Noncommutative tori and universal sets of non-binary quantum gates

Alexander Yu. Vlasov*

Federal Radiological Center (IRH), 197101, Mira Street 8, St.-Petersburg, Russia

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I. INTRODUCTION

Let $\mathcal{H}_l$ be a Hilbert space of quantum system with $l$ states and $\mathcal{H}_l^n = \mathcal{H}_l^\otimes n$ is an $l^n$-dimensional Hilbert space of $n$ systems expressed as $n$-th tensor power. For $l = 2$ element of $\mathcal{H}_l^2 (\mathcal{H}_l^2)$ is usually called a qubit(s). An algebra $\mathbb{C}(l^n \times l^n)$ of all complex $l^n \times l^n$ matrices corresponds to general linear transformations of $\mathcal{H}_l^n$ and a group of unitary matrices $U(l^n)$ corresponds to physically possible evolution. Because of the natural structure of tensor power it is possible to consider groups of transformations of subsystems $U(l^k) \cong U(l^n) \cap (\mathbb{C}(l^k \times l^k) \otimes 1_{l^{n-k}})$. Such transformations correspond to quantum gates. For $l = 2$ they are usually called $k$-qubits gates.

The problem of universality in quantum simulation and computation is related with the possibility of expression or approximation of arbitrary unitary transformation by composition of specific unitary transformations (quantum gates) from given set. In an earlier paper (Ref. 6) application of Clifford algebras to constructions of universal sets of binary quantum gates $U_k \in U(2^n)$ was shown. For application of a similar approach to non-binary quantum gates $U_k \in U(l^n)$ in present work is used rational noncommutative torus $\mathbb{T}_{n/l}^{2n}$. A set of universal non-binary two-gates is presented here as one example.

\*

E-mail: qbeat@mail.ru or alex@protection.spb.su

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II. NONCOMMUTATIVE TORUS $T^2_\theta$ AND QUANTUM ONE-GATES

Let us consider one-particle transformations. For two-dimensional case, $\mathcal{H}_2$, any two Pauli matrices, for example $\sigma_x$ and $\sigma_z$, generate full four-dimensional basis of $\mathbb{C}(2 \times 2)$, i.e., \( \{\sigma_x^2 = \sigma_y^2 = 1_2, \sigma_x, \sigma_z, \sigma_y = i\sigma_x\sigma_z\} \).

Analogously, two generators $U, V$ of noncommutative torus $T^2_\theta$ defined as
\[
UV = \exp(2\pi i \theta) VU, \quad VV^\dagger = UU^\dagger = 1,
\]
produce for rational $\theta = 1/l$ an algebra isomorphic to $\mathbb{C}(l \times l)$. The basis of the algebra are $l^2$ elements $U^m V^n$, $m, n = 0, \ldots, l - 1$.

Let us use the Weyl representation of $U$ and $V$ as the right cyclic shift operator and its Fourier transform:
\[
U_{kj} = \delta_{k+j,(mod \ l),j}, \quad V_{kj} = \exp(2\pi ik/l)\delta_{kj}.
\]
The representation and $U^m V^n$ basis are well known in quantum information science after application to the theory of quantum error correction.\(^\text{11}\)

To find transformation between basis $U^m V^n$ and canonical basis $E^{ab}$ of $\mathbb{C}(l \times l)$, there $(E^{ab})_{jk} = \delta_{aj}\delta_{bk}$, it is enough to use $E^{00} = 1/l \sum_{k=0}^{l-1} V^k$ together with $E^{ab} = U^{l-a} E^{00} U^b$.

Let us show that any $U^m V^n$ (except $1_1$ for $m = n = 0$) can be generated from $U$ and $V$ using only commutators $[A, B] \equiv (\text{ad } A) B \equiv AB - BA$. For $U$ and $V$ commutator is simply $[U, V] = (1 - \zeta) UV \propto UV$, where $\zeta = \exp(2\pi i/l)$. It is convenient to use “\(\text{ad}\)” for consecutive commutators, for example $\text{ad } A)^2 B \equiv [A, [A, B]]$, and symbol “proportional,” $A \propto B \Rightarrow A = \alpha B$, to avoid unessential nonzero complex multipliers $\alpha$.

Direct expressions for $l^2 - 1$ elements $U^m V^n$ are
\[
U^m V^n \propto (\text{ad } U)^{m-1(\mod l)} ((\text{ad } V)^{n-1(\mod l)} [U, V])
\]
where $0 \leq m, n < l$ are any pair of numbers except $m = n = 0$, $(\text{ad } U)^0$ or $(\text{ad } V)^0$ corresponds to absence of the term, and $-1(\mod l) = l - 1$.

