The averaging of non-local Hamiltonian structures in Whitham’s method.

Andrei Ya. Maltsev

L.D. Landau Institute for Theoretical Physics, 117940
ul. Kosygina 2, Moscow, maltsev@itp.ac.ru
and
SISSA-ISAS, Via Beirut 2-4 - 34014 Trieste, ITALY
maltsev@sissa.it

Abstract

We consider the \(m\)-phase Whitham’s averaging method and propose a procedure of “averaging” of non-local Hamiltonian structures. The procedure is based on the existence of a sufficient number of local commuting integrals of a system and gives a Poisson bracket of Ferapontov type for the Whitham’s system. The method can be considered as a generalization of the Dubrovin-Novikov procedure for the local field-theoretical brackets.

Introduction.

We consider the averaging of non-local Hamiltonian structures in Whitham’s averaging method. As it is well known, the Whitham’s method permits to obtain equations on the “slow” modulated parameters of exact periodic or quasi-periodic solutions of systems of partial differential equations and it was pointed out by Whitham \([1]\) that these equations can be written in the Lagrangian form if the initial system possesses a local Lagrangian structure. The Lagrangian formalism for the Whitham’s system is given in this case by the “averaging” of a local Lagrangian function, defined for the initial system, on the corresponding space of (quasi)-periodic solutions. Some basic questions concerning Whitham’s method can be found in \([1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]\).

B.A. Dubrovin and S.P. Novikov investigated also the question of the conservation of local field-theoretical Hamiltonian structures in Whitham’s method and suggested a procedure of “averaging” of a local field-theoretical Poisson bracket, giving a Poisson bracket of Hydrodynamic type for the Whitham system \([4, 7, 9]\), see also \([17]\).

The Jacobi identity for the averaged bracket and the invariance of the Dubrovin-Novikov procedure was proved by the author in \([18]\) (see also \([19]\)) using the Dirac restriction procedure of the initial bracket on the subspace of quasi-periodic “\(m\)-phase” solutions of the initial system. The connection between the procedure of Dubrovin and Novikov and the procedure of averaging of the Lagrangian function in the case when the initial local Hamiltonian structure just follows from the local Lagrangian one was also studied in \([20]\). Some extension of the averaging of “local” Hamiltonian structures for the case of discrete systems is also presented in \([21]\).

In the present work we consider the Poisson brackets having a non-local part
\[
\{ \varphi'(x), \varphi'(y) \} = \sum_{k \geq 0} B^{ij}_k(\varphi, \varphi_x, \ldots) \delta^{(k)}(x - y) + \sum_{k \geq 0} e_k S^{ij}_k(\varphi, \varphi_x, \ldots) \nu(x - y) S^{ij}_k(\varphi, \varphi_y, \ldots)
\]
where \(e_k = \pm 1\), \(\nu(x - y) = -\nu(y - x)\), \(\partial_x \nu(x - y) = \delta(x - y)\) and both sums contain a finite number of terms depending on a finite number of derivatives of \(\varphi\) with respect to \(x\).

Let us also point out here that the brackets (0.1) usually appear in the theory the so-called “integrable” hierarchies (see [22, 23, 24]), connected with the method of the inverse scattering problem.

The most general form of the non-local Hamiltonian operators (0.1) containing only \(\delta'(X - Y)\) and \(\delta(X - Y)\) in the local part and the quasi-linear fluxes \(S^\nu_{(k)\lambda}(U) U^\lambda_X\) of “hydrodynamic” type in the non-local one

\[
\{U^\nu(X), U^\mu(Y)\} = g^\nu\mu(U) \delta'(X - Y) + b^\nu\mu(U) U^\lambda_X \delta(X - Y) + \sum_{k \geq 0} e_k S^\nu_{(k)\lambda}(U) U^\lambda_X \nu(X - Y) S^\mu_{(k)\delta}(U) U^\delta_Y, \quad 1 \leq \nu, \mu, \lambda, \delta \leq N
\]

was suggested by E.V.Ferapontov in [27] as a generalization of the bracket introduced in [26] and is usually called a weakly non-local Poisson bracket of Hydrodynamic type. We will discuss here a possibility of “averaging” of the brackets (0.1) in the Whitham’s method to obtain the bracket of such “Hydrodynamic type” for the Whitham system.

As was shown by E.V. Ferapontov, the Hamiltonian operators of this type reveal a beautiful differential-geometrical structure following from the Jacobi identity of the bracket ([27, 28, 29, 30]). (In particular they can be obtained as the Dirac restriction of local differential-geometrical Poisson brackets on a submanifold with flat normal connection (28)).

The first example of the non-local bracket (of Mokhov-Ferapontov type, see [26]) for the Whitham’s system for NS equation in the one-phase case was constructed by M.V. Pavlov in [38] from a nice differential-geometrical consideration. After that there was set the question about a possibility of constructing of nonlocal Hamiltonian structures for Whitham’s system from the structures (0.1) for the initial one. As was mentioned above, the Hamiltonian operators (0.1) exist for many “integrable” systems like KdV and in the paper [30] (see also [39]) there was a discussion of the possibility of averaging of the non-local operators for KdV equation using the local bi-Hamiltonian structure and the recursion operator for two averaged local Poisson brackets. The corresponding calculations for the \(m\)-phase periodic solutions of KdV were made by V.L. Alekseev in [40].

Here we propose a general construction for the averaging of operators (0.1) in the Whitham’s method which gives a generalization of the Dubrovin-Novikov procedure for the case of presence of non-local terms in the bracket. Our procedure does not require a local bi-Hamiltonian structure and can be used in the general situation. Like in the procedure of Dubrovin and Novikov, we require here the existence of a sufficient number of commuting local integrals, generating local flows according to (0.1), and we also impose some conditions of “regularity” of the full family of \(m\)-phase solutions as in the local case (see [18]).
1 Some general properties of the non-local brackets.

Let us consider a non-local 1-dimensional Hamiltonian structure of the type:

$$\{\varphi^i(x), \varphi^j(y)\} = \sum_{k \geq 0} B_{k}^{ij}(\varphi, \varphi_x, \ldots) \delta^{(k)}(x - y) +$$

$$+ \sum_{k \geq 0} \tilde{S}_{(k)}^{ij}(\varphi, \varphi_x, \ldots) \nu(x - y) \tilde{T}_{(k)}^{ij}(\varphi, \varphi_y, \ldots) , \quad 1 \leq i, j \leq n$$

(1.1)

where we have finite numbers of terms in both sums depending on a finite number of derivatives of \(\varphi\) with respect to \(x\).

We will call a local translationally invariant Hamiltonian function a functional of the form:

$$H[\varphi] = \int \mathcal{P}_H(\varphi, \varphi_x, \ldots) dx$$

(1.2)

Here \(\nu(x - y)\) is the skew-symmetric function

$$\nu(x - y) = \frac{1}{2} \text{sgn} (x - y) , \quad D_x \nu(x - y) = \delta(x - y) ,$$

(1.3)

and \(\delta^{(k)}(x - y)\) is the \(k\)-th derivative of the delta-function with respect to \(x\).

We assume here that the bracket (1.1) is written in the “irreducible” form, which means that the number of terms in the second sum is the minimal possible and the sets \(\{\tilde{S}_{(k)}\}\) and \(\{\tilde{T}_{(k)}\}\) represent two linearly independent sets of vector-functions of the variables \((\varphi, \varphi_x, \ldots)\). From the skew-symmetry of the bracket (1.1) it follows then that the sets of \(\tilde{S}_{(k)}\) and \(\tilde{T}_{(k)}\) define actually the same linear space in the space of functions and it can be easily seen that the bracket (1.1) can be represented in the “canonical” form:

$$\{\varphi^i(x), \varphi^j(y)\} = \sum_{k \geq 0} B_{k}^{ij}(\varphi, \varphi_x, \ldots) \delta^{(k)}(x - y) + \sum_{k \geq 0} e_k \tilde{S}_{(k)}^{ij}(\varphi, \varphi_x, \ldots) \nu(x - y) \tilde{T}_{(k)}^{ij}(\varphi, \varphi_y, \ldots)$$

(1.4)

where \(e_k = \pm 1\).

Indeed, since the sets \(\{\tilde{S}_{(k)}\}\) and \(\{\tilde{T}_{(k)}\}\) span the same linear space we have just one finite-dimensional space, generated by fluxes (vector fields)

$$\varphi^i_{\tau_k} = \tilde{S}_{(k)}^{i}(\varphi, \varphi_x, \ldots)$$

and a symmetric (view the skew-symmetry of the bracket and the function \(\nu(x - y)\)) finite-dimensional constant 2-form, which describes their couplings in the non-local part of (1.1). So, we can write it in the canonical form according to its signature after some linear transformation of the flows \(\tilde{S}_{(k)}\) and \(\tilde{T}_{(k)}\) with constant coefficients.

We should also define in every case the functional space where we consider the action of the Hamiltonian operator (1.4) and this can depend on a concrete situation. The most natural thing is to consider the functional space \(\varphi(x)\) and the algebra of functionals \(I[\varphi]\), such that their variational derivatives, multiplied by the flows \(S_{(k)}(\varphi, \varphi_x, \ldots)\), give us rapidly decreasing functions as \(|x| \to \infty\).

Below we will use the functionals of the type

$$\int \sum_{p=1}^{n} \varphi^p(x) q_p(x) dx$$

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where \( q_p(x) \) are arbitrary smooth functions with compact supports, to examine the properties of the bracket \((1.4)\). For all the other functionals used in the considerations we will assume that they have a compatible with the bracket \((1.4)\) form in the sense discussed above.

We will assume here for simplicity that the functions \( B_{ij}^k \) and \( S_{(k)}^i \) represent analytic functions of their arguments (maybe in some open region of the values of \((\varphi, \varphi_x, \ldots)\)).

We will construct here a procedure, which gives us a bracket of Ferapontov type \(([27]-[30])\)
\[
\{U^\nu(X), U^\mu(Y)\} = g^\nu\mu(U) \delta'(X - Y) + b^\nu\mu(U) U^\lambda_X \delta(X - Y) + \\
+ \sum_{k \geq 0} e_k S_{(k)}^\nu(U) U^\lambda_X \nu(X - Y) S_{(k)}^\mu(U) U^\delta_Y, \quad 1 \leq \nu, \mu, \lambda, \delta \leq N \tag{1.5}
\]
from the initial bracket \((1.4)\) after the averaging on an appropriate family of exact \(m\)-phase solutions of a local system, which is supposed to be Hamiltonian with respect to the bracket \((1.4)\) with a local Hamiltonian function \(H\).

So, we consider here the Whitham’s method for the local fluxes (if they exist)
\[
\dot{\varphi}^i = Q^i(\varphi, \varphi_x, \ldots) \tag{1.6}
\]
generated by the Hamiltonian functions \((1.2)\) in the non-local Hamiltonian structure \((1.4)\).

Certainly, this situation can arise in general only for special Hamiltonian functions, so all the considerations here appeal as a rule to the “integrable systems” like KdV, NS, etc., where we have a lot of such functionals.

Let us now formulate some general theorem about the non-local part of the bracket \((1.4)\).

**Theorem 1.1**

*Suppose we have a non-local Hamiltonian operator written in the “canonical” form \((1.4)\), where all \( B_{ij}^k \) and \( S_{(k)}^i \) represent analytic functions of \((\varphi, \varphi_x, \ldots)\) in some open region of their values.

Then, for the same region of the values of \((\varphi, \varphi_x, \ldots)\):

1) The flows
\[
\dot{\varphi}^i = S_{(k)}^i(\varphi, \varphi_x, \ldots) \tag{1.7}
\]
commute with each other.

2) Any of the flows \((1.4)\) conserves the Hamiltonian structure \((1.4)\)*

*Proof.*

Let us consider the functional
\[
\int \sum_{p=1}^n \varphi^p(x) q_p(x) \, dx \tag{1.8}
\]
for some \( q_p(x) \) with compact supports and consider the Hamiltonian flow \( \xi^i(x) \), generated by \((1.8)\) according to \((1.4)\), i.e.
\[ \xi^i(x) = \sum_{k \geq 0} B^i_k(\varphi, \varphi_x, \ldots) \frac{d^k}{dx^k} q_j(x) + \]
\[ + \frac{1}{2} \sum_{k \geq 0} e_k S^i_{(k)}(\varphi, \varphi_x, \ldots) \left[ \int_{-\infty}^x S^j_{(k)}(\varphi, \varphi_z, \ldots) q_j(z) \, dz - \int_{x}^{\infty} S^j_{(k)}(\varphi, \varphi_z, \ldots) q_j(z) \, dz \right] \quad (1.9) \]
(we assume summation over the repeated indices).

For the Hamiltonian flow \( \xi^i(x) \) we should have:
\[ \left[ L_{\xi} \hat{J} \right]^{ij}(x, y) = 0 \quad (1.10) \]
where \( \hat{J} \) is the Hamiltonian operator (1.8) and \( L_{\xi} \) is the Lie-derivative, given by the expression:
\[ \left[ L_{\xi} \hat{J} \right]^{ij}(x, y) = \int \xi^s(z) \frac{\delta}{\delta \varphi^s(z)} J^{ij}(x, y) \, dz - \]
\[ - \int J^{sj}(z, y) \frac{\delta}{\delta \varphi^s(z)} \xi^i(x) \, dz - \int J^{is}(x, z) \frac{\delta}{\delta \varphi^s(z)} \xi^j(y) \, dz \]

Let us now consider the relation (1.10) for \( x \) and \( y \) larger than any \( z \) from the supports of \( q_p(z) \). Then we will have
\[ \xi^i(x) = \sum_{k \geq 0} e_k S^i_{(k)}(\varphi, \varphi_x, \ldots) \left[ \frac{1}{2} \int_{-\infty}^{\infty} S^p_{(k)}(\varphi, \varphi_w, \ldots) q_p(w) \, dw \right] \]
and
\[ \left[ L_{\xi} \hat{J} \right]^{ij}(x, y) = \sum_{k \geq 0} \hat{B}^{ij}_k(\varphi, \varphi_x, \ldots) \delta^{(k)}(x - y) + \]
\[ + \sum_{k \geq 0} e_k \hat{S}^i_{(k)}(\varphi, \varphi_x, \ldots) \nu(x - y) S^j_{(k)}(\varphi, \varphi_y, \ldots) + \]
\[ + \sum_{k \geq 0} e_k S^i_{(k)}(\varphi, \varphi_x, \ldots) \nu(x - y) \hat{S}^j_{(k)}(\varphi, \varphi_y, \ldots) - \]
\[ - \sum_{k \geq 0} (-1)^k \frac{d^k}{dy^k} \left( B^j_{k}(\varphi, \varphi_y, \ldots) \sum_{k' \geq 0} e_{k'} \frac{\delta S^i_{(k'})(\varphi, \varphi_x, \ldots)}{\delta \varphi^s(y)} \right) \frac{1}{2} \int_{-\infty}^{\infty} S^p_{(k')}(\varphi, \varphi_w, \ldots) q_p(w) \, dw - \]
\[ - \sum_{k \geq 0} B_{k}(\varphi, \varphi_x, \ldots) d^k \frac{d^k}{dx^k} \left( \sum_{k' \geq 0} e_{k'} \frac{\delta S^i_{(k')}(\varphi, \varphi_y, \ldots)}{\delta \varphi^s(x)} \right) \frac{1}{2} \int_{-\infty}^{\infty} S^p_{(k')}^{\prime}(\varphi, \varphi_w, \ldots) q_p(w) \, dw - \]
\[ - \int dz \sum_{k \geq 0} e_k S^s_{(k)}(\varphi, \varphi_z, \ldots) \nu(z - y) S^j_{(k)}(\varphi, \varphi_y, \ldots) \times \]
\[ \times \sum_{k' \geq 0} e_{k'} \frac{\delta}{\delta \varphi^s(z)} \left( S^i_{(k')}^{\prime}(\varphi, \varphi_x, \ldots) \left[ \frac{1}{2} \int_{-\infty}^{\infty} S^p_{(k')}^{\prime}(\varphi, \varphi_w, \ldots) q_p(w) \, dw \right] \right) - \]
− \int dz \sum_{k \geq 0} e_k S^i_k(\varphi, \varphi, \ldots) \nu(z - x) S^s_k(\varphi, \varphi, \ldots) \times \\
\times \sum_{k' \geq 0} e_{k'} \frac{\delta}{\delta \varphi^s(z)} \left( S^j_{(k')}(\varphi, \varphi, \ldots) \left[ \frac{1}{2} \int_{-\infty}^{\infty} S^p_{(k')}(\varphi, \varphi, \ldots) q_p(w) \, dw \right] \right)

where \dot{B}^i_k(\varphi, \varphi, \ldots) and \dot{S}^i_k(\varphi, \varphi, \ldots) are the derivatives of these functions with respect to the flow

\varphi^i_t = \sum_{k \geq 0} e_k S^i_k(\varphi, \varphi, \ldots) \left[ \frac{1}{2} \int_{-\infty}^{\infty} S^p_{(k')}(\varphi, \varphi, \ldots) q_p(w) \, dw \right]

(1.11)

Here we also used that \( x, y > \text{Supp } q_p \) when omitted the variational derivatives with respect to \( \varphi^a(x) \) and \( \varphi^a(y) \) of the non-local expressions containing the convolutions with \( q_p(w) \) (the 4-th and the 5-th terms).

