RANDOM INTEGRAL EQUATIONS ON TIME SCALES

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Abstract. In this paper, we present the existence and uniqueness of random solution of a random integral equation of Volterra type on time scales. We also study the asymptotic properties of the unique random solution.

Keywords: random integral equations, time scale, existence, uniqueness, stability.

Mathematics Subject Classification: 34N05, 45D05, 45R99.

1. INTRODUCTION

The random integral equations of Volterra type, as a natural extension of deterministic ones, arise in many applications and have been investigated by many mathematicians. For details, the reader may see the monograph [22, 27], the papers [7, 12, 21, 26] and references therein. For the general theory of integral equations see, the monographs [8, 11] and references therein. In recent years, it initiated the study of integral equations on time scales and obtained some significant results see [1, 16, 19, 25]. The stochastic differential equations on time scales was first studied by Sanyal in his Ph.D. Thesis [24]. For other results about stochastic processes see [23].

The aim of this paper is to obtain the general conditions which ensure the existence and uniqueness of a random solution of a random integral equation of Volterra type on time scales and to investigate the asymptotic behavior of such a random solution. The paper is organized as follows: in Section 2 we set up the appropriate framework on random processes on time scales. We also introduce some functional spaces within which the study of random integral equations can be developed. In Section 3 we present the existence and uniqueness of random solutions. Finally, we establish an asymptotic stability result.
2. PRELIMINARIES

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real number $\mathbb{R}$. Then the time scale $\mathbb{T}$ is a complete metric space with the usual metric on $\mathbb{R}$. Since a time scale $\mathbb{T}$ may or may not be connected, we need the concept of jump operators. The forward (backward) jump operator $\sigma(t)$ at $t \in \mathbb{T}$ for $t < \sup \mathbb{T}$ (respectively $\rho(t)$ for $t > \inf \mathbb{T}$) is given by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ (respectively $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$) for all $t \in \mathbb{T}$. If $\sigma(t) > t$, $t \in \mathbb{T}$, we say $t$ is right scattered. If $\rho(t) < t$, $t \in \mathbb{T}$, we say $t$ is left scattered. If $\sigma(t) = t$, $t \in \mathbb{T}$, we say $t$ is right-dense. If $\rho(t) = t$, $t \in \mathbb{T}$, we say $t$ is left-dense. Also, define the graininess function $\mu : \mathbb{T} \to [0, \infty)$ as $\mu(t) := \sigma(t) - t$. We recall that a function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous function if $f$ is continuous at every right-dense point $t \in \mathbb{T}$, and $\lim_{s \to t^-} f(s)$ exists and is finite at every left-dense point $t \in \mathbb{T}$. We remark that every rd-continuous function is Lebesgue $\Delta$-integrable (see [14]). A rd-continuous function $f : \mathbb{T} \to \mathbb{R}$ is called positively regressive if $1 + \mu(t)f(t) > 0$ for all $t \in \mathbb{T}$. We will denote by $\mathcal{R}^+$ the set of all positively regressive functions. In the following, assume that $\mathbb{T}$ is unbounded. Without lost the generality, assume that $0 \in \mathbb{T}$ and let $\mathbb{T}_0 = [0, \infty) \cap \mathbb{T}$. Also, assume that there exists a strictly increasing sequence $(t_n)_n$ of elements of $\mathbb{T}_0$ such that $t_n \to \infty$ as $n \to \infty$. Denote by $\mathcal{L}$ the $\sigma$-algebra of $\Delta$-measurable subsets of $\mathbb{T}_0$ and by $\lambda$ the Lebesgue $\Delta$-measure of $\mathcal{L}$. Having the measure space $(\mathbb{T}_0, \mathcal{L}, \lambda)$ one can introduce the Lebesgue-Bochner integral for functions from $\mathbb{T}_0$ to a Banach space by simply employing the standard procedure from measure theory (see [3, 18]). The Lebesgue-Bochner integral for functions from $\mathbb{T}_0$ to a Banach space was introduced by Neidhart in [18] and the Henstock-Kurzweil-Pettis integral was introduced by Cichoń in [10]. For details on the construction of the Lebesgue integral for real functions defined on a time scale, see [2, 4, 5, 9, 14, 15]. Further, let $(\Omega, \mathcal{A}, P)$ be a complete probability space. A function $x : \Omega \to \mathbb{R}$ is called a random variable if $\{\omega \in \Omega : x(\Omega) < a\} \in \mathcal{A}$ for all $a \in \mathbb{R}$. Let $1 \leq p < \infty$. A random variable $x : \Omega \to \mathbb{R}$ is said to be $p$-integrable if $\int_{\Omega} |x(\omega)|^p dP(\omega) < \infty$. Let $L^p(\Omega)$ be the space of all $p$-integrable random variables. Then $L^p(\Omega)$ is a vector space and the function $x \mapsto \|x\|_{L^p(\Omega)}$ defined by

$$
\|x\|_{L^p(\Omega)} = \left( \int_{\Omega} |x(\omega)|^p dP(\omega) \right)^{1/p}
$$

is a seminorm on $L^p(\Omega)$. If $x \in L^1(\Omega)$, then

$$
E[x] := \int_{\Omega} x(\omega) dP(\omega)
$$

is called the expected value of random variable $x$. A random variable $x$ is called a $P$-essentially bounded if there exists a $M > 0$ and $A \in \mathcal{A}$ with $P(A) = 0$ such that $|x(\omega)| \leq M$ for all $\omega \in \Omega \setminus A$. Let $L^\infty(\Omega)$ be the space of all $P$-essentially bounded random variables. Then

$$
\|x\|_{L^\infty(\Omega)} = P\text{-ess sup}_{\omega \in \Omega} |x(\omega)|
$$
is a seminorm on \( L^\infty(\Omega) \), where

\[
P\text{-ess sup}_{\omega \in \Omega} |x(\omega)| := \inf \{ M > 0 : |x(\omega)| \leq M \quad P\text{-a.e. } \omega \in \Omega \}.
\]

