Reconstruction of symmetric convex bodies from Ehrhart-like data

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Abstract

In a previous paper [2], we showed how to use the Ehrhart function \( L_P(s) \), defined by
\[
L_P(s) = \#(sP \cap \mathbb{Z}^d)
\]
for reconstruct a polytope \( P \). More specifically, we showed that, for rational polytopes \( P \) and \( Q \), if
\[
L_{P+w}(s) = L_{Q+w}(s)
\]
for all integer vectors \( w \), then \( P = Q \). In this paper we show the same result, but assuming that \( P \) and \( Q \) are symmetric convex bodies instead of rational polytopes.

1 Introduction

Given a polytope \( P \subseteq \mathbb{R}^d \), the classical Ehrhart lattice point enumerator \( L_P(t) \) is defined as
\[
L_P(t) = \#(tP \cap \mathbb{Z}^d), \quad \text{integer } t \geq 0.
\]
Here, \( \#(A) \) is the number of elements in \( A \) and \( tP = \{tx \mid x \in P\} \) is the dilation of \( P \) by \( t \). The above definition may be extended to allow for \( P \) to be an arbitrary convex body, and for \( t \) to be an arbitrary real number.

To minimize confusion, we will denote real dilation parameters with the letter \( s \), so that \( L_P(t) \) denotes the classical Ehrhart function and \( L_P(s) \) denotes the extension considered in this paper. So, for example, \( L_P(t) \) is just the restriction of \( L_P(s) \) to integer values.

It is clear from the definition that the classical Ehrhart function is invariant under integer translations; that is, for every polytope \( P \) and every integer vector \( w \), we have
\[
L_{P+w}(t) = L_P(t)
\]
for all \( t \). This is not true for the real Ehrhart function \( L_P(s) \). In fact, an earlier paper [2] shows that the list of functions \( L_{P+w}(s) \), for integer \( w \), is a complete set of invariants for full-dimensional rational polytopes. More precisely:

**Theorem 1.** Let \( P \) and \( Q \) be full-dimensional rational polytopes, and suppose that \( L_{P+w}(s) = L_{Q+w}(s) \) for all \( s > 0 \) and all integer \( w \). Then \( P = Q \).

It is also conjectured that the rationality hypothesis may be dropped. In this paper, we will show a special case of this conjecture, where we assume that the polytopes are symmetric (that is, \( x \in P \) if and only if \(-x \in P\)). In fact, we never use the fact that \( P \) and \( Q \) are polytopes; we just need convexity. So we have the following.
Theorem 2. Let \( K \) and \( H \) be symmetric convex bodies, and assume that 
\[ L_{K+w}(s) = L_{H+w}(s) \]
for all real \( s > 0 \) and all integer \( w \). Then \( K = H \).

2 Notation and basic results

The punchline is Alexandrov’s projection theorem. Let \( K \subseteq \mathbb{R}^d \) be a convex body (that is, a convex, compact set with nonempty interior). For any unit vector \( v \), we will denote by \( V_K(v) \) the \((d-1)\)-dimensional area of the orthogonal projection of \( K \) in \( \{v\}^\perp \).

For example, let \( K = [0,1] \times [0,1] \subseteq \mathbb{R}^2 \), \( v = (0,1) \) and \( v' = (\sqrt{2}/2, \sqrt{2}/2) \). Then \( V_K(v) = 1 \) and \( V_K(v') = \sqrt{2} \).

A convex body \( K \) is said to be symmetric if \( x \in K \) if and only if \( -x \in K \). An important reconstruction theorem for symmetric convex bodies is Aleksandrov’s projection theorem (see e.g. [1, p. 115]).

Theorem 3 (Aleksandrov’s projection theorem). Let \( K \) and \( H \) be two symmetric convex bodies in \( \mathbb{R}^d \) such that 
\[ V_K(v) = V_H(v) \]
for all unit vectors \( v \). Then \( K = H \).

So, the goal is to compute the function \( V_K \) using the Ehrhart functions \( L_{K+w} \). The two main tools are the Hausdorff distance and pseudopyramids.

