MOCK MODULAR FORMS AND CLASS NUMBER RELATIONS

MICHAEL H. MERTENS

ABSTRACT. In this paper, we prove an almost 40 year old conjecture by H. Cohen concerning the generating function of the Hurwitz class number of quadratic forms using the theory of mock modular forms. This conjecture yields an infinite number of so far unproven class number relations.

1. INTRODUCTION

Since the days of C.F. Gauß, it has been an important problem in number theory to determine the class numbers of binary quadratic forms. One aspect of this, which is also of interest regarding computational issues, are the so called class number relations. These express certain sums of class numbers in terms of more elementary arithmetic functions which are easier to understand and computationally more feasible. The first examples of these relations are due to L. Kronecker [17] and A. Hurwitz [15][16]:

Let $H(n)$ denotes the Hurwitz class number of a non-negative integer $n$ (cf. Section 2 for the definition). Then we have the relation

$$\sum_{s \in \mathbb{Z}} H(4n - s^2) + 2\lambda_1(n) = 2\sigma_1(n),$$

where

$$\lambda_k(n) := \frac{1}{2} \sum_{d|n} \min\left(d, \frac{n}{d}\right)^k$$

and $\sigma_k(n) := \sum_{d|n} d^k$ is the usual $k$-th power divisor sum.

This was further extended by M. Eichler in [11]. For odd $n \in \mathbb{N}$ we have

$$\sum_{s \in \mathbb{Z}} H(n - s^2) + \lambda_1(n) = \frac{1}{3}\sigma_1(n).$$

2000 Mathematics Subject Classification. 11E41, 11F37, 11F30.

Key words and phrases. class number relation, (mock) modular form, Appell-Lerch sum.

The author’s research is supported by the DFG-Graduiertenkolleg 1269 ”Global Structures in Geometry and Analysis”.

This paper is part of the author’s PhD-thesis, written under the supervision of Prof. Dr. K. Bringmann at the Universität zu Köln.
Other such examples of class number relations can be obtained, e.g. from the famous Eichler-Selberg trace formula for cusp forms on $SL_2(\mathbb{Z})$.

In 1975, H. Cohen [8] generalized the Hurwitz class number using Dirichlet’s class number formula (see e.g. [10]) to a number $H(r, n)$ which is closely related to the value of a certain Dirichlet $L$-series at $(1-r)$ and showed that for $r \geq 2$ the generating function

$$\mathcal{H}_r(\tau) := \sum_{n=0}^{\infty} H(r, n) q^n, \quad q = e^{2\pi i \tau}, \quad \text{Im}(\tau) > 0$$

is a modular form of weight $r + \frac{1}{2}$ on $\Gamma_0(4)$ ([8, Theorem 3.1]). This yields many interesting relations in the shape of (1.1) and (1.3) for $H(r, n)$.

The case $r = 1$, where $H(1, n) = H(n)$, was treated around the same time by D. Zagier ([21], [14, Chapter 2]): He showed that the function $\mathcal{H}(\tau) = \mathcal{H}_1(\tau)$ is in fact not a modular form but can be completed by a non-holomorphic term such that the completed function transforms like a modular form of weight $\frac{3}{2}$ on $\Gamma_0(4)$.

In more recent years, this phenomenon has been understood in a broader context: The discovery of the theory behind Ramanujan’s mock theta functions by S. Zwegers [23], J.H. Bruinier and J. Funke [6], K. Bringmann and K. Ono [3] and many, many others has revealed that the function $\mathcal{H}$ is an example of a weight $\frac{3}{2}$ mock modular form, i.e. the holomorphic part of a harmonic weak Maass form\footnote{In the literature the spelling “Maass form” is more common, although these functions are named after the German mathematician Hans Maass (1911-1992)} (see Section 2 for a definition). Using this theory, some quite unexpected connections to combinatorics occur, as for example in [2], where class numbers were related to ranks of so-called overpartitions.

In [8], Cohen considered the formal power series

$$S^1_4(\tau, X) := \sum_{n=0}^{\infty} \sum_{\substack{n \text{ odd} \in \mathbb{Z} \leq n}} \frac{H(n - s^2)}{1 - 2sX + nX^2} + \sum_{k=0}^{\infty} \lambda_{2k+1}(n) X^{2k}$$

From Zagier’s and his own results, as well as computer calculations, he conjectured that the following should be true.

**Conjecture 1.1.** (H. Cohen, 1975)

The coefficient of $X^\ell$ in the formal power series $S^1_4$ is a (holomorphic) modular form of weight $\ell + 2$ on $\Gamma_0(4)$.

The goal of this paper is to prove the following result.
Theorem 1.2. Conjecture 1.1 is true. Moreover, for \( \ell > 0 \) the coefficient of \( X^\ell \) in (1.4) is a cusp form.

This obviously implies new relations for Hurwitz class numbers which to the author's knowledge have not been proven so far. We give some of them explicitly in Corollary 4.2.

The main idea of the proof of Theorem 1.2 is to relate both summands in the coefficient of the above power series to objects which in accordance to the nomenclature in [9] should be called *quasi mixed mock modular forms*, complete them, such that they transform like modular forms and show that the completion terms cancel each other out. The same idea is also used in a very recent preprint [1] by K. Bringmann and B. Kane in which they also prove several identities for sums of Hurwitz class numbers conjectured by B. Brown et al. in [5].

The outline of this paper is as follows: the preliminaries and notations are explained in Section 2. Section 3 contains some useful identities and other lemmas which will be used in Section 4 to prove Cohen’s conjecture.

Since many of the proofs involve rather long calculations, we omit some of them here. More detailed proofs will be available in the author's PhD thesis [18].

2. Preliminaries

First, we fix some notation. For this entire paper, let \( \tau \) be a variable from the complex upper half plane \( \mathbb{H} \) and denote \( x := \text{Re}(\tau) \) and \( y := \text{Im}(\tau) \). As usual, we set \( q := e^{2\pi i \tau} \). The letters \( u, v \) denote arbitrary complex variables. The differential operators with respect to all these variables shall be renormalized by a factor of \( \frac{1}{2\pi i} \), thus we abbreviate

\[
D_t := \frac{1}{2\pi i} \frac{d}{dt}.
\]

An element of \( \text{SL}_2(\mathbb{Z}) \) is always denoted by \( \gamma = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \). For some natural number \( N \), we set as usual

\[
\Gamma_0(N) = \left\{ \gamma = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \text{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}.
\]

There are two different theta series occuring in this paper. One is the theta series of the lattice \( \mathbb{Z} \),

\[
\vartheta(\tau) := \sum_{n \in \mathbb{Z}} q^{n^2},
\]

while the other is the classical Jacobi theta series

\[
\Theta(v; \tau) := \sum_{\nu \in \frac{1}{2} + \mathbb{Z}} q^{\nu^2/2} e^{2\pi i \nu(v+1/2)}.
\]
Note that, e.g. in [23], the letter \( \vartheta \) stands for the Jacobi theta series.

