THE DELTA-UNLINKING NUMBER OF ALGEBRAICALLY SPLIT LINKS

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Abstract. It is known that algebraically split links (links with vanishing pairwise linking number) can be transformed into the trivial link by a series of local moves on the link diagram called delta-moves; we define the delta-unlinking number to be the minimum number of such moves needed. This generalizes the notion of delta-unknotting number, defined to be the minimum number of delta-moves needed to move a knot into the unknot. While the delta-unknotting number has been well-studied and calculated for prime knots, no prior such analysis has been conducted for the delta-unlinking number. We prove a number of lower and upper bounds on the delta-unlinking number, relating it to classical link invariants including unlinking number, 4-genus, and Arf invariant. This allows us to determine the precise value of the delta-unlinking number for algebraically split prime links with up to 9 crossings as well as determine the 4-genus for most of these links.

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1. Introduction

An m-component link is the isotopy class of an embedding of $\sqcup_m S^1 \to S^3$; a knot is a 1-component link. A link can be depicted as a diagram representing its projection onto the plane. The $\Delta$-move is the local move on a link diagram that transforms the region within a disk as in Figure 1 and leaves the rest of the diagram unchanged. The $\Delta$-move is known to be a unknotting move [MN89]; therefore, every knot $K$ can be deformed into the unknot via some sequence of $\Delta$-moves. The $\Delta$-unknotting number $u^\Delta(K)$ is the minimal number of $\Delta$-moves needed to deform $K$ into the unknot; it has been calculated for prime knots with up to 10 crossings [NNU98].

In a link, we allow the three strands of the $\Delta$-move to belong to any component(s) of the link; in the case that they all belong to the same component, it is called a self $\Delta$-move as in Figure 6.

We call two links (self) $\Delta$-equivalent if one link can be deformed into the other by (self) $\Delta$-moves involving any number of components.

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The $\Delta$-moves were first introduced in [MN89]. In a link, it is not hard to see that a $\Delta$-move preserves linking number. Murakami and Nakanishi proved the converse, giving the following classification of links up to $\Delta$-equivalence:

**Theorem 1** (Theorem 1.1 in [MN89]). Let $L = L_1 \sqcup L_2 \sqcup \cdots \sqcup L_m$ and $L' = L'_1 \sqcup L'_2 \sqcup \cdots \sqcup L'_m$ be ordered oriented $m$-component links. Then $L$ and $L'$ are $\Delta$-equivalent if and only if $\text{lk}(L_i, L_j) = \text{lk}(L'_i, L'_j)$ for $1 \leq i < j \leq m$, where $\text{lk}$ denotes the linking number.

Given $\Delta$-equivalent links $L$ and $L'$, we can define the (self) $\Delta$-Gordian distance $d_{\Delta G}(L, L')$ between the links to be the minimal number of (self) $\Delta$-moves needed to deform one link into the other. In particular, if an $m$-component link $L$ is algebraically split, that is, if $L$ has vanishing pairwise linking numbers, then $L$ is $\Delta$-equivalent to the trivial link with $m$-components, denoted $0^m$. We denote the distance between an algebraically split link $L$ and the trivial link by $u_{\Delta}(L)$, the $\Delta$-unlinking number.

In the case of knots, it has been shown that $u^\Delta$ and $d_G^\Delta$ relate to many other well-known knot invariants including the unknotting number and Arf invariant [MN89]. We generalize these relations to links (or in the case of the Arf invariant, to proper links) and relate $u^\Delta$ to other link invariants such as the 4-genus.

For instance, in Section 2 we show:

**Proposition 1.** Given an algebraically split link $L$,

$$u^\Delta(L) \geq \frac{1}{2} u(L)$$

where $u(L)$ denotes the unlinking number of $L$.

