Harmonic oscillator in twisted Moyal plane: eigenvalue problem and relevant properties

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Abstract

The paper reports on a study of a harmonic oscillator (ho) in the twisted Moyal space, in a well defined matrix basis, generated by the vector fields
\[ X_a = e^a_\mu(x) \partial_\mu = (\delta^\mu_a + \omega^{\mu \nu}_{ab} x^b) \partial_\mu, \]
which induce a dynamical star product. The usual multiplication law can be hence reproduced in the \( \omega^{\mu}_{ab} \) null limit. The star actions of creation and annihilation functions are explicitly computed. The ho states are infinitely degenerate with energies depending on the coordinate functions.

Keywords Harmonic oscillator, twisted Moyal plane, eigenvalue problem.

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1 Introduction

It is generally believed that the picture of spacetime as a manifold $M$ locally modelled on the flat Minkowski space should break down at very short distances of the order of the Planck length $l_p = (G\hbar/c^3)^{1/2}$. Limitations in the possible accuracy of localization of spacetime events should in fact be a feature of a quantum theory incorporating gravitation. The obtaining of a better understanding of physics at short distances and the cure of the problems occurring when trying to quantize gravity should lead to change the nature of spacetime in a fundamental way. This could be realized by implementing the noncommutativity through the coordinates which satisfy the commutation relations $[\hat{x}^\mu, \hat{x}^\nu] = iC^{\mu\nu}(\hat{x}) \neq 0$. In general, the function $C^{\mu\nu}(\hat{x})$ is unknown, but, for physical reasons, should vanish at large distances where we experience the commutative world and may be determined by experiments $[6]$ and $[15]$. The $\Theta$–deformation case which may at very short distances provide a reasonable approximation for $C^{\mu\nu}(\hat{x})$ is described by the commutation relation $[\hat{x}^\mu, \hat{x}^\nu] = i\Theta^{\mu\nu}$. $\Theta^{\mu\nu}$ is usually chosen in the form

$$\Theta = \begin{pmatrix}
0 & \Theta_1 & & & \\
-\Theta_1 & 0 & \Theta_2 & & \\
& 0 & 0 & \Theta_2 & \\
& & -\Theta_2 & 0 & \\
& & & \vdots & \ddots \\
& & & & \vdots \\
0 & & & & \Theta_{D/2} \\
& & & & -\Theta_{D/2} & 0
\end{pmatrix}$$

(1)

where $\Theta_j \in \mathbb{R}$, $j = 1, 2, \cdots, \frac{D}{2}$, have dimension of length square, $([\Theta_j] = [L]^2)$, $D$ denoting the spacetime dimension. The algebra of functions of such noncommuting coordinates can be represented by the algebra of functions on ordinary spacetime, equipped with a noncommutative $\star$–product. For a constant antisymmetric matrix $\Theta^{\mu\nu}$, this can be represented by the Groenewold-Moyal product:

$$(f \star g)(x) = \exp\left\{\frac{i\Theta^{\mu\nu}}{2} \partial_\mu \otimes \partial_\nu f(x) \otimes g(x)\right\}$$ \quad x \in \mathbb{R}^D_\Theta \quad \forall f, g \in C^\infty(\mathbb{R}^D_\Theta)$$

(2)

$m$ is the ordinary multiplication of functions, $C^\infty(\mathbb{R}^D_\Theta)$ - the space of suitable smooth functions on $\mathbb{R}^D_\Theta$ and $\mathbb{R}^D_\Theta$ - the $D$–dimensional Moyal space. This product can be generalized under the
form

\[(f \star g)(x) = m\left\{ e^{i\Theta_{ab}} X_a \otimes X_b f(x) \otimes g(x) \right\} \]

(3)

where \(X_a = e^\mu_a(x)\partial_\mu\) are vector fields. The commutation relation of coordinates then becomes

\[[x^\mu, x^{\nu}]_\star = i\Theta^{\mu\nu} e^\mu_a(x) e^{\nu}_b(x) = i\Theta^{\mu\nu}(x)\]

engendering a twisted scalar field theory where \(e^\mu_a\), and hence the \(\star\)-product itself, appear dynamical. See [1]-[12] for more details. Besides, the Leibniz rule extends to the commuting fields \(X_a\) as follows:

\[X_a(f \star g) = (X_a f) \star g + f \star (X_a g).\]

(4)

Recently [1], a formulation of dynamical noncommutativity, which allows for a consistent interpretation of position measurement and the solution of the problem of a noncommutative well has been put forward. This work addresses a study of a harmonic oscillator properties in the twisted Moyal plane.

The paper is organized as follows. In Section 2, using appropriate matrix basis and deforming the issue of a twisted product, we solve the resulting eigenvalue problem to find the states and the energy spectrum of the harmonic oscillator Hamiltonian. These states are infinitely degenerate. Some concluding remarks are pointed out in Section 3.

