On the distribution of the order and index of \( g(\text{mod } p) \) over residue classes

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Abstract

For a fixed rational number \( g \notin \{-1, 0, 1\} \) and integers \( a \) and \( d \) we consider the set \( N_g(a, d) \) of primes \( p \) for which the order of \( g(\text{mod } p) \) is congruent to \( a(\text{mod } d) \). For \( d = 4 \) and \( d = 3 \) we show that, under the Generalized Riemann Hypothesis (GRH), these sets have a natural density \( \delta_g(a, d) \) and compute it. The results for \( d = 4 \) generalise earlier work by Chinen and Murata. The case \( d = 3 \) was apparently not considered before.

1 Introduction

Let \( g \notin \{-1, 0, 1\} \) be a rational number (this assumption on \( g \) will be maintained throughout this paper). For \( u \) a rational number, let \( \nu_p(u) \) denote the exponent of \( p \) in the canonical factorisation of \( u \) (throughout the letter \( p \) will be used to indicate prime numbers). If \( \nu_p(g) = 0 \), then there exists a smallest positive integer \( k \) such that \( g^k \equiv 1(\text{mod } p) \). We put \( \text{ord}_g(p) = k \). This number is the (residual) order of \( g(\text{mod } p) \). The index of the subgroup generated by \( g \) mod \( p \) inside the multiplicative group of residues mod \( p \), \( [(\mathbb{Z}/p\mathbb{Z})^\times : \langle g(\text{mod } p) \rangle] \), is denoted by \( r_g(p) \) and called the (residual) index mod \( p \) of \( g \). Although \( \text{ord}_g(p) \) and \( r_g(p) \) satisfy the easy relation

\[
\text{ord}_g(p)r_g(p) = p - 1,
\]

the functions themselves fluctuate quite irregularly. Given this it comes perhaps not as a surprise that a simple question such as Artin’s primitive root conjecture (1927), which asserts that \( \{ p : r_g(p) = 1 \} \) is infinite if \( g \) is not a square, remains unsolved. On the assumption of the Riemann Hypothesis for a certain class of Dedekind zeta functions, however, this was proved by C. Hooley [11]. Many variations of Artin’s conjecture have been considered in the course of time, the most far reaching in [17].

Many authors studied the divisibility of the order by some prescribed integer...
The case \( d = 2 \) for example is closely related to the non-divisibility of certain integer sequences by a prescribed prime. We say that an integer sequence \( S = \{s_j\}_{j=1}^\infty \) is divisible by an integer \( m \), if there exists an integer \( k \) such that \( m | s_k \).

It is easy to see that for a prime \( p \) with \( \nu_p(g) = 0 \) the sequence \( S(g) = \{g^j + 1\}_{j=1}^\infty \) is divisible by \( p \) if and only if \( \text{ord}_g(p) \) is even. Hasse [9, 10] showed that the set of prime divisors of the aforementioned sequence has a Dirichlet density. It is not difficult to extend his argument to show that these sequences have a natural density (hereafter we merely write density instead of natural density) of prime divisors. For \( g = 10 \) a prime \( p \neq 2, 5 \) divides \( S(g) \) if and only if the period of the decimal expansion of \( 1/p \) is even, cf. [28]. Using some algebraic number theory these results can be extended to some other well-known sequences, cf. [1, 12, 25].

In all of these cases the density can be computed unconditionally and turns out to be a rational number. For example, the density of prime divisors of \( S(2) \) is \( 17/24 \).

Now let \( d > 2 \) be given. By similar methods the divisibility of the order by \( d \) or the coprimality of the order with \( d \) can be studied. In this direction we especially like to mention K. Wiertelak, who wrote many papers on this subject, starting in the seventies of the previous century. See [34] for his most recent paper. Again one can prove that the density of the set of such primes exists and is rational.

In the light of the extensive literature on the case where the order is divisible by \( d \), it is somewhat surprising that the question of how the order is distributed over the various residue classes mod \( d \) has up to this century only been considered for \( d = 2 \). The purpose of this paper and its sequel(s) is to address this question for various other values of \( d \). For the understanding of the general case it is in my viewpoint crucial to first study a particular case in detail, for which we take \( d = 4 \).

For \( d = 4 \) our main interest is in the set \( N_g(a, 4) \), but it turns out to be fruitful to consider \( N_g(1, 2^s; j, 4) \) and \( N_g(3, 4; j, 4) \) separately, where \( N_g(a_1, d_1; a_2, d_2)(x) \) counts the number of primes \( p \leq x \) satisfying \( \nu_p(g) = 0 \) for which \( p \equiv a_1 \pmod{d_1} \) and \( \text{ord}_g(p) \equiv a_2 \pmod{d_2} \). For convenience we denote \( N_g(0, 1; a, d)(x) \) by \( N_g(a, d)(x) \). Although the functions \( N_g(1, 2^s; 1, 4)(x) \) and \( N_g(1, 2^s; 3, 4)(x) \) are more complicated (see Theorem [6] to describe, they turn out to be asymptotically equal under GRH. For the more easily describable functions \( N_g(3, 4; 1, 4)(x) \) and \( N_g(3, 4; 3, 4)(x) \) (vide Lemma [13]), the asymptotic behaviour can be different.

For \( s|r \) the number field \( \mathbb{Q}(\zeta_r, g^{1/s}) \) will be denoted by \( K_{r,s} \). By \( \pi_L(x) \) we denote the number of rational primes \( p \leq x \) that are unramified in the number field \( L \) and split completely in \( L \). As usual we let \( \text{Li}(x) \) denote the logarithmic integral, that is \( \text{Li}(x) = \int_2^x \frac{dt}{\log t} \).
For the primes \( p \equiv 1(\text{mod } 2^s) \), \( s \geq 2 \), we find:

**Theorem 1** Write \( g = g_1/g_2 \) with \( g_1, g_2 \) integers. Let \( s \geq 2 \). For \( j = 0 \) and \( j = 2 \) we have

\[
N_g(1, 2^s; j, 4)(x) = \delta_g(1, 2^s; j, 4)\text{Li}(x) + O\left(\frac{x(\log \log x)^4}{\log^3 x}\right),
\]

where

\[
\delta_g(1, 2^s; 0, 4) = 2^{1-s} - \sum_{r \geq s} \left(\frac{1}{[K_{2r,2r-1} : \mathbb{Q}]} - \frac{1}{[K_{2r+1,2r-1} : \mathbb{Q}]}\right)
\]

\[
\delta_g(1, 2^s; 2, 4) = \sum_{r \geq s} \left(\frac{1}{[K_{2r,2r-1} : \mathbb{Q}]} - \frac{1}{[K_{2r+1,2r-1} : \mathbb{Q}]} - \frac{1}{[K_{2r,2r} : \mathbb{Q}]} + \frac{1}{[K_{2r+1,2r} : \mathbb{Q}]}\right).
\]

For \( j = 1 \) and \( j = 3 \) we have, under GRH,

\[
N_g(1, 2^s; j, 4)(x) = \frac{x}{2} - \frac{\log |g_1g_2|}{\log^{3/2} x} + O\left(\frac{x(\log \log x)^4}{\log^{3/2} x}\right),
\]

where

\[
\delta_g(1, 2^s; 1, 2) = \sum_{r \geq s} \left(\frac{1}{[K_{2r,2r} : \mathbb{Q}]} - \frac{1}{[K_{2r+1,2r} : \mathbb{Q}]}\right)
\]

is the density of the set \( N_g(1, 2^s; 1, 2) \) and the implied constant is absolute.

For the primes \( p \equiv 3(\text{mod } 4) \) we find:

**Theorem 2** Write \( g = g_1/g_2 \) with \( g_1 \) and \( g_2 \) integers. Let \( \psi_0, \psi_1 \) denote the principal, respectively non-principal character mod 4. Let \( h_{\psi_1}(v) = \sum_{d|v} \mu(d)\psi_1(v/d) \), where \( \mu \) denotes the Möbius function. We have

\[
\begin{cases}
N_g(3, 4; 0, 4)(x) = 0; \\
N_g(3, 4; 2, 4)(x) = \#\{p \leq x : p \equiv 3(\text{mod } 4), \left(\frac{g}{p}\right) = -1\}.
\end{cases}
\]

Assuming GRH we have, when \( j \) is odd,

\[
N_g(3, 4; j, 4)(x) = \frac{1}{2} \#\{p \leq x : p \equiv 3(\text{mod } 4), \left(\frac{g}{p}\right) = 1\}
\]

\[
+ (-1)^{\frac{j-1}{2}} \frac{\Delta_g}{4} \frac{x}{\log x} + O\left(\frac{x(\log \log x)^4}{\log^{3/2} x}\right),
\]

where

\[
\Delta_g = \sum_{\sqrt{2}v \in K_{2v,2v} \cap 2^v \mathbb{Z}} h_{\psi_1}(v) \left[\frac{1}{[K_{2v,2v} : \mathbb{Q}]} - \frac{1}{[K_{2v+1,2v} : \mathbb{Q}]}\right] - \sum_{\sqrt{2}v \in K_{2v,2v} \cap 2^v \mathbb{Z}} h_{\psi_1}(v) \left[\frac{1}{[K_{2v,2v} : \mathbb{Q}]}\right],
\]

and the implied constant is absolute.
It is clear that $N_g(j, 4)(x) = N_g(1, 4; j, 4)(x) + N_g(3, 4; j, 4)(x)$ and we leave it to the reader to add the estimates for the latter two quantities given in Theorem 1 respectively Theorem 2 (Corollary 1 provides an example).

In Section 5 we derive explicit versions of Theorem 1 and Theorem 2. For $s \geq 0$ it follows that $\delta_g(1, 2^s; j, 4)$ exists and is in $\mathbb{Q} + \mathbb{Q}A_{\psi_1}$, where

$$A_{\psi_1} = \prod_{p \equiv 3(\mod 4)} \left(1 - \frac{2p}{(p^2 + 1)(p - 1)}\right) = 0.643650679662525\ldots$$

As an example we mention the following corollary to Theorem 1 and Theorem 2 (for the notation $h$ and $D$ we refer to Lemma 1; for a non-zero real number $r$ we denote its sign by $\text{sgn}(r)$).

**Corollary 1 (GRH).** Suppose that $h = 1$ and $j$ is odd. Then $\delta_g(1, 4) = \delta_g(3, 4) = 1/6$ unless $D$ is divisible by 8 and has no prime divisor congruent to 1(mod 4), in which case we have

$$\delta_g(j, 4) = \begin{cases} \frac{1}{6} + \text{sgn}(g)A_{\psi_1}(-1)^{-1} \prod_{p \mid D} \frac{2p}{p^2 - p^2 - p - 1} & \text{if } D \neq 8; \\ \frac{7}{48} + \text{sgn}(g)A_{\psi_1}(-1)^{-1} & \text{if } D = 8. \end{cases}$$

In case $d = 3$ similar results to those for $d = 4$ are obtained in Section 6. In particular we will show that, under GRH, $\delta_g(1, 3^s; j, 3)$ exists for $s \geq 0$ and that $\delta_g(1, 3^s; j, 3) \in \mathbb{Q} \cup \mathbb{Q}A_{\xi_1}$, where

$$A_{\xi_1} = \prod_{p \equiv 2(\mod 3)} \left(1 - \frac{2p}{(p^2 + 1)(p - 1)}\right) = 0.173977122429634\ldots,$$

and $\xi_1$ denotes the non-principal character mod 3. The rational numbers involved we explicitly compute.

The analogous problem of studying the primes for which the index is congruent to $a$(mod $d$) turns out to be far easier (see Section 3). Nevertheless, at least for $d = 3$ and $d = 4$, we again find, under GRH, that these densities exist and are in $\mathbb{Q} \cup \mathbb{Q}A_{\xi_1}$, respectively $\mathbb{Q} \cup \mathbb{Q}A_{\psi_1}$.

Instead of requiring GRH it is enough to require that RH holds for every field $\mathbb{Q}(\zeta_r, g^{1/s})$ with $s|r$. Indeed, if a given result is under GRH we mean that we require RH to hold for every field that occurs in the proof of this result.

In a sequel to this paper by a slightly different method the case where $d$ is an odd prime power is investigated (but less explicitly), see [24].

The density $\delta_g(j, 4)$ with $g$ a positive integer that is not a pure power (i.e. $h = 1$ in the notation of Lemma 1), was first studied by Chinen and Murata in [3, 4, 5, 6, 7, 27], culminating (in [27]) in their proof of Corollary 1 for the case $g > 0$.

