Axion Electrodynamics from Infeld-van der Waerden formalisms

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In this article we will explore the spacetime phase structure inside the Infeld-van der Waerden formalisms, showing that Maxwell’s theory in this scenario leads to an axion-like phase electrodynamics, which suggests that this phase may originate the axion field.

I. INTRODUCTION

Nowadays, the axion plays an important role in developments in several areas of physics, such as QCD, in solving the strong CP symmetry problem [1, 2], applications in condensed matter [3] and string theory [4]. The axion is also a good candidate to explain cold dark matter [5–8], even though it is not directly observed in experiments. The axion/Maxwell coupling is given by the following lagrangean term

\[ \mathcal{L}_\alpha = \frac{1}{4} \alpha F_{\mu\nu} F^*_{\mu\nu}. \]

Since \( A_\mu \) is the electromagnetic gauge potential and \( \epsilon_{\mu\nu\sigma\rho} \) are the Levi-Civita tensor components, the objects \( \alpha, F_{\mu\nu} \triangleq 2\nabla_{[\mu} A_{\nu]} \) and \( F^*_{\mu\nu} \) represent, respectively, the axion pseudo-scalar, Maxwell’s tensor and its Hodge dual defined by \( F^*_{\mu\nu} \triangleq (1/2)\epsilon_{\mu\nu\sigma\rho} F_{\sigma\rho} \). This term is interpreted as a topological effect in U(1) gauge electrodynamic theory as it originates from a total derivative of a product between \( A_\mu \) and \( \partial_\mu A_\nu \). In trivial topology cases the term \( F_{\mu\nu} F^*_{\mu\nu} \) must vanish. We can use the local dual invariant electrodynamics (LDIE) formalism [9–12] developed by Tiwari to add the axion field into Maxwell’s electrodynamics. The LDIE accounts for the electromagnetic duality symmetry by adding magnetic sources. It is also based on Sudbery’s vector lagrangean [13], that considers \( F_{\mu\nu} \) as the fundamental dynamical variable. Visinelli [14, 15] has also obtained a dual axion electrodynamics using a scalar lagrangean, but the equations derived by him are not identical to Tiwari’s equations. It’s worth reporting that Visinelli’s equations assume the Tiwari’s pattern when a CP-preserving configuration is taken [15]. An important consequence of Tiwari’s works for

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us is the fact that magnetic sources must be part of the equations in order to preserve local duality invariance.

By using the 2-spinor index formalisms \[16–20\], geometric current for Infeld-van der Waerden electromagnetic fields have been defined in \[21, 22\]. It seems that geometric source equations follow similar structures to those obtained by Tiwari. In the 20’s, van der Waerden \[16\] provided an electron description based on Weyl’s representation and Infeld subsequently extended Waerden’s work \[17\]. Infeld together with van-der Waerden posteriorly established the geometrical theory for curved spacetime, thus founding the $\gamma\varepsilon$-formalisms for General Relativity \[18\]. Each $\gamma\varepsilon$ nomenclature is based on its spinor metric, being canonical in the $\varepsilon$-formalism and depending locally on the spacetime coordinates in the $\gamma$-formalism. In general, the spinor decomposition of $g_{\mu\nu}$ admits a local scale/phase freedom, such that in $\gamma$-formalism, its metric spinor takes into account this freedom.

The Infeld-van der Waerden formalisms have been studied recently by Cardoso \[23–25\] with alternative proposals, for example \[26\]. Sometimes classical world theories can be rewritten in its spinor form, since the $SL(2, \mathbb{C})$ linear group of unimodular complex $2 \times 2$ matrices has an homomorphism two to one with the orthochronous proper Lorentz group $L^\uparrow$. The Infeld-van Waerden formalism was based on the homomorphism between the Weyl and Lorentz groups. In fact, the Weyl group $W(2, \mathbb{C})$ is longer than $SL(2, \mathbb{C})$, so the phase invariance is expressed by

$$h(e^{i\theta}g) = h(g)$$

with $h(g) \in SO(1,3)$ and $h(g) \in W(2,\mathbb{C})$, how presented in \[27\]. The correspondence between world objects living in Minkowski spacetime and spinors living in the symplectic two-dimensional complex spinor spaces is done using $SL(2,\mathbb{C})$ matrices. The relation between the Minkowski metric tensor component $\eta_{\mu\nu}$ and a canonical symplectic spinor metric component $\varepsilon_{AB}$ is given by the fundamental Clifford algebra $\mathcal{Cl}(1,3)$. A remarkable fact is the existence of a phase freedom where $\eta_{\mu\nu}$ is kept unchanged by a gauge transformation $\varepsilon_{AB} \mapsto e^{i\Theta_i} \varepsilon_{AB}$. For curved spacetimes, the $SL(2, \mathbb{C})$ matrices are substituted by connecting objects $\Sigma^{AA'}_{\mu}$, which are generic Hermitian matrices that depend locally on the spacetime coordinates. The role of $SL(2, \mathbb{C})$ is played by the generalized Weyl gauge group, which implement such transformations in General Relativity \[28\]. Thus, on each spacetime point we can connect $g_{\mu\nu}$ to $\varepsilon_{AB}$ by using $\Sigma^{AA'}_{\mu}$. We have the freedom to transform the objects as follows

