Spontaneous breaking of the C, P, and rotational symmetries by topological defects in two extra dimensions

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Abstract

We formulate models of complex scalar fields in the space-time that has a two-dimensional sphere as extra dimensions. The two-sphere \( S^2 \) is assumed to have the Dirac-Wu-Yang monopole as a background gauge field. The nontrivial topology of the monopole induces topological defects, i.e. vortices, in the scalar field. When the radius of \( S^2 \) is larger than a critical radius, the scalar field develops a vacuum expectation value and creates vortices in \( S^2 \). Then the vortices break the rotational symmetry of \( S^2 \). We exactly evaluate the critical radius as \( r_q = \sqrt{|q|}/\mu \), where \( q \) is the monopole number and \( \mu \) is the imaginary mass of the scalar. We show that the vortices repel each other. We analyze the vacua of the models with one scalar field in each case of \( q = 1/2, 1, 3/2 \) and find that: when \( q = 1/2 \), a single vortex exists; when \( q = 1 \), two vortices sit at diametrical points on \( S^2 \); when \( q = 3/2 \), three vortices sit at the vertices of the largest triangle on \( S^2 \). The symmetry of the model \( G = U(1) \times SU(2) \times CP \) is broken to \( H_{1/2} = U(1)' \), \( H_1 = U(1)'' \times CP \), \( H_{3/2} = D_{3h} \), respectively. Here \( D_{3h} \) is the symmetry group of a regular triangle. We extend our analysis to the doublet scalar fields and show that the symmetry is broken from \( G_{\text{doublet}} = U(1) \times SU(2) \times SU(2)_f \times P \) to \( H_{\text{doublet}} = SU(2)' \times P \). Finally we obtain the exact vacuum solution of the model with the multiplet \( (q_1, q_2, \ldots, q_{2j+1}) = (j, j, \cdots, j) \) and show that the symmetry is broken from \( G_{\text{multiplet}} = U(1) \times SU(2) \times SU(2j + 1)_f \times CP \) to \( H_{\text{multiplet}} = SU(2)' \times CP' \).

Our results caution that a careful analysis of dynamics of the topological defects is required for construction of a reliable model that possesses such a defect structure.

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1 Introduction

In spite of outstanding success of the Standard Model of particle physics in most of experimental tests, there are several puzzles yet to be solved: the neutrino oscillation \cite{1}, the muon anomalous magnetic moment \cite{2}, supersymmetry breaking, the origin of the three generations, the fermion mass hierarchy, and the hierarchy problem of the electroweak, GUT, and Planck scales. Recently, the hypothesis that the space-time has compact extra dimensions \cite{3} is calling a renewed interest since it provides possible solutions to these puzzles. Here we cannot cover all the aspects of physics of extra dimensions. Instead, we concentrate on the aspects related to the origin of the generations and to supersymmetry breaking.

The challenges to explain the fermion masses and generations by higher-dimensional models have a long history. Hosotani \cite{4} found that dynamics of the Wilson loop in a multiply-connected space breaks gauge symmetry and generates fermion masses. This mechanism has been taken over to the string theory. He also showed \cite{5} that the monopole background in \( S^2 \) generates a left-right asymmetric structure of fermions. Akama, Rubakov, Shaposhnikov, and others, in their early attempts \cite{6}, proposed a hypothesis that our four-dimensional space-time is a soliton-like object embedded in the higher-dimensional space. Recently, Arkani-Hamed and Schmaltz \cite{7} built a model in that the Standard Model fields are confined to a thick wall in the extra dimensions, and explained the hierarchy of Yukawa couplings. Libanov and Troitsky \cite{8} built a model in that the fermions appear as zero modes trapped in the core of a topological defect in two extra dimensions, and attributed the number of the generations to the topological number of the vortex defect. They with Frère \cite{9} also built another model, which has a single vortex with topological number one but has three fermion generations. Kawamura \cite{10} suggested a mechanism of gauge symmetry breaking by the boundary condition in the orbifold \( S^1/Z_2 \), and recently Hall and Nomura \cite{11} completed a GUT model using this mechanism. Bando et al. \cite{12} showed that dynamics of Kaluza-Klein modes produces the hierarchy of Yukawa couplings via the power-law behavior of the renormalized couplings to the energy scale. Concerning dynamics in extra dimensions, Randjbar-Daemi, Salam and Strathdee \cite{13} proved that the \( SU(2) \)-invariant solutions of the Yang-Mills system in \( R^4 \times S^2 \) have tachyonic excitations and are unstable. Recently, Dvali, Randjbar-Daemi and Tabbash \cite{14} proposed a model in \( R^4 \times CP^1 \times CP^2 \) with the monopole and instanton background. Using the mechanism proved previously, they showed that the extra-dimensional components of the gauge fields become tachyonic and deduced the chiral fermion spectrum of the standard model.

On the other hand, recently one of the authors \cite{15} proposed a new mechanism of supersymmetry breaking by a topologically nontrivial boundary condition in one extra dimension \( S^1 \). In those models the translational symmetry in the direction of \( S^1 \) is spontaneously broken and the supersymmetry is subsequently broken. However, the model built on the one extra dimension is too simple to provide a realistic particle spectrum. Therefore it is desirable to construct models on higher extra dimensions and to analyze the patterns of symmetry breaking.

The purposes of this paper are threefold: the first one is to construct models that have a two-dimensional sphere \( S^2 \) as extra dimensions; the second is to analyze exhaustively the patterns of symmetries including the rotational symmetry of \( S^2 \); the third one is to study the topological structure of the vacuum.

Topological defects in the vacuum break spatial symmetries. For example, vortices in
the sphere break the rotational symmetry. Since the symmetry of the space governs the particle spectrum, it is also worth studying.

To convey our basic idea let us shortly review the model [16] that exhibits spontaneous breaking of the translational symmetry in one extra dimension. Assume that the space-time is a direct product of the Minkowski space with a circle, \( \mathbb{R}^{n} \times \mathbb{S}^{1} \), which is equipped with the coordinate system \((x^{0}, x^{1}, \ldots, x^{n-1}, \phi)\), and in which the points \(\{\phi + 2\pi m\}_{m=0, \pm 1, \pm 2, \ldots}\) are identified. The radius of \(\mathbb{S}^{1}\) is denoted by \(r\). The model consists of a complex scalar field \(f(x, \phi)\) and has the Lagrangian

\[
\mathcal{L} = g_{\mu\nu} \frac{\partial f^{*} \partial f}{\partial x^{\mu} \partial x^{\nu}} - \frac{1}{r^{2}} \frac{\partial f^{*} \partial f}{\partial \phi} + \mu^{2} f^{*} f - \lambda (f^{*} f)^{2}. \tag{1.1}
\]

This model is invariant under the translations along \(\mathbb{S}^{1}\),

\[
e^{-iPd} : f(x, \phi) \mapsto f(x, \phi - d), \tag{1.2}
\]

and also under the phase transformations,

\[
e^{-iQt} : f(x, \phi) \mapsto e^{-it} f(x, \phi). \tag{1.3}
\]

Usually one imposes the periodic boundary condition on the field as \(f(x, \phi + 2\pi) = f(x, \phi)\), but actually there is no a priori reason to impose it if the field \(f\) itself is not a direct observable. Instead we may impose the twisted boundary condition

\[
f(x, \phi + 2\pi) = e^{-i2\pi\alpha} f(x, \phi) \tag{1.4}
\]

with the real parameter \(\alpha\) (\(|\alpha| \leq 1/2\)). It should be noted that the condition (1.4) is compatible with both the symmetries, (1.2) and (1.3). So the symmetries are not explicitly broken even when the twisted boundary condition is imposed. The vacuum configuration

\[
\langle f(x, \phi) \rangle = v e^{-i\alpha \phi} \tag{1.5}
\]

minimizes the energy functional

\[
E = \int_{0}^{2\pi} d\phi r \left\{ \frac{1}{r^{2}} \frac{\partial f^{*} \partial f}{\partial \phi} - \mu^{2} f^{*} f + \lambda (f^{*} f)^{2} \right\} = 2\pi r \left\{ \frac{1}{r^{2}} \alpha^{2} v^{2} - \mu^{2} v^{2} + \lambda v^{4} \right\}. \tag{1.6}
\]

Then, if we put \(r_{c} := |\alpha|/\mu\), the minimum of \(E\) is realized by

\[
v^{2} = \begin{cases} 
0 & \text{for } r \leq r_{c}, \\
\left(1 - \frac{\alpha^{2}}{\mu^{2} r^{2}}\right) \frac{\mu^{2}}{2\lambda} & \text{for } r > r_{c}.
\end{cases} \tag{1.7}
\]

We can see that when the radius \(r\) of the circle is larger than the critical radius \(r_{c}\), the field \(f\) develops the non-vanishing non-constant vacuum expectation value (1.5) and accordingly the translational symmetry (1.2) is spontaneously broken. However, the vacuum (1.3) is still invariant under the transformations given by \(e^{-i(P+\alpha Q)d}\). Therefore the symmetry is broken from \(U(1) \times U(1)\) to \(U(1)\). The existence of the finite critical radius is a remarkable feature of this model.

Spontaneous breaking of translational symmetry has various physical implications; for example, it can provide a new mechanism of supersymmetry breaking as has been pointed out in the previous papers [15]. Because supercharges generate translations as

\[\{Q_{\alpha}, Q_{\beta}\} = 2\sigma^{\mu}_{\alpha\beta} P_{\mu}, \tag{1.8}\]
breaking of the translational symmetry inevitably implicates breaking of the supersymmetry. More concretely, the super-transformation of a fermionic field is given by

\[ \delta_\xi \psi(x) = i\sqrt{2} \sigma^\mu \xi \partial_\mu \varphi(x) + \sqrt{2} \xi F(x). \]  

(1.9)

The supersymmetry is broken if \( \langle F(x) \rangle \neq 0 \). This is a usual pattern of supersymmetry breaking and is called \( F \)-term breaking. But the supersymmetry can be broken also if \( \partial_\mu \langle \varphi(x) \rangle \neq 0 \). The non-vanishing expectation value of the derivative \( \partial_\mu \langle \varphi(x) \rangle \) implies breaking of the translational symmetry. Thus the translational symmetry breaking involves the supersymmetry breaking. One of the authors \[15\] constructed and analyzed concrete models that give rise to supersymmetry breaking via the twisted boundary condition in \( S^1 \).

The model mentioned above consists of only one complex scalar field in \( R^n \times S^1 \) and has the \( U(1) \) internal symmetry. Actually it is possible to construct models that have more fields and larger symmetries. One of the authors \[16\] constructed a class of models that have real \( n \)-component fields and \( O(n) \) symmetries and exhaustively studied the patterns of breaking of the translational symmetry and of the \( O(n) \) symmetries. However, models built on the space-time with higher extra dimensions than \( S^1 \) are more desirable for application to particle physics, because higher dimensional manifolds can involve larger symmetries and richer particle spectra, which are useful for construction of realistic models.

As a step to the exploration to higher extra dimensions, in the recent paper \[17\] we defined and studied a model that has the two-dimensional sphere \( S^2 \). Then the translational symmetry \( (1.2) \) of \( S^1 \) is naturally replaced by the rotational symmetry of \( S^2 \). Moreover, the twisted boundary condition \( (1.4) \) in \( S^1 \) is replace by the twisted patching condition in \( S^2 \), which is described as follows: the spherical coordinate of \( S^2 \) is denoted by \( (\theta, \phi) \). Let us introduce a pair of complex scalar fields, \( f_+ \) and \( f_- \), which are defined in the upper \( (0 \leq \theta < \pi) \) and the lower \( (0 < \theta \leq \pi) \) region of \( S^2 \), respectively. In the overlap of the two regions we impose the \textit{twisted patching condition}

\[ f_-(\theta, \phi) = e^{-im\phi} f_+(\theta, \phi), \quad 0 < \theta < \pi, \]  

(1.10)

on the fields. Here \( m \) is an integer. Eq. \( (1.10) \) is nothing but the gauge transformation that accompanies the Dirac-Wu-Yang monopole \( A_\pm = (1 \mp \cos \theta) d\phi \). In the body of this paper \( m \) is written as \( m = 2q \) and \( q \) is called the monopole charge. When \( m \neq 0 \), the vacuum expectation value \( \langle f_\pm(\theta, \phi) \rangle \) cannot be a non-vanishing constant over \( S^2 \) owing to the patching condition. Even if \( \langle f_\pm(\theta, \phi) \rangle \) is nonzero in some region, it should be zero at some points in \( S^2 \) for the topological obstruction. The zero points of \( \langle f_\pm(\theta, \phi) \rangle \) are called vortices. Thus the rotational symmetry is pinned down by the vortices.

In the previous paper \[15\] we studied in detail the model that has a complex scalar field in the monopole background in \( R^n \times S^2 \). There we proved the existence of a critical radius of \( S^2 \); when the radius of \( S^2 \) exceeds the critical radius, the field develops vortices and breaks the rotational symmetry. We also estimated the exact value of the critical radius. But our treatment was restricted to the model that has only one complex scalar field. Besides, discrete symmetries including \( C \) and \( P \) were missed from our consideration.

