Isotope Effect in Superconductors with Coexisting Interactions of Phonon and Nonphonon Mechanisms

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We examine the isotope effect of superconductivity in systems with coexisting interactions of phonon and nonphonon mechanisms in addition to the direct Coulomb interaction. The interaction mediated by the spin fluctuations is discussed as an example of the nonphonon interaction. Extended formulas for the transition temperature $T_c$ and the isotope-effect coefficient $\alpha$ are derived for cases (a) $\omega_{\text{up}} < \omega_D$ and (b) $\omega_{\text{up}} > \omega_D$, where $\omega_{\text{up}}$ is an effective cutoff frequency of the nonphonon interaction that corresponds to the Debye frequency $\omega_D$ in the phonon interaction. In case (a), it is found that the nonphonon interaction does not change the condition for the inverse isotope effect, i.e., $\mu^* > \lambda_{\text{ph}}/2$, but it modifies the magnitude of $\alpha$ markedly. In particular, it is found that a giant isotope shift occurs when the phonon and nonphonon interactions cancel each other largely. For instance, strong critical spin fluctuations may give rise to the giant isotope effect. In case (b), it is found that the inverse isotope effect occurs only when the nonphonon interaction and the repulsive Coulomb interaction, in total effect, work as repulsive interactions against the superconductivity. We discuss the relevance of the present result to some organic superconductors, such as $\kappa$-(ET)$_2$Cu(NCS)$_2$ and Sr$_2$RuO$_4$ superconductors, in which inverse isotope effects have been observed, and briefly to high-$T_c$ cuprates, in which giant isotope effects have been observed.

KEYWORDS: phonon-mediated pairing interactions, spin fluctuations, exotic superconductors, organic superconductor, Sr$_2$RuO$_4$, giant isotope shift, inverse isotope effect, negative isotope-effect coefficient

The mechanisms of the pairing interactions of nonphonon origins have been examined for anisotropic superconductivity. For example, spin fluctuation exchange interactions have been examined for the superfluid $^3\text{He}^{(1)}$ and exotic superconductors, such as heavy-fermion superconductors,$^2$ organic superconductors,$^3$–$^5$ high-$T_c$ cuprates,$^6$ and Sr$_2$RuO$_4$ superconductors.$^7$ In order to clarify the origin of the pairing interactions, the isotope effect has been observed in many superconductors. It is widely known that the isotope-effect coefficient $\alpha = 0.5$ observed in nontransition-metal superconductors, such as Hg and Sn, is evidence of phonon-mediated pairing interactions in those systems. Here, the isotope-effect coefficient $\alpha$ has been defined by

$$\alpha = \frac{\partial \ln T_c}{\partial \ln M}, \quad (1)$$

where $M$ is the relevant atomic mass. The value $\alpha = 0.5$ is directly derived from the BCS formula $T_c = 1.13 \omega_D e^{-1/\lambda}$ with $\omega_D \propto M^{-1/2}$ and $\lambda \propto M^0$.

However, deviation of $\alpha$ from 0.5 is not necessarily evidence of the presence of the pairing interaction of a nonphonon origin. For instance, in transition-metal superconductors, such as Ru and Os, the value of $\alpha$ largely deviates from 0.5 due to the strong repulsive Coulomb interaction $V_C$. In the presence of it, $\lambda$ is replaced with $\tilde{\lambda} \equiv \lambda - \mu^*$, where

$$\mu^* = \frac{\mu}{1 + \mu \ln(W/\omega_D)}, \quad (2)$$

$\mu \equiv V_C N(0)$ and $W$ denotes an effective cutoff energy of the Coulomb interactions of the order of the band width.

Hence, we obtain

$$\alpha = \frac{1}{2} \left[ 1 - \left( \frac{\mu^*}{\lambda} \right)^2 \right]. \quad (3)$$

In particular, the strong Coulomb repulsion such as $\lambda > \mu^* > \lambda/2$ gives rise to the inverse isotope effect ($\alpha < 0$). The anharmonicity of the lattice vibrations may also affect the isotope effect in compounds, such as PdH(D) ($\alpha \approx -0.25$)$^8$ The inverse isotope effect has been observed also in some organic superconductors$^9$–$^{15}$ and Sr$_2$RuO$_4$,$^{16}$ but the mechanisms is unknown.

