M Theory, Type IIA String
and 4D $N = 1$ SUSY
$SU(N_L) \otimes SU(N_R)$ Gauge Theory

Amit Giveon and Oskar Pelc

Racah Institute of Physics, The Hebrew University
Jerusalem, 91904, Israel
E-mail: giveon@vms.huji.ac.il, oskar@shum.cc.huji.ac.il

Abstract

$SU(N_L) \otimes SU(N_R)$ gauge theories are investigated as effective field theories on
$D_4$ branes in type IIA string theory. The classical gauge configuration is shown
to match quantitatively with a corresponding classical $U(N_L) \otimes U(N_R)$ gauge the-
ory. Quantum effects freeze the $U(1)$ gauge factors and turn some parameters into
moduli. The $SU(N_L) \otimes SU(N_R)$ quantum model is realized in M theory. Starting
with an $N = 2$ configuration (parallel $NS$ fivebranes), the rotation of a single $NS$
fivebrane is considered. Generically this leads to a complete lifting of the Coulomb
moduli space. The implications of this result to field theory and the dynamics of
branes are discussed. When the initial $M$ fivebrane is reducible, part of the Coulomb
branch may survive. Some such situations are considered, leading to curves describ-
ing the effective gauge couplings for $N = 1$ models. The generalization to models
with more gauge group factors is also discussed.
1. Introduction

The realization of supersymmetric gauge field theories as theories describing the low energy dynamics of branes in string theory [1], is an approach that led to much progress in the understanding of both subjects. To study $N = 4$ SUSY gauge theories in 3 dimension, a particularly simple and useful construction was introduced in [2], involving type IIB $D$ branes and $NS_5$ branes in flat spacetime. To consider $N = 1$ SUSY gauge theories in 4 dimension, a construction involving type IIA $D$ branes and $NS_5$ branes in flat spacetime was presented and studied in [3]. These constructions were generalized in several directions [4]-[40], leading to realizations of field theories in various dimensions and with various amount of unbroken supersymmetry. In particular, one may consider type IIA $D_4$ branes with boundaries on $NS_5$ and/or $D_6$ branes [3]. The field theory describing the low energy dynamics of the $D_4$ branes is a 4D gauge theory[1]: $N_c$ parallel $D_4$ branes suspended (in one of their directions) between two parallel $NS_5$ branes will lead to $N = 2$ SYM $U(N_c)$ theory; adding $D_6$ branes, or semi infinite $D_4$ branes, corresponds to fundamentals in the field theory, leading to $N = 2$ SQCD; more than two $NS_5$ branes lead to a gauge group with more $U(N)$ factors. A rotation of some of the branes breaks the supersymmetry further, and an appropriate rotation leads to $N = 1$ supersymmetric models. Such rotations correspond in field theory to turning on masses for some of the adjoint chiral superfields (chiral components in the $N = 2$ vector multiplets).

The above description was obtained in the limit of small string coupling constant, (where string perturbation theory is reliable). The branes are considered to be flat and one ignores interactions between them. One also considers energy scales below the string scale, so that all the massive modes of the string can be ignored. For a sufficiently small string coupling, the gauge coupling constant is also small and, consequently, one obtains correspondence with a classical field theory. Quantum corrections to the classical field theory correspond to quantum corrections in the dynamics of the branes. In particular, it was shown in [19] that a $D_4$ brane ending on a $NS_5$ brane causes the latter to bend. One of the implications to the corresponding field theory is that the $U(1)$ factor in the $U(N)$ gauge group is frozen. Another effect was identified through a comparison with known quantum results in field theory. It was shown in [11] that when the $NS_5$ branes are not parallel there are quantum forces between $D_4$ branes, corresponding in field theory to a dynamically generated superpotential.

The weakly coupled type IIA string is a particular limit of M theory. A systematic study of many aspects of the quantum behavior in a limit of M theory corresponding to a strong string coupling was initiated in [19]; the string coupling turns into a large 10'th spatial direction, and the $D_4$ and $NS_5$ branes configuration of the type IIA limit turns into a worldvolume of the M theory fivebrane $M_5$. This approach was followed also in [20]-[40]. One considers a small Planck length in 11 dimensions (relative to the characteristic scales of the configuration), and this justifies the use of a low energy (long wavelength) approximation. The fivebrane worldvolume is of the form $\mathbb{R}^{3+1} \times \Sigma$, where $\Sigma$ is a Riemannian manifold.

---

1 It is 4D rather than 5D, because the $D_4$ branes are finite in one of their spatial dimensions, so this dimension is invisible in the long wavelength approximation.
surface, and it can be determined by imposing appropriate asymptotic conditions, which represent the characterization of the quantum system considered. Once the curve is known, it provides some information. The space of solutions (corresponding to given asymptotic conditions), represents the moduli space of vacua of the quantum system, so one obtains in this way the number of vacua (or the dimension of the moduli space, when it is continuous). Note that this should be the quantum moduli space, possibly modified by a dynamically generated superpotential. This means that the M theory description is expected to take into account the effect that in type IIA string theory was interpreted as a force between $D_4$ branes (as mentioned above). The genus of $\Sigma$ is the number of massless Abelian gauge fields in the low energy effective field theory and when it is non vanishing, $\Sigma$ determines the effective gauge coupling – it is precisely the Seiberg-Witten (SW) curve [41] of the corresponding gauge theory. The curves for $N = 2$ models with a simple classical gauge group and fundamental matter were found [19, 21, 22], in complete agreement with the known results [42]-[46]. This method was also used to obtain unknown results. In particular, curves for several $N = 2$ SUSY models with product gauge groups were derived [19, 21, 22, 38].

In this work we continue the study of the realization of 4D SUSY gauge theories in the type IIA string and M theory. Some of our goals are to fill gaps in the “dictionary” relating string and field theoretical phenomena and to use this dictionary to obtain information about both subjects. We consider models based on $U(N_L) \times U(N_R)$ and $SU(N_L) \times SU(N_R)$ gauge groups, with bi-fundamentals and possibly also with fundamentals and/or adjoints. Such models are realized by three $NS_5$ branes connected by $D_4$ branes (and additional semi-infinite $D_4$ branes). We concentrate on branches of vacua with vanishing vev’s of the fundamentals. Other aspects of this model were considered in [8]. In section 2, we perform a field-theoretical analysis. We start with the classical analysis of both models, emphasizing the differences between them. We find that the main difference is that the Fayet-Iliopoulos parameters in the $U(N)$ models transform to moduli in the $SU(N)$ model. We also compute the resulting moduli spaces of vacua. Next, we consider quantum corrections to the moduli space, in situations that are investigated later in M theory. In section 3, we consider the brane configuration in type IIA string theory. First the “classical” description is given. It should correspond to the classical $U(N_L) \times U(N_R)$ model and we indeed display a detailed, quantitative, geometric identification of almost all the parameters and moduli of the field theory. We list the evidence for this identification. In particular, a perfect match is found in the moduli spaces. The only parameters that seem to have no geometrical manifestation are the Yukawa couplings of the bi-fundamentals. We then describe the modifications in this picture caused by quantum effects. In particular, the $U(1)$ factors are frozen, which means that the low energy effective theory has an $SU(N_L) \times SU(N_R)$ gauge group.

In section 4, we move to the strong string coupling limit – M theory – looking for the $M_5$ brane that corresponds to the type IIA brane configuration. We start with an $N = 2$ supersymmetric configuration – parallel $NS_5$ branes. In this configuration there is a Coulomb branch, parameterized by the vev’s of the adjoint fields. The general form of the corresponding SW curve was determined in [19]. We find the explicit curve, i.e.,
its dependence on the coordinates and moduli, by considering various limits. We then consider what happens when one of the branes is rotated (breaking the supersymmetry to \( N = 1 \)). We find that generically (i.e., when the curve is irreducible), the full Coulomb branch is lifted. This remains so for models with more \( \text{NS}_5 \) branes (more gauge group factors). In field theory this means, for example, that when a mass is given to the adjoint field of the first gauge factor, all the moduli are fixed, including those corresponding to the other adjoints. This differs from the classical situation, where only that part of the moduli space corresponding to the massive adjoint is lifted. One of the implications of this result is that a Coulomb branch can survive only when the curve is reducible, which means that the polynomial that defines it is factorizable. We explore this possibility and then investigate in detail one such situation in which the central \( \text{NS}_5 \) brane, detached from the rest of the configuration, is rotated. This leads to SW curves for \( N = 1 \) models with \( SU(N) \otimes SU(N) \) gauge group with fundamentals and bi-fundamentals (and no adjoints). Some of these curves were derived before by field-theoretical considerations \([53, 54]\) and we find complete agreement with them. For other models our results seem to be new. Finally, we return to the irreducible case and analyze in detail the rotation of the left \( \text{NS}_5 \) brane in the absence of fundamentals, obtaining explicit solutions for the curve. We check the limit of vanishing gauge coupling for the right factor and recover correctly the results for SQCD found in \([25]\). We conclude in section \( 5 \) with a discussion. In an appendix, we explain our strategy in the use of symmetries.

2. The Field Theoretical Models

In this section we present and analyze the field-theoretical models that emerge from the brane configurations discussed in the later sections.

2.1 The Classical \( U(N_L) \times U(N_R) \) Model

We consider an \( N = 1 \) supersymmetric gauge theory with a gauge group \( U(N_L) \times U(N_R) \), two bi-fundamentals and for each factor of the gauge group, an adjoint and several fundamentals (\( n_L \) and \( n_R \) respectively). The matter content is described in the following table:

| \( U(N_L) \) | \( \bar{N}_L^- \) | \( \bar{N}_L^+ \) | \( \text{Adj}_0 \oplus [1]_0 \) | \( [N_L]_+ \) | \( [\bar{N}_L^-]_0 \) | \( [1]_0 \) | \( [1]_0 \) | \( [\bar{N}_L^-]_0 \) |
| \( U(N_R) \) | \( \bar{N}_R^- \) | \( \bar{N}_R^+ \) | \( [1]_0 \) | \( [1]_0 \) | \( [\bar{N}_R^-]_0 \) | \( [\bar{N}_R^-]_0 \) |

Here \( a = 1 \ldots N_L \) and \( \bar{a} = 1 \ldots N_R \) are color indices, \( i = 1 \ldots n_L \) and \( \bar{i} = 1 \ldots n_R \) are "flavor" indices; \( [R]_q \) denotes a representation of \( U(N) = SU(N) \otimes U(1) \): a representation

\(^2\)Curves for \( N = 1 \) models are known only for a limited number of cases: \([17, 18, 42, 19, 54]\).
\[ R \] under \( SU(N) \) and a \( U(1) \) charge \( q \). The superpotential considered is
\[
W = \xi_L \text{tr} A_L + \frac{1}{2} \mu_L \text{tr} A_L^2 + \text{tr}(m_L \bar{Q}_L Q_L) + \text{tr}(\lambda_L \bar{Q}_L A_L Q_L) + \\
+ \xi_R \text{tr} A_R + \frac{1}{2} \mu_R \text{tr} A_R^2 + \text{tr}(m_R \bar{Q}_R Q_R) + \text{tr}(\lambda_R \bar{Q}_R A_R Q_R) + \\
+ m_F \text{tr}(\bar{F} F) + \kappa_L \text{tr}(\bar{F} A_L F) + \kappa_R \text{tr}(F A_R \bar{F})
\] (2.2)

(in matrix notation)\(^3\). We also allow Fayet-Iliopoulos (FI) terms for the \( U(1) \) vector superfields \( \text{tr} V_L, \text{tr} V_R \)
\[
\int d^2\theta d^2\bar{\theta}(\eta_L \text{tr} V_L + \eta_R \text{tr} V_R) , \quad \eta_L, \eta_R \in \mathbb{R} .
\]

We will determine the (classical) vacua of this model, restricting our attention to vacua with \( Q = 0 \). The \( D \)-term equations for a supersymmetric vacuum are
\[
[A_L, A_L^\dagger] + F F^\dagger - \bar{F} \bar{F} = -\eta_L I_{N_L} , \quad [A_R, A_R^\dagger] + \bar{F} \bar{F}^\dagger - F^\dagger F = -\eta_R I_{N_R} .
\] (2.3)

(where \( I_N \) is the \( N \) dimensional identity matrix) and they imply\(^4\):

- \( [A_L, A_L^\dagger] = [A_R, A_R^\dagger] = 0 \);
- \( \eta_L = -\eta_R =: \eta \);
- if \( N_L \neq N_R \) then \( \eta \) must vanish;
- \( \eta \) can always be chosen non-negative (possibly after a \( L \leftrightarrow R \) transformation);
- \( F \) and \( \bar{F} \) can be simultaneously diagonalized by a color rotation and then they assume the following form
\[
F = \text{diag}\{c_a\} , \quad \bar{F} = \text{diag}\{\bar{c}_a\} ,
\] (2.4)

with
\[
\bar{c}_a = +\sqrt{|c_a|^2 + \eta} , \quad c_a \in \mathbb{C} , \quad a = 1 \ldots N := \min(N_L, N_R) \quad (2.5)
\]

(for \( N_L \neq N_R \) there is also a block of zeros with an appropriate dimension).

The space of inequivalent \((F, \bar{F})\) configurations is, therefore, \( N \) (complex) dimensional. It can be parameterized by\(^5\) \( \{c_a\} \) or, equivalently, by \( \{b_a\} \), where
\[
b_a = c_a \bar{c}_a
\] (2.6)

\(^3\)In some situations, some of the parameters are redundant. For example, when \( \mu \neq 0 \), one can eliminate \( \xi \) by a (scalar) shift of \( A \), while when \( \mu = 0 \), such a shift can be used to eliminate \( m_F \). This will be also apparent in the corresponding brane configuration, to be described later.

\(^4\)For generic values of \( m_L, m_R \), this is actually implied by the equations of motion.

\(^5\)The vanishing of \( [A, A^\dagger] \) is explained in \(^55\); the rest of the above results can be obtained following the procedure described in the appendix of \(^53\) for a similar model.

\(^6\)More precisely, configurations \( \{c_a\} \) related by a permutation are gauge equivalent, so the moduli space is obtained by dividing the \( \{c_a\} \) space by the permutation group.
are the $N$ common eigenvalues of $\hat{F}F$ and $F\hat{F}$ (the extra eigenvalues, existing for $N_L \neq N_R$, all vanish). As usual \cite{50}, the moduli space can be parameterized also by gauge-invariant polynomials

$$T_l := \text{tr}(\hat{F}F)^l, \quad l = 1 \ldots N.$$  

The gauge choice (2.4) also shows explicitly that the gauge symmetry unbroken by $F, \hat{F}$ is generically $U(1)^N_D$ (diagonally embedded in $U(N_L) \otimes U(N_R)$). We adopt this gauge choice in the following classical analysis.

We now turn to the F-term equations $dW = 0$. We assume $\kappa \neq 0$ (for both left and right, as is the case for $N = 2$ supersymmetry) and, to simplify the analysis we “absorb” $\kappa$ into a redefinition of $A, \xi, \mu$ and $\lambda$. Practically this means that we set $\kappa = 1$ in the superpotential\footnote{$\kappa$ does not disappear! Rather it moves to the kinetic term of $A$.}  and, later, $\kappa$ can be recovered by

$$A \rightarrow \kappa A, \quad \xi \rightarrow \xi/\kappa, \quad \mu \rightarrow \mu/\kappa^2, \quad \lambda \rightarrow \lambda/\kappa.$$  

In the same way one can absorb $\lambda$ in $\tilde{Q}$ (assuming it is invertible), by setting $\lambda = 1$ (after absorbing $\kappa$) and recover it (before recovering $\kappa$) by\footnote{$\kappa$ and $\lambda$ can also be recovered using symmetries. This is explained in Appendix B.}

$$Q \rightarrow Q\lambda, \quad m \rightarrow \lambda^{-1}m.$$  

The equations obtained are

$$0 = m_F F + A_L F + FA_R,$$  

$$0 = m_F \hat{F} + \hat{F} A_L + A_R \hat{F},$$  

$$0 = \xi_L I_{N_L} + \mu_L A_L + \hat{F},$$  

$$0 = \xi_R I_{N_R} + \mu_R A_R + \hat{F}$$  

We now solve them for different choices of parameters.

2.1.1 $\mu_L \mu_R = 0$

Assume $\mu_R = 0$. From eq. (2.11) we obtain

$$\hat{F} F = -\xi_R I_{N_R}$$  

which implies that all $c_a$’s are determined by $\xi_R$, they are equal and if they do not vanish, then $N_L \geq N_R$.

