Scale-free quantitative unique continuation and equidistribution estimates for solutions of elliptic differential equations

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We consider elliptic differential operators on either the whole Euclidean space $\mathbb{R}^d$ or on subsets consisting of a cube $\Lambda_L$ of integer length $L$. For eigenfunctions of the operator, and more general solutions of elliptic differential equations, we derive several quantitative unique continuation results. The first result is of local nature and estimates the vanishing order of a solution. The second is a sampling result and compares the $L^2$-norm of a solution over a union of equidistributed $\delta$-balls in space with the $L^2$-norm on the entire space. In the case where the space $\mathbb{R}^d$ is replaced by a finite cube $\Lambda_L$ we derive similar estimates. A particular feature of our bound is that they are uniform as long as the coefficients of the operator are chosen from an appropriate ensemble, they are quantitative and explicit with respect to the radius $\delta$, they are $L$-independent and stable under small shifts of the $\delta$-balls. Our proof applies to second order terms which have slowly varying coefficients on the relevant length scale. The results can be also interpreted as special cases of uncertainty relations, observability estimates, or spectral estimates.

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1. Introduction

How much can the amplitude of a function oscillate over a domain? How unevenly can the mass of the function be concentrated on different regions in space? This is – phrased very sketchily – the question we study in this paper. The functions considered are solutions of elliptic partial differential inequalities in various domains. We are interested in a-priori estimates which do not depend on the individual function considered, and are uniform with respect to (certain) variations of the domain and the variable coefficients of the elliptic operator. We measure the oscillations of the amplitude in terms of local $L^2$-norms. There are three types of results:

(a) a quantitative unique continuation principle (or vanishing order estimate) for solutions of variable coefficient elliptic partial differential equations, or inequalities,

(b) a sampling theorem for solutions of variable coefficient elliptic partial differential inequalities on the entire Euclidean space $\mathbb{R}^d$, and

(c) an equidistribution theorem for solutions of variable coefficient elliptic partial differential inequalities on cubes in $\mathbb{R}^d$ of odd integer length $L$.

The first result concerns the vanishing order of a function and is in this sense local. However, as it is well known, global restrictions on the class of functions considered have a strong influence on the order of vanishing, cf. [DF88, TTV16].

The last two results (b) & (c) are closely related. The sampling theorem can be understood as a version of the equidistribution theorem in the case where the side length of the cube is infinite and thus it equals the whole of $\mathbb{R}^d$. Moreover, we derive similar estimates for linear combinations of (generalized) eigenfunctions of an elliptic partial differential operator, as long as the corresponding eigenvalues are sufficiently close together. Such functions obviously do not need to be solutions of a partial differential equation.

We were led to derive such estimates motivated by our previous studies of periodic or disordered physical systems modeled by partial differential operators. The coefficients of these operators are either periodic or stochastically homogeneous. In [RMV13] and [Kle13], see also [NTTV15, NTTV] for more general statements, such results have been derived for Schrödinger operators with periodic, quasiperiodic or random potentials. In this context they are a versatile tool for spectral analysis and can be exploited to establish Anderson localization for certain random models where this was not possible before. However, bounds of the type (a), (b), and (c) above appear also in other contexts. For instance in quantum ergodicity one is interested in delocalization and equidistribution properties of eigenfunctions [BML, AM15, Zel92, BSSP03], in control theory observability estimates and spectral inequalities play an important role, e.g. to estimate the control cost [LR95, RL12], on manifolds one studies the vanishing order of eigenfunctions of the Laplace-Beltrami or a Schrödinger operator, cf. [DF88, JL99, Kuk98, Bak13]. Finally, since our theorems can be viewed as scale-uniform quantitative uncertainty principles for certain low dimensional subspaces, there is a relation to uniform uncertainty principles in compressive sensing as well. We have no space to elucidate and dwell on these relations here, but refer to the survey paper [TTV16] for a detailed discussion.

For a result which is already established for the Laplace operator one might wonder whether there is a straightforward extension to variable coefficient elliptic partial differential operators. Indeed, for the questions at hand, if only the zero order term (interpreted as the potential) contains variable coefficients one can accommodate even local singularities, as demonstrated in
While we can use the proof strategies of [RMV13] and [Kle13], the key tool, namely a Carleman estimate which holds for the Laplacian (or a Schrödinger operator) does not hold verbatim for variable coefficient operators. If one is striving for an optimal type of a Carleman estimate in the sense of [EV03] or [BK05] one cannot use simply the Carleman weight function of the Laplacian for other elliptic partial differential operators. Rather, depending on Lipschitz and ellipticity constants of the variable coefficients, one has to choose an adapted weight function. This has been observed in [EV03] and quantitatively implemented in [NRT]. The latter refinement turns out to be crucial for the application in this note. This leads to the condition in our theorems, that the coefficients are only allowed to vary slowly on the length scale which is determined by the equidistributed set. One way to satisfy this condition is to choose a dense sampling rate, another one to choose the Lipschitz constant of the coefficients in the partial differential equation sufficiently small. It seems that with the existing Carleman estimates it is not possible to derive better results, cf. Remark 10 for more details.

The rest of the paper is organized in the following way: To illustrate our new results we resort to a comparison, and first consider the more transparent, but simpler case of Schrödinger equations. In the following section we state our new results, which are divided in two groups according to items (b), and (c) above. In §3.1 we spell out the main tool of our proof, namely a quantitative unique principle. It corresponds to item (a) in the list above. The remainder of Section 3 contains the proofs of the theorems in Section 2, whereas some technical aspects are deferred to an appendix. On the technical level an interesting part of the paper will be the Benchmark: Schrödinger operators

*Remark 8 and 10, where we discuss the innovations and limitations of our theorems and our approach.*

**Benchmark: Schrödinger operators**

The new results in the present paper are best understood when compared to what was recently established for the special case of (stationary) Schrödinger equations. In this case only the zero order term is variable and the results are simpler to formulate, which we do next. For $L > 0$ we denote by $\Lambda_L = (-L/2,L/2)^d \subset \mathbb{R}^d$ the cube with side length $L$, and by $\Delta_L$ the Laplace operator on $L^2(\Lambda_L)$ subject to either Dirichlet, Neumann, or periodic boundary conditions. Moreover, for a measurable and bounded $V : \mathbb{R}^d \to \mathbb{R}$ we denote by $V_L : \Lambda_L \to \mathbb{R}$ its restriction to $\Lambda_L$ given by $V_L(x) = V(x)$ for $x \in \Lambda_L$, and by $h_L = -\Delta_L + V_L \quad \text{on} \quad L^2(\Lambda_L)$ the corresponding Schrödinger operator. For $\Omega \subset \mathbb{R}^d$ open and $\psi \in L^2(\Omega)$ we denote by $\|\psi\| = \|\psi\|_\Omega$ the usual $L^2$-norm of $\psi$. If $\Gamma \subset \Omega$ we use the notation $\|\psi\|_\Gamma = \|\chi_\Gamma \psi\|_\Omega$. Moreover, we denote by $B(\rho) \subset \mathbb{R}^d$ the open ball in $\mathbb{R}^d$ with radius $\rho > 0$ and center zero, by $\overline{B}(x,\rho) \subset \mathbb{R}^d$ the open ball in $\mathbb{R}^d$ with radius $\rho > 0$ and center $x \in \mathbb{R}^d$.

**Definition 1.** Let $G > 0$ and $\delta > 0$. We say that a sequence $z_j \in \mathbb{R}^d$, $j \in (G\mathbb{Z})^d$, is $(G,\delta)$-equidistributed, if

$$\forall j \in (G\mathbb{Z})^d: \quad B(z_j,\delta) \subset \Lambda_G + j.$$  

Corresponding to a $(G,\delta)$-equidistributed sequence $z_j \in \mathbb{R}^d$, $j \in (G\mathbb{Z})^d$, we define for $L > 0$ the sets

$$S_\delta = \bigcup_{j \in (G\mathbb{Z})^d} B(z_j,\delta) \subset \mathbb{R}^d \quad \text{and} \quad S_{\delta,L} = \bigcup_{j \in (G\mathbb{Z})^d} B(z_j,\delta) \cap \Lambda_L \subset \Lambda_L.$$
see Fig. 1 for an illustration. Note that the sets $S_δ, S_δ,L$ depend on $G$ and the choice of the $(G, δ)$-equidistributed sequence.

![Image of Fig 1](image)

Figure 1: Illustration of $S_δ, S_δ,L ⊂ Λ ⊂ R^2$ for periodically (left) and non-periodically (right) arranged $δ$-equidistributed sequences.

**Theorem 2 ([NTTV15, NTTV]).** There is a constant $N = N(d)$, such that for all $G > 0$, all $δ ∈ (0, G/2)$, all $(G, δ)$-equidistributed sequences, all measurable and bounded $V : R^d → R$, all $L ∈ G\mathbb{N}$, all $E_0 ≥ 0$ and all $φ ∈ \text{Ran}(χ_{(−∞, E_0]}(h_L))$ we have

$$\|φ\|_{S_δ,L}^2 ≥ C^G_{\text{sfUC}}\|φ\|_{Λ_L}^2,$$

where

$$C^G_{\text{sfUC}} = C^G_{\text{sfUC}}(d, δ, b, \|V\|_\infty) := \left(\frac{δ}{G}\right)^{N\left(1+G^{4/3}\|V\|_\infty^{2/3}+G\sqrt{E_0}\right)}.$$

This extends previous results of [CHK07], [RMV13], [Kle13]. We denote by $Δ : W^{2,2}(R^d) → L^2(R^d)$ the Laplace operator on $R^d$.

**Theorem 3 ([TV16]).** There is a constant $N = N(d)$, such that for all $E_0, G > 0$, all $δ ∈ (0, G/2)$, all $(G, δ)$-equidistributed sequences, all measurable and bounded $V : R^d → R$, and all intervals $I ⊂ (−∞, E_0]$ with

$$|I| ≤ 2\gamma \quad \text{where} \quad \gamma^2 = \frac{1}{2G^4}\left(\frac{δ}{G}\right)^{N\left(1+G^{4/3}\|V\|_\infty+G\sqrt{E_0}\right)^{2/3}},$$

and all $ψ ∈ \text{Ran}(χ_I(−Δ + V))$ we have

$$\|ψ\|_{S_δ}^2 ≥ G^4\gamma^2\|ψ\|_{R^d}^2.$$

This is an adaptation of the main theorem of [Kle13] to the space $R^d$. Theorem 3 covers only short energy intervals $I$. An extension of this result to the case of arbitrary compact intervals $I$ is not immediate by looking at the proof of [NTTV15, NTTV], but using a generalized eigenfunction expansion it is likely that such an extension can be proven, which leads to the following
Conjecture 4. There is a constant $N = N(d)$, such that for all $E_0, G > 0$, all $\delta \in (0, G/2)$, all $(G, \delta)$-equidistributed sequences, all measurable and bounded $V : \mathbb{R}^d \to \mathbb{R}$, all $E_0 > 0$, and all $\phi \in \operatorname{Ran}(\chi_{(-\infty, E_0]}(-\Delta + V))$ we have

$$\|\phi\|^2_{S_\delta} \geq C_{\text{stUC}} \|\phi\|^2_{\mathbb{R}^d}$$

with $C_{\text{stUC}}$ as above.

