Abstract. It is one of the wonderful “coincidences” of the theory of finite groups that the simple group $G$ of order 25920 arises as both a symplectic group in characteristic 3 and a unitary group in characteristic 2. These two realizations of $G$ yield two $G$-covers of the moduli space $(\mathbb{P}^1)^6$ of configurations of six points on the projective line modulo $\text{PGL}_2$, via the 3- and 2-torsion of the Jacobians of the double and triple cyclic covers of $\mathbb{P}^1$ branched at those six points. Remarkably these two covers are isomorphic. This was proved over $\mathbb{C}$ by transcendental methods in [HW]. We give an algebraic proof valid over any field not of characteristic 2 or 3 that contains the cube roots of unity. We then explore the connection between this $G$-cover $S$ of $(\mathbb{P}^1)^6$ and the elliptic surface $y^2 = x^3 + \text{sextic}(t)$, whose Mordell-Weil lattice is $E_8$ with automorphisms by a central extension of $G$.

0. Introduction. The moduli spaces of the title all cover the moduli space $(\mathbb{P}^1)^6$ of unordered sextuples of distinct points on $\mathbb{P}^1$ modulo the action of $\text{PGL}_2$, or equivalently of sextic polynomials $S(t)$ without repeated roots modulo the action of $\text{GL}_2$. [A quintic counts as a sextic with a root at infinity; polynomials of lower degree would have repeated roots at infinity and are thus excluded.] To a configuration of six distinct points on $\mathbb{P}^1$ are associated two curves $C : u^2 = S(t)$ and $C' : v^3 = S(t)$ of genus 2 and 4, which are cyclic covers of $\mathbb{P}^1$ of degrees 2 and 3 branched at those six points. Two of our covers of $(\mathbb{P}^1)^6$ are obtained by adding full level-3 structure to $C$ and full level-2 structure to $C'$; that is, they are the moduli of genus-2 curves $C$ with a choice of generators for the 3-torsion subgroup of the Jacobian $J(C)$, and of curves $C'$ with a choice of generators for the 2-torsion subgroup of $J(C')$. The Weil pairing gives these torsion subgroups the structure of four-dimensional spaces over the finite fields $\mathbf{F}_3$ and $\mathbf{F}_4$, with respectively a symplectic and a unitary structure; thus the corresponding moduli spaces are normal covers of $(\mathbb{P}^1)^6$ with Galois groups $\text{PSp}_4(\mathbf{F}_3)$ and $\text{SU}_4(\mathbf{F}_2)$. It is one of the wonderful “coincidences” of the theory of simple finite groups that these two groups of order 25920 coincide; let $G$ be the finite group isomorphic with both $\text{PSp}_4(\mathbf{F}_3)$ and $\text{SU}_4(\mathbf{F}_2)$. (See page 26 of the ATLAS [C&] for this identification and for and further properties of $G$ that we shall use.) It is a remarkable fact that the two $G$-covers of $(\mathbb{P}^1)^6$ are also isomorphic. This is proved in [HW] (see also [H, Ch.5]) by transcendental methods: first replace $(\mathbb{P}^1)^6$ by the moduli space $\mathbb{P}^1_{(6)}$ of ordered sextuples of distinct points, which is an $S_6$ cover of $(\mathbb{P}^1)^6$; then regard both $G$-covers of $\mathbb{P}^1_{(6)}$ over $\mathbb{C}$ as quotients of the complex 3-ball by arithmetic groups acting freely; and prove that they are isomorphic by
computing enough invariants. Our two $G$-covers of \( (\mathbb{P}^1_6) \) are then quotients of these $G$-covers of $\mathbb{P}^1_6$ by the same action of $S_6$, are thus isomorphic as well. Of course this leaves completely mysterious the algebraic meaning of the identification between the two moduli spaces.

We obtain this identification algebraically via a third moduli space $S$: the space of sextic polynomials $S(t)$ together with all representations of $S$ as the difference between the square and the cube of polynomials of degree at most 3 and 2, which we shall call minimal representations of $S(t)$ as $y^2 - x^3$. By counting parameters we may surmise that the forgetful map from $S$ to $(\mathbb{P}^1_6)$ is a Galois cover, but it is not at all clear what the group should be.

We prove that the group is $G$ by constructing algebraic maps from $S$ to the two geometric $G$-covers of $(\mathbb{P}^1_6)$ and showing that the maps are isomorphisms. This of course yields an algebraic isomorphism between the $J(C)[3]$ and $J(C')[2]$ moduli spaces. We then give a further interpretation of $G$ as the group of automorphisms of a $\mathbb{Z}[e^{2\pi i/3}]$ lattice in $C^4$ isomorphic as a Euclidean lattice with $E_8$, which arises here as the Mordell-Weil lattice of the rational elliptic surface

$$\mathcal{E} = \mathcal{E}_S : y^2 = x^3 + S(t). \quad (1)$$

The 240 minimal vectors of this lattice correspond 2 : 1 with the 120 odd elements of order 2 in the Jacobian of $C'$; they correspond 3 : 1 with the 80 nontrivial elements of order 3 in the Jacobian of $C$; and they correspond 6 : 1 with the 40 representations of $S$ as the difference between the cube of a quadratic and the square of a cubic polynomial. Even the fact that there are always 40 such representations is far from well known, though it turns out that Clebsch [Cl] had already obtained this enumeration and, in collaboration with Jordan, also its relation with what we now call $J(C)[3]$.

After this paper was largely completed I found that the identification of the $J(C)[3]$ and $J(C')[2]$ moduli spaces via the minimal solutions of $y^2 = x^3 + S(t)$ was already obtained by van Geemen in [vG], and in essentially the same way. He was led to it via the study of theta functions. The interpretation of this result in terms of the Mordell-Weil lattice of (1), and the connections with other geometric ideas given or announced here, still appear to be new.

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1. The $\mathcal{M}_2$ and $\mathcal{M}_6^6$ pictures of $(\mathbb{P}^1_6)$ and their $G$-covers. Let $k$ be a field not of characteristic 2 or 3 containing a cube root of unity $\rho$. To six distinct points $t_1, \ldots, t_6$ on
\( \mathbb{P}^1(\bar{k}) \) permuted by \( \text{Gal}(\bar{k}/k) \) we associate two curves defined over \( k \): the genus 2 curve

\[
C : u^2 = S(t) = \epsilon \prod_{j=1}^{6} (t - t_j)
\]

up to quadratic twist \( \epsilon \in k^*/k^{*2} \), and the genus-4 superelliptic curve (called a “Picard curve” in [HW])

\[
C' : v^3 = S(t) = \varepsilon \prod_{j=1}^{6} (t - t_j)
\]

up to cubic twist \( \varepsilon \in k^*/k^{*3} \). If some \( t_j = \infty \) the corresponding factor \( t - t_j \) is replaced by 1.

Except for the twists \( \epsilon, \varepsilon \), these curves do not depend on the choice of coordinate on \( \mathbb{P}^1(k) \):
changing \( t \) to \( (at + b)/(ct + d) \) changes \( \prod_{j=1}^{6} (t - t_j) \) to

\[
\prod_{j=1}^{6} \left( \frac{at + b}{ct + d} \right)^{6} \prod_{j=1}^{6} \frac{t - t_j}{ct_j + d},
\]

in which the factor \( \left( \frac{(ad - bc)}{(ct + d)} \right)^{6} \) may be absorbed into \( u^2 \) or \( v^3 \), while the constant factor \( \prod_{j=1}^{6} (t_j + d) \) is absorbed into \( \epsilon \) or \( \varepsilon \). The reader may check that in the special cases \( t_j = \infty \) or \( ct_j + d = 0 \) the formula (4) still holds with our interpretation of a factor “\( t - \infty \)” as 1, and the factor \( ct_j + d \) replaced by \(-c\) if \( t_j = \infty \) and by \(- (at_j + b)\) if \( ct_j + d = 0 \).

It is well known that every curve of genus 2 is of the form \( C \) for some choice of \( t_j \) and \( \epsilon \), uniquely determined by the curve up to the action of \( \text{PGL}_2(k) \), and that every principally polarized abelian surface which is not a product of elliptic curves is the Jacobian of such a curve. In other words, it is known that \( \mathbb{P}^1 \) is identified with the moduli space \( \mathcal{M}_2 \) of curves of genus 2, and with the open subset of the moduli space \( \mathcal{A}_2 \) of principally polarized abelian surfaces, namely the subset parametrizing indecomposable surfaces.

It is also known, though not as widely, that the curves \( C' \) yield another pair of moduli interpretations of \( \mathbb{P}^1 \). Just as the genus-2 curves \( C \) have the hyperelliptic involution \( (t, u) \leftrightarrow (t, -u) \), the genus-4 curves have an automorphism of order 3 defined by

\[
\rho(t, v) = (t, \rho v).
\]

The fixed points of this involution are the six points above \( t = t_j \). By the Riemann-Hurwitz formula, an order-3 automorphism of any curve of genus 4 has six fixed points if and only if the quotient curve has genus 0. Now there are two kinds of genus-4 curves cyclically covering \( \mathbb{P}^1 \) with degree 3: the curves \( C' \), and curves of the form

\[
v^3 = \prod_{i=1}^{3} \frac{t - t_i}{t - t_j},
\]
The two kinds of curves are distinguished by the action of $\varrho$ on the four-dimensional space of holomorphic differentials on the curve. In both cases the fixed subspace is necessarily trivial because it is the space of holomorphic differentials on the quotient curve. But the action of $\varrho$ on $H^1(C')$ is diagonalized by the basis $(dt/v^2, t dt/v^2, t^2 dt/v^2, dt/v)$, the first three of whose vectors have eigenvalue $\varrho$ and the last has eigenvalue $\overline{\varrho}$; whereas the $\varrho$- and $\overline{\varrho}$-eigenspaces of the holomorphic differentials on (6) both have dimension 2. It follows that over an algebraically closed field the curves $C'$ are precisely those genus-4 curves with an order-3 automorphism $\varrho$ whose 1-, $\varrho$-, and $\overline{\varrho}$-eigenspaces have dimensions 0, 3, 1. When $k$ is not algebraically closed we must also impose the condition that the quotient curve have a rational point (and thus be identified with $P^1$).\[\text{[HW]}\]

