THE DEFOCUSING ENERGY-CRITICAL WAVE EQUATION WITH A CUBIC CONVOLUTION

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Abstract. In this paper, we study the theory of the global well-posedness and scattering for the energy-critical wave equation with a cubic convolution nonlinearity

\[ u_{tt} - \Delta u + (|x|^{-4} * |u|^2)u = 0 \]

in spatial dimension \( d \geq 5 \). The main difficulties are the absence of the classical finite speed of propagation (i.e. the monotonic local energy estimate on the light cone), which is a fundamental property to show the global well-posedness and then to obtain scattering for the wave equations with the local nonlinearity \( u_{tt} - \Delta u + |u|^{d-2}u = 0 \). To compensate it, we resort to the extended causality and utilize the strategy derived from concentration compactness ideas. Then, the proof of the global well-posedness and scattering is reduced to show the nonexistence of the three enemies: finite time blowup; soliton-like solutions and low-to-high cascade. We will utilize the Morawetz estimate, the extended causality and the potential energy concentration to preclude the above three enemies.

Key Words: wave-Hartree equation, Concentration compactness, Morawetz estimate, Extended causality, Scattering.

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1. Introduction

This paper is devoted to study the global well-posedness (GWP) and scattering for the defocusing energy-critical wave equation with a cubic convolution (wave-Hartree)

\[
\begin{align*}
\ddot{u} - \Delta u + f(u) &= 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \quad d \geq 5 \\
(u(0), \dot{u}(0)) &= (u_0(x), u_1(x)) \in \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d),
\end{align*}
\]

where \( u(t,x) \) is a real-valued function in spacetime \( \mathbb{R} \times \mathbb{R}^d \), \( f(u) = (V(\cdot) * |u|^2)u \) with \( V(x) = |x|^{-4} \), the dot denotes the time derivative and \( \ast \) stands for the convolution in \( \mathbb{R}^d \).

The terminology “Energy-critical” is due to the fact that both the energy \( E(u, \dot{u}) \) defined by

\[
E(u, \dot{u}) := \frac{1}{2} \int_{\mathbb{R}^d} (|\nabla u(x)|^2 + |\dot{u}(x)|^2) \, dx + \frac{1}{4} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(y)|^2|u(x)|^2}{|x-y|^4} \, dx \, dy
\]

and the equation (1.1) itself are invariant under the rescaling symmetry

\[
\lambda u(t, \lambda x) \mapsto \lambda^\frac{d-2}{2} u(\lambda t, \lambda x),
\]

for \( \lambda > 0 \). Note that the energy is conserved by the flow (1.1), hence we do not specify time in the notation.
On one hand, the scattering theory for the energy critical wave equation
\begin{equation}
\ddot{u} - \Delta u + \mu|u|^{2^*-2}u = 0, \quad 2^* = \frac{2d}{d-2},
\end{equation}
has been intensively studied in [1, 7, 8, 11, 33, 36]. When \( \mu = 1 \), which corresponds to the defocusing case, the theory of the global well-posedness and scattering has been studied by Grillakis [7], Kapitanski [8], Shatah-Struwe [33], Bahouri-Gérard [1], Tao [36] and the references cited therein. In particular, Tao in [36] derived an exponential type spacetime bound. The analogs for 3D quintic Schrödinger equation have been established by Colliander, Keel, Staffilani, Takaoka, and Tao [4] recently. For the focusing case: \( \mu = -1 \), and \( 3 \leq d \leq 5 \), recently Kenig and Merle [11] employed sophisticated “concentrated compactness + rigidity method” to obtain the dichotomy-type result under the assumption that \( E(u_0, u_1) < E(W, 0) \), where \( W \) denotes the ground state of the elliptic equation
\[ \Delta W + |W|^{4d-2}W = 0. \]
Thereafter, [2] extend the above result in [11] to higher dimensions.

On the other hand, the scattering theory for the Hartree equation
\[ i\dot{u} = -\Delta u + \mu(|x|^{-\gamma} * |u|^2)u \]
has been also studied by many authors (see [6, 17, 21–26]). For the subcritical cases in the defocusing case (i.e. \( 2 \leq \gamma < \min\{4, d\} \) and \( \mu = 1 \)), Ginibre and Velo [6] derived the associated Morawetz inequality and extracted an useful Birman-Solomjak type estimate to obtain the asymptotic completeness in the energy space. Nakanishi [30] improved the results by a new Morawetz estimate. For the critical case (\( \gamma = 4, \ d \geq 5 \)), Miao, Xu and Zhao [21] took advantage of a new kind of the localized Morawetz estimate to rule out the possibility of the energy concentration at origin and established the scattering results in the energy space for the radial data in dimension \( d \geq 5 \). We refer also to [22–26] for the general data and focusing case.

For the equation (1.1) with \( V(x) = |x|^{-\gamma} \), using the ideas of Strauss [34, 35] and Pecher [32], Mochizuki [28] showed that if \( d \geq 3, \ 2 \leq \gamma < \min\{d, 4\} \), then the global well-posedness and scattering results with small initial data hold in the energy space \( H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \). One may also refer to [27] which study a complete scattering theory of the Klein-Gordon equation with a cubic convolution for large data in the subcritical case. This paper is devoted to study a complete scattering theory of the equation (1.1) for the critical case (i.e. \( \gamma = 4, \ d \geq 5 \)) in the energy space \( \dot{H}^1_x(\mathbb{R}^d) \times L^2_x(\mathbb{R}^d) \).

Before stating the main results, we introduce some background materials.

**Definition 1.1** (solution). A function \( u : I \times \mathbb{R}^d \to \mathbb{R} \) on a nonempty time interval \( I \) containing zero is a strong solution to (1.1), if \( u, u_t \in C^0_t(J; \dot{H}^1_x(\mathbb{R}^d) \times L^2_x(\mathbb{R}^d)) \) and \( u \in L_{t,x}^{2/(d+1)}(J \times \mathbb{R}^d) \) for any compact \( J \subset I \) and for each \( t \in I \), it obeys the Duhamel’s formula:
\[ \begin{pmatrix} u(t) \\ \dot{u}(t) \end{pmatrix} = V_0(t) \begin{pmatrix} u_0(x) \\ u_1(x) \end{pmatrix} - \int_0^t V_0(t-s) \begin{pmatrix} 0 \\ f(u(s)) \end{pmatrix} ds, \]
where
\[ V_0(t) = \left( \dot{K}(t), K(t) \right), \quad K(t) = \frac{\sin(t\omega)}{\omega}, \quad \omega = \left( -\Delta \right)^{1/2}. \]

We refer to the interval \( I \) as the lifespan of \( u \). We say that \( u \) is a maximal-lifespan solution if the solution cannot be extended to any strictly large interval. We say that \( u \) is a global solution if \( I = \mathbb{R} \).

The solution lies in the space \( L^2(t, x) \) locally in time is natural since by Strichartz estimate (see Lemma 2.1 below), the linear flow always lies in this space. Also, the finiteness of the norm on maximal-lifespan implies the solution is global and scatters in both time directions in the sense of (1.9). In view of this, we define
\[ S_I(u) = \|u\|_{L^{2(d+1)}(I \times \mathbb{R}^d)} \overset{\Delta}{=} \|u\|_S(I) \]
as the scattering size of \( u \). Closely associated with the notion of scattering is the notion of blowup:

**Definition 1.2 (Blowup).** We call that a maximal-lifespan solution \( u : I \times \mathbb{R}^d \to \mathbb{R} \) blows up forward in time if there exists a time \( t_0 \in I \) such that \( S_{[t_0, \sup I]}(u) = +\infty \); similarly, \( u(t, x) \) blows up backward in time if \( S_{[\inf I, t_0]}(u) = +\infty \).

Our main result is the following global well-posedness and scattering result in the energy space.

**Theorem 1.1.** Let \( d \geq 5 \) and \((u_0, u_1) \in \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)\) be initial data with energy bound
\[ E(u_0, u_1) \leq E \]
for any constant \( E > 0 \). Then there exists a unique global solution \( u(t) \) of (1.1) which scatters in the sense that there exists solution \( v_\pm \) of the free wave equation
\[ \ddot{v} - \Delta v = 0 \]
with \((v_\pm(0), \dot{v}_\pm(0)) \in \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)\) such that
\[ \| (u(t), \dot{u}(t)) - (v_\pm(t), \dot{v}_\pm(t)) \|_{\dot{H}^1 \times L^2} \to 0, \quad \text{as} \quad t \to \pm \infty. \]

As we know, there is no pointwise criteria for the critical problem, GWP and scattering result are simultaneously solved in general. However, the study history of the \( \dot{H}^1 \)-critical wave equation shows us scattering result is later than global well-posedness.