2. Main results

Now we turn to the class of models which is treated in the theorems and proofs of the present paper. Let $d \in \mathbb{N}$ and $\mathcal{H}$ be the second order partial differential expression

$$\mathcal{H}u := -\operatorname{div}(A \nabla u) + b^T \nabla u + cu = -\sum_{i,j=1}^d \partial_i (a^{ij} \partial_j u) + \sum_{i=1}^d b_i \partial_i u + cu,$$

where $A : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ with $A = (a^{ij})_{i,j=1}^d$, $b : \mathbb{R}^d \to \mathbb{C}^d$, $c : \mathbb{R}^d \to \mathbb{C}$, and $\partial_i$ denotes the $i$-th weak derivative. We assume that $a^{ij} \equiv a^{ji}$ for all $i, j \in \{1, \ldots, d\}$, and that there are constants $\vartheta_1 \geq 1$ and $\vartheta_2 \geq 0$ such that for almost all $x, y \in \mathbb{R}^d$ and all $\xi \in \mathbb{R}^d$ we have

$$\vartheta_1^{-1} |\xi|^2 \leq \xi^T A(x) \xi \leq \vartheta_1 |\xi|^2 \quad \text{and} \quad \|A(x) - A(y)\|_\infty \leq \vartheta_2 |x - y|. \quad (1)$$

Moreover, we assume that $b, c \in L^\infty(\mathbb{R}^d)$. Here we denote by $|z|$ the Euclidean norm of $z \in \mathbb{C}^d$, and by $\|M\|_\infty$ the row sum norm of a matrix $M \in \mathbb{C}^{d \times d}$.

2.1. Sampling theorems on $\mathbb{R}^d$

Theorem 5 (Sampling Theorem). Assume

$$\varepsilon_1 := 1 - 33d(\sqrt{d} + 2)\vartheta_1^6 \vartheta_2 > 0. \quad (2)$$

Then for all measurable and bounded $V : \mathbb{R}^d \to \mathbb{R}$, all $\psi \in W^{2,2}(\mathbb{R}^d)$ and $\zeta \in L^2(\mathbb{R}^d)$ satisfying $|\mathcal{H}\psi| \leq |V\psi| + |\zeta|$ almost everywhere on $\mathbb{R}^d$, all $\delta \in (0, 1/2)$ and all $(1, \delta)$-equidistributed sequences we have

$$\|\psi\|^2_{S_\delta} + \delta^2 \|\zeta\|^2_{\mathbb{R}^d} \geq C_{\text{stUC}} \|\psi\|^2_{\mathbb{R}^d},$$

where

$$C_{\text{stUC}} = d_1 \left(\frac{\delta}{d_2}\right)^{d_3} \left(\frac{d_1}{d_2} + \frac{\|V\|^2_{L^2} + \|b\|^2_{L^\infty} + \|c\|^2_{L^\infty}}{d_2}\right) \ln \varepsilon_1$$

with

$$d_1 = K_2 \vartheta_1^{-31/2 - d} e^{-10\vartheta_1}, \quad d_2 = K_2 \vartheta_1^2, \quad \text{and} \quad d_3 = K_2 \vartheta_1^{25} e^{15\vartheta_1} (1 + \vartheta_2)^2,$$

where $K_2$ is a positive constant depending only on the dimension.

By scaling, see Appendix C, we obtain the following variant of Theorem 5 for $(G, \delta)$-equidistributed sequences.
Theorem 6. Let $G > 0$ and assume
\[\varepsilon_2 := 1 - 33\varepsilon d(\sqrt{d} + 2)\partial d^0 G \partial_2 > 0.\] (3)

Then for all measurable and bounded $V : \mathbb{R}^d \to \mathbb{R}$, all $\psi \in W^{2,2}(\mathbb{R}^d)$ and $\zeta \in L^2(\mathbb{R}^d)$ satisfying $|\mathcal{H}\psi| \leq |V\psi| + |\zeta|$ almost everywhere on $\mathbb{R}^d$, all $\delta \in (0, G/2)$ and all $(G, \delta)$-equidistributed sequences we have
\[\|\psi\|_{S_\delta}^2 + \delta^2 G^2 \|\zeta\|_{\mathbb{R}^d}^2 \geq C_{\text{sfUC}} \|\psi\|_{\mathbb{R}^d}^2,\]
where
\[C_{\text{sfUC}} = D_1 \left(\frac{\delta}{GD_2}\right)^{\frac{\partial_2}{2}} \left(1 + G^4\|V\|_{L^\infty}^2 + G^2\|b\|_{L^\infty}^2 + G^4\|\epsilon\|_{L^\infty}^2\right)^{-\ln \varepsilon_2}\]
with
\[D_1 = \frac{K_2}{(1 + G\partial_2)^2} e^{-10\partial_1}, \quad D_2 = K_2 \partial_1^2, \quad \text{and} \quad D_3 = K_2 \partial_1^{25} e^{15\partial_1} (1 + G\partial_2)^2,\]
where $K_2$ is a positive constant depending only on the dimension.

Obviously, $\varepsilon_1$ in (2) equals $\varepsilon_2$ with $G = 1$. In the same way, $C_{\text{sfUC}}$, $d_1$, $d_2$, and $d_3$ coincide with $C_{\text{sfUC}}$, $D_1$, $D_2$, and $D_3$ in the case $G = 1$.

Remark 7. Let us discuss condition (2) and (3), respectively. Since the ellipticity constant $\partial_1$ is at least one, it is required that the product $G \cdot \partial_2$ of the length scale and Lipschitz constant be small. (If the scale is set to $G = 1$ it means that $\partial_2$ should be small. Compare also condition (5) in Theorem 16.) This means that the variation of the coefficients of the second order term should be sufficiently small on the relevant scale $G$. Coinciding with physical intuition, it is possible to satisfy assumption (2) by increasing the sampling rate appropriately, i.e. by choosing the cubes $\Lambda_G + j$, which define the $(G, \delta)$-equidistribution property, sufficiently small.

A condition like (2) or (5) naturally appears in the literature on unique continuation. To satisfy it, authors usually assume that the radius/scale $R$, respectively $G$, is sufficiently small. To prove the unique continuation property this is no restriction, since then only small balls are of interest.

Remark 8 (Scale free unique continuation constant $C_{\text{sfUC}}$). To appreciate the theorem properly one wants to understand how the constant $C_{\text{sfUC}}$ depends on the model parameters. First of all, we see that it is polynomial in the small radius $\delta$, and that the exponent exhibited in (4) is an estimate on the vanishing order, e.g. as studied in [DF88, Kuk98]. Second, one sees that the estimate is uniform as the potential $V$ or the coefficients $c$ and $b$ vary over ensembles with uniformly bounded $L^\infty$-norm. No regularity properties, as encoded in Sobolev or total variation norms, play a role here. This is of importance, e.g., if one wants to derive results for random operators, where one is dealing not with one operator, but a whole family of them, and aims at uniform bounds. See [RMV13, Kle13, BK13, NTTV15, NTTV] for recent applications of this type. Note that the dependence of the exponent in $C_{\text{sfUC}}$ is at worst quadratic with respect to $\|b\|_{L^\infty}$, $\|c\|_{L^\infty}$, $\|V\|_{L^\infty}$. Thirdly, we see that Theorems 11 to 14 hold for all odd $L \in \mathbb{N}$ with a uniform constant $C_{\text{sfUC}}$ independent of $L$. This is actually the reason why we call it scale-free unique continuation constant. Fourthly, note that the bound is stable under small shifts of the $\delta$-balls inside the periodicity cells. All that needs to be satisfied is the geometric equidistribution property for the $\delta$-balls. This is crucial, if one is considering a model which is ideally periodic, but one wants to make sure that results do not break down if, more realistically,
small deviations from the ideal lattice structure are allowed. Then there is the dependence on the parameter $\varepsilon_2 > 0$ which measures the distance to the critical threshold value of zero. This is not a natural model parameter, but a quantity which reflects the limitations of our Carleman estimate approach. Finally, there are the model parameters $\vartheta_1$ and $\vartheta_2$. One sees that while the dependence of the exponent in $C_{\text{sUC}}$ on $\vartheta_2$ is still of polynomial nature, the parameter $\vartheta_1$, i.e. the ellipticity constant, enters in an exponential way. In Theorem 5.1 where the vanishing rate depends in a quadratic way on the sup-norm of the potential and in an exponential way on the coefficient functions of the second order part.

A particular case where the assumption $|\mathcal{H}\psi| \leq |V\psi| + |c|$ in Theorem 6 is satisfied, is the case of an eigenfunction $\psi$. More generally, we formulate a corollary of Theorem 6 for functions in the range of some spectral projector of a self-adjoint realization of the differential expression $\mathcal{H}$. We introduce the following assumption on the coefficient functions $b$ and $c$.

(SA): We have $b = \tilde{b}i$ and $c = \tilde{c} + \text{div} \tilde{b}/2$ for some bounded $\tilde{b}, \tilde{c} \in L^\infty(\mathbb{R}^d)$.

We define the differential operator $H : W^{2,2}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, $H\psi = \mathcal{H}\psi$. If assumption (SA) is satisfied, then $H$ is a self-adjoint operator in $L^2(\mathbb{R}^d)$.

**Theorem 9.** Let $G > 0$ and assume (SA) and Ineq. (3). Then for all $E \in \mathbb{R}$, all $\delta \in (0, G/2)$, all $(G, \delta)$-equidistributed sequences, and all $\psi \in \text{Ran } \chi_{[E-\gamma, E+\gamma]}(H)$ with

$$\gamma^2 = \frac{D_1}{G^4} \left( \frac{\delta}{GD_2} \right)^{\frac{D_3}{2}} \left[ 1 + G^{4/3} |E|^{2/3} + G^2 \|b\|_\infty^2 + G^4 \|c\|_\infty^{2/3} \right] - \ln \varepsilon_2$$

we have

$$\|\psi\|_{S_2}^2 \geq C_{\text{sUC}} \|\psi\|_{\mathbb{R}^d}^2 \quad \text{with} \quad C_{\text{sUC}} = D_1 \left( \frac{\delta}{GD_2} \right)^{\frac{D_3}{2}} \left[ 1 + G^{4/3} |E|^{2/3} + G^2 \|b\|_\infty^2 + G^4 \|c\|_\infty^{2/3} \right] - \ln \varepsilon_2$$

where $D_1$, $D_2$ and $D_3$ are given in Theorem 6.

**Remark 10** (Relation between the condition on the parameters $\vartheta_1$, $\vartheta_2$ and the constants $C_{\text{sUC}}$). It is natural to wonder whether the restriction (3), respectively (5), is necessary to derive a quantitative unique continuation estimate as it is expressed in Theorem 16 and Lemma 18, i.e. where we have a powerlike vanishing behavior and where the bound on vanishing order is a polynomial function of $\|b\|_\infty$, $\|c\|_\infty$, $\|V\|_\infty$, it seems that we have to use a Carleman weight function as in [NRT], see Theorem 20. In that case the restriction on the Lipschitz and ellipticity constant enters naturally.

