Quasicircles and quasiperiodic surfaces in pseudo-hyperbolic spaces

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Abstract We study in this paper quasiperiodic maximal surfaces in pseudo-hyperbolic spaces and show that they are characterised by a curvature condition, Gromov hyperbolicity or conformal hyperbolicity. We show that the limit curves of these surfaces in the Einstein Universe admits a canonical quasisymmetric parametrisation, while conversely every quasisymmetric curve in the Einstein Universe bounds a quasiperiodic surface in such a way that the quasisymmetric parametrisation is a continuous extension of the uniformisation; we give applications of these results to asymptotically hyperbolic surfaces, rigidity of Anosov representations and a version of the universal Teichmüller space.

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1 Introduction

We continue our study of complete maximal surfaces in pseudo-hyperbolic spaces of signature \((2, n)\) begun in [12,31].

The pseudo-hyperbolic space \(H^{2,n}\) is constructed in the following way: take a vector space \(E\) equipped with a quadratic form \(q\) of signature \((2, n + 1)\)—that is admitting a vector space of dimension 2 on which \(q\) is positive definite, while \(q\) is negative definite on the orthogonal of dimension \(n + 1\). Then the pseudo-hyperbolic space of signature \((2, n)\) is

\[
H^{2,n} := \left\{ u \in E \mid q(u) = -1 \right\} / \pm \text{Id}.
\]

For \(n = 0\), we obtain the Minkowski model of the hyperbolic space. The pseudo-hyperbolic space of \(H^{2,n}\) of signature \((2, n)\) inherits by restriction of the quadratic form \(q\) a pseudo-Riemannian metric of signature \((2, n)\) and constant sectional curvature \(-1\). Let \(G := \text{SO}_0(q)\) be the identity component of \(O(q)\), then \(G\) acts transitively by isometries on \(H^{2,n}\), turning \(H^{2,n}\) into a pseudo-Riemannian symmetric space.

The pseudo-hyperbolic space naturally embeds in \(P(E)\) and admits a boundary at infinity denoted \(\partial \infty H^{2,n}\) and called the Einstein Universe which is the quadric in \(P(E)\) associated to \(q\). This quadric admits a notion of positivity leading to the concept of positive loops and semi-positive loops: for instance, the intersection of the quadric with the projectivisation of a space of type \((2, 1)\) is a positive loop. Positivity in this context has been studied by [2,11,21].

One of the concepts introduced and studied in the current paper is that of quasisymmetric maps from the projective real line to the Einstein Universe. As for the case of maps into the complex projective line, quasisymmetric maps are defined using a cross-ratio. These maps are positive and enjoy similar properties as their counterparts in the complex projective line: they enjoy a uniform Hölder modulus of continuity—see Theorem 3.17—while equivariant ones are linked to Anosov representations.

A maximal surface in \(H^{2,n}\) is a spacelike surface on which the mean curvature vector vanishes everywhere. We proved in [31] that every complete maximal surface \(\Sigma\) has a limit curve \(\partial \infty \Sigma\) in \(\partial \infty H^{2,n}\) which is a semi-positive loop and, more importantly, that every such semi-positive loop is the boundary of a unique complete maximal surface.

Among homogeneous complete maximal surfaces, we have the totally geodesic hyperbolic planes and Barbot surfaces which are flat and orbits of diagonal subgroup of \(G\). Barbot surfaces are characterised by a rigidity theorem. By a result of Cheng [13, Theorem 2], any complete maximal surface has non positive induced curvature, while Barbot surfaces are the only flat ones. A slight and technical improvement of Cheng’s result gives
Theorem 1 (Cheng Curvature rigidity) Let $\Sigma$ be a complete maximal surface in $H^{2,n}$. If the intrinsic curvature of $\Sigma$ is zero at a point, it vanishes everywhere and $\Sigma$ is a Barbot surface.

For $n = 1$, this result was also obtained by Bonsante and Schlenker in [8]. This result sets the background of the main results of this article, which concern quasiperiodic maximal surfaces.

Let us introduce the space $\mathcal{M}(n)$ which consists of pairs $(x, \Sigma)$ where $\Sigma$ is a complete maximal surface and $x$ a point in $\Sigma$. By construction every maximal surface $\Sigma$ maps into $\mathcal{M}(n)$, by the map which associates $(x, \Sigma)$ to $x$. By a compactness theorem proved in [31], $G$ acts cocompactly on $\mathcal{M}(n)$ when the latter is equipped with the topology of smooth convergence on every compact. We will show that Barbot surfaces give rise to a unique point called the Barbot point in $\mathcal{M}(n)/G$.

We then define quasiperiodic maximal surfaces—and justify the terminology later—as surfaces whose images in $\mathcal{M}(n)/G$ have a closure that does not contain the Barbot point. The first goal of this paper is to give different characterisations of quasiperiodic maximal surfaces. Let us state here the most important ones.

Theorem A (Quasiperiodic surfaces) Let $\Sigma$ be a complete maximal surface in $H^{2,n}$. The following statements are equivalent:

(i) $\Sigma$ is quasiperiodic,
(ii) the induced metric on $\Sigma$ has curvature bounded above by some negative constant,
(iii) the induced metric on $\Sigma$ is Gromov hyperbolic,
(iv) $\Sigma$ is of conformal hyperbolic type and any uniformisation is biLipschitz,
(v) the limit curve $\partial_{\infty}\Sigma$ is the image of a quasisymmetric map.

These are generalisations of the phenomenon discovered for $\text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R})$ in [8] and $\text{PSL}(3, \mathbb{R})$ in [6] and in this last context we see that images of quasisymmetric maps in our sense are also the analogue of the boundary of hyperbolic convex sets in $\mathbb{P}(\mathbb{R}^3)$, as introduced by Benoist in [5].

The quasi-symmetric parametrisation of $\partial_{\infty}\Sigma$ is explicit and gives the second goal of this paper:

Theorem B (Extension of uniformisation) Let $\Sigma$ be a quasiperiodic maximal surface. Then any uniformisation of $\Sigma$ by the hyperbolic disk extends continuously to a quasisymmetric homeomorphism between the boundary of the hyperbolic disk to the boundary of $\Sigma$ in $\partial_{\infty}H^{2,n}$.

An important role is played in the proof by pseudo-Riemannian analogues of functions in hyperbolic spaces. First, a horofunction is the function associated to a non-zero lightlike vector $z_0$, whose projection belongs to $\partial_{\infty}\Sigma$, and defined by
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\[ h(x) := \log |\langle x, z_0 \rangle|. \]

When \( n = 0 \), one recovers usual horofunctions. Similarly, the \textit{spatial hyperbolic distance} between two points in spacelike position is given by

\[ \delta(x, y) := \cosh^{-1}(|\langle x, y \rangle|). \]

When \( n = 0 \), \( \delta \) is the hyperbolic distance, while in general \( \delta \) is not a distance in \( H^{2,n} \). We will however prove that the restriction of \( \delta \) to any complete maximal surface is a distance up to a universal constant—Corollary 7.8 to be compared with [20].

Associated to these functions are two other characterisations of quasiperiodic surfaces—Theorems 6.2 and 6.5 as well as the Rigidity Theorem 5.5.

As we already said, a pervasive tool in the paper is the use of the cocompact \( G \) action on the laminated space \( M(n) \). Dwelling on that, we also have a description of quasiperiodic surfaces as leaves in a lamination in Theorem 6.6, thus justifying the terminology quasiperiodic surface.

Finally, say a complete maximal surface is \textit{asymptotically hyperbolic} if its curvature converges to -1 at infinity, or equivalently by Gauß equation (Proposition 4.5), if the norm of the second fundamental form goes to zero. Then we have

**Theorem C** (Asymptotic hyperbolicity) \( C^1 \) spacelike curve in \( \partial_\infty H^{2,n} \) is quasisymmetric and bounds a complete maximal asymptotically hyperbolic surface.

As a corollary, we obtain

**Theorem D** (\( C^1 \) rigidity) An Anosov representation of a closed surface group in \( G \) whose limit curve is \( C^1 \) and spacelike factors through \( (O(2, 1) \times O(n))_0 \). In particular the limit curve is a circle.

One should compare with the fact that any Hitchin representation in \( SO(2, 3) \) always has a \( C^1 \)-timelike limit curve [28], while Potrie and Sambarino [34] has shown that if the limit map of a Hitchin representation is \( C^2 \) then the representation factors though an irreducible \( SL(2, \mathbb{R}) \).

This whole set of results represent a natural extension of the theory of uniformisation and quasisymmetric maps, and quite naturally the last section is devoted to a description of consequences of these results in the spirit of Bers’ universal Teichmüller space [7].

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1.1 Description of the paper

(i) Sect. 2 describes the Einstein Universe $\partial_{\infty}H^{2,n}$, which will be revealed as a bordification of the pseudo-hyperbolic space later. The Einstein Universe admits a conformally flat Minkowski structure, which is described using Minkowski charts and patches. This Minkowski structure is associated to a positive structure: positive triples, quadruples and loops. Associated to a positive triple are diamonds and their diamond distance which play the role of intervals in the real projective line—corresponding to the $n = 0$ case. Critical objects in our study are Barbot crowns which are semi-positive loops. Finally we define quasiperiodic loops.

(ii) Sect. 3 describes the cross-ratio associated to four generic points in $\partial_{\infty}H^{2,n}$ and introduces the important new concept of quasisymmetric maps. We describe cross-ratio in Minkowski charts and prove one of the main result of this paper: Theorem 3.17 which shows that quasisymmetric maps admit a Hölder modulus of continuity.

(iii) In Sect. 4 we describe the pseudo-hyperbolic space and maximal surfaces therein. We describe horofunctions and spatial distance. We spend some time recalling the classical Gauss, Codazzi and Ricci equations governing these maximal surfaces. We finally recall the compactness results of [31] that we express using the space of pointed maximal surfaces $\mathcal{M}(n)$ which is a laminated space on which the isometry group $G$ of $H^{2,n}$ acts cocompactly. We then describe Barbot surfaces which are the maximal surfaces associated to Barbot crowns and show their properties. We finally introduce quasiperiodic maximal surfaces which is one of the main object of study of the current paper.

(iv) In Sect. 5, we prove two Rigidity results that characterise Barbot surfaces. The first one, Proposition 5.3 stated as Theorem 1 in the introduction, shows that Barbot surfaces are extremal for a pointwise curvature condition, while the second one, Theorem 5.5, shows that Barbot surfaces are extremal for a pointwise condition involving either the gradient of an horofunction or the spatial distance. Proposition 5.3 is proved using a slightly different point of view than Cheng’s original proof.

(v) We use these Rigidity Theorems and the compactness results of [31] in Sect. 6 to give six alternative characterisations of quasiperiodic surfaces involving curvature, horofunctions, spatial distance, uniformi-
sation, Gromov hyperbolicity and laminations, some of these being stated in Theorem A.

(vi) We now move in Sect. 7 to prove Theorem 7.1. This result shows that the uniformisation of quasiperiodic surfaces extends in a precise and quantitative way to a quasisymmetric map. The proof involves the bounds on horofunctions given in the Rigidity Theorems as well as the study of Gromov products in this setting, that were also considered by Glorieux and Monclair [20].

(vii) As another application of our techniques, we study in Sect. 8 asymptotically hyperbolic maximal surfaces and prove Theorem C and D.

(viii) In the final Sect. 9 we show how our results can be used to define an analogue universal Teichmüller space and we discuss briefly the consequence of an announcement by Qiongling Li and Takuro Mochizuki for the \( n = 2 \) case well as a related conjecture.

(ix) For the sake of completeness, we give in an appendix a proof of a classical Bochner Formula.

2 Einstein Universe and positivity

In this section, we describe the geometry of the Einstein Universe \( \partial_\infty \mathbb{H}^{2,n} \) of signature \((1, n)\) and study the notion of positivity in \( \partial_\infty \mathbb{H}^{2,n} \). Although it is not relevant at this stage that the Einstein Universe is indeed the boundary of the pseudo-hyperbolic space \( \mathbb{H}^{2,n} \), we nevertheless stick (as in [31]) with the notation \( \partial_\infty \mathbb{H}^{2,n} \).

The Einstein Universe has been extensively studied and part of the material covered here can be found in [4, 12, 14, 31]. Although most of what we describe could be described as part of the general theory of parabolic spaces, Bruhat cells etc. we keep our approach elementary.

Throughout this paper, \( E \) will be a real \((n + 3)\)-dimensional vector space equipped with a signature \((2, n + 1)\) quadratic form \( q \) and \( G := \text{SO}_0(q) \) will be the connected component of the identity of the group of linear endomorphisms of \( E \) preserving \( q \).

2.1 The geometry of the Einstein Universe

The Einstein Universe is the quadric associated to \( q \):

\[
\partial_\infty \mathbb{H}^{2,n} := \{ x \in \mathbb{P}(E), \ q(x) = 0 \}.
\]

The group \( G \) acts transitively on \( \partial_\infty \mathbb{H}^{2,n} \) and the stabiliser of a point in \( \partial_\infty \mathbb{H}^{2,n} \) is a (disconnected maximal parabolic) subgroup \( P \).
We denote by $P_+(E) = (E \setminus \{0\})/\mathbb{R}_{>0}$ the set of linear rays in $E$. The double cover $\partial_{\infty}H_+^{2,n}$ of $\partial_{\infty}H^{2,n}$ is defined as the preimage of $\partial_{\infty}H^{2,n}$ in the double cover $P_+(E)$ of $P(E)$. The stabiliser in $G$ of a point in $\partial_{\infty}H_+^{2,n}$ is now isomorphic to the identity component $P_0$ of $P$.

**Definition 2.1 (Photons and circles)**

(i) A photon in $\partial_{\infty}H^{2,n}$ is the projectivisation of an isotropic 2-plane in $E$: at most one photon passes through two distinct points.

(ii) A spacelike circle is the intersection of $\partial_{\infty}H^{2,n}$ with the projectivisation of a space of signature $(2, 1)$: at most one circle passes through three distinct points.

The corresponding subsets of $\partial_{\infty}H_+^{2,n}$ are defined in an analogous way.

### 2.1.1 The symmetric space of $G$

The group $G$ acts transitively on the Grassmannian $\text{Gr}_{2,0}(E)$ of oriented positive definite 2-planes in $E$. The stabiliser of an element in $\text{Gr}_{2,0}(E)$ is isomorphic to $\text{SO}(2) \times \text{SO}(n + 1)$ and is thus a maximal compact subgroup of $G$, giving an identification between $\text{Gr}_{2,0}(E)$ and the (Riemannian) symmetric space of $G$.

Given a plane $U$ in $\text{Gr}_{2,0}(E)$, with orthogonal subspace $V$, the quadratic form

$$q_U := q|_U \oplus (-q|_V)$$

is positive definite on $E = U \oplus V$. In particular, any element $x$ in $\text{Gr}_{2,0}(E)$ induces an Euclidean scalar product on $E$, hence on the Lie algebra of $G$: the (non-invariant) scalar product of two elements $A$ and $B$ in the Lie algebra of $G$ is $\langle A, B \rangle_x = \text{tr}(AB^t)$, where $B^t$ denotes the transpose of $B$ with respect to the Euclidean scalar product on $E$ defined by $x$.

### 2.1.2 Conformally flat structure

The Einstein Universe $\partial_{\infty}H^{2,n}$ is naturally equipped with a (locally) conformally flat structure $\{g_{\text{Ein}}\}$ of signature $(1, n)$.

Indeed, the tangent space to $\partial_{\infty}H^{2,n}$ at $x$, is identified with the space $\text{Hom}(x, x^\perp/x)$. The restriction of $q$ to $x^\perp/x$ has signature $(1, n)$, providing $\text{Hom}(x, x^\perp/x)$ with a conformal class of quadratic form. This structure is conformally flat: any point in $\partial_{\infty}H^{2,n}$ has a neighbourhood which is conformal to a Minkowski space.

Observe that the notion of spacelike, timelike and lightlike vectors is meaningful in $\partial_{\infty}H^{2,n}$. For details, see [4, Section 3 and 4].
2.1.3 Product structure

Consider a positive definite 2-plane $U$ in $\text{Gr}_{2,0}(E)$ with orthogonal $W$, and let $q_U$ be the corresponding Euclidean quadratic form on $E$ defined in equation (1). Any isotropic ray in $\partial_\infty H^2_{+n}$ contains a unique point $(u, v) \in U \oplus W$ with $q_U(u) = q_U(v) = 1$. This gives a diffeomorphism

$$\partial_\infty H^2_{+n} \cong S^1 \times S^n$$

where $S^1 \subset U$ and $S^n \subset W$ are the unit spheres. In this coordinate system, the conformal metric of $\partial_\infty H^2_{+n}$ is given by

$$[g_{\text{Ein}}] = [g_{S^1} \oplus (-g_{S^n})],$$

where $g_{S^i}$ is the canonical metric on $S^i$ of curvature 1.

**Definition 2.2 (Visual distance and metric)** The visual metric associated to a plane $U$ in $\text{Gr}_{2,0}(E)$ is the Riemannian metric on $\partial_\infty H^2_{+n}$ induced by $q_U$, namely

$$g = g_{S^1} \oplus g_{S^n}.$$ 

This visual metric gives rise to a metric, also called visual, on $\partial_\infty H^2_{+n}$. The corresponding distances are called visual distances.

Finally, observe that isotropic 2-planes in $E$ are exactly graphs of linear maps $\varphi$ from $U$ to $V$ such that for any $u$ in $U$, we have $q_U(\varphi(u)) = q_U(u)$. In particular, photons in the splitting $\partial_\infty H^2_{+n} \cong S^1 \times S^n$ correspond to graph of isometries from $S^1$ to $S^n$.

2.2 Positive triples

We discuss now the important notion of positivity in $\partial_\infty H^2_{+n}$.

2.2.1 Motivation: the projective line

Let $V$ be a real vector space of dimension 2. In that context, let us define

(i) A positive triple in $P(V)$ to be a triple of pairwise distinct points.

(ii) A positive quadruple $(a, b, c, d)$ is a quadruple of pairwise distinct points such that $b$ and $d$ are not in the same connected component of $P(V) \setminus \{a, c\}$.

(iii) A map from $P(V)$ to itself to be positive if it is orientation preserving or orientation reversing.
2.2.2 Higher dimensions

We now describe the generalisation of positive triples to $\partial_\infty H^{2,n}$ for $n > 0$ and will do later the same for positive quadruples and positive maps. A pair of points $(a, b)$ in $\partial_\infty H^{2,n}$ is transverse, if $q(a_0, b_0) \neq 0$ where $a_0$ and $b_0$ are non zero vectors in $a$ and $b$ respectively (or equivalently, if $a$ and $b$ do not lie on a same photon).

**Definition 2.3 (Positive triple)** A triple of points $\tau = (a, b, c)$ of pairwise distinct points in $\partial_\infty H^{2,n}$ (or in $\partial_\infty H^{2,n+1}$) is positive if $W_\tau := a \oplus b \oplus c$ is a linear space of signature $(2, 1)$. We denote by $T(n)$ and $T_+(n)$ the set of positive triples in $\partial_\infty H^{2,n}$ and $\partial_\infty H^{2,n+1}$ respectively.

There is a unique spacelike circle passing through a positive triple. Conversely, any triple of pairwise distinct points in a spacelike circle is positive. We warn the reader that the terminology positive triples, though standard, is confusing: being a positive triple is invariant under all permutations.

2.2.3 Action of $G$ on positive triples

We have the following

**Lemma 2.4** If $n > 0$, then

(i) the action of $G$ on $T_+(n)$ has two orbits,
(ii) the action of $G$ on $T(n)$ is transitive and the stabiliser of a point is isomorphic to $SO(n)$.

**Proof** The group $G$ preserves a space orientation and a time orientation in $E$. This means that for any positive definite 2-plane $P$ in $E$, both $P$ and $P^\perp$ carry a natural orientation which is preserved by the action of $G$.

Recall that an orthonormal basis of $E$ is a basis $(e_1, \ldots, e_{n+3})$ such that $\langle e_i, e_j \rangle = \varepsilon_i \delta_{i,j}$, where $\varepsilon_i = 1$ for $i = 1, 2$ and $\varepsilon_i = -1$ otherwise. Such a basis will be called canonical is moreover $(e_1, e_2)$ and $(e_3, \ldots, e_{n+3})$ are positively oriented. The group $G$ acts simply transitively on the set of canonical bases.

Given a positive triple $(z_1, z_2, z_3)$ in $\partial_\infty H^{2,n}$ associated to a vector space $V$ of type $(2, 1)$, one can find an orthonormal set $(e_1, e_2, e_3)$ basis in $V$ so that

\[
\begin{align*}
  e_1 + e_3 & \quad \text{belongs to } z_1 \\
  -\frac{1}{2}e_1 + \frac{\sqrt{3}}{2}e_2 + e_3 & \quad \text{belongs to } z_2 \\
  -\frac{1}{2}e_1 - \frac{\sqrt{3}}{2}e_2 + e_3 & \quad \text{belongs to } z_3
\end{align*}
\]
and such a triple can be completed to an orthonormal basis of $E$. The only obstruction of extending $(e_1, e_2, e_3)$ to a canonical basis is the orientation of $(e_1, e_2)$. This proves item (i).

For item (ii), repeating the construction for a positive triple $\tau = (z_1, z_2, z_3)$ in $\partial_\infty H^{2,n}$, one notices that we have exactly two choices for $(e_1, e_2, e_3)$ corresponding to the two (amongst the eight possible) lifts of $\tau$ in $\partial_\infty H^{2,n}_+$ which lies on the two lifts of a positive circle. Only one of the two will give the correct orientation on the pair $(e_1, e_2)$ defined above. If $n > 0$, one can find an orthonormal basis $(e_4, \ldots, e_{n+3})$ of $(z_1 \oplus z_2 \oplus z_3)^\perp$ such that $(e_3, \ldots, e_{n+3})$ is positively oriented. So $G$ acts transitively on the space of positive triples in $\partial_\infty H^{2,n}$.

The stabiliser of the positive triple $(z_1, z_2, z_3)$ acts simply transitively on the space of canonical basis having $(e_1, e_2, e_3)$ for the first 3 vectors. This group is thus isomorphic to $SO(n)$.

\[ \square \]

2.2.4 Distances

Let us start with a proposition

**Proposition 2.5** There exists a $G$ equivariant map $\tau \mapsto \beta_\tau$ from the space $T(n)$ of positive triples in $\partial_\infty H^{2,n}$ to the symmetric space $Gr_{2,0}(E)$ of $G$.

**Proof** This is a direct consequence of Lemma 2.4 since the stabiliser of a positive triple is compact. Let us however give an explicit construction of this map.

Let $\tau = (a, b, c)$ be positive triple and $W_\tau := a \oplus b \oplus c$. Then the intersection of $W_\tau$ with $H^{2,n}$ is a hyperbolic plane $H_\tau$ and the points $a, b$ and $c$ are the vertices of an ideal hyperbolic triangle in $H_\tau$. Denote its barycenter by $b_\tau$. The point $b_\tau$ corresponds to a negative definite line $L$ in $W_\tau$, and we denote by $\beta_\tau$ its 2-dimensional spacelike orthogonal in $W_\tau$. We define the image of $\tau$ in $Gr_{2,0}(E)$ to be $\beta_\tau$. \[ \square \]

In particular, since $T(n)$ is a transitive $G$ space which fibres over $Gr_{2,0}(E)$, the symmetric space of $G$ by Paragraph 2.1.1

**Corollary 2.6** (Canonical distance) The space $T(n)$ of positive triples in $\partial_\infty H^{2,n}$ is a Riemannian homogeneous $G$-space.

**Definition 2.7** (Visual distance) The visual distance associated to a positive triple on $\partial_\infty H^{2,n}$ is the visual distance associated to $\beta_\tau$ according to Definition 2.2.
2.3 Minkowski patch and charts

In the sequel, $\mathbb{E}^{1,n}$ will be an affine space, whose underlying vector space is equipped with a quadratic form $q$ of signature $(1,n)$.

We explain in this paragraph that $\partial_\infty \mathbb{H}^{2,n}$ is isomorphic to the “conformal compactification” of $\mathbb{E}^{1,n}$. Let us first define

**Definition 2.8 (Minkowski patch and light cones)** Let $a$ be a point in $\partial_\infty \mathbb{H}^{2,n}$.

(i) The light cone $L(a)$ is the union of all photons passing though $a$.
(ii) The Minkowski patch of $a$ is $\text{Mink}(a):=\partial_\infty \mathbb{H}^{2,n} \setminus L(a)$.

One may recognise the description of a big Bruhat cell in the parabolic space $\partial_\infty \mathbb{H}^{2,n}$. We now give a natural conformal description of a Minkowski patch:

**Proposition 2.9** Let $a$ be a point in $\partial_\infty \mathbb{H}^{2,n}$ and $b$ be a point in $\text{Mink}(a)$. Consider $a_0$ and $b_0$ be non zero vectors in $a$ and $b$ respectively, with $\langle a_0, b_0 \rangle = 1$ and set $F:= (a \oplus b)^\perp$. Define $\psi_0$ as the map from $F$ to $E$ given by the formula

$$\psi_0(u) = u - b_0 + \frac{1}{2} q(u) a_0 ,$$

where $q = q|_F$. Then the projectivisation $\psi : F \rightarrow \mathbb{P}(E)$ of $\psi_0$ is a conformal diffeomorphism onto $\text{Mink}(a)$ that we call a canonical Minkowski chart.

**Proof** One easily checks that $q$ has signature $(1,n)$. Observe that $\psi_0$ takes value in the set of null vectors in $E$ whose scalar product with $a_0$ is non-zero. Thus $\psi$ takes values $\text{Mink}(a)$.

We now claim that $\psi$ is surjective: any point $x$ in $\text{Mink}(a)$ has a unique lift $x_0$ to $E$ with $\langle x_0, a_0 \rangle = 1$. Thus

$$x_0 = u - b_0 + \alpha a_0 ,$$

for some $\alpha$ in $\mathbb{R}$ and $u$ in $F$. The condition $q(x_0) = 0$ implies $2\alpha = q(u)$ and so $x_0 = \psi_0(u)$.

Finally, $\psi_0$ is conformal: given $u$ in $F$, we have

$$T_u \psi_0(v) = v + \langle u, v \rangle a_0 ,$$

so $(\psi_0^* q)(u) = q(u)$. Since the projection from a non degenerate vector subspace tangent to the null cone to its projectivisation is conformal by definition of the conformal structure on the Einstein Universe, $\psi$ itself is conformal. \(\square\)

This leads to the following definition
Definition 2.10 (Minkowski chart) Given a point \( a \) in \( \partial_\infty \mathbb{H}^{2,n} \), a Minkowski chart for \( a \) is a conformal diffeomorphism \( \phi \) from \( \mathbb{E}^{1,n} \) to \( \text{Mink}(a) \) so that \( \phi = \psi \circ A \) where \( \psi \) is the canonical Minkowski chart associated to \( b \) in \( \text{Mink}(a) \) and \( A \) is an affine conformal map from \( \mathbb{E}^{1,n} \) to \( F = (a \oplus b)^\perp \).

The following describes useful properties of Minkowski charts.

Proposition 2.11 Let \( a \) be a point in \( \partial_\infty \mathbb{H}^{2,n} \).
(i) Given a lightlike line in \( \mathbb{E}^{1,n} \), the closure of its image by a Minkowski chart is a photon intersecting \( L(a) \) in a unique point.
(ii) Given a spacelike line in \( \mathbb{E}^{1,n} \), the closure of its image by a Minkowski chart is a spacelike circle passing through \( a \).

Proof of (i) Let first prove the result for a canonical Minkowski chart (see Proposition 2.9 for the notations). Let \( L \) be a lightlike line in \( F \) parametrised by \( c : t \rightarrow tw \) with \( q(w) = 0 \). Thus
\[
\psi_0(c(t)) = tw - b_0 + \frac{1}{2} q(tw)a_0 = tw - b_0.
\]
Since \( \langle b_0, w \rangle = 0 \), the 2-plane \( U := \text{span}(b_0, w) \) is isotropic. Thus \( \psi(c(t)) \) is contained in the photon \( P(U) \). Moreover, the intersection of \( L(a) \) with \( U \) is reduced to the isotropic line spanned by the vector \( \frac{w}{\langle w, a_0 \rangle} - b_0 \).

Given two Minkowski charts \( \psi \) and \( \phi \) of \( \text{Mink}(a) \), the map \( \psi \circ \phi^{-1} \) extends to a conformal transformation of \( \partial_\infty \mathbb{H}^{2,n} \) fixing \( a \). Because elements in \( G \) maps photons to photons (as well as spacelike circles to spacelike circles), the results proved for \( \phi \) and \( L \) extend to all charts and lightlike lines.

Proof of (ii) Similarly to the proof of item (i), a spacelike linear line in \( F \) has the form \( c : t \rightarrow tw \) with \( q(w) > 0 \). Its image by \( \psi_0 \) is given by
\[
\psi_0(c(t)) = tw - b_0 + \frac{t^2}{2} q(w)a_0.
\]
Since \( w \) is positive definite, the vector space spanned by \( \{w, a_0, b_0\} \) has signature \((2, 1)\) and intersects \( L(a) \) at \( a \).

23.1 \( \tau \)-charts

Let \( \mathbb{R}^{1,n} \) be the vector space \( \mathbb{R}^{n+1} \) equipped with the quadratic form \( q_{1,n} = dx_1^2 - dx_2^2 - \cdots - dx_n^2 \) of signature \((1, n)\). We identify the stabiliser in \( \text{O}(1, n) \) of \((1, 0 \ldots, 0)\) with \( \text{O}(n) \).

