Assortative Mixing by Degree Makes a Network More Unstable

Markus Brede$^1$ and Sitabhra Sinha$^2$

$^1$CSIRO Centre for Complex Systems, Canberra, ACT 2601, Australia
$^2$The Institute of Mathematical Sciences, C.I.T. Campus, Taramani, Chennai - 600 113 India

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We investigate the role of degree correlation among nodes on the stability of complex networks, by studying spectral properties of randomly weighted matrices constructed from directed Erdős-Rényi and scale-free random graph models. We focus on the behaviour of the largest real part of the eigenvalues, $\lambda_{\text{max}}$, that governs the growth rate of perturbations about an equilibrium (and hence, determines stability). We find that assortative mixing by degree, where nodes with many links connect preferentially to other nodes with many links, reduces the stability of networks. In particular, for sparse scale-free networks with $N$ nodes, $\lambda_{\text{max}}$ scales as $N^{\frac{\alpha}{3}}$ for highly assortative networks, while for disassortative graphs, $\lambda_{\text{max}}$ scales logarithmically with $N$. This difference may be a possible reason for the prevalence of disassortative networks in nature.

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Whether complex systems are more stable than simpler ones has been a contentious issue for a long time. Stability of a $N$-dimensional dynamical system $\dot{x} = -x + F(x)$, (e.g., a network of $N$ nodes, each node $i$ associated with a variable $x_i$ whose time evolution is governed by a general nonlinear function $F$) is measured by the rate at which a small perturbation about an equilibrium state $x^*$ decays with time. This is determined by the largest real part, $\lambda_{\text{max}}$, of the eigenvalues of the Jacobian matrix $J_{ij} = -\delta_{ij} + \partial F_i/\partial x_j |_{x^*}$, representing the interactions within the system. If $\lambda_{\text{max}}$ is positive, the effect of a small perturbation will grow exponentially with time, and the system will be rapidly dislodged from the equilibrium state. In the absence of detailed knowledge about the interactions among the constituent elements of a complex system which determine $J_{ij}$, ensembles of randomly constructed matrices have been studied. Results from several studies of random matrices, symmetric as well as asymmetric, show that $\lambda_{\text{max}} \sim \sqrt{NC\sigma^2}$ where $N$ is the system size, $C$ is the connectivity (the probability that there is a link between any pair of nodes in the network) and $\sigma$ is a measure for interaction strength between the nodes. Assuming self regulation, e.g., $J_{ii} = -1$, it follows that a transition from stability to instability takes place as the network either grows in size or becomes more densely (or strongly) connected. This result, implying complexity promotes instability, has been shown to be remarkably robust with respect to various generalisations.

There is a direct correspondence between the nature of the matrices $J$ and the structure of the underlying directed network, because two nodes $i,j$ are connected iff $J_{ij} \neq 0$. Clearly, the random matrices investigated in these previous studies correspond to Erdős-Rényi (ER) random graphs. Recently, it has been pointed out in a series of studies that many real world networks differ from these purely random graphs, e.g., in having highly skewed degree distributions that typically follow a power law with the exponent $\gamma$ between $-2$ and $-3$. Previous studies have investigated spectral properties of undirected binary graphs with power law degree distribution, however, there are further detailed features that make real world networks appear fundamentally different from random graphs. In particular, in almost all biological networks, nodes of high degree tend to avoid being connected to other highly connected nodes, i.e. these networks show disassortative mixing. Understanding why networks exhibit these patterns is one of the challenges of current research.

In this paper we show that assortativity can significantly influence network stability. For this, we study spectral properties of randomly weighted matrix ensembles in which we introduce correlations between the matrix elements $J_{ij}$ that reflect graph properties found in real world networks. We first examine the growth of $\lambda_{\text{max}}$ with network size in matrix ensembles corresponding to graphs with Poisson degree distributions. Increasing the assortativity of such graphs results in increasing the largest eigenvalue, implying decrease in stability. Next, we study the behaviour of growing networks with scale free degree distribution. We observe the eigenvalue distribution (that departs significantly from the spectra of the corresponding adjacency matrices) and determine the scaling of $\lambda_{\text{max}}$ with network size. Again we find that assortative mixing leads to a loss of stability. Our results seem to imply that the disassortative mixing seen in biological networks could be a consequence of evolutionary optimisation of these networks to minimise the effect of dynamical fluctuations.