Of course it would be possible to use weight functions which are not of polynomial type. We believe the that in this case it would be possible to remove the conditions on the slow variation of the coefficients (5), respectively (3). However, it is unclear whether we could obtain with this approach quantities $C_{\text{qUC}}$ in Lemma 18 and $C_{\text{sUC}}$ in the subsequent theorems where in the exponent appear only linear and quadratic expressions of $\|V\|_\infty$, $\|b\|_\infty$ and $\|c\|_\infty$. We plan to investigate this approach in a sequel paper. Finally, one could try to find approaches different from Carleman estimates to obtain estimates on the vanishing order. In dimension one and two there should be enough alternative tools to achieve this goal.
2.2. Equidistribution theorems on $\Lambda_L$

In this section we will consider functions $\psi \in W^{2,2}(\Lambda_L)$ and $\zeta \in L^2(\Lambda_L)$, $L > 0$, satisfying $|\mathcal{H}\psi| \leq |V\psi| + |\zeta|$ almost everywhere on $\Lambda_L$. In order to define appropriate extensions of such functions we will assume that $\psi$ satisfies Dirichlet or periodic boundary conditions on the sides of $\Lambda_L$, and denote by $\mathcal{D}(\Delta^\text{Dir}_L)$ and $\mathcal{D}(\Delta^\text{per}_L)$ the domain of the Laplace operator on $\Lambda_L$ subject to Dirichlet or periodic boundary conditions, respectively. Moreover, depending on the boundary conditions, we introduce for $L > 0$ the following assumptions on the coefficients $a^{ij}$, $i,j \in \{1, \ldots, d\}$.

**E**

| \begin{array}{l}
| \text{(Dir)} \quad \text{For all } i,j \in \{1, \ldots, d\} \text{ with } i \neq j \text{ the coefficient function } a^{ij} \text{ vanishes on the sides of } \\
\Lambda_L.
| \text{(Per)} \quad \text{For all } i,j \in \{1, \ldots, d\} \text{ the coefficient function } a^{ij} \text{ satisfies periodic boundary conditions } \\
\text{on the sides of } \Lambda_L.
| \end{array} |

**Theorem 11** (Equidistribution Theorem). Let $L, G > 0$ and assume (3), $L/G \in \mathbb{N}$ is odd, and (Per). Then for all measurable and bounded $V : \Lambda_L \to \mathbb{R}$, all $\psi \in \mathcal{D}(\Delta^\text{per}_L)$ and $\zeta \in L^2(\Lambda_L)$ satisfying $|\mathcal{H}\psi| \leq |V\psi| + |\zeta|$ almost everywhere on $\Lambda_L$, all $\delta \in (0, G/2)$ and all $(G, \delta)$-equidistributed sequences we have

$$
\|\psi\|_{S_{\delta,L}}^2 + \delta^2 G^2 \|\zeta\|_{L^2}^2 \geq C_{\text{sfUC}} \|\psi\|_{\Lambda_L}^2,
$$

where $C_{\text{sfUC}} = C_{\text{sfUC}}(d, G, \delta, \vartheta_1, \vartheta_2, \|V\|_\infty, \|b\|_\infty, \|c\|_\infty)$ is given in Theorem 6.

**Theorem 12** (Equidistribution Theorem bis). Let $L, G > 0$ and assume (3), $L/G \in \mathbb{N}$ is odd, and (Dir). Then for all measurable and bounded $V : \Lambda_L \to \mathbb{R}$, all $\psi \in \mathcal{D}(\Delta^\text{Dir}_L)$ and $\zeta \in L^2(\Lambda_L)$ satisfying $|\mathcal{H}\psi| \leq |V\psi| + |\zeta|$ almost everywhere on $\Lambda_L$, all $\delta \in (0, G/2)$ and all $(G, \delta)$-equidistributed sequences we have

$$
\|\psi\|_{S_{\delta,L}}^2 + \delta^2 G^2 \|\zeta\|_{L^2}^2 \geq C_{\text{sfUC}} \|\psi\|_{\Lambda_L}^2,
$$

where $C_{\text{sfUC}} = C_{\text{sfUC}}(d, G, \delta, \vartheta_1, \vartheta_2, \|V\|_\infty, \|b\|_\infty, \|c\|_\infty)$ is given in Theorem 6.

We define for $L > 0$ the differential operators $H^\text{per}_L : \mathcal{D}(\Delta^\text{per}_L) \to L^2(\Lambda_L)$ and $H^\text{Dir}_L : \mathcal{D}(\Delta^\text{Dir}_L) \to L^2(\Lambda_L)$ by $H^\text{per}_L \psi = \mathcal{H}\psi$ and $H^\text{Dir}_L \psi = \mathcal{H}\psi$. If assumption (SA) from Section 2.1 is satisfied, then $H^\text{per}_L$ and $H^\text{Dir}_L$ are self-adjoint operators in $L^2(\Lambda_L)$. Here are two analogs of Theorem 9 for operators defined on boxes $\Lambda_L$.

**Theorem 13.** Let $L, G > 0$ and assume (3), (SA), $L/G \in \mathbb{N}$ is odd, and (Per). Then for all $E \in \mathbb{R}$, all $\delta \in (0, G/2)$, all $(G, \delta)$-equidistributed sequences, and all $\psi \in \text{Ran } \chi_{[E-\gamma, E+\gamma]}(H^\text{per}_L)$ with

$$
\gamma^2 = \frac{D_1}{G^4} \left( \frac{\delta}{GD_2} \right)^{-\frac{1}{2}} \left( 1 + G^{4/3} |E|^{2/3} + G^2 \|b\|_\infty^2 + G^{4/3} \|c\|_\infty^{2/3} \right)^{-1} \ln \varepsilon_2
$$

we have

$$
\|\psi\|_{S_{\delta,L}}^2 \geq \frac{C_{\text{sfUC}}}{2} \|\psi\|_{\Lambda_L}^2 \quad \text{with} \quad C_{\text{sfUC}} = D_1 \left( \frac{\delta}{GD_2} \right)^{-\frac{1}{2}} \left( 1 + G^{4/3} |E|^{2/3} + G^2 \|b\|_\infty^2 + G^{4/3} \|c\|_\infty^{2/3} \right)^{-1} \ln \varepsilon_2,
$$

where $D_1$, $D_2$ and $D_3$ are given in Theorem 6.
Theorem 14. Let $L, G > 0$ and assume $(3)$, $(SA)$, $L/G \in \mathbb{N}$ is odd, and $(\text{Dir})$. Then for all $E \in \mathbb{R}$, all $\delta \in (0, G/2)$, all $(G, \delta)$-equidistributed sequences, and all $\psi \in \text{Ran} \chi_{[-E, E]}(H_L^\text{Dir})$ with

$$
\gamma^2 = \frac{D_1}{G^4} \left( \Delta \frac{\delta}{GD_2} \right) \left( 1 + G^{4/3} |E|^{2/3} + G^2 \|b\|_\infty^2 + G^4/3 \|c\|_\infty^{2/3} \right) - \ln \varepsilon_2
$$

we have

$$
\|\psi\|^2_{S_{\delta,L}} \geq \frac{C_\text{qUC}}{2} \|\psi\|^2_{\Lambda_L} \quad \text{with} \quad C_\text{qUC} = D_1 \left( \Delta \frac{\delta}{GD_2} \right) \left( 1 + G^{4/3} |E|^{2/3} + G^2 \|b\|_\infty^2 + G^4/3 \|c\|_\infty^{2/3} \right) - \ln \varepsilon_2,
$$

where $D_1$, $D_2$, and $D_3$ are given in Theorem 6.

Remark 15 (Additional boundary conditions for the coefficients of $A$). In the equidistribution theorem there appear special boundary conditions on the coefficient matrix $A : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ which do not feature in the sampling theorem. The reason is, that in the proof we need to extend the differential equation and its solution to a larger set, in order to avoid dealing with regions near the boundary of the domain. It is a general principle that in such regions estimates on gradients and other derivatives are harder to obtain than in the interior. (In fact, we extend the equation to the whole of $\mathbb{R}^d$ for simplicity.) Now the extension from $\Lambda_L$ to $\mathbb{R}^d$ is done with a simple mirroring procedure, where we have to match all of the relevant interior and exterior derivatives. For a Laplace operator this is always possible. In the case of an elliptic differential operator with mixed derivatives present, this leads, in general, to an overdetermined system of linear equations. The conditions (Dir) and (Per) we impose eliminate some of the equations, so that under these conditions a solution to system of linear equations always exists. A more flexible way to extend the solutions outside of the domain $\Lambda_L$ could possibly allow the boundary restrictions (Dir), respectively (Per), to be lifted. However, we do not regard this as the primary challenge to improve our results in this paper, but subordinate to the question, whether one can allow coefficients with arbitrary large Lipschitz constants.

3. Proofs

3.1. Quantitative unique continuation

The first step in our proofs is the following quantitative unique continuation principle which we formulate next.

Theorem 16 (Quantitative unique continuation theorem). Let $D_0, R \in (0, \infty)$, $\delta \in (0, 2R)$ $K_V, \beta \in [0, \infty)$, and assume

$$
\varepsilon_0 := 1 - 33edR \delta_1 \delta_2 > 0.
$$

Then there is a constant $C_\text{qUC} = C_\text{qUC}(d, \delta_1, \delta_2, R, D_0, K_V, \|b\|_\infty, \|c\|_\infty, \delta, \beta) > 0$, such that for any $\Omega \subset \mathbb{R}^d$ open, $x \in \Omega$ and $\Theta \subset \Omega$ measurable and satisfying

$$
\Theta \subset \overline{B(x, R)} \setminus B(x, \delta/2) \quad \text{and} \quad B(x, 2e\delta_1 R + 2D_0) \subset \Omega,
$$

any measurable $V : \Omega \to [-K_V, K_V]$, any $\zeta \in L^2(\Omega)$ and $\psi \in W^{2,2}(\Omega)$ satisfying the differential inequality

$$
|\mathcal{H}\psi| \leq |V\psi| + |\zeta| \quad \text{a.e. on} \quad \Omega \quad \text{as well as} \quad \frac{\|\psi\|^2_{L^2}}{\|\psi\|^2_{H^1}} \leq \beta,
$$

\[\text{where } D_1, D_2, D_3 \text{ are given in Theorem 6.} \]
we have
\[ \|\psi\|_{B(x, \delta)}^2 + \delta^2 \|\zeta\|_{\Omega}^2 \geq C_{qUC} \|\psi\|_{\Theta}^2. \]

Note that \( \Theta \) and \( B(x, \delta) \) may overlap. The constant \( C_{qUC} \) is given explicitly in Eq. (17).

Remark 17. Inspired by \[BK05\] several related quantitative unique continuation principles have been proved in the literature on (random) Schrödinger operators, see \[GK13\], \[BK13\], \[RMV13\]. The application of these estimates are manifold, including a Wegner estimate for alloy type Schrödinger operators with small support, the log Hölder continuity of the integrated density of states for general Schrödinger operators, and localization on various energy/disorder regimes. Our result is an extension of these results to variable coefficient divergence type operators.

The inequalities are loosely related to so called three circle annuli inequalities as they are often used in the literature on control theory and harmonic analysis on compact manifolds, see for instance \[RL12, Bak13\].

In Appendix B we give an estimate on \( C_{qUC} \) under several assumptions on the parameters. This is formulated in the following lemma.

