SEM I - IN F I N I T E CO HOMOLOG Y OF $\mathcal{W}$-ALGEBRAS

Peter Bouwknegt\textsuperscript{1} \textsuperscript{†}, Jim McCarthy\textsuperscript{2} \textsuperscript{‡} and Krzysztof Pilch\textsuperscript{1} \textsuperscript{*}

\textsuperscript{1} Department of Physics and Astronomy  
University of Southern California  
Los Angeles, CA 90089-0484, USA

\textsuperscript{2} Department of Physics and Mathematical Physics  
University of Adelaide, Adelaide, SA 5001, Australia

Abstract

We generalize some of the standard homological techniques to $\mathcal{W}$-algebras, and compute the semi-infinite cohomology of the $\mathcal{W}_3$ algebra on a variety of modules. These computations provide physical states in $\mathcal{W}_3$ gravity coupled to $\mathcal{W}_3$ minimal models and to two free scalar fields.

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hepth@xxx/9302086  

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1. Introduction

An outstanding problem in $\mathcal{W}$-gravity is the computation of physical states. In particular, this has been studied as an application of BRST cohomology [1-7]. One of the ultimate goals is to achieve a better understanding of (quantum) $\mathcal{W}$-geometry through the computation of the ground ring and its associated symmetries [8]. From a mathematical point of view, the semi-infinite cohomology of a $\mathcal{W}$-algebra is interesting since some standard techniques for computing Lie algebra cohomology no longer apply due to nonlinearity of the algebra.

In this letter we will summarize various results on the computation of the BRST cohomology of $\mathcal{W}$-gravity coupled to $\mathcal{W}$-matter. Details and additional results will appear elsewhere [9]. Throughout the paper we will have the $\mathcal{W}_3$ algebra in mind, although some of the results will be formulated for more general $\mathcal{W}$-algebras while others can be generalized straightforwardly. We have chosen an approach that remains entirely within the context of $\mathcal{W}$-algebras. It is not inconceivable, though, that our results can also be deduced from analogous results for affine Lie algebras by quantum Drinfeld-Sokolov reduction.

In Section 2, apart from introducing notation, we establish some general results on the structure of $\mathcal{W}$-algebra modules. These results are needed for the cohomology computation, but are also interesting in their own right. In Section 3 we will calculate the cohomology of the BRST operator recently constructed in [4], on the product of an irreducible $\mathcal{W}_3$ minimal model module and a two-scalar Fock module (i.e. ‘$\mathcal{W}_3$ gravity coupled to a $\mathcal{W}_3$ minimal model’) for the states which satisfy the analogue of the Seiberg bound in ordinary gravity [10]. By specializing these results to the identity module ($c^M = 0$), where the BRST-operator reduces to the one constructed in [1], we find the physical states for ‘pure $\mathcal{W}_3$ gravity.’ In Section 4 we present some results concerning the cohomology of the BRST-operator on the product of two two-scalar Fock spaces (i.e. ‘$\mathcal{W}_3$ gravity coupled to two free scalar fields’ or the ‘$d = (2,2)$ $\mathcal{W}_3$ string’).

2. General results

The results in this section apply to the $\mathcal{W}$-algebras $\mathcal{W}[\mathfrak{g}]$ based on a simple simply-laced Lie algebra $\mathfrak{g}$ of rank $\ell$. We will use the notations $W$ for the Weyl group of $\mathfrak{g}$, $\hat{W}$ for the Weyl group of the (untwisted) affine Lie algebra $\hat{\mathfrak{g}}$, $P_+$ for the set of dominant integral weights of $\mathfrak{g}$, $\Delta_+$ for the set of positive roots of $\mathfrak{g}$, $Q_+ = \mathbb{Z}_+\Delta_+$ for the positive root lattice of $\mathfrak{g}$ and $\ell(w)$ for the length of a Weyl group element $w \in W$ (or $w \in \hat{W}$). For
background material on $\mathcal{W}$-algebras and a complete list of notations we refer the reader to the recent review [11] and references therein.

The algebra $\mathcal{W}[\mathfrak{g}]$ is generated by $L_n^{(i)}$, where $n \in \mathbb{Z}$ and $i$ runs over the orders of the independent Casimirs of $\mathfrak{g}$. In particular, $L_n = L_n^{(2)}$ are the generators of the Virasoro subalgebra. The Verma module $M(h^{(i)}, c)$ is defined, as usual, as the module induced from a highest weight vector $|h^{(i)}\rangle$ by the $L_n^{(i)}$, $n < 0$, and is labelled by the $L_0^{(i)}$ eigenvalues $h^{(i)}$ on the highest weight vector.

Let $F(\Lambda, \alpha_0)$ denote the Fock space of $\ell$ scalar fields $\phi^k(z)$, normalized such that $\phi^k(z)\phi^l(w) = -\delta^{kl} \ln(z - w)$, and coupled to a background charge $\alpha_0 \rho$, where $\rho$ is the principal vector of the Lie algebra $\mathfrak{g}$. The Fock space vacuum $|\Lambda\rangle$ is labelled by a vector $\Lambda$ in the weight lattice of $\mathfrak{g}$ such that $p^k|\Lambda\rangle = \Lambda^k|\Lambda\rangle$.

