Ontological models, preparation contextuality and nonlocality

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Abstract
The ontological model framework for an operational theory has generated much interest in recent time. The debate concerning reality of quantum state has been made more precise in this framework. With the introduction of generalized notion of contextuality in this framework, it has been shown that completely mixed state of a qubit is preparation contextual [Phys. Rev. A 71, 052108 (2005)]. Interestingly, this new idea of preparation contextuality has been used to reveal nonlocality of some ψ-epistemic models without any use of Bell’s inequality. In particular, nonlocality of a non maximally ψ-epistemic model has been demonstrated from preparation contextuality and steerability of the maximally entangled state of two qubits [Phys. Rev. Lett 110, 120401 (2013)]. In this paper, we generalize both these results. We, first, show that any mixed qubit state is preparation contextual. We, then, show that in a strictly non maximally ψ-epistemic model; nonlocality of any two-qubit pure entangled state follows from preparation contextuality and Schrödinger-GHJW steering.

I. INTRODUCTION

The nature of quantum state has been debated since the inception of quantum theory [1–4, 6, 7, 19]. Does it represent the physical reality or observer’s knowledge about the system? In the ontological models framework, introduced by Harrigan and Spekkens [8], this discussion has been made much more precise. An ontological model which aims to reproduce quantum predictions, can be of two types – ψ-ontic or ψ-epistemic [8]. The model is said to be ψ-epistemic if it considers the quantum mechanical state ψ to represent observer’s knowledge about the system. This is in contrast to ψ-ontic view point which considers ψ to represent reality of the system. The de-Broglie-Bohm model [9, 10] is an example of this later view point where ψ is given the ontic status. A ψ-ontic model can be incomplete, however, in the sense that the quantum mechanical state ψ does not provide the complete description of reality, all by itself. This is the case with the de-Broglie-Bohm model where although ψ is a representative of the reality, but it does not provide the complete reality of the system. The complete description is obtained when it is supplemented by the positions of the particles.

Not all the predictions of Quantum theory are compatible with that of a local-realistic theory. Quantum theory also shows contradiction with a noncontextual Hidden Variable Theory. The incompatibility of quantum theory with local-realism was first established by Bell [11]. Bell established this by means of an inequality which is violated by the singlet state of a pair of qubits. On the other hand Kochen and Specker (KS) have shown that any realistic interpretation for a quantum system described by Hilbert space of dimension greater than two must be contextual [12]. The notion of contextuality has been generalized recently by Spekkens [13] where he has demonstrated preparation contextuality of a completely mixed qubit state [13]. In a subsequent work, he, with Harrigan, has shown that in order to rule out the locality of any theory in which ψ has got an ontic status, one does not need Bell’s inequality, a straightforward argument based on preparation contextuality of completely mixed state of a qubit and steerability of maximally entangled state of two qubits suffice [8]. The authors of [8] further remark that Bell’s inequality is only necessary to rule out locality for ψ-epistemic hidden variable theories. Surprisingly, in a very recent development, Liefer and Maroney have demonstrated the nonlocality even for a ψ-epistemic model without any use of Bell’s inequality [14] provided the model is not maximally ψ-epistemic.

In this article we extend the proof of preparation contextuality for any mixed qubit state. We also show that the nonlocality proof of Liefer and Maroney runs even for any nonmaximally pure entangled state of two-qubit provided we make their notion of non-maximal ψ-epistemicity a bit more strong. This paper is organized as follows. In Section-II, we briefly describe the ontological model framework. We prove the preparation contextuality of any mixed qubit in Section-III. Section-IV deals with the nonlocality proof of any nonmaximally two-qubit pure entangled states in a (strictly) non-maximally ψ-epistemic ontological model.