Definition 2.12 (\( \tau \)-chart) Let \( \tau = (a, b, c) \) be a positive triple in \( \partial_\infty \mathbb{H}^{2,n} \). A \( \tau \)-chart is a Minkowski chart \( \psi_\tau \) for \( a \) whose source is \( \mathbb{R}^{1,n} \) and such that \( b = \psi_\tau(-1, 0, \ldots, 0) \) and \( c = \psi_\tau(1, 0, \ldots, 0) \).

We have the following...
**Proposition 2.13** Let \( \tau \) be a positive triple in \( \partial \mathcal{H}^{2,n} \). Up to the precomposition by an element of \( \mathcal{O}(n) \), there exists a unique \( \tau \)-chart.

**Proof** Let \( \tau = (a, b, c) \) a positive triple. Let \( \psi : \mathbb{E}^{1,n} \rightarrow \mathcal{M}(a) \) be a Minkowski chart. By Proposition 2.11, the affine segment \( \delta \) between \( \psi^{-1}(b) \) and \( \psi^{-1}(c) \) is spacelike. Let \( g \) be the conformal transformation that sends the segment \( \delta_0 = [(-1, 0, \ldots, 0), (1, 0, \ldots, 0)] \) to \( \delta \). Observe that \( \psi \circ g \) is again a Minkowski chart. It follows that \( \psi \circ g \) is a \( \tau \)-chart.

Let \( \phi \) be another \( \tau \)-chart. From Proposition 2.11, the map \( g = \psi^{-1} \circ \phi \) is a conformal transformation of \( \mathbb{E}^{1,n} \) fixing \( \delta_0 \) and 0 and so belonging to \( \mathcal{O}(n) \). \( \square \)

2.4 Diamonds and positive quadruples

We introduce in this section the notion of positive quadruples which is a special case of a theory developed in [22] based on [21].

2.4.1 Diamonds and diamond distance

**Proposition 2.14** Given two transverse points \( a \) and \( b \) in \( \partial \mathcal{H}^{2,n} \), the set of points \( c \) in \( \partial \mathcal{H}^{2,n} \) so that \( (a, b, c) \) is a positive triple has two connected components.

**Proof** Let \( \psi : \mathbb{R}^{1,n} \rightarrow \text{Mink}(a) \) be a Minkowski chart with \( \psi(0) = b \). It follows from Proposition 2.11 that the points \( c \) in \( \partial \mathcal{H}^{2,n} \) such that \( (a, b, c) \) is positive are exactly the image of spacelike vectors in \( \mathbb{R}^{1,n} \). This set is thus the union of two convex open cones. \( \square \)

**Definition 2.15** (Diamond) Each of these connected components is called a **diamond** defined by \( a \) and \( b \). Given a positive triple \( (a, c, b) \), the diamond defined by \( a \) and \( b \) containing \( c \) is denoted \( \Delta_c(a, b) \), while the diamond not containing \( c \) is denoted \( \Delta^*_c(a, b) \) (see Fig. 1).

![Fig. 1](diamonds.png) Diamonds in in a \( \tau \)-chart (the point \( a \) is at infinity)
When \( n = 0 \), diamonds are intervals. We now construct another metric on diamonds than the visual distance—as in Definition 2.7—associated to a positive triple and which is better designed for our purposes.

The Euclidean metric on \( \mathbb{R}^{1,n} \) is
\[
dx_1^2 + dx_2^2 + \ldots + dx_{n+1}^2.
\]

**Definition 2.16 (Diamond distance)** Let \( \tau = (a, b, c) \) be a positive triple in \( \partial_\infty H^{2,n} \), the diamond distance \( \delta_\tau \) on the diamond \( \Delta^*_a(b, c) \) is the image by (any) \( \tau \)-chart of the Euclidean metric in \( \mathbb{R}^{1,n} \).

**Proposition 2.17** There is a positive constant \( C \) so that given a positive triple \( \tau = (a, b, c) \), the metrics \( \delta_\tau \) and \( d_\tau \) are \( C \)-biLipschitz on \( \Delta^*_a(b, c) \).

**Proof** Indeed a diamond has finite diameter with respect to both the diamond and the visual distances, which are both Riemannian on \( \mathbb{R}^{1,n} \).

---

**2.5 Positive quadruples and positive maps**

**Definition 2.18 (Positive quadruple)** A quadruple \( (a, b, c, d) \) is positive if all sub-triples are positive and if furthermore (Fig. 2)
\[
\Delta_b(a, c) = \Delta^*_d(a, c).
\]

We can finally define positive maps:

**Definition 2.19 (Positive map and positive loop)** A positive map from a subset \( A \) of \( P(V) \) to \( \partial_\infty H^{2,n} \) is a map sending positive quadruples to positive quadruples. A positive loop in \( \partial_\infty H^{2,n} \) is the image of \( P(V) \) by a continuous positive map.

**Proposition 2.20** A continuous map from \( P(V) \) to \( \partial_\infty H^{2,n} \) sending positive triples to positive triples is positive.

**Proof** Let \( \varphi : P(V) \to \partial_\infty H^{2,n} \) be a continuous map mapping positive triples to positive triples. Let \( (a', b', c', d') \) be a positive quadruple in \( P(V) \) that is mapped to \( (a, b, c, d) \) in \( \partial_\infty H^{2,n} \).

Since pairs of distinct points in \( P(V) \) are mapped to transverse pairs in \( \partial_\infty H^{2,n} \), the image by \( \varphi \) of \( P(V) \setminus \{a', c'\} \) consists in two disjoint arcs \( \gamma_1 \) and

---

Fig. 2 A positive quadruple \( (a, b, c, d) \)
\( \gamma_2 \)—containing respectively \( b \) and \( d \)—contained in \( \Delta_b(a, c) \cup \Delta_b^*(a, c) \) both joining \( a \) and \( c \).

Using a \( \tau \)-chart—with \( \tau = (x, b, c) \)—we see that the union of the lightcones \( L(x) \) for \( x \) in \( \gamma_1 \) covers \( \Delta_b(a, c) \). Recall that any point \( y \) in \( \gamma_2 \) is transverse to any point \( x \) in \( \gamma_1 \) and thus cannot lie in \( L(x) \), since no element in \( L(x) \) is transverse to \( x \). It follows that \( \gamma_2 \)—and thus \( d \) —lies \( \Delta_b^*(a, c) \). Thus \( \Delta_a(a, c) = \Delta_b^*(a, c) \), and \( (a, b, c, d) \) is positive.

Thus in an equivalent way, we could have defined a positive loop in \( \partial_\infty \mathbb{H}^{2,n} \) as a topological circle for which any triple of pairwise distinct points is positive. This is the definition that we used in [31].

### 2.6 Semi-positive loops and Barbot crowns

For a triple of points in \( \partial_\infty \mathbb{H}^{2,n} \), being positive is not a closed condition. As a result, positive loops do not form a closed set. We recall now the notion of semi-positive loops introduced in [31] that are natural degeneration of positive loops.

#### 2.6.1 Semi-positivity

**Definition 2.21** (Semi-positive loop) A triple of points is non-negative if the vector space generated is not of type \((1, 2)\).

A semi-positive loop is the image of an injective continuous map from \( \mathbb{P}(V) \) to \( \partial_\infty \mathbb{H}^{2,n} \) (or to \( \partial_\infty \mathbb{H}^{2,n}_+ \)) sending positive triples to non-negative triples and sending at least one positive triple to a positive one.

We proved in [31, Lemma 2.8]:

**Proposition 2.22** The preimage in \( \partial_\infty \mathbb{H}^{2,n}_+ \) of a positive (respectively semi-positive loop) in \( \partial_\infty \mathbb{H}^{2,n} \) has two connected components, each of which is a positive (respectively semi-positive) loop.

As a result, semi-positive loops in \( \partial_\infty \mathbb{H}^{2,n}_+ \) are exactly lift of semi-positive loops in \( \partial_\infty \mathbb{H}^{2,n} \). Positive and semi-positive loops in \( \partial_\infty \mathbb{H}^{2,n}_+ \) can be characterized in terms of graphs. A map \( f \) between two metric spaces \((X, d_X)\) and \((Y, d_Y)\) is contracting if \( d_Y(f(x), f(y)) < d_X(x, y) \) for any distinct points \( x \) and \( y \) in \( X \). We proved in [31, Proposition 2.11]:

**Proposition 2.23** Let \( \Lambda \) be a topological circle in \( \partial_\infty \mathbb{H}^{2,n}_+ \) and consider a splitting \( \partial_\infty \mathbb{H}^{2,n}_+ \cong \mathbb{S}^1 \times \mathbb{S}^n \) (see Paragraph 2.1.3). Then

(i) The set \( \Lambda \) is a semi-positive loop if and only if it is the graph of a \( 1 \)-Lipschitz map from \( \mathbb{S}^1 \) to \( \mathbb{S}^n \) that is different from a global isometry.
(ii) The set $\Delta$ is positive if and only if it is the graph of a contracting map from $S^1$ to $S^n$.

We have the following:

**Corollary 2.24** If $\Delta$ is a semi-positive loop which is not positive, then it contains an arc with non empty interior which is the intersection of $\Delta$ with a photon.

**Proof** Up to taking a lift, we can assume that $\Delta \subset \partial_\infty H_+^{2,n}$. By the previous proposition, given a splitting $\partial_\infty H_+^{2,n} \cong S^1 \times S^n$, the loop $\Delta$ is the graph of a 1-Lipschitz map $f : S^1 \to S^n$ and since $\Delta$ is not positive there exists $x \neq y$ in $S^1$ such that $d_{S^1}(x, y) = d_{S^n}(f(x), f(y))$. The set

$$\sigma = \{ z \in S^1, \ d_{S^n}(f(x), f(z)) = d_{S^1}(x, z) \}$$

is a closed proper arc in $S^1$ with non-empty interior. Since photons are graph of isometries, the graph of $f$ above $\sigma$ is contained in a photon $\phi$. By construction, $\Delta \cap \phi = (\sigma, f(\sigma)) \subset S^1 \times S^n$.

Proposition 2.23 allows us to have a notion of convergence of semi-positive loops—as graph of 1-Lipschitz maps—and we have:

**Corollary 2.25** Let $\{ \Delta_k \}_{k \in \mathbb{N}}$ be a sequence of semi-positive loops in $\partial_\infty H_+^{2,n}$. Then

(i) The sequence $\{ \Delta_k \}_{k \in \mathbb{N}}$ subconverges to either a semi-positive loop or a photon.

(ii) If each $\Delta_k$ passes through a given positive triple $\tau$, then the sequence $\{ \Delta_k \}_{k \in \mathbb{N}}$ subconverges to a semi-positive loop passing through $\tau$.

**Proof** The first item comes from the fact that the space $M$ of 1-Lipschitz maps from $S^1$ to $S^n$ is compact. From Proposition 2.23, the graph of such a map in $S^1 \times S^n \cong \partial_\infty H_+^{2,n}$ is either a semi-positive loop or a photon.

For the second item, observe that passing through $\tau$ is a closed condition and that no photon can pass through a positive triple.

---

### 2.6.2 Barbot crowns

We define here an important semi-positive loop in $\partial_\infty H_+^{2,n}$, that we call Barbot crown, from Barbot’s work in [2].

**Definition 2.26** A Barbot crown $C$ has four cyclically ordered vertices in $\partial_\infty H_+^{2,n}$, spanning a four dimensional vector space, and four edges which are segment of photons joining consecutive vertices.
Let $C$ be a Barbot crown with vertices $(v_1, v_2, v_3, v_4)$ and let $F_C := v_1 \oplus \ldots \oplus v_4$. Then we denote by $A_C$ the Abelian subgroup of $G$ which, in the splitting $E = v_1 \oplus v_2 \oplus v_3 \oplus v_4 \oplus F_C^\perp$, is defined by the matrices
\[
a(\lambda, \mu) = \begin{pmatrix}
1/\lambda & 0 & 0 & 0 & 0 \\
0 & 1/\mu & 0 & 0 & 0 \\
0 & 0 & \lambda & 0 & 0 \\
0 & 0 & 0 & \mu & 0 \\
0 & 0 & 0 & 0 & \text{Id}_{F_C^\perp}
\end{pmatrix}.
\]

Then we have

**Proposition 2.27** If $(v_1, v_2, v_3, v_4)$ are the vertices of a Barbot crown $C$ labelled clockwise and $F_C := v_1 \oplus v_2 \oplus v_3 \oplus v_4$, then $F_C$ is of type $(2, 2)$ and the Barbot crown $C$ lies in the projective space of $F$.

All Barbot crowns are isomorphic under the action of $G$. Moreover a Barbot crown is globally invariant under the action of a Cartan subgroup of $G$.

**Proof** The 2-planes $P_i := v_i \oplus v_{i+1}$ (where $i \in \mathbb{Z}/4\mathbb{Z}$) are isotropic and the associated photons $\phi_i := P(P_i)$ contain the edges of $C$. The two 2-planes $v_i \oplus v_{i+2}$ have signature $(1, 1)$ and are orthogonal to each other. It follows that $F_C$ has signature $(2, 2)$, and that the Barbot crown $C$ is contained in $\partial_\infty \mathbb{H}^{2,n} \cap P(F_C)$ which is isomorphic under $G$ to $\partial_\infty \mathbb{H}^{2,1}$.

The subgroup $A_C$ preserves the Barbot crown and is a Cartan subgroup of $G$. Observe that the Barbot crown is uniquely defined by $A_C$ (up to cyclic transformation). Since all Cartan subgroups are conjugate, it follows that all Barbot crowns are isomorphic under the action of $G$. \qed

One of our main result is the following:

**Proposition 2.28** Let $\Lambda$ be a semi-positive loop in $\partial_\infty \mathbb{H}^{2,n}$. If $\Lambda$ is not positive, then the closure of its $G$-orbit contains a Barbot crown.

Before proving the proposition, let us prove a lemma.

**Lemma 2.29** Let $\Lambda$ be a semi-positive loop in $\partial_\infty \mathbb{H}^{2,n}$ contained in $M := \bigcup_{i=1}^4 \phi_i$ where the $\phi_i$ are the photons spanned by the vertices of a Barbot crown. Then $\Lambda$ is a Barbot crown.

**Proof** Since $\phi_i$ and $\phi_{i+2}$ do not intersect, while $\phi_i$ and $\phi_{i+1}$ only intersect at a point and all intersection points are distinct, the set $M$ is homeomorphic to the graph drawn in Fig. 3.

Then the image of a continuous injective map from $S^1$ to $M$ is either one of the photons or is made of four arcs, one in each photon. The first case corresponds to a photon, which is not a semi-positive loop, while the second case is a Barbot crown. \qed
Proof of Proposition 2.28 Since $\Lambda$ is not positive, there is a photon $\phi_1$ so that the intersection of $\phi_1$ with $\Lambda$ is a closed non trivial interval $\sigma$—see Corollary 2.24. Denote by $v_1$ and $v_2$ the extremities of this segment.

Let us pick two other points $v_3$ and $v_4$ in $\partial_\infty \mathbb{H}^{2,n}$ and not in $\phi_1$ so that $(v_1, v_2, v_3, v_4)$ generates a Barbot crown. Let $\phi_i$ be the photon passing through $v_i$ and $v_{i+1}$, and $M$ be the union of this four photons. 

In the splitting $E = v_1 \oplus v_2 \oplus v_3 \oplus v_4 \oplus F_\mathcal{C}^\perp$, let $g$ be the element in $G$ defined by the matrix $a(4, 2)$ using the notation (2). The action of $g$ on $\partial_\infty \mathbb{H}^{2,n}$ has the following dynamics:

(i) $g$ fixes $v_1, v_2, v_3, v_4$ and globally fixes $\sigma$ and $M$.
(ii) for any $x$ in $\partial_\infty \mathbb{H}^{2,n}$, any limit values of $\{g^k(x)\}_{k \in \mathbb{N}}$ lies in $M$.
(iii) for any $x$ in $\partial_\infty \mathbb{H}^{2,n}$ and not in $\phi_1$ any limit values of $\{g^k(x)\}_{k \in \mathbb{N}}$ is not in $\phi_1$.

By Corollary 2.25, up to extracting a subsequence, the sequence $\{g^k(\Lambda)\}_{k \in \mathbb{N}}$ converges to $\Lambda_\infty$ which is either a photon or a semi-positive loop and lies in $M$.

Since $\Lambda_\infty$ contains $\sigma$ by construction and an element not in $\phi_1$ by the third item above, $\Lambda_\infty$ is not a photon. Thus by Lemma 2.29 and the second item, $\Lambda_\infty$ is a Barbot crown.

\[ \square \]

2.7 Quasiperiodic loops and maps

Let us fix positive triples $\kappa_0$ and $\tau_0$ in $\mathbb{P}(V)$ and $\partial_\infty \mathbb{H}^{2,n}$ respectively (recall that $V$ is a 2-dimensional $\mathbb{R}$-vector space). We first introduce the notion of quasiperiodic maps and loops:

Definition 2.30 (Quasiperiodic maps and loops)
(i) Let $\xi$ be a a positive map from $\mathbf{P}(V)$ to $\partial_\infty \mathbf{H}^{2,n}$. The map $\xi$ is quasiperiodic, if for any sequence $\{\kappa_k\}_{k \in \mathbb{N}}$ of positive triples in $\mathbf{P}(V)$, if

(a) $h_k$ is the element of $\text{PSL}(V)$ so that $h_k(\kappa_0) = \kappa_k$,

(b) $g_k$ is an element in $G$ so that $g_k(\xi(\kappa_k)) = \tau_0$.

Then the sequence $\{g_k \circ \xi \circ h_k\}_{k \in \mathbb{N}}$ subconverges to a positive map.

(ii) A positive loop $\Lambda$ in $\partial_\infty \mathbf{H}^{2,n}$ is quasiperiodic if for any sequence $\{\tau_k\}_{k \in \mathbb{N}}$ of positive triples in $\Lambda$, if $g_k$ is an element in $G$ so that $g_k(\tau_k) = \tau_0$, then the sequence of renormalised curves $\{g_k(\Lambda)\}_{k \in \mathbb{N}}$ subconverges as a graph to a positive loop.

Observe that the image of a continuous quasiperiodic map is a quasiperiodic loop. As a byproduct of our constructions, we will show in corollary 7.3 that quasiperiodic loops admits a quasiperiodic parametrisation.

**Definition 2.31** (Pointed loops and spaces)

(i) A pointed semi-positive loop is a pair $(\Lambda, \tau)$ where $\Lambda$ is a semi-positive loop and $\tau$ is a positive triple in $\Lambda$. The space $\mathcal{L}(n)$ of those pairs is the space of semi-positive loops.

(ii) A pointed positive loop is a pair $(\Lambda, \tau)$ where $\Lambda$ is a positive loop and $\tau$ is a positive triple in $\Lambda$. The space $\mathcal{L}^+(n)$ of those pairs is the space of positive loops.

(iii) The forgetting map from $\mathcal{L}(n)$ to $\mathcal{T}(n)$ sends $(\Lambda, \tau)$ to $\tau$.

Since any semi-positive loop $\Lambda$ is a Lipschitz graph (see Proposition 2.23), $\mathcal{L}(n)$ can be embedded as a closed subset of the product of the space of Lipschitz graph with $\mathcal{T}(n)$. This embedding gives $\mathcal{L}(n)$, and hence $\mathcal{L}^+(n)$ the structure of a locally compact topological space on which the group $G$ acts continuously. Corollary 2.25 implies the following

**Proposition 2.32** The forgetting map is $G$-equivariant and proper. In particular the action of $G$ on $\mathcal{L}(n)$ is proper and $\mathcal{L}(n)/G$ is compact.

The space $\mathcal{L}_{\tau_0}(n)$ of pointed semi-positive loops $(L, \tau_0)$ is compact and the map $\pi_0$ from $\mathcal{L}_{\tau_0}(n)$ to $\mathcal{L}(n)/G$ is proper and surjective.

2.7.1 The Barbot locus

We define the Barbot locus in $\mathcal{L}(n)/G$ as the quotient of all pointed Barbot crowns by the action of $G$.

**Proposition 2.33** When $n$ equals 1, the Barbot locus is homeomorphic to a disjoint union of two circles. For $n$ larger than 1, the Barbot locus is homeomorphic to a circle.

**Proof** Let us first deal with the case $n = 1$. The action of $G \cong \text{SO}_0(2, 2)$ on the set of photons has two orbits, each corresponding to a ruling of the one
sheeted hyperboloid $\partial_\infty \mathbb{H}^{2,1}$. Two distinct photons intersect if and only if they belong to different families, and any point of $\partial_\infty \mathbb{H}^{2,1}$ is the intersection of a unique pair of photons. In particular, a Barbot crown is made of four photons $\phi_1, \ldots, \phi_4$ where $\phi_i$ and $\phi_{i+1}$ belong to different families.

We define now a $G$-invariant map $(\tau, C) \mapsto w_{\tau, C}$ sending a pointed Barbot crown $(\tau, C)$ to a point in $P(\mathbb{R}^2)$ and show that this map is two-to-one.

Let $(\tau, C)$ be a Barbot crown where $\tau = (x_1, x_2, x_3)$. By positivity, each edge contains at most one $x_i$. Let $c$ be the positive circle spanned by $\tau$.

**CLAIM:** Assume that $x_1, x_2$ and $x_3$ are interior points of the edges, then $c$ intersects $C$ in a fourth point $p_\tau \in c \setminus \{x_1, x_2, x_3\}$.

**Proof of the claim** It follows from [31, Lemma 2.8] that $c$ is disjoint from $P(\beta_\tau^1)$, where $\beta_\tau$ is the barycenter of $\tau$. So using Sect. 2.1.3, we get an identification between $\partial_\infty \mathbb{H}^{2,1} \setminus P(\beta_\tau^1)$ and $S^1 \times \mathbb{R}$ in which $c \cong S^1 \times \{0\}$ and photons in each family give respectively increasing and decreasing maps. Now if $x_i, x_j$ and $x_k$ are points in consecutive edges say $\phi_1, \phi_2$ and $\phi_3$, then by the intermediate value theorem, the photon $\phi_4$ intersects $S^1 \times \{0\}$ in a unique point $p_\tau$ in the interval $(x_k, x_j)$ of $S^1 \setminus \{x_i, x_j, x_k\}$. \hfill $\square$

Observe in the general case that at most one of the $x_i$—say $x_1$—is a vertex $v_k$: indeed assuming $x_2$ is a vertex, then by transversality $x_2 = x_{k+2}$, but then $C$ is included in $L(x_1) \cup L(x_2)$ and thus no element of $C$ can be transverse to both $x_1$ and $x_2$. Thus, if one of the $x_i$ is a vertex, we set unambiguously $p_\tau = x_i$.

Taking the unique projective identification between $c$ and $P(\mathbb{R}^2)$ sending $\tau$ to $(0, 1, \infty)$, the map $w : \tau \mapsto p_\tau$ defines a map $\tau \mapsto w_\tau$ from the Barbot locus to $P(\mathbb{R}^2)$.

Let us now prove that $w$ is two-to-one. The group $G$ acts simply transitively on the positive triples in $\partial_\infty \mathbb{H}^{2,1}$, so if $\tau_0$ is a fixed positive triple, any point in the Barbot locus has a unique representant which is a pointed Barbot crown $(\tau_0, C)$. Identify $P(\mathbb{R}^2)$ with the circle $c_0$ spanned by $\tau_0$ such that $(0, 1, \infty)$ is mapped to $\tau_0$. The result then follows from the fact that a Barbot crown passing through $\tau_0$ and a fourth point $p \in c_0$ is entirely given by a choice of a photon passing through a given point of $\tau_0$, and that there are exactly 2 such photons (one in each family).

Finally, for the case $n > 1$, we fix a signature $(2, 2)$ subspace of $E$. Each pointed Barbot crown in $\partial_\infty \mathbb{H}^{2,n}$ can be mapped to $P(F)$. The stabiliser $Stab_G(F) \cong (O(2, 2) \times O(n-1))_0$ does not act effectively on a Barbot crown in $F$ and the effective action is given by the index two subgroup of $O(2, 2)$ whose maximal compact is $SO(2) \times O(2)$. The element $(Id, -Id)$ intertwine the two family of photons in $\partial_\infty \mathbb{H}^{2,1}$ and thus identify the two disjoint circles representing the Barbot locus for $n = 1$. \hfill $\square$
Given a semi-positive loop $\Lambda$, let $T_\Lambda$ be the space of pairs $(\Lambda, \tau)$ with $\tau$ in $\Lambda$. We have a natural map from $T_\Lambda$ to $\mathcal{L}(n)/G$.

Now, as a consequence of Proposition 2.28, we obtain the following characterisation of quasiperiodic loops:

**Proposition 2.34** A positive loop $\Lambda$ is quasiperiodic if and only if the closure of the image of $T_\Lambda$ in $\mathcal{L}(n)/G$ does not contain a Barbot crown or, equivalently, if the image $T_\Lambda$ in $\mathcal{L}^+(n)/G$ is precompact.

### 3 Cross-ratio and quasisymmetric maps

We define in this section in Definition 3.1 a cross-ratio on $\partial^\infty\mathcal{H}^{2,n}$ which generalises the usual cross-ratio on the real projective line $\mathbb{P}(V)$. We characterise circles using the cross-ratio in Proposition 3.4.

This allows us to define quasisymmetric maps in Definition 3.5. We then show two properties of quasisymmetric maps: equicontinuity in Theorem 3.12 and the existence of a Hölder modulus of continuity Theorem 3.17. Formally the former is a consequence of the latter but we first have to prove equicontinuity.

#### 3.1 Cross-ratio and positivity

Equip the 2-dimensional vector space $V$ with a volume form $\omega$. The usual cross-ratio is defined for a quadruple of pairwise distinct points in $\mathbb{P}(V)$ by

\[
[x, y, z, t] := \frac{\omega(x_0, y_0)\omega(z_0, t_0)}{\omega(x_0, t_0)\omega(z_0, y_0)},
\]

where $x_0, y_0, z_0$ and $t_0$ are non-zero vectors in $x, y, z$ and $t$ respectively. Observe that the definition is independent on the choice of $x_0, y_0, z_0$ and $t_0$ and on the choice of $\omega$.

**Definition 3.1** (Cross-ratio) Given any quadruple $(x, y, z, t)$ of pairwise transverse points in $\partial^\infty\mathcal{H}^{2,n}$, we define the cross-ratio

\[
b(x, y, z, t) := \frac{\langle x_0, y_0 \rangle \langle z_0, t_0 \rangle}{\langle x_0, t_0 \rangle \langle z_0, y_0 \rangle},
\]

where $x_0, y_0, z_0$ and $t_0$ are non-zero vectors of $x, y, z$ and $t$. If $\xi$ is a map from $\mathbb{P}(V)$ to $\partial^\infty\mathcal{H}^{2,n}$, we write whenever defined for $(x, y, z, t)$ in $\mathbb{P}(V)$,

\[
b_\xi(x, y, z, t) := b(\xi(x), \xi(y), \xi(z), \xi(t)).
\]
We warn the reader that unlike the usual cross-ratio in \( \mathbb{P}(V) \), the value of \( b(x, y, z, t) \) does not characterise the positivity of a quadruple of pairwise transverse points in \( \partial_\infty H^{2,n} \). For instance, it can be shown—see Proposition 3.4—that under the canonical isomorphism \( \mathbb{P}(V) \cong \partial_\infty H^{2,0} \), we have \( b(x, y, z, t) = [x, y, z, t]^2 \). Hartnick and Strubel defined in [23] a “functorial” cross-ratio for Shilov boundary of Hermitian symmetric spaces, which we suspect could be used to define positivity of quadruples. However, we stick with our definition of cross-ratio which is easier to handle.

The following lemma gives an expression of the cross-ratio in a Minkowski chart:

**Lemma 3.2** Let \( (a, x, b, y) \) be a quadruple of pairwise transverse points in \( \partial_\infty H^{2,n} \). Then in any Minkowski chart for \( a \) with associated quadratic form \( q \) we have

\[
b(a, x, b, y) = \frac{q(b - y)}{q(b - x)}. \tag{3}
\]

**Proof** Observe that the right hand side of the inequality is invariant by conformal transformation. It follows that it is enough to prove formula (3) in the canonical chart of Mink \( (a) \) associated to \( b \) (see Proposition 2.9). Let us choose as in Proposition 2.9 \( a_0 \) in \( a \) and \( b_0 \) in \( b \) with \( \langle a_0, b_0 \rangle = 1 \). The Minkowski chart from \( (a \oplus b)^\perp \) to Mink \( (a) \) is given by

\[
\psi_0(u) = u - b_0 - \frac{1}{2}q(u)a_0.
\]

Then a straightforward computations give:

\[
\langle \psi_0(u), a_0 \rangle = -1 \text{ and } \langle \psi_0(u), b_0 \rangle = -\frac{1}{2}q(u).
\]

In particular,

\[
b(a, \psi_0(u), b, \psi_0(v)) = \frac{\langle \psi_0(u), a_0 \rangle \langle \psi_0(v), b_0 \rangle}{\langle \psi_0(v), a_0 \rangle \langle \psi_0(u), b_0 \rangle} = \frac{q(v)}{q(u)}.
\]

This concludes the proof since \( b_0 = \psi_0(0) \).

**Corollary 3.3** Let \( (a, b, c) \) be a positive triple in \( \partial_\infty H^{2,n} \) and let \( u \) and \( v \) be two points in \( \Delta_0^a(b, c) \) such that \( b(a, b, v, c) < b(a, b, u, c) \). Then any element \( x \) in \( \Delta_0^a(u, v) \) satisfies

\[
b(a, b, v, c) < b(a, b, x, c) < b(a, b, u, c).
\]
Proof Use the notations of the previous lemma with \( \tau = (a, b, c) \) and define a causal structure on \( \mathbb{R}^{1,n} \) (that is, an orientation of spacelike vectors) such that the vector from \( b \) to \( c \) is future pointing.

