DISTRIBUTION OF ZEROS OF MATCHING POLYNOMIALS OF HYPERGRAPHS

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Abstract. Let $H$ be a connected $k$-graph with maximum degree $\Delta \geq 2$ and let $\mu(H, x)$ be the matching polynomial of $H$. In this paper, we focus on studying the distribution of zeros of the matching polynomials of $k$-graphs. We prove that the zeros (with multiplicities) of $\mu(H, x)$ are invariant under a rotation of an angle $2\pi/\ell$ in the complex plane for some positive integer $\ell$ and $k$ is the maximum integer with this property. Let $\lambda(H)$ denote the maximum modulus of all zeros of $\mu(H, x)$. We show that $\lambda(H)$ is a simple root of $\mu(H, x)$ and

$$\Delta^\frac{1}{k} \leq \lambda(H) < \frac{k}{k-1}((k-1)(\Delta - 1))^{\frac{1}{k}}.$$  

To achieve these, we introduce the path tree $T(H, u)$ of $H$ with respect to a vertex $u$ of $H$, which is a $k$-tree, and prove that

$$\frac{\mu(H - u, x)}{\mu(H, x)} = \frac{\mu(T(H, u) - u, x)}{\mu(T(H, u), x)},$$

which generalizes the celebrated Godsil’s identity on the matching polynomial of graphs.

1. Introduction

1.1. Background. There are several polynomials associated with a hypergraph, such as Tutte polynomial [26, 27], the chromatic polynomial [46, 49], the characteristic polynomial [8], and the matching polynomial [7, 45, 50]. One of the most fundamental topics in hypergraph polynomial theory is to investigate the distribution of zeros of these polynomials. Further results on the location of zeros of graph polynomials can be found in [38, 43]. In this paper, we study the distribution of zeros of the matching polynomials of hypergraphs.

A $k$-uniform hypergraph $H = (V(H), E(H))$ consists of a vertex set $V(H)$ and an edge set $E(H)$, where each $e \in E(H)$ is a $k$-element subset of $V(H)$. In this paper, we use the term “$k$-graph” for the case of $k$-uniform hypergraphs with $k \geq 2$, and the term “graph” exclusively for $k = 2$. For a subset $M$ of $E(H)$, denote by $V(M)$ the set of vertices of $H$ contained in an edge in $M$. If no two distinct edges in $M$ share a common vertex, then $M$ is called a matching of $H$. The maximum number of edges in a matching of $H$ is called the matching number of $H$, denoted by $m(H)$. An $r$-matching is a matching of size $r$. We denote by $p(H, r)$ the number of $r$-matchings of $H$, with the convention that $p(H, 0) = 1$. 

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Recently, to investigate the spectral radius of adjacency tensor of $k$-trees, Su, Kang, Li, and Shan [45] introduced the following matching polynomial of a $k$-graph $\mathcal{H}$:

\begin{equation}
\mu(\mathcal{H}, x) = \sum_{r \geq 0} (-1)^r p(\mathcal{H}, r)x^{|V(\mathcal{H})| - kr},
\end{equation}

which is a minor adjustment based on the definition of matching polynomial introduced by Zhang, Kang, Shan and Bai [50], and Clark and Cooper [7]. However, most of their results are focused on the spectra of adjacency tensor of $k$-trees but not on the properties of the matching polynomial itself. This prompts us to explore more useful and interesting properties of the matching polynomial.

When $k = 2$ in (1.1), it is exactly the classical matching polynomial of a graph introduced by Heilmann and Lieb [24]. The matching polynomial is an important mathematical object and has been applied in different fields including spectral graph theory [10, 18, 35, 39], combinatorics [5, 17], statistical physics [23, 24], chemistry [1, 4, 20], and many others. We refer readers to the textbooks [19, 34] for more background and history on the matching polynomial theory, and see [38, 43] for related graph polynomials.

There are two fundamental theorems in matching polynomial theory mentioned here. The first one is Theorem 1.1, proved independently by Mowshowitz [40, Theorem 4] in 1972 and by Lovász and Pelikán [35, Theorem 1] in 1973, indicates that the characteristic polynomial and the matching polynomial are closely related. By applying this theorem, Mowshowitz [40] showed that there exist infinitely many pairs of non-isomorphic cospectral trees, and Schwenk [42] proved that almost all trees are not determined by their spectra.

Theorem 1.1 ([35, 40]). The characteristic polynomial of a tree coincides with its matching polynomial.

The other one is the following celebrated Heilmann-Lieb theorem [24], which gives a characterization of distribution of zeros of matching polynomials of graphs.

Theorem 1.2 (Heilmann–Lieb [24]). Let $G$ be a graph with maximum degree $\Delta(G) \geq 2$. Then the zeros (with multiplicities) of $\mu(G, x)$ are symmetrically distributed about the origin, and lie in the interval $(-2\sqrt{\Delta(G) - 1}, 2\sqrt{\Delta(G) - 1})$.

The bound $2\sqrt{\Delta(G) - 1}$ for zeros of matching polynomial of graphs in Theorem 1.2 is closely related to the second largest eigenvalues of graphs. If $G$ is a $d$-regular graph, then $d$ is always the largest adjacency eigenvalue of $G$, called the trivial eigenvalue of $G$. The well known Alon–Boppana Theorem [2] states that for every $d \geq 2$ and $\varepsilon > 0$, there are only finitely many $d$-regular graphs whose second largest eigenvalue is at most $2\sqrt{d - 1} - \varepsilon$. In addition, Friedman [14] proved that for every $\varepsilon > 0$, with high probability, random $d$-regular graphs have second largest eigenvalue smaller than $2\sqrt{d - 1} + \varepsilon$, which was conjectured by Alon [2]. Motivated by the above results, Lubotzky, Phillips, and Sarnark [37] introduced the concept of Ramanujan graphs: $d$-regular graphs whose nontrivial eigenvalues are between $-2\sqrt{d - 1}$ and $2\sqrt{d - 1}$. It plays an important role in the study of linear expander of graphs [25]. Based on the upper bound in Theorem 1.2, Marcus, Spielman, and Srivastava [39] showed that there are infinitely many bipartite Ramanujan graphs by the breakthrough technique called interlacing family. See [22, 39] for more details and related topics.

Over the years, several different proofs of real-rootedness of the matching polynomial in Theorem 1.2 were presented, see e.g. [6, 16, 19, 30]. It is worth mentioning that the original proofs of Theorem 1.1 in [35] and 1.2 in [24] are elementary, relying only on simple recurrence formulas for the matching polynomial. One of essential reasons is that the zeros of the matching polynomial of a graph are all real, and hence many methods in real-rooted polynomial theory can be applied. However, as we show in this paper, the matching polynomial of a $k$-graph must contain some nonreal zeros when $k \geq 3$. Therefore, some of previous techniques for studying the matching polynomial of a graph can’t be adapted to the $k$-graph setting.
1.2. **Our results.** Inspired by the above classical results on the matching polynomial of graphs and lack of study of the matching polynomial of $k$-graphs, in this paper, we investigate the distribution of zeros of the matching polynomial of $k$-graphs. For the sake of convenience, we use $\lambda(\mathcal{H})$ to denote the maximum modulus of all zeros of $\mu(\mathcal{H}, x)$. Our first result in this paper is the following theorem which can be viewed as a generalization of Theorem 1.1 for $k$-graphs.

**Theorem 1.3.** Let $\mathcal{T}$ be a $k$-tree. Then every zero of $\mu(\mathcal{T}, x)$ is an eigenvalue of the adjacency tensor of $\mathcal{T}$. Moreover, $\lambda(\mathcal{T})$ is a root of $\mu(\mathcal{T}, x)$ and equals to the spectral radius of the adjacency tensor of $\mathcal{T}$.

Theorem 1.3 establishes a relation between the zeros of the matching polynomial of a $k$-tree $\mathcal{T}$ and the eigenvalues of the adjacency tensor of $\mathcal{T}$. In particular, it implies that we may study $\lambda(\mathcal{T})$ in terms of the spectral radius of the adjacency tensor of $\mathcal{T}$, and vice versa. In addition to its own conclusion, Theorem 1.3 plays an important role in the proof of our main result. To state the main result, we need to introduce more notations. A real polynomial $f(x)$ is called $\ell$-symmetric if

$$f(x) = x^t g(x^\ell)$$

for some nonnegative integer $t$ and some real polynomial $g(x)$. In other words, $f(x)$ is $\ell$-symmetric if and only if its zeros remains invariant under a rotation of an angle $2\pi/\ell$ on the complex plane. The maximum number $\ell$ such that (1.2) holds is called the cyclic index of $f(x)$. Now, Theorem 1.2 implies that for a graph $G$ with $\Delta(G) \geq 2$, the cyclic index of $\mu(G, x)$ is $2$ and $\lambda(G) \leq 2\sqrt{\Delta(G) - 1}$.

We are now ready to state our main result.

**Theorem 1.4.** Let $\mathcal{H}$ be a connected $k$-graph with maximum degree $\Delta \geq 2$. Then the cyclic index of $\mu(\mathcal{H}, x)$ is $k$, $\lambda(\mathcal{H})$ is a simple root of $\mu(\mathcal{H}, x)$, and

$$\Delta^{\frac{1}{2}} \leq \lambda(\mathcal{H}) < \frac{k}{k - 1}((k - 1)(\Delta - 1))^{\frac{1}{2}}.$$

Theorem 1.4 extends Theorem 1.2 to $k$-graphs and provides a characterization of distribution of zeros of matching polynomials of $k$-graphs. In particular, it implies that the matching polynomial of a $k$-graph must contain some nonreal zeros when $k \geq 3$.

The second eigenvalue of hypergraphs was introduced by Friedman and Wigderson [12, 13]. Lenz and Mubayi [29] showed that a hypergraph satisfies some quasirandom properties if and only if it has a small second eigenvalue. In 2019, Li and Mohar [31] generalized Alon–Boppana Theorem to $k$-graphs, and showed that for any finite $d$-regular $k$-graph $\mathcal{H}$ on $n$ vertices, the second eigenvalue of $\mathcal{H}$ is at least

$$\frac{k}{k - 1}((k - 1)(d - 1))^{\frac{1}{2}} - o_n(1),$$

where $o_n(1)$ is a quantity that tends to zero for every fixed $d$ as $n \to \infty$. Similar to the important application of Theorem 1.2 in Ramanujan graphs, Theorem 1.4 is expected to be useful to extend Ramanujan graphs to hypergraphs.

Moreover, combining Theorem 1.3 and Theorem 1.4, we immediately obtain the following result due to Friedman [12] (see also [31] by Li and Mohar).

