A SIS REACTION-DIFFUSION MODEL
WITH A FREE BOUNDARY CONDITION AND
NONHOMOGENEOUS COEFFICIENTS

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Abstract. This paper is devoted to a spatial heterogeneous SIS model with the infected group equipped with a free boundary. Our main aim is to determine whether the disease is spreading forever or extinct eventually, and to illustrate, under the nonhomogeneous spatial environment, free boundaries can have a large influence on the infected behavior at the large time. For this purpose, we first introduce a basic reproduction number and then establish a spreading-vanishing dichotomy. Then by investigating the effect of the diffusion rate, initial domain and spreading speed on the asymptotic behavior of the infected group, we establish some sufficient conditions and even necessary and sufficient conditions for disease spreading or vanishing.

1. Introduction. Lots of researches have been made to study the dynamics of disease transmission in the field of theoretical epidemiology, and there are many SIS-type, SIR-type reaction-diffusion models established to describe the relationships between susceptible-infected or susceptible-infected-recovered populations [11, 18, 22, 23, 25, 31]. In recent years, more researchers have found that spatial diffusion and environmental heterogeneity both play an important role in affecting the dynamic behavior of infectious diseases, such as measles, tuberculosis, flu, etc., especially for vector-borne diseases, such as malaria, dengue fever, West Nile virus, etc. Plenty of discussions about the effects of spatial variations have been made in population models, prey-predator model and Lotka-Volterra competition system; See, for example, [2, 13, 14, 15, 16, 21].

Allen et al. [1] studied a SIS epidemic reaction-diffusion model of the form

\[
\begin{align*}
S_t &= d_S \Delta S - \frac{\beta(x)SI}{S+I} + \gamma(x)I, \quad x \in \Omega, \ t > 0, \\
I_t &= d_I \Delta I + \frac{\beta(x)SI}{S+I} - \gamma(x)I, \quad x \in \Omega, \ t > 0,
\end{align*}
\]

with homogeneous Neumann boundary condition

\[
\frac{\partial S}{\partial n} = \frac{\partial I}{\partial n} = 0, \quad x \in \Omega, \ t > 0,
\]

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where \(S(x,t)\) and \(I(x,t)\) represent the density of susceptible and infected individuals at location \(x\) and time \(t\), \(d_S\) and \(d_I\) represent the corresponding diffusion rates for the susceptible and infected populations, \(\beta(x)\) and \(\gamma(x)\) are positive H"older-continuous functions on \(\Omega\) and represent the rates of disease transmission and recovery at \(x\), respectively. Allen et al. [1] introduced a so-called basic reproduction number \(R_0\), which is given by

\[
R_0 = \sup_{\phi \in H^1_0(\Omega), \phi \neq 0} \left\{ \frac{\int_{\Omega} \beta \phi^2 \, dx}{\int_{\Omega} (d_I |\nabla \phi|^2 + \gamma \phi^2) \, dx} \right\}.
\]

Allen et al. [1] showed that the population density \((S,I)\) converges to a unique disease-free equilibrium \((S_0,0)\) when \(R_0 < 1\), and that there exists a unique positive endemic equilibrium \((S^*,I^*)\) when \(R_0 > 1\). This implies that, if there are some low risk spots in the domain \(\Omega\) (i.e., \(\beta(x) < \gamma(x)\) for some \(x \in \Omega\)), then the endemic equilibrium \(I^*\) will vanish through making the diffusion rate of the susceptible individuals approach zero. Wu et al. [29] recently investigated the following diffusive SIS epidemic model with mass action infection mechanism

\[
\begin{align*}
S_t &= d_S \Delta S - \beta(x)SI + \gamma(x)I, \quad x \in \Omega, \ t > 0, \\
I_t &= d_I \Delta I + \beta(x)SI - \gamma(x)I, \quad x \in \Omega, \ t > 0,
\end{align*}
\]

and described the asymptotic profiles of steady states as \(d_s \to 0\) or \(d_s \to \infty\).

In the real world, an outbreak of the disease at the initial time may occur at a small area, and then spreads into the total region. This spreading phenomenon always presents a linear fashion, namely, the spreading radius eventually exhibits a linear growth curve against time [20, 26]. Motivated by these phenomena, many researchers are attempting to consider a SIR epidemic model with a free boundary, which describes the spreading frontier of the disease. Du et al. [7] investigated the following single-species diffusive logistic model with a free boundary

\[
\begin{align*}
u_t - d \Delta \nu &= u(\alpha(r) - \beta(r)u), \quad t > 0, \ 0 < r < h(t), \\
u(r,0) &= 0, \quad u(t,h(t)) = 0, \quad t > 0, \\
h'(t) &= -\mu u_r(t,h(t)), \quad t > 0, \\
h(0) &= h_0, \quad u(0,r) = u_0(r), \quad 0 \leq r \leq h_0,
\end{align*}
\]

with the initial function \(u_0(r)\) satisfying

\[
u_0 \in C^2([0,h_0]), \quad u_0'(0) = u_0(h_0) = 0, \quad u_0 > 0 \text{ in } [0,h_0),
\]

where \(u(t,r)\) denotes the population density of a new or invasive species, \(r = |x|, x \in \mathbb{R}^N (N \geq 2)\), \(\Delta \nu = u_{rr} + \frac{N-1}{r} u_r\), \(r = h(t)\) is the moving boundary to be determined, \(h_0, \mu\) and \(d\) are positive constants, \(\alpha, \beta \in C^\infty([0,\infty))\) for some \(\nu_0 \in (0,1)\). Du et al. [7] established a spreading-vanishing dichotomy, that is, as time \(t \to \infty\), the population \(u(x,t)\) either successfully establishes itself in the new environment (called spreading), in the sense that \(h(t) \to \infty\) and \(u(x,t) \to a/b\), or the population fails to establish and vanishes eventually (called vanishing), namely \(h(t) \to h_\infty < \infty\) and \(u(x,t) \to 0\). Moreover, Du et al. [7] claimed that when \(h_0 < R^*\) (where \(R^*\) is a positive constant such that the principal eigenvalue \(\lambda_1(R^*) = 1\)), there exists a critical constant \(\mu^*\) such that if \(\mu \leq \mu^*\), the invasive species vanishes at the end, and that if \(\mu > \mu^*\), spreading happens.
Wang [27] investigated the following diffusive logistic equation with a free boundary and sign-changing coefficient

\[
\begin{cases}
    u_t - du_{xx} = u(m(x) - u), & t > 0, \quad 0 < r < h(t), \\
    B[u](t, 0) = 0, & u(t, h(t)) = 0, \quad t \geq 0, \\
    h'(t) = -\mu u_x(t, h(t)), & t \geq 0, \\
    h(0) = h_0, & u(0, r) = u_0(r), \quad 0 \leq r \leq h_0,
\end{cases}
\]

where \(B[u] = \alpha u - \beta u_x\), \(\alpha\) and \(\beta\) are nonnegative constants and satisfy \(\alpha + \beta = 1\). Wang [27] found that whether the population density \(u(t, x)\) becomes zero as \(t \to \infty\) or the positive steady-states solution \(u(x)\) exists is determined by the spreading speed \(\mu\) and diffusion rate \(d\). By constructing upper or lower solutions, Cao et al. [4] investigated the vanishing/spreading mechanism of the susceptible group for the following nonlocal SIS epidemic model with a free boundary

\[
\begin{align*}
    N_1 &= \Delta N + \sigma - \mu N, & x \in \mathbb{R}, \\
    I_1 &= \Delta I + \beta(N - I) \int_0^1 K(x, y)I(y, t)dy - (\mu + \gamma)I, & x \in (g(t), h(t)), \\
    N(x, 0) &= N_0(x), & I(x, 0) = I_0(x), & x \in \mathbb{R}, \\
    I(x, t) &= 0, & x \in \mathbb{R}/(g(t), h(t)), \\
    g'(t) &= -\mu_1 I_x(g(t), t), & g(0) = -h_0, \\
    h'(t) &= -\mu_1 I_x(h(t), t), & h(0) = h_0
\end{align*}
\]

for \(t \geq 0\).

A natural question is that: Would incorporation of spatial heterogeneity, diffusion, and free boundary condition lead to any new phenomenon in disease spreading? such a question not only gets involved with disease spread and control in reality, but also suggests new aspects and considerations for modelling spatial-temporal dynamics of infectious diseases. Motivated by the above works, in this paper we shall discuss the spreading mechanism of the following SIS reaction-diffusion disease model

\[
\begin{align*}
    S_t &= d\Delta S + \sigma - \mu S - \beta(x)SI + \gamma(x)I, & x \in \mathbb{R}, & t > 0, \\
    I_t &= d\Delta I - \mu I + \beta(x)SI - \gamma(x)I, & x \in (g(t), h(t)), & t > 0, \\
    I(x, t) &= 0, & x \in \mathbb{R}/(g(t), h(t)), & t \geq 0, \\
    S(x, 0) &= S_0(x), & I(x, 0) = I_0(x), & x \in \mathbb{R}, \\
    g'(t) &= -k I_x(g(t), t), & g(0) = -h_0, & t \geq 0, \\
    h'(t) &= -k I_x(h(t), t), & h(0) = h_0, & t \geq 0
\end{align*}
\]

where \(\sigma\) and \(\mu\) are positive constants representing environment carrying capability and natural mortality rate, respectively; \(\beta(x)\) and \(\gamma(x)\) are positive Hölder continuous functions accounting for connect infectious rate and recovery rate, respectively. Here, we discuss the SIS model in a heterogeneous environment, and the coefficients \(\beta\) and \(\gamma\) are both space-dependent. Moreover, system (1) possesses a free boundary with respect to \(I\), and \(x = g(t)\) and \(x = h(t)\) are the moving left and right boundaries to be defined. Ecologically, this model means that beyond the free boundaries of \((g(t), h(t))\), there is only susceptible, no infectious individuals. The two equations governing the free boundaries, \(g'(t) = -k I_x(g(t), t)\) and \(h'(t) = -k I_x(h(t), t)\), are a special case of the well-known Stefan condition, which has been established in [19] for the diffusive populations. The parameter \(k > 0\) can be understood as the diffusivity of the disease, i.e., the larger \(k\) is, the easier that the disease can transmit to a new area. In addition, we give some restrictions on the initial functions \(S_0\) and
in the sense that

\[ I_t^∞(x) = \lim \inf_{x \to -\infty} I^∞(x) = \lim \inf_{x \to -\infty} \beta(x), \quad \gamma^∞ = \lim \sup_{x \to +\infty} \gamma(x) = \lim \sup_{x \to +\infty} \gamma(x), \quad (3) \]

(H1): \( \beta^∞ > \mu + \gamma^∞ \), where

\[ \beta^∞ = \lim \inf_{x \to -\infty} \beta(x), \quad \gamma^∞ = \lim \sup_{x \to +\infty} \gamma(x) \]

(H2): \( \sigma \) is sufficient large and satisfies \( \sigma \geq \mu \).

Assumption (3) means that the environment is symmetric as \( x \to +\infty \) and \( x \to -\infty \) for \( \beta \) and \( \gamma \). Assumption (H1) implies that at the faraway place, there is a relatively loose condition for the spreading of the disease (high risk domain), which weakens the superfluous influence on the spreading-vanishing behavior for the disease from the faraway site. Assumption (H2) implies that the total loss of the individuals may not be too fast.

There are several reasons why we are particularly interested in such a system as (1). First of all, we want to find whether the spreading-vanishing dichotomy established in [7] still holds in our SIS model (1), that is, as time \( t \to \infty \), whether the infected population \( I(x,t) \) eventually occupies the whole area \( \mathbb{R} \) (called spreading) in the sense that \( -g(t), h(t) \to \infty \), and \( I(x,t) \to I^∞(x) \) for some positive function \( I^∞(x) \) defined on \( \mathbb{R} \), or the infected population \( I(x,t) \) eventually vanishes (called vanishing) in the sense that the spatial domain \( (g(t), h(t)) \) of infected population \( I(x,t) \) is eventually finite and \( I(x,t) \to 0 \) locally uniformly. Furthermore, it would be interesting to investigate the influence of the spreading speed \( k \). We expect to know whether there exists a critical value \( k^* \) such that the disease spreading happens when \( k > k^* \) while the disease will die out when \( k < k^* \). Secondly, in [1] and [29], the susceptible component is assumed to occupy the whole area \( \Omega \) from beginning to end, and the basic reproduction number \( R_0 \) plays a threshold role: the endemic equilibrium solution exists if and only if \( R_0 > 1 \). This result, however, may not hold now because at the initial time, the region where the susceptible component is located is a small part, and may translate into the whole area when the disease spreading occurs. This different phenomenon enables us to conjecture that even though the reproduction number is less than 1, the disease may spread to the whole area when the spreading speed is large. Thus, we may expect to obtain many different new results compared with the SIS models in [1, 29]. Thirdly, although Cao et al. [4] obtained many sufficient conditions ensuring the asymptotic behavior of the susceptible group by constructing upper and lower solutions, we prefer to find more generalized criteria based on the reproduction number. In particular, we shall employ a method related to the basic reproduction number to show that the susceptible component will die out when the initial region is sufficiently small as well as \( k \) is sufficiently small, and to establish a concrete upper bound for \( h_0 \). This improves the relevant results in [4], because the requirement in Theorem 4.2 of [4] is essentially a special case of our main result (Theorem 5.7). Last but not the least, no paper has been devoted to the influence of the diffusion rate \( d \) on the disease spreading/vanishing with a free boundary.