Now, we recall the history of the \( \dot{H}^1 \)-critical wave equation (1.4) with \( \mu = 1 \). In [7,5,34], by the finite propagation speed of wave equation, they considered the Cauchy problem with compact data. And without loss of generality, one can assume the solution is smooth. They showed the existence of the global smooth solution by ruling out the accumulation of the energy at any time, where they utilized the classical finite speed of propagation (i.e. the monotonic local energy estimate on the light cone)
\[ \int_{|x| \leq R-t} e(t, x) dx \leq \int_{|x| \leq R} e(0, x) dx, \quad t > 0 \]
where
\[ e(t, x) = \frac{1}{2} |\dot{u}|^2 + \frac{1}{2} |\nabla u|^2 + \frac{1}{2^4} |u|^2, \]
which is a fundamental property for the wave equations with local nonlinearities. By compactness argument, one can show the global existence and uniqueness of the energy solution. While the scattering result of the energy solution was solved ten years later [1] by making use of the concentration compactness idea. It also depends heavily on the monotonic local energy estimate on the light cone.

However, the wave-Hartree lacks the classical finite speed of propagation. The non-local property of Hartree term cause the essential difficulties for nonlinear pointwise estimates, this defeats our attempts to establish the same classical finite speed of propagation as above. As a substitute, one may resort to the causality (Theorem 3 in Menzala-Strauss [19]), however it holds only for the case \( V \in L^{d/3} + L^\infty \), which does not contain the energy critical case \( V(x) = |x|^{-4} \), the exponent \( d/3 \) stems from the estimate of the term \( \int u_t u(V \ast u^2) dx \) as we know that this term cannot be controlled by the energy if \( V \in L^p(\mathbb{R}^d) \) when \( p < \frac{d}{4} \). To overcome it, we make use of the finite speed of propagation of the free operators \( K(t) \) and \( \dot{K}(t) \) and the boundness of the local-in-time Strichartz estimate of the solution (the nonlinear interaction is actually the linear feedback), to establish the causality for the energy critical case \( V \in L^{d/4} + L^\infty \). See the detail in Subsection 2.3.

Since the wave-Hartree lacks the classical finite speed of propagation, we will not utilize the classical methods in [7,8,33] to prove the GWP first and then scattering for the wave equation with local nonlinearity. While, inspired by the strategy derived from concentration compactness ideas [10,11,16], we will show GWP and scattering result simultaneously. We remark that the method here also works for the local nonlinearity \( f(u) = |u|^{4d-4} u \).

Other than the classical finite speed of propagation, it is also not easy to verify that the Hartree nonlinearity satisfies some positive properties, e.g. \( G(u) = f(u)\bar{u} - 2 \int_0^{[u]} f(r) dr \), and it plays an important role in establishing some Morawetz-type estimates in [29]. We overcome this difficulty by using the symmetry property of \( V(x) \) and also establish Morawetz-type estimate by borrowing some strategies from [31].

1.1. Outline of the proof of Theorem 1.1. For each \( E > 0 \), let us define \( \Lambda(E) \) to be the quantity
\[ \Lambda(E) = \sup \left\{ \|u\|_{S(I)} : E(u, u_t) \leq E \right\} \]
where \( u \) ranges over all solutions to (1.1) on the spacetime slab \( I \times \mathbb{R}^d \) of energy less than \( E \) and
\[ E_{\text{crit}} = \sup \{ E : \Lambda(E) < +\infty \}. \]

Our goal in the following is to prove that \( E_{\text{crit}} = +\infty \). We argue by contradiction. If \( E_{\text{crit}} < +\infty \), then we will see that the failure of Theorem 1.1 is caused by a special class of solutions: on the other hand, these solutions have so many good properties that they do not exist. Thus we get a contradiction. While we will make some further reductions
later, the main property of the special counterexamples is almost periodicity modulo symmetries:

**Definition 1.3.** Let $d \geq 5$, a solution $u$ to (1.1) with maximal lifespan $I$ is called almost periodic modulo symmetries if $(u, u_t)$ is bounded in $\dot{H}^1_x \times L^2_x$ and there exist functions $N(t) : I \to \mathbb{R}^+$, $x(t) : I \to \mathbb{R}^d$ and $C(\eta) : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $t \in I$ and $\eta > 0$,

$$\int_{|x-x(t)| \geq \frac{C(\eta)}{N(t)}} (|\nabla u(t, x)|^2 + |u_t(t, x)|^2) \, dx \leq \eta$$

and

$$\int_{|\xi| \geq C(\eta) N(t)} (|\xi|^2 |\hat{u}(t, \xi)|^2 + |\hat{u}_t(t, \xi)|^2) \, d\xi \leq \eta.$$

We refer to the function $N(t)$ as the frequency scale function for the solution $u$, to $x(t)$ as the spatial center function, and to $C(\eta)$ as the compactness modulus function.

**Remark 1.1.** By Ascoli-Arzela Theorem, $u$ is almost periodic modulo symmetries if and only if the set

$$\left\{ \left( N(t)^{-\frac{d-2}{2}} u, N(t)^{-\frac{d}{2}} u \right) \left( t, x(t) + \frac{x}{N(t)} \right), t \in I \right\}$$

falls in a compact set in $\dot{H}^1_x \times L^2_x$. The following are consequences of this statement. If $u$ is almost periodic modulo symmetries, then

$$\int_{|x-x(t)| \geq \frac{C(\eta)}{N(t)}} |u(t, x)|^2 \, dx \leq \eta.$$

We are now ready to state the first major milestone in the proof of Theorem 1.1.

**Theorem 1.2** (Reduction to almost periodic solutions, [11,12]). Assume $E_{\text{crit}} < +\infty$. Then there exists a maximal-lifespan solution $u : I \times \mathbb{R}^d \to \mathbb{R}$ to (1.1) such that

1. $u$ is almost periodic modulo symmetries;
2. $u$ blows up both forward and backward in time;
3. $u$ has the minimal kinetic energy among all blowup solutions. More precisely, let $v : J \times \mathbb{R}^d \to \mathbb{R}$ be a maximal-lifespan solution which blows up in at least one time direction, then

$$\sup_{t \in I} \left\| (u(t), u_t(t)) \right\|_{\dot{H}^1_x \times L^2_x} \leq \sup_{t \in J} \left\| (v(t), v_t(t)) \right\|_{\dot{H}^1_x \times L^2_x}.$$

The reduction to almost periodic solutions is now a standard technique in the analysis of dispersive equations at critical regularity. Their existence was first proved in the pioneering work by Karaani [13] for the mass-critical NLS. Kenig and Merle [10] adapted the argument to the energy-critical NLS, and first applied this to study the wellposedness problem.

So far, we do not have any control on the frequency scale function $N(t)$. However, the following theorem shows that no matter how small the set of minimal kinetic energy blowup solution is, we will inevitably encounter at least one of the following three enemies. Thus the proof of Theorem 1.1 is reduced to showing the nonexistence of the three scenarios.
Theorem 1.3 (Three enemies, [15,24]). Suppose \( d \geq 5 \) is such that Theorem 1.1 fails, that is, \( E_{\text{crit}} \not< +\infty \). Then there exists a maximal-lifespan solution \( u : I \times \mathbb{R}^d \rightarrow \mathbb{R} \), which is almost periodic modulo symmetries, and \( S_I(u) = +\infty \). Moreover, we can also ensure that the lifespan \( I \) and the frequency scale function \( N(t) : I \rightarrow \mathbb{R}^+ \) match one of the following three scenarios:

1. (Finite time blowup) \( |\inf_I I| < +\infty \), or \( \sup I < +\infty \).
2. (Soliton-like solution) \( I = \mathbb{R} \) and \( N(t) = 1 \) for all \( t \in \mathbb{R} \).
3. (Low-to-high cascade) \( I = \mathbb{R} \) and
   \[
   \lim_{t \to \infty} N(t) = +\infty.
   \]

The reference given above discusses the energy-critical NL S; however, the result follows from Theorem 1.2 by the same arguments since they are essentially combinatorial and so apply to any dispersive equation, such as wave equation [16]. We will utilize the Morawetz estimate, the extended causality and the potential energy concentration to preclude the above three enemies. And one can adopt the proof of Lemma 5.18 in [14] to prove a similar result for wave equation that the almost periodic solutions satisfy the following local constancy property:

Lemma 1.1 (Local constancy). Let \( u \) be an almost periodic solution to (1.1) on \( I \).

