Isomonodromic deformations and Hurwitz spaces

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1 Introduction

Here we solve $N \times N$ Riemann-Hilbert (inverse monodromy) problems with all monodromy matrices having the structure of matrices of quasi-permutation (i.e. matrices which have only one non-zero element in each column and each row). Such Riemann-Hilbert problem may be associated to arbitrary Hurwitz space of algebraic curves $L$ of genus $g$ realized as $N$-sheeted covering over $\mathbb{CP}1$, and allows solution in terms of Szegő kernel on $L$. If we denote coordinate on $\mathbb{CP}1$ by $\lambda$ and projections of the branch points to complex plane by $\lambda_1, \ldots, \lambda_n$ then the solution of inverse monodromy problem of that type has the following form:

$$\Psi(\lambda)_{jk} = S(\lambda^{(j)}, \lambda^{(k)}_0) E_0(\lambda, \lambda_0), \quad j, k = 1, \ldots, N$$

where $\lambda^{(j)}$ is the point on $j$th sheet of $L$ having projection $\lambda$ on $\mathbb{CP}1$; $S(P, Q)$ is Szegő kernel on $L$:

$$S(P, Q) = \frac{1}{\Theta[p \mid q]} \frac{\Theta[p \mid q](U(P) - U(Q))}{E(P, Q)};$$

$E(P, Q)$ ($P, Q \in L$) is the prime-form on $L$ and $E_0(\lambda, \lambda_0) = (\lambda - \lambda_0)/\sqrt{d\lambda d\lambda_0}$ is the prime-form on $\mathbb{CP}1$; $p, q \in \mathbb{C}^g$ are two vectors such that the combination $Bp + q$ ($B$ is the matrix of $b$-periods on $L$) does not belong to theta-divisor ($\Theta$) on Jacobi variety $J(L)$.

Function $\Psi(\lambda)$ has determinant 1 and is normalized at $\lambda = \lambda_0$ by the condition $\Psi(\lambda = \lambda_0) = I$. It solves the inverse monodromy problem with quasi-permutation monodromy matrices which can be expressed in terms of $p, q$ and intersection indexes of certain contours on $L$. If parameter vectors $p$ and $q$ (and, therefore, also the monodromy matrices) don’t depend on $\{\lambda_j\}$, we fall in the framework of isomonodromy deformations; then the residues $A_j(\{\lambda_j\})$ of the function $\Psi \Psi^{-1}$ satisfy the Schlesinger system.

The associate $\tau$-function can be shown to be proportional to $\Theta[p \mid q](0)$ up to some factor which depends only on $\{\lambda_j\}$. In $N = 2$ case the factor can also be calculated explicitly (see [4]) to give

$$\tau(\{\lambda_j\}) = [\det A]^{-\frac{1}{2}} \prod_{j < k} (\lambda_j - \lambda_k)^{-\frac{1}{2}} \Theta[p \mid q](0|B). \quad (1.1)$$

where $n = 2g + 2; \lambda_1, \ldots, \lambda_{2g+2}$ are branch points on the hyperelliptic curve $L$; $A$ is the matrix of $a$-periods of non-normalized holomorphic differentials on this curve.

As it was demonstrated by Malgrange [4], the tau-function of Schlesinger system may be interpreted as determinant of certain Toeplitz operator. It was further argued by

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Palmer that the tau-function could also be interpreted as determinant of Cauchy-Riemann operator acting on certain class of matrix spinors with prescribed singularities at certain points on Riemann sphere. However, the non-standard type of the domain of the Cauchy-Riemann operators defined in this way makes it rather difficult to establish the links with more conventional framework of [3].

On the other hand, the Cauchy-Riemann determinants corresponding to compact Riemann surfaces were very actively exploited in the context of perturbative string theory in late 80’s (see [4, 5, 6, 7]. Comparison with formulas of works [4, 5, 6] shows that formula (1.1) coincides with the determinant of Cauchy-Riemann operator acting on spinors on which have twists $e^{2\pi ip_j}$ and $e^{2\pi i q_j}$ along cycles $a_j$ and $b_j$ respectively. Therefore, it seems tempting to speculate that this observation is also true for arbitrary curves; this should be a subject of further study.

Another result of these notes concerns the divisor $(\theta) \subset \mathbb{C}^n$ in the space of parameters $\{\lambda_j\}$ introduced by Malgrange. This is the divisor of zeros of $\tau$-function in $\mathbb{C}^n$, or, equivalently, divisor in $\{\lambda_j\}$-space where the solution of inverse monodromy problem with given monodromy data fails to exist. For our class of monodromy data we have

$$\{\lambda_j\} \in (\theta) \iff Bp + q \in (\Theta),$$

where $(\Theta)$ is theta-divisor on Jacobian $J(\mathcal{L})$.