**Theorem 1.5** ([12, 31]). Let $\mathcal{T}$ be a connected $k$-tree with maximum degree $\Delta \geq 2$. Let $\rho(\mathcal{T})$ be the spectral radius of the adjacency tensor of $\mathcal{T}$. Then

$$\rho(\mathcal{T}) < \frac{k}{k - 1}((k - 1)(\Delta - 1))^{\frac{1}{2}}.$$
The matching polynomial of a $k$-graph still obeys the vertex-deletion recursive formula, see [45] or Lemma 2.6, and the recursive formula is crucial when we use mathematical induction to obtain some properties of the matching polynomial.

The $\alpha$-normal method. In 2016, Lu and Man [36] introduced the $\alpha$-normal method to investigate the spectral radii of $k$-graphs, which is very powerful to deal with the eigenvalues of tensor [3, 50] and the $p$-spectral radii of $k$-graphs [33]. Informally, the method transforms the eigenvalue problem of the adjacency tensor of a $k$-graph $\mathcal{H}$ to the problem of weighted incidence matrix of $\mathcal{H}$ with certain restriction, which is easier to deal with especially for $k$-trees, see Section 3 for details. We will use this method to prove Theorem 1.3 with some additional techniques.

The path tree. The path tree (also called Godsil’s tree), introduced by Godsil [15], is considered as one of the most important and useful tools in matching polynomial theory. For a connected graph $G$ and a vertex $u \in V(G)$, the path tree $T(G, u)$ is a tree which has vertices as the paths in $G$ starting at $u$, where two such paths are adjacent if one is a maximal proper subpath of the other.

Godsil [15, Theorem 2.5] proved the following celebrated identity:

$$
\mu(G - u, x) = \frac{\mu(T(G, u) - u, x)}{\mu(T(G, u), x)},
$$

and further that $\mu(G, x)$ divides $\mu(T(G, u), x)$. As mentioned above, some of the previous proof-method in matching polynomial theory of graphs are ineffective for $k$-graphs. However, it is surprising that we can generalize the path tree from graphs to hypergraphs and prove that (1.4) is also true in the $k$-graph paradigm. This is the essential tool of the paper, especially for the proof of Theorem 1.4.

1.4. Organization of the paper. In Section 2, we introduce the path tree $T(\mathcal{H}, u)$ of a $k$-graph $\mathcal{H}$ with respect to a vertex $u$, and prove that (1.4) is also true in the $k$-graph paradigm. In Section 3, we prove Theorem 1.3 by using the $\alpha$-normal method with some additional techniques. In Section 4, we investigate the distribution of zeros of the matching polynomial. Based on (1.4) and Theorem 1.3, we prove that every zero of $\mu(\mathcal{H}, x)$ is an eigenvalue of the adjacency tensor of $T(\mathcal{H}, u)$, and the cyclic index of $\mu(\mathcal{H}, x)$ is $k$. Furthermore, we show that $\lambda(\mathcal{H})$ is a simple root of $\mu(\mathcal{H}, x)$, and is exactly the spectral radius of the adjacency tensor of $T(\mathcal{H}, u)$. Using these results, we establish the bounds (1.3) of $\lambda(\mathcal{H})$ and prove Theorem 1.4 at the end of Section 4. Finally, we conclude this paper with some further discussions and questions in Section 5.

2. Path trees of $k$-graphs

In order to prove Theorem 1.4, we introduce the path trees of $k$-graphs and generalize the Godsil’s identity (1.4) to $k$-graphs. Before proceeding, we introduce some notations and terminology. Let $v$ be a vertex of a $k$-graph $\mathcal{H}$. Denote by $N_\mathcal{H}(v)$ the set of all vertices of $\mathcal{H}$ adjacent to $v$ and by $E_\mathcal{H}(v)$ the set of all edges of $\mathcal{H}$ incident with $v$. The degree of $v$ is defined as $|E_\mathcal{H}(v)|$ and is denoted by $d_\mathcal{H}(v)$. The maximum degree and minimum degree of the vertices of $\mathcal{H}$ is denoted by $\Delta(\mathcal{H})$ and $\delta(\mathcal{H})$, respectively. For a subset $W$ of $V(\mathcal{H})$, let $\mathcal{H}[W]$ denote the subgraph of $\mathcal{H}$ induced by $W$, i.e., $V(\mathcal{H}[W]) = W$ and $E(\mathcal{H}[W]) = \{e \in E(\mathcal{H}) : e \subset W\}$. For the sake of convenience, we simply write $\mathcal{H} - W$ instead of $\mathcal{H}[V(\mathcal{H}) \setminus W]$, write $\mathcal{H} - v$ for $\mathcal{H} - \{v\}$, and use $\mathcal{H} - e$ to denote $\mathcal{H} - \{v_1, \ldots, v_k\}$ where $e = \{v_1, \ldots, v_k\}$ is an edge. Let $p = (v_0, e_1, v_1, e_2, v_2, \ldots, e_\ell, v_\ell)$ be a sequence of vertices and edges alternatively. $p$ is called a path in a $k$-graph $\mathcal{H}$ if the vertices and edges are distinct and $v_i, v_{i+1} \in e_{i+1}$ for $i = 0, 1, \ldots, \ell - 1$. $p$ is called a cycle if we only allow $v_0 = v_\ell$ in the definition of the path.

**Definition 2.1.** A path $p = (v_0, e_1, v_1, e_2, v_2, \ldots, e_\ell, v_\ell)$ is said to be nonbacktracking if $v_i \notin e_j$ for any $0 \leq i < j - 1 \leq \ell - 1$, equivalently, for any $2 \leq j \leq \ell$, $e_j$ contains no any vertex in the set \{${v_0, v_1, \ldots, v_{j-2}}$\}. 

A \( k \)-graph \( \mathcal{H} \) is connected if any pair of vertices of \( \mathcal{H} \) are connected by a path, and is a \( k \)-uniform hypertree (or simply \( k \)-tree) if \( \mathcal{H} \) is both connected and acyclic. In particular, we say that \( p \) begins at the vertex \( v_0 \) and terminates at the vertex \( v_\ell \) and the edge \( e_\ell \). We use \( v(p) \) and \( e(p) \) to denote the terminal vertex and the terminal edge of \( p \), respectively. A path \( p' \) is called a continuation of \( p \) if it is lengthened from \( p \) by attaching a new edge at \( v(p) \).

**Definition 2.2.** Let \( \mathcal{H} \) be a connected \( k \)-graph and \( u \) be a vertex of \( \mathcal{H} \). The path tree \( T(\mathcal{H}, u) \) with respect to \( u \) is a \( k \)-graph whose vertices are all nonbacktracking paths in \( \mathcal{H} \) starting at \( u \) and \( p_1 \) and its continuations \( p_2, \ldots, p_k \) form an edge of \( T(\mathcal{H}, u) \) if and only if \( e(p_2) = \cdots = e(p_k) \).

**Remark 2.3.** For \( k = 2 \), a nonbacktracking path is just a path. Therefore, the path trees of \( k \)-graphs is an extension of the path trees of graphs.

For a given vertex \( u \in V(\mathcal{H}) \), \( u \) itself is a path, and the corresponding vertex of \( T(\mathcal{H}, u) \) will be also referred to as \( u \). Observe that \( T(\mathcal{H}, u) \) is determined by \( \mathcal{H} \) and \( u \). Figure 2 shows an example of a path tree of the complete 3-graph on four vertices, depicted in Figure 1.

![Figure 1. The graph \( K_4^3 \).](image)

Two \( k \)-graphs \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) are called isomorphic, denoted by \( \mathcal{H}_1 \cong \mathcal{H}_2 \), if there exists a bijection \( \theta : V(\mathcal{H}_1) \rightarrow V(\mathcal{H}_2) \) such that \( \{v_1, \ldots, v_k\} \in E(\mathcal{H}_1) \) if and only if \( \{\theta(v_1), \ldots, \theta(v_k)\} \in E(\mathcal{H}_2) \). We call such bijection \( \theta \) an isomorphism from \( \mathcal{H}_1 \) to \( \mathcal{H}_2 \). The following lemma shows that a path tree is indeed a \( k \)-tree, and presents some important properties of path trees which will be used in later.

**Lemma 2.4.** Let \( \mathcal{H} \) be a connected \( k \)-graph and \( u \) be a vertex of \( \mathcal{H} \). Then we have the following:
1. \( T(\mathcal{H}, u) \) is a \( k \)-tree.
2. \( \mathcal{H} \) is a \( k \)-tree if and only if \( T(\mathcal{H}, u) \) is isomorphic to \( \mathcal{H} \).
(3) Let \( E_H(u,v) = E_H(u) \cap E_H(v) \) and \( e_H(u,v) = |E_H(u,v)| \). Then

\[
\mathcal{T}(H, u) - u \supset \bigoplus_{w \in N_H(u)} e_H(u,w) \mathcal{T}(H - u, w),
\]

where \( \bigoplus \) denotes the distinct union of \( k \)-graphs and \( tH \) denotes the disjoint union of \( t \) copies of \( H \).

**Proof.** (1) For simplicity, denote \( \mathcal{T} = \mathcal{T}(H, u) \). By the definition of \( \mathcal{T} \), \( \mathcal{T} \) has no cycle. Since \( \mathcal{T} \) is \( k \)-uniform and acyclic, it suffices to prove that \( \mathcal{T} \) is connected. We prove the connectedness recursively. Fix an arbitrary vertex \( p \in V(\mathcal{T}) \). If \( p' \) is a continuation of \( p \) with terminal edge \( e(p') = \{v(p), v(p'), v_3, \ldots , v_k\} \), i.e., \( e(p') \) is the new added edge in \( p' \), by the property of nonbacktracking paths of \( p' \), the nonbacktracking path \( (p, e(p'), v_i) \) is also a continuation of \( p \) for every \( 3 \leq i \leq k \). Thus the set

\[
\{p, p', (p, e(p'), v_3), \ldots , (p, e(p'), v_k)\}
\]

forms an edge of \( \mathcal{T} \). This implies the connectedness.

(2) If \( H \) is a \( k \)-tree, then there is a unique path joining \( u \) and \( v \) for each vertex \( v \in V(H) \). Now, the mapping that maps each vertex \( v \) to the unique path joining \( u \) to \( v \) is an isomorphism from \( H \) to \( \mathcal{T} \). On the other hand, if \( H \) contains some cycles, then \( \mathcal{T} \) is not isomorphic to \( H \) since \( \mathcal{T} \) is a \( k \)-tree by (1).

(3) The last statement is straightforward from the definition of \( \mathcal{T} \). \( \square \)

Let \( H \) be a \( k \)-graph with the components \( H_1, \ldots , H_s \). For each \( i = 1, \ldots , s \), let \( \mathcal{T}(H_i, v_i) \) be the path tree of \( H_i \) with respect to the vertex \( v_i \in V(H_i) \). We can naturally define the path tree of \( H \) with respect to \( v = (v_1, \ldots , v_s) \) to be the disjoint union of \( \mathcal{T}(H_1, v_1), \ldots , \mathcal{T}(H_s, v_s) \), denoted by \( \mathcal{T}(H, v) \). Note that

\[
\mu(H_1 \oplus H_2, x) = \mu(H_1, x) \mu(H_2, x)
\]

for disjoint union of \( k \)-graphs \( H_1 \) and \( H_2 \), so we usually restrict our attention to connected \( k \)-graphs.

