THE IRRATIONALITY MEASURE OF $\pi$ AS SEEN THROUGH THE EYES OF $\cos(n)$

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Abstract. For different values of $\gamma \geq 0$, analysis of the end behavior of the sequence $a_n = \cos(n)\gamma$ yields a strong connection to the irrationality measure of $\pi$. We show that if $\lim \sup |\cos n|^{\gamma^2} \neq 1$, then the irrationality measure of $\pi$ is exactly 2. We also give some numerical evidence to support the conjecture that $\mu(\pi) = 2$, based on the appearance of some startling subsequences of $\cos(n)^\gamma$.

1. Introduction

1.1. Irrationality Measure. For any irrational number $\alpha$, we can measure how well $\alpha$ can be approximated by rational numbers by studying the inequality

$$ \left| \alpha - \frac{p}{q} \right| < \frac{1}{q^\mu}, \quad \text{for some } \mu \in \mathbb{R}. \quad (1) $$

Definition 1.1 (Irrationality Measure). For $\alpha \in \mathbb{R}$, define the sets

$$ U(\alpha, \mu) = \left\{ \frac{p}{q} \in \mathbb{Q} : 0 < \left| \alpha - \frac{p}{q} \right| < \frac{1}{q^\mu} \right\}. $$

The irrationality measure of $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ (see [1], e.g.) is

$$ \mu(\alpha) = \inf\{\mu : |U(\alpha, \mu)| < \infty\}. $$

Our main result concerns the following set:

$$ \Phi = \{ \gamma \geq 0 : \lim \sup \{|\cos k|^{k^\gamma} \} = 1 \}. \quad (2) $$

Theorem 1.1. For $\Phi$ as defined in (2), we have $\sup \Phi = 2\mu(\pi) - 2$.

Corollary 1.2. If $\sup \Phi = 2$, then $\mu(\pi) = 2$. 

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It can be shown that the irrationality measure of any rational number is one and that the measure of every algebraic number (of order greater than 1) is 2. However, transcendental numbers may have any irrationality measure greater than or equal to 2. Upper bounds have been proven for many transcendental constants. For example, $\mu(\pi) \leq 7.6063$ [2] and $\mu(\ln 3) \leq 5.125$ [3]. It is widely believed that $\mu(\pi) = 2$, but this remains an open problem. Using Corollary 1.2, we provide some further numerical evidence that $\mu(\pi) = 2$ in [4].

Remark 1.1 (The relation of $\mu(\pi)$ to the end behavior of special sequences). The value of $\mu(\pi)$ is closely tied to the behavior and convergence of special sequences of transcendental numbers. For example, Alekseyev proved that the convergence of the Flint Hills series directly implies $\mu(\pi) < 2.5$, in [4].

The investigation in the present paper began with a question raised by a student regarding the behavior of the sequence $a_n = \cos(n)^n$. Since $|\cos n| < 1$, one might expect that raising this number to a large power would cause $a_n \to 0$. However, preliminary numerical investigations revealed that $\cos n$ this is not the case. Indeed, while most values of this sequence are extremely close to 0, certain subsequences form oscillations with amplitude 1. The top of Figure 1 shows the first million terms of the sequence, and the bottom shows a magnification of just the first 100,000 terms, which reveals several rapidly decaying subsequences. In this note, we examine the sequence $a_n = \cos(n)^\gamma$, for $\gamma \geq 0$, and show that is closely related to the value of $\mu(\pi)$.

2. PROOF OF THE MAIN RESULT

The proof will require a couple of technical lemmas, which we state and prove before proceeding to the proof of Theorem 1.1.

Definition 2.1. We define a function $Q(n) : \mathbb{Z} \to \mathbb{Z}$ such that for any integer $n$, $Q(n)$ maps to the integer minimizing $|n - Q(n)\pi|$. In other words, if $[x]$ denotes
the nearest integer to $x$ then
\[
Q(n) = \left\lfloor \frac{n}{\pi} \right\rfloor = \arg\min\{|n - m\pi| : m \in \mathbb{Z}\}.
\] (3)

Lemma 2.1. $\lim_{n \to \infty} \frac{n}{Q(n)} = \pi$.