Now each of $A_2$ and $A_4^s$ has natural arithmetic covers: the moduli spaces of principally polarized abelian surfaces or $\varrho$-fourfolds with additional (torsion) structure. We will thus obtain two families of natural covers of $(P^1_6^*)$. For instance, for each prime $p$ the moduli space of principally polarized abelian surfaces with full level-$p$ structure — that is, with a choice of generators for their $p$-torsion — is a Galois cover of $A_2$ with Galois group $\text{PSp}_4(F_p) = \text{Sp}_4(F_p)/\{\pm 1\}$ if the ground field contains the $p$-th roots of unity. \[\text{[Not all of } \text{GL}_4(F_p) \text{ because the Galois group must respect not only the group structure but also the Weil pairing on the } p\text{-torsion; we divide by } \{\pm 1\} \text{ to account for the quadratic twists.}]\] When $p = 2$, a full level-2 structure on $J(C)$ is just an ordering of the six Weierstrass points, because the 15 nontrivial 2-torsion points are represented by the differences between pairs of Weierstrass points; thus we obtain the cover of $(P^1_6^*)$ by the moduli space $P^1_6$ of ordered sextuples of distinct points on $P^1$ modulo $\text{PGL}_2$. Of course the Galois group of this cover is the symmetric group $S_6$, so we have recovered the identification of this group with the symplectic group $\text{Sp}_4(F_2)$. Likewise we obtain Galois covers of $A_4^s$ from the $p$-torsion points of $\varrho$-fourfolds, with the Galois group this time depending on whether $p$ is ramified, inert or split in $Q(\varrho)$. For the ramified prime 3 we obtain an intermediate cover from the 3-torsion points in the kernel of $\sqrt{-3} = \varrho - \overline{\varrho}$. These constitute a 4-dimensional space over $F_3$; the Weil pairing yields a quadratic form on this space, taking a $(\sqrt{-3})$-torsion point $P$ to the pairing of $P$ with any of the 3-torsion points $P'$ such that $P = \sqrt{-3}P'$. This quadratic form turns out to have Arf invariant 1,

\[\text{[If we worked over a field } k \text{ not containing } \varrho \text{ we would also have to require that the involution in } \text{Gal}(k(\varrho)/k) \text{ take } \varrho \text{ to } \varrho^{-1}.}\]

\[\text{[HW]}\]
so we obtain a cover of $A^6_2$ with Galois group $SO^+_4(F_3)$. Again the $(\sqrt{-3})$-torsion points on $J(C')$ are generated by the differences between the six points with $t = t_j$ (note that these are fixed by $\rho$, so in the kernel of $1 - \rho = \bar{\rho}\sqrt{-3}$), so once more we find the cover of $(P^1_6')$ by $P^1_{(6)}$, whose Galois group $S_6$ is this time identified with $SO^+_4(F_3)$. (See [C\&, p.4] for the realizations of $S_6$ as linear groups in characteristics 2 and 3.)

The next cases are $p = 3$ for $A_2$ and $p = 2$ for $A^6_2$. We already know that the former yields a cover of $A_2$ with Galois group $\text{PSp}_4(F_3) = G$. As to the latter, the even prime is inert in $\mathbb{Q}(\rho)$, so the 2-torsion points of $J(C')$ have the structure of a four-dimensional vector space over $F_2$. The Weil pairing is an alternating $F_2$-bilinear map on that space consistent with the $F_4$ structure (that is, such that $\langle P, P' \rangle = \langle \rho P, \rho P' \rangle$); this yields a natural unitary form on $J(C')[2]$, namely $v : P \mapsto \langle P, \rho P \rangle$. Thus the Galois group for the cover of $A^6_2$ by the moduli space of $\phi$-fourfolds with a full level-2 structure is $U_4(F_2)/F^*_4 = SU_4(F_2)$ is again $G$. (We divide by the center $F^*_4$ to account for the cubic twists; since the dimension 4 is coprime to $3 = \#F^*_4$, the resulting group $\text{PU}_4(F_2)$ is isomorphic with $SU_4(F_2)$. So we have again obtained, from two different curves $C, C'$ associated with a six-point configuration in $\mathbb{P}^1$ and full level structures of different levels, covers of $(P^1_6')$ with the same Galois group. But, unlike the case of the $S_6$-cover of $(P^1_6')$ by $P^1_{(6)}$, it is not immediately clear here that the two $G$-covers of $(P^1_6')$ are the same. This is what Hunt and Weintraub proved transcendentally in [HW], and what we next prove algebraically by identifying both spaces with the space of sextics $S(t)$ together with all minimal solutions $(x, y) \in k[t] \times k[t]$ of $y^2 = x^3 + S(t)$.

2. Torsion divisors and $S$. The action of $\text{GL}_2$ on sextic polynomials $S(t)$ also respects the minimal solutions of $y^2 = x^3 + S(t)$. Indeed, substituting $(at + b)/(ct + d)$ for $t$ in $y^2 = x^3 + S(t)$ and multiplying by $(ct + d)^6$ yields $y'^2 = x'^3 + S_1(t)$, where $x_1, y_1, S_1$ are the images of $x, y, S$ under $(a/b, c/d) \in \text{GL}_2$. Thus the set of minimal solutions of $y^2 = x^3 + S$ makes sense even if $S$ is not a sextic polynomial but a $\text{GL}_2$-orbit of such polynomials, or a point on $(P^1_6')$. In the sequel we assume for simplicity that no $t_j = \infty$, i.e. that $S(T)$ is of degree 6, not 5. The argument can be readily adapted to handle the case of a quintic; alternatively we may make all the $t_j$ finite by first applying a $\text{PGL}_2(k)$ transformation, which as already observed does not change the problem. Note that since $k \ni \rho$ and $k$ is not of characteristic 2 or 3 there are always more than six points in $\mathbb{P}^1(k)$ so such a transformation must exist.

**Theorem 1.** Let $y \in k[t]$ be a polynomial of degree at most 3 such that $y^2 - S(t)$ is a $k^*$-multiple of a cube in $k[t]$. Then the rational function $y(t) - u$ on $C$ has divisor $3D$ for some nonprincipal divisor $D$. Each nonzero 3-torsion element of $J(C)$ arises as the class of such a divisor $D$ for exactly one choice of $y$. 

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Remark: This theorem, and its following proof, turn out to have already been given (in older terminology, and only over $\mathbb{C}$) in [C].

Proof: On $C$ we have $S(t) = u^2$, so $\alpha x^3 = y^2 - S(t)$ factors as $\alpha x^3 = (y - u)(y + u)$. Now $y, S$ have no common roots: at such a root $x^3$ would vanish, so it would be a root of $x$ as well; but then $S = y^2 - \alpha x^3$ would vanish there to order at least 2, which is not allowed. Thus $y, u$ have no common zeros on $C$, and $u$ has a triple pole at each of the points at infinity while $y$ has a pole of order at most 3 there, and neither $y$ nor $u$ has any pole at a point of $C$ with $t \neq \infty$. Thus the $y - u$ and $y + u$ have no common zeros nor any finite poles, and at each point at infinity at least one of $y \pm u$ has a triple pole while the other has a pole of order $\leq 3$. Now $x$ is regular except for poles of order at most 2 at the points at infinity. Since $(y + u)(y - u) = \alpha x^3$, we conclude that the divisor of $y + u$ has the form $3D$ for some divisor $D$ on $C$. Since $3D$ is principal, the class $[D] \in J(C)$ of $D$ is a 3-torsion point on the Jacobian. But $D$ cannot itself be principal. Indeed, if it were, $(y + u)^{1/3}$ would be a rational function on $C$ of degree at most 2. But all such functions are in $k(t)$. Thus $y + u$ would also in $k(t)$, which is absurd because $y \in k(t)$ but $u \notin k(t)$. Thus $[D]$ is a nonzero 3-torsion point in the Jacobian, as claimed.

Now let the divisor $K$ on $C$ be the sum of the two points at infinity; this is the divisor of the differential $dt/u$ on $C$, and is thus canonical. Since $D$ includes at worst simple poles at the points at infinity, $D + K$ is an effective, non-canonical divisor of degree 2. Conversely, let $T$ be a nonzero 3-torsion point in $J(C)$. Then $K + T$ is a non-canonical divisor class of degree 2, and so by Riemann-Roch has a unique effective representative, which we may call $K + D$. Since $D \sim T$, the divisor $3D$ is principal, say the divisor of $f \in (k(C))^*$. Since $3K + 3D$ is effective, this function $f$ has at most triple poles at the points at infinity, and no other poles; thus it is of the form $y_1(t) + cu$ where $y_1$ is a polynomial of degree at most 3 and $c \in k$.

If $c$ vanished then $f$ would be a rational function of $t$ which, when considered as a function on $C$, would have all its zeros and poles of order divisible by 3. But then the same would be true of $f$ as a function on $\mathbb{P}^1$, because the cover $t : C \to \mathbb{P}^1$ has degree prime to 3. Thus $f$ would be a $k^*$-multiple of the cube of a rational function — and this rational function would have divisor $D$, which would therefore be principal, contradicting the hypothesis that $T \neq 0$.