$$g_{\mu\nu} \mapsto g_{\mu\nu} : \quad \varepsilon_{AB} \mapsto |\gamma|e^{i\Theta_i} \varepsilon_{AB} \quad \text{and} \quad \Sigma^{AA'}_{\mu} \mapsto |\gamma|^{-1} \Sigma^{AA'}_{\mu},$$

such that the metric invariance is extended by an extra scale transformation expressed by $|\gamma|$. 
Originally, this freedom has been interpreted as the geometrical origin of the electromagnetic potential, since it would lead to an imaginary part of the spinor connection trace $\gamma^{\mu A}$ in $\gamma$-formalism, such that it would satisfy Weyl’s principle of gauge invariance. Weyl was working on the relationship between the tetrad formalism for curved spacetime and the parameter $\lambda$ of the Dirac 4-spinor phase transformation: $\Psi \mapsto e^{i\lambda} \Psi$. He noticed that if the tetrad varies, $\lambda$ must vary too (for mathematical details, see [27, 28]). Infeld and van der Waerden, following Weyl’s steps, considered using the phase transformation of $\Psi$ to implement Dirac’s theory in curved spacetime.

In fact, given generic spin transformations: $\xi_A \mapsto \Lambda^{B} \xi_B$, $\zeta^A \mapsto \zeta^B \Lambda^{-1A}$ and complex conjugates, the quantity $\gamma^{\mu A}$ transforms as $\gamma^{\mu A} \mapsto \gamma^{\mu A} + \partial_{\mu} \ln \Delta_{A}$ [18, 20, 23, 24]. $\Lambda^{B}(x^{\alpha})$ is some group of spin transformations for curved spacetime and $\Delta_{A} \equiv \det(\Lambda_{A}^{B})$. In both formalisms, the Dirac spinor is taken to be $\Psi = (\psi_{A}, \chi^{A'})$, such that each 2-spinor transforms as those generic spinors. Taking into account the Weyl gauge group

$$\Lambda_{B}^{A} = e^{i\lambda(x^{\alpha})} \delta_{A}^{B},$$

useful to consider gauge invariance principle in generic backgrounds, we have $\psi_{A} \mapsto e^{i\lambda(x^{\alpha})} \psi_{A}$, $\chi^{A'} \mapsto e^{i\lambda(x^{\alpha})} \chi^{A'}$ and $\Delta_{A} = e^{2i\lambda}$. Thus, the imaginary part of $\gamma^{\mu A}$ transforms by the action of (4) as follows

$$\text{Im} \gamma^{\mu A} \mapsto \text{Im} \gamma^{\mu A} + 2 \partial_{\mu} \lambda.$$  

By defining the world vector $A_{\mu} \equiv -(1/2) \text{Im} \gamma^{\mu A}$, we can see that it transforms as

$$A_{\mu} \mapsto A_{\mu} - \partial_{\mu} \lambda,$$

which is exactly how the electromagnetic potential is supposed to transform according to the gauge invariance principle. Therefore, several authors identified $A_{\mu}$ with the electromagnetic potential vector. This is the reason why in this scenario the literature considered the electromagnetic fields to be intrinsic geometrical structures, instead of external physical fields.

However, as remembered by Penrose [29], this idea wasn’t consolidated due to physical reasons, such that the scale/phase should not be understood on its original form. Namely, on this interpretation, the formalism would imply a relation between electric charge and spin, since the scale/phase couples with each type of fermion. Unfortunately, the neutron disabled this elegant idea, as it has spin, but no electric charge. Explicitly in [25], the equations (5.3) show the couplings between Dirac spinors and $A_{\mu}$, since that $\beta$-terms carry it.

Furthermore, the interpretation of $A_{\mu}$ as an electromagnetic potential component has impaired some investigations of Maxwell’s theory in the $\gamma$-formalism. We will work with Maxwell’s theory
in the $\gamma$-formalism considering that (5) does not mean an electromagnetic potential, and thus our electromagnetic fields will be considered as external physical entities. Therefore, we are free to reinterpret the physical significance of the phase and scale, as well as their possible physical consequences.

Our main motivation will be to investigate the topological origin of the axion from $\gamma\varepsilon$-formalisms, since in different contexts the axion is related to a topological effect. It would be interesting to contextualize the axion electrodynamics in a spinorial structure in order to reveal aspects of adjacent geometry and physical phenomena. We will show that when the concept of geometrical source for Maxwell’s equations in spinorial form is applied, the mathematics suggests that it has a magnetic nature, thus implying an LDIE as a consequence.

