Local correlations in the 1D Bose gas from a scaling limit of the XXZ chain

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Abstract. We consider the $K$-body local correlations in the (repulsive) 1D Bose gas for general $K$, both at finite size and in the thermodynamic limit. Concerning the latter we develop a multiple integral formula which applies for arbitrary states of the system with a smooth distribution of Bethe roots, including the ground state and finite-temperature Gibbs states. In the cases $K \leq 4$ we perform the explicit factorization of the multiple integral. In the case of $K = 3$ we obtain the recent result of Kormos et al, whereas our formula for $K = 4$ is new. Numerical results are presented as well.

Keywords: correlation functions, form factors, quantum integrability (Bethe ansatz)

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1. Introduction

The delta-function interacting 1D Bose gas (also known as the Lieb–Liniger model or the quantum nonlinear Schrödinger equation) is one of the oldest and most important integrable models. Its study goes back to the papers [1,2] where it was shown that the spectrum can be obtained by the Bethe ansatz [3]. The thermodynamical properties of the model were determined in [4] using the method nowadays known as the thermodynamical Bethe ansatz (TBA). After these seminal papers tremendous effort was devoted to the calculation of correlation functions using various approaches [5]–[11]. One of the most important recent results is the exact determination of the long-distance behavior of correlations [12]–[18].

Apart from purely academic interest, the study of the 1D Bose gas was spurred by the recent success of experiments with cold atoms in quasi-one-dimensional
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traps [19]–[25]. A remarkable result was presented in [22], where the authors managed to measure exact predictions of the TBA (for further developments and open questions see [26]). In experimental situations the local correlations are of special interest, for example the three-body local correlation is related to the rate of particle loss [27,28,25] and to the third moment of the density fluctuations [29,30]. Moreover, even the four-body correlations might be accessible to experiment, as was recently demonstrated in a 3D experiment [24].

Concerning the general $K$-body local correlations (for the precise definition see the main text) there has been considerable theoretical progress, too. The $K=1$ case is simply given by the (linear) density of particles, whereas the $K=2$ case was related to the thermodynamical quantities of the model in [31]. Concerning the higher-body cases small-coupling and large-coupling expansions were performed in [31,32], whereas the exact ground state value of the three-body correlation was calculated in [33]. A new approach was initiated in [34,35], where an infinite integral series (also called the LeClair–Mussardo or LM series) was derived using a special non-relativistic limit of the sinh–Gordon model. The LM series applies for any $K$ and arbitrary temperature, including the ground state, and it can be considered as an effective large-coupling expansion of the quantity in question. The papers [36,37] considered the relation between the LM series and previous form factor calculations with the algebraic Bethe ansatz (ABA); in [37] it was shown that the LM series can be understood and proven within the ABA. However, there was one crucial problem: there were no explicit and general results available for the form factors entering the LM series; the numerical results in [34,35] were obtained using a truncation of the full series.

The important task of the exact summation of the LM series was performed for the first time in the recent article [38], where the authors evaluated the three-body correlation based on a well-supported conjecture for the corresponding form factors. To the best of our knowledge this is the first time that an exact, explicit and compact result was given for a non-trivial correlation of the 1D Bose gas, valid at arbitrary couplings and temperatures.

In the present work we contribute to the calculation of the $K$-body correlators using a different approach. Our strategy is the following. First we consider a related physical quantity (the so-called ‘emptiness formation probability’) on a generic XXZ spin chain and show that a special scaling limit of the spin chain [11,39] yields the desired correlations in the Bose gas (section 3). The matrix elements of the operator on the spin chain are calculated in section 4 borrowing results from the works [40,41]. We then perform the scaling limit towards the Bose gas in section 5, this way we obtain the form factors in a finite volume, with a finite number of particles. Finally, the thermodynamic limit is performed in the Bose gas (section 6) leading to the multiple integral (6.7), which is the main result of this work (see (6.10) for the dimensionless form).

In principle the multiple integrals could be evaluated for any $K$, but in practice this becomes more and more difficult with increasing $K$; therefore it is desirable to derive more compact results. In section 7 we show how to factorize the multiple integral in the cases $K \leq 4$. The results are the expressions (7.3), (7.10) and (7.12). In subsection 7.5 we also present examples of the numerical results.

Finally in section 8 we determine all form factors entering a modified form of the LM series, making it an explicit integral series for the $K$-body local correlations.
2. The Lieb–Liniger model

The second quantized form of the Hamiltonian is

\[
H_{\text{LL}} = \int_0^L dx \left( \partial_x \Psi^\dagger \partial_x \Psi + c \Psi^\dagger \Psi \Psi^\dagger \Psi \right). \tag{2.1}
\]

Here \( L \) is the size of the system, periodic boundary conditions are understood, and \( \Psi(x, t) \) and \( \Psi^\dagger(x, t) \) are canonical non-relativistic Bose fields satisfying

\[
[\Psi(x, t), \Psi^\dagger(y, t)] = \delta(x - y). \tag{2.2}
\]

The mass of the particles was set to \( m = 1/2 \), we used the convention \( \hbar = 1 \) and \( c > 0 \) is the coupling constant.

The eigenstates of the Hamiltonian (2.1) can be constructed using the Bethe ansatz [1,2,7]. The \( N \)-particle coordinate space wavefunction is given by

\[
\chi_N(\{p\}|\{x\}) = \frac{1}{\sqrt{N!}} \sum_{p \in S_N} \exp \left\{ i \sum_j x_j (p_j) \right\} \prod_{j > k} \left( \frac{(p_j - (p_k)^2 + ic(x_j - x_k))}{(p_j - (p_k))} \right), \tag{2.3}
\]

where \( p_j \) are the quasi-momenta of the particles and \( \epsilon(x) \) is the sign function.

Periodic boundary conditions force the quasi-momenta to be solutions of the Bethe ansatz equations

\[
e^{ip_jL} \prod_{k \neq j} \frac{p_j - p_k - ic}{p_j - p_k + ic} = 1. \tag{2.4}
\]

The energy and momentum of the multi-particle state are given by

\[
E_N = \sum_j p_j^2 \quad P_N = \sum_j p_j.
\]

The norm of the wavefunction (2.3) is [42,43]

\[
\mathcal{N}_{\text{LL}} = \int |\chi_N|^2 = \prod_{j < k} \frac{(p_j - p_k)^2 + c^2}{(p_j - p_k)^2} \times \det \mathcal{G}_{\text{LL}} \tag{2.5}
\]

with

\[
\mathcal{G}_{\text{LL}}^{jk} = \delta_{j,k} \left( L + \sum_{l=1}^N \varphi(p_j - p_l) \right) - \varphi(p_j - p_k) \tag{2.6}
\]

and

\[
\varphi(u) = \frac{2c}{u^2 + c^2}.
\]

We will be interested in the matrix elements of the operators

\[
\mathcal{O}_K = (\Psi^\dagger(0))^K (\Psi(0))^K.
\]
In coordinate space the matrix elements are given by the integrals
\[
\langle \phi_N | \mathcal{O}_K | \chi_N \rangle = \frac{N!}{K! (N-K)!} \int_0^L dx_1 \cdots dx_{N-K} \times \phi_N^*(0, \ldots, 0, x_1, \ldots, x_{N-K}) \chi_N(0, \ldots, 0, x_1, \ldots, x_{N-K}).
\]  
(2.7)

The expectation value of $\mathcal{O}_K$ describes the probability to have $K$ particles at the same point. It is useful to introduce the dimensionless quantities
\[
g_K = \frac{\langle \mathcal{O}_K \rangle}{n^K},
\]
where $n = N/L$ is the particle density. It can be shown by scaling arguments that in the thermodynamic limit $g_K$ only depends on the dimensionless parameters
\[
\gamma = \frac{c}{n} \quad \tau = \frac{T}{n^2},
\]
where $T$ is the temperature (we used the convention $k_B = 1$ for the Boltzmann constant).

In principle the form factors (2.7) could be obtained by performing the integrals in coordinate space, but this becomes increasingly complicated with growing $N$. Note also that the algebraic Bethe ansatz for the Bose gas does not lead to simple results either: the action of the field operators on Bethe states can be evaluated easily, but afterwards one would have to compute scalar products of Bethe states with a reduced set of $N-K$ particles, neither of which are on-shell, and there is no good formula for the scalar products of such states. One way out of these problems is to consider a related quantity (the ‘emptiness formation probability’) on the XXZ spin chain, where there are methods available to compute its matrix elements.

