Constructing balleans

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Abstract. A ballean is a set endowed with a coarse structure. We introduce and explore three constructions of balleans from a pregiven family of balleans: bornological products, bouquets, and combs. We analyze also the smallest and largest coarse structures on a set $X$ compatible with a given bornology on $X$.

Keywords. Ballean, coarse structure, bornological product, bouquet, comb.

1. Introduction

Given a set $X$, a family $\mathcal{E}$ of subsets of $X \times X$ is called a coarse structure on $X$, if

- each $E \in \mathcal{E}$ contains the diagonal $\Delta_X := \{(x, x) : x \in X\}$ of $X$;
- if $E, E' \in \mathcal{E}$, then $E \circ E' \in \mathcal{E}$ and $E^{-1} \in \mathcal{E}$, where $E \circ E' = \{(x, y) : \exists z \ ((x, z) \in E, \ (z, y) \in E')\}$, $E^{-1} = \{(y, x) : (x, y) \in E\}$;
- if $E \in \mathcal{E}$ and $\Delta_X \subseteq E' \subseteq E$, then $E' \in \mathcal{E}$.

Elements $E \in \mathcal{E}$ of the coarse structure are called entourages on $X$.

For $x \in X$ and $E \in \mathcal{E}$, the set $E[x] := \{y \in X : (x, y) \in E\}$ is called a ball of radius $E$ centered at $x$. Since $E = \bigcup_{x \in X} \{x\} \times E[x]$, the entourage $E$ is uniquely determined by the family of balls $\{E[x] : x \in X\}$. A subfamily $\mathcal{B} \subset \mathcal{E}$ is called a base of the coarse structure $\mathcal{E}$, if each set $E \in \mathcal{E}$ is contained in some $B \in \mathcal{B}$.

The pair $(X, \mathcal{E})$ is called a coarse space [11] or a ballean [8, 10]. In [8], every base of a coarse structure, defined in terms of balls, is called a ball structure. We prefer the name balleans not only by authors’ rights but also because a coarse space sounds like some special type of topological spaces. In fact, the balleans can be considered as non-topological antipodes of uniform topological spaces. Our compromise with [11] is in the usage of the name “coarse structure” in place of the ball structure.

In this paper, all balleans under consideration are supposed to be connected: for any $x, y \in X$, there is $E \in \mathcal{E}$ such that $y \in E[x]$. A subset $Y \subseteq X$ is called bounded, if $Y = E[x]$ for some $E \in \mathcal{E}$, and $x \in X$. The family $\mathcal{B}_X$ of all bounded subsets of $X$ is a bornology on $X$. We recall that a family $\mathcal{B}$ of subsets of a set $X$ is a bornology, if $\mathcal{B}$ contains the family $[X]^\omega \subset \mathcal{B}$ of all finite subsets of $X$, and $\mathcal{B}$ is closed under finite unions and taking subsets. A bornology $\mathcal{B}$ on a set $X$ is called unbounded, if $X \notin \mathcal{B}$.

Each subset $Y \subseteq X$ defines a subballean $(Y, \mathcal{E}|_Y)$ of $(X, \mathcal{E})$, where $\mathcal{E}|_Y = \{E \cap (Y \times Y) : E \in \mathcal{E}\}$. A subballean $(Y, \mathcal{E}|_Y)$ is called large, if there exists $E \in \mathcal{E}$ such that $X = E[Y]$, where $E[Y] = \bigcup_{y \in Y} E[y]$.

Let $(X, \mathcal{E})$, $(X', \mathcal{E}')$ be balleans. A mapping $f : X \to X'$ is called coarse (or macrouniform), if, for every $E \in \mathcal{E}$, there exists $E' \in \mathcal{E}'$ such that $f(E(x)) \subseteq E'(f(x))$ for each $x \in X$. If $f$ is a bijection...
such that $f$ and $f^{-1}$ are coarse, then $f$ is called an \textit{asymorphism}. If $(X, \mathcal{E})$ and $(X', \mathcal{E}')$ contain large asymphomorphic subballeans, then they are called \textit{coarsely equivalent}.

