Limit point buckling of a finite beam on a nonlinear foundation

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Abstract

In this paper, we consider an imperfect finite beam lying on a nonlinear foundation, whose dimensionless stiffness is reduced from 1 to $k$ as the beam deflection increases. Periodic equilibrium solutions are found analytically and are in good agreement with a numerical resolution, suggesting that localized buckling does not appear for a finite beam. The equilibrium paths may exhibit a limit point whose existence is related to the imperfection size and the stiffness parameter $k$ through an explicit condition. The limit point decreases with the imperfection size while it increases with the stiffness parameter. We show that the decay/growth rate is sensitive to the restoring force model. The analytical results on the limit load may be of particular interest for engineers in structural mechanics.

Keywords: Buckling, Imperfection, Finite beam, Nonlinear foundation, Limit point

I. INTRODUCTION

An elastic beam on a foundation is a model that can be found in a broad range of applications: railway tracks, buried pipelines, sandwich panels, coated solids in material, network beams, floating structures... The usual way to model the interaction between the beam and the foundation is to replace the latter with a set of independent springs whose restoring force is a linear [see e.g. 1–8] or a nonlinear [see e.g. 9–19] function of the local deflection of the beam. In both cases, the nonlinear effects, from the beam’s deformation and/or from the restoring force, play a crucial role in the buckling and the post-buckling behaviors. In particular, for a softening nonlinear foundation, the equilibrium curves of the beam may exhibit a maximum load (i.e. limit point) at which the structure loses its stability. Small imperfections, arising from various sources, usually have an appreciable effect
Fig. 1: Sketch of a beam on a nonlinear foundation. The beam has an imperfect shape $W_0$ and its lateral displacement is $W$. The compressive force is $P$ and the restoring force per unit length is $-\bar{P}$.

on this maximum load. The papers on deterministic imperfection sensitivity include those of [4, 5, 11, 15, 20-24] and extensive references for the stochastic imperfection sensitivity are compiled in [25]. As a general rule, the maximum load at which the beam becomes unstable diminishes with increasing imperfection size. Considering a finite beam on a bilinear/exponential foundation, [24] has shown the existence of a critical imperfection size $A_{0c}$ such that: if $A_0 < A_{0c}$, then the maximum load diminishes with the imperfection size, from the critical buckling load predicted by the classical linear analysis [see 10], for $A_0 = 0$, to the Euler load for $A_0 = A_{0c}$. In this case, the decay rate is sensitive to the restoring force model. For $A_0 > A_{0c}$, the maximum load is the Euler load (i.e. buckling load of a beam with no foundation).

In the present paper we aim to extend these results to two restoring force models with more general softening behaviors. We derive an analytical expression for $A_{0c}$ and study the evolution of the maximum load with the imperfection size and the stiffness reduction.

II. FORMULATION OF THE PROBLEM

We consider the effects of a compressive load $P$ on a beam of length $L$, with bending stiffness $EI$, lying on a foundation that provides a restoring force per unit length $\bar{P}$ (see Fig. 1). The beam and the foundation are assumed to be well bonded at their interface and remain bonded during deformation. Thus, interfacial slip or debonding is not considered. The mobilization of the foundation (also named the yield point) is noted $\Delta$, its linear stiffness $K_0$ and its nonlinear stiffness $K$.

In its initial configuration, the beam has an imperfect shape $W_0 = A_0 \sin (\pi X/L)$, where
Fig. 2: Dimensionless restoring force $p_k$. Red line: bi-linear model. Blue line: hyperbolic model. The stiffness ratio is $k \leq 1$.

$A_0$ is the imperfection size and $X$ is the longitudinal coordinate.

We introduce the characteristic length $L_c = (EI/K_0)^{1/4}$ and the non-dimensional quantities

$$l = \frac{L}{L_c}, \quad x = \frac{X}{L_c}, \quad w = \frac{W}{\Delta}, \quad w_0 = \frac{W_0}{\Delta}, \quad a_0 = \frac{A_0}{\Delta}, \quad \lambda = \frac{PL_c^2}{EI}, \quad k = \frac{K}{K_0}, \quad p_k = \frac{P}{K_0\Delta},$$

as the dimensionless beam length, longitudinal coordinate, lateral deflection (measured from the initial configuration), imperfection shape, imperfection size, compressive load, stiffness ratio and restoring force respectively.

