On functors from category of Giri algebras to category of convex spaces

Tomáš Crháč

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Abstract

In [1] the author asserts that the category of convex spaces is equivalent to the category of Eilenberg-Moore algebras over the Giri monad. Some of the statements employed in the proof of this claim have been refuted in our earlier paper [4]. Building on the results of that paper we prove that no such equivalence exists.

1 Introduction

The Giri monad \((\mathcal{G}, \eta, \mu)\) on the category of measurable spaces \(\text{Meas}\) has been introduced in [3]. The functor \(\mathcal{G}\) maps every measurable space \(X\) to the measurable space \(\mathcal{G}(X)\) of all probability measures on \(X\). The space \(\mathcal{G}(X)\) is endowed with the \(\sigma\)-algebra induced by the evaluations \(ev_K : \mathcal{G}(X) \to \mathbb{I}, \quad K \in \Sigma X,\)

where \(\Sigma X\) is the \(\sigma\)-algebra of \(X\), and \(\mathbb{I}\) is the unit interval. Every measurable map \(f : X \to Y\) is sent to the pushforward map \(\mathcal{G}(f)\) defined by

\[
\mathcal{G}(f)(\varphi) = \varphi \circ f^- \quad \forall \varphi \in \mathcal{G}(X),
\]

where \(f^-\) denotes the inverse image map. The unit \(\eta : X \to \mathcal{G}(X)\) of the Giri monad assigns the Dirac measure to every element of \(X\), and the counit \(\mu : \mathcal{G}^2(X) \to \mathcal{G}(X)\) is given by the integral

\[
\mu(\varphi)(K) = \int_{\xi \in \mathcal{G}(X)} \xi(K) d\varphi \quad \forall \varphi \in \mathcal{G}^2(X), K \in \Sigma X.
\]

The Eilenberg-Moore algebras over the Giri monad, called Giri algebras throughout this paper, will be written as pairs \((X, \kappa)\), where \(X\) is a measurable space, and \(\kappa : \mathcal{G}(X) \to X\) is the structure map. The category of Giri algebras will be denoted by \(\text{Meas}^\mathcal{G}\).

For a treatment of convex spaces we refer to [2] and especially draw the reader’s attention to Lemma 3.8, which is employed in our proof. Herein
we stick to the definition of the convex structure on a set $A$ by means of a family of binary operations

$$+_r : A \times A \to A, \quad r \in [0, 1],$$

called the \textit{convex combination operations}, satisfying the following axioms:

\textbf{unit law}

$$x +_0 y = x$$

\textbf{idempotency}

$$x +_r x = x$$

\textbf{parametric commutativity}

$$x +_r y = y +_{1-r} x$$

\textbf{deformed parametric associativity}

$$(x +_r y) +_s z = x +_t (y +_{s/t} z),$$

where $s > 0$ and $t = r + s - rs$.

Maps of convex spaces preserving the convex combination operations are called \textit{affine} and the category of convex spaces with affine maps as morphisms is denoted by $\text{Cvx}$.

\textbf{Theorem.} There is no covariant fully faithful functor from the category of Giri algebras to the category of convex spaces.

The proof of the theorem is deferred to Section 4. An outline of the reasoning reads as follows: we define a functor $\Phi$ from the category of Giri algebras to the category of convex spaces and show that the functor is not full. Next we prove that any fully faithful functor would have to be naturally isomorphic to $\Phi$, but then it cannot be full—a contradiction.

An immediate corollary is that the category of Giri algebras is not equivalent to the category of convex spaces, refuting the assertion of [1].

\section{Preliminaries}

As a Giri algebra, the unit interval $I$ is endowed with the Borel $\sigma$-algebra, and its structure map $E$ is given by

$$E(\varphi) = \int_{x \in I} x \, d\varphi \quad \forall \varphi \in \mathcal{G}(I).$$
More generally, for a set \( M \), let \( \mathbb{I}^M \) denote the set of all functions from the set \( M \) to the unit interval. As a measurable space, \( \mathbb{I}^M \) is endowed with the evaluation \( \sigma \)-algebra, i.e., the \( \sigma \)-algebra induced by the evaluation maps
\[
ev_m : \mathbb{I}^M \to \mathbb{I}, \quad \forall m \in M,
\]
where \( \ev_m(\alpha) = \alpha(m) \). The structure map \( E_M \) defined by
\[
E_M(\varphi)(m) = \int_{\mathbb{I}^M} \ev_m d\varphi \quad \forall \varphi \in \mathcal{G}(\mathbb{I}^M),
\]
makes \( (\mathbb{I}^M, E_M) \) a Giri algebra.

