Linear-Logic Based Analysis of Constraint Handling Rules with Disjunction

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Constraint Handling Rules (CHR) is a declarative committed-choice programming language with a strong relationship to linear logic. Its generalization CHR with Disjunction (CHR\(^{\lor}\)) is a multi-paradigm declarative programming language that allows the embedding of horn programs. We analyse the assets and the limitations of the classical declarative semantics of CHR before we motivate and develop a linear-logic declarative semantics for CHR and CHR\(^{\lor}\).

We show how to apply the linear-logic semantics to decide program properties and to prove operational equivalence of CHR\(^{\lor}\) programs across the boundaries of language paradigms.

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1. INTRODUCTION

A declarative semantics is a highly desirable property for a programming language. It offers a clean theoretical foundation for the language, allows to prove program properties such as correctness and operational equivalence and guarantees platform independence. Declarative programs tend to be shorter and clearer as they contain, ideally, only information about the modeled problem and not about control.

Constraint Handling Rules (CHR) [Frühwirth 1994; 1998; 2009] is a declarative committed-choice general-purpose programming language developed in the 1990s as a portable language extension to implement user-defined constraint solvers. Operationally, it mixes rule-based multiset rewriting over constraints with calls to a built-in constraint solver with at least rudimentary capabilities. It is Turing complete and it has been shown that every algorithm can be implemented in CHR with optimal time complexity [Sneyers et al. 2005]. Hence, it makes an efficient stand-alone general-purpose programming language.

Constraint Handling Rules with Disjunction (CHR\(^{\lor}\)) [Abdennadher and Schütz 1998] extends the inherently non-deterministic formalism of CHR with the possibility to include...
backtracking search and thus to embed horn programs. It features both don't-care and don't-know non-determinism. We can justly describe it as a multi-paradigm declarative programming language.

Owing to its heritage in logic programming and constraint logic programming, CHR features a declarative semantics in classical logic. We have shown that for certain classes of programs, the classical declarative semantics of CHR reflects the functionality of a program but poorly [Betz and Frühwirth 2005]. Operationally, CHR is a state transition system whereas the classical declarative semantics considers all states in a derivation as logically equivalent. Hence, the directionality of the rules, the inherent non-determinism of their execution and any change of state eludes this declarative semantics.

Linear logic is a sub-structural logical formalism [Girard 1987] that has been shown to bear a close relationship to concurrent committed-choice systems [Miller 1992; Fages et al. 2001]. It shows that it is well-suited to model the committed-choice rules of CHR. It furthermore allows a faithful embedding of classical logic, so we can straightforwardly embed the constraint theory underlying the built-in constraint solver into linear logic. Linear logic thus enables us to model the two reasoning mechanisms of CHR in a single formalism. Moreover, it shows that we can encode CHR$^\lor$ into linear logic in a way that preserves its characteristic dichotomy of don't-know and don't-care non-determinism.

In this article, we propose a linear-logic semantics for CHR and CHR$^\lor$ that incorporates all the features mentioned above. We found the semantics on the intuitionistic segment of linear logic as it suffices for our purpose while being easier to handle than the full segment. We propose two variants of the semantics. The first variant is based on introducing proper axioms in the sequent calculus of linear logic. The second variant is similar to the semantics previously published in Betz and Frühwirth [2005] and Betz [2007]. The first formulation allows for considerably more elegant proofs, in particular of its soundness and completeness. The second formulation allows to perform a broader range of reasoning tasks. As we formalize and prove the equivalence of both representations, so we can use either representation according to the respective application.

This article is structured as follows: 2 In Sect. 2, we recall the syntax and operational semantics of CHR. 3 In Sect. 3, we introduce the intuitionistic segment of linear logic. 4 In Sect. 4, we develop a linear-logic semantics for constraint handling rules, and we show its soundness and completeness with respect to the operational semantics. 5 In Sect. 5, we extend our semantics to CHR$^\lor$ and prove its soundness and completeness. We show that the linear-logic semantics allows in general for less precise reasoning over CHR$^\lor$ than over CHR. We then introduce a well-behavedness property for CHR$^\lor$ programs that amends this limitation. In Sect. 6, we show how our semantics can be applied to reason about program observables as well as to compare programs even across the boundaries of programming paradigms. In Sect. 7, we discuss related work before we conclude in Sect. 8.

2. CONSTRAINT HANDLING RULES

In this section, we recall the syntax and the operational semantics $\omega_e$ of Constraint Handling Rules.

2.1 The Syntax of CHR

We distinguish two disjoint classes of atomic constraints: atomic built-in constraints and atomic user-defined constraints. We denote the former as $c_b(\bar{t})$ and the latter as $c_u(\bar{t})$, where $c_b, c_u$ are $n$-ary constraint symbols and $\bar{t}$ is a sequence of $n$ terms. Built-in constraints and
user-defined constraint are possibly empty conjunctions of their respective atomic constraints. A conjunction of atomic constraints in general, irrespective of their class, is called a goal\(^1\). Empty goals and empty constraints are denoted as \(\top\).

The syntax of constraints is summarized in Def. 2.1.

**Definition 2.1 Constraint Syntax.** Let \(c_b(\overline{t})\), \(c_u(\overline{t})\) denote an \(n\)-ary atomic built-in or user-defined constraint, respectively, where \(\overline{t}\) is an \(n\)-ary sequence of terms:

- **Built-in constraint:** \(B \ ::= \top \mid c_b(\overline{t}) \mid B \land B'\)
- **User-defined constraint:** \(U \ ::= \top \mid c_u(\overline{t}) \mid U \land U'\)
- **Goal:** \(G \ ::= \top \mid c_u(\overline{t}) \mid c_b(\overline{t}) \mid G \land G'\)

\(\top\) stands for the empty constraint or the empty goal, respectively. The set of built-in constraints furthermore contains at least falsity \(\bot\), and the binary constraint \(\equiv\), standing for syntactic equality. For any two goals \(G, G'\), the goal equivalence relation \(\equiv\) denotes equivalence with respect to the associativity and commutativity of \(\land\) and the neutrality of the identity element \(\top\).

Both built-in and user-defined constraints are special cases of goals. The goal equivalence relation \(\equiv\) does not account for idempotence, thus implicitly imposing a multiset semantics on goals. For example, \(c_u(\overline{t}) \land c_u(\overline{t}) \not\equiv c_u(\overline{t})\). We denote the set of variables occurring in a goal \(G\) as \(\text{vars}(G)\).

A CHR program is a set of rules adhering to the following definition:

**Definition 2.2 Rule Syntax.** (1) A CHR rule is of the form

\[
r @ H_1 \setminus H_2 \Rightarrow G \mid B_u \land B_b\]

The rule head \(H_1 \setminus H_2\) consists of the kept head \(H_1\) and the removed head \(H_2\). Both \(H_1, H_2\) are user-defined constraints. At least one of them must be non-empty. The guard \(G\) is a built-in constraint. The rule body is of the form \(B_u \land B_b\), where \(B_u\) is a built-in constraint and \(B_b\) is a user-defined constraint. \(r\) serves as an identifier for the rule.

(2) The identifier \(r\) is operationally irrelevant and can be omitted along with the \(@\). An empty guard \(G = \top\) can be omitted along with the \(\Rightarrow\). A rule with an empty kept head \(H_1\) can be written as \(r @ H_2 \Leftrightarrow G \mid B_u \land B_b\). Such a rule is called a simplification rule. A rule where the removed head \(H_2\) is empty can be written as \(r @ H_1 \Rightarrow G \mid B_u \land B_b\). Such a rule is called a propagation rule. A rule where neither \(H_1\) nor \(H_2\) are empty is called a simulation rule.

(3) A variant of a rule \(r @ H_1 \setminus H_2 \Rightarrow G \mid B_u \land B_b\) with variables \(\overline{x}\) is of the form \((r @ H_1 \setminus H_2 \Rightarrow G \mid B_u \land B_b)[\overline{x} \mapsto \overline{y}]\) where \(\overline{y}\) is an arbitrary sequence of pairwise distinct variables.

(4) A CHR program is a set of CHR rules.

In anticipation of Section 2.2, we point out that propagation rules may cause trivial non-termination of programs as they do not in general eliminate the pre-condition of their firing. Hence, precautions have to be taken. We refer the reader to Abdennadher [1997] and Duck et al. [2004] for the most common approach based on keeping a history of applied rules and

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\(^1\) Note that the term goal is used in CHR for historical reasons and does not imply that program execution is understood as proof search.
to Betz et al. [2010] for a more recent approach based on finite representations of infinite program states and computations.

2.2 The Equivalence-Based Semantics $\omega_e$

In this section, we recall the operational semantics of CHR. Several formalizations of the operational semantics exist in the literature. We choose the so-called *equivalence-based semantics* $\omega_e$ as it contains all the elements that we represent in our linear-logic semantics while allowing for elegant proofs of theoretical properties.

Operationally, built-in and user-defined constraints are handled separately. For the handling of built-in constraints, CHR requires a so-called *predefined constraint handler* whereas user-defined constraints are handled by the actual user program. We assume that the predefined solver implements a *complete* and *decidable* first-order *constraint theory* $CT$ over the built-in constraints.

**Definition 2.3 Constraint Theory.** A constraint theory $CT$ is a decidable theory of intuitionistic logic over the built-in constraints. We assume that it is given as a set of formulas of the form

$$\alpha ::= \forall(\exists \bar{x}. B \rightarrow \exists \bar{x}'. B')$$

called $CT$-axioms where $B, B'$ are possibly empty built-in constraints and $\bar{x}, \bar{x}'$ are possibly empty sets of variables.

It should be noted that defining constraint theories explicitly over intuitionistic rather than full classical logic is non-standard. It is, however, an unproblematic decision because in the operational semantics only judgements over conjunctions of positive literals are considered. Furthermore, this decision allows us to restrict ourselves to the intuitionistic fragment of linear logic when translating constraint theories into linear logic.

CHR itself is a transition system over equivalence classes of program states, which are defined as follows:

**Definition 2.4 CHR State.** (1) A CHR state is a tuple of the form $S = \langle G; V \rangle$ where $G$ is a goal called *constraint store* and $V$ is a set of variables called *global variables*.

(2) For a CHR state $S = \langle U \land B; V \rangle$, where $U$ is a user-defined constraint and $B$ is a built-in constraint, we call

(a) $\tilde{l}_S ::= (\text{vars}(U) \cup \text{vars}(B)) \setminus V$ the *local variables* of $S$ and

(b) $\tilde{s}_S ::= \tilde{l}_S \setminus \text{vars}(U)$ the *strictly local variables* of $S$.

(3) A variant of a state $S = \langle G; V \rangle$ with local variables $\tilde{l}$ is a state $S'$ of the form $S' = \langle G[\tilde{l}/\bar{x}]; V \rangle$, where $\bar{x}$ is a sequence of pairwise distinct variables that do not occur in $V$.

The state transition system that formalizes the operational semantics builds on the following definition of equivalence between CHR states:

**Definition 2.5 Equivalence of CHR States.** In the following, let $U, U'$ denote arbitrary user-defined constraints, $B, B'$ built-in constraints, $G, G'$ goals, $V, V'$ sets of variables $v$ a variable and $t$ a term. State equivalence, written as $\cdot \equiv_e \cdot$, is the smallest equivalence relation over CHR states that satisfies all of the following conditions:

(1) *(Goal Transformation)*

$$G \equiv_e G' \Rightarrow \langle G; V \rangle \equiv_e \langle G'; V \rangle$$
(2) (Equality as Substitution)
\[ \langle U \land x \doteq t \land \exists B ; V \rangle \equiv \langle U[x/t] \land x \doteq t \land \exists B ; V \rangle \]

(3) (Application of CT) Let \( \overline{s}, \overline{s}' \) be the strictly local variables of \( \langle U \land B ; V \rangle, \langle U \land B' ; V \rangle \).
If \( CT \models \exists \overline{s}. B \leftrightarrow \exists \overline{s}'. B' \) then:
\[ \langle U \land B ; V \rangle \equiv \langle U \land B' ; V \rangle \]

(4) (Neutrality of Redundant Global Variables)
\[ x \notin \text{var}(G) \Rightarrow \langle G; \{x\} \cup V \rangle \equiv \langle G; V \rangle \]

(5) (Equivalence of Failed States) For all goals \( G, G' \) and all sets of variables \( V, V' \):
\[ \langle G \land \bot ; V \rangle \equiv \langle G' \land \bot ; V' \rangle \]

Where there is no ambiguity, we usually write \( \cdot \equiv \cdot \) rather than \( \cdot \equiv_e \cdot \).

While we generally impose a multiset semantics over goals, Definition 2.5.3 implicitly restores the set semantics for built-in constraints within states. When discussing pure CHR – as opposed to its generalization CHR\( ^\lor \) (cf. Sect.5) – we will usually consider states in the following normal form:

Definition 2.6 Normal Form of CHR States. A CHR state \( S \) is considered in normal form if it is of the form \( S = \langle U \land B ; V \rangle \) where \( U \) is a user-defined constraint called the user-defined store and \( B \) is a built-in constraint called the built-in store. Such a state is usually written in ternary notation: \( \langle U; B; V \rangle \).

Any state with an inconsistent built-in store is called a failed state as formalized in the following definition:

Definition 2.7 Failed State. Any CHR state \( S \equiv \langle U; \bot ; V \rangle \) for some \( G, V \) is called a failed state. We use \( S_\bot = \langle \top; \bot; \emptyset \rangle \) as the default representative for the set of failed states.

The following lemma states several properties following from Def. 2.5 that have been presented and proven in Raiser et al. [2009]:

Lemma 2.8 Properties of State Equivalence. The following properties hold in general:

(1) (Renaming of Local Variables)
\[ \langle U; B; V \rangle \equiv \langle U[x/y]; B[x/y]; V \rangle \]
for \( x \notin V \) and \( y \notin V \) and \( y \) does not occur in \( U \) or \( B \).

(2) (Partial Substitution) Let \( U[x \mapsto t] \) be a user-defined constraint where some occurrences of \( x \) are substituted with \( t \):
\[ \langle U; x \doteq t, B; V \rangle \equiv \langle U[x \mapsto t]; x \doteq t, B; V \rangle \]

(3) (Logical Equivalence) If
\[ \langle U; B; V \rangle \equiv \langle U'; B'; V' \rangle \]
then \( CT \models (\exists \overline{t}. U \land B) \leftrightarrow (\exists \overline{t}'. U' \land B') \), where \( \overline{t}, \overline{t}' \) are the local variables of \( \langle U; B; V \rangle, \langle U'; B'; V' \rangle \), respectively.
Lemma 2.8.1 allows us to assume without loss of generality that the local variables of any two specific states are renamed apart. Concerning Lemma 2.8.3, note that logical equivalence of \( \exists \bar{l}.U \land B \) and \( \exists \bar{l}'.U' \land B' \) is a necessary but not a sufficient condition for state equivalence. The linear logic semantics will enable us to formulate a similar condition that is both necessary and sufficient (cf. Sect. 4.2).

The task of deciding equivalence – and more so: non-equivalence – is not always trivial using the axiomatic definition. We quote Theorem 2.10 which gives a necessary, sufficient, and decidable criterion. It uses the following notion of matching:

**Definition 2.9 Matching of Constraints.** For user-defined constraints \( U = c_1(\bar{t}_1) \land \ldots \land c_n(\bar{t}_n), \ U' = c'_1(\bar{t}'_1) \land \ldots \land c'_m(\bar{t}'_m) \), the matching relation \( U \equiv U' \) holds if and only if \( n = m \) and there exists a permutation \( \sigma \) such that

\[
\bigwedge_{i=1}^n c_i(\bar{t}_i) \equiv c'_{\sigma(i)}(\bar{t}'_{\sigma(i)})
\]

The following theorem has been published and proven in [Raiser et al. 2009].

**Theorem 2.10 Criterion for \( \equiv \).** Consider CHR states \( S = \langle U; B; V \rangle, S' = \langle U'; B'; V \rangle \) with local variables \( \bar{l}, \bar{t} \) that have been renamed apart. Then \( S \equiv S' \) if and only if:

\[
CT \models \forall (B \rightarrow \exists \bar{t}' . ((U \equiv U') \land B')) \land \forall (B' \rightarrow \exists \bar{l} . ((U \equiv U') \land B))
\]

We define the notion of local variables of CHR rules, which is necessary for the definition of the operational semantics:

**Definition 2.11 Local Variables in Rules.** For a CHR rule \( r \circ H_1 \ \backslash \ H_2 \Leftrightarrow G \mid B_a \land B_b \),
we call the set

\[
\tilde{y}_r = vars(B_a, B_b, G) \setminus vars(H_1, H_2)
\]

the local variables of \( r \).

The transition system constituting the operational semantics of CHR is specified in the following definition:

**Definition 2.12 Transition System of \( \omega_c \).** CHR is a state transition system over equivalence classes of CHR states defined by the following transition rule, where \( (r \circ H_1 \ \backslash \ H_2 \Leftrightarrow G \mid B_a \land B_b) \) is a variant of a CHR rule whose local variables \( \tilde{y}_r \) are renamed apart from any variable in \( vars(H_1, H_2, U, B, V) \):

\[
\frac{r \circ H_1 \ \backslash \ H_2 \Leftrightarrow G \mid B_a \land B_b \quad CT \models \exists (G \land B)}{[(H_1 \land H_2 \land U; G \land B; V)] \mapsto^r [(H_1 \land B_c \land U; G \land B_b \land B; V)]}
\]

If the applied rule is obvious from the context or irrelevant, we write transition simply as \( \mapsto^r \). We denote its reflexive-transitive closure as \( \mapsto^* \). In the following, we sometimes write \( S \mapsto T \) instead of \( [S] \mapsto^* [T] \) to preserve clarity.

The required disjointness of the local variables \( \tilde{y}_r \) from all variables occurring in the pre-transition state outside \( G \) enforces that fresh variables are introduced for the local variables of the rule. When reasoning about programs, we usually refer to the following observables:
Definition 2.13 Computables States and Constraints. Let $S$ be a CHR state, $\mathbb{P}$ be a program, and $CT$ be a constraint theory. We distinguish three sets of observables:

- Computable states: $C_{\mathbb{P},CT}(S) := \{T \mid [S] \vdash^* [T]\}$
- Answers: $\mathcal{A}_{\mathbb{P},CT}(S) := \{T \mid [S] \vdash^* [T] \nvdash\}$
- Data-sufficient answers: $S_{\mathbb{P},CT}(S) := \{[\langle \top; \mathbb{B}; \mathbb{V} \rangle \mid [S] \vdash^* [(\top; \mathbb{B}; \mathbb{V})]\}$

For all three sets, if the respective constraint theory $CT$ is clear from the context or not important, it may be omitted from the identifier of the respective set.