II. ONTOLOGICAL MODEL AND ONTIC-EPISTEMIC CLASSIFICATION

We start with briefly discussing the ontological model framework for an operational theory as introduced by Harrigan and Spekkens in [8]. The goal of an operational theory is merely to specify the probabilities $p(k|M, P, T)$ of different outcomes $k \in K_M$ that may result from a measurement procedure $M \in \mathcal{M}$ given a particular

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preparation procedure \( P \in \mathcal{P} \), and a particular transformation procedure \( T \in \mathcal{T} \); where \( \mathcal{M}, \mathcal{P} \) and \( \mathcal{T} \) respectively denote the sets of measurement procedures, preparation procedures and transformation procedures; \( \mathcal{K}_M \) denotes the set of measurement results for the measurement \( M \). When there is no transformation procedure, we simply have \( p(k|M,P) \). The only restrictions on \( \{p(k|M,P)\}_{k \in \mathcal{K}_M} \) is that all of them are non-negative and \( \sum_{k \in \mathcal{K}_M} p(k|M,P) = 1 \) \( \forall \ M, P \). As an example, in an operational formulation of quantum theory, every preparation \( P \) is associated with a density operator on \( \mathcal{K}_M \) preparation procedures and transformation procedures; \( K_M \) is associated with a positive operator valued measure (POVM) \( \{E_k\} \), where \( E_k \geq 0 \) and \( \sum_k E_k = I \). The probability of obtaining outcome \( k \) is given by the generalized Born rule, \( p(k|M,P) = \text{Tr}(E_k \rho) \).

Whereas an operational theory does not tell anything about physical state of the system, in an ontological model of an operational theory, the primitives of description are the properties of microscopic systems. A preparation procedure is assumed to prepare a system with certain properties and a measurement procedure is assumed to reveal something about those properties. A complete specification of the properties of a system is referred to as the ontic state of that system. In an ontological model for quantum theory a particular preparation method (context) \( C_\rho \) of the quantum state \( \rho \) actually yields a probability distribution \( \mu(\lambda|\rho,C_\rho) \) over the ontic state \( \lambda \in \Lambda \), where \( \Lambda \) denotes the ontic state space. \( \mu(\lambda|\rho,C_\rho) \) is called the epistemic state associated with \( \rho \) and it must satisfy:

\[
\int_{\Lambda} \mu(\lambda|\rho,C_\rho)d\lambda = 1 \quad \forall \ \rho \text{ and } C_\rho.
\] (1)

The probability of obtaining the \( k \)'th outcome is given by the response function, \( \xi(k|\lambda,C_{E_k}) \), where \( C_{E_k} \) denotes the particular measurement method (context) used. When contextuality [2] is not relevant, the notations \( C_\rho \) and \( C_{E_k} \) will be omitted.

For the ontological model to reproduce quantum predictions for a quantum mechanical state \( \rho \), the following must be satisfied

\[
\int_{\Lambda} \xi(k|\lambda)\mu(\lambda|\rho)d\lambda = \text{Tr}(\rho E_k)
\] (2)

In special cases, when pure quantum states are associated with preparation and measurements are projective, the above relation reduces to

\[
\int_{\Lambda} \xi(\phi|\lambda)\mu(\lambda|\rho)d\lambda = |\langle \phi|\psi \rangle|^2.
\] (3)

An ontological model is considered to be \( \psi \)-ontic if the specification of the ontic state \( \lambda \) uniquely determines the quantum state \( \psi \) [8]. For this to be true, it is necessary that the preparations of any pair of different quantum states, \( \psi \) and \( \phi \), should yield ontic state distributions whose supports, \( \Lambda_\psi \) and \( \Lambda_\phi \), do not overlap. Hence, the epistemic states associated with distinct quantum states are completely non-overlapping in a \( \psi \)-ontic model. In other words, different quantum states pick out disjoint regions of \( \Lambda \). A variation of \( \psi \), therefore, implies a variation of reality.

