Bilinear Enhancements of Quandle Invariants

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Abstract

We enhance the quandle counting invariants of oriented classical and virtual links using a construction similar to quandle modules but inspired by symplectic quandle operations rather than Alexander quandle operations. Given a finite quandle $X$ and a vector space $V$ over a field, sets of bilinear forms on $V$ indexed by pairs of elements of $X$ satisfying certain conditions yield new enhanced multiset- and polynomial-valued invariants of oriented classical and virtual links. We provide examples to illustrate the computation of the invariants and to show that the enhancement is proper.

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1 Introduction

Quandles are algebraic structures with axioms encoding the Reidemeister moves in knot theory, analogous to the way the group axioms encode geometric symmetries. Finite quandles $X$ define non-negative integer valued invariants of oriented links in $\mathbb{R}^3$ via the set of quandle homomorphisms from the knot quandle $Q(K)$ of a knot or link $K$ to $X$. The elements of the homset can be represented as colorings of a diagram of the knot or link, i.e., assignments of elements of $X$ to the arcs in the diagram, in a way analogous to representation of linear transformations via matrices. More precisely, a different choice of diagram for the knot or link will give a different representation of the the same homset element in the same way that a different choice of basis yields a different matrix for the same linear transformation. The cardinality of the quandle homset is then an integer-valued invariant of the knot or link for each finite quandle $X$, known as the quandle counting invariant, denoted $\Phi^Z_X(K) = |\text{Hom}(Q(K), X)|$.

An enhancement of the quandle counting invariant can be defined whenever we have an invariant $\phi$ of quandle homset elements; then the multiset of $\phi$-values over the homset defines an invariant whose cardinality recovers $\Phi^Z_X(K)$ but is generally stronger. In several previous papers such as [5], the second listed author as well as others used quandle modules to enhance the quandle counting invariant. These are modules over commutative rings with identity which are invariants of quandle homset colors, very much like Alexander module invariants of the quandle-colored diagrams. Such a module can be interpreted as a set of bead colorings of a quandle-colored diagram with bead interaction rules at crossings representing a kind Alexander quandle operation deformed by the quandle homset colors.

In this paper we pursue a similar idea with role of Alexander quandles replaced with another type of quandle structure defined on modules over commutative rings with identity, namely symplectic quandles. The paper is organized as follows. In Section 2 we review the basics of quandle theory. In Section 3 we define our bilinear enhancements and use them to introduce an infinite family of polynomial-valued invariants of knots and links. In Section 4 we provide examples to illustrate the computation of the invariants and in particular show that the enhancement is proper, i.e, not determined by $\Phi^Z_X$. We conclude in Section 5 with questions for future research.

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2 Quandle Basics

We begin with a definition. See \[4,7,8\] for more.

**Definition 1.** A quandle is a set \( X \) with a binary operation \( \triangleright : X \times X \to X \) satisfying the conditions

(i) For all \( x \in X \), \( x \triangleright x = x \),

(ii) For all \( y \in X \), the map \( f_y : X \to X \) defined by \( f_y(x) = x \triangleright y \) is invertible, and

(iii) For all \( x, y, z \in X \), we have

\[
(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z).
\]

A quandle in which the maps \( f_y \) from axiom (ii) are involutions is called an involutory quandle or kei. We may write \( x \triangleright^{-1} y \) to refer to \( f_y^{-1}(x) \).

**Example 1.** Common examples of quandle structures include:

- Groups \( G \) with \( g \triangleright h = h^{-1}gh \), known as conjugation quandles,
- Groups \( G \) with \( g \triangleright h = h^{-1}gh \), known as core quandles,
- Modules over \( \mathbb{Z}[t^\pm 1] \) with \( x \triangleright y = tx + (1 - t)y \), known as Alexander quandles, and
- Modules \( M \) over a commutative ring \( R \) with a symplectic form \( \langle , \rangle : M \times M \to R \) with \( x \triangleright y = x + [x, y]y \), known as symplectic quandles.

