All about the Static Fermion Bags in the Gross-Neveu Model

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Abstract

We review in detail the construction of all stable static fermion bags in the 1 + 1 dimensional Gross-Neveu model with $N$ flavors of Dirac fermions, in the large $N$ limit. In addition to the well known kink and topologically trivial solitons (which correspond, respectively, to the spinor and antisymmetric tensor representations of $O(2N)$), there are also threshold bound states of a kink and a topologically trivial soliton: the heavier topological solitons (HTS). The mass of any of these newly discovered HTS’s is the sum of masses of its solitonic constituents, and it corresponds to the tensor product of their $O(2N)$ representations. Thus, it is marginally stable (at least in the large $N$ limit). Furthermore, its mass is independent of the distance between the centers of its constituents, which serves as a flat collective coordinate, or a modulus. There are no additional stable static solitons in the Gross-Neveu model. We provide detailed derivation of the profiles, masses and fermion number contents of these static solitons. For pedagogical clarity, and in order for this paper to be self-contained, we also included detailed appendices on supersymmetric quantum mechanics and on reflectionless potentials in one spatial dimension, which are intimately related with the theory of static fermion bags. In particular, we present a novel simple explicit formula for the diagonal resolvent of a reflectionless Schrödinger operator with an arbitrary number of bound states. In additional appendices we summarize the relevant group representation theoretic facts, and also provide a simple calculation of the mass of the kinks.

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1 Introduction

Many years ago, Dashen, Hasslacher and Neveu (DHN) [1], and following them Shei [2], used inverse scattering analysis [3] to find static fermion-bag [4, 5] soliton solutions to the large-$N$ saddle point equations of the Gross-Neveu (GN) [6] and of the 1 + 1 dimensional, multi-flavor Nambu-Jona-Lasinio (NJL) [7] models. In the GN model, with its discrete chiral symmetry, a topological soliton, the so called Callan-Coleman-Gross-Zee (CCGZ) kink [8], was discovered prior to the work of DHN. In this paper we will concentrate exclusively on the GN model.

One version of writing the action of the 1 + 1 dimensional GN model is

$$S = \int d^2 x \left\{ \sum_{a=1}^{N} \bar{\psi}_a \left( i \dot{\psi} - \sigma \right) \psi_a - \frac{1}{2g^2} \sigma^2 \right\}, \quad (1.1)$$

where the $\psi_a \ (a = 1, \ldots, N)$ are $N$ flavors of massless Dirac fermions, with Yukawa coupling to the scalar\(^1\) auxiliary field $\sigma(t,x)$.

An obvious symmetry of (1.1) with its $N$ Dirac spinors is $U(N)$. Actually, (1.1) is symmetric under the larger group $O(2N)$. It is easy to see this[1] in a concrete representation for $\gamma$ matrices, which we choose as the Majorana representation

$$\gamma^0 = \sigma_2, \quad \gamma^1 = i \sigma_3 \quad \text{and} \quad \gamma^5 = -\gamma^0 \gamma^1 = \sigma_1. \quad (1.2)$$

(Henceforth, in this paper we will use this representation for $\gamma$ matrices in all explicit calculations.)

In order to expose the $O(2N)$ symmetry, we write each Dirac spinor as

$$\psi_a = \phi_a + i \chi_a \quad (1.3)$$

where $\phi_a$ and $\chi_a$ are hermitean two-component Majorana spinors. In terms of (1.2) and (1.3) we can write the lagrangian in (1.1) (up to surface terms) as\(^2\)

$$\mathcal{L} = \sum_{a=1}^{N} \left[ i(\phi_a T \dot{\phi}_a + \chi_a T \dot{\chi}_a) - i(\phi_a T \sigma_1 \phi'_a + \chi_a T \sigma_1 \chi'_a) - \sigma (\phi_a T \sigma_2 \phi_a + \chi_a T \sigma_2 \chi_a) \right] - \frac{\sigma^2}{2g^2} \quad (1.4)$$

\(^1\)The fermion bag solitons in these models arise, as is well known, at the level of the effective action, after integrating the fermions out, and not at the level of the action (1.1).

\(^2\)Here we use the common notation where an overdot stands for a time derivative and a prime stands for a spatial derivative.
which is hermitean and non-vanishing, because all spinors are Grassmann valued. Evidently, (1.4) is invariant under orthogonal transformations of the $2N$ Majorana spinors $\phi_a$ and $\chi_a$.

The fact that the symmetry group of (1.1) is $O(2N)$ rather than $U(N)$, indicates that (1.1) is invariant against charge-conjugation. In the representation (1.2), charge-conjugation is realized simply by complex conjugation of the spinor$^3$

$$\psi^c(x) = \psi^*(x). \quad (1.5)$$

Thus, if $\psi = e^{-i\omega t}u(x)$ is an eigenstate of the Dirac equation

$$\left[i\partial - \sigma(x)\right] \psi = 0, \quad (1.6)$$

with energy $\omega$, then $\psi^*(x) = e^{i\omega t}u^*(x)$ is an energy eigenstate of (1.6), with energy $-\omega$. Therefore, the GN model (1.1) is invariant against charge conjugation, and energy eigenstates of (1.6) come in $\pm \omega$ pairs.

The remarkable discovery DHN made was that all self-consistent static bag configurations in the GN model were reflectionless. That is, the static $\sigma(x)$’s that solve the saddle point equations of the GN model are such that the reflection coefficient of the Dirac equation (1.6) vanishes identically for all momenta. In other words, a fermion wave packet impinging on one side of the potential $\sigma(x)$ will be totally transmitted (up to phase shifts, of course).

We note in passing that besides their role in soliton theory [3], reflectionless potentials appear in other diverse areas of theoretical physics [9, 10, 11, 12]. For a review, which discusses reflectionless potentials (among other things) in the context of supersymmetric quantum mechanics, see [13].

Since the works of DHN and of Shei, these fermion bags were discussed in the literature several other times, using alternative methods [14]. For a recent review on these and related matters (with an emphasis on the relativistic Hartree-Fock approximation), see [15].

$^3$The matrices $\gamma^0, \gamma^1$ are pure imaginary. Thus by taking the complex conjugate of the Dirac equation $\left[i\partial - eA - \sigma(x)\right] \psi = 0$ we conclude that $\left[i\partial + eA - \sigma(x)\right] \psi^* = 0$. 

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In many of these treatments, one solves the variational, saddle point equations by performing mode summations over energies and phase shifts. An alternative to such summations is to solve the saddle point equations by manipulating the resolvent of the Dirac operator as a whole, with the help of basic tools of Sturm-Liouville operator theory. The resolvent of the Dirac operator takes care of mode summation automatically.

Some time ago, such an alternative to mode summation techniques was developed [16, 17], which was based on the Gel’fand-Dikii (GD) identity [18] (an identity obeyed by the diagonal resolvent of one-dimensional Schrödinger operators, see e.g. (A.33)) \(^4\), to study fermion bags in the GN model[16] as well as other problems [17]. In a nut-shell, the method of [16, 17] is based on the fact that in certain models, the explicit form of the saddle-point equations for one-dimensional (or quasi one-dimensional) static, space-dependent field condensates, suggests certain parameter dependent ansätze for the diagonal resolvent of the Dirac (or Klein-Gordon) operator. This explicit construction of the diagonal resolvent is based on the GD identity as well as on simple dimensional analysis. All subsequent manipulations with these expressions involve the space dependent condensates directly, and given such an ansatz for the diagonal resolvent, one can construct a static space-dependent solution of the saddle-point equations in a straightforward manner, bypassing the need to work with the scattering data and the so called trace identities that relate them to the space dependent condensates.

That method was applied in [16, 19, 20] to study static fermion bags in the GN and NJL models and reproduced the static bag results of DHN and of Shei. It was also used in [21] to make approximate variational calculations of static fermion bags in the massive GN model [21]. Similar ideas were later used in [22] to calculate the free energy of inhomogeneous superconductors.

In this paper we study the entire spectrum of stable static fermion bags in the GN model in the large \(N\) limit, thus extending the “static part” of the work[1] of DHN.

\(^4\)For a simple derivation of the GD identity, see [17, 19].
We find that in addition to the well known kink and topologically trivial solitons, which correspond, respectively, to the spinor and antisymmetric tensor representations of $O(2N)$, the GN model bears also threshold bound states of a kink and a topologically trivial soliton - the heavier topological solitons (HTS)[23]. The mass of any of these newly discovered HTS’s is the sum of masses of its solitonic constituents, and it corresponds to the tensor product of their $O(2N)$ representations. Thus it is marginally stable (at least in the large $N$ limit). Furthermore, the mass of an HTS is independent of the distance between the centers of its constituents, which thus serves as a flat collective coordinate, or a modulus. There are no additional stable static solitons in the Gross-Neveu model.

The rest of the paper is organized as follows: In Section 2 we recall some basic facts about the GN model and its dynamics. In particular, we define the effective action $S_{\text{eff}}[\sigma]$, obtained from (1.1) after integrating the fermions out, and derive the generic saddle-point equation for $\sigma(x,t)$.

In Section 3 we focus on the simpler problem of finding extremal static $\sigma(x)$ configurations, and as we mentioned earlier, DHN have already shown[1] that the extremal static configurations are necessarily reflectionless. Our analysis employs extensively one dimensional supersymmetric quantum mechanics and the theory of reflectionless potentials. We apply these formalisms to construct a generic static reflectionless background $\sigma(x)$ (subjected to the appropriate boundary conditions at $x = \pm \infty$), which supports a given number of bound states of the Dirac equation (1.6) at some given energies, and the diagonal resolvent of the Dirac operator in (1.6) associated with it. We then use this explicit resolvent to calculate the energy functional associated with this reflectionless background, as well as its fermion number spectrum. A reflectionless background $\sigma(x)$ depends on a finite set of real parameters, and the effective action evaluated at such a $\sigma(x)$ configuration is an ordinary function of these parameters. By solving the ordinary extremum problem for this function we identify all static extremal fermion bags in the GN model.

After listing all extremal static $\sigma(x)$ configurations in Section 3, we identify, in
Section 4, the *stable* configurations, namely, those extremal bags which are protected by the conservation of topological charge and $O(2N)$ quantum numbers against decaying into lighter fermion bags. These are the well known CCGZ kink and DHN solitons, and in addition, our newly discovered marginally stable HTS.

Some general background material, as well as many technical details are relegated to the Appendices. In Appendix A we discuss (in some detail) the general properties of the resolvent of the Dirac operator in a static $\sigma(x)$ background and its relation with supersymmetric quantum mechanics. As is well known, the Dirac equation in any such background is equivalent to a pair of two isospectral Schrödinger equations in one dimension, namely, a realization of one-dimensional supersymmetric quantum mechanics. We use this underlying supersymmetry to express all four entries of the space-diagonal Dirac resolvent (i.e., the resolvent evaluated at coincident spatial coordinates) in terms of a single function: the diagonal resolvent of one of the two isospectral Schrödinger operators. Furthermore, using the underlying supersymmetry of the Dirac equation, we prove that the expectation value of the spatial component of the fermion current operator in the static $\sigma(x)$ background vanishes identically, as should be expected on physical grounds. We also prove an identity relating the expectation values of the fermion density and the pseudoscalar density operators in the $\sigma(x)$ background, and show that it has a simple interpretation in terms of bosonization. This appendix follows, in part, our recent discussion in [20].

