POISSON STRUCTURES ON COMPLEX FLAG MANIFOLDS ASSOCIATED WITH REAL FORMS

PHILIP FOTH AND JIANG-HUA LU

Abstract. For a complex semisimple Lie group $G$ and a real form $G_0$ we define a Poisson structure on the variety of Borel subgroups of $G$ with the property that all $G_0$-orbits in $X$ as well as all Bruhat cells (for a suitable choice of a Borel subgroup of $G$) are Poisson submanifolds. In particular, we show that every non-empty intersection of a $G_0$-orbit and a Bruhat cell is a regular Poisson manifold and we compute the dimension of its symplectic leaves.

Dedicated to Alan Weinstein on the occasion of his 60th Birthday.

1. Introduction.

Let $G$ be a connected and simply connected complex semisimple Lie group with Lie algebra $\mathfrak{g}$, and let $X$ be the variety of Borel subalgebras of $\mathfrak{g}$. In this paper we use a real form $\mathfrak{g}_0$ of $\mathfrak{g}$ to define a Poisson structure on $X$. This Poisson structure depends on a choice of a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ such that $\mathfrak{g}_0 \cap \mathfrak{b}$ is a maximally compact Cartan subalgebra of $\mathfrak{g}_0$. Instead of dealing with each real form individually, we fix a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$. Then, as is shown in [6], a real form $\mathfrak{g}_v$ of $\mathfrak{g}$ can be constructed from each Vogan diagram $v$ for $\mathfrak{g}$ such that $\mathfrak{g}_v \cap \mathfrak{b}$ is a maximally compact Cartan subalgebra of $\mathfrak{g}_v$. The corresponding Poisson structure on $X$ is denoted by $\Pi_v$.

Let $G_v$ be the real form of $G$ corresponding to $\mathfrak{g}_v$, and let $B$ be the Borel subgroup of $G$ with Lie algebra $\mathfrak{b}$. The Poisson structure $\Pi_v$ has the property that each $G_v$-orbit as well as each $B$-orbit in $X$ is a Poisson submanifold. The $B$-orbits in $X$ will be referred to as the Bruhat cells. We compute the rank of $\Pi_v$. In particular, if $G_v$-orbit $\mathcal{O}$ meets a Bruhat cell $\mathcal{C}$, they intersect transversally, and we find that all the symplectic leaves in $\mathcal{O} \cap \mathcal{C}$ have the same dimension, so $\mathcal{O} \cap \mathcal{C}$ is a regular Poisson manifold. Moreover, we show that all symplectic leaves in each connected component of $\mathcal{O} \cap \mathcal{C}$ are translates of each other by elements of a Cartan subgroup of $G_v$. We also show that the $G_v$-invariant Poisson cohomology for each open $G_v$-orbit in $X$ is isomorphic to the de Rham cohomology of $X$.

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Results similar to those presented here for the full flag manifold \( X = G/B \) are also valid for a partial flag manifold \( G/P \), where \( P \) is a parabolic subgroup of \( G \) containing \( B \). We will treat these more general cases as well as some further properties of \( \Pi_v \) in a future paper.

Throughout this paper, if \( V \) is a set and \( \sigma \) is an involution on \( V \), we will use \( V^\sigma \) to denote the fixed point set of \( \sigma \) in \( V \).

## 2. Real forms of \( g \) and Vogan diagrams

Let \( g \) be a complex simple Lie algebra. In this section we recall the classification of real forms of \( g \) by Vogan diagrams. Details can be found in \([6, \text{Chapter 6}]\).

Suppose that \( g_0 \) is a real form of \( g \) and that \( \tau_0 \) is the corresponding complex-conjugate linear involution on \( g \). Let \( \theta_0 \) be a Cartan involution of \( g_0 \), and let \( h_0 \) be a \( \theta_0 \)-stable maximally compact Cartan subalgebra of \( g_0 \). Set \( t_0 = h_0^{\theta_0} \) and \( a_0 = h_0^{-\theta_0} \) so that \( h_0 = t_0 + a_0 \). Let \( \gamma_0 \) be the complexification of \( \theta_0 \). Then the Cartan subalgebra \( h = h_0 + i h_0 \) of \( g \) is \( \gamma_0 \)-stable. Let \( \Delta \) be the root system for \((g, h)\). Since \( h_0 \) is a maximally compact Cartan subalgebra of \( g_0 \), there exists \( x_0 \in i t_0 \) that is regular for \( \Delta \). Define the subset \( \Delta^+ \) of positive roots in \( \Delta \) by \( \alpha \in \Delta^+ \) if and only if \( \alpha(x_0) > 0 \). Then \( \gamma_0(\Delta^+) = \Delta^+ \). Let \( \Sigma \subset \Delta^+ \) be the set of simple roots in \( \Delta^+ \). Then \( \gamma_0(\Sigma) = \Sigma \), so \( \gamma_0 \) gives rise to an involutive automorphism of the Dynkin diagram of \( g \). Let \( \mathcal{I} \) be the set of non-compact imaginary simple roots. The Vogan diagram of \( g_0 \) associated to the triple \((\theta_0, h_0, \Delta^+)\) is the Dynkin diagram \( D(g) \) of \( g \) together with an involutive automorphism \( \gamma_0 \) on \( D(g) \) and the vertices corresponding to the simple roots in \( \mathcal{I} \) painted black.

