Abelian topological groups without irreducible Banach representations

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Abstract

We exhibit abelian topological groups admitting no nontrivial strongly continuous irreducible representations in Banach spaces. Among them are some abelian Banach–Lie groups and some monothetic subgroups of the unitary group of a separable Hilbert space.

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1 Introduction

The classical 1943 Gelfand–Raïkov theorem states that every locally compact group has a complete system of irreducible strongly continuous representations in Hilbert spaces. Does this result admit a sensible generalization to wider classes of topological groups?

The most immediate problem is that even the ‘nicest’ non locally compact topological groups known to date not necessarily possess nontrivial representations in Hilbert spaces, irreducible or not — for example, there are abelian Banach–Lie
groups without weakly continuous Hilbert space representations [1]. If one allows for representations in more general Banach spaces, the situation becomes more favourable. In 1957 Teleman [19] proved that every Hausdorff topological group admits a faithful strongly continuous representation in a suitable Banach space by isometries. He further asked whether every topological group admits a complete system of strongly continuous irreducible representations in Banach spaces by isometries. The answer is ‘No:’ as noticed recently by the present author [16], no minimally almost periodic monothetic topological group admits nontrivial irreducible representations in Banach spaces by isometries. Can Teleman’s conjecture survive if one drops the restrictive requirement ‘by isometries’ and allows any strongly continuous Banach representations? The aim of this note is to show that the answer is still in the negative.

We exhibit a vast class of abelian topological groups admitting no infinite-dimensional continuous irreducible representations in Banach spaces: an abelian topological group is such whenever the torsion subgroup is everywhere dense in it. Clearly, every minimally almost periodic abelian topological group (that is, one without nontrivial continuous characters) contained in this class admits no nontrivial irreducible Banach representations. We observe that at least three previously known classes of examples of minimally almost periodic groups fall into this category. Among them are all minimally almost periodic abelian Banach–Lie groups, and even some monothetic subgroups of the unitary group of a separable Hilbert space with the strong operator topology, which fact shows that the unitary representations of some of the ‘nicest’ non locally compact groups known cannot in general be decomposed into irreducibles in any reasonable sense.

To substantiate the subject of this note, let us remark that infinite-dimensional irreducible Banach representation of the infinite cyclic group do exist indeed: every bounded linear operator on a Banach space having no invariant subspaces [3, 17, 8] leads to such a representation. It remains yet to be seen if there exists a minimally almost periodic topological group admitting an irreducible continuous Banach representation. The importance of this open problem (and especially its version for Hilbert space representations) stems from its obvious relevance to the Invariant Subspace Problem. Our present results suggest, however, that learning to produce representations of this kind might take more than a straightforward extension of the Gelfand–Raïkov theorem.
2 Main result

A representation $\rho$ of a topological group $G$ in a normed space $E$ is strongly continuous, or simply continuous, if it is continuous as a mapping $\rho: G \times E \to E$. Equivalently, $\rho$ is strongly continuous if it is continuous as a homomorphism $\rho: G \to \text{GL}(E)$, where the general linear group $\text{GL}(E)$ of $E$ is equipped with the strong operator topology. It is worth remembering that the strong operator topology is never a group topology on $\text{GL}(E)$, but it is such on the subgroup formed by all isometries, in particular, on the unitary group $U(\mathcal{H})$ of a Hilbert space $\mathcal{H}$.

Recall (see e.g. [12]) that the collection of all elements of an abelian group $G$ having finite order forms a subgroup of $G$ called the torsion subgroup, which we denote by $T(G)$. If every element of $G$ is of finite order, then $G$ is called a torsion group.

**Theorem 2.1** Let $G$ be a topological abelian group which is algebraically a torsion group. Then every continuous irreducible representation of $G$ in a Banach space is one-dimensional and indeed a continuous character.

**Proof.** Let $\rho$ be a continuous representation of $G$ in a Banach space $E$. Assume that for some $g \in G$, the operator $\rho_g$ is not a scalar multiple of the identity. Now a standard argument from operator theory (see e.g. [3], p. 277) shows that $\rho$ admits a proper subrepresentation. Indeed, since $(\rho_g)^n = I$ for some $n$, the bounded linear operator $\rho_g$ has an eigenvalue, $\lambda$, which is an $n$-th root of unity. (See e.g. [12], Sect. 12, or [3], Exercise 2, p. 35.) The space $F = \text{Ker}(\lambda I - \rho_g)$ is closed, proper (since $\rho_g \neq \lambda I$), and clearly invariant under every operator from $\mathcal{L}(E)$ commuting with $\rho_g$. (As operator theorists say, $F$ is hyperinvariant.) Since $G$ is abelian, the space $F$ is invariant under every operator $\rho_h$, $h \in G$, and $\rho|_F$ forms a proper Banach subrepresentation of $\rho$.