We will use the conventions $\hbar = c = 1$, as well as the metric signature (+ − − −). The index symmetry and antisymmetry will be indicated, respectively, by round and square brackets. The paper will be organized as follows. In section 2 we will review the axion electrodynamics while in the section 3, we must present the $\gamma\varepsilon$-formalisms. In the section 4 we will write Maxwell’s theory in both formalisms to show that on certain aspects we can identify $\alpha$ with $\Theta$.

II. LOCAL DUAL AXION ELECTRODYNAMICS

Originally postulated by Wilczek [30], axion electrodynamics can be taken into account by using the action $I = I^f + I^\alpha + I^s + I^k$, where $I^f$, $I^\alpha$, $I^s$ and $I^k$ represent, respectively, Maxwell’s, axion/Maxwell interaction, electric source and kinetic axion actions. Explicitly, $I$ is given by

$$I = -\frac{1}{4} \int F^\mu\nu F_{\mu\nu} d^4x + \frac{k}{4} \int \alpha F^\mu\nu F^*_{\mu\nu} d^4x + \int j_\mu A^\mu d^4x + \frac{1}{2} \int \partial_\mu \alpha \partial^\mu \alpha d^4x. \quad (7)$$

The object $j_\mu \doteq 4\pi (\rho, -j)$ represents the electric current density. The term $I^\alpha = \int L^\alpha d^4x$ is the axion coupled to the EM tensor, where in the 3-vector notation, we have

$$\frac{1}{4} \int \alpha F^{\mu\nu} F^*_{\mu\nu} d^4x = \int \alpha E \cdot B d^4x. \quad (8)$$

Taking the Euler-Lagrange equations in (7) with respect to $A_\mu$, we obtain

$$\partial^\mu F_{\mu\nu} = j_\nu - k (\partial^\mu \alpha) F^*_{\mu\nu} \quad \text{and} \quad \partial^\mu F^*_{\mu\nu} = 0. \quad (9)$$

By expanding (9), we find the modified non-homogeneous vector equations

$$\nabla \cdot E + k (\nabla \alpha) \cdot B = \rho, \quad \nabla \times B - k (\nabla \alpha) \times E - k (\partial t \alpha) B - \partial_t E = j. \quad (10)$$
This correction is measured by the coupling constant $k$, and its value is estimated as being $k \sim 10^{-11} \text{GeV}^{-1}$, which is very difficult to observe. Therefore, in the classical energy range, we must consider the limit $k \to 0$, where the Maxwell’s equations are recovered. From (7), we can take the variation with respect to the $\alpha$ field,

$$\Box \alpha = \eta^{\mu\nu} \partial_\mu \partial_\nu \alpha = k \mathbf{E} \cdot \mathbf{B}. \quad (11)$$

From the knowledge that $\mathbf{E} \cdot \mathbf{B} = \epsilon^{\mu\nu\rho\sigma} \partial_\mu A_\nu \partial_\rho A_\sigma$, we have

$$\partial^\mu \alpha = k \epsilon^{\mu\nu\rho\sigma} A_\nu \partial_\rho A_\sigma, \quad (12)$$

which shows that the axion’s influence can be interpreted as a topological effect.

If we postulate a magnetic source $m_\mu \doteq 4\pi (\rho, -\mathbf{m})$, Maxwell’s field equations become

$$\partial^\mu F_{\mu\nu} = j_\nu \quad \text{and} \quad \partial^\mu F^*_{\mu\nu} = m_\nu. \quad (13)$$

The so-called electric/magnetic duality occurs when $F_{\mu\nu} \mapsto F^*_{\mu\nu}$, $F^*_{\mu\nu} \mapsto -F_{\mu\nu}$, $j_\mu \mapsto m_\mu$ and $m_\mu \mapsto -j_\mu$ in (13). Originally, this duality means that $\mathbf{E} \mapsto \mathbf{B}$ and $\mathbf{B} \mapsto -\mathbf{E}$ in the vacuum, such as observed by Heaviside [31].

We can extend the duality to a more general rotation with an arbitrary constant rotation angle $\chi$. An extended duality transformation does not change the mathematical structure of (13), and thus it does not provide new solutions for the equations. A global duality rotation can be represented by the field transformations

$$F_{\mu\nu} \mapsto F_{\mu\nu} \cos \chi + F^*_{\mu\nu} \sin \chi \quad \text{and} \quad F^*_{\mu\nu} \mapsto F^*_{\mu\nu} \cos \chi - F_{\mu\nu} \sin \chi, \quad (14)$$

with the sources changing simultaneously as

$$j_\mu \mapsto j_\mu \cos \chi + m_\mu \sin \chi \quad \text{and} \quad m_\mu \mapsto m_\mu \cos \chi - j_\mu \sin \chi. \quad (15)$$

By taking the special case $\chi = \pi/2$, it is easy to see the usual electric/magnetic duality discussed previously.