2 Harmonic oscillator in twisted Moyal space

As a prelude to the construction of a matrix basis appropriate for this study, let us set up main algebraic relations pertaining to twisted noncommutative coordinate transformations.

2.1 Useful relations

We consider the following infinitesimal affine transformation

\[e^\mu_a(x) = \delta^\mu_a + \omega^\mu_{ab} x^b, \quad \omega^\mu_{ab} =: -\omega^\mu_{ba}, \text{ and } |\omega^\mu| < < 1.\]

(5)

In the sequel, we restrict the discussion to \(D = 2\), where \(e^\mu_a\) and \(\Theta^{ab}\) can be expressed as follows:

\[(e)^\mu_a = \begin{pmatrix} 1 + \omega_{12}^1 x^2 & \omega_{12}^2 x^1 \\ -\omega_{12}^1 x^2 & 1 - \omega_{12}^2 x^1 \end{pmatrix} \quad \text{and} \quad (\Theta)^{ab} = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} = \theta(J)^{ab}\]

(6)
where $J^{12} = -J^{21} = 1$, $J^{11} = J^{22} = 0$. At the first order of expansion,

$$e^{-1} =: \det(e^{\mu}_{\nu}) = 1 + \omega_{12}^1 x^2 - \omega_{12}^2 x^1$$

$$e =: \det(e^{\mu}_{\nu}) = 1 - \omega_{12}^1 x^2 + \omega_{12}^2 x^1.$$  

(7) 

(8) 

The $\star$–product of two Schwartz functions on $\mathbb{R}^2_{\Theta}$ can be written under the form

$$(f \star g)(x) = m \left[ \exp \left( \frac{i}{2} \theta e^{-1} J^{\mu\nu} \partial_{\mu} \otimes \partial_{\nu} \right) \right] (f \otimes g)(x)$$

(9) 

where $\mu, \nu = 1, 2$ and $\partial_{\mu} =: \frac{\partial}{\partial x^\mu}$. Using the twisted star product (9) one can see that

$$e^{ikx} \star e^{iqx} = e^{i(k+q)x} e^{-\frac{i}{2} \theta e^{-1} kJq}.$$ 

(10) 

The Fourier transform of $f, g \in \mathcal{S}(\mathbb{R}^2_{\Theta})$ can be written as

$$\hat{f}(k) = \int d^2 x \ e^{-ikx} f(x), \quad \hat{g}(q) = \int d^2 x \ e^{-iqx} g(x)$$

(11) 

with the functions inverse transform given by

$$f(x) = \frac{1}{(2\pi)^2} \int d^2 k \ e^{ikx} \hat{f}(k), \quad g(x) = \frac{1}{(2\pi)^2} \int d^2 q \ e^{iqx} \hat{g}(q).$$ 

(12) 

We can then redefine the twisted star product of two Schwartz functions $f, g$ as:

$$(f \star g)(x) = \frac{1}{(2\pi)^4} \int d^2 k d^2 q \ \hat{f}(k) \hat{g}(q) e^{ikx} \star e^{iqx}$$

$$= \frac{1}{(2\pi)^4} \int d^2 k d^2 q \ \hat{f}(k) \hat{g}(q) e^{i(k+q)x} e^{-\frac{i}{2} \theta e^{-1} kJq}$$

$$= \frac{1}{(2\pi)^4} \int d^2 k d^2 q \int d^2 y d^2 z \ f(y) g(z) e^{i(kx-y - \frac{1}{2} \theta e^{-1} Jq) e^{iq(x-z)}}$$ 

(13) 

Using the identity

$$\int d^2 k \ e^{ik(x-y - \frac{1}{2} \theta e^{-1} Jq)} = (2\pi)^2 \delta^{(2)}(x - y - \frac{1}{2} \theta e^{-1} Jq)$$ 

(14) 

and the variable change $q$ to $q' = \frac{1}{2} \theta e^{-1} Jq$, we arrive at the adapted form for the proof of the next Proposition [2.3]:

$$(f \star g)(x) = \left(\frac{e}{\pi \theta}\right)^2 \int d^2 y d^2 z \ f(x-y) g(z) e^{-2 \pi i (x-y) J(x-z)}$$

$$= \left(\frac{e}{\pi \theta}\right)^2 \int d^2 y d^2 z \ f(x-y) g(x-z) e^{-2 \pi i y Jz}$$

$$= \int d^2 z \frac{d^2 t}{(2\pi)^2} f(x - \frac{1}{2} \theta e^{-1} t) g(x - z) e^{-itJz}.$$ 

(15)
Proposition 2.1 If $f$ and $g$ are two Schwartz functions on $\mathbb{R}^2_\Theta$, then $f \ast g$ is also a Schwartz function on $\mathbb{R}^2_\Theta$.