In most cases, a nonzero isotope shift ($\alpha \neq 0$) suggests the presence of the phonon contribution to the pairing interactions. One might expect that isotope shift can be attributed to the change in normal state properties, such as the density of states. However, usually, such changes should be of the order of $(\Delta M/2M) (\omega_D/\epsilon_F)$, which is usually negligible except for very narrow band systems, where $\Delta M$ is the shift of the atomic mass. Therefore, the isotope effect is mainly attributed to a change in the pairing interactions.

Here, it could not be excluded a priori that $\alpha \neq 0$ occurs because the phonon interaction $V_{\text{ph}}(\mathbf{p}, \mathbf{p}')$ reduces the anisotropic pairing interaction, but it seems unlikely. The effective interaction for the anisotropic gap function proportional to $\gamma(\mathbf{p})$ is essentially proportional to the average of $V_{\text{ph}}(\mathbf{p}, \mathbf{p}') \gamma(\mathbf{p}) \gamma(\mathbf{p}')$ on the Fermi surface, where $\gamma(\mathbf{p})$ is a function for expressing the momentum dependence. Since $V_{\text{ph}}(\mathbf{p}, \mathbf{p}')$ tends to take larger negative values for smaller momentum transfers, i.e., for $\mathbf{p} \approx \mathbf{p}'$, where $\gamma(\mathbf{p}) \gamma(\mathbf{p}') > 0$, the above average must be negative. For example, in order to obtain a repulsive
effective interaction for $\gamma(p) \propto p_x$, $V_{ph}(p, p')$ must have a negative peak at $|p-p'| \sim 2k_{p'}$ rather than $|p-p'| \sim 0$, but it is unlikely. In fact, some explicit calculations have shown that the phonon interaction includes attractive components for anisotropic pairing.\textsuperscript{17–19} Therefore, the magnitude of the isotope shift of $T_c$ reflects, more or less, the degree of the positive contribution to the Cooper pair formation from the phonons.

We should note that spin triplet pairing does not necessarily exclude the phonon mechanisms of the pairing interaction. The triplet pairing interaction is usually weaker than the singlet pairing interaction in the phonon mechanism, but it could become dominant, for example, in the presence of a strong short-range Coulomb interaction,\textsuperscript{17–19} which strongly suppresses the isotropic pairing more than the anisotropic pairing. In this context, the large deviation of $\alpha$ from 0.5 is a natural consequence of the strong Coulomb repulsion. The layer structure also favors the anisotropic pairing interactions mediated by phonons.\textsuperscript{19}

In this paper, we examine the isotope effect in systems in which interactions of phonon and nonphonon mechanisms coexist. In particular, we examine the condition for the inverse isotope effect.

We assume that the nonphonon interaction also has an effective cutoff frequency $\omega_{np}$ that corresponds to the Debye frequency $\omega_D$ in the phonon interaction. This assumption is explained as follows for the interactions mediated by the spin fluctuations. The spin fluctuations have a characteristic frequency $\omega_{sf}$, which becomes smaller as one approaches the magnetic instability point for the critical slowing down. The structure of the susceptibility $\chi(q, \omega)$ reflects this characteristic frequency. The susceptibility $\chi(q, \omega)$ has a sharp peak around a momentum $q_0$ with a width corresponding to $\omega_{sf}$, in addition to a broad peak over the whole momentum space, which reflects the electron characteristic energy $\epsilon_F$. Since the effective interaction mediated by the spin fluctuations is essentially proportional to $\chi(q, \omega)$, it consists of two parts with different characteristic energies $\omega_{sf}$ and $\epsilon_F$. In the gap equation, these characteristic energies play roles of the effective cutoff energies of the two parts of the effective interaction. The part of the effective interaction with $\epsilon_F$ can be included in the repulsive Coulomb interaction, which also has a large effective cutoff energy of the same order. Therefore, we may regard that the interaction mediated by the spin fluctuations has an effective cutoff energy of the order of $\omega_{sf}$, namely, $\omega_{np} \sim \omega_{sf}$. In fact, it has been shown in a theoretical calculation\textsuperscript{5} that the contribution to the pairing interaction mainly comes from this sharp peak of $\chi$. In theoretical phase diagrams,\textsuperscript{5,20} the superconductivity by such a pairing interaction occurs only near the magnetic phase boundary. An explicit calculation, which illustrates the above argument, has been given in ref. 5.