When a random variable \( x \) is \( p \)-integrable or \( P \)-essentially bounded it is convenient to use notation \( \hat{x} \) to denote the equivalent class of random variables which coincide with \( x \) for \( P \)-a.e. \( \omega \in \Omega \). Let us denote by \( L^p(\Omega) \) the space of all equivalence classes of random variables that are \( p \)-integrable and by \( L^\infty(\Omega) \) the space of all equivalence classes of random variables that are \( P \)-essentially bounded. If \( x \in L^p(\Omega), 1 \leq p \leq \infty \), we denote by \( \hat{x} \) its equivalence class, that is, \( y \in \hat{x} \) if and only if \( y(\omega) = x(\omega) \) for \( P \)-a.e. \( \omega \in \Omega \). Moreover, we have that \( \|y\|_{L^p(\Omega)} = \|x\|_{L^p(\Omega)} \). Thus we can define a norm \( \| \cdot \|_{L^p(\Omega)} \) on \( L^p(\Omega) \) by means of the formula \( \|\hat{x}\|_{L^p(\Omega)} = \|x\|_{L^p(\Omega)}, 1 \leq p \leq \infty \). Then \( L^p(\Omega), 1 \leq p \leq \infty \), is a Banach space with respect to the norm \( \| \cdot \|_{L^p(\Omega)} \).

Since, for \( 1 \leq p \leq \infty \), \( L^p(\Omega) \) is a Banach space, then all elementary properties of the calculus (such as continuity, differentiability, and integrability) for abstract functions defined on a subset of \( \mathbb{T} \) with values into a Banach space remain also true for the functions defined a subset of \( \mathbb{T} \) with values into \( L^p(\Omega) \), \( 1 \leq p \leq \infty \).

Thereby, if \( X : \mathbb{T}_0 \rightarrow L^p(\Omega) \) is strongly measurable then the function \( t \mapsto \|X(t)\|_{L^p(\Omega)} \) is Lebesgue measurable on \( \mathbb{T}_0 \). Also, a strongly measurable function \( X : \mathbb{T}_0 \rightarrow L^p(\Omega) \) is Bochner \( \Delta \)-integrable on \( \mathbb{T}_0 \) if and only if the function \( t \mapsto \|X(t)\|_{L^p(\Omega)} \) is Lebesgue \( \Delta \)- integrable on \( \mathbb{T}_0 \) (see [3]).

Let \( 1 \leq p \leq \infty \). A function \( X : \mathbb{T}_0 \rightarrow L^p(\Omega) \) is called \( rd \)-continuous function if \( X \) is continuous at every right-dense point \( t \in \mathbb{T}_0 \), and \( \lim_{s \rightarrow t-} X(s) \) exists in \( L^p(\Omega) \) at every left-dense point \( t \in \mathbb{T}_0 \).

Of particular importance is the fact that every \( rd \)-continuous function \( X : \mathbb{T}_0 \rightarrow L^p(\Omega) \) is Bochner \( \Delta \)-integrable on \( \mathbb{T}_0 \) (see [3, Theorem 6.3]).

If \( X : \mathbb{T}_0 \rightarrow L^p(\Omega) \) is a strongly measurable function then for each fixed \( t \in \mathbb{T}_0 \), \( X(t) \in L^p(\Omega) \) is an equivalence class. If for each \( t \in \mathbb{T}_0 \) we select a particular function \( x(t, \cdot) \in X(t) \) then we obtain a function \( x(\cdot, \cdot) : \mathbb{T}_0 \times \Omega \rightarrow \mathbb{R} \) such that \( \omega \mapsto x(t, \omega) \) is a random variable for each \( t \in \mathbb{T}_0 \). This resulting function is called a representation of \( X \). In fact, such a representation is so called a random process. However, is not immediate that this representation function is even a \( L \times \mathcal{A} \)-measurable function. In this sense, we have the following result.

**Lemma 2.1.** (a) ([13, Theorem III.11.17]). Let \( (\mathbb{T}_0 \times \Omega, \mathcal{L} \times \mathcal{A}, \lambda \times P) \) be the product space of the measure space \( (\mathbb{T}_0, \mathcal{L}, \lambda) \) and \( (\Omega, \mathcal{A}, P) \). Let \( 1 \leq p \leq \infty \) and let \( X : \mathbb{T}_0 \rightarrow L^p(\Omega) \) be a Bochner \( \Delta \)-integrable function. Then there exists a \( \mathcal{L} \times \mathcal{A} \)-measurable function \( x(\cdot, \cdot) : \mathbb{T}_0 \times \Omega \rightarrow \mathbb{R} \) which is uniquely determined except a set of \( \lambda \times P \)-measure zero, such that \( \hat{x}(t, \cdot) = X(t) \) for \( \lambda \)-a.e. \( t \in \mathbb{T}_0 \). Moreover, \( x(\cdot, \omega) \) is Lebesgue \( \Delta \)-integrable on \( \mathbb{T}_0 \) for \( P \)-a.e. \( \omega \in \Omega \) and integral \( \int_{\mathbb{T}_0} x(t, \omega) \Delta t \), as a function of \( \omega \), is equal to the element \( \int_{\mathbb{T}_0} X(t) \Delta t \) of \( L^p(\Omega) \), that is,