## 2 Preliminaries

### 2.1 The index and algebraic number theory

In this section we recall some well-known arguments from the theory of primitive roots that are essential for an understanding of the rest of this paper.
The index can be easily related to algebraic number theory and by using (11) we then can get a grip on the order. Thus a prime $p$ that satisfies $k | r_q(p)$ must obviously satisfy $p \equiv 1 \pmod k$ and $q (p-1)/k \equiv 1 \pmod p$, in other words it must split completely in the field $\mathbb{Q}(\zeta_k, g^{1/k}) = K_{k,k}$. On the other hand a prime $p$ that satisfies the latter condition satisfies $k | r_q(p)$. Then, by the principle of inclusion and exclusion, we can describe for example the set of primes $p$ that satisfy $r_q(p) = k$. Note that $r_q(p) = k$ iff $k | r_q(p)$ and $gk \nmid r_q(p)$ for any prime $q$. Let $R_g(a, f; t)$ denote the set of primes $p$ with $p \equiv a \pmod f$ and $r_q(p) = t$. Let $R_g(a, f; t)(x)$ denote the number of primes $p \leq x$ in $R_g(a, f; t)$. Using the principle of inclusion and exclusion, we then find that

$$R_g(a, f; t)(x) = \sum_{n=1}^{\infty} \mu(n)\#\{p \leq x : p \equiv a \pmod f, (p, K_n, n/\mathbb{Q}) = \text{id}\}. \quad (4)$$

By $(p, K/\mathbb{Q})$ we denote the Frobenius symbol. We have $(p, K/\mathbb{Q}) = \text{id}$ iff $p$ is unramified and splits completely in $K$. Since sets of the form $\{p : p \equiv a \pmod f, (p, K_r, r/\mathbb{Q}) = \text{id}\}$, will occur rather frequently in the sequel, we will denote them by $S_g(a, f; r, n)$ and the corresponding counting function by $S_g(a, f; r, n)(x)$. Assuming GRH, it follows from (17) that $R_g(a, f; t)$ has a density.

Sofar this density has only been evaluated in terms of an Euler product (singular series) in the case $t = 1$ with $a$ and $f$ arbitrary [18], or in the case $f \mid 2$ and $t$ arbitrary [26, 31]. For example, for $t = 1$ and $2 \nmid f$ the density, under GRH, is a rational multiple of the Artin constant $\prod_p (1 - \frac{1}{p(p-1)})$ [11]. For an unified Galois theoretic treatment of finding Euler products for these cases see [18].

For our purposes such an evaluation of the density of $R_g(a, f; t)$ will, however, be irrelevant, an evaluation in terms of an infinite series will be sufficient. The tool to arrive at such an expression for the density is the Chebotarev density theorem:

**Theorem 3** (GRH). Let $K$ be an algebraic number field, let $L/K$ be a finite Galois extension and $C$ be a conjugacy class in $G = \text{Gal}(L/K)$. We let $\pi(x; L/K, C)$ denote the number of unramified prime ideals $p$ in $K$ such that $(p, L/K) = C$ and $Np \leq x$. Then, under RH for the field $L$ we have

$$\pi(x; L/K, C) = \frac{\#C}{\#G} \text{Li}(x) + O\left(\frac{\#C}{\#G} \sqrt{x \log(d_{L}[L: \mathbb{Q}] )}\right), \quad \text{as } x \to \infty, \quad (5)$$

where $d_L$ denotes the discriminant of $L$.

**Remark.** The proof of Theorem 3 is in essence due to Lagarias and Odlyzko, the present formulation is due to Serre [30, p. 133], who removed ‘un terme parasite’ in the formulation of Lagarias and Odlyzko ([13, Theorem 1.1]). In case $C = \text{id}$ the result was proved earlier by Lang [16]. For several variants of Artin’s primitive root conjecture Lang’s result is all one needs. There are also unconditional variants that certainly allow us to deduce that $\pi(x; L/K, C) \sim \frac{\#C}{\#G} \text{Li}(x)$, as $x$ tends to infinity.
For an arbitrary integer \( m \geq 1 \) let us see how Chebotarev’s density theorem can be used to estimate \( S_g(a, f; r, n)(x) \), where \( n|r \). To this end we consider the compositum of the fields \( \mathbb{Q}(\zeta_f) \) and \( K_{r,n} \), that is \( K_{[f,r],n} \), where by \([f,r]\) we denote the lowest common multiple of \( f \) and \( r \). Let \( K_1 \) and \( K_2 \) be number fields that are Galois. If there is an automorphism \( \sigma_1 \in \text{Gal}(K_1/\mathbb{Q}) \) and an automorphism \( \sigma_2 \in \text{Gal}(K_2/\mathbb{Q}) \) such that \( \sigma_1 = \sigma_2 \) on \( K_1 \cap K_2 \), then there is an unique \( \sigma \in K_1 \cdot K_2 \), the compositum of \( K_1 \) and \( K_2 \) such that \( \sigma|_{K_j} = \sigma_j \) for \( j = 1, 2 \). Now in order to apply the Chebotarev density theorem, we have to count the number of elements in the conjugacy class of \( \sigma \in \text{Gal}(K_{[f,r],n}/\mathbb{Q}) \), where \( \sigma \) is such that \( \sigma|_{\mathbb{Q}(\zeta_f)} = \sigma_{a,f} \), where \( \sigma_{a,f} \in \text{Gal}(\mathbb{Q}(\zeta_f)/\mathbb{Q}) \) is uniquely determined by \( \sigma_{a,f}(\zeta_f) = \zeta_f^q \), and \( \sigma_{a,f}|_{K_{r,n}} = id \). By the above remark such a \( \sigma \) exists, and is unique, if and only if \( \sigma|_{\mathbb{Q}(\zeta_f)\cap K_{r,n}} = id \). Note that a conjugate \( \tau \sigma \tau^{-1} \) acts trivially on \( K_{r,n} \) and can be regarded as an element of \( \text{Gal}(K_{[f,r],n}/K_{r,n}) \), which is a subgroup of the abelian group \((\mathbb{Z}/f\mathbb{Z})^*\). Hence \( \tau \sigma \tau^{-1} \) acts as \( \sigma \). We conclude that the conjugacy class has one element if \( \sigma|_{\mathbb{Q}(\zeta_f)\cap K_{r,n}} = id \) and zero otherwise. By this argument we expect from (6), assuming there is enough cancellation in the error terms, that the density of \( R_g(a; f; t) \) is given by

\[
\lim_{x \to \infty} \frac{R_g(a, f; t)(x)}{\pi(x)} = \sum_{n=1}^{\infty} \frac{\mu(n)c_1(a, f, nt)}{[K_{[f,nt],nt}: \mathbb{Q}]}, \tag{6}
\]

where

\[
c_1(a,f,nt) = \begin{cases} 1 & \text{if } \sigma_{a,f}^t|_{\mathbb{Q}(\zeta_f)\cap K_{n,nt}} = id; \\ 0 & \text{otherwise}, \end{cases}
\]

and \( \pi(x) = \sum_{\rho \leq x} 1 \). By [7] we know that (6) holds true, under GRH.

### 2.2 Field degrees and intersections

In order to explicitly evaluate certain densities in this paper, the following result will play a crucial role. The notations \( D, g_0, h \) and \( n_r \) will reappear again and again in the sequel. If \( a \) and \( b \) are integers, then by \((a,b)\) and \([a,b]\) we denote the greatest common divisor, respectively lowest common multiple of \( a \) and \( b \).

**Lemma 1** Write \( g = \pm g_0^h \), where \( g_0 \) is positive and not an exact power of a rational. Let \( D \) denote the discriminant of the field \( \mathbb{Q}(\sqrt{g_0}) \). Put \( m = D/2 \) if \( \nu_2(h) = 0 \) and \( D \equiv 4 \pmod{8} \) or \( \nu_2(h) = 1 \) and \( D \equiv 0 \pmod{8} \), and \( m = [2^{\nu_2(h)+2}, D] \) otherwise. Put

\[
n_r = \begin{cases} m & \text{if } g < 0 \text{ and } r \text{ is odd}; \\ [2^{\nu_2(hr)+1}, D] & \text{otherwise}. \end{cases}
\]

We have

\[
[K_{kr,k} : \mathbb{Q}] = [\mathbb{Q}(\zeta_{kr}, g^{1/k}) : \mathbb{Q}] = \frac{\varphi(kr)k}{\epsilon(kr,k)(k,h)},
\]

where, for \( g > 0 \) or \( g < 0 \) and \( r \) even we have

\[
\epsilon(kr,k) = \begin{cases} 2 & \text{if } n_r|kr; \\ 1 & \text{if } n_r \nmid kr, \end{cases}
\]

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and for $g < 0$ and $r$ odd we have

$$
\epsilon (kr, k) = \begin{cases} 
\frac{2}{3} & \text{if } n_r | kr; \\
\frac{1}{3} & \text{if } 2 | k \text{ and } 2^{\nu_2(h) + 1} \nmid k; \\
1 & \text{otherwise}. 
\end{cases}
$$

Proof. For $r = 1$ the result follows from Proposition 4.1 of [31] (see also the proof of Theorem 2.2 in [31]). For $r > 1$ the result follows from the case where $r = 1$ on noting that, with $\tilde{g} = g^r$, we have $\mathbb{Q}(\zeta_{kr}, g^{1/k}) = \mathbb{Q}(\zeta_{kr}, \tilde{g}^{1/kr})$. The distinction between $r$ even and $r$ odd arises in the case $g < 0$ since $\tilde{g}$ is then positive or negative, according to whether $r$ is even or odd, respectively. 

In our analytic considerations we need an upper bound for the discriminant of the field $K_{kr,k}$.

Lemma 2 The discriminant $D'$ of the field $K_{rk,k}$ satisfies

$$
\log |D'| \leq rk \left( \log(rk) + \log(|g_1g_2|) \right),
$$

where $g_0 = g_1/g_2$ and $g_1$ and $g_2$ are integers.

Proof. If $L_1/\mathbb{Q}$ and $L_2/\mathbb{Q}$ are two extension fields and $L$ is their compositum, then the associated discriminants (over $\mathbb{Q}$) satisfy $d_L|\cdot|d_{L_1}|d_{L_2}|L_1L_2$. From this we have the estimate

$$
\log |d_L| \leq [L_2 : \mathbb{Q}] \log |d_{L_1}| + [L_1 : \mathbb{Q}] \log |d_{L_2}|.
$$

(7)

It is well-known that the discriminant of the cyclotomic field $\mathbb{Q}(\zeta_m)$ and the field $\mathbb{Q}(g^{1/n})$ divide $m^{\varphi(m)}$, respectively $(ng_1g_2)^n$ (see e.g. [2]). On invoking these estimates, the result then follows from (7). 

From cyclotomy we recall the following well-known result.

Lemma 3 We have $\mathbb{Q}(\sqrt{g}) \subseteq \mathbb{Q}(\zeta_f)$ iff $\Delta | f$, where $\Delta$ denotes the discriminant of the field $\mathbb{Q}(\sqrt{g})$.

Proof. See e.g. [32]. 

In order to evaluate the densities for the modulus 4 we need the following result, which can be easily deduced from Lemma 1 and the previous lemma.

Lemma 4 Put $L_v = \mathbb{Q}(\zeta_8) \cap \mathbb{Q}(\zeta_{2v}, g^{1/2v})$. Let $v$ be odd.

If $h$ is odd and $D \nmid 8v$, then $L_v = \mathbb{Q}$.

If $h$ is odd and $D | 8v$, then

$$
L_v = \begin{cases} 
\mathbb{Q}(\sqrt{\text{sgn}(g)}) & \text{if } D \equiv 1 \text{ (mod 4)}; \\
\mathbb{Q}(\sqrt{-g}) & \text{if } D \equiv 4 \text{ (mod 8)}; \\
\mathbb{Q}(\sqrt{2 \cdot \text{sgn}(g)}) & \text{if } D \equiv 8 \text{ (mod 32)}; \\
\mathbb{Q}(\sqrt{2 \cdot \text{sgn}(-g)}) & \text{if } D \equiv 24 \text{ (mod 32)}. 
\end{cases}
$$

If $h$ is even, then $L_v = \mathbb{Q}(\sqrt{\text{sgn}(g)})$.
Lemma 5

Upon.

