Higher-Rank Supersymmetry
and
Topological Field Theory*

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Abstract

The $N = 2$ minimal superconformal model can be twisted yielding an example of topological conformal field theory. In this article we investigate a Lie theoretic extension of this process.

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1. Introduction

The twisting of the $N = 2$ minimal superconformal model \([1]\) gives rise to a fundamental class of topological field theories \([2]\). Topological conformal field theories (TCFT) realized as the topological version of certain $N = 2$ models exhibit remarkable properties such as the existence of the flat coordinate system and the prepotential \([3]\). It is thus quite interesting to ask to what extent one can generalize the relationship between TCFT and $N = 2$ superconformal theories. In this contribution we wish to point out that there is a natural Lie theoretic extension of the $N = 2$ superconformal algebra which makes us possible to construct a wider class of TCFT.

This paper is organized as follows. In sect. 2 we introduce the higher-rank generalization of the $N = 2$ minimal model. In sect. 3 we show that the original idea of the chirality in $N = 2$ theory can be extended in our model. In sect. 4, applying the twisting procedure, we discuss topological properties. In sect. 5 we sketch how to evaluate multiple integrations in order to demonstrate the BRST exactness of the twisted stress tensor. In sect. 6 we present some concluding remarks.

2. Higher-Rank Supersymmetry

The close resemblance between the $N = 2$ minimal model and the $SU(2)$ WZNW model has been well recognized. If one considers an arbitrary simple Lie algebra, motivated by the $SU(2)$ paradigm, the ‘higher-rank’ analog of the $N = 2$ minimal model appears \([4,5]\). This is our model based on which we shall construct TCFT. The model is described by a familiar system $(g_{-\text{parafermion}})_k \times \text{(boson)}^n$ with the special values of Coulomb gas parameters

$$
\alpha_+ = \sqrt{\frac{k+g}{kg}}, \quad \alpha_+ \alpha_- = -\frac{1}{k}, \quad \alpha_0 = \alpha_+ + \alpha_-.
$$

(2.1)

Here $g$ is a simple Lie algebra of rank $n$, the level $k$ is a positive integer, and $g$ is the dual Coxeter number of $g$. (We follow the convention of \([4]\).) The $N = 2$ minimal model corresponds to the case $g = A_1$.

The existence of ‘exotic supersymmetry’ in this system has been expected through the analysis of the branching relations associated with the coset model $g_k \oplus U(1)^n / U(1)^n$ \([1]\). The generators of this symmetry are given by\footnote{Throughout this paper we suppress normal orderings.}

$$
G^\alpha(z) := \psi_\alpha(z)e^{i\alpha_+ \cdot \varphi(z)}, \quad \alpha \in \Delta
$$

(2.2)
where $\Delta$ is the set of roots of $g$, $\psi_\alpha(z)$ are the generating parafermions, and $\varphi(z) = (\varphi_1(z), \ldots, \varphi_n(z))$ are free bosons with their two-point functions $\langle \varphi_a(z) \varphi_b(w) \rangle = -\delta_{ab} \log(z-w)$. The stress tensor consists of the parafermion piece $T_{\text{para}}(z)$ and the free boson piece

$$T(z) = T_{\text{para}}(z) - \frac{1}{2}(\partial \varphi(z))^2 = \sum_{n \in \mathbb{Z}} z^{-n-2} L_n .$$

(2.3)

The central charge $c$ is given by $c = \frac{kd}{k+g}$ with $d = \text{dim} \, g$. Since $\psi_\alpha(z)$ has a conformal weight $1 - \frac{\alpha^2}{2k}$, the conformal weight of $G^\alpha(z)$ is $1 + \frac{\alpha^2}{2g}$. There are also $U(1)^n$ currents which we normalize by

$$J^a(z) = \frac{i}{\alpha_+} \partial \varphi_a(z) = \sum_{n \in \mathbb{Z}} z^{-n-1} J^a_n , \quad a = 1, \ldots, n .$$

(2.4)

The chiral algebra generated by $G^\alpha(z)$, $T(z)$ and $J^a(z)$ is closed and associative by construction. We will refer to this symmetry as the higher-rank supersymmetry. The structure of this algebra is, in general, complicated due to nonlocality as well as nonlinearity, but, as we will see below, the twisting procedure can be studied without much difficulty since we do not need the whole algebra.