In Appendix B we summarize some useful properties of reflectionless Schrödinger hamiltonians and their resolvents. Following that summary of results from the literature, we use them to derive an explicit simple representation for the diagonal resolvent of a reflectionless Schrödinger operator with a prescribed number of bound states, which we have not encountered in the literature. We then use the formalism reviewed in that appendix to derive an explicit formula for a $\sigma(x)$ with a prescribed set of bound states. We also work out explicitly the cases of backgrounds with a single bound state and with two bound states, which are the cases most relevant to this work.
Appendix C contains some technical details concerning the derivation in Section 2 of the saddle point equation in the static case.

In Appendix D we discuss the quantization of a multiplet of $2N$ Majorana fields in a given static $\sigma(x)$ background and the related $O(2N)$ multiplet structure of bound states of fermions trapped in that background. We show there that each bound state in the corresponding Dirac-Majorana equation, at non-zero frequency, gives rise to an irreducible factor of an $O(2N)$ antisymmetric tensor, and that the zero mode, which exists if and only if $\sigma(x)$ has non-trivial topology, gives rise to a single factor of the spinor representation. Also included in that appendix are the necessary facts concerning the spinorial and antisymmetric tensor representations of $O(2N)$.

In Appendix E, we give a simple derivation of the mass formula of the CCGZ kink. Finally, in Appendix F, we present an alternative proof of Eq. (4.12), which is the basis for the complete classification of stable static solitons.
2 Dynamics of the Gross-Neveu Model

We now recall some basic facts about the dynamics of the GN model. The partition function associated with (1.1) is

\[ Z = \int D\sigma D\bar{\psi} D\psi \exp i \int d^2x \left\{ \bar{\psi} \left( i\partial - \sigma \right) \psi - \frac{1}{2g^2} \sigma^2 \right\} \]  

(2.1)

Integrating over the grassmannian variables leads to

\[ Z = \int D\sigma \exp \{ iS_{\text{eff}}[\sigma] \} \]

where the bare effective action is

\[ S_{\text{eff}}[\sigma] = -\frac{1}{2g^2} \int d^2x \sigma^2 - iN \text{Tr} \log \left( \frac{1}{i\partial - \sigma} \right) \]

(2.2)

and the trace is taken over both functional and Dirac indices.

The theory (2.2) has been studied in the limit \( N \to \infty \) with \( Ng^2 \) held fixed[6]. In this limit (2.1) is governed by saddle points of (2.2) and the small fluctuations around them. (In this paper, as in [1], we will consider only the leading term in the \( 1/N \) expansion, and thus will not compute the effect of the fluctuations around the saddle points.) The most general saddle point condition reads

\[ \frac{\delta S_{\text{eff}}}{\delta \sigma(x,t)} = -\frac{\sigma(x,t)}{g^2} + iN \text{Tr} \left[ \frac{1}{i\partial - \sigma} |x,t\rangle \langle x,t| \right] = 0. \]  

(2.3)

In particular, the non-perturbative vacuum of (1.1) is governed by the simplest large \( N \) saddle points of the path integral associated with it, where the composite scalar operator \( \bar{\psi}\psi \) develops a space-time independent expectation value. The relevant object to discuss in this context is the effective potential \( V_{\text{eff}} \) associated with (1.1), namely, the value of \( -S_{\text{eff}} \) for space-time independent \( \sigma \) configurations per unit time per unit length. \( V_{\text{eff}}(\sigma) \) has two degenerate (absolute) minima at \( \sigma = \pm m \neq 0 \), where \( m \) is fixed by the (bare) gap equation[6]

\[ -m + iNg^2 \int \frac{d^2k}{(2\pi)^2} \frac{1}{k - m} = 0 \]  

(2.4)

which yields the fermion dynamical mass

\[ m = \Lambda e^{-\frac{m}{Ng^2}}. \]  

(2.5)

\[ ^{5}\text{From this point to the end of the paper flavor indices are usually suppressed.} \]
Here $\Lambda$ is an ultraviolet cutoff. The mass $m$ must be a renormalization group invariant. Thus, the model is asymptotically free. We can get rid of the cutoff at the price of introducing an arbitrary renormalization scale $\mu$. The renormalized coupling $g_R(\mu)$ and the cutoff dependent bare coupling are then related through

$$\Lambda e^{-\frac{\pi}{Ng_2(\Lambda)}} = \mu e^{-\frac{\pi}{Ng_R^2(\mu)}}$$

in a convention where $Ng_R^2(m) = \pi$. Trading the dimensionless coupling $g_R$ for the dynamical mass scale $m$ represents the well-known phenomenon of dimensional transmutation.

In terms of $m$, we can write down the renormalized effective potential (in the large $N$ limit) as

$$V_{eff}(\sigma) = \frac{N}{4\pi} \sigma^2 \log \frac{\sigma^2}{em^2}.$$  

(2.7)

It is evidently symmetric under $\sigma \rightarrow -\sigma$, which generates (together with $\psi \rightarrow \gamma_5 \psi$ in (1.1)) the discrete (or $\mathbb{Z}_2$) chiral symmetry of the GN model.

This discrete symmetry is dynamically broken by the non-perturbative vacuum, and thus there is a kink solution [1, 8, 16], the CCGZ kink mentioned above, $\sigma(x) = m \tanh(mx)$, interpolating between the two degenerate minima $\sigma = \pm m$ of (2.7) at $x = \pm \infty$. Therefore, topology insures the stability of these kinks.

The CCGZ kink is one example of non-trivial excitations of the vacuum, which are described semiclassically by large $N$ saddle points of the path integral over (1.1) at which $\sigma$ develops space dependent, or even space-time dependent expectation values. These expectation values are the space-time dependent solution of (2.3), and have analogs in other field theories [12, 24]. Saddle points of this type are important also in discussing the large order behavior[25, 26] of the $\frac{1}{N}$ expansion of the path integral over (1.1).

These saddle points describe sectors of (1.1) that include scattering states of the (dynamically massive) fermions in (1.1), as well as a rich collection of bound states thereof. These bound states result from the strong infrared interactions, which polarize the vacuum inhomogeneously, causing the composite scalar $\bar{\psi}\psi$ field to form finite
action space-time dependent condensates. We may regard these condensates as one
dimensional fermion bags [4, 5] that trap the original fermions (“quarks”) into stable
finite action extended entities (“hadrons”).

Finding explicit space-time dependent solutions of (2.3) is a very difficult problem.
In [1], DHN managed to guess such a set of solutions which oscillate in time (and
thus are not merely boosts of static bags). Finding a systematic method to generate
space-time dependent solutions of equations like (2.3) is, of course, one of the basic
goals of quantum field theory.

2.1 Static Inhomogeneous $\sigma(x)$ Backgrounds

In this paper we focus on the easier problem of solving (2.3) for static inhomogeneous
condensates.

Static solutions of (2.3) are subjected to certain spatial asymptotic boundary
conditions. For the usual physical reasons, we set boundary conditions on our static
background fields such that $\sigma(x)$ starts from one of its vacuum expectation values
$\sigma = \pm m$ at $x = -\infty$, wanders along the $\sigma$ axis, and then relaxes back to one of its
vacuum expectation values at $x = +\infty$:

$$\sigma \xrightarrow{x \to \pm \infty} \pm m, \quad \sigma' \xrightarrow{x \to \pm \infty} 0.$$  \hspace{1cm} (2.8)

These four possible asymptotic behaviors determine the topological nature of the
condensate. The two cases in which the asymptotic values are of opposite signs are
topologically non trivial. The CCGZ kink (where, by definition, $\sigma(\infty) = m$ ) and
antikink ($\sigma(\infty) = -m$) are two such examples. The other two cases, i.e., the cases
with asymptotic values of equal signs, are topologically trivial.

The topological charge associated with these boundary conditions is simply

$$q = \frac{1}{2m} (\sigma(\infty) - \sigma(-\infty)) = \frac{1}{2m} \int_{-\infty}^{\infty} \partial_x \sigma(x),$$ \hspace{1cm} (2.9)

and can take on values $q = \pm 1$ for the topologically non trivial configurations, and
$q = 0$ for the topologically trivial ones.
As typical of solitonic configurations, we expect that $\sigma(x)$ tends to its asymptotic boundary values (2.8) at an exponential rate which is determined, essentially, by the mass gap $m$ of the model. It is in such static backgrounds $\sigma(x)$ that we have to invert the Dirac operator and calculate its resolvent in seeking static solutions of (2.3).

In addition to the CCGZ kink, there are also topologically trivial inhomogeneous condensates. These topologically trivial condensates are stable because of the binding energy released by the trapped fermions, and therefore cannot form without such binding. Thus, they are stable due to dynamics. In contrast, the stability of the CCGZ kink is guaranteed by topology already. It can form without binding fermions.

Note in passing, that the 1+1 dimensional NJL model, with its continuous symmetry, does not have a topologically stable soliton solution. Thus, unlike the GN model, the solitons arising in the NJL model can be stabilized only by binding fermions [2, 19]. This description agrees with the general physical picture drawn in [27].

2.1.1 The Energy of a Static Configuration and Supersymmetry

In order to study the extremum condition (2.3) on $S_{\text{eff}}$ around a static space dependent background $\sigma(x)$, we need to invert the Dirac operator

$$D \equiv \omega \gamma^0 + i\gamma^1 \partial_x - \sigma(x).$$

(2.10)

(We have naturally transformed $i\partial_x - \sigma(x)$ to the $\omega$ plane, since $\sigma(x)$ is static.) In particular, we have to find the diagonal resolvent of (2.10) in that background.

There is an intimate connection between the spectral theory of the Dirac operator (2.10) and one dimensional supersymmetric quantum mechanics [13, 16]. This connection is explained in detail Appendix A. In particular, the topological charge $q$ (2.9) coincides with the Witten index of the one dimensional supersymmetric quantum mechanics (see Eq. (A.22)).

The desired diagonal resolvent $\langle x | iD^{-1} | x \rangle$ is defined in (A.29) in Appendix A. It is shown there, that due to the underlying supersymmetric quantum mechanics, its
four entries
\[ \langle x | -i D^{-1} | x \rangle \equiv \begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix} \] (2.11)
are completely determined, in a simple way, in terms of any one of the off-diagonal entries, \( B \) or \( C \).