## 2 Schlesinger system and $\tau$-function

Let us fix the notations. Consider the following Riemann-Hilbert problem on $\mathbb{CP}1$: for a given set of $n + 1$ points $\lambda_0, \lambda_1, \ldots, \lambda_n \in \mathbb{C}$, construct a function $\Psi(\lambda) : \mathbb{CP}1 \setminus \{\lambda_1, \ldots, \lambda_n\} \to SL(N, \mathbb{C})$, which has the following properties:

- $\Psi(\lambda)$ is holomorphic on universal covering of $\lambda \in \mathbb{CP}1 \setminus \{\lambda_1, \ldots, \lambda_n\}$ and on some sheet of this covering $\Psi(\lambda_0) = I$.
- $\Psi(\lambda)$ has regular singular points at $\lambda = \lambda_j$, $j = 1, \ldots, n$ with given monodromy matrices $M_j \in SL(N, \mathbb{C})$.

If in addition to monodromy matrices we fix the logarithms of their eigenvalues, this RH problem is always solvable outside of submanifold of codimension 1 in the space of parameters $\{\lambda_j, M_j\}$. Outside of this submanifold function $\Psi$ satisfies the matrix differential equation

$$\frac{d\Psi}{d\lambda_j} = \sum_{j=1}^{n} \left( \frac{A_j}{\lambda - \lambda_j} - \frac{A_j}{\lambda_0 - \lambda_j} \right) \Psi$$

(2.2)

with certain matrices $A_j \in sl(N, \mathbb{C})$; eigenvalues $t_j^{(1)}, \ldots, t_j^{(N)}$ of $A_j$ are equal (up to the factor $2\pi i$) to the logarithms of eigenvalues of matrices $M_j$. We call the set $\{M_j, t_j^{(k)}\}$ the monodromy data.

If we impose the isomonodromy conditions, $dM_k/d\lambda_j = 0$, $j, k = 1, \ldots, n$, and assume that $t_j^{(l)} - t_j^{(s)} \notin \mathbb{Z}$ for any $l$ and $s$, then function $\Psi$ satisfies the following equations with respect to $\lambda_j$:

$$\frac{d\Psi}{d\lambda_j} = \left( \frac{A_j}{\lambda_0 - \lambda_j} - \frac{A_j}{\lambda - \lambda_j} \right) \Psi$$

(2.3)

Compatibility condition of (2.2) and (2.3) gives Schlesinger system for the residues $A_j$ as functions of poles $\{\lambda_k\}$.
In particular, if we choose \( \lambda_0 = \infty \), the Schlesinger system has the following form:

\[
\frac{\partial A_j}{\partial \lambda_k} = \frac{[A_j, A_k]}{\lambda_j - \lambda_k}, \quad j \neq k; \quad \frac{\partial A_j}{\partial \lambda_j} = -\sum_{k \neq j} \frac{[A_j, A_k]}{\lambda_j - \lambda_k}.
\]

The \( \tau \)-function of Schlesinger system is defined by the formula \([8]\):

\[
\frac{d}{d\lambda_j} \ln \tau = \frac{1}{2} \text{res}_{|\lambda = \lambda_j} \text{tr} \left( \Psi_{\lambda} \Psi^{-1} \right)^2 \equiv \sum_{j < k} \frac{\text{tr} A_j A_k}{\lambda_j - \lambda_k}.
\]

According to Malgrange \([1]\), the function \( \tau(\{\lambda_j\}) \) vanishes in the space \( \mathbb{C}^n \setminus \{\lambda_j = \lambda_k, j, k = 1, \ldots, n\} \) precisely on the submanifold where function \( \Psi \) corresponding to a given set of monodromies \( M_j \) and eigenvalues \( t_j^{(k)} \) fails to exist.

3 Riemann-Hilbert problems associated to hyperelliptic curves

In this section we give a modified version of construction proposed in \([9]\). Take \( n = 2g+2 \) and consider hyperelliptic curve \( L \) given by equation

\[
w^2 = \prod_{j=1}^{2g+2} (\lambda - \lambda_j).
\]

Let us define two \(-1/2\)-forms \( \varphi_{1,2} \) in fundamental polygon \( \hat{L} \) of \( L \) by the formulas:

\[
\varphi_1(P) = \Theta[p_1 q_1] (U(P) + U(D_1)) E(P, D_1)
\]

(3.6)

\[
\varphi_2(P) = \Theta[p_2 q_2] (U(P) + U(D_2)) E(P, D_2)
\]