Thus $f = y_1 + cu$ for some $c \in k^*$, and multiplying by $c^{-1}$ we find the unique function of the form $y + u$ whose divisor is $3D$ for some $D \sim T$. It remains to prove that $y^2 - S(t)$ is a $k^*$-multiple of a cube. Since the divisor of $y + u$ is divisible by 3, the same is true of the divisor of $y - u$, which is the image of $y + u$ under the hyperelliptic involution. But then the same is true of $(y - u)(y + u) = y^2 - S(t)$, considered as a function on $C$. As noted already this means that the divisor of $y^2 - S(t)$ considered as a function of $t$ is also divisible by 3, so $y^2 - S(t)$ is indeed of the form $\alpha x^3$ for some $x \in k[t]$ and we are done.
Corollary: i) There are $3^4 - 1 = 80$ polynomials $y \in \bar{k}[t]$ of degree at most 3 such that
$y^2 - S(t)$ is a cube in $\bar{k}[t]$.
ii) $\mathcal{S}$ is a $G$-cover of $(\mathbb{P}^1_0)$, isomorphic with the moduli space of curves of genus 2 with full
level-3 structure. □

Much the same analysis applies to the curves $C'$. The one difference here is that the divisor
$K_0$, consisting of the sum of the points (now three of them) at infinity, is no longer canonical:
the differential $dt/v^2$ has divisor $2K_0$, not $K_0$. Thus $K_0$ is a distinguished semicanonical
divisor, a.k.a. theta characteristic. (As with the divisor of $K$ on $C$, the divisor $K_0$ depends
on the choice of coordinate $t$ on $\mathbb{P}^1$, but the linear equivalence class of $K_0$ does not: if we
used instead a coordinate whose pole is at $\tau \neq \infty$ our divisor would be the fiber of $\tau$ under
the map $t : C' \rightarrow \mathbb{P}^1$, and the difference between these two $K_0$'s would be the divisor of
the rational function $t - \tau$.) Using $K_0$ we may identify 2-torsion elements $T$ of the Jacobian
with theta characteristics $K_0 + T$. Recall that there is an affine-linear quadratic form on
the theta characteristics which gives the parity of the space of sections; for genus 4, there
are $(2^8 + 2^4)/2 = 136$ even and $(2^8 - 2^4)/2 = 120$ odd theta characteristics. The divisor
$K_0$ has the two-dimensional space of sections generated by 1 and $t$, so it is an even theta
characteristic. None of the other 135 even thetas $\Theta$ has any nonzero sections: if it did then
by parity it would have at least 2 independent ones, whose ratio would be a degree-3 map
to $\mathbb{P}^1$ not of the form $(at + b)/(ct + d)$; but there are no such rational functions on $C'$. The
same argument shows that each odd theta has only a one-dimensional space of sections, and
thus has a unique effective representative.

We are now ready to state the analogue of Theorem 1 for $C'$:

**Theorem 2.** Let $x \in k[t]$ be a polynomial of degree at most 2 such that $x^3 - S(t)$ is a
$k^*$-multiple of a square in $k[t]$. Then the rational function $x(t) + v$ on $C'$ has divisor $2D$ for
some nonprincipal divisor $D$ such that $[D + K_0]$ is an odd theta characteristic. Each odd
theta characteristic arises in this way for exactly one choice of $x$.

**Proof:** On $C'$ we have $S(t) = v^3$, so $\beta y^2 = x^3 + S(t)$ factors as $\beta y^2 = (x + v)(x + \rho v)(x + \bar{\rho} v)$.
Now $x, S$ have no common roots: at such a root $y^2$ would vanish, so it would be a root of $y$ as
well; but then $S = \beta y^2 - x^3$ would vanish there to order at least 2, which is not allowed. Thus
$x, v$ have no common zeros on $C'$, and $v$ has double poles at each of the points at infinity
while $x$ has a pole of order at most 2 there, and neither $x$ nor $v$ has any pole at a point of $C'$
with $t \neq \infty$. Thus, of the three factors $x + v, x + \rho v,$ and $x + \bar{\rho} v$ of $\beta y^2,$ none has a pole
except at one of the points at infinity; each has poles at infinity of order at most 2; at each
point at infinity at most one factor may have a pole of order $< 2$; and no two have a common
zero. Now $y$ is regular except for poles of order at most 3 at the points at infinity. Since
$(x + v)(x + \rho v)(x + \bar{\rho} v) = \beta y^2$, we conclude that the divisor of $x + v$ has the form $2D$ for some
divisor $D$ on $C'$. Since $2D$ is principal, the class $[D] \in J(C')$ of $D$ is a 2-torsion point on the Jacobian. But $D$ cannot itself be principal. Indeed, if it were, $(x + v)^{1/2}$ would be a rational function on $C'$ of degree at most 3. But all such functions are in $k(t)$. Thus $x + v$ would also in $k(t)$, which is absurd because $x \in k(t)$ but $v \notin k(t)$. Thus $[D]$ is a nonzero 2-torsion point in the Jacobian, as claimed. Moreover $[D + K_0]$ is an odd theta characteristic, because $D + K_0$ is an effective semicanonical divisor not equivalent to $K_0$.

Conversely, let $T$ be a nonzero 2-torsion point in $J(C')$ such that $K_0 + T$ is an odd theta characteristic, and let $K_0 + D$ be the unique effective divisor in the class of $K_0 + T$. Since $D \sim T$, the divisor $2D$ is principal, say the divisor of $f \in (k(C'))^*$. Since $2K + 2D$ is effective, this function $f$ has at most double poles at the points at infinity, and no other poles; thus it is of the form $x_1(t) + cu$ where $x_1$ is a polynomial of degree at most 2 and $c \in k$. If $c$ vanished then $f$ would be a rational function of $T$ which, when considered as a function on $C'$, would have all its zeros and poles of order divisible by 2. But then the same would be true of $f$ as a function on $\mathbf{P}^1$, because the cover $t : C' \to \mathbf{P}^1$ has degree prime to 2. Thus $f$ would be a $k^*$-multiple of the square of a rational function — and this rational function would have divisor $D$, which would therefore be principal, contradicting the hypothesis that $T \neq 0$. Thus $f = x_1 + cv$ for some $c \in k^*$, and multiplying by $c^{-1}$ we find the unique function of the form $x + v$ whose divisor is $3D$ for some $D \sim T$. It remains to prove that $x^3 + S(t)$ is a $k^*$-multiple of a square. Since the divisor of $x + v$ is divisible by 2, the same is true of the divisors of $x + \rho v$ and $x + \bar{\rho} v$, which are the images of $x + v$ under $\rho$ and $\bar{\rho}^2$. But then the same is true of $(x + v)(x + \rho v)(x + \bar{\rho} v) = x^3 + S(t)$ considered as a function on $C'$. As noted already this means that the divisor of $x^3 + S(t)$ considered as a function of $t$ is also divisible by 2, so $x^3 + S(t)$ is indeed of the form $\beta y^2$ for some $y \in k[t]$ and we are done.

**Corollary:** i) There are 120 polynomials $x \in \bar{k}[t]$ of degree at most 2 such that $x^3 + S(t)$ is a square in $\bar{k}[t]$.

ii) $S$ is a $G$-cover of $(\mathbf{P}^1_6)$, isomorphic with the moduli space of curves $C'$ with full level-2 structure. $\square$

Combining the Corollaries to Theorems 1 and 2 we obtain

**Theorem 3.** i) Any polynomial $S(t)$ of degree 5 or 6 without repeated roots over an algebraically closed field not of characteristic 2 or 3 can be written as $y^2 - x^3$ for 240 pairs of polynomials $x(t), y(t)$ of degree at most 2 and 3 respectively.

ii) Over any field not of characteristic 2 or 3 that contains the cube roots of unity, the moduli space of curves $C$ with full level-3 structure and the moduli space of curves $C'$ with full level-2 structure are isomorphic $G$-covers of $(\mathbf{P}^1_6)$.

**Proof:** i) This follows from part (i) of the Corollary to either Theorem 1 or Theorem 2 since
240 = 3 \cdot 80 = 2 \cdot 120. (Fortunately the two computations agree!)

ii) This follows from parts (ii) of the Corollaries to Theorems 1 and 2, which identify both $G$-covers with $\mathcal{S}$. □

The reader may have already surmised that the distinction between even and odd theta characteristics is equivalent to the unitary structure on $J(C')[2]$. Indeed a theta characteristic $K_0 + T$ is even according as $\nu(T) = 0$ or 1. This can be proved as follows. Let $q$ be the affine-quadratic form on the theta characteristics $\Theta$ which is 0 on even and 1 on odd $\Theta$'s. It is known [M] that $q$ is compatible with the Weil pairing: if $T, T'$ are any 2-torsion points then their Weil pairing $\langle T, T' \rangle$ (written additively to take values in $F_2$ rather than $\{\pm 1\}$) is

$$q(\Theta) + q(\Theta + T) + q(\Theta + T') + q(\Theta + T + T')$$

for any choice of $\Theta$. Now take $\Theta = K_0$ and $T' = \rho T$. We have seen already that $q(K_0) = 0$. Moreover $q$ is $\rho$-invariant. Thus (6) reduces to

$$q(K_0 + T) + q(K_0 + \rho T) + q(K_0 + \rho^2 T) = 3q(K_0 + T) = q(K_0 + T). \quad (8)$$

Thus $q(K_0 + T) = \langle T, \rho T \rangle = \nu(T)$ as claimed.

3. The lattice $E_8^\rho$. Our proof of Theorem 3 does not entirely dispel the mystery of the coincidence of the two $G$-covers of $(P_6^1)$; even the fact that $\text{PSp}_4(F_3)$ and $\text{SU}_4(F_2)$ are isomorphic appears to emerge as an accident — in fact formulate our analysis so that the isomorphism $\text{PSp}_4(F_3) \cong \text{SU}_4(F_2)$ arises as a by-product! But there is a more satisfactory approach to this isomorphism via the low-dimensional representations of $G$ and its double cover $2G = \text{Sp}_4(F_3)$. (The existence of $2G$ cannot be so readily seen from the $\text{SU}_4(F_2)$ model of $G$: except for in characteristic 2 have trivial Schur multiplier [C&x, p.xvi]!)

A four-dimensional representation $V$ of $2G$ will also figure in a third description of our $G$-cover $\mathcal{S}$ of $(P_6^1)$ which will naturally yield both the $J(C)[3]$ and $J(C')[2]$ pictures of $\mathcal{S}$ via the reduction of $V$ mod 2 and 3 respectively.

It is known that $\text{Aut}(G)$ contains $G$ with index 2. When $G$ is viewed as $\text{PSp}_4(F_3)$, the outer automorphisms are linear transformations that multiply the symplectic form by $-1$, while in the $\text{SU}_4(F_2)$ viewpoint they are conjugate-linear transformations preserving the unitary form. Thus $\text{Aut}(G)$ appears as $\text{PGSp}_4(F_3)$ in characteristic 3 and as $\Sigma U_4(F_2)$ in characteristic 2.