Proof. This is immediate from Definition 2.1 by Diophantine approximation. \hfill \Box

Lemma 2.2. Let $\phi_0(x)$ and $\phi_1(x)$ be $\mathbb{R}$-valued functions satisfying
\[
\lim_{x \to \infty} |\phi_0(x)| = \infty \quad \text{and} \quad \lim_{x \to \infty} |\phi_1(x)| = \infty, \quad \text{as} \quad x \to \infty
\]
If $\phi_0(x) \neq 0$ for all $x \in \mathbb{R}$ and $\phi_0 = o(\phi_1)$, as $x \to \infty$, then
\[
\lim_{x \to \infty} \left(1 - \frac{1}{\phi_0(x)}\right) \phi_1(x) = 0.
\] (4)

Proof. Observe that $r(r-1)(r-2)\ldots(r-n+1) \leq r^n$, so comparing the Taylor series for $(1 + x)^r$ and $e^{rx}$ gives
\[
(1 + x)^r \leq e^{rx}.
\]
Now making the substitutions $x = -\frac{1}{\phi_0(x)}$ and $r = \phi_1(x)$, we obtain
\[
\left(1 - \frac{1}{\phi_0(x)}\right) \phi_1(x) \leq \exp\left(-\frac{\phi_1(x)}{\phi_0(x)}\right),
\]
and the right side tends to 0 as $x \to \infty$ because $\phi_0 = o(\phi_1)$. \hfill \Box

Proof of Theorem 1.1. We first show that
\[
\limsup \{|\cos k^\gamma|\}_{k \in \mathbb{N}} = 1, \quad \text{for all} \quad \gamma < 2\mu(\pi) - 2.
\] (5)

By Definition 1.1 there are infinitely many integers $p, q > 0$ such that:
\[
0 < \left|\pi - \frac{p}{q}\right| < \frac{1}{q^{\mu(\pi) - \varepsilon}}, \quad \varepsilon > 0.
\]

Observe that $|\cos x| \geq 1 - \frac{x^2}{2}$ for $x \in \mathbb{R}$. Choosing arbitrary integers $p$ and $Q(p)$ satisfying Definition 1.1 and combining the above inequalities, we have:
\[
|\cos p| = |\cos(p - \pi Q(p))| \geq 1 - \frac{(p - \pi Q(p))^2}{2} > 1 - \frac{1}{2Q(p)^{2(\mu(\pi) - 1 - \varepsilon)}}
\] (6)

Raising (6) to $p^\gamma$ and applying the Bernoulli inequality we have:
\[
|\cos p|^{p^\gamma} \geq \left(1 - \frac{1}{2Q(p)^{2(\mu(\pi) - 1 - \varepsilon)}}\right)^{p^\gamma} > 1 - \frac{p^\gamma}{2Q(p)^{2(\mu(\pi) - 1 - \varepsilon)}}
\] (7)

Observe $\limsup |\cos k^\gamma| \leq 1$. Let $\gamma < 2(\mu(\pi) - 1 - \varepsilon)$. We have $p^\gamma = o(Q(p)^{2(\mu(\pi) - 1 - \varepsilon)})$ because $Q(p) = O(p)$ by Lemma 2.1. Since we can choose arbitrarily large $p$ and $Q(p)$ satisfying (6), we can bring $|\cos p|^{p^\gamma}$ arbitrarily close to 1 by (7). This establishes (5).