As the transition system does not allow transitions from an empty user-defined store (nor from failed states), the data-sufficient answers $S_{\mathbb{P},CT}(S)$ are a subset of the answers $\mathcal{A}_{\mathbb{P},CT}(S)$ of any state $S$. The following property follows directly:

Property 2.14 Hierarchy of Observables. For any state $S$, program $\mathbb{P}$ and constraint theory $CT$, we have:

$$S_{\mathbb{P},CT}(S) \subseteq \mathcal{A}_{\mathbb{P},CT}(S) \subseteq C_{\mathbb{P},CT}(S)$$

Confluence is an important property in transition systems. We define it in the usual manner:

Definition 2.15 Confluence. A CHR program $\mathbb{P}$ is confluent if for all states $S, T, T'$ such that $[S] \vdash^* [T]$ and $[S] \vdash^* [T']$, there exists a state $T''$ such that $[T] \vdash^* [T'']$ and $[T'] \vdash^* [T''].$

Confluence restricts the number of possible answers to a query:

Property 2.16. Let $\mathbb{P}$ be a confluent CHR program. Then for every CHR state $S$, we have $|S_{\mathbb{P}}(S)| \in \{0, 1\}$ and $|A_{\mathbb{P}}(S)| \in \{0, 1\}$, where $|\cdot|$ denotes cardinality.

Proof sketch. We assume that for some states $S, T, T'$ and some confluent program $\mathbb{P}$, we have $S \vdash^* T \nvdash$ and $S \vdash^* T' \nvdash$ and $[T] \neq [T']$. Applying Def. 2.15 leads to a contradiction. □

A necessary, sufficient and decidable criterion for confluence has been given in Abdennadir et al. [1996]. Example 2.17 presents a standard CHR example program to illustrate our definitions.

Example 2.17. The following program implements a solver for the (user-defined) partial-order constraint $\leq$. Rule $r_1$ implements idempotence of identical constraints, $r_R$ implements reflexivity, $r_S$ symmetry and $r_T$ transitivity of the partial-order relation:

$$
\begin{align*}
  r_1 & \quad @ x \leq y \land x \leq y \iff x \leq y \\
  r_R & \quad @ x \leq x \iff \top \\
  r_S & \quad @ x \leq y \land y \leq x \iff x = y \\
  r_T & \quad @ x \leq y \land y \leq z \implies x \leq z 
\end{align*}
$$

The following is a sample derivation, starting from an initial state $S_0 = \langle a \leq b \land b \leq c \land c \leq a; \top; \{a, b, c\} \rangle$. According to the usual practice, all variables occurring in the initial state are global. Equivalence transformations are stated explicitly:

\[\text{\ldots} \]

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We will give the formal definition in terms of a sequent calculus. The calculus is based on binary sequents of the form

$$\Gamma \vdash \alpha$$

where $\Gamma$ is a multiset of formulas (written without braces) called *antecedent* and $\alpha$ is a formula called *consequent*. A sequent $\Gamma \vdash \alpha$ represents the fact that assuming the formulas in $\Gamma$, we can conclude $\alpha$. A *proof tree* – or simply *proof* – is a finite labeled tree whose nodes are labeled with sequents such that the relationship between every sequent node and

3.1 Definition

between internal and external choice and a faithful embedding of classical logic. In this section, we recall the intuitionistic fragment of linear logic, which is easier to handle than the full fragment but sufficient for our declarative semantics. It allows for a straightforward, faithful embedding of intuitionistic logic.

3. INTUITIONISTIC LINEAR LOGIC

Linear logic was introduced by Girard [1987]. Unlike classical logic, linear logic does not allow free copying or discarding of assumptions. It furthermore features a fine distinction between internal and external choice and a faithful embedding of classical logic. The set $C_{\mathcal{F},\mathcal{T}}(S_0)$ is infinite as the operational semantics $\omega_c$ allows potentially unlimited applications of $r_T$.

3.1 Definition

We will give the formal definition in terms of a *sequent calculus*. The calculus is based on binary sequents of the form

$$\langle a \leq b \land b \leq c \land c \leq a; \top; \{a, b, c\} \rangle \quad (1)$$

$$\vdash r_T \langle a \leq b \land b \leq c \land c \leq a; \top; \{a, b, c\} \rangle \quad (2)$$

$$\vdash r_T \langle a \leq b \land b \leq c \land c \leq a; \top; \{a, b, c\} \rangle \quad (3)$$

$$\vdash r_T \langle \top; a \equiv b \land a \equiv c; \{a, b, c\} \rangle \quad (4)$$

With respect to our observables, we have:

$$S_{\mathcal{F},\mathcal{T}}(S_0) = \mathcal{A}_{\mathcal{F},\mathcal{T}}(S_0) = \{ (\top; a \equiv b \land a \equiv c; \{a, b, c\}) \}$$

The set $C_{\mathcal{F},\mathcal{T}}(S_0)$ is infinite as the operational semantics $\omega_c$ allows potentially unlimited applications of $r_T$. 
its direct children corresponds to one of the inference rules of the calculus. We distinguish a special set of sequents called axioms. A proof tree is called complete if all its leaves are axioms. We call a sequent $\Gamma \vdash \alpha$ valid if there exists a complete proof tree $\pi$ with $\Gamma \vdash \alpha$ at the root.

The following two structural rules are common to many logical systems. They establish reflexivity and a form of transitivity of the judgement relation.

$\alpha \vdash \alpha$ \hspace{1cm} \frac{\Gamma \vdash \alpha, \Delta \vdash \beta}{\Gamma, \Delta \vdash \beta}$

The tokens of (intuitionistic) linear logic are commonly considered as representing resources rather than truths. This terminology reflects the fact that assumptions may not be copied nor discarded freely in linear logic, but must be used exactly once. From a different point of view, we might say that linear logic consumes assumptions in judgements and is aware of their multiplicities.

Multiplicative conjunction is distinguished from classical or intuitionistic conjunction as it lacks idempotence. Hence, $\alpha \otimes \beta$ represents exactly one instance of $\alpha$ and one instance of $\beta$. The formula $\alpha$ is not equivalent to $\alpha \otimes \alpha$. Multiplicative conjunction is introduced by the following inference rules:

$\frac{\Gamma, \alpha, \beta \vdash \gamma}{\Gamma, \alpha \otimes \beta \vdash \gamma}$ \hspace{1cm} $\frac{\Gamma \vdash \alpha \Delta \vdash \beta}{\Gamma, \Delta \vdash \alpha \otimes \beta}$

The constant $1$ represents the empty resource and is consequently the neutral element with respect to multiplicative conjunction.

$\frac{\Gamma \vdash \alpha}{\Gamma, 1 \vdash \alpha}$ \hspace{1cm} $\frac{\Gamma \vdash \alpha}{1 \vdash \alpha}$

Linear implication $\to$ allows the application of modus ponens where the preconditions of a linear implication are consumed on application. For example, the sequent $\alpha \otimes (\alpha \to \beta) \vdash \beta$ is valid whereas $\alpha \otimes (\alpha \to \beta) \vdash \alpha \otimes \beta$ is not. The following inference rules introduce $\to$:

$\frac{\Gamma \vdash \alpha \beta \Delta \vdash \gamma}{\Gamma, \alpha \to \beta \Delta \vdash \gamma}$ \hspace{1cm} $\frac{\Gamma \vdash \alpha}{\Gamma, \Delta \vdash \alpha \to \beta}$

The $!$ ("bang") modality marks stable facts or unlimited resources, thus recovering propositions in the classical (or intuitionistic) sense. Like an classical proposition, a banged resource may be freely copied or discarded. Hence, $!\alpha \otimes !(\alpha \to \beta) \vdash !(\alpha \otimes \beta)$ is a valid sequent. Four inference rules introduce the bang:

$\frac{\Gamma \vdash \alpha}{! \Gamma \vdash !\alpha}$ \hspace{1cm} $\frac{\Gamma, \alpha \vdash \beta}{\Gamma, !\alpha \vdash \beta}$ (Dereliction)

$\frac{\Gamma, !\alpha, !\alpha \vdash \beta}{\Gamma, !\alpha \vdash \beta}$ (Contraction) \hspace{1cm} $\frac{\Gamma \vdash \beta}{\Gamma, !\alpha \vdash \beta}$ (Weakening)

Example 3.1. We can model the fact that one cup of coffee ($c$) is one euro ($e$) as $!(e \to c)$. A "bottomless cup" is an offer including an unlimited number of refills. We assume
that any natural number of refills is possible. We model this as \(!e \rightarrow !c\). From this, we may judge that it is possible to get two cups of coffee for one euro: \(!e \rightarrow !c\) ⊢ e → c ⊗ c. Fig. 3.1 gives an exemplary proof tree, proving this judgement.

In classical (and intuitionistic) logic, internal choice is an aspect of conjunction, as exemplified by the judgement \(\alpha \land \beta \vdash \alpha\). This is inherited by the additive conjunction \(\&\) of linear logic. The formula \(\alpha \& \beta\) expresses a choice between \(\alpha\) and \(\beta\), i.e. the sequents \(\alpha \& \beta \vdash \alpha\) and \(\alpha \& \beta \vdash \alpha\) are valid, but \(\alpha \& \beta \vdash A \otimes B\) is not.

\[
\begin{align*}
\frac{\Gamma, \alpha \vdash \gamma}{\Gamma, \alpha \& \beta \vdash \gamma} \quad & (L\&_1) \\
\frac{\Gamma, \beta \vdash \gamma}{\Gamma, \alpha \& \beta \vdash \gamma} \quad & (L\&_2) \\
\frac{\Gamma \vdash \alpha}{\Gamma \vdash \alpha \& \beta} \quad & (R\&) \\
\end{align*}
\]

The \(\top\) ("top") is the resource that all other resources can be mapped to, i.e. for every \(\alpha\), the implication \(\alpha \rightarrow \top\) is a tautology. It is hence the neutral element with respect to additive conjunction.

\[
\frac{}{\Gamma \vdash \top} \quad (R\top)
\]

External choice is an aspect of classical (and intuitionistic) disjunction. In linear logic, it is represented by the additive disjunction \(\oplus\). Analogous to classical logic, \(\alpha \oplus \beta \vdash \alpha\) is not valid. However, \(!(\alpha \rightarrow \gamma)\), \(!(\beta \rightarrow \gamma)\), \(\alpha \oplus \beta \vdash \gamma\) is valid.

\[
\begin{align*}
\frac{\Gamma, \alpha \vdash \gamma \quad \Gamma, \beta \vdash \gamma}{\Gamma, \alpha \oplus \beta \vdash \gamma} \quad & (L\oplus) \\
\frac{\Gamma \vdash \alpha}{\Gamma \vdash \alpha \oplus \beta} \quad & (R\oplus_1) \\
\frac{\Gamma \vdash \beta}{\Gamma \vdash \alpha \oplus \beta} \quad & (R\oplus_2)
\end{align*}
\]

Analogous to falsity in the classical sense, absurdity \(\textbf{0}\) is a constant that yields every other resource. It is the neutral element with respect to \(\oplus\).

\[
\frac{}{\textbf{0} \vdash \alpha} \quad (L\textbf{0})
\]

Example 3.2. We assume that, besides coffee, the cafeteria offers also pie (\(p\)) at the price of one euro per piece: \(!e \rightarrow p\). We infer that for one euro, we have the choice between an arbitrary amount of coffee and a piece of pie: \(!e \rightarrow !c\), \(!e \rightarrow p\) \vdash e → (lc&p). Let us furthermore assume that rather than with euros, we can also pay with dollars (\(d\)) at a 1 : 1 ratio: \(!d \rightarrow !c\), \(!d \rightarrow p\). We may infer either one of one dollar or one euro buys us a choice between an arbitrary amount of coffee and one pie:

\[
!(e \rightarrow !c), !(e \rightarrow p), !(d \rightarrow !c), !(d \rightarrow p) \vdash (e \oplus d) \rightarrow (!c&p).
\]
We can extend intuitionistic linear logic into a first-order system with the quantifiers $\exists$ and $\forall$. Their introduction rules are the same as in classical logic. In the following rules, $t$ stands for an arbitrary term whereas $a$ stands for a variable that is not free in $\Gamma$, $\alpha$ or $\beta$:

\[
\frac{\Gamma, a[x/t] \vdash \beta}{\Gamma, \forall x.\alpha \vdash \beta} \quad (L\forall) \quad \frac{\Gamma \vdash \beta[x/a]}{\Gamma \vdash \forall x.\beta} \quad (R\forall)
\]

\[
\frac{\Gamma, a[x/a] \vdash \beta}{\Gamma, \exists x.\alpha \vdash \beta} \quad (L\exists) \quad \frac{\Gamma \vdash \beta[x/t]}{\Gamma \vdash \exists x.\beta} \quad (R\exists)
\]

### 3.2 Properties of Intuitionistic Linear Logic

The resulting first-order system allows for a faithful embedding of intuitionistic first order logic. This is widely considered one of the most important features of linear logic. The following translation from intuitionistic logic into intuitionistic linear logic is a variant of a translation proposed by Negri [1995]:

**Definition 3.3.** $(\cdot)^*$ is a translation from formulas of intuitionistic logic to formulas of intuitionistic linear logic, recursively defined by the following rules:

\[
p(t)^* := !p(t) \\
(\bot)^* := 0 \\
(\top)^* := 1 \\
(A \land B)^* := A^* \otimes B^* \\
(A \lor B)^* := A^* \oplus B^* \\
(A \Rightarrow B)^* := !(A^* \multimap B^*) \\
(\forall x.A)^* := \forall x.(A^*) \\
(\exists x.A)^* := \exists x.(A^*)
\]

$p(t)$ stands for an atomic proposition. The definition is extended to sets and multisets of formulas in the obvious manner. It has been proven in Negri [1995] that an intuitionistic sequent ($\Gamma \vdash_{\text{IL}} \alpha$) is valid if and only if ($\Gamma^* \vdash_{\text{ILL}} \alpha^*$) is valid in linear logic.

We distinguish two sorts of axioms in the sequent calculus. The (*Identity*) axiom and the constant axioms ($L\bot$), ($R\top$), ($L\forall$) and ($R\exists$) constitute the *logical axioms* of intuitionistic linear logic. All axioms we add to the system on top of these are called *non-logical axioms* or *proper axioms*. We usually use the letter $\Sigma$ to denote the set of proper axioms.

We express the fact that a judgement $\Gamma \vdash \alpha$ is provable using a non-empty set $\Sigma$ of proper axioms by indexing the judgement relation with the set of proper axioms: $\vdash_{\Sigma}$.

**Definition 3.4 Linear-Logic Equivalence.**

1. We call two linear-logic formulas $\alpha, \beta$ *logically equivalent* if both $\alpha \vdash \beta$ and $\beta \vdash \alpha$ are provable. We write this as $\alpha \equiv_{\Sigma} \beta$.

2. For any set of proper axioms $\Sigma$, we call two linear-logic formulas $\alpha, \beta$ *logically equivalent modulo* $\Sigma$ if both $\alpha \vdash_{\Sigma} \beta$ and $\beta \vdash_{\Sigma} \alpha$ are provable. We write this as $\alpha \equiv_{\Sigma} \beta$.

As a well-behaved logical system, linear logic features a cut-elimination theorem [Girard 1987]:

**Theorem 3.5 Cut Elimination Theorem.**

1. If a sequent $\Gamma \vdash \alpha$ has a proof $\pi$ that does not contain any proper axioms, then it has a proof $\pi'$ that contains neither proper axioms nor the (Cut) rule.
(2) If a sequent $\Gamma \vdash \Sigma \alpha$ has a proof $\pi$ containing proper axioms, then it has a proof $\pi'$ where the (Cut) rule is only used at the leaves such that one of its premises is an axiom.

A proof without any applications of (Cut) is called cut-free. A proof where (Cut) is only applied at the leaves is called cut-reduced.

A important consequence of cut elimination is the subformula property. We quote a weak formulation of the property, which will suffice for our purpose: Every formula $\alpha$ in a cut-free proof of a sequent $\Gamma \vdash \beta$ is a subformula of either $\Gamma$ or $\beta$, modulo variable renaming. In a cut-reduced proof of a sequent $\Gamma \vdash \Sigma \beta$, every formula $\alpha$ is a subformula of $\Gamma$ or $\beta$, modulo variable renaming, or there exists a proper axiom $(\Delta \vdash \gamma) \in \Sigma$ such that $\alpha$ is a subformula of $\Delta$ or $\gamma$, modulo variable renaming.

4. A LINEAR-LOGIC SEMANTICS FOR CHR

In this section, we motivate and develop the linear-logic semantics for Constraint Handling Rules. We firstly recall the classical declarative semantics in Sect. 4.1. Then we motivate and present a linear-logic semantics based on proper axioms in Sect. 4.2. We will henceforth call this the axiomatic linear-logic semantics for CHR. Its soundness with respect to the operational semantics is shown in Sect. 4.3. We continue in Sect. 4.4 by introducing the notion of state entailment, which we use to formulate and prove the completeness of our semantics in Sect. 4.5. Finally, in Sect. 4.6, we show an alternative linear-logic semantics that encodes programs and contraints theories into linear logic.

4.1 Analysis of the Classical Declarative Semantics

CHR is founded on a classical declarative semantics, which is reflected in its very syntax. In this section, we recall the classical declarative semantics and discuss its assets and limitations.

In the following, $\exists \bar{x}$ stands for existential quantification of all variables except those in $\bar{x}$, where $\bar{x}$ is a set of variables. The classical declarative semantics is given in the following table, where $(\cdot)^\dagger$ stand for translation to classical logic:

- **States:** $\langle U; B; \forall \gamma \rangle^\dagger$ := $\exists \bar{y}.(U \land B)$
- **Rules:** $(r @ H_1 \setminus H_2 \iff G \mid B)^\dagger$ := $\forall (G \rightarrow (H_1 \rightarrow (H_2 \leftrightarrow \exists \bar{y}.B)))$
- **Programs:** $\{R_1, \ldots, R_m\}^\dagger$ := $R_1^\dagger \land \ldots \land R_m^\dagger$

$\bar{y}$ denotes the local variables of the respective rule. The following lemma – cited from Frühwirth and Abdennadher [2003] – establishes the relationship between the logical readings of programs, constraint theories and states:

**Lemma 4.1 (Logical Equivalence of States).** Let $P$ be a CHR program and $S$ be a state. Then for all computable states $T_1$ and $T_2$ of $S$, the following holds: $P^\dagger, CT \models \forall (T_1^\dagger \iff T_2^\dagger)$.