\[
\int_{\mathbb{T}_0} x(t, \cdot) \Delta t = \left( \int_{\mathbb{T}_0} X(t) \Delta t \right)(\cdot).
\]
(b) ([13, Lemma III.11.16]). Let $1 \leq p < \infty$ and let $x(\cdot, \cdot) : \mathbb{T}_0 \times \Omega \to \mathbb{R}$ be a $\mathcal{L} \times \mathcal{A}$-measurable function such that $x(t, \cdot) \in L^p(\Omega)$ for $\lambda$-a.e. $t \in \mathbb{T}_0$. Then the function $X : \mathbb{T}_0 \to L^p(\Omega)$, defined by $X(t) = \hat{x}(t, \cdot)$, is strongly measurable on $\mathbb{T}_0$.

A $\mathcal{L} \times \mathcal{A}$-measurable function $x(\cdot, \cdot) : \mathbb{T}_0 \times \Omega \to \mathbb{R}$ will be called a measurable random process.

**Remark 2.2.** Let $x(\cdot, \cdot) : \mathbb{T}_0 \times \Omega \to \mathbb{R}$ be a measurable random process such that, for each fixed $t \in \mathbb{T}_0$, $x(t, \cdot) \in L^p(\Omega)$. If we denote $\hat{x}(t, \cdot)$ by $X(t)$, then $X(t) : \Omega \to \mathbb{R}$ is a random variable such that $X(t) \in L^p(\Omega)$ and $x(t, \omega) = X(t)(\omega)$ for $P$-a.e. $\omega \in \Omega$.

In the following, using a common abuse of notation in measure theory, we will denote $x(t, \cdot)$ by $X(t)$ for each fixed $t \in \mathbb{T}_0$. In this way, a measurable random process $x(\cdot, \cdot) : \mathbb{T}_0 \times \Omega \to \mathbb{R}$ such that $x(t, \cdot) \in L^p(\Omega)$ for all $t \in \mathbb{T}_0$ can be identified with a strongly measurable function $X : \mathbb{T}_0 \to L^p(\Omega)$.

Let us denote by $C_c = C(\mathbb{T}_0, L^p(\Omega))$ the space of continuous functions $X : \mathbb{T}_0 \to L^p(\Omega)$ with the compact open topology. We recall that if $K$ is a compact subset of $\mathbb{T}_0$ and $U$ is an open subset of $L^p(\Omega)$ and we put

$$S(K, U) = \{X : K \to L^p(\Omega) \mid X(K) \subset U\},$$

then the sets

$$S(K_1, \ldots, K_n; U_1, \ldots, U_n) = \bigcap_{i=1}^n S(K_i, U_i),$$

where $n \in \mathbb{N}$, form a basis for the compact open topology. In fact, this topology coincides with the topology of uniform convergence on any compact subset of $\mathbb{T}_0$. The space $C_c$ is a locally convex space [28] whose topology is defined by means of the following family of seminorms:

$$\|X\|_n = \sup_{t \in K_n} \|X(t)\|_{L^p(\Omega)},$$

where $K_n = [0, t_n] \subset \mathbb{T}_0$, $n \in \mathbb{N}$ and $(t_n)_n$ is a strictly increasing sequence of elements of $\mathbb{T}_0$ such that $t_n \to \infty$ as $n \to \infty$.

A distance function can be defined on $C_c$ by

$$d_c(X, Y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|X - Y\|_{L^p(\Omega)}}{1 + \|X - Y\|_{L^p(\Omega)}}.$$

The topology induced by this distance function is the same topology of uniform convergence on any compact subset of $\mathbb{T}_0$.

Further, consider a continuous function $g : \mathbb{T}_0 \to (0, \infty)$. By $C_g = C_g(\mathbb{T}_0, L^p(\Omega))$ we denote the space of all continuous functions from $\mathbb{T}_0$ into $L^p(\Omega)$ such that

$$\sup_{t \in \mathbb{T}_0} \left\{ \frac{\|X(t)\|_{L^p(\Omega)}}{g(t)} : t \in \mathbb{T}_0 \right\} < \infty.$$
Then
\[\|X\|_{C_g} := \sup_{t \in \mathbb{T}_0} \|X(t)\|_{L^p(\Omega)} \] is a norm of \(C_g\).

Lemma 2.3. \((C_g, \|\cdot\|_{C_g})\) is a Banach space.

