κ-generalization of Gauss’ law of error

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Abstract

Based on the κ-deformed functions (κ-exponential and κ-logarithm) and associated multiplication operation (κ-product) introduced by Kaniadakis (Phys. Rev. E 66 (2002) 056125), we present another one-parameter generalization of Gauss’ law of error. The likelihood function in Gauss’ law of error is generalized by means of the κ-product. This κ-generalized maximum likelihood principle leads to the so-called κ-Gaussian distributions.

Key words: Gauss’ law of error, κ-deformed function, κ-product
PACS: 02.50.Cw, 05.20.-y, 06.20.Dk

1 Introduction

Gaussian (or normal) distribution [1] is one of the most well-known and fundamental distributions in many fields of science, e.g., an error distribution in measurement, a probability distribution of a fluctuating physical quantity in statistical mechanics, which accounts for the successes of Boltzmann-Gibbs exponential distributions [2]. For statistically independent $N$ events, it is well-known that the limiting stable distribution becomes a Gaussian one for sufficiently large $N$. This is the consequence of central limit theorem [3], which is the most famous theorem in mathematical probability theory. Historically however the Gauss derivation, as known as Gauss’ law of error, played a very important role before establishing the central limit theorem.

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Quite recently one of the authors et al. [4] have shown a one-real-parameter \((q)\) generalization of Gauss’ law of error, in which the \(q\)-generalized maximum likelihood principle leads to the so-called \(q\)-Gaussian distribution, by utilizing the Tsallis \(q\)-deformed functions [5,6,7,8] and associated multiplication operation, i.e., so called \(q\)-product [9,10]. Tsallis’ entropy and \(q\)-deformed functions have been successfully applied in order to explain the ubiquitous existence of power-law behaviors in nature. Tsallis’ entropy \(S_q = (\int dx f(x)^q - 1)/(1 - q)\) is a one-real-parameter generalization of Boltzmann-Gibbs-Shannon entropy \(S_{BGS} = - \int dx \ f(x) \ln f(x)\) and employed for generalizing the traditional Boltzmann-Gibbs statistical mechanics, thermodynamics, and Jaynes’ information theory.

The \(q\)-deformed exponential and logarithmic functions are defined by

\[
\exp_q(x) \equiv [1 + (1 - q)x]_+^{\frac{1}{1-q}}, \\
\ln_q(x) \equiv \frac{x^{1-q} - 1}{1-q},
\]

respectively. Here \(q\) is a real parameter characterizing the deformation functions, and \([x]_+ \equiv \max(x, 0)\). In the \(q \to 1\) limit, the \(q\)-exponential and \(q\)-logarithmic functions reduce to the standard exponential and logarithmic functions, respectively. The following useful identity holds.

\[
\exp_q(x + y) = \exp_q(x) \cdot \exp_q\left(\frac{y}{1 + (1 - q)x}\right).
\]

As in the general usage for a Gaussian function, we here mean that any function of the general form \(\exp_q(-\alpha x^2)\) with a positive constant \(\alpha\) as a \(q\)-Gaussian function.

One of the most fundamental ingredients for proving the \(q\)-generalization of Gauss’ law of error is the recently introduced \(q\)-product [9,10], which is defined by

\[
x \otimes_q y \equiv \left[x^{1-q} + y^{1-q} - 1\right]_+^{\frac{1}{1-q}}.
\]

On the other hand, Kaniadakis [11,12] has introduced another one-parameter deformed exponential function

\[
\exp_{\{\kappa\}}(x) \equiv \left(\sqrt{1 + \kappa^2 x^2} + \kappa x\right)^{\frac{1}{\kappa}},
\]

and its inverse function

\[
\ln_{\{\kappa\}}(x) \equiv \frac{x^\kappa - x^{-\kappa}}{2\kappa},
\]

where \(\kappa\) is a real parameter and takes a value in the range \((-1, 1)\). Both \(\kappa\)-deformed functions are symmetric under interchange of \(\kappa \leftrightarrow -\kappa\). In the \(\kappa \to 0\)
limit, the $\kappa$-exponential and $\kappa$-logarithmic functions reduce to the standard 
exponential and logarithmic functions, respectively. 

Note that the $\kappa$-logarithmic function can be expressed in terms of the two 
$q$-logarithmic functions with the different $q$ indices as follows. 

$$
\ln_{\kappa}(x) = \frac{1}{2} \ln_{1+\kappa}(x) + \frac{1}{2} \ln_{1-\kappa}(x).
$$  \hspace{1cm} (7)

However the $\kappa$-exponential function cannot be expressed in terms of the $q$-
exponential functions. 

