Existence and behavior of steady solutions on an interval for general hyperbolic-parabolic systems of conservation laws

Benjamin Melinand∗ and Kevin Zumbrun†

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Abstract

We study the inflow-outflow boundary value problem on an interval, the analog of the 1D shock tube problem for gas dynamics, for general systems of hyperbolic-parabolic conservation laws. In a first set of investigations, we study existence, uniqueness, and stability, showing in particular local existence, uniqueness, and stability of small amplitude solutions for general symmetrizable systems. In a second set of investigations, we investigate structure and behavior in the small- and large-viscosity limits.

A phenomenon of particular interest is the generic appearance of characteristic boundary layers in the inviscid limit, arising from noncharacteristic data for the viscous problem, even of arbitrarily small amplitude. This induces an interesting new type of “transcharacteristic” hyperbolic boundary condition governing the formal inviscid limit.

1 Introduction

In this paper, inspired by recent results in [MZ19, BBZ21] on the “1D shock tube problem” for gas dynamics, i.e., steady flow on a bounded interval, with noncharacteristic inflow/outflow boundary conditions, we here begin the systematic study of the “generalized 1D shock problem” for arbitrary systems of hyperbolic-parabolic conservation laws.

Our goals are two-fold: first, to add context and larger foundation to the somewhat special analyses of [MZ19, BBZ21], and, second, to introduce what seems to be a family of new and interesting types of hyperbolic and hyperbolic-parabolic problems for general conservation laws.
1.1 Equations and assumptions

Following [SZ01, Zum10], we consider steady solutions of general viscous conservation laws:

\[ \partial_t f^0(U) + f(U)_x = (B(U)U)_x , \quad 0 < x < 1 , \quad t > 0, \]

where \( U \in \mathcal{U} \subset \mathbb{R}^n \),

\[
U = \begin{pmatrix} U_I \ U_{II} \end{pmatrix}, \quad f^0 = \begin{pmatrix} f^0_I \ f^0_{II} \end{pmatrix} f = \begin{pmatrix} f_I \ f_{II} \end{pmatrix} \in \mathbb{R}^r \times \mathbb{R}^{n-r}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & B_{22} \end{pmatrix}, \quad B_{22} \in M_{n-r}(\mathbb{R}),
\]

and the partial derivative \( df^0 \) is invertible and lower block triangular, without loss of generality

\[
df^0(U) = \begin{pmatrix} \text{Id}_r & 0 \\ (df^0_{II})_I & (df^0_{II})_{II} \end{pmatrix},
\]

with the boundary conditions

\[ U(0) = U_0 = \begin{pmatrix} U_{0I} \\ U_{0II} \end{pmatrix} \quad \text{and} \quad U_{II}(1) = U_{1II}. \]

We make the following assumptions:

\((H0)\) \( f \) and \( B \) are smooth.

\((H1)\) \( \Re \sigma((df^0_{II})_{II}^{-1} B_{22}(U)) > 0 \) for any \( U \), where \( \sigma(\cdot) \) denotes spectrum.

\((H2)\) The eigenvalues \( \sigma(df_I)_{II}(U) \) are real, and positive, for any \( U \).

Condition \((H1)\) corresponds to strict parabolicity of \((\ref{eq:conservation_law})\) with respect to \( U_{II} \), consistent with Dirichlet boundary conditions on \( U_{II}(0), U_{II}(1) \). Note that \( B_{22}(U) \) is necessary invertible.

The first part of Condition \((H2)\) corresponds to hyperbolicity of \((\ref{eq:conservation_law})\) with respect to variable \( U_I \). The second part of Condition \((H2)\) means that the flow moves from the left to the right, so that the hyperbolic, \( U_I \), part of the flow is completely entering the domain at the left boundary, consistent with Dirichlet boundary conditions on \( U_I(0) \).

In the following, we denote \( A^0(U) = df^0(U), \ A(U) = df(U), \) with

\[
A^0 = \begin{pmatrix} \text{Id}_r & 0 \\ A^0_{21} & A^0_{22} \end{pmatrix} , \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} , \quad A_{II} \in M_r(\mathbb{R}).
\]

1.2 Main results

In Section 2, we categorize essentially completely existence, uniqueness and stability of small-amplitude steady solutions for general viscous conservation laws \((\ref{eq:conservation_law})\) under steadily increasing assumptions, culminating with the physically natural assumption of existence of a convex entropy. We discuss afterward in a more partial and speculative way the
issues of global existence, uniqueness, and stability, that is, the situation for large-amplitude solutions. In particular, we outline a strategy based on entropy dissipation and Brouwer degree for global uniqueness of constant solutions and global existence of large-amplitude systems, generalizing and illuminating the one introduced in [BBZ21] in the context of gas dynamics.

In Section 3, we turn to the study of structure and behavior of steady solutions, in the small- and large-viscosity limits. Even for the simplest case of isentropic gas dynamics, there is a rich “zoo” of possible solution patterns in the inviscid limit, featuring standing shocks, left and right boundary layers, and, the most novel feature, a new type of characteristic boundary layer appearing not as a boundary case, but generically in “rarefying,” or “expansive” solutions. These induce in the formal inviscid limit a new type of “transcharacteristic” boundary condition accommodating perturbations on either side of a characteristic state, hence requiring different numbers of boundary conditions; see Section 3.1. This phenomenon occurs for any rarefying steady solution, even of arbitrarily small amplitude, hence must be dealt with in order to produce a hyperbolic description of dynamics.

1.3 Discussion and open problems

A very interesting finding of [BBZ21] is that nonuniqueness may hold for systems with convex entropy, in particular even for gas dynamics with an artificially devised equation of state. Yet, it appears (numerically) to hold for the standard polytropic equation of state. A very interesting open problem is to verify this analytically. Likewise, the proof or disproof of existence for more general equations of state, or more general equations such as magnetohydrodynamics (MHD) are important open problems. Existence of large-amplitude multi-D solutions is a challenging more long term goal, even for gas dynamics, that appears to require further ideas for its resolutions.

As regards structure and asymptotic behavior, a very interesting open problem is to determine the possible feasible hyperbolic configurations for general systems, similarly as done here for isentropic gas dynamics. The understanding of hyperbolic behavior for the limiting zero-viscosity dynamics is another challenging direction. The handling of characteristic boundary layers, and resulting transcharacteristic type boundary conditions is a particularly intriguing piece of this puzzle. Similarly, the understanding of large-viscosity behavior for general systems is another interesting direction for study.

The most intriguing open problem is one that has not been addressed at all here, namely structure of the corresponding multi-D solutions in the inviscid limit. This could be thought of as a combination of compressible Pouseille flow and the 1-D shock problem; the results should be very interesting indeed. In particular, even for small data, for which multi-D existence and uniqueness are known [KK97], the question of structure seems not to have yet been addressed. An especially interesting question seems to be what is the role in multi-d of the characteristic boundary layer configurations we have seen here in 1-D. Presumably such configurations are there in the construction of [KK97], but their numerical and asymptotic description is still wanting.
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2 Existence, uniqueness and spectral stability

2.1 The linear case

We assume in this part that \( df \) and \( B \) are both constant. We have the following proposition.

**Proposition 2.1.** If \( df \) and \( B \) are both constant and conditions (H0)-(H2) are satisfied, Problem (1.1) has a unique steady state that satisfies the boundary condition (1.2) if and only if

\[
\sigma \left( B^{-1}_{22} \left( A_{22} - A_{21} A_{11}^{-1} A_{12} \right) \right) \cap 2i\pi \mathbb{Z} \setminus \{0\} = \emptyset.
\]

**Remark 2.1.** Condition (2.1) is a compatibility condition between the parabolic and the hyperbolic part. A similar condition was assumed for the study of quasilinear noncharacteristic boundary layers (on the half-line) in, for instance, [Met06, Lemma 5.1.3] or [Met04]. For example, the following system does not satisfy Condition (2.1)

\[
U_t + \begin{pmatrix} 0 & 2\pi \\ -2\pi & 0 \end{pmatrix} U_x = U_{xx}, \quad 0 < x < 1,
\]

and any constant state \( \hat{U} \) must satisfy \( \hat{U}(0) = \hat{U}(1) \).