For \( \lambda \geq 0 \) and \( x \in \mathbb{R}^d \), denote by \( B_\lambda(x) \) the ball with radius \( \lambda \) centered at \( x \). If \( K \) is a convex body and \( \lambda \geq 0 \), define \( K_\lambda \) by
\[ K_\lambda = \bigcup_{x \in K} B_\lambda(x). \]

The Hausdorff distance \( \rho(K,H) \) between two convex bodies \( K \) and \( H \) is defined to be (Figure 1)
\[ \rho(K,H) = \inf\{ \lambda \geq 0 \mid K \subseteq H_\lambda \text{ and } H \subseteq K_\lambda \}. \]

It can be shown that the set of convex sets in \( \mathbb{R}^d \) is a metric space under the Hausdorff distance and that the Euclidean volume is continuous in this space (see e.g. [1, p. 9]), but we just need the following special case of this theory.

Lemma 4. Let \( K \) and \( A_1, A_2, \ldots \) be convex bodies. If \( \lim_{i \to \infty} \rho(K, A_i) = 0 \), then \( \lim_{i \to \infty} \text{vol } A_i = \text{vol } K \).

The concept of pseudopyramid was introduced in [2]. If \( K \) is a convex body, the pseudopyramid \( \text{ppyr } K \) is defined to be (Figure 2)
\[ \text{ppyr } K = \bigcup_{0 \leq \lambda \leq 1} \lambda K. \]

We will use the following lemma, which is a consequence of Lemma 1 of [2].

Lemma 5. Let \( K \) and \( H \) be convex bodies, and suppose that 
\[ L_K(s) = L_H(s) \]
for all \( s > 0 \). Then \( \text{vol } \text{ppyr } H = \text{vol } \text{ppyr } K \).
Figure 1: Hausdorff distance between two convex sets. The thick lines are the boundaries of the sets $K$ and $H$; the thin lines are the boundaries of the sets $K_\lambda$ and $H_\lambda$.

![Figure 1: Hausdorff distance between two convex sets.](image)

**Figure 2:** Pseudopyramid of a polytope.

![Figure 2: Pseudopyramid of a polytope.](image)

In other words, whenever we know $L_K$, we may assume we also know $\text{vol ppyr } K$.

**Proof.** Lemma 1 of [2] states that,\footnote{Tecnically, Lemma 1 of [2] only states this for polytopes, but the proof holds verbatim for convex bodies.} if $L_K(s) = L_H(s)$, then $L_{\text{ppyr } K}(s) = L_{\text{ppyr } H}(s)$. Since

$$\lim_{s \to \infty} \frac{L_{\text{ppyr } K}(s)}{s^d} = \text{vol ppyr } K$$

(and similarly for $H$), we have $\text{vol ppyr } K = \text{vol ppyr } H$. \qed
3 Pseudopyramid volumes and areas of projections

In this section, we will show how to compute the function $V_K$ in terms of the numbers $\text{vol_pypyr}(P + w)$ for integer $w$.

Given a convex body $K$, define its spherical projection $S(K)$ by (Figure 3)

$$S(K) = \left\{ \frac{x}{\|x\|} \mid x \in K \text{ and } x \neq 0 \right\}.$$

The connection between pseudopyramid volumes and areas of projections can be seen in Figure 3. The set $\|v\| S(K + v)$ is a dilation of the projection $S(K + v)$ of $K$. Note that the shape of $\|v\| S(K + v)$ “looks like” the orthogonal projection of $K$ in $\{v\}^\perp$; that is, the area of $\|v\| S(K + v)$ approximates $V_K(v)$.

If the pseudopyramid were an actual pyramid (with base $\|v\| S(K + v)$), then using the formula $v = \frac{Ah}{d}$ for the volume of a pyramid would allow us to discover what is the area of the projection, which would give an approximation to $V_K(v)$. We will show that this formula is true “in the infinity”; that is, using limits, we can recover the area of the projection using taller and taller pseudopyramids.