We start with the interactions between electrons,

$$V(p, p') = \sum_{\alpha} \sum_{k=1}^{n} V_{k}^\alpha (\xi_p, \xi_{p'}) \gamma_{\alpha}(p_{||}) \gamma_{\alpha}(p_{||'}),$$

where $p_{||}$ and $\xi_p$ denote the two-dimensional momentum coordinate on the Fermi surface and the electron energy measured from the Fermi surface with indices $k$ and $\alpha$ to express the kinds of interactions and the pairing symmetries. The symmetry functions $\gamma_{\alpha}(p_{||})$ are normalized with respect to the average on the Fermi surface.\textsuperscript{21} We consider a gap function proportional to $\gamma_{\alpha}(p_{||})$, retaining only terms with $\alpha$ in eq. (4). Introducing cutoff energies $\omega_k$, we put

$$V_{k}^\alpha (\xi, \xi') = V_{k0}^\alpha \theta(\omega_k - |\xi|) \theta(\omega_k - |\xi'|),$$

where $\omega_1 < \omega_2 < \omega_3 < \cdots < \omega_n$. We should note that $V_{k0}^\alpha$ for the repulsive Coulomb interaction can be nonzero even for anisotropic pairing, unless we assume only onsite repulsion. For eq. (5), the gap function is written as

$$\Delta_{\alpha}(p) = \sum_{k=1}^{n} \Delta_{k\alpha} \theta(\omega_k - |\xi_p|) \gamma_{\alpha}(p_{||}).$$

The linearized gap equation ($T_c$ equation) is written as

$$\Delta_{k\alpha} = \lambda_k \left( \sum_{k'=1}^{n} \Delta_{k'\alpha} l_{k'} + \sum_{k'=k}^{n} \Delta_{k'\alpha} l_{k} \right),$$

where we have defined $\lambda_k = -V_{k0}^\alpha N(0)$ and $l_k = \ln(2\omega_k^2/\omega_D^2)$ with the Euler constant $\gamma = 0.57721\ldots$. We write the transition temperature in the form

$$T_c = \frac{2\omega_1 e^{-1/\lambda_1}}{\pi},$$

defining the effective coupling constant $\lambda_1$. We define $\alpha_{k0} = -\partial \ln \omega_k / \partial \ln M$ for convenience.

First, we consider the case with $n = 2$. The $T_c$ equation is written as

$$\begin{pmatrix} \lambda_1^{-1} - l_1 & -l_1 \\ -l_1 & \lambda_2^{-1} - l_2 \end{pmatrix} \begin{pmatrix} \Delta_{1\alpha} \\ \Delta_{2\alpha} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. $$

Therefore, we obtain $\lambda_1 = \lambda_1 + \lambda_2^*$ and $\lambda_2 = \lambda_2/[1 - \lambda_2 \ln(\omega_1/\omega_1)]$. The isotope-effect coefficient is obtained as

$$\alpha = \alpha_{10} \left[ 1 - (\lambda_2^* \lambda_1 + \lambda_1^* \lambda_2)^2 \right] + \alpha_{20} (\lambda_2^* \lambda_1 + \lambda_1^* \lambda_2)^2. $$

If we take $V_{10}^\alpha$ and $V_{20}^\alpha$ as a phonon-mediated pairing interaction and the repulsive Coulomb interaction, respectively, then $\omega_1 = \omega_D \propto M^{-1/2}$ and $\omega_2 = W \propto M^0$. If the nonphonon interaction is one mediated by the spin fluctuations, $\omega_{np}^* \sim \omega_{sf}$, and case (a) occurs in proximity to the magnetic instability. We define $\lambda_{np} = -V_{np}^\alpha N(0)$, where $V_{ph}^\alpha$ and $V_{np}^\alpha$ denote the coupling constants of the phonon and nonphonon interactions, respectively.