Proof: It is immediate by induction on the formulas (4) and (15) using integration by parts. □

The tensor $\tilde{\Theta}^{\mu\nu}$ can be explicit as

$$
(\tilde{\Theta})^{\mu\nu} = (\Theta)^{\mu\nu} - (\Theta^{[\mu}_{ab} \omega^\nu])_{b} = \begin{pmatrix} 0 & \theta e^{-1} \\ -\theta e^{-1} & 0 \end{pmatrix}.
$$

(16)

The twisted Moyal product of fields generates some basic properties like the Jacobi identity

$$
[x^\mu, [x^\nu, x^\rho]]_\ast + [x^\rho, [x^\mu, x^\nu]]_\ast = \Theta^{b\mu} \Theta^{d[\nu}_{bd} = 0
$$

(17)

confering a Lie algebra structure to the defined twisted Moyal space, and

$$
x^\mu \ast f = x^\mu f + \frac{i}{2} \Theta^{ab} c^\mu_a c^\rho_b \partial_\rho f \quad \text{and} \quad f \ast x^\mu = x^\mu f - \frac{i}{2} \Theta^{ab} c^\mu_a c^\rho_b \partial_\rho f.
$$

(18)

The star brackets (anticommutator and commutator) of $x^\mu$ and $f$ can be immediately deduced as follows: $\{x^\mu, f\}_\ast = 2x^\mu f$, $[x^\mu, f]_\ast = i \Theta^{ab} c^\mu_a c^\rho_b \partial_\rho f$. The relations (18) can be detailed for $x^\mu$, $\mu = 1, 2$ as:

$$
x^1 \ast f = x^1 f + \frac{i}{2} \theta e^{-1} \partial_2 f \quad f \ast x^1 = x^1 f - \frac{i}{2} \theta e^{-1} \partial_2 f
$$

(19)

$$
x^2 \ast f = x^2 f - \frac{i}{2} \theta e^{-1} \partial_1 f \quad f \ast x^2 = x^2 f + \frac{i}{2} \theta e^{-1} \partial_1 f
$$

(20)

giving rise to the creation and annihilation functions

$$
a = \frac{x^1 + ix^2}{\sqrt{2}} \quad \bar{a} = \frac{x^1 - ix^2}{\sqrt{2}}
$$

(21)

with the commutation relation $[a, \bar{a}]_\ast = \theta e^{-1}$. It then becomes a matter of algebra to use the transformations of the vector fields $\partial_1$ and $\partial_2$ into $\partial_a = : \frac{\partial}{\partial a}$ and $\partial_{\bar{a}} = : \frac{\partial}{\partial \bar{a}}$ and vice-versa to infer $e^{-1} = 1 - a \omega - \bar{a} \bar{\omega}$ and $e = 1 + a \omega + \bar{a} \bar{\omega}$, where

$$
\omega = \frac{\omega^2_{12} + i \omega^1_{12}}{\sqrt{2}} \quad \text{and} \quad \bar{\omega} = \frac{\omega^2_{12} - i \omega^1_{12}}{\sqrt{2}}
$$

(22)

leading to useful relations

$$
\frac{\partial e^{-1}}{\partial a} = -\omega, \quad \frac{\partial e^{-1}}{\partial \bar{a}} = -\bar{\omega} \quad \text{and for} \quad k \in \mathbb{Z}, \quad \omega^k = \omega, \quad \bar{\omega}^k = \bar{\omega}.
$$

(23)
Expressing the twisted \(*\)-product (9) in terms of vectors fields \(\partial_a\) and \(\partial_{\bar{a}}\) as

\[
(f \ast g)(a, \bar{a}) = \mathfrak{m} \left[ \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^{n-k}}{k!(n-k)!} \left( \frac{1}{2} \theta e^{-1} \right)^{n} \times (\partial_a \otimes \partial_{\bar{a}})^{k}(\partial_{\bar{a}} \otimes \partial_a)^{n-k}(f \otimes g)(a, \bar{a}) \right]
\]

(24)

and using equations (19) and (20) (or independently (24)) yield

\[
a \ast f = \left( a + \frac{\theta e^{-1}}{2} \frac{\partial}{\partial a} \right) f \quad \bar{a} \ast f = \left( \bar{a} - \frac{\theta e^{-1}}{2} \frac{\partial}{\partial \bar{a}} \right) f
\]

(25)

\[
f \ast a = \left( a - \frac{\theta e^{-1}}{2} \frac{\partial}{\partial a} \right) f \quad f \ast \bar{a} = \left( \bar{a} + \frac{\theta e^{-1}}{2} \frac{\partial}{\partial \bar{a}} \right) f.
\]

(26)

Provided the above definitions, we can now introduce the notions of right and left harmonic oscillator states denoted by \(f^R_{m0}\) and \(f^L_{0n}\), respectively.