We also find a relationship with 4-genus:

**Theorem 2.** For $\Delta$-equivalent proper links $L, L'$,

$$d_G^\Delta(L, L') \geq |g_4(L) - g_4(L')|.$$  

And hence:

**Corollary 2.1.** Given an algebraically split link $L$,

$$u^\Delta(L) \geq g_4(L).$$

Moreover, we show the following in Section 3:

**Theorem 3.** Given $\Delta$-equivalent proper links $L, L'$ we have

$$d_G^\Delta(L, L') \equiv \text{Arf}(L) + \text{Arf}(L') \pmod{2}.$$  

It then immediately follows:

**Corollary 3.1.** Given an algebraically split link $L$,

$$u^\Delta(L) \equiv \text{Arf}(L) \pmod{2}.$$  

These and other such bounds allow us to determine $u^\Delta$ for algebraically split prime links up to 9 crossings; see Section 4. We also determine the 4-genus for nearly all of these links.

### 2. Lower bounds on $\Delta$-unlinking number

#### 2.1. Unknotting number and unlinking number

A $\Delta$-move is independent of the choice of orientation and mirroring [MN89]. In particular, if $L$ and $L'$ are $\Delta$-equivalent then so are their mirrors $mL$ and $mL'$. Thus $u^\Delta(L) = u^\Delta(mL)$. The local moves in Figure 2 are equivalent to $\Delta$-moves [TY02]; here the strands may belong to any component(s) of the link, except in the case
of the self $\Delta$-move, in which case they must all belong to the same component. These alternative representations of the $\Delta$-move are instrumental for finding $\Delta$-pathways between $\Delta$-equivalent paths.

Given links $L$ and $L'$, one may transform $L$ into $L'$ and then $L'$ into the trivial link, or vice versa. Thus $d^\Delta_G$ is a metric on a set of links with equivalent linking number. For algebraically split links, it then follows from the triangle inequality:

$$d^\Delta_G(L, L') \leq u^\Delta(L) + u^\Delta(L'), \quad u^\Delta(L) \leq d^\Delta_G(L, L') + u^\Delta(L'), \quad u^\Delta(L') \leq d^\Delta_G(L, L') + u^\Delta(L).$$

and so

$$|u^\Delta(L) - u^\Delta(L')| \leq d^\Delta_G(L, L') \leq u^\Delta(L) + u^\Delta(L').$$

A $\Delta$-move can be accomplished by two crossing changes; see Figure 4. Thus $d^\Delta_G(L, L') \leq 2d^\Delta_G(L, L')$ where $d_G(L, L')$ denotes the Gordian distance between $L$ and $L'$, that is, the minimal number of crossing changes needed to transform $L$ into $L'$. It immediately follows:

**Proposition 1.** Given an algebraically split link $L$,

$$u^\Delta(L) \geq \frac{1}{2} u(L)$$

where $u(L)$ denotes the unlinking number of $L$.

**Figure 2.** Moves equivalent to a $\Delta$-move.

**Figure 3.** The $\Delta$-inequality.

**Figure 4.** Two crossing changes are necessary to perform a $\Delta$-move.
This lower bound may achieve the $\Delta$-unlinking number for a link. For instance, consider the link $L_{9a2}$ (as seen in Figure 5). By [NO15], we know $u(L_{9a2}) = 3$ and so $u^\Delta(L_{9a2}) \geq 2$. Moreover, there exists a $\Delta$-pathway comprising only two $\Delta$-moves: a $\Delta$-move transforms $L_{9a2}$ into the split union $3_1\#0_1$ which is again transformed by a $\Delta$-move into the 2-component trivial link. The inequality of Proposition 1 may be strict; see table in Section 4.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5.png}
\caption{The link $L_{9a2}$.}
\end{figure}

**Proposition 2.** Given an algebraically split link $L = L_1 \sqcup L_2 \sqcup \cdots \sqcup L_m$, $u^\Delta(L) \geq u^\Delta(L_1) + u^\Delta(L_2) + \cdots + u^\Delta(L_m)$.