A realization of $\mathcal{W}[\mathfrak{g}]$ on the Fock space $F(\Lambda, \alpha_0)$ of $\ell$ scalar fields can be constructed by means of the quantum Drinfeld-Sokolov reduction. In particular, we have the following expression for the stress-energy tensor

$$ T(z) = -\frac{1}{2}(\partial \phi(z) \cdot \partial \phi(z)) - i\alpha_0 \rho \cdot \partial^2 \phi(z). \quad (2.1) $$

It generates a Virasoro subalgebra of central charge $c = \ell - 12\alpha_0^2|\rho|^2$, while

$$ h(\Lambda) \equiv h^{(2)}(\Lambda) = \frac{1}{2}(\Lambda, \Lambda + 2\alpha_0 \rho). \quad (2.2) $$

The free field realization also induces homomorphisms

$$ M(\Lambda, c) \xrightarrow{i'} F(\Lambda, \alpha_0) \xrightarrow{i''} M(\Lambda, c)^*, \quad (2.3) $$

where we have written $M(\Lambda, c)$ instead of $M(h^{(i)}(\Lambda), c)$, and $M(\Lambda, c)^*$ is the contragradient Verma module. It should be kept in mind, though, that the map $\Lambda \to (h^{(i)}(\Lambda))$ is not 1–1 but rather $(h^{(i)}(\Lambda)) = (h^{(i)}(\Lambda'))$ iff $\Lambda + \alpha_0 \rho = w(\Lambda' + \alpha_0 \rho)$ for some Weyl group element $w \in W$.

Let $M_N(\Lambda, c)$ and $F_N(\Lambda, \alpha_0)$ denote the subspaces at energy level $L_0 = h(\Lambda) + N$, and let $\{w_i\}$ and $\{v_i\}$ denote $p_\ell(N)$-dimensional bases of $M_N(\Lambda, c)$ and $F_N(\Lambda, \alpha_0)$, respectively. The maps $i'$ and $i''$ can be analyzed through the determinants $S'_N(\Lambda, \alpha_0)$ and $S''_N(\Lambda, \alpha_0)$ of the corresponding Shapovalov forms,

$$ S'_N(\Lambda, \alpha_0) = \det (\langle v_i | i'(w_j) \rangle), \quad S''_N(\Lambda, \alpha_0) = S'_N(w_0(\Lambda + \alpha_0 \rho) - \alpha_0 \rho, \alpha_0), \quad (2.4) $$

where $\langle \cdot | \cdot \rangle$ denotes the (canonical) form on $F(\Lambda, \alpha_0)$. We have used the isomorphism $F(\Lambda, \alpha_0)^* \cong F(w_0(\Lambda + \alpha_0 \rho) - \alpha_0 \rho, \alpha_0)$, where $w_0$ is the longest element in $W$. Defining $\alpha_\pm$ by $\alpha_0 = \alpha_+ + \alpha_-$, $\alpha_+ \alpha_- = -1$ we have
Theorem 2.1.

\[
\begin{align*}
\det S'_N(\Lambda, \alpha_0) &\sim \prod_{\alpha \in \Delta_+} \prod_{r, s \in \mathbb{N}, rs \leq N} ((\Lambda + \alpha_0 \rho, \alpha) - (r \alpha_+ + s \alpha_-))^{p_N(N-rs)}, \\
\det S''_N(\Lambda, \alpha_0) &\sim \prod_{\alpha \in \Delta_+} \prod_{r, s \in \mathbb{N}, rs \leq N} ((\Lambda + \alpha_0 \rho, \alpha) + (r \alpha_+ + s \alpha_-))^{p_N(N-rs)}.
\end{align*}
\]

The proof parallels the one given in [12] for the Virasoro algebra. The product of the two determinants is, of course, proportional to the usual Kac determinant of the Shapovalov form on \( M(\Lambda) \) (see e.g. [11] and references therein).

As an immediate consequence of Theorem 2.1 we have

**Corollary 2.2.**

(a) \( F(\Lambda, \alpha_0) \cong \begin{cases} M(\Lambda, c) & \text{if } (\Lambda + \alpha_0 \rho, \alpha) \notin (\mathbb{N} \alpha_+ + \mathbb{N} \alpha_-) \text{ for all } \alpha \in \Delta_+, \\
M(\Lambda, c)^* & \text{if } (\Lambda + \alpha_0 \rho, \alpha) \notin - (\mathbb{N} \alpha_+ + \mathbb{N} \alpha_-) \text{ for all } \alpha \in \Delta_+.
\end{cases} \)

In particular, if \( (\Lambda + \alpha_0 \rho, \alpha) \notin (\mathbb{N} \alpha_+ + \mathbb{N} \alpha_-) \) for all \( \alpha \in \Delta \), then \( M(\Lambda, c) \) (and thus also \( F(\Lambda, \alpha_0) \)) is irreducible.

