HIRZEBRUCH-RIEMANN-ROCH FOR GLOBAL MATRIX FACTORIZATIONS

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Abstract. We prove a Hirzebruch-Riemann-Roch type formula for global matrix factorizations. This is established by an explicit realization of the abstract Hirzebruch-Riemann-Roch type formula of Shklovsky. We also show a Grothendieck-Riemann-Roch type theorem.

1. Introduction

Let $k$ be a field of characteristic zero and let $G$ be either the group $\mathbb{Z}$ or $\mathbb{Z}/2$. We consider a $G$-graded dg enhancement $D_{dg}(X, w)$ of the derived category of matrix factorizations for $(X, w)$. Here $X$ is an $n$-dimensional nonsingular variety over $k$ and $w$ is a regular function on $X$. An object of $D_{dg}(X, w)$ is a $G$-graded vector bundle $E$ on $X$ equipped with a degree 1, $O_X$-linear homomorphism $\delta_E : E \to E$ such that $\delta_E^2 = w \cdot \text{id}_E$. The structure sheaf $O_X$ is by definition $G$-graded but concentrated in degree 0. The degree of $w$ is 2. If $w$ is nonzero, then $G$ is forced to be $\mathbb{Z}/2$. Assume that the critical locus of $w$ is set-theoretically in $w^{-1}(0)$ and proper over $k$.

The Hochschild homology $\text{HH}(D_{dg}(X, w))$ of $D_{dg}(X, w)$ is naturally isomorphic to

$$\mathbb{H}^{-s}(X, (\Omega^*_X, -dw));$$

see [10, 14]. The isomorphism is called the Hochschild-Kostant-Rosenberg (in short HKR) type isomorphism, denoted by $I_{HKR}$. Here $(\Omega^*_X, -dw)$ is a $G$-graded complex $\bigoplus_{p \in \mathbb{Z}} \Omega^p_X[p]$ with the differential $-dw$. Let $\text{ch}(E) \in \mathbb{H}^0(X, (\Omega^*_X, -dw))$ be the image of the categorical Chern character $\text{Ch}(E) \in \text{HH}_0(D_{dg}(X, w))$ of $E$ under $I_{HKR}$. In this paper we prove the following Hirzebruch-Riemann-Roch type formula.

**Theorem 1.1.** For matrix factorizations $P$ and $Q$ in $D_{dg}(X, w)$ we have

$$(1.1) \sum_{i \in G} (-1)^i \dim \mathbb{R}^i \text{Hom}(P, Q) = (-1)^{(n+1)/2} \int_X \text{ch}(P)^\vee \wedge \text{ch}(Q) \wedge \text{td}(X),$$

where $\text{td}(X) \in \bigoplus_{p \in \mathbb{Z}} H^0(X, \Omega^p_X[p])$ is the Todd class of $X$.
We explain notation in the above theorem. Firstly, the operation ∧ is the wedge product inducing
\[(1.2) \quad \mathbb{H}^*(X, (\Omega_X^\bullet, -dw)) \otimes \mathbb{H}^*(X, (\Omega_X^\bullet, dw)) \otimes \mathbb{H}^*(X, (\Omega_X^\bullet, 0)) \xrightarrow{\text{wedge}} \oplus_p H_c^*(X, \Omega_X^p[1]);\]
see §3.2 for details. Secondly, \(\int_X\) is the composition
\[(1.3) \quad \oplus_{p \in \mathbb{Z}} H_c^*(X, \Omega_X^p[1]) \xrightarrow{\text{proj}} H_c^0(X, \Omega_X^0[n]) \xrightarrow{\text{tr}_{X, k}} k\]
of the projection and the canonical trace map \(\text{tr}_X\) for the properly supported cohomology; see §3.6.2. Thirdly, \(\vee\) is induced from a chain map
\[(\Omega_X^\bullet, dw) \rightarrow (\Omega_X^\bullet, -dw), \text{ defined by } (-1)^{p} \text{id} : \Omega_X^p \rightarrow \Omega_X^p\]
in each component.

For a proper dg category \(\mathcal{A}\) there is an abstract Hirzebruch-Riemann-Roch formula (2.5) due to Shklyarov [18]. By an explicit realization of the formula for \(\mathcal{A} = D_{dg}(X, w)\) we will obtain Theorem 1.1. Let
\[\langle \cdot, \cdot \rangle_{\text{can}} : HH_* (\mathcal{A}) \otimes HH_* (\mathcal{A}^{\text{op}}) \rightarrow k\]
be the so-called canonical pairing for \(\mathcal{A}\). Here \(\mathcal{A}^{\text{op}}\) is the opposite category of \(\mathcal{A}\) and there is an isomorphism \(\mathcal{A}^{\text{op}} \cong D_{dg}(X, -w)\); see §3.1. This yields
\[(1.4) \quad \begin{array}{ccc}
HH_* (\mathcal{A}) & \xrightarrow{\text{IHKR}} & \mathbb{H}^{-*}(X, (\Omega_X^\bullet, -dw)) \\
\vee & & \vee \\
HH_* (\mathcal{A}^{\text{op}}) & \xrightarrow{\text{IHKR}} & \mathbb{H}^{-*}(X, (\Omega_X^\bullet, dw)),
\end{array}\]
where the left \(\vee\) is defined to make the diagram commute. Shklyarov’s formula says that the left-hand side of (1.4) is equal to \(\langle \text{Ch}(Q), \text{Ch}(P) \rangle_{\text{can}}\).

Therefore Theorem 1.1 is reduced to an explicit realization of the pairing.

**Theorem 1.2.** The canonical pairing \(\langle \cdot, \cdot \rangle_{\text{can}}\) under \(I_{\text{IHKR}}\) corresponds to
\[(-1)^{(n+1)/2} \int_X (\cdot \wedge \cdot \wedge \text{td}(X))\].

The following formula for \(\text{ch}(E)\) is established in [4] [9] [14]. Let \(\mathcal{U} = \{U_i\}_{i \in I}\) be an affine open covering of \(X\) and let \(\nabla_i\) be a connection of \(E|_{U_i}\). In the Čech hypercohomology \(\mathbb{H}^*(\mathcal{U}, (\Omega_X^\bullet, (\cdot)^1 dw \wedge))\)
\[\text{ch}(E) = \text{str} \exp(-\sum_{i \in I} (\nabla_i \wedge \nabla_j)_{i,j \in I, i < j}).\]
Here \(\text{str}\) means the supertrace and the products in the exponential are Alexander-Čech-Whitney cup products in the Čech complex \(\check{C}^*(\mathcal{U}, (\text{End}(E) \otimes \Omega_X^\bullet))\); see [3] for details.

Remarks on others’ related works are in order. When \(\mathbb{G} = \mathbb{Z}\) and \(X\) is a projective variety over \(\mathbb{C}\), there is a natural isomorphism between Hodge cohomology and the singular (or equivalently \(C^\infty\)-de Rham) cohomology of the associated complex manifold \(X^\text{an}\): \(\oplus_{p,q} H^q(X, \Omega_X^p)[p] \xrightarrow{\phi} H^*(X^\text{an}, \mathbb{C})\). Let \(\text{tw}\) be an automorphism of \(\oplus_{p,q} H^q(X, \Omega_X^p)[p]\) sending a \((p, q)\)-form \(\gamma^{p,q}\) to
(\frac{1}{2\pi i})^{p+q}, then \(\phi(\text{tw}(\text{ch}(E)))\) coincides with the topological Chern character \(\text{ch}_{\text{top}}(E)\) of \(E\). The right-hand side of \((1.1)\) becomes

\[
\int_{\mathcal{X}_{an}} \text{ch}_{\text{top}}(P^{\vee}) \text{ch}_{\text{top}}(Q) \tau_{\text{top}}(X).
\]

Here \(\int_{\mathcal{X}_{an}}\) denotes the usual integration and \(\tau_{\text{top}}(X)\) is the usual Todd class of \(X^{an}\). Hence Theorem \((1.1)\) is the usual Hirzebruch-Riemann-Roch theorem \([8]\). When \(G = \mathbb{Z}\) and \(k = \mathbb{C}\), Theorem \((1.1)\) is the O’Brian – Toledo – Tong theorem for algebraic coherent sheaves \([13]\). When \(G = \mathbb{Z}\), Theorem \((1.1)\) and its generalization Corollary \((3.3)\) coincide with Theorem 4 of Markarian \([12]\) and some works of Căldăruşă – Willerton \([2]\) and Ramadoss \([16]\), respectively.