In general, a Vogan diagram for \( g \) is defined to be a triple \((D(g), d, \mathcal{I})\), where \( D(g) \) is the Dynkin diagram of \( g \), \( d \) is an involutive automorphism of \( D(g) \), and \( \mathcal{I} \) is a subset of vertices of \( D(g) \) such that \( d(\alpha) = \alpha \) for each \( \alpha \in \mathcal{I} \). Every Vogan diagram for \( g \) comes from a real form of \( g \) (see below), although two different Vogan diagrams can come from isomorphic real forms. A non-redundant list of Vogan diagrams with the corresponding isomorphism class of real forms for all simple Lie algebras is given in \([6]\). Every Vogan diagram in the list in \([6]\) is normalized in the sense that at most one vertex is painted black.

For the purpose of defining Poisson structures on the variety of Borel subalgebras of \( g \), we now recall the explicit construction of a real form of \( g \) from a Vogan diagram \([6, \text{Theorem 6.88}]\). We need to fix the following data for \( g \).

Choose a Cartan subalgebra \( h \) of \( g \) and let \( \Delta \) be the root system for \((g, h)\). Fix a choice of positive roots \( \Delta^+ \) and let \( \Sigma \) be the basis of simple roots. Let \( \langle \cdot, \cdot \rangle \) be the Killing form of \( g \) and let root vectors \( \{E_\alpha : \alpha \in \Delta\} \) be chosen such that \( [E_\alpha, E_{-\alpha}] = H_\alpha \) for each \( \alpha \in \Delta^+ \), where \( H_\alpha \) is the unique element of \( h \) defined by \( \langle H, H_\alpha \rangle = \alpha(H) \) for all
Let \( H \in \mathfrak{h} \), and such that the numbers \( m_{\alpha,\beta} \) given by \( [E_\alpha, E_\beta] = m_{\alpha,\beta}E_{\alpha+\beta} \) when \( \alpha + \beta \in \Delta \) are real. Define a compact real form \( \mathfrak{k} \) of \( \mathfrak{g} \) as
\[
\mathfrak{k} = \text{span}_\mathbb{R}\{iH_\alpha, X_\alpha := E_\alpha - E_{-\alpha}, Y_\alpha := i(E_\alpha + E_{-\alpha})\},
\]
and let \( \theta \) be the complex conjugation of \( \mathfrak{g} \) defining \( \mathfrak{k} \). If \( d \) is an involutive automorphism of the Dynkin diagram of \( \mathfrak{g} \), define \( \gamma_d \) to be the unique automorphism of \( \mathfrak{g} \) satisfying \( \gamma_d(H_\alpha) = H_{d(\alpha)} \) and \( \gamma_d(E_\alpha) = E_{d(\alpha)} \) for each simple root \( \alpha \).

Given a Vogan diagram \( v \) for \( \mathfrak{g} \), not necessarily normalized, with the involutive diagram automorphism \( d \), let \( t_v \) be the unique element in the adjoint group of \( \mathfrak{g} \) such that
\[
\text{Ad}_{t_v}(E_\alpha) = \begin{cases} E_\alpha & \text{if } \alpha \text{ is a blank vertex in } v \\ -E_\alpha & \text{if } \alpha \text{ is a painted vertex in } v \end{cases}
\]
Define a complex conjugate linear involution
\[
\tau_v := \text{Ad}_{t_v} \circ \gamma_d \circ \theta.
\]

**Notation 2.1.** We use \( \mathfrak{g}_v = \mathfrak{g}^{\tau_v} \) to denote the real form of \( \mathfrak{g} \) defined by \( \tau_v \). Set \( \theta_v = \theta|_{\mathfrak{g}_v} \). Then \( \theta_v \) is a Cartan involution of \( \mathfrak{g}_v \), and \( \mathfrak{h}^{\tau_v} \) is a \( \theta_v \)-stable maximally compact Cartan subalgebra of \( \mathfrak{g}_v \), with \( \mathfrak{h} = \mathfrak{h}^{\tau_v} + i\mathfrak{h}^{\tau_v} \). The complexification of \( \tau_v \) is
\[
(2.1) \quad \gamma_v := \tau_v \theta = \theta \tau_v = \text{Ad}_{t_v} \gamma_d.
\]
Since \( \gamma_v(\Delta^+) = \Delta^+ \), the Vogan diagram of \( \mathfrak{g}_v \) associated to the triple \((\theta_v, \mathfrak{h}^{\tau_v}, \Delta^+)\) is \( v \).

One of the advantages of introducing the real form \( \mathfrak{g}_v \) is as follows. We say that a real subalgebra \( \mathfrak{l} \) of \( \mathfrak{g} \) is Lagrangian if its real dimension is equal to the complex dimension of \( \mathfrak{g} \) and if \( \text{Im} \ll x_1, x_2 \gg = 0 \) for all \( x_1, x_2 \in \mathfrak{l} \). A decomposition \( \mathfrak{g} = \mathfrak{l}_1 + \mathfrak{l}_2 \) is called a Lagrangian splitting if both \( \mathfrak{l}_1 \) and \( \mathfrak{l}_2 \) are Lagrangian. Let \( \mathfrak{n} \) be the subalgebra of \( \mathfrak{g} \) spanned by the set of all positive root vectors for \( \Delta^+ \). The following fact is easy to prove.

**Lemma 2.2.** Let \( \mathfrak{l}_d := \mathfrak{h}^{-\tau_v} + \mathfrak{n} \). Then \( \mathfrak{g} = \mathfrak{g}_v + \mathfrak{l}_d \) is a Lagrangian splitting of \( \mathfrak{g} \).

