A SHARP SQUARE FUNCTION ESTIMATE FOR THE CONE IN $\mathbb{R}^3$

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Abstract. We prove a sharp square function estimate for the cone in $\mathbb{R}^3$ and consequently the local smoothing conjecture for the wave equation in $2+1$ dimensions.

1. Introduction

This paper concerns the restriction theory of the cone in $\mathbb{R}^3$. Let $\Gamma$ be the truncated light cone
$$
\Gamma = \{ \xi_1^2 + \xi_2^2 = \xi_3^2, 1/2 \leq \xi_3 \leq 1 \},
$$
and let $N_{R^{-1}}(\Gamma)$ denote its $R^{-1}$-neighborhood. Cover $N_{R^{-1}}(\Gamma)$ by finitely overlapping sectors $\theta$ of angular width $R^{-1}/2$, where each sector is a rectangular box of dimensions $\sim R^{-1} \times R^{-1}/2 \times 1$. If $\hat{f}$ has support on $N_{R^{-1}}(\Gamma)$, we consider a set of functions $\{f_\theta\}$ such that: (a) $\hat{f}_\theta$ is supported on $\theta$ and (b) $f = \sum_\theta f_\theta$. For example\(^1\) here is a natural way to choose $\{f_\theta\}$: let $\psi_\theta$ be a smooth partition of unity subordinate to the covering $\{\theta\}$, and define $f_\theta$ by $\hat{f}_\theta = \hat{f}_\psi$. We have $f = \sum_\theta f_\theta$. We prove the following sharp square function estimate for this decomposition.

Theorem 1.1. (Square function estimate) For any $\epsilon > 0$, $R \geq 1$ and any function $f$ whose Fourier transform is supported on $N_{R^{-1}}(\Gamma)$, we have
$$
\|f\|_{L^4(\mathbb{R}^3)} \leq C\epsilon R^{\epsilon} \left( \sum_\theta |f_\theta|^2 \right)^{1/2}.
$$

This type of square function estimate was considered by Mockenhaupt \[18\] who proved that it implies the cone multiplier conjecture in $\mathbb{R}^3$, and by Mockenhaupt–Seeger–Sogge \[19\] (in a slightly different form) who essentially showed that it implies the local smoothing conjecture for the wave equation in $2+1$ dimensions. Here we recall the local smoothing conjecture, and we refer to \[18\] and \[17\] for more information about the cone multiplier conjecture. The local smoothing conjecture was formulated by Sogge in \[22\]. It is about bounding the $L^p$ norm of a solution to the wave equation in terms of the Sobolev norms of the initial data. Sogge \[22\] noticed that Bourgain’s proof of the boundedness of the circular maximal operator in \[1\] can be used to establish “local smoothing” estimates with nontrivial gain of regularity. Combining Theorem 1.1 with the arguments in \[19\] gives the following theorem.

Theorem 1.2. (Local smoothing in $2+1$ dimensions) Suppose that $u(x,t)$ is a solution of the wave equation in $2+1$ dimensions, with initial data $u(x,0) = u_0(x)$ and $\partial_t u(x,0) = u_1(x)$. Then for any $p \geq 4$, and any $\alpha > \frac{1}{2} - \frac{2}{p}$,
$$
\|u\|_{L^p(\mathbb{R}^2 \times [1,2])} \leq C_\alpha \left( \|u_0\|_{p,\alpha} + \|u_1\|_{p,-1+\alpha} \right).
$$

\(^1\)We remark that what we prove about $\{f_\theta\}$ in this paper is uniform as long as (a) and (b) are satisfied, i.e. does not depend on the particular choice of $\{f_\theta\}$. 

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The critical case of this Theorem is when \( p = 4 \) and \( \alpha \) is close to zero. Mockenhaupt, Seeger, and Sogge [19] proved that this estimate holds for \( p = 4 \) with \( \alpha > 1/8 \), and this was improved afterwards by several authors ([23], [26], [16]). In [25], Wolff proved the local smoothing conjecture for \( p > 74 \) in the full range of \( \alpha \). In that paper, Wolff introduced the idea of decoupling. His method was extended to higher dimensions by Laba-Wolff [15] and refined by Garrigós-Seeger [12][13] and Garrigós-Schlag-Seeger[11]. Then in [2], Bourgain and Demeter proved sharp decoupling estimates for cones in all dimensions, in particular proving the local smoothing conjecture in \( 2 + 1 \) dimensions for \( p \geq 6 \) in the full range of \( \alpha \). The sharp decoupling estimate for the cone does not, however, imply the full range of local smoothing estimates – at the end of the introduction we will discuss what the issue is. In a different direction, Lee and Vargas [17] proved a sharp \( L^3 \) square function estimate using multilinear restriction.

One new feature of our approach is that we prove a stronger estimate which works better for induction on scales. We need a little notation to state this estimate. First we recall the locally constant property of \( f \). For each sector \( \theta \), we let \( \theta^* \) denote the dual rectangular box: since \( \theta \) has dimensions \( 1 \times R^{-1/2} \times R^{-1/2} \), \( \theta^* \) has dimensions \( 1 \times R^{1/2} \times R \). We call such a \( \theta^* \) a plank. Recall that \( |f_{\theta}| \) is roughly constant on each translated copy of \( \theta^* \). In this paper we tile \( \mathbb{R}^3 \) with translated copies of \( \theta^* \). The restriction of \( f_{\theta} \) to one translated copy of \( \theta^* \) is called a wave packet. In addition to the sectors \( \theta \), we will consider larger angular sectors \( \tau \) with any angle between \( R^{-1/2} \) and 1. We write \( d(\tau) \) to denote this angle, which we call the aperture of \( \tau \).

For each \( \tau \), we define \( f_{\tau} = \sum_{\theta \subset \tau} f_{\theta} \), and we define \( \tau^* \) to be the dual rectangle to \( \tau \). If \( d(\tau) = s \), then \( \tau^* \) has dimensions \( 1 \times s^{-1} \times s^{-2} \), and \( |f_{\tau}| \) is roughly constant on each translated copy of \( \tau^* \).

Next we define \( U_{\tau} \) to be a scaled copy of \( \tau^* \) with diameter \( R \). If \( d(\tau) = s \), then \( U_{\tau} \) has dimensions \( Rs^2 \times Rs \times R \). Note that if \( \theta \subset \tau \) and if \( T \) is a translated copy of \( \theta^* \) which passes through the center of \( U_{\tau} \), then \( T \subset 10U_{\tau} \). For each \( \tau \), we tile \( \mathbb{R}^3 \) by translated copies of \( U_{\tau} \).

\[
\mathbb{R}^3 = \bigcup_{U \text{ a translated copy of } U_{\tau}} U.
\]

This tiling is natural because for each \( \theta \subset \tau \), each wave packet of \( f_{\theta} \) is contained in \( \sim 1 \) tile \( U \) in the tiling. We write \( \sum_{U \cap U_{\tau}} \) to denote the sum over all the translated copies \( U \) of \( U_{\tau} \) in the tiling of \( \mathbb{R}^3 \). Sometimes we want to keep track of the diameter \( R \) when talking about \( U_{\tau} \). In such cases we also write \( U_{\tau} = U_{\tau,R} \).

\[\text{Such kind of “locally constant” heuristic will be used a few times in the current paper. To justify this intuition one can use Corollary 4.3 in [3]. See also Lemma 6.1 and Lemma 6.2 in Section 6 of the current paper.}\]
If $U$ is a translated copy of $U_\tau$, then we define the square function $S_Uf$ associated with $U$ to be

$$S_Uf = (\sum_{\theta \subseteq \tau} |f_\theta|^2)^{1/2}.$$

We can now state our main estimate.

**Theorem 1.3.** Suppose that $f$ has Fourier support on $N_{R^{-1}}(\Gamma)$. Then

$$\|f\|_{L^4(\mathbb{R}^3)}^4 \leq C_r R^c \sum_{R^{-1/2} \leq s \leq 1} \sum_{d(\tau) = s} \sum_{U \supseteq U_\tau} |U|^{-1/2} \|S_Uf\|_{L^2}^4.$$

Here the sum over $s$ is over dyadic values of $s$ in the range $R^{-1/2} \leq s \leq 1$.

Let us take a moment to digest the right-hand side of this estimate. For this discussion, suppose that $f$ is essentially supported on one $B_R$. We start with the term where $s = R^{-1/2}$. In this case $\tau$ is one of the original sectors $\theta$ of aperture $R^{-1/2}$, $U_\tau$ is equal to $\theta^*$, and $|S_Uf| = |f_\theta|$. Since $|S_Uf| = |f_\theta|$ is roughly constant on $U$, $|U|^{-1} \|S_Uf\|_{L^2}^4 \approx |S_Uf|_{L^4}^4$. If the functions $f_\theta$ are essentially supported on disjoint regions, we would have $\|f\|_{L^4}^4 = \sum_{d(\theta) = R^{-1/2}} \sum_{U \supseteq U_\tau} \|S_Uf\|_{L^2}^4$, which matches the term $s = R^{-1/2}$ on the right-hand side of (1). Next consider the term where $s = 1$. In this case, there is only one $\tau$ which covers all of $\Gamma$, and the contribution to the right-hand side is essentially $|B_R|^{-1} \|S_{B_R}f\|_{L^2}^4 \sim |B_R|^{-1} \|f\|_{L^2(B_R)}^4$. If $f$ is roughly constant on the whole $B_R$, then we would have $\|f\|_{L^4(\mathbb{R}^3)}^4 \approx |B_R|^{-1} \|f\|_{L^2(B_R)}^4 \sim |B_R|^{-1} \|S_{B_R}f\|_{L^2(B_R)}^4$, which matches the term $s = 1$ on the right-hand side of (1). Finally we consider the intermediate values of $s$. It may happen that $f = f_\tau$ for some $\tau$ and that $f$ is essentially supported on a particular translated copy $U$ of $U_\tau$ and that $|f\|$ is roughly constant on $U$. In this case, $\|f\|_{L^4(\mathbb{R}^3)}^4 \approx |f_\tau\|_{L^4(U)}^4 \approx |U|^{-1} \|f\|_{L^2(U)}^4 \sim |U|^{-1} \|S_Uf\|_{L^2}^4$, which is the term corresponding to $U$ on the right-hand side of (1).

The proof of Theorem 1.3 is based on a new Kakeya-type estimate, which controls the overlapping of the planks in the wave packet decomposition of $f$.

**Lemma 1.4.** Suppose that $\hat{f}$ has support on $N_{R^{-1}}(\Gamma)$. Let $g$ denote the (squared) square function $g = \sum_{d(\theta) = R^{-1/2}} |f_\theta|^2$. Then

$$\int_{\mathbb{R}^3} |g|^2 \lesssim \sum_{R^{-1/2} \leq s \leq 1} \sum_{d(\tau) = s} \sum_{U \supseteq U_\tau} |U|^{-1} \|S_Uf\|_{L^2}^4.$$

Recall that each function $f_\theta$ is morally constant on the translated copies of $\theta^*$, where each $\theta^*$ is a $1 \times R^{1/2} \times R$ plank. The estimate in Lemma 1.4 is a Kakeya-type bound on the overlapping of these planks. The new feature of this estimate compared to previous Kakeya-type estimates is the structure of the right-hand side, which is designed to match the right-hand side of Theorem 1.3. The terms on the right-hand side keep track of how planks are packed into the rectangular boxes $U$. If the planks are spread out in the sense that each box $U$ does not contain too many planks, then it gives a strong bound.