So we have

0 = \left[ L^i_j \right]^\alpha_t(x, y) =

= \sum_{k \geq 0} \dot{B}^i_k(\varphi, \varphi, \ldots) \delta^k(x - y) + \sum_{k \geq 0} e_k \dot{S}^i_k(\varphi, \varphi, \ldots) \nu(x - y) S^j_{(k)}(\varphi, \varphi, \ldots) +

+ \sum_{k \geq 0} e_k S^i_k(\varphi, \varphi, \ldots) \nu(x - y) \dot{S}^j_{(k)}(\varphi, \varphi, \ldots) -

− \sum_{k \geq 0} (-1)^k \frac{d^k}{dy^k} \left( B^j_k(\varphi, \varphi, \ldots) \sum_{k' \geq 0} e_{k'} \frac{\delta S^j_{(k')}(\varphi, \varphi, \ldots)}{\delta \varphi^a(y)} \right) \frac{1}{2} \int_{-\infty}^{\infty} S^p_{(k')}(\varphi, \varphi, \ldots) q_p(w) \, dw -

− \sum_{k \geq 0} B^i_k(\varphi, \varphi, \ldots) \frac{d^k}{dx^k} \left( \sum_{k' \geq 0} e_{k'} \frac{\delta S^j_{(k')}(\varphi, \varphi, \ldots)}{\delta \varphi^a(x)} \right) \frac{1}{2} \int_{-\infty}^{\infty} S^p_{(k')}(\varphi, \varphi, \ldots) q_p(w) \, dw -

− \int dz \sum_{k \geq 0} e_k S^s_k(\varphi, \varphi, \ldots) \nu(z - y) S^j_{(k)}(\varphi, \varphi, \ldots) \times

\times \sum_{k' \geq 0} e_{k'} \frac{\delta S^j_{(k')}(\varphi, \varphi, \ldots)}{\delta \varphi^s(z)} \left[ \frac{1}{2} \int_{-\infty}^{\infty} S^p_{(k')}(\varphi, \varphi, \ldots) q_p(w) \, dw \right] -

− \int dz \sum_{k \geq 0} e_k S^i_k(\varphi, \varphi, \ldots) \nu(x - z) S^s_k(\varphi, \varphi, \ldots) \times

\times \sum_{k' \geq 0} e_{k'} \frac{\delta S^j_{(k')}(\varphi, \varphi, \ldots)}{\delta \varphi^s(z)} \left[ \frac{1}{2} \int_{-\infty}^{\infty} S^p_{(k')}(\varphi, \varphi, \ldots) q_p(w) \, dw \right] -

− \int dz \sum_{k \geq 0} e_k S^s_k(\varphi, \varphi, \ldots) \nu(z - y) S^i_k(\varphi, \varphi, \ldots) \times

\times \sum_{k' \geq 0} e_{k'} S^i_{(k')}(\varphi, \varphi, \ldots) \frac{1}{2} \left[ \frac{\delta}{\delta \varphi^s(z)} \int_{-\infty}^{\infty} S^p_{(k')}(\varphi, \varphi, \ldots) q_p(w) \, dw \right] -
that this identity can be written also in the form
\[ \int dz \sum_{k \geq 0} e_k S_{(k)}^i(\varphi, \varphi_x, \ldots) \nu(x-z) S_{(k)}^j(\varphi, \varphi_z, \ldots) \times \]
\[ \times \sum_{k' \geq 0} e_{k'} S_{(k')}^j(\varphi, \varphi_y, \ldots) \frac{1}{2} \left[ \frac{\delta}{\delta \varphi^k(z)} \int_{-\infty}^{\infty} S_{(k')}^p(\varphi, \varphi_w, \ldots) q_p(w) dw \right] \]
\[ \equiv \sum_{k \geq 0} e_k \left[ \frac{1}{2} \int_{-\infty}^{\infty} S_{(k)}^p(\varphi, \varphi_w, \ldots) q_p(w) dw \right] \cdot \left[ L_k J \right]^{ij}(x, y) + \]
\[ + \sum_{k, k' \geq 0} e_k e_{k'} S_{(k')}^i(\varphi, \varphi_x, \ldots) S_{(k)}^j(\varphi, \varphi_y, \ldots) \times \]
\[ \times \frac{1}{2} \int \left( S_{(k)}^s(\varphi, \varphi_z, \ldots) \nu(y-z) \frac{\delta}{\delta \varphi^k(z)} \left[ \int_{-\infty}^{\infty} S_{(k')}^p(\varphi, \varphi_w, \ldots) q_p(w) dw \right] \right) - \]
\[ - S_{(k')}^s(\varphi, \varphi_z, \ldots) \nu(x-z) \frac{\delta}{\delta \varphi^k(z)} \left[ \int_{-\infty}^{\infty} S_{(k)}^p(\varphi, \varphi_w, \ldots) q_p(w) dw \right] \right) dz \]

where \( [L_k J]^{ij}(x, y) \) represent the Lie derivatives of \( J \) with respect to the flows (1.7)
\[ \dot{\varphi}^i = S_{(k)}^i(\varphi, \varphi_x, \ldots) \]

Let us use again our condition \( x, y > \text{Supp } q_p \) and rewrite the above identity in the form
\[ 0 \equiv \sum_{k \geq 0} e_k \left[ \frac{1}{2} \int_{-\infty}^{\infty} S_{(k)}^p(\varphi, \varphi_w, \ldots) q_p(w) dw \right] \cdot \left[ L_k J \right]^{ij}(x, y) + \]
\[ + \sum_{k, k' \geq 0} e_k e_{k'} S_{(k')}^i(\varphi, \varphi_x, \ldots) S_{(k)}^j(\varphi, \varphi_y, \ldots) \times \]
\[ \times \frac{1}{4} \int \left( S_{(k)}^s(\varphi, \varphi_z, \ldots) \frac{\delta}{\delta \varphi^k(z)} \left[ \int_{-\infty}^{\infty} S_{(k')}^p(\varphi, \varphi_w, \ldots) q_p(w) dw \right] \right) - \]
\[ - S_{(k')}^s(\varphi, \varphi_z, \ldots) \frac{\delta}{\delta \varphi^k(z)} \left[ \int_{-\infty}^{\infty} S_{(k)}^p(\varphi, \varphi_w, \ldots) q_p(w) dw \right] \right) dz \]

Using the standard expression for the variational derivative and the integration by parts we obtain that this identity can be written also in the form
\[ 0 \equiv \sum_{k \geq 0} e_k \left[ \frac{1}{2} \int_{-\infty}^{\infty} S_{(k)}^p(\varphi, \varphi_w, \ldots) q_p(w) dw \right] \cdot \left[ L_k J \right]^{ij}(x, y) + \]
\[ + \sum_{k, k' \geq 0} e_k e_{k'} S_{(k')}^i(\varphi, \varphi_x, \ldots) S_{(k)}^j(\varphi, \varphi_y, \ldots) \frac{1}{4} \int_{-\infty}^{\infty} q_p(z) [S_{(k)}, S_{(k')}^p(z) ] dz \]

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where \([S(k), S(k')]\) is the commutator of the flows \((1.7)\), or
\[
0 \equiv \sum_{k \geq 0} e_k \left[ \frac{1}{2} \int_{-\infty}^{\infty} S^p_{(k)}(\varphi, \varphi, \ldots) q_p(w) \, dw \right] \cdot \left[ L_k J \right]^{ij}(x, y) + \tag{1.12}
\]
\[
+ \sum_{k > k' \geq 0} e_k e_{k'} \left( S^i_{(k')}(\varphi, \varphi_x, \ldots) S^j_{(k)}(\varphi, \varphi_y, \ldots) - S^i_{(k)}(\varphi, \varphi_x, \ldots) S^j_{(k')}(\varphi, \varphi_y, \ldots) \right) \times
\]
\[
\times \frac{1}{4} \int_{-\infty}^{\infty} q_p(z) \left[ S(k), S(k') \right]^p(z) \, dz
\]
for any \(q_p(z)\), such that \(x, y > Supp \, q_p(z)\).

As can be easily seen, the last term in \((1.12)\) represents the non-local part of \(\left[ L_\xi J \right]^{ij}(x, y)\), which does not contain the function \(\nu(x - y)\). The first term in \((1.12)\) also contains a non-local part, however, this part contains the function \(\nu(x - y)\). It is not difficult to see that this non-local part can in general be written in the “canonical” form
\[
\sum_{k \geq 0} e_k \left[ \frac{1}{2} \int_{-\infty}^{\infty} S^p_{(k)}(\varphi, \varphi, \ldots) q_p(w) \, dw \right] \sum_{s=1}^{Q} e'_s A^i_s(\varphi, \varphi_x, \ldots) \nu(x - y) A^j_s(\varphi, \varphi_y, \ldots) \tag{1.13}
\]
\((e'_s = \pm 1)\), where the functions \(A(\varphi, \varphi_x, \ldots)\) represent some linearly independent set of analytic vector-functions of \((\varphi, \varphi_x, \ldots)\). Let us prove now, that from the identity \((1.12)\) it follows actually that both the expressions \((1.13)\) and
\[
\sum_{k > k' \geq 0} e_k e_{k'} \left( S^i_{(k')}(\varphi, \varphi_x, \ldots) S^j_{(k)}(\varphi, \varphi_y, \ldots) - S^i_{(k)}(\varphi, \varphi_x, \ldots) S^j_{(k')}(\varphi, \varphi_y, \ldots) \right) \times
\]
\[
\times \frac{1}{4} \int_{-\infty}^{\infty} q_p(z) \left[ S(k), S(k') \right]^p(z) \, dz \tag{1.14}
\]
should be identically equal to zero.

Let us fix some value of \(x\) and consider the interval \(I = [x - \Delta, x + \Delta]\), such that \(x - \Delta, x + \Delta > Supp \, q_p(z)\). It is not difficult to see that for a linearly independent set of analytic functions \(A(\varphi, \varphi_y, \ldots)\) we can find an everywhere dense set \(S\) of analytic on the interval \(y \in [x - \Delta, x + \Delta]\) functions \(\varphi(y)\) (and infinitely smooth on the whole numerical axis), such that the functions \(A(\varphi, \varphi_y, \ldots)\) give a linearly independent set of analytic functions of \(y\) on the interval \(I\) for any \(\varphi(y) \in S\). It is easy to see also, that for any \(\varphi(y) \in S\) we can find a set of analytic functions \(\kappa_i(y)\) on the interval \(I\), such that the functions
\[
a_i(y) = A^i_s(\varphi, \varphi_y, \ldots) \kappa_i(y)
\]
still give a set of linearly independent analytic functions on \(I\).

According to Peano \((III)\), we can claim that there exists a point \(y_0 \in I\) such that the Wronskian
\[
W(y_0) = \begin{vmatrix}
  a_{(1)}(y_0) & a_{(1)y}(y_0) & \cdots & a_{(1)(Q-1)y}(y_0) \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{(Q)}(y_0) & a_{(Q)y}(y_0) & \cdots & a_{(Q)(Q-1)y}(y_0)
\end{vmatrix}
\]
\[8\]
is different from 0 at the point \( y_0 \). It is not difficult to see also, that we can assume actually that \( y_0 = x \), so we put \( W(x) \neq 0 \).

Let us introduce now infinitely smooth functions \( \zeta^0(y), \ldots, \zeta^{Q-1}(y) \) having the following properties:

1) All \( \zeta^l(y) \) are identically equal to zero outside the interval \( I \);
2) All \( \zeta^l(y) \) and all their derivatives \( \zeta_{s y}^l(y), \; s \geq 0 \), are equal to zero at the point \( y = x \);
3) \[
\int_{-\infty}^{+\infty} y^s \zeta^l(y) \, dy = 0 \ , \quad 0 \leq s < l ,
\]
4) The functions \( \hat{\zeta}^{(l)}(y) = \nu(x-y) \zeta^l(y) \) satisfy the relations
\[
\int_{-\infty}^{+\infty} y^s \hat{\zeta}^l(y) \, dy = 0 \ , \quad 0 \leq s \leq l
\]

Let us say again, that the functions \( \zeta^l(y) \) can be easily constructed and it is most convenient to represent them in the form shown at Fig. 1.

Let us consider now the convolutions (in \( y \)) of the full expression for \( [L_{\zeta\tilde{J}}]^{ij}(x,y) \) with the infinitely smooth functions \[
\kappa_j(y) \, C^l \, \hat{\zeta}^l(x + C(y-x)) \ , \quad l = 0, \ldots, Q-1
\]
and put \( C \to \infty \).

Easy to see that the local part of \( [\mathcal{L}_\xi \hat{J}]^{ij} (x, y) \) will give us identical zero in such convolutions due to the property (2) of the functions \( \zeta^l(y) \). In the same way, we will get zero in the limit \( C \to \infty \) in the non-local part (1.14) of \( [\mathcal{L}_\xi \hat{J}]^{ij} (x, y) \) according to the property (4) of the functions \( \hat{\zeta}^l(y) \). At the same time, the non-local part (1.13) will give us the values

\[
\sum_{k \geq 0} e_k \left[ \frac{1}{4} \int_{-\infty}^{\infty} S^p_{(k)}(\varphi, \varphi_w, \ldots) q_p(w) \, dw \right] \quad \frac{1}{4} \sum_{s=1}^{Q} e'_s A_{(s)}^i(\varphi, \varphi_x, \ldots) a_{(s)l}(x) \quad l = 0, \ldots, Q-1,
\]

in the limit \( C \to \infty \) according to the property (3) of the functions \( \zeta^l(y) \).

Coming back now to the property \( W(x) \neq 0 \) and assuming that in general

\[
\sum_{k \geq 0} e_k \left[ \frac{1}{2} \int_{-\infty}^{\infty} S^p_{(k)}(\varphi, \varphi_w, \ldots) q_p(w) \, dw \right] \neq 0
\]

for the linearly independent set \( \{ S_{(k)} \} \), we can claim now that the vanishing of the expression \( [\mathcal{L}_\xi \hat{J}]^{ij} (x, y) \) implies in fact the relations \( A_{(s)}^i(\varphi, \varphi_x, \ldots) = 0 \) for our chosen function \( \varphi(y) \in \mathcal{S} \).

