High-speed method for analyzing shielding current density in HTS with cracks: implementation of \( H \)-matrix method to GMRES

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Abstract. A high-speed analysis of a shielding current density in a high-temperature superconducting film containing cracks has been proposed. To this end, a numerical code has been developed for analyzing the shielding current density in an HTS film. In the code, the GMRES\((k)\) with the \( H \)-matrix method has been implemented to solve a linear-system obtained by discretizing an initial-boundary-value problem of a nonlinear integro-differential equation with respect to time and space. The results of the computations show that, if the Jacobi scaling is not applied to the linear-system, the Newton method does not converge at a certain time step. Therefore, the Jacobi scaling is an necessary tool for analyzing the shielding current density. Furthermore, by using the GMRES\((k)\) method with the \( H \)-matrix method as a linear-system solver, the time evolution of the shielding current can be analyzed with high speed. In conclusion, the speed of the GMRES\((k)\) with \( H \)-matrix method is about 221.3 times as fast as that of the \( LU \) decomposition.

Keywords: Finite element method, GMRES\((k)\) method, \( H \)-matrix method, High-temperature superconductor, Multiple cracks, Permanent magnet

1. Introduction

As is well known, a critical current density \( j_\text{C} \) is one of the most important parameters for engineering applications of high-temperature superconductors (HTSSs). In general, the standard four-probe method for measuring \( j_\text{C} \) has been employed. The procedures of the method are as follows: firstly, a large-area pad must be coated with silver to reduce the contact resistance between a HTS sample and the current lead. Secondary, the metallic coating requires the heat process. However, it may lead to the degradation of superconducting characteristics or to the damage of a sample surface. For this reason, contactless methods for measuring \( j_\text{C} \) have been so far desired.
For the purpose of contactlessly measuring \( j_C \) in an HTS film, Claassen et al. have proposed the inductive method [1]. By applying an ac current to a small coil placed just above an HTS film, they monitored a harmonic voltage induced in the coil. They found that, only when a coil current exceeds a threshold current, the third-harmonic voltage develops suddenly. They conclude that \( j_C \) can be evaluated from the threshold current. On the other hand, Mawatari et al. elucidated the inductive method on the basis of the critical state model [2]. From their results, they derived a theoretical formula between \( j_C \) and threshold current. In general, this method has been successfully employed as the measurement of the \( j_C \)-distributions in the film and as the detection of the cracks [3].

Ohshima et al. proposed another contactless method for measuring \( j_C \) [4, 5]. In the method, while moving a permanent magnet above an HTS film, the electromagnetic force acting on the film is measured. As a result, they found that the maximum repulsive force \( F_M \) is roughly proportional to \( j_C \). This means that \( j_C \) is estimated from the measured value of \( F_M \). This method is called the permanent magnet method. By using the method, the \( j_C \)-distributions are also determined [6]. In addition, the permanent magnet method has been successfully applied to the detection of cracks in an HTS film [7].

Recently, Hattori et al. have proposed a contactless and a high-speed measuring methods of \( j_C \) in an HTS film [8]. In the following, this is called a scanning permanent magnet method. In the method, a permanent magnet is moved along the HTS surface, and subsequently they measure an electromagnetic force \( F_z \) acting on the film. Consequently, they found that a \( j_C \)-distribution can be estimated from a \( F_z \)-distribution with a high-speed.

In order to numerically re-create the inductive method, the conventional/scanning permanent magnet methods, it is necessary to determine the time evolution of a shielding current density. As is well known, the macroscopic behavior of the shielding current density is expressed by the initial-boundary-value problem of a nonlinear integro-differential equation. By discretizing with respect to time and space, the problem is reduced to the simultaneous nonlinear equations at each time step. When we adopt the Newton method as a solver of the nonlinear equations [9, 10, 11, 12], we can obtain the linear-system of the dimension \( L \times L \). In the previous study, we require the operation count \( O(L^3) \) due to the use of the \( LU \) decomposition as the linear-system solver. We should consider the further high-speed method of the shielding current analysis for a large-scale problem.

The purpose of the present study is to propose the acceleration method for analyzing a shielding current density in an HTS film containing the cracks. For this purpose, we use the GMRES\((k)\) method as a solver of the linear-system and implement the \( H \)-matrix method [13, 14, 15, 16] to the GMRES. In addition, we investigate the performance of the GMRES\((k)\) with \( H \)-matrix method by simulating the scanning permanent magnet method.