In [25], Wolff connected Kakeya-type estimates for overlapping planks to incidence geometry problems in the spirit of the Szemerédi-Trotter problem. He adapted the cutting method from incidence geometry to this setting and he used it to estimate the overlaps of planks. He applied those geometric estimates at many scales to prove his results on local smoothing. In [2], Bourgain and Demeter apply multilinear Kakeya estimates at many scales to prove decoupling. In this paper, we apply Lemma 1.4 at many scales to prove Theorem 1.3.
Lemma 1.4 is proven using Fourier analysis. By Plancherel, $\|g\|_2^2 = \int |\hat{g}|^2$. Roughly speaking, we decompose the Fourier space, and the contributions of different regions to $\int |\hat{g}|^2$ correspond to the different terms on the right-hand side of Lemma 1.4. This approach to proving Kakeya-type estimates is based on some work of Orponen in projection theory [20] and is related to Vinh’s work [24] about incidence geometry over finite fields. It builds on [14], which applies similar ideas to rectangles and tubes instead of planks.

Our Theorem 1.3 and Lemma 1.4 have their “local” counterparts. To state those we introduce polynomially decaying weights that are essentially supported on a given box. For any box $B_R$ of diameter $R$, define the weight $w_{B_R,E}(x) = (1 + \text{dist}(x, B_R)/R)^{-E}$.

Take Theorem 1.3 as an example, for any $\text{supp} f \subset N_{R^{-1}}(\Gamma)$ we multiply $f$ and $f_\theta$ by a rapidly decaying bump function $\phi_R$ adapted to $B_R$ such that $|\hat{\phi}_R| > 1 > 0$ on $B_R$ and $\hat{\phi}_R$ is supported on the ball $B_{R^{-1}}$ centered at the origin. Invoking Theorem 1.3 for the decomposition $\phi_R f = \sum \theta \phi_R f_\theta$, we obtain the following “local” counterpart of Theorem 1.3:

**Theorem 1.5.** Suppose that $f$ has Fourier support on $N_{R^{-1}}(\Gamma)$. We take a weight $w_{B_R,E}(x)$ that decays arbitrarily fast (with $E$ arbitrarily large). Then

$$\|f\|_{L^4(B_R)}^4 \leq C_{\epsilon,E} R^\epsilon \sum_{R^{-1/2} \leq s \leq 1} \sum_{d(\tau) = s} \sum_{U \cap B_R} |U|^{-1} \|w_{B_R,E} \cdot S_U f\|_{L^2}^4.$$  

Here the sum over $s$ is over dyadic values of $s$ in the range $R^{-1/2} \leq s \leq 1$.

In the above theorem, the sum on the right-hand side is also “morally localized”. It is

$$\sum_{R^{-1/2} \leq s \leq 1} \sum_{d(\tau) = s} \sum_{U \cap B_R \subset \text{100B}_R} |U|^{-1/2} \|S_U f\|_{L^2}^4$$

plus some decaying error term.

While working on this project, we were strongly influenced by ideas related to decoupling, but the proof given here does not use the decoupling theorem per se. It does make use of a nice observation that Bourgain and Demeter used to reduce the decoupling theorem for the cone to the decoupling theorem for the paraboloid (See [2]). Similar ideas can also be traced back to the iteration argument of Pramanik-Seeger [21]). Instead of working with a truncated cone of height 1, Bourgain and Demeter worked with a truncated cone of height $1/K$ for a large constant $K$, denoted $\Gamma^{1/K}$. This shorter truncated cone can be approximated by a parabola at various scales. We will also work with $\Gamma^{1/K}$, allowing us to bring into play some estimates for the parabola.

As we mentioned above, sharp decoupling theorems do not imply the full range of local smoothing estimates or the square function estimate. Let us explain a little further what the issue is. The decoupling theorem for the cone gives the following bounds, which are sharp for every $p$ between 2 and $\infty$:

$$\|f\|_{L^p(\mathbb{R}^3)} \leq C_{\epsilon} R^\epsilon \left( \sum_{d(\tau) = R^{-1/2}} \|f_\theta\|_{L^p(\mathbb{R}^3)}^2 \right)^{1/2}$$

if $2 \leq p \leq 6,$

$$\|f\|_{L^p(\mathbb{R}^3)} \leq C_{\epsilon} R^\epsilon \left( \sum_{d(\tau) = R^{-1/2}} \|f_\theta\|_{L^p(\mathbb{R}^3)}^2 \right)^{1/2}$$

if $2 \leq p \leq 6,$
\[
\|f\|_{L^p(\mathbb{R}^3)} \leq C_R R^{\frac{1}{2}} \frac{1}{2^p} \left( \sum_{d(\theta) = R^{-1/2}} \|f_\theta\|_{L^p(\mathbb{R}^3)}^2 \right)^{1/2} \text{ if } p \geq 6.
\]

For any given \( p \), (4) implies local smoothing for that \( p \). But the inequality (4) cannot hold for any \( p < 6 \) because the power of \( R \) would be negative. The power of \( R \) in a decoupling inequality cannot be negative because of the following example: suppose that for each \( \theta \), \( |f_\theta| \) is approximately the characteristic function of \( B_R \), and at each point \( |f| \sim (\sum_\theta |f_\theta|^2)^{1/2} \). In this case, \( \|f\|_{L^p} \sim \left( \sum_\theta \|f_\theta\|_{L^p}^2 \right)^{1/2} \) for all \( p \). This example is not a counterexample for local smoothing, but to prove local smoothing for some \( p < 6 \) we have to do better than inequality (4) in some scenarios: for instance, if the supports of \( f_\theta \) are essentially disjoint at time 0. Roughly speaking, we need to improve the bound (4) when \( p < 6 \) and when each \( f_\theta \) is essentially supported on a sparse region of \( B_R \). Theorem 1.3 makes this precise.

There are similar issues in the problem of decoupling into small caps, which was studied in [9]. For instance, consider an exponential sum of the form

\[
f(x_1, x_2) = \sum_{j=1}^N a_j e \left( \frac{j}{N} x_1 + \frac{j^2}{N^2} x_2 \right), \text{ with } |a_j| \leq 1 \text{ for all } j.
\]

The decoupling theorem for the parabola gives a sharp bound on \( \|f\|_{L^p(B_{N^2})} \) for every \( p \). But suppose we want to bound \( \|f\|_{L^p(B_R)} \) for some \( R < N^2 \). If we divide the parabola into arcs \( \theta \) of length \( R^{-1/2} \), then each \( f_\theta \) is a sum of \( \sim NR^{-1/2} \) terms of (\#). It’s not hard to estimate the largest possible value of \( \|f_\theta\|_{L^p(B_R)} \) for each \( p \). Combining this bound for \( \|f_\theta\|_{L^p(B_R)} \) with decoupling gives an upper bound for \( \|f\|_{L^p(B_R)} \), but it isn’t sharp. When \( \|f_\theta\|_{L^p(B_R)} \) is close to its largest value, then \( |f_\theta| \) is concentrated on a sparse region of \( B_R \). The argument in [9] exploits this sparsity to improve the bound from decoupling and give sharp estimates for \( \|f\|_{L^p(B_R)} \) for every \( p \). The proof of the main theorem here builds on that proof.

The paper [9] also considers a decoupling problem in which the cone is divided into small squares instead of sectors. This problem was raised by Bourgain and Watt [5] in their work on the Gauss circle problem. The paper [9] shows that the square function estimate Theorem 1.1 implies a sharp estimate for this decoupling problem.

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2. Proof of the square function estimate from Theorem 1.3

In this section, we explain how Theorem 1.3 implies the square function estimate Theorem 1.1 and we discuss how the latter implies the local smoothing Theorem 1.2. First we recall the statement of Theorem 1.1:

**Theorem.** For any function $f$ whose Fourier transform is supported on $N_{R^{-1}}(\Gamma)$, we have

$$\|f\|_{L^4(\mathbb{R}^3)} \leq C_\Gamma R^s \left( \sum_{d(\theta) = R^{-1/2}} \|f_\theta\|^2 \right)^{1/2} \|f\|_{L^4(\mathbb{R}^3)}.$$

**Proof.** Let $U$ be a translated copy of $U_\tau$. Recall that

$$\|S_U f\|_{L^2} = \int_U \sum_{\theta \subset \tau} |f_\theta|^2.$$

By Cauchy-Schwarz,

$$\|S_U f\|_{L^4}^4 \leq |U| \int_U \left( \sum_{\theta \subset \tau} |f_\theta|^2 \right)^2.$$

Therefore,

$$\sum_{d(\tau) = s} \sum_{U \ni \tau} |U|^{-1} \|S_U f\|_{L^4}^4 \leq \sum_{d(\tau) = s} \int_{\mathbb{R}^3} \left( \sum_{\theta \subset \tau} |f_\theta|^2 \right)^2 \leq \int_{\mathbb{R}^3} \left( \sum_{\theta} |f_\theta|^2 \right)^2.$$ 

Essentially by [19], the square function estimate in Theorem 1.1 implies the local smoothing Theorem 1.2 for the wave equation in 2+1 dimensions. One technical difference is that the square function considered in [19] was the one in terms of “small caps” $\zeta$, $R^{-1/2}$-squares on $\Gamma$. Instead of the Littlewood–Paley estimate corresponding to equally spaced decompositions in $\mathbb{R}^2$ used in [19] (see (1.9) and the following first two lines on page 214 of [19]), one needs such an estimate for angular decompositions. In the $L^4$ case, such an angular square function estimate was proved by Córdoba (see ii) on the first page of [7]). Another proof by Carbery-Seeger could be found in [8].

3. Outline of the proof of the main theorem

In this section, we give an overview of the proof of Theorem 1.3 and outline the rest of the paper. First we review the statement of Theorem 1.3 and present it in a more detailed way.

Let $\Gamma$ be the truncated light cone $\Gamma = \{ \xi_1^2 + \xi_2^2 = \xi_3^2, 1/2 \leq |\xi_3| \leq 1 \}$. Let $f$ be a function whose Fourier transform is supported on $N_{R^{-1}}(\Gamma)$. We now precisely define the sectors $\theta$ and $\tau$ from the introduction. For each point $\xi \in \Gamma$ with $\xi_3 = 1$, we define a basis of $\mathbb{R}^3$ as follows: the core line direction is $c(\xi) = (\xi_1, \xi_2, 1)$, the normal direction is $n(\xi) = (\xi_1, \xi_2, -1)$, and the tangent direction is $t(\tau) = (-\xi_2, \xi_1, 0)$. Then we define a sector

$$\theta(\xi) := \{ \omega \in \mathbb{R}^3 : 1 \leq c(\xi) \cdot \omega \leq 2 \text{ and } |n(\xi) \cdot \omega| \leq R^{-1} \text{ and } |t(\xi) \cdot \omega| \leq R^{-1/2} \}.$$ 

We choose $10R^{1/2}$ evenly spaced $\xi$ in the circle $\Gamma \cap \{ \xi_3 = 1 \}$, and we let $S$ be the set of $\theta(\xi)$ for these $\xi$. It’s straightforward to check that these form a finitely overlapping cover of $N_{R^{-1}}(\Gamma)$. 

We can define sectors $\tau$ with larger angular aperture in a similar way. For any $s$ in the range $R^{-1/2} \leq s \leq 1$, and any $\xi$ as above, we can define

$$
\tau(s, \xi) := \{ \omega \in \mathbb{R}^3 : 1 \leq c(\xi) \cdot \omega \leq 2 \text{ and } |n(\xi) \cdot \omega| \leq s^2 \text{ and } |t(\xi) \cdot \omega| \leq s \}.
$$

Here $s = d(\tau)$ is the aperture of $\tau$ as described in the introduction.

For each dyadic $s$ in the range $R^{-1/2} \leq s \leq 1$, we choose $\sim s^{-1}$ evenly spaced $\xi$ in the circle, and let $S_s$ be the set of $\tau(s, \xi)$ for these $\xi$.

For each $\tau = \tau(s, \xi)$, we define a box $U_\tau$ as follows:

$$
U_\tau := \{ x \in \mathbb{R}^3 : |c(\xi) \cdot x| \leq Rs^2 \text{ and } |n(\xi) \cdot x| \leq R \text{ and } |t(\xi) \cdot x| \leq Rs \}.
$$

The box $U_\tau$ can also be described as the convex hull of the union of $\theta^*$ over all $\theta \subset \tau$. In other words, $U_\tau$ is the smallest rectangular box such that for any $\theta \subset \tau$, if a translated copy of $\theta^*$ intersects $U_\tau$, then it must lie in $10U_\tau$. We tile $\mathbb{R}^3$ by translated copies of $U_\tau$.