**Example 2.** An important example is the knot quandle \( Q(K) \) of an oriented knot or link \( K \). Given a diagram \( D \) of \( K \), \( Q(K) \) has a presentation with a generator \( a_i \) for each arc in \( D \) and a relation \( a_i \triangleright a_j = a_k \) at each crossing in \( D \) as shown.

\[
\begin{array}{c}
| a_i \\
\end{array}
\begin{array}{c}
| a_j \\
\end{array}
\begin{array}{c}
| a_i \triangleright a_j \\
\end{array}
\]

Then the knot quandle is the set of equivalence classes of quandle words in these generators, i.e. expressions defined recursively as generators \( a_j \) or expressions of the form \( w \triangleright z \) or \( w \triangleright^{-1} z \) where \( w, z \) are words, under the equivalence relation generated by the crossing relations together with the quandle axioms.

Now, let \( X \) be a finite quandle and \( D \) an oriented knot or link diagram representing an oriented knot or link \( K \). Then an assignment of elements of \( X \) to each generator \( a_i \) in \( D \) defines a quandle homomorphism \( f : Q(K) \to X \) if and only if the crossing relations \( f(a_i) \triangleright f(a_j) = f(a_k) \) are satisfied at every crossing. Such an assignment is known as an \( X \)-coloring of \( D \). In this way we can represent the quandle homset \( \text{Hom}(Q(K), X) \) via \( X \)-colorings of \( D \) and use this to compute the quandle homset by finding all colorings which satisfy the coloring condition at every crossing. Let us write \( D_f \) to denote the \( X \)-coloring of \( D \) determined by the homset element \( f \).

Performing a Reidemeister move on an \( X \)-colored diagram results in a unique \( X \)-coloring on the diagram following the move; thus, we can consider invariants of \( X \)-colored diagrams for any quandle \( X \). An invariant
φ of X-colored diagrams then yields an invariant of links called an enhancement by taking the multiset of φ-values over the quandle homset \( \mathrm{Hom}(Q(K), X) \). Examples are plentiful in the literature, starting with the quandle cocycle invariants introduced in \([3]\) and continuing with many more enhancements such as quantum enhancements \([10]\), quandle module enhancements \([2]\), quandle polynomial enhancements \([9]\) and many others.

### 3 Bilinear Enhancements

We begin with a definition.

**Definition 2.** Let \( X \) be a finite quandle and \( V \) a vector space over a field \( F \). An \( X \)-bilinear form on \( V \) is a choice of bilinear form \([\cdot, \cdot]_{x,y} : V \times V \to F\) for each ordered pair \( x, y \in X \) of quandle elements such that for all \( x, y, z \in X \) and \( a, b, c \in M \) we have

\[
\begin{align*}
[a, a]_{x, x} & = 0 \\
[a, b]_{x, y} & = [a, c]_{x, x} c, b + \sum_{y, z} [a, c]_{x, y} c, b + [b, c]_{y, z} c, c, y, z \\
[a, c]_{x, y} + [a, b]_{x, y} [b, c]_{y, z} & = [a, c]_{x, z} + [a, b]_{x, y} [b, c]_{y, z} \\
\end{align*}
\]

We will generally denote an \( X \)-bilinear form as \( \phi \). If the dimension of \( V \) is \( n \) and \( |X| = m \), we can specify such a \( \phi \) with an \( m \times m \) matrix of \( n \times n \) matrices \( B_{x, y} \) over \( F \) such that for \( \vec{u}, \vec{v} \in V \) and \( x, y \in X \) we have

\[ \phi(\vec{u}, \vec{v}) = \vec{u}^T B_{x, y} \vec{v}. \]

**Example 3.** Given any quandle \( X \) and \( F \)-vector space \( V \), we obtain a trivial example of an \( X \)-bilinear form by setting \([\cdot, \cdot]_{x,y} = 0\).

**Example 4.** As a (slightly) less trivial example, if \( X = V \) is a symplectic quandle with symplectic form \([\cdot, \cdot] \) then setting \([\cdot, \cdot]_{x,y} = [\cdot, \cdot] \) gives \( X \) the structure of an \( X \)-bilinear form.