2.2 The right and left states

Let \(f^R_{00} \in L^2(\mathbb{R}_\Theta^2)\) be the ho "right" fundamental state such that

\[a \ast f^R_{00} =: 0 \quad \text{with} \quad f^R_{00} = 2e^{-\frac{\theta \bar{a}}{e^{-1}}(1-\frac{1}{2}\bar{a}\omega)}.
\]

(27)

Then, \(f^R_{00}\) solves the eigenvalue problem \(H \ast f^R_{00} = \mathcal{E}^R_{00} f^R_{00}\) with the corresponding right fundamental eigenvalue \(\mathcal{E}^R_{00} = \frac{\theta}{2}(1 - 2\bar{a}\omega)\) of the self-adjoint unbounded twisted ho Hamiltonian operator

\[
H \ast(.) \quad : \quad \bar{a}a \ast(.) = \left[ \bar{a} \ast a + \frac{\theta e^{-1}}{2} \right] \ast(.) = \left[ a \ast \bar{a} - \frac{\theta e^{-1}}{2} \right] \ast(.)
\]

\[
= \frac{1}{2} \left[ \left( x^1 \right)^2 + \left( x^2 \right)^2 + \left( i\theta e^{-1}x^1 - \frac{\theta^2}{4}\omega^2 \right) \partial_2 \right.
\]

\[
- \left. \left( i\theta e^{-1}x^2 - \frac{\theta^2}{4}\omega^2 \right) \partial_1 - \frac{\theta^2}{4} e^{-2} \left( \partial_1^2 + \partial_2^2 \right) \right] \equiv \frac{1}{2} \mu_1
\]

(28)

defined in the domain

\[
\mathcal{D}(H\ast) = \left\{ f \in L^2(\mathbb{R}_\Theta^2) \mid f, f_{x^1}, f_{x^2} \in AC_{loc}(\mathbb{R}_\Theta^2); \frac{\mu_1}{2} f \in L^2(\mathbb{R}_\Theta^2) \right\}.
\]

(29)

\(AC_{loc}(\mathbb{R}_\Theta^2)\) denotes the set of the locally absolutely continuous functions on \(\mathbb{R}_\Theta^2\). Similarly, the fundamental left state \(f^L_{00} \in L^2(\mathbb{R}_\Theta^2)\) defined such that

\[
f^L_{00} \ast \bar{a} =: 0 \quad \text{with} \quad f^L_{00} = 2e^{-\frac{\theta a}{e^{-1}}(1-\frac{1}{2}\bar{a}\omega)}
\]

(30)
Proof the right states \( f_{00}^L \) \( H = \mathcal{E}_{00}^L f_{00}^L \) with the left fundamental eigenvalue \( \mathcal{E}_{00}^L = \frac{\theta}{2}(1 - 2a\omega) \) of the self-adjoint unbounded twisted ho Hamiltonian operator

\[
(\cdot) H =: (\cdot) \bar{a}a = (\cdot) \left[ a \ast a + \frac{\theta e^{-1}}{2} \right] = (\cdot) \left[ a \ast \bar{a} - \frac{\theta e^{-1}}{2} \right] = \frac{1}{2} \left[ (x^1)^2 + (x^2)^2 - \left( i \theta e^{-1} x^1 - \frac{\theta^2}{4} \omega_1^2 \right) \partial_2 \right. \\
+ \left. \left( i \theta e^{-1} x^2 - \frac{\theta^2}{4} \omega_2^2 \right) \partial_1 - \frac{\theta^2}{4} e^{-2} (\partial_1^2 + \partial_2^2) \right] \equiv \frac{1}{2} \mu_2 \quad (31)
\]
defined in the domain

\[
\mathcal{D}(\ast H) = \left\{ f \in L^2(\mathbb{R}_G^2) \mid f, f_{x_1}, f_{x_2} \in \mathcal{A}C_{loc}(\mathbb{R}_G^2); \frac{\mu_2}{2} f \in L^2(\mathbb{R}_G^2) \right\}.
\]

Then, the other states follow from the next statement.

**Proposition 2.2** The vectors \( f_{m0}^R \in L^2(\mathbb{R}_G^2) \) given for any \( m \in \mathbb{N} \) by

\[
f_{m0}^R = \frac{1}{\sqrt{m!\theta^m}} \left[ 2^m \bar{a}^m (1 + \frac{m \omega a}{2} - \frac{m a \bar{\omega}}{4}) - \frac{U_m \theta \omega \bar{a}^{m-1} a}{2} \right] f_{00}^R
\]
solve the eigenvalue problem \( H \ast f_{m0}^R = \mathcal{E}_{m0}^R f_{m0}^R \) with

\[
\mathcal{E}_{m0}^R = \frac{\theta}{2} \left[ 2m + 1 - m \omega - (3m + 2) \bar{a} \omega - \frac{m^2 \theta \omega}{4a} + \frac{\theta \omega U_m}{2^m a} \right], \quad m \in \mathbb{N}
\]

where

\[
U_m = (m - 1)2^{m-2} + \sum_{k=0}^{m-3} (k + 1)2^{k+1}, \quad m \geq 3, \quad U_{i \leq 1} = 0, U_2 = 1.
\]

**Proof:** The results are immediate by induction, performing similar analysis as in [8] to construct the right states \( f_{m0}^R \) such that \( \bar{a} \ast f_{m0}^R = \sqrt{\theta (m + 1)} f_{m+1,0}^R \).