**Proof of Proposition 2.1.** We can rewrite the problem as

\[
\begin{align*}
A_{11} U_I' + A_{12} U_{II}' &= 0, \\
A_{21} U_I' + A_{22} U_{II}' &= B_{22} U_{II}''.
\end{align*}
\]

Then, integrating, we get

\[
\begin{align*}
A_{11} U_I + A_{12} U_{II} &= C_1, \\
A_{21} U_I + A_{22} U_{II} + C_2 &= B_{22} U_{II}'.
\end{align*}
\]

where \( C_1 = A_{11} U_{I0} + A_{12} U_{II0} \) and \( C_2 \) is a constant that has to be determined. Then, since \( A_{11} \) is invertible, we obtain

\[
(2.2) \quad \begin{cases}
U_I = A_{11}^{-1} C_1 - A_{11}^{-1} A_{12} U_{II}, \\
U_{II}' = B_{22}^{-1} (A_{22} - A_{21} A_{11}^{-1} A_{12}) U_{II} + B_{22}^{-1} (C_2 + A_{21} A_{11}^{-1} C_1).
\end{cases}
\]

Denoting \( \tilde{A} = B_{22}^{-1} (A_{22} - A_{21} A_{11}^{-1} A_{12}) \) and \( \tilde{C} = B_{22}^{-1} (C_2 + A_{21} A_{11}^{-1} C_1) \), we solve

\[
\begin{align*}
U_{II}' &= \tilde{A} U_{II} + \tilde{C}, \\
U_{II}(0) &= U_{II0}.
\end{align*}
\]
We decompose $\hat{A}$ as $\hat{A} = P^{-1} \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix} P$ where $P$ and $F_2$ are invertible and $F_1$ is strictly upper triangular. For example, blockdiag$\{F_1, F_2\}$ could be taken to be a Jordan form for $\hat{A}$, with $F_1$ the real-eigenvalue part. We get that

$$U_{II}(1) = e^{\hat{A}}U_{II}(0) + P^{-1} \begin{pmatrix} f_0^1 e^{sF_1}ds \\ 0 \end{pmatrix} F_2^{-1} (e^{F_2} - I) P \tilde{C}.$$ 

Thus, we see that the map $C_2 \rightarrow U_{II}(1)$ is invertible if and only if $\sigma(\hat{A}) \cap 2i\pi \mathbb{Z}\backslash\{0\} = \emptyset$. □

### 2.2 Almost-constant steady states

We next study the existence of steady states for system (1.1)-(1.2) under the spectral assumption (2.1). We seek solutions $\tilde{U}$ of

$$B_{22}(U)U_1' = f_1(U) - f_{II}(U_0) + B_{22}(U_0)C_2, \quad U_{II}(0) = U_{0II} \quad f_1(U) = f_I(U_0)$$

is defined on $[0, 1]$. Thus, profiles are equivalent to roots $C_2$ of $\Phi(U_{II}, \cdot)$.

**Proposition 2.2.** Let $U_0 \in \mathbb{R}^n$ and assume conditions (H0)-(H2) are satisfied. Assume that Condition (2.1) is satisfied for $A = df(U_0)$ and $B = B(U_0)$. There exists $\delta > 0$ and $\varepsilon > 0$ such that for any $U_{II}$ with $|U_{0II} - U_{II}| \leq \delta$, there exists a unique solution $\tilde{U}$ of (2.3) satisfying

$$|\tilde{U}_{II}(0)| \leq \varepsilon.$$ 

Moreover, the solution is nondegenerate: i.e., corresponds to a nondegenerate root of $\Phi$.

**Remark 2.3.** We do not claim that for $U_{II}$ close enough to $U_{0II}$, there exists a unique solution of (2.3). The previous theorem only gives local uniqueness, even for small data.

**Proof of Proposition 2.2.** Let us fix $U_0 \in \mathbb{R}^n$s, taking without loss of generality $f_I(U_0) = 0$. First, we notice that for $U_{II} = U_{0II}, \tilde{U} \equiv U_0$ is a solution of (2.3). Then, by continuous dependence on parameters on the ODE, (H2) and the implicit function theorem on the constraint $f_I(U_I, U_{II}) = 0$, for $(U_{II}, C_2)$ close enough to $(U_{0II}, 0)$, one can express $U_I$ as a function of $U_{II}$ and the maximal solution $U$ of (2.4) is defined on $[0, 1]$. Therefore, we can define the map $\Phi$ on a neighborhood $V$ of $(U_{0II}, 0)$ in $\mathbb{R}^{n-r} \times \mathbb{R}^{n-r}$. The function $\Phi$ is $C^1$
on this domain and \( \Phi(U_{0I}, 0) = 0 \). Then, for any \( D \in \mathbb{R}^{n-r} \), \( d_2\Phi(U_0, 0) \cdot D = V_{II}(1) \), with \( V_{II} \) solving the variational equation

\[
B_{22}(U_0)V'_{II} = (A_{22} - A_{21}A^{-1}_{II}A_{12})(U_0)V_{II} + C_2 + (A_{21}A^{-1}_{II})(U_0)C_1 , \quad V_{II}(0) = 0,
\]

with \( C_1 = A_{11}(U_0)U_{I0} + A_{12}(U_0)U_{I10} \). As in the proof of Proposition 2.1, we can solve this ODE and, using Condition (2.1), we obtain that \( d_2\Phi(U_0, 0) \) is invertible. The result then follows from the implicit function theorem, as does nondegeneracy.

\[ \square \]

2.2.1 The Evans function

Linearizing (1.1) about a steady solution \( \hat{U} \), we obtain eigenvalue equations

\[
A^0V + (AV)_x = (BV)_x , \quad 0 < x < 1, \quad t > 0,
\]

\( A^0, A, B \) depending on \( x \), with homogeneous boundary conditions \( U(0) = 0, U_{II}(1) = 0 \). In particular, \( A^0_{II, I} = I_{n-r}, A^0_{I, II} = 0, A_{I, I} = df_1(\hat{U}) \), invertible, \( A_{I, II} = df_{II}(\hat{U}) \), and \( B = B(\hat{U}) \) with \( B_{II, II} \) invertible. Thus, denoting \( d/dx \) as \( ' \), we may rewrite (2.5) as a first-order system

\[
\begin{pmatrix}
A_{I, I}V_{I} \\
B_{II, II}V'_{II}
\end{pmatrix}' =
\begin{pmatrix}
\lambda A^{-1}_{II, I} & 0 & A'_{I, II} & A_{I, II}B^{-1}_{II, II} \\
0 & \lambda \alpha & 0 & \beta
\end{pmatrix}
\begin{pmatrix}
A_{I, I}V_{I} \\
B_{II, II}V'_{II}
\end{pmatrix}
\]

with

\[
\alpha = A^0_{II, II} + A_{II, I}A^{-1}_{I, I}A'_{II, I}, \quad \beta = (A_{II, II} + A_{II, I}A^{-1}_{I, I}A_{I, II})B^{-1}_{II, II},
\]

\[
\gamma = (A^0_{II, I} + A_{II, I}A^{-1}_{I, I}A_{I, I}^{-1}).
\]

The Evans function may thus be defined via a “shooting” construction, similarly as in [AGJ90, GZ98] for the whole line or [Rou01, SZ01] for the half-line case, as

\[
D(\lambda) := \det(Z_1, \ldots, Z_{n+r}) + |x=1,
\]

where \( Z_1, \ldots, Z_r \) are solutions with data \( Z_j(0) = (0, 0, B_{II}e_j) \) prescribed at \( x = 0 \) and \( Z_{r+1}, \ldots, Z_n \) and \( Z_{n+1}, \ldots, Z_{n+r} \) are solutions with data

\[
Z_j(1) = (A_{I, I}e_j, 0, 0), \quad j = r, \ldots, n-r
\]

and \( Z_{n+j}(1) = (0, 0, B_{II}e_j), \ j = 1, \ldots, r \) prescribed at \( x = 1 \). Evidently, \( D(\lambda) \) vanishes if and only if there exists a solution to the eigenvalue equation, hence spectral stability is equivalent to nonvanishing of the Evans function on \( \Re \lambda \geq 0 \).
2.2.2 Stability index, uniqueness, and Zumbrun-Serre/Rousset lemma

Clearly $D$ is real-valued for real $\lambda$. It is readily seen (see, e.g., SZ, Z1 in the half-line case) that $D(\lambda) \neq 0$ for $\lambda$ real and sufficiently large, hence we may define as in [GZ98] the Stability index

$$\mu := \text{sgn}D(0) \lim_{\lambda \to +\infty, \text{real}} \text{sgn}D(\lambda).$$

The index $\mu$ determines the parity of the number of roots of the Evans function with positive real part, with $+1$ corresponding to “even” and $-1$ to “odd”. Thus, $\mu = +1$ is a necessary condition for stability. The following result analogous to the Zumbrun-Serre/Rousset lemmas of [ZS99, Rou01] in the whole- and half-line case, relates low-frequency stability $D(0) \neq 0$ and stability index information to transversality of the steady profile solution of the standing-wave ODE (cf. [BBZ21, Lemma 6.1] for gas dynamics).