If $\mu_L$ also vanishes then a non-trivial value for $F$ (and, therefore, also for $\xi_{L,R}$) is possible only for $N_L = N_R$ and then $\xi_L = \xi_R =: \xi$. Therefore, we have two possible situations:

**Type 1:** $\mu_L = \mu_R = 0, \eta \neq 0$ and/or $\xi \neq 0$ ($N_L = N_R =: N$)
\( \tilde{F} \) is a non-vanishing multiple of the identity, therefore, eq. (2.10) leads to
\[
m_F + A_L + A_R = 0 \quad . 
\] (2.14)

The vacua can be parameterized by \( A_L \) (which can be diagonalized by a \( U(N)_D \) color rotation preserving \( F \) and \( \tilde{F} \)). The gauge symmetry is broken, generically, to \( U(1)^N_D \) (diagonally embedded in \( U(N)_L \otimes U(N)_R \)).

**Type 2:** \( \mu_L = \mu_R = 0, \eta = \xi = 0 \)

\( F \) and \( \tilde{F} \) vanish, so \( A_L \) and \( A_R \) are independent and can be diagonalized simultaneously by a \( U(N)_L \otimes U(N)_R \) color rotation. The vacua are parameterized by \( A_L \) and \( A_R \) and the gauge group is generically broken to \( U(1)^N_L \otimes U(1)^N_R \).

If \( \mu_L \) does not vanish then \( A_L \) is fixed by its equation of motion (2.11):
\[
A_L = -\frac{1}{\mu_L} (\xi_L I_{N_L} + F \tilde{F}) \quad . 
\] (2.15)

and we have also two possible situations:

**Type 3:** \( \mu_L \neq 0 = \mu_R, \eta \neq 0 (N_L = N_R) \) and/or \( \xi_R \neq 0 (N_L \geq N_R) \)

\( \tilde{F} \) does not vanish (and, moreover, has maximal rank), therefore eq. (2.10) leads to
\[
A_R = \left[ \frac{1}{\mu_L} (\xi_L - \xi_R) - m_F \right] I_{N_R} \quad . 
\] (2.16)

Thus, in this case, there is a unique vacuum, in which the gauge symmetry is broken to \( U(N_L - N_R)_L \otimes U(N_R)_D \) (the first factor is a subgroup of \( U(N)_L \) and the second is diagonally embedded in \( U(N)_L \otimes U(N)_R \)).

**Type 4:** \( \mu_L \neq 0 = \mu_R, \eta = \xi_R = 0 \)

\( F \) and \( \tilde{F} \) vanish, so \( A_R \) is free and can be diagonalized by a \( U(N_R) \) color rotation. It parameterizes the space of vacua and the gauge group is generically broken to \( U(N_L) \otimes U(1)^{N_R} \).

2.1.2 \( \mu_L \mu_R \neq 0 \)

If \( \mu_L \) and \( \mu_R \) do not vanish then both \( A_L \) and \( A_R \) are fixed by their equations of motion
\[
A_L = -\frac{1}{\mu_L} (\xi_L I_{N_L} + F \tilde{F}) \quad , \quad A_R = -\frac{1}{\mu_L} (\xi_R I_{N_R} + \tilde{F} F) \quad . 
\] (2.17)

\(^9\)The case of \( \mu_L = 0 \neq \mu_R \) is treated similarly, with identical results (exchanging \( L \leftrightarrow R \) but not \( F \leftrightarrow \tilde{F} \)).
Observe that the above equations imply that the unbroken gauge symmetry is the sub-group preserving $F$ and $\tilde{F}$ ($A_L$ and $A_R$ do not break the gauge symmetry further). Eq. (2.10) becomes
\[
(\hat{m}_F - \frac{1}{\mu} \tilde{c}_a c_a) \tilde{c}_a = 0 ,
\]
where
\[
\hat{m}_F : = m_F - \left( \frac{\xi_L}{\mu_L} + \frac{\xi_R}{\mu_R} \right) , \quad \frac{1}{\mu} : = \frac{1}{\mu_L} + \frac{1}{\mu_R}
\]
and this leads to two possible situations:

Type 5: $\hat{m}_F \neq 0$ and/or $\frac{1}{\mu} \neq 0$

In this case there is at most one non-vanishing solution $c$ to eq. (2.13), which means that $F\tilde{F}$ and $\tilde{F}F$ have (at most) one non-vanishing eigenvalue $b$ (determined uniquely by the parameters), with some multiplicity $r$ ($0 \leq r \leq \min(N_L, N_R)$). Therefore, in this situation there is a finite number of discrete vacua, parameterized by $r$. The gauge group is broken to $U(N_L - r)_L \otimes U(r)_D \otimes U(N_R - r)_R$. The range of values that $r$ may assume depends on the parameters:

$\eta \neq 0$ ($N_L = N_R = N$)

This leads to $\tilde{c} \neq 0$, therefore, eq. (2.18) implies that $r = N$ (a unique vacuum) and, moreover, if $\frac{1}{\mu} = 0 \neq \hat{m}_F$ then there is no vacuum at all.

$\frac{1}{\mu} = 0 \neq \hat{m}_F$ ($\eta = 0$)

This leads to $\tilde{c} = 0$, which means $r = 0$ (a unique vacuum).

$\frac{1}{\mu} \neq 0, \quad \eta = 0$

In this case all values of $r$ are allowed and there are $N + 1$ discrete vacua.

Type 6: $\hat{m}_F = \frac{1}{\mu} = 0$

In this case $F$ is unconstrained by eq. (2.13) and it parameterizes the space of vacua. The gauge symmetry is broken, generically, to $U(N_L - r)_L \otimes U(1)_D \otimes U(N_R - r)_R$, where $r$ is the rank of $\tilde{F}$.

2.2 The Classical $SU(N_L) \otimes SU(N_R)$ Model

We will also have to consider a variant of the above model, obtained by freezing the $N = 2$ vector multiplets corresponding to the two $U(1)$ factors in the gauge group. By this we mean that we replace each such $N = 2$ dynamical superfield by a fixed constant (which, by supersymmetry, is also independent of the Fermionic coordinates and, therefore, contributes only to the scalar component). The result is a model with an $SU(N_L) \otimes SU(N_R)$ gauge group, and almost the same matter content, the only difference being that $\text{tr} A$ is not a dynamical field, but rather a fixed parameter. We will now list the differences between the models:
• The modulus $\text{tr} \langle A \rangle$ becomes a parameter.

One can absorb $\text{tr} A$ in the mass parameters $m_L, m_R$ and $m_F$ ($\xi$ is shifted too, but see the next paragraph), so we can assume $\text{tr} A = 0$.

• In the superpotential, $\xi$ multiplies a constant and, therefore, could be ignored. But we re-introduce these terms with a different interpretation: $\xi$ becomes a Lagrange multiplier, enforcing the constraint $\text{tr} A = 0$. So we obtain the same equations (2.9, 2.12), the only difference being that we have also the above constraint, which is an additional equation, leading to the determination of $\xi$.

• The FI terms disappear (since there are no $U(1)$ gauge fields). Nevertheless, the D-term equations (2.3) remain the same, but with a different meaning: $\eta_L, \eta_R$ are no longer (FI) parameters but rather functions of the moduli – reflecting the requirement that the other terms in the equation should be orthogonal to the (traceless) generators of the gauge group and, therefore, must combine to a multiple of the identity matrix.

Observe that $\xi$ and $\eta$ undergo the same type of change (parameter $\rightarrow$ modulus). This is obviously related to the fact that they are in the same $SU(2)_R$ multiplet.

• For $Q = 0$, $A$, $F$ and $\tilde{F}$ can still be diagonalized simultaneously, but this time (compare with eq. (2.5))

$$\tilde{c}_a = e^{i \alpha} \sqrt{|c_a|^2 + \eta},$$

(2.19)

where $\alpha, \eta \in \mathbb{R}$. For $N_L = N_R =: N$, $\eta e^{i \alpha}$ is an additional complex modulus, so the space of inequivalent $(F, \tilde{F})$ configurations is $N + 1$ (complex) dimensional. This change in dimension is reflected by the existence of additional gauge invariant polynomials:

$$D := \det F, \quad \tilde{D} := \det \tilde{F}$$

($T_N \equiv \text{tr}(\tilde{F}F)^N$ is a polynomial function of $D, \tilde{D}$ and $T_l$ with $l < N$, so there are indeed $N + 1$ independent complex moduli).

The above changes lead obviously to a different space of parameters and moduli. To find it, we can use the same equations (2.9, 2.12), with a modified meaning: $\text{tr} A$ vanishes identically and, therefore, the trace of eqs. (2.11, 2.12) determines $\xi$:

$$T_1 \equiv \text{tr} \tilde{F}F = -N_L \xi_L = -N_R \xi_R.$$  

(2.20)

2.2.1 $\mu_L = \mu_R = 0$

Eqs. (2.11, 2.12) imply that both $\tilde{F}F$ and $F \tilde{F}$ are proportional to the identity. Therefore, for $N_L \neq N_R$, $F$ and $\tilde{F}$ must vanish and we have the same moduli space as for the previous model (type 2), parameterized by independent $A_L$ and $A_R$ (which, however, represent now less degrees of freedom, since they are traceless). We will call this branch of vacua the Coulomb branch. For $N_L = N_R$, $F$ and $\tilde{F}$ do not have to vanish and, for $m_F = 0$ we obtain an additional branch, parameterized by $A_L = -A_R$, $D$ and $\tilde{D}$ (for $m_F \neq 0$, $F$ and $\tilde{F}$ are forced to vanish by eqs. (2.9, 2.10)). This branch will be called the mixed branch.
\[ \text{2.2.2 } \mu_L \neq 0 = \mu_R \]

Eq. (2.12) implies that \( \tilde{F}F \) is a multiple of the identity (which means that it must vanish for \( N_L < N_R \)) and eq. (2.11) leads to

\[ A_L = -\frac{1}{\mu_L} \left( F \tilde{F} - I_{N_L} \frac{1}{N_L} T_1 \right) , \]

Therefore, for \( N_L < N_R \) we have a unique branch (again, as in the previous model – type 4), parameterized by \( A_R \) (with vanishing \( A_L, F \) and \( \tilde{F} \)), but for \( N_L \geq N_R \) there may be another branch, with non-vanishing \( F \) and/or \( \tilde{F} \) (rank \( N_R \)). In this case equations (2.9, 2.10) imply

\[ \mu_L m_F = \left( \frac{1}{N_R} - \frac{1}{N_L} \right) T_1 , \quad A_R = 0 , \]

so the situation is as follows: for \( N_L = N_R \) there is an additional branch if and only if \( m_F = 0 \) and then it is parameterized by \( D \) and \( \tilde{D} \) (\( A_L \) vanishes). For \( N_L > N_R \), \( \tilde{F}F \) is fixed by the parameters, but the overall phase of each of them (\( \alpha \) in eq. (2.19)) remains free, so we have a compact one (real) dimensional branch of vacua.

\[ \text{2.2.3 } \mu_L \mu_R \neq 0 \]

When \( \mu_L, \mu_R \) both do not vanish, \( A_L \) and \( A_R \) are determined by their equations of motion (as in (2.17))

\[ A_L = -\frac{1}{\mu_L} \left( F \tilde{F} - I_{N_L} \frac{1}{N_L} T_1 \right) , \quad A_R = -\frac{1}{\mu_R} \left( \tilde{F}F - I_{N_R} \frac{1}{N_R} T_1 \right) \]  

and then eq. (2.10) obtains the form

\[ 0 = \left( m_F + \frac{T_1}{\mu} - \tilde{c}_a c_a \right) \tilde{c}_a , \]

where

\[ \frac{1}{\mu} := \frac{1}{\mu_L} + \frac{1}{\mu_R} , \quad \frac{1}{\tilde{\mu}} := \frac{1}{N_L \mu_L} + \frac{1}{N_R \mu_R} . \]

We now specialize to the case \( N_L = N_R := N \). In this case, eq. (2.22) simplifies to

\[ 0 = \left[ m_F + \frac{1}{N \tilde{\mu}} (T_1 - N \tilde{c}_a c_a) \right] \tilde{c}_a . \]

For \( \frac{1}{\tilde{\mu}} \neq 0 \), there is at most one non-zero value \( \tilde{c}c \) for \( \tilde{c}_a c_a \), so eq. (2.23) simplifies further to

\[ \mu m_F = \frac{N - r}{N} \tilde{c}c , \]

where \( r \) is the multiplicity of \( \tilde{c}c \). This leads to the following solutions:
\[ \frac{1}{\mu} = 0 = m_F : \text{and } \tilde{F} \text{ are free}; \]

\[ \frac{1}{\mu} = 0 \neq m_F : \text{there is a unique vacuum } F = \tilde{F} = 0; \]

\[ \frac{1}{\mu} \neq 0 = m_F : \text{if } F \text{ and } \tilde{F} \text{ are multiples of the identity, so there is a continuous moduli space, parameterized by } D, \tilde{D}; \]

\[ \frac{1}{\mu} \neq 0 \neq m_F : \text{there are } N \text{ discrete vacua, parameterized by } r = 0 \ldots N - 1. \]

This is quite similar to the moduli space of the previous model, the main difference being the existence of a continuous moduli space of vacua also for \( \frac{1}{\mu} \neq 0 = m_F \).

### 2.3 The Quantum \( SU(N_L) \otimes SU(N_R) \) Model

In this section we discuss modifications in the moduli space of the \( SU(N_L) \otimes SU(N_R) \) models caused by quantum corrections. As in the classical analysis, we restrict our attention to vacua with \( Q = \tilde{Q} = 0 \).

#### 2.3.1 \( \mu_L = \mu_R = 0 \)

When the model has \( N = 2 \) supersymmetry (i.e., \( \mu = 0 \) and \( \kappa = \lambda = \sqrt{2} \)), the quantum corrections are severely restricted: both branches described in subsection 2.2.1 survive, the “mixed” branch retains the structure of a direct product and only the Coulomb parts are modified by quantum corrections\(^{10}\). Even when the Yukawa couplings are modified, so that the \( N = 2 \) supersymmetry is explicitly broken, the Coulomb branch is not lifted. To see this, one considers the \( R \)-symmetry with \( R_Q = R_F = 1 \) and \( R_A = 0 \). It is anomaly-free (i.e., \( R_{A_L} = R_{A_R} = 0 \)) and is respected by the classical superpotential, therefore, it must be respected also by the dynamically generated superpotential. This means that the full low energy effective superpotential is quadratic in \( F \) and \( Q \), and the Coulomb branch (with \( F = Q = 0 \); parameterized by \( A \)) is not lifted by quantum corrections.

#### 2.3.2 \( N_L = N_R := N \), \( \mu_L = -\mu_R := -\mu \neq 0 \)

In this case, at scales below \( \mu \), \( A_L \) and \( A_R \) are expected to decouple. When \( \mu \gg \Lambda_L, \Lambda_R \), the decoupling occurs at weak gauge coupling and can be analyzed semi-classically. Integrating out \( A_L, A_R \) by their equations of motion (see eqs. (2.21) and recall that we use the gauge choice (2.4), in which \( F \) and \( \tilde{F} \) commute)

\[ A_L = -A_R = B/\mu, \quad B := F\tilde{F} - \frac{1}{N} I_N \text{tr}(F\tilde{F}) \quad (2.24) \]

leads to the effective superpotential\(^{11}\)

\[ W = \text{tr}(m_L \tilde{Q}_L Q_L) + \text{tr}(m_R \tilde{Q}_R Q_R) + \frac{1}{\mu} \left[ \text{tr}(\tilde{Q}_L B Q_L) - \text{tr}(\tilde{Q}_R B Q_R) \right] , \quad (2.25) \]

\(^{10}\)This follows from the arguments presented in \([55]\).

\(^{11}\)Recall that we chose \( \lambda = \kappa = 1 \). Other values for them can be incorporated by the transformations (2.7,2.8). In particular, the condition for the existence of this branch is \( \mu_L/\kappa_L^2 = -\mu_R/\kappa_R^2 \).
so the resulting effective model, without adjoint fields, will have the tree-level superpotential (2.23). This branch is also not lifted by quantum corrections, since it is protected by the non-anomalous $R$ symmetry with $R(Q) = 1$ and $R(F) = 0$.

2.3.3 $\mu_L \neq 0 = \mu_R$

Finally, we discuss the situation, when only one of the adjoint fields – $A_L$ – is massive\footnote{We are grateful to David Kutasov for a discussion on this subsection.}. Consider, for simplicity, a model without fundamentals ($n_L = n_R = 0$). Integrating out $A_L$ (assuming, as before, $\mu_L \gg \Lambda_L, \Lambda_R$)

$$A_L = -\frac{1}{\mu_L} (F \tilde{F} - \frac{1}{N_L} \text{tr} F \tilde{F})$$

leads, at scales below $\mu_L$, to a model with a tree level superpotential

$$W_{\text{tree}} = -\frac{1}{2\mu_L} [\text{tr}(M^2) - (\text{tr} M)^2] + \text{tr} [(m_F + A_R) M] , \quad M := \tilde{F} F , \quad (2.26)$$

so the classical model in this case has a branch of vacua with $M = 0$, parameterized by $A_R$. Unlike the previous cases, here symmetry does not protect this branch from the dynamical generation of an $A_R$-dependent superpotential that would lift it. In fact we know that such a superpotential is generated in a closely related model. Taking $\Lambda_R = 0$ and $\frac{1}{\mu_L} = 0$, one obtains a model with gauge group $SU(N_L)$, under which $F, \tilde{F}$ transform as $N_R$ fundamentals and antifundamentals and $A_R$ transforms as $N^2_R - 1$ singlets. For $N_L < N_R$ this is almost “magnetic SQCD” of\footnote{We are grateful to David Kutasov for a discussion on this subsection.}. In the latter model, there are $N^2_R$ singlets $\hat{A}_R$, coupled to the meson $M$ by a superpotential $\text{tr} \hat{A}_R M$, so the only difference between the models is that one of the singlets of magnetic QCD (tr $\hat{A}_R$) is “frozen” here, becoming a parameter $m_F$. As is known, quantum effects indeed generate an $\hat{A}_R$-dependent superpotential in this model. In section\footnote{We are grateful to David Kutasov for a discussion on this subsection.}, we will show, using M theory, that this also happens for all the present models. We will return to the field theoretical consequences in section\footnote{We are grateful to David Kutasov for a discussion on this subsection.}. The explicit determination of the generated superpotential is left for future work.