(3.7)

where \( p, q \in \mathbb{C}^g; D_1 \) and \( D_2 \) are two arbitrary points of curve \( L; E(P, Q) \) is the prime form on \( L; \) initial point of the Abel map \( U(P) \) is chosen to be \( \lambda_1 \). Define auxiliary \( 2 \times 2 \) function \( \Phi(\lambda) \):

\[
\Phi_{kj}(\lambda) = \varphi_k(\lambda^{(j)}) ,
\]

where \( k, j = 1, 2; \lambda^{(j)} \) denotes point of \( L \) belonging to \( j \)th sheet and having projection \( \lambda \) on \( \mathbb{C}\mathbb{P}1 \). Define function \( \Psi(\lambda) \) by the formula

\[
\Psi(\lambda) = \sqrt{\frac{\det \Phi(\infty^1)}{\det \Phi(\lambda)}} \Phi^{-1}(\infty^1) \Phi(\lambda).
\]

(3.8)

The \( 1/2 \)-differentials in the denominator of prime-form in \( \varphi_{1,2} \) cancel out in expression for \( \Psi \); thus \( \Psi \) is a function i.e. 0-form on \( L \). The following theorem takes place, which is slightly modified version of the statement formulated in \([4]\).

**Theorem 3.1** Let us fix some points \( \lambda_1, \ldots, \lambda_{2g+2} \in \mathbb{C} \) and vectors \( p, q \in \mathbb{C}^g \). Consider hyperelliptic curve \( L \) \((3.5)\) with matrix of \( b \)-periods \( B \). Assume that \( \Theta[p^1 q] (0|B) \neq 0 \) i.e. vector \( Bp + q \) does not belong to theta-divisor. Then function \( \Psi(\lambda) \) defined by \((3.7),(3.8)\)  

\(^2\)In a different form solution of the same RH problem was obtained in \([4]\).
gives a solution to matrix Riemann-Hilbert problem with $\lambda_0 = \infty$ and singularities at the points $\lambda_1, \ldots, \lambda_{2g+2}$ with off-diagonal monodromies

$$M_j = \begin{pmatrix} 0 & -m_j \\ m_j & 0 \end{pmatrix},$$  

(3.9)

where

$$m_1 = 1, \quad m_2 = \exp\left\{-2\pi i \sum_{k=1}^{g} p_k \right\},$$

$$m_{2j+1} = -\exp\left\{2\pi i q_j - 2\pi i \sum_{k=j}^{g} p_k \right\},$$

$$m_{2j+2} = \exp\left\{2\pi i q_j - 2\pi i \sum_{k=j+1}^{g} p_k \right\},$$  

(3.10)

Proof. We can rewrite the expression for $\det \Phi$ using Fay identities [11]:

$$\Theta(z + U(c) - U(d))\Theta(z + U(d) - U(c))E(c, b)E(a, d) + \Theta(z + U(c) - U(b))\Theta(z + U(d) - U(a))E(c, a)E(d, b) = \Theta(z + U(c) + U(d) - U(a) - U(b))\Theta(z)E(c, d)E(a, b).$$

where $z \in \mathbb{C}^g; a, b, c, d$ are four arbitrary points of $\mathcal{L}$. After identification $-z \equiv \text{B}p + \text{q}, a \equiv D_1, b \equiv D_2, c \equiv P, d \equiv P^*$, the left-hand side of Fay identities gives $\det \Phi(P)$. Evaluating the right-hand side we obtain

$$\det \Phi(P) = \Theta\left[\frac{p}{q}\right](0)\Theta\left[\frac{p}{q}\right](U(D_1) + U(D_2))E(P, P^*)E(D_1, D_2).$$

(3.11)

Since function $\Psi$ is independent of $D_1$ and $D_2$, function $\Psi$ is undefined precisely at the points where the first prefactor vanishes i.e. vector $\text{B}p + \text{q}$ belongs to the theta-divisor ($\Theta$) on Jacobian $J(\mathcal{L})$. Outside of this singular variety function $\Psi$ is well-defined, non-singular and invertible in $\lambda$-plane outside of the points $\lambda_j$. At the points $\lambda_j$ it has regular singularities; expressions for monodromy matrices (3.10) follow from periodicity properties of theta-function.

If we assume that vectors $p$ and $q$ are $\{\lambda_j\}$-independent, functions $A_j(\{\lambda_k\}) \equiv \text{res}_{\lambda_j=\lambda_j} \Psi_{\lambda} \Psi^{-1}$ satisfy the Schlesinger system; corresponding tau-function is given by the following theorem.

**Theorem 3.2** [7] The tau-function of Schlesinger system corresponding to monodromy matrices (3.10) is given by

$$\tau(\{\gamma_j\}) = [\det A]^{-\frac{1}{2}} \prod_{j<k}(\lambda_j - \lambda_k)^{-\frac{1}{8}} \Theta\left[\frac{p}{q}\right](0|\text{B}).$$

(3.12)

i.e. coincides with determinant of Cauchy-Riemann operators $\partial_{P_{1/2}}^{p,q}$ on $\mathcal{L}$ [4, 5, 6].