Two models for the restoring force $p_k$ are considered in this article (see Fig. 2). The first one is

$$p_k (w) = -w - (1 - k) (\text{sgn} (w) - w) H (|w| - 1),$$

where sgn denotes the sign function and $H$ is the Heaviside function, defined as $H (|w| - 1) = 0$ if $|w| < 1$ and 1 if $|w| > 1$. This bi-linear restoring force refers to a foundation whose stiffness is instantaneously reduced from 1 to $k \leq 1$ when $w > 1$. The particular case $k = 1$
corresponds to a linear foundation. Reference [24] considered the particular case of $k = 0$. Here, we extend the study to $k \leq 1$, which leads to more general results.

To reflect the experimental tests on railway tracks performed by [26], also reported in [27, 28], who showed that the lateral friction force acting on a track is a smooth function of the lateral displacement, we introduce a hyperbolic profile defined as

$$p_k(w) = -kw - (1 - k) \tanh(w).$$  \hspace{1cm} (3)$$

This restoring force is a regularization of the bi-linear model as they share the same asymptotic behaviors.

We assume that $\lambda$ and $p_k$ are conservative forces, that strains are small compared to unity and that the kinematics of the beam is given by the classical Euler-Bernoulli assumption. The imperfection is also assumed to be small so that terms with higher powers of $w_0$ or its derivatives are neglected in the expression of the potential energy. Under these assumptions, the potential energy $V$ with low-order geometrically nonlinear terms is [see 10]

$$V = \int_0^l \left[ \frac{1}{2}w''^2 - \lambda \left( \frac{1}{2}w^{'2} + w_0'w' \right) - \int_0^w p_k(t) dt \right] dx,$$  \hspace{1cm} (4)$$

where a prime denotes differentiation with respect to $x$. The first term in the integral is the elastic bending energy, the second is the work done by the load $\lambda$, the last term is the energy stored in the elastic foundation.

The equilibrium states are given by the critical values of $V$. Assuming a simply supported beam, the boundary conditions are $w(0) = w(l) = 0$. Variations of (4) for an arbitrary kinematically admissible virtual displacement $\delta w$ yields the weak formulation of the equilibrium problem

$$\int_0^l \left[ w^{''''} + \lambda \left( w^{'''} + w_0^{'''} \right) - p_k(w) \right] \delta w \, dx = 0,$$  \hspace{1cm} (5)$$

which is equivalent to the stationary Swift-Hohenberg equation

$$w^{''''} + \lambda \left( w^{'''} + w_0^{'''} \right) - p_k(w) = 0,$$  \hspace{1cm} (6)$$
along with static boundary conditions \( w''(0) = w''(l) = 0 \).

This boundary value problem is nonlinear because of the restoring force and its solutions are highly sensitive to the length \( l \), as shown in [4]. Therefore, it is unrealistic to describe the behavior of the system over a large range of variation for \( l \). As done in [24], this study is restricted to a finite length beam where \( l < \sqrt{2\pi} \). For such values of \( l \) a classical linear analysis [see 10] shows that the first buckling mode is the most unstable one and appears for \( \lambda = \lambda_c + \lambda_e^{-1} \), where \( \lambda_e = (\pi/l)^2 \) is the Euler load.

### III. SOLVING METHODS

To solve (5) we apply a Galerkin method with a trigonometric test function \( w \) of amplitude \( y \): \( w = y \sin(\pi x/l) \), assuming that the deflection has the same shape as the first buckling mode and the initial imperfection. For more details about the principle of the method, the reader is referred to [24], where the procedure has already been used. In that paper, this method has been shown to be reliable in the prediction of the equilibrium paths of the system, for \( k = 0 \). We shall see in the present paper that it is actually reliable for any \( k \leq 1 \), thereby extending the results of [24].

The insertion of \( \delta w = \delta y \sin(\pi x/l) \) in (5) yields

\[
\int_0^l \sin\left(\frac{\pi}{l} x \right) \left[ w'''' + \lambda (w'' + w'') - p_k (w) \right] \, dx = 0.
\]

(7)

Splitting the restoring force in a linear and a nonlinear term \( N (w) \) leads to \( p_k (w) = -w - (1 - k) N (w) \). With this decomposition and \( w = y \sin(\pi x/l) \), (7) can be rewritten as

\[
\lambda_k = \frac{1}{a_0 + y} \left[ \lambda_c y + (1 - k) \frac{Q (y)}{\lambda_e} \right],
\]

