On the Hodge conjecture for quasi-smooth intersections in toric varieties

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Abstract
We establish the Hodge conjecture for some subvarieties of a class of toric varieties. First we study quasi-smooth intersections in a projective simplicial toric variety, which is a suitable notion to generalize smooth complete intersection subvarieties in the toric environment, and in particular quasi-smooth hypersurfaces. We show that under appropriate conditions, the Hodge conjecture holds for a very general quasi-smooth intersection subvariety, generalizing the work on quasi-smooth hypersurfaces of the first author and Grassi in Bruzzo and Grassi (Commun Anal Geom 28: 1773–1786, 2020). We also show that the Hodge Conjecture holds asymptotically for suitable quasi-smooth hypersurface in the Noether–Lefschetz locus, where “asymptotically” means that the degree of the hypersurface is big enough, under the assumption that the ambient variety $\mathbb{P}^{2k+1}_\Sigma$ has Picard group $\mathbb{Z}$. This extends to a class of toric varieties Otwinowska’s result in Otwinowska (J Alg Geom 12: 307–320, 2003).

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1 Introduction

A projective simplicial toric variety $\mathbb{P}^d_\Sigma$ satisfies the Hodge Conjecture, i.e., every cohomology class in $H^{p,q}(\mathbb{P}^d_\Sigma, \mathbb{Q})$ is a linear combination of algebraic cycles. On the one hand, by the Lefschetz hyperplane theorem, the Hodge conjecture holds true for every hypersurface and $p < \frac{d-1}{2}$ and by the hard Lefschetz theorem also for $p > \frac{d-1}{2}$. Moreover, by Theorem 1.1 in [3], when $p = \frac{d-1}{2}$, $d = 2k + 1$ and $\mathbb{P}^{2k+1}_\Sigma$ is an Oda variety with an ample class $\beta$ such that $k\beta - \beta_0$ is nef, where $\beta_0$ is the anticanonical class, the Hodge conjecture with rational coefficients holds for a very general hypersurface in the linear system $|\beta|$.

The notion of Oda varieties was introduced in [2]. Let us recall that the Cox ring of a toric variety $\mathbb{P}_\Sigma$ is graded over the class group $\text{Cl}(\mathbb{P}_\Sigma)$, and that there is an injection $\text{Pic}(\mathbb{P}_\Sigma) \to \text{Cl}(\mathbb{P}_\Sigma)$.

Definition 1.1 Let $\mathbb{P}_\Sigma$ be a toric variety with Cox ring $S$. $\mathbb{P}_\Sigma$ is said to be an Oda variety if the multiplication morphism $S^{\alpha_1} \otimes S^{\alpha_2} \to S^{\alpha_1 + \alpha_2}$ is surjective whenever the classes $\alpha_1$ and $\alpha_2$ in $\text{Pic}(\mathbb{P}_\Sigma)$ are ample and nef, respectively.

In [15] Mavlyutov proved a Lefschetz type theorem for quasi-smooth intersection subvarieties, and moreover using the “Cayley trick” he related the cohomology of a quasi-smooth subvariety $X = X_{f_1} \cap \cdots \cap X_{f_s} \subset \mathbb{P}^d_\Sigma$ to the cohomology of a quasi-smooth hypersurface $Y \subset \mathbb{P}^{d+s-1}_\Sigma$. This allows us to prove a Noether–Lefschetz type theorem, namely:

Theorem 2.5. Let $\mathbb{P}_\Sigma^d$ be an Oda projective simplicial toric variety. For a very general quasi-smooth intersection subvariety $X$ cut off by $f_1, \ldots, f_s$ such that $d + s = 2(\ell + 1)$ and

$$\sum_{i=1}^s \deg(f_i) - \beta_0$$

is nef, one has

$$H^{\ell+1-s,\ell+1-s}(X, \mathbb{Q}) = i^* \left( H^{\ell+1-s,\ell+1-s}(\mathbb{P}_\Sigma^d, \mathbb{Q}) \right).$$

From this one obtains the following result about the Hodge conjecture for quasi-smooth intersections.

Corollary 2.7. If $\mathbb{P}_\Sigma^d$ is an Oda projective simplicial toric variety, the Hodge Conjecture holds for a very general quasi-smooth intersection subvariety $X$ cut off by $f_1, \ldots, f_s$ such that $d + s$ is even and $\sum_{i=1}^s \deg(f_i) - \beta_0$ is nef.

