TOWARDS A HOLOGRAPHIC DUAL OF SQCD: 
HOLOGRAPHIC ANOMALIES AND HIGHER DERIVATIVE GRAVITY

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Abstract

We consider the holographic dual of SQCD in the conformal phase. It is based on a higher derivative gravity theory, which ensures the correct field theory anomalies. This is then related to a six dimensional gravity theory via $S^1$ compactification. Some speculations are then made about the correspondence, Seiberg duality, and the nature of confinement from a holographic perspective.

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1 Introduction

A good example of a four dimensional conformal field theory (CFT) is $\mathcal{N} = 1$ SQCD in the Seiberg conformal window [1]. This theory resides in the IR at a conformal fixed point and contains mesons, baryons and gauge invariant glue operators. In view of the success of AdS/CFT [2, 3, 4] at describing $\mathcal{N} = 4$ SYM in terms of a dual gravity theory, a natural question to ask is “what is the holographic dual of SQCD in its conformal phase [5, 6]?”. This is a more realistic theory as the conformal window implies that the number of flavours and the number of colours should be the same size. A logical place to begin this search is by considering the field theory external anomaly equations and their holographic counterpart. The standard two-derivative Ricci scalar action in five dimensions is incapable of capturing this anomaly, and one is forced to use higher curvature terms. We will look at adding a Lovelock action [7] which is the unique curvature squared term that can be added without spoiling perturbative unitarity of the underlying theory. In this paper we will use holographic anomalies to deduce the dual structure of $\mathcal{N} = 1$ four dimensional super conformal field theories (SCFT’s), in particular SQCD [1] in the conformal window.

The use of higher derivative (HD) theory, in particular for gravity, has recently been seen to be of use in various scenarios and theories [8, 9, 10, 11, 12, 13, 14]. In the context of holographic anomalies [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25] arising in AdS/CFT [2, 3, 4, 5, 6], one may use this with different interpretations. This is exactly analogous to the canonical case [2], but we choose to work backwards (from field theory requirements) to deduce what the string theory and brane setup is. The fact that this will apply in particular to Seibergs SQCD in the conformal window follows from the use of a brane setup that gives a low energy gauge theory. See [26, 27] for interesting discussions of an index in these theories.

The point of view that we take in this paper is that any theory (i.e. a classical or quantum action) formulated on $AdS_5$ can be used to generate boundary correlation functions on $\mathbb{R}^4$ of a CFT$_4$. If one believes in the holographic principle [28], then this seems to imply that SQCD should admit a string theory dual. This is a dynamical statement about the two theories. The anomalies associated to the boundary are different however. Although one must solve bulk field equations, one also has to give a meaning to the infinite volume of asymptotically $AdS_5$ space. The price you pay for this meaning is the breaking of symmetries of the boundary theory.

The outline of this paper is as follows. In section 2 we construct the HD gravity theory, based on the Lovelock action [7], which is capable of reproducing the Weyl anomaly on the boundary for differing central charges $c \neq a$. This relies on the Fefferman-Graham (FG) construction [16]. Note that this is not a string theory correction in $\alpha'$ in the usual sense, since the central charges are of the same size. We extend this also to the chiral anomaly of the $U(1)_R$ symmetry current. Again, HD gravity is seen to be essential.

In section 3 we construct the six dimensional gravity theory from which the five dimensional
one is obtained by KK reduction on $S^1$. If this is to admit a (non-critical) string theory description [29, 30, 31], then the Lovelock action is the unique HD term that can be added without spoiling perturbative unitarity of the underlying theory [32, 33, 34, 35]. Its effect is to introduce perturbative interactions of the graviton.

In section 4 we make some suggestions for realising a near horizon geometry of D-branes. The $S^1$ reduction arises due to taking the near horizon limit of a stack of D3-branes together with space-time filling D1-branes that are distributed in a homogeneous and isotropic fashion. We suggest a correspondence between Seibergs electric-magnetic duality and the use of electric and magnetic 2-forms in the D=6 supergravity which then may be interpreted in the string theory as the corresponding D1-branes. This leads one to propose a new duality: that D=4 SQCD in the conformal window with gauge group $SU(N_c)$ (for the electric theory) or $SU(N_f - N_c)$ (for the magnetic theory) is dual to a D=6, $\mathcal{N} = 1$ non-critical string theory on $AdS_5 \times S^1$ with $N_c$ 5-form flux or $(N_f - N_c)$ 5-form flux derived from a collective 2-form potential $B_2 \wedge B_2$ (these terms will be made clear in the paper later on). The weak form of this duality is at the level of supergravity which is the usual large $N_c$ limit but also now taking a similar large $N_f$ limit. This further leads one to a statement about confinement in the picture presented. Some remarks are made about the string theory formulation of this. In section 5 we conclude and make a few observations about some of the details that need to be filled in to make this proposal concrete.

