Tunable Measures for Information Leakage and Applications to Privacy-Utility Tradeoffs

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Abstract

In the first half of the paper, we introduce a tunable measure for information leakage called maximal $\alpha$-leakage. This measure quantifies the maximal gain of an adversary in refining a tilted version of its posterior belief of any (potentially random) function of a data set conditioning on a released data set. The choice of $\alpha$ determines the specific adversarial action ranging from refining a belief for $\alpha = 1$ to guessing the best posterior for $\alpha = \infty$, and for these extremal values maximal $\alpha$-leakage simplifies to mutual information and maximal leakage, respectively. For $\alpha \in (1, \infty)$ this measure is shown to be the Arimoto channel capacity. We show that maximal $\alpha$-leakage satisfies data processing inequalities and sub-additivity (composition property). In the second half of the paper, we use maximal $\alpha$-leakage as the privacy measure and study the problem of data publishing with privacy guarantees, wherein the utility of the released data is ensured via a hard distortion constraint. Unlike average distortion, hard distortion provides a deterministic guarantee of fidelity. We show that under a hard distortion constraint, both the optimal mechanism and the optimal tradeoff are invariant for any $\alpha > 1$, and the tunable leakage measure only behaves as either of the two extrema, i.e., mutual information for $\alpha = 1$ and maximal leakage for $\alpha = \infty$.

Index Terms

Mutual information, maximal leakage, maximal $\alpha$-leakage, Sibson mutual information, Arimoto mutual information, $f$-divergence, Privacy-utility tradeoff, hard distortion.

I. INTRODUCTION AND OVERVIEW

The measure and control of private information leakage is a recognized objective in communications, information theory, and computer science. Modern cryptography [1]–[3], for example, aims at designing and analyzing security systems that are believed to be impervious to computationally bounded adversaries. Alternatively, information-theoretic security studies settings wherein the utility of information between an adversary and the legitimate parties (e.g., the wiretap channel [4]–[6]) can be exploited to guarantee that no private information is leaked regardless of computational assumptions. An adversary that only observes the output of a (computationally) secure cipher or cannot overcome the information asymmetry in a wiretap-like setting does not, for all practical purposes, pose a privacy risk.

However, modern applications such as online data sharing, social networks, cloud-based services, and mobile computing have significantly increased the number of interfaces through which private information can leak. Services that require a user to disclose data in order to receive utility inevitably incur a privacy risk through unwanted inferences. For example, political preference can be reliably estimated from movie ratings [7], an online store can infer a medical condition by observing your shopping history [8], or social network users can be de-anonymized by tracking their interaction with peers [9], [10]. Moreover, practical implementations of cryptographic schemes are susceptible to so-called “side-channel attacks,” where sensitive information leaks through unexpected channels. For example, a malicious application may get timing characteristics [10], [12]. In these examples, an adversary that observes information leaking through a side channel can more reliably infer private data, such as a key or a plaintext.

In this paper, we provide two new metrics called $\alpha$-leakage and maximal $\alpha$-leakage that quantify information leakages through the lens of adversarial inference capabilities. Specifically, $\alpha$-leakage captures an adversary’s ability in inferring a specific private attribute in the dataset, and in contrast, maximal $\alpha$-leakage is for any arbitrary attribute of the dataset. These metrics can be applied to the aforementioned privacy and side-channel settings, and directly capture an adversary’s ability to infer (ranging from the most likely realization to the posterior distribution) for any information of original data from the released version or the one leaked via side-channels.

We evaluate the proposed measures of information leakage by using them as privacy metrics in privacy-guaranteed data publishing settings. In most non-trivial settings of data publishing, there is a fundamental tradeoff between privacy and utility: on the one hand, releasing data “as is” can lead to unwanted inferences of private information. On the other hand, perturbing or limiting the released data reduces its quality. The exact nature of the privacy-utility tradeoff (PUT) will depend to varying degrees on the distribution of the underlying data, as well as the chosen privacy metrics (e.g., differential privacy (DP) [13], [14], mutual information (MI) [15], probability of correctly guessing [16], [17], $f$-divergence-based leakage measures [18], [19], mutual information (MI) [15], probability of correctly guessing [16], [17], $f$-divergence-based leakage measures [18], [19].

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and maximal leakage (MaxL) [19]. Furthermore, most information-theoretic PUTs capture utility as a statistical average of desired measures [13]–[16], [18].

In contrast to statistical utilities, we introduce a new hard distortion metric to measure utility, which constrains the privacy mechanism so that the distortion between original and released datasets is bounded with probability 1. The concept of deterministic/hard utility has been considered in the form of ρ-recoverable functions in [17]. Differently, we bound the distortion of data itself instead of data functions, which naturally guarantees some recoverability of any arbitrary data functions. In addition, compared to average-case distortion constraints [20], the hard distortion metric is quite stringent but allows the data curator to make specific, deterministic guarantees on the fidelity of the released dataset to the original one. The deterministic guarantee can lead to more accurate statistical estimators, e.g., the empirical distribution estimation.

Using the aforementioned tunable measures of information leakage and hard distortion as privacy and utility measures, respectively, we precisely quantify the PUT for datasets that are entirely sensitive or contain both non-sensitive and sensitive private data, as shown in Figs. 1a and 1b, respectively.

A. Contributions and Organization

Our main contributions consist of the following:

• We introduce a tunable loss function, namely ρ-loss (1 ≤ ρ ≤ ∞), which captures logarithmic loss (log-loss) and 0-1 loss, respectively, for extremal values of ρ = 1 and ∞, respectively (Sec. III-A);

• Based on ρ-loss, we define two operational measures of information leakage: ρ-leakage and maximal ρ-leakage, and show that ρ-leakage is Arimoto mutual information and maximal ρ-leakage is MI for ρ = 1 and Arimoto channel capacity for ρ > 1. Note that maximal ρ-leakage captures MI and MaxL at extrema (Sec. III-B);

• Inspired by the fact that maximal ρ-leakage equals to the Arimoto channel capacity, we introduce a broader class of information-leakage measures based on f-divergences, which capture maximal ρ-leakage as a special case (Sec. III-C);

• We prove that maximal ρ-leakage satisfies several useful properties, including: (i) quasi-convexity, (ii) data-processing inequalities: post-processing inequality and linkage inequality, (iii) sub-additivity (iv) additivity for memoryless mappings (Sec. IV);

• In the context of privacy-guaranteed data publishing subject to a hard distortion utility constraint on data, we solve the resulting PUT problems exactly for maximal ρ leakage as well as its f-divergence-based variants (Sec. V-A), which restrict leakages about any arbitrary function of original data as shown in Fig. 1a. Focusing on ρ-leakage, which restricts leakage about specific sensitive data as shown in Fig. 1b, we provide an inner bound of the optimal PUT (Sec. V-B). In Sec. VI, we illustrate these results via two examples.

II. Preliminaries

We use capital letters to represent discrete random variables, and the corresponding capital calligraphic and lower-case letters represent their finite supports and the elements of the supports, respectively. For example, for a random variable X, its support is X with any possible realization x ∈ X. In addition, we use to log represent the natural logarithm, and [a, b] indicate the set of integers from a to b. We use ∥·∥ to indicate the cardinality of a set, e.g., |X|, and ∥·∥p to represent the p-norm of a vector, e.g., for α ≥ 1, ∥PX∥α = (Σx∈X Pα(x))1/α.

We begin by reviewing Rényi entropy and divergence [21], [22].
Definition 1. Given a distribution $P_X$, the Rényi entropy of order $\alpha \in (0,1) \cup (1, \infty)$ is defined as

$$H_\alpha(P_X) = \frac{1}{1-\alpha} \log \sum_x P(x)^\alpha,$$

and

$$= \frac{\alpha}{1-\alpha} \log \|P_X\|_\alpha, \quad (\alpha \geq 1).$$

Let $Q_X$ be a distribution over the support of $P_X$. The Rényi divergence (between $P_X$ and $Q_X$) of order $\alpha \in (0,1) \cup (1, \infty)$ is defined as

$$D_\alpha(P_X\|Q_X) = \frac{1}{\alpha-1} \log \left( \sum_x \frac{P_X(x)^\alpha}{Q_X(x)^{\alpha-1}} \right).$$

Both of the two quantities are defined by their continuous extensions for $\alpha = 1$ and $\infty$. Specifically, for $\alpha = \infty$, the two quantities are given by

$$H_\infty(P_X) = \min_x \log \frac{1}{P_X(x)},$$

which is called min-entropy, and

$$D_\infty(P_X\|Q_X) = \log \max_x \frac{P_X(x)}{Q_X(x)}.$$

For $\alpha = 1$, the Rényi entropy and divergence reduce to Shannon entropy and Kullback-Leibler divergence, respectively.

The $\alpha$-leakage and maximal $\alpha$-leakage metrics can be expressed in terms of Sibson MI [23] and Arimoto MI [24]. These quantities generalize the usual notion of MI. We review these definitions next.

Definition 2. Let discrete random variables $(X,Y) \sim P_{X,Y}$ with $P_X$ and $P_{Y\mid X}$ as the marginal and conditional distributions, respectively, and $Q_Y$ be an arbitrary distribution over the finite support $Y$. The Sibson mutual information of order $\alpha \in (0,1) \cup (1, \infty)$ is defined as

$$I_\alpha^S(X;Y) \triangleq \inf_{Q_Y} D_\alpha(P_{X,Y}\|P_X \times Q_Y) = \frac{\alpha}{\alpha-1} \log \sum_y \left( \sum_x P_X(x)P_{Y\mid X}(y\mid x)^\alpha \right)^{\frac{1}{\alpha}}.$$

The Arimoto mutual information of order $\alpha \in (0,1) \cup (1, \infty)$ is defined as

$$I_\alpha^A(X;Y) \triangleq H_\alpha(X) - H_\alpha^A(X\mid Y) = \frac{\alpha}{\alpha-1} \log \frac{\sum_y \left( \sum_x P_{X,Y}(x,y)^\alpha \right)^{\frac{1}{\alpha}}}{\sum_x \left( \sum_y P_X(x)^\alpha \right)^{\frac{1}{\alpha}}} = \frac{\alpha}{\alpha-1} \log \frac{\sum_y \|P_{X,Y}(\cdot,y)\|_\alpha}{\|P_X\|_\alpha}, \quad (\alpha \geq 1)$$

where $H_\alpha^A(X\mid Y)$ is Arimoto conditional entropy of $X$ given $Y$ defined as

$$H_\alpha^A(X\mid Y) = \frac{\alpha}{1-\alpha} \log \sum_y \left( \sum_x P_{X,Y}(x,y)^\alpha \right)^{\frac{1}{\alpha}}.$$

All of these quantities are defined by their continuous extension for $\alpha = 1$ or $\infty$.

Note that for $\alpha = 1$, both Sibson and Arimoto MIs reduce to Shannon’s MI; however, for $\alpha = \infty$, the Sibson MI is

$$I_\infty^S(X;Y) = \log \sum_y \max_x P_{Y\mid X}(y\mid x),$$

and the Arimoto MI is given by

$$I_\infty^A(X;Y) = \log \frac{\sum_y \max_x P_{X,Y}(x,y)}{\max_x P_X(x)}.$$
The expected loss in this case is the conditional cross-entropy, given by
\[ p = P_{X|Y}(x|y) \] with an observation \( Y = y \) and \( p = P_{X}(x) \) without any observation.

### III. Tunable Loss Function and Information Leakage Measures

In this section, we introduce a tunable loss function, namely \( \alpha \)-loss for \( \alpha \in [1, \infty] \), which simplifies to the logarithmic loss (log-loss) and 0-1 loss at the two extrema. Viewing information leakage through the lens of adversarial inference capabilities, we quantify the leakage via \( \alpha \)-loss, which the adversary intends to minimize, and define two tunable measures of information leakage, called \( \alpha \)-leakage and maximal \( \alpha \)-leakage, respectively.

#### A. \( \alpha \)-Loss Function

For a Markov chain \( X - Y - \hat{X} \), let \( \hat{X} \) be an estimator of \( X \) and \( P_{\hat{X}|Y} \) be a strategy for estimating \( X \) from \( Y \). We denote the probability of correctly estimating \( X = x \) given \( Y = y \) as \( P_{\hat{X}|Y}(x|y) \). The estimation strategy \( P_{\hat{X}|Y} \) is selected in order to minimize an expected loss metric. Denoting the loss function by \( \ell(x, y, P_{\hat{X}|Y}) \), the expected loss is given by \( \mathbb{E}[\ell(X, Y, P_{\hat{X}|Y})] \).

One formulation of the loss function is the probability of incorrectly guessing given by
\[ \ell_{0-1}(x, y, P_{\hat{X}|Y}) = 1 - P_{\hat{X}|Y}(x|y), \tag{12} \]
such that the expected loss \( \mathbb{E}[\ell_{0-1}(X, Y, P_{\hat{X}|Y})] \) is the expected probability of error. Here, the optimal strategy \( P^*_{\hat{X}|Y} \) is the standard maximal posterior (MAP) estimator given by
\[ P^*_{\hat{X}|Y}(x|y) = \begin{cases} 1, & x = \arg\max_{x \in \mathcal{X}} P_{X|Y}(x|y) \\ 0, & \text{otherwise} \end{cases}, \tag{13} \]
which makes the loss \( \ell_{0-1}(x, y, P^*_{\hat{X}|Y}) \) be either 0 or 1, and therefore, called 0-1 loss in the literature [27], [28]. The corresponding expected loss \( \mathbb{E}[\ell_{0-1}(X, Y, P^*_{\hat{X}|Y})] \) is the minimal expected probability of error.

To measure the uncertainty for the strategy \( P^*_{\hat{X}|Y} \), the log-loss (used, for example, in [27], [29]–[31]) is given by
\[ \ell_{\log}(x, y, P^*_{\hat{X}|Y}) = \log \frac{1}{P^*_{\hat{X}|Y}(x|y)}. \tag{14} \]
The expected loss in this case is the conditional cross-entropy, given by
\[ \mathbb{E}\left[\ell_{\log}(X, Y, P^*_{\hat{X}|Y})\right] = \sum_{x,y} P_{X,Y}(x, y) \log \frac{1}{P^*_{\hat{X}|Y}(x|y)}, \tag{15} \]
\[ = H(X|Y) + \sum_{y} P_Y(y) D(P_{X|Y=y} \| P^*_{\hat{X}|Y=y}). \tag{16} \]
Therefore, the optimal strategy is the true posterior distribution of \( X \) given \( Y \), i.e., \( P^*_{\hat{X}|Y} = P_{X|Y} \), which makes the expected loss in (16) become the conditional entropy \( H(X|Y) \). That is, the minimal expected log-loss is the true conditional entropy.
Note that both the 0-1 loss and log-loss functions are decreasing in the probability of correctly estimation $P_{X|Y}(x|y)$. Specifically, for $P_{X|Y}(x|y) = 1$, both the values of 0-1 loss and $\alpha$-loss are 0, and for $P_{X|Y}(x|y) = 0$, the values of 0-1 loss and log-loss become 1 and $\infty$, respectively. To allow a continuous quantification of the loss for $P_{X|Y}(x|y) = 0$ from 1 to $\infty$, we formally define a tunable loss function, namely $\alpha$-loss, as follows.