2.1. Mock Modular Forms. In this subsection we give some basic definitions about harmonic Maaß forms and mock modular forms (for details, cf. [6, 19] et al.). Therefore we fix some \( k \in \frac{1}{2} \mathbb{Z} \) and define for a smooth function \( f : \mathbb{H} \to \mathbb{C} \) the following operators:

1. The weight \( k \) slash operator by
\[
(f|_k \gamma)(\tau) = \begin{cases} 
(c\tau + d)^{-k} f \left( \frac{a\tau + b}{c\tau + d} \right), & \text{if } k \in \mathbb{Z} \\
\left( \frac{d}{a} \right) \varepsilon_d (c\tau + d)^{-k} f \left( \frac{a\tau + b}{c\tau + d} \right), & \text{if } k \in \frac{1}{2} + \mathbb{Z},
\end{cases}
\]
where \( \left( \frac{m}{n} \right) \) denotes the extended Legendre symbol in the sense of [20], \( \tau^{1/2} \) denotes the principal branch of the square root (i.e. \( -\frac{\pi}{2} < \text{arg}(\tau^{1/2}) \leq \frac{\pi}{2} \)), and
\[
\varepsilon_d := \begin{cases} 
1, & \text{if } d \equiv 1 \pmod{4} \\
i, & \text{if } d \equiv 1 \pmod{4}.
\end{cases}
\]
We shall assume \( \gamma \in \Gamma_0(4) \) if \( k \not\in \mathbb{Z} \).

2. The weight \( k \) hyperbolic Laplacian by \( \tau = x + iy \)
\[
\Delta_k := -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).
\]

Definition 2.1. Let \( f : \mathbb{H} \to \mathbb{C} \) be a smooth function and \( k \in \frac{1}{2} \mathbb{Z} \). We call \( f \) a harmonic weak Maaß form of weight \( k \) on some subgroup \( \Gamma \leq \text{SL}_2(\mathbb{Z}) \) (resp. \( \Gamma_0(4) \) if \( k \not\in \mathbb{Z} \)) of finite index if the following conditions are met:

1. \( (f|_k \gamma)(\tau) = f(\tau) \) for all \( \gamma \in \Gamma \) and \( \tau \in \mathbb{H} \)
2. \( (\Delta_k f)(\tau) = 0 \) for all \( \tau \in \mathbb{H} \)
3. \( f \) grows at most exponentially in the cusps of \( \Gamma \).

Proposition 2.2. ([19, Lemma 7.2, equation (7.8)]) Let \( f \) be a harmonic Maaß form of weight \( k \) with \( k > 0 \) and \( k \neq 1 \). Then there is canonical splitting
\[
f(\tau) = f^+(\tau) + f^-(\tau),
\]
where for some \( N \in \mathbb{Z} \) we have
\[
f^+(\tau) = \sum_{n=N}^{\infty} c_f^+(n) q^n
\]
and
\[
f^-(\tau) = c_f^-(0) \frac{(4\pi y)^{-k+1}}{k-1} + \sum_{n=1}^{\infty} c_f^-(n) \Gamma(k-1; 4\pi ny) q^{-n}.
\]
Here
\[ \Gamma(\alpha; x) = \int_{x}^{\infty} e^{-t^\alpha} dt \]
is the incomplete Gamma function.

**Definition 2.3.** (i) The functions \( f^+ \) and \( f^- \) in the above proposition are referred to as the holomorphic and non-holomorphic part of the harmonic weak Maass form \( f \).

(ii) A mock modular form is the holomorphic part of a harmonic weak Maass form.

There are several generalizations of mock modular forms, e.g. *mixed mock modular forms*, which are essentially products of mock modular forms and usual holomorphic modular forms. For details, we refer the reader to [9, Section 7.3].

2.2. **Class Numbers.** Let \( d \) be a non-negative integer with \( d \equiv 0, 3 \pmod{4} \). Then the class number for the discriminant \(-d\) is the number of \( \text{SL}_2(\mathbb{Z}) \) equivalence classes of primitive binary integral quadratic forms of discriminant \(-d\),
\[ h(-d) = \left| \left\{ Q = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \in \mathbb{Z}^{2 \times 2} \mid \det Q = d \text{ and } \gcd(a, b, c) = 1 \right\} / \text{SL}_2(\mathbb{Z}) \right|, \]
where of course \( \text{SL}_2(\mathbb{Z}) \) acts via \( (Q, \gamma) \mapsto \gamma^\text{tr} Q \gamma \).

The Hurwitz class number is now a weighted sum of these class numbers: Define \( w_3 = 3 \), \( w_4 = 2 \) and \( w_d = 1 \) for \( d \neq 3, 4 \). Then the Hurwitz class number is given by
\[ H(n) = \begin{cases} \frac{-1}{12} & \text{if } n = 0, \\ \sum_{f^2|n} \frac{h(-n/f^2)}{w_{n/f^2}} & \text{if } n \equiv 1, 2 \pmod{4}, \\ 0 & \text{otherwise}. \end{cases} \]

The generating function of the Hurwitz class number shall be denoted by
\[ \mathcal{H}(\tau) = \sum_{n=0}^{\infty} H(n) q^n. \]

We have the following result concerning a modular completion of the function \( \mathcal{H} \) which was already mentioned in the introduction (cf. [13, Chapter 2, Theorem 2]).

**Theorem 2.4.** Let
\[ \mathcal{R}(\tau) = \frac{1 + i}{16\pi} \int_{\tau}^{i\infty} \frac{\vartheta(z)}{(z + \tau)^{3/2}} dz. \]
Then the function

$$G(\tau) = \mathcal{H}(\tau) + R(\tau)$$

transforms under $\Gamma_0(4)$ like a modular form of weight $\frac{3}{2}$.

The idea of the proof is to write $\mathcal{H}$ as a linear combination of Eisenstein series of weight $\frac{3}{2}$, in analogy to the proof of [8, Theorem 3.1]. These series diverge, but using an idea of Hecke (cf. [13, §2]), who used it to derive the transformation law of the weight 2 Eisenstein series $E_2$, one finds the non-holomorphic completion term $R$.