For example, in terms of \( B \), one finds (see Eqs. (A.30), (A.35) and (A.38))
\[
\begin{align*}
A(x) &= D(x) = \frac{i[\partial_x + 2\sigma(x)]B(x)}{2\omega} \\
-\omega^2 C(x) &= \frac{1}{2}B'' + \sigma B' + (\sigma' + \omega^2)B.
\end{align*}
\] (2.12)

A quick way to introduce this underlying supersymmetry is the following: From the elementary identity
\[
\gamma_5 (i\partial - \sigma) \gamma_5 = -(i\partial + \sigma) \quad (2.13)
\]
it follows that \( \text{Tr} \log(i\partial - \sigma) = \frac{1}{2} \text{Tr} \log \left[ -(i\partial - \sigma)(i\partial + \sigma) \right] \). Thus, an alternative representation of (2.2) is [16]
\[
S_{\text{eff}}[\sigma] = -\frac{1}{2g^2} \int d^2x \sigma^2 - i \frac{N}{2} \text{Tr} \log \left( \partial^2 + \sigma^2 - i\gamma^\mu \partial_\mu \sigma \right). \quad (2.14)
\]
If, in addition, \( \sigma \) is time independent, (2.14) may be further simplified to
\[
S_{\text{eff}}[\sigma] = -\frac{T}{2g^2} \int dx \sigma^2 - i \frac{NT}{2} \int \frac{d\omega}{2\pi} \left[ \text{Tr} \log(H_b - \omega^2) + \text{Tr} \log(H_c - \omega^2) \right], \quad (2.15)
\]
where \( T = \int dt \) is a large temporal infrared cutoff, and
\[
H_b = -\partial_x^2 + \sigma^2 - \sigma' \quad \text{and} \quad H_c = -\partial_x^2 + \sigma^2 + \sigma' \quad (2.16)
\]
are the pair of isospectral Schrödinger operators defined in (A.6). In other words, they have the same spectrum of bound state energies, save, possibly, the existence of a bound state at \( \omega^2 = 0 \), which, if it exists, appears in the spectrum of only one of the operators. As explained in subsection A.1.1 of Appendix A, a bound state at
\[ \omega^2 = 0 \] exists only in topologically non-trivial \( \sigma(x) \) backgrounds. Due to the zero-mode mismatch of the spectra of \( H_b \) and \( H_c \), we will keep both \( \text{Tr} \log (H_b - \omega^2) \) and \( \text{Tr} \log (H_c - \omega^2) \) explicitly in our formulas for a while. We will take advantage of the isospectrality of \( H_b \) and \( H_c \) in the positive part of the spectrum at the appropriate places.

The operators \( H_b \) and \( H_c \) may be identified as the hamiltonians of the bosonic sector and of the fermionic sector of a one dimensional supersymmetric quantum mechanical system.

According to (A.31), the off-diagonal entries \( B(x) \) and \( C(x) \) in (2.11) are essentially the diagonal resolvents of \( H_b \) and \( H_c \):

\[
\frac{B(x)}{\omega} = \langle x | \frac{1}{H_b - \omega^2} | x \rangle \\
- \frac{C(x)}{\omega} = \langle x | \frac{1}{H_c - \omega^2} | x \rangle .
\] (2.17)

From (2.15) we may express the energy functional \( \mathcal{E}[\sigma(x)] \) of a static configuration \( \sigma(x) \) as

\[
\mathcal{E}[\sigma(x)] = -\frac{S_{eff}[\sigma]}{T} = \frac{1}{2g^2} \int dx \sigma^2 + i \frac{N}{2} \int \frac{d\omega}{2\pi} \left[ \text{Tr} \log (H_b - \omega^2) + \text{Tr} \log (H_c - \omega^2) \right].
\] (2.18)

This expression is divergent. We regulate it, as usual, by subtracting from it the divergent contribution of the vacuum configuration \( \sigma^2 = m^2 \) and by imposing a UV cutoff \( \Lambda \) on \( \omega \). Thus, the regulated (bare) energy functional associated with \( \sigma(x) \) is

\[
\mathcal{E}^{reg}[\sigma(x)] = \frac{1}{2g^2} \int dx (\sigma^2 - m^2) + i \frac{N}{2} \int \frac{d\omega}{2\pi} \left[ \text{Tr} \log (H_b - \omega^2) - \text{Tr} \log (H_{VAC} - \omega^2) \right] \\
+ i \frac{N}{2} \int \frac{d\omega}{2\pi} \left[ \text{Tr} \log (H_c - \omega^2) - \text{Tr} \log (H_{VAC} - \omega^2) \right],
\] (2.19)

where

\[
H_{VAC} = -\partial_x^2 + m^2
\] (2.20)
is the hamiltonian corresponding to the vacuum configuration. We are not done yet, as the integrals over $\omega$ in (2.19) still diverge logarithmically with the UV cutoff $\Lambda$. However, as will be clear from the explicit calculations in the next section, this divergence is canceled by the logarithmic $\Lambda$ dependence of the bare coupling $g^2$, as determined by (2.5) and (2.6).

The renormalized quantity $\mathcal{E}[\sigma(x)]$ thus defined in (2.19) is the mass of the static fermion bag.
3 Extremal Static Fermion Bags

Now that we have written the energy-functional (2.19) of a static configuration, or a fermion bag, our next step is to identify those fermion bags on which (2.19) is extremal.

The energy functional (2.19) is, in principle, a complicated and generally unknown functional of $\sigma(x)$ and of its derivatives. Thus, the extremum condition $\frac{\delta \mathcal{E}[\sigma]}{\delta \sigma(x)} = 0$, as a functional equation for $\sigma(x)$, seems untractable. The considerable complexity of the functional equations that determine the extremal $\sigma(x)$ configurations is the source of all difficulties that arise in any attempt to solve the model under consideration.

DHN found a way around this difficulty. They have discovered that the extremal static $\sigma(x)$ configurations are necessarily reflectionless [1]. Let us briefly recall the arguments of [1]. DHN used inverse scattering techniques [3] to express the energy functional $\mathcal{E}[\sigma]$ (2.19) in terms of the so-called “scattering data” associated with, e.g., the hamiltonian $H_b$ in (2.16), and thus with $\sigma(x)$. The scattering data associated with $H_b$ are [3] the reflection amplitude $r(k)$ of $H_b$ (where $k$ is the momentum of the scattering state), the number $K$ of bound states in $H_b$ and their corresponding energies $0 \leq \omega_n^2 \leq m^2, (n = 1, \cdots K)$, and also additional $K$ parameters $\{c_n\}$, where $c_n$ has to do with the normalization of the $n$th bound state wave function $\psi_n$.

More precisely, The $n$th bound state wave function, with energy $\omega_n^2$, must decay as $\psi_n(x) \sim \text{const.} \exp -\kappa_n x$ as $x \rightarrow \infty$, where

$$0 < \kappa_n = \sqrt{m^2 - \omega_n^2}. \quad (3.1)$$

If we impose that $\psi_n(x)$ be normalized, this will determine the constant coefficient as $c_n$. (With no loss of generality, we may take $c_n > 0$.)

Thus, to summarize, in the inverse scattering technique, one trades the independent variables $\sigma(x)$ for the scattering data. Then, one looks for scattering data that extremize $\mathcal{E}[\sigma]$.

The key point is that $\mathcal{E}[\sigma]$ depends on $r(k)$ only through certain dispersion inte-
grals involving \( \log(1 - |r(k)|^2) \). Thus, the saddle point condition
\[
\frac{\delta \mathcal{E}[\sigma]}{\delta r^*(k)} = 0, \quad k \geq 0
\]
is almost trivially solved by
\[
r(k) = 0.
\]
Moreover, there seem to be no other solutions of (3.2). In other words, extremal static \( \sigma(x) \) configurations are necessarily reflectionless.

This restriction on \( \sigma(x) \) is very powerful, since once the reflection amplitude, i.e., the function \( r(k) \), is eliminated out of the scattering data, all that remains is a discrete set of \( 2K \) real parameters:
\[
0 \leq \omega_1 < \omega_2 < \cdots < \omega_K < m, \quad c_1, c_2, \cdots, c_K.
\]
Thus, the space of all reflectionless backgrounds \( \sigma(x) \) is parametrized by a discrete set of real parameters. We still have to extremize the energy functional \( \mathcal{E}[\sigma] \) with respect to these parameters.

Note, that due to the elementary fact, that the spectrum of the one dimensional Schrödinger operator \( H_b \) cannot be degenerate, all the \( \omega_n \)'s must be different from each other. Thus, the inequalities in (3.4) are strict. (See also the remark following (B.3).)

We see that the formidable problem of finding the extremal \( \sigma(x) \) configurations of the energy functional \( \mathcal{E}[\sigma] \) (2.19), is reduced to the simpler problem of extremizing an ordinary function \( \mathcal{E}(\omega_n, c_n) = \mathcal{E}[\sigma(x; \omega_n, c_n)] \) with respect to the \( 2K \) parameters \( \{c_n, \omega_n\} \) that determine the reflectionless background \( \sigma(x) \). If we solve this ordinary extremum problem, we will be able to calculate the mass of the fermion bag.

In Appendix B we summarized (in some detail) the necessary facts about reflectionless potentials and the diagonal resolvents associated with them. (We will need these resolvents below.) In particular, Eqs. (B.38) and (B.39) of Appendix B tell us how the \( 2K \) parameters determine \( \sigma(x) \) explicitly. For the particular examples with \( K = 1 \) and \( K = 2 \), see (B.46) and (B.64), respectively.
3.1 Extremal Static Reflectionless Fermion Bags

The energy functional $E[\sigma]$, evaluated on a reflectionless $\sigma(x)$ configuration, is an ordinary function of the parameters (3.4) which define $\sigma(x)$. This function (when defined with the renormalized $E$ in (2.19))

$$M(\omega_n, c_n) = E[\sigma(x; \omega_n, c_n)] \quad (3.5)$$

is the mass of the corresponding fermion bag. Thus, the extremal bags are determined by solving

$$\frac{\partial M}{\partial \omega_n} = 0 \quad , \quad \frac{\partial M}{\partial c_n} = 0. \quad (3.6)$$

Let $\alpha$ be any one of the $2K$ parameters in (3.4). Then,

$$\frac{\partial M}{\partial \alpha} = \int_{-\infty}^{\infty} dx \delta E[\sigma] \frac{\delta \sigma(x)}{\delta \alpha}, \quad (3.7)$$

where $\frac{\delta E[\sigma]}{\delta \sigma(x)}$ is evaluated at the appropriate reflectionless $\sigma(x)$.