In [9], Tiwari finds the axion electrodynamics equations, which we will adapt as

$$\partial^\mu F_{\mu\nu} = j_\nu + (\partial^\mu \alpha) F^*_{\mu\nu} \quad \text{and} \quad \partial^\mu F^*_{\mu\nu} = m_\nu - (\partial^\mu \alpha) F_{\mu\nu}. \quad (16)$$

If we consider valid the Maxwell’s equations for which $m_\mu = 0$ in [9], we obtain from (16) the following expressions

$$(\partial^\mu \alpha) F^*_{\mu\nu} = 0 \quad \text{and} \quad (\partial^\mu \alpha) F_{\mu\nu} = m_\nu. \quad (17)$$
The first relation outlines the topological effect caused by an axion field: its presence produces a change on the electromagnetic field components. The last one can be interpreted as a mechanism which would impair the magnetic monopole observation, so that the axion would cancel its effects. In general, expressions (13) are invariant when \( \chi \) is taken with \( \chi \) constant, while (17) are invariant when \( \chi = \chi(x^\alpha) \), and, simultaneously, the axion transforms as

\[
\partial_\mu \alpha \rightarrow \partial_\mu \alpha + \partial_\mu \chi \quad \Leftrightarrow \quad \alpha \rightarrow \alpha + \chi + c, \tag{18}
\]

The eq. (18) gives the behavior of the axion under a duality rotation, with \( c \) an integration constant.

III. SPACETIME SCALE/PHE\-\( \)ASE INV\( \)ARIANCE FROM INFELD-VAN DER
WAERDEN FORMALISMS

The group (algebra) \( \mathbb{C}\ell(1,3) \) for Minkowski spacetime is represented in the 2-spinor index formalism as follows

\[
\sigma_{\mu AA'}\sigma_{\nu AA'} + \sigma_{\mu AA'}\sigma_{\nu AA'} = 2\eta_{\mu \nu}. \tag{19}
\]

The Einstein summation convention will be adopted and each spinor indices take either the values 0, 1 \((0', 1')\). The objects \((\sigma_{\mu AA'}) \in SL(2, \mathbb{C})\) are the normalized Pauli matrices and for each “world index” we have a \( 2 \times 2 \) hermitian matrix. The generalization of (19) for curved space is given by

\[
g_{\mu \nu}(x^\alpha) = \Sigma_{\mu AA'}^{(x^\alpha)}\Sigma_{\nu AA'}^{(x^\alpha)}, \tag{20}
\]

in which \( g_{\mu \nu} \) is the metric tensor component for a generic background, and the objects \( \Sigma_{\mu AA'}^{(x^\alpha)} \in \mathbb{C} \) are the Infeld-van der Waerden symbols \[18, 23, 32\], which are called connecting objects. The complex conjugation is denoted by \((\Sigma_{\mu AA'})^* = \Sigma_{\mu AA'}^{*}\). The relation between spinors and tensors is established by using the hermitian matrix set \( \Sigma \), for example \( v_\mu = \Sigma^{AA'}_{\mu} v_{AA'} \) and \( v_{AA'} = \Sigma^{\mu}_{AA'} v_\mu \). In order for us to obtain the object \( \Sigma_{\nu AA'} \) from \( \Sigma_{\mu AA'} \) it is necessary the use of the “metric” spinor to lower (or raise) the spinor indexes: \( \xi_A = \varepsilon_{BA}^B \xi_B, \xi^{A} = \varepsilon^{AB} \xi_{B} \). The object \( \varepsilon_{AB} \) is a skew-symmetric spinor component. In the matrix form, we have

\[
(\varepsilon_{AB}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{21}
\]

With the use of the metric spinor, we can rewrite (20) as follows

\[
g_{\mu \nu} = \Sigma_{\mu AA'}^{BB'} \varepsilon_{AB} \varepsilon_{A'B'}. \tag{22}
\]
A remarkable fact is that (22) is invariant under transformations (3).

In the pioneer paper [18], Infeld and van der Waerden created a more general formalism given by

\[ \gamma_{AB} ≡ |\gamma| e^{\Theta} \varepsilon_{AB} \quad \text{and} \quad \Upsilon_{\mu AB} ≡ |\gamma|^{-1} \Sigma_{\mu AB}, \]

in which $|\gamma|$ and $\Theta$ are real-valued functions of some spacetime coordinates $x^\mu$. The object $\gamma_{AB}$ is the fundamental object of the $\gamma$-formalism, while $\Upsilon_{\mu AB}$ is its connecting object. We will continue to present only aspects that will be directly used by us in our work. Detailed studies of the $\gamma\varepsilon$-formalisms are given in the references [23–25].