For coarse spaces $(X_\alpha, \mathcal{E}_\alpha)$, $\alpha \in \kappa$, their product is the Cartesian product $X = \prod_{\alpha \in \kappa} X_\alpha$ endowed with a coarse structure generated by the base consisting of the entourages

$$\{(x_\alpha)_{\alpha \in \kappa}, (y_\alpha)_{\alpha \in \kappa} \in X \times X : \forall \alpha \in \kappa \ (x_\alpha, y_\alpha) \in E_\alpha\},$$

where $(E_\alpha)_{\alpha \in \kappa} \in \prod_{\alpha \in \kappa} \mathcal{E}_\alpha$.

A class $\mathfrak{M}$ of balleans is called a \textit{variety}, if $\mathfrak{M}$ is closed under the formation of subballeans, coarse images, and Cartesian products. For the characterization of all varieties of balleans, see [7].

Given a family $\mathcal{F}$ of subsets of $X \times X$, we denote, by $\mathcal{E}$, the intersection of all coarse structures, containing each $F \cup \triangle_X$, $F \in \mathcal{F}$, and say that $\mathcal{E}$ is generated by $\mathcal{F}$. It is easy to see that $\mathcal{E}$ has a base of subsets of the form $E_1 \circ E_1 \circ \ldots \circ E_n$, where

$$E_1, \ldots, E_n \in \{F \cup F^{-1} \cup \{(x, y)\} \cup \triangle_X : F \in \mathcal{F}, \ x, y \in X\}.$$  

By a \textit{pointed ballean}, we shall understand a ballean $(X, \mathcal{E})$ with a distinguished point $e_* \in X$.

2. \textbf{Metrizability and normality}

Every metric $d$ on a set $X$ defines the coarse structure $\mathcal{E}_d$ on $X$ with the base $\{(x, y) : d(x, y) < n \} : n \in \mathbb{N}$. A ballean $(X, \mathcal{E})$ is called \textit{metrizable}, if there is a metric $d$ such that $\mathcal{E} = \mathcal{E}_d$.

\textbf{Theorem 1} ([5]). \textit{A ballean $(X, \mathcal{E})$ is metrizable, if and only if $\mathcal{E}$ has a countable base.}

Let $(X, \mathcal{E})$ be a ballean. A subset $U \subseteq X$ is called an \textit{asymptotic neighborhood} of a subset $Y \subseteq X$, if, for every $E \in \mathcal{E}$, the set $E[Y] \setminus U$ is bounded.

Two subsets $Y, Z$ of $X$ are called \textit{asymptotically disjoint} (separated), if, for every $E \in \mathcal{E}$, the intersection $E[Y] \cap E[Z]$ is bounded ($Y$ and $Z$ have disjoint asymptotic neighborhoods).

A ballean $(X, \mathcal{E})$ is called \textit{normal} [6], if any two asymptotically disjoint subsets of $X$ are asymptotically separated. Every ballean $(X, \mathcal{E})$ with a linearly ordered base of $\mathcal{E}$ is normal. In particular, every metrizable ballean is normal, see [6].

A function $f : X \to \mathbb{R}$ is called \textit{slowly oscillating}, if, for any $E \in \mathcal{E}$ and $\varepsilon > 0$, there exists a bounded subset $B$ of $X$ such that $\text{diam}(f(E[x])) < \varepsilon$ for each $x \in X \setminus B$.

\textbf{Theorem 2} ([6]). \textit{A ballean $(X, \mathcal{E})$ is normal, if and only if for any two disjoint asymptotically disjoint subsets $Y, Z$ of $X$, there exists a slowly oscillating function $f : X \to [0, 1]$ such that $f(Y) \subset \{0\}$ and $f(Z) \subset \{1\}$.}

For any unbounded bornology $\mathcal{B}$ on a set $X$, the cardinals

$$\text{add}(\mathcal{B}) = \min \{\mathcal{A} \subset \mathcal{B} : \bigcup \mathcal{A} \notin \mathcal{B}\},$$

$$\text{cov}(\mathcal{B}) = \min \{|\mathcal{C}| : \mathcal{C} \subset \mathcal{B}, \bigcup \mathcal{C} = X\}$$

and

$$\text{cof}(\mathcal{B}) = \min \{\mathcal{C} \subset \mathcal{B} : \forall B \in \mathcal{B} \exists C \in \mathcal{C} \ B \subset C\}$$

are called the \textit{additivity}, the \textit{covering number}, and the \textit{cofinality} of $\mathcal{B}$, respectively. It is well known (and easy to see) that $\text{add}(\mathcal{B}) \leq \text{cov}(\mathcal{B}) \leq \text{cof}(\mathcal{B})$.