Using now the properties of the set \( \mathcal{S} \) and the translational invariance of our Hamiltonian operator we conclude now that \( A_{(s)}^i(\varphi, \varphi_x, \ldots) \equiv 0 \) on the full set of functions which we consider. As a result, we can claim now that the non-local part (1.13) of the expression \( [\mathcal{L}_\xi \hat{J}]^{ij} (x, y) \) is in fact identical equal to zero. As a consequence, we can claim also the the non-local part (1.14) should be also identical zero on the full set of functions \( \varphi(x) \).

Looking now at the form of the term (1.14) we can see that it represents a sum of linearly independent tensor functions of \( (x, y) \), so we get that every coefficient, given by the integral

\[
\frac{1}{2} \int_{-\infty}^{\infty} q_p(z) \left[ S_{(k)}, S_{(k')} \right]^p(z) \, dz ,
\]

should be in fact identically equal to zero. In view of the arbitrariness of the functions \( q_p(z) \) we obtain then

\[
[S_{(k)}, S_{(k')}] \equiv 0
\]

From (1.12) we then have also for a linearly independent set of \( S_{(k)} \) and different \( q_p(w) \) that

\[
[\mathcal{L}_k \hat{J}]^{ij} (x, y) \equiv 0
\]

So we obtain the statements of the theorem.

*Theorem 1.1 is proved.*

It is also obvious that the statements of the theorem are valid for all the brackets (1.1) written in the “irreducible” form, since all \( \tilde{S}_{(k)} \) and \( \tilde{T}_{(k)} \) in this case are just linear combinations of the flows \( S_{(k)} \).

*Remark.*
Let us point here that the first statement of the Theorem for the non-local brackets \((1.5)\) of Ferapontov type was proved previously by E.V. Ferapontov in [27] using differential-geometrical considerations. In [27]-[30] also the full classification of the brackets \((1.5)\) from the differential geometrical point of view can be found.

It is easy to see now that the local functional of type \((1.2)\)
\[
I = \int P(\varphi, \varphi_x, \ldots) \, dx
\]
generates a local flow in the Hamiltonian structure \((1.1)\) if and only if the derivative of its density \(P(\varphi, \varphi_x, \ldots)\) with respect to any of the flows \((1.7)\) represents total derivative with respect to \(x\), i.e. there exist such \(Q_{(k)}(\varphi, \varphi_x, \ldots)\) that
\[
P_{\tau_k}(\varphi, \varphi_x, \ldots) \equiv \partial_x Q_{(k)}(\varphi, \varphi_x, \ldots)
\]
As was also pointed out by E.V. Ferapontov ([27]), this means that the integral \(I\) represents a conservation law for any of the systems \((1.7)\).

From the Theorem 1.1 we obtain now that the flows \((1.7)\) commute in fact with all the local Hamiltonian fluxes, generated by local functionals \((1.2)\), since they conserve in this case both the Hamiltonian structure and the corresponding Hamiltonian functions.

2 The Whitham method and the “regularity” conditions.

Now we come to Whitham’s averaging procedure (see [1]-[10]). Let us remind that in the \(m\)-phase Whitham’s method for systems \((1.6)\) we make a rescaling transformation \(X = \epsilon x, T = \epsilon t\) to obtain the system
\[
\epsilon \varphi_T^i = Q^i(\varphi, \epsilon \varphi_x, \epsilon^2 \varphi_{XX}, \ldots) \tag{2.1}
\]
Then we try to find functions
\[
S(X, T) = (S^1(X, T), \ldots, S^m(X, T))
\]
and \(2\pi\)-periodic with respect to each \(\theta^\alpha\) (\(\theta = (\theta^1, \ldots, \theta^m)\)) functions
\[
\Phi^i(\theta, X, T, \epsilon) = \sum_{k=0}^{\infty} \epsilon^k \Phi^i_{(k)}(\theta, X, T) ,
\]
such that the functions
\[
\varphi^i(\theta, X, T, \epsilon) = \sum_{k=0}^{\infty} \epsilon^k \Phi^i_{(k)} \left( \theta + \frac{S(X, T)}{\epsilon}, X, T \right) \tag{2.2}
\]
satisfy system \((2.1)\) at any \(\theta\) in any order of \(\epsilon\).

It follows then that \(\Phi^i_{(0)}(\theta, X, T)\) at any \(X\) and \(T\) defines an exact \(m\)-phase solution of \((1.6)\), depending on some parameters \(U = (U^1, \ldots, U^N)\) and initial phases \(\theta_0 = (\theta^1_0, \ldots, \theta^m_0)\) and, besides that, we have the relations
\[
S_T^\alpha = \omega^\alpha(U(X, T)) \quad , \quad S_X^\alpha = k^\alpha(U(X, T))
\]
where $\omega^\alpha(U)$ and $k^\alpha(U)$ are respectively the frequencies and the wave numbers of the corresponding $m$-phase solution of (1.6).

The compatibility conditions of system (2.1) in the first order of $\epsilon$ together with the relations

$$k^\alpha_T = \omega^\alpha_X$$

give us Whitham’s system of equations on the parameters $U(X,T)$, which represents a quasi-linear system of hydrodynamic type

$$U_T^\nu = V^\nu_\mu(U)U^\mu_X$$

(2.3)

Let us note here, that the representation of the modulated solutions of system (1.6) in the form (2.2) is in fact usually possible just in the one-phase situation. In the multi-phase case we can usually write down just the first term in the expansion (2.2), while the higher order corrections have in general more complicated form (see e.g. [11, 12, 13]). Let us say, however, that the Whitham system, defined as above, still plays the central role in description of the modulated solutions both in the one-phase and the multi-phase case.

The first procedure of averaging of local field-theoretical Poisson brackets was proposed in [4]-[9] by B.A. Dubrovin and S.P. Novikov. This procedure permits to obtain local Poisson brackets of Hydrodynamic type:

$$\{U^\nu(X), U^\mu(Y)\} = g^{\nu\mu}(U) \delta'(X - Y) + b^{\nu\mu}_\gamma(U) U^\gamma_X \delta(X - Y)$$

(2.4)

for Whitham’s system (2.3) from a local Hamiltonian structure

$$\{\varphi^i(x), \varphi^j(y)\} = \sum_{k \geq 0} B_{ij}^{(k)}(\varphi, \varphi_x, \ldots) \delta(k)(x - y)$$

for the initial system (1.6).

The method of Dubrovin and Novikov is based on the presence of $N$ (equal to the number of parameters $U^\nu$ of the family of $m$-phase solutions of (1.6)) local integrals

$$I^\nu = \int P^\nu(\varphi, \varphi_x, \ldots) dx ,$$

(2.5)

commuting with the Hamiltonian function (1.2) and with each other

$$\{I^\nu, H\} = 0 , \quad \{I^\nu, I^\mu\} = 0 ,$$

(2.6)

and can be described in the following way:

We calculate the pairwise Poisson brackets of the densities $P^\nu$ in the form

$$\{P^\nu(x), P^\mu(y)\} = \sum_{k \geq 0} A_{k}^{\nu\mu}(\varphi, \varphi_x, \ldots) \delta(k)(x - y)$$

where

$$A_0^{\nu\mu}(\varphi, \varphi_x, \ldots) \equiv \partial_x Q^{\nu\mu}(\varphi, \varphi_x, \ldots)$$
according to (2.6). Then the Dubrovin-Novikov bracket on the space of functions $U(X)$ can be written in the form

$$\{U^\nu(X), U^\mu(Y)\} = \langle A_1^{\nu\mu}(U) \delta'(X - Y) + \frac{\partial(Q^{\nu\mu})}{\partial U^\gamma} U^\gamma \delta(X - Y) \rangle$$

(2.7)

where $\langle \ldots \rangle$ means the averaging on the family of $m$-phase solutions of (1.6) given by the formula:

$$\langle F \rangle = \lim_{c \to \infty} \frac{1}{2c} \int_{-c}^{c} F(\varphi, \varphi_x, \ldots) \, dx = \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} F(\Phi, k^\alpha(U) \Phi_\theta^\alpha, \ldots) \, d^m \theta$$

(2.8)

Here we choose the parameters $U^\nu$ such that they coincide with the values of $I^\nu$ on the corresponding solutions

$$U^\nu = \langle P^\nu(x) \rangle$$

The Jacobi identity for the averaged bracket (2.7) in the general case was proved in [18] (for systems having also local Lagrangian formalism there was a proof in [20]).

Let us note here also that the procedure described above gives a Poisson bracket only if we average the initial Hamiltonian structure on a “full regular family” of $m$-phase solutions (see [17, 18, 42]). We will formulate actually more precise requirements when describe the averaging procedure in the non-local case.

Brackets (2.4) can be described from the differential-geometrical point of view. Thus, for a non-degenerated tensor $g^{\nu\mu}$ we have in fact that it should represent a flat contravariant metric and the values

$$\Gamma^\nu_{\mu\gamma} = -g_{\mu\lambda} b^\lambda_{\gamma}$$

should give the Levi-Civita connection for the metric $g_{\nu\mu}$ (with lower indices). The brackets (2.4) with a degenerated tensor $g^{\nu\mu}$ are more complicated but also have a nice geometrical structure (see [16]).

The non-local Poisson brackets (1.5) give a generalization of local Poisson brackets of Dubrovin and Novikov and are closely connected with the integrability of systems of hydrodynamic type, reducible to the diagonal form ([25]). Namely, any system reducible to the diagonal form and Hamiltonian with respect to the bracket (1.5) satisfies in fact (see [27]-[30]) the so-called “semi-Hamiltonian” property, introduced by S.P. Tsarev (25), and can be integrated by Tsarev’s “generalized hodograph method”. In [33] the investigation of possible equivalence of the “semi-Hamiltonian” properties introduced by Tsarev and the Hamiltonian properties with respect to the bracket (1.5) can be also found.

Let us also point out here that the questions of integrability of Hamiltonian systems, which can not be written in the diagonal form, were studied in [34]-[37].

The procedure of averaging of the non-local Poisson brackets in the Whitham method and the proof of the Jacobi identity for the averaged non-local bracket resemble the same things for the local brackets. However the formulas of averaging and the proof contain in fact some essential differences, so, we have to represent here special consideration for the non-local case.

---

1Strictly speaking this formula is valid for generic set of the wave numbers $k^\alpha$, but we should use in any case the second part of it for the averaged quantities to obtain the right procedure. (Here $k^\alpha$ are continuous parameters on the family of the $m$-phase solutions).
The $m$-phase solutions of (1.6)

\[ \varphi^i(x, t) = \Phi^i(\omega t + kx + \theta_0) , \]

where \( \omega = (\omega^1, \ldots, \omega^m) \), \( k = (k^1, \ldots, k^m) \), are defined by $2\pi$-periodic solutions of the system

\[ \omega^\alpha \Phi^i_{\theta^\alpha} - Q^i \left( \Phi, k^\alpha \Phi_{\theta^\alpha}, k^\alpha k^\beta \Phi_{\theta^\alpha \theta^\beta}, \ldots \right) = 0 , \quad (2.9) \]

depending on $\omega$ and $k$ as on parameters. So, we assume that for generic $\omega$ and $k$ we obtain from (2.9) a finite-dimensional submanifold \( \mathcal{M}_{\omega,k} \) (in the space of $2\pi$-periodic with respect to each $\theta^\alpha$ functions), parameterized by the initial phase shifts $\theta_0^\alpha$ and maybe also by some additional parameters $r^1, \ldots, r^h$.

Combining all such $\mathcal{M}_{\omega,k}$ at different $\omega$ and $k$ we obtain that the $m$-phase solutions of the system (1.6) can be parameterized by $N = 2m + h$ parameters $U^1, \ldots, U^N$, invariant with respect to the initial shifts of $\theta^\alpha$, and the initial phase shifts $\theta_0^\alpha$ after the choice of some “initial” functions $\Phi_{U^i}(\theta, U)$, corresponding to the zero initial phases. The joint of the submanifolds $\mathcal{M}_{\omega,k}$ at all $\omega$ and $k$ gives us a submanifold $\mathcal{M}$ in the space of $2\pi$-periodic with respect to each $\theta^\alpha$ functions, which corresponds to the full family of $m$-phase solutions of (1.6).

For the Whitham procedure we should now require some “regularity” properties of the system of constraints (2.9). Namely

(I) We require that the linearized system (2.9)

\[ \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \left( \omega^\alpha \delta_j^\beta \delta_{\theta^\alpha}(\theta - \theta') - \frac{\delta Q^i(\theta)}{\delta \varphi^j(\theta')} \right) \Psi^i(\theta') \, d^m\theta' = 0 \]

has for generic $\omega$ and $k$ exactly $h + m = N - m$ solutions (“right eigen vectors”) $\xi_{U^i}(\theta, r)$ at the corresponding “points” of $\mathcal{M}_{\omega,k}$, given by the vectors tangential to $\mathcal{M}_{\omega,k}$, i.e. the functions $\Phi_{U^i}(\theta, r, \omega, k)$ and $\Phi_{r^i}(\theta, r, \omega, k)$ (at the fixed values of $\omega$ and $k$).

(II) We also require that the number of linearly independent “left eigen vectors” $\kappa_{U^i}(\theta, r)$, orthogonal to the image of the introduced linear operator, is exactly the same $(N - m)$ as the number of the “right eigen vectors” $\xi_{U^i}(\theta, r)$ for generic $\omega$ and $k$. In addition, we will assume that the corresponding $\kappa_{U^i}(\theta, r)$ also depend continuously on the parameters $U^\nu$ on $\mathcal{M}$.

The requirements (I) and (II) are actually closely connected with the Whitham procedure and the asymptotic solutions (2.2). Indeed, it is not difficult to see that every $k$-th term in the expansion (2.2) is determined by the defined above linear system with a nontrivial right-hand part, depending on the previous terms of (2.2). For resolvability of these systems we have in any case to require the orthogonality of the right-hand part to all the “regular left eigen vectors” $\kappa_{U^i}(\theta, r)$, corresponding

\footnote{For the multiphase case ($m \geq 2$) it is essential that the closure of any orbit generated by the vectors $(\omega^1, \ldots, \omega^m)$ and $(k_1, \ldots, k_m)$ in the $\theta$-space is the full $m$-dimensional torus $\mathbb{T}^m$. For the case of “rationally-dependent” $\omega^1, \ldots, \omega^m$ and $k_1, \ldots, k_m$ and $m \geq 3$ we have that the operators (2.9) are independent on each of such closed submanifolds in $\mathbb{T}^m$ which can have the dimensionality $< m$. The functions from $\mathcal{M}_{\omega,k}$ can be found in this case from the additional requirement that they define also $m$-phase solutions for systems (1.7) (with some $\Omega^\alpha(\omega, k)$) and the systems generated by the functionals $I^\nu$ (see later) (also with some $\omega^\mu(\omega, k)$). All these requirements uniquely define the finite-dimensional spaces $\mathcal{M}_{\omega,k}$, which continuously depend on the parameters $\omega$ and $k$.}
to the zero eigen values. The corresponding orthogonality conditions in the first order of \( \epsilon \) together with the relations

\[
k_T = \omega_X
\]
give a system of \( (N - m) + m = N \) equations, which coincides (by definition) with the Whitham’s system of equations (2.3).

Let us now discuss the requirements (I) and (II) from the Hamiltonian point of view.

First of all, like in the procedure of Dubrovin and Novikov, for the procedure of averaging of the bracket (1.4) we need a set of integrals \( I^\nu, \nu = 1, \ldots, N \), satisfying the following requirements:

(A) Every \( I^\nu \) is a local functional

\[
I^\nu = \int P^\nu(\varphi, \varphi_x, \ldots) \, dx
\]
which generates a local flow

\[
\varphi^i_{t^\nu} = Q^i_{(\nu)}(\varphi, \varphi_x, \ldots)
\]
with respect to the bracket (1.4).