Let $T$ be the open subset of $|\beta|$ corresponding to quasi-smooth hypersurfaces, and let $\mathcal{H}^{2k} = R^{2k} \pi_* \mathcal{O}_X \otimes \mathcal{O}_T$ be the Hodge bundle on $T$; here $\pi : X \to T$ is the
tautological family on $T$, and $d = 2k + 1$. We restrict $\mathcal{H}^{2k}$ to a contractible open subset $U \subset T$. The bundle $\mathcal{H}^{2k}$ has a Hodge decomposition

$$\mathcal{H}^{2k} = \bigoplus_{p+q=2k} \mathcal{H}^{p,q}$$

but this is not holomorphic. On the other hand, the bundles that make up the Hodge filtration

$$F^p\mathcal{H}^{2k} = \bigoplus_{p=0}^{2k} \mathcal{H}^{2k-p,p}$$

are holomorphic; to see this one can use the period map (which in particular we write for $p = k$)

$$\mathcal{P}^{k,2k} : U \to \text{Grass}(b_k, H^{2k}(X_{u_0}, \mathbb{C}))$$

where $b_k = \dim F^k H^{2k}(X_{u_0}, \mathbb{C})$ for a fixed point $u_0 \in U$; this map sends $f \in U$ to the subspace $F^k H^{2k}(X_f, \mathbb{C}) \subset H^{2k}(X_f, \mathbb{C}) = H^{2k}(X_{u_0}, \mathbb{C})$. This map is holomorphic (see [14] and [5, Prop. 3.4]). But, by the very definition of the period map (see also [17], Section 10.2.1 for the smooth case)

$$F^k \mathcal{H}^{2k} \simeq (\mathcal{P}^{k,2k})^* U_k,$$

where $U_k$ is the tautological bundle on the Grassmannian Grass$(b_k, H^{2k}(X_{u_0}, \mathbb{C}))$, so that the bundles $F^k \mathcal{H}^{2k}$ are indeed holomorphic.

Pushing ahead the ideas developed in [5] and [4], let $\lambda_f$ be a nonzero class in the primitive cohomology $H^{k,k}(X_f, \mathbb{Q})/H^{k,k}(\mathbb{P}_{\Sigma}^{2k+1}, \mathbb{Q})$, and let $U$ be a contractible open subset of $T$ around $f$, so that $\mathcal{H}^{2k}_f$ is constant. Moreover, let $\lambda \in \mathcal{H}^{2k}(U)$ be the section defined by $\lambda_f$ and let $\tilde{\lambda}$ be its image in $(\mathcal{H}^{2k}/F^k \mathcal{H}^{2k})(U)$. One has

**Proposition 1.2** The local Noether–Lefschetz loci can be defined as

$$N_{k,\beta}^{k,\beta} := \{ G \in U \mid \tilde{\lambda}_G = 0 \}$$

where $\beta = \deg(f)$.

The following result is Theorem 1.2 in [4].

**Theorem.** Let $\mathbb{P}_{\Sigma}^{2k+1}$ be an Oda variety with an ample class $\beta$ such that $k\beta - \beta_0 = n\eta$, where $\beta_0$ is the anticanonical class, $\eta$ is a primitive ample class, and $n \in \mathbb{N}$. Let

$$m_\beta = \max \{ i \in \mathbb{N} \mid i\eta \leq \beta \}. \quad (1)$$

For every positive $c$ there is a positive $\delta$ such that for every $m \geq \max(\frac{1}{c}, m_\beta)$ and $\hat{d} \in [1, m\delta]$, and every nontrivial Hodge class $\lambda \in F^k \mathcal{H}^{2k}(U)$ such that
\[
\text{codim} N_{k, \lambda, U}^{k, \beta} \leq \frac{m_{k, \beta}^k}{k!},
\]

for every \( f \in N_{k, \lambda, U}^{k, \beta} \), there exists a \( k \)-dimensional variety \( V \subset X_f \) with \( \deg V \leq (1 + \epsilon)\tilde{d} \). Here \( \deg V \) is taken with respect to the ample divisor \( \eta \), i.e.,

\[
\deg V = [V] \cdot \eta^k.
\]

Based on this, in this paper we obtain the following result.

**Theorem 4.2.** Under the same hypotheses of the previous theorem, assume also that \( \text{Pic}(\mathbb{P}^{2k+1}) = \mathbb{Z} \). Then, if \( V \subset X_f \) is a nonempty quasi-smooth intersection subvariety of \( \mathbb{P}^{2k+1} \) for some \( f \in N_{k, \lambda, U}^{k, \beta} \), there exists \( c \in \mathbb{Q}^* \) such that \( \lambda_f = c \lambda_V \), where \( \lambda_V \) is the class of \( V \) in \( H_{\text{prim}}^{k, k}(X_f, \mathbb{Q}) \).