It is easy to check that $G$ is indeed a harmonic weak Maass form of weight $\frac{3}{2}$. As a mock modular form, the function $\mathcal{H}$ is rather peculiar since it is basically the only example of such an object which is holomorphic at the cusps of $\Gamma_0(4)$ (cf. [9, Section 7]).

2.3. Appell-Lerch Sums. In this subsection we are going to recall some general facts about Appell-Lerch sums. For details, we refer the reader to [22, 23].

**Definition 2.5.** Let $\tau \in \mathbb{H}$ and $u, v \in \mathbb{C} \setminus (\mathbb{Z}\tau + \mathbb{Z})$. The Appell-Lerch sum of level 1 is then the following expression:

$$A_1(u, v) = A_1(u, v; \tau) := a^{1/2} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{n(n+1)/2} b^n}{1 - a q^n}$$

where $a = e^{2\pi i u}$, $b = e^{2\pi i v}$, and $q = e^{2\pi i \tau}$.

In addition, we define the following real-analytic functions.

$$R(u; \tau) := \sum_{\nu \in \frac{1}{2} + \mathbb{Z}} \left\{ \text{sgn}(\nu) - E \left( \left( \nu + \frac{\text{Im} u}{y} \right) \sqrt{2y} \right) \right\} (-1)^{\nu-1/2} q^{-\nu^2/2} e^{-2\pi i \nu u},$$

$$E(t) := 2 \int_0^t e^{-\pi u^2} du = \text{sgn}(t) \left( 1 - \beta \left( t^2 \right) \right),$$

$$\beta(x) := \int_{-x}^{x} u^{-1/2} e^{-\pi u} du,$$

where for the second equality in (2.5) we refer to [23, Lemma 1.7].

This function $R$ has some nice properties, a few of which are collected in the following Propositions.

**Proposition 2.6.** (cf. [23, Proposition 1.9])

The function $R$ fulfills the elliptic transformation properties.
(i) \( R(u + 1; \tau) = -R(u; \tau) \)
(ii) \( R(u; \tau) + e^{-2\pi i u - \pi i \tau} R(u + \tau; \tau) = 2e^{-\pi i u - \pi i \tau/4} \)
(iii) \( R(-u) = R(u) \).

The following proposition has already been mentioned in \[4\]. The proof is a straightforward computation.

**Proposition 2.7.** The function \( R \) lies in the kernel of the renormalized Heat operator \( 2D_\tau + D_u^2 \), hence

\[
(2.7) \quad D_u^2 R = -2D_\tau R.
\]

We now define the non-holomorphic function

\[
\widehat{A}_1(u, v; \tau) = A_1(u, v; \tau) + \frac{i}{2} \Theta(v; \tau) R(u - v; \tau)
\]

which will henceforth be referred to as the completion of the Appell-Lerch sum \( A_1 \).

**Theorem 2.8.** ([22, Theorem 2.2])
The real-analytic function \( \widehat{A}_1 \) transforms like a Jacobi form of weight 1 and index \( \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \):

\[
(S) \quad \widehat{A}_1(-u, -v) = -\widehat{A}_1(u, v).
\]

\[
(E) \quad \widehat{A}_1(u + \lambda_1 \tau + \mu_1, v + \lambda_2 \tau + \mu_2) = (-1)^{(\lambda_1 + \mu_1)} a^{\lambda_1 - \lambda_2} b^{-\lambda_1} q^{\lambda_1^2/2 - \lambda_1 \lambda_2} \widehat{A}_1(u, v) \quad \text{for} \quad \lambda_i, \mu_i \in \mathbb{Z}.
\]

\[
(M) \quad \widehat{A}_1\left(\frac{u}{c\tau + d}, \frac{v}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right) = (c\tau + d) e^{\pi ic(-u^2 + 2uv)/(c\tau + d)} \widehat{A}_1(u, v; \tau)
\]

for \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \).

3. Some Lemmas

As we mentioned in the introduction, we would like to relate the two summands for each coefficient in the power series in Conjecture \[1.1\] to some sort of modular object. For that purpose, we recall the definition of Rankin-Cohen brackets as given in \[71\ p. 53\], which differs slightly (see below) from the original one in \[8\ Theorem 7.1\].

**Definition 3.1.** Let \( f, g \) be smooth functions defined on the upper half plane and \( k, \ell \in \mathbb{R}_{>0}, n \in \mathbb{N}_0 \). Then we define the \( n \)-th Rankin-Cohen bracket of \( f \) and \( g \) as

\[
[f, g]_n = \sum_{r+s=n} (-1)^r \binom{k+n-1}{r} \binom{\ell+n-1}{s} D^r f D^s g
\]
where for non-integral entries we define
\[
\binom{m}{s} := \frac{\Gamma(m + 1)}{\Gamma(s + 1)\Gamma(m - s + 1)}.
\]

Here, the letter $\Gamma$ denotes the usual Gamma function.

It is well-known (cf. [8, Theorem 7.1]) that if $f, g$ transform like modular forms of weight $k$ and $\ell$ resp. then $[f, g]_n$ transforms like a modular form of weight $k + \ell + 2n$ and that $[f, g]_0 = f \cdot g$. The interaction of the first Rankin-Cohen bracket, which itself fulfills the Jacobi identity of Lie brackets, and the regular product of modular forms give the graded algebra of modular forms the additional structure of a so-called Poisson algebra (cf. [7, p. 53]).

Note that our definition of the Rankin-Cohen bracket differs from the one in [8, Theorem 7.1] by a factor of $n!(−2\pi i)^n$ which guarantees that if $f$ and $g$ have integer Fourier coefficients, so does $[f, g]_n$.

**Lemma 3.2.** The coefficient of $X^{2k}$ in (1.4) is given by
\[
\frac{c_k}{2} \left( [\mathcal{K}, \vartheta]_k(\tau) - [\mathcal{K}, \vartheta]_k(\tau + \frac{1}{2}) \right) + \Lambda_{2k+1,\text{odd}}(\tau),
\]
where $c_k = k! \frac{\sqrt{\pi} \Gamma(k/2)}{\Gamma(k + 1/2)}$ and
\[
\Lambda_{\ell,\text{odd}}(\tau) := \sum_{n=0}^{\infty} \lambda_\ell(2n + 1)q^{2n+1}
\]
with $\lambda_\ell$ as in (1.2). The coefficient of $X^{2k+1}$ is identically 0.

