FRACTIONAL DERIVATIVES OF COMPOSITE FUNCTIONS AND THE CAUCHY PROBLEM FOR THE NONLINEAR HALF WAVE EQUATION

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Abstract. We show new results of wellposedness for the Cauchy problem for the half wave equation with power-type nonlinear terms. For the purpose, we propose two approaches on the basis of the contraction-mapping argument. One of them relies upon the $L^q_t L^\infty_x$ Strichartz-type estimate together with the chain rule of fairly general fractional orders. This chain rule has a significance of its own. Furthermore, in addition to the weighted fractional chain rule established in Hidano, Jiang, Lee, and Wang (arXiv:1605.06748v1 [math.AP]), the other approach uses weighted space-time $L^2$ estimates for the inhomogeneous equation which are recovered from those for the second-order wave equation. In particular, by the latter approach we settle the problem left open in Bellazzini, Georgiev, and Visciglia (arXiv:1611.04823v1 [math.AP]) concerning the local wellposedness in $H^{s_{\text{rad}}}(\mathbb{R}^n)$ with $s > 1/2$.

1. Introduction

This paper is concerned with the Cauchy problem for the nonlinear, first-order wave equation

$$
\begin{cases}
  i\partial_t u - \sqrt{-\Delta} u = F(u), & t > 0, x \in \mathbb{R}^n, \\
  u(0) = u_0.
\end{cases}
$$

This equation has rich mathematical problems, and it has recently gained much attention; to mention a few papers treating power-type nonlinear terms, see [2], [5], [10] for local/global wellposedness, [18] for finite-time blow up, [24], [2] for stability/instability of ground states, [5] for illposedness for low-regularity data, and [10] for the proof of various a priori estimates. The present paper also treats the power-type nonlinear term of the form $\lambda|u|^{p-1}u$ (which is algebraic if $p$ odd) or $\lambda|u|^p$ (which is algebraic if $p$ even) ($\lambda \in \mathbb{C} \setminus \{0\}$, $p > 1$), and discusses three problems left unexplored in the existing literature. They are: (i) wellposedness in the Sobolev spaces $H^s(\mathbb{R}^n)$ with $s \geq s_c := n/2 - 1/(p-1)$, especially when $s$ is strictly smaller than, and very close to $p$; (ii) local wellposedness in $H^{(1/2)+\varepsilon}_{\text{rad}}(\mathbb{R}^n)$ when $1 < p < 1 + 2/(n-1)$, which is the problem left open in [2]; (iii) global wellposedness for small data in $H^s_{\text{rad}}(\mathbb{R}^n)$ for $1 + 2/(n-1) < p < 1 + 2/(n-2)$, $s \in (s_c, 1]$.

Our proof of wellposedness in $H^s$ builds upon the $L^q_t L^\infty_x$ Strichartz-type estimates due to Klainerman and Machedon [23] (see also Proposition 1 of Fang and...
Wang [8]). Before the statement of the wellposedness, let us make the assumption on the nonlinear term $F(u)$.

**Definition 1.** We say $F(u)$ has the property $(F)_{k,p}$ if there exists $p > 1$ and $k \in \mathbb{N}$ with $k \leq p$ such that when it is considered as a function $\mathbb{R}^2 \to \mathbb{R}^2$, $F : \mathbb{C} \to \mathbb{C}$ is a $C^k$ function, satisfying $F^{(j)}(0) = 0$ for $0 \leq j \leq k$ and for all $z_1, z_2 \in \mathbb{C}$

$$|F^{(j)}(z_1) - F^{(j)}(z_2)| \leq C_j \begin{cases} |z_1 - z_2| (|z_1|, |z_2|) |p^{j-1} & j \leq k, p \geq j + 1 \\ |z_1 - z_2|^{p-j} & j = k, p \in [k, k+1] \end{cases}.$$

Let $U(t) := \exp(-itD)$ with $D = \sqrt{-\Delta}$. By $\lceil p \rceil$, we mean the smallest integer greater than or equal to $p$. Our first theorem concerning the problem (i) is as follows:

**Theorem 1.1.** Let $n \geq 1$, $p > 1$. When $F(u)$ is not algebraic, suppose that

$$p > \max \left( s_c, \frac{n-1}{2}, \frac{n+1}{4} \right),$$

and $F(u)$ has the property $(F)_{k,p}$ with $k = \lceil p \rceil - 1$ (i.e., $k < p \leq k + 1$ and $k \in \mathbb{N}$). Then the problem (1.1) is locally wellposed in $H^s$, provided that $s \geq s_c = n/2 - 1/(p-1)$, and $s > \max((n-1)/2, (n+1)/4)$ (we additionally assume $s < p$ when $F$ is not algebraic). More precisely, we have

1. for any $u_0 \in H^s$, there exists a $T_0 > 0$ depending on $n$, $p$, $u_0$, the constants $C_j$ in (1.2), such that there exists a unique solution

$$u \in L^\infty([0,T_0]; H^s) \cap C([0,T_0]; L^2) \cap L^q(0,T_0; L^\infty)$$

to the associated integral equation

$$u(t) = U(t)u_0 - i \int_0^t U(t - \tau)F(u(\tau))d\tau$$

for some $q \in (\max(4/(n-1), 2), \infty]$ with $q \geq p - 1$, $1/q \geq n/2 - s$. Moreover, $u$ actually belongs to $C([0,T_0]; H^s)$.

2. (Persistence of regularity) if $u_0 \in H^{s_1}$ with some $s_1 > s$ ($s_1 < p$ if $F(u)$ is not algebraic), then $u \in C([0,T_0]; H^{s_1})$.

3. (Critical wellposedness) in addition, when we can take $s = s_c$, that is, when $p > 5$ for $n = 2$, $p > 3$ for $n \geq 3$, and if $n \geq 8$ and $F(u)$ is not algebraic,

$$p > \frac{n + 2 + \sqrt{(n+2)(n-6)}}{4}$$

for $n \geq 8$,

so that we have $p > s_c$, the problem is critically local wellposed in $H^{s_c}$. In particular, there exists $\varepsilon_0 > 0$ such that $T_0 = \infty$ and the solution scatters in $H^{s_c}$, i.e.,

$$\exists u_0^{s_c} \in H^{s_c}, \lim_{t \to +\infty} \|u(t) - U(t)u_0^{s_c}\|_{H^{s_c}} = 0,$$

when $\|D^{s_c}u_0\|_{L^2(\mathbb{R}^n)} < \varepsilon_0$.

**Remark 1.1.** The similar result with additional assumption $p > \lceil s \rceil$ has been obtained in Dinh [5]. If the initial data $u_0$ is radially symmetric about the origin $x = 0$, the result similar to Theorem 1.1 obviously remains true for $s > \max(s_c, (n-1)/2)$ for $n \geq 3$ and $s > s_c$, $s > 1/2$ for $n = 2$. This improvement is due to the well-known fact that the range of admissible pairs $(q,r)$ improves for radially symmetric data,
see Lemma 3.1 below. Such a result of small data global existence for radially symmetric data in the case $n = p = 3$ has been already shown in Fujiwara, Georgiev and Ozawa [10].

**Remark 1.2.** The regularity assumptions $s \geq s_c$ and $s > (n+1)/4$ are sharp in general. Actually, by the connection of this problem with the nonlinear wave equations established in Section 6, we know that, when $p \geq 2$ with $n = 3, 4$ and $p > 1$ with $n \geq 5$, there exists $F(u)$ (e.g., $F = i\mathcal{R}u|p|$) such that the problem is ill-posed in $H^s$ for $s = \max(s_c, (n+1)/4) - \delta$ with arbitrary small $\delta > 0$. See Lindblad [25] for $n = 3$ and $p = 2$, Fang and Wang [7, Theorem 1.2] for $s \in (-n/2, s_c)$ when $s_c > 0$, [7, Theorem 1.3] for $s \in (\max(s_c, 0), (n+1)/4)$ when $s_c < (n+1)/4$ and $n \geq 5$, and [7, Corollary 1.1 and Theorem 1.4] for $s \in (s_c, (n+1)/4)$ when $s_c < (n+1)/4, p \geq 2$ and $n = 3, 4$. See also [5] for some ill-posed results with $s < s_c$, by applying the technique of Christ, Colliander and Tao [4].

In addition to the $L^n_t L^\infty_x$ Strichartz-type estimate, the proof of Theorem 1.1 uses the Ginibre-Ozawa-Velo type estimate on fractional derivatives of composite functions (see Lemma 2.3 below), by which we in particular obtain for some constant $C = C(n, p, s)$

$$\|D^s(|u|^p-1)v\|_{L^2(\mathbb{R}^n)} \leq C\|v\|_{L^\infty(\mathbb{R}^n)}^{p-1}\|D^sv\|_{L^2(\mathbb{R}^n)}$$

whenever $s \in (0, \min(p, n/2))$, $p > 1$. The way of showing the fractional chain rule (1.5) is inspired by Ginibre, Ozawa and Velo [11], and we closely follow the argument of the proof of Lemma 3.4 of [11]. In the contraction-mapping argument, we use (1.5) in combination with the basic estimate Lemma 3.1 for the inhomogeneous equation (3.1), and hence the $L^n_t L^\infty_x$ Strichartz-type norm naturally comes into play.