When \(G = \mathbb{Z}/2\), \(X\) is an open subscheme of \(\mathcal{X}_{an}\) containing the origin, and \(w\) has only one singular point at the origin, the composition of wedge products \((1.2)\) and \(\int_{\mathcal{X}}\) in \((1.3)\) is a residue pairing as shown in \([1, \text{Proposition 4.34}]\). Thus in this case, Theorem \((1.1)\) is the Polishchuk – Vaintrob theorem \([15, \text{Theorem 4.1.4}]\) and Theorem \((1.2)\) is the Brown – Walker theorem \([1, \text{Theorem 1.8}]\) proving a conjecture of Shklyarov \([19, \text{Conjecture 3}]\).

It is natural to consider the stacky version of Theorem \((1.1)\). It will be treated elsewhere \([3]\).

Conventions: Unless otherwise stated a dg category is meant to be a \(G\)-graded dg category over \(k\). For a variety \(X\) over \(k\), we write simply \(\Omega_{X}^{p}\) for the sheaf \(\Omega_{X}^{p}\) of relative differential \(p\)-forms of \(X\) over \(k\). For a homogenous element \(a\) in a \(G\)-graded \(k\)-space, \(|a|\) denotes the degree of \(a\). For a dg category \(\mathcal{A}\) we often write \(\text{Hom}_{\mathcal{A}}(x, y)\) instead of the Hom complex \(\text{Hom}_{\mathcal{A}}(x, y)\) between objects \(x, y\) of \(\mathcal{A}\). For a dg algebra \(A\), \(C(A)\) denotes the Hochschild \(G\)-graded complex \(\bigoplus_{n \geq 0} A \otimes A[1]^{\otimes n}\) with differential \(b\). Similarly for a dg category \(\mathcal{A}\), \(C(\mathcal{A})\) denotes the Hochschild complex of \(\mathcal{A}\); see for example \([1, 18]\). For \(a_{i} \in \mathcal{A}(x_{i+1}, x_{i}), i = 1, \ldots, n, x_{i} \in \mathcal{A}\), we write \(a_{0}[a_{1}| \ldots | a_{n}]\) for \(a_{0} \otimes s_{a_{1}} \otimes \ldots \otimes s_{a_{n}}\), where \(s\) is the suspension so that \(|sa| = |a| - 1\). The symbol \(\text{str}\) stands for the supertrace. By a coherent factorization for \((X, w)\) we mean a \(G\)-graded coherent \(\mathcal{O}_{X}\)-sheaf \(E\) with a curved differential \(\delta_{E}\) such that \(\delta_{E}^{2} = w \cdot \text{id}_{E}\).

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2. Abstract Hirzebruch-Riemann-Roch

Following mainly \([15, 18]\) we review the abstract Hirzebruch-Riemann-Roch theorem in the framework of Hochschild homology theory.

2.1. Categorical Chern characters. For a \(G\)-graded dg category \(\mathcal{A}\) over \(k\) let \(C(\mathcal{A})\) be the Hochschild complex of \(\mathcal{A}\). For \(x \in \mathcal{A}\), the identity morphism \(1_{x}\) of \(x\) is a 0-cycle element and hence it defines a class

\[
\text{Ch}(x) := [1_{x}] \in \text{HH}_{0}(\mathcal{A}) := H^{0}(C(\mathcal{A})),
\]
which is called the categorical Chern character of $x$. For an object $y$ of the
dg category $\text{Perf}\mathcal{A}$ of perfect right $\mathcal{A}$-modules, we also regard $\text{Ch}(y)$ as an
element of $HH_0(\mathcal{A})$ by the canonical isomorphism $HH_*(\text{Perf}\mathcal{A}) \cong HH_*(\mathcal{A})$.

2.2. K"unneth isomorphism. Let $\mathcal{A}$, $\mathcal{B}$ be dg categories. We define a
natural chain map over $k$

$C(\mathcal{A}) \otimes C(\mathcal{B}) \to C(\mathcal{A} \otimes \mathcal{B})$

$a_0[a_1, ..., a_n] \otimes b_0[b_1, ..., b_m] \mapsto (-1)^{|b_0|(|\Sigma_{i=1}^n (|a_i| - 1)|)} a_0 \otimes b_0 \sum \sigma \pm [c_{\sigma(1)}, ..., c_{\sigma(n+m)}],$

where $\sigma$ runs for all $(n, m)$-shuffles, $c_1 = a_1 \otimes 1, ..., c_n = a_n \otimes 1, c_{n+1} =
1 \otimes b_1, ..., c_{n+m} = 1 \otimes b_m$, and the rule of sign $\pm$ is determined by the
Koszul sign rule. Here after each shuffle, each 1 is uniquely replaced by an
appropriate identity morphism so that the outcomes make sense as elements
of the Hochschild complex $C(\mathcal{A} \otimes \mathcal{B})$. The Eilenberg-Zilber theorem says that
the chain map is a quasi-isomorphism. We call the induced isomorphism

$HH_*(\mathcal{A}) \otimes HH_*(\mathcal{B}) \to HH_*(\mathcal{A} \otimes \mathcal{B})$

the K"unneth isomorphism, denoted by K"unn.

2.3. The diagonal bimodule. We denote by $\mathcal{A}^{op}$ the opposite category of
$\mathcal{A}$. For $x \in \mathcal{A}$ we write $x^\vee$ for the object of $\mathcal{A}^{op}$ corresponding to $x$. Let
$\text{Com}_{dg}k$ be the dg category of complexes over $k$. The diagonal $\mathcal{A}$-$\mathcal{A}$-bimodule
$\Delta_\mathcal{A}$ of a dg category $\mathcal{A}$ is defined to be the dg functor

$\Delta_\mathcal{A} : \mathcal{A} \otimes \mathcal{A}^{op} \to \text{Com}_{dg}k; \ y \otimes x^\vee \mapsto \text{Hom}_\mathcal{A}(x, y).$

Since $\mathcal{A} \otimes \mathcal{A}^{op} \cong (\mathcal{A}^{op} \otimes \mathcal{A})^{op}$, $\Delta_\mathcal{A}$ is a right $\mathcal{A}^{op} \otimes \mathcal{A}$-module. Assume that
$\mathcal{A}$ is proper, i.e.,

$\sum_{i \in \mathbb{G}} \dim H^i(\text{Hom}_\mathcal{A}(x, y)) < \infty$

for all $x, y \in \mathcal{A}$. Then we may replace the codomain of $\Delta_\mathcal{A}$ by the dg category
$\text{Perf}k$ of perfect dg $k$-modules.

