D–dimensional massless particle with extended gauge invariance

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Abstract
We propose the model of D–dimensional massless particle whose Lagrangian is given by the N–th extrinsic curvature of world-line. The system has N + 1 gauge degrees of freedom constituting W–like algebra; the classical trajectories of the model are space-like curves which obey the conditions \( k_{N+a} = k_{N-a}, \ k_{2N} = 0, \ a = 1, \ldots, N-1, \ N \leq [(D-2)/2], \) while the first N curvatures \( k_i \) remain arbitrary. We show that the model admits consistent formulation on the anti-DeSitter space. The solutions of the system are the massless irreducible representations of Poincaré group with \( N \) nonzero helicities, which are equal to each other.

1 The model
About ten years ago M.Plyushchay proposed the beautiful Lagrangian system describing the four-dimensional massless particle with the helicity \( c \).\(^{(1)}\)

\[
S = c \int k_1 d\tilde{s}, \quad d\tilde{s} = |dx| \neq 0,
\]
(1)

where \( k_1 \) denotes the first curvature of a world-line, assumed to be a non-isotropic curve. Quantization of this model yields the massless irreducible representation of Poincaré group with any integer \( \square \) and half-integer \( \square \) helicity. This model possesses the gauge \( W_3 \) symmetry reflected in the gauge equivalence of its classical trajectories, which are space-like plane curves, i.e. \( k_1 \) is any, \( k_2 = 0 \).\(^{(3)}\) Another interesting property of Plyushchay’s model is that it admits formulation on the AdS space.\(^{(4)}\)

Is it possible to generalize the Plyushchay’s model in a \( D > 4 \) dimensional space-time?
This question is not so trivial, since in \( D > 6 \) massless particles are specified by \( [(D-2)/2] \) helicities (the weights of the little Lorentz group \( SO(D-2) \)), while the Plyushchay’s model in \( D > 6 \) has only one nonvanishing helicity. On the other hand, only those massless irreps are conformal invariant, whose all helicities are equal to each other (being any (half)integers in an even dimensional space-time, and 0,1/2 in the odd-dimensional one).\(^{(5)}\)

Systematically investigating the actions

\[
S = \int \mathcal{L}(k_1, \ldots, k_N) d\tilde{s},
\]
(2)

one find that the systems describing massless particles with fixed helicities are of the form

\[
S = c \int k_N ds, \quad d\tilde{s} = |dx| \neq 0
\]
(3)

where \( k_i \) denote the reparametrization invariants (extrinsic curvatures) of a world-line.\(^{(6)}\)

We establish the following interesting properties of this model (for more detail see \( \square \)):

- its classical solutions are space-like curves which obey the conditions \( k_{N+i} = k_{N-i}, \) where \( i = 1, \ldots, N \leq N_0 = [(D-2)/2], \) \( k_0 \equiv 0; \) the values of the first \( N \) curvatures remain arbitrary, so the model possesses \( N + 1 \) gauge degrees of freedom forming the algebra of the \( W \)-type;
- it describes a massless particle whose \( N \) helicities are equal to each other \( c_1 = c_2 = \ldots = c_N = c, \) and the remaining ones vanish;
- it admits a consistent formulation on the Anti-De Sitter spaces.
2 The analysis

In order to obtain the Hamiltonian formulation of the system, we should replace the action \([3]\) by the classically equivalent one, whose Lagrangian depends on the first-order derivatives, and then perform the Legendre transformation. For this purpose, it is convenient to use the recurrent equations for extrinsic curvatures, which follow from the Frenet equations for a moving frame \(\{e_a\}\) of nonisotropic curves:

\[
\dot{x} = se_1, \quad \dot{e}_a = sK_a^b e_b, \quad e_a e_b = \eta_{ab}, \quad a, b = 1, \ldots, D
\]

\[
K_{ab} + K_{ba} = 0, \quad K_{ab} = \begin{cases} \pm k_a, & \text{if } b = a \pm 1 \\ 0, & \text{if } b \neq a \pm 1 \end{cases}, \quad k_a \geq 0.
\]