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Proof. Here we give a version of the proof which is slightly simplified compared with the original version of (3). Taking into account the following identity valid for \(2 \times 2\) matrices,

\[
\frac{1}{2} \text{tr}(Ψ_λΨ^{-1})^2 = -\frac{1}{\det(Ψ_λ)} + \frac{1}{4} \left(\frac{\det(Ψ_λ)}{\det Ψ}\right)^2,
\]

we find that

\[
\frac{1}{2} \text{tr}(Ψ_λΨ^{-1})^2(λ) = -\frac{1}{\Theta [p^T_q]}(0) \frac{\partial^2 \{\Theta [p^T_q] (U(μ) - U(λ))\}}{\partial λ \partial μ} |_{μ = λ} - \frac{\partial^2 \{\ln E(λ, μ)\}}{\partial λ \partial μ} |_{μ = λ^*},
\]

Dependence of \(τ\)-function on vectors \(p\) and \(q\) is contained in the first term of right-hand side. This term can be further rewritten as

\[
\frac{1}{\Theta [p^T_q]}(0) \sum_{k,l=1}^{g} \frac{∂^2 \Theta [p^T_q](0)}{∂z_k∂z_l} \frac{dU_k}{dλ} \frac{dU_l}{dλ} = 4πi \sum_{k,l=1}^{g} \frac{∂ \ln \Theta [p^T_q](0)}{∂B_{lk}} \frac{dU_k}{dλ} \frac{dU_l}{dλ},
\]

where we used the heat equation for theta-function; \(z_k\) denotes the \(k\)th argument of theta-function. Dependence of matrix of \(b\)-periods on the branch points is given by the following equations [12, 13]:

\[
\frac{∂ Β_{kl}}{∂ λ_j} = πi \frac{∂ U_k}{∂ κ_j}(λ_j) \frac{∂ U_l}{∂ κ_j}(λ_j),
\]

where \(κ_j = \sqrt{λ - λ_j}\) is a local parameter near point \(λ_j\). On the other hand, value \(4πi \frac{∂ U_k}{∂ κ_j}(λ_j) \frac{∂ U_l}{∂ κ_j}(λ_j)\) is nothing but the residue of the rational function \(\frac{dU_k}{dλ}(λ) \frac{dU_l}{dλ}(λ)\) at \(λ = λ_j\). Continuing the calculation of the first term in (3.13) we get

\[
\sum_{j=1}^{g} \frac{1}{λ - λ_j} \frac{∂ \ln Θ [p^T_q]}{∂ λ_j}(0),
\]

and, therefore,

\[
τ = f(\{λ_j\})Θ [p^T_q](0),
\]

where function \(f\) does not carry any dependence on \(p\) and \(q\). Now, to determine function \(f\) we can choose vectors \(p\) and \(q\) in such a way that the \(τ\)-function may be explicitly calculated in elementary functions. One of possible choices of that kind is to take \(p, q\) to coincide with some even half-integer characteristic \(p^T, q^T\). We choose characteristic \(p_τ, q_τ\) to correspond to some subset \(T = \{i_1, \ldots, i_{g+1}\}\) of the set \(\{1, \ldots, 2g+2\}\) via the standard relation

\[
Βp^T + q^T = U(λ_{i_1}) + \ldots + U(λ_{i_{g+1}}) - K.
\]

According to Thomae formulas [11],

\[
Θ^4 [p^T_q](0) = \pm \frac{(\text{det} A)^2}{(2πi)^{2g}} \prod_{j,k \in T} (λ_j - λ_k) \prod_{j,k \notin T} (λ_j - λ_k),
\]

where \(A_{jk} = \int_{λ_k}^{λ_j} \frac{dλ}{w}\). Therefore, the \(τ\)-function (3.14) may be up to unessential overall constant factor rewritten as follows:

\[
τ = f(\{λ_j\})(\text{det} A)^{1/2} \prod_{λ_j, λ_k \in T} (λ_j - λ_k)^{1/4} \prod_{λ_j, λ_k \notin T} (λ_j - λ_k)^{1/4}.
\]
Alternatively, we can easily calculate the same \( \tau \)-function directly. Taking into account that 
\[
U(\lambda_1) = 0 \quad ; \quad U(\lambda_2) = \frac{1}{2} \sum_{k=1}^{g} e_k \quad ; \quad U(\lambda_{2j+1}) = \frac{1}{2} Be_j + \frac{1}{2} \sum_{k=j+1}^{g} e_j \quad ; \quad U(\lambda_{2j+2}) = \frac{1}{2} Be_j + \frac{1}{2} \sum_{k=j+1}^{g} e_j
\]
we find:
\[
q^T_{j+1} - q^T_j = \frac{1}{2} (\delta_{2j+2} + \delta_{2j+3} + 1) \quad ; \quad p^T_j = \frac{1}{2} (\delta_{2j+1} + \delta_{2j+2} + 1),
\]
where \( \delta_j = 1 \) for \( j \in T \) and \( \delta_j = 0 \) for \( j \not\in T \). Substituting these formulas to (3.10) we see that the monodromy matrices have the following form:
\[
M_j = i(-1)^{\delta_j + \delta_1} \sigma_1,
\]
where by \( \sigma_j, j = 1, 2, 3 \) we denote the standard Pauli matrices. By simultaneous similarity transformation which does not modify associate \( \tau \)-function this set of monodromy matrices may be transformed to the set of diagonal matrices
\[
\tilde{M}_j = i\sigma_3, \quad \lambda_j \in T; \quad \tilde{M}_j = -i\sigma_3, \quad \lambda_j \not\in T.
\]