The following theorem was proved in [10, 1.4].
Theorem 3. If the product $X \times Y$ of balleans $X, Y$ is normal, then
\[ \text{add}(B_X) = \text{cof}(B_X) = \text{cof}(B_Y) = \text{add}(B_Y). \]

Theorem 4. Let $X$ be the Cartesian product of a family $\mathcal{F}$ of metrizable balleans. Then the following statements are equivalent:

1. $X$ is metrizable;
2. $X$ is normal;
3. All but finitely many balleans from $\mathcal{F}$ are bounded.

Proof. We need only to show $(2) \Rightarrow (3)$. Assume the contrary. Then there exists a family $(Y_n)_{n<\omega}$ of unbounded metrizable balleans such that the Cartesian product $Y = \prod_{n<\omega} Y_n$ is normal. On the other hand, $\text{add}(B_Y) \leq \text{add}(B_{Y_0}) = \aleph_0$, and the standard diagonal argument shows that $\text{cof}(B_Y) > \aleph_0$, contradicting Theorem 3. \hfill $\square$

3. Bornological products

Let $\{ (X_\alpha, E_\alpha) : \alpha \in A \}$ be an indexed family of pointed balleans, and let $\mathcal{B}$ be a bornology on the index set $A$. For each $\alpha \in A$, we denote the distinguished point of the ballean $X_\alpha$ by $e_\alpha$.

The $\mathcal{B}$-product of the family of pointed balleans $\{ X_\alpha : \alpha \in A \}$ is the set
\[ X_\mathcal{B} = \{ (x_\alpha)_{\alpha \in A} \in \prod_{\alpha \in A} X_\alpha : \{ \alpha \in A : x_\alpha \neq e_\alpha \} \in \mathcal{B} \}, \]
endowed with the coarse structure $E_\mathcal{B}$, generated by the base consisting of the entourages
\[ \{ ((x_\alpha)_{\alpha \in A}, (y_\alpha)_{\alpha \in A}) \in X_\mathcal{B} \times X_\mathcal{B} : \forall \alpha \in B \ (x_\alpha, y_\alpha) \in E_\alpha \}, \]
where $B \in \mathcal{B}$ and $(E_\alpha)_{\alpha \in B} \in \prod_{\alpha \in B} E_\alpha$.

For the bornology $\mathcal{B} = \mathcal{P}_A$ consisting of all subsets of the index set $A$, the $\mathcal{B}$-product $X_\mathcal{B}$ coincides with the Cartesian product $\prod_{\alpha \in A} X_\alpha$ of the coarse spaces $(X_\alpha, E_\alpha)$.

If each $X_\alpha$ is the doubleton $\{0, 1\}$ with distinguished point $e_\alpha = 0$, then the $\mathcal{B}$-product is called the $\mathcal{B}$-macrocube on $A$. If $|A| = \omega$ and $\mathcal{B} = [A]^{<\omega}$, then we get the well-known Cantor macrocube, whose coarse characterization was given by Banakh and Zarichnyi in [2].

For relations between macrocubes and hyperballeans, see [3, 9].

Theorem 5. Let $\mathcal{B}$ be a bornology on a set, and let $X_\mathcal{B}$ be the $\mathcal{B}$-product of a family of unbounded metrizable pointed balleans. Then the following statements are equivalent:

1. $X_\mathcal{B}$ is metrizable;
2. $X_\mathcal{B}$ is normal;
3. $|A| = \omega$ and $\mathcal{B} = [A]^{<\omega}$.