**Lemma 2.4.** The zero-frequency limit $D(0)$ is equal to $\det B_{11,11}$ multiplied by the Jacobian determinant $\det(d\Psi(C))$ associated with problem [2.4] evaluated at root $C_2$; in particular,

$$\text{sgn}D(0) = \text{sgn} \det d\Psi(C).$$

**Proof.** This follows from the observation that for $\lambda = 0$ the eigenvalue equation reduces to the variational equation (2.2) for the steady profile, with $C_1 = 0$ and $C_2 = B_{11,11}U_{11}'(0)$ imposed by the homogeneous boundary conditions at $x = 0, 1$. Noting that $d\Psi(C_2)$ is equal to the value of the solution $U_{11}(1)$ of (2.2) with matrix-valued data $\tilde{C}_2 = C_2 = \text{Id}_r$, whereas $D(0)$ is equal to the determinant of the solution with matrix-valued data $U_{11}'(0) = B_{11,11}\text{Id}_r$, we have $D(0) = \det B_{11,11} \det d\Psi(C_2)$, giving the result by $\det B_{11,11} > 0$. 

2.3 Symmetrizable systems

The spectral condition (2.1) is satisfied for many physical systems, in particular ones that are symmetrizable in the sense of [KSS88]. We first have the following technical lemma.

**Lemma 2.5.** If $\tilde{A}$ is symmetric and $\tilde{B} + \tilde{B}^t > 0$, then $\sigma(\tilde{B}^{-1}\tilde{A}) \cap i\mathbb{R} \subset \{0\}$.

**Proof.** If $\tilde{B}v = i\tau \tilde{A}v$ for $\tau \neq 0$, we get

$$2i\tau \langle v, \tilde{A}v \rangle = \langle v, (\tilde{B} + \tilde{B}^t)v \rangle + \langle v, (\tilde{B} - \tilde{B}^t)v \rangle.$$

and since $\tilde{B} + \tilde{B}^t > 0$, $v = 0$. 

We recall that (1.1) is said to be symmetrizable if:

- There exists a smooth map $S : U \in \mathbb{R}^n \mapsto S(U)$ such that, for any $U \in \mathbb{R}^n$, $S(U)A^0(U)$ is symmetric positive definite, $S(U) = \begin{pmatrix} S_{11}(U) & 0 \\ 0 & S_{22}(U) \end{pmatrix}$, $S(U)A(U)$ is symmetric and $S_{22}(U)B_{22}(U) + (S_{22}(U)B_{22}(U))^t > 0$.

Then we have the following useful Lemma.
Lemma 2.6. Under assumption (H3), \( \sigma(B^{-1}_{22}(A_{22} - A_{21}A_{11}^{-1}A_{12})) \cap i\mathbb{R} \subset \{0\} \).

Proof. We note first that by assumption \( S_{22}A_{21} = (S_{11}A_{12})^t \). Then we write
\[
B^{-1}_{22}(A_{22} - A_{21}A_{11}^{-1}A_{12}) = (S_{22}B_{22})^{-1}(S_{22}A_{22} - (S_{11}A_{12})^t(S_{11}A_{11})^{-1}S_{11}A_{12})
\]
and the result follows from the previous lemma.

\[ \square \]

Theorem 2.7. For symmetrizable systems satisfying Conditions (H0)-(H3), almost-constant solutions of almost constant data exist and are locally unique, nondegenerate, and spectrally stable.

Proof. The existence, local uniqueness, and nondegeneracy are immediate consequences of Proposition 2.2 and Lemma 2.6. Furthermore, one can easily adapt [MZ19, Prop. 3.2] and prove that the spectrum of the linearized operator about a steady state \( \hat{U} \) only contains eigenvalues. We then consider the eigenvalue problem
\[
\lambda A^0(\hat{U})V + (A(\hat{U})V)_x = \left(B(\hat{U})V_x + dB(\hat{U})V \hat{U}_x\right)_x , \quad V(0) = 0 , \quad V_{II}(1) = 0.
\]
If \( \hat{U} \equiv U_0 \), one can check that
\[
\Re(\lambda) \left(S(U_0)A^0(U_0)V, V\right)_{L^2(0,1)} + (S_{22}(U_0)B_{22}(U_0)V_{IIx}, V_{IIx})_{L^2(0,1)} + \frac{1}{2} |\sqrt{S_{11}(U_0)A_{11}(U_0)V_{I}}(1)|^2 = 0.
\]
Noting, since \( A_{11}(U) \) is symmetric for the inner product associated to \( S_{11}(U) \), that Conditions (H2)-(H3) give \( S_{11}(U_0)A_{11}(U_0) > 0 \), we thus have that \( U_0 \) is spectrally stable.

Similarly, for almost-constant steady states (meaning \( \hat{U}_x \) small enough), using an appropriate Goodman-type estimate and Poincaré inequality, as e.g. in the proof of [MZ19, Prop. 3.4]), we easily get
\[
\Re(\lambda) \left(\varphi A^0(\hat{U})S(\hat{U})V, V\right)_{L^2(0,1)} \leq -\alpha (V, V)_{L^2(0,1)}
\]
for some \( \alpha > 0 \) and \( \varphi > 0 \) such that \( \varphi S(\hat{U})A(\hat{U}) + \varphi \left((S(\hat{U}))x A(\hat{U}) - S(\hat{U})(A(\hat{U})))x\right) < 0 \). Thus, we may again conclude that \( \hat{U} \) is spectrally stable. See [MZ19, Section 3] for similar computations in the isentropic case. Alternatively, one could conclude by continuous dependence on coefficients of solutions of the eigenvalue ODE, as in [Zum10, GMWZ09].

\[ \square \]

Remark 2.8. Note that, contrary to the whole line situation (see for instance [Kaw83, KS88, Zum04]), we do not assume a Kawashima’s genuine coupling condition in Theorem 2.7. The main reason behind this is that a steady state \( \hat{U} \) of a purely hyperbolic system \( \hat{B} \equiv 0 \) on an interval under assumptions (H0), (H2), (H3) is stable, as in this case \( F = F_I \) and \( F_I(U_0) = F_I(\hat{U}) \) enforces \( \hat{U} \) constant, and also all characteristics move from left to right, sweeping perturbations out of the domain, with full Dirichlet conditions at the left boundary. Even better, any solution of Problem 1.1-1.2 initially close enough to \( \hat{U} \) is equal to \( \hat{U} \) after a finite time. Thus, the hyperbolic part of the system need not be coupled to the parabolic part in order to achieve stability, but is stable even by itself.
Remark 2.9. Using the same kind of energy estimates, one can also prove the nonlinear stability of an almost-constant steady state. See [MZ19, Section 6] for similar considerations in the isentropic gas dynamic case.

Combining the argument of 2.1 with the results of Theorem 2.7, we may deduce not only nonvanishing of \(D(0)/\mu, d\Phi\) but useful sign information, included here for definiteness.

Proposition 2.10. For arbitrary solutions of symmmetrizable systems,

\[
\mu = \text{sgn}D(0) = \text{sgn} \det d\Phi(C_2).
\]

For constant solutions of symmmetrizable systems, corresponding to \(C_2 = C_2^*\),

\[
\mu = \text{sgn} \det d\Phi(C_2^*) = +1.
\]

Proof. From the argument of Proposition 2.1, we find that the profile map \(\Phi : C_2 \to U_{II}(1)\) satisfies \(\det d\Phi(C_2^*) = \det \int_0^1 e^{\tilde{A}(1-s)} ds \neq 0\) at the value \(C_2^*\) corresponding to a constant solution, where \(\tilde{A} = B_{22}^{-1}(A_{22} - A_{21}A_{11}^{-1}A_{12})\), and thus is invariant under homotopy within the class of symmetrizable systems, hence may take \(B_{22}\) without loss of generality to be a multiple of the identity, hence, by symmetrizability, \(\tilde{A}\) has real semisimple eigenvalues \(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_{n-r}\). Diagonalizing, we have by direct computation that \(\text{sgn} \det d\Phi(C_2^*) = \text{sgn} \det \int_0^1 e^{\tilde{\alpha}_j(1-s)} ds = \Pi_{j=1}^{n-r} \text{sgn} \int_0^1 e^{\alpha_j(1-s)} ds = 1\), hence \(\text{sgn} \det d\Phi(C_2^*) = \text{sgn} D(0) = +1\).

Using spectral stability, \(D(\lambda) \neq 0\) together with real-valuedness of the Evans function when restricted to the real axis, we may extend by a further homotopy in \(\lambda\) to conclude that the limit of \(\text{sgn} D(\lambda)\) as \(\lambda \to \infty\), real, is also +1. Summarizing, we have \(2.12\). By standard arguments [SZ01, Zum10], either WKB-type or using energy estimates, one has for general solutions of symmetrizable systems that \(D(\lambda) \neq 0\) for \(\lambda\) real and large, hence \(\lim_{\lambda \to +\infty} = +1\), giving \(2.11\), by homotopy in \(C_2\).