As mentioned above (see the problem (ii)), the second purpose of this paper is to prove the local wellposedness in $H^{1/2+\varepsilon}_{\text{rad}}(\mathbb{R}^n)$, when $p$ is $H^{1/2}$ subcritical, that is, $1 < p < 1 + 2/(n-1)$. The importance of studying the wellposedness in $H^{1/2}$ is obvious in view of the two conservation laws

$$\|u(t)\|_{L^2(\mathbb{R}^n)} = \|u_0\|_{L^2(\mathbb{R}^n)},$$

(1.6)

$$\frac{1}{2}\|D^{1/2}u(t)\|^2_{L^2(\mathbb{R}^n)} + \frac{\lambda}{p+1}\|u(t)\|^{p+1}_{L^{p+1}(\mathbb{R}^n)}$$

(1.7)

$$= \frac{1}{2}\|D^{1/2}u_0\|^2_{L^2(\mathbb{R}^n)} + \frac{\lambda}{p+1}\|u_0\|^{p+1}_{L^{p+1}(\mathbb{R}^n)}$$

for $F(u) = \lambda|u|^{p-1}u$ ($\lambda \in \mathbb{R}$). In spite of its importance, the local wellposedness in $H^{1/2}$ actually remains open, except the particular case $n = 1$ and $F(u) = -|u|^2u$, see Krieger, Lenzmann and Raphaël [24]. In the present paper, at the cost of imposing a little more regularity and radial symmetry on the data, we study the wellposedness in $H^{(1/2)+\varepsilon}_{\text{rad}}(\mathbb{R}^n)$. We shall prove

**Theorem 1.2.** Let $n \geq 2$, $p \in (1, 1+2/(n-1))$ and suppose that $F(u)$ satisfies

$$|F(u)| \leq C_0|u|^p, |F'(u)| \leq C_1|u|^{p-1},$$

(1.8)

Let $s \in (1/2, 1] \cap (1/2, n/2)$ and $s_1 \in (1/2, s]$. Then there exists a constant $c > 0$ depending on $n$, $p$, $s$, $s_1$, and the constant $C_0$ in (1.8) such that the Cauchy problem (1.1) with $u_0 \in H^{s_1}_{rad}(\mathbb{R}^n)$ admits a unique, radially symmetric solution satisfying

$$u \in H^s(0, T_\Lambda; H^s) \cap C([0, T_\Lambda], L^2),$$

(1.9)
\( r^{-(1-\delta)/2} D^\sigma u \in L^2((0, T_\Lambda) \times \mathbb{R}^n) \), \( \delta := 1 - \frac{n-1}{2} (p-1), \sigma = 0, 1 - s_1, s_1, s \),

where

\[
T_\Lambda := c\Lambda^{-(p-1)/\delta}, \quad \Lambda := \|D^{s_1} u_0\|^2_{L^2(\mathbb{R}^n)} \|D^{1-s_1} u_0\|^2_{L^2(\mathbb{R}^n)}.
\]

Remark 1.3. By (1.11), we see that if \( \|D^{s_1} u_0\|^2_{L^2(\mathbb{R}^n)} \|D^{1-s_1} u_0\|^2_{L^2(\mathbb{R}^n)} \to 0 \) for some \( s_1 \in (1/2, s) \), then \( T_\Lambda \to \infty \), regardless of \( \|u_0\|_{H^s} \).

This theorem resolves the problem left open in Bellazzini, Georgiev and Visciglia [2]. In order to show the wellposedness in \( H^s_{rad} \) along with the lower bound (1.11) on lifespan, we use the method of weighted space-time \( L^2 \) estimates together with the weighted fractional chain rules (see [13, Theorem 2.5]). The similar approach has already played a key role in getting the optimal lower bound on lifespan of small solutions to the second order semi-linear wave equations with the minimal-regularity radial data [13]. See also Keel, Smith and Sogge [20] for an earlier and influential result, proved by the approach based on weighted \( L^2 \)-estimates, concerning long-time existence of small, classical solutions to the second-order semi-linear wave equation. With the help of the boundedness of the Riesz transforms on \( L^2(\omega(x)dx) \) (\( \omega(x) \in A_2 \)), we recover the space-time \( L^2 \) estimates for the half-wave equation from those for the second order wave equation, see Proposition 3.2 below.

Concerning the problem (iii), we show:

**Theorem 1.3.** Let \( n \geq 2 \) and let \( F(u) \) satisfy (1.8) for some \( p \in (1 + 2/(n-1), 1 + 2/(n-2)) \), \( s \in (s_c, 1] \cap (s_c, n/2) \) and \( u_0 \in H^s_{rad}(\mathbb{R}^n) \). Also, let \( \delta \in (0, 1) \) satisfy

\[
1 - \delta = \left( \frac{n}{2} - s \right) (p-1)
\]

and let \( \delta' \) satisfy

\[
\delta < \delta' < \left( s - \frac{1}{2} \right) (p-1).
\]

Moreover, set

\[
s_1 := \frac{n}{2} - \frac{1 - \delta + \delta'}{p-1}
\]

so that \( s_1 \in (1/2, s_c) \). Then there exists an \( \varepsilon_0 > 0 \) such that if

\[
\|D^{s_1} u_0\|_{L^2(\mathbb{R}^n)} + \|D^s u_0\|_{L^2(\mathbb{R}^n)} < \varepsilon_0,
\]

then the Cauchy problem (1.1) admits a unique, radially symmetric solution satisfying

\[
u \in L^\infty(0, \infty; H^s) \cap C([0, \infty); L^2),
\]

\[
r^{-(1-\delta)/2} \langle r \rangle^{-\delta'/2} D^\sigma u \in L^2((0, \infty) \times \mathbb{R}^n), \quad \sigma = 0, s_1, s.
\]

Remark 1.4. As mentioned in Remark 1.1, global existence of small solutions for radially symmetric data in \( H^s_{rad} \) can be proved by the method using the Strichartz estimates for \( n = 2, p > 3 \).

Our proof of Theorem 1.3 uses global-in-time space-time \( L^2 \) estimates with the weight \( r^{-(1-\delta)/2} \langle r \rangle^{-\delta'/2} \), as in the study of the second order wave equations [14], [13]. See also the earlier papers [16], [15] where the weight \( r^{-(1-\delta)/2} \langle r \rangle^{-\delta'/2} \) was used for the proof of “almost global” existence.
We note that in view of Theorem 1.1 and Theorem 1.3, global existence of small solutions remains open when \(1 + 2/(n - 2) \leq p < 3\) \((n \geq 4)\). We expect that one may improve the result to the cases \(p > s_c\) and \(p > 1 + 2/(n - 1)\), with further efforts. However, for the cases with \(p < s_c\), e.g., \(p \in (7/5, 3/2)\) with \(n = 7\), the problem seems to be beyond the scope of the current technology.

We should also mention that Inui \([18, \text{Theorem 1.4}]\) proved finite-time blow up even for small data in the case \(F(u) = \lambda|u|^p\) with \(1 < p < 1 + 1/n\). Since we prove global existence for \(1 + 2/(n - 1) < p < 1 + 2/(n - 2)\) and small \(H^1\) radially symmetric data, it is an interesting problem whether \((1.1)\) with certain \(F(u)\) with power \(1 + 2/n \leq p \leq 1 + 2/(n - 1)\) has finite-time blow-up solutions even for small data. In this connection, it is also an interesting problem whether the lower bound of the lifespan \((1.11)\) is sharp. By establishing a connection between the half-wave problems and the nonlinear wave equations, we show that the lower bound is optimal, in general. Moreover, we could also handle the critical case.

**Theorem 1.4.** Let \(n = 2, 3, 4\), \(F(u) = i|\Re u|^p\) with \(p \in \{\max(1, (n - 1)/2, 1 + 2/(n - 1)\}, \text{and} s \in [(n - 1)/2, p]\) except \(s = 1/2\) and \(n = 2\). For any real-valued, radial function \(g \in C_0^\infty(\mathbb{R}^n)\) with \(\int g dx > 0\), there exist constants \(C, \varepsilon_0 > 0\) such that for any \(\varepsilon \in (0, \varepsilon_0)\), the Cauchy problem \((1.1)\) with data \(u_0 = \varepsilon g\) can not have solution \(u \in L^\infty(0, T; H^s) \cap C([0, T]; L^2)\) with

\[
L_\varepsilon = \begin{cases} 
\exp(C\varepsilon^{-(p-1)}) & p = 1 + \frac{2}{n-1}, \\
C\varepsilon^{-\frac{2}{(n-2)(n-1)}} & 1 < p < 1 + \frac{2}{n-1}.
\end{cases}
\]

In addition, let \(n \geq 2\), \(s \in (1/2, 1]\) \(\cap (1/2, n/2)\) and let \(F(u)\) satisfy \((1.8)\) with \(p = 1 + 2/(n - 1)\). There exist \(c, \varepsilon_1 > 0\), such that for any \(\varepsilon \in (0, \varepsilon_1)\), the Cauchy problem \((1.1)\) admits a solution in \(u \in L^\infty(0, T; H^s) \cap C([0, T]; L^2)\), where

\[
T_\varepsilon = \exp(c\varepsilon^{-(p-1)})
\]

for any radial data \(u_0 \in H^s\) with \(\varepsilon^2 = \|u_0\|_{H^s}^2 + \|u_0\|_{H^s} \|u_0\|_{H^{s-\delta}}\).

This paper is organized as follows. Section 2 is devoted to the proof of a fractional chain rule for general composite functions, Lemma 2.3. In Section 3, for half wave equations, we collect and study two class of space-time estimates, Strichartz type estimates, Lemma 3.1, and Morawetz type local energy estimates, Proposition 3.2. Such estimates have been extensively investigated for the wave equations, and have played an important role in our understanding of various linear and nonlinear wave equations. Equipped with the Strichartz type estimates, and the chain rule of fractional orders, we present the proof of Theorem 1.1 in Section 4. In Section 5, we prove Theorems 1.2 and 1.3, as well as the existence part in Theorem 1.4. The key to the proof is the local energy estimates Proposition 3.2, the radial Sobolev inequalities, and a weighted fractional chain rule of \([13, \text{Theorem 2.5}]\). Section 6 is devoted to the remaining part of Theorem 1.4, nonexistence of global solutions and upper bound of the lifespan, for the sample case of \(F(u) = i|\Re u|^p\) with \(1 < p \leq 1 + 2/(n - 1)\).

2. Fractional chain rule

In this section, we prove a version of the fractional chain rule, which has a significance of its own and plays an essential role in the proof of Theorem 1.1.
2.1. Function spaces. To begin with, let us recall some basic facts about the Besov/Sobolev spaces. At first, let \( \phi(x) \in C_0^\infty(\mathbb{R}^n) \), \( \phi \geq 0 \), with \( \phi = 1 \) for \( |x| \leq 1 \) and \( \phi = 0 \) for \( |x| \geq 2 \), we define the Littlewood-Paley projection operators \( S_j, P_j \), for \( f \in S' \) as follows

\[
\mathcal{F}(S_j f)(\xi) = \phi(2^{-j}\xi)(\mathcal{F}f)(\xi) , \quad P_j f = S_j f - S_{j-1} f ,
\]

where \( \mathcal{F} \) denotes the Fourier transform. We also define the inhomogeneous Littlewood-Paley projection operators by \( \tilde{P}_j \) for \( j \in \mathbb{N} \) as follows \( \tilde{P}_j = P_j \) for \( j \geq 1 \) and \( \tilde{P}_0 = 0 \).