If an ontological model fails to be \( \psi \)-ontic, then it is said to be \( \psi \)-epistemic, i.e., the class \( \psi \)-epistemic was merely defined as the complement of \( \psi \)-ontic. In a \( \psi \)-epistemic model, distinct quantum states are consistent with the same state of reality, i.e., variation of \( \psi \) does not necessarily imply a variation of reality. It is in this sense, quantum states are judged epistemic in such models.

There is possibility of trivial \( \psi \)-epistemic models, such as ones for which there is only a single pair of states, \( |\psi \rangle \) and \( |\phi \rangle \), for which \( \Lambda_\phi \) and \( \Lambda_\psi \) overlap. However, there are nontrivial \( \psi \)-epistemic models. Indeed, Aaronson et al. [16] have shown that there are maximally-nontrivial \( \psi \)-epistemic models, for which the supports \( \Lambda_\phi \) and \( \Lambda_\psi \) overlap for every non-orthogonal pair, \( |\psi \rangle \) and \( |\phi \rangle \). These models can be constructed for any finite dimensional Hilbert space. Very recently Maroney introduces the concept of degree of epistemicity of an ontological model [17]. In the ontological model the following elementary relation holds:

\[
\int_{\Lambda_\psi} \mu(\lambda|\psi)d\lambda = \int_{\Lambda_\psi} \xi(\phi|\lambda)\mu(\lambda|\psi)d\lambda
\]
\[
\leq \int_{\Lambda} \xi(\phi|\lambda)\mu(\lambda|\psi)d\lambda = |\langle \phi|\psi \rangle|^2
\] (4)

The first line follows from the fact that an ontic state \( \lambda \) that is compatible with the state preparation \( |\phi \rangle \) must assign value 1 to the response function \( \xi(\phi|\lambda) \). The above equation can be expressed as:

\[
\int_{\Lambda_\psi} \mu(\lambda|\psi)d\lambda = f(\phi,\psi)|\langle \phi|\psi \rangle|^2,
\] (5)

where \( 0 \leq f(\phi,\psi) \leq 1 \). In a \( \psi \)-ontic theory, \( f(\phi,\psi) = 0 \) for every pair of different quantum states. An ontological model will be called maximally \( \psi \)-epistemic if \( f(\phi,\psi) = 1 \) for all \( |\phi \rangle \) and \( |\psi \rangle \), otherwise the model will be called nonmaximally \( \psi \)-epistemic. The Kochen-Specker model for a spin-\( \frac{1}{2} \) system is a nice example of a maximally \( \psi \)-epistemic model. However, such a model is not possible for Hilbert spaces of dimensions three or more [17].

III. PREPARATION CONTEXTUALITY FOR MIXED QUBIT STATES

The traditional notion of contextuality [12] was generalized by Spekkens in [13] and has been discussed more elaborately in Ref. [18]. While the well known notion of KS-contextuality [12] is applicable for sharp (projective/Von-Neumann) measurements of quantum theory only, this generalized notion of contextuality is applicable to an arbitrary operational theory rather than just to quantum
theory and to arbitrary experimental procedures (preparation procedure, transformation procedure as well as unsharpen measurement procedure) rather than just to sharp measurements and also it is applicable to a broad class of ontological models of quantum theory rather than just to deterministic hidden variable models.

As mentioned in the previous section, the role of an operational theory is merely to specify the probabilities $p(k|M, P)$ of different outcomes $k$ that may result from a measurement procedure $M$ given a particular preparation procedure $P$. Given the rule for determining probabilities of outcomes, one can define a notion of equivalence among experimental procedures. Two preparation procedures are deemed equivalent if they yield the same long-run statistics for every possible measurement procedure, that is, $P$ is equivalent to $P'$ if:

$$p(k|M, P) = p(k|M, P') \; \forall \; M.$$ (6)

It might happen that the mere specification of the equivalence class of a procedure does not specify the procedure completely. The set of features of an experimental procedure which are not specified by specifying the equivalence class is called the context of the experimental procedure. An ontological model is called noncontextual if the representation of every experimental procedure in the model depends only on the equivalence class and not on its contexts.

