The Itô transform for a general class of pseudo-differential operators

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Abstract
We give an Itô formula for a general class of pseudo-differential operators.

1 Introduction
Let us recall what is the Itô formula for a purely discontinuous martingale $t \to M_t$ with values in $\mathbb{R}$ [1]. Let $f$ be a $C^2$ function on $\mathbb{R}$. We have

\[ f(M_t) = f(M_0) + \int_0^t f'(M_s^-)dM_s + \sum_{s \leq t} f(M_s) - f(M_s^-) - f'(M_s^-)\Delta M_s \] (1)

It is the generalization of the celebrated Itô formula for the Brownian motion $t \to B_t$ on $\mathbb{R}$ [1]

\[ f(B_t) = f(B_0) + \int_0^t f'(B_s)dB_s + 1/2 \int_0^t f''(B_s)ds \] (2)

A lot of stochastic analysis tools for diffusions were translated by Léandre in semi-group theory in [6], [7], [8], [10], [11], [14], [15], [16], [18]. Some basic tools of stochastic analysis for the study of jump processes were translated by Léandre in semi-group theory in [11], [12], [19]. For review on that, we refer to the review of Léandre [9], [17].

Léandre has extended the Itô formula for the Brownian motion to the case of some classical partial differential equations in [19], [21], [22], [23]. In such a case, there is until now no convenient measure on a convenient path space associated to this partial differential equation. In [23], we have extended the Itô formula for jump process for an integro-differential generator when there is until now no stochastic process associated. Jump processes are generically generated by pseudo-differential operators which satisfy the maximum principle [5].
In this paper, we give an Itô formula for a general class of positive elliptic pseudo differential operators. For material on pseudo-differential operators, we refer on [2], [3], [4] and [5]. Since the considerations below on pseudo-differential operators are more and less classical, we won’t enter in the technical details of the proof.

2 The two semi-groups

Let \( \hat{u} \) be the Fourier transform of a smooth function \( u \) on \( \mathbb{R}^d \). Let \( a(x, \xi) \) be a global symbol of order \( m \) on \( \mathbb{R}^d \). It is a smooth function from \( \mathbb{R}^d \times \mathbb{R}^d \) into \( \mathbb{C} \) such that for all \( k, k' \)

\[
\sup_{x \in \mathbb{R}^d} |D_x^k D_\xi^{k'} a(x, \xi)| \leq C_{k,k'} |\xi|^{-k'}
\]

We say that a global symbol of order \( m \) is elliptic if for \( |\xi| > M \)

\[
\inf_{x \in \mathbb{R}^d} |a(x, \xi)| \geq C_M |\xi|^m
\]

We consider the proper pseudodifferential operator associated to the symbol \( a \): the Fourier transform of \( L_0 u \) is given by

\[
\int_{\mathbb{R}^d} a(x, \xi) \hat{u}(\xi) d\xi
\]

We consider its adjoint \( L_0^* \) on \( L^2(dx) \) and we put \( L = L_0^* L_0 \).

All the considerations of [2] which were valid on a compact subset of \( \mathbb{R}^d \) are still true because (3) and (4) are valid globally. In particular, \( L \) is essentially selfadjoint on \( L^2(dx) \) and generates a contraction semi-group \( P_t \) on \( L^2(dx) \).

Let us consider a smooth function \( f \) from \( \mathbb{R}^d \) into \( \mathbb{R} \) with compact support and a smooth function \( v \) with compact support from \( \mathbb{R}^d \times \mathbb{R} \) into \( \mathbb{C} \). \((x, y)\) denotes the generic element of \( \mathbb{R}^d \times \mathbb{R} \). We consider the smooth function from \( \mathbb{R}^d \) into \( \mathbb{R} \)

\[
\hat{v}(x) = v(x, f(x))
\]

We consider the function \( \varpi \) from \( \mathbb{R}^d \times \mathbb{R} \) into \( \mathbb{C} \)

\[(x, y) \mapsto v(x, y + f(x))\]

We apply \( L \) to \( \varpi \), \( y \) being frozen. We get a function \( L\varpi \). We put

\[(\hat{L}v)(x, y) = (L\varpi)(x, y - f(x))\]

Definition 1 \( \hat{L} \) is called the Itô transform of \( L \).

We remark that \((x, y) \mapsto (x, y + f(x))\) is a diffeomorphism of \( \mathbb{R}^d \times \mathbb{R} \) which keeps the measure \( dx \otimes dy \) invariant. This shows:
Theorem 2 \( \hat{L} \) is positive symmetric on \( L^2(dx \otimes dy) \). It admits therefore a self-adjoint extension still denoted \( \hat{L} \). This self-adjoint extension generates a semigroup \( \hat{P}_t \) of contraction on \( L^2(dx \otimes dy) \).

We get

**Theorem 3 (Itô formula)** We have the relation for all smooth function \( v \) with compact support

\[
P_t(v)(x) = (\hat{P}_t(v))(x, f(x)) \tag{9}
\]

**Remark:** If we consider the generator \( L = \sum X_i^2 \) where the \( X_i \) are smooth vector fields, \( \hat{L} = \sum \hat{X}_i^2 \) where

\[
\hat{X}_i = (X_i, <X_i, df>) \tag{10}
\]

which corresponds to the generator of [19], [21], [22]. Analogous remark holds for the considerations of [23].