Then there exists \( \delta = \delta(u) \) such that for all \( t_0 \in I \),

\[
[t_0 - \delta N(t_0)^{-1}, t_0 + \delta N(t_0)^{-1}] \subset I
\]

and

\[
N(t) \sim N(t_0) \text{ uniformly for } t \in [t_0 - \delta N(t_0)^{-1}, t_0 + \delta N(t_0)^{-1}], \ t_0 \in I.
\]

We also need the following result in spirit of Lemma 5.21 in [14], which relates the frequency scale function of an almost periodic solution to its Strichartz norms. One may refer to [14] for the proof.

Lemma 1.2 (Spacetime bounds). Assume that \( u(t,x) \) is an almost periodic solution to (1.1) on a time interval \( J \). Then

\[
\| \nabla^\mu u \|_{L^q_t L^r_x(J \times \mathbb{R}^d)} \lesssim_u 1 + \int_J N(t) dt,
\]

where \( \mu \in [0,1] \), \( (q,r) \in \Lambda_1 \) and \( \Lambda_1 \) is defined by

\[
\Lambda_1 = \left\{ (q,r) : \frac{1}{q} + \frac{d}{r} = \frac{d}{2} - (1 - \mu), \ \frac{2}{q} \leq (d-1)(\frac{1}{2} - \frac{1}{r}), \ (q,r,d) \not= (2,+\infty,3) \right\}.
\]

The paper is organized as follows. In Section 2, we deal with the local theory for the equation (1.1) and the extended causality. In Section 3, we show the Morawetz estimate. We prove the potential energy concentration for the almost periodic solutions in Section 4. In Section 5, we preclude the global almost periodic solutions to (1.1) in the sense of Theorem 1.2. Finally in Section 6, we exclude the finite time blowup solutions to (1.1) in the sense of Theorem 1.3.
1.2. Notations. Finally, we conclude the introduction by giving some notations which will be used throughout this paper. To simplify the expression of our inequalities, we introduce some symbols \( \lesssim, \sim, \ll \). If \( X, Y \) are nonnegative quantities, we use \( X \lesssim Y \) or \( X = O(Y) \) to denote the estimate \( X \leq CY \) for some \( C \) which may depend on the critical energy \( E_{\text{crit}} \) but not on any parameter such as \( \eta \) and \( \rho \), and \( X \sim Y \) to denote the estimate \( X \lesssim Y \lesssim X \). We use \( X \ll Y \) to mean \( X \leq cY \) for some small constant \( c \) which is again allowed to depend on \( E_{\text{crit}} \). We will sometimes write \( a \) to denote \( a - \eta \) for arbitrarily small \( \eta > 0 \). We use \( C \gg 1 \) to denote various large finite constants, and \( 0 < c \ll 1 \) to denote various small constants. For any \( r, 1 \leq r \leq \infty \), we denote by \( \| \cdot \|_r \) the norm in \( L^r = L^r(\mathbb{R}^d) \) and by \( r' \) the conjugate exponent defined by \( \frac{1}{r} + \frac{1}{r'} = 1 \).

The Fourier transform on \( \mathbb{R}^d \) is defined by

\[
\hat{f}(\xi) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x) dx,
\]

giving rise to the fractional differentiation operators \( |\nabla|^s \) and \( \langle \nabla \rangle^s \), defined by

\[
|\nabla|^s f(\xi) := |\xi|^s \hat{f}(\xi), \quad \langle \nabla \rangle^s f(\xi) := \langle \xi \rangle^s \hat{f}(\xi),
\]

where \( \langle \xi \rangle := 1 + |\xi| \). This helps us to define the homogeneous and inhomogeneous Sobolev norms

\[
\|f\|_{H^s(\mathbb{R}^d)} := \||\xi|^s f\|_{L^2(\mathbb{R}^d)}, \quad \|f\|_{H^s(\mathbb{R}^d)} := \||\xi|^s f\|_{L^2(\mathbb{R}^d)}, \quad \|f\|_{\dot{W}^{s,p}(\mathbb{R}^d)} = \|\langle \nabla \rangle^s f\|_{L^p(\mathbb{R}^d)}.
\]

We will also need the Littlewood-Paley projection operators. Specifically, let \( \varphi(\xi) \) be a smooth bump function adapted to the ball \( |\xi| \leq 2 \) which equals 1 on the ball \( |\xi| \leq 1 \). For each dyadic number \( N \in 2^\mathbb{Z} \), we define the Littlewood-Paley operators

\[
\mathcal{P}_\leq N f(\xi) := \varphi(\frac{\xi}{N}) \hat{f}(\xi),
\]

\[
\mathcal{P}_> N f(\xi) := \left( 1 - \varphi(\frac{\xi}{N}) \right) \hat{f}(\xi),
\]

\[
\mathcal{P}_N f(\xi) := \left( \varphi(\frac{\xi}{N}) - \varphi(\frac{2\xi}{N}) \right) \hat{f}(\xi).
\]

Similarly we can define \( P_\leq N, P_\geq N, \) and \( P_{M \leq N} = P_{\leq N} - P_{\leq M} \), whenever \( M \) and \( N \) are dyadic numbers. We will frequently write \( f_{\leq N} \) for \( P_\leq N f \) and similarly for the other operators.

The Littlewood-Paley operators commute with derivative operators, the free propagator, and the conjugation operation. They are self-adjoint and bounded on every \( L^p_s \) and \( \dot{H}^s_s \) space for \( 1 \leq p \leq \infty \) and \( s \geq 0 \), moreover, they also obey the following Bernstein estimates

\[
\|\langle \nabla \rangle^s P_\leq N f\|_{L^p} \lesssim N^s \|P_\leq N f\|_{L^p},
\]

\[
\|P_N f\|_{L^q} \lesssim N^{\frac{n}{p} - \frac{s}{q}} \|P_N f\|_{L^p},
\]

where \( s \geq 0 \) and \( 1 \leq p \leq q \leq \infty \).
2. Preliminaries

2.1. The Strichartz estimates. In this section, we consider the Cauchy problem for the equation (1.1)

\[
\begin{cases}
\ddot{u} - \Delta u + f(u) = 0, \\
u(0) = u_0, \quad \dot{u}(0) = u_1.
\end{cases}
\]

The integral equation for the Cauchy problem (2.1) can be written as

\[
u(t) = K(t)u_0 + K(t)u_1 - \int_0^t K(t-s)f(u(s))ds,
\]

or

\[
\left( \begin{array}{c} u(t) \\ \dot{u}(t) \end{array} \right) = V_0(t) \left( \begin{array}{c} u_0(x) \\ u_1(x) \end{array} \right) - \int_0^t V_0(t-s) \left( \begin{array}{c} 0 \\ f(u(s)) \end{array} \right) ds,
\]

where

\[
V_0(t) = \left( \begin{array}{c} K(t), K(t) \\ \dot{K}(t), \ddot{K}(t) \end{array} \right), \quad K(t) = \frac{\sin(t\omega)}{\omega}, \quad \omega = (\Delta)^{1/2}.
\]

The Strichartz estimates involve the following definitions:

Definition 2.1 (Admissible pairs). A pair of Lebesgue space exponents \((q, r)\) are called wave admissible for \(\mathbb{R}^{1+d}\), or denote by \((q, r) \in \Lambda_0\) when \(q, r \geq 2\), and

\[
\frac{2}{q} \leq (d-1)\left(\frac{1}{2} - \frac{1}{r}\right), \quad \text{and} \quad (q, r, d) \neq (2, \infty, 3).
\]

For a fixed spacetime slab \(I \times \mathbb{R}^d\), we define the Strichartz norm

\[
\|u\|_{S^1(I)} := \sup \|\nabla |t|^\mu u\|_{L^q_t L^r_x(I \times \mathbb{R}^d)},
\]

where the supremum is taken over all admissible pairs \((q, r) \in \Lambda_0\) and numbers \(\mu \in [0, 1]\) obeying the scaling condition \(\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - (1 - \mu)\). We denote \(S^1(I)\) to be the closure of all test functions under this norm.

Now we recall the following Strichartz estimates.

Lemma 2.1 (Strichartz estimates, [5, 9, 18, 20]). Fix \(d \geq 5\). Let \(I\) be a compact time interval and let \(u : I \times \mathbb{R}^d \to \mathbb{R}\) be a solution to the forced wave equation

\[
u_{tt} - \Delta u + F_1 + F_2 = 0.
\]

Then for any \(t_0 \in I\),

\[
\|u\|_{S^1(I)} + \|
\partial_t u\|_{L^p_t L^q_x(I \times \mathbb{R}^d)} \lesssim \|\|(u(t_0), \partial_t u(t_0))\|_{H^1_t L^2_x} + \|\nabla |t|^\frac{d}{2} F_1\|_{L^2_{t,x}(I \times \mathbb{R}^d)} + \|F_2\|_{L^2_t L^2_x(I \times \mathbb{R}^d)}.
\]

Now we give a few basic estimates.