Moreover, if we have equality, then $L$ is self $\Delta$-equivalent to the trivial link.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure6.png}
\caption{Left to right: A self $\Delta$-move, a $\Delta$-move involving two components, and a $\Delta$-move involving three components.}
\end{figure}

**Proof.** Transforming the link $L$ into the trivial link requires unknotting each component. Note that only self $\Delta$-moves modify the knot type of any component of a link; see Figure 6. Moreover, a self $\Delta$-move only changes the knot type of a single component. The inequality follows.

Now, suppose we have equality. Then unknotting the components with self $\Delta$-moves is sufficient to obtain the trivial link. Thus $L$ is self $\Delta$-equivalent to the trivial link. \hfill \blacksquare

Observe, however, that the converse of the last part of the proposition fails. That is, there exist links which are self $\Delta$-equivalent to the trivial link that have $u^\Delta(L) > u^\Delta(L_1) + \cdots + u^\Delta(L_m)$. For instance, the Bing double of a knot is an algebraically split link and a boundary link [Cim06] and thus by Corollary 3.2 it is self $\Delta$-equivalent to the trivial link, but the components of a Bing double are unknotted.

2.2. 4-genus. Recall that the 4-genus $g_4(L)$ of a link $L = L_1 \sqcup L_2 \sqcup \cdots \sqcup L_m$ is defined as

$$g_4(L) = \min \left( \sum_{i=1}^{m} g(F_i) \mid F_1 \sqcup \cdots \sqcup F_m \hookrightarrow B^4, \ \partial F_i = L_i \right),$$

where the minimum is over smooth embeddings of the disjoint, oriented surfaces $F_1, F_2, \ldots, F_m$ in the 4-ball $B^4$. Meanwhile, the slice genus $g^*(L)$ is the minimal genus of a single such embedded surface that has $L$ as its boundary. In particular, $g_4(L) \geq g^*(L)$. 

Figure 7. A $\Delta$-move achieved by fusion with the Borromean rings.

Theorem 2. Given $\Delta$-equivalent links $L$ and $L'$, we have

$$d_{G}^{\Delta}(L, L') \geq |g_4(L) - g_4(L')|.$$

Proof. Suppose $d_{G}^{\Delta}(L, L') = n$. Since a $\Delta$-move can be represented as fusion with the Borromean rings [MN89] (see Figure 7), $L$ is the result of fusion of the split union of $L'$ with $n$ copies of the Borromean rings $B_1, B_2, \ldots, B_n$. Note that each component $L_i$ of $L$ is fused with a component of the Borromean rings $B_j$ exactly when the arc in the corresponding $\Delta$-move belongs to $L_i$. Thus there exist embeddings of disjoint, oriented surfaces $F_1, \ldots, F_m$ in $S^3 \times [0, 1]$ such that $F_i \cap (S^3 \times \{0\}) = L_i$ and $F_i \cap (S^3 \times \{1\})$ is the fusion of $L'_i$ with the components of the Borromean rings that correspond to arcs of $\Delta$-moves belonging to $L'_i$.

By fusing a component of the Borromean rings with itself, isotopying, then fusing the components back together, as in Figure 8 (cf. [SM83]), we see that each $B_i$ bounds three disjoint surfaces: one surface of genus 1 and two disks. On our surfaces $F_1, \ldots, F_m$ we can thus cap off the components of $B_i$, contributing $n$ to the total genus. See Figure 9. Also, we can cap off $L'_1, \ldots, L'_m$ with disjoint surfaces that each have total genus $g_4(L')$.

Figure 8. The Borromean rings have 4-genus of 1 from a genus 1 surface and two disks.
Thus the components of \( L \) bound disjoint, oriented surfaces with total genus \( d_G^\Delta(L, L') + g_4(L') \), giving

\[
g_4(L) \leq d_G^\Delta(L, L') + g_4(L').
\]

Since \( \Delta \)-moves are reversible, by symmetry we similarly have,

\[
g_4(L') \leq d_G^\Delta(L, L') + g_4(L).
\]

The result follows.

By letting \( L' \) be the trivial link in Theorem 2, we have the following corollary.