The organization of this paper as follows: in Section 2, we give the global existence, uniqueness, regularity and estimate of \((S,I,h,g)\). Section 3 is devoted to
some preliminary results on eigenvalue problems and basic reproduction numbers. In Section 4, we establish a spreading-vanishing dichotomy for system (1). Finally, some sufficient conditions for the disease vanishing or spreading are given in Section 5.

For convenience, we introduce the following notations. Denote by $L^2(\Omega)$ the Lebesgue space of integrable functions defined on $\Omega$, and let $H^k(\Omega)$ ($k \geq 0$) be the Sobolev space of the $L^2$-functions $f(x)$ defined on $\Omega$ whose derivatives $\frac{\partial^n f}{\partial x^n}$ ($n = 1, \ldots, k$) also belong to $L^2(\Omega)$. Denote the space $H^1_0(\Omega) = \{ u \in H^1(\Omega) : u(x) = 0 \text{ for all } x \in \partial \Omega \}$.

2. Global existence, uniqueness and estimate. We first consider the local existence and uniqueness, and then the global existence and uniqueness.

Theorem 2.1. For any given initial value $(S_0, I_0)$ satisfying (2), and some $\nu \in (0, 1)$, there exists a $T > 0$, such that the problem (1) has a unique solution $(S, I, g, h)$ satisfying

$$S \in C^{1+\nu, \frac{1+\nu}{2}}(D_T^\infty), \quad I \in C^{1+\nu, \frac{1+\nu}{2}}(D_T), \quad g \in C^{1+\nu}([0, T]), \quad h \in C^{1+\nu}([0, T]),$$

where

$$D_T^\infty \triangleq \{(x, t) : x \in \mathbb{R}, t \in [0, T]\},$$

$$D_T \triangleq \{(x, t) : x \in [g(t), h(t)], t \in [0, T]\}.$$

Furthermore, there exists a positive constant $C$ such that

$$\|S\|_{C^{1+\nu, \frac{1+\nu}{2}}(D_T^\infty)} + \|I\|_{C^{1+\nu, \frac{1+\nu}{2}}(D_T)} + \|g\|_{C^{1+\nu}([0, T])} + \|h\|_{C^{1+\nu}([0, T])} \leq C,$$

where $C$ depends only on $k$, $h_0$, $\|S_0\|_{C^2(\mathbb{R})}$ and $\|I_0\|_{C^2([-h_0, h_0])}$.

Proof. Using similar arguments as that in [4, 5, 8], we first straighten the free boundaries. Let $\xi$ be a function in $C^3([0, +\infty))$ such that

$$\xi(y) = 1 \text{ if } |y - h_0| < \frac{h_0}{2}, \quad \xi(y) = 0 \text{ if } |y - h_0| > \frac{h_0}{2}, \quad |\xi'(y)| < \frac{5}{h_0} \text{ for all } y.$$

Consider the following transformation $(y, t) \to (x, t)$

$$x = y + \xi(y)(h(t) - h_0) + \xi(-y)(g(t) + h_0), \quad -\infty < y < +\infty.$$

It is easy to see that when $|h(t) - h_0| \leq \frac{h_0}{8}$ and $|g(t) - h_0| \leq \frac{h_0}{8}$, the above transformation $x \to y$ is a diffeomorphism from $(-\infty, +\infty)$ onto $(-\infty, +\infty)$. Furthermore, it changes the left free boundary $x = g(t)$ to the fixed line $y = -h_0$ and the right free boundary $x = h(t)$ to the fixed line $y = h_0$. Then, a direct calculation gives us

$$\frac{\partial y}{\partial x} = \frac{1}{1 + \xi'(y)(h(t) - h_0) - \xi(-y)(g(t) + h_0)} \triangleq A(g(t), h(t), y),$$

$$\frac{\partial y}{\partial t} = -\xi(y)h'(t) - \xi(-y)g'(t) \triangleq B(g(t), h(t), y),$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\xi''(y)(h(t) - h_0) + \xi(-y)(g(t) + h_0)}{[1 + \xi'(y)(h(t) - h_0) - \xi(-y)(g(t) + h_0)]^3} \triangleq C(g(t), h(t), y),$$

If we set

$$S(x, t) = S(y + \xi(y)(h(t) - h_0) + \xi(-y)(g(t) + h_0), t) = u(y, t),$$

$$I(x, t) = I(y + \xi(y)(h(t) - h_0) + \xi(-y)(g(t) + h_0), t) = v(y, t),$$
then the parabolic operator \( L = \frac{\partial}{\partial t} - d\Delta \) can be rewritten as
\[
L S = u_t - A^2 d u_y y - (C d - B) u_y, \\
L I = v_t - A^2 d v_y y - (C d - B) v_y.
\]

Set
\[
\beta(y + \xi(y) (h(t) - h_0) + \xi(-y) (g(t) + h_0)) = \tilde{\beta}(y), \\
\gamma(y + \xi(y) (h(t) - h_0) + \xi(-y) (g(t) + h_0)) = \tilde{\gamma}(y).
\]

Then the free boundary problem (1) becomes
\[
\begin{cases}
  u_t - A^2 d u_y y - (C d - B) u_y = \sigma - \mu u - \tilde{\beta} uv + \tilde{\gamma} v, & y \in \mathbb{R}, \\
  v_t - A^2 d v_y y - (C d - B) v_y = -\mu v + \tilde{\beta} uv - \tilde{\gamma} v, & y \in (-h_0, h_0), \\
  v(y, t) = 0, & y \in \mathbb{R}/(-h_0, h_0), \\
  u(y, 0) = u_0(y) = S_0(x), & v(y, 0) = v_0(y) = I_0(x), & y \in \mathbb{R}, \\
  g'(t) = -kv_y (-h_0, t), & g(0) = -h_0, \\
  h'(t) = -kv_y (h_0, 0), & h(0) = h_0,
\end{cases}
\]
for \( t \geq 0 \), where \( A = A(g(t), h(t), y), B = B(g(t), h(t), y), C = C(g(t), h(t), y), \)
\( \tilde{\beta} = \beta(y) \) and \( \tilde{\gamma} = \gamma(y) \).

We denote \( \bar{h} = -kv_y(0) \) and \( \bar{g} = -kv_y(-h_0) \), and for \( 0 < T < \frac{h_0}{8(1 + |\bar{h}| + |\bar{g}|)} \),
define \( \tilde{D}_T = [-h_0, h_0] \times [0, T] \), and
\[
H_T = \{ h \in C^1[0, T] : h(0) = h_0, h'(0) = \bar{h}, \| h' - \bar{h} \|_{C([0, T])} \leq 1 \}, \\
G_T = \{ g \in C^1[0, T] : g(0) = -h_0, g'(0) = \bar{g}, \| g' - \bar{g} \|_{C([0, T])} \leq 1 \}, \\
U_T = \{ u \in C(D_T^\infty) | u(y, 0) = u_0(y), \| u - u_0 \|_{L^\infty(D_T^\infty)} \leq 1 \}, \\
V_T = \{ v \in C(D_T^\bar{h}) | v(y, 0) = v_0(y), \| v - v_0 \|_{L^\infty(D_T^\bar{h})} = \| v - v_0 \|_{C(D_T^\bar{h})} \leq 1 \}.
\]

Note that for \( h_1, h_2 \in H_T \) and \( g_1, g_2 \in G_T \), we have \( h_1(0) = h_2(0) = h_0, g_1(0) = g_2(0) = -h_0, \) and
\[
\| h_1 - h_2 \|_{C([0, T])} \leq T \| h_1' - h_2' \|_{C([0, T])}, \\
\| g_1 - g_2 \|_{C([0, T])} \leq T \| g_1' - g_2' \|_{C([0, T])}.
\]
So it is easy to see that \( \Gamma_T = U_T \times V_T \times H_T \times G_T \) is a complete metric space with the
metric
\[
d((u_1, v_1, g_1, h_1), (u_2, v_2, g_2, h_2)) = \| u_1 - u_2 \|_{L^\infty(D_T^\infty)} + \| v_1 - v_2 \|_{C(D_T^\bar{h})} \]
\[
+ \| g_1 - g_2 \|_{C([0, T])} + \| h_1 - h_2 \|_{C([0, T])}.
\]

Now, we shall prove the existence and uniqueness result by using the contraction mapping theorem. It follows from
\( 0 < T < \frac{h_0}{8(1 + |\bar{h}| + |\bar{g}|)} \) that for any given \((u, v, g, h) \in \Gamma_T, \)
\[
|h(t) - h_0| \leq T(1 + |\bar{h}|) < \frac{h_0}{8} \quad \text{and} \quad |g(t) + h_0| \leq T(1 + |\bar{g}|) < \frac{h_0}{8}.
\]
Thus the transformation \((y, t) \to (x, t)\) as well as \( A, B \) and \( C \) are well defined. In view of the standard \( L^p \) theory and the Sobolev imbedding theorem [17], for any
(u, v, g, h) ∈ Γ_T, the following initial boundary value problem

\[
\begin{aligned}
\tilde{u}_t - A^2 \tilde{u}_{yy} - (Cd - B)\tilde{u}_y &= \sigma - \mu u - \beta u v + \gamma v, & y \in \mathbb{R}, \\
\tilde{v}_t - A^2 \tilde{v}_{yy} - (Cd - B)\tilde{v}_y &= -\mu v + \beta u v - \gamma v, & y \in (-h_0, h_0), \\
\tilde{v}(y, t) &= 0, & y \in \mathbb{R}/(-h_0, h_0), \\
\tilde{u}(y, 0) = S_0(x), & \tilde{v}(y, 0) = I_0(x), & y \in \mathbb{R},
\end{aligned}
\]

for \( t \geq 0 \) admits a unique solution

\[
(\tilde{u}, \tilde{v}) \in C^{1+\nu, \frac{1+\nu}{2}}(\Gamma_T^\infty) \times C^{1+\nu, \frac{1+\nu}{2}}(\Delta_T)
\]
satisfying

\[
\|\tilde{u}\|_{C^{1+\nu, \frac{1+\nu}{2}}(\Gamma_T^\infty)} + \|\tilde{v}\|_{C^{1+\nu, \frac{1+\nu}{2}}(\Delta_T)} \leq C_1,
\]

where \( \nu = 1 - 2/p \) and \( 2 < p < \infty \) is a positive integer, \( C_1 \) only depends on \( h_0, \|S_0\|_{C^2(\mathbb{R})} \) and \( \|I_0\|_{C^2([-h_0, h_0])} \). Define

\[
\begin{aligned}
\tilde{h}(t) &= h_0 - k \int_0^t \tilde{v}_y(h_0, s)ds, \\
\tilde{g}(t) &= -h_0 - k \int_0^t \tilde{v}_y(-h_0, s)ds,
\end{aligned}
\]

This implies that

\[
\begin{aligned}
\tilde{h}'(t) &= -k\tilde{v}_y(h_0, t), & \tilde{h}'(0) = \tilde{h}, & \tilde{h}(0) = h_0, \\
\tilde{g}'(t) &= -k\tilde{v}_y(-h_0, t), & \tilde{g}'(0) = \tilde{g}, & \tilde{g}(0) = -h_0,
\end{aligned}
\]

Then

\[
\tilde{h}', \tilde{g}' \in C^{\tilde{\gamma}}([0, T]),
\]

and

\[
\|\tilde{h}'\|_{C^{\tilde{\gamma}}([0, T])}, \|\tilde{g}'\|_{C^{\tilde{\gamma}}([0, T])} \leq kC_1 \triangleq C_2.
\]

Now, we regard the above initial boundary value problem (8) with (10) as a functional map \( \mathfrak{F} : \Gamma_T \to C^{1+\nu, \frac{1+\nu}{2}}(\Gamma_T^\infty) \times C^{1+\nu, \frac{1+\nu}{2}}(\Delta_T) \times [C^{1+\tilde{\gamma}}([0, T])]^2 \) defined by

\[
\mathfrak{F}(u, v, g, h) = (\tilde{u}, \tilde{v}, \tilde{g}, \tilde{h}).
\]

If this map \( \mathfrak{F} \) is a contraction map, then it has a unique fixed point \((u^*, v^*, g^*, h^*)\) satisfying \( \mathfrak{F}(u^*, v^*, g^*, h^*) = (u^*, v^*, g^*, h^*) \), which implies that \((u^*, v^*, g^*, h^*)\) is a solution of (6).