In an ontological model for quantum theory, an equivalence class of preparation procedures is associated with a density operator $\rho$. Hence, the model will be called preparation non-contextual if it associates a single epistemic state $\mu(\lambda|\rho)$ with a given density operator, $\rho$, regardless of its context of preparation. Conversely a model is said to be preparation contextual if the epistemic state that it assigns to $\rho$ depends on the context of its preparation, i.e. there exist different contexts of preparation $C_\rho$ and $C'_\rho$ giving rise to the same density operator $\rho$ such that $\mu(\lambda|\rho, C_\rho) \neq \mu(\lambda|\rho, C'_\rho)$.

Spekkens proved the preparation contextuality for completely mixed state of a qubit [13]. In the following, we extend this result for any mixed qubit state.

**Lemma 1:** Any mixed state of a qubit is preparation contextual.

To prove the lemma, we adapt the proof of preparation contextuality for completely mixed state of a qubit as in Ref. [13]. We write below the a couple of features of the representations of preparation procedures in an ontological model as they will be used in the proof.

- **Feature 1:** If two preparation procedures are distinguishable with certainty in a single-shot measurement, then their associated probability distributions are non-overlapping, i.e.,

$$\Lambda_\phi \cap \Lambda_\psi = 0 \; \text{for all orthogonal pair } |\phi\rangle \text{ and } |\psi\rangle.$$ (7)

- **Feature 2:** A convex combination of preparation procedures is represented within an ontological model by a convex sum of the associated probability distributions.

**Proof:** A mixed qubit $\rho_q = \frac{1}{2}(I + \hat{n}\sigma)$, with $0 \leq |\hat{n}| (= q) < 1$, can have several decompositions. A different preparation procedure is associated with every such decomposition. Consider the following decompositions of a mixed state $\rho_n$ of a qubit:

$$\rho_n = \frac{1-q}{2} |\phi_n^+\rangle\langle \phi_n^+| + \frac{1+q}{2} |\phi_n\rangle\langle \phi_n|$$ (8)

$$= \frac{1-q}{2} (|\psi_a\rangle\langle \psi_a| + |\psi_a^+\rangle\langle \psi_a^+|) + q|\phi_n\rangle\langle \phi_n|$$ (9)

$$= \frac{1-q}{2} (|\psi_b\rangle\langle \psi_b| + |\psi_b^+\rangle\langle \psi_b^+|) + q|\phi_n\rangle\langle \phi_n|$$ (10)

$$= \frac{1-q}{2} (|\psi_c\rangle\langle \psi_c| + |\psi_c^+\rangle\langle \psi_c^+|) + q|\phi_n\rangle\langle \phi_n|$$ (11)

$$= \frac{1-q}{3} (|\psi_a\rangle\langle \psi_a| + |\psi_b\rangle\langle \psi_b| + |\psi_c\rangle\langle \psi_c|)$$

$$+ q|\phi_n\rangle\langle \phi_n|.$$

where $|\phi_n\rangle\langle \phi_n| = \frac{1}{2}(I + \hat{n}\sigma)$ and the vectors $|\psi_a\rangle, |\psi_b\rangle, |\psi_c\rangle$ are chosen from equatorial plane perpendicular to $\hat{n}$ in such a manner that the line joining the points corresponding to $|\psi_a\rangle, |\psi_a^+\rangle$ makes an angle 600 with the other two lines containing $|\psi_b\rangle, |\psi_b^+\rangle$ and $|\psi_c\rangle, |\psi_c^+\rangle$ respectively (Fig.1).

