Abstract

An Abelian sandpile model is considered on the Husimi lattice of triangles with an arbitrary coordination number $q$. Exact expressions for the distribution of height probabilities in the Self-Organized Critical state are derived.
1 Introduction.

To explain the temporal and spatial scaling in dynamical dissipative systems being in the Self-Organized Critical (SOC) state, Bak, Tang, and Weisenfeld have introduced cellular automaton models known as sandpiles [1].

Latter on, Dhar has studied sandpile models which can be described by the Abelian group [2]. For these Abelian sandpile models (ASM’s) some ensemble-average quantities such as the total number of configurations in the SOC state, distribution of height probabilities, two-point correlation functions, etc., have been calculated exactly on the square lattice [3-6].

Unfortunately, for the sandpile models formulated on the standard plane lattices it seems to be rather difficult to formulate a direct theoretical approach leading to the solution in closed form.

An appreciable insight can be obtained, however, by examining pseudolattices such as the Bethe lattice, Husimi lattice, and their generalizations.

Using this approach, Dhar and Majumdar [7] have obtained exactly a more complete set of sandpile characteristics for the Bethe lattice.

The aim of this article is to analyze the distribution of height probabilities of the ASM on the Husimi lattice with an arbitrary coordination number \( q \). We show that this model can be solved exactly by using the fractal structure of the lattice.

The outline of the article is as follows: in the next section, we define the lattice and the ASM on it. In section 3, we consider the Husimi lattice with \( q = 4 \) and find recursion relations for numbers of allowed configurations in the SOC state. In section 4, we compute exactly the distribution of height probabilities in this particular case. In section 5, we generalize our results to an arbitrary coordination number of the lattice.

2 Lattice and Model.

As a basic building block for constructing a cactus or a pure Husimi tree [8], we will take \( \gamma \) triangle plaquettes joined at the base site (root), see Fig.1a. This basic building block will be called the first-generation branch. To the second-generation branch, we attach \( 2\gamma \) basic building blocks at each free site of the first-generation branch (Fig.1b). Continuing this process we develop higher-generation branches. Then, at the final step, we take three \( n \)th-generation branches and connect their base sites by the triangle plaquette. As a result, we will get the cactus with the coordination number \( q = 2(\gamma + 1) \).

Let as define ASM on this connected graph of \( N \) sites as follows: to each site \( i \) (1 \( \leq i \leq N \)) we associate an integer \( z_i \) (1 \( \leq z_i \leq q \)) which is the height of a column of sand grains. The evolution of the system is specified by two rules:

i. Addition of a sand grain at a randomly chosen site \( i \) increases \( z_i \) by 1.

ii. The site \( i \) topples if the height \( z_i \) exceeds a critical value \( z_c = q \) and sand grains drop on the nearest neighbors.
The number of surface sites of the Husimi tree is comparable with the interior ones. Hence, the calculation of the thermodynamic limit of the bulk properties requires special care. In our work we define the height distribution of sand grains for the sites deep inside the tree. Using these interior sites one can construct an infinite lattice, for they have the same features. Therefore, we will consider the problem on the Husimi lattice rather than on the Husimi tree.

Any configuration \(\{z_i\}\) on the Husimi tree in which \(1 \leq z_i \leq q\) is a stable configuration under the toppling rule. These configurations can be divided into two classes: allowed and forbidden configurations [2].

In the SOC state, only allowed configurations have a nonzero probability. Any subconfiguration of heights \(F\) on a finite connected set of sites is forbidden if

\[
z_i \leq q_i,
\]

where \(q_i\) is a coordination number of a site \(i\) in the given subconfiguration \(F\) [4].

In turn, we can divide the allowed configurations on an \(n\)th-generation branch of the Husimi tree into three nonoverlapping classes: weakly allowed of the type 1 (\(W_1\)), weakly allowed of the type 2 (\(W_2\)) and strongly allowed (\(S\)) configurations.