The following result shows that this order of growth cannot be improved that in each case the field claimed to equal $L_v$ has the correct degree.

Let us first consider the case where $g > 0$. We may suppose that $h$ is odd and $D|8v$, since in the remaining cases the degree of $L_v$ is 1 and hence $L_v = \mathbb{Q}$. It remains to show that $i \in L_v$ if $D \equiv 4(\text{mod} \ 8)$, $\sqrt{2} \in L_v$ if $D \equiv 8(\text{mod} \ 32)$ and $\sqrt{-2} \in L_v$ if $D \equiv 24(\text{mod} \ 32)$.

i) $D \equiv 4(\text{mod} \ 8)$. In this case the discriminant of $\mathbb{Q}(\sqrt{-g}, D/4)$, divides $v$ and thus by Lemma 3, $\sqrt{-g} \in \mathbb{Q}(\zeta_v) \subseteq K_{2v,2v}$. Since also $\sqrt{-g} \in K_{2v,2v}$ it follows that $i \in K_{2v,2v}$.

ii) $D \equiv 8(\text{mod} \ 32)$. Now $\sqrt{2g} \in \mathbb{Q}(\zeta_v)$ and thus $\sqrt{2} \in K_{2v,2v}$.

iii) $D \equiv 24(\text{mod} \ 32)$. Now $\sqrt{-2g} \in \mathbb{Q}(\zeta_v)$ and thus $\sqrt{-2} \in K_{2v,2v}$.

Suppose $g < 0$. If $h$ is even, we have to show that $i \in \mathbb{Q}(\zeta_v, g^{1/2v}) = \mathbb{Q}(\zeta_v, \zeta_{4v} g_0^{h/2v})$. Since $(\zeta_{4v} g_0^{h/2v})^v \in \mathbb{Q}(i) \setminus \mathbb{Q}$, this is clear. As before we may now suppose that $h$ is odd and $D|8v$. It remains to show that $i \in L_v$ if $D \equiv 1(\text{mod} \ 4)$, $\sqrt{-2} \in L_v$ if $D \equiv 8(\text{mod} \ 32)$ and $\sqrt{2} \in L_v$ if $D \equiv 24(\text{mod} \ 32)$. Note that $(\zeta_{4v} g_0^{h/2v})^v$ is a rational multiple of $\sqrt{-g_0}$ and hence $\sqrt{-g_0} \in K_{2v,2v}$.

i) $D \equiv 1(\text{mod} \ 4)$. We have $\sqrt{-g_0} \in \mathbb{Q}(\zeta_v)$ and since $\sqrt{-g_0} \in K_{2v,2v}$, it follows that $i \in K_{2v,2v}$.

ii) $D \equiv 8(\text{mod} \ 32)$. We have $\sqrt{2g_0} \in \mathbb{Q}(\zeta_v)$ and since $\sqrt{-g_0} \in K_{2v,2v}$, it follows that $\sqrt{-2} \in L_v$.

iii) $D \equiv 24(\text{mod} \ 32)$. We have $\sqrt{-2g_0} \in \mathbb{Q}(\zeta_v)$ and since $\sqrt{-g_0} \in K_{2v,2v}$, it follows that $\sqrt{2} \in L_v$.

Lemma 4 allows one to establish the following property of $\Delta_g$.

Proposition 1 We have $\Delta_{-g} = -\Delta_g$.

Proof. If $h$ is even, then $\Delta_{-g} = \Delta_g = 0$. If $h$ and $v$ are odd and $8|D$ then $[\mathbb{Q}(\zeta_{2v}, g^{1/2v}) : \mathbb{Q}] = [\mathbb{Q}(\zeta_{2v}, (-g)^{1/2v}) : \mathbb{Q}]$ by Lemma 4 The result now follows easily on invoking Lemma 4.

2.3 Index $t$ revisited

In this section we extend some results of Murata [26], which he established for squarefree integers $\geq 2$, to arbitrary $g \in \mathbb{Q}\setminus\{-1,0,1\}$. However, the method of proof employed in this paper is rather different.

We will need that $\sum_{r>x} 1/(r \varphi(r)) = O(1/x)$ (for a proof see e.g. [19] Lemma 8.4), the following result shows that this order of growth cannot be improved upon.

Lemma 5 We have

$$
\sum_{n \leq x} \frac{1}{n \varphi(n)} = \prod_{p} \left(1 + \frac{p}{(p^2 - 1)(p - 1)}\right) - \frac{315 \zeta(3)}{2 \pi^4 x} + O\left(\frac{\log x}{x^2}\right).
$$
Proof. Landau [14] has shown that
\[
\sum_{n \leq x} \frac{1}{\varphi(n)} = \frac{315 \zeta(3)}{2\pi^4} \left\{ \log x + \gamma - \sum_p \frac{\log p}{p^2 - p + 1} \right\} + O \left( \frac{\log x}{x} \right), \tag{8}
\]
where \(\gamma\) denotes Euler’s constant. Using (8) and the Euler identity, the result follows on partial integration. Alternatively one can apply Landau’s method for establishing (8) to the sum \(\sum_{n \leq x} 1/(n\varphi(n))\).

The estimate
\[
\frac{1}{[K_{vt,vt} : \mathbb{Q}]} \leq \frac{2h}{vt\varphi(vt)}, \tag{9}
\]
ensures that
\[
A(g, t) := \sum_{v=1}^{\infty} \frac{\mu(v)}{[K_{vt,vt} : \mathbb{Q}]}
\]
converges absolutely.

**Lemma 6** We have \(\sum_{t \leq y} A(g, t) = 1 + O(\frac{h}{y})\), where the implied constant is absolute.

**Proof.** One easily checks that \(\sum_{t=1}^{\infty} A(g, t) = 1\). Using (9), \(\varphi(vt) \geq \varphi(v)\varphi(t)\) and Lemma 5, we infer that
\[
\sum_{t \geq y} A(g, t) = O \left( \sum_{t \geq y} \frac{h}{v t \varphi(vt)} \right) = O \left( \sum_{t \geq y} \frac{h}{t \varphi(t)} \sum_{v=1}^{\infty} \frac{1}{v \varphi(v)} \right) = O(\frac{h}{y}),
\]
where the implied constants are all absolute. \(\Box\)

**Theorem 4** (GRH). Write \(g = g_1/g_2\), with \(g_1\) and \(g_2\) integers. For \(t \leq x^{1/3}\) we have
\[
R_g(0, 1; t)(x) = A(g, t) \frac{x}{\log x} + O \left( \frac{x \log \log x}{\varphi(t) \log^2 x} + \frac{x \log |g_1g_2|}{\log^2 x} \right),
\]
where the implied constant is absolute.

**Proof.** Since the proof is carried out along the lines of Hooley’s proof [11], we only sketch it. Let
\[
M_g(x, y) = \#\{p \leq x : t|r_g(p), \text{ } qt \nmid r_g(p), \text{ } q \leq y \} \quad \text{and} \quad M_g(x, y, z) = \#\{p \leq x : t|\varphi(p), \text{ } y \leq q \leq z \},
\]
where \(q\) denotes a prime number. Note that
\[
R_g(0, 1; t)(x) = M_g(x, \tau_1) + O(M_g(x, \tau_1, \tau_2)) + O(M_g(x, \tau_2, \tau_3)) + O(M_g(x, \tau_3, \frac{x - 1}{t})).
\]

We take \(\tau_1 = \log x/6\), \(\tau_2 = \sqrt{x} \log^{-2} x\) and \(\tau_3 = \sqrt{x} \log x\). We use the starting observation that
\[
R_g(0, 1; t)(x) = \sum_{v=1}^{\infty} \mu(v) \pi_{K_{vt,vt}}(x).
\]
Using Lemma 2 and Theorem 3 we can estimate \( \pi_{K,v,v}(x) \), under GRH. Proceeding as Hooley did, we then obtain that both \( M_g(x, \zeta_1, \zeta_2) \) and \( M_g(x, \zeta_3, \frac{x-1}{t}) \) are of order \((\log |g_1g_2|)x \log^{-2}x\), where the implied constant is absolute. Furthermore, we obtain that \( M_g(x, \zeta_2, \zeta_3) = O(\frac{x}{\psi(x)} \log x) \), where again the implied constant is absolute. For the main term we find that

\[
M_g(x, \zeta_1) = A(g, t) \frac{x}{\log x} + O \left( \frac{x \log |g_1g_2|}{\log^2 x} \right),
\]

where the implied constant is absolute. On adding the various terms, the theorem follows.

The following result is a slight generalisation of Lemma 2.4 of [7].

**Lemma 7** (GRH). Let \( \psi(x) \) be a monotonous increasing positive function which satisfies

\[
\lim_{x \to \infty} \psi(x) = +\infty \text{ and } \psi(x) \ll (\log x)^{1/2}.
\]

Then we have

\[
\# \{ p \leq x : r_g(p) \geq \psi(x) \} \ll \log |g_1g_2| \frac{\pi(x)}{\psi(x)},
\]

where the constants implied by the \( \ll \)-symbol are absolute and \( g = g_1/g_2 \) with \( g_1 \) and \( g_2 \) integers.

**Proof.** Let \( y \) denote the largest integer not exceeding \( \psi(x) \). We have

\[
\# \{ p \leq x : r_g(p) \geq y \} = \{ p \leq x : \nu_p(g) = 0 \} - \bigcup_{t=1}^{y-1} R_g(0, 1; t)(x),
\]

where \( \bigcup_{t=1}^{y-1} \) is a disjoint union. The latter identity together with Theorem A, Lemma \( \text{[3]} \) and \( \text{[4]} \), yields

\[
\# \{ p \leq x : r_g(p) \geq y \} = \pi(x) + O(\log |g_1g_2|)
\]

\[
- \left( 1 + O \left( \frac{h}{y} \right) \right) \frac{x}{\log x} + O \left( \frac{x \log y \log \log x}{\log^2 x} \right) + O \left( \frac{xy \log |g_1g_2|}{\log^2 x} \right),
\]

where the implied constants are all absolute. On using the prime number theorem in the form \( \pi(x) = x/\log x + O(x \log^{-3/2} x) \), the result then follows. \( \square \)

### 2.4 On the convolution of the Möbius function with Dirichlet characters

Let \( \chi \) be a Dirichlet character of conductor \( f_\chi \) and order \( o_\chi \) (for definitions and basic facts on Dirichlet characters we refer the reader to Hasse [8]). Let \( G_d \) denote the group of characters defined on \((\mathbb{Z}/d\mathbb{Z})^*\). We have that \( G_d \cong (\mathbb{Z}/d\mathbb{Z})^* \). An important auxiliary function in this paper is the convolution, \( h_\chi = \mu * \chi \), of the Möbius function \( \mu \) with a Dirichlet character \( \chi \), that is \( h_\chi(n) = \sum_{d|n} \mu(d) \chi(n/d) \).

In this section we collect some auxiliary results involving \( h_\chi \). We note the following trivial result.
Lemma 8 The function $h_\chi$ is multiplicative. With the convention that $0^0 = 1$, it satisfies $h_\chi(1) = 1$ and $h_\chi(p^r) = \chi(p)^{r-1}[\chi(p) - 1]$.

In particular if $\chi$ is the trivial character mod $d$, then

$$h_\chi(v) = \begin{cases} 
\mu(v) & \text{if } v | d; \\
0 & \text{otherwise.}
\end{cases}$$

By using one of the orthogonality relations for Dirichlet characters, the following result is easily obtained.

Lemma 9 Let $a(\mod d)$ be a reduced residue class mod $d$. We have

$$\sum_{t=a(\mod d)} \mu(t) = \frac{1}{\varphi(d)} \sum_{\chi \in \mathbb{G}_d} \overline{\chi(a)} h_\chi(v),$$

where $\chi_k$ runs over the Dirichlet characters modulo $d$.

Note that the lemma expresses a non-multiplicative function as a linear combination of multiplicative functions. This will play an important role later on.

Let $r, s$ be non-negative integers. Put

$$C_\chi(h, r, s) = \sum_{(r, v) = 1, s | v} \frac{h_\chi(v)(h, v)}{v \varphi(v)}$$

and $A_\chi = \prod_{\chi(p) \neq 0} \left(1 + \frac{[\chi(p) - 1]p}{[p^2 - \chi(p)](p - 1)}\right)$.