Note also that both $q$- and $\kappa$-exponentials shows asymptotic power-law be-
haviors for large $x$: 

$$
\exp_q(x) \sim x \to \infty ((1 - q)x)^{\frac{1}{1-q}}, \hspace{1cm} (8)
$$

$$
\exp_{\kappa}(x) \sim x \to \infty (2\kappa x)^{\frac{1}{\kappa}}, \hspace{1cm} (9)
$$

whereas for a small value of $x$ both functions behave as the standard 
exponential functions. Only in the intermediate region they are slightly different 
each other. Although the difference is not so large, this means that if an ex-
perimential data is well fitted with $\kappa$-exponential function it should not be well 
ﬁtted with $q$-exponential one, and vice versa. Until now in the literature there 
are only a few experimental evidences which are well fitted with $\kappa$-exponential 
probability distributions: the flux distribution of the cosmic rays extends over 
13 decades in energy [13]; the rain events in meteorology (the number density 
of rain events versus the event size) [13,14]; and the analysis of the fracture 
problem (the relation between the length of a transversal cut of the conducting 
thin ribbon and the electrical resistance) [15].

Kaniadakis [13] had already introduced a deformed algebra based on his $\kappa$-
deformed functions before establishing the concept of the $q$-product [9,10]. His 
$\kappa$-deformed product [13] is defined by 

$$
x \otimes_{\kappa} y \equiv \left(\frac{x^\kappa - x^{-\kappa}}{2} + \frac{y^\kappa - y^{-\kappa}}{2} + \sqrt{1 + \left(\frac{x^\kappa - x^{-\kappa}}{2} + \frac{y^\kappa - y^{-\kappa}}{2}\right)^2}\right)^{\frac{1}{\kappa}}, \hspace{1cm} (10)
$$

which has the following properties

associativity  \hspace{1cm} $(x \otimes_{\kappa} y) \otimes_{\kappa} z = x \otimes_{\kappa} (y \otimes_{\kappa} z)$, \hspace{1cm} (11)

unit element  \hspace{1cm} $x \otimes_{\kappa} 1 = 1 \otimes_{\kappa} x = x$, \hspace{1cm} (12)

inverse element  \hspace{1cm} $x \otimes_{\kappa}(1/x) = (1/x) \otimes_{\kappa} x = 1$, \hspace{1cm} (13)

and satisfying the following relations
\[
\ln_{\{\kappa\}} (x \otimes y) = \ln_{\{\kappa\}}(x) + \ln_{\{\kappa\}}(y),
\]
\[
\exp_{\{\kappa\}}(x) \otimes \exp_{\{\kappa\}}(y) = \exp_{\{\kappa\}}(x + y).
\]

In this way the \(\kappa\)-product also has suitable properties for generalizing Gauss’ law of error. It is then natural to ask whether we can generalize Gauss’ law of error based on the \(\kappa\)-product. This paper presents a positive answer to the above question. We show the \(\kappa\)-generalization of Gauss’ law of error, in which the \(\kappa\)-generalized maximum likelihood principle leads to the \(\kappa\)-generalized Gaussian distribution. In the next section, we present the \(\kappa\)-generalized version of the maximum entropy principle in order to derive the \(\kappa\)-Gaussian distributions. In section 3, by utilizing the \(\kappa\)-product, the likelihood function in Gauss’ law of error is generalized. We then prove the solution of the \(\kappa\)-generalized maximum likelihood principle is the \(\kappa\)-Gaussian distribution. The last section is devoted to our concluding remarks.

2 Maximum \(\kappa\)-entropy principle

Kaniadakis’ \(\kappa\)-entropy [11] can be defined [16] by

\[
S_{\kappa} \equiv \kappa \left( 1 - \int dx f(x)^{1+\kappa} \right) + c_{-\kappa} \left( 1 - \int dx f(x)^{1-\kappa} \right),
\]
with \( c_{\kappa} = \frac{1}{2\kappa(1+\kappa)} \).

\( S_{\kappa} \) reduces to Boltzmann-Gibbs-Shannon entropy \( S_{\text{BGS}} = -\int dx f(x) \ln f(x) \) in the limit of \( \kappa \to 0 \).

Maximizing \( S_{\kappa} \) under the two constraints,

\[
\int_{-\infty}^{\infty} dx \: x^2 f(x) = \sigma^2, \quad \int_{-\infty}^{\infty} dx f(x) = 1,
\]

leads to

\[
\frac{\delta}{\delta f(x)} \left( S_{\kappa} - \beta \int_{-\infty}^{\infty} dx \: x^2 f(x) - \gamma \int_{-\infty}^{\infty} dx f(x) \right) = 0,
\]

where \( \beta \) and \( \gamma \) are Lagrange multipliers associated with the two constraints, respectively. It’s solution is the so-called \(\kappa\)-Gaussian distribution,

\[
f(x) = \exp_{\{\kappa\}} \left( -\beta x^2 - \gamma \right).
\]