As was pointed above we should require then that the local flows (1.7), defined by the bracket (1.4) in the “canonical” (or “irreducible”) form, conserve all the \( I^\nu \), i.e. the time derivatives of the corresponding \( P^\nu(\varphi, \varphi_x, \ldots) \) with respect to each of the flows (1.7) represent total derivatives with respect to \( x \)

\[
\frac{d}{dt^k} P^\nu(\varphi, \varphi_x, \ldots) \equiv \partial_x F^\nu_{(k)}(\varphi, \varphi_x, \ldots)
\]
for some functions \( F^\nu_{(k)}(\varphi, \varphi_x, \ldots) \).

(B) All \( I^\nu \) commute with each other and with the Hamiltonian function (1.2)

\[
\{ I^\nu, I^\mu \} = 0 \quad \{ I^\nu, H \} = 0
\]

(C) The averaged densities \( \langle P^\nu \rangle \)

\[
\langle P^\nu \rangle = \lim_{c \to \infty} \frac{1}{2c} \int_{-c}^c P^\nu(\varphi, \varphi_x, \ldots) \, dx = \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} P^\nu(\Phi, k^\alpha \Theta^\alpha, \ldots) \, d^m \theta
\]
can be regarded as independent coordinates \( U^1, \ldots, U^N \) on the family of \( m \)-phase solutions of (1.6).

From the requirements above we immediately obtain that the flows (2.11) commute with our initial system (1.6) and with each other.

From Theorem 1.1 we obtain also that the commutative flows (1.7), defined by the Poisson bracket, also commute with (1.6) and (2.11) since they conserve the corresponding Hamiltonian functions and the Hamiltonian structure (1.4).

\footnote{Here again we can use everywhere the second part of the formula (2.14) for the averaged values on \( \mathcal{M} \).}
Now we can consider the functionals

\[ I^\nu = \lim_{c \to \infty} \frac{1}{2c} \int_{-c}^{c} P^\nu(\varphi, \varphi_x, \ldots) \, dx \]  
(2.15)

and

\[ H = \lim_{c \to \infty} \frac{1}{2c} \int_{-c}^{c} P_H(\varphi, \varphi_x, \ldots) \, dx \]  
(2.16)

on the space of the quasiperiodic functions (with \( m \) wave numbers).

It is easy to see now that the local fluxes \((1.6), (1.7)\) and \((2.11)\), being considered on the space of the quasiperiodic functions, also conserve the values of \( I^\nu \) and \( H \) and commute with each other, since these properties can be expressed just as local relations containing \( \varphi, \varphi_x, \ldots \) and the time derivatives of the densities \( P^\nu(\varphi, \varphi_x, \ldots) \), \( P_H(\varphi, \varphi_x, \ldots) \) at the same point \( x \).

Now we can conclude that all the fluxes \((1.7)\) and \((2.11)\) leave invariant the family of \( m \)-phase solutions, given by \((2.9)\), and can generate on it only linear shifts of the initial phases \( \theta^a_0 \), which follows from the commutativity of the flows

\[ \varphi^i_{\alpha k}(\theta) = L^i_{(k)}(\varphi, k^\alpha \varphi_{\theta^a}, k^\beta \varphi_{\theta^b \theta^a}, \ldots) \]  
(2.17)

and

\[ \varphi^i_{\nu}(\theta) = Q^i_{(\nu)}(\varphi, k^\alpha \varphi_{\theta^a}, k^\alpha k^\beta \varphi_{\theta^a \theta^b}, \ldots) \]  
(2.18)

with the flows \( \varphi_{\alpha}^i = \varphi_{0^a}^i \) and

\[ \varphi_t^i = Q^i(\varphi, k^\alpha \varphi_{\theta^a}, k^\alpha k^\beta \varphi_{\theta^a \theta^b}, \ldots) \]

on the space of \( 2\pi \)-periodic with respect to each \( \theta^a \) functions and the conservation of the functionals \( I^\nu \) (i.e. \( U^\nu \) on \( \mathcal{M} \)) by the flows \((2.17)\) and \((2.13)\). (Here \( k^\alpha \) are \( m \) wave numbers of the function \( \varphi(x) \).) So, we obtain that our family of \( m \)-phase solutions of \((1.6)\) represents also a family of \( m \)-phase solutions for systems \((1.7)\) and \((2.11)\), and we can consider also the Whitham equations for these systems, based on the family \( \mathcal{M} \).

We can also conclude that in our situation the variational derivatives of the functionals \((2.13)\) and \((2.16)\) with respect to \( \varphi(\theta) \) at the points of the submanifold \( \mathcal{M} \) represent some linear combinations of the corresponding “left eigen vectors” \( \kappa_q(\theta + \theta_0, U) \) (see \([7]-[10]\) and references therein). Indeed, from the conservation of the functionals \((2.15)\) and \((2.16)\) by the flows \( \varphi_{\alpha}^i = \varphi_{0^a}^i \) and

\[ \varphi_t^i = Q^i(\varphi, k^\alpha \varphi_{\theta^a}, \ldots) \]

we can conclude that the convolution of their derivatives (with respect to \( \varphi^i(\theta) \)) with the system of constraints \((2.9)\) is identically zero for all the periodic functions with respect to all \( \theta^a \) and for any \( k^1, \ldots, k^m \) and \( \omega^1, \ldots, \omega^m \). So we can take the variational derivative of the corresponding expression with respect to \( \varphi^i(\theta^a) \) and then omit the second variational derivative of \( I^\nu \) and \( H \) according to the conditions \((2.9)\). After that we obtain that the variational derivatives of \( I^\nu \) and \( H \) are also orthogonal to the image of the linearized operator \((2.9)\) at the points of \( \mathcal{M} \) and so represent some linear combinations of \( \kappa_q(\theta + \theta_0, U) \) on \( \mathcal{M} \).
Lemma 2.1

Suppose we have the properties (A)-(C) and (I)-(II) for our family of \(m\)-phase solutions of \((1.6)\). Let us put

\[ U^\nu = \langle P^\nu \rangle = \bar{I}^\nu \]  

(2.19)
on the space \(\mathcal{M}\) and define the functions \(k^\alpha = k^\alpha(U)\) on the submanifold \(\mathcal{M}\).

Then the functionals \(K^\alpha = k^\alpha(\bar{I}[\varphi])\) on the space of \(2\pi\)-periodic with respect to each \(\theta^\alpha\) functions (and also on the space of quasiperiodic functions \(\varphi(x)\) with \(m\) wave numbers) have zero variational derivatives on the submanifold \(\mathcal{M}\).

Proof.

As we have from (II), the maximal number of linearly independent variational derivatives of \(\bar{I}^\nu\) on \(\mathcal{M}\) is equal to \(h + m = N - m\). So, we have \(m\) linearly independent relations

\[ \sum_{\nu=1}^{N} \lambda^\alpha_\nu(U) \frac{\delta \bar{I}^\nu}{\delta \varphi(\theta)} = 0 , \quad \alpha = 1, \ldots, m \]  

(2.20)

\((\varphi(\theta) = (\varphi^1(\theta), \ldots, \varphi^m(\theta)))\), considered at given \(k^1, \ldots, k^m\) at any point of \(\mathcal{M}\) (or in other words

\[ \sum_{\nu=1}^{N} \lambda^\alpha_\nu(U) \frac{\delta \bar{I}^\nu}{\delta \varphi(x)} = 0 , \quad \alpha = 1, \ldots, m \]  

(2.21)

when considered on the space of functions with \(m\) wave numbers.) We can use here the standard expression for the variational derivative and the formula \(2.14\) for \(\bar{I}^\nu\).

Since we can obtain a change of the values of these linear combinations of the functionals \(\bar{I}^\nu\) on \(\mathcal{M}\) only due to variations of \(k\) in \(2.14\) but not of \(\varphi^\alpha(\theta)\) (or in other words only if we have non-bounded variations of \(\varphi^\alpha(x)\) after the variations of the wave numbers) we have on \(\mathcal{M}\)

\[ \sum_{\nu=1}^{N} \lambda^\alpha_\nu(U) dU^\nu = \sum_{\beta=1}^{m} \mu^{(\alpha)}_\beta(U) dk^\beta(U) \]  

(2.22)

for some functions \(\mu^{(\alpha)}_\beta(U)\).

Since \(U^\nu\) represent independent coordinates on \(\mathcal{M}\), the matrix \(\mu^{(\alpha)}_\beta\) has the full rank and is reversible. So, we get the differentials \(dk^\beta\) as some linear combinations of differentials \(\sum_{\nu=1}^{N} \lambda^\alpha_\nu(U) dU^\nu\), corresponding to the functionals with zero derivatives on \(\mathcal{M}\)

\[ dk^\beta = \sum_{\alpha=1}^{m} (\mu^{-1})^\alpha_\beta \sum_{\nu=1}^{N} \lambda^\alpha_\nu(U) dU^\nu \]

So Lemma 2.1 now follows from \(2.20\).

Remark 1.

As can be seen from the proof of Lemma 2.1, the variational derivatives of \(\bar{I}^\nu\) on \(\mathcal{M}\) should span the full \((N - m)\)-dimensional linear space, generated by all \(\kappa(\theta + \theta_0, U)\), if we want to take \(\langle P^\nu \rangle\) as a set of independent coordinates on \(\mathcal{M}\). It is essential also that we consider the full family
of m-phase solutions, given by (2.9) at different $\omega$ and $k$, (but not its “submanifold”) and have $m$ independent relations (2.22) on $N$ differentials $dU^\nu$ from $m$ relations (2.20).

**Remark 2.**

Let us note here that the equations (2.21) were introduced at first by S.P. Novikov in [15] as the definition of the $m$-phase solutions for the KdV equation.

Let us now prove a technical lemma which we will need later.

**Lemma 2.2**

Let us introduce the additional densities

$$
\Pi^\nu_{i(k)}(\varphi, \varphi_x, \ldots) \equiv \frac{\partial P^\nu(\varphi, \varphi_x, \ldots)}{\partial \varphi_{kx}^i} \tag{2.23}
$$

for $k \geq 0$, where $\varphi_{kx}^i \equiv \partial^k \varphi^i / \partial x^k$.

Then on the submanifold $\mathcal{M}$ we have the relation

$$
\sum_{\nu=1}^{N} \frac{\partial k^\alpha}{\partial U^\nu} \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \sum_{p \geq 1} p \, k^{\beta_1}(U) \cdots k^{\beta_{p-1}}(U) \, \Phi_{(in)}^{i} \theta^\beta_1 \ldots \theta^\beta_{p-1} \, \theta^\gamma (\theta, U) \times \\
\times \Pi^\nu_{i(p)} (\Phi_{(in)}(\theta, U), k^\gamma \Phi_{(in)}(\theta, U), \ldots) \, d^m \theta \equiv \delta^\alpha_\beta \tag{2.24}
$$

at any $U$ and $\theta_0$.

**Proof.**

According to Lemma 2.1 we should not take into account variations of the form of $\Phi_{(in)}(\theta + \theta_0, U)$ when we consider infinitesimal changes of the values of the functionals $k^\alpha(\bar{I})$ on $\mathcal{M}$. So, the only source for a change of these functionals on $\mathcal{M}$ is the dependence on the wave numbers $k$ in the expressions

$$
\bar{I}^\nu = \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} P^\nu (\Phi_{(in)}, \varphi_{(in)}^\gamma, k^\gamma k^\delta \Phi_{(in)}(\theta, \gamma), \ldots) \, d^m \theta
$$

So, we can write

$$
d \left( k^\alpha(\bar{I}) \big| \mathcal{M} \right) = \sum_{\nu=1}^{N} \frac{\partial k^\alpha}{\partial U^\nu} (U) \frac{\partial \bar{I}^\nu(\varphi)}{\partial k^\beta} \big| _{\mathcal{M}} \, dk^\beta
$$

where the values of $\partial \bar{I}^\nu(\varphi)/\partial k^\beta$ on $\mathcal{M}$ are given by the integral expressions from (2.24). Since the values of the functionals $k^\alpha(\bar{I})$ on $\mathcal{M}$ coincide by definition with the wave numbers $k^\alpha$, we obtain the relation (2.24).

**Lemma 2.2 is proved.**

For the evolution of the densities $P^\nu(\varphi, \varphi_x, \ldots)$ according to our system (1.6) we can also write the relations

$$
\frac{d}{dt} P^\nu(\varphi, \varphi_x, \ldots) \equiv \partial_{\varphi} Q^\nu H(\varphi, \varphi_x, \ldots) \tag{2.25}
$$
and the Whitham’s system (2.25) can be also written in the following “conservative” form
\[ \frac{\partial T}{\partial \nu} U_\nu = \frac{\partial X}{\partial \nu} \langle Q^{\nu H} \rangle, \quad \nu = 1, \ldots, N \] (2.26)
for the parameters \( U_\nu = \langle P_\nu \rangle \), which gives an equivalent form of the Whitham equations.

The conservative form (2.26) of the Whitham’s system will be very convenient in our considerations of the averaging of Hamiltonian structures.

Let us now put some “regularity” requirements about the joint \( M \) of the submanifolds \( M_{\omega,k} \) for all \( \omega \) and \( k \), corresponding to the full set of the \( m \)-phase quasiperiodic solutions of the system (1.6).

(III) We require that \( M \) represents an \((N + m)\)-dimensional submanifold in the space of the \( 2\pi \)-periodic with respect to each \( \theta^a \) functions.

The property (III) means nothing but the fact that the shapes of the solutions of (2.9) are all different at different \( \omega \) and \( k \) in the space of the \( 2\pi \)-periodic vector-functions of \( \theta \) so that \( \omega \) and \( k \) can be reconstructed from them. It is easy to see that this requirement corresponds to the generic situation. We will use here the property (III) in our procedure of averaging of bracket (1.1).

We will work with the full family of \( 2\pi \)-periodic solutions of (2.9) which will also depend on the “slow” variables \( X \) and \( T \). To define the corresponding submanifold in the space of functions \( \varphi(\theta, X, T) \) we should extend the coordinates \( U_\nu = \langle P_\nu \rangle \) as functionals of \( \varphi(\theta) \) in the vicinity of our submanifold \( M \). This can be easily done (see [18]) in the following way:

Let introduce \( N \) different functionals
\[ A_\nu = \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} a^\nu(\varphi, \varphi^{\theta^a}, \varphi^{\theta^a\theta^\beta}, \ldots) \, d^m \theta, \]
such that their values \( \bar{A}_\nu \) are functionally independent on the functions from the submanifold \( M \).

Then we can express \( U_\nu = U_\nu(A) \) in terms of \( \bar{A}_\nu \) on \( M \) and after that extend them as the functionals \( U_\nu(A) \) on the space of \( 2\pi \)-periodic with respect to each \( \theta^a \) functions.

We can also expand the coordinates \( \theta_0^a \) (see [18]) in the vicinity of \( M \) by introduction of, say, functionals
\[ B_\alpha^a[\varphi(\theta)] = \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \varphi^{\theta^a}(\theta) \, \Phi_{(in)}(\theta, U[\varphi]) \, d^m \theta, \]
which have zero values for \( \varphi(\theta) = \Phi_{(in)}(\theta, U[\varphi]) \). In the generic situation we can locally express the values of \( \theta_0^a \) on \( M \) in terms of \( \bar{B}_\alpha^a \) and after that put \( \theta_0^a = \theta_0^a(B[\varphi]) \) in the corresponding local coordinate maps in the vicinity of \( M \).

Now we consider the system
\[ \varphi^i(\theta, X) - \Phi_{(in)}^i(\theta + \theta_0[\varphi], U[\varphi]) \equiv 0, \] (2.27)
where \( \theta_0^a[\varphi] \) and \( U_\nu[\varphi] \) are the functionals in the vicinity of \( M \), as a system of constraints, which defines \( M \) in the space of \( 2\pi \)-periodic with respect to each \( \theta^a \) functions.