A crucial point is that the covariant derivative leads to an important difference between the formalisms. In both formalisms the compatibility metric $\nabla_{\alpha} g_{\mu\nu} = 0$ implies that

\[ \nabla_\mu (\gamma_{AB} \gamma_{A'B'}) = 0 = \nabla_\mu (\varepsilon_{AB} \varepsilon_{A'B'}), \]

since $\nabla_\alpha \Upsilon_{\mu AB} = 0 = \nabla_\alpha \Sigma_{\mu AB}$ is valid for any hermitian matrix set [23]. Regardless of the formalism, the covariant derivative of some spinors $\xi^A$ and $\zeta^A$ follows, respectively,

\[ \nabla_\mu \xi^A = \partial_\mu \xi^A + \omega^{AB}_\mu \xi^B \quad \text{and} \quad \nabla_\mu \zeta_A = \partial_\mu \zeta_A - \omega^{AB}_\mu \zeta_B, \]

with $\omega^{AB}_\mu = (\partial_\mu B^A, \gamma_\mu A^A)$, being the spinor connection associated with $M_{AB} = (\varepsilon_{AB}, \gamma_{AB})$. The object $\gamma^{\mu A^A}$ can be written as [23]

\[ \gamma^{\mu A^A} = \partial_\mu \ln |\gamma| - 2i A_\mu, \]

with $A_\mu = -(1/2) \text{Im} \gamma^{\mu A^A}$ being a real world vector. The spinors $\xi^A$ and $\zeta_A$ transform under the action of the generalized Weyl gauge group, which can be expressed in the component form as

\[ \Lambda^B_A = \sqrt{\varsigma} e^{i\lambda} \delta^B_A, \]

in which $\varsigma > 0$ is a real function and $\lambda$ the gauge parameter of the group.

Additionally, in the $\varepsilon$-formalism in (24) we have $\nabla_\mu \varepsilon_{AB} = 0 = \nabla_\mu \varepsilon^{AB}$, while in the $\gamma$-formalism the covariant derivative of $\gamma_{AB}$ and $\gamma^{AB}$ leads to

\[ \nabla_\mu \gamma_{AB} = i\beta_\mu \gamma_{AB} \quad \text{and} \quad \nabla_\mu \gamma^{AB} = -i\beta_\mu \gamma^{AB}. \]

The world object $\beta_\mu$ is defined as

\[ \beta_\mu ≡ \partial_\mu \Theta + 2A_\mu. \]
When the spacetime is flat, we have that $\gamma_{\mu A}^B = 0$ such that $\gamma_{\mu A}^A = 0$. As a consequence, $A_\mu = 0$ and $\gamma = \pm 1^1$, so that $\beta_\mu$ is simply $\beta_\mu = \partial_\mu \Theta$. The gauge behaviors of $\partial_\mu \Theta$ and $A_\mu$ are given by

$$\partial_\mu \Theta \mapsto \partial_\mu \Theta + 2 \partial_\mu \lambda \quad \text{and} \quad A_\mu \mapsto A_\mu - \partial_\mu \lambda. \quad (30)$$

In the following, we will analyze the electromagnetic structure in spinor spaces considering both formalisms, thus comparing the theory with a magnetic source in the $\varepsilon$-formalism with the theory in the absence of source in the $\gamma$-formalism.

### IV. MAXWELL’S THEORY AND AXION-LIKE PHASE/MAXWELL COUPLING

#### A. Electromagnetic Fields

The Maxwell’s tensor and its Hodge dual can be rewritten in the 2-spinor notation as

$$2F_{AA'BB'} = M_{AB}f_{A'BB'} + M_{A'BB}f_{AB}, \quad \text{and} \quad 2F^*_{AA'BB'} = i(M_{AB}f_{A'BB'} - M_{A'BB}f_{AB}), \quad (31)$$

in which $f_{AB}$ is a symmetric object, $f_{AB} = f_{(AB)} \in \mathbb{C}$, called Maxwell spinor. The decomposition (31) is valid for any bivector $[23, 32, 33]$. For convenience, it is usual to define the complex object $F_{\mu\nu}^{(\pm)}$ as being $F_{\mu\nu}^{(\pm)} = F_{\mu\nu} \pm iF_{\mu\nu}^*$. If we take the spinor form of $F_{\mu\nu}^{(\pm)}$ and then use (31), we obtain

$$F_{AA'BB'}^{(+)} = M_{A'B'}f_{AB} \quad \text{and} \quad F_{AA'BB'}^{(-)} = M_{AB}f_{A'B'}. \quad (32)$$

If we contract both expressions in (31) and (32) by $M_{A'B'}$ and $M_{AB}$, we find

$$f_{AB} = F_{ABC'C'}^{(+)CC'} = (1/2) F_{ABC'}^{(+)} C' \quad \text{and} \quad f_{A'B'} = F_{A'B'C'C}^{(-)} = (1/2) F_{A'B'C'}^{(-)} C. \quad (33)$$

Now, considering Minkowski spacetime and then using the $\varepsilon$-formalism, we have [32]

$$(f_{AB}) = \begin{pmatrix} F_x - iF_y & -F_z \\ -F_x & F_y - iF_z \end{pmatrix}, \quad (34)$$

in which it was used the Penrose complex 3-vector $\mathbf{F} = \mathbf{E} - i\mathbf{B}$, with $\mathbf{E}$ and $\mathbf{B}$ being, respectively, the electric and magnetic fields.

