Extended Fractional Supersymmetric Quantum Mechanics

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Abstract

Recently, we presented a new class of quantum-mechanical Hamiltonians which can be written as the $F^{\text{th}}$ power of a conserved charge: $H = Q^F$ with $F = 2, 3, \ldots$. This construction, called fractional supersymmetric quantum mechanics, was realized in terms of a paragrassmann variable $\theta$ of order $F$, which satisfies $\theta^F = 0$. Here, we present an alternative realization of such an algebra in which the internal space of the Hamiltonians is described by a tensor product of two paragrassmann variables of orders $F$ and $F - 1$ respectively. In particular, we find $q$-deformed relations (where $q$ are roots of unity) between different conserved charges.

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1. Introduction

We have recently presented a new generalization of SUSY quantum mechanics which we call fractional supersymmetric (FSUSY) quantum mechanics.\cite{1} In such a construction, the Hamiltonian is expressed as the $F^{th}$ power of a conserved fractional supercharge: $H = Q^F$, with $[H, Q] = 0$ and $F = 2, 3, \ldots$. This is a generalization of the ordinary SUSY quantum mechanics ($F = 2$) different from the para-supersymmetric one\cite{2}. Moreover, since the Hamiltonian is the generator of time translation, the FSUSY transformations associated with $Q$ are the $F^{th}$ roots of time translations. These FSUSY transformations were also described in Ref. [1] and an invariant action was provided (a fractional superspace formulation is given in Ref. [3]).

In this letter, we present an alternative realization of the algebra defining the FSUSY quantum mechanics. In particular, we have two conserved fractional supercharges $Q_i$ satisfying the following FSUSY algebras ($i, j = 1, 2$):

$$Q_i^F = H, \quad [H, Q_i] = 0, \quad (1a)$$

$$[Q_i, Q_j]_{q^{i-j}} = 0 \quad (1b)$$

with

$$q^F = 1 \quad (q^n \neq 1 \text{ for } 0 < n < F) \quad (2)$$

and where we used the definition

$$[A, B]_\omega \equiv AB - \omega BA. \quad (3)$$

Note the $q$-deformed-like algebra of (1b). Note also that for $F = 2$, the algebras (1) reduces to the well-known superalgebra $\{Q_i, Q_j\} = 2\delta_{ij}H$.

In Ref. [1], the construction of FSUSY quantum mechanics of order $F$ (which we shall now call minimal as opposed to the extended version of the present work) was realized in terms of one paragrassmann variable $\theta$ of order $F$, which is such that $\theta^F = 0$. In matrix form, the Hamiltonians were represented by $F \times F$ matrices. In contrast, in the present extended version, the internal space of the quantum-mechanical systems is described by a tensor product of two paragrassmann variables of orders $F$ and $F - 1$ respectively (for $F > 2$). In matrix form, the Hamiltonians are realized in terms of $F(F - 1) \times F(F - 1)$ matrices, and they
have an $F(F-1)$-fold degeneracy (above the ground state). We will see that these Hamiltonians are also supersymmetric, that is, they can simultaneously be written as the square of a conserved supercharge. Here, we restrict ourself to the one dimensional case but the construction is directly applicable for some particular two dimensional systems\textsuperscript{[1]}.

In the Sect. 2, we introduce the generalized creation and annihilation operators (interpolating between ordinary bosonic and fermionic ones) which are used to describe the internal space of the Hamiltonians. In Sect. 3, we recall the usual construction of ordinary SUSY quantum mechanics. In Sect. 4, we present the extended FSUSY quantum mechanics of order 3, and its generalization to arbitrary order in Sect. 5.

2. Paragrassmann variables

In this section, we introduce generalized variables which interpolate between ordinary bosonic and fermionic ones. They can be interpreted either as generalized coordinates, or generalized creation and annihilation operators. The latter interpretation is more relevant in the present quantum-mechanical context, but the former point of view is used in Ref. [3] when discussing the fractional superspace formulation of FSUSY transformations. The notation, however, will be more reminiscent of the generalized coordinates interpretation.