**Definition 3 ($\alpha$-loss).** Let random variables $X$, $Y$ and $\hat{X}$ form a Markov chain $X - Y - \hat{X}$, where $\hat{X}$ is an estimator of $X$. The $\alpha$-loss of the strategy $P_{\hat{X}|Y}$ for estimating $X$ from $Y$ is

$$\ell_\alpha(x, y, P_{\hat{X}|Y}) = \frac{\alpha}{\alpha - 1} \left(1 - P_{\hat{X}|Y}(x|y)\right),$$

where $\alpha \in (1, \infty)$. It is defined by its continuous extension for $\alpha = 1$ and $\alpha = \infty$, respectively, and is given by

$$\ell_1(x, y, P_{\hat{X}|Y}) = \lim_{\alpha \to 1} \ell_\alpha(x, y, P_{\hat{X}|Y}) = \log \frac{1}{P_{\hat{X}|Y}(x|y)},$$

$$\ell_\infty(x, y, P_{\hat{X}|Y}) = \lim_{\alpha \to \infty} \ell_\alpha(x, y, P_{\hat{X}|Y}) = 1 - P_{\hat{X}|Y}(x|y).$$

Note that for $\alpha = 1$, the expression in (18) follows directly from the L'Hôpital's rule and $\alpha$-loss becomes the log-loss in (14); and for $\alpha = \infty$, the loss in (19) is exactly the 0-1 loss in (12). Fig. 2 plots the $\alpha$-loss function in (17) for different values of $\alpha$. From Fig. 2, we observe that $\alpha$-loss function is decreasing and convex in the probability of correctly guessing.

**Lemma 1.** For $1 \leq \alpha \leq \infty$, the minimal expected $\alpha$-loss is given by

$$\min_{P_{\hat{X}|Y}} \mathbb{E} \left[\ell_\alpha(X, Y, P_{\hat{X}|Y})\right] = \begin{cases} \frac{\alpha}{\alpha - 1} \left(1 - \exp\left(\frac{1-\alpha}{\alpha} H^A_\alpha(X|Y)\right)\right), & \alpha > 1 \\ H(X|Y), & \alpha = 1 \end{cases},$$

with the optimal estimation strategy given by

$$P^*_X(x|y) = \frac{P_{X|Y}(x|y)^\alpha}{\sum_{x \in X} P_{X|Y}(x|y)^\alpha}.$$

A detailed proof is in Appendix A. Note that in (20), $H^A_\alpha(X|Y)$ is Arimoto conditional entropy of $X$ given $Y$ in (9). For $\alpha = \infty$, the expression of $H^A_\infty(X|Y)$ is

$$H^A_\infty(X|Y) = \log \sum_y P_Y(y) \max_x P_{X|Y}(x|y),$$

such that $\exp(H_\infty(X|Y))$ is the maximal expected probability of correctly guessing $X$ from $Y$. Therefore, for $\alpha = \infty$, the minimal expected $\alpha$-loss is the minimal expected probability of error. In addition, the optimal estimation strategy in (21)
becomes the true posterior distribution of $X$ for $\alpha = 1$ and the MAP estimator for $\alpha = \infty^1$, respectively.

**Example 1.** Let the conditional probability distribution of $X$ given $Y = y$ be a binomial distribution with parameters $(n, p) = (20, 0.5)$, i.e., $P_{X|Y}(x|y) = \binom{20}{x}0.5^x0.5^{20-x}$ for $x \in [0, 20]$. Fig. 3 shows the optimal strategies in (21) for different values of $\alpha$. We observe from Fig. 3 that as $\alpha$ grows from 1 to $\infty$, the optimal strategy gradually eliminates the less likely values of $X$ (given $y$) and transforms from the true posterior distribution to the MAP estimator.

### B. $\alpha$-Leakage and Maximal $\alpha$-Leakage

Let $X$ and $Y$ represent the original data and released data, respectively, and let $U$ represent an arbitrary (potentially random) function of $X$ that the observer (a curious or malicious user of the released data $Y$) is interested in learning. In [19], Issa et al. introduced MaxL to qualify the maximal gain in an adversary’s ability of guessing $U$ after observing $Y$. We review the definition below.

**Definition 4 (19, Def. 1).** Given a joint distribution $P_{X,Y}$ on finite alphabets, the maximal leakage from $X$ to $Y$ is

$$L_{\text{Max}}(X \rightarrow Y) \triangleq \sup_{U \in X-Y} \log \frac{\max_{\hat{U}} E \left[ P_{\hat{U}|Y}(U|Y) \right]}{\max_u P_U(u)},$$

(23)

where $\hat{U}$ represents an estimator taking values from the same arbitrary finite support as $U$.

Note that the numerator of the logarithmic term in (23) is the maximal expected probability of correctly guessing $U$ with $Y$ given by

$$\max_{\hat{U}} E \left[ P_{\hat{U}|Y}(U|Y) \right] = \max_u \sum_y P_Y(y)P_{\hat{U}|Y}(u|y),$$

(24)

which is exactly the complement of the minimal expected 0-1 loss in guessing $U$ with $Y$. Similarly, the denominator is the complement of the minimal expected 0-1 loss in guessing $U$ without $Y$. Therefore, MaxL is a leakage measure related to 0-1 loss in (12).

In addition, in Def. 4, $U$ represents any (possibly random) function of $X$. The numerator represents the maximal probability of correctly guessing $U$ based on $Y$, while the denominator represents the maximal probability of correctly guessing $U$ without knowing $Y$. Thus, MaxL quantifies the maximal logarithmic gain in guessing any possible function of $X$ when an adversary has access to $Y$.

Analogously to the derivation of MaxL from 0-1 loss, we introduce $\alpha$-leakage and maximal $\alpha$-leakage based on $\alpha$-loss (under the assumptions of discrete random variables and finite supports). The formal definitions are as follows.

**Definition 5 ($\alpha$-Leakage).** Given a joint distribution $P_{X,Y}$ and an estimator $\hat{X}$ with the same support as $X$, the $\alpha$-leakage from $X$ to $Y$ is defined as

$$\mathcal{L}_\alpha(X \rightarrow Y) \triangleq \frac{\alpha}{\alpha - 1} \log \frac{\max_{\hat{X}} E \left[ P_{\hat{X}|Y}(X|Y)^{\frac{\alpha}{\alpha - 1}} \right]}{\max_{\hat{X}} E \left[ P_{\hat{X}}(X)^{\frac{\alpha}{\alpha - 1}} \right]},$$

(25)

for $\alpha \in (1, \infty)$ and by the continuous extension of (25) for $\alpha = 1$ and $\infty$.

Whereas $\alpha$-leakage captures how much an adversary can learn about $X$ from $Y$, we also wish to quantify the information leaked about any function of $X$ through $Y$. To this end, we define maximal $\alpha$-leakage below.

**Definition 6 (Maximal $\alpha$-Leakage).** Given a joint distribution $P_{X,Y}$ on finite alphabets $X \times Y$, the maximal $\alpha$-leakage from $X$ to $Y$ is defined as

$$L_{\alpha}^\text{max}(X \rightarrow Y) \triangleq \sup_{U \in X-Y} \mathcal{L}_\alpha(U; Y),$$

(26)

$$= \sup_{U \in X-Y} \lim_{\alpha' \rightarrow \alpha} \frac{\alpha'}{\alpha' - 1} \log \frac{\max_{\hat{U}} E \left[ P_{\hat{U}|Y}(U|Y)^{\frac{\alpha'}{\alpha' - 1}} \right]}{\max_{\hat{U}} E \left[ P_{\hat{U}}(U)^{\frac{\alpha'}{\alpha' - 1}} \right]},$$

(27)

where $1 \leq \alpha \leq \infty$, and $U$ represents any function of $X$ and takes values from an arbitrary finite alphabet.

Note that for $\alpha \geq 1$,

$$\max_{\hat{U}} E \left[ P_{\hat{U}|Y}(U|Y)^{\frac{\alpha}{\alpha - 1}} \right] = 1 - \frac{1}{\alpha} \min_{\hat{U}} E \left[ \ell_\alpha(U, Y, P_{\hat{U}|Y}) \right].$$

(28)

$^1$Note that if there are more than one realizations sharing the same maximal posterior belief, for $\alpha = \infty$ the optimal strategy in (21) will output these most likely values with the same probability.
Thus, there is a similar connection between maximal $\alpha$-leakage and $\alpha$-loss (in Def. 3) as that observed in (24) between $\text{MaxL}$ and 0-1 loss, and maximal $\alpha$-leakage quantifies an adversary’s capability to infer any function of data $X$ from the released $Y$.

Making use of the result in Lemma 1, the following theorem simplifies the expression of $\alpha$-leakage in (25).

**Theorem 1.** For $1 \leq \alpha \leq \infty$, $\alpha$-leakage defined in (25) simplifies to

$$\mathcal{L}_\alpha(X \rightarrow Y) = I_\alpha^a(X; Y).$$

(29)

From (28) and Lemma 1, we simplify the scaled logarithm of the ratio in (25) to Arimoto MI. A detailed proof is in Appendix B, where we show that Arimoto conditional entropy and Rényi entropy capture the inference uncertainties of an adversary for knowing $Y$ or not, respectively, and $\alpha$-leakage measures the decrease in the inference uncertainty by knowing $Y$.

Making use of the conclusion in Thm. 1, the following theorem gives equivalent expressions for maximal $\alpha$-leakage.

**Theorem 2.** For $1 \leq \alpha \leq \infty$, the maximal $\alpha$-leakage defined in (26) simplifies to

$$\mathcal{L}_\alpha^{\text{max}}(X \rightarrow Y) = \begin{cases} \sup_{P_X\in \mathcal{P}(X)} I_\alpha^a(\tilde{X}; Y), & 1 < \alpha \leq \infty \\ I(X; Y), & \alpha = 1 \end{cases}$$

(30a)

$$\mathcal{L}_\alpha^{\text{max}}(X \rightarrow Y) = \sup_{U \in \mathcal{P}(U)} I_\alpha^a(U; Y) \quad \alpha \in [1, \infty].$$

(30b)

where $P_X$ is a probability distribution over the support of $P_X$.

Note that maximal $\alpha$-leakage is essentially the Arimoto channel capacity (with a support-set constrained input distribution) for $\alpha > 1$ [24], which is used to characterize probabilities of decoding error for scenarios in which transmission rates are higher than channel capacity. The limit of maximal $\alpha$-leakage for $\alpha = 1$ gives the Shannon channel capacity. Recall that the limit of $\alpha$-loss in (17) leads to the log-loss (for $\alpha = 1$) and 0-1 loss (for $\alpha = \infty$) functions, respectively. Consequently, for $\alpha = 1$ and $\infty$, maximal $\alpha$-leakage simplifies to MI and $\text{MaxL}$, respectively.

A detailed proof for Thm. 2 is in Appendix C. We summarize key steps in the proof as follows: by applying Thm. 1, we write maximal $\alpha$-leakage as

$$\mathcal{L}_\alpha^{\text{max}}(X \rightarrow Y) = \sup_{U \in \mathcal{P}(U)} I_\alpha^a(U; Y) \quad \alpha \in [1, \infty].$$

(31)

For $\alpha = 1$, Arimoto MI is simply the Shannon’s MI, and combining with the data processing inequalities, (31) simplifies to $I(X; Y)$. Note that for $\alpha > 1$, Arimoto MI does not satisfy data processing inequalities. By using the facts that Arimoto MI and Sibson MI have the same supremum [25, Thm. 5] and that Sibson MI satisfies data processing inequalities [25, Thm. 3], we upper bound the supremum of (31) by $\sup_{P_X} I_\alpha^a(X; Y)$, and then, show that the upper bound can be achieved by a specific $U$ with $H(X|U) = 0$.

**Example 2.** Given a binary channel

$$P_{Y|X} = \begin{bmatrix} 1 - \rho_1 & \rho_1 \\ \rho_2 & 1 - \rho_2 \end{bmatrix},$$

(32)

where $\rho_1, \rho_2 \in [0, 1]$ are the crossover probabilities, maximal $\alpha$-leakage in (30) is given by

$$\mathcal{L}_\alpha^{\text{max}}(X \rightarrow Y) = \frac{\alpha}{\alpha - 1} \log \left( \left( 1 - \rho_1 \right)^\alpha \left( 1 - \rho_2 \right)^\alpha - \rho_1^\alpha \rho_2^\alpha \right) \frac{1}{\alpha - 1} + \left( 1 - \rho_1 \right)^\alpha - \rho_1^\alpha \right);$$

(33)

If $\rho_1 = \rho_2$, (33) simplifies to

$$\mathcal{L}_\alpha^{\text{max}}(X \rightarrow Y) = \frac{1}{\alpha - 1} \log \left( 1 - \rho_1 \right)^\alpha + \rho_1^\alpha \right) + \log 2;$$

(34)

which is exactly the $\alpha$-leakage for the binary symmetric channel with the uniform input distribution. Fig. 4 plots the values of maximal $\alpha$-leakage for example channels where $\rho_1 = \rho_2$ and $\rho_1 \neq \rho_2$, and shows that the ordering of leakages for the two channels varies with $\alpha$.

C. Leakage Measures Based on $f$-Divergence

We introduce two classes of information leakages derived from $f$-divergence, called $f$-leakage and maximal $f$-leakage. The $f$-leakage depends on the distribution of original data, and in contrast, maximal $f$-divergence only depends on the support of original data. We also show the relation between the $f$-divergence-based measures and maximal $\alpha$-leakage for $\alpha = 1$ and $\alpha > 1$, respectively.

Recall that for a convex function $f: \mathbb{R}_+ \rightarrow \mathbb{R}$ such that $f(1) = 0$, an $f$-divergence $D_f$ is a measure of the distance between two distributions given by

$$D_f(P_Y \| Q_Y) = \sum_y Q(y) f \left( \frac{P(y)}{Q(y)} \right).$$

(35)
Theorem 3. For Sibson MI [23, 25, 26].

convexity in the conditional distribution

inequality including (i) non-negativity [25, Sec. II-A], (ii) quasi-convexity

For discrete random variables

Given a joint distribution

Definition 7. Given a joint distribution

and the maximal f-leakage is defined as

where

Note that in Definition 7, maximal f-leakage ($L_f^{\text{max}}$) depends on the distribution of $X$ only through its support. In contrast, f-leakage ($L_f$) depends fully on the distribution of $X$. Both measures depend on the chosen mechanism $P_{Y|X}$.

Recall that for $\alpha = 1$, maximal $\alpha$-leakage is MI. Therefore, it is a special case of $L_f(X \to Y)$ in (36) with $f(t) = t \log t$. Furthermore, for $\alpha > 1$, maximal $\alpha$-leakage has a one-to-one relationship with a special case of $L_f^{\text{max}}$ in (37) for $f$ given by

$$f_\alpha(t) = \frac{1}{\alpha-1}(t^{\alpha}-1),$$

such that $D_f$ is the Hellinger divergence of order $\alpha$ [32]. The following lemma makes precise this observation.

Lemma 2. For discrete random variables $X$ and $Y$, the maximal $\alpha$-leakage ($\alpha > 1$) from $X$ to $Y$ can be written as

$$L_{\alpha}^{\text{max}}(X \to Y) = \frac{1}{\alpha-1} \log (1 + (\alpha - 1) L_f^{\text{max}}(X \to Y)),$$

where $L_{\alpha}^{\text{max}}(X \to Y)$ is the $L_f^{\text{max}}(X \to Y)$ in (37) for $f_\alpha$ given by (38) such that $D_f$ is the Hellinger divergence of order $\alpha$.

A detailed proof is in Appendix D.