Consider an allowed subconfiguration \(C\) on the \(n\)th-generation branch \(G_n\) with a root \(a\) (Fig.2a). The coordination number of the root \(a\) is \(q - 2\). Adding a vertex \(b\) to the \(G_n\), one defines a subgraph \(G' = G_n \cup b\). If the subconfiguration \(C' = C \cup b\) with \(z_b = 1\) on the \(G'\) is forbidden, then \(C\) is called the weakly allowed subconfiguration of type 1 (\(W_1\)). Thus, \(W_1\) can be locked by one bond.

Now add two vertices \(b\) and \(d\) to the \(G_n\) and consider a subconfiguration \(C'' = C \cup b \cup d\) on \(G'' = G_n \cup b \cup d\) (Fig.2b). If the subconfiguration \(C'' = C \cup b \cup d\) with \(z_b = 1\), \(z_d = 1\) on the \(G''\) is forbidden, then \(C\) is called the weakly allowed subconfiguration of type 2 (\(W_2\)).

Any allowed subconfigurations defined on the \(n\)th-generation branches that cannot be locked by one bond or by two bonds form a strongly allowed class (\(S\)).

It is important to note that any subconfiguration of the \(W_1\) type is also of the \(W_2\) type. To obtain the nonoverlapping classes, we always check the subconfiguration first to belong to the \(W_1\) type and only after that to the \(W_2\) type.

We will start with mentioning some forbidden subconfigurations that can occur on the Husimi tree.

Let us first consider the subconfiguration shown in Fig.3a. It is easy to check that \(C = C_1 \cup C_2\) will be forbidden if both \(C_1\) and \(C_2\) are of the \(W_1\) type. The next example of the forbidden subconfiguration is shown in Fig.3b where configurations \(C_1, C_2\) and \(C_3\) should be of the \(W_1\) or \(W_2\) type.
3 Recursion relations.

In this section, for simplicity, we will consider a Husimi tree with $\gamma = 1$ ($q = 4$). The $n$th-generation branch $G_n$ with a root $a$ (Fig.4) consists of two $(n-1)$th-generation branches $G_{n-1}^{(1)}$ and $G_{n-1}^{(2)}$ with roots $a_1$ and $a_2$, respectively. Vertices $a_1$ and $a_2$ are nearest neighbors of the root $a$. Let $N_{W_1}(G_n,i)$, $N_{W_2}(G_n,i)$ and $N_S(G_n,i)$ be the numbers of distinct $W_1$, $W_2$ and $S$ type subconfigurations on the $G_n$ with a given height $z_a = i$. Let us also introduce

$$N_{W_1}(G_n) = \sum_{i=1}^{4} N_{W_1}(G_n,i),$$

$$N_{W_2}(G_n) = \sum_{i=1}^{4} N_{W_2}(G_n,i),$$

$$N_S(G_n) = \sum_{i=1}^{4} N_S(G_n,i).$$

The number of allowed subconfigurations on the $G_n$ with a given root is

$$N(G_n,i) = N_{W_1}(G_n,i) + N_{W_2}(G_n,i) + N_S(G_n,i).$$

At the same time, $N_{W_1}(G_n)$, $N_{W_2}(G_n)$ and $N_S(G_n)$ can be expressed in terms of the full numbers of allowed subconfigurations on the two $(n-1)$th-generation branches $G_{n-1}^{(1)}$ and $G_{n-1}^{(2)}$:

$$N_{W_1}(G_n) = N_S(G_{n-1}^{(1)})N_S(G_{n-1}^{(2)}) + N_S(G_{n-1}^{(1)})N_{W_1}(G_{n-1}^{(2)}) + N_{W_1}(G_{n-1}^{(1)})N_S(G_{n-1}^{(2)}) +$$

$$N_S(G_{n-1}^{(1)})N_{W_2}(G_{n-1}^{(2)}) + N_{W_2}(G_{n-1}^{(1)})N_S(G_{n-1}^{(2)}) + N_{W_1}(G_{n-1}^{(1)})N_{W_2}(G_{n-1}^{(2)}) +$$