2.4 Systems with convex entropy

A system \((1.1)\) is said to have a convex entropy [Kaw83, KS88] if it has an entropy/entropy flux pair \((\eta, q)(f^0) : \mathbb{R}^n \to \mathbb{R}^2\) such that

\[
\frac{d^2 \eta}{(df^0)^2} > 0, \quad (d\eta/df^0)(df/df^0) = dq/df^0,
\]

and

\[
(df^0)^t \frac{d^2 \eta}{(df^0)^2} B + \left( (df^0)^t \frac{d^2 \eta}{(df^0)^2} B \right)^t \geq 0,
\]

with equality only on \(\ker B\). It is a theorem of [KS88] that existence of a convex entropy implies symmetrizability, i.e., reducibility by coordinate change to a system satisfying \((H3)\). Thus, we may deduce local uniqueness information for systems with a convex entropy already by reference to Theorem 2.7.
2.4.1 Global uniqueness for constant data

Arguing directly, we may obtain under a mild additional assumption, much more. Namely, assume as holds for most physical systems that

\[(2.15) \quad (df_I)_I \text{ symmetric (hence positive definite).} \]

For gas dynamics, magnetohydrodynamics, and elasticity, \((df_I)_I = uId_r\) and so \((2.15)\) is trivially satisfied. By \(\langle h, f_I(W + h, U_{II}) - f_I(W) \rangle = \langle h, \left( \int (df_I)(W + \theta h)d\theta \right)h \rangle > 0\), this yields the global solvability property

\[(2.16) \quad \text{For fixed } U_{II}, (df_I)(\cdot, U_{II}) \text{ is (globally) one-to-one.} \]

Alternatively, we may take \((df_I) + (df_I)_t > 0\), or just impose \((2.16)\) without reference to \((2.15)\). Then, we have the following global uniqueness result, for constant data only.

**Theorem 2.11.** For systems \((1.1)-(1.2)\) with a global convex entropy and satisfying \((H0)-(H3)\) and \((2.16)\), solutions of \((2.3)\) for constant data \(U_{0II} = U_{1II}\) are globally unique, nondegenerate (full rank), and spectrally stable, consisting exclusively of constant states.

**Proof.** Following [Lax73], we obtain multiplying \((1.1)\) by \(d\eta/df\) and using \((2.13)\)(ii) the equation

\[ \eta_t + q_x = (d\eta/df^0)(BU_x)_x = ((d\eta/df^0)BU_x)_x - (d\eta/df^0)B(U_x) = \langle df_0U_x, \frac{d^2\eta}{(df^0)^2}U_x \rangle. \]

By \((2.14)\), we have \(\langle df_0U_x, \frac{d^2\eta}{(df^0)^2}BU_x \rangle \geq 0\) with equality if and only if \(BU_x = 0\). Thus, integrating the steady equation from \(x = 0\) to \(x = 1\), we obtain

\[(2.17) \quad \langle q(U) - d\eta(U)B(U)U' \rangle_0^1 \leq 0, \]

with equality if and only if \(BU' \equiv 0\) or equivalently \(U'_{II} \equiv 0\).

On the other hand, integrating the \(U_I\) equation, we have \(f_I(U) \equiv \text{constant, whence, by } (H2)\) and \((2.16)\), \(U_I(0) = U_I(1)\), and so \(U(0) = U(1)\). Thus, \(q(U)|_0^1\) vanishes in \((2.17)\). At the same time, by addition of an arbitrary linear function, we may take \(\eta\) without loss of generality to satisfy \(d\eta(U(0)) = d\eta(U(1)) = 0\), whence the entire lefthand side of \((2.17)\) vanishes. Then, equality holds in \((2.17)\) and so we must have \(U'_{II} \equiv 0\) and therefore \(U_{II} \equiv U_{II}(0)\) and \(f_I \equiv f_I(U(0))\). Applying \((H2)\) and the implicit function theorem, we find that we may solve for a unique value of \(U_I\) in a neighborhood of \(U_I(0)\) as a function of the constants \(U_{II}\) and \(f_I(U)\). Since \(U_I(x)\) varies continuously starting at \(U_I(0)\), it can thus never escape this local neighborhood, and so \(U_I(x) \equiv U_I(0)\) as well. This gives global uniqueness of \(U \equiv U(0)\). Nondegeneracy and spectral stability follow by Theorem 2.7.

**Remark 2.12.** The assumption \((2.15)\) seems possibly related to the circle of ideas around entropy and symmetrizability. It would be very interesting to identify sharp criteria for \((2.15)\) assuming existence of a complex entropy; however, we have not found such.
2.4.2 Global existence for general data

Following [BBZ21], define the feasible set $C$ as the connected component of the set of parameters $C_2$ corresponding to constant solutions of the open set of $C_2$ for which the solution of (2.4) is defined on $[0, 1]$ and remains in the interior of its domain of definition $U$. If $\Phi$ is “proper” in the sense that the inverse image in $U$ of a compact set in $C$ is compact, then we may conclude general existence from the special uniqueness result of Theorem 2.11.

**Corollary 2.13.** For systems possessing a convex entropy for which $\Phi$ is proper in the above sense with respect to $U, C$, there exists a steady profile solution for any data $U(0), U_2(1)$.

**Proof.** The map $\Phi$ is continuous on $C$, by continuous dependence of solutions of ODE, for $\Phi$ proper in the above sense, one may thus define the Brouwer degree $d(\Phi, C, U_{II}(1))$ for each target image $U_{II}(1)$, considering left data $U(0)$ as a fixed parameter [Bro11, Mil65, Hir94, Pra06, DM21]. Recall that Brouwer degree of a proper map is invariant under homotopy, and, for regular values $U_{II}(1)$, for which $\Phi^{-1}$ consists of a finite set of isolated nondegenerate roots, is given by

\[
d(\Phi, C, U_{II}(1)) := \sum_{C_2 \in \Phi^{-1}(U_{II}(1))} \text{sgn det } d\Phi(C_2).
\]

This includes the case $\Phi^{-1}(U_{II}(1)) = \emptyset$, for which $d(\Phi, C, U_{II}(1)) = 0$, hence nonzero Brouwer degree implies existence of a solution.

By homotopy invariance, the degree may be computed at any such $U_{II}(1)$, in particular the value $U_{II}^*(1) = U_{II}(0)$ to which Theorem 2.11 applies. For this value, the inverse image of $\Phi$ consists of a single $C_2^*$ corresponding to a constant solution. Moreover, by Proposition 2.10, $C_2^*$ is an isolated nondegenerate root, with $\text{sgn det } d\Phi(C_2^*) = +1$, hence by (2.18) $d(\Phi, C, U_{II}^*(1)) = +1$. By homotopy invariance, therefore, $d(\Phi, C, U_{II}(1)) = +1$ for any $U_{II}(1) \in U_{int}$. implying existence of a solution by nonvanishing of the degree.

**Remark 2.14.** The condition that $\Phi$ be proper is quite strict in practice, as solutions may escape to infinity, etc. It was verified in [BBZ21] for the compressible Navier–Stokes equations with polytropic equation of state. It is a very interesting open question whether it holds for a general convex equation of state, or for more complicated physical systems such as viscoelasticity and magnetohydrodynamics (MHD).

**Remark 2.15.** It is worth noting that under solvability assumption (2.16), the shock tube problem reduces to a nonlinear elliptic problem in $U_{II}$, with Dirichlet boundary conditions, to which all of the theory of global existence and uniqueness for such may be applied. However, up to now we have not been able to make use of this connection; rather our studies of specific systems seem to be an interesting source of examples for elliptic theory.

2.4.3 Uniqueness and spectral stability

For large-amplitude solutions, neither uniqueness nor spectral stability appear to follow from the existence of a convex entropy. Indeed, numerical computations of [BBZ21] indicate that
for the common example of the compressible Navier–Stokes equation with an artificially chosen convex equation of state—hence possessing a convex entropy—both failure of local uniqueness, and failure of spectral stability may occur, with steady and Hopf bifurcations.

2.5 Conclusions

For small-amplitude data, local existence, uniqueness, and spectral stability may be deduced from the standard structural assumptions of symmetrizability. For systems with convex entropy, we obtain, further, from global uniqueness of solutions of constant data. Provided the mapping \( \Phi \) may to shown to be proper (in the sense described above), we obtain as a corollary large-amplitude existence; however, this appears possibly quite restrictive. The (numerical) gas-dynamical examples of [BBZ21] suggest that large-amplitude uniqueness and spectral stability in general do not hold for systems possessing a convex entropy. Thus, these questions must apparently be addressed either by problem-specific analysis, asymptotic limit, or numerical Evans function evaluation as in [MZ19, BBZ21].