Proof. Let \((X_n)\) be a Cauchy sequence in \(C_g\). Then for each \(\varepsilon > 0\) there exists a \(N = N(\varepsilon) > 0\) such that \(\|X_n - X\|_{C_g} < \varepsilon\) for all \(n, m \geq N\). Hence, by (2.1), it follows that
\[\|X_n(t) - X_m(t)\|_{L^p(\Omega)} < \varepsilon g(t), \tag{2.2}\]
for all \(t \in \mathbb{T}_0\) and \(n, m \geq N\). Since \(L^p(\Omega)\) is a complete metric space, it follows that, for any fixed \(t \in \mathbb{T}_0\), \((X_n(t))\) is a convergent sequence in \(L^p(\Omega)\). Therefore, for any fixed \(t \in \mathbb{T}_0\), there exists \(X(t) \in L^p(\Omega)\) such that \(X(t) = \lim_{n \to \infty} X_n(t)\) in \(L^p(\Omega)\). Moreover, it follows from (2.2) that \(X(t) = \lim_{n \to \infty} X_n(t)\) in \(L^p(\Omega)\), uniformly on any compact subset of \(\mathbb{T}_0\). Hence, \(X\) is a continuous function from \(\mathbb{T}_0\) into \(L^p(\Omega)\). Further, we show that \(X \in C_g\). Let us keep \(n\) fixed and take \(m \to \infty\) in (2.2). Then we obtain that \(X_n - X \in C_g\) for all \(n \geq N\). Since \(X = (X - X_n) + X_n\) and \(X - X_n, X_n \in C_g\), it follows that \(X \in C_g\). \(\square\)

Remark 2.4. The topology of \(C_g\) is stronger than the topology of \(C_c\). Indeed, if \(X_n \to X\) in \(C_g\) as \(n \to \infty\), then for each \(\varepsilon > 0\) there exists \(N = N(\varepsilon) > 0\) such that \(\|X_n(t) - X(t)\|_{L^p(\Omega)} < \varepsilon g(t)\), for all \(t \in \mathbb{T}_0\) and \(n \geq N(\varepsilon)\). Since \(g\) is bounded on any compact subset of \(\mathbb{T}_0\), it allows that \(X_n(t) \to X(t)\) as \(n \to \infty\), uniformly on any compact subset of \(\mathbb{T}_0\). In other words, convergence in \(C_g\) implies convergence in \(C_c\). If \(g(t) = 1\) on \(\mathbb{T}_0\), then \(C_g\) becomes the space \(\bar{C} = C(\mathbb{T}_0, L^p(\Omega))\) of all continuous and bounded functions from \(\mathbb{T}_0\) into \(L^p(\Omega)\). The norm on \(\bar{C}\) is given by
\[\|X\|_c = \sup_{t \in \mathbb{T}_0} \|X(t)\|_{L^p(\Omega)}.\]

Note that the following inclusions hold \(C \subset C_g \subset C_c\).

Let \((B, D)\) be a pair of Banach spaces such that \(B, D \subset C_c\) and let \(T\) be a linear operator from \(C_c\) to itself. The pair of Banach spaces \((B, D)\) is called admissible with respect to the operator \(T : C_c \to C_c\) if \(T(B) \subset D\) ([13]).

Remark 2.5. If the pair \((B, D)\) is admissible with respect to the linear operator \(T : C_c \to C_c\) then, by Lemma 2.1.1 from [21], it follows that \(T\) is a continuous operator from \(B\) to \(D\). Therefore, there exists a \(M > 0\) such that
\[\|TX\|_D \leq M \|X\|_B, \quad X \in B.\]
3. RANDOM INTEGRAL EQUATION OF VOLterra TYPE

In this section we study the existence and uniqueness of a random solution of a random integral equation of Volterra type.

\[ x(t, \omega) = h(t, \omega) + \lambda \int_{t_0}^{t} k(t, s, \omega) f(s, x(s, \omega), \omega) \Delta s, \quad t \in \mathbb{T}_0, \tag{3.1} \]

where \( P \text{-a.e. } \omega \in \Omega, \; x(\cdot, \cdot) : \mathbb{T}_0 \times \Omega \to \mathbb{R} \) is the unknown random process, \( h : \mathbb{T}_0 \times \Omega \to \mathbb{R} \) is a measurable random process, \( f : \mathbb{T}_0 \times \mathbb{R} \times \Omega \to \mathbb{R} \) is a random function, \( k : \Gamma \times \Omega \to \mathbb{R} \) is the random kernel, \( \lambda \in \mathbb{R}^* \), and \( \Gamma := \{(t, s) \in \mathbb{T}_0 \times \mathbb{T}_0 : t_0 \leq s \leq t < \infty \} \).

In what follows, we will use the notations \( X(t) = x(t, \cdot), \; H(t) = h(t, \cdot) \), \( K(t, s) = k(t, s, \cdot) \), \( F(t, X(t)) = f(t, x(t, \cdot), \cdot) \).

Let us consider the following assumptions:

(h1) \( K(t, s) \in L^\infty(\Omega) \) for all \( (t, s) \in \Gamma, \; K(\cdot, \cdot) : \Gamma \to L^\infty(\Omega) \) continuous in its first variable and \( rd \)-continuous in its second variable, there exists \( k_0 > 0 \) and \( \alpha > 0 \) with \( -\alpha \in \mathcal{R}^+ \) such that

\[ \|K(t, s)\|_{L^\infty(\Omega)} \leq k_0 e_{-\alpha}(t, \sigma(s)) \]

for \( (t, s) \in \Gamma \).

(h2) \( f(\cdot, x, \cdot) : \mathbb{T}_0 \times \Omega \to \mathbb{R} \) is a \( \mathcal{L} \times \mathcal{A} \)-measurable function for each \( x \in \mathbb{R} \), and there exist an \( a > 0 \) and a positive random variable \( L : \Omega \to \mathbb{R} \) such that \( P(\{\omega \in \Omega : L(\omega) > a\}) = 0 \) and

\[ |f(t, x, \omega) - f(t, y, \omega)| \leq L(\omega) |x - y| \]

for all \( t \in \mathbb{T}_0 \) and \( x, y \in \mathbb{R} \).