$$N_{W_2}(G_{n-1}^{(1)})N_{W_1}(G_{n-1}^{(2)}) + N_{W_2}(G_{n-1}^{(1)})N_{W_2}(G_{n-1}^{(2)}),$$

$$N_{W_2}(G_n) = N_S(G_{n-1}^{(1)})N_S(G_{n-1}^{(2)}) + N_S(G_{n-1}^{(1)})N_{W_1}(G_{n-1}^{(2)}) + N_{W_1}(G_{n-1}^{(1)})N_S(G_{n-1}^{(2)}) +$$

$$N_S(G_{n-1}^{(1)})N_{W_2}(G_{n-1}^{(2)}) + N_{W_2}(G_{n-1}^{(1)})N_S(G_{n-1}^{(2)}) + N_{W_1}(G_{n-1}^{(1)})N_{W_2}(G_{n-1}^{(2)}) +$$

$$N_{W_2}(G_{n-1}^{(1)})N_{W_1}(G_{n-1}^{(2)}) + N_{W_2}(G_{n-1}^{(1)})N_{W_2}(G_{n-1}^{(2)}),$$

$$N_S(G_n) = 2N_S(G_{n-1}^{(1)})N_S(G_{n-1}^{(2)}) + N_S(G_{n-1}^{(1)})N_{W_1}(G_{n-1}^{(2)}) + N_{W_1}(G_{n-1}^{(1)})N_S(G_{n-1}^{(2)}) +$$

$$2N_S(G_{n-1}^{(1)})N_{W_2}(G_{n-1}^{(2)}) + 2N_{W_2}(G_{n-1}^{(1)})N_S(G_{n-1}^{(2)}).$$

As we can see, expressions (3.3) and (3.6) are equal to each other. We will prove in section 5 that this holds for all $q$.
Let us define

\[ X = \frac{N_W}{N_S}, \tag{3.8} \]

where \( N_W \equiv N_{W_1} = N_{W_2} \).

From (3.5)-(3.7) one may find the following recursion relation:

\[ X(G_n) = \frac{1 + 2X(G_{n-1}) + 2X(G_{n-1}^{(2)}) + 3X(G_{n-1}^{(1)})X(G_{n-1}^{(2)})}{2 + 3X(G_{n-1}^{(1)}) + 3X(G_{n-1}^{(2)})}. \tag{3.9} \]

If we consider graphs \( G_{n-1}^{(1)} \) and \( G_{n-1}^{(2)} \) to be isomorphic, then \( N(G_{n-1}^{(1)}) = N(G_{n-1}^{(2)}) \) and equation (3.9) simplifies to

\[ X(G_n) = \frac{1}{2} (1 + X(G_{n-1})). \tag{3.10} \]

With the initial condition \( X(G_0) = 1/2 \), equation (3.10) has a simple solution

\[ X(G_n) = 1 - 2^{-(n+1)}. \tag{3.11} \]

On the infinite generation branches \( (n \to \infty) \) the ratio of the number of the \( W_1 \) type configurations or of the \( W_2 \) type to the strongly allowed ones tends to 1.

4 Distribution of height probabilities.

Consider now a randomly chosen site \( O \) deep inside the Husimi tree (Fig.5). The nearest neighbors of the site \( O \) are roots of the four \( n \)th-generation branches. For a given value \( z_0 = i \ (1 \leq i \leq 4) \) at the site \( O \), the number of allowed configurations \( N(i) \) is expressed via \( N(G_{n-1}^{(1)}), N(G_{n}^{(2)}), N(G_{n}^{(3)}), N(G_{n}^{(4)}), \) where again \( N(G_n^{(\alpha)}) = \sum_{i=1}^{4} N(G_n^{(\alpha)}, i), \alpha = 1, 2, 3, 4. \)