IV. PROPERTIES OF MAXIMAL $\alpha$-LEAKAGE

Thm. 1 shows that $\alpha$-leakage is exactly Arimoto MI, and therefore, several basic properties of $\alpha$-leakage have been shown including (i) non-negativity [25, Sec. II-A], (ii) quasi-convexity $^2$ in $P_{Y|X}$ given $P_X$ [33, Chapter 3.5], and (iii) post-processing inequality$^3$ [34, Cor. 1]. We now explore properties of maximal $\alpha$-leakage and show that its properties include: (i) quasi-convexity in the conditional distribution $P_{Y|X}$; (ii) data processing inequalities; (iii) sub-additivity (composition property [19]) and additivity for memoryless mechanisms.

The following theorem results from the expression of maximal $\alpha$-leakage in Thm. 2 as well as some known properties of Sibson MI [23, 25, 26].

Theorem 3. For $1 \leq \alpha \leq \infty$, maximal $\alpha$-leakage

$^2$For $\alpha \geq 1$ and $P_X$, the Arimoto MI $I^p_{\alpha}(X;Y)$ is the logarithm of a linear combination of the $p$-norm ($p = \alpha$) $\|P_{Y|X}(\cdot|x)\|_\alpha$. From [33, Chapter 3.5], we know a log-convex function is quasi-convex such that $I^p_{\alpha}(X;Y)$ is quasi-convex in $P_{Y|X}$ given $P_X$.

$^3$From the monotonicity of conditional Arimoto entropy [34, Cor. 1], one can derive that for a Markov chain $X - Y - Z$, $I^p_{\alpha}(X;Z) \leq I^p_{\alpha}(X;Y)$.
1. is quasi-convex in $P_{Y|X}$;
2. is monotonically non-decreasing in $\alpha$;
3. satisfies data processing inequalities: let random variables $X, Y, Z$ form a Markov chain, i.e., $X - Y - Z$, then
\[
\mathcal{L}_\alpha^{\text{max}}(X \to Z) \leq \mathcal{L}_\alpha^{\text{max}}(X \to Y) \leq \mathcal{L}_\alpha^{\text{max}}(Y \to Z).
\] (40a)
\[
\mathcal{L}_\alpha^{\text{max}}(X \to Z) \leq \mathcal{L}_\alpha^{\text{max}}(Y \to Z).
\] (40b)
4. satisfies
\[
\mathcal{L}_\alpha^{\text{max}}(X \to Y) \geq 0
\] (41)
with equality if and only if $X$ is independent of $Y$, and
\[
\mathcal{L}_\alpha^{\text{max}}(X \to Y) \leq \begin{cases} \log |\mathcal{X}| & \alpha > 1 \\ H(P_X) & \alpha = 1 \end{cases}
\] (42)
with equality if and only if $X$ is a deterministic function of $Y$.

A detailed proof is in Appendix E.

**Remark 1.** Note that:
- Since both MI and MaxL are convex in $P_{Y|X}$, $\mathcal{L}_1^{\text{max}}(X \to Y)$ and $\mathcal{L}_\infty^{\text{max}}(X \to Y)$ are convex in $P_{Y|X}$.
- From the monotonicity in Part 2, we can upper bound maximal $\alpha$-leakage as
\[
\mathcal{L}_\alpha^{\text{max}}(X \to Y) \leq \mathcal{L}_{\text{MaxL}}(X \to Y) = I_\infty^S(X; Y).
\] (43)
- The data processing inequalities in (40a) and (40b) are called post-processing inequality and linkage inequality, respectively [35], [36]. It is worth noting that not all information leakage metrics satisfy the linkage inequality [18], [36]. Examples include $\alpha$-leakage, maximal information leakage [15], probability of correctly guessing, and DP.

From Thm. 2, we know that for $\alpha > 1$, maximal $\alpha$-leakage is the supremum of Arimoto/Sibson MI over all possible distributions on the support of original data, and therefore, is a function of a conditional probability distribution. The following theorem lower bounds the supremum by a closed-form expression of the conditional probability distribution.

**Theorem 4 (Lower Bound).** For $1 < \alpha \leq \infty$, maximal $\alpha$-leakage is lower bounded by
\[
\mathcal{L}_\alpha^{\text{max}}(X \to Y) \geq \frac{\alpha}{\alpha - 1} \log \sum_{y \in \mathcal{Y}} \frac{\|P_{Y|X}(y|x)\|^\alpha}{|\mathcal{X}|^\frac{1}{\alpha}},
\] (44)
with equality if and only if for all $x_1, x_2 \in \mathcal{X}$, there is
\[
\sum_{y \in \mathcal{Y}} \frac{P_{Y|X}(y|x_1)^\alpha}{\|P_{Y|X}(y|x_1)\|^\alpha - 1} = \sum_{y \in \mathcal{Y}} \frac{P_{Y|X}(y|x_2)^\alpha}{\|P_{Y|X}(y|x_1)\|^\alpha - 1}.
\] (45)
A detailed proof is in Appendix F.

When data may be revealed multiple times (e.g., entering a password multiple times), it is essential to quantify how mechanisms designed with maximal $\alpha$ leakage compose in terms of total leakage. Consider two released versions $Y_1$ and $Y_2$ of $X$. The following theorem upper bounds maximal $\alpha$-leakage to an adversary who has access to both $Y_1$ and $Y_2$ simultaneously.

**Theorem 5 (Sub-additivity/Composition).** Given a Markov chain $Y_1 - X - Y_2$, we have ($\alpha \in [1, \infty]$)
\[
\mathcal{L}_\alpha^{\text{max}}(X \to Y_1, Y_2) \leq \sum_{i \in \{1, 2\}} \mathcal{L}_\alpha^{\text{max}}(X \to Y_i).
\] (46)
A detailed proof is in Appendix G.

The following theorem shows the additivity of maximal $\alpha$-leakage for memoryless mechanisms.

**Theorem 6 (Additivity for Memoryless Mechanisms).** For $\alpha \in [1, \infty]$ and an finite integer $n > 0$, let $X^n$ and $Y^n$ be $n$-length input and output, respectively, of a memoryless mechanism with no feedback, i.e.,
\[
P_{Y^n|X^n} = \prod_{i=1}^n P_{Y_i|X_i},
\] (47)
where $X_i$ and $Y_i$ represent the $i^{\text{th}}$ element of $X^n$ and $Y^n$, respectively, such that

\footnote{For $\alpha = \infty$, the $I_\infty^S(P_X, P_{Y|X})$ depends on the marginal distribution $P_X$ only through the support of $X$.}
(1) for \( \alpha > 1 \)
\[
\mathcal{L}_{\alpha}^{\max}(X^n \rightarrow Y^n) = \sum_{i=1}^{n} \mathcal{L}_{\alpha}^{\max}(X_i \rightarrow Y_i)
\] (48)

(2) for \( \alpha = 1 \)
\[
\mathcal{L}_{1}^{\max}(X^n \rightarrow Y^n) \leq \sum_{i=1}^{n} \mathcal{L}_{1}^{\max}(X_i \rightarrow Y_i)
\] (49)

with equality if and only if entries of \( X^n \) are mutually independent.

A detailed proof is in Appendix H.

V. PRIVACY-UTILITY TRADEOFF WITH A HARD DISTORTION CONSTRAINT

In a privacy-guaranteed data publishing setting, a data curator/provider uses a mapping called privacy mechanism to generate distorted versions of original data for releases. The privacy mechanism determines the fidelity of the released data. The higher the fidelity is, the more utility of the data is maintain, and meanwhile, the less privacy of the data is preserved. Therefore, a privacy-utility tradeoff (PUT) problem arises in the design of the privacy mechanism.

We consider the two different data publishing scenarios shown in Figs. 1a and 1b: the first where the entirety of the dataset \( X \) is considered private, and the second where the dataset consists of two parts \( X \) and \( Y \), where only \( S \) is considered private. For the first case (Fig. 1), we use maximal \( \alpha \)-leakage as the privacy metric, thereby limiting the inference of any private information about the dataset represented by the function \( U \). For the second case (Fig. 2), we use \( \alpha \)-leakage as the privacy metric, thereby limited the inference only of the specific private information represented by \( S \).

We measure utility in terms of a hard distortion metric, which constrains the privacy mechanism so that the distortion between each pair of original and released data is bounded with probability 1. Specifically, for the original and released data \( X, Y \) and a distortion function \( d(\cdot, \cdot) \), the utility guarantee is modeled as the hard distortion constraint \( d(X, Y) \leq D \) with probability 1, where \( D \) is the maximal permitted distortion. In other words, if a privacy mechanism \( P_{Y|x} \) satisfies the hard distortion constraint, for any possible input \( x \), all output \( y \) of the privacy mechanism must lie in a non-empty set \( B_D(x) \) given by
\[
B_D(x) \triangleq \{ y : d(x, y) \leq D \},
\] (50)
i.e., for any \( x \) with \( P_X(x) > 0 \), \( P_{Y|x}(y|x) = 0 \) if \( y \notin B_D(x) \). Thus, a mathematical model of the PUT problem is given by
\[
\begin{align}
\inf_{P_{Y|x} \in P_{Y|x}} & \mathcal{L}_{\alpha}(X \rightarrow Y) \\
\text{s.t.} & \quad d(X, Y) \leq D,
\end{align}
\] (51a)

where the set \( P_{Y|x} \) is the collection of stochastic matrices, and the superscript and subscript of \( \mathcal{L} \) depend on the privacy measure under consideration (see Sec. III for notation).

**Remark 2.** Note that given any input \( x \), the hard distortion constraint in (51b) will force the conditional probabilities of the outputs that are not in \( B_D(x) \) to be zero. Thus, this utility guarantee is incompatible with some privacy notions, which require each input to be mapped to all outputs with some positive probabilities; e.g., DP and any maximal \( f \)-leakage with \( f(0) = \infty \).

A. PUTs for Entirely Sensitive Datasets

For the privacy-guaranteed publishing of an entirely sensitive dataset shown in Fig. 1a, we use maximal \( \alpha \)-leakage as the privacy metric. From Section III-C, we know that maximal \( \alpha \)-leakage is a specific case of \( f \)-leakage and maximal \( f \)-leakage (in Def. 7) for \( \alpha = 1 \) and \( \alpha > 1 \), respectively. Hereby, we solve the PUT problems which minimize either \( f \)-leakage or maximal \( f \)-leakage, subject to a hard distortion constraint. By applying the relations between maximal \( \alpha \)-leakage and the \( f \)-divergence-based variants, we derive the optimal PUTs and optimal privacy mechanisms for the PUT problem with maximal \( \alpha \)-leakage as the privacy measure. We denote an optimal PUT as \( \text{PUT}_{\text{HD, } \mathcal{L}_{\alpha}} \), where HD and \( \mathcal{L}_{\alpha} \) in the subscript indicate the hard distortion and the involved privacy measure, respectively.

The following theorem characterizes the optimal PUT \( \text{PUT}_{\text{HD, } \mathcal{L}_{\alpha}} \) in (51) for the case that \( f \)-leakage is used as the privacy measure.

**Theorem 7.** For any \( f \)-leakage \( \mathcal{L}_f \) in (36) and a distortion function \( d(\cdot, \cdot) \) with \( B_D(x) \) in (50), the optimal PUT in (51) is given by
\[
\begin{align}
\text{PUT}_{\text{HD, } \mathcal{L}_{\alpha}}(D) &= \inf_{P_{Y|x} : d(X, Y) \leq D} \mathcal{L}_f(X; Y), \\
&= f(0) + \inf_{Q_Y} \mathbb{E}_{Q_Y} \left[ Q_Y(B_D(X)) \left( f\left( \frac{1}{Q_Y(B_D(X))} \right) - f(0) \right) \right].
\end{align}
\] (52)
Moreover, letting $Q^*_Y$ be the distribution achieving the infimum in (53), an optimal mechanism $P^*_Y|X$ is given by

$$P^*_Y|X(y|x) = \frac{1(d(x, y) \leq D)Q^*_Y(y)}{Q^*_Y(B_D(x))}.$$  \hfill (54)

A detailed proof in Appendix I. Note that as a result of the distribution dependence of the leakage measure $L_f$ in (36), the optimal tradeoff in (53) is an expected function of $X$.

Making use of maximal $f$-divergence as the privacy constraint, the optimal PUT in (51) is given by $PUT_{HD,L_f^\alpha}$ in the following theorem.

**Theorem 8.** For any maximal $f$-leakage $L_f^\max$ in (37), a distortion function $d(\cdot,\cdot)$ and $B_D(x)$ in (50), the optimal PUT in (51) is given by

$$PUT_{HD,L_f^\max}(D) = \inf_{P_Y|X:d(X,Y) \leq D} L_f^\max(X \rightarrow Y),$$  \hfill (55)

$$= q^* f((q^*)^{-1}) + (1 - q^*)f(0),$$  \hfill (56)

with $q^*$ defined as

$$q^* \triangleq \sup_{Q_Y} \inf_x Q_Y(B_D(x)).$$  \hfill (57)

Moreover, letting $Q^*_Y$ be the distribution achieving the supremum in (57), an optimal mechanism $P^*_Y|X$ is given by (54).

A detailed proof is in Appendix J.

**Remark 3.** The PUTs in (53) and (56) simplify to finding an output distribution $Q_Y$ that can be viewed as a “target” distribution, i.e., the optimal mechanism aims to produce this distribution as closely as possible, subject to the utility constraint. In particular, the resulting optimal mechanism (in (54)), for any input, distributes the outputs according to $Q_Y$ while conditioning the output to be within a ball of radius $D$ around the input. The optimization in (57) ensures that all inputs are uniformly masked while (53) provides average guarantees.

The next corollary characterizes the optimal tradeoff $PUT_{HD,L_f^\alpha}$ for maximal $\alpha$-leakage. Recall that for $\alpha = 1$, $L_1^\max$ equals $L_f$ with $f(t) = t \log t$. For $\alpha > 1$, from the one-to-one relationship between $L_\alpha^\max$ and $L_f^\max$ in (39), we know that finding $PUT_{HD,L_\alpha^\max}$ is equivalent to finding the optimal tradeoff $PUT_{HD,L_f^\max}$ in (55) for $L_f^\max = L_f^\max$.

**Corollary 1.** For maximal $\alpha$-leakage, the optimal PUT in (51) is given by

$$PUT_{HD,L_\alpha^\max}(D) = \inf_{P_Y|X:d(X,Y) \leq D} L_\alpha^\max(X \rightarrow Y),$$  \hfill (58)

$$= \begin{cases} 
\inf_{Q_Y} \mathbb{E} \log \frac{1}{Q_Y(B_D(X))}, & \alpha = 1 \\
- \log q^*, & \alpha > 1 
\end{cases} \hfill (59a)$$

where $q^*$ is defined in (57). Moreover, an optimal mechanism is given by (54), where for $\alpha = 1$, $Q_Y^*$ achieves the infimum in (59a); and for $\alpha > 1$, $Q_Y^*$ achieves the supremum in (57).

**Remark 4.** Note that subject to a hard distortion constraint, the optimal privacy mechanism is always given by (54). In particular, for maximal $\alpha$-leakage, the optimal mechanism as well as the optimal PUT are identical for all $\alpha > 1$.