If \( H \) is not connected, define \( \mathcal{T}(H, u) = \mathcal{T}(\hat{H}, u) \), where \( \hat{H} \) is the component of \( H \) containing \( u \). Combining (2.1), we immediately obtain the following lemma, which will be used to discuss the case for disconnected \( k \)-graphs.

**Lemma 2.5.** Let \( H \) be a \( k \)-graph and \( u \in V(H) \). Let \( \hat{H} \) be the component of \( H \) containing \( u \). We have the following:

\[
\frac{\mu(H - u, x)}{\mu(H, x)} = \frac{\mu(\hat{H} - u, x)}{\mu(\hat{H}, x)},
\]

and

\[
\frac{\mu(\mathcal{T}(H, u) - u, x)}{\mu(\mathcal{T}(H, u), x)} = \frac{\mu(\mathcal{T}(\hat{H}, u) - u, x)}{\mu(\mathcal{T}(\hat{H}, u), x)}.
\]

To present the main results of this section, we also need the following lemma that provides a recursive formula for the matching polynomial of \( k \)-graphs.

**Lemma 2.6** ([45]). Let \( H \) be a \( k \)-graph. For each vertex \( u \in V(H) \),

\[
\mu(H, x) = x \mu(H - u, x) - \sum_{e \in E_H(u)} \mu(H - e, x).
\]

The following two theorems play a crucial role in this paper, and the case \( k = 2 \) was proved by Godsil (see [15, 16] or [19, Chapter 6]).
Theorem 2.7. Let $\mathcal{H}$ be a $k$-graph. Then, for any vertex $v \in V(\mathcal{H})$, 

$$
\frac{\mu(\mathcal{H} - u, x)}{\mu(\mathcal{H}, x)} = \frac{\mu(T(\mathcal{H}, u) - u, x)}{\mu(T(\mathcal{H}, u), x)}.
$$

Proof. We prove (2.4) by induction on $|V(\mathcal{H})|$. If $|V(\mathcal{H})| = k$, we can assume that $\mathcal{H}$ is a $k$-tree with a single edge, so (2.4) holds by Lemma 2.4 (2).

Assume $|V(\mathcal{H})| > k$. For a given $e = \{u, v_2, \ldots, v_k\} \in E_{\mathcal{H}}(u)$, denote by 

$$
p(e) = \{(u), (u, e, v_2), \ldots, (u, e, v_k)\}
$$

the edge of $T(\mathcal{H}, u)$ corresponding to $e$. We first prove the following claim.

Claim:

$$
\frac{\mu(\mathcal{H} - e, x)}{\mu(\mathcal{H} - u, x)} = \frac{\mu(T(\mathcal{H}, u) - p(e), x)}{\mu(T(\mathcal{H}, u) - u, x)}.
$$

Proof of Claim: Let $V_i = \{u, v_2, \ldots, v_i\}$ for each $i = 2, \ldots, k$ and $V_1 = \{u\}$. By the induction hypothesis and Lemma 2.5, we have

$$
\frac{\mu(\mathcal{H} - e, x)}{\mu(\mathcal{H} - u, x)} = \prod_{i=1}^{k-1} \frac{\mu(\mathcal{H} - V_{i+1}, x)}{\mu(\mathcal{H} - V_i, x)} = \prod_{i=1}^{k-1} \frac{\mu(T(\mathcal{H} - V_i, v_{i+1}) - v_{i+1}, x)}{\mu(T(\mathcal{H} - V_i, v_{i+1}), x)}.
$$

Here, $\mathcal{H} - V_i$ may be disconnected for some $i$, and in this case we also obtain (2.6) by Lemma 2.5.

Let $p_i(e) = \{(u), (u, e, v_2), \ldots, (u, e, v_i)\}$ for each $i = 2, \ldots, k$ and $p_1(e) = \{u\}$. Then,

$$
\frac{\mu(T(\mathcal{H}, u) - p(e), x)}{\mu(T(\mathcal{H}, u) - u, x)} = \prod_{i=1}^{k-1} \frac{\mu(T(\mathcal{H}, u) - p_i(e), x)}{\mu(T(\mathcal{H}, u) - p_1(e), x)}.
$$

Combining (2.6) and (2.7), in order to prove (2.5), it suffices to show that

$$
\frac{\mu(T(\mathcal{H} - V_i, v_{i+1}) - v_{i+1}, x)}{\mu(T(\mathcal{H} - V_i, v_{i+1}), x)} = \frac{\mu(T(\mathcal{H}, u) - p_i(e), x)}{\mu(T(\mathcal{H}, u) - p_1(e), x)}
$$

for each $i = 1, \ldots, k - 1$. This fact follows from the observation that $T(\mathcal{H} - V_i, v_{i+1})$ is isomorphic to the component of $T(\mathcal{H}, u) - p_i(e)$ containing the path $(u, e, v_{i+1})$ as a vertex. This proves the claim.

By Lemma 2.6 and (2.5), one can deduce that

$$
\frac{\mu(\mathcal{H}, x)}{\mu(\mathcal{H} - u, x)} = \frac{x \mu(\mathcal{H} - u, x) - \sum_{e \in E_{\mathcal{H}}(u)} \mu(\mathcal{H} - e, x)}{\mu(\mathcal{H} - u, x)}
$$

(by Lemma 2.6)

$$
= x - \sum_{e \in E_{\mathcal{H}}(u)} \frac{\mu(\mathcal{H} - e, x)}{\mu(\mathcal{H} - u, x)}
$$

(by (2.5))

$$
= x - \sum_{e \in E_{\mathcal{H}}(u)} \frac{\mu(T(\mathcal{H}, u) - p(e), x)}{\mu(T(\mathcal{H}, u) - u, x)}
$$

$$
= x - \sum_{p \in E_{T(\mathcal{H}, u)}(u)} \frac{\mu(T(\mathcal{H}, u) - p, x)}{\mu(T(\mathcal{H}, u) - u, x)}
$$

$$
= \frac{x \mu(T(\mathcal{H}, u) - u, x) - \sum_{p \in E_{T(\mathcal{H}, u)}(u)} \mu(T(\mathcal{H}, u) - p, x)}{\mu(T(\mathcal{H}, u) - u, x)}
$$

(by Lemma 2.6)

$$
= \frac{\mu(T(\mathcal{H}, u), x)}{\mu(T(\mathcal{H}, u) - u, x)}.
$$

This completes the proof of the theorem. \(\square\)
**Theorem 2.8.** Let $\mathcal{H}$ be a $k$-graph and $u \in V(\mathcal{H})$. Then there exists a proper subforest $\mathcal{F}$ of $\mathcal{T}(\mathcal{H}, u)$ such that

$$\mu(\mathcal{H}, x) = \frac{\mu(\mathcal{T}(\mathcal{H}, u), x)}{\mu(\mathcal{F}, x)}.$$  

In particular, $\mu(\mathcal{H}, x)$ divides $\mu(\mathcal{T}(\mathcal{H}, u), x)$.

**Proof.** We prove the first statement by induction on $|V(\mathcal{H})|$. It is trivial for the case $|V(\mathcal{H})| = k$ by Lemma 2.4 (2).

Assume $|V(\mathcal{H})| > k$. By Theorem 2.7, we have

$$\mu(\mathcal{T}(\mathcal{H}, u), x) = \mu(\mathcal{H}, x) \frac{\mu(\mathcal{T}(\mathcal{H}, u) - u, x)}{\mu(\mathcal{H} - u, x)}.$$  

(2.9)

Thus, to prove the first statement, it suffices to prove that the second factor on the right side of (2.9) is a matching polynomial of a subforest of $\mathcal{T}(\mathcal{H}, u)$. Let $\mathcal{H}_1, \ldots, \mathcal{H}_s$ be the components of $\mathcal{H} - u$. Then,

$$\mu(\mathcal{H} - u, x) = \prod_{i=1}^{s} \mu(\mathcal{H}_i, x).$$

Let $w_i \in V(\mathcal{H}_i)$ be a neighbor of $u$ in $\mathcal{H}$ for each $i = 1, \ldots, s$, and let $W = \{w_1, \ldots, w_s\}$. By Lemma 2.4 (3), we have

$$\mu(\mathcal{T}(\mathcal{H}, u) - u, x) = \prod_{w \in N(\mathcal{H}, u)} \left( \mu(\mathcal{T}(\mathcal{H} - u, w), x) \right)^{e_H(u, w)}$$

$$= \left( \prod_{i=1}^{s} \mu(\mathcal{T}(\mathcal{H}_i - u, w_i), x) \right)^{e_H(u, w_i)} \prod_{w \in N(\mathcal{H}, u) \setminus W} \left( \mu(\mathcal{T}(\mathcal{H} - u, w), x) \right)^{e_H(u, w)}$$

$$= \mu(\mathcal{H}, x) \left( \prod_{i=1}^{s} \mu(\mathcal{T}(\mathcal{H}_i - u, w_i), x) \right)$$

where

$$\mathcal{H} = \left( \bigoplus_{i=1}^{s} (e_H(u, w_i) - 1) \right) \mathcal{T}(\mathcal{H} - u, w_i) \bigoplus \left( \bigoplus_{w \in N(\mathcal{H}, u) \setminus W} e_H(u, w) \mathcal{T}(\mathcal{H} - u, w) \right).$$

Therefore,

$$\frac{\mu(\mathcal{T}(\mathcal{H}, u) - u, x)}{\mu(\mathcal{H} - u, x)} = \mu(\mathcal{H}, x) \left( \prod_{i=1}^{s} \frac{\mu(\mathcal{T}(\mathcal{H}_i - u, w_i), x)}{\mu(\mathcal{H}_i, x)} \right).$$

Since $\mathcal{T}(\mathcal{H} - u, w_i)$ is a path tree of $\mathcal{H}_i$, by induction hypothesis, for each $i = 1, \ldots, s$, there exists subforest $F_i$ of $\mathcal{T}(\mathcal{H}_i - u, w_i)$ such that

$$\mu(F_i, x) = \frac{\mu(\mathcal{T}(\mathcal{H}_i - u, w_i), x)}{\mu(\mathcal{H}_i, x)}.$$  

Thus,

$$\frac{\mu(\mathcal{T}(\mathcal{H}, u) - u, x)}{\mu(\mathcal{H} - u, x)} = \mu(\mathcal{H}, x) \left( \prod_{i=1}^{s} \mu(F_i, x) \right).$$

Moreover, one can directly check that $\left( \bigoplus_{i=1}^{s} F_i \right) \bigoplus \mathcal{H}$ is a proper subforest of $\mathcal{T}(\mathcal{H}, u)$ and this implies the first statement. The ‘in particular’ statement easily follows from the first statement. $\Box$
3. Zeros of matching polynomials of \( k \)-trees

The aim of this section is to prove Theorem 1.3. We begin with the notations of eigenvalues of tensor and the \( \alpha \)-normal method.

