The heat conduction model involving two temperatures on the segment with Wentzell boundary conditions

N S Goncharov and G A Sviridyk
South Ural State University, pr. Lenina, 76 454080 Chelyabinsk, Russia
E-mail: goncharovns@susu.ru

Abstract. According to the theory of relatively $p$-bounded operators, we study the Heat Conduction model involving two temperatures for isotropic material, which describes, the rate of change of internal energy due to the movement of the heat flux form one medium to its complement with general Wentzell boundary conditions. In particular, we consider spectrum of one-dimensional Laplace operator on the segment $[0,1]$ with general Wentzell boundary conditions. We examine the relative spectrum in one-dimensional Heat Conduction equation involving two temperatures, and construct the resolving group in the Cauchy-Wentzell problem with general Wentzell boundary conditions. In the paper, these problems are solved under the assumption that the initial space is a contraction of the space $L^2(0,1)$.

1. Introduction
Let us consider the Cauchy-Wentzell problem

$$
\varphi(x,0) = v_0(x), x \in [0,1] \\
\varphi_{xx}(0,t) + \alpha_0 \varphi_x(0,t) + \alpha_1 \varphi(0,t) = 0, \\
\varphi_{xx}(1,t) + \beta_0 \varphi_x(0,t) + \beta_1 \varphi(1,t) = 0
$$

(1)

for the Heat Conduction equation involving two temperatures on the segment $[0,1]$

$$
c \varphi_t(x,t) = k \varphi_{xx}(x,t) + c a \varphi_{txx}(x,t) + r(x,t), (x,t) \in [0,1] \times \mathbb{R}_+,
$$

(2)

which describes for isotropic material the rate of change of internal energy due to the movement of the heat flux from one medium to its complement. Here $c$ is the specific heat at the fixed reference temperature $\varphi_0$ ($c$ as customary belongs to $\mathbb{R}_+$); the parameter $r$ is the heat supplied (per unit volume) from external world; the parameters $k, a \in \mathbb{R}_+$.

This model was considered in [1] in term of the theory of continuum thermodynamics [2]. In particular, since in [2] were proved the theorem related to the existence of two temperatures (radiative and conductive), the equation (2) we understand with the respect to the radiative temperatures $\varphi(x,t)$. The study of the initial-boundary value problem for equation (2) in Banach spaces associated with the abstract linear Sobolev type equation

$$
Lu_t = Mu + f,
$$

(3)

that’s why further we reduce the equation (2) to the usual form

$$
\lambda \varphi_t(x,t) - \varphi_{txx}(x,t) = \alpha \varphi_{xx}(x,t) + f(x,t), (x,t) \in [0,1] \times \mathbb{R}_+.
$$

(4)

Note also for the first time Wentzell boundary condition were considered in [3] in order to find diffusive processes for Markov processes homogeneous in time on the segment. Independently, these conditions were investigated in [4]. More general case were studied later in [5]. Namely, the domain belongs to $n$-dimensional Euclidean space which is a circle or a sphere, and semigroup is $C_0$-contracting and invariant under rotations. Further results of work [3] have been developed and generalized in many
works, for example, in papers [6, 7, 8, 9]. In particular, we refer to [10], since the structure of the space which we use is specified in there.

The purpose of this work is to research resolvability of problem (1) – (4) with Wentzell boundary conditions. The article contains two sections except introduction, conclusion, and the list of references. The relative spectrum of the operator Laplace with Wentzell boundary condition is found in the first section. The main results on resolvability of the Cauchy-Wentzell problem in the Heat Conduction model involving two temperatures are given in the second section.