For case (a), we put $\omega_1 = \omega_{np}, \omega_2 = \omega_D, \lambda_1 = \lambda_{np},$
and \( \lambda_0^* = \lambda_{ph}^* \). Thus, we obtain

\[
\alpha = \frac{1}{2} \left( \frac{\lambda_{ph}^*}{\lambda_{np} + \lambda_{ph}^*} \right)^2,
\]

which is positive definite. On the other hand, for case (b), we put \( \omega_1 = \omega_D \), \( \omega_2 = \omega_{np} \), \( \lambda_1 = \lambda_{ph} \), and \( \lambda_2 = \lambda_{np} \). Thus, we obtain

\[
\alpha = \frac{1}{2} \left[ 1 - \left( \frac{\lambda_{np}^*}{\lambda_{ph} + \lambda_{np}^*} \right)^2 \right],
\]

where \( \lambda_{np}^* = \lambda_{np}/[1 - \lambda_{np} \ln(\omega_{np}/\omega_1)] \). Therefore, \( \alpha \) could become negative only in case (b) \( (\omega_{np} > \omega_D) \) with \( \lambda_{ph} \alpha_{np} < 0 \).

Next, we examine the effect of the direct Coulomb interaction. The \( T_c \) equation with \( n = 3 \) can be solved in a similar manner to that in the above. It is obtained that \( \lambda_1 = \lambda_1' + \lambda_2^* \), \( \lambda_2^* = (\lambda_2 + \lambda_3^*)/[1 - (\lambda_2 + \lambda_3^*) l_{21}] \), and \( \lambda_3^* = \lambda_3/[1 - \lambda_{3,2}^*] \), where \( l_{kk'} \equiv \ln(\omega_k/\omega_{k'}) \). The transition temperature is given by eq. (8). Hence, we obtain

\[
\alpha = \alpha_1 C_1 + \alpha_2 C_2 + \alpha_3 C_3,
\]

where

\[
\begin{align*}
C_1 &= 1 - \left( \frac{\lambda_2^*}{\lambda_1 + \lambda_2^*} \right)^2, \\
C_2 &= \left( \frac{\lambda_2^*}{\lambda_1 + \lambda_2^*} \right)^2 \left[ 1 - \left( \frac{\lambda_3^*}{\lambda_2 + \lambda_3^*} \right)^2 \right], \\
C_3 &= \left( \frac{\lambda_2^*}{\lambda_1 + \lambda_2^*} \right)^2 \left( \frac{\lambda_1^*}{\lambda_2 + \lambda_3^*} \right)^2.
\end{align*}
\]

We examine cases (a) and (b) mentioned above, adding the direct Coulomb interaction \( V_{ph}^0 \) with \( \omega_3 = W \) and \( \lambda_3 = -\mu \).

In case (a), we obtain \( \ddot{\lambda}_1 = \lambda_{np} + \lambda_{ph}^* \) and

\[
\alpha = \frac{1}{2} \left( \frac{\lambda_{ph}^*}{\lambda_{np} + \lambda_{ph}^*} \right)^2 \left[ 1 - \left( \frac{\mu^*}{\lambda_{ph}^* - \mu^*} \right)^2 \right],
\]

where \( \lambda_{ph}^* = (\lambda_{ph} - \mu^*)/[1 - (\lambda_{ph} - \mu^*) \ln(\omega_D/\omega_{np})] \) with \( \mu^* \) defined by eq. (2). In this case, the sign of \( \alpha \) is determined only by the ratio \( \lambda_{ph}/\mu^* \). If we assume \( \mu^* > 0 \), there is no region of \( \lambda_{ph} < 0 \) where \( \alpha < 0 \) occurs. In Fig. 1 and eq. (15), it is also found that \( \alpha \) rapidly varies with \( \lambda_{ph} \) and \( \lambda_{np} \). The prefactor \( \lambda_{np}^* / (\lambda_{np} + \lambda_{ph}^*) \) in eq. (15) gives rise to a large value of \( \alpha \) when \( \lambda_{ph} \approx -\lambda_{np} \).