If \( i = 1 \), then each allowed configuration on the branches \( G_{n}^{(1)}, G_{n}^{(2)}, G_{n}^{(3)}, G_{n}^{(4)} \) cannot be of the \( W_1 \) type. It is also evident that two \( W_2 \) type configurations cannot occur on the neighboring branches \( G_{n}^{(1)} \) and \( G_{n}^{(2)} \) or \( G_{n}^{(3)} \) and \( G_{n}^{(4)} \). Hence, the number \( N(1) \) of allowed configurations with isomorphic branches is

\[ N(1) = [1 + 4X + 4X^2] \prod_{\alpha=1}^{4} N_S(G^{(\alpha)}). \tag{4.1} \]

Arguing similarly, one can get the following expressions for the numbers of allowed configurations for a given value at the site \( i \)

\[ N(2) = [1 + 8X + 12X^2] \prod_{\alpha=1}^{4} N_S(G^{(\alpha)}), \tag{4.2} \]
\[ N(3) = [1 + 8X + 22X^2 + 12X^3] \prod_{\alpha=1}^{4} N_S(G^{(\alpha)}) , \quad (4.3) \]

\[ N(4) = [1 + 8X + 22X^2 + 24X^3] \prod_{\alpha=1}^{4} N_S(G^{(\alpha)}) . \quad (4.4) \]

The probability \( P(i) \) of having the height \( i \) at site \( O \) is

\[ P(i) = \frac{N(i)}{N_{\text{total}}} , \quad (4.5) \]

where \( N_{\text{total}} = \sum_{i=1}^{4} N(i) \) is the total number of allowed configurations on the Husimi tree.

The system can reach the SOC state only in the thermodynamic limit. For the sites far from the surfaces in this limit \((n \to \infty)\) we have \( X = 1 \). Thus, from (4.1)-(4.3) we get

\[ P(1) = \frac{9}{128}, \quad P(2) = \frac{21}{128}, \quad P(3) = \frac{43}{128}, \quad P(4) = \frac{55}{128} . \quad (4.6) \]

The obtained values (4.6) for the height probabilities characterize the SOC state of the ASM on the Husimi lattice with \( q = 4 \).

5 Generalization to the Husimi lattice with an arbitrary coordination number \( q \).

We now intend to generalize results obtained in two previous sections. Before the recursion relations and expressions for the height probabilities will be written out, we want to prove that the number of the \( W_1 \) type configurations and the number of the \( W_2 \) type ones is equal on the \( n \)th-generation branch of the Husimi tree.

The numbers of weakly allowed configurations of both the types can be written as

\[ N_{W_1}(G_n) = N_{W_1}(G_n, 1) + N_{W_1}(G_n, 2) + \cdots + N_{W_1}(G_n, q), \quad (5.1) \]

\[ N_{W_2}(G_n) = N_{W_2}(G_n, 2) + \cdots + N_{W_2}(G_n, q - 1) + N_{W_2}(G_n, q). \quad (5.2) \]

It is easy to see that \( N_{W_1}(G_n, 1) = N_{W_2}(G_n, q) = 0 \). By definition, the \( W_1 \) type configurations can be locked by one bond. When the height of the root is increased by 1, it becomes of \( W_2 \) type. Similarly, decreasing the height of the root of \( W_2 \) type configuration by 1, we get the \( W_1 \) type. Thus for each configuration of the \( W_1 \) type there is a unique configuration of the \( W_2 \) type and vice versa. Therefore, we can conclude that (5.1) and (5.2) are equal.

For an arbitrary coordination number of the Husimi lattice, equation (3.10) generalizes to

\[ X(G_n) = \frac{1}{2\gamma} (1 + X(G_{n-1})) . \quad (5.3) \]
and with the initial condition $X(G_0) = 1/(2\gamma)$, it has the solution

$$X(G_n) = \frac{1}{2\gamma - 1} - \frac{1}{2\gamma - 1}(2\gamma)^{-(n+1)}. \quad (5.4)$$

For large enough $n$ we have $X(G_n) \xrightarrow{n \to \infty} 1/(2\gamma - 1)$.

