On the equivalency of Teleparallel Gravity (TG) and Einstein Gravity (EG)

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Abstract. Teleparallel gravity (TG) is one of formulation of gravity to bring Einstein gravity (EG) into gauge scenario, like the other interactions in nature. TG and EG are regarded as two equivalent theories. However, not all physicists have such view. They have also different way to view the equivalency. In this article, the discourse on the equivalency is discussed.

1. Introduction

Gravitation founded on universality principle: all of the things which has difference of mass and composition feel the same gravitation, and if free fall, has same path [1]. Einstein gravity based on spacetime with Lorentzian metric $g_{\mu\nu}$. The most fundamental field of EG is the metric field $g_{\mu\nu}$, other important field like connection (called Levi-Civita connection) and curvature are obtained from the field [2].

Based on gauge theory which formulated by Yang-Mills in 1954, physicists have an idea to bring gravity into gauge scenario, i.e. to make gravity as an interaction like the other interactions in nature, electromagnetic, strong, and weak interactions. One of the gauge gravitation theory is teleparallel gravity. Gravity in the theory is regarded instead of as geometry as a force. Levi-Civita connection playing important role in EG that has non vanishing curvature but vanishing torsion, is replaced with by Weitzenbök connection in TG that has curvature vanish but non vanishing torsion.

Fundamentally, although EG and TG are different in describing gravitational field, theory of teleparallel gravity and Einstein’s gravity is equivalent. It can be view from Lagrangian of EG and TG, and also the equivalent of teleparallel field equation and Einstein field equation [1]. Some of physicists believe that the theories is different. Garecki in [3] thought that there is nothing new in TG, EG is better than TG because EG is invariant of tetrad changing, but TG is not. Indeed, TG is only invariant under global Lorentz transformation. This opinion is in accordance with the thoughts of Combi and Romero [4]. Furthermore, Knox in [5] asked the theoreics is under-determination. But the geometry of TG is not clear, which one the geometry of spacetime is used, Weitzenbök spacetime or Riemannian spacetime like EG. It is because of splitting of gravitational effect and inertial effect.

On the other hand, some physicists is agree with the statement. The discussion on that discourse based on some articles: [1], [6], [7], and [8]. There is an adagium that two are equivalent, if they have the same prediction for every case. Meanwhile, not all of the theory is true for all of the case.
2. Einstein Gravity

Einstein Gravity is based on two fundamental principles: all of the law in physics is invariant under all coordinate systems and there is a reference where the gravitation vanishes locally, called inertial reference. Gravitational field equation of EG is:

\[ G_{\mu\nu} = 8\pi G T_{\mu\nu}, \]  

(1)

\( G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \) is Einstein tensor as potential of the gravitational field, \( R_{\mu\nu} \) and \( R \) are Ricci tensor and Ricci scalar related to Einstein tensor, and \( T_{\mu\nu} \) is energy-momentum tensor describing the source of gravity.

Based on the equation (1), gravitation is related to geometry of spacetime which is described by metric \( g_{\mu\nu} \) and connection \( \Gamma^\sigma_{\mu\nu} \). Generally, the metric and connection are independent. Metric is related how to measure distance and angel, whereas the connection give information about how vectors change under parallel transport.

In EG, the connection as the Levi-Civita connection \( (\Gamma^a_{\mu\nu})_{LC} \), which is metric connection and obtained from the metric. The characteristics of Levi-Civita connection: its torsion vanished \( \left( (\Gamma^a_{\mu\nu})_{LC} = 0 \right) \) because the connection is symmetry in the two last indices, but curvature is not \( \left( R^\sigma_{\mu\nu\lambda} \neq 0 \right) \). Lagrangian density equation for a gravitational field is

\[ \mathcal{L}_G = \sqrt{-g} R, \]  

(2)

with the boundary term is vanish and \( g = \det(g_{\mu\nu}) \).

3. Teleparallel Gravity

Teleparallel gravity (TG) is one of formulation of gravity to bring Einstein gravity (EG) into gauge scenario. TG views gravitation as a force, like the other interaction in the nature. Generally, the spacetime is showed in a coordinate system, for the next, the spacetime will be showed by using a tetrad field, \( e_a(x^\mu) \):

\[ \eta_{ab} = g(e_a, e_b) = g_{\mu\nu} e^\mu_a e^\nu_b. \]  

(3)

Latin indices \( a = 0,1,2,3 \) is related to coordinate of tangent space and Greek indices \( \mu = 0,1,2,3 \) is related to spacetime coordinate. Tetrad field forms a orthonormal basis for a tangent space in every point of spacetime with coordinate basis \( (x^\mu) \). There are two kinds of tetrads: holonomic \( (f_{ab} = 0) \) and non-holonomic \( (f_{ab} \neq 0) \) that related with commutation relation:

\[ [e_a, e_b] = f_{ab} e_c. \]  

(4)

In TG, non-holonomic tetrad is chosen to describe the gravitation. If the connection formed by the tetrad field, there will be a term that called spin connection, \( A^b_{\mu a} \)

\[ \nabla_\mu e_a = (\partial_\mu e^\rho_a + e^\nu_a \Gamma^\rho_{\mu\nu}) \partial_\rho = A^b_{\mu a} e_b. \]  

(5)