Lemma 2.2 (Product rule [3]). Let \(s \geq 0\), and \(1 < r, p_j, q_j < \infty\) such that \(\frac{1}{r} = \frac{1}{p_i} + \frac{1}{q_i} (i = 1, 2)\). Then,

\[
\|\nabla |s (fg)|\|_{L^q_x(\mathbb{R}^d)} \lesssim \|f\|_{L^{p_1}_x(\mathbb{R}^d)} \|\nabla |s g|\|_{L^{q_1}_x(\mathbb{R}^d)} + \|\nabla |s f|\|_{L^{p_2}_x(\mathbb{R}^d)} \|g\|_{L^{q_2}_x(\mathbb{R}^d)}.
\]

This together with Hardy-Littlewood-Sobolev inequality yields the following nonlinear estimate.
Lemma 2.3 (Nonlinear estimate). For $d \geq 5$, we have

$$
\| \nabla \frac{1}{2} \left( (|x|^{-4} \ast (uv)) u \right) \|_{L_{t,x}^{2(d+1)}(I \times \mathbb{R}^d)} + \| \nabla \frac{1}{2} \left( (|x|^{-4} \ast |u|^2) v \right) \|_{L_{t,x}^{2(d+1)}(I \times \mathbb{R}^d)} \lesssim \|u\|_{L_{X(I)}^2}^2 \|v\|_{L_{X(I)}^2},
$$

where $X(I)$ is defined to be

$$
X(I) = L_{t}^{d+1} L_{x}^{d^2-d-4} \left( \bigcap_{I} L_{t}^{2(d+1)} W_{2, \frac{1}{2} \cdot 2(d+1)-1} (I \times \mathbb{R}^d) \right).
$$

Proof. By Lemma 2.2 and Hardy-Littlewood-Sobolev inequality, we obtain

$$
\| \nabla \frac{1}{2} \left( (|x|^{-4} \ast (uv)) u \right) \|_{L_{t,x}^{2(d+1)}(I \times \mathbb{R}^d)} \lesssim \| |x|^{-4} \ast (uv) \|_{L_{t,x}^{2(d+1)}} \| \nabla \frac{1}{2} u \|_{L_{t,x}^{2(d+1)}} + \| \nabla \frac{1}{2} \left( (|x|^{-4} \ast (uv)) v \right) \|_{L_{t,x}^{2(d+1)}} \| u \|_{L_{t,x}^{2(d+1)}} \lesssim \|u\|_{L_{X(I)}^2}^2 \|v\|_{L_{X(I)}^2}.
$$

Similarly, we can estimate another term.

\[\Box\]

2.2. Stability. Closely related to the continuous dependence on the data, an essential tool for concentration compactness arguments is the stability theory. More precisely, given an approximate equation

$$
\tilde{u}_t - \Delta \tilde{u} = -f(\tilde{u}) + e
$$

to (1.1), with $e$ small in a suitable space and $(\tilde{u}_0, \tilde{u}_t)$ is close to $(u_0, u_t)$ in energy space, it is possible to show that solution $u$ to (1.1) stays very close to $\tilde{u}$. Note that the question of continuous dependence on the data corresponds to the case $e = 0$.

The following lemma for the nonlinear wave-Hartree equation is analogous to the nonlinear Schrödinger equation in [3]. For convenience, we state the lemma and sketch its proof.

Lemma 2.4 (Stability). Let $I$ be a time interval, and let $\tilde{u}$ be a function on $I \times \mathbb{R}^d$ which is a near-solution to (1.1) in the sense that

$$
\tilde{u}_t - \Delta \tilde{u} = -f(\tilde{u}) + e
$$

for some function $e$. Assume that

$$
\| \tilde{u} \|_{L_{t}^\infty(I; H_{x}^1(\mathbb{R}^d))} + \| \partial_t \tilde{u} \|_{L_{t}^\infty(I; L_{x}^2(\mathbb{R}^d))} \leq M,
$$

$$
\|u(t_0) - \tilde{u}_0(t_0), u_t(t_0) - \tilde{u}_t(t_0)\|_{H^1 \times L^2} \leq \epsilon
$$

for some constant $M, E > 0$, where $S(I)$ is defined in (1.6). Let $t_0 \in I$, and let $(u(t_0), u_t(t_0)) \in H^1 \times L^2$ be close to $(\tilde{u}(t_0), \tilde{u}_t(t_0))$ in the sense that

$$
\| (u(t_0) - \tilde{u}(t_0), u_t(t_0) - \tilde{u}_t(t_0)) \|_{H^1 \times L^2} \leq \epsilon
$$

for some constant $M, E > 0$, where $S(I)$ is defined in (1.6). Let $t_0 \in I$, and let $(u(t_0), u_t(t_0)) \in H^1 \times L^2$ be close to $(\tilde{u}(t_0), \tilde{u}_t(t_0))$ in the sense that

$$
\|u(t_0) - \tilde{u}_0(t_0), u_t(t_0) - \tilde{u}_t(t_0)\|_{H^1 \times L^2} \leq \epsilon
$$

for some constant $M, E > 0$, where $S(I)$ is defined in (1.6). Let $t_0 \in I$, and let $(u(t_0), u_t(t_0)) \in H^1 \times L^2$ be close to $(\tilde{u}(t_0), \tilde{u}_t(t_0))$ in the sense that

$$
\|u(t_0) - \tilde{u}_0(t_0), u_t(t_0) - \tilde{u}_t(t_0)\|_{H^1 \times L^2} \leq \epsilon
and assume also that the error term obeys

$$\|\nabla \frac{1}{2} e\|_{L^{2(d+1)}(I \times \mathbb{R}^d)} \leq \epsilon$$

(2.12)

for some small $0 < \epsilon < \epsilon_1 = \epsilon_1(M,E)$. Then, we conclude that there exists a solution $u : I \times \mathbb{R}^d \to \mathbb{R}$ with initial data $(u(t_0), u_t(t_0))$ at $t = t_0$, and furthermore

$$\|u - \tilde{u}\|_{S^1(I)} \leq C(M,E),$$

(2.13)

$$\|u\|_{S^1(I)} \leq C(M,E),$$

$$\|u - \tilde{u}\|_{S^1(I)} \leq C(M,E)\epsilon,$$

where $c$ is a positive constant that depends on $d$, $M$ and $E$, and $S^1(I)$ is defined in (2.5).

**Proof.** Since $\|\tilde{u}\|_{S^1(I)} \leq M$, we may subdivide $I$ into $C(M,\epsilon_0)$ time intervals $I_j$ such that

$$\|\tilde{u}\|_{S^1(I_j)} \leq \epsilon_0 \ll 1, \quad 1 \leq j \leq C(M,\epsilon_0).$$

By the Strichartz estimate and standard bootstrap argument, we have

$$\|\tilde{u}\|_{S^1(I_j)} \leq C(E), \quad 1 \leq j \leq C(M,\epsilon_0).$$

Summing up over all the intervals, we obtain that

$$\|\tilde{u}\|_{S^1(I)} \leq C(E).$$

(2.14)

In particular, we have

$$\|\tilde{u}\|_{X(I)} \leq C(E),$$

(2.15)

where $X(I)$ is defined in (2.7). This implies that there exists a partition of the right half of $I$ at $t_0$:

$$t_0 < t_1 < \cdots < t_N, \quad I_j = (t_j, t_{j+1}), \quad I \cap (t_0, \infty) = (t_0, t_N),$$

such that $N \leq C(L,\delta)$ and for any $j = 0, 1, \cdots, N - 1$, we have

$$\|\tilde{u}\|_{X(I_j)} \leq \delta \ll 1.$$

(2.16)

The estimate on the left half of $I$ at $t_0$ is analogue, we omit it.