If $U$ is a translated copy of $U_\tau$, then we define $S_U f$ by

$$
S_U f = \left( \sum_{\theta \subset \tau, d(\theta) = R^{-1/2}} |f_\theta|^2 \right)^{1/2} |U|.
$$

As written, this definition appears to depend upon $U$, $\tau$, and $R$. But in fact $R$ and $\tau$ can be read off from $U$. $R$ is the diameter of $U$. The aperture $d(\tau) = s$ can be read off from the dimensions of $U$, which are $Rs^2 \times Rs \times R$. And the direction $\xi$ of $\tau$ can be read off from the direction of $U$.

For example, suppose that $B_r$ is a cube of side length $r$, and consider $S_{B_r} f$. Here $R$ is the diameter of $B_r$, which is $r$. Since the dimensions of $B_r$ are $r \times r \times r$, the aperture of $\tau$ is $s = 1$. There is only one $\tau$ with aperture 1 - it covers all of $\Gamma$. Therefore,

$$
S_{B_r} f = \left( \sum_{\theta = r^{-1/2}} |f_\theta|^2 \right)^{1/2} |B_r|.
$$

In particular, $|S_{B_1} f|$ is just $|f|$ restricted to $B_1$.

We define $S(r, R)$ as the smallest constant such that

$$
\sum_{B_r \subset \mathbb{R}^3} |B_r|^{-1} \|S_{B_r} f\|_{L^2(B_r)}^4 \leq S(r, R) \sum_{R^{-1/2} \leq s \leq 1} \sum_{\tau \subset \mathbb{R}^3} |U|^{-1} \|S_U f\|_{L^2}^4
$$

holds for every function $f$ with supp $\hat{f} \subset N_{R^{-1}}(\Gamma)$. On the left-hand side of inequality (7), $\sum_{B_r \subset \mathbb{R}^3}$ means the sum over a tiling of $\mathbb{R}^3$ with $B_r$. On the right-hand side of inequality (7), the first sum $\sum_{R^{-1/2} \leq s \leq 1}$ means the sum over dyadic numbers $s$ between $R^{-1/2}$ and 1 and the second sum is the sum over $\tau$ in $S_s$. Note that $R$ is involved in the definition of $U_\tau$ and hence definition of $U$. By Hölder’s inequality, $S(r, R) < \infty$ for any $0 < r, R < \infty$. We will only consider $S(r, R)$ when $r < R$. Theorem [3] is equivalent to the bound $S(1, R) \leq C \cdot R^e$ since $|S_{B_1} f| = |f|$ on any $B_1$ and $|f|$ is morally constant on $B_1$. We will derive Theorem [3] from a series of bounds about $S(r, R)$.

In Section 4 we prove the Kakeya-type estimate Lemma [4] and we use it to prove

**Lemma 3.1.** For any $r \geq 10$, $r_1 \in [r, r^2]$,

$$
S(r_1, r^2) \leq C.
$$
Next we bring into play a trick from the proof of decoupling for the cone in \([2]\): instead of working with \(\Gamma\) we work with a subset of \(\Gamma\) that lies close to a parabola. We let \(P\) denote an arc of a parabola of length \(\sim 1\) lying in \(\Gamma\). For any \(K \geq 10\), we define \(\Gamma_{\frac{1}{K}}\) to be the \(1/K\)-neighborhood of \(P\) in \(\Gamma\). We will eventually choose \(K\) to be a large constant depending on \(\epsilon\) (which remains fixed as \(R \to \infty\)). The precise formula for \(\Gamma_{\frac{1}{K}}\) is designed to make Lorentz rescaling work in a clean way, and we give the formula in Section 5 when we discuss Lorentz rescaling. We can define a sector \(\tau \subset \Gamma_{\frac{1}{K}}\) and its aperture \(d(\tau)\) in the same way as before (again see Section 5 for equations). Then we define \(S_K(r, R)\) as the smallest constant such that (7) holds for every \(f\) with \(\text{supp} \hat{f} \subset N_{R^{-1}}(\Gamma_{\frac{1}{K}})\). Since \(\Gamma_{\frac{1}{K}} \subset \Gamma\), \(S_K(r, R) \leq S(r, R)\). On the other hand, since \(K\) will be a chosen constant, \(S_K(r, R)\) is almost equal to \(S(r, R)\) and we can use it equally well to prove Theorem 1.3.

If \(R = K\), then \(N_{R^{-1}}(\Gamma_{\frac{1}{K}})\) is the \(1/K\)-neighborhood of the parabolic arc \(P\), and the restriction theory for the parabola can be used to study \(S_K(1, K)\). In Section 6 we use this idea to prove the following lemma.

**Lemma 3.2.** For any \(K \geq 10\), any \(1 \leq r \leq K\), and any \(\delta > 0\), \(S_K(r, K) \leq C_4 K^{\delta}\).

Theorem 1.3 will follow by combining Lemma 3.1 and Lemma 3.2 with some Lorentz rescaling. The Lorentz rescaling is the analogue of the parabolic rescaling in the context of the cone. We review it in Section 5. We use the Lorentz rescaling in Section 7 to prove the following lemma, which relates \(S_K(r, R)\) for various values of \(r, R\):

**Lemma 3.3.** For any \(r_1 < r_2 \leq r_3\),

\[
S_K(r_1, r_3) \leq \log r_2 \cdot S_K(r_1, r_2) \max_{r_2^{-1/2} \leq s \leq 1} S_K(s^2 r_2, s^2 r_3).
\]

This lemma is an important motivation for working with \(S_K(r, R)\). It allows Lemma 3.1 and Lemma 3.2 to be applied at many different scales. A key point of studying Theorem 1.3 instead of trying to prove Theorem 1.4 directly is that it allows this multiscale analysis to come into play.

Assuming the lemmas, we now prove bounds on \(S_K(r, R)\) and use them to deduce Theorem 1.3.

**Proposition 3.4.** For any \(\epsilon > 0\), there exists \(K = K(\epsilon)\) so that for any \(1 \leq r \leq R\), we have

\[
S_K(r, R) \leq \widetilde{C}_\epsilon (R/r)^{\epsilon}.
\]

**Proof.** We apply induction on the scale \(R\). Also, for each \(R\), we apply induction on \(R/r\).

If \(R \leq K = K(\epsilon)\), then it is straightforward to check that \(S_K(r, R)\) is bounded by a constant \(C_4 K = \widetilde{C}_\epsilon\). Also, from the definition of \(S_K(r, R)\), it is straightforward to check that \(S_K(R, R) \leq 1\). These cases are the base of the induction.

If \(r \leq K^{1/2}\), we apply Lemma 3.3 with \(r_1 = r, r_2 = K,\) and \(r_3 = R\). Then we apply Lemma 3.2 and induction:

\[
S_K(r, R) \leq \log K \cdot S_K(r, K) \max_{K^{-1/2} \leq s \leq 1} S_K(s^2 K, s^2 R) \leq \log K \cdot C_4 \widetilde{C}_\epsilon K^\delta (R/K)^{\epsilon}.
\]

We choose \(\delta = \epsilon/4\), and then we choose \(K = K(\epsilon)\) large enough so that \(\log K \cdot C_\epsilon K^{-\epsilon/4} K^{\epsilon/4} \leq 1\), and the induction closes in this case.

If \(K^{1/2} \leq r \leq R^{1/2}\), then we apply Lemma 3.3 with \(r_1 = r, r_2 = r^2,\) and \(r_3 = R\), and then we apply Lemma 3.1 and induction:
\[ S_K(r, R) \leq 2 \log r \cdot S_K(r, r^2) \max_{r^{-1} \leq s \leq 1} S_K(s^2 r^2, s^2 R) \leq 2 \log r \cdot C\tilde{c}(\frac{R}{r})^\epsilon. \]

We choose \( K = K(\epsilon) \) large enough so that for all \( r \geq K^{1/2} \), we have \( 2 \log r \cdot C r^{-\epsilon} \leq 1 \), and the induction closes in this case.

If \( r > R^{1/2} \), then we apply Lemma 3.1 to get \( S_K(r, R) \lesssim 1 \). \( \square \)

Finally we show how Proposition 3.3 implies Theorem 1.3. Proposition 3.3 implies that for every \( \epsilon > 0 \), we can choose \( K = K(\epsilon) \) so that \( S_K(1, R) \leq C\epsilon R^\epsilon \) for all \( R \). Suppose that the support of \( \hat{f} \) is contained in \( N_{R^{-1}}(\Gamma_{\pm}) \subset B_3 \). Since \( |f| \) is morally constant on unit balls, we have\(^3\)

\[
\int_{\mathbb{R}^3} |f|^4 \lesssim \sum_{B_1 \subset \mathbb{R}^3} \|f\|^4_{L^2(B_1)} = \sum_{B_1 \subset \mathbb{R}^3} \|S_{B_1} f\|^4_{L^2(B_1)} \leq C\epsilon R^\epsilon \sum_{R^{-1/2} \leq s \leq 1} \sum_{d(\tau) = s} \sum_{U/\tau} |U|^{-1} \|SU f\|^4_{L^2}.
\]

This inequality is essentially Theorem 1.3 except that we assumed that \( \hat{f} \) is supported on \( N_{R^{-1}}(\Gamma_{\pm}) \) instead of \( N_{R^{-1}}(\Gamma) \). Since \( N_{R^{-1}}(\Gamma) \) can be covered by \( O(K) = O_{\epsilon}(1) \) affine copies of \( \Gamma_{\pm} \), we can reduce Theorem 1.3 to (8). Here are the details.

Let \( \{A_j\}_{1 \leq j \leq K} \) be a collection of linear transformations such that \( \Gamma \subset \bigcup A_j(\Gamma_{\pm}) \). Each \( A_j \) is a composition of a scaling by a factor \( \sim 1 \) and a rotation in the \((\xi_1, \xi_2)\)-plane. Similarly, we can arrange that \( N_{R^{-1}}(\Gamma) \subset \bigcup A_j(N_{R^{-1}}(\Gamma_{\pm})) \). Let \( \{\psi_j\} \) be a \( C^\infty \) partition of unity subordinate to this covering. This partition of unity only depends on \( K \). If \( f \) is a function whose Fourier transform is supported on \( N_{R^{-1}}(\Gamma) \), then \( \hat{f} = \sum_j \psi_j \hat{f} \). Define \( f_j \) by \( \hat{f}_j = \psi_j \hat{f} \) and \( \hat{f}_{j, \theta} = \psi_j \hat{f}_\theta \). The support of \( \hat{f}_j \) is contained in \( A_j(N_{R^{-1}}(\Gamma_{\pm})) \). Since (8) is invariant under rotations and approximately invariant under rescaling by a factor \( \sim 1 \), (8) holds for each function \( f_j \).

Now by the triangle inequality and Hölder’s inequality,

\[
\|f\|^4_{L^4(\mathbb{R}^3)} \lesssim \sum_j \|f_j\|^4_{L^4(\mathbb{R}^3)} \\
\lesssim K^3 C\epsilon R^\epsilon \sum_{j} \sum_{R^{-1/2} \leq s \leq 1} \sum_{d(\tau) = s} \sum_{U/\tau} |U|^{-1} \|SU f_j\|^4_{L^2} \\
\lesssim K^3 C\epsilon R^\epsilon \sum_{R^{-1/2} \leq s \leq 1} \sum_{d(\tau) = s} \sum_{U/\tau} |U|^{-1} (\sum_{j} \|SU f_j\|^2_{L^2})^2 \\
\lesssim K C\epsilon R^\epsilon \sum_{R^{-1/2} \leq s \leq 1} \sum_{d(\tau) = s} \sum_{U/\tau} |U|^{-1} \|SU f\|^4_{L^2}.
\]

To see the last inequality, note that \( f_{j, \theta} = f_\theta \ast \psi_j \) and \( \psi_j \) is rapidly decaying outside the ball of radius \( K \) centered at the origin. Hence, by Lemma 6.2 each \( \|f_{j, \theta}\|_{L^2(B_1)} \lesssim K \|f_\theta\|_{L^2(w_{B_1, E})} \) for any polynomially decaying weight \( w_{B_1, E} \). It suffices to take \( E \) large enough.