**B. PUTs for Datasets Containing Non-Sensitive Data**

For datasets containing both sensitive and non-sensitive data, indicated by $S$ and $X$, respectively, as shown in Fig 1b, the purpose of privacy protection is to limit information leakage of sensitive data while releasing non-sensitive data. We use $\alpha$-leakage from $S$ to $Y$ as the privacy measure, where $Y$ is the released version of $X$. Therefore, with $P_Y|S,X$ in the place of $P_Y|X$ in (51), we obtain the optimal PUT as

$$PUT_{HD,L_\alpha}(D) = \inf_{P_Y|S,X:d(X,Y) \leq D} L_\alpha(S; Y),$$  \hfill (60)

and for any $(s, x)$ with $P_{S,X}(s, x) > 0$, the non-empty set $B_D$ in (50) is given by

$$B_D(s, x) = \{y : d(x, y) \leq D\}. \hfill (61)$$

The following theorem lower bounds $PUT_{HD,L_\alpha}$.
Theorem 9. The minimal leakage PUT_{|\mathcal{L}|,\alpha} (1 \leq \alpha \leq \infty) in (60) is lower bounded by

\[
PUT_{|\mathcal{L}|,\alpha}(D) \geq \begin{cases} 
\sum_{s,x} P(s,x) \log \left( \frac{\max_{y \in B_D(s,x)} \sum_{s' \in S_D(y')} P(s')}{\sum_{s' \in S_D(y')} P(s')} \right)^{-1}, & \alpha = 1 \\
\log \sum_{s,x} P(s,x) \left( \frac{\max_{y \in B_D(s,x)} \sum_{s' \in S_D(y')} P(s')}{\sum_{s' \in S_D(y')} P(s')} \right)^{-1}, & \alpha = \infty \\
\frac{\alpha}{\alpha - 1} \log \sum_{s,x} P(s,x) \left( \frac{\max_{y \in B_D(s,x)} \sum_{s' \in S_D(y')} P(s')}{\sum_{s' \in S_D(y')} P(s')} \right)^{1-\frac{1}{\alpha}}, & \text{else},
\end{cases}
\]

where the nonempty set \( B_D(s,x) \) of \( y \) is given by (61), and the set \( S_D(y) \) of \( s \) for each \( y \) is defined as

\[
S_D(y) = \{ s : \exists x, P_{S,X}(s,x) > 0, d(x,y) \leq D \}. 
\]

The lower bound is tight if there exists an privacy mechanism \( P_{Y|S,X} \in \mathcal{P}_{Y|S,X}(D) \) such that

(i) given \( s,x \), for any \( y \) with \( P(y|s,x) > 0 \),

\[
\sum_{s' \in S_D(y)} P(s') = \max_{y' \in B_D(s,x)} \sum_{s' \in S_D(y')} P(s');
\]

(ii) given any \( y \) with \( P_Y(y) > 0 \), for any \( s \in S_D(y) \),

\[
\sum_{x : d(x,y) \leq D} P(y|s,x) P(x|s) = \frac{P_Y(y)}{\sum_{s' \in S_D(y)} P(s')},
\]

where \( P_Y \) is the marginal distribution of \( Y \) from the privacy mechanism \( P_{Y|S,X} \) and \( P_{S,X} \).

The proof details are in Appendix K.

Note that by using maximal \( \alpha \)-leakage as the privacy measure, the setting for publishing datasets consisting of sensitive and non-sensitive data can be generalized to restrict leakages about all functions of the sensitive data. This will be addressed in future work.

VI. APPLICATIONS: PUTs FOR HARD DISTORTION CONSTRAINT

In this section, we apply the results in Sec. V to data sets and present the optimal PUTs for two examples: (i) using absolute distance between types (empirical distributions) of binary data sets as the distortion function; (ii) discrete data sets with Hamming distortion.

A. Example 1: Binary Datasets with Hard Distortion on Types

When considering dataset disclosure under privacy constraints, a reasonable goal is to design privacy mechanisms that preserve the statistics of the original data set while preventing inference of each individual record (e.g., a sample or a row of the data set). Since the type (empirical distribution) of a dataset captures its statistics, we quantify distortion as the distance between the type of the original and released datasets. We use maximal \( \alpha \)-leakage to capture the gain of an adversary (with access to the released data set) in inferring any function of the original dataset.

Let \( X^n \) be a random dataset with \( n \) entries and \( Y^n \) be the corresponding released dataset generated by a privacy mechanism \( P_{Y^n|X^n} \). Entries of both \( X^n \) and \( Y^n \) are from the same alphabet \( \mathcal{X} \). Let \( P_{x^n} \) and \( P_{y^n} \) indicate the types of input dataset \( x^n \) and output dataset \( y^n \), respectively. We define the distortion function as the distance between types, given by

\[
d_{T}(x^n, y^n) = \max_{x \in \mathcal{X}} |P_{x^n}(x) - P_{y^n}(x)|,
\]

and therefore, obtain \( PUT_{|\mathcal{L}|,\alpha} \) as in (58) but with data sets \( X^n, Y^n \) in place of single letters \( X, Y \). Since types of \( n \)-length sequences take on only values that are multiples of \( \frac{1}{n} \), this distortion function \( d_{T} \) takes on values of the form \( \frac{m}{n} \), where \( m \in [0,n] \).

We concentrate on binary datasets, i.e., \( \mathcal{X} = \{0,1\} \). Note that for binary datasets, we can simply write \( d_{T}(x^n, y^n) = |P_{x^n}(1) - P_{y^n}(1)| \). For a \( n \)-length binary dataset, the number of types is \( n + 1 \). Therefore, all input and output datasets can be categorized into \( n + 1 \) type classes defined as

\[
T(i) \triangleq \{ x^n : nP_{x^n}(1) = i \}.
\]
An optimal privacy mechanism maps each input to the optimal tradeoff for the distortion function in (65), given integers \( n, m \) where \( 0 \leq m \leq n \), the optimal tradeoff for \( \alpha > 1 \) is

\[
\text{PUT}_{\text{HD,} \mathcal{L}_{\alpha}^{\max}} \left( \frac{m}{n} \right) = \min_{P_{Y^n|X^n}: \text{dist}(X^n, Y^n) \leq \frac{m}{n}} \mathcal{L}_{\alpha}^{\max} (X^n \rightarrow Y^n) \]

where \( \mathcal{L}_{\alpha}^{\max} (X^n \rightarrow Y^n) \) is given by (68). An optimal privacy mechanism maps all input data sets in a type class to a unique output data set which is feasible and belongs to a type class in the set \( \mathcal{T}^* \) given by

\[
\mathcal{T}^* \triangleq \left\{ T(j) : j = l + (2m+1)k, k \in [0, \left\lceil \frac{n+1}{2m+1} \right\rceil] \right\},
\]

where \( l = m \) if \( \left\lfloor \frac{n+1}{2m+1} \right\rfloor \leq \frac{m}{2m+1} \), and otherwise, \( l = n - \left( \left\lfloor \frac{n+1}{2m+1} \right\rfloor - 1 \right) (2m+1) \).

A detailed proof is in Appendix L. Let \( (n, m) = (9, 2) \) such that from Thm. 10, we have \( \text{PUT}_{\text{HD,} \mathcal{L}_{\alpha}^{\max}} \left( \frac{2}{9} \right) = 1 \) bit and \( \mathcal{T}^* = \{ T(2), T(7) \} \). Fig. 5 shows the optimal mechanism, which maps all input data sets in \( \{ T(i) : i \in [0, 4] \} \) (resp. \( \{ T(i) : i \in [5, 9] \} \)) to a unique output data set in \( T(2) \) (resp. \( T(7) \)) with probability 1.

### B. Example 2: Hard Hamming Distortion on Datasets

In the example, we consider hard Hamming distortion on datasets with entries from general finite alphabets. Formally, for datasets \( x^n, y^n \in \mathcal{X}^n \), we define the Hamming distortion function on data sets as

\[
d_H(x^n, y^n) = \frac{1}{n} \sum_{i=1}^{n} 1(x_i \neq y_i).
\]

Therefore, we obtain \( \text{PUT}_{\text{HD,} \mathcal{L}_{\alpha}^{\max}} \) as in (58) but with datasets \( X^n, Y^n \) in place of single letters \( X, Y \).

**Theorem 11.** For datasets from a finite alphabet \( \mathcal{X} \) and Hamming distortion function, for any integers \( n, m \) where \( 0 \leq m \leq n \), the optimal tradeoff for \( \alpha > 1 \) is

\[
\text{PUT}_{\text{HD,} \mathcal{L}_{\alpha}^{\max}} \left( \frac{m}{n} \right) = \min_{P_{Y^n|X^n}: \text{dist}(x^n, y^n) \leq \frac{m}{n}} \mathcal{L}_{\alpha}^{\max} (x^n \rightarrow y^n) \]

where \( \mathcal{L}_{\alpha}^{\max} (x^n \rightarrow y^n) \) is given by (72). An optimal privacy mechanism maps each input \( x^n \in \mathcal{X}^n \) uniformly to every feasible output, i.e., for all \( x^n, y^n \) where \( d_H(x^n, y^n) \leq \frac{m}{n} \), \( P_{Y^n|X^n}(y^n|x^n) = \frac{1}{\mathcal{X}^n} \) for \( \alpha = 1 \).

The key observation to reach the conclusion in Thm. 11 is that every output dataset is in the same number of feasible balls, such that a uniform distribution over the output space lead to equal probability for the feasible ball of each input data set. The proof details are in Appendix M. Fig. 6 illustrates the optimal mechanism in Thm. 11 for \( \mathcal{X} = \{0, 1, 2\} \) and \( (n, m) = (2, 1) \). Note that permuting items of a data set does not change the type but will lead to a non-zero Hamming distortion. The
Fig. 6: An optimal mechanism of (71) for \(\alpha > 1\) with \((n, m) = (2, 1)\) and \(\mathcal{X} = \{0, 1, 2\}\) where rows and columns are \(x^n\) and \(y^n\), respectively. Note that we color the conditional probabilities of feasible outputs (respect to the hard Hamming distortion) and their values are the same as 0.2 in the optimal mechanism.

distortion on types in (65) can be viewed as a relaxation of the Hamming distortion, in the sense that the set of feasible privacy mechanisms in (71) belongs to that in (67), i.e.,

\[
\left\{ P_{Y^n|X^n} : d_H(x^n, y^n) \leq \frac{m}{n} \right\} \subset \left\{ P_{Y^n|X^n} : d_T(x^n, y^n) \leq \frac{m}{n} \right\}.
\]

Therefore, for non-binary alphabets, the result in Thm. 11 upper bounds the minimal leakage in (67).

VII. Conclusion

Via \(\alpha\)-loss \((1 \leq \alpha \leq \infty)\), we have defined two tunable measures of information leakage: \(\alpha\)-leakage for a specific function of original data, and maximal \(\alpha\)-leakage for any arbitrary function of original data, and proven that: (i) \(\alpha\)-leakage equals to Arimoto mutual information for \(1 \leq \alpha \leq \infty\); (ii) for \(\alpha > 1\), maximal \(\alpha\)-leakage equals to Arimoto channel capacity; and for \(\alpha = 1\) and \(\alpha = \infty\) it simplifies to mutual information and maximal leakage, respectively. From properties of Arimoto mutual information, \(\alpha\)-leakage is known to be quasi-convex in the conditional distribution and satisfy post-processing inequality. For maximal \(\alpha\)-leakage, we have proven that it is quasi-convex in the conditional distribution, satisfies data processing inequalities and a composition property.

In the context of privacy-guaranteed data publishing, we have explored PUT problems for the proposed tunable leakage measures and hard distortion utility constraints. This utility constraint has the advantage that it allows the data curator/provider to make specific, deterministic guarantees on the quality of the released dataset. For maximal \(\alpha\)-leakage, we have shown that: (i) for all \(\alpha > 1\), we obtain the same optimal privacy mechanism and optimal PUT, both of which are independent of the distribution of the original data; (ii) for \(\alpha = 1\), the optimal mechanism differs and depends on the distribution of the original data. In other words, for this hard distortion measure, maximal \(\alpha\)-leakage behaves as either mutual information or maximal leakage. To further explore the effect of maximal \(\alpha\)-leakage for \(\alpha \in (1, \infty)\), we will address PUT problems with average utility measures in future work.

APPENDIX A: PROOF OF LEMMA 1

For \(\alpha \geq 1\), the minimal expected \(\alpha\)-loss in (16)

\[
\min_{P_X|Y} \mathbb{E} \left[ \ell_\alpha(X, Y, P_{X|Y}) \right] = \min_{P_X|Y} \frac{\alpha}{\alpha - 1} \left( 1 - \sum_{x, y} P_{X,Y}(x, y) P_{X|Y}(x|y)^{\frac{\alpha - 1}{\alpha}} \right)
\]

\[
= \frac{\alpha}{\alpha - 1} \left( 1 - \max_{P_X|Y} \sum_{x, y} P_{X,Y}(x, y) P_{X|Y}(x|y)^{\frac{\alpha - 1}{\alpha}} \right) \quad \text{(73)}
\]

\[
= \frac{\alpha}{\alpha - 1} \left( 1 - \sum_{y} P_{Y}(y) \frac{\max_{P_X|Y} P_{X|Y}(x|y)}{\sum_{x} P_{X|Y}(x|y) P_{X|Y}(x|y)^{\frac{\alpha - 1}{\alpha}}} \right). \quad \text{(74)}
\]
For each \( y \) with \( P_Y(y) > 0 \), the maximization in (73) can be explicitly write as
\[
\max_{P_{\hat{X}|Y}} \sum_{x \in X} P_{X|Y}(x|y) P_{\hat{X}|Y}(x|y)^{\frac{\alpha-1}{\alpha}}
\]
(76a)
subject to
\[
\sum_{x \in X} P_{\hat{X}|Y}(x|y) = 1
\]
(76b)
and
\[
P_{\hat{X}|Y}(x|y) \geq 0 \quad \text{for all } x \in X.
\]
(76c)

For \( 1 \leq \alpha \leq \infty \), the exponent \( \frac{\alpha-1}{\alpha} \geq 0 \) such that the problem in (76) is a convex program. Therefore, by using Karush-Kuhn-Tucker (KKT) conditions, we obtain the optimal value of (76) as
\[
\max_{P_{\hat{X}|Y}} \sum_{x \in X} P_{X|Y}(x|y) P_{\hat{X}|Y}(x|y)^{\frac{\alpha-1}{\alpha}} = \|P_{X|Y}(.|y)\|_{\alpha}
\]
(76d)
with the optimal solution \( P_{\hat{X}|Y}^*(x|y) \) as
\[
P_{\hat{X}|Y}^*(x|y) = \frac{P_{X|Y}(x|y)^{\alpha}}{\sum_{x \in X} P_{X|Y}(x|y)^{\alpha}} \quad \text{for all } x \in X.
\]
(76e)

For \( \alpha = 1 \), the optimal solution \( P_{\hat{X}|Y}^* = P_{X|Y} \). For \( \alpha = \infty \), we have
\[
\lim_{\alpha \to \infty} P_{\hat{X}|Y}^*(x|y) = \lim_{\alpha \to \infty} \left( \frac{P_{X|Y}(x|y)}{\max_{x \in X} P_{X|Y}(x|y)} \right)^{\alpha} = \begin{cases} \frac{1}{k(y)}, & x = \arg \max_x P_{X|Y}(x|y) \\ 0, & \text{otherwise,} \end{cases}
\]
(77)
where the integer \( k(y) \) indicates the cardinality of the set \( \{ x : x = \arg \max_x P_{X|Y}(x|y) \} \).