Observe that the function \( q(x - b) \) is strictly increasing along future pointing spacelike curves in \( \Delta_a^*(b, c) \) while the function \( q(x - c) \) is strictly decreasing. Since both are positive, Lemma 3.2 implies that the function \( f(x) := b(a, b, x, c) \) is strictly decreasing along future pointing spacelike curves.

The result follows from the fact that \( \Delta_a^*(u, v) \) is the union of all future pointing spacelike curves from \( u \) to \( v \).

The proof of the following result follows closely that of [29, Theorem 5.3]

**Proposition 3.4** A continuous map \( \xi \) whose image contains a positive triple is a circle map if and only if for all quadruples of pairwise distinct points

\[
\mathbf{b}_\xi(x, y, z, t) = [x, y, z, t]^2.
\]

**Proof** It follows from the hypothesis that \( \xi(x) \) and \( \xi(y) \) are transverse whenever \( x \) is different from \( y \). In particular \( \xi \) is injective.

Let \( (x_1, x_2, x_3) \) be a positive triple in \( \mathbf{P}(V) \) whose image by \( \xi \) is positive. It is enough to show that for any \( x_4 \) so that \( (x_1, x_2, x_3, x_4) \) is a positive quadruple, then \( \xi(x_4) \) belongs to the circle \( C \) that passes through \( \xi(x_1), \xi(x_2) \) and \( \xi(x_3) \). Let \( y_0 \) and \( z_0 \) be two points in \( \mathbf{P}(V) \) different from all the points \( x_i \), let also \( \eta \) be the circle map sending \( (x_1, x_2, x_3) \) to \( (\xi(x_1), \xi(x_2), \xi(x_3)) \). Denote by \( \xi_0(x_i), \xi_0(z_0) \) and \( \xi_0(y_0) \) some non zero vectors in \( \xi(x_i), \xi(z_0) \) and \( \xi(y_0) \).

Let us consider the \( 4 \times 4 \) matrix \( A(\xi) \) whose coefficients are \( a_{i,j} := \mathbf{b}_\xi(y_0, z_0, x_i, x_j) \). The determinant of this matrix is a non zero multiple of the Gram matrix \( G(\xi_0) \) whose coefficients are \( g_{i,j} = \langle \xi_0(x_i), \xi_0(x_j) \rangle \):

\[
\det(A(\xi)) = \frac{\prod_{i=1}^4 \langle \xi_0(y_0), \xi_0(z_0) \rangle^4}{\prod_{i=1}^4 \langle \xi_0(y_0), \xi_0(x_i) \rangle \prod_{j=1}^4 \langle \xi_0(z_0), \xi_0(x_j) \rangle} \det(G(\xi_0)).
\]

Thus, the determinant of \( A(\xi) \) is zero if and only if \( \text{span}\{\xi(x_1), \ldots, \xi(x_4)\} \) has dimension 3, that is if \( \xi(x_4) \) belongs to \( C \). In particular, \( \det(A(\eta)) = 0 \). Since our hypothesis implies that \( A(\xi) = A(\eta) \), it follows that \( \det(A(\xi)) = 0 \) and that \( \xi(x_4) \) belongs to \( C \).

\[ \square \]

### 3.2 Quasisymmetric maps

Recall that a homeomorphism \( \varphi \) of \( \mathbf{P}(V) \) is quasisymmetric if there exists constants \( A \) and \( B \) greater than 1 such that for any quadruple of pairwise
distinct points in $\mathbf{P}(V)$, we have

$$A^{-1} \leq ||[x, y, z, t]| \leq A \text{ implies } B^{-1} \leq ||[\varphi(x), \varphi(y), \varphi(z), \varphi(t)]|| \leq B .$$

Here we do not impose $\varphi$ to be orientation preserving (this is why we need the absolute value).

We now use our cross-ratio to define quasisymmetric maps from $\mathbf{P}(V)$ to $\partial_\infty \mathbf{H}^{2,n}$. This notion is closely related to both the classical notion of quasicircles for Kleinian groups and the notion of Sullivan maps developed in [27].

After some preliminaries involving the related notion in a Minkowski patch we prove the main results of this section concerning quasisymmetric maps: equicontinuity (Theorem 3.12) and Hölder property (Theorem 3.17).

**Definition 3.5 (Quasisymmetric)** Let $\Omega$ be a subset of $\mathbf{P}(V)$ whose closure is connected, $\xi$ a map from $\Omega$ to $\partial_\infty \mathbf{H}^{2,n}$ and $A, B$ be constants greater than 1. The map $\xi$ is $(A, B)$-quasisymmetric on $\Omega$ if it is positive and for all quadruple $(x, y, z, t)$ in $\Omega^4$ we have

$$A^{-1} \leq ||[x, y, z, t]| \leq A \text{ implies } B^{-1} \leq |b_\xi(x, y, z, t)| \leq B . \quad (4)$$

We will refer to $\Omega$ as the defining set of $\xi$.

We also say $\xi$ is quasisymmetric if it is $(A, B)$-quasisymmetric for some constants $A$ and $B$.

Finally, we call quasicircle the image of a quasisymmetric map defined on $\mathbf{P}(V)$.

3.3 Preliminaries

3.3.1 Choice of constants

The following lemma tells us the choice of $A$ is not relevant:

**Lemma 3.6** For any constants $A, B$ and $C$ greater than 1, there exists a constant $D$ such that any $(A, B)$-quasisymmetric map is $(C, D)$-quasisymmetric.

**Proof** If $C \leq A$, then we can take $D = B$. Let us thus assume $C > A$. Let $(x, y, z, t)$ be elements of $\Omega$ so that

$$C^{-1} \leq ||[x, y, z, t]| \leq C .$$

Let $N$ be any integer. Using the density of $\Omega$ in an interval and the cocycle property of the cross-ratio

$$b(a, c_0, d, c_2) = b(a, c_0, d, c_1) \cdot b(a, c_1, d, c_2)$$
we can find \(w_0, \cdots, w_N\) in \(\Omega\) with \(w_0 = y, w_N = t\), so that

\[
C^{-\frac{1}{N}} \leq |[x, w_i, z, w_{i+1}]| \leq C^\frac{1}{N}.
\]

In particular taking

\[
\frac{\log C}{\log A} \leq N \leq 1 + \frac{\log C}{\log A}
\]

we have

\[
C^\frac{1}{N} = \exp\left(\frac{\log(C)}{N}\right) \leq \exp(\log(A)) = A.
\]

Thus

\[
A^{-1} \leq |[x, w_i, z, w_{i+1}]| \leq A.
\]

It follows that

\[
B^{-1} \leq |b_\xi (x, w_i, z, w_{i+1})| \leq B.
\]

Using finally the cocycle property for \(b_\xi\), we have

\[
B^{-N} \leq |b_\xi (x, y, z, t)| \leq B^N.
\]

In particular, \(D := B^{1 + \frac{\log(C)}{\log(A)}} \geq B^N\) satisfies the required condition. \(\square\)

### 3.3.2 Quasisymmetry in a Minkowski patch

In a manifold equipped with a conformal structure of type \((1, n)\) the set of nonzero spacelike vectors is the union of two non intersecting convex cones. A space orientation is a continuous choice of one of this cone over the whole manifold.

Let \(F\) be a vector space equipped with a quadratic form \(q\) of signature \((1, n)\). In this case, the choice of one spacelike vector at one point defines the space orientation.

For \(\tau = (a, b, c)\) a positive triple in \(\partial_\infty \mathbb{H}^{2,n}\), we choose the space orientation for \(\text{Mink}(c)\) to be defined by \(b - a\). Then we define for any \(x\) in \(\text{Mink}(c)\) the set \(I_+(x)\) to be the cone of space like vectors defining the orientation, while \(I_-(x)\) is the cone of space like vectors defining the opposite orientation. Similarly \(J_+(x)\) and \(J_-(x)\) denote the closure of \(I_+(x)\) and \(I_-(x)\) respectively. Then

\[
\Delta_c^*(a, b) = I_+(a) \cap I_-(b).
\]
**Definition 3.7** *(Positive and semi-positive maps)* Let \( \Omega \) be a dense subset of an interval \( O \) of \( \mathbb{R} \). A map \( f \) from \( \Omega \) to \( F \) is

(i) **positive** if for all \( x > y \), \( f(x) \) belongs to \( I_+(f(y)) \)—equivalently if \( f(y) \) belongs to \( I_-(f(x)) \); in particular \( I_+(f(x)) \subset I_+(f(y)) \).

(ii) **Semi-positive** if for all \( x > y \), \( f(x) \) belongs to \( J_+(f(y)) \)—equivalently if \( f(y) \) belongs to \( J_-(f(x)) \); in particular \( J_+(f(x)) \subset J_+(f(y)) \).

We have the following

**Lemma 3.8** Let \( \Omega \) be a dense subset of an interval \( O \) and \( f \) be a semi-positive map from \( \Omega \) to \( F \). Then for any \( x \) in \( O \), the following limits exist

\[
f^+(x) := \lim_{y \to x \text{ and } y > x} f(y) \quad f^-(x) := \lim_{y \to x \text{ and } y < x} f(y).
\]

**Proof** If \( z \) and \( y \) belong to \( \Omega \) and \( z > y \), we have \( J_+(f(z)) \subset J_+(f(y)) \). Let

\[
W(x) := \bigcup_{y > x} J_+(f(y))
\]

then \( W(x) = J_+(w) \) for some unique element \( w \) in \( F \). We then define \( f^+(x) := w \). We construct symmetrically \( f^- \). The result follows. \( \Box \)

Given constants \( A, B \) greater than 1, we say that a map \( f \) from \( \Omega \) to \( F \) is \((A, B)\text{-quasisymmetric}\) if it is positive and if \( (x, y, z) \) are pairwise distinct in \( \Omega \) then

\[
\frac{1}{A} \leq \left| \frac{x - y}{z - x} \right| \leq A \quad \text{implies} \quad \frac{1}{B} \leq \frac{q(f(x) - f(y))}{q(f(z) - f(x))} \leq B. \quad (6)
\]

This slight abuse of language is justified by Lemma 3.2.

**Lemma 3.9** Given a semi-positive map \( f \) from a dense subset \( \Omega \) of an interval \( O \) to \( F \),

(i) the maps \( f^+ \) and \( f^- \) are semi-positive and respectively rightcontinuous and leftcontinuous.

(ii) If \( f^+ \) (respectively \( f^- \)) is continuous, then \( f^+ \) (respectively \( f^- \)) is the unique semi-positive continuous extension of \( f \) to \( O \).

(iii) Assume furthermore that \( f \) is \((A, B)\text{-quasisymmetric}\), then \( f^+ \) and \( f^- \) are both \((A, B)\text{-quasisymmetric}\).

**Proof** The only non trivial point is item (iii). We first show that if \( f \) is positive on \( \Omega \), then \( f^- \) (or equivalently \( f^+ \)) is positive on \( O \). Let us first observe that,
since $f^-$ is semi-positive, if $y < x < z$ and $x$ is in $O$, $y$ and $z$ in $O$, then by
density of $\Omega$

$$f(x) \in J_+(f^-(y)) \cap J_-(f^-(z)) =: J(y, z).$$

Thus $f^-(\{y, z\})$ is included in $J(y, z)$. Moreover since $\Omega$ is dense, there exists
$x_0$ and $w_0$ in $\Omega$ so that $J(y, z)$ contains

$$J_0 = J_+(f(x_0)) \cap J_-(f(w_0))$$

Since $f$ is positive, $J_0$—hence $J(y, z)$—contains a space like segment. In
particular $J(y, z)$ is not a lightlike line. This implies that $f^-(y)$ and $f^-(z)$
are not joined by a lightlike segment, and so $f^-$ is positive. The proof for $f^+$
is analogous.

We conclude the proof of the last item using that $\Omega$ is dense and the cross-
ratio is continuous.

\hfill $\Box$

3.3.3 Quasisymmetry in the Minkowski case and continuity

**Proposition 3.10** Let $f$ be a positive map from a dense subset $\Omega$ of an interval
$O$ to $F$. If $f$ is $(A, B)$-quasisymmetric, then $f$ extends uniquely on $O$ to a
continuous $(A, B)$-quasisymmetric map.

**Proof** Let us first assume that $f$ is defined on $O$ and is leftcontinuous. By
Lemma 3.9, it is enough to show that $f = f^+$, since then $f$ will be both right-
continuous and leftcontinuous and thus continuous. Let $x$ be in $O$, $a := f(x)$
and $a^+ := f^+(x)$. Our goal is to show that $a^+ = a$

**First step:** we have $q(a^+ - a) = 0$.

Let us consider $y_0^k = x + \frac{1}{k}$ and $y_1^k = x + \frac{2}{k}$, so that both sequences $\{y_0^k\}_{k \in \mathbb{N}}$
and $\{y_1^k\}_{k \in \mathbb{N}}$ converge to $x$ when $k$ goes to infinity and satisfy $x < y_0^k < y_1^k$.
Moreover

$$\frac{|y_0^k - y_1^k|}{|y_0^k - x|} = 1.$$ 

By the quasisymmetric property, the ratio

$$\frac{q(f(y_0^k) - f(y_1^k))}{q(f(y_0^k) - f(x))}$$

is bounded from below. Since

$$\lim_{k \to \infty} q(f(y_0^k) - f(y_1^k)) = q(a^+ - a^+) = 0$$
it follows that
\[ q(a^+ - a) = \lim_{k \to \infty} (q(f(y_k^0) - f(x))) = 0. \]

**SECOND STEP.** We now prove that \( a^+ = a \).

Assume \( a \neq a^+ \). By the first step the line \( L \) through \( a^+ \) and \( a \) is lightlike. For any \( z < x \), since \( f \) is positive \( \langle f(z), a \rangle \) is non zero. Thus \( f(z) \) does not belong to \( L \) and moreover if \( P_z \) be the plane containing \( f(z) \) and \( L \) then the restriction of \( q \) to \( P_z \) is non degenerate and therefore has signature \((1, 1)\). Let \( L'_z \) be the unique lightlike line in \( P_z \) through \( f(z) \) and not parallel to \( L \). Since \( f \) is leftcontinuous,
\[
\lim_{z \to x, z < x} f(z) = a.
\]

Then if \( g(z) \) is the intersection of \( L'_z \) with \( L \) (see Fig. 4). We have
\[
\lim_{z \to x, z < x} g(z) = a.
\]

We now observe that if \( b \) is in \( L'_z \) and \( c \) in \( L \), then
\[ q(b - c) = 2(b - g(z) | g(z) - c). \]

Thus
\[ \frac{q(f(z) - a^+)}{q(f(z) - a)} = \frac{\langle f(z) - g(z) | g(z) - a^+ \rangle}{\langle f(z) - g(z) | g(z) - a \rangle} = \frac{g(z) - a^+}{g(z) - a} \tag{7} \]
where the latter ratio is the ratio of three points in an affine line.

Let us now choose a sequence \( \{x_k\}_{k \in \mathbb{N}} \) in \( O \) converging to \( x \) and so that \( x_k > x \). Then for all \( z \) strictly smaller than \( x \),
\[
\lim_{k \to \infty} \frac{x_k - z}{x - z} = 1.
\]
Thus we have
\[ \frac{g(z) - a^+}{g(z) - a} = \frac{q(f(z) - a^+)}{q(f(z) - a)} = \lim_{k \to \infty} \frac{q(f(z) - f(x_k))}{q(f(z) - f(x))} \leq B. \]

where the last inequality comes from the \((A, B)\)-quasisymmetry. Since \(\lim_{z \to x, z < x} g(z) = a\) it follows that \(\lim_{z \to x, z < x} g(z) = a^+\). Thus \(a = a^+\) and this concludes the proof of the particular case when \(f\) is leftcontinuous and defined on \(O\).

Turning to the general case, let \(f\) be an \((A, B)\)-quasisymmetric map defined on a dense subset of \(O\). Using item \((iii)\) of Lemma 3.9 and the previous discussion we obtain that \(f^−\) is continuous. Thus \(f^−\) is the unique continuous extension of \(f\) to \(O\) by the second item of Lemma 3.9. This concludes the proof of the general case.

3.3.4 Limit of quasisymmetric maps in the Minkowski case

We now prove

**Proposition 3.11** Let \(\{f_k\}_{k \in \mathbb{N}}\) be a sequence of \((A, B)\)-quasisymmetric maps from \(\mathbb{R}\) to \(F\). Assume that we have two points \(y\) and \(z\) so that \(\{f_k(y)\}_{k \in \mathbb{N}}\) and \(\{f_k(z)\}_{k \in \mathbb{N}}\) converges respectively to \(a\) and \(b\) so that \(q(a - b) > 0\).

Then after extracting a subsequence \(\{f_k\}_{k \in \mathbb{N}}\) converges uniformly to a positive \((A, B)\)-quasisymmetric map \(f\) on the interval \([y, z]\).

**Proof** Let us choose a dense countable subset \(\Omega\) of \([y, z]\) containing \(y\) and \(z\). For \(k\) large enough, we observe that \(f_k\) takes values in a compact neighbourhood of \(I_+(a) \cap I_-(b)\). Then, using Cantor’s diagonal argument and extracting subsequences, we obtain a map \(f\) defined on \(\Omega\) so that \(\{f_k\}_{k \in \mathbb{N}}\) converges pointwise to \(f\) on \(\Omega\). Observe that \(f\) is semi-positive. Using the \((A, B)\)-quasisymmetry and Lemma 3.6, we deduce that for any \(x\) in \(\Omega\) and \([y, z]\) the ratios

\[ \frac{q(f_k(y) - f_k(z))}{q(f_k(y) - f_k(x))} \]

are uniformly bounded. Since the numerator converges to a non zero value it follows that \(q(f(y) - f(x)) > 0\). Repeating the argument we obtain that for all distinct \(x\) and \(t\) in \(\Omega\), then \(q(f(t) - f(x)) > 0\). In other words, \(f\) is positive. Obviously \(f\) is \((A, B)\)-quasisymmetric. Thus by Proposition 3.10, \(f\) extends uniquely to a continuous positive \((A, B)\)-quasisymmetric map on \([y, z]\), extension that we also denote \(f\).

Let us prove that \(\{f_k\}_{k \in \mathbb{N}}\) converges to \(f\) uniformly on \([y, z]\). Let \(\{x_k\}_{k \in \mathbb{N}}\) be a sequence of points in \([y, z]\) converging to \(x\). Let \(w\) and \(t\) in \(\Omega\) and \([y, z]\),
with \( w < x < t \). Let \( c \) be a limit value of \( \{ f_k(x_k) \}_{k \in \mathbb{N}} \). By semi-positivity of the functions \( f_k \) for \( k \) large enough \( f_k(x_k) \) belongs to

\[
J_+(f_k(w)) \cap J_-(f_k(t)).
\]

Since \( \{ f_k(w) \}_{k \in \mathbb{N}} \) and \( \{ f_k(w) \}_{k \in \mathbb{N}} \) converge to \( f(w) \) and \( f(t) \) respectively, \( c \) belongs to

\[
J(w, t) := J_+(f(w)) \cap J_-(f(t)).
\]

Since \( f \) extends to a continuous function, it follows

\[
\lim_{w, t \to x} J(w, t) = \{ f(x) \}
\]

thus \( c = f(x) \). We have shown that \( \lim_{k \to \infty} f_k(y_k) = f(x) \). This concludes the proof of uniform convergence. 

\[ \square \]

### 3.4 Equicontinuity

We now come back to quasisymmetric maps from \( \mathbb{P}(V) \) to \( \partial_\infty \mathbb{H}^{2,n} \). Fix \( \kappa_0 \) and \( \tau_0 \) positive triples in \( \mathbb{P}(V) \) and \( \partial_\infty \mathbb{H}^{2,n} \) respectively. Recall that distances between positive triples is given in Definition 2.6.

**Theorem 3.12** (Equicontinuity) Every sequence of \((A, B)\)-quasisymmetric maps sending \( \kappa_0 \) to \( \tau_0 \) possesses a subsequence that converges uniformly to a continuous \((A, B)\)-quasisymmetric map. Equivalently, the action by precomposition and postcomposition of \( \text{PSL}(V) \times G \) on the space of \((A, B)\) quasisymmetric maps is cocompact.

Here are several corollaries, the first one is immediate, the next ones are proved right after the Equicontinuity Theorem.

**Corollary 3.13** A quasisymmetric map is quasiperiodic.

**Corollary 3.14** For every constants \( A, B, C \) greater than 1, there exists a constant \( D \) such that if \( \xi \) is \((A, B)\)-quasisymmetric, if \((a, b, c, d)\) is a quadruple in \( \mathbb{P}(V) \), then

\[
C^{-1} \leq |b(\xi(a), \xi(b), \xi(c), \xi(d))| \leq C \text{ implies } D^{-1} \leq |[a; b; c; d]| \leq D.
\]

**Corollary 3.15** For every constants \( A, B \), greater than 1, there exists a constant \( D \) such that if \( \phi \) is a homeomorphism of \( \mathbb{P}(V) \) so that both \( \xi \) and \( \xi \circ \phi \) are \((A, B)\)-quasisymmetric, then \( \phi \) is \((A, D)\) quasisymmetric.
**Corollary 3.16** Let $A, B$ be constant greater than 1, then for any positive real $a$ there is a positive real $b$ so that the following holds. If $\xi$ be a $(A, B)$-quasisymmetric map, then for any positive triples $\kappa_1$ and $\kappa_2$, we have

$$d(\kappa_1, \kappa_2) \leq a \implies d(\xi(\kappa_1), \xi(\kappa_2)) \leq b.$$  

**Proof of the Equicontinuity Theorem 3.12** Let $\{\xi_k\}_{k \in \mathbb{N}}$ be a sequence of quasisymmetric map sending a triple $\kappa_0 = (x_0, y_0, z_0)$ in $P(V)$ to a positive triple $\tau_0 = (a_0, b_0, c_0)$.

Let us identify $P(V) \setminus \{x_0\}$ with $\mathbb{R}$, so that $y_0$ and $z_0$ are identified with $-1$ and $1$ respectively.

Using the $\tau_0$-chart of Mink($a_0$), we obtain a sequence $\{f_k\}_{k \in \mathbb{N}}$ of maps from the interval $[-1, 1]$ to $\mathbb{R}^{1,n}$ so that the two sequences $\{f_k(-1)\}_{k \in \mathbb{N}}$ and $\{f_k(1)\}_{k \in \mathbb{N}}$ are constant.

By Lemma 3.2, the maps $f_k$ are $(A, B)$-quasisymmetric as maps from $[-1, 1]$ to $\mathbb{R}^{1,n}$. By Proposition 3.11, we can extract a subsequence so that $\{f_k\}_{k \in \mathbb{N}}$ converges uniformly on $[-1, 1]$ to an $(A, B)$ quasi symmetric map in the Minkowski sense. Using Lemma 3.2 again, $\{\xi_k\}_{k \in \mathbb{N}}$ converges uniformly on the interval $\Delta_{x_0}^*[y_0, z_0]$ to a map which is $(A, B)$-quasisymmetric. Cycling through $(x_0, y_0, z_0)$ and using the fact that

$$P(V) = \Delta_{x_0}^*[y_0, z_0] \cup \Delta_{y_0}^*[x_0, z_0] \cup \Delta_{z_0}^*[x_0, y_0]$$

we complete the proof of the theorem. \hfill \Box

**Proof of Corollary 3.14** Let us fix some positive constants $A$ and $B$, and let

$$\mathcal{B} := \{ (\xi, a, b, c, d) \mid \xi \text{ is } (A, B)\text{-quasisymmetric }, (a, b, c, d) \in P(V)^4 \}. $$

Then $\mathcal{B}$ is equipped with an action of $H := PSL(V) \times G$ given by

$$(h, g) \cdot (\xi, a, b, c, d)) = (g \circ \xi \circ h^{-1}, h(a), h(b), h(c), h(d)).$$

This action is cocompact by the Equicontinuity Theorem 3.12. Let $C$ be a constant greater than 1 and

$$\mathcal{B}(C) := \{ (\xi, a, b, c, d) \in \mathcal{B} \text{ such that } C^{-1} \leq |b(\xi(a), \xi(b), \xi(c), \xi(d))| \leq C \}. $$

Observe that $\mathcal{B}(C)$ is closed and $H$-invariant. Thus its projection in $\mathcal{B}/H$ is compact. Let finally, for any positive integer $n$,

$$\mathcal{B}_n(C) := \{ (\xi, a, b, c, d) \in \mathcal{B}(C) \text{ such that } n^{-1} < |[a; b; c; d]| < n \}. $$

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Since the union for all \( n \) in \( \mathbb{N} \) of the sets \( B_n(C) \) is equal to \( B(C) \) and that \( B_n(C) \) is open in \( B(C) \) and \( H \)-invariant, it follows by compactness that there exists some \( p \) so that \( B_p(C) \) is equal to \( B(C) \). Thus we obtain the corollary by taking \( D = p \).

Proof of Corollary 3.15 Assume that the cross-ratio of the quadruple \((a, b, c, d)\) in \( P(V)^4 \) is in \([A, A^{-1}]\) then the cross-ratio of \( \xi \circ \phi(a, b, c, d) \) is in \([B, B^{-1}]\), since \( \xi \circ \phi \) is \((A, B)\)-quasisymmetric. Hence, by Corollary 3.14 and since \( \xi \) is \((A, B)\)-quasisymmetric, there exists a constant \( D \) only depending on \( A \) and \( B \) so that the cross-ratio of \( \phi(a, b, c, d) \) is in \([D, D^{-1}]\). Thus \( \phi \) is \((A, D)\)-quasisymmetric.

Proof of Corollary 3.16 Let us fix some constants \( A \) and \( B \) greater than 1, and let \( a \) be positive. Let

\[
A(a) = \{ (\xi, \tau_0, \tau_1) \mid \xi \text{ is } (A, B)\text{-quasisymmetric} \\
\tau_0 \tau_1 \text{ are positive triples in } P(V) \text{ with } d(\tau_0, \tau_1) \leq a \}.
\]

As in the previous corollary \( A(a) \) is equipped with an action of \( H := \text{PSL}(V) \times G \), and this action is cocompact by the Equicontinuity Theorem. In particular since the function which associates to \((\xi, \tau_0, \tau_1)\), the real \( d(\xi(\tau_0), \xi(\tau_1)) \) is continuous and \( H \)-invariant, it is bounded. This concludes the proof of the corollary.

3.5 Hölder property

Recall any positive triple \( \tau \) in \( \partial_\infty H^{2,n} \) defines a visual distance \( d_\tau \) on \( \partial_\infty H^{2,n} \) (see Definition 2.7).

Theorem 3.17 (Hölder Modulus of Continuity) For any constants \( A, B \) greater than 1, there exist positive constants \( M \) and \( \alpha \) with the following property: if \( \xi \) is an \((A, B)\)-quasisymmetric map from a dense subset of \( P(V) \) to \( \partial_\infty H^{2,n} \), then \( \xi \) extends uniquely to a map from \( P(V) \) to \( \partial_\infty H^{2,n} \) such that if \( \tau_0 \) is any positive triple in \( P(V) \), then for all \( x \) and \( y \) in \( P(V) \)

\[
d_\xi(\tau_0)(\xi(x), \xi(y)) \leq M \cdot d_{\tau_0}(x, y)^\alpha.
\]

Observe that equicontinuity is a formal consequence of this theorem but we actually use equicontinuity in the proof of this theorem.

3.5.1 A contraction lemma

Lemma 3.18 (Contraction Lemma) For any \( B > 1 \), if \( \tau = (a, b, c) \) and \( \tau' = (a, x, y) \) are positive triples in \( \partial_\infty H^{2,n} \) satisfying
(i) the diamond $\Delta^*_a(x, y)$ is included in $\Delta^*_a(b, c)$,
(ii) if we have the inequalities

$$B^{-1} \leq b(a, b, x, c) \leq B, \quad B^{-1} \leq b(a, b, y, c) \leq B$$

then, on $\Delta^*_a(x, y)$, the diamond distances $\delta_\tau$ and $\delta_\tau'$ (see Definition 2.16) satisfy

$$\delta_\tau \leq \frac{B - 1}{B + 1} \cdot \delta_\tau'.$$

**Proof** Let $g$ in $G$ so that $g\tau = \tau'$. Then $g$ defines an isometry

$$(\Delta^*_a(b, c), \delta_\tau) \to (\Delta^*_a(x, y), \delta_\tau').$$

The statement is thus equivalent to the fact that $g$ is $\frac{B^{-1}}{B + 1}$-contracting when seen as a map from $(\Delta^*_a(b, c), \delta_\tau)$ to itself.

Fix a $\tau$-chart $\psi_\tau : \mathbb{R}^{1,n} \to \text{Mink}(a)$ and identify $\mathbb{R}^{1,n}$ with its image by $\psi_\tau$. Denote respectively by $q$ the quadratic form of signature $(1, n)$ and $q_\tau$ the Euclidean quadratic form on $\mathbb{R}^{1+n}$. To lighten notations, we write $\Delta = \Delta^*_a(b, c)$.

Since $g$ fixes the point $a$, it is a conformal transformation of the Minkowski patch $\text{Mink}(a)$ which we identify using the $\tau$-chart as a conformal transformation $g'$ of $\mathbb{R}^{1,n}$. We may thus write

$$g' = g_0 + u$$

where $u$ is a translation and $g_0$ belongs to $H := \mathbb{R}^* \times \text{SO}(q)$.

The subgroup $K$ of $H$ fixing the first coordinate is both a maximal compact subgroup of $H$ and a subgroup of $\text{SO}(q_\tau)$. Observe also that $K$ preserves $\Delta$.

Let $A$ be the subgroup of $H$, fixing the last $n - 1$ coordinates. Then $A$ is a Cartan subgroup of $H$. Using the Cartan decomposition $H = KAK$, we can write $g_0 = k_0\alpha k_1$, where $k_0$ and $k_1$ belongs to $K$ and $\alpha$ to $A$. In particular,

$$\|g'\| = \|g_0\| = \|\alpha\|$$

where the norm is computed with respect to $q_\tau$.