In this paper we first examine the symmetries of the model of one complex scalar field in \( R^n \times S^2 \) with the monopole background in detail. We then find that the model has a \( Z_2 \) symmetry, which is analogous to the \( CP \) symmetry. We show that the rotational and the discrete symmetries are spontaneously broken when the radius of \( S^2 \) is larger than the critical radius. We calculate the locations of the vortices of the stable vacuum. We also
clarify the structure of Nambu-Goldstone bosons that are associated with the symmetry breaking. Furthermore we extend our argument to models that have more scalar fields and larger symmetries. If the matter multiplet has the charge \((q_1, q_2, \ldots, q_n)\) and if \(\sum_i q_i = 0\), then the monopole gauge field can be embedded in an \(SU(n)\) gauge field, which is free from the twisted patching condition. Moreover, when \((q_1, q_2, \cdots, q_{2j+1}) = (j, j, \cdots, j)\), we obtain the exact vacuum and show that a modified rotational symmetry is left unbroken. Thus the patterns of symmetry breaking strongly depend on the matter contents of the model. Although our study is motivated by an attempt to get a new mechanism of supersymmetry breaking, our model is not yet made supersymmetric, or does not exhibit supersymmetry breaking, either. We should declare that our study is restricted to bosonic fields at the present stage.

This paper is organized as follows: in Section 2 we define the model of one complex scalar field in \(R^n \times S^2\) and characterize its complete symmetry as \(G = U(1) \times SU(2) \times CP\). We find that the model is not invariant under \(C\) and \(P\) separately but it is actually invariant under the combined \(CP\) transformation. We also estimate the exact critical radius as \(r_q = \sqrt{|q|}/\mu\), where \(q\) is the monopole number and \(\mu\) is the imaginary mass of the scalar. Although this part is a repetition of the previous paper \([7]\), we include it here to make the present paper self-contained. We calculate the approximate vacuum in each case of \(q = 1/2, 1, 3/2\) concretely and find that: when \(q = 1/2\), a single vortex exists; when \(q = 1\), two vortices sit at opposite two points on \(S^2\); when \(q = 3/2\), three vortices sit at the vertices of the largest triangle on \(S^2\). Then we observe that the symmetry is broken to \(H_{1/2} = U(1)'\), \(H_1 = U(1)' \times CP\), \(H_{3/2} = D_{3h}\) for each case. Here \(D_{3h}\) is the symmetry of a regular triangle including the reflection transformation. Since our approximate calculation is based on the variational method, we give an argument to justify the method.

In Section 3 we formulate the model of doublet fields with monopole charges \((q_1, q_2) = (q, -q)\). Then we show that the scalar and the gauge fields are embedded in \(SU(2)\) and are transformed into fields that are single-valued over the whole \(S^2\). The model has the symmetry \(G_{\text{doublet}} = U(1) \times SU(2) \times SU(2)f \times P\). Charge conjugation is included in \(SU(2)f\). Then we obtain the exact vacuum of the \(q = 1/2\) doublet model and show that the symmetry is broken to \(H_{1/2} = SU(2)' \times P\) when \(r > (\sqrt{2}/\mu)^{-1}\). In this model vortices do not appear and a modified rotational symmetry remains unbroken.

In Section 4 we consider models that consist of arbitrary multiplet fields with \((q_1, q_2, \cdots, q_n)\). Then we prove that the fields become free from the Dirac singularity if \(\sum_i q_i = 0\). We show that a specific model with the charge multiplet \((q_1, q_2, \cdots, q_{2j+1}) = (j, j, \cdots, j)\) has the symmetry \(G_{\text{multiplet}} = U(1) \times SU(2) \times SU(2j+1)f \times CP\). We obtain its exact vacuum solution and show that the symmetry is broken to \(H_{\text{multiplet}} = SU(2)' \times CP'\) when \(r > \sqrt{3}/\mu\).

In conclusion we speculate about generalizations and applications of this work.

## 2 Singlet models

### 2.1 Definitions and symmetries

First, we define a model, which exhibits spontaneous breaking of the rotational symmetry. Our model is defined in the space-time \(R^n \times S^2\), where \(R^n\) is an \(n\)-dimensional Minkowski space and \(S^2\) is a two-dimensional sphere of the radius \(r\). The Cartesian coordinate of \(R^n\) is denoted by \((x^0, x^1, \cdots, x^{n-1})\) while the spherical coordinate of \(S^2\) is denoted by \((\theta, \phi)\). The space \(R^n\) is equipped with the metric \(g_{\mu\nu} = \text{diag}(+1, -1, \cdots, -1)\). Our model consists of a
complex scalar field \( f \) in \( \mathbb{R}^n \times S^2 \) with a background gauge field \( A \) in \( S^2 \). The gauge field \( A \) is fixed to be the Dirac-Wu-Yang monopole [13], which is defined as follows: two open sets of \( S^2 \), \( U_+ = \{(\theta, \phi)|\theta \neq \pi\} \) and \( U_- = \{(\theta, \phi)|\theta \neq 0\} \), cover \( S^2 = U_+ \cup U_- \). The monopole field \( A \) is described by the pair of 1-forms \( (A_+, A_-) \),

\[
A_{\pm}(\theta, \phi) = (\pm 1 - \cos \theta) d\phi \quad \text{in } U_{\pm}.
\]  

The associated magnetic field is \( B = dA_+ = dA_- = \sin \theta \, d\theta \wedge d\phi \). Then the total magnetic flux is given by \( \int B = 4\pi \). The complex scalar field \( f \) is described by a pair of fields \( (f_+, f_-) \), where \( f_{\pm} \) is a smooth function over \( U_{\pm} \), respectively. The covariant derivative of \( f \) is defined by

\[
Df_\pm = df_\pm - iqA_\pm f_\pm \quad \text{in } U_{\pm}
\]  

with the coupling constant \( q \). It is represented as the product \( q = eg \) of the electric charge \( e \) and the magnetic charge \( g \) that is usually defined by the magnetic flux \( \int B = 4\pi g \). However, we conveniently make the constant \( q \) absorb both \( e \) and \( g \) into its definition and we simply call \( q \) the magnetic number or the magnetic charge. The pairs of the fields, \( (A_+, A_-) \) and \( (f_+, f_-) \), are patched together by the gauge transformation [13],

\[
A_- = A_+ - 2d\phi, \quad f_- = e^{-2iq\phi} f_+ \quad \text{in } U_+ \cap U_-.
\]  

Because the fields \( f_{\pm} \) must each be single-valued, \( 2q \) must be an integer. The action of our model is

\[
S_A[f] = \int d^n x \, d\theta d\phi \, r^2 \sin \theta \left\{ g^\mu_\nu \frac{\partial f_\pm}{\partial x^\mu} \frac{\partial f_\pm}{\partial x^\nu} - \frac{1}{r^2} \left| \frac{\partial f_\pm}{\partial \theta} \right|^2 \right. 
\]

\[
- \frac{1}{r^2 \sin^2 \theta} \left| \frac{\partial f_\pm}{\partial \phi} - iq(\pm 1 - \cos \theta) f_\pm \right|^2 + \mu^2 f_\pm^* f_\pm - \lambda (f_\pm^* f_\pm)^2 \}.
\]  

where \( \mu^2, \lambda > 0 \) are real parameters.

This model has a global symmetry

\[
G = U(1) \times SU(2) \times CP,
\]  

as being seen below. The \( U(1) \) symmetry is defined as a family of the transformations

\[
e^{-iQt} : f_\pm \mapsto e^{-iqt} f_\pm
\]  

with \( t \in \mathbb{R} \). The generator of the transformation is \( Q = q \). On the other hand, the elements of \( SU(2) \) act on \( S^2 \) as rotation transformations. Of course, the explicit form of the background gauge field \( A_{\pm} = (\pm 1 - \cos \theta) d\phi \) is not invariant under arbitrary rotations. To leave it invariant, a rotation transformation is to be complemented by a gauge transformation. Such a gauge transformation can be calculated as follows: the spherical coordinate \( (\theta, \phi) \) is assigned to a point \( p \in S^2 \). We introduce two maps \( s_{\pm} : U_{\pm} \to SU(2) \) by

\[
s_{\pm}(p) := e^{-i\sigma_3 \phi/2} e^{-i\sigma_2 \theta/2} e^{\pm i\sigma_3 \phi/2},
\]  

where \( \{\sigma_1, \sigma_2, \sigma_3\} \) are the Pauli matrices. Note that \( s_{\pm}(p) \) are smooth in \( U_{\pm} \), respectively. These maps have the property

\[
s_{\pm}(p) \cdot \sigma_3 \cdot s_{\pm}(p)^{-1} = \sigma_1 \sin \theta \cos \phi + \sigma_2 \sin \theta \sin \phi + \sigma_3 \cos \theta.
\]  

(2.8)
Because
\[
 s_\pm^{-1} ds_\pm = -\frac{i}{2} \left\{ \sigma_1 (\mp \sin \phi d\theta - \cos \phi \sin \theta d\phi) + \sigma_2 (\cos \phi d\theta \mp \sin \phi \sin \theta d\phi) + \sigma_3 (\mp 1 + \cos \theta) d\phi \right\},
\]
we get
\[
 A_\pm = -i \text{tr} (\sigma_3 \cdot s_\pm^{-1} ds_\pm). \tag{2.10}
\]
Let indices \(\alpha\) and \(\beta\) denote either + or −. Suppose that a point \(p \in U_\alpha\) is transformed to \(g^{-1}p \in U_\beta\) by an element \(g \in SU(2)\). Then we define an element of \(SU(2)\) by
\[
 W_{\alpha\beta}(g; p) := s_\alpha(p)^{-1} \cdot g \cdot s_\beta(g^{-1}p), \tag{2.11}
\]
which is called the Wigner rotation in the context of the representation theory, for example, in \([12]\) and \([13]\). Since \(W_{\alpha\beta} \cdot \sigma_3 \cdot W_{\alpha\beta}^{-1} = \sigma_3\), the value of the Wigner rotation has a form
\[
 W_{\alpha\beta}(g; p) = e^{-i\sigma_3 \omega/2} \tag{2.12}
\]
with a real number \(\omega\), which defines a function \(\omega_{\alpha\beta}(g; p)\). Since Eq. (2.11) implies
\[
 s_\beta(g^{-1}p) = g^{-1} \cdot s_\alpha(p) \cdot W_{\alpha\beta}(g; p), \tag{2.13}
\]
the gauge field rotated by \(g \in SU(2)\) becomes
\[
 A_{\beta}(g^{-1}p) = -i \text{tr} (\sigma_3 \cdot s_\beta^{-1}(g^{-1}p)ds_\beta(g^{-1}p)) = -i \text{tr} \left( \sigma_3 \cdot W_{\alpha\beta}(g; p)^{-1} s_\alpha(p)^{-1} g d(g^{-1} s_\alpha(p) W_{\alpha\beta}(g; p)) \right) = -i \text{tr} \left( \sigma_3 \cdot s_\alpha(p)^{-1} ds_\alpha(p) \right) - i \text{tr} \left( \sigma_3 \cdot W_{\alpha\beta}(g; p)^{-1} dW_{\alpha\beta}(g; p) \right) = A_\alpha(p) - d\omega_{\alpha\beta}(p). \tag{2.14}
\]
Therefore, a sequence of the rotation by \(g \in SU(2)\) and the gauge transformation by \(\omega_{\alpha\beta}(g; p)\),
\[
 \varphi(g) : f_\alpha(p) \mapsto f'_\alpha(p) = e^{i \varphi_{\alpha\beta}(g; p)} f_\beta(g^{-1}p), \tag{2.15}
\]
leaves the monopole gauge field invariant as
\[
 A_\alpha(p) \mapsto A'_\alpha(p) = A_{\beta}(g^{-1}p) + d\omega_{\alpha\beta}(g; p) = A_\alpha(p). \tag{2.16}
\]
Under this transformation the action (2.4) is also left invariant. It is easily verified that the Wigner rotations satisfy
\[
 W_{\alpha\beta}(g; p)W_{\beta\gamma}(g'; g^{-1}p) = W_{\alpha\gamma}(gg'; p). \tag{2.17}
\]
For the identity transformation \(e \in SU(2)\) we have
\[
 W_{\pm+}(e; p) = s_-(p)^{-1} \cdot s_+(p) = e^{i\sigma_3 \phi}, \tag{2.18}
\]
which implies
\[
 \omega_{\pm+}(e; p) = -2\phi \tag{2.19}
\]
in the place of (2.12). Hence (2.15) reproduces the patching condition \(f_- = e^{-2iq\phi} f_+\), as it should be.
Although calculation of the concrete value of the Wigner rotation (2.11) is cumbersome, to give a definite example we calculate it for \( g = e^{-i\sigma_3\gamma/2} \), which is a rotation around the \( z \)-axis by the angle \( \gamma \). The point \( p = (\theta, \phi) \in U_\pm \) is then transformed to \( g^{-1}p = (\theta, \phi-\gamma) \in U_\pm \), and from the definitions (2.7) and (2.11) we get
\[
W_{\pm\pm}(e^{-i\sigma_3\gamma/2}; p) = e^{\mp i\sigma_3\gamma/2},
\]
which implies
\[
\omega_{\pm\pm}(e^{-i\sigma_3\gamma/2}; p) = \pm\gamma = \text{constant}
\]
in (2.12). The transformation (2.15) now becomes
\[
\varphi(e^{-i\sigma_3\gamma/2}) : f_\pm(\theta, \phi) \mapsto e^{\pm iq\gamma}f_\pm(\theta, \phi - \gamma),
\]
For later use, let us calculate the Wigner rotation for the rotation around the \( x \)-axis by the angle \( \pi \), which is represented by \( g = e^{-i\sigma_1\pi/2} = -i\sigma_1 \). The point \( p = (\theta, \phi) \in U_\pm \) is then transformed to \( g^{-1}p = (\pi - \theta, -\phi) \in U_\mp \). Then the corresponding Wigner rotation is calculated to be
\[
W_{\pm\mp}(e^{-i\sigma_1\pi/2}; p) = -i\sigma_3 = e^{-i\sigma_3\pi/2},
\]
and therefore we have
\[
\omega_{\pm\mp}(e^{-i\sigma_1\pi/2}; p) = \pi
\]
in the place of (2.12). The transformation (2.15) now becomes
\[
\varphi(e^{-i\sigma_1\pi/2}) : f_\pm(\theta, \phi) \mapsto e^{iq\pi}f_\mp(\pi - \theta, -\phi).
\]
For later use we would like to present infinitesimal transformations. The generators \( \{L_1, L_2, L_3\} \) of the \( SU(2) \) transformations (2.15) are defined by
\[
- iL_k f_\alpha(p) := \left. \frac{d}{d\gamma} \varphi(e^{-i\sigma_k\gamma/2})f_\alpha(p) \right|_{\gamma=0}.
\]
Then their concrete forms are expressed as
\[
L_1 f_\pm = i \left[ \sin \phi \frac{\partial}{\partial \theta} + \cos \phi \frac{\partial}{\partial \phi} \right] f_\pm,
\]
\[
L_2 f_\pm = i \left[ -\cos \phi \frac{\partial}{\partial \theta} + \sin \phi \frac{\partial}{\partial \phi} \right] f_\pm,
\]
\[
L_3 f_\pm = i \left[ -\frac{\partial}{\partial \phi} \pm iq \right] f_\pm.
\]
They satisfy \([L_j, L_k] = i\epsilon_{jkl}L_\ell\) and exactly agree with the angular momentum operators given by Wu and Yang [18]. We make also the linear combinations
\[
L_+ f_\pm := (L_1 + iL_2) f_\pm
\]
\[
= e^{i\phi} \left[ \frac{\partial}{\partial \theta} + \frac{i}{\sin \theta} \left( \cos \theta \frac{\partial}{\partial \phi} + iq(1 \mp \cos \theta) \right) \right] f_\pm,
\]
\[
L_- f_\pm := (L_1 - iL_2) f_\pm
\]
\[
= e^{-i\phi} \left[ \frac{\partial}{\partial \theta} + \frac{i}{\sin \theta} \left( \cos \theta \frac{\partial}{\partial \phi} + iq(1 \mp \cos \theta) \right) \right] f_\pm.
\]
We should mention the discrete symmetry that our model possesses. We can show that although either parity \( P \) or charge conjugation \( C \) is not symmetry of the model, their combination \( CP \) is its symmetry.
Let us define the parity transformation of our model. Actually, parity transforms a model with the monopole number \( q \) to another model with \(-q\). Parity is inversion of \( S^2 \) as