**Corollary 2.1.** Given an algebraically split link \( L \),

\[
u^\Delta(L) \geq g_4(L).
\]

Note that the bound also holds in the topological category, since any smooth embedding of a surface is locally flat and hence \( g_4^{\text{top}}(L) \leq g_4(L) \).

![Figure 9. Capping off sets of Borromean rings, each contributing 1 to the 4-genus.](image)

3. Other methods for determining \( \Delta \)-unlinking number

3.1. **Arf Invariant.** Recall, a link \( L = L_1 \sqcup L_2 \sqcup \cdots \sqcup L_m \) is called a proper link if

\[
\sum_{1 \leq i < j \leq m} \text{lk}(L_i, L_j) \equiv 0 \pmod{2}.
\]

Robertello showed that the Arf invariant is well-defined for proper links \( L \) \cite{Hos84, Rob65}. In particular, if a proper link \( L \) cobounds a planar surface with a knot \( K \) then we may define \( \text{Arf}(L) := \text{Arf}(K) \).

**Theorem 3.** Given \( \Delta \)-equivalent proper links \( L \) and \( L' \), we have

\[
d_G^\Delta(L, L') \equiv \text{Arf}(L) + \text{Arf}(L') \pmod{2}.
\]
Proof. Suppose $d_G^\Delta(L, L') = n$. Representing the $\Delta$-move as band fusion with the Borromean rings [MN89] (see Figure 7), $L'$ is the result of the fusion of the split union of $L$ with $n$ copies of the Borromean rings $B_1, B_2, \ldots, B_n$. Thus, for each component of $L$, we can construct disjoint surfaces $F_1, F_2, \ldots, F_m$ embedded in $S^3 \times [0, 1]$. Now, we may fuse the components of $L'$ to obtain a knot $K$. But $L' = L \sqcup B_1 \sqcup B_2 \sqcup \cdots \sqcup B_n$. Thus, there exists a planar surface cobounded by $K$ and $L'$ and hence $\operatorname{Arf}(K) = \operatorname{Arf}(L') = \operatorname{Arf}(L \sqcup B_1 \sqcup B_2 \sqcup \cdots \sqcup B_n)$. Then

$$\operatorname{Arf}(L') \equiv \operatorname{Arf}(L) + \operatorname{Arf}(B_1) + \operatorname{Arf}(B_2) + \cdots + \operatorname{Arf}(B_n) \pmod 2.$$ 

And since $\operatorname{Arf}(B_i) = 1$, we conclude

$$\operatorname{Arf}(L') + \operatorname{Arf}(L) \equiv n \pmod 2.$$ 

By letting $L'$ be the trivial link in Theorem 3, we obtain the following corollary.

**Corollary 3.1.** Given an algebraically split link $L$,

$$u^\Delta(L) \equiv \operatorname{Arf}(L) \pmod 2.$$ 

**Example 1.** The link $L9a40$ has $g_4(L9a40) = 2$. Thus by Corollary 2.1, $u^\Delta(L9a40) \geq 2$; however, since $\operatorname{Arf}(L9a40) = 1$, by Corollary 3.1 we have $u^\Delta(L9a40) \geq 3$. In fact, there is a path of three $\Delta$-moves transforming $L9a40$ into the trivial link $0_1^2$: $L9a40 \xleftarrow{\Delta} mL7a4 \xrightarrow{\Delta} mL5a1 \xleftarrow{\Delta} 0_1^2$. See Figure 10.

![Figure 10. A sequence of $\Delta$-moves unlinking $L9a40$.](image)

Note it immediately follows from Corollary 3.1 that a $\Delta$-move necessarily changes the link type of a proper link. This is not the case for non-proper links such as the Hopf link; see Figure 11.

![Figure 11. A Hopf link transformed into itself by a $\Delta$-move.](image)

### 3.2. Milnor’s Invariants

A self $\Delta$-move is a $\Delta$-move that only involves arcs from the same component of a link, as in Figure 6(a). We have the following classification of links up to self $\Delta$-equivalence:

**Corollary 3.2** (Corollary 1.5 in [Yas09]). A link $L$ is self $\Delta$-equivalent to a trivial link if and only if $\check{\mu}_L(I) = 0$ for any $I$ with $r(I) \leq 2$. 