2.4. The canonical pairing. For a proper dg category $\mathcal{A}$, the canonical
pairing $\langle \cdot, \cdot \rangle_{\text{can}}$ is defined as the composition

$HH_*(\mathcal{A}) \times HH_*(\mathcal{A}^{op}) \xrightarrow{\text{K"unn}} HH_*(\mathcal{A} \otimes \mathcal{A}^{op}) \xrightarrow{\Delta_*} HH_*(\text{Perf}k) \cong k,$

where $\Delta_*$ is the homomorphism in Hochschild homology level induced from
the dg functor $\Delta_\mathcal{A}$. Here we use the canonical isomorphism $HH_*(\text{Perf}k) \cong k$
making a commuting diagram for $C \in \text{Perf}k$

\begin{equation}
HH_* (\text{Perf}k) \xrightarrow{\cong} HH_*(k) \cong k, \quad \text{natural} \quad \text{str} \quad \text{ZEnd}(C)
\end{equation}
where \( \text{ZEnd}(C) \) is the graded \( k \)-space of closed endomorphisms of \( C \) and \( \text{str} \) denotes the supertrace. Since \( HH_\ast(\text{Perf}k) = HH_0(\text{Perf}k) \), the pair \( \langle \gamma, \gamma' \rangle \) for \( \gamma \in HH_p(A), \gamma' \in HH'_p(A^{\text{op}}) \) can be nontrivial only when \( p + p' = 0 \).

2.5. **A proposition.** Let \( A \) be a proper dg category. Let \( M \) be a perfect right \( A^{\text{op}} \otimes B \) module, in other words, a perfect \( A-B \)-bimodule. Denote by \( T_M \) the dg functor

\[
\otimes_A M : \text{Perf}A \rightarrow \text{Perf}B \quad \text{sending} \; N \mapsto N \otimes_A M
\]

and denote the induced map in Hochschild homology by

\[
(T_M)_\ast : HH_\ast(A) \rightarrow HH_\ast(B).
\]

**Proposition 2.1.** \([18, \text{Proposition 4.2}]\) If we write \( \text{Ch}(M) = \sum t_i \otimes t_i \in HH_\ast(A^{op}) \otimes HH_\ast(B) \cong HH_\ast(A^{op} \otimes B) \cong HH_\ast(\text{Perf}(A^{op} \otimes B)) \) via the Künneth isomorphism and the canonical isomorphism, then for every \( \gamma \in HH_p(A) \) we have

\[
(T_M)_\ast(\gamma) = \sum_i \langle \gamma, t_i \rangle_{\text{can}} t_i \in HH_p(B).
\]

**Proof.** The proof given in \([18]\) also works for dg categories. \( \square \)

Furthermore assume that \( A \) is smooth, i.e., the diagonal bimodule \( \Delta_A \) is perfect. Then the Hochschild homology of \( A \) is finite dimensional and hence Proposition 2.1 can be rewritten as a commuting diagram

\[
\begin{array}{ccc}
HH_\ast(A) & \xrightarrow{\langle \cdot, \cdot \rangle_{\text{can}}} & HH_\ast(A^{\text{op}}) \times HH_\ast(A^{\text{op}}) \times HH_\ast(B) \\
(T_M)_\ast & \downarrow{\text{Ch}(M)} & \downarrow{\text{Ch}(M)} \\
HH_\ast(B) & & \end{array}
\]

Since \( T_{\Delta_A} = \text{id}_A \), the above diagram for \( M = \Delta_A \) shows that \( \langle \cdot, \cdot \rangle_{\text{can}} \) is non-degenerate and the canonical pairing is characterized as follows.

Since \( \Delta_A \in \text{Perf}(A^{op} \otimes A) \), via the Künneth isomorphism we can write

\[
\text{Ch}(\Delta_A) = \sum_i T_i \otimes T_i, \; \text{for some} \; T_i \in HH_\ast(A^{op}), \; T_i \in HH_\ast(A).
\]

Then \( \langle \cdot, \cdot \rangle_{\text{can}} \) is a unique non-degenerate \( k \)-bilinear map \( \langle \cdot, \cdot \rangle : HH_\ast(A) \times HH_\ast(A^{op}) \rightarrow k \) satisfying

\[
\sum_i \langle \gamma, T_i \rangle \langle T_i, \gamma' \rangle = \langle \gamma, \gamma' \rangle, \; \text{for every} \; \gamma \in HH_\ast(A), \gamma' \in HH_\ast(A^{op}).
\]

2.6. **The chain map \( \triangledown \).** Define an isomorphism of complexes

\[
\triangledown : (C(A), b) \rightarrow (C(A^{op}), b)
\]

\[
a_0[a_1][a_2][a_3] \rightarrow (-1)^{n+1} \sum_{1 \leq i < j \leq n} a_i a_j a_i a_j (-1)^{|a_i| - 1} a_0 a_1 \cdots a_n.
\]

**Remark 2.2.** Using the quasi-Yoneda embedding and the HKR-type isomorphism it is straightforward to check that the chain map \( \triangledown \) in § 2.6 fits in diagram \([1,4]\); see for example \([4]\).
2.7. **Abstract generalized HRR.** For a proper dg category $\mathcal{A}$ we may consider a sequence of natural maps

\[(2.3) \quad HH_*(\mathcal{A}) \otimes HH_*(\mathcal{A}) \xrightarrow{\sim} HH_*(\mathcal{A}) \otimes HH_*(\mathcal{A}^{op}) \xrightarrow{\text{Kinn}} HH_*(\mathcal{A} \otimes \mathcal{A}^{op}) \xrightarrow{\Delta_*} HH_*(\text{Perf}k) \cong k.
\]

For two closed endomorphisms $b \in \text{End}_\mathcal{A}(y)$, $a \in \text{End}_\mathcal{A}(x)$, we define an endomorphism $Lb \: \Rightarrow Ra$ of $\text{Hom}_\mathcal{A}(x, y)$ by sending $c$ to $p_1 q | a | c | b \Rightarrow c | a$. Note that $\Delta_*(b \otimes a) = Lb \otimes Ra$. Hence from (2.3) and (2.1) we obtain

\[(2.4) \quad \text{str}(Lb \otimes Ra) = \langle [b], [a] \rangle_{\text{can}},
\]

Here $[b], [a]$ are the homology classes in $HH_0(\mathcal{A})$ represented by $b, a$, respectively.

2.8. **Abstract HRR.** When $a = 1_x$, $b = 1_y$, (2.4) yields the abstract Hirzebruch-Riemann-Roch theorem [15, 18] for Hochschild homology:

\[(2.5) \quad \sum_{i \in \mathbb{Z}} (-1)^i \dim H^i(\text{Hom}_\mathcal{A}(x, y))) = \langle \text{Ch}(y), \text{Ch}(x)^\vee \rangle_{\text{can}}.
\]

This tautological HRR theorem can be useful when one expresses the right-hand side of (2.5) in an explicit form.

### 3. Proofs of Theorems

In this section we prove Theorems 1.1 and 1.2. As in § 1 let $X$ be an $n$-dimensional nonsingular variety over $k$ and $w$ is a function on $X$ such that the critical locus of $w$ is in $w^{-1}(0)$ and proper over $k$.

