MR \hookrightarrow C([0, T]; H^s_0(\Omega)) \) and the coercivity (2) yields \( MR \subset L^\infty(0, T; H^s_0(\Omega)) \). Now, it is well known that \( C([0, T]; L^2(\Omega)) \cap L^\infty(0, T; H^s_0(\Omega)) \subset C([0, T]; H^s_0(\Omega)-weak) \) (see, for instance, Lemma 3.3 of [20]). Then, as \( t \to t \) for fixed \( t \),
\[
a_{s} \| u(t) - u(\tau) \|^2_{H^s_0(\Omega)} \leq \int_{\Omega} \mathbb{A}(u(t) - u(\tau)) \cdot (u(t) - u(\tau)) + \mu \| u(t) - u(\tau) \|^2_{L^2(\Omega)}
\]
\[
= 2 \int_{\Omega} \mathbb{A}(u(t) - u(\tau)) \cdot u(t) + \int_{\Omega} [\mathbb{A}u(\tau) \cdot u(\tau) - \mathbb{A}u(t) \cdot u(t)] + \mu \| u(t) - u(\tau) \|^2_{L^2(\Omega)}.
\]

The three terms tend to 0: the first one by the weak continuity of \( u(\cdot) \) in \( H^s_0(\Omega) \), the second one by the continuity of the map \( \tau \mapsto \int_{\Omega} \mathbb{A}u(\tau) \cdot u(\tau) \), and the third one again by the embedding \( MR \hookrightarrow C([0, T]; L^2(\Omega)) \). \( \square \)

Recall, for instance, from [17, footnote on p. 244, and Example 1 pp. 479–480] and [6, Theorem 1.1(iii)], the following known maximal regularity result: For all \( f \in L^2(0, T; L^2(\Omega)) \), \( u_0 \in H^s_0(\Omega) \), there exists a unique solution to the autonomous problem
\[
\begin{align*}
\mathbb{u} \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; L^{2,\mathbb{A}}) \cap C([0, T]; H^s_0(\Omega)), \\
u'(t) + \mathbb{A}u(t) = f(t) \quad \text{for a.e. } t \in [0, T], \\
u(0) = u_0.
\end{align*}
\]

We consider now a linear nonautonomous problem, obtained by a multiplicative perturbation.

Theorem 1. Let \( Y = Y(t, x) : [0, T] \times \Omega \to \mathbb{R}^{m \times n} \) be a matrix-valued function with invertible values that is coercive and satisfies (4). Then, for every \( f \in L^2(0, T; L^2(\Omega)) \), \( u_0 \in H^s_0(\Omega) \), there exists a unique solution of the problem
\[
\begin{align*}
\mathbb{u} \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; L^{2,\mathbb{A}}) \cap C([0, T]; H^s_0(\Omega)), \\
u'(t) + Y(t, \cdot)\mathbb{A}u(t) = f(t) \quad \text{for a.e. } t \in [0, T], \\
u(0) = u_0.
\end{align*}
\]
Moreover, there exists a constant \( c = c(\gamma, \tilde{\gamma}, a_s, a^*, \mu, T) > 0 \) independent of \( f \) and \( u_0 \) such that
\[
\|u\|_{MR} \leq c \left( \|f\|_{L^2(0,T;L^2(\Omega))} + \|u_0\|_{H^1_0(\Omega)} \right)
\]
for each solution \( u \) of (16)

**Proof.** We use the method of continuity (cf. Section 5.2 of [28]) as in the proof of Theorem 3.2 of [3]. For every \( \lambda \in [0,1] \), consider the function \( Y_\lambda := (1-\lambda)I + \lambda Y \) for the identity matrix \( I \) of dimension \( m \times m \) and the bounded operator
\[
B_\lambda : MR \to L^2(0,T;L^2(\Omega)) \times H^1_0(\Omega)
\]
given by
\[
B_\lambda u = (u' + \lambda A u, u_0).
\]
Then, \( B(\lambda) := B_\lambda, B : [0,1] \to \mathcal{L}(MR,L^2(0,T;L^2(\Omega)) \times H^1_0(\Omega)) \) (where \( \mathcal{L} \) denotes the space of linear bounded operators), is continuous and \( B_0 \) is invertible by the maximal regularity result for the linear autonomous problem (15). Therefore, by Theorem 5.2 of [28], it suffices to prove the a priori estimate
\[
\|u\|_{MR} \leq \|B_\lambda u\| = c_1 \left( \|u' + \lambda A u\|_{L^2(0,T;L^2(\Omega))} + \|u_0\|_{H^1_0(\Omega)} \right) \quad \forall \lambda \in [0,1], \forall u \in MR,
\]
for some constant \( c_1 = c_1(\gamma, \tilde{\gamma}, a_s, a^*, T) > 0 \), which gives (17) for \( \lambda = 1 \).

Let \( \lambda \in [0,1] \). Let \( u \in M^R \) be such that
\[
u' + \lambda A u = f \quad \text{and} \quad u(0) = u_0.
\]
Then, multiplying the equation by \( [Y_\lambda^+]^{-1}u'(t) \), where \( [Y_\lambda^+]^{-1} \) is the inverse of the adjoint of \( Y_\lambda \), we have, for almost every \( t \in [0,T] \),
\[
\int_{\Omega} [Y_\lambda^+]^{-1}u'(t) \cdot u'(t) + \int_{\Omega} \lambda A u \cdot [Y_\lambda^+]^{-1}u'(t) = \int_{\Omega} f \cdot [Y_\lambda^+]^{-1}u'(t),
\]
which by Lemma 1 and the Cauchy–Schwarz inequality, gives
\[
\int_{\Omega} [Y_\lambda^+]^{-1}u'(t) \cdot u'(t) + \frac{1}{2} \frac{d}{dt} \int_{\Omega} \lambda A u \cdot u'(t) \leq \frac{\tilde{\gamma}}{2} \int_{\Omega} [Y_\lambda^+]^{-1}f(t)^2 + \frac{1}{2\gamma} \int_{\Omega} [u'(t)]^2.
\]
Integrating over time on \( ]0,t[ \) for every finite \( t \in ]0,T[ \) and using the estimate (4), it follows by (2) that
\[
\frac{1}{2\gamma} \int_{0}^{t} \|u'(\tau)\|_{L^2(\Omega)}^2 d\tau + \frac{a_s}{2} \|u(t)\|_{H^1_0(\Omega)}^2 \leq \frac{a^*}{2} \|u_0\|_{H^1_0(\Omega)}^2 + \frac{\mu}{2} \|u(t)\|_{L^2(\Omega)}^2 + \frac{\tilde{\gamma}}{2\gamma} \int_{0}^{t} \|f(\tau)\|_{L^2(\Omega)}^2 d\tau.
\]
Observe that by the Cauchy–Schwarz inequality,
\[
\|u(t)\|_{L^2(\Omega)}^2 = \|u(0)\|_{L^2(\Omega)}^2 + \int_{0}^{t} \frac{d}{d\tau} \|u(\tau)\|_{L^2(\Omega)}^2 d\tau
\]
\[
= \|u_0\|_{L^2(\Omega)}^2 + 2 \int_{0}^{t} \int_{\Omega} u(\tau) \cdot u'(\tau) d\tau
\]
\[
\leq \|u_0\|_{L^2(\Omega)}^2 + 2\mu\tilde{\gamma} \int_{0}^{t} \|u(\tau)\|_{L^2(\Omega)}^2 d\tau + \frac{1}{2\mu\tilde{\gamma}} \int_{0}^{t} \|u'(\tau)\|_{L^2(\Omega)}^2 d\tau,
\]

(19)
and so we have

\[
\frac{1}{2\gamma} \int_0^t \|u'(\tau)\|^2_{L^2(\Omega)} \, d\tau + a_s \|u(t)\|^2_{H^1_0(\Omega)} \leq a^* \|u_0\|^2_{H^1_0(\Omega)} + \mu \|u_0\|^2_{L^2(\Omega)} + \frac{\gamma}{\gamma^2} \|f\|^2_{L^2(0,T;L^2(\Omega))} + 2\mu^2 \gamma \int_0^t \|u(\tau)\|^2_{H^1_0(\Omega)} \, d\tau \leq a^* \|u_0\|^2_{H^1_0(\Omega)} + \mu \|u_0\|^2_{L^2(\Omega)} + \frac{\gamma}{\gamma^2} \|f\|^2_{L^2(0,T;L^2(\Omega))} + 2\mu^2 \gamma c_S \int_0^t \|u(\tau)\|^2_{H^1_0(\Omega)} \, d\tau,
\]

(20)

where \(c_S\) is the constant for the Sobolev embedding \(H^1_0(\Omega) \hookrightarrow L^2(\Omega)\). Applying the integral form of Grönwall’s lemma to the second term on the left-hand side, there exists a constant \(c_2 = c_2(\gamma, \bar{\gamma}, a^*, a^*, \mu, T) > 0\) such that

\[
\sup_{t \in [0,T]} \|u(t)\|^2_{H^1_0(\Omega)} \leq c_2 \left( \|u_0\|^2_{H^1_0(\Omega)} + \|f\|^2_{L^2(0,T;L^2(\Omega))} \right).
\]

Inserting this into (20), we obtain that

\[
\int_0^T \|u'(\tau)\|^2_{L^2(\Omega)} \, d\tau \leq c_3 \left( \|u_0\|^2_{H^1_0(\Omega)} + \|f\|^2_{L^2(0,T;L^2(\Omega))} \right)
\]

for some constant \(c_3 = c_3(\gamma, \bar{\gamma}, a^*, a^*, \mu, T) > 0\).