3. The Type IIA Brane Configuration

Our next step is to construct a configuration of branes in Type IIA string theory that will reproduce the results obtained in the previous section.

3.1 The Classical Description

We will consider (classically) flat branes in flat 10 dimensional spacetime: solitonic five-branes (Denoted $NS_5$) and Dirichlet 4-branes and 6-branes (denoted $D_4$ and $D_6$ respectively). The $D_4$ branes will have boundaries on $NS_5$ branes. The orientation of branes is
described in the following table

| Type  | \(x^\mu, \mu = 0, 1, 2, 3\) | \(v := x^4 + ix^5\) | \(s = (x^6 + ix^{10})/R_{10}\) | \(x^{7}\) | \(w := x^8 + ix^9\) |
|-------|-----------------------------|-----------------|-----------------------------|--------|-------------------|
| NS\(_5\) | — | — | — | — | — |
| \(D_4\) | — | — | • | • | • |
| \(D_6\) | — | • | [—] | • | — |

(we will also have \(NS_5\) and \(D_6\) branes rotated in the \((v, w)\) space; they will be described shortly). In the table, a dash ‘—’ represents a direction along which the brane is extended and a bullet ‘•’ represents a direction transverse to the brane. For the \(D_4\) brane, ‘[—]’ means that the brane does not extend along the full \(x^6\) axis, since at least one of its sides ends on a \(NS_5\) brane. We will have both “finite” and “semi-infinite” \(D_4\) branes in the \(x^6\) direction.

In the next section we will reinterpret the type IIA string theory discussed here as M theory on \(\mathbb{R}^{10} \times S^1\). We, therefore, incorporated in the table also the extension of the branes in the 11th (compact) dimension. \(x^{10}\) is the corresponding coordinate and \(2\pi R_{10}\) is its periodicity.

The “finite” \(D_4\) branes (i.e., those that are restricted from both sides in the \(x^6\) direction by a \(NS_5\) brane) are the only branes in the configuration that have a finite extent in the “internal space” \((x^i, i = 4 \ldots 9)\), therefore, at a low enough energy, the dynamics of the configuration is approximately the dynamics of the finite \(D_4\) branes in the background of a fixed configuration of all other branes – an effective field theory on the world volume of the finite \(D_4\) branes. Because of the finite extent in the \(x^6\) direction, this dynamics will be effectively 4 dimensional.

A collection of branes of the above types preserves \(1/4\) of the original 10 dimensional \(N = 2\) supersymmetry [2], therefore, the corresponding effective 4 dimensional field theory will have \(N = 2\) supersymmetry. A \(NS_5\) brane with a different orientation then that described above will break generically all the remaining supersymmetry. However there is a way to rotate a \(NS_5\) brane in such a way that half of the remaining supersymmetry is preserved and the resulting 4 dimensional field theory has \(N = 1\) supersymmetry. In particular, one can perform the following rotation:

\[
\begin{pmatrix}
  v \\
  w
\end{pmatrix}
\rightarrow
\begin{pmatrix}
  \cos \theta & -\sin \theta \\
  \sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
  v \\
  w
\end{pmatrix}
\]

(3.2)

Indeed, this rotation can be described as an \(SU(2)\) transformation:

\[
\begin{pmatrix}
  x^4 + ix^8 \\
  x^5 - ix^9
\end{pmatrix}
\rightarrow
\begin{pmatrix}
  e^{i\theta} \\
  e^{-i\theta}
\end{pmatrix}
\begin{pmatrix}
  x^4 + ix^8 \\
  x^5 - ix^9
\end{pmatrix}
\]

and this guarantees [28] the preservation of \(N = 1\) supersymmetry. Obviously, \(w\) in (3.2) can be replaced by \(e^{i\varphi}w, \varphi \in \mathbb{R}\) (or equivalently \(v \rightarrow e^{-i\varphi}v\)), leading to a different rotation, so a rotation of this kind is parameterized by two real angles. Identical considerations lead to the possibility to rotate a \(D_6\) brane, using the transformation (3.2). In fact, one
can use the transformation (3.2) to rotate an arbitrary number of NS\(_5\) branes and D\(_6\) branes (each with different rotation parameters) without spoiling \(N = 1\) supersymmetry.

At this stage we can specify the precise configuration of branes that will be considered:

- Three NS\(_5\) branes, denoted by \(S_L, S_M, S_R\), according to their order along the \(x^6\) axis. We choose \(S_M\) to be parallel to the \(v\) plane, while the other two are rotated with rotation parameters \((\theta_L, \varphi_L)\) and \((-\theta_R, \varphi_R)\) (note the sign difference).

- \((n_L, N_L, N_R, n_R)\) D\(_4\) branes extended in the \(x^6\) direction in the intervals
  \[-\infty \leftrightarrow S_L \leftrightarrow S_M \leftrightarrow S_R \leftrightarrow +\infty\]
  respectively.

### 3.2 Identification of the Effective Field Theory

We argue that the effective field theory on the D\(_4\) branes is indeed the one presented in section 2. Moreover, we suggest a quantitative identification between most of the parameters and moduli of the field theory and the parameters determining the brane configuration (see figures 2 and 3). To specify the location of \(S_L\) and \(S_R\) in the \(v\) and \(w\) planes, we choose a reference point on each brane (it is denoted by star “*” in the figures) and refer to its coordinates. The arbitrariness of this choice will be discussed below. We now list the identification. The parameters are

\(\Delta v\) represents \(T_{st} \Delta v\), which is the mass of a string of length \(\Delta v\). \(w\) represents \(T_{st}^{3/2} w\) and, therefore, \(\tan \theta\) represents \(T_{st}^{3/2} \tan \theta\).

\(^{13}\)Parts of this identification already appeared in the literature. We combine them with some new elements and obtain an almost complete picture.

\(^{14}\)In the following identification we assume unit string tension \(T_{st}\). \(T_{st}\) can be recovered by dimensional analysis. For example, \(\Delta v\) represents \(T_{st} \Delta v\), which is the mass of a string of length \(\Delta v\). \(w\) represents \(T_{st}^{3/2} w\) and, therefore, \(\tan \theta\) represents \(T_{st}^{3/2} \tan \theta\).
Figure 2: Projection on the \((v, w)\) subspace. The displacements described are for an \(S_L \leftrightarrow S_R\) \(D_4\) brane (denoted by a bullet \(\bullet\)).

- **v-position**: (here the \(D_4\) branes are the semi-infinite ones)
  
  \[
  m_F = (S_L \text{ brane}) - (S_R \text{ brane}) \; , \\
  -m_L = (D_{4L} \text{ brane}) - (S_L \text{ brane}) \; , \\
  m_R = (D_{4R} \text{ brane}) - (S_R \text{ brane}) \; ;
  \]

- **w-position**:
  
  \[
  \xi_L = (S_L \text{ brane}) - (S_M \text{ brane}) \; , \\
  \xi_R = (S_R \text{ brane}) - (S_M \text{ brane}) \; ;
  \]

- **s-position**: \((\tau = \frac{g}{2\pi} + \frac{i4\pi}{g^2})\)
  
  \[
  i\tau_L = (S_L \text{ brane}) - (S_M \text{ brane}) \; , \\
  -i\tau_R = (S_R \text{ brane}) - (S_M \text{ brane}) \; ;
  \]

- **orientation of the \(NS_5\) branes**:
  
  \[
  \mu_L = e^{i\varphi_L} \tan \theta_L \; , \\
  \mu_R = e^{i\varphi_R} \tan \theta_R
  \]

and the moduli are (here the \(D_4\) branes are the finite ones):

- **v-position**:
  
  \[
  A_L = (D_{4L} \text{ brane}) - (S_L \text{ brane}) \; , \\
  -A_R = (D_{4R} \text{ brane}) - (S_R \text{ brane}) \; ;
  \]

- **w-position**:
  
  \[
  -F = (D_{4L} \text{ brane}) - (S_M \text{ brane}) \; , \\
  -\tilde{F} F = (D_{4R} \text{ brane}) - (S_M \text{ brane}) \; .
  \]
Also, the FI parameters $\eta_L$ and $\eta_R$ are related to the $x^7$-positions of $S_L$ and $S_R$, respectively, relative to $S_M$. When the fundamentals are realized by $D_6$ branes (see below), $\tilde{Q}Q$ is related to the $w$ positions of $D_4$ branes connecting $D_6$ branes and the Yukawa couplings $\lambda$ are related to the rotation of the $D_6$ branes. The only other parameters that do not appear in the above list are the Yukawa couplings $\kappa$ of the bi-fundamentals. Actually, in deriving the above relations, we used the results of the previous section, where we set $\kappa = \lambda = 1$. As explained there, $\lambda$ and $\kappa$ can be recovered by using eqs. (2.8) and (2.7), so in the above list each field-theoretical quantity actually represents the modified expression obtained after recovering $\lambda$ and $\kappa$. For example, the orientation of the $NS_5$ branes is related to $e^{i\varphi} \tan \theta = \mu/\kappa^2$. As long as we consider configurations that preserve $N = 2$ supersymmetry, $\kappa$ and $\lambda$ are fixed, so nothing essential is missing. But when some of the branes are rotated, $N = 2$ supersymmetry is broken and the corresponding behavior of the Yukawa coupling may be important. In particular, qualitative changes are expected when they vanish.

In the following we list the evidence for the above identification.

**Field Content**

The field content was identified already in previous works [2, 9]. $N = 2$ vector multiplets originate from strings connecting two $D_4$ branes which extend between the *same* $NS_5$ branes, while $N = 2$ hypermultiplets originate from the other 4-4 strings. The identification of the vector field content leads also to the identification of the gauge symmetry $U(N_L) \otimes U(N_R)$.

Note that we could replace the semi-infinite $D_4$ branes by $D_6$ branes, obtaining the same matter content. For example, the $D_4$ branes extended to $+\infty$ can be replaced by $D_6$ branes located between $S_M$ and $S_R$ (along the $x^6$ axes). Indeed, when such a $D_6$ brane is moved to the right and crosses $S_R$, a $D_4$ brane is generated that extends between the $D_6$ and $S_R$ branes, so a semi-infinite $D_4$ brane can be viewed as connected to a $D_6$ brane at infinity. When the crossing $NS_5$ and $D_6$ branes are orthogonal to each other, this motion seems to be irrelevant in the effective field theory on the $D_4$ branes, allowing to switch between these two representations. In most of the discussion it will not be necessary to introduce $D_6$ branes.

**Parameters and Moduli**

There is a difference in the expected geometric realization of field-theoretical parameters and moduli: given the fields, their dynamics is dictated by the action, characterized by its parameters. In the brane realization of the field theory, the dynamics is that of the finite $D_4$ brane and it is dictated by the geometric configuration of the infinite – non-dynamical – branes. Therefore, this is what the parameters of the field theory should describe geometrically. The action determines, in particular, the moduli space of vacua, and this corresponds to the possible geometric locations of the finite – dynamical – $D_4$

\[^{15}\text{The relevance of this motion in other situations was discussed in [8] and [12].}\]

\[^{16}\text{The $D_6$ branes are essential for the description of the Higgs branch of the field theory on the $D_4$ branes, but we do not consider this branch of vacua in the present work.}\]
branes. Therefore the moduli of the field theory should determine aspects of the geometric configuration that involve only the finite $D_4$ branes.

This distinction between parameters and moduli is obeyed in the identification described above.

**Dimensionality and Redundancy**

One can verify that the dimensionality of the parameters and moduli indeed agrees with the one of the geometric quantities with which they are identified\(^\text{17}\). In this context one should observe that there is a redundancy in the description of both the field theory and the brane configuration. In the brane configuration this is the choice of a reference point on $S_L$ and $S_R$. One can verify that a shift of this reference point has exactly the same effect as a (scalar) shift of $A_L$ and $A_R$, respectively, accompanied by a change of parameters that leaves the action invariant, up to an additive constant. A shift in $A_L$, for example, should be accompanied by a shift of $m_F$ and $m_L$ and, if $\mu_L \neq 0$, also of $\xi_L$. Such a shift can be used to set to zero some redundant parameters (see footnote \(^\text{3}\)). All this is reflected exactly in the geometric description.

**The Equations of Motion and the Moduli Space of Vacua**

The geometric identification implies naturally qualitative and quantitative relations between the various parameters and moduli (these were already used in the discussion of redundancy above). In the field theory, these relations are the equations of motion derived from the action, therefore, the identification of the relations from the geometry leads, at least to a large extent, to the identification of the action and, in particular, the superpotential. With this in mind, it is worthwhile to identify the geometric origin of the equations of motion:

- The restrictions on $\eta$, obtained from the D-term equations, follow directly from the requirement that the $D_4$ branes should have $x^7 = \text{const}.$.

- A non-zero eigenvalue for $\tilde{F}$ corresponds to an $S_L \leftrightarrow S_R$ $D_4$ brane, therefore, the left and right $D_4$ branes from which it is formed should have the same $v$ and $w$ (see fig. \(^\text{2}\)). The $w$ position implies that $\tilde{F}F$ and $F\tilde{F}$ must have common non-zero eigenvalues (which follows from the D-term equations) and the $v$ position implies that for such an eigenvalue, $m_F + A_L + A_R = 0$ (which follows from eqs. \(^\text{2.9,2.10}\)).

- From a diagram in the $(v, w)$ space (fig \(^\text{3}\)) one obtains, for an $S_L \leftrightarrow S_R$ $D_4$ brane, $\xi + \mu A + \tilde{c}c = 0$, which is precisely the content of eqs. \(^\text{2.11,2.12}\). In particular, $A$ vanishes for $\mu = \infty$ and is free for $\mu = 0$.

Finally, one can classify all the possible brane configurations. The result agrees completely with the field-theoretical analysis and one observes the six types of situations described in section \(^\text{2}\) (see figure \(^\text{3}\)). Actually, once the equations of motion are identified geometrically,

\(^{17}\)Of course, only gauge invariant quantities can have a geometric meaning, so in $A_L, A_R, \tilde{F}F, F\tilde{F}$, only the eigenvalues should be considered. Moreover, in the identification of $m_L, m_R$, we assume that they are diagonal.
Figure 3: The possible brane configurations. The types of configurations correspond to the types of vacua found in section 2.
the moduli spaces should obviously agree too.

\( N = 2 \) Supersymmetry

In the brane configuration described above, 4 dimensional \( N = 2 \) supersymmetry is broken iff the \( NS_5 \) branes are not all parallel. This is identified with non-vanishing masses for the adjoints, which indeed break the supersymmetry to \( N = 1 \). Changes in the Yukawa couplings \( \kappa, \lambda \) also break \( N = 2 \) supersymmetry, but these parameters are not represented geometrically in the above configuration\(^1\).

R-Symmetry

A rotation in the “internal space” (i.e., not acting on \( x^\mu, \mu = 0, 1, 2, 3 \)) rotates also \( \theta_\alpha \) – the Fermionic coordinate of the four dimensional superspace and, therefore, corresponds to an \( R \)-symmetry of the field-theoretical model. The relations suggested above are compatible with the following identification: \( SU(2)_{789} \) (rotations in the (789) space) is the \( SU(2)_R \) symmetry and \( U(1)_{45} \) and \( U(1)_{89} \) (rotations in the \( v \) plane and \( w \) plane respectively) are the \( U(1)_R \) symmetries characterized by the following charges:

|       | \( v \) | \( w \) | \( \theta_\alpha \) | \( F, \tilde{F} \) | \( Q, \tilde{Q} \) | \( \xi \) | \( \mu \) | \( m_F \) | \( m \) | \( \kappa \) | \( \lambda \) | \( \Lambda^b \) |
|-------|--------|--------|-----------------|----------------|----------------|--------|--------|--------|--------|--------|--------|--------|
| \( R_{45} \) | 2      | 2      | 1               | 2              | -2             | 2      | 2      |        |        |        |        |
| \( R_{89} \) | 2      | 1      | 1               | 1              | 2              | 2      |        |        |        |        |        |

(3.3)

(the charges are left↔right symmetric, so we dropped the \( L, R \) subscripts). Observe that \( \kappa \) and \( \lambda \) are neutral, so the transformations (2.7, 2.8) do not affect the \( R \) charges above. For later convenience, we include in the table the charge of the instanton factor \( \Lambda^b \), representing the chiral anomaly. The one-loop beta function coefficients are \( b_{L,R} = 2N_{L,R} - (N_{R,L} + n_{L,R}) \).