3.1. Eigenvalues of the adjacency tensor of \( k \)-graphs. A real tensor (also called hypermatrix) \( \mathcal{A} = (a_{i_1 \ldots i_k}) \) of order \( k \) and dimension \( n \) refers to a multi-dimensional array with entries \( a_{i_1 \ldots i_k} \in \mathbb{R} \) for all \( i_j \in [n] = \{1, \ldots, n\} \) and \( j \in [k] \). Clearly, if \( k = 2 \), then \( \mathcal{A} \) is a square matrix of dimension \( n \). Let \( \mathcal{I} = (i_{i_1 \ldots i_k}) \) be the identity tensor of order \( k \) and dimension \( n \), that is, \( i_{i_1 \ldots i_k} = 1 \) if \( i_1 = \cdots = i_k \in [n] \) and \( i_{i_1 \ldots i_k} = 0 \) otherwise.

Let \( \mathcal{A} = (a_{i_1 \ldots i_k}) \) be a tensor of order \( k \) and dimension \( n \). For a vector \( x = (x_1, \ldots, x_n)^\top \in \mathbb{C}^n \), denote by \( x[k] = (x_1^k, \ldots, x_n^k)^\top \) and let \( \mathcal{A}x^{k-1} \) be a vector in \( \mathbb{C}^n \) whose \( i \)th component is defined in the following

\[
(\mathcal{A}x^{k-1})_i = \sum_{i_2, \ldots, i_k \in [n]} a_{i_1 i_2 \ldots i_k} x_{i_2} \cdots x_{i_k}.
\]

In 2005, Lim [32] and Qi [41] independently introduced the eigenvalues of tensors. For some \( \lambda \in \mathbb{C} \), if the polynomial system

\[
\mathcal{A}x^{k-1} = \lambda x^{k-1},
\]

has a solution \( x \in \mathbb{C}^n \setminus \{0\} \), then \( \lambda \) is called an eigenvalue of \( \mathcal{A} \) and \( x \) is an eigenvector of \( \mathcal{A} \) associated with \( \lambda \).

The determinant of \( \mathcal{A} \), denoted by \( \det \mathcal{A} \), is defined as the resultant of the polynomial system \( \mathcal{A}x^{k-1} \) [21] and the characteristic polynomial \( \phi_A(x) \) of \( \mathcal{A} \) is defined as \( \det(x\mathcal{I} - \mathcal{A}) \) [41]. It is proved in [41] that \( \lambda \) is an eigenvalue of \( \mathcal{A} \) if and only if it is a root of \( \phi_A(x) \).

Let \( \mathcal{H} \) be a \( k \)-graph on \( n \) vertices \( v_1, \ldots, v_n \). The adjacency tensor of \( \mathcal{H} \) [8] is defined as \( \mathcal{A}(\mathcal{H}) = (a_{i_1 \ldots i_k}) \), an order \( k \) \( n \) dimensional tensor, where

\[
a_{i_1 i_2 \ldots i_k} = \begin{cases} \frac{1}{(k-1)!}, & \text{if } \{v_{i_1}, \ldots, v_{i_k}\} \in E(\mathcal{H}); \\ 0, & \text{otherwise}. \end{cases}
\]

In this paper, the eigenvalues of a \( k \)-graph \( \mathcal{H} \) always refer to those of its adjacency tensor. The spectral radius of \( \mathcal{H} \) is defined as

\[
\rho(\mathcal{H}) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathcal{A}(\mathcal{H})\},
\]

which is exactly the spectral radius of \( \mathcal{A}(\mathcal{H}) \).

The next two lemmas are the applications of the Perron–Frobenius theorem of weakly irreducible nonnegative tensors [11, 47, 48].

**Lemma 3.1** ([8, 28]). Let \( \mathcal{H} \) be a connected \( k \)-graph. Then \( \rho(\mathcal{H}) \) is an eigenvalue of \( \mathcal{A}(\mathcal{H}) \) with an corresponding eigenvector having all elements positive, and \( \rho(\mathcal{G}) < \rho(\mathcal{H}) \) for any proper subgraph \( \mathcal{G} \) of \( \mathcal{H} \).

We use \( H \setminus e \) to denote the \( k \)-graph obtained from \( H \) by deleting \( e \) along with resultant isolated vertices.

**Lemma 3.2** (Theorem 13, [51]). Let \( \mathcal{H} \) be a \( k \)-graph and \( e \in E(\mathcal{H}) \) be an edge containing two degree-one vertices. If \( \lambda \) is an eigenvalue of \( \mathcal{A}(\mathcal{H} \setminus e) \), then \( \lambda \) is an eigenvalue of \( \mathcal{A}(\mathcal{H}) \).

An edge \( e \) of \( \mathcal{H} \) is called a pendant edge if it contains exactly \( k - 1 \) vertices of degree one. If \( F \) is a subtree of a \( k \)-tree \( T \), it is known in [7] that there exists a sequence of edges \( (e_1, \ldots, e_s) \) such that, for any \( i \) \( (1 \leq i \leq s) \), \( e_i \) is a pendant edge of \( T_{i-1} \), where \( T_0 = T \), \( T_i = T_{i-1} \setminus e_i \), and \( T_s = F \).

From this fact and Lemma 3.2, we immediately obtain the following result.

**Corollary 3.3.** If \( F \) is a subtree of a \( k \)-tree \( T \), then every eigenvalue of \( \mathcal{A}(F) \) is an eigenvalue of \( \mathcal{A}(T) \).
3.2. The $\alpha$-normal method for $k$-graphs. In 2016, Lu and Man [36] introduced the $\alpha$-normal method to investigate the spectral radii of $k$-graphs. The method is a powerful tool to deal with the eigenvalues of tensor [3, 50] and the $p$-spectral radii of $k$-graphs [33].

Let $\mathcal{H}$ be a $k$-graph. A weighted incidence matrix $B$ of $\mathcal{H}$ is a complex matrix whose rows are indexed by $V(\mathcal{H})$ and columns are indexed by $E(\mathcal{H})$ such that for any $v \in V(\mathcal{H})$ and any $e \in E(\mathcal{H})$, the entry $B(v, e) \neq 0$ if and only if $v \in e$.

**Definition 3.4** ([50]). Let $\mathcal{H}$ be a $k$-graph, and let $\{\alpha_e\}_{e \in E(\mathcal{H})}$ be an ordered sequence consisting of complex numbers such that $\alpha_e \neq 0$ for any $e \in E(\mathcal{H})$. We call $\mathcal{H}$ generalized $\{\alpha_e\}_{e \in E(\mathcal{H})}$-normal with respect to $B$ if there exists a weighted incidence matrix $B$ of $\mathcal{H}$ such that

(C1) $\sum_{e \in E_H(v)} B(v, e) = 1$ for any $v \in V(\mathcal{H})$.

(C2) $\prod_{e \in E} B(v, e) = \alpha_e$ for any $e \in E(\mathcal{H})$.

If $\alpha_e = \alpha$ for any $e \in E(\mathcal{H})$, then $\mathcal{H}$ is called generalized $\alpha$-normal. Moreover, $\mathcal{H}$ is called consistently generalized $\alpha$-normal with respect to $B$ if $\mathcal{H}$ is generalized $\alpha$-normal and

(C3) For any cycle $v_0 e_1 v_1 \cdots e_\ell v_\ell (= v_0)$ of $\mathcal{H}$,

$$\prod_{i=1}^{\ell} \frac{B(v_i, e_i)}{B(v_{i-1}, e_i)} = 1.$$ 

In what follows, we simply say that $\mathcal{H}$ is generalized $\{\alpha_e\}_{e \in E(\mathcal{H})}$-normal or generalized $\alpha$-normal if the weighted incidence matrix $B$ is clear from the context.

**Lemma 3.5** (Lemma 2.2, [50]). Let $\mathcal{H}$ be a connected $k$-graph. Then, a nonzero number $\lambda$ is an eigenvalue of $\mathcal{H}$ corresponding to an eigenvector with all the elements nonzero if and only if $\mathcal{H}$ is consistently generalized $\alpha$-normal with $\alpha = \lambda^{-k}$.

3.3. Proof of Theorem 1.3. This subsection is devoted to proving Theorem 1.3. For this, let us first introduce some more notations and terminology which we need. Let $\mathcal{T}$ be a $k$-forest and $\bar{\mu}(\mathcal{T}, \{x_e\}_{e \in E(\mathcal{T})})$ denote the following multivariate polynomial with respect to indeterminates $\{x_e\}_{e \in E(\mathcal{T})}$:

\[
\bar{\mu}(\mathcal{T}, \{x_e\}_{e \in E(\mathcal{T})}) = \sum_{M \in \mathcal{M}(\mathcal{T})} (-1)^{|M|} \prod_{e \in E(M)} x_e.
\]

(3.1)

If $\Delta(\mathcal{T}) \geq 2$, then there exists a vertex $u$ such that $d_{\mathcal{T}}(u) \geq 2$ and it is incident with at most one non-pendent edge in $\mathcal{T}$. Denote by $\mathcal{N}(\mathcal{T})$ all such vertices of $\mathcal{T}$. Take a vertex $u \in \mathcal{N}(\mathcal{T})$. Denote by $P_{\mathcal{T}}(u)$ the set consisting of $d_{\mathcal{T}}(u) - 1$ chosen pendent edges incident to $u$, and denote by $f$ the remaining edge incident to $u$. Let $\{\alpha_e\}_{e \in E(\mathcal{T})}$ be an ordered sequence consisting of nonzero complex numbers. The triple $(\mathcal{T}, u, P_{\mathcal{T}}(u))$ is called admissible with respect to $\{\alpha_e\}_{e \in E(\mathcal{T})}$ if

$$\sum_{g \in P_{\mathcal{T}}(u)} \alpha_g \neq 1.$$

Otherwise, it is called inadmissible.

The following result suggests that if a $k$-tree $\mathcal{T}$ and a sequence $\{\alpha_e\}_{e \in E(\mathcal{T})}$ satisfy some restrictions, then the ‘root’ of $\bar{\mu}(\mathcal{T}, \{x_e\}_{e \in E(\mathcal{T})})$ and the ‘generalized $\{\alpha_e\}_{e \in E(\mathcal{T})}$-normal’ of $\mathcal{T}$ obey a recursive rule. It is noticeable that the proof idea of the following result derives from the proof of [50, Theorem 3.1].