Finally, since

\[
\int_0^T \|A\, u(\tau)\|^2_{L^2(\mathbb{R}^n)} \, d\tau \leq \frac{1}{\gamma^2} \int_0^T \|Y\, A\, u(\tau)\|^2_{L^2(\mathbb{R}^n)} \, d\tau = \frac{1}{\gamma^2} \int_0^T \|u'(\tau) - f(\tau)\|^2_{L^2(\Omega)} \, d\tau,
\]

the MR norm of \(u\) can be estimated giving (17) and by Lemma 1, the proof is complete. \(\square\)

**Remark 1.** If \(\mu = 0\) in (2), Theorem 1 is a special case of Theorem 1.1 of [6].

Next, we identify local and nonlocal vectorial operators to which we can apply Theorem 1.

**Example 1: Local operators**

As a first example of \(A\), we consider the local operator \(L\), such that the \(i\)-th component of \(L\,u\) is given by

\[
(L\,u)^i = -\sum_{\alpha,\beta,j} \partial_{\alpha}(A_{ij}^{\alpha\beta} \partial_{\beta} u^j) + b_i u^i
\]

(8)

for \(b_i \in L^\infty(\Omega)\), and \(A_{ij}^{\alpha\beta} \in L^\infty(\Omega)\) is a bounded, coercive tensor symmetric in \(\alpha,\beta\) with \(0 < a_\alpha \leq a^*\), that is,

\[
\sum_{\alpha,i} a_\alpha |\Xi_i|^2 \leq \sum_{\alpha,\beta,i,j} A_{ij}^{\alpha\beta} \Xi_i^\alpha \Xi_j^\beta \leq a^* \sum_{\alpha,i} |\Xi_i^\alpha|^2 \text{ for all } \Xi \in \mathbb{R}^{m \times n}.
\]

Then, the linear nonautonomous problem for \(f \in L^2(0,T;L^2(\Omega))\) and \(u_0 \in H^1_0(\Omega)\) given by

\[
u'(t) + Y(t,\cdot)\, L\, u(t) = f(t), \text{ for a.e. } t \in [0,T],
\]

has a solution \(u \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;L^2_\Sigma)\). This extends the results of [3].

Furthermore, we can explicitly write out the space \(L^2_\Sigma\) for the following special case of \(A = L\) using the following proposition:
Proposition 1 (Theorem 4.9 of [27]). Let \( \Omega \) be an open domain in \( \mathbb{R}^n \). Suppose in addition that \( A_{ij}^{\alpha\beta} \in C^{0,1}_{\text{loc}}(\Omega) \) is continuous up to the boundary of \( \overline{\Omega} \) and satisfies the Legendre–Hadamard condition

\[
\sum_{\alpha,\beta,I,J} A_{ij}^{\alpha\beta} \nu_\alpha \nu_\beta \xi_I \xi_J \geq a |\xi|^2 |\nu|^2 \quad \forall \xi \in \mathbb{R}^m, \nu \in \mathbb{R}^n
\] (21)

for some constant \( a > 0 \). Then, each weak solution of \( \mathcal{L}u = f \) with \( f \in L^2_{\text{loc}}(\Omega) \) is in \( H^2_{\text{loc}}(\Omega) \).

If, in addition, by Theorem 6 of [21], \( \Omega \) is bounded with \( C^{1,1} \) boundary and \( A_{ij}^{\alpha\beta} \in C^{0,1}(\overline{\Omega}) \), we can extend the regularity result up to the boundary of \( \Omega \) for the unique solution of the homogeneous Dirichlet problem for \( f \in L^2(\Omega) \), so that the unique solution \( u \in H^1_0(\Omega) \) lies in \( H^2(\Omega) \).

It is well known (see, for instance, section 5 of [25]) that the Legendre–Hadamard condition (21) for tensors \( A \) continuous up to the boundary of \( \overline{\Omega} \) implies the coercivity condition in (2), which we recall, is given by Gårding’s inequality

\[
(\mathcal{L}u, u) + \mu \|u\|^2_{L^2(\Omega)} \geq a_* \|u\|^2_{H^1_0(\Omega)} \quad \forall u \in H^1_0(\Omega). \quad \text{(2nd condition of (2))}
\]

(Recall also that this is not true if \( u \) does not have support in \( \overline{\Omega} \).) Therefore, as a corollary, we have the following:

Corollary 1. Suppose \( \mathcal{L} \) is of the form (8) such that \( A_{ij}^{\alpha\beta} \) is locally Lipschitz and continuous up to the boundary of \( \overline{\Omega} \) satisfying the Legendre–Hadamard condition (21), then the linear nonautonomous Cauchy problem for \( f \in L^2(0,T;L^2(\Omega)) \) and \( u_0 \in H^1_0(\Omega) \) given by

\[
u'(t) + Y(t, \cdot) \mathcal{L}u(t) = f(t), \quad \text{for a.e. } t \in [0,T],
\]

has a solution \( u \in H^1(0,T;L^2(\Omega)) \cap C([0,T];H^1_0(\Omega)) \cap L^2(0,T;H^2_{\text{loc}}(\Omega)) \).

If, in addition, \( \Omega \) is bounded with \( C^{1,1} \) boundary and \( A_{ij}^{\alpha\beta} \in C^{0,1}(\overline{\Omega}) \), then \( u \in H^1(0,T;L^2(\Omega)) \cap C([0,T];H^1_0(\Omega)) \cap L^2(0,T;H^2(\Omega)) \).

Example 2: Anisotropic fractional operators

We can also consider the vectorial operators \( \mathcal{L}_A^s : H^s_0(\Omega) \rightarrow H^{-s}(\Omega) \) for \( 0 < s \leq 1 \), given by

\[
\langle \mathcal{L}_A^s u, v \rangle = \sum_{\alpha,\beta,I,J} \int_{\mathbb{R}^n} A_{ij}^{\alpha\beta} D^s_{\alpha} u_I D^s_{\beta} v_J
\] (9)

with \( A_{ij}^{\alpha\beta} \) as in Example 1. Here, \( D^s_{\alpha} \) coincides with the classical derivative \( \partial_{\alpha} \) in the classical case of \( s = 1 \), and \( \mathcal{L}_A^s \) in (9) reduces to \( \mathcal{L} \) in (8) when \( s = 1 \).

Then, the linear nonautonomous problem for \( f \in L^2(0,T;L^2(\Omega)) \) and \( u_0 \in H^s_0(\Omega) \) given by

\[
u'(t) + Y(t, \cdot) \mathcal{L}_A^s u(t) = f(t), \quad \text{for a.e. } t \in [0,T]
\]

has a solution \( u \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^s_{\text{loc}}(\Lambda)) \).

In the particular case when \( A^{\alpha\beta} \) is given by a diagonal constant matrix, \( \mathcal{L}_A^s \) corresponds to a system of equations defined with the fractional Laplacian.

Recall that the fractional Laplacian is defined, for \( u \in H^s_0(\Omega) \), by

\[
(-\Delta)^s u(x) = c_{n,2s} P.V. \int_{\mathbb{R}^n} u(x) - u(y) |x - y|^{n+2s} dy.
\] (22)

Then, we have the following:
Proposition 2 (Theorem 7.1 of [29], or Theorem 4.1 and Remark 7 of [10]). Suppose $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain.

Let $f \in L^2(\Omega)$ and $s \in ]\frac{1}{2}, 1[$. Then the solution to the homogeneous Dirichlet problem

$$(-\Delta)^s u = f \text{ in } \Omega, \quad u = 0 \text{ in } \Omega^c$$

lies in the Besov space

$$u \in \dot{B}^{s+1/2}_{2,\infty}(\Omega) \subset H^{s+1/2-\epsilon}(\Omega)$$

for any positive $\epsilon$.

Consider the vectorial fractional Laplacian $(-\Delta)^s_m$ defined by

$$(-\Delta)^s_m = \begin{bmatrix} c_1(-\Delta)^s & 0 \\ \vdots & \ddots \\ 0 & c_m(-\Delta)^s \end{bmatrix}$$

for constants $c_1, \ldots, c_m > 0$. Then, applying Proposition 2 component-wise, we have the following:

Corollary 2. Suppose $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain. The linear nonautonomous Cauchy problem for $f \in L^2(0,T;L^2(\Omega))$ and $u_0 \in H^s_0(\Omega)$ given by

$$u'(t) + Y(t,\cdot)(-\Delta)^s_m u(t) = f(t), \quad \text{for a.e. } t \in ]0,T[$$

has a solution $u \in H^1(0,T;L^2(\Omega)) \cap C([0,T];H^s_0(\Omega)) \cap L^2(0,T;H^{s+1/2-\epsilon}(\Omega))$ for any positive $\epsilon$.

Example 3: Anisotropic nonlocal operators

Next, we consider the anisotropic nonlocal operator $L^s_A : H^s_0(\Omega) \to H^{-s}(\Omega)$ for $0 < s < 1$

$$L^s_A u = P.V. \int_{\mathbb{R}^n} A(x,y) \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy \quad (10)$$

defined for a symmetric, bounded, coercive matrix kernel $A(x,y)$ such that (2) holds. Then, once again, the linear nonautonomous problem for $f \in L^2(0,T;L^2(\Omega))$ and $u_0 \in H^s_0(\Omega)$ given by

$$u'(t) + Y(t,\cdot)L^s_A u(t) = f(t), \quad \text{for a.e. } t \in ]0,T[$$

has a solution $u \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;L^2_{L^s_A})$.

