System Identification Algorithm for Non-Uniformly Sampled Data

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Abstract

Considerable effort has been devoted to the development of algorithms for identification of parsimonious discrete time models from noisy input/output data sets since this facilitates controller design. Several methods, such as nuclear norm minimization, have been used to provide approximate solutions to this non-convex problem. However, even though the field of continuous time system identification is now mature, results on parsimonious model identification of continuous time systems are still very limited. In this paper, an atomic norm minimization method is proposed for this purpose that can handle non-uniformly sampled data without preprocessing. The proposed approach provides an efficient way to use noisy, non-uniformly sampled data to determine a reliable, low-order continuous time model. Numerical performance is illustrated using academic examples and simulated behavioral data from a smoking cessation study.

Keywords

Continuous time system identification; non-uniformly sampled data; parsimonious system identification; randomized system identification algorithm

1. INTRODUCTION

Continuous time models have generally been preferred to analyze the characteristics of the systems Rao and Unbehauen (2006). Moreover some more recent applications, such as the design of medical/behavioral treatment Bekiroglu et al. (2016); Lagoa et al. (2014), have

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significant challenges that can only be effectively addressed by using continuous time models. For instance, the behavioral process data sets (e.g. Bekiroglu et al. (2016); Lagoa et al. (2014)), measured with ecological momentary assessment (EMA), contain a significant amount of noise and non-uniform samples. Since, to develop efficient personalized treatments, one needs a model that approximates subject behavior, there is a need for the development of system identification algorithms that can accommodate noisy, sparse, non-uniform measurements.

Motivated by these difficulties, we develop a computationally efficient approach to the problem of continuous time system identification aimed at obtaining low-order models. More precisely, given noisy and non-uniformly sampled data, we develop an algorithm that utilizes both available a-priori information and such data sets to identify “simple” models. This proposed approach is based on the idea of describing the response of a Linear Time Invariant (LTI) continuous time system as a linear sum of suitably chosen atoms. Existing results (see Yilmaz et al. (2013); Bekiroglu et al. (2014); Shah et al. (2012)) have shown, in a discrete time setting and for a suitable set of atoms, that minimizing the atomic norm subject to bounds on approximation error does result in the identification of low-order models that are compatible with the collected data.

Motivated by these results, in this paper, we propose an atomic norm approach to the problem of low-order continuous time system identification. To be able to efficiently solve the resulting optimization problem, the following assumptions are made: i) the system’s poles are located in a known compact region of a complex plane and ii) the input is a (known) finite combination of exponentials. The second assumption is only needed for developing efficient closed form expressions of some quantities used in the algorithm. Note that the second assumption is not restrictive since in many settings (including, for example, medical/behavioral treatment studies) the inputs are low complex ones such as combinations of steps and/or impulses; e.g., see Bekiroglu et al. (2016); Lagoa et al. (2014). Finally, we propose a randomized Frank-Wolfe algorithm to efficiently solve the resulting optimizations problems.

1.1 Previous Work

There are two main time domain system identification algorithms in the literature for continuous time systems Rao and Unbehauen (2006). i) The indirect approach: First the discrete time model is identified with uniformly sampled input/output data and then this model is converted to the continuous time model, see e.g., Ohta and Kawai (2004); Mercère et al. (2007). This approach is attractive since the available data are discrete in general, and continuous time system identification is a mathematically difficult problem. However, although converting a discrete time model to a continuous time model is theoretically possible with a bilinear transformation, such process leads to a numerical problem when the sampling rate is high. Furthermore uniformly sampled data are required for this indirect approach but not required for our proposed method. ii) The direct approach: In this approach, the continuous time model is directly identified from discrete data by approximating the derivative to construct input-output equation. In other words, the higher order derivatives are approximated with a state variable filter or a causal, stable, realizable
linear operator, see e.g. Johansson (2009); Garnier (2011); Young (1981). Recently it has been shown that this method can handle non-uniformly sampled data if the filtered data are sampled uniformly Garnier (2011). Different than the direct approach, the method proposed uses the available non-uniformly sampled data without filtering or derivative estimation to estimate the uniformly sampled impulse response of the continuous system.

Sparse system identification is mainly studied for discrete time models (see, e.g. Yilmaz et al. (2013)). However recently a sparse continuous time model identification algorithm has been considered under the uniform sampling requirement Yue et al. (2016). Different than Yue et al. (2016), in this paper, it is shown that the sparse continuous time system identification problem can be solved very efficiently from non-uniformly sampled data.