Since \( K \) is a constant only depending on \( \epsilon \), this gives Theorem 1.3.

\(^3\)Strictly speaking, one need to apply Lemma 6.1 and Lemma 6.2 to justify the first “\( \lesssim \)” in inequality (8). This is similar to the arguments in Section 6 where we do in full details.
4. A Kakeya-type estimate

In this section, we prove the Kakeya-type estimate Lemma 1.4 and we use it to prove Lemma 3.1.

Proof of Lemma 1.4. If supp \( \hat{f} \subset N_{r^{-2}}(\Gamma) \), then we define the (squared) square function \( g \) of \( f \) as
\[
g = \sum_{a(\theta) = r^{-1}} |f_a|^2.
\]
The Fourier transform of \( |f_a|^2 \) is supported on the Minkowski sum \( \tilde{\theta} = \theta + (-\theta) \).

The set \( \tilde{\theta} \) is itself a plank of dimensions \( \sim r^{-2} \times r^{-1} \times 1 \) centered at the origin. Notice that while the original sectors \( \theta \) are disjoint, the planks \( \tilde{\theta} \) are not disjoint. The way that they overlap plays an important role in the proof.

We need a little notation to proceed, and we start by recalling the definition of the sector \( \theta(\xi) \).

For each point \( \xi \in \Gamma \) with \( \xi_3 = 1 \), we define a basis of \( \mathbb{R}^3 \) as follows: the core line direction is \( c(\xi) = (\xi_1, \xi_2, 1) \), the normal direction is \( n(\xi) = (\xi_1, \xi_2, -1) \), and the tangent direction is \( t(\tau) = (-\xi_2, \xi_1, 0) \). Then we define a sector
\[
\theta(\xi) := \{ \omega \in \mathbb{R}^3 : 1 \leq c(\xi) \cdot \omega \leq 2 \text{ and } |n(\xi) \cdot \omega| \leq r^{-2} \text{ and } |t(\xi) \cdot \omega| \leq r^{-1} \}.
\]

The Minkowski sum \( \tilde{\theta}(\xi) = \theta(\xi) + (-\theta(\xi)) \) is approximately equal to the following rectangular box:
\[
\tilde{\theta}(\xi) \sim \{ \omega \in \mathbb{R}^3 : |c(\xi) \cdot \omega| \leq 1 \text{ and } |n(\xi) \cdot \omega| \leq r^{-2} \text{ and } |t(\xi) \cdot \omega| \leq r^{-1} \}.
\]

The overlapping of the boxes \( \tilde{\theta} \) is best described in terms of similar rectangular boxes at smaller scales. For any dyadic \( \sigma \) in the range \( r^{-1} \leq \sigma \leq 1 \), and any \( \xi \) as above, we define a box \( \Theta(\sigma, \xi) \) by
\[
\Theta(\sigma, \xi) = \{ \omega : |\omega \cdot c(\xi)| \leq \sigma^2 \text{ and } |\omega \cdot n(\xi)| \leq r^{-2} \text{ and } |\omega \cdot t(\xi)| \leq r^{-1} \sigma \}.
\]

Notice that \( \Theta(1, \xi) \) is essentially equal to \( \tilde{\theta}(\xi) \), and for \( \sigma < 1 \), \( \Theta(\sigma, \xi) \subset \tilde{\theta}(\xi) \). At the other extreme, \( \Theta(r^{-1}, \xi) \) is essentially the ball of radius \( r^{-2} \) centered at the origin, regardless of \( \xi \).

If we intersect \( \Theta(\sigma, \xi) \) with the slab \( \{(1/2) \sigma^2 \leq \omega_3 \leq \sigma^2 \} \), then it lies in the \( r^{-2} \)-neighborhood of the light cone. Let \( \Gamma(\sigma^2) \) denote the part of the light cone where \( (1/2) \sigma^2 \leq \omega_3 \leq \sigma^2 \). Each \( \Theta(\sigma, \xi) \cap \{(1/2) \sigma^2 \leq \omega_3 \leq \sigma^2 \} \) is a sector of \( N_{r^{-2}}(\Gamma(\sigma^2)) \), just as \( \theta \) is a sector of \( N_{r^{-2}}(\Gamma) \).

The number of such sectors needed to cover \( N_{r^{-2}}(\Gamma(\sigma^2)) \) is \( \sim \sigma r \). If \( |\xi - \xi'| > \sigma^{-1} r^{-1} \), then \( \Theta(\sigma, \xi) \cap \Theta(\sigma, \xi') \cap \{(1/2) \sigma^2 \leq \omega_3 \leq \sigma^2 \} \) is empty. Conversely, if \( |\xi - \xi'| < \sigma^{-1} r^{-1} \) then \( \Theta(\sigma, \xi) \cap \{(1/2) \sigma^2 \leq \omega_3 \leq \sigma^2 \} \) is comparable to \( \Theta(\sigma, \xi') \cap \{(1/2) \sigma^2 \leq \omega_3 \leq \sigma^2 \} \). By symmetry, the same holds when we intersect with \( -\sigma^2 \leq \omega_3 \leq -(1/2) \sigma^2 \) at the other side of the light cone. Now by convexity, we conclude that if \( |\xi - \xi'| \leq \sigma^{-1} r^{-1} \), then \( \Theta(\sigma, \xi) \subset \tilde{\Theta}(\sigma, \xi') \).

For each dyadic \( \sigma \) in the range \( r^{-1} \leq \sigma \leq 1 \), let \( CP_{\sigma} \) be a set of \( \sim \sigma r \) planks of the form \( \Theta(\sigma, \xi) \) with the directions \( \xi \) evenly spaced in the circle. (The letters \( CP \) stand for centered plank.) The size of \( CP_{\sigma} \) is chosen so that for any \( \Theta(\sigma, \xi) \), we can choose \( \Theta(\sigma, \xi') \in CP_{\sigma} \) so that \( \Theta(\sigma, \xi) \subset 2 \Theta(\sigma, \xi') \).

We define \( CP \) as a union over dyadic scales: \( CP = \cup_{r^{-1} \leq \sigma \leq 1} CP_{\sigma} \).

For a given \( \tilde{\theta}(\xi) \) and a given scale \( \sigma \), there are \( \sim 1 \) many \( \Theta(\sigma, \xi') \in CP_{\sigma} \) with \( \Theta \subset 2 \tilde{\theta} \). On the one hand, \( \Theta(\sigma, \xi) \subset \tilde{\theta}(\xi) \), and we can choose \( \Theta(\sigma, \xi') \subset CP_{\sigma} \) so that \( \Theta(\sigma, \xi') \subset 2 \Theta(\sigma, \xi) \). On the other hand, \( \tilde{\theta}(\xi) \cap N_{r^{-2}}(\Gamma(\sigma^2)) \) is essentially equal to the sector \( \Theta(\sigma, \xi) \cap \{(1/2) \sigma^2 \leq \omega_3 \leq \sigma^2 \} \), and so \( 2 \tilde{\theta}(\xi) \) contains \( \Theta(\sigma, \xi') \) only if \( |\xi - \xi'| \lesssim \sigma^{-1} r^{-1} \).
Now for each scale $\sigma$, for each $\theta = \theta(\xi) \in \mathbf{S}$, we associate one $\Theta = \Theta(\sigma, \xi) \in \mathbf{CP}_\sigma$ with $|\xi - \xi'| \leq \sigma^{-1}r^{-1}$. For each $\Theta \in \mathbf{CP}_\sigma$, we let $\mathbf{S}_\Theta$ be the set of all $\theta \in \mathbf{S}$ which are associated with $\Theta$. So for each $\sigma$, $\mathbf{S} = \bigcup_{\Theta \in \mathbf{CP}_\sigma} \mathbf{S}_\Theta$. If $\theta \in \mathbf{S}_\Theta$, then $\Theta \subset 2\tilde{\Theta}$.

Let $\Omega = \cup_{\Theta \in \mathbf{S}} \Theta \sim \cup_{\Theta \in \mathbf{CP}_\sigma, \Theta}$. Since $(|f_\theta|^2)^\wedge$ is supported on $\tilde{\Theta}$, it follows that $\tilde{g}$ is supported on $\tilde{\Omega}$. We break $\Omega$ into pieces associated with different scales $\sigma$ as follows. We define $\Omega_{\leq \sigma} = \cup_{\Theta \in \mathbf{CP}_\sigma, \Theta} \Theta$. Then we define $\Omega_\sigma = \Omega_{\leq \sigma} \setminus \Omega_{\leq \sigma/2}$ if $\sigma > r^{-1}$, and we define $\Omega_{r^{-1}} = \Omega_{\leq r^{-1}}$, so that

$$\Omega = \bigsqcup_{1 \leq \sigma \leq r^{-1}} \Omega_\sigma.$$ (Here $\bigsqcup$ denotes a disjoint union, and the union is over dyadic $\sigma$.)

Now if $\omega \in \Omega_\sigma$, we bound $|\tilde{g}(\omega)|$ as follows:

$$|\tilde{g}(\omega)| = \left| \sum_\theta \Theta(\sigma, \xi) \cap \{ |f_\theta|^2 \}^\wedge(\omega) \right| \leq \sum_{\Theta \in \mathbf{CP}_\sigma} \left| \sum_{\theta \in \mathbf{S}_\Theta} \Theta(\sigma, \xi) \cap \{ |f_\theta|^2 \}^\wedge(\omega) \right|.

**Lemma 4.1.** If $\Theta \in \mathbf{CP}_\sigma$ makes a non-zero contribution to the right-hand side of (10), then $\omega \in 4\Theta$.

*Proof.* Suppose that $\sum_{\theta \in \mathbf{S}_\Theta} \Theta(\sigma, \xi) \cap \{ |f_\theta|^2 \}^\wedge(\omega)$ is non-zero. Then we must have $\omega \in \tilde{\Theta}$ for some $\theta \in \mathbf{S}_\Theta$. Suppose $\theta = \theta(\xi)$ and $\Theta = \Theta(\sigma, \xi')$. We know that $|\xi - \xi'| \leq \sigma^{-1}r^{-1}$ and so $\Theta(\sigma, \xi) \subset 2\Theta$.

We claim that $\tilde{\theta} \cap \Omega_{\leq \sigma}$ is contained in $2\Theta(\sigma, \xi)$. This will finish the proof, because $\omega \in \tilde{\Theta} \cap \Omega_{\leq \sigma} \subset 2\Theta(\sigma, \xi) \subset 4\Theta(\sigma, \xi')$.

