Online Frank-Wolfe with Unknown Delays

Yuanyu Wan
National Key Laboratory for Novel Software Technology, Nanjing University, Nanjing 210023, China

Wei-Wei Tu
4Paradigm Inc., Beijing 100000, China

Lijun Zhang
National Key Laboratory for Novel Software Technology, Nanjing University, Nanjing 210023, China

Abstract

The online Frank-Wolfe (OFW) method has gained much popularity for online convex optimization due to its projection-free property. Previous studies showed that for convex losses, OFW attains $O(T^{3/4})$ regret over general sets and $O(T^{2/3})$ regret over strongly convex sets, and if losses are strongly convex, these bounds can be improved to $O(T^{2/3})$ and $O(\sqrt{T})$, respectively. However, they assumed that each gradient queried by OFW is revealed immediately, which may not hold in practice. In this paper, we consider a more practical setting where gradients arrive with arbitrary and unknown delays, and propose delayed OFW which generalizes OFW to this setting. The main idea is to perform an update similar to OFW after receiving any gradient, and play the latest decision for each round. We first show that for convex losses, delayed OFW achieves $O(T^{3/4} + dT^{1/4})$ regret over general sets and $O(T^{2/3} + dT^{1/3})$ regret over strongly convex sets, where $d$ is the maximum delay. Furthermore, we prove that for strongly convex losses, delayed OFW attains $O(T^{2/3} + d\log T)$ regret over general sets and $O(\sqrt{T} + d\log T)$ regret over strongly convex sets. Compared with regret bounds in the non-delayed setting, our results imply that the proposed method is robust to a relatively large amount of delay.

1. Introduction

Online convex optimization (OCO) has become a leading paradigm for online learning due to its capability to model various problems from diverse domains such as online routing, online matrix completion, and online advertisement (Hazan, 2016). In general, it is formulated as a structured repeated game between a player and an adversary. In each round $t$, the player first chooses a decision $x_t$ from a convex decision set $\mathcal{K} \subseteq \mathbb{R}^n$, where $n$ is the dimensionality. Then, the adversary selects a convex function $f_t(x) : \mathcal{K} \mapsto \mathbb{R}$, and the player suffers a loss $f_t(x_t)$. The player aims to choose decisions such that the regret

$$R(T) = \sum_{t=1}^{T} f_t(x_t) - \min_{x \in \mathcal{K}} \sum_{t=1}^{T} f_t(x)$$

is sublinear in the number of total rounds $T$. Online gradient descent (OGD) is a standard method for OCO, which enjoys an $O(\sqrt{T})$ regret bound for convex losses (Zinkevich, 2003) and an $O(\log T)$ regret bound for strongly convex losses (Hazan et al., 2007). However, it needs to compute a projection onto the decision set to ensure the feasibility of each decision, which is computationally expensive for complicated decision sets (Hazan and Kale, 2012).
To tackle this computational issue, Hazan and Kale (2012) proposed the online Frank-Wolfe (OFW) method, which has become one of the most commonly used algorithms for OCO over complicated decision sets. The main advantage of OFW is its projection-free property: instead of performing the projection operation, it utilizes a linear optimization step to select a feasible decision, which could be much more efficient. For example, in the problem of online matrix completion with a bounded trace norm, the linear optimization step is at least an order of magnitude faster than the projection operation (Hazan and Kale, 2012). Attracted by the projection-free property, much recent research effort has been devoted to the regret analysis of OFW. Specifically, it has been showed that for convex losses, OFW achieves an $O(T^{3/4})$ regret bound over general sets (Hazan and Kale, 2012; Hazan, 2016) and an $O(T^{2/3})$ regret bound over strongly convex sets (Wan and Zhang, 2021). If losses are strongly convex, these regret bounds over general and strongly convex sets can be improved to $O(T^{2/3})$ and $O(\sqrt{T})$, respectively (Wan and Zhang, 2021; Garber and Kretzu, 2021).

However, the existing regret analysis of OFW assumed that the gradient $\nabla f_t(x_t)$ is revealed immediately after making the decision $x_t$. Although this assumption is commonly used to establish theoretical results for the standard OCO, it is not necessarily satisfied in many practical applications, where the feedback could be delayed. For example, in the problem of online routing over communication networks (Awerbuch and Kleinberg, 2008), the decision is a specific routing path for a packet, and the feedback for the decision is the latency of the packet, which is revealed only after the packet arrives its destination. In the problem of online advertisement (McMahan et al., 2013; He et al., 2014), the decision is to serve an ad to a user, and the feedback for the decision is whether the ad is clicked or not, which cannot be determined unless the user has seen the ad for a certain period.

To address the above limitation, this paper proposes delayed OFW, which generalizes OFW to a more practical setting, where the gradient $\nabla f_t(x_t)$ arrives at the end of round $t + d_t - 1$, and $d_t \geq 1$ denotes an arbitrary and unknown delay. Compared with OFW that requires the gradient $\nabla f_t(x_t)$ for updating the decision $x_t$, the main change of our delayed OFW is to update the decision similar to OFW after receiving any gradient, and play the latest decision for each round. Our theoretical contributions are summarized as follows.

- First, we show that for convex losses, our delayed OFW achieves an $O(T^{3/4} + dT^{1/4})$ regret bound over general sets, where $d$ is the maximum delay, which matches the $O(T^{3/4})$ regret bound in the non-delayed setting as long as $d$ does not exceed $O(\sqrt{T})$.
- Second, if decision sets are strongly convex, we show that our delayed OFW achieves an improved regret bound of $O(T^{2/3} + dT^{1/3})$ for convex losses, which matches the $O(T^{2/3})$ regret bound in the non-delayed setting as long as $d$ does not exceed $O(T^{1/3})$.
- Third, we prove that for strongly convex losses, our delayed OFW attains an $O(T^{2/3} + d\log T)$ regret bound over general sets, which matches the $O(T^{2/3})$ regret bound in the non-delayed setting as long as $d$ does not exceed $O(T^{2/3}/\log T)$.
- Fourth, if decision sets are strongly convex, we prove that our delayed OFW attains an improved regret bound of $O(\sqrt{T} + d\log T)$ for strongly convex losses, which matches the $O(\sqrt{T})$ regret bound in the non-delayed setting as long as $d$ does not exceed $O(\sqrt{T}/\log T)$. 

2
To the best of our knowledge, this is the first effort that investigates the effect of arbitrary and unknown delays on the performance of projection-free OCO algorithms. Moreover, we notice that there exist studies that generalized OGD to the delayed setting, but they only obtained regret bounds which are worse than those in the non-delayed setting (Quanrud and Khashabi, 2015; Wan et al., 2022). By contrast, our results imply that delayed OFW is robust to a relatively large amount of delay.

2. Related Work

In this section, we briefly review related work on projection-free OCO algorithms, and OCO under delayed feedback.

2.1 Projection-free OCO Algorithms

The OFW method (Hazan and Kale, 2012; Hazan, 2016) is the first projection-free OCO algorithm, which is an online extension of the classical Frank-Wolfe method (Frank and Wolfe, 1956; Jaggi, 2013). For convex losses, OFW first chooses an arbitrary $x_1 \in \mathcal{K}$, and then iteratively updates its decision by the following linear optimization step

$$v_t \in \arg \min_{x \in \mathcal{K}} \langle \nabla F_t(x_t), x \rangle$$

$$x_{t+1} = x_t + \sigma_t(v_t - x_t)$$

where $F_t(x)$ is a surrogate loss function defined as

$$F_t(x) = \eta \sum_{i=1}^{t} \langle \nabla f_i(x_i), x \rangle + \|x - x_1\|^2_2$$

and $\eta, \sigma_t$ are two parameters. With appropriate parameters, it can attain an $O(T^{3/4})$ regret bound for convex losses.

If losses are convex and smooth, Hazan and Minasyan (2020) proposed a randomized projection-free method, which is based on a classical OCO method called follow the perturbed leader (Kalai and Vempala, 2005), and achieved an expected regret bound of $O(T^{2/3})$. Recently, Wan and Zhang (2021) proved that OFW can achieve an $O(T^{2/3})$ regret bound for strongly convex losses. Specifically, to utilize the strong convexity of losses, Wan and Zhang (2021) redefined $F_t(x)$ in (2) to

$$F_t(x) = \sum_{i=1}^{t} \left( \langle \nabla f_i(x_i), x \rangle + \frac{\beta}{2} \|x - x_i\|^2_2 \right)$$

where $\beta$ is the modulus of the strong convexity. The same regret bound was concurrently established by Garber and Kretzu (2021).

Moreover, various projection-free algorithms and improved regret bounds have been provided for OCO with special decision sets. When decision sets are polyhedral, Garber and Hazan (2016) proposed a variant of OFW, which achieves an $O(\sqrt{T})$ regret bound for convex losses and an $O(\log T)$ regret bound for strongly convex losses. Later, Levy and Krause (2019) proposed a projection-free method over smooth sets, which replaces the projection
operation in OGD with a fast approximate one, and achieved the same regret bounds as in Garber and Hazan (2016). If decision sets are strongly convex, Wan and Zhang (2021) showed that OFW can attain an $O(T^{2/3})$ regret bound for convex losses by setting the parameter $\sigma_t$ as

$$\sigma_t = \arg\min_{\sigma \in [0,1]} \langle \sigma (v_t - x_t), \nabla F_t(x_t) \rangle + \sigma^2 \|v_t - x_t\|^2_2$$

and an $O(\sqrt{T})$ regret bound for strongly convex losses by setting the parameter $\sigma_t$ as

$$\sigma_t = \arg\min_{\sigma \in [0,1]} \langle \sigma (v_t - x_t), \nabla F_t(x_t) \rangle + \frac{\beta t \sigma^2}{2} \|v_t - x_t\|^2_2.$$  

\subsection{OCO under Delayed Feedback}

The first work about OCO under delayed feedback was due to Weinberger and Ordentlich (2002), which focused on a special case with a fixed and known delay $d'$, i.e., the feedback for each decision $x_t$ is received at the end of round $t + d' - 1$. They proposed a technique that can convert any traditional OCO algorithm for the non-delayed setting into this delayed setting. Specifically, their technique is to run $d$ instances of a traditional OCO algorithm, where each instance is used every $d'$ rounds, which is feasible because the feedback for any decision $x_t$ must be available for selecting the decision $x_{t+d'}$. If the traditional OCO algorithm enjoys a regret bound of $R(T)$ for the non-delayed setting, they proved that this technique achieves a regret bound of $dR(T/d')$. As a result, by combining with OGD, the regret bound of this technique is $O(\sqrt{d'T})$ for convex losses and $O(d' \log T)$ for strongly convex losses. Later, Langford et al. (2009) studied the same delayed setting, and showed that simply performing each gradient descent step in the original OGD with a delayed gradient can also achieve the $O(\sqrt{d'T})$ regret bound for convex losses and the $O(d' \log T)$ regret bound for strongly convex losses.

Joulani et al. (2013) further extended the technique in Weinberger and Ordentlich (2002) to handle a different delayed setting, where each feedback is delayed by arbitrary rounds, but the time-stamp is knowable when it is received. For this setting, their technique is able to attain a regret bound of $dR(T/d)$ by combining with a traditional OCO algorithm with $R(T)$ regret, where $d$ is the maximum delay. Note that the existing projection-free OCO algorithms can also be combined with the techniques in Weinberger and Ordentlich (2002) and Joulani et al. (2013). However, in practical problems, each feedback may be delayed by arbitrary and unknown rounds, and its time-stamp could also be unknown, which limits the application of these techniques.

To deal with arbitrary and unknown delays, Quanrud and Khashabi (2015) proposed a delayed variant of OGD for convex losses, and established a regret bound of $O(\sqrt{S_d})$, where $S_d$ is the sum of delays over $T$ rounds. Similar to the method in Langford et al. (2009), the main idea of Quanrud and Khashabi (2015) is to perform a gradient descent step with the sum of delayed gradients received at each round. Wan et al. (2022) further extended the delayed variant of OGD for strongly convex losses by using an appropriate step size, and established a regret bound of $O(d \log T)$. We also note that Joulani et al. (2016) proposed adaptive online algorithms that enjoy data-dependent regret bounds with arbitrary and unknown delays, but they assumed that the feedback for the decision $x_t$ is the entire loss function $f_t(x)$, which could be unavailable in practical applications.
2.3 Discussions
Despite these great progresses about the above two families of algorithms, it remains unclear how arbitrary and unknown delays affect the performance of projection-free OCO algorithms. This paper takes the first step towards understanding this effect by studying OFW with arbitrary and unknown delays, and shows that our delayed OFW is robust to a relatively large amount of delay. Moreover, if the time-stamp of the delayed feedback is knowable, one may combine the technique in Joulani et al. (2013) with OFW. For convex losses, this way attains an $O(d^{1/4}T^{3/4})$ regret bound over general sets and an $O(d^{1/3}T^{2/3})$ regret bound over strongly convex sets. If losses are strongly convex, it attains an $O(d^{1/3}T^{2/3})$ regret bound over general sets and an $O(\sqrt{dT})$ regret bound over strongly convex sets. However, these regret bounds are worse than those achieved by our delayed OFW.