Applying the optimal solution \( P_{\hat{X}|Y}^* \) to (75), we have
\[
\min_{P_{\hat{X}|Y}} \mathbb{E} \left[ \ell_{\alpha}(X, Y, P_{\hat{X}|Y}) \right] = \begin{cases} \frac{\alpha}{\alpha - 1} \left( 1 - \sum_y P_{X,Y}(X|y) \right), & \alpha > 1 \\ \sum_{x,y} P_{X,Y}(x,y) \log \frac{1}{P_{X|Y}(x|y)}, & \alpha = 1 \\ \frac{\alpha}{\alpha - 1} \left( 1 - \exp \left( \frac{1-\alpha}{\alpha} H_{\alpha}^A(X|Y) \right) \right), & \alpha > 1 \\ H(X|Y), & \alpha = 1. \end{cases}
\]
(78)
(79)

**APPENDIX B: PROOF OF THEOREM 1**

The expression (25) can be explicitly written as
\[
\mathcal{L}_{\alpha}(X \to Y) = \lim_{\alpha' \to \alpha} \frac{\alpha'}{\alpha' - 1} \log \left( \frac{\max_{P_{X|Y}} \sum_{x,y} P_{X,Y}(x,y) \left( P_{\hat{X}|Y}(x|y) \right)^{\frac{\alpha'-1}{\alpha'}}}{\max_{P_{\hat{X}}} \sum_x P_X(x) P_{\hat{X}}(x)^{\frac{\alpha'-1}{\alpha'}}} \right).
\]
(80)

To simplify the expression in (80), we need to solve the two maximizations in the logarithm. From (28), we know that to solve the maximization in the numerator equals to find the minimal expected \( \alpha \)-loss. Making use of the result in Lemma 1, we have that for \( \alpha' \in (1, \infty) \),
\[
\max_{P_{X|Y}} \sum_{x,y} P_{X,Y}(x,y) P_{\hat{X}|Y}(x|y)^{\frac{\alpha'-1}{\alpha'}} = \exp \left( \frac{1-\alpha'}{\alpha'} H_{\alpha'}^A(X|Y) \right).
\]
(81)

Similarly, by applying KKT conditions to the maximization in the denominator, we have that for \( \alpha' \in (1, \infty) \)
\[
\max_{P_{\hat{X}}} \sum_x P_X(x) P_{\hat{X}}(x)^{\frac{\alpha'-1}{\alpha'}} = \exp \left( \frac{1-\alpha'}{\alpha'} H_{\alpha'}(X) \right).
\]
(82)

Therefore, we have for \( \alpha' \in (1, \infty) \)
\[
\mathcal{L}_{\alpha}(X \to Y) = \frac{\alpha'}{\alpha' - 1} \log \exp \left( \frac{1-\alpha'}{\alpha'} \left( H_{\alpha'}^A(X|Y) - H_{\alpha'}(X) \right) \right) = I_{\alpha'}^A(X; Y).
\]
(83)
From the continuous extensions of Arimoto MI for \( \alpha = 1 \) and \( \infty \), respectively, we have that for \( 1 \leq \alpha \leq \infty \), \( \alpha \)-leakage equals to Arimoto MI.

\[
L_{\alpha}^\max (X \to Y) = \sup_{U \to X \to Y} I_{\alpha}(U;Y). \tag{84}
\]

If \( \alpha = 1 \), we have
\[
L_{1}^\max (X \to Y) = \sup_{U \to X \to Y} I(U;Y) \leq I(X;Y) \tag{85}
\]
where the inequality is from data processing inequalities of MI [37, Thm 2.8.1].

If \( \alpha = \infty \), we have
\[
L_{\infty}^\max (X \to Y) = \sup_{U \to X \to Y} \frac{\sum P_Y(y) \max_u P_{U|Y}(u|y)}{\max_u P_U(u)} \tag{86}
\]
which is exactly the expression of MaxL, and therefore, we have [19, Thm. 1]
\[
L_{\infty}^\max (X \to Y) = \log \sum_y \max_x P_{Y|X}(y|x). \tag{87}
\]

For \( \alpha \in (1, \infty) \), we provide an upper bound for \( L_{\alpha}^\max (X \to Y) \), and then, give an achievable scheme as follows.

**Upper Bound:** We have an upper bound of \( L_{\alpha}^\max (X \to Y) \) as
\[
L_{\alpha}^\max (X \to Y) = \sup_{U \to X \to Y} I_{\alpha}(U;Y) \tag{88a}
\]
\[
\leq \sup_{P_{X|U}: P_{X|U} \leq P_X} \sup_{P_U} I_{\alpha}(\tilde{U};Y) \tag{88b}
\]
\[
= \sup_{P_{X|U}: P_{X|U} \leq P_X} \sup_{P_U} I_{\alpha}(\tilde{U};Y) \tag{88c}
\]
\[
= \sup_{P_X \leq P_X} I_{\alpha}(\tilde{X};Y) \tag{88d}
\]
\[
= \sup_{P_X \leq P_X} I_{\alpha}(\tilde{X};Y) \tag{88e}
\]
where \( P_{\tilde{X}} \ll P_X \) means the alphabet of \( P_{\tilde{X}} \) is a subset of that of \( P_X \). The inequality in (88b) holds because the supremum of Arimoto MI over all \( P_{\tilde{U}, \tilde{X}} \) on \( U \times X \) is no less than that (in (88a)) over these \( P_{U,X} \) constrained by the \( P_X \). The equations in (88c) and (88e) result from that Arimoto MI and Sibson MI of order \( \alpha > 0 \) have the same supremum [25, Thm. 5]; and (88d) obeys the data processing inequalities [25, Thm. 3].

**Lower Bound:** We lower bound (84) by consider a random variable \( U \) such that \( U \to X \to Y \) is a Markov chain and \( H(X|U) = 0 \). Specifically, let the alphabet \( \mathcal{U} \) consist of \( \mathcal{U}_U \), a collection of \( U \) mapped to a \( x \in X \), i.e., \( \mathcal{U} = \cup_{x \in X} \mathcal{U}_x \) with \( U = u \in \mathcal{U}_x \) if and only if \( X = x \). Therefore, for the specific variable \( U \), we have
\[
P_{Y|U}(y|u) = \begin{cases} P_{Y|X}(y|x) & \text{for all } u \in \mathcal{U}_x \\ 0 & \text{otherwise.} \end{cases} \tag{89}
\]
Construct a probability distribution \( P_{\tilde{X}} \) over \( X \) from \( P_U \) as
\[
P_{\tilde{X}}(x) = \frac{\sum_{u \in \mathcal{U}_x} P_U(u) \alpha}{\sum_{x \in X} \sum_{u \in \mathcal{U}_x} P_U(u) \alpha} \quad \text{for all } x \in X. \tag{90}
\]
Thus,
\[
I_{\alpha}(U;Y) = \frac{\alpha}{\alpha - 1} \log \frac{\sum_{y \in Y} \left( \frac{\sum_{x \in X} \sum_{u \in \mathcal{U}_x} P_{Y|U}(y|u) \alpha P_U(u) \alpha}{\sum_{x \in X} \sum_{u \in \mathcal{U}_x} P_U(u) \alpha} \right)^{\frac{1}{\alpha}}}{\left( \frac{\sum_{y \in Y} \left( \frac{\sum_{x \in X} \sum_{u \in \mathcal{U}_x} P_{Y|U}(y|u) \alpha P_U(u) \alpha}{\sum_{x \in X} \sum_{u \in \mathcal{U}_x} P_U(u) \alpha} \right)^{\frac{1}{\alpha}}} \right)^{\frac{1}{\alpha}}}} \tag{91}
\]
\[ L^\max_{\alpha}(X \to Y) = \sup_{U \subseteq \mathcal{X} \to \mathcal{Y}} \inf_{P_{U} \in \mathcal{P} \mathcal{X}} D_{\alpha}(P_{X,Y} || P_{X} \times Q_{Y}) \]

\[ = \frac{1}{\alpha - 1} \log (1 + (\alpha - 1) \sup_{P_{X} \in \mathcal{P} \mathcal{X}} \inf_{Q_{Y}} H_{\alpha}(X \to Y)) \]

\[ = \frac{1}{\alpha - 1} \log (1 + (\alpha - 1) \mathcal{L}_{\alpha}(X \to Y)) \]

Therefore, combining (88) and (95), we obtain (30a).

**APPENDIX D: PROOF FOR LEMMA 2**

Define the convex function

\[ f_{\alpha}(t) = \frac{1}{\alpha - 1} (t^\alpha - 1), \]

then for the two distributions \( P \) and \( Q \) over the support \( \mathcal{Y} \), we have a \( f \)-divergence \( H_{\alpha}(P \| Q) \), which is the Hellinger divergence of order \( \alpha \) [32], given by

\[ H_{\alpha}(P \| Q) = \frac{1}{\alpha - 1} \left( \sum_{y \in \mathcal{Y}} P(y)^\alpha Q(y)^{1 - \alpha} - 1 \right). \]

Therefore, the Rényi divergence can be written in terms of the Hellinger divergence as

\[ D_{\alpha}(P || Q) = \frac{1}{\alpha - 1} \log (1 + (\alpha - 1) H_{\alpha}(P || Q)). \]

Thus, since \( z \to \frac{1}{\alpha - 1} \log (1 + (\alpha - 1) z) \) is monotonically increasing in \( z \) for \( \alpha > 1 \), we can write maximal \( \alpha \)-leakage can be written as

\[ \mathcal{L}^\max_{\alpha}(X \to Y) = \sup_{P_{X}} \inf_{Q_{Y}} D_{\alpha}(P_{X,Y} || P_{X} \times Q_{Y}) \]

\[ = \frac{1}{\alpha - 1} \log \left( 1 + (\alpha - 1) \sup_{P_{X}} \inf_{Q_{Y}} H_{\alpha}(X \to Y) \right) \]

\[ = \frac{1}{\alpha - 1} \log \left( 1 + (\alpha - 1) \mathcal{L}_{H_{\alpha}}(X \to Y) \right). \]

That is, for \( \alpha > 1 \) maximal \( \alpha \)-leakage is a monotonic function of the Hellinger divergence-based measure.

**APPENDIX E: PROOF OF THEOREM 3**

**The proof of part 1:** We know that for \( \alpha \geq 1 \), \( I_{\alpha}^{S}(X;Y) \) is quasi-convex \( P_{Y|X} \) for given \( P_{X} \) [37, Thm. 2.7.4], [26, Thm. 10]. In addition, the supremum of a set of quasi-convex functions is also quasi-convex, i.e., if the function \( f(a,b) \) is quasi-convex in \( b \) for any given \( a \), the supremum \( \sup_{a} f(a,b) \) is also quasi-convex in \( b \) [33]. Therefore, maximal \( \alpha \)-leakage in (30) is quasi-convex \( P_{Y|X} \) for given \( P_{X} \).

**The proof of part 2:** Let \( \beta > \alpha \geq 1 \), and \( P_{X,\alpha} = \arg \sup_{P_{X}} I_{\alpha}^{S}(P_{X}, P_{Y|X}) \) for given \( P_{Y|X} \), such that

\[ \mathcal{L}^\max_{\alpha}(X \to Y) = I_{\alpha}^{S}(P_{X,\alpha}, P_{Y|X}) \]

\[ \leq I_{\beta}^{S}(P_{X,\alpha}, P_{Y|X}) \]

\[ \leq \sup_{P_{X}} I_{\beta}^{S}(P_{X}, P_{Y|X}) \]

\[ = \mathcal{L}^\max_{\beta}(X \to Y) \]

where (103) results from that \( I_{\alpha}^{S} \) is non-decreasing in \( \alpha \) for \( \alpha > 0 \) [26, Thm. 4], and the equality in (104) holds if and only if \( P_{X,\alpha} = \arg \sup_{P_{X}} I_{\beta}(P_{X}, P_{Y|X}) \).

**The proof of part 3:** Let random variables \( X, Y \) and \( Z \) form the Markov chain \( X - Y - Z \). Making use of that Sibson MI of order \( \alpha > 1 \) satisfies data processing inequalities [25, Thm. 3], i.e.,

\[ I_{\alpha}^{S}(X;Z) \leq I_{\alpha}^{S}(X;Y) \]

\[ I_{\alpha}^{S}(X;Z) \leq I_{\alpha}^{S}(Y;Z) \]

\[ = \mathcal{L}^\max_{\alpha}(X \to Y) \]

\[ \leq \mathcal{L}^\max_{\beta}(X \to Y) \]

\[ \leq \mathcal{L}^\max_{\alpha}(X \to Y) \]

\[ = \mathcal{L}^\max_{\beta}(X \to Y) \]

Therefore, combining (88) and (95), we obtain (30a).
we prove that maximal $\alpha$-leakage satisfies data processing inequalities as follows. We first prove (40a). Let $P^*_X = \arg \sup_{P_X} I^*_\alpha(P_X, P_{Z|X})$. For the Markov chain $X - Y - Z$, we have

\[
\mathcal{L}^\alpha_{\alpha}(X \rightarrow Z) = I^*_\alpha(P_X, P_{Z|X}) \leq I^*_\alpha(P_X, P_Y|X) \leq \sup_{P_X} I^*_\alpha(P_X, P_Y|X) = \mathcal{L}^\alpha_{\alpha}(X \rightarrow Y)
\]

(108) (109) (110) (111)

where the inequality in (109) results from (106). Similarly, the inequality in (40b) can be proved directly from (107).

