METRIC MEAN DIMENSION OF FLOWS
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Abstract. The present paper is devoted to investigating the metric mean dimension theory of continuous flows. We introduce the notion of metric mean dimension of continuous flows to characterize the complexity of flow with infinite topological entropy and derive some relevant elementary properties, including the classical Lindenstrauss-Weiss inequality and Abramov type formula of metric mean dimension.

To connect ergodic theory and metric mean dimension theory, we establish variational principles for metric mean dimension of continuous flows in terms of local entropy function and Brin-Katok local entropy and of a class of special flow called uniformly Lipschitz flow in terms of mean Rényi information dimension, Shapira's entropy and Katok's entropy.

1. Introduction

By a pair $(X, \phi)$ we mean a continuous flow, where $X$ is a compact metrizable topological space $X$ with a metric $d$, $\phi : X \times \mathbb{R} \to X$ is a continuous mapping so that $\phi_{t+s} = \phi_t \circ \phi_s$ for all $t, s \in \mathbb{R}$ and $\phi_t(x) := \phi(x, t)$ denotes the homeomorphism on $X$. Given $t \in \mathbb{R}$, a Borel probability measure $\mu$ on $X$ is said to be $\phi_t$-invariant if $\mu(B) = \mu(\phi_t(B))$ for any Borel set $B$, and a $\phi_t$-invariant measure is said to be $\phi_t$-ergodic if for any Borel measurable set $B$ with $\phi_t(B) = B$ has measure 0 or 1. A Borel probability measure $\mu$ on $X$ is said to be $\phi$-invariant if $\mu$ is $\phi_t$-invariant for all $t \in \mathbb{R}$, and a $\phi$-invariant measure is said to be $\phi$-ergodic if any Borel measurable set $B$ with $\phi_t(B) = B$ for all $t \in \mathbb{R}$ has measure 0 or 1. By $\mathcal{M}(X)$, $\mathcal{M}_{\phi}(X)$, $\mathcal{M}_{\phi_t}(X)$, $\mathcal{E}_{\phi}(X)$, $\mathcal{E}_{\phi_t}(X)$ we denote the sets of all Borel probability measures on $X$, all $\phi_t$-invariant probability measures, all $\phi_t$-ergodic probability measures, all $\phi$-ergodic probability measures, respectively.

Mean dimension introduced by Gromov [Gro99] is a new topological invariant for topological dynamical systems, which a vital role in dealing with the embedding problems of dynamical systems [LW00, Gut15, GLT16, Gut17, GT20, GJ20, Tsu20]. To connect the mean dimension

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theory and entropy theory, Lindenstrauss and Weiss [LW00] introduced the notion of metric mean dimension and proved that metric mean dimension is an upper bound of mean dimension. Some applications of metric mean dimension can be found in [Tsu18, GS20]. It is well-known that the classical variational principle [Wal82] provides a vital bridge between ergodic theory and topological dynamics, which states that topological entropy is the supremum of measure-theoretic entropy over some invariant measures. Inspired by the classical variational principle, one can inject ergodic theoretic ideas into mean dimension theory by establishing some new variational principles for metric mean dimension. However, some changes are needed in the context of infinite entropy systems. For example, the roles of topological entropy and measure-theoretic entropy are replaced by metric mean dimension and the different entropy-like quantities related to measure-theoretic entropy up to an error $\epsilon > 0$, respectively. In 2018, Lindenstrauss and Tsukamoto’s [LT18] pioneering work shows that there exists variational principle for metric mean dimension in terms of rate-distortion function that comes from information theory. Later, the authors further proved that the variational principles for metric mean dimension still hold by changing rate-distortion function into Katok’s entropy [VV17], mean Rényi information function [GS21], Shapira’s entropy, Brin-Katok local entropy and local entropy function [Shi21]. Besides, the setting of metric mean dimension in different systems and the corresponding variational principle are exhibited in [JRS21, Wu21, CRV22, CDZ22, CL21, CL22].

The present paper aims to establish variational principles for metric mean dimension in the framework of continuous flows. On the one hand, there exists intrinsical differences between the ergodic theory of flow and its discrete samples. For instances, in general an invariant probability measure for time one map is not invariant for flow, and an ergodic probability measure for flow is not necessarily ergodic for time one map. Hence one can not directly derive variational principle for metric mean dimension of flows based the previous work. On the other hand, Abramov entropy formulas show that both topological entropy and measure-theoretic entropy of $\phi_t$, $t \in \mathbb{R}$, are equal to the absolute value of $t$ times their corresponding entropies of $\phi_1$, yet for different discrete samples of flow the metric mean dimension of the phase space and the measure-theoretic metric mean dimension of invariant measures may allow different speeds to approximate their (infinite) entropy along with the time. Therefore, the two obstacles lead to some significant difficulties when establishing variational principles for metric mean dimension of continuous flows.

To overcome the first obstacle, we introduce the notion of local $\epsilon$-entropy function of flows inspired by [YZ07, Shi21] to establish variational principle for metric mean dimension in terms of the local $\epsilon$-entropy function of flows. More precisely, we have the following.
Theorem 1.1. Let \((X, \phi)\) be a continuous flow with a metric \(d\). Then

\[
\overline{\text{mdim}}_M(\phi, X, d) = \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{x \in X} h_d(x, \epsilon, \phi) \\
\underline{\text{mdim}}_M(\phi, X, d) = \liminf_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{x \in X} h_d(x, \epsilon, \phi),
\]

where \(\overline{\text{mdim}}_M(\phi, X, d)\) and \(\underline{\text{mdim}}_M(\phi, X, d)\) denote the upper and lower metric mean dimensions of \(X\), respectively. \(h_d(x, \epsilon, \phi)\) is local \(\epsilon\)-entropy function at \(x\) with respect to \(\phi\).

Besides, motivated by the work [Bow73, FH12, W21, CLS21, YCZ22] we introduce the notion of Bowen metric mean dimension on subsets of continuous flows, which allows us to establish variational principle for Bowen metric mean dimension for compact subsets in terms of Brin-Katok local entropy. Then we extend the variational principle for metric mean dimension on subset to the whole phase space. We state it as follows.

Theorem 1.2. Let \((X, \phi)\) be a continuous flow with a metric \(d\). Then

\[
\overline{\text{mdim}}_M(\phi, X, d) = \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{\mu \in \mathcal{M}(X)} h_{BK}^\mu(\phi, \epsilon) \\
\underline{\text{mdim}}_M(\phi, X, d) = \liminf_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{\mu \in \mathcal{M}(X)} h_{BK}^\mu(\phi, \epsilon),
\]

where \(h_{BK}^\mu(\phi, \epsilon), h_{BK}^\mu(\phi, \epsilon)\) are upper and lower Brin-Katok local \(\epsilon\)-entropies of \(\phi\), respectively. The result is also true for \(\overline{\text{mdim}}_M(\phi, X, d)\) by changing \(\limsup_{\epsilon \to 0}\) into \(\liminf_{\epsilon \to 0}\).

To overcome the second obstacle, we need an auxiliary condition on the flow to offset the differences in the topological and measure-theoretic aspects for the flows and its discrete samples, which was also considered in [CL21, CL22]. In this case, we can restrict our attentions to deal with the first obstacle since many abundant results about the variational principles for metric mean dimension of time-one map can be applied. Finally, we can establish variational principles for metric mean dimension of a class of special continuous flow.