In other words, \( \lambda_f \) is algebraic.

In his paper [11] A. Dan proves a form of our Theorem 4.2 for smooth hypersurfaces in odd-dimensional projective spaces \( \mathbb{P}^{2k+1} \) which is not asymptotic. Although our result is more general in two ways, as we consider quasi-smooth intersections in *toric varieties* with \( h^{k, k} = 1 \) (for instance, weighted or fake projective spaces); however, our result is asymptotic.

In Sect. 3 we give an extension of the notion of Gorenstein ideal to Cox rings; this may have some interest on its own.

## 2 Very general quasi-smooth intersections

Let \( f_1, \ldots, f_s \) be homogeneous polynomials in the Cox ring \( S = \mathbb{C}[x_1, \ldots, x_n] \) of \( \mathbb{P}^d_{\Sigma} \). Their zero locus \( V(f_1, \ldots, f_s) \) defines a closed subvariety \( X \subset \mathbb{P}^d_{\Sigma} \). Let \( U(\Sigma) = \mathbb{A}^n - Z(\Sigma) \), where \( Z(\Sigma) \) is the irrelevant locus, i.e., \( Z(\Sigma) = \text{Spec} B \), where \( B \) is the irrelevant ideal.

**Definition 2.1** [15] \( X \) is a codimension \( s \) quasi-smooth intersection if \( V(f_1, \ldots, f_s) \cap U(\Sigma) \) is either empty or a smooth intersection subvariety of codimension \( s \) in \( U(\Sigma) \).

This notion generalizes that of smooth complete intersection in a projective space. For \( s = 1 \) it reduces to the notion of *quasi-smooth hypersurface*, see Def. 3.1 in [1]. If we regard \( \mathbb{P}^d_{\Sigma} \) as an orbifold, then an intersection of hypersurfaces \( X_{f_1} \cap \cdots \cap X_{f_s} \) is quasi-smooth when it is a sub-orbifold of \( \mathbb{P}^d_{\Sigma} \), see Prop 1.3 [15]; heuristically, “\( X \) has only singularities coming from the ambient variety.”

We also have a Lefschetz type theorem in this context.

**Proposition 2.2** ([15] Proposition 1.4) Let \( X \subset \mathbb{P}^d_{\Sigma} \) be a closed subset, defined by homogeneous polynomials \( f_1, \ldots, f_s \in B \). Then the natural map \( i^* : H^i(\mathbb{P}^d_{\Sigma}) \rightarrow H^i(X) \) is an isomorphism for \( i < d - s \) and an injection for \( i = d - s \). In particular, this is true if the hypersurfaces cut by the polynomials \( f_i \) are ample.
Hence if \( p \neq \frac{d-s}{2} \) every cohomology class in \( H^{p,p}(X) \) is a linear combination of algebraic cycles. So let us see what happens when \( p = \frac{d-s}{2} \). The idea is to relate the Hodge structure of a quasi-smooth intersection variety \( X = X_{f_1} \cap \cdots \cap X_{f_s} \) in \( \mathbb{P}^d \) with the Hodge structure of a quasi-smooth hypersurface \( Y \) in a toric variety \( \mathbb{P}^{d+s-1}_{X,\Sigma} \) whose fan depends on \( X \) and \( \Sigma \).

**Proposition 2.3** Let \( X = X_{f_1} \cap \cdots \cap X_{f_s} \) be quasi-smooth intersection subvariety in \( \mathbb{P}^d_{\Sigma} \) cut off by homogeneous polynomials \( f_1 \ldots f_s \). There exists a projective simplicial toric variety \( \mathbb{P}^{d+s-1}_{X,\Sigma} \) and a quasi-smooth hypersurface \( Y \subset \mathbb{P}^{d+s-1}_{X,\Sigma} \) such that for \( p \neq \frac{d+s-1}{2}, \frac{d+s-3}{2} \),

\[
H^{p-1,d+s-1-p}_{\text{prim}}(Y) \simeq H^{p-s,d-p}_{\text{prim}}(X).
\]