The invariants $F_{\mu\nu}F^{\mu\nu}$ and $F^*_{\mu\nu}F^{\mu\nu}$ are rewritten in the spinor form as

$$2F_{\mu\nu}F^{\mu\nu} = -2F^*_{\mu\nu}(F^*)^{\mu\nu} = \text{Re} [f_{AB}f^{AB}] \quad \text{and} \quad 2F^*_{\mu\nu}F^{\mu\nu} = \text{Im} [f_{AB}f^{AB}]. \quad (35)$$

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1. In general $\ln |\gamma| = \text{const.}$ Here we will assume $\gamma = \pm 1$. 
Maxwell’s action \( (7) \) in Minkowski background can be rewritten using \( (35) \), i. e.,

\[
I^f_M = -\frac{1}{8} \int \mathcal{L}^f_M d^4x_M \quad \text{with} \quad \mathcal{L}^f_M \equiv \Re \left[ M^{AC} M^{BD} f_{AB} f_{CD} \right]. \quad (36)
\]

We must note the definition of the Maxwell’s lagrangean in the \( M \)-formalism given above. The volume element \( d^4x_M \) is written in the \( M \)-formalism as follows

\[
d^4x_M = i (M_{AC} M_{BD} M_{A'B'} M_{B'C'} - M_{AD} M_{BC} M_{A'C'} M_{B'D'}) dx^{A'A'} dx^{BB'} dx^{CC'} dx^{DD'}. \quad (37)
\]

The object \( \epsilon_{AA'B'B'CC'D'D'} \) is the spinor form of the Levi-Civita tensor component. If we take a closer look at the equality \( \gamma_{AB} \gamma_{C'D'} = \epsilon_{AB} \epsilon_{C'D'} \), we can notice that it implies that \( d^4x_\gamma = d^4x_\varepsilon \), such that we will take into consideration only the lagrangean term \( \mathcal{L}^f_M \) of \( (36) \).

The objects \( (32) \) are useful to obtain Maxwell’s spinor equations, since each one of them carries \( f_{AB} \) and \( f_{A'B'} \) in a unique way. In what follows, we will review Maxwell’s equations with the electric and magnetic sources in the \( \varepsilon \)-formalism. Later, we will explore the Maxwell’s theory in the \( \gamma \)-formalism without magnetic sources to show a coupling between the EM tensor and the spacetime phase.

**B. Maxwell’s Theory in the \( \varepsilon \)-formalism**

Let us then begin by rewriting Maxwell’s equations in the \( \varepsilon \)-formalism. We will use the equations with electric and magnetic sources in curved spacetime, i. e.,

\[
\nabla^\mu F_{\mu\nu} = j_\nu \quad \text{and} \quad \nabla^\mu F^*_{\mu\nu} = m_\nu. \quad (38)
\]

Also, we will consider the complex version of equations \( (38) \), by using \( F_{\mu\nu}^{(\pm)} \), such that

\[
\nabla^\mu F_{\mu\nu}^{(\pm)} = j_\nu \pm im_\nu. \quad (39)
\]

If we take \( (39) \) in its spinor form and later substitute \( (32) \), we will have the following spinor expression

\[
\nabla^B_{A'} f_{AB} = j_{AA'} + im_{AA'}, \quad (40)
\]

with its complex conjugate. The expression \( (40) \) and its complex conjugate represent Maxwell’s equations with magnetic sources in the \( \varepsilon \)-formalism. Since one is the complex conjugate of the other, we have, effectively, only one single spinor expression.
We must note that the global duality rotation (14)-(15) of (4 0) assumes the complex form

\[ f_{AB} \mapsto e^{-\chi i} f_{AB} \quad \text{and} \quad j_{AA'} + im_{AA'} \mapsto e^{-\chi i} (j_{AA'} + im_{AA'}), \]

(41)

with \( \chi \) constant. In the \( \varepsilon \)-formalism, the field lagrangean (36) can be written as

\[ \mathcal{L}^f_\varepsilon = \text{Re} \left[ \varepsilon^{AC} \varepsilon^{BD} f_{AB} f_{CD} \right]. \]

(42)

By using (34) in (42), we can verify that

\[ \mathcal{L}^f_\varepsilon = 2 \left( B \cdot B - E \cdot E \right), \]

(43)

such that the result is in accordance with Maxwell’s lagrangean. Another important relation is the imaginary term in the \( \varepsilon \)-formalism, i.e.,

\[ \text{Im} \left[ \varepsilon^{AC} \varepsilon^{BD} f_{AB} f_{CD} \right] = 4E \cdot B, \]

(44)

which is in accordance with \( F^*_{\mu\nu} F^{\mu\nu} \). The expression (40) will be used indirectly to define the geometric magnetic source and to get to our axion-like phase electrodynamics.