(b) For \( \alpha_0^2 \leq -4 \) or, equivalently, \( c \geq c_{\text{crit}} - \ell = \ell + 48|\rho|^2 \) we have

\[
F(\Lambda, \alpha_0) \cong \begin{cases} M(\Lambda, c) & \text{for } i(\Lambda + \alpha_0 \rho) \in \eta D_+, \\
M(\Lambda, c)^* & \text{for } -i(\Lambda + \alpha_0 \rho) \in \eta D_+,
\end{cases}
\]

where \( D_+ = \{ \lambda \in h_\mathbb{C}^* \mid (\lambda, \alpha) \in \mathbb{R}_+, \forall \alpha \in \Delta_+ \} \) denotes the fundamental Weyl chamber, and \( \eta = \text{sign}(-i\alpha_0) \).

Part (a) of the theorem follows from the absence of zeros in the corresponding determinants, whilst part (b) follows from (a) by observing that \( -i \alpha_\pm \in \eta \mathbb{R}_+ \) for \( c \geq c_{\text{crit}} - \ell \) (see [13] for \( \mathfrak{g} = \mathfrak{sl}(2) \)).

The embedding structure of the Fock spaces \( F(\Lambda, \alpha_0) \) follows, in principle, from Theorem 2.1. It is quite complicated in general, except for \( c = \ell \) (i.e. \( \alpha_0 = 0 \)) where we have the following result

**Theorem 2.3.** If \( w \in W \) such that \( w\Lambda \in P_+ \) then

\[
F(\Lambda) = \bigoplus_{\substack{\beta \in Q_+ \\
w\Lambda + \beta \in P_+}} m(w\Lambda; \beta) L(w\Lambda + \beta), \tag{2.5}
\]

where, for \( \Lambda \in P_+ \) and \( \beta \in Q_+ \), the multiplicity \( m(\Lambda; \beta) \) (with which the \( c = \ell \) irreducible \( \mathcal{W} \)-module \( L(\Lambda + \beta) \) occurs in the direct sum decomposition of \( F(\Lambda) \)) is equal to the
multiplicity of the weight $\Lambda$ in the irreducible finite dimensional representation of $\mathfrak{g}$ with highest weight $\Lambda + \beta$.

For $\mathfrak{sl}(3)$ we have the following generating function for these multiplicities

$$
\sum_{\beta \in Q^+_+} m(\Lambda; \beta)e^{\beta} = \frac{1}{(1 - e^{\alpha_1})(1 - e^{\alpha_2})(1 - e^{\alpha_3})}
- \frac{e^{(\Lambda + \rho, \alpha_1)\alpha_2}}{(1 - e^{\alpha_2})(1 - e^{\alpha_3})(1 - e^{\alpha_1 + 2\alpha_2})}
- \frac{e^{(\Lambda + \rho, \alpha_2)\alpha_1}}{(1 - e^{\alpha_1})(1 - e^{\alpha_3})(1 - e^{2\alpha_1 + \alpha_2})}.
$$

Complete reducibility of the $c = \ell$ Fock spaces $F(\Lambda)$ follows, as usual, from the existence of a positive definite hermitian form on $F(\Lambda)$. The rest of the theorem is proved by the standard construction of a set of singular vectors in $F(\Lambda)$ using screening operators. Completeness of this set then follows by comparing the characters on both sides of (2.5).

In the course of the cohomology computation we will need to know resolutions of the irreducible $\mathcal{W}$-modules $L(\Lambda)$. Completely degenerate modules occur for $\alpha^2_+ = p'/p$, where $p, p'$ are two relatively prime positive integers. If we label $\Lambda$ through $\Lambda = \alpha_+ \Lambda^{(+)} + \alpha_- \Lambda^{(-)}$, then the set of completely degenerate modules given by $\Lambda^{(+)} \in P^{p-h^\vee}_+, \Lambda^{(-)} \in P^{p'-h^\vee}_+$ constitute the spectrum of the so-called $\mathcal{W}$ minimal models. [Here, $P^k_+$ denotes the set of integrable weights of level $k$ of the affine Lie algebra $\hat{\mathfrak{g}}$.]