3.3 Quantum effects

So far, we considered the classical description of branes. There are, however, quantum effects that change the situation described above considerably.

3.3.1 The Bending of \( NS_5 \) Branes

Classically, a \( NS_5 \) brane is flat and the branes discussed in this work have constant \( s \). However, as observed in \(^{19}\), when a \( D_4 \) brane ends on a \( NS_5 \) brane, the \( NS_5 \) brane bends. For example, when a single \( D_4 \) brane at \( v = 0 \) ends on the left side of a \( NS_5 \) brane extending classically in the \( v \) direction, the \( s \) coordinate of the \( NS_5 \) brane behaves asymptotically as \( s = \log v \). One implication of this effect is that the \( U(1)_{45} \) symmetry of the classical \( NS_5 \) brane is broken quantum mechanically and this fits well with the \( U(1)_R \) anomaly known in field theory \(^{11}\). Another consequence of this bending, that

---

\(^1\)As remarked above, when the fundamentals are realized by \( D_6 \) branes (and not by semi-infinite \( D_4 \) branes), \( \lambda \) corresponds to the rotation of the \( D_6 \) branes, which indeed breaks the \( N = 2 \) supersymmetry.
is more important to us, is the freezing of the $N = 2$ vector multiplets corresponding to the $U(1)$ gauge factors. To understand this, one should observe that, since the $NS_5$ branes are bent, with the “center of bending” determined by the location of the $D_4$ branes, a change in this location will be accompanied by a change in the location of the $NS_5$ brane, including the asymptotic parts, and it can be shown \[19\] that such a change will cost an infinite amount of (kinetic) energy. Therefore, the modulus corresponding to the average location of the $D_4$ branes is frozen and, by $N = 2$ supersymmetry, this happens to the whole $N = 2$ vector multiplet to which it belongs. This argument was given for an $N = 2$ supersymmetric configuration, however, since this effect is seen to be related to the local interaction between a $NS_5$ brane and the $D_4$ branes ending on it, it is not expected to depend on an existence of other, remote, branes that may break the $N = 2$ supersymmetry. This freezing means that, while classically we obtained an effective $U(N_L) \otimes U(N_R)$ field theory, the incorporation of the quantum effects will lead to a realization of a quantized $SU(N_L) \otimes SU(N_R)$ field theory.

### 3.3.2 Inter-brane Forces

Another quantum effect in brane dynamics was discovered in [11]. Investigation of the moduli space of $N = 1$ SQCD led to the observation that $D_4$ branes between two non-parallel $NS_5$ branes interact with each other and with some other $D_4$ branes connected to the same $NS_5$ branes. Consider, for example, a $D_4$ brane between two $NS_5$ branes $S_L, S_R$ which are rotated with respect to each other by $90^\circ$. If there is another $D_4$ brane between $S_L$ and a $D_6$ brane parallel to $S_L$, then the two $D_4$ branes will be either attracted to each other (when they end on opposite sides of $S_L$), or repelled from each other (when they end on the same side). This interaction has infinite range; it is a Coulomb-like interaction in the co-dimension of the intersection of the $D_4$ brane with the $NS_5$ brane. Other $D_6$ branes can “screen” these forces in some situations, as described in [11]. The effect of this interaction is to lift moduli spaces of vacua, when they exist, since vacua correspond to configurations in which all branes are in equilibrium. We will discuss brane forces in the present models in section 5. Before that, we will gain some relevant information from an analysis using M-theory.

### 4. Analysis in M Theory

In this section, we follow the approach initiated in [19], interpret the type IIA brane configuration described in the previous section as a fivebrane in M theory and exploit the resulting simplifications to obtain information about the corresponding field theory and also about brane dynamics.

We start by describing how the description given in this section is related to that of the previous section\[19\]. There are three independent length scales relevant to this discussion:

\[19\] We use here arguments given at [19, 20].
the string length scale \( l_{st} \) (inversely related to the string tension), the length \( l_6 \) of the (finite) \( D_4 \) branes and the radius \( R_{10} \) of the 11th dimension. In terms of these, the string coupling constant \( g_{st} \), the (inverse) gauge coupling constant \( \text{Im} \tau \) and the 11 dimensional plank length \( l_{pl} \) are given by

\[
g_{st} = \frac{R_{10}}{l_{st}}, \quad \text{Im} \tau = \frac{l_6}{R_{10}}, \quad l_{pl}^3 = l_{st}^2 R_{10}.
\]  

(4.1)

The realization of the field theory by a type IIA brane configuration, as described in the previous section, is performed in the limit of small \( g_{st} \), (i.e. \( R_{10} \ll l_{st} \)), where string perturbation theory is reliable. One also considers length scales above \( l_{st} \) (so that all the massive modes of the string can be ignored) and above \( l_6 \) (so that the effective field theory is 4 dimensional). For a sufficiently small string coupling, the gauge coupling is small at the string scale and significant quantum effects in the gauge theory appear only at very low energies, where complications of string theory, including gravitation, are expected to be negligible. This leads to the conclusion that the dynamics is reliably described by a gauge field theory or, in other words, that this brane configuration indeed realizes a gauge field theory and, therefore, can be used to investigate its properties.

In the present section, we consider a different limit: \( l_{pl} \ll R_{10}, l_6 \). In this range of parameters \( g_{st} \) is large and string perturbation theory is not reliable. However, one can use instead the dual – M theory – description of the model. If all the characteristic scales are large with respect to \( l_{pl} \) (as \( R_{10} \) and \( l_6 \) are), the low energy (long wavelength) approximation to M theory is expected to be sufficient.

In the passage between the two limits one keeps \( \tau \) fixed, so classically we keep considering the same effective field theory. There is by now much evidence that also in the quantum theory many aspects do not change (even when there is only \( N = 1 \) supersymmetry \[25, 26, 27, 30, 31, 34, 37, 35, 40\]). In particular, this method is expected to reproduce correctly the aspects that we will consider: the vacuum structure and the low energy gauge coupling in a Coulomb phase.

### 4.1 The Corresponding M Theory Configuration

Both NS\(_5\) branes and \( D_4 \) branes of type IIA string theory, correspond to the same object in M theory – a fivebrane, therefore, the brane configuration described in section 3 corresponds, in M theory, to a single fivebrane, with a non-trivial topology. The worldvolume of this fivebrane is of the form \( \mathbb{R}^{3+1} \times \Sigma \), where \( \mathbb{R}^{3+1} \) is the 4 dimensional spacetime, parametrized by \( x^\mu, \mu = 0, 1, 2, 3 \) and \( \Sigma \) is a two dimensional surface in the internal space \( S \), parametrized by \( v, w \) and \( s \) (each connected component of the curve has fixed \( x^7 \), so this coordinate will not be important in the following). Moreover, to obtain (at least) \( N = 1 \) supersymmetry in the 4 dimensional effective field theory, \( \Sigma \) is required \[59, 60\] to be a complex Riemann surface in the complex structure induced by the coordinateas \( v, w, s \) of \( S \). This also implies that \( \Sigma \) is smooth and generically it will have no singularities, therefore, for \( l_{pl} \) small enough, the low energy (long wavelength) approximation of M theory is justified.
As explained in Ref. [19] (extending results from [21]), the resulting low energy effective 4 dimensional field theory contains \( g \) abelian gauge fields, where \( g \) is the genus of \( \Sigma \), and \( \Sigma \) is the Seiberg-Witten curve for this theory, i.e., it determines the low energy effective gauge coupling. The \( N = 2 \) supersymmetric brane configuration of section 3 leads to a fivebrane worldvolume with genus \( N_L + N_R - 2 \), supporting the claim that this is the low energy limit of an \( SU(N_L) \otimes SU(N_R) \) (and not of \( U(N_L) \otimes U(N_R) \)) gauge theory.

### 4.2 The Curve With \( N = 2 \) Supersymmetry

Ref. [19] considers an \( N = 2 \) supersymmetric version of the model discussed here (corresponding to \( \mu = 0 \) and \( \kappa = \lambda = \sqrt{2} \) in the present notation). The dependence of the corresponding curve on the coordinates \((v, t)\) was determined (by requiring an appropriate asymptotic behavior) to be

\[
C_L Q_L(v) t^3 - P_L(v) t^2 + P_R(v) t - C_R Q_R(v) = 0 \quad , \quad t = t_0 e^{-s} \quad , \quad (4.2)
\]

where \( Q_L, P_L, P_R, Q_R \) are polynomials of degree \( n_L, N_L, N_R, n_R \) respectively and can be chosen to be with leading coefficient equal to 1 (\( t_0 \) is a constant, possibly dimensionfull).

To use this curve as a starting point for deformations, we must identify explicitly its dependence on the parameters and moduli. We will show that the curve is

\[
G(v, t) := \Lambda_L^{b_L} Q_L(v) t^3 - P_L(v) t^2 + P_R(v) t - \Lambda_R^{b_R} Q_R(v) = 0 \quad (4.3)
\]

where

\[
P_L(v) \overset{SC}{=} \det(v - v_L - \langle A_L \rangle) \quad , \quad P_R(v) \overset{SC}{=} \det(v - v_R + \langle A_R \rangle) \quad , \quad (4.4)
\]

\[
Q_L(v) = \det(v - v_L + m_L/\sqrt{2}) \quad , \quad Q_R(v) = \det(v - v_R - m_R/\sqrt{2}) \quad , \quad (4.5)
\]

and \( \Lambda_L^{b_L}, \Lambda_R^{b_R} \) are the instanton factors (the one-loop beta function coefficients being \( b_{L,R} = 2N_{L,R} - (N_{R,L} + n_{L,R}) \)). The definition of \( P_L, P_R \) deserves some explanation. Consider, for example, \( P_L \) and take \( v_L = 0 \) (by shifting \( v \)). It is a polynomial in \( v \)

\[
P_L(v) = \sum_{i=0}^{N_L} s_{Li} v^{N_L-i} = \prod_{a=1}^{N_L}(v - A_{La}) \quad ,
\]

with the coefficients \( s_{Li} \) being the moduli parameterizing the space of vacua. Alternatively, one can take the roots \( \{A_{La}\} \) of \( P_L \) as the coordinates, with the understanding that they should be identified under permutation. What eq. (4.4) means is that in the semi-classical region \( \{A_{La}\} \) are the eigenvalues of \( \langle A_L \rangle \) or, equivalently, that \( P_L(v) \) is the characteristic polynomial of \( \langle A_L \rangle \). We will use this notation also later, \( ^{SC} \) meaning “semi-classically equal”. Note that this is an equality only up to terms vanishing in the semi-classical region.

To derive eq. (4.3), one considers first the limits of vanishing gauge couplings. For example, a vanishing \( \Lambda_R^{b_R} \) corresponds to the right \( NS_5 \) brane \( \langle S_R \rangle \) being taken to \( x_6 \to \infty \).
(recall that the gauge coupling $\tau$ is proportional to the length of the finite $D_4$ branes). This is achieved by taking $C_R = 0$: 

$$t[C_L Q_L(v)t^2 - P_L(v)t + P_R(v)] = 0$$

the first factor corresponds to a $NS_5$ brane at $t = 0$ (which is indeed $x_0 = \infty$), while the second factor should correspond to a curve for an $SU(N_L)$ gauge theory with $n_L + N_R$ flavors. A Change of variables $t \rightarrow t/C_L Q_L$ and then $y = 2t - P_L$ leads to the curve

$$y^2 = P_L^2 - 4C_L Q_L P_R$$

which is the familiar curve for this model \cite{14, 13}, and this leads to the identification of $C_L, P_L, Q_L$ and $P_R$, up to terms proportional to $\Lambda_R^{bl}$. At this stage it would be natural to choose $v_L = 0$ (which also implies $v_R = -m_F/\sqrt{2}$), but the freedom to shift $v$ leads to a general $v_L$. The limit of vanishing $\Lambda_L^{bl}$ leads similarly to the identification\footnote{Note that in these works $\lambda = \sqrt{2}$ was absorbed in $m: m/\sqrt{2} \rightarrow m$.} \footnote{To obtain the curve in the familiar form, one should choose $v_R = 0$ and change variables $v \rightarrow -v$. We also use the freedom to redefine $(-1)^{n_L+n_R} \Lambda_R^{br} \rightarrow \Lambda_R^{br}$.} \footnote{This is always true at weak coupling and, therefore, is exactly true in the limit $M \rightarrow \infty$, since we assume asymptotic freedom.} \footnote{This is always true at weak coupling and, therefore, is exactly true in the limit $M \rightarrow \infty$, since we assume asymptotic freedom.} of $C_R$ and $Q_R$.

At this stage the curve is identified up to terms proportional to $\Lambda_L^{bl} \Lambda_R^{br}$. To show that there are no such terms, consider an $SU(\hat{N}_L) \otimes SU(\hat{N}_R)$ model, with $\hat{N} = N + 1$ and choose moduli of the form

$$\hat{A}_{La} = \begin{cases} A_{La} + M & a < \hat{N}_L \\ -N_L M & a = \hat{N}_L \end{cases}, \quad \hat{A}_{Ra} = \begin{cases} A_{Ra} - M & a < \hat{N}_R \\ N_R M & a = \hat{N}_R \end{cases},$$

with a large $M$ (which will eventually be taken to infinity). At the scale $M$ the gauge symmetry is broken to $SU(N_L) \otimes SU(N_R)$ and $A_{La}, A_{Ra}$ are the moduli of the resulting model. Assuming asymptotic freedom ($b_L, b_R > 0$), for $M$ large enough the symmetry breaks at weak coupling and can be analyzed semi-classically. $M$ contributes, through the moduli, to the effective mass of the fundamentals (as can be seen by examining the superpotential), so if the mass parameters are kept constant the effective mass will diverge. To cancel this effect, we shift also the masses

$$\hat{m}_L = m_L - \sqrt{2} M \quad , \quad \hat{m}_R = m_R + \sqrt{2} M$$

(note that $M$ does not contribute to the mass of the bi-fundamentals, so we can keep $m_F$ constant). Finally, the matching of scales is\footnote{This is always true at weak coupling and, therefore, is exactly true in the limit $M \rightarrow \infty$, since we assume asymptotic freedom.}

$$\hat{A}_L^{bl} = \frac{(N_L M)^2}{N_R M} \Lambda_L^{bl}, \quad \hat{A}_R^{br} = \frac{(N_R M)^2}{N_L M} \Lambda_R^{br}, \quad \hat{A}_{La} = \frac{N_L^2}{N_R} M \Lambda_L^{bl} \quad , \quad \hat{A}_{Ra} = \frac{N_R^2}{N_L} M \Lambda_R^{br},$$

where the numerators correspond to the heavy gauge bosons and the denominators correspond to heavy fundamentals (coming from the bi-fundamentals). This process should be reflected correctly in the curve. The curve for the $SU(\hat{N}_L) \otimes SU(\hat{N}_R)$ model is

$$\hat{A}_L^{bl} \hat{Q}_L(\hat{v}) \hat{t}^3 - \hat{P}_L(\hat{v}) \hat{t}^2 + \hat{P}_R(\hat{v}) \hat{t} - \hat{A}_R^{br} \hat{Q}_R(\hat{v}) = 0$$
and using the scale matching, one obtains
\[ \Lambda^L \hat{Q}_L(\hat{v}) t^3 - \frac{1}{N_L M} \hat{P}_L(\hat{v}) t^2 + \frac{1}{N_R M} \hat{P}_R(\hat{v}) t - \Lambda^R \hat{Q}_R(\hat{v}) = 0 \quad , \quad t = \frac{N_R \hat{t}}{N_L} . \] (4.6)

From eqs. (4.4, 4.5) we deduce
\[ \hat{P}_L(\hat{v}) \overset{\text{SC}}{=} (v - v_L + N_L M) P_L(v) \quad , \quad \hat{Q}_L(\hat{v}) \overset{\text{SC}}{=} Q_L(v) \quad , \quad \hat{v} = v + M \]
(and the same for \( P_R, Q_R \)), so the terms in eq. (4.6) must be finite in the limit \( M \to \infty \). These terms are functions of \( \hat{v}, \hat{A}, \hat{m} \) and \( \hat{\Lambda}^b \), which all depend polynomially on \( M \), so if we assume that \( \hat{P} \) and \( \hat{Q} \) depend polynomially on their arguments, we conclude that a term proportional to \( \Lambda^L \hat{A}^L \Lambda^R \hat{A}^R \) cannot appear in them (since such a term would be at least quadratic in \( M \)). Observe that corrections to \( P \) which are linear in \( \Lambda^b \) are still allowed, so the moduli are defined only up to terms linear in \( \Lambda^b \). At this stage we determined completely the curve for the “up” model, but now one can break the symmetry, as described above, and find that the curve for the “down” model also does not have \( \Lambda^L \Lambda^R \) terms and, therefore, is given by eq. (4.3).