Proof. To see that $(2) \Rightarrow (3)$, repeat the proof of Theorem 4. \hfill $\square$

Theorem 6. Let $\mathcal{B}$ be a bornology on a set $A$, and let $X_\mathcal{B}$ be the $\mathcal{B}$-product of a family $\{ X_\alpha : \alpha \in A \}$ of bounded pointed balleans which are not singletons. The coarse space $X_\mathcal{B}$ is metrizable, if and only if the bornology $\mathcal{B}$ has a countable base.
Proof. Apply Theorem 1.

Let \( X \) be a macrocube on a set \( A \), and let \( Y \) be a macrocube on a set \( B \), \( A \cap B = \emptyset \). Then \( X \times Y \) is a macrocube on \( A \cup B \), and, by Theorem 3, \( X \times Y \) needs not to be normal.

**Question 1.** How can one detect whether a given macrocube is normal? Is a \( \mathcal{B} \)-macrocube on an infinite set \( A \) normal provided that \( \mathcal{B} \neq \mathcal{P}_A \) is a maximal unbounded bornology on \( A \)?

Let \( \{ X_n : n < \omega \} \) be a family of finite balleans, \( \mathcal{B} = [\omega]^{<\omega} \). By \cite{10}, the \( \mathcal{B} \)-product of the family \( \{ X_n : n < \omega \} \) is coarsely equivalent to the Cantor macrocube.

**Question 2.** Let \( \{ X_\alpha : \alpha \in A \} \) be a family of finite (bounded) pointed balleans, and let \( \mathcal{B} \) be a bornology on \( A \). How can one detect whether a \( \mathcal{B} \)-product of \( \{ X_\alpha : \alpha \in A \} \) is coarsely equivalent to some macrocube?

4. Bouquets

Let \( \mathcal{B} \) be a bornology on a set \( A \), and let \( \{ (X_\alpha, \mathcal{E}_\alpha) : \alpha \in A \} \) be a family of pointed balleans. The subballean

\[
\bigvee_{\alpha \in A} X_\alpha := \{ (x_\alpha)_{\alpha \in A} \in X_\mathcal{B} : |\{ \alpha \in A : x_\alpha \neq e_\alpha \}| \leq 1 \}
\]

of the \( \mathcal{B} \)-product \( X_\mathcal{B} \) is called the \( \mathcal{B} \)-bouquet of the family \( \{ (X_\alpha, \mathcal{E}_\alpha) : \alpha \in A \} \). The point \( e = (e_\alpha)_{\alpha \in A} \) is the distinguished point of the ballean \( \bigvee_{\alpha \in A} X_\alpha \).

For every \( \alpha \in A \), we identify the ballean \( X_\alpha \) with the subballean \( \{ (x_\beta)_{\beta \in A} \in X_\mathcal{B} : \forall \beta \in A \setminus \{\alpha\} \ x_\beta = e_\beta \} \) of \( \bigvee_{\alpha \in A} X_\alpha \). Under such identification, \( \bigvee_{\alpha \in A} X_\alpha = \bigcup_{\alpha \in A} X_\beta \) and \( X_\alpha \cap X_\beta = \{ e \} = \{ e_\alpha \} = \{ e_\beta \} \) for any distinct indices \( \alpha, \beta \in A \).

Applying Theorem 1, we can prove the following two theorems.

**Theorem 7.** Let \( \mathcal{B} \) be a bornology on a set \( A \), and let \( \{ X_\alpha : \alpha \in A \} \) be a family of unbounded pointed metrizable balleans. The \( \mathcal{B} \)-bouquet \( \bigvee_{\alpha \in A} X_\alpha \) is metrizable, if and only if \( |A| = \omega \) and \( \mathcal{B} = [A]^{<\omega} \).

**Theorem 8.** Let \( \mathcal{B} \) be a bornology on a set \( A \), and let \( \{ X_\alpha : \alpha \in A \} \) be a family of bounded pointed balleans, which are not singletons. The \( \mathcal{B} \)-bouquet \( \bigvee_{\alpha \in A} X_\alpha \) is metrizable, if and only if the bornology \( \mathcal{B} \) has a countable base.