**Proof.** As in [8, Theorem 6.1] we define for a modular form $f$ with $f(\tau) = \sum_{n=0}^{\infty} a(n)q^n$ of weight $k$ and an integer $D \neq 0$ the series
\[
S_D^f(\tau; X) := \sum_{n=0}^{\infty} \left( \sum_{s \in \mathbb{Z}, s^2 \leq n} \frac{a \left( \frac{n-s^2}{|D|} \right)}{1 - sX + nX^2} \right) q^n,
\]
where we assume $a(n) = 0$ if $n \not\in \mathbb{N}_0$. From p. 283 in [8] we immediately get the formula
\[
S_D^f(\tau; X) = \sum_{n=0}^{\infty} \frac{\sqrt{\pi} \Gamma \left( n + k - \frac{1}{2} \right)}{\Gamma \left( n + \frac{1}{2} \right) \Gamma \left( k - \frac{1}{2} \right)} [g, \vartheta]_n(\tau),
\]
where $g(\tau) := f(|D|\tau)$. This yields the assertion by plugging in $f = \mathcal{K}$, $k = \frac{3}{2}$, and $D = 1$. □
Since in the Rankin-Cohen brackets that we consider here, we have linear combinations of products of derivatives of a mock modular form and a regular modular form, one could call an object like this a quasi mixed mock modular form.

**Lemma 3.3.** For odd $k \in \mathbb{N}$, the function $\Lambda_{k, odd}$ can be written as a linear combination of derivatives of Appell-Lerch sums. More precisely

$$\Lambda_{k, odd} = \frac{1}{2} \left( D_v^k A_{1, odd} \right) \left( 0, \tau + \frac{1}{2}; 2\tau \right),$$

where we define

$$A_{1, odd}^{odd}(u, v; \tau) := a^{1/2} \sum_{n \in \mathbb{Z}, n \ odd} \frac{(-1)^n q^{n(n+1)/2} b^n}{1 - a q^n}$$

$$= \frac{1}{2} \left( A_1(u, v; \tau) - A_1(u, v + \frac{1}{2}; \tau) \right),$$

where again $a = e^{2\pi i u}$ and $b = e^{2\pi i v}$.

**Proof.** First we remark that the right-hand side of the identity to be shown is actually well-defined because as a function of $u$, $A_1(u, v; \tau)$ has simple poles in $\mathbb{Z} \tau + \mathbb{Z}$ (cf. [23, Proposition 1.4]) which cancel out if the sum is only taken over odd integers. Thus the equation actually makes sense.

Then we write $\Lambda_{k, odd}$ as a $q$-series

$$2 \Lambda_{k, odd}(\tau) = \sum_{\ell = 0}^{\infty} \sum_{m = 0}^{\infty} \min(2\ell + 1, 2m + 1) q^{(2\ell+1)(2m+1)}$$

$$= 2 \sum_{\ell = 0}^{\infty} \left( (2\ell + 1) b^{2\ell+1} q^{(2\ell+1)^2} \right) + \sum_{\ell = 0}^{\infty} (2\ell + 1) q^{(2\ell+1)^2}$$

$$= 2 \sum_{\ell = 0}^{\infty} (2\ell + 1) q^{(2\ell+1)^2} \left( \frac{1}{1 - q^{2(2\ell+1)}} - 1 \right) + \sum_{\ell = 0}^{\infty} (2\ell + 1) q^{(2\ell+1)^2}$$

$$= \sum_{\ell = 0}^{\infty} (2\ell + 1) \left( q^{(2\ell+1)^2} + q^{(2\ell+1)^2 + 2(2\ell+1)} \right).$$

This is easily seen to be the same as $\left( D_v^k A_{1, odd} \right) \left( 0, \tau + \frac{1}{2}; 2\tau \right)$. \hfill $\square$

**Remark 3.4.** Now we can write down completions for each summand in (3.3) and thus we see that the function

$$\frac{c_k}{2} \left( \langle \mathcal{G}, \vartheta \rangle_k(\tau) - \langle \mathcal{G}, \vartheta \rangle_k(\tau + \frac{1}{2}) \right) + \left( D_v^{2k+1} A_{1, odd} \right) \left( 0, \tau + \frac{1}{2}; 2\tau \right)$$
transforms like a modular form of weight $2k + 2$. Because the Fourier coefficients of the holomorphic parts grow polynomially, they are holomorphic at the cusps as well.

Thus it remains to show that the non-holomorphic parts given by

$$ (3.4) \quad \frac{c_k}{2} (\mathcal{R}, \vartheta)_{k}(\tau) - [\mathcal{R}, \vartheta]_{k}(\tau + \frac{1}{2}) $$

and

$$ i \frac{1}{4} \left( \sum_{\ell=0}^{2k+1} \binom{2k + 1}{\ell} (-1)^{\ell} (D_u R)(-\tau - \frac{1}{2}; 2\tau)(D_v^{2k-\ell+1}\Theta)(\tau + \frac{1}{2}; 2\tau) \right) $$

are indeed equal up to sign and that the function in (3.3) is modular on $\Gamma_0(4)$.

This shows that we will need some specific information about the derivatives of the Jacobi theta series and the $R$-function evaluated at the torsion point $(\tau + \frac{1}{2}, 2\tau)$.

A simple and straightforward calculation gives us the following result.

Lemma 3.5. For $r \in \mathbb{N}_0$ one has

$$ (3.6) \quad (D_r \Theta)(\tau + \frac{1}{2}; 2\tau) = -q^{-1/4} \sum_{s=0}^{\left\lfloor \frac{r}{2} \right\rfloor} \binom{r}{2s} \left( -\frac{1}{2} \right)^{r-2s} (D_r^s \Theta)(\tau) $$

and

$$ (3.7) \quad (D_r \vartheta)(\tau + 1; 2\tau) = iq^{-1/4} \sum_{s=0}^{\left\lfloor \frac{r}{2} \right\rfloor} \binom{r}{2s} \left( -\frac{1}{2} \right)^{r-2s} (D_r^s \vartheta)(\tau + \frac{1}{2}), $$

with $\Theta$ as in (2.2) and $\vartheta$ as in (2.1).

Lemma 3.6. The following identities are true:

$$ (3.8) \quad R(-\tau - \frac{1}{2}; 2\tau) = iq^{1/4} $$

$$ (3.9) \quad R(-\tau - 1; 2\tau) = -q^{1/4} $$

$$ (3.10) \quad (D_u R)(-\tau - \frac{1}{2}; 2\tau) = -\frac{1 + i}{4\pi} q^{1/4} \int_{-\tau}^{i\infty} \frac{\vartheta(z)}{(z + \tau)^{3/2}} dz - \frac{i}{2} q^{1/4} $$

$$ (3.11) \quad (D_u R)(-\tau - 1; 2\tau) = -\frac{1 + i}{4\pi} q^{1/4} \int_{-\tau}^{i\infty} \frac{\vartheta(z + \frac{1}{2})}{(z + \tau)^{3/2}} dz + \frac{1}{2} q^{1/4}. $$
Proof. The identities (3.8) and (3.9) follow directly by applying the transformation properties (iii), (i), and (ii) of $R$ in Proposition 2.6.