(h3) \( F(t, 0) \in L^p(\Omega) \) for all \( t \in \mathbb{T}_0 \) and there exists \( \beta \in (0, \alpha) \) with \( -\beta \in \mathcal{R}^+ \) such that

\[ r := \sup_{t \in \mathbb{T}_0} \frac{\|F(t, 0)\|_{L^p(\Omega)}}{e_{-\beta}(t, 0)} < \infty. \]

In what follows, consider \( g(t) := e_{-\beta}(t, 0), \; t \in \mathbb{T}_0, \) where \( 0 < \beta < \alpha \). Also, we will use the notation \( C_{\beta} \) instead of \( C_y \).

**Lemma 3.1.** If (h2) and (h3) hold, then

\[ \sup_{t \in \mathbb{T}_0} \frac{\|F(t, X(t))\|_{L^p(\Omega)}}{e_{-\beta}(t, 0)} \leq a \|X\|_{C_{\beta}} + r < \infty \]

(3.2)

for every \( X \in C_{\beta} \), and

\[ \|F(t, X(t)) - F(t, Y(t))\|_{L^p(\Omega)} \leq a \|X(t) - Y(t)\|_{L^p(\Omega)} \]

(3.3)

for all \( t \in \mathbb{T}_0 \) and \( X, Y \in C_{\beta} \).
Proof. If we denote \( \{ \omega \in \Omega : L(\omega) \leq a \} \) by \( \Omega_a \), then from (h2) we have that \( P(\Omega_a) = 1 \). If \( X, Y \in C_\beta \), using the Minkowski’s inequality, (h2) and (h3), we have
\[
\| F(t, X(t)) \|_{L^p(\Omega)} = \| f(t, x(t, \cdot), \cdot) \|_{L^p(\Omega)} \leq \\
\leq \left( \frac{1}{p} \right) \left( \int_{\Omega} |f(t, x(t, \omega), \omega) - f(t, 0, \omega)|^p dP(\omega) \right) + \left( \int_{\Omega} |f(t, 0, \omega)|^p dP(\omega) \right) \leq \\
\leq \left( \frac{1}{p} \right) \left( \int_{\Omega} |L(\omega)|^p |x(t, \omega)|^p dP(\omega) \right) + \| F(t, 0) \|_{L^p(\Omega)} \leq \\
\leq a \| X(t) \|_{L^p(\Omega)} + \| F(t, 0) \|_{L^p(\Omega)}.
\]
Dividing both sides of the last inequality by \( e^{-\beta}(t, 0) > 0 \) and taking the supremum with respect to \( t \in \mathbb{T}_0 \), we obtain (3.2). Also,
\[
\| F(t, X(t)) - F(t, X(t)) \|_{L^p(\Omega)} = \| f(t, x(t, \cdot), \cdot) - f(t, y(t, \cdot), \cdot) \|_{L^p(\Omega)} = \\
= \left( \int_{\Omega} |f(t, x(t, \omega), \omega) - f(t, y(t, \omega), \omega)|^p dP(\omega) \right) \leq \\
\leq \left( \frac{1}{p} \right) \left( \int_{\Omega} |L(\omega)|^p |x(s, \omega) - y(s, \omega)|^p dP(\omega) \right) \leq a \| X(t) - Y(t) \|_{L^p(\Omega)} .
\]

Remark 3.2. It follows from Lemma 3.1 that \( F(t, X(t)) \in L^p(\Omega) \) for all \( t \in \mathbb{T}_0 \) and \( X \in C_\beta \). Moreover, (3.2) implies that the function \( t \mapsto F(t, X(t)) \) belong to \( C_\beta \) for all \( X \in C_\beta \).

Lemma 3.3. Let us consider the integral operator \( \mathcal{T} : C_c \rightarrow C_c \) defined by
\[
(\mathcal{T}X)(t) = \int_0^t K(t, s)X(s)\Delta s, \quad t \in \mathbb{T}_0. \tag{3.4}
\]
If (h1) holds, then \( \mathcal{T}(C_\beta) \subset C_\beta \).

Proof. Let \( X \in C_\beta \). We have that
\[
\| (\mathcal{T}X)(t) \|_{L^p(\Omega)} \leq \int_0^t \| K(t, s)X(s) \|_{L^p(\Omega)} \Delta s \leq \int_0^t \| K(t, s) \|_{L^\infty(\Omega)} \| X(s) \|_{L^p(\Omega)} \Delta s = \\
= \int_0^t \| K(t, s) \|_{L^\infty(\Omega)} \frac{\| X(s) \|_{L^p(\Omega)}}{e^{-\beta}(s, 0)} e^{-\beta}(s, 0) \Delta s \leq \\
\leq \| X \|_{C_\beta} \int_0^t \| K(t, s) \|_{L^\infty(\Omega)} e^{-\beta}(s, 0) \Delta s.
\]
Take into account (h1), we infer that

$$\int_0^t \|K(t,s)\|_{L^\infty(\Omega)} e^{-\beta(s,0)} \Delta s \leq k_0 \int_0^t e^{-\alpha(t,\sigma)} e^{-\beta(s,0)} \Delta s = \frac{k_0}{\alpha - \beta} [e^{-\beta(t,0)} - e^{-\alpha(t,0)}].$$

