A Note on Ghost-Free Matter Couplings in Massive Gravity and Multi-Gravity

Kurt Hinterbichler$^1$ and Rachel A. Rosen$^2$

$^1$Perimeter Institute for Theoretical Physics, 31 Caroline St. N, Waterloo, Ontario, Canada, N2L 2Y5
$^2$Physics Department and Institute for Strings, Cosmology, and Astroparticle Physics, Columbia University, New York, NY 10027, USA

We consider a recently proposed non-minimal matter coupling in massive gravity and multi-gravity. We argue that, when formulated in terms of unconstrained vielbeins, this matter coupling is free of the Boulware-Deser ghost to all orders away from the decoupling limit.

I. INTRODUCTION AND SUMMARY

Recent years have seen great progress in constructing and understanding effective field theories of massive gravity [1–3] and its close cousins bi-gravity [4] and multi-gravity [5] (see [6, 7] for reviews). Part of this quest has been to determine if, beyond the mass terms of de Rham, Gabadadze and Tolley (dRGT) [2], there are other interactions in these theories that are free of the instability known as the Boulware-Deser ghost [8]. In particular, there has been interest in non-minimal couplings to matter [9–23].

In [11], a coupling to matter using more than one metric was introduced. It was argued that, although there is a phenomenologically interesting regime of validity for which the ghost is absent, this matter coupling causes the reappearance of the Boulware-Deser ghost above the strong coupling scale of the theory. In this work we consider an analogous matter coupling expressed in terms of unconstrained vielbeins rather than metrics. We argue that the vielbein version of the matter coupling preserves the primary constraint necessary to remove the Boulware-Deser ghost, to all orders beyond the decoupling limit. In the presence of the matter coupling, the vielbein formulation is not equivalent to the metric formulation, so there is no discrepancy between our results and those of [11].

II. NON-CANONICAL MATTER COUPLING

Consider the ghost-free bi-gravity theory of [4] written in terms of two vielbeins $e^A_{\mu}$ and $\tilde{e}^A_{\mu}$ as in [5]. In $D$ dimensions, the theory includes an Einstein-Hilbert term for each vielbein,

$$
\epsilon_{A_1 \cdots A_D} R^{A_1 A_2} \wedge e^{A_3} \wedge \cdots \wedge e^{A_D},
\epsilon_{A_1 \cdots A_D} \tilde{R}^{A_1 A_2} \wedge \tilde{e}^{A_3} \wedge \cdots \wedge \tilde{e}^{A_D},
$$

where $R^{AB}$ and $\tilde{R}^{AB}$ are the curvature two-forms corresponding to $e^A_{\mu}$ and $\tilde{e}^A_{\mu}$ respectively. In addition, the ghost-free bi-gravity theory contains a linear combination of $D + 1$ non-derivative terms [2–5],

$$
\epsilon_{A_1 \cdots A_D} e^{A_1} \wedge \cdots \wedge e^{A_n} \wedge \tilde{e}^{A_{n+1}} \wedge \cdots \wedge \tilde{e}^{A_D},
$$

for $n = 0, 1, \cdots, D$. This first and last of these are cosmological constants, while the others induce genuine interactions among the vielbeins.

Let us couple this theory to some scalar matter sector that does not contain higher derivative terms. In particular, let us consider minimal coupling to a composite vielbein $\tilde{e}^A_{\mu}$ that is a function of the two vielbeins $e^A_{\mu}$ and $\tilde{e}^A_{\mu}$. Diffeomorphism invariance of the matter action among the vielbeins.

III. MATTER COUPLING DECOMPOSITION

Consider the ghost-free matter coupling [5] written in terms of vielbeins $e^A_{\mu}$ and $\tilde{e}^A_{\mu}$ as in [5]. In $D$ dimensions, the theory includes an Einstein-Hilbert term for each vielbein,

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$$

for $n = 0, 1, \cdots, D$. This first and last of these are cosmological constants, while the others induce genuine interactions among the vielbeins.
From (3), we can read off four equations that determine $\tilde{N}$, $\tilde{N}^i$, $\tilde{e}_i^a$ and $\tilde{a}^a$:

$$\tilde{e}_i^a \tilde{a}^a = e_i^a v_a, \quad \tilde{e}_i^a \left( \delta^a_b + \frac{1}{\gamma + 1} \tilde{v}_b \tilde{v}^a \right) = e_i^b \left( \delta^a_b + \frac{1}{\gamma + 1} v_b v^a \right) + \alpha \tilde{e}_i^a,$$

$$\tilde{N} \tilde{v}^a + \tilde{N}^i \tilde{e}_i^a \left( \delta^a_b + \frac{1}{\gamma + 1} \tilde{v}_b \tilde{v}^a \right) = -N \tilde{v}^a + N^i \tilde{e}_i^a \left( \delta^a_b + \frac{1}{\gamma + 1} v_b v^a \right) + \alpha \tilde{N} \tilde{e}_i^a. \quad (7)$$

The key point is that the first two equations of (7) are entirely independent of the lapse and shift variables, and the second two are linear in the lapse and shift variables. The first two can be solved for $\tilde{e}_i^a$ and $\tilde{a}^a$ and the result will depend only on $e_i^a$, $\tilde{e}_i^a$ and $v^a$. Plugging the solution into the second two and solving for $\tilde{N}$ and $\tilde{N}^i$, the solutions for $\tilde{N}$ and $\tilde{N}^i$ will be linear in $N$, $N^i$, $\tilde{N}$ and $\tilde{N}^i$, as desired.