Of course it is possible to suggest simpler expression for particular values of $m$ and $n$, but Eq. (2.3) shows also application of a third element $W \propto UV$:
\[
UW = \zeta WU, \quad VW = \zeta VW.
\]
It is convenient to define
\[
W = \zeta^{(l-1)/2} UV, \quad W^l = 1_l.
\]

Similar with case $l = 2$ with $(\sigma_x, \sigma_y, \sigma_z)$, any pair between $(U, V, W)$ may be used for generation of a full algebra due to Eq. (2.3) together with possibility to express initial pair $(U, V)$ from $(U, W)$ or $(V, W)$:
\[
V \propto (\text{ad } U)^{l-1} W, \quad U \propto (\text{ad } V)^{l-1} W,
\]
but for $l \geq 3$ there is a special property that should be taken into account. Let us use notation $A \rightarrow_\zeta B$ for $AB = \zeta BA$. The definition of relation “$\rightarrow_\zeta$” is asymmetric for $l \geq 3$.

It is clear from the diagram:

that for different sets with three operators the relation may be transitive or not.
For example, $U \to \zeta V$ from Eq. (2.1), $U \to \zeta W$ and $W \to \zeta V$ from Eq. (2.4) and so we have transitive relation $U \to \zeta W \to \zeta V$, i.e., ordering. Let us call it $\zeta$-order for certainty. On the other hand, it is simply to check $W^\dagger \to \zeta U$ and $V \to \zeta W^\dagger$ and here is some cyclic graph. The cyclic case is more symmetric, because all pairs are equivalent.

For the ordered case it is not so, because $\zeta$-order produces a canonical map to a subset of natural numbers, i.e., indexes, and it is convenient for construction of noncommutative torus $\mathbb{T}_{1/l}^{2n}$, $\zeta$-ordered by definition: $\mathbb{T}_k \to \zeta \mathbb{T}_j$ for $k < j$ [see Eq. (1.2)].

Because of the principle here is used the following definition for generators of $\mathbb{T}_{1/l}^{2n}$:

$$T_0 \equiv U, \quad T_1 \equiv W, \quad T_k \equiv U, \quad T_{k+1} \equiv W,$$

where $U$ is defined in Eq. (2.2) and $W$ in Eq. (2.3).

### III. REPRESENTATIONS OF NONCOMMUTATIVE TORI $\mathbb{T}_{1/l}^{2n}$

Let us use notation $T_x \equiv U$, $T_y \equiv W$, $T_z \equiv V$, where $U, V, W$ are defined in Eqs. (2.2, 2.5). There is $\zeta$-order $T_x \to \zeta T_y \to \zeta T_z$, i.e.,

$$T_x T_y = \zeta T_y T_z, \quad T_y T_z = \zeta T_z T_x, \quad T_x T_z = \zeta T_z T_x.$$  

(3.1)

It is possible to introduce $2n$ generators of $\mathbb{T}_{1/l}^{2n}$ as

$$T_{2k} = \mathbb{1}_l \otimes \cdots \otimes \mathbb{1}_l \otimes T_x \otimes T_z \otimes \cdots \otimes T_z,$$

(3.2a)

$$T_{2k+1} = \mathbb{1}_l \otimes \cdots \otimes \mathbb{1}_l \otimes T_y \otimes T_z \otimes \cdots \otimes T_z,$$

(3.2b)

in direct analogy with construction of Clifford algebras.

It is clear that different products of $T_k$ generate full matrix algebra $\mathbb{C}(l^2)$, because $T_x$ and $T_y$ generate $\mathbb{C}(l \times l)$. Let us prove that generators Eq. (3.2) satisfy definition Eq. (1.2) of noncommutative torus $\mathbb{T}_{1/l}^{2n}$.

First, $T_0^\dagger = \mathbb{1}$ because $T_x^\dagger = T_y^\dagger = T_z^\dagger = \mathbb{1}_l$.

To prove that $T_k \to \zeta T_j$ for any $k < j$, it is enough to consider a few cases (here “$T$” means “any element” and “$\zeta$” marks $\zeta$-order of only a pair of noncommutative terms in the tensor products):

**Case 1:** $T_{2k} \to \zeta T_{2k+1}$, $k \geq 0$

$$T_{2k} = \mathbb{1}_l \otimes \cdots \otimes \mathbb{1}_l \otimes T_x \otimes T_z \otimes \cdots \otimes T_z,$$

$$T_{2k+1} = \mathbb{1}_l \otimes \cdots \otimes \mathbb{1}_l \otimes T_y \otimes T_z \otimes \cdots \otimes T_z.$$  

**Case 2:** $T_{2k} \to \zeta T_{2k+j+1}$, $k \geq 0, j > 0$

$$T_{2k} = \mathbb{1}_l \otimes \cdots \otimes \mathbb{1}_l \otimes T_x \otimes T_z \otimes \cdots \otimes T_z,$$

$$T_{2k+1+j} = \mathbb{1}_l \otimes \cdots \otimes T \otimes T_z \otimes T_z \otimes \cdots \otimes T_z.$$  