**Lemma 1.** Let \( \overline{0} : M \to \mathbb{I} \) be the zero constant function. If the set \( M \) is uncountable, then \( \{ \overline{0} \} \) is not measurable in \( \mathbb{I}^M \).

**Proof.** Let \( A \) be the free convex space over \( M \). Recall that \( A \) is the set of maps \( x \in \mathbb{I}^M \) with \( x^{-1}(0) \) cofinite in \( M \) and \( \sum_{m \in M} x(m) = 1 \), and \( A \) has the pointwise convex structure. Consider the set \( \text{Cvx}(A, \mathbb{I}) \) of affine maps, with the evaluation \( \sigma \)-algebra. Then the map
\[
f : \mathbb{I}^M \to \text{Cvx}(A, \mathbb{I})
\]
given by
\[
f(\alpha)(x) = \sum_{m \in M} \alpha(m)x(m) \quad \forall \alpha \in \mathbb{I}^M, x \in A
\]
yields an isomorphism of measurable spaces, and \( f(\overline{0}) \) is the constant zero function \( A \to \mathbb{I} \). Now the conclusion follows from in [4, Lemma 4]. \( \square \)

Viewing the unit interval as a convex space, it is endowed with the usual convex structure
\[
x +_r y = (1 - r)x + ry \quad \forall r \in [0, 1] \quad \forall x, y \in \mathbb{I}.
\]

Recall that for every pair of elements \( x, y \) of a convex space \( A \), the path map \( \pi_{x,y} : \mathbb{I} \to A \) is defined by \( \pi_{x,y}(r) = x +_r y \), and it is the unique affine map satisfying
\[
\pi_{x,y}(0) = x \& \pi_{x,y}(1) = y.
\]

Let us now turn the attention to the two point object \( \mathbf{2} = \{0, 1\} \). Its convex structure is determined by
\[
0 +_r 1 = \begin{cases} 
0 & \text{if } r < 1, \\
1 & \text{otherwise}.
\end{cases}
\]

This convex space allows for the definition of the characteristic function, which we denote by \( \chi \). In what follows we are going to characterize the sets whose characteristic function is affine.
A subset of a convex space \(A\) is said to be \textit{convex} if it is closed under the convex combination operations. The empty set, all singletons, \(A\) itself, and the image and the inverse image of a convex set under an affine map are all convex.

**Lemma 2.** Let \(K\) be a subset of a convex space \(A\). Then the characteristic function \(\chi_K : A \rightarrow 2\) is affine if and only if \(K\) is convex and satisfies the following coconvexity condition:

\[
(2) \quad x +_r y \in K \implies x \in K \quad \forall r < 1 \quad \forall x, y \in A.
\]

**Proof.** First assume that \(\chi_K\) is affine. Then \(K\) is the inverse image of the set \(\{1\} \subseteq 2\), hence convex. For \(r < 1\) and \(x, y \in A\) with \(x +_r y \in K\) we have

\[
\chi_K(x) +_r \chi_K(y) = \chi_K(x +_r y) = 1,
\]

which is possible only if \(\chi_K(x) = 1\), thus \(x \in K\).

Let us on the other hand suppose that \(K\) is convex and satisfies condition (2). We have the following cases:

(i) \(x, y \in K\). Then also \(x +_r y \in K\), and

\[
\chi_K(x +_r y) = 1 = 1 +_r 1 = \chi_K(x) +_r \chi_K(y).
\]

(ii) \(x, y \notin K\). From (2) it follows that the complement of \(K\) is convex, so that \(x +_r y \notin K\), and

\[
\chi_K(x +_r y) = 0 = 0 +_r 0 = \chi_K(x) +_r \chi_K(y).
\]

(iii) \(x \notin K, y \in K\). For \(r = 1\) we have

\[
\chi_K(x +_r y) = \chi_K(y) = \chi_K(x) +_r \chi_K(y),
\]

whereas for \(r < 1\) it follows from (2) that \(x +_r y \notin K\), thus

\[
\chi_K(x +_r y) = 0 = 0 +_r 1 = \chi_K(x) +_r \chi_K(y).
\]