3. Main Results
In this section, we first introduce necessary preliminaries including the problem setting, definitions, and assumptions. Then, we present our delayed OFW and the corresponding theoretical guarantees for convex and strongly convex losses, respectively.

3.1 Preliminaries
We consider the problem of OCO with unknown delays (Quanrud and Khashabi, 2015; Wan et al., 2022). Similar to the standard OCO, in each round $t = 1, \ldots, T$, the player first chooses a decision $x_t$ from the decision set $K$, and then the adversary selects a convex function $f_t(x)$. However, different from the standard OCO, the gradient $g_t = \nabla f_t(x_t)$ is revealed at the end of round $t + d_t - 1$, where $d_t \geq 1$ denotes an arbitrary and unknown delay. As a result, the player actually receives gradients $\{g_k | k \in F_t\}$ at the end of round $t$, where

$$F_t = \{k | k + d_k - 1 = t\}.$$

Then, we recall the standard definition for strongly convex functions (Boyd and Vandenberghe, 2004).

**Definition 1** A function $f(x) : K \to \mathbb{R}$ is called $\beta$-strongly convex over $K$ if it holds that

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\beta}{2} \| y - x \|_2^2$$

for all $x, y \in K$.

Next, we introduce the definition for strongly convex sets (Garber and Hazan, 2015).

**Definition 2** A convex set $K \subseteq \mathbb{R}^n$ is called $\beta_K$-strongly convex if it holds that

$$\gamma x + (1 - \gamma) y + \gamma (1 - \gamma) \frac{\beta_K}{2} \| x - y \|_2^2 z \in K$$

for any $x, y \in K$, $\gamma \in [0, 1]$, and $z \in \mathbb{R}^n$ with $\| z \|_2 = 1$. 

The strong convexity of decision sets has been widely utilized in previous studies (Levitin and Polyak, 1966; Dunn, 1979; Garber and Hazan, 2015; Rector-Brooks et al., 2019; Kerdreux et al., 2021). Common examples of strongly convex sets include various balls induced by $\ell_p$ norms, Schatten norms, and group norms (Garber and Hazan, 2015).

Finally, we introduce two assumptions, which are commonly used in studies about OCO (Shalev-Shwartz (2011); Hazan (2016)).

**Assumption 1** The gradients of all loss functions are bounded by $G$, i.e., it holds that

$$\|\nabla f_t(x)\|_2 \leq G$$

for any $x \in K$ and $t \in [T]$.

**Assumption 2** The diameter of the decision set $K$ is bounded by $D$, i.e., it holds that

$$\|x - y\|_2 \leq D$$

for any $x, y \in K$.

### 3.2 Delayed OFW for Convex Losses

As presented in (1) and (2), the original OFW for convex losses requires the gradient $g_t$ before making the decision $x_{t+1}$. However, in the problem of OCO with unknown delays, this requirement is not necessarily satisfied, because the player actually receives gradients $\{g_k | k \in F_t\}$ at each round $t$, which may not contain $g_t$. To address this limitation, we propose a natural generalization of OFW to this delayed problem, which is described as follows.

The main idea is to update the decision similar to OFW for each received gradient, and play the latest decision for each round. In this way, there exist some intermediate decisions that are not really played. To facilitate presentations, we introduce an additional notation $y_\tau$ to denote the $\tau$-th intermediate decision. Moreover, we denote the sum of $\tau$ received gradients by $\bar{g}_\tau$. Initially, we choose an arbitrary vector $y_1 \in K$ and set $\tau = 1$, $\bar{g}_0 = 0$. At each round $t = 1, \ldots, T$, we play the latest decision $x_t = y_\tau$ and query the gradient $g_t = \nabla f_t(x_t)$. Then, we receive delayed gradients that are queried in a set of rounds $F_t$, and perform the following steps for each received gradient.

Specifically, for any $k \in F_t$, inspired by (2) of the original OFW, we first compute $g_\tau = \bar{g}_{\tau-1} + g_k$ and define

$$F_\tau(y) = \eta \langle \bar{g}_\tau, y \rangle + \|y - y_1\|_2^2.$$

Then, similar to (1) of the original OFW and (4) utilized by Wan and Zhang (2021), we perform the following update

$$v_\tau \in \arg \min_{y \in K} \langle \nabla F_\tau(y_\tau), y \rangle$$

$$y_{\tau+1} = y_\tau + \sigma_\tau (v_\tau - y_\tau)$$

where the parameter $\sigma_\tau$ is set by a line search rule

$$\sigma_\tau = \arg \min_{\sigma \in [0,1]} \langle \sigma (v_\tau - y_\tau), \nabla F_\tau(y_\tau) \rangle + \sigma^2 \|v_\tau - y_\tau\|_2^2. \quad (6)$$
Algorithm 1 Delayed OFW for Convex Losses

1: **Input:** $\eta$
2: **Initialization:** choose an arbitrary vector $y_1 \in K$ and set $\tau = 1, \bar{g}_0 = 0$
3: for $t = 1, 2, \ldots, T$ do
4: Play $x_t = y_{\tau}$ and query $g_t = \nabla f_t(x_t)$
5: Receive a set of delayed gradients $\{g_k | k \in F_t\}$
6: for $k \in F_t$ do
7: $\bar{g}_{\tau} = \bar{g}_{\tau-1} + g_k$
8: Define $F_\tau(y) = \eta(\bar{g}_\tau, y) + \|y - y_1\|_2^2$
9: $v_\tau \in \arg\min_{y \in K} \langle \nabla F_\tau(y_\tau), y \rangle$
10: $y_{\tau+1} = y_\tau + \sigma_\tau (v_\tau - y_\tau)$ with $\sigma_\tau$ in (6)
11: $\tau = \tau + 1$
12: end for
13: end for

Finally, we update $\tau = \tau + 1$ so that $\tau$ still indexes the latest intermediate decision.

The detailed procedures are summarized in Algorithm 1, which is named as delayed OFW for convex losses. Define $d = \max\{d_1, \ldots, d_T\}$.

We first establish the following theorem with respect to the regret of Algorithm 1 over general sets.

**Theorem 1** Let $x^*$ be an arbitrary vector in the set $K$. Under Assumptions 1 and 2, Algorithm 1 with $\eta = D \sqrt{2G(T+2)^{1/4}}$ has

$$\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x^*) = O(T^{3/4} + dT^{1/4}).$$

Theorem 1 shows that our Algorithm 1 can achieve an $O(T^{3/4} + dT^{1/4})$ regret bound for convex losses and general sets with unknown delays. This bound matches the $O(T^{3/4})$ regret bound of OFW for convex losses and general sets in the non-delayed setting (Hazan and Kale, 2012; Hazan, 2016) as long as $d$ does not exceed $O(\sqrt{T})$.

Then, we further consider strongly convex sets, and establish the following theorem.

**Theorem 2** Let $x^*$ be an arbitrary vector in the set $K$. Suppose Assumptions 1 and 2 hold, and the decision set is $\beta_K$-strongly convex, Algorithm 1 with $\eta = D \frac{2}{2G(T+2)^{2/3}}$ has

$$\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x^*) = O(T^{2/3} + dT^{1/3}).$$

Theorem 2 shows that our Algorithm 1 can achieve an $O(T^{2/3} + dT^{1/3})$ regret bound for convex losses and strongly convex sets with unknown delays. First, this bound improve the above $O(T^{3/4} + dT^{1/4})$ regret bound for convex losses and general sets, when the term about $d$ is not dominant. Second, this bound matches the $O(T^{2/3})$ regret bound of OFW for convex losses and strongly convex sets in the non-delayed setting (Wan and Zhang, 2021) as long as $d$ does not exceed $O(T^{1/3})$. 

7
Algorithm 2 Delayed OFW for Strongly Convex Losses

1: **Input:** $\beta$
2: **Initialization:** choose an arbitrary vector $y_1 \in \mathcal{K}$ and set $\tau = 1$, $\bar{g}_0 = 0$
3: for $t = 1, 2, \ldots, T$ do
4: Play $x_t = y_\tau$ and query $g_t = \nabla f_t(x_t)$
5: Receive a set of delayed gradients $\{g_k | k \in F_t\}$
6: for $k \in F_t$ do
7: $\bar{g}_\tau = \bar{g}_{\tau - 1} + g_k$
8: Define $F_\tau(y) = \langle \bar{g}_\tau, y \rangle + \sum_{i=1}^{\tau} \frac{\beta}{2} \|y - y_i\|^2$
9: $v_\tau \in \argmin_y \langle \nabla F_\tau(y), y \rangle$
10: $y_{\tau + 1} = y_\tau + \sigma_\tau (v_\tau - y_\tau)$ with $\sigma_\tau$ in (7)
11: $\tau = \tau + 1$
12: end for
13: end for

3.3 Delayed OFW for Strongly Convex Losses

We proceed to handle $\beta$-strongly convex losses by slightly modifying Algorithm 1. Recall that in the standard OCO without delays, the critical idea of utilizing the strong convexity of losses is to replace the surrogate loss function in (1) by that in (3) (Wan and Zhang, 2021). The main difference is that the regularization term in (3) is about all historical decisions, instead of only the initial decision.

Inspired by (3), we first redefine $F_\tau(y)$ in Algorithm 1 to

$$F_\tau(y) = \langle \bar{g}_\tau, y \rangle + \sum_{i=1}^{\tau} \frac{\beta}{2} \|y - y_i\|^2.$$ 

Second, since $F_\tau(y)$ is modified, similar to (5) used by Wan and Zhang (2021), we adjust the line search rule to

$$\sigma_\tau = \argmin_{\sigma \in [0, 1]} \langle \sigma (v_\tau - y_\tau), \nabla F_\tau(y_\tau) \rangle + \frac{\beta \sigma^2}{2} \|v_\tau - y_\tau\|^2.$$ (7)

The detailed procedures are summarized in Algorithm 2, which is named as delayed OFW for strongly convex losses.

Then, we establish the following theorem about the regret of Algorithm 2 over general sets.

**Theorem 3** Let $x^*$ be an arbitrary vector in the set $\mathcal{K}$. Suppose Assumptions 1 and 2 hold, and all losses are $\beta$-strongly convex, Algorithm 2 has

$$\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x^*) = O(T^{2/3} + d \log T).$$

Theorem 3 shows that our Algorithm 2 can achieve an $O(T^{2/3} + d \log T)$ regret bound for strongly convex losses and general sets with unknown delays. First, this bound is better
than the regret bound in Theorem 1 that is achieved by only using the convexity condition. Second, the $O(d \log T)$ term in this bound is better than the $O(dT^{1/3})$ term in the regret bound presented by Theorem 2 that has benefited from the strong convexity of decision sets. Third, this bound matches the $O(T^{2/3})$ regret bound of OFW for strongly convex losses and general sets in the non-delayed setting (Garber and Kretzu, 2021; Wan and Zhang, 2021) as long as $d$ does not exceed $O(T^{2/3}/\log T)$.

Furthermore, we establish the following theorem with respect to the regret of Algorithm 2 over strongly convex sets.

**Theorem 4** Let $x^*$ be an arbitrary vector in the set $K$. Suppose Assumptions 1 and 2 hold, all losses are $\beta$-strongly convex, and the decision set is $\beta_K$-strongly convex, Algorithm 2 has

$$
\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x^*) = O(\sqrt{T} + d \log T).
$$

Theorem 4 shows that our Algorithm 2 can achieve an $O(\sqrt{T} + d \log T)$ regret bound for strongly convex losses and strongly convex sets with unknown delays. First, this bound is better than the above $O(T^{2/3} + d \log T)$ regret for strongly convex losses and general sets. Second, this bound matches the $O(\sqrt{T})$ regret bound of OFW for strongly convex losses and strongly convex sets in the non-delayed setting (Wan and Zhang, 2021) as long as $d$ does not exceed $O(\sqrt{T} / \log T)$.

4. Theoretical Analysis

In this section, we first introduce necessary preliminaries for our analysis, and then prove Theorems 1 and 3. The omitted proofs can be found in the appendix.