**Theorem 9.** A bornological bouquet of any family of pointed normal balleans is normal.

**Proof.** Let \( \mathcal{B} \) be a bornology on a non-empty set \( A \), and let \( X \) be the \( \mathcal{B} \)-bouquet of pointed normal balleans \( X_\alpha, \alpha \in A \). Given two disjoint asymptotically disjoint sets \( Y, Z \subset X \), we shall construct a slowly oscillating function \( f : X \to [0, 1] \) such that \( f(Y) \subset \{0\} \) and \( f(Z) \subset \{1\} \). The definition of a coarse structure on the \( \mathcal{B} \)-bouquet ensures that, for every \( \alpha \in A \), the subsets \( Y \cap X_\alpha \) and \( Z \cap X_\alpha \) are asymptotically disjoint in the coarse space \( X_\alpha \) which is identified with the subspace \( \{ (x_\beta) \in X : \forall \beta \in A \setminus \{\alpha\} \ x_\beta = e_\beta \} \) of the \( \mathcal{B} \)-bouquet \( X \). By the normality of \( X_\alpha \), there exists a slowly oscillating function \( f_\alpha : X_\alpha \to [0, 1] \) such that \( f_\alpha(Y \cap X_\alpha) \subset \{0\} \) and \( f_\alpha(Z \cap X_\alpha) \subset \{1\} \). Changing the value of \( f_\alpha \) at the distinguished point \( e_\alpha \) of \( X_\alpha \), we can assume that \( f_\alpha(e_\alpha) = f_\beta(e_\beta) \) for any \( \alpha, \beta \in A \). Then the function \( f : X \to [0, 1] \) defined by \( f(X_\alpha) = f_\alpha \) for \( \alpha \in A \) is slowly oscillating and has the desired property: \( f(Y) \subset \{0\} \) and \( f(Z) \subset \{1\} \). By Theorem 2, the ballean \( X \) is normal. \qed
5. Combs

Let \((X, \mathcal{E})\) be a ballean, and let \(A\) be a subset of \(X\). Let \(\{(X_\alpha, \mathcal{E}_\alpha) : \alpha \in A\}\) be a family of pointed ballean with marked points \(e_\alpha \in X_\alpha\) for \(\alpha \in A\).

The bornology \(B_X\) of the ballean \((X, \mathcal{E})\) induces a bornology \(B := \{B \in B_X : B \subset A\}\) on the set \(A\). Let \(\bigvee_{\alpha \in A} X_\alpha\) be the \(B\)-bouquet of the family of pointed ballean \(\{(X_\alpha, \mathcal{E}_\alpha) : \alpha \in A\}\), and let \(e\) denote the distinguished point of the bouquet \(\bigvee_{\alpha \in A} X_\alpha\).

For every \(\alpha \in A\), we identify the ballean \(X_\alpha\) with the subballean \(\{(x_\beta)_{\beta \in A} \in \bigvee_{\alpha \in A} X_\alpha : \forall \beta \in A \setminus \{\alpha\} \ x_\beta = e_\beta\}\) of \(\bigvee_{\alpha \in A} X_\alpha\). Then \(\bigvee_{\alpha \in A} X_\alpha = \bigcup_{\alpha \in A} X_\alpha\) and \(X_\alpha \cap X_\beta = \{e\} = \{e_\alpha\} = \{e_\beta\}\) for any distinct indices \(\alpha, \beta \in A\).

The subballean

\[ X_{\alpha} \leadsto X_\alpha := (X \times \{e\}) \cup \bigcup_{\alpha \in A} \{\{\alpha\} \times X_\alpha\} \]

of the ballean \(X \times \bigvee_{\alpha \in A} X_\alpha\) is called the \textit{comb} with handle \(X\) and spines \(X_\alpha, \alpha \in A \subset X\). We identify the handle \(X\) and the spines \(X_\alpha\) with the subsets \(X \times \{e\}\) and \(\{\alpha\} \times X_\alpha\) in the comb \(X_{\alpha} \leadsto X_\alpha\).