(iv) \(x \in K, y \notin K\). Then employing the previous case we have

\[
\chi_K(x +_r y) = \chi_K(y +_{1-r} x)
\]

\[
= \chi_K(y) +_{1-r} \chi_K(x) = \chi_K(x) +_r \chi_K(y).
\]

\(\square\)

As a measurable space, \(2\) is endowed with the discrete \(\sigma\)-algebra. The function \(h : G(2) \rightarrow I\) defined by

\[
h(\varphi) = \varphi(\{1\}) \quad \forall \varphi \in G(2)
\]

is an isomorphism of the Giri algebras \((G(2), \mu_2)\) and \((I, E)\). Moreover, the path map \(\pi_{0,1} : I \rightarrow 2\) is measurable, and the composition

\[
(3) \quad \epsilon_2 = \pi_{0,1} \circ h
\]

provides a structure map, making \((2, \epsilon_2)\) a Giri algebra.
3 Convex structure induced by structure map

Let \( X \) be a measurable space. The underlying set of \( G(X) \) has a natural pointwise convex structure

\[
(\varphi +_r \psi)(K) = \varphi(K) +_r \psi(K) \quad \forall K \in \Sigma X.
\]

Given any measurable map \( f : X \to Y \), a simple computation shows that \( G(f) \) is affine with respect to the pointwise convex structures, defined as above, on \( G(X) \) and \( G(Y) \).

For a Giri algebra \((X, \kappa)\), the convex structure induced on \( X \) by \( \kappa \) is defined by

\[
\alpha +_r \beta = \kappa(\eta_X(\alpha) +_r \eta_X(\beta)), \quad \text{for all } r \in [0,1] \text{ and } \alpha, \beta \in X.
\]

It is obvious that the operations given by (5) satisfy the unit law, idempotency, and parametric commutativity axioms. The deformed parametric associativity axiom follows from

\[
\kappa(\varphi +_r \psi) = \kappa(\varphi) +_r \kappa(\psi) \quad \forall \varphi, \psi \in G(X),
\]

which is in turn an easy consequence of the equality

\[
\varphi +_r \psi = \mu_X(\eta_X(\varphi) +_r \eta_X(\psi)),
\]

where the (pointwise) convex combination operation on the right is carried out in \( G^2(X) \).

For all morphisms \( f : (X, \kappa) \to (Y, \lambda) \) we furthermore have

\[
f(\alpha +_r \beta) = f\left(\kappa(\eta_X(\alpha) +_r \eta_X(\beta))\right) \\
= \lambda\left(G(f)(\eta_X(\alpha) +_r \eta_X(\beta))\right) \\
= \lambda\left(G(f)(\eta_X(\alpha)) + G(f)(\eta_X(\beta))\right) \\
= \lambda(\eta_Y(f(\alpha)) +_r \eta_Y(f(\beta))) \\
= f(\alpha) +_r f(\beta),
\]

so that \( f \) is affine with respect to the convex structures induced on \( X \) and \( Y \) by the structure maps \( \kappa \) and \( \lambda \), respectively.

As a result of the considerations above, it is possible to define a covariant functor

\[
\Phi : \text{Meas}^G \to \text{Cvx}
\]

so that \( \Phi(X, \kappa) \) is the set \( X \) with the convex structure induced by \( \kappa \), and \( \Phi(f) = f \) for all morphisms of Giri algebras.
Lemma 3. The functor $\Phi$ enjoys the following properties:

(i) $\Phi(\mathbb{I}^M, E_M)$ has the pointwise convex structure;

(ii) $\Phi(2, \varepsilon_2)$ has the usual convex structure on $2$, given by equality (1).

Proof. Both assertions follow from straightforward computations:

(i) For all $\alpha, \beta \in \mathbb{I}^M$ and $m \in M$ we have

$$
(\alpha +_r \beta)(m) = E_M(\eta_I^M(\alpha) +_r \eta_I^M(\beta))(m)
= \int_{\mathbb{I}^M} ev_m d(\eta_I^M(\alpha) +_r \eta_I^M(\beta))
= \left( \int_{\mathbb{I}^M} ev_m d\eta_I^M(\alpha) \right) +_r \left( \int_{\mathbb{I}^M} ev_m d\eta_I^M(\beta) \right)
= \alpha(m) +_r \beta(m).
$$

(ii) With $h$ as in the definition (3) of the map $\varepsilon_2$ we have

$$
0 +_r 1 = \pi_{0,1}(h(\eta_2(0)) +_r h(\eta_2(1))) = \pi_{0,1}(0 +_r 1) = \pi_{0,1}(r).
$$

Lemma 4. The functor $\Phi$ is not full.