It is known that the inhomogeneous Besov spaces are Banach spaces, for all \( s < n/q, r \)

\[
\|f\|_{B^s_{q,r}} = \|2^{js}P_j f\|_{L^r} , \quad \|f\|_{B^s_{q,r}} = \|2^{js}\tilde{P}_j f\|_{L^r} .
\]

where we have \( f = \sum_{j \in \mathbb{Z}} P_j f \) in \( S' \).

For \( s \in \mathbb{R} \), \( q, r \in [1, \infty] \), we define the Besov (semi)norms for \( f \in S' \),

\[
\|f\|_{B^s_{q,r}} = \|2^{js}P_j f\|_{L^r} , \quad \|f\|_{B^s_{q,r}} = \|2^{js}\tilde{P}_j f\|_{L^r} .
\]

Based on the Besov (semi)norm, we define Besov spaces by

\[
B^s_{q,r} = \{f \in S'; \|f\|_{B^s_{q,r}} < \infty\} , \quad \dot{B}^s_{q,r} = \{f \in S'_h; \|f\|_{\dot{B}^s_{q,r}} < \infty\} .
\]

It is obvious that \( \dot{B}^s_{q,r} \) is included in the space of tempered distributions vanishing at infinity,

\[
S'_0 := \{f \in S'; \lim_{N \to -\infty} S_N f = 0 \}
\]

where we have \( f = \sum_{j \in \mathbb{Z}} P_j f \) in \( S' \).

For \( s \in \mathbb{R} \), \( q, r \in [1, \infty] \), we define the Leibniz rule for such spaces,

\[
\|f\|_{H^s} = \|\xi^s \mathcal{F}(f)(\xi)\|_{L^2} , \quad \|f\|_{H^s} = \|(1 + |\xi|^2)^{s/2} \mathcal{F}(f)(\xi)\|_{L^2} .
\]

Based on these (semi)norms, we define \( L^2 \) based Sobolev spaces by

\[
H^s = \{f \in S'; \|f\|_{H^s} < \infty\} , \quad \dot{H}^s = \{f \in S'; \mathcal{F}(f) \in L^1_{loc}, \|f\|_{\dot{H}^s} < \infty\} .
\]

It is known that the inhomogeneous Sobolev spaces are Hilbert spaces, for all \( s \in \mathbb{R} \), and the homogeneous Sobolev spaces \( \dot{H}^s \) are Hilbert spaces, if and only if \( s < n/2 \), see, e.g., [1, Proposition 1.34, page 26]. Moreover, \( \dot{B}^s_{2,2} = \dot{H}^s \) if \( s < n/2 \) and we denote \( \|f\|_{\dot{B}^s_{q,r}} = \|f\|_{\dot{B}^s_{q,r}} \) with property \( f \in \dot{B}^s_{q,r} .
\]

Here, for future reference, let us record the fractional Leibniz rule for such spaces, see, e.g., [1, Corollary 2.54].
Lemma 2.1 (Fractional Leibniz rule). Let $s > 0$ with \((2.3)\). Then $\dot{B}^s_{q,r} \cap L^\infty$ is an algebra. Moreover, there exists a constant $C$, depending only on the dimension $n$, such that
\[
\|fg\|_{\dot{B}^s_{q,r}} \leq \frac{C^{s+1}}{s} \left( \|f\|_{L^\infty} \|g\|_{\dot{B}^s_{q,r}} + \|f\|_{\dot{B}^s_{q,r}} \|g\|_{L^\infty} \right).
\]
In particular, we have
\[
\|fg\|_{\dot{\dot{H}}^s} \lesssim \|f\|_{L^\infty} \|g\|_{\dot{H}^s} + \|f\|_{\dot{\dot{H}}^s} \|g\|_{L^\infty}, \quad 0 < s < n/2,
\]
and for algebraic $F(u)$ with power $p > 1$,
\[
\|F(u)\|_{\dot{\dot{H}}^s} \lesssim \|u\|^{p-1}_{L^\infty} \|u\|_{\dot{H}^s}, \quad 0 < s < n/2.
\]

For Besov spaces, we recall the following difference characterization, see, e.g., \[3, \text{Theorem 6.2.5, 6.3.1}\]. Notice, however, that our statement for the homogeneous Besov space is more precise, as the proof of \((2.11)\) in \[3, \text{Theorem 6.3.1}\] has implicitly used the assumption $f = \sum P_j f$, that is, $f \in S'_0$.

Lemma 2.2 (Difference characterization of the Besov spaces). For $s \in (0, m)$, $m \in \mathbb{N}$, $q, r \in [1, \infty]$, we have for any $u \in S'$,
\[
\|u\|_{\dot{B}^s_{q,r}} \leq \|t^{-s}\| \Delta_m^y u(x)\|_{L^\infty_{\mathbb{R}^m} L^r(\mathbb{R}^n)} \|u\|_{L^r((0,\infty),dt/t)},
\]
\[
\|u\|_{\dot{B}^s_{q,r}} \simeq \|u\|_{L^q} + \|t^{-s}\| \Delta_m^y u(x)\|_{L^\infty_{\mathbb{R}^m} L^r(\mathbb{R}^n)} \|u\|_{L^r((0,\infty),dt/t)},
\]
where
\[
\Delta_m^y u(x) = (\tau_y - I)^m u(x), \quad (\tau_y - I) u(x) = u(x + y) - u(x).
\]
Moreover, if $u \in S'_0$, we have
\[
\|u\|_{\dot{B}^s_{q,r}} \gtrsim \|t^{-s}\| \Delta_m^y u(x)\|_{L^\infty_{\mathbb{R}^m} L^r(\mathbb{R}^n)} \|u\|_{L^r((0,\infty),dt/t)}.
\]

2.2. Fractional chain rule. With the definitions of the Besov/Sobolev spaces, we are ready to state the fractional chain rule.

Lemma 2.3. Suppose that $F(u)$ has the property $(F)_{k,p}$ (see Definition 1). Then for any $s \in (0, \min(k+1, p))$ and $q, r \in [1, \infty]$, we have
\[
\|F(u)\|_{\dot{B}^s_{q,r}} \lesssim \|u\|^{p-1}_{L^\infty} \|u\|_{\dot{B}^s_{q,r}}, \quad u \in \dot{B}^s_{q,r} \cap L^\infty,
\]
\[
\|F(u)\|_{\dot{B}^s_{q,r}} \lesssim \|u\|^{p-1}_{L^\infty} \|u\|_{\dot{B}^s_{q,r}}, \quad u \in B^s_{q,r} \cap L^\infty.
\]
In addition, if $r \in [1, 2]$, $q \in [1, \infty)$, $s \in (0, n/q)$ and $s \geq n/q - n/2$, then $F(u) \in S'_h$ and so $F(u) \in \dot{B}^s_{q,r}$. In particular, with $q = r = 2$, we have
\[
\|F(u)\|_{\dot{H}^s} \lesssim \|u\|^{p-1}_{L^\infty} \|u\|_{\dot{H}^s}, \quad s \in (0, \min(k+1, p, n/2)),
\]
\[
\|F(u)\|_{H^s} \lesssim \|u\|^{p-1}_{L^\infty} \|u\|_{H^s}, \quad s \in (0, \min(k+1, p)).
\]
Remark 2.1. The estimates of this form have been well-known for many special situations. Typical examples include $0 < s \leq \lfloor p \rfloor$ or $k = 1$. See \[11, \text{Lemma 3.4}\] for the case $k = 1$ (actually, our proof is inspired by the proof in \[11\]). The estimates for $H^s$ ($s > 0$) or $W^{s,q}$ ($s \in \mathbb{N}$) have been well-known for the case $s \leq \lfloor p \rfloor$, see e.g. \[31, \text{Lemma A.9} \], \[17, \text{Corollary 6.4.5}\]. Estimates involving $L^p$ based Sobolev spaces are also known for many cases, see, e.g., \[19, \text{Lemma A2} \], \[32, \text{Chapter 2 Proposition 5.1} \] and references therein for case $0 < s < 1 < p$. See also \[13, \text{Theorem 2.5} \] for a
recent work on weighted fractional chain rule. There are also estimates for Hölder continuous case $0 < s < p < 1$, see [21, Lemma A.12].

Remark 2.2. The inhomogeneous estimates (2.13) and (2.15) have been known, see, e.g., [30, Theorem 2, page 325]. It seems to the authors that the homogeneous estimates (2.12) and (2.14) may be new, although a weaker version of (2.12), with $L^\infty$ replaced by $L^\infty \cap \dot{B}^0_{\infty,2}$ and $q, r \geq 2$, has been available from [29, Lemma 2.2, page 402], which was also proven by enhancing the proof in [11] for $k = 1$.

2.3. Preparation. The basic idea of proof is to exploit the difference characterization of the Besov spaces, Lemma 2.2. For that purpose, we introduce some notations and exploit the properties of the difference operators.