Theorem 1.3. Let \((X, \phi)\) be a uniformly Lipschitz flow with a metric \(d\). Then for every \(F(\mu, \epsilon) \in \mathcal{D}\)

\[
\overline{\text{mdim}}_M(\phi, X, d) = \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{\mu \in \mathcal{E}_\epsilon(X)} F(\mu, \epsilon) \\
\underline{\text{mdim}}_M(\phi, X, d) = \liminf_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{\mu \in \mathcal{E}_\epsilon(X)} F(\mu, \epsilon),
\]
where $F(\mu, \epsilon)$ is chosen from the candidate set
\[
\mathcal{D} = \left\{ \inf_{\text{diam}(P) \leq s} h_\mu(\phi_1, P), \inf_{\text{diam}(U) \leq \epsilon} h^S_\mu(\phi_1, U, \delta), \inf_{\text{diam}(U) \leq \epsilon} h^K_\mu(\phi_1, U, \delta) \right\},
\]
\[
\delta \in (0, 1), \text{ and the infimums in } \mathcal{D} \text{ range over all finite Borel measurable partitions } P \text{ of } X \text{ with diameter at most } \epsilon, \text{ all finite open covers } U \text{ of } X \text{ with diameter at most } \epsilon, \text{ respectively.}
\]

We remark that if the scale function $S = \log \frac{1}{\epsilon}$ and the potential function $f = 0$, then the condition $A$ can be removed in [CL21, Theorem C] and the supremum can only range over all ergodic measures of $\phi$ by Theorem 1.3.

The rest of this paper is organized as follows. In section 2, we introduce the notions of the metric mean dimension of flows on the whole phase space and on subsets, and we prove Theorems 1.1 and 1.2. In section 3, we give the proof of Theorem 1.3.

2. Proofs of Theorem 1.1 and 1.2

2.1. Metric mean dimension of flows. In this subsection, we introduce the notion of metric mean dimension of continuous flows defined by spanning set and separated set and prove Theorem 1.1.

Let $t \in \mathbb{R}$, $n \in \mathbb{N}$ and $x, y \in X$. We respectively define the $t$-th Bowen metric with respect to $\phi$, $n$-th Bowen metric with respect to $\phi_t$ as
\[
d_t(x, y) := \sup_{s \in [0, t]} d(\phi_s x, \phi_s y),
\]
\[
d_{n, \phi_t}(x, y) := \max_{j \in \{0, \ldots, n-1\}} d(\phi_{tj} x, \phi_{tj} y).
\]

Then the $(t, \epsilon, \phi)$-ball of $x$ is given by $B_t(x, \epsilon, \phi) = \{ y \in X : d_t(x, y) < \epsilon \}$, and the $(n, \epsilon, \phi_t)$-ball of $x$ is given by $B_{n, \phi_t}(x, \epsilon) = \{ y \in X : d_{n, \phi_t}(x, y) < \epsilon \}$. Clearly, the sets $B_t(x, \epsilon, \phi)$ and $B_{n, \phi_t}(x, \epsilon)$ are open due to the continuity of $\phi$.

Fix a non-empty subset $Z \subset X$ and $\epsilon > 0$. A set $E \subset X$ is a $(t, \epsilon)$-spanning set of $Z$ if for any $x \in Z$, there exists $y \in E$ such that $d_t(x, y) < \epsilon$. The smallest cardinality of $(t, \epsilon)$-spanning set of $Z$ is denoted by $r_t(\phi, Z, d, \epsilon)$. A set $F \subset Z$ is a $(t, \epsilon)$-separated set of $Z$ if $d_t(x, y) \geq \epsilon$ for any $x, y \in F$ with $x \neq y$. The largest cardinality of $(t, \epsilon)$-separated set of $Z$ is denoted by $s_t(\phi, Z, d, \epsilon)$.

Put $r(\phi, Z, d, \epsilon) = \limsup_{t \to \infty} \frac{1}{t} \log r_t(\phi, Z, d, \epsilon)$ and $s(\phi, Z, d, \epsilon) = \limsup_{t \to \infty} \frac{1}{t} \log s_t(\phi, Z, d, \epsilon)$. By a standard method [Wal82], we have $r(\phi, X, d, \epsilon) \leq s(\phi, X, d, \epsilon) \leq r(\phi, X, d, \frac{\epsilon}{2})$.

**Definition 2.1.** Let $(X, \phi)$ be a continuous flow with a metric $d$. We define upper and lower metric mean dimensions of $X$ with respect to $\phi$
as

\[ \text{mdim}_M(\phi, X, d) = \limsup_{\epsilon \to 0} \frac{r(\phi, X, d, \epsilon)}{\log \frac{1}{\epsilon}} = \limsup_{\epsilon \to 0} \frac{s(\phi, X, d, \epsilon)}{\log \frac{1}{\epsilon}}, \]

\[ \text{mdim}_M(\phi, X, d) = \liminf_{\epsilon \to 0} \frac{r(\phi, X, d, \epsilon)}{\log \frac{1}{\epsilon}} = \liminf_{\epsilon \to 0} \frac{s(\phi, X, d, \epsilon)}{\log \frac{1}{\epsilon}}. \]

One can define metric mean dimensions of \( X \) with respect to \( \phi_t \), denoted by \( \text{mdim}_M(\phi_1, X, d) \), \( \text{mdim}_M(\phi_1, X, d) \), by similarly defining the quantities \( r(\phi_1, X, d, \epsilon) \), \( s(\phi_1, X, d, \epsilon) \) using \( d_{n, \phi_t} \) metric. The quantities \( \text{mdim}_M(\phi, X, d) \) and \( \text{mdim}_M(\phi, X, d) \) depend on the metric on \( X \) and hence are not topological invariant. Recall that the topological entropy of continuous flows introduced by Bowen and Ruelle [BR75] is given by

\[ h_{\text{top}}(\phi, X) = \lim_{\epsilon \to 0} r(\phi, X, d, \epsilon) = \sup_{\epsilon > 0} r(\phi, X, d, \epsilon). \]

Hence the metric mean dimension of finite entropy system is zero, and this shows that metric mean dimension is a useful quantity to characterize flow that admits infinite topological entropy.

Next, we derive some basic properties related to metric mean dimension of flow which will be used in the subsequent proof.

**Proposition 2.2.** Let \((X, \phi)\) be a continuous flow with a metric \( d \). Then for every \( \tau > 0 \)

\[ \text{mdim}_M(\phi, X, d) = \limsup_{\epsilon \to 0} \limsup_{n \to \infty} \frac{r_{n\tau}(\phi, X, d, \epsilon)}{n\tau \log \frac{1}{\epsilon}}, \]

\[ \text{mdim}_M(\phi, X, d) = \liminf_{\epsilon \to 0} \limsup_{n \to \infty} \frac{r_{n\tau}(\phi, X, d, \epsilon)}{n\tau \log \frac{1}{\epsilon}}. \]

A flow naturally induces a discrete topological dynamical system \((X, \phi_t), t \in \mathbb{R}\). The following proposition examines the relationship between the metric mean dimension of a flow and its discrete samples, which can be regard as an analogue of Abramov measure-theoretic entropy formula [Abr59].

The following argument is also used by Li and Cheng [CL21, Definition 4.1] to study the scaled pressure of fixed-point free continuous flows.

**Definition 2.3.** A continuous flow \((X, \phi)\) is said to be a uniformly Lipschitz flow if for any \( t_0 > 0 \), there exists \( L(t_0) > 0 \) such that for any \( \epsilon > 0 \) and \( x, y \in X \) with \( d(x, y) \leq \frac{\epsilon}{L(t_0)} \), one has

\[ d(\phi_s(x), \phi_s(y)) < \epsilon \]

for all \( s \in [0, t_0] \).
Proposition 2.4. Let \((X, \phi)\) be a continuous flow with a metric \(d\). Then for every \(\tau \in \mathbb{R}_+ \setminus \{0\}\)
\[
\frac{1}{\tau} \operatorname{mdim}_M(\phi, X, d) \leq \operatorname{mdim}_M(\phi, X, d),
\]
\[
\frac{1}{\tau} \operatorname{mdim}_M(\phi, X, d) \leq \operatorname{mdim}_M(\phi, X, d),
\]
and the the equalities hold if \((X, \phi)\) is a uniformly Lipschitz flow.