It is easy to see that the latter series is absolutely convergent. Note that $h_\chi(v) \leq 2^{\omega(v)}$, where $\omega(v)$ denotes the number of distinct prime divisors of $v$. Note also that for every $\epsilon > 0$ we have $|h_\chi(v)| \leq 2^{\omega(v)} \leq \sum_{d | v} 1 \ll v^\epsilon$ and $\varphi(v) \gg v^{1-\epsilon}$.

From this the absolute convergence easily follows. Since $h_\chi$ is a multiplicative function, we can invoke Euler’s identity. After some tedious but easy calculations this then yields the following result.

Lemma 10 Let $h, r, s \geq 1$ be integers. Let $\chi$ be a Dirichlet character mod $d$.

Put $e_p = \nu_p(h)$.

i) If $(r, s) = 1$, then

$$C_\chi(h, r, s) = \prod_{p | r, s} \left(1 + p^{1-e_p} \left[\frac{[\chi(p) - 1]}{p-1} \left(\frac{p^{e_p} - \chi(p) p^{e_p}}{p - \chi(p)} + \frac{\chi(p) p^{e_p}}{p^2 - \chi(p)}\right)\right]\right)$$

$$\prod_{\nu_p(s) = 1} p^{1-e_p} \left[\frac{\chi(p) - 1}{p-1} \left(\frac{p^{e_p} - \chi(p) p^{e_p}}{p - \chi(p)} + \frac{\chi(p) p^{e_p}}{p^2 - \chi(p)}\right)\right]$$

$$\prod_{\nu_p(s) \geq e_p + 1} \chi(p)^{\nu_p(s) - 1} \left[\frac{\chi(p) - 1}{p-1} \left(\frac{p^{e_p} - \chi(p) p^{e_p}}{p - \chi(p)} + \frac{\chi(p) p^{e_p}}{p^2 - \chi(p)}\right)\right]$$

$$\prod_{2 \leq \nu_p(s) \leq e_p} \left[\frac{\chi(p) - 1}{p-1} \left(\frac{\chi(p) p^{\nu_p(s) - 1} - \chi(p) p^{\nu_p(s)}}{1 - \chi(p)/p} + \frac{p^{1-e_p} \chi(p) p^{e_p}}{p^2 - \chi(p)}\right)\right].$$

In particular $C_\chi(h, r, s) = cA_\chi = cC_\chi(1, d, 1)$, with $c \in \mathbb{Q}(\zeta_d)$.

ii) If $(r, s) > 1$, then $C_\chi(h, r, s) = 0$. 

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Remark. In this paper we only need to evaluate $C_\chi(h, r, s)$ in the case where the largest odd divisor of $s$ is squarefree and $\nu_2(s) \geq e_2 + 1$, in which case it assumes a bit simpler form than the general one given in Lemma 10.

Only the primes $p$ with $\chi(p) \neq 1$ contribute to $A_\chi$. Note that $A_\chi \neq 0$. If $\chi$ is a principal character, then $A_\chi = 1$. If $\chi$ is real, then $A_\chi > 0$. If $\chi$ is a Dirichlet character and $\chi'$ is its associated primitive Dirichlet character, then clearly $A_\chi$ equals $A_{\chi'}$ with some local factors involving only $\zeta_{o_\chi}$ left out. Hence $A_\chi = c' A_{\chi'}$ with $c' \in \mathbb{Q}(\zeta_{o_\chi})$. Also note that $\overline{A_\chi} = A_{\overline{\chi}}$.

The constants $A_\chi$ are the basic constants in this paper. They have a product expansion in terms of special values of Dirichlet $L$-series \[21\, 22\]. This result is related to the denominator identities arising in the theory of Lie algebras \[23\]. These expansions can be used to evaluate $A_\chi$ with high numerical accuracy \[21\]. The values of $A_{\phi_1}$ and $A_{\xi_1}$ given in the introduction are taken from Table 3 of \[21\].

Another result involving $h_\chi$ needed is the following (where the sums are over the integers $v \geq 1$).

**Lemma 11** Let $r, s$ be integers with $s|r$ and $g > 0$. Let $\chi$ be a Dirichlet character. Then, if $g > 0$ or $g < 0$ and $s$ is even,

$$
\sum_{(r,v)=1} \frac{h_\chi(v)}{[K_{sv,v} : \mathbb{Q}]} = \frac{1}{\varphi(s)} \left( C_\chi(h, r, 1) + C_\chi(h, r, \frac{n_s}{n_s, s}) \right).
$$

When $g < 0$ and $s$ is odd, the latter sum equals

$$
\frac{1}{\varphi(s)} \left( C_\chi(h, r, 1) - \frac{1}{2} C_\chi(h, r, 2) + \frac{1}{2} C_\chi(h, r, 2^{\nu_2(h)+1}) + C_\chi(h, r, \frac{n_s}{n_s, s}) \right).
$$

**Proof.** For $g > 0$ or $g < 0$ and $2|s$, the proof easily follows from the identity

$$
\sum_{(r,v)=1} \frac{h_\chi(v)}{[K_{sv,v} : \mathbb{Q}]} = \sum_{(r,v)=1} \frac{h_\chi(v)(h, v)}{v \varphi(sv)} + \sum_{n_s|sv} \frac{h_\chi(v)(h, v)}{v \varphi(sv)},
$$

which on its turn is an easy consequence of Lemma 11. The proof of the remaining case is similar. \qed

### 2.5 Preliminaries specific to the case $d = 4$

#### 2.5.1 Even order

Let $s \geq 2$. It is not difficult to estimate $N_g(1, 2^s; 0, 4)(x)$. To this end we consider the set of primes $p \equiv 1(\text{mod } 2^s)$ such that $\text{ord}_g(p) \neq 0(\text{mod } 4)$. Let $r = \nu_2(p - 1)$. Note that $r \geq s$. Now $p$ satisfies $\text{ord}_g(p) \neq 0(\text{mod } 4)$ if and only if $g^{(p-1)/2^{r-1}} \equiv 1(\text{mod } p)$, that is if and only if $p$ splits completely in $K_{2r-2r-1}$, but not completely in $K_{2r+1, 2r-1}$. By Chebotarev’s density theorem we expect that $N_g(1, 2^s; 0, 4)$ has a density as given in \[4\]. Indeed, this can be shown unconditionally. Wiertelak’s work \[33\, Theorem 2\] goes beyond this and shows that we even have an estimate for $N_g(1, 2^s; 0, 4)(x)$ as given by \[2\].

Similarly we can easily estimate $N_g(1, 2^s; 1, 4)(x) + N_g(1, 2^s; 3, 4)(x)$ which equals $N_g(1, 2^s; 1, 2)(x)$ and $N_g(1, 2^s; 2, 4)(x)$. See Theorem 11 for the outcome.
2.5.2 Odd order

We let \( \psi_0, \psi_1 \) denote the trivial respectively non-trivial character mod 4. The starting point of our analysis is the following easy result.

**Lemma 12** Let \( s \geq 1 \). For \( j \) odd we have

\[
N_g(1, 2^s; j, 4)(x) = \sum_{r \geq s} \sum_{t \equiv j \pmod{4}} \# \{ p \leq x : p \equiv 1 + 2^r \pmod{2^{r+2}}, \ r_g(p) = 2^r t \} \\
+ \sum_{r \geq s} \sum_{t \equiv -j \pmod{4}} \# \{ p \leq x : p \equiv 1 + 3 \cdot 2^r \pmod{2^{r+2}}, \ r_g(p) = 2^r t \},
\]

\[
N_g(3, 8; j, 4)(x) = \sum_{t \equiv j \pmod{4}} \# \{ p \leq x : p \equiv 3 \pmod{8}, \ r_g(p) = 2t \} 	ext{ and}
\]

\[
N_g(7, 8; j, 4)(x) = \sum_{t \equiv -j \pmod{4}} \# \{ p \leq x : p \equiv 7 \pmod{8}, \ r_g(p) = 2t \}.
\]

**Proof.** We only prove the assertion regarding \( N_g(1, 2^s; j, 4)(x) \), the other assertions being easier to prove. For every prime \( p \equiv 1 \pmod{2^s} \) there exists an unique \( r \geq s \) such that either \( p \equiv 1 + 2^r \pmod{2^{r+2}} \) or \( p \equiv 1 + 3 \cdot 2^r \pmod{2^{r+2}} \). We assume that we are in the first case, the other case being dealt with similarly. Using (11) we note that \( \text{ord}_g(p) \equiv j \pmod{4} \) if and only if \( r_g(p) = 2^r t \) for some \( t \geq 1 \) with \( t \equiv j \pmod{4} \). \( \Box \)

The densities of the sets appearing in Lemma 12 can be determined, under GRH, on invoking (6). Assuming the densities add up, we then arrive at the conjecture that the densities are as stated in Theorem 6 (cf. Section 4).

An alternative approach is obtained on first resumming the expressions in Lemma 12 and only then applying Lenstra’s machinery [17], which is what is explained next.

Let \( s \geq 1 \). From Lemma 12 and (11) we deduce that, when \( j \) is odd,

\[
N_g(1, 2^s; j, 4)(x) = \sum_{r \geq s} \sum_{t \equiv j \pmod{4}} \sum_{n=1}^{\infty} \mu(n)S_g(1 + 2^r, 2^{r+2}, nt2^r, nt2^r)(x)
\]

\[+ \sum_{r \geq s} \sum_{t \equiv -j \pmod{4}} \sum_{n=1}^{\infty} \mu(n)S_g(1 + 3 \cdot 2^r, 2^{r+2}, nt2^r, nt2^r)(x). \tag{10}\]

Note that the fields that arise in the \( S_g \) occurring in (10) are of the form \( K_{v2^r, v2^r} \). On grouping together the contributions involving the various \( K_{v2^r, v2^r} \) the triple sums can be reduced to double sums. To this end we first note that we can restrict to the case where \( n \) is odd, since if \( n \) is even and \( m \) is odd, \( S_g(1 + m \cdot 2^r, 2^{r+2}, nt2^r, nt2^r) \) is empty (then the condition on the Frobenius symbol implies that \( p \equiv 1 \pmod{2^{r+1}} \)). On putting \( v = nt \) the summation is then over all odd \( v \geq 1 \). As weighing factors we then get sums as in Lemma 9 with \((a, f) = (1, 4)\) and \((a, f) = (3, 4)\). On applying Lemma 9 and noting that for
odd $v$, $h_{\psi_0}(v) = 1$ if $v = 1$ and $h_{\psi_0}(v) = 0$ otherwise, we then obtain that $N_g(1, 2^s; j, 4)(x) = \frac{1}{2}I_1 + \frac{(-1)^{\nu_1}}{2}I_2$, where

$$I_1 = \sum_{r \geq s} \left[ \# \{ p \leq x : (p, K_{2^r, 2^r}/\mathbb{Q}) = id \} - \# \{ p \leq x : (p, K_{2^{r+1}, 2^r}/\mathbb{Q}) = id \} \right]$$

and

$$I_2 = \sum_{r \geq s} \sum_{2^uv} h_{\psi_1}(v) \left[ S_g(1 + 2^r, 2^{r+2}; v^{2r'}, v^{2r'})(x) - S_g(1 + 3 \cdot 2^r, 2^{r+2}; v^{2r'}, v^{2r'})(x) \right].$$

(Note that the latter double sum can be simplified to a single sum. On doing so we find that $I_2$ equals

$$\sum_{2^uv} h_{\psi_1}(w_{odd}) \left[ S_g(1 + 2^{\nu_2}(w), 2^{\nu_2(w)+2}; w, w)(x) - S_g(1 + 3 \cdot 2^{\nu_2}(w), 2^{\nu_2(w)+2}; w, w)(x) \right],$$

where $w_{odd}$ is the largest odd divisor of $w$.) Note that if we add $N_g(1, 2^s; 1, 4)(x)$ and $N_g(1, 2^s; 3, 4)(x)$ we obtain $I_1$, which is a well-known result.

If $s \geq 2$, then the Chebotarev density theorem implies, unconditionally, that for $v$ odd and $r \geq s$,

$$S_g(1 + 2^r, 2^{r+2}; v^{2r'}, v^{2r'})(x) \sim S_g(1 + 3 \cdot 2^r, 2^{r+2}; v^{2r'}, v^{2r'})(x), \quad \text{as } x \to \infty. \quad (11)$$

Thus we might expect that $I_2$ behaves like an error term and that, consequently, $N_g(1, 2^s; 1, 4)(x) \sim I_1/2$ as $x$ tends to infinity. Theorem 10 shows that this is indeed true, under GRH.