\[
P : S^2 \to S^2; \quad (\theta, \phi) \mapsto (\pi - \theta, \pi + \phi).
\]

(2.32)

Parity changes appearance of the patching conditions (2.3) as

\[
A_-(\pi - \theta, \pi + \phi) = A_+(\pi - \theta, \pi + \phi) - 2d(\pi + \phi)
\]

\[
= A_+(\pi - \theta, \pi + \phi) - 2d\phi,
\]

(2.33)

\[
f_-(\pi - \theta, \pi + \phi) = e^{-2iq(\pi + \phi)} f_+(\pi - \theta, \pi + \phi)
\]

\[
= e^{-2iq\phi} e^{-2iq\pi} f_+(\pi - \theta, \pi + \phi),
\]

(2.34)

which can be rearranged as

\[
-A_+(\pi - \theta, \pi + \phi) = -A_-(\pi - \theta, \pi + \phi) - 2d\phi,
\]

(2.35)

\[
e^{-iq\pi} f_+(\pi - \theta, \pi + \phi) = e^{2iq\phi} e^{iq\pi} f_-(\pi - \theta, \pi + \phi).
\]

(2.36)

If we define the parity transformation of the fields as

\[
\varphi_P : A_\pm(\theta, \phi) \mapsto -A_{\mp}(\pi - \theta, \pi + \phi),
\]

(2.37)

\[
\varphi_P : f_\pm(\theta, \phi) \mapsto e^{\pm iq\pi} f_{\mp}(\pi - \theta, \pi + \phi),
\]

(2.38)

it changes the monopole number from \( q \) to \(-q\) as seen by comparing (2.3) with (2.36). Besides, the monopole background field is left invariant as

\[
A_\pm(\theta, \phi) \mapsto -(\mp 1 - \cos(\pi - \theta))d(\pi + \phi)
\]

\[
= (\pm 1 - \cos \theta)d\phi
\]

\[
= A_\pm(\theta, \phi).
\]

(2.39)

Charage conjugation is defined simply by complex conjugation

\[
\varphi_C : f_\pm \mapsto f^*_\pm.
\]

(2.40)

Then the patching condition (2.3) becomes

\[
f^*_- = e^{2iq\phi} f^*_+.\]

(2.41)

Therefore charge conjugation is not symmetry of the model, either. Instead it transforms a model with the monopole number \( q \) to another model with \(-q\).

Because the combined \( CP \) transformation turns the sign of the monopole number back to the original, \( CP \) is symmetry of the model. For later use we write down the \( CP \) transformation explicitly as

\[
\varphi_{CP} = \varphi_C \circ \varphi_P: f_\pm(\theta, \phi) \mapsto e^{\mp iq\pi} f^*_+(\pi - \theta, \pi + \phi).
\]

(2.42)

### 2.2 Rotational symmetry breaking

Assume that the monopole number \( q \) is not zero and that the fields \((f_+, f_-)\) are continuous functions. Then it is proved that if the field \( f \) is rotationally invariant, it must vanish identically as \( f \equiv 0 \) over \( S^2 \). The contraposition says that if \( f \) is not identically zero, \( f \).
cannot be rotationally invariant over \( S^2 \). Actually, even if \( f \) takes nonzero values in some region of \( S^2 \), the value of \( f(\theta, \phi) \) must vanish at some points in \( S^2 \). It can be shown that the number of the zero points of \( f \) is \( 2|q| \). Hence the zero points pin down the rotational symmetry. The zero points are called vortices \([22]\).

Here we describe the outline of the proof of the theorem about the number of zero points. The continuous function \( f_+ : U_+ \to C \) defines a family of loops \( \{ \ell_+^{\theta} : S^1 \to C \mid 0 \leq \theta \leq \pi/2 \} \) by \( \ell_+^{\theta}(\phi) := f_+(\theta, \phi) \). Similarly, the continuous function \( f_- : U_- \to C \) defines a family of loops \( \{ \ell_-^{\theta} : S^1 \to C \mid \pi/2 \leq \theta \leq \pi \} \) by \( \ell_-^{\theta}(\phi) := f_-(\theta, \phi) \). Since \( f_\pm \) are single-valued over \( U_\pm, \ell_+^{\theta} \) and \( \ell_-^{\theta} \) are shrunk loops. Suppose that \( f_+ \) does not vanish in the upper hemisphere, \( 0 \leq \theta \leq \pi/2 \). Then the loops \( \{ \ell_+^{\theta}(\phi) = f_+(\theta, \phi) \mid 0 \leq \theta \leq \pi/2 \} \) never touch nor cross the zero in the complex plane. Therefore, the loop \( \ell_+^{\pi/2} \) does not wind around the zero. Accordingly, \( \ell_+^{\pi/2}(\phi) = f_-(\pi/2, \phi) = e^{-2iq\phi}f_+(\pi/2, \phi) = e^{-2iq\phi}\ell_+^{\pi/2}(\phi) \) runs around the zero in the complex plane \( 2q \) times in the clockwise direction. In the limit \( \theta \to \pi \) the loop \( \ell_+^{\theta} \) continuously shrinks into a point in the complex plane. Hence the loop \( \ell_+^{\theta} \) crosses the zero \( 2|q| \) times during the change of \( \theta \) over \( \pi/2 \leq \theta \leq \pi \). Therefore \( f_- \) has \( 2|q| \) zero points. We may relax the assumption and allow \( f_+ \) to have zero points in \( U_+ \). We can similarly show that the total number of zero points of \( f_\pm \) is equal to \( 2|q| \). Then the theorem about the number of zero points is proved.

Let us turn to the theorem about rotational symmetry breaking. The statement that \( f \) is rotationally invariant means that both \( f_+ \) and \( f_- \) remain invariant under the transformations \([2.13]\) by \( SU(2) \). Assume that \( f_\pm \) are rotationally invariant. We have already shown that \( f_\pm \) vanish at some point in \( S^2 \) if \( q \neq 0 \). Hence \( f_\pm \) should vanish everywhere because the group \( SU(2) \) acts on \( S^2 \) transitively. Accordingly, the rotationally invariant \( f_\pm \) must be identically zero. The proof is over.

The last theorem tells that if the scalar field exhibits a nonzero vacuum expectation value \( \langle f \rangle \), the rotational symmetry is necessarily broken. Now we would like to examine a condition for rotational symmetry breaking. In this paper we analyze the model only at the classical level. Moreover, since the translational symmetry in \( R^6 \) is kept unbroken, what we need to find is the vacuum configuration \( \langle f(\theta, \phi) \rangle \) that minimizes the classical energy functional

\[
E = \int d\theta d\phi r^2 \sin \theta \left\{ \frac{1}{r^2} \left| \frac{\partial f_+}{\partial \theta} \right|^2 + \frac{1}{r^2 \sin^2 \theta} \left| \frac{\partial f_+}{\partial \phi} - iq(\pm 1 - \cos \theta) f_+ \right|^2 - \mu^2 f_+^* f_+ + \lambda (f_+^* f_+)^2 \right\}.
\]

(2.43)

The variation of the gradient energy with respect to \( f_+^* \) gives the Laplacian coupled to the monopole,

\[
-\Delta_q f_+ := - \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \left( \frac{\partial}{\partial \phi} - iq(\pm 1 - \cos \theta) \right) \right] f_+.
\]

(2.44)

The eigenvalue problem of the monopole Laplacian was solved by Wu and Yang \([18]\) and the solutions are summarized in a neat way by Coleman \([21]\); its eigenfunctions are expressed in terms of the matrix elements of unitary representations of \( SU(2), D_{mq}^{ij}(\theta, \phi, \psi) = \)

---

\(^\dagger\) To be precise, what can be shown is that the sum of indices of all the zero points is equal to \( 2q \). We can assign an integer index to each vortex by counting the winding number of the scalar field around the vortex with taking its sign into account. Then we can assert that the sum of the indices is \( 2q \). However, the number of vortices may appear more or less than \( 2|q| \) by pair creation of vortex and anti-vortex or by duplication of vortices.
\langle j, m | e^{-iJ_3 \phi} e^{-iJ_2 \theta} e^{-iJ_3 \psi} | j, q \rangle, \\
\text{as} \\
f_{\pm}(\theta, \phi) = D_{mq}^j(\theta, \phi, \mp \phi) = \langle j, m | e^{-iJ_3 \phi} e^{-iJ_2 \theta} e^{\pm iJ_3 \phi} | j, q \rangle = e^{-i(m \mp q) \phi} d_{mq}^j(\theta), \\
(2.45) \\
which belongs to the eigenvalue \\
\epsilon_j = j(j + 1) - q^2, \quad (j = |q|, |q| + 1, |q| + 2, \cdots). \\
(2.46) \\
Each eigenvalue is degenerated with respect to the index \( m \), which has a range \\
m = -j, -j + 1, \cdots, j - 1, j. 
Since the lowest eigenvalue of the Laplacian is \( \epsilon_{|q|} = |q| \), the lower bound of 
the energy is given by \\
E \geq (|q| - \mu^2 r^2) \int d\theta d\phi \sin \theta |f_{\pm}|^2 + \lambda r^2 \int d\theta d\phi \sin \theta |f_{\pm}|^4. \\
(2.47) \\
If \( |q| - \mu^2 r^2 > 0 \), the RHS of the inequality (2.47) is positive definite. Then the minimum 
of \( E \) is realized only by the trivial vacuum \( f \equiv 0 \). On the other hand, if \( |q| - \mu^2 r^2 < 0 \), it is possible to find a field \( f \) 
that makes the energy negative. For example, let us adopt the eigenfunction (2.45) of the lowest 
eigenvalue, \( f_{\pm}(\theta, \phi) = c e^{-i(m \mp q) \phi} d_{mq}^j(\theta) \), where \( c \) is a 
complex number. The energy of this field configuration can be written into the form \\
E = -a_2 |c|^2 + a_4 |c|^4 \\
(2.48) \\
with coefficients \( a_2, a_4 > 0 \). In particular, we have \( a_2 = 4\pi(2|q| + 1)^{-1}(\mu^2 r^2 - |q|) \). Thus, 
for \( 0 < |c|^2 < a_2/a_4 \), this field realizes \( E < 0 \). Therefore, the minimum value of \( E \) must be 
negative and it is realized by a nontrivial vacuum \( f \neq 0 \). Thus we conclude that the rotational symmetry is spontaneously broken when the radius \( r \) of \( S^2 \) is larger than the 
critical radius \( r_q \), i.e. \\
r > r_q := \frac{\sqrt{|q|}}{\mu}. \\
(2.49) \\
We may rephrase this condition in terminology of superconductivity physics [22]. In the 
context of Landau-Ginzburg theory, the field \( f \) is regarded as the order parameter and the 
energy (2.43) is regarded as the free energy. If the field \( f \) develops a nonzero condensation 
in the applied magnetic field, it creates vortices. The radius of the core of a vortex is 
characterized by the coherence length \( \xi = M_\sigma^{-1} = (\sqrt{2}\mu)^{-1} \). Here \( M_\sigma \) is the so-called Higgs 
mass. Then the condition for rotational symmetry breaking (2.49) can be expressed as \\
r > \sqrt{2|q|} \xi. \\
(2.50) \\
Namely, when the radius of the sphere becomes larger than that of the vortex core, the 
condensation occurs and the vortices begin to appear. On the other hand, when the radius 
is smaller, the magnetic field is too strong to admit the superconducting state, and hence 
the system remains the normal state.