We only show (3.10), since (3.11) then also follows from the obvious fact that $R(u; \tau + 1) = e^{-\pi i/4}R(u; \tau)$. From the definition of $R$ in (2.4) and (2.5) we see that

$$(D_u R)(-\tau - \frac{1}{2}; 2\tau) = iq^{1/4} \sum_{n \in \mathbb{Z}} \left( \frac{1}{\sqrt{4\pi}} e^{-4\pi n^2 y} - \text{sgn}(n)(n + \frac{1}{2}) \beta(4n^2 y) \right) q^{-n^2},$$

with $\beta$ as in (2.6). Note that for convenience, we define $\text{sgn}(0) := 1$.

By partial integration one gets for all $x \in \mathbb{R}_\geq 0$ that

$$\beta(x) = \frac{1}{\pi} x^{-1/2} e^{-\pi x} - \frac{1}{2\sqrt{\pi}} \Gamma(-\frac{1}{2}; \pi x),$$

where again, $\Gamma(\alpha; x)$ denotes the incomplete Gamma function. Using the well-known fact that for $\tau \in \mathbb{H}$ and $n \in \mathbb{N}$ it holds that

$$\int_{\pi - i\infty}^{\pi + i\infty} \frac{e^{2\pi i nz}}{(-i(z + \tau))^{3/2}} dz = i(2\pi n)^{1/2} q^{-n} \Gamma(-\frac{1}{2}; 4\pi ny)$$

we get the assertion by a straightforward calculation. \hfill \Box

Now we take a closer look at (3.5).

Lemma 3.7. For all $k \in \mathbb{N}_0$ it holds true that

\begin{align*}
\sum_{\ell=0}^{2k+1} (-1)^{\ell} \binom{2k + 1}{\ell} (D_u R)(-\tau - \frac{1}{2}; 2\tau)(D_v^{2k-\ell+1} \vartheta)(\tau + \frac{1}{2}; 2\tau) \\
= q^{-1/4} \sum_{m=0}^{k-m} \sum_{\ell=0}^{k-m} \left[ \frac{1}{2} (D_u R)(-\tau - \frac{1}{2}; 2\tau) + \frac{2(k - \ell - m) + 1}{2\ell + 1} (D_u^{2\ell+1} R)(-\tau - \frac{1}{2}; 2\tau) \right] \\
\times b_{k,\ell,m} \left( \frac{1}{2} \right)^{2(k-\ell-m)} (D_\tau^m \vartheta)(\tau)
\end{align*}

and

\begin{align*}
\sum_{\ell=0}^{2k+1} (-1)^{\ell} \binom{2k + 1}{\ell} (D_u R)(-\tau - 1; 2\tau)(D_v^{2k-\ell+1} \vartheta)(\tau + 1; 2\tau) \\
= -iq^{-1/4} \sum_{m=0}^{k-m} \sum_{\ell=0}^{k-m} \left[ \frac{1}{2} (D_u R)(-\tau - 1; 2\tau) + \frac{2(k - \ell - m) + 1}{2\ell + 1} (D_u^{2\ell+1} R)(-\tau - 1; 2\tau) \right] \\
\times b_{k,\ell,m} \left( \frac{1}{2} \right)^{2(k-\ell-m)} (D_\tau^m \vartheta)(\tau + \frac{1}{2}),
\end{align*}
where

\[ b_{k,\ell,m} := \frac{(2k + 1)!}{(2\ell)!(2m)!(2(k - \ell - m) + 1)!} = \frac{2k + 1}{2\ell, 2m, 2(k - \ell - m) + 1}. \]

**Proof.** Again, we only show the former equation, the latter follows from the transformation laws. For simplicity, we omit the arguments of the functions considered.

We obtain

\[
\begin{align*}
\sum_{\ell=0}^{2k+1} & (-1)^{\ell} \binom{2k + 1}{\ell} (D_u^\ell R)(D_v^{2k-\ell+1}\Theta) \\
= & \sum_{\ell=0}^{k} \binom{2k + 1}{2\ell} (D_u^{2\ell} R)(D_v^{2(k-\ell)+1}\Theta) - \sum_{\ell=0}^{k} \binom{2k + 1}{2\ell + 1} (D_u^{2\ell+1} R)(D_v^{2(k-\ell)}\Theta) \\
= & q^{-1/4} \left[ \sum_{\ell=0}^{k} \sum_{m=0}^{k-\ell} \binom{2k + 1}{2\ell} \binom{2(k-\ell)}{2m} \left(-\frac{1}{2}\right)^{2(k-\ell-m)+1} (D_u^{2\ell} R)(D_v^m \vartheta) \\
- & \sum_{\ell=0}^{k} \sum_{m=0}^{k-\ell} \binom{2k + 1}{2\ell + 1} \binom{2(k-\ell)}{2m} \left(-\frac{1}{2}\right)^{2(k-\ell-m)+1} (D_u^{2\ell+1} R)(D_v^m \vartheta) \right] \\
= & q^{-1/4} \sum_{\ell=0}^{k} \sum_{m=0}^{k-\ell} \left[ \frac{1}{2} (D_u^{2\ell} R) + \frac{2(k - \ell - m) + 1}{2\ell + 1} (D_u^{2\ell+1} R) \right] b_{k,\ell,m} \left(\frac{1}{2}\right)^{2(k-\ell-m)} (D_v^m \vartheta).
\end{align*}
\]

Interchanging the sums gives the desired result. \(\square\)

**Corollary 3.8.** Conjecture [1.1] is true if the identity

(3.12)

\[ (D_v^m \mathcal{R})(\tau) = -\frac{i}{4} q^{-1/4} (-1)^m \sum_{\ell=0}^{m} \left[ \frac{1}{2} (D_u^{2\ell} R)(-\tau - \frac{1}{2}; 2\tau) \\
+ \frac{2(m - \ell) + 1}{2\ell + 1} (D_u^{2\ell+1} R)(-\tau - \frac{1}{2}; 2\tau) \right] \left(\frac{2m + 1}{2\ell}\right) \left(\frac{1}{4}\right)^{m-\ell} \]

holds true for all \(m \in \mathbb{N}_0\) and the function in (3.3) is modular on \(\Gamma_0(4)\).
Proof. Lemma 3.7 gives us that Conjecture 1.1 holds true if the identity (3.13)
\[ c_k(-1)^{k-m} \frac{k+\frac{1}{2}}{m} \left( \begin{array}{c} k - \frac{1}{2} \\ k - m \end{array} \right) D_r^{k-m} \mathcal{R}(\tau) \]
\[ = -\frac{i}{4} q^{-1/4} \sum_{\ell=0}^{k-m} \left[ \frac{1}{2} (D_u 2^\ell R)(-\tau - \frac{1}{2}; 2\tau) + \frac{2(k - \ell - m) + 1}{2\ell + 1} (D_u 2^{\ell+1} R)(-\tau - \frac{1}{2}; 2\tau) \right] \]
\[ \times b_{k,\ell,m} \left( \frac{1}{2} \right)^{2(k-\ell-m)} \]
does as well.

