On Multiple-Access Systems with Queue-Length Dependent Service Quality

Daewon Seo, Graduate Student Member, IEEE, Avhishek Chatterjee, Lav R. Varshney, Senior Member, IEEE

Abstract—It is commonly observed that higher workload lowers job performance. We model the workload as a queueing process and study the information-theoretic limits of reliable communication through a system with queue-length dependent service quality. The goal is to investigate a multiple-access setting, where transmitters dispatch encoded symbols over a system that is a superposition of continuous-time $\text{GI}_k/\text{GI}/1$ queues, and a noisy server, whose service quality depends on the queue-length and processes symbols in order of arrival.

We first determine the capacity of single-user queue-length dependent channels and further characterize the best and worst dispatch and service processes for $\text{GI}/\text{M}/1$ and $\text{M}/\text{GI}/1$ queues, respectively. Then we determine the multiple-access channel capacity using point processes. In particular, when the number of transmitters is large and each arrival process is sparse, the superposition of arrivals approaches a Poisson point process. In characterizing the Poisson approximation, we show that the capacity of the multiple-access system converges to the capacity of a single-user $\text{M}/\text{GI}/1$ queue-length dependent system. The speed of convergence bound in the number of users is explicitly given. Further, the best and worst server behaviors of $\text{M}/\text{GI}/1$ queues from the single-user case are preserved in the sparse multi-access case.

Index Terms—multiple-access channel, quality of service, Poisson point process

I. INTRODUCTION

Overloading often lowers job performance, including human workers and even machine systems [2]–[5], e.g., due to stress. Queueing theory well models the arriving workload and thus is popularly used to model manufacturing, traffic, telecommunication, and other systems. On the other hand, information-theoretic models are used to analyze the limits of reliable communication in the presence of noise. This work combines those two areas and studies reliable information processing in a queueing system.

Motivated by applications in crowdsourcing, multimedia communication, and stream computing, we had previously brought some notions of reliability into queueing by establishing the capacity of single-user systems with queue-length dependent service quality [6]. There, a sequence of coded symbols was sent using an arrival process, processed by an unreliable queueing server, and returned to a destination for decoding. The level of channel noise was a function of the queue length. Note that unlike timing channels [7], [8], information was conveyed only in the symbols.

Since there are often multiple input streams in the motivating applications rather than just one, e.g., due to multihoming, here we consider a scenario where multiple transmitter-destination pairs want to send information reliably and therefore dispatch coded symbols on arrival processes. A particular motivational setting is driver-assisted autonomous trucks [9], where a human driver remotely monitors multiple semi-autonomous trucks and steps in (i.e., processes information) only when the autonomous algorithm cannot handle.

Fig. 1 presents such a multiple-access setting, where before entering a single central processor, the multiple arrival processes are superposed. Once coded symbols arrive at the central queue processor, they are served in a First Come, First Serve (FCFS) manner, and returned to the intended receiver. Note that if there is a single central receiver, the topology reduces to a multiple-access channel. As before, a distinguishing aspect of this work is that reliability of the central server depends on queue-length arising from the superposed arrival process.

Our previous results [6] for capacity of single-user queue-length dependent quality considered time-slotted (i.e., discrete-time) queues and further optimized the server for Geo/1 queues or optimized a dispatcher for GI/Geo/1 queues, under given reliability requirements.

Here, we consider the superposition of multiple arrival processes in a continuous-time setting. Before proceeding, we first study the capacity of the continuous-time single-user case, and also specify the best and worst dispatch processes for a GI/M/1 queue, and service processes for a M/GI/1 queue with additional conditions. Then, the capacity expression of the multiple-access setting is given in terms of the stationary distribution of queue-length seen by each user’s departures. Surprisingly, our results show there is no loss in capacity due
to multiple-access interference.

As superposition of non-Poisson arrivals is in general intractable, we also consider the large-user asymptotic by introducing a random marked point process (RMPP, or simply PP) approach \[10\], \[11\] and apply the superposition convergence to a Poisson point process \[12\]. The latter states the superposition of a large number of sparse arrivals is approximately Poisson. Building on this result, we prove that the capacity for \( \sum_k \text{Gl}_k / \text{Gl}/1 \) queues, where \( \Sigma_k \) stands for the superposition, converges to that for single-user \( \text{M}/\text{Gl}/1 \) queues. In other words, even though individuals are non-Poisson arrivals, sending information as if a single-user \( \text{M}/\text{Gl}/1 \) queue is asymptotically optimal. It also implies the best and worst services obtained for a single-user \( \text{M}/\text{Gl}/1 \) queue are preserved.

Like our FCFS model, where the server only processes a single job at a time and waiting jobs interfere with the processing job via increased queue-length, multiple-access interference in queueing through a processor sharing model is considered in \[13\], \[14\]. Large-user asymptotics also appear in the many-access channel \[15\] which studies the Gaussian multiple access channel capacity in terms of number of users.

The remainder of the paper is organized as follows. Sec. [II] introduces the queue-length dependent channel and some definitions of point processes. Sec. [III] describes the capacity of single-user case for continuous-time queueing workloads as well as its best and worst behaviors. Main contributions of this paper are discussed in Sec. [IV] which states capacity formulas for a general \( K \)-user system and its Poisson approximation when users are sparse. Sec. [V] concludes.

II. PRELIMINARIES AND SYSTEM MODEL

A. Point Processes

We use a PP approach to queueing systems, enabling us to derive analytical properties. Let us define an RMPP \( \Phi = \Phi(t) \) as follows.

Definition 1: Let \( \mathcal{B} \) be the Borel \( \sigma \)-algebra of \( \mathbb{R} \). Given a mark space \( \mathcal{M} \) and its sigma-algebra \( \sigma(\mathcal{M}) \), consider a marked counting measure \( N(B \times M) \) where \( B \in \mathcal{B} \) and \( M \in \sigma(\mathcal{M}) \) such that \( N(B \times M) < \infty \) for any bounded \( B \).

The \( \mathcal{N}, \sigma(\mathcal{N}) \) be the set of all such counting measures and its smallest \( \sigma \)-algebra, respectively. Then, a random marked point process (RMPP, or simply a point process (PP)), \( \Phi(t) \) is a random element from \( (\Omega, \mathcal{F}, P) \) to \( (\mathcal{N}, \sigma(\mathcal{N})) \).

For queueing applications, the mark usually denotes a random service time at the server or the time required to finish each job. Hence, \( \mathcal{M} = \mathbb{R}_+ \) and since only the \( /\text{Gl}/1 \) queue is considered in this work, each mark is i.i.d. from some distribution \( P^{S} \). Since all randomness from arrival and service times is captured in the RMPP, any queue response such as queue-length or waiting time is a deterministic function of the RMPP.

Two equivalent representations of a PP are especially useful in this paper. Suppose the mark space is empty, i.e., \( \mathcal{M} = \emptyset \) for illustration. However, the following representations can be easily extended to RMPPs with a non-empty mark space. The first representation is to use an inter-arrival time representation, induced by Dirac delta functions.

Letting \( \{T_i \in \mathbb{R}_+ \}_{i \in \mathbb{Z}} \) be a non-decreasing random sequence,

\[
\Phi(t) \Leftrightarrow \sum_{i=-\infty}^{\infty} \delta_{T_i} \Leftrightarrow (\ldots, A_{-1}, A_0, A_1, \ldots),
\]

where \( A_i := T_i - T_{i-1} \geq 0 \). So \( T_i \) indicates the time epoch when the \( i \)th arrival comes. The case for i.i.d. \( A_i \) is called a renewal process, which arises in Sec. [III].

The other representation is by a random counting measure, which is useful especially in Sec. [IV]. Note that

\[
N(B) = \int \sum_{i=-\infty}^{\infty} 1_B \delta_{T_i} dt,
\]

that is, the number of arrivals in \( B \), for any bounded \( B \subset \mathbb{B} \) uniquely determines \( \Phi(t) \). Here \( 1_B = 1_B(t) \) is the indicator function with criterion \( \{t \in B\} \) and we write \( 1_B \Phi \) to stand for the restricted RMPP on \( B \).

A time shift operation is denoted by \( T_\tau \Phi(t) = \Phi(t + \tau) \), enabling definitions of stationarity and ergodicity.

Definition 2 (Stationarity, Def. 1.2.1 [10]): An RMPP \( \Phi \) is stationary if the probability measure \( P \) is invariant with respect to the time shift \( T_\tau \), i.e., for any set \( Z \in \sigma(\mathcal{N}) \),

\[
P(T_\tau Z) = P(Z) \text{ for all } \tau \in \mathbb{R}.
\]

Definition 3 (Ergodicity, Def. 1.2.5 [10]): A stationary RMPP \( \Phi \) (or its probability measure \( P \)) is ergodic if any set \( Z \in \sigma(\mathcal{N}) \) satisfying \( T_\tau Z = Z \) for all \( \tau \in \mathbb{R} \) implies either \( P(Z) = 0 \) or \( 1 \).