3.1.1 The Extremum Conditions on the Mass $M(\omega_n, c_n)$, Its Flat Directions, and Collective Coordinates.

In order to calculate $\frac{\delta E[\sigma]}{\delta \sigma(x)}$ around a generic $\sigma(x)$ we start with the elementary identity

$$\delta \text{Tr} \log \left( -\partial_x^2 + V - \omega^2 \right) = \int_{-\infty}^{\infty} dx \langle x| \frac{1}{-\partial_x^2 + V - \omega^2} |x \rangle \delta V(x), \quad (3.8)$$

and apply it to (2.19), with $H_b$ and $H_c$ defined in (2.16). For convenience, let us record here the corresponding potentials,

$$V_b = \sigma^2 - \sigma' \quad \text{and} \quad V_c = \sigma^2 + \sigma'. \quad (3.9)$$

Thus, using (2.17) (or (A.31)), which relate the diagonal resolvents of $H_b$ and $H_c$, respectively, to the entries $B(x)$ and $C(x)$ in the diagonal resolvent (2.11), we arrive at

$$\delta E[\sigma] = \frac{1}{g^2} \int dx \sigma \delta \sigma + i \frac{N}{2} \int \frac{d\omega}{2\pi} \int dx \left[ \left( \frac{B(x)}{\omega} \right) \delta V_b(x) - \left( \frac{C(x)}{\omega} \right) \delta V_c(x) \right]. \quad (3.10)$$
By substituting (3.10) into (3.7) we obtain
\[ \frac{\partial M}{\partial \alpha} = \int_{-\infty}^{\infty} dx \left\{ \frac{1}{2g^2} \frac{\partial \sigma^2}{\partial \alpha} + \frac{iN}{2} \int \frac{d\omega}{2\pi} \left[ \left( \frac{B(x)}{\omega} \right) \frac{\partial V_b(x)}{\partial \alpha} - \left( \frac{C(x)}{\omega} \right) \frac{\partial V_c(x)}{\partial \alpha} \right] \right\}. \] (3.11)

As was mentioned above, the diagonal resolvent of the Dirac operator in reflectionless \( \sigma(x) \) backgrounds was evaluated in Appendix B (see section B.2). Thus, our next step is to substitute \( B(x) \) and \(-C(x)\) from (B.32) in (3.11). Let us do this explicitly for \( B(x) \). According to (B.32)
\[ B(x) = \langle x | \frac{\omega}{H_b - \omega^2} | x \rangle = \frac{\omega}{2\sqrt{m^2 - \omega^2}} \left( 1 - 2 \sum_{n=1}^{K} \frac{\kappa_n \psi_n^2}{\omega^2 - \omega_n^2} \right), \] (3.12)
where \( \psi_n \) are the (normalized) bound state wave functions of \( H_b \) (and we have also used \( k = i\sqrt{m^2 - \omega^2} \)). Thus,
\[ \int_{-\infty}^{\infty} dx \left( \frac{B(x)}{\omega} \right) \frac{\partial V_b(x)}{\partial \alpha} = \frac{1}{2\sqrt{m^2 - \omega^2}} \int_{-\infty}^{\infty} dx \left( 1 - 2 \sum_{n=1}^{K} \frac{\kappa_n \psi_n^2}{\omega^2 - \omega_n^2} \right) \frac{\partial V_b(x)}{\partial \alpha} = \]
\[ \frac{1}{2\sqrt{m^2 - \omega^2}} \left[ \int_{-\infty}^{\infty} dx \frac{\partial V_b(x)}{\partial \alpha} - 2 \sum_{n=1}^{K} \kappa_n \langle \psi_n | \frac{\partial V_b}{\partial \alpha} | \psi_n \rangle \right]. \] (3.13)

From first order perturbation theory we know that
\[ \langle \psi_n | \frac{\partial V_b}{\partial \alpha} | \psi_n \rangle = \langle \psi_n | \frac{\partial H_b}{\partial \alpha} | \psi_n \rangle = \frac{\partial \omega_n^2}{\partial \alpha}. \] (3.14)
Thus, we may simplify (3.13) further and obtain
\[ \int_{-\infty}^{\infty} dx \left( \frac{B(x)}{\omega} \right) \frac{\partial V_b(x)}{\partial \alpha} = \frac{1}{2\sqrt{m^2 - \omega^2}} \left[ \int_{-\infty}^{\infty} dx \frac{\partial V_b(x)}{\partial \alpha} - 2 \sum_{n=1}^{K} \kappa_n \left( \frac{\partial \omega_n^2}{\partial \alpha} \right) \right]. \] (3.15)

Recall that \( H_b \) and \( H_c \) are isospectral, save, possibly, a zero energy ground state, which can appear in the spectrum of only one of these operators. Since the zero energy ground state (if it exists) is the only difference in the energy spectra of \( H_b \) and \( H_c \), and since, in any case, it obviously does not contribute to the sum in (3.15), we conclude that the contribution of \(-C(x)\) to (3.11) is the same as (3.15), but with \( V_c \) instead of \( V_b \) in the first term in (3.15).
Thus, combining the contributions of of $B(x)$ and $-C(x)$ into (3.11) we obtain

$$\frac{\partial M}{\partial \alpha} = \left( \frac{1}{2g^2} + iN \int \frac{d\omega}{2\pi} \frac{1}{2\sqrt{m^2 - \omega^2}} \right) \int_{-\infty}^{\infty} dx \frac{\partial \sigma^2}{\partial \alpha} - iN \sum_{n=1}^{K} \int \frac{d\omega}{2\pi} \frac{1}{\sqrt{m^2 - \omega^2}} \frac{\kappa_n \left( \frac{\partial \omega_n^2}{\partial \alpha} \right)}{\omega^2 - \omega_n^2}$$

(3.16)

(where in the first term we have used $V_b + V_c = 2\sigma^2$).

Consider now the integrals over $\omega$ in (3.16). We will show in Appendix C (see (C.4)) that the bare gap equation may be written as

$$m g^2 + iN \int \frac{d\omega}{2\pi} \frac{m}{\sqrt{m^2 - \omega^2}} = 0.$$  

Thus, the first term in (3.16) vanishes, and

$$\frac{\partial M}{\partial \alpha} = -iN \sum_{n=1}^{K} \int \frac{d\omega}{2\pi} \frac{1}{\sqrt{m^2 - \omega^2}} \frac{\kappa_n \left( \frac{\partial \omega_n^2}{\partial \alpha} \right)}{\omega^2 - \omega_n^2}.$$  

(3.17)

The remaining integral over $\omega$ is UV-convergent. Thus, renormalization of the coupling constant $g^2$, based on the gap equation (see (2.5)), also renders (3.16) finite.

Next, observe from (3.17), that if $\alpha$ is one of the coefficients $c_k$ in (3.4), then

$$\frac{\partial M}{\partial c_k} = 0$$  

(3.18)

identically! This does not produce any condition on the $c_k$’s. The energy functional $\mathcal{E}[\sigma]$ (2.19), evaluated on a reflectionless $\sigma(x; \omega_n, c_n)$, is independent of the coefficients $c_k$ that do affect the shape of $\sigma(x)$. The $c_k$’s are thus flat directions of $\mathcal{E}[\sigma]$ in the space of all reflectionless $\sigma(x)$ configurations. In fact, we show in Appendix B, that the $c_k$’s (or more precisely, their logarithms) are collective translational coordinates of the fermion bag $\sigma(x)$ (see e.g., (B.46) and (B.64)), which is a familiar concept in soliton and instanton physics. One of these coordinates, corresponds, of course, to global translations of the bag as a whole.

The full effective action functional $S_{\text{eff}}[\sigma(x,t)]$ contains, in principle, all information on dynamics of space-time dependent $\sigma(x,t)$ bags, or solitons. However, it is a complicated and almost intractable object. Some progress may be achieved, perhaps, by trying to extend the periodic time-dependent solution of (2.3) which was
guessed by DHN in [1]. As another small step towards understanding time dependent soliton dynamics from $S_{\text{eff}}$, one might consider elevating the $c_k$’s, which determine the shape of static reflectionless bags, into slowly varying functions of time, and thus study soliton dynamics in the framework of an adiabatic approximation, in the spirit of [28, 29].

Let us return to (3.17), and consider the remaining possibility $\alpha = \omega_n$. In this case we obtain

$$\frac{\partial M}{\partial \omega_n} = -iN \int \frac{d\omega}{2\pi} \frac{1}{\sqrt{m^2 - \omega^2}} \frac{2\kappa_n \omega_n}{\omega^2 - \omega_n^2},$$

(3.19)

which has to vanish for extremal bags, and thus produces an equation to determine $\omega_n$. Note that (3.19) involves only one particular $\omega_n$. Thus, the extremal $\omega_n$’s are determined independently of each other, except for the constraint that they cannot coincide.

We observe from (3.19) that $\omega_1 = 0$ is a possible solution of $\frac{\partial M}{\partial \omega_n} = 0$. Thus, there are extremal bags for which $H_b$ (or $H_c$) has a zero energy ground state. As explained in Appendix A (see subsection A.1.1), a zero mode $\omega_1 = 0$ occurs as a solution of (3.19) if and only if $\sigma(x)$ has non-trivial topology. All other solutions $\omega_n^2$ of $\frac{\partial M}{\partial \omega_n} = 0$ must be strictly positive.

In our calculations so far we used DHN’s result that extremal static $\sigma(x)$ configurations are necessarily reflectionless, and looked for the extremal reflectionless configurations. As a consistency check of our calculations, we verify in Appendix C that vanishing of the right hand side of (3.19) guarantees that the corresponding reflectionless $\sigma(x)$ is indeed extremal among all possible static configurations.

3.1.2 Solution of the Extremum Conditions and Computation of the Soliton’s Mass.

In order to determine the extremal $\sigma(x)$ configurations, we have to determine the $\omega_n$’s from (3.19). With no loss of generality we will assume henceforth that $\omega_n > 0$.  

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6According to calculations in the next subsection, the integral $\int \frac{d\omega}{2\pi \sqrt{m^2 - \omega^2}} \frac{1}{\sqrt{m^2 - \omega^2}}$ (along the appropriate contour) is finite. Thus, the right-hand side of (3.19) vanishes at $\omega_1 = 0$. 

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In order to determine the non-vanishing $\omega_n$’s, we have to evaluate the (UV finite) integral
\[
I(\omega_n) = \int_{C} \frac{d\omega}{2\pi i} \frac{1}{\sqrt{m^2 - \omega^2}} \frac{1}{\omega^2 - \omega_n^2}.
\] (3.20)

To this end we have to choose the proper contour $C$, and thus we have to invoke our understanding of the physics of fermions: The Dirac equation $(i\gamma^\alpha - \sigma(x))\psi = 0$ (Eq. (1.6)) in the background $\sigma(x)$ under consideration has a pair of charge-conjugate bound states at $\pm \omega_n$. These are the simple poles in (3.20). The continuum states appear as the two cuts along the real axis with branch points at $\pm m$ (see Figure 1).

The bound states at $\pm \omega_n$ are to be considered together due to the charge conjugation invariance of the GN model, as we discuss in Appendix D (see in particular subsection D.2.2). Due to Pauli’s principle, we can populate each of the bound states $\pm \omega_n$ with up to $N$ (non-interacting) fermions. In such a typical multiparticle state, the negative frequency state is populated by $N - h_n$ fermions and the positive frequency state contains $p_n$ fermions. In the parlance of Dirac’s hole theory, we have thus created a many fermion state, with $p_n$ particles and $h_n$ holes occupying the pair of charge-conjugate bound states at energies $\pm \omega_n$. Let us name this many body state a $(p_n, h_n)$ configuration.

Since in this paper we are interested only in solitons in their ground state, we must take all states in the Dirac sea completely filled, and no positive energy scattering states.

Mathematically, we thus have to let $C$ enclose the cut on the negative $\omega$ axis $N$ times, and then go $N - h_n$ times around the pole at $-\omega_n$, and $p_n$ times around the pole at $\omega_n$, as shown in Figure 1.
Fig. 1: The contour $C$ in the complex $\omega$ plane in Eq. (3.20). The continuum states appear as the two cuts along the real axis with branch points at $\pm m$, and the bound states are the poles at $\pm \omega_n$. The contour wraps $N$ times around the cut on the negative $\omega$ axis, and then go $N - h_n$ times around the pole at $-\omega_n$, and $p_n$ times around the pole at $\omega_n$.