**Example 5.** Let \( X \) be the quandle structure on the set \( \{1, 2, 3\} \) defined by the operation table

| ⧵ | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 1 | 1 | 2 |
| 2 | 2 | 2 | 1 |
| 3 | 3 | 3 | 3 |

and let \( V = (\mathbb{Z}_2)^2 \). Then one can check that the array of matrices

\[
\begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

defines an \( X \)-bilinear form on \( V \).

Definition 2 is motivated by the notion of giving a secondary coloring to the arcs in an \( X \)-colored oriented...
Definition 3. Let $X$ be a finite quandle, $V$ an $F$-vector space and $\phi$ an $X$-bilinear form. A $(X, \phi)$-coloring of an oriented link diagram $D$ is an assignment of an element of $X$ and an element of $V$ to each arc in $D$ such that the bead coloring rule is satisfied at every crossing in $D$.

We can now state our main result.

Proposition 1. If an oriented $X$-colored link diagram has $\phi$-coloring before a Reidemeister move, there is a unique $\phi$-coloring of the diagram after the move which agrees with the original coloring outside the neighborhood of the move.

Proof. This is a matter of checking the statement for a generating set of oriented Reidemeister moves as in [11]; indeed, the axioms defining $X$-bilinear forms are chosen precisely with this property in mind. We apply the above bead coloring rule to the Reidemeister moves to generate the conditions given in Definition 2. Let us use the generating set of Reidemeister moves shown:
Each type I move gives condition (i); we illustrate with one of the four:

For the type II Reidemeister moves we need to check that the two coloring rules are consistent; i.e., we need

\[ a = c - [c,b]_{x\triangleright y,y}b = a + [a,b]_{x,y}b - [a + [a,b]_{x,y}b,b]_{x\triangleright y,y}b, \]

so it suffices to show that

\[ [a,b]_{x,y} = [a + [a,b]_{x,y}b,b]_{x\triangleright y,y}. \]

The case of condition (ii) where \( y = z \) and \( b = c \) implies

\[ [a,b]_{x,y} = [a + [a,b]_{x,y}b,b + [b,b]_{y,y}b]_{x\triangleright y,y}\triangleright y, \]

which, by condition (i) and \( y\triangleright y = y \), gives

\[ [a,b]_{x,y} = [a + [a,b]_{x,y}b,b]_{x\triangleright y,y} \]

as required.

Conditions (ii) and (iii) are required by the all-positive Reidemeister III moves, the last move in the generating set.

Starting with our bead coloring rule and using bilinearity of \([,]_{x,y}\) we have

\[ d = a + [a,b]_{x,y}b \]
\[ e = b + [b,c]_{y,z}c \]
\[ f = d + [d,c]_{x\triangleright y,z}c \]
\[ = a + [a,b]_{x,y}b + [a + [a,b]_{x,y}b,c]_{x\triangleright y,z}c \]
\[ = a + [a,b]_{x,y}b + [a,c]_{x\triangleright y,z}c + [a,b]_{x,y}b [b,c]_{x\triangleright y,z}c \]
\[ = a + [a,b]_{x,y}b + ([a,c]_{x\triangleright y,z} + [a,b]_{x,y}b [b,c]_{x\triangleright y,z})c. \]
Moreover,

\[
\begin{align*}
g &= a + [a, c]_{x, z} c \\
e &= b + [b, c]_{y, z} c \\
f &= g + [g, e]_{x \triangledown z, y \triangledown z} e \\
&= a + [a, c]_{x, z} c + [a + [a, c]_{x, z} c, b + [b, c]_{y, z} c]_{x \triangledown z, y \triangledown z} (b + [b, c]_{y, z} c) \\
&= a + Ab + ([a, c]_{x, z} + [b, c]_{y, z} A) c
\end{align*}
\]

where

\[
A = [a + [a, c]_{x, z} c, b + [b, c]_{y, z} c]_{x \triangledown z, y \triangledown z}.
\]

Then comparing coefficients in \( f \) yields condition (ii) and condition (iii) in Definition 2. \( \square \)

**Corollary 2.** Let \( X \) be a quandle, \( V \) an \( \mathbb{F} \)-vector space and \( \phi \) and \( X \)-bilinear form. Then for any \( X \)-colored oriented knot or link diagram \( D_f \), the number of \( (X, \phi) \)-colorings of \( D_f \) is unchanged by \( X \)-colored Reidemeister moves.