B. System Model

Multiple users intend to send messages to respective targeted receivers. To do that, the \( k \)th user picks an encoded sequence of symbols \( \{X(k)_i\} \)—each symbol is drawn from finite space \( X \)—and dispatches it over an independent stationary renewal arrival process with inter-arrival time distribution \( P^A(k) \). Those arrivals are superposed just before entering a \( /\text{Gl}/1 \) queue. The server follows FCFS service discipline with i.i.d. service time according to \( P^S \). Assume that the waiting room is unlimited.

Since the server is unreliable, the symbol is corrupted to \( Y(k)_n \in \mathcal{Y}^n \) randomly, where \( \mathcal{Y} \) is also finite. The transition probability, denoted by \( W = W_Q \), is dependent on \( Q \), the queue-length at the moment just before the symbol’s departure, excluding the job being serviced. That is, the channel at time \( t \) is \( W_Q := P_{Y|X,Q} \), where \( Q \) is the queue-length seen by departure. In this sense we say the system is queue-length dependent. Departing symbols are labeled and delivered to the intended receiver. Since symbols are encoded against channel noise, receivers can decode the sequence to recover the original information. We assume there is a central coordination mechanism that reveals each transmitter’s dispatching process to all other transmitters, but not realizations.

We use \( \sum(\cdot) \) to denote superposition, so the queue of interest is written as \( \sum_k \text{Gl}_k / \text{Gl}/1 \). The queues are assumed

\footnote{Throughout this paper, symbol (common in information theory) and job (or customer, common in queueing theory) are interchangeable.}
always stable, i.e., superposed arrival rate \( \lambda \) and service rate \( \mu \) satisfy traffic intensity \( \rho := \frac{\lambda}{\mu} < 1 \). Also we suppose some technical assumptions on arrivals and service: 1) arrivals and service processes are simple, i.e., \( P_k^{A}(t) = 0 \) for all \( k \), \( P_k^{B}(t) = 0 \); 2) at least one of \( \{P_k^{A}(t)\}_{k=1}^{K} \) and \( P_k^{B}(t) \) is continuous and strictly positive on \( \mathbb{R} \).

We assume causal knowledge of arrival and departure realizations, i.e., the encoders do not know them, but the decoders do. Also all \( P_k^{A} \) are available to transmitters, but not their realizations.

III. CONTINUOUS-TIME SINGLE-USER QUEUE-CHANNEL

This section investigates the capacity of single-user queue-length dependent channels like \cite{6}, but in continuous-time.

A. Coding Theorem for GI/GI/1 Queues

Consider a simple renewal arrival process \( \Phi(t) \) with arrival rate \( \lambda \), i.e., the \( i \)th inter-arrival time \( A_i \sim P^A \) i.i.d. with \( \lambda = 1/\mathbb{E}[A_1] \). Recall that the service quality (channel) of the \( i \)th job depends only on the queue-length seen by the \( i \)th departure (i.e., just before \( i \)th departure), denoted \( Q_1 \). We first express capacity using the information spectrum \cite{16, 17}; see \cite{16, 17} for notation of various information functionals.

**Proposition 4:** For a simple renewal PP \( \Phi(t) \) with rate \( \lambda = 1/\mathbb{E}[A_1] \),

\[
C(\Phi) = \sup_{P_X} \mathbb{I}(X;Y|Q) \quad \text{[bits/sym]} \tag{1}
\]

\[
= \sup_{P_X} \lambda \mathbb{I}(X;Y|Q) \quad \text{[bits/time]}.
\]

**Proof:** See \cite{6} Prop. 1.

**Lemma 5:** For each simple renewal PP \( \Phi \), there exists a unique stationary distribution \( \pi \) such that if \( Q_1 \sim \pi \), then any \( Q_i \sim \pi \). Furthermore, for any measurable \( f : \mathbb{Z}_+ \rightarrow \mathbb{R}_+ \),

\[
\frac{1}{n} \sum_{i=1}^{n} f(Q_i) \rightarrow \mathbb{E}_\pi[f(Q)] \quad \text{as } n \to \infty \text{ almost surely.}
\]

**Proof:** Consider an arrival time instance when the system is empty, i.e., no job in the queue, no job at the instance of an arrival. At this instance, a new cycle of queueing begins from the empty state. So let us consider the queue-length process seen by arrivals, \( \{Q_i\}_{i \in \mathbb{Z}} \). In GI/GI/1 queues, the cycles are i.i.d. and so are called regenerative cycles \cite[Chap. VI]{18}, denoted by \( \{R_i \in \mathbb{Z}_+\}_{i \in \mathbb{Z}} \). Also, \( \rho < 1 \) implies \( \mathbb{E}[R] < \infty \) and these cycles are repeated infinitely many times. We know the limiting distribution of \( Q \), say \( \bar{\pi} \), exists and is ergodic so for any measurable nonnegative function \( f \),

\[
\mathbb{E}_\pi[f(\hat{Q})] = \frac{1}{\mathbb{E}[R]} \mathbb{E}\left[ \sum_{i: \text{inside of } R} f(\hat{Q}_i) \right] = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(\hat{Q}_i).
\]

Next, suppose the queue is in steady-state. Since the beginning and end of cycles are empty-state, whenever there is an arrival, there is a corresponding departure in the cycle. Thus, \( \hat{Q} \overset{d}{=} Q \), i.e., \( \bar{\pi} = \pi \). Therefore, for any measurable nonnegative function \( f \),

\[
\mathbb{E}_\pi[f(\hat{Q})] = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(\hat{Q}_i) = \mathbb{E}_\pi[f(Q)].
\]

This completes the proof.

Combining Prop. 4 and Lem. 5 we have a simpler capacity expression in terms of expectation over \( Q \), or equivalently in terms of stationary distribution \( \pi(Q) \).

**Theorem 6:** For GI/GI/1 queues, the capacity formula \cite{1} can be further simplified to

\[
C(\Phi) = \sup_{P_X} \mathbb{E}[I(P_X, W_Q)] = \sup_{P_X} \sum_{q=0}^{\infty} \pi(q) I(P_X, W_q) \tag{2}
\]

in bits per job, and

\[
C(\Phi) = \lambda \sup_{P_X} \mathbb{E}[I(P_X, W_Q)] = \sup_{P_X} \lambda \sum_{q=0}^{\infty} \pi(q) I(P_X, W_q) \tag{3}
\]

in bits per time. Therefore, it is easy to see that the capacity over all renewal PPs with stable arrival rate \( \lambda < \mu \) is

\[
C = \sup_{\lambda \in (0, \mu)} \sup_{P^A} \mathbb{E}[I(P_X, W_Q)] \quad \text{[bits/time].}
\]

**Proof:** See \cite{6} Thm. 1] with generalization to general discrete channels.

**Remark 7:** In this work, we assume a simple transmitter that does not know arrival and departure realizations, which implies channel state information is unavailable. If the channel state information is available without delay, the capacity formula follows immediately as

\[
C(\Phi) = \lambda \sup_{P_X} \mathbb{E}[I(P_X, W_Q)] \quad \text{[bits/time].} \tag{4}
\]

Thus, we can see that when the capacity-achieving distributions are all identical with some \( P^A_X \), such as binary symmetric channels or binary erasure channels, the transmitter simply picks \( P^*_X \) even without the channel state information and achieves \cite{4} by the codebook identical with no channel state information. Channel state feedback even without delay does not improve capacity in this case.

A closed-form expression of \( \pi(Q) \) is unknown in general, but is known for some special types of queues. Let us rewrite \cite{2} for two special types of queues GI/M/1 and M/GI/1, and consider per symbol capacity since per time capacity follows by multiplying by \( \lambda \).

**Theorem 8 (** GI/M/1 queues **):** Let \( A^*(\cdot) \) be the Laplace-Stieltjes transform of \( P^A(t) \) and define \( \sigma^* \) as the unique solution of \( \sigma = A^*(\mu(1 - \sigma)) \) in \((0, 1)\). Then, the capacity of GI/M/1 queues is given by

\[
C(\Phi) = \sup_{P_X} \mathbb{E}[I(P_X, W_Q)] \quad \text{[bits/sym]},
\]

where \( \pi(q) = (1 - \rho)/(1 - z) \) and \( q \).

**Proof:** See App. A

**Theorem 9 (** M/GI/1 queues **):** The capacity of M/GI/1 queues is given by

\[
C(\Phi) = \sup_{P_X} \mathbb{E}[I(P_X, W_Q)] \quad \text{[bits/sym]},
\]

where \( \pi(q) = \text{inverse of probability generating function} \).

\[
\Pi(z) = \frac{1 - \rho(1 - z)K(z)}{K(z) - z},
\]
and $K(z)$ is the probability generating function of $k_q$ with

$$k_q = \int_0^\infty P^S(t) \frac{e^{-\lambda t} t^q}{q!} dt.$$

**Proof:** See App. B.

**Example:** Consider an M/M/1 queue and a binary symmetric channel (corresponding to binary classification) with queue-length dependent transition probability $\epsilon_q$. Then, we know that $\pi(q) = (1 - \rho)^q$ and Thm. 8 shows that

$$C = \lambda \sum_{q=0}^\infty \pi(q)(1 - H_2(\epsilon_q))$$

[bits/time],

where $H_2(\cdot)$ is the binary entropy function. Fig. 2 shows the capacity curves for different rates.