As whenever two density operators are orthogonal in the vector space of operators, the associated preparation procedures can be distinguished with certainty in a single shot measurement and as whenever two preparation procedures are distinguishable with certainty in a single shot measurement, their associated probability distributions are non-overlapping (Feature 1), we have

$$\mu(\lambda|\phi_a)\mu(\lambda|\phi_a) = 0$$ (14)

$$\mu(\lambda|\psi_a)\mu(\lambda|\psi_a) = 0$$ (15)

$$\mu(\lambda|\psi_b)\mu(\lambda|\psi_b) = 0$$ (16)

$$\mu(\lambda|\psi_c)\mu(\lambda|\psi_c) = 0$$ (17)

The above six decompositions (Eqs.(8)-(13)) of $\rho_n$ are associated with six different preparation procedures $C_{\phi_a}, C_{\psi_a}, ..., C_{\psi_c}$ respectively.

As in an ontological model a convex combination of preparation procedures is represented by a convex sum
of associated probability distributions (Feature 2), so

\[ \mu(\lambda|\rho_n, C_{\phi_1^\pm \phi_n}) = \frac{1-q}{2} \mu(\lambda|\phi_n^\pm) + \frac{1+q}{2} \mu(\lambda|\phi_n) \]  
(18)

\[ \mu(\lambda|\rho_n, C_{\psi_a \psi_n^\pm}) = \frac{1-q}{2} \mu(\lambda|\psi_a) + \mu(\lambda|\psi_n^\pm) + q\mu(\lambda) \]  
(19)

\[ \mu(\lambda|\rho_n, C_{\psi_b \psi_n^\pm}) = \frac{1-q}{2} \mu(\lambda|\psi_b) + \mu(\lambda|\psi_n^\pm) + q\mu(\lambda) \]  
(20)

\[ \mu(\lambda|\rho_n, C_{\psi_c \psi_n^\pm}) = \frac{1-q}{2} \mu(\lambda|\psi_c) + \mu(\lambda|\psi_n^\pm) + q\mu(\lambda) \]  
(21)

\[ \mu(\lambda|\rho_n, C_{\psi_{a,b,c} \psi_n^\pm}) = \frac{1-q}{3} \left[ \mu(\lambda|\psi_a) + \mu(\lambda|\psi_b) + \mu(\lambda|\psi_c) \right] + q\mu(\lambda|\psi_n) \]  
(22)

As mentioned before, the assumption of preparation

\[ \mu(\lambda|\rho_n) = \frac{1-q}{2} \mu(\lambda|\phi_n^+ \phi_n^-) + \frac{1+q}{2} \mu(\lambda|\phi_n) \]  
(24)

\[ \mu(\lambda|\rho_n) = \frac{1-q}{2} [\mu(\lambda|\psi_a) + \mu(\lambda|\psi_b^\pm)] + q\mu(\lambda|\phi_n) \]  
(25)

\[ \mu(\lambda|\rho_n) = \frac{1-q}{2} [\mu(\lambda|\psi_b) + \mu(\lambda|\psi_c^\pm)] + q\mu(\lambda|\phi_n) \]  
(26)

\[ \mu(\lambda|\rho_n) = \frac{1-q}{2} [\mu(\lambda|\psi_c) + \mu(\lambda|\psi_a^\pm)] + q\mu(\lambda|\phi_n) \]  
(27)

\[ \mu(\lambda|\rho_n) = \frac{1-q}{3} [\mu(\lambda|\psi_a^\pm) + \mu(\lambda|\psi_b^\pm) + \mu(\lambda|\psi_c^\pm)] + q\mu(\lambda|\phi_n) \]  
(28)

\[ \mu(\lambda|\rho_n) = \frac{1-q}{3} [\mu(\lambda|\psi_a^\pm) + \mu(\lambda|\psi_b^\pm) + \mu(\lambda|\psi_c^\pm)] + q\mu(\lambda|\phi_n) \]  
(29)