The curve (4.3) implies the identification of charges for the coordinates \( v, t \) (in complete agreement with the type IIA analysis). In particular, the mass dimensions are \( [v] = 1 \) and \( [t] = N_R - N_L \).

In the next subsections, we will consider the rotation of a \( \text{NS}_5 \) brane. Such a rotation will break \( N = 2 \) supersymmetry and, consequently, one cannot exclude \textit{a priori} the possibility that the Yukawa couplings \( \kappa \) and \( \lambda \) will vary. It will turn out that the curve (4.3) will continue to play a role in these scenarios and then it will be important to reintroduce \( \kappa \) and \( \lambda \) into the curve. This dependence is completely determined by symmetries (after a consistent choice of charges for the coordinates is made) and the result is (see Appendix A) that one should perform the following redefinitions
\[ A \to \frac{1}{\sqrt{2}} \kappa A \quad , \quad \mu \to 2 \mu / \kappa^2 \quad , \quad m \to \kappa \lambda^{-1} m \quad , \quad \Lambda^b \to \Lambda^b \det(\lambda / \kappa)(\kappa / \sqrt{2})^{2N} . \] (4.7)

To simplify the notation, we will continue to use the form (4.3) of the curve with the understanding that each parameter in it actually represents an expression containing also \( \kappa \) and/or \( \lambda \) factors, as described in eq. (4.7).

### 4.3 Rotation of a \( \text{NS}_5 \) Brane

In this subsection we explore the possibilities to rotate a \( \text{NS}_5 \) brane, \textit{i.e.,} we look for M-theory fivebranes that will correspond, in the type IIA description, to a configuration obtained from the \( N = 2 \) supersymmetric one (described in the previous subsection) by a rotation of \textit{one} of the \( \text{NS}_5 \) branes in the \( (v, w) \) plane.

Consider, for example, a configuration with \( S_L \) rotated by an angle \( \theta \) and \( n_L = n_R = 0 \). The corresponding quantum system will be characterized by the asymptotic behavior of
the fivebrane. In the present case, there will be three parts of the curve that extend to infinity in the internal space, corresponding to the three $NS_5$ branes. In each of these parts $v$ extends to infinity (assuming $\theta \neq \frac{1}{2}\pi$). The $w$ behavior is determined by the chosen orientation of the $NS_5$ branes and the $t$ behavior is determined by the bending of the $NS_5$ brane, because of the $D_4$ branes attached to it [19] (see subsection 3.3). This leads to the following behavior:

$$S_L: \quad t \sim v^{N_L}/C_L, \quad w \sim \mu v, \quad \mu = e^{i\varphi} \tan \theta;$$

$$S_M: \quad t \sim v^{N_R-N_L}, \quad w \to w_M;$$

$$S_R: \quad t \sim C_Rv^{-N_R}, \quad w \to w_R,$$

where $C_L, C_R, w_M$ and $w_R$ are parameters (the absence of a corresponding $C_M$ parameter in $S_M$ reflects a choice of normalization of $t$). Their physical meaning is undetermined at this stage and will be identified later\(^\text{23}\). Note that classically, one would expect $w_M = w_R = 0$, however we will see that this is impossible.

We look for a Riemann surface $\Sigma$, with the asymptotic behavior \(^\text{4.8}\. Each such surface will correspond to a vacuum of the quantum system characterized by these asymptotic conditions. We first observe that $\Sigma$ can be compactified by adding to it the three points at $v \to \infty$. Indeed, in each of the asymptotic regions $v$ is a local holomorphic coordinate: $t$ and $w$ are single valued functions of $v$ (this is a reflection of the fact that each such region corresponds to a single $NS_5$ brane). This means that each $v = \infty$ point has a neighborhood holomorphically parameterized by $1/v$ and by adding these neighborhoods to $\Sigma$ we obtain a compact Riemann surface (which will be denoted by the same symbol $\Sigma$). Next consider the nature of $w$ as a function on $\Sigma$. This is a coordinate of the embedding of $\Sigma$ in the (internal part of) spacetime and as such, it is obviously single-valued on $\Sigma$. Moreover, from the asymptotic conditions \(^\text{4.8}\. we see that it has a single singular point (where it diverges) and this point is a simple pole (as can be seen, using the holomorphic coordinate $1/v$). If $\Sigma$ is an irreducible surface (i.e., not a union of surfaces), then this implies that $w$ is a (holomorphic) bijection between $\Sigma$ and the $w$ plane (this is why $w_M$ and $w_R$ cannot both vanish). We therefore arrive at the following conclusions:

- $\Sigma$ has a genus 0;
- $w$ is a global coordinate on $\Sigma$, which means that the surface can be described by two functions

$$v = V(w), \quad t = T(w)$$

(rather than two equations).

While the second conclusion is important technically (and will be used later), the first one is the physically significant one, because it implies that the above rotation makes all the vector fields massive. For SQCD this is obvious, because the rotation corresponds to $C_L$ and $C_R$ will, indeed, be identified with the corresponding parameters in eq. \(^\text{4.2}\), but this is not needed at this stage.

\(^\text{24}\)We use here arguments appearing in \cite{ref} for SQCD ($SU(N)$ gauge group).
adding a mass to the adjoint, and this lifts the Coulomb phase also classically. However, in the present model there are also $A_R$ moduli (corresponding to the location of the right finite $D_4$ branes in the $v$ direction) which, in the classical analysis, are not affected by such a rotation and parameterize a branch with massless gauge fields. We will see later that this branch collapses to a discrete set of vacua.

This is a general phenomenon. Note that it arises rather directly from the fact that a single $NS_5$ brane is being rotated. It, therefore, remains valid also for $n_L, n_R \neq 0$ and, moreover, for models with more gauge group factors (i.e., more $NS_5$ branes). It implies that a non-zero mass $\mu$ for a single adjoint (the first or the last in the chain) lifts the Coulomb branch parameterized by all the adjoints.

The above discussion refers to the case of an irreducible curve. This case will be further analyzed in subsection 4.5. From the above discussion we see that the only possibility for a Coulomb phase to survive a $NS_5$ brane rotation (i.e., turning on a mass for one of the adjoints) is if the curve is reducible. Then the rotation will not affect the whole curve (in fact, only one irreducible component will be affected), and the unaffected components may have non-vanishing genus and give rise to massless vector fields in the effective 4 dimensional field theory. Such brane configurations are the subject of the next subsection.

4.4 A Reducible Curve

If the curve $\Sigma$ is reducible (i.e. a union of several Riemann surfaces), there is exactly one irreducible component containing the rotated $NS_5$ brane and the above discussion refers to it without any changes. For the other components, the asymptotic conditions characterizing them are independent of $\mu$, therefore, these components are $\mu$-independent, i.e., unaffected by the rotation. $w$ is bounded in these components and, therefore, constant (this constant is determined by the asymptotic conditions). The genus of these components is not restricted by the above considerations, therefore, such a configuration can correspond to a Coulomb branch. The effective gauge coupling in such a branch will be $\mu$-independent.

Assuming that the curve depends smoothly on $\mu$, if it is reducible after rotation, it will be reducible also before rotation, so we can analyze this kind of rotation possibilities looking at the unrotated curve, eq. (4.3). Consider an irreducible component $\Sigma_0$ of this curve. It may contain $l$ $NS_5$ branes, where $l = 0, 1, 2, 3$. In the $l = 0$ case, $v$ is bounded and, therefore, constant so this case corresponds to a flat, infinite, $D_4$ brane. There is no $NS_5$ to rotate here, so this case is irrelevant for the present discussion\(^{25}\). The $l = 3$ case is essentially the same as the irreducible case (that will be discussed later), with reduced $n_L, N_L, N_R$ and $n_R$ (this is the complement of the $l = 0$ case). The $l = 2$ case is mathematically identical to the description of an $SU(N)$ model and the rotation of one of the two $NS_5$ branes was described in \(^{25, 26, 27}\). We will encounter such a situation as a special case in subsection 4.5.

\(^{25}\)Recalling that the semi infinite $D_4$ branes can be seen as ending on $D_6$ branes at infinity, it is clear that this is a root of a Higgs (or mixed) branch where a $D_4$ brane extends between two $D_6$ branes and can “slide” in the (789) directions.
Finally, consider the $l = 1$ case, in which the rotated $NS_5$ brane is detached from the remaining two. The component to be rotated has one region of diverging $v$ and there we have (according to the asymptotic conditions) $t \sim C v^k$ for some (integral) $k$. Therefore, $v$ is a global holomorphic coordinate and the curve is described by a holomorphic function $t = T_0(v)$. If this function is bounded, then it is constant, implying also $k = 0$, and this corresponds to the detachment of a flat $NS_5$ brane. For $n_L = n_R = 0$ this is the only possibility. For non-vanishing $n_L$ and/or $n_R$, there are more possibilities, corresponding to a $NS_5$ brane connected to semi-infinite $D_4$ branes. In the following, we concentrate on the detachment of flat $NS_5$ branes.

4.4.1 A Flat $NS_5$ Brane

The simplest possibility for the curve to be reducible is the case of a flat $NS_5$ brane detaching from the rest of the curve. This corresponds to $G$ in eq. (4.3) being divisible, as a polynomial, by $(t - t_0)$ for some ($v$-independent) $t_0$. Equivalently, the equation

$$\Lambda^b_L Q_L(v)t_0^3 - P_L(v)t_0^2 + P_R(v)t_0 - \Lambda^b_R Q_R(v) = 0 \quad (4.10)$$

should be an identity in $v$. Comparing the leading powers of $v$, we conclude that at least two of the 4 polynomials in eq. (4.10) should have the highest degree. Assuming $b_L, b_R > 0$ (asymptotic freedom), we obtain two different possibilities:

- $n_L < N_L = N_R > n_R$

Comparing powers of $v$, one obtains the conditions

$$P_L - \Lambda^b_L Q_L = P_R - \Lambda^b_R Q_R =: \bar{P} \quad (4.11)$$

and when they are satisfied, $G$ becomes

$$G = (t - 1)(\Lambda^b_L Q_L t^2 - \bar{P} t + \Lambda^b_R Q_R) \quad (4.12)$$

which corresponds to the detachment of the central $NS_5$ brane. The condition (4.11) for the detachment implies that $P_L, P_R$ (and $\bar{P}$) must coincide in the classical limit (and, in particular, $m_F = 0$), which means that classically the finite $D_4$ branes from both sides of $S_M$ must be aligned with each other. This is exactly the criterion obtained from a “classical” analysis, in both the geometric (Type IIA brane) and algebraic (field-theoretical) descriptions. The quantum corrections to this criterion are, in most cases, shifts of the moduli, which are insignificant in the present context, since we characterized them only classically. For $n_L = N_L - 1$ and/or $n_R = N_R - 1$ there is also a shift in the condition on $m_F$:

$$\frac{1}{\sqrt{2}} m_F \equiv v_L - v_R = \frac{1}{N}(\epsilon_R \Lambda_R - \epsilon_L \Lambda_L) \quad (4.13)$$

\[26\]This is the type 1 situation of section 2 and $P_L = P_R$ corresponds to $A_L = -A_R$ and $m_F = 0$.

\[27\]This will become significant in the investigation of the connections to other models and branches, where the moduli have a more unambiguous definition.
(where $\epsilon = 1$ when $n = N - 1$ and vanishes otherwise). We interpret this shift as follows: classically, a detachment situation is a root of a branch parameterized by $\det \langle F \rangle$ and $\det \langle \tilde{F} \rangle$, therefore, one expects that the (exact quantum) mass of $F$ should vanish there (indeed, at this point the curve degenerates, reflecting the existence of massless states). This suggests that the above value for $m_F$ is the classical value needed to cancel the quantum corrections to the mass, so that the final mass will vanish.

- $n_L = N_L > N_R > n_R$

Comparing powers of $v$, one obtains the conditions

$$P_L - Q_L = \Lambda_L^{b_L}(P_R - \Lambda_R^{b_R}Q_R) =: \Lambda_L^{b_L} \bar{P}_R \quad (4.14)$$

and when they are satisfied, $G$ becomes

$$G = (\Lambda_L^{b_L}t - 1)(Q_Lt^2 - \bar{P}_Rt + \Lambda_R^{b_R}Q_R) \quad (4.15)$$

which corresponds to the detachment of the left $NS_5$ brane. The condition (4.14) for the detachment implies that $P_L$ and $Q_L$ must coincide in the classical limit, which means, as in the previous case, that classically the $D_4$ branes from both sides of the detaching $NS_5$ brane must be aligned with one another. $\bar{P}_R$ coincides with $P_R$ classically, and the quantum difference is only in the moduli.

The detachment of the right $NS_5$ brane is treated analogously.

### 4.4.2 Rotating a Flat Central $NS_5$ Brane

We consider now the rotation of a (detached) flat central $NS_5$ brane, which is possible, as we saw, only for $N_L = N_R =: N$. According to the identification in subsection 3.2, such a rotation corresponds, classically, to $\mu_L = -\mu_R =: -\sqrt{2}\mu \neq 0$ and $m_F = 0$. As discussed in subsection 3.3, at scales below $\mu$, $A_L$ and $A_R$ decouple, leading to a model with a tree level superpotential

$$W = \text{tr}(m_L \tilde{Q}_L Q_L) + \text{tr}(m_R \tilde{Q}_R Q_R) + \frac{\sqrt{2}}{\mu} \left[\text{tr}(\tilde{Q}_L B Q_L) - \text{tr}(\tilde{Q}_R B Q_R)\right] \quad , (4.16)$$

with

$$B := F\tilde{F} - \frac{1}{N} I_N \text{tr}(F\tilde{F})$$

(and $A_L^{\text{sc}} = -A_R^{\text{sc}} = B/\mu$), so the resulting effective model, without adjoint fields, will have the tree-level superpotential (4.16). This model has vacua parameterized by $F$ and $\tilde{F}$ and we have shown that they are not lifted by quantum corrections.

---

28Observe that in this case $b_R > 0$ implies $N_R - n_R \geq 2$, so the sub-leading terms of $\bar{P}_R$ and $P_R$ coincide.

29This differs from eq. (2.25) because here $\kappa$ and $\lambda$ are set to $\sqrt{2}$ and not 1.
The curve (the unrotated part; see eq. (4.12)) becomes

$$\Lambda_L^{\prime b_L} Q_L^t t^2 - \bar{P}^t + \Lambda_R^{\prime b_R} Q_R = 0$$

(4.17)

where\(^{30}\)

$$w = \mu v \quad , \quad \bar{P}'(w) := \mu N \bar{P}^{SC} \mu^N \det(v - A_L)^{SC} \det(w - B) \quad ,$$

$$Q_L'(w) = Q_L(v) = \det(w/\mu + m_L/\sqrt{2}) \quad , \quad Q_R'(w) = Q_R(v) = \det(w/\mu - m_R/\sqrt{2})$$

and

$$\Lambda_L^{\prime b_L} = \mu^N \Lambda_L^{b_L} \quad , \quad \Lambda_R^{\prime b_R} = \mu^N \Lambda_R^{b_R}$$

(4.18)

are the usual matching relations for the scales. The geometric interpretation of eq. (4.17) is as follows: the relation $w/v = \mu$ identifies $w$ as the coordinate of the $(89)$ plane (according to the classical analysis, which is valid in the semi-classical – asymptotic – region of the curve), so actually $S_L$ remains in the original orientation ($v$ plane) and the rest of the curve is rotated with respect to it. Observe that for fixed $A$, $m$ and $\Lambda^b$, the projection of the curve on the $v$ plane (which is what eq. (4.12) describes) remains unchanged. One consequence of this is that the effect of non-vanishing $\mu$ can be undone by a holomorphic change of coordinates $w - \mu v \rightarrow v$, so the low-energy effective gauge coupling is independent of $\mu$ (for given $A$, $m$ and $\Lambda^b$). Another consequence is that strictly speaking, the effect of changing $\mu$ is not a rotation but, rather a stretching of the curve in the $w$ direction. From this it is also clear that taking $\mu$ to infinity holding $A$, $m$ and $\Lambda^b$ fixed will not lead to a curve in the $w$ plane. This is the geometrical origin of the change of scales (4.18). Holding $A^{\prime b'}$ (and $B$) fixed, corresponds, roughly to stretching the curve in the $v$ direction and this is the way to obtain a finite curve for $\mu \rightarrow \infty$. For $n_L = n_R = 0$, this is an exact statement\(^{31}\), so the projection of the curve on the $w$ axis is independent of $\mu$, for given $A^{\prime b'}$ and $B$ and, therefore, so is the effective gauge coupling (as can be verified by a shift $v - w/\mu \rightarrow v$).