2 The General D=5 Gravity Theory

The philosophy we take as our starting point is the want to calculate correlation functions of a CFT$_4$, by using a bulk theory formulated on $AdS_5$. We are specifically interested in using a bulk description that captures the boundary external anomaly equations

$$\langle T^i(x) \rangle = \frac{c}{16\pi^2} [W_{ijkl}]^2 - \frac{a}{16\pi^2} [\tilde{R}_{ijkl}]^2 + \frac{b}{16\pi^2} [V^{ij}]^2, \quad (2.1)$$
$$\langle \partial_i R^i(x) \rangle = \frac{p}{24\pi^2} R_{ijkl} \tilde{R}_{ijkl} + \frac{q}{9\pi^2} V_{ij} V^{ij}. \quad (2.2)$$

Here $T_{ij}$ is the stress-energy tensor and $J_i$ is a global $U(1)$ chiral current. The metric $g_{ij}$ couples to $T_{ij}$, whilst the $U(1)$ gauge field $V_i$ couples to the current (here $V_{ij} = \partial_i V_j$). See [36] for a review of the relevant ideas. If we require the theory to have $\mathcal{N} = 1$ supersymmetry (which will be the focus of this paper), then the coefficients are fixed to be $b = c$, $p = c - a$, and $q = 5a - 3c$.

The operator insertions are given by variations of a renormalised action $S[V_i, g_{ij}]_{\text{ren}}$ (see [37])

$$\delta S[V_i, g_{ij}]_{\text{ren}} = \int d^4x [\delta g^{ij} \langle T_{ij} \rangle + \delta V_i \langle J^i \rangle], \quad (2.3)$$

when we choose the variations to be a Weyl rescaling and a local $U(1)$ transformation.

By the standard AdS/CFT dictionary, the dual bulk fields are given by the metric $G_{MN}$ and a local $U(1)$ gauge field $A_M$. In what follows we will use the first order Cartan formalism [38]
as this gives a more elegant formulation and clarifies certain aspects. We consider a theory, initially without supersymmetry, and impose restrictions as and when is necessary. The action can be split into the following contributions:

\[ S = S[E] + S[A] + S[E \wedge A] + S[E^{-1}, A]. \] (2.4)

Each of these contributions are given respectively by

\[ S[E] = \frac{\alpha_1}{16\pi G_5} \int_{\mathcal{M}_5} \epsilon_{abcde}(R^{ab} \wedge R^{cd} \wedge E^e) \]
\[ + \frac{\alpha_2}{16\pi G_5} \int_{\mathcal{M}_5} \epsilon_{abcde}(R^{ab} \wedge E^c \wedge E^d \wedge E^e), \] (2.5)

\[ S[A] = \frac{\alpha_4}{16\pi G_5} \int_{\mathcal{M}_5} dA \wedge dA \wedge A, \] (2.6)

\[ S[E \wedge A] = \frac{\alpha_5}{16\pi G_5} \int_{\mathcal{M}_5} (R^{ab} \wedge R_{ab} \wedge A), \] (2.7)

and

\[ S[E^{-1}; A] = \frac{\alpha_6}{16\pi G_5} \int_{\mathcal{M}_5} (dA) \wedge^* (dA). \] (2.8)

In the above \( R^{ab} \) is the curvature 2-form and \( E^a \) is the fünfbein, together with a torsion-less connection \( \omega^{ab} \) from which the curvature 2-form is constructed (for a review on writing higher derivative gravity in this form see [34]). In addition the numbers \( \alpha_1, \cdots, \alpha_6 \) are arbitrary until we impose further symmetries (notably supersymmetry) on the bulk parent theory. Some comments are in order here about each contributing piece. Firstly, the part \( S[E] \) consists of purely gravitational elements and is simply the Gauss-Bonnet density continued from \( D = 4 \) to \( D = 5 \), the Ricci scalar, and the cosmological constant (where \( \Lambda = -\alpha_3 \)). Writing this in terms of the metric we have

\[ S[E] = \frac{1}{16\pi G_5} \int_{\mathcal{M}_5} d^5X \sqrt{G} [\alpha_2 R - \Lambda + \alpha_1 [(R_{MNPQ})^2 - 4(R_{MN})^2 + R^2]]. \] (2.9)