(8)

where the subscript \( k \) denotes the dependance of \( \lambda \) on the parameter \( k \). The function \( Q \) takes into account the nonlinear behavior of the restoring force and is given by

\[
Q (y) = \frac{2}{l} \int_0^l \sin\left(\frac{\pi}{l} x \right) N \left( y \sin\left(\frac{\pi}{l} x \right) \right) \, dx.
\]

(9)
For the restoring force models (2) and (3), \( Q \) is negative and decreases monotonically to the asymptote \(-y + 4/\pi\) as \( y \to +\infty\). Thus \( \lambda_k \) is maximum for \( k = 1 \) (linear foundation) and has an horizontal asymptote \( \lambda_k^\infty = \lambda_c + k\lambda_e^{-1} \) as \( y \to +\infty\).

Equilibrium paths predicted by (8) are traced out in the plane \( (y = \max (w), \lambda) \) by gradually incrementing \( y \) and evaluating \( \lambda_k, k \) and \( a_0 \) being fixed. Predictions are compared with a numerical solution of (6), using MATLAB’s boundary value solver \textit{bvp4c} [this is a finite difference code that implements a collocation formula, details of which can be found in 29].

IV. RESULTS

The equilibrium paths predicted by the Galerkin method and the numerical solution are shown in Fig. 3. A perfect agreement in the predictions is found for both restoring force models (the relative error between the two methods being less than 0.1%). Since the Galerkin method was initiated with a test function having the same shape as the imperfection, we conclude that the deflection is just an amplification of the initial curvature. In other words, in the range \( l < \sqrt{2}\pi \), no localized buckling is observed for a beam on a bi-linear or hyperbolic foundation. This behavior has also been reported by [4] for a linear foundation, showing a tendency toward localization when increasing the beam length.

As expected, the equilibrium paths traced out for the hyperbolic restoring force are below those traced out for the bi-linear force, the hyperbolic profile modeling a softer foundation than the bi-linear one. However, the choice of the restoring force has little influence on the shape of the equilibrium paths.

For small \( a_0 \), the equilibrium paths first increase to a maximum \( \lambda_m \) that is smaller (or equals to in the case of \( a_0 = 0 \)) than the buckling load \( \lambda_c \). Then, the paths asymptotically decrease to \( \lambda_k^\infty \). In the case of \( a_0 = 0 \), \( k = 1 \), they remain equal to \( \lambda_c \). For high \( a_0 \), the equilibrium paths increase monotonically to the asymptote \( \lambda_k^\infty \leq \lambda_c \). The asymptotic value \( \lambda_k^\infty = \lambda_c \) is reached for \( a_0 = 0 \) and \( k = 1 \).

Note that for \( k < -\lambda_e^2 \), \( \lambda_k^\infty \) is negative, so that equilibrium states with \( \lambda < 0 \) are predicted (see Fig. 3(a)). Physically, when \( k < -\lambda_e^2 \) the restoring force \( p_k \) may become negative so that springs are compressed, pushing up the beam. In this situation, the restoring force has a destabilizing effect on the beam. To counteract this effect, a tensile force \( \lambda < 0 \)
Fig. 3: Equilibrium paths of a finite length beam on a nonlinear foundation. Circles: numerical predictions. Lines: Galerkin solution. In red: bi-linear restoring force model. In blue: hyperbolic model. (a) $k = -2$, (b) $k = 0$, (c) $k = 0.75$, (d) $k = 1$. On each subfigure, the equilibrium paths are plotted (from top to bottom) for $a_0 = 0$, $a_0 = 0.0238$, $a_0 = 0.595$ and $a_0 = 1.19$, as shown in (d). The length of the beam is $l = 3$. 
Fig. 4: Maximum load $\lambda_m$ that the beam may support versus the imperfection size $a_0$ and the foundation stiffness ratio $k$. In red: bi-linear restoring force model. In blue: hyperbolic model. Vertical dotted lines correspond to the critical imperfection sizes $a_{0c}$ for $k = 0.75$ and $k = 0$. The length of the beam is $l = 3$.

$$\max. \text{ load} = \frac{4(1-k)}{\pi \lambda_c^2 + k}$$

$$\lambda_c = (\pi/l)^2$$

$$\lambda_{0c} = \lambda_c + \lambda_c^{-1}$$

$$\lambda_{\infty} = \lambda_c + k\lambda_c^{-1}$$

Fig. 5: Diagram of existence of a limit point for an imperfect finite beam on a bi-linear/hyperbolic foundation. $k$ is the stiffness ratio of the foundation and $a_0$ the imperfection size. $\lambda_c = (\pi/l)^2$ is the Euler load, $\lambda_c = \lambda_c + \lambda_c^{-1}$ and $\lambda_{\infty} = \lambda_c + k\lambda_c^{-1}$. 
has to be applied.

