Analytical model of a 3D beam dynamics in a wakefield device

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In this paper we suggest an analytical model and derive a simple formula for the beam dynamics in a wakefield structure of arbitrary cross-section. Result could be applied to the estimation of an upper limit of the projected beam size in devices such as the dechirper and the wakefield striker. Suggested formalism could be applied to the case when slices of the beam are distributed along an arbitrary 3D line.

I. INTRODUCTION

Effects of beam instabilities or beam breakup (BBU) effects resulting from parasitic wake fields limit considerably the intensity of the beam that can be transported through accelerator components. Simulation of the BBU is one of the main problems in accelerator physics. The most accurate and up-to-date approach to the problem is a direct particle tracking, which requires a lot of computational resources and leads to a demand of a quick estimate of beam dynamics and tools to crosscheck complicated simulations. Analytical models that allowed to analyze BBU was initially developed in [1–3] and recently in [4]. The most complete up to date solution was derived by Delayen in a series of publications [5–7].

A new theoretical approach that can be used for obtaining direct analytical formulas for the transverse wakefields has recently been developed [8]. As a result, an upper limit for the transverse and longitudinal wake force in a beam pipe of arbitrary cross-section has been derived. Using a very simple form of an upper limit for the wake force derived in [8], we start from an approximate equation of motion, and derive solution for the transverse dynamics of a pencil like beam (beam transverse size is much smaller then the aperture). In contrast to [5] we consider simplified expression for the transverse wake field and show that solution in our case could be generalized to a beam distributed along 3D line. In a special case when all slices of the bunch has the same initial displacement along the central axis and the beam is not twisted, our result are in complete agreement with [5].

The derived formula could be applied, for example, to the estimation of the upper limit of the projected beam size in a devices such as the dechirper [9–11] and the wakefield striker [12].

II. MODEL DESCRIPTION

We consider an electron beam traveling at a speed close to the speed of light along the axis of the wake field structure (Fig1) with the initial displacement from the center of the structure. To be consistent with the notation of the paper [8] we introduced complex transverse velocity momentum and force as

\[
V_\perp = V_x + iV_y, \quad p_\perp = p_x + ip_y, \quad F_\perp = F_x + iF_y.
\]

Equation of transverse motion could be written as

\[
\frac{\partial \gamma m_e V_\perp}{\partial t} = F_\perp,
\]

here \(\gamma\) is the relativistic gamma factor and \(m_e\) is the electron mass at rest. We assume our beam to be ultra-relativistic \(V_z \approx c\) and consequently \(z \approx ct\), hence

\[
\frac{\partial p_\perp}{\partial t} \approx c \frac{\partial p_\perp}{\partial z}.
\]

For the relativistic momentum \(p_\perp = \gamma m_e V_\perp\), we have

\[
\frac{\partial \gamma}{\partial z} V_\perp + \gamma \frac{\partial V_\perp}{\partial z} = \frac{F_\perp}{m_e c}.
\]

We assume that energy losses are small and could be neglected \(d\gamma/dz \approx 0\). In that case, we have

\[
\frac{\partial^2 \omega}{\partial z^2} = \frac{F_\perp}{\gamma m_e c^2}.
\]

Here \(\omega = x + iy\) is the complex vector of the transverse positions.

![Figure 1. Schematic diagram of the transverse motion of ultra relativistic particle beam.](image-url)
Figure 2. Schematic diagram of the conformal mapping.

The transverse Lorentz force acting on the bunch is usually casted in a framework of transverse wakefield. The transverse force is taken at the point of the bunch we have:

\[ G_\perp(\omega, \omega_0, s, s_0) \lesssim (s - s_0) \theta(s - s_0) \left[ \tilde{A} + \tilde{B} \delta \omega_0(s_0) + \tilde{C} \delta \omega(s)^* \right]. \]  

\[ G_\perp(\omega, \omega_0, s, s_0) \lesssim (s - s_0) \theta(s - s_0) \left[ A + B \omega(s) + C \omega(s)^* \right]. \]  

Consequently for the case of \( \delta \omega = \delta \omega_0 = \tilde{\omega} \) when transverse force is taken at the point of the bunch we have:

\[ G_\perp(s, s_0) \lesssim (s - s_0) \theta(s - s_0) \left[ A + B \tilde{\omega}(s) + C \tilde{\omega}(s)^* \right]. \]  

We consider a model of a pencil like bunch assuming that the transverse size of the bunch is much smaller than the vacuum chamber aperture size. In this case, transverse motion of different longitudinal slices of the bunch (slices by \( s \) coordinate see Fig.1) is mainly defined by the motion of the center of mass of the slice. So we do not take into account effects of slice shape modification, treating each slice as a point particle.