(22) But there is no distribution which is compatible with

Eqs. (14)-(17) and Eqs. (24)-(29). To see this let us
denote the probability of occurrence of a fixed \( \lambda_0 \) for dif-
ferent preparations by \( \mu(\lambda_0|\phi_n), \ldots, \mu(\lambda_0|\psi_n^\pm) \). To satisfy
Eq. (14), one out of the pair \( \mu(\lambda_0|\phi_n) \) and \( \mu(\lambda_0|\phi_n^+) \)
must be zero. Same is true for the pairs \{\( \mu(\lambda_0|\phi_n), \mu(\lambda_0|\phi_n^-) \),
\{\( \mu(\lambda_0|\phi_b), \mu(\lambda_0|\phi_b^+) \)\} and \{\( \mu(\lambda_0|\phi_c), \mu(\lambda_0|\phi_c^+) \)\}
der in order to satisfy Eqs. (15)-(17). Thus, in all, we have sixteen
different situations. Each of these situations implies

\[ \mu(\lambda_0|\phi_n) = \mu(\lambda_0|\phi_n^+) = \mu(\lambda_0|\phi_n^-) = \mu(\lambda_0|\phi_b) \]
\[ = \mu(\lambda_0|\phi_b^+) = \mu(\lambda_0|\phi_c) = \mu(\lambda_0|\phi_c^+) \]  
(30)

We analyze validity of Eq. (30) in only a few of such sit-
uations as its validity in the remaining situations follows
similarly.

(a) \( \mu(\lambda_0|\phi_n) = \mu(\lambda_0|\phi_n^+) = \mu(\lambda_0|\phi_n^-) = \mu(\lambda_0|\phi_b) = 0; \)
This makes Eq. (28) to read as \( \mu(\lambda_0|\rho_n) = 0 \)
which further imply \( \mu(\lambda_0|\phi_n^+) = 0, \mu(\lambda_0|\phi_n^-) = 0, \mu(\lambda_0|\phi_b^+) = 0, \mu(\lambda_0|\phi_b^-) = 0 \) from Eqs. (24), (25),
(26), (27) respectively.

(b) \( \mu(\lambda_0|\phi_n) = \mu(\lambda_0|\phi_n^+) = \mu(\lambda_0|\phi_b) = \mu(\lambda_0|\phi_c) = 0; \)
For this case, Eqs. (25) and (28) give \( \mu(\lambda_0|\phi_a) = 0 \)
and thus the case reduces to (a).

The above argument does not depend on particular
choice of \( \lambda \), and hence Eq. (30) holds for all \( \lambda \)’s. Thus we conclude that for the density matrix \( \rho_n \) preparation
noncontextual assignment of ontic state \( \lambda \) is not possible.

Unlike the various proofs of measurement contextuality
using different resolutions of identity in terms of one di-
mensional projectors [19–21], the meaning of preparation
contextuality needs further elaboration. Two extreme
cases that may be implied by preparation contextuality
are the following:

(1) Distribution of the ontic variables \( \lambda \) for a mixed
state are different for its different preparation pro-
cedures while distribution of the ontic variables \( \lambda \)
ise same for every preparation of a pure state;

(2) Distribution of \( \lambda \) for pure states may depend on
the context created by its different preparation pro-
cedures but the distribution of the ontic variables \( \lambda \)
for the mixed state remains same for all preparation
procedures.
For the first case, the nonlocality of any pure entangled state of two qubits in some \( \psi \)-epistemic models could follow from Schrödinger-GHJW steering [22, 23]. But, in general, the possibility of the second conclusion can not be discarded and hence the preparation contextuality of \( \psi \)-epistemic models as shown above, can not be directly used to reveal nonlocality. Of course, further assumption in the \( \psi \)-epistemic model may help to link preparation contextuality of qubit density matrix with nonlocality of two qubits pure entangled state and the next section discusses this possibility.