4.1 Preliminaries

Because of the effect of delays, there may exist some gradients that arrive after round $T$. Although our Algorithms 1 and 2 do not need to use these gradients, they are useful for the analysis. As a result, in our analysis, we further set $x_t = y_\tau$ and perform steps 5 to 12 of Algorithms 1 and 2 for any $t = T + 1, \ldots, T + d - 1$. In this way, all gradients $g_1, g_2, \ldots, g_T$ queried by our algorithms are utilized, which produces decisions $y_1, y_2, \ldots, y_{T+1}$.

Then, let $\tau_t = 1 + \sum_{i=1}^{t-1} |F_i|$ for any $t \in [T + d]$. It is not hard to verify that our Algorithms 1 and 2 ensure that

$$
x_t = y_{\tau_t} \tag{8}
$$

for any $t \in [T + d - 1]$. Next, we define

$$
\mathcal{I}_t = \begin{cases} 
\emptyset, & \text{if } |F_t| = 0, \\
\{\tau_t, \tau_t + 1, \ldots, \tau_{t+1} - 1\}, & \text{otherwise}. 
\end{cases} \tag{9}
$$

Let $s = \min \{t | t \in [T + d - 1], |F_t| > 0\}$. It is not hard to verify that

$$
\cup_{t=s}^{T+d-1} \mathcal{I}_t = [T] \text{ and } \mathcal{I}_i \cap \mathcal{I}_j = \emptyset, \forall i \neq j \tag{10}
$$

9
and

$$\bigcup_{i=1}^{T+d-1} F_t = [T] \text{ and } F_i \cap F_j = \emptyset, \forall i \neq j. \quad (11)$$

To facilitate the analysis, we further denote the time-stamp of the \( \tau \)-th gradient used in the update of our Algorithms 1 and 2 by \( c_\tau \). To help understanding, one can imagine that our Algorithms 1 and 2 also implement \( c_\tau = k \) in their step 7. If \( |F_t| \neq 0 \), we have

$$\{c_{\tau_1}, \ldots, c_{\tau_{t+1}}\} = F_t. \quad (12)$$

Moreover, by using this notation, \( F_\tau(y) \) defined in Algorithms 1 and 2 is respectively equivalent to

$$F_\tau(y) = \eta \sum_{i=1}^{\tau} \langle g_{c_i}, y \rangle + \|y - y_1\|_2^2 \quad (13)$$

and

$$F_\tau(y) = \sum_{i=1}^{\tau} \langle g_{c_i}, y \rangle + \sum_{i=1}^{\tau} \frac{\beta}{2} \|y - y_i\|_2^2. \quad (14)$$

### 4.2 Proof of Theorem 1

Let \( t' = t + d - 1 \) for any \( t \in [T] \). According to the convexity of \( f_t(x) \), we have

$$f_t(x_t) - f_t(x^*) \leq \langle g_t, x_t - x^* \rangle = \langle g_t, x_t - x_t' \rangle + \langle g_t, x_t' - x^* \rangle \leq \langle g_t, x_t' - x^* \rangle + \|g_t\|_2 \|x_t - x_t'\|_2 \leq \langle g_t, x_t' - x^* \rangle + G \|x_t - x_t'\|_2 = \langle g_t, y_{\tau_t} - x^* \rangle + G \|y_{\tau_t} - y_{\tau_t'}\|_2$$

where the last inequality is due to Assumption 1, and the last equality is due to (8).

By summing over \( t = 1, \ldots, T \), we have

$$\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x^*) \leq \sum_{t=1}^{T} \langle g_t, x_t - x^* \rangle \leq \sum_{t=1}^{T} \langle g_t, y_{\tau_t} - x^* \rangle + \sum_{t=1}^{T} G \|y_{\tau_t} - y_{\tau_t'}\|_2. \quad (15)$$

Then, we bound the first term in the right side of (15) as follows

$$\sum_{t=1}^{T} \langle g_t, y_{\tau_t} - x^* \rangle = \sum_{t=1}^{T+d-1} \sum_{i \in F_t} \langle g_t, y_{\tau_t+1} - x^* \rangle = \sum_{t=8}^{T+d-1} \sum_{i \in F_t} \langle g_{c_i}, y_{\tau_t} - x^* \rangle$$

$$= \sum_{t=8}^{T+d-1} \sum_{i \in F_t} \langle g_{c_i}, y_{\tau_t} - x^* \rangle = \sum_{t=8}^{T+d-1} \sum_{i \in F_t} \langle g_{c_i}, y_{\tau_t} - x^* \rangle$$

$$= \sum_{t=8}^{T+d-1} \sum_{i \in F_t} \langle g_{c_i}, y_{\tau_t} - x^* \rangle + \sum_{t=8}^{T+d-1} \sum_{i \in F_t} \langle g_{c_i}, y_{\tau_t} - y_i \rangle$$
where the first equality is due to (11), the second equality is due to \( i + d_i - 1 = t \) for any \( i \in F_t \), the third equality is due to (12), and the last equality is due to (9) and (10).

Then, due to

\[
(g_{c_t}, y_{\tau_t} - y_t) \leq \|g_{c_t}\|_2 \|y_{\tau_t} - y_t\|_2 \leq G \|y_{\tau_t} - y_t\|_2
\]

we further have

\[
\sum_{t=1}^{T} (g_{c_t}, y_{\tau_t'} - x^*) \leq \sum_{t=1}^{T} (g_{c_t}, y_t - x^*) + \sum_{t=s}^{T-d-1} \sum_{i=\tau_t}^{\tau_{t+1}-1} G \|y_{\tau_t} - y_i\|_2.
\]

(16)

By combining (15) and (16), we have

\[
\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x^*) \leq A + B + C.
\]

(17)

Next, we proceed to bound terms \( A, B, \) and \( C \) defined above. Specifically, we first establish the following bound for the sum of terms \( A \) and \( C \) by carefully analyzing the distance \( \|y_{\tau_t} - y_{\tau_t'}\|_2 \) in the term \( A \) and the distance \( \|y_{\tau_t} - y_i\|_2 \) in the term \( C \).

**Lemma 1** Let \( y^*_t = \text{argmin}_{y \in \mathcal{K}} F_{t-1}(y) \) for any \( t \in [T+1] \), where \( F_t(y) \) is defined in (13). Suppose Assumption 1 and 2 hold, and there exist some constants \( \gamma > 0 \) and \( 0 < \alpha \leq 1 \) such that

\[
F_{t-1}(y_t) - F_{t-1}(y^*_t) \leq \gamma (t+2)^{-\alpha}
\]

for any \( t \in [T+1] \), Algorithm 1 ensures

\[
\sum_{t=1}^{T} G \|y_{\tau_t} - y_{\tau_t'}\|_2 + \sum_{t=s}^{T-d-1} \sum_{i=\tau_t}^{\tau_{t+1}-1} G \|y_{\tau_t} - y_i\|_2 \leq 3dGD + 4Gd\sqrt{T} + \frac{8G\sqrt{T}}{2 - \alpha} T^{1-\alpha/2} + \frac{3\eta G^2 dT}{2}.
\]

Note that Lemma 1 introduces an assumption about \( y_t \) and \( F_{t-1}(y) \). According to our Algorithm 1, \( y_t \) is actually generated by approximately minimizing \( F_{t-1}(y) \) with a linear optimization step. Therefore, by following the analysis of the original OFW (Hazan, 2016), we show that this assumption can be satisfied with \( \gamma = 8D^2 \) and \( \alpha = 1/2 \).

**Lemma 2** Let \( y^*_t = \text{argmin}_{y \in \mathcal{K}} F_{t-1}(y) \) for any \( t \in [T+1] \), where \( F_t(y) \) is defined in (13). Under Assumptions 1 and 2, for any \( t \in [T+1] \), Algorithm 1 with \( \eta = \frac{D}{\sqrt{2G(T+2)^{3/4}}} \) has

\[
F_{t-1}(y_t) - F_{t-1}(y^*_t) \leq \frac{8D^2}{\sqrt{t+2}}
\]

Then, by combining Lemmas 1 and 2, we have

\[
A + C \leq (3 + 8\sqrt{2})Gd + \frac{32\sqrt{2}GD}{3} T^{3/4} + \frac{3\eta G^2 dT}{2} \leq (3 + 8\sqrt{2})Gd + \frac{32\sqrt{2}GD}{3} T^{3/4} + \frac{3Gd T^{1/4}}{2\sqrt{2}}
\]

(18)

\[
= O(T^{3/4} + dT^{1/4})
\]
where the second inequality is due to \( \eta = \frac{D}{\sqrt{2G(T+2)^{3/4}}} \).

Furthermore, by following the analysis of the original OFW (Hazan, 2016), we establish an upper bound for the term \( B \).

**Lemma 3** Under Assumptions 1 and 2, for any \( \mathbf{x}^* \in \mathcal{K} \), Algorithm 1 with \( \eta = \frac{D}{\sqrt{2G(T+2)^{3/4}}} \) ensures

\[
\sum_{t=1}^{T} \langle \mathbf{g}_{ct}, \mathbf{y}_t - \mathbf{x}^* \rangle \leq \frac{11\sqrt{2GD(T+2)^{3/4}}}{3} + \frac{GDT^{1/4}}{\sqrt{2}}.
\]

From Lemma 3, we have

\[ B = O(T^{3/4}). \] (19)

Then, by combining (17), (18), and (19), we have

\[
\sum_{t=1}^{T} f_t(\mathbf{x}_t) - \sum_{t=1}^{T} f_t(\mathbf{x}^*) = O(T^{3/4} + dT^{1/4}).
\]

### 4.3 Proof of Theorem 3

Since \( f_t(\mathbf{x}) \) is \( \beta \)-strongly convex, we have

\[
\sum_{t=1}^{T} f_t(\mathbf{x}_t) - \sum_{t=1}^{T} f_t(\mathbf{x}^*) \leq \sum_{t=1}^{T} \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}^* \rangle - \frac{\beta}{2} \| \mathbf{x}_t - \mathbf{x}^* \|_2^2. \] (20)

Then, we note that the first term in the right side of (20) can be bounded by reusing (15) and (16). Specifically, we have

\[
\sum_{t=1}^{T} f_t(\mathbf{x}_t) - \sum_{t=1}^{T} f_t(\mathbf{x}^*) \leq A + B + C - \sum_{t=1}^{T} \frac{\beta}{2} \| \mathbf{x}_t - \mathbf{x}^* \|_2^2
\]

\[
= A + C + \sum_{t=1}^{T} \langle \mathbf{g}_ct, \mathbf{y}_t - \mathbf{x}^* \rangle - \sum_{t=1}^{T} \frac{\beta}{2} \| \mathbf{x}_t - \mathbf{x}^* \|_2^2
\] (21)

where terms \( A \), \( B \), and \( C \) are defined in (15) and (16).

Next, we consider the last term in the right side of (21). For any \( \mathbf{y}_t, \mathbf{x}_t, \mathbf{x}^* \in \mathcal{K} \), we have

\[
\| \mathbf{y}_t - \mathbf{x}^* \|_2^2 = \| \mathbf{y}_t - \mathbf{x}_t \|_2^2 + \| \mathbf{x}_t - \mathbf{x}^* \|_2^2 + 2 \langle \mathbf{y}_t - \mathbf{x}_t, \mathbf{x}_t - \mathbf{x}^* \rangle
\]

\[
\leq \| \mathbf{y}_t - \mathbf{x}_t \|_2^2 + \| \mathbf{x}_t - \mathbf{x}^* \|_2^2 + 2 \| \mathbf{y}_t - \mathbf{x}_t \|_2 \| \mathbf{x}_t - \mathbf{x}^* \|_2
\]

\[
\leq 3D \| \mathbf{y}_t - \mathbf{x}_t \|_2 + \| \mathbf{x}_t - \mathbf{x}^* \|_2^2
\]

where the last inequality is due to Assumption 2.
By combining the above inequality and (8) with (21), we have
\[
\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x^*) \leq A + C + \sum_{t=1}^{T} \left( \langle g_{t}, y_t - x^* \rangle - \frac{\beta}{2} \|y_t - x^*\|_2^2 \right) \tag{22}
\]
\[
+ \sum_{t=1}^{T} \frac{3\beta D}{2} \|y_t - y_{\tau_t}\|_2.
\]

Then, we proceed to establish upper bounds for terms $A$, $C$, and $E$ by carefully analyzing the distance $\|y_{\tau_t} - y_{\tau_{t'}}\|_2$ in the term $A$, the distance $\|y_{\tau_t} - y_t\|_2$ in the term $C$, and the distance $\|y_t - y_{\tau_t}\|_2$ in the term $E$.