We now describe the reason why we think of this chiral algebra as the higher-rank extension of the $N = 2$ algebra. The crucial ingredient in the $N = 2$ model is the spectral flow \cite{8} and the chiral ring \cite{9}. These properties are in fact realized in our present model. Let us first consider the spectral flow. The existence of $U(1)^n$ currents implies an $n$-parameter spectral flow. We denote the flow operator by $U_{\bar{\eta}}$ with $\bar{\eta}$ being an $n$-dimensional flow vector. One finds an inner automorphism of the algebra

$$U_{\bar{\eta}} L_n U_{\bar{\eta}}^{-1} = L_n + \bar{\eta} \cdot J_n + \frac{|\bar{\eta}|^2}{2\alpha_+^2} \delta_{n,0} ,$$

$$U_{\bar{\eta}} J^a_n U_{\bar{\eta}}^{-1} = J^a_n + \frac{\bar{\eta}^a}{\alpha_+} \delta_{n,0} ,$$

$$U_{\bar{\eta}} G^\alpha_r U_{\bar{\eta}}^{-1} = G^\alpha_{r+\bar{\eta} \cdot \alpha} ,$$

(2.5)

See, however, Appendix where we find a connection to the Zamolodchikov-Fateev spin-$4/3$ current algebra, when $g = A_2$. 

\[2\]
where
\[ G_\alpha(z) = \sum_r z^{-r-\left(1+\frac{\alpha^2}{2g}\right)} G_\alpha^r. \] (2.6)

Specializing \( \bar{\eta} = \frac{\eta}{g} \rho \) with \( \rho \) being half the sum of the positive roots of \( g \) yields a one-parameter flow
\[
\mathcal{U}_{\bar{\eta}} L_n \mathcal{U}_{\bar{\eta}}^{-1} = L_n + \frac{\eta}{g} \rho \cdot J_n + \frac{c}{24} \eta^2 \delta_{n,0},
\]
\[
\mathcal{U}_{\bar{\eta}} J_n \mathcal{U}_{\bar{\eta}}^{-1} = J_n^{\alpha} + \frac{c}{d} \eta \rho \delta_{n,0},
\] (2.7)
\[
\mathcal{U}_{\bar{\eta}} G_\alpha \mathcal{U}_{\bar{\eta}}^{-1} = G_\alpha^{\rho+\frac{\eta}{g} \rho \delta_{n,0}},
\]

where the strange formula \( 12\rho^2 = gd \) has been used. As in \( N = 2 \) theories this flow is important when we consider the analogs of the chiral ring and the Ramond ground states of the present system in the next section.

3. Chiral Primary Fields

In \( N = 2 \) superconformal field theory chiral primary fields play a distinguished role \[3\]. In particular, they become BRST invariant physical observables after twisting \[1, 2]–\[4\]. To find analogs of chiral primary fields, for which we use the same terminology in the following, let us define the NS sector as the set of fields \( A_{NS}(z) \) which are local with respect to the fractional-spin currents \( G_\alpha(z) \). Then
\[
G_\alpha(z) A_{NS}(0) = \sum_{r \in \mathbb{Z} - \frac{\alpha^2}{2g}} z^{-r-\left(1+\frac{\alpha^2}{2g}\right)} \left(G_\alpha^r A_{NS}\right)(0). \] (3.1)
The NS primary states are annihilated by all the positive modes, i.e. \( L_n, J_n^\alpha \) and \( G_\alpha^{\rho+\frac{\eta}{g} \rho} \) with \( n > 0 \). Chiral fields \( \Phi \) are those in the NS sector obeying\[3\]
\[
\oint dz \ G^{\alpha_i}(z) \Phi(0) = 0, \quad i = 1, \ldots, n, \] (3.2)

for each simple root \( \alpha_i \) of \( g \). Chiral primary fields satisfy both (3.2) and the primary condition. These are explicitly obtained as
\[
\Phi_\Lambda(z) = \phi_\Lambda(z) e^{-i\alpha \cdot \Lambda \cdot \varphi(z)}, \quad \Lambda \in \mathbb{P}^k_+, \] (3.3)