Since $-\alpha, -\beta \in \mathbb{R}^+$ and $-\alpha < -\beta$, then (see [6, Corollary 2.10]) we have that $e^{-\beta(t,0)} > e^{-\alpha(t,0)}$, $t \in T_0$, and it follows that

$$\int_0^t \|K(t,s)\|_{L^\infty(\Omega)} e^{-\beta(s,0)} \Delta s \leq \frac{k_0}{\alpha - \beta} e^{-\beta(t,0)}, \quad t \in T_0. \quad (3.5)$$

Consequently,

$$\|(TX)(t)\|_{L^p(\Omega)} \leq \frac{k_0}{\alpha - \beta} \|X\|_{C^\beta} e^{-\beta(t,0)}, \quad t \in T_0,$$

and thus $TX \in C^\beta$ for every $X \in C^\beta$, that is, $T(C^\beta) \subset C^\beta$.  

**Remark 3.4.** Since, by Lemma 3.3, the pair $(C^\beta, C^\beta)$ is admissible with respect to the linear operator $T : C^c \to C^c$ then, by Remark 2.5, it follows that $T$ is a continuous operator from $C^\beta$ to $C^\beta$. Therefore, there exists a $M > 0$ such that

$$\|TX\|_{C^\beta} \leq M \|X\|_{C^\beta}, \quad X \in C^\beta.$$

In fact, it easy to see that $M = \frac{k_0}{\alpha - \beta}$ is the norm of $T$ as a linear operator from $C^\beta$ into $C^\beta$.

A solution $X \in C^\beta$ of the integral equation (3.1) is called *asymptotically exponentially stable* if there exists a $\rho > 0$ and a $\beta > 0$ such that $-\beta \in \mathbb{R}^+$ and

$$\|X(t)\|_{L^p(\Omega)} \leq \rho e^{-\beta(t,0)}, \quad t \in T_0.$$

**Remark 3.5.** The admissibility concept is related to stability in various senses (see [17]). Let $T : C^c \to C^c$ be a linear operator. Roughly speaking we say that the pair of function spaces $B, D \subset C^c$ is admissible with respect to the equation

$$X = H + TX, \quad (3.6)$$

if this equation has its solution in the space $D$, for each $H \in D$. Therefore, if we choose $D = C^\beta$ and if $X \in C^\beta$ is a solution of the equation (3.6), then there exists a $\rho > 0$ such that $\|X\|_{C^\beta} \leq \rho$. Using (2.1) we infer that

$$\|X(t)\|_{L^p(\Omega)} \leq \rho e^{-\beta(t,0)}$$

for all $t \in T_0$, that is, the solution of the equation (3.6) is asymptotically exponentially stable. For several results concerning the admissibility theory for Volterra integral equations see [11].
These preliminaries being completed, we shall state the following result.

**Theorem 3.6.** If the assumptions (h1)–(h3) hold and \( H \in C_\beta \), then the integral equation \((3.1)\) has a unique asymptotically exponentially stable solution, provided that \( |\lambda| aM < 1 \), where \( M > 0 \) is the norm of the operator \( T \).

**Proof.** Let us consider the operator \( \mathcal{V} : C_\beta \rightarrow C_c \) defined by

\[
(\mathcal{V}X)(t) = H(t) + \lambda \int_0^t K(t,s)F(s,X(s))\Delta s, \quad t \in T_0.
\]

(3.7)

Then we can rewrite the operator \( \mathcal{V} \) as

\[
(\mathcal{V}X)(t) = H(t) + \lambda \mathcal{T}G(t), \quad t \in T_0,
\]

(3.8)

where \( G(t) := F(t,X(t)) \), \( t \in T_0 \) and \( \mathcal{T} \) is the operator given by \((3.4)\). Since by Remark 3.2 and Lemma 3.1 we have that

\[
\|G\|_{C_\beta} \leq a \|X\|_{C_\beta} + r,
\]

then

\[
\|\mathcal{T}G(t)\|_{L^p(\Omega)} \leq bMe^{-\beta(t,0)}, \quad t \in T_0,
\]

(3.9)

where \( b := a \|X\|_{C_\beta} + r \). From \((3.8)\) and \((3.9)\) we obtain that

\[
\|(\mathcal{V}X)(t)\|_{L^p(\Omega)} \leq \|H(t)\|_{L^p(\Omega)} + b|\lambda|M e^{-\beta(t,0)},
\]

for all \( t \in T_0 \). Dividing both sides of the last inequality by \( e^{-\beta(t,0)} > 0 \) and taking the supremum with respect to \( t \in T_0 \), it follows that

\[
\|\mathcal{V}X\|_{C_\beta} \leq \|H\|_{C_\beta} + b|\lambda|M,
\]