Let $I^\pm$ be the two lightlike lines in the plane defined by the first two coordinates. Then $\|\alpha\| = \max(\lambda_+ , \lambda_-)$ where $\lambda_\pm$ are the eigenvalues of $\alpha$ on $I^\pm$. Observe now that

$$\lambda^\pm = \frac{\ell (\alpha(I^\pm \cap \Delta))}{\ell (I^\pm \cap \Delta)} = \frac{\ell (\alpha(I^\pm \cap \Delta))}{\sqrt{2}}$$

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where $\ell$ is the length associated to $q_\tau$, and the last inequality comes from the fact that

$$\ell (I^\pm \cap \Delta) = \sqrt{2}.$$

Since $K$ globally fixes $\Delta$ and preserves $\ell$, and the translations preserve $\ell$, we have

$$\ell (\alpha(I^\pm \cap \Delta)) = \ell ((k_0\alpha k_1)(I^\pm \cap \Delta)) = \ell (((k_0\alpha k_1 + u) \cdot I^\pm) \cap ((k_0\alpha k_1 + u) \cdot \Delta)) = \ell (g'(I^\pm \cap \Delta)).$$

Let $\Delta' := g' \cdot \Delta = \Delta_a^*(x, y)$, we get

$$\|g'\| \leq \sup \left\{ \frac{\ell(\phi \cap \Delta')}{\sqrt{2}} \mid \phi \text{ lightlike line} \right\}. \quad (9)$$

Corollary 3.3 implies that

$$\Delta' \subset C_B := \{ w \in \Delta, \ B^{-1} \leq b(a, b, \psi_\tau(w), c) \leq B \}.$$ 

Thus the result follows from the inequality (9) and the following CLAIM: any lightlike segment contained in $C_B$ has length less that $\frac{B-1}{B+1} \sqrt{2}$.

Let us prove this claim. Let $\phi$ be a lightlike ray intersecting the lightcone of $b$ and $c$ in $p$ and $q$ respectively (see Fig. 5). For $u = q - p$, the intersection between $\phi$ and $\Delta$ is parametrised by

$$\phi(t) = p + tu, \ t \in [0, 1].$$
From Lemma 3.2, we have

$$b(a, b, \phi(t), c) = \frac{q(\phi(t) - c)}{q(\phi(t) - b)}.$$  

Writing $\phi(t) - b = tu + (p - b)$ and using the fact that $u$ and $(p - b)$ are isotropic (and similarly for $q(\phi(t) - c)$), we obtain

$$\begin{align*}
q(\phi(t) - c) &= -2(1 - t)\langle u, q - c \rangle \\
q(\phi(t) - b) &= 2t \langle u, p - b \rangle.
\end{align*}$$

We get

$$b(a, b, \phi(t), c) = \lambda \cdot \frac{1 - t}{t} \quad \text{with} \quad \lambda = \frac{\langle u, q - c \rangle}{\langle u, p - b \rangle}.$$  

Observe that $\lambda$ is positive. Thus $\phi \cap C_B$ is parametrised by the compact segment $J_\phi$ of $]0, 1[,$ defined by

$$J_\phi = \left[ \frac{1}{B/\lambda + 1}, \frac{1}{1/B\lambda + 1} \right].$$

Hence

$$\frac{\ell(\phi \cap C_B)}{\ell(\phi \cap \Delta)} = \left| \frac{1}{B/\lambda + 1} - \frac{1}{1/B\lambda + 1} \right| = \frac{(B^2 - 1)\lambda}{(B + \lambda)(\lambda B + 1)} \leq \frac{B - 1}{B + 1}.$$  

It follows that

$$\ell(\phi \cap C_B) \leq \frac{B - 1}{B + 1} \ell(\phi \cap \Delta) \leq \frac{B - 1}{B + 1} \sqrt{2}.$$  

This completes the proof of the claim and thus of the contraction lemma. \qed

3.5.2 Proof of Theorem 3.17

The proof follows the same scheme as the proof of the corresponding theorem in [27]. We now assume the hypothesis of Theorem 3.17 and prove two preliminary lemmas.

Let us first show that it is enough to prove the statement for $x$ and $y$ so close with respect to $\tau_0$, more precisely

Lemma 3.19 Assume that there exists positive $\varepsilon$, $M_0$ and $\alpha_0$, so that if $d_{\tau_0}(x, y) \leq \varepsilon$, then

$$d_{\xi(\tau_0)}(\xi(x), \xi(y)) \leq M_0 \cdot d_{\tau_0}(x, y)^{\alpha_0}.$$  

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Then there exists positive $M$ and $\alpha$ so that for all $x$, $y$ and $\tau_0$,

$$d_{\xi(\tau_0)}(\xi(x), \xi(y)) \leq M \cdot d_{\tau_0}(x, y)^\alpha.$$ 

**Proof** Indeed, let $k$ so that $\varepsilon \geq \frac{1}{k} \cdot \text{diam}(P(V))$. Let $x$ and $y$ in $P(V)$. Using a geodesic between $x$ and $y$, let $(x_0, \ldots, x_k)$ so that $d_{\tau_0}(x_i, x_{i+1}) = \frac{1}{k}d_{\tau_0}(x, y)$ and $x_0 = x, x_k = y$. It follows that $d_{\tau_0}(x_i, x_{i+1}) \leq \frac{1}{k} \text{diam}(P(V)) \leq \varepsilon$. Thus

$$d_{\xi(\tau_0)}(\xi(x), \xi(y)) \leq \sum_{i=1}^{k} d_{\xi(\tau_0)}(\xi(x_{i-1}), \xi(x_i)) \leq M \sum_{i=1}^{k} d_{\tau_0}(x_{i-1}, x_i)^\alpha$$

$$\leq Mk \left( \frac{d_{\tau_0}(x, y)}{k} \right)^\alpha = (Mk^{1-\alpha})d_{\tau_0}(x, y)^\alpha.$$ 

\[\Box\]

Let $\gamma = \gamma_{t_0}$ be the geodesic in $H^2$ starting at $t_0$ and orthogonal to the geodesic between $x_0$ and $y_0$. Let us now prove an elementary fact from hyperbolic geometry

**Lemma 3.20** Let $A > 1$ be any positive number. There exists a nested sequence of intervals

$$\Delta^*_t(x_0, y_0) = \Delta_0 \supset \Delta_1 \supset \ldots \supset \Delta_N$$

with $\Delta_i = \Delta^*_t(x_i, y_i)$ and such that

(i) $x$ and $y$ belong to $\Delta_N$.

(ii) $x_{i+1}$ and $y_{i+1}$ belong to the set of those $u$ so that $\frac{1}{A_0} \leq [t_0, x_i, u, y_i] \leq A$.

(iii) $N > -C \log(d_{\tau_0}(x, y))$ for some constant $C$ only depending on $A$.

**Proof** Denote by $z_0$ the intersection of $\gamma$ with the geodesic through $x_0$ and $y_0$, and by $p \in \gamma$ is the barycenter of $(t_0, x, y)$. As remarked in [27, Equation (43)], there is a universal constant $C_0 > 0$ such that

$$d_{H^2}(p, z_0) \geq -C_1 \log \left( d_{\tau_0}(x, y) \right).$$

There exists $\delta > 0$, only depending on $A$, such that if $(a, b)$ and $(u, v)$ are the end points of two geodesics at a distance $\delta$ orthogonal to $\gamma$, with $(t_0, a, u, v, b)$ going counter-clockwise, then

$$[t_0, a, u, b] = \frac{1}{A} [t_0, a, v, b] = A.$$
Let $z_1, \ldots, z_N$ be the points on $\gamma$ such that $d_{H^2}(z_i, z_{i+1}) = \delta$ for all $i = 0, \ldots, N - 1$ and $d_{H^2}(z_N, p) < \delta$. Hence

$$d_{H^2}(z_0, p) < N\delta.$$}

Thus $N > -C_0 \log (d_{t_0}(x, y))$, with $C_0$ only depending on $A$. The diamond $\Delta_i = \Delta_{t_0}^*(x_i, y_i)$, with $x_i$ and $y_i$ the intersections of $\partial_{\infty} H^2$ with the geodesic orthogonal to $\gamma$ passing through $z_i$ satisfy the required properties (see Fig. 6).

**Proof of Theorem 3.17** We first consider a special case.

**Symmetric case:** We first assume that $x$ and $y$ are symmetric with respect to $\gamma = \gamma_{t_0}$ and contained in the diamond $\Delta_{t_0}^*(x_0, y_0)$—recall that in $P(V)$ the diamond $\Delta_{t_0}^*(x_0, y_0)$ is the interval with extremities $x_0$ and $y_0$ not containing $t_0$.

Recall that (see Remark 2.17) the visual distance $d_{(u, v, w)}$ is uniformly biLipschitz to the diamond distance $\delta_{(u, v, w)}$ on $\Delta_u^*(v, w)$: there exists $F > 1$, so that for $x, y$ in $\Delta_u^*(v, w)$,

$$F^{-1}d_{(u, v, w)}(x, y) \leq \delta_{(u, v, w)}(x, y) \leq Fd_{(u, v, w)}(x, y).$$

Fix $\Delta_i = \Delta_{t_0}^*(x_i, y_i)$ as in Lemma 3.20 for $A$ so that $\xi$ is $(A, B)$-quasisymmetric. Let $\kappa_i := (a_0, b_i, c_i) := \xi(t_0, x_i, y_i)$. Since $\xi$ is $(A, B)$-quasisymmetric, any pair of diamonds $\Delta_{a_0}^*(b_i, c_i)$ and $\Delta_{a_0}^*(b_{i+1}, c_{i+1})$ satisfies the conditions of Lemma 3.18, so we have $\delta_{\kappa_i} \leq \lambda \delta_{\kappa_{i+1}}$ on $\Delta_{a_0}^*(b_N, c_N)$.
with \( \lambda = \frac{B-1}{B+1} < 1 \). We get

\[
\delta_{k_0}(\xi(x), \xi(y)) \leq \lambda^N \delta_{k_N}(\xi(x), \xi(y)) \leq 2\lambda^{-C \log(d_{\tau_0}(x,y))} = 2d_{\tau_0}(x,y)^{-C \log(\lambda)}.
\]

where we used \( \delta_{k_N}(\xi(x), \xi(y)) \leq 2 \) since \( \xi(x) \) and \( \xi(y) \) both belongs to \( \Delta_{\omega_0}(b_N, c_N) \). Thus

\[
d_{k_0}(\xi(x), \xi(y)) \leq M_0 (d_{\tau_0}(x,y))^\alpha
\]

with \( M_0 = 2F^{-1} \) and \( \alpha = -C \log(\lambda) \).

We have thus proved that if \( x \) and \( y \) are both elements of the interval \( \Delta^{*}_{\tau_0}(x_0, y_0) \) and symmetric with respect to \( \gamma \), then

\[
d_{\xi(\tau_0)}(\xi(x), \xi(y)) \leq M_1 d_{\tau_0}(x,y)^\alpha.
\]

**General case:** Let us move to the general case. Let us first observe that there exists some (universal) \( \varepsilon \) so that if \( d_{\tau_0}(x, y) \leq \varepsilon \), then using a rotation \( R \) by the barycenter of \( \tau_0 \), we can find \( \tau = R(\tau_0) \) so that the symmetry condition above is satisfied, for \( x, y \) and \( \tau \). It follows that

\[
d_{\xi(\tau)}(\xi(x), \xi(y)) \leq M_1 d_{\tau}(x,y)^\alpha.
\]

Now we notice that there exists a constant \( L \) only depending on \( H^2 \), so that \( d(\tau, \tau_0) \leq L \). By Condition (8) of corollary 3.16 \( d(\xi(\tau), \xi(\tau_0)) \leq M \), it follows, by the continuity of the dependence of the visual distance with the positive triple that there exists constants \( C \) and \( D \) only depending on \( A \) and \( B \), so that

\[
C^{-1}d_{\tau} \leq d_{\tau_0} \leq Cd_{\tau} \quad D^{-1}d_{\xi(\tau)} \leq d_{\xi(\tau_0)} \leq Dd_{\xi(\tau)}.
\]

Thus for \( \varepsilon \) as above and \( x, y, \tau_0 \) so that \( d_{\tau_0}(x, y) \leq \varepsilon \) we have

\[
d_{\xi(\tau_0)}(\xi(x), \xi(y)) \leq M_2 d_{\tau_0}(x,y)^\alpha.
\]

with \( M_2 = M_1 C^\alpha D \). We complete the proof of the theorem by using Lemma 3.19. \( \square \)

**3.5.3 The equivariant case**

In this paragraph, we explain the relation between quasisymmetric maps and the theory of maximal representations of surface groups into \( G \).

Let \( \Gamma \) be a cocompact Fuchsian subgroup of \( \text{PSL}(V) \) and \( \rho \) a morphism from \( \Gamma \) to \( G \). We first have
Proposition 3.21 Any positive continuous equivariant map is quasisymmetric.

Proof This is a direct consequence of the cocompactness of the action of $\Gamma$ on the set of positive quadruples in $P(V)$ whose cross-ratios are in $[A^{-1}, A]$ for $A > 1$.

Since $G$ is of Hermitian type we can define as in [9,11] maximal representations. From [10, Corollary 6.3], $\rho$ is maximal if and only there exists an equivariant positive continuous map $\xi : P(V) \to \partial_\infty H^{2,n}$. As a corollary

Proposition 3.22 Let $\rho$ be a representation from $\Gamma$ into $G$. There exists a $\rho$-equivariant quasisymmetric map from $P(V)$ into $\partial_\infty H^{2,n}$ if and only if $\rho$ is maximal.

In particular, the space of $G$-orbits of quasisymmetric maps from $P(V)$ to $\partial_\infty H^{2,n}$ contains the space $\text{Rep}^\text{max}(\Gamma, G)$ of maximal representations of any cocompact Fuchsian group $\Gamma$.

4 Maximal surfaces in pseudo-hyperbolic spaces

We describe the pseudo-hyperbolic space $H^{2,n}$ in Paragraph 4.1, maximal surfaces in Paragraph 4.2, the equations governing them in Paragraph 4.3 and finally the space of all maximal surfaces in Paragraph 4.4 with its compactness properties coming from [31].

4.1 The pseudo-hyperbolic space

Recall that $E$ is a $(n + 3)$-dimensional real vector space equipped with a signature $(2, n + 1)$ quadratic form $q$. The pseudo-hyperbolic space is defined by

$$H^{2,n} := \{ x \in E , \ q(x) = -1 \}/\{\pm \text{Id} \}.$$ 

We will in the sequel freely identify $H^{2,n}$ with its projection in $P(E)$. The space $H^{2,n}$ is equipped with a natural pseudo-Riemannian metric $g$ of signature $(2, n)$ and constant curvature $-1$. The group $G$ acts transitively on $H^{2,n}$ preserving this metric, turning $H^{2,n}$ into a pseudo-Riemannian symmetric space of $G$. The boundary of $H^{2,n}$ in $P(E)$ is the Einstein Universe $\partial_\infty H^{2,n}$.

4.1.1 Pointed hyperbolic plane

Given a 3-dimensional linear subspace $F$ of $E$ of signature $(2, 1)$, the projective plane $P(F)$ intersects $H^{2,n}$ in a totally geodesic subspace isometric to the hyperbolic disk, called a hyperbolic plane.
A pointed hyperbolic plane is then a pair $P = (q, H)$ where $H$ is a hyperbolic plane and $q$ a point in $H$. Equivalently, a pointed hyperbolic plane corresponds to an orthogonal decomposition $E = q \oplus U \oplus W$ where $U$ is a positive definite 2-plane such that $H = P (q \oplus U) \cap H^{2,n}$ and $q$ is a line.

In [31, Section 3.1] given any pointed hyperbolic plane $P = (q, H)$ we showed the existence of a warped projection $\pi_P$ from $H^{2,n}$ to $H$ with the following properties:

(i) The projection is natural: $\pi_{gP} (gx) = g\pi_P (x)$ for any element $g$ in $G$.
(ii) For any $z$ in $H$, $\pi_P (z) = z$.
(iii) The preimage of a point is diffeomorphic to a projective space of dimension $n$ and has type $(0, n)$.

The map sending a pointed hyperbolic plane $P = (q, H)$ to $T_q H$ gives an isomorphism between the space of pointed hyperbolic planes and the set $G(H^{2,n})$ of pairs $(x, P)$ where $x \in H^{2,n}$ and $P$ is a positive definite 2-plane in $T_x H^{2,n}$. Observe that $G(H^{2,n})$ is a $G$-homogeneous space and the stabiliser of a point in $G(H^{2,n})$ is compact. In particular, $G(H^{2,n})$ carries a natural Riemannian metric whose associated distance will be denoted by $d_G$ (see Paragraph 2.1.1). The map $(x, P(F)) \mapsto x^\perp \cap F$ defines a Riemannian fibration from $G(H^{2,n})$ to $Gr_{2,0} (E)$, the symmetric space of $G$.

Finally, note that if $\tau$ is a positive triple in $\partial_\infty H^{2,n}$, then there exists a unique pointed hyperbolic plane $P_\tau = (q_\tau, H_\tau)$ such that $H_\tau$ is the hyperbolic plane containing $\tau$ in its boundary and $q_\tau$ is the barycenter of the triple, seen as the ideal vertices of a triangle. The map $\tau \mapsto (q_\tau, H_\tau)$ defines a proper $G$-equivariant map from the set $T(n)$ of positive triples in $\partial_\infty H^{2,n}$ to $G(H^{2,n})$.

### 4.1.2 Spatial distance

Let $x$, $y$ be two distinct points in $H^{2,n}$. The complete geodesic between $x$ and $y$ is given by the intersection of $H^{2,n}$ and the projective line $P(x \oplus y)$. We call the pair $(x, y)$ acausal if the geodesic between them is spacelike, meaning that the induced metric is positive definite. This happens exactly when $q$ restricts to a signature $(1, 1)$ quadratic form on $x \oplus y$. The following was proved in [19][Proposition 3.2]

**Lemma 4.1** Let $x$, $y$ be two distinct points in $H^{2,n}$ and $x_0$, $y_0$ be vectors in $x$ and $y$ respectively satisfying $q(x_0) = q(y_0) = -1$. The pair $(x, y)$ is acausal if and only if $|\langle x_0, y_0 \rangle|$ is greater than 1. Moreover, in this case, the length of the geodesic between $x$ and $y$ is equal to $\cosh^{-1}(|\langle x_0, y_0 \rangle|)$.

Then the spatial distance is the map $\omega$ from $H^{2,n} \times H^{2,n}$ to $\mathbb{R}$ so that

$$\omega(x, y) = \cosh^{-1}(|\langle x_0, y_0 \rangle|)$$

(10)
if the pair \((x, y)\) is acausal and 0 otherwise. By Lemma 4.1, \(\delta\) restricts to the usual hyperbolic distance on any hyperbolic plane but we will see that \(\delta\) fails to satisfy the triangular inequality.

4.1.3 Horofunctions

Given a non-zero isotropic vector \(z_0\) in \(E\), we define the associated horofunction, on \(H^{2, n} \setminus P(z_0^{\perp})\) is given by

\[
    h_{z_0}(x) = \log |\langle x_0, z_0 \rangle| ,
\]

where \(x_0\) is a vector in \(x\) with \(q(x_0) = -1\). Rescaling \(z_0\) by a non-zero multiplicative constant changes \(h_{z_0}\) by an additive constant. In particular, the gradient of \(h_{z_0}\) only depends on the class of \(z_0\) in \(\partial_\infty H^{2, n}\). Observe that, given a point \(x \in H^{2, n} \setminus P(z_0^{\perp})\), the gradient of \(h_{z_0}\) is given by

\[
    (\nabla h_{z_0})_x = \frac{\pi(z_0)}{\langle x_0, z_0 \rangle} ,
\]

where \(\pi\) is the orthogonal projection on \(T_xH^{2, n}\) and \(x_0\) is such that \(q(x_0) = -1\) and \(\langle x_0, z_0 \rangle > 0\).

When \(n = 0\), the function \(h_{z_0}\) is the classical Busemann function of hyperbolic geometry.

4.2 Spacelike surfaces in \(H^{2, n}\)

Recall that a spacelike surface in \(H^{2, n}\) is an immersion of a connected 2-dimensional manifold \(\Sigma\) (without boundary) into \(H^{2, n}\) whose induced metric is positive definite. Such a surface is called complete if the induced metric is complete.

The following is proved in [31, Proposition 3.10]

**Proposition 4.2** Let \(\Sigma\) be a complete spacelike surface in \(H^{2, n}\). Then

(i) any pair of distinct points on \(\Sigma\) is acausal.

(ii) The boundary \(\partial_\infty \Sigma\) of \(\Sigma\) in \(P(E)\) is a semi-positive loop contained in \(\partial_\infty H^{2, n}\).

As a result, if \(\Sigma\) is a complete spacelike surface in \(H^{2, n}\), the restriction \(\partial_\Sigma\) of the spatial distance to \(\Sigma\) is non-zero on any pair of distinct points. Observe however that in general \(\partial_\Sigma\) fails to be a distance: the triangle inequality fails on triples \((x, y, z)\) such that \(x \oplus y \oplus z\) has signature \((1, 2)\).

Another important aspect of complete spacelike surfaces in \(H^{2, n}\) is the fact that they are entire graphs. More specifically, for any pointed hyperbolic plane
\( P = (q, H) \) and complete maximal surface \( \Sigma \), the warped projection \( \pi_P \) restricts to a diffeomorphism from \( \Sigma \) to \( H \). We refer to [31, Section 3.1] for more details. As a corollary

**Corollary 4.3** Let \( \Sigma \) be a complete spacelike surface and \( P = (q, H) \) be a pointed hyperbolic plane. There exists a unique point in \( \Sigma \) whose warped projection on \( H \) is equal to \( q \).

Observe that in particular, a complete spacelike surface is always properly embedded. In what follows, we will always assume that our spacelike surfaces are embedded.

### 4.2.1 Second fundamental forms and shape operators

Let \( \Sigma \) be a spacelike surface in \( H_2^n \). The normal bundle \( N \Sigma \) of \( \Sigma \) is the orthogonal to \( T \Sigma \). The Levi–Civita connection \( \nabla \) of \( H_2^n \) decomposes in the splitting \( TH_2^n = T \Sigma \oplus N \Sigma \) as

\[
\nabla = \begin{pmatrix}
\nabla^T & B \\
II & \nabla^N
\end{pmatrix}.
\]

(12)

The forms \( II \), seen as an element of \( \Omega^1(\Sigma, \text{Hom}(T \Sigma, N \Sigma)) \), and \( B \), seen as an element of \( \Omega^1(\Sigma, \text{Hom}(N \Sigma, T \Sigma)) \), are called the second fundamental form and the shape operator respectively. The second fundamental form is symmetric, that is satisfies \( II(x, y) = II(y, x) \) for any vectors \( x, y \) of \( T \Sigma \). In particular, \( II \) can be considered as a section of \( \text{Sym}^2(T^* \Sigma) \otimes N \Sigma \).

The connections \( \nabla^T \) and \( \nabla^N \) are unitary with respect to the induced metrics on \( T \Sigma \) and \( N \Sigma \) respectively. Finally, if \( x, y \) are sections of \( T \Sigma \) and \( \xi \) is a section of \( N \Sigma \), then derivating \( \langle y, \xi \rangle = 0 \) in the direction \( x \) gives the classical relation

\[
\langle II(x, y), \xi \rangle = -\langle y, B(x, \xi) \rangle.
\]

(13)

### 4.3 Maximal surfaces and their fundamental equations

Below is some standard material that we recall for completeness and to emphasise our sign conventions.

**Definition 4.4** A spacelike surface \( \Sigma \) in \( H_2^n \) is called maximal if \( \text{tr}_{g_1} II = 0 \) where \( g_1 \) is the induced metric on \( \Sigma \).

Maximal surfaces are critical points for the area functional (see [31, Section 3.3] for more details).
Let $\Sigma$ be a complete maximal surface in $H^{2,n}$ and fix an orthonormal framing $(e_1, e_2)$ of $T\Sigma$. Define the norm of the second fundamental form as
\[
\|\Pi\|^2 = -\sum_{i=1}^{2} \langle \Pi(e_i, e_j), \Pi(e_i, e_j) \rangle,
\]
and observe that since the restriction of $g$ is negative definite on $N\Sigma$, we have $\|\Pi\|^2 \geq 0$.

**Proposition 4.5** (Gauß equation) Using the above notations, if $\Sigma$ is a complete maximal surface in $H^{2,n}$ and $K_\Sigma$ is its intrinsic curvature, then
\[
K_\Sigma = -1 + \frac{1}{2} \|\Pi\|^2. \tag{14}
\]

**Proof** Using the decomposition of the Levi–Civita connection $\nabla$ of $H^{2,n}$ given in equation (12), we have
\[
\langle \nabla_x \nabla_y z, t \rangle = \langle \nabla^T_x \nabla^T_y z, t \rangle + \langle B(x, \Pi(y, z)), t \rangle,
\]
where $x, y, z$ and $t$ are sections of $T\Sigma$. Using the convention
\[
R(a, b)c = \nabla_a \nabla_b c - \nabla_b \nabla_a c - [a, b]c
\]
and equation (13), we get
\[
\langle R(x, y)z, t \rangle = \langle R^T(x, y)z, t \rangle - \langle \Pi(y, z), \Pi(x, t) \rangle + \langle \Pi(x, z), \Pi(y, t) \rangle,
\]
where $R^T$ is the curvature tensor of $\nabla^T$. The result then follows from applying the equation to $x = t = e_1$ and $y = z = e_2$, using $\Pi(e_1, e_1) = -\Pi(e_2, e_2)$ and the fact that $\nabla$ has sectional curvature $-1$. 

Denote by $\omega$ the area form of $\Sigma$ (with its induced metric). Observe first that there exists a unique endomorphism $\varphi$ of $N\Sigma$ so that for any $U$ and $V$ in $N\Sigma$, $u$ and $v$ in $T\Sigma$,
\[
\langle B(v, U), B(u, V) \rangle - \langle B(u, U), B(v, V) \rangle = \omega(u, v) \langle \varphi(U), V \rangle. \tag{15}
\]

**Proposition 4.6** (Ricci equation) Using the above notations, the curvature tensor $R_N$ of $\nabla^N$ is given by
\[
R_N = \omega \otimes \varphi, \tag{16}
\]
Moreover if $(e_1, e_2)$ is an orthonormal basis of $T\Sigma$ and $\alpha$ and $\beta$ are the sections of $N\Sigma$ defined by

$$\alpha := \Pi(e_1, e_1) \quad \beta := \Pi(e_1, e_2)$$

then

$$\langle \varphi(\alpha), \beta \rangle = -2\|\alpha \wedge \beta\|^2 := -2 \left( \|\alpha\|^2 \|\beta\|^2 - \langle \alpha, \beta \rangle^2 \right). \quad (17)$$

**Proof** Given $u$ and $v$ sections of $T\Sigma$, Eq. (12) gives

$$\langle \nabla_u \nabla_v U, V \rangle = \langle \nabla_u^N \nabla_v^N U, V \rangle + \langle \Pi(u, B(v, U)), V \rangle = \langle \nabla_u^N \nabla_v^N U, V \rangle - \langle B(v, U), B(u, V) \rangle.$$

This gives Ricci equation:

$$0 = \langle R(u, v)U, V \rangle = \langle R_N(u, v)U, V \rangle - \langle B(v, U), B(u, V) \rangle + \langle B(u, U), B(v, V) \rangle = \langle R_N(u, v)U, V \rangle - \omega(u, v)\varphi(U, V).$$

The second statement follows from using the relation

$$B(e_1, \alpha) = \sum_{i=1}^2 \langle B(e_1, \alpha), e_i \rangle \cdot e_i$$

$$= -\sum_{i=1}^2 \langle \alpha, \Pi(e_1, e_i) \rangle \cdot e_i = -\langle \alpha, \alpha \rangle \cdot e_1 - \langle \alpha, \beta \rangle \cdot e_2$$

and similar formulas for the other terms. \qed

For the last equation, we denote by $D$ the connection on $\Hom(T\Sigma, N\Sigma)$ induced by $\nabla^T$ and $\nabla^N$. The $D$-exterior derivative is the map

$$d^D \Omega^1(\Sigma, \Hom(T\Sigma, N\Sigma)) \rightarrow \Omega^2(\Sigma, \Hom(T\Sigma, N\Sigma))$$

defined by

$$d^D \theta(x, y) := D_x(\theta(y)) - D_y(\theta(x)) - \theta([x, y]).$$

**Proposition 4.7** (Codazzi equation) Using the same notations as above, seeing $\Pi$ as an element of $\Omega^1(\Sigma, \Hom(T\Sigma, N\Sigma))$, we have $d^D\Pi = 0$. 

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Proof. This is obtained using that \( \langle R(x, y)z, \xi \rangle = 0 \) for any sections \( x, y, z \) of \( T \Sigma \) and \( \xi \) of \( N \Sigma \), together with the formula

\[
\langle \nabla_x \nabla_y z, \xi \rangle = \left( \Pi(x, \nabla_y^T z), \xi \right) + \left( \nabla_x^N (\Pi(y, z)), \xi \right).
\]

\( \square \)

### 4.4 The space of all maximal surfaces

We denote by \( \mathcal{M}(n) \) the set of all pointed complete maximal surfaces in \( H^{2,n} \)

\[
\mathcal{M}(n) := \{ (x, \Sigma) \mid x \in \Sigma, \; \Sigma \text{ is maximal complete} \}.
\]  

(18)

Convergence of pointed spacelike surfaces on every compact induces a natural topology on those spaces (see [31, Appendix A] for more details).