On the other hand, in case (b), we obtain \( \ddot{\lambda}_1 = \lambda_{ph} + \lambda_{np}^* \) and

\[
\alpha = \frac{1}{2} \left[ 1 - \left( \frac{\lambda_{np}^*}{\lambda_{ph} + \lambda_{np}^*} \right)^2 \right],
\]

where \( \lambda_{np}^* = (\lambda_{np} - \mu^*)/[1 - (\lambda_{np} - \mu^*) \ln(\omega_{np}/\omega_D)] \) and in this case \( \mu^* = \mu/[1 + \mu \ln(W/\omega_{np})] \). It is found that \( \lambda_{ph} \lambda_{np}^* < 0 \) is a necessary condition for \( \alpha < 0 \). Since it is reasonable to assume \( \lambda_{ph} > 0 \) even for anisotropic pairing as argued above, an inverse isotope effect suggests that the nonphonon interaction and the direct Coulomb interaction, in total effect, work as repulsive interactions against the superconductivity \( (\lambda_{np}^* < 0) \) when \( \omega_{np} > \omega_D \). Figure 2 shows the phase diagram obtained using eq. (16).

Now, we describe the general solution of eq. (7) with any positive integer \( n \). Equation (7) is rewritten as \( \Delta_k = \lambda_k F_k \), where we have defined \( F_k \equiv \sum_{k'=1}^{k-1} l_{kk'} \Delta_{k'} + l_k \Delta_k \) with \( \Delta_k = \sum_{n'=k}^{n} \Delta_{n'} \). On the other hand, it is easily proved from eq. (7) with \( k + 1 \sim n \) that an equation of the form \( \Delta_{k+1} = \lambda_{k+1}^* F_k \) holds for \( k \geq 1 \) with a recurrence formula

\[
\lambda_k = \frac{\lambda_{k+1} + \lambda_{k+1}^*}{1 - (\lambda_k + \lambda_{k+1}^*) l_{k,k-1}}
\]

for \( k \geq 2 \). Hence, it holds that \( \Delta_1 = \Delta_1 + \Delta_2 = (\lambda_1 + \lambda_2^*) l_1 \Delta_1 \). Therefore, \( T_c \) is given by eq. (8) with \( \ddot{\lambda}_1 = \lambda_1 + \lambda_2^* \). Hence, from eq. (1), the general formula for \( \alpha \) is obtained as

\[
\alpha = \sum_{k=1}^{n} C_k \alpha_{k0}
\]

with \( C_k = \Delta_k = \lambda_{k+1} \) for \( k \leq n - 1 \) and \( C_n = \lambda_n \), where \( \Delta_k = \sum_{j=1}^{k} \lambda_{j+1}^* / (\lambda_j + \lambda_{j+1}^*) \) for \( k \geq 2 \) and \( \lambda_1 = 1 \). For example, if the interactions \( V_k^0 \) with \( k = 1, \ldots, n - 1 \) are phonon interactions with \( \alpha_{k0} = 1/2 \), it is found that \( \alpha \approx 1/2 \) for large \( n \), since \( \Delta_n \ll 1 \).
Let us discuss the inverse isotope effect observed in clean samples of Sr$_2$RuO$_4$ with $T_c > 0.94T_{c0}$, where $T_{c0}$ is the transition temperature in the clean limit. From the present result for case (b), which is probably the case for Sr$_2$RuO$_4$, the inverse isotope effect suggests that the superconductivity in this compound is induced by the phonon-mediated pairing interaction.\(^{24}\)

This result is consistent with the recent report that the antiferromagnetic fluctuation does not seem to play a constructive role in the triplet superconductivity in Sr$_2$RuO$_4$.\(^{25}\) The impurity concentration dependence of the isotope-effect coefficient\(^{16}\) can also be reproduced within a theory based on a phonon mechanism.\(^{21}\) Many novel mechanisms have been proposed for the superconductivity in this compound.\(^{20}\) However, there is no conclusive evidence for the nonphonon mechanism at the present. The present theory does not consider the effect of anharmonicity of lattice vibrations. However, if such effect is essential for the inverse isotope effect, we also obtain the same conclusion that the phonon interaction is dominant.