---

3 We note that there are as many similar sets of conditions as the order of the Weyl group of \( g \) corresponding to different choices of the simple root system. Hence for each such choice one can repeat the whole story in the sequel.
where $\phi^\Lambda_\Lambda$ is a parafermionic primary field with a conformal weight
\[
\frac{1}{2(k + g)}(\Lambda + 2\rho) \cdot \Lambda - \frac{\Lambda^2}{2k},
\]
and $P^k_+$ is the set of dominant weights appearing in the level $k$ WZNW model. Although one could consider $\Phi_\Lambda$ for any values of the Coulomb gas parameters as a deformation of the highest weight primary field of the WZNW model, one special feature arising from (2.1) is that the conformal weight and $U(1)^n$ charges of $\Phi_\Lambda$ are linearly related:
\[
L_0|\Phi_\Lambda\rangle = \frac{1}{g}\rho \cdot J_0|\Phi_\Lambda\rangle,
\]
with
\[
J_0^a|\Phi_\Lambda\rangle = \frac{g\Lambda^a}{k + g}|\Phi_\Lambda\rangle.
\]
Then one can adopt the conventional argument (found e.g. in [9]) and deduce that there are no short distance singularities between two chiral primary fields:
\[
\Phi_\Lambda(z)\Phi_{\Lambda'}(w) \sim \Phi_{\Lambda + \Lambda'}(w),
\]
where we should understand that $\Phi_{\Lambda + \Lambda'}(w) = 0$ if $\Lambda + \Lambda' \not\in P^k_+$. Thus our chiral primary fields define a finite nilpotent ring.

Let us apply the flow (2.7) with $\eta = 1$ to the NS sector. The chiral primary states $|\Phi_\Lambda\rangle$ are then mapped onto the R ground states which have a conformal weight $h$ and $U(1)^n$ charges $q$ given by
\[
h = \frac{c}{24}, \quad q = \frac{g}{k + g}(\Lambda + \rho) - \rho.
\]
These states are indeed responsible for the non-vanishing contribution to the ‘generalized index’ [4].

4. Topological Version

Starting with the higher-rank supersymmetric model introduced in the previous sections let us now construct TCFT. The first step is to twist the model through the redefinition of the stress tensor
\[
T(z) \to \hat{T}(z) = T(z) + \frac{1}{g}\rho \cdot J(z).
\]
The new stress tensor $\hat{T}(z)$ satisfies the Virasoro algebra with a vanishing central charge. Among the supercurrents $G^\alpha(z)$ with $\alpha \in \Delta$ we see that the currents $G^{\alpha_j}(z)$ with simple roots $\alpha_j$ acquire conformal weights 1 with respect to the new stress tensor. Hence BRST operators in the topological version will be constructed from $G^{\alpha_j}(z)$. The BRST structure of the theory is governed by a directed graph essentially determined by the affine Weyl group. One associates a BRST charge to each arrow, the explicit form of which is given by a multiple integral of $G^{\alpha_j}$ and depends on the location of the arrow in the graph.

In TCFT the stress tensor is expressed as a BRST commutator

$$\hat{T}(z) = \{Q, G(z)\}. \quad (4.2)$$

For $G(z)$ we take the current $G^{-\theta}(z)$ whose conformal weight is 2 with respect to $\hat{T}(z)$ and the explicit expression of the BRST commutator was proposed in [10]:

$$\hat{T}(z) = (\text{const.}) T_s(z)$$

$$T_s(z) := \int \cdots \int_{\Gamma} \prod_{i=1}^{n} \prod_{j=1}^{a_i} dz_i^{(j)} G_{\alpha_i}^{\alpha^i}(z_i^{(j)}) G^{-\theta}(z), \quad (4.3)$$

where $\theta = \sum_{i=1}^{n} a_i \alpha_i$ is the highest root of $g$. The contour $\Gamma$ of the integrals must be chosen appropriately [11]. The evaluation of $T_s(z)$ will be studied in the next section.

The physical states in our topological theory are realized as the non-vanishing BRST cohomology. Though the analysis of the BRST complex is quite complicated, the index calculation based on the branching relation shows that the chiral primary fields $\Phi_{\Lambda}(z)$, $\Lambda \in P_k^+$ turn out to be the basic physical observables. In fact the BRST invariance of $\Phi_{\Lambda}(z)$ immediately follows from the chirality condition (3.2) since the action of BRST charges on $\Phi_{\Lambda}(z)$ is given by

$$0 = \{Q_{(j)}, \Phi_{\Lambda}(z)\} = \int_z dw G_{\alpha^j}^{\alpha^j}(w) \Phi_{\Lambda}(z), \quad j = 1, \ldots, n. \quad (4.4)$$

All these states have zero conformal weights with respect to $\hat{T}(z)$ by virtue of (3.5). Thus the physical states are labeled by the $U(1)^n$ charge (3.6).