In the present model the magnetic field is assumed to be a fixed background and is not 
allowed to change. If the reaction to the magnetic field is taken into account, the Meissner 
effect should be observed. We postpone study of the dynamics of the gauge field in the 
sphere to future work.
2.3 Vacuum configurations

We will now calculate the concrete vacuum configuration of the scalar field. We will obtain approximate solutions of the scalar field by a variational method for three cases with the monopole number \( q = 1/2, 1, 3/2 \). Let us explain briefly our method of calculation. A generic field which satisfies the patching condition (2.3) can be expanded in a series of the eigenfunctions (2.45) as

\[
f_{q}^{\pm} (\theta, \phi) = \sum_{j=|q|}^{\infty} \sum_{m=-j}^{j} c_{m}^{j} D_{m,q}^{j}(\theta, \phi, \mp \phi). \tag{2.51}
\]

Now we use the lowest approximation for it; we restrict the series to the leading terms with \( j = |q| \) as

\[
f_{q}^{\pm} (\theta, \phi) = \sum_{m=-|q|}^{|q|} c_{m} D_{m,q}^{|q|}(\theta, \phi, \mp \phi) \tag{2.52}
\]

and substitute it into the energy functional (2.43). Then the coefficients \( \{ c_{m} \} \) are adjusted to minimize the energy. The minimizer is the vacuum configuration. In the next subsection this lowest approximation will be justified.

2.3.1 \( q=1/2 \)

Let us begin a concrete calculation for the case of the smallest charge \( q = 1/2 \). In the expansion (2.52) we put \( c_{1/2} = -ve^{i\gamma/2} \sin(\beta/2) \) and \( c_{-1/2} = ve^{-i\gamma/2} \cos(\beta/2) \) with the real parameters \( (v, \beta, \gamma) \). Then we get

\[
f_{1/2}^{\pm} (\theta, \phi) = ve^{\pm i\phi/2} \left[ -e^{-i(\phi-\gamma)/2} \sin(\beta/2) \cos(\theta/2) + e^{i(\phi-\gamma)/2} \cos(\beta/2) \sin(\theta/2) \right]. \tag{2.53}
\]

Thus it can easily be seen that the value of \( f_{1/2}^{1/2} \) vanishes at the point \( (\theta, \phi) = (\beta, \gamma) \). Since the position of the zero point of \( f_{1/2}^{1/2} \) can be moved to the north pole of \( S^2 \) by an appropriate \( SU(2) \) rotation, we can set \( (\beta, \gamma) = (0, 0) \) without loss of generality. Then the vacuum field is given by

\[
\langle f_{1/2}^{1/2}(\theta, \phi) \rangle = ve^{(1\pm 1)\phi/2} \sin(\theta/2), \tag{2.54}
\]

and the single vortex is located at the north pole of \( S^2 \). The configurations of vortices for various \( q \) are shown in Fig.1. The energy (2.43) is then calculated as

\[
E = -2\pi \left( \mu^{2} r^{-2} - \frac{1}{2} \right) v^{2} + \frac{2}{3} \cdot 2\pi \lambda r^{2} v^{4}. \tag{2.55}
\]

Hence the energy is minimized by

\[
v^{2} = \begin{cases} 
0 & \text{for } r \leq \sqrt{1/2} \mu^{-1}, \\
\left( 1 - \frac{1}{2\mu^{2} r^{-2}} \right) \frac{3\mu^{2}}{4\lambda} & \text{for } r > \sqrt{1/2} \mu^{-1}.
\end{cases} \tag{2.56}
\]

This result is to be compared with the vacuum expectation value

\[
\langle f^{0} \rangle^{2} = \frac{\mu^{2}}{2\lambda}, \tag{2.57}
\]

for the case of \( q = 0 \).
It can be verified that the vacuum field configuration (2.54) is invariant under the combined transformation of the rotation (2.22) of the angle \( \gamma \) around the \( z \)-axis with the phase rotation (2.6) \( t = -\gamma e^{i\gamma/2} } } \). 

\[
e^{i\gamma/2} \varphi(\sigma_3 e^{-i\gamma/2}) : f_\pm^{1/2}(\theta, \phi) \mapsto e^{i\gamma/2} e^{\pm i\gamma/2} f_\pm^{1/2}(\theta, \phi - \gamma). \quad (2.58)
\]

This transformation \( e^{i\gamma/2} \varphi(\sigma_3 e^{-i\gamma/2}) \) is equal to \( e^{-i(L_3 - Q)\gamma} \). On the other hand, it is obvious that \( \langle f_\pm^{1/2} \rangle \) is not invariant under \( \text{CP} \). Therefore, we conclude that when \( q = 1/2 \) and \( r > (\sqrt{2}\mu^{-1})^{-1} \), the symmetry \( U(1) \times SU(2) \times CP \) is spontaneously broken to the subgroup \( U(1)' \) that is generated by \( L_3 - Q \). Accordingly there are three massless Nambu-Goldstone bosons, which couple to the generators \( \{ X_1, X_2, X_3 \} = \{ L_1, L_2, L_3 + Q \} \) of the broken symmetry. However, the Nambu-Goldstone boson coupling to the charge \( Q \) would be absorbed into the gauge boson via the Higgs mechanism if the gauge field has its own dynamical degrees of freedom. The scalar field \( f(x^0, x^1, \ldots, x^{n-1}, \theta, \phi) \) is expanded around the vacuum (2.54) as

\[
f(x^0, x^1, \ldots, x^{n-1}, \theta, \phi) = \langle f(\theta, \phi) \rangle - i \sum_j X_j \langle f(\theta, \phi) \rangle \pi_j(x^0, x^1, \ldots, x^{n-1}) \\
+ f_{\text{massive}}(x^0, x^1, \ldots, x^{n-1}, \theta, \phi). \quad (2.59)
\]

The three real fields \( \{ \pi_1, \pi_2, \pi_3 \} \) describe the Nambu-Goldstone bosons. We can calculate each mode by operating each generator, (2.29), (2.30), (2.31), on the vacuum (2.54) as

\[
L_+ \langle f_\pm^{1/2} \rangle = 0, \quad (2.60)
\]

\[
L_- \langle f_\pm^{1/2} \rangle = -ve^{i(\pm 1)\phi/2} \cos(\theta/2), \quad (2.61)
\]

\[
(L_3 + Q) \langle f_\pm^{1/2} \rangle = ve^{i(1\pm 1)\phi/2} \sin(\theta/2), \quad (2.62)
\]

\[
(L_3 - Q) \langle f_\pm^{1/2} \rangle = 0. \quad (2.63)
\]

Eq.(2.63) reflects the invariance \( e^{-i(L_3 - Q)\gamma} \langle f_\pm^{1/2} \rangle = \langle f_\pm^{1/2} \rangle \). Using Eq.(2.60) we rewrite the expansion (2.54) as

\[
f_\pm^{1/2}(x, \theta, \phi) = \langle f_\pm^{1/2}(\theta, \phi) \rangle - i/2 L_- \langle f_\pm^{1/2} \rangle (\pi_1 + i\pi_2)(x) - i(L_3 + Q) \langle f_\pm^{1/2} \rangle \pi_3(x) \\
+ f_{\text{massive}} \langle x, \theta, \phi \rangle. \quad (2.64)
\]
Besides, the massive modes are further expanded as

\[
(f_{\text{massive}}^{1/2})_\pm(x, \theta, \phi) = \frac{1}{2} L_-(f_{\pm}^{1/2}) (\sigma^1 + i\sigma^2)(x) + (L_3 + Q)(f_{\pm}^{1/2}) \sigma^3(x) + \sum_{j=3/2}^{\infty} \sum_{m=-j}^{j} D_{m,1/2}^j(\theta, \phi, \mp \phi) S_j^m(x)
\]

(2.65)

with real scalars \(\sigma^j(x)\) and complex scalars \(S_j^m(x)\).

We would like to put a remark about a relation of our argument with the Coleman theorem \[23\], which forbids spontaneous breaking of continuous symmetries in two dimensions. Our model is built on higher dimensions than two; it is in the direct product space \(\mathbb{R}^n \times S^2\) of the Minkowski space \(\mathbb{R}^n\) with the extra \(S^2\). Thus the Coleman theorem is not applicable to our model.

2.3.2 \(q=1\)

Next let us consider the case of \(q = 1\). Then a generic scalar field \(f\) has two vortices on \(S^2\). The lowest expansion (2.52) now becomes

\[
f_\pm^1(\theta, \phi) = \frac{1}{2} \left[ c_{+1} e^{-i \phi} (1 + \cos \theta) + c_0 \sqrt{2} \sin \theta + c_{-1} e^{i \phi} (1 - \cos \theta) \right] e^{\pm i \phi}
\]

\[
= \frac{1}{\sqrt{2}} \left[ c_x (i \sin \phi - \cos \theta \cos \phi) + c_y (-i \cos \phi - \cos \theta \sin \phi) + c_z \sin \theta \right] e^{\pm i \phi}.
\]

(2.66)

We have put \(c_{\pm 1} = \mp (c_x \pm ic_y) / \sqrt{2}\) and \(c_0 = c_z\). Under the \(SU(2)\) transformation (2.15), the vector \((c_x, c_y, c_z)\) obeys the triplet representation. By using the \(U(1) \times SU(2)\) transformations, (2.6) and (2.15), we can turn the coefficients into the form

\[
(c_x, c_y, c_z) = (0, iv \sin \alpha, v \cos \alpha),
\]

(2.67)

with the real parameters \(v\) and \(\alpha\) \((0 \leq \alpha \leq \pi/4)\). Then the field (2.66) becomes

\[
f_\pm^1(\theta, \phi) = \frac{1}{\sqrt{2}} v \left[ (\cos \alpha \sin \theta + \sin \alpha \cos \phi) - i \sin \alpha \cos \theta \sin \phi \right] e^{\pm i \phi}.
\]

(2.68)

Thus two zero points of \(f^1\) are located at \((\theta, \phi) = (\sin^{-1}(\tan \alpha), \pi)\). So, the relative displacement of two vortices is controlled by the parameter \(\alpha\).

Substituting the trial function (2.68) into (2.43), we evaluate the total energy as

\[
E = -\frac{4\pi}{3} (\mu^2 r^2 - 1) v^2 + \frac{8\pi}{15} \left\{ 1 + \frac{1}{4} (1 - \cos 4\alpha) \right\} \lambda r^2 v^4.
\]

(2.69)

The minimum of the potential is realized when \(\cos 4\alpha = 1\), i.e. \(\alpha = 0\). Then the two vortices are located at opposite two points on \(S^2\). On the other hand, the maximum of the potential is realized when \(\cos 4\alpha = -1\), i.e. \(\alpha = \pi/4\). Then the two vortices coincide. Thus we observe that the vortices repel each other. The minimum of the energy is realized by \(\alpha = 0\) with

\[
v^2 = \begin{cases} 
0 & \text{for } r \leq \mu^{-1}, \\
(1 - \frac{1}{\mu^2 r^2}) \frac{5\mu^2}{4\lambda} & \text{for } r > \mu^{-1}
\end{cases}
\]

(2.70)
The vacuum configuration \((2.68)\) then becomes
\[
\langle f_\pm^1(\theta, \phi) \rangle = \frac{1}{\sqrt{2}} v \sin \theta e^{\pm i\phi},
\]
and has two vortices at the north and the south poles of \(S^2\), respectively. This configuration \((2.74)\) is invariant under the rotations around the \(z\)-axis \((2.22)\), which is generated by \(L_3\). Under \(CP\) given in \((2.42)\), \(\langle f^1 \rangle\) is transformed as
\[
(\varphi_{CP}\langle f^1 \rangle)_\pm(\theta, \phi) = -\frac{1}{\sqrt{2}} v e^{\pm i(\pi + \phi)} \sin(\pi - \theta) = \frac{1}{\sqrt{2}} v e^{\pm i\phi} \sin \theta.
\]
Therefore, \(\langle f^1 \rangle\) is \(CP\) invariant. Thus, we conclude that when \(q = 1\) and \(r > \mu^{-1}\), two vortices appear and settle down at two diammetrical points on \(S^2\). The symmetry \(U(1) \times SU(2) \times CP\) is spontaneously broken to \(U(1)'' \times CP\), where \(U(1)''\) is generated by \(L_3\). In the broken phase three Nambu-Goldstone bosons, which couple to the generators \(\{Q, L_1, L_2\}\), appear. The field \(f\) can be expanded in a way similar to \((2.59)\) as
\[
f_\pm(x, \theta, \phi) = \langle f^1_\pm(\theta, \phi) \rangle - \frac{i}{2} L_+(f^1_\pm)(\pi^1 - i\pi^2)(x) - \frac{i}{2} L_-(f^1_\pm)(\pi^1 + i\pi^2)(x)
\]
\[-iQ(f^1_\pm)\pi^0(x) + (f^1_{\text{massive}})_\pm(x, \theta, \phi),
\]
where \(\{\pi^0, \pi^1, \pi^2\}\) are the Nambu-Goldstone fields. Each mode is calculated as
\[
Q\langle f^1_\pm \rangle = \frac{1}{\sqrt{2}} v e^{\pm i\phi} \sin \theta,
\]
\[
L_+\langle f^1_\pm \rangle = -\frac{1}{\sqrt{2}} ve^{i(1\pm)\phi}(1 - \cos \theta),
\]
\[
L_-\langle f^1_\pm \rangle = -\frac{1}{\sqrt{2}} ve^{i(-1\pm)\phi}(1 + \cos \theta),
\]
\[
L_3\langle f^1_\pm \rangle = 0.
\]
The last equation \((2.74)\) confirms the invariance \(e^{-iL_3\gamma}\langle f^1_\pm \rangle = \langle f^1_\pm \rangle\). However, the boson of \(\pi^0\), which couples to the charge \(Q\), would disappear via the Higgs mechanism, if the gauge field has its own dynamical degrees of freedom. Furthermore, the massive modes are given by
\[
(f^1_{\text{massive}})_\pm(x, \theta, \phi) = \frac{1}{2} L_+\langle f^1_\pm \rangle (\sigma^1 - i\sigma^2)(x) + \frac{1}{2} L_\mp\langle f^1_\pm \rangle (\sigma^1 + i\sigma^2)(x)
\]
\[+Q\langle f^1_\pm \rangle \sigma^0(x) + \sum_{j=2}^{\infty} \sum_{m=-j}^{j} D^j_{m,1}(\theta, \phi, \mp\phi)S^m_j(x)
\]
with real scalars \(\sigma^j(x)\) and complex scalars \(S^m_j(x)\).