Lemma 18. Let \( 2D_0 = R \geq 1, \delta < 2 \) and \( \varepsilon_0 > 0 \). Then,
\[ C_{qUC} \geq C_1 \left( \frac{\delta}{C_2 R} \right)^{\frac{C_2}{\Theta}} \left( 1 + \|V\|_{\infty}^2 + \|b\|_{\infty}^2 + \|c\|_{\infty}^2 \right) \ln \varepsilon_0 + \ln \beta, \]
with
\[ C_1 = \frac{K_1 \psi_1^{-31/2} e^{-10\psi_1}}{(1 + \psi_2)(\vartheta_1 + \vartheta_2)}, \quad C_2 = 10e\psi_1^2, \quad \text{and} \quad C_3 = K_1 \psi_1^{25} e^{15\psi_1} (1 + \vartheta_2)^2, \]
where \( K_1 \) is a positive constant depending only on the dimension.

The explicit form of the constant \( C_{qUC} \) may appear complicated and, indeed, merits a detailed discussion. Among others it gives an estimate on the vanishing order of \( \psi \). Since \( C_{qUC} \) shares its important features with the constant \( C_{sfUC} \) appearing in the next theorem, we will not discuss \( C_{qUC} \) separately but refer to Remark 8.

Now the proof of Theorems 5 to 14 follow.

3.2. Carleman estimate and Cacciopoli inequality

We start with a formulation of the the Cacciopoli inequality, which may be found in \[RMV13\] in the case where \( \mathcal{H} = -\Delta \).

Lemma 19 (Cacciopoli inequality). Let \( \Omega \subset \mathbb{R}^d \) be open, \( V : \Omega \rightarrow \mathbb{R} \) bounded and measurable, \( \zeta \in L^2(\Omega) \), \( \psi \in W^{2,2}(\Omega) \) satisfying \( |\mathcal{H}\psi| \leq |V\psi| + |\zeta| \) almost everywhere on \( \Omega \), \( 0 \leq r_1 < r_2 \), \( r \in (0, \infty) \), \( S = B(r_2) \setminus B(r_1) \), \( S^+ = B(r_2 + r) \setminus B(r_1 - r) \) and assume \( S^+ \subset \Omega \). Then there is an absolute constant \( C' \geq 1 \) such that
\[ \int_S \nabla \psi^T A \nabla \psi \leq \left( 2\|V\|_{\infty}^2 + 1 + 2\|b\|_{\infty}^2 + \frac{8\vartheta_1^2 C'}{r_2^2} + 2\|c\|_{\infty} \right) \int_{S^+} |\psi|^2 + 2 \int_{S^+} |\zeta|^2. \]
For the prefactors appearing on the right hand side we use the symbol \( F_r = F_r(V, b, c, \vartheta_1) \).
Proof. We use $0 \leq ||x-y||^2 = ||x||^2 + ||y||^2 - 2 \operatorname{Re}(x,y)$, Green’s theorem and Cauchy Schwarz, and obtain for all real-valued $\eta \in C_c^\infty(\Omega)$

$$\|\eta \mathcal{H}\psi\|^2 + \|\eta \psi\|^2 \geq 2 \operatorname{Re}(\langle \mathcal{H}\psi, \eta^2 \psi \rangle)$$

$$= 4 \operatorname{Re}(\eta \nabla \psi, A \psi \nabla \eta) + 2 \nabla \psi, \eta^2 A \nabla \psi \rangle + 2 \operatorname{Re}(\langle b^T \nabla \psi, \eta^2 \psi \rangle + 2 \operatorname{Re}(c \psi, \eta^2 \psi)$$

$$\geq 2 \nabla \psi, \eta^2 A \nabla \psi \rangle - 4 \|\eta \nabla \psi\| \|\psi A \nabla \eta\| - 2 \|\eta b^T \nabla \psi\| \|\eta \psi\| + 2 \operatorname{Re}(c \psi, \eta^2 \psi).$$

Since $2ab \leq sa^2 + s^{-1}b^2$, we have for all $s, t > 0$

$$\|\eta \mathcal{H}\psi\|^2 + \|\eta \psi\|^2 \geq - \frac{2}{s} \|\psi A \nabla \eta\|^2 + (2-2s) \nabla \psi, \eta^2 A \nabla \psi \rangle - \frac{1}{t} \|\eta b^T \nabla \psi\|^2 - t \|\eta \psi\|^2 + 2 \operatorname{Re}(c \psi, \eta^2 \psi).$$

We choose $s = 1/4$, $t = 2 \|b\|^2$, and obtain by using $\|\eta b^T \nabla \psi\|^2 \leq \|b\|^2 \|\eta \nabla \psi\|^2$

$$(\nabla \psi, \eta^2 A \nabla \psi \rangle \leq \|\eta \mathcal{H}\psi\|^2 + (1 + 2 \|b\|_\infty^2) \|\eta \psi\|^2 + 8 \|\psi A \nabla \eta\|^2 + 2 \|c\|_\infty \langle \psi, \eta^2 \psi \rangle.$$

(6)

Now we choose a radially symmetric function $\eta \in C_c^\infty(\Omega)$ with $\eta \in [0,1]$, $\operatorname{supp} \eta = S^+$, $\eta \equiv 1$ on $S$, and $|\nabla \eta| \leq \sqrt{C'/r}$ with some absolute constant $C' > 0$. Hence, by our assumption $|\mathcal{H}\psi| \leq |V\psi| + |\zeta|$, ellipticity, and our Lipschitz condition, the statement of the lemma follows from Ineq. (6).

Now we cite a Carleman estimate from [NRT]. A particular feature of this Carleman estimate is that the weight function and the corresponding constants are given explicitly. For us it is of particular importance to track the dependence on the ellipticity and Lipschitz constants $\vartheta_1$ and $\vartheta_2$, respectively. Carleman estimates of this type have been given before in [EV03, KSU11] and [BK05]. However, they did not derive the dependence on the weight function and the constants in the Carleman estimate on $\vartheta_1$ and $\vartheta_2$. Or they concerned only the pure Laplacian in the first place.

For $\mu, \rho > 0$ we introduce a function $w_{\rho,\mu} : \mathbb{R}^d \to [0, \infty)$ by

$$w_{\rho,\mu}(x) := \varphi(\sigma(x/\rho)),$$

where $\sigma : \mathbb{R}^d \to [0, \infty)$ and $\varphi : [0, \infty) \to [0, \infty)$ are given by

$$\sigma(x) := (x^TA_0^{-1}x)^{1/2}, \quad \text{and} \quad \varphi(r) := r \exp \left( - \int_0^r \frac{1-e^{-\mu t}}{t} dt \right).$$

Note that the function $w_{\rho,\mu}$ satisfies

$$\forall x \in B_\rho: \quad \frac{\vartheta_1^{1/2} |x|}{\rho \mu_1} \leq \frac{\sigma(x)}{\rho \mu_1} \leq w_{\rho,\mu}(x) \leq \frac{\sigma(x)}{\rho} \leq \frac{\sqrt{\vartheta_1}|x|}{\rho},$$

where

$$\mu_1 = \begin{cases} e^{\sqrt{\vartheta_1} \mu} & \text{if } \sqrt{\vartheta_1} \mu \leq 1, \\ e^{\sqrt{\vartheta_1}/\mu} & \text{if } \sqrt{\vartheta_1}/\mu > 1. \end{cases}$$

**Theorem 20 ([NRT]).** Let $\rho > 0$ and $\mu > 33d \vartheta_1^{1/2} \vartheta_2 \rho$. Then there are constants $\alpha_0 = \alpha_0(\rho, \vartheta_1, \vartheta_2, \mu, \|b\|_\infty, \|c\|_\infty) > 0$ and $C = C(\vartheta_1, \vartheta_2, \mu) > 0$, such that for all $\alpha \geq \alpha_0$ and all $u \in W^{2,2}(\mathbb{R}^d)$ with support in $B(\rho) \setminus \{0\}$ we have

$$\int_{\mathbb{R}^d} (\alpha \rho^2 w_{\rho,\mu}^{-1-2\alpha} \nabla u^T A \nabla u + \alpha^3 w_{\rho,\mu}^{-1-2\alpha} |u|^2) \leq C \rho^4 \int_{\mathbb{R}^d} w_{\rho,\mu}^{-2-2\alpha} |\mathcal{H}u|^2.$$
Remark 21. Upper bounds for the constants $C$ and $\alpha_0$ are known explicitly, see [NRT]. In the case where $b$ and $c$ are identically zero, the conclusion of Theorem 20 holds with $C = \hat{C}$ and $\alpha_0 = \hat{\alpha}_0$ satisfying the upper bounds

$$\hat{C} \leq 2d^2 \vartheta_1^6 e^{4\mu\sqrt{\bar{\nu}}} \mu_1^4 (3\mu^2 + (9\rho\vartheta_2 + 3)\mu + 1) \mu^{-1}$$

and

$$\hat{\alpha}_0 \leq 11d^4 \vartheta_1^{33/2} e^{6\mu\sqrt{\bar{\nu}}} \mu_1^6 (3\rho\vartheta_2 + \mu + 1)^2 (1 + \mu(\mu + 1)\mu^{-1}),$$

where $C_{\mu} = \mu - 33d\vartheta_1^{11/2}\vartheta_2\rho$. In the general case where $b, c \in L^\infty(B_\rho)$ the conclusion of the theorem holds with

$$C = 6\hat{C} \quad \text{and} \quad \alpha_0 = \max \left\{ \hat{\alpha}_0, C\rho^2 \|b\|_\infty^2 \vartheta_1^{3/2} + C^{1/3} \rho^{4/3} \|c\|_\infty^{2/3} \vartheta_1^2 \right\}.$$

3.3. Proof of Theorem 16

Proof of Theorem 16. We follow [RMV13, Proof of Theorem 3.1]. For convenience we assume $x = 0$, hence $\Omega \supset B(2e\vartheta_1R + 2D_0)$. The general case follows by translation. We choose a function $\eta : \mathbb{R}^d \to [0, 1]$, $\eta \in C_c^\infty(\Omega)$, depending only on $|x|$ satisfying

$$\eta(x) = \begin{cases} 0 & \text{if } x \in B(\delta/4) \cup B(2e\vartheta_1R + D_0)^c, \\ 1 & \text{if } x \in B(2e\vartheta_1R) \setminus B(\delta/2), \end{cases}$$

and

$$\max (\|\nabla \eta\|_\infty, \|\Delta \eta\|_\infty) \leq \begin{cases} (M/\delta)^2 & \text{if } x \in B(\delta/2) \setminus B(\delta/4), \\ (M/D_0)^2 & \text{if } x \in B(2e\vartheta_1R + D_0) \setminus B(2e\vartheta_1R), \end{cases} \quad (7)$$

where $M = M(d) \in (0, \infty)$ is a constant depending only on the dimension. See Fig. 2 for an illustration of the geometric setting. Note that $\eta \equiv 1$ on $\Theta$ by assumption. Recall that $\varepsilon_0 = 1 - 33eRd\vartheta_1^{11/2}\vartheta_2$ and set

$$\rho := 2e\vartheta_1R + 2D_0 \quad \text{and} \quad \mu := 33d\rho\vartheta_1^{11/2}\vartheta_2 + \frac{\rho\varepsilon_0}{2eR\sqrt{\vartheta_1}},$$