If $s = 1$, however, then (10) does not necessarily hold true for every $r \geq s$. It thus makes sense to consider $N_g(3, 4; j, 4)(x)$ for $j$ odd separately. We then obtain analogous expressions to those for $I_1$ and $I_2$, but instead of summing over $r \geq s$ we take $r$ to equal one:

$$N_g(3, 4; j, 4)(x) = \frac{1}{2} \# \{ p \leq x : p \equiv 3 \pmod{4}, \left( \frac{q}{p} \right) = 1 \} + \frac{(-1)^{\nu_1}}{2} \sum_{2^uv} h_{\psi_1}(v) \left[ S_g(3, 8; 2v, 2v)(x) - S_g(7, 8; 2v, 2v)(x) \right].$$

Using that $p$ splits completely in $\mathbb{Q}(\sqrt{-2})$ iff $p \equiv 1 \pmod{8}$ or $p \equiv 3 \pmod{8}$ and that $p$ splits completely in $\mathbb{Q}(\sqrt{2})$ iff $p \equiv \pm 1 \pmod{8}$, we obtain the following result. We present it with a more succinct proof. For a number field $L$ we let $\pi_L(x)$ denote the number of rational primes $p \leq x$ that split completely in $L$.

**Lemma 13** Let $j$ be odd. For every $x$ we have

$$N_g(3, 4; j, 4)(x) = \frac{1}{2} \# \{ p \leq x : p \equiv 3 \pmod{4}, \left( \frac{q}{p} \right) = 1 \} + \frac{(-1)^{\nu_1}}{2} \sum_{2^uv} h_{\psi_1}(v)A_v(x),$$

where $A_v(x) = \pi_{K_{2v, 2v}(\sqrt{-2})}(x) - \pi_{K_{2v, 2v}(\sqrt{2})}(x)$. 

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Proof. Let us consider only the case where \( j \equiv 1 \pmod{4} \), the remaining case being dealt with similarly. Note that in both the left hand side and the right hand side of the identity that is to be established only primes \( p \) satisfying \( p \equiv 3 \pmod{4} \) and \( \nu_p(g) = 0 \) are counted. Now let \( p \) be a prime such that \( p \equiv 3 \pmod{4} \), \( \nu_p(g) = 0 \) and \( p \leq x \). We shall show that it is counted with the same multiplicity in both the left and the right hand side of the identity that is to be established, thus finishing the proof.

Since by assumption \( \nu_p(g) = 0 \), there exists a largest integer \( k \) such that \( g^{\frac{k-1}{2}} \equiv 1 \pmod{p} \). Note that in both the left and the right hand side only primes \( p \) with \( k \) even are counted. Thus we may write \( k = 2k_1 \). Note that \( k_1 \) must be odd. Let us assume that \( p \equiv 3 \pmod{8} \). Then \( p \) is counted on the right hand side with weight

\[
\frac{1}{2} + \frac{1}{2} \sum_{m|k_1} h_{\psi_1}(m) = \frac{1}{2} + \frac{1}{2} (\psi_1 * \mu * 1)(k_1) = \frac{1 + \psi_1(k_1)}{2}.
\]

Thus the weight is 1 if \( k_1 \equiv 1 \pmod{4} \) and 0 otherwise. In other words the weight is 1 iff \( \ord_g(p) \equiv 1 \pmod{4} \).

The case where \( p \equiv 7 \pmod{8} \) is dealt with similarly.

Using Chebotarev’s density theorem we now expect that

\[
\lim_{x \to \infty} \sum_{2 \nmid \nu} \frac{h_{\psi_1}(v)A_v(x)}{\pi(x)} = \sum_{2 \nmid \nu} \left( \frac{h_{\psi_1}(v)}{[K_{2v,2v}(\sqrt{-2}) : \mathbb{Q}]} - \frac{h_{\psi_1}(v)}{[K_{2v,2v}(\sqrt{2}) : \mathbb{Q}]} \right).
\]

Let us denote the quantity on the right hand side by \( \tilde{\Delta}_g \). Note that

\[
\tilde{\Delta}_g = \frac{1}{2} \sum_{\sqrt{-2} \in K_{2v,2v} : \mathbb{Q}} h_{\psi_1}(v) - \frac{1}{2} \sum_{\sqrt{2} \in K_{2v,2v} : \mathbb{Q}} h_{\psi_1}(v) = \frac{\Delta_g}{2}.
\]

Theorem 3 shows that this heuristic holds true, under GRH.

3 The distribution of the index over residue classes

The problem of the distribution of the index over residue classes is far easier than that of the distribution of the order. However, the answers to both problems turn out to have some features in common.

Let \( a \) and \( d \) be integers. Under GRH it follows from Pappalardi’s work \[29\] that the density, \( \rho_g(a,d) \), of the set of primes \( p \) such that \( r_g(p) \equiv a \pmod{d} \) exists and equals

\[
\rho_g(a,d) = \sum_{t \equiv a \pmod{d}} \sum_{v=1}^{\infty} \frac{\mu(v)}{[K_{vt,vt} : \mathbb{Q}]}.
\]

Using this the following result is then easily deduced.
Theorem 5 (GRH). Let $a$ and $d$ be arbitrary natural numbers. Put $\delta = d/(a,d)$. Then the density of the primes $p$ with $r_g(p) \equiv a(mod \ d)$, $\rho_g(a,d)$, exists and satisfies

$$\rho_g(a,d) = \sum_{\chi \in G_d} c_\chi A_\chi, \text{ with } c_\chi \in \mathbb{Q}(\zeta_\chi).$$

Furthermore, $\zeta_\chi = \overline{\zeta_\chi}$. The number $c_\chi$ can be explicitly computed.

Proof. On putting $w = (a,d)$ and $\alpha = a/w$ we obtain

$$\rho_g(a,d) = \sum_{t \equiv \alpha(mod \ \delta)} \sum_{\mu(v)} \sum_{v=1}^{\infty} \mu(v) [K_{vt,vtw} : \mathbb{Q}].$$

Writing $vt = v_1$ and invoking Lemma 9 we then obtain

$$\rho_g(a,d) = 1/\varphi(\delta) \sum_{\chi \in G_d} \chi(\alpha) \sum_{v=1}^{\infty} h_\chi(v_1) [K_{v_1w,v_1w} : \mathbb{Q}].$$

Let $g > 0$. By Lemma 11 we have

$$\sum_{v=1}^{\infty} h_\chi(v) [K_{vw,vw} : \mathbb{Q}] = \sum_{v=1}^{\infty} h_\chi(v)(h, vw) \varphi(vw) + \sum_{v=1}^{\infty} h_\chi(v)(h, vw) \varphi(vw) = J_1 + J_2.$$

We can rewrite $J_1$ as

$$J_1 = (h, w) \sum_{v=1}^{\infty} h_\chi(v)(h, vw) \varphi(vw) / (h, w) \varphi(vw),$$

where the argument of the sum is easily seen to be multiplicative in $v$. If $p \nmid hw f_\chi$, then the local factor at $p$ in the Euler product for $J_1$ equals that of $A_\chi$ and so $J_1 = c_\chi A_\chi$, where $c_\chi \in \mathbb{Q}(\zeta_\chi)$. Rewriting the condition $n_1 | vw$ as $n_1 / (n_1, w) | v$, we see that also $J_2 = c_\chi A_\chi$, where $c_\chi \in \mathbb{Q}(\zeta_\chi)$. A similar argument can be used in the case where $g < 0$, cf. the proof of Lemma 11.

On using that $\overline{h_\chi} = h_\chi$ and $\overline{A_\chi} = A_\chi$ it follows that $\zeta_\chi = \overline{\zeta_\chi}$. \qed

A special case occurs when $d | a$. Then $\rho_g(a,d)$ is the density of primes $p \leq x$ such that the index of $g(mod \ p)$ is divisible by $d$, that is $\rho_g(a,d)$ is the density of primes $p \leq x$ that split completely in $K_{d,d}$. By an unconditional version of the Chebotarev density theorem we then infer

Proposition 2 The density $\rho_g(0,d)$ exists and satisfies

$$\rho_g(0,d) = 1/[K_{d,d} : \mathbb{Q}].$$

In the three examples some special cases of Theorem 5 are discussed.

Example 1. (GRH). We consider the case where $(a,d) = 1$ and $g > 0$. Then, using Lemma 11 we obtain that

$$\rho_g(a,d) = 1/\varphi(d) \sum_{\chi \in G_d} \chi(a) \left( C_\chi(h, 1, 1) + C_\chi(h, 1, n_1) \right).$$
Example 2. (GRH). We assume that \((a, d) = 1\) and \(g = 2\). Note that \(D = 8\) and hence \(n_1 = 8\). On invoking the formula of Example 1 with \(h = 1\) and \(n_1 = 8\), we obtain, using Lemma 10:

\[
\rho_2(a, d) = \frac{1}{\varphi(d)} \sum_{\chi \in \mathbb{G}_d} \chi(a) \cdot \delta_{\chi},
\]

with

\[
\delta_{\chi} = \left( \frac{1}{2} + \frac{\chi(2)((\chi(2))^2 - \chi(2) + 12)}{8(4 - \chi(2))} \right) \prod_{p > 2} \left( 1 + \frac{p(\chi(p) - 1)}{(p - 1)(p^2 - \chi(p))} \right).
\]

(This corrects a typo in Corollary 8 of [29]). Alternatively we can write

\[
\rho_2(a, d) = \frac{1}{\varphi(d)} \prod_{p \mid d} \left( 1 - \frac{1}{p(p-1)} \right) \sum_{\chi \in \mathbb{G}_d} \chi(a) \left( 1 + \frac{\chi(4)(\chi(2) - 1)}{8(\chi(2) + 2)} \right) A_{\chi}.
\]

Example 3 (GRH). Let \(g = 2\) and \(d = 3\). Using Example 2 we compute \(\rho_2(\pm 1, 3) = \frac{5}{12} \pm \frac{5}{16} A_{\xi_1}\), that is \(\rho_2(1, 3) = 0.471034\ldots\) and \(\rho_2(2, 3) = 0.362298\ldots\).

(This corrects the values claimed in [29, p. 386].) From the latter two values we infer that \(\delta_2(0, 3) = 1/6\). By Proposition 2 it follows that we even have unconditionally that \(\delta_2(0, 3) = 1/6\). Up to \(p_{10^6} = 1299709\) the approximations \(0.16589, 0.47127\) and \(0.36283\) for \(\rho_2(0, 3), \rho_2(1, 3)\) respectively \(\rho_2(2, 3)\) are found.

4 Proof of the main results

In our formulation of the main results stated in the introduction, we have added the (known) case \(d = 2\) for completeness. As already pointed out in Section 2.5.1 these cases have been well studied. The best known error terms are due to Wiertelak, cf. [34]. The densities for these cases can be explicitly evaluated using Lemma 1. Since the interested reader can easily carry this out herself, we abstained from writing down the rather lengthy (because of case distinctions) outcome. For some further elaboration on these cases see Section 2.5.1.

Our proof for the remaining cases has an analytic and algebraic component, with the analytic component being captured by the following result.

Theorem 6 (GRH). Let \(s \geq 1\). Let \(\psi_1\) be the non-principal character modulo 4. For \(r \geq 1\) let \(\sigma_{1,r}, \sigma_{-1,r} \in \text{Gal}(\mathbb{Q}(\zeta_{2^r+2})/\mathbb{Q})\) be the automorphisms that are uniquely determined by \(\sigma_{1,r}(\zeta_{2^r+2}) = \zeta_{2^r+2}^{1+3^2r}\), respectively \(\sigma_{-1,r}(\zeta_{2^r+2}) = \zeta_{2^r+2}^{1+3^2r}\).

For \(j = -1\) and \(j = 1\) let

\[
c_j(r, tn) = \begin{cases} 1 & \text{if } \sigma_{j,r} \mid_{\mathbb{Q}(\zeta_{2^r+2})} \cap \mathbb{Q}(\zeta_{2^r+2}^{1/2^{r+1}n}) = \text{id}; \\ 0 & \text{otherwise}. \end{cases}
\]

For \(j = 1\) and \(j = 3\) we have

\[
N_g(1, 2^s; j, 4)(x) = \delta_g(1, 2^s; j, 4) \frac{x}{\log x} + O \left( \log |g_1 g_2| \frac{x}{\log^{3/2} x} \right),
\]

\[
T_g(1, 2^s; j, 4)(x) = \delta_g(1, 2^s; j, 4) \frac{x}{\log x} + O \left( \log |g_1 g_2| \frac{x}{\log^{3/2} x} \right).
\]
From Lemma 12 and (4) we deduce that
\[ N \]
and the implied constant is absolute.