**Proof** One constructs \( \mathbb{P}^{d+s-1}_{X,\Sigma} \) via the so-called “Cayley trick”. Let \( L_1, \ldots, L_s \) be the line bundles associated to the quasi-smooth hypersurfaces \( X_1, \ldots, X_s \), and so let \( \mathbb{P}(E) \) be the projective bundle of \( E = L_1 \oplus \cdots \oplus L_s \). It turns out that \( \mathbb{P}(E) \) is a \( d+s-1 \)-dimensional projective simplicial toric variety whose Cox ring is

\[
\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_s]
\]

where \( S = \mathbb{C}[x_1, \ldots, x_n] \) is the Cox ring of \( \mathbb{P}^d_{X,\Sigma} \). The hypersurface \( Y \) is cut off by the polynomial \( F = y_1f_1 + \cdots + y_sf_s \) and is quasi-smooth by Lemma 2.2 in [15]. Moreover, combining Theorem 10.13 in [1] and Theorem 3.6 in [15], we have that

\[
H^{p-1,d+s-1-p}_{\text{prim}}(Y) \simeq R(F)_{(d+s-p)\beta-\beta_i} \simeq H^{p-s,d-p}_{\text{prim}}(X)
\]

for \( p \neq \frac{d+s-1}{2}, \frac{d+s-3}{2} \) as desired. \( \square \)

Here \( R(F) \) is the Jacobian ring of \( Y \), i.e., the quotient of the Cox ring

\[
R(F) = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_s]/J(F),
\]

where \( J(F) \) is the ideal generated by the derivatives of \( F \), see [1].

**Remark 2.4** With the same notation of Proposition 2.3, note that we have a well defined map

\[
\phi : |\beta_1| \times \cdots \times |\beta_s| \to |\beta|
\]

\[
(f_1, \ldots, f_s) \mapsto f_1y_1 + \cdots + f_sy_s.
\]

Moreover, by the Noether-Lefschetz theorem \( NL_p \) is a countable union of closed sets \( \bigcup_i C_i \) and hence \( \bigcup \phi^{-1}(C_i) \) is too.

We have a Noether-Lefschetz type theorem, namely,
**Theorem 2.5** Let \( \mathbb{P}_d^\Sigma \) be an Oda projective simplicial toric variety. Then for a very general quasi-smooth intersection subvariety \( X \) cut off by \( f_1, \ldots, f_s \) such that \( d + s = 2(l + 1) \) and \( \sum_{i=1}^s \deg(f_i) - \beta_0 \) is nef, one has that

\[
H^{l+1-s,l+1-s}(X, \mathbb{Q}) = i^*(H^{l+1-s,l+1-s}(\mathbb{P}_d^\Sigma, \mathbb{Q}))
\]

So we get a natural generalization of the Noether-Lefschetz loci.

**Definition 2.6** The Noether-Lefschetz locus \( NL_{\beta_1, \ldots, \beta_s} \) of quasi-smooth intersection varieties is the locus of \( s \)-tuples \( (f_1, \ldots, f_s) \) such that \( X = X_{f_1} \cap \ldots \cap X_{f_s} \) is quasi-smooth intersection with \( f_i \in |\beta_i| \) and \( H^{l+1-s,l+1-s}(X, \mathbb{Q}) \neq i^*(H^{l+1-s,l+1-s}(\mathbb{P}_d^\Sigma, \mathbb{Q})) \).

Now we consider the Hodge conjecture for very general quasi-smooth intersection subvarieties in \( \mathbb{P}_d^\Sigma \).

**Corollary 2.7** If \( \mathbb{P}_d^\Sigma \) is a Oda projective simplicial toric variety, the Hodge Conjecture holds for a very general quasi-smooth intersection subvariety \( X \) cut off by \( f_1, \ldots, f_s \) such that \( d + s = 2(l + 1) \) and \( \sum_{i=1}^s \deg(f_i) - \beta_0 \) is nef.

**Proof** First note that by Thereom 4.1 in [12] the projective simplicial toric variety \( \mathbb{P}_{2l+1}^\Sigma \) is Oda and since \( X \) is very general the quasi-smooth hypersurface \( Y \) is very general as well. So applying the Noether-Lefschetz theorem one has that \( h^0_{prim}(Y) = 0 = h^0_{prim}(X) \) or equivalently every \( (l + 1 - s, +1 - s) \) cohomology class is a linear combination of algebraic cycles.

\[ \square \]

### 3 Cox-Gorenstein ideals

We shall need a partial generalization of Macaulay’s theorem (see e.g. Thm. 6.19 in [18] for the classical theorem). This generalization is basically contained in the work of Cox and Cattani-Cox-Dickenstein [7, 9].