Proof. Fix \(\tau > 0\) and \(\epsilon > 0\). Choose a subsequence \(n_k\) that converges to \(\infty\) satisfying
\[
\lim_{k \to \infty} \frac{\log r_{n_k}(\phi, X, d, \epsilon)}{n_k} = r(\phi, X, d, \epsilon)
\]
and hence choose a subsequence \(t_k\) such that \(n_k \tau \leq t_k < (n_k + 1) \tau\). Note that \(B_{t_k}(x, \epsilon, \phi) \subset B_{n_k}(x, \epsilon, \phi)\). Then
\[
\frac{r_{n_k}(\phi, X, d, \epsilon)}{t_k} \leq \frac{r_{n_k}(\phi, X, d, \epsilon)}{n_k} \frac{n_k}{t_k}.
\]
Since \(\lim_{k \to \infty} \frac{t_k}{n_k} = \tau\), we get
\[
\frac{r(\phi, X, d, \epsilon)}{\tau} \leq r(\phi, X, d, \epsilon)
\]
by letting \(k \to \infty\), which implies the desired results.

Assume that \((X, \phi)\) is a uniformly Lipschitz flow. Then \(d(\phi_s x, \phi_s y) < \epsilon\) for all \(0 \leq s \leq \tau\) if \(x, y \in X\) with \(d(x, y) < \epsilon\). Hence \(B_{n_k}(x, \epsilon, \phi) \subset B_{n_k}(x, \epsilon, \phi)\) for every \(n \in \mathbb{N}\). Using the fact that \(r(\phi, X, d, \epsilon) = \limsup_{n \to \infty} \frac{r_{n_k}(\phi, X, d, \epsilon)}{n_k}\), we obtain that
\[
\frac{r(\phi, X, d, \epsilon)}{\tau} \geq r(\phi, X, d, \epsilon).
\]
This completes the proof. □

We interpret why uniformly Lipschitz flow is needed for obtaining the converse inequalities. By the continuity of \(\phi\), there exists \(\delta(\epsilon) > 0\) such that \(d(x, y) < \delta(\epsilon)\) so that \(d(\phi_s x, \phi_s y) < \epsilon\) for all \(s \in [0, \tau]\). Similarly, we have
\[
\frac{r(\phi, X, d, \epsilon)}{\tau} \geq r(\phi, X, d, \epsilon).
\]
According to the definition of metric mean dimension, we can formulate the following inequality
\[
\frac{\log \frac{1}{\delta(\epsilon)}}{\log \frac{1}{\epsilon}} \cdot \frac{r(\phi, X, d, \delta(\epsilon))}{\tau \log \frac{1}{\delta(\epsilon)}} \geq \frac{r(\phi, X, d, \epsilon)}{\log \frac{1}{\epsilon}}.
\]
However, we fail to determine whether \(\limsup_{\epsilon \to 0} \frac{\log \frac{1}{\delta(\epsilon)}}{\log \frac{1}{\epsilon}} = 1\) or not even if we require \(\delta(\epsilon) \leq \epsilon\).

Recall that the mean dimension of a real continuous flow \((X, \phi)\) introduced by Gutman and Jin [GJ20] is given by
\[
\operatorname{mdim}(X, \phi) = \lim \lim_{\epsilon \to 0} \frac{\operatorname{Wdim}_\epsilon(X, d_n)}{n},
\]
where \(\operatorname{Wdim}_\epsilon(X, d_n)\) is defined by \((d_n, \epsilon)\)-embedding mapping. Replacing \(d_n\) by the metric \(d_{n, \phi_1}\) one can similarly define the mean dimension \(\operatorname{mdim}(X, \phi_1)\) of \(\phi_1\). With the help of the classical Lindenstrauss-Weiss’s inequality and Proposition 2.4, we can extend this inequality to continuous flows.
Corollary 2.5. Let $(X, \phi)$ be a continuous flow with a metric $d$. Then
\[
\text{mdim}(X, \phi) \leq \text{mdim}_M(\phi, X, d) \leq \text{mdim}_M(\phi, X, d).
\]

Proof. By [GJ20, Proposition 2.5], we have $\text{mdim}(X, \phi) = \text{mdim}(X, \phi_1)$. Together with the fact $\text{mdim}(X, \phi_1) \leq \text{mdim}_M(X, \phi_1, d)$ [LW00, Theorem 4.2] and Proposition 2.4, we get the desired result.

The following proposition reveals that metric mean dimension is a local quantity, and the metric mean dimension of the whole phase space can be computed by arbitrarily small (compact) subsets in sense of diameter.

Proposition 2.6. Let $(X, \phi)$ be a continuous flow with a metric $d$. Suppose that $X$ is a finite union of closed subset $K_i$, $i = 1, \ldots, N$. Then for every $\epsilon > 0$
\[
r(\phi, X, d, \epsilon) = \max_{1 \leq j \leq N} r(\phi, K_j, d, \epsilon).
\]
Consequently, $\text{mdim}_M(\phi, X, d) = \max_{1 \leq j \leq N} \text{mdim}_M(\phi, K_j, d)$.

Proof. Fix $\epsilon > 0$. Clearly, one has $r(\phi, X, d, \epsilon) \geq \max_{1 \leq j \leq N} r(\phi, K_j, d, \epsilon)$. For each $t > 0$, there exists $j_{t, \epsilon} \in \{1, \ldots, N\}$ such that $r_{t}(\phi, K_{j_{t, \epsilon}}, d, \epsilon) = \max_{1 \leq j \leq N} r_{t}(\phi, K_j, d, \epsilon)$. Choose a subsequence $t_k$ that converges to $\infty$ and $j_{t_k} \in \{1, \ldots, N\}$ (only depends on $\epsilon$) so that $r(\phi, X, d, \epsilon) = \lim_{k \to \infty} r_{t_k}(\phi, X, d, \epsilon)$ and $r_{t_k}(\phi, K_{j_{t_k}}, d, \epsilon) = \max_{1 \leq j \leq N} r_{t_k}(\phi, K_j, d, \epsilon)$ for all $k \in \mathbb{N}$. This yields that
\[
r(\phi, X, d, \epsilon) \leq r(\phi, K_{j_{t_k}}, d, \epsilon) \leq \max_{1 \leq j \leq N} r(\phi, K_j, d, \epsilon).
\]

In [YZ07] Ye and Zhang introduced the notion of local entropy function to study the uniform entropy points and proved that $h_{\text{top}}(\phi_1, X) = \sup_{x \in X} h(\phi_1, x)$, where $h_{\text{top}}(\phi_1, X)$ and $h(\phi_1, x)$ are the topological entropy of $X$, the local entropy function at $x$ with respect to $\phi_1$, respectively. We extend this notion to continuous flows to establish an analogous variational principle for metric mean dimension.