This part of the action is constructed solely from the \( E^a \) 1-form, and not its inverse, together with the invariant tensor \( \epsilon_{abcde} \) of the tangent space Lorentz group \( SO(1, 4) \). The second contribution \( S[A] \) is the Cherns-Simons term, which was studied in the context of \( N = 4 \) SYM in [39]. This will give rise to the \( V_{ij} \widetilde{V}^{ij} \) contribution in the current anomaly. The third term \( S[E \wedge A] \) is a mixed term. It is normally invisible in standard gravity duals involving just the Ricci scalar because the central charges must necessarily coincide. This will give the other piece \( R_{ijkl} \widetilde{R}^{ijkl} \) in the current anomaly. The last term \( S[E^{-1}; A] \) involves the inverse of the metric which is required for gauge field kinetic term in the Weyl anomaly. So except for the last term, it is very much
like a generalised Yang-Mills theory, where terms involving the inverse of the gauge connection are explicitly excluded. This general form of the action is then relevant for the anomaly analysis and correlator calculations. The reason for not including other higher derivative terms such as $R^2$, is due to requiring perturbative unitarity of the higher dimensional parent theory from which this action descends. This will be discussed later.

2.1 The Weyl Anomaly

As our starting point, we reconsider the vacuum solutions found in [18]. They consider a general higher derivative gravity theory (that is an action containing the Weyl tensor squared and the Ricci tensor squared, in addition to the Lovelock term), and find that the equations of motion for this system admit a maximally symmetric space (in particular $AdS_5$) as a solution. The length scale, $L$, of $AdS_5$ is then determined in terms of the parameters of this action. For the case we are considering the size is found to be

$$-\Lambda L^4 + 12 \alpha_2 L^2 - 24 \alpha_1 = 0. \quad (2.10)$$

Since $L^2 \in \mathbb{R}^+$, the discriminant must be positive semi-definite:

$$\frac{3 \alpha_2^2}{2 \alpha_1} \geq \Lambda. \quad (2.11)$$

These parameters will be related to the central charges in the boundary CFT.

Having found a ground state, perturbations can be setup using the Fefferman-Graham form of the metric [15, 17]:

$$ds^2 = \frac{1}{z^2}(L^2 dz^2 + g_{ij}(z, x) dx^i dx^j). \quad (2.12)$$

This allows one to make an analysis of the near boundary physics [18]. Indeed, the coefficients $\alpha_1, \alpha_2, \Lambda$ can now be determined in terms of the central charges $c, a$ in equation (2.1). Putting in the numbers, one finds

$$\alpha_1 = \frac{G_5}{\pi L}(c - a), \quad (2.13)$$

$$\alpha_2 = \frac{4G_5}{\pi L^3}(3c - a), \quad (2.14)$$

$$\Lambda = \frac{4G_5}{\pi L^5}(26c - 4a). \quad (2.15)$$

By the inequality (2.11), one finds a non-trivial relationship amongst the two central charges

$$14c^2 - 3ac + a^2 \geq 0. \quad (2.16)$$

This relation might be expected to be a gravitational version of the Seiberg conformal window. To see this we take the known values of the central charges for SQCD in the IR:
\[ c_{IR} = \frac{1}{16} \left( 7N_c^2 - 2 - 9N_f^4/N_c^2 \right), \]  
\[ a_{IR} = \frac{1}{16} \left( 6N_c^2 - 3 - 9N_f^4/N_c^2 \right). \]

In principle for the inequality to hold, there is a non-trivial inequality between \( N_f \) and \( N_c \). However, taking the large \( N_c \) and large \( N_f \) limit one finds that the inequality is satisfied regardless of the values of \( N_c \) and \( N_f \)

\[ 12 \left( 6 - \frac{9N_c^2}{N_f^2} \right)^2 + 25 \left( 6 - \frac{9N_c^2}{N_f^2} \right) + 14 \geq 0. \]

The conformal window seems to be independent of requiring a real space at least in the large-\( N \) limit. At the level of five dimensional supergravity, this is a statement of consistency.

In fact, the near boundary analysis performed in [18] is very interesting because the scale anomaly automatically satisfies the Weyl consistency conditions [40, 41], and seems to arise from the maximal symmetry of the ground state (see also [21] for a related discussion). It would be interesting to study this further.