**Lemma 4** Let $y_t^* = \text{argmin}_{y \in \mathcal{K}} F_{t-1}(y)$ for any $t = 2, \ldots, T + 1$, where $F_t(y)$ is defined in (14). Suppose Assumption 1 and 2 hold, all losses are $\beta$-strongly convex, and there exist some constants $\gamma > 0$ and $0 \leq \alpha < 1$ such that $F_{t-1}(y_t) - F_{t-1}(y_t^*) \leq \gamma(t-1)^\alpha$ for any $t = 2, \ldots, T + 1$, Algorithm 1 ensures
\[
\sum_{t=1}^{T} \frac{3\beta D}{2} \|y_t - y_{\tau_t}\|_2 \leq 3dD\sqrt{2\beta \gamma} + \frac{6D\sqrt{2\beta \gamma}}{1 + \alpha} T^{(1+\alpha)/2} + 3\beta dD^2 + 3D(G + \beta D) d \ln T
\]
and
\[
\sum_{t=1}^{T} G\|y_{\tau_t} - y_{\tau_{t'}}\|_2 + \sum_{t=s}^{T+d-1} \sum_{t'=t_{\tau_t}}^{t_{\tau_{t'}}-1} G\|y_{\tau_t} - y_t\|_2
\]
\[
\leq 3dGD + \frac{4G(G + \beta D) d (1 + \ln T)}{\beta} + 4dG \sqrt{\frac{2\gamma}{\beta}} + \sqrt{\frac{2\gamma}{\beta}} \frac{8G}{1 + \alpha} T^{(1+\alpha)/2}.
\]

Note that Lemma 4 also introduces an assumption about $y_t$ and $F_{t-1}(y)$. According to our Algorithm 2, $y_t$ is actually generated by approximately minimizing $F_{t-1}(y)$ with a linear optimization step. Therefore, by following the analysis of OFW for strongly convex losses (Wan and Zhang, 2021), we show that this assumption is satisfied with $\gamma = 16(G + 2\beta D)^2 / \beta$ and $\alpha = 1/3$.

**Lemma 5** Let $y_t^* = \text{argmin}_{y \in \mathcal{K}} F_{t-1}(y)$ for any $t = 2, \ldots, T + 1$, where $F_t(y)$ is defined in (14). Suppose Assumption 1 and 2 hold, and all losses are $\beta$-strongly convex, for any $t = 2, \ldots, T + 1$, Algorithm 2 has
\[
F_{t-1}(y_t) - F_{t-1}(y_t^*) \leq \frac{16(G + 2\beta D)^2(t-1)^{1/3}}{\beta}.
\]

By combining Lemmas 4 and 5, we have
\[
A + C + E = O \left( T^{(1+\alpha)/2} + d \log T \right) = O(T^{2/3} + d \log T).
\]

Furthermore, by following the analysis of OFW for strongly convex losses (Wan and Zhang, 2021), we establish an upper bound for the term $B'$.
Lemma 6 Suppose Assumption 1 and 2 hold, and all losses are $\beta$-strongly convex, for any $x^* \in K$, Algorithm 2 ensures

$$
\sum_{t=1}^{T} \left( \langle g_t, y_t - x^* \rangle - \frac{\beta}{2} \| y_t - x^* \|_2^2 \right) \leq \frac{6\sqrt{2}(G + 2\beta D)^2 T^{2/3}}{\beta} + \frac{2(G + 2\beta D)^2 \ln T}{\beta} + (G + \beta D)D.
$$

From Lemma 6, we have

$$
B' = O(T^{2/3}).
$$

Finally, by combining (22), (23), and (24), we have

$$
\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x^*) = O(T^{2/3} + d \log T).
$$

5. Conclusion

In this paper, we propose delayed OFW for OCO with arbitrary and unknown delays. For convex losses, we show that it attains an $O(T^{3/4} + d T^{1/4})$ regret bound over general sets and an $O(T^{2/3} + d T^{1/3})$ regret bound over strongly convex sets. When losses are strongly convex, we further prove that it can attain an $O(T^{2/3} + d \log T)$ regret bound over general sets and an $O(\sqrt{T} + d \log T)$ regret bound over strongly convex sets. Compared with existing regret bounds of OFW in the non-delayed setting, our results demonstrate that delayed OFW is robust to a relatively large amount of delay.

References

Baruch Awerbuch and Robert Kleinberg. Online linear optimization and adaptive routing. *Journal of Computer and System Sciences*, 74(1):97–114, 2008.

Stephen Boyd and Lieven Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.

Joseph C. Dunn. Rates of convergence for conditional gradient algorithms near singular and nonsingular extremals. *SIAM Journal on Control and Optimization*, 17(2):187–211, 1979.

Marguerite Frank and Philip Wolfe. An algorithm for quadratic programming. *Naval Research Logistics Quarterly*, 3(1–2):95–110, 1956.

Dan Garber and Elad Hazan. Faster rates for the frank-wolfe method over strongly-convex sets. In *Proceedings of the 32nd International Conference on Machine Learning*, pages 541–549, 2015.

Dan Garber and Elad Hazan. A linearly convergent conditional gradient algorithm with applications to online and stochastic optimization. *SIAM Journal on Optimization*, 26(3):1493–1528, 2016.
Dan Garber and Ben Kretzu. Revisiting projection-free online learning: the strongly convex case. In *Proceedings of the 24th International Conference on Artificial Intelligence and Statistics*, pages 3592–3600, 2021.

Elad Hazan. Introduction to online convex optimization. *Foundations and Trends in Optimization*, 2(3–4):157–325, 2016.

Elad Hazan and Satyen Kale. Projection-free online learning. In *Proceedings of the 29th International Conference on Machine Learning*, pages 1843–1850, 2012.

Elad Hazan and Edgar Minasyan. Faster projection-free online learning. In *Proceedings of the 33rd Annual Conference on Learning Theory*, pages 1877–1893, 2020.

Elad Hazan, Amit Agarwal, and Satyen Kale. Logarithmic regret algorithms for online convex optimization. *Machine Learning*, 69(2):169–192, 2007.

Xinran He, Junfeng Pan, Ou Jin, Tianbing Xu, Bo Liu, Tao Xu, Yanxin Shi, Antoine Atallah, Ralf Herbrich, Stuart Bowers, and Joaquin Q. Candela. Practical lessons from predicting clicks on ads at facebook. In *Proceedings of the 8th International Workshop on Data Mining for Online Advertising*, pages 1–9, 2014.

Martin Jaggi. Revisiting frank-wolfe: Projection-free sparse convex optimization. In *Proceedings of the 30th International Conference on Machine Learning*, pages 427–435, 2013.

Pooria Joulani, András György, and Csaba Szepesvári. Online learning under delayed feedback. In *Proceedings of the 30th International Conference on Machine Learning*, pages 1453–1461, 2013.

Pooria Joulani, András György, and Csaba Szepesvári. Delay-tolerant online convex optimization: Unified analysis and adaptive-gradient algorithms. *Proceedings of the 30th AAAI Conference on Artificial Intelligence*, pages 1744–1750, 2016.

Adam Kalai and Santosh Vempala. Efficient algorithms for online decision problems. *Journal of Computer and System Sciences*, 71(3):291–307, 2005.

Thomas Kerdreux, Lewis Liu, Simon Lacoste Julien, and Damien Scieur. Affine invariant analysis of frank-wolfe on strongly convex sets. In *Proceedings of the 38th International Conference on Machine Learning*, pages 5398–5408, 2021.

John Langford, Alexander J. Smola, and Martin Zinkevich. Slow learners are fast. In *Advances in Neural Information Processing Systems 22*, pages 2331–2339, 2009.

Evgeny S. Levitin and Boris T. Polyak. Constrained minimization methods. ***USSR Computational mathematics and mathematical physics*, 6:1–50, 1966.

Kfir Y. Levy and Andreas Krause. Projection free online learning over smooth sets. In *Proceedings of the 22nd International Conference on Artificial Intelligence and Statistics*, pages 1458–1466, 2019.
Appendix A. Proof of Lemma 1

We first note that $F_t(y)$ is 2-strongly convex for any $t = 0, \ldots, T$, and Hazan and Kale (2012) have proved that for any $\beta$-strongly convex function $f(x)$ over $\mathcal{K}$ and any $x \in \mathcal{K}$, it holds that

$$\frac{\beta}{2}\|x - x^*\|^2 \leq f(x) - f(x^*)$$

(25)

where $x^* = \arg\min_{x \in \mathcal{K}} f(x)$.

Then, we consider the term $\sum_{t=1}^{T} G\|y_{\tau_t} - y_{\tau_t'}\|_2$. If $T \leq 2d$, we have

$$\sum_{t=1}^{T} G\|y_{\tau_t} - y_{\tau_t'}\|_2 \leq TGD \leq 2dGD$$

(26)
where the first inequality is due to Assumption 2. If $T > 2d$, we have
\begin{align*}
\sum_{t=1}^{T} G \| y_{\tau_t} - y_{\tau_t'} \|_2 &= \sum_{t=1}^{2d} G \| y_{\tau_t} - y_{\tau_t'} \|_2 + \sum_{t=2d+1}^{T} G \| y_{\tau_t} - y_{\tau_t'} \|_2 \\
&\leq 2dGD + \sum_{t=2d+1}^{T} G \left( \| y_{\tau_t} - y_{\tau_t'} \|_2 + \| y_{\tau_t} - y_{\tau_t'} \|_2 + \| y_{\tau_t'} - y_{\tau_t} \|_2 \right).
\end{align*}
(27)

Because of (25), for any $t \in [T + 1]$, we have
\begin{align*}
\| y_t - y_t^* \|_2 &\leq \sqrt{F_{t-1}(y_t) - F_{t-1}(y_t^*)} \leq \sqrt{(t + 2)^{-\alpha/2}}
\end{align*}
(28)
where the last inequality is due to $F_{t-1}(y_t) - F_{t-1}(y_t^*) \leq \gamma(t + 2)^{-\alpha}$.

Moreover, for any $i \geq \tau_t$, we have
\begin{align*}
\| y_{\tau_t}^* - y_{\tau_t} \|_2^2 &\leq F_{t-1}(y_{\tau_t}^*) - F_{t-1}(y_t^*) = F_{t-1}(y_{\tau_t}^*) - F_{t-1}(y_t^*) + \left( \eta \sum_{k=\tau_t}^{i-1} g_k, y_{\tau_t}^* - y_t^* \right) \\
&\leq \eta \left( \sum_{k=\tau_t}^{i-1} g_k \right) \| y_{\tau_t}^* - y_t^* \|_2 \leq \eta G(i - \tau_t) \| y_{\tau_t}^* - y_t^* \|_2
\end{align*}
(29)
where the first inequality is still due to (25) and the last inequality is due to Assumption 1.

Because of $t' = t + d_t - 1 \geq t$, we have $\tau_{t'} \geq \tau_t$. Then, from (29), we have
\begin{align*}
\| y_{\tau_t}^* - y_{\tau_{t'}} \|_2 &\leq \eta G(\tau_{t'} - \tau_t) = \eta G \sum_{k=t}^{t'-1} |F_k|.
\end{align*}
(30)

Then, by substituting (28) and (30) into (27), if $T > 2d$, we have
\begin{align*}
\sum_{t=1}^{T} G \| y_{\tau_t} - y_{\tau_{t'}} \|_2 &\leq 2dGD + \sum_{t=2d+1}^{T} G \left( \sqrt{(\tau_t + 2)^{-\alpha/2}} + \eta G \sum_{k=t}^{t'-1} |F_k| + \sqrt{(\tau_{t'} + 2)^{-\alpha/2}} \right) \\
&\leq 2dGD + \sum_{t=2d+1}^{T} 2G \sqrt{(\tau_t + 2)^{-\alpha/2}} + \eta G^2 \sum_{t=2d+1}^{T} \sum_{k=t}^{t'-1} |F_k| \\
&\leq 2dGD + \sum_{t=2d+1}^{T} 2G \sqrt{(\tau_t - 1)^{-\alpha/2}} + \eta G^2 \sum_{t=2d+1}^{T} \sum_{k=t}^{t'-1} |F_k|
\end{align*}
(31)

where the second inequality is due to $(\tau_t + 2)^{-\alpha/2} \geq (\tau_{t'} + 2)^{-\alpha/2}$ for $\tau_t \leq \tau_{t'}$ and $\alpha > 0$.

To bound the second term in the right side of (31), we introduce the following lemma.