For fixed $y \in \mathbb{R}^n$ with $|y| < t$, let

$$u_{(m)}(x, \lambda) = [\Pi_{j=1}^m (I + \lambda_j \Delta_y^i)]u(x) = \sum_{\alpha \in \{0,1\}^m} \lambda^\alpha \Delta^\alpha_{\alpha[1]} u(x), \lambda \in [0, 1]^m.$$ 

By definition, we see that $\tau_y \Delta^\alpha_{\alpha[1]} u(x) = \Delta^\alpha_{\alpha[1]} u(x) + \Delta^\alpha_{\alpha[1]} u(x)$, and so

$$\sup_{|y| \leq t} |\tau_y \Delta^\alpha_{\alpha[1]} u(x)| \leq |\Delta^\alpha_{\alpha[1]} u(x)| + |\Delta^\alpha_{\alpha[1]} u(x)|, m \geq 1,$$


(2.16)

$$\|u_{(m)}(x, \lambda)\|_{L^\infty} \leq \|u\|_{L^\infty}, \lambda \in [0, 1]^m, m \geq 0.$$ 

Moreover, for any $\lambda \in [0, 1]^m$, $|y| < t$, $k \geq 1$ and $m \geq 0$, we have

$$\Delta^\alpha_{\alpha[1]} u_{(m)}(x, \lambda) = \sum_{\alpha \in \{0,1\}^m} \lambda^\alpha \Delta^\alpha_{\alpha[1]} u(x) = O(\sum_{j=0}^m |\Delta^\alpha_{\alpha[1]} u(x)|),$$


(2.18)

$$\tau_y \Delta^\alpha_{\alpha[1]} u_{(m)}(x, \lambda) = \sum_{\alpha \in \{0,1\}^m} \lambda^\alpha \tau_y \Delta^\alpha_{\alpha[1]} u(x) = O(\sum_{j=0}^{m+1} |\Delta^\alpha_{\alpha[1]} u(x)|).$$


(2.19)

For any $\alpha \in \{0,1\}^m$, let $A_\alpha = \{j : \alpha_j = 0\}$,

$$\partial^\alpha_{\alpha[1]} u_{(m)}(x, \lambda) = \Delta^\alpha_{\alpha[1]} u_{(m-|\alpha|)}(x, \lambda_\alpha \{\lambda_j\}_{j \in \Lambda_\alpha}).$$

Notice that

$$\Delta^\alpha_{\alpha[1]} (F \circ u)(x) = F(u(x + y)) - F(u(x))$$

$$= \int_0^1 \frac{d}{d\lambda} F(u(x) + \lambda \Delta^\alpha_{\alpha[1]} u(x))d\lambda$$

$$= \int_0^1 \frac{d}{d\lambda} F(u(x), \lambda))d\lambda,$$

then for any $k \geq 1$, we have

$$\Delta^\alpha_{\alpha[1]} (F \circ u)(x) = \int_{[0,1]^k} \frac{d^k}{d\lambda_1d\lambda_2 \cdots d\lambda_k} F(u(\lambda))(x, \lambda)d\lambda.$$
By using (2.20), we get

$$
\Delta^y_k (F \circ u)(x) = \sum_{1 \leq l < k, \sum_{\alpha_m = 1}^k \alpha_m \in \{0,1\}^k \setminus 0} \int_{[0,1]^k} F^{(l)}(u(k_j)(x, \lambda)) \Pi_{m=1}^l \Delta^{y_{|\alpha_m|}} u(k-|\alpha_m|)(x, \{\lambda_j\}_{j \in A_{\alpha_m}}) d\lambda
$$

$$
= \sum_{1 \leq l < k, \sum_{\alpha_m = 1}^k \alpha_m \in \{0,1\}^k \setminus 0} \int_{[0,1]^k} F^{(k)}(u(k)(x, \lambda)) \Pi_{m=1}^k \Delta^y u(k-1)(x, \{\lambda_j\}_{j \neq m}) d\lambda
$$

Based on this fact, we know that

$$
\Delta^y_{k+1} (F \circ u)(x) = \Delta^y_k \int_{[0,1]^k} F^{(k)}(u(k)(x, \lambda)) \Pi_{m=1}^k \Delta^y u(k-1)(x, \{\lambda_j\}_{j \neq m}) d\lambda
$$

$$
+ \Delta^y_k \sum_{1 \leq l < k, \sum_{\alpha_m = 1}^k \alpha_m \in \{0,1\}^k \setminus 0} \int_{[0,1]^k} F^{(l)}(u(k)(x, \lambda)) \Pi_{m=1}^l \Delta^{y_{|\alpha_m|}} u(k-|\alpha_m|)(x, \{\lambda_j\}_{j \neq m}) d\lambda
$$

$$
= \int_{[0,1]^k} F^{(k)}(u(k)(x, y, \lambda)) - F^{(k)}(u(k)(x, \lambda)) \Pi_{m=1}^k \Delta^y u(k-1)(x, \{\lambda_j\}_{j \neq m}) d\lambda
$$

$$
+ \sum_{l=1}^k \int_{[0,1]^k} F^{(k)}(u(k)(x, y, \lambda)) \Pi_{m=1}^{l-1} \Delta^y u(k-1)(x, \{\lambda_j\}_{j \neq m}) x \delta_{\lambda_j \neq l}(x) \Pi_{m=l+1}^k \Delta^y u(k-1)(x, \{\lambda_j\}_{j \neq m}) d\lambda
$$

$$
+ \sum_{1 \leq l < k, \sum_{\alpha_m = 1}^k \alpha_m \in \{0,1\}^k \setminus 0} \int_{[0,1]^k} (\Delta^y_l F^{(l)}(u(k)(x, \lambda))) \Pi_{m=1}^l \Delta^{y_{|\alpha_m|}} u(k-|\alpha_m|)(x, \{\lambda_j\}_{j \in A_{\alpha_m}}) d\lambda
$$

$$
+ \sum_{1 \leq l < k, \sum_{\alpha_m = 1}^k \alpha_m \in \{0,1\}^k \setminus 0} \int_{[0,1]^k} (\Delta^{y_{|\alpha_m|+1}} u(k-|\alpha_m|))(x, \{\lambda_j\}_{j \in A_{\alpha_m}}) \Pi_{m=0}^l \Delta^{y_{|\alpha_m|}} u(k-|\alpha_m|)(x, \{\lambda_j\}_{j \in A_{\alpha_m}}) d\lambda
$$

where we have used the assumption (1.2) with \( z_2 = 0 \), (2.16)-(2.19).
2.4. Proof of Lemma 2.3. With help of the previous properties, we could apply Lemma 2.2 to give the proof of Lemma 2.3. Here we write down the proof for (2.12) only and leave the similar proof for (2.13) to the interested reader.

Actually, by (1.2) and (2.17)-(2.18), we have
\[
\left| F^{(k)}(u_k(x + y, \lambda)) - F^{(k)}(u_k(x, \lambda)) \right| \Pi_{m=1}^{k} \Delta^{y}_m u_{(k-1)}(x, \{\lambda_j\}_{j \neq m}) \right| \\
\leq C \sum_{j=1}^{k} |\Delta^{y}_j u(x)| \left\{ \begin{array}{l}
|u|_{L_{p-k-1}^\infty(x, \lambda)}^{p-k} |\Delta^{y}_j u(x)|, \quad p \geq k + 1 \\
|\Delta^{y}_j u(x)|^{p-k}, \quad k \leq p \leq k + 1
\end{array} \right.
\]
for any \( \lambda \in [0, 1]^m \). In combination with the previous relation, we get
\[
\Delta^{y}_{k+1}(F(u))(x) \lesssim \sum_{1 \leq j \leq k+1} \|u\|_{L_{p-j}^\infty \Pi_{m=1}^{j} \Delta^{y}_m u(x)}, \quad p \geq k + 1,
\]
and
\[
\Delta^{y}_{k+1}(F(u))(x) \lesssim \sum_{j=1}^{k+1} |\Delta^{y}_j u(x)|^p + \sum_{1 \leq j \leq k, j \neq m} \|u\|_{L_{p-j}^\infty \Pi_{m=1}^{j} \Delta^{y}_m u(x)}
\]
when \( k \leq p \leq k + 1 \).

For the first case, \( 0 < s < k+1 \leq p \), we observe that for any \( \sum_{m=1}^l a_m \geq k+1 \) and \( a_m \in [1, k+1] \) with \( l \in [1, k+1] \), there exists \( b_m \in [1, a_m] \) such that \( \sum_{m=1}^l b_m = k+1 \) and \( 0 < sb_m/(k+1) < b_m \leq a_m \). Using the difference characterization of the Besov space, (2.8), we get
\[
\|F(u)\|_{B^{r}_{p,q}} \lesssim \|t^{-s}|\Delta^{y}_{k+1}(F(u))(x)\|_{L_{p-k}^\infty B^{r}_{p,q}} L^{-s}(0, \infty, dt/t)
\]
\[
\lesssim \sum_{1 \leq j \leq k+1, a_m \in [1, k+1]} \|u\|_{L_{p-j}^\infty \Pi_{m=1}^{j} \Delta^{y}_m u(x)} \|t^{-s} B^{r}_{p-k} \|_{L^{r-1}((0, \infty), dt/t)}
\]
\[
\lesssim \sum_{1 \leq j \leq k+1, a_m \geq k+1} \|u\|_{L_{p-j}^\infty \Pi_{m=1}^{j} \Delta^{y}_m u(x)} \|t^{-s} B^{r}_{p-k} \|_{L^{r-1}((0, \infty), dt/t)}
\]
\[
\lesssim \sum_{l=1}^\infty \sum_{m=1}^l b_m = k+1, b_m \geq 1 \|u\|_{L_{p-l}^\infty \Pi_{m=1}^{l} \Delta^{y}_m u(x)} \|t^{-s} B^{r}_{p-k} \|_{L^{r-1}((0, \infty), dt/t)}
\]
\[
\lesssim \|u\|_{L_{p-k}^\infty} \|u\|_{B^{r}_{p,q}},
\]
where we have used the interpolation inequality
\[
\|u\|_{B^{\theta}_{q,r}} \lesssim \|u\|_{B^{\theta}_{q,r}} \|u\|_{B^{\theta}_{\infty,\infty}}, \quad \theta \in (0, 1), s > 0, q, r \in [1, \infty],
\]
and the trivial embedding $\|u\|_{\dot{B}^{q,r}_{\infty}} \lesssim \|u\|_{L^\infty}$.