2. Relative spectrum of the Laplace operator with Wentzell boundary conditions
Let us consider the differential operator
\[ A\varphi(x) = \varphi''(x), \quad x \in [0, 1] \]  
with general Wentzell boundary conditions
\[ A\varphi(0) + \alpha_0 \varphi'(0) + \alpha_1 \varphi(0) = 0, \]  
\[ A\varphi(1) + \beta_0 \varphi'(1) + \beta_1 \varphi(1) = 0. \]  
By formulas (5) – (7) we define the linear operator \( A: \text{dom} \ A \subset \mathcal{G} \to \mathcal{G} \). Here \( \mathcal{G} \) is a space \( (L^2[0, 1], dx|_{(0,1)} \oplus \eta ds|_{(0,1)}) \) with the norm
\[
\| \varphi \|_G^2 = \int_0^1 |\varphi(x)|^2 \, dx + \eta_0 |\varphi(0)|^2 + \eta_1 |\varphi(1)|^2,
\]
where \( dx \) is a Lebesgue measure on the segment \( (0, 1) \); \( ds \) is a point measure at the boundary; \( \eta_0 = \frac{1}{-\alpha_1}, \eta_1 = \frac{1}{\beta_1} \), where \( \alpha_1 < 0 < \beta_1 \), are positive weights. The full construction of the space \( \mathcal{G} \) is given in [10]. We consider also the linear manifold \( \text{dom} \ A = \{ \varphi \in C^2[0, 1]: \text{conditions (6), (7) are fulfilled} \} \) as the domain of the operator \( A \).

Lemma 1. Let the operator \( A \) be defined by formulas (5) – (7). Then

(i) \( \text{dom} \ A = \{ \varphi \in C^2[0, 1]: \text{conditions (6), (7) are fulfilled} \} \) is a Banach space with regard to the norm \( \| \varphi \|_{C^2[0,1]} \);

(ii) \( \text{dom} \ A \) is densely embedded in \( \mathcal{G} \);

(iii) \( A \in \mathcal{L}(\text{dom} \ A; \mathcal{G}) \).

Let us given an idea of the proof. Statement (i) is obviously, since \( \text{dom} \ A \) forms a subspace closed in \( C^2[0, 1] \). Statement (ii) obviously follows the fact that the operator of embedding \( \mathcal{G}: C^2[0, 1] \to \mathcal{G} \) is compact. Statement (iii) is obviously.

We consider the spectral problem for the operator \( A \) with general Wentzell boundary conditions. Prove the following theorem.

Theorem 1. Suppose that the operator \( A \) satisfies the conditions of Lemma 1. Then \( A \) has a real, discrete, finite multiplicity spectrum with the unique limit point at infinity.

Proof. It follows from [10] that the operator \( A \) on \( \mathcal{G} \) is essentially self-adjoint. This means that the spectrum of the operator \( A \) is real. Let us define the spectrum of the operator \( A \) and find its resolvent. We have \( (\lambda I - A) \varphi = f(x), x \in [0, 1] \) for \( f \in C^2[0, 1] \).

Consider the case of \( \lambda < 0 \). Solve the differential equation with general Wentzell boundary conditions by classical methods, and obtain the resolvent of the following form
\[
\varphi(x) = (\lambda I - A)^{-1} f = R_\lambda f = C_1 \cos(\sqrt{-\lambda} x) + C_2 \sin(\sqrt{-\lambda} x) + \int_0^x \frac{f(s)}{\sqrt{-\lambda}} \sin(\sqrt{-\lambda} (x-s)) \, ds.
\]
Write the coefficients \( C_1 = \frac{A_0}{B} \) and \( C_2 = \frac{A_1}{B} \) for the resolvent, where
\[
B = (\lambda + \alpha_1)(\lambda \sin(\sqrt{-\lambda}) + \beta_0 \sqrt{-\lambda} \cos(\sqrt{-\lambda}) + \beta_1 \sin(\sqrt{-\lambda})) - \alpha_0 \sqrt{-\lambda} (\lambda \cos(\sqrt{-\lambda}) - \beta_0 \sqrt{-\lambda} \sin(\sqrt{-\lambda}) + \beta_1 \cos(\sqrt{-\lambda})).
\]
\[ A_0 = f(0)\left( \lambda \sin(\sqrt{-\lambda}) + \beta_0 \sqrt{-\lambda} \cos(\sqrt{-\lambda}) + \beta_1 \sin(\sqrt{-\lambda}) \right) - \alpha_0 \sqrt{-\lambda} (f(1) - \int_0^1 \frac{f(s) \lambda}{\sqrt{-\lambda}} \sin(\sqrt{-\lambda} (1 - s)) \, ds - \beta_0 \int_0^1 f(s) \cos(\sqrt{-\lambda} (1 - s)) \, ds - \beta_1 \int_0^1 \frac{f(s)}{\sqrt{-\lambda}} \sin(\sqrt{-\lambda} (1 - s)) \, ds), \]