**Case 3:** $T_{2k+1} \to \zeta T_{2k+1+j}$, $k \geq 0, j > 0$

$$T_{2k+1} = \mathbb{1}_l \otimes \cdots \otimes \mathbb{1}_l \otimes T_y \otimes T_z \otimes \cdots \otimes T_z,$$

$$T_{2k+1+j} = \mathbb{1}_l \otimes \cdots \otimes T \otimes T_z \otimes T_z \otimes \cdots \otimes T_z.$$
IV. GENERATION OF $T_{1/l}^{2n}$ BY COMMUTATORS

Let us prove that for $l > 2$ it is possible to generate $T_{1/l}^{2n}$ using only commutators of $2n$ elements $T_k$. The case with $l = 2$, $\mathcal{C}(2n, \mathbb{C}) \cong T_{1/l}^{2n}$ was considered in earlier work, and it was shown that $2n$ generators are not enough and it is necessary to add any element of third or fourth order.

Here is presented a proof that for $l > 2$, $2n$ generators are enough. Let us instead of $T_i^nT_j^{n_1}\cdots T_k^{n_k}$ write simply $T(n_i, n_j, \ldots, n_k)$ if it is possible without lost of clarity. Sequences of indexes are always chosen ordered.

Let us prove that for $l > 2$ it is possible to generate

$$T(n_i, n_j) \propto (\text{ad} T_i)^{n_i-1}(\text{ad} T_j)^{n_j-1}(\text{mod } l)[T_i, T_j].$$

(4.1)

Let all cases with $T(n_i, \ldots, n_k)$, $2 \leq k < 2n$, $i_1 < i_2 < \cdots < i_k$, be proved and it is necessary to generate all $T(n_i, \ldots, n_k)$ with $i_k < j \leq 2n$. There are a few different cases:

**Case 1**: $\Sigma(n_i, \ldots, n_k) \mod l \neq 0$:

$$T(n_i, \ldots, n_k) \propto (\text{ad} T_j)^{n_j} T(n_i, \ldots, n_k).$$

(4.2)

**Case 2**: $\Sigma(n_i, \ldots, n_k) \mod l = 0$ and Eq. (4.2) vanishes.

**Case 2.1**: $\exists n_i \in (n_i, \ldots, n_i)$, $n_i \neq n_j$:

$$T(n_i, \ldots, n_k) \propto [T_i, (\text{ad} T_j)^{n_j} T(n_i, \ldots, n_i - 1, \ldots, n_k)].$$

(4.3)

**Case 2.2**: $\nexists n_i \in (n_i, \ldots, n_i)$, $n_i \neq n_j$, i.e., $n_i = \ldots = n_k = n$:

**Case 2.2.1**: $2n_j \neq l$:

$$T(n_i, \ldots, n_k, n_j) \propto [T(n_i, \ldots, n_i - k), T(n_k, n_j)].$$

(4.4)

**Case 2.2.2**: $2n_j = l$; let $n_i = n_i' + n_i''$:

$$T(n_i, \ldots, n_k, n_j) \propto [T(n_i, \ldots, n_i'), T(n_i'', n_j)].$$

(4.5)

The cases include all possible variables and so the suggestion is proved by recursion and all $l^n - 1$ possible products of generators except of $1$ can be represented using commutators.

V. UNIVERSAL SET OF QUANTUM TWO-GATES

Elements $T_k$ have up to $n$ non-unit terms in tensor product Eq. (4.2). Here is described construction with no more than two terms. It is used for description of a universal set of quantum two-gates and also has direct analog with two-qubit gates.

Let us consider $B_0 = T_0$, $B_j = T_j T_j^\dagger_{1/l}$, $1 \leq j < 2n$. It is possible to generate full $T_{1/l}^{2n}$ using the $2n$ elements: $T_1 \propto [T_0, B_1], T_i \propto [B_i, T_i-1], \forall i > 1$, and so it is possible to generate recursively all $T_i$ and use construction of $T_{1/l}^{2n}$ described above.

Using Eq. (4.2) it is possible to write expressions for $B_j$:

$$B_0 = T_0 = 1 \otimes (n-1) \otimes T_x,$$

$$B_{2k+1} = T_{2k+1} T_{2k}^\dagger \propto 1 \otimes (n-k-1) \otimes T_x \otimes 1 \otimes \mathbb{I}^k,$$

$$B_{2k+2} = T_{2k+2} T_{2k+1} \propto 1 \otimes (n-k-2) \otimes T_x \otimes T_x^\dagger \otimes \mathbb{I}^k,$$

(5.1a, 5.1b, 5.1c)

with $k = 0, \ldots, n-1$ (or $n-2$).

To produce a universal set of quantum one- and two-gates it is enough to use constructions of unitary matrices mentioned in the Introduction:

$$G_k = e^{i\tau(B_k + B_k^\dagger)}, \quad F_k = e^{i\tau(B_k - B_k^\dagger)}.$$ 

(5.2)

It is possible to choose $\tau$ to express an arbitrary matrix with given precision as product of matrices Eq. (5.2).
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