**Lemma 7** Let $\tau_t = 1 + \sum_{i=1}^{t-1} |F_i|$ for any $t \in [T + d]$. If $T > 2d$, for $0 < \alpha \leq 1$, we have
\[ \sum_{t=2d+1}^{T} (\tau_t - 1)^{-\alpha/2} \leq d + \frac{2}{2 - \alpha} T^{1-\alpha/2}. \]
(32)
For the third term in the right side of (31), if $T > 2d$, we have
\[ \sum_{t=2d+1}^{T} \sum_{k=t}^{t'-1} |F_k| \leq \sum_{t=1}^{T} \sum_{k=t}^{t'+1} |F_k| \leq |F_t| = \sum_{k=0}^{d} \sum_{t=1+k}^{d} |F_k| \leq \sum_{t=1}^{d} \sum_{t=2}^{T+d-1} |F_k| = dT \tag{33} \]
where the second inequality is due to
\[ t' - 1 < t' = t + d_t - 1 \leq t + d - 1. \]

By substituting (32) and (33) into (31) and combining with (26), we have
\[ \sum_{t=1}^{T} G_{\tau_t} \leq 2dG\sqrt{\gamma} + 2G_{\delta}d\sqrt{\gamma} + \frac{4G\sqrt{\gamma}}{2 - \alpha} T^{1-\alpha/2} + \eta G^2 dT. \tag{34} \]

Then, for the term $\sum_{t=s}^{T+d-1} \sum_{i=\tau_t}^{\tau_t+1-1} G_{\tau_t} - y_i^* \| y_i - y_i^* \|_2$, we have
\[ \sum_{t=s}^{T+d-1} \sum_{i=\tau_t}^{\tau_t+1-1} G_{\tau_t} - y_i^* \| y_i - y_i^* \|_2 \]
\[ \leq |F_s| G + \sum_{t=s+1}^{T+d-1} \sum_{i=\tau_t}^{\tau_t+1-1} G_{\tau_t} (\| y_i - y_i^* \|_2 + \| y_i^* - y_i^* \|_2 + \| y_i^* - y_i^* \|_2) \]
\[ \leq |F_s| |G + \sum_{t=s+1}^{T+d-1} \sum_{i=\tau_t}^{\tau_t+1-1} G_{\tau_t} (\sqrt{\gamma} (\tau_t + 2)^{-\alpha/2} + \eta G (i - \tau_t) + \sqrt{\gamma} (i + 2)^{-\alpha/2}) \]
\[ \leq |F_s| G + \sum_{t=s+1}^{T+d-1} \sum_{i=\tau_t}^{\tau_t+1-1} 2G_{\sqrt{\gamma} (\tau_t + 2)^{-\alpha/2} + \eta G^2 \sum_{t=s+1}^{T+d-1} \sum_{k=0}^{T+d-1 \tau_t+1-1} k \]
\[ \leq |F_s| G + \sum_{t=s+1}^{T+d-1} \sum_{i=\tau_t}^{\tau_t+1-1} 2G_{\sqrt{\gamma} (\tau_t + 1)^{-\alpha/2} + \eta G^2 \sum_{t=s+1}^{T+d-1} \sum_{k=0}^{T+d-1 \tau_t+1-1} k \]
where the first inequality is due to Assumption 2, the second inequality is due to (28) and (29), and the third inequality is due to $(\tau_t + 2)^{-\alpha/2} \geq (i + 2)^{-\alpha/2}$ for $	au_t \leq i$ and $\alpha > 0$.

Moreover, for any $t \in [T + d - 1]$ and $k \in F_t$, since $1 \leq d_k \leq d$, we have
\[ t - d + 1 \leq k = t - d_k + 1 \leq t \]
which implies that
\[ |F_t| \leq t - (t - d + 1) + 1 = d. \tag{36} \]
Then, it is easy to verify that
\[ \tau_t+1 - \tau_t - 1 < \tau_t+1 - \tau_t = |F_t| \leq d. \]
Therefore, by combining with (35), we have

\[
\sum_{t=s}^{T+d-1} \sum_{i=\tau_t}^{\tau_{t+1}-1} G\|y_{\tau_t} - y_i\|_2
\leq dGD + \sum_{t=s+1}^{T+d-1} \sum_{i=\tau_t}^{\tau_{t+1}-1} 2G\sqrt{\gamma}(\tau_t - 1)^{-\alpha/2} + \eta G^2 \sum_{t=s}^{T+d-1} \frac{|F_t|^2}{2}
\]

\[
\leq dGD + \sum_{t=s+1}^{T+d-1} \sum_{i=\tau_t}^{\tau_{t+1}-1} 2G\sqrt{\gamma}(\tau_t - 1)^{-\alpha/2} + \eta G^2 \sum_{t=s}^{T+d-1} \frac{d|F_t|}{2}
\]

\[
= dGD + \sum_{t=s+1}^{T+d-1} \sum_{i=\tau_t}^{\tau_{t+1}-1} 2G\sqrt{\gamma}(\tau_t - 1)^{-\alpha/2} + \frac{\eta G^2 dT}{2}.
\]

(37)

Furthermore, we introduce the following lemma.

**Lemma 8** Let \( \tau_t = 1 + \sum_{i=1}^{t-1} |F_i| \) for any \( t \in [T+d] \) and \( s = \min \{ t | t \in [T + d - 1], |F_t| > 0 \} \). For \( 0 < \alpha \leq 1 \), we have

\[
\sum_{t=s+1}^{T+d-1} \sum_{i=\tau_t}^{\tau_{t+1}-1} (\tau_t - 1)^{-\alpha/2} \leq d + \frac{2}{2 - \alpha} T^{1-\alpha/2}.
\]

(38)

By substituting (38) into (37), we have

\[
\sum_{t=s}^{T+d-1} \sum_{i=\tau_t}^{\tau_{t+1}-1} G\|y_{\tau_t} - y_i\|_2 \leq dGD + 2G\sqrt{\gamma}d + \frac{4G\sqrt{\gamma}T^{1-\alpha/2}}{2 - \alpha} + \frac{\eta G^2 dT}{2}.
\]

(39)

We complete the proof by combing (34) and (39).

**Appendix B. Proof of Lemma 2**

At the beginning of this proof, we recall the standard definition for smooth functions (Boyd and Vandenberghe, 2004).

**Definition 1** A function \( f(x) : \mathcal{K} \to \mathbb{R} \) is called \( \alpha \)-smooth over \( \mathcal{K} \) if for all \( x, y \in \mathcal{K} \), it holds that

\[
f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{\alpha}{2} \|y - x\|_2^2.
\]

It is not hard to verify that \( F_t(y) \) is 2-smooth over \( \mathcal{K} \) for any \( t \in [T] \). This property will be utilized in the following.

For brevity, we define \( h_t = F_{t-1}(y_t) - F_{t-1}(y_t^*) \) for \( t = 1, \ldots, T + 1 \) and \( h_t(y_{t-1}) = F_{t-1}(y_{t-1}) - F_{t-1}(y_t^*) \) for \( t = 2, \ldots, T + 1 \).

For \( t = 1 \), since \( y_1 = \arg\min_{y \in \mathcal{K}} \|y - y_1\|_2^2 \), we have

\[
h_1 = F_0(y_1) - F_0(y_1^*) = 0 \leq \frac{8D^2}{\sqrt{3}} = \frac{8D^2}{\sqrt{t + 2}}.
\]

(40)
Then, for any \( T + 1 \geq t \geq 2 \), we have

\[
\begin{align*}
    \tilde{h}_t(y_{t-1}) &= F_{t-1}(y_{t-1}) - F_{t-1}(y^*_t) \\
    &= F_{t-2}(y_{t-1}) - F_{t-2}(y^*_t) + \langle \eta g_{c_{t-1}}, y_{t-1} - y^*_t \rangle \\
    &\leq F_{t-2}(y_{t-1}) - F_{t-2}(y^*_t) + \langle \eta g_{c_{t-1}}, y_{t-1} - y^*_t \rangle \\
    &\leq h_{t-1} + \eta \|g_{c_{t-1}}\|_2 \|y_{t-1} - y^*_t\|_2 \\
    &\leq h_{t-1} + \eta \|g_{c_{t-1}}\|_2 \|y_{t-1} - y^*_t\|_2 + \eta \|g_{c_{t-1}}\|_2 \|y^*_{t-1} - y^*_t\|_2 \\
    &\leq h_{t-1} + \eta \|g_{c_{t-1}}\|_2 \|y_{t-1} - y^*_t\|_2 + \eta \|g^*_{t-1} - y^*_t\|_2
\end{align*}
\]

where the first inequality is due to \( y^*_{t-1} = \arg\min_{y \in \mathcal{K}} F_{t-2}(y) \) and the last inequality is due to Assumption 1.

Moreover, for any \( T + 1 \geq t \geq 2 \), we note that \( F_{t-2}(x) \) is also 2-strongly convex, which implies that

\[
\|y_{t-1} - y^*_{t-1}\|_2 \leq \sqrt{F_{t-2}(y_{t-1}) - F_{t-2}(y^*_{t-1})} \leq \sqrt{h_{t-1}}
\]

where the first inequality is due to (25).

Similarly, for any \( T + 1 \geq t \geq 2 \)

\[
\|y^*_{t-1} - y^*_t\|_2 \leq F_{t-1}(y^*_{t-1}) - F_{t-1}(y^*_t) \\
= F_{t-2}(y^*_{t-1}) - F_{t-2}(y^*_t) + \langle \eta g_{c_{t-1}}, y^*_{t-1} - y^*_t \rangle \\
\leq \eta \|g_{c_{t-1}}\|_2 \|y^*_{t-1} - y^*_t\|_2
\]

which implies that

\[
\|y^*_{t-1} - y^*_t\|_2 \leq \eta \|g_{c_{t-1}}\|_2 \leq \eta G.
\]

By combining (41), (42), and (43), for any \( T + 1 \geq t \geq 2 \), we have

\[
\tilde{h}_t(y_{t-1}) \leq h_{t-1} + \eta G \sqrt{h_{t-1}} + \eta^2 G^2.
\]

Then, for any \( T + 1 \geq t \geq 2 \), since \( F_{t-1}(y) \) is 2-smooth, we have

\[
\tilde{h}_t = F_{t-1}(y_t) - F_{t-1}(y^*_t) \\
= F_{t-1}(y_{t-1} + \sigma_{t-1}(v_{t-1} - y_{t-1})) - F_{t-1}(y^*_t) \\
\leq h_t(y_{t-1}) + \langle \nabla F_{t-1}(y_{t-1}), \sigma_{t-1}(v_{t-1} - y_{t-1}) \rangle + \sigma_{t-1}^2 \|v_{t-1} - y_{t-1}\|_2^2.
\]

Moreover, according to Algorithm 1, we have

\[
\sigma_t = \arg\min_{\sigma \in [0,1]} \langle \sigma(v_t - y_t), \nabla F_t(y_t) \rangle + \sigma^2 \|v_t - y_t\|_2^2
\]

for any \( t \in [T] \).

Therefore, for \( t = 2 \), by combining (44) and (45), we have

\[
\begin{align*}
    h_2 &\leq h_1 + \eta G \sqrt{h_1} + \eta^2 G^2 + \langle \nabla F_1(y_1), \sigma_1(v_1 - y_1) \rangle + \sigma_1^2 \|v_1 - y_1\|_2^2 \\
    &\leq h_1 + \eta G \sqrt{h_1} + \eta^2 G^2 = \frac{D^2}{2(T + 2)^{3/2}} \leq 4D^2 = \frac{8D^2}{\sqrt{T + 2}}
\end{align*}
\]

20
where the second inequality is due to (46), and the first equality is due to (40) and \( \eta = \frac{D}{\sqrt{2G(T+2)^{3/4}}} \).