Note that \( \gamma \) cannot be factored out from the argument of the \(\kappa\)-exponential function unless \( \kappa = 0 \). This is one of the most different properties of the \(\kappa\)-exponential function against the \(q\)-exponential function. In fact, applying the similar argument to the \(q\)-deformed functions, we can obtain the \(q\)-Gaussian
function \( f_q(x) = \exp_q (-\beta' x^2 - \gamma') \), where two Lagrange multiplier \( \beta' \) and \( \gamma' \) are introduced. Then, due to the property of Eq. (3), \( \gamma' \) can be factored out as

\[
f_q(x) = \exp_q (-\beta' x^2 - \gamma') = \exp_q(-\gamma') \exp_q \left( \frac{-\beta' x^2}{1 - (1 - q) \gamma'} \right).
\] (21)

In Fig. 1, \( \kappa \)-Gaussian functions are plotted for some different \( \kappa \) values.

![\( \kappa \)-Gaussian functions](image)

**Fig. 1.** \( \kappa \)-Gaussian functions Eq. (20) for some values of \( \kappa \) with \( \beta = 1 \). For each curve \( \gamma \) is determined so that the normalization is satisfied. \( \kappa = 0 \) corresponds to the standard Gaussian function.

### 3 \( \kappa \)-generalization of Gauss’ law of error

Let us consider the same situation as conventional Gauss’ law of error [1,4], i.e., we get \( n \) observed values:

\[
x_1, x_2, \cdots, x_n \in \mathbb{R}
\] (22)

as the results of \( n \) measurements for certain observations. Each observed value \( x_i \) \( (i = 1, \cdots, n) \) is a result of the measurement of identically distributed random variable \( X_i \) \( (i = 1, \cdots, n) \). There exists a true value \( x \) satisfying the *additive* relation:

\[
x_i = x + e_i \quad (i = 1, \cdots, n),
\] (23)
where each of $e_i$ is an error in each observation of the true value $x$. Thus, for each $X_i$, there exists a random variable $E_i$ such that $X_i = x + E_i$ ($i = 1, \cdots, n$). Every $E_i$ has the same probability density function $f$ which is differentiable, because $X_1, \cdots, X_n$ are identically distributed random variables (i.e., $E_1, \cdots, E_n$ are also so).

In order to prove the theorem for the $\kappa$-generalization of Gauss’ law of error, we use the following lemma. Although the proof, which can be found in Ref. [4], is simple and compact, we here show it for the sake of being self-contained.

**Lemma** Let $\phi$ be a continuous function from $\mathbb{R}$ into itself and satisfying that $\sum_{i=1}^{n} \phi(e_i) = 0$ for every $n \in \mathbb{N}$ and $e_1, \cdots, e_n \in \mathbb{R}$ with $\sum_{i=1}^{n} e_i = 0$. Then there exists $a \in \mathbb{R}$ such that $\phi(e) = ae$.

**Proof.** In the case $n = 2$, we can easily see that $\phi(-e) = -\phi(e)$ for every $e = e_1, e_2 \in \mathbb{R}$. Moreover, in the case that $n = 3$ we have $\phi(e_1 + e_2) = \phi(e_1) + \phi(e_2)$ for every $e_1, e_2 \in \mathbb{R}$. From this result and continuity of $\phi(e)$, it is easy to show that $\phi(e)$ must be a linear function of $e$, which prove the lemma. $\blacksquare$

**Theorem** For a given set of the data $x_1, x_2, \cdots, x_n$, if the likelihood function $L_{\{\kappa\}}(\theta)$ of a variable $\theta$, which is defined by

\[
L_{\{\kappa\}}(\theta) = L_{\{\kappa\}}(x_1, x_2, \cdots, x_n; \theta) \equiv f(\theta - x_1) \otimes_{\kappa} f(\theta - x_2) \otimes_{\kappa} \cdots \otimes_{\kappa} f(\theta - x_n),
\]  

(24)

takes the maximum at

\[
\theta = \theta^* \equiv \frac{1}{n} \sum_{i=1}^{n} x_i,
\]

(25)

then the probability density function $f$ must be a $\kappa$-Gaussian distribution:

\[
f(\theta - x_i) = \exp_{\{\kappa\}} \left( -a_\kappa (\theta - x_i)^2 + C_\kappa \right).
\]

(26)

where $a_\kappa$ is a $\kappa$-dependent positive constant, and $C_\kappa$ is a $\kappa$-dependent normalization factor.