**B. Optimization of Capacity**

This subsection considers optimization of the capacities for GI/M/1 and M/GI/1 queues given in Thms. 8 and 9. To do so, we impose two conditions such that

1) $P_\lambda^X$ achieves the capacity for all $W_q$.

2) At such $P_\lambda^X$, the system becomes more unreliable as $q$ increases in a step-down manner, i.e., for some $b \in \mathbb{Z}_+$,

$$I(P_\lambda^X, W_0) = \cdots = I(P_\lambda^X, W_b) > I(P_\lambda^X, W_{b+1}) = \cdots .$$

Note that condition 1) covers $|F|$-ary symmetric or $|F|$-ary erasure channels since $P_\lambda^X$ is uniform. Such channels model multi-label classification via crowdsourcing platform [19] in that events $\{X \neq Y\}$ in a symmetric channel and $\{Y = \text{erasure}\}$ in an erasure channel model 'misclassification' and 'I don’t know' answers of a crowdworker, respectively. In particular, introducing the step change in noise allows us to find the best and worst server behaviors explicitly. It is natural in applications (including non-human applications) for the server to be more unreliable as the queue gets longer, see [6] for modeling details.

**Corollary 10:** Fix arrival rate $\lambda$. For GI/M/1 queues, the best inter-arrival distribution is deterministic, i.e., $P^A(t)$ only has a unit point mass at $t = \lambda^{-1}$.

**Proof:** For the sake of brevity, let $c_b := I(P_\lambda^X, W_b)$ and $c_{b+1} := I(P_\lambda^X, W_{b+1})$. Then, the capacity is written as

$$C(\Phi) = \sum_{q=0}^\infty \pi(q) I(P_\lambda^X, W_q)$$

$$= \sum_{q=0}^b (1 - \sigma^*)^q I(P_\lambda^X, W_q) + \sum_{q=b+1}^\infty (1 - \sigma^*)^q c_{b+1}$$

$$= c_b (1 - \sigma^*)^{b+1} + c_{b+1} (\sigma^*)^{b+1}$$

As $c_b > c_{b+1}$, maximizing $C(\Phi)$ with given $\lambda$ is equivalent to minimizing $\sigma^*$. Note that $\sigma^*$ is the unique fixed point of $\sigma = A^\ast(\mu(1 - \sigma))$ and at $\sigma = 0$ and 1,

$$\int_0^\infty P^A(t) e^{-\mu t(1-\sigma)} dt \bigg|_{\sigma=0} = \int_0^\infty P^A(t) e^{-\mu t} dt > 0$$

$$\int_0^\infty P^A(t) e^{-\mu t(1-\sigma)} dt \bigg|_{\sigma=1} = \int_0^\infty P^A(t) dt = 1.$$

Furthermore, $A^\ast(\mu(1 - \sigma))$ is strictly convex in $\sigma$ since

$$\frac{\partial}{\partial \sigma} A^\ast(\mu(1 - \sigma)) > 0, \quad \frac{\partial^2}{\partial \sigma^2} A^\ast(\mu(1 - \sigma)) > 0.$$  

Due to Jensen’s inequality, we obtain

$$A^\ast(\mu(1 - \sigma)) = \int_0^\infty P^A(t) e^{-\mu t(1-\sigma)} dt$$

$$\geq e^{-\mu E[A](1-\sigma)} = e^{-\frac{\lambda}{\lambda-1}(1-\sigma)},$$

where the equality is attained only when $A = \lambda^{-1}$ almost surely. It means that when $P^A$ is deterministic, the curve $A^\ast(\mu(1 - \sigma)) = e^{-\frac{\lambda}{\lambda-1}(1-\sigma)}$ lower bounds all other curves so that achieves the smallest fixed point. Therefore, the deterministic inter-arrival distribution achieves the greatest capacity.

**Corollary 11:** Fix arrival rate $\lambda$. For GI/M/1 queues, cramming inter-arrivals asymptotically minimize the capacity, i.e., $P^A(t; \epsilon, \delta)$ asymptotically achieves the smallest capacity as $\epsilon, \delta \to 0$, where

$$P^A(t; \epsilon, \delta) = \begin{cases} 1 - \epsilon & \text{if } t = \delta \\ \epsilon & \text{if } t = \frac{1}{\lambda-1} \delta (1-\epsilon) \\ 0 & \text{otherwise.} \end{cases}$$

**Proof:** Similar to the proof of Cor. 10, it is sufficient to show that $\sigma^*$ is maximized, i.e., when $P^A$ is cramming $A^\ast(\mu(1 - \sigma))$ upper bounds all other curves. We know that for any $P^A$, 

$$A^\ast(\mu(1 - \sigma)) = \int_0^\infty P^A(t) e^{-\mu t(1-\sigma)} dt$$

$$\leq \int_0^\infty P^A(t) dt = 1.$$
On the other hand, note that the cramming inter-arrival distribution asymptotically achieves the upper bound as $\epsilon, \delta \to 0$ so that it maximizes the fixed point solution $\sigma^*$. Also notice that the location of $\epsilon$ point mass is determined to satisfy mean constraint $E[A] = \lambda^{-1}$.

**Corollary 12:** Fix service rate $\mu$. For $\mathcal{M}/\mathcal{G}/1$ queues with service quality stepping down at $b = 0$, i.e.,

$$I(P^*_X, W_0) > I(P^*_X, W_1) = I(P^*_X, W_2) = \cdots,$$

the capacity is constant among all service distributions.

**Proof:** When the threshold $b = 0$, let $c_0 := I(P^*_X, W_0)$ and $c_1 := I(P^*_X, W_1)$. Since the capacity is given by

$$C(\pi) = \pi(0)c_0 + (1 - \pi(0))c_1 = c_1 + \pi(0)(c_0 - c_1),$$

so $\pi(0)$ completely determines the capacity. On the other hand, by the inverse $Z$-transform relation,

$$\pi(0) = \Pi(0) = 1 - \rho.$$

Thus, the capacity is constant over all $P^S$ of service rate $\mu$.

**Corollary 13:** Fix service rate $\mu$. For $\mathcal{M}/\mathcal{G}/1$ queues with service quality stepping down at $b = 1$, i.e.,

$$I(P^*_X, W_0) = I(P^*_X, W_1) > I(P^*_X, W_2) = \cdots,$$

the capacity is maximized when the service is deterministic. On the other hand, the capacity is asymptotically minimized by cramming service, i.e., $P^S (t; \epsilon, \delta)$ asymptotically minimizes the capacity as $\epsilon, \delta \to 0$, where

$$P^S (t; \epsilon, \delta) = \begin{cases} 1 - \epsilon & \text{if } t = \delta \\ \epsilon & \text{if } t = \frac{t}{e^\epsilon - (1-\epsilon)\delta} \\ 0 & \text{otherwise.} \end{cases}$$

**Proof:** Let $c_0 := I(P^*_X, W_0), c_2 := I(P^*_X, W_2)$ for simplicity. Then the capacity is given by

$$C = (\pi(0) + \pi(1))c_0 + (1 - \pi(0) - \pi(1))c_2.$$ 

Since $c_0 > c_2$, it is apparent that the capacity is maximized (resp. minimized) when $\pi(0) + \pi(1)$ is maximized (resp. minimized). Also note that

$$\pi(0) = 1 - \rho = 1 - \frac{\lambda}{\mu},$$

$$\pi(1) = \frac{\Pi(z) - \pi(0)}{z} = \left. \frac{(1 - \rho)(1 - \rho)K(z)}{K(z) - \rho} - \pi(0) \right|_{z=0} = \frac{(1 - \rho)(1 - K(z))}{K(z) - \rho} = \left. \frac{1 - \rho}{K(0)} \right|_{z=0}.$$ 

Since $\pi(0) + \pi(1) = \frac{1 - \rho}{K(0)}$, the best (resp. the worst) service distribution should minimize (resp. maximize) $K(0) = k_0$. Recall the expression of $k_0$,

$$k_0 = \int_0^\infty P^S(t)e^{-\lambda t}dt.$$ 

The same arguments of [5] and [6] imply that the deterministic service distribution $P^S(t) = \delta_{\mu^{-1}}$ maximizes the capacity, and

$$P^S(t; \epsilon, \delta) = \begin{cases} 1 - \epsilon & \text{if } t = \delta \\ \epsilon & \text{if } t = \frac{t}{e^\epsilon - (1-\epsilon)\delta} \\ 0 & \text{otherwise.} \end{cases}$$

asymptotically minimizes the capacity as $\epsilon, \delta \to 0$.

Cor. 13 is also of interest when the number of users is large and each arrival process is sparse, see Sec. [IV].

**IV. MULTIUSER INPUT $\sum_k G_k/\mathcal{G}/1$ QUEUES**

Recall the system model in Sec. [II-B]. Since $K$ users simultaneously dispatch encoded symbols, each user sees a different queue-length distribution from that for single-user systems; thus, capacity changes. We characterize the individual and sum capacities for the $K$-user scenario in terms of $\pi_{kk}(Q)$, the stationary queue-length distribution seen by user $k$’s departures. Note that $K, k$ denote total number of users and a specific $k$th user, respectively. Since the superposition process is in general intractable, we obtain asymptotics of capacity using Poisson approximation when component PPs are independent and sparse.