3 Asymptotic limits

3.1 Small-viscosity/large interval asymptotics

In either the vanishing-viscosity limit, or the large-interval limit \([0, X]\), \( X \to +\infty \) after rescaling back to the unit interval \([0,1]\), we are led to consider in place of (1.1)

\[
(3.1) \quad \partial_t f^0(U) + f(U)_x = \varepsilon (B(U)U_x)_x, \quad 0 < x < 1,
\]

with \( \varepsilon = \frac{1}{X} \), \( \varepsilon \to 0^+ \), and the steady profile equation is

\[
(3.2) \quad f(U)' = \varepsilon (B(U)U')',
\]

Formally setting \( \varepsilon = 0 \) in (3.2), we obtain \( f(U) \equiv \text{constant} \), or \( U \equiv \text{constant} \) on smooth portions, separated by standing shock and boundary layers. This indicates a rich “zoo” of possible steady solution structures. Some examples from the isentropic gas dynamics case are displayed in Figure 1.

**Hyperbolic structure.** It is readily deduced that the limiting configurations depicted in Figure 1 are in fact the only possibilities for the isentropic case. For, \( \rho u \equiv \text{constant} \) imposes \( \rho, u > 0 \) throughout the limiting pattern, whence all states have either one or two positive characteristics \( \alpha = u \pm c \), where \( c \) is sound speed. This in turn implies, by general results of [MZ03] that, as rest points of the scalar steady profile ODE, they are attractors or repellors, respectively. A nontrivial nondegenerate boundary layer at the left endpoint \( x = 0 \) must terminate at \( x = 0^+ \) at a rest point, which must therefore be an attractor; at the right endpoint, \( x = 1^- \) a repellor. Interior shocks must connect a repellor on the left with a saddle on the right. It follows that nontrivial boundary layers and interior shocks cannot coexist, but occur only separately. Moreover, there can occur at most one nondegenerate boundary layer, either at the left or the right endpoint. The final possibility completing our
zoo of possible configurations is a double boundary layer configuration, for which the end point must necessarily be degenerate, corresponding to a “sonic”, or “characteristic” point where $u = c$. Shocks or boundary layers connecting to a nondegenerate rest point decay exponentially; those connecting to a degenerate rest point decay algebraically. For further discussion of boundary layer structure for the compressible Navier-Stokes equations, see, e.g., [SZ01, GMWZ05].

For the nonisentropic case, we again have $\rho, u > 0$, imposing in this instance that state have either two or three positive characteristics $\alpha = u - c, u, u + c$; as rest points of the steady profile equation, these correspond to saddle points or repellors, respectively. Similar analysis to the above yields again that left boundary layers and shocks cannot coincide; however, there is the new possibility of patterns consisting of a left boundary layer plus a right boundary layer, or an interior shock plus a right boundary layer, as nontrivial right boundary layers may connect to either repellors or saddles in the nonisentropic case.

**Rigorous asymptotics.** A very interesting open problem would be to carry out the zero-viscosity limit rigorously, in preparation for the more complicated dynamics of the 2d shock tube problem. One might hope also to understand the spectra of such wave patterns as the approximate direct sum of the spectra of component layers, as would follow, for example, by the methods of [Zum10, Zum11] if the components layers remained appropriately spatially separated in the limit. A first apparently nontrivial step, of interest in its own right, is to show for given boundary data existence and uniqueness of feasible limiting patterns as described in Section 3.1.
We illustrate this for the (surprisingly complicated) case of isentropic gas dynamics:

\begin{equation}
\rho_t + (pu)_x = 0, \quad (pu)_t + (pu^2 + p(\rho))_x = \nu u_{xx}; \quad p', p'' > 0.
\end{equation}

**Proposition 3.1.** For isentropic gas dynamics, there exists for each boundary data a unique limiting configuration consisting of: (i) a single boundary layer at left or right endpoint; (ii) a single interior standing shock; or (iii) a double boundary-layer connecting from each boundary to a “characteristic point” \( u = c \), with \( c = \sqrt{\rho'(\rho)} \) denoting sound speed. These determine a unique inviscid solution up to location of the interior shock. The unique viscous steady solution converges to an inviscid one with particular interior shock location as viscosity \( \nu \to 0 \) at sharp rates \( \nu^{1/p} \) in \( L^p \), \( 1 < p < \infty \) and \( \nu \log \nu \) in \( L^1 \).

**Proof.** Referring to [MZ19], we have that the profile ODE for \( (\hat{\rho}, \hat{u})(x) \) is given by profile ODE

\begin{equation}
mu \hat{\rho}' = \hat{\rho}^2 (b - \psi(\hat{\rho})), \quad \hat{u} = m/\hat{\rho},
\end{equation}

where \( \nu \) is coefficient of viscosity; \( m = \rho(0)u(0) > 0; \psi(\rho) := \frac{m^2}{\rho} + p(\rho) \) is convex, with limit \( +\infty \) as \( \rho \to 0, +\infty \); and \( b \) is a constant determined implicitly by \( \rho(1) = m/u(1) \).

Evidently, \( \psi \) is minimized at the characteristic point \( \rho_* \) where \( m^2 / \rho^2 = \rho'(\rho) \), or \( u^2 = c^2 \), whence \( u = c \) by positivity of \( u \) and \( c \). For \( b > \psi_* := \psi(\rho_*) \), there exist nondegenerate rest points \( \rho_- < \rho_* < \rho_+, \) with \( \hat{\rho}' > 0 \) for \( \rho_- < \hat{\rho} < \rho_+ \) and \( \hat{\rho}' < 0 \) for \( \hat{\rho} < \rho_- \) and \( \hat{\rho} > \rho_+ \). For \( b = \psi_* \), there is a single degenerate rest point at \( \hat{\rho} = \rho_* \), with \( \hat{\rho}' < 0 \) for all \( \hat{\rho} \neq \rho_* \). Checking \( \psi''(\rho_*) = -2m^2/\rho_*^3 - \rho''(\rho_*) < 0 \), hence to lowest order the flow near \( \rho_* \) is of Riccati type, \( \hat{\rho}' \sim -\nu^{-1} \rho^2 \). For \( b < \psi_* \), there are no rest points and \( \hat{\rho}' \) is uniformly negative. See Figure 2 for a typical phase portrait.

**(case \( \rho(0) \geq \rho(1) \))** It follows that for any \( \rho(0) > \rho(1) > \rho_* \), there exists a unique decreasing right viscous boundary layer with value \( \rho(1) \) at \( x = 1 \) and converging at exponential rate \( e^{-\nu|1|^/\nu} \) as \( x \to -\infty \) to the lefthand state \( \rho(0) \). Similarly, for any \( \rho_* > \rho(0) > \rho(1) \), there is a unique decreasing left viscous boundary layer with value \( \rho(0) \) at \( x = 0 \) and converging as \( e^{-\nu|0|^/\nu} \) as \( x \to +\infty \) to \( \rho(1) \). For \( \rho(0) \geq \rho_* \geq \rho(1) \), there are unique decreasing characteristic right and left viscous boundary layers connecting \( \rho(1) \) and \( \rho(0) \) to the characteristic point \( \rho_* \), with convergence at algebraic rate \( |x|/\nu \)^{-1}.

**(case \( \rho(0) \leq \rho(1) \))** Similarly, for \( \rho(0) < \rho(1) < \rho_* \), there is a unique increasing and exponentially converging right viscous boundary layer from \( \rho(1) < \phi_+(b) \) to \( \rho(0) = \rho_-(b) \) for some \( b > \psi_* \), while for \( \rho_* < \rho(0) < \rho(1) \), there is a unique increasing and exponentially converging left viscous boundary layer from \( \rho(0) > \phi_-(b) \) to \( \rho(1) = \rho_+(b) \) for some \( b > \psi_* \). For \( \rho(0) < \rho_* < \rho(1) \), finally, there exists either a unique increasing and exponentially converging right viscous boundary layer from \( \rho(1) < \phi_+(b) \) to \( \rho(0) = \rho_-(b) \) for some \( b > \psi_* \), a unique increasing and exponentially converging left viscous boundary layer from \( \rho(0) > \rho_-(b) \) to \( \rho(1) = \rho_+(b) \) for some \( b > \psi_* \), or an exponentially converging stationary interior shock from \( \rho(0) = \rho_-(b) \) to \( \rho(1) = \rho_+(b) \) for some \( b > \psi_* \).