2. Governing Equations

In the scanning, a cylindrical permanent magnet is placed in relation to a film surface at a constant distance \( d \). It is moved in the direction parallel to the surface. We suppose a rectangle-shaped HTS film of length \( l \), width \( w \), and thickness \( b \).

Throughout the present study, we use the Cartesian coordinate system \( \langle O : e_x, e_y, e_z \rangle \),
where the z-axis is parallel to the thickness direction and the origin \(O\) is the centroid of the film. In terms of the coordinate system, the symmetry axis of the permanent magnet is expressed as \((x, y) = (x_m, y_m)\). Moreover, the movement of the magnet is assumed as \(x_m(t) = -vt + l/2\), where \(v\) is the magnitude of the scanning velocity. For the purpose of characterizing the strength of the magnet, we use a magnetic flux density \(B_F\) at \((x, y, z) = (0, 0, b/2)\) for the case with \(v = 0\) mm/s.

We suppose that a rectangle cross-section of the film is denoted by \(\Omega\) and the film includes \(M\) pieces of the crack. If the multiple cracks are contained in the film, \(\Omega\) has not only the outer boundary \(C_0\) but the inner boundaries \(C_i (i = 1, 2, \cdots, M)\). Otherwise, the boundary of \(\Omega\) consists of the outer boundary \(C_0\) only. Also, a normal unit vector and a tangential unit vector on \(C_i\) are denoted by \(n\) and \(t\), respectively.

As is well known, YBCO superconductors have a strong crystallographic anisotropy: shielding current flow in the \(c\)-axis direction differs from that in the \(ab\)-plane, and the flow along \(c\)-axis is almost negligible. Here, the \(c\)-axis is the direction along \(z\), and it is perpendicular to the \(ab\)-plane. On the basis of the fact, we assume the thin-layer approximation [9]: the thickness of the HTS film is sufficiently thin that a shielding current density can hardly flow in the thickness direction.

Under the above assumptions, the shielding current density \(j\) can be written as

\[
j = \frac{2}{b} \nabla S \times e_z, \tag{1}\]

and the time evolution of the scalar function \(S(x, t)\) is governed by the following integro-differential equation [9]:

\[
\frac{\partial}{\partial t} (\hat{W}S) + (\nabla \times E) \cdot e_z = -\frac{\partial}{\partial t} \langle B \cdot e_z \rangle, \tag{2}\]

where a bracket \(\langle \rangle\) is an average operator over the thickness of the HTS film and is defined by

\[
\langle f \rangle \equiv \frac{1}{b} \int_{-b/2}^{b/2} f dz. \tag{3}\]

In addition, \(B\) and \(E\) denote an applied magnetic flux density and an electric field, respectively. \(\hat{W}\) is defined by

\[
\hat{W}S \equiv \int_{\Omega} Q(|x - x'|)S(x', t)d^2x' + \frac{2}{b} S(x, t). \tag{4}\]

Here, both \(x\) and \(x'\) are position vectors in the \(xy\)-plane. The explicit form of a function \(Q(r)\) is given by

\[
Q(r) = -\frac{1}{\pi b^2} \left( \frac{1}{r} - \frac{1}{\sqrt{r^2 + b^2}} \right). \tag{5}\]

It must be noted that \(Q(r)\) becomes singular at \(r = 0\). Therefore, it is clearly necessary to evaluate the first right-hand side of (5) as an improper integral. The high-performance method of this integral is described in Ref. [10].
Table 1: The geometrical and the physical parameters

| Symbol | Quantity Description | Value       |
|--------|----------------------|-------------|
| \( l \) | length of HTS        | 36 mm       |
| \( w \) | width of HTS         | 12 mm       |
| \( b \) | thickness of HTS     | 1 \( \mu \)m |
| \( j_C \) | critical current density | 2.5 MA/cm\(^2\) |
| \( E_C \) | critical electrical field | 1 mV/m |
| \( \alpha \) | degree of power law  | 20          |
| \( d \) | distance between magnet and film | 0.5 mm |
| \( R \) | radius of magnet     | 0.8 mm      |
| \( H \) | height of magnet     | 2 mm        |
| \( v \) | velocity of magnet   | 10 cm/s     |
| \( B_F \) | magnetic flux density | 0.1 T       |
| \( y_m \) | y-coordinate of magnet | 0 mm      |
| \( L_c \) | crack size           | 2.4 mm      |
| \( x_c \) | x-coordinate of crack | \( \pm 6 \) mm |
| \( y_c \) | y-coordinate of crack | 0 mm       |

As is well known, the shielding current density \( j \) is closely related to the electric field \( E \). The relation can be written in the form

\[
E = E(|j|)\left(\frac{j}{|j|}\right),
\]  

(6)

As a function \( E(j) \), we adopt the power law

\[
E(j) = E_C\left(\frac{j}{j_C}\right)^\alpha,
\]  

(7)

where \( E_C \) is the critical electric field and \( \alpha \) is a degree of power law.