### 2.3.3 \(q=3/2\)

Finally, let us examine the case of \(q = 3/2\). Then the number of vortices is three. The lowest expansion \((2.52)\) now becomes
\[
f^3/2_{\pm}(\theta, \phi) = \left[ c_{3/2} e^{-\pm 3i\phi/2} \cos^3(\theta/2) + c_{1/2} \sqrt{3} e^{-i\phi/2} \cos^2(\theta/2) \sin(\theta/2)
\right.
\[+ c_{-1/2} \sqrt{3} e^{i\phi/2} \cos(\theta/2) \sin^2(\theta/2) + c_{-3/2} e^{3i\phi/2} \sin^3(\theta/2) \]e^{\pm 3i\phi/2}.
\]
By substituting it into (2.43), the energy functional is evaluated as

\[
E = -\pi \left( \mu^2 r^2 - \frac{3}{2} \right) \left( |c_{3/2}|^2 + |c_{1/2}|^2 + |c_{-1/2}|^2 + |c_{-3/2}|^2 \right) \\
+ \frac{4}{35} \pi r^2 \lambda \left( 5|c_{3/2}|^4 + 3|c_{1/2}|^4 + 3|c_{-1/2}|^4 + 5|c_{-3/2}|^4 \\
+ 10|c_{3/2}|^2|c_{1/2}|^2 + 4|c_{3/2}|^2|c_{-1/2}|^2 + |c_{3/2}|^2|c_{-3/2}|^2 \\
+ 9|c_{1/2}|^2|c_{-1/2}|^2 + 4|c_{1/2}|^2|c_{-3/2}|^2 + 10|c_{-1/2}|^2|c_{-3/2}|^2 \\
+ 3c_{1/2}c_{-1/2}c_{3/2}c_{-3/2} + 3c_{1/2}^*c_{-1/2}c_{3/2}^*c_{-3/2} \\
+ 2\sqrt{3} c_{1/2} c_{3/2} c_{1/2}^* + 2\sqrt{3} c_{1/2} c_{3/2} c_{1/2}^* \\
+ 2\sqrt{3} c_{1/2} c_{3/2} c_{1/2}^* + 2\sqrt{3} c_{1/2} c_{3/2} c_{1/2}^* \right).
\]

(2.80)

The minimum of \( E \) is then found at \((c_{3/2}, c_{1/2}, c_{-1/2}, c_{-3/2}) = (-v, 0, 0, v)\), which is unique up to the \(U(1) \times SU(2)\) symmetry. Thus the vacuum is

\[
\langle J_{\pm}^{3/2}(\theta, \phi) \rangle = v \begin{cases} 
-e^{-3i\phi/2} \cos^3(\theta/2) + e^{3i\phi/2} \sin^3(\theta/2) & e^{\pm 3i\phi/2} \\
e^{-i\phi/2} \cos(\theta/2) - e^{i\phi/2} \sin(\theta/2) & e^{\pm i\phi/2}
\end{cases}
\]

(2.81)

with

\[
v^2 = \begin{cases} 
0 & \text{for } r \leq \sqrt{3/2} \mu^{-1}, \\
\left( 1 - \frac{3}{2\mu^2 r^2} \right) \frac{35\mu^2}{44\lambda} & \text{for } r > \sqrt{3/2} \mu^{-1}.
\end{cases}
\]

(2.82)

We can read off the location of zero points from (2.81); they are located at \(\phi = 0, 2\pi/3, 4\pi/3\) on \(\theta = \pi/2\). Namely, the vortices are located at the vertices of the largest equilateral triangle on the equator as shown in Fig.1(c). Then the vacuum energy is evaluated as

\[
E_{\text{vac}} = -\left( \mu^2 r^2 - \frac{3}{2} \right) \frac{35\pi}{44\lambda v^2}.
\]

(2.83)

For comparison let us calculate the energy of the configuration in that the three vortices coincide at the north pole of \(S^2\). Such a configuration is specified by \((c_{3/2}, c_{1/2}, c_{-1/2}, c_{-3/2}) = (0, 0, 0, v')\). Then the energy (2.80) is

\[
E = -\pi \left( \mu^2 r^2 - \frac{3}{2} \right) v'^2 + \frac{20}{35} \pi r^2 \lambda v'^4.
\]

(2.84)

The minimal of \( E \) is realized by

\[
v'^2 = \left( 1 - \frac{3}{2\mu^2 r^2} \right) \frac{7\mu^2}{8\lambda} \quad \text{for } r > \sqrt{3/2} \mu^{-1}
\]

(2.85)

and the minimal value is

\[
E_{\text{coincide}} = -\frac{11}{20} \cdot \left( \mu^2 r^2 - \frac{3}{2} \right)^2 \frac{35\pi}{44\lambda r^2}.
\]

(2.86)

Hence we can see that \( E_{\text{coincide}} > E_{\text{vac}} \). Therefore the vortex whose topological number is three is unstable and decays into three separated vortices. This result corresponds to Type II superconductor, in which vortices repel each other and form the Abrikosov lattice [24].
Jacobs and Rebbi \cite{24} gave a detailed analysis of interaction of vortices of the Abelian Higgs model in $R^2$.

Now we describe symmetry of the vacuum. The rotation around the $z$-axis (2.22) of the angle $\gamma = 2\pi/3$ followed by the $U(1)$ rotation (2.4) of the phase $t = 2\pi/3$, transforms the scalar field as

$$\varphi_R := e^{-i\pi/2} \varphi(e^{-i\sigma_3\pi/3}) : f^{3/2}_{\pm}(\theta, \phi) \mapsto f^{3/2}_{\pm}(\theta, \phi - 2\pi/3). \quad (2.87)$$

This composite transformation $\varphi_R$ generates the cyclic group $\mathbb{Z}_3$. Actually, the configuration (2.81) remains invariant under the operation of $\varphi_R$. On the other hand, the $\pi$-rotation around the $x$-axis (2.25) followed by the $U(1)$ rotation (2.4) of the phase $t = \pi/3$ transforms the scalar field as

$$\varphi_T := e^{-i\pi/2} \varphi(e^{-i\sigma_1\pi/2}) : f^{3/2}_{\pm}(\theta, \phi) \mapsto e^{i\pi} f^{3/2}_{\pm}(\pi - \theta, -\phi). \quad (2.88)$$

This composite transformation $\varphi_T$ generates another cyclic group $\mathbb{Z}_2$. It can easily be verified that the configuration (2.81) remains invariant also under the operation of $\varphi_T$. Note that the two operations $\varphi_R$ and $\varphi_T$ do not commute each other; they generate a nonabelian group $D_3$, which is called the dihedral group of the order three, i.e. the symmetry group of a regular triangle.

Moreover, the vacuum for $q = 3/2$ has another discrete symmetry that originates from $CP$. The vacuum configuration $\langle f^{3/2} \rangle$ is transformed by $CP$ as

$$\langle \varphi_{CP}(f^{3/2}) \rangle_{\pm}(\theta, \phi) = i \mu [e^{-3i\phi/2} \cos^3(\theta/2) + e^{3i\phi/2} \sin^3(\theta/2)] e^{\pm 3i\phi/2}. \quad (2.89)$$

Hence $\langle f^{3/2} \rangle$ is not invariant under $CP$. To turn it into the original form we apply the rotation around the $z$-axis (2.22) with $\gamma = \pi$ and the phase rotation (2.6) with $t = 2\pi/3$ to get

$$e^{-i\pi/2} \varphi(e^{-i\sigma_3\pi/2}) : f^{3/2}_{\pm}(\theta, \phi) \mapsto -e^{3i\pi/2} f^{3/2}_{\pm}(\theta, \phi - \pi). \quad (2.90)$$

Then we can verify the invariance

$$\langle e^{-i\pi/2} \varphi(e^{-i\sigma_3\pi/2}) \circ \varphi_{CP}(f^{3/2}) \rangle_{\pm}(\theta, \phi) = \langle f^{3/2} \rangle(\theta, \phi). \quad (2.91)$$

Note that the parity followed by the $\pi$-rotation around the $z$-axis is equivalent to the reflection by a mirror perpendicular to the $z$-axis:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \xrightarrow{P} \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix} e^{-i\sigma_3\pi/2} \xrightarrow{P} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \quad (2.92)$$

Thus the symmetry of the vacuum of $q = 3/2$ includes the reflection symmetry.

We thus conclude that when $q = 3/2$ and $r > \sqrt{3/2} \mu^{-1}$, three vortices appear and settle down at three points separated furthest each other. The symmetry $U(1) \times SU(2) \times CP$ is spontaneously broken to the discrete nonabelian group $D_{3h}$. Here $D_{3h}$ denotes the dihedral group of the order three with the horizontal reflection. In plain words, $D_{3h}$ is the symmetry group of a regular triangle without orientation. Hence there are four Nambu-Goldstone bosons, which couple to the generators $\{Q, L_1, L_2, L_3\}$ of the broken symmetry. The field $f$ can be expanded in a way similar to (2.53) as

$$f^{3/2}_{\pm}(x, \theta, \phi) = \langle f^{3/2}_{\pm}(\theta, \phi) \rangle - i/2 L_+ (f^{3/2}_{\pm}(\pi^1 - i\pi^2)(x) - i/2 L_-(f^{3/2}_{\pm}(\pi^1 + i\pi^2)(x)
- i/2 (L_3 - Q) (f^{3/2}_{\pm}(\pi^3 - \pi^0)(x) - i/2 (L_3 + Q) (f^{3/2}_{\pm}(\pi^3 + \pi^0)(x)
+ (f^{3/2}_{\text{massive}})_{\pm}(x, \theta, \phi), \quad (2.93)$$
where \( \{\pi^0, \pi^1, \pi^2, \pi^3\} \) are massless fields associated with the modes

\[
(L_3 - Q)\langle f^{3/2}_\pm\rangle &= 3v e^{-3i\phi/2}e^{\pm3i\phi/2}\cos^3(\theta/2), \quad (2.94) \\
L_+\langle f^{3/2}_\pm\rangle &= 3v e^{-i\phi/2}e^{\pm3i\phi/2}\cos^2(\theta/2)\sin(\theta/2), \quad (2.95) \\
L_-\langle f^{3/2}_\pm\rangle &= -3v e^{i\phi/2}e^{\pm3i\phi/2}\cos(\theta/2)\sin^2(\theta/2), \quad (2.96) \\
(L_3 + Q)\langle f^{3/2}_\pm\rangle &= 3v e^{3i\phi/2}e^{\pm3i\phi/2}\sin^3(\theta/2). \quad (2.97)
\]

Actually, the boson of \( \pi^0 \), which couples to \( Q \), would be absorbed into the gauge boson via the Higgs mechanism.

Our results are summarized as follows: let \( G \) be the original symmetry of the model and \( H_q \) be the symmetry of the vacuum for the monopole number \( q \). Then we have observed the pattern of symmetry breaking

\[
G = U(1) \times SU(2) \times CP \rightarrow \begin{cases} 
H_{1/2} = U(1)' & \text{for } r > \sqrt{1/2} \mu^{-1} \\
H_1 = U(1)' \times CP & \text{for } r > \mu^{-1} \\
H_{3/2} = D_{3h} & \text{for } r > \sqrt{3/2} \mu^{-1}
\end{cases} \quad (2.98)
\]

where \( U(1)' = \{e^{-i(L_3-Q)\gamma}\} \) and \( U(1)' = \{e^{-iL_3\gamma}\} \).

Before closing this subsection, we should make a comment on \( H_q \) for \( q > 3/2 \). Although it is difficult to continue to find vacuum configurations for \( q > 3/2 \) even in the lowest approximation \( (2.52) \), it seems reasonable to speculate that the unbroken symmetry \( H_q \) is discrete for \( q \geq 3/2 \). This is expected from the observations that \( 2q \) vortices appear for \( r > r_q \) and that the potential between vortices is repulsive. Then, three separated vortices are enough to break the continuous symmetry \( G \) to a discrete symmetry.

### 2.4 Stability of the vacuum

In the previous subsection we solved the problem to minimize the energy functional \( (2.43) \) by the variational method within the restricted function space \( (2.52) \), which is the eigenspace belonging to the lowest eigenvalue of the monopole Laplacian \( (2.44) \). Here we would like to clarify validity of our analysis.

The first point to be declared is that the critical radius \( (2.49) \) is exact in the context of the classical field theory. We did not recourse any approximation in the evaluation of the critical radius.