For the second case, $0 < s < p$ and $p \in [k, k + 1]$, since $0 < s < p \leq k + 1$, all terms except $\sum_{j=1}^{k+1} |\Delta_j^y u(x)|^p$ could be handled in the same way. Then, we have

$$
\|F(u)\|_{\dot{B}^{q,r}_{p,s}} \lesssim \|u\|_{L^\infty}^{p-1} \|u\|_{\dot{B}^{q,r}_{p,s}} + \sum_{j=1}^{k+1} \|t^{-s} \|\Delta_j^y u(x)\|^p \|L_{q,r}^\infty \|L^p((0,\infty),dt/t)
$$

$$
\lesssim \|u\|_{L^\infty} \|u\|_{\dot{B}^{q,r}_{p,s}} + \sum_{j=1}^{k+1} \|t^{-s/p} \|\Delta_j^y u(x)\|^p \|L^{p,q,r}((0,\infty),dt/t)
$$

$$
\lesssim \|u\|_{L^\infty} \|u\|_{\dot{B}^{q,r}_{p,s}} + \|u\|_{\dot{B}^{q,r}_{p,s}}^p
$$

To complete the proof of Lemma 2.3, it remains to prove that $F(u) \in S'_h$ if $r \in [1, 2]$, $q \in [1, \infty)$, $s \in (0, n/q)$ and $s \geq n/q - n/2$. Actually, under the additional assumption on $(s, q, r)$, as $\|F(u)\|_{\dot{B}^{q,r}_{p,s}} < \infty$, we see that there is $w \in S'_h$ such that

$$
w = \sum_j P_j F(u) \in \dot{B}^{q,r}_{p,s} \subset S'_h
$$

(see, e.g., [1, Remark 2.24, page 66]), which gives us $F(u) - w = P(x)$ for some polynomial $P(x)$. By Sobolev embedding, for $g_0 \in [2, \infty)$ with $\frac{n}{q_0} = \frac{n}{q} - s$, we get

$$
u, w \in \dot{B}^{g_0}_{q,r} \subset B^{g_0}_{q_0,r} \subset B^{g_0}_{q_0,2} \subset L^{g_0}.
$$

As $F(0) = 0$, $u \in L^\infty$ and $F'(u) \in L^\infty$, then $F(u) \in L^{g_0}$. Thus

$$
P(x) = F(u) - w \in L^{g_0}
$$

and so $P(x) \equiv 0$, which proves $F(u) = w \in S'_h$.

3. Space-time estimates for half wave equations

In this section, for half wave equations:

$$
i \partial_t u - \sqrt{-\Delta} u = G, t > 0, u(0) = u_0,
$$

we collect and study two class of space-time estimates, Strichartz type estimates and Morawetz type estimates. For the wave equations, these estimates have been extensively investigated and have played an important role in our understanding of various linear and nonlinear wave equations.

The Strichartz type estimates for the half wave equations could be easily covered by what have been developed for wave equations. On the other hand, it is less trivial that many Morawetz type estimates could also be recovered from the corresponding estimates for wave equations.

3.1. Strichartz type estimates. We need the $L^q_t L^\infty_x$ Strichartz type estimates. More precisely, we use:

**Lemma 3.1.** Let $n \geq 2$ and $q \in (4, \infty)$ ($n = 2$), $q \in (2, \infty)$ ($n \geq 3$). Let $\sigma := n/2 - 1/q$. Then there exists $C > 0$, which is independent of $T > 0$, such that we have the inequality

$$
\|U(t)f\|_{L^q_t L^\infty_x} \leq C \|f\|_{\dot{H}^\sigma(R^n)}.
$$
In particular, for any solutions to (3.1), we have

\[(3.3) \quad \|u(t)\|_{\dot{H}^{s}(\mathbb{R}^n)} \leq \|u_0\|_{\dot{H}^{s}(\mathbb{R}^n)} + \int_{0}^{t} \|G(\tau)\|_{\dot{H}^{s}(\mathbb{R}^n)}d\tau, \quad t > 0, s \in [0,n/2) \]

and,

\[(3.4) \quad \|u\|_{L_{t}^{q}L_{x}^{\infty}} \leq \|U(t)u_0\|_{L_{t}^{q}L_{x}^{\infty}} + C\|G\|_{L_{t}^{q}\dot{H}^{s}}, \quad T > 0.\]

Furthermore, if \(u_0(x)\) and \(G(t, x)\) are radially symmetric about the origin \(x = 0\), then the estimates (3.2) and (3.4) remain true also for \(q \in (2, 4)\) (\(n = 2\)) and for \(q = 2\) (\(n \geq 3\)). Here and in what follows, we will use the notation \(\|f\|_{L^{q}(0, T; X)} := \|f\|_{L^{q}(0, T; L^{q}(\mathbb{R}^{3}))}\).

**Proof.** The Strichartz-type estimate (3.2) is proved by Escobedo and Vega [6] for \(n = 3\), Klainerman and Machedon [23] for \(n \geq 2\) (see also Proposition 1 of [8]). The estimate (3.3) is a consequence of the unitarity of \(U(t)\) on \(L^{2}(\mathbb{R}^{n})\). Also, the estimate (3.4) follows directly from (3.2). In the presence of radial symmetry, the improvement of the range of admissible pairs \((q, r)\) was pointed out by Klainerman and Machedon for the solutions to the second order equation \(\Box u = 0\) in three space dimensions. They proved

\[\|D^{-1} \sin tDg\|_{L^{2}(0, \infty; L^{\infty}(\mathbb{R}^{3}))} \leq C\|g\|_{L^{2}(\mathbb{R}^{3})}\]

for radially symmetric \(g \in L^{2}\). In [8], Fang and Wang performed estimates directly for \(U(t)f\), and proved (3.2) for \(q \in (2, \infty)\) (\(n = 2\)), \(q \in [2, \infty)\) (\(n \geq 3\)).

**Remark 3.1.** As explained above, it is proved in [8] that the inequality (3.2) holds for \(q = 2\) and all \(n \geq 3\) when \(u_0\) is radially symmetric, which settles the conjecture made by Klainerman in [22] (see Remark 1 there).

3.2. **Local energy estimates.** As is well-known, the local energy estimates (which are also known as KSS type estimates, Morawetz (radial) estimates) are indispensable for many problems for the wave equations. In this subsection, for solutions to the half-wave equations, we prove the analogs of the local energy estimates.

**Proposition 3.2.** Let \(n \geq 2\) and \(0 < \delta \leq 1\). There exists a constant \(C > 0\) depending only on \(n, \delta, \) and \(T > 0\), such that any solutions to (3.1) satisfy

\[(3.5) \quad \|u\|_{L_{t}^{q}L_{x}^{2} + T^{-\delta/2}\|u^{-1-\delta/2}\|_{L_{t}^{2}L_{x}^{2}} \leq C\|u_0\|_{L^{2}(\mathbb{R}^{n})} + CT^{\delta/2}\|g\|_{L_{t}^{q}L_{x}^{2}},\]

\[(3.6) \quad \|u\|_{L_{t}^{\infty}L_{x}^{2} + (\ln(2 + T))^{-1/2}\|u^{-1-\delta/2}\|_{L_{t}^{2}L_{x}^{2}} \leq C\|u_0\|_{L^{2}(\mathbb{R}^{n})} + C(\ln(2 + T))^{1/2}\|g\|_{L_{t}^{q}L_{x}^{2}}.\]

In addition, if \(0 < \delta < \delta'\), \(\delta \leq 1\), and \(1 + \delta' - \delta < n\), there exists a constant \(C > 0\) depending on \(n, \delta, \delta'\) such that the solution to (3.1) satisfies

\[(3.7) \quad \|u\|_{L_{t}^{\infty}(0, \infty; L^{2}(\mathbb{R}^{n}))) + \|u^{-1-\delta/2}\|_{L^{2}(0, \infty; \mathbb{R}^{n})} \leq C\|u_0\|_{L^{2}(\mathbb{R}^{n})} + C\|g\|_{L^{2}(0, \infty; \mathbb{R}^{n})}.\]

At first, let us record a version of local energy estimates for the standard wave operator \(\Box = \partial_{t}^{2} - \Delta\).
Lemma 3.3. Let \( n \geq 2 \), we have
\[
\left\| (\partial_t, \nabla_x)v \right\|_{L^\infty L^2 L^{2,1/2} L^2} \lesssim \left\| (\partial_t, \nabla_x)v(0) \right\|_{L^2} + \left\| \Box v \right\|_{L^\infty L^2+L^{2,1/2} L^2}.
\]
Here, with \( q \in [1, \infty] \), \( \left\| f \right\|_{L^q L^2} = \left\| 2^q f \right\|_{L^q L^2, \tau(t \in \mathbb{R}, \tau \geq 2)}. \) In particular, we have
\[
\| \partial v \|_{L^\infty L^2} + T^{-\delta/2} \| r^{-(1-\delta)/2} \partial v \|_{L^2 L^2} \\
\leq C \| \partial v(0) \|_{L^2} + CT^{\delta/2} \| r^{-(1-\delta)/2} \Box v \|_{L^2 L^2},
\]
(3.10)
\[
\| \partial v \|_{L^\infty (0, \infty \times L^2(\mathbb{R}^n))} + \| r^{-(1-\delta)/2} \partial v \|_{L^2((0, \infty) \times \mathbb{R}^n)} \\
\leq C \| \partial v(0) \|_{L^2} + C \| r^{-(1-\delta)/2} \Box v \|_{L^2((0, \infty) \times \mathbb{R}^n)},
\]
(3.11)

The estimate (3.8) for \( n \geq 3 \) could be proved by multiplier method, see, e.g., [27, 34]. The estimate for \( n = 2 \) is obtained in [28], even for small perturbation of the flat metric. Local energy estimates have rich history and we refer [28, 26] for more exhaustive history of such estimates.

The estimates (3.9)-(3.11) are also known as KSS type estimates [20, 16, 27, 12, 15], which are known to be implied by the local energy estimates (3.8), see, e.g., [20, 27, 34] and [33, Lemma 1.5].

To obtain Proposition 3.2 from Lemma 3.3, we need to check that the weight functions, \( w \), behave well under various operations, and it amounts to property \( w^2 \in A_2 \).

Lemma 3.4 (Lemma 2.7 of [13]). Let \( n \geq 1 \) and let \( w(x) := r^{-(1-\delta)/2} (r)^{\delta'/2}, \) with \( 0 \leq 1-\delta \leq 1-\delta' < n \). Then \( w^2 \in A_1 \), where by \( A_p \) we mean the Muckenhoupt \( A_p \) class with \( p \in [1, \infty] \). In particular, it applies to all of the three weight functions occurring in Proposition 3.2.