Proof. Let $M$ be an arbitrary uncountable set. In view of Lemma 3 it is possible to write the characteristic map of $\{0\}$ as

$$
\chi_{\{0\}} : \Phi(\mathbb{I}^M, E_M) \to \Phi(2, \varepsilon_2),
$$

where $0$ is the constant zero function. The set $\{0\}$ is convex and a simple argument shows that it satisfies the coconvexity condition (2), thus $\chi_{\{0\}}$ is affine due to Lemma 2.

From Lemma 1 it however results that $\chi_{\{0\}}$ is not measurable.

Lemma 5. Let $(X, \kappa)$ be an arbitrary Giri algebra. Then all path maps $\pi_{\alpha,\beta} : \mathbb{I} \to \Phi(X, \kappa)$ belong to $\text{Meas}^{\mathbb{G}}((\mathbb{I}, E), (X, \kappa))$.

Proof. Note that from equality (6) it follows that the convex structure induced on $\mathbb{G}(X)$ by $\mu_X$ is simply the pointwise convex structure defined
by (4), so that for all $\varphi, \psi \in \mathcal{G}(X)$ the path map $\pi_{\varphi, \psi}$ is measurable, and

$$
\mu_X(\mathcal{G}(\pi_{\varphi, \psi})(\omega))(K) = \int_{\xi \in \mathcal{G}(X)} \xi(K) d\mathcal{G}(\pi_{\varphi, \psi})(\omega)
= \int_{r \in I} \pi_{\varphi, \psi}(r)(K) d\omega
= \int_{r \in I} (\varphi(K) + r \psi(K)) d\omega
= (\varphi(K) \int_{r \in I} (1 - r) d\omega) + (\psi(K) \int_{r \in I} r d\omega)
= \varphi(K) + e_{\langle \omega \rangle} \psi(K)
= \pi_{\varphi, \psi}(E(\omega))(K),
$$

thus $\pi_{\varphi, \psi} \in \text{Meas}^{\mathcal{G}}((I, E), (\mathcal{G}(X), \mu_X))$. Since

$$
\pi_{\alpha, \beta} = \kappa \circ \pi_{\eta_X(\alpha), \eta_X(\beta)},
$$

we conclude that $\pi_{\alpha, \beta} \in \text{Meas}^{\mathcal{G}}((I, E), (X, \kappa))$. \qed

4 Proof of the theorem

Let $F : \text{Meas}^{\mathcal{G}} \to \text{Cvx}$ be a covariant functor, and assume by contradiction that $F$ is full and faithful.

The set $\text{Meas}^{\mathcal{G}}(1, 0)$, where $0$ and $1$ are initial and terminal objects, respectively, is empty. Hence $\text{Cvx}(F(1), F(0))$ is also empty and it follows that $F(0)$ is an initial convex algebra and that $F(1) \neq \emptyset$. For arbitrary $x, y \in F(1)$, let $\overline{x}$ and $\overline{y}$ denote the constant maps $F(1) \to F(1)$ with the values $x$ and $y$, respectively. Since the functor $F$ is full, there are morphisms $u, v \in \text{Meas}^{\mathcal{G}}(1, 1)$ such that $F(u) = \overline{x}$ and $F(v) = \overline{y}$, but as $1$ is a terminal Giri algebra, $u$ must be equal to $v$, thus $\overline{x} = \overline{y}$, and $x = y$. This shows that $F(1)$ is a singleton, and thus a terminal object in $\text{Cvx}$.

The unique element of $F(1)$ will be denoted by $\ast$ in the sequel. For every Giri algebra $(X, \kappa)$ we now define a map $\tau_{X, \kappa} : \Phi(X, \kappa) \to F(X, \kappa)$ by

$$
\tau_{X, \kappa}(\alpha) = F(\overline{\alpha})(\ast) \quad \forall \alpha \in X,
$$

where $\overline{\alpha} : 1 \to X$ denotes the constant map with value $\alpha$. We claim that $\tau$ is a natural isomorphism, but then, as a consequence of Lemma 4, the functor $F$ is not full, a contradiction.