The declarative semantics of CHR must be distinguished from LP languages and related paradigms as CHR is not based on the notion of execution as proof search. Declaratively, execution of a CHR program means stepwise transformation of the information contained in the state under logical equivalence as defined by the program’s logical reading $P^\dagger$ and the constraint theory $CT$. Founding CHR on such a declarative semantics is an obvious choice for several reasons:

Firstly, the notion of execution as proof search naturally implies a notion of search. This stands in contrast to the committed-choice execution of CHR. Furthermore, the forward-reasoning approach faithfully captures the one-sided variable matching between rule heads.
and constraints in CHR, as opposed to unification. For example, a CHR state \( \langle p(x); \top; \emptyset \rangle \) (where \( x \) is a variable) does not match with the rule head \( (p(a) \iff \ldots) \) (where \( a \) is a constant) just as we cannot apply modus ponens on a fact \( \exists x. p(x) \) and an implication \( (p(a) \rightarrow \ldots) \). In contrast, an LP goal \( p(x) \) would be unified with a rule head \( (p(a) \leftarrow \ldots) \), accounting for the fact that application of the rule might lead to a proof of an instance of \( p(x) \).

There are, however, several limitations to the classical declarative semantics of CHR, which shall be discussed in the following:

**Directionality.** One limitation lies in the fact that the classical declarative semantics does not capture the inherent directionality of CHR rules. Rather, all states within a computation are considered logically equivalent. Consider e.g. the minimal CHR program

\[
a \leftrightarrow b
\]

In this program, we can compute a state \( \langle b; \top; \emptyset \rangle \) from a state \( \langle a; \top; \emptyset \rangle \) but not vice versa. This is not captured in its logical reading \( a \leftrightarrow b \) which e.g. implies \( b \rightarrow a \). The classical declarative semantics cannot be used e.g. to show that the state \( \langle a; \top; \emptyset \rangle \) is not a computable state \( \langle b; \top; \emptyset \rangle \).

**Dynamic Change.** Any program state that does not only contain declarative information about a supposedly static object world but also meta-information about the program state eludes the semantics. Consider the following program which computes the minimum of a set:

\[
\text{min}(x), \text{min}(y) \iff x \leq y \mid \text{min}(x)
\]

On a fixed-point execution, the program correctly computes the minimum of all arguments of \( \text{min} \) constraints found in the store at the beginning of the computation. Its logical reading, however, is unhelpful at best:

\[
\forall x, y. x \leq y \rightarrow (\text{min}(x) \land \text{min}(y) \iff \text{min}(x))
\]

**Deliberate Non-Determinism.** Any program that makes deliberate use of the inherent non-determinism of CHR has a misleading declarative semantics as well. Consider the following program, which simulates a coin throw in an appropriate probabilistic semantics of CHR (cf. Frühwirth et al. [2002]). (Note that \( \text{coin} \) is a variable, \( \text{head} \) and \( \text{tail} \) are constants.)

\[
\text{throw}(\text{coin}) \iff \text{coin} \oplus \text{head} \\
\text{throw}(\text{coin}) \iff \text{coin} \oplus \text{tail}
\]

The logical reading of this program implies \( \forall \text{coin}. (\text{coin} \oplus \text{head} \iff \text{coin} \oplus \text{tail}) \). From this follows \( \text{head} \oplus \text{tail} \) and – since \( \text{head} \) and \( \text{tail} \) are distinct constants – falsity \( \bot \). The program’s logical reading is thus inconsistent, trivially implying anything.

**Multiplicities.** Finally, while CHR faithfully keeps track of the multiplicities of constraints, this aspect eludes the classical semantics. Consider the idempotence rule from Example 2.17, which removes multiple occurrences of the same constraint:

\[
r_t \odot x \leq y, x \leq y \iff x \leq y
\]

The logical reading of this rule is a tautology, falsely suggesting that the rule is redundant:

\[
\forall x, y. (x \leq y \land x \leq y \iff x \leq y)
\]
In conclusion, the classical declarative semantics is a powerful tool to prove the soundness and a certain notion of completeness of any program whose states contain only model information about a static object world and no explicit meta-information. It faithfully captures the logical theory behind those programs. However, it is not adequate to capture the logic behind programs that deal with any form of meta-information, make deliberate use of non-determinism or rely on the multiplicities of constraints. As it does not capture the inherent directionality of CHR rules, it is not suitable to prove safety conditions, i.e., to show that a certain intermediate or final state cannot be derived from a certain initial state.

4.2 The Axiomatic Linear-Logic Semantics for CHR

Our linear-logic semantics is based on two observations: Firstly, the difference in behaviour between built-in and user-defined constraints in CHR resembles the difference between linear and banged atoms in linear logic. Secondly, the application of simplification rules on user-defined constraints resembles the application of modus ponens in linear logic.

Building on the first observation, we define an adequate representation of CHR constraints in linear logic. Translation to linear logic will be denoted as \((\cdot)^L\). For atomic constraints, the choice is obvious:

\[
\begin{align*}
\text{Atomic built-in constraints:} & \quad c_b(\bar{t})^L := \text{lic}(\bar{t}) \\
\text{Atomic user-defined constraints:} & \quad c_u(\bar{t})^L := c_u(\bar{t}) \\
\text{Falsity:} & \quad \bot^L := 0 \\
\text{Empty constraint/goal:} & \quad \top^L := 1 \\
\end{align*}
\]

Classical conjunction is mapped to multiplicative conjunction for both built-in and user-defined constraints.

\[
(G_1 \land G_2)^L := G_1^L \otimes G_2^L
\]

This mapping is motivated by the fact that multiplicative conjunction is aware of multiplicities and has no notion of weakening, thus capturing the multiset semantics of user-defined constraints. For any built-in constraint \(B\), the mapping equals the translation quoted in Def. 3.3: \(B^L = B^*\). Accordingly, we map the empty goal \(\top\) to \(\mathbf{1}\) and falsity \(\bot\) to \(0\). The translation of CHR states is analogous to the classical case:

\[
\langle U; B; V \rangle^L := \exists_v.U^L \otimes B^L
\]

The translation of constraints, goals, and states is summed up in Fig. 4.2.

**Proper axioms.** The constraint theory \(CT\), the interaction between equality constraints (which are by definition built-in) and user-defined constraints, and programs are translated to proper axioms. Firstly, we define a set of proper axioms encoding the constraint theory as well as modelling the interaction between equality \(=\) and user-defined constraints.

**Definition 4.2\((\Sigma_{CT})\).** For built-in constraints \(B, B'\) and sets of variables \(\bar{x}, \bar{x}'\) such that
The existential quantification of the local variables $\bar{y}_r$ corresponds to the fact that these variables are by definition disjoint from $\text{vars}(H_1, H_2, \cup, \exists, \forall)$, assuring that fresh variables are introduced for the local variables of the rule. Fig. 4.2 sums up the three sets of proper axioms, represented as inference rules.

In anticipation of the soundness theorem presented in Sect. 4.3, we give an example of a CHR derivation and show that it corresponds to a valid linear logic judgement:

**Example 4.5.** Let $\mathcal{P}$ be the partial-order constraint solver from Example 2.17 and let $CT$ be a minimal constraint theory. We observe that under $\mathcal{P}$, we have:

$$(3 \leq a; a = 3; \emptyset) \rightarrow^* (\top; \top; \emptyset)$$

This corresponds to the judgement $(3 \leq a; a = 3; \emptyset)^L \vdash_\Sigma (\top; \top; \emptyset)^L$ or $\exists a.(3 \leq a \otimes !a = 3) +_\Sigma 1$, respectively, where $\Sigma = \Sigma_{CT} \cup \Sigma_{\exists} \cup \Sigma_{\forall}$. The following is a proof of this judgement:

$$
\begin{array}{c}
\begin{array}{c}
3 \leq x \otimes 1 \leq 3 \\
\begin{array}{c}
\vdash 3 \leq x \otimes 1 \leq 3 \\
\vdash 1 \oplus 1 \leq 3 \\
\vdash 3 \leq x \otimes 1 \leq 3 \rightarrow 1 \oplus 1 \leq 3 \\
\vdash 3 \leq x \otimes 1 \leq 3 \rightarrow \top \otimes 1 \leq 3 \\
\end{array}
\end{array}
\end{array}
$$

The sequent $1 \otimes !x \equiv 3 \rightarrow 1$ is a tautology and as such could be derived without proper axioms, but it is also trivially included in $\Sigma_{CT}$.

While the soundness result for our semantics is straightforward, defining completeness is not quite as simple. Consider the following example:
Example 4.6. In the proof tree given in Example 4.5 we use the following proper axiom from $\Sigma_{CT}$:

$$1 \otimes \not x \equiv 3 \vdash 1$$

This implies:

$$\langle \top; x \equiv 3; \{x\} \rangle \vdash_\Sigma L \vdash \langle \top; \top; \emptyset \rangle L$$

We observe, however, that $\langle \top; x \equiv 3; \{x\} \rangle \mapsto^* \langle \top; \top; \emptyset \rangle$ is untrue.

In the following section, we prove the soundness of our semantics. In Sect. 4.4, we develop the notion of state entailment. We will apply this notion to specify and prove a completeness result in Sect. 4.5.

4.3 Soundness of the Linear Logic Semantics

In this section, we prove the soundness of the axiomatic linear-logic semantics for CHR with respect to the operational semantics.

**Lemma 4.7** ($\equiv \Rightarrow \vdash_L$). Let CT be a constraint theory and $\Sigma = \Sigma_{CT} \cup \Sigma_{\not=}$ for arbitrary $\Sigma_{CT}$. For arbitrary CHR states $S, T$, we have:

$$S \equiv \Sigma \Rightarrow S_L \vdash_T L$$

**Proof sketch.** We prove that state equivalence $S \equiv \Sigma \Rightarrow$ implies linear judgement $S \vdash T$ by showing that every of the conditions given for $S \equiv \Sigma$ in Def. 2.5 implies $S \vdash T$: Def. 2.5.1 implies linear judgement since multiplicative conjunction is associative, commutative and invariant w.r.t. $\top$, thus corresponding to goal equivalence. For Def. 2.5.2, linear judgement is guaranteed, as $\Sigma_{\not=}$ allows us to prove $\exists \not y. B \otimes \not x \equiv 3 \otimes t \equiv \bar{t} \otimes B$. For Def. 2.5.3, it is similarly guaranteed by $\Sigma_{CT}$. Def. 2.5.4 implies linear judgement since the addition or removal of a global variable not occurring in a state does not change the logical reading of the state. W.r.t. Def. 2.5.5, linear judgement holds since $\varphi \otimes 0 \vdash \psi$ is valid for any $\varphi, \psi$. All the above arguments can be shown to apply in the reverse direction as well, thus proving compliance with the implicit symmetry of $\equiv \Sigma$. The implicit reflexivity and transitivity of state equivalence comply with linear judgement due to the (Identity) and (Cut) rules. \qed

Theorem 4.8 states the soundness of our semantics.

**Theorem 4.8 Soundness.** Let $P$ be a program, CT be a constraint theory, and $\Sigma = \Sigma_{\not=} \cup \Sigma_{CT} \cup \Sigma_{=}$. Then for arbitrary states $S, T$, we have:

$$S \mapsto^* T \Rightarrow S_L \vdash_T L$$

**Proof.** Let $S, T$ be states such that $S \mapsto^* T$. According to Def. 2.12, there exists a variant of a rule with fresh variables $r @ H_1 \setminus H_2 \Rightarrow G | B_a \land B_b$ and states $S' = \langle H_1 \land H_2 \land V; G \land B; V \rangle$, $T' = \langle B_a \land H_1 \land U; B_b \land G \land B; V \rangle$ such that $S' \equiv S$ and $T' \equiv T$. Consequently, $\Sigma_{\not=}$ contains:

$$H_1^L \otimes H_2^L \otimes G^L \vdash \exists_x. (B_a^L \otimes B_b^L \otimes G^L)$$

From which we prove:

$$\exists \not y. H_1^L \otimes H_2^L \otimes G^L \otimes U \otimes B \vdash \exists \not y. H_1^L \otimes \exists \not y. (B_a^L \otimes B_b^L \otimes G^L) \otimes U \otimes B$$

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The local variables $\bar{y}_r$ of $r$ are by Def. 2.12 disjoint from $\text{vars}(H_1, H_2, U, B, V)$. Hence, we have:

$$\exists \cdot \forall . H_1 \otimes H_2 \otimes G \otimes U \otimes B \vdash \exists \cdot \forall . H_1 \otimes H_2 \otimes B_{\bar{y}} \otimes B \otimes U \otimes B$$

This corresponds to $S' \vdash T'$. Lemma 4.7 proves that $S \vdash T$. As the judgement relation $\vdash$ is transitive and reflexive, we can generalize the relationship to the reflexive-transitive closure $S \leftrightarrow^* T$.  

4.4 State Entailment

In this section, we define the notion of entailment, which we will use to formulate our theorem of completeness. We present it alongside various properties that follow from it and that will be used in upcoming sections.

Definition 4.9. State entailment, written as $\vdash$, is the smallest partial-order relation over equivalence classes of CHR states that satisfies the following conditions:

1. (Weakening of the Built-In Store) For states $\langle U; B; V \rangle$, $\langle U; B'; V \rangle$ with local variables $\bar{x}, \bar{x}'$ such that $CT \models \forall (\exists \bar{x}. B \rightarrow \exists \bar{x}'. B')$, we have:
   $$\left[\langle U; B; V \rangle \right] \vdash \left[\langle U; B'; V \rangle \right]$$

2. (Omission of Global Variables)
   $$\left[\langle U; B; \{x \cup V\} \rangle \right] \vdash \left[\langle U; B; V \rangle \right]$$

To simplify notation, we often write $S \vdash S'$ instead of $[S] \vdash [S']$. Theorem 4.10 gives a decidable criterion for state entailment. The criterion requires that the global variables of the entailed state are contained in the global variables of the entailing state. This is never a problem, as we may choose representatives of the respective equivalence classes that satisfy the condition.

Theorem 4.10. Criterion for $\vdash$. Let $S = \langle C; B; V \rangle$, $S' = \langle C'; B'; V' \rangle$ be CHR states with local variables $\bar{l}, \bar{P}$ that have been renamed apart and where $\forall' \subseteq \forall$. Then we have:

$$[S] \vdash [S'] \iff CT \models \forall (B \rightarrow \exists \bar{P}.((U \equiv \forall' \cap B')))$$

Proof. '$\Rightarrow$': We show that the explicit axioms of entailment, as well as the implicit conditions reflexivity, anti-symmetry and transitivity comply with the criterion:

Def. 4.9.1. We assume w.l.o.g. that the strictly local variables of $\langle U; B; V \rangle$, $\langle U; B'; V' \rangle$ are renamed apart. We observe that $(U \equiv \forall)$ is a tautology for any $U$. Hence, from $CT \models \forall (\exists \bar{x}. B \rightarrow \exists \bar{x}'. B')$ follows $CT \models \forall (\exists \bar{x}. B \rightarrow \exists \bar{P}.((U \equiv \forall) \wedge B'))$, which proves: $CT \models \forall (B \rightarrow \exists \bar{P}.((U \equiv \forall) \wedge B'))$

Def. 4.9.2. Let $\bar{l}$ be the local variables of $\langle U; B; \{x \cup V\} \rangle$. For any $x$ we have: $CT \models \forall (B \rightarrow \exists \bar{l}.((U \equiv \forall) \wedge B))$

Reflexivity. Let $\langle U; B; V \rangle, \langle U'; B'; V' \rangle$ be CHR states such that $\left[\langle U; B; V \rangle \right] = \left[\langle U'; B'; V' \rangle \right]$, i.e. $\langle U; B; V \rangle \equiv \langle U'; B'; V' \rangle$. Assuming that the local variables $\bar{l}, \bar{P}$ have been named apart, Thm. 2.10 implies $CT \models \forall (B \rightarrow \exists \bar{P}.((U \equiv \forall') \wedge B'))$.

Anti-Symmetry. Let $\langle U; B; V \rangle, \langle U'; B'; V' \rangle$ be CHR states with local variables $\bar{l}, \bar{P}$ such that $CT \models \forall (B \rightarrow \exists \bar{P}.((U \equiv \forall') \wedge B'))$ and $CT \models \forall (B' \rightarrow \exists \bar{P}.((U \equiv \forall) \wedge B))$. By Thm. 2.10, we have that $\langle U; B; V \rangle \equiv \langle U'; B'; V' \rangle$ and hence $\left[\langle U; B; V \rangle \right] = \left[\langle U'; B'; V' \rangle \right]$. 

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Transitivity. Let \( (U; B; \overline{V}) \), \( (U'; B'; \overline{V'}) \), \( (U''; B''; \overline{V''}) \) be CHR states where the local variables \( \overline{I}, \overline{I'}, \overline{I''} \) have been renamed apart and such that \( CT \models \forall (B \Rightarrow \exists \overline{I'} \cdot ((U=U') \land B')) \) and \( CT \models \forall (B' \Rightarrow \exists \overline{I''} \cdot ((U'=U'') \land B'')) \). Therefore, \( CT \models \forall (B \Rightarrow \exists \overline{I''} \cdot ((U=U') \land \exists \overline{I'} \cdot ((U'=U'') \land B'')) \). As the sets of local variables are disjoint, we get \( CT \models \forall (B \Rightarrow \exists \overline{I''} \cdot ((U=U') \land (U'=U'') \land B'')) \) and finally
\[
CT \models \forall (B \Rightarrow \exists \overline{I''} \cdot ((U=U') \land (U'=U'') \land B''))
\]

\( \equiv \leftarrow \) Let \( S = (U; B; \overline{V}) \), \( S' = (U'; B'; \overline{V'}) \) be CHR states with local variables \( \overline{y}, \overline{y'} \) that have been renamed apart and such that \( \overline{V'} \subseteq \overline{V} \) and \( CT \models \forall (B \Rightarrow \exists \overline{y'} \cdot ((U=U') \land B')) \). We apply Def. 4.9.1 to infer: \( S \rhd (U; (U=U') \land B'; \overline{V}) \). By Def. 2.5.2 and Def. 2.5.3, we get \( S \rhd (U'; B'; \overline{V}) \). Since \( \overline{V'} \subseteq \overline{V} \), several applications of Def. 4.9.2 give us \( S \rhd (U'; B'; \overline{V'}) \equiv S' \). □

Corollary 4.11 is a direct consequence of Theorem 2.10 and Theorem 4.10. It establishes the relationship between state equivalence and state entailment.

**Corollary 4.11 (\( \equiv \Rightarrow \rhd \)).** For arbitrary CHR states \( S, T \), state equivalence \( S \equiv_e T \) holds if and only if both \( S \rhd T \) and \( T \rhd S \) hold.

Lemma 4.12 establishes an important relationship between state transition and state entailment.

**Lemma 4.12.** Let \( S, U, T \) be CHR states. If \( S \rhd U \) and \( U \rightarrow^r T \) then there exists a state \( V \) such that \( S \rightarrow^r V \) and \( V \rhd T \).