Let \( \mathfrak{a} = \text{span}_\mathbb{R}\{1H_\alpha : \alpha \in \Sigma\} \), and let \( \mathfrak{t} = \mathfrak{i} \mathfrak{a} \). We note that since
\[
\mathfrak{h}^{-\tau_v} = \mathfrak{h}^{-\gamma_d \circ \theta} = \mathfrak{t}^{-\gamma_d} + \mathfrak{a}^{-\gamma_d},
\]
the Lagrangian complement \( \mathfrak{l}_d \) of \( \mathfrak{g}_v \) depends only on \( d \), and in the case when \( d = 1 \), we have \( \mathfrak{l}_d = \mathfrak{a} + \mathfrak{n} \). Note that \( \mathfrak{h}^{-\tau_v} = \mathfrak{h}^{\gamma_d \circ \theta} = \mathfrak{t}^{\gamma_d} + \mathfrak{a}^{\gamma_d} \) also depends only on \( d \).

**Remark 2.3.** Recall [2, Definition 6.10] that two real forms \( \tau_1 \) and \( \tau_2 \) are said to be in the same inner class if there exists \( g \in \text{Int}(\mathfrak{g}) \), the adjoint group of \( \mathfrak{g} \), such that \( \tau_1 = \text{Ad}_g \tau_2 \). Inner classes of real forms are in one-to-one correspondence with involutive automorphisms of the Dynkin diagram of \( \mathfrak{g} \) [2 Proposition 6.12]. Let \( d \) be an involutive automorphism of \( D(\mathfrak{g}) \). Then as \( v \) runs over the collection of all Vogan diagrams with \( d \) as the diagram automorphism, the real form \( \mathfrak{g}_v \) runs over all \( \text{Int}(\mathfrak{g}) \)-conjugacy classes of real forms of \( \mathfrak{g} \) in the inner class corresponding to \( d \).
3. The Poisson structure $\Pi_v$ on $X$.

Let $\mathfrak{g}$ be a complex semi-simple Lie algebra, and let $X$ be the variety of all Borel subalgebras of $\mathfrak{g}$. We keep the notation from Section 2. Let $v$ be a Vogan diagram for $\mathfrak{g}$ and $\mathfrak{g}_v = \mathfrak{g}^{\tau_v}$ be the real form of $\mathfrak{g}$ constructed in Section 2. Let $G$ be the connected and simple connected Lie group with Lie algebra $\mathfrak{g}$. Without any risk of confusion, we shall also denote by $\tau_v$ the lift of $\tau_v$ from $\mathfrak{g}$ to $G$, and we set $G_v = G^{\tau_v}$. It follows from [3, Theorem 8.2, p. 320] that the group $G_v$ is connected.

In this section, we will start with a Vogan diagram $v$ for $\mathfrak{g}$ and define a Poisson structure $\Pi_v$ on $X$ such that every $G_v$-orbit in $X$ is a Poisson submanifold. This Poisson structure comes from an identification of $X$ with the $G$-orbit through $t + \mathfrak{n}$ inside the variety $\mathcal{L}$ of Lagrangian subalgebras of $\mathfrak{g}$, which was studied in [3]. We now recall the relevant details.

Set $n = \dim \mathfrak{g}$ and let $\text{Gr}_{\mathbb{R}}(n, \mathfrak{g})$ be the Grassmannian of real $n$-dimensional subspaces of $\mathfrak{g}$. The set $\mathcal{L}$ of all Lagrangian subalgebras of $\mathfrak{g}$ is naturally a real subvariety of $\text{Gr}_{\mathbb{R}}(n, \mathfrak{g})$.

The natural action of $G$ on $\text{Gr}_{\mathbb{R}}(n, \mathfrak{g})$ gives rise to a Lie algebra anti-homomorphism $\kappa$ from $\mathfrak{g}$ to the Lie algebra of vector fields on $\text{Gr}_{\mathbb{R}}(n, \mathfrak{g})$, whose extension from $\wedge^2 \mathfrak{g}$ to the space of bi-vector fields on $\text{Gr}_{\mathbb{R}}(n, \mathfrak{g})$ will also be denoted by $\kappa$. Given a Lagrangian splitting $\mathfrak{g} = \mathfrak{l}_1 + \mathfrak{l}_2$, we define the element $R_{\mathfrak{l}_1, \mathfrak{l}_2} \in \wedge^2 \mathfrak{g}$ by:

\[(3.1) \quad \langle R_{\mathfrak{l}_1, \mathfrak{l}_2}, (x_1 + \xi_1) \wedge (x_2 + \xi_2) \rangle = \langle \xi_2, x_1 \rangle - \langle \xi_1, x_2 \rangle, \quad x_1, x_2 \in \mathfrak{l}_1, \xi_1, \xi_2 \in \mathfrak{l}_2,
\]

where $\langle , \rangle = \text{Im} \ll , \gg$. Set $\Pi_{\mathfrak{l}_1, \mathfrak{l}_2} = \frac{1}{2} \kappa(R_{\mathfrak{l}_1, \mathfrak{l}_2})$. Clearly, $\Pi_{\mathfrak{l}_1, \mathfrak{l}_2}$ is tangent to every $G$-orbit in $\text{Gr}_{\mathbb{R}}(n, \mathfrak{g})$, so it is tangent to $\mathcal{L}$.

**Theorem 3.1.** [3, Theorems 2.14 and 2.18] The bi-vector field $\Pi_{\mathfrak{l}_1, \mathfrak{l}_2}$ restricts to a Poisson structure on $\mathcal{L}$. If $L_1$ and $L_2$ are the connected subgroups of $G$ with Lie algebras $\mathfrak{l}_1$ and $\mathfrak{l}_2$ respectively, then all the $L_1$- as well as $L_2$-orbits in $\mathcal{L}$ are Poisson submanifolds with respect to $\Pi_{\mathfrak{l}_1, \mathfrak{l}_2}$.