The associate function \( \Psi \) may be chosen to be diagonal: \( \Psi(\lambda) = \text{diag}(\varphi_0(\lambda), \varphi_0^{-1}(\lambda)) \) with
\[
\varphi_0(\lambda) = \prod_{j \in T} (\lambda - \lambda_j)^{1/4} \prod_{j \not\in T} (\lambda - \lambda_j)^{-1/4},
\]
which leads to the following formula for \( \tau \)-function:
\[
\tau = \prod_{j,k \in T} (\lambda_j - \lambda_k)^{1/8} \prod_{j,k \not\in T} (\lambda_j - \lambda_k)^{1/8} \prod_{j \in T, k \not\in T} (\lambda_j - \lambda_k)^{-1/8}. \quad (3.16)
\]

Comparing (3.13) and (3.16) we get
\[
f(\{\lambda_j\}) = (\det A)^{-1/2} \prod_{j<k} (\lambda_j - \lambda_k)^{-1/8},
\]
proving (3.12).

4 Solution of matrix Riemann-Hilbert problems in terms of Szegö kernel on algebraic curves

Consider non-singular algebraic curve \( L \) defined by polynomial equation
\[
f(\lambda, w) = 0
\]
of degree \( N \) in \( w \). Hurwitz space is the moduli space of curves of fixed genus \( g \) and fixed number of sheets \( N \). In analogy to Dubrovin [13] we shall in addition fix the types of ramification at all branch points. Denote projections of branch points on \( \lambda \)-plane by \( \lambda_j, j = 1, \ldots, n \) (admitting little inaccuracy we shall also call \( \lambda_j \) the branch points). If we denote multiplicities of branch points \( \lambda_1, \ldots, \lambda_n \) by \( m_1, \ldots, m_n \) respectively, the genus of \( L \) is given by Riemann-Hurwitz formula
\[
g = \frac{1}{2} \sum_{j=1}^{n} m_j - N + 1.
\]
We consider Hurwitz space $H$ consisting of the curves which can be obtained from $L$ by variation of branch points $\lambda_j$ without changing their type of ramification. Assume that the normalization point $\lambda_0$ does not coincide with any of $\lambda_j$.

To each Hurwitz space $H$ we can associate solution of certain RH problem with singularities at points $\lambda_j$ and quasi-permutation monodromy matrices (For brevity we call any matrix which has only one non-vanishing element in each row and only one non-vanishing element in each column the quasi-permutation matrix.)

Denote by $\pi: L \to \mathbb{C}P^1$ the projection of $L$ to $\lambda$-plane. Let us denote by $l_1, \ldots, l_n$ the natural basis in $H^0(\mathbb{C}P^1 \setminus \{\lambda_1, \ldots, \lambda_n\}, \mathbb{Z})$. As a starting point of all $l_j$ we choose $\lambda_0$. Consider $\pi^{-1}(l_j)$. This is a set of $N$ non-intersecting contours $l_j^{(k)}$, $k = 1, \ldots, N$ on $L$, where by $l_j^{(k)}$ we denote contour starting at $\lambda_0^{(k)}$. Denote the endpoint of $l_j^{(k)}$ by $\lambda_0^{(k')}$ with some $k' = k'(k)$. If $\lambda_0^{(k)}$ is not a branch point, then $k = k'$, and contour $l_j^{(k)}$ is closed; if $\lambda_0^{(k)}$ is a branch point, then $k \neq k'$ and contour $l_j^{(k)}$ is non-closed.

Assume now that point $\lambda_0$ does not belong to the set of projections of basic cycles $(a_j, b_j)$ on $\mathbb{C}P^1$. Introduce intersection indexes

$$\alpha_j^{(k)} = l_j^{(k)} \circ a_s, \quad \beta_j^{(k)} = l_j^{(k)} \circ b_s$$ (4.17)

where $j = 1, \ldots, n$; $s = 1, \ldots, g$; $k = 1, \ldots, N$ (4.18)

Choose on $L$ a canonical basis of cycles $(a_j, b_j)$, $j = 1, \ldots, g$. Introduce the basis of holomorphic 1-forms $dU_j$ on $L$ normalized by $\oint_{a_j} dU_k = \delta_{jk}$, matrix of $b$-periods $B$ and the Abel map $U(P)$, $P \in L$. Denote initial point of Abel map by $P_0$.