We conclude that if $\rho$ is irreducible, then each $\rho_g$, $g \in G$ is a scalar multiple of the identity and $\dim E = 1$. Since $\rho(G)$ is a torsion multiplicative subgroup of $\mathbb{C}^\times$, it is contained in $\mathbb{T} = U(1)$ and $\rho$ is a character.

**Corollary 2.2** Let the torsion subgroup of an abelian topological group $G$ be everywhere dense in it. Then every continuous irreducible representation of $G$ in a Banach space is one-dimensional and indeed a continuous character.

**Proof.** The restriction of a continuous irreducible representation $\rho$ of $G$ in a Banach space $E$ to the torsion subgroup $T(G)$ is irreducible because $T(G)$ is everywhere
dense in $G$. Therefore, $E = \mathbb{C}$ and $\rho|_{T(G)}$ is a continuous character by Theorem 2.1. Its extension to a continuous homomorphism $G \to \mathbb{T} \subset \text{GL}(\mathbb{C})$ is unique and must therefore coincide with $\rho$. We conclude that $\rho$ is also a continuous character, Q.E.D.

### 3 Connected torsion groups

The following observation was made by A.A. Markov [13].

**Proposition 3.1** An abelian torsion group equipped with a connected group topology is minimally almost periodic.

**Proof.** The image of $G$ under every continuous character $\chi: G \to \mathbb{T}$ is connected and consists of elements of finite order, which means that $\chi(G) = \{e_T\}$ and $\chi$ is trivial.

It follows immediately from Theorem 3.4 and Proposition 3.1 that all such topological groups actually enjoy an (at least, formally) stronger property.

**Corollary 3.2** An abelian torsion group equipped with a connected group topology admits no nontrivial irreducible continuous Banach representations.

This result leads to our first class of abelian topological groups admitting no irreducible representations in Banach spaces.

**Examples 3.3** In [13] Markov had constructed the first ever example of a connected group topology on an infinite torsion group (whose elements have order 2). He then used the above observation 3.1 to show that $G$ is minimally almost periodic. Later Graev [10] showed how to produce numerous such examples much more easily as varietal free topological groups: using the modern terminology [14], every Graev free topological group, $F_{\mathbb{A}_n}(X)$ on a nontrivial connected topological space $X$ formed in the variety $\mathbb{A}_n$ of all topological groups of finite period $n$ is nontrivial (moreover, contains $X$ as a topological subspace).

Still one more way to generate such groups at will is to embed any torsion group $G$ equipped with the discrete topology into a path-connected topological group of all $G$-valued step functions on $[0,1]$ equipped with the topology of convergence in measure, using the construction of Hartman and Mycielski [11].

Our next result follows immediately from Corollary 2.2 and Proposition 3.1.
Corollary 3.4 Let the torsion subgroup of an abelian topological group $G$ be connected and everywhere dense in $G$. Then $G$ admits no nontrivial strongly continuous irreducible representations in Banach spaces.

Example 3.5 Here is an interesting source of rather natural examples of topological groups possessing the properties stated in Corollary 3.4, which construction seems to have never been explored before. For a pointed topological space $X = (X, \ast)$ denote by $A(X)$ the Graev free abelian topological group on $X$, that is, an abelian topological group algebraically free over $X \setminus \{\ast\}$ and containing $X$ as a closed topological subspace in such a way that every continuous mapping $f$ from $X$ to an abelian topological group $G$, sending $\ast$ to the identity, gives rise to a unique continuous homomorphism $\bar{f}: A(X) \to G$ with $f = \bar{f} \circ i$. (See e.g. [15].) Let $L(X)$ be the free locally convex space on $X = (X, \ast)$, that is, a locally convex space containing $X \setminus \{\ast\}$ as a Hamel basis and a closed topological subspace in such a way that every continuous mapping $f$ from $X$ to a locally convex space $E$, sending $\ast$ to zero, gives rise to a unique continuous linear operator $\bar{f}: L(X) \to E$ with $f = \bar{f} \circ i$. (See [6].) The identity mapping $id_X : X \to X$ extends to a canonical continuous homomorphism $i : A(X) \to L(X)$, which is an embedding of $A(X)$ into the additive topological group of $L(X)$ as a closed topological subgroup. (It follows from the corresponding result for the free Markov abelian topological group and the free Markov locally convex space, see [20] and [21].) Denote the factor-group $L(X)/A(X)$ by $\mathbb{T}(X)$; it is nontrivial whenever $X$ is such. (For example, $\mathbb{T}(\{0, 1\}) \cong \mathbb{T}$.) If $X$ is connected, then the torsion subgroup of $\mathbb{T}(X)$ is connected, being algebraically generated by the image under the quotient homomorphism $L(X) \to \mathbb{T}(X)$ of the connected set $\bigcup_{n=1}^{\infty} \frac{1}{n} X$.