To complete the Weyl anomaly matching, consider equation (2.8) written in terms of the metric

\[ S[E^{-1}; A] = \frac{\alpha_6}{16\pi G_5} \int_{\mathcal{M}_5} d^5 X \sqrt{G} G^{AB} G^{CD} F_{AC} F_{BD}. \]  

Putting this on the ground state solution \( \mathcal{M}_5 = AdS_5(L) \) then gives

\[ S[A] = \frac{\alpha_6}{16\pi G_5} \int d^4 x dz L \frac{z^3}{z^4} \sqrt{g(z,x)} z^4 g^{ij}(z,x) g^{kl}(z,x) F_{ik}(z,x) F_{jl}(z,x). \]

Assuming that \( g_{ij}(z,x) \) and \( F_{ij}(z,x) \) admit power series expansions in the radial coordinate as in [20], with \( g_{ij}(z,x) = g_{ij}(x) + z g_{ij}^{(1)} + \cdots \) and \( F_{ij}(z,x) = V_{ij}(x) + z V_{ij}^{(1)} + \cdots \) we isolate the logarithmic divergence to be

\[ S[A] = \frac{L \alpha_6}{16\pi G_5} \ln \epsilon \int d^4 x \sqrt{g} g^{ij}(x) V_{ij}(x) V_{ij}(x). \]  

Using the scale transformations \( \delta g_{ij} = 2\delta \sigma g_{ij}, \delta \epsilon = 2\delta \sigma \epsilon \) as in [18], one finds

\[ \delta S[A] = \frac{L \alpha_6}{16\pi G_5} \delta \sigma \int d^4 x \sqrt{g} V^{ik}(x) V_{jl}(x) \]
\[ = 2\sigma \int d^4 x \sqrt{g}(T_i^k). \]

This determines \( \alpha_6 = c G_5 / \pi L \).
2.2 Chiral Anomalies

Having understood the origin of the Weyl anomaly, one can now ask how the corresponding chiral anomaly looks like. Consider first the Chern-Simons term:

\[ S[A] = \frac{\alpha_4}{16\pi G_5} \int_{M_5} dA \wedge dA \wedge A. \]  

(2.25)

Of course, this does not require a metric as the Weyl anomaly did. Writing this in components in the (FG) coordinates, we can make the following gauge transformation

\[ \delta S[A] = \frac{\alpha_4}{16\pi G_5} \int_{AdS_5} d^4x dz \epsilon^{ijkl} F_{ij} F_{kl} \delta A_z \]  

(2.26)

where \( \delta A_z = \partial_z \delta \lambda(z,x) \). Assuming again the power expansions of the field strengths, this can be integrated directly to give the anomaly

\[ \delta S[A] = \frac{\alpha_4}{16\pi G_5} \delta \lambda \int d^4x (\epsilon^{ijkl} V_{ij} V_{kl}). \]  

(2.27)

From this the anomaly coefficient can be read off as \( \alpha_4 = 16G_5(5a - 3c)/9\pi \).

In a similar fashion, we can match the gravitational contribution coming from equation (2.7). One makes the same decomposition and variation

\[ \delta S[E \wedge A] = \frac{\alpha_5}{16\pi G_5} \delta \lambda \int d^4x dz (\epsilon^{ijkl} R_{ijmn} R_{kl}^{mn} \delta A_z), \]  

(2.29)

and therefore using again \( \delta A_z = \partial_z \delta \lambda \),

\[ \delta S[E \wedge A] = \frac{\alpha_5}{16\pi G_5} \delta \lambda \int d^4x (\epsilon^{ijkl} R_{ijmn} R_{kl}^{mn}). \]  

(2.30)

This fixes the last parameter to be \( \alpha_5 = 2G_5(c - a)/3\pi \). Having done this we have uniquely fixed the supergravity theory which reproduces the field theory anomalies through holographic renormalisation. One can now ask about the origin of this theory. Note also that the chiral anomaly arises in a different way to that of the Weyl anomaly; there it is associated with the divergence of the radial cutoff, which is related to the scale transformation. Here the integration is done and the gauge transformation parameter is already manifest.

3 The D=6 Parent Theory

In the standard \( \mathcal{N} = 4 \) SYM duality, the R-symmetry arises as the isometry group of the the \( S^5 \) which the IIB string theory is compactified on. The group is \( SO(6) \) which admits \( SU(4) \) as a covering group and permits fermions and supersymmetry to be realised. Clearly for \( \mathcal{N} = 1 \) SCFT’s we want the group to be \( U(1)_R = SO(2) \) and thus the simplest choice is for the compact space to be \( S^1 \). This means that the dual string theory (if it exists) should be a non-critical \( D = 6 \) string theory on \( AdS_5 \times S^1 \).
At this point the higher derivative gravity theory makes its entrance. As shown in [32], the dimensionally continued Euler density from \( D = 4 \) dimensions (where it is topological) to \( D = 6 \) is the unique term which ensures perturbative unitarity when expanded around Minkowski space-time. The first non-trivial terms enter at cubic order and thus are graviton self interactions. So suppose one calculates perturbative scattering amplitudes for string theory in \( D = 6 \). If we demand unitarity for the graviton then this Lovelock action is the unique term. Further, it doesn’t represent a string theory correction in \( \alpha' \) in the usual sense i.e. it is not a loop correction. It is just another tree level interaction term at the same scale as for the usual Einstein-Hilbert term set by the gravitational constant \( G_6 \). This seems to imply that in order to write down a consistent string theory in six dimensions, one is forced to use a higher derivative theory for all the associated space-time fields, and to introduce other interaction terms (an example will be given in the next section). This should change the nature of self interactions of fields and possibly also interactions amongst one another.