3.1. **A geometric realization of $\Delta_\mathcal{A}$.** Let $\mathcal{A} = D_{dg}(X, w)$. It is proper and smooth. There is the duality functor

\[D : \mathcal{A}^{op} \to D_{dg}(X, -w); (E, \delta_E) \mapsto (\text{Hom}_{\mathcal{O}_X}(E, \mathcal{O}_X), \delta_E^\vee),\]

which is an isomorphism. Hence we have the HKR type isomorphism

\[HH_*(\mathcal{A}^{op}) \cong \mathbb{H}^{-\circ}((\Omega^*_X, dw)).\]

Let $X'$ be another nonsingular variety with a global function $w'$. Assume that the critical locus of $w'$ is proper over $k$ and located on the zero locus of $w'$. Let $\mathcal{B}$ denote $D_{dg}(X', w')$. Let $\tilde{w} := w \otimes 1 - 1 \otimes w'$ a global function on $Z := X \times X'$. We consider a dg functor

\[\Psi : D_{dg}(Z, -\tilde{w}) \to \text{Perf}(\mathcal{A}^{op} \otimes \mathcal{B})\]

defined by letting

\[\Psi(z) : \mathcal{A} \otimes \mathcal{B}^{op} \to \text{Com}_{dg}(k);
\]

\[y \otimes x^\vee \mapsto \text{Hom}_{D_{dg}(Z, -\tilde{w})}(D(y) \boxtimes x, z)\]
for \( y \in \mathcal{A} \), \( x \in \mathcal{B} \), \( z \in D_{dg}(Z, -\bar{w}) \). Since \( \mathcal{A}^{op} \otimes \mathcal{B} \) are saturated by \([10]\), we may apply Proposition 3.4 of \([17]\) to see that \( \Psi(z) \) is indeed a perfect right \( \mathcal{A}^{op} \otimes \mathcal{B} \)-module.

Let \( f : X \to X' \) be a proper morphism such that \( f^*w' = w \). Then there is a dg functor
\[
\mathbb{R}f_* : D_{dg}(X, w) \to D_{dg}(X', w')
\]
by derived pushforward; see \([5\), § 2.2]. Define
\[
\Delta_{\mathbb{R}f_*} : \mathcal{A} \otimes \mathcal{B}^{op} \to \text{Com}_{dg}k; \ y \otimes x^\vee \to \text{Hom}_B(x, \mathbb{R}f_*y).
\]
Again by Proposition 3.4 of \([17]\), we see that \( \Delta_{\mathbb{R}f_*} \) is a perfect right \( \mathcal{A}^{op} \otimes \mathcal{B} \)-module. Let \( \Gamma_f \subset X \times X' \) denote the graph of \( f \). Since \( Z = X \times X' \) is nonsingular, there is an object \( \mathcal{O}_{\Gamma_f}^{\tilde{w}} \) in \( D_{dg}(Z, -\bar{w}) \) which is quasi-isomorphic to the coherent factorization \( \mathcal{O}_{\Gamma_f} \) for \( (Z, \bar{w}) \). Since
\[
\Delta_{\mathbb{R}f_*}(y \otimes x^\vee) = \mathcal{B}(x, \mathbb{R}f_*y) \cong \text{Hom}_{D_{dg}(Z, -\bar{w})} (D(y) \boxtimes x, \mathcal{O}_{\Gamma_f}^{\tilde{w}}),
\]
by the projection formula \([5\), § 2.2], \( \Delta_{\mathbb{R}f_*} \) and \( \Psi(\mathcal{O}_{\Gamma_f}^{\tilde{w}}) \) are isomorphic in the derived category of right \( \mathcal{A}^{op} \otimes \mathcal{B} \)-modules. Hence \( \text{Ch}(\Delta_{\mathbb{R}f_*}) = \text{Ch}(\Psi(\mathcal{O}_{\Gamma_f}^{\tilde{w}})) \).

Consider a dg functor
\[
\boxtimes : \mathcal{A}^{op} \otimes \mathcal{B} \to D_{dg}(Z, -\bar{w}); u^\vee \otimes v \mapsto D(u) \boxtimes v.
\]
The following commutative diagram of natural isomorphisms transforms the abstract terms to the concrete terms:

\[
\begin{array}{c}
\begin{array}{ccc}
\text{HH}_\#(\mathcal{A}^{op}) \otimes \text{HH}_\#(\mathcal{B}) & \xrightarrow{=} & \text{HH}_\#(\mathcal{A}^{op} \otimes \mathcal{B}) \\
\text{Kunn} & & \text{Yoneda} \\
\downarrow & & \downarrow \\
\text{HH}_\#(D_{dg}(Z, -\bar{w})) & \xrightarrow{=} & \text{HH}_\#(\text{Perf}(\mathcal{A}^{op} \otimes \mathcal{B})) \\
\end{array}
\end{array}
\]

The commutativity of the triangle is straightforward. The commutativity of the rectangle can be seen as follows. Using the Mayer-Vietoris sequence argument, we reduce it to the case when \( X \) and \( X' \) are affine. We further reduce it to the curved smooth algebra case. In the curved smooth algebra case, the commutativity of a corresponding diagram for Hochschild complexes of the second kind is straightforward; see for example \([4]\).

We conclude that
\[
\text{ch}(\Delta_{\mathbb{R}f_*}) = \text{ch}(\mathcal{O}_{\Gamma_f}^{\tilde{w}}) \in \mathbb{H}^0(\Omega^*_X, d\bar{w})
\]
by the compatibility of the K"unneth isomorphisms and the HKR type isomorphisms in \((3.1)\). In particular for \( f = \text{id}_X \) we have
\[
\text{ch}(\Delta_A) = \text{ch}(\mathcal{O}_{\Delta X}^{\tilde{w}}) \in \mathbb{H}^0(\Omega^*_X, d\bar{w})
\]
if the subscript \( \Delta_X \) denote \( \Gamma_{\text{id}_X} \).
3.2. Some definitions.

**Definition 3.1.** Considering a vector bundle $F$ as an object in the derived category of coherent sheaves on $X$, we have the categorical Chern character of $F$ and hence $\text{ch}(F) \in \bigoplus_p H^0(X, \Omega^p_X[p]) \cong I^{HKR}_{HH_0}(D^b(\text{coh}(X)))$. Using this and the Todd class formula in terms of Chern roots we define $\text{td}(F) \in \bigoplus_p H^0(X, \Omega^p_X[p])$, which we call the Todd class of $F$ valued in Hodge cohomology. We write $\text{td}(X)$ for $\text{td}(T_X)$, called the Todd class of $X$. Similarly, we define the $i$-th Chern class $c_i(F)$ of $F$ valued in Hodge cohomology.

**Definition 3.2.** Let $w_i \in \Gamma(X, \mathcal{O}_X)$, $i = 1, 2$ and let $Z_i$ be the critical locus of $w_i$. The wedge product $\wedge$ of twisted Hodge cohomology classes is defined by the composition of

$$
\bigoplus_{q_1 + q_2 = q \in \mathbb{G}} H^{q_1}(X, (\Omega^*_{X, dw_1}) \otimes H^{q_2}(X, (\Omega^*_{X, dw_2}))
\xrightarrow{\text{K"unneth}} H^q(X^2, (\Omega^*_{X, dw_1}) \boxtimes (\Omega^*_{X, dw_2}))
\xrightarrow{\Delta^*_X} H^q(X, (\Omega^*_{X, dw_1}) \boxtimes_{\mathcal{O}_X} (\Omega^*_{X, dw_2}))
\xrightarrow{\text{wedge}} H^q_{Z_1 \cap Z_2}(X, (\Omega^*_{X, dw_1} + dw_2)).
$$

Here $\Delta^*_X$ is the pullback of the diagonal morphism $X \to X \times X$. We sometimes omit the symbol $\wedge$ for the sake of simplicity.

**Definition 3.3.** Let $Z$ denote the critical locus of $w$. Consider a sequence of maps

$$(3.4) \quad H^*(X, (\Omega^*_{X, -dw})) \times H^*(X, (\Omega^*_{X, dw})) \xrightarrow{\langle \cdot, \cdot \rangle} H^*_Z(X, (\Omega^*_{X, 0}))
\xrightarrow{\langle \text{td}(X) \rangle} H^*_Z(X, (\Omega^*_{X, 0})) \xrightarrow{\text{proj}} H^0_Z(X, \Omega^n_X[n]) \to H^0_c(X, \Omega^n_X[n]) \xrightarrow{(-1)^{\left(\frac{n+1}{2}\right)}b} k.$$

Denote the composition $(-1)^{\left(\frac{n+1}{2}\right)}\int_X \langle \cdot \wedge \cdot \wedge \text{td}(X) \rangle$ by $\langle \cdot, \cdot \rangle$.