Next, we show that
\[
\limsup \{|\cos k^\gamma|\}_{k \in \mathbb{N}} = 0, \quad \text{for all} \quad \gamma > 2\mu(\pi) - 2.
\] (8)

\footnote{Note that $[x]$ is not the same as the floor or ceiling function.}
First, observe that $|\cos x| \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}$ for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Here, the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is chosen arbitrarily; any interval which is small enough to ensure the inequality would suffice (the inequality does not hold for all $x$). Fix $\varepsilon > 0$ and let $\gamma = 2(\mu(\pi) - 1 + \varepsilon)$. Then for arbitrary integers $p$ and $Q(p)$, $|p - \pi Q(p)| \leq \frac{p}{2}$, we have:

$$|\cos p|^{p^\gamma} \leq \left(1 - \frac{(p - \pi Q(p))^2}{2} + \frac{(p - \pi Q(p))^4}{24}\right)^{p^\gamma}$$

(9)

By Definition 1.1 for any real number $\lambda > 0$, there exists an integer $N$ such that

$$|p - q\pi| > \frac{1}{q^{\mu(\pi)-1+\lambda}}, \quad \text{for all } p, q > N.$$ (10)

Since $1 - \frac{x^2}{2} + \frac{x^4}{24}$ is monotonically increasing on $[-\frac{\pi}{2}, 0)$ and monotonically decreasing on $(0, \frac{\pi}{2})$, for $p, Q(p) > N$, we have:

$$|\cos p|^{p^\gamma} \leq \left(1 - \frac{(p - \pi Q(p))^2}{2} + \frac{(p - \pi Q(p))^4}{24}\right)^{p^\gamma} \leq \left(1 - \frac{1}{2Q(p)^{2(\mu(\pi)-1+\lambda)}} + \frac{1}{24Q(p)^{4(\mu(\pi)-1+\lambda)}}\right)^{p^\gamma}$$

(11)

Observe that if we define a function $\phi(p)$ such that:

$$\frac{1}{\phi(p)} = \frac{12Q(p)^{2(\mu(\pi)-1+\lambda)} - 1}{24Q(p)^{4(\mu(\pi)-1+\lambda)}}$$

Then $\phi(p) = O(Q(p)^{2(\mu(\pi)-1+\lambda)})$. If $\gamma > 2(\mu(\pi) - 1 + \lambda)$, then we have $\phi(Q(p)) = o(p^\gamma)$. Since we can choose arbitrarily large $p$ and $Q(p)$ satisfying (10), by Lemma 2.2 we can bring (9) arbitrarily close to 0.

Thus $\limsup \{ |\cos k|^{k^\gamma} \}_{k \in \mathbb{N}} = 0$ for all $\gamma > 2(\mu(\pi) - 1 + \lambda)$. Since our choice of $\lambda$ is arbitrary but greater than zero, this establishes (8). By (5) and (8), the proof is complete.

3. Further investigations

**Theorem 3.1.** $\limsup |\cos n|^{n^2} \geq \exp(-\pi^2/2) \approx 0.007192$.

**Proof.** By Diophantine approximation, (6) becomes:

$$|\cos p| = |\cos(p - \pi Q(p))| \geq 1 - \frac{(p - \pi Q(p))^2}{2} > 1 - \frac{1}{2Q(p)^2}$$

(12)

Raising (12) to the $p^2$, we obtain

$$|\cos p|^{p^2} \geq \left(1 - \frac{1}{2Q(p)^2}\right)^{p^2}.$$ (13)

Fix $\varepsilon > 0$. Since $\frac{p}{Q(p)} \to \pi$ by Lemma 2.1 we have $\pi^2 - \varepsilon < \frac{p^2}{Q(p)^2} < \pi^2 + \varepsilon$ for all sufficiently large $p$, whence

$$\left(1 - \frac{\pi^2 - \varepsilon}{p^2}\right)^{p^2} > \left(1 - \frac{p^2}{Q(p)^2}\right)^{p^2} > \left(1 - \frac{\pi^2 + \varepsilon}{p^2}\right)^{p^2}.$$
It follows that the limit of the right side of (13) is \( \exp(-\pi^2/2) \). □