\[ A_1 = (\lambda + \alpha_3) (f(1) - \int_0^1 \frac{f(s) \lambda}{\sqrt{-\lambda}} \sin(\sqrt{-\lambda} (1 - s)) \, ds - \beta_0 \int_0^1 f(s) \cos(\sqrt{-\lambda} (1 - s)) \, ds - \beta_1 \int_0^1 \frac{f(s)}{\sqrt{-\lambda}} \sin(\sqrt{-\lambda} (1 - s)) \, ds) - f(0) (\lambda \cos(\sqrt{-\lambda}) - \beta_0 \sqrt{-\lambda} \sin(\sqrt{-\lambda}) + \beta_1 \cos(\sqrt{-\lambda})). \]

The resolvent operator \( R_2 \) is the sum of a two-dimensional operator (a linear combination of sine and cosine) and an integral operator of Hilbert-Schmidt type. A two–dimensional operator is finite-dimensional, and hence compact, since the coefficients \( C_1 \) and \( C_2 \) depend continuously on \( f \) in the metric of \( \mathcal{H} \). Hence, the operator \( R_2 = (\lambda \mathbb{I} - A)^{-1} \) is compact in \( \mathcal{H} \) as the sum of finite-dimensional and compact operators. By Hilbert’s theorem, \( R_2 \) has a discrete, finite multiplicity spectrum with the unique limit point at 0.

Let us show that the operator \( A \) has a discrete, finite multiplicity spectrum with the unique limit point at infinity. Fix an arbitrary eigenvalue \( \lambda_0 \) of the operator \( R_2 \) and express the eigenvalues of the operator \( A \) through the eigenvalues of the resolvent \( R_2 \). We obtain \( R_2 f = \lambda_0 f \), where \( f \) is the eigenvector of the resolvent. By acting with the operator \( (\lambda \mathbb{I} - A) \) on both parts of the equality and dividing by \( \lambda_0 \) (\( \lambda_0 \neq 0 \)), we get the expression

\[ Af = \left( \lambda \mathbb{I} - \frac{1}{\lambda_0} \right) f \]

which shows how the eigenvalues of the original and resolvent operators are related. Due to the behavior of the spectrum of the operator \( R_2 \), we proved that for \( \lambda < 0 \) the operator \( A \) has a discrete, finite multiplicity spectrum with the unique limit point at infinity.

Similarly, consideration of the case \( \lambda > 0 \) by the Sturm-Liouville method shows that the set of eigenvalues is finite or empty depending on the conditions on the coefficients in (6) – (7).

Consider the case \( \lambda = 0 \). Find sufficient conditions for the set of eigenvalues of the operator \( A \). Note that if the coefficients in (6) – (7) satisfy the equality

\[ \alpha_0 \beta_1 = \alpha_3 (\beta_0 + \beta_1), \]

then \( \lambda = 0 \) belongs to the set of eigenvalues of the operator \( A \). The theorem is proved.