For $\mathfrak{I} \in \mathcal{L}$, let $\mathfrak{n}(\mathfrak{I})$ be the normalizer subalgebra of $\mathfrak{I}$ in $\mathfrak{l}_1$. Let $\mathfrak{m}(\mathfrak{I})$ be the annihilator of $\mathfrak{n}(\mathfrak{I})$ in $\mathfrak{l}_1$, i.e. $\mathfrak{m}(\mathfrak{I}) = \{ x \in \mathfrak{l}_1 : \langle x, y \rangle = 0 \ \forall y \in \mathfrak{n}(\mathfrak{I}) \} \subseteq \mathfrak{l}_1$, and let $\mathcal{V}(\mathfrak{I}) = \mathfrak{n}(\mathfrak{I}) + \mathfrak{m}(\mathfrak{I})$.

**Proposition 3.2.** [3, Theorem 2.21] [2, Corollary 7.3] For each $\mathfrak{I} \in \mathcal{L}$, the space $\mathcal{V}(\mathfrak{I})$ is a Lagrangian subalgebra of $\mathfrak{g}$. The co-dimension of the symplectic leaf of $\Pi_{\mathfrak{l}_1, \mathfrak{l}_2}$ through $\mathfrak{I}$ in the orbit $L_1 \cdot \mathfrak{I}$ is equal to $\dim(\mathcal{V}(\mathfrak{I}) \cap \mathfrak{l}_2)$.

**Notation 3.3.** Let $v$ be a Vogan diagram for $\mathfrak{g}$. We denote by $\Pi_v$ the Poisson structure on $\mathcal{L}$ defined by the Lagrangian splitting $\mathfrak{g} = \mathfrak{g}_v + \mathfrak{l}_d$ in Lemma 2.2. Let $H$, $N$, and $B$ be respectively the connected subgroups of $G$ with Lie algebras $\mathfrak{h}$, $\mathfrak{n}$, and $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$, so $B = HN$. Identify the $G$-orbit through $t + \mathfrak{n} \in \mathcal{L}$ with $G/B \cong X$. The induced Poisson structure on $X$ will also be denoted by $\Pi_v$. Let $H^{-\gamma_d \circ \theta} = \{ h \in H : \gamma_d \circ \theta(h) = h^{-1} \}$ and let $L_d = H^{-\gamma_d \circ \theta} N$. By the Bruhat lemma, orbits of $L_d$ in $X \cong G/B$, which are the same
as the $N$-orbits in $X$, are labeled by the elements in the Weyl group $W$ of $\Delta$. We refer to these $N$-orbits as the Bruhat cells in $X$.

By [3, Theorem 2.18], we have

**Proposition 3.4.** Each $G_v$-orbit in $X$ as well as each Bruhat cell in $X$ is a Poisson submanifold with respect to $\Pi_v$.

When $v$ is the Vogan diagram with $d = 1$ and no vertex painted, we have $\tau_v = \theta$, so $g_v = \mathfrak{k}$. The Poisson structure $\Pi_v$ in this case was first introduced in [11] and [13], and it has the property that its symplectic leaves are precisely the Bruhat cells (hence the name “Bruhat Poisson structure” in [11]). In [3] and [10] this Poisson structure was related to some earlier work of Kostant [7] and of Kostant-Kumar [8] on the Schubert calculus on $X$.

The splitting $\mathfrak{g} = \mathfrak{g}_v + \mathfrak{l}_d$ naturally defines a Lie bialgebra structure on $\mathfrak{g}_v$ and therefore a Poisson Lie group structure on $G_v$. All the $G_v$-orbits in $\mathcal{L}$ become $G_v$-Poisson homogeneous spaces [3, 9]. We remark that in [1], Andruskiewitsch and Jancsa classified non-triangular Lie bialgebra structures on $\mathfrak{g}_v$ using Belavin-Drinfeld triples. The one defined by the splitting $\mathfrak{g} = \mathfrak{g}_v + \mathfrak{l}_d$ comes from the standard Belavin-Drinfeld triple. We refer to [1] for details.

**Example.** Here we take $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and

$$\mathfrak{g}_v = \mathfrak{su}(1, 1) = \left\{ \begin{pmatrix} 1x & y + 1z \\ y - 1z & -1x \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$  

Then $d = 1$ and $\mathfrak{l}_d = \mathfrak{a} + \mathfrak{n}$ consists of upper triangular matrices in $\mathfrak{sl}(2, \mathbb{C})$ with real diagonal entries. Identify $G/B$ with $\mathbb{P}^1$ via the action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [w_0 : w_1] = [aw_0 + bw_1 : cw_0 + dw_1]$$

of $G$ on $\mathbb{P}^1$ and by taking $[1 : 0] \in \mathbb{P}^1$ as the basepoint. There are two Bruhat cells: the zero-dimensional basepoint $[1 : 0]$, and the other being the rest:

$$U_1 = \mathbb{P}^1 \setminus \{[1 : 0]\} = \{[w_0 : w_1], w_1 \neq 0\}.$$  

In terms of the holomorphic coordinate $z$ on $U_1$ given by $z = w_0/w_1$ the Poisson structure $\Pi_v$, up to a scalar multiple, is given by:

$$\Pi_v = \imath(1 - |z|^2) \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \bar{z}}.$$  

Setting $u = 1/z$, we see that in the $u$-coordinate on the open set

$$U_0 = \{[w_0 : w_1] \in \mathbb{P}^1, w_0 \neq 0\} = \{[1 : u], u \in \mathbb{C}\},$$  

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In terms of the holomorphic coordinate $z$ on $U_1$ given by $z = w_0/w_1$ the Poisson structure $\Pi_v$, up to a scalar multiple, is given by:

$$\Pi_v = \imath(1 - |z|^2) \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \bar{z}}.$$  