An heuristic argument in favour of the truth of the latter theorem is easily given. From Lemma 12 and (4) we deduce that \( N_g(1, 2^s; j, 4)(x) \) equals
\[
\sum_{r \geq s} \sum_{t \equiv j \pmod{4}} \sum_{n=1}^{\infty} \mu(n)\{ p \leq x : p \equiv 1 + 2^r \pmod{2^r+2}, (p, K_{2^r t, 2^r t}/Q) = id \} + \\
\sum_{r \geq s} \sum_{t \equiv -j \pmod{4}} \sum_{n=1}^{\infty} \mu(n)\{ p \leq x : p \equiv 1 + 3 \cdot 2^r \pmod{2^r+2}, (p, K_{2^r t, 2^r t}/Q) = id \}.
\]
The density of the inner sums is given, under GRH, by (4). Assuming the densities add up and there is sufficiently cancellation in the error terms, we then arrive at the heuristic that the densities should be as claimed. Our proof of Theorem 6 is in the same spirit:

**Proof of Theorem 6**. Let us denote the triple sum in the formulation of the result by \( \sum \sum \sum \). All constants implied by the O-symbols in this proof will be absolute. The first formula of Lemma 12 can be more compactly written as
\[ N_g(1, 2^s; j, 4)(x) = \sum_{r \geq s} \sum_{2^r} R_g(1 + (2 - \psi_1(jt))2^r, 2^{r+2}, 2^r t)(x). \]

On retaining only the primes with \( r_g(p) \leq y \), we obtain
\[ N_g(1, 2^s; j, 4)(x) = T_1(y) + O(\#\{ p \leq x : r_g(p) \geq y \}), \tag{12} \]
where
\[ T_1(y) = \sum_{r \geq s} \sum_{2^r \leq y} R_g(1 + (2 - \psi_1(jt))2^r, 2^{r+2}, 2^r t)(x). \]
The function \( R_g(1 + (2 - \psi_1(jt))2^r, 2^{r+2}, 2^r t)(x) \) can be estimated in the same way as \( R_g(0, 1; t)(x) \) (see the proof of Theorem 4). We find that, under GRH,
\[
R_g(1 + (2 - \psi_1(jt))2^r, 2^{r+2}, 2^r t)(x) = \text{Li}(x) \sum_{n=1}^{\infty} \frac{\mu(n)\delta_{\psi_1(jt)}(r, t)}{|Q(\zeta_{2^{r+2}}, \zeta_{2^r t}, g^{1/2^r t}) : Q|} \\
+ O \left( \frac{\log(|g_1 g_2|) \cdot x}{\log^2 x} \right) + O \left( \frac{x \log \log x}{\phi(2^r t) \log^2 x} \right).
\]
This, when substituted in (12), yields
\[
T_1(y) = \text{Li}(x) \sum_{r \geq s} \sum_{2^r \leq y} + O \left( \frac{h \text{Li}(x)}{y} \right) + O \left( \frac{x \log y \log \log x}{\log^2 x} \right) + O \left( \frac{xy \log |g_1 g_2|}{\log^2 x} \right),
\]

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where we used \( \sigma \) and

\[
\sum_{r \geq s} \sum_{2^t t \geq y} \sum_{n=1}^{\infty} \, \frac{h}{2^t t n \varphi(2^t t n)} = O \left( \sum_{m \geq y} \sum_{n=1}^{\infty} \frac{h}{mn \varphi(mn)} \right) = O \left( \frac{h}{y} \right),
\]

cf. the proof of Lemma \( \text{[8]} \). On taking \( y = \sqrt{\log x} \) in \( \text{[12]} \), the result then follows on invoking Lemma \( \text{[7]} \) with \( \psi(x) = \sqrt{\log x} \).

The algebraic part is a consequence of the following result.

**Lemma 14** Let \( n \) be squarefree and \( t \) be odd. Denote the intersection of \( \mathbb{Q}(\zeta_8) \) and \( K_{2nt,nt} \) by \( L_{nt} \).

i) If \( r = 1, 2 \mid h n, 8 \mid D \) and \( D \mid 8 n t \), then

\[
L_{nt} = \mathbb{Q}(\sqrt{2} \cdot \text{sgn}(g)), \quad c_1(r, t n) = \frac{1 - \text{sgn}(g)}{2}, \quad c_{-1}(r, t n) = \frac{1 + \text{sgn}(g)}{2},
\]

if \( D \equiv 8(\text{mod } 32) \) and

\[
L_{nt} = \mathbb{Q}(\sqrt{2} \cdot \text{sgn}(-g)), \quad c_1(r, t n) = \frac{1 + \text{sgn}(g)}{2}, \quad c_{-1}(r, t n) = \frac{1 - \text{sgn}(g)}{2},
\]

if \( D \equiv 24(\text{mod } 32) \).

ii) We have \( c_1(r, t n) \neq c_{-1}(r, t n) \) if and only if \( r = 1, 2 \mid h n, 8 \mid D \) and \( D \mid 8 n t \).

**Proof.** i). Under the hypothesis of part i) we can apply Lemma \( \text{[11]} \) to infer that \( L_{nt} \), the field intersection in the definition of \( c_{\pm 1}(1, t n) \), equals \( \mathbb{Q}(\sqrt{2} \cdot \text{sgn}(g)) \) if

\( D \equiv 8(\text{mod } 32) \) and \( \mathbb{Q}(\sqrt{2} \cdot \text{sgn}(-g)) \) in the remaining case \( D \equiv 24(\text{mod } 32) \).

Writing \( \sqrt{2} = \zeta_8 + \zeta_8^{-1} \) one calculates that \( \sigma_{\pm 1,1}(\sqrt{2}) = \mp \sqrt{2} \), \( \sigma_{\pm 1,1}(\sqrt{-2}) = \pm \sqrt{-2} \). From this observation and the definition of \( c_{\pm 1}(r, t n) \) the result follows at once.

ii). ‘\( \leftarrow \)’. Follows by part i). ‘\( \Rightarrow \)’. The intersection of the fields \( \mathbb{Q}(\zeta_2^{2r+2}) \) and \( \mathbb{Q}(\zeta_8, g^{1/2} t n) \) is abelian and, since \( 4 \mid nt \), is contained in \( \mathbb{Q}(\zeta_{2r+1}, \sqrt{2}) \) or \( \mathbb{Q}(\zeta_{2r+1}, \sqrt{-2}) \). As \( \sigma_{1,r}(\zeta_{2r+1}) = \sigma_{-1,r}(\zeta_{2r+1}) \) for every \( r \geq 1 \), we deduce that \( \mathbb{Q}(\zeta_2^{2r+2}) \cap \mathbb{Q}(\zeta_8, g^{1/2} t n) \) must contain at least one element from \( \{\sqrt{-2}, \sqrt{2}\} \).

Now let us consider how \( \sigma_{1,r} \) and \( \sigma_{-1,r} \) act on \( \sqrt{2} = (\zeta_8 + \zeta_8^{-1}) \). For \( r \geq 2 \) we have \( \sigma_{1,r}(\sqrt{2}) = \sigma_{-1,r}(\sqrt{2}) \), since then \( \sigma_{1,r}(\zeta_8) = \sigma_{-1,r}(\zeta_8) \). Thus we must have \( r = 1 \). If \( n \) is even, then \( i \in \{\zeta_2^{2r+2} \cap \mathbb{Q}(\zeta_8, g^{1/2} t n) \} \) and since \( \sigma_{1,1}(i) = -i \), we infer that \( c_{\pm 1}(r, t n) = 0 \). Thus \( n \) is odd.

From the above discussion it follows that \( \mathbb{Q}(\zeta_8) \cap \mathbb{Q}(\zeta_{2nt}, g^{1/2}nt) \) with \( nt \) odd must contain \( \sqrt{2} \) or \( \sqrt{-2} \). By Lemma \( \text{[11]} \) this leads then to the further restrictions (apart from \( r = 1 \) and \( 2 \nmid n \)): \( 2 \mid h, 8 \mid D \) and \( D \mid 8 nt \).

**Proof of Theorems** \( \text{[1]} \) and \( \text{[2]} \). The claims in the cases where the order is even or \( 1(\text{mod } 2) \) are already known. The results for the remaining cases are a straightforward consequence of Theorem \( \text{[3]} \) and Lemma \( \text{[14]} \). If \( s \geq 2 \), then \( r \geq 2 \) in the triple sum for the densities. By Lemma \( \text{[14]} \) we then have that \( c_{-1}(t, r n) = c_1(t, r n) \) and hence \( \delta_g(1, 2^s; 1, 4) = \delta_g(1, 2^s; 3, 4) \). On noting that \( N_g(1, 2^s; 1, 4)(x) + N_g(1, 2^s; 3, 4) = N_g(1, 2^{s+1}; 1, 2)(x) \) the proof of Theorem \( \text{[1]} \) is
then completed.

Since for ‘most’ \( t \) and \( n \) we have \( c_1(1, tn) = c_{-1}(1, tn) \) (Lemma 14) it is natural to compute the difference \( \delta_g(3, 4; 1, 4) - \delta_g(3, 4; 3, 4) \). Since \( \delta_g(3, 4; 1, 4) + \delta_g(3, 4; 3, 4) \) is easily evaluated, we are then done. We proceed by filling in the details.

Proceeding as in the proof of Theorem 6 we infer that

\[
\delta_g(3, 4; j, 4) = \delta_g(1, 2; j, 4) - \delta_g(1, 4; j, 4) = \sum_{\substack{t=1 \atop 2\nmid t}}^{\infty} \sum_{n=1}^{\infty} \frac{\mu(n)c_{\psi_1(it)}(1, tn)}{[\mathbb{Q}(\zeta_8, \zeta_{2tn}, g^{1/2tn}) : \mathbb{Q}]}
\]

(now only the terms with \( r = 1 \) contribute and hence the triple sum is reduced to a double sum). From the latter formula and the fact that \( c_1(1, tn) = c_{-1}(1, tn) \) for \( t \) is odd and \( n \) is even (see Lemma 14), we infer that

\[
\delta_g(3, 4; 1, 4) - \delta_g(3, 4; 3, 4) = \sum_{\substack{2\nmid v \atop 2\nmid n}} \frac{\mu(n)c_{\psi_1(v)}(1, tn) - c_{-\psi_1(v)}(1, tn)}{[\mathbb{Q}(\zeta_8, \zeta_{2v}, g^{1/2v}) : \mathbb{Q}]}
\]

On writing \( nt = v \) we obtain

\[
\delta_g(3, 4; 1, 4) - \delta_g(3, 4; 3, 4) = \sum_{\substack{2\nmid v \atop 2\nmid n}} \frac{h_{\psi_1(v)}[c_1(1, v) - c_{-1}(1, v)]}{[\mathbb{Q}(\zeta_8, \zeta_{2v}, g^{1/2v}) : \mathbb{Q}]}
\]

On invoking Lemma 14 the latter sum is seen to equal \( \Delta_g/2 \). We now infer from Theorem 6 that

\[
N_g(3, 4; 1, 4)(x) - N_g(3, 4; 3, 4)(x) = \frac{\Delta_g}{2} \text{Li}(x) + O\left(\log \left|g_1g_2\right| \frac{x}{\log^{3/2} x}\right),
\]

where the implied constant is absolute. On noting that

\[
N_g(3, 4; 1, 4)(x) + N_g(3, 4; 3, 4)(x) = \#\{p \leq x : p \equiv 3(\text{mod } 4), \left(\frac{g}{p}\right) = 1\},
\]

the result easily follows.

5 Explicit evaluation of the densities

In this section we explicitly evaluate the densities (under GRH), computed in Theorem 1 and Theorem 2.

Using Lemma 11 Theorem 14 can be made completely explicit. For reasons of space we restrict ourselves to describing the situation for a ‘generic’ \( g \).

Theorem 7 (GRH). Let \( s \geq 2 \). If \( h \) is odd and \( D \) contains an odd prime factor, then

\[
\delta_g(1, 2^s; j, 4) = \begin{cases} 2^{1-s} - \frac{2}{3} \cdot 4^{1-s} & \text{if } j = 0; \\ 4^{1-s}/6 & \text{if } j = 1; \\ 4^{1-s}/3 & \text{if } j = 2; \\ 4^{1-s}/6 & \text{if } j = 3. \end{cases}
\]
Proof. The conditions on \( h \) and \( D \) ensure that the degrees \([K_{m,n} : \mathbb{Q}]\) occurring in the sums in Theorem 1 are equal \( \varphi(m)n \). It then remains to sum some geometric series.