Let us take a look at correlation functions. Since the stress tensor takes the BRST exact form (1.3), correlators of the basic physical operators are independent of their world sheet positions, and hence there is no notion of distance in the theory. This is a familiar phenomenon observed in twisted $N = 2$ theories [1–3]. We also note that upon twisting we have coupled the system to a ‘background charge at infinity’. The selection rule of $U(1)^n$ charges arising from this features our topological field theory. We intend to further discuss the properties of correlation functions, including perturbed behaviors, in a subsequent paper [12].
5. Multiple Integrals

In this section we briefly comment on the validity of (4.3). For simplicity, consider the case $g = A_n$ and $k = 1$. By a standard calculation we obtain

$$T_s(z) = \int \prod_{j=1}^{n} dz_j \left[ \prod_{j=1}^{n-1} (z_j - z_{j+1})(z_1 - z)(z_n - z) \right]^{-\alpha_+^2}$$

$$\times \exp \left[ i\alpha_+ \sum_{j=1}^{n} \alpha_j \cdot (\varphi(z_j) - \varphi(z)) \right],$$

where $\alpha_+^2 = \frac{n+2}{n+1}$ and we have used the fact $\theta = \sum_{j=1}^{n} \alpha_j$. If we make a change of variables $z_j \to z + \prod_{l=1}^{j} u_l$, $j = 1, \ldots, n$ and integrate over $u_1$, we find, after standard manipulations of contours, that

$$T_s(z) \propto \prod_{j=2}^{n} \left[ \oint du_j u_j^{(n+1)-1} (1 - u_j)^{-\frac{1}{n+1}-1} \right]$$

$$\times \left[ -\frac{1}{2} \left( \alpha_1 \cdot \partial \varphi(z) + \sum_{j=2}^{n} (u_2 \cdots u_j) \alpha_j \cdot \partial \varphi(z) \right)^2 \right. \left. + \frac{i}{2} (n+1) \alpha_0 \left( \alpha_1 \cdot \partial^2 \varphi(z) + \sum_{j=2}^{n} (u_2 \cdots u_j)^2 \alpha_j \cdot \partial^2 \varphi(z) \right) \right].$$

By repeatedly applying the recursion property of the beta function,

$$B(a, b) = \int dt t^{a-1} (1 - t)^{b-1}, \quad B(a + 1, b) = \frac{a}{a + b} B(a, b),$$

it is easy to see that

$$T_s(z) \propto -\frac{1}{2} \sum_{i,j=1}^{n} (\omega_i \cdot \omega_j) (\alpha_i \cdot \partial \varphi)(\alpha_j \cdot \partial \varphi)(z) + i\alpha_0 \rho \cdot \partial^2 \varphi(z),$$

where $\omega_1, \ldots, \omega_n$ are the fundamental weights of $A_n$ and

$$\omega_i \cdot \omega_j = \min(i, j) - \frac{ij}{n+1},$$

$$2\rho = \sum_{i=1}^{n} i(n + 1 - i) \alpha_i.$$
Finally using the formula
\[ \sum_{i,j=1}^{n} (\omega_i \cdot \omega_j)(\alpha_i)^a(\alpha_j)^b = \delta_{ab}, \] (5.6)
which follows from \( \sum_{a=1}^{n}(\alpha_i)^a(\omega_j)^a = \delta_{ij} \) and \( \sum_{i=1}^{n}(\alpha_i)^a(\omega_i)^b = \delta_{ab} \) we arrive at
\[ T_k(z) \propto -\frac{1}{2}(\partial \varphi(z))^2 + i\alpha_0 \rho \cdot \partial^2 \varphi(z). \] (5.7)

For other Lie algebras we encounter more complicated integrals. For instance, in the case \( g = D_n \) we have to deal with lower moments with respect to a two-variable Selberg density to derive the desired result. Further details of the calculation will be reported elsewhere [12].

### 6. Concluding Remarks

We have seen that our higher-rank supersymmetric model is a fairly natural extension of the \( N = 2 \) minimal model and its topological version possesses all the properties characteristic to TCFT.

Let us point out some interesting issues which remain to be properly understood. One may notice that the modified stress tensor \( \hat{T}(z) \) is that describing the \( g_k \oplus g_0/g_k \simeq g_k/g_k \) coset theory. According to recent works [13,14] it has been established that correlation functions in topological \( g_k/g_k \) model yield the fusion algebra of the WZNW model \( g_k \). Furthermore, this result can be understood in terms of the deformed chiral ring of topological \( N = 2 \) theory [15,16]. Thus it will be significant to clarify whether the present chiral primary ring, after certain deformation, has any relevance to the \( g_k \) fusion algebra [12].