3.1. Cosine Identities. We manipulate the following identity to obtain insight on the sequence:

\[
\prod_{k=1}^{n} \cos \theta_k = \frac{1}{2^n} \sum_{e \in S} \cos(e_1 \theta_1 + \cdots + e_n \theta_n),
\]

where \( S = \{1, -1\}^n \). If \( \theta_k = n \) for all \( k \) and we let \( i \) be the number of times 1 appears in \( e \), this becomes

\[
\cos(n)^n = \frac{1}{2^n} \sum_{e \in S} \cos((e_1 + \cdots + e_n)n) = \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} \cos((2i - n)n).
\]

Observe that for \( \cos(n)^n \), we have:

\[
\cos(n)^n = \frac{1}{2^n} \sum_{i=0}^{n^2} \binom{n^2}{i} \cos((2i - n^2)n).
\]

(14)

Applying the cosine approximation and writing \( Q(n) = \left\lceil \frac{n}{\pi} \right\rceil \), we have:

\[
\cos(n)^n \geq \frac{1}{2^n} \sum_{i=0}^{n^2} \binom{n^2}{i} \left( 1 - \frac{(2i - n^2)^2(n - \pi \left\lceil \frac{n}{\pi} \right\rceil)^2}{2} \right)
\]

\[
= \frac{1}{2^n} \sum_{i=0}^{n^2} \left( \binom{n^2}{i} - \binom{n^2}{i} \frac{(2i - n^2)^2}{2} (n - \pi \left\lceil \frac{n}{\pi} \right\rceil)^2 \right)
\]

\[
= 1 - \frac{n^2(1 - \frac{\pi}{\left\lceil \frac{n}{\pi} \right\rceil})^2}{2 \cdot 2^n} \sum_{i=0}^{n^2} \binom{n^2}{i} (2i - n^2)^2
\]

(15)

From (15), we can see that the existence of a lower bound (as we pick larger and better \( n \) approximating multiples of \( \pi \)) is determined by the growth rate of the second term. If the rate is \( \mathcal{O}(1) \), then there is a positive lower bound. If the growth rate is \( o(1) \), then the lower bound is 1, and the limsup is also 1.
Figure 3. $\gamma = 1$ plotted from 0 to 200000. $\gamma = 1.5$ plotted from 0 to $10^6$.

4. NUMERICAL RESULTS

Computationally, we find values of $|\cos k|^\gamma$ very close to one for values of $k$ on the order of $10^8$, with values of $\gamma$ very close to 2. When we set $\gamma = 2$, we do not find values of $k$ close to one. This is evidence that $\mu(\pi) = 2$, but a proof has evaded us so far. Various values of $\gamma$ are plotted in Figure 2 and Figure 3. We see that for larger values of $\gamma$, points near one get more sparse but still reach values very close to 1. In the case of $\gamma = 1$, there are interesting subsequences, some of which persist quasiperiodically or die out entirely. Investigations of these subsequences are an interesting subtopic which is open to exploration. Figure 4 shows plots of $\gamma$ near and above 2.

When $\gamma$ is just below two, we still see subsequences that are close to 1, but they become more sparse as $\gamma$ is closer to 2. In the case where $\gamma = 2.1$, we see that the values approach zero, and do not appear to take on values close to one in the range sampled, which is expected if $\mu(\pi) = 2$.

5. SUBSEQUENCES OF $(\cos n)^n\gamma$

Various interesting subsequences appear in $(\cos n)^n\gamma$, and the following theorems shed light on the topic.

**Theorem 5.1.** For all $0 \leq \gamma < 2(\mu(\pi) - 1)$, for any real number $\alpha \in (0, 1)$, there exist infinitely many arbitrarily long arithmetic progressions $S$ such that for all $a_n \in S$, $|\cos a_n|^\alpha > \alpha$.