#### 4.4.1 Cocompact actions

Let \( G(H^{2,n}) \) be the Riemannian manifold defined in Paragraph 4.1. As a consequence of results in [31], we have

**Theorem 4.8** (Compactness Theorem) The map from \( \mathcal{M}(n) \) to \( G(H^{2,n}) \) sending \( (x, \Sigma) \) to \( T_x \Sigma \) is proper. The group \( G \) acts properly, continuously and cocompactly on \( \mathcal{M}(n) \).

The first assertion is a rewriting of the first item in [31, Theorem 6.1] while the second assertion is an immediate application of the first using the fact that \( G \) acts properly and transitively on \( G(H^{2,n}) \).

Finally, we remark the cocompactness of the action of \( G \) on some spaces related to \( \mathcal{M}(n) \), again as a consequence of [31, Theorem 6.1].

**Corollary 4.9** The group \( G \) acts cocompactly and continuously on the spaces \( \mathcal{N}_p(n) \) given by

\[
\mathcal{N}_p(n) := \{ (x, \Sigma, z_1, \ldots, z_p), \; \text{with} \; x \in \Sigma \; \text{and} \; z_i \in \Sigma \cup \partial_\infty \Sigma \}.
\]

**Proof** Let consider the compactification \( \overline{H}^{2,n} := H^{2,n} \cup \partial_\infty H^{2,n} \) of \( H^{2,n} \) in the projective model. Observe first that the diagonal action of \( G \) on \( \mathcal{M}(n) \times \left( \overline{H}^{2,n} \right)^p \) is cocompact, since the action of \( G \) on \( \mathcal{M}(n) \) is cocompact. Thus the corollary follows from the third item in [31, Theorem 6.1] which guarantees that \( \mathcal{N}_p(n) \) is a closed set in \( \mathcal{M}(n) \times \left( \overline{H}^{2,n} \right)^p \). \( \square \)
4.4.2 Laminated space

We end this section by describing a natural product structure on $\mathcal{M}(n)$ that turns it into a laminated space, or a Riemann surface lamination—see appendix in [35]. This structure will be comes from the graph property of complete maximal surfaces (see Sect. 4.2).

Fix a pointed hyperbolic plane $P = (q, H)$ and recall that any complete maximal surface is a graph above $H$ with respect to the warped projection $\pi$ from $H^{2,n}$ to $H$. Let

$$\mathcal{M}_0 = \{ (y, \Sigma) \in \mathcal{M}(n) \mid \pi(y) = q \}.$$ 

Then the map from $\mathcal{M}(n)$ to $\mathcal{M}_0 \times H$ so that the image of $(x, \Sigma)$ is $((y, \Sigma), \pi(x))$—where $y$ is the unique point of $\Sigma$ projecting to $q$—is a homeomorphism. Moreover, leaves in $\mathcal{M}(n)$ correspond to sets of the form $\{\ast\} \times H$, in other words maximal surfaces.

We may thus consider $\mathcal{M}(n)$ as a laminated space whose leaves identify with maximal surfaces. Observe also that the natural $G$ action on $\mathcal{M}(n)$ preserves the leaves. In the sequel, we will freely identify maximal complete surface $\Sigma$ with the leaf it corresponds to in $\mathcal{M}(n)$.

4.5 Loops and surfaces

The next theorem is the main result of [31, Theorem B]:

**Theorem 4.10** (Asymptotic Plateau problem) Any semi-positive loop $\Lambda$ in $\partial_\infty H^{2,n}$ bounds a unique complete maximal surface $\Sigma(\Lambda)$ in $H^{2,n}$.

We will now extend this correspondence to a map from the space $\mathcal{L}(n)$ of pointed semi-positive loops to $\mathcal{M}(n)$. Recall from Sect. 4.1 that any positive triple $\tau$ defines a pointed hyperbolic plane $P_\tau = (q_\tau, H_\tau)$. Moreover, by Corollary 4.3, if $\Sigma$ is a complete maximal surface, there is a unique point $b_\tau$ on $\Sigma$ that projects to $q_\tau$. This defines

**Definition 4.11** (Barycenter map) Using the above notations, the barycenter map from $\mathcal{L}(n)$ to $\mathcal{M}(n)$ is the map $B$ which associates to $\tau$ the pointed maximal surface $(b_\tau, \Sigma(\Lambda))$.

We now prove

**Proposition 4.12** (Barycenter map) The barycenter map is $G$-equivariant, continuous, proper and surjective.

First step: proof of $G$-equivariance, continuity and properness The $G$-equivariance is obvious and the continuity is a direct consequence of the second
item in Theorem [31, Theorem 6.1]. It follows that the induced map $B_0$ from $L(n)/G$ to $\mathcal{M}(n)/G$ is continuous. The fact that the map $B$ is proper is now a consequence of some general topology argument, using the following facts

(i) The group $G$ acts properly on $L(n)$ and $\mathcal{M}(n)$.
(ii) The map $B_0$ from $L(n)/G$ to $\mathcal{M}(n)/G$ is continuous,
(iii) The spaces $L(n)/G$ to $\mathcal{M}(n)/G$ are compact,
(iv) The projection from $\pi_0$ from $L(n)_{\tau_0}$ to $L(n)/G$ is proper (see Proposition 2.32).

Let us show then that $B$ is proper: let $K$ be a compact in $\mathcal{M}(n)$, let $K_1$ be the projection of $K$ in $\mathcal{M}(n)/G$, let $K_2$ its preimage in $L(n)/G$ and $K_3 = B(\pi_0^{-1}(K_2))$. We have that $K_1$ is compact by continuity, $K_2$ is compact since it is a closed set (by continuity of $B_0$) in the compact space $L(n)/G$, and finally $K_3$ is compact by properness and continuity.

Then by properness of the action of $G$

$$K_4 := \{ g \in G \mid gK_3 \cap K \neq \emptyset \}$$

is compact. Then one sees that $B^{-1}(K)$ is a closed subset of $K_4 \cdot K$, and is therefore compact. \hfill \Box

Second step: proof of surjectivity Assume first that $\Lambda$ is positive. Then the set $T_\Lambda$ of positive triples contained in $\Lambda$ identifies with the set of pairwise distinct triples of points in $\Lambda$. The boundary $\partial T_\Lambda$ of $T_\Lambda$ in $\Lambda^3$ is made of triples such that at least two of the elements of the triple are equal. We have a projection $\pi$ from $\partial T_\Lambda$ to $\Lambda$ given by

$$\pi(t, s, s) = \pi(s, t, s) = \pi(s, s, t) = s.$$ 

Let us now prove

**Lemma 4.13** Let $\Lambda$ be a positive loop and $\{\tau_k\}_{k \in \mathbb{N}}$ a sequence in $T_\Lambda$ converging to $\tau_\infty$ in $\partial T_\Lambda$. Then the sequence $\{q_{\tau_k}\}_{k \in \mathbb{N}}$ of barycenters in $H^{2,n}$ converges to $\pi(\tau_\infty)$ in $\partial_\infty H^{2,n}$.

**Proof** Let $p$ be a vector in $E$ such that $P(p^\perp)$ is disjoint from $\Lambda$. The existence of $p$ is guaranteed by [31, Proposition 2.15 (iii)]. Let $F := p^\perp$. Then in the splitting

$$E = F \oplus \mathbb{R}p$$

any point in $\Lambda$ has the form $[(x, 1)]$ for a unique $x$ in $F$. Let $\tau = (t^1, t^2, t^3)$ be a positive triple in $\Lambda$. We write $t^i = [u^i] \in P(E)$ where $u^i = (s^i, 1) \in E$
with \( s^i \) in \( F \). Observe that \( \langle u^i, u^i \rangle = 0 \) while \( \langle u^i, u^{i+1} \rangle > 0 \). Let

\[
\lambda^i = \sqrt{\frac{\langle u^j, u^k \rangle}{\langle u^i, u^j \rangle \langle u^i, u^k \rangle}} ,
\]

where \( (i, j, k) \) are pairwise distinct.

We claim that the barycenter \( q_\tau \) of \( \tau \) is the line spanned by

\[
v := \lambda^1 u^1 + \lambda^2 u^2 + \lambda^3 u^3 .
\]

In fact, the \( \lambda^i \) are defined in such a way that \( \langle \lambda^i u^i, \lambda^j u^j \rangle = 1 \) for any \( i \neq j \).

So for \( (i, j, k) \) pairwise distinct, the line \( \delta \) between \( t^i \) and \( [v] \) intersects the geodesic \( \gamma \) between \( t^j \) and \( t^k \) along \( x = [u^j + u^k] \), and \( T_x \gamma = [u^i - u^k] \) is orthogonal to \( T_x \delta = [-t^i + t^j + t^k] \). So \([v] = q_\tau \).

Observe that for \( (i, j, k) \) pairwise distinct, we have

\[
\frac{\lambda^i}{\lambda^j} = \frac{\langle u^j, u^k \rangle}{\langle u^i, u^k \rangle} .
\]

Assume now that we have a sequence \( \{\tau_k\}_{k \in \mathbb{N}} \), where \( \tau_k = (t^1_k, t^2_k, t^3_k) \) where

\[
\lim_{k \to \infty} t^1_k = \lim_{k \to \infty} t^2_k = t_\infty = [(s_\infty, 1)] \neq w_\infty = \lim_{k \to \infty} t^3_k .
\]

Then

\[
\lim_{k \to \infty} \langle u^1_k, u^2_k \rangle = 0  \quad \lim_{k \to \infty} \langle u^2_k, u^3_k \rangle = a \neq 0  \quad \lim_{k \to \infty} \langle u^1_k, u^3_k \rangle = b \neq 0 .
\]

Thus

\[
\lim_{k \to \infty} \lambda^2_k = 0 , \quad \lim_{k \to \infty} \lambda^2_k = \lim_{k \to \infty} \lambda^1_k = \infty \quad \text{and} \quad \lim_{k \to \infty} \frac{\lambda^2_k}{\lambda^1_k} = \frac{a}{b} \neq 0 .
\]

Then the barycenter

\[
q_{\tau_k} = \left[ \left( \frac{\lambda^1_k s^1_k + \lambda^2_k s^2_k + \lambda^3_k s^3_k}{\lambda^1_k + \lambda^2_k + \lambda^3_k} , 1 \right) \right]
\]

converges to \( t_\infty = [(s_\infty, 1)] \).

The case when \( \{t^1_k\}_{k \in \mathbb{N}} , \{t^2_k\}_{k \in \mathbb{N}} \) and \( \{t^3_k\}_{k \in \mathbb{N}} \) converge to the same point \( t_\infty \) is simpler: in this case, the sequence of ideal hyperbolic triangles with vertices \( t^1_k , t^2_k \) and \( t^3_k \) converges to the point \( t_\infty \). Since any barycenter \( b_{\tau_k} \) is contained in the ideal triangle, the result follows.
We obtain the following corollary:

**Corollary 4.14** Assume \( \Lambda \) is positive. If \( \{ \tau_k \}_{k \in \mathbb{N}} \) is a sequence in \( T_{\Lambda} \) converging to \( \tau_\infty \) in \( \partial T_{\Lambda} \), then the sequence \( \{ b_{\tau_k} \}_{k \in \mathbb{N}} \) in \( \Sigma(\Lambda) \) converges to \( \pi(\tau_\infty) \).

**Proof** Since we already know that the barycenter map is proper, the sequence \( \{ b_{\tau_k} \}_{k \in \mathbb{N}} \) leaves every compact set. Let us extract a subsequence so that \( \{ b_{\tau_k} \}_{k \in \mathbb{N}} \) accumulates at a point \( y \) in \( \Lambda \). Assume that \( y \) and \( \pi(\tau_\infty) \) are distinct.

Recall that the geodesic joining \( q_{\tau_k} \) and \( b_{\tau_k} \) is timelike. Going to the limit, we obtain a timelike or lightlike geodesic joining \( y \) to \( \pi(\tau_\infty) \). However, \( \Lambda \) is positive, and thus any two distinct points are joined by a spacelike geodesic. This implies that \( y = \pi(\tau_\infty) \).

We now prove the surjectivity of \( B \) from \( T_{\Lambda} \) to \( \Sigma(\Lambda) \) when \( \Lambda \) is positive: as a consequence of the previous corollary the barycenter map extends continuously to a map from \( \Lambda^3 \) to \( \Sigma(\Lambda) \cup \Lambda \). Fixing \( x_0 \) in \( \Lambda \), let

\[
D := \{(x_0, y, z) \mid (x_0, y, z) \text{ is positively oriented}\}.
\]

The set \( D \) is an open disk: indeed let us identify \( P(V) \setminus \{x_0\} \) with \( \mathbb{R} \), then

\[
D = \{x_0\} \times U \quad U := \{(x, y) \in \mathbb{R}^2 \mid x > y\}.
\]

Moreover the injection of \( D \) in \( T_{\Lambda} \) extends to a continuous map \( f \) from \( \partial D \) to \( \partial T_{\Lambda} \) sending a closed interval \( I \) to \( x_0 \), and is an homeomorphism from the complementary of \( I \) to the complementary of \( x_0 \). It follows that the barycenter map restricted to \( \partial D \) has degree 1 as a map with values in \( \Lambda \). Thus the barycenter map from \( D \) with values in \( \Sigma(\Lambda) \) is surjective.

To prove in general that \( B \) is surjective, we first observe that its image is closed since \( B \) is continuous and proper. Secondly, the image is dense: by our first discussion, it contains all pairs \((x, \Sigma)\) so that the boundary of \( \Sigma \) is positive.

\[
\mathbb{4.6} \quad \text{Barbot surfaces}
\]

A *Barbot surface* is a maximal surface whose boundary at infinity is a Barbot crown. Some of the results explained here can also be found in [8] (where Barbot surfaces are called horospheres). Let us start with an explicit description of a Barbot surface. Let \( z_0, z_1, z_2, z_3 \) be a quadruple of vectors so that

\[
\langle z_i, z_i \rangle = \langle z_i, z_{i+1} \rangle = 0 \quad \langle z_i, z_{i+2} \rangle = -\frac{1}{4}.
\]

\[
(19)
\]
In particular if $v_i$ are the lines generated by $z_i$, then $(v_0, v_1, v_2, v_3)$ are the vertices of a Barbot crown $C$. Let also $A_C$ be the abelian subgroup defined by equation (2). Then we have

**Proposition 4.15** The Barbot surface for a barbot crown $C$ is the orbit $V := A_C \cdot x$, where $x = z_0 + z_1 + z_2 + z_3$.

**Proof** Observe first that $\langle x, x \rangle = -1$. Let us now consider the following curves which are drawn on $V$.

$$
\alpha(t) := e^t z_0 + z_1 + e^{-t} z_2 + z_3 \quad \beta(t) := z_0 + e^t z_1 + z_2 + e^{-t} z_3.
$$

Then

$$
\begin{align*}
\alpha(0) &= x & \dot{\alpha}(0) &= z_0 - z_2 & \ddot{\alpha}(0) &= z_0 + z_2 \\
\beta(0) &= x & \dot{\beta}(0) &= z_1 - z_3 & \ddot{\beta}(0) &= z_1 + z_3 \\
\langle \dot{\alpha}(0), \dot{\beta}(0) \rangle &= 0 & \| \ddot{\alpha}(0) \|^2 &= \| \ddot{\beta}(0) \|^2 &= \frac{1}{2}.
\end{align*}
$$

Denote by $H$ the mean curvature vector. Then

$$
\frac{1}{2} H = \Pi(\dddot{\alpha}(0), \dddot{\alpha}(0)) + \Pi(\dddot{\beta}(0), \dddot{\beta}(0))
$$

is the normal part of $\dddot{\alpha}(0) + \dddot{\beta}(0)$. It follows that $H = 0$, hence by homogeneity that $V$ is a maximal surface. It is then easy to show that the boundary of $V$ in the projective model is the Barbot crown $C$.

The following summarises some properties of Barbot surfaces.

**Proposition 4.16** We have

(i) A Barbot surface is an orbit of a Cartan subgroup in $G$.
(ii) All Barbot surfaces are equivalent under the action of $G$.
(iii) The induced metric on a Barbot surface is flat.
(iv) At any point of a Barbot surface, there exists a horofunction whose square of the norm of the gradient is 2.
(v) Let $\Sigma$ be a Barbot surface, $d_\Sigma$ the induced distance on $\Sigma$ and $\bar{d}$ the spatial distance defined in Eq. (10), then

$$
\sup_{x, y \in \Sigma} \left( \frac{\bar{d}(x, y)}{d_\Sigma(x, y)} \right) = \sqrt{2}.
$$

As a consequence of the first two items, Barbot surfaces define a unique point in $\mathcal{M}(n)/G$ that we call the Barbot point and denote $m_0$. 

---

**Note:** The Springer logo is present in the bottom right corner of the page.
Proof The first item follows by Proposition 4.15. The second follows from the fact that Cartan subgroups are conjugate. The third one from the fact that it is a transitive orbit of $\mathbb{R}^2$ acting by isometries.

For the fourth item, let us consider the horofunction $h$ associated to the lightlike vector $Z:=z_0$. Then

$$h(\alpha(t)) = -t + \log\left(\frac{1}{4}\right) \quad h(\beta(t)) = \log\left(\frac{1}{4}\right).$$

Thus $\langle \nabla h, \dot{\alpha}(0) \rangle = -1$ and $\langle \nabla h, \dot{\beta}(0) \rangle = 0$. It follows that $\nabla h = -2\dot{\alpha}(0)$ and

$$\|\nabla h\|^2 = 2.$$

For the last item, by the transitive action of $\mathbb{R}^2$, it suffices to prove the result for a fixed $x$. Let $\lambda, \mu$ be two constants and define

$$\gamma(t) = e^{\lambda t}z_0 + e^{\mu t}z_1 + e^{-\lambda t}z_2 + e^{-\mu t}z_3.$$

Observe that $\gamma(t)$ is a geodesic. Let us assume, without loss of generality that $\lambda = \max\{\pm\lambda, \pm\mu\}$. Since $q(\dot{\gamma}(t)) = \frac{1}{2}(\lambda^2 + \mu^2)$ we have

$$d_\Sigma(\gamma(0), \gamma(t)) = t\sqrt{\frac{\lambda^2 + \mu^2}{2}}.$$

Moreover we have

$$|\langle \gamma(0), \gamma(t) \rangle| = \frac{1}{2}(\cosh(\lambda t) + \cosh(\mu t)).$$

It follows that the function

$$\frac{\dot{d}(\gamma(0), \gamma(t))}{d_\Sigma(\gamma(0), \gamma(t))} = \frac{\cosh^{-1}|\langle \gamma(0), \gamma(t) \rangle|}{d_\Sigma(\gamma(0), \gamma(t))}$$

is increasing and its limit as $t$ goes to infinity is equal to $\sqrt{\frac{2}{1 + \frac{\mu^2}{\lambda^2}}}$. The result follows.

The following characterisation of Barbot surfaces from [8, Lemma 5.5] will be used

Lemma 4.17 (Bonsante–Schlenker) If $\Sigma$ is a complete flat maximal surface in $\mathbb{H}^{2,1}$, then $\Sigma$ is a Barbot surface.
4.7 Periodic and quasiperiodic surfaces

We now follow a path similar to the one we followed in Sect. 2.7 for loops in the Einstein Universe.

Definition 4.18 Let \( \Sigma \) be a complete maximal surface. We say

(i) The surface \( \Sigma \) is periodic if it is invariant by a discrete subgroup of \( G \) isomorphic to the fundamental group of a closed surface of genus at least 2.

(ii) The surface \( \Sigma \) is quasiperiodic if the closure of its \( G \)-orbit in \( \mathcal{M}(n) \) does not contain a Barbot surface.

We have the following

Proposition 4.19 Periodic surfaces are quasiperiodic.

Proof Observe that a maximal surface is quasiperiodic if and only if its image is precompact in \( (\mathcal{M}(n)/G)\setminus\{m_0\} \) where \( m_0 \) is the Barbot point. By definition, a periodic surface corresponds to a compact leaf in \( \mathcal{M}(n)/G \) which is different from \( b \) (since the only groups acting freely and cocompactly on a Barbot surface are abelian). The result follows.

Remark 4.20 It was proved in [12] that periodic surfaces are in one-to-one correspondence with maximal representations of (closed) surface groups in \( G \).

As a corollary of Proposition 4.12

Corollary 4.21 A complete maximal surface is quasiperiodic if and only if its boundary at infinity is quasiperiodic.

5 Rigidity Theorems

In this section we show that Barbot surfaces play a special role in the theory. Cheng [13] proved that

Theorem 5.1 (Cheng) Let \( \Sigma \) be a complete maximal surface in \( \mathbb{H}^{2,n} \). Then the intrinsic curvature of \( \Sigma \) is non positive. If \( \Sigma \) is flat then \( \Sigma \) is a Barbot surface.

As a consequence of this theorem and Gauß equation, we get an \( a \) priori bound on the norm of the second fundamental form.

Corollary 5.2 Let \( \Sigma \) be a complete maximal surface in \( \mathbb{H}^{2,n} \). Then the norm of the second fundamental form \( \Sigma \) is no greater than 2.
The corollary is an improvement over Ishihara bounds [26] in which the norm of the second fundamental form by $2n$ in $H^{2,n}$. Note that Ishihara’s bounds are more general and optimal for maximal complete spacelike submanifolds of dimension $p$ in $H^{p,q}$ for $q < p$, where he obtains the bounds $pq$.

We improve the statement of Cheng to a pointwise rigidity result—which is not stated as such in [13] but should also follow by his techniques;

**Proposition 5.3** (Curvature rigidity) Let $\Sigma$ be a complete maximal surface in $H^{2,n}$. Then if the intrinsic curvature $\Sigma$ is zero at a point, it vanishes everywhere and $\Sigma$ is a Barbot surface.

Our proof relies on our compactness results rather than on Omori maximum principle as well as a holomorphic interpretation/Bochner formula which makes the computations in [13] more transparent. We have a similar corollary

**Corollary 5.4** (Rigidity for the second fundamental form) Let $\Sigma$ be a complete maximal surface in $H^{2,n}$. Then if this norm is 2 at a point, it is 2 everywhere and $\Sigma$ is a Barbot surface.

We have a similar rigidity result in term of the norms of gradient of horofunctions and spatial distance. Observe that although the gradient of the spatial distance to a point $z$ in $\Sigma$ is not defined at $z$, we can extend continuously its norm by imposing the value 1 at $z$.

**Theorem 5.5** (Horofunction and spatial distance rigidity theorem) Let $\Sigma$ be a complete maximal surface in $H^{2,n}$ with intrinsic curvature $K$. Let $c = \inf (-K_\Sigma)$ and let $z$ be a point in $\Sigma \cup \partial \Sigma$. Let $h_z$ be

- The horofunction associated to $z$, if $z$ belongs to $\partial \Sigma$;
- The function spatial distance to $z$, if $z$ belongs to $\Sigma$.

Then

(i) The square of the norm of the gradient of $h_z$ is not less than 1 and no greater than $2 - c$.

(ii) If for some $z$, the square of the norm of the gradient of $h_z$ is equal to 2 at a point, then $\Sigma$ is a Barbot surface.

Recall that by the Cheng’s Theorem, $c$ is non negative.

**5.1 Rigidity for the curvature**

We first translate the problem into holomorphic terms. Then in the next subsection, we use Bochner’s formula of Proposition A.1 to prove Proposition 5.3. Finally, in the last subsection, we prove Theorem 5.5.
5.1.1 A holomorphic picture

Let $\Sigma$ be a complete maximal surface in $H^{2,n}$ with induced metric $g_T$, normal bundle $(N\Sigma, g_N)$ and second fundamental form $\Pi$.

Denote by $K$ the canonical bundle over $\Sigma$, that is, the holomorphic line bundle $\text{Hom}_\mathbb{C}(T^{1,0}\Sigma, \mathbb{C})$. Define the complex vector bundle

$$E := K^2 \otimes N^C \Sigma,$$

where $N^C \Sigma = N \Sigma \otimes \mathbb{C}$.

The metrics $g_T$ on $T\Sigma$ and $-g_N$ on $N\Sigma$ induce a positive definite metric $g$ on the bundle $\text{Sym}^2(T^*\Sigma) \otimes N\Sigma$. The Hermitian extension of $g$ restricts to a Hermitian metric $h$ on $E$.

Similarly, the connections $\nabla^T$ and $\nabla^N$ on $T\Sigma$ and $N\Sigma$ (see equation (12)) define a unitary connection $\nabla^E$ on $(E, h)$ whose $(0, 1)$-part is a Dolbeaut operator $\bar{\partial}_E$ (see Appendix 1 for definitions). In this setting, we have the following:

**Lemma 5.6** Using the same notations as above, there is a holomorphic section $\sigma$ of $E$ such that the second fundamental form of $\Sigma$ is the real part of $\sigma$. Moreover, the intrinsic curvature $K_\Sigma$ of $\Sigma$ satisfies

$$K_\Sigma = -1 + \|\sigma\|^2_h.$$

**Proof** The complexification $\phi_C$ of a section $\phi$ of $\text{Sym}^2(T^*\Sigma) \otimes N\Sigma$ decomposes under types as

$$\phi_C = \phi^{2,0} + \phi^{1,1} + \overline{\phi^{2,0}}.$$

Observe that $\phi^{2,0}$ is a section of $E$.

Let $(e_1, e_2)$ be a local orthonormal framing of $T\Sigma$. Let $\partial_z = \frac{1}{2}(e_1 - ie_2)$ the associated section of $K^*$ and $dz$ the section of $K$ dual to $\partial_z$. We have

$$\phi^{1,1} = \phi(\partial_z, \overline{\partial_z}) \, dz \, d\bar{z} = \frac{1}{2} \text{tr}_{g_T}(\phi) dz \, d\bar{z}.$$

Applying the previous decomposition to $\Pi$, one sees that $\Sigma$ is maximal if and only if $\Pi^{1,1} = 0$ or in other words denoting $J$ the complex structure on $\Sigma$

$$\Pi(X, Y) + \Pi(JX, JY) = 0.$$

In particular, $\Pi$ is the real part of the section $\sigma := 2\Pi^{2,0}$ of $E$ or in other words for $X$ and $Y$ in $T\Sigma$,

$$\sigma(X, Y) = \Pi(X, Y) - i\Pi(JX, Y).$$
The Codazzi equation (Proposition 4.7) says that
\[ d^D \Pi = 0. \]
This last equation is equivalent to
\[ \overline{\partial}_E \sigma = 0 \]
the holomorphicity of \( \sigma \). Indeed
\[
\left( \overline{\partial}_E \right)_X \sigma(X, Y) = \nabla^E_J X \sigma(X, Y) - i \nabla^E_X \sigma(X, Y) \\
= \nabla_J X \Pi(X, Y) - i \nabla_X \Pi(J X, Y) \\
- i \left( \nabla_X \Pi(X, Y) - i \nabla_X \Pi(J X, Y) \right) \\
= \left( \nabla_J X \Pi(X, Y) - \nabla_X \Pi(J X, Y) \right) \\
- i \left( \nabla_J X \Pi(J X, Y) + \nabla_X \Pi(X, Y) \right) \\
= d^D \Pi(J X, X)(Y) - i d^D \Pi(J X, X)(J Y)
\]
using in the last equation that \( \Pi(J X, Y) = \Pi(X, J Y) \).

Using a local frame \( (e_1, e_2) \) and defining \( \alpha = \Pi(e_1, e_2) \) and \( \beta = \Pi(e_1, e_2) \), we have
\[
\sigma = 2 \Pi(\partial_z, \partial_z) dz^2 = (\alpha - i \beta) dz^2, \\
\|\sigma\|^2_h = -g_N(\alpha, \alpha) - g_N(\beta, \beta) = \frac{1}{2} \|\Pi\|^2.
\]
The result now follows from Gauß equation (Proposition 4.5).

We now use Bochner formula to have the following corollary

**Corollary 5.7** Let \( \Sigma \) be a complete maximal surface. Assume that \( x \) in \( \Sigma \) is
a maximum of the curvature. Then \( K_\Sigma(x) \leq 0 \). If furthermore at this point \( x \),
\( K_\Sigma(x) = 0 \), then \( \Sigma \) is a Barbot surface.

**Proof** We use the notation introduced in the previous paragraph. Let
\[ f := \frac{1}{2} \|\sigma\|^2_h \]. Bochner formula (see Proposition A.1) gives
\[
\Delta f = h(R_E(e_1, e_2)\sigma, J \sigma) + \|\nabla^E \sigma\|^2_h.
\]
Let \( (e_1, e_2) \) be a local orthonormal frame of \( T \Sigma \). Let \( \partial_z = \frac{1}{2}(e_1 - ie_2) \) be the
local section of the complexified bundle of \( T \Sigma \). Let \( dz \) be the section of \( K \) dual
to the section \( \partial_z \). Let \( \alpha = \Pi(e_1, e_1), \beta = \Pi(e_1, e_2) \). Then we have
\[
R_E(e_1, e_2)\sigma = R_E(e_1, e_2)((\alpha - i \beta) dz^2)
\]
We have $R_T(e_1, e_2) \partial_z = -iK_\Sigma \cdot \partial_z$ and hence $R_K(e_1, e_2) dz = iK_\Sigma \cdot dz$. On the other hand, if $h_N$ is the Hermitian extension of the (positive definite) metric $(-g_N)$ to $N^C \Sigma$, and using Ricci equation (Proposition 4.6), we have

$$h_N(R_N(e_1, e_2)(\alpha - i\beta), i(\alpha - i\beta)) = -2(R_N(e_1, e_2)\alpha, \beta) = 4\|\alpha \wedge \beta\|^2.$$  

This gives, using Lemma 5.6 for the first equality

$$\frac{1}{2} \Delta K_\Sigma = \Delta f = 2K_\Sigma \|\sigma\|^2_h + 4\|\alpha \wedge \beta\|^2 + \|\nabla^E \sigma\|^2_h \geq 2K_\Sigma (1 + K_\Sigma). \quad (21)$$

At a maximum $x$ of $K_\Sigma$, we have $\Delta K_\Sigma \leq 0$ and thus get $K_\Sigma(x) \leq 0$. This shows the first part of the corollary.