Let \( S \) be the Cox ring of a complete simplicial toric variety \( \mathbb{P}_\Sigma \). This is graded over the effective classes in the class group \( Cl(\mathbb{P}_\Sigma) \) and [8]

\[
S^\alpha \simeq H^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(\alpha)).
\]

As \( \mathcal{O}_{\mathbb{P}_\Sigma}(\alpha) \) is coherent and \( \mathbb{P}_\Sigma \) is complete, each \( S^\alpha \) is finite-dimensional over \( \mathbb{C} \); in particular, \( S^0 \simeq \mathbb{C} \).

**Lemma 3.1** For every effective \( N \in Cl(\mathbb{P}_\Sigma) \), the set of classes \( \alpha \in Cl(\mathbb{P}_\Sigma) \) such that \( N - \alpha \) is effective is finite.

**Proof** Since the torsion submodule of \( Cl(\mathbb{P}_\Sigma) \) is finite, we may assume that \( Cl(\mathbb{P}_\Sigma) \) is free. Then the exact sequence

\[ \square \]
splits, and we may identify \( \text{Cl}(\mathbb{P}_\Sigma) \) with a free subgroup of \( \text{Div}_\mathbb{T}(\mathbb{P}_\Sigma) \), generated by a subset \( \{ D_1, \ldots, D_r \} \) of \( \mathbb{T} \)-invariant divisors. A class in \( \text{Cl}(\mathbb{P}_\Sigma) \) is effective if and only its coefficients on this basis are nonnegative, whence the claim follows.

We shall give a definition of Cox-Gorenstein ideal of the Cox rings which generalizes to toric varieties the definition given by Otwinowska in [16] for projective spaces. Let \( B \subset S \) be the irrelevant ideal, and for a graded ideal \( I \subset B \), denote by \( V_I(B) \) the corresponding closed subscheme of \( \mathbb{P}_\Sigma \).

**Definition 3.2** A graded ideal \( I \) of \( S \) contained in \( B \) is said to be a Cox-Gorenstein ideal of socle degree \( N \in \text{Cl}(\mathbb{P}_\Sigma) \) if

1. there exists a \( \mathbb{C} \)-linear form \( \Lambda \in (S^N)^\vee \) such that for all \( \alpha \in \text{Cl}(\mathbb{P}_\Sigma) \)

\[
I^\alpha = \{ f \in S^\alpha \mid \Lambda(fg) = 0 \text{ for all } g \in S^{N-\alpha} \};
\]

2. \( V_I(B) = \emptyset \).

**Remark 3.3** Cox-Gorenstein ideals need not be Artinian. Property 2 in this definition replaces that condition.

**Proposition 3.4** Let \( R = S/I \). If \( I \) is Cox-Gorenstein then

1. \( \dim_{\mathbb{C}} R^N = 1 \);
2. the natural bilinear morphism

\[
R^\alpha \times R^{N-\alpha} \to R^N \cong \mathbb{C}
\]

is nondegenerate whenever \( \alpha \) and \( N - \alpha \) are effective.

**Proof**

1. From eq. (2) we see that the sequence

\[
0 \to M \to \text{Div}_\mathbb{T}(\mathbb{P}_\Sigma) \to \text{Cl}(\mathbb{P}_\Sigma) \to 0
\]

is exact.
2. Define \( \Phi : R^\alpha \times R^{N-\alpha} \to \mathbb{C} \) as \( \Phi(x, y) = \Lambda(\bar{x}\bar{y}) \), where \( \bar{x}, \bar{y} \) are pre-images of \( x, y \) in \( S \). One easily checks that this is well defined and that via the isomorphism \( R^N \cong k \) it coincides with the pairing (3). Now if \( x \in R^\alpha \) and \( \Phi(x, y) = 0 \) for all \( y \in R^{N-\alpha} \) then \( \Lambda(\bar{x}\bar{y}) = 0 \) for all \( \bar{y} \in S^{N-\alpha} \) so that \( \bar{x} \in I^\alpha \), i.e., \( x = 0 \).
Let \( f_0, \ldots, f_d \) be homogeneous polynomials, \( f_i \in S^\alpha_i \), where \( d = \dim \mathbb{P}_\Sigma \) and each \( \alpha_i \) is ample, and let \( N = \sum \alpha_i - \beta_0 \), where \( \beta_0 \) is the anticanonical class of \( \mathbb{P}_\Sigma \). Assume that the \( f_i \) have no common zeroes in \( \mathbb{P}_\Sigma \), i.e., \( V_\Sigma(I) = \emptyset \) if \( I = (f_0, \ldots, f_d) \).