Let us introduce function $\Psi(\lambda)$ in analogy to (3.8):

$$\Psi(\lambda) = \left[\frac{|\det \Phi(\lambda_0)|}{|\det \Phi(\lambda)|}\right]^{1/N} \Phi^{-1}(\lambda_0) \Phi(\lambda).$$ (4.19)

Function $\Phi(\lambda)$ is defined as follows:

$$\Phi(\lambda)_{kj} = \frac{\lambda - \mu}{\sqrt{d \lambda d \mu}} \varphi_k(\lambda^{(j)}),$$ (4.20)

where by $\lambda^{(j)}$ we denote the point of $j$th sheet of curve $L$ having projection $\lambda$ on $\mathbb{C}P^1$; $\mu \in \mathbb{C}$ is an arbitrary point. Here $\varphi_k(P)$ are holomorphic spinors on $L$. To define them choose an arbitrary set of $N$ positive non-special divisors $D_k$, $k = 1, \ldots, N$ of degree $N - 1$ each i.e $D_k = \sum_{j=1}^{N-1} D_k^j$. Take

$$\varphi_k(P) = \Theta \left[ \frac{\Theta^*}{\Theta} \right] (U(P) + U(D_k) - C) \prod_{j=1}^{N-1} E(P, D_k^j) \prod_{j=1}^{N} E(P, \mu^{(j)})$$ (4.21)

where $p, q \in \mathbb{C}^g$; $E(P, Q)$ is the prime-form; $C \equiv \sum_{k=1}^{N} U(\lambda^{(k)})$.

It is clear that vector $C$ does not depend on $\lambda$; it depends only on the choice of initial point of Abel map $P_0$. This follows from the fact that for any holomorphic 1-form $dU(P)$ on $L$ the sum $\sum_{j=1}^{N} dU(\lambda^{(j)})$ is holomorphic 1-form on $\mathbb{C}P^1$, therefore identically vanishing. The function $\varphi_k(\lambda^{(j)})/\sqrt{d \lambda}$ behaves near branch point $\lambda_j$ as $\tau_j^{-m_j/2}$ where $\tau_j = (\lambda - \lambda_j)^{1/(m_j+1)}$ is the local parameter near $\lambda_j$. Therefore, to completely define this function on $\hat{L}$, one has to define system of contours on $L$ which connect the branch points with odd $m_j$, and where functions $\varphi_k(P)/\sqrt{d \lambda}$ change sign. Denote this system of contours by $L$. 

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Theorem 4.1 Suppose that $\Theta \left[ \frac{p}{q} \right] (0) \neq 0$. Then function $\Psi$ (4.13) is independent of the choice of divisors $D_k$ and point $\mu$ and solves the RH problem on $\mathbb{CP}^1$ with quasi-permutation matrices $M_j$ which can be expressed in terms of vectors $p$ and $q$.

Proof. By counting number of poles and zeros it is easy to check that $\det \Phi(\lambda)$ does not vanish outside of branch points $\lambda_j$. The spinors $\varphi_k(P)$ ([21]) transform as follows under the analytical continuation along basic cycles:

$$T_{a_j}[\varphi_k(P)] = e^{2\pi i p_j} \varphi_k(P), \quad T_{b_j}[\varphi_k(P)] = e^{-2\pi i q_j} \varphi_k(P). \quad (4.22)$$

When we consider analytical continuation of $\psi(\lambda^{(k)})/\sqrt{d\lambda}$ along contour $I_j^{(k)}$ from $\lambda^{(k)}_0$ to $\lambda^{(k)}_0$, we come to the value $\psi(\lambda^{(k')}_0)/\sqrt{d\lambda}$ up to the factor which is collected from crossing the contours $\{a_j, b_j\}$ and contour $L$, where this function has jumps. Denote by $I_j^{(k)}$ the intersection index of $I_j^{(k)}$ and $L$. Then the total factor we collect along contour $\lambda_j^{(k)}$ is $\exp \left\{ \pi i I_j^{(k)} + 2\pi i \sum_{s=1}^g \alpha_{js} q_s + \beta_{js} p_s \right\}$, where intersection indeces $\alpha_{js}$ and $\beta_{js}$ are given by (4.18). Therefore, monodromy matrices corresponding to our $\Psi$, have the following form

$$(M_j)_{kl} = \exp \left\{ \pi i I_j^{(k)} + 2\pi i \left[ \sum_{s=1}^g \alpha_{js} q_s + \beta_{js} p_s \right] \right\} \delta_{k(k'),l} \quad (4.23)$$

($\delta_{ab}$ is the Kronecker symbol); obviously, this is a matrix of quasi-permutation. Independence of function $\Psi$ on the choice of divisors $D_k$ and point $\mu$ follows from uniqueness of solution of Riemann-Hilbert problem with given $\{M_j\}$ and $\{I_j^{(s)}\}$.