We introduce a paragrassmann variable $\theta$ of order $M$, and its derivative $\partial \equiv \partial/\partial \theta$, which satisfy

\begin{align}
\theta^M &= 0, \quad \partial^M = 0, \quad M = 1, 2, \ldots \quad (4a) \\
(\theta^{M-1} \neq 0, \partial^{M-1} \neq 0) \quad (4b)
\end{align}

In order to be able to recover the 3 different limits which we describe below (fermionic, bosonic and “null”), we take the generalized commutation relation between $\theta$ and $\partial$ to be

\[ [\partial, \theta]_q = \alpha(1 - q) \quad (5) \]

where we have used the definition (3), and where $\alpha$ is a free parameter and $q \in \mathbb{C}$ a primitive $M^{th}$ root of unity:

\[ q^M = 1 \quad (q^n \neq 1 \text{ for } 0 < n < M). \quad (6) \]
By a *primitive* root, we mean a root satisfying the condition in parentheses; for instance, $q \neq \pm 1$ for $M = 4$. [We will see below that the condition (6) is actually a consequence of (4) and (5).] First, note that the null limit $M = 1$ ($q = 1$), that is $\theta = \partial = 0$, is well-defined since the r.h.s of (5) is zero for $q = 1$ (and for finite $\alpha$). In previous works\[^4\] on paragrassmann variables, the r.h.s of (5) was chosen to be 1 instead of $\alpha(1- q)$, which is inconsistent in the limit $M = 1$ (which we will use here). Second, note that we recover the ordinary grassmann case ($q = -1$) for $M = 2$, i.e., $\{\partial, \theta\} = 2\alpha$. Third, for some choices of $\alpha$, we also recover (within factors) the bosonic case ($q = 1$) for $M \to \infty$. For instance\[^\dagger\] for $\alpha = M$ we find that the r.h.s of (5) is finite and non-zero:

$$\lim_{M \to \infty} M(1 - q) = -2\pi i. \quad (7)$$

Strictly speaking, this is the bosonic limit for the first root $q = \exp(2\pi i/M)$. In the context of a fractional superspace formalism\[^3\], we can choose to work with real $\theta$ and $\partial$, whereupon the consistency of the relation (5) under hermitian conjugation implies that $\alpha$ must be real. Moreover, a real $\alpha$ leads to a real r.h.s of (10) and a real factor in the r.h.s of (28). In the following sections, we will not be concerned with the $M \to \infty$ limit, so we let $\alpha$ remain unfixed.

The definition (5) implies

$$[\partial, \theta^n]_q = \alpha(1 - q^n) \theta^{n-1}. \quad (8)$$

Setting $n = M$ in (8) demonstrates that the consistency of the formalism requires the condition (6). To see the bosonic limit of that formula, rewrite $\alpha(1 - q^n)$ as

$$\alpha(1 - q^n) = \alpha(1 - q)(1 + q + q^2 + \ldots + q^{n-1}) \quad (9)$$

which leads, for $\alpha = M$, to $(-2\pi i)n$ in the limit $M \to \infty$. The definition (5) also implies

$$\sum_{i=0}^{M-1} \partial^{M-1-i} \theta^{M-1} \partial^i = M\alpha^{M-1}, \quad (10)$$

\[^\dagger\] Other choices are for instance $\alpha = (M - 1)^2/M$ or simply $\alpha = M/4$, which lead simultaneously to a well-defined bosonic limit and to the correct factor for the fermions, i.e. $\{\partial, \theta\} = 1$.  

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as well as the same relation with $\theta$ and $\partial$ interchanged. Also note that for a given $q$ of order $M$, one has

$$\sum_{i=1}^{M-1} (1 - q^i) = \prod_{i=1}^{M-1} (1 - q^i) = M. \quad (11)$$

We shall also need the operator $B_{(M)}$ defined as

$$B_{(M)} = \sum_{i=0}^{\infty} c_i \theta^i \partial^i = c_0 + \sum_{i=1}^{M-1} c_i \theta^i \partial^i \quad (12a)$$

with

$$c_0 = (1 - M)/2 \quad \text{and} \quad c_i = [\alpha^i(1 - q^i)]^{-1} \quad (i = 1, 2, \ldots, M - 1). \quad (12b)$$

This operator has the following properties:

$$[B_{(M)}, \theta] = \theta, \quad [B_{(M)}, \partial] = -\partial. \quad (13)$$

Note that for $\alpha = M$ and $M \to \infty$, we have $c_j \to (i/2\pi)\delta_{j,1}$.