Assume now that $K_\Sigma(x) = 0$. Thus on a neighbourhood of $x$ we have $\Delta K_\Sigma \geq 4K_\Sigma$. By the Strong Maximum Principle [18, Theorem 3.5], $K_\Sigma = 0$ everywhere. Moreover, by equation (21), we have that $\|\alpha \wedge \beta\|^2 = 0$ and $\nabla^E \sigma = 0$ everywhere.

This implies that $\alpha$ and $\beta$ are colinear and the line bundle $L := \text{span}\{\alpha, \beta\}$ is a parallel subbundle of $N \Sigma$. Hence, the subbundle $V$ of type $(2, 2)$ of the flat trivial $E$ whose fibre at a point $x$ is $L_x \oplus T_x \Sigma \oplus x$ is parallel with respect to the flat trivial connection and in particular constant. Thus $\Sigma$ is a flat maximal surface contained in the totally geodesic copy of $H^{2,1}$ given by $P(V) \cap H^{2,n}$. By Bonsante–Schlenker Lemma 4.17, $\Sigma$ is a Barbot surface.

5.1.2 Proof of the Curvature Rigidity Proposition 5.3

Recall that by Theorem 4.8 the space $\mathcal{M}(n)/\mathbb{G}$ is compact. Thus the continuous function $\mathcal{M}(n)/\mathbb{G} \to \mathbb{R}$, $(x, \Sigma) \mapsto K_\Sigma(x)$ reaches its maximum at a point $(x_0, \Sigma_0)$. By Corollary 5.7 $K_{\Sigma_0}(x_0) \leq 0$, hence for all $(x, \Sigma)$, we have $K_\Sigma(x) \leq K_{\Sigma_0}(x_0) \leq 0$.

5.2 Horofunction Rigidity

Consider a complete maximal surface $\Sigma$ in $H^{2,n}$, $z$ a point in $\Sigma \sqcup \partial_\infty \Sigma$ and $z_0$ a non zero vector in $\Sigma$. Then any point $x$ in $\Sigma$ has a unique lift to $E$, that we still denote by $x$, such that $q(x) = -1$ and $\langle z_0, x \rangle < 0$. In the sequel, we will implicitly use this canonical lift to define scalar products. We prove the
next proposition in Paragraph 5.2.3. Then the Horofunction Rigidity Theorem in 5.2.4.

**Proposition 5.8** (Gradient bounds) Let $\Sigma$ be a complete maximal surface in $H^{2,n}$ and $z$ be a point in $\Sigma \sqcup \partial_\infty \Sigma$. Assume a point $x$ in $\Sigma$ is a critical point of $\|\nabla h_z\|$. Then we have the following inequality

$$\|\nabla h_z\|^2 \leq 2 + K_\Sigma. \quad (22)$$

In particular $\|\nabla h_z\|^2 \leq 2$ with equality and only if $K_\Sigma = 0$.

### 5.2.1 The gradient

Let $z$ be a point in $\Sigma \sqcup \partial_\infty \Sigma$, seen as a line in $E$, and let $z_0$ be a non zero vector in the line $z$. Then

**Lemma 5.9** Let $x$ be a point in $\Sigma$ not equal to $z$, then for any $u$ in $T_x \Sigma$,

$$d_x h_z(u) = -\frac{\langle u, z_0 \rangle}{\sqrt{\langle x, z_0 \rangle^2 + \langle z_0, z_0 \rangle}} \quad \|\nabla h_z\| \geq 1,$$

where the gradient is taken along $\Sigma$.

**Proof** If $z$ is a point of $\partial_\infty \Sigma$ and $z_0$ is a non zero vector in $z$, then $h_z(x) = \log(-\langle x, z_0 \rangle)$. It follows that

$$d_x h_z(u) = \frac{\langle u, z_0 \rangle}{\langle x, z_0 \rangle} = -\frac{\langle u, z_0 \rangle}{\sqrt{\langle x, z_0 \rangle^2 + \langle z_0, z_0 \rangle}}.$$

Assume now that $z$ belongs to $\Sigma$ and let again $z_0$ be a non zero vector in $z$. Since the derivative of $\cosh^{-1}$ at a point $x$ is $(x^2 - 1)^{-\frac{1}{2}}$, we get for $z$ in $\Sigma$,

$$d_x h_z(u) = -\frac{\langle u, z_0 \rangle}{\sqrt{\langle x, z_0 \rangle^2 - 1}} = -\frac{\langle u, z_0 \rangle}{\sqrt{\langle x, z_0 \rangle^2 + \langle z_0, z_0 \rangle}}.$$

Let us now write

$$z_0 = -\langle x, z_0 \rangle x + z_0^T + z_0^N$$

where $z_0^T$ and $z_0^N$ are the tangential and normal component of $z_0$ at $x$. Then

$$q(z_0^T) \geq q(z_0^T) + q(z_0^N) = q(z_0^T + z_0^N) = q(z_0 + \langle x, z_0 \rangle x) = \langle z_0, z_0 \rangle + \langle x, z_0 \rangle^2.$$
It follows that
\[ \| \nabla h_z \|^2 = \frac{q(z_T^T)}{\langle z_0, z_0 \rangle + \langle x, z_0 \rangle^2} \geq 1. \]

5.2.2 Some preliminary lemmas

For \( \Sigma, z_0 \) and \( x \) as above, we define \( \beta_x \in \text{Sym}^2(T^*_x \Sigma) \) by
\[ \beta_x(u, v) = \frac{\langle \Pi(u, v), z_0 \rangle}{\langle x, z_0 \rangle}, \]
where \( \Pi \) is the second fundamental form of \( \Sigma \). Observe that \( \beta_x \) only depends on the projectivisation \( z \) of \( z_0 \). In the sequel, we write \( h := h_z \).

**Lemma 5.10** Let \( \Sigma \) be a spacelike surface. For any point \( x \) in \( \Sigma \), the Hessian
\[ \nabla^2 h \] of \( h \) at \( x \) on \( \Sigma \) satisfies
\[ \nabla^2 x h = \phi_z \cdot (g - dh \otimes dh + \beta_x) \]
where \( \otimes \) denotes the symmetric tensor product, \( g \) is the induced metric on \( \Sigma \) and
\[ \phi_z(x) = -\frac{\langle x, z_0 \rangle}{\sqrt{\langle x, z_0 \rangle^2 + \langle z_0, z_0 \rangle}} \]
and in particular \( \phi_z = 1 \) if \( z \) is in \( \partial_\infty \Sigma \).

**Proof** Let \( (x_t)_{t \in (-\epsilon, \epsilon)} \) be a smooth geodesic in \( \Sigma \) with \( x_0 = x \). We have
\[ \left. \frac{d^2}{dt^2} \right|_{t=0} h(x_t) = -\frac{\langle D_x \dot{x}, z_0 \rangle}{\sqrt{\langle x, z_0 \rangle^2 + \langle z_0, z_0 \rangle}} + \frac{\langle \dot{x}, z_0 \rangle^2}{\left(\sqrt{\langle x, z_0 \rangle^2 + \langle z_0, z_0 \rangle}\right)^3}, \]
where \( D \) is the Levi–Civita connection of \( E \). The decomposition of \( D \) in the splitting \( E = \langle x \rangle \oplus T_x \Sigma \oplus N_x \Sigma \) gives
\[ (D_a b)_x = \langle a, b \rangle x + (\nabla_a b)_x + \Pi(a, b), \]
where \( \nabla \) is the Levi–Civita connection of \( \Sigma \). Thus
\[ \left. \frac{d^2}{dt^2} \right|_{t=0} h(x_t) = -\frac{\langle \dot{x}, \dot{x} \rangle \langle x, z_0 \rangle}{\sqrt{\langle x, z_0 \rangle^2 + \langle z_0, z_0 \rangle}} - \frac{\langle \Pi(\dot{x}, \dot{x}), z_0 \rangle}{\sqrt{\langle x, z_0 \rangle^2 + \langle z_0, z_0 \rangle}}. \]
which is what we wanted to prove.

\textbf{Lemma 5.11} Let $\Sigma$ be a spacelike surface. If $x$ is a critical point of the function $\|\nabla h\|^2$ in $\Sigma$ and $x$ is different from $z$, then

$$\beta_x(\nabla h, \nabla h) = \|\nabla h\|^2 (\|\nabla h\|^2 - 1).$$

\textbf{Proof} Let $(e_1, e_2)$ be an orthonormal framing of $T \Sigma$. Then for any vector field $X$ on $\Sigma$, we have at $x$

$$0 = \frac{1}{2} X \cdot (\|\nabla h\|^2) = \sum_{i=1}^2 (X \cdot dh(e_i)) dh(e_i)$$

$$= \sum_{i=1}^2 \nabla^2 h(X, dh(e_i)e_i) = \nabla^2 h(X, \nabla h).$$

Applying the previous lemma to $X = \nabla h$ gives

$$0 = \|\nabla h\|^2 - \|\nabla h\|^4 + \beta_x(\nabla h, \nabla h).$$

And thus the result.

\textbf{Lemma 5.12} Assume that $\Sigma$ is maximal and $u$ is a tangent vector in $\Sigma$ at $x$.

$$\beta_x(u, u)^2 \leq \|u\|^4 (\|\nabla h\|^2 - 1)(1 + K_\Sigma(x)). \tag{23}$$

\textbf{Proof} Let us use the notation $\|u\|^2 = |\langle u, u \rangle|$. In the orthogonal splitting $E = \mathbb{R} \cdot x \oplus T_x \Sigma \oplus N_x \Sigma$, write $z_0 = -\langle x, z_0 \rangle x + z_0^T + z_0^N$. It follows that

$$\|z_0^T\|^2 - \|z_0^N\|^2 = \langle z_0, z_0 \rangle + \langle x, z_0 \rangle^2 \leq \langle x, z_0 \rangle^2. \tag{24}$$

This implies that

$$\frac{\|z_0^N\|^2}{\langle x, z_0 \rangle^2} \leq \frac{\|z_0^T\|^2}{\langle z_0, z_0 \rangle + \langle x, z_0 \rangle^2} - 1 = \|\nabla h\|^2 - 1. \tag{25}$$

Now

$$\beta_x(u, u)^2 = \frac{\langle \Pi(u, u), z_0^N \rangle^2}{\langle x, z_0 \rangle^2} \leq \|\Pi(u, u)\|^2 \frac{\|z_0^N\|^2}{\langle x, z_0 \rangle^2}$$
\[ \|\nabla h\|^2 (\|\nabla h\|^2 - 1). \]

From Gauß equation (Proposition 4.5), and since \( \Sigma \) is maximal, we have
\[ \|\Pi(u, u)\|^2 \leq \|\Pi(u, u)\|^2 + \|\Pi(Ju, u)\|^2 = (1 + K_\Sigma)\|u\|^2. \]

where \( J \) is the complex structure on \( \Sigma \). The result follows. \( \square \)

### 5.2.3 Proof of Proposition 5.8

Applying Lemma 5.11 we obtain that
\[ \beta_h(\nabla h, \nabla h) = \|\nabla h\|^2 (\|\nabla h\|^2 - 1). \]

Then using Lemma 5.12 for \( u = \nabla h \), we obtain
\[ (\|\nabla h\|^2 (\|\nabla h\|^2 - 1))^2 \leq \|\nabla h\|^4 (\|\nabla h\|^2 - 1)(1 + K_\Sigma(x)). \] (26)

Since \( \|\nabla h\| \geq 1 \) by Lemma 5.9, we get
\[ (\|\nabla h\|^2 - 1)^2 \leq (\|\nabla h\|^2 - 1)(1 + K_\Sigma(x)). \] (27)

Assuming now that \( \|\nabla h\| > 1 \), we get
\[ \|\nabla h\|^2 \leq 2 + K_\Sigma(x). \] (28)

This gives the result since \( K_\Sigma(x) \leq 0 \) by the Curvature Rigidity Theorem 5.3. Observe that Inequality (28) is also true whenever \( \|\nabla h\| = 1 \) since \( K_\Sigma(x) \geq -1 \). This concludes the proof of the proposition.

### 5.2.4 Proof of the Horofunction and Spatial Distance Rigidity Theorem

By Corollary 4.9, the action of \( G \) on the space \( \mathcal{N}_1(n) \) of triples \((x, \Sigma, z)\), where \((x, \Sigma)\) is a complete maximal surface and \( z \) a point in \( \Sigma \cup \partial_\infty \Sigma \), is cocompact. Then the function
\[ \Phi \ (x, \Sigma, z) \mapsto \|\nabla h_z(x)\|^2 \]

where \( h \) is the horofunction associated to \( z \) is continuous and \( G \) invariant. It follows that \( \Phi \) is bounded.

Let now \((x, \Sigma, z)\) be a point in \( \mathcal{N} \) at which the maximum of \( \Phi \) is reached. Observe then that \( x \) is a critical point of \( \|\nabla h(x)\|^2 \) along \( \Sigma \).

The result now follows from Proposition 5.8 and the Curvature Rigidity Proposition 5.3.
6 Characterisations of quasiperiodicity

In this section, we give different characterisation of quasiperiodic maximal surfaces using our previous results

6.1 Curvature bounds

**Theorem 6.1** (Curvature characterisation) A complete maximal surface \( \Sigma \) in \( H^{2,n} \) is quasiperiodic if and only if there exists a positive constant \( c \) so that the intrinsic curvature of \( \Sigma \) satisfies \( K_\Sigma < -c \).

**Proof** Remark that the function \( x \mapsto K_\Sigma(x) \) is continuous and \( G \)-invariant from \( \mathcal{M}(n) \) to \( \mathbb{R} \). By compactness of \( \mathcal{M}/G \) (see Theorem 4.8), it reaches its maximum on the closure \( C \) of the image of \( \Sigma \) in \( \mathcal{M}(n)/G \). By Theorem 5.5, this maximum is 0 if and only if \( C \) contains a Barbot surface. The result follows.

6.2 Horofunction bounds

Similarly

**Theorem 6.2** (Horofunction characterisation) A surface is quasiperiodic if and only if there exists a positive real \( h_0 \) strictly less than 1 so the square of norm of the gradient of any horofunction is bounded by \( 1 + h_0 \)

**Proof** Let \( \mathcal{N}_1(n) \) be the space of of triples \( (x, \Sigma, z) \) where \( (x, \Sigma) \) is a complete maximal surface and \( z \) a point in \( \partial_\infty \Sigma \). By Corollary 4.9, the action of \( G \) on the space \( \mathcal{N}_1(n) \) is cocompact. Denoting \( h_\Sigma \) the horofunction associated to \( z \) in \( \partial_\infty \Sigma \), the function associating to \( (x, \Sigma, z) \) the value \( \| \nabla h_\Sigma(x) \|^2 \) is continuous and \( G \)-invariant. Thus it reaches its maximum on the closure \( C \) of the preimage of \( \Sigma \) in \( \mathcal{N}_1(n)/G \). By Theorem 5.5, this maximum is 2 if and only if \( C \) contains a Barbot surface. The result follows.

6.3 Conformal characterisation

**Theorem 6.3** (Conformal characterisation) A complete maximal surface \( \Sigma \) is quasiperiodic if and only if there exists a conformal biLipschitz \( \Phi \) map from \( H^2 \) to \( \Sigma \).

**Proof** We have already seen that if \( \Sigma \) is quasiperiodic, its curvature \( K_\Sigma \) is less than a negative constant \( -c \).

By the classical Ahlfors–Schwarz–Pick Lemma [12, Theorem A.1], since \( K_\Sigma > -1 \), any conformal map from \( H^2 \) to \( \Sigma \) is length increasing.
Similarly, because $K_{\Sigma} \leq -c$, any conformal map from $\Sigma$ to $H_c^2$ is length increasing (here $H_c^2$ is the disk equipped with the metric $\frac{1}{c}g_{H_c^2}$ of constant curvature $-c$). The uniformisation thus gives $g_{H^2} \leq g_{\Sigma} \leq \frac{1}{c}g_{H_c^2}$.

Conversely, if $\Sigma$ were not quasiperiodic, there would be a sequence of point $\{x_k\}_{k \in \mathbb{N}}$ so that the sequence of pointed maximal surface $\{(x_k, \Sigma)\}_{k \in \mathbb{N}}$ converges to a Barbot surface. Then the sequence of Riemannian surface $\{(x_k, \Sigma)\}_{k \in \mathbb{N}}$ would converge uniformly on every compact to a flat metric. In particular the flat metric would be quasiisometric to a hyperbolic metric which is impossible.

6.4 Gromov hyperbolicity

**Theorem 6.4** (Gromov Hyperbolicity) A complete maximal surface $\Sigma$ is quasiperiodic if and only if the induced Riemannian metric is Gromov-hyperbolic.

This result is to be compared to some of Benoist–Hulin results for $\text{SL}(3, \mathbb{R})$ [6].

**Proof** By the curvature characterisation, we see that a quasiperiodic complete maximal surface has curvature bounded from above by a negative constant and hence is Gromov hyperbolic.

Conversely, if a surface is not quasiperiodic, there would be a sequence of point $\{x_k\}_{k \in \mathbb{N}}$ so that the sequence of pointed maximal surface $\{(x_k, \Sigma)\}_{k \in \mathbb{N}}$ converges to a Barbot surface. Then the sequence of Riemannian surface $\{(x_k, \Sigma)\}_{k \in \mathbb{N}}$ would converge uniformly on every compact to a flat metric. This would guarantee the existence of arbitrarily thick triangles and contradict Gromov hyperbolicity.

6.5 Characterisation using the spatial distance

Recall that from Proposition 4.2, complete maximal surfaces are acausal. In particular, it is natural to compare the induced distance with the spatial distance $\bar{d}$ defined in Definition 10. Our characterisation is the following

**Theorem 6.5** (Spatial distance) Let $\Sigma$ be a complete maximal surface, then we have the following inequality

$$d \leq \bar{d} \leq \sqrt{2}d. \quad (29)$$

Moreover, $\Sigma$ is quasiperiodic if and only if there exists $c < \sqrt{2}$ so that

$$\bar{d} \leq cd. \quad (30)$$
Proof of Theorem 6.5 The inequality $d \leq \bar{d}$ is consequence of Lemma 5.9 or equivalently to the fact that the warped projection on a hyperbolic plane is length increasing [31, Lemma 3.7].

The inequality $\bar{d} \leq \sqrt{2}d$ comes from the Horofunction and Spatial Distance Rigidity Theorem 5.5.

If $\Sigma$ is not quasiperiodic, then we can find a sequence $\{x_k\}_{k \in \mathbb{N}}$ in $\Sigma$ so that $\{(x_k, \Sigma)\}$ converges (up to the action of $\mathcal{G}$) to $(x, \Sigma_0)$ where $\Sigma_0$ is a Barbot surface. It follows by Proposition 4.16, that we have

$$\sup \left( \frac{\bar{d}}{d} \right) \geq \sqrt{2}. \quad (31)$$

Conversely assume that $\Sigma$ is such that inequality (31) is satisfied. Then, for any positive $\varepsilon$, we can find a geodesic $\gamma$ and a positive $t$, so that, we have

$$\bar{d}(\gamma(0), \gamma(t)) \geq \sqrt{2}(1 - \varepsilon)t.$$

Then setting $y = \gamma(0)$ and $f : x \mapsto \bar{d}(y, x)$, we have

$$\int_{\varepsilon t}^{t} df(\gamma') \, dt = \int_{0}^{t} df(\gamma') \, dt - \int_{0}^{\varepsilon t} df(\gamma') \, dt \geq t\sqrt{2}(1 - \varepsilon) - t\sqrt{2}\varepsilon \geq (1 - 2\varepsilon)\sqrt{2}t$$

since the gradient of $f$ has norm less than $\sqrt{2}$ by Theorem 5.5. Thus there exists $x = \gamma(s)$, with $s$ in $[\varepsilon t, t]$ so that

$$\| \nabla \bar{d}^{y}(x) \| \geq \sqrt{2} \frac{1 - 2\varepsilon}{1 - \varepsilon}.$$

Since $\varepsilon$ is arbitrary, we can find sequences $\{x_k\}_{k \in \mathbb{N}}$ and $\{y_k\}_{k \in \mathbb{N}}$ so that

$$\lim_{k \to \infty} \| \nabla \bar{d}^{y_k}(x_k) \| \geq \sqrt{2}.$$

Let $f_k(x) = \bar{d}(x, y_k)$, then after extracting a subsequences, we can assume that $(x_k, \Sigma, y_k)$ converges to $(x, \Sigma_0, z)$ in $\mathcal{N}_1(n)$. Thus $\{f_k - f_k(x_k)\}_{k \in \mathbb{N}}$ converges to a function $h_z$ which is, up to some additive constant, either a horofunction or the spatial distance to a point in $\Sigma$, and so that $\| \nabla h_z(x) \|^2 \geq 2$.

It follows by Theorem 5.5 that $\Sigma_0$ is a Barbot surface and thus that $\Sigma$ is not quasiperiodic. \qed

6.6 Lamination characterisation

Here is another interpretation of quasiperiodic leaves.
Theorem 6.6 (Laminated interpretation) A complete maximal surface $\Sigma$ is quasiperiodic if and only if there exists

(i) a compact space $F$ laminated by hyperbolic Riemann surfaces.

(ii) a $H^{2,n}$-bundle $H$ over $F$ equipped with a flat connection along the leaves whose parallel transport is transversely continuous in the smooth topology along the leaves.

(iii) a section of $H$, lifting all leaves to maximal complete surfaces so that $\Sigma$ is one of these lifts.

The case when $F$ is a compact surface is the periodic case and it is customary to describe the compact laminated case as quasiperiodic.

Proof Let $\Gamma$ be a cocompact torsion free lattice in $G$. The space $M(n)/\Gamma$ is then laminated and we take as $F$ the closure of the image of $\Sigma$ in $M(n)/\Gamma$. The bundle $H$ is the induced bundle from the trivial bundle with the $\Gamma$-action. All leaves in $F$ are hyperbolic because they are uniformised by the hyperbolic disk by the Conformal Characterisation Theorem 6.3.

Conversely, such a leaf in a space $F$ is quasiperiodic since its closure of its $G$ orbit in $M(n)$ only contains complete surfaces which are hyperbolic, whereas Barbot surfaces are parabolic.

7 Extension of uniformisation

Let us fix a point $x_0$ in $H^2$ and identify $\partial_\infty H^2$ with $P(V)$.

Theorem 7.1 (Extension of uniformisation) For any positive constant $c$, there exists some constants $A$ and $B$ with the following property:

Assume that $(x, \Sigma)$ is a quasiperiodic surface whose curvature is bounded by $-c$. Then there exists a continuous map $\Psi$ from $H^2 \cup \partial_\infty H^2$ to $\Sigma \cup \partial_\infty \Sigma$ which respects interiors and boundaries, is conformal in the interior and $(A, B)$-quasisymmetric on the boundary and so that $\Psi(x_0) = x$.

Such a map $\Psi$ is called an extension of uniformisation for $(x, \Sigma)$. Recall that any quasiperiodic surface has curvature bounded by some negative constant by Theorem 6.1. The next result tells us extensions behave well under limits.

Theorem 7.2 (Boundary compactness) Let $\{(x_k, \Sigma_k)\}_{k \in \mathbb{N}}$ be a sequence of quasiperiodic surfaces converging to a quasiperiodic surface $(x, \Sigma)$. Assume that there is a positive constant $c$ so that the curvature of all surfaces $\Sigma_k$ is bounded from above by $-c$. Let $\{\Psi_k\}_{k \in \mathbb{N}}$ be the corresponding extensions of conformal parametrisation. Then $\{\Psi_k\}_{k \in \mathbb{N}}$ subconverges uniformly to an extension of conformal parametrisation of $(x, \Sigma)$.

Here is an interesting corollary of this construction whose statement does not involve maximal surfaces. Note that we have not been able to give a direct
proof of this corollary, without using the maximal surface solution to the asymptotic Plateau problem.

**Corollary 7.3** Every quasiperiodic loop admits a quasisymmetric—hence quasiperiodic—parametrisation.

**Proof** Indeed a quasiperiodic loop bounds a quasiperiodic surface, and we have just shown that the boundary at infinity of such a quasiperiodic surface admits a quasisymmetric parametrisation. □

We then define an \((A, B)\)-quasicircle to be a quasiperiodic loop such that any extension of uniformisation is \((A, B)\)-quasisymmetric. We then have some converse to Theorem 7.1

**Corollary 7.4** For any constants \(A\) and \(B\) greater than 1, there exists a positive \(c\) so that if the boundary of a complete maximal surface has an \((A, B)\)-quasisymmetric parametrisation then the curvature of \(\Sigma\) is bounded by \(-c\) from above.

**Proof** The space of pairs \((\xi, \tau)\) where \(\xi\) is an \((A, B)\)-quasisymmetric map and \(\tau\) a positive triple in \(P(V)\) is equipped with a cocompact action of \(H := \text{PSL}(V) \times G\). Denote by \(\Sigma(\xi)\) the solution to the asymptotic Plateau problem for \(\xi(P(V))\). The barycenter map \(B\) being continuous by Proposition 4.12, then map which associates to \((\xi, \tau)\) the curvature of \(B(\Sigma(\xi), b_{\tau})\) is continuous, \(H\)-invariant reaches it maximum at a pair \((\xi_0, \tau_0)\). This maximum is negative by the Rigidity Proposition 5.3. Since \(B\) is surjective, the result is proved. □

**Corollary 7.5** For any constants \(A\) and \(B\) greater than 1, there exists \(C\) so that the image of an \((A, B)\)-quasisymmetric map is an \((A, C)\)-quasicircle.

**Proof** We may as well assume that the quasisymmetric map sends a given positive triple to a fixed positive triple. Since the curvature of any complete maximal surface whose boundary admits an \((A, B)\)-quasisymmetric parametrisation bounded by \(-c\), the result is a consequence of the Uniformisation Theorem 7.1. □

We also have another corollary that answers a question asked to us by François Guéritaud, showing that quasisymmetric maps are extension of quasiisometries with respect to the spatial distance, thus generalising the corresponding fact in the context of rank 1 symmetric spaces.

**Corollary 7.6** Every quasisymmetric map from \(\partial_\infty H^2\) to \(\partial_\infty H^{2,n}\), is a continuous extension of a map \(f\) from \(H^2\) to \(\partial_\infty H^{2,n}\) such that there exists some positive constants \(A\) and \(B\) so that for all \(x\) and \(y\) in \(H^2\)

\[ A^{-1}d(x, y) - B \leq \partial(f(x), f(y)) \leq Ad(x, y) + B. \]
However we do not know whether the converse to that corollary is true.

**Proof of Corollary 7.6** Let $\xi$ be a quasisymmetric map from $\mathbf{P}(V)$ to $\partial_{\infty}\mathbb{H}^{2,n}$, and let $\Lambda = \xi(\mathbf{P}(V))$. By Theorem 7.1, and identifying $\mathbf{P}(V)$ with $\partial_{\infty}\mathbb{H}^{2}$, there is a quasisymmetric parametrisation $\xi_{0}$ of $\Lambda$ which is a continuous extension of a conformal map $f_{0}$ from $\mathbb{H}^{2}$ to the maximal surface $\Sigma(\Lambda)$. By Theorem 6.3, the map $f_{0}$ is biLipschitz. Thus by Theorem 6.5, there is a constant $C$ greater than 1 such that for any $x, y$ in $\mathbb{H}^{2}$

$$C^{-1}d(x, y) \leq \delta(f_{0}(x), f_{0}(y)) \leq Cd(x, y).$$

By Corollary 3.15, there is a quasisymmetric homeomorphism $\phi$ of $\partial_{\infty}\mathbb{H}^{2}$ such that $\xi = \xi_{0} \circ \phi$. By the Douady–Earle extension Theorem [15, Theorem 2], $\phi$ is a continuous extension of a quasiisometry $\Phi$ from $\mathbb{H}^{2}$ to itself. Thus $\xi$ is the continuous extension of $f := f_{0} \circ \Phi$ which satisfies the required condition. □

### 7.1 Control of Gromov products

We study quantities—also considered by Glorieux and Monclair [19]—related to Gromov products when $n = 0$.

We prove three propositions that provide a control on these quantities.

In the sequel, we will lift any complete maximal surface in $\mathbb{H}^{2,n}$ to $\{x \in E, q(x) = -1\}$. This will allow us to define the scalar product $\langle x, y \rangle$ between two points $x, y$ in $\Sigma$. Observe that the quantity $|\langle x, y \rangle|$ is independent on the lift. Similarly for $z, w$ in $\partial_{\infty}\Sigma$ the quantity—or **Gromov product**—

$$\left| \frac{\langle z, w \rangle}{\langle z, x \rangle \langle x, w \rangle} \right| := \left| \frac{\langle z_{0}, w_{0} \rangle}{\langle z_{0}, x \rangle \langle x, w_{0} \rangle} \right|$$

is well defined and independent on the lift of $\Sigma$ and the choice of the non zero vector $z_{0}, w_{0}$ in $z$ and $w$ respectively.