Resolutions of the $c < \ell$ minimal modules $L(\Lambda^{(+)}, \Lambda^{(-)})$ in terms of Fock spaces were conjectured in [14,15]. They can be obtained by performing a quantum Drinfeld-Sokolov reduction on the corresponding free field resolution for the underlying affine Lie algebra $\hat{\mathfrak{g}}$. We will not repeat them here. For the resolutions in terms of Verma modules we propose

**Conjecture 2.4.** Let $L(\Lambda^{(+)}, \Lambda^{(-)})$, $\Lambda^{(+)} \in P^{p-h^\vee}_+, \Lambda^{(-)} \in P^{p'-h^\vee}_+$ be an irreducible $\mathcal{W}$ minimal model module. We have a resolution $(C^{(i)}L(\Lambda^{(+)}, \Lambda^{(-)}), d')$, $i \leq 0$, of $L(\Lambda^{(+)}, \Lambda^{(-)})$ with terms

$$
C^{(i)}L(\Lambda^{(+)}, \Lambda^{(-)}) \cong \bigoplus_{\{w \in \hat{W} \mid \ell(w) = -i\}} M(\alpha_+(w(\Lambda^{(+)} + \rho) - \rho) + \alpha_- \Lambda^{(-)})..
$$

Note that the validity of Conjecture 2.4 is not at all obvious. First of all, there will be singular vectors in $M(\Lambda)$ beyond the ones that are used to build the resolution of Conjecture 2.4. Secondly, since the generator $W_0$ will in general not be diagonalizable on Verma modules [16], there is no a priori reason why all the terms in the resolution should consist of modules that are induced from one-dimensional representations of the abelian
subalgebra \( \{ L_0^{(i)} \} \), i.e. Verma modules. Nevertheless, the various examples we have gone through numerically were not in contradiction with the above conjecture.

For \( c = \ell \) irreducible modules \( L(\Lambda) \) the situation is rather more problematic. Naively, one might still expect Verma module resolutions analogous to the one in Conjecture 2.4 (apart from the fact that now the resolution runs over the finite Weyl group). This naive guess turns out to be wrong, precisely because of the problems mentioned above. A study of examples suggests that one can still construct resolutions of \( L(\Lambda) \), but now including generalized Verma modules (modules induced from nontrivial representations of the subalgebra \( \{ L_0^{(i)} \} \)). We will return to this in Section 4 and [9].

In contrast, the \( c = \ell \) analogue of the Fock space resolution seems straightforward

**Conjecture 2.5.** Let \( \Lambda \in P_+ \). We have a resolution \((C^{(i)}(L(\Lambda)), d')\) of the \( c = \ell \) irreducible module \( L(\Lambda) \) with terms

\[
C^{(i)}(L(\Lambda)) \cong \bigoplus_{\{w \in W | \ell(w) = -i\}} F(\Lambda + \rho) - \rho).
\]

### 3. \( W_3 \) gravity coupled to \( W_3 \) minimal models

From now on we will restrict the discussion to the \( W_3 \) algebra generated by \( L_n \) and \( W_n = L_n^{(3)} \), \( n \in \mathbb{Z} \). Let \( V^M \) and \( V^L \) be two arbitrary positive energy \( W_3 \) modules, and let \( F^{gh(1)} \) and \( F^{gh(2)} \) denote the Fock space of a set of first order anticommuting (ghost) fields \( (b^{[1]}, c^{[1]}) \) and \( (b^{[2]}, c^{[2]}) \) of conformal dimensions \((2, -1)\) and \((3, -2)\), respectively.

Consider the BRST operator \( d = \oint dz 2\pi i J(z) \) acting on \( V^M \otimes V^L \otimes F^{gh(1)} \otimes F^{gh(2)} \), where \([4,5]\)

\[
J = c^{[2]}(1/\sqrt{\beta^M} W^M - i/\sqrt{\beta^L} W^L) + c^{[1]}(T^M + T^L + \frac{1}{2} T^{gh(1)} + T^{gh(2)}) + (T^M - T^L)b^{[1]} c^{[2]} \partial c^{[2]} + \mu \partial b^{[1]} c^{[2]} \partial^2 c^{[2]} + \nu b^{[1]} c^{[2]} \partial^3 c^{[2]}.
\]

(3.1)

Here \( \mu = \frac{2}{3} \nu = \frac{1}{10 \beta^M} (1 - 17 \beta^M) \), while \( \beta = 16/(22 + 5c) \). The operator is nilpotent provided \( c^M + c^L = 100 \), so that on defining \( \alpha_\pm = 1/2 (a_0^M \mp i a_0^L) \) we have \( \alpha_+ \alpha_- = -1 \). The cohomology will be denoted by \( H(d, V^M \otimes V^L) \). It is graded by ghost number \( gh(\cdot) \), where \( gh(c^{[1]}) = gh(c^{[2]}) = -gh(b^{[1]}) = -gh(b^{[2]}) = 1 \), and the ghost number of the physical

\[1\] Since, for the most part, we will be using only generic properties of the \( W_3 \) BRST operator we expect most of the results to generalize immediately to other \( W \)-algebras.
vacuum $|0\rangle_{gh}$ (annihilated by all positively-moded ghost oscillators, as well as by the zero modes $b_0^{[1]}$ and $b_0^{[2]}$) in the ghost sector is chosen to be zero.