The reduced equation of the Heat Conduction involving two temperatures

\[ \lambda \varphi_t(x, t) - \varphi_{txx}(x, t) = \alpha \varphi_{xx}(x, t) + f(x, t), (x, t) \in [0, 1] \times \mathbb{R}_+ \]

can be considered as a non-homogeneous Sobolev type equation \( Lu = Mu + f \), where the operators \( L = \lambda - A \in L(dom A; \mathcal{H}), M = \alpha A \in L(dom A; \mathcal{H}); \) the function \( f = f(x, t) \in C^2([0, 1] \times \mathbb{R}_+; \mathcal{H}) \).

In order to solve the Cauchy-Wentzell problem (8) – (9), we find the L-spectrum operator of \( M \). Since the \( L \)-resolvent of the operator \( M \) takes the form

\[ (\mu L - M)^{-1} = (\mu (\lambda - A) - \alpha A)^{-1} = (\mu + \alpha)^{-1} \left[ \frac{\mu \lambda}{\mu + \alpha} - A \right]^{-1} \]

with \( \mu + \alpha \neq 0 \), then \( \mu \) belongs to relative spectrum \( \sigma^L(M) \) iff

\[ \mu = \frac{\alpha \sigma(A)}{\lambda - \sigma(A)}. \]

Therefore, according to Theorem 1, with \( \mu + \alpha \neq 0 \), we have a discrete, finite L-spectrum \( \sigma^L(M) \) of the operator \( M \) with the limit point \( -\alpha \) at infinity.
Consider the case $\mu + \alpha = 0$. With $\lambda = 0$ we have $\sigma^L(M) = \{-\alpha\}$. With $\alpha \neq 0$, and $\sigma^L(M) = \{0\}$, if $\alpha = 0$. We described the $L$-spectrum of the operator $M$, getting the following corollary of Theorem 1.

Corollary 1. The $L$-spectrum of the operator $M$ in the Heat Conduction equation involving two temperatures with Wentzell boundary conditions is discrete, finite multiplicity, with the limit point $-\alpha$ at infinity.

3. The Cauchy-Wentzell problem in the Heat Conduction model involving two temperatures

Let us consider the Cauchy-Wentzell problem in the previously introduced space $\mathcal{G}$ on the segment $[0, 1]$ for the reduced equation of Heat Conduction involving two temperatures

$$\lambda \varphi_t(x, t) - \varphi_{xx}(x, t) + \alpha \varphi_x(x, t) + f(x, t), (x, t) \in [0, 1] \times \mathbb{R}_+.$$  

By Corollary 1, the operator $M$ is $L, \sigma$-bounded, therefore, the following theorem holds.

Theorem 2. Suppose that the operator $A$ satisfies the conditions of Lemma 1, and $f \in \mathcal{G}$ is a fixed vector. Then

(i) if $\lambda \notin \sigma(A)$, then for any $v_0 \in \text{dom } A$ and $f \in \mathcal{G}$ there exists the unique solution $\varphi \in \mathcal{C}^2(\mathbb{R}; \text{dom } A)$ to problem (8) – (9), which has the form

$$\varphi(x, t) = \sum_{k=1}^{\infty} \alpha_k \frac{e^{\alpha_k t} - 1}{\alpha_k^2} \varphi(x) + \sum_{k=1}^{\infty} \frac{\alpha_k}{\alpha_k^2} \varphi_k(x);$$

(ii) if $\lambda \in \sigma(A)$, then for any $f \in \mathcal{G}$ and $v_0 \in \mathcal{G}$ there exists the unique solution $\varphi \in \mathcal{C}^2(\mathbb{R}; \mathcal{G})$ to problem (8) – (9), which has the form

$$\varphi(x, t) = -\frac{1}{\alpha \lambda} \sum_{\lambda = \lambda_k}^{\infty} \varphi(x) + \sum_{\lambda \neq \lambda_k}^{\infty} \frac{\alpha_k}{\alpha_k^2} \varphi_k(x)$$

Proof. The proof of this theorem depends on the kernel of the operator $L$ and consists in applying either the classical theorem for a non-homogeneous differential operator equation, or Sviridyuk’s theorem.