In [1, 7, 9] it is shown that for each \( G \in S^N \) one can define a meromorphic \( d \)-form \( \xi_G \) on \( \mathbb{P}_\Sigma \) by letting

\[
\xi_G = \frac{G \Omega}{f_0 \cdots f_d}
\]

where \( \Omega \) is a Euler form on \( \mathbb{P}_\Sigma \). The form \( \xi_G \) determines a class in \( H^d(\mathbb{P}_\Sigma, \mathcal{O}) \), where \( \mathcal{O} \) is the canonical sheaf of \( \mathbb{P}_\Sigma \) (the sheaf of Zariski \( d \)-forms on \( \mathbb{P}_\Sigma \)), and in turn the trace morphism \( \text{Tr}_{\mathbb{P}_\Sigma} : H^d(\mathbb{P}_\Sigma, \mathcal{O}) \to \mathbb{C} \) associates a complex number to \( G \), so we can define \( \Lambda \in (S^N)^\vee \) as

\[
\Lambda(G) = \text{Tr}_{\mathbb{P}_\Sigma}(\xi_G) \in \mathbb{C}.
\]

Finally, we can prove a toric version of Macaulay’s theorem.

**Theorem 3.5** The linear map defined in Eq. (4) satisfies the condition in Definition 3.2. Therefore, the ideal \( I = (f_0, \ldots, f_d) \) is a Cox-Gorenstein ideal of socle degree \( N \).

**Proof** By Theorem 6 in [7] the map \( \Lambda \) establishes an isomorphism \( R^N \cong \mathbb{C} \). Hence, if \( f \in S^a \) is such that \( \Lambda(fg) = 0 \) for all \( g \in S^{N-a} \), then \( fg \in I^a \), which implies \( f \in I^a \). On the other hand, it is clear that \( \Lambda(fg) = 0 \) if \( f \in I^a \) and \( g \in S^{N-a} \).

Another example is given in terms of toric Jacobian ideals. For every ray \( \rho \in \Sigma(1) \) we shall denote by \( v_\rho \) its rational generator, and by \( x_\rho \) the corresponding variable in the Cox ring. Recall that \( d \) is the dimension of the toric variety \( \mathbb{P}_\Sigma \), while we denote by \( r = \#\Sigma(1) \) the number of rays. Given \( f \in S^6 \) one defines its toric Jacobian ideal as

\[
J_0(f) = \left( x_\rho \frac{\partial f}{\partial x_\rho}, \ldots, x_\rho \frac{\partial f}{\partial x_\rho} \right).
\]

We recall from [1] the definition of nondegenerate hypersurface and some properties (Def. 4.13 and Prop. 4.15).

**Definition 3.6** Let \( f \in S(\Sigma)^\beta \), with \( \beta \) an ample Cartier class. The associated hypersurface \( X_f \) is nondegenerate if for all \( \sigma \in \Sigma \) the affine hypersurface \( X_f \cap O(\sigma) \) is a smooth codimension one subvariety of the orbit \( O(\sigma) \) of the action of the torus \( \mathbb{T}^d \).

**Proposition 3.7**

1. Every nondegenerate hypersurface is quasi-smooth.
2. If \( f \) is generic then \( X_f \) is nondegenerate.

The following is part of Prop. 5.3 in [9], with some changes in the terminology.
Proposition 3.8 Let \( f \in S(\Sigma)^{\beta} \), and let \( \{ \rho_1, \ldots, \rho_d \} \subset \Sigma(1) \) be such that \( \nu_{\rho_1}, \ldots, \nu_{\rho_d} \) are linearly independent.

1. The toric Jacobian ideal of \( f \) coincides with the ideal
\[
\left( f, x_{\rho_1} \frac{\partial f}{\partial x_{\rho_1}}, \ldots, x_{\rho_d} \frac{\partial f}{\partial x_{\rho_d}} \right).
\]

2. The following conditions are equivalent:
   
   (a) \( f \) is nondegenerate;
   
   (b) the polynomials \( x_{\rho_i} \frac{\partial f}{\partial x_{\rho_i}}, i = 1, \ldots, r, \) do not vanish simultaneously on \( X_f \);
   
   (c) the polynomials \( f \) and \( x_{\rho_i} \frac{\partial f}{\partial x_{\rho_i}}, i = 1, \ldots, d, \) do not vanish simultaneously on \( X_f \).

3. If moreover \( \beta \) is ample and \( f \) is nondegenerate, then \( J_0(f) \) is a Cox-Gorenstein ideal of socle degree \( N = (d + 1)\beta - \beta_0 \), where \( \beta_0 \) is the anticanonical class of \( \mathbb{P}^d_\Sigma \).