In the sequel, the module $V^L$, representing the $\mathcal{W}_3$ gravity (‘Liouville’) sector, will be assumed to be a Fock module $F(\Lambda^L, \alpha_0^L)$, while for the matter sector we will be interested in either irreducible modules $L(\Lambda)$ or Fock modules $F(\Lambda^M, \alpha_0^M)$. Our strategy will be to reduce the computation to that of the cohomology on a product of a Verma and contragradient Verma module. In the $\mathcal{W}_3$ gravity sector we will therefore restrict to states satisfying $-i(\Lambda^L + \alpha^L_0) \in \eta^L D_+$, and apply Corollary 2.2 (b). In the matter sector we will represent irreducible modules $L(\Lambda)$ by their resolution in terms of Verma modules (Conjecture 2.4), and compute the BRST cohomology using the resulting double complex (see, e.g. [17] for some background material). In the case of $c^M = 2$ Fock modules, we will first use the decomposition of the Fock space into irreducible modules (Theorem 2.3), and then proceed as above.

A few comments are in order. To compute analogous cohomologies $H(d, V^M \otimes V^L)$ in the Virasoro or affine Lie algebra case it often turns out to be convenient to use the triangular decomposition $g \cong n_+ \oplus h \oplus n_-$, and pass to the cohomology relative to the Cartan subalgebra $h$. This relative cohomology $H(g, h; V^M \otimes V^L)$ can then be studied by means of a spectral sequence whose first term is given by (see e.g. [17] and references therein for more details)

$$E_1 \cong (H(n_-, V^M) \otimes H(n_+, V^L))_h.$$

The absolute cohomology is then eventually recovered from the relative one by a long exact sequence.

This procedure, however, does not work for $\mathcal{W}$-algebras for a variety of reasons. First of all, due to the nonlinear terms in their defining relations, $\mathcal{W}$-algebras do not have, strictly speaking, a triangular decomposition. As a consequence there does not exist a BRST operator corresponding to some nilpotent subalgebra, and hence the ‘splitting’ (3.2) cannot make sense for $\mathcal{W}$-algebras. Also, the passage to the cohomology relative to the abelian subalgebra generated by $\{L^\text{tot}_0, W^\text{tot}_0\}$, where $L^\text{tot}_0 = \{d, b_0^{[1]}\}$, $W^\text{tot}_0 = \{d, b_0^{[2]}\}$, is problematic. Although the subset of states annihilated by $L^\text{tot}_0, W^\text{tot}_0, b_0^{[1]}$ and $b_0^{[2]}$ do form a subcomplex, the complicated expression for $W^\text{tot}_0$ clearly makes it hard to determine this subcomplex explicitly (i.e. to write down a basis of states), and it is therefore unsuitable for calculations. And, even if one could determine this relative cohomology, it is not clear
how to deduce from it the absolute cohomology due to the nondiagonalizability of \( W_{0}^{tot} \). It is, for instance, no longer obvious that the absolute cohomology should be concentrated on the states annihilated by \( W_{0}^{tot} \). However, one can still consider the cohomology relative to \( L_{0}^{tot} \) — a step which in fact is necessary as it effectively reduces the computation to one on a finite-dimensional complex. Note that the problems above are somewhat reminiscent of those occurring for the semi-infinite cohomology of the super-Virasoro algebra.

As outlined before, the computation of the cohomology \( H(d, L(\Lambda^{(+)}, \Lambda^{(-)}) \otimes F(\Lambda^{L}, \alpha_{0}^{L})) \) in the case of states satisfying \( -i(\Lambda^{L} + \alpha_{0}^{L} \rho) \in \eta^{L} D_{+} \) can be reduced to a computation of the cohomology on (contragradient) Verma modules, for which we have the following technical result

**Theorem 3.1.** The cohomology \( H(d, M(\Lambda^{M}) \otimes M(\Lambda^{L})^{*}) \) is nonvanishing iff \( w(\Lambda^{M} + \alpha_{0}^{M} \rho) = -i(\Lambda^{L} + \alpha_{0}^{L} \rho) \) for some \( w \in W \), in which case it is spanned by \( \{ v, c_{0}^{[1]} v, c_{0}^{[2]} v, c_{0}^{[1]} c_{0}^{[2]} v \} \) where \( v \) denotes the highest weight vector of \( M(\Lambda^{M}) \otimes M(\Lambda^{L})^{*} \).

Despite the apparent difficulties in dealing with \( W \) algebras mentioned above, it turns out that the proof of Theorem 3.1 is not too difficult, and can be given in complete analogy to the standard proof of e.g. \( H(n, M) \cong \mathcal{C} \) in the Virasoro or affine Lie algebra case. Namely, by introducing a filtration whose corresponding spectral sequence reduces to the Koszul complex in the first term and collapses after that (cf. e.g. [18]).

One can easily verify that the ‘quartet’ structure of Theorem 3.1 will persist in the cohomology \( H(d, L(\Lambda^{(+)}, \Lambda^{(-)}) \otimes M(\Lambda^{L})^{*}) \) by examining the zig-zag procedure in the double complex of BRST cohomology and resolution of \( L(\Lambda^{(+)}, \Lambda^{(-)}) \). In the rest of the paper we will therefore adopt the terminology ‘prime state’ for the lowest ghost number state of each quartet of states of ghost numbers \( (G, G+1, G+1, G+2) \), and only mention results for these prime states, leaving it implicitly understood that to every prime state there is an associated quartet.