Since $\varepsilon_0 > 0$ by assumption, we have $\mu > 33d\rho\vartheta_1^{11/2}\vartheta_2$. Hence we can apply the Carleman estimate from Theorem 20 with these choices of $\rho$ and $\mu$ to the function $u = \eta \psi$ and obtain for all $\alpha \geq \alpha_0 = \alpha_0(d, \rho, \vartheta_1, \vartheta_2, \mu)$

$$I_1 := \int_{B(\rho)} \alpha^3 w^{-1-2\alpha} |\eta \psi|^2 \leq \rho^4 C \int_{B(\rho)} w^{2-2\alpha} |\mathcal{H}(\eta \psi)|^2,$$

where $C = C(d, \vartheta_1, \rho\vartheta_2, \mu) > 0$ and $w = w_{\rho, \mu}$. Recall that $w$ satisfies

$$\forall x \in B_\rho: \quad \frac{\vartheta_1^{-1/2} |x|}{\rho \mu_1} \leq \frac{\sigma(x)}{\rho \mu_1} \leq w_{\rho, \mu}(x) \leq \frac{\sigma(x)}{\rho} \leq \frac{\sqrt{\vartheta_1} |x|}{\rho}. \quad (8)$$
The Leibniz rule and \((a + b + c)^2 \leq 3(a^2 + b^2 + c^2)\) yields that \(I_1\) is bounded by

\[
I_1 = \rho^4 C \int_{B(\rho)} w^{2-2\alpha} \left| \mathcal{H}_c \eta \psi + (\mathcal{H}_\psi \eta) \eta + 2 \sum_{i,j=1}^{d} a^{ij} (\partial_i \eta) (\partial_j \psi) \right|^2 \\
\leq 3\rho^4 C \int_{B(\rho)} w^{2-2\alpha} \left( |\mathcal{H}_c \eta|^2 |\psi|^2 + |\mathcal{H}_\psi \eta|^2 |\eta|^2 + 4 \left| \sum_{i,j=1}^{d} a^{ij} (\partial_i \eta) (\partial_j \psi) \right|^2 \right),
\]

where \(\mathcal{H}_c \eta = - \text{div}(A \nabla \eta) + b^T \nabla \eta\). The pointwise estimate \(|\mathcal{H}\psi| \leq |V\psi| + |\zeta|\), \(\|V\|_{\infty} \leq K_V\), and \(w \leq \sqrt{\frac{1}{\Omega}}\) on \(B(\rho)\) gives

\[
\int_{B(\rho)} w^{2-2\alpha} |\mathcal{H}_\psi|^2 \eta^2 \leq 2K_V^2 \frac{\delta_{1/3}'}{\delta_{1/3}} \int_{B(\rho)} w^{2-2\alpha} |\eta|^2 + 2 \int_{B(\rho)} w^{2-2\alpha} |\eta\zeta|^2.
\]

From Ineq. (9) and Ineq. (10) we obtain for all \(\alpha \geq \alpha_0\)

\[
\left[ \frac{\alpha^3}{3\rho^4 C} - 2K_V^2 \frac{\delta_{1/3}'}{\delta_{1/3}} \right] \int_{B(\rho)} (w^{-1-2\alpha} |\eta\psi|^2) \\
\leq \int_{B(\rho)} w^{2-2\alpha} \left( |\mathcal{H}_c \eta|^2 |\psi|^2 + 4 \left| \sum_{i,j=1}^{d} a^{ij} (\partial_i \eta) (\partial_j \psi) \right|^2 + 2|\eta\zeta|^2 \right) =: I_2 + 2 \int_{B(\rho)} w^{2-2\alpha}|\eta\zeta|^2.
\]

Note that

\[
2 \int_{B(\rho)} w^{2-2\alpha}|\eta\zeta|^2 \leq 2 \left( \frac{4\rho\lambda_1 \sqrt{\delta_{1/3}}}{\delta} \right)^{2\alpha-2} \|\zeta\|_{\Omega}^2,
\]

Figure 2: Cutoff function \(\eta\) and geometric setting of Theorem 16. Here, \(R_2 = 2e\theta_1 R\) and \(R_3 = 2e\theta_1 R + 2D_0 = \rho\).
which will be used later. Additionally to \( \alpha \geq \alpha_0 \) we choose
\[
\alpha \geq \sqrt[3]{16\rho^4CK^2V^3/2} =: \alpha_1.
\]

This ensures that
\[
I_2 + 2 \int_{B(\rho)} w^{2-2\alpha} |\eta \zeta|^2 \geq \frac{5}{24} \rho^4 C \int_{B(\rho)} (w^{-1-2\alpha} |\eta\psi|^2).
\]

Since \( \eta \equiv 1 \) on \( \Theta \) and by our bound on the weight function we have
\[
I_2 + 2 \int_{B(\rho)} w^{2-2\alpha} |\eta \zeta|^2 \geq \frac{5}{24} \rho^4 C \left( \frac{\rho}{\sqrt{\partial_1 R}} \right)^{1+2\alpha} \|\psi\|^2_{\Theta}.
\]

(12)

Now we turn to an upper bound on \( I_2 \). Since \( a^{ij} = a^{ji} \) and \( A(x) = (a^{ij}(x))_{i,j=1}^d \) is positive definite for almost all \( x \in B(\rho) \), we can apply Cauchy Schwarz and obtain
\[
\left| \sum_{i,j=1}^d a^{ij}(\partial \eta)(\partial \psi) \right|^2 \leq \left( \sum_{i,j=1}^d a^{ij}(\partial \eta)(\partial \eta) \right) \left( \sum_{i,j=1}^d a^{ij}(\partial \eta)(\partial \psi) \right) \leq \vartheta_1 |\nabla \eta|^2 (\nabla \psi^T A \nabla \psi).
\]

Since \( \mathcal{H}_c \eta \neq 0 \) only on \( \text{supp} \nabla \eta \) we have
\[
I_2 \leq \int_{\text{supp} \nabla \eta} w^{2-2\alpha} \left( |\mathcal{H}_c \eta|^2 |\psi|^2 + 4\vartheta_1 |\nabla \eta|^2 (\nabla \psi^T A \nabla \psi) \right).
\]

We split the integral according to the two components
\[
B_1 = \left\{ x \in \mathbb{R}^d : \frac{\delta}{4} \leq |x| \leq \frac{\delta}{2} \right\} \quad \text{and} \quad B_2 = \left\{ x \in \mathbb{R}^d : 2e\vartheta_1 R \leq |x| \leq 2e\vartheta_1 R + D_0 \right\}
\]

of \( \text{supp} \nabla \eta \) and obtain by using the property (7) of the function \( \eta \)
\[
I_2 \leq \int_{B_1} w^{2-2\alpha} \left( (\mathcal{H}_c \eta)^2 |\psi|^2 + 4\vartheta_1 \left( \frac{M}{\delta} \right)^4 \nabla \psi^T A \nabla \psi \right)
\]

\[\quad + \int_{B_2} w^{2-2\alpha} \left( (\mathcal{H}_c \eta)^2 |\psi|^2 + 4\vartheta_1 \left( \frac{M}{D_0} \right)^4 \nabla \psi^T A \nabla \psi \right).
\]

On \( B_1 \) we use the general bound (8) on the weight function \( w \). In order to estimate the weight function on \( B_2 \), we note that for \( r \geq 1/\mu \) we have by using \( 1 - e^{-\mu t} \leq \mu t \) and \( 1 - e^{-\mu t} < 1 \) the bound
\[
\varphi(r) \geq r \exp \left( -\int_0^{1/\mu} \mu dt \right) \exp \left( -\int_0^r \frac{1}{t} dt \right) = \frac{1}{e\mu}.
\]

Since \( \sigma(x) \geq \vartheta_1^{-1/2} |x| \), this implies that for all \( x \in B_\rho \) with \( |x| \geq \sqrt{\vartheta_1 \rho}/\mu \) we have
\[
w(x) = \varphi(\sigma(x/\rho)) \geq \frac{1}{e\mu}.
\]

(13)

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Since $2e\vartheta_1 R \geq \sqrt{\vartheta_1} \rho / \mu$ by our choice of $\mu$, we can use the bound (13) for all $x \in B_2$. Hence, for $\alpha \geq 1 =: \alpha_2$ we arrive at

\[
I_2 \leq \left( \frac{4\rho \vartheta_1 \sqrt{\vartheta_1}}{\delta} \right)^{2\alpha - 2} \left( \frac{M}{\delta} \right)^4 \int_{B_1} \left( (\mathcal{H}_c \eta)^2 |\psi|^2 + 4\vartheta_1 \left( \frac{M}{\delta} \right)^4 \nabla \psi^T A \nabla \psi \right)
+ (e\mu)^{2\alpha - 2} \int_{B_2} \left( (\mathcal{H}_c \eta)^2 |\psi|^2 + 4\vartheta_1 \left( \frac{M}{D_0} \right)^4 \nabla \psi^T A \nabla \psi \right).
\]

Now we use the pointwise estimate

\[
|\mathcal{H}_c \eta|^2 \leq 3\vartheta_1^2 |\Delta \eta|^2 + 3\vartheta_1^2 (2d - 1)^2 \frac{|\nabla \eta|^2}{|x|^2} + 3(\vartheta_2 d^2 + \|b\|_\infty^2) |\nabla \eta|^2,
\]

see Appendix A, and obtain again by using the property (7) of the function $\eta$ that $I_2$ is bounded from above by

\[
\left( \frac{4\rho \vartheta_1 \sqrt{\vartheta_1}}{\delta} \right)^{2\alpha - 2} \left( \frac{M}{\delta} \right)^4 \int_{B_1} \left[ 3\vartheta_1^2 + \frac{768\vartheta_1^2 d^2}{\delta^2} + 3(\vartheta_2 d^2 + \|b\|_\infty^2) |\psi|^2 + 4\vartheta_1 \nabla \psi^T A \nabla \psi \right]
+ (e\mu)^{2\alpha - 2} \left( \frac{M}{D_0} \right)^4 \int_{B_2} \left[ 3\vartheta_1^2 + \frac{12\vartheta_1^2 d^2}{(2e\vartheta_1 R)^2} + 3(\vartheta_2 d^2 + \|b\|_\infty^2) \nabla \psi^T A \nabla \psi \right].
\]