**Lemma 3.6.** Let $\mathcal{T}$ be a $k$-tree with $\Delta(\mathcal{T}) \geq 2$ and $\{\alpha_e\}_{e \in E(\mathcal{T})}$ be an order sequence consisting of nonzero complex numbers. Assume that $(\mathcal{T}, u, P_{\mathcal{T}}(u))$ is admissible with respect to $\{\alpha_e\}_{e \in E(\mathcal{T})}$. Let
\( \hat{T} \) be the \( k \)-forest obtained from \( T \) by deleting all edges in \( P_T(u) \) and the resultant isolated vertices. Set \( \hat{\alpha}_e = \alpha_e \) for any \( e \in E(\hat{T}) \setminus \{ f \} \) and \( \hat{\alpha}_f = \frac{-\alpha_f}{g \in P_T(u)} \). Then the following assertions hold.

1. If \( \{ \alpha_e \}_{e \in E(T)} \) is a root of \( \hat{\mu}(T, \{ x_e \}_{e \in E(\hat{T})}) \), then \( \{ \hat{\alpha}_e \}_{e \in E(\hat{T})} \) is a root of \( \hat{\mu}(\hat{T}, \{ x_e \}_{e \in E(\hat{T})}) \).

2. If \( \hat{T} \) is generalized \( \{ \hat{\alpha}_e \}_{e \in E(\hat{T})} \)-normal, then \( T \) is generalized \( \{ \alpha_e \}_{e \in E(T)} \)-normal.

Proof. (1) Assume that \( \{ \alpha_e \}_{e \in E(T)} \) is a root of \( \hat{\mu}(T, \{ x_e \}_{e \in E(\hat{T})}) \), and write \( \chi_M = \prod_{e \in E(M)} \alpha_e \) for any matching \( M \) of \( T \). Then we have

\[
\sum_{M \in \mathcal{R}} (-1)^{|M|} \chi_M + \sum_{M \in \mathcal{S}} (-1)^{|M|} \chi_M + \sum_{M \in \mathcal{M}(T) \setminus (\mathcal{R} \cup \mathcal{S})} (-1)^{|M|} \chi_M = 0,
\]

where \( \mathcal{R} \) is the set of all matchings of \( T \) containing \( f \), and \( \mathcal{S} \) is the set of all matchings of \( T \) containing none of edges incident to \( u \) in \( T \). Moreover, (3.2) is equivalent to

\[
\sum_{M \in \mathcal{R}(\hat{T})} (-1)^{|M|} \chi_M + \sum_{M \in \mathcal{S}(\hat{T})} (-1)^{|M|} \chi_M - \left( \sum_{g \in P_T(u)} \chi_g \right) \left( \sum_{M \in \mathcal{S}(\hat{T})} (-1)^{|M|} \chi_M \right) = 0,
\]

where \( \mathcal{R}(\hat{T}) \) is the set of all matchings of \( \hat{T} \) containing \( f \), and \( \mathcal{S}(\hat{T}) = \mathcal{M}(\hat{T}) \setminus \mathcal{R}(\hat{T}) \). Rearranging (3.3), we have

\[
\sum_{M \in \mathcal{R}(\hat{T})} (-1)^{|M|} \chi_M + \left( 1 - \sum_{g \in P_T(u)} \alpha_g \right) \sum_{M \in \mathcal{S}(\hat{T})} (-1)^{|M|} \chi_M = 0.
\]

Note that \( \sum_{g \in P_T(u)} \alpha_g \neq 1 \), so (3.4) is equivalent to say that

\[
\sum_{M \in \mathcal{R}(\hat{T})} (-1)^{|M|} \chi_M + \left( 1 - \sum_{g \in P_T(u)} \alpha_g \right) \sum_{M \in \mathcal{S}(\hat{T})} (-1)^{|M|} \chi_M = 0.
\]

This implies that \( \{ \hat{\alpha}_e \}_{e \in E(\hat{T})} \) is a root of \( \hat{\mu}(\hat{T}, \{ x_e \}_{e \in E(\hat{T})}) \), as desired.

(2) Assume that \( \hat{T} \) is generalized \( \{ \hat{\alpha}_e \}_{e \in E(\hat{T})} \)-normal with respect to \( \hat{B} \). Define a weighted incidence matrix \( \hat{B} \) of \( \mathcal{T} \) by

\[
\hat{B}(v, e) = \begin{cases} 1 & \text{if } v \in V(\hat{T}) \setminus \{ u \}, e \in E(\hat{T}) \text{ and } v \in e; \\ \frac{1}{1 - \sum_{g \in P_T(u)} \alpha_g} \prod_{e \in E(M)} \alpha_e & \text{if } v = u \text{ and } e = f; \\ \alpha_e & \text{if } v = u \text{ and } e \in P_T(u); \\ 1 & \text{if } v \neq u, v \in e \text{ and } e \in P_T(u); \\ 0 & \text{if } e \notin v. 
\end{cases}
\]

Then one may check that \( \mathcal{T} \) is generalized \( \{ \alpha_e \}_{e \in E(T)} \)-normal with respect to \( B \), as desired.

\( \square \)

**Remark 3.7.** Note that the converses of (1) and (2) in Lemma 3.6 also hold by their proof. However, these facts are not needed in this paper.

**Lemma 3.8.** Let \( \mathcal{T} \) be a \( k \)-star and \( \alpha_e \neq 0 \) for any \( e \in E(T) \). If \( \{ \alpha_e \}_{e \in E(T)} \) is a root of \( \mu(T, \{ x_e \}_{e \in E(T)}) \), then \( \mathcal{T} \) is generalized \( \{ \alpha_e \}_{e \in E(T)} \)-normal.

Proof. Clearly, \( \mu(T, \{ x_e \}_{e \in E(T)}) = 1 - \sum_{e \in E(T)} x_e \). If \( \{ \alpha_e \}_{e \in E(T)} \) is a root of \( \mu(T, \{ x_e \}_{e \in E(T)}) \), then \( \sum_{e \in E(T)} \alpha_e = 1 \). Let \( u \in V(T) \) such that the degree of \( u \) equals \( \Delta(T) \). Define a weighted
Then $\mathcal{T}$ is generalized $\{\alpha_e\}_{e \in E(\mathcal{F})}$-normal with respect to $B$. The proof is completed. \hfill \Box

By Lemma 3.8, we immediately get the following result which will be used in later.

**Corollary 3.9.** Let $\mathcal{T}$ be a $k$-tree with $\Delta(\mathcal{T}) \geq 2$ and $\alpha_e \neq 0$ for any $e \in E(\mathcal{T})$. If $(\mathcal{T}, u, P_T(u))$ is inadmissible for some $u \in \mathcal{N}(\mathcal{T})$, then the $k$-star $\mathcal{F}$ induced by $P_T(u)$ is generalized $\{\alpha_e\}_{e \in E(\mathcal{F})}$-normal.

The following result is the key in the proof of Theorem 1.3 which suggests that if $\alpha$ is a root of $\mu(\mathcal{T}, x)$, then there exists a subtree $\mathcal{F}$ of $\mathcal{T}$ such that $\mathcal{F}$ is generalized $\alpha$-normal.

**Lemma 3.10.** Let $\mathcal{T}$ be a $k$-tree, and let $\alpha_e \neq 0$ for any $e \in E(\mathcal{T})$. If $\{\alpha_e\}_{e \in E(\mathcal{T})}$ is a root of $\hat{\mu}(\mathcal{T}, \{x_e\}_{e \in E(\mathcal{T})})$, then there exists a subtree $\mathcal{F}$ of $\mathcal{T}$ such that $\mathcal{F}$ is generalized $\{\alpha_e\}_{e \in E(\mathcal{F})}$-normal.

**Proof.** If $\mathcal{T}$ contains only one edge, it follows from Lemma 3.8. Assume that $\Delta(\mathcal{T}) \geq 2$ and $\{\alpha_e\}_{e \in E(\mathcal{T})}$ is a root of $\hat{\mu}(\mathcal{T}, \{x_e\}_{e \in E(\mathcal{T})})$. Our sketchy strategy will be to construct a sequence $(T_1, \ldots, T_s)$ consisting of subtrees of $\mathcal{T}$ by applying Lemma 3.6 (1) repeatedly. Then, based on $T_s$, we repeatedly apply Lemma 3.6 (2) to obtain the required subtree $\mathcal{F}$ such that $\mathcal{F}$ is generalized $\{\alpha_e\}_{e \in E(\mathcal{F})}$-normal. Firstly, we provide an algorithm to construct $(T_1, \ldots, T_s)$, see Algorithm 1.

We are now ready to obtain the required subtree $\mathcal{F}$ such that $\mathcal{F}$ is generalized $\{\alpha_e\}_{e \in E(\mathcal{F})}$-normal. Note that the algorithm will stop when $\mathcal{N}(T_s) = \emptyset$ or $(T_s, u_s, P_{T_s}(u_s))$ is inadmissible with respect to $\{\alpha_e\}_{e \in E(\mathcal{T})}$. Based on this, we divide the remaining proof into two cases.

**Case 1:** $\mathcal{N}(T_s) = \emptyset$. This implies that $T_s$ contains only one edge. In this case, we will show that $\mathcal{F} = \mathcal{T}$. Consider the sequence $(\mathcal{T} = T_1, \ldots, T_s)$, It follows from Step 10 of Algorithm 1 that $(T_i, u_i, P_{T_i}(u_i))$ is admissible with respect to $\{\alpha_e^i\}_{e \in E(T_i)}$ for any $1 \leq i \leq s - 1$. Note that $\{\alpha_e^i\}_{e \in E(T_i)}$ is a root of $\hat{\mu}(T_i, \{x_e\}_{e \in E(T_i)})$. Thus, we can conclude that $\{\alpha_e^i\}_{e \in E(T_i)}$ is a root of $\hat{\mu}(T_i, \{x_e\}_{e \in E(T_i)})$ for $2 \leq i \leq s$ by applying repeatedly Lemma 3.6 (1). Since $T_s$ contains only one edge, by Lemma 3.8, $T_s$ is generalized $\{\alpha_e^s\}_{e \in E(T_s)}$-normal. Applying Lemma 3.6 (2) repeatedly to $T_i$ for each $i$ from 2 to $s$, we conclude that $T_i$ is generalized $\{\alpha_e^i\}_{e \in E(T_i)}$-normal for $1 \leq i \leq s - 1$. The assertion follows if we take $\mathcal{F} = T_1$.