3.1 Approximating spherical projections

For convex bodies $K$, the set $S(K)$ is a manifold\(^2\). If $K$ does not contain the origin in its interior, then $S(K)$ may be parameterized with a single coordinate system; that is, there is a set $U \subseteq \mathbb{R}^{d-1}$ and a continuously differentiable function $\varphi : U \rightarrow S(K)$ which is a bijection between $U$ and $S(K)$. Since we want to

\(^2\) Technically, $S(K)$ will be a manifold-with-corners (see [3, p. 137]). However, their interiors relative to the $(d - 1)$-dimensional sphere $S^{d-1}$ are manifolds, and since we’re dealing with areas there will be no harm in ignoring these boundaries.
Figure 4: The spherical projection $\mu S(K + \mu v)$, when projected orthogonally to the plane $x_d = 0$ (the set $K_\mu$), approaches the volume of the projection $K'$. 

move $P$ towards infinity, this shall always be the case if the translation vector is long enough. In this case, we define its area to be [3, p. 126]

$$\text{area } S(K) = \int_U \|D_1 \varphi \times \cdots \times D_{d-1} \varphi\| = \int_U \left\| \frac{\partial \varphi}{\partial x_1} \times \cdots \times \frac{\partial \varphi}{\partial x_{d-1}} dx_1 \cdots dx_{d-1} \right\|.$$

The following theorem states that the spherical projection approximates, in a sense, the orthogonal projection, for large enough translation vectors.

**Theorem 6.** Let $v$ be a unit vector and $K \subseteq \mathbb{R}^d$ a convex body. Then

$$\lim_{\mu \to \infty} \mu^{d-1} \text{area } S(P + \mu v) = V_K(v).$$

**Proof.** By rotating all objects involved if needed, we may assume that $v = (0, \ldots, 0, 1)$. Let $N$ be large enough that $K \subseteq B_N(0)$; we'll assume that $\mu > N$, so that $K + \mu v$ lies strictly above the hyperplane $x_d = 0$.

Let $K'$ be the orthogonal projection of $K$ into $\{v\}^\perp$. We'll think of $K'$ as being a subset of $\mathbb{R}^{d-1}$. Denote by $K_\mu$ the projection of the set $\mu S(K + \mu v)$ on $\mathbb{R}^{d-1}$ (Figure 4); that is, first project $K + \mu v$ to the sphere with radius $\mu$, then discard the last coordinate. Note this is similar to projecting it to the hyperplane $x_d = 0$. We'll show that, as $\mu$ goes to infinity, both the Hausdorff distance between $K_\mu$ and $K'$ and the difference between the volume of $K_\mu$ and the area of $\mu S(K + \mu v)$ tend to zero.

First, let's bound the Hausdorff distance between $K'$ and $K_\mu$. If $x \in K + \mu v$, then $x$ gets projected to a point $x_0 \in K'$ by just discarding the last coordinate;
however, to be projected to a point \( x_1 \in K_\mu \), first we replace \( x \) by \( x' = \frac{\mu}{\|x\|} x \) to get a point \( x' \in \mu S(K + \mu v) \), and then the last coordinate of \( x' \) is discarded. Note that \( x_1 = \frac{\mu}{\|x\|} x_0 \); therefore, the distance between these two points is

\[
\| x_0 - x_1 \| = \left| 1 - \frac{\mu}{\|x\|} \right| \| x_0 \| = \frac{\| x \| - \mu}{\|x\|} \| x_0 \|
\]

We have \( x \in K + \mu v \subseteq B_N(\mu v) \), so \( \mu - N \leq \|x\| \leq \mu + N \). As \( v = (0, \ldots, 0, 1) \) and \( x_0 \) is \( x \) without its last coordinate, we have \( \|x_0\| \leq N \) (because, in \( \mathbb{R}^{d-1} \), we have \( x_0 \in B_N(0) \)). So, the distance between \( x_0 \) and \( x_1 \) is at most \( \frac{N^2}{\mu + N} \).

Every point in \( K' \) and in \( K_\mu \) is obtained through these projections. This means that, given any point \( x_0 \) in one of the sets, we may find another point \( x_1 \) in the other set which is at a distance of at most \( \frac{N^2}{\mu + N} \) from the former, because we can just pick a point \( x \) whose projection is \( x_0 \); then its other projection \( x_1 \) will be close to \( x_0 \). Thus

\[
\rho(K', K_\mu) \leq \frac{N^2}{\mu + N},
\]

so by Theorem 4 the volumes of \( K_\mu \) converges to \( \text{vol} K' \).