Proof of Proposition 3.2. The idea is to reduce the required estimates to what are known for solutions to the second-order wave equation. Let \( u_0 \in A_1 \) be any of the three weight functions occurring in Proposition 3.2, and the estimates to be proven could be rewritten as follows
\[
\left\| u \right\|_{L^\infty L^2} + \left\| u_0 u \right\|_{L^2 L^2} \leq C \| u_0 \|_{L^2} + C \| w_0^{-1} G \|_{L^2 L^2}, n \geq 2.
\]

Now, for the solution \( u \) to (3.1), let us introduce the auxiliary equation
\[
i \partial_t v + \sqrt{-\Delta} v = -u, \quad v(0) = 0.
\]
Then, we see that \( v \) satisfies
\[
(\partial_t^2 - \Delta)v = -(i \partial_t - \sqrt{-\Delta})(i \partial_t + \sqrt{-\Delta}) v = (i \partial_t - \sqrt{-\Delta}) u = G
\]
and
\[
\partial_t v(0) = -\frac{1}{i} (u(0) + \sqrt{-\Delta} v(0)) = i u_0.
\]
Based on Lemma 3.3, we get
\[
\| \partial v \|_{L^\infty L^2} + \| u_0 \partial v \|_{L^2 L^2} \leq C \| u_0 \|_{L^2} + C \| w_0^{-1} G \|_{L^2 L^2}, n \geq 2.
\]
By (3.13), we see that the proof of (3.12) is reduced to
\[
\| u_0 \sqrt{-\Delta} v \|_{L^2((0, T) \times \mathbb{R}^n)} \lesssim \| u_0 \nabla v \|_{L^2((0, T) \times \mathbb{R}^n)}.
\]
However, as \( w_0^2 \in A_1 \subset A_2 \) by Lemma 3.4, we know that
\[ R_j = \sqrt{-\Delta}^{-1} \partial_j, \]
the \( j \)-th Riesz transform, is bounded on \( L^2(w_0^2 dx) \), and so
\[
\| w_0 \Delta w \|_{L^2(\mathbb{R}^n)} = \| w_0 (-(\Delta)^{-1/2}(-\Delta)w) \|_{L^2(\mathbb{R}^n)} \\
\leq \sum_{j=1}^n \| w_0 R_j \partial_j w \|_{L^2(\mathbb{R}^n)} \leq C \sum_{j=1}^n \| w_0 \partial_j w \|_{L^2(\mathbb{R}^n)},
\]
which completes the proof of (3.15). The proof of Proposition 3.2 is finished.

4. Proof of Theorem 1.1

The proof of Theorem 1.1 requires two ingredients. They are the \( L^q_t L^\infty_x \) Strichartz-type estimates, Lemma 3.1, and the chain rule of fractional orders, Lemma 2.3. We will give the proof for non-algebraic nonlinearity in the following, and the proof in the case of algebraic nonlinearity follows the same lines of proof, with (2.14) replaced by (2.7).

4.1. Subcritically local wellposedness in \( H^s \). Define the nonlinear mapping
\[
\Phi[u](t) := U(t)u_0 - i \int_0^t U(t - \tau)F(u(\tau))d\tau.
\]
The existence of solution is equivalent to seeking a fixed point of \( \Phi \). The proof of local wellposedness for \( H^s \) with \( s > n/2 \) and \( q = \infty \) is routine consequence of classical energy argument, and we will focus only on the case \( s \in (0, n/2) \) with \( q < \infty \) (and so is \( n \geq 2 \)).

Thanks to the assumption \( p > s, k = \lfloor p \rfloor - 1 \) and \( q \geq p - 1 \), we get by (3.3) and (2.15), that, for any \( t \in [0, T] \) and \( s_0 \in \{0, s\} \),
\[
\| \Phi[u](t) \|_{H^{s_0}} \leq \| u_0 \|_{H^{s_0}} + C_1 \int_0^t \| u(\tau) \|_{L^\infty}^{p-1} \| u(\tau) \|_{H^s} d\tau \\
\leq \| u_0 \|_{H^{s_0}} + C_1 T^{1 - \frac{p-1}{p}} \| u \|_{L^\infty_t L^p_x}^{p-1} \| u \|_{L^\infty_t H^s}.
\]
As \( q \in (\max(4/(n-1), 2), \infty) \) and \( 1/q \geq n/2 - s \), we have \( H^s \subset H^{n/2-1/q} \), and by (3.2) and (3.4),
\[
\| \Phi[u] \|_{L^q_t L^\infty_x} \leq \| U(t)u_0 \|_{L^q_t L^\infty_x} + C_2 T^{1 - \frac{p-1}{p}} \| u \|_{L^\infty_t L^\infty_x}^{p-1} \| u \|_{L^\infty_t H^s} \\
\leq C_3 \| u_0 \|_{H^s} + C_2 T^{1 - \frac{p-1}{p}} \| u \|_{L^\infty_t L^\infty_x}^{p-1} \| u \|_{L^\infty_t H^s}.
\]
We also have
\[
\| \Phi[u](t) - \Phi[v](t) \|_{L^2} \leq C_4 T^{1 - \frac{p-1}{p}} (\| u \|_{L^q_t L^\infty_x} + \| v \|_{L^q_t L^\infty_x})^{p-1} \| u(t) - v(t) \|_{L^p_t L^2}.
\]
For the case \( q > p - 1 \), setting
\[
T = c \| u_0 \|_{H^s}^{-\frac{p-1}{q(p-1)}}
\]
for some small \( c > 0 \) such that
\[
C_1 T^{1 - \frac{p-1}{p}} (2C_3 \| u_0 \|_{H^s})^{p-1} \leq 1/2, C_2 T^{1 - \frac{p-1}{p}} (2C_3 \| u_0 \|_{H^s})^{p-1} \leq C_3/2, \quad C_4 T^{1 - \frac{p-1}{p}} (4C_3 \| u_0 \|_{H^s})^{p-1} \leq 1/2.
\]
then we see that
\[ \|\Phi[u]\|_{L^p_T H^s} \leq 2\|u_0\|_{H^s}, \|\Phi[u]\|_{L^p_T L^\infty_x} \leq 2C_3\|u_0\|_{H^s}, \]
\[ \|\Phi[u](t) - \Phi[v](t)\|_{L^2} \leq \frac{1}{2}\|u(t) - v(t)\|_{L^p_T L^2}. \]
That is, we have proved that \( \Phi \) is a contraction mapping on the complete metric space
\[ (4.1) \quad X_1 := \{ u \in L^\infty(0, T; H^s) \cap C([0, T]; L^2) \cap L^p_T L^\infty_x, u(0) = u_0, \]
\[ \|u(t)\|_{H^s} \leq 2\|u_0\|_{H^s}, \|u\|_{L^p_T L^\infty_x} \leq 2C_3\|u_0\|_{H^s} \}
with the metric \( d(u, v) := \|u - v\|_{L^\infty_T L^2} \). The unique fixed point \( u \in X_1 \) is the unique solution we are looking for.

We can easily verify the fact that \( u \in C([0, T]; H^s) \). Indeed, we know that \( U(t)u_0 \in C(\mathbb{R}; H^s) \) and \( \|F(u(\tau))\|_{H^s} \in L^1([0, T]) \), which yields that
\[ \int_0^t U(t - \tau)F(u(\tau))d\tau \in C([0, T]; H^s), \]
we find the right hand side of \( (4.1) \) is in \( C([0, T]; H^s) \) and so is \( u \in C([0, T]; H^s) \).

In addition, if \( u_0 \in H^{s_1} \), observing that \( (4.2) \) with \( s_0 = s_1 \) also holds with some new constant \( C'_1 \). Then see that \( \Phi \) is also a contraction mapping on the complete metric space
\[ (4.1) \quad X_2 := \{ u \in L^\infty(0, T; H^{s_1}) \cap C([0, T]; L^2) \cap L^p_T L^\infty_x, u(0) = u_0, \]
\[ \|u(t)\|_{H^{s_1}} \leq 2\|u_0\|_{H^{s_1}}, \|u\|_{L^p_T L^\infty_x} \leq 2C_3\|u_0\|_{H^{s_1}} \}
for the same kind of \( T \) and the metric \( d(u, v) := \|u - v\|_{L^\infty_T L^2} \), which essentially verifies the persistence of regularity property. The details are left to the interested reader.

4.2. Critically local wellposedness in \( H^{s_c} \). For the critical case \( s = s_c \in (0, n/2, \), we are forced to set \( q = p - 1 \in (\max(4/(n - 1), 2), \infty) \). For this situation, \( (4.2) \) and \( (4.3) \) could be replaced by
\[ |(4.8) \|\Phi[u](t)\|_{H^{s_0}} \leq \|u_0\|_{H^{s_0}} + C_1\|u\|^{p-1}_{L^{p-1}_t L^\infty_x} \|u\|_{L^p_T H^{s_0}}, \forall t \in [0, T], s_0 \in \{0, s_c\}, \]
\[ (4.9) \|\Phi[u]\|_{L^p_T L^\infty_x} \leq \|U(t)u_0\|_{L^p_T L^\infty_x} + C_2\|u\|^{p-1}_{L^{p-1}_t L^\infty_x} \|u\|_{L^p_T H^{s_0}}. \]

Let \( A > 0 \) be a small constant satisfying
\[ (4.10) \quad C_1(2A)^{p-1} \leq \frac{1}{2}, \quad A + C_2(2A)^{p-1}(2\|u_0\|_{H^{s_c}}) \leq 2A, \quad C_4(4A)^{p-1} \leq \frac{1}{2}, \]
By the Strichartz estimate \( (3.2) \),
\[ \|U(t)u_0\|_{L^{p-1}(0, \infty; L^\infty_x)} \leq C_3\|u_0\|_{H^{s_c}} < \infty. \]
Thus, for this small constant \( A \), we can choose \( T_0 > 0 \) so that
\[ (4.11) \quad \|U(t)u_0\|_{L^{p-1}_0 L^\infty_x} \leq A \]
thanks to the absolute continuity of the integral. If we define
\[ (4.12) \quad X_3 := \{ u = u(t, x) : u \in L^\infty_T H^{s_c} \cap C([0, T_0]; L^2) \cap L^{p-1}_t L^\infty_x, u(0) = u_0, \]
\[ \|u\|_{L^\infty_T L^2} \leq 2\|u_0\|_{L^2}, \|u\|_{L^{p-1}_T H^{s_c}} \leq 2\|u_0\|_{H^{s_c}}, \|u\|_{L^{p-1}_T L^\infty_x} \leq 2A \]
with the metric $d(u, v) := \|u - v\|_{L^{\infty}_x L^2_t}$, then $X$ is a complete metric space and we see from (4.8), (4.9) and (4.5) that $\Phi$ is a contraction mapping on $X_3$.