### 3. The XXZ chain and its special scaling limit

The XXZ spin chain with $M$ sites and periodic boundary conditions is given by the following Hamiltonian:
\[
H = J \sum_{j=1}^M (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta (S_j^z S_{j+1}^z - 1/4)) + h \sum_{j=1}^M S_j^z.
\]  
(3.1)

This model is also solvable by the Bethe ansatz [3], [44]–[46]. The $N$-particle eigenstates are given by
\[
| \phi_N \rangle = \frac{1}{\sqrt{N!}} \sum_{y_1=1}^{L} \cdots \sum_{y_N=N}^{L} \phi_N(\{\lambda\}| y_1, \ldots, y_N) \sigma^{-}_{y_1} \cdots \sigma^{-}_{y_N} | 0 \rangle.
\]  
(3.2)

Here $|0\rangle$ is the reference state with all spins up and $y_j$ are the positions of the down spins. The amplitudes are
\[
\phi_N(\{\lambda\}|y) = \frac{1}{\sqrt{N!}} \sum_{\mathcal{P} \in S_N} \prod_{1 \leq m < n \leq N} \frac{\sinh(\mathcal{P}_m \lambda - \mathcal{P}_n \lambda + \epsilon(y_n - y_m) \eta)}{\sinh(\mathcal{P}_m \lambda - \mathcal{P}_n \lambda)} 
\times \prod_{l=1}^{N} F(\mathcal{P}_{l} \lambda_{l}, y_l),
\]  
(3.3)

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where $\lambda_j$ are the rapidities of the magnons and the amplitude $F(\lambda, y)$ is given by

$$ F(\lambda, y) = \frac{1}{\sinh(\lambda - \xi_y)} \prod_{j=1}^{y-1} \frac{\sinh(\lambda - \xi_j + \eta)}{\sinh(\lambda - \xi_j)}. \quad (3.4) $$

The parameter $\eta$ is related to the anisotropy:

$$ \Delta = \cosh \eta. $$

In (3.4) we introduced inhomogeneities $\xi_j$ for the sites of the spin chain; they will be used as a technical tool to obtain the form factors in section 4. The physical limit consists of setting all $\xi_j \rightarrow \eta/2$. The expression (3.3) is a seemingly over-complicated way to write down the wavefunction, because it is valid at arbitrary values of the variables $y_j$ and not only in the region $y_1 < \cdots < y_M$. We used this form to have an exact agreement with the conventions used in (2.3).

The Bethe equations follow from the periodicity of the wavefunction and they are

$$ d(\lambda_j) \prod_{k \neq j} \frac{\sinh(\lambda_j - \lambda_k + \eta)}{\sinh(\lambda_j - \lambda_k - \eta)} = 1, \quad (3.5) $$

where

$$ d(\lambda) = \prod_{k=1}^{M} \frac{\sinh(\lambda - \xi_k)}{\sinh(\lambda - \xi_k + \eta)}. \quad (3.6) $$

In the normalization (3.2) and (3.3) the norm of the wavefunction is given by

$$ N_{XXZ} = \sum_{y_1} \cdots \sum_{y_N} |\phi(y_1, \ldots, y_N)|^2 $$

$$ = (-\sinh \eta)^{-N} \prod_{j<k} f(\lambda_j, \lambda_k) f(\lambda_k, \lambda_j) \times \det \mathcal{G}_{XXZ} \quad (3.7) $$

with

$$ \mathcal{G}_{XXZ}^{jk} = \delta_{j,k} \left( \frac{d'(\lambda_j)}{d(\lambda_j)} + \sum_{l \neq j} \varphi_{XXZ}(\lambda_j - \lambda_l) \right) - \varphi_{XXZ}(\lambda_j - \lambda_k). \quad (3.8) $$

The kernel $\varphi_{XXZ}$ is given by

$$ \varphi_{XXZ}(u) = \frac{-\sinh \eta}{\sinh(u + \eta/2) \sinh(u - \eta/2)}. \quad (3.9) $$

One-particle momenta and energies are given by the formulæ

$$ e^{ip(\lambda)} = \frac{\sinh(\lambda + \eta/2)}{\sinh(\lambda - \eta/2)} \quad e(\lambda) = J \frac{\sinh^2 \eta}{\cos(2\lambda) - \cosh \eta} - h.$$
3.1. Towards the Lieb–Liniger model

There is a special scaling limit of the XXZ chain which yields the physical quantities of the Lieb–Liniger model [47,39,11]. In order to obtain the Bose gas in a finite volume $L$ one has to set

$$\eta = i\pi - i\varepsilon, \quad M = \frac{c}{\varepsilon^2}L$$

and let $\varepsilon \to 0$ (here $c$ is the coupling constant of the Bose gas). The number $N$ of the magnons has to be kept fixed and the rapidities of the particles have to be scaled as

$$\lambda_j = p_j \frac{\varepsilon}{c}.$$ 

After the limiting procedure the magnons can be identified as the particles of the Bose gas with rapidity $p_j$. It can be shown that under an appropriate scaling of the parameters $J$ and $h$

$$e(\lambda) = J \frac{\sinh^2 \eta}{\cos(2\lambda) - \cosh \eta} - h \to p^2 - \mu,$$

where $\mu$ is the chemical potential in the Bose gas. However, this will not be needed in the following; we will consider the Bethe wavefunctions and the form factors of local operators. In the following we assume that the homogeneous limit $\xi_j \to \eta/2$ is performed first on the spin chain, and the limit towards the Bose gas is taken afterward.

Taking the scaling limit of the Bethe equations (3.5) results in

$$(-1)^M e^{i\nu_j} \prod_{k \neq j} \frac{\nu_j - \nu_k - i\varepsilon}{\nu_j - \nu_k + i\varepsilon} = 1. \quad (3.10)$$

For the sake of simplicity we only consider even chains so that no twist appears in the Bethe equations.

The limiting form of the Bethe wavefunction can be taken by setting $x_j = \frac{\varepsilon^2}{c} y_j$ and keeping $x_j$ finite, which will correspond to the position of the particles of the Bose gas. The wavefunction then is

$$\Psi_N(x|\nu) = \frac{1}{\sqrt{N!}} \sum_{\rho \in \sigma_N} \prod_{m>n} \frac{(P\nu)_m - (P\nu)_n + i\varepsilon(x_m - x_n)c}{(P\nu)_m - (P\nu)_n} \prod_{l=1}^N F((P\nu)_l, x_l), \quad (3.11)$$

where

$$F(\nu, x) = e^{-i\nu x} (-1)^{\nu}. \quad (3.12)$$

Apart from factors of $(-1)$ the above expression is equal to the complex conjugate of the Bethe wavefunction (2.3). It can be argued that the factors of $(-1)^{\nu}$ do not affect the calculation of form factors of local operators. Indeed, for any coordinate space calculation one has to take the product of two wavefunctions with the down spins placed at prescribed positions. Depending on the operator in question an overall factor of $(-1)$ may remain, but the position-dependent factors of $(-1)^{\nu}$ always cancel. For the operators considered in this paper every such factor cancels, therefore they will be neglected in the following.

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Due to the relation between $y$ and $x$ it is expected that the norm of the wavefunction behaves as
\[ N_{\text{XXZ}} \rightarrow \left( \frac{c}{\varepsilon^2} \right)^N N_{\text{LL}}. \]  
Comparing the formulae (2.5) and (3.7) we obtain the same scaling: the Gaudin determinants behave as
\[ \det G_{\text{XXZ}} \rightarrow \left( \frac{i\varepsilon}{\varepsilon} \right)^N \det G_{\text{LL}} \]
and the prefactors contribute an extra $(i\varepsilon)^{-N}$.

### 3.2. The emptiness formation probability

We are interested in the local operators $E_{j}^{\alpha\beta}$ acting on site $j$ with matrix elements
\[ (E_{j}^{\alpha\beta})_{kl} = \delta_{k,\alpha}\delta_{l,\beta}. \]
In particular we consider the composite operator
\[ s_K = E_{-1}^{-}E_{-2}^{-} \cdots E_{-K}^{-}. \]  
When sandwiched between two states, this operator forces $K$ particles to occupy the first $K$ sites. The expectation value of $s_K$ (or sometimes its spin reverse) is called the ‘emptiness formation probability’.