IV. NONLOCALITY OF 2-QUBIT PURE ENTANGLLED STATES

Leifer and Maroney have demonstrated the nonlocality of a non maximally \( \psi \)-epistemic model without using Bell’s inequality [14]. In order to show the nonlocality of such a model, they have first shown preparation contextuality of a completely mixed qubit. The nonlocality of a non maximally \( \psi \)-epistemic model has then been demonstrated by the use of the fact that Alice can remotely prepare different decompositions of Bob’s reduced density matrix when they share a maximally entangled state of two qubits.

Here we pose the question in a different way: whether this approach of Leifer and Maroney, namely, showing nonlocality by preparation contextuality and steerability, can be extended to demonstrate nonlocality of any two-qubit pure entangled state? Interestingly, we find an affirmative answer to this question provided the non maximally \( \psi \)-epistemic model satisfy certain condition.

We consider non-maximally \( \psi \)-epistemic models where for every \( \psi \), there exists at least one \( ? \) \( \phi \) such that

\[
\int_{\Lambda_\phi} \mu(\lambda|\psi)\,d\lambda < |\langle\phi|\psi\rangle|^2. \tag{31}
\]

We now proceed to show the preparation contextuality for mixed states of a qubit in this model as this preparation contextuality will subsequently be used to show nonlocality of such a model.

Consider the following two preparation procedures of a mixed qubit \( \rho_n = \frac{1}{2}(I + \vec{n} \cdot \vec{\sigma}) \) (0 \( \leq |\vec{n}| (= q) < 1\):

(I) Where either of the two preparations \( C_{\psi_n} \) and \( C_{\phi_n} \) corresponding to orthogonal quantum mechanical states \( |\phi_n\rangle \) and \( |\phi_n^\perp\rangle \) are implemented with respective probabilities \( (1 + q)/2 \) and \( (1 - q)/2 \). Here \( |\phi_n\rangle \langle \phi_n| = \frac{1}{2} (I + \vec{n} \cdot \vec{\lambda}) \).

(II) Two preparations \( C_{\psi_n} \) and \( C_{\chi_n} \) corresponding to non orthogonal quantum mechanical states \( |\psi_n\rangle \) and \( |\chi_n\rangle \) are implemented with probabilities \( r \) and \( 1 - r \) respectively (Fig. 2). Here \( |\psi_n\rangle \) is chosen in such a manner that inequality (31) holds for the pair \( |\psi_n\rangle \) and \( |\phi_n\rangle \).

Using (3) for \( |\langle\phi|\psi\rangle|^2 \), we get from (31)

\[
\int_{\Lambda_{\phi_n}} \mu(\lambda|\psi_n)\,d\lambda < \int_{\Lambda} \xi(\phi_n|M, \lambda)\mu(\lambda|\psi_n)\,d\lambda \tag{32}
\]

As \( \xi(\phi_n|M, \lambda) = 1 \), almost everywhere on \( \Lambda_{\phi_n} \); hence

\[
\int_{\Lambda_{\phi_n}} \xi(\phi_n|M, \lambda)\mu(\lambda|\psi_n)\,d\lambda < \int_{\Lambda} \xi(\phi_n|M, \lambda)\mu(\lambda|\psi_n)\,d\lambda \tag{33}
\]

This means that there is a set \( \Omega \) of ontic states of non zero measure such that:

(i) \( \Lambda_{\phi_n} \cap \Omega = \emptyset \),

(ii) \( \Omega \) is assigned nonzero probability by \( C_{\psi_n} \),

(iii) and \( \xi(\phi_n|M, \lambda) > 0 \) for \( \lambda \in \Omega \).