Setting $u = 1/z$, we see that in the $u$-coordinate on the open set

$$U_0 = \{[w_0 : w_1] \in \mathbb{P}^1, w_0 \neq 0\} = \{[1 : u], u \in \mathbb{C}\},$$
we have
\[ \Pi_v = i(|u|^2 - 1)|u|^2 \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial \bar{u}}. \]
Thus \( \Pi_v \) vanishes precisely at the basepoint \([1 : 0]\) and at every point of the form \([z : 1]\) with \(|z| = 1\). If we identify \( \mathbb{P}^1 \) with the unit sphere \( S^2 \) in \( \mathbb{R}^3 \) via:
\[ (3.2) \quad \mathbb{P}^1 \longrightarrow S^2 : [w_0, w_1] \mapsto \left( \frac{2 \text{Re}(w_0\bar{w}_1)}{|w_0|^2 + |w_1|^2}, \frac{2 \text{Im}(w_0\bar{w}_1)}{|w_0|^2 + |w_1|^2}, \frac{|w_0|^2 - |w_1|^2}{|w_0|^2 + |w_1|^2} \right), \]
then we see that \( \Pi_v \) vanishes at the “North pole” \((0, 0, 1)\) and at every point on the Equator \( x_3 = 0 \). Under this identification, there are exactly three orbits of \( SU(1, 1) \) on \( S^2 \): the Northern hemisphere, the Equator, and the Southern hemisphere. Each one of these three orbits is clearly a Poisson submanifold.

4. Symplectic leaves of \( \Pi_v \) in \( X \).

Suppose that \( \mathcal{O} \) is a \( G_v \)-orbit in \( X \) and \( \mathcal{C} \) is a Bruhat cell such that \( \mathcal{O} \cap \mathcal{C} \neq \emptyset \). Since \( g = g_v + l_h \), \( \mathcal{O} \) and \( \mathcal{C} \) intersect transversally. By Proposition 3.4, \( \mathcal{O} \cap \mathcal{C} \) is a Poisson submanifold of \( \Pi_v \). In this section we show that \((\mathcal{O} \cap \mathcal{C}, \Pi_v)\) is a regular Poisson manifold and we compute the dimension of its symplectic leaves.

It is well-known [14] that there are only finitely many \( G_v \)-orbits in \( X \). We first recall from [12] some facts about these orbits.

Let \( N_G(\mathfrak{h}) \) be the normalizer subgroup of \( \mathfrak{h} \) in \( G \). Set
\[ \mathcal{Z} = \{ g \in G : g^{-1}\tau_v(g) \in N_G(\mathfrak{h}) \}. \]
Then \( H \) acts on \( \mathcal{Z} \) from the right by right multiplication, and \( G_v \) acts on \( \mathcal{Z} \) from the left by left multiplication. Let \( Z \) be the double coset space
\[ Z = G_v \backslash \mathcal{Z} / H. \]
For each \( z \in \mathcal{Z} \), choose any \( g_z \in \mathcal{Z} \) in the double coset \( z \) and define \( \mathcal{O}_z \) to be the \( G_v \)-orbit in \( X \) through \( g_zB \in X \cong G/B \). Clearly, \( \mathcal{O}_z \) is independent of the choice of \( g_z \). According to [12] Theorem 6.1.4], the map \( z \mapsto \mathcal{O}_z \) is a one-to-one correspondence between the set \( Z \) and the set of \( G_v \)-orbits in \( X \). Let \( W = N_G(\mathfrak{h})/H \) be the Weyl group. Thus we also have the map
\[ \varphi : Z \longrightarrow W : z = G_vg_zH \mapsto g_z^{-1}\tau_v(g_z)H \in W. \]
According to [12] Theorem 6.4.2], the codimension of the \( G_v \)-orbit \( \mathcal{O}_z \) in \( X \) equals \( l(\varphi(z)) \), where \( l \) is the length function on the Weyl group \( W \). We also introduce the map:
\[ \sigma_z = \varphi(z)\tau_v : \mathfrak{h} \longrightarrow \mathfrak{h}. \]
For any \( g_z \) in the double coset \( z \), we also have \( \sigma_z = \text{Ad}^{-1}_{g_z} \circ \tau_v \circ \text{Ad}_{g_z} \), so \( \sigma_z \) is an involution.
Assume now that \( z \in Z \) and \( w \in W \) are such that \( O_z \cap C_w \neq \emptyset \), where \( C_w \) is the Bruhat cell in \( X \) corresponding to \( w \), i.e. the \( N \)-orbit through \( w \in G/B \). Then \( \dim_{\mathbb{R}} C_w = 2l(w) \), and since \( O_z \) and \( C_w \) intersect transversally, we have

\[
\dim(O_z \cap C_w) = 2l(w) - l(\varphi(z)).
\]

Define now

\[
\delta_{z,w} = \dim(h^{w_{\sigma_z}w^{-1}} \cap h^{-\tau_w}).
\]

**Theorem 4.1.** Each symplectic leaf in the intersection \( O_z \cap C_w \) has dimension equal to

\[
\dim(O_z \cap C_w) - \delta_{z,w} = 2l(w) - l(\varphi(z)) - \delta_{z,w}.
\]