We will show that the operator $s_K$ scales to $O(K)$ in the limiting procedure. In the coordinate Bethe ansatz its $N$-particle form factors are given by
\[
\langle \{\lambda\}|s_K|\{\mu\} \rangle = \frac{N!}{K!(N-K)!} \sum_{y_1,\ldots,y_{N-K}=K+1}^{M} \phi_N^*(\lambda|1,\ldots,K,y_1,\ldots,y_{N-K}) \times \phi_N(\mu|1,\ldots,K,y_1,\ldots,y_{N-K}).
\]  
This formula has to be compared to (2.7). It is easy to see that the scaling limit of the Bethe wavefunctions works even if a fixed number of particles are placed on the first few sites:
\[ \phi_N(\mu|1,\ldots,K,y_1,\ldots,y_{N-K}) \rightarrow \chi_N(p|0,0,\ldots,0,x_1,\ldots,x_{N-K}). \]
Therefore the un-normalized form factor will behave as
\[ \left( \frac{\varepsilon^2}{c} \right)^{M-K} \langle \{\lambda\}|s_K|\{\mu\} \rangle \rightarrow \langle \{p\}|\mathcal{O}_K|\{k\} \rangle^*. \]
where it is understood that
\[ \frac{c}{\varepsilon}\lambda_j \rightarrow p_j \quad \frac{c}{\varepsilon}\mu_j \rightarrow k_j. \]
For the normalized form factors we get
\[ \left( \frac{\varepsilon^2}{c} \right)^{-K} \frac{\langle \{\lambda\}|s_K|\{\mu\} \rangle}{\sqrt{\langle \lambda|\lambda \rangle \langle \mu|\mu \rangle}} \rightarrow \frac{\langle \{p\}|\mathcal{O}_K|\{k\} \rangle^*}{\sqrt{\langle p|p \rangle \langle k|k \rangle}}. \]  
Our strategy is to obtain explicit determinant formulae for the matrix elements (3.15) and to take the scaling limit according to (3.16).
4. Form factors in the XXZ chain

In this section we compute explicit determinant formulae for the matrix elements (3.15) in the framework of the algebraic Bethe ansatz (ABA). Mostly we will use the results of the papers [48, 40, 41]; the only difference between the present approach and the traditional methods is that here the homogeneous limit \( \xi_j \rightarrow \eta/2 \) is taken explicitly before performing the thermodynamic limit or the limit towards the Bose gas.

The central object in ABA is the monodromy matrix, a \( 2 \times 2 \) matrix in the so-called auxiliary space with operator-valued entries which act on the Hilbert space of the spin chain:

\[
T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}.
\]

It is built from the so-called local \( L \)-matrices:

\[
T(u) = L_M(u) \cdots L_1(u),
\]

where

\[
L_j(u) = R_0(u - \xi_j).
\]

Here \( j \) refers to the quantum space of the spin at site \( j \), \( 0 \) refers to the auxiliary space and the parameters \( \xi_j \) are identical to the inhomogeneities already introduced in (3.4).

The operator \( R(u) \) is the \( R \) matrix of the XXZ type:

\[
R(u) = \frac{1}{\sinh(u + \eta)} \begin{pmatrix} \sinh(u) & \sinh(\eta) \\ \sinh(\eta) & \sinh(u) \end{pmatrix}.
\]

The trace of the monodromy matrix is called the transfer matrix:

\[
\tau(u) = A(u) + D(u).
\]

In the homogeneous limit \( \xi_j = \eta/2 \) it is related to the Hamiltonian (3.1) at \( \hbar = 0 \) as

\[
H \sim \frac{d}{du} \tau(u) \bigg|_{u=\eta/2} + \text{const}.
\]

The normalization (4.1) of the \( R \) matrix results in the following vacuum eigenvalues:

\[
A(u)|0\rangle = |0\rangle \quad D(u)|0\rangle = d(u)|0\rangle,
\]

with \( d(\lambda) \) given by (3.6).

In the framework of ABA the Bethe states are

\[
|0\rangle \prod_{j=1}^N C(\lambda_j) \quad \text{and} \quad \prod_{j=1}^N B(\mu_j)|0\rangle.
\]

They are eigenstates of the transfer matrix if the rapidities satisfy the Bethe equations (3.5). Apart from an overall normalization factor they are identical to the states given by the coordinate wavefunctions (3.2).

In order to compute the matrix elements of \( s_K \) in the framework of ABA the local operators have to be expressed in terms of the entries of the transfer matrix. This problem
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\[ E_{j}^{\alpha\beta} = \prod_{k=1}^{j-1} (A + D)(\xi_k) \times T^{\alpha\beta}(\xi_j) \times \prod_{k=j+1}^{M} (A + D)(\xi_k). \] (4.2)

Applying this formula to operators \( E_{j}^{--} \) on neighboring sites one gets

\[ s_K = D(\xi_1)D(\xi_2)\cdots D(\xi_K) \prod_{l=K+1}^{M} (A + D)(\xi_l). \] (4.3)

Evaluated on Bethe states equation (4.3) yields

\[ \langle 0 | \prod_{j=1}^{N} C(\lambda_j) s_K \prod_{j=1}^{N} B(\mu_j) | 0 \rangle = \langle 0 | \prod_{j=1}^{N} C(\lambda_j) D(\xi_1)D(\xi_2)\cdots D(\xi_K) \prod_{j=1}^{N} B(\mu_j) | 0 \rangle \]

\[ \times \prod_{j=1}^{K} \frac{1}{t(\xi_j, \{\mu\})} \] (4.4)

with \( t(u, \{\mu\}) \) being the corresponding eigenvalue of the transfer matrix. Evaluated at the inhomogeneities it is

\[ t(\xi_j, \{\mu\}) = \prod_{k=1}^{N} \frac{\sinh(\mu_k - \xi_j + \eta)}{\sinh(\mu_k - \xi_j)}. \] (4.5)

In (4.4) we also used the fact that

\[ \prod_{l=1}^{M} (A + D)(\xi_l) = 1. \]

The action of multiple \( D \) operators on the dual state results in [41]

\[ \langle 0 | \prod_{j=1}^{N} C(\lambda_j) D(\xi_1)D(\xi_2)\cdots D(\xi_K) \]

\[ = \sum_{\{\lambda^+\} \cup \{\lambda^-\}} \prod_{\lambda^+ < \xi} \frac{\sinh(\xi - \xi_j)}{\sinh(\xi - \xi_j) \sinh(\lambda^+_j - \lambda^+_o)} \times \det t(\lambda^+_i, \xi_j) \times \prod_{\lambda^-_p} f(\lambda^-_p, \lambda^-_o) \]

\[ \times \prod_{l=1}^{K} d(\lambda^+_l) \langle 0 | \prod_{o} C(\lambda^-_o) C(\xi_1) \cdots C(\xi_K) \]

with

\[ t(\lambda, \xi_j) = \frac{\sinh \eta}{\sinh(\lambda - \xi) \sinh(\lambda - \xi + \eta)}, \quad f(\lambda_j, \lambda_k) = \frac{\sinh(\lambda_j - \lambda_k + \eta)}{\sinh(\lambda_j - \lambda_k)}. \]

The scalar product of an arbitrary state and a Bethe state is [50]

\[ \langle 0 | \prod_{j} C(\lambda_j) \prod_{j} B(\mu_j) | 0 \rangle = \prod_{j,k} \frac{\sinh(\mu_j - \lambda_k + \eta)}{\sinh(\lambda_j - \lambda_k) \sinh(\mu_k - \mu_j)} \times \det S, \] (4.6)

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\[
S_{jk} = t(\mu_j, \lambda_k) - d(\lambda_k) t(\lambda_k, \mu_j) \prod_{t=1}^{N} \frac{\sinh(\lambda_k - \mu_t + \eta)}{\sinh(\lambda_k - \mu_t - \eta)}.
\]

Specializing this to the present case

\[
\langle \prod_{o} C(\lambda_o^-) C(\xi_1) \cdots C(\xi_K) \prod_{j} B(\mu_j) | 0 \rangle
= \frac{\prod_{j,k} \sinh(\mu_j - \lambda_k^- + \eta) \prod_{j,o} \sinh(\mu_j - \xi_o + \eta)}{\prod_{j<k} \sinh(\mu_j - \mu_j) \prod_{j<k} \sinh(\lambda_j^- - \lambda_k^-) \prod_{j<k} \sinh(\xi_j - \xi_k) \prod_{j,k} \sinh(\xi_j - \lambda_k^-)} \times \det U.
\]

Here

\[
U_{jl} = t(\mu_j, \xi_l) \quad \text{for } l = 1 \cdots K
\]

\[
U_{j,K+l} = t(\mu_j, \lambda_l^-) - d(\lambda_l^-) t(\lambda_l^-, \mu_j) \prod_{o=1}^{N} \frac{\sinh(\lambda_l^- - \mu_o + \eta)}{\sinh(\lambda_l^- - \mu_o - \eta)} \quad \text{otherwise.} \quad (4.7)
\]