The Frenet equations in the Euclidean space read

\[
\dot{x} = se_1, \quad \dot{e}_a = sk_a e_{a+1} - sk_{a-1} e_{a-1}, \quad e_0 = e_{D+1} \equiv 0, \quad k_0 = k_D = 0.
\]

It is easy to verify that for the transition to the Frenet equations for non-isotropic curves in the pseudo-Euclidean space, we should make, for some index \(a\), the substitution,

\[
(e_a, sk_a, sk_{a-1}, s) \rightarrow (ie_a, isk_a, isk_{a-1}, (-i)^a s).
\]

The choice \(a = 1\) means the transition to a time-like curve, while \(a = 2, \ldots, D\)-to space-like ones. Thus, throughout the paper we use the Euclidean signature.

Taking into account the expressions \([5]\) one can replace the initial Lagrangian (in arbitrary time parametrization \(d\tilde{s} = sdt\), \(s = |\dot{x}|\)) by the following one

\[
\mathcal{L} = c\sqrt{e_N^2 - (sk_N^2)^{N-1} + p(\dot{x} - se_1) + \sum_{i=1}^{N-1} p_i (\dot{e}_i - sk_i e_{i+1} + sk_{i-1} e_{i-1})} - s \sum_{i,j=1}^{N} d_{ij} (e_i e_j - \delta_{ij})
\]

where \(s, k_i, d_{ij}, p_{i-1}, e_i\) are independent variables, \(k_0 = 0, p_0 = e_0 = 0\).

Performing the Legendre transformation for this Lagrangian (see, for details, \([4]\)), one gets the Hamiltonian system

\[
\omega = dp \wedge dx + \sum_{i=1}^{N} dp_i \wedge de_i,
\]

\[
\mathcal{H} = s \left[ p e_1 + \sum_{i=1}^{N} k_{i-1} \phi_{i-1,i} + \frac{1}{2s} (\Phi_{N,N} - c^2) + \sum_{i,j=1}^{N} d_{ij} (e_i e_j - \delta_{ij}) \right],
\]

with the primary constraints

\[
e_i e_j - \delta_{ij} \approx 0,
\]

\[
p_N e_N \approx 0, \quad p_N e_{N-2} \approx 0,
\]

\[
p e_1 \approx 0,
\]

\[
\phi_{i-1,i} \equiv p_{i-1} e_i - p_i e_{i-1} \approx 0, \quad \Phi_{N,N} \equiv p_N^2 - \sum_{i=1}^{N} (p_N e_i)^2 \approx c^2.
\]

It is convenient to introduce the new variables, instead of \(p_i\),

\[
p_i^\perp \equiv p_i - \sum_{j=1}^{N} (p_i e_j) e_j, \quad p_i^\perp p_j^\perp \equiv \Phi_{i,j}, \quad p_i^\perp e_j = 0,
\]

\[
\phi_{i,j} \equiv p_i e_j - p_j e_i,
\]

\[
\chi_{ij} = p_i e_j, \quad i \geq j.
\]

Since the constraints \([4]\) are conjugated to \(\chi_{ij}\) and commute with \(p_i^\perp\) and \(\phi_{ij}\), we impose, without loss of generality, the gauge conditions \(\chi_{ij} \approx 0\) fixing the values of \(d_{ij}\)

\[
\chi_{ij} \approx 0 : \Rightarrow 2d_{i,j} = \delta_{ij} \delta_{i} \delta_{N} k_{N c}.
\]
Notice that in this formulation $s$ and $sk_1$ play the role of Lagrangian multipliers, so the primary constraints produce either secondary ones, or the explicit relations on the first $N$ curvatures.