For a given order $M$ which is prime, we can actually introduce $M - 2$ other fractional derivatives. We thus have $M - 1$ derivatives, which we write as $\partial_i$ with $i = 1, 2, \ldots, M - 1$. With this notation, $\partial = \partial_1$. They satisfy

$$\partial_i^M = 0, \quad \partial_i^{M-1} \neq 0, \quad [\partial_i, \theta]_{q^i} = \alpha(1 - q^i) \quad (14)$$

and have the properties

$$[\partial_1, \partial_{M-1}]_{q^{-1}} = 0. \quad (15)$$

The relation (14) implies both (8) with the substitution $q \to q^i$, and (10) with the substitution $\partial \to \partial_i$. Note that in the SUSY limit ($M = 2; q = -1$) we are left with only one derivative which satisfies $[\partial, \partial]_{q^{-1}} = \{\partial, \partial\} = 2 \partial^2 = 0$ as it must.

For a non-prime $M$, the situation is more complicated. For instance, for $M = 4$, setting $i = 2$ in (15) implies $\partial_2^2 = 0$, which contradicts (14). In other words, $q^i$ is not a primitive root for $i = 2$. Therefore, for a non-prime $M$, there are fewer than $M - 1$ derivatives $\partial_i$, but at least two: $\partial \equiv \partial_1$ and $\hat{\partial} \equiv \partial_{M-1}$. (In the following, we will use only these two derivatives.)
An $M \times M$ matrix realization of $\theta$ and $\partial$ is given by

$$\theta = \begin{pmatrix} 0 & a_1 & 0 & 0 & 0 \\ 0 & 0 & a_2 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & a_{M-1} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \partial = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ b_1 & 0 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & b_{M-1} & 0 \end{pmatrix}$$

(16a)

with the constraint (no summation on $i$)

$$a_i b_i = \alpha (1 - q^i).$$

(16b)

Note that in general $\partial \neq \theta^\dagger$. As for $\hat{\partial}$, it is given by

$$\hat{\partial} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \hat{b}_1 & 0 & 0 & 0 & 0 \\ 0 & \hat{b}_2 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \hat{b}_{M-1} & 0 \end{pmatrix} \quad \text{with} \quad \hat{b}_i = -q^i b_i.$$  

(17)

In this matrix realization, $B_{(M)}$ is found to be the third component of the spin-$(M - 1)/2$ representation of the rotational group:

$$B_{(M)} = J_3^{[(M-1)/2]}.$$  

(18)

3. Ordinary SUSY quantum mechanics

Let us first recall the usual construction of one-dimensional SUSY quantum mechanics. We introduce the bosonic operators $a$ and $a^\dagger$:

$$a = [p + i W(x)]/\sqrt{2}, \quad a^\dagger = [p - i W(x)]/\sqrt{2}$$

(19)

which satisfy

$$[a^\dagger, a] = \frac{d}{dx} W(x) \equiv W'(x)$$

(20)

where $p = -id/dx$. We need an ordinary grassmann variable $\theta$ and its derivative $\partial$, which are interpreted as fermionic creation and annihilation operators. For $M = 2$, the commutation relations (4-6) give

$$\{\partial, \theta\} = 2\alpha, \quad \theta^2 = \partial^2 = 0.$$  

(21)
The conserved supercharge $Q$ and Hamiltonian $H$ satisfying

$$Q^2 = H, \quad [H, Q] = 0,$$  \hspace{1cm} (22)

are given by

$$Q = \partial a + e \theta a^\dagger$$  \hspace{1cm} (23a)

$$H = \frac{1}{2} (p^2 + W^2) + W' \cdot S$$  \hspace{1cm} (23b)

where

$$S = -\frac{1}{2} + e \, \theta \partial = B(2)$$  \hspace{1cm} (24)

and $e^{-1} = 2\alpha$. With the realization (16), we find

$$S = \frac{1}{2} \sigma_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  \hspace{1cm} (25)

Thus the Hamiltonian (23b) describes a “spin-1/2” particle moving in a potential $W^2/2$ and a “magnetic field” $W'$. Moreover, with the choice $\alpha = 1/2$ ($e = 1$), we may choose $\theta = \sigma_+ \text{ and } \partial = \sigma_-$, i.e., $\partial = \theta^\dagger$.