An application of Lemma 19 with $r_1 = \delta/4$, $r_2 = \delta/2$ and $r = \delta/2$ for the first summand and $r_1 = 2e\vartheta_1 R$, $r_2 = 2e\vartheta_1 R + D_0$ and $r = D_0/2$ for the second summand gives by using $D_0 \geq \delta$ and $2e\vartheta_1 R \geq \delta$

\[
I_2 \leq \left( \frac{4\rho \vartheta_1 \sqrt{\vartheta_1}}{\delta} \right)^{2\alpha - 2} \left( \frac{M}{\delta} \right)^4 \left[ 3\vartheta_1^2 + \frac{768\vartheta_1^2 d^2}{\delta^2} + 3(\vartheta_2 d^2 + \|b\|_\infty^2) + 4\vartheta_1 F_{\delta/2} \right] \|\psi\|_{B(\delta)}^2
+ (e\mu)^{2\alpha - 2} \left( \frac{M}{D_0} \right)^4 \left[ 3\vartheta_1^2 + \frac{12\vartheta_1^2 d^2}{(2e\vartheta_1 R)^2} + 3(\vartheta_2 d^2 + \|b\|_\infty^2) + 4\vartheta_1 F_{D_0/2} \right] \|\psi\|_\Omega^2
+ \left( \frac{4\rho \vartheta_1 \sqrt{\vartheta_1}}{\delta} \right)^{2\alpha - 2} \left( \frac{M}{\delta} \right)^4 16\vartheta_1 \|\zeta\|_\Omega^2.
\]

where $F_t = F_t(V, b, c, \vartheta_1)$, $t > 0$, is defined in Lemma 19. Next we want to use

\[
(e\mu)^{2\alpha - 2} \left( \frac{M}{D_0} \right)^4 \left[ 3\vartheta_1^2 + \frac{768\vartheta_1^2 d^2}{(2e\vartheta_1 R)^2} + 3(\vartheta_2 d^2 + \|b\|_\infty^2) + 4\vartheta_1 F_{D_0/2} \right] \|\psi\|_\Omega^2 \leq \frac{3}{24} \frac{\alpha^2}{\rho \sqrt{\vartheta_1 R}} \left( \frac{\rho}{\sqrt{\vartheta_1 R}} \right)^{1 + 2\alpha} \|\psi\|_{B(\delta)}^2.
\]

by choosing $\alpha$ sufficiently large. Since $\|\psi\|_\Omega^2 / \|\psi\|_{B(\delta)}^2 \leq \beta$, this is satisfied if

\[
\alpha^3 \left( \frac{\rho}{\sqrt{\vartheta_1 R \mu}} \right)^{2\alpha} \geq \frac{8\rho^3 \sqrt{\vartheta_1 R} \beta}{\vartheta_1^2 \mu^2} \left( \frac{M}{D_0} \right)^4 \left[ 3\vartheta_1^2 + \frac{768\vartheta_1^2 d^2}{(2e\vartheta_1 R)^2} + 3(\vartheta_2 d^2 + \|b\|_\infty^2) + 4\vartheta_1 F_{D_0/2} \right].
\]

(15)
Since we want to verify Ineq. (15) by choosing \( \alpha \) sufficiently large, we now argue that 
\[
\frac{\rho}{(\sqrt{\vartheta_1}eR\mu)} > 1.
\]
Indeed, by our assumption \( \varepsilon_0 = 1 - 33ed\vartheta_1^2eR > 0 \) we have
\[
\mu = 33d\vartheta_1^{11/2}\vartheta_2\rho + \frac{\rho\varepsilon_0}{2\sqrt{\vartheta_1}eR} = \frac{33}{2}d\vartheta_1^{11/2}\vartheta_2\rho + \frac{\rho}{2\sqrt{\vartheta_1}eR} < \frac{\rho}{\sqrt{\vartheta_1}eR}.
\]
Hence,
\[
\frac{\rho}{\sqrt{\vartheta_1}eR\mu} > 1,
\]
which ensures that there exists \( \alpha_3 \) such an Ineq. (15) is satisfied for all \( \alpha \geq \max\{1, \alpha_3\} \). A possible choice of \( \alpha_3 \) is
\[
\alpha_3 := \left( 2\ln\left( \frac{\rho}{\sqrt{\vartheta_1}eR\mu} \right) \right)^{-1} \ln \left( \frac{8C\rho^3\sqrt{\vartheta_1}R\beta}{e^2\mu^2} \right)^4 \left( \frac{M}{D_0} \right)^{4} \times \left[ 3d_1^2 + \frac{3\vartheta_2^2d^2}{(2\varepsilon_0^2R)^2} + 3(\vartheta_2 d^2 + \|b\|_\infty)^2 + 4\vartheta_1 F_{D_0/2} \right].
\]
Hence we can subsume the second summand of the right hand side of Ineq. (14) into the lower bound of Ineq. (12) and obtain by using Ineq. (11) and \( M \geq 1 \) for all \( \alpha \geq \max\{\alpha_0, \alpha_1, \alpha_2, \alpha_3\} \)
\[
\frac{1}{12} C^2 \rho^3 \left( \frac{\rho}{\sqrt{\vartheta_1}R} \right)^{1+2\alpha} \|\psi\|_B^2 \leq \left( 4\rho\mu_1 \sqrt{\vartheta_1} \right)^{2\alpha - 2} \left( \frac{M}{\delta} \right)^4 \left[ 3d_1^2 + \frac{768\vartheta_2^2d^2}{\delta^2} + 3(\vartheta_2 d^2 + \|b\|_\infty)^2 + 4\vartheta_1 F_{\delta/2} \right] \|\psi\|_{B(\delta)}^2 \leq \left( 4\rho\mu_1 \sqrt{\vartheta_1} \right)^{2\alpha - 2} \left( \frac{M}{\delta} \right)^4 \left( 18d_1^2 \right) \|\zeta\|_{B(\delta)}^2.
\]
Now we use the lower bound \( \alpha_3 \geq 1 \) and
\[
\frac{18d_1^2}{3\vartheta_1^2 + \frac{768\vartheta_2^2d^2}{\delta^2} + 3(\vartheta_2 d^2 + \|b\|_\infty)^2 + 4\vartheta_1 F_{\delta/2}} \leq \delta^2,
\]
and obtain the statement of the theorem with
\[
C_{q\text{UC}} := \frac{4\mu_1^2 \sqrt{\vartheta_1} \delta^2 (3R\rho CM^2)^{-1}}{3\vartheta_1^2 + 768\vartheta_2^2d^2\delta^{-2} + 3(\vartheta_2 d^2 + \|b\|_\infty)^2 + 4\vartheta_1 F_{\delta/2}} \left( \frac{\delta}{4\mu_1 \sqrt{\vartheta_1}R} \right)^{2\alpha^*}
\]
where \( \alpha^* := \max\{\alpha_0, \alpha_1, \alpha_2, \alpha_3\} \).

\[\Box\]

### 3.4. Proof of Theorem 5 and Theorem 9

We follow [RMV13, Proof of Theorem 2.1]. For \( L > 0 \) and \( x \in \mathbb{R}^d \) we denote by \( \Lambda_L(x) = (-L/2, L/2)^d + x \) the cube of side length \( 2L \) centered at \( x \), and for \( a \in \mathbb{R} \) by \( |a| \) the smallest integer larger than or equal to \( a \).

**Proof of Theorem 5.** Fix \( \psi \in W^{2,2}(\mathbb{R}^d) \) and \( \zeta \in L^2(\mathbb{R}^d) \). We say that a site \( k \in \mathbb{Z}^d \) is dominating if
\[
\int_{\Lambda_1(k)} |\psi|^2 \geq \frac{1}{2T^d} \int_{\Lambda_T(k)} |\psi|^2 \quad \text{with} \quad T = \left[ 2(\sqrt{a} + 2) (2\varepsilon_0 + 1) \right],
\]
where
and otherwise weak. We denote by $W \subset \mathbb{R}^d$ the union of unit cubes centered at weak sites and by $D \subset \mathbb{R}^d$ the union of unit cubes centered at dominating sites. Then

$$
\int_W |\psi|^2 = \sum_{k \in \mathbb{Z}^d, k \text{ is weak}} \int_{\Lambda_1(k)} |\psi|^2 < \frac{1}{2T^d} \sum_{k \in \mathbb{Z}^d, k \text{ is weak}} \int_{\Lambda_T(k)} |\psi|^2
$$

$$
\leq \frac{1}{2T^d} \sum_{k \in \mathbb{Z}^d} \int_{\Lambda_T(k)} |\psi|^2 \leq \frac{T^d}{2T^d} \int_{\mathbb{R}^d} |\psi|^2 = \frac{1}{2} \int_{\mathbb{R}^d} |\psi|^2.
$$

(19)

Since $D$ is the complement of $W$ in $\mathbb{R}^d$, we have

$$
2 \int_D |\psi|^2 > \int_{\mathbb{R}^d} |\psi|^2.
$$

(20)

For a dominating site $k \in \mathbb{Z}^d$ we define its right near-neighbor by

$$
k^+ = k + 2e_1 \quad \text{where} \quad e_1 = (1, 0, \ldots, 0)^T \in \mathbb{R}^d.
$$

For each dominating site $k \in \mathbb{Z}^d$ we want to apply Theorem 16 with

$$
\beta = 2T^d, \quad R = \sqrt{d} + 2, \quad D_0 = R_2, \quad \Omega = \Lambda_T(k), \quad x = z_{k^+} \quad \text{and} \quad \Theta = \Lambda_1(k).
$$

Therefore, we have to check whether the assumptions of Theorem 16 are satisfied for these specific choices. Assumption (5) of Theorem 16 is satisfied by Assumption (2). Note that $\Theta = \Lambda_1(k)$ is obviously disjoint from the open ball $B(z_{k^+}, \delta/2)$, and there exists $a \in \Lambda_1(k)$ with $|a - z_{k^+}| = 2$. Thus, for each $b \in \Lambda_1(k)$ we have $|b - z_{k^+}| \leq |b - a| + |a - z_{k^+}| = \sqrt{d} + 2 = R$. Hence, $\Theta \subset B(z_{k^+}, R) \setminus B(z_{k^+}, \delta/2)$. In order to verify $B(x, 2e_1R + 2D_0) \subset \Omega$ we note that for each $y \in B(z_{k^+}, 2e_1R + 2D_0)$ we have

$$
|k - y| \leq |k - k^+| + |k^+ - z_{k^+}| + |z_{k^+} - y| < 2 + \sqrt{d}/2 + (a + 1)R \leq T/2,
$$

and therefore, $B(z_{k^+}, 2e_1R + 2D_0) \subset B(k, T/2) \subset \Lambda_T(k)$. Finally, $\|\psi\|_\Omega^2 / \|\psi\|_\Theta^2 \leq \beta$ since $k$ is dominating. Thus, for every dominating site $k$ Theorem 16 gives

$$
\|\psi\|_B(z_{k^+}, \delta)^2 + \delta^2 \|\zeta\|_\Omega^2 \geq C_{\text{qUC}} \|\psi\|_{\Lambda_1(k)}^2
$$

where $C_{\text{qUC}}$ is given in Theorem 16 with the above choices of the parameters. Since $B(z_{k^+}, \delta) \subset \Lambda_1(k)$ for all $k \in \mathbb{Z}^d$, we obtain by summing over all dominating $k \in \mathbb{Z}^d$

$$
\|\psi\|_S^2 + T^d \delta^2 \|\zeta\|_{\mathbb{R}^d}^2 \geq C_{\text{qUC}} \|\psi\|_{D_0}^2 > \frac{C_{\text{qUC}}}{2} \|\psi\|_{\mathbb{R}^d}^2.
$$

From Lemma 18 we infer a lower bound on $C_{\text{qUC}}/(2T^d)$ leading to the stated constant $c_{\text{UUC}}$. \qed

The proof of Theorem 6 is postponed to Appendix C.