It can be shown that a comb \(X_{\alpha} \leadsto X_\alpha\) carries the smallest coarse structure such that the identity inclusions of the ballean \(X\) and \(X_\alpha, \alpha \in A\), into \(X_{\alpha} \leadsto X_\alpha\) are macrouniform.

**Theorem 10.** The comb \(X_{\alpha} \leadsto X_\alpha\) is metrizable, if the ballean \(X\) and \(X_\alpha, \alpha \in A\), are metrizable, and, for each bounded set \(B \subset X\), the intersection \(A \cap B\) is finite.

**Proof.** Applying Theorem 7, we conclude that the bouquet \(\bigvee_{\alpha \in A} X_\alpha\) is metrizable. Then the comb \(X_{\alpha} \leadsto X_\alpha\), being a subspace of the metrizable ballean \(X \times \bigvee_{\alpha \in A} X_\alpha\), is metrizable. \(\square\)

By analogy with Theorem 9, we can prove

**Theorem 11.** The comb \(X_{\alpha} \leadsto X_\alpha\) is normal, if the ballean \(X\) and \(X_\alpha, \alpha \in A\), are normal.

6. Coarse structures determined by bornologies

Let \(B\) be a bornology on a set \(X\). We say that a coarse structure \(\mathcal{E}\) on \(X\) is \textit{compatible} with \(B\), if \(B\) coincides with the bornology \(B_X\) of all bounded subsets of \((X, \mathcal{E})\).

The family of all coarse structures compatible with a given bornology \(B\) has the smallest and largest elements \(\downarrow B\) and \(\uparrow B\).

The smallest coarse structure \(\downarrow B\) is generated by the base consisting of the entourages \((B \times B) \cup \Delta_X\), where \(B \in B\).

The largest coarse structure \(\uparrow B\) consists of all entourages \(E \subseteq X \times X\) such that \(E^{-1}[B] \cup E[B] \in B\) for every \(B \in B\).

An unbounded ballean \((X, \mathcal{E})\) is called

- \textit{discrete}, if \(\mathcal{E} = \downarrow B_X\),
- \textit{ultradiscrete}, if \(X\) is discrete, and its bornology \(B_X\) is maximal by inclusion in the family of all unbounded bornologies on \(X\);
- \textit{maximal}, if its coarse structure is maximal by inclusion in the family of all unbounded coarse structures on \(X\);
• relatively maximal, if $\mathcal{E} = \uparrow B_X$.

It can be shown that an unbounded ballean $(X, \mathcal{E})$ is discrete, iff, for every $E \in \mathcal{E}$, there exists a bounded set $B \subset X$ such that $E[x] = \{x\}$ for each $x \in X \setminus B$. In [10, Chapter 3], discrete balleans are called pseudodiscrete.

It is clear that each maximal ballean is relatively maximal. For maximal balleans, see [10, Chapter 10]. For any regular cardinal $\kappa$, the ballean $(\kappa, \uparrow [\kappa]^{< \kappa})$ is maximal.

Each ultradiscrete ballean is both discrete and relatively maximal.

A ballean $(X, \mathcal{E})$ is called ultranormal, if $X$ contains no two unbounded asymptotically disjoint subsets. By [10, Theorem 10.2.1], every unbounded subset of a maximal ballean is large, which implies that each maximal ballean is ultranormal. A discrete ballean is ultranormal, if and only if it is ultradiscrete.

**Example 1.** For every infinite set $X$, there exists a bornology $\mathcal{B}$ on $X$ such that $\downarrow \mathcal{B} = \uparrow \mathcal{B}$, but the ballean $(X, \downarrow \mathcal{B}) = (X, \uparrow \mathcal{B})$ is not ultradiscrete. Consequently, the ballean $(X, \downarrow \mathcal{B}) = (X, \uparrow \mathcal{B})$ is discrete and relatively maximal, but not ultranormal.

**Proof.** By Theorem 3.1.6 [4], there are two free ultrafilters $p, q$ on $X$ such that, for every function $f : X \to X$ and any $P \in p$ and $Q \in q$, we have $f(P) \notin q$ and $f(Q) \notin p$. We put $B = \{B \subseteq X : B \notin p, B \notin q\}$ and note that $B$ is a bornology on $X$.