With this we approximate the particle density distribution \( \rho(x, y, s) \) as

\[ \rho(x, y, s) = \delta(x(s)) \omega(y(s)) \rho_1(s). \]  

Here \( \rho_1(s) \) is the longitudinal charge distribution and \( \omega \) is the Dirac delta function. Substitution of (7) into (6) gives

\[ F_\perp(\omega(s), s) = \int G_\perp(\omega(s), \omega_0(s), s, s_0) \rho_1(s_0) ds_0. \]  

It was shown in [8] that the upper limit of a Green’s function \( G_\perp \) for an arbitrary cross section of longitudinally homogenous vacuum channel could be written in terms of conformal mapping function \( f(\omega, \omega_0) \) that maps the cross section area on a unit disk (see Fig.2). Here \( \omega = x + iy \) is the point of test particle and \( \omega_0 = x_0 + iy_0 \) point of the source particle. Mapping is arranged in such a way that \( \omega_0 \) corresponds to the center of the unit disk. Corresponding formula in complex notation could be written according to [8] as

\[ G_\perp(\omega, \omega_0, s, s_0) \leq \frac{4eQ\theta(s - s_0)}{a^2} (s - s_0) f''(\omega, \omega_0)^* f'(\omega_0, \omega_0). \]  

Here \( Q \) is the total charge of the bunch, \( e \) is the electron charge, and \( a \) is the characteristic size of the aperture. Force is given in CGS units following notations of the [8].

For small displacements \( \delta \omega \) and \( \delta \omega_0 \) from the initial position \( \omega_0 \) and point \( \omega \) we may linearize \( G_\perp \) as

\[ G_\perp(\omega, \omega_0, s, s_0) \lesssim (s - s_0) \theta(s - s_0) \left[ \tilde{A} + \tilde{B} \delta \omega_0(s_0) + \tilde{C} \delta \omega(s)^* \right], \]  

\[ G_\perp(\omega, \omega_0, s, s_0) \lesssim (s - s_0) \theta(s - s_0) \left[ A + B \omega(s) + C \omega(s)^* \right]. \]  

Consequently for the case of \( \delta \omega = \delta \omega_0 = \tilde{\omega} \) when transverse force is taken at the point of the bunch we have:

\[ G_\perp(s, s_0) \lesssim (s - s_0) \theta(s - s_0) \left[ A + B \tilde{\omega}(s) + C \tilde{\omega}(s)^* \right]. \]  

We the substitute [15] into the equation [5] with [5] and arrive at

\[ \frac{\partial^2 \tilde{\omega}(s, z)}{\partial z^2} = [A + C \tilde{\omega}(s, z)^*] \int_0^s \rho_1(s_0)(s - s_0) ds_0 + B \int_0^s \tilde{\omega}(s_0, z) \rho_1(s_0)(s - s_0) ds_0. \]  

Here for the sake of convenience we introduced \( A = \frac{\tilde{A}}{\gamma_0 m_c c}, B = \frac{\tilde{B}}{\gamma_0 m_c c} \) and \( C = \frac{\tilde{C}}{\gamma_0 m_c c} \).

Equation (16) is the main equation of the suggested model. We note that this equation and coefficients in this equation are valid for arbitrary waveguide with arbitrary material. Complex function \( \omega \) that gives solution to this equation is essentially un upper bound for any real trajectory. Real trajectory will be bounded by the limiting trajectory as \( |\omega_r| \leq |\omega| \).

III. SOLUTION OF THE DYNAMIC PROBLEM

For the further analysis we assume that \( |\tilde{B}| >> |\tilde{C}| \) and uniform charge distribution \( \rho_1(s) = 1/l, s \in [0, l] \).

According to [16], [8] and [14] equation of motion could be reduced to:

\[ \frac{\partial^2 \tilde{\omega}(s, z)}{\partial z^2} = \frac{As^2}{2} + B \int_0^s \tilde{\omega}(s_0, z)(s - s_0) ds_0. \]  

with initial conditions:

\[ \tilde{\omega}(s, 0) = l_0(s), \quad \frac{\partial \tilde{\omega}(s, 0)}{\partial z} = v_0(s), \]
and assuming uniform charge distribution \( \rho_z(s) = 1/l \) for \( s \in [0,l] \). We define forward and inverse Laplace transformations as \[13\]

\[
\mathcal{L}[f(s)] = \int_0^\infty e^{-ps} ds
\]

and

\[
\mathcal{L}^{-1}[F(p)] = \frac{1}{2\pi i} \lim_{l \to \infty} \int_{c-iR}^{c+iR} e^{sp} F(p) dp,
\]

where \( c \) is a real number such that the line \( p = c \) in the complex plane avoid singularities of \( F(p) \). We perform Laplace transformation by \( s \) on both sides of equation \[17\]. On the left hand side we have:

\[
\mathcal{L} \left[ \frac{\partial^2 \phi(s,z)}{\partial z^2} \right] = \mathcal{L} \left[ \frac{\partial^2}{\partial z^2} \mathcal{L} \left[ \phi(s,z) \right] \right].
\]