Similarly, the study of the ho left states provides the following result.

**Proposition 2.3** The vectors \( f_{0n}^L \in L^2(\mathbb{R}_G^2) \) given for any \( n \in \mathbb{N} \) by

\[
f_{0n}^L = \frac{1}{\sqrt{n!\theta^n}} \left[ 2^n a^n (1 + \frac{n \omega a}{2} - \frac{n \omega \bar{a}}{4}) - \frac{U_n \theta \omega a^{n-1} \bar{\omega}}{2} \right] f_{00}^L
\]
solve the eigenvalue problem \( f_{0n}^L \ast H = \mathcal{E}_{0n}^L f_{0n}^L \) with

\[
\mathcal{E}_{0n}^L = \frac{\theta}{2} \left[ 2n + 1 - n \omega - (3n + 2) \omega a - \frac{n^2 \theta \omega}{4a} + \frac{\theta \omega U_n}{2^n a} \right], \quad n \in \mathbb{N}.
\]

**Proof:** It uses the same procedure as previously, but with the construction of the left states \( f_{0n}^L \) such that \( f_{0n}^L \ast a = \sqrt{\theta (n + 1)} f_{0,n+1}^L \). □
Besides \( \lim_{\omega, \bar{\omega} \to 0} \mathcal{E}_{m0}^R = \theta \left( m + \frac{1}{2} \right) \) and \( \lim_{\omega, \bar{\omega} \to 0} \mathcal{E}_{0n}^L = \theta \left( n + \frac{1}{2} \right) \) corresponding to the usual Moyal \( \star \) product spectrum of the harmonic oscillator Hamiltonian \( H \).

All these results show that the ho right and left states as well as their respective energy spectrum are expressible in terms of the space deformation constant \( \theta \) and of an additional piece inherent to the nature of the induced infinitesimal transformation through the parameter \( \omega \) and its conjugate. Besides, a noteworthy feature of these states is the following.

**Proposition 2.4** The right and left fundamental states defined by \( f_{00}^{(m)R} =: a^{m+1} \star f_{m0}^R \) and \( f_{00}^{(n)L} =: f_{0n}^L \star \bar{a}^{n+1} \) are given by the following expressions:

\[
\begin{align*}
  f_{00}^{(m)R} &= -\frac{\sqrt{m!}\theta^{m+2}}{8} \sum_{j=1}^{m} \frac{(m+4j+1)}{(m-j)!} \bar{\omega} f_{00}^R \\
  f_{00}^{(n)L} &= f_{00}^{(n)R}(\bar{a} \leftrightarrow a, \bar{\omega} \leftrightarrow \omega) \\
                 &= -\frac{\sqrt{n!}\theta^{n+2}}{8} \sum_{j=1}^{n} \frac{(n+4j+1)}{(n-j)!} \omega f_{00}^L
\end{align*}
\]

which, in the usually Moyal product case, are reduced to 0. Besides, the twisted harmonic oscillator states \( f_{m0}^R \) and \( f_{m0}^L \) are degenerate with respect to the rules

\[
\begin{align*}
  a^{m+2} \star f_{m0}^R &= 0 \quad \text{and} \quad f_{0n}^L \star \bar{a}^{n+2} = 0 \quad \forall \ m \geq 1.
\end{align*}
\]

The proof is straightforward. See appendix A.

**Remark 2.1**

1. \( f_{00}^R \) and \( f_{00}^L \) are the twisted fundamental states restoring, in the limit of ordinary Moyal space, the fundamental state given by \( 2e^{-2\phi_0} \). For the analysis purpose, we call \( f_{00}^R \) and \( f_{00}^L \) the normal twisted fundamental states.

2. The states \( f_{m0}^R \) correspond to twisted right \( m+1 \) particles states, reducing, in the usual case, to right \( m \) particles states, while the states \( f_{0n}^L \) represent the twisted left \( n+1 \) particles states.