**Proof.** Let \( S = (U; B; \overline{V}) \) and let \( \overline{y}_S, \overline{y}_U, \overline{y}_T \) be the local variables of \( S, U, T \). By definition, \( U \rightarrow^r T \) implies that there is a variant of a CHR rule \( r \vdash (H_1 \land H_2 \Rightarrow G | B_b \land B_a) \) such that \( [U] = [(H_1 \land H_2 \land \overline{U}; G \land B; \overline{V})] \) and \( [T] = [(H_1 \land B_a \land \overline{U}; G \land B_b \land \overline{B}; \overline{V})] \).

Now let \( V = (H_1 \land B_a \land \overline{U}; G \land B_b \land \overline{B} \land (U\equiv(H_1 \land H_2 \land \overline{U}) \land \overline{B}; \overline{V}) \). From \( S \rhd [U] \) follows by Thm. 4.10: \( CT \models \forall (B \Rightarrow \exists \overline{y}_U \cdot ((U\equiv(H_1 \land H_2 \land \overline{U}) \land G \land \overline{B})))) \). Assuming w.l.o.g. that \( \overline{y}_U \cap \overline{y}_T = \emptyset \), we can apply Def. 2.5.3 to get \( S \equiv (U; B \land G \land \overline{B} \land (U\equiv(H_1 \land H_2 \land \overline{U}); \overline{V}) \) and then \( S \equiv (H_1 \land H_2 \land \overline{U}; B \land G \land \overline{B} \land (U\equiv(H_1 \land H_2 \land \overline{U}); \overline{V}) \). According to Def. 2.12, we have \( S \rightarrow^r V \). We apply Def. 4.9 to show that \( V \rhd T \). □

In anticipation of Section 4.5, the following example shows how the notion of entailment fills the gap between the computability relation between states and the judgement relation between their respective linear-logic readings.

**Example 4.13.** In Example 4.6, we showed that the following judgement, which does not correspond to any transition in CHR, is provable in our sequent calculus system:

\[
\langle T; x \equiv 3; \{x\} \rangle \vdash \langle T; T; \emptyset \rangle
\]

We observe that the two states are connected by the entailment relation:

\[
\langle T; x \equiv 3; \{x\} \rangle \vdash\bowtie \langle T; T; \emptyset \rangle
\]

In the following section, we will show that state entailment precisely covers the discrepancy between transitions in a CHR program and judgements in its corresponding sequent calculus system as exemplified in Example 4.6.
4.5 Completeness of the Axiomatic Semantics

The notion of merging is an important tool for the proofs in this section. We define it as follows:

Definition 4.14 (\(\cdot \circ \cdot\)). Let \(S = \langle U; B; V \rangle, S' = \langle U'; B'; V \rangle\) be CHR states that share the same set of global variables and whose local variables are renamed apart. Their merging is defined as:

\[ S \circ S' := \langle U \land U'; B \land B'; V \rangle \]

The following property assures that we can without loss of generality assume the existence of \(S \circ T\) for any two states \(S, T\):

Property 4.15. For any CHR states \(S, T\), there exist states \(S', T'\) such that \(S \equiv S', T \equiv T'\) such that \(S' \circ T'\) exists.

Proof sketch. Lemma 2.8.1 allows to rename the local variables apart, and Def. 2.5.4 allows the union of their respective sets of global variables.

Lemma 4.16 states two properties of merging that will be used in upcoming proofs:

Lemma 4.16 Properties of \(\cdot \circ \cdot\). Let \(S, S', T\) be CHR states such that both \(S \circ T\) and \(S' \circ T\) exist. The following properties hold:

1. \(S \supset S' \implies S \circ T \supset S' \circ T\)
2. \(S \supseteq S' \implies S \circ T \supseteq S' \circ T\)

Proof. Lemma 4.16.1: We assume w.l.o.g. that the states \(S, S', T\) share the same set of global variables. Let \(S = \langle U; B; V \rangle, S' = \langle U'; B'; V \rangle, T = \langle U_T; B_T; V \rangle\) with local vars \(\bar{l}, \bar{p}, \bar{r}\). From \(S \supset S'\) follows by Thm. 2.10: \(CT \models \forall (B \land \bar{B}_T \rightarrow \exists \bar{l} \bar{p} \bar{r}. ((U \equiv U') \land (B \equiv B'))\). As \(U_T \equiv U_T\) is a tautology, we get \(CT \models \forall (B \land B_T \rightarrow \exists \bar{l} \bar{p} \bar{r}. ((U \equiv U') \land (U_T \equiv U_T) \land (B \equiv B' \land B_T))\) which proves \(S \circ T \supset S' \circ T\).

Lemma 4.16.2: We assume w.l.o.g. that the states \(S, S', T\) share the same set of global variables. According to Def. 2.12, there exists a variant of a CHR rule \(r @ H_1 \land H_2 \Leftarrow G | B_a \land B_b\) such that \(S \equiv \langle H_1 \land H_2 \cup U; G \land B; V \rangle\) and \(S' \equiv \langle H_1 \land H_2 \cup U; G \land B_b \land B; V \rangle\). By Prop. 4.15, there exists a state \(T' = \langle U; B'; V \rangle\) such that \(T' \equiv T\) whose local variables are renamed apart from those of \(S\) and \(T\). By Def. 2.12, we get \(S \circ T \supseteq S' \circ T\).

Lemma 4.17 sets the stage for the completeness theorem:

Lemma 4.17. Let \(\pi\) be some cut-reduced proof of a sequent \(S^L \vdash_T T^L\), where \(S, T\) are arbitrary CHR states and \(\Sigma = \Sigma_T \cup \Sigma_G\) for a program \(P\) and a constraint theory \(CT\). Any formula \(\alpha\) in \(\pi\) is either of the form \(\alpha = S^L_\alpha\) where \(S_\alpha\) is a CHR state or of the form \(\alpha = c_\alpha(\bar{t})\) where \(c_\alpha(\bar{t})\) is some built-in constraint.

Proof. We observe that both the root of \(\pi\) and all proper axioms in \(\Sigma\) are of the form \(U^L_1 \vdash U^L_2\) where \(U_1, U_2\) are CHR states. The subformula property hence guarantees that every formula \(\alpha\) in \(\pi\) is a subformula of the logical reading \(U^L\) of some CHR state \(U\). The general form of such a logical reading is \(U^L = \exists \bar{l}_1, \ldots, \exists \bar{l}_n.(c_{\bar{l}_1}(\bar{t}_1) \land \ldots \land c_{\bar{l}_n}(\bar{t}_n)) \land (\exists \bar{c}_{\bar{l}}(\bar{r}) \land \ldots \land c_{\bar{l}_n}(\bar{r}))\) where \(\bar{l}_1, \ldots, \bar{l}_n\) are the local variables of \(U\), \(c_{\bar{l}_1}(\bar{t}_1), \ldots, c_{\bar{l}_n}(\bar{t}_n)\) are user-defined constraints and \(c_{\bar{l}_1}(\bar{r}), \ldots, c_{\bar{l}_n}(\bar{r})\) are its built-in constraints. We observe that any subformula \(\alpha\) of \(U^L\) is either of the form \(\alpha = S^L_\alpha\) for some CHR state \(S_\alpha\) or of the form \(\alpha = c_\alpha(\bar{t})\), where \(c_\alpha(\bar{t})\) is a built-in constraint.
The completeness of our semantics is formulated in Theorem 4.18:

**Theorem 4.18 Completeness.** Let $S, T$ be CHR states, $P$ be a CHR program, $CT$ be a constraint theory and let $\Sigma = \Sigma_P \cup \Sigma_{CT} \cup \Sigma_\pi$. If $S^L \vdash_\Sigma T^L$, then there exists a state $T'$ such that $S \vdash T'$ and $T' \triangleright T$ in $P$.

**Proof.** To preserve of clarity, we will omit the set $\Sigma$ of proper axioms from the judgement symbol $\vdash$. Throughout the proof, $D_n(U, V)$ denotes the fact that for CHR states $U, V$, there exist states $U_1, \ldots, U_n$ such that:

$$ U \leftrightarrow U_1 \ldots \leftrightarrow U_n \triangleright V \quad (\ast) $$

Consequently, $D_n(U, V)$ equals $U \triangleright V$.

Secondly, we define an operator on formulas analogous to merging on states: For any two (possibly empty) sequences of variables $\bar{x}, \bar{y}$ and quantifier-free formulas $\alpha, \beta$ let $\exists \, \bar{x}. \alpha \land \exists \, \bar{y}. \beta := \exists \, \bar{x, y}. \alpha \lor \beta$. We observe that for arbitrary CHR states $U, V$ where $U \triangleright V$ exists, we have $U^L \lor V^L \equiv (U \triangleright V)^L$. In the following, we assume w.l.o.g. that all existentially quantified variables in the antecedent of a sequent occurring in $\pi$ are renamed apart. Hence, for every two formulas of the form $U^L, V^L$ occurring in the antecedent of one sequent in $\pi$, both $U \triangleright V$ and $U^L \lor V^L$ exist.

We introduce a completion function $\eta$, defined by the following table, where $U$ is a CHR state, $c_b(\bar{t})$ is a built-in constraint and $\Gamma \vdash \alpha$ is a sequent:

| $\eta(U^L)$ | $\eta(c_b(\bar{t}))$ | $\eta(\gamma, \Gamma)$ | $\eta(\Gamma \vdash \alpha)$ | $\eta(\vdash \alpha)$ |
|-------------|---------------------|-------------------|------------------|------------------|
| $U^L$       | $!c_b(\bar{t})$     | $\eta(\gamma) \land \eta(\Gamma)$ | $\eta(\Gamma \vdash \alpha)$ | $1 \vdash \eta(\alpha)$ |

For a sequent $\Gamma \vdash \alpha$, we call $\eta(\Gamma \vdash \alpha)$ the $\eta$-completion of $\Gamma \vdash \alpha$. From Lemma 4.17 follows that for every sequent $\Gamma \vdash \alpha$ in $\pi$, its $\eta$-completion $\eta(\Gamma \vdash \alpha)$ is of the form $U^L \lor V^L$ for some CHR states $U, V$. For example,

$$ \eta(\exists y. c_u(x, y), x \not\equiv 1 \vdash \exists z. c_u(1, z)) = \exists y. c_u(x, y) \land \forall x \equiv 1 \vdash \exists z. c_u(1, z) = (c_u(x, y) \land x \equiv 1; \{x\})^L \lor (c_u(1, z); T; \emptyset)^L $$

We show by induction over the depth of $\pi$ that for every such $U^L \lor V^L$, we have $D_n(U, V)$, where $n$ is the number of $\Sigma_\pi$-axioms in the proof of $U^L \lor V^L$.

**Base case:** In case the proof of $S^L \lor T^L$ consists of a single leaf, it is either an instance of a (Identity), (R1), or (L0), or a proper axiom $(\Gamma \vdash \alpha) \in (\Sigma_\pi \cup \Sigma_{CT} \cup \Sigma_\pi)$.

--- (Identity), (R1), (L0):

$$ \frac{}{\alpha \vdash \alpha} \quad (\text{Identity}) \quad \frac{1 \vdash (R1)}{\emptyset \vdash \alpha} \quad (L0) $$

In the case of (Identity), we have $\eta(\alpha \vdash \alpha) = U^L \vdash U^L$ for some CHR state $U^L$. In the case of (R1), we have $\eta(\vdash 1) = U^L \vdash U^L$ for $U^L = (\top; T; \emptyset)$. As the entailment relation is reflexive, we have $D_n(U, U)$. In the case of (L0), we have $\eta(0 \vdash \alpha) = U^L \lor V^L$ where $U \equiv S_\pi$. By Def. 2.5.5 and Def. 4.9.1, we have that $U^L \vdash V^L$ and therefore $D_n(U, V)$.

--- For a proper axiom $(\Gamma \vdash \alpha) \in (\Sigma_\pi \cup \Sigma_{CT})$ we have $\Gamma \vdash \alpha = U^L \lor V^L$ where $U, V$ are CHR states such that $U \triangleright V$ and therefore $D_n(U, V)$.
—For a proper axiom \((\Gamma \vdash \alpha) \in \Sigma\) we have \(\Gamma \vdash \alpha = U^L \vdash V^L\) where \(U, V\) are CHR states such that \(U \leftrightarrow V\) and therefore \(D_1(U, V)\).

**Induction step:** We distinguish nine cases according to which is the last inference rule applied in the proof. Cut reduction implies that it must be one of (Cut), \((L\bar{\otimes})\), \((R\otimes)\), (Weakening), (Dereliction), \((\text{Contraction})\), \((R!)\), \((L\bar{\otimes})\), and \((R\otimes)\).

\((L\bar{\otimes}), (\text{Dereliction}), (R!)\): For (Dereliction) and \((R!)\), the banged formula must be an atomic built-in constraint \(c(b)\):

\[
\begin{align*}
\Gamma, \alpha, \beta &\vdash \gamma \quad (L \otimes) \\
\Gamma, \alpha \otimes \beta &\vdash \gamma \\
\Gamma, c(b) &\vdash \beta \quad \text{(Dereliction)} \\
\Gamma &\vdash c(b) \quad (R!)
\end{align*}
\]

Since \(\eta(\alpha, \beta) = \eta(\alpha \otimes \beta)\) and \(\eta(c(b)) = \eta(c(b))\), each of these rule is invariant to the \(\eta\)-completion of the sequent, thus trivially satisfying the hypothesis.

\((L1)\):

\[
\Gamma \vdash \alpha \quad (L1)
\]

We assume that \(S_\Gamma = \langle C_\bar{\Gamma}, B_\bar{\Gamma}, V_\bar{\Gamma} \rangle\) and \(S_\beta\) are CHR states such that \(S_\Gamma^L = \eta(\Gamma), S_\beta^L = \eta(\beta),\) and \(D_n(S_\Gamma, S_\beta)\). Then by Def. 2.5.3, we have \(D_n(S_\Gamma, S_\beta)\) where \(S_\Gamma^L = \langle U_\Gamma, B_\Gamma \land^ L, V_\Gamma \rangle\). As \(S_\Gamma^L = \eta(\Gamma, 1)\), this proves the hypothesis.

\((\text{Weakening})\): By Lemma 4.17, we have that the introduced formula is of the form \(!c(b)(\bar{\delta})\).

\[
\Gamma \vdash \beta \quad (\text{Weakening})
\]

We assume that \(S_\Gamma = \langle U_\Gamma, B_\Gamma, V_\Gamma \rangle\) and \(S_\beta\) are CHR states such that \(S_\Gamma^L = \eta(\Gamma), S_\beta^L = \eta(\beta)\) and \(D_n(S_\Gamma, S_\beta)\). Furthermore, let \(U = \langle U, B \land c(b)(\bar{\delta}); V \rangle\). Since \(U^L = \eta(\Gamma, !c(b)(\bar{\delta}))\) and \(U \triangleright S_\Gamma, \text{Lemma 4.12} \text{ proves the hypothesis.}\)

\((\text{Contraction})\): By the subformula property, we have that the contracted formula is of the form \(!c(b)(\bar{\delta})\).

\[
\Gamma, !c(b)(\bar{\delta}), !c(b)(\bar{\delta}) &\vdash \beta \quad (\text{Contraction})
\]

Since \(\langle U; B \land c(b)(\bar{\delta}); V \rangle \triangleright \langle U; B \land c(b)(\bar{\delta}) \land c(b)(\bar{\delta}); V \rangle\) we prove the hypothesis analogously to (Weakening).

\((R\otimes): The subformula property implies that the joined formulas must be CHR states \(U^L\) and \(V^L\) without local variables:

\[
\Gamma \vdash U^L \quad \Delta \vdash V^L \quad (R\otimes)
\]

Let \(S_\Gamma, S_\beta\) be CHR states such that \(S_\Gamma^L = \eta(\Gamma), S_\beta^L = \eta(\Delta)\). The induction hypothesis gives us \(D_n(S_\Gamma, U)\) and \(D_m(S_\beta, V)\) for some \(n, m\). By Lemma 4.16.1 and Lemma 4.16.2 we have \(D_n(S_\Gamma \bowtie S_\beta, U \bowtie V)\) and \(D_m(U \bowtie S_\Delta, U \bowtie V)\). By Lemma 4.12, we get \(D_{n+m}(S_\Gamma \bowtie S_\Delta, U \bowtie V)\).

\((\text{Cut}): Since \pi is a cut-reduced proof and all axioms are of the form \(U_1^L \vdash U_2^L\), the eliminated formula must be the logical reading of a CHR state U:

\[
\Gamma \vdash U^L \quad U_1^L, \Delta \vdash \beta \quad (\text{Cut})
\]

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Let $S_1, S_2, S_3$ be CHR states such that $S_1^L = \eta(\Gamma)$, $S_2^L = \eta(\Delta)$, and $S_3^L = \eta(\beta)$. The induction hypothesis gives us $D_\eta(S_1, U)$ and $D_m(U \circ S_2, S_3)$. Applying Lemma 4.16, we get $D_m(S_1 \circ S_2, U \circ S_3)$. By Lemma 4.12, we get $D_m(S_1 \circ S_2, U \circ S_3)$ which proves the hypothesis.

---(L∃): In the preconditional sequent, the quantified variable $x$ is by definition replaced by a fresh constant $a$ that does not occur in $\Gamma, \alpha, \beta$:

$$\frac{\Gamma, \alpha [x/a] \vdash \beta}{\Gamma, \exists x. \alpha \vdash \beta} (L∃)$$

Let $U = \langle U [x/a] ; \exists [x/a] ; \forall \cup \{a\} \rangle$ and $S_\beta$ be CHR states such that $U^L = \eta(\Gamma, \alpha [x/a])$, $S_\beta^L = \eta(\beta)$, and $x \notin \forall$. The definition of state equivalence gives us $U \equiv \langle \forall; \beta \wedge x = a; \forall \cup \{a\} \rangle$. Furthermore, we have $\eta(\Gamma, \exists x. \alpha) = (\exists [x/a] ; \forall)^L$. By the induction hypothesis, we have states $U_1, \ldots, U_n$ such that $U \mapsto_{\forall} U_1 \mapsto_{\forall} \ldots \mapsto_{\forall} U_n \triangleright S_\beta$ where $U_i = \langle \forall; \beta \wedge x = a; \forall \cup \{a\} \rangle$ for $i \in [1, \ldots, n]$. Neither the binding $x$ is a nor the set of global variables affect rule applicability. Hence, we can construct an analogous derivation $\langle \exists [x/a] ; \forall \rangle \mapsto_{\forall} U'_1 \mapsto_{\forall} \ldots \mapsto_{\forall} U'_n$ where $U'_i = \langle \exists [x/a] ; \forall \rangle$ for $i \in [1, \ldots, n]$. Since $U_n \triangleright S_\beta$ and $a$ must not occur in $\beta$, we also have $\langle \exists [x/a] ; \forall \rangle \triangleright S_\beta$. Therefore, we have $D_m((\exists [x/a] ; \forall), S_\beta)$. As $\eta(\Gamma, \exists x. \alpha) = (\exists [x/a] ; \forall)^L$, this proves the hypothesis.