3.1 The D=6 Supergravity Theory

In usual perturbative string theory, the low energy effective action can be deduced by calculating scattering amplitudes and then writing down a classical action which reproduces them at tree level [42]. For the non-critical string theory we are considering, this should be the D=6 supergravity with eight supercharges [43]. The supergravity multiplets that are relevant for us are the graviton multiplet \((G_{MN}, \Psi^M_{\alpha}, B_2^-)\), the tensor multiplet \((B_2^+, \lambda^\alpha, \phi)\) and the vector multiplet \((V, \psi)\). Here \( B_2^+ \) has a self dual field strength and \( B_2^- \) an anti-self dual field strength. For later use define the 2-from potential as \( B_2 := B_2^+ + B_2^- \). There is also the hypermultiplet \((\chi, q^Y)\) which are not needed for the following discussion (The scalars parameterise a quaternionic Kähler manifold).

Guided by our knowledge of the \( D = 5 \) theory presented in the previous section, we can now write an action in six dimensions which will upon compactification on \( S^1 \) of radius \( l \), give the previous \( D = 5 \) theory. We want to consider a higher derivative theory of \( D = 6 \) supergravity written using differential forms. Let \( E^\pi = E_\pi^A(Y)dY^A \) be the sechsbein 1-forms, where \( Y^A \) are coordinates on \( M_6 \), and the curvature two form is \( R^{\Pi\Sigma} \). Firstly there are the curvature squared terms (in the following equations \( \mathcal{A}_i \) are coefficients which need to be fixed by supersymmetry)

\[
S[R^2] = \frac{\mathcal{A}_1}{16\pi G_6} \int_{M_6} \epsilon_{\Delta\Theta\Lambda\Xi\Sigma}(R^{\Delta\Theta} \wedge R^{\Lambda\Xi} \wedge E^{\Pi} \wedge E^{\Sigma}) \\
+ \frac{\mathcal{A}_2}{16\pi G_6} \int_{M_6} (R^{\Delta\Theta} \wedge R_{\Delta\Theta} \wedge dV) \\
+ \frac{\mathcal{A}_3}{16\pi G_6} \int_{M_6} (d\phi \wedge^* d\phi^*)(d\phi \wedge^* d\phi) \tag{3.1}
\]

When compactified using an ansatz \( dV = dA + \Lambda \wedge d\theta \) for the 1-form, one obtains equation (2.7). Similarly the last term in equation (2.5) can be so obtained. Next there are the usual kinetic
\[ S[E, V, B, \phi] = \frac{A_3}{16\pi G_6} \int_{M_6} \epsilon_{\Delta \Theta \Lambda \Xi \Pi \Sigma} (R^{\Delta \Theta} \wedge E^\Lambda \wedge E^\Xi \wedge E^\Pi \wedge E^\Sigma) \]
\[ + \frac{A_3}{16\pi G_6} \int_{M_6} (dV \wedge^* dB \wedge dB + d\phi \wedge^* d\phi). \] (3.2)

The kinetic term for the \( V \) gives the kinetic term for \( A \) field in five dimensions, equation (2.8).

There is a topological term
\[ S[V] = \frac{A_4}{16\pi G_6} \int_{M_6} dV \wedge dV \wedge dV \] (3.3)
which gives the Chern-Simons term equation (2.6). There are also a set of interaction terms between the fields, an example of which are
\[ S[B, \phi] = \frac{A_5}{16\pi G_6} \int_{M_6} d(B_2 \wedge B_2) \wedge d\phi \]
\[ + \frac{A_6}{16\pi G_6} \int_{M_6} d(B_2 \wedge B_2) \wedge^* d(B_2 \wedge B_2) \]
\[ + \frac{A_7}{16\pi G_6} \int_{M_6} d^6Y \sqrt{G} \left[(dB_2)_{MNP}dB^{MNP}\right]^2 \] (3.4)

and will be a relevance when considering an ensemble of D1-branes in the D3-branes. The second two pieces in equation (3.4) are examples of new interaction terms mentioned in the last section that should be introduced when dealing with a HD gravity theory. Upon compactification the gravitational constants are related by \( G_6 = 2\pi lG_5 \) (the radius of the \( S^1 \) has been set equal to one), and the coefficients \( A_i \) can then be related to the coefficients \( \alpha_i \) used in the previous section. The pieces considered so far are the ones relevant for the initial holographic duality and the action we consider is the following
\[ S[E; parent] = S[R^2] + S[E, V, B, \phi] + S[V] + S[B, \phi], \] (3.5)