Thus, the integral (3.20) depends also on $h_n$ and $p_n$. In this way, we obtain

$$I(\omega_n, p_n, h_n) = \frac{1}{2\omega_n \kappa_n} \left( \frac{2N}{\pi} \arctan \frac{\omega_n}{\kappa_n} + p_n - (N - h_n) \right)$$

(3.21)

(with $\kappa_n = \sqrt{m^2 - \omega_n^2}$). The first term in (3.21) is the contribution of the part of $C$ wrapped around the negative cut, and the other two terms come, respectively, from the poles at $\omega_n$ and $-\omega_n$. The function $I(\omega_n, p_n, h_n)$ depends on $h_n$ and $p_n$ only through their sum

$$\nu_n = p_n + h_n,$$

(3.22)

the total number of particles and holes in the pair of charge conjugate bound states at $\pm \omega_n$. Pauli’s Principle allows all $0 \leq \nu_n \leq 2N$. However, it turns out that the allowed range for $\nu_n$ in extremal configurations is restricted to only half that range:

Substituting (3.21) into (3.19) we obtain

$$\frac{\partial M}{\partial \omega_n} = 2\kappa_n \omega_n I(\omega_n, \nu_n) = \left( \frac{2N}{\pi} \arctan \frac{\omega_n}{\kappa_n} + \nu_n - N \right).$$

(3.23)

Thus, the extremal value $\omega_n^*$ satisfies

$$\frac{\omega_n^*}{\kappa_n^*} = \frac{\omega_n^*}{\sqrt{m^2 - \omega_n^*}} = \cot \left( \frac{\pi \nu_n}{2N} \right) \geq 0,$$

(3.24)

rendering

$$\omega_n^* = m \cos \left( \frac{\pi \nu_n}{2N} \right).$$

(3.25)

---

This expression, for $h_n = 0$, is quoted, e.g., in [1, 16].
Thus, since $\omega_n^* \geq 0$, we must have\(^8\) $\nu_n \leq N$.

However, the end-points of this range cannot be realized physically. Concerning $\nu_n = 0$, note that if the pair of energy levels corresponding to (3.25) are empty, then $\omega_n^* = m$ plunges into the continuum and ceases to be a bound state. Thus, the self-consistent, extremal energy levels (3.25) must trap fermions in order to exist. This fact will be related to soliton stability in Section 4 (see also the discussion around (3.32) in this section). At the other end-point $\nu_n = N$, we obtain $\omega_n^* = 0$. As we have already mentioned (recall the discussion in subsection A.1.1 in Appendix A, which we have already referred to following (2.16) and (3.19)), $\omega_n^* = 0$ occurs if and only if $\sigma(x)$ has non-trivial topology, and does not depend on dynamical details such as the value of $\nu_n$. Thus, $\nu_n = N$ cannot occur in a topologically trivial soliton. What is perhaps less obvious, is that we cannot have $\nu_n = N$ in a topologically non-trivial soliton either. To clarify this assertion, consider an extremal topologically non-trivial soliton with $0 = \omega_1^* < \omega_2^* < \cdots < \omega_K^*$. Then, form a sequence of such solitons in which $\nu_2$ increases. As $\nu_2$ tends to $N$, $\omega_2^*$ tends to coalesce with $\omega_1^* = 0$. This cannot happen, because the spectrum $\{\omega_n^{*2}\}$ of the one-dimensional Schrödinger operator $H_b$ (or $H_c$) is not degenerate. Putting all these facts together we conclude that $\nu_n$ is restricted to the range

$$0 < \nu_n < N.$$  \hspace{1cm} (3.26)

We would like at this point to integrate (3.23) and find the mass $M(\omega_1, \ldots, \omega_K)$. To this end it is convenient to introduce \cite{1} the angular parameter $\theta_n$ through the relations

$$\omega_n = m \cos \theta_n \quad \text{and} \quad \kappa_n = m \sin \theta_n,$$  \hspace{1cm} (3.27)

where positivity of $\omega_n$ and $\kappa_n$ implies $0 \leq \theta_n \leq \frac{\pi}{2}$. In particular, note that the extremal value $\omega_n^*$ in (3.25) corresponds to

$$\theta_n^* = \frac{\pi \nu_n}{2N}.$$  \hspace{1cm} (3.28)

---

\(^8\)This can be already seen from (3.23). From the actual computation of the contribution of the part of $\mathcal{C}$ wrapped around the negative cut, the first term in (3.21), it is clear that $\arctan \frac{\omega_n}{\kappa_n} \geq 0$. Thus, $\frac{\partial M}{\partial \omega_n} = 0$ has a solution only in the range (3.26).
Recall that all the $\omega_n$’s which parametrize a given static soliton, must be different from each other (see the paragraph following (3.4)). Consequently, all the $\nu_n$ must be different from each other, and so must be the $\theta^*_n$’s.

In terms of $\theta_n$, we may rewrite (3.19) as

$$\frac{\partial M}{\partial \omega_n} = \frac{1}{m \sin \theta_n} \frac{\partial M}{\partial \theta_n} = \nu_n - \frac{2N\theta_n}{\pi}.$$  

(3.29)

Finally, we can integrate (3.29) back, to obtain $M(\theta_1, \cdots \theta_K)$ explicitly. Thus,

$$M(\theta_1, \cdots, \theta_K) = m \sum_{n=1}^{K} \left[ \left( \nu_n - \frac{2N\theta_n}{\pi} \right) \cos \theta_n + \frac{2N}{\pi} \sin \theta_n \right] + \text{const.}$$  

(3.30)

Note that the mass (3.30) depends only on the $\theta_n$’s which correspond to strictly positive $\omega_n$’s. The integration constant in (3.30) depends on the topology of $\sigma(x)$, and will be determined below according to physical considerations.

Eq. (3.30) is the mass of a fermion bag with an arbitrary reflectionless profile $\sigma(x)$. We are interested only in extremal configurations. Substituting (3.28) in (3.30), we obtain the mass of the extremal bag as

$$M(\theta^*_1, \cdots, \theta^*_K) = \frac{2Nm}{\pi} \sum_{\omega_n > 0}^{K} \sin \left( \frac{\pi \nu_n}{2N} \right) + \text{const.}$$  

(3.31)

The extremal value (3.25) of the parameter $\omega_n$ is determined by the total number $\nu_n$ of particles and holes trapped in the bound states of the Dirac equation at $\pm \omega_n$, and not by the numbers of trapped particles and holes separately. As we explain in Appendix D (see subsection D.2.2), this fact is a manifestation of the underlying $O(2N)$ symmetry, which treats particles and holes symmetrically. It indicates that this pair of bound states gives rise to an $O(2N)$ antisymmetric tensor multiplet of rank $\nu_n$ of soliton states. The states in this multiplet correspond to all the possible $(p_n, h_n)$ configurations, with $\nu_n = p_n + h_n$ fixed, subjected to Pauli's principle. There are $\sum_{k=0}^{\nu_n} C_N^k C_N^{\nu_n-k} = C_{2N}^{\nu_n} = (2N)!/\nu_n!(2N-\nu_n)!$ such states, precisely the dimension of an $O(2N)$ antisymmetric tensor of rank $\nu_n$. Due to the restriction (3.26) only tensors of ranks $0 < \nu_n < N$ are allowed.
The antisymmetric tensors of higher ranks (in the range \( N < \nu_n \leq 2N \)), each of which is dual to an antisymmetric tensor in the allowed range (3.26), are not realized as *extremal* fermion bags (their ranks cannot be interpreted as the total number of particles and holes trapped by the bag). Note, however, as a mathematical fact, that the mass formulas (3.30) and (3.31) are invariant against the duality transformation of tensorial ranks \( \nu_n \rightarrow 2N - \nu_n \), performed simultaneously with \( \theta^*_n \rightarrow \pi - \theta^*_n \). This is equivalent to \( \omega^*_n \rightarrow -\omega^*_n \). Thus, a \((p_n, h_n)\) configuration belonging to an antisymmetric tensor multiplet of rank \( 0 < \nu_n < N \) is mapped in this way onto an \((N - h_n, N - p_n)\) configuration belonging to the dual antisymmetric tensor of rank \( 2N - \nu_n \). Evidently, any two dual antisymmetric tensor multiplets have the same dimension, and make identical contributions to the mass formulas. Thus, these duality transformations may be used to extend the domain of definition of (3.30) and (3.31) to all \( O(2N) \) antisymmetric tensors \( 0 \leq \nu_n \leq 2N \).

The bound state at \( \omega_1 = 0 \), which arises if and only if \( \sigma(x) \) has non-trivial topology, behaves differently from the previous case. Due to its topological origin, this mode is stable and cannot flow away from zero as the other parameters which define \( \sigma(x) \) vary. In particular, it remains unchanged as the number of trapped fermions varies, in contrast with (3.25). Since this bound state is unpaired, it can trap up to \( N \) fermions. Thus, it gives rise to a \( 2^N \) dimensional multiplet of soliton states, which, as we show in subsection D.2.1, is the spinorial representation of \( O(2N) \) [30].

Let us determine now the integration constant in (3.31). As was mentioned earlier, this integration constant depends on the topological charge carried by the corresponding extremal \( \sigma(x) \) configuration. We have already mentioned in subsection 2.1, that topologically trivial solitons are stable because of the binding energy released by the trapped fermions, and cannot form without such binding. Thus, the binding energy

\[
B(\theta_1^* \cdots, \theta_K^*) = m \sum_{n=1}^{K} \nu_n - M(\theta_1^* \cdots, \theta_K^*)
\]

of a topologically trivial soliton, which measures the stability of the soliton, should tend to zero as the number of trapped fermions tends to zero. In other words, if each of the pairs of levels \( \pm \omega_n \) traps a small number \( \nu_n << N \) of fermions, we expect that
$M(\theta_1^*, \cdots, \theta_K^*) \simeq m \sum_{n=1}^{K} \nu_n$. Imposing this physical constraint on (3.31) determines the integration constant (in the topologically trivial case) to be zero. This cannot be true in the topologically non-trivial sector, since the the integration constant in (3.31) is evidently the minimal mass in that sector. The latter is the mass of the topologically non-trivial soliton which has a single bound state at $\omega_1 = 0$, namely the CCGZ kink (or antikink). Its mass is

$$M_{\text{kink}} = \frac{Nm}{\pi},$$

(3.33)
as quoted in [8] (see also Eq. (3.30) in [1]), and is independent of the number $n_0$ of fermions trapped in the bound state at $\omega_1 = 0$, as we mentioned in the previous paragraph. In Appendix E we will present our own derivation of (3.33), based on (2.19) and on a simple dimensional argument. In addition, we will provide below (see the passage following (3.36)) a simple physical argument for the correctness of (3.33).

Thus, gathering all these facts together, we conclude that

$$M(\nu_1, \cdots, \nu_K; q) = \frac{2Nm}{\pi} \sum_{n=1}^{K} \delta_{\omega_n, 0} \sin \left( \frac{\pi \nu_n}{2N} \right) + \frac{Nm}{\pi} \delta_{|q|, 1},$$

(3.34)

where $q$ is the topological charge (2.9) carried by the soliton (and all the ranks $\nu_n$ are different from each other).

From (3.33) and (3.34) we see that the kink, as well as other extremal solitons (whose ranks $\nu_n$ are a finite fraction of $N$), are heavy objects. Their masses are of $O(N)$. In fact, this should be expected, based on our experience with soliton physics in weakly interacting field theories: in this paper we study the GN model (1.1) in the large $N$ limit, in which $N \sim g^{-2}$, and thus all soliton masses are of $O(g^{-2})$.