Note, at the hyperbolic (inviscid, \( \nu = 0 \)) level, that left and right characteristic boundary layers may be concatenated to form a “double boundary layer” with intermediate characteristic state \( \rho_* \), connecting \( \rho(0) > \rho_* \) to \( \rho(1) < \rho_* \). In all other cases, the data \( \rho(0) \) and \( \rho(1) \)
may be connected by a single element consisting of a left or right, increasing or decreasing boundary layer, or, in the limiting case, a single stationary interior shock.

Combining these elements, we obtain a unique feasible inviscid limit, up to arbitrary positioning of the interior standing shock. A refined analysis taking into account exponential decay rates $e^{-r_0|x|/\nu}$ and $e^{-r_1|x|/\nu}$ of the viscous shock as $x \to -\infty$ and $x \to +\infty$, respectively, combined with the linearization $[(\bar{u}^2 - \bar{c}^2)\Delta \rho] = 0$ of Rankine-Hugoniot relation $[m^2/\rho + p(\rho)] = 0$, where $[\cdot]$ denotes jump across the shock and $\Delta \rho$ denotes change in $\rho$ due to truncation of the (infinite-extent) viscous shock at boundaries $x = 0, 1$, we obtain asymptotics $r_0x_s = r_1(1 - x_s) + O(\nu)$, or

$$ x_s(\nu) = \frac{r_1}{r_0 + r_1} + O(\nu) $$

as $\nu \to 0$ toward the limiting shock location $x_s(0) = r_0/(r_0 + r_1)$. We omit the details.

Putting these pieces back together, we obtain convergence of viscous to inviscid solution in $L^1$ at rate $O(\nu)$ for solutions containing shocks and noncharacteristic boundary layers. The analysis of double layer solutions formed by concatenation of characteristic left and right boundary layers is somewhat more involved. Noting that $\psi \sim (\rho - \rho_*)^2$ near $\rho_*$ in the boundary layer profile ODE, and splitting into regions where $(\rho - \rho_*)^2 \lesssim b$ and $(\rho - \rho_*)^2 \gtrsim b$, we find after a brief calculation that $|\rho - \rho_*|$ varies from 0 to $\sqrt{b} \sim \nu$ on an $x$-interval of order-one length and afterward grows like $\nu/x$, giving a total $L^1$ convergence error $\sim \nu + \int_0^1 (\nu/x)dx \sim \nu \log \nu$. Other $L^p$ norms, $1 < p < \infty$, go similarly.

**Hyperbolic dynamics.** A very interesting question is the hyperbolic description of dynamics in the inviscid limit, in particular whether one may extend the formal description of Gisclon–Serre [GS94, Gis96, GMWZ09] for the half-line $[0, +\infty)$, based on viscous profiles,
3 ASYMPTOTIC LIMITS

to the case of a bounded interval. In this approach, one imposes on the interior of the set the usual hyperbolic description, with jump conditions given by the requirement that there exist a corresponding viscous shock profile connecting the endstates of the jump. Analogously, one requires at the boundary that solutions lie in the “reachable set” of limiting states as $x \to +\infty$ of a viscous boundary layer profile satisfying the viscous boundary conditions at $x = 0$. For noncharacteristic boundary layers, this description may be rigorously validated under very general circumstances \cite{Gis96, GMWZ09}, both in the sense that it provides a well-posed hyperbolic problem, and that the solution of this problem is the limit of viscous solutions.

In the present, bounded interval case, the appearance of characteristic boundary layers complicates matters. For illustration, considering again the case of isentropic gas dynamics \cite[(3.3)]{3.3} with viscous inflow/outflow boundary conditions specifying

\begin{equation}
(\rho(0), u(0), u(1)) = (\rho_0, u_0, u_1); \quad \rho_0, u_0, u_1 > 0.
\end{equation}

Then, the classification of boundary layers in the proof of Proposition \cite[(3.1)]{3.1} shows that the left inviscid boundary conditions induced by “reachability” criterion of Gisclon-Serre are given for “subsonic” states $u_0 \leq \sqrt{p'(\rho_0)}$ by the full Dirichlet conditions

\begin{equation}
(\rho(0), u(0)) = (\rho_0, u_0),
\end{equation}

but for “supersonic” states $u_0 \geq \sqrt{p'(\rho_0)}$ by the “one-sided” criteria

\begin{equation}
\rho(0)u(0) = m_0 := \rho_0u_0, \quad u(0) \geq \sqrt{p'(\rho_0)}.
\end{equation}

As supersonic characteristics travel to the right, scenario \cite[(3.8)]{3.8} is not expected to occur, except possibly at time $t = 0$.

Meanwhile, for subsonic states $u(1) \leq \sqrt{p'(\rho(0))}$ (which serve only as repellors in backward $x$ flow, so admit only trivial, constant boundary-layer connections), the induced right inviscid boundary conditions are given by the single condition

\begin{equation}
(3.9) \quad u(1) = u_1,
\end{equation}
in agreement with the number of incoming (hyperbolic) characteristic modes to the domain. For supersonic states $u(1) \geq \sqrt{p'(\rho_1)}$, for which all characteristic modes leave the domain, the state $(\rho(1), u(1))$ is an attractor in backward $x$ for the boundary layer profile ODE with appropriate constant $b$, connecting to all states $\rho \leq \rho(1)^\dagger$, where $(\rho(1)^\dagger, u(1)^\dagger)$ denotes the conjugate point connected to $(\rho(1), u(1))$ by a standing viscous shock profile, satisfying $\rho(1)^\dagger > \rho(1), u(1)^\dagger < u(1)$. Thus, there are no imposed right boundary conditions, in agreement with the number (zero) of incoming modes, other than the “range” condition

\begin{equation}
(3.10) \quad \rho(1)u(1) = m_1 := \rho_1u_1, \quad u(1)^\dagger \leq u_1
\end{equation}
on the set of allowable $u(1)$. This is always satisfied for $u(1) \leq u_1$, but for $u(1) > u_1$ imposes a strictly stronger upper bound on $\rho(1)$ than subcharacteristicity.
As all conditions are open ones, for solutions lying near a noncharacteristic steady solution, the operant boundary conditions are of the standard type considered in hyperbolic initial boundary value theory, hence in this local setting the formal description of Gisclon-Serre may indeed be extended sensibly to the case of a bounded interval, as follows also by the original observations of [GS94, Gis96, GMWZ09]. However, conditions (3.8) and (3.10), when in effect, are not consistent with standard hyperbolic flow, but rather a new “obstacle” type boundary condition. Thus, both near the characteristic double layer type solutions, for which (3.8) may enter, or in the global, large-amplitude setting where (3.10) comes into play, it seems a very interesting open problem whether such a boundary condition can determine a reasonable hyperbolic flow. We emphasize for the case of double layer solutions that even small perturbations of the background steady state lead to consideration of nonstandard boundary condition (3.8). This might be called “subcharacteristic type,” bridging as it does between sub- and supercharacteristic boundary conditions, and seems one of the more interesting aspects of the investigations of this section.

More generally, the questions of global existence for large-amplitude data for the viscous problem on the interval, and convergence to a vanishing viscosity limit seem very interesting, especially in the isentropic case where the entropy-based method of compensated compactness has so been successful on the whole-line problem [CP10, DiP83, Ser86]. One may ask in particular whether or not boundary entropy conditions as in [DF88] might agree with the standard and nonstandard boundary conditions derived by viscous profile considerations above.

### 3.2 The standing shock limit

A simple case in which the zero-viscosity limit can be completely carried out is that of the “standing-shock limit” generalizing the study of [Zum10] in the case of the half-line. This consists of the study of a stationary viscous n-shock \( \tilde{U}^\varepsilon(x) = \tilde{U}\left(\frac{1}{2}(x \pm \frac{1}{2})\right) \) of (3.1), solving (3.2) for all \( \varepsilon > 0 \), with respect to its “own” boundary conditions, i.e.

\[
U_0 = \tilde{U}^\varepsilon(0), \quad U_1^{II} = \tilde{U}^\varepsilon(1).
\]

We consider this for the general class of system (1.1),(1.2) under assumptions (H0)-(H3) plus the additional assumption used in [Zum10]:

- (H4) the eigenvalues of \( df(U_0) \) and \( df(U_1) \) are nonzero.