---(R∃): By definition, the quantified variable $x$ substitutes an arbitrary term $t$.

$$\frac{\Gamma \vdash \beta[x/t]}{\Gamma \vdash \exists x. \beta} (R∃)$$

Let $S_1, U, V$ be CHR states such that $S_1^L = \eta(\Gamma)$, $U^L = \eta(\exists x. \beta)$, and $V^L = \eta(\forall x. \beta)$. By the induction hypothesis we have $D_\eta(S_1, U)$ for some $n$. Let $V = \langle \exists [x/t] ; \forall \rangle$ and $U = \langle \exists [x/t] ; \forall \cup \{x\} \rangle$. We have $U \equiv \langle \exists [x/t] ; \forall \rangle \triangleright \langle \exists [x/t] ; \forall \cup \{x\} \rangle \equiv V$, and therefore, $D_\eta(S_1, V)$.

Finally, we have $D_N(S, T)$ for some $N$, i.e. there exist states $S_1, \ldots, S_N$ such that:

$$S \equiv S_1 \mapsto \ldots \mapsto S_N \triangleright T$$

It follows that for $T' = S_N$, we have $S \mapsto^* \hat{T}$ and $\hat{T} \triangleright T$. □

Lemma 4.19 states that when excluding the proper axioms in $\Sigma$, logical judgement implies state entailment:

**Lemma 4.19** ($\mapsto \vdash$). For arbitrary CHR states $S, T$, entailment $S \triangleright T$ holds if and only if the judgement $S^L \vdash T^L$ is provable for $\Sigma = \Sigma_{CT} \cup \Sigma_\exists$.

**Proof sketch.** $\Rightarrow$: We apply Thm. 4.18 to the empty program $\overline{P} = \emptyset$.

$\Rightarrow'$: We prove that all conditions in Def. 4.9 comply with the judgement relation $\mapsto$: For Def. 4.9(i), $CT \models \forall (\exists \rightarrow \exists')$ implies that $\Sigma_{CT}$ contains an axiom $\exists \rightarrow \exists'$. Hence, we can prove $\exists \rightarrow \exists \cup \beta \rightarrow \exists \rightarrow \exists \cup \beta'$. For Def. 4.9(ii), it is valid since $S_L \vdash \exists x.S^L$ holds for any $S^L$. Concerning the implicit conditions of a partial order relation, reflexivity and anti-symmetry hold for the judgement relation $\vdash$ as well and anti-symmetry is a natural consequence of Def. 3.4. □

Theorem 4.20 defines the relationship between state equivalence and the linear-logic semantics. It is a direct consequence of Corollary 4.11 and Lemma 4.19 and therefore goes without proof:

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Theorem 4.20 (\(\equiv \Leftrightarrow \vdash\)). Let \(CT\) be a constraint theory and \(\Sigma = \Sigma_{CT} \cup \Sigma_{-}\). For arbitrary CHR states \(S, T\), we have:

\[ S \equiv T \iff S^L \vdash^\Sigma T^L \]

The following example illustrates the completeness theorem:

Example 4.21. We consider the partial-order program \(P\) given in Example 2.17 and a minimal constraint theory \(CT\). For \(\Sigma = \Sigma_P \cup \Sigma_{CT} \cup \Sigma_{-}\), we have

\[ a \leq b \otimes b \leq c \otimes c \leq a \vdash^\Sigma \lnot a \equiv b \]

which equals:

\[ \langle a \leq b \land b \leq c \land c \leq a; \top; \langle a, b, c \rangle \rangle^L_{\Sigma} \vdash^\Sigma \langle \top; a \equiv b; \langle a, b \rangle \rangle^L_{\Sigma} \]

This corresponds to:

\[ \langle a \leq b \land b \leq c \land c \leq a; \top; \langle a, b, c \rangle \rangle \vdash^* \langle \top; a \equiv b \land a \equiv c; \langle a, b, c \rangle \rangle \Rightarrow \langle \top; a \equiv b; \langle a, b \rangle \rangle \]

4.6 Encoding Programs and Constraint Theories

In the axiomatic linear-logic semantics presented in Sect. 4.2 to Sect. 4.5, only states are represented in logical judgements. Both programs and constraint theories disappear into the proper axioms of a sequent calculus system and hence are not objects of logical reasoning.

In this section, we show how to encode programs and constraint theories into logical judgements, enabling us to reason directly about them as well. In Sect. 6.2, we will use this encoding to decide operational equivalence of programs. As a further benefit, a complete encoding of programs and constraint theories assures the existence of cut-free proofs for the respective judgements and ensure compatibility with established methods for automated proof search methods relying on this property.

As usual, \((\cdot)^L\) stands for translation into linear logic.

Encoding of Constraint Theories. The constraint theory \(CT\) itself is encoded according to the translation quoted in Def. 3.3. Furthermore, for every \(n\)-ary user-defined constraint symbol \(c_u\) and every \(j \in \{1, \ldots, n\}\), we add the following formula to the translation of the theory, where \(x_1, \ldots, x_n\) and \(y\) are variables:

\[ !\forall(c_u(x_1, \ldots, x_j, \ldots, x_n) \otimes (!x_j \equiv y) \dashv c_u(x_1, \ldots, y, \ldots, x_n) \otimes (!x_j \equiv y) \]

We obtain the following encoding of constraint theories:

Definition 4.22 (\(CT^L\)). Let \(CT\) be a constraint theory. Its linear-logic reading \(CT^L\) is given as:

\[ CT^L := CT^* \cup \bigcup_{c_u/n} \forall(c_u(\ldots, x_j, \ldots) \otimes !x_j = y) \dashv c_u(\ldots, y, \ldots) \otimes !x_j = y) \]

Encoding of \(\Sigma_P\). The translation of CHR rules follows the same lines as the encoding of the \(CT\) axioms:

Definition 4.23 (\(R^L, \Sigma^L\)). (1) Let \(R = r \otimes H_1 \setminus H_2 \dashv G \mid B_a \wedge B_b\) be a CHR rule with local variables \(\bar{y}_r\). Then its linear-logic reading \(R^L\) is defined as:

\[ R^L := \forall(H_1^L \otimes H_2^L \otimes G^L) \dashv H_1^L \otimes \exists\bar{y}_r(B_a^L \otimes B_b^L \otimes G^L) \]
(2) Let $P = \{R_1, \ldots, R_n\}$ be a CHR program. Then its linear-logic reading $P^L$ is defined as:

$$P^L := \bigcup_{R \in P} R^L$$

For the encoding semantics, the following soundness and completeness theorem holds:

**Theorem 4.24 Soundness and Completeness.** Let $S, T$ be CHR states. There exists a state $U$ such that

$$S \leftrightarrow^* U \text{ and } U \vdash T$$

in a program $P$ and a constraint theory $CT$ if and only if

$$P^L, CT^L \vdash \forall(S^L \rightarrow T^L)$$

**Proof.** We prove Thm. 4.24 by showing that any proof tree in the axiomatic semantics can be transformed into a proof tree in the encoding semantics and vice versa. To ensure of clarity, we will omit the set of proper axioms from the judgement symbol.

**Axiomatic to encoding:** We assume a proof $\pi$ of a sequent $S^L \vdash T^L$ in the axiomatic semantics. We replace every axiom $\exists X, \|B^L \vdash \exists Y, B'^L$ in $\Sigma_{CT}$ by a sub-tree proving $CT \vdash \|B^L \vdash \exists Y, B'^L$. The same is done for every equality axiom in $\Sigma$. Similarly, every axiom $H^L_1 \otimes H^L_2 \otimes G^L + H^L_1 \otimes \exists \cdot P^L, (B^L_1 \otimes B'^L_1 \otimes G^L)$ in $\Sigma$ is replaced with a sub-tree proving $P^L, H^L_1 \otimes H^L_2 \otimes G^L + H^L_1 \otimes \exists \cdot P^L, (B^L_1 \otimes B'^L_1 \otimes G^L)$. We propagate the thus introduced instances of $CT^L$ and $P^L$ throughout the proof tree, thus producing a proof $\pi'$ of $CT^L, \ldots, CT^L, P^L, \ldots, P^L, S^L \vdash T^L$.

We insert $\pi'$ into:

$$\frac{\pi'}{CT^L, P^L, S^L \vdash T^L} \quad \text{market contraction}\)

$$\frac{CT^L, P^L \vdash S^L \rightarrow T^L}{CT^L, P^L \vdash \forall(S^L \rightarrow T^L)} \quad \text{market weakening} $$

**Encoding to axiomatic:** Let $\otimes$ stand for element-wise multiplicative conjunction of a set and let $\pi$ be a proof of a sequent $CT^L, P^L \vdash \forall(S^L \rightarrow T^L)$ in the encoding semantics.

For every $\forall(X, B^L \rightarrow \exists Y, B'^L) \in CT^L$, we have $\forall(X, B^L \rightarrow \exists Y, B'^L)$ where $\Sigma = \Sigma_{CT} \cup \Sigma$. Hence, there exists a proof $\pi_{CT}$ of $\forall(X, B^L \rightarrow \exists Y, B'^L)$. Similarly, there exists a proof $\pi_{P}$ of $\forall(Z, \otimes P^L)$.

$$\frac{\pi_{CT}}{\frac{\frac{\frac{\pi_{P}}{CT^L, \otimes P^L \vdash \forall(S^L \rightarrow T^L)}}{CT^L, \otimes P^L \vdash \forall(S^L \rightarrow T^L)}}{CT^L \vdash \forall(S^L \rightarrow T^L)}}{\forall(S^L \rightarrow T^L)} \quad \text{mark weakening}$$

As we can transform the respective proof tree from the axiomatic to the encoding semantics and vice versa, the two representations are equivalent. $\square$

5. A LINEAR-LOGIC SEMANTICS FOR CHR$^\triangledown$

In this section, we extend our linear logic semantics to CHR with Disjunction (CHR$^\triangledown$), a common extension of CHR. To avoid ambiguity, we will henceforth use the term pure
CHR to refer to the regular segment of CHR without disjunction.

We will firstly recall the syntax and semantics of CHR$^\vee$ in Sect. 5.1. Then we define an equivalence-based formalization of its operational semantics in Sect. 5.2, analogous to $\omega_e$ for pure CHR. In Sect. 5.3, we apply this equivalence-based formalization to define a linear-logic semantics for CHR$^\vee$ and proof its soundness and completeness. In Sect. 5.4, we show that in the case of CHR$^\vee$, the linear-logic semantics has less desirable properties than for pure CHR: Concretely, linear-logic based reasoning over CHR$^\vee$ programs produces in general less precise results than over CHR programs. We then introduce the well-behavedness properties of compactness and analyticness which amend this limitation.

5.1 Introduction to CHR$^\vee$

CHR$^\vee$ has a richer syntax than pure CHR: The definition of goals is extended by the disjunction operator $\vee$. Alluding to its operational meaning, we may also refer to $\vee$ as the split operator. We also introduce the notion of configuration, which can be read as a disjunction of CHR states, and we extend the definition of goal equivalence to account for distributivity.

**Definition 5.1 Goals, States, Configurations.** We adapt the definitions of goal and state, and we define configuration as follows:

- Built-in constraint: $B ::= \top | c_a(\bar{t}) | B \land B'$
- User-defined constraint: $U ::= \top | c_a(\bar{t}) | U \land U'$
- CHR$^\vee$ goal: $G ::= \top | c_u(\bar{t}) | c_b(\bar{t}) | G \land G' | G \lor G'$
- CHR$^\vee$ state: $S ::= \langle G; V \rangle$
- Configuration: $\bar{S} ::= \varepsilon | S | S \lor \bar{S}$

For any two goals $G, G'$, goal equivalence $G \equiv G'$ denotes equivalence between goals with respect to associativity and commutativity of $\land$, the neutrality of $\top$ with respect to $\land$, and the distributivity of $\land$ over $\lor$. $\varepsilon$ stands for the empty configuration, which is operationally equivalent to a failed state $S\bot$.

A goal which does not contain disjunctions is called flat. A state $\langle G; V \rangle$ where $G$ is flat is also called flat. A configuration $\bar{S}$ is called flat if it is empty or consists only of flat states.

Allowing $\land$ to distribute over $\lor$ guarantees that every goal is equivalent to its disjunctive normal form (DNF). We do not allow the opposite law of distributivity. For example, we have $G_1 \land (G_2 \lor G_3) \equiv_G (G_1 \land G_2) \lor (G_1 \land G_3)$ but $G_1 \lor (G_2 \land G_3) \not\equiv_G (G_1 \lor G_2) \land (G_1 \lor G_3)$. Thus any finite goal has only a finite number of equivalent representations.

In CHR$^\vee$, we use the same definition for state equivalence as in pure CHR. However, as the definition of goal equivalence is extended, this implicitly carries over to state equivalence. For example: $\langle G_1 \land (G_2 \lor G_3); V \rangle \equiv \langle (G_1 \land G_2) \lor (G_1 \land G_3); V \rangle$.

As in goals, CHR$^\vee$ allows disjunctions in rule bodies. The clear separation between user-defined constraints and built-in constraints in the rule body no longer applies. This is reflected in the following definition:

**Definition 5.2 CHR$^\vee$ Rules.** A CHR$^\vee$ rule is of the form

$$r @ H_1 \setminus H_2 \Leftrightarrow G | B$$

The kept head $H_1$ and the removed head $H_2$ are user-defined constraints. The guard $G$ is a built-in constraint. The rule body $B$ is a CHR$^\vee$ goal. $r$ serves as an identifier for the rule and may be omitted along with the $@$. An empty guard may be omitted along with the $|$.
We observe that restricting CHR\(^\lor\) to the segment without disjunction restores pure CHR. Hence, pure CHR is a subset of CHR\(^\lor\). The operational semantics of CHR\(^\lor\) has originally been defined in [Abdennadher and Schütz 1998]. An additional transition rule called Split resolves disjunctions by branching the computation. Adjusted to our syntax, we express that transition rule as follows:

\[
\text{Split: } \langle G_1 \lor G_2; V \rangle \mapsto \text{sp} \langle G_1; V \rangle \lor \langle G_2; V \rangle
\]

We can straightforwardly adapt the operational semantics \(\omega_e\) to the syntax of CHR\(^\lor\).

Adding one rule to handle equivalence transformations of states and two more rules to handle composition of configurations gives us the following operational semantics for CHR\(^\lor\):

**Definition 5.3 Operational Semantics of CHR\(^\lor\).** CHR\(^\lor\) is a state transition system over configurations defined by the following transition rules, where \((r @ H_1 \setminus H_2 \Leftrightarrow G \mid B)\) is a variant of a CHR\(^\lor\) rule whose local variables \(\bar{y}_r\) are renamed apart from any variable occurring in \(\text{vars}(H_1, H_2, G, V)\):

**Apply:**

\[
\frac{\langle r \; @ \; H_1 \setminus H_2 \Leftrightarrow G \mid B \rangle \quad \text{CT} \models \exists (G \land B)}{\langle H_1 \land H_2 \land G \land \mathcal{G}; V \rangle \leftarrow^r \langle H_1 \land G \land B \land \mathcal{G}; V \rangle}
\]

**Split:**

\[
\frac{\langle G_1 \lor G_2; V \rangle \mapsto^p \langle G_1; V \rangle \lor \langle G_2; V \rangle}{\text{StateEquiv: } S' \equiv S} \quad \frac{S \mapsto^r T \quad T \equiv T'}{S' \mapsto^r T'}
\]

**CompLeft:**

\[
\frac{S \mapsto^r T \quad S \lor T}{S' \mapsto^r T'}
\]

**CompRight:**

\[
\frac{T \lor S \mapsto^r T' \quad T' \lor S'}{S \mapsto^r T'}
\]

If the applied rule is obvious from the context or irrelevant, we write transition simply as \(\mapsto\). We denote its reflexive-transitive closure as \(\mapsto^*\).

The following example shows a possible computation in CHR\(^\lor\):

**Example 5.4.** Consider the following CHR\(^\lor\) program:

\[
r_1 @ \text{bird} \Leftrightarrow \text{albatross} \lor \text{penguin}
\]

\[
r_2 @ \text{penguin} \land \text{flies} \Leftrightarrow \perp
\]

Running this program with the initial state \(\langle \text{bird} \land \text{flies}; \emptyset \rangle\) produces the following fixed-point computation:

\[
\begin{align*}
\langle \text{bird} \land \text{flies}; \emptyset \rangle & \mapsto^r \langle \langle \text{albatross} \land \text{flies}; \emptyset \rangle \lor \langle \text{penguin} \land \text{flies}; \emptyset \rangle \rangle \\
& \mapsto^p \langle \langle \text{albatross} \land \text{flies}; \emptyset \rangle \lor \langle \text{penguin} \land \text{flies}; \emptyset \rangle \rangle \\
& \mapsto^r \langle \langle \text{albatross} \land \text{flies}; \emptyset \rangle \lor \langle \perp; \emptyset \rangle \rangle
\end{align*}
\]

The first transition step is justified by the **Apply** as well as the **StateEquiv** transition rule. The last transition step is justified by **Apply** and **CompLeft**.

**5.2 An Equivalence-Based Operational Semantics for CHR\(^\lor\)**

While the operational semantics presented in Sect. 5.1 precisely formalizes the execution of a CHR\(^\lor\) program, it is of limited use for program analysis. For example, we would
intuitively assume that two configurations should be considered equivalent if they differ only in the order of their member states.

In this section, we propose a notion of equivalence of configurations, we show its compliance with rule application and we propose a formalization of the operational semantics based on equivalence classes of configurations.

Definition 5.5 Equivalence of Configurations. Equivalence of configurations, denoted as $\cdot \equiv \cdot$, is the smallest equivalence relation over configurations satisfying all of the following properties:

1. **Associativity and Commutativity:**
   \[ S \lor T \equiv T \lor S \quad \text{and} \quad (S \lor T) \lor U \equiv S \lor (T \lor U) \]

2. **State Equivalence**
   \[ S \equiv S' \implies S \lor T \equiv S' \lor T \]

3. **Neutrality of Failed States:**
   \[ S_\bot \lor T \equiv T \]

4. **Split:**
   \[ [(G_1 \lor G_2; V)] \lor T \equiv [(G_1; V)] \lor [(G_2; V)] \lor T \]

Compliance of configuration equivalence with rule application is formalized as follows:

**Property 5.6 Compliance with Rule Application.** Let $S, S', T$ be arbitrary configurations such that $S \equiv S'$ and $S \to^* T$. Then there exists a $T'$ such that $T \equiv T'$ and $S' \to^* T'$.