### 3.1.3 Summary: Masses, Profiles and $O(2N)$ Quantum Numbers of Extremal Fermion Bags

We complete the task of determining the extremal static soliton configurations by combining the results of the previous discussion and the results of the analysis made in sections B.2, B.3 and D.2 in the appendices. We summarize our findings as follows:
(a) topologically trivial fermion bags, $q = 0$

An extremal static, topologically trivial soliton, bearing $K$ pairs of bound states, has its $K \omega_n$’s in (3.4) determined by (3.25), with the $K$ moduli $c_n$ left as arbitrary parameters. Its mass is given by (3.34) with $q = 0$. Its profile $\sigma(x)$ is computed in Appendix B (see section B.2) and is given by (B.38). According to Appendix D (see subsection D.2.2) we should refer to this object more precisely as an $O(2N)$ multiplet of solitonic states. This multiplet transforms under $O(2N)$ as the direct product of the $K$ individual antisymmetric tensors associated with the $K \omega_n$’s.

In particular, the case $K = 1$ with $\omega_1 = m \cos \left( \frac{\pi \nu}{2N} \right) > 0$, in which the pair of bound states at $\pm \omega_1$ trap $\nu_1 = \nu$ fermions, is the so-called DHN soliton. Its mass is

$$M(\nu) = \frac{2Nm}{\pi} \sin \left( \frac{\pi \nu}{2N} \right)$$  \hspace{1cm} (3.35)

and its profile is given by (B.46):

$$\sigma(x) = m + \kappa \tanh(\kappa x) - \kappa \tanh \left[ \kappa x + \frac{1}{2} \log \left( \frac{m + \kappa}{m - \kappa} \right) \right]$$  \hspace{1cm} (3.36)

(where we have set $\sigma(\infty) = m$ and $x_0 = 0$ in (B.46)), with $\kappa = m \sin \left( \frac{\pi \nu}{2N} \right)$. These formulas agree with the results quoted in Eqs. (3.27) and (3.28) of [1]. The profile (3.36) is that of a bound state of a kink and an antikink, with interkink distance $\frac{1}{2\kappa} \log \left( \frac{m + \kappa}{m - \kappa} \right)$. This interkink distance diverges in the limit $\kappa \to m$, i.e., when $\nu \to N$, the inaccessible end-point of (3.26). Heuristically, we can understand this divergence as a necessary compromise between dynamics and topology: As $\nu$ tends to $N$, $\omega = m \cos \left( \frac{\pi \nu}{2N} \right)$ tends to zero, which is not in the spectrum of the corresponding Dirac operator $i\partial - \sigma(x)$, since $\sigma(x)$ is topologically trivial. The best our soliton can do is to split into infinitely separated kink and antikink. Probing any finite neighborhood around either of these defects may lead us to the illusion that we are in the topologically non-trivial sector.

One might perhaps interpret this as a situation in which the infinitely separated kink and antikink are bound at threshold. (This is subjected to the plausible assumption that there are no long-range intersoliton forces in this model, since the mass gap is finite, and there are no gauge interactions.) Thus, (3.35) must tend, in the limit $\nu \to N$, to $2M_{kink}$, from which we see that $M_{kink} = \frac{Nm}{\pi}$, in accordance with (3.33).
(b) topologically nontrivial fermion bags, $q = \pm 1$

An extremal static, topologically non-trivial $\sigma(x)$ configuration, with $2K$ parameters (3.4), has an unpaired bound state at $\omega_1 = 0$, and additional $K-1$ pairs of bound states at $\pm \omega_n$, which are determined by (3.25). Its mass is given by (3.34) with $|q| = 1$, and its profile is determined as plus or minus the profile given by (B.39) (depending on the topological charge $q$ it carries). As explained in detail in Appendix D (see subsections D.2.1 and D.2.2), it gives rise to a multiplet of solitonic states which transform under $O(2N)$ as the direct product of one factor of the $2^N$ dimensional (reducible) spinor representation of $O(2N)$ (associated with the bound state at $\omega_1 = 0$), and the $K-1$ individual antisymmetric tensors associated with the remaining non-vanishing $\omega_n$’s. In particular, the case $K = 1$ corresponds to the CCGZ kink (or antikink) with mass (3.33) and profile $\sigma(x) = \pm mtanh(mx)$ (see (B.49)). These kinks are thus pure $O(2N)$ isospinors [30].

### 3.2 Fermion Number Content of the Solitonic Multiplets.

Expressions for the expectation values of fermion bilinear operators in a static $\sigma(x)$ background are derived in Appendix A (see Section A.3). They are given as dispersion integrals over linear combinations of the entries of the diagonal resolvent (2.11). In particular, according to (A.43), (A.44) and (A.48), the regularized fermion density in a given background $\sigma(x)$ is

$$
\langle j^0(x) \rangle_{\text{reg}} = iN \int \frac{d\omega}{2\pi} \left[ (B(x) - B_{\text{VAC}}) - (C(x) - C_{\text{VAC}}) \right],
$$

(3.37)

where we have subtracted the UV-divergent vacuum contributions. For evaluating $\langle j^0(x) \rangle_{\text{reg}}$ in reflectionless $\sigma(x)$ backgrounds, we will substitute, $B(x)$ and $-C(x)$ from (B.32) (or equivalently, the expression (3.12) for $B(x)$, and its analog for $-C(x)$) in (3.37). For extremal reflectionless $\sigma(x)$ backgrounds, the focus of our interest in this paper, the non-vanishing $\omega_n$’s are determined by (3.25). Thus, for each $\omega_n > 0$, we will assume that the fermions which occupy the pair of energy levels $\pm \omega_n$ form a $(p_n, h_n)$ configuration, with $p_n + h_n = \nu_n$, in accordance with (3.22). If $\omega_1 = 0$ (i.e., if $\sigma(x)$ has non-trivial topology), we assume that this level is occupied
by $0 \leq n_0 \leq N$ fermions.

Let us evaluate the contribution of the $B$-term in (3.37) explicitly. From (3.12) and (A.44) we obtain this contribution as

$$\sum_{n=1}^{K} \int_{C_n} \frac{d\omega}{2\pi i} \frac{\omega}{\sqrt{m^2 - \omega^2}} \frac{\kappa_n \psi_{Bn}^2}{\omega_n^2 - \omega^2} ,$$

(3.38)

where $\psi_{Bn}$ is the $n$th bound state wave function of $H_b$, and $C_n$ is the contour in Fig. 1 (with the obvious exception, that if $\omega_1 = 0$, then the contour $C_1$ wraps $n_0$ times around the pole at $\omega = 0$). For each $\omega_n > 0$, the contribution to (3.38) coming from the integral around the left cut in Fig. 1 is $-\frac{N}{2} \psi_{Bn}^2$, the contribution from the pole at $-\omega_n$ is $\frac{N-h_n}{2} \psi_{Bn}^2$, and that from the pole at $\omega_n$ is $\frac{p_n-h_n}{2} \psi_{Bn}^2$. Adding these three terms, we obtain the total contribution of the pair of levels at $\pm \omega_n$ to (3.38) simply as $\frac{p_n-h_n}{2} \psi_{Bn}^2$. If $\sigma(x)$ has non-trivial topology, and carries topological charge $q = 1$, then according to the discussion following Eqs. (A.18), (A.19), only $H_b$ has a normalizable zero mode $\psi_{B0}$, and we must set $\omega_1 = 0$ in (3.38). In this case, the contribution to (3.38) is $\left( -\frac{N}{2} + n_0 \right) \psi_{B0}^2$, where the first term comes from the integral around the left cut in Fig. 1 (as before), and the second term comes from the pole at $\omega = 0$.

Combining all these results together, we obtain that the $B$-term in (3.37) yields

$$\left( -\frac{N}{2} + n_0 \right) \psi_{B0}^2 \delta_{q,1} + \sum_{n=1}^{K} \frac{p_n-h_n}{2} \psi_{Bn}^2 .$$

(3.39)

Similarly, since $-C(x)$ is given by an expression analogous to (B.32), the $C$-term in (3.37) yields

$$\left( -\frac{N}{2} + n_0 \right) \psi_{C0}^2 \delta_{q,-1} + \sum_{n=1}^{K} \frac{p_n-h_n}{2} \psi_{Cn}^2 ,$$

(3.40)

where $\psi_{Cn}$ is the $n$th bound state wave function of $H_c$. (Recall from the discussion following Eqs. (A.18), (A.19) that if the topological charge $q = -1$, only $H_c$ has a normalizable zero mode $\psi_{C0}$.) Finally, combining (3.39) and (3.40) together, we obtain

$$\langle j^0(x) \rangle_{\text{reg}} = \left( -\frac{N}{2} + n_0 \right) \left( \psi_{B0}^2(x) \delta_{q,1} + \psi_{C0}^2(x) \delta_{q,-1} \right) + \sum_{n=1}^{K} (p_n-h_n) \frac{\psi_{Bn}^2(x) + \psi_{Cn}^2(x)}{2} .$$

(3.41)
Integrating over $x$, we obtain the expectation value of the total fermion number $N_f$ in the background of the extremal fermion bag, simply as

$$\langle N_f \rangle = \left( n_0 - \frac{N}{2} \right) \delta_{|q|,1} + \sum_{\omega_n > 0}^{K} (p_n - h_n).$$  \hspace{1cm} (3.42)

The terms in (3.41), associated with the positive $\omega_n$’s, have simple physical interpretation:

$$N_{f,\text{val}}^{(n)} = p_n - h_n = \nu_n - 2h_n,$$  \hspace{1cm} (3.43)

the number of particles minus the number of holes, is the *valence* fermion number of the $(p_n, h_n)$ configuration, which occupies the pair of bound states at $\pm \omega_n$, and

$$\frac{1}{2} (\psi_{Bn}^2(x) + \psi_{Cn}^2(x))$$

is the probability density to find any of these fermions in a small neighborhood of the point $x$. The $(p_n, h_n)$ configuration is a member of an $O(2N)$ antisymmetric tensor multiplet of rank $\nu_n < N$ (recall (3.26)). Thus, as we scan through all states in this multiplet, we see that $N_{f,\text{val}}^{(n)}$ has a symmetric spectrum

$$-\nu_n \leq N_{f,\text{val}}^{(n)} \leq \nu_n,$$  \hspace{1cm} (3.44)

in accordance with charge conjugation invariance.

Interpretation of the first term in (3.41), associated with the zero mode, is more delicate, and exhibits an interesting physical phenomenon. As in the previous case, $(\psi_{B0}^2(x) \delta_{q,1} + \psi_{C0}^2(x) \delta_{q,-1})$ is the probability density to find a fermion, trapped in the zero mode bound state, in a small neighborhood of the point $x$. However, unlike the previous case, the coefficient of this probability density is the valence number $n_0$ of fermions trapped in the zero mode, *shifted* by the contribution $-\frac{N}{2}$ of the vacuum (the filled Dirac sea). Thus, the fermion number associated with the zero mode is

$$N_f^{(0)} = n_0 - \frac{N}{2}.$$  \hspace{1cm} (3.45)

This quantity coincides with the fermion number operator $N_f^{(\text{spinor})}$ (D.16) which we constructed explicitly for the $O(2N)$ spinor representation in Appendix D.