Let $\epsilon > 0$ and $x \in X$. We define the $\epsilon$-local entropy function at $x$ with respect to $\phi$ as
\[
h_d(x, \epsilon, \phi) = \inf \{ r(\phi, K, d, \epsilon) : K \text{ is a closed neighborhood of } x \}.
\]

Proof of Theorem 1.1. Fix $\epsilon > 0$. Then the inequality $\sup_{x \in X} h_d(x, \epsilon, \phi) \leq r(\phi, X, \epsilon)$ is clear. Let $\{\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_{n_1}\}$ be the closed cover of $X$ with diameter (with respect to $d$) at most $1$. By Proposition 2.6, there exists $1 \leq j_1 \leq n_1$ such that $r(\phi, X, d, \epsilon) = r(\phi, \tilde{B}_{j_1}, d, \epsilon)$. Cover the closed ball $\tilde{B}_{j_1}^1$ with the family $\{\tilde{B}_{j_1}^2, \tilde{B}_{j_2}^2, \ldots, \tilde{B}_{j_2}^2\}$ of closed subsets of $\tilde{B}_{j_1}^1$. 

whose diameter is at most $\frac{1}{2}$. Applying Proposition 2.6 again there exists $1 \leq j_2 \leq n_2$ such that $r(\phi, X, d, \epsilon) = r(\phi, \bar{B}_{j_2}^1, d, \epsilon) = r(\phi, \bar{B}_{j_2}^2, d, \epsilon)$. Following this procedure, for every $k \in \mathbb{N}$ there is a closed ball $\bar{B}_{j_k}^k$ with diameter at most $\frac{1}{k}$ such that $r(\phi, X, d, \epsilon) = r(\phi, \bar{B}_{j_k}^k, d, \epsilon) \leq r(\phi, X, d, \epsilon) = r(\phi, \bar{B}_{j_k}^k, d, \epsilon)$. Set $\bigcap_{k \geq 1} \bar{B}_{j_k}^k = \{x_0\}$. Then for any closed neighborhood $K$ of $x_0$ we can choose sufficiently large $k_0 \in \mathbb{N}$ such that $\bar{B}_{j_k}^{k_0} \subset K$. So $r(\phi, X, d, \epsilon) = r(\phi, \bar{B}_{j_k}^{k_0}, d, \epsilon) \leq r(\phi, K, d, \epsilon)$, which implies that $r(\phi, X, d, \epsilon) \leq h_d(x_0, \epsilon, \phi) \leq \sup_{x \in X} h_d(x, \epsilon, \phi)$. This completes the proof. \hfill \Box

2.2. Bowen metric mean dimension on subsets. In this subsection, we introduce the notion of Bowen metric mean dimension and prove Theorem 1.2.

Inspired by the work of [Bow73, FH12, W21, CLS21, YCZ22] we define the Bowen metric mean dimension of flows on arbitrary non-empty subset $Z \subset X$ by means of Carathéodory-Pesin structure [Pes97], which allows us to generalize the definition of metric mean dimension to the subset of $X$ which is not necessarily compact or $T$-invariant.

**Definition 2.7.** Let $Z \subset X$ be a non-empty subset and $\epsilon > 0, N \in \mathbb{N}, \lambda \in \mathbb{R}$. Put

$$M(\phi, d, Z, \lambda, N, \epsilon) = \inf \left\{ \sum_{i \in I} e^{-n_i \lambda} \right\},$$

where the infimum is taken over all finite or countable covers $\{B_{n_i}(x_i, \epsilon, \phi)\}_{i \in I}$ of $Z$ with $n_i \geq N$.

Since $M(\phi, d, Z, \lambda, N, \epsilon)$ is non-decreasing when $N$ increases, so the limit $M(\phi, d, Z, \lambda, \epsilon) = \lim_{N \to \infty} M(\phi, d, Z, \lambda, N, \epsilon)$ exists. It is readily to check that the quantity $M(\phi, d, Z, \lambda, \epsilon)$ has a critical value of parameter $\lambda$ jumping from $\infty$ to $0$. The critical value is defined by

$$M(\phi, d, Z, \lambda, \epsilon) = \inf \{ \lambda : M(\phi, d, Z, \lambda, \epsilon) = 0 \} = \sup \{ \lambda : M(\phi, d, Z, \lambda, \epsilon) = \infty \}.$$

We define Bowen upper metric mean dimension of $\phi$ on the set $Z$ as

$$\text{mdim}^B_{M}(\phi, Z, d) = \limsup_{\epsilon \to 0} \frac{M(\phi, d, Z, \epsilon)}{\log \frac{1}{\epsilon}}.$$

One can similarly Bowen lower metric mean dimension $\text{mdim}^B_{M}(\phi, Z, d)$ of $\phi$ on the set $Z$. Replacing the ball $B_n(x, \epsilon, \phi)$ by $B_n(x, \epsilon, \phi_1)$ in Definition 3.7 we can define the quantities $M(\phi_1, d, Z, \lambda, N, \epsilon), M(\phi_1, d, Z, \lambda, \epsilon), M(\phi_1, d, Z, \epsilon)$ and the Bowen upper and lower metric mean dimensions $\text{mdim}^B_{M}(\phi_1, Z, d), \text{mdim}^B_{M}(\phi_1, Z, d)$ of $\phi_1$ on $Z$. We sometimes omit $d$ in above quantities when $d$ is clear.
Proposition 2.8. Let \((X, \phi)\) be a uniformly Lipschitz flow with a metric \(d\). Then for any non-empty subset \(Z \subset X\)
\[
\text{mdim}_M(\phi, Z, d) = \text{mdim}_M(\phi_1, Z, d), \quad \text{mdim}_M(\phi, Z, d) = \text{mdim}_M(\phi_1, Z, d).
\]

When \(Z = X\), one can adapt the method of [Bow73] to show the metric mean dimension of \(X\) defined by Carathéodory-Pesin structure and spanning set are equivalent. Here, we provide a detailed proof for sake of readers.

Proposition 2.9. Let \((X, \phi)\) be a continuous flow with a metric \(d\). Then
\[
\text{mdim}_M(\phi, X) = \text{mdim}_M(\phi, X),
\]
\[
\text{mdim}_M(\phi, X) = \text{mdim}_M(\phi, X).
\]

Proof. It suffices to show the first equality and the second one can be proved in similar manner. The inequality \(\text{mdim}_M(\phi, X) \leq \text{mdim}_M(\phi, X)\) follows by using the fact every \((n, \epsilon)\)-spanning set of \(X\) with the minimal cardinality \(r_n(\phi, X, d, \epsilon)\) is also a finite open cover of \(X\).

Fix \(\epsilon > 0\) and let \(s > M(\phi, d, X, \epsilon)\). Then there exist \(N_0\) and a finite open cover \(\{B_{t_i}(x_i, \epsilon, \phi)\}_{i \in I}\) of \(X\) with \(t_i \geq N_0\) and \(\sum_{i \in I} e^{-s t_i} < 1\). (If one gets a countable open cover of \(X\), noticing that the compactness of \(X\) by discarding some open sets the sum is still less than 1.) It follows that
\[
\sum_{k=1}^{\infty} \sum_{j_1, \ldots, j_k \in I} e^{-s(t_{j_1} + \cdots + t_{j_k})} = \sum_{k=1}^{\infty} \left( \sum_{i \in I} e^{-s t_i} \right)^k < \infty.
\]

Let \(M = \max_{i \in I} t_i\) and define \(c_0 = 0, c_i = \sum_{s=0}^{i-1} t_{j_s}, j_s \in I, i \geq 1\). For every sufficiently large \(N \geq N_0\), we consider the family
\[
\mathcal{F}_N := \{ \bigcap_{i=0}^k \phi^{-c_i} B_{t_{j_i}}(x_{t_{j_i}}, \epsilon, \phi) : k \geq 0, N \leq \sum_{i=0}^k t_{j_i} < N + M \}.
\]

Assume that each \(A \in \mathcal{F}_N\) is not empty. Then we can choose \(x_A \in A\) such that \(A \subset B_N(x_A, 2\epsilon, \phi)\). One can check that \(X \subset \bigcup_{A \in \mathcal{F}_N} B_N(x_A, 2\epsilon, \phi)\).