We take for $V$ one of the two irreducible representations of $2G$ of dimension 4; the choice does not matter, because these representations (which are each other’s contragredient) are exchanged by an outer automorphism of $G$. This representation is defined over $\mathbb{Q}(\rho)$; the image of $2G$ in $\text{GL}(V)$ (actually $\text{SL}(4)$), extended by the three-element group $(\rho)$ of $\mu_4$-multiples of the identity, is a complex reflection group — #32 in the list of [ST p.301] —
generated by what the ATLAS calls “triflections”: linear transformations of order 3 with a codimension-1 fixed subspace. The nontrivial central element of $G$ acts on $V$ by multiplication by $-1$; thus the exterior square $W = \wedge^2 V$ is a 6-dimensional representation of $G$. It turns out that this representation is irreducible and defined over $\mathbb{Q}$. This time $W$ is the unique representation of its dimension; thus the action of $G$ on $W$ extends to Aut($G$). It turns out that the image of Aut($G$) in GL($W$) is again a reflection group, #35 in [ST], which is to say the Weyl group $W(E_6)$. Kneser observed in [K] that reducing the action of Aut($G$) on the $E_6$ lattice mod 3 and 2 identifies Aut($G$) with PGSp$_4(F_3)$ and $\Sigma U_4(F_2)$ in these groups’ guises as the orthogonal groups SO$_5(F_3)$ and SO$_6^-(F_2)$. Indeed $E_6/3E_6^+$ is and $E_6/2E_6$ are orthogonal spaces of dimensions 5 and 6 over $F_3$ and $F_2$ respectively, both with actions of $G$ respecting the quadratic form; but the simple groups SO$_5(F_3)$ and SO$_6^-(F_2)$ are barely large enough to accommodate copies of Aut($G$), so the reduction maps from Aut($G$) to these two groups must be isomorphisms. In particular the isomorphism between SO$_5(F_3)$ and SO$_6^-(F_2)$ follows. Kneser explains most of the sporadic isomorphisms between classical simple groups in the same fashion. But he does not observe an alternative explanation along the same lines.

Remarkably $G$ is involved in yet another complex reflection group, the 5-dimensional #33 defined over $\mathbb{Q}$. The reductions of this representation mod 3 and 2 again yield the isomorphism PGSp$_4(F_3) \cong SU_4(F_2)$. We do not pursue this here because this representation does not enter into our investigation of $S$. 

We call the lattice $E_8^\rho$ because it is obtained from the $E_8$ root lattice by choosing a 3-cycle in $W(E_8)$ acting on $E_8$ with trivial fixed space. There is a unique conjugacy class of such 3-cycles; identifying one with $\rho$ gives $E_8$ the structure of a four-dimensional lattice $E_8^\rho$ over $\mathbb{Z}(\rho)$. Its group of automorphisms is the subgroup of $W(E_8)$ commuting with $\rho$, which is isomorphic with $(2G) \times \langle \rho \rangle$. We choose the isomorphism so that $E_8^\rho \otimes Q(\rho)$ is our representation $V$ of $2G$ rather than its contragredient. We can also describe $E_8^\rho$ explicitly as the sublattice of $\mathbb{Z}[\rho]$ consisting of vectors congruent mod $(\sqrt{-3})$ to a linear combination of $(1,1,1,0)$ and $(1,-1,0,1)$, i.e. to a vector in the “tetracode” in $F_4$. (That is, $E_8^\rho$ is the $\mathbb{Z}[\rho]$-lattice obtained from the tetracode by “Construction A.”) The inner product $\langle \vec{z}, \vec{z}' \rangle$ of $\vec{z} = (z_1, z_2, z_3, z_4)$ with $\vec{z}' = (z_1', z_2', z_3', z_4')$ is $\frac{2}{3} \sum_{j=1}^{4} z_j \bar{z}_j$; if $\vec{z}, \vec{z}' \in E_8^\rho$ then $(\vec{z}, \vec{z}') \in (2/\sqrt{3})E[\rho]$. The roots (nonzero vectors of minimal norm) are the $2 \cdot 3 \cdot 4 = 24$ multiples of unit vectors by $\pm \mu_3 \sqrt{-3}$, and the $3^3(3^2 - 1) = 216$ minimal lifts of the 8 nonzero vectors of the tetracode. (Each such vector has 1 zero and 3 nonzero coordinates; in a minimal lift, 0 lifts to 0, and ±1 lifts to one of the three choices $\pm \mu_3$..) This adds up to the familiar count of $24 + 216 = 240$ roots. For each root $\vec{r}$ we obtain a triflection $x \mapsto x + \frac{1}{2}(\rho - 1)(x, \vec{r})r$, and these triflections generate Aut($E_8^\rho$) = $(2G) \times \langle \rho \rangle$. Reducing mod 2, we obtain an $F_4$-vector space $E_8^\rho/2E_8^\rho$ of dimension 4 with a Hermitian form $\frac{1}{2}(\cdot, \cdot) \mod 2$. Thus the group

\footnote{Remarkably $G$ is involved in yet another complex reflection group, the 5-dimensional #33 defined over $\mathbb{Q}(\rho)$. The reductions of this representation mod 3 and 2 again yield the isomorphism PGSp$_4(F_3) \cong SU_4(F_2)$. We do not pursue this here because this representation does not enter into our investigation of $S$.}
\[ \text{Aut}_1(E^\rho_8) \] of linear isometries of \( E^\rho_8 \) of determinant 1 maps to a subgroup of \( SU_4(\mathbb{F}_2) \); the kernel consists of the identity and multiplication by \(-1\), so we find \( \text{Aut}_1(G)/\{\pm 1\} \) as a subgroup of \( SU_4(\mathbb{F}_2) \). Reducing mod \( (\sqrt{-3}) \), we obtain an \( \mathbb{F}_3 \)-vector space \( E^\rho_8/\sqrt{-3}E^\rho_8 \) of dimension 4 with a symplectic form \( \sqrt{-3}(\cdot, \cdot) \) mod \( (\sqrt{-3}) \). Thus \( \text{Aut}_1(E^\rho_8) \) maps to \( \text{Sp}_4(\mathbb{F}_3) \), this time with trivial kernel (since the scalar multiplications by \( \rho, \bar{\rho} \) are excluded from \( \text{Aut}_1 \)).

We thus find \( \text{Aut}_1(G)/\{\pm 1\} \) as a subgroup of \( \text{PSp}_4(\mathbb{F}_3) \). Since \( SU_4(\mathbb{F}_2) \) and \( \text{PSp}_4(\mathbb{F}_3) \) are barely large enough to accommodate \( \text{Aut}_1(E^\rho_8)/\{\pm 1\} \), these two inclusions must be isomorphisms. We have thus explained the isomorphism of \( SU_4(\mathbb{F}_2) \) with \( \text{PSp}_4(\mathbb{F}_3) \) by regarding these two groups as the mod-2 and mod-3 manifestation of the characteristic-zero object \((E^\rho_8, 2.G)\). This explanation extends to \( \text{Aut}(G) \): the outer automorphisms of \( S_8 \) by conjugate-linear isometries such as \((z_1, z_2, z_3, z_4) \mapsto (\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4)\). Such maps descend to conjugate-linear automorphisms of \( E^\rho_8/2E^\rho_8 \), and thus extend \( SU_4(\mathbb{F}_2) \) to \( \Sigma U_4(\mathbb{F}_2) \): on \( E^\rho_8/\sqrt{-3}E^\rho_8 \), they act linearly but reverse the symplectic pairing, thus extending \( \text{Sp}_4(\mathbb{F}_3) \) to \( \text{GSp}_4(\mathbb{F}_3) \) and \( \text{PSp}_4(\mathbb{F}_3) \) to \( \text{PGSp}_4(\mathbb{F}_3) \). Therefore \( \Sigma U_4(\mathbb{F}_2) \cong \text{Aut}(G) \cong \text{PGSp}_4(\mathbb{F}_3) \).

We next consider the reduction of the 240 roots mod 2 and \( \sqrt{-3} \). Since \( \frac{1}{3}(r, r) = 1 \) for every root \( r \), all the roots belong to one of the 120 odd classes in \( E^\rho_8/2E^\rho_8 \); clearly \( r \) and \( -r \) are in the same class, and conversely if \( r, r' \) are roots congruent mod \( 2E^\rho_8 \) then \( r' = \pm r \) because at least one of \( \frac{1}{3}(r \pm r') \) is a lattice vector of norm \( < 2 \). But there are \( 240 = 2 \cdot 120 \) roots; since each odd class mod \( 2E^\rho_8 \) accounts for at most two of them, each of the 120 odd classes must be represented by a pair \( \pm r \) of opposite roots. Modulo \( \sqrt{-3} \), no root is congruent to zero and every root \( r \) is congruent to \( r, pr, \) and \( \bar{p}r \), and to no other roots \( r' \) lest

\[
4 = \frac{|r - r'|^2}{\sqrt{-3}} + \frac{|pr - r'|^2}{\sqrt{-3}} + \frac{|ar{p}r - r'|^2}{\sqrt{-3}} \tag{9}
\]

be the sum of three positive even integers. Thus each of the 80 nonzero classes in \( E^\rho_8/\sqrt{-3}E^\rho_8 \) contains at most 3 roots, and since there are \( 240 = 3 \cdot 80 \) roots we again conclude that each nonzero class represents a triple \( \mu_3 r \) of roots. Of course these counts \( 240 = 3 \cdot 80 = 2 \cdot 120 \) are highly suggestive of the counts in the first parts of the Corollaries to Theorem 1 and 2; we make the connection in the next section. Note that we could also have obtained these results from the identification of \( \text{Aut}(E^\rho_8)/\{\pm 1\} \) with \( SU_4(\mathbb{F}_2) \) and \( \text{PSp}_4(\mathbb{F}_3) \), together with the fact that the unitary and symplectic groups act transitively on odd and nonzero vectors respectively.

We conclude our description of the lattice \( E^\rho_8 \) by noting that it could also have been defined directly from the representation of \( 2.G \) on \( \mathcal{V} \), without recourse to the \( E_8 \) root lattice: the representation is globally irreducible, and thus has a unique \((2.G)\)-stable lattice, see [G, T].