The primary constraints \((11),(12)\) produce the following set of constraints, which are of the first class

\[
\begin{align*}
pe_i &\approx 0, \quad pp_i \approx 0, \quad p^2 \approx 0, \\
\phi_{ij} &\approx 0, \quad \Phi_{ij} - c^2 \delta_{ij} \approx 0
\end{align*}
\]

so, the dimension of phase space is

\[
D_{\text{phys}} = 2(D-1) + N(2D-3N-5).
\]

From the expressions \((15)\) which provide the model with the mass-shell and transversality conditions, we conclude that the nontrivial classical solutions of the system \((3)\) are the space-like curves with $N \leq N_0 = \left\lfloor (D-2)/2 \right\rfloor$.

Comparing the equations of motion with the Frenet formulae \((5)\), one finds that the space-like vectors \((e_i, p^i/c = e_{2N+1-i})\) define the first $2N$ elements of moving frame, while $p$ defines its $(2N+1)$-th, isotropic, element. We also get the following relations on curvatures

\[
k_{N+i} = k_{N-i}, \quad i = 1, \ldots, N, \quad k_0 \equiv 0.
\]

The first $N$ curvatures remain arbitrary, hence, the system possesses $N+1$ gauge degrees of freedom.

Let introduce the complex variables

\[
z_i = \frac{p_i + i|c|e_i}{\sqrt{2}|c|}
\]

in which the Hamiltonian system reads

\[
\omega = dp \wedge dx + i \sum_i dz_i \wedge d\bar{z}_i,
\]

\[
\mathcal{H} = \frac{i}{2c} \left[ i\sqrt{2}p(\bar{z}_1 - z_1) + i \sum_{i=1}^{N-1} k_i(z_i\bar{z}_{i+1} - z_{i+1}\bar{z}_i) + k_N(z_N\bar{z}_N - c^2) \right],
\]

while the full set of constraints takes the form

\[
\begin{align*}
z_i\bar{z}_j - |c|\delta_{ij} &\approx 0, \\
z_i z_j &\approx 0, \\
pz_i &\approx 0, \\
p^2 &\approx 0.
\end{align*}
\]

In these coordinates the equations of motion become holomorphic one

\[
\begin{align*}
\dot{x} &= i(z_1 - \bar{z}_1), \\
\dot{z}_i &= -i\delta_{i,1} - p + k_{i-1}z_i - k_{i-2}z_{i-2}, \\
\dot{z}_N &= -i\delta_{1,N} + ik_Nz_N - k_{N-1}z_{N-1}, \\
\dot{p} &= 0,
\end{align*}
\]

which can be considered as the implicit indicator of the $W$–algebraic origin of a system’s gauge symmetries \[8\]. Another argument is that the equations of motion for $z_i$ can be rewritten in the form \[8\]

\[
\hat{L}z_i = \sum_{i=0}^N \lambda_i(k_{i+1}) \frac{dz_1}{d\tau^i} = 0.
\]
Hence, the model under consideration possesses conformal symmetry if:

\[ N \text{ even}, \quad \text{iff} \quad N = 1, 2 \] 

Due to Siegel [5], the (massless) irreducible representations of the system possess conformal invariance, if

\[ c_1 = \ldots = c_N = c \neq 0, \quad c_{N+1} = \ldots = c_{N_0} = 0. \] 

To quantize the system, we should choose the polarization

\[ \hat{z}_i = \frac{\partial}{\partial z_i}, \quad \hat{x} = \frac{\partial}{\partial p}, \quad \hat{z}_i = z_i, \quad \hat{p} = p. \]

Then, after standard manipulations we get that the wave function is of the form

\[ \Psi(p, z_i) = \psi(p) A_{\{1\}A_{\{2\}}\ldots A_{\{N\}}} z^{A_{\{1\}}} z^{A_{\{2\}}} \ldots z^{A_{\{N\}}}, \]

where

\[ A^{(i)} = A_1^{(i)} \ldots A_c^{(i)}, \quad z_i^{A^{(i)}} = z_1^{A^{(i)}} \ldots z_i^{A_{\{c\}}}, \quad |c| = 1, 2, 3, \ldots. \]