It should also be mentioned that the higher-rank supersymmetric models are realized in the critical limit of solvable vertex-type lattice models [17]. Quite recently, Saleur [18] has observed an interesting structure of solvable lattice models whose critical behaviors are described by the \( N = 2 \) minimal model. It seems important to seek a similar correspondence in our higher-rank setting.

Finally we note that the parallelism with \( N = 2 \) supersymmetry is yet to be completed. What is crucially missing in higher-rank supersymmetry is its possible connection to certain geometry if there is any. It would be very exciting if one can find a geometric interpretation which could be an analog of the deep relation between Ricci-flat Kähler manifolds and \( N = 2 \) superconformal field theories [19].
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Appendix.

The Zamolodchikov-Fateev spin-$4/3$ algebra $[20]$ is generated by two spin $4/3$ currents $\psi(z)$ and $\psi^*(z)$ which satisfy the operator product expansions

$$
\psi(z)\psi(w) \sim \frac{\lambda}{(z-w)^{4/3}} \left[ \psi^*(w) + \frac{1}{2} (z-w) \partial \psi^*(w) + \cdots \right],
$$

$$
\psi^*(z)\psi^*(w) \sim \frac{\lambda}{(z-w)^{4/3}} \left[ \psi^*(w) + \frac{1}{2} (z-w) \partial \psi^*(w) + \cdots \right], \quad (A.1)
$$

$$
\psi(z)\psi^*(w) \sim \frac{1}{(z-w)^{8/3}} \left[ I + \frac{8}{3c} (z-w)^2 T(w) + \cdots \right].
$$

The associativity gives a constraint $[20]$

$$
9\lambda^2 c = 4(8 - c), \quad (A.2)
$$

which can be parametrized as

$$
c = 2 \left( 1 - \frac{12}{v(v+4)} \right), \quad \lambda^2 = \frac{1}{3} \left( \frac{v+2}{(v-2)(v+6)} \right). \quad (A.3)
$$

The minimal unitary series corresponds to $v = 2, 3, \ldots$ and is realized by the GKO coset $(A_1)_4 \oplus (A_1)_{v-2}/(A_1)_{v+2}$.

We present here another realization of this algebra. Consider $G^\alpha(z), \alpha \in \Delta$, defined in (2.2) for $g = A_2$. Note that all these currents have spin $4/3$. The level-$k$ $A_2$-parafermion currents $\psi^\alpha(z)$ generate the algebra $[37]$

$$
\psi^\alpha(z)\psi^\beta(w) \sim \frac{K_{\alpha,\beta}}{(z-w)^{1+\alpha+\beta/k}} \left[ \psi^\alpha+\beta(w) + \frac{1}{2} (z-w) \partial \psi^\alpha+\beta(w) + \cdots \right],
$$

$$
\psi^\alpha(z)\psi^-\alpha(w) \sim \frac{1}{(z-w)^{2-\alpha^2/k}} \left[ I + \frac{k+3}{3k} (z-w)^2 T_{\text{para}}(w) + \cdots \right], \quad (A.4)
$$
where $K_{\alpha,\beta} = K_{\beta,\alpha}$, $K_{\alpha_1,\alpha_2} = K_{\alpha_1,-\theta} = K_{\alpha_2,-\theta} = 1/\sqrt{k}$ and otherwise zero. It is then straightforward to check that

$$\psi(z) = \frac{1}{\sqrt{3}} \sum_{\alpha=\alpha_1,\alpha_2,-\theta} G^\alpha(z),$$

$$\psi^*(z) = \frac{1}{\sqrt{3}} \sum_{\alpha=\alpha_1,\alpha_2,-\theta} G^{-\alpha}(z),$$

$$T(z) = T_{\text{para}}(z) - \frac{1}{2}(\partial \varphi(z))^2,$$

satisfy (A.1) with

$$c = \frac{8k}{k+3}, \quad \lambda = \frac{2}{\sqrt{3k}},$$

The associativity is clear by construction. This construction is rather reminiscent of making an $N = 1$ superconformal generator out of two $N = 2$ superconformal generators.

\[4\] For $k = 1$, this realization coincides with that considered in [21].
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