4. The elliptic surface \( \mathcal{E} \) and its Mordell-Weil lattice. We next identify \( E^\rho_8 \) with the
group of $\bar{k}(t)$-rational points on the elliptic curve

$$\mathcal{E} : y^2 = x^3 + S(t)$$

for each sextic $S \in k[t]$ without repeated roots. To do this we must show not only that $\mathcal{E}(\bar{k}(t))$ is a free abelian group of rank 8 but also specify an action of $\rho$ and a quadratic form and an action of $\rho$ on that group. We will have $\rho$ act by complex multiplication (CM), and take for the quadratic form the canonical height $\hat{h}$ on the points of an elliptic curve over a function field. We describe these extra structures (CM and $\hat{h}$) on $\mathcal{E}(\bar{k}(t))$ in turn, and then show that the resulting $\mathbb{Z}[\rho]$-lattice is isometric with $E_0^\rho$.

That $\mathcal{E}$ has complex multiplication by $\mathbb{Z}[\rho]$ means that $\mathbb{Z}[\rho]$ acts on $\mathcal{E}$ by endomorphisms (a.k.a. isogenies, i.e. algebraic maps commuting with the group law). Giving $\mathcal{E}$ a $\mathbb{Z}[\rho]$ action means exhibiting an endomorphism $\rho$ satisfying the minimal equation $\rho^2 + \rho + 1 = 0$, with addition defined via the group law on $\mathcal{E}$. Such an endomorphism is $\rho : (x, y) \mapsto (\rho x, y)$: the three images $(x, y)$, $(\rho x, y)$, $(\bar{\rho} x, y)$ of a generic point $(x, y)$ on $\mathcal{E}$ under $1, \rho, \rho^2$ have the same $y$-coordinate, and thus are the intersection of $\mathcal{E}$ with a line and sum to zero in the group law.

We chose $(x, y) \mapsto (\rho x, y)$ rather than $(x, y) \mapsto (\bar{\rho} x, y)$ so that $\rho$ multiplies by $\rho$ the invariant differential $\omega = dx/y$ on $\mathcal{E}$; by linearity it follows that $\phi^* \omega = \omega$ for every endomorphism $\phi \in \mathbb{Z}[\rho]$. We note for future use that any $\phi \in \mathbb{Z}[\rho]$, considered as a map $\phi : \mathcal{E} \to \mathcal{E}$, has degree $\phi \bar{\phi}$. Every elliptic curve in characteristic other than 2 or 3 that has CM by $\mathbb{Z}[\rho]$ can be written as $y^2 = x^3 + a_6$ for some nonzero $a_6$, and so becomes isomorphic with

$$E_0 : Y^2 = X^3 + 1$$

once we extract a sixth root of $a_6$. Thus in our case $\mathcal{E}$ becomes isomorphic with the constant curve $E_0$ when we extend $k(t)$ to the function field of the curve

$$C'' : u^6 = S(t).$$

This is a curve of genus 10 (a smooth plane sextic) which cyclically covers both $C$ and $C'$ in degrees 3 and 2 respectively. The identification of $\mathcal{E}$ with $E_0$ over $k(C'')$ takes $(x, y)$ to $(w^{-2}x, w^{-3}y)$. Thus the $k(t)$-rational points of $\mathcal{E}$ are identified with the subgroup of $\mathcal{E}_0(k(C''))$ consisting of the $k(C'')$-rational points $P$ of $E_0$ whose image under the generator of $\text{Gal}(k(C'')/k(t))$ taking $w$ to $-\rho w$ is $-\rho P$. Note that $\mathcal{E}_0(k(C''))$ may be regarded as the set of maps $P : C'' \to E_0$, which inherits an abelian group structure from the group law of $E_0$; in this viewpoint $\mathcal{E}(k(t))$ consists of those maps $P$ for which the diagram

$$\begin{array}{ccc}
C'' & \xrightarrow{P} & E_0 \\
\downarrow & & \downarrow (-\rho) \\
C'' & \xrightarrow{P} & E_0
\end{array}$$

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commutes using \((t, w) \mapsto (t, -\rho w)\) for the arrow \(C'' \to C'''.\) Likewise we may regard the \(\tilde{k}(t)\)-rational points of \(E\) as maps from \(C'''\) to \(E_0\) defined over \(\tilde{k}\) for which that diagram commutes.

We next review the canonical height on \(E\); see [3], Ch.VIII §9 (pp.227–233]) for the canonical height on a general elliptic curve over a global field, and [2], [8], [9] for the present case of an elliptic curve, especially one of constant \(j\)-invariant, over a function field.

Let \(E\) be any elliptic curve over a function field \(F = k(C_0),\) with \(C_0\) a curve over the field of constants \(k.\) Choose a Weierstrass model for \(E,\) with coordinates \(x, y.\) The naïve height on \(E\) is a function \(h\) on its group \(E(F)\) of rational points whose value at a point measures the point’s complexity: the zero point has naïve height 0, and a nonzero point \((x, y)\) has naïve height

\[
h(x, y) := \max(\deg x, \frac{2}{3} \deg y).
\]

(Here “\(\deg\)” is the degree of an element of the function field considered as a rational map to \(\mathbb{P}^1.\) Note that \(h\) depends on the choice of Weierstrass model for \(E,\) though the naïve heights associated with different models differ only by \(O(1).\) The canonical or Néron-Tate height on the curve does not depend on the choice of model. It is a \(\mathbb{Q}\)-valued quadratic form \(\hat{h}\) on \(E(F),\) positive-definite on \(E(F) \otimes \mathbb{R},\) which is within \(\pm O(1)\) of the naïve height \(h.\) These two properties uniquely characterize \(\hat{h}.\) It follows that if \(C_1\) is a curve covering \(C_0\) with degree \(d,\) and \(F'/F\) the corresponding field extension with \(F' = k(C_1)\) and \([F' : F] = d,\) then for any \(P \in E(F)\) its height \(\hat{h}_{F'}(P)\) as a point of \(E(F')\) is \(d\) times its height \(\hat{h}_F(P)\) as a point of \(E(F)\). Indeed, from the definition \((1)\) it is clear that the naïve heights \(h_F, h_{F'}\) satisfy \(h_{F'} = dh_F,\) so the restriction of \(d^{-1}\hat{h}_{F'}\) to \(E(F)\) satisfies both criteria for \(\hat{h}_F.\)

There are other equivalent characterizations more suitable for computing \(\hat{h}.\) Tate showed that for any endomorphism \(\phi : E \to E\) of degree \(> 1\) the canonical height is the unique function \(\hat{h} : E(F) \to \mathbb{R}\) such that \(\hat{h} = h + O(1)\) and \(\hat{h}(\alpha P) = (\deg \alpha)\hat{h}(P)\) for every \(P \in E(F).\)

(Usually one takes for \(\phi\) the multiplication-by-\(n\) map for some small \(n > 1\) such as \(n = 2,\) but we’ll be able to exploit \(\phi \notin \mathbb{Z}\) as well.) Néron described \(\hat{h}\) in terms of intersection theory on \(E\) considered as an elliptic surface over the constant field of \(F,\) which yields a formula for \(\hat{h}(P) - h(P)\) as a finite sum of terms each depending on the reduction of \(P\) at one of the places where \(E\) has bad reduction. From either characterization it follows that if \(E\) is a curve defined over \(k,\) so points \(P \in E(F)\) are equivalent to rational maps \(P : C_0 \to E,\) then the height of such a point equals twice the degree of \(P;\) that is, the canonical height equals the naïve height. (Since \(x, y\) are functions of degree 2, 3 on \(E,\) the definition \((1)\) gives \(h(P) = 2 \deg(P).\)) In Tate’s viewpoint we see this by noting that \(h\) already satisfies the condition \(h(\alpha P) = (\deg \alpha)h(P),\) because the degree of rational maps is multiplicative under composition. In Néron’s approach this follows because there are no places of bad reduction.
and thus no contributions to the sum for \( \hat{h} - h \).

In our case \( \mathbb{I} \) of the curve \( E \) over \( k(t) \) or \( \bar{k}(t) \) we likewise show that \( \hat{h} = h \):

**Proposition.** The canonical height of any nonzero \( P \in E(\bar{k}(t)) \) with coordinates \( (x, y) \) is given by \( \hat{h}(x, y) = \max(\deg x, \frac{2}{3} \deg y) \); that is, the naïve and canonical heights on \( E(\bar{k}(t)) \) are equal.

As in \( \mathbb{I} \) we can give several proofs: one using the relationship with maps \( C'' \to E_1 \), one using Tate’s characterization, and one using Néron’s formula. We give the first and third proof, leaving the second as an exercise; each of the three introduces ideas that will later figure in our determination of \( E(\bar{k}(t)) \) and its connection with \( S \). The following observations regarding the naïve height will be helpful throughout. Let \( P : (x(t), y(t)) \) be a nonzero point on \( E(\bar{k}(t)) \). Let \( \tau_r \in \mathbb{P}^1(\bar{k}) \) be the points at which either \( x \) or \( y \) has a pole, with \( t_0 = \infty \). For \( j > 0 \), since \( S = y^2 - x^3 \) is regular at \( t_j \in \bar{k} \), the functions \( x^3, y^2 \) must have poles of the same order at \( \tau_r \), so \( x, y \) ave poles of order \( 2n_r, 3n_r \) at \( \tau_r \) for some integer \( n_r \). At \( t_0 = \infty \), \( y^2 - x^3 \) has a pole of order 5 or 6, so at least one of \( x^3, y^2 \) has a pole of order 6, and if one has a pole of order > 6 then both do and the pole orders are equal. In this last case \( x, y \) have poles of order \( 2n_0, 3n_0 \) at \( t = \infty \) for some integer \( n_0 > 1 \); otherwise we define \( n_0 = 1 \). Then

\[
\hat{h}(P) = \max(\deg x, \frac{2}{3} \deg y) = 2 \sum_r n_r. \tag{15}
\]

We proceed to our proofs of the Proposition:

**Proof 1:** Extending \( k(t) \) to \( k(C'') \) we have identified \( E \) with the constant curve \( E_0 \) and associated to \( P : (x, y) \) the rational map \( (t, w) \mapsto (w^{-2}x(t), w^{-3}y(t)) \) from \( C'' \) to \( E_0 \). We calculate the degree of this map by finding the preimages of the origin of \( E_0 \) and their multiplicities. We find that the preimages are the points with \( t = \tau_r \) for some \( r \), with multiplicity \( 6n_r \) if \( \tau_r = t_j \) for some \( j \) and \( n_r \) if not. In either case \( \tau_r \) contributes \( 6n_r \) to the total, because there are six points of \( C'' \) with \( t = \tau_r \) unless \( \tau_r = t_j \) in which case there is only one. Thus the degree of the map is \( 6 \sum_r n_r = 3h(P) \), and its height as a point of \( E(k(C'')) \) is twice that degree, or \( 6h(P) \). Since \( [k(C'') : k(t)] = 6 \) we divide by 6 to obtain the claimed formula \( \hat{h}(P) = \max(\deg x, \frac{2}{3} \deg y) \) for the height of \( P \) as a point of \( E(\bar{k}(t)) \).