**Proof.** We use Proposition \ref{prop:3.2} to compute dimensions of the symplectic leaves in \( O_z \cap C_w \). Let \( x = g_zB \in X \) be a point in \( O_z \cap C_w \), where \( g_z \in Z \) lies in the double coset \( z \). Let \( I_x = \text{Ad}_{g_z}(t + n) \in L \). Let \( n(I_x) = g_v \cap \text{Ad}_{g_z}(h + n) \) be the normalizer subalgebra of \( I_x \) in \( g_v \), \( m(I_x) \) be the annihilator subspace of \( n(I_x) \) in \( I_x \), and \( \mathcal{V}(I_x) = n(I_x) + m(I_x) \). We claim that \( \mathcal{V}(l_x) = \text{Ad}_{g_z}(h^{\sigma_z} + n) \). Indeed, it follows from the definition of \( \sigma_z \) that

\[
\text{Ad}_{g_z}(h^{\sigma_z}) \subset g_v \cap \text{Ad}_{g_z}(h + n) = n(I_x).
\]

It is also clear that \( \text{Ad}_{g_z}n \subset m(I_x) \), so

\[
\text{Ad}_{g_z}(h^{\sigma_z} + n) \subset n(I_x) + m(I_x) = \mathcal{V}(I_x).
\]

Since both \( \text{Ad}_{g_z}(h^{\sigma_z} + n) \) and \( \mathcal{V}(I_x) \) have the same dimension, they must coincide.

Let now \( S_x \) be the symplectic leaf of \( \Pi_v \) in \( X \) through \( x \). By Proposition \ref{prop:3.2}, the codimension of \( S_x \) in \( O_z \) is equal to \( \dim(\mathcal{V}(I_x) \cap I_d) \). Let \( \dot{w} \in N_G(h) \) be a representative of \( w \) in \( K \). Since \( x \in C_w \), there exist \( n \in N \) and \( b \in B \) such that \( g_z = n\dot{w}b \). Then we have

\[
\mathcal{V}(I_x) \cap I_d = (\text{Ad}_{n\dot{w}b}(h^{\sigma_z} + n)) \cap (h^{-\tau_v} + n) = \text{Ad}_{n}( (\text{Ad}_{\dot{w}}(h^{\sigma_z} + n)) \cap (h^{-\tau_v} + n)) = \text{Ad}_{n} \left( h^{w_{\sigma_z}w^{-1}} \cap h^{-\tau_v} + (\text{Ad}_{\dot{w}}n) \cap n \right),
\]

where in the last line we have the direct sum of vector spaces. Since

\[
\dim(\text{Ad}_{\dot{w}}n) \cap n = \dim_{\mathbb{R}} X - \dim_{\mathbb{R}} C_w,
\]

we have

\[
\dim(\mathcal{V}(I_x) \cap I_d) = \delta_{z,w} + \dim_{\mathbb{R}} X - \dim_{\mathbb{R}} C_w,
\]

and thus

\[
\dim S_x = \dim O_z - \dim(\mathcal{V}(I_x) \cap I_d) = \dim(O_z \cap C_w) - \delta_{z,w}.
\]

\[\square\]

Note, that the number \( \delta_{z,w} \) depends only on \( d \) and the two Weyl group elements \( \varphi(z) \) and \( w \). Define \( d : W \to W \) by \( d(w) = \gamma_d w \gamma_d \). Following \cite{12}, we say that \( w \in W \) is a
a twisted involution if $d(w) = w^{-1}$. Denote by $I_d$ the set of all $d$-twisted involutions in $W$. Clearly, every $\varphi(z)$ is in $I_d$. The Weyl group $W$ acts on $I_d$ by

$$w_1 \ast w = w_1 w d(w_1^{-1}) \quad \text{for} \quad w_1 \in W, \quad w \in I_d,$$

and the set $\varphi(Z) \subset I_d$ is $W$-invariant. In fact, the $W$-action on $G/H$, given by $w \cdot gH = gw^{-1}H$, commutes with the left action of $G_v$ by left multiplication, and thus induces a left action of $W$ on $Z$, which we denote by $w \cdot z$ for $w \in W$ and $z \in Z$. It is also easy to see that $\varphi : Z \to W$ is $W$-equivariant, i.e. $\varphi(w \cdot z) = w \ast \varphi(z)$ for all $w \in W$ and $z \in Z$. Similarly, the involution $\tau_v : G \to G$ gives rise to an involution on $Z$ which depends only on $d$. Denote this involution by $z \to d(z)$. Then we also have $\varphi(d(z)) = d\varphi(z) = \varphi(z)^{-1}$.

As maps on $\mathfrak{h}$, we see that $w_{\sigma_d} w^{-1} = (w \ast \varphi(z)) \tau_v$. Thus we also have:

$$\delta_{z, w} = \dim(\mathfrak{h}^{(w \ast \varphi(z)) \tau_v} \cap \mathfrak{h}^{-\tau_v}).$$

**Corollary 4.2.** 1) When $w \ast \varphi(z) = 1$, symplectic leaves of $\Pi_v$ in $O_z \cap C_w$ are precisely its connected components.

2) Every open orbit $O_z$ has an open symplectic leaf $O_z \cap C_{w_0}$, where $w_0$ is the longest element in $W$;

3) If $d = 1$, symplectic leaves in an open orbit $O_z$ are precisely the connected components of intersections of Bruhat cells with $O_z$.

**Proof.** 1) When $w \ast \varphi(z) = 1$, we have $\delta_{z, w} = 0$, so every symplectic leaf in $O_z \cap C_w$ is open in $O_z \cap C_w$.

2) Since $C_{w_0}$ is dense in $X$, it intersects with every open orbit $O_z$. Since an orbit $O_z$ is open if and only if $\varphi(z) = 1$, statement 2) follows from 1) and the fact that $w_0$ commutes with $d$. The fact that $C_{w_0} \cap O_z$ is connected follows from the observation that $O_z$ is a connected open complex submanifold of $X$ and thus $O_z \cap (X \setminus C_{w_0})$ is a divisor in $O_z$.