4 Abelian Banach–Lie groups

Corollary 4.1 A minimally almost periodic abelian Banach–Lie group admits no nontrivial irreducible continuous Banach representations.

Proof. A minimally almost periodic abelian Banach–Lie group is necessarily connected (for otherwise it would possess a discrete abelian factor group) and therefore a quotient group of the additive group of a Banach space $E$ (the underlying space of the Banach–Lie algebra of $G$) modulo a discrete subgroup, $D$. Denote by $C$ the union of all one-parameter subgroups passing through elements of $D \setminus \{0\}$ (that is,
by the cone spanned by $D$). The closed subgroup of $E$ generated by $C$ is in fact a closed linear subspace, $F$, and assuming $F \neq E$ leads to a contradiction, since the quotient group $E/F$ is a topological quotient group of $G$ and the additive group of a non-degenerate Banach space, hence not minimally almost periodic. It means that $F = E$ and as a corollary, the image of $C$ generates an everywhere dense subgroup of $G$. Since the image of every one-parameter subgroup of $E$ passing through a point of $D$ is topologically isomorphic to the group $\mathbb{T} = U(1)$, the torsion subgroup of $G$ is everywhere dense in $G$. Now Theorem 2.1 together with minimal almost periodicity imply the result.

**Examples 4.2** A large class of examples of abelian Banach–Lie groups admitting no continuous characters are known, cf. [18], [4], [1], [2]. Every such group is in fact monothetic, that is, contains an everywhere dense cyclic subgroup. According to Corollary 4.1, every such Banach–Lie group admits no irreducible Banach representations.

## 5 Levi groups

A topological group $G$ is called a Levi group [8, 9, 16] if $G$ contains an increasing chain of compact subgroups $G_i$, $i \in \mathbb{N}$, having an everywhere dense union in $G$ and such that whenever $\liminf \mu_i(A_i) > 0$ for some $A_i \subseteq G_i$, one has $\lim \mu_i(V A_i \cap G_i) = 1$ for every neighbourhood of identity $V$, where $\mu_i$ denotes the normalized Haar measure on $G_i$.

**Corollary 5.1** An abelian Levi group admits no irreducible continuous Banach representations.

**Proof.** Since the torsion subgroup of every compact abelian group is everywhere dense, one concludes that the torsion subgroup of $\bigcup_i G_i$ is everywhere dense in $G$ and Theorem 2.1 applies. The known minimal almost periodicity of Levi groups accomplishes the proof. (Every Levi group $G$ is extremely amenable, that is, has a fixed point in every compact $G$-space, see [8] and also [16]. This property obviously implies minimal almost periodicity; whether the converse is true, is unknown as of August 1997.)

**Example 5.2** A concrete example of a Levi group $G$ (due to Glasner and, independently, Furstenberg and B. Weiss) is the abelian group $L_1(X, \mathcal{S})$ of all measurable
$\mathbb{S}^1$-valued complex functions on a nonatomic Lebesgue measure space $X$ equipped with the metric

$$d(f,g) = \int_X |f(x) - g(x)| \, d\mu(x),$$

where $\mathbb{S}^1$ is identified with the multiplicative subgroup of all complex numbers of modulus 1. As the compact subgroups $G_i$ of $G$, one can choose tori of increasing finite dimension formed by step functions corresponding to a sequence of refining partitions of $X$. An ingenious argument \cite{9} shows that this group is also monothetic.

6 Monothetic unitary groups

There are at least two known examples of minimally almost periodic monothetic topological groups admitting a faithful strongly continuous unitary representation in a Hilbert space. One of them is the Levy group $L_1(X, \mathbb{S}^1)$, having a faithful continuous unitary representation by multiplication operators in the Hilbert space $L_2(X)$ \cite{4}. The other known example belongs to Banaszczyk \cite{2}, Th. 5.1, who constructed monothetic Banach–Lie groups (modelled over the spaces $l_p$) without continuous characters but admitting a faithful unitary representation.

According to Corollary 5.1 and Corollary 4.1, the topological groups of the above type admit no nontrivial irreducible Banach representations. The same property is obviously shared by their images in the unitary group $U(H)$ equipped with the strong operator topology. We have established the following.

**Corollary 6.1** There exist monothetic topological subgroups $G$ of the unitary group $U(H)$ of the separable Hilbert space equipped with the strong operator topology that admit no nontrivial irreducible strongly continuous Banach representations.

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