We can simplify this a little further: We have
\[ c_k \left( \frac{k+\frac{1}{2}}{m} \right) \left( \frac{k - \frac{1}{2}}{k - m} \right) = \left( \frac{k}{m} \right) \frac{\sqrt{\pi} \Gamma(k + \frac{3}{2})}{\Gamma(k - m + \frac{3}{2}) \Gamma(m + \frac{1}{2})} \]
and using Legendre’s duplication formula for the Gamma function we obtain after a little calculation that
\[ \frac{\left( \frac{1}{2} \right)^{2(k-\ell-m)} b_{k,\ell,m}}{c_k \left( \frac{k+\frac{1}{2}}{m} \right) \left( \frac{k - \frac{1}{2}}{k - m} \right)} = \left( 2(k - m) + 1 \right) \left( \frac{1}{4} \right)^{k-\ell-m} \]
and hence the corollary. \(\square\)

Remark 3.9. Since
\[ \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix} \Gamma_0(4) \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix} = \Gamma_0(4) \]
we see that for any (not necessarily holomorphic) modular form \( f \) of even weight \( k \) on \( \Gamma_0(4) \), the function \( g = f|_k \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix} \) is a modular form of the same weight on \( \Gamma_0(4) \) as well. In particular, this applies to \([\mathcal{G}, \vartheta]|_k \) for all \( k \in \mathbb{N}_0 \).

Lemma 3.10. The second summand in (3.3) transforms like a modular form on \( \Gamma_0(4) \).

Proof. Looking at the \((2\ell + 1)\)-st derivative of \( \tilde{A}_1^{odd}(0, v; \tau) \) with respect to \( v \), one immediately sees that this has the modular transformation properties of a Jacobi form of weight \( 2\ell + 2 \) and index 0 on \( \text{SL}_2(\mathbb{Z}) \). By [12, Theorem 1.3] it follows that \( A(\tau) := (D_v 2^\ell \tilde{A}_1^{odd})(0, \tau + \frac{1}{2}; \tau) \) transforms (up to some power of \( q \)) like a modular form of weight \( \ell + 1 \) on the group
\[ \Gamma := \left\{ \gamma \in \text{SL}_2(\mathbb{Z}) \mid \frac{a - 1}{2} + \frac{c}{2} \in \mathbb{Z} \text{ and } \frac{b}{2} + \frac{d - 1}{2} \in \mathbb{Z} \right\}. \]
We are interested in \((D^2_\nu A_{1,\text{odd}})(0, \tau + \frac{1}{2}; \tau) = \frac{1}{2\pi i} A|_{2\ell+2} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\) and since one easily checks that
\[
\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \Gamma_0(4) \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \leq \Gamma,
\]
the assertion follows. □

4. A Proof of Cohen’s Conjecture

We now prove Theorem 1.2 using Corollary 3.8. The proof is an induction on \(m\). Since the base case \(m = 0\) gives an alternative proof of the class number relation \((1.3)\) by Eichler, we give this as a proof of an additional theorem.

**Theorem 4.1.** (M. Eichler, 1955, [11])

For odd numbers \(n \in \mathbb{N}\) we have the class number relation

\[
\sum_{s \in \mathbb{Z}} H(n - s^2) + \lambda_1(n) = \frac{1}{3} \sigma_1(n).
\]

**Proof.** Let

\[
F_2(\tau) = \sum_{n=0}^{\infty} \sigma_1(2n + 1)q^{2n+1}.
\]

We recall that this function is a modular form of weight 2 on \(\Gamma_0(4)\) (cf. eg. [8, Proposition 1.1]).

Plugging in \(m = 0\) into (3.12) gives us the equation

\[
R(\tau) = -\frac{i}{4} q^{-1/4} \left[ \frac{1}{2} R(-\tau - \frac{1}{2}; 2\tau) + (D_u R)(-\tau - \frac{1}{2}; 2\tau) \right].
\]

This equality holds true by Lemma 3.6. Hence we know by Corollary 3.8, Lemma 3.3, Remark 3.9 and Lemma 3.10 that

\[
\frac{1}{2} \left( \mathcal{H}(\tau) \vartheta(\tau) - \mathcal{H}(\tau + \frac{1}{2}) \vartheta(\tau + \frac{1}{2}) \right) + \Lambda_{1,\text{odd}}(\tau)
\]

is indeed a holomorphic modular form of weight 2 on \(\Gamma_0(4)\) as well.

Since the space of modular forms of weight 2 on \(\Gamma_0(4)\) is 2-dimensional, the assertion follows by comparing the first two Fourier coefficients of the function above and \(\frac{1}{3} F_2(\tau)\).

The proof of this given in [11] involves topological arguments about the action of Hecke operators on the Riemann surface associated to \(\Gamma_0(2)\) on the one hand and arithmetic of quaternion orders on the other.
Note that the knowledge of Eichler’s class number relation does not necessarily imply that the non-holomorphic parts of our considered mixed mock modular forms cancel.

**Proof of Theorem 1.2.** The base case of our induction is treated above, thus suppose that (3.12) holds true for one \( m \in \mathbb{N}_0 \).

For simplicity, we omit again the argument \((-\tau - \frac{1}{2}; 2\tau)\) in the occurring \( R \) derivatives.

By the induction hypothesis we see that \[ D^{m+1}_\tau R(\tau) = D_\tau(D^m_\tau R(\tau)) \]

\[ = -\frac{i}{4}q^{-1/4}(1)^m \left\{ -\frac{1}{4} \sum_{\ell=0}^m \left[ \frac{1}{2}(D^2_{D_u R} + \frac{2(m - \ell)}{2\ell + 1}) \cdot \left( \frac{2m + 1}{2\ell} \right) \frac{1}{4} \right]^{m-\ell} \right. \]

\[ + \sum_{\ell=0}^m \left[ \frac{1}{2}(D_\tau D^2_{D_u R}) + \frac{2(m - \ell)}{2\ell + 1} \cdot \left( \frac{2m + 1}{2\ell} \right) \frac{1}{4} \right]^{m-\ell} \} \].