Note that for $N$ odd, $N_f^{(0)}$ is not an integer! Eq.(3.45) thus exhibits fractional fermion number [31], a phenomenon which occurs because of the non-trivial topology
of the background interacting with the fermions, and is independent of other details of the background (such as $\sigma(x)$ being reflectionless). Furthermore, it is valid for any value of $N$, and has nothing to do with the large $N$ limit. Charge conjugation invariance of the GN model restricts the fractional part of $N_f^{(0)}$ to be either 0 or $\frac{1}{2}$. Indeed, it is $N_f^{(0)}$ as defined in (3.45) which acquires a symmetric spectrum $-\frac{N}{2} \leq N_f^{(0)} \leq \frac{N}{2}$, as required by charge conjugation invariance, and not the valence piece $n_0$ alone. For more details on fractionalization of fermion number in quantum field theory, see [33].

\footnote{The difference of any two eigenvalues of $N_f^{(0)}$ must be an integer, of course, and charge conjugation invariance implies that if $n$ is an eigenvalue, so is $-n$.}

\footnote{In other soliton bearing quantum field theoretic models, fermion number induced by solitons may acquire other rational, or even irrational values [32].}


4 Investigating Stability of Extremal Static Fermion Bags

The extremal static soliton multiplets which we encountered in the previous section, correspond, in the limit \( N \to \infty \), to exact eigenstates of the hamiltonian of the GN model. However, at large but finite \( N \), we expect some of these states to become unstable and thus to acquire small widths, similarly to the behavior of baryons in QCD with a large number of colors [34]. The latter are also solitonic objects and are analogous to the “multi-quark” bound states of the GN model.

Furthermore, we can imagine perturbing the GN action \((1.1)\) by a small perturbation, which is a singlet under all the discrete and continuous symmetries of the model (e.g., by adding to \((1.1)\) a term \( \epsilon \int d^2x \sigma^{2n} \), and ask which of the extremal fermion bags of the previous section are stable against such perturbations.

Under these circumstances, all possible decay channels of a given soliton multiplet must conserve, in addition to energy and momentum, \( O(2N) \) quantum numbers and topological charge.

It turns out that non-trivial results concerning stability may be established without getting into all the details of decomposing \( O(2N) \) representations, by imposing a simple necessary condition on the spectrum of the fermion number operator \( N_f \) in the multiplets involved in a given decay channel. As we have learned so far, a given static soliton multiplet is a direct product of \( O(2N) \) antisymmetric tensors and, for topologically non-trivial solitons, a factor of the spinor representation. The decay products of this soliton also correspond to a direct product of antisymmetric tensors and spinors.

We have also learned that the spectrum of \( N_f \) in the antisymmetric tensor and spinor representations is symmetric, namely, \(-N_{f,\text{max}}^f \leq N_f \leq N_{f,\text{max}}^f\). When we compose two such representations \( D_1, D_2 \), the spectrum of \( N_f \) in the composite representation \( D_1 \otimes D_2 \), will obviously have the range \(-N_{f,\text{max}}^f(D_1) - N_{f,\text{max}}^f(D_2) \leq N_f(D_1 \otimes D_2) \leq N_{f,\text{max}}^f(D_1) + N_{f,\text{max}}^f(D_2)\). In particular, each of the possible (i.e., integer or half-integer) eigenvalues in this range, will appear in at least one irreducible representation in the decomposition of \( D_1 \otimes D_2 \). More generally, the spectrum of \( N_f(D_1 \otimes D_2 \cdots \otimes D_L) \)
will have the range $|N_f(D_1 \otimes \cdots \otimes D_L)| \leq N_f^{\text{max}}(D_1) + \cdots + N_f^{\text{max}}(D_L)$.

Consider now a decay process, in which a parent static soliton, which belongs to a (possibly reducible) representation $D_{\text{parent}}$, decays into a bunch of other solitons, such that the collection of all irreducible representations associated with the decay products is $\{D_1, \ldots, D_L\}$ (in which a given irreducible representation may occur more than once). By $O(2N)$ symmetry, the representation $D_{\text{parent}}$ must occur in the decomposition of $D_1 \otimes D_2 \cdots \otimes D_L$. Thus, according to the discussion in the previous paragraph, if this decay process is allowed, we must have

$$N_f^{\text{max}}(D_{\text{parent}}) \leq N_f^{\text{max}}(D_1) + \cdots + N_f^{\text{max}}(D_L),$$

which is the necessary condition for $O(2N)$ symmetry we sought for. (Obviously, similar necessary conditions arise for the other $N-1$ components of the highest weight vectors of the representations involved.) The decay process under consideration must respect energy conservation, i.e., it must be exothermic. Thus, we supplement (4.1) by the requirement

$$M_{\text{parent}} \geq \sum_{\text{products}} M_k$$

(4.2)
on the masses $M_i$ of the particles involved.

For each of the static soliton multiplets discussed in the previous section, we will scan through all decay channels and check which of these decay channels are necessarily closed, simply by requiring that the two conditions (4.1) and (4.2) be mutually contradictory.

The oscillating time dependent DHN solitons [1] may appear as decay products in some of the channels. However, for a given assignment of $O(2N)$ quantum numbers of the final state, they will occur at a higher mass than the final state containing only static solitons$^{11}$. Thus, if a decay channel into a final state containing only static solitons is necessarily closed (in the sense that (4.1) and (4.2) are mutually contradictory).

\textsuperscript{11}In [1], DHN showed that the oscillating solitons occur in $O(2N)$ hypermultiplets of antisymmetric tensors. Such a hypermultiplet is determined by a principal quantum number $\nu_p$ ($0 < \nu_p \leq N$), which determines the common mass $\frac{2\pi \nu_p}{N} \sin \left(\frac{\pi \nu_p}{N}\right)$ of all the states in it. For $\nu_p$ even, the hypermultiplet contains antisymmetric tensors of ranks $0, 2, \ldots \nu_p$ (each with multiplicity 1). For $\nu_p$ odd, it contains tensors of ranks $1, 3, \ldots, \nu_p$. 

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contradictory), then all decay channels into final states with the same $O(2N)$ quantum numbers, which contain time dependent solitons, are necessarily closed as well. Thus, it is enough to scan only through decay channels into final states made purely of static solitons\textsuperscript{12}.

For all the DHN multiplets ($K = 1, q = 0$) and for the CCGZ kink multiplet ($K = 1, |q| = 1$), we will find in this way that all decay channels are necessarily closed, thus establishing their stability. That these are stable multiplets is well known, of course [1, 8]: they are the lightest solitons, given their $O(2N)$ and topological quantum numbers. Our new result is the marginal stability of the heavier, topologically non-trivial solitons, with a single pair of non-zero bound states ($K = 2, |q| = 1$) [23]. For all other static soliton multiplets, there are always decay channels in which (4.1) and (4.2) are mutually consistent. In such cases, more detailed analysis is required to establish stability or instability of the soliton under consideration. Nevertheless, given that the heavier topologically non-trivial soliton ($K = 2, |q| = 1$) is already at the threshold of stability, we conjecture that all these other multiplets are unstable.

### 4.1 Investigating Stability of Topologically Trivial Solitons

Consider the decaying parent soliton to be a topologically trivial static soliton with $K$ pairs of bound states at non-zero energies, corresponding to the direct product of $K$ antisymmetric tensor representations of ranks $\tilde{\nu}_1, \ldots, \tilde{\nu}_K$. The corresponding angular variables $\Theta_1, \ldots, \Theta_K$ are determined by (3.28), namely, $\Theta_n = \frac{\pi \tilde{\nu}_n}{2N}$. Thus, according to (3.34) and (3.44), the mass of this soliton is $M(\tilde{\nu}_1, \ldots, \tilde{\nu}_K) = \frac{2Nm}{\pi} \sum_{n=1}^{K} \sin \Theta_n$, and the maximal fermion number eigenvalue occurring in this representation is $N_f^{max}(D_{\text{parent}}) = \sum_{n=1}^{K} \tilde{\nu}_n$.

Following the strategy which we laid above, we shall now scan through all imaginable decay channels of this parent soliton (into final states of purely static solitons), and identify those channels which are necessarily closed.

\textsuperscript{12}This statement is surely true if DHN’s time dependent solitons exhaust all the stable time dependent configurations of the model. However, even if there are time dependent solitons in the GN model beyond those discovered by DHN, it is most likely that they will also be heavier than the corresponding static states with the same $O(2N)$ quantum numbers.
We first scan through all decay channels into topologically trivial solitons. Thus, assume that the parent soliton under consideration decays into a configuration of lighter topologically trivial solitons, with quantum numbers of the direct product of $L$ antisymmetric tensor representations $\nu_1, \ldots, \nu_L$ (and corresponding angular parameters $\theta_1, \ldots, \theta_L$). The way in which these $L$ multiplets are arranged into extremal fermion bags is of no consequence to our discussion. Thus, we discuss all decay channels consistent with these quantum numbers in one sweep.

The necessary conditions (4.1) and (4.2) imply

$$\sum_{n=1}^{K} \Theta_n \leq \sum_{i=1}^{L} \theta_i, \quad \sum_{n=1}^{K} \sin \Theta_n \geq \sum_{i=1}^{L} \sin \theta_i,$$

(4.3)

where all $0 < \Theta_n, \theta_i < \frac{\pi}{2}$. The two pairs of boundary hypersurfaces in (4.3) are

$$\Sigma_1, \tilde{\Sigma}_1 : \quad \theta_1 + \cdots + \theta_L = \Theta_1 + \cdots + \Theta_K$$

$$\Sigma_2, \tilde{\Sigma}_2 : \quad \sin \theta_1 + \cdots + \sin \theta_L = \sin \Theta_1 + \cdots + \sin \Theta_K,$$

(4.4)

where $\Sigma_{1,2}$ are hypersurfaces in $\theta$-space and $\tilde{\Sigma}_{1,2}$ are the corresponding hypersurfaces in $\Theta$-space.

The two necessary conditions in (4.3) will contradict each other, thereby protecting the parent soliton against decaying through the channel under consideration, if and only if the hypersurface $\Sigma_2$ lies (in the positive orthant) between the origin and the hyperplane $\Sigma_1$. When will this happen?

To answer this question it is useful to split

$$\sin \Theta_1 + \cdots + \sin \Theta_K = s + \sin \beta,$$

(4.5)

where

$$s = \text{integral part of } [\sin \Theta_1 + \cdots + \sin \Theta_K] \quad \text{and}$$

$$\sin \beta = \text{fractional part of } [\sin \Theta_1 + \cdots + \sin \Theta_K].$$

(4.6)

Thus, the integer $0 \leq s \leq K$, and $0 \leq \beta \leq \frac{\pi}{2}$.