Suppose in addition, $m = n$ and the kernel $A(x,y)$ is a measurable matrix of the form

$$A(x,y) = \frac{\hat{a}(x-y)}{|x-y|^{n+2s}} \chi_{C \cap B_r(0)}(x-y) \left( \frac{x-y}{|x-y|} \otimes \frac{x-y}{|x-y|} \right), \quad (23)$$

where $\hat{a}$ is an even, coercive, and bounded function such that $0 < a_s \leq \hat{a} \leq a^* < \infty$ for some constants $a_s, a^* > 0$, $0 < r \leq \infty$, and $C$ is a double cone with apex, that is,

$$C = \left\{ h \in \mathbb{R}^n \setminus \{0\} : \frac{h}{|h|} \in \mathcal{O} \cup (-\mathcal{O}) \text{ for any open subset } \mathcal{O} \text{ of the unit sphere } S^{d-1} \right\}$$

with positive Hausdorff measure.
Defining the space $H^{s}_{\text{loc}}(\Omega)$ by $\{u \in L^2(\Omega) : \eta u \in H^s(\Omega) \forall \eta \in C_c^\infty(\Omega)\}$, by Theorem 3.1 of [32], we have the following local regularity result for $L^s_A$ for this special case:

**Proposition 3.** For $0 < s < 1$, let $\Omega \subset \mathbb{R}^n$ be an open set, and $f \in L^2(\Omega)$ extended by 0 outside. Then the weak solution to

$$L^s_A u = f \text{ in } \Omega, \quad u = 0 \text{ in } \Omega^c$$

lies in $H^{2s}_{\text{loc}}(\Omega)$. Moreover, for any $\eta \in C_c^\infty(\Omega)$, there exists a constant $C$ such that

$$\|\eta u\|_{H^{2s}(\mathbb{R}^n)} \leq C\|f\|_{L^2(\Omega)}.$$ 

**Remark 2.** Observe that $L^s_A$ can be viewed as the nonlocal version of Example 1, as explained in pp. 1304–1305 of [32]. See also Lemma 3.1 of [24].

As a corollary, we once again have the following:

**Corollary 3.** The linear nonautonomous Cauchy problem for $f \in L^2(0, T; L^2(\Omega))$ and $u_0 \in H^s_0(\Omega)$ given by

$$u'(t) + Y(t, \cdot)L^s_A u(t) = f(t), \quad \text{for a.e. } t \in [0, T],$$

has a solution $u \in H^1(0, T; L^2(\Omega)) \cap C([0, T]; H^s_0(\Omega)) \cap L^2(0, T; H^{2s}_{\text{loc}}(\Omega))$.

### 3 THE NONLINEAR PROBLEM $\sigma \leq s \leq 1$

We next consider the quasilinear vectorial problem, when $0 < \sigma < s \leq 1$, extending the nonlocal vectorial problem with no source function considered in [33], as well as the vectorial semilinear case in [1, 39] and [4]. This also generalizes [3] to systems of the form (1) defined in a bounded or unbounded open set $\Omega \subset \mathbb{R}^n$. We will apply the Schaefer fixed point theorem, which is a generalization of the Leray–Schauder fixed point theorem to locally convex spaces, to the approximating bounded subsets $\Omega_k \subset \Omega$, as in Theorem 4.1 of [3], so that the regularity of the boundary of $\Omega$ can be ignored.

Assume

$$D(A) = L^2_A \subset H^{s+\theta}_{\text{loc}}(\Omega) \text{ for some } \theta \geq 0. \tag{24}$$

Note that this assumption is weaker than the one given in assumption (4.1) of [3], and allows us to cover the fractional derivatives as well.

Then, we have the following main result:

**Theorem 2.** Suppose $\Omega \subset \mathbb{R}^n$ is an open set. Let $A$ satisfy (24) for $\theta \geq 0$, and

$$\Pi : [0, T] \times \Omega \times \mathbb{R}^m \times \mathbb{R}^q \to \mathbb{R}^{m \times m}$$

be a matrix-valued function $\Pi = \Pi(t, x, u, p)$ with invertible values satisfying

$$\gamma |\xi|^2 \leq \Pi \xi \cdot \xi \quad \text{and} \quad P \xi \cdot \xi^* \leq \gamma |\xi||\xi^*|, \quad 0 < \gamma \leq \gamma^*, \quad \text{for all } \xi, \xi^* \in \mathbb{R}^m, \tag{4}$$

such that $\Pi$ is continuous in $u$ and $p$ for almost every $(t, x)$. Let

$$f : [0, T] \times \Omega \times \mathbb{R}^m \times \mathbb{R}^q \to \mathbb{R}^m$$

be a solution.
be a measurable vector function that is continuous in $u$ and $p$ for almost every $(t, x)$, satisfying
\[ |f(t, x, u, p)| \leq F(t, x) + \Lambda_1 |u| + \Lambda_2 |p| \quad \text{for some } F \in L^2(0, T; L^2(\Omega)), \Lambda_1, \Lambda_2 \geq 0. \tag{25} \]

Then for every $u_0$ such that $u_0 \in H^s_0(\Omega)$, there exists
\[ u \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; L^2^\alpha) \cap C([0, T]; H^s_0(\Omega)), \tag{26} \]

solving the problem
\[ u'(t) + \Pi(t, \cdot, u, \Sigma u) \mathbb{A} u(t) = f(t, \cdot, u, \Sigma u) \quad \text{for a.e. } t \in ]0, T[, \]
\[ u(0) = u_0, \tag{27} \]

where $\Sigma$ represents fractional derivatives of order $\sigma$ with $0 < \sigma < s \leq 1$, if $\vartheta = 0$, and for $0 < \sigma < s \leq 1$, if $\vartheta > 0$. Moreover, there exists a constant $c' = c'(\gamma, \hat{\gamma}, a^s, \alpha^*, \Lambda_1, \Lambda_2, T) > 0$ such that for every solution $u$ of (27),
\[ \|u\|_{MR} \leq c'(\|F\|_{L^2(0, T; L^2(\Omega))} + \|u_0\|_{H^s_0(\Omega)}). \tag{28} \]

Remark 3. In the general quasilinear case without very strong restrictions on $\Pi$ and on $f$, that is, excluding small perturbations of the linear problem (16), we cannot expect the uniqueness of the solutions. This was observed in the classical gradient case in [3].

Remark 4. This extends the results for the classical derivatives with $s = \vartheta = 1$ so that $s + \vartheta = 2$ with $\sigma = 1$ as considered in [3] for the scalar problem, as well as [4] and [33] for the semilinear vectorial problem and quasilinear vectorial problem, respectively. In particular, we can consider fractional or nonlocal derivatives of any order $\sigma \leq s \leq 1$. This generalizes the classical gradient, and is conceptually similar to the ideas of Boussandel (see [11] and [12]), where he considers the classical gradient weighted by a measure.

For general operators $\mathbb{A}$ satisfying (2), the theorem applies with $\vartheta = 0$ and $\sigma < s \leq 1$. For special operators satisfying (24) with $\vartheta > 0$, we can consider derivatives of order $\sigma$ up to and including $\sigma = s$ for $s \leq 1$, as in Section 3.2.2. This includes the classical vectorial operator $L$ with $s = \vartheta = 1$ as given in Proposition 1, as well as the nonlocal vectorial operator $L^s_\alpha$ with $\vartheta = s < 1$ in Proposition 3. Observe that the latter includes the case of the fractional Laplacian $(-\Delta)_m^s$ of Proposition 2, with $0 < \vartheta < \frac{1}{2}$ for $s > \frac{1}{2}$ and $\vartheta = s$ for $s < \frac{1}{2}$. Furthermore, when $\mathbb{A} = (-\Delta)_m^s$, $\Sigma u$ can involve both the fractional and the nonlocal derivatives. This is because the fractional Laplacian $(-\Delta)^s$ can be represented by both the fractional gradient $D^s$ and the nonlocal derivative $D^\vartheta = (-\Delta)^{s/2}$, that is, $(-\Delta)^s u = -D^s \cdot D^s u = (-\Delta)^{s/2}(-\Delta)^{s/2} u$, when considered in $\mathbb{R}^n$.

We shall also use the Schaefer fixed point theorem, as it is reproduced in Theorem 2.2 of [3], which we state here for reference. Note that if $E$ is a Banach space, this theorem reduces to the Leray–Schauder fixed point theorem (see, for instance, Theorem 11.3 of [28]).

**Theorem 3 (Schaefer Fixed Point Theorem).** Let $E$ be a complete locally convex vector space and let $S : E \to E$ be a continuous mapping. Assume that there exists a continuous seminorm $p : E \to \mathbb{R}^+$, a constant $R > 0$, and a compact set $K \subset E$ such that the Schaefer set
\[ \mathcal{S} = \{u \in E : u = \lambda Su \text{ for some } \lambda \in [0, 1]\} \]
is included in
\[ C := \{u \in E : p(u) < R\} \]
such that \[ SC \subset \mathcal{K}. \]

Then \( S \) has a fixed point.

Next, we need to extend the definition of \( \Sigma \) to the space \( H^\sigma_{loc}(\Omega) \). We shall use the following lemma, where the proof is the direct consequence of the continuity of the restriction and the extension of the operator in Sobolev spaces for smooth domains (see, for instance, [19]).

**Lemma 2.** Let \( \mathcal{O} \) be a Lipschitz bounded open set such that \( \overline{\mathcal{O}} \subset \Omega \subset \mathbb{R}^n \). Then for a function \( \mathbf{v} \in H^\sigma_{loc}(\Omega) \), there is an extension \( \tilde{\mathbf{v}} \in H^\sigma(\mathbb{R}^n) \) of \( R_\mathcal{O} \mathbf{v} \), the restriction of \( \mathbf{v} \) to \( \mathcal{O} \). In addition, this extension satisfies

\[
\| \Sigma \tilde{\mathbf{v}} \|_{L^2(0,T;[L^2(\mathbb{R}^n)])} \leq C_{\mathcal{O}} \| \mathbf{v} \|_{L^2(0,T;H^\sigma(\mathcal{O}))}
\]

for some constant \( C_{\mathcal{O}} \) depending on \( \mathcal{O} \).