We can see now that the linearized system (2.27)
\[ \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} (L^i_{j,\theta_0}[\theta, \theta']) \delta \Phi^j(\theta') \, d^m \theta' = 0, \]
where
\[
L^i_{j U, \theta_0} (\theta, \theta') \equiv \delta^i_j \delta (\theta - \theta') - \sum_{\alpha=1}^m \Phi^\alpha_{(in) \theta^\alpha} (\theta + \theta_0[\varphi], U[\varphi]) \times \frac{\delta \theta^\alpha_0 [\varphi]}{\delta \varphi^j (\theta')} - \sum_{\nu=1}^N \Phi^i_{(in) \nu} (\theta + \theta_0[\varphi], U[\varphi]) \times \frac{\delta U^\nu_0 [\varphi]}{\delta \varphi^j (\theta')}
\]
has at any point \((U, \theta_0)\) of \(M\) exactly \(N + m\) solutions \(\tilde{\xi}_{(p) U, \theta_0} (\theta)\), corresponding to the tangential to \(M\) vectors \(\Phi_{(in) \theta^\alpha}\) and \(\Phi_{(in) \nu}\), \(\alpha = 1, \ldots, m\), \(\nu = 1, \ldots, N\).

It is evident also that all the “left eigenvectors” \(\tilde{\kappa}_{(p) U, \theta_0} (\theta)\), orthogonal to the image of \(\hat{L}\), are given by the variational derivatives \(\delta \theta^\alpha_0 [\varphi] / \delta \varphi^i (\theta)\) and \(\delta U^\nu_0 [\varphi] / \delta \varphi^i (\theta)\).

From the invariance of the submanifold \(M\) with respect to the flows (1.7) and (2.11) we can also write here the relations
\[
\frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} L^i_{j U, \theta_0} (\theta, \theta') \cdot \left( \Phi^\alpha_{(in)} (\theta' + \theta_0, U), k^\alpha \Phi^i_{(in) \theta^\alpha} (\theta' + \theta_0, U), \ldots \right) \left. d^m \theta' \equiv 0 \right) (2.28)
\]
and
\[
\frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} L^i_{j U, \theta_0} (\theta, \theta') \cdot \left( \Phi^\alpha_{(in)} (\theta' + \theta_0, U), k^\alpha \Phi^i_{(in) \theta^\alpha} (\theta' + \theta_0, U), \ldots \right) \left. d^m \theta' \equiv 0 \right) (2.29)
\]
for any \(i, k\) and \(\nu\) at any point \((U, \theta_0)\) of \(M\), where \(k^\alpha = k^\alpha [\Phi]\) can be considered now as the values of the corresponding functional on \(M\).

We now introduce the space of functions \(\varphi (\theta, X, T)\), depending on “slow” parameters \(X\) and \(T\) and 2\(\pi\)-periodic with respect to each \(\theta^\alpha\). Systems (2.27), considered independently at different \(X\), give us a system of constraints defining the submanifold \(M'\) in the space of functions \(\varphi (\theta, X)\), corresponding to \(m\)-phase solutions of (1.6) depending on the additional parameters \(X\) and \(T\).

It will be actually convenient to introduce also the “modified” constraints (2.27)
\[
G^i_{U, \theta_0} (\theta, X) = \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} L^i_{j U, \theta_0} (\theta, \theta') \cdot \varphi^j (\theta') - \Phi^i_{(in)} (\theta' + \theta_0 [\varphi], U [\varphi]) \left. d^m \theta' \equiv 0 \right) (2.30)
\]
and take \(U^\nu (X), \theta_0^\nu (X)\) and \(G^i_{U, \theta_0} (\theta, X)\), such that
\[
\frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \tilde{\kappa}_{(p) U} (\theta + \theta_0 (X)) \cdot G^i_{U, \theta_0} (\theta, X) \left. d^{m} \theta \equiv 0 \right) , \quad p = 1, \ldots, N + m \right) (2.31)
\]
as coordinates in the vicinity of \(M'\) instead of the \(\varphi^i (\theta, X)\). It is easy to see also that we can find uniquely \(\varphi^i (\theta, X)\) from the relations
\[
\frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} L^i_{j U, \theta_0} (\theta, \theta') \cdot \varphi^j (\theta') - \Phi^i_{(in)} (\theta' + \theta_0, U) \left. d^m \theta' \equiv G^i_{U, \theta_0} (\theta, X) \right)
\]
and the values of \(U^\nu (X)\) and \(\theta_0^\nu (X)\) under the conditions (2.31). 4

\(4\)This system of constraints is different from the system introduced in [18].
We will need also another coordinate system in the vicinity of $\mathcal{M}'$, which differs from the described above by a transformation, depending on the small parameter $\epsilon$ and singular at $\epsilon \to 0$. Namely, we recall our integrals (2.5)

$$I^\nu = \int \mathcal{P}^\nu(\varphi, \varphi_x, \ldots) dx,$$

make a transformation $X = \epsilon x$ and define the functionals

$$J^\nu(X) = \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \mathcal{P}^\nu(\varphi(\theta, X), \epsilon \varphi_x(\theta, X), \ldots) d^m \theta$$

(2.32)

on the space of $2\pi$-periodic with respect to each $\theta^\alpha$ functions $\varphi(\theta, X)$.

Let us also introduce the functionals

$$\theta_0^\alpha(X) = \theta_0^\alpha(X) - \theta_0^\alpha(X_0) - \frac{1}{\epsilon} \int_{X_0}^X k^\alpha(J(X')) dX'$$

(2.33)

for some fixed point $X_0$. We have identically

$$\theta_0^\alpha(X_0) \equiv 0$$

(2.34)

As was shown in [18], we can also obtain the values of $U^\nu(X)$ and $\theta_0^\alpha(X)$ from $J^\nu(X)$, $\theta_0^\alpha(X)$ and $\theta_0^\alpha(X_0)$ on $\mathcal{M}'$ as formal series in powers of $\epsilon$ and we will have for these series

$$U^\nu(X)[J, \theta_0^\alpha] = J^\nu(X) + \sum_{k \geq 1} \epsilon^k u_0^\nu(J, J_0, \theta_0^\alpha, \ldots)$$

(2.35)

$$\theta_0^\alpha(X)[J, \theta_0^\alpha] = \theta_0^\alpha(X) + \theta_0^\alpha(X_0) + \frac{1}{\epsilon} \int_{X_0}^X k^\alpha(J(X')) dX'$$

(2.36)

The form of the relation (2.35) will be important in our considerations, so we reproduce here the calculations from [18].

We remind that the values $J^\nu(X)$, $\theta_0^\alpha(X)$, $\theta_0^\alpha(X_0)$ and $U^\nu(X)$ are connected on $\mathcal{M}'$ by the relations (the definition of $J^\nu(X)$):

$$J^\nu(X) = \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \mathcal{P}^\nu(\Phi(\theta + s(X, \epsilon), U(X)), \epsilon \partial_\theta \Phi(\theta + s(X, \epsilon), U(X)), \ldots) d^m \theta =$$

$$= \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \mathcal{P}^\nu(\Phi(\theta + s(X, \epsilon), U(X)), k^\alpha(J) \partial_{\theta^\alpha} \Phi(\theta + s(X, \epsilon), U(X)), \ldots) d^m \theta +$$

$$+ \sum_{k \geq 1} \epsilon^k \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \mathcal{P}^\nu(\Phi(\theta + s(X, \epsilon), U(X)), \ldots) d^m \theta ,$$
where
\[ s(X, \epsilon) \equiv \theta_0^*(X) + \theta_0(X_0) + \frac{1}{\epsilon} \int_{X_0}^{X} k(J(X')) dX' \]
and \( \mathcal{P}^\nu_{(k)}(\Phi_{(in)}(\theta + s(X, \epsilon), \ldots) \) are local densities depending on \( \Phi_{(in)}(\theta + s(X, \epsilon), U(X)) \) and their derivatives with respect to \( U^\nu \) and \( \theta^\alpha \) with the coefficients of type: \( U_X(X), U_{XX}(X), \ldots, k(J), \partial_X k(J), \partial_X^2 k(J), \ldots, \) and \( \theta_0^* \)(X), \( \theta_0^{**} \)(X), \ldots, given by the collecting together these terms, having the general multiplier \( \epsilon^k \). The term corresponding to the zero power of \( \epsilon \) is written separately.

After the integration with respect to \( \theta \), which removes the singular at \( \epsilon \to 0 \) phase shift \( \theta_0 \) in the argument of \( \Phi_{(in)} \), we obtain on \( \mathcal{M}' \):
\[ J^\nu(X) = \zeta^\nu(J, U) + \sum_{k \geq 1} \epsilon^k \zeta^\nu_{(k)}(U, U_X, \ldots, U_{XX}, J, J_X, \ldots, J_{XX}, \theta_0^*, \ldots, \theta_0^{*k}) \quad (2.37) \]
The sum in (2.37) contains a finite number of terms. The functions \( \zeta^\nu_{(k)} \) and \( \zeta^\nu \) are the integrated with respect to \( \theta \) functions \( \mathcal{P}^\nu_{(k)} \) and \( \mathcal{P}^\nu \) respectively.

So, since
\[ \zeta^\nu(J, U) = \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \mathcal{P}^\nu(\Phi_{(in)}(\theta, U), k^a(J) \Phi_{(in)\theta^a}(\theta, U), \ldots) d^n\theta \]
we obtain that the system
\[ J^\nu(X) = \zeta^\nu(J, U)(X) \quad (2.38) \]
is satisfied by the solution \( J^\nu(X) \equiv U^\nu(X) \) according to the definition of the parameters \( U^\nu \). Since we suppose that system (2.38) has a generic form we will assume that (locally) this is the only solution and put \( J^\nu(X) = U^\nu(X) \) in the zero order of \( \epsilon \).

After that we can resolve system (2.37) by iterations, taking on the initial step \( U^\nu(X) = J^\nu(X) \). The substitution of (2.35) into (2.37) under the condition of the non-singularity of matrix \( \| \frac{\partial \zeta^\nu(J, U)}{\partial U^\nu} \|_{U^\nu} \) will sequentially define the functions \( u^\nu_{(k)} \). So we obtain the relations (2.35) and (2.36).

Now we can take also the values of \( J^\nu(X), \theta_0^{*\alpha}(X), \theta_0^*(X_0) \) and \( G^i_{(\nu)[\varphi,\theta_0(\varphi)]}(\theta, X) \) with the restrictions (2.34) and also
\[ \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \mathcal{R}_{(\nu)[\varphi, (X)]} \left( \theta + \theta_0^*(X) + \theta_0(X_0) + \frac{1}{\epsilon} \int_{X_0}^{X} k(J(X')) dX' \right) G^i_{(\nu)[\varphi, \theta_0(\varphi)]}(\theta, X) d^n\theta \equiv 0 \]
as coordinates in the vicinity of \( \mathcal{M}' \).

We define now a Poisson bracket on the space of functions \( \varphi(\theta, X) \) by the formula
\[ \{ \varphi^i(\theta, X), \varphi^j(\theta', Y) \} = \sum_{k \geq 0} B^i_{k}^{(j)}(\varphi, \epsilon \varphi_X, \ldots) \epsilon^k \delta^{(k)}(X - Y) \delta(\theta - \theta') + \]
\[ + \frac{1}{\epsilon} \delta(\theta - \theta') \sum_{k \geq 0} e_k S^i_{(k)}(\varphi, \epsilon \varphi_X, \ldots) \nu(X - Y) S^j_{(k)}(\varphi, \epsilon \varphi_Y, \ldots) \quad (2.40) \]
which is just a rescaling of the bracket (1.4), multiplied by $\delta(\theta - \theta')$. We normalize here the $\delta$-function $\delta(\theta - \theta')$ by $(2\pi)^m$.

The pairwise Poisson brackets of the constraints $G^i_{[U,\theta_0]}(\theta, X)$ on $\mathcal{M}'$ can be written in the form

$$\{G^i(\theta, X), G^j(\theta', Y)\}|_{\mathcal{M}'} = \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \tilde{L}_k^j(U(X), \theta_0(X))|_{\mathcal{M}'} d^m \tau d^m \sigma \quad (2.41)$$

(we can omit the Poisson brackets of $L_i^k$ and $L_i^j$ on $\mathcal{M}'$ and also the brackets of the functionals $\theta_0^\nu[\varphi]$ and $U^\nu[\varphi]$ from $\Phi_{(in)}$ in (2.30) since they are multiplied by the convolutions of the corresponding $L$-operators with the “right eigen vectors” $\Phi_{(in)\theta^\alpha}$ and $\Phi_{(in)U^\nu}$, which are zero on $\mathcal{M}'$).

Brackets (2.41) evidently satisfy the orthogonality conditions:

$$\frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \tilde{L}_k^j(U[J, \theta_0](X)) \left( \theta + \theta_0^\nu(X) + \theta_0(X_0) + \frac{1}{\epsilon} \int_{X_0}^X k(J(X'))dX' \right) \times$$

$$\times \left\{ \{G^i(\theta, X), G^j(\theta', Y)\} |_{\mathcal{M}'} d^m \theta \right. = 0 \quad (2.42)$$

$$\frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \{G^i(\theta, X), G^j(\theta', Y)\} |_{\mathcal{M}'} \times$$

$$\times \tilde{L}_k^j(U[J, \theta_0](Y)) \left( \theta + \theta_0^\nu(Y) + \theta_0(X_0) + \frac{1}{\epsilon} \int_{X_0}^Y k(J(Y'))dY' \right) \left. d^m \theta' \right. = 0 \quad (2.43)$$

for $q = 1, \ldots, N + m$ in the coordinates $J(X)$, $\theta_0^\nu(X)$ and $\theta_0(X_0)$ on the submanifold $\mathcal{M}'$.

We note now that every derivative with respect to $X$ or $Y$ appears in the bracket (2.40) with the multiplier $\epsilon$ but, being applied to the functions

$$\varphi^j(\theta, X) = \Phi_{(in)}^j \left( \theta + \theta_0^\nu(X) + \theta_0(X_0) + \frac{1}{\epsilon} \int_{X_0}^X k(J(X'))dX', U[J, \theta_0](X) \right) \quad (2.44)$$

on $\mathcal{M}'$, contains the nonzero at $\epsilon \to 0$ term $k^\alpha(J) \partial / \partial \theta^\alpha$. Now we formulate the statement about the structure of the bracket (2.40) on $\mathcal{M}'$ in the coordinates $J(X)$, $\theta_0^\nu(X)$ and $\theta_0(X_0)$.

**Lemma 2.3**

The pairwise Poisson brackets of constraints $G^i_{[U,\theta_0]}(\theta, X)$ on $\mathcal{M}'$ have no singular terms at $\epsilon \to 0$ and no non-local terms in the zero order of $\epsilon^0$ at any fixed coordinates $J^\nu(X)$, $\theta_0^\alpha(X)$ and $\theta_0(X_0)$ (such that $U(X) = U[J, \theta_0^\nu](X)$, $\theta_0(X) = \theta_0[J, \theta_0^\nu(\theta_0(X_0))](X)$).

**Proof.**

The first statement is evident for the local part of bracket (2.40), since any differentiation with respect to $X$ in it appears with the multiplier $\epsilon$ and has the regular at $\epsilon \to 0$ form $k^\alpha(J(X)) \partial / \partial \theta^\alpha + O(\epsilon)$, being applied to the functions of the form (2.44). So, we should check only the non-local part of (2.40), which contains the multiplier $\epsilon^{-1}$ in it. But, according to the relation (2.28) and also (2.35), we have that the terms arising on the both sides of $\nu(X - Y)$ (the convolutions of $\hat{L}$ with $S(k)\hat{\Phi}, k^\alpha\Phi_{\theta^\alpha}, \ldots)$) are of order of $\epsilon$ on $\mathcal{M}'$ in the coordinates $J^\nu(X)$ and $\theta_0^\alpha(X)$. So, we obtain
that all the non-local part of (2.41) is of the order of $\epsilon$ on $\mathcal{M}'$ at any fixed coordinates $J^\nu(X)$, $\theta^\alpha_0(X)$ and $\theta^\alpha_0(X_0)$.