**Proof sketch.** Element states of a configuration are handled independently of each other, making associativity and commutativity idempotent to rule application. Equivalence transformation of states complies due to the StateEquiv rule. Failed states do not allow rule application. Any application of the Split axiom hindering rule application can be reversed by application of the Split transition.

The compliance property allows us to define an operational semantics based on equivalence classes of configurations using only a single transition rule. In analogy to the equivalence-based semantics $\omega_e$ for pure CHR, we will refer to this operational semantics as $\omega_\lor$.

Definition 5.7 Transition System of $\omega_\lor$. CHR is a state transition system over equivalence classes of configurations. It is defined by the following transition rule, where $(r @ H_1 \setminus H_2 \Leftarrow G | B)$ is a variant of a CHR rule whose local variables $\bar{y}_r$ are renamed apart from any variable occurring in $\text{vars}(H_1, H_2, G, V)$:

\[
\frac{r @ H_1 \setminus H_2 \Leftarrow G | B \quad CT \models \exists(G \land \exists)}{[(H_1 \land H_2 \land G \land \exists; V) \lor T] \to^* [(H_1 \land G \land B \land \exists; V) \lor T]}$

If the applied rule is obvious from the context or irrelevant, we write transition simply as $\to^*$. We denote its reflexive-transitive closure as $\to^*$. 

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Analogously to pure CHR, we define a notion of confluence:

**Definition 5.8 Confluence.** A CHR\(^\gamma\) program \(\mathcal{P}\) is called confluent, if for arbitrary configurations \(S, T, U\) such that \([S] \rightarrow^* [T]\) and \([S] \rightarrow^* [U]\), there exists a configuration \(V\) such that \([T] \rightarrow^* [V]\) and \([U] \rightarrow^* [V]\).

Furthermore, we define three sets of observables based on equivalence classes of configurations:

**Definition 5.9 Observables.** Let \(S\) be a CHR state, \(\mathcal{P}\) be a program, and \(CT\) be a constraint theory. We distinguish the following sets of observables:

- **Computable config.:** \(\bar{C}_{\mathcal{P}, CT}(S) := \{[T] \mid [S] \rightarrow^* [T]\}\)
- **Answer:** \(\bar{A}_{\mathcal{P}, CT}(S) := \{[T] \mid [S] \rightarrow^* [T] \neq \perp\}\)
- **Data-sufficient answer:** \(\bar{S}_{\mathcal{P}, CT}(S) := \{\left\{\langle T; B; \forall_1 \rangle \lor \ldots \lor \langle T; B_n; \forall_n \rangle \right\} \mid [S] \rightarrow^* \left\{\langle T; B_1; \forall_1 \rangle \lor \ldots \lor \langle T; B_n; \forall_n \rangle \right\}\}

Note that the parameters for all three sets are states rather than configurations, as we assume that every computation starts from a singular state. For all three sets, if the constraint theory \(CT\) is clear from the context or not important, we may omit it from the respective identifier.

Analogously to Property 6.13, we have a hierarchy of observables:

**Property 5.10 Hierarchy of Observables.** For any state \(S\), program \(\mathcal{P}\) and constraint theory \(CT\), we have:

\[
\bar{S}_{\mathcal{P}, CT}(S) \subseteq \bar{A}_{\mathcal{P}, CT}(S) \subseteq \bar{C}_{\mathcal{P}, CT}(S)
\]

The following example illustrates our definitions:

**Example 5.11.** We recur to the program from Example 5.4.

\[
\text{bird} \; \Leftrightarrow \; \text{albatross} \lor \text{penguin} \\
\text{penguin} \land \text{flies} \; \Rightarrow \; \bot
\]

Using \(\omega^\gamma\), we can construct the following derivation starting from the initial state \(S_0 = \langle \text{bird} \land \text{flies}; \emptyset \rangle\):

\[
\begin{array}{l}
\langle \text{bird} \land \text{flies}; \emptyset \rangle \\
\rightarrow_v \langle (\text{albatross} \lor \text{penguin}) \land \text{flies}; \emptyset \rangle \\
= \langle \text{albatross} \land \text{flies}; \emptyset \rangle \lor \langle \text{penguin} \land \text{flies}; \emptyset \rangle \\
\rightarrow_v \langle (\text{albatross} \land \text{flies}) \lor (\bot; \emptyset) \rangle \\
= \langle (\text{albatross} \land \text{flies}) \rangle
\end{array}
\]

In comparison with Example 5.4, we now obtain our result with one less transition. More importantly, our transition system consists of only one transition rule now. The equivalence relation over configurations allows us to omit the failed state from the final configuration, producing a more elegant representation of the answer.

With respect to the observables, we have \(\bar{C}_{\mathcal{P}}(S_0) = \{[S_0], \{(\text{albatross} \lor \text{penguin}) \land \text{flies}; \emptyset\}, \{(\text{albatross} \land \text{flies})\}\}, \bar{A}_{\mathcal{P}}(S_0) = \{\{(\text{albatross} \land \text{flies})\}\}, \text{and } \bar{S}_{\mathcal{P}}(S_0) = \emptyset.

### 5.3 Extending the Linear-Logic Semantics to CHR\(^\gamma\)

In this section, we develop a linear-logic semantics for CHR\(^\gamma\), based on the equivalence-based operational semantics \(\omega^\gamma\).
This is logically equivalent to:

\[ \neg \exists \bar{x}, \exists \bar{y}, \neg \exists \bar{z}, \exists \bar{w} \]

We gain the insight that don’t-care non-determinism in CHR is already implicitly mapped to additive conjunction & in linear logic.

Mapping the split connective ∨ to multiplicative disjunction ⊗ is an obvious choice, as: (1) ⊗ distributes over ⊗, (2) absurdity 0 – representing failed states – is neutral with respect to ⊗, and (3) ⊗ complements &, which represents committed choice. Hence we preserve the clear distinction between the two types of non-determinism. We furthermore adapt the translations of states and programs to the syntax of CHRY, thus obtaining the semantics given in Fig. 4.

5.3.2 Soundness of the Linear Logic Semantics for CHRY. In this section, we prove the soundness of our semantics with respect to \( \omega^N \). At first, we show that configuration equivalence implies logical judgement:

**Lemma 5.12** \( \equiv \Rightarrow \vdash \).

1. For goals \( G_1, G_2 \) such that \( G_1 \equiv_G G_2 \), we have \( G_1 \vdash_G G_2 \).
2. For CHRY states \( S_1, S_2 \) and an arbitrary constraint theory \( CT \) such that such that \( S_1 \equiv S_2 \), we have \( S_1 \vdash C \) and \( S_2 \vdash C \) where \( \Sigma = \Sigma_{CT} \).
(3) For configurations \(\bar{S}_1, \bar{S}_2\) and an arbitrary constraint theory \(CT\) such that such that \(\bar{S}_1 \equiv \bar{S}_2\), we have \(\bar{S}_1 \vdash_\Sigma \bar{S}_2\) where \(\Sigma = \Sigma_{CT}\).

**Proof.** Lemma 5.12.1: The property holds, as \(\otimes\) is associative, commutative, has the neutral element 1 and distributes over \(\oplus\). Lemma 5.12.2: Proof is analogous to Lemma 4.7. Lemma 5.12.3: We consider the properties given in Def. 5.5 – Def. 5.5.1: For all \(\alpha, \beta, \gamma\), we have \(\alpha \oplus \beta \equiv \beta \oplus \alpha\) and \((\alpha \oplus \beta) \oplus \gamma \equiv \alpha \oplus (\beta \oplus \gamma)\). Def. 5.5.2: The property follows from Lemma 5.12.2. Def. 5.5.3: For all \(\alpha\), we have \(0 \oplus \alpha \equiv \alpha\). Def. 5.5.4: For all \(\alpha, \beta, \gamma, \forall\), we have \((\exists\forall, \alpha \oplus \beta) \oplus \gamma \equiv (\exists\forall, \alpha) \oplus (\exists\forall, \beta) \oplus \gamma\). □

Theorem 4.8 states the soundness of the axiomatic linear-logic semantics for CHR\(^y\).

**Theorem 5.13 Soundness.** For any CHR\(^y\) program \(\mathcal{P}\), constraint theory \(CT\) and configurations \(\bar{U}, \bar{V}\),

\[
[M] \mapsto^* [\bar{V}] \implies \bar{U} \vdash_\Sigma \bar{V}
\]

where \(\Sigma = \Sigma_\mathcal{P} \cup \Sigma_{CT}\).

**Proof.** Let \(\bar{U}, \bar{V}\) be configurations such that \(\bar{U} \mapsto^* \bar{V}\). According to Def. 2.12, there exists a variant of a rule with fresh variables \((r \otimes H_1 \setminus H_2 \Rightarrow G \setminus B)\) and configurations \(\bar{U} = (H_1 \setminus H_2 \wedge G \wedge \forall; \forall) \lor \bar{U}', \bar{V} = (B_H \wedge H_1 \wedge B_B \wedge G \wedge \forall; \forall) \lor \bar{V}'\) such that \(\bar{U} \equiv \bar{U}'\) and \(\bar{V} \equiv \bar{V}'\). Consequently, \(\Sigma_{\mathcal{P}}\) contains:

\[
H^l_1 \otimes H^l_2 \otimes G^l \lor \exists\forall, (B^l \otimes G^l)
\]

Analogous to the proof of Thm. 4.8, we proceed to:

\[
\exists\forall, H^l_1 \otimes H^l_2 \otimes G^l \lor \otimes \exists\forall, H^l_1 \otimes G^l \otimes B^l \otimes \otimes
\]

And then to:

\[
(\exists\forall, H^l_1 \otimes H^l_2 \otimes G^l \lor \otimes) \lor \exists\forall, (H^l_1 \otimes G^l \otimes B^l \otimes \otimes) \otimes \tilde{\otimes}
\]

This corresponds to \(\bar{U}^l \vdash_\Sigma \bar{V}^l\). Lemma 4.7 then proves that \(\bar{U}^l \vdash_\Sigma \bar{V}^l\). As the judgement relation \(\vdash_\Sigma\) is transitive and reflexive, the relationship can be generalized to the reflexive-transitive closure \(\bar{U} \mapsto^* \bar{V}\). □

5.3.3 **Configuration Entailment.** Analogously to state entailment for pure CHR, we define a notion of **configuration entailment** to characterize the discrepancy between transitions in a CHR\(^y\) program and judgements in its corresponding sequent calculus system and thus to completeness of the linear-logic semantics:

**Definition 5.14 Entailment of Configurations.** Entailment of configurations, denoted as \(\cdot \models \cdot\), is the smallest reflexive-transitive relation over equivalence classes of configurations satisfying the following conditions:

1. **Weakening:** For any state \(S\) and configuration \(\tilde{T}\):

\[
[S] \models [S \lor \tilde{T}]
\]

2. **Redundance of Stronger States:** For any CHR\(^y\) states \(S_1, S_2, T\) such that \(S_1 \triangleright S_2\):

\[
[S_1 \lor S_2 \lor \tilde{T}] \models [S_2 \lor \tilde{T}]
\]

The following property follows from the definition:
Property 5.15 ($\triangleright \Rightarrow$). For $\text{CHR}^\triangleright$ states $S_1, S_2$ such that $S_1 \triangleright S_2$:

$$[S_1 \lor T] \triangleright [S_2 \lor \bar{T}]$$

Proof. $[S_1 \lor \bar{T}] \triangleright [S_2 \lor S_1 \lor \bar{T}] = [S_1 \lor S_2 \lor \bar{T}] \triangleright [S_2 \lor \bar{T}]$.

Lemma 5.16 corresponds to Lemma 4.12 for the case of pure $\text{CHR}$.

Lemma 5.16 Exchange of $\Rightarrow$ and $\triangleright$. Let $\bar{S}, \bar{U}, \bar{T}$ be configurations. If $\bar{S} \triangleright \bar{U}$ and $\bar{U} \Rightarrow \bar{T}$ then there exists a configuration $\bar{V}$ such that $\bar{S} \Rightarrow^* \bar{V}$ and $\bar{V} \triangleright \bar{T}$.

Proof. Firstly, we consider hypothesis with respect to the axioms of configuration entailment (cf. Def. 5.14):

Def. 5.14.1. Assume that $[\bar{S}] \triangleright [S \lor \bar{S}] \Rightarrow^* [\bar{T}]$. It follows that either (i) $[S] \Rightarrow^r [S']$ and $[\bar{T}] = [S' \lor S]$ or (ii) $[\bar{S}] \Rightarrow^r [S']$ and $[T] = [S \lor \bar{S}]$. In case (i), we have $[V] = [S]$ and $[\bar{S}] \triangleright [S \lor \bar{S}]$.

Def. 5.14.2. Assume that $[S_1 \lor S_2 \lor \bar{S}] \triangleright [S_2 \lor \bar{S}] \Rightarrow^r [\bar{T}]$ where $[S_1] \triangleright [S_2]$. It follows that either (i) $[S_2] \Rightarrow^r [S_2']$ and $[\bar{T}] = [S_2' \lor \bar{S}]$ or (ii) $[\bar{S}] \Rightarrow^r [S_2']$ and $[\bar{T}] = [S_2 \lor \bar{S}]$. In case (i), we have $[V] = [S_2]$ and $[\bar{S}] \triangleright [S \lor \bar{S}]$.

Hence, we get $[\bar{V}] = [S_1' \lor S_2' \lor \bar{S}]$ and $[S_1 \lor S_2 \lor \bar{S}] \Rightarrow^r [S_1' \lor S_2 \lor \bar{S}] \Rightarrow [\bar{V}] = [S_2' \lor \bar{S}]$.

In case (ii), we have $[V] = [S_1 \lor S_2 \lor S' \lor \bar{S}]$ and $[S_1 \lor S_2 \lor \bar{S}] \Rightarrow^r [S_1 \lor S_2 \lor S'] \triangleright [S_2 \lor \bar{S}] = [T]$.

For the reflexive closure of these axioms, the hypothesis is true as $[\bar{S}] = [\bar{U}]$ implies $[\bar{V}] = [\bar{T}]$. For their transitive closure, it follows by induction. Hence, the hypothesis holds for configuration entailment in general.

5.3.4 Completeness of the Linear-Logic Semantics for $\text{CHR}^\triangleright$. Lemma 5.17 sets the stage for the completeness theorem. Its proof is analogous to the proof of Lemma 4.17 and will be omitted here:

Lemma 5.17. Let $\pi$ be a cut-reduced proof of a sequent $\bar{S}^L \vdash \bar{T}^L$ where $\bar{S}, \bar{T}$ are arbitrary configurations. Any formula $\alpha$ in $\pi$ is either of the form $\alpha = S^L_{\alpha}$ or of the form $\alpha = \text{vars}(\alpha)$. It should be noted that the configuration $S^L_{\alpha}$ is a configuration and $c_{\alpha}(i)$ is a built-in constraint.

It should be noted that the configuration $S^L_{\alpha}$ is not necessarily unique, i.e. more than one configuration might map to a specific formula. For example, let formula $\alpha = c_{\alpha}(i)$. We then have $\langle c_{\alpha}(i); \text{vars}(c_{\alpha}(i)) \rangle^L = (\langle c_{\alpha}(i); \text{vars}(c_{\alpha}(i)) \rangle \lor (c_{\alpha}(i); \text{vars}(c_{\alpha}(i))))^L = \alpha$.

However, we have by Def. 5.5.4 that $S^L = T^L \Rightarrow S \triangleright T$.

Theorem 5.18 Completeness of the Semantics for $\text{CHR}^\triangleright$. Let $\bar{S}, \bar{T}$ be configurations, let $\Sigma$ be a program and $\text{CT}$ be a constraint theory. Then the sequent $\bar{S} \triangleright \bar{T}$ is provable in a sequent calculus with proper axioms $\Sigma = \Sigma_{CT} \cup \Sigma_n \cup \Sigma_\Sigma$ if and only if there exists a configuration $\bar{U}$ such that $\bar{S} \Rightarrow^* \bar{U}$ and $\bar{U} \triangleright \bar{T}$.

Proof. To preserve clarity, we will omit the set of proper axioms from the judgement symbol. Furthermore, $\models \bar{U}, \bar{V}$ denotes the fact that for configurations $\bar{U}, \bar{V}$, there exist configurations $\bar{U}_1, \ldots, \bar{U}_n$ for some $n$ such that:

$$\bar{U} \Rightarrow \bar{U}_1 \Rightarrow \cdots \Rightarrow \bar{U}_n \triangleright \bar{V}$$

Entailment $\bar{U} \triangleright \bar{V}$ implies $\models \bar{U}, \bar{V}$. We define $\triangleright$ as in the proof of Thm. 4.18.
Let \( \pi \) be a cut-reduced proof of \( \bar{S}^L \vdash \bar{T}^L \). We assume w.l.o.g. that all existentially quantified variables in the antecedent of a sequent in \( \pi \) are renamed apart. We define \( \eta \) as an extension of the completion function from the proof of Thm. 4.18 to configurations:

\[
\begin{align*}
\eta(\bar{S}^L) &::= \bar{S}^L \\
\eta(c_\beta(\bar{f})) &::= c_\beta(\bar{f}) \\
\eta(\alpha, \Gamma) &::= \eta(\alpha) \otimes \eta(\Gamma) \\
\eta(\Gamma \vdash \alpha) &::= \eta(\Gamma) \vdash \eta(\alpha) \quad \text{for non-empty } \Gamma \\
\eta(\Gamma) &::= 1 \vdash \eta(\alpha)
\end{align*}
\]

From Lemma 5.17 follows that for every sequent \( \Gamma \vdash \alpha \) in \( \pi \), we have \( \eta(\Gamma \vdash \alpha) = U^L \vdash V^L \) for some configurations \( U, V \). We show by induction over the depth of \( \pi \) that for every such \( U^L \vdash V^L \), we have \( D(U, V) \).

**Base case:** In case the proof of \( \bar{S}^L \vdash \bar{T}^L \) consists in a leaf, it is an instance of (Identity), (L0), (R1), or a proper axiom \((\Gamma \vdash \alpha) \in (\Sigma_a \cup \Sigma_{CT} \cup \Sigma_\pi)\). We apply the same arguments as in the proof of Thm. 4.18.