Every directed graph can be represented by its adjacency matrix $A_{ij}$, where $A_{ij} = 1$ if there is a link from $j$ to $i$; $0$ otherwise. Following May, we generate the matrix $J_{ij}$ from $A_{ij}$ of a network with desired properties, by replacing the non-zero elements of the latter matrix with entries drawn from a Gaussian distribution having mean zero and variance $\sigma^2$. Note that, as the re-
changed, we can generate networks with a specific value distribution fixed. As this leaves the assortativity unchanged, while keeping the degree possibility. Fig. 1 illustrates the dependence of assortativity on assortative network, the steps leading to higher (lower) assortativity are accepted with increased (decreased) probability. Note that, changing the assortativity by this rewiring procedure also increases the clustering coefficient, while keeping the degree distribution fixed. As this leaves the assortativity unchanged, we can generate networks with a specific value of $c$ while keeping $a$ constant. In this case, there is no statistically significant change of $\langle \lambda_{\text{max}} \rangle$ with $c$ (data not shown). Combining this result with Fig. 1 allows us to conclude that high assortativity leads to lower stability for networks with Poisson degree distribution.

Next, we look at random matrices constructed from scale-free graph substrates. The networks are generated by adding at each time step a new node with $m$ links that connect to existing nodes following the Barabási-Albert (BA) preferential scheme presented in Ref. [12]. To avoid ‘dynamic correlations’ resulting from the BA procedure, we subsequently randomise the networks using the link swapping technique discussed in Ref. [20]. Corresponding $J_{ij}$ matrices are constructed as before. In Fig. 2 a comparison of numerical data for the spectral densities of unrandomized and randomised BA networks demonstrates that the dynamical correlations in the former indeed have an impact on the spectral density. While both spectral densities have exponentially decaying tails in the BA case, but not after randomisation. The comparison illustrates that the ‘dynamic correlations’ in the BA network have an effect on the spectrum.

First, we consider matrix ensembles constructed from graphs with Poisson degree distributions. We start with an ER graph (for which $a \approx 0$) and then introduce assortativity by applying the method proposed in Ref. [19]. The algorithm essentially consists of performing a series of rewiring procedures, such that, to obtain an assortative network, the steps leading to higher (lower) assortativity are accepted with increased (decreased) probability. In the BA case, but not after randomisation.

FIG. 1: Dependence of the average of largest real part of the eigenvalues, $\langle \lambda_{\text{max}} \rangle$, on assortativity for Poisson degree distributed networks of size $N = 100$. Data points represent averages over 1000 independent runs. The inset shows the tail of the real axis projection of the spectrum for random matrices derived from non-assortative ($a \approx 0$, filled circles) and strongly assortative ($a = 0.8$, open circles) networks. The tail becomes more extended for strongly assortative networks.

FIG. 2: (Top) Real axis projection of the spectral densities (shifted to the right by $\Delta \lambda = 1$) for matrices constructed from BA networks (filled circles) and randomised BA networks (open circles) with $N = 5000$, $m = 10$, and $\sigma = 0.2$. The left inset illustrates the exponential tail of the spectrum, while the right inset shows that the spectrum follows a power law $P(\lambda) \sim \lambda^{-1/2}$ for small $\lambda$ in the BA case, but not after randomisation. The comparison illustrates that the ‘dynamic correlations’ in the BA network have an effect on the spectrum. The data represent averages over 100 independent runs.
the number of links per node added, $m$, becomes a linear function of network size. Fig. 3 shows the scaling of the average of largest real part of the eigenvalues, $\langle \lambda_{\text{max}} \rangle$, with network size $N$. For three types of growing networks, viz., having exponential (i.e., no preferential attachment) and scale-free degree distribution with exponents $\gamma = -3$ (the original BA preferential attachment algorithm) and $\gamma \approx -2.1$ (Bianconi networks, constructed after a modification of the BA procedure following Ref. [22]), we find the same scaling behaviour as in ER graphs, $\lambda_{\text{max}} \sim N^{1/2}$, which contrasts with the relation $\lambda_1 \sim N^{1/4}$ reported for adjacency matrices of large undirected BA networks [12]. Variation among different network topologies are chiefly expressed in different finite size behaviour and proportionality constants.