The initial boundary to eq. (2) is given by

\[
S = 0 \text{ at } t = 0.
\]  

(8)

Boundary conditions are set as follows:

\[
S = 0 \text{ on } C_i,
\]  

(9)

\[
\frac{\partial S}{\partial s} = 0 \text{ on } C_i,
\]  

(10)

\[
\oint_{C_i} E \cdot t ds = 0,
\]  

(11)

where \( s \) is an arclength along \( C_i \). The boundary condition (11) can be obtained by rewriting the integral form of Faraday’s law. By solving the initial-boundary-value problem of (2), the
time evolution of the shielding current density can be determined for the film containing the multiple cracks.

Throughout the present study, we assume that the two cracks are contained in the HTS film and they have the same size and shape. In addition, the crack shapes and positions are the line segments and parallel to the $x$-axis at the center point $(x, y) = (x_c, y_c)$, respectively. Also, let $L_c$ be the crack size. Values of the geometrical and the physical parameters are tabulated in Table 1.

In Figs. 1 (a) and (b), we show the examples of a shielding current density $j$ distribution and an electromagnetic force $F_z$. This figure indicates that the shielding current density $j$ flows along the cracks. Furthermore, the constant electromagnetic force $F_z$ is observed for the case where there exists no crack. Otherwise, the value of $F_z$ changes drastically near the cracks and the film edges.

Figure 1: The examples of (a) the distribution of the shielding current density $j$ at $x_m = 4.5$ mm and (b) the dependence of the electromagnetic force $F_z$ on the magnet position $x_m$. Here, the green circle indicates the orthographic projection of the permanent magnet and the blue thick line shows the cracks. Furthermore, the red arrow is the direction of the shielding current density $j$.  

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3. Numerical method and its performance

3.1. Discretization

The initial-boundary-value problem of (2) is discretized with respect to space and time by using the finite element method (FEM) and the backward Euler method, respectively. In the present study, we divide the region $\Omega$ into a set of square elements $\Omega_1, \Omega_2, \cdots, \Omega_{N_e}$, i.e.,

$$\Omega \approx \bigcup_{e=1}^{N_e} \Omega_e \quad (12)$$

where $\Omega_e$ is $e$th element. Furthermore, the number of nodes is denoted by $N$. The inner boundary $C_i$ ($i = 1, 2, \cdots, M$) is characterized by the number $K$ of the line segments. On the other hand, the time domain $[0, \ell = v]$ is equally divided into $N_t$ steps. In the following, the time step size is denoted by $\Delta t = [v/N_t]$, and its value is given by $N_t = 1200$.

Firstly, let us discretize the initial-boundary-value problem of (2) with respect to time. By means of the backward Euler method, the initial-boundary-value problem of (2) is transformed into the following nonlinear boundary-value problem [11]:

\[
\begin{align*}
\mu_0 \dot{W} S + \Delta t (\nabla \times E) \cdot e_z &= u \quad \text{in} \quad \Omega \\
S &= 0 \quad \text{on} \quad C_0 \\
\frac{\partial S}{\partial s} &= 0 \quad \text{on} \quad C_i \\
\int_{C_i} E \cdot t ds &= \phi_i \\
\gamma_i(E) &= 0
\end{align*}
\]

Here, $u$ is defined by $u \equiv \mu_0 \dot{W} s^{(n-1)} - (B^{(n)} \cdot e_z - \langle B^{(n-1)} \cdot e_z \rangle)$, where the superscript $(n)$ is the value at time $t = t^{(n)} (\equiv n \Delta t)$. On the other hand, $\gamma_i(E)$ is a numerically determined value of $\int_{C_i} E \cdot t ds$. Note that, if Eq. (17) does not exist, an electromotive force $\phi_i$ on $C_i$ develops [11]. This means that the boundary condition (11) is not satisfied. For this reason, we add Eq. (17) to the nonlinear boundary-value problem (*1). Therefore, the solution of (*1) is not only $S^{(n+1)}$ but also $\phi^{(n+1)}$.