Now, restricting the Liouville momentum to \( -i(\Lambda^{L} + \alpha_{0}^{L} \rho) \in \eta^{L} D_{+} \), using Corollary 2.2 and using the resolution of Conjecture 2.4 for \( L(\Lambda^{(+)}, \Lambda^{(-)}) \), Theorem 3.1 gives the following result

\[ 2 \text{ They have to be ‘generalized zero eigenstates’ of } W_{0}^{tot}, \text{ though, i.e. there should exist an } N \text{ such that } (W_{0}^{tot})^{N} | \psi \rangle = 0. \]

\[ 3 \text{ We have adopted the terminology ‘prime state’ from [6], where it was introduced in a different context.} \]
Theorem 3.2. For $\alpha_\pm = \frac{1}{2}(\alpha_0^M \mp i \alpha_0^L)$, consider $-i(\Lambda^L + \alpha_0^L \rho) \in \eta^L D_+$, where $\eta^L = \text{sign}(-i\alpha_0^L)$. Then $H(d, L(\Lambda^{(+)}), L(\Lambda^{(-)})) \neq 0$ iff there exists a $w \in \hat{W}$ and a $w' \in W$ such that $w'((\alpha_+ w(\Lambda^{(+)}) + \alpha_-(\Lambda^{(-)} + \rho)) = -i(\Lambda^L + \alpha_0^L \rho)$, in which case there is a (unique) prime state of ghost number $-\ell(w)$.

The number of affine Weyl group elements with a specified length $\ell$, or alternatively the number of prime physical states with ghost number $-\ell$, can be read off from the Poincaré series of $\hat{W}$ (see e.g. [19]) which for $\hat{sl}(3)$ reads

$$P(t) = \sum_{w \in \hat{W}} t^{\ell(w)} = \frac{(1 + t + t^2)}{(1 - t)^2} = 1 + 3 \sum_{n \geq 1} nt^n. \quad (3.3)$$

The level at which this prime state occurs is given by $(\Lambda^{(+)}) + \rho - w(\Lambda^{(+)}) + \Lambda^{(-)} + \rho$.

In particular, if we take the identity representation $1 \cong L(0,0)$ at $c^M = 0$ (i.e. $p' = 4, p = 3$), the BRST operator (3.1) reduces to the one of [1], which has been used in recent discussions of two-scalar gravity (pure $\mathcal{W}_3$ gravity) [6]. In this case Theorem 3.2 yields one $G = 0$ prime state at energy level 0, three $G = -1$ prime states at energy levels 1, 1, 2, six $G = -2$ prime states at energy levels 3, 3, 4, 4, 5, 5, nine $G = -3$ prime states at energy levels 4, 6, 6, 8, 8, 9, 9, 11, 11, and so forth. Some of these states have been explicitly constructed in [6]. As a check on Conjecture 2.4 we have explicitly constructed some of the remaining ones.

4. The $\mathcal{W}_3$ string

In this section we summarize some results on the cohomology of a tensor product of two Fock space modules, $F(\Lambda^M, \alpha_0^M) \otimes F(\Lambda^L, \alpha_0^L)$. A striking difference between the present problem and its analogue for the Virasoro algebra is that the direct approach developed in [20], which employs a spectral sequence arising from a natural degree on the set of oscillators in the Fock spaces, seems to be difficult to implement: although it is still rather straightforward to identify a spectral sequence whose first term gives the usual Virasoro semi-infinite cohomology, the subsequent terms of this sequence are of such complexity that a direct analysis seems impossible without additional insights into the structure of the BRST complex. For that reason we will instead follow the path of Section 3 and reduce the computation to that of the cohomology of Verma modules.
For generic values of the momenta $\Lambda^M$ and $\Lambda^L$ one expects to find only level 0 (‘tachyonic’) states in the cohomology [4], which is the analogue of the similar result for the Virasoro algebra [21,20]. More precisely, we parametrize the momenta $\Lambda^M$ and $\Lambda^L$ by
\[
\Lambda^M + \alpha^M_0 \rho = \alpha_+ \Lambda^{(+)} + \alpha_- \Lambda^{(-)},
\]
\[-i(\Lambda^L + \alpha^L_0 \rho) = \alpha_+ \Lambda^{(+)} - \alpha_- \Lambda^{(-)}.
\]
We call them generic iff there is no positive root $\alpha \in \Delta_+$ such that
\[
(\Lambda^{(+)}, \alpha) \in \mathbb{Z}, \quad (\Lambda^{(-)}, \alpha) \in \mathbb{Z}, \quad (\Lambda^{(+)}, \alpha)(\Lambda^{(-)}, \alpha) > 0.
\]

Then we have

**Theorem 4.1.** For generic momenta $\Lambda^M$ and $\Lambda^L$ as defined above, $H(d, F(\Lambda^M, \alpha^M_0) \otimes F(\Lambda^L, \alpha^L_0)) \neq 0$ iff there exists $w \in W$ such that $w(\Lambda^M + \alpha^M_0 \rho) = -i(\Lambda^L + \alpha^L_0 \rho)$, in which case it is spanned by the states $v$, $c^{[1]}_0 v$, $c^{[2]}_0 v$ and $c^{[1]}_0 c^{[2]}_0 v$, where $v = |\Lambda^M \rangle \otimes |\Lambda^L \rangle \otimes |0 \rangle_{gh}$ is the physical vacuum.