3.3. **Proof of Theorem 1.2.** Since $\text{td}(X)$ is invertible, the nondegeneracy of $\langle \cdot, \cdot \rangle$ follows from Serre’s duality; see [6, § 4.1]. Therefore it is enough to show that $\langle \cdot, \cdot \rangle$ satisfies (2.2) under the HKR-type isomorphism in (3.1).

Recalling (3.3), we write

$$
\text{ch}(\mathcal{O}_{\Delta}^\wedge) = \sum_i t^i \otimes t_i \in \bigoplus_{q \in \mathbb{Z}} H^q(X, (\Omega^*_{X, dw}) \otimes H^{-q}(X, (\Omega^*_{X, -dw})).
$$

For $\gamma \in H^*(\Omega^*_{X, -dw})$ and $\gamma' \in H^*(\Omega^*_{X, dw})$, we have

$$(3.5) \quad \sum_i \langle t^i \otimes t_i, \gamma' \rangle = \int_{X \times X} \langle \gamma \otimes \gamma' \rangle \wedge \text{ch}(\mathcal{O}_{\Delta}^\wedge) \wedge (\text{td}(X) \otimes \text{td}(X)),$$

since $\int_X \otimes k \int_X = \int_{X \times X} \circ \text{K"unneth}$. 

Since $O_{\Delta_X}$ is supported on the diagonal $\Delta_X \subset X \times X$, we will apply the deformation of $X \times X$ to the normal cone of $\Delta_X$. The normal cone is isomorphic to the tangent bundle $T_X$ of $X$. Let $\pi$ denote the projection $T_X \to X$. We claim a sequence of equalities

$$\text{RHS of (3.5)} \overset{(\dagger)}{=} \int_{T_X} \pi^*(\gamma \wedge \gamma') \wedge \text{ch}(\text{Kos}(s)) \wedge \pi^*\text{td}(X)^2$$
$$\overset{(\dagger\dagger)}{=} (-1)^{(n+1)} \int_X (\gamma \wedge \gamma' \wedge \text{td}(X)) = \langle \gamma, \gamma' \rangle,$$

whose proof will be given below. Here $s$ is the 'diagonal' section of $\pi^*T_X$ defined by $s(v) = (v, v) \in \pi^*T_X$ for $v \in T_X$ and $\text{Kos}(s)$ is the Koszul complex $(\bigwedge^* \pi^*T^*_X, \iota_s)$ associated to $s$.

For $(\dagger)$ consider the deformation space $M^\circ$ of $X \times X$ to the normal cone of the diagonal $\Delta_X$; see [7]. It is a variety with morphisms $h : M^\circ \to X \times X$ and $pr : M^\circ \to \mathbb{P}^1$, satisfying that (i) the preimages of general points of $\mathbb{P}^1$ are $X \times X$, (ii) the preimage of a special point $\infty$ of $\mathbb{P}^1$ is the normal cone $N_{\Delta_X/X^2} = T_X$, (iii) $pr$ is a flat morphism, and (iv) $h|_{T_X}$ coincides with the composition $\Delta \circ \pi$.

The morphism $\Delta \times \text{id}_{\mathbb{A}^1} : X \times \mathbb{A}^1 \to X \times X \times \mathbb{A}^1$ extends to a closed immersion $f : X \times \mathbb{P}^1 \to M^\circ$. For a closed point $p$ of $\mathbb{P}^1$ let $M^\circ_p$ denote the fiber $pr^{-1}(p)$ and consider the commuting diagram

\[
\begin{array}{cccc}
X & \xrightarrow{f} & X \times \mathbb{P}^1 & \xleftarrow{f^\circ} X \\
\Downarrow \Delta = f_0 & & \Downarrow f & \Downarrow f_x = \text{zero section} \\
X^2 & \xrightarrow{g_0} & M^\circ & \xrightarrow{g_x} M^\circ_p = T_X \\
\Downarrow 0 & & \Downarrow pr & \Downarrow \Delta \circ \pi \\
\mathbb{P}^1 & \xleftarrow{h} & X^2 & \\
\end{array}
\]

with three fiber squares. Since $X \times \mathbb{P}^1$ and $M^\circ_p$ are Tor independent over $M^\circ$, we have

$$\text{(3.6)} \quad \mathbb{L}_{g_0^*f_*O_{X \times \mathbb{P}^1}} \sim (f_p)_*O_X,$$

i.e., they are quasi-isomorphic as coherent factorizations for $(M_p, -h^*\tilde{w}|_{M_p})$. Note that $h^*\tilde{w}|_{T_X \otimes O_X} = 0$. Since $s$ is a regular section with the zero locus $X \subset T_X$, two factorizations $(f_\infty)_*O_X$ and $\text{Kos}(s)$ are quasi-isomorphic to each other as coherent factorizations for $(T_X, 0)$:

$$\text{(3.7)} \quad (f_\infty)_*O_X \sim \text{Kos}(s).$$
For $\rho = (\gamma \otimes \gamma') \cdot (\text{td}(X) \otimes \text{td}(X))$, we have a sequence of equalities

\[
\int_{X \times X} \rho \wedge \text{ch}((f_0)_* \mathcal{O}_X)
= \int_{X \times X} \rho \wedge \text{ch}(Lg_0^*f_*\mathcal{O}_{X \times \mathbb{P}^1}) \quad \text{by (3.6)}
= \int_{X \times X} g_0^*(h^*\rho \wedge \text{ch}(f_*\mathcal{O}_{X \times \mathbb{P}^1})) \quad \text{by the functoriality of ch}
= \int_{T_X} g_0^*(h^*\rho \wedge \text{ch}(f_*\mathcal{O}_{X \times \mathbb{P}^1})) \quad \text{by Lemma 3.7}
= \int_{T_X} \pi^*\Delta^*\rho \wedge \text{ch}(\text{Kos}(s)) \quad \text{by (3.6) & (3.7)},
\]

which shows (†).

The equality (††) immediately follows from some basic properties of the proper pushforward (3.11) in Hodge cohomology: the functoriality (3.18), the projection formula (3.19), and (3.21).

### 3.4. Proof of Theorem 1.1

For $\alpha \in \oplus_{i \in \mathbb{G}} \mathbb{R}^{i \text{End}}(P)$ and $\beta \in \oplus_{i \in \mathbb{G}} \mathbb{R}^{i \text{End}}(Q)$, let us define

\[
L_\beta \circ R_\alpha : \oplus_{i \in \mathbb{G}} \mathbb{R}^{i \text{Hom}}(P, Q) \rightarrow \oplus_{i \in \mathbb{G}} \mathbb{R}^{i \text{Hom}}(P, Q), \quad c \mapsto (-1)^{||\alpha|||c|} \beta \circ c \circ \alpha.
\]

Since $\alpha$ and $\beta$ are cycle classes of $C(A)$, they can be considered as elements of $HH_*(A)$. We denote by $\tau(\alpha)$, $\tau(\beta)$ be the image of $\alpha$, $\beta$ under the HKR map. The map $\tau$ is sometimes called the boundary-bulk map. Combining (2.4) and Theorem 1.2 we obtain this.

**Corollary 3.4.** (The Cardy Condition) We have

(3.8) \[
\text{str}(L_\beta \circ R_\alpha) = (-1)^{\binom{n+1}{2}} \int_X \tau(\beta) \wedge \tau(\alpha)^\vee \wedge \text{td}(X).
\]

In particular, Theorem 1.1 holds.

Corollary (3.8) is the matrix factorization version of Theorem 16 of [2] and the explicit Cardy condition in [16].

Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an affine open covering of $X$ and let $\nabla_i$ be a connection of $P|_{U_i}$, which always exists. By [4, 9, 14] the following formula for $\tau(\alpha)$ in the Čech cohomology $\check{H}^0(\mathcal{U}, (\Omega^\bullet_X, dw))$ is known:

\[
\tau(\alpha) = \text{str} \left( (\exp(-(\nabla_i, \delta_E))_i - (\nabla_i - \nabla_j)_{i < j})\check{\alpha} \right),
\]

where $\check{\alpha}$ is a Čech representative of $\alpha$. Here we recall that $\Omega^\bullet_X = \oplus_{p=0}^n \Omega^p_X[p]$ is $\mathbb{G}$-graded.

In the local case, i.e., $X$ is an open neighborhood of the origin 0 in $\mathbb{A}^n_k$ and $w$ has a critical point only at 0 with $w(0) = 0$, we can relate the canonical
pairing with a residue pairing. Let \( x = (x_1, \ldots, x_n) \) be a local coordinate system and let \( \partial_i w = \frac{\partial w}{\partial x_i} \). Proposition 4.34 of [1] shows that

\[
\int_X (\tau(\beta) \wedge \tau(\alpha)\wedge) = \text{Res}_{x=0} \left[ \frac{g(x)f(x)}{\partial_1 w, \ldots, \partial_n w} \right]
\]

for \( \tau(\alpha) = f(x)dx_1 \ldots dx_n, \tau(\beta) = g(x)dx_1 \ldots dx_n \) in \( \Omega_X^*/dw \wedge \Omega_X^{-1} \). Hence from Theorem 1.2 and \( \tau(\alpha') = \tau(\alpha)^\wedge \) we immediately obtain this.

**Corollary 3.5.** [1, 15] In the local case we have

\[
\langle \beta, \alpha'^\wedge \rangle_{can} = (-1)^{\left(\begin{array}{c} n+1 \\ 2 \end{array}\right)} \text{Res}_{x=0} \left[ \frac{g(x)f(x)}{\partial_1 w, \ldots, \partial_n w} \right].
\]

The corollary above reproves a conjecture of Shklyarov [19, Conjecture 3].

### 3.5. GRR type theorem.

Consider the proper morphism \( f : X \to X' \) in § 3.1, inducing the dg functor \( \mathbb{R}f_* \) and the module \( \Delta_{\mathbb{R}f_*} \in \text{Perf}(A^{op} \otimes B) \). They together make a commutative diagram

\[
\begin{array}{ccc}
A := D_{dg}(X, w) & \xrightarrow{\mathbb{R}f_*} & B := D_{dg}(X', w') \\
Yone & \xrightarrow{T_{\Delta_{\mathbb{R}f_*}}} & \text{Perf}B.
\end{array}
\]

The paring defined by the composition

\[
\mathbb{H}^*(X', (\Omega_{X'}^*, -dw')) \otimes \mathbb{H}^*(X', (\Omega_{X'}^*, dw')) \xrightarrow{\triangle} \mathbb{H}^*_c(X', (\Omega_{X'}^*, 0)) \xrightarrow{[X'_0, k]}
\]

is nondegenerate by the Serre duality; see [6, § 4.1]. Using the paring we define the pushforward for \( q \in \mathbb{C} \)

\[
\int_f : \mathbb{H}^q(X, (\Omega_X^*, -dw)) \to \mathbb{H}^q(X', (\Omega_{X'}^*, -dw'))
\]

by the projection formula requirement

\[
\int_{X'} (\int_f \alpha) \wedge \beta = \int_X \alpha \wedge f^* \beta
\]

for every \( \beta \in \mathbb{H}^{-q}(X', (\Omega_{X'}^*, dw')) \).

Let \( n = \dim X \) and \( m = \dim X' \). Denote by \( HH(\mathbb{R}f_*) \) the map in Hochschild homology level from \( \mathbb{R}f_* \). Let \( K_0(A), K_0(B) \) be the Grothendieck group of the homotopy category of \( A, B \), respectively.
Theorem 3.6. The diagram

\[
\begin{array}{ccc}
K_0(A) & \xrightarrow{\mathbb{R}f_*} & K_0(B) \\
\text{Ch} & & \text{Ch} \\
HH_*(A) & \xrightarrow{HH(\mathbb{R}f_*)} & HH_*(B) \\
I_{HKR} & & I_{HKR} \\
\mathbb{H}^-(X, (\Omega^*_X, -dw)) & \xrightarrow{(1-\frac{1}{2})\int_f \wedge \text{td}(T_f)} & \mathbb{H}^-(X', (\Omega^*_X, -dw'))
\end{array}
\]

is commutative. Here \(\text{td}(T_f) := \text{td}(X)/f^*\text{td}(X')\) and \(\# = \begin{pmatrix} n+1 \\ 2 \end{pmatrix} - \begin{pmatrix} m+1 \\ 2 \end{pmatrix}\).

Proof. By the definition of categorical Chern characters the upper rectangle is commutative. Consider \(\gamma \in HH_*(A)\). Let \(\alpha := I_{HKR}(\gamma)\) and \(\alpha' := I_{HKR}(HH(\mathbb{R}f_*)(\gamma))\). If we write \(ch(\Delta_{Rf_*}) = \sum_i T^i \otimes T^i \in H^*(X, (\Omega^*_X, dw)) \otimes H^*(X, (\Omega^*_X, -dw))\), then by Proposition 2.1 and Theorem 1.2 we have for \(\beta \in H^-(X', (\Omega^*_X, dw'))\)

\[
(3.9) \int_{X'} \alpha' \wedge \beta \wedge \text{td}(X') = (-1)^{\begin{pmatrix} n+1 \\ 2 \end{pmatrix}} \sum_i \int_X \alpha \wedge T^i \wedge \text{td}(X) \int_{X'} T_i \wedge \beta \wedge \text{td}(X').
\]

By (3.2) and a normal-cone deformation argument as in § 3.3 we have

RHS of (3.9)

\[
= (-1)^{\begin{pmatrix} n+1 \\ 2 \end{pmatrix}} \int_{X \times X'} (\alpha \otimes \beta) \wedge ch(\Omega^\wedge_{X'} \otimes \text{td}(X) \otimes \text{td}(X'))
\]

\[
= (-1)^{\begin{pmatrix} n+1 \\ 2 \end{pmatrix}} \int_{f^*T_{X'}} \pi^*(\alpha \wedge f^*\beta \wedge \text{td}(f^*T_{X'}) \wedge \text{td}(X)) \wedge ch(\text{Kos}(s))
\]

\[
= (-1)^{\frac{1}{2}} \int_X \alpha \wedge f^*\beta \wedge \text{td}(X) = (-1)^{\frac{1}{2}} \int_{X'} \left( \int_f \alpha \wedge \text{td}(X) \right) \wedge \beta,
\]

where \(\pi\) denotes the projection \(f^*T_{X'} \to X\) and \(s\) is the diagonal section of \(\pi^* f^*T_{X'}\) on \(f^*T_{X'}\). Hence LHS of (3.9) equals \((-1)^{\frac{1}{2}} \int_{X'} (\int_f \alpha \wedge \text{td}(X)) \wedge \beta\), which shows the commutativity of the lower rectangle. \(\square\)

3.6. Pushforward in Hodge cohomology. We collect some properties of pushforwards in Hodge cohomology that are used in § 3.3 For lack of a suitable reference we provide their proofs.

Throughout this subsection \(f : X \to Y\) will be a morphism between varieties \(X, Y\) with varieties \(n, m\) respectively. Let \(d = n - m\).