### 3.1 Proof of Theorem 2

Let \((\Omega_k)_k\) be an increasing sequence of open bounded subsets of \( \mathbb{R}^n \) with Lipschitz boundaries such that \( \overline{\Omega}_k \subset \Omega \) and \( \bigcup_{k \in \mathbb{N}} \Omega_k = \Omega \). Consider the locally convex space

\[
E := L^2(0,T;H^\sigma_{loc}(\Omega))
\] := \{ \mathbf{u} \in L^2_{loc}(0,T,\mathbb{R}^n) : \mathbf{u}\big|_{0,T,\times\Omega_k} \in L^2(0,T;H^\sigma(\Omega_k)) \text{ for every } k \in \mathbb{N} \},
\]

which is a Fréchet space for the sequence of seminorms given by \( \| \cdot \|_{L^2(0,T;H^\sigma(\Omega_k))} \), \( k \in \mathbb{N} \), as defined in (13) for each \( \Omega_k \).

Recall that for a Lipschitz open bounded set \( \mathcal{O} \subset \mathbb{R}^n \) (cf. Theorem 7.26 of [28]), the Sobolev embedding \( H^\sigma'(\mathcal{O}) \hookrightarrow H^\sigma(\mathcal{O}) \) is compact for \( \sigma < \sigma' \) by the Rellich–Kondrachov theorem. Then, by Aubin–Lions lemma (Lemma II.7.7 of [42]), we have the compact embedding

\[
H^1(0,T;L^2(\mathcal{O})) \cap L^2(0,T;H^\sigma(\mathcal{O})) \hookrightarrow L^2(0,T;H^\sigma(\mathcal{O})).
\]

(29)

Since \( L^2_{\mathcal{A}} \subset H^{s+\delta}_{loc}(\Omega) \) by Assumption (24), applying (29) for \( \sigma' = s + \delta \) for \( \delta \geq 0 \) for the open bounded sets \( \Omega_k \), it follows that

\[
MR = H^1(0,T;L^2(\Omega)) \cap L^2(0,T;L^2_{\mathcal{A}}) \hookrightarrow L^2(0,T;H^\sigma_{loc}(\Omega)) = E
\]

(30)

for \( \sigma < s + \delta \) is also compact, for any set \( \Omega \subset \mathbb{R}^n \).

Fix \( T \) and \( \mathbf{u}_0 \in H^s_{0}(\Omega) \). We first show that, for each \( k \), to the following problem for \( \mathbf{u} = \mathbf{u}(t,x) \) in \( 0,T,\times\Omega \),

\[
\mathbf{u} \in MR = H^1(0,T;L^2(\Omega)) \cap L^2(0,T;L^2_{\mathcal{A}}),
\]

\[
\mathbf{u}'(t) + \Pi(t,\cdot,\chi_k \mathbf{u},\Sigma \mathbf{u}) \mathcal{A} \mathbf{u}(t) = \chi_k \mathbf{f}(t,\cdot,\mathbf{u},\Sigma \mathbf{u}) \quad \text{for a.e. } t \in ]0,T[,
\]

\[
\mathbf{u}(0) = \mathbf{u}_0
\]

admits a solution \( \mathbf{u} = \mathbf{u}_k \) such that, we have

\[
\| \mathbf{u} \|_{MR} \leq c \left( \| \mathbf{F} \|_{L^2(0,T;L^2(\Omega))} + \| \mathbf{u}_0 \|_{H^s_0(\Omega)} \right)
\]

(28)
for some constant $c' = c'(y, \hat{y}, \sigma, \sigma^*, a^*, \mu, \Lambda_1, \Lambda_2, T) > 0$ independent of $k$. Here, $\chi_k = \chi_{\Omega_k}(x)$ denotes the scalar characteristic function, which is 1 if $x \in \Omega_k$ and 0 otherwise.

For each fixed $k \in \mathbb{N}$ and for every $\mathbf{v} \in E$, we extend $\mathbf{v}$ to $\mathbb{R}^n$ as in Lemma 2, with $\mathcal{S} = \Omega_k$ and we set

$$
\Pi_{\mathbf{v},k}(t,x) := \Pi(t,x, \chi_{\Omega_k}(x)\mathbf{v}(t,x), \Sigma \tilde{\mathbf{v}}(t,x)), \quad \text{and}
$$

$$
\mathbf{f}_{\mathbf{v},k}(t,x) := \chi_{\Omega_k}(x)\mathbf{f}(t,x, \mathbf{v}(t,x), \Sigma \tilde{\mathbf{v}}(t,x)).
$$

Then, $\Pi_{\mathbf{v},k}$ inherits the same properties as $\Pi$, while $\mathbf{f}_{\mathbf{v},k}$ is measurable and satisfies

$$
\|\mathbf{f}_{\mathbf{v},k}\|_{L^2(0,T;L^2(\Omega))}^2 \leq 2 \int_0^T \int_{\Omega_k} F(t,x)^2 + \Lambda_1^2 |\mathbf{v}|^2 + \Lambda_2^2 |\Sigma \tilde{\mathbf{v}}|^2 
$$

$$
\leq 2\|F\|_{L^2(0,T;L^2(\Omega))}^2 + 2\Lambda_1^2 \|\mathbf{v}\|^2_{L^2(0,T;L^2(\Omega_k))} + 2C_k^2 \Lambda_2^2 \|\Sigma \tilde{\mathbf{v}}\|^2_{L^2(0,T;H^2(\Omega_k))} < \infty
$$

for some constant $C_k = C(\Omega_k)$ as in (3'). Then, by Theorem 1, there exists a unique solution $\mathbf{u} = : T_k \mathbf{v} \in MR$ of the problem

$$
\mathbf{u}'(t) + \Pi_{\mathbf{v},k}(t,\cdot)\mathbb{A}\mathbf{u}(t) = \mathbf{f}_{\mathbf{v},k}(t,\cdot) \quad \text{for a.e. } t \in ]0,T[, \quad \text{and}
$$

$$
\mathbf{u}(0) = \mathbf{u}_0,
$$

satisfying the inequality

$$
\|\mathbf{u}\|_{MR} \leq c \left( \|\mathbf{f}_{\mathbf{v},k}\|_{L^2(0,T;L^2(\Omega))} + \|\mathbf{u}_0\|_{H^1(\Omega)} \right) 
$$

$$
\leq c_4 \left( \|F\|_{L^2(0,T;L^2(\Omega))} + \|\mathbf{v}\|_{L^2(0,T;H^2(\Omega_k))} + \|\mathbf{u}_0\|_{H^1(\Omega)} \right)
$$

(33)

for some constant $c_4 = c_4(c,k,\Lambda_1,\Lambda_2)$, where $c$ is the same constant from Theorem 1. In this way, we have defined an operator $T_k : E \rightarrow MR \subset E$.

Next, let $\mathbf{v}_i \rightarrow \mathbf{v}$ in $E$, and denote $\mathbf{u}_i = T_k \mathbf{v}_i$ and $\mathbf{u} = T_k \mathbf{v}$. We want to show that $T_k$ is continuous, that is, $\mathbf{u}_i \rightarrow \mathbf{u}$ in $E$. Since $(\mathbf{u}_i)_i$ is bounded in $MR$ by estimate (33), which is uniform in $i$ for fixed $k$, and since $MR$ is a Hilbert space, we may assume, after passing to a subsequence, that there exists a $\mathbf{w} \in E$ such that

$$
\mathbf{u}_i \rightarrow \mathbf{w} \quad \text{in } MR.
$$

(34)

Passing to a further subsequence, we may in addition assume that

$$
\mathbf{u}_i' \rightarrow \mathbf{w}' \quad \text{in } L^2(0,T;L^2(\Omega)), \quad \text{and}
$$

$$
\mathbb{A}\mathbf{u}_i \rightarrow \mathbb{A}\mathbf{w} \quad \text{in } L^2(0,T;L^2(\Omega)).
$$

(35)

(36)

We show that $\mathbf{w} = \mathbf{u}$. By definition, $\mathbf{v}_i \rightarrow \mathbf{v}$ in $E$, which means that for every $k$, $\mathbf{v}_i \rightarrow \mathbf{v}$ in $L^2(0,T;H^2_0(\Omega_k))$. Passing to a further subsequence and using a diagonalization argument, there exists a function $V_k \in L^2([0,T]\times\Omega_k)$ such that

$$
(\mathbf{v}_i, \Sigma \tilde{\mathbf{v}}_i) \rightarrow (\mathbf{v}, \Sigma \tilde{\mathbf{v}}) \quad \text{a.e. on } [0,T]\times\Omega, \quad \text{and}
$$

$$
|\mathbf{v}_i| + |\Sigma \tilde{\mathbf{v}}_i| \leq V_k \quad \text{a.e. on } [0,T]\times\Omega_k, \quad \forall i \in \mathbb{N},
$$

(37)
by the continuity of $\Sigma$, which involves the $\partial, D^\sigma$, and $D^F$ operators. For instance, specifically for the operator $D^\sigma$, for each $\Omega_k$, since $\bar{v}_i = v_i$ in $\Omega_k$ and $v_i \to v$ in $E$,

$$\int_{\Omega} (D^\sigma_j \bar{v}) \varphi = - \int_{\mathbb{R}^n} \bar{v}_j (D^\sigma \varphi) - \int_{\mathbb{R}^n} \bar{v} (D^\sigma_j \varphi) = \int_{\Omega} (D^\sigma_j \bar{v}) \varphi \quad \forall \varphi \in C^\infty_0(\Omega),$$

for each $j = 1, \ldots, n$, which means

$$D^\sigma \bar{v}_i \to D^\sigma \bar{v}.$$ 

By the continuity of $\Pi$ and $f$, we have

$$\Pi_{v_i,k}(t, x) := \Pi(t, x, \chi_k v_i, \Sigma \bar{v}_i) \to \Pi(t, x, \chi_k v, \Sigma \bar{v}) =: \Pi_{v,k}(t, x),$$

and

$$f_{v_i,k}(t, x) := \chi_{\Omega_k}(x) f(t, x, v_i, \Sigma \bar{v}_i) \to \chi_{\Omega_k}(x) f(t, x, v, \Sigma \bar{v}) =: f_{v,k}(t, x) \quad \text{a.e. on } ]0, T[ \times \Omega.$$ 