Let us denote the set of \( (X, \phi) \)-colorings of \( D_f \) by \( C^\phi_{X}(D_f) \).

**Definition 4.** Let \( X \) be a quandle, \( M \) a module over a commutative ring with identity \( R \), and \( \phi = \{ [\cdot, \cdot]_{x, y} \mid x, y \in X \} \) an \( X \)-bilinear form on \( M \). Given an \( X \)-coloring of a diagram \( D \) of an oriented link, we define the \( X \)-bilinear enhanced multiset of \( D \) to be the multiset of bead colorings of \( D_f \) where \( D_f \) ranges over the set of \( X \)-colorings of \( D \),

\[
\Phi^\phi_{X,M}(D) = \{ C^\phi_{X}(D_f) \mid f \in \text{Hom}(\mathcal{Q}(D), X) \}
\]

and we define the \( X \)-bilinear enhanced polynomial of \( D \) to be

\[
\Phi^\phi_{X}(D) = \sum_{f \in \text{Hom}(\mathcal{Q}(D), X)} u |C^\phi_{X}(D_f)|.
\]

Moreover, for any knot or link \( L \) we define \( \Phi^\phi_{X,M}(L) \) and \( \Phi^\phi_{X}(L) \) to be \( \Phi^\phi_{X,M}(D) \) and \( \Phi^\phi_{X}(D) \) respectively where \( D \) is any diagram of \( L \).

Then, the above shows that we have the following corollary.

**Corollary 3.** For any finite quandle \( X \), \( \mathbb{F} \)-vector space \( V \) and \( X \)-bilinear form \( \phi \), the multiset \( \Phi^\phi_{X,M}(L) \) and polynomial \( \Phi^\phi_{X}(L) \) are invariant under Reidemeister moves and hence invariants of oriented links.

**Remark 1.** We note that evaluating \( \Phi^\phi_{X}(L) \) at \( u = 1 \) yields the quandle counting invariant \( |\text{Hom}(\mathcal{Q}(L), X)| \), so the new polynomial invariant is an enhancement of the quandle counting invariant. In the next section we will show that the enhancement is proper, i.e., not determined by the quandle counting invariant in general.

**Remark 2.** We note also that nothing in the definition of \( \Phi^\phi_{X,M}(L) \) or \( \Phi^\phi_{X}(L) \) depends on the supporting surface of having genus zero, so both are also well-defined invariants of virtual links.

### 4 Examples

In this section we collect a few examples of the new invariants.
Example 6. Let $F = \mathbb{Z}_2$ and $V = F^2$. Let $X$ and $\phi$ be the quandle and $X$-bilinear form given in Example 5. Then the pictured Hopf link $L2a1$ has five $X$-colorings as shown:

Then putting beads on each arc, we have bead-coloring equations for the first $X$-coloring

$$
\begin{align*}
\phi(a) &= a + [a,b]_1 \ b \\
\phi(b) &= b + [b,a]_1 \ a
\end{align*}
$$

Writing $a = (a_1,a_2)$ and $b = (b_1,b_2)$, these become

$$
\begin{align*}
a_1 &= a_1 + (a_1b_2 + a_2b_1)b_1 \\
a_2 &= a_2 + (a_1b_2 + a_2b_1)b_2
\end{align*}
$$

We solve these equations to find the number of valid bead colorings for this $X$-coloring. If $b = (0,0)$ all four choices of $a \in (\mathbb{Z}_2)^2$ are solutions; if $b = (1,1)$ we need $a = (1,1)$ or $a = (0,0)$; if $b = (1,0)$ we have solutions $a = (1,0)$ and $a = (0,0)$ and if $b = (0,1)$ then $a = (0,1)$ and $a = (0,0)$ are solutions. Thus we have 10 bead colorings for this $X$-coloring and so its contribution to the enhanced polynomial is $u^{10}$.