First, we prove that \( \mathfrak{F} \) maps \( \Gamma_T \) into itself. From (9) and (11) we have

\[
\begin{aligned}
\|\tilde{u} - u_0\|_{C(\Gamma_T^\infty)} &\leq \|\tilde{u} - u_0\|_{C^{0, \frac{\nu+1}{2}}(\Gamma_T^\infty)} T^{\frac{\nu+1}{2}} \\
&\leq \|\tilde{u} - u_0\|_{C^{1+\nu, \frac{1+\nu}{2}}(\Gamma_T^\infty)} T^{\frac{\nu+1}{2}} \leq C_1 T^{\frac{\nu+1}{2}}, \\
\|\tilde{v} - v_0\|_{C(\Delta_T)} &\leq \|\tilde{v} - v_0\|_{C^{0, \frac{\nu+1}{2}}(\Delta_T)} \leq \|\tilde{v} - v_0\|_{C^{1+\nu, \frac{1+\nu}{2}}(\Delta_T)} T^{\frac{\nu+1}{2}} \leq C_1 T^{\frac{\nu+1}{2}},
\end{aligned}
\]

and

\[
\begin{aligned}
\|\tilde{h}' - \tilde{h}_0\|_{C([0, T])} &\leq \|\tilde{h}'\|_{C^{\tilde{\gamma}}([0, T])} T^{\tilde{\gamma}} \leq kC_1 T^{\tilde{\gamma}}, \\
\|\tilde{g}' - \tilde{g}_0\|_{C([0, T])} &\leq \|\tilde{g}'\|_{C^{\tilde{\gamma}}([0, T])} T^{\tilde{\gamma}} \leq kC_1 T^{\tilde{\gamma}}.
\end{aligned}
\]

Thus, if \( T \leq \min\{(kC_1)^{-2/\nu}, C_1^{-2/(1+\nu)}\} \), then \( \mathfrak{F} \) maps \( \Gamma_T \) to \( \Gamma_T \).
Next, we show that $\mathcal{F}$ is a contraction mapping on $\Gamma_T$ for sufficiently small $T > 0$. For any $(\vec{u}_1, \vec{v}_1, g_1, h_1) = \mathcal{F}(u_1, v_1, g_1, h_1)$ and $(\vec{u}_2, \vec{v}_2, g_2, h_2) = \mathcal{F}(u_2, v_2, g_2, h_2)$ with $(u_1, v_1, g_1, h_1), (u_2, v_2, g_2, h_2) \in \Gamma_T$, denote $\hat{u} = \vec{u}_1 - \vec{u}_2$ and $\hat{v} = \vec{v}_1 - \vec{v}_2$, then from (9) we have

\[
\begin{cases}
\hat{u}_t - A^2 \hat{u}_{yy} - (Cd - B)\hat{u}_y = F_1, & y \in \mathbb{R}, \ t > 0, \\
\hat{v}_t - A^2 \hat{v}_{yy} - (Cd - B)\hat{v}_y = F_2, & y \in (-h_0, h_0), \ t > 0, \\
\hat{v}(y, t) = 0, & y \in \mathbb{R}/(-h_0, h_0), \ t \geq 0, \\
\hat{u}(y, 0) = 0, \ \hat{v}(y, 0) = 0, & y \in \mathbb{R},
\end{cases}
\]

where

\[
F_1 = A^2(h_1, g_1, y) - A^2(h_2, g_2, y) |d\hat{u}_{1,yy}|
\]

\[
+ [C(h_1, g_1, y)d - C(h_2, g_2, y)d + B(h_2, g_2, y) - B(h_1, g_1, y)]|\hat{u}_{1,y}| - \mu(u_1 - u_2) - \beta(h_1, g_1)u_1v_1 + \beta(h_2, g_2)u_2v_2
\]

\[
+ \gamma(h_1, g_1)v_1 - \gamma(h_2, g_2)v_2,
\]

and

\[
F_2 = A^2(h_1, g_1, y) - A^2(h_2, g_2, y) |d\hat{v}_{1,yy}|
\]

\[
+ [C(h_1, g_1, y)d - C(h_2, g_2, y)d + B(h_2, g_2, y) - B(h_1, g_1, y)]|\hat{v}_{1,y}| - \mu(u_1 - u_2) + \beta(h_1, g_1)u_1v_1 - \beta(h_2, g_2)u_2v_2
\]

\[
- \gamma(h_1, g_1)v_1 + \gamma(h_2, g_2)v_2,
\]

Since

\[
||\hat{u}||_{C^{1+, \frac{1}{2}}(\bar{D}_T)} + ||\hat{v}||_{C^{1+, \frac{1}{2}}(\bar{D}_T)} \leq C_1,
\]

we can use the standard $L^p$ theory and the Sobolev imbedding theorem again to get

\[
||\hat{u}_1 - \hat{u}_2||_{C^{1+, \frac{1}{2}}(\bar{D}_T)} + ||\hat{v}_1 - \hat{v}_2||_{C^{1+, \frac{1}{2}}(\bar{D}_T)}
\]

\[
\leq C_3 \left(||u_1 - u_2||_{L^\infty(\bar{D}_T)} + ||v_1 - v_2||_{C(\bar{D}_T)}\right)
\]

\[
+ C_3 \left(||g_1 - g_2||_{C([0, T])} + ||h_1 - h_2||_{C([0, T])}\right),
\]

where $C_3$ depends on $A$, $B$, $C$, $C_1$ and $k$. On the other hand, directly taking the derivative of (10) and making use of (11) yield

\[
||\hat{h}_1 - \hat{h}_2||_{C^0(\bar{D}_T)} + ||\hat{g}_1 - \hat{g}_2||_{C^0(\bar{D}_T)} \leq 2k \left(||\hat{v}_{1,y} - \hat{v}_{2,y}||_{C^0(\bar{D}_T)}\right).
\]

Combine (12) and (13), then we obtain

\[
||\hat{u}_1 - \hat{u}_2||_{C^{1+, \frac{1}{2}}(\bar{D}_T)} + ||\hat{v}_1 - \hat{v}_2||_{C^{1+, \frac{1}{2}}(\bar{D}_T)}
\]

\[
+ ||\hat{h}_1 - \hat{h}_2||_{C^0(\bar{D}_T)} + ||\hat{g}_1 - \hat{g}_2||_{C^0(\bar{D}_T)}
\]

\[
\leq C_4 \left(||u_1 - u_2||_{L^\infty(\bar{D}_T)} + ||v_1 - v_2||_{C(\bar{D}_T)}\right)
\]

\[
+ C_4 \left(||g_1 - g_2||_{C([0, T])} + ||h_1 - h_2||_{C([0, T])}\right),
\]

where $C_4$ depends on $A$, $B$, $C$, $C_1$ and $k$.
where $C_3$ only depends on $C_3$ and $k$. Now, using similar arguments to the discussion below equation (11) we can find a sufficiently small $T > 0$ such that
\[
\|\tilde{u}_1 - \tilde{u}_2\|_{C(\bar{D}_T^\infty)} + \|v_1 - v_2\|_{C(\bar{D}_T^\infty)} + \|\tilde{h}_1 - \tilde{h}_2\|_{C([0,T])} + \|\tilde{g}_1 - \tilde{g}_2\|_{C([0,T])}
\leq T^{\frac{1}{2}} \left( \|u_1 - u_2\|_{C^1(\bar{D}_T^\infty)} + \|v_1 - v_2\|_{C^1(\bar{D}_T^\infty)} \right)
+ T\tilde{\gamma} \left( \|\tilde{h}_1 - \tilde{h}_2\|_{C^\infty([0,T])} + \|\tilde{g}_1 - \tilde{g}_2\|_{C^\infty([0,T])} \right)
\leq \frac{1}{2} \|u_1 - u_2\|_{L^\infty(\bar{D}_T^\infty)} + \|v_1 - v_2\|_{C(\bar{D}_T^\infty)}
+ \frac{1}{2} \|g_1 - g_2\|_{C([0,T])} + \|h_1 - h_2\|_{C([0,T])}.
\]
This means that for an appropriate $T$, $\tilde{\gamma}$ is a contraction mapping on $\Gamma_T$ and has a unique fixed point $(u,v,g,h)$ in $\Gamma_T$. This implies $(u,v,g,h)$ is a solution of (6). Otherwise, by the Schauder estimates, we get the additional regularity for $(u,v,g,h)$, i.e., $u \in C^{1+\nu,\frac{1}{\nu}}(\bar{D}_T^\infty)$, $v \in C^{1+\nu,\frac{1}{\nu}}(\bar{D}_T^\infty)$, $g \in C^{1+\nu}(0,T)$ and $h \in C^{1+\nu}(0,T)$. Thus $(u,v,g,h)$ is a unique local classical solution of problem (6). The proof is completed. \(\square\)

**Theorem 2.2.** Let $(S,I,g,h)$ be a bounded solution of problem (1) for $t \in (0,T)$ with some $T \in (0, +\infty)$. Then there exist positive constants $C_1$, $C_2$ and $C_3$ independent of $T$ such that
\[
0 < S(x,t) \leq C_1 \quad \text{for } x \in \mathbb{R}, \ t \in (0,T),
0 < I(x,t) \leq C_2 \quad \text{for } x \in (g(t), h(t)), \ t \in (0,T),
0 < -g'(t) \leq C_3, \ 0 < h'(t) \leq C_3 \quad \text{for } t \in (0,T).
\]

**Proof.** It follows from (4) that $S(x,t)$ and $I(x,t)$ are bounded on $\mathbb{R} \times (0,T)$. Then, considering the initial value condition (2), and applying the strong maximum principle to $S$ and $I$, we have
\[
S(x,t) > 0 \text{ for } x \in \mathbb{R}, \ t \in (0,T) \quad \text{and} \quad I(x,t) > 0 \text{ for } x \in (g(t), h(t)), \ t \in (0,T).
\]

Furthermore, applying the strong maximum principle to the following equations
\[
\begin{align*}
I_x(g(t),t) &> 0, \quad I_x(h(t),t) < 0 \quad \text{for } t \in (0,T).
\end{align*}
\]
Hence $g'(t) < 0$ and $h'(t) > 0$ for $t \in (0,T)$.

Set $N(x,t) = S(x,t) + I(x,t)$ for $(x,t) \in \mathbb{R} \times (0, +\infty)$, then from (1) and (2) we see that $N(x,t)$ satisfies
\[
\begin{align*}
N_t &= d\Delta N + \sigma - \mu N, \quad x \in \mathbb{R}, \ t \in (0,T),
N(x,0) &= N_0(x) = S_0(x) + I_0(x),
N_0(x) &\in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) \quad \text{and} \quad N_0(x) > 0 \text{ for } x \in \mathbb{R},
\end{align*}
\]
It follows from the comparison principle that $N(x,t) \leq \tilde{N}(t)$ for $x \in (g(t), h(t))$ and $t \in (0,T)$, where
\[
\tilde{N}(t) \triangleq \frac{\sigma}{\mu} + (\|N_0\|_{L^\infty} - \frac{\sigma}{\mu}) e^{-\mu t},
\]
which is the solution of the Cauchy problem

\[ \dot{N}_t = \sigma - \mu \dot{N}, \quad t > 0, \quad \dot{N}(0) = \|N_0\|_{L^\infty}. \quad (16) \]

By the comparison principle in [6], we have

\[ S(x,t) \leq N(x,t) \leq \bar{N}(t) \leq \max_{t\in[0,\infty)} N(t) \triangleq C_1 \quad \text{for } x \in (g(t),h(t)), \quad t \in (0,T), \]

from which we also have \( I(x,t) \leq C_2 \) for \( x \in (g(t),h(t)) \) and \( t \in (0,T) \).