Let

$$\gamma(t) = u(t) - \tilde{u}(t),$$

(2.17)

and

$$\left(\begin{array}{c}
\gamma_j(t) \\
\gamma_j(t)
\end{array}\right) = V_0(t - t_j) \left(\begin{array}{c}
\gamma(t_j) \\
\gamma(t_j)
\end{array}\right), \quad 0 \leq j \leq N - 1,$$

(2.18)

then $\gamma$ satisfies the following difference equation

$$\begin{cases}
(\partial_u - \Delta)\gamma = - (V \ast |\tilde{u}|^2)\gamma - 2 (V \ast (\gamma \tilde{u})) \tilde{u} - 2 (V \ast (\gamma \tilde{u})) \gamma \\
- (V \ast |\gamma|^2) \tilde{u} - (V \ast |\gamma|^2) \gamma - e \\
\gamma(t_j) = \gamma_j(t_j), \quad \gamma(t_j) = \gamma_j(t_j),
\end{cases}$$

where $c$ is a positive constant that depends on $d$, $M$ and $E$, and $S^1(I)$ is defined in (2.5).
which implies that

\[
\begin{align*}
(\gamma(t)) &= (\gamma_j(t)) + \int_{\gamma_j}^t \mathbf{V}_0(t-s)\left(\begin{array}{c} 0 \\ e\rho(\gamma)(s) \end{array} \right) ds, \\
(\dot{\gamma}_j(t)) &= \int_{t_j}^{t_{j+1}} \mathbf{V}_0(t-s)\left(\begin{array}{c} 0 \\ e\rho(\gamma)(s) \end{array} \right) ds.
\end{align*}
\]

It follows from Lemma 2.1 and Lemma 2.3 that

\[
\|\gamma - \gamma_j\|_{X(t_j)} + \|\gamma_{j+1} - \gamma_j\|_{X(t)} \lesssim \sum_{k=1}^{3} \|\gamma\|_{X(t_j)}^{k-3} \|\tilde{u}\|_{X(t_j)}^{3-k} + \|\nabla|e|e^2 \|_{2(d+1)}^{(d+1)/3} (I_j \times \mathbb{R}^d)
\]

(2.19)

Therefore, assuming that

\[
\|\gamma\|_{X(t_j)} \leq \delta \ll 1, \quad \forall \ j = 0, 1, \ldots, N - 1,
\]

then by (2.16) and (2.19), we have

\[
\|\gamma\|_{X(t_j)} + \|\gamma_{j+1}\|_{X(t_{j+1}, t_N)} \leq C\|\gamma_j\|_{X(t_j, t_N)} + \epsilon,
\]

(2.21)

for some absolute constant \( C > 0 \). By (2.12) and iteration on \( j \), we obtain

\[
\|\gamma\|_{X(t)} \leq (2C)^N \epsilon \leq \frac{\delta}{2},
\]

(2.22)

if we choose \( \epsilon_1 \) sufficiently small. Hence the assumption (2.20) is justified by continuity in \( t \) and induction on \( j \). Then repeating the estimate (2.19) once again, we can get the other Strichartz estimates on \( u \).

Using the above lemma as well as its proof, one easily derives the following local well-posedness theorem for (1.1).

**Theorem 2.1 (Local well-posedness).** Assume that \( d \geq 5 \). Then, given \( (u_0, u_1) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \) and \( t_0 \in \mathbb{R} \), there exists a unique maximal-lifespan solution \( u : I \times \mathbb{R}^d \to \mathbb{R} \) with initial data \( (u(t_0), u_r(t_0)) = (u_0, u_1) \). This solution also has the following properties:

1. (Local existence) \( I \) is an open neighborhood of \( t_0 \).
2. (Blowup criterion) If \( \inf(I) \) is finite, then \( u \) blows up forward in time in the sense of Definition 1.2. If \( \inf(I) \) is finite, then \( u \) blows up backward in time.
3. (Scattering) If \( \sup(I) = +\infty \) and \( u \) does not blow up forward in time, then \( u \) scatters forward in time in the sense of Definition 1.9. Conversely, given \( (v_+, \dot{v}_+) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \), there is a unique solution to (1.1) in a neighborhood of infinity so that (1.9) holds.
4. There exists \( \delta = \delta(d, \|(u_0, u_1)\|_{H^1 \times L^2}) \) such that if

\[
\|K(t-t_0)u_0 + K(t-t_0)u_1\|_{S(t)} < \delta,
\]
then, there exists a unique solution \( u : I \times \mathbb{R}^d \to \mathbb{R} \) to (1.1), with \((u, \dot{u}) \in C(I; \dot{H}^1 \times L^2)\), and
\[
\|u\|_{S(I)} \leq 2\delta, \quad \|u\|_{\dot{S}(I)} < +\infty.
\]

(5) **Small data global existence** If \( \|(u_0, u_1)\|_{\dot{H}^1 \times L^2} \) is sufficiently small (depending on \( d \)), then \( u \) is a global solution which does not blow up either forward or backward in time. Indeed, in this case
\[
S_{\mathbb{R}}(u) \lesssim \|(u_0, u_1)\|_{\dot{H}^1 \times L^2}.
\]

### 2.3. The Extended Causality

In this subsection, we show a kind of the finite speed of propagation named as causality to control the spatial center function \( x(t) \), which extends the result in Menzala-Strauss [19]. As stated in the introduction, for the wave-Hartree equation, we cannot show the monotone local energy estimate on the light cone. And as a substitute, Menzala-Strauss [19] show a kind of the finite speed of propagation named as causality for the case \( V \in L^\frac{d}{d-4} + L^\infty \).

In fact, the causality can be improved, this relies on two important observations: one point is that the linear operators \( K(t) \) and \( \dot{K}(t) \) still enjoy the finite speed of propagation, the other point is that the Hartree term acted by the cut-off function can be viewed as the linear feedback of the cutoff solution in the cut-off Duhamel formulae (see (2.25)) due to the short-time Strichartz-norm boundness of the solution. The former allows the cutoff function to go cross the linear operators \( K(t) \) and \( \dot{K}(t) \) and act directly on data and nonlinearity, while the latter suggests us to iterate the solution just as the Gronwall inequality not as the bootstrap argument. Based on the above discussions, we can extend the exponent range of the causality to energy critical case \( V \in L^\frac{d}{d-4} + L^\infty \).

**Lemma 2.5 (Extended Causality).** Assume that the data \((u_0, u_1) \in \dot{H}^1 \times L^2\) have the compact support, i.e
\[
\text{supp } u_0, \quad \text{supp } u_1 \subset \{ x \in \mathbb{R}^d : |x| \leq R \}
\]
for some constant \( R > 0 \) and \((u(t), \dot{u}(t)) \in C([0, T_+(u_0, u_1)), \dot{H}^1 \times L^2)\) is the finite energy solution of the equation (1.1) with initial data \((u_0, u_1)\). Then it holds that
\[
u(t, x) = 0, \quad a.e. \quad x \in \{ x \in \mathbb{R}^d : |x| > R + t \}, \quad \forall \ t \in [0, T_+(u_0, u_1)).
\]

**Proof.** Let \[
\chi_t(x) = \begin{cases} 
1, & |x| > R + t, \\
0, & |x| \leq R + t. 
\end{cases}
\]

From Duhamel’s formula (2.24) and the finite speed of propagation for the linear operators \( K(t) \) and \( \dot{K}(t) \), one has
\[
\chi_t(x) u(t) = \chi_t(x) \dot{K}(t) u_0 + \chi_t(x) K(t) u_1 - \chi_t(x) \int_0^t K(t - s) f(u(s)) ds
\]
\[
= -\chi_t(x) \int_0^t K(t - s)(|x|^{-4} * |u(s)|^2) \chi_s(x) u(s) ds.
\]
For compact \([0, T] \subset \subset [0, T_+ (u_0, u_1))\), by the Strichartz estimate, we have
\[
\left\| \chi_t(x) u(t) \right\|_{L^6_t([0,T],L^\infty_x)} \lesssim \left\| \left( |x|^{-4} * |u|^2 \right) \chi_t(x) u \right\|_{L^1_t([0,T],L^2_x)} \lesssim \|u\|^2 \left\| \chi_t(x) u \right\|_{L^6_t([0,T],L^\infty_x)}.
\]

For \([0, T] \subset \subset [0, T_+ (u_0, u_1))\), we can divide the interval \([0, T]\) into \([0, T] = \bigcup_{j=1}^J I_j\), such that
\[
\left\| u \right\|_{L^6_t(I_j, L^\infty_x)} \leq \eta,
\]
where \(\eta\) is a small positive constant. And so
\[
\left\| \chi_t(x) u(t) \right\|_{L^6_t(I_j, L^\infty_x)} \leq C \eta \left\| \chi_t(x) u(t) \right\|_{L^6_t(I_j, L^\infty_x)},
\]
thus
\[
\left\| \chi_t(x) u(t) \right\|_{L^6_t(I_j, L^\infty_x)} = 0.
\]
Summing up over all the intervals, we obtain that
\[
(2.26) \quad \left\| \chi_t(x) u(t) \right\|_{L^6_t([0,T],L^\infty_x)} = 0.
\]

On the other hand, from the Strichartz estimates, we get
\[
\left\| \chi_t(x) u(t) \right\|_{L^6_t([0,T],H^1)} \lesssim \|u\|^2 \left\| \chi_t(x) u \right\|_{L^6_t([0,T],L^\infty_x)} \left\| \chi_t(x) u \right\|_{L^6_t([0,T],L^\infty_x)} = 0.
\]
This together with (2.26) yields that
\[
\chi_t(x) u(t) \equiv 0, \text{ a.e. } x \in \mathbb{R}^d.
\]
Therefore
\[
u(t) \equiv 0, \text{ a.e. } x \in \{x : |x| > R + |t|\}.
\]
\[
\blacksquare
\]

3. Morawetz-type Estimate

In this section, our task is to establish a useful Morawetz estimate by choosing a suitable multiplier, which plays an important role in excluding the almost periodic solutions to (1.1). Noting that the nonlinearity of (1.1) is a convolution term, we need use the quantity \(|u\theta|^2/r + |u|^2/r^3\) to estimate \(|u|^2\) inspired by Nakanishi [31]. Compared with the local nonlinearity, we explore a certain symmetry in nonlinearity to get a positive integral quantity (this helps us to weak the requirement that the integrand is positive) to deal with the nonlocal nonlinearity. The embedding theorem for polar coordinates given in [31] is a bridge connecting the whole space \(\mathbb{R}^d\) with spherical surface \(S^{d-1}\) in the proof.