Moreover, by the growth assumption on $f$ in (25) and uniform domination of $v_i$ by $V_k$ in (37), we have

$$|f_{v_i,k}| \leq F + (\Lambda_1 + \Lambda_2) V_k \quad \text{a.e. in } ]0, T[ \times \Omega, \quad \forall i \in \mathbb{N}. \quad (38)$$

Recall that, for every $i \in \mathbb{N}$, $u_i$ satisfies the problem

$$u_i' + \Pi_{v_i,k} A u_i = f_{v_i,k}. \quad (39)$$

By the dominated convergence theorem and (38),

$$f_{v_i,k} \to f_{v,k} \quad \text{strongly in } L^2(0, T; L^2(\Omega)).$$

Also, by the dominated convergence theorem, since $\Pi_{v_i,k}$ is uniformly bounded as in (4), we have, for every $\varphi \in L^2(0, T; L^2(\Omega))$,

$$
\Pi_{v_i,k}^* \varphi \to \Pi_{v,k}^* \varphi \quad \text{in } L^2(0, T; L^2(\Omega)).
$$

By (36), it follows that for every $\varphi \in L^2(0, T; L^2(\Omega))$,

$$\int_0^T \int_{\Omega} \Pi_{v_i,k} A u_i \cdot \varphi = \int_0^T \int_{\Omega} A u_i \cdot \Pi_{v_i,k}^* \varphi \to \int_0^T \int_{\Omega} A w \cdot \Pi_{v,k}^* \varphi = \int_0^T \int_{\Omega} \Pi_{v,k} A w \cdot \varphi,$$

or equivalently

$$\Pi_{v_i,k} A u_i \to \Pi_{v,k} A w \quad \text{weakly in } L^2(0, T; L^2(\Omega)). \quad (40)$$

Therefore, taking $i \to \infty$ in (39) gives

$$u'(t) + \Pi_{v,k} A w(t) = f_{v,k}(t) \quad \text{in } \Omega \text{ for a.e. } t \in ]0, T[.$$ 

Since $MR \hookrightarrow C([0, T]; H^1_0(\Omega))$ by Lemma 1, the weak convergence of $w \to u$ in $MR$ gives

$$w(0) = \lim_{i \to \infty} u_i(0) = u_0.$$ 

But $u$ is also the solution of the problem (32), which is unique by Theorem 1, so $w = u$. 


Since $\mathbf{u}_i \rightharpoonup \mathbf{u}$ in $\text{MR}$, by the compact embedding $\text{MR} \hookrightarrow E$, we obtain, passing to a subsequence if necessary,

$$\mathbf{u}_i \rightharpoonup \mathbf{u} \quad \text{strongly in } E,$$

so $\mathcal{T}_k$ is continuous.

In the next step, we show that there exists a nonnegative constant depending on $\gamma, \check{\gamma}, a_*, a^*, \mu, \Lambda_1, \Lambda_2$, and $T$ independent of $k$ such that for every element $\mathbf{u}$ in the Schaefer set

$$\delta_k = \{ \mathbf{v} \in E : \mathbf{v} = \alpha \mathcal{T}_k \mathbf{v} \text{ for some } \alpha \in [0, 1] \} \subset \text{MR},$$

estimate (28) holds.

Assume that $\mathbf{u}_k = \alpha_k \mathcal{T}_k(\mathbf{u}_k) \in L^2(0, T; H^s_0(\Omega))$ for some $\alpha_k \in [0, 1]$, that is, $\mathbf{u}_k$ satisfies

$$u_k'(t) + \Pi(t, \cdot, \chi_k \mathbf{u}_k, \Sigma \mathbf{u}_k)\mathbf{u}_k(t) = \alpha_k \chi_k f(t, \cdot, \mathbf{u}_k, \Sigma \mathbf{u}_k) \quad \text{for a.e. } t \in [0, T],$$

and

$$\mathbf{u}_k(0) = \mathbf{u}_0.$$

Multiplying the equation by $[\Pi u_k^*]^{-1} u_k'(t)$, where $\Pi u_k = \Pi(t, \cdot, \chi_k \mathbf{u}_k, \Sigma \mathbf{u}_k)$, and integrating over $\Omega$, we obtain, by Lemma 1 and the Cauchy–Schwarz inequality,

$$\int_{0}^{T} \left| \int_{0}^{t} [\Pi u_k^*]^{-1} u_k'(r) \cdot u_k'(t) \, dr \right| \leq \frac{\bar{\gamma}}{2} \int_{0}^{T} \left| [\Pi u_k^*]^{-1} f_k(r) \right|^2 \, dr + \frac{1}{2\bar{\gamma}} \int_{0}^{T} \left| u_k'(r) \right|^2 \, dr \leq \frac{\bar{\gamma}}{2} \int_{0}^{T} \left| f_k(r) \right|^2 \, dr + \frac{1}{2\bar{\gamma}} \int_{0}^{T} \left| u_k'(r) \right|^2 \, dr,$$

by the positivity of $\Pi$ and since $\alpha_k \in [0, 1]$. We have denoted $f_k := f(t, x, \mathbf{u}_k, \Sigma \mathbf{u}_k)$. Making use of the coercivity and boundedness of $\Pi u_k$ in (4), we integrate over time on $[0, T]$ for every finite $t \in [0, T]$ to obtain, by (2) and (19), that

$$\frac{1}{\bar{\gamma}} \int_{0}^{t} \left| u_k'(r) \right|^2 \, dr + a_* \left| \mathbf{u}_k(t) \right|_{H^s_0(\Omega)}^2 \leq a_* \left| \mathbf{u}_0 \right|_{H^s_0(\Omega)}^2 + \mu \left| \mathbf{u}_k(t) \right|_{L^2(\Omega)}^2 + \frac{1}{2\bar{\gamma}} \int_{0}^{t} \left| f_k(r) \right|_{L^2(\Omega)}^2 \, dr \leq a_* \left| \mathbf{u}_0 \right|_{H^s_0(\Omega)}^2 + 2\mu^2 \bar{\gamma} \int_{0}^{t} \left| \mathbf{u}_k(r) \right|_{L^2(\Omega)}^2 \, dr + \frac{1}{2\bar{\gamma}} \int_{0}^{t} \left| \mathbf{u}_k'(r) \right|_{L^2(\Omega)}^2 \, dr \leq a_* \left| \mathbf{u}_0 \right|_{H^s_0(\Omega)}^2 + 2\mu^2 \bar{\gamma} \int_{0}^{t} \left| \mathbf{u}_k(r) \right|_{L^2(\Omega)}^2 \, dr + \frac{1}{2\bar{\gamma}} \int_{0}^{t} \left| \mathbf{u}_k'(r) \right|_{L^2(\Omega)}^2 \, dr \leq a_* \left| \mathbf{u}_0 \right|_{H^s_0(\Omega)}^2 + \frac{\bar{\gamma}}{2} \| F \|_{L^2(0; L^2(\Omega))}^2 \leq a_* \left| \mathbf{u}_0 \right|_{H^s_0(\Omega)}^2 + \frac{\bar{\gamma}}{2} \| F \|_{L^2(0; L^2(\Omega))}^2 + \frac{\Lambda_1^2 \bar{\gamma}}{2} \int_{0}^{t} \left| \mathbf{u}_k(r) \right|_{H^s_0(\Omega)}^2 \, dr + \frac{\Lambda_2^2 \bar{\gamma}}{2} \int_{0}^{t} \left| \Sigma \mathbf{u}_k(r) \right|_{L^2(\Omega)}^2 \, dr + \frac{\bar{\gamma}}{2} \int_{0}^{t} \left| u_k'(r) \right|_{L^2(\Omega)}^2 \, dr.$$

Here, we have used the following estimate:

$$\| \Sigma \mathbf{u} \|_{L^2(0, T; L^2(\Omega))} \leq c_2 \| \mathbf{w} \|_{L^2(0, T; H^s_0(\Omega))} \leq c_2 c_S \| \mathbf{u} \|_{L^2(0, T; H^s_0(\Omega))},$$

where $c_2$ is the constant from (3), and $c_S$ is the constant for the Sobolev embedding $H^s_0(\Omega) \hookrightarrow H^s_0(\Omega)$ for $\sigma \leq s$, so that they depend only on the structure of $\Sigma$ and $\Omega$ and not on $\mathbf{w} \in L^2(0, T; H^s_0(\Omega))$. Then, applying Grönwall’s lemma, we can argue as in the proof of Theorem 1 to get estimate (28) for every $\mathbf{u}_k \in \delta_k$, uniformly in $k$. 
This means that $S_k$ is bounded in $MR$. By the definition of the $MR$ norm, this implies that there exists an $R > 0$ such that

$$S_k \subset C_k := \{ v_k \in E : \| v_k \|_{L^2(0,T;H^s(\Omega_k))} < R \},$$

because clearly $\| \cdot \|_{L^2(0,T;H^s(\Omega_k))} \leq \| \cdot \|_{L^2(0,T;H^s(\Omega))}$. It follows from the definition of $T_k$ and (33) that $T_k C_k$ is contained in a bounded subset of $MR$. By compactness of the embedding (30), $T_k C_k$ is contained in a compact subset of $E$. Therefore, by Schaefer’s fixed point theorem (Theorem 3), the mapping $T_k$ admits a fixed point $u_k$ such that $u_k \in MR$. By the definition of $T_k$, this element $u_k$ is a solution of the problem (31), and since $u_k \in S_k$, $u_k$ satisfies (28).