**Proposition 7.7** There exists a positive constant $M_{1}$ such that for any pointed complete maximal surface $(x, \Sigma)$ in $\mathcal{M}(n)$ and points $z, w$ in $\Sigma \cup \partial_{\infty}\Sigma$, we have

$$\left| \frac{\langle z, w \rangle}{\langle z, x \rangle \langle x, w \rangle} \right| \leq M_{1}.$$

**Proof** Let $\mathcal{N}_{2}(n)$ of quadruples $(x, \Sigma, z, w)$, where $(x, \Sigma)$ is a pointed quasiperiodic maximal surface and $z$ and $w$ are points in $\Sigma \cup \partial_{\infty}\Sigma$. Let $P_{x}$ be the affine hyperplane $\{v \mid \langle v, x \rangle = -1\}$. Let $\pi_{x}$ be the radial projection on
the hyperplane. By [31, Proposition 2.27 and Proposition 2.5 (iii)] the radial projection of \( \Sigma \cup \partial \Sigma \) is bounded in \( P_x \). It follows that the function

\[
F : (x, \Sigma, z, w) \mapsto \left| \frac{\langle z, w \rangle}{\langle z, x \rangle \langle x, w \rangle} \right| = \left| \langle \pi_x(z), \pi_x(w) \rangle \right|
\]

is well defined, \( \mathbb{G} \)-invariant and continuous on \( \mathcal{N}_2(n) \). Since the action of \( \mathbb{G} \) is cocompact on \( \mathcal{N}_2(n) \) by Corollary 4.9, \( F \) is bounded and the result follows.

\[\square\]

**Corollary 7.8** There exists a constant \( C \) only depending on \( n \), so that for all complete maximal surface \( \Sigma \) in \( \mathbb{H}^{2,n} \), for all triple of points \( x, y, z \) in \( \Sigma \)

\[\bar{d}(x, y) \leq \bar{d}(x, z) + \bar{d}(z, y) + C.\]

in particular, the function \( \bar{d}_0 \) so that \( \bar{d}_0(x, y) = \bar{d}(x, y) + C \) if \( x \) is different than \( y \) and \( \bar{d}_0(x, x) = 0 \) is a distance.

This final remark was pointed out to us by Indira Chatterji.

**Proof** Since \( \cosh(\bar{d}(x, y)) = |\langle x, y \rangle| \) and \( \frac{1}{2} e^u \leq \cosh(u) \leq e^u \) for any positive \( u \), we get

\[
\left| \frac{\langle z, w \rangle}{\langle z, x \rangle \langle x, w \rangle} \right| \geq \frac{1}{2} \exp \left( \bar{d}(z, w) - \bar{d}(x, w) - \bar{d}(x, z) \right). 
\]

Thus by Proposition 7.7, we get \( \bar{d}(z, w) - \bar{d}(x, w) - \bar{d}(x, z) \leq C := \log(2M_1) \).

\[\square\]

We prove the next two propositions in Paragraph 7.1.4

**Proposition 7.9** For any positive \( R \) and \( K \), there is positive \( M_2 \) such that for any quasiperiodic maximal surface \( \Sigma \), if \( \eta \) is an oriented \( K \)-quasigeodesic of extremities \( z \) and \( w \) in \( \Sigma \) such that \( x \) is at within a distance at most \( R \) from \( \eta \), then

\[
\left| \frac{\langle z, w \rangle}{\langle z, x \rangle \langle x, w \rangle} \right| \geq M_2.
\]

We also need the converse property

**Proposition 7.10** For any positive \( c \), any positive \( K \) and \( B \), then for any quasiperiodic maximal surface \( \Sigma \) with curvature bounded by \(-c\) and any \( K \)-quasigeodesic arc \( \eta \) in \( \Sigma \) with extremities \( w \) and \( z \), then if \( x \) in \( \Sigma \) is so that

\[
\left| \frac{\langle z, w \rangle}{\langle z, x \rangle \langle x, w \rangle} \right| \geq B
\]
then $d_{\Sigma}(x, \eta) \leq A$.

### 7.1.1 Some a priori bounds

In this paragraph, we do not require the complete maximal surface to be quasiperiodic.

**Lemma 7.11** There exists a constant $M_1$ so that for any three points $(a, b, c)$ on a complete maximal surface $\Sigma$, we have the inequality

$$
\frac{1}{M_1} \exp\left(-\sqrt{2}d_{\Sigma}(b, c)\right) \leq \left| \frac{\langle a, b \rangle}{\langle a, c \rangle} \right| \leq M_1 \exp\left(\sqrt{2}d_{\Sigma}(b, c)\right).
$$

**Proof** Indeed, by Proposition 7.7, we have

$$
\frac{1}{M_1} \left| \frac{\langle a, b \rangle}{\langle a, c \rangle} \right| \leq |\langle b, c \rangle| = \cosh(\delta(b, c)) \leq \exp(\delta(b, c)) \leq \exp\left(\sqrt{2}d_{\Sigma}(b, c)\right).
$$

where we used Theorem 6.5 for the last inequality. This proves the right inequality, the second one comes from reversing the role of $b$ and $c$. \qed

**Corollary 7.12** There exists a constant $M$ so that for any three points $(a, b, c)$ on a maximal surface, we have the inequality

$$
\frac{1}{M} \exp\left(-2\sqrt{2}d_{\Sigma}(b, c)\right) \leq \left| \frac{\langle a, b \rangle}{\langle b, c \rangle\langle a, c \rangle} \right| \leq M \exp\left(\sqrt{2}d_{\Sigma}(b, c)\right).
$$

**Proof** Indeed

$$
\left| \frac{\langle a, b \rangle}{\langle b, c \rangle\langle a, c \rangle} \right| \geq \frac{\exp\left(-\sqrt{2}d_{\Sigma}(b, c)\right)}{M_1|\langle b, c \rangle|} \geq \frac{2\exp\left(-\sqrt{2}d_{\Sigma}(b, c)\right)}{M_1 \exp\left(\sqrt{2}d_{\Sigma}(b, c)\right)} = \frac{2}{M_1} \exp\left(-2\sqrt{2}d_{\Sigma}(b, c)\right)
$$

where we used the previous lemma for the first inequality and Theorem 6.5 for the last. Similarly

$$
\left| \frac{\langle a, b \rangle}{\langle b, c \rangle\langle a, c \rangle} \right| \leq \frac{M_1 \exp\left(\sqrt{2}d_{\Sigma}(b, c)\right)}{|\langle b, c \rangle|} \leq M_1 \exp\left(\sqrt{2}d_{\Sigma}(b, c)\right)
$$

where we used the previous lemma for the first inequality and the inequality $|\langle b, c \rangle| \geq 1$ for the second. \qed
7.1.2 Minima of horofunction

**Lemma 7.13** Let $\Sigma$ be a quasiperiodic maximal surface. Then the restriction of any horofunction to a complete geodesic in $\Sigma$ has no local maximum.

**Proof** By the Horofunction Characterisation Theorem 6.2 there is $h_0 < 1$, such that for any $z$ in $\partial_\infty \Sigma$ we have $\|\nabla h_z\|^2 \leq 1 + h_0$—where $h_z$ is the horofunction associated to $z$. Consider a geodesic $\gamma$ in $\Sigma$ parametrised by arc-length, and let $x$ be a critical point of $h_z$ on $\gamma$. The tangent vector $\gamma'$ at $x$ satisfies $dh(\gamma') = 0$. Using Lemma 5.10 with the same notations, we have

$$\nabla^2_x h_z(\gamma', \gamma') = 1 + \beta_x(\gamma', \gamma') \geq 1 - \sqrt{(1 + K_\Sigma(x))(\|\nabla h_z\|^2 - 1)} \geq 1 - \sqrt{h_0} > 0,$$

where we used Lemma 5.12 in the second inequality, our hypothesis in the third and the fact that $0 \leq 1 + K_\Sigma \leq 1$ by Proposition 5.3. Thus the second derivative of $h_z$ at a critical point is positive and the result follows. \hfill $\Box$

7.1.3 Gromov products on geodesics

Let us concentrate on geodesics for the induced metric.

**Lemma 7.14** There exists a positive $M_3$ such that for any quasiperiodic maximal surface $\Sigma$, any oriented geodesic $\gamma$ in $\Sigma$, if $s$ and $t$ are positive then

$$\left| \frac{\langle \gamma(-s), \gamma(t) \rangle}{\langle \gamma(-s), \gamma(0) \rangle \langle \gamma(0), \gamma(t) \rangle} \right| \geq M_3. \quad (34)$$

**Proof** Suppose the result wrong. Then there exists a sequence $\{(x_k, \gamma_k, \Sigma_k)\}_{k \in \mathbb{N}}$, where $(x_k, \Sigma_k)$ is a pointed quasiperiodic surface, $\gamma_k$ is a complete geodesic with $\gamma_k(0) = x_k$, as well as sequences $\{t_k\}_{k \in \mathbb{N}}$ and $\{s_k\}_{k \in \mathbb{N}}$ of real numbers such that

$$\lim_{k \to \infty} \left| \frac{\langle \gamma_k(-s_k), \gamma_k(t_k) \rangle}{\langle \gamma_k(-s_k), x_k \rangle \langle x_k, \gamma_k(t_k) \rangle} \right| = 0. \quad (35)$$

Observing that by Corollary 7.12,

$$\left| \frac{\langle \gamma_k(-s_k), \gamma_k(t_k) \rangle}{\langle \gamma_k(-s_k), x_k \rangle \langle x_k, \gamma_k(t_k) \rangle} \right| \geq \frac{1}{M} \sup \left\{ \exp(-2\sqrt{2}s_k), \exp(-2\sqrt{2}t_k) \right\}$$

this implies that $\{s_k\}_{k \in \mathbb{N}}$ and $\{t_k\}_{k \in \mathbb{N}}$ both converge to $\infty$. \hfill $\Box$
For \((x, \Sigma)\) in \(\mathcal{M}(n)\), an oriented geodesic \(\gamma\) passing through \(x\) is given by a linear ray \(T_x \gamma \subset T_x \Sigma\). In particular, and one can find a sequence \(\{g_k\}_{k \in \mathbb{N}}\) in \(G\) such that \(g_k(x_k, \gamma_k, \Sigma_k)\) subconverges to \((x, \gamma, \Sigma)\) with \((x, \Sigma)\) quasiperiodic.

To make life simpler, we assume the sequence \(\{g_k\}_{k \in \mathbb{N}}\) is constant and equal to the identity. Let us then consider

\[ u_k = \frac{1}{\langle x_k, \gamma_k(t_k) \rangle} \gamma_k(t_k), \quad v_k = \frac{1}{\langle x_k, \gamma_k(-s_k) \rangle} \gamma_k(-s_k). \]

Up to extracting a subsequence, \(\{u_k\}_{k \in \mathbb{N}}\) and \(\{v_k\}_{k \in \mathbb{N}}\) converge respectively to lightlike vectors \(u, v\). By [31, Theorem 6.1], \(u\) and \(v\) gives rise in the projective compactification to elements of \(\partial_\infty \Sigma\). By assertion (35), \(\langle u, v \rangle = 0\). However since \(\Sigma\) is quasiperiodic, \(\partial_\infty \Sigma\) is positive, thus \(u = v\).

Let \(\{w_k\}_{k \in \mathbb{N}}\) be a sequence of lightlike vectors converging to \(u\) so that \(w_k\) belongs to \(\partial_\infty \Sigma_k\) and normalised so that \(\langle x_k, w_k \rangle = 1\). Let \(h_k\) be the horofunction associated to \(w_k\). Then

\[
\lim_{k \to \infty} h_k(\gamma_k(-s_k)) = \lim_{k \to \infty} \log |\langle w_k, v_k \rangle| = -\infty
\]

\[
\lim_{k \to \infty} h_k(\gamma_k(t_k)) = \lim_{k \to \infty} \log |\langle w_k, u_k \rangle| = -\infty
\]

\[
\lim_{k \to \infty} h_k(\gamma_k(0)) = \lim_{k \to \infty} \log |\langle x_k, w_k \rangle| = 0.
\]

It follows that for \(k\) large enough, \(h_k\) has a local maximum on \(\gamma_k([-s_k, t_k])\). This contradicts Lemma 7.13.

**Corollary 7.15** For any \(R\) there is a positive constant \(M_4\) such that if \(\Sigma\) is a quasiperiodic maximal surface and \(\gamma\) is an oriented geodesic in \(\Sigma\) such that \(x\) is at within a distance \(R\) of \(\gamma\), both \(s\) and \(t\) are positive, then

\[
\frac{\langle \gamma(-s), \gamma(t) \rangle}{\langle \gamma(-s), x \rangle \langle x, \gamma(t) \rangle} \geq M_4.
\]

**Proof** Let assume (after a change of time) that \(d(x, \gamma(0))\) is less than \(R\). Then

\[
\frac{\langle \gamma(-s), \gamma(t) \rangle}{\langle \gamma(-s), x \rangle \langle x, \gamma(t) \rangle} = \frac{\langle \gamma(-s), \gamma(0) \rangle}{\langle \gamma(-s), x \rangle} \cdot \frac{\langle \gamma(t), \gamma(0) \rangle}{\langle \gamma(t), x \rangle} \cdot \frac{\langle \gamma(-s), \gamma(0) \rangle}{\langle \gamma(-s), \gamma(t) \rangle} \cdot \frac{\langle \gamma(t), \gamma(0) \rangle}{\langle \gamma(t), x \rangle} \geq M_3 \frac{\langle \gamma(-s), \gamma(0) \rangle}{\langle \gamma(-s), x \rangle} \cdot \frac{\langle \gamma(t), \gamma(0) \rangle}{\langle \gamma(t), x \rangle} \geq M_3 \frac{\langle \gamma(-s), \gamma(0) \rangle}{M_2} \exp(-2\sqrt{2}R) \geq M_3 \frac{M_1}{M_2} \exp(-2\sqrt{2}R).
\]

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Where we used Lemma 7.14 for the first inequality and Lemma 7.11 for the second.

7.1.4 Proof of Propositions 7.9 and 7.10

Proof of Proposition 7.9 Let \( \gamma \) be the geodesic between \( \eta(-s) \) and \( \eta(t) \) and observe that \( \gamma \) is at distance at most \( K + R \) from \( x \). The result now follows from Corollary 7.15.

Proof of Proposition 7.10 By the quasiisometry property, it is enough to prove the proposition whenever \( \eta \) is a geodesic arc. In other words if

\[
\frac{\langle z, w \rangle}{\langle z, x \rangle \langle x, w \rangle} \geq B \tag{36}
\]

then \( d_\Sigma(x, y) \leq A \) for some constant \( A \) and some \( y \) on the arc \( \eta \). Since the curvature of \( \Sigma \) is bounded by \( -c \), where \( c \) is positive, there exists a point \( y \) in \( \eta \) which is \( R \)-close to both geodesic arcs \([x, w]\) and \([x, z]\), for \( R \) only depending on \( c \). In particular, by Corollary 7.15 we have

\[
\left| \frac{\langle x, w \rangle}{\langle x, y \rangle \langle y, w \rangle} \right| \geq M_4 \quad \text{and} \quad \left| \frac{\langle x, z \rangle}{\langle x, y \rangle \langle y, z \rangle} \right| \geq M_4. \tag{37}
\]

Multiplying both inequalities and then inequality (36), we get

\[
\left| \frac{\langle z, w \rangle}{\langle x, y \rangle^2 \langle y, w \rangle \langle y, z \rangle} \right| \geq BM_4^2. \tag{38}
\]

Thus

\[
|\langle x, y \rangle|^2 \leq \frac{1}{BM_2^2} \left| \frac{\langle z, w \rangle}{\langle y, w \rangle \langle y, z \rangle} \right| \leq \frac{M_1}{BM_4^2}. \tag{39}
\]

where we used Proposition 7.7 in the last inequality. As a conclusion, using Theorem 6.5

\[
d_\Sigma(x, y) \leq \sqrt{2} \delta(x, y) = \sqrt{2} \cosh^{-1}(|\langle x, y \rangle|) \leq A := \sqrt{2} \cosh^{-1}\left(\frac{\sqrt{BM_1}}{BM_4}\right).
\]

The result follows.
7.2 Construction of a quasisymmetric map

We now build in the second paragraph a candidate to be a quasisymmetric map and proves its property.

7.2.1 A boundary coincidence

We will need

**Proposition 7.16** Let \( \{ x_k \}_{k \in \mathbb{N}} \) and \( \{ y_k \}_{k \in \mathbb{N}} \) be two sequences of points on a quasiperiodic maximal surface \( \Sigma \) so that \( d_{\Sigma}(x_k, y_k) \) is uniformly bounded. Assume that \( \{ x_k \}_{k \in \mathbb{N}} \) converges to \( u \) in \( \partial_{\infty} \Sigma \). Then \( \{ y_k \}_{k \in \mathbb{N}} \) also converges to \( u \).

**Proof** Let \( h \) be the horofunction associated to a non zero vector \( u_0 \) in \( u \). Since \( \| \nabla h \|^2 < 2 \) and \( d_{\Sigma}(x_k, y_k) \) is uniformly bounded, then \( |h(x_k) - h(y_k)| \) is uniformly bounded. Since \( \lim_{k \to \infty} h(x_k) = -\infty \), we have \( \lim_{k \to \infty} h(y_k) = -\infty \) hence \( \lim_{k \to \infty} \langle u_0, y_k \rangle = 0 \). After extracting a subsequence, we may assume that \( \{ y_k \}_{k \in \mathbb{N}} \) converges to a point \( v \) in \( \Sigma \cup \partial_{\infty} \Sigma \). Let us fix a point \( z_0 \) in \( \Sigma \), then

\[
\lim_{k \to \infty} \frac{1}{\langle z_0, y_k \rangle} y_k = v_0
\]

where \( v_0 \) is the non zero vector in \( v \) so that \( \langle v_0, z_0 \rangle = 1 \). Recall that on a maximal surface \( |\langle z_0, y_k \rangle| \geq 1 \), thus

\[
|\langle u_0, v_0 \rangle| \leq \lim_{k \to \infty} |\langle u_0, y_k \rangle| = 0.
\]

Since \( \Sigma \) is quasiperiodic maximal, \( \partial_{\infty} \Sigma \) is positive, so \( u_0 = v_0 \). This proves the result. \( \square \)

7.2.2 A quasisymmetric map

Assume \( \Sigma \) is a quasiperiodic maximal surface. By Theorem 6.3, the uniformisation is biLipschitz and thus a quasiisometry. In particular there is a constant \( K \) so that any geodesic in \( H^2 \) is mapped under the uniformisation to a \( K \)-quasigeodesic.

Let us choose once and for all a fixed point \( x_0 \) in \( \Sigma \). All the geodesics we consider here are assumed to be complete. Because \( \Sigma \) is properly embedded, any geodesic \( \gamma \) in \( H^2 \) has a properly embedded image. Thus for any point \( x \) in \( \partial_{\infty} H^2 \), fix a geodesic \( \gamma \) converging to \( x \) and let us consider the (non empty) set \( B_x \) of limit values in \( \partial_{\infty} H^{2,n} \) of all sequences converging to \( x \) along \( \gamma \).
Observe that $B_x$ is connected: the set of limit values of a function defined on some interval is connected. Hence $B_x$ a closed interval in $\partial_\infty \Sigma$. Observe now

**Proposition 7.17** Fixing $x$, $B_{\gamma,x} = B_{\eta,x}$.

**Proof** Choose parametrisations of $\gamma$ and $\eta$ so that

$$\lim_{s \to \infty} d_{H^2}(\gamma(s), \eta(s)) = 0.$$ 

Now if $\gamma(t_k) \to u$, then since $d_\Sigma(\eta(k), \gamma(t_k)) \to 0$, by Proposition 7.16, we also have $\eta(t_k) \to u$. This proves the result. \qed

Accordingly, we drop the reference $\gamma$ and write $B_{\gamma,x} = B_x$. Let us choose a map

$$\Phi : \partial_\infty H^2 \to \partial_\infty \Sigma,$$

such that for any point $x$ in $\partial_\infty H^2$, the point $\Phi(x)$ belongs to $B_x$.

From the quasiperiodicity of $\Sigma$, we know that $\partial_\infty \Sigma$ is positive. In particular, for any pairwise distinct points $x, y, z$ and $t$ in $\partial_\infty \Sigma$, the cross-ratio $b(x, y, z, t)$ is well-defined (see Definition 3.1). Finally, denote by $[,] , [ ,] , [ ,]$ the cross-ratio on $\partial_\infty H^2$ obtained by identifying $\partial_\infty H^2$ with $P(V)$.

**Proposition 7.18** Let $\Sigma$ and $\Phi$ be as above. If $(a, b, c, d)$ is a quadruple of pairwise distinct points in $\partial_\infty H^2$, then $(\Phi(a), \Phi(b), \Phi(c), \Phi(d))$ are pairwise transverse. Moreover, for every $A$ there is a $B$ such that

$$A^{-1} < ||[a, b, c, d]| < A \implies B^{-1} < |b(\Phi(a), \Phi(b), \Phi(c), \Phi(d))| < B.$$ 

**Proof** Since $\Phi$ takes value in $\partial_\infty \Sigma$ which is positive, transversality of the image of pairwise distinct points in $\partial_\infty H^2$ is equivalent to the injectivity of $\Phi$.

For any $A$, there exists a constant $R$, such that for any any quadruple $(a, b, c, d)$ of points in $\partial_\infty H^2$ with

$$A^{-1} \leq ||[a, b, c, d]| \leq A$$

we can find a point $p$ in $\Sigma$ within a distance $R$ from the four geodesics $\gamma_{ab}, \gamma_{cd}, \gamma_{ad}$ and $\gamma_{cb}$, where $\gamma_{ij}$ is the quasigeodesic in $\Sigma$ which is the image by the uniformisation of the geodesic between $i$ and $j$ in $\partial_\infty H^2$ (where $i, j \in \{a, b, c, d\}$). Since by Proposition 7.17, we have $B_i = B_{\gamma_{ij},i}$, we can find sequences $\left\{t_{ij}^k\right\}_{k \in \mathbb{N}}$, for all distinct pairs $i, j \in \{a, b, c, d\}$ so that

$$\lim_{k \to \infty} \gamma_{ij}(t_{ij}^k) = \Phi(j) \text{ and } \lim_{k \to \infty} \gamma_{ij}(-t_{ij}^k) = \Phi(i)$$
Applying Proposition 7.9 we get that for all distinct \( i, j \) we get

\[
\left| \frac{\langle \Phi(i), \Phi(j) \rangle}{\langle p, \Phi(i) \rangle \langle p, \Phi(j) \rangle} \right| = \lim_{k \to \infty} \left| \frac{\langle \gamma_{ij}(-t_{ij}^k), \gamma_{ab}(t_{ij}^k) \rangle}{\langle p, \gamma_{ij}(-t_{ij}^k) \rangle \langle p, \gamma_{ij}(t_{ij}^k) \rangle} \right| \geq M_2. \tag{40}
\]

In particular, \( \Phi \) is injective. Moreover, from Proposition 7.7 we have

\[
\left| \frac{\langle \Phi(i), \Phi(j) \rangle}{\langle p, \Phi(i) \rangle \langle p, \Phi(j) \rangle} \right| \leq M_1. \tag{41}
\]

Let us write \((\alpha, \beta, \gamma, \delta) = (\Phi(a), \Phi(b), \Phi(c), \Phi(d))\) and observe that

\[
\frac{\langle \alpha, \beta \rangle \langle \gamma, \delta \rangle}{\langle \alpha, \delta \rangle \langle \gamma, \beta \rangle} = \frac{\langle \alpha, \beta \rangle}{\langle p, \beta \rangle \langle p, \alpha \rangle} \cdot \frac{\langle \gamma, \delta \rangle}{\langle p, \gamma \rangle \langle p, \delta \rangle} \cdot \frac{\langle p, \gamma \rangle \langle p, \beta \rangle}{\langle \gamma, \beta \rangle} \cdot \frac{\langle p, \alpha \rangle \langle p, \delta \rangle}{\langle \alpha, \delta \rangle}.
\]

Then the inequalities (41) and (40) yields

\[
\left( \frac{M_2}{M_1} \right)^2 \leq \frac{\langle \alpha, \beta \rangle \langle \gamma, \delta \rangle}{\langle \alpha, \delta \rangle \langle \gamma, \beta \rangle} \leq \left( \frac{M_1}{M_2} \right)^2.
\]

Thus the result is proved with \( B = \left( \frac{M_1}{M_2} \right)^2 \). \( \square \)

**Proposition 7.19** The map \( \Phi \) is positive.

**Proof** From Proposition 7.18, for any choice of \( \Phi, \Phi \) maps transverse quadruples to transverse quadruples.

Let \( H_0 \) be the totally geodesic hyperbolic plane in \( H^{2,n} \) tangent to \( \Sigma \) at our preferred point \( x_0 \). Recall that the positive curve \( \partial_\Sigma \Sigma \) is a graph over any \( \partial_\infty H_0 \) and let \( p \) be the warped projection \( \partial_\Sigma \Sigma \to \partial_\infty H_0 \)—see Proposition 2.23. Observe that a quadruple \((x, y, z, t)\) in \( \partial_\infty \Sigma \) is positive if and only if its projection \((p(x), p(y), p(z), p(t))\) is cyclically oriented in \( \partial_\infty H_0 \).

Recall that \( \Phi(x) \) is any point in the connected subset \( B_x \) defined in the beginning of Paragraph 7.2.2. The set \( p(B_x) \) is an interval \( I_x \). By injectivity of \( \Phi \) - for any choice of \( \Phi \)—the intervals \( I_x \) and \( I_y \) are disjoint of \( x \) is different from \( y \).

Let \((a, b, c, d)\) be a positive quadruple in \( \partial_\infty H^2 \). To prove that

\[
(p(\Phi(a)), p(\Phi(b)), p(\Phi(c)), p(\Phi(d)))
\]

is cyclically oriented—and hence that \((\Phi(a), \Phi(b), \Phi(c), \Phi(d))\) is positive—it is then enough to show that there exist points \( z(a), z(b), z(c) \) and \( z(d) \) in \( B_a, B_b, B_c \) and \( B_d \) respectively so that \((p(z(a)), p(z(b)), p(z(c)), p(z(d)))\) is cyclically oriented. This is what we are going to prove now.
Let \( \alpha, \beta, \gamma \) and \( \delta \) be the four geodesics—with respect to the hyperbolic distance—in \( \mathbb{H}^2 \) joining \( x_0 \) to \( a, b, c \) and \( d \) respectively. These arcs are pairwise disjoint (except at the origin) and cyclically ordered. The projection on \( H_0 \) of these four arcs are four non intersecting embedded arcs which are cyclically ordered. Let now take \( \alpha_k, \beta_k, \gamma_k \) and \( \delta_k \) be points in \( \alpha, \beta, \gamma \) and \( \delta \) respectively whose projection in \( H_0 \) has distance to \( x_0 \) exactly \( k \). These four points are cyclically oriented on the circle at distance \( k \) from \( x_0 \). We may extract a subsequence so that \( \{\alpha_k\}_{k \in \mathbb{N}}, \{\beta_k\}_{k \in \mathbb{N}}, \{\gamma_k\}_{k \in \mathbb{N}} \) and \( \{\delta_k\}_{k \in \mathbb{N}} \) also converges to points that we denote \( z(a), z(b), z(c) \) and \( z(d) \) respectively which belong to \( B_a, B_b, B_c \) and \( B_d \) respectively. Their projections \( p(z(a)), p(z(b)), p(z(c)) \) and \( p(z(d)) \) to \( \partial_\infty H_0 \) is then cyclically oriented. This completes the proof that the quadruple \( (\Phi(a), \Phi(b), \Phi(c), \Phi(d)) \) is positive. The proof follows. \( \square \)

7.2.3 Proof of Theorems 7.1 and 7.2

As a consequence of Propositions 7.19 and 7.18, \( \Phi \) is quasisymmetric. Hence \( \Phi \) is Hölder, so continuous, and so uniquely defined.

Both Theorem 7.1—more precisely that \( \Phi \) is a continuous extension—and 7.2 now follows from the following assertion:

Let \( \{z_k\}_{k \in \mathbb{N}} \) a sequence of points in \( \mathbb{H}^2 \) converging to \( z \) in \( \partial_\infty \mathbb{H}^2 \), then \( \{\Psi_k(z_k)\}_{k \in \mathbb{N}} \) converges to \( \Phi(z) \), where \( \{\Psi_k\}_{k \in \mathbb{N}} \) is a sequence of uniformisation as in Theorem 7.2.

Let then \( \{z_k\}_{k \in \mathbb{N}} \) a sequence of points in \( \mathbb{H}^2 \) converging to \( z \) in \( \partial_\infty \mathbb{H}^2 \), so that \( \{\Psi_k(z_k)\}_{k \in \mathbb{N}} \) converges to a point \( u \).

By construction, there exists a sequence \( \{y_k\}_{k \in \mathbb{N}} \) of points in \( \mathbb{H}^2 \), so that \( \{\Psi_k(y_k)\}_{k \in \mathbb{N}} \) converges to \( \Phi(z) \). Using the fact \( \{\Psi_k\}_{k \in \mathbb{N}} \) converges on every compact of \( \mathbb{H}^2 \), we can find a sequence \( n_k \) going to \( \infty \), so that \( \{\Psi_{n_k}(y_k)\}_{k \in \mathbb{N}} \) converges to \( \Phi(z) \). Extracting subsequences and renaming, we have constructed two sequences \( \{z_k\}_{k \in \mathbb{N}} \) and \( \{y_k\}_{k \in \mathbb{N}} \) so that \( \{\Psi_k(z_k)\}_{k \in \mathbb{N}} \) converges to a point \( u \), and \( \{\Psi_k(y_k)\}_{k \in \mathbb{N}} \) converges to \( \Phi(z) \). The assertion now follows from the following lemma:

**Lemma 7.20** Let \( \{z_k\}_{k \in \mathbb{N}} \) and \( \{y_k\}_{k \in \mathbb{N}} \) be sequences of points in \( \mathbb{H}^2 \) converging to the same point \( z \) in \( \partial_\infty \mathbb{H}^2 \). Then \( \{\Psi_k(z_k)\}_{k \in \mathbb{N}} \) and \( \{\Psi_k(y_k)\}_{k \in \mathbb{N}} \) converge to the same point in \( \partial_\infty \Sigma \).