Here $\tilde{\mu}_L(I)$ denotes Milnor’s $\tilde{\mu}$ invariants which measure the higher order linking of a link, introduced in \cite{Mil54, Mil57}. For an $m$-component link, the multiindex $I = \{i_1, i_2, \ldots, i_n\}$ takes values $1 \leq i_1, i_2, \ldots, i_n \leq m$, possibly repeated; $r(I)$ denotes the maximum number of times the indices $i_k$ repeats a value.

In particular, for 2-component algebraically split links, if $\tilde{\mu}_L(1122) \neq 0$, then $L$ is not self $\Delta$-equivalent to the trivial link. It then follows from Proposition 2 that $\Delta^\Delta(L) > \Delta^\Delta(L_1) + \Delta^\Delta(L_2)$ and thus $\Delta^\Delta(L) \geq \Delta^\Delta(L_1) + \Delta^\Delta(L_2) + 1$, improving the lower bound for $\Delta^\Delta(L)$.

We can calculate $\tilde{\mu}_L(1122)$ for a link $L$ using the link’s Alexander polynomial.

**Theorem 4** (Theorem 2 in \cite{Stu90}). A 2-component link $L$ has Alexander polynomial of the form

$$\Delta_L(x, y) = (x-1)(y-1)f(x, y)$$

and thus $|\tilde{\mu}_L(1122)| = |f(1, 1)|$.

**Example 2.** The link $L_{9a2}$ has Alexander polynomial \cite{BM}

$$\Delta_{L_{9a2}}(x, y) = \frac{(x-1)(y-1)(y^4 - y^3 + y^2 - y + 1)}{\sqrt{xy}^{5/2}}$$

and thus $|\tilde{\mu}_{L_{9a2}}(1122)| = |f(1, 1)| = 1$. Hence, $\Delta^\Delta(L_{9a2}) \geq \Delta^\Delta(L_{9a2_1}) + \Delta^\Delta(L_{9a2_2}) + 1 = 2$ since one of the components is an unknot and the other is a trefoil (which has $\Delta$-unknotting number 1). In fact, $\Delta^\Delta(L_{9a2}) = 2$ since there exists the following $\Delta$-pathway:

$L_{9a2} \xrightarrow{\Delta} 3_1 \# 0_1 \xrightarrow{\Delta} 0_1^2$.

3.3. **L9a18.** Some algebraically split links require additional methods to determine the $\Delta$-unlinking number. For instance, the link $L_{9a18}$ can be transformed into the trivial link $0_1^2$ by three $\Delta$-moves via the pathway

$L_{9a18} \xrightarrow{\Delta} L_{7a4} \xrightarrow{\Delta} L_{5a1} \xrightarrow{\Delta} 0_1^2$.

Moreover, since $\text{Arf}(L_{9a18}) = 1$, we conclude from Corollary 3.1 that $\Delta^\Delta(L_{9a18})$ is 1 or 3.

Suppose $\Delta^\Delta(L_{9a18}) = 1$. Since $|\tilde{\mu}_{L_{9a18}}(1122)| = 3 \neq 0$, by Corollary 3.2 we know $L_{9a18}$ is not self $\Delta$-equivalent to the trivial link. Thus the $\Delta$-move must contain two strands belonging to one component of $L_{9a18}$ and one distinguished strand belonging to the other. As $L_{9a18}$ is invertible, we may deform the diagram via ambient isotopy such that the distinguished strand belongs to a component that is an unknotted circle in the diagram.

When a link $L$ has an unknotted circular component, it can be represented as a knot $K_L$ on a punctured diagram, or equivalently a knot in the solid torus; see Figure 12.

Call a $\Delta$-move a toroidal $\Delta$-move if one arc belongs to an unknotted circular component and the other two arcs belong to the other component of a 2-component link. Then, the toroidal $\Delta$-move has a corresponding move in the punctured diagram or solid torus as depicted in Figure 13.