The second point to be examined is accuracy of the approximate solutions of the vacuum, like \( (2.54), (2.71), (2.81) \). We calculated them by the variational method with restricting the full function space \( (2.51) \) to the lowest eigenspace \( (2.52) \) of the monopole Laplacian. This restriction is a good approximation if the functions belonging to the lowest eigenvalue \( \epsilon_{|q|} \) give negative contribution to the quadratic term in \( (2.43) \), whereas those belonging to the next eigenvalue \( \epsilon_{|q|+1} \) give positive contribution to it. More explicitly, referring to the eigenvalues \( (2.49) \), the necessary condition for the validity is

\[
\epsilon_{|q|} - \mu^2 q^2 < 0 < \epsilon_{|q|+1} - \mu^2 q^2, \quad \text{or} \quad \frac{\sqrt{|q|}}{\mu} < r < \frac{\sqrt{3|q|+2}}{\mu}. \quad (2.99)
\]

In other words, the restricted variational method is valid when the radius of \( S^2 \) is larger than the critical radius but not too large.

The third point to be questioned is the stability of the vacuum against perturbations by higher eigenvalue functions. The approximation is improved by including higher-order
terms in the expansion (2.51). Now a question arises: if we include some of or all of higher terms in the expansion (2.51), does the better-approximated or the precise vacuum have the same symmetry as the lowest-approximated vacuum has?

The answer is affirmative: when higher-order terms are included in the trial function (2.51), but if the radius of $S^2$ is in the range (2.99), the vacuum calculated by the higher expansion has the same symmetry as the vacuum calculated by the lowest approximation has.

The above statement is proved as follows: first, note that the space of the scalar fields $f$ provides a unitary representation of $U(1) \times SU(2)$ by obeying the transformations (2.4) and (2.13). Let $f^{(0)}$ be the solution obtained by the lowest approximation like (2.54), (2.71), (2.81). Assume that $f$ is a solution obtained by the higher-order approximation. Then define $f^{(1)}$ as a correction to $f^{(0)}$, i.e.

$$f = f^{(0)} + f^{(1)}.$$  \hfill (2.100)

Let $H$ be the subgroup of $U(1) \times SU(2)$ that preserves $f^{(0)}$ invariant. Then we decompose $f^{(1)}$ into a component $(f^{(1)})_{||}$ that is in the identity representation of $H$, and its orthogonal complement $(f^{(1)})_{\perp}$ as

$$f^{(1)} = (f^{(1)})_{||} + (f^{(1)})_{\perp}. \hfill (2.101)$$

When (2.100) is substituted, the energy functional (2.43) is symbolically written as

$$E[f^{(0)} + f^{(1)}] = E[f^{(0)}] + \frac{\delta E}{\delta f}[f^{(0)}] \cdot f^{(1)} + \frac{1}{2} \frac{\delta^2 E}{\delta f^2}[f^{(0)}] \cdot (f^{(1)})^2 + \cdots, \hfill (2.102)$$

which is regarded as a polynomial with respect to $f^{(1)}$. The second term of the RHS is

$$\frac{\delta E}{\delta f}[f^{(0)}] \cdot f^{(1)} = \int d\theta d\phi \sin \theta \left\{ f^{(0)*}(-\Delta_q - \mu^2 r^2 + 2\lambda r^2 |f^{(0)}|^2)f^{(1)} + f^{(1)*}(-\Delta_q - \mu^2 r^2 + 2\lambda r^2 |f^{(0)}|^2)f^{(0)} \right\}. \hfill (2.103)$$

Since $f^{(0)}$ is invariant under the actions of $H$, $(-\Delta_q - \mu^2 r^2 + 2\lambda r^2 |f^{(0)}|^2)f^{(0)}$ is also invariant. Therefore, the term linear in the orthogonal component $(f^{(1)})_{\perp}$ vanishes as

$$\frac{\delta E}{\delta f}[f^{(0)}] \cdot (f^{(1)})_{\perp} = 0. \hfill (2.104)$$

Moreover, the third term of the RHS of (2.102) is

$$\frac{1}{2} \frac{\delta^2 E}{\delta f^2}[f^{(0)}] \cdot (f^{(1)})^2 = \int d\theta d\phi \sin \theta \left\{ f^{(1)*}(-\Delta_q - \mu^2 r^2)f^{(1)} + \lambda r^2 \left\{ 2|f^{(0)}|^2|f^{(1)}|^2 + (f^{(0)*}f^{(1)} + f^{(0)}f^{(1)*})^2 \right\} \right\}. \hfill (2.105)$$

If the radius of $S^2$ is in the range (2.99), the first term of $f^{(1)*}(-\Delta_q - \mu^2 r^2)f^{(1)}$ in the RHS is positive definite for $(f^{(1)})_{\perp}$ since it is orthogonal to the space of the lowest-eigenvalue functions. It is clear that the second term $\lambda r^2 \{ \cdots \}$ is also positive definite. Thus we conclude that the quadratic form (2.105) is positive for any $(f^{(1)})_{\perp}$, and deduce that the lowest-approximated vacuum $f^{(0)}$ is stable against the symmetry-breaking perturbation by $(f^{(1)})_{\perp}$. Hence in the vacuum $f$ the orthogonal component vanishes, i.e. $(f^{(1)})_{\perp} = 0$. This implies that the perturbed vacuum $f = f^{(0)} + (f^{(1)})_{||}$ remains invariant under the actions of $H$. The proof is over.
3 Doublet models

3.1 Embedding the doublet model in the SU(2)

Parity and charge conjugation change the sign of monopole charge as seen above. This fact suggests that a model constructed from a pair of matter fields with opposite charges can be invariant under both $P$ and $C$. Moreover, in the doublet matter model the gauge field of the Dirac monopole can be embedded into an $SU(2)$ gauge field, which is free from the Dirac singularity. Hence topology of the gauge field becomes trivial and the previous theorem that guarantees breaking of the rotational symmetry does not apply to the doublet model. So there is a possibility to restore the rotational symmetry in the doublet model. In the following we will examine structure of the doublet model in detail.

Let us consider a doublet of scalar fields $(f^q, f^{-q})$. Actually they have four components $(f^q_+, f^q_-, f^{-q}_+, f^{-q}_-)$, where $f^q_\pm$ is a smooth function defined in each domain $U_\pm$. They are related by the patching condition (2.3) as

$$f^{-q}_+ = e^{-2iq\sigma_3\phi} f^q_+, \quad f^{-q}_- = e^{2iq\phi} f^q_-,$$

or equivalently, by

$$\begin{pmatrix} f^q_+ \\ f^{-q}_- \end{pmatrix} = e^{-2iq_3\phi} \begin{pmatrix} f^q_+ \\ f^q_- \end{pmatrix} \quad \text{in } U_+ \cap U_-.
$$

We take the same gauge field (2.1) to define the covariant derivative

$$D \begin{pmatrix} f^q_+ \\ f^{-q}_- \end{pmatrix} = d \begin{pmatrix} f^q_+ \\ f^{-q}_- \end{pmatrix} - i q A_\pm \sigma_3 \begin{pmatrix} f^q_+ \\ f^{-q}_- \end{pmatrix}.
$$

Let us introduce two maps $\tau_\pm^q : U_\pm \to SU(2)$ by

$$\tau_\pm^q(\theta, \phi) := e^{i\sigma_1\pi/2} e^{i\sigma_3\phi} e^{i\sigma_2\theta/2} e^{\mp iq\sigma_3\phi}.
$$

Then we have

$$(\tau^q_\pm)^{-1} \cdot \tau^q_+= e^{-2iq\sigma_3\phi}
$$

and rewrite (3.2) as

$$\begin{pmatrix} f^q_+ \\ f^{-q}_- \end{pmatrix} = (\tau^q_\pm)^{-1} \cdot \tau^q_+ \begin{pmatrix} f^q_+ \\ f^{-q}_- \end{pmatrix}.
$$

Thus

$$\tilde{F} := \tau^q_+ \begin{pmatrix} f^q_+ \\ f^{-q}_- \end{pmatrix} = \tau^q_+ \begin{pmatrix} f^q_+ \\ f^{-q}_- \end{pmatrix}
$$

becomes a function that is well-defined and smooth over the whole $S^2 = U_+ \cup U_-$. By substituting

$$\begin{pmatrix} f^q_+ \\ f^{-q}_- \end{pmatrix} = (\tau^q_\pm)^{-1} \tilde{F}
$$

into (3.3) we get

$$D(\tau^q_\pm)^{-1} \tilde{F} = (\tau^q_\pm)^{-1} \left( d + (\tau^q_\pm)d(\tau^q_\pm)^{-1} - iq A_\pm \tau^q_\pm \sigma_3 (\tau^q_\pm)^{-1} \right) \tilde{F}
$$

Then the covariant derivative of the doublet $\tilde{F}$ is written as

$$D\tilde{F} = (d - iq\tilde{A})\tilde{F}$$
with the $SU(2)$ gauge field $\tilde{A}$ that is defined by

$$
-iq\tilde{A} := \tau_+^q d(\tau_+^q)^{-1} - iqA_+ \tau_+^q \sigma_3 (\tau_+^q)^{-1}
= \frac{1}{2} i\sigma_1 (-\sin 2q\phi d\theta - 2q\cos \theta \cos 2q\phi \sin \theta d\phi)
+ \frac{1}{2} i\sigma_2 (\cos 2q\phi d\theta - 2q\cos \theta \sin 2q\phi \sin \theta d\phi) + \frac{1}{2} i\sigma_3 2q \sin^2 \theta d\phi.
$$

(3.11)

This result is significant; $\tilde{A}$ is well-defined over the whole $S^2 = U_+ \cup U_-$ and is free from the Dirac singularity. If we introduce the orthogonal frame

$$
e^q_r := \begin{pmatrix} \sin \theta \cos 2q\phi \\ \sin \theta \sin 2q\phi \\ \cos \theta \end{pmatrix}, \quad e^q_\theta := \begin{pmatrix} \cos \theta \cos 2q\phi \\ \cos \theta \sin 2q\phi \\ -\sin \theta \end{pmatrix}, \quad e^q_\phi := \begin{pmatrix} -\sin 2q\phi \\ \cos 2q\phi \\ 0 \end{pmatrix},
$$

(3.12)

the resulted gauge field (3.11) is rewritten as

$$
\tilde{A} = -\frac{1}{2q} \sigma \cdot (e^q_\phi d\theta - 2q e^q_\theta \sin \theta d\phi).
$$

(3.13)

Thus the doublet model is reconstructed from the globally single-valued fields, $\tilde{F}$ and $\tilde{A}$. This result manifests the mathematical fact that the direct sum of the vector bundles of the monopole numbers $\pm q$ becomes a trivial bundle. In particular, when $q = 1/2$, the gauge field (3.13) can be simplified as

$$
A_{\text{Hosotani}} = -\frac{1}{r^2} \sigma \cdot (x \times dx),
$$

(3.14)

where $x = (x_1, x_2, x_3)$ denotes the Cartesian coordinates. This expression coincides with the monopole field that has been introduced by Hosotani [5]. If we define a covering map of the sphere by

$$
\pi_{2q} : S^2 \to S^2, \quad (\theta, \phi) \mapsto (\theta, 2q\phi),
$$

(3.15)

then using the pullback by $\pi_{2q}$ we can see that

$$
2q\tilde{A} = \pi_{2q}^* A_{\text{Hosotani}}.
$$

(3.16)

The field strength accompanying with $\tilde{A}$ is calculated to be

$$
\tilde{B} = d\tilde{A} - iq\tilde{A} \wedge \tilde{A} = -\sigma \cdot e^q_\theta \sin \theta d\theta \wedge d\phi.
$$

(3.17)

### 3.2 Symmetries of the doublet model

A model is defined by specifying the action or the energy functional. We take the energy functional

$$
E = \int d\theta d\phi r^2 \sin \theta \left\{ |Df|^2 + |Df^{-q}|^2 - \mu^2 (|f|^2 + |f^{-q}|^2) + \lambda (|f|^2 + |f^{-q}|^2)^2 \right\}
$$

(3.18)

for the doublet model. Let us examine the symmetries of this model.

First, this model is invariant under the global $U(1)$ transformation

$$
\begin{pmatrix} f_+^q \\ f_-^q \end{pmatrix} \mapsto \begin{pmatrix} e^{-iqt} f_+^q \\ e^{iqt} f_-^q \end{pmatrix}.
$$

(3.19)
Second, it is invariant also under the \( SU(2) \) rotations of \( S^2 \), which are given by (2.15).

Third, combined with the exchange \( f^q \leftrightarrow f^{-q} \), parity and charge conjugation become

\[
\psi_P : \begin{pmatrix} f^q \quad f^{-q} \\ f_\pm^- \quad f_\pm^+ \end{pmatrix} (\theta, \phi) \mapsto \begin{pmatrix} (1 \pm 1)q f_\pm^- \quad (1 \pm 1)q f_\pm^+ \end{pmatrix} (\pi - \theta, \pi + \phi), \tag{3.20}
\]

\[
\psi_C : \begin{pmatrix} f^q \quad f^{-q} \\ f_\pm^- \quad f_\pm^+ \end{pmatrix} (\theta, \phi) \mapsto \begin{pmatrix} (f_\pm^q)^* \quad (f_\pm^-)^* \end{pmatrix} (\theta, \phi). \tag{3.21}
\]

The multiplication by phase \((-1)^q\) is included in \( \psi_P \) to ensure that \((\psi_P)^2 = \text{identity}\). Then the energy functional (3.18) is invariant under both \( P \) and \( C \). For later reference, we write down the \( CP \) transformation;

\[
\psi_{CP} = \psi_C \circ \psi_P : \begin{pmatrix} f^q \quad f^{-q} \\ f_\pm^- \quad f_\pm^+ \end{pmatrix} (\theta, \phi) \mapsto \begin{pmatrix} (1 \pm 1)q(f_\pm^q)^* \quad (1 \pm 1)q(f_\pm^-)^* \end{pmatrix} (\pi - \theta, \pi + \phi). \tag{3.22}
\]

Fourth, our model has another \( SU(2) \) symmetry accompanying with the doublet structure. If we regard the doublet

\[
\begin{pmatrix} f^q \\ (f^{-q})^* \end{pmatrix} \tag{3.23}
\]

as independent variables, the both fields have the same charge \( q \), and therefore the doublet admits the \( SU(2) \) transformations, under which the energy functional (3.18) is left invariant. This symmetry is denoted by \( SU(2)_f \) to remind us of the flavor symmetry.