3. There are an infinite number of twisted right \( m-k \) particles states and an infinite number of twisted left \( n-k \) particles states given by \( a^{k+1} \star f_{m0}^R \) and \( f_{0n}^L \star \bar{a}^{k+1} \), respectively.
2.3 Matrix basis of the theory

The usual construction of a matrix basis \([8]\) exploits the \(\ast\)-multiplication of \(f_{m0}^R\) with \(f_{0n}^L\), i.e.

\[
L^2(\mathbb{R}_n^2) \ni \bar{b}^{(2)}_{mn} = \chi(\omega, \bar{\omega}, \theta, m, n) \frac{a^n \ast f_{0n}^L \ast f_{m0}^R}{\sqrt{m!n!\theta^{m+n}}}.
\]

(41)

Without loss of generality, we set the normalization constant \(\chi(\omega, \bar{\omega}, \theta, m, n) = 1\) by convention. The corresponding eigenvalue problems are given by

\[
H \ast b^{(2)}_{mn} = \mathcal{E}_{m0}^R \mu^{(2)}_{mn} \quad \text{and} \quad b^{(2)}_{mn} \ast H = \mathcal{E}_{0n}^L \mu^{(2)}_{mn}
\]

(42)

while the \(\ast\)-actions of the annihilation and creation functions \(a\) and \(\bar{a}\) are reproduced as follows:

\[
\bar{a} \ast b^{(2)}_{mn} = \sqrt{\theta(m+1)}b^{(2)}_{m+1,n} \quad \text{and} \quad b^{(2)}_{mn} \ast a = \sqrt{\theta(n+1)}b^{(2)}_{m,n+1},
\]

with the basis fundamental state \(b^{(2)}_{00} = f_{00}^R \ast f_{00}^L\) satisfying the expected requirements \(a \ast b^{(2)}_{00} = 0, \quad b^{(2)}_{00} \ast \bar{a} = 0\). Given the \((1, 1)\)-particles states defined by \(L^2(\mathbb{R}_n^2) \ni \Lambda_{mn}^{1,1} = a^m \ast b^{(2)}_{mn} \ast \bar{a}^n\), their twisted spectrums can be computed from the eigenvalue problems \(H \ast \Lambda_{mn}^{1,1} = \mathcal{E}_{m0}^R \Lambda_{mn}^{1,1}\) and \(\Lambda_{mn}^{1,1} \ast H = \mathcal{E}_{0n}^L \Lambda_{mn}^{1,1}\) to get, depending on the right and left Hamiltonian \(\ast\)-actions,

\[
\mathcal{E}_{\Lambda_{m0}^{1,1}}^R = \frac{\theta}{2} \left[ 1 - \frac{\bar{a}\omega}{2} \left( \sum_{j=1}^{m} \frac{m+j+1}{(m-j)!} + 4 \right) \right], \quad m > 0
\]

(43)

\[
\mathcal{E}_{\Lambda_{0n}^{1,1}}^L = \mathcal{E}_{\Lambda_{m0}^{1,1}}^R (\bar{a} \leftrightarrow a, \omega \leftrightarrow \bar{\omega}) = \frac{\theta}{2} \left[ 1 - \frac{a\omega}{2} \left( \sum_{j=1}^{n} \frac{n+j+1}{(n-j)!} + 4 \right) \right], \quad n > 0.
\]

(44)

For \(\omega = 0\) and \(\bar{\omega} = 0\), these energies are reduced to the usual Moyal space matrix basis right and left fundamental energies. As needed, the Wick rotation can be used to ensure the real value of the energy. In the same vein, one can define the single twisted \((m-k+1)\) right particles states by \(a^k \ast f_{m0}^R =: \Lambda_{m0}^{m-k+1} \in L^2(\mathbb{R}_n^2)\) corresponding to the energy values obtained from the right Hamiltonian \(\ast\)-action by

\[
\mathcal{E}_{\Lambda_{m0}^{m-k+1}}^R = \frac{\theta}{2} \left\{ 2m - 2k + 1 - (m-k)a\omega \\
+ \frac{\bar{a}\omega}{2} \left[ (m-k-1)(m-k)! \sum_{j=1}^{k} \frac{m+j+1}{(m-j)!} \right. \\
- (m-k)(m+4k+6) - 4 \right\} - \frac{(m-k)(m-2k)\theta\omega}{4\bar{a}} \\
+ \frac{(m-k+1)(m-k)\theta\omega U_m}{m2^{m+1}a} \right\} \quad m \geq k > 0.
\]

(45)
By analogy, the single twisted \((n - l + 1)\) left particles states \(f_{0n}^L \star \bar{a}^l =: \Lambda_{0n}^{n-l+1} \in L^2(\mathbb{R}^2_\Theta)\) are associated with the left action energy values

\[
\mathcal{E}^L_{\Lambda_{0n}^{n-l+1}} = \mathcal{E}^R_{\Lambda_{n0}^{n-l+1}} \left( \omega \leftrightarrow \bar{\omega}, \bar{a} \leftrightarrow a \right)
\]

\[
= \theta \left\{ 2n - 2l + 1 - (n - l)\bar{a}\bar{\omega} 
+ \frac{a\omega}{2} \left[ (n - l - 1)(n - l)! \sum_{j=1}^{l} \frac{n + 4j + 1}{(n - j)!} 
- (n - l)(n + 4l + 6) - 4 \right] - \frac{(n - l)(n - 2l)\theta\bar{\omega}}{4a} 
+ \frac{(n - l + 1)(n - l)\theta\omega \mathcal{U}_n}{n2^{n+1}a} \right\}, \quad n \geq l > 0. \tag{46}
\]