Following [5], we can use the $\tilde{N}$ equation of motion to eliminate the boost parameter $v^a$ and get an action that remains linear in $N$, $N^i$ and $\tilde{N}$. $N$, $\tilde{N}$ then enforce the first class primary constraints of overall diffeomorphism invariance, and $\tilde{N}$ enforces the extra primary constraint that eliminates the ghost.

In $D = 2$ we can be more explicit. The zweibeins are parametrized as

$$e^A_\mu = \begin{pmatrix} N \sqrt{1 + v^2} + N^i \bar{e}^i v^i & N v + N^i \bar{e} \sqrt{1 + v^2} \\ e \bar{v} & e \sqrt{1 + v^2} \end{pmatrix}, \quad \tilde{e}^A_\mu = \begin{pmatrix} \tilde{N} & \tilde{N}^i \tilde{e}^i \\ 0 & \tilde{e} \end{pmatrix}, \quad \tilde{\bar{e}}^A_\mu = \begin{pmatrix} \tilde{N} \sqrt{1 + \bar{v}^2} + \tilde{N} \tilde{\bar{e}} \tilde{\bar{v}} & \tilde{N} \tilde{\bar{v}} + \tilde{N} \tilde{\bar{e}} \sqrt{1 + \bar{v}^2} \\ \tilde{\bar{e}} \tilde{\bar{v}} & \tilde{\bar{e}} \sqrt{1 + \bar{v}^2} \end{pmatrix}. \quad (8)$$

Equations (7) become

$$\tilde{e} \tilde{v} = e v, \quad \tilde{e} \sqrt{1 + \bar{v}^2} = e \sqrt{1 + v^2} + \alpha \tilde{e}, \quad \tilde{N} \sqrt{1 + \bar{v}^2} + \tilde{N} \tilde{e} \tilde{v} = N \sqrt{1 + v^2} + N^i e v + \alpha \tilde{N}, \quad \tilde{N} \tilde{v} + \tilde{N} \bar{e} \sqrt{1 + \bar{v}^2} = N v + N^i e \sqrt{1 + v^2} + \alpha \tilde{N} \bar{e}. \quad (9)$$

Solving the first two of (9), $e$ and $v$ are independent of all lapses and shifts:

$$\tilde{e} = \sqrt{e^2 + \alpha^2 \bar{e}^2 + 2 \alpha e \bar{e} \sqrt{1 + \bar{v}^2}}, \quad \tilde{v} = \frac{e v}{\sqrt{e^2 + \alpha^2 \bar{e}^2 + 2 \alpha e \bar{e} \sqrt{1 + \bar{v}^2}}} \quad (10)$$

Using this in the final two expressions of (9), we have

$$\tilde{N} = \frac{N e + a^2 N \bar{e} + a (N^i - \bar{N} \tilde{\bar{e}}) e \bar{e} + \alpha (N \tilde{e} + \bar{N} e) \sqrt{1 + v^2}}{\sqrt{e^2 + \alpha^2 \bar{e}^2 + 2 \alpha e \bar{e} \sqrt{1 + \bar{v}^2}}} \quad (11)$$

Thus, the unconstrained vielbein formulation and the metric formulation of [11] are different theories. We argue that only the unconstrained vielbein theory, in which the symmetrization constraint is determined dynamically, is ghost-free at all scales. Choosing to impose the usual symmetrization constraint, as in e.g. [13, 15], rather than letting it be determined dynamically in the presence of matter, results in a theory which is equivalent to the ghostly metric formulation of [11]. Note, however, that because the ghost in the metric formulation appears above the strong coupling scale of the effective theory, this formulation is also acceptable if one takes the strong-coupling scale to be the cutoff at which new physics enters. It is only if we allow for the possibility that the strong coupling scale is not a new-physics cutoff and that the vielbein description has a strongly-coupled range of validity above the mass of the would-be ghost of the metric formulation that the difference plays a role.

We have focused on the bi-gravity case, but our arguments go through in massive gravity as well, for which the second metric is frozen to some fiducial metric. In addition, it is straightforward to extend the argument to theories of many interacting spin-2 fields $e^A_\mu(\chi)$, with the

\footnote{We are grateful to Claudia de Rham for discussions on this.}
effective vielbein given by the linear combination
\[ \tilde{e}^A_\mu = \sum_I \alpha(I) e^A_\mu(I) . \]  
(13)

The lapse and shift of the composite vielbein defined in this way will be linear in the lapses and shifts of the constituent vielbeins.

III. OUTLOOK

We have seen that, in the unconstrained vielbein formulation, bi-gravity and multi-gravity theories with matter multiply coupled to a linear combination of the various vielbeins have the primary constraint which removes the Boulware-Deser ghost to all orders. Because these couplings are not dynamically equivalent to the analogous couplings in the metric formulation, their cosmological and solar system phenomenology may be different and interesting to study.

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