Therefore, the form factor of the inhomogeneous chain is given by

\[
\langle \{\lambda\} | s_K | \{\mu\} \rangle = \prod_{j=1}^{K} \frac{1}{t(\xi_j)} \times \sum_{\{\lambda^-\} \cup \{\lambda^+\} \subset \{\lambda\}} \prod_{l<k} \sinh(\xi_i - \xi_j) \sinh(\lambda_j^+ - \lambda_k^-) \prod_{o,p} f(\lambda^+_p, \lambda^-_o) \times \prod_{l} d(\lambda^+_l)
\]

\[
= \frac{\prod_{j,k} \sinh(\mu_j - \lambda_k^- + \eta) \prod_{j,o} \sinh(\mu_j - \xi_o + \eta)}{\prod_{j<k} \sinh(\mu_j - \mu_j) \prod_{j<k} \sinh(\lambda_j^- - \lambda_k^-) \prod_{j<k} \sinh(\xi_j - \xi_k) \prod_{j,k} \sinh(\xi_j - \lambda_k^-)} \times \det t(\lambda_j^+, \xi_j) \times \det U. \quad (4.8)
\]

We now perform the homogeneous limit \( \xi_j \to \eta/2 \) following the method of [51]. For the matrix \( M \) we get

\[
\lim_{\eta \to 0} \frac{\det M}{\prod_{j>k} \sinh(\xi_j - \xi_k)} = \frac{1}{\prod_{\alpha=1}^{m-1} \alpha!} \det H
\]

with

\[
H_{jl} = \left[ \left( \frac{\partial}{\partial \xi} \right)^{l-1} t(\lambda_k, \xi) \right]_{\xi = \eta/2}.
\]

It is advantageous to use the form

\[
t(\lambda, \xi) = \frac{\cosh(\lambda - \xi)}{\sinh(\lambda - \xi)} - \frac{\cosh(\lambda - \xi + \eta)}{\sinh(\lambda - \xi + \eta)}.
\]

Taking the derivatives one is free to replace [52]

\[
\left( \frac{\partial}{\partial \xi} \right)^{l-1} t(\lambda, \xi) \to (-1)^{l-1}(l-1)! \left[ \left( \frac{\cosh(\lambda - \xi)}{\sinh(\lambda - \xi)} \right)^l - \left( \frac{\cosh(\lambda - \xi + \eta)}{\sinh(\lambda - \xi + \eta)} \right)^l \right].
\]

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The same steps can be performed for the corresponding elements of the matrix \( U \). Finally the homogeneous limit is

\[
\langle \{ \lambda \} | s_K | \{ \mu \} \rangle = \sum_{\{ \lambda^+ \} \cup \{ \lambda^- \} \atop |\{ \lambda^+ \}||\{ \lambda^- \}| = K} \prod_{o,p} \sinh(\lambda_p^+ + \eta/2) \prod_{j < l} \sinh(\lambda_j^+ - \lambda_l^+) \prod_{o,p} f(\lambda_p^+, \lambda_o^-) \times \prod_{l} d(\lambda_l^+) \\
\times \prod_{j,k} \sinh(\mu_j - \lambda_k^- + \eta) \prod_j \sinh^K(\mu_j - \eta/2) \prod_l \sinh^K(\eta/2 - \lambda_l) \times \det O \det V. \tag{4.9}
\]

with

\[
O_{jl} = \left[ \left( \frac{\cosh(\lambda_j^+ - \eta/2)}{\sinh(\lambda_j^+ - \eta/2)} \right)^l - \left( \frac{\cosh(\lambda_j^+ + \eta/2)}{\sinh(\lambda_j^+ + \eta/2)} \right)^l \right] \tag{4.10}
\]

and

\[
V_{jl} = \left[ \left( \frac{\cosh(\mu_j - \eta/2)}{\sinh(\mu_j - \eta/2)} \right)^l - \left( \frac{\cosh(\mu_j + \eta/2)}{\sinh(\mu_j + \eta/2)} \right)^l \right] \text{ for } l = 1 \cdots K
\]

\[
V_{j,K+l} = t(\mu_j, \lambda_j^-) - d(\lambda_j^-) t(\lambda_j^-, \mu_j) \prod_{o=1}^{N} \frac{\sinh(\lambda_j^- - \mu_o + \eta)}{\sinh(\lambda_j^- - \mu_o - \eta)} \text{ otherwise.} \tag{5.11}
\]

5. The scaling limit of the form factors

Here we take the scaling limit of the formula (4.9) to obtain the matrix elements of \( O_K \) in the Bose gas. We substitute

\[
\eta = i\pi - i\varepsilon \quad \lambda = \frac{\varepsilon}{c} p \quad \mu = \frac{\varepsilon}{c} k.
\]

It is straightforward to calculate the limiting values of the prefactors, but the determinants need special care. The elements of \( O \) are

\[
O_{jl} = \left[ \left( \frac{\sinh(\lambda_j^+ + i\varepsilon/2)}{\cosh(\lambda_j^+ + i\varepsilon/2)} \right)^l - \left( \frac{\sinh(\lambda_j^+ - i\varepsilon/2)}{\cosh(\lambda_j^+ - i\varepsilon/2)} \right)^l \right]. \tag{5.1}
\]

The leading terms will be

\[
O_{jl} \to t \left( \frac{\varepsilon}{c} \right)^{l-1} \left( p_j^+ \right)^{l-1} \varepsilon,
\]

which yields

\[
\det O \to K! (i\varepsilon)^K \left( \frac{\varepsilon}{c} \right)^{K(K-1)/2} \det \left[ (p_j^+)^{l-1} \right] = K! (i\varepsilon)^K \left( \frac{\varepsilon}{c} \right)^{(k-1)K/2} \prod_{j>l} (p_j^+ - p_l^+).
\]

One can use the same expansion for the first \( K \) columns of the matrix \( V \).

To obtain the proper normalization note that in ABA the norm of the Bethe state scales as

\[
\langle 0 | \prod_{j} C(\lambda_j) \prod_{j} B(\lambda_j) | 0 \rangle \to c^N \prod_{j<l} (p_j - p_l)^2 + c^2 \prod_{j<l} (p_j - p_l)^2 \times \det G_{LL}.
\]

This differs from (2.5) by the overall factor of \( c^N \).

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Collecting all the factors and performing a complex conjugation we find
\[
\langle \{p\}|O_K|\{k\}\rangle = e^{K-N(K!)}^2 \sum_{\{p^+\cup\{p^-\}} \prod_{o,l} f(p_o^+, p_l^-) \prod_l e^{-iLP_l^+} \\
\times \frac{\prod_{j,l}(k_j - p_l^- + ic)}{\prod_{j<k}(p_j^- - p_k^-)} \times \det Z,
\]
with
\[
Z_{jl} = (k_j)^{l-1} \quad \text{for} \quad l = 1 \cdots K
\]
\[
Z_{j,K+l} = t(k_j, p_l^-) - e^{-iLP_l^-} t(p_l^-, k_j) \prod_{o=1}^N \frac{(p_l^- - k_o + ic)}{(p_l^- - k_o - ic)} \quad \text{otherwise}.
\]

Here we used
\[
t(u) = \frac{ic}{u(u+ic)}, \quad f(u) = \frac{u + ic}{u}.
\]

Equation (5.2) refers to the normalization where the norms of the states \(|\{p\}\rangle\) and \(|\{k\}\rangle\) are given by (2.5).

The result (5.2) is valid whenever the set \(\{k\}\) satisfies the Bethe equations. In the case when \(\{p\}\) is also a solution but different from \(\{k\}\) we obtain the form factors
\[
F^K_N(\{p\}, \{k\}) = e^{K-N(K!)}^2 \sum_{\{p^+\cup\{p^-\}} \prod_{o,l} f(p_o^+, p_l^-) \prod_l e^{-iLP_l^+} \\
\times \frac{\prod_{j,l}(k_j - p_l^- + ic)}{\prod_{j<k}(p_j^- - p_k^-)} \times \det V,
\]
with
\[
V_{ jl} = (k_j)^{l-1} \quad \text{for} \quad l = 1 \cdots K
\]
\[
V_{j,K+l} = t(k_j, p_l^-) + t(p_l^-, k_j) \prod_{o=1}^N \frac{(p_l^- - k_o + ic)(p_l^- - p_o - ic)}{(p_l^- - k_o - ic)(p_l^- - p_o + ic)} \quad \text{otherwise}.
\]

This is a new result of the present work. In the case of \(K = 1\) (5.4) yields an alternative representation for the form factors of the density operator, which were previously determined in [50, 8, 53].