**Proof** We work by contradiction. Then we would have, after extracting subsequences that \( \{\Psi_k(x_k)\}_{k \in \mathbb{N}} \) and \( \{\Psi_k(y_k)\}_{k \in \mathbb{N}} \) converges to different points \( u \) and \( v \) in \( \partial_\infty \Sigma \). In particular the quantity

\[
\left| \frac{(\Psi_k(z_k), \Psi_k(x_0)) (\Psi_k(x_0), \Psi_k(y_k))}{(\Psi_k(z_k), \Psi_k(y_k))} \right|
\]

\( \square \)
is bounded above by a positive constant. By Proposition 7.10, this implies that the distance between \( x_0 \) and the geodesic arc between \( y_k \) and \( z_k \) is uniformly bounded from above. This would imply that \( \{y_k\}_{k \in \mathbb{N}} \) and \( \{z_k\}_{k \in \mathbb{N}} \) converges to different points. This is our contradiction and concludes the proof of the lemma. \( \square \)

8 Smooth spacelike curves and asymptotic hyperbolicity

Let us say a complete maximal surface \( \Sigma \) is *asymptotically hyperbolic* if the norm of the second fundamental form of \( \Sigma \) goes uniformly to zero as one approaches infinity, or equivalently if its intrinsic curvature goes to -1. We may also use the terminology *asymptotically totally geodesic surfaces*.

Using the following theorem, we obtain asymptotically geodesic complete maximal surface.

**Theorem 8.1** (Asymptotically Hyperbolic) Any \( C^1 \) spacelike positive loop in \( \partial_\infty H^{2,n} \) is quasiperiodic and bounds a complete maximal asymptotically hyperbolic surface.

Here is a corollary

**Corollary 8.2** (Smooth rigidity) Let \( \rho \) be a maximal representation of a closed surface group in \( G \). The limit curve of \( \rho \) is \( C^1 \) and spacelike if and only if \( \rho \) factors through a subgroup isomorphic to \( (O(2, 1) \times O(n))_0 \).

**Proof** The periodicity of the limit curves implies that the associated maximal surface \( \Sigma \) is periodic. If the limit curve is \( C^1 \), \( \Sigma \) is asymptotically totally geodesic, thus it is totally geodesic by cocompactness. The representation then preserves a totally geodesic hyperbolic plane, the stabiliser of which is isomorphic to \( (O(2, 1) \times O(n))_0 \). The converse is obvious. \( \square \)

One should compare with the smooth rigidity obtained by [20] in the case of AdS-quasi-Fuchsian groups (which are in general not surface groups). Glorieux–Monclair also have announced a similar result for \( H^{p,q} \)-convex cocompact representations that are deformations of cocompact lattices in \( SO(p, 1) \).

Remark that limit curves of Hitchin representations in \( SO(2, 3) \) have \( C^1 \) limit curves that are lightlike everywhere. In particular, the \( C^1 \) rigidity fails without the spacelike condition for \( n \geq 2 \).

8.0.1. Smooth curves and quasisymmetry Let us start with a proposition of independent interest

**Proposition 8.3** If \( \xi \) is a \( C^1 \) spacelike positive map from \( P(V) \) to \( \partial_\infty H^{2,n} \), then it is quasisymmetric. Moreover if \( \{(x^1_k, x^2_k, x^3_k, x^4_k)\}_{k \in \mathbb{N}} \) is a sequence of
positive quadruples in \( \partial_\infty \mathbb{H}^{2,n} \), so that there exists a point \( x_0 \) so that for any \( i \) in \( \{1, 2, 3, 4\} \), \( \{x^i_k\}_{k \in \mathbb{N}} \) converges to \( x_0 \), then

\[
\lim_{k \to \infty} \left( \frac{b^\xi(x^i_k, x^2_k, x^3_k, x^4_k)}{[x^1_k, x^2_k, x^3_k, x^4_k]^2} \right) = 1.
\]

**Proof.** Let us first show that the map

\[
\Phi : (x, y, z, t) \mapsto \frac{|b^\xi(x, y, z, t)|}{[x, y, z, t]^2}
\]

defined for a quadruple of pairwise distinct points in \( \mathbb{P}(V) \) extends continuously to \( (\mathbb{P}(V))^4 \) to a nowhere vanishing function. Let \( f : V \setminus \{0\} \to E \setminus \{0\} \) be a smooth \( \mathbb{R}^* \) equivariant lift of \( \xi \). Consider a smooth path \( (x(t))_{t \in (-\varepsilon, \varepsilon)} \) in \( V \) with \( x(t) = x_0 + t \dot{x}_0 \) where \( \omega(x_0, \dot{x}_0) = 1 \) so that

\[
\omega(x(t), x(s)) = s - t.
\]

Write \( u(t) = f(x(t)) \) and \( u(0) = u_0 \). Then

\[
u(t) - u(s) = \int_s^t \dot{u}(x)dx = (t - s)\dot{u}(s) + o(t - s).
\]

Thus

\[
\lim_{(s, t) \to 0, s \neq t} \left( \frac{\langle u(t), u(s) \rangle}{\omega(x(t), x(s))^2} \right) = \lim_{(s, t) \to 0, s \neq t} \left( \frac{q(u(t) - u(s))}{2\omega(x(t), x(s))^2} \right)
\]

\[
= \lim_{s \to 0} \frac{1}{2} q(\dot{u}_s) = \frac{1}{2} q(\dot{u}_0)
\]

where we used in the last equality that \( \dot{u} \) is continuous. Since \( f \) is a spacelike immersion, \( \dot{u}(t) \) is a non-zero spacelike vector. Thus the function \( \psi(x, y) = \langle f(x), f(y) \rangle (\omega(x, y))^{-2} \) defined for \( x \neq y \) in \( V \setminus \{0\} \) extends continuously when \( x = y \) to a nowhere vanishing function. Thus the \( (\mathbb{R}^*)^4 \)-invariant function on \( (V \setminus \{0\})^4 \)

\[
\frac{\psi(x, y)\psi(z, t)}{\psi(x, t)\psi(z, y)}
\]

pushes down to a nowhere vanishing continuous function on \( (\mathbb{P}(V))^4 \) extending \( \Phi \).
It follows by compactness of \( P(V)^4 \) that \( \Phi \) is bounded below and above by positive constants: we have \( A \) so that for all quadruples \((x, y, z, t)\)

\[
\frac{1}{A} \leq \frac{|b_\xi(x, y, z, t)|}{[x, y, z, t]^2} \leq A.
\]

Hence \( \xi \) is quasisymmetric. This completes the proof of the first part.

For the second statement, let us use the same notation. Then

\[
\left( \frac{b_\xi(x_k^1, x_k^2, x_k^3, x_k^4)}{[x_k^1, x_k^2, x_k^3, x_k^4]^2} \right) = \frac{\psi(x_0^1, x_0^2)\psi(x_0^3, x_0^4)}{\psi(x_0^1, x_0^4)\psi(x_0^3, x_0^2)}.
\]

Thus by continuity of \( \psi \),

\[
\lim_{k \to \infty} \left( \frac{b_\xi(x_k^1, x_k^2, x_k^3, x_k^4)}{[x_k^1, x_k^2, x_k^3, x_k^4]^2} \right) = \frac{\psi(x_0^1, x_0^2)\psi(x_0^3, x_0^4)}{\psi(x_0^1, x_0^4)\psi(x_0^3, x_0^2)} = 1.
\]

\[\Box\]

8.0.2. Collapsing triples Let us say a sequence \( \{\kappa_k\}_{k \in \mathbb{N}} \) of positive triples in \( P(V) \) is collapsing if \( \{\kappa_k\}_{k \in \mathbb{N}} \) converges to a point.

**Proposition 8.4** Let \( \xi \) be a \( C^1 \) spacelike positive map from \( P(V) \) to \( \partial_\infty \mathbb{H}^{2,n} \), and let \( \{\kappa_k\}_{k \in \mathbb{N}} \) be a sequence of collapsing triples in \( P(V) \). Let \( \tau_k = \xi(\kappa_k) \) and

(i) \( g_k \) an element in \( G \) so that \( g_k(\tau_k) = \tau_0 \)

(ii) \( h_k \) an element in \( \text{SL}(V) \) so that \( h_k(\kappa_0) = \kappa_k \).

(iii) Let finally \( \xi_k = g_k \circ \xi \circ h_k \).

Then the sequence of quasisymmetric maps \( \{\xi_k\}_{k \in \mathbb{N}} \) subconverges to a circle map \( \xi_\infty \).

**Proof** Since \( \xi \) is \( C^1 \) and spacelike, it is quasisymmetric by Proposition 8.3. It follows by the Equicontinuity Theorem 3.12, that we may extract a converging subsequence from \( \{\xi_k\}_{k \in \mathbb{N}} \) converging to a limit that we denote \( \xi_\infty \).

The sequence \( \{h_k\}_{k \in \mathbb{N}} \) goes to infinity in \( \text{PSL}(V) \) and thus using the Cartan decomposition and may assume after extracting a subsequence that it is proximal: there exists a point \( y_0 \), so that for any \( z \) different from \( y_0 \), the sequence \( \{h_k(y_0)\}_{k \in \mathbb{N}} \) converges to some \( x_0 \).

In particular, for any quadruple \((x, y, z, t)\) of pairwise distinct points different from \( y_0 \)

\[
\frac{b_{\xi_\infty}(x, y, z, t)}{[x, y, z, t]^2} = \lim_{k \to \infty} \left( \frac{b_{\xi_k}(x, y, z, t)}{[x, y, z, t]^2} \right).
\]
where in the last equation we used the second assertion of Proposition 8.3, and
in the others the invariance of the cross-ratio by $\text{SL}(V)$ and $G$.

Using the continuity of $\xi_\infty$, we deduce that $\xi_\infty$ is a positive map so that
\[
b_{\xi_\infty}(x, y, z, t) = [x, y, z, t]^2
\]
hence that $\xi_\infty$ is a circle map by Proposition 3.4. \hfill \Box

\subsection*{8.1 Proof of Theorem 8.1}

Consider $\Lambda$ be a $C^1$ spacelike positive loop in $\partial_\infty \mathbb{H}^{2,n}$, $\xi$ a $C^1$ parametrisation of
$\Lambda$ and $\Sigma(\Lambda)$ the solution to the asymptotic Plateau problem. Given a sequence
$\{x_k\}_{k \in \mathbb{N}}$ of points in $\Sigma$ converging to $y_0$ in $\Lambda$, Proposition 4.12 implies that we
can find a sequence $\{\tau_k\}_{k \in \mathbb{N}}$ of positive triples in $\Lambda$ so that $B(\Lambda, \tau_k) = (x_k, \Sigma)$,
where $B$ is the barycenter map. By Corollary 4.14, we can moreover choose
$\{\tau_k\}_{k \in \mathbb{N}}$ to be collapsing to $y_0$.

By Proposition 8.4, we can find a sequence $\{g_k\}_{k \in \mathbb{N}}$ in $G$ such that
$\{g_k(\Lambda, \tau_k)\}_{k \in \mathbb{N}}$ converges to $(\Lambda_0, \tau_0)$, where $\Lambda_0$ is the circle through $\tau_0$. By
continuity of $B$, the sequence $\{g_k(x_0, \Sigma)\}$ converges to the pointed hyperbolic
plane $(x_0, \Sigma_0) = B(\tau_0, \Lambda_0)$. As a result, the curvature of $\Sigma$ goes uniformly
to $-1$ on any compact containing $x_k$.

\section*{9 Relationship with the universal Teichmüller space}

In this final section, we construct in the first paragraph the analogue in our
setting of the Bers’ universal Teichmüller space $T(\mathbb{H}^2)$, [7]. In the second
paragraph, we describe a natural map from this space to $T(\mathbb{H}^2)$ that generalises the picture for closed surfaces proved in [12]. Finally, in the last two paragraphs, we propose an analogue of the Hitchin map, and state a conjecture related to works of Qiongling Li and Takuro Mochizuki [33].

\subsection*{9.1 An analogue of the universal Teichmüller space}

Let $QS_n$ the space of quasisymmetric maps from $P(\mathbb{R}^2)$ to $\partial_\infty \mathbb{H}^{2,n}$. The group
$\text{PSL}_2(\mathbb{R})$ acts by precomposition on $QS_n$ while $G$ acts by postcomposition.
We denote $QS_n := QS_n/G$ and observe now that $\text{PSL}_2(\mathbb{R})$ acts on $QS_n$ by precomposition. We equip $QS_n$ with the $C^0$ topology and $QS_n$ with the quotient topology.
Let also $T(H^2)$ be the group of quasisymmetric homeomorphisms of $P(\mathbb{R}^2)$ preserving $(0, 1, \infty)$ and observe $T(H^2)$ acts by precomposition on $QS_n/G$. In particular that $QS_0$ is a principal $T(H^2)$ space, in other words a torsor over $T(H^2)$, which is classically called the universal Teichmüller space as introduced by Bers in [7].

We refer to [17] for a survey on the universal Teichmüller space and [32] for further references, in particular for the complex Banach manifold structure—which is finer than the topology that we just introduced.

Our proposal is to consider $QS_n$ as an analogue of the universal Teichmüller space in our setting.

Let us first explain in which sense one wants to consider $QS_n$ as a universal (higher) Teichmüller space and first recall the $n = 0$ case.

Let $S$ be closed surface of hyperbolic type and let $\Gamma$ be its fundamental group. The choice of a point $x$ in the Teichmüller space $T(S)$ of $S$ and a triple $\tau_0$ in $\partial_\infty \Gamma$, defines an identification $f_x$ of $\partial_\infty$ with $P(R^2)$ with $f_x(\tau_0) = (0, 1, \infty)$ extending a holomorphic map $\phi_x$ from the universal cover of $S$—equipped with the complex structure coming from $x$—to $H^2$. Thus we obtain a map

$$\Phi_{\tau_0} : \left\{ \begin{array}{c} T(S) \times T(S) \to T(H^2) \\ (x, y) \mapsto f_y \circ f_x^{-1}. \end{array} \right.$$ 

Fixing a point $x$ in Teichmüller space and a triple in $\partial_\infty \Gamma$, we obtain the construction of an embedding from $T(S)$ to the Bers universal Teichmüller space $T(H^2)$.

More generally if $Q_\rho(S)$ denotes the vector bundle over the Teichmüller space $T(S)$ whose fiber over a point $x$ is $H^0((S, x), K^p_x)$, and $H^0_b(H^2, K^p)$ the vector space of bounded $p$-holomorphic differentials on $H^2$, we obtain a map

$$\Phi_{\tau_0}^\rho : \left\{ \begin{array}{c} T(S) \times Q_\rho(S) \to T(H^2) \times H^0_b(H^2, K^p) \\ (x, (y, q)) \mapsto (f_y \circ f_x^{-1}, (\phi_y^\ast)(q)). \end{array} \right.$$ 

Recalling that $T(H^2) = QS_0$, the picture describing $\Phi_{\tau_0}$ generalizes for any $n$. By [10] an element $\rho$ in $\text{Rep}_{\text{max}}(\Gamma, G)$ gives rise to a positive map $F_\rho$ from $\partial_\infty \Gamma$ to $\partial_\infty H^{2,n}$, which is unique up to postcomposing by an element in $G$. So we get a map

$$\Psi_{\tau_0} : \left\{ \begin{array}{c} T(S) \times \text{Rep}_{\text{max}}(\Gamma, G) \to QS_n \\ (y, \rho) \mapsto F_\rho \circ f_y^{-1}. \end{array} \right.$$ 

Then the following relation is satisfied:

$$\Psi_{\tau_0}(x, \rho) \circ \Phi_{\tau_0}^\rho(y, x) = \Psi_{\tau_0}(y, \rho),$$
that reflects the action of $T(\mathbb{H}^2)$ on $QS_n$ described in the beginning of the section. Hence fixing a point in Teichmüller space and a triple in $\partial_\infty \Gamma$, we obtain an embedding of $\text{Rep}^{\text{max}}(\Gamma, \mathbb{G})$ in $QS_n$, thus justifying our proposal of considering $QS_n$ as a universal (higher) Teichmüller space.

9.2 A fibration

The action of $T(\mathbb{H}^2)$ is transitive on $QS_0$. In our setting, we obtain from the action of $T(\mathbb{H}^2)$ a fibration that we describe now. Let $\Lambda_n$ be the space of pointed quasicircles in $\partial_\infty \mathbb{H}^{2,n}$ equipped with the $C^0$ topology. We have a continuous surjective map

$$p : \begin{cases} QS_n \to \Lambda_n \\ \xi \mapsto (\xi(\mathbb{P}(\mathbb{R}^2)), \xi(0, 1, \infty)) \end{cases}.$$ 

Observe that $\Lambda_0/G$ is a point. We have the following:

**Proposition 9.1** The map $p$ from $QS_n$ to $\Lambda_n$ constructed above is $G$-equivariant and a principal $T(\mathbb{H}^2)$-bundle. It carries a natural $G$-equivariant section $\mu$. This section is continuous by restriction on the set of $(A, B)$-quasicircles.

**Proof** By corollary 3.15, the action of $T(\mathbb{H}^2)$ is simply transitive on the fibre of $p$, giving it the structure of a principal $T(\mathbb{H}^2)$-bundle.

By Theorem 7.1, any pointed quasicircle $(\tau, \Lambda)$ comes with a preferred quasisymmetric parametrisation $\mu(\tau, \Lambda)$ that we see as an element of $QS_n$—which is the extension of the uniformisation of the quasiperiodic surface $\Sigma(\Lambda)$ sending $(0, 1, \infty)$ to $\tau$. The $G$-equivariance is direct from the construction. By Theorem 7.2 and Corollary 7.4, the restriction of the section $\mu$ on the set of $(A, B)$-quasicircles is continuous.

It follows from the $G$-equivariance of $p$ and the fact that $G$ acts properly on $L(n)$—thus on $\Lambda_n$—that the continuous action of $G$ on $QS_n$ is also proper. In particular, the quotient $QS_n$ is a Hausdorff topological space. It would be interesting to see whether the fibration $p$ could make sense as a fibration of (conjectural) complex Banach manifolds.

9.3 A map to the universal Teichmüller space

Using the principal $T(\mathbb{H}^2)$ bundle structure on $QS_n$ and the section $\mu$, we can define a map

$$\pi_{H^2} : QS_n \to T(\mathbb{H}^2).$$
by setting $\pi_{H^2}(\xi) = \varphi$, where $\varphi$ is such that $\xi \circ \varphi = \mu(p(\xi))$. For $n = 0$, this map is a homeomorphism. Moreover

**Proposition 9.2** The restriction of the projection $\pi_{H^2}$ to the space of $(A, B)$-quasisymmetric maps is continuous. Moreover the image of the set of $(A, B)$-quasisymmetric maps is included in the space of $(A, D)$-quasisymmetric homeomorphisms where $D$ only depends on $A$ and $B$.

**Proof** For any constants $A$ and $B$ greater than 1, there exist positive constants $C$ and $D$ with the following properties. Given an $(A, B)$-quasisymmetric map $\xi$, its image is an $(A, C)$ quasicircle by Corollary 7.5. Then the reparametrisation $\pi_{H^2}$ is an $(A, D)$-quasisymmetric homeomorphism by Corollary 3.15. Then the result follows by the Equicontinuity Theorem 3.12. \[ \qed \]

This map is an infinite dimensional analogue of the following finite dimensional case. When $S$ is a closed surface of negative Euler characteristic, it was proved in \[12\] for $n \geq 2$ and \[3\] for $n = 1$ that the space $\text{Rep}_{\max}(S, G)$ of maximal representations of $\pi_1(S)$ into $G$ parametrises the space of $\pi_1(S)$-invariant maximal surfaces in $H^{2,n}$. We thus obtain a map

$$
\pi_S : \text{Rep}_{\max}(S, G) \to T(S) \tag{42}
$$

sending a representation to the complex structure of its induced metric. Then we have the commutative diagram

\[
\begin{array}{ccc}
T(S) \times \text{Rep}_{\max}(S, G) & \xrightarrow{\Psi_{\tau_0}} & QS_n \\
\downarrow{(\text{Id},\pi_S)} & & \downarrow{\pi_{H^2}} \\
T(S) \times T(S) & \xrightarrow{\Phi_{\tau_0}} & T(H^2)
\end{array}
\]

### 9.4 A universal Hitchin map

The (finite dimensional) Hitchin map for $G$-Higgs bundles as defined in \[16\], extending the classical picture discovered by Hitchin in \[24\], also has an an infinite dimensional analogue in our setting.

When $S$ is a closed surface of negative Euler characteristic the Hitchin map associates a holomorphic quartic differential $q_4$ in $H^0(X, K_X^4)$ to any maximal representation $\rho : \pi_1(S) \to G$, where $X = \pi_S(\rho)$ (where $\pi_S$ is defined in Eq. (42)). The Hitchin map defines a map

$$
h_S : \text{Rep}_{\max}(S, G) \longrightarrow Q_4(S) \tag{43}
$$

It is proved in \[24\] that $h_S$ is proper and related to integrable systems.
The quartic differential corresponding to \( h_S(\rho) \) has an explicit description in terms of pseudo-hyperbolic geometry. If \( \Sigma \) is the maximal surface in \( H^{2,n} \) preserved by \( \rho(\pi_1(S)) \), the second fundamental form of \( \Sigma \) is the real part of a holomorphic section \( \sigma \) of \( K^2 \otimes N^C \Sigma \), where \( N^C \Sigma \) is the complexification of the normal bundle of \( \Sigma \)—see Paragraph 5.1.1. The tensor \( q_4 = g_N(\sigma, \sigma) \) is the holomorphic quartic differential associated to \( h_S(\rho) \). In fact, with the right choice of basis for Ad-invariant homogeneous polynomials, the quartic differential associated to a Higgs bundle \( (E, \Phi) \) equals \( \text{tr}(\Phi^4) \). Given the Higgs bundle description of equivariant maximal surfaces as in [12, Proof of Proposition 3.17], we have that \( q_4 = 4\beta^\dagger \beta \). But [12, Remark 3.18] implies that \( \sigma = 2\beta \) and so \( q_4 = g_N(\sigma, \sigma) \).

In our setting, we can thus associate a holomorphic quartic differential \( q_4 \) to any complete maximal surface \( \Sigma \) in \( H^{2,n} \). Since maximal surfaces have bounded geometry, such a quartic differential is bounded with respect to the induced metric. When \( \Sigma \) is quasiperiodic, by Theorem 6.3, the uniformisation gives a biLipschitz map between \( H^2 \) and \( \Sigma \), so \( q_4 \) is bounded with respect to the hyperbolic metric. Denoting by \( H^0_b(\text{H}^2, K^4) \) the vector space of holomorphic quartic differential on \( \text{H}^2 \) that are bounded with respect to the hyperbolic metric, we finally obtain a map

\[
h_{\text{H}^2} : QS_n \longrightarrow T(\text{H}^2) \times H^0_b(\text{H}^2, K^4).
\]

This map is again related to the map described in equation (43) through the commutative diagram.

When we wrote our first draft of the paper, Qiongling Li informed us she has a proof of the fact that \( h_{\text{H}^2} \) is proper. Her proof relies on our Theorem A, Theorem 7.2 and her recent work with Takuro Mochizuki [33, Proposition 3.12].

### 9.5 The Hitchin–Li–Mochizuki section

The case \( n = 2 \) is of special interest. In this case, the group \( \text{SO}_0(2, 3) \) being \( \mathbb{R} \)-split, Hitchin result [25] guarantees the existence of a Hitchin section \( \sigma_S \) of the Hitchin map \( h_S \) whose image consists of Hitchin representations.
These Hitchin representations being positive, they are associated to equivariant positive—hence quasisymmetric—loops in $\partial_\infty H^{2,n}$ by [10].

More precisely, Hitchin constructed in [25] a section $\sigma_S$ of the map $h_S$ defined in equation (43). His construction, together with the proof of Labourie’s conjecture for $\text{SO}_0(2,3)$ (see [30]) implies that $\sigma_S$ is a diffeomorphism onto its image, which corresponds to the Hitchin component. Finding the maximal surface corresponding to $(x, q_4)$ in $Q_4$ is equivalent to solving the corresponding Toda equations—see [1, Section 2] for more details.

In a recent preprint [33], Qiongling Li and Takuro Mochizuki solve the Toda equations for a given bounded holomorphic quartic differential on $H^2$. In particular, they prove in [33, Theorem 1.8] that the corresponding maximal surface is complete and conformally biLipschitz to the hyperbolic disc. Thus combining their work with Theorem 6.3, we get

**Proposition 9.3** For $n = 2$, the Hitchin map $h_H^2$ defined above admits a section $\sigma_{H^2}$—that we call the Hitchin–Li–Mochizuki section—related to the section $\sigma_S$ through the following commutative diagram.

$$
\begin{array}{ccc}
T(S) \times \text{Rep}^{\text{max}}(S, G) & \xrightarrow{\psi_{\sigma}} & QS_n \\
(Id, \sigma_S) & \downarrow & \sigma_{H^2} \\
T(S) \times Q_4(S) & \xrightarrow{\phi_{\sigma}} & T(H^2) \times H^0_b(H^2, K^4)
\end{array}
$$

We finish with the following conjecture

**Conjecture 9.4** The image of the Hitchin–Li–Mochizuki section consists of quasisymmetric maps whose image is $C^1$ and lightlike everywhere.

**Appendix A. Hermitian bundles and Bochner formula**

In this appendix we recall the basic definitions of Hermitian vector bundles over a Riemann surface and prove the classical Bochner’s formula and refer to [36, Chapter III] for more details.

**A.1 Hermitian vector bundles**

Consider a Riemann surface $X$. A holomorphic vector bundle over $X$ is a is complex vector bundle $E$ over $X$ equipped with a Dolbeaut operator,

$$\overline{\partial}_E : \Omega^0(X, E) \longrightarrow \Omega^{0,1}(X, E),$$

satisfying the Leibniz rule $\overline{\partial}_E(f \sigma) = \overline{\partial} f \otimes \sigma + f \cdot \overline{\partial}_E \sigma$ where $f$ is a function and $\sigma$ a section of $E$. We say that $\sigma$ is a holomorphic section if $\overline{\partial}_E \sigma = 0$. 
Observe that, if $\nabla$ is a connection on a complex vector bundle, then $\nabla^{0,1}$ (the $(0, 1)$-part of $\nabla$) is a Dolbeaut operator. In this case, a holomorphic section is a section $\sigma$ such that for any vector field $x$ we have

$$\nabla_j x \sigma = J \nabla_x \sigma,$$

where $j$ and $J$ are respectively the complex structure on $TX$ and $E$.

A Hermitian metric $h$ on $E$ is a section of the bundle $E^* \otimes \overline{E}^*$ (where $\overline{E}^*$ is the bundle of $\mathbb{C}$-antilinear form on $E$) which is positive definite and satisfies $h(\sigma_1, \sigma_2) = \overline{h(\sigma_2, \sigma_1)}$.

A connection $\nabla$ on $(E, h)$ is unitary if

$$dh(\sigma_1, \sigma_2) = h(\nabla\sigma_1, \sigma_2) + h(\sigma_1, \nabla\sigma_2).$$

If $(E, h)$ is a holomorphic vector bundle equipped with a Hermitian metric, there is a unique unitary connection $\nabla$ such that $\nabla^{0,1} = \overline{\partial}_E$. The connection $\nabla$ is called the Chern connection.

Finally, if $\nabla$ is a unitary connection, its curvature $R$ is the tensor in $\Omega^2(X, \text{End}(E))$ defined by

$$R(x, y)\sigma = \nabla_x \nabla_y \sigma - \nabla_y \nabla_x \sigma - \nabla_{[x, y]} \sigma.$$

Observe that $R(x, y)$ is skew-hermitian with respect to $h$.

### A.2 Bochner formula

We now prove the Bochner formula used in Sect. 5.1.2.

Let $E$ be a holomorphic vector bundle over a Riemann surface $X$, $h$ a Hermitian metric on $E$. We denote by $\langle ., . \rangle$ the corresponding pairing and by $\nabla$ the Chern connection.

Fix a Riemannian metric $g$ on $X$ and denote by $\Delta$ the corresponding Laplace–Beltrami operator.

**Proposition A.1** (Bochner formula) *Using the same notations as above, if $\sigma$ is a holomorphic section of $E$ and $f := \frac{1}{2} \Vert \sigma \Vert^2$, then*

$$\Delta f = \langle R(e_1, e_2)\sigma, J\sigma \rangle + \Vert \nabla\sigma \Vert^2,$$

*where $R$ is the curvature of $\nabla$, and $(e_1, e_2)$ is an orthonormal framing of $TX$.*

**Proof** Let $u, v$ be sections of $TX$, then $df(u) = h(\nabla_u \sigma, \sigma)$, and

$$\nabla^2 f(u, v) = \langle \nabla_u \nabla_v \sigma, \sigma \rangle + \langle \nabla_v \sigma, \nabla_u \sigma \rangle,$$

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\[ \Delta f = \text{tr}_g(\nabla^2 f) = \sum_{i=1}^{2} \langle \nabla_{e_i} \nabla_{e_i} \sigma, \sigma \rangle + \|\nabla \sigma\|^2. \]

Using Eq. (44) and the fact that \( J \) is \( \nabla \)-parallel, we have

\[ \langle \nabla_{e_1} \nabla_{e_1} \sigma, \sigma \rangle = -\langle \nabla_{e_1} \nabla_{e_2} \sigma, \sigma \rangle = -\langle J \nabla_{e_1} \nabla_{e_2} \sigma, \sigma \rangle = \langle \nabla_{e_1} \nabla_{e_2} \sigma, J \sigma \rangle. \]

For the last equation, we used the \( \langle J \alpha, J \beta \rangle = \langle \alpha, \beta \rangle \). The result follows from a similar computation for \( \langle \nabla_{e_2} \nabla_{e_2} \sigma, \sigma \rangle \).

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