**Proof.** Taking the $\kappa$-logarithm of the both side of the likelihood function $L_{\{\kappa\}}(\theta)$ in Eq. (24) leads to

\[
\ln_{\{\kappa\}} \left(L_{\{\kappa\}}(\theta) \right) = \sum_{i=1}^{n} \ln_{\{\kappa\}} \left(f(\theta - x_i)\right),
\]

(27)

where the property (14) is used. Differentiating the above formula Eq. (27) with respect to $\theta$, we have

\[
\frac{d}{d\theta} \ln_{\{\kappa\}} \left(L_{\{\kappa\}}(\theta) \right) = \sum_{i=1}^{n} \frac{d}{d\theta} \ln_{\{\kappa\}} \left(f(\theta - x_i)\right).
\]

(28)
When $\theta = \theta^*$ the likelihood function $L_{\{\kappa\}}(\theta)$ takes the maximum, so that

$$\sum_{i=1}^{n} \frac{d}{d\theta} \ln_{\{\kappa\}} f(\theta - x_i) \bigg|_{\theta=\theta^*} = 0. \tag{29}$$

Let $e_i$ and $\phi_{\kappa}(e)$ be defined by

$$e_i \equiv \theta - x_i, \quad (i = 1, \cdots, n), \quad \phi_{\kappa}(e) \equiv \frac{d}{d\theta} \ln_{\{\kappa\}} f(e) \bigg|_{\theta=\theta^*}, \tag{30}$$

respectively. Then Eq. (29) can be rewritten to

$$\sum_{i=1}^{n} \phi_{\kappa}(e_i) = 0. \tag{31}$$

On the other hand, from Eq. (25) , we obtain

$$\sum_{i=1}^{n} (\theta^* - x_i) = \sum_{i=1}^{n} e_i = 0. \tag{32}$$

Now our problem is reduced to determining the function $\phi_{\kappa}(e)$ simultaneously satisfying Eqs. (31) and (32). By using the lemma, $\phi_{\kappa}(e)$ should be proportional to $e$, i.e.,

$$\phi_{\kappa}(e) = -2a_{\kappa}e, \tag{33}$$

where $a_{\kappa}$ is a $\kappa$-dependent positive constant, and the factor $-2$ is introduced for the sake of the simplicity of the final result. Integrating Eq. (33) w.r.t $e$ gives

$$\ln_{\{\kappa\}} f(e) = -a_{\kappa}e^2 + C_{\kappa}, \tag{34}$$

where $C_{\kappa}$ is a normalization factor which is $\kappa$-dependent through $a_{\kappa}$. Thus we obtain

$$f(e) = \exp_{\{\kappa\}} \left( -a_{\kappa}e^2 + C_{\kappa} \right). \tag{35}$$

A normalizable distribution requires that $a_{\kappa}$ should be positive, and the normalization then determines the factor $C_{\kappa}$.

Finally let us confirm that the extremum of the likelihood function at $\theta = \theta^*$ is the maximum. By using Eq. (33), we obtain

$$\frac{d^2}{d\theta^2} \ln_{\{\kappa\}} \left( L_{\{\kappa\}}(\theta) \right) = \frac{d}{d\theta} \left( \frac{d}{d\theta} \phi_{\kappa}(\theta - x_i) \right) = \frac{d}{d\theta} (-2a_{\kappa}(\theta - x_i)) = -2a_{\kappa}. \tag{36}$$

Since $a_{\kappa}$ is positive, $L_{\{\kappa\}}(\theta^*)$ is the maximum. ■
4 Concluding remarks

We have shown two different methods to derive the $\kappa$-Gaussian distribution, i.e., both maximum entropy and maximum likelihood principles are generalized by utilizing $\kappa$-deformed functions and associated multiplication operation ($\kappa$-product). Both $\kappa$-generalized methods reduce to the standard ones, respectively, in the limit of $\kappa \to 0$. As the $q$-product is the most fundamental ingredient in the $q$-generalization of Gauss’ law of error [4], the $\kappa$-product enables us generalizing Gauss’ law of error for the $\kappa$-Gaussian.

As seen in both $q$- and $\kappa$-generalizations of the maximum likelihood principle, we remark that for any set of one-parameter generalizations of exponential and logarithmic functions we can generalize the likelihood function by introducing a suitable multiplication operation or product. Let us here denote this product as $\otimes_a$, where $a$ stands for a real-parameter of the deformed functions. By using this product we can define the generalized likelihood function, which is similar to Eq. (24) but $\kappa$-products are replaced with the products $\otimes_a$. The associated Gaussian distribution can be obtained from this $a$-parameter generalization of Gauss’ law of error.

The difference among $q$-, $\kappa$-, and other generalizations of Gauss’ law of error apparently lies in the definition of each generalized likelihood function in terms of the generalized products. For the standard case, the standard products of each distribution $f(\theta - x_i)$ define the original Gauss likelihood function $L(\theta)$, which is the $\kappa \to 0$ limit of Eq. (24). It is well-known that the standard products of $f$ means all $f$ are statistically independent. Unfortunately until now there is no consensus for interpreting any generalized likelihood function nor generalized product, although we feel that a certain kind of correlation is relevant [17].

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