**Proposition 3.1 (Morawetz estimate).** Let \(u\) be a solution to (1.1) on a spacetime slab \(I \times \mathbb{R}^d\). then we have
\[
(3.1) \quad \int_I \int_{\mathbb{R}^d} \frac{|u|^{2^*}}{|x|} \, dx \, dt \leq C(E),
\]
where \(2^* = \frac{2d}{d-2}\) and \(E\) is the energy \(E(u_0, u_1)\).
Proof. Let \( \psi = u_r + \frac{(d-1)u}{2|x|} \), then we have
\[
\text{Re}\{((\partial_t - \Delta)u + (V*|u|^2)u)\psi\} = \text{Re}\partial_t(\bar{u}\psi) + \text{Re}\nabla \cdot \{ - (\nabla u\bar{\psi}) + \theta \ell(u) + \frac{1}{2}|u|^2
abla (\frac{(d-1)}{4|x|}) \}
+ |u|^2 + \frac{(d-1)(d-3)|u|^2}{4r^3} - \frac{1}{2}\theta \cdot \nabla (V*|u|^2)|u|^2,
\]
where
\[
\ell(u) = \frac{1}{2} (-|\dot{u}|^2 + |\nabla u|^2 + (V*|u|^2)|u|^2),
\]
\[
r = |x|, \quad \theta = \frac{x}{|x|},
\]
\[u_r = \theta \cdot \nabla u, \quad u_\theta = \nabla u - \theta u_r.\]

Integrating the above equality with respect to \((t, x)\) over
\[W = \{(t, x) | 0 < a \leq t \leq b, \ x \in \mathbb{R}^d\},\]
we obtain that
\[
\int_W \left[ \frac{|u|^2}{r} + \frac{(d-1)(d-3)|u|^2}{4r^3} - \frac{1}{2}|u|^2 \frac{x}{|x|} \cdot \nabla (V*|u|^2) \right] dx dt \leq C \int_{\mathbb{R}^d} \bar{u} \psi dx |_{t=b} - |_{t=a}.
\]
Since
\[- \left( \frac{x}{|x|} - \frac{y}{|y|} \right) \nabla V(x - y) = 4 \frac{|x||y| - x \cdot y}{|x - y|^6} \left( \frac{1}{|x|} + \frac{1}{|y|} \right) \geq 0,
\]
we have
\[- \int_a^b \int_{\mathbb{R}^d} \left( \frac{x}{|x|} - \frac{y}{|y|} \right) \nabla V(x - y)|u(x)|^2|u(y)|^2 dy dx dt \geq 0.
\]
This implies that
\[
- \frac{1}{2} \int_W |u|^2 \frac{x}{|x|} \cdot \nabla (V*|u|^2) dx dt \geq 0.
\]
Substituting (3.3) into (3.2) and making use of the Hardy inequality, we obtain that
\[
\int_W \left[ \frac{|u|^2}{r} + \frac{(d-1)(d-3)|u|^2}{4r^3} \right] dx dt \leq C \int_{\mathbb{R}^d} \bar{u} \psi dx |_{t=b} - |_{t=a}
\leq C (||\dot{u}||_2^2 + ||\nabla u||_2^2)
\leq CE.
\]

On the other hand, we have
\[
\int_{\mathbb{R}^d} \frac{|u|^2}{r} dx = \int_0^\infty r^{-1} \int_{S^{d-1}} |u(r\theta)|^2 d\theta r^{d-1} dr = \int_0^\infty r^{d-2} ||u||_L^2(r) dx \leq \int_0^\infty r^{d-2} ||u||_L^2(r) dx.
\]
Note that the following interpolation and the Sobolev embedding
\[
[H^1(S^{d-1}), L^p(S^{d-1})]_{\frac{d}{d-\frac{2}{d}}} = H^\sigma(S^{d-1}) \hookrightarrow L^q(S^{d-1}),
\]
where
\[
\begin{align*}
\beta &= \frac{2(d-1)}{d-2}, \quad \sigma = \frac{d-2}{d}, \\
\frac{1}{q} &= \frac{1}{d-\frac{2}{d}} + \frac{1}{q} = \left(1 - \frac{2}{d}\right) \frac{1}{d} + \frac{2}{d},
\end{align*}
\]
it follows that
\begin{equation}
\int_{\mathbb{R}^d} \frac{|u|^2}{r} \, dx \leq \int_0^\infty r^{d-2} \|u\|_{H_{\theta}^1(S^{d-1})}^2 \|u\|_{L_{\theta}^r(S^{d-1})} \, dr.
\end{equation}

Next we need the following Sobolev embedding for the polar coordinates given by Proposition 3.7 in [31].

Lemma 3.1. Let $1 \leq p < d$. Then
\begin{equation}
\dot{W}^{1,p} \hookrightarrow L_{\theta}^{\frac{p}{d-p}} L_r^{\infty, \nu} \hookrightarrow L_{\theta}^{\infty, \nu} L_r^\beta,
\end{equation}
where \( \beta = \frac{(d-1)p}{d-p} \), \( \nu = \frac{d-p}{p} \) and \( L_{\theta}^{\infty, \nu} = \{ u : \|u\|_{L_{\theta}^{\infty, \nu}} = \|r^p u(r)\|_{L_{\theta}^{\infty}} < \infty \} \).

Applying this lemma with \( p = 2 \) to (3.7), we have
\begin{equation}
\int_{\mathbb{R}^d} \frac{|u|^2}{r} \, dx \leq \|r^2 u^\theta\|_{L_r^\beta}^4 \int_0^\infty r^{-3} \|u\|_{H_{\theta}^1(S^{d-1})}^2 \, dr
\leq \|\nabla u\|_{L_2^d}^4 \int_{\mathbb{R}^d} \left( \frac{|u|^2}{r^3} + \frac{|u_\theta|^2}{r} \right) \, dx.
\end{equation}
Integrating (3.9) with respect to time \( t \) and using (3.4), we deduce that
\begin{equation}
\int_a^b \int_{\mathbb{R}^d} \frac{|u|^2}{r} \, dx \, dt \leq C(E).
\end{equation}
This concludes Proposition 3.1.

4. The potential energy concentration

In this section, we will prove the potential energy concentration for the almost periodic solutions. The analog for the energy super-critical wave equation was originally shown by Killip-Visan [16]. Here we give a sketch of the proof.

Proposition 4.1 (potential energy concentration). Assume that \( u(t, x) \) is an almost periodic solution to (1.1) with maximal lifespan \( I \). Then there exists a constant \( C = C(u) \) such that
\begin{equation}
\int_I \int_{|x - x(t)| \leq C/N(t)} |u(t, x)|^{\frac{2d}{d-2}} \, dx \, dt \gtrsim \frac{1}{|J|} |J|
\end{equation}
uniformly for all intervals \( J = [t_1, t_2] \subset I \) with \( t_2 \geq t_1 + N(t_1)^{-1} \).

Before we prove the above lemma, we recall a lemma in [16]. It says that although it is not possible to obtain the lower bounds on the norm of \( u(t) \) for a single time \( t \) as the nonlinear wave equation relies on two independent initial data, this phenomenon must be rare as follows.