In what follows, we shall show that \( 0 < h(t) \leq M^{-1} < h(t) < x(h(t)) \) for \( t \in (0,T) \), where \( C_3 \) is independent of \( T \). Here we just give the proof about \( h'(t) \), because the proof about \( g'(t) \) is similar. Define

\[ \Omega_M \triangleq \{(x,t) : 0 < t < T, h(t) - M^{-1} < x < h(t)\} \]

and the function

\[ w(x,t) \triangleq C_1[2M(h(t) - x) - M^2(h(t) - x)^2], \]

where \( M \) is a positive constant to be chosen later. By direct calculation we have

\[
\begin{align*}
w_x &= 2C_1 M h'(t)[1 - M(h(t) - x)] \leq 0, \\
w_{xx} &= -2MC_1[1 - M(h(t) - x)], \\
w_{xxx} &= -2C_1 M^2,
\end{align*}
\]

and

\[ -\mu I + \beta(x)SI - \gamma(x)I \leq \bar{\beta} C_1^2, \quad \bar{\beta} = \max_{x \in \mathbb{R}} \beta(x). \]

Thus, if \( M^2 \geq \bar{\beta} C_1/2d \) then

\[ w_t - dw_{xx} \geq 2dC_1 M^2 \geq \bar{\beta} C_1^2 \quad \text{in } \Omega_M. \]

Now, we compare the values of \( w \) and \( I \) on the boundary of \( \Omega_M \). First, we have

\[ w(h(t), t) = 0 = I(h(t), t) \]

and

\[ w(h(t) - M^{-1}, t) = C_1 \geq I(h(t) - M^{-1}, t) \]

for \( t \in (0,T) \). Then we compare \( w(x,0) \) with \( I_0(x) \). Note that

\[ w_x(x,0) = -2MC_1[1 - M(h(0) - x)] \leq -C_1 M \]

for \( x \in [h_0 - (2M)^{-1}, h_0] \). Hence, if we choose \( M \) to be

\[ M = \max \left\{ \sqrt{\frac{\bar{\beta} C_1}{2d}}, \frac{4\|I_0\|_{C^1([-h_0,h_0])}}{3C_1} \right\}, \quad (17) \]

then

\[ w_x(x,0) \leq \frac{4\|I_0\|_{C^1([-h_0,h_0])}}{3} \leq I_0'(x) \]

for \( x \in [h_0 - (2M)^{-1}, h_0] \). Note that \( w(h_0,0) = I_0(h_0) = 0 \), then we have \( w(x,0) \geq I_0(x) \) for \( x \in [h_0 - (2M)^{-1}, h_0] \). Moreover, for \( x \in [h_0 - M^{-1}, h_0 - (2M)^{-1}] \), we have

\[ w(x,0) \geq \frac{3}{4} C_1, \]

\[ |I_0'(x)| \leq \|I_0\|_{C^1([-h_0,h_0])} = \frac{3}{4} C_1 M, \]

\[ I_0(x) \leq M^{-1} |I_0'(x)| \leq \frac{3}{4} C_1. \]
Thus, if \( M \) satisfies (17), then \( w(x, 0) \geq I_0(x) \) for \( x \in [h_0 - M^{-1}, h_0] \). Hence, using
the maximum principle, we have \( w(t, h(t)) = 0 \) for \((x, t) \in \Omega_M \). Note that
\( w(h(t), t) = I(h(t), t) = 0 \), then we have
\[
[w(h(t), t) - I(h(t), t)]_x < 0, \quad I_x(h(t), t) > w_x(h(t), t) = -2MC_1,
\]
which implies that
\[
h'(t) = -kI_x(h(t), t) < 2kMC_1 \triangleq C_2.
\]
It is easy to see that \( C_2 \) is independent of \( T \). Therefore, we complete the proof. \( \square \)

**Theorem 2.3.** System (1)-(2) has a unique solution \((S, I)\), which exists for all
\( t \in (0, \infty) \).

**Proof.** Let \([0, t_{\max}]\) be the maximal time interval in which the solution exists. Suppose
on the contrary that \( t_{\max} < \infty \). By Theorem 2.2, there exist positive constants
\( C_1, C_2 \) and \( C_3 \) such that for \( t \in (0, t_{\max}) \)
\[
0 < S(x, t) \leq C_1 \quad \text{for } x \in \mathbb{R}, \quad t \in (0, t_{\max}),
0 < I(x, t) \leq C_2 \quad \text{for } x \in (g(t), h(t)), \quad t \in (0, t_{\max}),
0 < -g'(t) \leq C_3, \quad 0 < h'(t) \leq C_3 \quad \text{for } t \in (0, t_{\max}).
\]
So we can find an appropriate pair of \( \theta_0 \) and \( T_0 \) such that \( \theta_0 \in (0, t_{\max}) \), \( \theta_0 + T_0 > t_{\max} \) and
\[
0 < T_0 < \frac{h(\theta_0)}{8(1 + |g'(\theta_0)| + |h'(\theta_0)|)}.
\]
Now consider \( \theta_0 \) as the initial time, straighten the free boundaries for (1) again as
the first part of the proof of Theorem 2.1, and use the standard parabolic regularity,
then we can find \( C_4 > 0 \) depending only on \( \theta_0, T_0, C_1, C_2 \) and \( C_3 \) such that
\[
\|S(\cdot, t)\|_{C^2(\mathbb{R})}, \quad \|I(\cdot, t)\|_{C^2([g(t), h(t)])} \leq C_4
\]
for \( t \in [\theta_0, t_{\max}] \). By the proof of Theorem 2.1, we can find a constant \( \tau > 0 \)
depending only on \( C_i \) \((i = 1, 2, 3, 4)\) such that for arbitrary \( t^* \in [\theta_0, t_{\max}] \), the
solution of problem (1) with initial time \( t^* \) can be extended uniquely to the time
\( t^* + \tau \). But this contradicts the assumption and hence the proof is completed. \( \square \)

3. Eigenvalue problems and basic reproduction numbers. Usually, a basic
reproduction number can be employed to characterize the dynamics of the temporal
and spatial spread of the disease. In this section, we will define a basic reproduction
number based on the definition for Dirichlet boundary conditions. For this purpose,
we first consider the following eigenvalue problem
\[
\left\{ \begin{array}{ll}
d \Delta \phi + \alpha(x)\phi + \lambda \phi = 0 & \text{in } \Omega, \\
\phi = 0 & \text{on } \partial \Omega,
\end{array} \right.
\]
where \( d \) is a positive constant, \( \Omega \subset \mathbb{R} \) is a bounded domain with \( \partial \Omega \) of class \( C^{2+\nu} \),
and \( \alpha(x) \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) is positive somewhere in \( \Omega \). Let \( \lambda_1(d, \alpha, \Omega) \) denote
the principal eigenvalue of the problem (18). It is well known that \( \lambda_1(d, \alpha, \Omega) \) exists
uniquely and the associated eigenfunction, denoted by \( \phi_1 \), can be selected to be
positive on \( \Omega \) and normalized by \( \|\phi_1\|_{L^2(\Omega)} = 1 \). Note that the operator \( \Delta \) is
self-adjoint, then \( \lambda_1(d, \alpha, \Omega) \) can be characterized by the following variational form
\[
\lambda_1(d, \alpha, \Omega) = \inf_{\phi \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_{\Omega} [d |\nabla \phi|^2 - \alpha(x) \phi^2] \, dx}{\int_{\Omega} \phi^2 \, dx}.
\]
The following proposition collects some important properties of $\lambda_1(d, \alpha, \Omega)$. For a proof, see e.g. p. 95 in [3] or p. 69 in [24].

**Proposition 1** ([3, 24]). The principal eigenvalue $\lambda_1(d, \alpha, \Omega)$ of (18) depends smoothly on $d > 0$ and continuously on $\alpha \in L^\infty(\Omega)$. Moreover, it has the following properties:

(i): $\lambda_1(d, \alpha, \Omega)$ is a strictly monotone increasing function of $d$, and a strictly monotone decreasing function of $\alpha$;

(ii): $\lambda_1(d, \alpha, \Omega) \to \lambda_1 = -\max_{x \in \Omega} \alpha(x)$ as $d \to 0$;

(iii): $\lambda_1(d, \alpha, \Omega) \to +\infty$ as $d \to +\infty$.

In view of Proposition 1, we have the following result.

**Corollary 1.** If $\alpha(x)$ is positive on an open subset of $\Omega$, then there exists $d^*(\alpha, \Omega) > 0$ such that $\lambda_1(d, \alpha, \Omega) < 0$ if $0 < d < d^*(\alpha, \Omega)$, that $\lambda_1(d, \alpha, \Omega) = 0$ if $d = d^*(\alpha, \Omega)$ and that $\lambda_1(d, \alpha, \Omega) > 0$ if $d > d^*(\alpha, \Omega)$.

**Proposition 2** (Corollary 2.3 of [3]). For $\Omega_1, \Omega_2 \subset \mathbb{R}$ with $\Omega_1 \subset \Omega_2$, there is $\lambda_1(d, \alpha, \Omega_1) \geq \lambda_1(d, \alpha, \Omega_2)$, with strict inequality if $\Omega_1 \setminus \Omega_2$ is an open set.

Next, we deduce a basic reproduction number related to the eigenvalue problem

$$\begin{cases}
d\Delta \phi + (\beta \nu - \mu - \gamma)\phi + \lambda \phi = 0 & \text{in } \Omega, \\
\phi = 0 & \text{on } \partial \Omega.
\end{cases}$$

(20)

By [12, Theorem 3.2], we have the next two propositions.

**Proposition 3.** Define the basic reproduction number $R_0(\Omega)$ for (1) by

$$R_0(\Omega) = \sup_{\phi \in H_0^1(\Omega) \setminus \{0\}} \left\{ \frac{\int_{\Omega} \beta \phi^2 \, dx}{\int_{\Omega} d|\nabla \phi|^2 + (\mu + \gamma)\phi^2 \, dx} \right\}.$$

Then

(i): $R_0(\Omega)$ is a positive and monotone decreasing function of $d > 0$;

(ii): $R_0(\Omega) \to \max \{\frac{\sigma(\beta(x))}{\mu(x) + \gamma(x)} : x \in \Omega\}$ as $d \to 0$;

(iii): $R_0(\Omega) \to 0$ as $d \to +\infty$;

(iv): $R_0(\Omega) > 1$ when $\lambda_1(d, \beta \nu - \mu - \gamma, \Omega) < 0$, $R_0(\Omega) = 1$ when $\lambda_1(d, \beta \nu - \mu - \gamma, \Omega) = 0$, and $R_0(\Omega) < 1$ when $\lambda_1(d, \beta \nu - \mu - \gamma, \Omega) > 0$.

**Proposition 4.** If $\beta \sigma(x) = \mu - \gamma$ changes the sign, then there exists a threshold value $d^* \in [0, \infty)$ such that $R_0(\Omega) > 1$ for $d < d^*$ and $R_0(\Omega) < 1$ for $d > d^*$. If $\beta \sigma(x) = \mu - \gamma \leq 0$ for all $x \in \Omega$, we have $R_0(\Omega) < 1$ for all $d > 0$.

**Proposition 5.** For $\Omega_1, \Omega_2 \subset \mathbb{R}$ with $\Omega_1 \subset \Omega_2$, there is $R_0(\Omega_1) \leq R_0(\Omega_2)$, with strict inequality if $\Omega_1 \setminus \Omega_2$ is an open set.

The proof of Proposition 5 is similar to that of Corollary 2.3 of [3], and hence is omitted. Notice that the domain for the free boundary problem (1) is changing with $t$, then the basic reproduction number is not a constant and will vary as time $t$ changes. Now we introduce the following basic reproduction number $R_0^F(t)$ for the free boundary problem (1)

$$R_0^F(t) = \sup_{\phi \in W_0^{1,2}(\Omega(t),b(t))) \cap \{\phi \neq 0\}} \left\{ \frac{\int_{\Omega(t)} \phi^2 \, dx}{\int_{\Omega(t)} d|\nabla \phi|^2 + (\mu + \gamma)\phi^2 \, dx} \right\}.$$
Lemma 3.1. $R^F_t(t)$ is strictly monotone increasing function of $t$, that is, if $t_1 < t_2$, then $R^F_{t_1}(t_1) < R^F_{t_2}(t_2)$. In addition, if $-g(t) \to \infty$ and $h(t) \to \infty$ as $t \to \infty$, then

$$\lim_{t \to \infty} R^F_0(t) \geq \frac{\sigma \beta_{\infty}}{\mu (\mu + \gamma_{\infty})}.$$ 

Proof. It follows from Theorem 2.2 that $-g'(t) > 0$ and $h'(t) > 0$ for $t \in [0, \infty)$, which means that $-g(t)$ and $h(t)$ are strictly increasing functions. So by Proposition 5, $R^F_t(t)$ is strictly monotone increasing function of $t$. Assume that $-g(t) \to \infty$ and $h(t) \to \infty$ as $t \to \infty$. It follows from $(H_1)$ that for any given $\epsilon > 0$ there exists a positive constant $x_0$ such that

$$\beta(x) > \beta_{\infty} - \epsilon, \quad \gamma(x) < \gamma_{\infty} + \epsilon,$$

for $|x| > x_0$. Moreover, there exists $T_0$ such that $-g(t) < x_0$ and $h(t) > x_0$ for all $t \geq T_0$. For each fixed $t \in [T_0, \infty)$, let $\psi_t(\cdot) \in C^2[g(t), h(t)]$ be such that

- $\psi_t(x) = 1$ for $x \in \left[ g(t) + \frac{3}{4}h(t) - \frac{3}{4} \right],$
- $\psi_t(x) = 0$ for $x \in \left( g(t), (g(t) + 1) \right] \cup \left[ h(t) - \frac{1}{4}, h(t) \right],$
- $|\psi_t'(x)| \leq 4$ for $x \in [g(t), g(t) + 1] \cup [h(t) - 1, h(t)].$