**Proof 2** (sketch): Use Tate’s characterization, taking for \( \phi \) the isogeny

\[
\sqrt{-3} = \rho - \bar{\rho} : (x, y) \mapsto (\frac{3x^3 - 4y^2}{3x^2}, \sqrt{-3} \frac{9x^3y - 8y^3}{9x^3}) \tag{16}
\]

or

\[
2 : (x, y) \mapsto (\frac{9x^4 - 8xy^2}{4y^2}, -\frac{27x^6 - 9x^3y^2 + y^4}{8y^3}). \tag{17}
\]

That is, show that these multiply the naïve height by 3 and 4 respectively. This can be done directly by counting poles of the coordinates of \( \sqrt{-3} \cdot (x, y) \) and \( 2 \cdot (x, y) \). (One can also invoke
the computation in Proof 1 together with the fact that $\sqrt{-3}$ and 2, considered as isogenies of $E_0$, have degrees 3 and 4 respectively.)

Proof 3: In Néron’s approach the terms in the formula for $\hat{h} - h$ are indexed by places where $E$ reduces to a decomposable curve. We will show that $\mathcal{E}$ as no such places and thus that $\hat{h} - h = 0$. Specifically, we’ll show that the singular reductions of $\mathcal{E}$ are six cubics $y^2 = x^3$, with Kodaira type II indicating a cusp singularity at the origin but only one component, at the roots $t_j$ of $S(t)$. This is clear for finite $t$, where $E$ is smooth as a surface (since the $t_j$ are distinct) so the reduction of the Néron model is obtained by just specializing $t$, with no further blowing up necessary. At $t = \infty$ the coefficient $S(t)$ has a pole, so we change coordinates to $(x', y') = (x/t^2, y/t^3)$ satisfying $y'^2 = x'^3 + (S(t)/t^6)$, which at $t = \infty$ is again smooth as a surface and so reduces to a cuspidal cubic if some $t_j = \infty$ and to an elliptic curve if not. $\blacksquare$

Corollary: i) The canonical height of every point $P \in \mathcal{E}(\overline{k}(t))$ is an even integer, which is positive unless $P$ is the zero point.
ii) The points $P$ of height 2 are those coming from minimal solutions of $y^2 = x^3 + S(t)$. iii) The torsion subgroup of $\mathcal{E}(\overline{k}(t))$ is trivial.

Proof: i) By the Proposition it is enough to prove this for the naïve height. The height of the zero point is 0. In [13] we exhibited the naïve height of a nonzero point $P$ as a sum of nonnegative even integers $2n_r$, including the positive $2n_0$. Thus $h(P)$, and so also $\hat{h}(P)$, is a positive even integer as claimed.

ii) If $\sum n_r = 2$ then $n_0 = 1$ and all other $n_r = 0$. Thus $x, y$ have no poles at finite $t$, and have poles of orders at most 2, 3 at $t = \infty$. Conversely it is clear that if $x, y$ are polynomials of degrees at most 2, 3 with $y^2 = x^3 + S$ then $P : (x, y)$ is a rational point on $\mathcal{E}$ with $h(P) = 2$, and thus by the Proposition also with $\hat{h}(P) = 2$.

iii) If $P$ is a torsion point then $nP = 0$ for some $n > 1$, so $\hat{h}(P) = n^{-2} \hat{h}(nP) = 0$. By part (i) it follows that $P = 0$. $\square$

On general principles, the group of $\overline{k}(t)$-rational points of $\mathcal{E}$ is finitely generated. The Proposition and its Corollary tell us that this group is torsion-free and has the structure of an even $\mathbb{Z}[\rho]$-lattice $L_S$, a.k.a. the Mordell-Weil lattice of $\mathcal{E}$, in the unitary space $(\mathcal{E}(\overline{k}(t))) \otimes \mathbb{Z}[\rho] \mathbb{C} = L_S \otimes \mathbb{Z}[\rho] \mathbb{C}$, whose minimal nonzero points are the minimal solutions of $y^2 = x^3 + S(t)$. We already know that there are 240 such, so the following theorem should come as no surprise:

**Theorem 4.** For every sextic polynomial $S(t)$ without repeated roots, the lattice $L_S$ is isomorphic with $E_8^\rho$.

Proof: It is enough to prove that $L_S$ is isomorphic with $E_8$ as a $\mathbb{Z}$-lattice, because $E_8^\rho$ is
$E_8$ with its unique structure as a $\mathbb{Z}[\rho]$-lattice. We determine $L_S$ as a $\mathbb{Z}$-lattice by relating it with the Néron-Severi group $\text{NS}(E)$ with $E$ considered as a surface over $\bar{k}$. To do this we need a model of $E$ smooth everywhere, including $t = \infty$. But we have obtained such a model in the course of proving the $\hat{h} = h$ proposition. This model is the union of the two open subsets $t \neq \infty$ and $t \neq 0$, each of which is represented as a surface in $\mathbb{A}^1 \times \mathbb{P}^2$: for $t \neq \infty$, using coordinates $t$, $(x : y : 1)$; for $t \neq 0$, using $1/t$, $(x/t^2 : y/t^3 : 1)$. We saw already that this surface has no reducible fibers and that its only singular fibers are reductions of Type II at the six points $t = t_j$. It follows as in [Sh] that $E$ is a $\bar{k}$-rational elliptic surface, so that its Néron-Severi group is the even unimodular lattice of signature $(1, 9)$, of which the zero section and the fibers contribute a unimodular sublattice of signature $(1, 1)$ ("hyperbolic plane"), and therefore the Mordell-Weil lattice is even unimodular of rank 8. Thus $L_S \cong E_8$ as claimed.

Remark: We could also have proved that $L_S$ has $\mathbb{Z}[\rho]$-rank 4 using the description of $L_S$ as maps to $E_0$ from $C''$, or equivalently with its Jacobian $J(C'')$, that make (13) commute. Now the for $0 < s < 6$ the $(-\rho)^s$-eigenspace of the action of $(t, w) \mapsto (t, -\rho w)$ on $H^1(C'')$ consists of the differentials $P(t) dt/\omega^{6-s}$ with $P \in \mathbb{C}[t]$ of degree at most $4 - s$, and thus has dimension $5 - s$. For $s = 3$ these are the differentials pulled back from $H^1(C)$ via the triple cover $C'' \to C$; for $s = 2$ and $s = 4$ they are the pullbacks of $H^1(C')$ via the double cover $C'' \to C'$. Thus $L_S$ consists of the annihilator in $\text{Hom}(J(C''), E_0)$ of the images of $J(C)$, $J(C')$ in $J(C'')$. The quotient of $J(C'')$ by the abelian variety generated by $J(C)$ and $J(C')$ is an abelian variety $J_0(C'')$ of dimension $g(C'') - g(C) - g(C') = 10 - 4 - 2 = 4$ equipped with an endomorphism inherited from $(t, w) \mapsto (t, -\rho w)$ that multiplies all its holomorphic differentials by $-\rho$. Thus $J_0(C'')$ is an abelian fourfold isogenous with $E_0^4$, and the rank of $\text{Hom}(J_0(C''), E_0)$ equals that of $\text{Hom}(E_0^4, E_0) = \mathbb{Z}[\rho]^4$. Moreover, at least in characteristic zero we see that $J_0(C'')$, and thus the lattice $L_S$, must be independent of $S$, because all components of the moduli space of fourfolds isogenous with $E_0^4$ have genus zero. But it seems much harder to prove that $L_S \cong E_0^6$ using this approach.

By Theorem 4 the lattice of $\bar{k}(t)$-points on $E$ is independent of the choice of $S$. However, the field extension $k_S$ of $k$ needed to define the points of $E(\bar{k}(t))$ does depend on $S$. It is clear that this extension is finite, because $E(\bar{k}(t))$ is generated by finitely many points. Moreover it is a normal extension, because Galois conjugation in $k_S$ permutes the solutions of the equation (14) defined over $k$. Furthermore, $\text{Gal}(k_S/k)$ respects the group law and canonical height on $L_S$. Thus we may regard $L_S$ as a $\text{Gal}(k_S/k)$ module, and obtain a map from $\text{Gal}(k_S/k)$ (or even from $\text{Gal}(\bar{k}/k)$) to $\text{Aut}(L_S) \cong \text{Aut}(E_0^6) = \mu_3 \times (2G)$. We have seen already that, at least modulo the center $\mu_3$ of $\text{Aut}(E_0^6)$, this Galois representation depends only on the $\text{PGL}_2(k)$-orbit of $S$; in the $L_S$ viewpoint, this is because if $S, S'$ are sextics equivalent under
some \( g \in \text{PGL}_2(k) \) then \( g : \mathbb{P}^1 \to \mathbb{P}^1 \) lifts to an isomorphism from \( \mathcal{E}_S \) to \( \mathcal{E}_{S'} \), using the models of these elliptic surfaces constructed during the proof of Theorem 4. Our work in §2 shows that the representation \( \text{Gal}(k_S/k) \to G \) is equivalent to the Galois representations on \( J(C)[3]/\{ \pm 1 \} \) and \( J(C')[2] \), and is surjective onto \( G \) for generic \( S \). But the curves \( C, C' \) also figure in the arithmetic of \( \mathcal{E} \) as an elliptic curve over \( k(t) \): their function fields are the extensions of \( k(t) \) generated by the \( (\sqrt{-3}) \)- and 2-torsion points of the curve. This is clear from our formulas \([16,17]\) for the isogenies \( \sqrt{-3}, 2 : \mathcal{E} \to \mathcal{E} : \) the nontrivial \( (\sqrt{-3}) \)-torsion points are those with \( x = 0 \) and thus \( y = \pm \sqrt{3} = \pm u \), while the nontrivial 2-torsion points have \( y = 0 \) and \( x = -\sqrt{3} = -\rho^*v \). The 3- and 2-torsion groups of \( J(C) \) and \( J(C') \) then arise naturally in the descent on \( \mathcal{E} \) via the isogenies \( \sqrt{-3} \) and 2, and allow us to extend the maps of Theorems 1 and 2 from the 240 roots to \( J(C)[3] \) and \( J'(C)[2] \) to isomorphisms \( L_S/\sqrt{-3} L_S \cong J(C)[3] \), \( L_S/2 L_S \cong J'(C)[2] \). We thus obtain our final explanation of the equivalence between the \( G \)-covers of \( (\mathbb{P}^1_6) \) coming from \( J(C)[3] \) and \( J'(C)[2] \) as with the mod-3 and mod-2 manifestations of \( G \) itself, the two constructions of the \( G \)-cover \( S/\mathbb{P}^1_6 \) are now revealed as the mod-\( (\sqrt{-3}) \) and mod-2 manifestations of the projective representation of \( \text{Gal}(k/k) \) on \( L_S \). We next exhibit these descent maps; see \([5]\), Ch. VIII] for the general theory of descent of which they are two special cases.