Finally, we extend the result to show that (27) admits a solution. For every $k \in \mathbb{N}$, we choose a solution $u_k$ of the problem (31). Since every such solution is an element of $S_k$ and satisfies the estimate (28), which is independent of $k$, the sequence $(u_k)_k$ is bounded in $MR$. Since $MR$ is a Hilbert space, we may assume (after passing to a subsequence) that there exists a limit $u \in E$ such that $u_k \rightharpoonup u$ in $MR$. By the compactness of the embedding (30), passing to a subsequence again if necessary, we obtain, through a diagonalization argument, that

$$u_k' \rightharpoonup u' \quad \text{in} \quad L^2(0,T;L^2(\Omega)),
$$

$$Au_k \rightharpoonup Au \quad \text{in} \quad L^2(0,T;L^2(\Omega)),
$$

$$(\chi_k u_k, \Sigma u_k) \rightarrow (u, \Sigma u) \quad \text{a.e. on } ]0,T[ \times \Omega, \text{ and}
$$

$$|\chi_k u_k| + |\Sigma u_k| \leq U \quad \text{a.e. on } ]0,T[ \times \Omega, \forall k \in \mathbb{N},
$$

for some $U \in L^2_{loc}(]0,T[ \times \Omega)$.

By continuity of $\Pi$ and $f$, since $\Omega_k$ is increasing to $\Omega$, we have a.e. on $]0,T[ \times \Omega$:

$$\Pi(t,x,\chi_k u_k, \Sigma u_k) \rightarrow \Pi(t,x,u, \Sigma u), \quad \text{and}
$$

$$\chi_{\Omega_k}(x)f(t,x,u_k, \Sigma u_k) \rightarrow f(t,x,u, \Sigma u).$$

By the uniform boundedness of $f u_k, k$ in (38) and the domination of $u_k$ by $U$ in (43), we have

$$|\chi_{\Omega_k}(x)f(t,x,u_k, \Sigma u_k)| \leq F + (\Lambda_1 + \Lambda_2)U \quad \text{a.e. on } ]0,T[ \times \Omega, \forall k \in \mathbb{N}.
$$

Also, as in (40), the convergences in (43) imply that

$$\Pi(t,x,\chi_k u_k, \Sigma u_k)A u_k \rightharpoonup \Pi(t,x,u, \Sigma u)A u \quad \text{weakly in} \quad L^2(0,T;L^2(\Omega)).
$$

Therefore,

$$\chi_{\Omega_k}(x)f(t,x,u_k, \Sigma u_k) = u_k' + \Pi(t,x,\chi_k u_k, \Sigma u_k)A u_k \quad \text{converges weakly in} \quad L^2(0,T;L^2(\Omega)).
$$

On the other hand, for every $\varphi \in L^2(0,T;C_c(\Omega))$ compactly supported in $]0,T[ \times \Omega$, we have

$$\int_0^T \int_{\Omega} \chi_{\Omega_k}(x)f(t,x,u_k, \Sigma u_k) \cdot \varphi \rightarrow \int_0^T \int_{\Omega} f(t,x,u, \Sigma u) \cdot \varphi
$$

by the dominated convergence theorem. Since compactly supported functions are dense in $L^2(0,T;L^2(\Omega))$, we have the weak convergence

$$\chi_{\Omega_k}(x)f(t,x,u_k, \Sigma u_k) \rightharpoonup \chi_{\Omega}(x)f(t,x,u, \Sigma u) = f(t,x,u, \Sigma u) \quad \text{weakly in} \quad L^2(0,T;L^2(\Omega)).
Letting $k \to \infty$ in the problem (31), we therefore obtain that $u$ satisfies the original problem

$$u' + \Pi(t, x, u, \Sigma u) A u = f(t, x, u, \Sigma u) \quad \text{a.e. in } [0, T] \times \Omega.$$  

Furthermore, invoking the continuity $MR \hookrightarrow C([0, T]; H^s_0(\Omega))$ by Lemma 1 as before, $u_k(0) \to u(0)$ in $H^s_0(\Omega)$, so $u(0) = u_0$. Thus, $u$ is a solution to the problem (28). Furthermore, since estimate (42) is independent of $k$, we can pass to the limit to obtain estimate (28).

**Remark 5.** It is also possible to consider a different nonlocal vectorial operator $A u = (A_1 u^1, \ldots, A_m u^m)$ for each equation in the system

$$u' + \Pi(t, x, u, \Sigma u) A u = f(t, x, u, \Sigma u) \quad \text{in } [0, T] \times \Omega,$$

for $A_i$ given by (possibly different) scalar operators satisfying (2), which may be of the form (8), (9), or (10), and for $\Pi$ satisfying the same assumptions.

**Remark 6.** The results in Theorem 2 can in fact be extended to the inhomogeneous Dirichlet boundary problem $u = g$ in $]0, T[ \times \partial \Omega$.

Indeed, writing $MR(\mathbb{R}^n)$ for

$$MR(\mathbb{R}^n) := H^s(0, T; L^2(\mathbb{R}^n)) \cap \{ u \in H^s(\mathbb{R}^n) : A u \in L^2(\mathbb{R}^n) \},$$

let $g \in MR(\mathbb{R}^n) \cap L^2(0, T; H^{s+\varphi}(\mathbb{R}^n)) \cap C([0, T]; H^s(\mathbb{R}^n))$, such that $g(0) \in H^s(\mathbb{R}^n)$. Considering $\tilde{u} = u - g$, we can solve the problem for $\tilde{u} \in MR(\Omega)$, for the corresponding translated problem.

### 3.2 Examples

#### 3.2.1 Quasilinear system with the classical Laplacian

As a first example, we consider the classical Laplacian $\Delta$ in the case of $s = 1$ as in Example 5.1 of [3], extended to the case of a system of equations.

Consider the vectorial Laplacian $(-\Delta)_m$ defined by

$$(-\Delta)_m = \begin{bmatrix} -c_1 \Delta & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -c_m \Delta \end{bmatrix}$$

for constants $c_1, \ldots, c_m > 0$. Then, applying Theorem 2 for $A = (-\Delta)_m$, we have the following:

**Corollary 4.** Suppose $\Pi$ and $f$ satisfy the assumptions of Theorem 2 with $\Omega$ being an open bounded domain with $C^{1,1}$ boundary. Then, writing for the gradient $\partial = (\partial_1, \ldots, \partial_n)$, for every $u_0 \in H^1_0(\Omega)$, the nonlinear problem given by

$$\begin{cases} u'(t) + \Pi(t, \cdot, u, \partial u)(-\Delta)_m u(t) = f(t, \cdot, u, \partial u) & \text{for a.e. } t \in [0, T], \\ u = 0 & \text{a.e. on } [0, T] \times \partial \Omega, \\ u(0, \cdot) = u_0(\cdot) & \text{a.e. in } \Omega, \end{cases}$$

has a solution $u \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega)) \cap C([0, T]; H^1(\Omega))$ for any $T \in ]0, \infty[$. Moreover, there exists a constant $c' = c'(\gamma, \Lambda_1, \Lambda_2, T) > 0$ such that every solution $u$ satisfies

$$\|u\|_{MR} \leq c'(\|F\|_{L^2(0, T; L^2(\Omega))} + \|u_0\|_{H^1(\Omega)}).$$

In particular, this extends the results of [3] to a system of equations.
3.2.2 Approximation models for interacting nonlocal diffusive species populations

We also consider a nonlocal version of cross-diffusive systems modeling two interacting species, given for $0 < s \leq 1$ by

\[
\begin{align*}
u' &= -D_1(u, v, \Sigma u, \Sigma v)(-\Delta)^s u + R_1(u, v, \Sigma u, \Sigma v), \quad x \in \Omega, t > 0, \\
v' &= -D_2(u, v, \Sigma u, \Sigma v)(-\Delta)^s v + R_2(u, v, \Sigma u, \Sigma v), \quad x \in \Omega
\end{align*}
\]

\[ u(0, x) = u_0(x), \quad v(0, x) = v_0(x), \quad x \in \Omega, \]

\[ u(t, x) = v(t, x) = 0 \quad x \in \Omega^c, t > 0, \tag{45} \]

where $\Sigma$ has order $\sigma$ with $0 < \sigma < s$. The diffusion coefficients $D_1$ and $D_2$ are bounded and strictly positive, describing a controlled nonlocal nonlinear spreading of the biological population, which is dependent both on the size and density of the species itself and the other species. The interaction between the two species is both in terms of space competition with regard to diffusion, as well as in the nonlinear bounded reaction terms $R_1(u, v)$ and $R_2(u, v)$. Here, the spreading can be represented by any of the operators in Section 2, and can be both nonlocal, as described by the nonlocal operator $\mathcal{L}^s_{\lambda_1}$, or local, given by the classical operator $\mathcal{L}$. Such systems with constant diffusion coefficients may appear in activator–inhibitor systems with linear or sublinear kinetic functions (see, for instance, Chapter 9 of [45]).