Lemma 2.3 is proved.

Let us formulate now the last “regularity” property of the submanifold $\mathcal{M}'$ with respect to the Poisson structure (2.40).

We consider in the coordinates $J^\nu(X)$, $\theta^\alpha_0(X)$ and $\theta^\alpha_0(X_0)$ on $\mathcal{M}'$ a linear non-homogeneous system on the functions

$$f_{j,[J,\theta_0^0]}(\theta' + \theta^\alpha_0(Y) + \theta_0(X_0) + \frac{1}{\epsilon} \int_{X_0}^Y k(J(Y'))dY', Y, \epsilon)$$

having the form

$$\frac{1}{(2\pi)^m} \int_{0}^{2\pi} \cdots \int_{0}^{2\pi} \{G^i_{[U,\theta_0^0]}(\theta, X), G^j_{[U,\theta_0^0]}(\theta', Y)\}|_{\mathcal{M}'} \times$$

$$\times f_j(\theta' + \theta^\alpha_0(Y) + \theta_0(X_0) + \frac{1}{\epsilon} \int_{X_0}^Y k(J(Y'))dY', Y, \epsilon) d^m\theta' dY =$$

$$= \{G^i_{[U,\theta_0^0]}(\theta, X), F[\varphi](\epsilon)\}|_{\mathcal{M}'}$$

(2.45)

where $F[\varphi](\epsilon)$ is a functional, defined in the vicinity of $\mathcal{M}'$.

After all differentiations with respect to $X$ we can omit the term

$$\theta^\alpha_0(X) + \theta_0(X_0) + \frac{1}{\epsilon} \int_{X_0}^X k^\alpha(J(X'))dX'$$

which appears in all functions depending on $\theta$ and $X$ in (2.45), and then consider the system (2.45) at the zero order of $\epsilon$.

From Lemma 2.3 we have that in the zero order of $\epsilon$ the brackets $\{G^i(\theta, X), G^j(\theta', Y)\}$ on $\mathcal{M}'$ do not include non-local terms, containing $\nu(X - Y)$. For the derivatives with respect to $X$, which arise with the multiplier $\epsilon$ from the local terms of $\{\varphi^k(\tau, X), \varphi^*(\sigma, Y)\}|_{\mathcal{M}'}$, we should take in the zero order of $\epsilon$ only the main part $k^\alpha(J) \partial/\partial\theta^\alpha$. So, in the zero order of $\epsilon$ we obtain from (2.45) just linear systems of integro-differential equations with respect to $\theta$ and $\theta'$ on the functions $f_j(\theta', X)$, independent at different $X$. We have also that the right-hand side of (2.45) satisfies at any $X$ and $\epsilon$ the compatibility conditions (2.42) (let us remind that $U^\nu[J,\theta_0^0]$ are the asymptotic series at $\epsilon \to 0$).

(IV) We require that the system (2.45) is resolvable on $\mathcal{M}'$ for any $F[\varphi](\epsilon)$ in the class of $2\pi$-periodic with respect to all $\theta^\alpha$ functions and its solutions can be represented in the form of regular at $\epsilon \to 0$ asymptotic series

$$f_{j,[J,\theta_0^0]}(\theta, Y, \epsilon) = \sum_{n \geq 0} \epsilon^n f_{j,[J,\theta_0^0]}^{(k)}(\theta, Y)$$

for regular at $\epsilon \to 0$ right-hand sides of (2.45).\(^5\)

\(^5\)Let us say that this requirement is satisfied for a wide class of Poisson brackets (1.4), however, it is not necessary
The condition (IV) is responsible for the Dirac restriction of the bracket (2.40) on the submanifold \( \mathcal{M}' \).

Now we prove a statement which will be very important for our averaging procedure.

**Lemma 2.4**

The Poisson brackets of the functionals \( \theta_0^\alpha(X) \) with \( J^\nu(Y) \) are of order of \( \epsilon \) at \( \epsilon \to 0 \) on \( \mathcal{M}' \) at any fixed coordinates \( J^\nu(X), \theta_0^\alpha(X) \) and \( \theta_0^\alpha(X_0) \):

\[
\{ \theta_0^\alpha(X), J^\nu(Y) \} |_{\mathcal{M}'} = O(\epsilon), \quad \epsilon \to 0
\] (2.46)

**Proof.**

First we note that the Poisson brackets of \( \varphi^i(\theta, X) \) with the functionals \( J^\nu(Y) \) can be written in the form

\[
\{ \varphi^i(\theta, X), J^\nu(Y) \} = \sum_{k \geq 0} C_k^{i\nu}(\varphi(\theta, X), \epsilon \varphi_X(\theta, X), \ldots, X - Y) +
\]

\[
+ \sum_{k \geq 0} \epsilon_k S_k^i(\varphi(\theta, X), \epsilon \varphi_X(\theta, X), \ldots) \nu(X - Y) \left( F_{k}^{\nu}(\varphi(\theta, Y), \epsilon \varphi_Y(\theta, Y), \ldots) \right) dY
\]

for some \( C_k^{i\nu}(\varphi, \epsilon \varphi_X, \ldots) \) and \( F_{k}^{\nu}(\varphi, \epsilon \varphi_Y, \ldots) \) (we have integrated with respect to \( \theta' \)).

So, the flow generated by the functional \( \int q(Y) J^\nu(Y) dY \) (where \( q(Y) \) has a compact support) can be written as

\[
\varphi^i = \sum_{k \geq 0} C_k^{i\nu}(\varphi, \epsilon \varphi_X, \ldots) \epsilon_k q_k(X) +
\]

\[
+ \sum_{k \geq 0} \epsilon_k S_k^i(\varphi, \epsilon \varphi_X, \ldots) \int \nu(X - Y) q(Y) \frac{d}{dY} F_{k}^{\nu}(\varphi, \epsilon \varphi_Y, \ldots) dY
\]

\[
= \sum_{k \geq 0} C_k^{i\nu}(\varphi, \epsilon \varphi_X, \ldots) \epsilon_k q_k(X) + \sum_{k \geq 0} \epsilon_k S_k^i(\varphi, \epsilon \varphi_X, \ldots) F_{k}^{\nu}(\varphi, \epsilon \varphi_X, \ldots) q(X) -
\]

\[
- \sum_{k \geq 0} \epsilon_k S_k^i(\varphi, \epsilon \varphi_X, \ldots) \int \nu(X - Y) F_{k}^{\nu}(\varphi, \epsilon \varphi_Y, \ldots) q_Y(Y) dY
\] (2.47)

As can be easily seen, the local terms of (2.47) have the form

\[
q(X) \left[ C_0^{i\nu}(\varphi, \epsilon \varphi_X, \ldots) + \sum_{k \geq 0} \epsilon_k S_k^{i\nu}(\varphi, \epsilon \varphi_X, \ldots) F_{k}^{\nu}(\varphi, \epsilon \varphi_X, \ldots) \right] + O(\epsilon)
\]

fulfilled in the general case. It can be actually shown that this requirement can be significantly weakened and replaced by resolvability of system (2.45) just for everywhere dense set of parameters \( U[\varphi] \) on \( \mathcal{M}' \), using the approach represented in [42, 43]. We will, however, use here the assumption, formulated above, since the methods used in [42, 43] require in fact noticeably longer considerations.
where the term in the brackets is just the flow, generated by the functional

\[
\frac{1}{(2\pi)^m} \int \int_{0}^{2\pi} \int_{0}^{2\pi} P^\nu(\varphi, \epsilon \varphi_X, \ldots) d^m \theta \, dX
\]

In the non-local part of (2.47) (the last expression) we have the convolution of the “slow” functions \(q_Y(Y)\) with the rapidly oscillating \(F^\nu_{(k)}(\varphi, \epsilon \varphi_Y, \ldots)\), where \(\varphi'(\theta, Y)\) has the form (2.44). So, in the leading order of \(\epsilon\) we can neglect the dependence on \(\theta\) of the last integral of (2.47) and take the averaged with respect to \(\theta\) values \(\langle F^\nu_{(k)} \rangle\) on \(M'\) instead of the exact \(F^\nu_{(k)}(\varphi, \epsilon \varphi_Y, \ldots)\) in the integral expression in (2.47).

After that we obtain, that the non-local term of (2.47) gives us in the zero order of \(\epsilon\) a linear combination of the flows \(S_{(k)}(\varphi, \epsilon \varphi_X, \ldots)\), considered on the functions

\[
\varphi^i(\theta, X) = \Phi^i_{(in)} \left( \theta + \theta^*_0(X) + \theta_0(X_0) + \frac{1}{\epsilon} \int_{X_0}^{X} k(J(X')) \, dX' \right)
\]

at any fixed point \(X\).

From the invariance of the submanifold \(M\) with respect to the flows (2.17) and (2.18) we can conclude now that the flow (2.47), being considered at the points of \(M'\) with fixed coordinates \(J(X), \theta^*_0(X), \theta_0(X_0)\) in the zero order of \(\epsilon\), leaves \(M'\) invariant and generates on it a linear evolution of the initial phases

\[
\theta^*_0(X) = \theta^*_0(X) + \theta^*_0(X_0) + \frac{1}{\epsilon} \int_{X_0}^{X} k(J(X')) \, dX'
\]

with some frequencies \(\Omega^\nu_{[\theta]}(X)\). Here we use the formula (2.35) for \(U[J, \theta^*_0]\) and we can claim now that the Poisson brackets of the functionals \(\theta^*_0(X)\) with \(\int q(Y) J^\nu(Y) dY\) at the points of \(M'\) with fixed coordinates \(J^\nu(X), \theta^*_0(X)\) and \(\theta_0(X_0)\) have the form

\[
\{\theta^*_0(X), \int q(Y) J^\nu(Y) dY\} = \Omega^\nu_{[\theta]}[J, \theta^*_0](X) + O(\epsilon)
\]

(2.48)

Let us now prove the relation

\[
\{k^\alpha(J(X)), \int q(Y) J^\nu(Y) dY\} = \epsilon \frac{d}{dX} \Omega^\nu_{[\theta]}[J, \theta^*_0](X) + O(\epsilon^2)
\]

(2.49)

at the points of \(M'\) with fixed values of \(J^\nu(X), \theta^*_0(X)\) and \(\theta_0(X_0)\).

Using again the relation (2.35) we can write for (2.47) at the points of \(M'\)

\[
\varphi^i = \Omega^\beta_{[\theta]}(X) \Phi^i_{(in)} \left( \theta + \theta^*_0(X) + \theta_0(X_0) + \frac{1}{\epsilon} \int_{X_0}^{X} k(J(X')) \, dX', U[J, \theta^*_0](X) \right) + \\
+ \epsilon \eta^i \left( \theta + \theta^*_0(X) + \theta_0(X_0) + \frac{1}{\epsilon} \int_{X_0}^{X} k(J(X')) \, dX', [J, \theta^*_0] \right)
\]

(2.50)

where \([J, \theta^*_0]\) means a regular at \(\epsilon \to 0\) dependence on \(J, J_X, \theta^*_0, \ldots\).
We are interested in the evolution of the functionals
\[ J^\mu(X) = \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} P^\mu(\varphi, \epsilon\varphi_X, \ldots) \ d^m\theta \]

We have
\[
\frac{d}{dt} J^\mu(X) = \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \left( \frac{\partial P^\mu}{\partial \varphi^i} \varphi^i_t + \frac{\partial P^\mu}{\partial \varphi^i_X} \varphi^i_{tX} + \frac{\partial P^\mu}{\partial \varphi^i_{XX}} \varphi^i_{tXX} + \ldots \right) d^m\theta = 
\]
where the densities \( \Pi^\mu_{(i)} \) were introduced in (2.23).

It is easy to see that (2.50) does not change \( J^\mu(X) \) at the zero order of \( \epsilon \) and we can also state that the terms of the order of \( \epsilon \) in (2.50) (i.e. \( \epsilon\eta^i(\theta + \ldots, X) \)) are unessential for the evolution of \( k(J(X)) \) on \( M' \) at the order of \( \epsilon \). Indeed, their contribution to the evolution of \( J^\mu(X) \) in the order of \( \epsilon \) is
\[
\epsilon \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \left( \Pi^\mu(\theta + \ldots, X) + \Pi^\mu \theta \right) d^m\theta 
\]
where we should take only the main term \( k^\gamma(J(X)) \partial/\partial\theta^\gamma \) for the derivatives \( \epsilon \partial/\partial\theta \) in the formula (2.51). After the integration by parts we have for this contribution
\[
\epsilon \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \left( \Pi^\mu(\theta + \ldots, X) \right) d^m\theta 
\]

But after the substitution of the main part of \( \varphi^i(\theta, X) \)
\[
\Phi^i_{(in)} \left( \theta + \theta^*_0(X) + \theta_0(X_0) + \frac{1}{\epsilon} \int_X X dX, J(X) \right) 
\]
(according to (2.35)) into the densities \( \Pi^\mu_{(i)}(\varphi, \epsilon\varphi_X, \ldots) \) we obtain in the leading order of \( \epsilon \) the convolution of \( \eta(\theta, X) \) with the variational derivative of the functional \( I^\mu \), introduced in (2.15), with respect to \( \varphi(\theta, X) \). Our statement follows now from Lemma 2.1, which claims that the variational derivatives of the functionals \( k^\alpha(\tilde{I}[\varphi]) \) are identically equal to zero on the space of \( m \)-phase solutions of (1.6).

Consider now the first term of (2.50). We have that the evolution of \( J^\mu(X) \), which is responsible for the evolution of \( k(J) \), is given by the expression
\[
\frac{d}{dt} J^\mu(X) = \Omega_{(\mu)}^\nu(X) \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \left( \frac{\partial P^\mu}{\partial \varphi^i} \Phi^i_{(in)}(\theta + s(X, \epsilon), U[J, \theta^*_0](X)) + \right) \left( \frac{\partial P^\mu}{\partial \varphi^i_X} \Phi^i_{(in)}(\theta + s(X, \epsilon), U[J, \theta^*_0](X)) + \right) \ d^m\theta + 
\]
\[
+ \epsilon \left( \Omega_{(\mu)}^\nu(X) \right) \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \left( \frac{\partial P^\mu}{\partial \varphi^i_X} + \frac{\partial P^\mu}{\partial \varphi^i_X} \frac{\partial}{\partial X} + 3 \frac{\partial P^\mu}{\partial \varphi^i_{XX} X} \frac{\partial^2}{\partial X^2} + \ldots \right) \times \Phi^i_{(in)}(\theta + s(X, \epsilon), U[J, \theta^*_0](X)) d^m\theta + O(\epsilon^2) , 
\]
where \( s(X, \epsilon) \equiv \theta_0^*(X) + \theta_0(X_0) + \frac{1}{\epsilon} \int_{X_0}^{X} k(J(X')) dX' \).