In view of the definition of $R^F_0(t)$, we obtain

$$R^F_0(t) = \sup_{\phi \in W^{1,2}_0((g(t), h(t))), \phi \neq 0} \left\{ \frac{\sigma \int_{g(t)}^{h(t)} d|\nabla \phi|^2 + (\mu + \gamma) \phi^2 dx}{\int_{g(t)}^{h(t)} d\psi_t^2 dx} \right\}$$

$$\geq \frac{\sigma}{\mu} \int_{g(t)}^{h(t)} d|\nabla \psi_t|^2 + (\mu + \gamma) \psi_t^2 dx$$

$$\geq \frac{\sigma}{\mu} \left( \int_{g(t)}^{h(t) + 1} \int_{h(t) - 1}^{h(t)} + \int_{h(t) - 1}^{h(t)} \right) \beta \psi_t^2 dx + \frac{\sigma}{\mu} \left( \int_{g(t)}^{h(t) + 1} \int_{h(t) - 1}^{h(t)} + \int_{h(t) - 1}^{h(t)} \right) \beta dx$$

$$\geq \frac{\sigma}{\mu} \left( \int_{g(t)}^{h(t) + 1} \int_{h(t) - 1}^{h(t)} + \int_{h(t) - 1}^{h(t)} \right) \beta dx$$

$$\geq \frac{32d + (h(t) - 2x_0 - g(t))(\mu + \gamma_{\infty} + \epsilon) + 2x_0(\mu + \max_{x \in [-x_0, 0]} \gamma(x))}{\mu (\mu + \gamma_{\infty})}$$

for $t \in [T_0, \infty)$. Moreover,

$$\liminf_{t \to \infty} R^F_0(t) \geq \liminf_{t \to \infty} \frac{\sigma (h(t) - 2x_0 - g(t) - 2)(\beta_{\infty} - \epsilon) + 2x_0(\mu + \max_{x \in [-x_0, 0]} \gamma(x))}{\mu (\mu + \gamma_{\infty})}$$

$$= \frac{\sigma \beta_{\infty} - \epsilon}{\mu (\mu + \gamma_{\infty})}.$$ 

Note that $R^F_0(t)$ is monotonic and $\epsilon$ is arbitrary, then $\lim_{t \to \infty} R^F_0(t) \geq \frac{\sigma \beta_{\infty}}{\mu (\mu + \gamma_{\infty})}$. \hfill \square

Remark 1. Under the assumptions $(H1)$ and $(H2)$, if $-g(t) \to \infty$ and $h(t) \to \infty$ as $t \to \infty$ then $\lim_{t \to \infty} R^F_0(t) > 1.$
4. Sharp criteria for spreading and vanishing. Our main purpose of this section is to establish a spreading-vanishing dichotomy theorem for the free boundary problem (1). We shall show that if the domain \((g(t), h(t))\) of \(I(x, t)\) is finite in the end, then \(I(x, t) \to 0\) locally uniformly as \(t \to \infty\), which implies the disease will die out. Conversely, if the domain \((g(t), h(t))\) spreads to the whole area, then each solution \((S, I)\) for system (1) tends to some positive steady state.

In view of Theorems 2.2 and 2.3, we have \(g'(t) < 0\) and \(h'(t) > 0\) for \(t \in \mathbb{R}\). Hence, \(g(t)\) is strictly decreasing but \(h(t)\) is strictly increasing with respect to \(t\), and both of their limits exist. Here we define

\[
g_\infty = \lim_{t \to +\infty} g(t), \quad h_\infty = \lim_{t \to +\infty} h(t).
\]

**Theorem 4.1.** If \(-\infty < g_\infty < h_\infty < \infty\), then \(\lim_{t \to \infty} \|I(\cdot, t)\|_{C([g(t), h(t)])} = 0\), and \(\lim_{t \to \infty} S(x, t) = \sigma/\mu\) locally uniformly for \(x \in (-\infty, +\infty)\).

**Proof.** We first prove that \(\lim_{t \to \infty} \|I(\cdot, t)\|_{C([\hat{g}(t), h(t)])} = 0\). Suppose on the contrary that \(\limsup_{t \to \infty} \|I(\cdot, t)\|_{C([\hat{g}(t), h(t)])} = \delta > 0\) for some positive constant \(\delta\). Then we can find a sequence \((x_k, t_k) \in [g(t), h(t)] \times (0, \infty)\) with \(t_k \to \infty\) as \(k \to \infty\) such that

\[
I(x_k, t_k) = \frac{\delta}{2} \quad \text{for all } k \in \mathbb{N}.
\]

Note that \(x_k \in [g(t_k), h(t_k)] \subseteq [g_\infty, h_\infty]\) and \(-\infty < g_\infty < h_\infty < \infty\), then by passing to a subsequence if necessary, there exists \(x_0 \in [g_\infty, h_\infty]\) such that \(x_k \to x_0\) as \(k \to \infty\). Define two sequences of functions sequence as follows

\[
I_k(x, t) = I(x, t + t_k), \quad S_k(x, t) = S(x, t + t_k)
\]

for \(x \in [g(t + t_k), h(t + t_k)]\) and \(t \in [-t_k, \infty)\). In view of Theorem 2.1, for any \(\nu \in (0, 1)\) there exists a constant \(\tilde{C}\) such that the solution \((S, I, g, h)\) of (1) satisfies

\[
\|S\|_{C^{1+\nu}([g(t), h(t)] \times [0, \infty))} < \tilde{C}, \quad \|I\|_{C^{1+\nu}([g(t), h(t)] \times [0, \infty))} < \tilde{C},
\]

and

\[
\|g\|_{C^{1+\nu}([0, \infty))} < \tilde{C}, \quad \|h\|_{C^{1+\nu}([0, \infty))} < \tilde{C}.
\]

Following the standard parabolic regularity, there exists a subsequence \(\{(S_{k_i}, I_{k_i})\}\) of \(\{(S_k, I_k)\}\) such that \((S_{k_i}, I_{k_i}) \to (\hat{S}, \hat{I})\) as \(k_i \to \infty\), where \((\hat{S}, \hat{I})\) is a solution to the following system

\[
\begin{align*}
S_t - d\Delta S &= \sigma - \mu S - \beta(x)SI + \gamma(x)I, \quad x \in (g_\infty, h_\infty), \quad t \in (0, \infty), \\
I_t - d\Delta I &= -\mu I + \beta(x)SI - \gamma(x)I, \quad x \in (g_\infty, h_\infty), \quad t \in (0, \infty).
\end{align*}
\]

Note that

\[
\hat{I}(x_0, 0) = \lim_{k_i \to \infty} I_{k_i}(x_k, 0) = \lim_{k_i \to \infty} I(x_k, t_k) \geq \frac{\delta}{2}.
\]

It follows from the maximum principle that \(\hat{I} > 0\) in \((g_\infty, h_\infty) \times (0, \infty)\). So, by the Hopf boundary lemma at the points \((g_\infty, 0)\) and \((h_\infty, 0)\), where \(\hat{I}(g_\infty, 0) = \hat{I}(h_\infty, 0) = 0\), there exist two positive constants \(c_1\) and \(c_2\) such that

\[
\hat{I}_x(g_\infty, 0) = -c_1 < 0, \quad \hat{I}_x(h_\infty, 0) = -c_2 < 0.
\]

Thus, for all large enough \(k_i\), we have

\[
\begin{align*}
I_x(g(t_{k_i}), t_{k_i}) &= \partial_x I_{k_i}(g(t_{k_i}), 0) < -\frac{c_1}{2} < 0, \\
I_x(h(t_{k_i}), t_{k_i}) &= \partial_x I_{k_i}(h(t_{k_i}), 0) < -\frac{c_2}{2} < 0,
\end{align*}
\]
Thus, it follows from the first equation of (1) that
\[
\lim_{t \to \infty} h'(t_k) = -kI_x(h(t_k), t_k) \geq \frac{kc_1}{2} > 0,
\]
\[
g'(t_k) = kI_x(g(t_k), t_k) \leq -\frac{kc_2}{2} < 0.
\]
On the other hand, it follows from \(-\infty < g_\infty < h_\infty < \infty\) that \(\lim_{t \to \infty} g'(t) = \lim_{t \to \infty} h'(t) = 0\), which is a contradiction. Hence,
\[
\lim_{t \to \infty} ||I(\cdot, t)||_{C((g(t), h(t)))} = 0.
\]
Thus, it follows from the first equation of (1) that \(\lim_{t \to \infty} S(x, t) = \sigma/\mu\) locally uniformly for \(x \in \mathbb{R}\).

The following comparison principle is similar to that in [4] and [8].

**Lemma 4.2.** Suppose that \(T \in (0, \infty), h, \tilde{h}, g, \bar{g} \in C^1([0, T]), \tilde{N}, \bar{N} \in C^{2,1}(\mathbb{R} \times (0, T)) \cap C(\mathbb{R} \times [0, T]), \tilde{I}, \bar{I} \in C(D^\prime_T) \cap C^{2,1}(D^\prime_T), I \in C(D^\prime_T) \cap C^{2,1}(D^\prime_T)\) satisfy

\[
\begin{align*}
\tilde{N}_t - d\Delta \tilde{N} & \geq \sigma - \mu \tilde{N}, & x & \in \mathbb{R}, \\
\tilde{I}_t - d\Delta \tilde{I} & \geq -\mu \tilde{I} + \beta(x)(\tilde{N} - \tilde{I}) - \gamma(x)\tilde{I}, & x & \in (\tilde{g}(t), \tilde{h}(t)), \\
\bar{N}_t - d\Delta \bar{N} & \leq -\mu \bar{N} - \beta(x)(\bar{N} - \bar{I}) - \gamma(x)\bar{I}, & x & \in (\bar{g}(t), \bar{h}(t)), \\
\bar{I}_t - d\Delta \bar{I} & \leq -\mu \bar{I} + \beta(x)(\bar{N} - \bar{I}) - \gamma(x)\bar{I}, & x & \in (\bar{g}(t), \bar{h}(t)), \\
I(x, t) & = 0, & x & \in \mathbb{R}/(\bar{g}(t), \bar{h}(t)), \\
\tilde{g}(t) & \leq -kI_x(\tilde{g}(t), t), & \tilde{h}'(t) & \geq -kI_x(\tilde{h}(t), t), \\
\bar{g}(t) & \geq kI_x(\bar{g}(t), t), & \bar{h}'(t) & \leq -kI_x(\bar{h}(t), t),
\end{align*}
\]

for \(0 < t \leq T\) and
\[
\tilde{N}(x, 0) \geq N_0(x) \geq N(x, 0), \quad \tilde{I}(x, 0) \geq I_0(x) \geq I(x, 0)
\]
for \(x \in \mathbb{R}\) and \(\tilde{g}(0) \leq -h_0 \leq \bar{g}(0), \tilde{h}(0) \geq \bar{h}(0)\), where
\[
D^\prime_T = \{(x, t) \in \mathbb{R}^2 : \tilde{g}(t) < x < \tilde{h}(t), 0 < t \leq T\},
\]
\[
D^\prime_T = \{(x, t) \in \mathbb{R}^2 : \bar{g}(t) < x < \bar{h}(t), 0 < t \leq T\}.
\]

Then the unique solution \((S, I; g, h)\) of free boundary problem (1)-(2) satisfies
\[
\tilde{N}(x, t) \leq N(x, t) \leq \bar{N}(x, t), \quad \tilde{I}(x, t) \leq I(x, t) \leq \bar{I}(x, t),
\]
and
\[
g(t) \geq \tilde{g}(t) \geq \bar{g}(t), \quad h(t) \leq \tilde{h}(t) \leq \bar{h}(t)
\]
for \(x \in \mathbb{R}\) and \(0 < t \leq T\).

The proof of Lemma 4.2 is similar to that of Lemma 3.5 in [8] and hence is omitted.

**Theorem 4.3.** Under the assumptions (H1) and (H2), if \(h_\infty = -g_\infty = \infty\) then \((S, I) \to (\tilde{S}, \tilde{I})\) locally uniformly as \(t \to \infty\) for \(x \in \mathbb{R}\), where \(\tilde{S} = \frac{\sigma}{\mu} - \tilde{I} > 0\) and \(\tilde{I}\) is the unique positive steady state of the following equation