**The proof of part 4:** For $\alpha = 1$, we have

\[
\mathcal{L}^\alpha_{1}(X \rightarrow Y) = I(X; Y) \geq 0,
\]

(112) with equalities if and only if $X$ is independent of $Y$ [37]. For $1 < \alpha \leq \infty$, referring to (6) and (30a) we have

\[
\mathcal{L}^\alpha_{\alpha}(X \rightarrow Y) = \sup_{P_X} \frac{\alpha}{\alpha - 1} \log \sum_y \left( \sum_x P_X(x) P_Y(y|x)^{\alpha} \right)^{\frac{1}{\alpha}} \geq \sup_{P_X} \frac{\alpha}{\alpha - 1} \log \sum_y \left( \sum_x P_X(x) P_Y(y|x)^{\alpha} \right)^{\frac{1}{\alpha}} = \sup_{P_X} \frac{\alpha}{\alpha - 1} \log 1 = 0,
\]

(113) (114) (115) where (114) results from applying Jensen’s inequality to the convex function $f : t \rightarrow t^\alpha$ ($t \geq 0$), such that the equality holds if and only if given any $y \in \mathcal{Y}$, $P_Y(y|x)$ are the same for all $x \in \mathcal{X}$, such that

\[
P_Y(y|x) = P_Y(y) \quad x \in \mathcal{X}, y \in \mathcal{Y}
\]

(116) which means $X$ and $Y$ are independent, i.e., $P_Y(y|x)$ is a rank-1 row stochastic matrix. For $\alpha = 1$, from (30b) we know $\mathcal{L}^\alpha_{\infty}(X \rightarrow Y) = I(X; Y)$. Therefore,

\[
\mathcal{L}^\alpha_{\alpha}(X \rightarrow Y) - H(X) = \sum_{x,y} P(x,y) \log \frac{P(x|y)}{P(y)} - \sum_{x} P(x) \log \frac{1}{P(x)} = \sum_{x,y} P(x,y) \log \frac{P(x|y)}{P(y)} - \sum_{x,y} P(x,y) \log \frac{1}{P(x)} = \sum_{x,y} P(x,y) \log P(x|y) \leq 0,
\]

(117) (118) (119) with equality if and only if for all $x, y \in \mathcal{X} \times \mathcal{Y}$, the conditional probability $P_X(y|x)$ is either 1 or 0. That is, $\mathcal{L}^\alpha_{\alpha}(X \rightarrow Y) \leq H(X)$ with equality if and only if $X$ is a deterministic function of $Y$ [38, Lem. 1]. For $1 < \alpha \leq \infty$, from the monotonocity of maximal $\alpha$-leakage in $\alpha$ and (30a), we have

\[
\mathcal{L}^\alpha_{\alpha}(X \rightarrow Y) \leq \mathcal{L}^\alpha_{\infty}(X \rightarrow Y)
\]

(120)

\[
= \log \sum_{y} \max_{x} P_Y(y|x)
\]

(121)

\[
\leq \log \sum_{y} \prod_{x} P_Y(y|x) = \log |\mathcal{X}|.
\]

(122) where the equality in (122) holds if and only if for every $y \in \mathcal{Y}$, $\sum_{x} P(y|x) = \max_{x} P(y|x)$, i.e., $X$ is a deterministic function of $Y$. To prove that for $\alpha \in (1, \infty)$, the upper bound in (122) is achievable, we construct a mapping $P_{X \rightarrow Y}$ such that $X$ is a deterministic function of $Y$. That is, for every $y \in \mathcal{Y}$, there exists an unique $x_y \in \mathcal{X}$ such that $P(y|x) = 1$. Therefore, we have $x_y = \arg_{x} P_{X \rightarrow Y}(y|x) > 0$. For $\alpha \in (1, \infty)$, from (6) and (30b) we have

\[
\mathcal{L}^\alpha_{\alpha}(P_{X \rightarrow Y}) = \sup_{P_X} \frac{\alpha}{\alpha - 1} \log \sum_{y} \left( \sum_{x} P_X^\frac{\alpha}{\alpha - 1}(x) P_{X \rightarrow Y}(y|x) \right) = \sup_{P_X} \frac{\alpha}{\alpha - 1} \log \prod_{x \in \mathcal{X}} P_X^\frac{\alpha}{\alpha - 1}(x);
\]

(123) (124)
in addition, since the function maximized in (124) is symmetric and concave in $P_X$, it is Schur-concave in $P_X$, and therefore, the optimal distribution of $X$ achieving the supreme in (124) is uniform. Thus,

$$L_{\alpha}^{\text{max}}(P_X \rightarrow Y) = \log |\mathcal{X}|, \quad 1 < \alpha \leq \infty.$$  \hspace{1cm} (125)

Therefore, maximal $\alpha$-leakage achieves its maximal value $\log |\mathcal{X}|$ and $H(P_X)$ for $\alpha > 1$ and $\alpha = 1$, respectively, if and only if $X$ is a deterministic function of $Y$.

\section*{APPENDIX F: PROOF FOR THEOREM 4}

To prove Thm. 4, we define a divergence function $k_\alpha$ for $\alpha > 1$ and provide a lower bound for its sum in the following definition and lemma, respectively.

\textbf{Definition 8.} Given two discrete distributions $P_Y$ and $Q_Y$ over the support $\mathcal{Y}$, a divergence function $k_\alpha$ for $\alpha > 1$ is defined as

$$k_\alpha(P_Y \parallel Q_Y) \triangleq \sum_y Q_Y(y) \left( \frac{P_Y(y)}{Q_Y(y)} \right)^\alpha.$$  \hspace{1cm} (126)

In addition, the function $k_\alpha(P_Y \parallel Q_Y)$ is jointly convex in $(P_Y, Q_Y)$, such that $k_\alpha(P_Y \parallel Q_Y) \geq 1$ with equality if and only if $P_Y = Q_Y$.

\textbf{Lemma 3.} Let $K$ be a positive integer with $K < \infty$. Given a group of distributions $\{P_k : k \in [1, K]\}$ and an arbitrary distribution $P$ on a discrete set $\mathcal{Y}$, there is

$$\sum_{k=1}^{K} k_\alpha(P_k \parallel P) \geq \sum_{k=1}^{K} k_\alpha(P_k \parallel P_c) = \left( \sum_{y} \left( \sum_{k=1}^{K} P_k(y)^\alpha \right)^\frac{1}{\alpha} \right)^\alpha,$$  \hspace{1cm} (127)

with equality if and only if $P = P_c$, where $P_c$ is given by

$$P_c(y) = \frac{1}{Z} \left( \sum_{k=1}^{K} P_k(y)^\alpha \right)^\frac{1}{\alpha}, \quad \alpha \in [1, \infty]$$  \hspace{1cm} (128)

where $Z$ is the constant as

$$Z = \sum_{y} \left( \sum_{k=1}^{K} P_k(y)^\alpha \right)^\frac{1}{\alpha},$$  \hspace{1cm} (129)

which guarantees that $P_c$ is a distribution.

\textit{Proof.} From the definition of $k_\alpha$ in (8), we have

$$\sum_{k=1}^{K} k_\alpha(P_k \parallel P) - \sum_{k=1}^{K} k_\alpha(P_k \parallel P_c)$$

$$= \sum_{y} \left( \sum_{k=1}^{K} P_k(y)^\alpha \right) \left( P(y)^{1-\alpha} - P_c(y)^{1-\alpha} \right)$$

$$\geq \sum_{y} Z^\alpha P_c(y)^\alpha \left( P(y)^{1-\alpha} - P_c(y)^{1-\alpha} \right)$$

$$= Z^\alpha \sum_{y} \left( P_c(y)^\alpha \left( P(y)^{1-\alpha} - P_c(y)^{1-\alpha} \right) \right)$$

$$= Z^\alpha (k_\alpha(P_c \parallel P) - 1) \geq 0$$

with equality if and only if $P = P_c$. In addition, making use of the expression of $P_c$ and $Z$ in (128) and (129), respectively, we have

$$\sum_{k=1}^{K} k_\alpha(P_k \parallel P_c)$$

$$= \sum_{k=1}^{K} \sum_{y} P_c(y) \left( \frac{P_k(y)}{P_c(y)} \right)^\alpha$$

$$= \sum_{k=1}^{K} \sum_{y} Z^{\alpha-1} K \left( \sum_{k=1}^{K} P_k(y)^\alpha \right) \frac{P_k(y)^\alpha}{\sum_{k=1}^{K} P_k(y)^\alpha}$$  \hspace{1cm} (135)
\[ L_{\alpha}^{\text{max}}(X \rightarrow Y) = \sum_y K z^{\alpha - 1} \left( \sum_{k=1}^K \frac{P_k(y)^{\alpha}}{K} \right)^{\frac{1}{\alpha}} \]
\[ = K \left( \sum_y \left( \sum_{k=1}^K \frac{P_k(y)^{\alpha}}{K} \right)^{\frac{1}{\alpha}} \right)^{\alpha} \]
\[ = \left( \sum_y \sum_{k=1}^K P_k(y)^{\alpha} \right)^{\frac{1}{\alpha}} \cdot \]  

Making use of the results in Lemma 3, we prove Thm. 4 as follows.

**Proof.** From Thm. 2, we have that for \( \alpha > 1 \)
\[ L_{\alpha}^{\text{max}}(X \rightarrow Y) = \sup_{P_X} I_{\alpha}^S(\hat{X}, Y) = \sup_{P_X} \inf_{Q_Y} D_{\alpha}(P_X P_Y |X| P_X Q_Y) \]
\[ = \sup_{P_X} \inf_{Q_Y} \frac{1}{\alpha - 1} \log \sum_x P_X(x)k_\alpha(P_{Y|X=x}||Q_Y). \]

For \( \alpha > 1 \), the function \( f: t \rightarrow \frac{1}{\alpha - 1} \log t \) is increasing in \( t \geq 0 \). Therefore, we simplify the optimization in (141) as
\[ \sup_{P_X} \inf_{Q_Y} \sum_x P_X(x)k_\alpha(P_{Y|X=x}||Q_Y) \]
and provide a lower bound of (142) as follows. Since the divergence function \( k_\alpha \) is joint convex in the pair of distributions, the objective function in (142) is joint convex in \((P_{Y|X}, Q_Y)\) for fixed \( P_X \), and linear in \( P_X \) for fixed \((P_{Y|X}, Q_Y)\). Therefore, the max-min equals to the min-max as followed:
\[ \sup_{P_X} \inf_{Q_Y} \sum_x P_X(x)k_\alpha(P_{Y|X=x}||Q_Y) = \inf_{Q_Y} \sup_{P_X} \sum_x P_X(x)k_\alpha(P_{Y|X=x}||Q_Y) \]
\[ = \inf_{Q_Y} \max_{x} k_\alpha(P_{Y|X=x}||Q_Y) \]
\[ \geq \inf_{Q_Y} \sum_x k_\alpha(P_{Y|X=x}||Q_Y) \]
\[ \geq \frac{\sum_x k_\alpha(P_{Y|X=x}||P_c)}{\mid \mathcal{X} \mid} \]
\[ = \frac{1}{\mid \mathcal{X} \mid} \left( \sum_y \|P_{Y|X}(y|\cdot)\|_\alpha \right) \alpha \]  

where the inequality in (146) is directly from (127) in Lemma 3 with equality if and only if
\[ Q_Y(y) = P_c(y) = \frac{1}{Z} \|P_{Y|X}(y|\cdot)\|_\alpha, \]  
with the constant \( Z = \sum_y \|P_{Y|X}(y|\cdot)\|_\alpha \). Therefore, for any \( P_{Y|X} \), we have
\[ L_{\alpha}^{\text{max}}(X \rightarrow Y) \geq \frac{\alpha}{\alpha - 1} \log \sum_{x} \|P_{Y|X}(y|\cdot)\|_\alpha \mid \mathcal{X} \mid \]
with equality if and only if the \( P_{Y|X} \) guarantees that the divergence function \( k_\alpha(P_{Y|X=x}||P_c) \) are the same for all \( x \in \mathcal{X} \), i.e., the \( P_{Y|X} \) satisfies (45). \( \square \)

**APPENDIX G: PROOF OF THEOREM 5**

Let \( \mathcal{Y}_1 \) and \( \mathcal{Y}_2 \) be the alphabets of \( Y_1 \) and \( Y_2 \), respectively. For any \((y_1, y_2) \in \mathcal{Y}_1 \times \mathcal{Y}_2\), due to the Markov chain \( Y_1 - X - Y_2 \), the corresponding entry of the conditional probability matrix of \((Y_1, Y_2)\) given \( X \) is
\[ P(y_1, y_2 | x) = P(y_1 | x) P(y_2 | x y_1) = P(y_1 | x) P(y_2 | x). \]

Therefore, for \( \alpha \in (1, \infty) \)
\[ L_{\alpha}^{\text{max}}(X \rightarrow Y_1, Y_2) = \sup_{P_X} \frac{\alpha}{\alpha - 1} \log \sum_{y_1, y_2} \left( \sum_x P_X(x) P_{Y_1, Y_2|X}(y_1, y_2 | x)^{\alpha} \right)^{\frac{1}{\alpha}} \]

(150)
Let $K(y_1) = \sum_{x \in \mathcal{X}} P_X(x) P_{Y_1|X}(y_1|x) \alpha$, for all $y_1 \in \mathcal{Y}_1$, such that we can construct a set of distributions over $\mathcal{X}$ as

$$P_{\mathcal{X}}(x|y_1) = \frac{P_X(x) P_{Y_1|X}(y_1|x) \alpha}{K(y_1)}.$$  

(152)

Therefore, from (151), $\mathcal{L}_{\alpha}^{\max}(X \to Y_1, Y_2)$ can be rewritten as

$$\mathcal{L}_{\alpha}^{\max}(X \to Y_1, Y_2) = \sup_{P_{\mathcal{X}}} \frac{\alpha}{\alpha - 1} \log \sum_{y_1, y_2 \in \mathcal{Y}_1 \times \mathcal{Y}_2} \left( \sum_{x \in \mathcal{X}} K(y_1) P_{\mathcal{X}}(x|y_1) P_{Y_2|\mathcal{X}}(y_2|x) \right)^{\frac{1}{\alpha}}.$$  

(153)

$$= \sup_{P_{\mathcal{X}}} \frac{\alpha}{\alpha - 1} \log \sum_{y_1, y_2} \left( \sum_{x} P_X(x) P_{Y_1|X}(y_1|x) \alpha \left( \sum_{x} P_{\mathcal{X}}(x|y_1) P_{Y_2|\mathcal{X}}(y_2|x) \right)^{\frac{1}{\alpha}} \right)$$  

(154)

$$= \sup_{P_{\mathcal{X}}} \frac{\alpha}{\alpha - 1} \log \sum_{y_1} \left( \sum_{x} P_X(x) P_{Y_1|X}(y_1|x) \alpha \sum_{y_2} \left( \sum_{x} P_{\mathcal{X}}(x|y_1) P_{Y_2|\mathcal{X}}(y_2|x) \right)^{\frac{1}{\alpha}} \right)$$  

(155)

$$\leq \sup_{P_{\mathcal{X}}} \frac{\alpha}{\alpha - 1} \log \left( \sum_{y_1} \left( \sum_{x} P_X(x) P_{Y_1|X}(y_1|x) \alpha \right)^{\frac{1}{\alpha}} \left( \max_{y_1} \sum_{y_2} \left( \sum_{x} P_{\mathcal{X}}(x|y_1) P_{Y_2|\mathcal{X}}(y_2|x) \right)^{\frac{1}{\alpha}} \right) \right)$$  

(156)

$$= \sup_{P_{\mathcal{X}}} \frac{\alpha}{\alpha - 1} \log \left( \sum_{y_1} \left( \sum_{x} P_X(x) P_{Y_1|X}(y_1|x) \alpha \right)^{\frac{1}{\alpha}} \sum_{y_2} \left( \sum_{x} P_{\mathcal{X}}(x|y_1) P_{Y_2|\mathcal{X}}(y_2|x) \right)^{\frac{1}{\alpha}} \right)$$  

(157)

$$\leq \sup_{P_{\mathcal{X}}} \frac{\alpha}{\alpha - 1} \log \sum_{y_1} \left( \sum_{x} P_X(x) P_{Y_1|X}(y_1|x) \alpha \right)^{\frac{1}{\alpha}} + \sup_{P_{\mathcal{X}}} \frac{\alpha}{\alpha - 1} \log \sum_{y_2} \left( \sum_{x} P_{\mathcal{X}}(x) P_{Y_2|\mathcal{X}}(y_2|x) \right)^{\frac{1}{\alpha}}$$  

(158)

$$= \mathcal{L}_{\alpha}^{\max}(X \to Y_1) + \mathcal{L}_{\alpha}^{\max}(X \to Y_2),$$  

(159)

where $y_1^\star$ in (157) is the optimal $y_1$ achieving the maximum in (156). Therefore, the equality in (156) holds if and only if, for all $y_1 \in \mathcal{Y}_1$,

$$\sum_{y_2} \left( \sum_{x} P_X(x|y_1) P_{Y_2|\mathcal{X}}(y_2|x) \right)^{\frac{1}{\alpha}} = \sum_{y_2} \left( \sum_{x} P_{\mathcal{X}}(x|y_1^\star) P_{Y_2|\mathcal{X}}(y_2|x) \right)^{\frac{1}{\alpha}};$$  