For a common setup, consider a triangular array of independent, stationary, and renewal (thus, ergodic) PPs $\Phi_{kk}, K \in \mathbb{Z}_+, k \in [1 : K]$. Also suppose each PP has an inter-arrival distribution $\lambda_{kk}$ with arrival rate $\lambda_{kk}$, not necessarily identical. Let us also assume second-moment finiteness of inter-arrival times, which is necessary to prove Lem. [16]

$$\mathbb{E}[P^S_{kk}[A^2]] < \infty \text{ for all } k \in [1 : K].$$

A. Coding Theorem for $K$-user Channels

Let $\Phi_{kk}$ be the superposition arrival process of $K$th-row components, i.e, $\Phi_{k} := \sum_{k=1}^{K} \Phi_{kk}$. Note that the component PPs are stationary and ergodic.

The next lemma proves the superposition process is stationary and ergodic as well.

**Lemma 14:** Suppose each $\Phi_{kk}, k \in [1 : K]$ is independent, stationary, and ergodic. Then, $\Phi_{kk}$ is also stationary and ergodic.

**Proof:** First prove the stationarity. Take an arbitrary bounded Borel set $B$ and let $B' = T: B$ be the time-shifted set by $\tau \in \mathbb{R}$. Consider the counting measure representation; then due to independence, $N_K(B) = \sum_k N_{kk}(B)$ and

$$N_K(B) = \sum_k N_{kk}(B) = \sum_k N_{kk}(B') = N_{kk}(B'),$$

where (a) is due to the stationarity of individual PPs and (b) is due to independence of individual PPs. As $\tau \in \mathbb{R}$ is arbitrary, stationarity is shown.

Next show the ergodicity. Suppose $\Phi_{kk}$ is not ergodic: then, by Def. [5] there exists a $Z \in \sigma(N)$ such that for any $\phi_K \in Z$ and $\tau \in \mathbb{R}$, it holds that $T \phi_K \in Z$, however, $0 < P_K[Z] < 1$. 


As $Z$ is closed under any time-shift operation, we can write for $\phi_K \in Z$,
\[ \phi_K = \left( \sum_k \phi_{Kk} \right) \in Z \Rightarrow \]
\[ \phi'_K := T_r \phi_K = T_r \sum_k \phi_{Kk} = \left( \sum_k T_r \phi_{Kk} \right) \in Z \quad \forall r \in \mathbb{R}. \]

(8)

Now consider $P_K[Z]$. Let $Z_k$ be the collection of $\phi_{Kk}$ consisting some $\phi \in Z$. As $\phi_{Kk}$ is a component of $\phi_K$, $T_r \phi_{Kk}$ is also a component of $\phi'_K$ by (8) so that $Z_k$ is also closed.

Since each $\Phi_{Kk}$ is stationary and ergodic, $P_{Kk}[Z_k]$ is either 0 or 1. However, because $0 < P_K[Z] = \prod_k P_{Kk}[Z_k] < 1$ by independence, there is a contradiction. Therefore, $\Phi_K$ is ergodic.

Let $Q_{i(K)}$ be the queue-length process seen by the superposed departures. The next lemma further guarantees that the stationary distribution $\pi_K$ exists and $Q_{i(K)}$ is ergodic since $\Phi_K$ is stationary and ergodic from Lem.\[14\].

Lemma 15 (Chap. 2.2 [10]): If the input PP $\Phi$ of the queue $\cdot/\text{GI}/1$ with traffic intensity $\rho < 1$ is stationary and ergodic, then the queue-length distribution seen by departures is also stationary and ergodic. Furthermore, the stationary distribution is independent of the initial state.

Now let us consider individual ‘seen by departures’ processes. Let $Q_{i(k)}$, $\pi_K$ be the queue-length process seen by user $k$’s departures and its stationary distribution. The following lemma proves the existence of $\pi_K$ and its ergodicity.

Lemma 16: Suppose (7) holds. Then, for each $k \in [1 : K]$, the stationary distribution $\pi_K$ exists. Furthermore, for any measurable $f : Z_+ \rightarrow \mathbb{R}_+$, $\frac{1}{\tau} \sum_{i=1}^{n} f(Q_{i(k)}) \rightarrow \mathbb{E}_{\pi_K}[f(Q)]$ as $n \rightarrow \infty$ almost surely.

Proof: See App.\[C\].

As before, Lem.\[16\] allows a simpler capacity expression. Let $C_{\text{ind}}(\Phi_{Kk}), C_{\text{sum}}(\Phi_{K})$ be the $k$th user’s individual capacity and their sum capacity. The following theorem only describes per job capacity, but, per time capacity is immediate by multiplying by individual and sum arrival rates, respectively.

Theorem 17:
\[ C_{\text{ind}}(\Phi_{Kk}) = \mathbb{E}_{\pi_{Kk}}[I(P_X, W_Q)] \quad [\text{bits/sym}], \]
\[ C_{\text{sum}}(\Phi_{K}) = \mathbb{E}_{\pi_{K}}[I(P_X, W_Q)] = \sum_{k=1}^{K} w_k C_{\text{ind}}(\Phi_{Kk}) \quad [\text{bits/sym}], \]
where $w_k := \lambda_{Kk}/\sum_{j=1}^{K} \lambda_{Kj}$.

Proof: Since individual $\{\pi_{Kk}\}$ are stationary and ergodic, the first statement follows.

To show the second statement, notice that
\[ C_{\text{sum}}(\Phi_{K}) \leq \mathbb{E}_{\pi_{K}}[I(P_X, W_Q)] \]
holds. In addition, since $\pi_{K}(q) = \sum_{k} w_k \pi_{Kk}(q)$, the equality holds.

Unlike typical multiple-access settings, it is interesting to note that the per time sum capacity is simply a sum of per time individual capacities, which means that greedy individuals do not degrade optimality in sum information rate. This follows since once arrival processes are fixed, symbol noise levels are also fixed by queue-length. The server processes one symbol at a time, therefore, adding more (or reducing) information in a user’s codeword does not increase (or decrease) interference levels.

B. Poisson Approximation

In the previous subsection, we obtained the multiple-access capacity formula for general $\sum_{k} \text{GI}/\text{GI}/1$ queues. However, a more explicit expression is unavailable even for an $\cdot/\text{F acknowledge}/\text{F symmetric}$ channel or an erasure channel, unless the queue is $\sum_{k} \text{M/\text{GI}/1}$. This is because the superposition of $K$ independent renewal PPs is not necessarily renewal and is renewal if and only if individual PPs are Poisson [20] (thus, the superposition process is also Poisson). So the tractability of the superposition process is limited. Although it is intractable, when $K$ is large and individual PPs are sparse (formally defined in Def.\[18\] below) we can approximate the superposition process by a Poisson PP.

Consider a triangular array of i.i.d., stationary, ergodic, and renewal PPs, $\{\Phi_{Kk}\}$, where $K \in \mathbb{Z}_+$ and $k \in [1 : K]$. Individual processes are assumed to be sparse as given below. The superposition process of row PPs is denoted by $\Phi_{K} := \sum_{k} \Phi_{Kk}$ with corresponding probability measure $P_{K}$. Let $N_{Kk}(B)$ be the counting measure corresponding to $\Phi_{Kk}$, i.e., the number of events of $\Phi_{Kk}(t)$ in $B \in \mathfrak{B}$. Also let $N_{K}(B)$ be the number of events of $\Phi_{K}$ in $B$, so $N_{K}(B) = \sum_{k} N_{Kk}(B)$. Then we can derive that $N_{K}(B)$ converges to the Poisson distribution of intensity measure $\lambda[B]$ where $\cdot$ is the Lebesgue measure, or equivalently, $\Phi_{K}(t)$ converges to the Poisson process, say $\Phi^{*}(t)$, with probability measure $P^{*}$, under the sparsity condition. Let $N^{*}$ be the counting measure for the Poisson PP, i.e., for any bounded $B \in \mathfrak{B}$,
\[ \mathbb{P}[N^{*}(B) = j] = \frac{1}{j!}(\lambda[B])^j e^{-\lambda[B]} . \]

Definition 18: For a given bounded $B \in \mathfrak{B}$, the triangular processes are said to be sparse with sum rate $\lambda_{K} := \sum_{k} \lambda_{Kk} + \frac{a(K,B)}{|B|}$ if,
\[ \lambda_{Kk} := \frac{\mathbb{P}[N_{Kk}(B) = 1]}{|B|}, \]
\[ g_1(K,B) := \sum_{k=1}^{K} \sum_{j=2}^{\infty} j \mathbb{P}[N_{Kk}(B) = j] \rightarrow 0 \quad \text{as} \quad K \rightarrow \infty, \]
\[ g_2(K) := \max_{k \in [1 : K]} \lambda_{Kk} \rightarrow 0 \quad \text{as} \quad K \rightarrow \infty. \]

The next lemma shows that $\Phi_{K}$ locally converges to $\Phi^{*}$ on $B$ in total variation sense. The lemma holds for any bounded $B \in \mathfrak{B}$, but we focus on a bounded interval $B = [a,b]$. Proof is based on so called Poisson approximation and available in various forms, e.g., [12], but we give a more detailed proof with explicit convergence speed. Let $\lambda^{*}_{K} := \sum_{k} \lambda_{Kk}$.