**Figure 12.** The link $L_{9a18}$ can be represented as a knot in a solid 1-torus.
There is a simple numerical invariant of a knot $K$ in the solid torus, denoted $\beta_1(K)$, defined by lifting $K$ to its infinite cyclic cover and calculating the linking number $\text{lk}(K_0, K_1)$ [Bat15]. Figure 14 shows the lift for $K_{L9a18}$ from which we determine $|\beta_1(K_{L9a18})| = 3$.

Since a toroidal $\Delta$-move changes two crossings, it will change $\beta_1$ by at most 2. Hence, as the trivial link has vanishing $\beta_1$, $K_{L9a18}$ cannot be one toroidal $\Delta$-move away from the trivial link. Hence $u^\Delta(L9a18) \neq 1$ so we conclude $u^\Delta(L9a18) = 3$.

4. Table of $\Delta$-unlinking numbers

We tabulate here the algebraically split prime links with their $\Delta$-unlinking numbers. The Arf invariants are from [Mon12]. The unlinking numbers are from [NO15]. The Rolfsen names are from Knot Atlas [BM]. The 4-genus lower bound was calculated using Corollary 1.5 in [Pow17]. We determined the exact value of the 4-genus of many of the links by band summing to construct explicit upper bounds. The $\bar{\mu}(1122)$ invariant was calculated using Theorem 4. The column headers are consistent with the notation in the text, but for clarity we have, in order: link name (Thistlethwaite and Rolfsen), $\Delta$-unlinking number, half of the unlinking number, sum of delta-unknotting numbers, Arf invariant, 4-genus, Milnor $\bar{\mu}$ invariant, and the method(s) used to calculate the delta-unlinking number.
Table 1. \(\Delta\)-unlinking number and certain invariants for algebraically split links up to 9 crossings.