It should be noted that the \( \psi_C \) transformation is not an independent symmetry as seen in the following: when it acts on the doublet (3.23), (3.21) is equivalently written as

\[
\psi_C : \begin{pmatrix} f^q \\ (f_-^q)^* \end{pmatrix} (\theta, \phi) \mapsto \begin{pmatrix} (f^-_q)^* \\ f^q_\pm \end{pmatrix} (\theta, \phi). \tag{3.24}
\]

But this is nothing but the transformation

\[
\psi_C = \sigma_1 = i e^{-i\pi/2} = e^{i\pi/2} e^{-i\pi/2}, \tag{3.25}
\]

which is an element of \( U(1) \times SU(2)_f \). Thus we conclude that the symmetries of the doublet model (3.18) are

\[
G_{\text{doublet}} = U(1) \times SU(2)_f \times SU(2)_f. \tag{3.26}
\]

### 3.3 \( q=1/2 \) doublet

For the doublet model of \( q = 1/2 \) we can find an exact vacuum configuration, which realizes the lowest energy of the functional (3.18). The exact vacuum is given by

\[
\begin{pmatrix} f^1_\pm \\ (f_-^{1/2})^* \end{pmatrix} (\theta, \phi) = v \begin{pmatrix} D^{1/2}_1 \quad D^{1/2}_{-1} \\ D^{1/2}_{1} \quad D^{1/2}_{-1} \end{pmatrix} (\theta, \phi, \mp \phi) = v e^{\mp i\phi/2} \begin{pmatrix} e^{-i\phi/2} \cos(\theta/2) \\ e^{i\phi/2} \sin(\theta/2) \end{pmatrix}. \tag{3.27}
\]

Since these are the eigenfunctions for the lowest eigenvalue of the monopole Laplacian, the configuration (3.27) realizes the lowest value of the kinetic energy term in (3.18). Moreover, since \(|f^{1/2}|^2 + |f^{-1/2}|^2 = v^2\) is constant, (3.27) can realize the lowest value of the potential energy term in (3.18). Thus (3.27) is the exactly lowest energy state. Then the total energy (3.18) is evaluated as

\[
E = 4\pi \left\{ \frac{1}{2} - \mu^2 r^2 \right\} v^2 + \lambda r^4 v^4 \tag{3.28}
\]
which is minimized by
\[ v^2 = \begin{cases} 
0 & \text{for } r \leq \sqrt{1/2} \mu^{-1}, \\
\left(1 - \frac{1}{2\mu^2 r^2}\right)\mu^2 \frac{2\lambda}{2} & \text{for } r > \sqrt{1/2} \mu^{-1}.
\end{cases} \tag{3.29} \]

By applying the embedding map (3.7) we can rewrite the doublet solution (3.27) in a global form as
\[ \tilde{F} = \tau^{1/2} \left( \begin{array}{c} f_1^{1/2} \\
f_{-1/2}^{1/2} \end{array} \right) \]
\[ = v e^{i\sigma_1 \pi/2} e^{i\sigma_3 \phi/2} e^{i\sigma_2 \theta/2} e^{i\sigma_3 \phi/2} \left( e^{\pm i\phi/2} e^{-i\phi/2} \cos(\theta/2) \right) \]
\[ \left( e^{\mp i\phi/2} e^{-i\phi/2} \sin(\theta/2) \right) \]
\[ = v \left( \begin{array}{c} 0 \\
i \end{array} \right). \tag{3.30} \]

Thus the field \( \tilde{F} \) is reduced to a constant function and has no vortices in \( S^2 \). Hence it is naturally expected that the vacuum is invariant under the rotations.

Let us examine the preserved symmetry of the vacuum (3.27). For \( q = 1/2 \), parity (3.20) becomes
\[ \psi_P : \left( \begin{array}{c} f_1^{1/2} \\
f_{-1/2}^{1/2} \end{array} \right) \mapsto \left( \begin{array}{c} \pm f_{-1/2}^{-1/2} \\
\mp f_{1/2}^{-1/2} \end{array} \right) (\pi - \theta, \pi + \phi), \tag{3.31} \]
and it can be easily seen that the vacuum (3.27) is invariant under \( \psi_P \). Apparently, the vacuum is not invariant under either the \( SU(2) \) rotations or the \( SU(2)_f \) flavor transformations. However, it can be shown that the vacuum is actually invariant under actions of the diagonal elements of \( SU(2) \times SU(2)_f \), which are expressed as
\[ \psi_g : \left( \begin{array}{c} f_\alpha^{1/2} \\
f_{-\alpha}^{-1/2} \end{array} \right) (p) \mapsto e^{i\omega_{\alpha\beta}(g;p)/2} g \left( \begin{array}{c} f_\alpha^{1/2} \\
f_{-\alpha}^{-1/2} \end{array} \right) (g^{-1}p), \quad g \in SU(2). \tag{3.32} \]

We call this symmetry the flavored rotational symmetry because it transforms the flavor doublet and simultaneously rotates the point \( p \) in the sphere. We will give the proof of the invariance later in a more general context. These results are summarized as follows: the vacuum of the \( q = 1/2 \) doublet model has the symmetry
\[ H_{1/2 \text{ doublet}} = SU(2)' \times P. \tag{3.33} \]

It is to be noted that the \( C \) symmetry (3.21) is broken spontaneously in this model.

## 4 Multiplet models

In this section we discuss general aspects of models that have several matter fields in the monopole background field.

### 4.1 Disentanglement of the Dirac singularity

Remember that the domains \( U_\pm \) of the sphere \( S^2 \) are defined as \( U_\pm = \{ (\theta, \phi) \in S^2 | \cos \theta \neq \pm 1 \} \). Generally we can construct multiplets of scalar fields
\[ F_\pm = \left( \begin{array}{c} f_\pm^{q_1} \\
f_\pm^{q_2} \\
\vdots \\
f_\pm^{q_n} \end{array} \right), \tag{4.1} \]
where \( \{2q_i\}_{i=1,\ldots,n} \) are integers. Each multiplet \( F_\pm \) is a single-valued continuous function in each domain \( U_\pm \) and they are patched by the gauge transformation

\[
F_- = T(\phi)F_+ = \begin{pmatrix}
  e^{-2iq_1\phi} & 0 & \cdots & 0 \\
  0 & e^{-2iq_2\phi} & 0 & \\
  \vdots & \ddots & \ddots & \vdots \\
  0 & 0 & \cdots & e^{-2iq_n\phi}
\end{pmatrix} F_+ \quad (4.2)
\]

in \( U_+ \cap U_- \). If we put the diagonal matrix \( Q = \text{diag}(q_1, q_2, \cdots, q_n) \), we may write \( T(\phi) = e^{-2iQ\phi}. \) Then the covariant derivative of \( F_\pm \) is defined by

\[
DF_\pm = (d - iQA_\pm)F_\pm \quad (4.3)
\]

with the monopole gauge field \( A_\pm = (\pm 1 - \cos \theta)d\phi \). We may replace \( f^q \) by its conjugate field \( (f^q)^* \), which has a monopole charge \(-q\). Note that the fields \( (F_+, A_+) \) are ill-defined at the south pole, \( \theta = \pi \), of the sphere, whereas \( (F_-, A_-) \) are ill-defined at the north pole, \( \theta = 0 \). In the previous section, we studied the doublet model with \( (q_1, q_2) = (q, -q) \) and rewrote the model in terms of globally well-defined fields, which are free from singularities, by embedding the gauge field in the \( SU(2) \) group. This argument is applicable to general models with larger multiplets. We can prove that if

\[
\sum_{i=1}^n q_i = 0, \quad (4.4)
\]

then the monopole gauge field is embedded in the \( SU(n) \) group, and that the fields are transformed into globally single-valued ones.

Let us prove the above statement. The matrix-valued function \( T(\phi) \) in (4.2) is a map \( T : S^1 \to U(n) \). If the condition (4.4) is satisfied, \( T(\phi) \) is a continuous map \( T : S^1 \to SU(n) \). Because the first homotopy group of \( SU(n) \) is trivial, namely \( \pi_1(SU(n)) = 0 \), there exists a continuous map

\[
T_+ : [0, \pi] \times S^1 \to SU(n); \quad (\theta, \phi) \mapsto T_+ (\theta, \phi) \quad (4.5)
\]

such that \( T_+(0, \phi) = 1 \) and \( T_+(\pi, \phi) = T(\phi) \). Now we let \( T_+ \) operate on \( F_+ \) and define

\[
\tilde{F} = T_+ F_+ : [0, \pi] \times S^1 \to \mathbb{C}^n. \quad (4.6)
\]

Then we have \( \tilde{F}(0, \phi) = F_+(0, \phi) \) and \( \tilde{F}(\pi, \phi) = T(\phi)F_+(\pi, \phi) = F_-(\pi, \phi) \), and hence \( \tilde{F} \) becomes a single-valued continuous function \( \tilde{F} : S^2 \to \mathbb{C}^n \). The covariant derivative of \( F_+ \)

(4.3) implies

\[
DF_+ = (d - iQA_+)F_+ = (d - iQA_+)T_+^{-1}\tilde{F} = T_+^{-1}(d + T_+dT_+^{-1} - iT_+QQT_+^{-1}A_+)\tilde{F} \quad (4.7)
\]

Hence the covariant derivative of the multiplet \( \tilde{F} \) is written as

\[
D\tilde{F} = (d - i\tilde{A})\tilde{F} \quad (4.8)
\]

with the \( SU(n) \) gauge field \( \tilde{A} \) that is defined by

\[
\tilde{A} = iT_+dT_+^{-1} + T_+QQT_+^{-1}A_. \quad (4.9)
\]

Of course, \( \tilde{A} \) is well-defined over \( U_+ \). In the limit \( \theta \to \pi \), we have \( T_+ \to T = e^{-2iQ\phi} \) and

\[
\tilde{A} \to QT^{-1}A_+ + iTdT^{-1} = Q(A_+ - 2d\phi) = QA_-. \quad (4.10)
\]
therefore, $\hat{A}$ is well-defined over the whole $S^2$. Thus we arrive at the consequence that the fields $(F_\pm, A_\pm)$ are transformed to the globally defined fields $(\hat{F}, \hat{A})$.

To illustrate the above theorem we may consider a multiplet of charges $(q_1, q_2, \cdots, q_{2j}, q_{2j+1}) = (j, j-1, \cdots, -j+1, -j)$ for an integer or half-integer $j$. Then the matrix $Q$ is identified with the generator $J_3$ of $SU(2)$ in the spin $j$ representation. If we take

$$T_+(\theta, \phi) = e^{iJ_2 \theta} e^{iJ_3 \phi} e^{-iJ_2 \theta} e^{-iJ_3 \phi}, \quad (4.11)$$

it has the desired properties; $T_+(0, \phi) = 1$ and $T_+(\pi, \phi) = e^{-2iQ \phi}$. In this case, the gauge field $\hat{A}$ is embedded in the subgroup $SU(2) \subset SU(n)$.