**Induction step:** As \( \pi \) is cut-reduced, the final inference rule either has to be one of (Cut), (L\( \otimes \)), (R\( \otimes \)), (L1), (Weakening), (Dereliction), (Contraction), (R!), (L\( \exists \)) and (R\( \exists \)), or one of (L\( \exists \)), (R\( \exists \)) or (R\( \exists \)). In the former case, we can follow the same arguments as in the proof of Thm. 4.18. In the following, we consider (L\( \otimes \)) and (R\( \otimes \)).

\[ -(L\otimes): \]

\[ \frac{\Gamma, \alpha \vdash \gamma \quad \Gamma, \beta \vdash \gamma}{\Gamma, \alpha \oplus \beta \vdash \gamma} \quad (L\otimes) \]

Let \( G_a, G_b \) be goals, let \( S_T = \langle G; \forall \rangle \) be a state and let \( \bar{S}_\beta \) be a configuration such that \( G_a^L = \eta(\alpha), G_b^L = \eta(\beta), S_T^L = \eta(\Gamma) \) and \( \eta(\gamma) \). Let furthermore \( \bar{y}_a = \text{vars}(G_a) \) and \( \bar{y}_b = \text{vars}(G_b) \). Hence, \( \eta(\Gamma, \alpha) = \langle G \land G_a; \forall \cup \bar{y}_a \rangle^L, \gamma, \eta(\Gamma, \beta) = \langle G \land G_b; \forall \cup \bar{y}_b \rangle^L, \) and \( \eta(\Gamma, \alpha \oplus \beta) = \langle G \land (G_a \lor G_b); \forall \cup \bar{y}_a \cup \bar{y}_b \rangle^L \). The induction hypothesis gives us \( D(\langle G \land G_a; \forall \cup \bar{y}_a \rangle^L, \bar{S}_\gamma) \) and \( D(\langle G \land G_b; \forall \cup \bar{y}_b \rangle^L, \bar{S}_\gamma) \). By Def. 5.5.4 we have that \( \eta(\Gamma, \alpha \oplus \beta) \equiv \langle G \land G_a; \forall \cup \bar{y}_a \rangle \lor \langle G \land G_b; \forall \cup \bar{y}_b \rangle \). Finally by Lemma 5.16, we get \( D(\langle G \land (G_a \lor G_b); \forall \cup \bar{y}_a \cup \bar{y}_b \rangle^L, \bar{S}_\gamma) \).

\[ -(R\otimes_1), (R\otimes_2): \]

\[ \frac{\Gamma \vdash \alpha}{\Gamma \vdash \alpha \oplus \beta} \quad (R\otimes_1) \quad \frac{\Gamma \vdash \beta}{\Gamma \vdash \alpha \oplus \beta} \quad (R\otimes_2) \]

We consider (R\( \otimes_1 \)): By the subformula property, there exist configurations \( \bar{S}_T, \bar{S}_a, \bar{S}_b \) such that \( \bar{S}_T^L = \eta(\Gamma), \bar{S}_a^L = \eta(\alpha), \) and \( \bar{S}_b^L = \eta(\beta) \). By the induction hypothesis, we have \( D(\bar{S}_T, \bar{S}_a) \). By Def. 5.14.1, we have \( \bar{S}_a \triangleright (\bar{S}_a \lor \bar{S}_b) \) and therefore \( D_a(\bar{S}_T, \bar{S}_a \lor \bar{S}_b) \). (The proof for (R\( \otimes_2 \)) works analogously.)

Finally, we have \( D(\bar{S}, T) \), i.e. there exist configurations \( \bar{S}_1, \ldots, \bar{S}_n \) such that:

\[ \bar{S} \leftrightarrow \bar{S}_1 \ldots \leftrightarrow \bar{S}_n \triangleright T \]

It follows that for \( \bar{U} = S_n \), we have \( \bar{S} \leftrightarrow^* \bar{U} \) and \( \bar{U} \triangleright T \). \( \square \)

**Lemma 5.19 (\( \triangleright \leftrightarrow \)).** For configurations \( \bar{S}, T \), we have \( [\bar{S}] \triangleright [T] \) if and only if \( \bar{S}^L \vdash_T T^L \) where \( \Sigma = \Sigma_{CT} \cup \Sigma_\pi \).

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Fig. 5. The linear-logic encoding semantics for CHR\(\lor\).

\[\text{Rules: } (H_1 \setminus H_2 \Leftrightarrow G) \implies \forall(H_1^L \otimes H_2^L \otimes G^L \rightarrow H_1^L \otimes \exists y.(B^L \otimes G^L))\]

\[\text{Programs: } \{R_1, \ldots, R_n\}^L := \{R_1^L, \ldots, R_n^L\}\]

**Proof.** (\('\equiv\') Follows from Thm. 5.18 by assuming an empty program \(P = \emptyset\).

(\(\Rightarrow\)) We consider the axioms for configuration entailment in Def. 5.14: W.r.t. axiom (1), \([\hat{T}] \mathbin{\Downarrow} [S \lor \hat{T}]\) implies \(T^L \vdash (S \lor \hat{T})^L\) since \(\beta \vdash \alpha \oplus \beta\). For Def. 5.14.2, \([S_1] \mathbin{\Downarrow} [S_2]\) implies \(S_1^L \rhd \Sigma S_2^L\) by Lemma 4.7. From a proof of \(S_1^L \rhd \Sigma S_2^L\), we can construct a proof of \(S_1^L \oplus S_2^L \vdash T^L \rhd \Sigma S_2^L \oplus T^L\). As \(\rhd \Sigma\) is furthermore reflexive and transitive, the hypothesis is reduced to Lemma 5.12.

Analogously to the encoding semantics for pure CHR, we define an encoding semantics for CHR\(\lor\). The translation of states and configurations is unchanged from the axiomatic semantics. The translation of constraint theories is the same as in the encoding semantics for pure CHR. The translation of rules and programs is updated to the syntax of CHR\(\lor\) as shown in Fig. 5.3.4.

The soundness and completeness of the encoding semantics is proven analogously to Theorem 5.18:

**Theorem 5.20 Soundness and Completeness of the Encoding Semantics.** Let \(\hat{S}, \hat{T}\) be configurations. There exists a configuration \(\hat{U}\) such that

\[\hat{S} \rightarrow^\ast \hat{U} \text{ and } \hat{U} \rhd \hat{T}\]

in a program \(P\) and a constraint theory \(CT\) if and only if

\[P^L, CT^L \vdash (\forall \hat{S}^L \rightarrow \hat{T}^L)\]

As the encoding semantics is logically equivalent to the one proposed in Betz [2007], Theorem 5.20 also proves the equivalence of the axiomatic linear-logic semantics with that earlier semantics.

### 5.4 Congruence and Analyticness

The operational semantics \(\omega_r\) for pure CHR features the pleasant property that state equivalence coincides with mutual entailment of states (cf. Corollary 4.11). In this section, we show that the property of mutual configuration entailment, henceforth called **congruence of configurations**, does not in general coincide with configuration equivalence.

To overcome this limitation, we introduce a well-behavedness property on configurations – **compactness** – and one on CHR\(\lor\) programs – **analyticness** – which guarantee that congruence coincides with equivalence.

**Definition 5.21 Congruence of Configurations.** Given a constraint theory \(CT\), two configurations \(\hat{S}, \hat{T}\) are considered **congruent** if \(\hat{S} \mathbin{\Downarrow} \hat{T}\) and \(\hat{T} \mathbin{\Downarrow} \hat{S}\). Congruence of \(\hat{S}\) and \(\hat{T}\) is denoted as \(\hat{S} \equiv \hat{T}\).

Congruence of configurations does not generally comply with rule applications as the following example shows.
Example 5.22 Non-Compliance with Rule Application. By compliance, we mean the property that for arbitrary configurations $S, S', T$ such that $S \equiv \varnothing$ and $S \rightarrow^* T$, there exists a $T'$ such that $S' \rightarrow T'$ and $T' \equiv \bar{T}$.

Let $S = \langle c_a(X) \rangle$ and $T = \langle c_a(0) \rangle \lor \langle c_a(X) \rangle$ be configurations. As $\langle c_a(0) \rangle \triangleright \langle c_a(X) \rangle$, we have congruence: $S \blacktriangleright \bar{T}$. Now consider the following minimal CHR program:

$$r @ c_a(0) \iff d_a(0)$$

We observe that we have $T \rightarrow^* \langle d_a(0) \rangle \lor \langle c_a(X) \rangle$ whereas $\bar{S}$ is an answer configuration i.e. it does not allow any further transition. We thus observe that congruence of configurations is not in general compliant with rule application.

However, we can make a somewhat weaker statement about the relationship between congruence and rule application:

Property 5.23 Weak Compliance with Rule Application. Let $S, S', T$ be configurations such that $S \blacktriangleright \bar{S}'$. Then $S \rightarrow^* T$ implies that there exists a $T'$ such that $S' \rightarrow T'$ and $T' \blacktriangleright T$.

Proof. $S \blacktriangleright \bar{S}'$ implies $S' \blacktriangleright \bar{S}$. Furthermore, we have $S \rightarrow^* T$. Hence, Lemma 5.16 proves $S' \rightarrow^* T'$ and $T' \blacktriangleright T$. $\Box$

As the congruence relation does not strongly comply with rule application, it is not appropriate as a general equivalence relation over configurations. On the other hand, from Lemma 5.19 follows that congruence of configurations coincides with logical equivalence over the respective linear-logic readings:

Property 5.24. For arbitrary configurations $S, T$, we have $S \blacktriangleright T \iff S \equiv T$.

Hence, any reasoning over $\text{CHR}^\equiv$ via the linear-logic semantics is necessarily modulo congruence. In order to allow precise logical reasoning over $\text{CHR}^\equiv$, we identify a segment of $\text{CHR}^\equiv$ where congruence and equivalence of configurations coincide. Firstly, we introduce the notion of compactness:

Definition 5.25 Compactness. A configuration $\bar{S}$ is called compact if it does not have a representation $S' \equiv \varnothing$ $\bar{S}$ of the form $S' = S_1 \lor S_2 \lor \bar{S}''$ where $S_1, S_2$ are flat states such that $S_1 \neq S_2$ and $S_1 \triangleright S_2$.

We extend the compactness property to equivalence classes of configurations in the obvious manner. The following lemma states that compactness guarantees that congruence and equivalence coincide.

Lemma 5.26. Let $S, T$ be compact configurations such that $S \blacktriangleright T$. Then $S \equiv T$.

Proof. Considering Def. 5.5, we observe that every configuration $\bar{S}$ has a representation of the form $\bar{S} \equiv \varnothing S_1 \lor \ldots \lor S_n$, where $S_i = \langle \mathbb{U}_i \land \mathbb{V}_i \lor \_ \rangle$ for $i \in \{1, \ldots, n\}$. By Def. 5.14, any two configurations $\bar{S}, \bar{T}$ where $\bar{S} \triangleright \bar{T}$ have representations $\bar{S} \equiv S_1 \lor \ldots \lor S_n, \bar{T} \equiv S_1 \lor \ldots \lor T_m$ such that for every $S_i$ where $S_i \neq S_L$, we have the exists a $T_j$ such that $S_i \triangleright T_j$).

As $\bar{S} \blacktriangleright \bar{T}$, we have representations $\bar{S} \equiv S_1 \lor \ldots \lor S_n, \bar{T} \equiv S_1 \lor \ldots \lor T_m$ such that for every consistent $S_i$, we have a $T_j$ such that $S_i \triangleright T_j$, and for every consistent $T_j$ there is an $S_i$ such that $S_i \triangleright T_j$. It follows that for every consistent $S_i$, we have $T_j, S_k$ such that...
$S_i \triangleright T_j \triangleright S_k$. As $\bar{S}$ is compact, $S_i \triangleright S_j$ implies $S_i \equiv S_k$ and furthermore $S_i \equiv T_j$. As $\bar{T}$ is compact, there is exactly one $T_j$ such that $S_i \equiv T_j$.

Since every consistent $S_i$ has a unique corresponding state $T_j$ with $S_i \equiv T_j$ and vice versa, Def. 5.5 implies that $\bar{S} \equiv \bar{T}$. □

We furthermore introduce a well-behavedness property for CHR$^\vee$ programs which guarantees compactness of derived configurations by assuring that disjoint member states of a configuration have contradicting built-in states. It appears that a large number of practical CHR$^\vee$ programs satisfy this property.

**Definition 5.27 Analytic Program.** A CHR$^\vee$ program is called analytic if for any flat state $S$ and configuration $\bar{T}$ where $[S] \mapsto [\bar{T}]$, we have that $\bar{T}$ is compact.

We give a sufficient (although not necessary) criterion for analyticness of CHR$^\vee$ programs:

**Lemma 5.28 Criterion for Analyticness.** Let $\mathcal{P}$ be a CHR$^\vee$ program consisting of rules $R_1, \ldots, R_n$. Assume that every rule $R_i$ is of the form $r @ H_1 \setminus H_2 \Leftarrow G | (U_1 \wedge B_1) \lor \ldots \lor (U_m \wedge B_m)$ such that $CT \not\vdash \exists (B_i \wedge B_j)$ for every $i, j \in \{1, \ldots, n\}$. Then $\mathcal{P}$ is analytic.

**Proof.** We assume a single rule application $S \mapsto^* \bar{T}$ where the applied rule be of the form $R_i = r @ H_1 \setminus H_2 \Leftarrow G | (U_1 \wedge B_1) \lor \ldots \lor (U_m \wedge B_m)$ such that $CT \not\vdash \exists (B_i \wedge B_j)$ for $i, j \in \{1, \ldots, n\}$.

It follows that for every $T_1 = \langle U_1; B_1; V_1 \rangle, T_2 = \langle U_2; B_2; V_2 \rangle$ such that $\bar{T} \equiv T_1 \lor T_2 \lor \bar{T}'$, we have $CT \not\vdash \exists (B_1 \wedge B_2)$. It follows by Lemma 4.10 that $T_1 \not\triangleright T_2$.

As the built-in store grows monotonically stronger, correctness for the transitive closure of $\mapsto$ follows by induction. For the reflexive closure it follows from the fact that the state $S$ is trivially a compact configuration. □

### 6. APPLICATION

In this section, we outline how our results can be applied to reason over programs and their respective observables. We separate it into two broad application domains: In Section 6.1, we discuss the relationship between the linear-logic semantics and program observables. In Section 6.2, we show how we can compare the operational semantics of programs by means of their linear-logic semantics.

#### 6.1 Reasoning About Observables

In this section, we show how to apply our results to reason about observables in both pure CHR and CHR$^\vee$. We will first discuss pure CHR in detail and then show how the results are generalized to CHR$^\vee$.

**6.1.1 Reasoning About Observables in Pure CHR.** We define two sets of observables based on the linear logic semantics, paralleling the observable sets of computable states and data-sufficient answers.

**Definition 6.1.** Let $\mathcal{P}$ be a pure CHR program, $CT$ a constraint theory, and $S$ an initial state. Assuming that $\Sigma = \Sigma_{\mathcal{P}} \cup \Sigma_{CT} \cup \Sigma_{\mathcal{P}}$, we distinguish two sets of observables based on the linear logic semantics:

\[
\begin{align*}
\mathcal{L}^C_{\mathcal{P}}(S) & := \{T | \mathcal{P}^L, CT^L, S^L \vdash T^L\} \\
\mathcal{L}^C_{CT}(S) & := \{\langle T; \mathcal{B}; \mathcal{V} \rangle | \mathcal{P}^L, CT^L, S^L \vdash (T; \mathcal{B}; \mathcal{V})^L\}
\end{align*}
\]
If the constraint theory $CT$ is clear from the context or not important, we write the sets as $L^C_T(S)$, $L^S_T(S)$.

The following definition and property establish the relationship between the logical observables $L^C_T$ and $L^S_T$ and the operational observables $C_T$ and $S_T$.

**Definition 6.2 Lower Closure of $\triangleright$.** For any set $\mathcal{S}$ of equivalence classes of CHR states, $\forall \mathcal{S} := \{ [T] | \exists S \in \mathcal{S} \triangleright [T] \}$

The following property follows directly from Theorem 4.24:

**Property 6.3 Relationship Between Observables.** For a pure CHR program $P$, a constraint theory $CT$, and an initial state $S$, we have:

$\forall C^D_{P, CT}(S)$

$\forall S^S_{P, CT}(S)$

From this relationship follow several properties that we can use to reason about the operational semantics. Firstly, in order to prove that a state $S$ cannot develop into a failed state, it suffices to show that there exists any state $T$, such that $[T]$ is not contained in $C(S)$:

**Property 6.4 Exclusion of Failure.** Under a program $P$, a constraint theory $CT$, and a CHR state $S$ if there exists a state $T$ such that $T \notin L^C_{P, CT}(S)$ then $S \notin C_{P, CT}(S)$.

Secondly, we can guarantee data-sufficient answers for a state $S$, if we can prove the empty resource 1 in linear logic:

**Property 6.5 Assuring Data-Sufficient Answers.** (1) Under a program $P$, a constraint theory $CT$, and a CHR state $S$, if $\langle \top; T; \emptyset \rangle \in L^D_{P, CT}(S)$ then $S$ has at least one data-sufficient answer.

(2) If $P$ is furthermore confluent, $S$ has exactly one data-sufficient answer.

**Proof sketch.** The first property follows from the fact that for any data-sufficient state $\langle \top; T; \emptyset \rangle$, we have $\langle \top; T; \emptyset \rangle \triangleright \langle \top; T; \emptyset \rangle$. The second property follows from Prop. 2.16. □

Finally, if a specific state does not follow in linear logic, it is guaranteed not to follow in the operational semantics:

**Property 6.6 Safety Properties.** For a program $P$, a constraint theory $CT$, and any two CHR states $S, T$, if $S' \notin L^C_{T, CT}(S)$ then $S' \notin C_{T, CT}(S)$.

**Example 6.7.** This example shows how to exploit the completeness of our semantics to prove safety properties for CHR programs. By safety property, we mean a problem of non-existence of a derivation between two CHR states. The general form of a safety property is $[T] \notin C_S(S)$.

We implement the $n$-Dining-Philosophers Problem for an arbitrary number of philosophers and we show using the phase semantics that the program can never reach a state in which any two philosophers directly neighboring each other are eating at the same time.