In an ontological model, a convex combination of preparation procedures is represented by a convex sum of the associated probability distributions (Feature 2). Hence, if the preparation procedures I and II are represented by \( C_{\phi_n, \psi_n} \) and \( C_{\phi_n, \chi_n} \) respectively, we have

\[
\mu(\lambda|\rho_n, C_{\phi_n, \psi_n}) = \frac{1+q}{2} \mu(\lambda|\phi_n) + \frac{1-q}{2} \mu(\lambda|\phi_n^\perp) \tag{34}
\]

\[
\mu(\lambda|\rho_n, C_{\phi_n, \chi_n}) = r \mu(\lambda|\psi_n) + (1-r) \mu(\lambda|\chi_n) \tag{35}
\]

The region \( \Omega \) is assigned zero probability by \( C_{\phi_n} \) as \( \Lambda_{\phi_n} \cap \Omega = \emptyset \) and \( C_{\phi_n} \) must assign zero probability to any set of ontic states that assigns nonzero probability to \( \phi_n \) in a measurement of any orthonormal basis that contains it.

Hence the region \( \Lambda_{\psi_n, \chi_n} \cap \Omega = \emptyset \) (where \( \Lambda_{\psi_n, \chi_n} = \Lambda_{\psi_n} \cup \Lambda_{\chi_n} \)) is the support of \( \mu(\lambda|\rho_n, C_{\psi_n, \chi_n}) \) is assigned zero probability by preparation I whereas preparation II assigns non zero probability to it. Thus \( \mu(\lambda|\rho_n, C_{\psi_n, \chi_n}) \) and \( \mu(\lambda|\rho_n, C_{\phi_n, \psi_n}) \) are distinct distributions of ontic states of the system. On the contrary, the assumption of preparation noncontextuality demands a distribution associated with a preparation procedure to depend only on the density operator associated with that procedure and not on the particular decomposition of the density matrix [13, 18].

FIG. 2. The two decompositions of mixed state of a qubit \( \rho_n \) corresponding to preparation procedures (I) and (II).
This preparation contextuality can be used to show nonlocality of a nonmaximally pure entangled state of two qubit. Consider a situation where two far separated parties Alice and Bob share a nonmaximally pure entangled state of two qubits. The reduced density matrix of Bob’s system is then given by

\[
\rho_n = \frac{1 + q}{2} |\phi_n\rangle\langle\phi_n| + \frac{1 - q}{2} |\phi_n^\dagger\rangle\langle\phi_n^\dagger| \tag{36}
\]

\[
= r|\psi_n\rangle\langle\psi_n| + (1 - r)|\chi_n\rangle\langle\chi_n| \tag{37}
\]

Now, according to the Gisin-Hughston-Jozsa-Wootters (GHJW) theorem [22, 23], Alice can remotely prepare Bob’s system either in (36) or in (37). However, as shown above, these two ensembles cannot correspond to the same probability distribution over ontic states. Thus, the distribution on Bob’s side depends on Alice’s choice of measurement, which implies nonlocality.

V. CONCLUSION

The concept of steering was first introduced by Schrödinger in 1935 [24]. Though this concept was disturbing for him, but it has no direct implication for nonlocality. Steering has attracted much attention in recent years [25–27]. Whereas the nonlocality of \(\psi\)-ontic models is not so clear. It seems that nonlocality of a general \(\psi\)-epistemic model will not follow from steering due to the possibility of the second implication of preparation contextuality (mentioned in Section-III). But, interestingly, in a recent development, it has been shown that just a little loss of epistemicity (in the sense that overlap of the ontic supports of at least one pair of nonorthogonal pure states is strictly less than the quantum overlap) makes nonlocality of such models to follow from steering [14]. Extension of this loss of epistemicity (i.e., for every pure state there exist at least another nonorthogonal pure state such that overlap of their ontic supports is strictly less than the quantum overlap) makes it possible to prove even Gisin-like theorem [28] for a two-qubit pure entangled state by using steering. Gisin’s theorem states that any pure bipartite entangled state violates a Bell’s inequality and is thus nonlocal [28]. Whether the present approach of showing nonlocality with the help of steering, but without any use of Bell’s inequality, can be extended to higher dimensional states remains open. This is because of the impossibility of reducing the problem to two qubits system, as was done in [28], in such a way to bypass the second possible implication of preparation contextuality (as mentioned in Section-III).

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