3) follows directly from 1). \( \square \)

Consider now the group $H^{\tau_v} = H \cap G_v$. Since the centralizer of $\mathfrak{h}^{\tau_v}$ in $G_v$ also centralizes $\mathfrak{h}$, we see that $H^{\tau_v}$ is the Cartan subgroup of $G_v$ corresponding to the Cartan subalgebra $\mathfrak{h}^{\tau_v}$. Then according to [3, Proposition 7.90] the group $H^{\tau_v}$ is connected.

The Poisson structure $\Pi_v$ on $X$ is $H^{\tau_v}$-invariant. Indeed, let $R \in \wedge^2 \mathfrak{g}$ be the element given in [2, (1)] for $I_1 = \mathfrak{g}_v$ and $I_2 = I_d$. We can also represent $R$ as $R = \sum_i \xi_i \wedge y_i$, where $\{y_i\}$ is a basis of $\mathfrak{g}_v$, and $\{\xi_i\}$ is the dual basis of $I_2$ with respect to the pairing between $\mathfrak{g}_v$ and $I_2$ given by $\langle \cdot, \cdot \rangle$, the imaginary part of the Killing form on $\mathfrak{g}$. If $h \in H^{\tau_v}$, then $\{\text{Ad}_h y_i\}$ is a basis of $\mathfrak{g}_v$, and $\{\text{Ad}_h \xi_i\}$ is its dual basis. Thus $\text{Ad}_h R = R$.

Assume now that $z \in Z$ and $w \in W$ are such that $O_z \cap C_w \neq \emptyset$. Clearly, $H^{\tau_v}$ leaves $O_z \cap C_w$ invariant. Since the Poisson structure $\Pi_v$ is $H^{\tau_v}$-invariant, if $S_z$ is the symplectic
Proposition 4.3. For any $x \in X$, the set $F_x$ is a connected component of $\mathcal{O}_z \cap C_w$.

Proof. It is easy to see that if $F_{x_1} \cap F_{x_2} \neq \emptyset$, then $F_{x_1} = F_{x_2}$. The statement would follow once we prove that $F_x$ is an open subset of $\mathcal{O}_z \cap C_w$ for each $x$.

Let $x = g_z B \in \mathcal{O}_z \cap C_w$ with $g_z \in Z$ in the double coset $z$. For $y \in \mathfrak{h}^{rev}$, let $X_y$ be the vector field on $X$ generating the action of $\exp(t y) \in H_0^{rev}$ on $X$. We claim that $X_y(x) \in T_x S_x$ if and only if $y \in p(\mathfrak{h}_{(w(z))}^{rev})$, where $p : \mathfrak{h} \to \mathfrak{h}^{rev}$ is the projection with respect to the decomposition $\mathfrak{h} = \mathfrak{h}^{rev} + \mathfrak{h}^{-rev}$. Assume the claim. Then since the kernel of the map $p : \mathfrak{h}_{(w(z))}^{rev} \to \mathfrak{h}^{rev}$ has dimension $\dim(\mathfrak{h}_{(w(z))}^{rev} \cap \mathfrak{h}^{-rev}) = \delta_{z,w}$, the image of the map

$$J_x : \mathfrak{h}^{rev} \to T_x \mathcal{O}_z / T_x S_x : y \mapsto X_y(x) + T_x S_x$$

has dimension equal to $\dim(\mathfrak{h}^{rev}) - \dim(\mathfrak{h}_{(w(z))}^{rev}) + \delta_{z,w} = \delta_{z,w}$. Thus $J_x$ is onto, and the $H_0^{rev}$-orbit in $\mathcal{O}_z \cap C_w$ through $x$ is transversal to the symplectic leaf $S_x$. It follows that $F_x$ is open in $\mathcal{O}_z \cap C_w$.

It remains to prove the claim. Denote also by $p : \mathfrak{g} \to \mathfrak{g}_v$ the projection with respect to the decomposition $\mathfrak{g} = \mathfrak{g}_v + \mathfrak{l}_\delta$, and let $q$ be the projection $q : \mathfrak{g} \to \mathfrak{g}_v / \mathfrak{g}_v \cap \text{Ad}_{g_z} \mathfrak{h} \cong T_x \mathcal{O}_z$. Then by [7, Corollary 7.3], we have $T_x S_x = (q \circ p)(\mathcal{V}(l_x))$, where, as in the proof of Theorem 4.1, $\mathcal{V}(l_x) = \text{Ad}_{g_z}(\mathfrak{h}_{\mathcal{O}_z} + \mathfrak{n})$. Let $y \in \mathfrak{h}^{rev}$. If $X_y(x) \in T_x S_x$, then there exist $y_1 \in \mathfrak{l}_\delta$ and $y_2 \in \mathfrak{g}_v$ with $y_1 + y_2 \in \mathcal{V}(l_x)$ such that $y - y_2 \in \mathfrak{g}_v \cap \text{Ad}_{g_z} \mathfrak{h} \subset \mathcal{V}(l_x)$. Thus $y + y_1 = y - y_2 + y_1 + y_2 \in \mathcal{V}(l_x)$. Write $y_1 = \xi_1 + u_1$, where $\xi_1 \in \mathfrak{h}^{-rev}$ and $u_1 \in \mathfrak{n}$. Then there exist $\xi_2 \in \mathfrak{h}_{\mathcal{O}_z}$ and $u_2 \in \mathfrak{n}$ such that $y + \xi_1 + u_1 = \text{Ad}_{g_z}(\xi_2 + u_2)$. Write $g_z = n \cdot w \cdot b$, where $n \in N, b \in B$, and $w$ is a representative of $w$ in $K$. Write $\text{Ad}_{n^{-1}}(y + \xi_1 + u_1) = y + \xi_1 + u_1'$ and $\text{Ad}_b(\xi_2 + u_2) = \xi_2 + u_2'$, where $u_1', u_2' \in \mathfrak{n}$. Then we have