We defined \( \Omega(p,z) \) as

\[
\Omega(p,z) := \mathcal{L} \left[ \phi(s,z) \right] .
\]

Hence, we have

\[
\frac{\partial^2 \Omega(p,z)}{\partial z^2} = B \mathcal{L} \left[ \int_0^s \phi(s_0,z)(s-s_0) ds_0 \right] .
\]

Using the convolution-multiplication theorem for the Laplace transformation \[13\]:

\[
\mathcal{L} \left[ \int_0^s g(s)f(s-t)dt \right] = \mathcal{L}[g(s)] \mathcal{L}[f(s)],
\]

we rewrite the integral on the right side of \[17\] in a Laplace image as

\[
\frac{\partial^2 \Omega(p,z)}{\partial z^2} = \frac{A}{p^3} + \frac{B}{p^2} \Omega(p,z)
\]

with the Laplace image of the integral kernel \( \mathcal{L}[s] \) given as

\[
\mathcal{L}[s] = -\frac{1}{p} s e^{-ps} \biggl|_0^\infty + \int_0^\infty \frac{e^{-ps}}{p} ds = \frac{1}{p^2}.
\]

The general and partial solutions to the above equation are given as

\[
\Omega_g(p,z) = f_1(p) e^{\frac{\sqrt{B}z}{p}} + f_2(p) e^{-\frac{\sqrt{B}z}{p}}
\]

\[
\Omega_{pl}(p,z) = -\frac{A}{Bp}
\]

where \( f_1 \) and \( f_2 \) are functions of \( p \). We define Laplace image for the initial conditions \[18\] as:

\[
L_0(p) = \mathcal{L} [l_0(s)]
\]

and

\[
V_0(p) = \mathcal{L} [v_0(s)] .
\]

Therefore, the solution in the Laplace image is given by

\[
\Omega(p,z) = -\frac{A}{Bp} + \left( \frac{A}{B} + pL_0(p) \right) \cosh \left( \frac{\sqrt{B}z}{p} \right) + \frac{pV_0(p)}{\sqrt{B}} \sinh \left( \frac{\sqrt{B}z}{p} \right).
\]

We introduce notations

\[
\Omega_l(s,z) = \mathcal{L}^{-1} \left[ \cosh \left( \frac{\sqrt{B}z}{p} \right) \right]
\]

and

\[
\Omega_{vl}(s,z) = \mathcal{L}^{-1} \left[ \sinh \left( \frac{\sqrt{B}z}{p} \right) \right].
\]

Next we focus on computing functions \( g_l(s,z) \). We note that \( \frac{1}{p} \cosh \left( \frac{\sqrt{B}z}{p} \right) \) has an essential singularity at \( p = 0 \). We use the fact that \( \cosh(x) \) is an order one entire function. As a consequence it could be represented by its Taylor series everywhere including small vicinity of the point \( x = \infty \), thus we use identity

\[
\frac{\cosh \left( \frac{\sqrt{B}z}{p} \right)}{p} = \frac{1}{p} \sum_{n=0}^\infty \frac{1}{(2n)!} \left( \frac{\sqrt{B}z}{p} \right)^{2n}.
\]

Using the definition of the inverse Laplace transformation \[20\] we write for \( g_l(s,z) \):

\[
g_l(s,z) =
\]

\[
= \frac{1}{2\pi i} \lim_{l \to \infty} \int_{c-iR}^{c+iR} e^{sp} \sum_{n=0}^\infty \frac{1}{(2n)!} \left( \frac{\sqrt{B}z}{p} \right)^{2n} dp.
\]

As far as Taylor series converges uniformly we can interchange sum and integral and arrive at

\[
g_l(s,z) =
\]

\[
= \frac{1}{2\pi i} \sum_{n=0}^\infty \frac{1}{(2n)!} \lim_{l \to \infty} \int_{c-iR}^{c+iR} e^{sp} \left( \frac{\sqrt{B}z}{p} \right)^{2n} dp.
\]

Let us now consider the integral

\[
I_1 = \lim_{l \to \infty} \int_{c-iR}^{c+iR} e^{sp} \left( \frac{\sqrt{B}z}{p} \right)^{2n} dp.
\]

We chose integration path as shown on Fig.\[3\] According to Jordan’s lemma \[13\] we have for \( \gamma_1 \) part of the contour

\[
\left| \int_{\gamma_1} e^{sp} \left( \frac{\sqrt{B}z}{p} \right)^{2n} dp \right| \leq \frac{B^n \pi}{s} z^{2n}.
\]
and for the integral along \( \gamma_2 \) and \( \gamma_3 \) paths

\[
\int_{\gamma_2, \gamma_3} e^{\sqrt{Bz} \frac{2n}{p}} dp \leq ce^{\gamma} B^{n+2n} \frac{p}{R}. \tag{39}
\]

From (38) and (39) in the limit \( R \to \infty \) we see that paths \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) do not contribute to the integral so one may write

\[
I_1 = \int e^{sp} \left[ \sqrt{Bz} \frac{2n}{p} \right] \frac{dp}{p}. \tag{40}
\]

with \( \gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4 \).