In order to the nonlinear boundary-value problem (*1), we adopt the Newton method, and subsequently we obtain the linear boundary-value problem. In addition, discretizing this problem by means of the FEM, we get a linear-system as follows:

\[
\begin{bmatrix}
B(S) & C & F(\phi) \\
C^T & O & O \\
D^T(S) & O & O
\end{bmatrix}
\begin{bmatrix}
\delta S \\\n\lambda \\\n\delta \phi
\end{bmatrix} =
\begin{bmatrix}
g(S, \phi) \\
0 \\
-h(S)
\end{bmatrix},
\]

Here, $S$ is the $N$-vector obtained from the scalar function $S(x, t)$, and a $N$-by-$N$ matrix $B(S)$ is defined by $B(S) = W + J(S)$. A symmetric matrix $W$ is determined from $\dot{W}$ and $J(S)$ is the Jacobian matrix. A feature of the matrices $W$ and $J(S)$ is a dense matrix and a sparse matrix, respectively. Furthermore, the dimension of the matrix $C$ is $N$-by-$M$, and that of $F(\phi), D(S)$
are $N$-by-$L$. Also, the matrices and the vector, $C$, $F(\phi)$, $D(S)$ and $h(S)$, are obtained from the information of the cracks.

In order to obtain the solutions, $S$ and $\phi$, of the problem (*1), the two procedures are performed as follows:

(i) By solving the linear-system (18), we get the corrections $\delta S$ and $\delta \phi$.

(ii) By using $S^{(j)} \leftarrow S^{(j-1)} + \delta S$ and $\phi^{(j)} \leftarrow \phi^{(j-1)} + \delta \phi$ approximate solutions, $S$ and $\phi$, are updated.

Here, the superscript $(j)$ is the number of iterations for the Newton method. The two procedures are repeated until the termination condition:

$$ \max \left( \frac{\|\delta S\|}{\|S\|}, \frac{\|\delta \phi\|}{\|\phi\|} \right) \leq \varepsilon $$

is satisfied. Here, $\varepsilon$ is a constant and $\| \|$ denotes the maximum norm.

![Figure 2: The residual history of the Newton method at $x_m = -5.55$ mm for the case with $N = 2821$.](image)

In the previous study, we used the $LU$ decomposition to solve the linear system (18). As is well known, this solver is unsuitable for the large-scale problem because it has an operation count $O(N^3)$. In order to achieve a high-speed analysis of shielding current density in an HTS film, we adopt the GMRES($k$) method for solving (18). Throughout the present study, the restart coefficient $k$ of the GMRES($k$) method is given by $k = N/5$. Additionally, the convergence determination of GMRES($k$) and that of the Newton method are fixed as $\varepsilon_G = 10^{-5}$ and $\varepsilon = 10^{-7}$, respectively.

Note that the Newton method does not converge at a certain time step. In Fig. 2, we show a residual norm of the Newton method. We see from this figure that the residual history decreases with the number of iterations for $j \leq 4$, whereas the value becomes constant for $j > 5$. The reason is that the coefficient matrix of (18) is ill-conditioned because the maximum value of the matrix $D(S)$ is drastically smaller than that of the matrix $W$. 

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Figure 3: Dependence of total number of iterations for the GMRES\(^{\sf(k)}\) method on the time steps for the case with \(N = 2821\).

Figure 4: The residual history of the GMRES\(^{\sf(k)}\) method for the case with \(x_m = -9\) mm. Here, \(N = 2821\).

### 3.2. Precondition

To solve the problem in §3.1, we apply the Jacobi scaling to the coefficient matrix of (18). In this scaling, a typical linear-system \(Ax = b\) is transformed to \(A^*x^* = b^*\), where \(A^*, x^*,\) and \(b^*\) are expressed as \(A^* = L^{-1}AR^{-1}\), \(x^* = Rx\), \(b^* = L^{-1}b\). The diagonal elements \(L_{ii}\) and \(R_{jj}\) of the matrices \(L\) and \(R\) are given by \(L_{ii} = \max |a_{ij}|\) and \(R_{jj} = \max |a_{ij}|/L_{ii}\), respectively.