$$y + \xi_1 + u_1' = \text{Ad}_{\tilde{w}}(\xi_2 + u_2').$$

Since $y + \xi_1, \text{Ad}_{\tilde{w}} \xi_2 \in \mathfrak{h}$ and $u_1', \text{Ad}_{\tilde{w}} u_2' \in \mathfrak{n} + \mathfrak{n}_-$, we have $y + \xi_1 = \text{Ad}_{\tilde{w}} \xi_2 \in \mathfrak{h}_{(w(z))}^{rev}$. Thus $y \in p(\mathfrak{h}_{(w(z))}^{rev})$. Conversely, if $y \in \mathfrak{h}^{rev}$ is such that $y + \xi_1 \in \mathfrak{h}_{(w(z))}^{rev} = \text{Ad}_{\tilde{w}} \mathfrak{h}_{\mathcal{O}_z}$ for some $\xi_1 \in \mathfrak{h}^{-rev}$, write $y + \xi_1 = \text{Ad}_w \xi_2$ for $\xi_2 \in \mathfrak{h}_{\mathcal{O}_z}$. Let $\text{Ad}_{b^{-1}} \xi_2 = \xi_2 + u_2$ for some $u_2 \in \mathfrak{n}$. We then have

$$\text{Ad}_n(y + \xi_1) = \text{Ad}_{n \cdot w \cdot b}(\xi_2 + u_2) \in \mathcal{V}(l_x).$$

On the other hand, let $\text{Ad}_n(y + \xi_1) = y + \xi_1 + u_1$ with $u_1 \in \mathfrak{n}$. We see that $y = p(\text{Ad}_n(y + \xi_1))$ so $X_y(x) \in T_x S_x$. \hfill \Box
5. Invariant Poisson cohomology of open orbits.

Let \( O_z \) be a \( G_v \)-orbit in \( X \) equipped with the Poisson structure \( \Pi_v \). Then \((O_z, \Pi_v)\) is a Poisson homogeneous space for the Poisson Lie group \( G_v \). The \( G_v \)-invariant Poisson cohomology of \((O_z, \Pi_v)\), denoted by \( H^\bullet_{\Pi_v, G_v}(O_z) \), is defined as the cohomology of the cochain complex \((\chi^\bullet_{G_v}(O_z), \partial_\Pi_v)\), where \( \chi^\bullet_{G_v}(O_z) \) is the space of all \( G_v \)-invariant complex multi-vector fields on \( O_z \), \( d_\Pi_v(V) = [\Pi_v, V] \), and \([\cdot, \cdot]\) is the Schouten bracket of the multi-vector fields.

**Proposition 5.1.** When \( O_z \) is an open \( G_v \)-orbit in \( X \), the \( G_v \)-invariant Poisson cohomology \( H^\bullet_{\Pi_v, G_v}(O_z) \) is isomorphic to the de Rham cohomology of \( X \).

**Proof.** As in the proof of Theorem 4.1 let \( x = g_z B \in X \) be an arbitrary point in \( O_z \), where \( g_z \in Z \) is in the coset \( z \), and let \( \mathcal{V}(l_z) = \text{Ad}_{g_z}(h^{\mathfrak{a}_z} + \mathfrak{n}) \). Since \( O_z \) is open, the stabilizer subalgebra of \( \mathfrak{g}_v \) at \( x \) is \( \mathfrak{g}_v \cap \mathcal{V}(l_z) = \text{Ad}_{g_z}(h^{\mathfrak{a}_z}) \). By [9, Theorem 7.5], the \( G_v \)-invariant Poisson cohomology \( H^\bullet_{\Pi_v, G_v}(O_z) \) is isomorphic to the relative Lie algebra cohomology of the Lie algebra \( \mathcal{V}(l_z) \otimes \mathbb{C} \) relative to the subalgebra \( (\text{Ad}_{g_z}(h^{\mathfrak{a}_z})) \otimes \mathbb{C} \). Thus the \( G_v \)-invariant Poisson cohomology is isomorphic to the \( \mathfrak{h} \)-invariant part of the Lie algebra cohomology of the direct sum Lie algebra \( \mathfrak{n} \oplus \mathfrak{n} \) with coefficients in \( \mathbb{C} \), which by Kostant’s theorem [7], is isomorphic to the de Rham cohomology of \( X \). \( \square \)

6. Remarks.

We have constructed a Poisson structure \( \Pi_v \) on \( X \) for each Vogan diagram \( v \) for \( \mathfrak{g} \) (which is not necessarily normalized). In particular, each Bruhat cell \( C_w \) in \( X \) carries the Poisson structure \( \Pi_v \). It would be interesting to study connections between the Poisson structures for different \( v \). Especially interesting are the properties of \( \Pi_v \) that depend only on the inner class \( d \) of the real form \( \mathfrak{g}_v \). We also remark that the Poisson structure \( \Pi_v \) is defined on the whole variety \( L \) of Lagrangian subalgebras of \( \mathfrak{g} \). We have only been looking at the restriction of \( \Pi_v \) to a particular \( G \)-orbit, namely the \( G \)-orbit through the Lagrangian subalgebra \( \mathfrak{t} + \mathfrak{n} \). There are many other interesting \( G \)-orbits in \( L \), such as the \( G \)-orbit through a given real form of \( \mathfrak{g} \). It would be interesting to study the properties of the Poisson structure \( \Pi_v \) on these orbits as well as on their closures with respect to both the classical topology and the Zariski topology.

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