To show that $\downarrow \mathcal{B} = \uparrow \mathcal{B}$, we need to check that, for any entourage $E \in \uparrow \mathcal{B}$, the set $Y = \{x \in X : E[x] \neq \{x\}\}$ belongs to the bornology $\mathcal{B}$. To derive a contradiction, we assume that $Y \notin \mathcal{B}$. For every $x \in Y$, we choose a point $f(x) \in E[x] \setminus \{x\}$. By Zorn’s Lemma, there exists a maximal subset $Z \subset Y$ such that $Z \cap f(Z) = \emptyset$. By the maximality of $Z$, there exists a point $y \in Y \setminus Z$, we get $f(y) \in Z$ and, hence, $f(Y \setminus Z) \subset Z$. It follows from $Y \notin \mathcal{B}$ that $Z \notin \mathcal{B}$, a desired contradiction.

The case $X \setminus Z \notin \mathcal{B}$ can be considered by analogy.

Since $X$ can be written as the union $X = P \cup Q$ of two disjoint unbounded sets $P \in p, Q \in q$, the ballean $(X, \uparrow \mathcal{B})$ is not ultradiscrete and not ultranormal.

By a bornological space, we understand a pair $(X, \mathcal{B}_X)$ consisting of a set $X$ and a bornology $\mathcal{B}_X$ on $X$. A bornological space $(X, \mathcal{B}_X)$ is unbounded, if $X \notin \mathcal{B}_X$. For two bornological spaces $(X, \mathcal{B}_X)$ and $(Y, \mathcal{B}_Y)$, their product is a bornological space $(X \times Y, \mathcal{B})$ endowed with the bornology

$$\mathcal{B}_{X \times Y} = \{B \subset X \times Y : B \subset B_X \times B_Y \text{ for some } B_X \in \mathcal{B}_X, B_Y \in \mathcal{B}_Y\}.$$  

The following theorem allows us to construct many examples of bornological spaces $(X, \mathcal{B})$ for which the coarse space $(X, \uparrow \mathcal{B})$ is not normal.

**Theorem 12.** Let $(X \times Y, \mathcal{B})$ be the product of two unbounded bornological spaces $(X, \mathcal{B}_X)$ and $(Y, \mathcal{B}_Y)$. If $\text{cov}(\mathcal{B}_Y) < \text{add}(\mathcal{B}_X)$, then the coarse space $(X \times Y, \uparrow \mathcal{B})$ is not normal.

**Proof.** Fix any point $(x_0, y_0) \in X \times Y$. Assuming that $\text{cov}(\mathcal{B}_Y) < \text{add}(\mathcal{B}_X)$, we shall prove that, for a coarse structure, $\mathcal{E}$ on $X \times Y$ is not normal, if $\mathcal{E}$ has the following three properties:

1. $\mathcal{E}$ is compatible with the bornology $\mathcal{B}$;
2. for any $B_Y \in \mathcal{B}_Y$, there exists $E \in \mathcal{E}$ such that $X \times B_Y \subset E[X \times \{y_0\}]$;
3. for any $B_X \in \mathcal{B}_X$, there exists $E \in \mathcal{E}$ such that
\[ B_X \times Y \subset E[\{x_0\} \times Y]. \]

It is easy to see that the coarse structure $\hat{\mathcal{B}}$ has these three properties.

By the definition of the cardinal $\kappa = \text{cov}(\mathcal{B}_Y)$, there is a family $\{Y_\alpha\}_{\alpha \in \kappa} \subset \mathcal{B}_Y$ such that $\bigcup_{\alpha \in \kappa} Y_\alpha = Y$.