\[
d\Delta \tilde{I} + \left(\frac{\sigma \beta(x)}{\mu} - \mu - \gamma(x) - \beta(x)\tilde{I}\right) \tilde{I} = 0, \quad x \in \mathbb{R}.
\]
Proof. If \((S,I)\) is the solution of \((1)-(2)\), then \(N(x,t) = S(x,t) + I(x,t)\) satisfies \((15)\). By [17, p. 320, Theorem 5.1], the Cauchy problem \((15)\) admits a unique positive solution \(N(x,t)\) for \((x,t) \in \mathbb{R} \times [0,\infty)\), which has the Schauder estimate
\[
\|N\|_{C^{1+\nu,1/2}([0,T],\mathbb{R})} \leq C_1(\|\sigma\| + \|N_0\|_{C^{1+\nu,1/2}(\mathbb{R})}) \leq C_2.
\]
Note that \(-\Delta\) has only nonnegative real eigenvalues, then system \((15)\) has a globally asymptotically stable steady-state solution \(\hat{N} = \frac{\sigma}{\mu}\), which is the unique bounded solution of
\[
d\Delta \hat{N} + \sigma - \mu \hat{N} = 0, \quad x \in \mathbb{R}. \tag{23}
\]
Furthermore,
\[
\lim_{t \to \infty} N(x,t) = \frac{\sigma}{\mu}
\]
exponentially.
When \(h_\infty = -g_\infty = \infty\), the stationary problem for \((1)\) is
\[
\begin{cases}
d\Delta S + \sigma - \mu S - \beta(x)SI + \gamma(x)I = 0, & x \in \mathbb{R}, \\
d\Delta I - \mu I + \beta(x)SI - \gamma(x)I = 0, & x \in \mathbb{R}.
\end{cases} \tag{24}
\]
Substituting \(\hat{S}\) with \(\frac{\sigma}{\mu} - \hat{I}\) into \((24)\) yields
\[
d\Delta \hat{I} + \left(\frac{\sigma \beta(x)}{\mu} - \mu - \gamma(x) - \beta(x)\hat{I}\right)\hat{I} = 0, \quad x \in \mathbb{R}.
\]
The existence and uniqueness of a positive solution of \((22)\) follow from Theorem 2.3 of [9]. Next, we show that \(I \to \hat{I}\) as \(t \to \infty\). The method is inspired by the works in [10, 9, 7]. In view of \(h_\infty = -g_\infty = \infty\) and Remark 1, we have \(\lim_{t \to \infty} R_0^h(t) > 1\). Hence, there exists \(t_0 > 0\) such that \(R_0^h(t) > 1\) for \(t > t_0\). Furthermore, by Proposition 1, there exists sufficient small \(\epsilon > 0\) such that \(\lambda_1(d, \beta(\frac{\sigma}{\mu} - \epsilon) - \mu - \gamma_1, [g(t), h(t)]) < 0\) for \(t > t_0\). Since \(N(\cdot, t) \to \sigma/\mu\) as \(t \to \infty\), we can find \(t_1 > 0\) such that for \(t > t_1\)
\[
\|S(\cdot, t) + I(\cdot, t) - \frac{\sigma}{\mu}\|_{C(\mathbb{R})} < \epsilon.
\]
Let \(T = \max\{t_0, t_1\}\), and consider the following Dirichlet problem
\[
\begin{cases}
\hat{T}_x = d\Delta \hat{T} - \mu \hat{T} + \beta(x)(\frac{\sigma}{\mu} - \epsilon - \hat{I})\hat{T} - \gamma(x)\hat{T}, & x \in (g(T), h(T)), \\
\hat{T}(x,t) = 0, & x \in \mathbb{R}/(g(T), h(T)), \\
\hat{T}(x,T) = I(x,T), & x \in (g(T), h(T))
\end{cases} \tag{25}
\]
for \(t \geq T\). It follows from \(\lambda_1(d, \beta(\frac{\sigma}{\mu} - \epsilon) - \mu - \gamma_1, [g(t), h(t)]) < 0\) that solution \(\hat{T}(x,t)\) of \((25)\) locally uniformly approaches to a positive function \(\tilde{I}(x)\) as \(t \to \infty\), which is a unique positive solution of the following equation
\[
d\Delta \tilde{I} - \mu \tilde{I} + \beta(x)(\frac{\sigma}{\mu} - \epsilon - \hat{I})\hat{T} - \gamma(x)\tilde{I} = 0, \quad x \in (g(T), h(T)).
\]
On the other hand, using comparison principle (Lemma 4.2) we have \(I(x,t) \geq \tilde{I}(x,t)\) for \(t > T\), and so \(\lim_{t \to \infty} I(x,t) \geq \tilde{I}(x)\) for \(x \in \mathbb{R}\).
Consider the following boundary blow-up problem
\[
\begin{cases}
\hat{T}_x = d\Delta \hat{T} - \mu \hat{T} + \beta(x)(\frac{\sigma}{\mu} + \epsilon - \hat{T})\hat{T} - \gamma(x)\hat{T}, & x \in (g(T), h(T)), \\
\hat{T}(x,t) = \infty, & x = g(T), x = h(T), \\
\hat{T}(x,T) \geq I(x,T), & x \in (g(T), h(T))
\end{cases} \tag{26}
\]
for \( t \geq T \). We also know that solution \( \bar{I}(x,t) \) of (26) locally uniformly approaches to a positive function \( \bar{I}(x) \) as \( t \to \infty \), which is a unique positive solution of
\[
d \Delta \bar{I} - \mu \bar{I} + \beta(x)\left(\frac{\sigma}{\mu} + \varepsilon - \bar{I} - \gamma(x)\bar{I}\right) = 0, \quad x \in (g(T), h(T))
\]
By the comparison principle (Lemma 4.2) we have \( I(x,t) \leq \bar{I}(x,t) \) for \( t \geq T \), and so \( \limsup_{t \to \infty} I(x,t) \leq \bar{I}(x) \) for \( x \in \mathbb{R} \).

For any given strictly increasing sequence \( \{t_n\}_{n=1}^{\infty} \subset (T, \infty) \) satisfying \( t_n \to \infty \), \( -g(t_n) \to \infty \) and \( h(t_n) \to \infty \) as \( n \to \infty \), consider the following two systems
\[
\begin{aligned}
L_t &= d \Delta L - \mu L + \beta(x)\left(\frac{\sigma}{\mu} - \varepsilon - L - \gamma(x)L\right), \quad x \in (g(t_n), h(t_n)), \\
\bar{I}(x,t) &= 0, \quad x \in \mathbb{R}/(g(t_n), h(t_n)), \\
L(x,t_n) &= I(x,t_n), \quad x \in (g(t_n), h(t_n)),
\end{aligned}
\]
and
\[
\begin{aligned}
T_t &= d \Delta T - \mu T + \beta(x)(2 - \varepsilon - T) - \gamma(x)T, \quad x \in (g(t_n), h(t_n)), \\
\bar{T}(x,t) &= \infty, \quad x = g(t_n), x = h(t_n), \\
\bar{T}(x,t_n) &= I(x,t_n), \quad x \in (g(t_n), h(t_n)).
\end{aligned}
\]
for \( t \geq t_n \). Du and Ma [10] has shown that system (27) (respectively, (28)) admits a unique positive solution, denoted by \( I_{n} \) (respectively, \( T_{n} \)). Then as \( n \to \infty \), \( I_{n} \) increases to the unique positive solution \( \bar{I}_{t+} \) of the following equation
\[
d \Delta I - \mu I + \beta(x)\left(\frac{\sigma}{\mu} - \varepsilon - I - \gamma(x)I\right) = 0, \quad x \in \mathbb{R}.
\]
So, we have \( \liminf_{t \to \infty} I(x,t) \geq \bar{I}_{t+}(x) \) for \( x \in \mathbb{R} \). As \( n \to \infty \), \( T_{n} \) decreases to the unique positive solution \( \bar{T}_{t+} \) for
\[
d \Delta I - \mu I + \beta(x)\left(\frac{\sigma}{\mu} + \varepsilon - I - \gamma(x)I\right) = 0, \quad x \in \mathbb{R}.
\]
So, we have \( \limsup_{t \to \infty} I(x,t) \leq \bar{T}_{t+}(x) \) for \( x \in \mathbb{R} \). Then, by the arbitrarily smallness of \( \varepsilon \), we have \( \limsup_{t \to \infty} I(\cdot,t) \leq \bar{I} \leq \liminf_{t \to \infty} I(\cdot,t) \), which implies that \( \lim_{t \to \infty} I(\cdot,t) = \bar{I} \) uniformly in any bounded subset of \( \mathbb{R} \).

Finally, we argue by contradiction to show that \( \bar{S} \) is strictly positive on \( \mathbb{R} \). Suppose that \( \bar{S}(x) = 0 \) for some \( x \in \mathbb{R} \). Then \( \bar{I}(x) = \frac{x}{\mu} \), as \( \bar{I} \) achieves its maximum on \( \mathbb{R} \) at \( x \), i.e., \( \Delta \bar{I}(x) \leq 0 \). But from (22),
\[
d \Delta \bar{I} - (\mu + \gamma(x)) \frac{x}{\mu} < 0,
\]
a contradiction. The proof is completed. \( \square \)

**Lemma 4.4.** There exist two constants \( g^* < 0 \) and \( h^* > 0 \) such that either \( g(t) \leq g^* \) or \( h(t) \geq h^* \) implies that \( R_0((g(t),h(t))) > 1 \).

**Proof.** The proof is similar to that of Lemma 3.1. We only deal with the existence of \( g^* \), because the existence of \( h^* \) can be treated analogously. From Proposition 5 we know that \( R_0((g(t),h(t))) \geq R_0((g(t),h_0)) \). For convenience, assume that \( g(t) \to -\infty \) as \( t \to \infty \) and \( \psi(x) \in C^2[g(t),h_0] \) satisfy
\[
\begin{aligned}
\psi_t(x) &= 1 \quad \text{for} \quad x \in \left[ g(t) + \frac{3}{4}, h_0 - \frac{3}{4} \right], \\
\psi_t(x) &= 0 \quad \text{for} \quad x \in \left[ g(t), g(t) + \frac{1}{4} \right] \cup \left[ h_0 - \frac{1}{4}, h_0 \right], \\
|\psi(x)| &\leq 4 \quad \text{for} \quad x \in [g(t), g(t) + 1] \cup [h_0 - 1, h_0].
\end{aligned}
\]
Similar to the proof of Lemma 3.1, it follows from (H1) and (H2) that
\[ \liminf_{t \to \infty} R_0((g(t), h_0)) = R_0((-\infty, h_0)) \geq \frac{\sigma (\beta_\infty - \epsilon)}{\mu (\mu + \gamma_\infty + \epsilon)} > 1. \] (29)
Due to the strict monotonicity of \( R_0 \) with respect to its domain, (29) implies that if \( R_0((0, h_0)) \geq 1 \), then \( g^* \) can be an arbitrary number from \((-\infty, 0]\), and if \( R_0((0, h_0)) < 1 \), there exists a unique negative constant \( g^* \) such that \( R_0((g^*, h_0)) = 1 \). Thus, if \( g(t) \leq g^* \) then \( R_0((g(t), h(t))) > R_0((g(t), h_0)) \geq R_0((g^*, h_0)) = 1 \). Thus, the proof is completed.

Lemma 4.5. If \( h_\infty < \infty \) or \( g_\infty > -\infty \), then \(-\infty < g_\infty < h_\infty < \infty \)

Proof. Without loss of generality, we assume that \( h_\infty < \infty \). Now, we show that \( g_\infty > -\infty \) by the contrary. Assume that \( g_\infty = -\infty \), then it follows from Lemma 4.4 that there exists \( T > 0 \) such that \( g(T) = g^* \) and \( R_0((g(T), h(T))) > 1 \). Since \( R_0(t) \) is strictly increasing function of \( t \) by Lemma 3.1, we have
\[ R_0((g(T), h(t))) > R_0((g(T), h(T))) > 1 \text{ for } t > T, \text{ and hence } R_0((g_\infty, h_\infty)) > 1. \]
In view of Proposition 3(iv), we have \( \lambda_1(d, \beta_\infty - \mu - \gamma, (g_\infty, h_\infty)) < 0 \). Using a similar argument as the proof of Theorem 4.3, we see that the solution \( I(x, t) \) of (1) satisfies \( \lim_{t \to \infty} I(x, t) = \hat{I}(x) \) uniformly, where \( \hat{I}(\cdot) \) is the unique positive solution of
\[ d\hat{I} + \left( \frac{\sigma \beta(x)}{\mu} - \mu - \gamma(x) - \beta(x) \hat{I} \right) \hat{I} = 0, \quad x \in (-\infty, h_\infty]. \]

By the Hopf boundary lemma, \( \hat{I}'(h_\infty) < 0 \). On the other hand, since \( h_\infty < \infty \), we have
\[ 0 = \lim_{t \to \infty} h'(t) = -\lim_{t \to \infty} kI_x(h(t), t) = \hat{I}'(h_\infty), \]
which contradicts \( \hat{I}'(h_\infty) < 0 \). So, we have \( g_\infty > -\infty \) and complete the proof.

Combining Theorems 4.1, 4.3 and Lemma 4.5, we immediately obtain the following spreading-vanishing dichotomy:

Theorem 4.6. Let \((S, I, g, h)\) be the solution to problem (1). Then, the following alternatives hold:

(i) Spreading: \(-g_\infty = h_\infty = \infty \) and \( \lim_{t \to +\infty} \left( S(x, t), I(x, t) \right) = \left( \hat{S}(x), \hat{I}(x) \right) \)
locally uniformly for \( x \in \mathbb{R} \), where \( \hat{I} \) is the unique positive solution of the stationary problem (22), and \( \hat{S}(x) = \frac{\beta_\infty}{\mu} - \hat{I} \) is a positive function; or

(ii) Vanishing: \(-\infty < g_\infty < h_\infty < \infty \) and \( \lim_{t \to +\infty} \| I(\cdot, t) \|_{C(\{g(t), h(t)\})} = 0 \)
and \( \lim_{t \to +\infty} \| S(x, t) \| = \sigma/\mu \) locally uniformly for \( x \in \mathbb{R} \).

5. Sufficient condition for disease vanishing or spreading.

Lemma 5.1. If \(-\infty < g_\infty < h_\infty < \infty \), then \( \lim_{t \to \infty} R_0^F(t) \leq R_0((g_\infty, h_\infty)) \leq 1 \).