The paper is organized as follows. In section 2 the notation, the system identification problem, and a brief discussion on input assumption are given. Section 3 presents the convex relaxation of the proposed problem. Section 4 discusses the iterative randomized algorithm to solve the relaxed optimization problem with a discussion on the estimation of uniformly sampled impulse response. In section 5, examples are presented.

2. PRELIMINARIES

2.1 Notation

Scalars are denoted by lower case letters (e.g., x), vector-valued signals by lowercase boldface letters (e.g., \( \mathbf{x} \in \mathcal{L}^p \)), where \( \mathcal{L}^p \), \( 0 \leq p \leq \infty \) denotes the usual \( \mathcal{L}^p \) space with the norm \( \| \cdot \|_p \), transpose of the vector is denoted as (e.g., \( \mathbf{x}^T \)), the real numbers by \( \mathbb{R} \), the complex numbers by \( \mathbb{C} \), and the exponential for each element of vector \( \mathbf{x} \) by \( e^x \). For a signal \( \mathbf{x}, \mathbf{x}(t_i) \) denotes the value of the signal at time \( t_i \). For a complex number \( p \), \( \text{Re}(p) \) denotes the real part of \( p \) and \( \text{Im}(p) \) denotes the imaginary part of \( p \). Also \( p^* \) represents the complex conjugate of complex number \( p \). For an origin centered closed half disc in left half plane in \( \mathbb{C} \) with radius \( \rho \) is denoted by \( \{ \rho \in \mathbb{C} \text{ or } \rho \in \mathbb{R} : \| \rho \| \leq \rho \text{ and } \text{Re}(\rho) \leq 0 \} \). \( \text{conv}(S) \) represents the convex hull of the set \( S \). Furthermore the convolution of the signals \( g \) and \( u \) is denoted by \( g * u \). The inverse Laplace transform is denoted by \( \mathcal{L}^{-1} \).

2.2 Problem Statement

In this section, the system identification problem and assumptions for our algorithm are presented. Assuming initial conditions are equal to zero and the observations are corrupted by noise; i.e., the output of the system is given by

\[
y(t) = (g * u)(t) + \varepsilon(t) = \int_{-\infty}^{\infty} g(t, \tau) u(\tau) d\tau + \varepsilon(t),
\]

(1)

where \( g \) represents the impulse response of \( G(s) \), \( u \) represents the input signal, and \( y(t) \) is output at time \( t \) corrupted by some noise \( \varepsilon \). However, available experimental measurements consist of
where time domain data $y(t_i)$ is the measured output at non-uniformly sampled time $t_i$ corrupted by some noise $e$. To represent the non-uniform measured time, the following vector is defined

$$t = \{t_0, t_1, t_2, \ldots, t_m\}, \text{ where } t_0 < t_1 < t_2 < \cdots < t_m.$$ 

**Problem:** Given

- set $\mathbb{S}_\rho = \{ p \in \mathbb{C} \text{ or } p \in \mathbb{R} : |p| \leq \rho \text{ and } \Re(p) \leq 0 \}$ which contains the poles of the system to be identified,
- input signal $u$ with “simple” closed-form description,
- output noisy measurements $y(t_i)$, and
- measurement noise bound $e$, (e.g., $\|e\|_2 \leq e_{\max}$),

determine the most parsimonious model $\tilde{G}(s)$ that approximates the true plant $G(s)$ while explaining available input-output data within the given bound on estimation error.

**Remark 1**—(On Stability). The compact set $\mathbb{S}_\rho$ covers only the stable poles. The unstable poles are omitted since some of the unstable poles’ responses might grow very fast and cause a numerical problem.

**Remark 2**—(On Zero Initial Condition). The nonzero initial condition can be effortlessly identified with a slight modification on the atomic norm used in this paper. System identification with a nonzero initial condition is actually a block sparsity problem. Again for clarity of exposition, we concentrate on the zero initial condition case.

### 2.3 Discussion on Input Assumption

As it is discussed in the introduction, continuous-time system identification algorithms can be mainly classified into two categories. One is an indirect identification algorithm based on subspace methods (e.g. Ohta and Kawai (2004)). Another is a direct identification algorithm category based on the approximation of differentiation (e.g. Johansson (2009)). In the direct estimation, the differential operator needs to be approximated because of implementation problems. Here we have the same problem: how to calculate the convolution integral. Therefore, without having uniformly sampled data, the calculation of convolution is not possible. However, if one has a simple closed form for the input, the convolution integrals can be easily computed.