The first term here after the substitution of exact \( \varphi^i \) in the form

\[
\varphi^i(\theta, X) = \Phi^i_{(in)\theta^0}(\theta + s(X, \epsilon), U[J, \theta^0_1](X))
\]
on \( \mathcal{M}' \), as can be easily seen, is just

\[
\mathcal{Q}_{[q]}^\mu(\theta) \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{\partial}{\partial \theta^\beta} \mathcal{P}^\mu_{(in)}(\Phi_{(in)}(\theta, \Phi_{(in)X}(\cdots), \cdots)) d^m \theta \equiv 0,
\]

while the second term on \( \mathcal{M}' \) in the leading order of \( \epsilon \) is equal to

\[
\epsilon \left( \mathcal{Q}_{[q]}^\mu(\theta) \right) \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \sum_{p \geq 1} p \times \left( \Phi^i_{(in)}(\theta + s(X, \epsilon), J(X)), \ldots \right) \times k^{\alpha_1}(J(X)) \ldots k^{\alpha_{p-1}}(J(X)) \Phi^i_{(in)\theta^0\theta^0 \ldots \theta^0}(\theta + s(X, \epsilon), J(X)), \ldots \right) \times d^m \theta,
\]

which coincides with the integral expression from (2.24) in Lemma 2.2. So, from Lemma 2.2 we have that the summation of (2.52) with \( \partial k^{\alpha}/\partial J^\mu \) is equal to \( \epsilon \left( \mathcal{Q}_{[q]}^\mu(\theta) \right) \right) \delta^\alpha_\beta \) and we obtain that

\[
\frac{\partial}{\partial t} k^{\alpha}(J) = \epsilon \frac{\partial}{\partial X^\alpha} \mathcal{Q}_{[q]}^{\mu}(\theta) + O(\epsilon^2),
\]

i.e. the relation (2.49).

Now, using (2.48) and (2.49), we can write that

\[
\{\theta_0^\alpha(X), \int q(Y) J^\nu(Y) dY\} = \{\theta_0^\alpha(X) - \theta_0^\alpha(X_0) - \frac{1}{\epsilon} \int_{X_0}^{X} k^{\alpha}(J(X')) dX', \int q(Y) J^\nu dY\} = O(\epsilon)
\]

for any \( q(Y) \) in our coordinates on \( \mathcal{M}' \).

So, we have

\[
\{\theta_0^\alpha(X), J^\nu(Y)\}|_{\mathcal{M}'} = O(\epsilon)
\]
at any fixed coordinates \( J^\nu(X), \theta_0^\alpha(X) \) and \( \theta_0^\alpha(X_0) \).

Lemma 2.4 is proved.

### 3 Averaging procedure.

Let us now describe the averaging procedure of the Poisson bracket (1.4) on the family of \( m \)-phase solutions of (1.6) under the conditions of “regularity” formulated above.

**Theorem 3.1**
Let us have a Poisson bracket (1.4) and a local system (1.6) generated by a local Hamiltonian function

\[ H = \int \mathcal{P}_H(\varphi, \varphi, \ldots) \, dx \]

which has \( N(\geq 2m) \)-parametric full family of \( m \)-phase solutions modulo \( m \) initial phase shifts \( \theta_0^\alpha \).

Let us have \( N \) commuting local translationally invariant integrals

\[ I^\nu = \int \mathcal{P}^\nu(\varphi, \varphi, \ldots) \, dx \]

\[ \{ I^\nu, H \} = 0, \quad \{ I^\nu, I^\mu \} = 0, \]

which generate local flows according to the Poisson bracket (1.4) and the averaged densities of which can be taken as parameters \( U^\nu \) on the space of \( m \)-phase solutions of (1.6) (the conditions (A)-(C)).

Then under the “regularity” conditions (I)-(IV) for the space of \( m \)-phase solutions of (1.6) we can construct a Poisson bracket of Ferapontov type (1.5) for the “slow” parameters \( U^\nu(X) \) by the following procedure:

We calculate the pairwise Poisson brackets of \( \mathcal{P}^\nu(\varphi, \varphi, \ldots) \) in the form

\[ \{ \mathcal{P}^\nu(\varphi, \varphi, \ldots), \mathcal{P}^\mu(\varphi, \varphi, \ldots) \} = \sum_{k \geq 0} A^\nu_0(\varphi, \varphi, \ldots) \delta^{(k)}(x - y) + \]

\[ + \sum_{k \geq 0} e_k \left( F^\nu_k(\varphi, \varphi, \ldots) \right)_x \nu(x - y) \left( F^\mu_k(\varphi, \varphi, \ldots) \right)_y \]

(where is a finite number of terms in the both sums). Here we have the total derivatives of the functions \( F^\nu_k \) and \( F^\mu_k \) with respect to \( x \) and \( y \) as a corollary of the fact that both \( I^\nu \) and \( I^\mu \) generate local flows according to the Poisson bracket (1.4). From the commutativity of the set \( \{ I^\nu \} \) we have also

\[ A^\nu_0(\varphi, \varphi, \ldots) + \sum_{k \geq 0} e_k \left( F^\nu_k(\varphi, \varphi, \ldots) \right)_x F^\mu_k(\varphi, \varphi, \ldots) \equiv \partial_x Q^\nu\mu(\varphi, \varphi, \ldots) \quad (3.1) \]

for some functions \( Q^\nu\mu(\varphi, \varphi, \ldots) \).

Then for the “slow” coordinates \( U^\nu(X) = \langle \mathcal{P}^\nu \rangle(X) \) we can define a Poisson bracket by the formula

\[ \{ U^\nu(X), U^\mu(Y) \} = \]

\[ = \left[ \langle A^\nu_1^\mu \rangle(X) + \sum_{k \geq 0} e_k \left( \langle F^\nu_k \rangle F^\mu_k - \langle F^\nu_k \rangle \langle F^\mu_k \rangle \right)(X) \right] \delta'(X - Y) + \]

\[ + \left[ \frac{\partial(Q^\nu\mu)}{\partial X}(X) - \sum_{k \geq 0} e_k \frac{\partial \langle F^\nu_k \rangle}{\partial X}(X) \langle F^\mu_k \rangle(X) \right] \delta(X - Y) + \]

\[ + \sum_{k \geq 0} e_k \frac{\partial \langle F^\nu_k \rangle}{\partial X} \nu(X - Y) \frac{\partial \langle F^\mu_k \rangle}{\partial Y}, \quad (3.2) \]

29
where the averaged values are the functions of \( U(X) \) and \( U(Y) \) at the corresponding points \( X \) and \( Y \).

Bracket \((3.2)\) satisfies the Jacobi identity and is invariant with respect to the choice of the set \( \{I^1, \ldots, I^N\} \), satisfying (A)-(C), if the choice of these integrals is not unique, i.e.

if \( U^\nu = \langle P^\nu \rangle, \ \tilde{U}^\nu = \langle \tilde{P}^\nu \rangle \) and \( \{ U^\nu(X), U^\mu(Y) \}, \ \{ \tilde{U}^\nu(X), \tilde{U}^\mu(Y) \} \) are the brackets \((3.2)\), constructed with the aid of the sets \( \{ I^\nu \} \) and \( \{ \tilde{I}^\nu \} \) respectively, then

\[
\{ \tilde{U}^\nu(X), \tilde{U}^\mu(Y) \} = \frac{\partial \tilde{U}^\nu}{\partial U^\lambda}(X) \{ U^\lambda(X), U^\sigma(Y) \} \frac{\partial \tilde{U}^\mu}{\partial U^\sigma}(Y)
\]

\text{Proof.}

The most difficult part is to prove the Jacobi identity for the bracket \((3.2)\). For this we use the Dirac restriction of the Poisson bracket \((2.40)\) on the submanifold \( M' \) with the coordinates \( \nu^\alpha(X), \theta^\alpha_0(X) \) and \( \nu_0^\alpha(X_0) \) on it. According to the Dirac restriction procedure we should find for \( \nu^\alpha(X), \theta^\alpha_0(X) \) and \( \nu_0^\alpha(X_0) \) the corrections of the form

\[
V^\nu(X) = \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} v^\nu_J(J, \theta^\alpha_0, \theta_0(X_0))(X, \theta^\nu, Z) G^J(\theta^\nu, Z) \text{ d}^m \theta \text{ d}Z,
\]

\[
W^\alpha(X) = \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} w^\alpha_J(J, \theta^\alpha_0, \theta_0(X_0))(X, \theta^\nu, Z) G^J(\theta^\nu, Z) \text{ d}^m \theta \text{ d}Z,
\]

and

\[
O^\nu = \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} o^\nu_J(J, \theta^\alpha_0, \theta_0(X_0))(\theta^\nu, Z) G^J(\theta^\nu, Z) \text{ d}^m \theta \text{ d}Z,
\]

such that the fluxes, generated in the Hamiltonian structure \((2.40)\) by the “functionals” \( J^\nu(X) + V^\nu(X), \ \theta^\alpha_0(X) + W^\alpha(X) \) and \( \nu_0^\alpha(X_0) + O^\nu \), leave \( M' \) invariant, i.e.

\[
\frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \{ G^i(\theta, Y), G^j(\theta^\nu, Z) \} \mid_{M'} \times v^\nu_J(X, \theta^\nu, Z) \text{ d}^m \theta \text{ d}Z = \{ G^i(\theta, Y), J^\nu(X) \} \mid_{M'} \quad (3.3)
\]

\[
\frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \{ G^i(\theta, Y), G^j(\theta^\nu, Z) \} \mid_{M'} \times w^\alpha_J(X, \theta^\nu, Z) \text{ d}^m \theta \text{ d}Z = \{ G^i(\theta, Y), \theta^\alpha_0(X) \} \mid_{M'} \quad (3.4)
\]

\[
\frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \{ G^i(\theta, Y), G^j(\theta^\nu, Z) \} \mid_{M'} \times o^\nu_J(\theta^\nu, Z) \text{ d}^m \theta \text{ d}Z = \{ G^i(\theta, Y), \nu^\nu_0(X_0) \} \mid_{M'} \quad (3.5)
\]

After that we put for the Dirac restriction on \( M' \)

\[
\{ J^\nu(X), J^\mu(Y) \}_{D} = \{ J^\nu(X) + V^\nu(X), J^\mu(Y) + V^\mu(Y) \} \mid_{M'} = \{ J^\nu(X), J^\mu(Y) \} \mid_{M'} -
\]

\[
- \frac{1}{(2\pi)^{2m}} \int_0^{2\pi} \cdots \int_0^{2\pi} v^\nu_J(X, \theta, Z) \times v^\mu_J(Y, \theta', Z') \times \{ G^i(\theta, Z), G^j(\theta', Z') \} \mid_{M'} \text{ d}^m \theta \text{ d}^m \theta' \text{ d}Z \text{ d}Z'
\]

and, in the same way,
\[ \{ J^\nu(X), \theta^\alpha_0(Y) \}_D = \{ J^\nu(X), \theta^\alpha_0(Y) \} |_{\mathcal{M}'} - \]
\[ - \frac{1}{(2\pi)^{2m}} \int_0^{2\pi} \cdots \int_0^{2\pi} v_0^\nu(X, \theta, Z) \times w_0^\alpha(Y, \theta', Z') \times \{ G^i(\theta, Z), G^j(\theta', Z') \} |_{\mathcal{M}'} \ d^n\theta \ d^n\theta' \ dZ \ dZ' \quad (3.7) \]
\[ \{ \theta^\alpha_0(X), \theta^\beta_0(Y) \}_D = \{ \theta^\alpha_0(X), \theta^\beta_0(Y) \} |_{\mathcal{M}'} - \]
\[ - \frac{1}{(2\pi)^{2m}} \int_0^{2\pi} \cdots \int_0^{2\pi} w_0^\alpha(X, \theta, Z) \times w_0^\beta(Y, \theta', Z') \times \{ G^i(\theta, Z), G^j(\theta', Z') \} |_{\mathcal{M}'} \ d^n\theta \ d^n\theta' \ dZ \ dZ' \quad (3.8) \]

(And so on).

After calculation of the brackets in (3.3)-(3.5) and the substitution of \( \varphi(\theta, X) \) in the form (2.44), we obtain regular at \( \epsilon \to 0 \) systems for the functions \( \tilde{v}_j^\nu(X, \theta, Z, \epsilon) \), \( \tilde{w}_j^\alpha(X, \theta, Z, \epsilon) \) and \( \tilde{\sigma}_j^\alpha(\theta, Z, \epsilon) \), such that

\[ v_j^\nu(X, \theta, Z, \epsilon) = \tilde{v}_j^\nu \left( X, \theta + \theta^\nu_0(Z) + \theta_0(X_0) + \frac{1}{\epsilon} \int_{X_0}^{Z} k(J(Z')) dZ', \epsilon \right) \]
\[ w_j^\alpha(X, \theta, Z, \epsilon) = \tilde{w}_j^\alpha \left( X, \theta + \theta^\alpha_0(Z) + \theta_0(X_0) + \frac{1}{\epsilon} \int_{X_0}^{Z} k(J(Z')) dZ', \epsilon \right) \]

and
\[ \sigma_j^\alpha(\theta, Z, \epsilon) = \tilde{\sigma}_j^\alpha \left( \theta + \theta^\alpha_0(Z) + \theta_0(X_0) + \frac{1}{\epsilon} \int_{X_0}^{Z} k(J(Z')) dZ', \epsilon \right) \],

which coincide with the system (2.43).

From the arguments analogous to those used in Lemma 2.3 and the fact that the flows, generated by the functionals \( J^\nu(X) \), leave invariant the submanifold \( \mathcal{M}^\theta \) at the zero order of \( \epsilon \) (at fixed coordinates \( J(X), \theta^\alpha_0(X), \theta_0(X_0) \)) we have also that the right-hand sides of these systems are regular at \( \epsilon \to 0 \) in these coordinates.

So, according to (IV), we can find the functions \( \tilde{v}_j^\nu, \tilde{w}_j^\alpha \) and \( \tilde{\sigma}_j^\alpha \) in the form of regular at \( \epsilon \to 0 \) asymptotic series. (The functions \( v_j^\nu(X, \theta, Z, \epsilon), w_j^\alpha(X, \theta, Z, \epsilon) \) and \( \sigma_j^\alpha(\theta, Z, \epsilon) \) are not uniquely defined but it is easy to show that this does affect the Dirac restriction of the bracket (2.40) on \( \mathcal{M}' \) according to the formulas (3.6)-(3.8)).

Besides that, as was mentioned above, the flows (2.47), generated by the functionals \( \int q(X) J^\nu(X) dX \) on the functions (2.44), leave invariant the submanifold \( \mathcal{M}' \) at the zero order of \( \epsilon \) and generate a linear evolution of the initial phases. So, we can conclude that the right-hand side of the linear system (3.3) contains no zero powers of \( \epsilon \) and we should start the expansion for \( \tilde{v}_j^\nu(X, \theta, Z, \epsilon) \) from the first power.

Now we have
\[ \tilde{v}_j^\nu[J, \theta^\alpha_0, \theta_0(X_0)](X, \theta, Z, \epsilon) = \sum_{k \geq 1} \epsilon^k \tilde{v}_j^\nu(k)[J, \theta^\alpha_0, \theta_0(X_0)](X, \theta, Z) \]
\[ \tilde{w}_j^\alpha[J, \theta^\alpha_0, \theta_0(X_0)](X, \theta, Z, \epsilon) = \sum_{k \geq 0} \epsilon^k \tilde{w}_j^\alpha(k)[J, \theta^\alpha_0, \theta_0(X_0)](X, \theta, Z) \]
\[ \delta_j^\alpha[J, \theta_0, \theta_0(X_0)](\theta, Z, \epsilon) = \sum_{k \geq 0} \epsilon^k \delta_j^\alpha(k)[J, \theta_0, \theta_0(X_0)](\theta, Z) \]

According to the relations above and Lemma 2.3 we can see now that the corrections to the values \( \{J^\nu(X), J^\mu(Y)\}_{M'} \) and \( \{\theta_0^\alpha(X), J^\mu(Y)\}_{M'} \) due to the Dirac restriction on \( M' \) are of order of \( O(\epsilon^2) \) and \( O(\epsilon) \) respectively.