Proof. By the contradiction we suppose that \( R_0((g_\infty, h_\infty)) > 1 \). Then
\[ R_0((g_\infty, h_\infty)) = \lim_{t \to \infty} R_0^F(t) > 1. \]
Note that \( R_0(\Omega) \) is continuous on the domain \( \Omega \). Then there exists a \( T \) such that for \( t \geq T \), \( (g(T), h(T)) \subset (g(t), h(t)) \subset (g_\infty, h_\infty) \) and
\[ 1 < R_0^F(T) \leq R_0((g_\infty, h_\infty)) \]
which means that \( \lambda_1(d, \beta_\infty - \mu - \gamma, (g(T), h(T))) < 0 \). Thus, there exists sufficient small \( \epsilon > 0 \) such that \( \lambda_1(d, \beta_\infty - \epsilon - \mu - \gamma, (g(T), h(T))) < 0 \). From Theorem
Theorem 5.2. Let \( \epsilon \) be such that \( \epsilon \) is sufficiently small such that
\[
S_t - dS - \sigma + \mu S + \beta(x)S - \gamma(x)I \leq -\mu\epsilon + \left( \beta\mu - \gamma \right) \epsilon_1 \phi(x) \leq 0,
\]
if we let \( \epsilon_1 \) be sufficiently small such that \( \left( \beta(\mu^2 - \epsilon) - \gamma \right) \epsilon_1 \phi(x) \leq |\mu\epsilon| \). Moreover,
\[
I_t - dI + \mu I - \beta(x)S + \gamma(x)I \leq -\left( \beta(\mu^2 - \epsilon) - \gamma \right) \epsilon_1 \phi(x) = \epsilon_1 \lambda_1 \phi(x) < 0.
\]
Let \( N = S + I \), then \((N, I)\) satisfies the following auxiliary problem
\[
\begin{cases}
N_t \leq dN + \sigma - \mu x, & x \in \mathbb{R}, \\
I_t \leq dI - \mu I + \beta(x)N - \gamma(x)I, & x \in (g(T), h(T)), \\
S(x, 0) = \frac{\sigma}{\mu} - \epsilon, & I(x, 0) = \epsilon_1 \phi(x), \\
I(x, t) = 0, & x \in \mathbb{R}, \end{cases}
\]
for \( t \geq T_1 \). It follows from Lemma 4.2 and the comparison principle that
\[
\liminf_{t \to \infty} ||I(\cdot, t)||_{C([g(T), h(T)))} \leq \limsup_{t \to \infty} ||I(\cdot, t)||_{C([g(T), h(T)))} = ||\epsilon_1 \phi(x)||_{C([g(T), h(T))]} > 0.
\]
However, it follows from Theorem 4.6 that \( \lim_{t \to \infty} ||I(\cdot, t)||_{C([g(T), h(T))]} = 0 \), which is a contradiction. The proof is completed.

From Lemma 5.1, we obtain the following result.

**Theorem 5.2.** If \( R_0((-h_0, h_0)) \geq 1 \), then \(-g_\infty = h_\infty = \infty\).

**Proof.** Since that \((-h_0, h_0) \subset (g_\infty, h_\infty)\), we have
\[
R_0((g_\infty, h_\infty)) > R_0((-h_0, h_0)) \geq 1.
\]
Then from Lemma 5.1, we get \(-g_\infty = h_\infty = \infty\).

From Lemma 4.4, we immediately obtain the following results about \( g^* \) and \( h^* \) and the spatial domain.

**Theorem 5.3.** If \( |h_0| \geq \min\{h^*, -g^*\} \), then \(-g_\infty = h_\infty = \infty\), where \( g^* \) and \( h^* \) are defined as that in Lemma 4.4.

**Corollary 2.** If \( R_0((-h_0, h_0)) \geq 1 \), then \( \lim_{t \to \infty} I(x, t) = \tilde{I}(x) \) and \( \lim_{t \to \infty} S(x, t) = \tilde{S}(x) \) locally uniformly for \( x \in \mathbb{R} \), where \( \tilde{I} \) is the unique positive solution of the stationary problem (22), \( \tilde{S}(x) = \frac{\sigma}{\mu} - \tilde{I} \) is a positive function.
There are many results on fixed boundary situations. For example, Allen et al. [1], Wu and Zou [29] found that $R_0((-h_0, h_0)) > 1$ or $R_0((-h_0, h_0)) < 1$ is a distinct critical condition for the disease spreading or vanishing. In this paper, however, $R_0((-h_0, h_0)) > 1$ is a sufficient condition ensuring the disease spreading, but $R_0((-h_0, h_0)) < 1$ cannot guarantee the disease vanishing. In what follows, we shall show that even though $R_0((-h_0, h_0)) < 1$, the disease may extinct or survive, which is invariably determined by $I_0$, $k$ and $d$.

**Theorem 5.4.** If $R_0((-h_0, h_0)) < 1$ and $k$ is sufficiently small, then $-\infty < g_{\infty} < h_{\infty} < \infty$.

**Proof.** First we rewrite system with $N(x, t) = S(x, t) + I(x, t)$ as

$$
\begin{aligned}
N_t = d\Delta N + \sigma - \mu N, & \quad x \in \mathbb{R}, \\
I_t = d\Delta I - \mu I + \beta(x)(N - I)I - \gamma(x)I, & \quad x \in (g(t), h(t)), \\
I(x, t) = 0, & \quad x \in \mathbb{R}/(g(t), h(t)), \\
N(x, 0) = N_0(x) - S_0(x) + I_0(x), & \quad I(x, 0) = I_0(x), \quad x \in \mathbb{R}, \\
g'(t) = -kI_x(g(t), t), & \quad g(0) = -h_0, \\
h'(t) = -kI_x(h(t), t), & \quad h(0) = h_0
\end{aligned}
$$

(32)

for $t \geq 0$. It follows from $R_0((-h_0, h_0)) < 1$ that $\lambda_1(d, \beta; \mu - \gamma, [-h_0, h_0]) > 0$, and hence there exists $\epsilon > 0$ small enough such that $\lambda_1(d, \beta; \mu + \epsilon - \gamma, [-h_0, h_0]) > 0$. Note that $N(x, t) \to \frac{N_0}{t}$ locally uniformly and exponentially as $t \to \infty$ for $x \in \mathbb{R}$, there exists $T > 0$ such that $N(x, t) \leq \frac{N_0}{t} + \epsilon$ for $t \geq T$. Also note that $\lambda_1(\Omega)$ is continuous with respect to its domain, we can find a constant $\tau$ such that $\lambda_1(d, \beta; \mu + \epsilon - \gamma, [-h_0 - \tau, h_0 + \tau]) > 0$. Thanks to (5), if $k \leq \tau/CT$ with $C$ is defined as in (5), then $\lambda_1^* \equiv \lambda_1(d, \beta; \mu + \epsilon - \gamma, [g(T), h(T)]) > 0$. As in [27], we define

$$
s(t) = 1 + 2\delta - \delta e^{-\eta t}, \quad \overline{N}(x, t) = \frac{\sigma}{\mu} + \epsilon, \quad t \geq T,
$$

and

$$
\overline{I}(x, t) = \begin{cases}
K e^{\eta t} \varphi\left(\frac{\sigma t}{s(t)}\right), & g(T)s(t) \leq x \leq h(T)s(t), \quad t \geq T, \\
0, & x > h(T)s(t) \text{ or } x < g(T)s(t), \quad t \geq T,
\end{cases}
$$

where $0 < \delta, \eta < 1$ and $K > 0$ are constants to be determined later, $\varphi(x)$ is a positive eigenfunction associated with $\lambda_1^*$. Denote $y = x/s(t)$, a direct calculation yields

$$
\begin{aligned}
\overline{I}_t - d\Delta \overline{I} + \mu \overline{I} - \beta(x)(\overline{N} - \overline{I}) + \gamma(x)\overline{I} \\
\geq \overline{I}_t - d\Delta \overline{I} + \mu \overline{I} - \beta(x)\overline{N}\overline{I} + \gamma(x)\overline{I} \\
= K e^{\eta t} \left(-\eta \varphi(y) - \varphi'(y) \frac{xs'(t)}{s^2(t)} - d\varphi''(y) \frac{1}{s^2(t)} + \mu \varphi(y)\right) \\
+ K e^{\eta t} \left(-\beta(x) \frac{\sigma}{\mu} + \epsilon \right) \varphi(y) + \gamma(x) \varphi(y) \\
= K e^{\eta t} \varphi(y) \left(-\eta - \frac{\varphi'(y) y \delta e^{-\eta t}}{\varphi(y)} - (\mu - \beta(x) \frac{\sigma}{\mu} + \epsilon + \gamma(y)) \right) \\
- \frac{1}{s^2(t)} (\mu - \beta(x) \frac{\sigma}{\mu} + \epsilon + \gamma(y)) + \lambda_1^* \frac{\varphi(y)}{s^2(t)}
\end{aligned}
$$

for $g(T)s(t) \leq x \leq h(T)s(t), \quad t \geq T$. 


Since \( \varphi'(g(T)) > 0 \), and \( \varphi'(h(T)) < 0 \), it is easy to see that there exists \( M > 0 \) such that
\[
x \varphi'(x) \leq M \varphi(x) \quad \text{for} \quad g(T) \leq x \leq h(T).
\] (33)

Note that \( \mu - \beta(x)(\frac{\sigma}{\mu} + \epsilon) + \gamma(x) \) is uniformly continuous in \([3g(T), 3h(T)]\), then, for any given \( 0 < \epsilon_1 \ll 1 \), there exists \( 0 < \delta_0(\epsilon_1) \ll 1 \) such that, for all \( 0 < \delta \leq \delta_0(\epsilon_1) \)
\[
  \left| \frac{d}{dt} \left( \frac{x}{s(t)} \right) \right| \leq \epsilon_1, \quad (34)
\]
for \( g(T)s(t) \leq x \leq h(T)s(t), t \geq T \). Then from (33) and (34) we have
\[
  T_{t} - d \Delta \eta - \beta(x)(N - T)\eta - \gamma(x)\eta T + K e^{-\eta t} \varphi(y)(-\eta - M\eta - \epsilon_1 + \frac{\lambda^2}{4}) > 0 \quad (35)
\]
for \( g(T)s(t) \leq x \leq h(T)s(t), t \geq T \) provided that \( \eta \) and \( \epsilon_1 \) are sufficiently small.

Since \( \varphi(x) > 0 \) in \([g(T), h(T)]\), we can choose \( K \) sufficiently large such that
\[
  T(x, T) = K e^{-\eta T} \varphi\left( \frac{x}{s(T)} \right) > I(x, T) \quad \text{for} \quad g(T) \leq x \leq h(T). \quad (36)
\]

Note that \( T_x(g(T)s(t), t) = K e^{-\eta t} \varphi'(g(T)) \frac{1}{s(T)} \), \( T_x(h(T)s(t), t) = K e^{-\eta t} \varphi'(h(T)) \frac{1}{s(T)} \)
and \( g(T)s'(t) = g(T)\eta_0 e^{-\eta t} \), \( h(T)s'(t) = h(T)\eta_0 e^{-\eta t} \), then we can find \( k_0 > 0 \), for example,
\[
  k_0 = \min \left\{ \frac{-g(T)\eta_0(1 + 2\delta)}{K \varphi'(g(T))}, \frac{-h(T)\eta_0(1 + 2\delta)}{K \varphi'(h(T))} \right\},
\]
such that for \( 0 < k < k_0 \)
\[
  g(T)s'(t) \leq -k T_x(g(T)s(t), t), \quad h(T)s'(t) \geq -k T_x(h(T)s(t), t) \quad \text{for} \quad t \geq T. \quad (37)
\]

Now from (35), (36) and (37), and Lemma 4.2 we have

\[
  N(x, t) \leq N(x, T), \quad I(x, t) \leq T_x(x, T), \quad g(t) \geq g(T)s(t) \quad \text{and} \quad h(t) \leq h(T)s(t)
\]
for \( x \in \mathbb{R} \), \( t \geq T \). Thus,
\[
  g_\infty \geq \lim_{t \to \infty} g(T)s(t) = g(T)(1 + 2\delta) > -\infty,
\]
\[
  h_\infty \leq \lim_{t \to \infty} h(T)s(t) = h(T)(1 + 2\delta) < \infty
\]
for all \( 0 < k < k_0 \). The proof is completed. \( \square \)

**Lemma 5.5.** If \( -\infty < g_\infty < h_\infty < \infty \), then \( h_\infty < h^* \) and \( g_\infty > g^* \), where \( g^* \) and \( h^* \) are defined as that in Lemma 4.4.

**Theorem 5.6.** Assume that \( R_0((-h_0, h_0)) < 1 \), then there exists \( k_1 > 0 \) depending on \( I_0 \) and \( h_0 \) such that \( -g_\infty = h_\infty = \infty \) if \( k \geq k_1 \).

**Proof.** The idea of this proof comes from [28, Lemma 3.2], we aim to prove that there exists \( k_1 > 0 \) such that \( g_\infty < g^* \) and \( h_\infty > h^* \) when \( k \geq k_1 \), where \( g^* \) and \( h^* \) are defined as that in Lemma 4.4. Then by Lemma 5.5, we see that the spreading occurs.