In general, it seems that networks with degree distributions having a less extended tail are more stable. This appears to be supported by Gershgorin’s theorem [22], which gives the bound: $\lambda_{\text{max}} < R_{\text{max}} - 1$, where $R_{\text{max}} = \sum_{i \neq j} |J_{ij}|$ is related essentially to the largest degree, $k_{\text{max}}$, of the network. From $N \sum_{i,j} k^{-\gamma} \leq 1$, where $k$ are node degrees (see, e.g., Ref. [22]), one obtains the following dependence of $k_{\text{max}}$ on $N$ for the different networks: $k_{\text{max}} \sim N^{0.91}$ (for the Bianconi network), $k_{\text{max}} \sim N^{1/2}$ (for the BA network) and $k_{\text{max}} \sim \log(N)$ (for the exponential network). This ordering according to decreasing growth rates of $k_{\text{max}}$ with $N$ agrees very well with the ordering of the network types according to decreasing stability in Fig. 3.

As already mentioned, in the preceding case we considered the network connectivity $C$ to be constant as $N$ increases. However, most real networks are sparse, i.e. $C \sim N^{-1}$ (see, e.g., Ref. [10]). Therefore, in the second case, we measure $\langle \lambda_{\text{max}} \rangle$ in networks with increasing size, while keeping $m$ constant (resulting in decreasing connectivity $C$ with increasing $N$). We now compare the behaviour of $\lambda_{\text{max}}$ for assortative and disassortative networks, constructed using the same method as before. For the results reported in this paper, we choose $m = 2$, for which we verify numerically that, after randomisation, the resulting networks are still connected. Optimising for a high assortativity, however, tends to fracture this component, e.g., we find that networks with $a \approx 0.25$ typ-
ically have a giant component that comprises only 80% of the links in the network. Note that, this fraction of links comprising the largest connected cluster, is independent of the number of nodes $N$.

Fig. 4 (top) shows the scaling of $\langle \lambda_{\text{max}} \rangle$ with $N$, for constant $m$. As in the case of the ER graph-based matrices, we again find that high assortativity leads to lower stability. For the assortative scale-free networks with $a \approx 0.25$ we find that $\lambda_{\text{max}} \sim N^{0.2}$, whereas for the uncorrelated and disassortative networks, $\lambda_{\text{max}} \sim \log(N^{0.12})$ for large $N$. This difference in the scaling behaviour is suggestive of a huge difference in stability for large network sizes.

The reason for the different scaling of $\lambda_{\text{max}}$ in assortative and disassortative networks seems to be found in different scaling behaviours of the excess densities $f_{\text{real}}$ of real eigenvalues [Fig. 4 (bottom)]. For random matrices based on an ER graph topology, it is known that $f_{\text{real}} \propto N^{-1/2}$ for large $N$. For uncorrelated BA networks we find that $f_{\text{real}} \propto N^{-\delta}$, $\delta = 0.45 \pm 0.05$, which is very close to the value for ER graphs. However, for assortative networks, we observe $\delta = 0.1 \pm 0.05$, which results in a much more gradual decline than in disassortative networks. Hence, for the same network size $N$, more eigenvalues lie on the real axis in assortative networks and thus the probability of having higher maximum real parts is increased. We also note that, as in the case of matrices based on graphs with Poisson degree distribution [Fig. 4 (inset)], the tails at the spectrum edge show a more gradual decay for assortative networks, compared to uncorrelated and disassortative graphs (not shown).

In this paper, we have presented an analysis of randomly weighted matrices whose corresponding networks are constructed by starting from well known graph models such as ER and scale-free networks, and then using a link swapping algorithm to generate degree-correlated network ensembles with a desired assortativity. As the most important result of the paper, we find that assortative mixing by degree tends to severely destabilise networks. While the largest real part of the eigenvalues scales as a power of the network size in highly assortative networks, in the disassortative case we only observe a logarithmic dependence.

The question has been raised why almost all biological networks are disassortatively mixed. It has been suggested that disassortativity has its origin in the restriction that no two nodes have more than one link between them. However, a recent study shows that while this mechanism does indeed give rise to the kind of correlations actually observed in certain networks (e.g., the Internet), only a part of all the measured correlations can be accounted in this way. Other studies have pointed out that assortative mixing has a big impact on the percolation behaviour of networks. This implies that disassortative networks might have been favoured because perturbations have a reduced chance to propagate. Our results add another viewpoint to this issue. The results reported here imply that disassortative networks are more resistant to the effect of dynamical fluctuations than assortative networks. One may therefore speculate that, an evolutionary drive towards systems with reduced intensity of fluctuation (homeostasis) might be one of the reasons for the prevalence of disassortative networks in the biological world today.

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