In Fig. 2, we also show the residual history of the Newton method with the Jacobi scaling. We see from this figure that, for the case with the Jacobi scaling, the number of iterations for the Newton method is only 4 times. In addition, we can obtain the solutions at all time steps. From this result, this scaling is an necessary tool for analyzing the shielding current density. However, there remains the question: whether we need to the Jacobi scaling each time steps?

In order to answer the above question, we examine the total number of iterations for the
GMRES($k$) method. Hereafter, the total number of iterations is denoted by $I_T$. Also, we here treat the following two cases:

1. Implementation of the scaling at the only first time step (i.e. $t = 0$).
2. Implementation of the scaling at the each time steps.

In Fig. 3, we show the dependence of $I_T$ on the time steps. We see from this figure that the two cases of the number of iterations hardly change from the 1st step to the 8th step. On the other hand, for the case 2, the value of $I_T$ increases with the time steps of 9th step or more. Particularly, at the 300th time step, the number of iterations for the case 2 is about 1.57 times as great as that for the case 1 (see Fig. 4). In conclusion, although we can obtain the solutions using smaller number of the GMRES iterations for the case 2, the Jacobi scaling requires the operation count $O(N^2)$. Therefore, it is not advisable to perform the matrix scaling for the all time steps. These results imply that we use the scaling every few times.

Let us consider an application of the matrix scaling at every $k_{PR}$ time steps. To this end, we calculate the CPU time $\tau^*$ as a function of $k_{PR}$, and the behavior is depicted in Fig. 5. Here, $\tau_{opt}$ is defined by $\tau_{opt} \equiv \min_{k_{PR}} \tau^*$. This is figure that the values of $\tau_{opt}$ decrease with increasing $k_{PR}$ for $k_{PR} \geq 10$. On the other hand, $\tau_{opt}$ increases with $k_{PR}$. From this result, we found that the optimization value of $k_{PR}$ exists for the case with $8 \lesssim k_{PR} \lesssim 20$. In the following, we use $k_{PR} = 10$.

### 3.3. $H$-matrix method

As is well known, the GMRES($k$) method requires a matrix-vector product $Au$ at each iteration, where $u$ indicates an arbitrary vector. In the present study, the coefficient matrix of (18) corresponds to $A$. An important point is that the $N$-by-$N$ matrix $W$ is a symmetric dense matrix, whereas the matrices $J(S), C, D(S)$, and $F(\phi)$ are a space matrix. In addition,
the dimension $N$ is drastically larger than the dimensions $M$ and $K$. For reasons mentioned above, it takes time to calculate a matrix-vector product $Wu$ as compared with $J(S)u$, $Cu$. Hence, we only use the $H$-matrix method [13, 14, 15, 16] for accelerating the computation of $Wu$.

In the $H$-matrix method, a cluster tree is formed by dividing the FEM nodes. Firstly, a rectangle called a bounding box is set to include all nodes (see Fig. 6 (a)). This is called the root cluster $T^{(0)}$. Hereafter, the superscript of the cluster $T$ shows a level of the cluster, and the subscript denotes the number of cluster for each level. Next, the bounding box is divided into two boxes at the long side, and subsequently two clusters, $T^{(1)}_1$ and $T^{(1)}_2$, can be determined (see Fig. 6 (b)). The division of the cluster is repeated until an inequality $N_c \leq N_{\text{min}}$ is satisfied (see Figs. 6 (c) and (d)). Here, $N_c$ indicates the number of nodes in the cluster. We get the cluster tree of the level 3 (see Fig. 7). Finally, in order to sort the row and column number, the matrix $W$ is transformed into $W^*$ by using $W^* = PWP^T$, where permutation matrix $P$.

By using the cluster tree, a hierarchical matrix $H$ can be constructed. Firstly, the coefficient matrix $W^*$ is divided into the submatrix $W^*_{(\sigma,\tau)}$. Here, $\sigma$ and $\tau$ are clusters with row and column number, respectively. Next, a low-rank matrix is applying to the submatrix by means of the adaptive cross approximation [13]. In the approximation, the submatrix becomes the
Figure 7: A cluster tree for the level 3. Here, \( x_i \) shows \( i \)th node in the cluster (e.g., the cluster \( T_1^{(2)} \) contains the nodes \( x_1 \) and \( x_3 \)).