Converting by \( x \to \frac{x}{\varepsilon} \) to the large-interval limit and following the arguments of [Zum10] word for word, we find that, away from \( \lambda = 0 \), the spectra of the linearized operator about \( \tilde{U}^X(x) := \tilde{U}(\varepsilon x) \) on \( \Re \lambda \geq 0 \) approaches, as \( \varepsilon = \frac{1}{X} \to 0^+ \), the direct sum of the spectra of the viscous shock \( \tilde{U} \) as a solution on the whole line plus the spectra of the constant boundary layers on the half-lines \((0, +\infty)\) and \((-\infty, 1)\) determined by the values of \( \tilde{U}^\varepsilon \) at

---

1By (H3) the eigenvalues of \( df(U) \) are real and semi-simple since \( df(U) \) is symmetric for the inner product associated to \( S(U) \).
0 and 1 with the boundary conditions for the steady problem at $x = 0$ and $x = 1$. As the latter constant layers have been shown to be spectrally stable [GMWZ05], this implies that the spectra of $\hat{U}^X$ converges away from $\lambda = 0$ to that of $\hat{U}$ as $X \to \infty$. Rescaling, we find that, outside $B(0, c\varepsilon^{-1})$, any $c > 0$, the spectra of $\hat{U}^\varepsilon$ are well-approximately by $\varepsilon^{-1}$ times the spectra of $\hat{U}$. We record this observation as the following proposition.

**Proposition 3.2** (Spectral decomposition). For viscous $n$-shock solutions $\hat{U}$ of systems (1.1) satisfying (H0)–(H4), the corresponding standing-shock family $\bar{U}^\varepsilon$ contains no spectra $\Re \lambda \geq 0$ outside a ball $B(0, c\varepsilon^{-1})$ for $\varepsilon > 0$ sufficiently small, for any choice of $c > 0$, if and only if $\bar{U}$ is spectrally stable, i.e., has no spectra $\Re \lambda \geq 0$ with $\lambda \neq 0$. In particular, if $\bar{U}$ is spectrally unstable, then $\hat{U}^\varepsilon$ is spectrally unstable for $\varepsilon$ sufficiently small.

Proposition 3.2 gives no information about the corresponding stability index and uniqueness or nonuniqueness. However, this is provided definitively by the following result.

**Proposition 3.3** (Nonvanishing of the stability index). For viscous $n$-shock solutions $\bar{U}$ of systems (1.1) satisfying (H0)–(H4), the Evans function $D^\varepsilon$ associated with the corresponding standing-shock family $\bar{U}^\varepsilon$ satisfies $D^\varepsilon(0) \neq 0$ for $\varepsilon > 0$ sufficiently small.

**Proof.** We only sketch the proof, which belongs more to the circle of ideas in [Zum10] than those of the present paper. We first write the eigenvalue system in “flux” variables $(u_{II}, F)$ as

\[
\begin{align*}
U_{II}' &= B_{22}(U)^{-1}(F_{II} + A_{11}U_I), \\
F' &= \lambda A^0U,
\end{align*}
\]

where $F := B(\hat{U})U' - AU$ and $U_I = A_{11}^{-1}(F_I + A_{12}U_{II})$. This yields for $\lambda = 0$ in the second equation the simple dynamics $F \equiv \text{constant}$.

Next, we observe that the Evans function may be written equivalently as

\[
D^\varepsilon(\lambda) = \det \begin{pmatrix} U_{II}^1 & \cdots & U_{II}^r & 0 \\ F^1 & \cdots & F^r & I \end{pmatrix} \bigg|_{x=1},
\]

where $\begin{pmatrix} U_{II}^j \\ F^j \end{pmatrix}$, $j = 1, \ldots, r$ denote the solutions of (3.11) with initial conditions

\[
\begin{pmatrix} U_{II}^1 \\ F^1 \\ \vdots \\ U_{II}^r \\ F^r \end{pmatrix}(0) = \begin{pmatrix} 0 \\ B(0) \\ \vdots \\ 0 \\ I_r \end{pmatrix}
\]

at $x = 0$. In turn, we may view this as a Wronskian

\[
D^\varepsilon(\lambda) = \det \begin{pmatrix} U_{II}^1 & \cdots & U_{II}^r & U_{II}^{r+1} & \cdots & U_{II}^{n+r} \\ F^1 & \cdots & F^r & F^{r+1} & \cdots & F^{n+r} \end{pmatrix} \bigg|_{x=1},
\]
where \( \left( \frac{U^j_{r+1}}{F^j_{r+1}} \right) \), \( j = r+1, \ldots, n+r \) denote the solutions of (3.11) with initial conditions

\[
\begin{pmatrix}
U^j_{r+1} \\
F^j_{r+1}
\end{pmatrix} = \begin{pmatrix} 0 \\
I_n
\end{pmatrix}
\]

at \( x = 1 \).

By Abel’s theorem, vanishing or nonvanishing of the Wronskian (3.13) at \( x = 1 \) is determined by vanishing or nonvanishing at any \( x \in [0, 1] \). By the analysis of [Zum10], we find that, at \( x = c \) for any \( c > 0 \) sufficiently small, the solutions \( \left( \frac{U^j_{r+1}}{F^j_{r+1}} \right), j = 1, \ldots, r \) originating from \( x = 0 \) converge exponentially in \( X := 1/\varepsilon \) to the limiting subspace of solutions of (3.11) on the whole line decaying at \( x = -\infty \), which may be identified by the property \( F \equiv 0 \), hence also \( \det(U^r_{1+1}, \ldots, U^r_{n+1}) \neq 0 \). Recalling by the simple dynamics for \( \lambda = 0 \) that

\[
(F^{r+1}, \ldots, F^{r+n}) \equiv \left( B(0) \begin{pmatrix} 0 \\
I_r
\end{pmatrix} \right),
\]

we find that the Wronskian at \( x = c \) converges exponentially in \( X = \varepsilon^{-1} \) to

\[
\det(U^r_{1+1}, \ldots, U^r_{n+1})|_{x=c} \neq 0,
\]

hence \( D^\varepsilon(0) \neq 0 \) for \( \varepsilon > 0 \) sufficiently small. For further details, see [Zum10].

**Remark 3.4.** The result of Proposition 3.3, though proved by similar techniques, stands in striking contrast to the results of [Zum10, SZ01] in the half-line case, where the stability index was seen to change sign as parameters were varied for (full) polytropic gas dynamics, despite stability of the underlying whole-line shock profile. The standing-shock construction can nonetheless be useful in seeking instability/bifurcation, however, through bifurcation from unstable to stable background shock; this strategy is used successfully, e.g., in [BBZ21].

### 3.3 Large-viscosity/small interval asymptotics

It seems interesting to ask also what are the asymptotics of solutions as viscosity goes to infinity instead of zero, or, equivalently, interval length goes to zero with viscosity fixed.

#### 3.3.1 Case study: full gas dynamics

For full gas dynamics [BBZ21, Eq. (2.6)], (2.3) becomes

\[
\frac{\alpha u'}{u_0} = c_1 + u + \Gamma \frac{e}{u}, \quad \frac{\nu e'}{u_0} = c_2 - c_1 u - \frac{1}{2} u^2 + e,
\]

where \( u > 0 \) is fluid velocity, \( e > 0 \) specific internal energy, and

\[
c_1 = \frac{\alpha}{u_0} u'(0) - u_0 - \Gamma \frac{e_0}{u_0}, \quad c_2 = \frac{\nu}{u_0} e'(0) + \alpha u'(0) - e_0 - \frac{1}{2} u_0^2 - \Gamma e_0.
\]
3 ASYMPTOTIC LIMITS

(Here, we have taken without loss of generality $\rho_0, u_0 = 1$, so that by the steady mass conservation equation, $\rho u \equiv 1$, where $\rho$ is fluid density [BBZ21].)