We notice that integrand has a pole of the \( 2n+1 \) order at the point \( p = 0 \). Hence, by the residue theorem (see for example [15, 16]), we have

\[
I_1 = 2\pi i \lim_{p \to 0} \frac{d^2 e^{sp}}{dp^2}. \tag{41}
\]

We combine (39) with (41) and arrive at

\[
g_l(s, z) = \sum_{n=0}^{\infty} B^n (sz) \frac{2n}{(2n+1)!}! (2n)!\tag{42}
\]

Following the same steps as above one may arrive at the equation for the \( g_v(s, z) \) in the form

\[
g_v(s, z) = \sum_{n=0}^{\infty} B^n (sz) \frac{2n}{(2n+1)!}! (2n)!\tag{43}
\]

The above series expansions may be expressed in terms of the modified Bessel function \( I_0(x), I_1(x) \) and Bessel function \( J_0(x), J_1(x) \).

We consider Taylor series (see for example [12]) for \( I_0(x), I_1(x) \)

\[
I_0(x) = \sum_{n=0}^{\infty} \frac{(x/2)^{2n}}{(n!)^2}, \tag{44}
\]

\[
I_1(x) = \sum_{n=0}^{\infty} \frac{(x/2)^{2n+1}}{n!(n+1)!}. \tag{45}
\]

and for \( J_0(x), J_1(x) \)

\[
J_0(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(x/2)^{2n}}{n!(n+1)!}, \tag{46}
\]

\[
J_1(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(x/2)^{2n+1}}{n!(n+1)!}. \tag{47}
\]

As far as this series has infinite radius of convergence series representation is exact for any \( x \). Using substitution \( x = 2B^{1/4} \sqrt{s}z \), combining (44), (45) and taking into account (42), (43) we finally arrive at

\[
g_l(s, z) = \frac{1}{2} \left[ I_0(2B^{1/4} \sqrt{s}z) + J_0(2B^{1/4} \sqrt{s}z) \right], \tag{48}
\]

\[
g_v(s, z) = \frac{\sqrt{B}}{2} \left[ I_1(2B^{1/4} \sqrt{s}z) + J_1(2B^{1/4} \sqrt{s}z) \right]. \tag{49}
\]

We notice that

\[
\mathcal{Z}^{-1}[p\mathcal{Z}[f(s)]] = f'(s) + f(0). \tag{50}
\]

Here prime denotes total derivative by \( s \).

Applying inverse Laplace transformation to (31) and using first multiplication theorem [15] for the Laplace transformation with (47), (49) and (50) we arrive at

\[
\mathcal{L}^{-1}[s\mathcal{L}[f(s)]] = A + \left( \frac{l_0(0) + A}{B} \right) g_l(s, z) + v_0(0) g_v(s, z) + \int_0^s (s - s_0) g_v(s_0, z) ds_0 + \int_0^s v_0'(s - s_0) g_v(s_0, z) ds_0 + \frac{s}{\sqrt{B}}. \tag{51}
\]

Equation (48) gives the solution to the equation (17) with initial condition (18). We would like to mention that formula (48) gives a solution to a 2D problem. Motion along each individual coordinate could be simply derived from (48) by taking real and imaginary part in equation (48): \( x(s, z) = \Re[\omega(s, z)] \) and \( y(s, z) = \Im[\omega(s, z)] \). For the 1D case (setting constant \( B \) to be a pure real number), assuming that the initial conditions does not depend on \( s \), the formula above gives exactly the same result as was derived earlier by Delayen in [5] using different method.

IV. CONCLUSION

We have rigorously derived an analytical formula for the evolution of a bunch that is distributed along a 3D line. Formalism and the model that is developed in this paper is complimentary to the previous results [8]. We have arrived at the same formula using a different method and demonstrated how this formula could be extended to a 3D case.

Final equation (48) could be used to estimate upper limit of the bunch deformation and applicable to the
structures of arbitrary cross-section shape. As a consequence of the theory developed in [8] formula also remains valid for the arbitrary slowdown material (dielectric, corrugation, linear plasma).

We would like to mention that this study is a first step to a more general analysis. In current study we neglected "quadruple" component of the transverse force \( \tilde{C} = 0 \) in formula (15).

Our next step is to consider omitted term in the equation of transverse motion and to analyze different possibilities of suppression instability—for example the self-damping using the BNS damping condition [13].

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