**Case 2:** $(T_s, u_s, P_{T_s}(u_s))$ is inadmissible with respect to $\{\alpha_e^s\}_{e \in E(\mathcal{T})}$. Similarly, we can conclude that $\{\alpha_e^i\}_{e \in E(T_i)}$ is a root of $\hat{\mu}(T_i, \{x_e\}_{e \in E(T_i)})$ for $2 \leq i \leq s$. Since $(T_s, u_s, P_{T_s}(u_s))$ is inadmissible with respect to $\{\alpha_e^s\}_{e \in E(\mathcal{T})}$, it follows from Corollary 3.9 that the $k$-star $\hat{\mathcal{F}}$ induced by the edge set $P_{T_s}(u_s)$ is generalized $\{\alpha_e^s\}_{e \in E(\hat{\mathcal{F}})}$-normal.

Let $\mathcal{F}'$ be the subforest of $\mathcal{T}$ induced by $\cup_{i=1}^{s} P_{T_i}(u_i)$, and let $\mathcal{F}$ be the component of $\mathcal{F}'$ containing $P_{T_s}(u_s)$. Next, we will show that $\mathcal{F}$ is exactly a desired subtree. Clearly, there exists a sequence of natural numbers $1 < a_1 < a_2 < \cdots < a_\ell = s$ such that

$$E(\mathcal{F}) = \cup_{i=1}^{\ell} P_{T_{a_i}}(u_{a_i}).$$

Denote by $\mathcal{F}_{a_i}$ the subforest of $\mathcal{T}$ induced by edge set $\cup_{j=1}^{i} P_{T_{a_j}}(u_{a_j})$ for any $1 \leq i \leq \ell$. We can get a sequence of $k$-trees as follows

$$\mathcal{F} = \mathcal{F}_{a_1}, \mathcal{F}_{a_2}, \ldots, \mathcal{F}_{a_{\ell-1}}, \mathcal{F}_{a_\ell} = \hat{\mathcal{F}}.$$ 

Indeed, $\mathcal{F}_{a_i}$ can be obtained from $\mathcal{F}_{a_{i-1}}$ by deleting all edges in $P_{T_{a_{i-1}}}(u_{a_{i-1}})$ and the resultant isolated vertices for any $2 \leq i \leq \ell$. 

\begin{align*}
B(v,e) &= \begin{cases} 
\alpha_e, & \text{if } v = u \text{ and } e \in E(\mathcal{T}); \\
1, & \text{if } v \neq u, v \in e \text{ and } e \in E(\mathcal{T}) \\
0, & \text{if } v \notin e. 
\end{cases}
\end{align*}
Algorithm 1: A search for \((T_1, \ldots, T_s)\)

**Input**: A \(k\)-tree \(T\) and a sequence \(\{\alpha_e\}_{e \in E(T)}\) with \(\alpha_e \neq 0\) for any \(e \in E(T)\).

**Output**: A sequence of subtrees \((T_1, \ldots, T_s)\) with \(\{\alpha_e^{(1)}\}_{e \in E(T_1)}, \ldots, \{\alpha_e^{(s)}\}_{e \in E(T_s)}\).

1. \(i = 1\), set \(T_1 = T\) and \(\{\alpha_e^{(1)}\}_{e \in E(T)} = \{\alpha_e\}_{e \in E(T)}\);
2. repeat
   3. while \((T_i, \{\alpha_e^{(i)}\}_{e \in E(T_i)})\) with \(i \geq 1\) do
   4.     if \(N(T_i) = \emptyset\), then
   5.         set \(s = i\), stop and output \((T_1, \ldots, T_s)\) with \(\{\alpha_e^{(1)}\}_{e \in E(T_1)}, \ldots, \{\alpha_e^{(s)}\}_{e \in E(T_s)}\);
   6.     else
   7.         if there exists a vertex \(u_i\) in \(N(T_i)\) such that \((T_i, u_i, P_{T_i}(u_i))\) is inadmissible with respect to \(\{\alpha_e^{(i)}\}_{e \in E(T_i)}\), then
   8.             set \(s = i\), stop and output \((T_1, \ldots, T_s)\) with \(\{\alpha_e^{(1)}\}_{e \in E(T_1)}, \ldots, \{\alpha_e^{(s)}\}_{e \in E(T_s)}\);
   9.         else
   10.            for any \(u \in N(T_i)\), \((T_i, u, P_{T_i}(u))\) is admissible with respect to \(\{\alpha_e^{(i)}\}_{e \in E(T_i)}\);
   11.               choose one vertex \(u_i\) in \(N(T_i)\);
   12.               set \(c_i = \sum_{g \in P_{T_i}(u_i)} \alpha_g^{(i)}\);
   13.               set \(T_{i+1}\) to be the \(k\)-tree obtained from \(T_i\) by deleting all edges in \(P_{T_i}(u_i)\) and the resultant isolated vertices, where \(P_{T_i}(u_i)\) is the set consisting of \(d_{T_i}(u_i) - 1\) chosen pendent edges incident to \(u_i\) in \(T_i\);
   14.               define \(\{\alpha_e^{(i+1)}\}_{e \in E(T_{i+1})}\) as follows. Set
   15.                 \[\alpha_e^{(i+1)} = \frac{\alpha_e^{(i)}}{1 - c_i},\]
   16.               and \(\alpha_e^{(i+1)} = \alpha_e^{(i)}\) for any \(e \in E(T_{i+1})\) \(\setminus\) \(\{f_i\}\), where \(f_i\) is the only edge incident to \(u_i\) in \(T_i\) but none of \(P_{T_i}(u_i)\);
   17.               replace \(i\) by \(i + 1\) and return \((T_i, \{\alpha_e^{(i+1)}\}_{e \in E(T_{i+1})})\); \(\triangleright\) repeat for \(i\)
   18.     end
3. until output \((T_1, \ldots, T_s)\) with \(\{\alpha_e^{(1)}\}_{e \in E(T_1)}, \ldots, \{\alpha_e^{(s)}\}_{e \in E(T_s)}\);

As mentioned above, \(F_{a_\ell}\) is generalized \(\{\alpha_e^{(1)}\}_{e \in E(F_{a_\ell})}\)-normal. If \(a_\ell - 1 = a_{\ell - 1}\), by Lemma 3.6 (2), it is clear that \(F_{a_{\ell - 1}}\) is generalized \(\{\alpha_e^{(\ell - 1)}\}_{e \in E(F_{a_{\ell - 1}})}\)-normal. If not, we write the continuous natural numbers between \(a_{\ell - 1}\) and \(a_\ell\) as
\[
a_{\ell - 1} < a_{\ell - 1} + 1 < \ldots < a_\ell - 1 < a_\ell.
\]
Recall that Step 14 of Algorithm 1 suggests that for any \(1 \leq i \leq s - 1\), \(\alpha_e^{(i)} = \alpha_e^{(i+1)}\) for any \(e \in E(T_{i+1})\) unless \(e = f_i\), as well as the subgraph of \(T\) induced by \(f_i \cup P_{T_i}(u_i)\) is a \(k\)-star. Note that \(E(F)\) and \(\cup_{a_{\ell - 1} + 1}^{a_\ell - 1} P_{a_i}(u_{a_i})\) are disjoint. Thus,
\[
\alpha_{a_{\ell - 1} + 1}^{(1)} = \ldots = \alpha_{a_\ell - 1}^{(1)} = \alpha_{a_\ell}^{(1)} = \alpha_{a_\ell}
\]
for any \(e \in E(F_{a_\ell})\). Therefore, \(F_{a_{\ell - 1}}\) is generalized \(\{\alpha_e^{(\ell - 1 + 1)}\}_{e \in E(F_{a_{\ell - 1}})}\)-normal. It follows from Lemma 3.6 (2) that \(F_{a_{\ell - 1}}\) is generalized \(\{\alpha_e^{(\ell - 1)}\}_{e \in E(F_{a_{\ell - 1}})}\)-normal. Repeating the above discussion to \(F_{a_i}\) for each \(i\) from \(\ell - 1\) to 2, we have that \(F_{a_1}\) is generalized \(\{\alpha_i^{(1)}\}_{e \in E(F_{a_1})}\)-normal.

To finish the proof, it suffices to prove
\[
\alpha_{a_1}^{(1)} = \alpha_e \text{ for any } e \in E(F_{a_1}).
\]
It is obvious for $a_1 = 1$. If $a_1 \neq 1$, then for any $2 \leq j \leq a_1$, by the algorithm, we have

$$\alpha^j_e = \alpha^{j-1}_e$$

for any $e \in E(T_j) \setminus \{f_{j-1}\}$.

Since $E(F_{a_1}) \subseteq E(T_{a_1})$, and $f_j \notin E(F_{a_1})$ for any $1 \leq j \leq a_1 - 1$, we conclude (3.6) from (3.7). This completes the proof.

\[\square\]

**Lemma 3.11.** Let $T$ be a $k$-tree with $k \geq 3$. Then $0$ is an eigenvalue of $A(T)$.

**Proof.** Fix a vertex $u \in V(T)$. Let $u \in \mathbb{R}^{|V(T)|}$ be the vector whose $u$-th entry is 1 and other entries are 0. By the assumption '$k \geq 3$', one may check that the vector $u$ is an eigenvector of $A(T)$ associated with the eigenvalue 0, as desired. \[\square\]

**Theorem 3.12.** Let $T$ be a $k$-tree with the matching polynomial $\mu(T, x)$. Then every zero of $\mu(T, x)$ is an eigenvalue of the adjacency tensor of $T$.

**Proof.** If $k = 2$, then the result follows from Theorem 1.1. Assume that $k \geq 3$. By Lemma 3.11, 0 is an eigenvalue of $A(T)$. Let $\lambda \neq 0$ be a root of $\mu(T, x)$. Then

$$\mu(T, \lambda) = \lambda^{|V(T)|} \sum_{r \geq 0} (-1)^r p(T, r) \left(\lambda^{-k}\right)^r = 0.$$ 

Thus, $\{\lambda^{-k}\}_{e \in E(T)}$ is a root of $\tilde{\mu}(T, \{x_e\}_{e \in E(T)})$. By Lemma 3.10, there exists a subtree $F$ of $T$ such that $F$ is generalized $\lambda^{-k}$-normal. Hence $\lambda$ is an eigenvalue of $A(F)$ by Lemma 3.5. Moreover, by Corollary 3.3, $\lambda$ is an eigenvalue of $A(T)$. Therefore, every nonzero root of $\mu(T, x)$ is an eigenvalue of $A(T)$. This completes the proof. \[\square\]

We are now ready to prove Theorem 1.3.

**Proof of Theorem 1.3.** Note that the first statement of Theorem 1.3 is exactly Theorem 3.12. Thus we only need to prove that $\lambda(T)$ is a root of $\mu(T, x)$ and equals to the spectral radius of the adjacency tensor of $T$.

If $\lambda$ is a root of $\mu(T, x)$, then $\lambda$ is an eigenvalue of $A(T)$ by Theorem 3.12. Thus $|\lambda| \leq \rho(T)$ by the definition of spectral radius. Hence we have $\lambda(T) \leq \rho(T)$. We next prove that $\rho(T)$ is a root of $\mu(T, x)$.