4 Asymptotic Hodge conjecture

In this section we prove Theorem 4.2. Let us recall part of the notation and assumptions of [4]. Let \( \mathbb{P}^{2k+1}_\Sigma \) be an Oda variety with an ample Cartier class \( \beta \) such that \( k\beta - \beta_0 = n\eta \), where \( \beta_0 \) is the anticanonical class, \( \eta \) is a primitive ample class and \( n \in \mathbb{N} \).

We need to define a pre-order in the group
\[
N^1(\mathbb{P}^{2k+1}_\Sigma) = \text{Pic}(\mathbb{P}^{2k+1}_\Sigma) \otimes \mathbb{Q} / \text{numerical equivalence},
\]
by letting \( \alpha < \alpha' \) if \( \alpha' - \alpha \) is an effective class.

Let \( X_f \subset [\beta] \) be a quasi-smooth hypersurface in the Noether-Lefschetz locus associated to a nontrival Hodge class \( \lambda \in F^k \eta^l \mathcal{H}^2k(U) \). Again, its degree is computed by intersecting with the ample class \( \eta \), i.e., \( \deg X_f = [X_f] \cdot \eta \). Let \( r \) be number of rays of \( \Sigma \), so that \( r \geq 2(k + 1) \). Assuming that \( n \) is big enough, it follows from Proposition 4.7 or Theorem 6.1 in [4] that there exists a \( k \)-dimensional subvariety \( V \) of \( X_f \) satisfying the following conditions:

- \( \deg V \leq 2\delta m_\beta \) with \( 0 < \delta < \frac{1}{4(r-(k+1))} \) (the number \( m_\beta \) was defined in Eq. (1));
- the graded ideals \( I_V \) and
\[
E = \{ g \in S^r \mid \sum_{i=1}^{b} \lambda_i \int_{\text{Tub}_{g}} \frac{gh_{\Omega_0} f^{k+1}}{\mathcal{H}^{2k}} = 0 \text{ for all } h \in S^{N'} \},
\]
coincide in degree less than or equal to \((m_\beta - 2 - (r - j) \deg V)\eta\) for some \(j\), with \(0 < j < r\). Here \(\text{Tub}(-)\) is the adjoint of the residue map, and \(N = (k + 1)\beta - \beta_0\) is the socle degree of the Cox-Gorenstein ideal \(E\), while

\[
\lambda_f = \left(\sum_{i=1}^b \lambda_i y_i\right)^{p^d}
\]

is the Poincaré dual of some rational combination of the homology cycles \(y_i\) generating \(H_{2k}(X_f, \mathbb{Q})\). Moreover, via the isomorphism \(T(J'U) \simeq S^\beta\), the degree \(\beta\) summand \(E^\beta\) of \(E\) is identified with the tangent space \(T_f N_{X_f/k}\), to the Noether-Lefschetz locus, so that \(E^\beta\) contains the degree \(\beta\) part \(J(f)^\beta\) the Jacobian ideal of \(f\).

**Lemma 4.1** The toric Jacobian ideal \(J_0(f)\) is contained in \(E\).

**Proof** \(J_0(f) \subset J(f)\), so that \(J_0(f)^\beta \subset J(f)^\beta \subset E^\beta\), and since \(J_0(f)\) is generated in degree \(\beta\), one has \(J_0(f) \subset E\). \(\square\)

We denote by \(\lambda_V\) the class of \(V\) in \(H^{k,k}_{\text{prim}}(X_f, \mathbb{Q})\). In the following theorem we assume that \(\text{Pic}(\mathbb{P}^{2k+1}_\Sigma) = 1\), i.e., that \(\mathbb{P}^{2k+1}_\Sigma\) is a (possibly fake) weighted projective space [6, 13] (cf. [10] Exer. 5.1.13). This implies that \(h^{p,q}(\mathbb{P}^{2k+1}_\Sigma) = 1\) for all \(p\).

**Theorem 4.2** If \(V\) is a quasi-smooth intersection subvariety, there exists \(c \in \mathbb{Q}^*\) such that \(\lambda_f = c \lambda_V\).