Lemma 19: Fix a bounded $B \in \mathfrak{B}$ of interest and let $\Phi_{K}$ be Poisson PP with intensity $\lambda_{K}^{*}[B]$. Suppose individual PPs of the triangular array are sparse with sum rate $\lambda_{K}$. Then, $N_{K}(B) \rightarrow N_{K}^{*}(B)$ in total variation. Furthermore,
the speed of convergence is \( O(g(K, B)) \), where \( g(K, B) := \max \{ g_1(K, B), |B|^2 g_2(K) \} \).

Proof: See App. [D]

The next corollary is especially useful in the next subsection, where each user sends symbols on i.i.d. renewal arrivals.

**Corollary 20:** Further, suppose component PPs in a row of the triangular array are identically distributed, and \( \lambda^*_K = \lambda \) for all \( K \), i.e., Poisson PPs corresponding to each row are identical. Then, \( d_{TV}(N(B)_K, N^*(B)) \to 0 \) as \( K \to \infty \) with speed \( O(g_1(K, B), |B|^2 K^{-1}) \), where \( N^* \) is the counting measure for the Poisson PP with intensity \( \lambda \).

### C. Capacity Approximation

We reformulate input processes of the queue as two-sided RMPPs to streamline proofs and arguments. Recall that the mark space \( M = \mathbb{R}_+ \) and service times are drawn i.i.d. from \( P_S \). Suppose that the RMPPs begin at \( t = -T \) for large \( T > 0 \) and the queue is initially empty. Since all randomness of queueing is captured by the RMPP, any queue-state process is a deterministic function of \( \Phi(t) \) and initial state \( \theta_{-T} \). For example, discrete-time queue state processes, such as queue-length seen by arrivals or departures, can be expressed as \( z(i, \Phi, \theta_{-T}) \) for some deterministic function \( z \).

As we have seen previously, the process of queue-length seen by departures \( \{Q_i \}_{i \in \mathbb{Z}} \) is of interest. Note that

\[
Q_i(\Phi) = h(i, \Phi, \theta_{-T})
\]

for some deterministic function \( h \).

Consider the case of Cor. 20 where users’ individual arrivals are i.i.d. and corresponding Poisson sum rate is identically \( \lambda^*_K = \lambda \) for all \( K \). As corresponding Poisson PPs are identically distributed regardless of row \( K \), row index \( K \) for Poisson related quantities is dropped. Define \( Q^{(K)}_i \) to be the queue-length process seen by \( i \)th arrival of the \( K \)-user superposition process. Similarly let \( Q^{(K)}_i \) be the corresponding process for the Poisson PP \( \Phi^* = \Phi^* K \) for all \( K \). Then, the continuity theorem holds due to the locally convergence property above. Here, \( TV \) denotes local convergence of PP on \( B \in \mathbb{B} \) in total variation. For random variables, \( TV \) is the usual total variational convergence.

**Lemma 21:** For any \( \epsilon > 0 \), we can take a large interval \( B = B(\epsilon) \in \mathbb{B} \) that yields

\[
d_{TV}(Q^{(K)}_i, Q^{(K)}_i(\epsilon)) \leq 2 \epsilon + O(g(K, B)),
\]

where \( g(K, B) = \max \{ g_1(K, B), |B|^2 g_2(K) \} \). In other words, \( Q^{(K)}_i(\epsilon) \stackrel{TV}{\to} Q^{(K)}_i(\epsilon) \).

**Proof:** See App. [E]

Recall notations that \( \pi_{KK}, \pi_K \) denote the stationary queue-length distributions seen by individual user’s and superposed departures, respectively. As individual users are symmetric, \( \pi_{KK} \) are identical and in addition \( \pi_{KK} = \pi_K \) for all \( K \).

Since each arrival has only a few arrivals on \( B \) (with high probability), we implicitly suppose the transmission is repeated many times to achieve block code performance.

Let \( c_{\text{max}} := \sup_q \min \pi_{x} I(P_X, W_q) \), which is \( c_{\text{max}} \leq \log |\mathcal{X}| \) clearly. The final approximation follows.

**Theorem 22:** Let \( C(\Phi^*) \) be the single-user capacity of M/G/1 queue with arrival rate \( \lambda \), derived in Thm. [9]. Consider \( K \) users with sparse individual PPs \( \Phi_{KK} \). Then, under superposition, the sum capacity \( C^*_{\text{sum}}(\Phi_K) \) at arrival rate \( \lambda \) is approximated by the single-user capacity \( C(\Phi^*) \) as

\[
|C^*_{\text{sum}}(\Phi_K) - C(\Phi^*)| \leq c_{\text{max}} (4 \epsilon + O(g(K, B))) \quad \text{[bits/sym],}
\]

\[
|C^*_{\text{sum}}(\Phi_K) - C(\Phi^*)| \leq \frac{g_1(K, B)}{|B|} c_{\text{max}} + \lambda c_{\text{max}} (4 \epsilon + O(g(K, B))) \quad \text{[bits/time].}
\]

**Proof:** As \( \pi_K = \pi_{KK} \) for all \( K \), individuals can send information at rate

\[
C(\Phi_K) = \sum_q \pi_{kk}(q) I(P_X, W_q) \quad \text{[bits/sym]},
\]

the sum rate is also \( C(\Phi_{KK}) \) in bits per symbol sense. On the other hand, the stationary distribution \( \pi_K \) differs from the stationary distribution for Poisson, say \( \pi^* \), at most

\[
2 \epsilon + O(g(K, B))
\]

in total variation. This implies

\[
|C^*_{\text{sum}}(\Phi_K) - C(\Phi^*)| = \sum_{q=0}^{\infty} \pi^*(q) |\pi^*(q) - \pi_K(q)| \leq \sum_{q=0}^{\infty} \pi^*(q) - \pi_K(q) = c_{\text{max}} \cdot 2d_{TV}(Q^{(K)}_i, Q^{(K)}_i). \]

To obtain the second bound, recall that actual sum arrival rate of the superposition process deviates from \( \lambda \) by \( \frac{g_1(K, B)}{|B|} \).

Therefore,

\[
|C^*_{\text{sum}}(\Phi_K) - C(\Phi^*)| = \left| \left( \lambda + \frac{g_1(K, B)}{|B|} \right) \sum_q \pi_K(q) I(P_X, W_q) - \lambda \sum_q \pi^*(q) I(P_X, W_q) \right|
\]

\[
\leq \frac{g_1(K, B)}{|B|} c_{\text{max}} + \lambda c_{\text{max}} \cdot 2d_{TV}(Q^{(K)}_i, Q^{(K)}_i)
\]

\[
\leq \frac{g_1(K, B)}{|B|} c_{\text{max}} + \lambda c_{\text{max}} (4 \epsilon + O(g(K, B))) \quad \text{[bits/time].}
\]

Thm. 22 only considers the sum capacity, however, it is clear from the proof that individual per symbol capacity remains unchanged, and per time capacity is properly scaled, i.e.,

\[
|C_{\text{ind}}(\Phi_{KK}) - C(\Phi^*)| \leq \frac{g_1(K, B)}{K|B|} c_{\text{max}} + \frac{\lambda}{K} c_{\text{max}} (4 \epsilon + O(g(K, B))) \quad \text{[bits/time].}
\]

Therefore, the best and worst server results in Cor. 13 also apply to the superposition arrivals asymptotically as \( K \to \infty \).
**Corollary 23:** Suppose the conditions in Sec. 13-B hold. Then, for the K-user setting with sparse individuals, the results in Cor. 13 still hold asymptotically, that is, when the service quality steps down at $b = 1$, the sum and individual capacities are maximized when the service is deterministic. On the other hand, the sum and individual capacities are asymptotically minimized by cramming service.

**V. DISCUSSION**

In this paper, we have presented the capacity of queue-length dependent service quality in a multiple-access setting, motivated by the facts that 1) overloading lowers system performance, 2) workload comes from multiple independent sources. We modeled the workload and its processing as a queueing process with noisy server and characterized capacity of single-user and multiple-access systems.

We first obtain the capacity in multi-letter form, however, ergodicity of the workload enables us to derive the single-letter expressions in Thms. 6 and 17. Unlike typical multiple-access problems, information rate in codewords does not change other users’ performance as in Thm. 17. This is because other jobs affects channels only through their arrival processes, not through information rate. Furthermore, when the number of users is large and each arrival process is sparse, the individual and sum capacities are asymptotically close to the single-user capacity of $M/GI/1$ queues, and thus, the best (resp. the worst) service in single-user is also the best (resp. the worst) in multiple-access as well.