Formally taking $\alpha, \nu$ to infinity, and dropping $O(1)$ terms, we obtain limiting equations

$$\frac{\alpha}{u_0} u' = c_1, \quad \frac{\nu}{u_0} e' = c_2 - c_1 u \quad \text{with} \quad c_1 = \frac{\alpha}{u_0} u'(0), \quad c_2 = \frac{\nu}{u_0} e'(0),$$

or

$$\bar{u}' = u'(0), \quad \frac{1}{u_0} e' = \bar{e}'(0) - \frac{\alpha}{\nu} \bar{u}'(0)\bar{u},$$

giving exact solution

$$\bar{u}(x) = u_0 + x(u_1 - u_0), \quad \bar{e}(x) = x\left(e_1 - \frac{\alpha}{\nu} \frac{u_1^2 - u_0^2}{2}\right) + \frac{\alpha}{\nu} (u_1 - u_0)(u_0 x + (u_1 - u_0)x^2/2).$$

For $\varepsilon > 0$, set $E_\varepsilon = \{(\varepsilon, e) \in \mathbb{R}^2, \varepsilon < u, e < \frac{1}{\varepsilon}\}.$

**Proposition 3.5.** For any fixed $\varepsilon > 0$ and profile (3.18) contained in $E_\varepsilon$, holding $r = \nu\alpha > 0$ fixed, and taking $\alpha, \nu$ to infinity, there is for sufficiently large $(\alpha, \nu)$ a unique steady profile of (3.15) lying in $E_\varepsilon$, converging to the formal limit (3.18) in $H^1[0,1]$ at rate $O(\alpha^{-1}).$

**Proof.** Put back into second-order form, (3.15) becomes (cf. [BBZ21 Eq. (2.5)])

$$\left(\frac{\alpha}{u_0} u'\right)' = (u + \Gamma\frac{e}{u})', \quad \left(\frac{\nu}{u_0} e' + \frac{\alpha}{u_0} uu'\right)' = (\Gamma e + e + \frac{1}{2}u^2)',$$

Writing $u = \bar{u} + w, e = \bar{e} + z,$ we have, substituting in (3.19), substacting the corresponding equation for $\bar{u}$, dividing by $\alpha,$ the elliptic perturbation equations

$$\left(\frac{1}{u_0} w'\right)' = (O(\alpha^{-1}))', \quad \left(\frac{r}{\alpha u_0} e' + \frac{1}{u_0}(\bar{u}w' + \bar{w}u')' = +(O(\alpha^{-1}))',$$

with homogeneous data $(w, z)(0) = (0, 0), (w, z)(1) = (0, 0),$ where estimate $O(\alpha^{-1})$ (by Sobolev embedding) remains valid so long as $\|w\|_{H^1[0,1]}$ is sufficiently small.

With this structure, it is straightforward to carry out a contraction mapping argument for $w$ in a sufficiently small ball $B$ in $H^1[0,1],$ considering the right-hand side as input and the solution of the block-triangular operator on the left-hand side as image, and showing by energy estimates that the resulting operator is contractive on $B$ with contraction constant $O(\alpha^{-1}) < 1,$ giving existence and uniqueness by the Banach fixed point theorem of a solution in $B,$ i.e., small in $H^1[0,1].$ Thus, the righthand side of (3.20) is $O(\alpha^{-1}),$ and, solving, we have $\|(w, z)\|_{H^1[0,1]} = O(\alpha^{-1})$ as well. Recalling that $(w, z)$ by definition is the difference between exact and limiting solutions, we are done.

**Remark 3.6.** The large-viscosity analysis for isentropic gas dynamics is similar but much simpler, yielding $\bar{u}$ as the line interpolating between $u(0)$ and $u(1).$ This is to be compared with the rich small-viscosity behavior depicted in Figure 7.
3.3.2 General case

Now, let us consider the general case, under assumption (2.16) and, possibly after a coordinate change, $B_{22}$ symmetric positive definite: this is possible for example for systems with convex entropy [KS88]. Adding a variable viscosity $\nu$ in (1.1) gives

$$\partial_t f^0(U) + f(U)_x = \nu (B(U)U_x)_x,$$

leading to the steady problem

$$(3.21) \quad \nu B_{22}(U)U_{11}' = f_{11}(U) - f_{11}(U_0) + \nu \tilde{C}_2, \quad U_{11}(0) = U_{011}, \quad f_{11}(U) = f_{11}(U_0),$$

where $\tilde{C}_2 = B_{22}(U_0)C_2 = B_{22}U_{11}'(0)$, and functions of $U$ are viewed as functions on $U_{11}$ alone, through the dependence of $U_I$ on $U_{11}$ imposed by relation (2.16).

As in the previous case, we expect that solutions will converge to the solution of the formal limiting equations obtained by multiplying by $\nu^{-1}B_{22}(\bar{U}_{11})^{-1}$ and taking $\nu \to \infty$, or

$$(3.22) \quad \bar{U}_{11}' = B_{22}(\bar{U}_{11})^{-1}\tilde{C}_2, \quad \bar{U}_{11}(0) = U_{011}, \quad f_{11}(\bar{U}) = f_{11}(U_0).$$

A first question, partially answered in the next lemma, is whether (3.22) admits a solution.

**Lemma 3.7.** Let $0 < \beta_0 < B_{22}^{-1} < \beta_1$ on a subset $\tilde{U}$ of the domain of definition $U$ of (3.21), and set $\theta = \cos^{-1}(\beta_0/\beta_1)$. If the truncated cone $\tilde{T}$ of angle $\theta$ based at $U_{011}$ and centered about the segment from $U_{011}$ to $U_{111} + (\beta_1/\beta_0)(U_{111} - U_{011})$ is contained in $\tilde{U}$, then (i) any steady solution $\tilde{U}$ valued in $\tilde{U}$ of the formal limiting equation (3.22) lies also in $\tilde{T}$, and (ii) there is at least one steady solution $\tilde{U}$ lying in $\tilde{T}$, hence valued in $U$.

**Proof.** By $\beta_0 < B_{22}^{-1} < \beta_1$, for any vector $V$ the angle $\theta$ between $V$ and $B_{22}^{-1}V$ satisfies

$$\cos \theta = \frac{(V, B_{22}^{-1}V)}{|V||B_{22}^{-1}V|} \geq \frac{\beta_0}{\beta_1},$$

with length of $B_{22}^{-1}V$ lying between $\beta_0|V|$ and $\beta_1|V|$.

Likewise, so long as $U_{11}$ remains in $\tilde{U}$, for $0 \leq x \leq 1$ the vector

$$U_{11}(x) - U_{11}(0) = \left( \int_0^x B_{22}^{-1}(U_{11}(y))dy \right) \tilde{C}_2 = B_{\text{ave}}(x\tilde{C}_2),$$

by the corresponding bounds $\beta_0 < B_{\text{ave}} < \beta_1$, lies within the same angle $\theta$ of $\tilde{C}_2$, with length between $x\beta_0|\tilde{C}_2|$ and $x\beta_1|\tilde{C}_2|$. If $U_{11}$ is a solution of (3.22), i.e., $U_{11}(0) = U_{011}$ and $U_{11}(1) = U_{122}$, then this implies that $\tilde{C}_2$ is within angle $\theta$ of $U_{111} - U_{011}$, and thus $U_{11}(x)$, lying within angle $\theta$ of $\tilde{C}_2$, lies in the truncated cone $\tilde{T}$.

On the other hand, for $\tilde{C}_2$ lying within the narrower truncated cone $\tilde{T}_2$ of angle $\theta$ around $V = U_{111} - U_{011}$, and length between $|V|/\beta_1$ and $|V|/\beta_2$, we have by the same estimates that $U_{11}$ stays within the interior of $\tilde{T}$ for $0 \leq x \leq 1$, hence $\Phi(\tilde{C}_2) = U_{11}(1) - U_{011}$ is well-defined. Varying $\tilde{C}_2$ around the boundary of $\tilde{T}_2$, we find by the same angle estimate that the degree of $\Phi$ about 0 is +1, giving existence of at least one steady profile in $\tilde{T}$.

\[\square\]
Remark 3.8. By a constant change of coordinates, we may take

$$B_{22}(U_{II}) \to \tilde{B}_{22}(U_{II}) = B_{22}(U_{0II})^{-1/2}B_{22}(U_{II})B_{22}(U_{0II})^{-1/2},$$

in particular $\tilde{B}_{22}(U_{0II}) = \text{Id}$, to improve the condition number $\beta_1/\beta_0$. For $B_{22} \equiv \text{constant, as for gas dynamics, this yields } \tilde{B}_{22} \equiv \text{Id}$, giving an exact solution $U_{II}(1) = \tilde{B}_{22}^{-1}\tilde{C}_2$. In general, we do not see that there is necessarily a solution of (3.21) lying in $\tilde{U}$, even “nice” domains arising in applications, nor that solutions of (3.21) must necessarily be unique.

Mimicking our treatment of the gas dynamical case, we may write (3.21) in elliptic form

$$(\nu B_{22}(U)U_{II}')' = (f_{II}(U))', \quad U_{II}(0) = U_{0II}, \quad f_{I}(U) = f_{I}(U_0),$$

then, defining $W = \tilde{U}_{II} - \bar{U}_{II}$ to be the difference between exact and limiting solutions, rewrite as an elliptic perturbation system

$$(B_{22}(\bar{U})W' + (dB_{22}(\bar{U})W)\bar{U}_{II}')' = ((dB_{22}(\bar{U})W)' + (O(\nu^{-1}))'),$$

with Dirichlet boundary conditions $W(0) = W(1) = 0$. If, as in the gas-dynamical case, the operator on the lefthand side is uniformly elliptic, then obtain an existence/convergence result as in Proposition 3.5 by the same sort of contraction mapping argument used there. However, we do not see in general why this should be true, except for small data $|\bar{U}'| \ll 1$.

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