To obtain the mean value of \(O_K\) we take the limit \(\{p\} \to \{k\}\) in (5.2) and divide by the norm (2.5) resulting in
\[
\langle O_K \rangle_N = (K!)^2 \sum_{\{p^+\cup\{p^-\}} \prod_{j,l}^N \frac{p_j^+ - p_l^+}{(p_j^+ - p_l^+)^2 + c^2} \times \frac{\det H}{\det G^{LL}}.
\]
The elements of $\mathcal{H}$ are given by

$$
\mathcal{H}_{j,l} = \begin{cases} 
(p_j)^l & \text{for } l = 1 \cdots K \\
G_{j,l}^{LL} & \text{for } l = K + 1 \cdots N.
\end{cases}
$$

(5.7)

Here it is understood that in both $G_{j,l}^{LL}$ and $\mathcal{H}$ the ordering of the rapidities is given by $\{p\} = \{\{p^+\}, \{p^-\}\}$. The matrix $\mathcal{H}$ differs from $G_{j,l}^{LL}$ only in those columns which belong to the subset $\{p^+\}$.

In the case of $K = 1$ the above formula results in

$$
\langle O_1 \rangle = \frac{N}{L}
$$

as it should. To prove this, note that the sums of the columns of $G_{j,l}^{LL}$ are equal to $L$ in every row. Therefore every $\mathcal{H}$ gives

$$
\mathcal{H} = \frac{1}{L} G_{j,l}^{LL}.
$$

6. Expectation values in the thermodynamic limit

In this section we evaluate the thermodynamic limit of (5.6). We consider a Bethe state $|\Omega\rangle$ in a large volume $L$ with a large number of particles such that the particle density $n = N/L$ is fixed. In the thermodynamic limit one defines the density of roots $\rho^{(r)}(p)$ and holes $\rho^{(h)}(p)$ and the total density $\rho(p) = \rho^{(r)}(p) + \rho^{(h)}(p)$. This latter function satisfies the Lieb equation

$$
\rho(p) = \frac{1}{2\pi} + \int_{-\infty}^{\infty} dq \varphi(p-q) f(q) \rho(q).
$$

(6.1)

Here

$$
f(p) = \frac{\rho^{(r)}(p)}{\rho(p)}
$$

(6.2)

is a distribution function characterizing the state in question. In thermal equilibrium $f(p) = (1 + e^{\varepsilon(p)})^{-1}$ where $\varepsilon(p)$ is the so-called pseudo-energy, which is a solution of the TBA equation

$$
\varepsilon(p) = \frac{p^2 - \mu}{T} - \int_{-\infty}^{\infty} \frac{dp'}{2\pi} \varphi(p - p') \log(1 + e^{-\varepsilon(p')}).
$$

(6.3)

At zero temperature we recover the ground state distribution

$$
f(p) = \begin{cases} 
1 & |p| \leq \Lambda \\
0 & |p| > \Lambda
\end{cases}
$$

(6.4)

with $\Lambda$ being the Fermi rapidity. The particle number is always given by the formula

$$
n = \frac{N}{L} = \int f(p) \rho(p).
$$

We proceed to calculate the thermodynamic limit of (5.6) using the techniques of [40]. The ratio of determinants is calculated as

$$
\frac{\det \mathcal{H}}{\det G_{j,l}^{LL}} = \det \left( (G_{j,l}^{LL})^{-1} \mathcal{H} \right).
$$
The resulting matrix on the rhs will be equal to the identity matrix except for those columns belonging to the set \( \{ p^+ \} \). These elements can be evaluated by transforming the action of \( G^{LL} \) into an integral equation. This results in

\[
\frac{\det \mathcal{H}}{\det G^{LL}} = \prod_o \frac{1}{2\pi L \rho(p^+_o)} \times \det I,
\]

where

\[
I_{jl} = h^{(l-1)}(p^+_j).
\]

Here \( h^{(0)}(u) \) is the solution of the linear integral equation

\[
h^{(0)}(l) = p_l + \int_{-\infty}^{\infty} \frac{dq}{2\pi} \varphi(p - q) f(q) h^{(1)}(q).
\]  

(6.5)

Note that in this normalization \( h^{(0)}(p) = 2\pi \rho(p) \) and \( f(p) h^{(0)}(p) = 2\pi \rho^{(r)}(p) \).

As a final step one integrates over the rapidities \( p^+_j \) and in the thermodynamic limit one gets

\[
\langle \Omega | \mathcal{O}_K | \Omega \rangle = K! \int \frac{dp_1}{2\pi} \cdots \frac{dp_K}{2\pi} \prod_p f(p_o) \prod_{j \neq l} \frac{p_j - p_l}{(p_j - p_l)^2 + c^2} \times \det I.
\]  

(6.6)

This formula is the main result of our paper. Its explicit factorization is performed in the cases \( K = 2\ldots 4 \) in section 7.

For practical purposes it is useful to derive the dimensionless multiple integral for the quantity

\[
g_K = \frac{\langle \mathcal{O}_K \rangle}{n^K},
\]

which only depends on the dimensionless coupling constant \( \gamma = c/n \) and the dimensionless version of the distribution functions \( f(p) \). We define

\[
q = \frac{p}{c} \quad f(q) = f(p = qc).
\]

In the finite-temperature case \( f(q) = (1 + e^{\tilde{\varepsilon}(q)})^{-1} \), where \( \tilde{\varepsilon}(q) \) is the solution of the dimensionless equation

\[
\tilde{\varepsilon}(q) = -\alpha + \frac{q^2 \gamma^2}{\tau} - \int_{-\infty}^{\infty} \frac{dq'}{2\pi} \frac{2}{(q - q')^2 + 1} \log(1 + e^{-\tilde{\varepsilon}(q')}),
\]  

(6.8)

with

\[
\alpha = \frac{\mu}{T} \quad \tau = \frac{T}{n^2}.
\]
Defining the dimensionless functions $\tilde{h}^{(l)}(q)$ as

$$\tilde{h}^{(l)}(q) = q^l + \int_{-\infty}^{\infty} \frac{dq'}{2\pi} \frac{2}{(q' - q)^2 + 1} f(q')\tilde{h}^{(l)}(q')$$

(6.9)

we find $\tilde{h}^{(l)}(q) = c^l h^{(l)}(p = qc)$. Thus the dimensionless multiple integral formula is expressed as

$$g_K = (K!)^2 \gamma^K \int \frac{dq_1}{2\pi} \cdots \frac{dq_K}{2\pi} \prod_{j>l} \frac{q_j - q_l}{(q_j - q_l)^2 + 1} \prod_{j=1}^{K} f(q_j)\tilde{h}^{(j-1)}(q_j).$$

(6.10)

### 6.1. The $c \to 0^+$ limit

We take the small-coupling limit of the dimensionful formula (6.6) by sending $c \to 0$ and keeping $n$ fixed. The limiting form of the kernel $\varphi(u)$ is given by

$$\varphi(u) \to 2\pi \delta(u).$$

(6.11)

Therefore the solution of the integral equations (6.5) is

$$h^{(l)}(p) = \frac{p^l}{1 - f(p)}.$$

The determinant in (6.6) has the limiting value

$$\det I \to \prod_j \frac{1}{1 - f(p)} \prod_{j>k} (p_j - p_k)$$

resulting in

$$\langle O_K \rangle \to K! \prod_{j=1}^{K} \left[ \int \frac{dp_j}{2\pi} \frac{f(p_j)}{1 - f(p_j)} \right].$$

Note that

$$\langle O_1 \rangle = n = \int \frac{dp}{2\pi} \frac{f(p)}{1 - f(p)}.$$

therefore

$$g_K = \frac{\langle O_K \rangle}{n^K} \to K!$$

as it should be for free bosons.

The above calculation only applies if $f(p) < 1$, therefore the result is not valid for the ground state. In fact

$$\lim_{c \to 0} \lim_{T \to 0} g_K = 1.$$

We have checked this property numerically for $K \leq 4$ (see subsection 7.5). To prove it analytically from (6.6) one has to carefully analyze the sub-leading corrections to the
integral equation (6.5) with the weight function (6.4). We leave this problem for further research.

6.2. The $c \to \infty$ limit

Here we derive the leading term in the large-coupling expansion of $g_K$. In the $c \to \infty$ limit the kernel $\varphi(p)$ is of order $1/c$, therefore to leading order

$$h^{(l)}(p) = p^l$$

and

$$\langle O_K \rangle \to \frac{K!}{c^K(K-1)} \int \frac{dp_1}{2\pi} \cdots \frac{dp_K}{2\pi} \prod_o f(p_o) \prod_{j>l} (p_j - p_l)^2.$$ 

Evaluating this formula for the ground state gives

$$g_K = \frac{K!}{2^K} \left( \frac{\pi}{\gamma} \right)^{K(K-1)} \times \int_{-1}^{1} dx_1 \cdots dx_K \prod_{j>l} (x_j - x_l)^2.$$ 

This result was already obtained in the papers [31,32,34,35].