(160)

and the equality in (158) holds if and only if the optimal solutions $P_{\mathcal{X}}^\star$ and $P_{\mathcal{X}}^\star$ of the two maximizations in (158) satisfy, for all $x \in \mathcal{X},$

$$P_{\mathcal{X}}^\star(x) = \frac{P_X^\alpha(x) P_{Y_1|X}(y_1^\star|x)}{\sum_{x \in \mathcal{X}} P_X^\alpha(x) P_{Y_1|X}(y_1^\star|x)}.$$  

(161)

Now we consider $\alpha = 1$. For $Y_1 - X - Y_2$, we have

$$I(Y_2; X|Y_1) \leq I(Y_2; X).$$  

(162)

From Thm. 2, there is

$$\mathcal{L}_{1}^{\max}(X \to Y_1, Y_2) = I(X; Y_1) + I(X; Y_2|Y_1)$$  

(163)

$$\leq I(X; Y_1) + I(X; Y_2)$$  

(164)

$$= \mathcal{L}_{1}^{\max}(X \to Y_1) + \mathcal{L}_{1}^{\max}(X \to Y_2).$$  

(165)

For $\alpha = \infty$, we also have

$$\mathcal{L}_{\infty}^{\max}(X \to Y_1, Y_2) = \log \sum_{y_1, y_2 \in \mathcal{Y}_1 \times \mathcal{Y}_2} \max_{x \in \mathcal{X}} P(y_1|x) P(y_2|x)$$  

(166)

$$\leq \log \sum_{y_1, y_2 \in \mathcal{Y}_1 \times \mathcal{Y}_2} \left( \max_{x \in \mathcal{X}} P(y_1|x) \right) \left( \max_{x \in \mathcal{X}} P(y_2|x) \right)$$  

(167)

$$= \log \sum_{y_1 \in \mathcal{Y}_1} \max_{x \in \mathcal{X}} P(y_1|x) + \log \sum_{y_2 \in \mathcal{Y}_2} \max_{x \in \mathcal{X}} P(y_2|x)$$  

(168)

$$= \mathcal{L}_{\infty}^{\max}(X \to Y_1) + \mathcal{L}_{\infty}^{\max}(X \to Y_2).$$  

(169)
APPENDIX H: PROOF OF THEOREM 6

For $\alpha > 1$, a function $f(t) = \frac{\alpha}{\alpha-1} \log t$ is monotonically increasing in $t > 0$. Therefore, To solve maximal $\alpha$-leakage from $X^n$ to $Y^n$, i.e.,

$$\mathcal{L}_\alpha^{\max}(X^n \to Y^n) = \sup_{P_{X^n}} \alpha \log \sum_{y^n} \left( \sum_{x^n} P(x^n) P(y^n|x^n)^\alpha \right)^{-\frac{1}{\alpha}}$$

(170)

it is sufficient to consider

$$\sup_{P_{X^n}} \sum_{y^n} \left( \sum_{x^n} P(x^n) P(y^n|x^n)^\alpha \right)^{-\frac{1}{\alpha}}$$

(171)

For a memoryless $P_{Y^n|X^n}$ with no feedback, we simplify (171) as

$$\sup_{P_{X^n}} \sum_{y^n} \left( \sum_{x^n} P(x^n) P(y^n|x^n)^\alpha \right)^{-\frac{1}{\alpha}} = \prod_{i=1}^n \sum_{y_i} \left( \sum_{x_i} P(x_i) P(y_i|x_i)^\alpha \right)^{-\frac{1}{\alpha}}$$

(172)

For a memoryless $P_{Y^n|X^n}$ with no feedback and is memoryless, respectively;

the equality in (175) holds if and only if the source is memoryless, i.e., $P_{X^n|X_{i-1},\ldots,X_1} = P_{X_1}$ for all $i \in [1,n]$;

both (176) and (177) are from the distributive property of multiplication.

Therefore, we have for $\alpha > 1$,

$$\sup_{P_{X^n}} I^S_\alpha(X^n;Y^n) = \sum_{i=1}^n \sup_{P_{X_i}} I^S_\alpha(X_i;Y_i)$$

(179)

That is,

$$\mathcal{L}_\alpha^{\max}(X^n \to Y^n) = \sum_{i=1}^n \mathcal{L}_\alpha^{\max}(X_i \to Y_i)$$

(180)

For $\alpha = 1$, we have

$$I(X^n;Y^n) = \sum_{i,j=1}^n I(X_i;Y_j|X_{i-1},\ldots,X_1,Y_{j-1},\ldots,Y_1)$$

(181)
\[= \sum_{i,j=1}^{n} I(X_i; Y_j | X_{i-1}, \ldots, X_1) \quad (182)\]
\[= \sum_{i=1}^{n} I(X_i; Y_i | X_{i-1}, \ldots, X_1) \quad (183)\]
\[\leq \sum_{i=1}^{n} I(X_i; Y_i) \quad (184)\]

where

- (181) is from the chain rule of MI;
- (182) and (183) are from the facts that the mechanism has no feedback and is memoryless, respectively;
- from [37, (2.122)], we know that for a Markov chain \((182)\) and \((183)\) are from the facts that the mechanism has no feedback and is memoryless, respectively;
- \((186)\) follows from the fact that \(I(X; Y) \leq I(X; Z)\) with equality if and only if \(I(X; Z) = 0\). Therefore, since for any \(i \in [1, n]\) \((X_{i-1}, \ldots, X_1) - X_i - Y_i\), the equality in \((175)\) holds if and only if the source is memoryless, i.e., \(P_{X_i|X_{i-1}, \ldots, X_1} = P_{X_i}\) for all \(i \in [1, n]\). \(\square\)

**APPENDIX I: PROOF OF THEOREM 7**

Given \(P_X\), the collection of stochastic matrices is denoted as \(\mathcal{P}_{Y|X}\). The feasible ball \(B_D(x)\) around \(x\) is defined in (50). For the distribution dependent PUT in (52), we have

\[
\text{PUT}_{\text{HD}, \ell_f}(D) = \inf_{P_{Y|X} \in \mathcal{P}_{Y|X}} \inf_{Q_Y \leq D} D_f(P_{Y|X} P_X \| P_X \times Q_Y) \quad (185)
\]

\[
= \inf_{Q_Y} \sum_{x \in X} P_X(x) \inf_{P_{Y|X} | Y = x \leq D} \sum_{y \in Y} Q_Y(y) f \left( \frac{P_{Y|X}(y|x)}{Q_Y(y)} \right) \quad (186)
\]

\[
= \inf_{Q_Y} \sum_{x \in X} P_X(x) \inf_{P_{Y|X} | Y = x \leq D} \left( \sum_{y \in Y} Q_Y(y) f \left( \frac{P_{Y|X}(y|x)}{Q_Y(y)} \right) \right) \quad (187)
\]

\[
= \inf_{Q_Y} \sum_{x \in X} P_X(x) \left( \sum_{y \in Y} f \left( \frac{P_{Y|X}(y|x)}{Q_Y(y)} \right) \right) \quad (188)
\]

\[
\geq \inf_{Q_Y} \sum_{x \in X} P_X(x) \left( f(0) + \sum_{y \in Y} P_{Y|X}(y|x) \frac{Q_Y(y)}{Q_Y(B_D(x))} \right) \quad (189)
\]

\[
= f(0) + \inf_{Q_Y} \sum_{x \in X} P_X(x) \left( Q_Y(B_D(x)) \left( f \left( \frac{1}{Q_Y(B_D(x))} \right) - f(0) \right) \right) \quad (190)
\]

where

- (186) follows from the fact that \(D_f(P_{Y|X} P_X \| P_X \times Q_Y)\) is convex in \((P_{Y|X}, Q_Y)\) for fixed \(P_X\);
- (189) is directly from the hard distortion constraint \(d(X; Y) \leq 0\) such that for any \(y \notin B_D(x)\) \(P_{Y|X}(y|x) = 0\), and therefore, \(\sum_{y \notin B_D(x)} P_{Y|X}(y|x) = 1\);
- (190) is from the Jensen’s inequality such that

\[
\sum_{y \in B_D(x)} \frac{Q_Y(y)}{Q_Y(B_D(x))} f \left( \frac{P_{Y|X}(y|x)}{Q_Y(y)} \right) \geq f \left( \sum_{y \in B_D(x)} \frac{Q_Y(y)}{Q_Y(B_D(x))} \frac{P_{Y|X}(y|x)}{Q_Y(y)} \right) \quad (192)
\]

\[
= f \left( \frac{\sum_{y \in B_D(x)} P_{Y|X}(y|x)}{Q_Y(B_D(x))} \right) = f \left( \frac{1}{Q_Y(B_D(x))} \right), \quad (193)
\]
with equality if and only if there is a mechanism \( P_{Y|X} \) satisfying
\[
\frac{P_{Y|X}(y|x)}{Q_Y(y)} = \mathbb{1}(y \in B_D(x)) \quad \text{for fixed } x \in \mathcal{X}. \tag{194}
\]

Note that \( f : \mathbb{R}_+ \to \mathbb{R} \) is a convex function, such that the function \( tf\left(\frac{1}{t}\right) \) is convex in \( t \in \mathbb{R}_+ \). Therefore, the objective function in (191) is convex in \( Q_Y \). In addition, the feasible region of \( Q_Y \) in (191) \( Q_Y \) is the probability distribution simplex over the set \( \{B_D(x), x \in \mathcal{X}\} \). For finite supports \( \mathcal{X} \) and \( \mathcal{Y} \) of \( X \) and \( Y \), respectively, the set \( \{B_D(x), x \in \mathcal{X}\} \) is a compact, and therefore, the infimum in (191) is achievable.

\[\square\]

### Appendix J: Proof of Theorem 8

Given \( P_X \), the collection of stochastic matrices is denoted as \( \mathcal{P}_{Y|X} \). The feasible ball \( B_D(x) \) around \( x \) is defined in (50). For the distribution independent PUT in (55), we have
\[
\text{PUT}_{\text{ID}, t_m}(D) = \inf_{P_Y \in \mathcal{P}_{Y|X}} \sup_{P_X \in \mathcal{P}_{Y|X}} \inf_{Q_Y \in B_D(x)} D_f(P_X P_{Y|X} \parallel P_X \times Q_Y) \tag{195}
\]
\[
= \inf_{Q_Y} \sup_{P_X} \inf_{P_Y \in \mathcal{P}_{Y|X}} D_f(P_X P_{Y|X} \parallel P_X \times Q_Y) \tag{196}
\]
\[
= \inf_{Q_Y} \sup_{P_X} \sum_{x \in \mathcal{X}} P_X(x) \inf_{P_Y \in \mathcal{P}_{Y|X} \parallel Q_Y} \sum_{y \in \mathcal{Y}} Q_Y(y) \frac{P_X(y|x)}{Q_Y(y)} \tag{197}
\]
\[
= \inf_{Q_Y} \sup_{P_X} \sum_{x \in \mathcal{X}} P_X(x) \frac{Q_Y(B_D(x))}{Q_Y(B_D(x))} \frac{1}{Q_Y(B_D(x))} + Q_Y(B_D(x)^c) f(0) \tag{199}
\]
\[
\geq \inf_{Q_Y} \sup_{P_X} \sum_{x \in \mathcal{X}} P_X(x) \frac{Q_Y(B_D(x))}{Q_Y(B_D(x))} \frac{1}{Q_Y(B_D(x))} + Q_Y(B_D(x)^c) f(0) \tag{200}
\]
\[
= \inf_{Q_Y} \sup_{P_X} g(Q_Y(B_D(x))) \tag{201}
\]
\[
= \sup_{Q_Y} \inf_{Q_Y} g(Q_Y(B_D(x))) \tag{202}
\]
\[
\text{where}
\]
- (196) and (198) follow from the fact that \( D_f(P_X P_{Y|X} \parallel P_X \times Q_Y) \) is linear in \( P_X \) for fixed \( (P_{Y|X}, Q_Y) \) and convex in \( (P_{Y|X}, Q_Y) \) for fixed \( P_X \),
- (201) follows from the convexity of \( f \) and Jensen’s inequality. The equality holds if and only if there exists a mechanism \( P_{Y|X} \) satisfying (194),
- (203) results from \( q \triangleq Q_Y(B_D(x)) \) and
\[
g(q) \triangleq q f(q^{-1}) + (1 - q) f(0). \tag{205}
\]
Due to the convexity of \( f \), we have \( f(q^{-1}) - f(0) \leq f'(q^{-1})(q^{-1} - 0) \), from which, the derivative \( g'(q) = f(q^{-1}) - q^{-1} f'(q^{-1}) - f(0) \leq 0 \). Therefore, the function \( g \) in (205) is non-increasing, such that (204) is be simplified as \( g(q^*) \), where \( q^* \) is given by
\[
q^* \triangleq \sup_{Q_Y} \inf_{Q_Y} g(Q_Y(B_D(x))) \tag{206}
\]
Note that the feasible region of \( Q_Y \) in (206) \( Q_Y \) is the probability distribution simplex over the set \( \{B_D(x), x \in \mathcal{X}\} \). For finite supports \( \mathcal{X} \) and \( \mathcal{Y} \) of \( X \) and \( Y \), respectively, the set \( \{B_D(x), x \in \mathcal{X}\} \) is a compact, and therefore, the supremum in (206) is achievable.

\[\square\]
From Thm.1, we know that for $\alpha \geq 1$, $\alpha$-leakage $L_\alpha(S;Y)$ equals to Arimoto MI $I_\alpha^A(S;Y)$. Since $I_\alpha^A(S;Y) = H_\alpha(S) - H_\alpha^N(S|Y)$ and $H_\alpha(S)$ is independent of $P_{Y|S,X}$, to minimize $I_\alpha^A(S;Y)$ with respect to $P_{Y|S,X}$ can be simplified to maximize $H_\alpha^N(S|Y)$. In addition, for $\alpha > 1$, the function $g : t \to t^{\frac{1}{\alpha}} \log t$ is a monotonically non-increase function in $t > 0$. Therefore, the problem in (60) can be simplified to

$$\inf_{P_{Y|S,X} \in (X,Y) \leq D} \sum_{y} \left( \sum_{s} P(s, y)^\alpha \right)^{\frac{1}{\alpha}}.$$  \quad (207)

The hard distortion on $X$ and $Y$ in (60) determines a collection of feasible $x$ and therefore $s$ for each $y$. We define the two collections for each $y \in \mathcal{Y}$ as

$$\mathcal{X}_D(y) \triangleq \{ x \in \mathcal{X} : d(x, y) \leq D \},$$  \quad (208)

$$\mathcal{S}_D(y) \triangleq \{ s \in \mathcal{S} : \exists x \in \mathcal{X}_D(y), P_{S|X}(sx) > 0 \}.$$  \quad (209)

Note that both set defined above are independent of the privacy mechanism $P_{Y|S,X}$.