Then, for any \( t = 3, \ldots, T+1 \), by defining \( \sigma'_{t-1} = 2/\sqrt{t+1} \) and assuming \( h_{t-1} \leq \frac{8D^2}{\sqrt{t+1}} \), we have

\[
\begin{align*}
&h_t \leq h_t(y_{t-1}) + (\nabla F_{t-1}(y_{t-1}), \sigma'_{t-1}(v_{t-1} - y_{t-1})) + (\sigma'_{t-1})^2\|v_{t-1} - y_{t-1}\|^2_2 \\
&\leq h_t(y_{t-1}) + (\nabla F_{t-1}(y_{t-1}), \sigma'_{t-1}(y^*_t - y_{t-1})) + (\sigma'_{t-1})^2\|v_{t-1} - y_{t-1}\|^2_2 \\
&\leq (1 - \sigma'_{t-1}) h_t(y_{t-1}) + (\sigma'_{t-1})^2\|v_{t-1} - y_{t-1}\|^2_2 \\
&\leq (1 - \sigma'_{t-1})(h_{t-1} + \eta G \sqrt{h_{t-1}} + \eta^2 G^2) + (\sigma'_{t-1})^2 D^2 \\
&\leq (1 - \sigma'_{t-1}) h_{t-1} + \eta G \sqrt{h_{t-1}} + \eta^2 G^2 + (\sigma'_{t-1})^2 D^2 \\
&\leq \left(1 - \frac{2}{\sqrt{t+1}}\right) \frac{8D^2}{\sqrt{t+1}} + \frac{2D^2}{(T+2)^{3/4}(t+1)^{1/4}} + \frac{D^2}{2(t+1)} + \frac{4D^2}{t+1} \tag{48}
\end{align*}
\]

where the first inequality is due to (45) and (46), the second inequality is due to \( v_{t-1} \in \text{argmin}_{y \in \mathcal{K}} (\nabla F_{t-1}(y_{t-1}), y) \), the third inequality is due to the convexity of \( F_{t-1}(y) \), the fourth inequality is due to (44), and the last inequality is due to

\[
\left(1 - \frac{1}{\sqrt{t+1}}\right) \frac{1}{\sqrt{t+1}} \leq \frac{1}{\sqrt{t+2}} \tag{49}
\]

for any \( t \geq 0 \). Note that (49) can be derived by dividing \((t+1)\sqrt{t+2}\) into both sides of the following inequality

\[
\sqrt{t+1} + \sqrt{t+2} \leq (\sqrt{t+1} + 1)\sqrt{t+1} - \sqrt{t+2} \leq t + 1 + \sqrt{t+1} - \sqrt{t+2} \leq t + 1.
\]

By combining (40), (47), and (48), we complete this proof.

Appendix C. Proof of Lemma 3

In the beginning, we define \( y^*_t = \text{argmin}_{y \in \mathcal{K}} F_{t-1}(y) \) for any \( t \in [T+1] \), where \( F_t(y) = \eta \sum_{i=1}^{t} \langle g_{c_i}, y \rangle + \|y - y_1\|^2_2 \).

Let \( y^* \in \text{argmin}_{y \in \mathcal{K}} \sum_{t=1}^{T} \langle g_{c_t}, y \rangle \). It is easy to verify that

\[
\sum_{t=1}^{T} \langle g_{c_t}, y_t - x^* \rangle \leq \sum_{t=1}^{T} \langle g_{c_t}, y_t - y^*_t \rangle = \sum_{t=1}^{T} \langle g_{c_t}, y_t - y^*_t \rangle + \sum_{t=1}^{T} \langle g_{c_t}, y^*_t - y^* \rangle. \tag{50}
\]
Therefore, we will continue to upper bound the right side of (50). By applying Lemma 2, we have
\[
\sum_{t=1}^{T} \langle \mathbf{g}_t, \mathbf{y}_t - \mathbf{y}_t^* \rangle \leq \sum_{t=1}^{T} \| \mathbf{g}_t \| \| \mathbf{y}_t - \mathbf{y}_t^* \|_2 \leq \sum_{t=1}^{T} G \sqrt{F_{t-1}(\mathbf{y}_t) - F_{t-1}(\mathbf{y}_t^*)}
\]
(51)
where the second inequality is due to (25) and Assumption 1, and the last inequality is due to \( \sum_{t=1}^{T} (t + 2)^{-1/4} \leq 4(T + 2)^{3/4}/3 \).

Then, to bound \( \sum_{t=1}^{T} \langle \mathbf{g}_t, \mathbf{y}_t^* - \mathbf{y}^* \rangle \), we introduce the following lemma.

**Lemma 9** (Lemma 6.6 of Garber and Hazan (2016)) Let \( \{f_t(y)\}_{t=1}^{T} \) be a sequence of loss functions and let \( \mathbf{y}_t^* \in \arg\min_{y \in \mathcal{K}} \sum_{i=1}^{t} f_i(y) \) for any \( t \in [T] \). Then, it holds that
\[
\sum_{t=1}^{T} f_t(\mathbf{y}_t^*) - \min_{y \in \mathcal{K}} \sum_{t=1}^{T} f_t(y) \leq 0.
\]

To apply Lemma 9, we define \( \tilde{f}_t(y) = \eta \langle \mathbf{g}_t, y \rangle + \| y - \mathbf{y}_1 \|_2^2 \) and \( \tilde{f}_t(y) = \eta \langle \mathbf{g}_t, y \rangle \) for any \( t \geq 2 \). Note that \( F_t(y) = \sum_{i=1}^{t} \tilde{f}_i(y) \) and \( \mathbf{y}_{t+1}^* = \arg\min_{y \in \mathcal{K}} F_t(y) \) for any \( t = 1, \ldots, T \). Then, by applying Lemma 9 to \( \{\tilde{f}_t(y)\}_{t=1}^{T} \), we have
\[
\sum_{t=1}^{T} \tilde{f}_t(\mathbf{y}_{t+1}^*) - \sum_{t=1}^{T} \tilde{f}_t(\mathbf{y}^*) \leq 0
\]
which implies that
\[
\eta \sum_{t=1}^{T} \langle \mathbf{g}_t, \mathbf{y}_{t+1}^* - \mathbf{y}^* \rangle \leq \| \mathbf{y}^* - \mathbf{y}_1 \|_2^2 - \| \mathbf{y}_2^* - \mathbf{y}_1 \|_2^2.
\]

According to Assumption 2, we have
\[
\sum_{t=1}^{T} \langle \mathbf{g}_t, \mathbf{y}_{t+1}^* - \mathbf{y}^* \rangle \leq \frac{1}{\eta} \| \mathbf{y}^* - \mathbf{y}_1 \|_2^2 \leq \frac{D^2}{\eta}.
\]

Then, we have
\[
\sum_{t=1}^{T} \langle \mathbf{g}_t, \mathbf{y}_t^* - \mathbf{y}^* \rangle = \sum_{t=1}^{T} \langle \mathbf{g}_t, \mathbf{y}_{t+1}^* - \mathbf{y}^* \rangle + \sum_{t=1}^{T} \langle \mathbf{g}_t, \mathbf{y}_t^* - \mathbf{y}_{t+1}^* \rangle \leq \frac{D^2}{\eta} + \sum_{t=1}^{T} \| \mathbf{g}_t \| \| \mathbf{y}_t^* - \mathbf{y}_{t+1}^* \|_2
\]
\[
\leq \frac{D^2}{\eta} + \eta TG^2 \leq \sqrt{2}GD(T + 2)^{3/4} + \frac{GDT^{1/4}}{\sqrt{2}}
\]
(52)
where the second inequality is due to (43) and Assumption 1, and the last inequality is due to \( \eta = \frac{D}{\sqrt{2}G(T + 2)^{3/4}} \).

By substituting (51) and (52) into (50), we complete the proof.
Appendix D. Proof of Lemma 4

If $T \leq 2d$, it is easy to verify that
\[
\sum_{t=1}^{T} \frac{3\beta D}{2} \|y_t - y_{\tau_t}\|_2 \leq \frac{3\beta TD^2}{2} \leq 3\beta d^2
\]
where the first inequality is due to Assumption 2.

Then, if $T > 2d$, we have
\[
\sum_{t=1}^{T} \frac{3\beta D}{2} \|y_t - y_{\tau_t}\|_2 = \frac{3\beta D}{2} \sum_{t=1}^{2d} \|y_t - y_{\tau_t}\|_2 + \frac{3\beta D}{2} \sum_{t=2d+1}^{T} \|y_t - y_{\tau_t}\|_2 \leq 3\beta d^2 + \frac{3\beta D}{2} \sum_{t=2d+1}^{T} (\|y_t - y_{\tau_t}\|_2 + \|y_{\tau_t} - y_{\tau_{t-1}}\|_2) .
\]

Because $F_{t-1}(y)$ is $(t-1)\beta$-strongly convex for any $t = 2, \ldots, T+1$, we have
\[
\|y_t - y_{\tau_t}\|_2 \leq \sqrt{\frac{2(F_{t-1}(y_t) - F_{t-1}(y_{\tau_t}))}{(t-1)\beta}} \leq \sqrt{\frac{2\gamma}{(t-1)^1-\alpha\beta}}
\]
where the first inequality is due to (25) and the second inequality is due to $F_{t-1}(y_t) - F_{t-1}(y_{\tau_t}) \leq \gamma(t-1)^\alpha$.

Before considering $\|y_{\tau_t} - y_{\tau_{t-1}}\|_2$, we define $\tilde{f}_t(y) = \langle g_{\alpha_t}, y \rangle + \frac{\beta}{2} \|y - y_{\tau_t}\|_2^2$ for any $t = 1, \ldots, T$.

Note that $F_t(y) = \sum_{t=1}^{t} \tilde{f}_t(y)$. Moreover, for any $x, y \in K$ and $t = 1, \ldots, T$, we have
\[
|\tilde{f}_t(x) - \tilde{f}_t(y)| = |\langle g_{\alpha_t}, x - y \rangle + \frac{\beta}{2} \|x - y_{\tau_t}\|_2^2 - \frac{\beta}{2} \|y - y_{\tau_t}\|_2^2|
\begin{align*}
&= |\langle g_{\alpha_t}, x - y \rangle + \frac{\beta}{2} \|x - y_{\tau_t} + y - y_{\tau_t} - x + y\|_2^2|
\end{align*}
\leq \|g_{\alpha_t}\| \|x - y\|_2 + \frac{\beta}{2} (\|x - y_{\tau_t}\|_2^2 + \|y - y_{\tau_t}\|_2^2) \|x - y\|_2
\leq (G + \beta D) \|x - y\|_2
\]
where the last inequality is due to Assumptions 1 and 2.

Because of (25), for any $i \geq j > 1$, we have
\[
\|y_{\tau_i} - y_{\tau_j}\|_2^2 \leq \frac{2(F_{i-1}(y_{\tau_i}) - F_{i-1}(y_{\tau_j}))}{(i-1)\beta}
\begin{align*}
&= \frac{2(F_{j-1}(y_{\tau_i}) - F_{j-1}(y_{\tau_j})) + 2\sum_{k=j}^{i-1} (\tilde{f}_k(y_{\tau_i}) - \tilde{f}_k(y_{\tau_j}))}{(i-1)\beta}
\end{align*}
\leq \frac{2\sum_{k=j}^{i-1} (G + \beta D) \|y_{\tau_i} - y_{\tau_j}\|_2}{(i-1)\beta}
\leq \frac{2(i-j)(G + \beta D) \|y_{\tau_i} - y_{\tau_j}\|_2}{(i-1)\beta}
\]

23
where the last inequality is due to \( y_j^* = \arg\min_{y \in \mathcal{K}} F_j(y) \) and (56).

Note that all gradients queried at rounds \( 1, \ldots, t-d \) must arrive before round \( t \). Therefore, for any \( t \geq 2d + 1 \), we have \( t_1 = 1 + \sum_{k=1}^{t-1} |F_k| \geq t - d + 1 > t - d \) and

\[
\|y_t^* - y_{t_1}^*\|_2 \leq \frac{2(t - t_1)(G + \beta D)}{(t - 1)\beta} \leq \frac{2d(G + \beta D)}{(t - 1)\beta}
\]

(58)

where the first inequality is due to \( t \geq t_1 > 1 \) and (57).

By combining (54) with (55) and (58), if \( T > 2d \), we have

\[
\sum_{t=1}^{T} 3\beta D^2 + \frac{3\beta D}{2} \sum_{t=2d+1}^{T} \left( \frac{2\gamma}{(t-1)^{1-\alpha}\beta} + \frac{2d(G + \beta D)}{(t - 1)\beta} + \frac{2\gamma}{(t_1 - 1)^{1-\alpha}\beta} \right)
\]

\[
\leq 3\beta d D^2 + 3\beta D \sum_{t=2d+1}^{T} \sqrt{\frac{2\gamma}{(t_1 - 1)^{1-\alpha}\beta}} + \frac{2\gamma}{(t_1 - 1)^{1-\alpha}\beta} + 3D(G + \beta D)d \sum_{t=2}^{T} \frac{1}{t}
\]

\[
\leq 3\beta d D^2 + 3\beta D \sum_{t=2d+1}^{T} \sqrt{\frac{2\gamma}{(t_1 - 1)^{1-\alpha}\beta}} + 3D(G + \beta D)d \ln T
\]

\[
\leq 3\beta d D^2 + 3d D \sqrt{2\beta \gamma} + \frac{6D \sqrt{2\beta \gamma}}{1 + \alpha} T^{(1+\alpha)/2} + 3D(G + \beta D)d \ln T
\]

where the second inequality is due to \((\tau_1 - 1)^{1-\alpha} \leq (t - 1)^{1-\alpha}\) for \( t \geq \tau_1 \) and \( \alpha < 1 \), and the last inequality is due to Lemma 7 and \( 0 < 1 - \alpha \leq 1 \).