For $n_L = n_R = 0$ we obtain (at scales below $\mu$), an $N = 1$ supersymmetric $SU(N)^2$ gauge group, with two bi-fundamentals and a vanishing superpotential. This model was analyzed in \[^{53}\] by purely field theoretical methods. The analysis led to a curve that is equivalent with (4.17) (with a modified definition of moduli). A more general family of models, corresponding to $n_L \neq 0 = n_R$ and a superpotential

$$W = \sum_l \text{tr}[h_l \check{Q}_L (F \check{F})^l Q_L]$$

was analyzed in \[^{54}\]. When $h_l$ vanishes for $l \neq 0$, these models coincide with the present models (with $h_0 = m_L$) and, again, so do the curves. The curves for $n_L n_R \neq 0$ and/or non vanishing Yukawa coupling (finite $\mu$) obtained here, seem to be unknown so far.

\(^{30}\) $m_R$ may be corrected quantum mechanically: $m_R \rightarrow m_R - m_F$, where $m_F$ is given by eq. (4.13).

\(^{31}\) When there are fundamentals, their mass parameters $m_L, m_R$ determine the asymptotic $v$ coordinate of the semi-infinite $D_4$ branes, so if they are kept constant when changing $\mu$, then the asymptotic $w$ coordinate changes and in the limit $\mu \rightarrow \infty$ these $D_4$ branes escape to infinity. To fix this, one should recover $\lambda$ (using (4.7)) and rescale it, holding $\lambda/\mu$ fixed.
Consider now a large vev for $F$
\[
\langle F \rangle = c I_N , \quad c \gg \Lambda_L, \Lambda_R
\]
(note, however, that we still assume that $\mu$ is larger than anything else so, in particular, $\mu \gg c$). In gauge invariant coordinates, this means a large $D := \det F = c^N$. At the scale $c$ the gauge symmetry is broken to $SU(N)_D$ and if $c \gg \Lambda_L, \Lambda_R$, this can be analyzed semi-classically. At scales below $c$ we have, effectively, an $SU(N)_D$ model with an adjoint $A_D = 1/N \text{tr} \tilde{F}$, two singlets $\text{tr} F, \text{tr} \tilde{F}$ and fundamentals. The corresponding curve, as obtained from (4.17) is
\[
t^2_D - P_D t + \Lambda_D^{bd} Q_D = 0 ,
\]
where
\[
x = w/c \quad , \quad t_D = t/D
\]
\[
P_D(x) := \tilde{P}' / D^{SC} \equiv \det(w - B) / D^{SC} \equiv \det(x - A_D) ,
\]
\[
\Lambda_D^{bd} = \Lambda_L^{b,l} \Lambda_R^{d,r} / D^2 \quad , \quad Q_D(v_D) = Q_L Q_R|_{v = \frac{w}{c} x} .
\]
For $\mu \to \infty$, $Q_D \to \det m_L \det m_R$ and we recover the known curve for $SU(N)$ with an adjoint, fundamentals and vanishing Yukawa coupling [12].

4.5 An Irreducible Curve

In this subsection we return to the consideration of a rotation of a $NS_5$ brane when the curve is irreducible. We restrict ourselves to $n_L = n_R = 0$ and consider the rotation of the left $NS_5$ brane. As shown at the beginning of this section, the corresponding curve can be described by two functions
\[
v = V(w) \quad , \quad t = T(w)
\]
and they are characterized by the asymptotic conditions [18]. For $V(w)$ the conditions are that it has two simple poles at $w = w_M$ and at $w = w_R$ while $V \sim w/\mu$ for $w = \infty$. This leads to the general form
\[
V = \frac{p(w)}{\mu(w - w_M)(w - w_R)} , \quad p(w) = w^3 + p_1 w^2 + p_2 w + p_3 ,
\]
where the polynomial $p(w)$ does not vanish at $w_M$ and $w_R$.

Now the conditions for $T(w)$ imply, up to multiplicative constants
\[
S_L : \quad t \sim w^{N_L} , \quad \text{for } w \to \infty ;
\]
\[
S_M : \quad t \sim (w - w_M)^{N_L - N_R} , \quad \text{for } w \to w_M ;
\]
\[
S_R : \quad t \sim (w - w_R)^{N_R} , \quad \text{for } w \to w_R
\]
\[
^{32}\text{The poles are simple because, as discussed above, } v \text{ is a local holomorphic coordinate in the neighborhood of each pole.}
\]
and this leads to the general form

\[ T = C(w - w_M)^{N_L - N_R}(w - w_R)^{N_R} \ , \ C \neq 0 \] \hspace{1cm} (4.23)

Substituting formulas (4.21) and (4.23) into the asymptotic conditions (4.8) leads to the following relations:

\[ S_L : \quad C = (C_L \mu^{N_L})^{-1} \ ; \]
\[ S_M : \quad p(w_M)^{N_L - N_R} = \frac{C_L \mu_{b_L} b_L}{(w_M - w_R)^{b_R}} \ ; \] \hspace{1cm} (4.24)
\[ S_R : \quad p(w_R)^{N_R} = C_L C_R \mu^{N_L + N_R}(w_R - w_M)^{b_R} \ . \]

Given \( p(w_M) \) and \( p(w_R) \), \( V(w) \) is essentially determined, therefore, we have the following situation:

\( N_L \neq N_R : \) The asymptotic behavior defines (for any \( w_M \neq w_R \)) \( N_R | N_L - N_R | \) solutions;

\( N_L = N_R =: N : \) \( w_M \) and \( w_R \) are restricted by \( (w_M - w_R)^N = C_L \mu^N \) and when this is satisfied, there are \( N \) one (complex) parameter families of solutions.

To obtain more information about the curve, we use the following observation \([25]\). Let \( \Sigma_0 \) be the projection of the curve \( \Sigma \) on the \((t, v)\) subspace. We expect that \( \Sigma \) depends continuously on \( \mu \) in the neighborhood of \( \mu = 0 \) and this leads also to the expectation that \( w \) is a single valued function on \( \Sigma_0 \) (since it is the constant function for \( \mu = 0 \)). This means that \( \Sigma \) has the following description:

\[ G(v, t) = 0 \ , \ w = W(v, t) \]

(where \( G = 0 \) defines \( \Sigma_0 \)). At this point we have to specify more precisely what we are looking for: we look for a \textit{one parameter} family (parameterized by \( \mu \)) of deformations of the curve (4.3), which means that \( \mu \) is the only additional parameter determining \( \Sigma \) beyond those in (4.3). From table (3.3) we see that \( \mu \) and \( w \) are the only quantities in \( \Sigma \) that carry a \( U(1)_{89} \) charge. Now, since \( G \) is independent of \( w \), \( U(1)_{89} \) symmetry implies that it is also independent of \( \mu \) and continuity in \( \mu \) implies that \( \Sigma_0 \) is exactly the curve for \( \mu = 0 \), i.e., \( G \) is given by eq. (4.3) \([34]\). One consequence of this is the identification

\[ C_L = \Lambda_L^{b_L} \ , \quad C_R = \Lambda_R^{b_R} \]

(as can be seen by comparing the asymptotic conditions to the corresponding behavior of the curve). But much more information comes from the fact that

\[ G(v, t)|_{v=v,t=T} = 0 \]

\[ ^{33} \text{There is one free parameter left in } p(w), \text{ which corresponds to the arbitrariness in the choice of origin for } v. \]

\[ ^{34} \text{Such a phenomenon was already observed explicitly in subsection } 4.4.2. \]
(where $V$ and $T$ are taken from eqs. \([1.21,1.23]\)) is an identity in $w$. After the substitution $v = V, t = T$, this identity takes the following form:

$$0 = [CLt^3 - P_Lt^2 + PRt - CR]_{v = V, t = T}(w - w_M)^{b_R} \mu^{N_R}$$

$$= \mu^{N_R} CLC^3(w - w_M)^{b_L}(w - w_R)^{3N_R}$$

$$- \mu^{N_R-N_L} C^2(w - w_R)^{b_L} \sum_{k=0}^{N_L} l_k(w - w_M)^k(w - w_R)^k p(w)^{N_L-k}$$

$$+ C \sum_{k=0}^{N_R} r_k(w - w_M)^k(w - w_R)^k p(w)^{N_R-k} - \mu^{N_R} CR(w - w_M)^{b_R} ,$$

where $l_k, r_k$ are the coefficients in $P_L, P_R$ respectively:

$$P_L(v) = \sum_{0}^{N_L} l_k v^{N_L-k} , \quad P_R(v) = \sum_{0}^{N_R} r_k v^{N_R-k} \quad (4.26)$$

(note that $l_0 = r_0 = 1$ while $l_1 = -N_L v_L$ and $r_1 = -N_R v_R$). Expanding in powers of $w$, one obtains a set of equations for the parameters in eqs. \([1.21,1.23\) \] and \([1.26\). From now on we assume, for simplicity, $b_L, b_R \geq 2$ (otherwise there are additional terms in the expressions given below). Furthermore, we choose $w_R = 0$ (by a shift of $w$). There are 4 equations that do not depend on the moduli $l_k, r_k, k > 1$. These are the coefficients of $w^{2(N_L+N_R)}, w^{2(N_L+N_R)+1}, w^1$ and $w^0$ and they lead, correspondingly, to

$$C = (\mu^{N_L} \Lambda_L^{b_L})^{-1} , \quad (4.27)$$

$$p_1 = \mu v_L - b_L N_L w_M , \quad (4.28)$$

$$p_2 = -\mu v_R w_M - b_R N_R w_M , \quad (4.29)$$

$$p_3^{N_R} = \mu^{N_L+N_R} \Lambda_L^{b_L} \Lambda_R^{b_R} (-w_M)^{b_R} , \quad (4.30)$$

which means

$$V - v_L = \frac{w^3 - \frac{b_L}{N_L} w_M w^2 + \mu (v_L - v_R) w_M w + \left(1 - \frac{b_R}{N_R} w_M w \right) p_3}{\mu(w - w_M) w} . \quad (4.31)$$

We could also expand around $w_M$ (i.e., set $w_M = 0$), obtaining other 4 moduli-independent equations. Only one of them (the coefficient of $(w - w_M)^0$) contains new information (beyond eqs. \([1.27,1.30\)). Translating back to the choice $w_R = 0$, one obtains

$$p(w_M)^{N_L-N_R} = \frac{(\mu \Lambda_L)^{b_L}}{(w_M)^{b_R}} . \quad (4.32)$$

The implications of this relation depend on whether $N_L = N_R$ or not:
$N_L \neq N_R$: Eqs. (4.28,4.29) imply

$$p(w_M) = \mu(v_L - v_R)w_M^2 + (N_L - N_R) \left( \frac{p_3}{N_R} - \frac{w_M^3}{N_L} \right).$$

(4.33)

Combining this with eqs. (4.30) and (4.32) leads to

$$m_F/\sqrt{2} \equiv v_L - v_R$$

(4.34)

$$= \left( \frac{\mu N_L^b_L}{w_M^{N_L}} \right)^{\frac{1}{N_L-N_R}} + (N_L - N_R) \left[ \frac{w_M}{\mu N_L} - \frac{1}{N_R} \left( \frac{\mu N_L^b_L \Lambda_L^b_R}{(-w_M)^N} \right)^{\frac{1}{N_R}} \right],$$

so $w_M$ (or, more generally, $w_M - w_R$) can be seen as representing $m_F$ in the asymptotic conditions (4.8).

$N_L = N_R =: N$: Eq. (4.32) fixes $w_M$:

$$w_M^N = (\mu \Lambda_L)^N$$

(4.35)

and $m_F$ remains free. In other words, in this case the parameter $m_F$ is not reflected in the asymptotic conditions.

The other equations following from the identity (4.25) determine (uniquely) the moduli $l_k, r_k$ in the $N = 2$ curve, i.e., the points in the moduli space of the $N = 2$ configurations from which a rotation is possible. Note that in general there are much more equations then variables, so one may expect that $m_F$ is also constrained or even that there are no solutions at all. This has been checked for some non-trivial cases and it was found that $m_F$ remains free. Presumably, this is true in general. To clarify what exactly is left unsettled, we return to the question of the conditions for a rotation. So far we have shown that the vanishing genus of $\Sigma$ (which translates to the complete degeneration of $\Sigma_0$) is a necessary condition for a rotation. We now argue that it is also sufficient \textsuperscript{35}. Indeed, the functions $V(w), T(w)$ define a normalization of $\Sigma_0$, i.e., a map from a Riemann surface to $\Sigma_0$, which is bi-holomorphic everywhere, except at (the inverse image of) singular points of $\Sigma_0$. Such a normalization exists for any algebraic curve in $\mathbb{C}^2$ \textsuperscript{32}. Apparently, we constrain this normalization by the asymptotic conditions (4.8), but in fact they are always satisfied: the $t \leftrightarrow v$ relations follow from the asymptotic behavior of $\Sigma_0$ (derivable from eq. (4.3)) and the $w \leftrightarrow v$ relation only means that the normalization is to a genus 0 surfaces, which was already shown to be necessary. Therefore, every curve in the family (4.3) is rotatable iff it is completely degenerate. What remains unknown is if such curves exist for any given value of $m_F$.

To summarize, by the identification of the projection of $\Sigma$ on the $(v, t)$ subspace, we recovered correctly (see eqs. (4.27,4.30,4.32)) the relations (4.24) obtained from the asymptotic conditions (4.8). We obtained, however, additional equations: eqs. (4.28) and

\textsuperscript{35} We are grateful to Yaron Oz for helping to clarify this point.
that determine $p_1$ and $p_2$ (as functions of $m_F$) up to the freedom to shift $v$ and, for $N_L \neq N_R$, also correlate between $w_M$ and $m_F$; and other equations that determine the moduli $l_k, r_k$. As a result, we obtain, for any value of $m_F$, (at most) finite number of discrete solutions (vacua).

One can identify the situation when the central NS$_5$ brane ($S_M$) detaches from the rest of the curve. As discussed at the beginning of subsection 4.4, this is possible only for $N_L = N_R =: N$, and then the detached $S_M$ is flat: constant $t$ and $w$ (since in the present model there are no semi-infinite $D_4$ branes). In the present description this happens whenever $p(w_M) = 0$, and by eqs. (4.33) and (4.35) this means $v_L = v_R$ (as for the $\mu = 0$ case, discussed in subsection 4.4.1). Setting $v_L = v_R = 0$, one obtains

$$p(w) = (w - w_M)(w^2 - \frac{p_3}{w_M})$$

leading to a degenerate curve:

$$(w - w_M)(w^2 - \mu v w - \frac{p_3}{w_M}) = 0 \quad , \quad t = \left(\frac{w}{\mu \Lambda_L}\right)^N .$$

The first factor corresponds to the flat $S_M$: $w = w_M, t = 1$. Substituting

$$w_M = \epsilon' \mu \Lambda_L \quad , \quad p_3 = -\epsilon' \mu^3 \Lambda_L^2 \Lambda_R \quad , \quad \epsilon^N = \epsilon'^N = 1$$

in the second factor (using eqs. (4.33,4.30)), one obtains

$$w v = \epsilon \mu \Lambda_L \Lambda_R + w^2/\mu \quad , \quad t = \left(\frac{w}{\mu \Lambda_L}\right)^N ,$$

which is identical to the curve of $N = 1$ SYM with a massive adjoint [25], where $\mu$ is the mass of the adjoint and $\Lambda = \sqrt{\Lambda_L \Lambda_R}$ is the scale.