**Proposition 2.5** The energy spectrums \((45)\) and \((46)\) of the mixed twisted \((m - k + 1)\) right and \((n - l + 1)\) left particles states \(L^2(\mathbb{R}^2_\Theta) \ni \Lambda_{mn}^{m-k+1,n-l+1} =: a^k \star f_{m0}^R \star f_{0n}^L \star \bar{a}^l\) solve the following respective eigenvalue problems:

\[
H \star \Lambda_{mn}^{m-k+1,n-l+1} = \mathcal{E}^R_{\Lambda_{m0}^{m-k+1}} \Lambda_{mn}^{m-k+1,n-l+1} \tag{47}
\]

\[
\Lambda_{mn}^{m-k+1,n-l+1} \star H = \mathcal{E}^L_{\Lambda_{0n}^{n-l+1}} \Lambda_{mn}^{m-k+1,n-l+1}. \tag{48}
\]

We readily recover the spectrums \((43)\) and \((44)\) by replacing \(m = k\) and \(n = l\) in the relations \((45)\) and \((46)\). Of course, in the limit regime, these energies also well reproduce the ordinary Moyal plane \((m - k)\) right and \((n - l)\) left particles energies:

\[
\lim_{\omega,\bar{\omega} \to 0} \mathcal{E}^R_{\Lambda_{m0}^{m-k+1}} = \frac{\theta}{2} \left[ 2(m - k) + 1 \right] \tag{49}
\]

\[
\lim_{\omega,\bar{\omega} \to 0} \mathcal{E}^L_{\Lambda_{0n}^{n-l+1}} = \frac{\theta}{2} \left[ 2(n - l) + 1 \right] \tag{50}
\]

respectively.

### 3 Concluding remarks

We have investigated the main properties of harmonic oscillator in the framework of a dynamical noncommutativity realized through a twisted Moyal product. The construction of the appropriate matrix basis has introduced an \(x\)-dependence in the definition of the star product. We have computed the states and energies of the twisted harmonic oscillator. The degeneracy states and
their energy have been explicitly derived. All examined quantities easily acquire good physical properties when $\omega^2_{12}$ and $x^1$ are transformed into $i\omega^2_{12}$ and $ix^1$, respectively. Furthermore, the ordinary Moyal space tools are well recovered as expected by setting $\omega = 0$ and $\bar{\omega} = 0$.

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**Appendix A: Right and left $\star-$actions of the creation and annihilation functions onto the ho states**

In this part, we derive the right and left $\star-$actions of the creation and annihilation functions onto the ho states.

\[
a \star f^R_{m0} = \frac{m2^{m-1}\bar{a}^{m-1}}{\sqrt{m!}\theta^{m-2}}(1 + \frac{m-2}{2}a\omega - \frac{m+5}{4}\bar{a}\bar{\omega})f^R_{00} - \frac{(m-1)\theta\omega U_m\bar{a}^{m-2}}{4\sqrt{m!}\theta^{m-2}}f^R_{00}
\]

\[
a^2 \star f^R_{m0} = \frac{m(m-1)2^{m-2}\bar{a}^{m-2}}{\sqrt{m!}\theta^{m-4}}\left[1 + \frac{m-4}{2}a\omega\right]f^R_{00} - \frac{(m+9)(m-1) + m + 5}{4(m-1)}a\omega f^R_{00} - \frac{(m-1)(m-2)\theta\omega U_m\bar{a}^{m-3}}{8\sqrt{m!}\theta^{m-4}}f^R_{00}
\]

\[
\vdots
\]

\[
a^k \star f^R_{m0} = \frac{m(m-1)\cdots(m-k+1)2^{m-k}\bar{a}^{m-k}}{\sqrt{m!}\theta^{m-2k}}\left[1 + \frac{m-2k}{2}a\omega\right]f^R_{00} - \frac{\bar{a}\omega}{4}(m-k)!\sum_{j=1}^{k} \frac{(m+4j+1)}{(m-j)!}f^R_{00} - \frac{(m-1)(m-2)\cdots(m-k)\theta\omega U_m\bar{a}^{m-k-1}}{2^{k+1}\sqrt{m!}\theta^{m-2k}}f^R_{00}
\]

where $k \leq m$

\[
\vdots
\]