(3.10)

and so \( \mathcal{V}X \in C_\beta \) for all \( X \in C_\beta \). Further, we show that the operator \( \mathcal{V} \) is a contraction on \( C_\beta \). Indeed, using \((3.3)\) and \((3.5)\), we have

\[
\|(\mathcal{V}X)(t) - (\mathcal{V}Y)(t)\|_{L^p(\Omega)} \leq |\lambda| \int_0^t \|K(t,s)[F(s,X(s)) - F(s,Y(s))]\|_{L^p(\Omega)} \Delta s \leq
\]

\[
\leq |\lambda| \int_0^t \|K(t,s)\|_{L^\infty(\Omega)} \|F(s,X(s)) - F(s,Y(s))\|_{L^p(\Omega)} \Delta s \leq
\]

\[
\leq a|\lambda| \int_0^t \|K(t,s)\|_{L^\infty(\Omega)} \frac{\|X(s)-Y(s)\|_{L^p(\Omega)}}{e^{-\beta(s,0)}} e^{-\beta(s,0)} \Delta s \leq
\]

\[
\leq a|\lambda| \|X - Y\|_{C_\beta} \int_0^t \|K(t,s)\|_{L^\infty(\Omega)} e^{-\beta(s,0)} \Delta s \leq
\]

\[
\leq \frac{a|\lambda|k_0}{\alpha - \beta} \|X - Y\|_{C_\beta} e^{-\beta(t,0)} =
\]

\[
= a|\lambda| M \|X - Y\|_{C_\beta} e^{-\beta(t,0)}.
\]
Thus
\[
\| (\mathcal{V}X)(t) - (\mathcal{V}Y)(t) \|_{L^p(\Omega)} \leq a |\lambda| M \|X - Y\|_{C_\beta}
\]
for all \( t \in T_0 \), and so
\[
\| \mathcal{V}X - \mathcal{V}Y \|_{C_\beta} \leq a |\lambda| M \|X - Y\|_{C_\beta},
\]
with \( a |\lambda| M < 1 \), that is, \( \mathcal{V} \) is a contraction on \( C_\beta \). From Banach’s Fixed Point Theorem, it follows that there exist a unique solution \( X \in C_\beta \) of the integral equation (3.1). From Remark 3.5, we infer that the solution is asymptotically exponentially stable.

**Corollary 3.7.** If all the hypotheses of Theorem 3.6 hold for \( \beta = 0 \), then the integral equation (3.1) has a unique solution \( X \in C \).

**Corollary 3.8.** If all the hypotheses of Theorem 3.6, then the solution of the integral equation (3.1) is asymptotically stable in mean, that is, \( E[|X(t)|] \to 0 \) as \( t \to \infty \).

**Proof.** Since \( -\beta < 0 \), then \( e^{-\beta t} \) decreases monotonically towards zero as \( t \to \infty \), and therefore \( \|X(t)\|_{L^p(\Omega)} \to 0 \) as \( t \to \infty \). Since \( E[|X(t)|^p] = \|X(t)\|^p_{L^p(\Omega)} \) then, using the Jensen’s inequality, we infer that \( E[|X(t)|] \to 0 \) as \( t \to \infty \). □

**Remark 3.9.** Let \( T_0 = [0, \infty) \). Then, for \( g(t) = q(t) = e^{-\beta t} \), \( t \geq 0 \), we obtain Theorem 2.2 from [7]. For \( p = 2 \) and \( f(t, x, \omega) = f(t, x) \), we obtain Theorem 3.1 from [26]. Let \( T_0 = \mathbb{N} \). Then, for \( p = 2 \) and \( f(t, x, \omega) = f(t, x) \), we obtain Theorem 5.3.1 from [27].

In what follows, using the concept of admissibility, we prove a general result of the existence and uniqueness for the integral equation (3.1). From this result it is possible to derive many existence results, by particularizing the spaces \( B \) and \( D \).

Let us consider the integral equation (3.1) under the following conditions:

(\( \hat{h}1 \)) \( K(t, s) \in L^\infty(\Omega) \) for all \( (t, s) \in \Gamma, K(\cdot, \cdot) : \Gamma \to L^\infty(\Omega) \) continuous in its first variable and rd-continuous in its second variable.

(\( \hat{h}2 \)) \( B, D \subset C_c \) are Banach spaces stronger than \( C_c \) such that the pair \( (B, D) \) is admissible with respect to the linear operator \( T : C_c \to C_c \) defined by (3.4).

(\( \hat{h}3 \)) For each \( X \in D \), the function \( t \mapsto F(t, X(t)) \) belong to \( B \), and the operator \( \mathcal{G} : D \to B \), defined by \( (\mathcal{G}X)(t) = F(t, X(t)) \) for all \( t \in T_0 \), satisfies the Lipschitz condition
\[
\| \mathcal{G}X - \mathcal{G}Y \|_B \leq a \|X - Y\|_D
\]
for all \( X, Y \in D \) and some \( a > 0 \).

**Theorem 3.10.** If the assumptions (\( \hat{h}1 \))–(\( \hat{h}3 \)) hold and \( H \in D \), then the integral equation (3.1) has a unique solution \( X \in D \), provided that \( |\lambda| aM < 1 \), where \( M > 0 \) is the norm of the operator \( T \).
Proof. Let us consider the operator $V : D \rightarrow C_c$ defined by $VX = H + \lambda TGX$. Since the pair $(B, D)$ is admissible with respect to the linear operator $T$, it follows from Remark 2.5 that there exists a $M > 0$ such that $\|TX\|_D \leq M \|X\|_B$ for all $X \in B$. Using (h3) and the fact that $H \in D$ it follows from Minkowski’s inequality that

$$\|VX\|_D \leq \|H\|_D + |\lambda| M \|GX - G0\|_B \leq \|H\|_D + a |\lambda| M \|X\|_D + |\lambda| M \|G0\|_B < \infty,$$

that is, $VX \in D$ for all $X \in D$. Next, all $X, Y \in D$ we have that $VX - VY = \lambda T(GX - GY)$. Obviously, $GX - GY \in B$ and $VX - VY \in D$. It follows that

$$\|VX - VY\|_D \leq |\lambda| M \|GX - GY\|_B \leq |\lambda| aM \|X - Y\|_D,$$

with $|\lambda| aM < 1$, that is, $V$ is a contraction on $D$. From Banach’s Fixed Point Theorem, it follows that there exist a unique solution $X \in D$ of the integral equation (3.1).

\[\square\]

Remark 3.11. If $T_0 = [0, \infty)$, we obtain Theorem 2.4 from [7]. For $p = 2$ and $f(t, x, \omega) = f(t, x)$, we obtain Theorem 2.1.2 from [27]. If $T_0 = \mathbb{N}$, then, for $p = 2$ and $f(t, x, \omega) = f(t, x)$, we obtain Theorem 5.1.2 from [27].