We deal first with the 3-isogeny \( \sqrt{-3} \). We observed that its kernel is generated by the 3-torsion point \( (0, u) \) with \( u^2 = S(t) \). This the function \( y - u \in (k(C))(\mathcal{E}) \) is a Weil function on \( \mathcal{E} \): its divisor is \( 3((0, u)) - 3(0) \). Moreover \( y - u \) is locally a cube at its triple pole. Evaluation of \( y - u \) mod cubes at points other than \( 0 \) and \( (0, u) \) then extends to a homomorphism from \( E(k(C)) \) to \( k(C)^*/k(C)^* \) whose kernel is \( \sqrt{-3} E(k(C)) \). We claim that \( \sqrt{-3} E(k(C)) \cap E(k(t)) = \sqrt{-3} E(k(t)) \): otherwise we have \( Q = \sqrt{-3} Q_1 \) with \( Q \in E(k(t)) \) but \( Q_1 \) defined not over \( k(t) \) but only over \( k(C) \); but then the other preimages of \( Q \) under \( \sqrt{-3} \) would be the translates of \( Q_1 \) by \( (0, \pm u) \), which are also defined over \( k(C) \), and we would obtain a cubic extension of \( k(t) \) split by the quadratic extension \( k(C)/k(t) \), which is impossible. Thus the restriction of our homomorphism from \( E(k(C)) \) to \( L_S = E(k(t)) \) yields an injection from \( L_S/\sqrt{-3} L(S) \) to \( k(C)^*/k(C)^* \). Since the divisor of \( y - u \) on \( \mathcal{E} \) is divisible by 3, and \( \mathcal{E} \) has no reducible fibers, the divisor of \( y(t) - u \) on \( C \) is also divisible by 3; call it \( 3D \). (This generalizes the argument of Theorem 1, and can also be shown as was done there without explicitly invoking the fibers of \( \mathcal{E} \), though the key point that \( S \) has distinct roots amounts to the same thing; in the context of descent the condition that the zero or pole multiplicity of \( y(t) - u \) be a multiple of 3 at each point \( (t_0, u_0) \) of \( C \) is the condition of local triviality at \( t_0 \) of a principal homogeneous space for the isogeny \( \sqrt{-3} \).) Then \( \{D\} \) is a 3-torsion point on \( J(C) \), and the map \( (x, y) \mapsto \{D\} \) is an injection from \( L_S/\sqrt{-3} L(S) \) to \( J(C)[3] \). It is clear that the restriction of this descent map to a root agrees with the nonzero element of \( J(C)[3] \) associated with the root in Theorem 1.
Theorem 5. The maps from the 240 representations of $S(t)$ as $y^2 - x^3$ to $J(C)[3]$ and $J(C)[2]$ obtained in Theorems 1 and 2 extend to surjective homomorphisms from $L_S$ to $J(C)[3]$ and $J(C)[2]$ whose kernels are $\sqrt{-3} L_S$ and $2 L_S$ respectively; these homomorphisms are the descent maps for the isogenies $\sqrt{-3}$ and 2 on $E$. The moduli space $S$ is the moduli space for sextics $S(t)$ equipped with an isomorphism of $L_S$ with $E_6^d$; reducing $E_6^d$ mod $\sqrt{-3}$ and 2 recovers the $J(C)[3]$ and $J(C)[2]$ constructions of $S$.

5. Complements and coming attractions.

Orbits of the point stabilizer and the height pairing. The simple group $G$ acts transitively on the 40 representations of $S$ as the difference between the cube of a quadratic and the square of a cubic polynomial; but its action is not doubly transitive: the orbits of a point stabilizer have sizes 1, 12, 27. It is easy to see this from either the mod-3 or the mod-2 definitions of $G$: in each case the choice of a line in a symplectic $F_4^d$ or an odd line in a unitary $F_4^d$
divides the remaining 39 such lines into 12 orthogonal to the first and 27 not orthogonal to it. But Clebsch already obtained the $12 + 27$ partition in [3], though he knew neither the Weil pairing on $J(C)[3]$ nor the relevance of $C'$. How, then, could he distinguish the orthogonal from the non-orthogonal pairs? In effect he did it using the height pairing! Given a root $r \in E_8^g$, the $6 \cdot 39$ roots not proportional to $r$ consist of $6 \cdot 12$ orthogonal to $r$ and $6 \cdot 27$ not orthogonal to it; if $r, r'$ are roots neither proportional nor orthogonal to each other then their inner product $(r, r')$ is one of $\mu_6 \cdot 2/\sqrt{-3}$. Now $(r, r')$ is determined by the norms of $r + r', r + \rho r'$, and $r + \rho^2 r'$, which are even integers whose sum is 12 (cf. (1)). If $r, r'$ are not proportional then these norms are positive and $< 8$; clearly $(r, r') = 0$ if and only if each norm equals 4, and so $(r, r') \neq 0$ if and only if the norms are 2, 4, 6 in some order. In the $E_8$ picture, $r, r'$ are minimal solutions $(x(t), y(t))$ and $(x'(t), y'(t))$ of $y^2 = x^3 + S$, and $r + \rho r'$ is a solution of $y^2 = x^3 + S$ whose coordinates are obtained from $(x, y)$ and $(\rho^s x', y')$ by the group law on $E$. The degrees of these coordinates as functions of $t$ then determine the norm of $r + \rho r'$ via our Proposition. The coordinates of $r + \rho r'$ involve the slope quotient $(y - y')/(x - \rho^s x')$, and thus hinge on the distribution of the zeros of $y - y'$ among the zeros of $x - \rho^s x'$. (Since $y^2 - y'^2 = x^3 - x'^3$, the zeros of the three quadratic polynomials $x - \rho^s x'$ are the same as the zeros of the two cubics $y \pm y'$.) We find that $(r, r') = 0$ exactly when each of $x - \rho^s x'$ contains just one of the linear factors of $y - y'$, which is exactly Clebsch’s condition for distinguishing $6 \cdot 12$ of the remaining $6 \cdot 39$ minimal solutions.

**Cubic curves tangent to six given concurrent lines.** The problem of enumerating solutions of $y^2 = x^3 + S(t)$ has also appeared in [Ty], where they arose in yet another geometric guise: the forty cubic curves tangent to the six lines $t = t_j$ in the $(s, t)$ plane, concurrent at the point $(1 : s : t) = (0 : 1 : 0)$ at infinity. The connection of these cubics with $J(C)[3]$ is then seen as a special case of the construction of the last section of [EX]. We explain these connections next.

On a generic plane cubic $C$ in the $(s, t)$-plane, $t$ is a rational function of degree 3, and thus has six ramified points, each with one simple and one double preimage on $C$, corresponding to the six lines through $(0 : 1 : 0)$ tangent to $C$. Now fix six distinct points $t_j \in \mathbb{P}^1$ and consider curves $C$ of genus 1 with a degree-3 map $t : C \to \mathbb{P}^1$ ramified at the $t_j$. By the Riemann existence theorem for branched covers of $\mathbb{P}^1$, such curves are in 1:1 correspondence with 6-tuples of involutions in the symmetric group $S_3$ whose product is the identity and which generate a transitive subgroup of $S_3$, modulo conjugation in the subgroup they generate. Ignoring the transitivity condition and the conjugations we find $3^5 = 243$ such 6-tuples: the first five involutions may be chosen arbitrarily, and uniquely determine the sixth. Of those 243, the transitivity condition excludes only those for which all six involutions are the same, leaving an adjusted total of $243 - 3 = 240$. Each of these must generate all of $S_3$, so our final
count of triple covers of $\mathbf{P}^1$ ramified at $t = t_j$ is $240/3! = 40$.

We can directly relate these 40 covers with with two of our three pictures of $\mathcal{S}$. First, they naturally biject with solutions of $y^2 = x^3 + S(t)$ in polynomials $x(t), y(t)$ of degrees at most 2, 3, up to equivalence $(x, y) \sim (\mu_3 x, \pm y)$. Second, they naturally biject with pairs $\pm D$ of nontrivial 3-torsion points on the Jacobian of $C : u^2 = S(t)$.