The field equations contain derivatives of the metric only up to second order, i.e. terms like \( \partial^4G_{MN} \) are absent. This is significant because it ensures that we avoid the appearance of new classes of solutions which would involve non-linear differential equations with three or four derivatives of the metric. The Lovelock action makes the field equations more nonlinear, but still of second order. One can then still hope to find brane solutions of a similar form to the ones that are well known. The question now as to whether this system admits \( AdS_5 \times S^1 \) as a ground state solution is relevant. Some preliminary results are given in appendix A.
4 Some Speculations about the Near Horizon Geometry, Seiberg Duality and Confinement

Having seen much of the field theory structure given in terms of a D=6 supergravity theory, it would be interesting to have microscopic description given in terms of D-branes and strings and a near horizon geometry as in [2]. We want to consider a system of $N_c$ D3-branes in the usual 4 directions $x^i \in \mathbb{R}^4$ and $N_f$ anti D1-branes that have been distributed in the world-volume of the D3-branes in a homogeneous and isotropic way. This will have the effect of preserving the Minkowski isometries and only having flux through transverse space. We assume that the D1-branes and the D3-branes are interacting. An example of an interaction would be

$$S_{\text{interaction}}^1 = \int A_4 \wedge B_2$$

(here $A_4$ is dual to $\phi$). However, this is not gauge invariant and doesn’t require a metric. It is appropriate for a single or a stack of D1-branes. Another piece already encountered in equation (3.4) is

$$S_{\text{interaction}}^2 = \int dA_4 \wedge * (B_2 \wedge B_2).$$

and is the relevant term for considering the D3-filling D1-branes. It is then better to consider the 4-form $B_2 \wedge B_2$ as describing the collection of D1-branes rather than just $B_2$, as this will give only a local density with respect to the D3-brane. This can be made precise and draws on the simple case one encounters in usual electrostatics.

One can see that the number of D3-branes (in fact the electric charge!) is given by

$$Q_3 = \int_{S^1} * dA_4 = +N_c .$$

The D1-branes are more interesting. The electric charge is found by considering the collective potential $B_2 \wedge B_2$ (see appendix B) of the collection of anti-D1 branes and integrating it over the same $S^1$. If $B_2$ is electric then

$$Q_1^E = \int_{S^1} * d(B_2 \wedge B_2) = (-)N_f,$$

and if it is magnetic

$$Q_1^M = \int_{S^1} * d(B_2 \wedge B_2) = 0.$$

This gives a nice holographic interpretation of Seiberg duality in four dimensions in terms of six dimensional strings. We propose that the magnetic strings in six dimensions correspond to the *electric theory* with gauge group $SU(N_c)$, whilst the *magnetic theory* with gauge group $SU(N_f - N_c)$ corresponds to electric strings. One could also consider a stack of $(N_f - N_c)$ D3-branes and $N_f$ electric or magnetic strings. In this case electric strings correspond to the electric gauge theory, and magnetic strings to the magnetic gauge theory. It is just a convention choice of what one calls electric or magnetic between the theories. In this duality we are seeing mesons
and glueballs, since we are in the IR, rather than the more usual quarks and gluons encountered in the UV. This is obvious because we have been talking about Seiberg duality. The UV is not as interesting because we just have free quarks and gluons that are not interacting. This is a large $N_c$, large $N_f$ duality at the level of supergravity (exactly the same as Maldacenas original proposal [2]), and a novel feature seems to be that one can in principle describe the strongly coupled gauge theory (either the electric or magnetic theory) in terms of the weakly coupled supergravity theory, with a different radius of curvature $L$. The total D-brane charge is related to $L$ as in normal brane solutions. This is born out in the D=5 theory, where we know that the size of the $AdS_5$ radius and the cosmological constant are related and give the boundary field theory central charges. Therefore changing the size of the $AdS_5$ radius should change the values of the central charges in precisely the way dictated by Seiberg duality. It will be interesting to make this precise by finding an exact solution.

We have seen that it is essential to use the HD action to get the right field theory anomalies. This term is at the same scale as the usual Einstein-Hilbert term and so it is not correct to neglect it. One may legitimately ask what is the string theory that has this HD D=6 supergravity theory as its low energy limit. Here we make some speculations only.