| Link \( L \) | Thistlethwaite | Rolfsen | \( u \Delta(L) \) | \( \frac{1}{2} u(L) \) | \( \sum u \Delta(L_i) \) | Arf\((L)\) | \( g_4(L) \) | \( |\bar{\mu}_L(1122)| \) | Method(s) |
|---|---|---|---|---|---|---|---|---|---|
| \( L_{5a1} \) | \( 5_1 \) | 1 | 0.5 | 0 | 1 | 1 | 1 | Prop 1 |
| \( L_{6a4} \) | \( 6_2 \) | 1 | 1 | 0 | 1 | 1 | – | Prop 1 |
| \( L_{7a1} \) | \( 7_1 \) | 1 | 1 | 0 | 1 | 1 | 1 | Prop 1 |
| \( L_{7a3} \) | \( 7_3 \) | 3 | 1 | 1 | 1 | 2 | 2 | Cor 2.1 Cor 3.1 |
| \( L_{7a4} \) | \( 7_4 \) | 2 | 1 | 0 | 0 | 1 | 2 | Prop 1 Cor 3.1 |
| \( L_{7n2} \) | \( 7_5 \) | 2 | 0.5 | 1 | 0 | 1 | 1 | Prop 1 Cor 5.1 |
| \( L_{8a1} \) | \( 8_1 \) | 1 | 1 | 0 | 1 | 1 | 1 | Prop 1 |
| \( L_{8a2} \) | \( 8_2 \) | 1 | 0.5 | 1 | 1 | 1 | 0 | Prop 1 |
| \( L_{8a4} \) | \( 8_4 \) | 1 | 0.5 | 1 | 1 | 1 | 0 | Prop 1 |
| \( L_{8n2} \) | \( 8_{15} \) | 2 | 0.5 | 1 | 0 | 1 | 1 | Prop 1 Cor 3.1 |
| \( L_{9a1} \) | \( 9_2 \) | 1 | 1 | 0 | 1 | 1 | 1 | Prop 1 |
| \( L_{9a2} \) | \( 9_3 \) | 2 | 1.5 | 1 | 0 | 2 | 1 | Prop 1 |
| \( L_{9a3} \) | \( 9_5 \) | 2 | 1 | 1 | 0 | 1 | 1 | Prop 1 Cor 3.1 |
| \( L_{9a4} \) | \( 9_{18} \) | 4 | 1 | 2 | 0 | 2 | 2 | Cor 3.1 Sec 3.2 |
| \( L_{9a8} \) | \( 9_{25} \) | 3 | 1 | 1 | 1 | 1 or 2 | 2 | Cor 3.1 Sec 3.2 |
| \( L_{9a9} \) | \( 9_{27} \) | 2 | 1 | 0 | 0 | 1 or 2 | 2 | Prop 1 Cor 3.1 |
| \( L_{9a10} \) | \( 9_{36} \) | 3 | 1.5 | 2 | 1 | 1 or 2 | 2 | Prop 1 Cor 3.1 |
| \( L_{9a14} \) | \( 9_{43} \) | 4 or 6 | 1.5 | 3 | 0 | 3 | 3 | Cor 2.1 Cor 3.1 |
| \( L_{9a15} \) | \( 9_{45} \) | 3 or 5 | 1.5 | 2 | 1 | 2 | 3 | Cor 2.1 Cor 3.1 |
| \( L_{9a17} \) | \( 9_{47} \) | 2 or 4 | 1.5 | 1 | 0 | 2 | 3 | Cor 2.1 Cor 3.1 |
| \( L_{9a18} \) | \( 9_{50} \) | 3 | 1 | 0 | 1 | 1 | 3 | Sec 3.3 |
| \( L_{9a35} \) | \( 9_5 \) | 1 | 1 | 0 | 1 | 1 | 3 | Prop 1 |
| \( L_{9a38} \) | \( 9_{32} \) | 2 | 0.5 | 0 | 0 | 1 or 2 | 4 | Prop 1 Cor 3.1 |
| \( L_{9a40} \) | \( 9_{42} \) | 3 | 1 | 0 | 1 | 2 | 5 | Cor 2.1 Cor 3.1 |
| \( L_{9a42} \) | \( 9_{41} \) | 1 | 1 | 0 | 1 | 1 | 3 | Prop 1 |
| \( L_{9a53} \) | \( 9_{52} \) | 1 | 1 | 0 | 1 | 1 | – | Prop 1 |
| \( L_{9a54} \) | \( 9_{53} \) | 3 | 1.5 | 0 | 1 | 2 or 3 | – | Prop 1 Cor 3.1 |
| \( L_{9n2} \) | \( 9_{36} \) | 4 | 1 | 2 | 0 | 1 | 2 | Cor 3.1 Sec 3.2 |
| \( L_{9n3} \) | \( 9_{47} \) | 3 | 0.5 | 2 | 1 | 1 | 1 | Prop 2 Cor 3.1 |
| \( L_{9n5} \) | \( 9_{44} \) | 5 | 1 | 3 | 1 | 2 | 2 | Cor 3.1 Sec 3.2 |
| \( L_{9n6} \) | \( 9_{25} \) | 4 | 1 | 3 | 0 | 2 | 1 | Prop 2 Cor 3.1 |
| \( L_{9n8} \) | \( 9_{26} \) | 3 | 1 | 2 | 1 | 2 | 1 | Prop 2 Cor 3.1 |
| \( L_{9n25} \) | \( 9_{18} \) | 2 | 1 | 0 | 0 | 1 | – | Prop 1 Cor 3.1 |
| \( L_{9n27} \) | \( 9_{31} \) | 2 | 0.5 | 0 | 0 | 2 | – | Prop 1 Cor 9.1 |
We can also display the minimal $\Delta$-pathways as a tree. In Figure 15, each edge represents one $\Delta$-move and the path from a link $L$ to the trivial link is a pathway of minimal length $u^\Delta(L)$. Indeterminate cases are represented by a dashed edge. Recall $L\#L'$ denotes a split union between $L$ and $L'$.

Figure 15. A tree exhibiting $\Delta$-pathways of minimal length to the trivial link.
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