By the Theorem of Schwarz, the partial derivatives interchange and thus the total differential \( D_\tau \) is given by

\[ D_\tau(D^\ell_u R(-\tau - \frac{1}{2}; 2\tau)) = -(D^{\ell+1}_u R)(-\tau - \frac{1}{2}; 2\tau) + 2(D_\tau D^\ell_u R)(-\tau - \frac{1}{2}; 2\tau). \]

Now Proposition 2.7 implies that the above equals

\[ -\frac{i}{4}q^{-1/4}(1)^{m+1} \left\{ \sum_{\ell=0}^m \left[ \frac{1}{8}(D^2_{D_u R}) + \frac{1}{4} \cdot \left( \frac{2(m - \ell)}{2\ell + 1} + \frac{1}{2} \right) \cdot \left( \frac{2m + 1}{2\ell} \right) \right]^{m-\ell} \right. \]

\[ + \sum_{\ell=1}^m \left[ \frac{1}{2}(D^2_{D_u R}) + \frac{2(m - \ell)}{2\ell + 1} \cdot \left( \frac{2m + 1}{2\ell} \right) \right]^{m-\ell} \left. \right\} \]

\[ = -\frac{i}{4}q^{-1/4}(1)^{m+1} \left\{ \left[ \frac{1}{2} + \frac{(2m + 3)}{2}(D_u R) \right] \left( \frac{1}{4} \right)^{m+1} \right. \]

\[ + \sum_{\ell=1}^m \left[ \frac{1}{2}(D^2_{D_u R}) + \frac{2(m - \ell)}{2\ell + 1} \cdot \left( \frac{2m + 1}{2\ell} \right) \right]^{m-\ell} \left. \right\} \]

\[ + \left[ \frac{2m + 3}{4m + 2} \cdot (D^2_{D_u R}) + \frac{1}{2m + 1}(D^2_{D_u R}) \right] \left( \frac{2m + 1}{2m} \right) \} \].
It is easily seen that the last summand equals
\[
\left[ \frac{1}{2} (D_u^{2m+2} R) + \frac{1}{2m+3} (D_u^{2m+3} R) \right] \cdot (2m+3)
\]
and a direct but rather tedious calculation gives that
\[
\left( \frac{2m+1}{2\ell} \right) \cdot \left( \frac{1}{2} + \frac{2(2m-\ell) + 5}{4\ell - 2} \cdot \frac{(2\ell - 1)(2\ell)}{(2(m-\ell) + 2)(2(m-\ell) + 3)} \right) = \frac{1}{2} \left( \frac{2m+3}{2\ell} \right)
\]
and
\[
\left( \frac{2m+1}{2\ell} \right) \cdot \left( \frac{2(m+\ell) + 3}{2\ell + 1} + \frac{2(m-\ell) + 3}{2\ell - 1} \cdot \frac{(2\ell - 1)(2\ell)}{(2(m-\ell) + 2)(2(m-\ell) + 3)} \right)
\]
\[
= \frac{2(m-\ell) + 3}{2\ell + 1} \left( \frac{2m+3}{2\ell} \right).
\]
In summary, we therefore get
\[
D^{m+1}_\tau \mathcal{R}(\tau) = -\frac{i}{4} q^{-1/4} (-1)^{m+1} \sum_{\ell=0}^{m+1} \left[ \frac{1}{2} (D_u^{2\ell} R)(-\tau - \frac{1}{2}; 2\tau) \right.
\]
\[
+ \left. \frac{2(m-\ell) + 3}{2\ell + 1} (D_u^{2\ell+1} R)(-\tau - \frac{1}{2}; 2\tau) \right] \cdot \left( \frac{2m+3}{2\ell} \right) \left( \frac{1}{4} \right)^{m-\ell+1}
\]
which proves Conjecture 1.1.

The fact that we actually get a cusp form can be seen in the following way:

By [8, Corollary 7.2] we see that the function \( \tau \mapsto \frac{i}{2} ([\mathcal{H}, \vartheta]_k(\tau) - [\mathcal{H}, \vartheta]_k(\tau + \frac{1}{2})) \) is a non-holomorphic cusp form if \( k \geq 1 \). We use the same argument as there to see that \( (D_u^{2\ell+1} \tilde{A}_1^{odd})(0, \tau + \frac{1}{2}; 2\tau) \) is a cusp form as well. Because we know by Lemma 3.10 that we have for \( \gamma \in \text{SL}_2(\mathbb{R}) \) that
\[
(D_v^{2\ell+1} \tilde{A}_1^{odd})|_{2\ell+2\gamma} = D_v^{2\ell+1} (\tilde{A}_1^{odd}|_{1\gamma}),
\]
and by definition \( (D_v^{2\ell+1} \tilde{A}_1^{odd})(0, \tau + \frac{1}{2}; 2\tau) \) vanishes at the cusp \( i\infty \) for all \( \ell \in \mathbb{N}_0 \). So by the above equation it vanishes at every cusp of \( \Gamma_0(4) \). \( \square \)
Corollary 4.2. By comparing the first few Fourier coefficients of the modular forms in Theorem 1.2 one finds for all odd $n \in \mathbb{N}$ the following class number relations

$$
\sum_{s \in \mathbb{Z}} (4s^2 - n) H (n - s^2) + \lambda_3 (n) = 0,
$$

$$
\sum_{s \in \mathbb{Z}} g_4(s,n)H (n - s^2) + \lambda_5 (n) = -\frac{1}{12} \sum_{n=x^2+y^2+z^2+t^2} \mathcal{Y}_4(x, y, z, t),
$$

$$
\sum_{s \in \mathbb{Z}} g_6(s,n)H (n - s^2) + \lambda_7 (n) = -\frac{1}{3} \sum_{n=x^2+y^2+z^2+t^2} \mathcal{Y}_6(x, y, z, t),
$$

$$
\sum_{s \in \mathbb{Z}} g_8(s,n)H (n - s^2) + \lambda_9 (n) = -\frac{1}{70} \sum_{n=x^2+y^2+z^2+t^2} \mathcal{Y}_8(x, y, z, t)
$$

where $g_\ell(n, s)$ is the $\ell$-th Taylor coefficient of $(1 - sX + nX^2)^{-1}$ and $\mathcal{Y}_d(x, y, z, t)$ is a certain harmonic polynomial of degree $d$ in 4 variables. Explicitly, we have