If the Ricci scalar is expanded out to third order (with $G_{AB} = \eta_{AB} + \sqrt{G_6} h_{AB}$), then it corresponds to the usual 3-graviton string vertex

$$\int d^D X \sqrt{G} \sim \int d^D X h(\partial h)(\partial h) \sqrt{G_6} \quad \text{(4.6)}$$

The string amplitude can be recast into the form of the classical gravity action above. This is a perturbative definition and therefore it is admissible to consider an interaction which reproduces the Lovelock term as a low energy description. This term would have a dimensionful coupling constant but this is fine as we believe that a gravity action based curvature terms will only be a low energy effective description and thus renormalisation is not a problem. It also necessitates introducing other sets of interactions to preserve supersymmetry of the form we are anticipating in the D=6 HD supergravity we have partially written down. The hope would then be that those will modify the $\beta$-functional equations sufficiently to have a consistent string theory in D=6. Perhaps then a more general non-linear sigma model in two dimensions is what is needed to describe these string theories. This also seems to be related to the observation made in [18] that the Weyl Anomaly for $\mathcal{N} = 4$ SYM can be obtained with or without the usual Einstein Hilbert term. Whilst the Ricci scalar is important for unitarity requirements one can then see that HD gravity in the $D = 10$ could also make sense. It is only that in this case the large amount of supersymmetry can render HD terms to be unnecessary since we have perturbative consistency.
5 Conclusions and Outlook

It seems that the proposed HD theory has the right elements to be the dual of SQCD in its
conformal phase. By construction, the theory gives the correct holographic anomalies and one
can hope that there exists a well defined near horizon geometry. Of course much of this is a
provisional proposal, and it remains to fill in many of the technical details to make it a ‘bona
fide’ duality. On the technical side it will be necessary to fix the higher derivative action uniquely
by supersymmetry once the higher derivative terms for the other fields e.g. $B_2, \phi, \text{etc}$ have been
included. Next, it will be necessary to demonstrate that there are brane type solutions. We
would like to speculate here that the use of the complex coordinates $z$ and $\bar{z}$ (the coordinates
of the space transverse to the branes) will be important. One will encounter here expressions
involving objects like $\partial_z \partial_{\bar{z}} F(z, \bar{z})$ etc, and it looks possible to invoke the full power of complex
analysis to try and get solutions to these nonlinear equations. Further, one will be able to form
holomorphic and anti-holomorphic integrals for conserved charges. In fact setting up a detailed
dictionary as in the $\mathcal{N} = 4$ case, objects like the moduli space of the SQCD that one would look
at via brane probing could be given very elegant descriptions in terms of holomorphic integrals

$$
\langle tr X^2 \rangle = \frac{1}{2\pi i} \oint \frac{dz}{z} O(z)
$$

(5.1)

where the left hand side is a VEV for some field theory operator, whilst $O(z)$ is some supergravity
object. Yet again one can see something very reminiscent of Seiberg's holomorphy ideas.

One of the most exciting areas that could be opened up is more realistic phenomenology. One
can expect to apply similar ideas as in [45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55] to gravity duals
which are much closer to normal QCD. In fact if it is possible to calculate in the supergravity
some quantities in an expansion in $N_c/N_f$, it may be possible by considering ratios to really
make some quantitative comparisons with more usual field theory methods.

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of this work.
A The $AdS_5 \times S^1$ Solution in Lovelock Gravity

Here we demonstrate that the $AdS_5 \times S^1$ solution is an acceptable ground state of the Lovelock gravity together with a higher derivative scalar field. The action we consider is

$$S\left[E^6 \right] = \frac{1}{16\pi G_5} \int_{M_6} d^6Y \sqrt{G} \left[ R + \mathcal{A}[(R_{MNPQ})^2 - 4(R_{MN})^2 + R^2] \right]$$
$$+ \int_{M_6} d^6Y \sqrt{G} \left[ -(G^{MN} \partial_M \Phi \partial_N \Phi) - \mathfrak{B}(G^{MN} \partial_M \Phi \partial_N \Phi)^2 \right]. \quad (A.1)$$