Since by Theorem 1 the Laplace operator has a real, discrete, finite multiplicity spectrum having the limit point at $-\infty$, and $\{\lambda_k : k \in \mathbb{N}\}$ are eigenvalues of the Laplace operator, which are numbered in non-increasing order taking into account the multiplicity, and correspond to its eigenfunctions $\{\psi_k : k \in \mathbb{N}\}$. Then for $v \in \mathcal{G}$ we have

$$R_\mu(A)v = (\mu \mathbb{I} - A)^{-1}v = \sum_{k=1}^{\infty} \frac{\langle v, \psi_k \rangle}{\mu - \lambda_k} \psi_k(x)$$

and, therefore,

$$R_\mu^L(M)v = (\mu L - M)^{-1}v = \sum_{k=1}^{\infty} \frac{\langle v, \psi_k \rangle}{\mu(\lambda - \lambda_k) - \alpha \lambda_k} \psi_k$$

Termwise integration is admissible, since the series uniform convergences by the norm of the space $\text{dom } A$. Therefore, substituting the $L$-resolvent (10) of the operator $M$ and applying the residue theorem, we obtain the corresponding expressions (i), (ii).
4. Conclusion
We constructed the resolvent group in the Cauchy-Wenzell problem. To this end, we used the Sviridyuk’s theory, and the space, the structure of which is specified in [10]. Further, we plan to continue the results of the paper by applying the Wentzell boundary conditions in directions related to [11, 12].

Acknowledgments
This work was supported by Act 211 Government of the Russian Federation, contact no. 02.A03.21.0011

References
[1] Chen P J and Gurtin M E 1968 On a theory of heat conduction involving two temperatures Zeitschrift für Angewandte Mathematik und Physik Vol 19 614–627
[2] Gurtin M E and Williams W O 1967 An axiomatic foundation for continuum thermodynamics Archive for Rational Mechanics and Analysis Vol 26 pp 83–117
[3] Wentzell A D 1956 Semigroups of operators corresponding to a generalized differential operator of second order Doklady Academii Nauk SSSR Vol 111 269–272 (In Russ.)
[4] Feller W 1957 Second order differential operators and their lateral conditions Illinois Journal of Mathematics Vol 1 4 459–504
[5] Wentzell A D 1959 On boundary conditions for multidimensional diffusion processes Theory of Probability and its Applications Vol 4 164–177
[6] Favini A, Goldstein G R, Goldstein J A and Romanelli S 2007 Classification of general Wentzell boundary conditions for fourth order operators in one space dimension Journal of Mathematical Analysis and Applications Vol 333 1 219–235
[7] Coclite G M, Favini A, Gal C G, Goldstein G.R., Goldstein J A, Obrecht E and Romanelli S 2009 The role of Wentzell boundary conditions in linear and nonlinear analysis Advances in Nonlinear Analysis : Theory, Methods and Applications Vol 3 279–292
[8] Gal C G 2005 Sturm-Liouville operator with general boundary conditions Electronic Journal of Differential Equations Vol 2005 120 1–17
[9] Favini A, Goldstein G R and Goldstein J A 2003 The Laplacian with generalized Wentzell boundary conditions Progress in Nonlinear Differential Equations and Their Applications Vol 55 169–180
[10] Favini A, Goldstein G R, Goldstein J A and Romanelli S 2002 The Heat Equation with generalized Wentzell boundary condition Journal of Evolution Equations Vol 2 1–19
[11] Favini A, Zagrebin S A and Sviridyuk G A 2018 Multipoint initial-final value problems for dynamical Sobolev-type equations in the space of “Noises” Vol 2018 128 1–10
[12] Kitaeva O G, Shafranov D E and Sviridyuk G A 2019 Exponential dichotomies in Barenblatt--Zheltov-Kochina model in spaces of differential forms with “Noise” Bulletin of the South Ural State University (Series: Mathematical Modeling, Programming and Computer Software) Vol 12 2 47–57