From (32) we consider the following auxiliary free boundary problem
\[
  \begin{align*}
    u_t &= d \Delta u - \mu u, & x \in \mathbb{R},
    
    v_t &= d \Delta v - (\mu + \gamma(x))v, & x \in (m(t), n(t)),
    
    v(x, t) &= 0, & x \in \mathbb{R}/(m(t), n(t)),
    
    m'(t) &= -kv_x(m(t), t), \quad m(0) = -h_0,
    
    n'(t) &= -kv_x(n(t), t), \quad n(0) = h_0,
    
    u(x, 0) &= N_0(x) = S_0(x) + I_0(x), \quad v(x, 0) = I_0(x), \quad x \in \mathbb{R}
  \end{align*}
\] (38)
for \( t \geq 0 \), where \( m(t), n(t) \in C^1(0, +\infty) \). The existence and uniqueness of the solution to (38) can be obtained by the similar proof for system (1) in Theorems 2.1-2.3. So, system (38) has a unique global solution \((u, v, m, n)\) satisfying \( m'(t) < 0 \) and \( n'(t) > 0 \) for \( t > 0 \). Note that what we are interested in is the influence of speed \( k \), then we denote solutions by \((N^k, I^k, g^k, h^k)\) and \((u^k, v^k, m^k, n^k)\) instead of \((N, I, g, h)\) and \((u, v, m, n)\). By Lemma 4.2, we have

\[
N^k(x, t) \geq u^k(x, t), \quad I^k(x, t) \geq v^k(x, t), \quad g^k(t) \leq m^k(t) \quad \text{and} \quad h^k(t) \geq n^k(t) \quad (39)
\]

for \( t \geq 0 \) and \( x \in [m^k(t), n^k(t)] \). In what follows, we are going to prove that for all large enough \( k \),

\[
m^k(2) \leq g^* \quad \text{and} \quad n^k(2) \geq h^* \quad (40)
\]

First, smooth functions \( m(t) \) and \( n(t) \) can be chosen to satisfy \(-m(0) = n(0) = h_0/2\), \( m'(t) < 0 \), \( n'(t) > 0 \), \( m^2(2) = g^* \), and \( n^2(2) = h^* \). We then consider the following initial-boundary value problem

\[
\begin{align*}
  u_t &= d \Delta u - \mu u, & x \in \mathbb{R}, & t > 0, \\
  v_t &= d \Delta v - (\mu + \gamma(x)) v, & x \in (m(t), n(t)), & t > 0, \\
  g(x, t) &= 0, & x \in \mathbb{R}/(m(t), n(t)), & t \geq 0, \\
  u(x, 0) &= u_0(x), & x \in \mathbb{R}, \\
  v(x, 0) &= v_0(x), & x \in (-h_0/2, h_0/2).
\end{align*}
\]

Here we assume that the initial value \((u_0, v_0)\) satisfies

\[
\begin{align*}
  0 < u_0(x) &\leq u_0(x) \quad \text{on} \quad (-\infty, +\infty), \\
  0 < v_0(x) &\leq v_0(x) \quad \text{on} \quad [-h_0/2, h_0/2], \\
  v_0(-h_0/2) &= v_0(h_0/2) = 0, \quad v_0'(-h_0/2) > 0, \quad v_0'(h_0/2) < 0.
\end{align*}
\]

The standard theory for parabolic equations ensures that (41) has a unique positive solution \((u, v)\). Moreover, it follows from the Hopf boundary lemma that \( v_0'(m(t), t) > 0, v_0'(n(t), t) < 0 \) for all \( t \in [0, 2] \). Due to the choice of \( m(t), n(t) \) and \((u_0(x), v_0(x))\), on bounded closed set \([0, 2]\) there is a constant \( k_1 > 0 \) such that for all \( k \geq k_1 \),

\[
m'(t) \geq -k v_0'(m(t), t), \quad n'(t) \leq -k v_0'(n(t), t) \quad \text{for} \quad 0 \leq t \leq 2.
\]

Thus, note that \( m(0) = -h_0/2 > m^k(0) \) and \( n(0) = h_0/2 < n^k(0) \), then using Lemma 4.2 again, we have

\[
u^k(x, t) \geq u(x, t), \quad v^k(x, t) \geq v(x, t), \quad m^k(t) \leq m(t) \quad \text{and} \quad n^k(t) \geq n(t)
\]

for all \( 0 \leq t \leq 2 \), \( x \in [m(t), n(t)] \). In particular,

\[
m^k(2) \leq m^k(2) = g^*, \quad n^k(2) \geq n^k(2) = h^*,
\]

which together with (39) and (40), implies that

\[
g_\infty = \lim_{t \to \infty} g^k(t) < g^k(2) \leq g^*, \quad h_\infty = \lim_{t \to \infty} h^k(t) > h^k(2) \leq h^*.
\]

This yields the desired result. The proof is completed. \( \square \)

Here we stress that the condition that \( R_0((-h_0, h_0)) < 1 \) in Theorem 5.4 can be replaced by some restrictions on \( h_0, \beta(x), \gamma(x) \) or \( d \). The following theorem is an example.

**Theorem 5.7.** If \( h_0 \leq \frac{\beta}{2} \sqrt{\frac{d}{\beta C}} \) and \( k \) is sufficiently small, then \(-\infty < g_\infty < h_\infty < \infty\), where \( \beta^* = \max_{x \in \mathbb{R}} \beta(x) \) and \( C \) is defined as in Theorem 2.1.
Proof. In view of Theorem 5.4, it suffices to show that $R_0((-h_0, h_0)) < 1$. First, we consider the following eigenvalue problem

\[
\begin{cases}
  d\Delta \phi + \beta^* C \phi + \lambda \phi = 0, & x \in (-h_0, h_0), \\
  \phi = 0, & x = -h_0 \text{ and } x = h_0.
\end{cases}
\]

(44)

It is well-known that $\lambda_1(d, \beta^* C, (-h_0, h_0)) > 0$ if and only if $h_0 \leq \frac{\pi}{2} \sqrt{\frac{d}{\beta C}}$. Then it follows from Proposition 1 (i) and $\beta(x) \frac{\pi}{2} - \mu - \gamma(x) \leq \beta^* C$ for $x \in (-\infty, +\infty)$ that $\lambda_1(d, \beta^* C, (-h_0, h_0)) \geq \lambda_1(d, \beta^* C, (-h_0, h_0)) > 0$, which, together with Proposition 3 (d), means that $R_0((-h_0, h_0)) < 1$. This, together with Theorem 5.4, yields the desired result.

Theorem 5.7 gives out a sufficient condition ensuring $R_0((-h_0, h_0)) < 1$. That is to say, $-\infty < g_{\infty} < h_{\infty} < \infty$ provided that both $h_0$ and $k$ are sufficiently small. Cao et al. [4] stated that $-\infty < g_{\infty} < h_{\infty} < \infty$ if

$$h_0 \leq \frac{1}{4 \sqrt{\beta C}} \quad \text{and} \quad k \leq \frac{1}{8 M^*},$$

where $M^* = \frac{1}{4} ||I_0||_{L^\infty}$. It is easy to see that the above restriction is stronger than our requirements in Theorem 5.7, which implies that we not only unify but also improve the main results obtained by Cao et al. [4] who investigated the disease vanishing by constructing a suitable upper solution. For completeness, we can apply the same method to the proof of Theorem 4.2 of [4] to our system (1) and obtain the following result.

**Theorem 5.8.** If $k \leq \frac{1}{8 \sigma^2}$ and $h_0 \leq \frac{1}{4 \sqrt{\beta C}}$ then $-\infty < g_{\infty} < h_{\infty} < \infty$, where $C$ is the super bound of $N(x, t)$ obtained in Theorem 2.1, $\beta^* = \max_{x \in \mathbb{R}} \beta(x)$ and $C^* = \frac{1}{4} ||I_0||_{L^\infty}$.

**Proof.** The main idea is to construct a suitable upper solution to (32) and then to use Lemma 4.2. We define

$$N(x, t) = C,$$

$$T(x, t) = \begin{cases}
  C^* e^{-\eta t} \left(1 - \frac{x^2}{\pi \tau^2(t)}\right), & -s(t) \leq x \leq s(t), \\
  0, & x > s(t) \text{ or } x < -s(t),
\end{cases}$$

where $s(t) = 2h_0(2 - e^{-\eta t})$ for all $t \geq 0$. The rest is to prove that under the assignment $\eta = \frac{1}{16 h_0^2}$, $k \leq \frac{1}{8 \sigma^2}$ and $h_0 \leq \frac{1}{4 \sqrt{\beta C}}$, $(N, T)$ is a upper solution to (32); For more details, see the similar proof of Theorem 5.4 and Theorem 4.2 in [4].

**Remark 2.** When $R_0((-h_0, h_0)) < 1$. Theorems 5.4 and 5.6 say that the disease will vanish if the boundary translation speed $k$ is small enough and disease spreading happens if $k$ is large enough. Unfortunately, we cannot suggest a threshold value for $k$ determining wether the disease is spreading or vanishing as stated in [7] who gave out a threshold value $k^*$ such that the vanishing occurs when $k \leq k^*$, and spreading happens when $k > k^*$. The main reason is that the desired spreading parameter $k$ may determined by the accurate domain $(g(t), h(t))$ satisfying $R_0((g(t), h(t))) = 1$. The constants $g^*$ and $h^*$ given in Lemma 4.4 may essentially be the right domain satisfying $R_0((g^*, h^*)) = 1$. However, the desired pair of $g^*$ and $h^*$ is not uniquely determined, that is to say, there maybe exist many different $g^*$ and $h^*$ satisfying $R_0((g^*, h^*)) = 1$ since the disease has two spreading frontiers, which have different spreading speeds respectively. Hence, we cannot find a critical value for $k$. 

Finally, we consider the influence of the coefficient $d$ in system (1). Usually, the diffusion rate $d$ may urge the disease to be vanishing or spreading, or controls the asymptotic profiles of the endemic equilibrium. Here, we care more about the former.

**Theorem 5.9.** If the set $H = \{x \in [-h_0, h_0] : \sigma \beta(x) > \mu^2 + \mu \gamma(x)\}$ is not empty, then there exists $d^* > 0$ such that $-g_\infty = h_\infty = \infty$ provided that either $d \leq d^*$ or $d > d^*$ and $k$ is sufficiently large.

**Proof.** Note that $R_0((-h_0, h_0))$ is continuously dependent on $d$, then we rewrite $R_0((-h_0, h_0))$ as $R_0(d, (-h_0, h_0))$. From Proposition 3 (ii) we have

$$R_0(d, (-h_0, h_0)) \to \max\{\sigma \beta(x)/[\mu^2 + \mu \gamma(x)] : x \in [-h_0, h_0]\} > 1$$

as $d \to 0$, since $H$ is not empty. Then by Proposition 3 (i), there exists a positive constant $d^*$ such that $R_0(d^*, (-h_0, h_0)) = 1$, and $R_0(d, (-h_0, h_0)) > 1$ for $d < d^*$. Thus, following Theorem 5.2, we can get the desired result. When $d > d^*$, then $R_0(d, (-h_0, h_0)) < 1$, which together with Theorem 5.6 implies that $-g_\infty = h_\infty = \infty$ when $k$ is sufficiently large. The proof is completed.

**Theorem 5.10.** If $\sigma \beta(x) < \mu^2 + \mu \gamma(x)$ for all $x \in [-h_0, h_0]$, and $k$ is sufficiently small, then $-\infty < g_\infty < h_\infty < \infty$.

**Proof.** Considering Proposition 3 (ii) and (iii), $\sigma \beta(x) < \mu^2 + \mu \gamma(x)$ for all $x \in [-h_0, h_0]$ means that $R_0((-h_0, h_0)) < 1$ for all $d > 0$. Thus, following Theorem 5.4, we complete the proof.

**Remark 3.** The condition $\sigma \beta(x) < \mu^2 + \mu \gamma(x)$ on $[-h_0, h_0]$ actually implies that $[-h_0, h_0]$ is a low-risk domain. Theorem 5.10 tells that if the whole region $[-h_0, h_0]$ is a low-risk domain, the strategy of limiting the mobility of infected individuals can succeed in eradicating the disease.

**Theorem 5.11.** If $k$ is sufficiently small, then there exists $d_0 > 0$ such that $-\infty < g_\infty < h_\infty < \infty$ provided $d > d_0$.

**Proof.** Using Proposition 3 (i) and (iii), we see that there exists a positive constant $d_0$ such that $R_0(d, (-h_0, h_0)) < 1$ for all $d > d_0$. The last proof follows from Theorem 5.4.

**Remark 4.** With a mild modification, system (1) can be regarded as an invasion model, i.e., the indigenous species $S$ occupied the whole area, and the invasive species $I$ at the initial time only took up a little part $(-h_0, h_0)$, the relationship is competitive. Theorems 5.9 and 5.11 show that the lower mobility, larger initial domain and higher expansion speed can make the invasive species successfully survive eventually no matter whether the initial basic reproductive number is less than 1 or not. Conversely, the faster movement rate and smaller initial domain always make the invasive species die out, because the faster mobility and smaller domain may lead to a rapid loss of individuals across $\partial \Omega$ under the Dirichlet boundary condition. In many similar reaction-diffusion systems with fixed boundary conditions, however, the basic reproductive number being less than 1 always implies that the invasive species will die out eventually.
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