By Lemma 3.1, $\rho(T)$ is an eigenvalue of $A(T)$ with an corresponding eigenvector having all elements positive. Applying this fact to Lemma 3.5, we obtain that $H$ is generalized $\alpha$-normal with $\alpha = \rho(T)^{-k}$. Moreover, [50, Lemma 3.1] suggests that if $F$ is generalized $\{\alpha_e\}_{e \in E(F)}$-normal, then $\{\alpha_e\}_{e \in E(F)}$ is a root of $\tilde{\mu}(F, \{x_e\}_{e \in E(F)})$. We conclude that $\{\rho(T)^{-k}\}_{e \in E(T)}$ is a root of the polynomial

$$\tilde{\mu}(T, \{x_e\}_{e \in E(T)}) = \sum_{M \in M(T)} (-1)^{|M| \prod_{e \in E(M)} x_e}.$$ 

This implies that $\rho(T)$ is a root of the polynomial

$$x^{|V(T)|} \sum_{M \in M(T)} (-1)^{|M| x^{-k|M|} = \mu(T, x)},$$

as desired. This completes the proof. \[\square\]

4. The Distribution of Zeros of the Matching Polynomial

This section is devoted to studying the distribution of zeros of the matching polynomial. We mainly prove that every zero of $\mu(H, x)$ is an eigenvalue of the adjacency tensor of $T(H, u)$ and the cyclic index of $\mu(H, x)$ is exactly equal to $k$. Furthermore, we show that $\lambda(H)$, the maximum modulus of all zeros of $\mu(H, x)$ is a simple root of $\mu(H, x)$ and is exactly the spectral radius of the
adjacency tensor of $T(H, u)$. Using these facts, we provide the lower and upper bounds of $\lambda(H)$ and complete the proof of Theorem 1.4.

4.1. Zeros of the matching polynomials of $k$-graphs. Theorems 1.1 and 2.8 establish a bridge between the zeros of the matching polynomial of $H$ and the eigenvalues of the path tree $T(H, u)$. Next, we will prove that every zero of $\mu(H, x)$ is an eigenvalue of the adjacency tensor of $T(H, u)$ and the cyclic index of $\mu(H, x)$ is equal to $k$.

**Lemma 4.1** ([45]). Let $H$ be a $k$-graph. Then,

$$
\frac{d}{dx} \mu(H, x) = \sum_{v \in V(H)} \mu(H - v, x).
$$

**Theorem 4.2.** Let $H$ be a connected $k$-graph and $u \in V(H)$. Then every zero of $\mu(H, x)$ is an eigenvalue of $A(T(H, u))$. Moreover, $\lambda(H)$ is a simple root of $\mu(H, x)$ and equals to $\rho(T(H, u))$.

**Proof.** By Theorem 2.8, $\mu(H, x)$ divides $\mu(T(H, u), x)$, so every zero of $\mu(H, x)$ is a zero of $\mu(T(H, u), x)$. By Lemma 2.4 (1), $T(H, u)$ is a tree. Thus every zero of $\mu(T(H, u), x)$ is an eigenvalue of $A(T(H, u))$ by Theorem 1.3. We therefore deduce that every zero of $\mu(H, x)$ is an eigenvalue of $A(T(H, u))$. This proves the first statement.

We next prove that $\lambda(H)$ is a root of $\mu(H, x)$ and equals to $\rho(T(H, u))$. By Theorem 2.8, there exists a proper subforest $F$ of $T(H, u)$ such that

$$
(4.1) \quad \mu(H, x)\mu(F, x) = \mu(T(H, u), x),
$$

which implies that $\lambda(T(H, u)) = \max\{\lambda(H), \lambda(F)\}$. By Lemma 3.1, we have $\rho(F) < \rho(T(H, u))$. Besides, by Theorem 1.3, we have $\lambda(F) = \rho(F)$ and $\lambda(T(H, u)) = \rho(T(H, u))$. Thus $\lambda(F) < \lambda(T(H, u))$. We conclude that

$$
(4.2) \quad \lambda(H) = \lambda(T(H, u)) = \rho(T(H, u)).
$$

It remains to prove that $\lambda(H)$ is a simple root of $\mu(H, x)$. For every $v \in V(H)$, we claim $\lambda(H) > \lambda(H - v)$. Let $H_1, \ldots, H_s$ be the components of $H - v$. Without loss of generality, one may assume

$$
(4.3) \quad \lambda(H_1) = \max_{1 \leq i \leq s} \lambda(H_i).
$$

By (2.1), we have $\lambda(H - v) = \lambda(H_1)$. By (4.2), we further have $\lambda(H) = \rho(T(H, v))$ and $\lambda(H - v) = \lambda(H_1) = \rho(T(H_1, v_1))$, where $v_1$ is a vertex of $V(H_1)$ such that $v_1 \in N_H(v)$. Observe that $T(H_1, v_1)$ is a proper subtree of $T(H, v)$. Thus $\rho(T(H, v)) > \rho(T(H_1, v_1))$ by Lemma 3.1. This implies $\lambda(H) > \lambda(H - v)$, as desired.

Note that the leading coefficient of $\sum_{v \in V(H)} \mu(H - v, x)$ is positive. It follows from above claim that $\sum_{v \in V(H)} \mu(H - v, x)$ is positive whenever $x \geq \lambda(H)$. Therefore, $\lambda(H)$ is not a root of $\frac{d}{dx} \mu(H, x)$ by Lemma 4.1, and hence the root $\lambda(H)$ of $\mu(H, x)$ is simple. This completes the proof of the theorem.

As an application of Theorem 4.2, we have the following theorem.

**Theorem 4.3.** Let $H$ be a connected $k$-graph. Then the cyclic index of $\mu(H, x)$ is equal to $k$.

**Proof.** For the $k$-th root of unity $\xi$, one can check the following equalities:

$$
\mu(H, \xi) = \sum_{r \geq 0} (-1)^r p(H, r)(\xi x)^{|V(H)|-kr} = \xi^{|V(H)|} \mu(H, x).
$$

This implies that $\mu(H, x)$ is $k$-symmetric, so $k \leq c$, where $c$ is the cyclic index of $\mu(H, x)$.

Since $\mu(H, x)$ is $c$-symmetric and $\lambda(H)$ is a simple root of $\mu(H, x)$, we have that $\lambda(H)e^{2\pi i}, j = 0, 1, \ldots, c-1$ are zeros of $\mu(H, x)$. By Theorem 4.2, they are eigenvalues of a path tree $T(H, u)$ of $H$. 


with modulus $\rho(T(H, u))$. Let $d$ be the cyclic index of the characteristic polynomial of $A(T(H, u))$. By Theorem 2.6 and Eq. (2.7) in [9], $T(H, u)$ has exactly $d$ distinct eigenvalues with modulus $\rho(T(H, u))$. Thus, $d \geq c$. Furthermore, [9, Corollary 4.3] states that $d|k$ for any $k$-graph. Therefore, $d = k = c$, that is, the cyclic index of $\mu(H, x)$ is exactly $k$. \[ \square \]

**Remark 4.4.** From the proof of Theorem 4.3, we also get that $\mu(H, x)$ has exactly $k$ distinct zeros with modulus $\lambda(H)$ and they are equally distributed on complex plane, equivalently, they are $\lambda(H)e^{\frac{2\pi j}{k}}$, $j = 0, 1, \ldots, k - 1$. We therefore conclude that $\mu(H, x)$ is $\ell$-symmetric if and only if $\ell$ divides $k$.

4.2. The largest zero of the matching polynomial. In this subsection, we present lower and upper bounds for $\lambda(H)$ and complete the proof of Theorem 1.4.

**Lemma 4.5.** Let $H$ be a connected $k$-graph. For any subgraph $G$ of $H$, we have $\lambda(G) \leq \lambda(H)$ with equality holds if and only if $G = H$.

**Proof.** Without loss of generality, we may assume that $G$ is connected. Otherwise, we can get the result by considering each components of $G$. Let $u$ be a vertex of $G$. Since $G$ is a subgraph of $H$ containing $u$, $T(G, u)$ is a subgraph $T(H, u)$. By Theorem 4.2 and Lemma 3.1, we have

$$\lambda(G) = \rho(T(G, u)) \leq \rho(T(H, u)) = \lambda(H)$$

with equality holds if and only if $T(G, u) = T(H, u)$. Obverse that $T(G, u) = T(H, u)$ if and only if $G = H$. The result follows. \[ \square \]

**Corollary 4.6.** Let $H$ be a connected $k$-graph with maximum degree $\Delta$. Then $\lambda(H) \geq \Delta^\frac{k}{2}$ with equality if and only if all edges of $H$ share a common vertex.

**Proof.** Let $S_\Delta$ be the $k$-star with maximum degree $\Delta$, that is, a graph consisting of $\Delta$ edges sharing a common vertex. Clearly,

$$\mu(S_\Delta, x) = x^{\Delta k - \Delta - k + 1}(x^k - \Delta),$$

and thus $\lambda(S_\Delta) = \Delta^\frac{k}{2}$.

Let $u$ be a vertex of $H$ with $d_H(u) = \Delta$. Then, $T(H, u)$ contains $S_\Delta$ as a subgraph. By (4.2) and Lemma 4.5, we have

$$\lambda(H) = \lambda(T(H, u)) \geq \lambda(S_\Delta) = \Delta^\frac{k}{2}$$

with equality holds if and only if $T(H, u) = S_\Delta$. If all edges of $H$ have a common vertex $u$, then $T(H, u) = S_\Delta$ by the definition of path trees. Conversely, if $T(H, u) = S_\Delta$, then $H$ has exactly $\Delta$ edges as $|E(H)| = |E(T(H, u))| = \Delta$ and $\Delta(H) = \Delta$, which implies that all edges of $H$ have a common vertex. \[ \square \]

**Lemma 4.7.** Let $H$ be a connected $k$-graph with maximum degree $\Delta$ and $u$ be a vertex. For any integer $\xi \geq \max\{\Delta, 2\}$, if $d_H(u) < \xi$, then the inequality

$$\frac{\mu(H, x)}{\mu(H - u, x)} > ((k - 1)(\xi - 1))^\frac{k}{\xi}$$

holds for $x \geq \frac{\Delta}{k}(k - 1)(\xi - 1)^\frac{1}{\xi}$.