Next we consider the limit $\mu \rightarrow \infty$. We have already seen that if the parameters of the initial $N = 2$ curve are all held fixed, the projection of the curve on the ($v, t$) subspace remains unchanged, so $\mu$ does not parameterize a rotation but rather a stretching in the $w$ direction. From this it is clear that the $\mu \rightarrow \infty$ limit of such a procedure will be singular [33]. The correspondence to field theory, where $\mu$ is identified as the mass $\mu_L$ of $A_L$, leads to the resolution of this problem: in this process $A_L$ is decoupled, so we should hold fixed the combination $\Lambda_L^{b_L} = \mu_L \Lambda_{L_b}$ (the instanton factor of the model without $A_L$) and not $\Lambda_L^{b_L}$. When $N_L = N_R =: N$, one obtains (see eq. (4.35))

$$w_M^N = \Lambda_L^{2N}$$

and when $N_L \neq N_R$, eq. (4.34) becomes

$$v_L - v_R = \left(\frac{\Lambda_L^{b_L}}{w_M^{b_L}}\right)^{\frac{1}{N_L-N_R}} + (N_L - N_R) \left[\frac{w_M}{\mu N_L} - \frac{1}{N_R} \left(\frac{\Lambda_L^{b_L} \Lambda_R^{b_R}}{(-w_M)^N_L}\right)^{\frac{1}{N_R}}\right] , \quad (4.38)$$

\footnote{Such a situation was already discussed in subsection 4.4.2.}
so in both cases \( w_M \) and \( v_L - v_R \) are finite in the limit \( \mu \to \infty \). The curve takes the form

\[
V - v_L = \frac{w^3 - \frac{b_R}{N_L}w_M w^2 + \mu (v_L - v_R) w_M w + \left( \frac{b_R}{N_R} w - w_M \right) \mu q}{\mu (w - w_M) w},
\]

\[
T = (w - w_M)^{N_L - N_R} w^{N_R} / \Lambda_L^{v_L'},
\]

where

\[
q := -\frac{P_3}{\mu w_M} = \left[ \Lambda_L^{v_L'} \Lambda_R^{b_R} (-w_M)^{N_R - N_L} \right]^{1/N_R},
\]

so in the \( \mu \to \infty \) limit, \( T \) remains unchanged, while \( V \) becomes

\[
V = v_L + \frac{(v_L - v_R) w_M + (1 - \frac{N_L}{N_R}) q}{w - w_M} + \frac{q}{w}.
\]

Each of the three terms corresponds, roughly, to the tree \( \text{NS}_5 \) branes. For \( N_L = N_R \) one can use the explicit expressions (4.36), to obtain

\[
V = v_L + \frac{\epsilon' \Lambda_L^2 m_F / \sqrt{2}}{w - \epsilon' \Lambda_L^2} + \frac{\epsilon' \Lambda_R^2}{w} \Lambda_L^{v_L'}, \quad T = \left( \frac{w}{\Lambda_L^2} \right)^N.
\]

It is interesting to compare the above curves to those corresponding to \( N = 1 \) SQCD with \( N_L \) colors and \( N_R \) flavors. For a uniform quark mass \( m \), the curve is (in an appropriate choice of coordinates)

\[
V = v_L - \frac{w_m}{w + w_-}, \quad T = (w + w_-)^{N_L - N_R} w^{N_R} / \Lambda_L^{v_L'},
\]

where

\[
w_- = \left( m^{N_R - N_L} \Lambda^{v_L'} \right)^{1/N_L}.
\]

Identifying \( \Lambda' = \Lambda_L', m = v_L - v_R, w_- = -w_M \), the curves coincide for \( q = 0 \). For finite \( q \), the main difference between the curves is the absence (in the SQCD curve) of the third term of \( V - \) the term that corresponds to the right \( \text{NS}_5 \) brane (\( S_R \)). The absence of this term is understood as follows: in the type IIA brane description, the difference between the configurations is that \( S_R \) is replaced by \( N \) \( D_6 \) branes (each connected to one right \( D_4 \) brane). The third term in \( V \) describes the extension of \( S_R \) in the \( v \) direction and its absence in the second curve corresponds to the fact that the \( D_6 \) branes are at fixed \( v \). Now, the limit \( q \to 0 \) (for fixed \( w_M \) and \( \Lambda_L' \)) is the limit \( \Lambda_R \to 0 \) which, in the brane description, corresponds to taking \( S_R \) to \( x^6 \to \infty \). The coincidence of the curves means that in this limit the right \( D_4 \) branes “forget” what brane they are attached to. Another significant point in this comparison is that the quark mass is uniform. This means that all the \( D_6 \) branes are in the same \( v \) positions. Classically, this implies that all the right \( D_4 \) branes coincide and in our model this means \( \langle A_R \rangle = 0 \). This is in agreement with the fact that the \( N = 2 \) curve we rotated has genus 0, since this situation occurs for \( \langle A_R \rangle \) of the
order of $\Lambda_R$. Finally, in SQCD, in the limit $m \to 0$ there are vacua only for $N_L \leq N_R$, so it is interesting to see what is the corresponding behavior in our model. Setting $v_L = v_R$ (and $\mu = \infty$) in eq. (4.38), one obtains $w_M \sim \Lambda_R^{N_R-N_L}$, which indeed diverges in the limit $\Lambda_R \to 0$, unless $N_L \leq N_R$.

5. Discussion

In the previous sections we analyzed the same field theoretical model using: a direct (field theory) approach, weakly coupled type IIA string theory and an M theory limit corresponding to a strongly coupled type IIA string. In this section we discuss some conclusions that can be drawn from the results.

5.1 The Frozen $U(1)$ Factors

The freezing of the $U(1)$ gauge factors, described in subsection 3.3.1, raises a puzzle. As we saw in the comparison of the two classical models, the freezing of the $U(1)$ vector multiplets has an opposite effect on other quantities: $\xi$ and $\eta$ effectively change from being parameters to being moduli (related to vev’s of $F$ and $\tilde{F}$). Classically, we identified these parameters as the relative location of the $NS_5$ branes in the (789) directions. Keeping this identification seems to conflict with the "principle" that moduli should not correspond to a motion of infinite objects (this was, after all, the principle that led to the identification of this effect in the first place!). So one may deduce from this principle that this identification is wrong quantum mechanically, but this approach raises other problems. The FI parameters are left with no geometrical interpretation, which is acceptable (we already have such parameters – the Yukawa couplings of the bi-fundamentals). On the other hand, the (789) location of the $NS_5$ branes is left with no imprint in the effective gauge theory, implying that it is either irrelevant (which would be surprising) or fixed by quantum effects.

We may look for guidance in the results we obtained in M theory. In subsection 4.4.1, where we considered parallel $NS_5$ branes (corresponding to $\mu_L = \mu_R = 0$), we found detachment of a (flat) brane exactly in situations where we expected it from classical considerations. In particular, $S_M$ was detached exactly where one would expect the root of the Higgs branch, parameterized by $D^{SC} \det \langle F \rangle$ and $\tilde{D}^{SC} \det \langle \tilde{F} \rangle$ (compare to subsection 2.2.1). Once a (flat) $NS_5$ brane is detached, it is obviously free to move in the (789) directions. Moreover, once the detached brane is moved, the Coulomb branch is truncated, so this motion cannot be completely irrelevant. It does not influence the effective $U(1)$ gauge coupling, but neither are Higgs moduli, at least as long as the $N = 2$ supersymmetry is unbroken. In subsection 4.3, where we considered $\mu_L \neq 0 = \mu_R$,

37This problem is not specific to the present model and appears already in SQCD.
38Similarly, one can easily recognize the detachment of $S_L$ as the root of the baryonic branch parameterized by $Q_L$ moduli, although we did not consider these branches in this work.
we observed a detachment of $S_M$ for $N_L = N_R$, again, exactly where one expects the root of the Higgs branch (compare with subsection 2.2.2).

So the results from M theory seem to suggest that the (789) location of $S_M$ realizes, also in the quantum situation, the FI parameters, which are related (together with another, non geometrical, degree of freedom) to $D$ and $\tilde{D}$. This is also supported by symmetries and quantitative relations. But if this is true, we are back to the original problem: how can a modulus correspond to infinite motion. In principle, this problem can be resolved in two ways: either the above moduli are somehow frozen, or the $S_M$ brane is not infinite. In fact, we already know a situation with the same apparent problem. When the fundamental fields are realized by $D_6$ branes, there is a branch of vacua, (parameterized by $Q$ moduli) corresponding to the (789) locations of a $D_4$ branes extended between left $D_6$ branes and right $D_6$ branes. When these $D_6$ branes are taken to $-\infty$ and $+\infty$ respectively, the $Q$-moduli seem to correspond to the motion of infinite $D_4$ branes! Perhaps the $NS_5$ branes can also be made finite in a similar fashion. This deserves a further investigation.

5.2 Lifting the Coulomb Branch by Adjoint Masses

One of the main issues that were investigated in this work is the effect of turning on masses for the adjoint fields. In the classical models (realized in type IIA string theory) all possibilities were considered, with full agreement between geometric (type IIA) and algebraic (field theoretical) results. In the quantum models the classical branches of vacua may be lifted by dynamically generated superpotentials. In the type IIA description this is expected to correspond to forces between branes, as explained in subsection 3.3.2. Two kinds of situations were considered. Some comprehensive results were obtained using M theory. In the following we will summarize the results and discuss their implications both in field theory and in the dynamics of branes.

5.2.1 $N_L = N_R$, $\mu_L = -\mu_R$

This situation is realized by a rotation of a detached $S_M$ (the central $NS_5$ brane) and was investigated in subsection 4.4.2. Classically, there is a branch of vacua parameterized by $\hat{F}F$ moduli. One finds that this branch survives also in the quantum model. In field theory, this follows from the conservation of an $R$ symmetry, and in type IIA string theory this is because all $D_4$ branes extend between parallel $NS_5$ branes ($S_L$ and $S_R$) and, therefore, there are no forces between them. So we find agreement of all approaches.

Integrating out the massive adjoint fields, one obtains a model with a Yukawa-like coupling between the fundamentals and the bi-fundamentals (see eq. (4.10))

$$\frac{1}{\mu} \text{tr}(\hat{F}BQ) \quad , \quad B := F\tilde{F} - \frac{1}{N} I_N \text{tr}(F\tilde{F}) \quad ,$$

so this procedure leads to the SW curves for this family of $N = 1$ models. For some of these models these curves were derived previously, using field theoretical methods [53, 54].

\[39\] In the classical analysis we found a (smaller) Higgs phase also for $N_L > N_R$. We do not see this branch in the M theory analysis so, presumably, it is lifted by quantum corrections.
In all these cases, our results are in agreement with the previous ones. For \( n_L n_R \neq 0 \) our results seem to be new.

Even when the results are known, the M theory approach often provides an alternative understanding (as is already the case with the SW curve itself). For example, the change of variables and the matching of scales necessary to obtain a finite result in the limit of infinite mass follows naturally from the fact that for a fixed scale, the “rotation” is actually a stretching. Another example is the fact that the effective \( U(1) \) gauge coupling is independent of \( \mu \) and \( \text{tr} \tilde{F} F \). Geometrically this is obvious, since these quantities describe the location and orientation of the detached \( S_M \), which has nothing to do with the gauge fields.

5.2.2 \( \mu_L \neq 0 = \mu_R \)

This situation is realized by the rotation of \( S_L \) (the left \( \text{NS}_5 \) brane) and we analyzed in detail, in M theory (subsections 4.3 and 4.5), the case with no fundamentals \( (n_L = n_R = 0) \). We found that the rotation is possible for any value of \( m_F \) (the mass of the bi-fundamentals), and it leads to a complete lifting of the \( N = 2 \) Coulomb branch, leaving only discrete vacua with no massless gauge fields. Moving \( S_R \) to the extreme right (which corresponds to vanishing coupling of the \( SU(N_R) \) gauge factor) we recovered (for infinite \( \mu \)) the curve for \( N = 1 \) SQCD (with a uniform quark mass). This agreement should be regarded as a successful consistency check.

As remarked in subsection 4.3, the complete lifting of the Coulomb branch appears in a quite general class of models: one can consider a chain of an arbitrary number of \( \text{NS}_5 \) branes connected by \( D_4 \) branes, and what we found is that generically (\( i.e. \), when the curve is irreducible) all parts of the moduli space are lifted, including those related to remote \( D_4 \) branes. This is a quantum effect: returning, for concreteness, to the case of two gauge factors, classically the \( A_R \) moduli remain free when one gives mass to \( A_L \) (see subsection 2.2.2), and this is supported by the (classical) type IIA description, where the right \( D_4 \) branes are suspended between parallel \( \text{NS}_5 \) branes (\( S_M \) and \( S_R \)) and, therefore, free to move in the \( v \)-direction.

**Consequences for Field Theory:**

From the field-theoretical point of view (considered in subsection 2.3), this effect is expected to correspond, at scales below \( \mu_L \), to the dynamical generation of a superpotential which depends on \( A_R \) moduli and, in this way, constrains them. In the limit of vanishing \( \Lambda_R \) and \( \kappa_R \) this superpotential should reduce to that of \( N = 1 \) SQCD [63] (see also [25]). But when either of these parameters is non-vanishing, \( A_R \) dependence is possible. Unlike the previous case \( (\mu_L = -\mu_R) \), here the symmetries do not prevent this possibility and we have shown in subsection 2.3 that there is a related model – magnetic SQCD – in which such a superpotential is generated. The results from M theory mean that this is so also in the present models.

There is another framework in which one can investigate the effect of a mass for the adjoint: the low energy \( N = 2 \) description for \( \mu_L = 0 \). It is interesting to see what our results mean in this framework. One considers small \( \mu_L \) as a perturbation in the low energy
\( N = 2 \) description. Each point in the moduli space has a neighborhood parameterized (holomorphically) by moduli \( L_a, R_{\bar{a}} \), which are vev’s of the scalar \( N = 2 \) superpartners of the massless gauge fields (these are either the eigenvalues of \( A_L \) and \( A_R \), or related to them by a duality transformation). The perturbation is \( \mu_L U \), where \( U \) is a function of the moduli, that coincides with \( \frac{1}{2} \text{tr}(A_L^2) \) in the classical limit. At a generic point there are no other massless fields and the equations of motion require that the gradient of \( U \) vanishes:

\[
\partial_a U = \partial_{\bar{a}} U = 0
\]

which is (at least generically) false. To obtain a vacuum, one needs, for every non-vanishing component of the gradient, a massless \( N = 2 \) hypermultiplet \((\tilde{q}, q)\) charged electrically under the corresponding gauge field (this is what dictates the choice of moduli). Then, the superpotential looks like

\[
W = \mu_L U + \sqrt{2} \tilde{q} A q + \ldots
\]

(where \( A \) is the corresponding modulus), leading to the equations

\[
\mu_L \frac{\partial}{\partial A} U = -\tilde{q} q \, , \quad \tilde{q} A = A q = 0
\]

and implying \( A = 0 \).

Naively, one would expect that \( U \) is independent of \( R_{\bar{a}} \) (as is true classically), leaving the right part of the Coulomb branch unlifted. But since these parts are connected by the bi-fundamentals \( \tilde{F}, F \), this is not necessarily true. Without such a restriction, generically all the components of the gradient do not vanish, implying that all the moduli are lifted.

We found that this is what happens, and this means that quantum effects “mix” the right and left moduli, leading to a dependence of \( U \) on the \( A_R \) moduli. This is related to the left\(\leftrightarrow\)right mixing in \( P_L, P_R \) (see eq. (4.3)), which is possible in the parts proportional to the instanton factors (recall that those parts were not determined by the analysis in section 4).

**Consequences for Brane Dynamics**:

The results from M theory have also implications on the dynamics of branes. Dynamically generated superpotentials in field theory correspond to forces acting on the \( D_4 \) branes \[11\]. This is briefly described in subsection 3.3.2. Such forces reduce the possible equilibrium configurations of these branes and, therefore, lead to the truncation of the moduli space. Once the equilibrium configurations are known, they encode information about the forces that lead to them.

In the present case, in the equilibrium configurations, as found in M theory (and translated to weakly coupled type IIA language), all the right \( D_4 \) branes are grouped together and the displacement between them and the left \( D_4 \) branes (in the \( v \) direction) is a free parameter (corresponding to the mass \( m_F \) of the bi-fundamental in field theory). The question is, what are the forces that lead to such equilibrium configurations. Following

\[40\text{We are grateful to David Kutasov for a discussion on this issue.}\]
there is a Coulomb-like attraction between the left and right $D_4$ branes. We should also impose the constraint found in [19], that the average location of the $D_4$ branes is fixed. This constraint can be understood as a force between the $D_4$ branes and the $NS_5$ branes on which they end. Combining these forces, one obtains the correct vacuum structure (to show this, one uses the fact that a Coulomb force is a single valued function of the displacement). Note that without the constraint, the right $D_4$ branes would all be concentrated at the same point (on the central $NS_5$ brane) as the left $D_4$ branes, which would mean that the mass $m_f$ for the bi-fundamental is fixed to vanish. This is fully consistent with the corresponding field-theoretical description: the constraint corresponds to the freezing of the $U(1)$ factor, so ignoring it leads to a $U(N)$ gauge theory, where $m_f$ is a modulus (or, more precisely, absorbed by a modulus $\text{tr}A_R$) which is expected to be determined by the (quantum) equations of motion (as are the other parts of $A_R$). So we conclude that the constraint fixing the average position of the $D_4$ branes should be imposed.

When there are more than three $NS_5$ branes, we learn from the M theory analysis that there are forces also on the rightmost $D_4$ branes, in spite of the fact that they are in direct contact (i.e. share a common $NS_5$ brane) only with $D_4$ branes that extend between parallel $NS_5$ branes. We know that there are no forces between such $D_4$ branes before the rotation. However, once a $NS_5$ brane is rotated, there are forces acting on any $D_4$ brane connected to the rotated $NS_5$ brane through other $NS_5$ branes and $D_4$ branes (alternatively, there are “forces” acting on any part of an $M_5$ brane connected to a “rotated” part; the “non-locality” in the action of forces in the type IIA description is thus implied by the holomorphicity of the fivebrane worldvolume in the other M theory limit). When the chain is “cut” into two disconnected components, only the $D_4$ branes in the connected part of the rotated $NS_5$ brane are affected by the rotation, so this force is transferred through the brane configuration (through the $M_5$ brane) and not through the bulk. This situation reminds a set of paramagnetic bodies put along a line. Normally, there is no force between them, but when a magnet touches the body at the left end of the line, this body is magnetized and attracts the body to its right and so on. When the magnet is taken away, the forces disappear. Perhaps the forces between $D_4$ branes that extend between parallel $NS_5$ branes can be formulated as such induced magnetization (or polarization). The simplest scenario that leads to the correct vacuum structure would be that the rotated $NS_5$ brane induces (iteratively) attractive forces between $D_4$ branes extending between the same $NS_5$ branes. In a configuration with $N = 2$ supersymmetry, the absence of these forces is understood as a result of exact cancellations implied by the extended supersymmetry. Such cancellations cannot be expected when this symmetry is broken. It would be interesting to check this issue in other situations.