Moving to another queueing metric, we can also consider the case that service quality relies on the queue-length seen by arrivals. This might arise when customers in a hurry are the source of errors. The basic results of this paper, however, remain unchanged since for single-user case, distributions of queue-length seen by arrivals and departures are identical. The distributions seen by arrivals and departures are nonidentical for multiple-access settings, but, the approach in App. C still applies by a stopped process at arrival moments. Waiting time dependent service quality may arise in quantum information processing.

**APPENDIX A**

**PROOF OF THM. 8 (GI/1/1 QUEUES)**

The parts that deal with finding $\pi$ are standard, but are presented for completeness.

In the case of GI/1/1 queues, it is easier to derive the queue-length distribution seen by $i$th arrival (i.e., just prior to arrivals), say $\hat{Q}_i$, than $Q_i$ because of memoryless property of the server. From the same argument as in the proof of Lem. 5, we know that generic random variable $\hat{Q} \sim Q$ when they are stationary, so we will consider $\hat{Q}$ instead of $Q$.

Notice that $\hat{Q}_{n+1} = (\hat{Q}_n - \beta_i + 1)_+$, where $\beta_i$ is the number of jobs completed during the inter-arrival time $A_{n+1}$. As $\{A_n\}$ is i.i.d., it does not depend on the past history of the queue and neither does $\beta_i$. Therefore, $\hat{Q}_n$ forms a discrete-time Markov chain.

Define $\ell_q$ to be the probability of $q$ job completions between two consecutive arrivals, i.e.,

$$\ell_q := P[\beta_n = q | \hat{Q}_n \geq q] = \int_0^\infty p^A(t) \frac{e^{-\mu t} (\mu t)^q}{q!} dt.$$  \hspace{1cm} (10)

Then, the transition matrix $[P]$ is given by

$$[P] = \begin{bmatrix}
1 - \ell_0 & \ell_0 & 0 & 0 & \cdots \\
1 - \ell_0 - \ell_1 & \ell_1 & 0 & 0 & \cdots \\
1 - \ell_0 - \ell_1 - \ell_2 & \ell_2 & \ell_1 & 0 & \cdots \\
& \vdots & \vdots & \vdots & \ddots
\end{bmatrix},$$

and the stationary distribution relationship $\hat{\pi} = \hat{\pi}(P)$ yields

$$\hat{\pi}(0) = \sum_{q=0}^{\infty} \hat{\pi}(q) \left(1 - \sum_{i=0}^{q} \ell_i\right),$$

$$\hat{\pi}(i) = \sum_{q=0}^{\infty} \ell_q \hat{\pi}(i + q - 1) \text{ for } i > 1. \hspace{1cm} (11)$$

As the stationary distribution is unique, it suffices to show that $\hat{\pi}(q) = \hat{\pi}(0) \sigma^q$ for some $\sigma < 1$. Substituting $\hat{\pi}(q) = \hat{\pi}(0) \sigma^q$ into (11), we have

$$\sigma = \sum_{q=0}^{\infty} \ell_q \sigma^q =: B(\sigma). \hspace{1cm} (12)$$

Note that $B(0) = \ell_0 > 0$, $B(1) = 1$, and $B(\sigma)$ is convex over $\sigma \in [0, 1]$ since $B'(\sigma), B''(\sigma) \geq 0$. There are two possible cases: no fixed point in $(0, 1)$ or a unique fixed point in $(0, 1)$. Recall $B(\sigma)$ is a probability generating function of $\ell_q$ and thus, $B'(1) = \frac{\mu}{\lambda} = \rho^{-1} > 1$ since it is the number of job completions normalized by inter-arrivals. Therefore, the latter is the only possibility and the fixed point in $(0, 1)$ is unique. Let $\sigma^*$ denote the solution.

On the other hand, substituting (10) into (12),

$$\sigma = \sum_{q=0}^{\infty} \ell_q \sigma^q = \sum_{q=0}^{\infty} \left(\int_0^\infty p^A(t) \frac{e^{-\mu t} (\mu t)^q}{q!} dt\right) \sigma^q$$

$$= \int_0^\infty p^A(t) e^{-\mu t} \sum_{q=0}^{\infty} \frac{(\mu t)^q}{q!} dt$$

$$= \int_0^\infty p^A(t) e^{-\mu t(1-\sigma)} dt$$

$$= A^*(\mu(1-\sigma)),$$

where $A^*(\cdot)$ is the Laplace-Stieltjes transform of $p^A(t)$. Hence, the fixed point solution $\sigma^*$ is the unique root of

$$\sigma = A^*(\mu(1-\sigma)).$$

As $\sum_q \hat{\pi}(q) = 1$, it is easy to see $\hat{\pi}(0) = 1 - \sigma^*$. Therefore,

$$\pi(q) = \hat{\pi}(q) = (1 - \sigma^*) \sigma^q.$$

Such a matrix is called a lower Hessenberg matrix.
APPENDIX B
PROOF OF THM. 9 (M/GI/1 QUEUES)

The parts that deal with finding \( \pi \) are standard [22], but are presented for completeness.

To derive \( \pi(\Omega) \) in closed form, we will first show that \( \{Q_i\} \) forms a Markov chain, and then represent the stationary distribution in terms of \( P^S \).

Let \( Q_{n+1} \) be the queue-length seen by \((n+1)\)th departure. Then, we observe that
\[
\begin{align*}
Q_{n+1} = \begin{cases} 
Q_n + \alpha_{n+1} - 1 & \text{if } Q_n \geq 1, \\
\alpha_{n+1} & \text{if } Q_n = 0,
\end{cases}
\]
where \( \alpha_{n+1} \) is the number of jobs arriving during the service time of \((n+1)\)th job. Since \( \alpha_{n+1} \) is independent of past history \( \{Q_n, Q_{n-1}, \ldots, Q_1\} \), we know that \( \{Q_n\} \) forms a discrete-time Markov chain. Furthermore, it is time-homogeneous as inter-arrivals and services are i.i.d.

Denote the transition probability of the Markov chain by \( p_{ij} := P[Q_{n+1} = j | Q_n = i] \). Then,
\[
p_{ij} = \begin{cases} 
P[j - i + 1 \text{ arrivals during service}] & \text{if } i \geq 1, \\
P[j \text{ arrivals during service}] & \text{if } i = 0.
\end{cases}
\]

We obtain \( p_{ij} \) by marginalizing joint probability. Since the number of arrivals is Poisson,
\[
P[q \text{ arrivals during service}] = \int_0^\infty P[S = t \text{ and } q \text{ arrivals}] dt
\]
\[
= \int_0^\infty P^S(t)P[q \text{ arrivals}] dt
\]
\[
= \int_0^\infty P^S(t) e^{-\lambda t} \frac{(\lambda t)^q}{q!} dt.
\]
Letting \( k_q := P[q \text{ arrivals during service}] \), brevity, the transition matrix is given as follows [3]
\[
[P] = \begin{bmatrix}
k_0 & k_1 & k_2 & k_3 & \cdots \\
k_0 & k_1 & k_2 & k_3 & \cdots \\
0 & k_0 & k_1 & k_2 & \cdots \\
0 & 0 & k_0 & k_1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

Solving the stationary distribution identity \( \pi = \pi[P] \),
\[
\pi(q) = \pi(0)k_q + \sum_{j=1}^{q+1} \pi(j)k_{q-j+1}.
\]

Multiplying the equations of each \( q \) by \( z^q \) and summing over \( q = 0, 1, \ldots \), we have
\[
\Pi(z) = \frac{\pi(0)(1-z)K(z)}{K(z) - z},
\]
where \( \Pi(z), K(z) \) are probability generating functions of \( \pi(q) \) and \( k_q \), that is,
\[
\Pi(z) = \sum_{q=0}^\infty \pi(q)z^q \quad \text{and} \quad K(z) = \sum_{q=0}^\infty k_qz^q.
\]

Such a matrix is called an upper Hessenberg matrix.

Note that \( K(1) = \sum q_kq = 1 \). By l’Hôpital’s rule at \( z = 1 \), we have \( \pi(0) = 1 - K'(1) \). Since \( k_q \) is the normalized number of arrivals during service time, the first moment \( K'(1) = \rho = \frac{1}{\mu} \), which implies \( \pi(0) = 1 - \rho \). Therefore,
\[
\Pi(z) = \frac{(1-\rho)(1-z)K(z)}{K(z) - z}.
\]

APPENDIX C
PROOF OF LEM. 16

To prove the ‘seen by departures’ result, we start from continuous-time ergodicity in [23]. We first take a continuous-time piecewise-deterministic Markov process [24]. Then, since it is strong Markov, the stopped process at user \( k \) departures forms a stationary and ergodic discrete-time Markov chain. Suppose that once job processing is completed and the job departs at time \( t \), the next job enters the server at time \( t+\Delta \).