4. Extended FSUSY quantum mechanics of order 3

To describe the internal space of SUSY and minimal FSUSY$^{[1]}$ Hamiltonians, only one type of variable is used: a paragrassmann variable of order $F$. For the extended FSUSY case, the situation is different since two different types of internal-space variables are needed simultaneously: one paragrassmann variable of order $F$, and another of order $F - 1$; the variables of different order commute with each other. (The tensor product of these two types of variables may equivalently be seen as only one internal-space variable.) Thus, for the case $F = 3$, we need both a set of ordinary grassmann variables $(\tilde{\theta}, \tilde{\partial})$ [which satisfy (21), i.e., (4-6) with $M = 2$], and a set of paragrassmann variables $(\theta, \partial)$ of order 3 [which satisfy (4-6) with $M = 3$]. Hence we have:

$$\tilde{\theta}^2 = \tilde{\partial}^2 = 0, \quad \{\tilde{\partial}, \tilde{\theta}\} = 2\alpha'$$  \hspace{1cm} (26a)

$$\theta^3 = \partial^3 = 0, \quad [\partial, \theta]_q = \alpha(1 - q)$$  \hspace{1cm} (26b)

$$q^3 = 1 \quad (q \neq 1)$$  \hspace{1cm} (26c)

$$[\theta, \tilde{\theta}] = [\theta, \tilde{\partial}] = [\partial, \tilde{\theta}] = [\partial, \tilde{\partial}] = 0.$$  \hspace{1cm} (26d)
[Except on $W(x)$, the prime does not indicate a derivative.] In particular, from (26bc), we have
\[ \partial^2 \theta^2 + \partial \theta^2 \partial + \theta^2 \partial^2 = 3\alpha^2 \] (27)
and
\[ \partial^2 \theta + \partial \theta \partial + \theta \partial^2 = (3\alpha) \partial \] (28)
(as well as the same equations with the interchange $\theta \leftrightarrow \partial$). The formula (27), which is a particular case of (10), is useful in the present FSUSY context, whereas the formula (28) is used in rewriting\[ in rewriting\] the PSUSY quantum mechanics\[ in rewriting\] in terms of these paragrassmann variables\[ of these paragrassmann variables\].

We define the “square root” of the bosonic operator $a$ of (19) as
\[ a^{1/2} \equiv \tilde{\partial} + e' \tilde{\theta} a, \quad (a^{1/2})^2 = a, \] (29)
with $(e')^{-1} = 2\alpha'$. Now we introduce the FSUSY Hamiltonian $H$ of order 3 and the associated conserved charge $Q$ which satisfy
\[ Q^3 = H, \quad [H, Q] = 0. \] (30)
They are given by
\[ Q = \partial a^{1/2} + e \theta^2 a^\dagger \] (31a)
\[ H = \frac{1}{2}(p^2 + W^2) + W' \cdot S \] (31b)
where
\[ S = -\frac{1}{2} + ee' \partial \theta^2 \partial \cdot \tilde{\partial} \tilde{\theta} + e \theta^2 \partial^2 \] (32a)
\[ = \frac{1}{2}[B_{(3)} - B_{(2)} + B_{(3)}^2 \cdot B_{(2)}] \] (32b)
and with $e^{-1} = 3\alpha^2$. With the particular realization (16) and understanding the dot in (32) as the tensor product of the $3 \times 3$ matrix by the $2 \times 2$ matrix, we get for $S$ the following $6 \times 6$ matrix:
\[ S = \frac{1}{2} \begin{pmatrix} \sigma_3 & 0 & 0 \\ 0 & -\sigma_3 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} \] (33)