C. Maxwell’s Theory in the \( \gamma \)-formalism

Let us then consider the electromagnetic equations in the \( \gamma \)-formalism without magnetic sources., i.e., \( m_{\nu} = 0 \) in (38). The complex version will then yield

\[ \nabla^\mu F^{(\pm)}_{\mu\nu} = j_\nu. \]

(45)

If we rewrite (45) in its spinor form and use (31), we will obtain

\[ \nabla^B_{\ A} f_{AB} = j_{AA'} + i \beta^B_{\ A} f_{AB}, \]

(46)

and its complex conjugate. The \( \beta \)-term emerges due the eigenvalue equations (28). Now, comparing (40) with (46), we are inspired to consider a formal analogy which leads us to announce the \( \beta \)-term as a geometric magnetic current, i.e.,

\[ \beta^B_{\ A} f_{AB} \doteq m_{AA'}. \]

(47)

The idea of formally defining a geometrical source in the \( \gamma \)-formalism can be found in refs. [21, 22].

With the definition being established, we want to rewrite (47) in the world notation. First, we note that \( \beta^B_{\ A} f_{AB} = -\beta^{BB'} f_{AA'BB'}^{(+)} \) which implies that \( \beta^\mu F^{(+)}_{\mu AA'} \) so that when we apply \( \Upsilon_{\nu}^{AA'} \), we have

\[ \beta^\mu F^{(\pm)}_{\mu\nu} = m_\nu, \]

(48)
in which the minus sign emerges when we take the complex conjugate of (47). Thus, expression (48) provides a world extension for Maxwell’s theory. The full world equations are given by (45) and (48), and we can decompose (48) as

$$\beta^\mu F_{\mu\nu} = m_\nu \quad \text{and} \quad \beta^\mu F^*_{\mu\nu} = 0.$$  (49)

Let us then write equations (49) in Minkowski spacetime. By recalling that in this case $\beta_\mu = \partial_\mu \Theta$, expressions (49) will become

$$(\partial^\mu \Theta) F_{\mu\nu} = m_\nu \quad \text{and} \quad (\partial^\mu \Theta) F^*_{\mu\nu} = 0.$$  (50)

We must note the identical structure of (49) and (17). Thus, our system assumes the LDIE form where the axion field is identified with the spacetime phase! Using the 3-vector notation, Maxwell’s equations are supplemented by

$$(\nabla \Theta) \cdot E = \rho, \quad (\nabla \Theta) \cdot B = 0, \quad (\nabla \Theta) \times E + (\partial_t \Theta) B = 0, \quad (\nabla \Theta) \times B - (\partial_t \Theta) E = m.$$  (51)

To conclude our study, we will analyze some aspects of local duality and Maxwell lagrangean.

D. Maxwell’s Lagrangean and Local Duality Symmetry in Flat Spacetime

We will work only on Minkowski background. We must consider also the case without electric sources, i.e., $j_{AA'} = 0$. Consequently, we can rewrite expression (46) as follows

$$\varepsilon^{BC} \partial_C f_{AB} = i\varepsilon^{BC} (\partial_C \Theta) f_{AB}.$$  (52)

A local duality rotation in the spinor notation is taken simply as $f_{AB} \mapsto e^{-i(x^\mu)} f_{AB}$ in (11), with $\chi$ being a function of the spacetime coordinates. Applying the transformation in (52), we can observe that the invariance of (52) is obtained if $\partial_{AA'} \Theta$ transforms simultaneously as

$$\partial_{AA'} \Theta \mapsto \partial_{AA'} \Theta - \partial_{AA'} \chi \iff \Theta \mapsto \Theta - \chi + q,$$  (53)

with $q$ an integration constant. The phase transformation given in (53) implies that the metric spinor transforms under local duality rotation as

$$\gamma_{AB} \mapsto e^{i(q - \chi)} \gamma_{AB},$$  (54)

since $\tilde{\gamma}_{AB} = e^{i\Theta} \varepsilon_{AB}.$

Let us then consider Maxwell’s lagrangean, given in (36) written in the $\gamma$-formalism, i.e.,

$$\mathcal{L}_\gamma^f = \text{Re} \left[ \gamma^{AC} \gamma^{BD} f_{AB} f_{CD} \right].$$  (55)
Inserting (54) in (55) and taking \( \tilde{f}_{AB} \), we have
\[
\tilde{\gamma}_A^C \tilde{\gamma}_B^D \tilde{f}_{AB} \tilde{f}_{CD} = \cos (-q \gamma_{AC} \gamma_{BD}) f_{AB} f_{CD},
\]
(56)
such that Maxwell’s lagrangean is local duality invariant when \( q = 0, \pm \pi, \pm 2\pi, \ldots \), or simply
\[
\tilde{\mathcal{L}}_\gamma^f = \mathcal{L}_\gamma^f, \quad \text{if} \quad q = n\pi, \quad n \in \mathbb{Z}.
\]
(57)