For $\alpha \in (1, \infty)$, we have

$$\inf_{P_{Y|S,X} \in (X,Y) \leq D} \sum_{y} \left( \sum_{s} P(s, y)^\alpha \right)^{\frac{1}{\alpha}} = \inf_{P_{Y|S,X} \in (X,Y) \leq D} \sum_{y} \left( \sum_{s \in \mathcal{S}_D(y)} P(s, y)^\alpha \right)^{\frac{1}{\alpha}}$$  \quad (210)

$$= \inf_{P_{Y|S,X} \in (X,Y) \leq D} \sum_{y} \left( \sum_{s \in \mathcal{S}_D(y)} P(s, y)^\alpha \right)^{\frac{1}{\alpha}} \left( \sum_{s \in \mathcal{S}_D(y)} \frac{P(s)^\alpha}{P(s')^\alpha} \right)^{\frac{1}{\alpha}}$$  \quad (211)

$$\geq \inf_{P_{Y|S,X} \in (X,Y) \leq D} \sum_{y} \left( \sum_{s \in \mathcal{S}_D(y)} P(s, y)^\alpha \right)^{\frac{1}{\alpha}} \left( \sum_{s \in \mathcal{S}_D(y)} \frac{P(s)^\alpha}{P(s')^\alpha} \right)^{\frac{1}{\alpha}}$$  \quad (212)

$$= \inf_{P_{Y|S,X} \in (X,Y) \leq D} \sum_{y} \left( \sum_{s \in \mathcal{S}_D(y)} P(s, y)^\alpha \right)^{\frac{1}{\alpha}}$$  \quad (213)

$$= \inf_{P_{Y|S,X} \in (X,Y) \leq D} \sum_{y} \left( \sum_{s \in \mathcal{S}_D(y)} P(s, y)^\alpha \right)^{\frac{1}{\alpha}} P(s)^\alpha P(x, y|s)$$  \quad (214)

$$= \inf_{P_{Y|S,X} \in (X,Y) \leq D} \sum_{y} \left( \sum_{s \in \mathcal{S}_D(y)} P(s, y)^\alpha \right)^{\frac{1}{\alpha}} P(s)^\alpha P(x, y|s)$$  \quad (215)

$$\geq \inf_{P_{Y|S,X} \in (X,Y) \leq D} \sum_{y} \left( \sum_{s \in \mathcal{S}_D(y)} P(s, y)^\alpha \right)^{\frac{1}{\alpha}} P(s)^\alpha P(x, y|s)$$  \quad (216)

$$= \sum_{s,x} P(s)^\alpha P(x|s) \left( \max_{y \in \mathcal{S}_D(y)} \sum_{s' \in \mathcal{S}_D(y)} P(s')^\alpha \right)^{\frac{1}{\alpha} - 1}$$  \quad (217)

where

- (213) is directly from is from the concavity of the function $g_1 : t \to t^{\frac{1}{\alpha}}$ ($\alpha > 1$) and Jensen’s inequality. The equality holds if and only if the optimal $P_{Y|S,X}^*$ achieving the infimum satisfies that for all $s \in \mathcal{S}_D(y),$

$$P_{Y|S,X}^*(y|s) = \frac{P_{Y}^*(y)}{\sum_{s' \in \mathcal{S}_D(y)} P_{S}(s')},$$  \quad (218)

where $P_{Y}^*$ is the probability distribution of $Y$ derived from $P_{Y|S,X}^*$.

- in (215), $B_D(s, x)$ is the feasible ball defined in (61).

- the equality in (216) holds if and only if for any $(s, x)$, all $y$ with $P^*(y|s, x) > 0$ lead to the same $\sum_{s \in \mathcal{S}_D(y)} P(s').$

- the equality in (217) is from the fact that the function $g : t \to t^{\frac{1}{\alpha} - 1}$ is monotonically non-increasing in $t > 0$ for $\alpha \geq 1.$
Similarly, for $\alpha = \infty$, we have

$$
\inf_{P_{Y|S,X} : d(X,Y) \leq D} \sum_y P_Y(y) \max_s P_{S|Y}(s|y) = \inf_{P_{Y|S,X}} \sum_y P_Y(y) \max_s P_{S|Y}(s|y) \left( \sum_{X_D(y)} P_{S,X|Y}(s,x|y) \right)
$$

$$\geq \inf_{P_{Y|S,X}} \sum_y P(y) \left( \sum_{S_D(y)} \sum_{x' \in S_D(y)} P(s') \sum_{X_D(y)} P(s,x|y) \right)
$$

$$= \inf_{P_{Y|S,X}} \sum_{s,x} \sum_{B_D(s,x)} P(s) \left( \sum_{S_D(y)} \sum_{x' \in S_D(y)} P(s,x,y) \right)
$$

$$\geq \inf_{P_{Y|S,X}} \sum_{s,x} \sum_{B_D(s,x)} P(s,x,y) \min_{y \in B_D(s,x)} \frac{P(s)}{\sum_{s' \in S_D(y)} P(s')}
$$

$$= \sum_{s,x} P(s,x) P(s) \left( \max_{y \in B_D(s,x)} \sum_{s' \in S_D(y)} P(s') \right)^{-1}.
$$

Note that the sufficient and necessary conditions for the equalities in (220) and (222) hold are the same as that for (213) and (216), respectively.

For $\alpha = 1$, $\mathcal{L}_{\alpha}(S \rightarrow Y) = I^A(S;Y) = I(S;Y)$, such that

$$\text{PUT}_{\mathcal{L}_{\alpha}}(D) = \inf_{P_{Y|S,X} : d(X,Y) \leq D} \sum_{s,y} P(s,y) \log \frac{P(s,y)}{P(s)P(y)}
$$

$$= \inf_{P_{Y|S,X}} \sum_{y} \sum_{S_D(y)} \left( \left( \sum_{X_D(y)} P(s,x,y) \right) \log \frac{\sum_{X_D(y)} P(s,x,y)}{P(s)P(y)} \right)
$$

$$\geq \inf_{P_{Y|S,X}} \sum_{y} \left( \left( \sum_{S_D(y)} \sum_{x' \in X_D(y)} P(s,x,y) \right) \log \frac{\sum_{S_D(y)} \sum_{X_D(y)} P(s,x,y)}{P(s)P(y)} \right)
$$

$$\geq \inf_{P_{Y|S,X}} \sum_{y} \sum_{S_D(y)} \sum_{x' \in X_D(y)} P(s,x,y) \log \frac{1}{\sum_{s' \in S_D(y)} P(s')}
$$

Note that the inequality in (227) is from log-sum inequality in [37, Thm. 2.7.1], and the sufficient and necessary conditions for the equalities in (227) and (228) hold are the same as that for (213) and (216), respectively.

\[\Box\]

**APPENDIX L: PROOF OF THEOREM 10**

Define the distortion ball for the type-distance distortion in (65) as

$$B_m(x^n) \triangleq \left\{ y^n : |P_{x^n} - P_{y^n}| \leq \frac{m}{n} \right\}.
$$

From Corollary 1, to find an optimal mechanism $P_{Y^n|X^n}^*$, we need to find an output distribution $Q^*_Y$ which optimizes (57) with $x^n$ and $y^n$ in place of $x$, $y$.

Note that for the hard distortion $|P_{x^n} - P_{y^n}| \leq \frac{m}{n}$, all datasets in a type class share the same group of feasible output datasets, and this feasible group can be represented by output type classes. Therefore, for any $x^n \in T(i)$ ($i \in [0,n]$), we rewrite $B_m(x^n)$ as

$$B_m(x^n) = B_m(T(i)) \triangleq \bigcup_{|i-j| \leq m} T(j).
$$

(230)
We define an distribution $Q_T$ of type classes for outputs as

$$Q_T(T(j)) \triangleq \sum_{y^n \in T(j)} Q_{Y^n}(y^n), \text{ for } j \in [0, n],$$

such that

$$q^* = \sup_{Q_T} \inf_{i \in [0, n]} Q_T(B_m(T(i))).$$

The optimal distribution $Q_T$ is determined by both upper and lower bounding $q^*$ in (232). The upper bound is determined by restricting the optimization in (232) to a judicious choice of a small set of input types. The lower bound is a constructive scheme.

We define an index set $I_T \subset [0, n]$ for types as

$$I_T \triangleq \left\{ l + (2m + 1)k : k \in \left[ 0, \left\lfloor \frac{n + 1}{2m + 1} \right\rfloor - 1 \right] \right\}$$

where $l = m$ if $\left\lfloor \frac{n + 1}{2m + 1} \right\rfloor \leq \frac{m + n + 1}{2m + 1}$, and otherwise, $l = n - \left( \left\lfloor \frac{n + 1}{2m + 1} \right\rfloor - 1 \right)(2m + 1)$. From the expression of $I_T$ in (233), we observe that: (i) the difference between adjacent elements is $2m + 1$; (ii) for the first and last elements,

- if $\left\lfloor \frac{n + 1}{2m + 1} \right\rfloor \leq \frac{m + n + 1}{2m + 1}$ holds, the first element is $m$ and the last element is

$$m + (2m + 1) \left( \left\lfloor \frac{n + 1}{2m + 1} \right\rfloor - 1 \right) = (2m + 1) \left\lfloor \frac{n + 1}{2m + 1} \right\rfloor - m - 1 \in [n - m, n],$$

due to the inequalities $\frac{n + 1}{2m + 1} \leq \frac{m + n + 1}{2m + 1}$;

- if $\frac{n + 1}{2m + 1} > \frac{m + n + 1}{2m + 1}$ holds, the last element is $n$ and the first element is

$$n - \left( \left\lfloor \frac{n + 1}{2m + 1} \right\rfloor - 1 \right)(2m + 1) = n + 2m + 1 - \left\lfloor \frac{n + 1}{2m + 1} \right\rfloor(2m + 1) \in [0, m),$$

due to the inequalities $\frac{n + 1}{2m + 1} + 1 - \frac{1}{2m + 1} \geq \frac{n + 1}{2m + 1}$ for $n \in \mathbb{Z}_+$. Therefore, it is not difficult to see that feasible balls of input type classes indexed by $I_T$ are a partition of the set of all type classes, i.e.,

$$B_m(T(i_1)) \cap B_m(T(i_2)) = \emptyset \quad i_1, i_2 \in I_T,$$

$$\bigcup_{j \in [0, n]} T(j) = \bigcup_{i \in I_T} B_m(T(i)).$$

Therefore, the problem in (232) is upper bounded by

$$q^* \leq \sup_{Q_T} \inf_{i \in I_T} Q_T(B_m(T(i)))$$

$$\leq \sup_{Q_T} \frac{1}{|I_T|} \sum_{i \in I_T} Q_T(B_m(T(i)))$$

$$= \sup_{Q_T} \left( \left\lfloor \frac{n + 1}{2m + 1} \right\rfloor \right)^{-1} \sum_{j \in [0, n]} Q_T(T(j))$$

$$= \left( \left\lfloor \frac{n + 1}{2m + 1} \right\rfloor \right)^{-1},$$

where

- the inequality in (238) is from that the average probability of $B_m(T(i))$ over $i \in I_T$ is no less than the minimal probability of $B_m(T(i))$ for $i \in I_T$;

- the equality in (239) is from that the cardinality of $I$ defined in (233) is $\left\lfloor \frac{n + 1}{2m + 1} \right\rfloor$;

- the equality in (240) is from that for any distribution over types $T(j)$ with $j \in [0, n]$, the sum of $Q_T(T(j))$ over $j \in [0, n]$ is 1.

To lower bound $q^*$, we construct a distribution $Q_T'$ as

$$Q_T'(T(j)) = \begin{cases} \left( \left\lfloor \frac{n + 1}{2m + 1} \right\rfloor \right)^{-1} & j \in I_T \\ 0 & \text{otherwise.} \end{cases}$$
By (236) for each \( i \in [0, n] \), there is a unique \( j \) satisfying \( |i - j| \le m \). Therefore, we lower bound (232) by

\[
q^* \ge \inf_i Q_{T'}(B_m(T(i)))
\]

(242)

\[
= \inf_i Q_{T'} \left( \bigcup_{|i-j| \le m} T(j) \right)
\]

(243)

\[
= \left( \left\lceil \frac{n + 1}{2m + 1} \right\rceil \right)^{-1},
\]

(244)

where the equality in (244) hold because for any \( i \in [0, n] \), there is only one \( j \in I_T \) satisfying \( |i - j| \le m \) such that the union in (243) has exactly one element in it.

Therefore, \( q^* = \left( \left\lceil \frac{n + 1}{2m + 1} \right\rceil \right)^{-1} \) and the \( Q_{T'} \) defined in (241) achieves the optimum in (232). Thus, we can derive an optimal \( Q_{T'}^* \), which assigns the same non-zero probability to only one dataset of each type classes indexed by \( I_T \), i.e., \( Q_{T'}^*(y^n) = q^* \) for one \( y^n \in T(j) \) for each \( j \in I_T \). Therefore, from (54) we have the corresponding optimal privacy mechanism, which maps all input datasets in one input type class to one feasible output dataset with probability 1.

\[ \square \]

APPENDIX M: PROOF OF THEOREM 11

For the Hamming distortion function on data sets in (70), the feasible ball \( B_m(x^n) \) of any data set \( x^n \in X^n \) is given by

\[
B_m(x^n) = \left\{ y^n \in X^n : d_H(x^n, y^n) \le \frac{m}{n} \right\}.
\]

(245)

For each \( x^n \in X^n \), the number of data sets having different values at exactly \( k \) different positions is \( \binom{n}{k} (|X| - 1)^k \). Therefore, the number of elements in its feasible ball \( B_m(x^n) \) is

\[
|B_m(x^n)| = \sum_{i=0}^{m} \binom{n}{i} (|X| - 1)^i,
\]

(246)

Note that the cardinality \( |B_m(x^n)| \) in (246) of a feasible ball is independent of the input data set. We denote the cardinality as \( N_{ball} \), i.e., \( N_{ball} \triangleq |B_m(x^n)| \). Due to the symmetric property of the Hamming distortion on data sets in (70), i.e., for any two data sets \( x^n_1, x^n_2 \in X^n \), \( x^n_1 \in B_D(x^n_2) \) if and only if \( x^n_2 \in B_D(x^n_1) \), we know that each output data set is in exactly \( N_{ball} \) different feasible balls (the example in Fig. 6 may help to figure out the above relationships). Therefore,

\[
q^* = \sup_{Q_{Y^n}} \inf_{x^n \in X^n} Q_{Y^n}(B_m(x^n))
\]

(247)

\[
\le \sup_{Q_{Y^n}} \frac{1}{|X^n|} \sum_{x^n \in X^n} Q_{Y^n}(B_m(x^n))
\]

(248)

\[
= \sup_{Q_{Y^n}} \frac{1}{|X^n|} \sum_{x^n \in X^n} \sum_{y^n \in B_m(x^n)} Q_{Y^n}(y^n)
\]

(249)

\[
= \sup_{Q_{Y^n}} \frac{1}{|X^n|} \sum_{x^n \in X^n} Q_{Y^n}(y^n)
\]

(250)

\[
= \sup_{Q_{Y^n}} \frac{1}{|X^n|} \sum_{y^n \in X^n} \sum_{y^n \in B_m(x^n)} N_{ball} Q_{Y^n}(y^n)
\]

(251)

\[
= \frac{N_{ball}}{|X^n|}
\]

(252)

where

- the equality in (248) holds if and only if for an arbitrary pair of data sets \( x^n_1, x^n_2 \), there is

\[
Q_{Y^n}(B_D(x^n_1)) = Q_{Y^n}(B_D(x^n_2)),
\]

(253)

which can be satisfied by a uniform distribution over \( X^n \), i.e., \( Q_{Y^n} = \frac{1}{|X^n|} \).

- the equality in (251) holds because, for each \( y^n \), the number of sequences \( x^n \) where \( d_H(x^n, y^n) \le \frac{m}{n} \) is exactly \( N_{ball} \).

\[ \square \]

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