4.3. Small data scattering in $H^{s_c}$. When the initial data satisfies

$$\|u_0\|_{\dot{H}^{s_c}} \leq A/C_3,$$

we see that we could set $T_0 = \infty$ and so is the global existence of $u \in L^\infty_t H^{s_c} \cap C^1_t L^2_t \cap L^{p-1}_t L^\infty_x$. It remains to prove scattering for the global solution.

By Lemma 2.3, $F(u) \in L^1(0, \infty; H^{s_c})$. Using the basic inequality for vector-valued integrable functions, we have

$$\| \int_0^\infty U(-\tau)F(u(\tau))d\tau \|_{H^{s_c}} \leq C\| F(u)\|_{L^1(t, \infty; H^{s_c})},$$

which converges to 0 as $t \to +\infty$. In other words, we have proven that

$$u_0 - i \int_0^t U(-\tau)F(u(\tau))d\tau \to u_0 - i \int_0^\infty U(-\tau)F(u(\tau))d\tau := u_0^+, \quad \text{in } H^{s_c}.$$ 

Thus, we see that

$$\|u(t) - U(t)u_0^+\|_{H^{s_c}} = \|U(t) \int_0^\infty U(-\tau)F(u(\tau))d\tau\|_{H^{s_c}} = \| \int_0^\infty U(-\tau)F(u(\tau))d\tau\|_{H^{s_c}} \to 0,$$

as $t \to +\infty$. This completes the proof of Theorem 1.1.

5. Proof of Theorems 1.2 and 1.3

In this section, we prove Theorems 1.2 and 1.3, as well as the existence part in Theorem 1.4. The key to the proof is the local energy estimates Proposition 3.2, as well as the radial Sobolev inequalities, and the following weighted fractional chain rule of [13, Theorem 2.5]

Lemma 5.1 (Fractional chain rule). Let $s \in (0, 1)$, $q \in (1, \infty)$. Assume $F : \mathbb{R}^k \to \mathbb{R}^l$ is a $C^1$ map, satisfying $F(0) = 0$ and

$$|F'(\tau v + (1 - \tau)w)| \leq \mu(\tau)|G(v) + G(w)|,$$

with $G > 0$ and $\mu \in L^1([0, 1])$. If $w_1^q, (w_1 w_2)^q \in A_q$ and $w_2^{-1} \in A_1$, we have

$$\|w_1 w_2 D^s F(u)\|_{L^q_x} \leq \|w_1 D^s u\|_{L^q_x} \|w_2 G(u)\|_{L^{q'}}.$$

In particular, for $s \in [0, 1]$, if $w^2 \in A_1$, then we have

$$\|w^{-1} D^s F(u)\|_{L^2_x} \leq \|w D^s u\|_{L^2_x} \|w^{-2} G(u)\|_{L^{2'}}.$$

We notice that we have added the trivial cases $s = 0, 1$ to (5.3), due to the fact that if $w^2 \in A_2$,

$$\|w Du\| \simeq \|w \nabla u\|.$$
5.1. Proof of Theorem 1.2. Let \( n \geq 2, \ p \in (1, 1+2/(n-1)) \), \( \delta = 1 - \frac{2}{n-1}(p-1) \in (0, 1) \) and \( w(x) := |x|^{-(1-\delta)/2} \). By Lemma 3.4, \( w^2 \in A_1 \) and we can apply (5.3) with \( G(u) = |u|^{p-1} \); for \( s \in [0, 1] \),
\[
\| w^{-1} D^s F(u) \|_{L^2} \lesssim \| w D^s u \|_{L^2} \| w^{-2} G(u) \|_{L^\infty} \lesssim \| w D^s u \|_{L^2} \| w^{-2/(p-1)} u \|_{L^\infty}^{p-1}.
\]
Then, applying (3.5), we see that the map \( F \) defined in (4.1) satisfies
\[
\| F[u] \|_{X^s} := \| F[u] \|_{L^2_{(0,T)} H^s_x} + T^{-\delta/2} \| w D^s F[u] \|_{L^2_{(0,T)} L^2_x} \lesssim \| u_0 \|_{H^s(\mathbb{R}^n)} + T^{\delta/2} \| w^{-1} D^s G \|_{L^2_{(0,T)} L^2_x}
\]
and so, for fixed \( s \in (1/2, 1) \cap (1/2, n/2) \), \( s_1 \in (1/2, s) \), we have for \( \sigma = 0, 1 - s_1, s_1, s \),
\[
\| F[u] \|_{X^{s_1}} \leq C_5 (\| u_0 \|_{H^s(\mathbb{R}^n)} + T^{\delta/2} \| w D^s u \|_{L^2_{(0,T)} L^2_x} \| w^{-2/(p-1)} u \|_{L^\infty}^{p-1}).
\]
Note that \( w^{-2/(p-1)} = |x|^{(1-\delta)/(p-1)} = |x|^{(n-1)/2} \). Therefore we can use the radial Sobolev lemma of [13] (see (2.2) of [13] which is a direct consequence of the “end-point” trace lemma of [9] and the real interpolation) to get
\[
\| u \|_{L^\infty(\mathbb{R}^n)} \lesssim C \| D^s u \|_{L^2(\mathbb{R}^n)}^{1/2} \| D^{1-s_1} u \|_{L^p(\mathbb{R}^n)}^{1/2}.
\]
It is at this place that we need the assumption of radial symmetry in our proof.

Let us define the space of functions where we carry out the contraction-mapping argument: we set
\[
X(s, s_1, T) := \{ u = u(t,x), \text{ radially symmetric about } x = 0 : \| u \|_{X^s} \leq 2C_5 \| u_0 \|_{H^s(\mathbb{R}^n)} \text{ for } \sigma = 0, 1 - s_1, s_1, s \}
\]
Here, \( C_5 \) is the constant appearing in (5.4). Equipped with the metric \( d(u,v) := T^{-\delta/2} \| w(u-v) \|_{L^2_{(0,T)}} + \| u-v \|_{L^\infty_{(0,T)} L^2(\mathbb{R}^n)} \), \( X(s, s_1, T) \) is a complete metric space. By (5.4) and (5.5), we find that the mapping \( F[u] \) satisfies for \( \sigma = 0, 1 - s_1, s_1, s \)
\[
\| F[u] \|_{X^s} \leq C_5 \| u_0 \|_{H^s(\mathbb{R}^n)} + CT^{\delta/2} \| u \|_{X^{s_1}} (\| u \|_{X^{s_1}} (\| u \|_{X^{s_1-1}} + \| v \|_{X^{s_1}} (\| u \|_{X^{s_1-1}} + \| v \|_{X^{s_1}} (\| u \|_{X^{s_1-1}} + \| v \|_{X^{s_1}} (\| u \|_{X^{s_1-1}} + \| v \|_{X^{s_1}} (\| u \|_{X^{s_1-1}} + \| v \|_{X^{s_1}} (\| u \|_{X^{s_1-1}} + \| v \|_{X^{s_1}} (\| u \|_{X^{s_1-1}} + \| v \|_{X^{s_1}} (\| u \|_{X^{s_1-1}} + \| v \|_{X^{s_1}} (\| u \|_{X^{s_1-1}} + \| v \|_{X^{s_1}} \}^{(p-1)/2}.
\]
(5.7)

(5.8)

for any \( u, v \in X(s, s_1, T) \). Choosing \( T > 0 \) as in (1.11) for some small constant \( c > 0 \), then
\[
T^{\delta/2} (\| D^{s_1} u_0 \|_{L^2(\mathbb{R}^n)}^{1/2} \| D^{1-s_1} u_0 \|_{L^p(\mathbb{R}^n)})^{p-1} \ll 1
\]
and we easily see that \( F[u] \) maps \( X(s, s_1, T) \) into itself and it is a contraction-mapping there. In this way, we can prove the local wellposedness in \( H^s_{\text{rad}} \) when \( n \geq 2, s \in (1/2, 1] \cap (1/2, n/2) \) and \( 0 < p-1 < 2/(n-1) \). The proof of Theorem 1.2 is finished.

5.2. Small data long time existence, Theorems 1.3 and 1.4. The proof of small data global existence, Theorem 1.3, as well as the small data long time existence part of Theorem 1.4, proceeds along the same lines, with trivial modifications.