\[
a^m \star f^R_{m0} = \frac{m!}{\sqrt{m!}\theta^{-m}} \left[1 - \frac{maw}{2} - \frac{\bar{a}\bar{\omega}}{4}\sum_{j=1}^{m} \frac{(m+4j+1)}{(m-j)!}\right]f^R_{00}
\]
Similarly, \( f_{0n}^L \ast \check{a}^2 \), \( f_{0n}^L \ast \check{a}^3 \), \ldots \( f_{0n}^L \ast \check{a}^n \), can be computed as

\[
\begin{align*}
&f_{0n}^L \ast \check{a}^1 = \frac{n^2 a^{n-1}}{\sqrt{n!} \theta^{n-2}} \left( 1 + \frac{n + 5}{4} \check{a} \omega + \frac{n + 4}{2} \check{a} \omega \right) f_{00}^L \\
&- \frac{(n + 9)(n + 1) + n + 5}{4(n + 1)} \omega \left( 1 + \frac{n + 5}{4} \check{a} \omega \right) f_{00}^L \\
&- \frac{(n + 1)(n + 2) \check{a} \omega U_n}{8 \sqrt{n!} \theta^{n-4}} f_{00}^L \\
&\vdots \\
&f_{0n}^L \ast \check{a}^k = \frac{n + 2}{\sqrt{n} \theta^{n-2k}} \left[ 1 + \frac{n + 2}{2} \check{a} \omega \right] \\
&- \frac{a \omega}{4(n - k)} \left( \sum_{j=1}^{(n + 4j + 1)} \frac{(n + 4j + 1)}{(n - j)!} \right) f_{00}^L \\
&- \frac{(n - 1)(n - 2) \check{a} \omega U_n a^{n-k}}{2^{k+1} \sqrt{n!} \theta^{n-2k}} f_{00}^L,
\end{align*}
\]

where \( k \leq n \).

\[
\begin{align*}
&f_{0n}^L \ast \check{a}^n = \frac{n!}{\sqrt{n!} \theta^{-n}} \left[ 1 - \frac{n \check{a} \omega}{2} - \frac{a \omega}{4} \sum_{j=1}^{(n + 4j + 1)} \frac{(n + 4j + 1)}{(n - j)!} \right] f_{00}^L \\
&f_{0n}^L \ast \check{a}^{n+1} = -\frac{\sqrt{n!} \theta^{n+2} \omega}{8} \sum_{j=1}^{(n + 4j + 1)} \frac{(n + 4j + 1)}{(n - j)!} \omega f_{00}^L \propto f_{00}^L \\
&f_{0n}^L \ast \check{a}^{n+2} = 0.
\end{align*}
\]

**Appendix B: Useful identities**

\[
\begin{align*}
\partial^k_{\check{a}} f_{00}^R &= \left[ k(k - 1) \omega \left( -\frac{2 \check{a}}{\theta} \right)^{k-1} + \left( -\frac{2 \check{a}}{\check{a} \omega - 2} \right)^k \right] f_{00}^R, \\
\partial_{\check{a}}^k f_{00}^R &= \left[ k(k - 1) \omega \left( -\frac{2 \check{a}}{\theta} \right)^{k-1} + \left( -\frac{2 \check{a}}{\check{a} \omega - 2} \right)^k \right] f_{00}^R \\
\partial^k_{\check{a}} f_{00}^L &= \left[ k(k - 1) \omega \left( -\frac{2 \check{a}}{\theta} \right)^{k-1} + \left( -\frac{2 \check{a}}{\check{a} \omega - 2} \right)^k \right] f_{00}^L.
\end{align*}
\]
\[ \partial_k f_{00}^L = \left[ \frac{k(k-1)}{2} \omega \left( -\frac{2\theta}{k} \right)^{k-1} + \left( -\frac{2\theta}{\theta e^{-1}} \right)^k \right] f_{00}^L. \quad (58) \]

\[ \partial_a \left( -\frac{2a}{\theta e^{-1}} \right)^l = kl\omega \frac{l!}{(l-k+1)!} \left( -\frac{2}{\theta} \right)^{k-1} \left( -\frac{2\theta}{\theta} \right)^{l-k+1} \]
\[ + \frac{l!}{(l-k)!} \left( -\frac{2}{\theta} \right)^{k-1} \left( -\frac{2a}{\theta} \right)^{l-k} e^l \quad (59) \]

\[ \partial_a \left( -\frac{2\theta}{\theta e^{-1}} \right)^l = kl\bar{\omega} \frac{l!}{(l-k+1)!} \left( -\frac{2}{\theta} \right)^{k-1} \left( -\frac{2\theta}{\theta} \right)^{l-k+1} \]
\[ + \frac{l!}{(l-k)!} \left( -\frac{2}{\theta} \right)^{k} \left( -\frac{2\theta}{\theta} \right)^{l-k} e^l. \quad (60) \]

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