Assume that $\mathcal{E}$ is a coarse structure on $X \times Y$ satisfying conditions (1)-(3). First, we check that the sets $X \times \{y_0\}$ and $\{x_0\} \times Y$ are asymptotically disjoint in $(X \times Y, \mathcal{E})$. Given any entourage $E \in \mathcal{E}$, we should prove that the intersection $E[X \times \{y_0\}] \cap E[\{x_0\} \times Y]$ is bounded. By condition (1), for every $\alpha \in \kappa$, the bounded set $E^{-1}[E[\{x_0\} \times Y_\alpha]]$ is contained in the product $B_\alpha \times Y$ for some bounded set $B_\alpha \in \mathcal{B}_X$. Since $\kappa < \text{add}(\mathcal{B}_X)$, the union $B_{<\kappa} := \bigcup_{\alpha \in \kappa} B_\alpha$ belongs to the bornology $\mathcal{B}_X$. Given any point $(u, v) \in E[X \times \{y_0\}] \cap E[\{x_0\} \times Y]$, we find $x \in X$ and $y \in Y$ such that $(u, v) \in E[(x, y_0)] \cap E[(x_0, y)]$. Since $Y = \bigcup_{\alpha \in \kappa} Y_\alpha$, there exists $\alpha \in \kappa$ such that $y \in Y_\alpha$. Then $(x, y_0) \in E^{-1}[E[(x_0, y)]] \subset E^{-1}[E[\{x_0\} \times Y_\alpha]] \subset B_\alpha \times Y \subset B_{<\kappa} \times Y$ and, hence, $(u, v) \in E[(x, y_0)] \subset E[B_{<\kappa} \times \{y_0\}]$, which implies that the intersection
\[ E[X \times \{y_0\}] \cap E[\{x_0\} \times Y] \subset E[B_{<\kappa} \times \{y_0\}] \]
is bounded in $(X \times Y, \mathcal{E})$.

Assuming that the coarse space $(X \times Y, \mathcal{E})$ is normal, we can find disjoint asymptotic neighborhoods $U$ and $V$ of the asymptotically disjoint sets $X \times \{y_0\}$ and $\{x_0\} \times Y$. By condition (2), for every $\alpha \in \kappa$, there exists an entourage $E_\alpha \in \mathcal{E}$ such that $X \times Y_\alpha \subset E_\alpha[X \times \{y_0\}]$. Since $U$ is an asymptotic neighborhood of the set $X \times \{y_0\}$ in $(X \times Y, \mathcal{E})$, the set $(X \times Y) \setminus U \subset E_\alpha[X \times \{y_0\}] \setminus U$ is bounded in $(X \times Y, \mathcal{E})$. Now, condition (1) implies that $(X \times Y_\alpha) \setminus U \subset D_\alpha \times Y$ for some bounded set $D_\alpha \in \mathcal{B}_X$.

We claim that the family $\{D_\alpha\}_{\alpha \in \kappa}$ is cofinal in $\mathcal{B}_X$. Indeed, given any bounded set $D \in \mathcal{B}_X$, we use condition (3) and find an entourage $E \in \mathcal{E}$ such that $D \times Y \subset E[\{x_0\} \times Y]$. Since $V$ is an asymptotic neighborhood of the set $\{x_0\} \times Y$, the set $E[\{x_0\} \times Y] \setminus V$ is bounded in $(X \times Y, \mathcal{E})$, and condition (1) ensures that it has a bounded projection onto $Y$. Since $Y \notin \mathcal{B}_Y$, we can find a point $y \in Y$ such that $X \times \{y\}$ is disjoint with $E[\{x_0\} \times Y] \setminus V$. Find $\alpha \in \kappa$ with $y \in Y_\alpha$. Then $(X \times \{y\}) \cap E[\{x_0\} \times Y] \subset V$ and, hence,
\[ D \times \{y\} \subset (X \times y) \cap E[\{x_0\} \times Y] \subset (X \times Y_\alpha) \cap V \subset (X \times Y_\alpha) \setminus U \subset D_\alpha \times Y, \]
which yields the desired inclusion $D \subset D_\alpha$. Therefore,
\[ \text{cof}(\mathcal{B}_X) \leq |\{D_\alpha\}_{\alpha \in \kappa}| \leq \kappa = \text{cov}(\mathcal{B}_Y) < \text{add}(\mathcal{B}_X), \]
which contradicts the known inequality $\text{add}(\mathcal{B}_X) \leq \text{cof}(\mathcal{B}_X)$.

\[ \square \]

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