The evolution of $\lambda_m$ versus $a_0$ is shown in Fig. 4. A gradual drop in the maximum load admissible by the structure from $\lambda_c$ to $\lambda_k^\infty$ is observed when increasing $a_0$ (resp. decreasing $k$). This gradual drop is highly sensitive to the restoring force model. A log scale applied on Fig. 4 shows that, for small imperfection sizes, the decay rate does not depend on $k$: $\lambda_m - \lambda_c$ scales as $-a_0$, for the bi-linear model and as $-a_0^{2/3}$ for the hyperbolic model.

For $a_0$ larger than a critical value $a_{0c}$, the equilibrium paths do not have a limit point anymore. Actually, a path with no limit point may be seen as a path with a limit point at $(\infty, \lambda_k^\infty)$. Thereby, $a_{0c}$ may be obtained from (8) by enforcing $y \to \infty$ in $d\lambda_k/dy = 0$. Both restoring force models leads to

$$a_{0c} = \frac{4}{\pi} \frac{1 - k}{\lambda_c^2 + k},$$

whose dimensional equivalent form is

$$A_{0c} = \frac{4}{\pi^4} \frac{(K_0 - K) \Delta}{EI/L^4 + K}.$$  \hspace{1cm} (11)

The critical imperfection size predicted by [24] is therefore recovered in the particular case $K = 0$, showing that $A_{0c}$ only depends on the limiting plateau $K_0 \Delta$ of the restoring force [as stated in 30].

Finally, since $a_0 > 0$, equation (10) shows that if $k < -\lambda_e^2$ then the equilibrium paths always have a limit point $\lambda_k^\infty < \lambda_m < \lambda_c$.

V. CONCLUSION

In this paper, we considered the buckling of an imperfect finite beam on a bi-linear/hyperbolic foundation. The imperfection has been introduced as an initial curvature of size $a_0$ and the foundation stiffness ratio as a parameter $k \leq 1$, extending the result of [24] derived for $k = 0$.

Equilibrium paths of the beam have been predicted using a Galerkin method initiated with a single trigonometric function which has the same shape as the imperfection. Predictions compare well with a numerical solution and lead to the conclusion that only periodic
buckling can arise for a finite beam on a bi-linear/hyperbolic foundation, as also observed by [4] for an linear foundation.

We have shown the existence of a critical imperfection size $a_{0c} = 4 (1 - k) [\pi (\lambda_c^2 + k)]^{-1}$, independent of the restoring force model, such that:

- if $a_0 < a_{0c}$, then the maximum load diminishes with increasing imperfection size, from $\lambda_c = \lambda_c + \lambda_c^{-1}$, for $a_0 = 0$, to $\lambda_c + k\lambda_c^{-1}$ for $a_0 = a_{0c}$, $\lambda_c = (\pi/l)^2$ being the Euler load. The decay rate has been shown to be sensitive to the restoring force model. In the limit of small $a_0$, $\lambda_m - \lambda_c \sim -a_0$ for the bi-linear model and $\lambda_m - \lambda_c \sim -a_0^{2/3}$ for the hyperbolic model.

- If $a_0 > a_{0c}$, then the maximum load simply corresponds to $\lambda_c + k\lambda_c^{-1}$.

Finally, we have shown that if $k < -\lambda_c^2$ then an imperfect finite beam on a bi-linear/hyperbolic foundation can support a compressive load larger than $\lambda_{k}^\infty$, and smaller than $\lambda_c$, whatever the imperfection size is. This feature is highly interesting for an engineer since $a_0$ is usually hard to evaluate. The main results from this study are summarized in Fig. 5.

In the present paper, a bi-linear restoring force model for the foundation has been used but plasticity effects that would emerge from loading/unloading cycles have not been considered. Future works will have to highlight the way those effects could modify the maximum load that the beam can support. A basic model would consist of considering a permanent deflection as an imperfection whose size would grow up after each cycle. In that case, from the present study, it is expected a decrease of the maximum load after each cycle, at least as long as the accumulated deflection remains smaller than a threshold equivalent to $a_{0c}$.

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