To check the claim, we have to understand the geometry of the set $\Omega_{\leq \sigma}$. To picture the set $\Omega_{\leq \sigma}$, we found it helpful to consider the intersection of $\Theta(\sigma, \xi)$ with the plane $\omega_3 = h$. We assume $|h| \leq \sigma^2$ — otherwise the intersection is empty. The intersection is a rectangle with dimensions $r^{-1} \sigma \times \sqrt{2}r^{-2}$, and with the long side tangent to the circle of radius $h$ around the origin. Therefore, $\Theta(\sigma, \xi) \cap \{ \omega_3 = h \}$ is contained in the annulus

$$\{ h^2 \leq \omega_1^2 + \omega_2^2 \leq h^2 + r^{-2} \sigma^2 \}.$$ (11)

If we rotate $\xi$, the rectangle $\Theta(\sigma, \xi) \cap \{ \omega_3 = h \}$ rotates also, and the union of these rotated rectangles over all $\xi$ is equal to the annulus (11). Therefore, $\Omega_{\leq \sigma}$ is (approximately) given by

$$\Omega_{\leq \sigma} = \{ \omega : |\omega_3| \leq \sigma^2 \text{ and } \omega_3^2 \leq \omega_1^2 + \omega_2^2 \leq \omega_3^2 + r^{-2} \sigma^2 \}.$$ (12)

But the intersection of $\tilde{\Theta}(\xi)$ with the region (12) is approximately $\Theta(\sigma, \xi)$, and so $\tilde{\theta}(\xi) \cap \Omega_{\leq \sigma} \subset 2\Theta(\sigma, \xi)$ as claimed. \qed

Using Lemma 4.1 we can rewrite inequality (10): if $\omega \in \Omega_\sigma$, then

$$|\tilde{g}(\omega)| \leq \sum_{\Theta \in \mathbf{CP}_\sigma} \left| \sum_{\theta \in \mathbf{S}_\Theta} \Theta(\sigma, \xi) \cap \{ |f_\theta|^2 \}^\wedge(\omega) \right|.$$ (13)

**Lemma 4.2.** For any $\omega \in \Omega_\sigma$, the number of $\Theta \in \mathbf{CP}_\sigma$ so that $\omega \in 4\Theta$ is bounded by a constant $C$. 

Proof. Building on the description of $\Omega_{<\sigma}$ in (12) above, we see that if $|h| \leq \sigma^2/4$, then $\Omega_{\sigma} \cap \{\omega_3 = h\}$ is approximately given by

$$\{h^2 + (1/4)r^{-2}\sigma^2 \leq \omega_1^2 + \omega_2^2 \leq h^2 + r^{-2}\sigma^2\}. \tag{14}$$

If $\sigma^2/4 \leq |h| \leq \sigma^2$, then $\Omega_{\sigma} \cap \{\omega_3 = h\}$ is approximately given by

$$\{h^2 \leq \omega_1^2 + \omega_2^2 \leq h^2 + r^{-2}\sigma^2\}. \tag{15}$$

Let $C_{h,\rho}$ be the circle defined by $\omega_3 = h$ and $\omega_1^2 + \omega_2^2 = \rho^2$ with $|h| \leq \sigma^2$ and $\rho$ chosen such that $C_{h,\rho}$ lies in (13) or (15). These circles cover $\Omega_{\sigma}$. For any $\xi$, we will compute in the next two paragraphs that the fraction of $C_{h,\rho}$ contained in $4\Theta(\sigma,\xi)$ is $\lesssim \sigma^{-1}r^{-1}$. There are $\sim \sigma r$ different $\Theta(\sigma,\xi) \subset \mathbb{CP}_\sigma$. By circular symmetry, each frequency $\omega \in C_{h,\rho}$ lies in $4\Theta$ for approximately the same number of $\Theta \in \mathbb{CP}_\sigma$, and so each frequency $\omega$ lies in $4\Theta$ for $\leq C$ different $\Theta \in \mathbb{CP}_\sigma$.

We first do the case $|h| \leq \sigma^2/4$. Recall that $\Theta(\sigma,\xi) \cap \{\omega_3 = h\}$ is a rectangle with dimensions $r^{-1}\sigma \times r^{-2}$ which is tangent to the circle of radius $|h|$. Suppose for now that $r^{-1}\sigma \leq |h|$. If $A, B$ are the two endpoints of this rectangle and $O$ is the origin, then the angle $AOB$ is approximately $r^{-1}\sigma/|h|$. The angle between the rectangle $\Theta \cap \{\omega_3 = h\}$ and the circle $C_{h,\rho}$ is approximately equal to the angle $AOB$. Therefore, the arc length of $4\Theta \cap C_{h,\rho}$ is bounded by

$$\text{Length } (4\Theta \cap C_{h,\rho}) \lesssim r^{-1}\sigma^{-1}|h|. \tag{16}$$

Since the length of $C_{h,\rho}$ is $2\pi\rho \sim |h|$, the fraction of $C_{h,\rho}$ contained in $4\Theta$ is $\lesssim r^{-1}\sigma^{-1}$ as desired.

If $|h| < r^{-1}\sigma$, then the angle $AOB$ is $\sim 1$, and the length of $4\Theta \cap C_{h,\rho}$ is approximately $r^{-2}$. In this case the length of $C_{h,\rho}$ is $2\pi\rho \sim r^{-1}\sigma$, and so the fraction of $C_{h,\rho}$ covered by $4\Theta$ is still $\lesssim r^{-1}\sigma^{-1}$.

Finally, suppose that $\sigma^2/4 \leq |h| \leq \sigma^2$. In this case $4\Theta \cap C_{h,\rho}$ has arc length $\sim \sigma r^{-1}$ (the long side of the rectangle $\Theta \cap \{\omega_3 = h\}$). Since the length of $C_{h,\rho}$ is $2\pi\rho \sim |h| \sim \sigma^2$, the fraction of $C_{h,\rho}$ covered by $4\Theta$ is again $\lesssim \sigma^{-1}r^{-1}$. \hfill $\square$

Remark. A frequency $\omega \in \Omega_{\sigma}$ lies on two rather different $\Theta \in \mathbb{CP}_\sigma$, and maybe also on other $\Theta$ neighboring these two – this is because a point outside a circle lies on two lines tangent to the circle.

Applying Cauchy–Schwarz to (13) and using Lemma 4.2 we see that if $\omega \in \Omega_{\sigma}$, then

$$|\hat{g}(\omega)|^2 \lesssim \sum_{\Theta \in \mathbb{CP}, \omega \in 4\Theta} \left| \sum_{\theta \in S_{\omega}} (|f_\theta|^2)^{\omega}(\omega) \right|^2. \tag{16}$$

We let $\eta_\omega$ be a smooth function which is $\geq 1$ on $4\Theta$ and decays rapidly outside $4\Theta$. Summing over all dyadic $\sigma$, we see that for every frequency $\omega$,

$$|\hat{g}(\omega)|^2 \lesssim \sum_{\Theta \in \mathbb{CP}} \left| \eta_\omega(\omega) \sum_{\theta \in S_{\omega}} (|f_\theta|^2)^{\omega}(\omega) \right|^2. \tag{16}$$

Now we integrate and use Plancherel, giving

$$\int |g|^2 \lesssim \sum_{\Theta \in \mathbb{CP}} \int |\hat{\eta}_\omega| \ast \sum_{\theta \in S_{\omega}} |f_\theta|^2. \tag{16}$$
Now we can choose $\eta_\Theta$ so that $|\eta_\Theta^\vee(x)| \lesssim |\Theta^*|^{-1}$ for all $x$, and $\eta_\Theta^\vee$ is supported on $\Theta^*$. Therefore, it is natural to break up the right integral into translated copies of $\Theta^*$:

$$\int |g|^2 \lesssim \sum_{\Theta \in CP_{U/\Theta^*}} \sum_{\theta \in S_\Theta} \int_U |\eta_\Theta^\vee \ast \sum_{\theta \in S_\Theta} |f_\theta|^2|^2.$$

In the last integral, for each $x \in U$, we have

$$|\eta_\Theta^\vee \ast \sum_{\theta \in S_\Theta} |f_\theta|^2|^2 \lesssim |U|^{-1} \int \eta_{\Upsilon} \sum_{\theta \in S_\Theta} |f_\theta|^2,$$

where $\eta_{\Upsilon}(x) = \max_{y \in x + \Theta^*} |\eta_\Theta^\vee(x - y)|$ is a smooth bump function with height 1 supported on $2U$.

Therefore,

$$\int |g|^2 \lesssim \sum_{\Theta \in CP_{U/\Theta^*}} \sum_{\theta \in S_\Theta} |U|^{-1} \left( \int \eta_{\Upsilon} \sum_{\theta \in S_\Theta} |f_\theta|^2 \right)^2.$$

We associate $\Theta(\sigma, \xi)$ to $\tau(\sigma^{-1}r^{-1}, \xi)$. This gives a bijection from $CP_{\sigma}$ to $S_s$ with $s = \sigma^{-1}r^{-1}$. If $\Theta(\sigma, \xi) \subset 2\Theta(\xi')$, then we saw above that $|\xi - \xi'| \lesssim \sigma^{-1}r^{-1}$, and so $\Theta(\xi') \subset 4\tau(\sigma^{-1}r^{-1}, \xi)$. In particular, if $\theta \in S_\Theta$, then $\theta < 4r$. Also $\Theta(\sigma, \xi)^*$ is comparable to $U_{\tau(\sigma^{-1}r^{-1}, \xi)}$, which we can see by comparing the definition of $U_{\tau}$ in [19] with the definition of $\Theta$ in [19]. Rewriting the last inequality in terms of $\tau \in S_s$ instead of $\Theta \in CP_{\sigma}$, we get

$$\int |g|^2 \lesssim \sum_{r^{-1} \leq s \leq 1} \sum_{\tau \in S_s} \sum_{U/\Upsilon} |U|^{-1} \left( \int \eta_{\Upsilon} \sum_{\theta \in \tau} |f_\theta|^2 \right)^2.$$

By the definition of $S_U f$,

$$\sum_{U/\Upsilon} \left( \int \eta_{\Upsilon} \sum_{\theta \in \tau} |f_\theta|^2 \right)^2 \lesssim \sum_{U/\Upsilon} \|S_U f\|_{L^2}^4.$$

Plugging this in, we get

$$\int |g|^2 \lesssim \sum_{\Theta \in CP_{U/\Theta^*}} \sum_{\theta \in S_\Theta} |U|^{-1} \|S_U f\|_{L^2}^4.$$

This proves Lemma 4.3. 

We use this Kakeya-type estimate to prove Lemma 3.1. First we recall the statement. Unwinding the definition of $S(r, R)$, Lemma 3.1 says

**Lemma.** If $\hat{f}$ is supported on $N_{r^{-2}}(\Gamma)$ and $r_1 \in [r, r^2]$, then with $U$ having diameter $r^2$,

$$\sum_{B_{r_1} \subset \mathbb{R}^3} |B_{r_1}|^{-1} \|S_{B_{r_1}} f\|_{L^2(B_{r_1})}^4 \lesssim \sum_{r^{-1} \leq s \leq 1} \sum_{d(r) = s} \sum_{U/\Upsilon} |U|^{-1} \|S_U f\|_{L^2}^4.$$

In the proof, we use a trick known as the “local $L^2$ orthogonality”. We state it in the end of this section as Lemma 4.3 and one can see for example Proposition 6.1 in [3] for a proof.
Lemma 4.3 (Local orthogonality lemma, essentially Proposition 6.1 in [3]). In $\mathbb{R}^n$, let $f = \sum_\theta f_\theta$ such that $\text{supp} f_\theta \subset \theta$ in the Fourier space. Assuming $r > 0$ such that every point $\xi$ lies in $\leq M$ many $N_{r^{-1}}(\theta)$. For any $E > 0$,

$$\| f \|_{L^2(B_r)}^2 \lesssim_{M,E} \sum_{\theta \in I} \| f_\theta \|_{L^2(w_{B_r}, E)}^2.$$

5. The Lorentz rescaling

Lorentz transformations are the symmetries of our problem, and they’ve been used in many earlier papers on this topic (cf. for instance [25] and [2]). Here we review the Lorentz rescaling and check the properties that we will need in our rescaling argument in the next two sections.

The piece $\Gamma_{-\nu_3}$ is defined to work well with Lorentz transformations, and we now record the formula. This formula and the Lorentz rescaling generally look nicest in a rotated coordinate system where the light cone is given by the equation $2\nu_1 \nu_3 = \nu_2^2$. Here $\nu_2 = \xi_1$, $\nu_1 = 2^{-1/2}(\xi_3 - \xi_2)$ and $\nu_3 = 2^{-1/2}(\xi_3 + \xi_2)$. In these coordinates, if we intersect the light cone with the plane $\nu_3 = 1$ then we get the parabola $\nu_1 = (1/2)\nu_2^2$. So the light cone is actually the cone over a parabola.