Now, let’s relate \( \text{vol} K_\mu \) with \( \mu^{d-1} \text{area} S(K + \mu v) \). If \( y = (y_1, \ldots, y_d) \) is a point in \( \mu S(K + \mu v) \), we know that \( \|y\| = \mu \) and that \( y_d > 0 \) (because we’re assuming \( \mu > N \)). Therefore, if we define \( \varphi : K_\mu \to \mu S(K + \mu v) \) by

\[
\varphi(y_1, \ldots, y_{d-1}) = (y_1, \ldots, y_{d-1}, \sqrt{\mu^2 - y_1^2 - \cdots - y_{d-1}^2}),
\]

then \( \varphi \) will be a differentiable bijection between \( K_\mu \) and \( \mu S(K + \mu v) \), so that \( \varphi \) is a parametrization for \( \mu S(K + \mu v) \).

For the partial derivatives, we have \( \frac{\partial \varphi_d}{\partial y_j} = [i = j] \) if \( i < d \); that is, the partial derivatives behave like the identity. For \( i = d \), we have

\[
\frac{\partial \varphi_d}{\partial y_j} = \frac{y_j}{\sqrt{\mu^2 - y_1^2 - \cdots - y_{d-1}^2}}.
\]

Now, by definition of \( N \), we have

\[
\left| \frac{\partial \varphi_d}{\partial y_j} \right| \leq \frac{N}{\sqrt{\mu^2 - N^2}},
\]

so the vectors \( D_i \varphi \) converge uniformly to \( e_i \) for large \( \mu \). Since the generalized cross product is linear in each entry, the vector \( D_1 \varphi \times \cdots \times D_{d-1} \varphi \) converges uniformly to \( e_d \), and thus the number

\[
|\text{vol} K_\mu - \text{area} \mu S(K + \mu v)| = \left| \int_{K_\mu} 1 - \int_{K_\mu} \|D_1 \varphi \times \cdots \times D_{d-1} \varphi\| \right|
\]

\[
\leq \int_{K_\mu} 1 - \|D_1 \varphi \times \cdots \times D_{d-1} \varphi\|.
\]

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converges to zero.

Now, since \( \text{area} (\mu S(K + \mu v)) = \mu^{d-1} \text{area} S(K + \mu v) \), combining these two convergence results gives the theorem. \(\square\)

### 3.2 Limit behavior of pseudopyramids

Now we’ll show how to use the pseudopyramids to compute these projections.

Let \( K \) be a pseudopyramid. Define the outer radius \( R(K) \) of \( K \) to be the smallest number such that the ball of radius \( R(K) \) around the origin contains \( K \). That is,

\[
R(K) = \inf \{ R \geq 0 \mid K \subseteq B_R(0) \}.
\]

Define the front shell of \( K \) to be the set of points in the boundary of \( K \) which are not contained in any facet passing through the origin; that is, the set of points \( x \) in the boundary of \( K \) such that \( \lambda x \) is contained in the interior of \( K \) for all \( 0 < \lambda < 1 \). Define, then, the inner radius \( r(K) \) of \( K \) to be the largest number such that the ball of radius \( r(K) \) around the origin contains no points of the front shell of \( K \) (Figure 5). Note that this is equivalent to \( r(K) S(K) \) to be contained in \( K \); that is,

\[
r(K) = \sup \{ r \geq 0 \mid r S(K) \subseteq K \}.
\]

We leave the following lemma to the reader. It relates the inner and outer radii with the area of the spherical projection.

**Proposition 7.** Let \( K \subseteq \mathbb{R}^d \) be a convex body which does not contain the origin. Then

\[
\frac{\text{vol ppyr } K}{R(\text{ppyr } K)^d} \leq \frac{\text{area } S(K)}{d} \leq \frac{\text{vol ppyr } K}{r(\text{ppyr } K)^d}.
\]

This lemma, combined with Lemma 6, shows how to calculate the volume of the orthogonal projection knowing only the volumes of the pseudopyramids.