3.6.1. Definition of \(f_*\). Suppose that \(f\) is a proper locally complete intersection (l.c.i) morphism. Let \(E\) be a perfect complex on \(Y\). Denote by

\[
\tau_f : \mathbb{R}f_* f^! E \to E
\]
the duality map in the derived category $D^{+}_{qc}(\mathcal{O}_Y)$ of cohomologically bounded below quasi-coherent sheaves; see for example [11, § 4]. Since $f$ is l.c.i, $f^!\mathcal{O}_Y$ is taken to be an invertible sheaf up to shift and there is a canonical isomorphism $\mathbb{L}f^*E \otimes f^!\mathcal{O}_Y \cong f^!E$. For $q \in \mathbb{Z}$, let $\delta \in \mathbb{H}^q(X, f^!E)$, which can be considered as a map $\delta : \mathcal{O}_X[-q] \to f^!E$ in the derived category. We have a composition of maps

$$\mathcal{O}_Y[-q] \xrightarrow{\text{natural}} \mathbb{R}f_*f^*\mathcal{O}_Y[-q] \xrightarrow{\mathbb{R}f_*(\delta)} \mathbb{R}f_*f^!E \xrightarrow{\tau_f} E,$$

denoted by $f_*(\delta)$. This yields a homomorphism

$$f_* : \mathbb{H}^q(X, f^!E) \to \mathbb{H}^q(Y, E).$$

Let $g : Y \to Z$ be a proper l.c.i. morphism between varieties. The uniqueness of adjunction implies the functoriality of the pushforward

$$(g \circ f)_* = g_* \circ f_* : \mathbb{H}^q(X, (g \circ f)^!F) \to \mathbb{H}^q(Z, F)$$

for $F$ in $D^{+}_{qc}(\mathcal{O}_Z)$.

3.6.2. Definitions of $\mathcal{F}_f$ and $\mathcal{F}_X$. Let $f : X \to Y$ be a morphism between nonsingular varieties. For $p \geq 0$ with $p - d \geq 0$ we have a natural homomorphism

$$\Omega^p_X[q] \cong \bigwedge^{n-p} T_X[q] \otimes f^*\Omega^p_Y[-d] \otimes f^!\mathcal{O}_Y$$

$$\to \bigwedge^{n-p} f^*T_Y[q] \otimes f^*\Omega^p_Y[-d] \otimes f^!\mathcal{O}_Y \cong f^*\Omega^{p-d}_Y[q - d] \otimes f^!\mathcal{O}_Y.$$

denoted by $\mathcal{F}_f$.

We define Hodge cohomology with proper supports along $f$ as the direct limit:

$$H^q_{cf}(X, \Omega^p_X) := \lim_{\rightarrow} H^q_Z(X, \Omega^p_X),$$

where $Z$ runs over all closed subvarieties of $X$ that are proper over $Y$. By Nagata’s compactification and the resolution of singularities there is a nonsingular variety $\bar{X}$ including $X$ as an open subvariety and a proper morphism $\bar{f} : \bar{X} \to Y$ extending $f$. Recall the fact that if $Z$ is a closed subvariety of $X$ that is proper over $Y$, then $Z$ is a closed subvariety of $\bar{X}$. Let

$$\text{nat} : H^q_{\bar{f}^!}(X, \Omega^p_X) \to H^q_{\bar{f}_2^*}(\bar{X}, \Omega^p_{\bar{X}})$$

be the natural map where $(\bar{z}_1, \bar{z}_2)$ is either $(c, c)$ or $(cf, \emptyset)$. We define the pushforward (for $p \geq 0$ with $p - d \geq 0$)

$$(3.11) \quad \int_f : H^q_{\bar{f}_1^!}(X, \Omega^p_X) \to H^q_{\bar{f}_2^*}(Y, \Omega^{p-d}_Y); \gamma \mapsto \bar{f}_*(\mathcal{F}_f(\text{nat}(\gamma))).$$

Using the functoriality (3.10), we note that $\int_f$ is independent of the choices of $\bar{X}$, an open immersion $X \hookrightarrow \bar{X}$, and an extension $\bar{f}$. When $Y = \text{Spec} \mathbb{C}k$,
If \( v : X' \to X \) be a proper morphism between nonsingular varieties, we have the natural pullback map
\[
v^* : H^q_c(X, \Omega^p_X) \to H^q_c(X', \Omega^p_{X'}). \]

3.6.3. **Base change I.** Consider a fiber square diagram of varieties
\[
\begin{array}{ccc}
X' & \xrightarrow{u} & X \\
\downarrow{g} & & \downarrow{f} \\
Y' & \xrightarrow{u} & Y
\end{array}
\]  
Assume that \( f \) is a flat, proper, l.c.i morphism. Then from the base change [11 § 4.4] we obtain a base change formula, for \( \delta \in H^q_c(X, f^!\mathcal{O}_Y) \)
\[
g_* (\mathbb{L}v^*(\delta)) = \mathbb{L}u^* (f_*(\delta))
\]
in \( H^q(Y', \mathcal{O}_{Y'}) \). Here \( \mathbb{L}v^*(\delta) \in H^q(X', g^!\mathcal{O}_Y) \) is the naturally induced map
\[
\mathcal{O}_{X'}[-q] \to \mathbb{L}v^* f^!\mathcal{O}_Y \cong g^! \mathbb{L}u^* \mathcal{O}_Y = g^! \mathcal{O}_Y,
\]
in the derived category.

Furthermore suppose that all varieties \( X, Y, X' \) are nonsingular and \( Y' \) is a closed point of \( Y \). Then for \( \gamma \in H^d_c(X, \Omega^d_{X'}) \) we easily check that
\[
v^*(\gamma) = \mathbb{L}v^* (\mathcal{E}_f(\gamma))
\]
in \( H^d_c(X', \Omega^d_{X'}) = H^0_c(X', g^!\mathcal{O}_{Y'}) \). Hence (3.13) for \( \delta = \mathcal{E}_f(\gamma) \) means that
\[
\int_{X'} v^*(\gamma) = u^* (\int_f \gamma).
\]

3.6.4. **Base change II.** Let \( Y \) be a connected nonsingular complete curve and let \( Y' \) be a closed point of \( Y \). Consider the fiber square diagram (3.12) of nonsingular varieties. Assume that \( f \) is flat but possibly non-proper.

**Lemma 3.7.** For \( \gamma \in H^d_c(X, \Omega^d_{X'}) \) we have
\[
\int_{X'} v^*(\gamma) = \int_f \gamma \in k.
\]

**Proof.** By Nagata’s compactification \( f \) is extendible to a proper flat morphism \( \bar{f} : \bar{X} \to Y \) with an open immersion \( X \hookrightarrow \bar{X} \). By the resolution of singularities we can make that \( \bar{X} \) is nonsingular and the closure \( \bar{X}' \) of \( X' \) in \( \bar{X} \) is also nonsingular. Let \( \bar{v} : \bar{f}^{-1}(Y') \to \bar{X} \) be the induced morphism and let \( \bar{v}_o := \bar{v}|_{\bar{X}'} \). Thus we have a commutative diagram
\[
\begin{array}{ccc}
\bar{X}' & \xrightarrow{\bar{v}} & \bar{X} \\
\downarrow{\bar{g}} & & \downarrow{\bar{f}} \\
Y' & \xrightarrow{u} & Y
\end{array}
\]
with a fiber square. To show (3.15) we may assume \( \gamma \in H^d_Z(X, \Omega^d_X) \) for some complete subvariety \( Z \) of \( X \). Let \( \text{nat} \) denote the natural map \( H^d_Z(X, \Omega^d_X) \to H^d(X, \Omega^d_X) \). Then by the support condition of \( \gamma \) we have
\[
(3.17) \quad (\tilde{g}_o)_*\tilde{v}^*(\mathcal{F}_f(\text{nat}(\gamma))) = \tilde{g}_o\tilde{v}^*(\mathcal{F}_f(\text{nat}(\gamma))) \in k.
\]
Since \( \tilde{v}^*(\text{nat}(\gamma)) = L\tilde{v}^*(\mathcal{F}_f(\text{nat}(\gamma))) \) under \( \Omega^d_X \approx g^!\mathcal{O}_Y \), LHS of (3.17) becomes \( \int_X \tilde{v}^*(\text{nat}(\gamma)) \), which equals to LHS of (3.15) by the support condition of \( \gamma \). On the other hand by (3.13), RHS of (3.17) becomes \( u^*\int_f \text{nat}(\gamma) \), which equals to RHS of (3.15) by the support condition and \( H^0(Y, \mathcal{O}_Y) = k \).