### 2.4 Optimization Problem

Any finite dimensional LTI system with proper rational transfer function $G(s)$ whose poles are in $\mathbb{S}_\rho$ can be approximated via partial fractional expansion
Therefore the identification problem that we address in this paper can be formulated as follows: Let \( g_p(t) \) be defined as

\[
G(s) \approx \frac{B(s)}{A(s)} = \sum_{i=1}^{n} \frac{c_i}{s - p_i}, \quad p_i \in \mathbb{S}_\rho \text{ and } c_i \in \mathbb{C}. \tag{2}
\]

Solve

\[
\min_c \text{cardinality}\{c : c \neq 0\} \tag{3}
\]

\[
\text{s.t.} \sum_{i=0}^{m} \left[ \sum_{p \in \mathbb{S}_\rho} c_p (g_p * u)(t_i) - y(t_i) \right]^2 \leq \varepsilon.
\]

Although the solution of the optimization problem above gives the most parsimonious model that is consistent with available input/output data sets, there are reasons why we cannot solve it directly.

1. Maximizing sparsity subject to constraints is an NP-hard problem Donoho (2006).
2. The number of the poles in \( \mathbb{S}_\rho \) is (uncountably) infinite.

In the next sections, we present a convex relaxation of the optimization problem (3) and a continuous time system identification algorithm based on atomic norm minimization aimed at addressing these challenges.

3. CONVEX RELAXATION

In this section we provide a “tight” convex relaxation of the problem described above. As a first step, we start by defining the sets of elementary impulse and input responses.

3.1 Set of Elementary Impulse Responses

We start by defining a set of elementary impulse response that is used in the atomic norm based relaxation of the parsimonious system identification problem. Although the ones in (2) could in principle be used, the fact they do not take directly into account that one has real valued impulse responses results in numerical problems. Hence, we propose the following set of real valued elementary impulse responses
\[ \mathcal{F}_1 = \{ \pm \alpha_p^1 (e^{pt} + e^{-yt}) : p \in \mathbb{S}_p \text{ and } p \in \mathbb{C} \} \]

\[ \mathcal{F}_2 = \{ \pm \alpha_p^2 (-je^{pt} + je^{-yt}) : p \in \mathbb{S}_p \text{ and } p \in \mathbb{C} \} \]

\[ \mathcal{F}_3 = \{ \pm \alpha_p^3 (e^{pt}) : p \in \mathbb{S}_p \text{ and } p \in \mathbb{R} \}, \]

where \( \alpha_p \)s are scaling factors. Due to the space limitation only the formulas for scaling factors are presented here, and the mathematical reasoning of this scaling factors and their calculations can be found in Yilmaz et al. (2013). Basically the scaling factors are used to normalize the Hankel nuclear norm of each element in the set \( \mathcal{F} \). This leads to

\[ M = \frac{\text{Re} \left( \frac{I - e^{2it_m p}}{1 - e^{2ip}} \right) - \frac{1 - \left| e^{2it_m p} \right|^2}{1 - \left| e^p \right|^2} - \text{Re} \left( e^{2ip} (e^{2it_m p} \ast \frac{I - e^{2it_m p}}{1 - e^{2ip}}) \right) + \frac{|e^{2it_m p}|^2 + 2 \frac{1 - |e^{2it_m p}|^2}{1 - |e^p|^2}}}{1 - |e^p|^2}, \]

\[ \text{Pd} = 2M \left[ \frac{1 - e^{2it_m p}}{1 - e^{2ip}} \right] - \left[ \frac{1 - \left| e^{2it_m p} \right|^2}{1 - \left| e^p \right|^2} \right]^2 \]

\[ \alpha_p^1 = \left( \frac{2 \left( \text{Re} \left( \frac{1 - e^{2it_m p}}{1 - e^{2ip}} \right)^2 \right) + \left( \frac{1 - \left| e^{2it_m p} \right|^2}{1 - \left| e^p \right|^2} \right)^2 + 2 \sqrt{\text{Pd}}}{2} \right)^{-1} \]