\[ ^\dagger \] In the context of PSUSY quantum mechanics\[ of these paragrassmann variables\], the paragrassmann variables of order 3 are traditionally taken to satisfy $\partial^2 \theta + \theta \partial^2 = 2\partial$ and $\partial \theta \partial = 2\partial$. However, only the sum of these two relations, i.e. (28), is really needed.
where $\sigma_3$ is the third Pauli matrix given in (25). Thus, in this matrix form, the Hamiltonian (31b) is diagonal and is a sum of 3 ordinary SUSY Hamiltonians, each of them describing a spin-1/2 particle in a potential $W^2/2$ and interacting with a “magnetic field” $\pm W'$. With a similarity transformation, it is easy to change the sign of the middle $\sigma_3$ of (33) [by interchanging the two middle diagonal elements]. Therefore, the Hamiltonian (31b) is also supersymmetric; that is, it can also be written as the square of a charge: $H = \hat{Q}^2$ with $\hat{Q} = T[(\hat{\partial} \otimes 1_3)a + e'(\hat{\theta} \otimes 1_3)a^\dagger]T^{-1}$ where $1_3$ is the $3 \times 3$-identity in the space of the variables of order 3, and $T$ is some similarity transformation.

Now we want to introduce another conserved fractional charge $\hat{Q}$ of order 3. For that purpose, we need the second fractional derivative $\hat{\partial}$ which satisfies

$$\hat{\partial}^3 = 0, \quad [\hat{\partial}, \theta]_{q-1} = \alpha(1 - q^{-1}). \quad (34)$$

It has the property

$$[\hat{\partial}, \partial]_q = 0. \quad (35)$$

The relations (34) and (35) are particular cases of (14) and (15). Introducing $\hat{Q}$ as

$$\hat{Q} = (-1)^{1/3}[\hat{\partial}a^{1/2} - e\theta^2a^\dagger], \quad (36)$$

we have the following algebra

$$\hat{Q}^3 = H, \quad [H, \hat{Q}] = 0, \quad (37a)$$

$$[\hat{Q}, Q]_q = 0. \quad (37b)$$

The relation (37b) is valid only in matrix realizations, that is, with $\hat{\partial}$ given by (17) (in addition to the matrix realizations of $\theta$ and $\partial$). With the notation $Q_1 \equiv Q$ and $Q_2 \equiv \hat{Q}$, we can rewrite (37) as in (1) with $F = 3$. We can also introduce 3 “ladder” operators which cube to zero (see next section).

5. Extended FSUSY quantum mechanics of arbitrary order

Now we turn to the general case. To distinguish between paragrassmann variables of different order, we introduce the notation $\partial_{(M)}$, $\theta_{(M)}$ and $\alpha_{(M)}$ for the variables $(\partial, \theta)$ of order $M$ and the associated $\alpha$ defined through $[\partial_{(M)}, \theta_{(M)}]_q =
\[ \alpha_M (1-q) \text{ with } q^M = 1. \] We shall now need the "\( p \)th root" of the bosonic operator \( a \) of (19):

\[ a^{1/p} \equiv \partial_{(p)} + e' \theta_{(p)}^{-1} - (a^{1/p})^p = 1, \tag{38} \]

with \( (e')^{-1} = p \alpha_{(p)}^{-1} \). Note that in order to recover \( a^{1/1} = a \), we need the null limit \( M = 1 \) of (4). This will be useful when looking for the SUSY limit of the general case.