The field equations are

$$R_{MN} - \frac{1}{2} G_{MN} R - \frac{3}{2} G_M N E(6) + [RR]_{MN} = T_{MN}, \quad (A.2)$$
$$\partial_M [\sqrt{G} G^{MN} \partial_N \Phi] + 2\mathfrak{B} \partial_M [\sqrt{G} G^{MN} \partial_N (\Phi(\partial \Phi)^2)] = 0, \quad (A.3)$$

where

$$T_{MN} = \partial_M \Phi \partial_N \Phi - \frac{1}{2} G_{MN}(\partial \Phi)^2 + \mathfrak{B}(\partial \Phi)^2 [2\partial_M \Phi \partial_N \Phi - \frac{1}{2} G_{MN}(\partial \Phi)^2] \quad (A.4)$$

and

$$[RR]_{MN} \equiv 2RR_{MN} - 4R_{MA}R^A_N - 4R^{AB}R_{AMBN} + 2R_{MABC}R^{ABC}_N, \quad (A.5)$$
$$(\partial \Phi)^2 \equiv G^{MN} \partial_M \Phi \partial_N \Phi, \quad (A.6)$$
$$E[D] \equiv (R_{MNPQ})^2 - 4(R_{MN})^2 + R^2. \quad (A.7)$$

Consider the ansatz

$$ds^2 = ds^2[AdS_5(L)] + l^2 d\theta^2, \quad (A.8)$$
$$\partial_\theta \Phi = k. \quad (A.9)$$

We require that the Riemann tensor is a maximally symmetric space in the $AdS_5$ directions (with coordinates $\mu, \nu$), whilst is vanishes for any $S^1$ coordinate $\theta$:

$$R^{\mu\nu}_{\lambda\sigma} = - \frac{2}{L^2} [\delta^\mu_\lambda \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\lambda], \quad (A.10)$$
$$R^{\theta A}_{BC} = 0. \quad (A.11)$$

Then in D-dimensions

$$R^{\mu\nu}_{\lambda\sigma} R_{\mu\nu}^{\lambda\sigma} = \frac{8}{L^2} (D)(D - 1), \quad (A.12)$$
$$R^{\mu\nu} R_{\mu\nu} = \frac{4}{L^2} (D)(D - 1)^2, \quad (A.13)$$
$$R^2 = \frac{4}{L^4} (D)^2(D - 1)^2, \quad (A.14)$$
$$E[D] = \frac{4}{L^4} [2D(D - 1) - 4D(D - 1)^2 + D^2(D - 1)^2]. \quad (A.15)$$

The scalar field equation is satisfied with this ansatz, whilst the metric field equations become

$$(R + \mathcal{A} E[5]) = - \left( \frac{k^2}{l^2} \right) \left[ 1 + \frac{\mathfrak{B} k^2}{l^2} \right], \quad (A.16)$$
for the $θθ$ component and

$$(2R + A E[5]) = \left( \frac{k^2}{l^2} \right) \left[ 2 + \frac{3}{4B} \frac{k^2}{l^2} \right], \quad (A.17)$$

for the trace (this amounts to considering the $μν$ equation since it is proportional to the AdS$_5$ metric). A slight rearrangement gives

$$R = \frac{k^2}{l^2} \left[ 3 + 4 \frac{3}{4B} \frac{k^2}{l^2} \right], \quad (A.18)$$

$$E[5] = -\frac{k^2}{l^2} \left[ 4 + 7 \frac{3}{4B} \frac{k^2}{l^2} \right]. \quad (A.19)$$

These can then be related in an obvious way to the the length scale $L$ of the AdS$_5$. If supersymmetry can be made manifest in this system, we should be able to relate $A$ to $B$.

### B Distributing Charge

Here we wish to clarify the role of $B_2 ∧ B_2$ describing the brane distribution. To this end it is instructive to consider an example from electrostatics in $\mathbb{R}^4$. Consider an infinite line in the $z$-direction onto which one keeps placing units of charge $q_i$. The total charge is given by $\sum_i q_i = \int_{-\infty}^{+\infty} dz Q(z)$. In the static case we know the density function $Q(z)$ is constant, and that the electric field has $E_z = 0$ as the boundary condition on the line. The vector potential is given by $A_i = (Φ, A)$. So the question one can ask is “what is the charge density in terms of $A_i$?”.

From usual electrostatics $Q = ∫S1 (dθr)(E_\theta)$ Solving Maxwell’s equations (in cylindrical polar coordinates) we have $Q = ∫S1 (dθr)e^{θzt} A_z(∂rΦ) = ∫S1 *(A ∧ dA)$, where $A_z = 1$. The object $A ∧ dA$ describes how the point charges are distributed over the line. Similarly for the D1 branes with potential $B_2$, the relevant object is $B_2 ∧ dB_2$ when they fill the D3-space. Since we can ‘pull out’ a ‘d’, one sees that the object $B_2 ∧ B_2$ is the potential that the ensemble of D1-branes describe. This cannot be done for the charges on the line since $A ∧ A = 0$!

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