Now $\Gamma_{-\nu_3}$ is defined as follows.

$$\Gamma_{-\nu_3} = \{2\nu_1 \nu_3 = \nu_2^2, 1 - \frac{1}{K} \leq \nu_3 \leq 1, \frac{\nu_2}{\nu_3} \leq 1 \}. $$

For any real number $\eta$ with $|\eta| < 1$ and $0 < s < 1$ satisfying $-1 \leq \eta \pm s \leq 1$, we can define a surface sector $\Lambda \subseteq \Gamma_{-\nu_3}$ by

$$\Lambda = \Lambda(\eta, s) = \{ (\nu_1, \nu_2, \nu_3) \in \Gamma_{-\nu_3} : \frac{|\nu_2|}{\nu_3} - \eta < s \}.$$  

Here $s$ is the aperture of $\Lambda$, also denoted by $d(\Lambda)$. 

Proof of Lemma 4.3. As in Lemma 1.4 let $g = \sum_{d(\theta) = r^{-1}} |f_\theta|^2$. The functions $f_\theta$ have essentially disjoint Fourier support. Hence, fixing $E$ sufficiently large as before, by the local $L^2$ orthogonality Lemma 1.4 on each $B_{r_1}$:

$$\| S_{B_{r_1}} f \|_{L^2(B_{r_1})}^2 = \int_{B_{r_1}} \sum_{d(\tau) = r_1^{-1/2}} |f_\tau|^2 \lesssim \int_{\mathbb{R}^3} w_{B_{r_1}, E} \sum_{d(\tau) = r_1^{-1/2}} \sum_{\theta \subset \tau} |f_\theta|^2 = \int_{\mathbb{R}^3} w_{B_{r_1}, E} g.$$

By Cauchy–Schwarz, we get

$$|B_{r_1}|^{-1} \| S_{B_{r_1}} f \|_{L^2(B_{r_1})}^2 \lesssim \int_{\mathbb{R}^3} w_{B_{r_1}, E/2} |g|^2.$$

Summing over $B_{r_1}$ tiling $\mathbb{R}^3$,

$$\sum_{B_{r_1} \subset \mathbb{R}^3} |B_{r_1}|^{-1} \| S_{B_{r_1}} f \|_{L^2(B_{r_1})}^2 \lesssim \int_{\mathbb{R}^3} |g|^2.$$

Lemma 1.4 bounds $\int_{\mathbb{R}^3} |g|^2$ by the right-hand side of (17). □

In the end of this section, we record the local $L^2$ orthogonality lemma.
Each surface sector $\Lambda$ is closely associated to a sector $\tau = \tau(\Lambda)$, which is a rectangular box containing $\Lambda$ with smallest comparable dimensions. The sector $\tau(\Lambda)$ is approximately the convex hull of $\Lambda$. Similarly, starting with any sector $\tau$, there is an associated surface sector $\Lambda_\tau = \tau \cap \Gamma_{\uparrow}$.

The aperture of $\Lambda$ and the aperture of $\tau$ are approximately the same.

Note that $\tau$ and the $R^{-1}$-neighborhood $N_{R^{-1}}(\Lambda)$ of $\Lambda$ is quite different. The former, being a rectangular box, is usually much bigger than the latter.

For any surface sector $\Lambda \subset \Gamma_{\uparrow}$ there is a Lorentz transformation $L$ which maps $\Lambda$ diffeomorphically onto $\Gamma_{\uparrow}$. (The precise definition of $\Gamma_{\uparrow}$ was arranged to make this work.) The formula for $L$ is as follows.

Let $L : \Lambda(d(\Lambda), \eta) \to \Gamma_{\uparrow}$ be defined as (away from $\{z = 0\}$):

$$
\begin{align*}
\frac{\nu_1}{\nu_3} &\mapsto \frac{1}{d(\Lambda)} \left( \frac{\nu_1}{\nu_3} - \eta_A \right), \\
\frac{\nu_2}{\nu_3} &\mapsto \frac{1}{d(\Lambda)} \left( \frac{\nu_2}{\nu_3} - \eta_A \cdot \frac{\nu_2}{\nu_3} + \frac{\nu_3^2}{2} \right).
\end{align*}
$$

We can see that $L$ is actually a linear transformation:

$$
\begin{align*}
\nu_3 &\mapsto \nu_3, \\
\nu_2 &\mapsto \frac{1}{d(\Lambda)}(\nu_2 - \eta_A \nu_3), \\
\nu_1 &\mapsto \frac{1}{d(\Lambda)}(\nu_1 - \eta_A \nu_2 + \frac{\nu_3^2}{2}).
\end{align*}
$$

This linear transformation $L$ is called a Lorentz rescaling.

Suppose that $\tau$ is a sector with $d(\tau) = s$, and let $\Lambda = \Lambda_{\tau}$. We then study the rescaling map $L$ defined in (20). We will need to keep track of how this change of variables affects the characters in our inequalities, like sectors $\tau' \subset \tau$ and the regions $U_{\tau}$.

First, if $\Lambda' \subset \Lambda$ is a smaller surface sector, then $\mathcal{L}(\Lambda')$ is a surface sector of aperture $\sim s^{-1}d(\tau')$.

More precisely, since $\Lambda' \subset \Lambda$, we have

$$
[\eta_A - d(\Lambda'), \eta_A + d(\Lambda')] \supseteq [\eta_A - d(\Lambda), \eta_A + d(\Lambda)].
$$

By the above definition of $L$, we can see that $\mathcal{L}(\Lambda')$ is defined as

$$\{(\nu_1, \nu_2, \nu_3) \in \Gamma : \frac{\nu_2}{\nu_3} \in \left[\frac{1}{d(\Lambda)}(\eta_A - \eta_A) - \frac{d(\Lambda')}{d(\Lambda)} \frac{1}{d(\Lambda)}(\eta_A - \eta_A) + \frac{d(\Lambda')}{d(\Lambda)}\right]\}.$$

We see that (21) implies the above range of $\nu_2/\nu_3$ is in $[-1, 1]$, and that $\mathcal{L}(\Lambda')$ is a surface sector of aperture $\frac{d(\Lambda')}{d(\Lambda)}$ lying inside the whole $\Gamma_{\uparrow} = \mathcal{L}(\Lambda)$.

Next we consider how $L$ affects sectors $\tau' \subset \tau$. Suppose that $\Lambda_{\tau'}$ is a surface sector associated to $\tau'$. Note that $\tau'$ is approximately the convex hull of $\Lambda_{\tau'}$. Since taking convex hulls commutes with linear transformations, we see that $\mathcal{L}(\tau')$ is approximately the convex hull of $\mathcal{L}(\Lambda_{\tau'})$, which is a sector of aperture $\sim s^{-1}d(\tau')$.

Next we consider $\mathcal{L}(N_{R^{-1}}(\Lambda))$ for some $R > s^{-2}$. The $R^{-1}$-neighborhood of $\Gamma_{\uparrow}$ is covered by sectors $\theta \subset \tau$ with $d(\theta) = R^{-1/2}$. Therefore, $\mathcal{L}(N_{R^{-1}}(\Lambda))$ is covered by sectors $\mathcal{L}(\theta)$ with aperture $\sim s^{-1}R^{-1/2}$. The union of these sectors is the $s^{-2}R^{-1}$-neighborhood of $\Gamma_{\uparrow}$. In summary $\mathcal{L}(N_{R^{-1}}(\Lambda))$ is approximately $N_{s^{-2}R^{-1}}(\Gamma_{\uparrow})$.

Next we consider how the adjoint transformation, $\mathcal{L}^*$, behaves on physical space. It is standard that the adjoint transformation behaves naturally with respect to taking duals, so, if $\theta$ is a sector, then we have $\mathcal{L}(\theta)^* = \mathcal{L}^*(\theta^*)$. 
Finally we consider how $L^*$ affects the sets $U_\tau$. This is a good moment to remark that the set $U_\tau$ also depends on the radius $R$. In the argument up to this point, $R$ has been fixed, so we did not need to record the dependence. But in the next two sections, we will work with several different choices of $R$, so we need to record it. For this reason we write $U_{\tau,R}$ instead of $U_\tau$ when we would like to keep track of $R$. Let us recall the definition of $U_{\tau,R}$ from (5). If $\tau = \tau(s,\xi)$, then

$$U_{\tau,R} := \{ x \in \mathbb{R}^3 : |c(\xi) \cdot x| \leq Rs^2 \text{ and } |n(\xi) \cdot x| \leq R \text{ and } |t(\xi) \cdot x| \leq Rs \}.$$  

There is an equivalent more conceptual definition, which is useful for understanding $L^*(U_{\tau,R})$.

$$U_{\tau,R} := \text{Convex Hull } (\cup_{\theta \subset \tau, d(\theta) = R^{-1/2} \theta^*}) = \text{Convex Hull } (\cup_{\theta \subset \mathcal{L}(\tau'), d(\theta) = s^{-1/2} \theta^*}) = U_{\mathcal{L}(\tau'), s^2 R}. \tag{23}$$

Now let $\tau$ again denote a fixed sector with $d(\tau) = s$ and let $L$ be the Lorentz rescaling that takes $\Lambda_\tau$ to $\Gamma_{\frac{1}{2}}$.

**Lemma 5.1.** For any sector $\tau' \subset \tau$ and any $R \geq s^{-2}$,

$$L^*(U_{\tau',R}) = U_{\mathcal{L}(\tau'), s^2 R}.$$  

**Proof.**

$$L^*(U_{\tau',R}) = \text{Convex Hull } (\cup_{\theta \subset \tau', d(\theta) = R^{-1/2} \theta^*})$$

$$= \text{Convex Hull } (\cup_{\theta \subset \mathcal{L}(\tau'), d(\theta) = R^{-1/2} \theta^*})$$

$$= \text{Convex Hull } (\cup_{\theta \subset \mathcal{L}(\tau'), d(\theta) = s^{-1/2} \theta^*}) = U_{\mathcal{L}(\tau'), s^2 R}. \tag*{□}$$

We have now gathered enough background about Lorentz rescaling to carry out our Lorentz rescaling argument in the next two sections.

6. **The Proof of Lemma 3.2**

In this section, we prove Lemma 3.2. First we prove several lemmas about the “locally constant property” of $f_\theta$.

**Lemma 6.1.** Let $\theta$ be a compact convex set. If $\text{supp} \hat{f}_\theta \subset \theta$ and $T_\theta := \theta^*$, then there exists a positive function $\eta_{T_\theta}$ essentially supported on $100 T_\theta$ and rapidly decays away from it and $\|\eta_{T_\theta}\|_{L^1} \lesssim 1$ such that

$$|f_\theta| \leq \sum_{T \parallel T_\theta} c_T \chi_T \leq |f_\theta| * \eta_{T_\theta} \tag{24}$$

where $c_T$ is defined as $\max_{x \in T} |f_\theta|(x)$.

**Proof.** We bound $|f_\theta|$ by

$$|f_\theta| \leq \sum_{T \parallel T_\theta} c_T \chi_T. \tag{25}$$

Let $\phi_\theta$ be a smooth bump function supported on $2\theta$ and $\phi_\theta = 1$ on $\theta$. Since $\text{supp} \hat{f}_\theta \subset \theta$, we have $\hat{f}_\theta = \hat{f}_\theta \phi_\theta$ and $f_\theta = f_\theta * \phi_\theta^\vee$. Let $\eta_{T_\theta}(x) = \max_{t \in x + 10 T_\theta} |\phi_\theta^\vee|(t)$. Since $\phi_\theta^\vee$ is a function essentially supported on $T_\theta = \theta^*$ and $\|\phi_\theta^\vee\|_{L^1} \sim 1$, so is $\eta_{T_\theta}$. 