Now, considering the duality rotation of \( f_{AB} \) and (54) along with expressions (31) and (32), the electromagnetic spinor objects transform as
\[
2 \tilde{F}_{AA'BB'}^+ = e^{i q} \gamma_{AB} f_{A'B'} + e^{-i q} \gamma_{A'B'} f_{AB}, \quad \tilde{F}_{AA'BB'}^{(+)} = e^{-i q} \tilde{F}_{AA'BB'}^{(+)},
\]
\[
2 \tilde{F}_{AA'BB'}^- = i (e^{i q} \gamma_{AB} f_{A'B'} - e^{-i q} \gamma_{A'B'} f_{AB}), \quad \tilde{F}_{AA'BB'}^{(-)} = e^{i q} \tilde{F}_{AA'BB'}^{(-)}.
\]
(58)

We must note that (58) unchanges when \( q = m\pi \), with \( m \in 2\mathbb{Z} \). In order to complete our analysis, we will consider the axion lagrangean \( \mathcal{L}^\alpha \) given by the interaction term
\[
\mathcal{L}^\alpha = \frac{1}{4} \alpha \mathbf{E} \cdot \mathbf{B}.
\]
(59)

The notation \( \mathcal{L}^\phi \) in (59) can denote an axion-like coupling between \( \mathbf{E} \cdot \mathbf{B} \) and any field \( \phi \). In the spinor form, we can rewrite (59) in the \( \varepsilon \)-formalism as follows
\[
\mathcal{L}_\varepsilon^\alpha = \alpha \Im \left[ \varepsilon^{AC} \varepsilon^{BD} f_{AB} f_{CD} \right],
\]
(60)
in which we used (44). We must also note that
\[
\Re \left[ \varepsilon^{AC} \gamma_{BD} f_{AB} f_{CD} \right] = \Re \left[ \varepsilon^{AC} \varepsilon^{BD} f_{AB} f_{CD} \right] \cos (2\Theta) + \Im \left[ \varepsilon^{AC} \varepsilon^{BD} f_{AB} f_{CD} \right] \sin (2\Theta).
\]
(61)

Now, combining (61) with (55) and (12), expanding the trigonometric functions and noting (60), we have
\[
\mathcal{L}_\gamma^f = \mathcal{L}_\varepsilon^f + 2 \mathcal{L}_\varepsilon^\Theta + \sum_{j=1}^{\infty} \mathcal{L}_{\varepsilon}^{2j},
\]
(62)
with \( \mathcal{L}^{2j}_\varepsilon \) given by
\[
\mathcal{L}^{2j}_\varepsilon = (-1)^j \frac{(2\Theta)^{2j}}{(2j)!} \left[ \mathcal{L}_\varepsilon^f + \frac{2}{2j+1} \mathcal{L}_\varepsilon^\Theta \right].
\]
(63)

Thereby, we can observe by \( \mathcal{L}^\Theta_\varepsilon \) in (62) the axion-like coupling between the spacetime phase, with \( f_{AB} \).
Once a small angle $\delta \Theta \simeq 0$ is taken, we have $\cos (2\delta \Theta) \simeq 1$ and $\sin (2\delta \Theta) \simeq 2\delta \Theta$ in (61) or $L^2_{\varepsilon} \simeq 0$ in (62), which together with the fact that they are respectively even and odd functions, yields

$$L^f_\gamma (\pm 2\delta \Theta) \simeq L^f_\varepsilon \pm 2L^\delta \Theta.$$  (64)

Since our argument is $2\Theta$, we must make an adjustment so that $\Theta$ runs effectively from 0 to $\pi$. This result agrees with the axion models! Physically, it means that the axion varies from normal to topological insulator [34].

V. CONCLUSIONS AND OUTLOOK

Based on Tiwari’s work about LDIE, we have showed that, from the Infeld-van der Waerden formalism it is possible to identify the axion field as the geometric term $\Theta$. So, by restricting the equations for a flat background, we derived the interaction Maxwell-axion term purely from Maxwell’s theory. It is interesting to note also that in the $\gamma$-formalism the equations are valid for more general spaces, which may allow for the development of axion-related works in gravitational contexts, such as the ones involving cold dark matter.

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