\[ \alpha_p^2 = \left( \frac{2 \left( \left( \frac{1 - \left| e^{2it_m p} \right|^2}{1 - \left| e^p \right|^2} \right)^2 - \text{Re} \left( \left( \frac{1 - e^{2it_m p}}{1 - e^{2ip}} \right)^2 \right) \right) + 2 \sqrt{\text{Pd}}}{2} \right)^{-1} \]

\[ \alpha_p^3 = \frac{1 - e^{2ip}}{1 - e^{p(2t_m + 2)}}, \quad (4) \]

where \( t_m \) is the time that the final sample is obtained. Note that if the terms \( e^{2ip} \) and \( |e^p|^2 \) are equal to 1, then we perturb these terms slightly to deal with the numerical problem. Finally, define

\[ \mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_3. \]
At this point, one should note that choosing the “right” scaling factors is still an open problem. However, in our experience, the choice described above works well in practice.

3.2 Atom Set

We are now ready to define the set of atoms that is used in our approach. This set of atoms is composed of the responses of elementary systems to the input at the sample times. More precisely, the atom set is defined as

\[ \mathcal{A} = \{ a \in \mathbb{R}^{m+1} : a = [(u * h)(t_0)(u * h)(t_1) \cdots (u * h)(t_m)]^T, h \in \mathcal{H} \} \]

where \((u * h)(t)\) is the value of the convolution of \(u\) and \(h\) at time \(t\). Note that, given the fact that we assume the knowledge of a “simple” closed-form description of the input, elements of \(\mathcal{A}\) are easily computable.

Finally, one can define a map between the elements of \(\mathcal{A}\) and \(\mathcal{K}\) in closed form. This map is denoted by \(\mathcal{K} : \mathcal{A} \rightarrow \mathcal{K}\) and satisfies

\[ h = \mathcal{K}(a) \text{ if } a_i = (h * \delta)(t_{i-1}) \text{ for all } i = 1, 2, \ldots, m+1. \quad (5) \]

3.3 Convex Relaxation of the System Identification Problem

Given the discussion in Section 2.4, the output of a finite dimensional linear time invariant system can be arbitrarily approximated by linear combinations of elementary input responses. In other words, we have

\[ y = [y(t_0)y(t_1) \cdots y(t_m)]^T \approx \sum_{a \in \mathcal{A}} c_\alpha a \]

where equality holds if the system does not have any repeated poles. Hence the problem of parsimonious system identification can be formulated as finding estimated system input response

\[ x = \sum_{a \in \mathcal{A}} c_\alpha a \]

and corresponding impulse response

\[ g = \sum_{a \in \mathcal{A}} c_\alpha \mathcal{K}(a) \]

such that \(x\) closely matches the collected output data and the least number of elements of \(\mathcal{A}\) are used in the expressions above. In order to obtain a “tight” convex relaxation of such problem, we consider the following atomic norm
With this definition at hand, and given the vector of observed outputs $y = [y(t_0) \ y(t_1) \ \cdots \ y(t_m)]^T$, a natural convex relaxation of the parsimonious system identification problem is

$$\min_x \frac{1}{2} \| y - x \|_2^2$$

s.t. $\| x \|_a^f \leq \tau$ \hspace{1cm} (6)

$$x = \sum_{a \in \mathcal{A}} c_a a.$$ 

4. IDENTIFICATION ALGORITHM

The optimization problem (6) is a particular case of problems of the generic atomic norm constraint convex problem studied in Yilmaz et al. (2013):

$$\min_x f(x)$$ \hspace{1cm} (7)

subject to $\| x \|_a^f \leq \tau$,

where $f(x)$ is a convex and smooth function.

However, there are two reasons why we cannot directly apply well known algorithms, such as Frank-Wolfe algorithm, to the problem of parsimonious system identification: i) the atomic norm used in our formulation, to the best of our knowledge, does not have a computable closed form and ii) optimization problem (6) provides an non-uniformly sampled estimate of the response of the system which is not suitable for identification.

To overcome this limitation, in this paper we present a modification of the algorithm proposed in Yilmaz et al. (2013) that addresses the concerns mentioned above. Introducing a randomized search for the atoms allows us to have a convergent feasible estimate without having to compute the value of the atomic norm at each iteration. The second concern is addressed by having two variables that are simultaneously updated by the algorithm:

- $x_k$ - estimate of the system’s response at the (non-uniform) sample times $t_0, t_1, \cdots, t_{m}$

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• \[ \hat{g}_k = [\hat{g}_k(0) \ \hat{g}_k(T_s) \ \hat{g}_k(2T_s) \cdots \hat{g}_k(NT_s)] \] - evenly sampled estimate of the system’s impulse response.