For any $T/T_0$, 
\[
\max_{x \in T} |f_\theta|(x) \leq \max_{x \in T} \int |f_\theta|(y) \phi^*_\theta(x-y) \, dy \\
\leq \min_{x \in T} \int |f_\theta|(y) \eta_{T_0}(x-y) \, dy
\]
because for each $y$, \(\max_{x \in T} |\phi^*_\theta|(x-y) \leq \min_{x \in T} \max_{t \in \mathbb{R}} |\phi^*_\theta|(t)\). 
\[\square\]

**Lemma 6.2.** Let \(\eta_{T_0}\) be defined as in Lemma 6.1 and $T/T_0$, then there exists a positive function $w_T = 1$ on $10T$ and rapidly decaying away from it such that for any $1 \leq p < \infty$, 
\[
\int_T (|f_\theta| * \eta_{T_0})^p \lesssim_p \int |f_\theta|^p w_T.
\]

**Proof.** The function $\eta_{T_0}$ satisfies 
\[
\eta_{T_0} \leq \sum_{T/T_0} C_T \chi_T
\]
where $C_T$ is the characteristic function of the Minkowski sum $T \subset nT$ for any large integer $N > 0$ and $n(T, T_0)$ is the smallest $n \geq 1$ such that $T \subset nT_0$.

By Cauchy–Schwarz, 
\[
\int_T (|f_\theta| * \eta_{T_0})^p \leq \int_T \left( \sum_{T'/T_0} |f_\theta| * C_{T'} \chi_{T'} \right)^p
\]
\[
= \int_T \left( \sum_{T'/T_0} n(T', T_0)^{4(p-1)} \cdot n(T', T_0)^{4(p-1)} |f_\theta| * C_{T'} \chi_{T'} \right)^p
\]
\[
\lesssim \left( \sum_{T'/T_0} n(T', T_0)^{-4} \right)^{p-1} \cdot \sum_{T'/T_0} n(T', T_0)^{4(p-1)} \int_T (|f_\theta| * C_{T'} \chi_{T'})^p
\]
\[
\lesssim \sum_{T'/T_0} n(T', T_0)^{4(p-1)} \int_T (|f_\theta| * C_{T'} \chi_{T'})^p.
\]

Let $\chi_{T-T'}(x)$ be the characteristic function of the Minkowski sum $T - T' := T + (-T')$. Then by Young’s inequality, 
\[
\int_T (|f_\theta| * C_{T'} \chi_{T'})^p \leq \int (||f_\theta| \chi_{T-T'}| * (C_{T'} \chi_{T'}))^p
\]
\[
\lesssim_N n(T', T_0)^{-pN} \int_{T-T'} |f_\theta|^p
\]
It suffices to choose $w_T(x) \sim_N \sum_{T/T_0} n(\tilde{T}, T_0)^{-N} \chi_{\tilde{T}}(x)$.
\[\square\]

**Corollary 6.3.** If $U$ is tiled by $T/T_0$, then for any $1 \leq p < \infty$, 
\[
\int_U (|f_\theta| * \eta_{T_0})^p \lesssim_p \int |f_\theta|^p w_U
\]
where $w_U \geq 0$ is essentially supported on $10U$ and rapidly decays away from it.
Lemma 6.5. Let \( \theta_1, \theta_2 \subset \tau \) be two sectors of aperture \( d(\theta_1) = d(\theta_2) = K^{-1/2} \), and \( \text{dist}(\theta_1, \theta_2) \sim d(\tau) = s > K^{-1/2} \), then for any functions \( \supp f_{\theta_1} \subset N_{\tilde{K}} \Gamma_{\tilde{K}} \cap \theta_1 \) and \( \supp f_{\theta_2} \subset N_{\tilde{K}} \Gamma_{\tilde{K}} \cap \theta_2 \),

\[
\sum_{B_{K/2} \subset \mathbb{R}^3} |B_{K/2}|^{-1} \| f_{\theta_1} \|_{L^2(B_{K/2})}^2 \| f_{\theta_2} \|_{L^2(B_{K/2})}^2 \lesssim s^{-1} \sum_{B_K \subset \mathbb{R}^3} |B_K|^{-1} \left( \int_{B_K} |f_{\theta_1}|^2 w_{B_K} \right) \left( \int_{B_K} |f_{\theta_2}|^2 w_{B_K} \right).
\]

Proof. We do a “two dimensional bilinear Kakeya” style proof.

Since \( \supp \tilde{f}_{\theta_j} \subset N_{\tilde{K}} \Gamma_{\tilde{K}} \cap \theta_j \) for \( j = 1, 2 \), the Fourier support of \( f_{\tilde{\theta}} \) lies inside a box \( \tilde{\theta}_j \) of dimensions \( K^{-1/2} \times K^{-1} \times K^{-1} \) with a common \( K^{-1} \)-side on the \( \nu_3 \)-direction (Recall the \((\nu_1, \nu_2, \nu_3)\)-coordinate system and the equation of \( \Gamma_{\tilde{K}} \) from Section 5). And \( \tilde{T}_{\theta_j} := \tilde{T}_j \) becomes a slab of dimensions \( K^{1/2} \times K \times K \). Since \( \text{dist}(\tilde{\theta}_1, \tilde{\theta}_2) = \text{dist}(\theta_1, \theta_2) = s \), for each \( T_1 \parallel T_2 \parallel T_{\theta_j} \) and \( T_1, T_2 \subset B_K \), we have \( |T_1 \cap T_2| \sim s^{-1} K^{1/2} \cdot (s^{-1} K^{1/2}) \cdot K = s^{-1} K^{2} \). Hence the key inequality \( |T_1 \cap T_2| \sim s^{-1} |B_K|^{-1} |T_1||T_2| \) holds.

Now we are ready to bound

\[
\sum_{B_{K/2} \subset \mathbb{R}^3} |B_{K/2}|^{-1} \| f_{\theta_1} \|_{L^2(B_{K/2})}^2 \| f_{\theta_2} \|_{L^2(B_{K/2})}^2 \leq \sum_{B_{K/2} \subset \mathbb{R}^3} |B_{K/2}| c_T^2 c_T^2.
\]

(Lemma 6.1) \( \lesssim s^{-1} |B_K|^{-1} \left( \int_{B_K} \sum_{T_{\tilde{\theta}_j}} \sum_{T_{\tilde{\theta}_j}} c_{T_{\tilde{\theta}_j}}^2 \chi_{T_{\tilde{\theta}_j}} \right) \left( \int_{B_K} \sum_{T_{\tilde{\theta}_j}} \sum_{T_{\tilde{\theta}_j}} c_{T_{\tilde{\theta}_j}}^2 \chi_{T_{\tilde{\theta}_j}} \right) \)

\[
\lesssim s^{-1} |B_K|^{-1} \left( \int_{B_K} \sum_{T_{\tilde{\theta}_j}} \sum_{T_{\tilde{\theta}_j}} \left( |f_{\theta_1}| * \eta_{T_{\tilde{\theta}_j}} \right)^2 \right)^{1/2} 
\]

\[
\lesssim s^{-1} |B_K|^{-1} \left( \int_{B_K} \sum_{T_{\tilde{\theta}_j}} \sum_{T_{\tilde{\theta}_j}} \left( |f_{\theta_2}| * \eta_{T_{\tilde{\theta}_j}} \right)^2 \right)^{1/2} 
\]

(Corollary 6.3) \( \lesssim s^{-1} |B_K|^{-1} \left( \int_{B_K} \sum_{T_{\tilde{\theta}_j}} \sum_{T_{\tilde{\theta}_j}} |\eta_{T_{\tilde{\theta}_j}}|^2 \right) \left( \int_{B_K} \sum_{T_{\tilde{\theta}_j}} \sum_{T_{\tilde{\theta}_j}} |\eta_{T_{\tilde{\theta}_j}}|^2 \right) 
\]

By Hölder’s inequality.

Lemma 6.6. Let \( f \) be a function whose Fourier transform is supported on the \( \frac{1}{K} \)-neighborhood of \( \Gamma_{\frac{1}{K}} \). For any \( \delta > 0 \) and any \( r < K \),

\[
\| f \|_{L^4(\mathbb{R}^3)}^4 \leq C_4 K^d \sum_{K^{-1/2} \leq s \leq K^{-1/2}} \sum_{|\tau| = s} \sum_{U \parallel U_\tau} |U|^{-1} \| S_U f \|_{L^2}.
\]

Proof. We induct on \( K \). The base case \( K \lesssim 1 \) is easy by Hölder’s inequality.

Let \( 1 \ll K_0 \ll K^{d/10} \). We tile \( N_{\frac{1}{K}}(\Gamma_{\frac{1}{K}}) \) with sectors \( \tau \) of aperture \( \frac{1}{K_0} \) and width \( \frac{1}{K} \) and decompose \( f = \sum_{\tau} \| f_{\tau} \|_{L^1(\mathbb{R}^3)} \).
Now $N_\frac{1}{K}(\Gamma_{\frac{1}{K}})$ is the $\frac{1}{K}$-neighborhood of an arc of a parabola of length 1, and each $\tau$ is the $\frac{1}{K}$-neighborhood of an arc of the parabola of length $\frac{1}{K_0}$.

The Bourgain-Guth argument \cite{4} says the following. At each point $f(x) = \sum \tau f(x)$. Let $\tau^*$ satisfy $\max_{\tau} |f_{\tau^*}|(x) = |f_{\tau^*}||x|$. If $|f_{\tau^*}|(x) \geq 1/10 |f|(x)$, then $|f|^4(x) \lesssim \sum_{\tau} |f_{\tau^*}|(x)$. Otherwise, there exists a $\tau^{**}$ such that $\dist(\tau^{**}) \geq 1/K_0$ and $|f_{\tau^{**}}|(x) \geq |f_{\tau^*}||x| \geq 1/K_0 |f|(x)$. Hence,

$$|f|^4 \lesssim \sum_{d(\tau) = 1/K_0} |f_{\tau^*}|^4 + K^4 \sum_{\dist(\tau_1, \tau_2) \geq 1/K_0} |f_{\tau_1}|^2 |f_{\tau_2}|^2.$$

For the integral of the first term, we rescale $\tau$ to be the $K^{-1}K_0$-neighborhood of $\Gamma_{1/K}$ (the rescaling argument here is similar to the one in the proof of Lemma \ref{3.3} in Section \ref{7}, which we will do with full details), then we apply the induction hypothesis on the scale $K/K_0^2 < K$.

For the integral of the second term, we decompose $\tau_j = \sum_{\theta_j \subset \tau_j, d(\theta_j) = K^{-1/2} f_{\theta_j}, j = 1, 2}$. The functions $f_{\theta_1}, f_{\theta_2}$ are essentially orthogonal because they have almost disjoint Fourier support, as in the Fefferman-Córdoba proof of restriction for the parabola \cite{10} \cite{6}.

We recall the statement of Lemma \ref{3.2} Unwinding the definition of $S_K(r, K)$ it says the following:

**Proposition 6.7.** Let $f$ be a function whose Fourier transform is supported on the $\frac{1}{K}$-neighborhood of $\Gamma_{\frac{1}{K}}$. For any $\delta > 0$ and any $r \leq K$,

$$\sum_{B_r \subset \mathbb{R}^3} |B_r|^{-1} \|SB_r f\|^4_{L^2(B_r)} \leq C_\delta K^\delta \sum_{K^{-1/2} \leq s \leq 1} \sum_{d(\tau) = s U \setminus U_{\tau, K}} |U|^{-1} \|SU f\|^4_{L^2}.$$

**Proof.** We take advantage that $\Gamma_{\frac{1}{K}}$ is well-approximated by a parabola at the scale $1/K$ and use an approach similar to Fefferman-Córdoba’s to bound the left-hand side of \cite{20} by (essentially) the left-hand side of \cite{29}.