Therefore,
\[
\text{mdim}_M(\phi, X, d, 2\epsilon) e^{-s N} \leq \#\mathcal{F}_N \cdot e^{-s N}
\]
\[
\leq \sum_{k \geq 0} \sum_{N \leq t_{j_1} + \cdots + t_{j_k} < N + M} e^{-s(t_{j_1} + \cdots + t_{j_k})} \cdot e^{s M} < \infty.
\]

Then \(\text{r}(\phi, X, d, 2\epsilon) \leq s\). Letting \(s \to M(\phi, d, X, \epsilon)\) we get \(\text{r}(\phi, X, d, 2\epsilon) \leq M(\phi, d, X, \epsilon)\). This shows the converse inequality. \(\square\)

Next, we peruse a variational principle for Bowen metric mean dimension of flows on compact subsets.
Definition 2.10. Let \( f : X \to \mathbb{R} \) be a bounded function, and let \( N \in \mathbb{N}, \epsilon > 0, \lambda \in \mathbb{R} \). Put
\[
W(\phi, d, f, \lambda, N, \epsilon) = \inf \left\{ \sum_{i \in I} c_i e^{-n_i \lambda} \right\},
\]
where the infimum is taken over all finite or countable families \( \{ B_n(x_i, \epsilon, \phi) \}_{i \in I} \) with \( 0 < c_i < \infty, x_i \in X \) and \( n_i \geq N \), so that \( \sum_{i \in I} c_i \chi_{B_n(x_i, \epsilon, \phi)} \geq f \),
where \( \chi_A \) denotes the characteristic function of \( A \).

Let \( Z \subset X \) be a non-empty subset. Set
\[
W(\phi, d, Z, \lambda, N, \epsilon) := W(\phi, d, \chi_Z, \lambda, N, \epsilon).
\]

Let \( W(\phi, d, Z, \lambda, \epsilon) = \lim_{N \to \infty} W(\phi, d, Z, \lambda, N, \epsilon) \). It is readily to check that the quantity \( W(\phi, d, Z, \lambda, \epsilon) \) has a critical value of parameter \( \lambda \) jumping from \( \infty \) to \( 0 \). The critical value is defined by
\[
W(\phi, d, Z, \epsilon) := \inf \left\{ \lambda : W(\phi, d, Z, \lambda, \epsilon) = 0 \right\} = \sup \left\{ \lambda : W(\phi, d, Z, \lambda, \epsilon) = \infty \right\}.
\]

We define the weighted Bowen upper metric mean dimension of \( \phi \) on the set \( Z \) as
\[
\overline{\text{mdim}}_{WB}(\phi, Z, d) = \limsup_{\epsilon \to 0} \frac{W(\phi, d, Z, \epsilon)}{\log \frac{1}{\epsilon}}.
\]

By adapting the proof of [FH12, Proposition 3.2], the following proposition shows that the weighted Bowen metric mean dimension coincides with Bowen metric mean dimension.

Proposition 2.11. Let \( Z \subset X \) be a non-empty subset. Then for any \( s \geq 0, \epsilon > 0, \delta > 0 \) and \( N \in \mathbb{N} \)
\[
M(\phi, d, Z, s, N, 6\epsilon) \leq W(\phi, d, Z, s, N, \epsilon) \leq M(\phi, d, Z, s, N, \epsilon).
\]
Consequently, \( \overline{\text{mdim}}_{M}(\phi, Z, d) = \overline{\text{mdim}}_{WB}(\phi, Z, d) \).

To produce a Borel probability measure on \( X \), we also need the following Frostman’s lemma of flow by slightly modifying the proof of [FH12, Lemma 3.4].

Lemma 2.12. Let \( K \) be a non-empty compact subset of \( X \) and \( s \geq 0, \epsilon > 0, N \in \mathbb{N} \). Set \( c := W(\phi, d, K, s, N, \epsilon) > 0 \). Then there exists a Borel probability measure \( \mu \in M(X) \) such that \( \mu(K) = 1 \) and for any \( x \in X, n \geq N \),
\[
\mu(B_n(x, \epsilon, \phi)) \leq \frac{1}{c} e^{-sn}.
\]

Let \( \mu \in \mathcal{M}(X) \). We define upper and lower Brin-Katok local \( \epsilon \)-entropies of \( \phi \) as
\[ \overline{h}_\mu^{BK}(\phi, \epsilon) = \int \limsup_{t \to \infty} -\frac{\log \mu(B_t(x, \epsilon, \phi))}{t} d\mu, \]

\[ \underline{h}_\mu^{BK}(\phi, \epsilon) = \int \liminf_{t \to \infty} -\frac{\log \mu(B_t(x, \epsilon, \phi))}{t} d\mu, \]

The following theorem provides a variational principle for metric mean dimension on compact subsets, which is an analogue of variational principle for Bowen topological entropy on subsets [FH12].

**Theorem 2.13.** Let \((X, \phi)\) be a continuous flow with a metric \(d\) and \(K\) be a non-empty compact subset of \(X\). Then

\[ \overline{\text{mdim}}_M^{B}(\phi, K, d) = \limsup_{\epsilon \to 0} \frac{1}{\log \epsilon} \sup \{ \overline{h}_\mu^{BK} (\phi, \epsilon) : \mu \in \mathcal{M}(X), \mu(K) = 1 \}, \]

\[ \underline{\text{mdim}}_M^{B}(\phi, K, d) = \liminf_{\epsilon \to 0} \frac{1}{\log \epsilon} \sup \{ \underline{h}_\mu^{BK} (\phi, \epsilon) : \mu \in \mathcal{M}(X), \mu(K) = 1 \}. \]

**Proof.** Fix \(\epsilon > 0\), \(\mu(K) = 1\) and assume that \(\overline{h}_\mu^{BK}(\phi, \epsilon) > 0\). Let \(s < \overline{h}_\mu^{BK}(\phi, \epsilon)\). By a standard method, one can find a Borel set \(E \subset K\) and \(N \in \mathbb{N}\) such that \(\mu(E) > 0\) and \(-\frac{\log \mu(B_{n}(x, \epsilon, \phi))}{n} > s\) for any \(x \in E\) and \(n \geq N\).

Let \(\{B_n(x_i, \frac{\epsilon}{2}, \phi)\}_{i \in I}\) be a finite or countable cover of \(E\) with \(n_i \geq N\). We assume that \(B_n(x_i, \frac{\epsilon}{2}, \phi) \cap E \neq \emptyset\) for every \(i \in I\). Choose \(y_i \in B_n(x_i, \frac{\epsilon}{2}, \phi) \cap E\) for each \(i \in I\). Then \(\cup_{i \in I} B_n(y_i, \epsilon, \phi) \supseteq E\). This yields that

\[ \sum_{i \in I} e^{-sn_i} \geq \sum_{i \in I} \mu(B_n(y_i, \epsilon, \phi)) \geq \mu(E) > 0. \]

Therefore, \(M(\phi, d, K, \frac{\epsilon}{2}) \geq M(\phi, d, E, \frac{\epsilon}{2}) \geq s\). Letting \(s \to \overline{h}_\mu^{BK}(\phi, \epsilon)\), we obtain that \(\overline{h}_\mu^{BK}(\phi, \epsilon) \leq M(\phi, d, K, \frac{\epsilon}{2})\) for every \(\mu \in \mathcal{M}(X)\).