REFERENCES

[1] M. Adıvar, N.Y. Raffoul, Existence results for periodic solutions of integro-dynamic equations on time scales, Ann. Mat. Pura Appl. 188 (2009) 4, 543–559.

[2] R. Agarwal, M. Bohner, D. O’Regan, A. Peterson, Dynamic equations on time scales: a survey, J. Comput. Appl. Math. 141 (2002), 1–26.

[3] B. Aulbach, L. Neidhart, Integration on measure chains, Proc. of the Sixth International Conference on Difference Equations, B. Aulbach, S. Elaydi, G. Ladas, eds., Augsburg, Germany 2001, pp. 239–252.

[4] M. Bohner, A. Peterson, Dynamic Equations on Time Scales: an Introduction with Applications, Birkhäuser, Boston, 2001.

[5] M. Bohner, A. Peterson, Advances in Dynamic Equations on Time Scales, Birkhäuser, Boston, 2003.

[6] E. Akin-Bohner, M. Bohner, F. Akin, Pachpatte inequalities on time scales, JIPAM. J. Inequal. Pure Appl. Math. 6 (2005) 1, 1–23.

[7] N.U. Ahmed, K.L. Teo, On the stability of a class of nonlinear stochastic systems, J. Information and Control 20 (1972), 276–293.

[8] T.A. Burton, Volterra integral and differential equations, vol. 202 of Mathematics in Science and Engineering, Elsevier B.V., Amsterdam, 2nd ed., 2005.

[9] A. Cabada, D.R. Vivero, Expression of the Lebesgue $\Delta$-integral on time scales as a usual Lebesgue integral; application to the calculus of $\Delta$-antiderivatives, Math. Comput. Modelling 43 (2006), 194–207.
[10] M. Cichoń, *On integrals of vector-valued functions on time scales*, Commun. Math. Anal. **1** (2011) 11, 94–110.

[11] C. Corduneanu, *Integral Equations and Stability of Feedback Systems*, Academic Press, New York-London, 1973.

[12] B.C. Dhage, S.K. Ntouyas, *Existence and attractivity results for nonlinear first order random differential equations*, Opuscula Math. **30** (2010) 4, 411–429.

[13] N. Dunford, J.T. Schwartz, *Linear Operators I*, Interscience, New York, 1958.

[14] G.Sh. Guseinov, *Integration on time scales*, J. Math. Anal. Appl. **285** (2003), 107–127.

[15] S. Hilger, *Analysis on measure chains – a unified approach to continuous and discrete calculus*, Results Math. **18** (1990), 18–56.

[16] T. Kulik, C.C. Tisdell, *Volterra integral equations on time scales: basic qualitative and quantitative results with applications to initial value problems on unbounded domains*, Int. J. Difference Equ. **3** (2008) 1, 103–133.

[17] J.L. Massera, J.J. Schäffer, *Linear Differential Equations and Function Spaces*, Academic Press, New York, 1966.

[18] L. Neidhart, *Integration im Rahmen des Maßkettenkalküls*, Diploma Thesis, University of Augsburg, 2001.

[19] A. Sikorska-Nowak, *Integrodifferential equations on time scales with Henstock-Kurzweil-Pettis delta integrals*, Abstr. Appl. Anal. **1** (2010), 1–17.

[20] D.B. Pachpatte, *On a nonstandard Volterra type dynamic integral equation on time scales*, Electron. J. Qual. Theory Differ. Equ. **72** (2009), 1–14.

[21] W.J. Padgett, C.P. Tsokos, *On a stochastic integro-differential equation of Volterra type*, SIAM J. Appl. Math. **23** (1972), 499–512.

[22] A.T. Bharucha-Reid, *Random Integral Equations*, Academic Press, New York, 1972.

[23] S. Sanyal, *Mean square stability of Itô Volterra dynamic equation*, Nonlinear Dyn. Syst. Theory **11** (2011) 1, 83–92.

[24] S. Sanyal, *Stochastic Dynamic Equations*, Ph.D. Thesis, Missouri University of Science and Technology, Rolla, Missouri, 2008.

[25] C.C. Tisdell, A. Zaidi, *Basic qualitative and quantitative results for solutions to nonlinear dynamic equations on time scales with an application to economic modelling*, Nonlinear Anal. **68** (2008) 11, 3504–3524.

[26] C.P. Tsokos, M.A. Hamdan, *Stochastic asymptotic exponential stability of stochastic integral equations*, J. Appl. Prob. **9** (1972), 169–177.

[27] C.P. Tsokos, W.J. Padgett, *Random Integral Equations with Applications to Life Sciences and Engineering*, Academic Press, New York, 1974.

[28] K. Yosida, *Functional Analysis*, 6th ed., Springer-Verlag, 1980.
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