**Proof of Theorem 9.** We follow [Kle13, Proof of Theorem 1.1]. Set $V \equiv E$ and $\zeta = (H - E)\psi$. Then the assumption $|H\psi| \leq |E\psi| + ||(H - E)\psi||$ of Theorem 6 is satisfied by the triangle inequality. By using $||H - E)||\psi||^2 \leq \gamma^2 \|\psi\|^2$, we obtain the inequality

$$
C_{\text{UUC}} \|\psi\|_{\mathbb{R}^d}^2 \leq \|\psi\|_S^2 + \delta^2 G^2 ||(H - E)\psi||_{\mathbb{R}^d}^2 \leq \|\psi\|_S^2 + \delta^2 G^2 \gamma^2 \|\psi\|_{\mathbb{R}^d}^2.
$$

The result follows, since $\delta < G/2$ and $C_{\text{UUC}} - \delta^2 G^2 \gamma^2 \geq (3/4)C_{\text{UUC}}$. \qed
3.5. Proof of Theorem 11 to 14

Proof of Theorem 11 and 12. Here we follow the main lines of the proof of Theorem 5 up to certain minor changes. This is due to the fact that the function $\psi$ is defined just on $\Lambda_L$ instead of $\mathbb{R}^d$. We assume $G = 1$. The general case $G > 0$ follows by scaling, similar as explained in Appendix C. We denote by $\mathcal{H}_L$ the differential expression restricted to $\Lambda_L$, i.e. the coefficients $a_{ij} : \Lambda_L \to \mathbb{R}$, $b : \Lambda_L \to \mathbb{R}^d$ and $c : \Lambda_L \to \mathbb{C}$ are restricted on $\Lambda_L$. Our assumption on $\psi$ is equivalent to $|\mathcal{H}_L \psi| \leq |V \psi| + |\zeta|$ almost everywhere on $\Lambda_L$. We want to extend $\psi$, $V$, $\zeta$ and the coefficients of $\mathcal{H}_L$ in such a way, that the same inequality holds almost everywhere on $\mathbb{R}^d$.

In the case of Theorem 11, i.e. periodic boundary conditions, we extend the function $\psi$ $L$-periodically in each direction. Namely, let $e_p = (0, \ldots, 0, 1, 0, \ldots, 0)^T \in \mathbb{R}^d$, $p = 1, \ldots, d$, be the standard basis in $\mathbb{R}^d$, then

$$\psi(x + Le_p) = \psi(x).$$

In the same way we extend all the coefficients of $\mathcal{H}_L$, the potential $V$ and the function $\zeta$.

In the case of Theorem 12, i.e. Dirichlet boundary conditions, the extensions are different and made by symmetric and antisymmetric reflections with respect to the sides of $\Lambda_L$. Namely, assume that functions $\psi$, $V$, $\zeta$, $a_{ij}$, $b_i$, $c$ are defined on $\Lambda_L(m)$, $m = (m_1, \ldots, m_d) \in L\mathbb{Z}^d$. Then the above functions are extended on the neighboring boxes $\Lambda_L(m \pm Le_p)$, $p = 1, \ldots, d$, as follows:

$$\psi(x \pm Le_p) = -\psi(x + 2(mpL - x_p)e_p),$$

$$a_{ij}(x \pm Le_p) = a_{ij}(x + 2(mpL - x_p)e_p) \quad \text{if} \quad i \neq p, \quad j \neq p,$$

$$a_{pp}(x \pm Le_p) = a_{pp}(x + 2(mpL - x_p)e_p),$$

$$a_{pj}(x \pm Le_p) = a_{pj}(x \pm Le_p) = -a_{pj}(x + 2(mpL - x_p)e_p) \quad \text{if} \quad p \neq j,$$

$$b_i(x \pm Le_p) = -b_i(x + 2(mpL - x_p)e_p) \quad \text{if} \quad i \neq p,$$

$$b_p(x \pm Le_p) = b_p(x + 2(mpL - x_p)e_p),$$

$$c(x \pm Le_p) = c(x + 2(mpL - x_p)e_p),$$

$$V(x \pm Le_p) = V(x + 2(mpL - x_p)e_p),$$

$$\zeta(x \pm Le_p) = \zeta(x + 2(mpL - x_p)e_p).$$

Due to the assumption (Per) or (Dir) and the corresponding boundary condition of $\psi$, the extended $\psi$ is locally in $W^{2,2}(\mathbb{R}^d)$, satisfies $|\mathcal{H}_L \psi| \leq |V \psi| + |\zeta|$ almost everywhere on $\mathbb{R}^d$, and the coefficients of $\mathcal{H}$ satisfy the ellipticity and Lipschitz condition (1). Moreover, by construction of our extension of $\psi$ we have

$$\sum_{k \in \Lambda_L \cap \mathbb{Z}^d} \int_{\Lambda_T(k)} |\psi|^2 = T^d \int_{\Lambda_L} |\psi|^2. \quad (21)$$

Now we proceed as in the proof of Theorem 5. We define dominating sites as in (18). Instead of $W$, we introduce set $W(L)$ as the union of all unit cubes centered at weak sites $k$ located inside $\Lambda_L$. By $D(L)$ we denote the union of unit cubes centered at the dominating sites located inside $\Lambda_L$. Then employing (21), in the same way as in (19) we get

$$\int_{W(L)} |\psi|^2 = \sum_{k \in \mathbb{Z}^d \cap \Lambda_L \text{, } \text{ } k \text{ is weak}} \int_{\Lambda_1(k)} |\psi|^2 \leq \frac{1}{2T^d} \sum_{k \in \Lambda_L \cap \mathbb{Z}^d, \text{ } \text{ } k \text{ is weak}} \int_{\Lambda_T(k)} |\psi|^2 \leq \frac{1}{2T^d} \sum_{k \in \Lambda_L \cap \mathbb{Z}^d} \int_{\Lambda_T(k)} |\psi|^2 = \frac{1}{2} \int_{\Lambda_L} |\psi|^2.$$

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Hence, we arrive at the analogue of estimate (20):

$$2 \int_{D(L)} |\psi|^2 > \int_{\Lambda_L} |\psi|^2.$$  

Now all other arguments in the proof of Theorem 5 are reproduced literally and we obtain for all dominating sites $k \in \Lambda_L \cap \mathbb{Z}^d$

$$\|\psi\|_{B(\delta +, \delta)} + \delta^2 \|\zeta\|^2_{\Lambda_T(k)} \geq C_{qUC}\|\psi\|^2_{\Lambda_1(k)},$$

where $C_{qUC}$ is the constant from Theorem 16 with $\beta = 2T^d$, $R = \sqrt{d} + 2$, and $D_0 = R/2$. By summing over all dominating sites $k \in \Lambda_L \cap \mathbb{Z}^d$ and thanks to the oddness of $L$ and (21) we obtain

$$\|\psi\|_{\mathbb{S}_{\delta,L}}^2 + \delta^2 T^d \|\zeta\|^2_{\Lambda_T(k)} \geq \sum_{k \in \mathbb{Z}^d \cap \Lambda_L} \left( \|\psi\|^2_{B(\delta +, \delta)} + \delta^2 \|\zeta\|^2_{\Lambda_T(k)} \right) \geq C_{qUC}\|\psi\|^2_{D(L)} > \frac{C_{qUC}}{2}\|\psi\|^2_{\Lambda_L}.$$

From Lemma 18 we infer a lower bound on $C_{qUC}/(2T^d)$ leading to the stated constant $C_{\mu UC}$. □

Proof of Theorem 13 and 14. We follow [Kle13, Proof of Theorem 1.1]. Depending on the boundary condition $\bullet \in \{\text{per, Dir}\}$, we set $V \equiv E$ and $\zeta = (H^* \psi - E)\psi$. Then the assumption $|H\psi| \leq |E\psi| + |\zeta|$ of Theorem 11 and 12 is satisfied by the triangle inequality. By using $\|H^* - E\psi\|^2 \leq \gamma^2 \|\psi\|^2$, we obtain the inequality

$$C_{\mu UC}\|\psi\|^2_{\Lambda_L} \leq \|\psi\|^2_{\mathbb{S}_{\delta,L}} + (\delta/G)^2 G^2 \|H - E\psi\|^2_{\Lambda_1} \leq \|\psi\|^2_{\mathbb{S}_{\delta}} + (\delta/G)^2 G^2 \gamma^2 \|\psi\|^2_{\Lambda_L}.$$  

The result follows, since $\delta < G/2$ and $C_{\mu UC} - \delta^2 G^2 \gamma^2 \geq (3/4)C_{\mu UC}$. □

A. An estimate for rotational symmetric functions

Let $\eta \in C^\infty_c(\mathbb{R})$, where $\eta = \zeta \circ \sigma$ with $\sigma(x) = |x|$ and $\zeta : \mathbb{R} \to [0,1]$ some profile function. We want to estimate $\langle \mathcal{H}\eta \rangle^2$ pointwise. For the first and second derivatives we have

$$\frac{\partial \eta}{\partial x_i} = \zeta'(\sigma) \frac{x_i}{|x|}, \quad \frac{\partial^2 \eta}{\partial x_i \partial x_j} = \zeta''(\sigma) \frac{x_i x_j}{|x|^2} + \zeta'(\sigma) \left( \frac{\delta_{ij}}{|x|} - \frac{x_i x_j}{|x|^3} \right).$$

For the gradient and the Laplacian of $\eta$ there holds

$$|\nabla \eta| = |\zeta'(\sigma)| \quad \text{and} \quad \Delta \eta = \zeta''(\sigma) + \zeta'(\sigma) \frac{d-1}{|x|}.$$  

First we estimate $|\mathcal{H}_0 \eta| := |- \text{div}(A \nabla \eta)|$:

$$|\mathcal{H}_0 \eta| \lesssim \sum_{i,j} a_{ij} \left( \zeta''(\sigma) \frac{x_i x_j}{|x|^2} + \zeta'(\sigma) \left( \frac{\delta_{ij}}{|x|} - \frac{x_i x_j}{|x|^3} \right) \right) + \sum_{i,j} (\partial_i a_{ij}) \zeta'(\sigma) \frac{x_j}{|x|}$$

$$=: S_1 + S_2$$

For the summand $S_2$ we have

$$S_2 \leq \partial_2 |\zeta'(\sigma)| \sum_{i,j} \frac{|x_j|}{|x|} \leq \partial_2 d^2 |\zeta'(\sigma)| = \partial_2 d^2 |\nabla \eta|.$$  

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For the first summand $S_1$ we have

\[
S_1 \leq \sum_{i,j} a^{ij} \zeta''(\sigma \frac{x_i x_j}{|x|^2}) + \sum_{i,j} a^{ij} \zeta'(\sigma) \left( \frac{\delta_{ij}}{|x|} - \frac{x_i x_j}{|x|^3} \right)
\]

By Assumption 1 we have $\sum_{i,j} a^{ij} x_i x_j \leq \vartheta_1 |x|^2$. This gives

\[
S_1 \leq \vartheta_1 |\zeta''(\sigma)| + \frac{|\zeta'(\sigma)|}{|x|} \sum_{i,j} a^{ij} \left( \frac{\delta_{ij} - x_i x_j}{|x|^3} \right).
\]