### 3 Proof of the Itô formula

**Lemma 4** If \( v \) is a smooth function on \( \mathbb{R}^d \times \mathbb{R} \) whose all derivatives belong to \( L^2 \), \( \hat{P}_t v \) is still a smooth function whose all derivatives belong to \( L^2 \).

**Proof:** Let

\[
\overline{L}_1 = \hat{L} + \left(-\frac{\partial^2}{\partial y^2}\right)^{m/2} \tag{11}
\]

\( \overline{L} \) commute with \( \hat{L} \). Therefore, for all \( k \)

\[
(\overline{L}_1^k)\hat{P}_t = (\hat{P}_t)(\overline{L}_1^k) \tag{12}
\]

If \( v \) satisfies the hypothesis, \( \hat{P}_t v \) belongs to the domain of \( \overline{L}_1^k \). But \( \overline{L} \) is the transform of

\[
\hat{L} = L + \left(-\frac{\partial^2}{\partial y^2}\right)^{m/2} \tag{13}
\]

under the change of variable \( (x, y) \to (x, y + f(x)) \). Therefore \( \hat{P}_t v \) belongs to the domain of \( \hat{L}^k \). The result arises by Garding inequality.

Let \( \phi \) be a smooth function from \( \mathbb{R}^d \) into \([0, 1]\), equals to 0 if \( |\xi| \geq 2 \) and equals to 1 if \( |\xi| \leq 1 \). We consider the global symbol

\[
a_{\lambda}(x, \xi) = \phi(\xi/\lambda)a(x, \xi) \tag{14}
\]

and the operator \( L_{0,\lambda}, L_{0,\lambda}^* \) associated to it. Classically

\[
L_{0,\lambda} u(x) = \int_{\mathbb{R}^d} K_{\lambda}(x, y)u(y)dy \tag{15}
\]

\[
L_{0,\lambda}^* u(x) = \int_{\mathbb{R}^d} \overline{K}_{\lambda}(y, x)u(y)dy \tag{16}
\]
Lemma 5 If $u$ is smooth whose all derivative belong to $L^2$, then $(L_0 - L_{0,\lambda})u$ tends to zero as well as all his derivatives and in $L^2$ when $\lambda \to \infty$. The same holds for $(L_0^\ast - L_{0,\lambda}^\ast)u$.

Proof: $(L_0 - L_{0,\lambda})u$ is given by the oscillatory integral
\[
\int \int_{\mathbb{R} \times \mathbb{R}^d} \exp\{2\pi i < x - y | \xi > (1 - \phi(\xi/\lambda))a(x, \xi)u(y)dyd\xi \} \quad (17)
\]
The result holds by integrating by parts in $y$. Analog statement work for $(L_0^\ast - L_{0,\lambda}^\ast)u$.

Proof of the Itô formula: We put
\[
L_{\lambda} = L_{0,\lambda}^\ast L_{0,\lambda} \quad (18)
\]
$L_{\lambda}$ is a continuous operator acting on bounded continuous function on $\mathbb{R}^d$ endowed with its uniform norm. The same is true for its Itô transform $\hat{L}_{\lambda}$. Therefore $L_{\lambda}$ generates a semi-group $P_{\lambda,t}$ on bounded continuous functions on $\mathbb{R}^d$. $\hat{L}_{\lambda}$ generates a semi-group $\hat{P}_{\lambda,t}$ on bounded continuous functions on $\mathbb{R}^d \times \mathbb{R}$. Moreover if $u$ and $v$ are bounded continuous,
\[
P_{\lambda,t}u = \sum \frac{1}{n!} L^n_{\lambda} u \quad (19)
\]
and
\[
\hat{P}_{\lambda,t}v = \sum \frac{1}{n!} \hat{L}^n_{\lambda} v \quad (20)
\]
But
\[
L^n_{\lambda} \hat{v}(x) = (\hat{L}^n_{\lambda} v)(x, f(x)) \quad (21)
\]
Therefore
\[
P_{\lambda,t} \hat{v}(x) = (\hat{P}_{\lambda,t}v)(x, f(x)) \quad (22)
\]
But $(\hat{P}_{\lambda,t} - \hat{P}_t)(v)$ is solution of the parabolic equation
\[
- \frac{d}{dt}v_t = \hat{L}_{\lambda}v_t + (\hat{L}_{\lambda,t} - \hat{L})\hat{P}_t v \quad (23)
\]
with initial condition 0. The result arises from the two previous lemma, by the method of variation of constants since $P_{\lambda,t}$ is a semi-group of contraction on $L^2(dx \otimes dy)$. This shows that for $\lambda \to \infty$
\[
\hat{P}_{\lambda,t}v \to \hat{P}_t v \quad (24)
\]
in $L^2(dx \otimes dy)$. Similarly, in $L^2(dx)$
\[
P_{\lambda,t} \hat{v} \to P_t \hat{v} \quad (25)
\]
We remark that $\hat{L}_{\lambda}$ commute with $\overline{L}_{1}$. Therefore
\[
(\overline{L}_{1}^\ast)(\hat{P}_{\lambda,t} - \hat{P}_t)v = (\hat{P}_{\lambda,t} - \hat{P}_t)(\overline{L}_{1}^\ast v) \quad (26)
\]
By a similar argument to the proof of lemma (4), we can show that the convergence in (24) and (25) works for the uniform topology and not in $L^2$ only. This shows the result. ♦
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