This model can also be obtained as an approximation of Lotka–Volterra–type models, where the reaction terms are obtained from linearizing quadratic terms describing predator–prey, competition or cooperation interactions. For instance, the Shigesada–Kawasaki–Teramoto system (see, for instance, [36] and [14]) given by

\[ u' + D(-\Delta)^s u = R \quad x \in \Omega \]

for $u = (u_1, u_2)$ with diffusion matrix

\[
D = \begin{bmatrix}
d_1 + q \rho_{11} u_1^q + \rho_{12} u_2^q \\
q \rho_{13} (D^s u_1)^q + \rho_{14} (D^s u_2)^q \\
q \rho_{21} u_2^q - u_1^q \\
q \rho_{22} u_2^q + \rho_{23} (D^s u_2)^q + \rho_{24} (D^s u_1)^q
\end{bmatrix}
\]

and reaction term

\[ R = (R_1, R_2), \quad R_i = (a_{1,i} - b_{1,i} u_1 - c_{1,i} u_2) u_i + (a_{2,i} - b_{2,i} D^s u_1 - c_{2,i} D^s u_2) D^s u_i \]

can be approximated via the logistic function, which is a bounded nonlinearity,

\[ u_i \sim \frac{u_i}{1 + \varepsilon |u_i|} =: \tilde{u}_i \quad \varepsilon > 0, \]

\[ D^s u_i \sim \frac{D^s u_i}{1 + \varepsilon |D^s u_i|} =: \overline{D^s u_i} \quad \varepsilon > 0, \]

to obtain the system

\[ u' + D_{approx}(-\Delta)^s u = R_{approx} \quad x \in \Omega, \]
where $D_{\text{approx}}$ is of the form

$$
D_{\text{approx}} = \begin{bmatrix}
  d_1 + q\rho_{11}\hat{u}_1^q + \rho_{12}\hat{u}_2^q + q\rho_{13}(\hat{D}u_1)^q + \rho_{14}(\hat{D}u_2)^q \\
  q\rho_{21}\hat{u}_1^{q-1}u_2 \\
  d_2 + q\rho_{22}\hat{u}_2^q + \rho_{21}\hat{u}_1^q + q\rho_{23}(\hat{D}u_2)^q + \rho_{24}(\hat{D}u_1)^q \\
\end{bmatrix}
$$

and

$$
R_{\text{approx}} = \begin{bmatrix}
  (a_{1,1} - b_{1,1}\hat{u}_1 - c_{1,1}\hat{u}_2)u_1 + (a_{2,1} - b_{2,1}\hat{D}u_1 - c_{2,1}\hat{D}u_2)D^s u_1 \\
  (a_{1,2} - b_{1,2}\hat{u}_1 - c_{1,2}\hat{u}_2)u_2 + (a_{2,2} - b_{2,2}\hat{D}u_1 - c_{2,2}\hat{D}u_2)D^s u_2 \\
\end{bmatrix}
$$

so that $D_{\text{approx}}$ is bounded, under appropriate assumptions on $\rho_{ij}$ and $u$, and $R_{\text{approx}}$ has a linear growth on $u_1$ and $u_2$ for any fixed $\epsilon$, fulfilling our assumptions.

Supposing $D_{\text{approx}}$ satisfies (4), which is obtained by taking the sufficient assumption that $u_1, u_2$ are positive and that the diffusion coefficients $\rho_{ij} > 0$ are bounded, and the derivatives are of order $s$, by Theorem 2, this problem admits a global solution $u$ in

$$
u \in H^s(0,T;L^2(\Omega)) \cap L^2(0,T;H^s(\Omega)) \cap C([0,T];H^s_0(\Omega)),
$$

where $s = s + n - 1$. This means that we can consider a more general case of cross-diffusion involving nonlocal operators, and obtaining a regularity result that is comparable to the classical cases in Theorem B of [36] and Theorem 1.3 of [9].

### 4 THE NONLINEAR PROBLEM $s < \sigma < 2s \leq 2$ WITH $\Omega$ BOUNDED

In this section, we want to further extend the result to higher order derivatives $\sigma > s > 0$. In particular, $\sigma$ may be greater than 1, generalizing the scalar quasilinear diffusion equations in the classical case in [3]. Here, we focus on the classical elliptic operator $L$ as given in Example 1 of Section 2 defined by (8), as well as the nonlocal fractional Laplacian defined in Example 2 of Section 2 defined by (22), since we have additional regularity results for those cases. Then, by the results of [21], and [29] and [10], we know that there exists a unique solution to the Dirichlet problem associated with $L$ and with $(-\Delta)^s$, given by Propositions 1 and 2, respectively. Therefore, the spaces $L^2_L$ and $L^2_{(-\Delta)^s_m}$ are well defined. Furthermore, it is clear that $L$ and $(-\Delta)^s_m$ are bounded and $L^2(\Omega)$-coercive.

We first recall the following Poincaré inequality concerning the embedding of $H^s_0(\Omega)$ in $L^2(\Omega)$. See, for instance, Theorem 2.9 of [8] for the fractional case $0 < s < 1$.

**Lemma 3** (Poincaré inequality). Let $s \in [0,1]$. Then, for any open bounded set $\Omega \subset \mathbb{R}^n$, there exists a constant $c_p > 0$ depending only on $\Omega$, $n$, and $s$ such that

$$
c_p\|u\|_{L^\infty(\Omega)} \leq \|D^s u\|_{L^2(\mathbb{R}^n)}
$$

for all $u \in H^s_0(\Omega)$. In particular, in $H^s_0(\Omega)$, we have the equivalence of the norms $\|D^s u\|_{L^2(\mathbb{R}^n)}$ and $\|u\|_{H^s_0(\Omega)}$.

Assume $A$ satisfies $L^s_A := \{u \in H^s_0(\Omega) : A u \in L^2(\Omega) \} \subset H^{s'}(\Omega)$ for some $s < s' \leq 2s$ for $\Omega$ bounded and Lipschitz domain, that is, there exist a constant $C_A > 0$ and $\mu' \geq 0$ such that

$$
\|u\|_{L^2(0,T;H^{s'}_0(\Omega))} \leq C_A \|A u\|_{L^2(0,T;L^2(\Omega))} + \mu'\|u\|_{L^2(0,T;L^2(\Omega))}.
$$

(46)
In particular, this holds with \( \sigma' = 2 \) for \( A = L \), and \( \sigma' = s + \frac{1}{2} \) for \( A = (-\Delta)^s_m \), where \( \frac{1}{2} < s < 1 \). Therefore, applying the compact embedding (29), we obtain that

\[
MR = H^1(0,T;L^2(\Omega)) \cap L^2(0,T;L^2_A) \hookrightarrow L^2(0,T;H^\sigma(\Omega)) = E
\]

(47)
is compact for any open Lipschitz bounded set \( \Omega \subseteq \mathbb{R}^n \), for any \( \sigma < \sigma' \).

Also, by the Sobolev embeddings, there exists a Sobolev constant \( 0 < c_S < 1 \) depending on \( s < \sigma' \leq 2s \), \( \sigma \) and \( \Omega \), such that

\[
c_S \|v\|^2_{L^2(0,T;H^\sigma(\Omega))} \leq \|v\|^2_{L^2(0,T;H^{\sigma'}(\Omega))} \quad \forall v \in L^2(0,T;H^{\sigma'}(\Omega)).
\]

(48)

Then, assuming \( f : ]0,T[ \times \Omega \times \mathbb{R}^m \times \mathbb{R}^q \rightarrow \mathbb{R}^m \) satisfies the assumptions of Theorem 2 such that

\[
|f(t,x,u,p)| \leq F(t,x) + \Lambda_1 |u| + \Lambda_2 |p|^\alpha
\]

(49)

for some \( F \in L^2(0,T;L^2(\Omega)) \), \( \Lambda_1, \Lambda_2 \geq 0 \), such that either

(i) \( 0 < \alpha < 1 \), or
(ii) \( \alpha = 1 \) with

\[
0 < \Lambda_2 \leq \sqrt{\frac{c_S}{C_A}},
\]

(50)

we have the following result:

**Theorem 4.** Suppose \( \Omega \subseteq \mathbb{R}^n \) is a Lipschitz bounded open set. Let \( \Pi : ]0,T[ \times \Omega \times \mathbb{R}^m \times \mathbb{R}^q \rightarrow \mathbb{R}^{m \times m} \) satisfy the assumptions of Theorem 2, and \( f : ]0,T[ \times \Omega \times \mathbb{R}^m \times \mathbb{R}^q \rightarrow \mathbb{R}^m \) satisfy the assumptions (49) above for either condition (i) or (ii). Suppose, in addition, that \( A \) satisfies (46) for some \( s < \sigma' \leq 2s \leq 2 \). Then, for any \( \sigma < \sigma' \) and every \( u_0 \) such that \( u_0 \in H^\sigma_0(\Omega) \cap H^\sigma(\Omega) \), there exists

\[
u \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;L^2_A) \cap L^2(0,T;H^\sigma(\Omega)) \cap C([0,T];H^\sigma_0(\Omega))
\]

(51)
solving the problem

\[
u'(t) + \Pi(t,\cdot,u,\Sigma u)A\nu(t) = f(t,\cdot,u,\Sigma u) \quad \text{for a.e. } t \in ]0,T[,
\]

\[
u(0) = u_0,
\]

(52)

where \( \Sigma \) represents fractional gradients \( D^\sigma \) of order \( \sigma \), which may be greater than 1. Moreover, there exists a constant \( c'' = c''(\Omega,\gamma,\sigma,\alpha^*,C_A,\Lambda_1,\Lambda_2,T,\alpha) > 0 \) such that for every solution \( \nu \) of (52),

\[
\|\nu\|_{MR} \leq c'' \left( \|F\|^2_{L^2(0,T;L^2(\Omega))} + \|u_0\|^2_{H^\sigma_0(\Omega)} \right).
\]

(53)

**Proof.** Most of the proof follows the argument of Theorem 2, this time applying the Leray–Schauder fixed point theorem for the fixed point constructed in the Banach space \( E \) in (47), for \( \sigma < \sigma' \), where \( MR \) is compactly embedded. In particular, this means that we do not have to consider the sequence of sets \( \Omega_k \), and we can directly consider the compact map \( T \) defined by \( u = : T \phi \in MR \) of the problem

\[
u'(t) + \Pi_\nu(t,\cdot)A\nu(t) = f_\nu(t,\cdot) \quad \text{for a.e. } t \in ]0,T[,
\]

(53)

and

\[
u(0) = u_0,
\]

(54)
where we denote \( f_\varphi(t, x) = f(t, x, \varphi(t, x), \Sigma \varphi(t, x)) \) and \( \Pi_\varphi(t, x) = \Pi(t, x, \varphi(t, x), \Sigma \varphi(t, x)) \). A major modification lies in the proof that the Leray–Schauder set

\[
\mathcal{S} = \{ u \in \mathcal{E} : u = \lambda \mathcal{T} u \text{ for some } \lambda \in [0, 1] \}
\]  

is bounded. In particular, the proof of the a priori estimate in (42) needs to be modified for the case of \( \sigma \geq s \).