The sampling interval \( T_s \) used in the definition of \( \hat{g}_k \) needs to be small enough as to avoid aliasing; i.e., \( T_s \) should be small enough so that one can uniquely recover \( \hat{g}_k(t) \) from its samples \( \hat{g}_k(nT_s) \). Given the definition of the set of allowable poles \( \mathcal{S}_\rho \) for the impulse response, one can easily compute a bound on \( T_s \). The result below is a direct consequence of the Nyquist sampling theorem and the proof is omitted for the sake of brevity.

**Lemma 1**

Assume that a signal \( g(t) \) has all of its poles in the set \( \mathcal{S}_\rho \). Then we can uniquely recover \( g(t) \) from its samples \( g(nT_s) \), \( n = 0, 1, \ldots \) if

\[
T_s \leq \frac{\pi}{\rho}.
\]

We are now ready to present the proposed identification algorithm:

**Algorithm 1**

Randomized Frank-Wolfe Algorithm

```
1: \( x_{initial} \leftarrow a_0 \) for arbitrary \( a_0 \in \mathcal{A} \) and \( \hat{g}_{initial} \leftarrow \tau g_0 \) where \( \hat{g}_0 = [h_0(0)h_0(T_s) \cdots h_0(NT_s)]^T \), \( h_0 = \mathcal{K}(a_0) \)
2: for \( k = 0,1,2,3,\ldots \) do
3:     Randomly pick uniformly \( p_k \in \mathcal{S}_\rho \).
4:     Let \( \mathcal{A}(p_k) \) be set of all atoms with a pole at \( p_k \).
5:     \( a_k \leftarrow \arg\min_{a \in \mathcal{A}(p_k)} \langle \nabla f(x), a \rangle \) \quad \( a_k \in \mathcal{A} \)
6:     \( h_k \leftarrow \mathcal{K}(a_k) \) \quad \( h_k \in \mathcal{A} \)
7:     \( a_{k+1} \leftarrow \min_{\alpha \in [0,1]} f(x_k + \alpha [a_k - x_k]) \)
8:     \( x_{k+1} \leftarrow x_k + a_k [a_k - x_k] \)
9:     \( \hat{g}_{k+1} \leftarrow \hat{g}_k + a_k [h_k(T_s) \cdots h_k(NT_s)]^T - \hat{g}_k \)
10: end for
```

This is computationally very efficient because:

- the iterations in the algorithm only entail inner products with \( x \) and \( a \) and the computation of the gradient with respect to \( x \), a computational burden proportional to the number of sampling times \( m \),
- the estimate of the impulse response \( \hat{g}_k \) does not play any role in any of the computations except by being updated once the atom \( a_k \) is chosen,
- the computation of the value of the output of the map \( \mathcal{K} \) in step 6 is trivial since we know the pole(s) of \( a_k \).
Remark 3

Note that uniformly sampled impulse response needs to be long enough to identify the continuous time model. Therefore, only a sufficient number of measures need to be stored in the memory for Ho’s algorithms.

Next, each step of the algorithm is explained. In step 1, a random atom is picked and scaled by \( \tau \) to initialize a random input response \( x \) and an impulse response \( \bar{g} \). Then a random search for an atom that gives the steepest descent direction is employed in steps 3–5. More specifically, in step 3 a random pole is picked in \( S_\rho \) and the corresponding sampled atoms for this pole are calculated in step 4. In step 5 the gradient \( (\nabla f(x_k) = (x_k - y)) \) is calculated for the objective function in (6). Note that the non-uniformly sampled output \( y \) and its approximation \( x_k \) are used in this step, especially for gradient calculation. The uniformly sampled atom \( a_k \) does not play a role in these calculations. Then the random non-uniformly sampled atoms chosen in step 4 are checked with this gradient vector, and the optimum atom \( a_k \) is found very efficiently. Also the \( h_k \) at iteration \( k \) is calculated based on the pole of \( a_k \).