By Assumption 1 we have $\vartheta_1^{-1} \leq a_{ii} \leq \vartheta_1$ for all $i \in \{1, \ldots, d\}$ and $\vartheta_1^{-1} \leq |x|^{-2} \sum_{i,j} a^{ij} x_i x_j \leq \vartheta_1$. Hence,

\[
S_1 \leq \vartheta_1 |\zeta''(\sigma)| + \frac{|\zeta'(\sigma)|}{|x|} d \vartheta_1
= \vartheta_1 \left| \Delta \eta - \zeta'(\sigma) \frac{d - 1}{|x|} \right| + |\nabla \eta| \frac{d \vartheta_1}{|x|}
\leq \vartheta_1 |\Delta \eta| + \vartheta_1 |\nabla \eta| \frac{d - 1}{|x|} + |\nabla \eta| \frac{d \vartheta_1}{|x|}
= \vartheta_1 |\Delta \eta| + \vartheta_1 |\nabla \eta| (2d - 1).
\]

Putting everything together we obtain

\[
|\mathcal{H}_0 \eta| \leq \vartheta_1 |\Delta \eta| + \vartheta_1 (2d - 1) \frac{|\nabla \eta|}{|x|} + \vartheta_2 d^2 |\nabla \eta|.
\]

Hence,

\[
|\mathcal{H}_c \eta| \leq |\mathcal{H}_0 \eta| + |b^T \nabla \eta| \leq \vartheta_1 |\Delta \eta| + \vartheta_1 (2d - 1) \frac{|\nabla \eta|}{|x|} + (\vartheta_2 d^2 + \|b\|_\infty) |\nabla \eta|
\]

and

\[
|\mathcal{H}_c \eta|^2 \leq 3 \vartheta_1^2 |\Delta \eta|^2 + 3 \vartheta_1^2 (2d - 1)^2 \frac{|\nabla \eta|^2}{|x|^2} + 3 (\vartheta_2 d^2 + \|b\|_\infty)^2 |\nabla \eta|^2.
\]

**B. The constant $C_{qUC}$**

We derive an explicit bound on $C_{qUC}$ in the special case $2D_0 = R \geq 1$, and $\delta \leq 2$. In this special case we have $\varepsilon_0 \in (0, 1], \mu \in (\sqrt{\mu_1}, (5/2)\sqrt{\mu_1})$, $\mu_1 \in (1, (5/2)e^{\vartheta_1})$, $\rho = (2e\vartheta_1 + 1)R \leq (2e+1)\vartheta_1 R$, and $C_{\mu} \in (\sqrt{\mu_1} \varepsilon_0, (3/2)\sqrt{\mu_1} \varepsilon_0)$. By $K$ we denote constants depending only on the dimension which may change from line to line. For the constant $C$ from the Carleman estimate we have the upper bound

\[
C \leq K \varepsilon_0^{-1} \vartheta_1^1 \varepsilon_0^{10\vartheta_1} (1 + \vartheta_2) R.
\]

For the constant $F_{\delta/2}$ we have by using $\delta \leq 2$ and $x \leq 1 + x^2$

\[
F_{\delta/2} \leq \frac{3 + 2 (\|V\|_\infty^2 + \|b\|_\infty^2 + \|c\|_\infty^2) + 8 \vartheta_1^2 C'}{\delta^2/4} \leq K \frac{\vartheta_1^2}{\delta^2} (1 + \|V\|_\infty^2 + \|b\|_\infty^2 + \|c\|_\infty^2).
\]

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Hence, for the term

\[ T_1 = \frac{4\mu^2 \sqrt{\hat{\sigma}_0^2} (3R\rho C M^4)^{-1}}{3\vartheta_1^2 + 768\vartheta_1^4 d^2 \vartheta_2^2 + 3(\vartheta_2 d^2 + \|b\|_\infty)^2 + 4\vartheta_1 F_{\beta/2}} \]

in \( C_{qUC} \), we obtain the lower bound

\[ T_1 \geq K \frac{\sqrt{\hat{\sigma}_0^2}}{R^3(1 + \vartheta_1)(\vartheta_1 + \vartheta_2)^4(1 + \|V\|_\infty^2 + \|b\|_\infty^2 + \|c\|_\infty^2)} \]

For the constant \( C_{qUC} \) we obtain using \( \delta^4/R^3 \geq (\delta/(10e\theta_1^2 R))^4 \), \( (1 + x)^{-1} = (e^{-1}) \ln(1+x) \) and \( \ln(1+x) \leq 3x^{1/3} \) with \( x = \|V\|_\infty^2 + \|b\|_\infty^2 + \|c\|_\infty^2 \), we have

\[ C_{qUC} \geq K \frac{\theta_1^{-31/2} e^{-10\theta_1 \epsilon_0}}{(1 + \vartheta_1)(\vartheta_1 + \vartheta_2)^4(10e\theta_1^2 R)} \]

Next we give an upper bound on \( \alpha^* \). For \( \alpha_0 \) we have

\[ \alpha_0 \leq K \hat{\theta}_1^{25} e^{15\theta_1^2} (1 + \vartheta_2^2) R^2 \epsilon_0^{-1} \]

Hence,

\[ \alpha_0 \leq K \theta_1^{25} e^{15\theta_1^2} (1 + \vartheta_2^2) \epsilon_0^{-1} R^3 (1 + \|b\|_\infty^2 + \|c\|_\infty^2/3) \]

For \( \alpha_1 \) we have

\[ \alpha_1 \leq K \theta_1^{37/6} e^{10\theta_1^2} (1 + \vartheta_2^2) \epsilon_0^{-1/3} (\|V\|_\infty^2 R^5)^{1/3} \]

Since

\[ \frac{\sqrt{\hat{\sigma}_1} e R \mu}{\rho} = 33 d e R \theta_1^2 \vartheta_2 + \epsilon_0/2 = 1 - \epsilon_0/2 \]

we find

\[ \ln \left( \frac{\rho}{\sqrt{\hat{\sigma}_1} e R \mu} \right) = \ln \left( \frac{1}{1 - \epsilon_0/2} \right) \geq \epsilon_0/2 \]

Hence, using \( F_{D0/2} \leq K \theta_1^3 (1 + \|V\|_\infty^2 + \|b\|_\infty^2 + \|c\|_\infty^2) \) we obtain as an upper bound for \( \alpha_3 \)

\[ \alpha_3 \leq \epsilon_0^{-1} \ln \left( K \beta R \theta_1^{19} e^{10\theta_1^2} (1 + \vartheta_2^2) \epsilon_0^{-1} (\|V\|_\infty^2 + \|b\|_\infty^2 + \|c\|_\infty^2 + 1) \right) \]

Hence,

\[ \alpha^* \leq 1 + K \theta_1^{25} e^{15\theta_1^2} (1 + \vartheta_2^2) \epsilon_0^{-1} (1 + \|b\|_\infty^2 + \|c\|_\infty^2/3 + \|V\|_\infty^2) R^3 + \epsilon_0^{-1} \ln \left( K \beta R \theta_1^{19} e^{10\theta_1^2} (1 + \vartheta_2^2) \epsilon_0^{-1} (\|V\|_\infty^2 + \|b\|_\infty^2 + \|c\|_\infty^2 + 1) \right) \]

We now use \( \ln(1+x) \leq 3x^{1/3} \) for \( x \geq 0 \), \( \sum a_i^{1/3} \leq \sum a_i^{1/3} \) and \( \epsilon_0 \geq (\delta/(10e\theta_1^2 R))^{-1} \epsilon_0 \), and obtain

\[ C_{qUC} \geq K \frac{\theta_1^{-31/2} e^{-10\theta_1 \epsilon_0}}{(1 + \vartheta_1)(\vartheta_1 + \vartheta_2)^4(10e\theta_1^2 R)} \gamma \]

where

\[ \gamma = K \epsilon_0^{-1} \theta_1^{25} e^{15\theta_1^2} (1 + \vartheta_2^2) \left( 1 + \|V\|_\infty^{2/3} + \|b\|_\infty^2 + \|c\|_\infty^{2/3} \right) R^3 + \ln \beta - \ln \epsilon_0 \]
C. Scaling argument (deduction of Theorem 6 from Theorem 5)

Since we proved Theorem 6, 11 and 12 only in the special case $G = 1$, we show in this appendix how the general case $G > 0$ can be obtained by scaling. We restrict this discussion to Theorem 6 for functions on $\mathbb{R}^d$. The argument applies in the same way to Theorem 11 and 12 for functions on the cube $A_L$.

We fix a $(G, \delta)$-equidistributed sequence with $G > 0$ and $\delta \in (0, G/2)$. Let $g : \mathbb{R}^d \to \mathbb{R}^d$ be given by $g(x) = Gx$ and $\tilde{\psi} = \psi \circ g$. We want to estimate

$$\int_{S_\delta} |\psi|^2 = G^d \int_{S_\delta/G} |\tilde{\psi}|^2$$

from below. Note that $S_\delta/G$ corresponds to some $(1, \delta/G)$-equidistributed sequence. Let $\tilde{a}^ij = a^ij \circ g$, $\tilde{b} = G(b \circ g)$, $\tilde{c} = G^2(c \circ g)$, $\tilde{V} = G^2(V \circ g)$, $\tilde{\zeta} = G^2(\zeta \circ g)$ and

$$\tilde{\mathcal{H}} = \sum_{i,j=1}^d \partial_i \tilde{a}^ij \partial_j + \tilde{b}^T \nabla + \tilde{c}.$$

Then, by the chain rule and our assumption $|\mathcal{H}\psi| \leq |V\psi| + |\zeta|$, we have almost everywhere on $\mathbb{R}^d$ the inequality

$$|\tilde{\mathcal{H}}\tilde{\psi}| = G^2|\mathcal{H}(\psi) \circ g| \leq G^2|V\psi| + G^2|\zeta \circ g| = |\tilde{V}\tilde{\psi}| + |\tilde{\zeta}|.$$

Let $\tilde{A} = (\tilde{a}^ij)_{i,j=1}^d$ and $\tilde{\theta}_2 = G\theta_2$. The coefficients of $\tilde{\mathcal{H}}$ satisfy for almost all $x \in \mathbb{R}^d$ and all $\xi \in \mathbb{R}^d$

$$\theta_1^{-1}|\xi|^2 \leq \xi^T \tilde{A}(x) \xi \leq \theta_1|\xi|^2, \quad \|\tilde{A}(x) - \tilde{A}(y)\|_{\infty} \leq \tilde{\theta}_2|x - y|,$$

$$\|\tilde{b}\|_{\infty} \leq G\|b\|_{\infty}, \quad \|\tilde{c}\|_{\infty} \leq G^2\|c\|_{\infty}, \quad \text{and} \quad \|\tilde{V}\|_{\infty} \leq G^2\|V\|_{\infty}.$$ Hence we can apply Theorem 5 to the functions $\tilde{\psi}$ and $\tilde{\zeta}$ and obtain

$$\|\tilde{\psi}\|_{S_\delta}^2 = G^d \|\tilde{\psi}\|_{S_\delta/G}^2 \geq C_{\text{sfUC}} G^d \|\tilde{\psi}\|_{\mathbb{R}^d}^2 - (\delta/G)^2 G^d \|\tilde{\zeta}\|_{\mathbb{R}^d}^2 = C_{\text{sfUC}} \|\psi\|_{\mathbb{R}^d}^2 - (\delta/G)^2 G^4 \|\zeta\|_{\mathbb{R}^d}^2.$$

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