Let us take a continuous-time Markov process \( Z(t) := (L(t), A(t), S(t)) \in \mathcal{Z} \), where
\[
\begin{align*}
\bullet & \quad L(t) \text{ is the vector of transmitter jobs in order of their arrivals including the job in the server. If the system is empty, } \{L(t) = 0\}. \text{ Otherwise, } L(t) = (\ell_0, \ell_1, \ell_2, \ldots) \in [1 : K]^{Q(t)+1}, \text{ where } Q(t) \text{ is the queue-length at time } t. \\
\bullet & \quad A(t) \in \mathbb{R}_+^K \text{ is the residual arrival time vector whose component } A_k(t) \text{ indicates the remaining time until the next arrival of } k\text{'s user.} \\
\bullet & \quad S(t) \in (\mathbb{R}_+ \cup \infty)^K \text{ is the residual service time vector whose component } S_k(t) \text{ indicates residual service time if user } k\text{'s job is being served, infinite otherwise.}
\end{align*}
\]

Under condition [7], this is Harris recurrent so that there exists the stationary distribution \( \hat{\pi} \) and the following holds [23 Thm. 6.4]: For any \( g : \mathcal{Z} \mapsto \mathbb{R}_+ \),
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t g(Z(s)) ds = \mathbb{E}_\pi[g(Z)] \text{ almost surely.} \quad (13)
\]

Fix a user \( k \) and take a sequence of stopping times \( (t_1, t_2, \ldots) \) such that \( t_n := \min\{t > t_{n-1} : S_k(t) > 0, S_k(t) = 0\} \) (assume \( t_0 < 0 \) for simplicity), i.e., the sequence of hitting times at which user \( k \)th job departs. Take a small \( \Delta > 0 \) and two indicators \( g_1 := 1\{S_k(t) \leq \Delta\}, g_2 := 1\{|L(t)| = q + 1, S_k(t) \leq \Delta\} \). Since either inter-arrival time distributions or service time distribution is continuous, we know that \( \hat{\pi} \) is also continuous. Therefore, (13) implies
\[
\lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} g_1(Z(s)) ds \approx \Delta \cdot \hat{\pi}\{Z(t) : S_k(t) = 0\},
\]
\[
\lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} g_2(Z(s)) ds \approx \Delta \cdot \hat{\pi}\{Z(t) : Q(t) = q, S_k(t) = 0\}.
\]
Taking \( \Delta \to 0 \) and using the fact that the queue-length is a deterministic function of \( L(t) \), it follows that the stationary distribution exists and
\[
\pi_{Kk}(q) := \frac{\hat{\pi}\{Z(t) : |L(t)| = q + 1, S_k(t) = 0\}}{\hat{\pi}\{Z(t) : S_k(t) = 0\}}. \quad (14)
\]

Next show the ergodicity. Define two samplings
\[
\begin{align*}
h_1(Z(t)) & := 1\{S_k(t) \leq \Delta\}, \\
h_2(Z(t)) & := 1\{S_k(t) \leq \Delta\}f(q(t)),
\end{align*}
\]
and note that

$$\lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} h_1(Z(s))ds = \lim_{n \to \infty} \frac{n \Delta}{t_n} = \lambda_{Kk} \Delta$$

and

$$\lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} h_2(Z(s))ds = \lim_{n \to \infty} \frac{1}{t_n} \sum_{i=1}^{n} f(q(t_i)) \Delta = \lim_{n \to \infty} \frac{n}{t_n} \sum_{i=1}^{n} f(q(t_i)), \quad \lim_{n \to \infty} \frac{n}{t_n} \to \lambda_{Kk} \Delta$$

where \( \lim_{n \to \infty} \frac{n}{t_n} \to \lambda_{Kk} \Delta \) is assumed due to the system stability. Then,

$$\lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} h_2(Z(s))ds = \lim_{n \to \infty} \frac{1}{t_n} \sum_{i=1}^{n} f(q(t_i)). \quad (15)$$

Also letting \( \Delta \to 0 \) and applying (13) to the left side of (15),

$$\lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} h_2(Z(s))ds = \frac{\mathbb{E} \pi_{Kk}}{\mathbb{E} \pi_{Kk}} \sum_{q=0}^{\infty} f(q) \times \pi(Z(t) : S_k(t) = 0, |L(t)| = q + 1)$$

$$= \sum_{q=0}^{\infty} f(q) \times \pi(Z(t) : S_k(t) = 0, |L(t)| = q + 1)$$

$$= \sum_{q=0}^{\infty} f(q) \pi_{Kk}(q) = \mathbb{E}_{\pi_{Kk}}[f(Q)]. \quad (16)$$

Since \( Q_i = Q(t_i) \), the following holds from (15) and (16),

$$\lim_{n \to \infty} \frac{1}{t_n} \sum_{i=1}^{n} f(q_i) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(q(t_i)) = \mathbb{E}_{\pi_{Kk}}[f(Q)]$$

almost surely.

**APPENDIX D**

**Proof of Lem. 19**

We restricted to PPs over a bounded \( B \) so \( \Phi_K, \Phi^*_K \) both have no events outside of \( B \). Therefore it is sufficient to show that for all \( B' \in \mathcal{B} \) such that \( B' \subseteq B \),

$$d_{TV}(N_{Kk}(B'), N_{Kk}^*(B')) \to 0 \text{ as } K \to \infty.$$

Note that Poisson processes are infinitely divisible, so we can split into \( K \) independent Poisson PPs \( \{\Phi^*_k\}_{k \in \{1:K\}} \) with intensity \( \lambda_{Kk} \). Let \( N_{Kk} \) be the counting measure of \( \Phi_{Kk} \). From the Poisson distribution and its Taylor expansion when \( |B| \lambda_{Kk} \) is small:

$$\mathbb{P}[N_{Kk}^*(B) = 1] = |B| \lambda_{Kk} + O(|B|^2 \lambda_{Kk}^2),$$

$$\mathbb{P}[N_{Kk}(B) \geq 2] = O(|B|^2 \lambda_{Kk}^2).$$

Hence, total variational distance between individual PPs is computed as follows, where argument \( B \) is omitted for simplicity.

$$2d_{TV}(N_{Kk}, N_{Kk}^*)$$

$$= \sum_{j \in \mathbb{Z}_+} |\mathbb{P}[N_{Kk} = j] - \mathbb{P}[N_{Kk}^* = j]|$$

$$= \left| (1 - \mathbb{P}[N_{Kk} \geq 1]) - (1 - \mathbb{P}[N_{Kk}^* \geq 1]) \right|$$

$$= \sum_{j \geq 1} \left| \mathbb{P}[N_{Kk} = j] - \mathbb{P}[N_{Kk}^* = j] \right|$$

$$= \left| \mathbb{P}[N_{Kk}^* = 1] + \mathbb{P}[N_{Kk}^* \geq 2] - \mathbb{P}[N_{Kk} = 1] \right|$$

$$- \mathbb{P}[N_{Kk} \geq 2] \right| + \sum_{j \geq 1} \left| \mathbb{P}[N_{Kk} = j] - \mathbb{P}[N_{Kk}^* = j] \right|$$

$$\leq \left| \mathbb{P}[N_{Kk}^* = 1] - \mathbb{P}[N_{Kk} = 1] \right| + \mathbb{P}[N_{Kk}^* \geq 2]$$

$$+ \mathbb{P}[N_{Kk} \geq 2] \right| + \sum_{j \geq 1} \left| \mathbb{P}[N_{Kk} = j] - \mathbb{P}[N_{Kk}^* = j] \right|$$

$$\leq \mathbb{P}[N_{Kk}^* = 1] - |B| \lambda_{Kk} + O(|B|^2 \lambda_{Kk}^2) + \mathbb{P}[N_{Kk} \geq 2]$$

$$+ \sum_{j \geq 1} \left| \mathbb{P}[N_{Kk} = j] - \mathbb{P}[N_{Kk}^* = j] \right|$$

$$\leq O(|B|^2 \lambda_{Kk}^2) + \mathbb{P}[N_{Kk} \geq 2]$$

$$+ \sum_{j \geq 1} \left| \mathbb{P}[N_{Kk} = j] - \mathbb{P}[N_{Kk}^* = j] \right|$$

$$\leq O(|B|^2 \lambda_{Kk}^2) + \mathbb{P}[N_{Kk} \geq 2] + \mathbb{P}[N_{Kk} \geq 2] + \mathbb{P}[N_{Kk}^* \geq 2]$$

$$\leq O(|B|^2 \lambda_{Kk}^2) + 2\mathbb{P}[N_{Kk} \geq 2],$$

where (a) follows from the triangle inequality; (b) follows from (9) and the Taylor expansion; (c) follows from the Taylor expansion; (d) follows from the triangle inequality; and (e) follows from the Taylor expansion.

Now we bound total variation between two sums of independent random variables as follows.

$$d_{TV}(N_{Kk}, N_{Kk}^*)$$

$$\leq \sum_{k \in \{1:K\}} d_{TV}(N_{Kk}, N_{Kk}^*)$$

$$\leq \sum_{k \in \{1:K\}} O(|B|^2 \lambda_{Kk}^2) + \sum_{k \in \{1:K\}} \mathbb{P}[N_{Kk} \geq 2]$$

$$\leq c|B|^2 \cdot \sum_{k \in \{1:K\}} \lambda_{Kk} \left( \max_{k \in \{1:K\}} \lambda_{Kk} \right) + \sum_{k \in \{1:K\}} \mathbb{P}[N_{Kk} \geq 2]$$

$$= c|B|^2 \cdot \lambda_{K}^* \cdot g_2(K) + \sum_{k \in \{1:K\}} \mathbb{P}[N_{Kk} \geq 2],$$

where (a) follows from the total variation inequality for product measures, and (b) follows from the above derivation.