Condition $\Theta \left[ \frac{p}{q} \right] (0) \neq 0$ of the theorem guarantees the non-vanishing of $\det \Phi(\lambda)$ in (4.19). Namely, for arbitrary $N$ points $P_j \in \mathcal{L}$ we can prove that

$$\det_{N \times N} \{ \Theta \left[ \frac{p}{q} \right] (U(P_j) + U(D_k)) - C \} \prod_{j=1}^{N-1} E(P_j, D_j)$$

$$= F(\mu, \{\lambda_j\}, \{D_k\}) \Theta \left[ \frac{p}{q} \right] \left( \sum_{j=1}^N U(P_j) - C \right) \prod_{j,k=1}^N E(P_j, P_k) \quad (4.24)$$

for some $\{P_j\}$-independent section $F$. The proof of formula (4.24) may be obtained in a standard way. First, it is easy to prove that the r.h.s. and l.h.s. are sections of the same bundle on $\mathcal{L}$ with respect to each $P_j$. Then we check that positions of zeros of l.h.s. and r.h.s. with respect to each $P_j$ coincide. Choosing $P_j = \lambda^{(j)}_0$ we get $\sum_{j=1}^N U(P_j) = C$; therefore, $\det \Phi(\lambda)$ is proportional to $\Theta \left[ \frac{p}{q} \right] (0)$ as in $2 \times 2$ case (3.1). Thus function $\Psi$ (4.19) is undefined if $\Theta \left[ \frac{p}{q} \right] (0) = 0$ i.e.

$$Bp + q \in (\Theta)$$

where $(\Theta)$ is theta-divisor on Jacobian of $\mathcal{L}$.

The previous construction of function $\Psi$ may be simplified by choosing $\mu = \lambda_0$, and $D_k = \sum_{j \neq k} \lambda^{(j)}_0$. 

8
Corollary 4.1 Suppose that $\Theta [P^p_q](0) \neq 0$. Then function $\Psi(\lambda)$ with components

$$\Psi(\lambda)_{kj} = \frac{1}{\Theta [P^p_q](0)} \frac{\Theta [P^p_q](U(\lambda^{(j)}) - U(\lambda^{(k)}))}{E(\lambda^{(j)}, \lambda^{(k)})} \frac{\lambda - \lambda_0}{\sqrt{d\lambda d\lambda_0}}$$

(4.25)

belongs to $\text{SL}(N, \mathbb{C})$ for any $\lambda \in \mathbb{C}$, is non-singular on $\mathbb{C}$ outside of points $\lambda = \lambda_j$, satisfies normalization condition $\Psi(\lambda_0) = I$ and solves Riemann-Hilbert problem with monodromy matrices $[\lambda^{(k)}]$. 

Remark 4.1 Formula (1.25) may be rewritten in terms of Szegö kernel on $\mathcal{L}$:

$$S(P, Q) = \frac{1}{\Theta [P^p_q](0)} \frac{\Theta [P^p_q](U(P) - U(Q))}{E(P, Q)}$$

(4.26)

which is $(1/2, 1/2)$ differential on $\mathcal{L} \times \mathcal{L}$, as follows:

$$\Psi(\lambda)_{kj} = S(\lambda^{(j)}, \lambda^{(k)}) E_0(\lambda, \lambda_0)$$

(4.27)

where $E_0(\lambda, \lambda_0) = (\lambda - \lambda_0)/\sqrt{d\lambda d\lambda_0}$ is the prime-form on $\mathbb{C}P^1$.

Proof of the Corollary. For any two sets $P_1, \ldots, P_N$ and $Q_1, \ldots, Q_N$ we have the following identity (see [1], p.33):

$$\det\{S(P_j, Q_k)\} = \frac{\Theta [P^p_q]\left(\sum_{j=1}^{N}(U(P_j) - U(Q_j))\right)}{\Theta [P^p_q](0)} \frac{\prod_{j<k} E(P_j, P_k) E(Q_k, Q_j)}{\prod_{j,k} E(P_j, Q_k)}$$

(4.28)

analogous to (1.24). Choosing $P_j \equiv \lambda^{(j)}$ and $Q_k \equiv \lambda^{(k)}$ and using the basic properties of prime-form we conclude that $\det \Psi(\lambda) = 1$. Normalization condition $\psi_j \left(\lambda^{(k)}\right) = \delta_{jk}$ is the corollary of asymptotic expansion of prime form:

$$E(P, Q) = \frac{z(P) - z(Q)}{\sqrt{dz(P)dz(Q)}} (1 + o(1))$$

as $P \to Q$, where $z(P)$ is a local parameter.