7. Factorization of the multiple integrals

In this section we perform the factorization of the multiple integral formula (6.7) in the cases $K = 2$–$4$.\footnote{The case $K = 1$ is trivial and it simply yields the particle density as it should.} In order to keep the formulae as short as possible we will use the following notation:

$$\int \frac{d\tilde{p}}{2\pi} \cdots = \int \frac{dp}{2\pi} f(p) \cdots.$$ 

Moreover we will suppress the dependence of the mean value on the state $|\Omega\rangle$ and we write simply $\langle O_K \rangle$. The dependence on $|\Omega\rangle$ is carried out by the functions $f(p)$ and $h^{(l)}(p)$. We found it more convenient to work with the dimensionful formula (6.7), because this way non-trivial checks of dimensional analysis can be performed at each step of the calculation. The dimensionless formulae can be obtained as explained in 7.5.

The main idea behind the factorization procedure is simple: at each step the number of the integrals can be reduced by one using the integral equation (6.5), whenever the prefactors are such that the corresponding variable is present only in one denominator:

$$\frac{1}{(p_j - p_k)^2 + c^2} = \frac{1}{2c} \varphi(p_j - p_k).$$

Except from the case $K = 2$ this is not the case. Instead the prefactors have to be divided into several terms, in each of which one of the integrals can be performed. This is a non-trivial task with growing complexity as $K$ increases. In the following we present a case-by-case study up until $K = 4$. 

\[ \text{doi:10.1088/1742-5468/2011/11/P11017} \]
7.1. $K = 2$

One has

$$\langle O_2 \rangle = 4 \left[ \int \frac{d\tilde{p}_1 d\tilde{p}_2}{(p_1 - p_2)^2 + c^2} h^{(0)}(p_1) h^{(1)}(p_2) \right.$$

$$\left. - \int d\tilde{p}_1 d\tilde{p}_2 \frac{p_1}{(p_1 - p_2)^2 + c^2} h^{(0)}(p_1) h^{(1)}(p_2) \right].$$

In the first term one integrates over $p_1$ first, but in the second term over $p_2$ first. Using the integral equation (6.5) one gets

$$\langle O_2 \rangle = 2 \int d\tilde{p}_2 p_2 h^{(1)}(p_2)(h^{(0)}(p_2) - 1) - \int d\tilde{p}_1 p_1 h^{(0)}(p_1)(h^{(1)}(p_1) - p_1)$$

$$= \frac{2}{c} \int dp \left( p^2 h^{(0)}(p) - ph^{(1)}(p) \right).$$

(7.1)

This is in agreement with formula (10) of [38]. As was already explained in [34, 35, 38], in the finite-temperature case (7.1) agrees with the result obtained from the Hellmann–Feynman theorem. We also note that (7.1) can be proven for an arbitrary weight function $f(p)$ using the Hellmann–Feynman for a single state; one has to repeat the arguments of appendix D of [37].

For future use we define

$$\{ n, m \} = \int d\tilde{p} \, p^n h^{(m)}(p) = \int \frac{dp}{2\pi} f(p) \, p^n h^{(m)}(p).$$

(7.2)

It can be shown using the iterative solution to (6.5) that in general

$$\{ n, m \} = \{ m, n \}.$$

Using this notation we write

$$\langle O_2 \rangle = \frac{2}{c} (\{ 0, 2 \} - \{ 1, 1 \}).$$

(7.3)

Also, it is useful to derive a general formula which will be used often:

$$\int d\tilde{x} d\tilde{y} \frac{x^\alpha - y^\alpha}{(x - y)^2 + c^2} \left( h^{(\beta)}(x) h^{(\gamma)}(y) - h^{(\beta)}(y) h^{(\gamma)}(x) \right)$$

$$= \frac{1}{c} (\{ \gamma, \alpha + \beta \} - \{ \beta, \alpha + \gamma \}).$$

(7.4)

7.2. $K = 3$

The mean value is given by

$$\langle O_3 \rangle = 36 \int d\tilde{p}_1 \cdots d\tilde{p}_3 \prod_{j>i} \frac{p_j - p_i}{(p_j - p_i)^2 + c^2} h^{(0)}(p_1) h^{(1)}(p_2) h^{(2)}(p_3).$$

(7.5)

In this form neither of the integrals can be done directly. Instead, one has to divide the prefactors into several terms such that in each of them one integral can be performed. One way to do this is as follows. We define

$$D(x, y, z) = \frac{y - x}{(x - y)^2 + c^2} \frac{1}{(y - z)^2 + c^2}.$$
Then we find the identity

\[
\prod_{j>1} \frac{p_j - p_i}{(p_j - p_i)^2 + c^2} = \frac{1}{3} [D(x, y, z) + D(y, z, x) + D(z, x, y)] - D(y, x, z) - D(x, y, z) - D(z, y, x). \tag{7.6}
\]

For simplicity we used the variables \(x, y, z\) on the rhs instead of \(p_1, p_2, p_3\). Equation (7.6) can be proven as follows. The rhs is a completely antisymmetric function of the variables \(x, y, z\) and it has exactly the same poles as the function on the lhs. Therefore it has to be equal to the lhs multiplied by a symmetric polynomial. By power counting it is shown that this polynomial is a pure number. This number is found to be 1 by simple manipulations after sending \(c \to 0\) on both sides.

Substituting (7.6) into (7.5), performing one integral in each term using (6.5), changing variables accordingly and observing the cancellation of the terms including three \(h\) functions we find

\[
\langle O_3 \rangle = \frac{6}{c^3} \int \frac{d\tilde{x} d\tilde{y}}{(x - y)^2 + c^2} \times \left[ y^2 (h_1(x)h_0(y) - h_0(x)h_1(y)) \right.
+ y(h_0(x)h_2(y) - h_2(x)h_0(y))
+ \left. (h_2(x)h_1(y) - h_1(x)h_2(y)) \right]. \tag{7.7}
\]

Using (7.4) the last line of (7.7) yields

\[
\frac{6}{c} \int d\tilde{x} d\tilde{y} \frac{y - x}{(x - y)^2 + c^2} (h_2(x)h_1(y) - h_1(x)h_2(y)) = \frac{6}{c^2} (\{2, 2\} - \{1, 3\}).
\]

For the second line of (7.7) we can drop the term proportional to \(xy\) to find

\[
\frac{6}{c} \int d\tilde{x} d\tilde{y} \frac{y^2}{(x - y)^2 + c^2} (h_0(x)h_2(y) - h_2(x)h_0(y)) = \frac{3}{c^2} (\{0, 4\} - \{2, 2\}).
\]

The first line of (7.7) is more involved. We have to compute

\[
\frac{3}{c} \int d\tilde{x} d\tilde{y} \frac{(y - x)(y^2 + x^2)}{(x - y)^2 + c^2} (h_1(x)h_0(y) - h_0(x)h_1(y)). \tag{7.8}
\]

We write

\[
\frac{(y - x)(y^2 + x^2)}{(x - y)^2 + c^2} = \frac{y - x}{3} + \frac{c^2}{3} \frac{x - y}{(x - y)^2 + c^2} + \frac{2}{3} \frac{y^3 - x^3}{(x - y)^2 + c^2}. \tag{7.9}
\]

The first term in (7.9) gives

\[
2 \frac{\{0, 1\}^2 - \{0, 0\}\{1, 1\}}{c^2}.
\]

The second and third terms in (7.9) can be evaluated using (7.4).

Putting everything together

\[
\langle O_3 \rangle = \frac{1}{c^2} (-4\{1, 3\} + 3\{2, 2\} + \{0, 4\} + \{0, 2\} - \{1, 1\})
+ \frac{2}{c} (\{0, 1\}^2 - \{0, 0\}\{1, 1\}). \tag{7.10}
\]

This is in accordance with formula (11) of [38].

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7.3. $K = 4$

We define

$$D_4(x, y, z, u) = \frac{1}{((x - y)^2 + c^2)((y - z)^2 + c^2)((z - u)^2 + c^2)}.$$ 

Then we find

$$\prod_{j > l} \frac{p_j - p_l}{(p_j - p_l)^2 + c^2} = \frac{1}{12} \sum_{P \in \sigma_4} (-1)^{|P|} D_4(P). \quad (7.11)$$

This equation can be proven through the same steps as in the case of (7.6).