Therefore, the first term vanishes at speed \( O(|B|^2 g_2(K)) \), the second term \( \sum_k \mathbb{P}[N_{Kk} \geq 2] \to 0 \) at speed \( O(g_1(K, B)) \). So the overall speed of convergence is given by \( O(g(K, B)) \), where \( g(K, B) := \max\{g_1(K, B), |B|^2 g_2(K)\} \).
Finally, for all subsets $B' \subset B$ with $B' \in \mathcal{B}$, we can repeat the above argument, but the speed of convergence still holds since $g_1(K, B') \leq g_1(K, B)$ and $|B'|g_2(K) \leq |B|g_2(K)$.

**APPENDIX E**

**PROOF OF LEM. [21]**

We will first restrict the superposed RMPP on some $B$, and then apply the data processing inequality (also known as monotone theorem in some literature [25]) to show $Q_i^{(K)} \to TV Q_i^*$. Without loss of generality, we only consider some arbitrary $i$th system whose arrival was at $t_i > 0$.

Let us introduce *empty points* [10]. When $\phi(t)$ is a specific realization of $\Phi(t)$, an arrival time instance $e_j(\phi)$ at which there is no job in the system (in the queue and in the server both) is called an empty point.\(^4\) List $e_j(\phi)$ in order

$$\cdots < e_{-1}(\phi) < e_0(\phi) \leq 0 < e_1(\phi) < \cdots$$

The $j$th empty point implies that the queue state after $t = e_j(\phi)$ is completely determined only by arrivals after $e_j(\phi)$. Then, we know that $e_0(\Phi_K) \to TV e_0(\Phi^*)$ with speed $O(g(K, B))$ by data processing inequality and thus, $e_j(\Phi_K) \to TV e_j(\Phi^*)$ for any $j$ by stationarity.

Take a set of $\mathcal{P}$ realizations $A_{u_1} := \{ \phi : -u_1 < e_0(\phi) \leq 0 \}$. Since $e_0(\Phi_K) \to TV e_0(\Phi^*)$, for arbitrary $\epsilon_1 > 0$ it is possible to take $u_1, K_0$ such that for all $K \geq K_0$,

$$P_K[A_{u_1}] > 1 - \epsilon_1 \quad \text{and} \quad P^*[A_{u_1}] > 1 - \epsilon_1.$$

Also, take a set $A_{u_2} := \{ \phi : 0 < t_i(\phi) < u_2 \}$. Thus it is immediate that for arbitrary $\epsilon_2 > 0$ we can take $u_2 > 0$ such that $P_K[A_{u_2}] > 1 - \epsilon_2$ and $P^*[A_{u_2}] > 1 - \epsilon_2$.

Let $q(i, \phi)$ be the queue-length seen by $i$th departure of $\phi$, and $u := \max(u_1, u_2)$. By the property of the empty point and $A_{u_1}, A_{u_2}$,

$$P^*[\phi : q(i, \phi) = q(i, 1_{[-u, u]})) \geq P^*[A_{u_1} \cap A_{u_2}] > 1 - \epsilon, \quad P_K[\phi : q(i, \phi) = q(i, 1_{[-u, u]})) \geq P_K[A_{u_1} \cap A_{u_2}] > 1 - \epsilon.$$

Setting $B = [-u, u)$, we can bound total variation as follows.

$$d_{TV}(Q_i(\Phi_K), Q_i(\Phi^*)) \leq d_{TV}(Q_i(\Phi_K), Q_i(1_B \Phi_K)) + d_{TV}(Q_i(1_B \Phi_K), Q_i(1_B \Phi^*)) + d_{TV}(Q_i(1_B \Phi^*), Q_i(\Phi^*)) \leq 2\epsilon + d_{TV}(Q_i(1_B \Phi_K), Q_i(1_B \Phi^*)) \leq 2\epsilon + d_{TV}(1_B \Phi_K, 1_B \Phi^*) \leq 2\epsilon + O(g(K, B)).$$

where (a) follows from the triangle inequality; (b) follows from the property of empty point; (c) follows from the data processing inequality since $Q_i(\cdot)$ is a function of a PP; and (d) follows from Lem. [19]. Since $\epsilon_1, \epsilon_2$ are arbitrary, the statement is proved.

\(^4\)This is different from the regenerative cycles, introduced in Sec. III. Since we are considering arbitrary superposition process $\Phi$ that is not renewal in general, so $e_j(\Phi)$ is not regenerative.

**REFERENCES**

[1] D. Seo, A. Chatterjee, and L. R. Varshney, “On multiuser systems with queue-length dependent service quality,” in Proc. 2018 IEEE Int. Symp. Inf. Theory, Jun. 2018, pp. 341–345.

[2] B. Schwartz, “Queues, priorities, and social process,” Soc. Psychol., vol. 41, no. 1, pp. 3–12, Mar. 1978.

[3] D. C. Dugdale, R. Epstein, and S. Z. Pantilat, “Time and the patient-physician relationship,” J. Gen. Intern. Med., vol. 14, no. S1, pp. S34–S40, Jan. 1999.

[4] R. W. Derler and J. R. Richards, “Overcrowding in the nation’s emergency departments: Complex causes and disturbing effects,” Ann. Emerg. Med., vol. 35, no. 1, pp. 63–68, Jan. 2000.

[5] M. Jamal, “Job stress and job performance controversy revisited: An empirical examination in two countries,” Int. J. Stress Management, vol. 14, no. 2, pp. 175–187, May 2007.

[6] A. Chatterjee, D. Seo, and L. R. Varshney, “Capacity of systems with queue-length dependent service quality,” IEEE Trans. Inf. Theory, vol. 63, no. 6, pp. 3950–3963, Jun. 2017.

[7] V. Anantharam and S. Verdú, “Bits through queues,” IEEE Trans. Inf. Theory, vol. 42, no. 1, pp. 4–18, Jan. 1996.

[8] A. S. Bedekar and M. Azizoglu, “The information-theoretic capacity of discrete-time queues,” IEEE Trans. Inf. Theory, vol. 44, no. 2, pp. 446–461, Mar. 1998.

[9] S. Higginbotham, “Autonomous trucks need people,” IEEE Spectr., vol. 56, no. 3, p. 21, Mar. 2019.

[10] P. Franken, D. König, U. Arndt, and V. Schmidt, Queues and Point Processes. New York: John Wiley & Sons, 1982.

[11] D. J. Daley and D. Vere-Jones, An Introduction to the Theory of Point Processes. New York: Springer, 1998.

[12] O. Kallenberg, Random Measures, Theory and Applications. Cham, Switzerland: Springer, 2017.

[13] I. E. Telatar and R. G. Gallager, “Combining queuing theory with information theory for multiaccess,” IEEE J. Sel. Areas Commun., vol. 13, no. 6, pp. 963–969, Aug. 1995.

[14] S. Raj, E. Telatar, and D. Tse, “Job scheduling and multiple access,” in Advances in Network Information Theory, P. Gupta, G. Kramer, and A. J. van Wijngaarden, Eds. Providence: DIMACS, American Mathematical Society, 2004, pp. 127–137.

[15] X. Chen, T.-Y. Chen, and D. Guo, “Capacity of Gaussian many-access channels,” IEEE Trans. Inf. Theory, vol. 63, no. 6, pp. 3516–3539, Jun. 2017.

[16] S. Verdú and T. S. Han, “A general formula for channel capacity,” IEEE Trans. Inf. Theory, vol. 40, no. 4, pp. 1147–1157, Jul. 1994.

[17] T. S. Han, Information-Spectrum Methods in Information Theory. Berlin: Springer, 2003.

[18] A. Asmussen, Applied Probability and Queues, 2nd ed. New York, USA: New York: Springer-Verlag, 2003.

[19] A. Vempaty, L. R. Varshney, and P. K. Varshney, “Reliable crowdsourcing for multi-class labeling using coding theory,” IEEE J. Sel. Topics Signal Process., vol. 8, no. 4, pp. 667–679, Aug. 2014.

[20] S. M. Samuels, “A characterization of the Poisson process,” J. Appl. Probab., no. 1, pp. 72–85, Mar. 1974.

[21] K. Jagannathan, A. Chatterjee, and P. Mandayam, “Qubits through queues: the capacity of channels with waiting time dependent errors,” in Proc. 25th National Conf. Commun. (NCC’19), Feb. 2019.

[22] L. Kleinrock, Queuing Systems, Volume II: Computer Applications. John Wiley & Sons, Inc., 1976.

[23] J. Dai and S. Meyn, “Stability and convergence of moments for multi-class queueing networks via fluid limit models,” IEEE Trans. Autom. Control, vol. 40, no. 11, pp. 1889–1904, Nov. 1995.

[24] M. H. A. Davis, “Piecewise-deterministic Markov processes: A general model for multiaccess,” in Advances in Network Information Theory, P. Gupta, G. Kramer, and A. J. van Wijngaarden, Eds. Providence: DIMACS, American Mathematical Society, 2004, pp. 127–137.