### 5.3 More gauge group factors

The considerations of this work extend naturally to models with more gauge group factors (realized by brane configurations with more $NS_5$ branes). This was already discussed in the context of the lifting of the Coulomb branch. The discussion of a degenerate $M_5$ brane
can also be generalized in this direction, leading to the determination of SW curves for more \(N = 1\) SUSY models. For example, the detachment of a flat \(NS_5\) brane (one of the middle ones; in the absence of \(D_6\) branes) is possible whenever there are two adjacent factors with the same number of colors. The rotation of the detached brane translates in field theory to turning on masses for the corresponding two adjoint fields, and when these fields are integrated out one obtains an \(N = 1\) model where not all the vector superfields have a chiral \(N = 2\) superpartner.

**Acknowledgment:** We are grateful to Shmuel Elitzur, David Kutasov, Yaron Oz and Eliezer Rabinovici for helpful discussions. This work is supported in part by BSF – American-Israel Bi-National Science Foundation, and by the Israel Science Foundation founded by the Israel Academy of Sciences and Humanities – Centers of Excellence Program. A.G. thanks the Theory Division at CERN for hospitality.

**Note Added:**
After completion of this work, we received the preprint \[64\] which also considers the Coulomb branch of some \(N = 1\) SUSY gauge models with product groups by type IIA brane configurations.

**Appendix A. Symmetries**

The classical \(SU(N_L) \otimes SU(N_R)\) model with vanishing superpotential has a large group of global symmetries, which include non-Abelian factors \(SU(n_L)^2 \otimes SU(n_R)^2\), and \(U(1)\) factors described in the following table:\[41\]

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
\theta & q_{\theta} & q_F & q_{
L} & q_{AL} & q_{AR} & q_L & q_{LV} & q_R & q_{RV} \\
\hline
\theta & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
q_{\theta} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
q_F & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
q_{LV} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
q_R & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
q_{RV} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
\Lambda_{L}^{bL} & \Lambda_{R}^{bL} & \Lambda_{L}^{bR} & \Lambda_{R}^{bR} \\
\hline
-2(n_L + N_R) & -2(n_R + N_L) & 2N_R & 2N_L \\
\hline
2N_L & 2N_R & 2n_L & 2n_R \\
\hline
\end{array}
\]

where \(\theta\) is the Fermionic coordinate of superspace and \(\Lambda_{L}^{bL}, \Lambda_{R}^{bR}\) are the instanton factors of the two gauge groups. The charges of the instanton factors \(\Lambda_{L}^{bL}, \Lambda_{R}^{bR}\) are chosen to

\[\text{For } N_L = N_R \text{ and } n_L = n_R \text{ there is also a } Z_2 \text{ symmetry exchanging } L \leftrightarrow R.\]
compensate the anomalies. To determine them, one needs the Dynkin indices $k$ of the $SU(N)$ gauge groups: $k = N$ for the adjoint and $k = \frac{1}{2}$ for the fundamental. Similarly, the charges of the parameters in the action can be determined by the requirement that the (classical) action is invariant (recall that this means that the superpotential has R-charge $q_{\theta} = 2$).

These symmetries can be used to constrain the possible form of superpotentials and SW curves. Here we describe our strategy of using them.

- Conservation of $q_{\theta F}, q_{\theta LV}$ and $q_{\theta RV}$ (the “baryon numbers”) means that the corresponding fields must appear in pairs $\bar{F}^a F^b_a, \bar{Q}^i L_a Q^L_i b, \bar{Q}^i R_a Q^R_i b$.

- In most parts of this work we set the Yukawa couplings $\kappa$ and $\lambda$ to fixed values (either 1 or $\sqrt{2}$). The conservation of $q_{\theta AL}, q_{\theta AR}, q_{\theta L} + q_{\theta LV}$ and $q_{\theta R} + q_{\theta RV}$ can be used to recover $\kappa_L, \kappa_R, \lambda_L$ and $\lambda_R$ respectively. In fact, an efficient way to insure the conservation of the above charges is to use only neutral quantities. The prescription described in eqs. (2.7,2.8), can be interpreted as the identification of a neutral expression that reduces to the correct quantity when $\kappa$ and $\lambda$ are set to 1. For example $\mu_L$ represents the $(q_{\theta AL}$-neutral) expression $\mu_L/\kappa_L^2$. In the quantum theory additional quantities appear – the instanton factors and the coordinates of the SW curve. With the above approach, the coordinates are chosen to be neutral and the instanton factors (for $\kappa = \lambda = 1$) represent

$$\Lambda^b \to \kappa^{2N-n} \Lambda^b \det \lambda \quad \text{(A.2)}$$

When the Yukawa couplings are fixed to a value different from 1, the transformations recovering them should be modified in an obvious way. See, for example, eq. (4.7).

- We are left with two unused symmetries. To be useful when $\kappa$ and $\lambda$ are fixed, these symmetries should be chosen not to act on $\kappa$ and $\lambda$. One such choice is $R_{45}$ and $R_{89}$, defined in table (3.3).

References

[1] For a review, see for example: J. Polchinski, *TASI Lectures on D-Branes*, hep-th/9611050, and references therein.

[2] A. Hanany and E. Witten, *Type IIB Superstrings, BPS Monopoles, And Three-Dimensional Gauge Dynamics*, hep-th/9611230, Nucl. Phys. B492 (1997) 152.

[3] S. Elitzur, A. Giveon, D. Kutasov, *Branes and N=1 Duality in String Theory*, hep-th/9702014, Phys. Lett. B400 (1997) 269.

[4] J. de Boer, K. Hori, H. Ooguri, Y. Oz and Z. Yin, *Mirror Symmetry in Three-Dimensional Gauge Theories, SL(2,Z) and D-Brane Moduli Spaces*, hep-th/9612131.
[5] J. de Boer, K. Hori, Y. Oz and Z. Yin, Branes and Mirror Symmetry in N=2 Supersymmetric Gauge Theories in Three Dimensions, hep-th/9702154.

[6] H. Ooguri and C. Vafa, Geometry of N=1 Dualities in Four Dimensions, hep-th/9702180.

[7] J. L. F. Barbón, Rotated Branes and N=1 Duality, hep-th/9703051.

[8] N. Evans, C. V. Johnson and A. D. Shapere, Orientifolds, Branes, and Duality of 4D Gauge Theories, hep-th/9703210.

[9] J.H. Brodie and A. Hanany, Type IIA Superstrings, Chiral Symmetry, and N=1 4D Gauge Theory Dualities, hep-th/9704043.

[10] A. Brandhuber, J. Sonnenschein, S. Theisen and S. Yankielowicz, Brane Configurations and 4D Field Theory Dualities, hep-th/9704044.

[11] S. Elitzur, A. Giveon, D. Kutasov, E. Rabinovici and A. Schwimmer, Brane Dynamics and N=1 Supersymmetric Gauge Theory, hep-th/9704110.

[12] O. Aharony and A. Hanany, Branes, Superpotentials and Superconformal Fixed Points, hep-th/9704170.

[13] R. Tatar, Dualities in 4D Theories with Product Gauge Groups from Brane Configurations, hep-th/9704198.

[14] I. Brunner and A. Karch, Branes and Six Dimensional Fixed Points, hep-th/9705022.

[15] C. Ahn and R. Tatar, Geometry, D-branes and N=1 Duality in Four Dimensions with Product Gauge Group, hep-th/9705106.

[16] A. Hanany and A. Zaffaroni, Chiral Symmetry from Type IIA Branes, hep-th/9706047.

[17] C. Ahn, K. Oh and R. Tatar, Branes, Geometry and N=1 Duality with Product Gauge Groups of SO and Sp, hep-th/9707027.

[18] N. Evans, Softly Broken SQCD: in the Continuum, on the Lattice, on the Brane, talk presented at SUSY 97, hep-th/9707197.

[19] E. Witten, Solutions Of Four-Dimensional Field Theories Via M Theory, hep-th/9703166.

[20] B. Kol, 5d Field Theories and M Theory, hep-th/9705031.

[21] K. Landsteiner, E. Lopez and D. A. Lowe, N=2 Supersymmetric Gauge Theories, Branes and Orientifolds, hep-th/9705199.
A. Brandhuber, J. Sonnenschein, S. Theisen and S. Yankielowicz, *M Theory And Seiberg-Witten Curves: Orthogonal and Symplectic Groups*, hep-th/9705232.

A. Marshakov, M. Martellini and A. Morozov, *Insights and Puzzles from Branes: 4d SUSY Yang-Mills from 6d Models*, hep-th/9706050.

A. Fayyazuddin and M. Spalinski, *The Seiberg-Witten Differential From M-Theory*, hep-th/9706087.

K. Hori, H. Ooguri and Y. Oz, *Strong Coupling Dynamics of Four-Dimensional N=1 Gauge Theories from M Theory Fivebrane*, hep-th/9706082.

E. Witten, *Branes And The Dynamics Of QCD*, hep-th/9706109.

A. Brandhuber, N. Itzhaki, V. Kaplunovsky, J. Sonnenschein and S. Yankielowicz, *Comments on the M Theory Approach to N=1 SQCD and Brane Dynamics*, hep-th/9706127.

C. V. Johnson, *From M-theory to F-theory, with Branes*, hep-th/9706155.

A. Hanany and K. Hori, *Branes and N=2 Theories in Two Dimensions*, hep-th/9707192.

A. Hanany, M. J. Strassler and A Zaffaroni, *Confinement and Strings in MQCD*, hep-th/9707241.

S. Nam, K. Oh and S.-J. Sin, *Superpotentials of N=1 Supersymmetric Gauge Theories from M-theory*, hep-th/9707247.

M. Henningson and P. Yi, *Four-dimensional BPS-spectra via M-theory*, hep-th/9707251.

T. Nakatsu, K. Ohta, T. Yokono and Y. Yoshida, *Higgs Branch of N=2 SQCD and M theory Branes*, hep-th/9707258.

M. Schmaltz and R. Sundrum, *N = 1 Field Theory Duality from M-theory*, hep-th/9708015.

J. de Boer and Y. Oz, *Monopole Condensation and Confining Phase of N=1 Gauge Theories Via M Theory Fivebrane*, hep-th/9708044.

A. Mikhailov, *BPS States and Minimal Surfaces*, hep-th/9708068.

C. Csaki and W. Skiba, *Duality in Sp and SO Gauge Groups from M Theory*, hep-th/9708082.

K. Landsteiner and E. Lopez, *New Curves from Branes*, hep-th/9708118.
[39] N. Evans and M. Schwetz, *The Field Theory of Non-Supersymmetric Brane Configurations*, hep-th/9708124.

[40] C. Ahn, K. Oh and R. Tatar, *Sp(Nc) Gauge Theories and M Theory Fivebrane*, hep-th/9708127.

[41] N. Seiberg and E. Witten, *Monopole Condensation and Confinement in N = 2 Supersymmetric Yang-Mills Theory*, hep-th/9407087, Nucl. Phys. B426 (1994) 19; *Monopoles, Duality and Chiral Symmetry Breaking in N = 2 Supersymmetric QCD*, hep-th/9408099, Nucl. Phys. B431 (1994) 484.

[42] A. Hanany and Y. Oz, *On the Quantum Moduli Space of Vacua of N = 2 Supersymmetric SU(Nc) Gauge Theories*, hep-th/9505073, Nucl. Phys. B452 (1995) 283.

[43] P. C. Argyres, M. R. Plesser and A. Shapere, *The Coulomb Phase of N=2 Supersymmetric QCD*, hep-th/9505100, Phys. Rev. Lett. 75 (1995) 1699.

[44] J.A. Minahan and D. Nemeschansky, *Hyperelliptic Curves for Supersymmetric Yang-Mills*, hep-th/9507032, Nucl. Phys. B464 (1996) 3.

[45] A. Hanany, *On the Quantum Moduli Space of Vacua of N = 2 Supersymmetric Gauge Theories*, hep-th/9509176, Nucl. Phys. B466 (1996) 85.

[46] P.C. Argyres and A.D. Shapere, *The Vacuum Structure of N = 2 Super-QCD with Classical Gauge Groups*, hep-th/9509173, Nucl. Phys. B461 (1996) 437.

[47] K. Intriligator and N. Seiberg, *Phases of N = 1 Supersymmetric Gauge Theories in Four Dimensions*, hep-th/9408153, Nucl. Phys. B431 (1994) 551; *Duality, Monopoles, Dyons, Confinement and Oblique Confinement in Supersymmetric SO(Nc) Gauge Theories*, hep-th/9503179, Nucl. Phys. B444 (1995) 125.

[48] S. Elitzur, A. Forge, A. Giveon and E. Rabinovici, *More Results in N = 1 Supersymmetric Gauge Theories*, hep-th/9504080, Phys. Lett. B353 (1995) 79; *Effective Potentials and Vacuum Structure in N = 1 Supersymmetric Gauge Theories*, hep-th/9509130, Nucl. Phys. B459 (1996) 160; *Summary of Results in N = 1 Supersymmetric SU(2) Gauge Theories*, hep-th/9512140, Nucl. Phys. Proc. Suppl. 49 (1996) 174.

[49] S. Elitzur, A. Forge, A. Giveon, K. Intriligator and E. Rabinovici, *Massless Monopoles Via Confining Phase Superpotentials*, hep-th/9603051, Phys. Lett. B379 (1996) 121.
[50] A. Kapustin, *The Coulomb Branch of N = 1 Supersymmetric Gauge Theory With Adjoint and Fundamental Matter*, hep-th/9611049, Phys. Lett. B398 (1997) 104.

[51] T. Kitao, S. Terashima and S.-K Yang, *N = 2 Curves and a Coulomb Phase in N = 1 SUSY Gauge Theories With Adjoint and Fundamental Matters*, hep-th/9701009, Phys. Lett. B399 (1997) 75.

[52] A. Giveon, O. Pelc and E. Rabinovici, *The Coulomb Phase in N=1 Gauge Theories With a LG-Type Superpotential*, hep-th/9701045, Nucl. Phys. B499 (1997) 100.

[53] C. Csaki, J. Erlich, D. Freedman and W. Skiba, *N=1 Supersymmetric Product Group Theories in the Coulomb Phase*, hep-th/9704067.

[54] M. Gremm, *The Coulomb branch of N=1 supersymmetric SU(N_c) x SU(N_c) gauge theories*, hep-th/9707077.

[55] P. C. Argyres, M. R. Plesser and N. Seiberg, *The Moduli Space of N=2 SUSY QCD and Duality in N=1 SUSY QCD*, hep-th/9603042, Nucl. Phys. B471 (1996) 159.

[56] M.A. Luty and W. Taylor IV, *Varieties of vacua in classical supersymmetric gauge theories*, hep-th/9506098, Phys. Rev. D53 (1996) 3399.

[57] N. Seiberg, *Electric-Magnetic Duality in Supersymmetric Non-Abelian Gauge Theories*, hep-th/9411149, Nucl. Phys. B435 (1995) 129.

[58] M. Berkooz, M.R. Douglas and R.G. Leigh, *Branes Intersecting at Angles*, hep-th/9606139, Nucl. Phys. B480 (1996) 265.

[59] K. Becker, M. Becker and A. Strominger, *Fivebranes, Membranes and Non-Perturbative String Theory*, hep-th/9507158, Nucl. Phys. B456 (1995) 130.

[60] K. Becker, M. Becker, D. R. Morrison, H. Ooguri, Y. Oz and Z. Yin, *Supersymmetric Cycles in Exceptional Holonomy Manifolds and Calabi-Yau 4-Folds*, hep-th/9608116, Nucl. Phys. B480 (1996) 225.

[61] E. Verlinde, *Global Aspects of Electric-Magnetic Duality*, hep-th/9506011, Nucl. Phys. B455 (1995) 211.

[62] See, for example, P. A. Griffiths, *Introduction to Algebraic Curves* (English Translation), American Mathematical Society 1989.

[63] For a review, see for example: K. Intriligator and N. Seiberg, *Lectures on Supersymmetric Gauge Theories and Electric-Magnetic Duality*, hep-th/9509066, Nucl. Phys. Proc. Suppl. 45BC (1996) 1, and references therein.

[64] J. Lykken, E. Poppitz and S. P. Trivedi, *Chiral Gauge Theories from D-Branes*, hep-th/9708134.