$$
g_4(s,n) = (16s^4 - 12ns^2 + n^2),
$$

$$
g_6(s,n) = (64s^6 - 80s^4n + 24s^2n^2 - n^3),
$$

$$
g_8(s,n) = (256s^8 - 448s^6n + 240s^4n^2 - 40s^2n^3 + n^4),
$$

and

$$
\mathcal{Y}_4(x, y, z, t) = (x^4 - 6x^2y^2 + y^4),
$$

$$
\mathcal{Y}_6(x, y, z, t) = (x^6 - 5x^4y^2 - 10x^4z^2 + 30x^2y^2z^2 + 5x^2z^4 - 5y^2z^4),
$$

$$
\mathcal{Y}_8(x, y, z, t) = (13x^8 + 63x^6y^2 - 490x^6z^2 + 63x^4y^2z^2 - 315x^4y^2t^2 + 1435x^4z^4 - 630x^4z^2t^2 + 315x^2y^2z^4 + 1890x^2y^2z^2t^2 - 616x^2z^6 + 315x^2z^4t^2 - 315t^2y^2z^4 + 22z^8).
$$

The first two of the above relations were already mentioned in [8].

Remark 4.3. The formula (1.3) looks indeed very similar to the Eichler-Selberg trace formula as given in [8], so one might ask whether our result gives a similar trace formula for Hecke operators on the subspace $S_k^{\text{odd}}(\Gamma_0(4))$ of cusp forms of weight $k$ on $\Gamma_0(4)$ with only odd $q$-powers in their Fourier expansion (this space is of course Hecke invariant). Unfortunately, computer experiments showed that this is in fact not the case: As soon as $\dim S_k^{\text{odd}}(\Gamma_0(4)) \geq 2$, i.e. $k \geq 10$, the cusp form we get is not a multiple of the generating function of traces of Hecke operators.
Acknowledgements

The author would like to thank Prof. Dr. Kathrin Bringmann for suggesting this topic as part of his PhD thesis [18]. He also thanks his colleagues at the Universität zu Köln, especially Dr. Ben Kane, Maryna Viazovska, and René Olivetto, for many fruit- and helpful discussions.

References

[1] K. Bringmann and B. Kane, Sums of class numbers and mixed mock modular forms, preprint, arXiv:1305.0112 [math.NT] (2013).
[2] K. Bringmann and J. Lovejoy, Overpartitions and class numbers of binary quadratic forms, Proc. Natl. Acad. Sci. USA 106 (2009), 5513–5516.
[3] K. Bringmann and K. Ono, The $f(q)$ mock theta function conjecture and partition ranks, Invent. Math. 165 (2006), 243–266.
[4] K. Bringmann and S. Zwegers, Rank-Crank-Type PDEs and Non-Holomorphic Jacobi Forms, Math. Res. Lett. 17 (2010), 589–600.
[5] B. Brown, N. J. Calkin, T. B. Flowers, K. James, E. Smith, and A. Stout, Elliptic Curves, Modular Forms, and Sums of Hurwitz Class Numbers, J. of Number Theory 128 (2008), 1847–1863.
[6] J. H. Bruinier and J. Funke, On two geometric theta lifts, Duke Math. J. 1 (2004), no. 125, 45–90.
[7] J. H. Bruinier, G. van der Geer, G. Harder, and D. Zagier, The 1-2-3 of Modular Forms, Springer-Verlag Berlin Heidelberg, 2008.
[8] H. Cohen, Sums Involving the Values at Negative Integers of L-Functions of Quadratic Characters, Math. Ann. 217 (1975), 271–285 (English).
[9] A. Dabolkhar, S. Murthy, and D. Zagier, Quantum Black holes, Wall Crossing, and Mock Modular Forms, preprint, arXiv:1208.4074 [hep-th] (2012).
[10] H. Davenport and H. L. Montgomery, Multiplicative Number Theory, Graduate Texts in Mathematics, vol. 74, Springer-Verlag, 2000.
[11] M. Eichler, On the Class Number of Imaginary Quadratic Fields and the Sums of Divisors of Natural Numbers, J. Ind. Math. Soc. 15 (1955), 153–180.
[12] M. Eichler and D. Zagier, The Theory of Jacobi Forms, Birkhäuser Boston, Basel, Stuttgart, 1985.
[13] E. Hecke, Theorie der Eisensteinschen Reihen höherer Stufe und ihre Anwendung auf Funktionentheorie und Arithmetik, Abh. Math. Sem. Univ. Hambg. 5 (1927), 199–224, (Werke, pp. 461–486).
[14] F. Hirzebruch and D. Zagier, Intersection Numbers of Curves on Hilbert Modular Surfaces and Modular Forms of Nebentypus, Inv. Math. 36 (1976), 57–113.
[15] A. Hurwitz, Über Relationen zwischen Klassenzahlen binärer quadratischer Formen von negativer Determinante, Berichte der königlich sächsischen Gesellschaft der Wissenschaften zu Leipzig, mathematisch-physikalische Klasse 36 (1884), 193–197, (Mathematische Werke Bd. 2, pp. 1–4).
[16] ———, Über Relationen zwischen Klassenzahlen binärer quadratischer Formen von negativer Determinante, Math. Ann. 25 (1885), 157–196, (Mathematische Werke Bd. 2, pp. 8–50).
[17] L. Kronecker, Über die Anzahl der verschiedenen Klassen quadratischer Formen von negativer Determinante, Journal Reine Angew. Math. 57 (1860), 248–255, (Werke, Bd. IV, pp. 185–195).
[18] M. H. Mertens, Mock Modular Forms and Class Numbers of Quadratic Forms, Ph.D. thesis, Universität zu Köln, expected 2014.
[19] K. Ono, Unearthing the visions of a master: harmonic Maass forms and number theory, Current Developments in Mathematics 2008 (2009), 347–454.
[20] G. Shimura, Modular Forms of Half Integral Weight, Ann. of Math. (2) 97 (1973), no. 3, 440–481.
[21] D. Zagier, Nombres de classes et formes modulaires de poids 3/2, C. R. Acad. Sci. Paris Sér. A-B 281 (1975), no. 21, Ai, A883–A886.
[22] S. Zwegers, Multivariable Appell Functions, 2010.
[23] ———, Mock Theta Functions, Ph.D. thesis, Universiteit Utrecht, 2002.

Mathematisches Institut der Universität zu Köln, Weyertal 86-90, D-50931 Köln, Germany
E-mail address: mmertens@math.uni-koeln.de
URL: http://www.mi.uni-koeln.de/~mmertens