On the other hand, without loss of generality we assume \(W(\phi, d, K, s, \epsilon) > 0\). Let \(W(\phi, d, K, \epsilon) > s\). Then there exists \(N_0\) such that \(c := W(\phi, d, K, s, N_0, \epsilon) > 0\). By Lemma 2.12, there exists a \(\mu \in \mathcal{M}(X)\) so that \(\mu(K) = 1\) and

\[ \mu(B_n(x, \epsilon, \phi)) \leq \frac{1}{c} e^{-ns} \]

for any \(x \in K\) and \(n \geq N_0\). It follows that \(\overline{h}_\mu^{BK}(\phi, \epsilon) \geq s\). Letting \(s \to W(\phi, d, K, \epsilon)\) and then by Proposition 2.11 we know that

\[ M(\phi, d, K, 6\epsilon) \leq \sup \{ \underline{h}_\mu^{BK}(\phi, \epsilon) : \mu \in \mathcal{M}(X), \mu(K) = 1 \}. \]

Together with the two facts, we get the desired result. \(\square\)

**Proof of Theorem 1.2.** Fix \(\epsilon > 0\), \(\mu(X) = 1\). Let \(s < \overline{h}_\mu^{BK}(\phi, \epsilon)\). By a standard method, there exists a Borel subset \(E \subset X\) such that \(\mu(E) > 0\) and \(\limsup_{n \to \infty} -\frac{\log \mu(B_n(x, \epsilon, \phi))}{n} > s\) for all \(x \in E\). Put \(E_n = \{x \in E : \mu(B_n(x, \epsilon, \phi)) \leq e^{-ns}\}\). Then \(E = \cup_{n \geq N} E_n\) for every \(N \geq 1\). Hence
there exists \( n \geq N \) (depends on \( N \)) so that \( \mu(E_n) > 0 \) for each \( N \geq 1 \). Let \( F_n \subset E_n \) be a \((n, \epsilon)\)-separated set of \( E_n \) with the largest cardinality \( s_n(\phi, E_n, d, \epsilon) \). Then

\[
0 < \mu(E_n) \leq \sum_{x \in F_n} \mu(B_n(x, \epsilon, \phi)) \leq s_n(\phi, E_n, d, \epsilon) \cdot e^{-ns}.
\]

This implies that \( s_n(\phi, X, d, \epsilon) \geq \mu(E_n) e^{ns} \). Note that \( \mu(E_n) \leq 1 \). We finally get \( s(\phi, X, d, \epsilon) \geq s \). Letting \( s \to 0 \)

\[
\overline{h}_\mu(\phi, \epsilon) \leq s(\phi, X, d, \epsilon).
\]

By Proposition 2.9 and Theorem 2.13, one has Theorem 1.2. □

3. Proof of Theorem 1.3

In this section, we recall that the definitions of mean Rényi information dimensions [Pes97, GS21] and Katok’s entropy [Kat80] of invariant measures and give the proof of Theorem 1.3.

Throughout this section, \( L := L(1) \) is given as Definition 2.3.

3.1. Mean Rényi information dimension. Let \( t \in \mathbb{R}, \mu \in \mathcal{M}_\phi(X) \) and \( P \) be a finite Borel measurable partition of \( X \). By \( \text{diam} P = \sup_{A \in P} \text{diam}(A, d) \) we denote the diameter of \( P \). By \( P^n \) we denote the \( n\)-th join of the partitions \( P, \phi^{-t}P, ..., \phi^{-(n-1)}P \) with respect to \( \phi_t \) formed by the set \( A_{i_0} \cap \phi^{-t}A_{i_1} \cap \cdots \phi^{-(n-1)}A_{i_{n-1}} \) with \( A_{i_j} \in P, j = 0, ..., n - 1 \). The measure-theoretic entropy of \( \mu \) with respect to \( \phi_t \) is defined by \( h_\mu(\phi_t) = \sup_P h_\mu(\phi_t, P) \) with supremum over all finite Borel partitions of \( X \), where \( h_\mu(\phi_t, P) := \lim_{n \to \infty} \frac{H_\mu(P^n)}{n} \) and \( H_\mu(P^n) \) denotes the partition entropy of \( P^n \). The Abramov’s measure-theoretic entropy formula [Abr59] reveals that for every \( t \in \mathbb{R} \)

\[
h_\mu(\phi_t) = |t|h_\mu(\phi_1).
\]

The upper Rényi information dimension [GS21] of \( \mu \in \mathcal{M}_\phi(X) \) is defined by

\[
\overline{\text{MRID}}(X, \phi_1, d, \mu) = \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \inf_{\text{diam}(P) \leq \epsilon} h_\mu(\phi_1, P),
\]

where the infimum is taken over all finite Borel partitions of \( X \) with diameter at most \( \epsilon \).

3.2. Katok’s entropy. Let \( \mu \in \mathcal{M}(X), \epsilon > 0, t > 0 \) and \( \delta \in (0, 1) \). Put

\[
R^\delta(\phi, \epsilon, \phi) = \min\{ \#E : E \subset X \text{ and } \mu(\cup_{x \in E} B_t(x, \epsilon, \phi)) > \delta \}.
\]
We define the upper and lower Katok’s entropies of $\mu$ with respect to $\phi$ as
\[
\overline{h}_\mu^K(\epsilon, \delta, \phi) = \limsup_{t \to \infty} \frac{1}{t} \log R^\delta_\mu(t, \epsilon, \phi),
\]
\[
\underline{h}_\mu^K(\epsilon, \delta, \phi) = \liminf_{t \to \infty} \frac{1}{t} \log R^\delta_\mu(t, \epsilon, \phi).
\]
Similarly, one can define the quantities $R^\delta_\mu(n, \epsilon, \phi_1)$, $\overline{h}_\mu^K(\epsilon, \delta, \phi_1)$, $\underline{h}_\mu^K(\epsilon, \delta, \phi_1)$ by using Bowen balls $B_n(x, \epsilon, \phi_1)$. Sun [Sun01, Theorem A] proved that for any $\delta \in (0, 1)$,
\[
\lim_{\epsilon \to 0} \overline{h}_\mu^K(\epsilon, \delta, \phi) = \lim_{\epsilon \to 0} \underline{h}_\mu^K(\epsilon, \delta, \phi) = h_\mu(\phi_1).
\]
Besides, Katok’s entropy can be defined by open covers [Sha07] analogous to the definition of the topological entropy of a finite open cover.

Let $U$ be a finite open cover of $X$, $\delta \in (0, 1)$ and $\mu \in \mathcal{M}(X)$. By $\text{diam}(U) = \max_{U \in \mathcal{U}} \text{diam}(U, d)$ we denote the diameter of $U$. By $\text{Leb}(U)$ we denote the Lebesgue number of $U$, that is, the largest positive number $\delta > 0$ such that every open ball of radius $\delta$ is contained in some element of $U$. Let $U^n$ denote the $n$-th join of open covers $U, \phi^{-1}U, \ldots, \phi^{-(n-1)}U$ with respect to $\phi_1$ formed by the open set $U_{i_0} \cap \phi^{-1}U_{i_1} \cap \cdots \phi^{-(n-1)}U_{i_{n-1}}$ with $U_{i_j} \in \mathcal{U}, j = 0, \ldots, n - 1$. Set
\[
N_\mu(U, \delta) = \min \{ \#U' : \mu(U) > \delta \text{ and } U' \text{ is a subfamily of } U \}.
\]
We define
\[
\underline{h}_\mu^S(\phi_1, U, \delta) = \liminf_{n \to \infty} \frac{\log N_\mu(U^n, \delta)}{n},
\]
\[
\overline{h}_\mu^S(\phi_1, U, \delta) = \limsup_{n \to \infty} \frac{\log N_\mu(U^n, \delta)}{n}.
\]
Shapira [Sha07, Theorem 4.2] showed if $\mu \in \mathcal{E}_{\phi_1}(X)$, then the limit
\[
\lim_{n \to \infty} \frac{\log N_\mu(U^n, \delta)}{n}
\]
exists and does not depend on the choice of $\delta \in (0, 1)$.

Let $U$ be a finite open cover so that $\text{diam}(U) \leq \epsilon$ and $\text{Leb}(U) \geq \frac{\epsilon}{4}$. Indeed, such an open cover exists. Let $\{B(x, \frac{\epsilon}{2}) : x \in E\}$ be a finite open cover of $X$ with open $\frac{\epsilon}{2}$-ball (with respect to $d$). The family $\{B(x, \frac{\epsilon}{2}) : x \in E\}$ satisfies it. Using this fact, we have the following.