By combining (53) with the above inequality, we have

\[
\sum_{t=1}^{T} \frac{3\beta D}{2} \|y_t - y_{t_1}\|_2 \leq 3\beta d D^2 + 3d D \sqrt{2\beta \gamma} + \frac{6D \sqrt{2\beta \gamma}}{1 + \alpha} T^{(1+\alpha)/2} + 3D(G + \beta D)d \ln T.
\]

Then, we proceed to bound the term \( \sum_{t=s}^{T+1} (\tau_{t+1} - 1) |G||y_{t_1} - y_i|_2 \). We first have

\[
\sum_{t=s}^{T+d-1} \sum_{i=\tau_t}^{\tau_{t+1}-1} \|y_{t_1} - y_i\|_2
\]

\[
= \sum_{i=\tau_s}^{\tau_{t+1}-1} \|y_{t_1} - y_i\|_2 + \sum_{i=\tau_t}^{T+1} \sum_{i=\tau_t}^{\tau_{t+1}-1} \|y_{t_1} - y_i\|_2
\]

\[
\leq dD + \sum_{t=s+1}^{T+d-1} \sum_{i=\tau_t}^{\tau_{t+1}-1} \|y_{t_1} - y_i\|_2
\]

\[
\leq dD + \sum_{t=s+1}^{T+d-1} \sum_{i=\tau_t}^{\tau_{t+1}-1} (\|y_{t_1} - y_{t_1}^*\|_2 + \|y_{t_1}^* - y_i^*\|_2 + \|y_i^* - y_i\|_2)
\]

where the first inequality is due to Assumption 2 and \( \tau_{s+1} - \tau_s = |F_s| \leq d \).
where the first inequality is due to Assumption 2. By combining (61) with (60), we have

$$
\sum_{t=s}^{T+d-1} \sum_{i=\tau_t}^{\tau_{i+1}-1} \|y_{\tau_t} - y_i\|_2 \\
\leq dD + \sum_{t=s+1}^{T+d-1} \sum_{i=\tau_t}^{\tau_{i+1}-1} \left( \sqrt{\frac{2\gamma}{(\tau_t - 1)^{1-\alpha}}} + \frac{2(i - \tau_t)(G + \beta D)}{(i - 1)\beta} + \sqrt{\frac{2\gamma}{(i - 1)^{1-\alpha}}} \right) \\
\leq dD + \sum_{t=s+1}^{T+d-1} \sum_{i=\tau_t}^{\tau_{i+1}-1} \left( 2\sqrt{\frac{2\gamma}{(\tau_t - 1)^{1-\alpha}}} + \frac{2(i - \tau_t)(G + \beta D)}{(i - 1)\beta} \right) \\
\leq dD + 2d\sqrt{\frac{2\gamma}{\beta}} + \sqrt{\frac{2\gamma}{\beta}} \frac{4}{1 + \alpha} T^{(1+\alpha)/2} + \sum_{t=s+1}^{T+d-1} \sum_{i=\tau_t}^{\tau_{i+1}-1} \frac{2d(G + \beta D)}{(i - 1)\beta} \\
$$

(60)

where the first inequality is due to \((\tau_t - 1)^{1-\alpha} \leq (i - 1)^{1-\alpha}\) for \(0 < \tau_t - 1 \leq i - 1\) and \(\alpha < 1\), and the last inequality is due to Lemma 8, \(0 < 1 - \alpha \leq 1\), and

\[i - \tau_t \leq \tau_{i+1} - \tau_t \leq |F_t| \leq d.\]

Recall that we have defined

$$
\mathcal{I}_t = \begin{cases} 
\emptyset, \text{ if } |F_t| = 0, \\
\{\tau_t, \tau_t + 1, \ldots, \tau_{t+1} - 1\}, \text{ otherwise.}
\end{cases}
$$

It is not hard to verify that

$$
\bigcup_{t=s}^{T+d-1} \mathcal{I}_t = \{|F_t| + 1, \ldots, T\} \text{ and } \mathcal{I}_i \cap \mathcal{I}_j = \emptyset, \forall i \neq j.
$$

(61)

By combining (61) with (60), we have

$$
\sum_{t=s}^{T+d-1} \sum_{i=\tau_t}^{\tau_{i+1}-1} \|y_{\tau_t} - y_i\|_2 \leq dD + 2d\sqrt{\frac{2\gamma}{\beta}} + \sqrt{\frac{2\gamma}{\beta}} \frac{4}{1 + \alpha} T^{(1+\alpha)/2} + \sum_{t=|F_s|+1}^{T} \frac{2d(G + \beta D)}{(t - 1)\beta} \\
\leq dD + 2d\sqrt{\frac{2\gamma}{\beta}} + \sqrt{\frac{2\gamma}{\beta}} \frac{4}{1 + \alpha} T^{(1+\alpha)/2} + \sum_{t=2}^{T} \frac{2d(G + \beta D)}{(t - 1)\beta} \\
\leq dD + 2d\sqrt{\frac{2\gamma}{\beta}} + \sqrt{\frac{2\gamma}{\beta}} \frac{4}{1 + \alpha} T^{(1+\alpha)/2} + \frac{2d(G + \beta D)(1 + \ln T)}{\beta}.
$$

(62)

Next, we proceed to bound the term \(\sum_{t=1}^{T} G\|y_{\tau_t} - y_{\tau_{t'}}\|_2\). If \(T \leq 2d\), it is easy to verify that

$$
\sum_{t=1}^{T} \|y_{\tau_t} - y_{\tau_{t'}}\|_2 \leq TD \leq 2dD
$$

(63)

where the first inequality is due to Assumption 2.
Then, if \( T > 2d \), we have

\[
\sum_{t=1}^{T} \|y_{\tau t} - y_{\tau_t'}\|_2 \\
= \sum_{t=1}^{2d} \|y_{\tau t} - y_{\tau_t'}\|_2 + \sum_{t=2d+1}^{T} \|y_{\tau t} - y_{\tau_t'}\|_2 \\
\leq 2dD + \sum_{t=2d+1}^{T} (\|y_{\tau t} - y^*_{\tau t}\|_2 + \|y^*_{\tau_t} - y_{\tau_t'}\|_2 + \|y^*_{\tau_t'} - y_{\tau_t'}\|_2) \\
\leq 2dD + \sum_{t=2d+1}^{T} \frac{2\gamma}{(\tau_t - 1)^{1-\alpha}\beta} + \sum_{t=2d+1}^{T} \frac{2(G + \beta D)\sum_{k=t}^{t'-1} |F_k|}{\beta \sum_{i=1}^{k} |F_i|}.
\]

where the second inequality is due to (55) and (57), and the last inequality is due to \( \tau_{t'} \geq \tau_t \)

Then, we introduce the following lemma.

**Lemma 10** Let \( h_{k} = \sum_{i=1}^{k} |F_i| \). If \( T > 2d \), we have

\[
\sum_{t=2d+1}^{T} \sum_{k=t}^{t'-1} \frac{|F_k|}{h_k} \leq d + d \ln T.
\]

By applying Lemmas 7 and 10 to (64) and combining with (63), we have

\[
\sum_{t=1}^{T} \|y_{\tau t} - y_{\tau_t'}\|_2 \leq 2dD + 2d \sqrt{\frac{2\gamma}{\beta}} + \sqrt{\frac{2\gamma}{\beta}} \frac{4}{1+\alpha} T^{(1+\alpha)/2} + \frac{2(G + \beta D)d(1 + \ln T)}{\beta}.
\]

Finally, by combining (62) and (65), we complete this proof.

**Appendix E. Proof of Lemmas 5 and 6**

Recall that \( F_{\tau}(y) \) defined in Algorithm 2 is equivalent to that defined in (14). Let \( \hat{f}_t(y) = \langle g_{c_t}, y \rangle + \frac{\beta}{2} \|y - y_t\|_2^2 \) for any \( t = 1, \ldots, T \), which is \( \beta \)-strongly convex. Moreover, as proved in (56), functions \( \hat{f}_1(y), \ldots, \hat{f}_T(y) \) are \( (G + \beta D) \)-Lipschitz over \( K \) (see the definition of Lipschitz functions in Hazan (2016)).

Then, because of \( \nabla \hat{f}_t(y) = g_{c_t} \), it is not hard to verify that decisions \( y_1, \ldots, y_{T+1} \) in our Algorithm 2 are actually generated by performing OFW for strongly convex losses (see Algorithm 2 in Wan and Zhang (2021) for details) on functions \( \hat{f}_1(y), \ldots, \hat{f}_T(y) \). Note
that when Assumption 2 holds, and functions $\tilde{f}_1(y), \ldots, \tilde{f}_T(y)$ are $\beta$-strongly convex and $G'$-Lipschitz, Lemma 6 of Wan and Zhang (2021) has already shown that

$$F_{t-1}(y_t) - F_{t-1}(y_t^*) \leq \frac{16(G' + \beta D)^2(t - 1)^{1/3}}{\beta}$$

for any $t = 2, \ldots, T + 1$. Therefore, our Lemma 5 can be derived by simply substituting $G' = G + \beta D$ into the above inequality.

Moreover, when Assumption 2 holds, and functions $\tilde{f}_1(y), \ldots, \tilde{f}_T(y)$ are $\beta$-strongly convex and $G'$-Lipschitz, Theorem 3 of Wan and Zhang (2021) has already shown that

$$\sum_{t=1}^{T} \tilde{f}_t(y_t) - \sum_{t=1}^{T} \tilde{f}_t(x^*) \leq \frac{6\sqrt{2}(G' + \beta D)^2T^{2/3}}{\beta} + \frac{2(G' + \beta D)^2 \ln T}{\beta} + G'D.$$

We notice that $\sum_{t=1}^{T} \left( (g_{t-1}, y_t - x^*) - \frac{\beta}{2} \|y_t - x^*\|^2 \right) = \sum_{t=1}^{T} \tilde{f}_t(y_t) - \sum_{t=1}^{T} \tilde{f}_t(x^*)$. Therefore, our Lemma 6 can be derived by simply substituting $G' = G + \beta D$ into the above inequality.

**Appendix F. Proof of Lemma 7**

Since the gradient $g_1$ must arrive before round $d + 1$, for any $T \geq t \geq 2d + 1$, it is easy to verify that

$$\tau_t = 1 + \sum_{i=1}^{t-1} |F_i| \geq 1 + \sum_{i=1}^{d+1} |F_i| \geq 2.$$

Moreover, for any $i \geq 2$ and $(i+1)d \geq t \geq id + 1$, since all gradients queried at rounds $1, \ldots, (i-1)d + 1$ must arrive before round $id + 1$, we have

$$\tau_t = 1 + \sum_{i=1}^{t-1} |F_i| \geq (i-1)d + 2. \quad (66)$$

Then, we have

$$\sum_{t=2d+1}^{T} (\tau_t - 1)^{-\alpha/2} \leq \sum_{t=2d+1}^{[T/d]d} (\tau_t - 1)^{-\alpha/2} + \sum_{t=[T/d]d+1}^{T} (\tau_t - 1)^{-\alpha/2}$$

$$\leq \sum_{i=2}^{[T/d]-1} \sum_{t=1}^{(i+1)d} (\tau_t - 1)^{-\alpha/2} + d \leq d + \sum_{i=2}^{[T/d]-1} d((i-1)d + 1)^{-\alpha/2}$$

$$\leq d + \sum_{i=2}^{[T/d]-1} d((i-1)d)^{-\alpha/2} = d + \sum_{i=2}^{[T/d]-1} d^{1-\alpha/2}(i-1)^{-\alpha/2}$$

$$\leq d + \sum_{i=1}^{[T/d]} d^{1-\alpha/2}i^{-\alpha/2} \leq d + \frac{2}{2 - \alpha} T^{1-\alpha/2}$$

where the first inequality is due to $(\tau_t - 1)^{-\alpha/2} \leq 1$ for $\alpha > 0$ and $\tau_t \geq 2$, and the second inequality is due to (66) and $\alpha > 0$. 27
Appendix G. Proof of Lemma 8

Because of \( \tau_{t} = 1 + \sum_{i=t}^{T} |F_{i}| \), we have

\[
\sum_{t=s+1}^{T+d-1} \sum_{i=t_{2}}^{T+d-1} (\tau_{t} - 1)^{-\alpha/2} = \sum_{t=s+1}^{T+d-1} \frac{|F_{t}|}{(\sum_{i=s}^{T} |F_{i}|)^{\alpha/2}} \leq \sum_{t=s+1}^{T+d-1} \frac{|F_{t}|}{(\sum_{i=s}^{T} |F_{i}|)^{\alpha/2}} + \sum_{t=s+1}^{T+d-1} d \left( \frac{1}{(\sum_{i=s}^{T} |F_{i}|)^{\alpha/2}} - \frac{1}{(\sum_{i=s}^{T} |F_{i}|)^{\alpha/2}} \right) \]

where the first inequality is due to (36) and \((\sum_{i=s}^{T} |F_{i}|)^{\alpha/2} \leq (\sum_{i=s}^{T} |F_{i}|)^{\alpha/2}\).