Let us now restrict to the case of $d = (2,2)$ $\mathcal{W}_3$ string, namely set $c^M = 2$ (and $\alpha_\pm = \pm 1$). The structure of the cohomology then simplifies considerably because of the following observation

**Theorem 4.2.** The cohomology spaces $H^{(n)}(d, F(\Lambda^M, 0) \otimes F(\Lambda^L, \alpha^L_0))$ carry a fully reducible representation of $sl(3)$. The $sl(3)$ generators are explicitly given by the zero modes of a level-1 Frenkel-Kac-Segal vertex operator construction in terms of matter fields only.

Moreover, if we further restrict the Liouville momenta to the region $-i(\Lambda^L + \alpha^L_0 \rho) \in \eta^L D_+$, we can combine Corollary 2.2 and Theorem 2.3 and reduce the computation of $H(d, F(\Lambda^M, 0) \otimes F(\Lambda^L, \alpha^L_0))$ to that of $H(d, L(\Lambda) \otimes M(\Lambda^L)^*)$ for the $c = 2$ irreducible modules $L(\Lambda)$, which appear in the decomposition of $F(\Lambda^M, 0)$. The latter cohomology can in principle be determined as in Section 3 (cf. Theorem 3.2) provided one knows resolutions of $L(\Lambda)$’s in terms of Verma modules.

As we have discussed in Section 2, the natural assumption, that those resolutions are similar to the ones in terms of Fock spaces given by Conjecture 2.5, turns out to be incorrect! In fact, by assuming the ‘naive’ resolutions one would not reproduce all the physical states that have been constructed explicitly in [7], most notably those which by the descent equations [8,7] give rise to the $sl(3)$ currents above.
To clarify this we studied explicitly singular vectors in $c = 2$ Verma modules $M(\Lambda)$ for low lying dominant integral weights $\Lambda \in P_+$. Specifically, we have constructed all singular vectors and determined their embedding patterns up to level eight in the Verma modules $M(\Lambda)$ for $\Lambda$ given by (in Dynkin labels) $(0, 0), (1, 0), (0, 1), (2, 0), (0, 2)$ and $(1, 1)$. From those explicit examples, the details of which will be presented in [9], the following picture emerges. Singular vectors are not eigenstates of $W_0$ in general, rather they fall into indecomposable representations of the subalgebra $\{L_0, W_0\}$. (The simplest example of this phenomenon occurs for level one singular vectors in the Verma module with the weight $(0, 0)$, and has already been discussed in [16].) Thus one is led to consider ‘generalized Verma modules.’ We will denote by $M(\Lambda)_\nu$ the generalized Verma module with the highest weight subspace spanned by such an indecomposable $\nu$-dimensional representation, with the $L_0$ eigenvalue $h(\Lambda)$, and the generalized $W_0$ eigenvalue $w(\Lambda)$, i.e. $w(\Lambda)$ is a $\nu$-times degenerate root of the characteristic equation.

Assuming that the submodules of (generalized) Verma modules are generated by singular vectors, an embedding pattern then determines a resolution of the irreducible module $L(\Lambda)$ in terms of (generalized) Verma modules. An independent consistency check is provided by examining the resulting characters.

To illustrate this let us consider resolutions of the irreducible module $L(1, 0)$ as an example. [To simplify the notation we will denote the weights by their Dynkin labels.] The Fock space resolution of Conjecture 2.5 reads

$$0 \rightarrow F(3, 2) \rightarrow F(4, 0) \oplus F(1, 3) \rightarrow F(0, 2) \oplus F(2, 1) \rightarrow F(1, 0) \rightarrow 0 \quad (4.3)$$

while that in terms of (generalized) Verma modules is

$$0 \rightarrow M(3, 2) \rightarrow M(1, 3) \oplus M(3, 2)_2 \rightarrow M(4, 0) \oplus M(2, 1) \oplus M(1, 3)_2 \rightarrow M(0, 2) \oplus M(2, 1)_2 \rightarrow M(1, 0) \rightarrow 0 \quad (4.4)$$