**Lemma 8.** Let \( K \subseteq \mathbb{R}^d \) be a convex body, and \( v \) any unit vector. Then

\[
\lim_{\mu \to \infty} \frac{\text{vol ppyr } (K + \mu v)}{\mu} = \frac{V_K(v)}{d}.
\]
Proof. Let $N$ be large enough so that $K \subseteq B_N(0)$. For any $\mu$, as $v$ is a unit vector, we have $K + \mu v \subseteq B_{N+\mu}(0)$, so

$$R(\text{ppyr}(K + \mu v)) \leq \mu + N.$$ 

Since all the points in the front shell of $\text{ppyr}(K + \mu v)$ are points of $K$, all of them must have norm greater or equal to $\mu - N$. Therefore, no origin-centered ball with radius smaller than that can contain these points. Thus,

$$r(\text{ppyr}(K + \mu v)) \geq \mu - N.$$ 

Using these two inequalities and Proposition 7 gives

$$\frac{\text{vol ppyr}(K + \mu v)}{(\mu + N)^d} \leq \frac{\text{area } S(K + \mu v)}{d} \leq \frac{\text{vol ppyr}(K + \mu v)}{(\mu - N)^d},$$

which may be rewritten as

$$\frac{(\mu - N)^d \mu^{d-1} \text{area } S(K + \mu v)}{\mu^d} \leq \frac{\text{vol ppyr}(K + \mu v)}{\mu} \leq \frac{(\mu + N)^d \mu^{d-1} \text{area } S(K + \mu v)}{\mu^d}.$$ 

Now Theorem 6 and the squeeze theorem finish the proof. $\square$

For example, for $K = [0,1]^2$ and $v = (1,0)$, we have $\text{vol ppyr}(K + \mu v) = 1 + \frac{\mu}{2}$, so $\lim_{\mu \to \infty} \frac{\text{vol ppyr}(K + \mu v)}{\mu} = \frac{1}{\pi}$, which is precisely one-half of the area of $[0] \times [0,1]$, the projection $K'$ of $K$ on the $y$-axis. This highlights that, for large $\mu$, the pseudopyramid $\text{ppyr}(K + \mu v)$ “behaves like” an actual pyramid, with height $\mu$ and base $K'$.

### 3.3 Piecing everything together

**Theorem 2.** Let $K$ and $H$ be symmetric convex bodies, and assume that $L_{K+w}(s) = L_{H+w}(s)$ for all real $s > 0$ and all integer $w$. Then $K = H$.

**Proof.** By Lemma 5, we have $\text{vol ppyr}(K + w) = \text{vol ppyr}(H + w)$ for all integer $w$.

If $w$ is a nonzero integer vector, let $v = \frac{w}{\|w\|}$; then, by Lemma 8, we have

$$\frac{V_K(v)}{d} = \lim_{\mu \to \infty} \frac{\text{vol ppyr}(K + \mu v)}{\mu} = \lim_{\mu \to \infty} \frac{\text{vol ppyr}(H + \mu v)}{\mu} = \frac{V_H(v)}{d}.$$ 

This shows that, whenever $v$ is a multiple of a rational vector, we have $V_K(v) = V_H(v)$. Since the function $V_K$ and $V_H$ are continuous, we have $V_K = V_H$, and thus by Aleksandrov’s projection theorem we conclude that $K = H$. $\square$
4 Final remarks

Theorems 1 and 2 both assume that $L_{K+w}(s) = L_{H+w}(s)$ for all integer $w$ and real $s > 0$, and both conclude that $K = H$. The first theorem assume that the objects being considered are rational polytopes, but no symmetry condition is imposed; the second theorem assumes that the objects are symmetric, but otherwise permits arbitrary convex bodies.

This suggest the following common generalization of these theorems:

**Conjecture 9.** Let $K$ and $H$ be any convex bodies, and assume that $L_{K+w}(s) = L_{H+w}(s)$ for all real $s > 0$ and all integer $w$. Then $K = H$.

References

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