Theorem 2 shows that the sets \( N_g(3,4; j, 4) \) considered there have a density, \( \delta_g(3,4; j, 4) \), under GRH. The case where \( j \) is even is trivial and left to the reader.

**Theorem 8 (GRH).** Let \( g \in \mathbb{Q}\setminus\{-1, 0, 1\} \), Write \( g = \pm g_0^h \), where \( g_0 > 0 \) is not a power of a rational number. For any prime \( p \) define \( e_p \) by \( p^{e_p} || h \).

*If \( h \) is even, then \( \delta_g(3,4; j, 4) = (1 + \text{sgn}(g))/8.\)

Next, let \( h \) be odd. Then \( \delta_g(3,4; 1, 4), \delta_g(3,4; 3, 4) = 1/8, \) unless \( D \), the discriminant of the quadratic field \( \mathbb{Q}(\sqrt{g_0}) \), is divisible by 8 and has no prime divisor congruent to 1(mod 4), in which case we have

\[
\delta_g(3,4; j, 4) = \frac{1}{8} + \text{sgn}(g) \frac{(-1)^{j+1}}{8} P_1P_2P_3,
\]

where

\[
P_1 = \prod_{p \equiv j \pmod{8}, p || h, p \nmid 3} \left( \frac{2[p^{e_p} - (-1)^{e_p}]}{p^{e_p-1}(p^2-1)} + \frac{2p(-1)^{e_p}}{p^{e_p}(p^2+1)(p-1)} \right),
\]

\[
P_2 = \prod_{p \equiv j \pmod{8}, p || h, p \equiv 3 \pmod{4}} \left( 1 - \frac{2[p^{e_p} - (-1)^{e_p}]}{p^{e_p-1}(p^2-1)} + \frac{2p(-1)^{e_p+1}}{p^{e_p}(p^2+1)(p-1)} \right), \text{ and}
\]

\[
P_3 = \prod_{p \equiv j \pmod{8}, p || h, p \equiv 3 \pmod{4}} \left( 1 - \frac{2p}{(p^2+1)(p-1)} \right),
\]

and thus in particular \( \text{sgn} (\delta_g(3,4; 3, 4) - \delta_g(3,4; 1, 4)) = \text{sgn}(g) \), since the local factors of \( P_1P_2P_3 \) are all positive.

**Corollary 2 (GRH).** For \( j = 0, 1, 2, 3 \) we have \( \delta_g(3,4; j, 4) = c_1(j) + c_2(j)A_{\psi_1} \) with \( c_1(j) \geq 0 \) and \( c_2(j) \) rational numbers. We have \( c_2(0) = c_2(2) = 0 \) and \( c_2(1) = -c_2(3) \). For a ‘generic’ \( g \) all \( c_2(j) \) will be zero.

**Remark.** Using the result going back to Landau that there are \( O(x/\sqrt{\log x}) \) integers \( n \leq x \) having only prime divisors \( p \) with \( p \equiv 3(\text{mod}\ 4) \) [15] pp. 641-669, it is easily inferred that the number of non-generic integers \( g \) with \( |g| \leq x \) is \( O(x/\sqrt{\log x}) \).

**Corollary 3 (GRH).** Suppose that \( h = 1 \). Then \( \delta_g(3,4; 1, 4) = \delta_g(3,4; 3, 4) = 1/8 \) unless \( D \) is divisible by 8 and has no prime divisor congruent to 1(mod 4), in which case we have

\[
\delta_g(3,4; j, 4) = \frac{1}{8} + \text{sgn}(g)A_{\psi_1} \frac{(-1)^{j+1}}{8} \prod_{p \equiv j} \frac{2p}{p^3 - p^2 - p - 1}.
\]
Corollary 4 (GRH). Let \( j \) be odd. Then
\[
\delta_g(3,4;j,4) + \delta_{-g}(3,4;j,4) = \frac{1}{4}.
\]

An alternative proof of the latter corollary is obtained on combining Proposition 1 with Theorem 2.

Proof of Theorem 8. The proof is easily deduced from Theorem 2.

If \( h \) is even, then \( \Delta_g = 0 \) and the result follows on noticing that the density of the set of primes \( p \leq x \) with \( p \equiv 3 \pmod{4} \) and \( (g/p) = 1 \) equals \((1 + \text{sgn}(g))/4\).

Next assume that \( h \) is odd. Then the set appearing in the formula for \( N_g(3,4;j,4)(x) \) given in Theorem 2 has density \( 1/4 \). If \( 8 \nmid D \) it follows from Lemma 4 that the summation conditions are never met and hence \( \Delta_g = 0 \). So we may assume that \( 8 \mid D \). By Lemma 4 the \( v \)'s appearing in the two summations are divisible by \( D/8 \) and thus if \( D \) has a prime divisor \( p \) with \( p \equiv 1 \pmod{4} \), then \( h_{\psi_1}(D/8) = 0 \) by Lemma 8 and hence \( h_{\psi_1}(v) = 0 \) for these \( v \), which shows that \( \Delta_g = 0 \).

It remains to deal with the case where \( 8 \mid D \) and \( D \) contains no prime divisor \( p \) with \( p \equiv 1 \pmod{4} \). Using Lemma 1 and Lemma 4 we infer that
\[
\Delta_g = \text{sgn}(g)(-1)^{D/8} \frac{P_1 P_2 P_3}{2} \prod_{p \mid D \text{ and } p \equiv 3 \pmod{4}} (-1) = -\text{sgn}(g) \frac{P_1 P_2 P_3}{2}.
\]

An easy analysis shows that the local factors in the products \( P_1 \), \( P_2 \) and \( P_3 \) are all non-negative.

\[ \square \]

6 Modulus 3

The case \( d = 3 \) can be dealt with along the lines of the case \( d = 4 \), hence we suppress most details of the proofs. Our starting point is the following analog of Theorem 6.

Theorem 9 (GRH). Let \( \xi_0, \xi_1 \) be the principal, respectively non-principal character modulo 3. For \( r \geq 1 \) let \( \sigma_{1,r}, \sigma_{-1,r} \in \text{Gal}(\mathbb{Q}(\zeta_{3^r+1})/\mathbb{Q}) \) be the automorphisms that are uniquely determined by \( \sigma_{1,r}(\zeta_{3^r+1}) = \zeta_{3^{r+1}+3^r}, \text{ respectively } \sigma_{-1,r}(\zeta_{3^r+1}) = \zeta_{3^{r+1}+2 \cdot 3^r} \). For \( j = -1 \) and \( j = 1 \) let
\[
c_j(r, tn) = \begin{cases} 1 & \text{if } \sigma_{j,r} |_{\mathbb{Q}(\zeta_{3^r+1}) \cap \mathbb{Q}(\zeta_{3^r+1} g^{1/3^r} \sqrt{tn})} = \text{id}; \\ 0 & \text{otherwise.} \end{cases}
\]

Let \( s \geq 1 \). For \( j = 1 \) and \( j = 2 \) we have
\[
N_g(1, 3^s; j, 3)(x) = \delta_g(1, 3^s; j, 3) \frac{x}{\log x} + O \left( \log |g_1 g_2| \frac{x}{\log^{3/2} x} \right),
\]

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where

\[ \delta_g(1, 3^s; j, 3) = \sum_{r \geq s} \sum_{t \equiv 1 \pmod{3}} \sum_{n=1}^{\infty} \frac{\mu(n)c'_\xi(jt)(r, tn)}{[Q(\zeta^{3^r+1}, \zeta^{3^rtn}, g^{1/3^rtn}) : Q]}, \]

and the implied constant is absolute.

It is very easy to see that \( c'_1(r, tn) = c'_{-1}(r, tn) \) for \( r \geq 1 \) and thus for \( s \geq 1 \) we infer that, under GRH, \( \delta_g(1, 3^s; 1, 3) = \delta_g(1, 3^s; 2, 3) \). Since (unconditionally)

\[ \delta_g(1, 3^s; 0, 3) = \frac{3^{1-s}}{2} - \sum_{r \geq s} \left( \frac{1}{[K_{3^r, 3^r} : Q]} - \frac{1}{[K_{3^r+1, 3^r} : Q]} \right), \]

we then easily deduce, using Lemma 1, the following result.

**Theorem 10 (GRH).** Let \( e_3 = \nu_3(h) \) and \( s \geq 1 \). If \( e_3 \leq s \), then

\[ \begin{cases} 
\delta_g(1, 3^s; 0, 3) = \frac{3^{1-s}}{2} - 3^{2+e_3-2s}/8; \\
\delta_g(1, 3^s; 1, 3) = \frac{3^{2+e_3-2s}}{16}; \\
\delta_g(1, 3^s; 2, 3) = \frac{3^{2+e_3-2s}}{16}.
\end{cases} \]

If \( e_3 > s \), then

\[ \begin{cases} 
\delta_g(1, 3^s; 0, 3) = \frac{3^{1-e_3}}{8}; \\
\delta_g(1, 3^s; 1, 3) = \frac{3^{1-s}}{4} - 3^{1-e_3}/16; \\
\delta_g(1, 3^s; 2, 3) = \frac{3^{1-s}}{4} - 3^{1-e_3}/16.
\end{cases} \]

The reason that we cannot take \( s = 0 \) in Theorem 9 is that \( \sigma'_{-1,0} \) does not give rise to an automorphism of \( Q(\zeta_3) \). On the other hand \( \sigma'_{1,0} \) does and thus we can define \( c'_1(0, tn) \) as in Theorem 9.

Let \( j \in \{1, 2\} \). Since \( N_g(0, 1; j, 3)(x) = N_g(1, 3; j, 3)(x) + N_g(2, 3; j, 3)(x) + O(1) \), and \( N_g(1, 3; j, 3)(x) \) is covered in Theorem 9 it remains to deal with \( N_g(2, 3; j, 3)(x) \). Note that

\[ N_g(2, 3; j, 3)(x) = \sum_{t \equiv j \pmod{3}} \#\{p \leq x : p \equiv 2 \pmod{3}, r_g(p) = t}. \]

Reasoning as in Theorem 9 we then find that, under GRH, we have

\[ N_g(2, 3; j, 3)(x) = \delta_g(2, 3; j, 3) \frac{x}{\log x} + O \left( \log |g_{1, 2}| \frac{x}{\log^{3/2} x} \right), \tag{13} \]

where

\[ \delta_g(2, 3; j, 3) = \sum_{t \equiv j \pmod{3}} \sum_{3|\nu} \frac{\mu(n)c'_\xi(0, tn)}{[Q(\zeta_3, \zeta^{3\nu}, g^{1/3\nu}) : Q]} \]

\[ = \frac{1}{2} \sum_{3|\nu} \frac{(h_{\xi_0}(v) + \xi_1(j)h_{\xi_1}(v))c'_\xi(0, v)}{[Q(\zeta_3, \zeta^{1/v}, g^{1/v}) : Q]} \]

\[ = \frac{1}{4} + \frac{\xi_1(j)}{4} \sum_{3|\nu, \zeta_3 \notin K_{v, v}} \frac{h_{\xi_1}(v)}{[K_{v, v} : Q]}, \]

and the implied constant in (13) is absolute. In the derivation of the second equality we used Lemma 9 and in the derivation of the latter equality we used the trivial observation that \( c'_1(0, v) = 1 \) if \( \zeta_3 \notin K_{v, v} \). To sum up we obtained the following theorem.
Theorem 11 (GRH). The estimate holds with
\[ \delta_g(2, 3; j, 3) = \frac{\xi_0(j)}{4} + \frac{\xi_1(j)}{4} \sum_{\zeta_3 \not\in K_{v,v}} \frac{h_{\xi_1}(v)}{[K_{v,v} : Q]} \]
and an absolute implied constant.

(Since the condition \( \zeta_3 \not\in K_{v,v} \) implies \( 3 \not| v \), the latter condition can be dropped, in principle.) Theorem 11 is the density version of the following lemma.