**Proposition 3.1.** For every $\epsilon > 0$, $\mu \in \mathcal{M}(X)$ and $\delta \in (0, 1)$, we have
\[
\underline{h}_\mu^K(2\epsilon, \delta, \phi) \leq \inf_{\text{diam}(U) \leq \epsilon} \underline{h}_\mu^S(\phi_1, U, \delta) \leq \overline{h}_\mu^K(\epsilon, \delta, \phi),
\]
\[
\overline{h}_\mu^K(2\epsilon, \delta, \phi) \leq \inf_{\text{diam}(U) \leq \epsilon} \overline{h}_\mu^S(\phi_1, U, \delta) \leq \overline{h}_\mu^K(\epsilon, \delta, \phi).
\]
Proof of Theorem 1.3. It suffices to show
\[
\overline{\text{mdim}}_M(\phi, X, d) = \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{\mu \in \mathcal{E}_\phi(X)} F(\mu, \epsilon).
\]

We divide the proof into three steps.

Step 1 we show that the variational principle holds for
\[
F(\mu, \epsilon) = \inf_{\text{diam}(P) \leq \epsilon} h_\mu(\phi_1, P).
\]

Note that \(\overline{\text{mdim}}_M(\phi, X, d) = \overline{\text{mdim}}_M(\phi_1, X, d)\) by Proposition 2.4. Together with \([GS21, \text{Theorem 3.1}]\), one has
\[
\overline{\text{mdim}}_M(\phi, X, d) \geq \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{\mu \in \mathcal{E}_\phi(X)} \inf_{\text{diam}(P) \leq \epsilon} h_\mu(\phi_1, P).
\]

Fix \(\epsilon > 0\). Let \(\mu \in \mathcal{E}_{\phi_1}(X)\) and let \(t \in \mathbb{R}\). We define
\[
\mu_t(B) := \mu(\phi_t(B)), m(B) := \int_0^1 \mu_t(B) dt,
\]
for any Borel measurable set \(B\). Then \(\mu_t \in \mathcal{E}_{\phi_1}(X)\) and \(m \in \mathcal{E}_\phi(X)\). Since \(\phi_{-t}\) is the invertible measure-preserving transformation for the systems \((X, \mu_t, \phi_1)\) and \((X, \mu, \phi_1)\), then \(h_\mu(\phi_1, \phi_t(\xi)) = h_\mu(\phi_1, \xi)\) for any finite Borel partition \(\xi\).

Let \(\xi\) be a finite Borel partition of \(X\) with diameter less than \(\epsilon/L\). Notice that \(h_\mu(\phi_1, \xi)\) is a concave function with respect to \(\mu\). By Jensen’s inequality,
\[
h_{\mu_t}(\phi_1, \xi) \geq \int_0^1 h_\mu(\phi_1, \xi) dt.
\]
So there is a \(t_0 \in [0, 1]\) so that
\[
h_{\mu_t}(\phi_1, \xi) \geq h_{\mu_{t_0}}(\phi_1, \xi) = h_\mu(\phi_1, \phi_{t_0}(\xi)) \geq \inf_{\text{diam}(P) \leq \epsilon} h_\mu(\phi_1, P).
\]
It follows that \(\sup_{\mu \in \mathcal{E}_{\phi_1}(X)} \inf_{\text{diam}(P) \leq \epsilon} h_\mu(\phi_1, P) \leq \sup_{m \in \mathcal{E}_\phi(X)} \inf_{\text{diam}(\xi) \leq \epsilon/L} h_m(\phi_1, \xi)\).

By \([GS21, \text{Remark 3.6}]\) and Proposition 2.4, we finish the proof of step 1.

Step 2 By Proposition 3.1 it suffices to show for every \(\delta \in (0, 1)\), the variational principles hold for
\[
F(\mu, \epsilon) = \{l^K_\epsilon(\epsilon, \delta, \phi_1), \overline{l^K_\mu}(\epsilon, \delta, \phi_1)\}.
\]

Fix \(\delta \in (0, 1)\). Let \(\epsilon > 0\) and \(n \in \mathbb{N}\). Using the Proposition 2.4 and the fact that \(R^K_\mu(n, \epsilon, \phi_1) \leq r_n(\phi_1, X, d, \epsilon)\) for every \(\mu \in \mathcal{E}_\phi(X)\), then
\[
\overline{\text{mdim}}_M(\phi, X, d) \geq \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{\mu \in \mathcal{E}_\phi(X)} \overline{l^K_\mu}(\epsilon, \delta, \phi_1)
\]
\[
\geq \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{\mu \in \mathcal{E}_\phi(X)} l^K_\mu(\epsilon, \delta, \phi_1).
\]
Again, let \( \mu \in \mathcal{E}_\phi(X) \) and let \( t \in \mathbb{R} \). We define

\[
\mu_t(B) := \mu(\phi_t(B)), m(B) := \int_0^1 \mu_t(B) dt,
\]

for any Borel measurable set \( B \). Then \( \mu_t \in \mathcal{E}_\phi(X) \) and \( m \in \mathcal{E}_\phi(X) \). Let \( E_n \subset X \) with \( \#E_n = R^\delta_{\mu}(n+1, \epsilon, \phi) \). Then

\[
m(\bigcup_{x \in E_n} B_{n+1}(x, \epsilon, \phi)) = \int_0^1 \mu_t(\bigcup_{x \in E_n} B_{n+1}(x, \epsilon, \phi)) dt > \delta.
\]

Therefore, there exists a \( t_0 \in [0, 1] \) so that \( \mu_{t_0}(\bigcup_{x \in E_n} B_{n+1}(x, \epsilon, \phi)) = \mu((\bigcup_{x \in E_n} \phi_{t_0} B_{n+1}(x, \epsilon, \phi))) > \delta \). Since \( \phi_{t_0}(B_{n+1}(x, \epsilon, \phi)) \subset B_n(\phi_{t_0} x, \epsilon, \phi_1) \), then \( \mu((\bigcup_{x \in E_n} \phi_{t_0} B_{n}(\phi_{t_0} x, \epsilon, \phi_1))) > \delta \). This shows that \( R^\delta_{\mu}(n, \epsilon, \phi_1) \leq R^\delta_{\mu}(n+1, \epsilon, \phi) \) for every \( n \in \mathbb{N} \), which yields that

\[
\mathcal{H}^K(\epsilon, \delta, \phi_1) \leq \mathcal{H}^K(\epsilon, \delta, \phi) \leq \sup_{m \in M^\phi(X)} \mathcal{H}^K_m(\epsilon, \delta, \phi_1)
\]

\[
\leq \sup_{m \in \mathcal{E}_\phi(X)} \mathcal{H}^K_m(\epsilon/L, \delta, \phi_1),
\]

By [Shi21, Proposition 6.3] and Proposition 2.4, we obtain

\[
\text{mdim}_M(\phi, X, d) \leq \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{\mu \in \mathcal{E}_\phi(X)} \mathcal{H}^K(\epsilon, \delta, \phi_1).
\]

**Step 3** we show that the variational principle holds for

\[
F(\mu, \epsilon) = \mathcal{H}^K_{\mu}(\phi, \epsilon).
\]

By Theorem 1.2,

\[
\text{mdim}_M(\phi, X, d) \geq \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{\mu \in \mathcal{E}_\phi(X)} \mathcal{H}^K_{\mu}(\phi, \epsilon).
\]

Fix \( \epsilon > 0 \) and \( \mu \in \mathcal{E}_\phi(X) \). Notice that \( \phi_s(B_{t+s}(x, \epsilon, \phi)) \subset B_t(\phi_s x, \epsilon, \phi) \) for all \( s, t \geq 0 \). Then by the ergodicity of \( \mu \), \( \mathcal{H}^K_{\mu}(\phi, x, \epsilon) \) is a constant for \( \mu\text{-a.e.} x \in X \) and equals to \( \mathcal{H}^K_{\mu}(\phi, \epsilon) \). Let \( \delta > 0 \) and \( s > \mathcal{H}^K_{\mu}(\phi, \epsilon) \). Put

\[
X_N := \{ x \in X : -\frac{\log \mu(B_n(x, \epsilon, \phi))}{n} < s, \forall n \geq N \}.
\]