Figure 8: A hierarchical matrix \( H \) for the case with \( N = 2821 \) and \( N_{\text{min}} = N/50 \).

low-lank matrix for the case where a following admissibility condition is satisfied:

\[
\min(\text{diag}(\sigma), \text{diag}(\tau)) \leq \eta \text{ dist}(\sigma, \tau). \tag{20}
\]

Here, \( \text{diag}(\sigma) \) and \( \text{dist}(\sigma, \tau) \) are diagonal of the cluster \( \tau \) and a distance between the bounding boxes \( \sigma \) and \( \tau \), respectively. As a result, each submatrix is approximated as follows:

\[
W_{(\sigma,\tau)}^* \approx UV^T,
\]

where the dimension of the matrices, \( U \) and \( V \), is \( p \)-by-\( r \) and \( q \)-by-\( r \), respectively. In addition, an approximate rank is denoted by \( r \). An operation count of the matrix-vector product is \( O(r(p + q)) \) for the \( H \)-matrix method, whereas the usually operation is \( O(pq) \). Therefore, a low-rank approximation shows a high speed only for an inequality \( r < pq/(p + q) \).

In Fig. 8, we show an example of a hierarchical matrix \( H \). In this figure, the rainbow color shows the low-rank matrix, whereas the white color is the full-rank matrix. It is found that the number of the low-lank matrix is larger than that of the full-rank matrix. Moreover,
Figure 9: Dependence of the CPU time on the number $N$ of nodes. Here, \( \square \): LU decomposition, \( \triangle \): GMRES($k$) method, and \( \Box \): GMRES($k$) with $H$-matrix method.

The dimension of the largest submatrix is about 705, and an approximate rank $r$ becomes 12 (see Fig. 8). For this case, we get the operation counts 497000 and 14000, respectively. Consequently, the operation count decreases drastically by the $H$-matrix method.

Under the numerical methods, let us investigate a performance of the simultaneous linear-system solvers. To this end, we use the $LU$ decomposition, the GMRES($k$) method, and the GMRES($k$) with $H$-matrix method as the solvers. The parameters of the $H$-matrix method are given by $N_{\text{min}} = N/100$, $\eta = 1.4$.

In Fig. 9, we show the dependence of the CPU time $\tau^*$ on the number $N$ of nodes. We see from this figure that the CPU time $\tau^*$ monotonously increases with $N$. In addition, we obtain the proportional relationship between the CPU time and the FEM nodes. The relationship is expressed as $\tau^* \propto N^\alpha$. The results of the computations show that we can get $\alpha = 3.4$, $\alpha = 2.5$, and $\alpha = 2.1$ for the case with the $LU$ decomposition, the GMRES($k$) method and the GMRES($k$) with $H$-matrix method, respectively. We conclude that the GMRES($k$) method is a powerful tool for analyzing the shielding current density with high-speed. Furthermore, the speed of the GMRES($k$) method is accelerated by using the $H$-matrix method.

In order to express the speed of the $H$-matrix method quantitatively, we define two types of speedup ratio: $\tau_L/\tau_H$ and $\tau_G/\tau_H$. Here, $\tau_L$, $\tau_G$, and $\tau_H$ is the CPU time of the $LU$ decomposition, the GMRES($k$) method, and the GMRES($k$) with $H$-matrix method, respectively. If the two types of speedup ratio are 1 or more, the $H$-matrix method is high-speed.

In Fig. 10 (a), we show the dependence of the speedup ratio $\tau_L/\tau_H$ on the number $N$ of nodes. We see from this figure that the speedup ratio monotonously increases with $N$, and the $H$-matrix method is always faster than the $LU$ decomposition. Especially, the speed of the GMRES($k$) with $H$-matrix method is about 221.3 times as fast as that of the $LU$ decomposition at $N \approx 4 \times 10^3$. On the other hand, the $H$-matrix method is faster than the GMRES($k$) method for $N \gtrsim 10^3$ (see Fig. 10 (b)). The speed of the GMRES($k$) method can be accelerated by a factor of about 2.8 at about $N = 10^4$.
Figure 10: Dependence of the speedup ratio (a) $\tau_L/\tau_H$ and (b) $\tau_G/\tau_H$ on the number $N$ of nodes. Here, $\tau_L$, $\tau_G$, and $\tau_H$ is the CPU time of the LU decomposition, the GMRES($k$) method, and the GMRES($k$) with $H$-matrix method, respectively.