**Proof.** We prove the statement by induction on $n = |V(H)|$. If $n = k$, then $H$ is the graph consisting of a single edge and thus one may verify that

$$\frac{\mu(H, x)}{\mu(H - u, x)} = \frac{x^k - 1}{x^{k-1}} = x - \frac{1}{x^{k-1}} > ((k - 1)(\xi - 1))^\frac{k}{\xi}$$
holds for \( x \geq \frac{k}{k-1}((k-1)(\xi - 1))^{\frac{1}{k}} \), since
\[
x - \frac{1}{x^{k-1}} \geq \frac{k}{k-1}((k-1)(\xi - 1))^{\frac{1}{k}} - \frac{1}{(\xi k^{k-1} - 1)^{\frac{k-1}{k}}}.
\]
\[
= ((k-1)(\xi - 1))^{\frac{1}{k}} \left( 1 + \frac{1}{k-1} \left( 1 - \frac{(k-1)^{k-1}}{(\xi - 1)k^{k-1}} \right) \right)
\]
\[
> ((k-1)(\xi - 1))^{\frac{1}{k}}.
\]

We now assume that \( n \geq k + 1 \). It follows from the connectedness of \( H \) that \( \Delta \geq 2 \), and thus \( \Delta \leq \xi \). For any edge \( e = \{u, v_2, \ldots, v_k\} \in E_H(u) \), let \( V_i(e) = \{u, v_2, \ldots, v_i\} \) for each \( i \geq 2 \) and \( V_1(e) = \{u\} \). Note that for any \( i \in [n-1] \), we have that \( \Delta(H - V_i(e)) \leq \Delta \) and \( d_{H-V_i}(v_{i+1}) < \Delta \). By induction hypothesis and (2.2), we derive that for any \( i \in [n-1] \),
\[
\frac{\mu(H - V_{i+1}(e), x)}{\mu(H - V_i(e), x)} \leq \frac{1}{((k-1)(\xi - 1))^2}
\]
holds for \( x \geq \frac{k}{k-1}((k-1)(\xi - 1))^{\frac{1}{k}} \). Thus for any \( e \in E_H(u) \),
\[
(4.4) \quad \frac{\mu(H - e, x)}{\mu(H - u, x)} = \prod_{i=1}^{k-1} \frac{\mu(H - V_{i+1}(e), x)}{\mu(H - V_i(e), x)} \leq \frac{1}{((k-1)(\xi - 1))^2}
\]
while \( x \geq \frac{k}{k-1}((k-1)(\xi - 1))^{\frac{1}{k}} \).

By Lemma 2.6 and (4.4), combining the fact \( d_H(u) \leq \xi - 1 \), we have that
\[
(4.5) \quad \frac{\mu(H, x)}{\mu(H - u, x)} = x - \sum_{e \in E_H(u)} \frac{\mu(H - e, x)}{\mu(H - u, x)}
\]
\[
= x - \sum_{e \in E_H(u)} \prod_{i=1}^{k-1} \frac{\mu(H - V_{i+1}(e), x)}{\mu(H - V_i(e), x)}
\]
\[
> \frac{k}{k-1}((k-1)(\xi - 1))^{\frac{1}{k}} - \frac{\xi - 1}{((k-1)(\xi - 1))^2}
\]
\[
= ((k-1)(\xi - 1))^{\frac{1}{k}}
\]
holds for \( x \geq \frac{k}{k-1}((k-1)(\xi - 1))^{\frac{1}{k}} \). This completes the proof. \( \Box \)

**Theorem 4.8.** Let \( G \) be a connected \( k \)-graph with maximum degree \( \Delta \geq 2 \). Then
\[
\lambda(G) < \frac{k}{k-1}((k-1)(\Delta - 1))^{\frac{1}{k}}.
\]

**Proof.** Let \( H \) be a \( k \)-graph with maximum degree \( \Delta \geq 2 \). By Theorem 4.2, \( \lambda(H) \) is the largest real zero of \( \mu(H, x) \), so it suffices to show that \( \mu(H, x) > 0 \) for \( x \geq \frac{k}{k-1}((k-1)(\Delta - 1))^{\frac{1}{k}} \). We prove it by induction on \( n \). Let \( u \) be a vertex of \( H \) with \( d_H(u) = \delta(H) \). For \( n = k + 1 \), it is clear that \( H \) consists of two edges sharing \( k - 1 \) vertices. Thus we have
\[
\mu(H - u, x) = x^{k-1} > 0
\]
when \( x \geq \frac{k}{\kappa}(k - 1)(\Delta - 1))^{\frac{1}{k}} > 1 \). In addition, by Lemma 2.6,

\[
\frac{\mu(H, x)}{\mu(H - u, x)} = x - \sum_{e \in E_H(u)} \frac{\mu(H - e, x)}{\mu(H - u, x)} \\
= x - \sum_{e \in E_H(u)} \frac{x}{x^k - 1}
\]

\[
\geq \frac{k}{k - 1} ((k - 1)(\Delta - 1))^{\frac{1}{k}} \left(1 - \frac{\Delta}{(k - 1)(\Delta - 1) - 1}\right) \\
= \frac{k}{k - 1} ((k - 1)(\Delta - 1))^{\frac{1}{k}} \left(\frac{k^k(\Delta - 1) - (k - 1)^{k-1}(\Delta + 1)}{k^k(\Delta - 1) - (k - 1)^{k-1}}\right)
\]

\[
\geq 0
\]

for \( x \geq \frac{k}{\kappa}(k - 1)(\Delta - 1))^{\frac{1}{k}} \). This implies the statement for \( n = k + 1 \).

Assume \( n > k + 1 \). For any edge \( e = \{u, v_2, \ldots, v_k\} \in E_H(u) \), denote \( V_i(e) = \{u, v_2, \ldots, v_i\} \) for each \( i \geq 2 \) and \( V_1(e) = \{u\} \). By Lemma 4.7 and the fact \( d_{H-V_i}(v_{i+1}) < \Delta \), for any \( i \in [n - 1] \), the inequality

\[
\frac{\mu(H - V_{i+1}(e), x)}{\mu(H - V_i(e), x)} \leq \frac{1}{((k - 1)(\Delta - 1))^{\frac{k}{k}}}
\]

holds for \( x \geq \frac{k}{\kappa}(k - 1)(\Delta - 1))^{\frac{1}{k}} \). Thus, for any edge \( e \in E_H(u) \),

\[
\frac{\mu(H - e, x)}{\mu(H - u, x)} = \prod_{i=1}^{k-1} \frac{\mu(H - V_{i+1}(e), x)}{\mu(H - V_i(e), x)} \leq \frac{1}{((k - 1)(\Delta - 1))^\frac{k-1}{k}}
\]

when \( x \geq \frac{k}{\kappa}(k - 1)(\Delta - 1))^{\frac{1}{k}} \). By Lemma 2.6 and (4.6),

\[
\frac{\mu(H, x)}{\mu(H - v, x)} = x \frac{\mu(H - v, x) - \sum_{e \in E_H(v)} \mu(H - e, x)}{\mu(H - v, x)} \\
= x - \sum_{e \in E_H(v)} \frac{\mu(H - e, x)}{\mu(H - v, x)}
\]

\[
> \frac{k}{k - 1} ((k - 1)(\Delta - 1))^{\frac{1}{k}} - \frac{\Delta}{((k - 1)(\Delta - 1))^\frac{k-1}{k}}
\]

\[
= \frac{k(\Delta - 1) - \Delta}{((k - 1)(\Delta - 1))^\frac{k-1}{k}}
\]

\[
\geq 0
\]

when \( x \geq \frac{k}{\kappa}(k - 1)(\Delta - 1))^{\frac{1}{k}} \). Note that \( \mu(H - v, x) \neq 0 \) for any \( x \geq \frac{k}{\kappa}(k - 1)(\Delta - 1))^{\frac{1}{k}} \) by induction hypothesis. The result follows.

We now have all the tools to prove Theorem 1.4.

**Proof of Theorem 1.4.** Let \( H \) be a connected \( k \)-graph with maximum degree \( \Delta \geq 2 \). By Theorem 4.3, the cyclic index of \( \mu(H, x) \) is \( k \), and by Theorem 4.2, \( \lambda(H) \) is a simple root of \( \mu(H, x) \). Finally, the inequality (1.3) follows from Corollary 4.6 and Theorem 4.8. Thus the proof is completed. \( \square \)
5. Concluding Remarks

In this paper, we present a fundamental characterization of the distribution of zeros of the matching polynomials of k-graphs and generalize some results on the classical matching polynomial to k-graphs. Note that most of the results in this paper can be generalized to the multivariate weighted k-graphs, a k-graph $H = (V, E)$ associated with an edge-weighted function $w : E \rightarrow \mathbb{C}$ and a vertex-indeterminate $x = \{x_v\}_{v \in V}$, with some appropriate adjustment. For the sake of simplicity, we chose not to pursue that direction in detail.

There is another interesting function related to the matching polynomial, the matching generating function of a k-graph $H$, which is defined by

$$m(H, x) = \sum_{r \geq 0} p(H, r)x^r.$$

Note that

$$\mu(H, x) = \sum_{r \geq 0} (-1)^r p(H, r)x^{|V(H)|-kr} = x^{|V(H)|} \sum_{r \geq 0} p(H, r)(-x^{-k})^r.$$

Thus we have

$$\mu(H, x) = x^{|V(H)|}m(H, -x^{-k}).$$

Therefore, we can obtain some results similar to Theorem 2.7 and Theorem 2.8 for the matching generating functions.

As mentioned in Section 1, the result of Li and Mohar [31] indicates that for a connected k-graph $H$ with maximum degree $\Delta$, the threshold bound

$$k \left( (k - 1)(\Delta - 1) \right)^{\frac{1}{k - 1}}$$

plays an important role in studying the second eigenvalue of $H$. Besides, Theorem 1.5 states that this value is exactly an upper bound of the spectral radius of a connected k-tree with maximum degree $\Delta \geq 2$. In fact, by combining Theorem 4.2 and Theorem 1.5, we may obtain another proof for Theorem 1.4. Therefore, in the current setting, Theorem 1.4 can be viewed as a new version of Theorem 1.5 from the view point of matching polynomials. The main idea of [39] seems to imply that the former is more essential than the latter in the study of the second eigenvalues of hypergraphs and Ramanujan hypergraphs.

A sequence $a_0, a_1, \cdots, a_n$, of real numbers is said to be logarithmically concave (or log-concave for short) if $a_i^2 \geq a_{i-1}a_{i+1}$ for all $1 \leq i \leq n - 1$. Many important sequences in combinatorics are known to be log-concave. We refer the reader to a survey by Stanley [44] for various examples and more background. Applying the rooted-rootedness of the matching polynomial in Theorem 1.2, Heilmann and Lieb [24] prove that the matching number sequence $\{p(G, r)\}_{r \geq 0}$ of a graph $G$ is log-concave. However, for $k \geq 3$, the rooted-rootedness for matching polynomials of k-graphs is invalid as proved in Theorem 1.4. Thus, it would be interesting to study the log-concave property of the matching number sequence of a k-graph.

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