### 4.2 Exact solutions

In the subsection 3.3 we obtained the exact vacuum solution of the $q = 1/2$ doublet model. A similar method can be used to obtain a series of exact solutions of larger multiplet models as shown below. Let us consider a multiplet $F$ of fields with degenerated charges $(q_1, q_2, \cdots, q_{2j}, q_{2j+1}) = (j, j, \cdots, j, j)$ for a fixed $j > 0$ and define the model by the energy functional

$$E = \int d\theta d\phi r^2 \sin \theta \left\{ DF^\dagger DF - \mu^2 F^\dagger F + \lambda (F^\dagger F)^2 \right\}. \quad (4.12)$$

Apparently, this energy is invariant under the flavor transformations,

$$F_\pm \mapsto TF_\pm, \quad T \in SU(2j+1). \quad (4.13)$$

This symmetry is denoted by $SU(2j+1)$. The energy functional is invariant also under the $CP$ transformation

$$\varphi_{CP} : F_\pm(\theta, \phi) \mapsto e^{i\pi} F_\pm^\dagger(\phi - \theta, \theta + \phi), \quad (4.14)$$

whose definition is brought from (2.42). Thus the model has the symmetry

$$G_{\text{multiplet}} = U(1) \times SU(2) \times SU(2j+1) \times CP. \quad (4.15)$$

We can immediately write down the exact lowest energy state as

$$F_\pm(\theta, \phi) = \begin{pmatrix} f_j \\ f_{j-1} \\ \vdots \\ f_{-j+1} \\ f_{-j} \end{pmatrix} \pm \begin{pmatrix} D_{j, j} \\ D_{j-1, j} \\ \vdots \\ D_{-j+1, j} \\ D_{-j, j} \end{pmatrix} \begin{pmatrix} \theta, \phi \end{pmatrix} = v \begin{pmatrix} D_{j, j} \\ D_{j-1, j} \\ \vdots \\ D_{-j+1, j} \\ D_{-j, j} \end{pmatrix} \begin{pmatrix} \theta, \phi, \mp \phi \end{pmatrix}, \quad (4.16)$$

which is a natural generalization of the doublet solution (3.27) to the larger multiplet. To see that it minimizes the energy, first, remember that the matrix elements $\{D_{m,j}^j\}$ are eigenfunctions belonging to the lowest eigenvalue of the monopole Laplacian. Hence they minimize the kinetic term in (4.12). Second, note that the squared norm $F^\dagger F = v^2$ is constant for (4.16). Hence it minimizes the potential term in (4.12) by a suitable choice of $v$. By substituting (4.16) into (4.12) we evaluate the energy as

$$E = 4\pi \left\{ (j - \mu^2 r^2) v^2 + \lambda r^2 v^4 \right\}, \quad (4.17)$$
which is minimized by
\[ v^2 = \begin{cases} 
0 & \text{for } r \leq \sqrt{7/\mu}, \\
\left(1 - \frac{j}{\mu^2 r^2}\right)\frac{\mu^2}{2\lambda} & \text{for } r > \sqrt{7/\mu}.
\end{cases} \tag{4.18} \]

A remaining question is to ask the symmetry of the vacuum. The answer is that the vacuum (4.16) has the symmetry
\[ H_{\text{multiplet}} = SU(2)' \times CP'. \tag{4.19} \]

In the following we identify the symmetry \( H_{\text{multiplet}} \). We write the representation matrix element of \( SU(2) \) as \( D^j_{mm'}(g) = (j, m|g|j, m') \). Then the component field of (4.16) is written as
\[ (f_m)_{\pm}(p) = v D^j_{mj}(s_{\pm}(p)) \tag{4.20} \]
with use of the maps \( s_{\pm} : U_\pm \to SU(2) \) defined in (2.7). Under the action of \( g \in SU(2) \), it is transformed to
\[ (f_m)_{\beta}(g^{-1} p) = v D^j_{mj}(s_{\beta}(g^{-1} p)) = v D^j_{mj}(g^{-1} \cdot s_\alpha(p) \cdot s_\alpha(p)^{-1} g s_\beta(g^{-1} p)) = v \sum_{m', q'} D^j_{mm'}(g^{-1}) D^j_{m'q'}(s_\alpha(p)) D^j_{q'j}(s_\alpha(p)^{-1} g s_\beta(g^{-1} p)), \tag{4.21} \]
where \( \alpha \) and \( \beta \) denote + or −. By the definition of the Wigner rotation, namely by (2.11) and (2.12), we get
\[ D^j_{q'j}(s_\alpha(p)^{-1} g s_\beta(g^{-1} p)) = (j, q'|W_\alpha(g; p)|j, j) = (j, q'|e^{-ij\omega_{\alpha\beta}}|j, j) = \delta_{q'j} e^{-ij\omega_{\alpha\beta}}. \tag{4.22} \]
From (4.20), (4.21) and (4.22) we deduce
\[ (f_m)_{\beta}(g^{-1} p) = \sum_{n'} D^j_{mm'}(g^{-1}) (f_{n'})_\alpha(p) e^{-ij\omega_{\alpha\beta}}, \tag{4.23} \]
or
\[ e^{ij\omega_{\alpha\beta}} \sum_{m'} D^j_{mm'}(g)(f_{m'})_\beta(g^{-1} p) = (f_m)_\alpha(p), \tag{4.24} \]
Hence the configuration (4.16) is invariant under the combined transformation by the rotation \( g \in SU(2) \) with the flavor transformation \( D^j_{mm'}(g) \in SU(2j + 1)_f \). The elements \( \{D^j_{mm'}(g)\} \) forms a subgroup \( SU(2)_f \subset SU(2j + 1)_f \). Namely, the diagonal part \( SU(2)' = \text{diag}(SU(2) \times SU(2)_f) \) is identified as the symmetry group of the vacuum. In particular, when \( j = 1/2 \), the diagonal transformation (4.24) realizes the transformation (3.32) of the doublet as mentioned before. We call \( SU(2)' \) the flavored rotational symmetry again.

From the definition (2.7) we deduce that
\[ s_{\pm}(\theta, \phi)e^{i\sigma_3 \pi/2} = e^{-i\sigma_3 \phi/2} e^{-i\sigma_2 \theta/2} e^{i\sigma_3 \phi/2} \cdot i\sigma_1 = e^{-i\sigma_3 \phi/2} \cdot i\sigma_1 \cdot e^{i\sigma_2 \theta/2} e^{-i\sigma_3 \phi/2} = e^{-i\sigma_3 \phi/2} (-i\sigma_3) (-i\sigma_2) e^{i\sigma_2 \theta/2} e^{-i\sigma_3 \phi/2} = e^{-i\sigma_3 \phi/2} e^{-i\sigma_3 \pi/2} e^{-i\sigma_2 \theta/2} e^{i\sigma_3 \phi/2} = s_\mp(\pi - \theta, \pi + \phi) e^{\mp i\sigma_3 \pi/2}, \tag{4.25} \]
namely,

\[ s_\pm(\theta, \phi) = s_\mp(\pi - \theta, \pi + \phi) e^{\pm i\sigma_3 \pi/2} e^{-i\sigma_1 \pi/2}. \]  

(4.26)

Hence, we have

\[ D^j_{mj}(s_\pm(\theta, \phi)) = \sum_{m', q'} D^j_{mm'}(s_\mp(\pi - \theta, \pi + \phi)) D^j_{m'q'}(e^{\pm i\sigma_3 \pi/2}) D^j_{q'j}(e^{-i\sigma_1 \pi/2}). \]  

(4.27)

It is an easy exercise to verify that the representation matrix elements of \( SU(2) \) satisfy

\[ D^j_{mm'}(s_\mp(\pi - \theta, \pi + \phi)) = e^{i\pi(m-m')}(D^j_{-m,-m})^*(s_\mp(\pi - \theta, \pi + \phi)), \]  

\[ D^j_{m'q'}(e^{\pm i\sigma_3 \pi/2}) = \delta_{m'q'} e^{\pm iq' \pi}, \]  

\[ D^j_{q'j}(e^{-i\sigma_1 \pi/2}) = \delta_{q',j} e^{-ij\pi}. \]  

(4.28)

(4.29)

(4.30)

By substituting them into (4.27), we get

\[ D^j_{mj}(s_\pm(\theta, \phi)) = e^{im\pi} e^{ij\pi}(D^j_{-m,j})^*(s_\mp(\pi - \theta, \pi + \phi)). \]  

(4.31)

By the definition of \( CP \) in \( (1.14) \), the above equation can be written as

\[ (D^j_{mj} \circ s_\pm)(\theta, \phi) = e^{im\pi} \varphi_{CP}(D^j_{-m,j} \circ s_\pm)(\theta, \phi). \]  

(4.32)

Therefore, if we define the flavor conjugate transformation by

\[ \varphi_F : F_\pm = \begin{pmatrix} f_j \\ f_{j-1} \\ \vdots \\ f_{-j+1} \\ f_{-j} \end{pmatrix} \pm \rightarrow F'_\pm = \begin{pmatrix} e^{ij\pi} f_{-j} \\ e^{i(j-1)\pi} f_{-j+1} \\ \vdots \\ e^{-i(j-1)\pi} f_{j-1} \\ e^{-ij\pi} f_j \end{pmatrix}, \]  

(4.33)

which is an element of the group \( U(1) \times SU(2j + 1)_f \), then we have

\[ (D^j_{mj} \circ s_\pm)(\theta, \phi) = (\varphi_F \circ \varphi_{CP})(D^j_{mj} \circ s_\pm)(\theta, \phi). \]  

(4.34)

Thus we see that the vacuum configuration (4.20) is invariant under the transformation \( \varphi_F \circ \varphi_{CP} \). This symmetry is denoted by \( CP' \). Thus we accomplish identification of the symmetry of the vacuum as \( H_{\text{multiplet}} = SU(2)_f \times CP' \).

This result is remarkable in comparison with the result of an ordinary model in the flat space-time. Let us consider a model that consists of a multiplet of scalar fields \( F(x) = (f_j(x), \cdots, f_{-j}(x)) \) in the flat space-time. Suppose that the model has the global \( SU(2j + 1) \) symmetry under which \( F(x) \) obeys the fundamental representation of \( SU(2j + 1) \), and that the fields develop nonzero vacuum expectation values. If the translational symmetry is preserved, the expectation value \( \langle F(x) \rangle \) is independent of the space-time point \( x \), and hence the symmetry is broken from \( SU(2j + 1) \) to \( SU(2j) \). This is an ordinary pattern of symmetry breaking caused by the fundamental Higgs field. But, in our model in \( S^2 \), the one fundamental scalar multiplet breaks the rotation-flavor symmetry from \( SU(2) \times SU(2j + 1) \) to \( SU(2)' \). It may be worthwhile to remember that this peculiar pattern of symmetry breaking is caused by the monopole gauge field.
4.3 Phase structures of generic models

In regard to a general model that consists of the multiplet (4.1) with an arbitrary charge pattern \((q_1, q_2, \cdots, q_n)\), we do not yet have a complete consequence on its phase structure. However, we can deduce the following result confidently. In such a general model, each field \(f_i\) has different mass parameter \(\mu_i\) and different critical radius \(r_i = \sqrt{|q_i|/\mu_i}\). Let us set \(r_s = \min \{r_i\}_{i=1,\ldots,n}\). We can say that when the radius \(r\) of \(S^2\) is smaller than \(r_s\), namely \(r < r_s\), all the fields have vanishing vacuum expectation values and hence the symmetry is not broken. When \(r > r_s\), the field \(f_s\) begins to develop a nonzero vacuum expectation value, and the symmetry is broken.

If the model has an increasing sequence of critical radii \(r_1 < r_2 < \cdots\), it may exhibit a pattern of symmetry breakings that develops step by step when the radius of the sphere increases. But the detail of the symmetry pattern depends on the other parameters of the model. At the present stage we do not have a consequence applicable to such a general class of models.

5 Conclusion

We have studied the scalar field in the monopole background in \(R^n \times S^2\) with the charge \(q = 1/2, 1, 3/2, \cdots\). We showed that the field develops \(2|q|\) vortices when the radius of \(S^2\) is larger than the critical radius \(r_q = \sqrt{|q|/\mu}\). Then the rotational symmetry of \(S^2\) is broken. The vortices repel each other and settle down at the furthest separated points on \(S^2\). We found that in the doublet model with the charge \((q, -q)\) the vortices do not appear and the modified rotational symmetry is left unbroken. We also obtained the exact vacuum configuration of the model with the multiplet \((q_1, q_2, \cdots, q_{2j}, q_{2j+1}) = (j, j, \cdots, j, j)\) and found that the rotation-flavor symmetry \(SU(2) \times SU(2j + 1)\) is spontaneously broken to the flavored rotational symmetry \(SU(2)\)'s. We classified the patterns of symmetry breaking of these models.

We would like to put remarks on further developments of this work. The two-sphere, which is the extra dimensions of the present model, is a kind of homogeneous space \(S^2 = SU(2)/U(1)\). We have seen that the model defined in \(S^2\) has the \(SU(2)\) symmetry and the \(U(1)\) gauge field. As one of the directions for developments, we can construct models in higher dimensional manifolds than \(S^2\), which have larger global symmetries than \(SU(2)\) and larger gauge symmetries than \(U(1)\). For example, the \(n\)-dimensional sphere \(S^n\) is also a homogeneous space, which is written as \(S^n = SO(n + 1)/SO(n)\). More generally, a homogeneous space \(M = G/H\) induces a gauge field associated with the nonabelian gauge group \(H\) as has been shown in [25]. The model defined in \(G/H\) can have various topological defects other than vortices and exhibit more complicated patterns of symmetry breaking.

It is strongly desirable to include fermions in our formulation for application to realistic models. Fermions open ways for further development: chiral fermions, the generation structure, supersymmetry, its breaking, flavor mixing, and so on. When there is a topological defect in extra dimensions, zero modes of the Dirac operator are trapped in the defect and become chiral fermions in the four dimensions. Then the number of fermion generations coincides with the topological number of the defect. Our analysis indicates that a degenerated vortex with a large topological number is unstable and decays into separated vortices. Thus we need more careful analyses on dynamics of topological defects to build more realistic models. In particular, the back reaction of the gauge field, which is known as the Meissner...
effect, should be considered.

In this paper we assume the existence of the monopole background in the extra dimensions. We also assume the negative square mass $-\mu^2$ of the scalar field. Of course, their origins are to be pursued further. We would like to mention studies by other people, which are related to our problem. Hosotani showed that when fermions live in the space-time $R^n \times S^2$, the vacuum energy is lowered by the monopole background. Accordingly, the monopole can be generated dynamically. Moreover, Randjbar-Daemi, Salam and Strathdee proved that the monopole solutions of the classical Einstein-Maxwell theory are stable. On the other hand, the same authors showed that a solution of the classical Einstein-Yang-Mills theory in $R^4 \times S^2$ is unstable if the solution is invariant under the rotations of $S^2$. Although these models do not directly solve our problem about the origins of the monopole background and of the tachyonic mode, they give insights about dynamics in higher dimensions.

Moreover, finding a new mechanism of supersymmetry breaking is the original motivation of this work. To realize it we need to equip fermions in the extra dimensions in a supersymmetric manner. Furthermore, the twisted boundary condition could provide a new mechanism of the flavor mixing; if the eigenstates of the boundary condition differ from the eigenstates of the interaction, then the mixing occurs. We call this phenomenon the topological mixing mechanism and leave it for future study. Finally, it is an interesting question to ask cosmological implications of the critical radius.

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