Indeed, we obtain the bound on \( \| u \|_{L^2(0, T; H^\sigma_0(\Omega))} \) for \( \sigma \geq s \) as follows: Multiplying Equation (41) by \( \mathcal{A} u \) and integrating over \( \Omega \), we obtain, by the bounds (4) and making use of Lemma 1 and the Cauchy–Schwarz inequality, for a.e. \( t > 0 \),

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega u \cdot \mathcal{A} u + \gamma \| \mathcal{A} u \|_{L^2(\Omega)}^2 \leq \int_\Omega u' \cdot \mathcal{A} u + \int_\Omega \Pi u \mathcal{A} u \cdot \mathcal{A} u = \int_\Omega f \cdot \mathcal{A} u
\]

Integrating over time on \([0, t]\) for any \( t \leq T \), it follows by (2) that

\[
ad_s \| u \|_{L^2(0, T; H^{s}_0(\Omega))}^2 + \gamma \int_0^t \| \mathcal{A} u \|_{L^2(\Omega)}^2 \leq a^s \| u_0 \|_{H^s_0(\Omega)}^2 + \mu \| u(t) \|_{L^2(\Omega)}^2 + \frac{1}{\gamma} \int_0^t \| f \|_{L^2(\Omega)}^2,
\]

and so, taking the supremum over \( t \in [0, T] \) and making use of (19), we have

\[
a d_s \| u \|_{L^2(0, T; L^2(\Omega))}^2 + \gamma \| \mathcal{A} u \|_{L^2(0, T; L^2(\Omega))}^2 \leq a^s \| u_0 \|_{H^s_0(\Omega)}^2 + \mu \| u_0 \|_{L^2(\Omega)}^2 + 2 \mu ^2 \gamma \| u \|_{L^2(0, T; L^2(\Omega))}^2
\]

Considering only the term \( \| \mathcal{A} u \|_{L^2(0, T; L^2(\Omega))}^2 \) on the left-hand side of the inequality and applying Assumptions (46) and (25) then gives

\[
\frac{\gamma}{C_\mathcal{A}} \| u \|_{L^2(0, T; H^\sigma(\Omega))}^2 \leq \frac{\mu'}{C_\mathcal{A}} \| u \|_{L^2(0, T; L^2(\Omega))}^2 + a^s \| u_0 \|_{H^s_0(\Omega)}^2 + \mu \| u_0 \|_{L^2(\Omega)}^2
\]

Next, we argue as in estimate (42) to control the \( H^1(0, T; L^2(\Omega)) \)-norm of \( u \), and there exists a constant \( c_5 = c_5(\gamma, \gamma, a_s, a^s, \mu, A_1, A_2, T) > 0 \) such that

\[
\frac{\gamma}{C_\mathcal{A}} \| u \|_{L^2(0, T; H^{s}(\Omega))}^2 \leq c_5 \left( \| u_0 \|_{H^s_0(\Omega)}^2 + \| F \|_{L^2(0, T; L^2(\Omega))}^2 \right) + \frac{A_2^2}{\gamma} \int_0^T \| D^2 u \|_{L^2(\mathbb{R}^n)}^{2\sigma},
\]

Therefore, by (48),

\[
\frac{c_5 \gamma}{C_\mathcal{A}} \| u \|_{L^2(0, T; H^{s}(\Omega))}^2 \leq c_5 \left( \| u_0 \|_{H^s_0(\Omega)}^2 + \| F \|_{L^2(0, T; L^2(\Omega))}^2 \right) + \frac{A_2^2}{\gamma} \| u \|_{L^2(0, T; H^s(\Omega))}^{2\sigma}.
\]
By the Assumption (49) with either Condition (i) with $\alpha < 1$ or Condition (ii) with $\alpha = 1$ and $\Lambda_2 < \frac{\gamma \sqrt{S}}{C_A}$, we obtain an a priori bound on the term $\|u\|_{L^2(0,T;H^2(\Omega))}$.

Also, as in estimate (42), the Leray–Schauder set $\mathcal{S}$ given by (54) is bounded in $MR$. Making use of the Aubin–Lions compactness lemma $MR \hookrightarrow \tilde{E}$, by the Leray–Schauder principle, $T$ has a fixed point $u$ satisfying (51) and (53) solving the problem (52).

**Remark 7.** The theorem holds with $\sigma' = 2$ for $\mathbb{A} = \mathbb{L}$ as well as with $\sigma' = s + \frac{1}{2}$ for $\mathbb{A} = (-\Delta)^s_m$ for $\frac{1}{2} < s < 1$, with derivatives of order $s < \sigma < \sigma'$, which may possibly be of order greater than or equal to 1. In the case of the Riesz fractional gradient $D^s$, they are defined by (5), while for nonlocal derivatives $D^s$, they strongly depend on the kernel $\alpha_s$ in (6) and are not in general well defined. Nevertheless, when $D^s = (-\Delta)^{s/2}$ is the $s/2$-fractional Laplacian, since they may be also defined for all $s > 0$, we can extend Theorem 4 to $\Sigma$ involving these nonlocal derivatives with appropriate modifications in the proof.

**Remark 8.** As in Remark 6, the results in Theorem 4 can also be extended to the inhomogeneous Dirichlet boundary problem $u = g$ in $]0, T[ \times \Omega^\circ$, for $g \in MR(\mathbb{R}^n) \cap L^2(0, T; H^{s+\theta}(\mathbb{R}^n)) \cap C([0, T]; H^s(\mathbb{R}^n))$ such that $g(0) \in H^s(\mathbb{R}^n)$.

As a result, we can consider quasilinear diffusion equations and systems with derivatives of order $\sigma > s$ such that $\sigma$ may be greater than 1, generalizing the results of [3, 4] and [33]. This provides many useful applications, particularly in advection–diffusion systems, as seen in Section 4.1.2.

### 4.1 Examples

#### 4.1.1 A system with the classical Laplacian with $D^\sigma$-quasilinearity $1 < \sigma < 2$

As a first application, we take a relook at the vectorial classical Laplacian $(-\Delta)_m$ in Section 3.2.1, this time, with the $D^\sigma$ fractional derivatives for any $1 < \sigma < 2$ as defined in (7).

**Corollary 5.** Let $\Omega$ be a bounded domain with $C^{1,1}$-boundary. Suppose $\Pi$ and $f$ satisfy the assumptions of Theorem 4 with $f$ fulfilling either Conditions (i) or (ii) of (49). Then, for $1 < \sigma < 2$ and every $u_0 \in H^1_0(\Omega) \cap H^\sigma(\Omega)$, the nonlinear problem given by

$$u'(t) + \Pi(t, \cdot, u, D^\sigma u)(-\Delta)_m u(t) = f(t, \cdot, u, D^\sigma u) \quad \text{for a.e. } t \in ]0, T[, $$

has a solution $u \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega)) \cap C([0, T]; H^1_0(\Omega))$ satisfying (53) for any $T \in ]0, \infty[$.

In particular, this further extends the results of [3] to include the fractional derivatives $D^\sigma$ of order $1 < \sigma < 2$.

#### 4.1.2 Anisotropic advection–diffusion fractional equations for $s > 1/2$

Our last application is a semilinear anisotropic advection–diffusion system of equations. Such a system may be useful to transport models with fractional diffusion and is inspired, in the scalar case, by the 2-dimensional forced subcritical surface quasi-geostrophic flows with nonlocal dissipation (see, for instance, [16]) and the 2-dimensional Navier–Stokes equation.

Suppose $s > \frac{1}{2}$. Let $\mathbf{u}(t, x)$ be a bounded velocity field in $]0, T[ \times \Omega$ in a bounded $\Omega \subset \mathbb{R}^n$ such that

$$\|\mathbf{u}\|_{L^\infty(\Omega \times \Omega)} \leq C_\# < \infty, \quad C_\# \text{ depending on } \Omega, \gamma, s \text{ and } \mathbb{A} \text{ as in (50)}. \quad (55)$$
For $f \in L^2(0,T;L^2(\Omega))$ and $u_0 \in H^1_0(\Omega) \cap H^1(\Omega)$, the equation is given by

$$u'(t,x) + \Pi A u(t,x) = - \sum_{\alpha=1}^n v^\alpha(t,x) \partial_{\alpha} u(t,x) + f(t,x,u), \quad (t,x) \in ]0,T[ \times \Omega$$

$$u(t,x) = 0, \quad (t,x) \in ]0,T[ \times \Omega^c,$$

$$u(0,x) = u_0(x), \quad x \in \Omega,$$

where $A = (-\Delta)^s_m$ or $L$. Observe that this means that since $\frac{1}{2} < s < 1$, we have a convective term given by the classical gradient of $u$.

Since $v$ is bounded as in (55), we can apply Theorem 4 with $\sigma = 1$ and with the source function given by the term $- \sum_{\alpha} v^\alpha(t,x) \partial_{\alpha} u(t,x) + f(t,x,u)$, such that (49) is satisfied with $\alpha = 1$. As a result, the problem admits a global solution

$$u \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \cap C([0,T];H^s_0(\Omega))$$

with $1 = \sigma < \sigma' \leq 2s = 2$ for $A = L$ and $1 = \sigma < \sigma' \leq s + \frac{1}{2}$ for $A = (-\Delta)^s_m$ with $\frac{1}{2} < s < 1$.

Furthermore, we do not require that $v$ is divergence-free, which means that our result applies to compressible fluids as well. Such a result is new, as far as we know, since $v$ is different from those considered in other works such as [44] and [18]. However, by (55), $v$ must be bounded, which is a severe restriction, and therefore, in general, it may not cover the subcritical quasi-geostrophic model where $v$ is given by the vorticity function of the Riesz transform of $u$.

Moreover, limited by the elliptic regularity of $(-\Delta)^s$ in Proposition 2, we are only able to consider the subcritical $s > \frac{1}{2}$ case, and unable to obtain the critical $s = \frac{1}{2}$ or the supercritical $s < \frac{1}{2}$ cases.

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