Lemma 15 The quantity \( N_g(2, 3; j, 3)(x) \) equals
\[ \frac{\xi_0(j)}{2} \# \{ p \leq x : p \equiv 2 \pmod{3} \} + \frac{\xi_1(j)}{2} \sum_{3 | v} h_{\xi_1}(v)[\pi_{K_{v,v}}(x) - \pi_{K_{3v,v}}(x)]. \]

Proof. Similar to that of Lemma 13.

On noting that
\[ \frac{1}{2} \sum_{\zeta_3 \not\in K_{v,v}} \frac{h_{\xi_1}(v)}{[K_{v,v} : Q]} = \sum_{3 | v} h_{\xi_1}(v) \left( \frac{1}{[K_{v,v} : Q]} - \frac{1}{[K_{3v,v} : Q]} \right), \]
and invoking Lemma 11 and Lemma 10 we obtain the following three colloraries of Theorem 11.

Corollary 5 (GRH). Put \( \epsilon = 1 \) if \( 3 \not| D \) and \( \epsilon = -1 \) otherwise. If \( g > 0 \), then
\[ \delta_g(2, 3; j, 3) = \frac{\xi_0(j)}{4} + \frac{\xi_1(j)}{4} \left( C_{\xi_1}(h, 3, 1) + \epsilon C_{\xi_1}(h, 3, \frac{n_1}{(3, n_1)}) \right). \]
If \( g < 0 \), then \( \delta_g(2, 3; j, 3) \) equals
\[ \frac{\xi_0(j)}{4} + \frac{\xi_1(j)}{4} \left( C_{\xi_1}(h, 3, 1) - \frac{C_{\xi_1}(h, 3, 2)}{2} + \frac{C_{\xi_1}(h, 3, 2^{\nu_2(h)+1})}{2} + \epsilon C_{\xi_1}(h, 3, \frac{n_1}{(n_1, 3)}) \right). \]

Corollary 6 (GRH). Recall that \( e_p = \nu_p(h) \). Define \( \Omega(n) = \sum_{p | n} \nu_p(n) \). Define
\[ P'_1 = \prod_{p | D, p \equiv 2 \pmod{3}} \left( 2p^{-\nu_p} - (1)^{\nu_p} \right) \left( \frac{2p(-1)^{e_p}}{p^{e_p-1}(p^2-1)} + \frac{2p(-1)^{e_p}}{p^{e_p}(p^2+1)(p-1)} \right) \quad \text{and} \]
\[ P'_2 = \prod_{p \equiv 2 \pmod{3}} \left( 1 - \frac{2p(-1)^{e_p}}{p^{e_p-1}(p^2-1)} + \frac{2p(-1)^{e_p+1}}{p^{e_p}(p^2+1)(p-1)} \right). \]

Put \( \epsilon_1 = 0 \) if \( D \) has a prime divisor \( q \) that satisfies \( q \equiv 1 \pmod{3} \) and \( \epsilon_1 = 1 \) otherwise. If \( g > 0 \), then
\[ \delta_g(2, 3; j, 3) = \frac{\xi_0(j)}{4} + \frac{\xi_1(j)}{4} C_{\xi_1}(h, 3, 1) + \epsilon_1 \frac{\xi_1(j)}{5} (-1)^{\Omega(n_1)2^{e_2}+2-2^{e_2}(n_1)} P'_1 P'_2. \]
If \( g < 0 \), then
\[ \delta_g(2, 3; j, 3) = \frac{\xi_0(j)}{4} + \frac{\xi_1(j)}{4} \left( 1 - \frac{[2^{e_2} - (1)^{e_2}]}{3 \cdot 2^{e_2-1}} - \frac{2^{e_2-2}(-1)^{e_2}}{5} \right) C_{\xi_1}(h, 6, 1) + \epsilon_1 \frac{\xi_1(j)}{5} (-1)^{\Omega(n_1)2^{e_2}+2-2^{e_2}(n_1)} P'_1 P'_2. \]
Corollary 7 (GRH). Suppose $h = 1$. We have
\[
\delta_g(2, 3; j, 3) = \frac{\xi_0(j)}{4} + \frac{\xi_1(j)}{4} A \xi_1 \left( 1 + \epsilon_1(-1)^{\Omega(n)} 2^{4 - 2\nu_2(n)} \prod_{p \mid D \atop p > 3} \frac{2p}{p^3 - p^2 - p - 1} \right).
\]

Remark. Note that
\[
C_\xi(h, 3, 1) = \prod_{p \equiv 2 (\text{mod } 3)} \left( 1 - \frac{2[p^p - (-1)^{p+1}]}{p^{p+1} - 1} + \frac{2p(-1)^{p+1}}{p^p(p^2 + 1)(p-1)} \right).
\]

A somewhat tedious analysis of Corollary 6 together with Theorem 10 yields the following size comparison of $\delta_g(2, 3; 1, 3)$ with $\delta_g(2, 3; 2, 3)$ and of $\delta_g(1, 3)$ with $\delta_g(2, 3)$.

Proposition 3 (GRH). If $g > 0$ and $h$ is even, then $\delta_g(2, 3; 1, 3) \leq \delta_g(2, 3; 2, 3)$, otherwise $\delta_g(2, 3; 1, 3) \geq \delta_g(2, 3; 2, 3)$. We have $\delta_g(2, 3; 1, 3) = \delta_g(2, 3; 2, 3)$ iff $Q(\sqrt{\eta}) = Q(\sqrt{3})$ and $\nu_2(h) \in \{0, 2\}$. The same result holds with $\delta_g(2, 3; j, 3)$ replaced by $\delta_g(j, 3)$.

7 On the generic behaviour of $\delta_g(a, d), d = 3, 4$

If $g$ is not a square or -1, then an old heuristic model predicts that the number of primes $p \leq x$ such that $g$ is a primitive root mod $p$ should be asymptotically equal to $\sum_{p \leq x} \varphi(p-1)/(p-1)$, where $\varphi(p-1)/(p-1)$ is the density of primitive roots in $\mathbb{Z}_p^*$. It is easily proved, see e.g. [20], that on average $\varphi(p-1)/(p-1)$ is equal to the Artin constant $A$, that is
\[
\lim_{x \to \infty} \frac{1}{\pi(x)} \sum_{p \leq x} \frac{\varphi(p-1)}{p-1} = A = 0.37395 \ldots.
\]

From the work of Hooley [11] it can be deduced that under GRH for a positive proportion of all $g$ the above heuristic is false.

Let $\delta(p; a, d) = \sum_{r \equiv a (\text{mod } d)} \varphi(r)/(p-1)$, then $\delta(p; a, d)$ is the density of elements in the multiplicative group of the finite field $\mathbb{F}_p$ with order congruent to $a(\text{mod } d)$. A (naive) heuristic prediction for $N_g(a, d)(x)$ is then provided by
\[
\sum_{p \leq x} \delta(p; a, d). It can be shown that $\lim_{x \to \infty} \sum_{p \leq x} \delta(p; a, d)/\pi(x) = \delta(a, d)$ exists [21]. For $d = 3, 4$ some computation [21] shows that
\[
\delta(a, 3) = \begin{cases} \frac{3}{8} & \text{if } a \equiv 0 (\text{mod } 3); \\ \frac{5}{16} + \frac{A \xi_1}{4} & \text{if } a \equiv 1 (\text{mod } 3); \\ \frac{5}{16} - \frac{A \xi_1}{4} & \text{if } a \equiv 2 (\text{mod } 3); \end{cases}
\]

and
\[
\delta(a, 4) = \begin{cases} \frac{1}{8} & \text{if } a \text{ is even}; \\ \frac{3}{6} & \text{if } a \text{ is odd}. \end{cases}
\]

On comparing this computation with our conditional results for $\delta_g(a, 3)$ and $\delta_g(a, 4)$ we obtain the following result.
Proposition 4 (GRH). Let $d = 3, 4$ be fixed. There are at most $O(x/\sqrt{\log x})$ integers $|g| \leq x$ for which $\delta_g(a, d) \neq \delta(a, d)$ for some integer $a$. In particular, for almost all integers $|g| \leq x$ we have $\delta_g(a, d) = \delta(a, d)$ for every integer $a$.

This proposition shows that for fixed $d = 3, 4$ it makes sense to call an integer $g$ generic if $\delta_g(a, d) = \delta(a, d)$ for every integer $a$.

In a similar vein we have:

Proposition 5 (GRH). Let $d = 3, 4$ be fixed. If $|D(g)|$ tends to infinity as $g$ ranges over a set of rationals $g$ for which $h = 1$, then $\delta_g(a, d)$ tends to $\delta(a, d)$.

The latter two results seem to hold for other values of $d$ as well (cf. Table 2 of [21]), with $O(x/\sqrt{\log x})$ replaced by $o(x)$. I might return to this in a sequel.

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8 Tables

We illustrate our results by some examples (assuming GRH).

| $g$  | $g_0$ | $h$ | $\delta_g(1, 3) - \delta_g(2, 3)$       | numerical                | experimental         |
|------|-------|-----|----------------------------------------|--------------------------|----------------------|
| -14^4 | 14    | 4   | $3A_{\xi_1}/4$                        | +0.13048284...          | +0.13045317          |
| -196  | 14    | 2   | $A_{\xi_1}$                           | +0.17397712...          | +0.17399131          |
| -3^8  | 3     | 8   | $15A_{\xi_1}/16$                      | +0.16310355...          | +0.16310903          |
| -3    | 3     | 1   | $5A_{\xi_1}/2$                        | +0.43494280...          | +0.43499017          |
| -2    | 2     | 1   | $3A_{\xi_1}/8$                        | +0.06524142...          | +0.06525031          |
| 3     | 3     | 1   | 0                                      | 0                       | +0.00001393          |
| 9     | 3     | 2   | $-5A_{\xi_1}/2$                       | -0.43494280...          | -0.43502303          |
| 6561  | 3     | 4   | 0                                      | 0                       | -0.00001895          |
| 4     | 2     | 2   | $-7A_{\xi_1}/4$                       | -0.30445996...          | -0.30442279          |
| 5     | 5     | 1   | $67A_{\xi_1}/94$                      | +0.12400497...          | +0.12397327          |
| 25    | 5     | 2   | $-151A_{\xi_1}/94$                    | -0.27947388...          | -0.27952119          |
| 49    | 7     | 2   | $-3A_{\xi_1}/2$                       | -0.26096568...          | -0.26097396          |
| 2401  | 7     | 4   | $-A_{\xi_1}/2$                        | -0.08698856...          | -0.08697494          |

| $g$  | $g_0$ | $h$ | $\delta_g(1, 4) - \delta_g(3, 4)$       | numerical                | experimental         |
|------|-------|-----|----------------------------------------|--------------------------|----------------------|
| -216 | 6     | 3   | $9A_{\psi_1}/28$                       | +0.20688771...          | +0.20686925          |
| -9   | 3     | 2   | $0$                                    | 0                       | +0.00000068          |
| -81  | 3     | 4   | 0                                      | 0                       | -0.00000232          |
| 2    | 2     | 1   | $-A_{\psi_1}/4$                        | -0.16091266...          | -0.16088852          |
| 4    | 2     | 2   | 0                                      | 0                       | +0.00001122          |
| 8    | 2     | 3   | $-A_{\psi_1}/28$                       | -0.02298752...          | -0.02301736          |
| 512  | 2     | 9   | $-3A_{\psi_1}/28$                      | -0.06896257...          | -0.06897632          |
| 216  | 6     | 3   | $-9A_{\psi_1}/28$                      | -0.20688771...          | -0.20687020          |
| 2048 | 2     | 11  | $-489A_{\psi_1}/2396$                  | -0.13136276...          | -0.13134226          |
| 6^9  | 6     | 9   | $-A_{\psi_1}/4$                        | -0.16091266...          | -0.16088478          |
| 6^{27} | 6    | 27  | $-23A_{\psi_1}/84$                     | -0.17623768...          | -0.17620628          |

The number in the column ‘experimental’ arose on taking the density difference over the first $10^8$ primes, not letting the primes $p$ for which the order of $g$ mod $p$ is not defined contribute to either $\delta_g(1, 3)$ or $\delta_g(2, 3)$ (in Table 1), or $\delta_g(1, 4)$ and $\delta_g(3, 4)$ in Table 2. Thus, for example, in the column headed ‘experimental’ in Table 1 the numbers

$$N_g(1, 3)(p_{10^8}) - N_g(2, 3)(p_{10^8})$$

are recorded (recall that $p_{10^8} = 2038074743$). The last decimals in the columns headed ‘numerical’ and ‘experimental’ are not rounded.
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