3.6.5. **Projection formula.** Let \( X, Y, Z \) be nonsingular varieties and let \( f : X \to Y, g : Y \to Z \) be morphisms. Let \( d' = \dim Y - \dim Z \). The uniqueness of adjunction implies the functoriality of the pushforward, for \( p \geq 0 \) with \( p - d \geq 0 \) and \( p - d - d' \geq 0 \)
\[
(3.18) \quad \int_{g \circ f} = \int_g \circ \int_f : H^p_c(X, \Omega^p_X) \to H^{p-d-d'}_c(Z, \Omega^{p-d-d'}_Z).
\]

Let \( f : X \to Y \) be a (possibly non-proper) morphism between nonsingular varieties. Then for \( \gamma \in H^d_{c,f}(X, \Omega^d_X) \) and \( \sigma \in H^q(Y, \Omega^q_Y) \) the projection formula
\[
(3.19) \quad \int_f (f^*\sigma \wedge \gamma) = \sigma \wedge \int_f \gamma
\]
holds in \( H^q(Y, \Omega^q_Y) \). This can be verified as follows. We may assume that \( f \) is proper. Consider the commuting diagram

\[
\begin{array}{cccccc}
\Omega^q_Y & \xrightarrow{\mathbb{R}f_*\Omega^p_X} & \mathbb{R}f_*\Omega^p_X & \xrightarrow{\mathbb{R}f_*(\gamma \wedge \sigma)} & \mathbb{R}f_*(\mathcal{F}_f) & \xrightarrow{\mathbb{R}f_*(\mathcal{F}_f)} \Omega^p_Y \otimes \mathbb{R}f_*f^!\mathcal{O}_Y \\
\sigma \downarrow & & \mathbb{R}f_*f^*\sigma \downarrow & & \mathbb{R}f_*(f^*\gamma \wedge \sigma) \downarrow & & \mathbb{R}f_*(f^*\sigma \wedge \gamma) \\
\mathcal{O}_Y[-q] & \xrightarrow{\mathbb{R}f_*\mathcal{O}_X} & \mathbb{R}f_*\mathcal{O}_X & \xrightarrow{\mathbb{R}f_*(\mathcal{F}_f(\gamma \wedge \sigma))} & \mathbb{R}f_*(\mathcal{F}_f(\gamma \wedge \sigma))
\end{array}
\]

We note that the composition of the maps in the top horizontal line is \( \text{id}_{\Omega^q_Y} \otimes \mathbb{R}f_*f^!\mathcal{O}_Y \) using the generic smoothness of \( f \) and local coordinate systems for compatible bases of \( \Omega^d_X \) and \( \Omega^q_Y \). The clockwise compositions of maps starting from \( \mathcal{O}_Y[-q] \) followed by \( \tau_f \) yields LHS of (3.19) and the counterclockwise compositions of maps followed by \( \tau_f \) yields RHS of (3.19).

3.6.6. **Some computations.** Let \( Q \) be the tautological quotient bundle on the projective space \( \mathbb{P}^n \). We want to compute \( \int_{\mathbb{P}^n} \) of the top Chern class \( c_n(Q) \in H^0(\mathbb{P}^n, \Omega^d_{\mathbb{P}^n}[n]) \). The class \( c_n(Q) \) is equal to \((-1)^n c_1(\mathcal{O}(-1))^n \). Let \( U_i = \{ x_i \neq 0 \} \) where \( x_0, \ldots, x_n \) are homogeneous coordinates. On each \( U_i \), we may identify \( \mathcal{O}(-1) \) with the \( i \)-th component of \( \mathcal{O}_{\mathbb{P}^n}^{\mathbb{P}^{n+1}} \) by the tautological monomorphism \( \mathcal{O}(-1) \to \mathcal{O}_{\mathbb{P}^n}^{\mathbb{P}^{n+1}} \). This yields connections \( \nabla_i \) on \( \mathcal{O}(-1)|_{U_i} \). Let \( z_i = x_i/x_0 \). Note that \( \nabla_0 - \nabla_i = -\frac{dz_i}{z_i} \). Hence \( \nabla_i - \nabla_j = \frac{dz_i}{z_i} - \frac{dz_j}{z_j} \) on
By the $n$-th fold Alexander-Čech-Whitney cup product of a Čech representative $(\nabla_i - \nabla_j)_{i < j}$ of $c_1(\mathcal{O}(-1))$ we conclude that $c_n(Q)$ is representable by a Čech cycle

$$( -1)^{\binom{n+1}{2}} \frac{dz_1 \ldots dz_n}{z_1 \ldots z_n} \in \Omega^n_{\mathcal{P}^n} (U_0 \cap \ldots \cap U_n).$$

Here the sign contribution of $\binom{n}{2}$ among $\binom{n+1}{2}$ comes from the exchanges of odd Čech ‘elements’ and differential one forms $\frac{dz_i}{z_i}$; see [1, 4]. Thus

$$(3.20) \quad \int_{\mathcal{P}^n_k} c_n(Q) = (-1)^{\binom{n}{2}} \text{res} \left[ \frac{dz_1 \ldots dz_n}{z_1 \ldots z_n} \right] = (-1)^{\binom{n+1}{2}}.$$

Let $E$ be a rank $n$ vector bundle on a nonsingular variety $X$ and let $\pi : E \to X$ be the projection. We have the diagonal section $s$ of $\pi^*E$ by letting $s(e) = (e, e)$. Let $\bar{\pi} : \mathbb{P}(E \oplus \mathcal{O}_X) \to X$ be the projection, which is a proper extension of $\pi$:

$$\mathbb{P}(E \oplus \mathcal{O}_X) = \mathbb{P}(E) \cup E \supset E \supset \mathbb{P}(\mathcal{O}_X) = X.$$ 

Let $Q$ be the tautological quotient bundle on $\mathbb{P}(E \oplus \mathcal{O}_X)$. It has a section $\bar{s}$ by the composition $\mathcal{O} \xrightarrow{(0, -\text{id})} \pi^* E \oplus \mathcal{O} \xrightarrow{\text{quot}} Q$. Note that the zero locus $\bar{s}$ is $\mathbb{P}(\mathcal{O}_X)$, since $(0, -\text{id})$ is factored through the kernel of $\text{quot}$ exactly on $\mathbb{P}(\mathcal{O}_X)$. Note that the composition $\text{quot} \circ (\text{id}, 0)|_E : \pi^* E \to Q|_E$ is an isomorphism sending $s$ to $\bar{s}|_E$. Therefore we have

$$(3.21) \quad \int_{\bar{\pi}} \chi(\text{Kos}(s)) \text{td}(\pi^*E) = \int_{\bar{\pi}} \chi(\text{Kos}(\bar{s})) \text{td}(Q)$$

$$= \int_{\bar{\pi}} c_n(Q) \quad \text{(by letting } \bar{s} = 0) = (-1)^{\binom{n+1}{2}} \quad \text{(by (3.14) \& (3.20))}.$$

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