**Proof.** The proof relates closely to the proof of [5]. We modify (7) to obtain:

$$
|\cos(np)|^{(np)\gamma} \geq \left(1 - \frac{(n(p - \pi Q(p))^2}{2}\right)^{(np)\gamma} > 1 - \frac{n^{\gamma + 2}p^{\gamma}}{2Q(p)^{2(\mu(\pi) - 1 - \epsilon)}}
$$

By (1), for any $n \in \mathbb{N}$, we can bring $|\cos(np)|^{(np)\gamma}$ arbitrarily close to 1, which is greater than $\alpha$. Also observe that for any $a \in \mathbb{N}$, $a < n$ we have:

$$
\frac{n^{\gamma + 2}p^{\gamma}}{2Q(p)^{2(\mu(\pi) - 1 - \epsilon)}} > \frac{a^{\gamma + 2}p^{\gamma}}{2Q(p)^{2(\mu(\pi) - 1 - \epsilon)}}
$$

Thus for any series of length $n \in \mathbb{N}$, there exists infinitely many integers $p$ such that the sequence $\{\cos(kp)|^{(kp)\gamma}, k = 0, 1, 2, ..., n\}$ is bounded by $\alpha \in (0, 1)$. This completes the proof. \qed
Definition 5.1. We refer to the subsequences which Theorem 5.1 guarantees to exist as persistent subsequences.

The terminology of Definition 5.1 stems from the idea that these values of \( \cos(n_k) \) are sufficiently close to 1 that not even raising them to the (typically extremely large) number \( n_k \) will result in a number near 0.

Numerical experimentation reveals that each peak in Figure 1 corresponds to a separate persistent subsequence

\[
\cos(n_k)^{n_k}, \quad \text{where } n_k = 355k + (3 + 22j), \quad k \in \mathbb{N},
\]

for some fixed \( j \in \mathbb{N} \). The appearance of \( p = 355 \) is due to its role in the continued fraction expansion of \( \pi \) and the fact that \( \frac{355}{113} \) is the best rational approximation of \( \pi \) with denominator less than 16604, and the appearance of 22 is also likely due to \( \pi \approx \frac{22}{7} \). This suggests that other persistent subsequences may be found by looking at other exceptionally accurate fractional approximations to \( \pi \), for example: \( \frac{52163}{16604} \), \( \frac{833719}{265381} \) and \( \frac{42208400}{13435511} \). Indeed, we find persistent subsequences for \( n_k = 833719k \) and \( n_k = 833719k + 42208400 \), as in Figure 5.

Each persistent subsequence appears to have a single peak, of a shape very similar to a Gaussian distribution (for those whose peak is not too close to 0). Curve matching in MATLAB reveals that these peaks match well. For example, the persistent subsequence \( \cos(n_k)^{n_k} \) with \( n_k = 644 + 355k \) has a coefficient of...
determination $R^2 = 0.9983$ with the Gaussian $0.9978 e^{-\left(\frac{x-200.8}{8.84}\right)^2}$. However, these curves are not truly Gaussian; the following theorem derives the general peak shape we observe in persistent subsequences and is matched against a sample subsequence in Figure 6.

**Theorem 5.2.** For any arithmetic progression $a_n = pn + d$, the sequence $\{|\cos a_n|^{\alpha_n}\}$ lies on the curve

$$f(x) = |\cos((p - \pi Q(p))\alpha x + d - \pi Q(d))|^{x^\gamma}.$$  \hspace{1cm} (17)

**Proof.** The proof begins similarly to the proof of Theorem 5.1. Let $S$ be an arithmetic progression of the form $S_n = pn + d$, where $p, d \in \mathbb{Z}$. Then we have:

$$|\cos(pn + d)|^{(pn + d)^\gamma} = |\cos(n(p - \pi Q(p)) + (d - \pi Q(d)))|^{(pn + d)^\gamma}$$

The domain can be extended into the reals via the transformation $\alpha(x) = \frac{x - d}{p}$, and this yields (17). \hfill \Box

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Figure 6. $|\cos(8.49137 \cdot 10^{-8} x - 0.0353982)|^{-1.2}$ (red) plotted against the sequence $|\cos(a_n)|^{a_n^{-2}}$ (blue).