Finding the optimum step size \( \alpha \in [0, 1] \) at step 7 requires a second order polynomial minimization, where the closed form formula for the optimum \( \alpha^* \) is

\[
\alpha^* = \max(0, \min(\alpha_u, 1)) \quad \text{where} \\
\alpha_u = \frac{(x_k - y)^T (\tau a_k - x_k)}{(x_k - y)^T (x_k - y)}.
\]

Finally a state space realization of the system can be obtained from this estimated uniformly sampled impulse response \( \bar{g} \) via Ho’s algorithm (or via subspace methods).

Theorem 1

(Yılmaz et al. (2013)) The randomized Frank-Wolfe algorithm satisfies \( f(x_k) \rightarrow f^* \) with probability 1 as \( k \rightarrow \infty \).

5. EXAMPLES

In this section, the performance of the proposed method is illustrated by utilizing a synthetic data (step response) and simulated behavioral data from smoking cessation study.

5.1 Step Response Example

In this example, a second-order continuous time system is randomly generated while ensuring the poles of the system are located in \( S_5 \). The step response of this system is created by utilizing a nonuniform time vector that is generated as an increasing random walk. Then the non-uniformly sampled step response is contaminated by normally distributed noise \( e \in A(0, 0.1) \). For this example, the atoms are chosen from the set of the step responses \( A \). The sampling rate for impulse response is set at 0.01 second (\( t_s = 100 \geq 5/
\( \tau \)). In this example, we chose \( \tau = 2.3 \). The measured, true, and estimated step responses are depicted in figure 1.

The true and estimated impulse signals are given in figure 2.

5.2 Smoking Cessation Trial Example

The intervention design for behavioral problems is usually handled by employing some statistical tools to generalize the solution for all patients. However, dynamical modeling allows researchers to inform development of a person-based intervention instead of finding a generalized treatment. Clinicians could design a personalized treatment for each patient (e.g., using model predictive control, as in Bekiroglu et al. (2016)) if its dynamical model is known. On the other hand, as mentioned in the introduction, most behavioral data have some remarkable challenges for system identification. In this study, we develop a system identification algorithm that could overcome these difficulties. This is very important for personalized intervention design because if one knows the model of the subject response, the reaction to the treatment can be investigated by analyzing the impulse response of the patient specific models.

In this example, a simulated patient EMA data from a behavioral study is utilized to illustrate how the proposed algorithm could be used to estimate a model that approximates patient response from this type of data sets. We used data generated by a statistical algorithm which closely resembles collected data from smoking cessation studies. In this simulated study, a treatment is given randomly, and the measurements are collected while treatment is being provided. One of the major problems in the modeling of behavioral processes is defining the inputs and outputs. In the literature, treatments are usually assumed to be an input, and then the effects of this input can be measured with different outcomes (i.e., Bekiroglu et al. (2016) and references therein). Therefore, in this paper we assumed that the treatment is given constantly (step) as an input for this specific patient and the effects of the treatment is measured on craving with a single item on an 11-point response scale (e.g. see Figure 3). Finally, estimated impulse response is given in Figure 4. In this example, we chose \( \tau = 3 \) and \( \rho = 4 \).

In figure 3, the non-uniformly simulated data and its step response estimation are given. In figure 4, the estimated impulse response of this specific person is plotted. This impulse response shows that the treatment has an impact on craving once it is given to the patient, and its effect fades within approximately 4 hours. Therefore, the clinicians could give a second set of treatments to the patient after first 4 hours based on this impulse response analysis.

6. CONCLUSION

In this study, a new efficient parsimonious continuous time system identification method is introduced. Different from available approaches, this method uses non-uniformly sampled data directly to provide a parsimonious estimate of the impulse response of a continuous time system. An atomic norm approach is employed to promote sparsity. A set of atoms for a response of an input with known modes is designed for LTI system identification, and a
randomized version of the Frank-Wolfe algorithm is modified to solve the proposed optimization problem. Our results show that one can denoise and estimate the impulse response of a continuous time system reliably and efficiently from a noisy and non-uniformly sampled output data. The proposed approach provides a systematic way of estimating a model from a patient’s medical/behavioral data. This is exemplified with a simulated behavioral data set. Future work should investigate an atomic set for nonzero initial conditions. Systematic procedures for determining the value of $\tau$ from a priori information on the data also need to be investigated.

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Fig. 1. Step Responses
Fig. 2.
Impulse Responses
Fig. 3.
Treatment Responses
Fig. 4.
Estimated Impulse Responses