Lemma 4.1 \( (L_{\theta}^{\frac{d}{d-2}}) \)-norm nontrivially, [16]. Let \( u(t, x) \) be an almost periodic solution to (1.1) on a time interval \( I \). Then, for any \( A > 0 \), there exists \( \eta = \eta(u, A) > 0 \) such that
\begin{equation}
|\{ t \in [t_0, t_0 + AN(t_0)^{-1}] \cap I : \|u(t)\|_{L_{\theta}^{\frac{d}{d-2}}} \geq \eta \}| \geq \eta N(t_0)^{-1}.
\end{equation}
The proof of Proposition 4.1} In view of Lemma 4.1 it suffices to show (4.1) for intervals of the form \([t_0, t_0 + \delta N(t_0)^{-1}]\) for some small fixed \(\delta > 0\). We denote \(J(t_0) := [t_0, t_0 + \delta N(t_0)^{-1}]\).

Furthermore, (4.3) can be reduced to prove that there exists \(C = C(u)\) such that

\[
\int_{J(t_0)} \int_{|x-x(t)| \leq C/N(t)} |u|^{2d} dx dt \gtrsim_u N(t_0)^{-1} \lesssim_u |J(t_0)|,
\]

since \(J(t_0) = [t_0, t_0 + \delta N(t_0)^{-1}]\). On the other hand, it follows from Bernstein’s inequality that

\[
\int_{\mathbb{R}^d} \left| \nabla \frac{d}{2d} P_{\leq C N(t)} \left( \chi_{|x-x(t)| \leq C/N(t)} u \right) \right|^{\frac{2d}{d-2}} dx \lesssim N(t) \int_{|x-x(t)| \leq C/N(t)} |u|^{\frac{2d}{d-2}} dx,
\]

where \(\chi_{\Omega}\) is the smooth cutoff function associated to the bounded set \(\Omega\). Therefore, (4.3) can be further reduced to show

\[
\int_{J(t_0)} \int_{\mathbb{R}^d} \left| \nabla \frac{d}{2d} P_{\leq C N(t)} \left( \chi_{|x-x(t)| \leq C/N(t)} u \right) \right|^{\frac{2d}{d-2}} dx dt \gtrsim u 1.
\]

First, by Sobolev embedding and Lemma 4.1, we get

\[
\left\{ t \in J(t_0) : \int_{\mathbb{R}^d} \left| \nabla \frac{d}{2d} u(t, x) \right|^r dx \gtrsim_u 1 \right\} \gtrsim_u N(t_0)^{-1}, \quad r = \frac{2d^2}{(d+1)(d-2)},
\]

and we denote the above set by \(I_0 \subset J(t_0)\). From (1.12) and Sobolev embedding, we know that for any \(\epsilon > 0\), there exists \(C = C(\epsilon)\) such that

\[
\int_{|x-x(t)| \geq C(N(t))} \left| \nabla \frac{d}{2d} u(t, x) \right|^r dx < \epsilon.
\]

This together with (4.3) yields that

\[
\int_{|x-x(t)| \leq C(N(t))} \left| \nabla \frac{d}{2d} u(t, x) \right|^r dx \gtrsim_u 1, \quad \forall \ t \in I_0.
\]

For \(t \in I_0\), we have by Hölder’s inequality

\[
1 \lesssim_u \left( \int_{|x-x(t)| \leq C(N(t))} \left| \nabla \frac{d}{2d} u(t, x) \right|^r dx \right)^{\frac{d-1}{d}} \lesssim N(t)^{-1} \int_{|x-x(t)| \leq C(N(t))} \left| \nabla \frac{d}{2d} u(t, x) \right|^{\frac{2d}{d-2}} dx
\]

which implies that

\[
N(t) \lesssim_u \int_{\mathbb{R}^d} \left| \nabla \frac{d}{2d} u(t, x) \right|^{\frac{2d}{d-2}} dx, \quad \forall \ t \in I_0.
\]

Integrating the above inequality in \(t \in I_0\) and using Lemma 4.2 we obtain

\[
1 \lesssim \int_{I_0} N(t) dt \lesssim \int_{J(t_0)} \int_{\mathbb{R}^d} \left| \nabla \frac{d}{2d} u(t, x) \right|^{\frac{2d}{d-2}} dx dt \lesssim 1 + \int_{J(t_0)} N(t) dt \lesssim u 1.
\]

Hence, by Strichartz estimate and the standard bootstrap argument, one has

\[
\left\| \nabla \frac{d}{2d} u \right\|_{L^4_t L^{\frac{2d}{d-2}}(J(t_0) \times \mathbb{R}^d)} \lesssim 1.
\]
On the other hand, it follows from \( (1.14) \) and Sobolev embedding that

\[
\sup_{t \in J(t_0)} \left\{ \int_{|x-x(t)| \geq CN(t)} |u|^\frac{2d}{d-2} dx + \int_{\mathbb{R}^d} |P_{G_{CN(t)}} u|^{\frac{2d}{d-2}} dx \right\} \leq \epsilon.
\]

Interpolating this with \( (1.11) \), we estimate

\[
\int_{J(t_0)} \int_{\mathbb{R}^d} \left[ \| \nabla \frac{d}{2} P \geq CN(t) u \|^{\frac{2d}{d-2}} + \| \nabla \frac{d}{2} (\chi_{|x-x(t)| \geq CN(t)} u) \|^{\frac{2d}{d-2}} \right] dx dt \leq \epsilon^\theta.
\]

Combining with the lower bound in \( (4.10) \), we get

\[
(4.12) \quad \int_{J(t_0)} \int_{\mathbb{R}^d} \| \nabla \frac{d}{2} P \leq CN(t) (\chi_{|x-x(t)| \leq CN(t)} u) \|^{\frac{2d}{d-2}} dx dt \gtrsim 1,
\]

and so \( (4.10) \) follows. Thus, we conclude the proof of Lemma 4.1.

5. The soliton-like solution and frequency cascade solution

In this section, we will use the Morawetz estimate and the potential energy concentration established in the above section to preclude the soliton-like solution and low-to-high cascade solution in the sense of Theorem 1.3. More precisely, we will prove

**Theorem 5.1.** There are no global solutions to \( (1.1) \) that are soliton-like or low-to-high in the sense of Theorem 1.3.

**Proof.** It follows from Proposition 4.1 that there exists a constant \( C = C(u) \) such that

\[
(5.1) \quad \int_{J} \int_{|x-x(t)| \leq CN(t)} |u(t,x)|^{\frac{2d}{d-2}} dx dt \gtrsim_u |J|
\]

uniformly for all intervals \( J = [t_1, t_2] \subset \mathbb{R} \) with \( t_2 \geq t_1 + N(t_1)^{-1} \).

On the other hand, by the extend causality in Lemma 2.5 and \( (1.14) \), we have the following control on the spatial center function \( x(t) \)

\[ |x(t_1) - x(t_2)| \leq |t_1 - t_2| + CN(t_1)^{-1} + CN(t_2)^{-1}, \forall t_1, t_2 \in \mathbb{R}. \]

The similar result for the energy-supercritical wave equation can be found in [10]. Translating space so that \( x(0) = 0 \), we obtain by the Morawetz estimate \( (3.1) \), \( N(t) \geq 1 \) and \( (5.1) \), for \( T \geq 1 \),

\[
C(E) \geq \int_{0}^{T} \int_{\mathbb{R}^d} \frac{|u(t,x)|^{\frac{2d}{d-2}}}{|x|} dx dt
\]

\[
\geq \int_{0}^{T} \int_{|x-x(t)| \leq \frac{C}{N(t)}} \frac{|u(t,x)|^{\frac{2d}{d-2}}}{|x|} dx dt
\]

\[
\gtrsim_u \sum_{k=0}^{[T-1]} \int_{k+1}^{k} \int_{|x-x(t)| \leq \frac{C}{N(t)}} |u(t,x)|^{\frac{2d}{d-2}} dx dt
\]

\[
\gtrsim_u \sum_{k=0}^{[T-1]} \frac{1}{1+k} \ln (1 + T).
\]

Choosing \( T \) sufficiently large depending on \( u \), we derive a contradiction. Thus, we exclude the global almost periodic solutions in the sense of Theorem 1.3.
6. The finite time blowup

In this section, we preclude the finite time blowup solution in the sense of Theorem 1.3. We adopt the Morawetz estimate to get a contradiction. Without loss of generality, we assume that \( \sup I = 1 \).

First, we remark that the function \( N(t) \) tends to infinity as \( t \) approaches the blow-up time. In the context of the nonlinear wave equation with local nonlinearity \( f(u) = |u|^{2-}u \) this property is given in [11], while for the energy-supercritical wave equation, see [12][16].