4. Conclusion

We have proposed the high-speed analysis FEM code to obtain the time evolution of the shielding current density with the HTS film containing the cracks. In the code, we adopt the GMRES($k$) method with the Jacobi scaling. Moreover, in order to accelerate the shielding current analysis, we also use the $H$-matrix method. The conclusions are summarized as follows:

1) By using the Jacobi scaling, the convergence of the Newton method can be greatly improved. When the Jacobi scaling is not performed, the Newton method does not converge at a certain time step. This scaling is an necessary tool for analyzing the shielding current density.
2) By using the GMRES($k$) with the $H$-matrix method as a linear-system solver, the time evolution of the shielding current can be analyzed with high speed. In particular, the speed of the GMRES($k$) with $H$-matrix method is about 221.3 times as fast as that of the $LU$ decomposition.

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References

[1] J. H. Claassen, M. E. Reeves, R. J. Soulen, Jr.: A contactless method for measurement of the critical current density and critical temperature of superconducting films, *Rev. Sci. Instrum.*, 62:4 (1991), 996–1004.

[2] Y. Mawatari, H. Yamasaki, Y. Nakagawa: Critical current density and third-harmonic voltage in superconducting films, *Appl. Phys. Lett.*, 81:13 (2002), 2424–2426.

[3] S. B. Kim: The defect detection in HTS films on third-harmonic voltage method using various inductive coils *Physica C*, 463–465 (2007), 702–706.

[4] S. Ohshima, K. Takeishi, A. Saito, M. Mukaida, et al.: A simple measurement technique for critical current density by using a permanent magnet, *IEEE Trans. Appl. Supercond.*, 15:2 (2005), 2911–2914.

[5] A. Saito, K. Takeishi, Y. Takano, T. Nakamura, et al.: Rapid and simple measurement of critical current density in HTS thin films using a permanent magnet method, *Physica C*, 426 (2005), 1122–1126.

[6] S. Ikuno, T. Takayama, A. Kamitani, K. Takeishi, et al.: Analysis of measurement method for critical current density by using permanent magnet, *IEEE Trans. Appl. Supercond.*, 19:3 (2009), 3591–3594.

[7] S. Ohshima, K. Umezu, K. Hattori, H. Yamada, et al.: Detection of critical current distribution of YBCO-coated conductors using permanent magnet method, *IEEE Trans. Appl. Supercond.*, 21:3 (2011), 3385–3388.

[8] K. Hattori, A. Saito, Y. Takano, T. Suzuki, et al.: Detection of smaller $J_c$ region and damage in YBCO coated conductors by using permanent magnet method, *Physica C*, 471:21/22 (2011), 1033–1035.
[9] A. Kamitani, and S. Ohshima: Magnetic shielding analysis of axisymmetric HTS plates in mixed state, *IEICE Trans. Electron.*, E82-C:5 (1999), 766–773.

[10] T. Takayama, A. Kamitani, A. Tanaka, S. Ikuno: Numerical simulation of shielding current density in HTS: Application of high-performance method for calculating improper integral *Physica C*, 469:15-20, (2009) 1439–1442.

[11] A. Kamitani and T. Takayama, Numerical simulation of shielding current density in high-temperature superconducting film: Influence of film edge on permanent magnet method, *IEEE Trans. Appl. Supercond.*, 48:2 (2012), 727–730.

[12] T. Takayama and A. Kamitani: Numerical investigation on crack detection in HTS film: Accuracy of scanning permanent magnet method, *IEEE Trans. Appl. Supercond.*, 24:3 (2014), Art. ID. 9001505.

[13] M. Bebendorf: Approximation of boundary element matrices, *Numerische Mathematik*, 86 (2000), 565–589.

[14] S. Kurz, O. Rain, S. Rjasanow, The adaptive cross approximation technique for the 3-D boundary element method, *IEEE Transactions on Magnetics*, 38 (2002), 421–424.

[15] Y. Takahashi, C. Matsumoto, S. Wakao: Large-scale and fast nonlinear magnetostatic field analysis by the magnetic moment method with the adaptive cross approximation, *IEEE Trans. Magn.*, 43, (2007) 1277–1280.

[16] V. Le-Van, B. Bannwarth, A. Carpentier, O. Chadebec, et al.: The adaptive cross approximation technique for a volume integral equation method applied to nonlinear magnetostatic problems, *IEEE Trans. Magn.*, 50 (2014) Art. No. 7010904.