One has to evaluate

$$\langle O_4 \rangle = 24^2 \int d\tilde{p}_1 \cdots d\tilde{p}_4 \prod_{j < l} \frac{p_j - p_l}{(p_j - p_l)^2 + c^2} h^{(0)}(p_1)h^{(1)}(p_2)h^{(2)}(p_3)h^{(3)}(p_4)$$

$$= 48 \int d\tilde{p}_1 \cdots d\tilde{p}_4 \frac{1}{((p_1 - p_2)^2 + c^2)((p_2 - p_3)^2 + c^2)((p_3 - p_4)^2 + c^2)}$$

$$\times \sum_{P \in \sigma_4} (-1)^{|P|} \prod_{l=1}^4 h^{(P_l-1)}(p_l).$$

Performing the integrals over $p_1$ and $p_4$ and using the symmetries one gets

$$\langle O_4 \rangle = \frac{12}{c^2} \sum_{P \in \sigma_4} (-1)^{|P|} \int d\tilde{p}_2 d\tilde{p}_3 p_2^{P_2-1} p_3^{P_3-1} ((p_2 - p_3)^2 + c^2)^{-1} h^{(P_2-1)}(p_2)h^{(P_3-1)}(p_3).$$

There are in total six different combinations and we treat them one by one. We define

$$\langle O_4 \rangle = \frac{12}{c^2} \sum_{\sigma = 1}^6 K_{\sigma},$$

with $K_{\sigma}$ representing one of the six combinations. The first three cases are evaluated easily:

$$K_1 = \int dx dy \frac{y - x}{(x - y)^2 + c^2}(h^{(2)}(x)h^{(3)}(y) - h^{(3)}(x)h^{(2)}(y)) = \frac{1}{c}(\{2, 4\} - \{3, 3\})$$

$$K_2 = \int dx dy \frac{y^2 - x^2}{(x - y)^2 + c^2}(h^{(3)}(x)h^{(1)}(y) - h^{(1)}(x)h^{(3)}(y)) = \frac{1}{c}(\{3, 3\} - \{1, 5\})$$

$$K_3 = \int dx dy \frac{y^3 - x^3}{(x - y)^2 + c^2}(h^{(1)}(x)h^{(2)}(y) - h^{(2)}(x)h^{(1)}(y)) = \frac{1}{c}(\{1, 5\} - \{2, 4\})$$

The remaining three case are more complicated because we have to separate factors of the form

$$\frac{x^\alpha y^\beta - x^\beta y^\alpha}{(x - y)^2 + c^2} \quad \alpha, \beta > 0.$$ 

The next case is

$$K_4 = \int dx dy \frac{xy^2 - yx^2}{(x - y)^2 + c^2}(h^{(0)}(x)h^{(3)}(y) - h^{(3)}(x)h^{(0)}(y)).$$

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Here we use
\[
\frac{xy^2 - x^2y}{(x - y)^2 + c^2} = \frac{x - y}{3} + \frac{c^2}{3} \frac{y - x}{(x - y)^2 + c^2} + \frac{1}{3} \frac{y^3 - x^3}{(x - y)^2 + c^2}
\]
leading to
\[
K_4 = \frac{2}{3}((0, 1)\{0, 3\} - \{0, 0\}\{1, 3\}) + \frac{c}{3}(\{0, 4\} - \{1, 3\}) + \frac{1}{3c}(\{0, 6\} - \{3, 3\}).
\]

The next case is
\[
K_5 = \int d\tilde{x} d\tilde{y} \frac{xy^3 - yx^3}{(x - y)^2 + c^2}(h^{(2)}(x)h^{(0)}(y) - h^{(0)}(x)h^{(2)}(y)).
\]
Here we use
\[
\frac{xy^3 - x^3y}{(x - y)^2 + c^2} = \frac{1}{2} \left[ x^2 - y^2 - c^2 \frac{x^2 - y^2}{(x - y)^2 + c^2} + \frac{y^4 - x^4}{(x - y)^2 + c^2} \right]
\]
which gives
\[
K_5 = ((2, 2)\{0, 0\} - \{0, 2\}^2) + \frac{c}{2}(\{2, 2\} - \{0, 4\}) + \frac{1}{2c}(\{2, 4\} - \{0, 6\}).
\]
Finally, the last term is
\[
K_6 = \int d\tilde{x} d\tilde{y} \frac{x^2y^3 - y^2x^3}{(x - y)^2 + c^2}(h^{(0)}(x)h^{(1)}(y) - h^{(1)}(x)h^{(0)}(y)).
\]
Here we write
\[
\frac{x^2y^3 - x^3y^2}{(x - y)^2 + c^2} = \frac{1}{5} \left[ \frac{2c^2}{3}(x - y) + (x^3 - y^3) + 2(x^2y - y^2x) \\
+ \frac{2c^4}{3} \frac{y - x}{(x - y)^2 + c^2} + \frac{5c^2}{3} \frac{y^3 - x^3}{(x - y)^2 + c^2} + \frac{y^5 - x^5}{(x - y)^2 + c^2} \right]
\]
leading to
\[
K_6 = \frac{4c^2}{15}((0, 1)^2 - \{0, 0\}\{1, 1\}) + \frac{2}{5}(\{0, 3\}\{0, 1\} - \{1, 3\}\{0, 0\}) \\
+ \frac{4}{5}(\{0, 2\}\{1, 1\} - \{0, 1\}\{1, 2\}) + \frac{2c^3}{15}(\{0, 2\} - \{1, 1\}) + \frac{c}{3}(\{0, 4\} - \{1, 3\}) \\
+ \frac{1}{5c}(\{0, 6\} - \{1, 5\}).
\]

Putting everything together
\[
\langle O_4 \rangle = \frac{2}{5c^3}[8c^3((0, 1)^2 - \{0, 0\}(0, 1) + 32c((0, 1)(0, 3) - \{0, 0\}(1, 3)) \\
+ 24c((0, 2)(1, 1) - \{0, 1\}(1, 2)) + 30c((0, 0)(2, 2) - \{0, 2\})^2) \\
+ 4c((0, 2) - \{1, 1\}) + 5c^2((0, 4) - 4(1, 3) + 3(2, 2)) \\
+ 6(0, 6) - 6(1, 5) + 15(2, 4) - 10(3, 3)].
\]

This is a new result of the present work.

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7.4. Galilei invariance

The expectation values $\langle O_K \rangle$ are Galilei invariant, and it is useful to check this property in our final formulae. This constitutes a highly non-trivial check, as was already remarked in [38].

In our calculations we did not restrict ourselves to symmetric distributions; the weight functions $f(p)$ can be arbitrary. Therefore, to check Galilei invariance it is enough to consider an infinitesimal boost $b$. This boost yields the following infinitesimal transformations:

$$p \rightarrow p + b, \quad h^{(j)}(p) \rightarrow h^{(j)}(p) + b j h^{(j-1)}(p).$$

It is then readily seen that (6.7) is invariant due to the antisymmetry of the prefactors.

We also performed the check on our factorized formulae. The transformation rules for the quantities $\{\alpha, \beta\}$ are

$$\{\alpha, \beta\} \rightarrow \{\alpha, \beta\} + b[\alpha\{\alpha - 1, \beta\} + \beta\{\alpha, \beta - 1\}].$$

Using this rule we have checked that the variation of equations (7.3), (7.10) and (7.12) indeed vanishes.

7.5. Dimensionless formulae and numerical results

The dimensionless versions of formulae (7.3), (7.10) and (7.12) are obtained simply by setting $c = 1$, multiplying with an overall factor of $\gamma^K$, and replacing

$$\{n, m\} \rightarrow \int dq q^n \tilde{h}^{(m)}(q),$$

(7.13)

To numerically evaluate the factorized formulae the following steps have to be performed:

• Solve the TBA equation (6.8) iteratively. The parameter $\alpha$ can be fixed by requiring $g_1 = \gamma\{0, 0\} = 1$.

• Solve the linear integral equations (6.9) for $\tilde{h}^{(i)}(q)$.

• Evaluate (7.3), (7.10) and (7.12).

We performed this procedure for a wide range of the parameters $\gamma$ and $\tau$. The quantity $g_4$ shows the same qualitative behavior as $g_2$ and $g_3$ [38]: it is an increasing function of $\tau$ and a decreasing function of $\gamma$, with the limiting values given by

$$\lim_{\gamma \rightarrow 0} g_4 = \lim_{\tau \rightarrow \infty} g_4 = 4! = 24, \quad \lim_{\gamma \rightarrow 0} \lim_{\tau \rightarrow 0} g_4 = 1.$$

To demonstrate the numerical results we present the ground state values of $g_2$, $g_3$ and $g_4$ in figure 1, whereas the temperature dependence of $g_4$ is shown in figure 2 for the intermediate couplings $\gamma = 0.1$, 1 and 10.

At $T = 0$ the first term in the small-coupling expansion of $g_K$ is given by [32]

$$g_K = 1 - \frac{K(K - 1)}{\pi} \sqrt{\gamma} + O(\gamma).$$

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Figure 1. The ground state values of the $K$-body local correlations for $K \leq 4$ as a function of the dimensionless coupling $\gamma$ ($g_1 = 1$ by definition). The exact values are represented by the solid lines, whereas the dashed lines show the empirical formula (7.14).