The situation is rather different in the organic superconductors. The inverse isotope effect was also observed in the organic superconductors, such as $\kappa$-(ET)$_2$Cu(NCS)$_2$,\(^{10,11,13}\) $\beta$-(ET)$_2$I$_3$,\(^{11,12}\) and $\kappa$-(${\mathrm ET})_2$Ag(CF$_3$)$_2$(1-bromo-1,2-dichloroethane),\(^{9}\) while a normal isotope effect was observed in $\kappa$-(ET)$_2$Cu[N(CN)$_2$]Br.\(^{11,14,15}\) In contrast to Sr$_2$RuO$_4$, case (a) ($\omega_{\text{ph}} < \omega_D \lesssim W$) may be realized in the organic superconductors even if $\xi_{\text{mag}} = 2a \sim 3a$, because they have narrower band widths of 0.2eV $\sim$ 0.5eV. The coefficient $\alpha$ varies rapidly, as shown in Fig. 1, when $\lambda_{\text{ph}}$ changes, whichever $\lambda_{\text{ph}} > 0$ or $< 0$. Thus, possibilities of the nonphonon pairing interactions are not excluded in the organic superconductors with $\alpha < 0$.

In high-$T_c$ cuprates La$_{2-\delta}$Sr$_\delta$CuO$_4$, it was observed by Crawford et al.\(^{27}\) that $\alpha$ increases near the antiferromagnetic boundary, where $T_c$ decreases. In cuprates, the presence of the strong spin fluctuations has often been pointed out especially for underdoped samples. For example, we have discussed a possible relevance of the strong spin fluctuation to the suppression of $T_c$ in an underdoped region.\(^{28}\) Therefore, the giant isotope shift in the present result for case (a) may explain the experimental results. A more detailed discussion on this subject would be presented in a separate paper.

In conclusion, we have proposed a model of superconductors with coexisting interactions of phonon and nonphonon origins. In the presence of the Coulomb interaction, the extended formulas for $T_c$ and $\alpha$, and a phase diagram are obtained. It is found that the inverse isotope effect occurs due to the cancellation of the attractive and repulsive interactions, where one of them needs to be the phonon interaction, while the other the nonphonon interaction with a larger effective cutoff energy than the Debye frequency. When the phonon interaction is attractive, any additional nonphonon pairing interaction reduces the value of $\alpha$, but does not make $\alpha$ negative. It is also found that a strong critical spin fluctuation with $\omega_{\text{sf}} < \omega_D$ could give rise to a giant isotope shift.