To prove the injectivity of $\tau_{X, \kappa}$, consider arbitrary $\alpha, \beta \in X$. From $\tau_{X, \kappa}(\alpha) = \tau_{X, \kappa}(\beta)$ it follows that $F(\overline{\alpha}) = F(\overline{\beta})$, but then $\alpha = \beta$ as $F$ is faithful. In order to prove that $\tau_{X, \kappa}$ is surjective, take arbitrary $x \in F(X, \kappa)$. Since $F$ is full, there exists $\alpha \in X$ such that $F(\overline{\alpha})(\ast) = x$, i.e., $\tau_{X, \kappa}(\alpha) = x$. 

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Next we show that \( \tau \) is natural. Let \( u : (X, \kappa) \to (Y, \lambda) \) be a morphism of Giri algebras. Then for all \( \alpha \in X \) we have

\[
F(u) (\tau_{X,\kappa}(\alpha)) = F(u) (F(\pi)(\ast)) \\
= F(u \circ \pi)(\ast) = F\left(\frac{u(\alpha)}{u(\alpha)}\right)(\ast) = \tau_{Y,\lambda}(u(\alpha)).
\]

The crux of the proof thus depends in showing that all \( \tau_{X,\kappa} \) are affine. Since \( \tau_{I,E} \) is bijective, we may, for all \( r \in [0,1] \), define operations \( \oplus_{r} \) by the formula

\[
x \oplus_{r} y = \tau_{I,E}^{-1}(\tau_{I,E}(x) +_{r} \tau_{I,E}(y)) \quad \forall x, y \in \mathbb{I}.
\]

Straightforward calculations show that this family of operations defines a convex structure on \( \mathbb{I} \). Moreover, for all \( u \in \text{Meas}^{\mathcal{G}}(\mathbb{I}, \mathbb{I}) \) we have

\[
\tau_{I,E}(u(x \oplus_{r} y)) = (\tau_{I,E} \circ u)\left(\tau_{I,E}^{-1}(\tau_{I,E}(x) +_{r} \tau_{I,E}(y))\right) \\
= F(u) (\tau_{I,E}(x) +_{r} \tau_{I,E}(y)) \\
= F(u) (\tau_{I,E}(x)) +_{r} F(u) (\tau_{I,E}(y)) \\
= \tau_{I,E}(u(x)) +_{r} \tau_{I,E}(u(y)),
\]

hence, since \( \tau_{I,E} \) is injective,

\[
u(x \oplus_{r} y) = u(x) \oplus_{r} u(y).
\]

From Lemma 5 it follows that for all \( a, b \in \mathbb{I} \) the path maps \( \pi_{a,b} \) are elements of \( \text{Meas}^{\mathcal{G}}(\mathbb{I}, \mathbb{I}) \), thus

\[
\pi_{a,b}(x \oplus_{r} y) = \pi_{a,b}(x) \oplus_{r} \pi_{a,b}(y),
\]

and we apply [2, Lemma 3.8] to conclude that \( \oplus_{r} \) and \( +_{r} \) in fact coincide, but then

\[
\tau_{I,E}(x +_{r} y) = \tau_{I,E}(x) +_{r} \tau_{I,E}(y).
\]

For every Giri algebra \((X, \kappa)\) and \( \alpha, \beta \in X \) we therefore have

\[
\tau_{X,\kappa}(\alpha +_{r} \beta) = \tau_{X,\kappa}(\pi_{a,b}(r)) \\
= F(\pi_{a,b})(\tau_{I,E}(r)) \\
= F(\pi_{a,b})(\tau_{I,E}(0 +_{1} \tau_{I,E}(1))) \\
= F(\pi_{a,b})(\tau_{I,E}(0)) +_{r} \tau_{I,E}(1)) \\
= \tau_{X,\kappa}(\pi_{a,b}(0)) +_{r} \tau_{X,\kappa}(\pi_{a,b}(1)) \\
= \tau_{X,\kappa}(\alpha) +_{r} \tau_{X,\kappa}(\beta),
\]

which concludes the proof.
References

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