Here \( \psi(p) \) is the tensor of \( cN \)-th rank, whose symmetries are given by the \( N \times c \) Young tableau

\[
\begin{array}{cccc}
A_1^{(i)} & \ldots & \ldots & A_{\{c\}}^{(i)} \\
A_2^{(i)} & \ldots & \ldots & A_{\{c\}}^{(i)} \\
\vdots & & & \vdots \\
A_{\{c\}}^{(i)} & \ldots & \ldots & A_{\{c\}}^{(i)}
\end{array}
\]

In addition, the tensor \( \psi(p) \) should satisfy the transversality and mass-shell conditions:

\[ p^2 \psi(p) A_{\{1\}} A_{\{2\}} \ldots A_{\{N\}} = 0, \quad p^A \psi(p) A_{\{1\}} A_{\{2\}} \ldots A_{\{N\}} = 0. \tag{26} \]

Due to Siegel [5], the (massless) irreducible representations of the system possess conformal invariance, if \( N = N_0 \), where \( c_1 = \ldots = c_{N_0} \) is (half)integer for even \( D \) and \( c_N = 0, 1/2 \) for odd \( D \). Hence, the model under consideration possesses conformal symmetry if \( D = 2p, \quad N = (D-2)/2 \), since the helicities are integers.

Let us reformulate our model on the spaces with constant curvature. On the curved spaces the Frenet equations read

\[
\frac{dx}{sdr} = e_1, \quad \frac{de_a}{sdr} = k_a e_{a+1} - k_{a-1} e_{a-1},
\]

where

\[
\frac{D}{dr} = \frac{d}{dr} \hat{\Gamma}(\hat{x}), \quad e_a e_b = \delta_{ab}, \quad e_0 = e_{D+1} = 0,
\]

while \( (\hat{\Gamma})^A_B = \Gamma^A_{BC} \hat{x}^C, \quad \Gamma^A_{BC} \) are the Christoffel symbols of the metric \( g_{AB}(x) \) of underlying manifold.

Performing the manipulations, similar to the flat case, we get the Hamiltonian system

\[
\mathcal{H} = \hat{s} \left[ \pi e_1 + \sum_{i=1}^N k_{i-1} e_{i-1,i} + \frac{k_N N}{2c} (\Phi_{N,N} - c^2) + \sum_{i,j=1}^N e_i e_j \delta_{ij} \right],
\]
whose primary constraints are given by the expressions

\[
\begin{align*}
\pi e_1 & \approx 0, \\
e_i e_j - \delta_{ij} & \approx 0, \\
p_N e_N & \approx 0, \quad p_N e_{i-2} \approx 0, \\
\phi_{n-1,i} & \equiv p_{i-1} e_i - p_i e_{i-1} \approx 0, \\
\Phi_{N,N} & \equiv p_N^2 - \sum_{i=1}^N (p_N e_i)^2 \approx c^2. 
\end{align*}
\]  

(29)

where \( \pi \equiv p - \Gamma \), \( \Gamma_A \equiv \sum_{i=1}^N \Gamma_{AB}^C p(i) e_i C e_i B \).

It is easy to see that the Poisson brackets of the functions \( \phi_{ij}, \Phi_{ij} \) remain unchanged, as well as those with \( \pi a \), where \( a = e_i, p_i, \pi \). On the other hand,

\[
\{ \pi a, \pi b \} = \sum_{i=1}^N R(p_i|e_i, a, b),
\]

where \( R( , , , ) \) is the curvature tensor of underlying manifold.

Taking into account the expression for the Riemann tensor of the constant curvature spaces, we conclude that the constraint algebra of the system (28) is isomorphic to the one on the constant curvature spaces (8).

Therefore, the model (3) admits the consistent formulation on the anti-De Sitter spaces.

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