The fundamental difference between TG and EG is their geometry; the Levi-Civita connection in EG with non vanishing curvature (but vanishing torsion), is replaced with Weitzenböck connection with non vanishing torsion (but vanishing curvature). For case that \( A^b_{\mu a} = 0 \),
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That is called absolute condition or distant parallelism. Based on the equation (6), the torsion of Weitzenböck connection is

\[
(T_{\mu}^a)_{WZ} = \partial_{\mu}e^a_{\mu} - \partial_{\mu}e^a_{\mu}.
\]  

(7)

The Lagrangian density equation for gravitational field in TG is

\[
\mathcal{L}_G(e_{a\mu}) = -\frac{e}{16\pi G} S^{abc} T_{abc},
\]  

(8)

\[e = \det(e_{\mu}^a)\]

and

\[
S^{abc} = \frac{1}{4}(T^{abc} + T^{bac} - T^{cab}) + \frac{1}{2}(\eta^{ac}T^b - \eta^{ab}T^c).
\]  

(9)

The actions based on Lagrangian given in (8) and (2) are equivalent up to boundary term.

4. Lorentz Transformation

TG as a gauge theory that is related with a group translation (called Lorentz group). A holonomic tetrad which transformed by global Lorentz transformation (point-independent)

\[\Lambda_b^a \neq \Lambda_b^a(x),\]

(10)

will become still as a new holonomic tetrad. Beside of that, the holonomic tetrad that transformed by local Lorentz transformation (point-dependent):

\[\Lambda_b^a = \Lambda_b^a(x),\]

(11)

will change it, becomes a new non-holonomic tetrad. A equivalent class of tetrad \(\{e_{a}^\mu\}\) that is related to a global Lorentz transformation, has a same Weitzenböck connection. Two equivalent classes of tetrad \(\{e_{a}^\mu\}\) and \(\{\tilde{e}_{a}^\mu\}\) which are related to a local Lorentz transformation, have different Weitzenböck connection.

Now we focus on the torsion that related to these transformation. The torsion from the Weitzenböck connection has not vanish because of the non-holonomic tetrad which based on the equation (7). The unique thing about the torsion is wheather the torsion of \(e_{a}^\mu\) transformed by global or local Lorentz transformation that becomes new torsion from the new tetrad \(\tilde{e}_{a}^\mu\), and full field equation (12)

\[
(T_{\mu}^a)_{WZ} = \Lambda_b^a (T_{\mu}^a)_{WZ}.
\]  

(12)

The equation (12) showed that if the global or local Lorentz transformation is written as by

\[\Lambda_b^a = \tilde{e}_{\mu}^b e_{a}^\mu,\]

(13)

so the other torsion will be known. It means that the torsions which is related to global or local Lorentz transformation have the same physical meaning.

5. Inertia Reference

One of the principle of EG is about inertia reference, where is free gravitation. Generally, the equation of particle motion in that reference is:
The equation (14) is a geodesic equation for a particle’s worldline and shows that gravitational term \((\Gamma^a_{\mu\nu})_{\text{LC}}\) included in the geometry.

Chosen a tetrad, there is a reference in Minkowski spacetime with Lorentz metric \(\eta_{\mu\nu}\): 

\[
\delta_a = \delta^\mu_a \partial_\mu ,
\]

with \(e^\mu_a = \delta^\mu_a\) is a holonomic tetrad. Based on the Newton’s first equation:

\[
\frac{d^2 x^\mu}{dt^2} = 0.
\]

The equation (5.3) can be showed in the holonomic reference:

\[
\frac{d v^a}{dt} = 0,
\]

with \(v^a = \delta^a_\mu \frac{dx^\mu}{dt} = \delta^a_\mu u^\mu\) is a velocity-4. It means that the inertia reference is related to a holonomic tetrad. Because of the local Lorentz transformation, the equation (5.4) becomes:

\[
\frac{du^c}{dt} + \frac{1}{2} \left( f^c_{\ a} + f^c_{\ b} - f^c_{\ ab} \right) u^a u^b = 0.
\]

The equation (18) means that there is a new term in the equation of motion which describes force or the gravitational effect.

6. Conclusion

TG views gravitation as an interaction like the other interactions in nature, electromagnetic, strong, and weak interactions. The fundamental difference between TG and EG is their geometry; the Levi-Civita connection in EG with non-vanishing curvature (but vanishing torsion), is replaced with Weitzenböck connection with non-vanishing torsion (but vanishing curvature). Although, the Lagrangian of TG and EG are equivalent up to boundary term. Because of that, almost physicists say that TG and EG are equivalent.

There is something unique about TG. The torsions which is related to global or local Lorentz transformation have the same physical meaning because we can get the other from the one. The local Lorentz transformation of an holonomic tetrad becomes a new non-holonomic tetrad. So, the second term of the equation of motion of the new tetrad describes force or the gravitational effect.

References

[1] Aldrovandi R and Pereira J G 2013 Teleparallel Gravity: An Introduction (New York: Springer)
[2] Wald R M 1984 General Relativity (London: Universitas of Chicago Press)
[3] Garecki J 2010 arXiv:1010.2654v3[gr-qc]
[4] Combi L and Romero G E 2017 Ann. Phys. (Berlin) 2017 1700175
[5] Knox E 2011 Stud. Hist. Philos. Mod. Phys. 42 264
[6] Arcos H I and Pereira J G 2004 Int. J. Mod. Phys. D 13 2193
[7] Arcos H I, de Andrade V C and Pereira J G 2004 Int. J. Mod. Phys. D 13 807
[8] Sousa A A, Moura J S and Pereira R B 2010 Braz. J. Phys. 40 (1) 1