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1) P. W. Anderson and W. F. Brinkman: Phys. Rev. Lett. 30 (1973) 1108; S. Nakajima: Prog. Theor. Phys. 50 (1973) 1101.
2) D. J. Scalapino, E. Loh, Jr. and J. E. Hirsch: Phys. Rev. B 34 (1986) 8190; K. Miyake, S. Schmitt-Rink and C. M. Varma: Phys. Rev. B 34 (1986) 6554.
3) V. J. Emery: Synth. Met. 13 (1986) 21.
4) M. T. Beal-Monod, C. Bourbonnais and V. J. Emery: Phys. Rev. B 34 (1986) 7716.
5) H. Shimahara: J. Phys. Soc. Jpn. 58 (1989) 1735; H. Shimahara: Proceeding of the Physics and Chemistry of Organic Superconductors, eds. G. Saito and S. Kagoshima (Springer-Verlag, Berlin, Heidelberg, New York, 1990) p. 73.
6) As works on the spin fluctuation mechanism proposed at an early stage after the discovery of the high-$T_c$ cuprates, for example, see [K. Miyake, T. Matsuura, K. Sano and Y. Nagaoka: J. Phys. Soc. Jpn. 57 (1988) 722] and ref. 20.
7) T. M. Rice and M. Sigrist: J. Phys. Condens. Matter 7 (1995) L643; I. I. Mazin and D. J. Singh: Phys. Rev. Lett. 79 (1997) 733; Phys. Rev. Lett. 82 (1999) 4324.
8) B. M. Klein and R. E. Cohen: Phys. Rev. B 45 (1992) 12405, and references therein.
9) J. A. Schlueter et al.: Physica C 265 (1996) 163.
10) A. M. Kini et al.: Physica C 264 (1996) 81.
11) See references in refs. 9 and 10.
12) C.-P. Heidmann et al.: Physica B 143 (1986) 357; K. Andres et al.: Physica B 143 (1986) 334.
13) K. Oshima et al.: J. Phys. Soc. Jpn. 57 (1988) 730.
14) H. Ito et al.: J. Phys. Soc. Jpn. 60 (1991) 3230.
15) M. Tokumoto et al.: J. Phys. Soc. Jpn. 60 (1991) 1426.
16) Z. Q. Mao et al.: Phys. Rev. B 63 (2001) 144514.
17) I. F. Foulkes and B. L. Gyorgyi: Phys. Rev. B 15 (1977) 1395.
18) H. Shimahara and M. Kohmoto: Europhys. Lett. 57 (2002) 247.
19) H. Shimahara and M. Kohmoto: Phys. Rev. B 65 (2002) 174502.
20) H. Shimahara and S. Takada: J. Phys. Soc. Jpn. 57 (1988) 1044.
21) H. Shimahara: cond-mat/0304516; to be published in J. Phys. Soc. Jpn. 72, No. 8 (2003).
22) A similar formula of $T_c$ has been obtained by Yamaji in a model of s-wave pairing superconductivity induced by the pairing interaction mediated by intramolecular vibration modes. [K. Yamaji: Solid State Commun. 61 (1987) 413; T. Ishiguro and K. Yamaji: Organic Superconductors (Springer, Berlin, Heidelberg, 1990)].
23) Since $\omega_D \sim 410$ K\(^{29}\) and $\phi_F \sim 1.5$ eV,\(^{29,30}\) case (a) is realized when $\omega_D/\phi_F \lesssim 0.024$. It corresponds to the peak width of the $\chi(q, \omega)$ $\Delta q \lesssim 0.024 \times \pi/a$, where $a$ denotes a length scale of the order of the lattice constants. Then, the magnetic correlation length $\xi_{\text{mag}}$ needs to satisfy $\xi_{\text{mag}} \sim 1/\Delta q \sim 14 \times a$, which would not be realized in Sr$_2$RuO$_4$.
24) If $\lambda_{\text{ph}} < 0$ occurs in Sr$_2$RuO$_4$, we obtain the reverse conclusion. As far as the author’s knowledge, however, there is no study which shows $\lambda_{\text{ph}} < 0$ in Sr$_2$RuO$_4$ at the present.
25) N. Kikugawa and Y. Maeno: Phys. Rev. Lett. 89 (2002) 117001.
26) Many theories have been proposed for Sr$_2$RuO$_4$. In addition to ref. 7, see also, for example, T. Kuwabara and M. Ogata: Phys. Rev. Lett. 85 (2000) 4588; M. Sato and M. Kohmoto: J. Phys. Soc. Jpn. 69 (2000) 3505; T. Nomura and K. Yamada: J. Phys. Soc. Jpn. 69 (2000) 3678; M. E. Zhitomirsky and T. M. Rice: Phys. Rev. Lett. 87 (2001) 057001, and references therein.
27) M. K. Crawford et al.: Phys. Rev. B 41 (1990) 282.
28) H. Shimahara, Y. Hasegawa and M. Kohmoto: J. Phys. Soc. Jpn. 69 (2000) 1598.
29) Y. Maeno et al.: J. Phys. Soc. Jpn. 66 (1997) 1405.
30) T. Oguchi: Phys. Rev. B 51 (1995) R1385; D. J. Singh: Phys. Rev. B 52 (1995) R1385.