Let us consider separately the case $\lambda_0 = \infty$. In this case the above formulas should be slightly modified.

Corollary 4.2 Suppose that $\Theta [P^p_q](0) \neq 0$. Define function $\Psi(\lambda)$ with components

$$\Psi(\lambda)_{kj} = \frac{1}{\Theta [P^p_q](0)} \frac{\Theta [P^p_q](U(\lambda^{(j)}) - U(\infty^{(k)}))}{\Theta[S]\left(U(\lambda^{(j)}) - U(\infty^{(k)})\right)} \frac{\sqrt{dW(\lambda^{(k)})}}{\sqrt{dW(\lambda^{(j)})}}$$

(4.29)

where $[S]$ is an arbitrary non-degenerate odd half-integer characteristic and $dW(P) = \sum_{j=1}^{g} \frac{\partial \Theta[S]}{\partial z_j}(0) dU_j$. Then function $\Psi(\lambda)$ belongs to $\text{SL}(N, \mathbb{C})$ for any $\lambda \in \mathbb{C}$, is non-singular on $\mathbb{C}$ outside of points $\lambda = \lambda_j$, satisfies normalization condition $\Psi(\infty) = I$ and solves Riemann-Hilbert problem with monodromy matrices $[\lambda^{(k)}]$.

If we now assume that vectors $p$ and $q$ don’t depend on $\{\lambda_j\}$, matrices $M_j$ also don’t carry any $\{\lambda_j\}$-dependence and the isomonodromy deformation equations take place.
Theorem 4.2 Assume that vectors $p$ and $q$ don’t depend on $\{\lambda_j\}$. Then functions

$$A_j(\{\lambda_j\}) \equiv \text{res}_{\lambda = \lambda_j} \{\Psi_\lambda \Psi^{-1}\}$$

(4.30)

where function $\Psi(\lambda)$ is defined in (4.29), satisfy Schlesinger system outside of hyperplanes $\lambda_k = \lambda_j$ and submanifold of codimension one, on which vector $Bp + q$ belongs to theta-divisor $(\Theta)$ on $\mathcal{L}$.

Let us discuss now the calculation of corresponding $\tau$-function. It is known [1] that the $\tau$-function vanishes outside of the hyperplanes $\lambda_j = \lambda_k$ precisely at those points where the Riemann-Hilbert problem does not have a solution; together with explicit calculations in $2 \times 2$ case this suggests that the tau-function should be proportional to $\Theta \left[ \begin{array}{cc} p \\ q \end{array} \right] (0)$. Explicit calculation shows that this is really the case, and, moreover, this factor contains the whole dependence of $\tau$ on vectors $p$ and $q$. So,

$$\tau = f(\{\lambda_j\}) \Theta \left[ \begin{array}{cc} p \\ q \end{array} \right] (0)$$

with some function $f$ depending only on $\{\lambda_j\}$. Explicit calculation of function $f$ is possible in some special cases, like the curves of $Z_N$ class [14].

Taking into account the coincidence of the $\tau$-function in $2 \times 2$ case with determinant of Cauchy-Riemann operator acting on $1/2$-forms $w(P)$ on $\mathcal{L}$ satisfying boundary conditions $w(P + a_j) = e^{2\pi ip_j} w(P)$, $w(P + b_j) = e^{-2\pi iq_j} w(P)$ i.e.

$$\tau = \det \bar{\partial}_{1/2}$$

(4.31)

it is tempting to suggest that this coincidence takes place for arbitrary curves; then function $f$ would coincide (see [4, 5, 6]) with $[\det \bar{\partial}_0]^{-1/2}$ where operator $\bar{\partial}_0$ acts on 0-forms on $\mathcal{L}$.

Remark 4.2 It is clear that $\det \bar{\partial}_{1/2}$, as well as $\tau$-function, vanishes if $Bp + q \in (\Theta)$, since in this case $1/2$-form $\Theta \left[ \begin{array}{cc} p \\ q \end{array} \right] (U(P) - U(Q))/E(P, Q)$ for any $Q \in \mathcal{L}$ belongs to its kernel.

Another argument suggesting possible coincidence of $\tau$ and $\det \bar{\partial}_{1/2}$ in general case arises from consideration of Palmer [3]. It is also relevant to notice that close link between Cauchy-Riemann determinants and tau-functions arising in the theory of KP equation was mentioned in [15].

Finally, following [1], denote the divisor of zeros of $\tau$-function in $\mathbb{C}^n$ by $(\vartheta)$. Then we get the following relationship between Malgrange’s divisor $(\vartheta)$ and theta-divisor $(\Theta)$ on Jacobian $J(\mathcal{L})$:

$$\{\lambda_j\} \in (\vartheta) \iff Bp + q \in (\Theta),$$

Acknowledgements I thank John Harnad, Alexey Kokotov and Alexandr Orlov for important discussions at different stages of this work.

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