Since the smallest aperture in this proposition is $K^{-1/2}$, we use $\theta$ to denote a sector on $\Gamma_{\frac{1}{K}}$ of aperture $K^{-1/2}$ in the current proof.

Let $A_1, \ldots, A_{1000}$ be disjoint sets of $\theta$ such that each $\theta$ is in one of them and:

Within each $A_j$, if the Minkowski sum $(\theta_1 + \theta_2) \cap (\theta_1' + \theta_2') \neq \emptyset$, then $(\theta_1, \theta_2) = (\theta_1', \theta_2')$ or $(\theta_1', \theta_2')$.

\(^5\)Alternatively, one can blackbox the $L^4$ angular square function estimate by Córdoba \cite{7} and have a slightly shorter proof. We present a self-contained proof here.
contained in the $\frac{2}{K}$–neighborhood of the parabola $\nu_2^2 = 2\nu_1$, and the projection of each $\theta$ is the corresponding cap inside that neighborhood. We use $\text{Error}$ to denote a number (the “error term”) whose absolute value is $\leq 4K^{-1}$. If $x_1 + x_2 = a + \text{Error}$ and $x_1^2 + x_2^2 = b + \text{Error}$ with $a, b \leq 2$, then $(x_1 - x_2)^2 = 2b - a^2 + 7\text{Error}$. Hence $|x_1 - x_2| = \sqrt{|2b - a^2|} + 3\sqrt{\text{Error}}$. This would imply that the pair $(x_1, x_2)$ is determined by the pair $(a, b)$, up to a swap in order and up to changing within 100 adjacent caps $\theta$.

We use $\tau$ to denote caps with aperture $r^{-1/2} \geq K^{-1/2}$ in the current proof. Consider the decomposition $f_j = \sum_{\theta \in A_j} f_0$ and let $f_{j, \tau}$ denote caps with aperture $\tau$ inside that neighborhood. We use $\text{Error}$ withing 100 adjacent caps $\theta$.

By the property (*) and Plancherel, we have for a fixed $j$,

$$\int_{\mathbb{R}^3} |f_j|^4 = \int_{\mathbb{R}^3} |\sum_{\tau} f_{j, \tau}|^4$$

$$= \int_{\mathbb{R}^3} \sum_{\tau_1, \tau_2, \tau_3, \tau_4: (\text{supp} f_{j, \tau_1} + \text{supp} f_{j, \tau_2}) \cap (\text{supp} f_{j, \tau_3} + \text{supp} f_{j, \tau_4}) \neq \emptyset} f_{j, \tau_1} f_{j, \tau_2} f_{j, \tau_3} f_{j, \tau_4}$$

$$= \int_{\mathbb{R}^3} \sum_{\tau_1, \tau_2} n_{\tau_1, \tau_2} |f_{j, \tau_1} f_{j, \tau_2}|^2$$

$$\sim \int_{\mathbb{R}^3} (\sum_{\tau} |f_{j, \tau}|^2)^2$$

(31)

where $n_{\tau_1, \tau_2} = 1$ if $\tau_1 = \tau_2$ and $n_{\tau_1, \tau_2} = 4/2 = 2$ if $\tau_1 \neq \tau_2$.

By (31) we have

$$\sum_{B_r \subset \mathbb{R}^3} |B_r|^{-1} \|S_{B_r} f\|^4_{L^2(B_r)} \leq \sum_{j=1}^{1000} \sum_{B_r \subset \mathbb{R}^3} |B_r|^{-1} \|S_{B_r} f_j\|^4_{L^2(B_r)}$$

$$\leq \sum_{j=1}^{1000} \sum_{B_r \subset \mathbb{R}^3} \|S_{B_r} f_j\|^4_{L^4(B_r)}$$

$$= \sum_{j=1}^{1000} \int_{\mathbb{R}^3} (\sum_{\tau} |f_{j, \tau}|^2)^2$$

$$\sim \sum_{j=1}^{1000} \int_{\mathbb{R}^3} |f_j|^4$$

(Lemma 6.6) $\leq C_\delta K^4 \sum_{j=1}^{1000} \sum_{K^{-1/2} \leq s \leq 1} d(\tau) = s U/U_\tau, K$$

$$\|U\|^{-1} \|S_U f_j\|^4_{L^2} \leq C_\delta K^4 \sum_{K^{-1/2} \leq s \leq 1} d(\tau) = s U/U_\tau, K$$

$$\|U\|^{-1} \|S_U f\|^4_{L^2} \leq C_\delta K^4 \sum_{K^{-1/2} \leq s \leq 1} d(\tau) = s U/U_\tau, K$$

7. The proof of Lemma 3.3

Now we prove Lemma 3.3 using the Lorentz rescaling. First we recall the statement.
Lemma. For any $r_1 < r_2 \leq r_3$,

$$S_K(r_1, r_3) \leq \log r_2 \cdot S_K(r_1, r_2) \max_{r_2^{-1/2} \leq s \leq 1} S_K(s^2 r_2, s^2 r_3).$$

Proof. Suppose that $\hat{f}$ is supported on $N_{r_3^{-1}}(\Gamma_\pm)$. To bound $S_K(r_1, r_3)$, we need to bound

$$\sum_{B_{r_1} \subset \mathbb{R}^3} |B_{r_1}|^{-1} \|S_{B_{r_1}} f\|_{L^2(B_{r_1})}^2.$$

We can apply the definition of $S_K(r_1, r_2)$ and get

$$\sum_{B_{r_1} \subset \mathbb{R}^3} |B_{r_1}|^{-1} \|S_{B_{r_1}} f\|_{L^2(B_{r_1})}^4 \leq S_K(r_1, r_2) \sum_{r_2^{-1/2} \leq s \leq 1} \sum_{d(\tau) = s} \sum_{U \cap U_r, r_2} |U|^{-1} \|S_{U} f\|_{L^2(U)}^4.$$

Recall that if $U/\|U_r, \tau|$, then $S_{U} f = (\sum_{d(\tau') = r^{-1/2}} |f_{\tau'}|^2)^{1/2}_U$. In particular, $S_{U} f = (\sum_{d(\tau') = r^{-1/2}} |f_{\tau'}|^2)^{1/2}_U$.

Using Lorentz rescaling, we will prove the following lemma:

**Lemma 7.1.** For any sector $\tau$ with $d(\tau) = s$,

(32)

$$\sum_{U \cap U_r, r_2} |U|^{-1} \|S_{U} f\|_{L^2(U)}^4 \leq S_K(s^2 r_2, s^2 r_3) \sum_{r_2^{-1/2} \leq s' \leq s} \sum_{d(\tau') = s'} \sum_{U \cap U_r, r_3} |U|^{-1} \|S_{U} f\|_{L^2(U)}^4.$$

We defer the proof of Lemma 7.1 to the end of this section. If we plug Lemma 7.1 and expand everything, then we get Lemma 3.3

$$\sum_{B_{r_1} \subset \mathbb{R}^3} |B_{r_1}|^{-1} \|S_{B_{r_1}} f\|_{L^2(B_{r_1})}^4 \leq \log r_2 S_K(r_1, r_2) \max_{r_2^{-1/2} \leq s \leq 1} S_K(s^2 r_2, s^2 r_3) \sum_{r_3^{-1/2} \leq s' \leq 1} \sum_{d(\tau') = s'} \sum_{U \cap U_r, r_3} |U|^{-1} \|S_{U} f\|_{L^2(U)}^4.$$

The factor $\log r_2$ appears here for the following reason: after we expand, each sector $\tau'$ will appear at most $\log r_2$ times, because $\tau'$ lies in $\tau$ for at most $\log r_2$ sectors $\tau$ with $r_2^{-1/2} \leq d(\tau) \leq 1$.

**Proof of Lemma 7.1.** The definition of $S_K(s^2 r_2, s^2 r_3)$ says that if $\hat{h}$ is supported on $N_{s^{-2} r_3^{-1}}(\Gamma_\pm)$, then

$$\sum_{B_{s^2 r_2} \subset \mathbb{R}^3} |B_{s^2 r_2}|^{-1} \|S_{B_{s^2 r_2}} h\|_{L^2(B_{s^2 r_2})}^4 \leq S_K(s^2 r_2, s^2 r_3) \sum_{s^{-1/2} \leq d(\tau') \leq 1} \sum_{U \cap U_r, r_3} |U'|^{-1} \|S_{U'} h\|_{L^2(U')}^4.$$

On the other hand, Lemma 7.1 says that if $\tau$ is a sector of $\Gamma_\pm$ with $d(\tau) = s$, and $\hat{f}_{\tau}$ is supported on $N_{r_3^{-1}}(\Gamma_\pm) \cap \tau$, then

(34)

$$\sum_{U \cap U_r, r_2} |U|^{-1} \|S_{U} f\|_{L^2(U)}^4 \leq S_K(s^2 r_2, s^2 r_3) \sum_{r_3^{-1/2} \leq s' \leq s} \sum_{d(\tau') = s'} \sum_{U \cap U_r, r_3} |U|^{-1} \|S_{U} f\|_{L^2(U)}^4.$$

To connect them, we begin with a Lorentz transformation $\mathcal{L}$ so that $\mathcal{L} : \tau \cap \Gamma_\pm \rightarrow \Gamma_\pm$ is a diffeomorphism. This $\mathcal{L}$ is constructed in Section 4 where it is shown that $\mathcal{L}$ takes $N_{s^{-2} r_3^{-1}}(\Gamma_\pm) \cap \tau$ to $N_{s^{-2} r_3^{-1}}(\Gamma_\pm)$. Now we define $h$ by $\hat{h} = \hat{f}_{\tau}(\mathcal{L}^{-1}(\cdot))$. Moreover let $\hat{h}_{\tau''} = \hat{f}_{\tau}(\mathcal{L}^{-1}(\cdot))$ where
\[
\mathcal{L}(\tau') = \tau'' \text{, see the item (1) below. We see that } \hat{h} \text{ is supported on } N_{s^{-2}r_3^{-1}}(\Gamma_{\mathcal{L}}) \text{ and so } h \text{ obeys } k. \text{ When we unwind the Lorentz transformations, we claim that } k \text{ becomes } k, \text{ which proves the lemma. To see that this unwinding works as desired, we check how each piece transforms.}
\]

1. If \( \tau' \subset \tau \) is a sector of \( \Gamma_{\mathcal{L}} \) with aperture \( d(\tau') \), then \( \mathcal{L}(\tau') \) is a sector \( \tau'' \) of \( \Gamma_{\mathcal{L}} \) with \( d(\tau'') = s^{-1}d(\tau') \), as we showed in Section 5. In particular, \( \mathcal{L} \) transforms a \( \theta' \subset \tau \) with aperture \( d(\theta') = r_3^{-1/2} \) into a sector with aperture \( s^{-1}r_3^{-1/2} \), which appears in the definition of \( S_{U,r,h} \).

2. \( \mathcal{L}^*(U_{\tau',r_3}) = U_{\tau'',s^2r_3} \). Since \( \tau'' = \mathcal{L}(\tau') \), this follows from Lemma 5.1.

3. \( \mathcal{L}^*(U_{\tau,r_2}) = B_{s^2r_2} \). Note that \( \mathcal{L}(\tau) \) is the sector corresponding to all of \( \Gamma_{\mathcal{L}} \), which is essentially the unit ball. We will denote this sector just by \( B_1 \). By Lemma 5.1, \( \mathcal{L}^*(U_{\tau,r_2}) = U_{B_1,s^2r_2} \). By definition, the right-hand side is the convex hull of the union of \( \theta'^* \) over all sectors \( \theta^* \) of aperture \( \sim s^{-1}r_2^{-1} \), and this is approximately the ball of radius \( s^2r_2 \).

4. The Jacobian factors from the change of variables work out the same on the left-hand side and the right-hand side. Since both sides involve a volume to the power \(-1 \) times an \( L^2 \) norm to the power \( 4 \), the Jacobian factors are the same on both sides of the inequality. \( \Box \)

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