According to the relation (2.35) we can also substitute the values \( J^\nu(X) \) instead of \( U^\nu[J, \theta_0^\alpha](X) \) in the leading order of \( \epsilon \) as the arguments of the averaged functions on \( M' \).

Now we should substitute the functions \( \varphi^i(\theta, X), \varphi^j(\theta, Y) \) on \( M' \) in the form

\[ \varphi^i(\theta, X) = \Phi^i_{(in)} (\theta + \theta_0^\alpha(X) + \theta_0(X_0) + \frac{1}{\epsilon} \int_{X_0}^X k(J(X'))dX', U[J, \theta_0^\alpha](X)) \] (3.11)

and

\[ \varphi^j(\theta, Y) = \Phi^j_{(in)} (\theta + \theta_0^\alpha(Y) + \theta_0(X_0) + \frac{1}{\epsilon} \int_{X_0}^Y k(J(Y'))dY', U[J, \theta_0^\alpha](Y)) \] (3.12)

respectively.

It is easy to see that the local part of (3.10) gives us the expression
\[
\frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} d^m \theta \ A^\nu_\mu \left( \Phi_{(in)}^i(\theta + s(X), U[J, \theta_0^*(X)], \epsilon \frac{\partial}{\partial X} \Phi_{(in)}^i(\theta + s(X), U[J, \theta_0^*(X)], \ldots) \right) \times \\
\times \delta(X - Y) + \\
\epsilon (A^\nu_\mu(J(X))) \delta'(X - Y) + O(\epsilon^2)
\]  

in the coordinates \( J(X), \theta_0^*(X) \) and \( \theta_0(X_0) \) on \( \mathcal{M}' \), where \( s(X) \equiv \theta_0^*(X) + \theta_0(X_0) + \frac{1}{\epsilon} \int_{X_0}^X k(J(X')) dX' \).

Here we used only the main part of (3.111) and its derivatives in the second term of (3.13) and replaced \( U^\nu(X) \) by \( J^\nu(X) \) according to (2.35) in the arguments of the averaged functions modulo the higher orders of \( \epsilon \).

The non-local part of (3.10) gives for (3.9) the following equalities:

\[
\int \int dX dY \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{1}{\epsilon} \sum_{k \geq 0} c_k \epsilon q^\nu(X) \frac{\partial F^\nu_{(k)}}{\partial X} \left( \Phi_{(in)}(\theta + s(X), U(X)), \ldots \right) \times \\
\times \nu(X - Y) \epsilon q^\mu(Y) \frac{\partial F^\mu_{(k)}}{\partial Y} \left( \Phi_{(in)}(\theta + s(Y), U(Y)), \ldots \right) d^m \theta =
\]

\[
= \int \int dX dY \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \sum_{k \geq 0} c_k \epsilon \left[ q^\nu(X) \nu(X - Y) q^\mu(Y) \right] \times \\
\times F^\nu_{(k)} \left( \Phi_{(in)}(\theta + s(X), U(X)), \ldots \right) F^\mu_{(k)} \left( \Phi_{(in)}(\theta + s(Y), U(Y)), \ldots \right) d^m \theta =
\]

\[
= \int \int dX dY \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \sum_{k \geq 0} c_k \epsilon \left[ q^\nu(X) q^\mu(Y) - q^\nu(X) q^\mu(Y) \delta(X - Y) - q^\nu(X) q^\mu(Y) \delta'(X - Y) \right] \times \\
\times F^\nu_{(k)} \left( \Phi_{(in)}(\theta + s(X), U(X)), \ldots \right) F^\mu_{(k)} \left( \Phi_{(in)}(\theta + s(Y), U(Y)), \ldots \right) d^m \theta =
\]

\[
= \int \int dX dY \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \sum_{k \geq 0} c_k \epsilon q^\nu_{(X)}(X) \nu(X - Y) q^\mu_{(Y)}(Y) \times \\
\times F^\nu_{(k)} \left( \Phi_{(in)}(\theta + s(X), U(X)), \ldots \right) F^\mu_{(k)} \left( \Phi_{(in)}(\theta + s(Y), U(Y)), \ldots \right) d^m \theta +
\]

\[
+ \epsilon \sum_{k \geq 0} c_k \int \left( q^\nu(X) q^\mu(X) - q^\nu(X) q^\mu(X) \right) \langle F^\nu_{(k)} F^\mu_{(k)} \rangle (J(X)) dX -
\]

\[
- \epsilon \sum_{k \geq 0} c_k \int q^\nu(X) q^\mu(X) \langle F^\nu_{(k)} F^\mu_{(k)} \rangle (J(X)) dX -
\]

\[
- \epsilon \int q^\nu(X) q^\mu(X) \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \sum_{k \geq 0} c_k F^\nu_{(k)} \left( \Phi_{(in)}(\theta + s(X), U(X)), \ldots \right) \times
\]

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\[ \times \frac{\partial}{\partial X} F_{(k)}^\mu (\Phi_{(in)}(\theta + s(X), U(X)), \ldots) \, d^m \theta \, dX \quad + \quad O(\epsilon^2) \quad (3.14) \]

where we used the integration by parts for the generalized functions.

We can see now that in the first term of the expression above in the both regions \( X > Y \) and \( X < Y \) we have the convolution with respect to \( X \) and \( Y \) of the “slow” functions \( q_\alpha^X(X) q_\alpha^Y(Y) \) with the rapidly oscillating expressions

\[ \langle F_{(k)}^\nu (\Phi_{(in)}(\theta + s(X, \epsilon), J(X)), \ldots) F_{(k)}^\mu (\Phi_{(in)}(\theta + s(Y, \epsilon), J(Y)), \ldots) \rangle \]

in the main order of \( \epsilon \). Here \( \langle \ldots \rangle \) means the averaging with respect to phases \( \theta^\alpha \). Now, since small \( \Delta X \) and \( \Delta Y \) lead to the changes of phase equal to \( \frac{\epsilon}{\epsilon} k^\alpha (J(X)) \Delta X + O((\Delta X)^2) \) and \( \frac{\epsilon}{\epsilon} k^\alpha (J(Y)) \Delta Y + O((\Delta Y)^2) \), it is not very difficult to see that in the sense of “generalized” limit (i.e. in the sense of the convolutions with the “slow” functions \( X \) and \( Y \)) we can replace these oscillating expressions in the main order of \( \epsilon \) just by their mean values

\[ \sum_{k \geq 0} e_k \langle F_{(k)}^\nu \rangle (J(X)) \langle F_{(k)}^\mu \rangle (J(Y)) \]

where \( \langle \ldots \rangle \) means the averaging on the space of \( m \)-phase solutions.

As for the last term of (3.14), we recall that its sum with the expression arising from the first term of the local part in (3.13)

\[ \int dX q^\nu(X) q^\mu(X) \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} d^m \theta \left[ A^\nu_0 (\Phi_{(in)}(\theta + s(X), U(X)), \ldots) \right. \]

\[ - \left. \sum_{k \geq 0} e_k F_{(k)}^\nu (\Phi_{(in)}(\theta + s(X), U(X)), \ldots) \epsilon \frac{\partial}{\partial X} F_{(k)}^\mu (\Phi_{(in)}(\theta + s(X), U(X)), \ldots) \right] \]

is equal according to (3.11) to

\[ \epsilon \int \left( \frac{\partial \langle Q^{\nu \mu} \rangle (J(X))}{\partial X} - \sum_{k \geq 0} e_k \frac{\partial \langle F_{(k)}^{\nu \mu} \rangle (J(X))}{\partial X} \right) q^\nu(X) q^\mu(X) \, dX \]

in the leading order of \( \epsilon \).

So, we can write now:

\[ \{ \int q^\nu(X) J^\nu(X) \, dX; \int q^\mu(Y) J^\mu(Y) \, dY \} |_{M'} = \]

\[ = \epsilon \int \left[ q^\nu(X) q_\alpha^X(X) \langle A^\nu_1 (J(X)) \rangle - q_\alpha^X(X) q^\mu(X) \sum_{k \geq 0} e_k \langle F_{(k)}^{\nu \mu} \rangle (J(X)) \right. \]

\[ + \left. q^\nu(X) q^\mu(X) \partial_X \left( \langle Q^{\nu \mu} \rangle (J(X)) - \sum_{k \geq 0} e_k \langle F_{(k)}^{\nu \mu} \rangle (J(X)) \right) \right] \, dX + \]

\[ + \epsilon \int \int \sum_{k \geq 0} e_k q_\alpha^X(X) \langle F_{(k)}^{\nu \mu} \rangle (J(X)) \, \nu(X - Y) \langle F_{(k)}^{\mu \nu} \rangle (Y) \, q_\alpha^Y(Y) \, dXdY \quad + \quad O(\epsilon^2) \]

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After the integration by parts (in the sense of generalized functions) we can write this expression in the following “canonical” form:

\[
\{ \int q''(X)J''(X)dX, \int q''(Y)J''(Y)dY \}_{|_{M'}} = \\
= \epsilon \left( A''_{1}(J(X)) + \sum_{k \geq 0} e_k \left( F''_{(k)} - F''_{(k)} \right) (J(X)) \right) q''(X)q''(Y) dX + \\
+ \epsilon \int \left( \frac{\partial Q''}{\partial X} - \sum_{k \geq 0} e_k \frac{\partial F''_{(k)}}{\partial X} \right) q''(X)q''(Y) dX + \\
+ \int \sum_{k \geq 0} e_k q''(X) \frac{\partial F''_{(k)}}{\partial X} \nu(Y - X) \frac{\partial F''_{(k)}}{\partial Y} q''(Y) dX dY + O(\epsilon^2)
\]

which corresponds to the bracket

\[
\{ J''(X), J''(Y) \}_{|_{M'}} = \\
= \epsilon \left( A'' (J(X)) + \sum_{k \geq 0} e_k \left( F''_{(k)} - F''_{(k)} \right) (J(X)) \right) \delta''(X - Y) + \\
+ \epsilon \left( \frac{\partial Q''}{\partial X} - \sum_{k \geq 0} e_k \frac{\partial F''_{(k)}}{\partial X} \right) \delta''(X - Y) + \\
+ \epsilon \sum_{k \geq 0} e_k \frac{\partial F''_{(k)}}{\partial X} \nu(Y - X) \frac{\partial F''_{(k)}}{\partial Y} + O(\epsilon^2)
\]

for the functionals \( J''(X) \).

So, according to Lemma 2.4 and the remarks above we obtain for the Dirac restriction on \( M' \)

\[
\{ \theta'^{\alpha}_{0}(X), J''(Y) \}_{D} = O(\epsilon)
\]

and the relations \(3.15\) for the brackets \( \{ J''(X), J''(Y) \}_{D} \) in the coordinates \( J(X), \theta^\alpha_0(X) \) and \( \theta_0(X_0) \).

It is evident also that the Dirac brackets \( \{ J''(X), J''(Y) \}_{D} \) on \( M' \) do not depend in any order of \( \epsilon \) on the common initial phase \( \theta_0(X_0) \) because of the invariance of \( J''(X) \), the bracket \(2.40\) and the submanifold \( M' \) with respect to the common shifts of \( \theta^\alpha_0 \).

The dependence of \( \{ J''(X), J''(Y) \}_{D} \) on \( J(X), J(X), \theta^\alpha_0(X), \ldots \) is regular at \( \epsilon \to 0 \) and, as can be easily seen from \(3.15\), we have no dependence of \( \theta^\alpha_0 \) in the first order of \( \epsilon \).

So, it is easy to see now that the Jacobi identities for the bracket \( \{ \ldots, \ldots \}_{D} \) on \( M' \) with coordinates \( J(X), \theta^\alpha_0(X) \) and \( \theta_0(X_0) \), written for the fields \( J''(X), J''(Y) \) and \( J''(Z) \) in the order of \( \epsilon^2 \), coincide with the corresponding Jacobi identities for the bracket \(3.2\).

So we proved the Jacobi identity for the bracket \(3.2\).

The skew-symmetry of the bracket \(3.2\) is just a trivial corollary of the skew-symmetry of \(2.40\).

We now prove the invariance of the bracket \(3.2\) with respect to the choice of the integrals \( I'' \). The proof is just the same as in the local case and we will just reproduce it here.
Under the condition (IV) we have the unique restriction of the Poisson bracket (2.40) on the submanifold \( M' \) with the coordinates \( J(X), \theta^*_0(X), \theta_0(X_0) \) in the form of formal series at \( \epsilon \to 0 \).

So, the two restrictions of (2.40), obtained in the coordinates \( (J^\nu(X), \theta^*_0(X), \theta_0(X_0)) \) and

\[
(J^\nu(X), \theta^*_0(X), \theta_0(X_0))
\]

and

\[
(\tilde{J}^\nu(X), \tilde{\theta}^*_0(X), \theta_0(X_0))
\]

corresponding to the sets \{\( I^\nu \)\} and \{\( \tilde{I}^\nu \)\} (satisfying (A)-(C)) respectively, should transform one into another after the corresponding transformation of coordinates

\[
\tilde{J}^\nu(X) = \tilde{j}^\nu_{(0)}(J(X)) + \sum_{k \geq 1} \epsilon^k \tilde{j}^\nu_{(k)}[J, \theta^*_0](X)
\]

\[
\tilde{\theta}^*_0(X) = \tilde{\theta}^*_0[J, \theta^*_0, \epsilon](X)
\]
on \( M' \).

Now we note that the leading term of (3.15), coinciding with the bracket (3.2), transforms according to the transformation

\[
\tilde{J}^\nu(X) = \tilde{j}^\nu_{(0)}(J(X))
\]

which corresponds to the substitution \( \tilde{U}^\nu(X) = \tilde{U}^\nu(U(X)) \) on \( M' \) view the relation (2.35). So, we obtain the second part of the theorem.

*Theorem 3.1 is proved.*

*Remark.*

From Theorem 3.1 it also follows in particular that the procedure (3.2) is insensitive to the addition of the total derivatives with respect to \( x \) to the densities \( P^\nu(\varphi, \varphi_x, \ldots) \). This fact, however, can be also obtained from an elementary consideration of the definition of bracket (3.2).

*Theorem 3.2*

The Hamiltonian functions

\[
\bar{H}^\nu = \int U^\nu(X)dX
\]

and

\[
\bar{H} = \int \langle P_H \rangle(U(X))dX
\]
generate view (3.2) local commuting flows, which give us the Whitham equations for the systems (2.11) and (1.1) respectively.

*Proof.*

It is easy to check by direct substitution that any of \( \bar{H}^\mu \) generates the “conservative” form

\[
U^\nu_T = \partial_X \langle Q^\nu \rangle(U)
\]

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of the Whitham’s system for the corresponding flow (2.11). It is easy to see also that this flow conserves any of $\bar{H}^\nu$ and so all $\bar{H}^\nu$ and $\bar{H}^\mu$ commute view the bracket (3.2). The same property for the Hamiltonian function $\bar{H}$ (and also for the integral of the averaged density of any local integral $I$, commuting with $H$ and $I^\nu$ and generating a local flow view (1.4)) can be now obtained from the invariance of (3.2) with respect to the set $\{I^\nu\}$, since we can use the Hamiltonian function $\bar{H}$ in the form (1.2) as one of the integrals instead of any of $I^\nu$.

Theorem 3.2 is proved.

We can also see that the functionals $\bar{H}^\nu$ give us conservation laws for our Whitham system.

From the Theorem 1.1 it follows also that the flows

$$U^\nu = \partial_X\langle F^\nu_{(h)}\rangle(U)$$

commute with all the local flows, generated by local functionals $\int h(U)dX$ in the Hamiltonian structure (3.2), and it can be also seen that they give us the Whitham’s equations for the corresponding flows (1.7).

It can be easily seen also that the described procedure can be applied in the same way to the brackets (1.1) written also in the “irreducible” form and not only in the “canonical” one.

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