Then \( \mu(\bigcup_{N \geq 1} X_N) = 1 \) and hence there exists \( N_0 \) such that \( \mu(X_N) > \delta \) for any \( N \geq N_0 \) by the continuity of \( \mu \). Let \( N \geq N_0 \) and \( E_N \subset X_N \) be a \((N, 2\epsilon)\)-separated set of \( X_N \) with the largest cardinality. Then the balls \( \{B_N(x, \epsilon, \phi), x \in E_N\} \) are pairwise disjoint. This yields that

\[
1 \geq \mu(\bigcup_{x \in E_N} B_N(x, \epsilon, \phi)) = \sum_{x \in E_N} \mu(B_N(x, \epsilon, \phi)) \geq \#E_N \cdot e^{-sN}.
\]

Therefore, one has \( R^\delta_{\mu}(N, 2\epsilon, \phi) \leq e^{sN} \) for any \( N \geq N_0 \). Consequently, \( \mathcal{H}^K_{\mu}(2\epsilon, \delta, \phi_1) \leq \mathcal{H}^K_{\mu}(2\epsilon, \delta, \phi) \leq \mathcal{H}^{BK}_{\mu}(\phi, \epsilon) \leq \mathcal{H}^{BK}_{\mu}(\phi_1, \epsilon/L) \). By Step 2, we
get
\[ \overline{\text{mdim}}_M(\phi, X, d) \leq \limsup_{\epsilon \to 0} \frac{1}{\log \frac{1}{\epsilon}} \sup_{\mu \in \mathcal{E}_\phi(X)} R^B_{\mu}(\phi, \epsilon). \]

Finally, we finish this paper with three questions as follows.

**Question 1** Is it possible that the "\( \leq \)" can be strict in Proposition 2.4?

**Question 2** Does there exist the metric \( d \) compatible with the topology of \( X \) such that \( \text{mdim}(X, \phi) = \text{mdim}_M(\phi, X, d) \)?

**Question 3** Can the condition of uniformly Lipschitz flow be removed in Theorem 1.3?

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**References**

[Abr59] L. Abramov, On the entropy of a flow, *Dokl. Akad. Nauk SSSR* **128** (1959), 873-875.

[Bow73] R. Bowen, Topological entropy for noncompact sets, *Trans. Amer. Math. Soc.* **184** (1973), 125-136.

[BR75] R. Bowen and D. Ruelle, The ergodic theory of Axiom A flows, *Invent. Math.* **16** (1975), 181-202.

[BK83] M. Brin and A. Katok, On local entropy, Geometric dynamics (Rio de Janeiro), *Lecture Notes in Mathematics*, Springer, Berlin, **1007** (1983), 30-38.

[CDZ22] E. Chen, D. Dou and D. Zheng, Variational principles for amenable metric mean dimensions, *J. Diff. Equ.* **319** (2022), 41-79.

[CT06] T. M. Cover and J. A. Thomas, Elements of Information Theory, second edition, Wiley, New York, 2006.

[CLS21] D. Cheng, Z. Li and B. Selmi, Upper metric mean dimensions with potential on subsets, *Nonlinearity* **34** (2021), 852-867.

[CL22] D. Cheng and Z. Li, Upper metric mean dimensions for impulsive semi-flows, *J. Diff. Equ.* **311** (2022), 81-97.
[CL21] D. Cheng and Z. Li, Scaled pressure of continuous flows, *Nonlinearity* 34 (2021), 7829-7841.

[CRV22] M. Carvalho, F. Rodrigues and P. Varandas, A variational principle for the metric mean dimension of free semigroup actions, *Ergod. Th. Dynam. Syst.* 42 (2022), 65-85.

[FH12] D. J. Feng and W. Huang, Variational principles for topological entropies of subsets, *J. Funct. Anal.* 263 (2012), 2228-2254.

[Gro99] M. Gromov, Topological invariants of dynamical systems and spaces of holomorphic maps: I, *Math. Phys. Anal. Geom.* 4 (1999), 323-415.

[GT20] Y. Gutman and M. Tsukamoto, Embedding minimal dynamical systems into Hilbert cubes, *Invent. Math.* 221 (2020), 113-166.

[Gut15] Y. Gutman, Mean dimension and Jaworski-type theorems, *Proc. Lond. Math. Soc.* 111 (2015), 831-850.

[GLT16] Y. Gutman, E. Lindenstrauss and M. Tsukamoto, Mean dimension of $Z^k$ actions, *Geom. Funct. Anal.* 26 (2016), 778-817.

[Gut17] Y. Gutman, Embedding topological dynamical systems with periodic points in cubical shifts, *Ergodic Theory Dynam. Syst.* 37 (2017), 512-538.

[GJ20] Y. Gutman and L. Jin, Mean dimension and an embedding theorem for real flows, *Fund. Math.* 251 (2020), 161-181.

[GS20] Y. Gutman and A. Śpiewak, Metric mean dimension and analog compression, *IEEE Trans. Inform. Theory* 66 (2020), 6977-6998.

[GS21] Y. Gutman and A. Śpiewak, Around the variational principle for metric mean dimension, *Studia Math.* 261 (2021), 345-360.

[JRS21] T. Jacobus, F B. Rodrigues and M V. Silva, Some variational principles for the metric mean dimension of a semigroup action, arXiv:2107.01968

[Kat80] A. Katok, Lyapunov exponents, entropy and periodic orbits for diffeomorphisms, *Publ. Math. Inst. Hautes Études Sci.* 51 (1980), 137-173.

[LW00] E. Lindenstrauss and B. Weiss, Mean topological dimension, *Israel J. Math.* 115 (2000), 1-24.

[LT18] E. Lindenstrauss and M. Tsukamoto, From rate distortion theory to metric mean dimension: variational principle, *IEEE Trans. Inform. Theory* 64 (2018), 3590-3609.

[LT19] E. Lindenstrauss and M. Tsukamoto, Double variational principle for mean dimension, *Geom. Funct. Anal.* 29 (2019), 1048-1109.
[Pes97] Y.B. Pesin, Dimension theory in dynamical systems, University of Chicago Press, 1997.

[Sun01] W. Sun, Entropy of orthonormal $n$-frame flows, *Nonlinearity* **14** (2001), 829-842.

[Sha07] U. Shapira, Measure theoretical entropy of covers, *Israel J. Math.* **158** (2007), 225-247.

[SZ12] J. Shen and Y. Zhao, Entropy of a flow on non-compact sets, *Open Syst. Inf. Dyn.* **19** (2012), 1250015, 10 pp.

[Shi21] R. Shi, On variational principles for metric mean dimension, to appear in *IEEE Trans. Inform. Theory*, **68** (2022), 4282-4288.

[Tsu18] M. Tsukamoto, Mean dimension of the dynamical system of Brody curves, *Invent. Math.* **211** (2018), 935-968.

[Tsu20] M. Tsukamoto, Double variational principle for mean dimension with potential, *Adv. Math.* **361** (2020), 106935, 53 pp.

[VV17] A. Velozo and R. Velozo, Rate distortion theory, metric mean dimension and measure theoretic entropy, arXiv:1707.05762.

[Wal82] P. Walter, An introduction to ergodic theory, Springer-Verlag, New York, 1982.

[W21] T. Wang, Variational relations for metric mean dimension and rate distortion dimension, *Discrete Contin. Dyn. Syst.* **27** (2021), 4593-4608.

[Wu21] W. Wu, On relative metric mean dimension with potential and variational principles, to appear in *J. Dynam. Diff. Equat*. DOI:10.1007/s10884-022-10175-w.

[YZ07] X. Ye and G. Zhang, Entropy points and applications, *Trans. Amer. Math. Soc.* **359** (2007), 6167-6186.

[YCZ22] R. Yang, E. Chen and X. Zhou, Bowen’s equations for upper metric mean dimension with potential, *Nonlinearity* **35** (2022), 4905-4938.

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