**Lemma 6.1** (the lower bound of \( N(t) \)). Let \( u : I \times \mathbb{R}^d \to \mathbb{R} \) be an almost periodic solution to (1.1) with maximal lifespan \( I \), and \( \sup I = 1 \). Then there exist \( \varepsilon > 0 \) and \( C > 0 \) so that for all \( t \in (1 - \varepsilon, 1) \)

\[
N(t) \geq \frac{C}{1-t}
\]

**Proof.** We prove by contradiction. If not, then there exists a sequence \( t_n \to 1 \) so that

\[
N(t_n)(1-t_n) < \frac{1}{n}, \quad \forall \ n \in \mathbb{N}.
\]

For any \( n \in \mathbb{N} \), we denote

\[
(v_{0,n}, v_{1,n}) = \left( \frac{1}{N(t_n)^{\frac{d+2}{2}}} u(t_n, x(t_n) + \frac{x}{N(t_n)}), \frac{1}{N(t_n)^{\frac{d}{2}}} u_t(t_n, x(t_n) + \frac{x}{N(t_n)}) \right),
\]

and let \( v_n(t, x) \) be the solution to (1.1) with Cauchy data \( (v_{0,n}, v_{1,n}) \) with maximal lifespan \( I_n \). Then

\[
v_n(t, x) = \frac{1}{N(t_n)^{\frac{d+2}{2}}} u\left( \frac{t}{N(t_n)} + t_n, x(t_n) + \frac{x}{N(t_n)} \right).
\]

Hence, the scaling and space translation symmetries imply that we have

\[
\sup I_n = N(t_n)(1-t_n).
\]

Noting that \( u \) is an almost periodic solution, we can choose \( (f, g) \in \dot{H}^1 \times L^2 \) so that

\[
(v_{0,n}, v_{1,n}) \to (f, g), \text{ in } \dot{H}^1 \times L^2, \quad \text{as } n \to \infty.
\]

If we let \( \delta_0(d, \|f, g\|_{\dot{H}^1 \times L^2}) > 0 \) be the small constant as in Theorem 2.1 then there exists an open interval \( 0 \in J \subset \mathbb{R} \) small enough so that

\[
\| \dot{K}(t)f + K(t)g \|_{S(J)} < \frac{\delta_0}{3}.
\]

On the other hand, from Strichartz estimate, we know that as \( n \to +\infty \)

\[
\| \dot{K}(t)(f - v_{0,n}) + K(t)(g - v_{1,n}) \|_{S(J)} \lesssim \| (f - v_{0,n}, g - v_{1,n}) \|_{\dot{H}^1 \times L^2} \to 0.
\]

This together with (6.3) yields that there exists \( N \) large enough such that for all \( n > N \),

\[
\| \dot{K}(t)v_{0,n} + K(t)v_{1,n} \|_{S(J)} < \frac{2\delta_0}{3}.
\]
Thus, we have by Theorem 2.1
\[ \|v_n\|_{S(I)} \lesssim \delta_0 \]
which implies \( J \subset I_n \), and so \( \frac{1}{2} \sup J \in I_n \). However, this contradicts with
\[ \lim_{n \to \infty} \sup I_n = \lim_{n \to \infty} N(t_n)(1-t_n) = 0. \]
Therefore, we conclude this lemma. \( \square \)

**Remark 6.1.** Assume that \( u \) is an almost periodic solution with maximal lifespan \( I \), and \( \sup I = 1 \). Then, by the similar argument as (4.2) in [10], we have the control for \( x(t) \) by
\[ |x(t_1) - x(t_2)| \leq |t_1 - t_2| + \frac{C(\eta)}{N(t_1)} + \frac{C(\eta)}{N(t_2)}, \quad \forall t_1, t_2 \in [0, 1). \]
This together with Lemma 6.1 yields that
\[ |x(t_1) - x(t_2)| \leq |t_1 - t_2| + \frac{C(\eta)}{N(t_1)} + \frac{C(\eta)}{N(t_2)} \to 0, \quad \text{as} \quad t_1, t_2 \to 1^- \]
which means the limit \( \lim_{t \to 1^-} x(t) \) exists. Thus, the space translation symmetry implies that we can assume
\[ \lim_{t \to 1^-} x(t) = 0. \]

A second step is to show that the finite time blow-up in the sense of Theorem 1.3 must have compact support. More precisely, we have

**Lemma 6.2.** Let \( u : I \times \mathbb{R}^d \to \mathbb{R} \) be an almost periodic solution to (1.1) with maximal lifespan \( I \), \( \sup I = 1 \), and \( \lim_{t \to 1^-} x(t) = 0. \) Then
\[ \supp(u(t, \cdot), \partial_t u(t, \cdot)) \subset B(0, 1 - t), \quad \forall t \in [0, 1). \]

**Proof.** Since the proof is similar to [16], we just give a sketch of proof here for the sake of completeness.

It suffices to prove that for all \( s \in (0, 1) \)
\[ \int_{|x| > 1-s} |\nabla_{t,x} u(s, x)|^2 dx + \int_{|x| > 1-s} |u(s, x)|^2^* dx = 0. \]
This can be reduced to show that for all \( \eta, \varepsilon > 0 \),
\[ \int_{|x| > 1-s+\eta} |\nabla_{t,x} u(s, x)|^2 dx + \int_{|x| > 1-s+\eta} |u(s, x)|^2^* dx < C\varepsilon. \]
Furthermore, this follows from the following two facts
\[
\left\{ \begin{array}{l}
\{ x : |x| > 1 - s + \eta \} \subset \{ x : |x - x(t)| \geq \frac{C(\eta)}{N(t)} + (t-s) \}, \quad t \text{ close to} 1, s < t, \\
\int_{|x-x(t)| \geq \frac{C(\eta)}{N(t)} + (t-s)} |\nabla_{t,x} u(s, x)|^2 dx + \int_{|x-x(t)| \geq \frac{C(\eta)}{N(t)} + (t-s)} |u(s, x)|^2^* dx < C\varepsilon.
\end{array} \right.
\]
It is easy to check the second fact by the compactness [1,12] and the extended causality.
Now we turn to prove the first fact. From Lemma 6.1 and (6.6), we know that there exists $M = M(\varepsilon, \eta)$ and a sequence $t_n \to 1-$ such that for all $n \geq M$

$$|x(t_n)| + \frac{C(\varepsilon)}{N(t_n)} < \frac{\eta}{2}.$$ 

Hence, if $|x| > 1 - s + \eta$, then for $n \geq N$

$$|x - x(t_n)| \geq |x| - |x(t_n)| \geq 1 - s + \eta - \frac{\eta}{2} \geq t_n - s + \frac{C(\varepsilon)}{N(t_n)}.$$

Thus, if $t$ is close to 1, then

$$\{x : |x| > 1 - s + \eta\} \subset \{x : |x - x(t)| \geq \frac{C(\varepsilon)}{N(t)} + (t - s)\},$$

and so the fact follows. Therefore, we complete the proof of this lemma.

Now we can utilize the above properties and the Morawetz estimate to exclude the finite time blow-up solutions in the sense of Theorem 1.3.

**Theorem 6.1** (Absence of finite-time blowup solutions). There are no finite-time blowup solutions to (1.1) in the sense of Theorem 1.3.

**Proof.** We argue by contradiction. Assume that there exists a solution $u : I \times \mathbb{R}^d \to \mathbb{R}$ which is a finite time blowup in the sense of Theorem 1.3 Without loss of generality, we may assume sup$I = 1$.

For $T \in (\frac{1}{2}, 1)$, using Morawetz estimate, Lemma 6.2 $\text{supp}(u(t, \cdot), \partial_t u(t, \cdot)) \subset B(0, 1 - t)$, Lemma 6.1 $N(t) \geq \frac{C}{t - 1}$ and Proposition 4.1 we derive that

$$C(E) \geq \int_{\frac{T}{2}}^{T} \int_{\mathbb{R}^d} \frac{|u(t,x)|^{\frac{2d}{d-2}}}{|x|} dx dt$$

$$\geq \int_{\frac{T}{2}}^{T} \frac{1}{1-t} \int_{\mathbb{R}^d} |u(t,x)|^{\frac{2d}{d-2}} dx dt$$

$$\geq \sum_{k=1}^{[\log_2(1-T)]} 2^k \int_{1-2^{-k}}^{1-2^{-(k+1)}} \int_{\mathbb{R}^d} |u(t,x)|^{\frac{2d}{d-2}} dx dt$$

$$\geq \sum_{k=1}^{[\log_2(1-T)]} 1 \simeq \int_{\frac{T}{2}}^{T} \frac{1}{1-t} dt.$$ 

And so we derive a contradiction by taking $T$ close to 1 depending on $u$. Thus, we exclude the global almost periodic solutions in the sense of Theorem 1.3.

Therefore, we conclude Theorem 1.1

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