The latter bijection was given in \cite{E&}; more generally, for any distinct points $t_1, \ldots, t_{2g+2}$ in $\mathbf{P}^1$, a bijection was constructed between genus-$(g - 1)$ triple covers of $\mathbf{P}^1$ simply ramified at the $t_j$ and pairs of nontrivial 3-torsion points on the Jacobian of the genus-$g$ hyperelliptic curve $u^2 = \prod_{j=1}^{2g+2}(t - t_j)$. This curve is the discriminant of the triple cover; its compositum (a.k.a. fiber product over $\mathbf{P}^1$) with the triple cover is an $S_3$ cover of the line which is an unramified cyclic cubic cover of the hyperelliptic curve. By geometric Kummer theory such covers of any curve correspond bijectively with pairs of nontrivial 3-torsion points on the curve’s Jacobian. For a hyperelliptic curve one readily recovers from its cyclic cubic cover a triple cover of $\mathbf{P}^1$ ramified at the curve’s Weierstrass points. Given the $t_j$ there are thus $(3^g - 1)/2$ such triple covers, whose coefficients generate a field extension of generic Galois group contained in $\text{PSp}_{2g}(\mathbf{F}_3)$. The count of $(3^g - 1)/2$ can also be obtained as in the previous paragraph by solving equations in $S_3$. Before \cite{E&} it was already known \cite{Coh} that the Galois group is $\text{PSp}_{2g}(\mathbf{F}_3)$, but only by combining a transcendental monodromy computation with a difficult group-theoretical characterization of the permutation representation of $\text{PSp}_{2g}(\mathbf{F}_3)$ on $\mathbf{P}^{2g-1}(\mathbf{F}_3)$. In \cite{E&} the bijection with 3-torsion in the Jacobian was used to explain the symplectic structure in terms of the Weil pairing on the 3-torsion.

On the other hand, it is noted in \cite{Coh} that from $C$ and $t : C \to \mathbf{P}^1$ one may construct a solution of $y^2 = x^3 + S(t)$. Any function of degree 3 on a curve $C$ of genus 1 is the quotient of two sections of a divisor $D$ of degree 3; by Riemann-Roch, $D$ has a three-dimensional space of sections, which identifies $C$ with a cubic curve in the $(s, t)$ plane up to affine-linear transformations preserving $t$, i.e. up to transformations of the form $(t, s) \mapsto (t, as + bt + c)$. Using these transformations we put our cubic in the form $s^3 - 3x(t)s + 2y(t) = 0$ where the polynomials $x, y$ are of degree at most 2 and 3 respectively. This form is unique up to scalings $(s, x, y) \mapsto (\lambda s, \lambda^2 x, \lambda^3 y)$. The ramified points are the values of $t$ at which the discriminant of this cubic in $s$ vanishes, i.e. the roots of the sextic $y^2 - x^3$. Scaling by $\lambda$ multiplies $y^2 - x^3$ by $\lambda^6$.

Thus there is a $\lambda$, unique up to multiplication by $\mu_6$, that yields $y^2 - x^3 = S(t)$. Conversely, from a solution to $y^2 - x^3 = S(t)$ we recover a plane cubic $C : s^3 + 3x(t)s + 2y(t) = 0$ with a degree-3 cover $t : C \to \mathbf{P}^1$ ramified at the $t_j$, with two solutions $(x, y)$ producing the same curve if and only if they are equivalent under $(x, y) \sim (\mu_3 x, \pm y)$. At this point in \cite{Coh}, Cohen cites Clebsch for the enumeration of solutions of $y^2 - x^3 = S(t)$; indeed \cite{Coh} was the paper that alerted us to Clebsch’s work on this problem. It is easy to see that the pair of
3-torsion points of $J(C)$ constructed from $y^2 - x^3 = S(t)$ in Theorem 1 is the same as the pair obtained from the cubic $s^3 + 3x(t)s + 2y(t) = 0$ using the construction of $\mathbb{F}_k$.

**Ground fields without roots of unity.** We have assumed throughout that our ground field $k$ contains the cube roots of unity. If we work over a field, such as $\mathbb{Q}$, that does not contain them but still has characteristic other than 2 or 3, then we can still define the moduli spaces $(\mathbb{P}^1_s)$ and $\mathcal{S}$, but the latter space’s function field must contain $\mu_3$. This can be seen from each of our three pictures of $\mathcal{S}$: in the $J(C')[2]$ and Mordell-Weil pictures, via the action of $\rho$ on $C'$ and $\mathcal{E}$; in the $J(C)[3]$ picture, via the Weil pairing. [For the latter, it might seem that the sign ambiguity in $J(C)[3]$ may frustrate the extraction of a cube root of unity from the level-3 structure. But the only ambiguity is the possibility of multiplying all of $J(C)[3]$ by $-1$, and if $P, Q$ are 3-torsion points whose Weil pairing $\langle P, Q \rangle$ is $\rho$ then $\langle -P, -Q \rangle = \rho$ also.] The field extension $k(\mathcal{S})$ of $k((\mathbb{P}^1_s))$ then has Galois group $\text{Aut}(G)$, with the outer automorphism of $G$ inducing the Galois involution of $k(\rho)/k$.

**Twists, and $G$ vs. $\text{Aut}(E_6^G)$.** From a sextic $S(t) \in k[t]$ without repeated roots we find, from the fiber of $\mathcal{S}$ above the associated point of $(\mathbb{P}^1_s)$, a normal extension $k'$ of $k$ of degree at most 25920 and a map $\text{Gal}(k'/k) \to G$. But this $k'$ is not the field of definition $k_S$ of $L_S$: for each minimal solution $(x, y)$ of $y^2 - x^3 = S(t)$ the coordinates of $x^3, y^2$ are contained in $k'$, but not necessarily the coordinates of $x, y$ themselves. Indeed it might seem that to obtain $k'$ from $k$ we might need to extract a different sixth root for each of the 40 minimal representations of $S(t)$ as the difference between a square and a cube. But in the $E_6^G$ picture it is clear that a single cyclic sextic extension suffices to obtain $k_S$ from $k'$: the group $G$ is the quotient of $\text{Aut}(E_6^G)$ by its center $\mu_6$. Note, however, that the extension $k_S/k'$ is not visible at the level of moduli spaces: multiplying $S$ by $\lambda \neq 0$ does not change the associated point of $(\mathbb{P}^1_s)$ and thus gives rise to the same extension $k'$, but multiplies each of $x^3, y^2$ by $\lambda$ and thus twists the extension $k_S/k'$ by $\sqrt[6]{\lambda}$. Clearly $k_S/k'$ is the compositum of two extensions of $k'$, the first obtained by adjoining the coefficients of the $x$-coordinates of the roots, the second obtained by adjoining the coefficients of the $y$-coordinates. The former extension is either trivial or cyclic cubic, and the latter is either trivial or quadratic. Since the double cover $2G$ of $G$ does not split, the quadratic cover, though normal over $k$, cannot be obtained from $k_S$ by adjoining the square root of an element of $k$. On the other hand the index-2 subgroup of $\text{Aut}(E_6^G)$ does split as $\mu_4 \times G$, so the subfield of $k_S$ obtained by adjoining each root’s $x$ coefficients is the compositum of $k'$ with a cyclic cubic extension of $k$. It might be surmised that this extension is $k(\sqrt[6]{c\Delta(S)})$ for some $c \in \mathbb{Q}^*$, where $\Delta(S)$ is the discriminant of $S$; since $\Delta(\lambda S) = \lambda^{10}\Delta(S)$, this surmise would behave correctly under scaling of $S$. In a future paper we shall show that in fact $k(\sqrt[6]{\Delta(S)})$ (with $c = 1$) is the correct extension.

**Geometry of $\mathcal{S}$ and its fiber products with $(\mathbb{P}^1_s)$ and $(\mathbb{A}^1_s)$.** We have given three descriptions
of the moduli space $S$ but have not described its geometry. Burkhardt [B] had already in effect identified $S$ with an open set in his quartic hypersurface in $\mathbb{P}^4$, which is the zero locus of the degree-4 invariant of the five-dimensional representation of $G$ mentioned in footnote [B]. See also [H, p.190]. This hypersurface is rational [H, p.184–5], a fact Hunt attributed to Todd (1936), with an explicit birational map to $\mathbb{P}^3$ first given by Baker six years later. The fiber product over $(\mathbb{P}^1_6)$ of $S$ with $\mathbb{P}^1_{(6)}$ — that is, the moduli space of genus-2 curves $C$ with full level-6 structure, or equivalently of curves $C'$ with full level-$2\sqrt{3}$ structure — is an open set in an algebraic threefold of general type [HW]. Recall that this moduli space, call it $S'$, arose in the course of Hunt and Weintraub’s identification of the $G$-covers of $(\mathbb{P}^1_6)$ coming from $J(C)[3]$ and $J(C')[2]$; it is an $S_6$ cover of $S$. We have investigated an intermediate, non-Galois cover of $(\mathbb{P}^1_6)$, which is the fiber product of $S$ with the moduli space of configurations of six points of $\mathbb{P}^1$ one of which is distinguished. We may put this point at infinity, and consider the remaining five as points on the affine line $\mathbb{A}^1$; in keeping with our earlier notation we thus call this moduli space $(\mathbb{A}^1_5)$. Using the $E$ picture of $S$, and specializing the formulas of [H1], we have shown that the resulting $G$-cover of $(\mathbb{A}^1_5)$ is still rational, and is geometrically even nicer than $S$: it is the complement of hyperplanes in $\mathbb{P}^3 = P(V)!$ The representation of $2G$ on $V$ yields the action of $G$ on this projective space, and the excluded hyperplanes are the $40$ orthogonal complements of the roots of $E_8^p$. We shall show this in a future paper; the computations there will also verify the claim in the previous paragraph that $k(\sqrt[3]{\Delta(S)}) \subseteq k_S$.

**Degenerations.** In the same paper we shall also describe the behavior of the configuration of the $t_j$, and thus also of $J(C)[3]$, $J(C')[2]$ and $E$, as we approach the excluded hyperplanes. For instance, we find that as two of the five finite $t_j$ approach one another, the point on $\mathbb{P}^3$ approaches one of the hyperplanes, while as some $t_j \to \infty$ the point approaches one of the $40$ (projectivizations of) roots, which must be blown up to detect the configuration of the remaining four $t_j$. Perhaps most strikingly, we obtain the moduli space $A_2(3)$ of principally polarized abelian surfaces with full level-3 structure by blowing up $P^3$ at the $90$ lines such as $z_1 = z_2 - z_3 = 0$ that contain four of the $40$ projectivized roots: $A_2(3)$ is the complement in that blown-up $P^3$ of the proper transforms of the $40$ hyperplanes.

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