**Proof** We divide the proof in three steps.

**Step I:** \(\lambda_V \neq 0\). For clarity, for every cohomology class of a subvariety we denote in the cohomology of which ambient variety we consider it (so we write \([V]_{X_f}\) and \([V]_{\mathbb{P}^{2k+1}_\Sigma}\)). Since \(V \subset X_f\) is a regular embedding we have

\[
[V]_{X_f}^2 = \int_V c_k(N_{V/X_f}) = \int_V \left[ c(N_{V/\mathbb{P}^{2k+1}_\Sigma})/c(N_{X_f/\mathbb{P}^{2k+1}_\Sigma} | V) \right]_k
\]

\[
[8pt] = \int_{\mathbb{P}^{2k+1}_\Sigma} [V]_{\mathbb{P}^{2k+1}_\Sigma} \cup \Xi_k
\]

where \(\Xi_k\) is the component in \(H^{k-k}(\mathbb{P}^{2k+1}_\Sigma)\) of

\[
\Xi = \frac{\prod_i (1 + A_i)}{1 + [X_f]_{\mathbb{P}^{2k+1}_\Sigma}};
\]

here \(A_1, \ldots, A_{2k+1}\) are the classes in \(\text{Cl}(\mathbb{P}^{2k+1}_\Sigma)\) of the hypersurfaces that cut the quasi-smooth intersection subvariety \(V\). The claim is proved by contradiction: if \([V]_{X_f}\) is the restriction of a class in \(H^{k,k}(\mathbb{P}^{2k+1}_\Sigma)\), i.e.,

\[
[V]_{X_f} = b \eta_{X_f}^k
\]

for some \(b\), then comparing this with (6) we obtain

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where $m_k$ is defined by $\Xi_k = m_k \eta^k$. But (7) cannot be true when $\deg X_j$ is big enough.

**Step II.** Let $E_{\text{alg}}$ and $E$ be the Cox-Gorenstein ideals associated to $\lambda_V$ and $\lambda_f$, respectively, as in Eq. (5). To prove the theorem it is enough to show that $E = E_{\text{alg}}$. Note that $I_V + J_0(f)$ is contained in $E$ and $E_{\text{alg}}$. Moreover, since $V \subset X_f$, and $f$ is quasi-smooth, there exist $K_1, \ldots, K_{k+1} \in B$ such that $f = A_1 K_1 + \ldots + A_{k+1} K_{k+1}$ and $(A_1, \ldots, A_{k+1}, K_1, \ldots, K_{k+1})$ is a Cox-Gorenstein ideal with socle degree $N$; this will follow from the next step, which concludes the proof.

**Step III.** It is enough to show that every Cox-Gorenstein ideal $I$ of socle degree $N$ containing $I_V + J_0(f)$ also contains $(A_1, \ldots, A_{k+1}, K_1, \ldots, K_{k+1})$. By assumption

$$
(A_j, j \in \{1, \ldots, k + 1\}, \sum_{j=1}^{k+1} x_j \frac{\partial A_j}{\partial x_i} K_j, i \in 1, \ldots, r) \subset I.
$$

Let us see that $K_j \in I$ for every $j \in \{1, \ldots, k + 1\}$. Let $M \in \text{Mat}(r \times (k + 1))$ be the matrix $[x_j \frac{\partial A_j}{\partial x_i}]$ and $K$ the column $(K_j)_{j \in \{1, \ldots, k+1\}}$. Let $I \subset \{1, \ldots r\}$ with cardinality $k + 1$ and let $M_I$ be the matrix obtained extracting the $i \in I$-arrows of $M$. We have that $\sum_{j=1}^{k+1} x_j \frac{\partial A_j}{\partial x_i} K_j = (MK)_i = (M_I K)_i$; multiplying by the adjoint of $M_I$ we get that $\det(M_I)K_j \in I$ for all $j \in \{1, \ldots, k + 1\}$. On one hand the ideal $(I : K_j)$ contains the ideal

$$
J = I_V + \langle \det M_I | I \subset \{1, \ldots, r\}, \#I = k + 1 \rangle.
$$

Since $V$ is a smooth complete intersection subvariety, it follows that $J$ is base point free, and therefore it contains a complete intersection Cox-Gorenstein ideal $J'$ by the toric Macaulay theorem, Theorem 3.5. Since $J$ is generated in degree less than or equal to $(\deg V) \eta$, we can take $J'$ with the same property. It follows that

$$
soc(J') \leq 2(k + 1)(\deg V) \eta - \beta_0 \leq 2rm_\beta \delta \eta - \beta_0.
$$

On the other hand if $K_j \notin I$ then $(I : K_j)$ contains a Cox-Gorenstein ideal with socle degree

$$
N - \deg K_j \geq N - \beta = k \beta - \beta_0;
$$

then comparing the above two inequalities and keeping in mind that $r \geq 2(k + 1)$, we get

$$
\delta \geq \frac{1}{2r} \geq \frac{1}{4(r - (k + 1))},
$$

which is absurd. \qed

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