Figure 2. The quantity $g_4$ as a function of the dimensionless temperature $\tau$ for intermediate couplings. In the $\tau \to 0$ limit the three curves approach small (but non-zero) values which are shown in figure 1.

We found that the empirical formula

$$g_K \approx \exp \left(-\frac{K(K-1)}{\pi} \sqrt{\gamma} \right)$$

(7.14)

gives a surprisingly good approximation and can be used for practical purposes even at $\gamma \sim 1$. The predictions of (7.14) are also plotted in figure 1. It is expected that (7.14) holds with a good approximation even for higher $K$.

It would be useful to compare the exact numerical values for $g_3$ and $g_4$ cases to the various approximations available in the literature [31, 32] including the large-coupling
expansion both at zero and finite temperatures. This is out of the scope of the present work and is left for further research.

8. Mean values in the LeClair–Mussardo formalism

In this section we elaborate on the LeClair–Mussardo formalism, which is an alternative approach to obtain expectation values of local operators leading to an infinite integral series [54, 55, 34, 35, 37]. For our present purposes the following form of the series is the most convenient [37]:

$$
\langle O_K \rangle = \sum_{N} \frac{1}{N!} \int \frac{dp_1}{2\pi} \cdots \frac{dp_N}{2\pi} \left( \prod_{j=1}^{N} f(p_j) \omega(p_j) \right) F_{N,s}^{K}(p_1, \ldots, p_N), \tag{8.1}
$$

where

$$
\omega(p) = \exp \left( - \int \frac{dp'}{2\pi} f(p') \varphi(p - p') \right)
$$

and $f(p)$ is defined in (6.2). The form factors appearing in the above series are defined as

$$
F_{N,s}^{K}(p_1, \ldots, p_N) = \prod_{j<k} \frac{(p_j - p_k)^2}{(p_j - p_k)^2 + c^2} \times \lim_{\epsilon \to 0} F_{N}^{K}(\{p_j + \epsilon\}, \{p_j\}), \tag{8.2}
$$

where the form factor on the rhs is given by (5.4). This prescription is also called the ‘symmetric evaluation of the diagonal limit’: note that this limit is different from the way we obtained the mean value (5.6) because in (5.4) the Bethe equations were substituted into the matrix element before taking the diagonal limit. Therefore the object in (8.2) does not depend on the volume $L$. Note also that the lhs refers to a normalization where the norm of the Bethe state is given simply by the Gaudin determinant $G^{LL}$.

Alternatively (8.1) can be expressed as [37, 34, 35]

$$
\langle O_K \rangle = \sum_{N} \frac{1}{N!} \int \frac{dp_1}{2\pi} \cdots \frac{dp_N}{2\pi} \left( \prod_{j=1}^{N} f(p_j) \right) F_{N,c}^{K}(p_1, \ldots, p_N). \tag{8.3}
$$

Here $F_{N,c}^{K}(p_1, \ldots, p_N)$ are the so-called connected evaluations of the diagonal form factors. Their precise definition and the relation to $F_{N,s}^{K}(p_1, \ldots, p_N)$ can be found in [55, 37]. The series (8.3) was originally developed in [54] in the framework of integrable quantum field theories. Later it was used in [34, 35, 38] to compute the quantities $g_K$ up to $K = 3$. However, for higher $K$ the results (8.1)–(8.3) are only formal because the form factors themselves were not calculated previously. In [34, 35] a prescription was given of how to obtain the connected evaluation using a special non-relativistic limit of certain form factors of the sinh–Gordon model. However, the actual calculation becomes more and more demanding with higher $K$ and $N$.

We fill this gap here by calculating the explicit results for $F_{N,s}^{K}(p_1, \ldots, p_N)$ for arbitrary $K$ and $N$: we take the symmetric diagonal limit of the form factor (5.4). Note that every singularity of the form factor is included in the matrix $Z$ and even the elements of $Z$ can

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be evaluating easily. Taking the limit and multiplying with the prefactors we obtain

$$F_{N,s}^K(p_1,\ldots,p_N) = (K!)^2 \sum_{\{p^+\}_u \cup \{p^-\}_l = K} \left[ \prod_{j>l} \frac{p_j^+-p_l^+}{(p_j^+-p_l^+)^2+c^2} \right] \times \det Y.$$  \hspace{1cm} (8.4)

The elements of $Y$ are given by

$$Y_{j,l} = (p_j)^{l-1} \quad \text{if} \quad p_l \in \{p^+\}$$

$$Y_{j,l} = \delta_{j,l} \left( \sum_{o=1}^{N} \varphi(p_j-p_o) \right) - \varphi(p_j-p_l) \quad \text{if} \quad p_l \in \{p^-\}.$$  \hspace{1cm} (8.5)

Note that

$$Y = \mathcal{H}|_{L=0},$$

where $\mathcal{H}$ is the matrix defined in (5.7).

With these results the series (8.1) can be considered an explicit representation of the mean value.

It would be desirable to have a general recipe for the re-summation of the series, which would be an alternative way to obtain factorized formulae like (7.10) and (7.12). However, this is far from being easy. The simpler cases $K=1$ and $2$ were already calculated in [34,35]. The highly non-trivial case of $K=3$ was considered in [38], where the authors evaluated the series (8.3) (and obtained the result (7.10) for the first time) based on the following conjecture for the quantities $F_{3,c}^3$:

$$F_{N,c}^3 = \frac{1}{2c^2} \sum_{p} \varphi_{12} \varphi_{23} \cdots \varphi_{N-1,N} (p_{1N}^3 - p_{12}^3 - p_{23}^3 - \cdots - p_{N-1,N}^3).$$  \hspace{1cm} (8.6)

Here we check this formula in the first two cases. In the simplest case of $N=3$ our formula (8.4) gives

$$F_{3,s}^3 = F_{3,c}^3 = 36 \prod_{j>l} \frac{p_{jl}^3}{p_{jl}^3+c^2}.$$ 

This was already calculated in [34,35] and is in agreement with (8.6). In the case of $N=4$ one has to use the following relation between the symmetric and connected evaluations [55]:

$$F_{4,s}^3(p_1,p_2,p_3,p_4) = F_{4,c}^3(p_1,p_2,p_3,p_4) + \sum_j F_{3,c}^3(\hat{p}_j) \times \left( \sum_{k \neq j} \varphi_{jk} \right).$$

Here $\hat{p}_j$ means that $p_j$ is not present among the arguments of the form factor. We used the program Mathematica to express $F_{4,c}^3$, using the above relation and we found agreement with (8.6). This is a highly non-trivial check of the conjecture (8.6); a proof for arbitrary $N$ is not known.

Finally we note that in the simpler cases of $K=1$ and $2$ we evaluated (8.4) and found exact agreement with the corresponding formulae of appendix D in [37].

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9. Conclusions

We developed multiple integral formulae for the local correlations in the 1D Bose gas. The final results for the expectation value $\langle O_K \rangle$ is given by equation (6.7), whereas the dimensionless formula for $g_K$ is given by (6.10).

In section 7 we performed the explicit factorization of the multiple integrals in the cases $K = 2–4$; for $K = 3$ we obtained the recent result of [38] whereas our formula for $K = 4$ is new. Our method of factorization relies only on the integral equation (6.5) defining the auxiliary functions entering the multiple integral. Therefore the process works for arbitrary distribution of Bethe roots and not only for the ground state or the finite-temperature Gibbs states.

The general recipe of how to perform the factorization for $K > 4$ is not known. The strategy is clear: at each step the prefactors have to be manipulated in such a way that the number of integrals can be reduced by one using the integral equation (6.5). We believe that this can always be done and it would be interesting to develop a general algorithm for this process.

An alternative way to obtain the mean values $\langle O_K \rangle$ would be to take the thermodynamic limit on the XXZ spin chain first and to perform the scaling limit towards the Bose gas afterwards. The advantage of this approach would be that on the spin chain the factorization of the multiple integral formulae for the elements of the reduced density matrix is by now well understood (see [56] and references therein). In fact, we attempted to take the scaling limit of the factorized results of [57] concerning the emptiness formation probability. However, this turned out to be cumbersome already in the case $K = 2$. Thus it seems that the direct approach of the present paper is more advantageous, at least for the small values of $K$ considered here.

Another alternative way would be to sum up the LeClair–Mussardo series (8.1) or (8.3). The diagonal form factors entering (8.1) are given explicitly by (8.4). Therefore the remaining task is purely combinatorial: one has to expand the sums of determinants appearing in (8.4) and put the resulting expression in a form which is amenable for resummation. Again, this is a formidable problem, the solution of which is not yet known.

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