We assume that $CT$ includes the constraint theory for natural numbers.

$$
\begin{align*}
\text{fork}(x) \land \text{fork}(y) & \iff y = x + 1 \mod n \land \text{eat}(x) \\
\text{eat}(x) & \iff y = x + 1 \mod n \land \text{fork}(x) \land \text{fork}(y) \\
\text{putfork}(0) & \iff \top \\
\text{putfork}(n) & \iff n \geq 1 \land n_1 = n - 1 \land \text{fork}(n_1) \land \text{putfork}(n_1)
\end{align*}
$$
We want to prove that two philosophers (among $n$ philosophers) which are seated side by side cannot be eating at the same time. This can be formalized by the following safety property (we naturally assume there are at least two philosophers):

$$\forall n, i. (\text{eat}(i) \land \text{eat}(j); j = i + 1 \text{ mod } n; \emptyset) \notin L^C_P((\text{put fork}(n); \top; \emptyset))$$

Showing that a certain state is not included in $L^C_P(S)$, or – more generally – that a certain linear-logic judgement is not valid is in general not trivial. Having an automated theorem prover try all possible inference rules exhaustively is an option. In [Haemmerlé and Betz 2008], a method to prove safety properties using the phase semantics of linear logic has been proposed. At this point, it shall suffice to state that we can show:

$$P^L, CT^L \not\vDash \exists n, i. (\text{put fork}(n) \rightarrow \text{eat}(i) \otimes \text{eat}(j) \otimes ! \text{ } (j = i + 1))$$

This proves that no two philosophers seated side by side can be eating at the same time.

6.1.2 Generalization to CHR $\lor$. As for pure CHR, we define two sets of linear logic observables, paralleling the sets of computable configurations and data-sufficient answer configurations.

**Definition 6.8.** Given a CHR $\lor$ program $P$, a constraint theory $CT$, and an initial state $S$, we distinguish two sets of observables based on the linear logic semantics:

$$L^C_{P, CT}(S) ::= \{ [\overline{T}] | P^L, CT^L, S^L \vdash \overline{T} \}$$

$$L^S_{P, CT}(S) ::= \{ \langle \top; B_1; V_1 \rangle \lor \ldots \lor \langle \top; B_n; V_n \rangle | P^L, CT^L, S^L \vdash (\langle \top; B_1; V_1 \rangle \lor \ldots \lor \langle \top; B_n; V_n \rangle) \}$$

The relationship between the logical observables and the operational observables is parallel to pure CHR, though generalized to the lower closure of configuration entailment.

**Definition 6.9 Lower Closure of $\triangleright$.** For any set $\mathbb{S}$ of equivalence classes of CHR states,

$$\nabla \mathbb{S} ::= \{ [\overline{T}] | \exists \overline{S} \in \mathbb{S}. [S] \triangleright [\overline{T}] \}$$

By Theorem 5.20, we then have:

**Property 6.10 Relationship Between Observables.** For a CHR $\lor$ program $P$, a constraint theory $CT$, and an initial state $S$, we have:

$$L^C_{P, CT}(S) = \nabla C_{P, CT}(S)$$

$$L^S_{P, CT}(S) = \nabla S_{P, CT}(S)$$

Furthermore, each of Property 6.4, Property 6.5, and Property 6.6 have their obvious counterparts in CHR $\lor$.

6.2 Comparison of Programs

In this section, we put special emphasis on the comparison of CHR and CHR $\lor$ programs across programming paradigms. Hence, we will not treat pure CHR in an isolated manner but as a subset of CHR $\lor$. Note also that we use the encoding rather than the axiomatic formulation of our semantics in this section.

We define three notions of operational equivalence, each one corresponding to one set of observables as introduced in Section 2.2.
Definition 6.11 Operational Equivalence. (1) Two CHR programs $P_1, P_2$ are operationally $S$-equivalent under a given constraint theory $CT$ if for any state $S$, we have $\bar{S}_{P_1, CT}(S) = \bar{S}_{P_2, CT}(S)$.

(2) Two CHR programs $P_1, P_2$ are operationally $A$-equivalent under a given constraint theory $CT$ if for any state $S$, we have $\mathcal{A}_{P_1, CT}(S) = \mathcal{A}_{P_2, CT}(S)$.

(3) Two CHR programs $P_1, P_2$ are operationally $C$-equivalent under a given constraint theory $CT$ if for any state $S$, we have $\bar{C}_{P_1, CT}(S) = \bar{C}_{P_2, CT}(S)$.

We will mainly focus on $C$-equivalence and $S$-equivalence. What we call $A$-equivalence has been researched extensively in the past (cf. Abdennadher et al. [1999]). It shows in this section that the linear-logic semantics is not adequate to reason about $A$-equivalence.

Definition 6.12 Logical Equivalence of Programs. Two CHR programs $P_1, P_2$ are called logically equivalent under a given constraint theory $CT$ if $CT^L \vdash \otimes P_1 \leadsto \otimes P_2$, where the unary operator $\otimes$ stands for element-wise multiplicative conjunction and $\otimes P_1 \leadsto \otimes P_2$ is shorthand for $(\otimes P_1 \leadsto \otimes P_2) \& (\otimes P_2 \leadsto \otimes P_1)$.

The following proposition relates $C$- and $S$-equivalence.

Proposition 6.13. Operational $S$-equivalence is a necessary but not a sufficient condition for $C$-equivalence.

Proof. To show that $S$-equivalence is a necessary condition, we assume two $C$-equivalent programs $P_1, P_2$. For every state $S$, we have $\bar{C}_{P_1}(S) = \bar{C}_{P_2}(S)$. As each $S_{P_i}$ is the projection of $\bar{C}_{P_i}(S)$ to configurations with empty user-defined stores, we also have $S_{P_1}(S) = S_{P_2}(S)$.

To show that $S$-equivalence is not a sufficient condition, consider the following two programs:

\[
P_1 = \{ \quad a(x) \iff b(x) \\
           b(x) \iff x \neq 0 \quad \}\qquad P_2 = \{ \quad a(x) \iff x \neq 0 \\
           b(x) \iff x \neq 0 \quad \}\n\]

Both programs ultimately map every $a(x)$ and $b(x)$ to $x \neq 0$. Hence, they are $S$-equivalent. For $S = (a(x); \emptyset)$ and $T = (b(x); \emptyset)$ we have $[T] \in C_{P_1}(S)$ but $[T] \notin C_{P_2}(S)$. Hence, the programs are not $C$-equivalent. \qed

We can show that operational $C$-equivalence implies logical equivalence of programs.

Proposition 6.14. Let $P_1, P_2$ be two $C$-equivalent CHR programs under $CT$. Then $CT^L \vdash \otimes P_1 \leadsto \otimes P_2$.

Proof. Since $P_1$ and $P_2$ are $C$-equivalent, we have that $\bar{C}_{P_1}(S) = \bar{C}_{P_2}(S)$ for all $S$. For every rule $R = (r \in H_1 \setminus H_2 \iff G | B) \in P_2$, we have by Def. 5.7: $[H_1 \amp B \amp G; \bar{x}] \in C_{P_2}$ where $\bar{x} = \text{vars}(H_1 \amp H_2 \amp G)$ and then by our hypothesis $[H_1 \amp B \amp G; \bar{x}] \in \bar{C}_{P_1}$. Therefore, we get $CT^L \vdash \otimes P_1 \leadsto \otimes P_2$. Applying this to all rules $R \in P_2$, we show $CT^L \vdash \otimes P_1 \leadsto \otimes P_2$. Analogously, we get $CT^L \vdash \otimes P_2 \leadsto \otimes P_1$. \qed

The reverse direction does not hold in general as the following example shows:
Example 6.15. Let the constraint theory $CT$ contain at least the theory of natural numbers. Compare the following two programs:

\[ P_1 = \{ c(x) \iff x \geq 1 \} \quad P_2 = \{ c(x) \iff T \} \]

\[ c(x) \iff x \geq 1 \]

The greater-or-equal constraint $\geq$ is a built-in constraint. Hence, it is translated as $(x \geq 1)^\Sigma = (x \geq 1)$. As $(x \geq 1)^\Sigma \vdash 1$, we have $\bigotimes P_1^\Sigma \vdash \bigotimes P_2^\Sigma$. We observe that $S_{P_1}((c(x); x)) = \{(x \geq 1); x\}$ and $S_{P_2}((c(x); x)) = \{(x \geq 1); x\}$. As the sets are not equal, $P_1$ and $P_2$ are not operationally $S$-equivalent and hence, by Prop. 6.13, not $C$-equivalent.

However, if we restrict ourselves to analytic, confluent programs, we can show that logical equivalence of programs implies operational $S$-equivalence:

**Proposition 6.16.** Let $P_1, P_2$ be two analytic confluent CHR$^\nu$ programs such that $CT^L \vdash \bigotimes P_1 \leftrightarrow \bigotimes P_2$. Then $P_1, P_2$ are $S$-equivalent.

**Proof.** As both $P_1$ and $P_2$ are confluent, we have $|S_{P_1,CT}(S)| \in \{0, 1\}$ for any state $S$ and $i \in \{1, 2\}$, where $| \cdot |$ denotes cardinality. If $|S_{P_i,CT}(S)| = 0$ then $|\nabla S_{P_i,CT}(S)| = 0$. Otherwise, $|\nabla S_{P_i,CT}| \geq 1$. In the former case, our proposition is trivially true since $S_{P_i,CT} = \emptyset$. In the following, we assume $|S_{P_i,CT}| = 1$.

Logical equivalence implies that $\tilde{L}_{P_i,CT}^C(S) = \tilde{L}_{P_i,CT}^C(S)$ for all $S$. Since $\tilde{L}^C$ is the projection of $\tilde{L}^C$ to configurations with empty user-defined stores, we also have $\tilde{L}_{P_i,CT}^S = \tilde{L}_{P_i,CT}^S(S)$ and hence $\nabla \tilde{S}_{P_i,CT}(S) = \nabla \tilde{S}_{P_i,CT}(S)$.

Since $|S_{P_i,CT}(S)| = 1$ for $i \in \{1, 2\}$, each lower closure $\nabla \tilde{S}_{P_i,CT}(S)$ has a maximum $[M_j] \in \nabla \tilde{S}_{P_i,CT}(S)$ such that $\forall [S] \in \nabla \tilde{S}_{P_i,CT}(S), [M_j] \triangleright [S]$ and $S_{P_i,CT}(S) = \{[M_j]\}$. As $\nabla \tilde{S}_{P_i,CT}(S) = \nabla \tilde{S}_{P_i,CT}(S)$, we have $M_1 \equiv M_2$. As both programs are analytic, we furthermore have that $M_1, M_2$ are compact. Hence, we have $M_1 \equiv M_2$ and therefore: $S_{P_1,CT}(S) = S_{P_2,CT}(S)$. \qed

The following example shows that logical equivalence does not imply operational $A$-equivalence:

Example 6.17. We consider the program $P = \{ c(x) \iff c(x) \}$ and the empty program $P_0 = \emptyset$.

As the logical reading $P^L = \forall c(x) \iff c(x))$ of $P$ is a logical tautology, it follows that $P^L \vdash^\Sigma P_0^L$ for any $\Sigma$. Yet, for $S = (c(x); T; \emptyset)$, we have $\mathcal{A}_P(S) = \emptyset$ whereas $\mathcal{A}_P(S) = [S]$. Therefore $\mathcal{A}_P(S) \neq \mathcal{A}_P(S)$.

The following final example shows how we can apply the linear-logic semantics to compare programs across programming paradigms.

Example 6.18. We begin with the following classic Prolog program which implements a ternary `append` predicate for lists, where the third argument is the concatenation of the first two:

\[
\text{append}(x, y, z) \leftarrow x \left[ \cdot \right] \land y \left[ \cdot \right] \\
\text{append}(x, y, z) \leftarrow x \left[ h_1 \right] \land z \left[ h_2 \right] \land \text{append}(l_1, y, l_2)
\]

We can embed this program into CHR$^\nu$ by explicitly stating the don’t-know non-
determinism using the $\lor$ operator.

$$P_1 = \{ \text{append}(x, y, z) \iff (x=\[\] \land y=z) \lor (x=[h]_1 \land z=[h]_2 \land \text{append}(l_1, y, l_2)) \}$$

The linear-logic reading of the embedded program looks as follows:

$$P^L_1 = \{ \forall x, y, z. (\text{append}(x, y, z) \rightarrow \exists l_1, l_2, h \left( (\forall x=\[\] \land y=z) \lor (\forall x=[h]_1 \land z=[h]_2 \land \text{append}(l_1, y, l_2)) \) \}$$

Secondly, we write a program to implement the $\text{append}$ predicate the way it would be expected in CHR:

$$P_2 = \{ \text{append}([\], y, z) \iff y=z \\\text{append}([h]_1, y, z) \iff z=[h]_2 \land \text{append}(l_1, y, l_2) \}$$

The two programs are not per se $S$-equivalent. Consider their behaviour in case the first argument of $\text{append}$ is bound to anything else than a list. For $S_0 = (\text{append}(3, x, y):\emptyset)$, we have $\mathcal{S}_P(S_0) = \{\bot\}$ but $\mathcal{S}_P(S_0) = \emptyset$.

Now let us assume that the first argument is always bound to a list. We can model this by the following formula:

$$\varphi = \forall (\text{append}(x, y, z) \rightarrow \text{append}(x, y, z) \otimes (\forall x=\[\] \lor l_1. l_1 \land x=[h]_1))$$

It shows that $CT^L, \varphi \vdash \bigotimes P_1 \dashv \vdash \bigotimes P_2$. Hence, under the assumption that the first argument is always bound to a (non-empty or empty) list, the two programs are operationally $S$-equivalent.

Moreover, we observe that $\varphi$ is equivalent to the logical reading of the CHR $\lor$ rule $R_\varphi$:

$$R_\varphi = (r @ \text{append}(x, y, z) \iff \text{append}(x, y, z) \land (x=\[\] \lor x=[h]_1))$$

Moreover $CT^L, \varphi \vdash \bigotimes P_1 \dashv \vdash \bigotimes P_2$ implies that $CT^L, \vdash (\bigotimes P_1 \otimes \varphi) \dashv \vdash (\bigotimes P_2 \otimes \varphi)$ Hence, the programs $P'_1 = P_1 \cup R_\varphi$ and $P'_2 = P_2 \cup R_\varphi$ are operationally $S$-equivalent (without any further assumptions).

7. RELATED WORK

From its advent in the 1980ies, linear logic has been studied in relationship with programming languages.

Common linear logic programming languages such as LO[Andreoli and Pareschi 1990], Lolli[Hodas and Miller 1991], LinLog[Andreoli 1992], and Lygon[Harland et al. 1996] rely on generalizations of backward-chaining backtracking resolution of horn clauses.

The earliest approach at defining a linear-logic semantics for a committed-choice programming language that we are aware of has been proposed in [Zlatuska 1993]. The corresponding language is indeed a fragment of pure CHR without multiple heads and with substantial restrictions on the use of built-in constraints.

The linear-logic programming language LolliMon, proposed in [López et al. 2005], integrates backward-chaining proof search with committed-choice forward reasoning. It is an extension of the aforementioned language Lolli. Thesequent calculus underlying Lolli is extended by a set of dedicated inference rules. The corresponding connectives are syntactically detached from Lolli’s own connectives and operationally they are processed within
a monad. The actual committed-choice behavior comes by the explicit statement in the operational semantics, that these inference are to be applied in a committed-choice manner during proof search. With respect to Lolli, committed comes thus comes at the cost of giving up the general notion of execution as proof search, although it is retained outside the monad.

The class LCC of linear logic concurrent constraint programming languages [Fages et al. 2001] has a close relationship with CHR, although the former is based on agents whereas the latter is based on rules. Similar to CHR, LCC languages are non-deterministic and execution is committed-choice. The linear logic semantics of LCC is similar to our linear logic semantics for pure CHR and, as far as the two are comparable, it features similar results for soundness and completeness. Unlike CHR\(^{\lor}\) however, LCC has no notion of disjunction.

Furthermore, Fages et al. have proposed the so-called frontier semantics [Fages et al. 2001] for LCC, in which the committed-choice operator is interpreted analogously to the disjunction operator \(\lor\) in CHR\(^{\lor}\). In the linear-logic interpretation of the frontier semantics, it is correspondingly mapped to the multiplicative disjunction \(\&\). However, the frontier semantics does not constitute a distinct programming language but is viewed as a tool to reason about properties of LCC programs. Hence, committed choice never co-exists with disjunction as in the linear logic semantics for CHR\(^{\lor}\). Rather, the two are viewed as different interpretations of the same connective for different purposes.

More recently, Simmons et al. proposed the linear logic-based committed-choice programming language Linear Logical Algorithms [Simmons and Pfenning 2008]. While the language itself corresponds to a segment of pure CHR, the aim of the work is to define a cost semantics for algorithms that feature non-deterministic choices.

8. CONCLUSION
In this article, we have presented a detailed analysis of the relationship between both pure CHR and CHR\(^{\lor}\) with intuitionistic linear logic and we have shown its applications from reasoning about programs observables to deciding operational equivalence of multi-paradigm CHR\(^{\lor}\) programs.

Our first main contribution is the linear-logic semantics for the segment of pure CHR. It encodes both CHR programs and constraint theories to proper axioms of the sequent calculus. We have shown that equivalence of CHR states coincides with logical equivalence of the logical readings of state. Furthermore, we have introduced the notion of state entailment, which precisely characterizes the discrepancy between the transition relation between states in CHR and judgements between their corresponding logical readings. It is a key notion for the study and the application of our semantics.

Our second main contribution is the definition of a linear-logic semantics for CHR\(^{\lor}\). This semantics maps the dualism between don’t-care and don’t-know non-determinism in CHR\(^{\lor}\) to the dualism of internal and external choice in linear logic. Analogously to pure CHR, we have defined a notion of configuration entailment to characterize the discrepancy between state transition and logical judgement.

We have shown that the linear-logic semantics for CHR\(^{\lor}\) has somewhat less desirable properties than the one for pure CHR. Concretely, mutual configuration entailment does not coincide with configuration equivalence. This makes linear-logic based reasoning over CHR\(^{\lor}\) in general more imprecise. However, we have presented a well-behavedness prop-
property for CHR$^\lor$ – analyticness – that amends this limitation.

As our third main contribution, we have shown how to apply our results to reason about CHR and CHR$^\lor$ programs. We have defined sets of linear-logic based observables that correspond with the usual program observables of computable state and data-sufficient answer by means of state entailment or configuration entailment, respectively. We have presented criteria to prove various program properties, foremost safety properties, which consist in the non-computability of a specific state from a certain initial state. Furthermore, we have given a criterion to prove operational equivalence with respect to data-sufficient answers for multi-paradigm programs.

As a further contribution, we have for the first time defined an equivalence relation over configurations and shown its compliance with rule application. Based on this relation, we have defined an elegant formalization of the operational semantics of CHR$^\lor$ based on equivalence classes of configurations. The equivalence-based semantics provides a language to express properties of programs such as operational equivalence across the boundaries of programming paradigms.

Our results entail a wide range of possible future work. An obvious line of future work lies in the application of established methods for automated proof search in linear logic to reason about CHR and CHR$^\lor$ programs. As significant effort has been put in the current result on amending the discrepancy between linear judgement and the semantics of CHR, it furthermore suggests itself to investigate whether a “purer” formalism to reason about CHR could be extracted from linear logic that avoids these discrepancies.

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