More precisely, let
\[
w := \left\{ \begin{array}{ll}
p^{-(1-\delta)/2} (p')^{-\delta/2} & p = 1 + 2/(n-1) \\
p^{-1} & p \in (1 + 2/(n-1), 1 + 2/(n-2))
\end{array} \right.
\]
and
\[
A_T := \left\{ \begin{array}{ll}
\ln(2 + T) & p = 1 + 2/(n-1) \\
1 & p \in (1 + 2/(n-1), 1 + 2/(n-2)).
\end{array} \right.
\]
Here, we choose $\delta' > \delta$ to be fixed such that Lemma 3.4 applies for $w$. Then we have for any $T \in (0, \infty)$ and $\sigma \in [0, 1]$,
\[
\|\Phi[u]\|_{X^s} \lesssim \|u\|_{\dot{H}^s(\mathbb{R}^n)} + A_T \|wD^\alpha u\|_{L^2_t L^2_x} \|w^{-\frac{s}{2+1}} u\|_{L^p_t L^\infty_x}
\]
\[
\lesssim \|u_0\|_{\dot{H}^s(\mathbb{R}^n)} + A_T^2 \|u\|_{X^s} \|w^{-\frac{s}{2+1}} (u, v)\|_{L^p_t L^\infty_x},
\]
and
\[
\|\Phi[u] - \Phi[v]\|_{X^s} \lesssim A_T^2 \|u - v\|_{X^s} \|w^{-\frac{s}{2+1}} (u, v)\|_{L^p_t L^\infty_x},
\]
where
\[
\|u\|_{X^s} := \|u\|_{L^p_t \dot{H}^s_x} + A_T^{-1} \|wD^\alpha u\|_{L^2_t L^2_x}.
\]
Based on these estimates, the proof of small data long time existence is easily reduced to the radial Sobolev lemma, (5.5) and
\[
\|\rho^{-n/2-s} u\|_{L^\infty_x} \lesssim \|u\|_{\dot{H}^s}, \quad s \in (1/2, n/2)
\]
(see, e.g., [9, 13, (2.2)]).

6. NONEXISTENCE OF GLOBAL SOLUTIONS AND UPPER BOUND OF THE LIFESPAN

In this section, we give the remaining part of Theorem 1.4, nonexistence of global solutions and upper bound of the lifespan, for the case of $1 < p \leq 1 + 2/(n-1)$ and $F(u) = i|\Re u|^p$, that is
\[
i\partial_t u - \sqrt{-\Delta} u = i|\Re u|^p.
\]
As we have mentioned in the introduction, the idea is to establish a connection between the half-wave problems and the nonlinear wave equations. The procedure is similar to that of Proposition 3.2. Introduce $v$ in the way similar to (3.13), that is,
\[
i\partial_t v + \sqrt{-\Delta} v = iu, \quad v(0) = 0.
\]
Then, we see that $v$ satisfies
\[
(\partial_t^2 - \Delta) v = -(i\partial_t - \sqrt{-\Delta})(i\partial_t + \sqrt{-\Delta}) v = -i(i\partial_t - \sqrt{-\Delta}) u = |\Re u|^p
\]
and
\[
v(0) = 0, \quad \partial_t v(0) = \frac{1}{i} (iu(0) - \sqrt{-\Delta} v(0)) = u_0.
\]
For any real-valued, radial function $g \in C_0^\infty(\mathbb{R}^n)$ with $\int g dx > 0$, we set the initial data to be $u_0 = \varepsilon g$ with $\varepsilon > 0$. Then we see that $v$ is real-valued and so is $\sqrt{-\Delta} v$ (as $\sqrt{-\Delta} v = \sqrt{-\Delta} \hat{v}$). Thus
\[
- i\partial_t v + \sqrt{-\Delta} v = -i\hat{u}, \quad \partial_t v = \frac{u + \hat{u}}{2} = \Re u, \quad \sqrt{-\Delta} v = i \frac{u - \hat{u}}{2} = -\Im u,
\]
and so
\[
(\partial_t^2 - \Delta) v = |\partial_t v|^p, \quad v(0) = 0, \quad \partial_t v(0) = \varepsilon g.
\]
In summary, we have seen that for any solution $u$ to (6.1) with data $u_0 = \varepsilon g$, the solution $v$ to (6.2) satisfies (6.5). Then it is well-known that, for $p \in (1, 1+2/(n-1)]$, (6.5) could not admit global solutions for any $\varepsilon > 0$, and the lifespan $T$ satisfies $T \leq L_\varepsilon$ with $L_\varepsilon$ given by (1.18), for some constant $C > 0$, see, e.g., [36] and references therein.

For completeness, we give here a simple proof by using the method of test functions, in the current setting, as appeared in [35], [37] for wave equations.
Assume that \( u \in L^\infty(0, T; H^s) \cap C([0, T]; L^2) \subset C([0, T]; H^{s-\delta}) \) (for any \( \delta > 0 \)) is a solution to the equation (6.1) with data \( u_0 = \varepsilon g \). Then our task is to show that \( T < L_\varepsilon \). As \( v \) is the solution to (6.2), we have \( v \in L^\infty(0, T; H^s) \cap C([0, T]; H^{s-\delta}) \).

Recalling (6.4), we have

\[
(6.6) \quad v \in L^\infty(0, T; H^{s+1}) \cap C([0, T]; H^{s+1-\delta}) \cap C^1([0, T]; H^{s-\delta}).
\]

Recall that, by using the similar proof of Theorem 1.1 with radial Strichartz estimates Lemma 3.1, the problem (6.5) with radial data is locally wellposed in \( L^\infty H^{s+1} \cap Lip H^s \) with \( s \in [s_c, p) \) and \( s \geq (n-1)/2 \) (except \( s = 1/2 \) and \( n = 2 \)). As we have \( p \leq 1 + 2/(n-1) \), \( s_c \leq 1/2 \) and so the problem is locally wellposed for \( s \in (1/2, p) \) for \( n = 2 \) and \( s \in [(n-1)/2, p) \) for \( n = 3, 4 \). In particular, there is \( R > 0 \) such that \( g \) is supported in \( B_R \), and so

\[
\text{supp } v \subset \{(t, x) \in [0, T] \times \mathbb{R}^n, |x| \leq t + R \}.
\]

Let

\[
\phi(t, x) = \int_{S^n-1} e^{x \cdot \omega - t} d\omega.
\]

Then, it satisfies

\[
\Delta \phi = \phi, \partial_t \phi = -\phi, \partial_t^2 \phi = \phi.
\]

Moreover, it is well-known that

\[
(6.7) \quad 0 < c(1 + |x|)^{-n-1} e^{x} \leq \phi \leq C(1 + |x|)^{-n-1} e^{x}.
\]

For the convenience of the reader, let us give a proof. The estimate for \( r := |x| \leq 1 \) is trivial and we need only to consider \( r \gg 1 \). In this case,

\[
r^{-n-1} e^{r} \phi(t, x) = |S^{n-2}| \int_0^\pi e^{r \cos \theta} r^{-n-1} (\sin \theta)^{n-2} d\theta
\]

\[
= |S^{n-2}| \int_0^{\pi/2} e^{r \cos \theta} r^{-n-1} (\sin \theta)^{n-2} d\theta + o(1).
\]

Then, as \( 2\theta / \pi \leq \sin \theta \leq \theta \) and \( \theta^2 / \pi \leq 1 - \cos \theta \leq \theta^2 / 2 \) for \( \theta \in [0, \pi / 2] \), we have

\[
\int_0^\pi e^{r \cos \theta} r^{-n-1} (\sin \theta)^{n-2} d\theta \leq \int_0^{\pi/2} e^{-r \theta^2 / \pi} r^{-n-1} \theta^{n-2} d\theta \leq \left( \frac{\pi}{2} \right)^{n-1} \int_0^\infty e^{-r \theta^2 / \pi} \theta^{n-2} d\theta,
\]

\[
\int_0^\pi e^{r \cos \theta} r^{-n-1} (\sin \theta)^{n-2} d\theta \geq \int_0^\pi e^{-r \theta^2 / \pi} r^{-n-1} \left( \frac{2\theta}{\pi} \right)^{n-2} d\theta = \left( \frac{2}{\pi} \right)^{n-2} \int_0^\pi e^{-r \theta^2 / \pi} \theta^{n-2} d\theta,
\]

and so is (6.7).

Introducing a cutoff function \( \psi \) which is identity on \( B_{T+R} \), then we have \( v_t = \psi v_t \in C([0, T]; H^{s-\delta}) \subset C([0, T]; L^p) \) for \( p \in (1, 1+2/(n-1)) \), \( \int \phi |v_t|^p dx = \int \phi \psi v_t^p dx \in C([0, T]) \). As \( \Delta v \in C([0, T]; H^{s-1-\delta}) \), by (6.5), we have

\[
\int v_{tt} \phi dx = \int \phi (\Delta v + |v_t|^p) dx \in C([0, T])
\]

and so \( \int \phi v dx \in C^2, \int \phi v_t dx \in C^1 \).

With \( \square = \partial_t^2 - \Delta \), we have \( \square \phi = 0, \)

\[
\phi \square v - v \square \phi = \frac{d}{dt}(\phi \partial_t v - v \partial_t \phi) - \nabla \cdot (\phi \nabla v - v \nabla \phi),
\]
and so
\[(6.8)\]
\[F' = \frac{d}{dt} \int \phi(\partial_t v + v)dx = \frac{d}{dt} \int (\phi \partial_t v - v \partial_t \phi)dx = \int \phi \Box v - v \Box \phi dx = \int \phi |v|^p dx,\]
where \(F(t) := \int \phi(\partial_t v + v)dx \in C^1\). On the other hand, let \(H(t) := \int \phi \partial_t vdx \in C^1\),
\[(6.9)\]
\[\int \phi \Box v - v \Box \phi dx = \int (\phi v_{tt} - v \phi_{tt})dx = H' - \int (\phi_t v_t + v \phi_t)dx = H' + \int \phi(v_t - v)dx.\]

Let \(G := \int \phi(v_t - v)dx = 2H - F\). Then we have \(F' = H' + G \geq 0\),
\[G' = 2H' - F' \geq 2H' - 2F' = -2G\]
and so
\[G(t) \geq G(0)e^{-2t} = \varepsilon e^{-2t} \int g\phi dx > 0.\]

Thus, as \(F'(t) = \int \phi |v|^p dx,\)
\[F \leq 2H \lesssim \left( \int \phi |v|^p dx \right)^{1/p} \left( \int \phi dx \right)^{1 - 1/p} \lesssim F'(t)^{1/p}(t + R)^{\frac{2}{p - 1} - \frac{1}{1 - 1/p}},\]
that is, there exists \(c > 0\) such that
\[(6.10)\]
\[F'(t) \geq cF^p(t + R)^{-\frac{n-1}{2}(\frac{n-1}{2} - 1)}, \forall t \in [0, T], F(0) = \varepsilon \int g\phi \geq c\varepsilon.\]

Based on \((6.10)\), it is a standard argument to conclude \(T < L_\varepsilon\) for some \(C > 0\). The proof is finished.

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