Let \( h_{t} = \sum_{i=s}^{t} |F_{i}| \) for any \( t = s, \ldots, T + d - 1 \). Since \( 0 < \alpha \leq 1 \), it is not hard to verify that

\[
\sum_{t=s+1}^{T+d-1} \frac{|F_{t}|}{(\sum_{i=s}^{T} |F_{i}|)^{\alpha/2}} = \sum_{t=s+1}^{T+d-1} \frac{|F_{t}|}{(h_{t})^{\alpha/2}} = \sum_{t=s+1}^{T+d-1} \int_{h_{t-1}}^{h_{t}} \frac{1}{(h_{t})^{\alpha/2}} dx \leq \int_{h_{s}}^{h_{T+d-1}} \frac{1}{x^{\alpha/2}} dx \leq \frac{2}{2 - \alpha} T^{1-\alpha/2}.
\]

Finally, we complete this proof by combining (67) with (68).

Appendix H. Proof of Lemma 10

It is not hard to verify that

\[
\sum_{t=2d+1}^{T} \sum_{k=t}^{T} \frac{|F_{k}|}{h_{k}} \leq \sum_{t=s}^{T} \sum_{k=t}^{T} \frac{|F_{k}|}{h_{k}} \leq \sum_{t=s}^{T} \sum_{k=t}^{T+d-1 \alpha/2} \frac{|F_{k}|}{h_{k}} = \sum_{t=s}^{T+d-1} \frac{|F_{t}|}{h_{t}} \leq \sum_{t=s}^{T+d-1} \frac{|F_{t}|}{h_{t}} = d \sum_{t=s}^{T+d-1} \frac{|F_{t}|}{h_{t}}
\]

where the first inequality is due to \( s \leq d < 2d + 1 \), and the second inequality is due to \( T' - 1 = t + d - 2 < t + d - 1 \).

Moreover, we have

\[
\sum_{t=s}^{T+d-1} \frac{|F_{t}|}{h_{t}} = \frac{|F_{s}|}{h_{s}} + \sum_{t=s+1}^{T+d-1} \int_{h_{t-1}}^{h_{t}} \frac{1}{x^{\alpha/2}} dx \leq \frac{|F_{s}|}{h_{s}} + \sum_{t=s+1}^{T+d-1} \int_{h_{t-1}}^{h_{t}} \frac{1}{x^{\alpha/2}} dx = \frac{|F_{s}|}{h_{s}} + \int_{h_{s}}^{h_{T+d-1}} \frac{1}{x^{\alpha/2}} dx = 1 + \ln \frac{T}{|F_{s}|} \leq 1 + \ln T
\]

where the last equality is due to \( h_{s} = |F_{s}| \) and \( h_{T+d-1} = T \).

Finally, we complete this proof by combining the above two inequalities.
Appendix I. Proof of Theorem 2

This proof is similar to that of Theorem 1, and we still need to use (17) and Lemma 1. The main difference is that Lemmas 2 and 3 can be improved by utilizing the strong convexity of decision sets (Wan and Zhang, 2021). Specifically, we introduce the following two lemmas.

**Lemma 11** Let $y^*_t = \arg\min_{y \in \mathbb{K}} F_{t-1}(y)$ for any $t \in [T+1]$, where $F_t(y)$ is defined in (13). Suppose Assumptions 1 and 2 hold, and the decision set is $\beta_K$-strongly convex, for any $t \in [T+1]$, Algorithm 1 with $\eta = \frac{D}{2G(T+2)^{2/3}}$ has

$$F_{t-1}(y_t) - F_{t-1}(y^*_t) \leq \frac{\gamma}{(t+2)^{2/3}}$$

where $\gamma = \max(4D^2, 4096/(3\beta_K^2))$.

**Lemma 12** Suppose Assumptions 1 and 2 hold, and the decision set is $\beta_K$-strongly convex, for any $x^* \in \mathbb{K}$, Algorithm 1 with $\eta = \frac{D}{2G(T+2)^{2/3}}$ ensures

$$\sum_{t=1}^T \langle g_{c_t}, y_t - x^* \rangle \leq \frac{11}{4} G \sqrt{\gamma} (T + 2)^{2/3}$$

where $\gamma = \max(4D^2, 4096/(3\beta_K^2))$.

Recall terms $A$, $B$, and $C$ defined in (15) and (16). By combining Lemmas 1 and 11, we have

$$A + C = O(T^{2/3} + dT^{1/3}).$$

(69)

From Lemma 12, we have

$$B = O(T^{2/3})$$

(70)

By combining (17), (69), and (70), we complete this proof.

Appendix J. Proof of Lemmas 11 and 12

Recall that $F_t(y)$ defined in Algorithm 1 is equivalent to that defined in (13). Then, let $\tilde{f}_t(y) = \langle g_{c_t}, y \rangle$ for any $t = 1, \ldots, T$. According to Assumption 1, for any $t = 1, \ldots, T$ and $x, y \in \mathbb{K}$, we have

$$|\tilde{f}_t(x) - \tilde{f}_t(y)| \leq \|g_{c_t}\|_2\|x - y\|_2 \leq G\|x - y\|_2$$

which implies that $\tilde{f}_t(y)$ is $G$-Lipschitz over $\mathbb{K}$ (see the definition of Lipschitz functions in Hazan (2016)). Because of $\nabla \tilde{f}_t(y_t) = g_{c_t}$, it is not hard to verify that decisions $y_1, \ldots, y_{T+1}$ in our Algorithm 2 are actually generated by performing OFW with line search (see Algorithm 1 in Wan and Zhang (2021) for details) on functions $\tilde{f}_1(y), \ldots, \tilde{f}_T(y)$.

We first note that when functions $\tilde{f}_1(y), \ldots, \tilde{f}_T(y)$ are $G$-Lipschitz, and the decision set is $\beta_K$ strongly convex and satisfies Assumption 2, Lemma 1 of Wan and Zhang (2021) has already shown that for any $t \in [T+1]$, OFW with line search and $\eta = \frac{D}{2G(T+2)^{2/3}}$ ensures

$$F_{t-1}(y_t) - F_{t-1}(y^*_t) \leq \frac{\gamma}{(t+2)^{2/3}}$$
where \( \gamma = \max(4D^2, 4096/(3\beta^2_K)) \). Therefore, our Lemma 11 can be directly derived from Lemma 1 of Wan and Zhang (2021).

Moreover, when functions \( \tilde{f}_1(y), \ldots, \tilde{f}_T(y) \) are \( G \)-Lipschitz, and the decision set is \( \beta_K \) strongly convex and satisfies Assumption 2, Theorem 1 of Wan and Zhang (2021) has already shown that OFW with line search and \( \eta = \frac{D}{2G(T+2)^{3/2}} \) ensures

\[
\sum_{t=1}^{T} \tilde{f}_t(y_t) - \sum_{t=1}^{T} \tilde{f}_t(x^*) \leq \frac{11}{4} G \sqrt{\gamma} (T + 2)^{2/3}
\]

where \( \gamma = \max(4D^2, 4096/(3\beta^2_K)) \). We notice that

\[
\sum_{t=1}^{T} \langle g_c, y_t - x^* \rangle = \sum_{t=1}^{T} \tilde{f}_t(y_t) - \sum_{t=1}^{T} \tilde{f}_t(x^*).
\]

Therefore, our Lemma 12 can be derived by simply using Theorem 1 of Wan and Zhang (2021).

### Appendix K. Proof of Theorem 4

This proof is similar to that of Theorem 3, and we still need to use (22) and Lemma 4. The main difference is that Lemmas 5 and 6 can be improved by utilizing the strong convexity of decision sets (Wan and Zhang, 2021). Specifically, we introduce the following two lemmas.

#### Lemma 13
Let \( y_t^* = \arg\min_{y \in K} F_t-1(y) \) for any \( t = 2, \ldots, T + 1 \), where \( F_t(y) \) is defined in (14). Suppose Assumption 1 and 2 hold, all losses are \( \beta \)-strongly convex, and the decision set is \( \beta_K \)-strongly convex, for any \( t = 2, \ldots, T + 1 \), Algorithm 2 has

\[
F_{t-1}(y_t) - F_{t-1}(y_t^*) \leq \gamma
\]

where \( \gamma = \max \left( \frac{4(G + 2\beta D)^2}{\beta}, \frac{288\beta}{\beta^2 K} \right) \).

#### Lemma 14
Suppose Assumption 1 and 2 hold, all losses are \( \beta \)-strongly convex, and the decision set is \( \beta_K \)-strongly convex, for any \( x^* \in K \), Algorithm 2 ensures

\[
\sum_{t=1}^{T} \left( \langle g_c, y_t - x^* \rangle - \frac{\beta}{2} \|y_t - x^*\|^2 \right) \leq \gamma \sqrt{2T} + \frac{\gamma \ln T}{2} + (G + \beta D)D
\]

where \( \gamma = \max \left( \frac{4(G + 2\beta D)^2}{\beta}, \frac{288\beta}{\beta^2 K} \right) \).

By combining Lemmas 4 and 13, with \( \alpha = 0 \), we have

\[
A + C + E = O \left( T^{(1+\alpha)/2} + d \log T \right) = O(\sqrt{T} + d \log T)
\]

where terms \( A, C, \) and \( E \) are defined in (15), (16), and (22).

Then, from Lemma 14, we have

\[
B' = O(\sqrt{T})
\]
where $B'$ is defined in (22).

Finally, by combining (22), (71), and (72), we have

$$
\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x^*) = O(\sqrt{T} + d \log T).
$$

**Appendix L. Proof of Lemmas 13 and 14**

Recall that $F_t(y)$ defined in Algorithm 2 is equivalent to that defined in (14). Let $\tilde{f}_t(y) = \langle g_{ct}, y \rangle + \frac{\beta}{2} \|y - y_t\|^2_2$ for any $t = 1, \ldots, T$, which is $\beta$-strongly convex. Moreover, as proved in (56), functions $\tilde{f}_1(y), \ldots, \tilde{f}_T(y)$ are $(G + \beta D)$-Lipschitz over $K$ (see the definition of Lipschitz functions in Hazan (2016)).

Then, because of $\nabla \tilde{f}_t(y_t) = g_{ct}$, it is not hard to verify that decisions $y_1, \ldots, y_{T+1}$ in our Algorithm 2 are actually generated by performing OFW for strongly convex losses (see Algorithm 2 in Wan and Zhang (2021) for details) on functions $\tilde{f}_1(y), \ldots, \tilde{f}_T(y)$. Note that when Assumption 2 holds, functions $\tilde{f}_1(y), \ldots, \tilde{f}_T(y)$ are $\beta$-strongly convex and $G'$-Lipschitz, and the decision set is $\beta_K$-strongly convex, Lemma 2 of Wan and Zhang (2021) has already shown that

$$
F_{t-1}(y_t) - F_{t-1}(y^*_t) \leq \max \left( \frac{4(G' + \beta D)^2}{\beta}, \frac{288\beta}{\beta_K^2} \right)
$$

for any $t = 2, \ldots, T + 1$. Therefore, our Lemma 13 can be derived by simply substituting $G' = G + \beta D$ into the above inequality.

Moreover, when Assumption 2 holds, and functions $\tilde{f}_1(y), \ldots, \tilde{f}_T(y)$ are $\beta$-strongly convex and $G'$-Lipschitz, Theorem 2 of Wan and Zhang (2021) has already shown that

$$
\sum_{t=1}^{T} \tilde{f}_t(y_t) - \sum_{t=1}^{T} \tilde{f}_t(x^*) \leq \gamma \sqrt{2T} + \frac{\gamma \ln T}{2} + G'D
$$

where $\gamma = \max \left( \frac{4(G' + \beta D)^2}{\beta}, \frac{288\beta}{\beta_K^2} \right)$. We notice that

$$
\sum_{t=1}^{T} \left( \langle g_{ct}, y_t - x^* \rangle - \frac{\beta}{2} \|y_t - x^*\|^2_2 \right) = \sum_{t=1}^{T} \tilde{f}_t(y_t) - \sum_{t=1}^{T} \tilde{f}_t(x^*).
$$

Therefore, our Lemma 6 can be derived by simply substituting $G' = G + \beta D$ into the above inequality.