A similar analysis reveals three more contributions of $u^{10}$ by other $X$-colorings; however, the monochromatic $X$-coloring by the quandle element 3 has bead coloring equations $a = a$ and $b = b$, so there are 16 total bead colorings and this $X$-coloring contributes $u^{16}$. Hence, we obtain invariant value

$$
\Phi_X^\phi(L2a1) = u^{16} + 4u^{10}.
$$

Example 7. Using python, we computed $\Phi_X^\phi(L)$ for a choice of orientation for each of the prime links with up to seven crossings as found at the knot atlas [1] with respect to the quandle $X$ and $X$-bilinear form on $V = (\mathbb{Z}_2)^2$ in Example 5; the results are in the table.

| $\Phi_X^\phi(L)$       | $L$                           |
|------------------------|-------------------------------|
| $u^{16} + 4u^{10}$     | $L2a1, L6a2, L7a6$            |
| $5u^{16}$              | $L6a3, L7a5$                  |
| $2u^{10} + 7u^{16}$    | $L7a2, L7a3, L7n1, L7n2$     |
| $5u^{16} + 4u^{10}$    | $L4a1, L5a1, L7a4$           |
| $9u^{16}$              | $L6a1, L7a1$                  |
| $7u^{64} + 8u^{22}$    | $L6n1, L7a7$                  |
| $7u^{64} + 8u^{28}$    | $L6a5$                        |
| $19u^{64} + 8u^{40}$   | $L6a4$                        |

In particular, this table shown several examples of links with the same number of $X$-colorings distinguished by the bead-coloring information, and we have shown that the enhancement is proper.
Example 8. For our final example, we computed the invariant for the same set of links with the same quandle as in Example 7 but with a different choice of bilinear enhancement over the same vector space, namely

$$\phi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ & & \vdots \\ & & 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ & & \vdots \\ & & 0 & 0 \\ 0 & 0 \\ & & \vdots \\ & & 0 & 0 \\ 0 & 0 \\ & & \vdots \\ & & 0 & 0 \\ 0 & 0 \end{bmatrix}. $$

The results are collected in the table.

| $\Phi^\phi_X(L)$ | $L$ |
|------------------|-----|
| $5u^{19}$        | $L2a_1, L6a_2, L7a_6$ |
| $5u^{16}$        | $L6a_3, L7a_5$ |
| $4u^{16} + 5u^{10}$ | $L4a_1, L5a_1, L7a_4$ |
| $4u^{10} + 5u^{16}$ | $L7a_2, L7a_3, L7n_1, L7n_2$ |
| $9u^{16}$        | $L6a_1, L7a_1$ |
| $6u^{10} + 9u^{22}$ | $L6n_1, L7a_7$ |
| $6u^{10} + 9u^{28}$ | $L6a_5$ |
| $18u^{64} + 9u^{40}$ | $L6a_4$ |

5 Questions

We end with some questions for future research.

The bilinear enhancement idea could be generalized in a number of ways, all of which could provide interesting new families of invariants. What extra conditions would be needed to replace the understand bead coefficient with a second bilinear form depending on the quandle colors? How about generalizations to other knot-theoretic categories such as spatial graphs, handlebody-knots or surface-links? Alternatively, we could replace the quandle colorings with invariant colorings by other algebraic structures such as groups, biquandles or tribrackets.

As with all such families of invariants, it is interesting to ask what precise information about the knot or link is being extracted by these invariants. For example, in [6] the quandle colorings of a two-component link by a specific family of quandles were shown to be measuring the linking number of the link. This and many of other families of coloring invariants are waiting for similar connections with other invariants and properties of knots and links to be found.

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