Now we introduce the FSUSY Hamiltonian \( H_{(F)} \) of order \( F \) and the associated conserved charge \( Q_{(F)} \):

\[ Q_{(F)} = \partial_{(F)} a^{1/(F-1)} + e' \theta_{(F)}^{-1} a^+, \quad e^{-1} = F \alpha_{(F)}^{-1} \tag{39a} \]

\[ H_{(F)} = \frac{1}{2} (p^2 + W^2) + W' \cdot S_{(F)} \tag{39b} \]

where

\[ S_{(F)} = \frac{1}{2} + ee' \sum_{i=1}^{F-1} \left\{ \partial_{(F-1)}^{-1} \theta_{(F)}^{-1} \partial_{(F)}^{-1} \sum_{j=0}^{i-1} \partial_{(F-1)}^{F-2-j} \theta_{(F-1)}^{-2} \partial_{(F-1)}^{j} \right\}. \tag{40} \]

They satisfy the following fractional supersymmetry algebra:

\[ Q_{(F)}^F = H_{(F)}, \quad [H_{(F)}, Q_{(F)}] = 0. \tag{41} \]

Note how we recover the SUSY case for \( F = 2 \) using \( a^{1/1} = a \) in (39a). With the realization (16), we get for \( S \) an \( F(F-1) \times F(F-1) \) diagonal matrix with an equal number of \( \pm 1 \) entries. Therefore, the Hamiltonian (39b) is also SUSY and the same discussion as for the \( F = 3 \) case applies. In particular, the spectrum of the harmonic oscillator is \( F(F-1) \) degenerate except for the ground state which has half the degeneracy.

Now we drop the \( (F) \) subscripts. Using the other fractional derivatives (14), one may write \( F - 1 \) conserved charges (fewer for a non-prime \( F \)) satisfying (41); it suffices to replace \( \partial \) in (39a) by \( \partial_i \). Rather, let us introduce the following charge (valid for both prime and non-prime \( F \)):

\[ \hat{Q} = (-1)^{1/F} \left[ \hat{\partial} a^{1/(F-1)} - e \theta_{(F)}^{-1} a^+ \right] \tag{42} \]

where we recall the notation \( \hat{\partial} \equiv \partial_{F-1} \). We have the following algebra:

\[ \hat{Q}^F = H, \quad [H, \hat{Q}] = 0, \tag{43a} \]
\[ [\hat{Q}, Q]_q = 0. \] (43b)

The \( q \)-deformed relation (43b) holds only if one imposes the additional constraint \( \theta^{F-1}(\partial + q \hat{\theta}) = 0 \), which is automatically fulfilled in the matrix realization (16) and (17). For \( F = 2 \), this constraint is trivial since \( \hat{\theta} = \partial \) and \( q = -1 \).

Using the notation \( Q_1 \equiv Q \) and \( Q_2 \equiv \hat{Q} \), we may combine the equations (41) and (43) either as in (1), or in the following way \((r_i = 1, 2)\):

\[ \{Q_{r_1}, Q_{r_2}, \ldots, Q_{r_F}\} = F \delta_{r_1 r_2 \ldots r_F} H \] (44)

where \( \delta_{r_1 r_2 \ldots r_F} = \prod_{i=1}^{F-1} \delta_{r_i, r_{i+1}} \) and where we used the following definition of the multilinear product:

\[ \{X_1, X_2, \ldots, X_F\} \equiv X_1 X_2 \ldots X_F + \text{cyclic permutations of the } X_i. \] (45)

[So, the right hand side of (45) contains \( F \) terms.] This product is a generalization of the anti-commutator. Note that from (44), we recover for \( F = 2 \) the well-known superalgebra

\[ \{Q_i, Q_j\} = 2 \delta_{ij} H, \quad i, j = 1, 2. \] (46)

We can also introduce \( F \) “ladder” operators. They are given by

\[ Q_i = Q + q^i (-1)^{-1/F} \hat{Q}, \quad i = 1, 2, \ldots, F \] (47)

and satisfy

\[ \{Q_{r_1}, Q_{r_2}, \ldots, Q_{r_F}\} = F (1 - q^{r_1+r_2+\ldots+r_F}) H. \] (48)

In particular, we have \( Q_i^F = 0 \) (but \( Q_i^{F-1} \neq 0 \)). For \( F = 2 \), we recover the SUSY case:

\[ Q_2^2 = 0, \quad \{Q_+, Q_-\} = 4H, \] (49)

with \( Q_\pm = Q \pm i\hat{Q} \) (i.e., with the notation \( Q_+ \equiv Q_2 \) and \( Q_- \equiv Q_1 \)). In the construction (47-48), we can include the case \( F = 1 \).
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