In the same way as for the $q = 4$ case, we can calculate the height probabilities of the sites deep inside the lattice

$$P(i) = \frac{(2\gamma + 1)^{\gamma+1}}{2^{\gamma+2}(\gamma + 1)^2(\gamma + 3)^\gamma} \times \sum_{n_1=0}^{i-1} \sum_{n_2=0}^{(i-1)/2 \text{ even}} \frac{\gamma + 1)!}{(\gamma + 1 - n_1 - n_2)! \cdot n_1! \cdot n_2!} 2^{n_1} 3^{n_2} (2\gamma + 1)^{-(n_1+n_2)}, \quad (5.5)$$

where $n_1 + n_2 \leq \gamma + 1$ and $1 \leq i \leq q$.

The probability of the height to be equal 1 has a simple form

$$P(1) = \frac{(2\gamma + 1)^{\gamma+1}}{2^{\gamma+2}(\gamma + 1)^2(\gamma + 3)^\gamma}, \quad (5.6)$$

and tends to 0 when $\gamma \to \infty$.

In this article, we have exactly calculated the distribution of height probabilities in the Self-Organized Critical state of the ASM on the Husimi lattice of triangles. The next step of our investigations will be the Husimi lattice with square plaquettes where we intend to derive exact expressions for two-point correlation functions and critical exponents of avalanches. The choice of the Husimi lattice gives us hope to find the relationship between the chaos and SOC state, as some spin models formulated on this lattice show the chaotic behavior [9, 10].

**Acknowledgments**

We are grateful to N.S. Ananikian and V.B. Priezzhev for fruitful discussions.

One of us (R.R.S.) was partially supported by German Bundesministerium für Forschung and Technologie under the grant N 211-5291 YPI.

**References**

[1] P. Bak, C. Tang, and K. Weisenfeld, Phys.Rev.Lett. 59 (1987) 381; Phys.Rev. A38 (1988) 364.

[2] D. Dhar, Phys.Rev.Lett. 64 (1990) 1613.

[3] S.N. Majumdar and D. Dhar, J.Phys. A24 (1991) L357.
[4] V.B. Priezzhev, J.Stat.Phys. 74 (1994) 955.

[5] E.V. Ivashkevich, J.Phys. A27 (1994) 3643.

[6] S.A. Janowsky and C.A. Laberge, J.Phys. A26 (1993) L973.

[7] D. Dhar and S.N. Majumdar, J.Phys. A23 (1990) 4333.

[8] J.W. Essam and M.E. Fisher, Rev.Mod.Phys. 42 (1972) 272.

[9] J.L. Monroe, J.Stat.Phys. 65 (1991) 255; J.Stat.Phys. 67 (1992) 1185.

[10] N.S.Ananikian, R.R.Lusiniants, K.A.Oganessyan, Miramare-Trieste IC/94/326; N.S.Ananikian, K.A.Oganessyan, preprint YERPHI-1418(5)-94, Phys.Lett. A200 (1995) 205.
Figure 1: (a) A first-generation branch which consists of $\gamma$ triangle plaquettes joined at the base site. (b) A second-generation branch.

Figure 2: (a) A $n$th-generation branch $G_n$ and vertex $b$ form a subgraph $G'$. (b) Now two vertexes $b$ and $d$ and the $G_n$ form a subgraph $G''$. 
Figure 3: Two examples of forbidden subconfigurations that can occur on the Husimi tree.

Figure 4: The $n$th-generation branch $G_n$ with two nearest $(n - 1)$th-generation branches $G^{(1)}_{n-1}$ and $G^{(2)}_{n-1}$. 
Figure 5: A site $O$ with height $i$ is located deep inside the lattice and surrounded by the four $n$th-generation branches $G^{(1)}_{n-1}, G^{(2)}_{n-1}, G^{(3)}_{n-1}, G^{(4)}_{n-1}$.