Finally, an analogue of Theorem 3.1 for generalized Verma modules holds [9], so that the cohomology can now be computed as in Section 3 by considering a double complex. In all the examples we have considered, one finds that although (generalized) Verma module resolutions can extend beyond the (minus) third term, as in (4.3), the resulting (prime) physical states have ghost numbers $G$ between $-3$ and 0. The cohomology states at $G = -3$ form the ground ring of the theory [8]. The following conjecture gives a lower bound on the number of elements of the ground ring.
Conjecture 4.3. Given a pair of integral weights \((\Lambda^M, -i\Lambda^L)\), let \(w \in W\) be such that \(w\Lambda^M \in P_+\). Then there are ground ring elements \(\phi_{(\Lambda^M, -i\Lambda^L)}\) provided \(w\Lambda^M + \beta = -i\Lambda^L\) for some \(\beta \in Q_+\). The multiplicity is given by \(m(w\Lambda^M; \beta)\), and the energy level at which it occurs is given by \(\frac{1}{2}|\Lambda^L + 2\rho|^2 - \frac{1}{2}|\Lambda^M|^2\).

This conjecture follows from the decomposition of the Fock space module in Theorem 2.3 and the fact that each irreducible module \(L(\Lambda)\) appears to have the Verma module \(M(r_3(\Lambda + \rho) - \rho)\) in its resolution, which gives rise to a \(G = -3\) state. Also all the ground ring elements at level less than 8 are obtained in this way. It appears that at levels 8 and higher there are additional generators in the ground ring, so a complete characterization of all \(G = -3\) still remains an open problem.

Among the states above, in particular we find one state at energy level 4, namely \(\phi_{(0,0)}\), which is the \(SL(2, \mathbb{R})\) vacuum. Then there are six states at energy level 6

\[
\begin{align*}
x_1 &= \phi_{(\Lambda_1, \Lambda_1)}, & x_2 &= \phi_{(\Lambda_2 - \Lambda_1, \Lambda_1)}, & x_3 &= \phi_{(-\Lambda_2, \Lambda_1)}, \\
y_1 &= \phi_{(-\Lambda_1, \Lambda_2)}, & y_2 &= \phi_{(\Lambda_1 - \Lambda_2, \Lambda_2)}, & y_3 &= \phi_{(\Lambda_2, \Lambda_2)};
\end{align*}
\]

which transform in the \(3 + \bar{3}\) under \(sl(3)\). (Note that \(x_1\) is precisely the state arising from the resolution \([1.4]\).) The states \(x_1, x_3, y_1, y_3\) were explicitly constructed in [7]. The remaining two can be obtained by acting with the \(sl(3)\) generators of Theorem 4.2.

Consider now the ring of operators corresponding to the states \(x_1, \ldots, y_3\). By examining the cohomology at level 8 one concludes that, after a suitable normalization, their product must satisfy a constraint \(x_1y_1 + x_2y_2 + x_3y_3 = 0\). Assuming that there are no vanishing relations other than those dictated by the absence of cohomology at certain momenta, we find the ring generated by those six operators to consist precisely of elements listed in Conjecture 4.3. It carries a representation of \(sl(3)\) under which it decomposes into a direct sum of irreducible representations, every inequivalent irreducible finite-dimensional representation occurring exactly once (this is called a ‘model space’ for \(sl(3)\)). The appearance of this ‘model space’ seems to have been anticipated in [22]. This ring is easily seen to be equivalent to the ring of harmonic polynomials (traceless tensors with both covariant and contravariant indices) on the Euclidean 6-plane [23].

We have also verified that all higher ghost number states in the regime \(-i(\Lambda^L + \alpha_0^L\rho) \in \eta^L D_+\) explicitly constructed in [7] arise from those generalized resolutions.

\footnote{We have also checked explicitly that, in the notation of [7], \(J_{\alpha_3}\) maps \(x_i\) onto \(\gamma_{\alpha_0}\), even without the addition of BRST exact terms.}
Finally, let us make some remarks on the general case. In the Virasoro case there exists an isomorphism (as a Virasoro module)

\[ F(\Lambda^M, \alpha^M_0) \otimes F(\Lambda^L, \alpha^L_0) \cong F(\Lambda^{M'}, \alpha^{M'}_0) \otimes F(\Lambda^{L'}, \alpha^{L'}_0), \]

(4.6)

whenever the labels are related by an \(SO(2,\mathcal{C})\) rotation [24,21]. As a consequence, the cohomologies on these spaces are isomorphic. In particular, the Fock space cohomologies for \(c^M < 1\) can be calculated from the \(c^M = 1\) cohomology \(H(d, F(\Lambda^M, 0) \otimes F(\Lambda^L, \alpha^L_0))\) by \(SO(2,\mathcal{C})\) rotation. Given two two-scalar Fock \(W\)-modules, their tensor product is not a \(W\) module, so a naive generalization of (4.6) does not exist. One could look for more general vector space isomorphisms that intertwine with the BRST operator (these would necessarily have to act on the ghost Fock spaces too). We have not found any. Nevertheless, the explicit construction of one-parameter sets of physical states [7] suggests that the cohomology \(H(d, F(\Lambda^M, \alpha^M_0) \otimes F(\Lambda^L, \alpha^L_0))\) at different values of \(c^M\) might still be isomorphic.

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