The variables $\theta_i$ are restricted to the $L$ dimensional hypercube $[0, \frac{\pi}{2}]^L$ in the positive orthant. The hypersurface $\Sigma_2$ intersects the boundaries of this cube along
(curved) sub-simplexes whose vertices are

\[ \theta_i^{(v)} = \frac{\pi}{2} (\delta_{ii_1} + \delta_{ii_2} + \ldots + \delta_{ii_s}) + \beta \delta_{ii_{s+1}} , \quad (i = 1, \ldots, L) \]  
(4.7)

with all possible choices of \( s + 1 \) coordinates \( i_1, \ldots, i_{s+1} \) out of \( L \) (as the reader can convince herself or himself by working out explicitly a few low dimensional examples). All these vertices lie on the hypersurface

\[ \Sigma_{s\beta} : \quad \theta_1 + \cdots + \theta_L = \beta + \frac{\pi}{2} s . \]  
(4.8)

The curved hypersurface \( \Sigma_2 \) is dented towards the origin. In the positive orthant it lies between the hypersurface \( \Sigma_{s\beta} \) and the origin. Thus, the two necessary conditions in (4.3) will contradict each other as required, if and only if \( \Sigma_{s\beta} \) lies between \( \Sigma_1 \) and the origin, namely, when the parameters of the parent soliton are restricted according to

\[ \Theta_1 + \cdots + \Theta_K \geq \beta + \frac{\pi}{2} s . \]  
(4.9)

At the same time, these parameters are subjected to (4.5), which picks up a particular hypersurface \( \tilde{\Sigma}_2 \) in the space of the \( \Theta \)'s. Thus, the desired contradiction of the two necessary conditions in (4.3) occurs for parent solitons which correspond to points on the intersection of \( \tilde{\Sigma}_2 \) and (4.9). Let us determine this intersection:

The hypersurface \( \tilde{\Sigma}_2 \) intersects the boundaries of the \( K \) dimensional hypercube \([0, \frac{\pi}{2}]^K \) (the range of the \( \Theta_n \)'s) along (curved) sub-simplexes whose vertices are

\[ \Theta_n^{(v)} = \frac{\pi}{2} (\delta_{nn_1} + \delta_{nn_2} + \ldots + \delta_{nn_s}) + \beta \delta_{nn_{s+1}} , \quad (n = 1, \ldots, K) \]  
(4.10)

with all possible choices of \( s + 1 \) coordinates \( i_1, \ldots, i_{s+1} \) out of \( K \) (similarly to the intersection of \( \Sigma_2 \) and the boundary of the \( L \) dimensional cube \([0, \frac{\pi}{2}]^L \)). All these vertices lie on the hypersurface

\[ \tilde{\Sigma}_{s\beta} : \quad \Theta_1 + \cdots + \Theta_K = \beta + \frac{\pi}{2} s , \]  
(4.11)

which is, of course, the boundary of (4.9). Furthermore, \( \tilde{\Sigma}_2 \) is dented towards the origin and lies (in the positive orthant) between the origin and \( \tilde{\Sigma}_{s\beta} \). Thus, we conclude that the desired intersection is precisely the set of all vertices defined by (4.10).
Each such vertex represents a soliton which cannot decay through the channel under consideration, and is thus potentially stable. More precisely, all these vertices correspond to the same soliton, since the coordinates of these vertices are just permutations of each other, and thus all of them correspond to the same set of parameters, in which

\[ s \text{ of the } \Theta' \text{s are degenerate and equal to } \frac{\pi}{2} \]
\[ \text{one of the } \Theta' \text{s is equal to } \beta , \text{ and} \]
\[ \text{the remaining } K - (s + 1) \Theta' \text{s are null.} \quad (4.12) \]

In Appendix F we provide an alternative proof of (4.12).

Does such a soliton exist? To answer this question let us recall a few basic facts: The parent soliton under discussion is topologically trivial. As such, it must bind fermions to be stabilized, and none of its bound state energies may vanish. Thus, all the ranks occurring in it must satisfy \( 0 < \tilde{\nu}_n < N \), in accordance with (3.26). Finally, recall that all the \( \omega_n \)'s which parametrize a given static soliton, must be different from each other (see the paragraph following (3.28)). Gathering all these facts we deduce from (3.27) and (3.28) that all the \( \Theta_n \)'s must be different from each other, and furthermore, that each \( \Theta_n \neq 0, \frac{\pi}{2} \).

Thus, the only physically realizable parent solitons, which are necessarily stable against the decay channel in question, correspond to \( s = 0 \) and \( K = s + 1 = 1 \). These are, of course, the static DHN solitons with profile (3.36) and mass (3.35).

A corollary of our analysis so far is that DHN solitons are stable against decaying into \( L_f \) free fermions or antifermions\(^{13}\) (i.e., \( L_f \) fundamental \( O(2N) \) representations) plus \( L - L_f \) solitons corresponding to higher antisymmetric tensor representations. In particular, it is stable against complete evaporation into free fermions. This fact manifests itself in the binding energy function (3.32), which for a DHN soliton of rank \( \nu = \frac{2N\Theta}{\pi} \) is

\[ B(\Theta) = \frac{2Nm}{\pi}(\Theta - \sin \Theta) . \quad (4.13) \]

\(^{13}\)Strictly speaking, this argument is valid only for values of \( L_f \) which are a finite fraction of \( N \), since our mass formula (3.34) is the leading order in the \( 1/N \) expansion, while removing a finite number of particles from the parent soliton is a perturbation of the order \( 1/N \) relative to (3.34).
$B(\Theta)$ is positive and increases monotonically in the physical range $0 < \Theta < \frac{\pi}{2}$, and so does the binding energy per fermion, $\frac{B(\Theta)}{\nu} = m \left(1 - \frac{\sin \Theta}{\Theta}\right)$. This is in contrast with binding in nuclei, where the binding energy per nucleon saturates. The DHN soliton becomes ever more stable as it traps more fermions (up to the maximal number $\nu = N$). This is the so-called “mattress effect”, familiar from the physics of fermion bags.

All the DHN solitons ($K = 1, q = 0$) have masses below the kink-antikink threshold $\frac{2\kappa m \pi}{\nu}$. Thus they cannot decay into pairs of topologically non-trivial solitons. The topologically trivial solitons with $K \geq 2$, on the other hand, may have masses above the kink-antikink threshold, but there is not much interest in looking for such heavier solitons which are stable against decaying into pairs of topological defects, since we have already established that they are not necessarily protected against decaying into topologically trivial solitons\textsuperscript{14}.

To summarize, we have thus covered all possible decay channels of a given static topologically trivial soliton into a configuration of other solitons (static or time dependent). We have found that the only static solitons, which are necessarily stable against decaying into any of these channels (in the sense that (4.1) and (4.2) will always contradict each other), are just the DHN solitons. This is of course, well known. It follows almost trivially from the fact that if $0 \leq \theta_n \leq \frac{\pi}{2}, \forall n$ and $0 \leq \sum_n \theta_n \leq \frac{\pi}{2}$, then $\sin \left(\sum_n \theta_n\right) \leq \sum_n \sin \theta_n$.

\subsection*{4.2 Investigating Stability of Topologically Non-Trivial Solitons}

The CCGZ kink and antikink are the lightest states with topological charge $q = \pm 1$. Thus they are stable. Are there any other stable topologically non-trivial solitons?

To answer this question we will apply our strategy to specify all topologically non-trivial solitons which are necessarily stable. Thus, consider such a generic parent soliton. It has a zero energy bound state $\omega_1 = 0$ with its corresponding spinorial

\textsuperscript{14}For what it worths, it can be shown, by going through the same analysis as in the previous case, that there are no decay channels into pairs of topological solitons, against which any of the $K \geq 2, q = 0$ solitons (with mass above the kink-antikink threshold) is necessarily protected.
representation factor, and additional $K - 1$ pairs of bound states at non-zero $\omega_n$'s, with the direct product of their $K - 1$ antisymmetric tensor representations of ranks $\tilde{\nu}_1, \ldots, \tilde{\nu}_{K-1}$. The corresponding angular variables $\Theta_1, \ldots, \Theta_{K-1}$ are determined by (3.28), namely, $\Theta_n = \frac{\pi \tilde{\nu}_n}{2N}$. Thus, according to (3.34), (3.44) and (3.45), the mass of this soliton is $M(\tilde{\nu}_1, \ldots, \tilde{\nu}_{K-1}) = \frac{Nm}{\pi} + \frac{2Nm}{\pi} \sum_{n=1}^{K-1} \sin \Theta_n$, and the maximal fermion number eigenvalue occurring in this representation is $N_{f_{\text{max}}}^\nu(D_{\text{parent}}) = \frac{N}{2} + \sum_{n=1}^{K-1} \tilde{\nu}_n$.

The parent soliton may decay into a final state, which will contain, in addition to a lighter soliton with the same topological charge, $P$ pairs of topologically non-trivial solitons (with opposite topological charges), and a bunch of topologically trivial solitons. The final state of the decay is thus most generally a direct product of some $L$ antisymmetric tensor representations $\nu_1, \ldots, \nu_L$ (and corresponding angular parameters $\theta_1, \ldots, \theta_L$), and $2P + 1$ spinorial representations. As in the previous case, the way in which the $L$ antisymmetric tensors are arranged into extremal fermion bags is of no consequence to our discussion.

The necessary conditions (4.1) and (4.2) now imply

$$\sum_{n=1}^{K-1} \Theta_n \leq \frac{\pi}{2} P + \sum_{i=1}^{L} \theta_i, \quad \sum_{n=1}^{K-1} \sin \Theta_n \geq P + \sum_{i=1}^{L} \sin \theta_i,$$

(4.14)

(where all $0 < \Theta_n, \theta_i < \frac{\pi}{2}$). These conditions are identical to (4.3), with $K$ there replaced here by $K - 1$ and with $s \geq P$. Going through the same analysis as we did for the topologically trivial soliton, we conclude that the two necessary conditions (4.14) contradict each other only when $s = 0$ (which means $P = 0$), and $K - 1 \leq s + 1 = 1$.

Thus, in addition to the CCGZ kink ($K = 1$), we have discovered a heavier stable, topologically non-trivial soliton ($K = 2$). As in [23], we shall refer to it as the "Heavier Topological Soliton" (HTS), which is a bound state of a DHN soliton and a kink or an antikink. Its mass is

$$M_{HTS,\nu} = \frac{Nm}{\pi} + \frac{2Nm}{\pi} \sin \left(\frac{\pi \nu}{2N}\right),$$

(4.15)

where $\nu$ is the rank of its antisymmetric tensor factor, which is tensored with the $2^N$ dimensional spinor. Thus, $M_{HTS,\nu}$ coincides with the sum of masses of a CCGZ kink and a DHN soliton of rank $\nu$. Drawing further the analogy between the solitons
discussed in this Review and baryons in QCD with large \( N_{\text{color}} \), the HTS would correspond to a dibaryon.

The profile of this HTS (for boundary conditions corresponding to \( q = 1 \)) is given by (B.66) and (B.69), which we repeat here for convenience:

\[
\sigma(x) = m + \frac{2\kappa}{1 + \frac{m + \kappa}{m - \kappa} e^{2\kappa(x - y_0)}}
\]

\[
- 2(m + \kappa) \frac{1 + \frac{m + \kappa}{(m - \kappa)^2} \kappa e^{2m(x - x_0)} + \frac{m + \kappa}{(m - \kappa)^2} m e^{2\kappa(x - y_0)}}{1 + \left(\frac{m + \kappa}{m - \kappa}\right)^2 e^{2m(x - x_0)} + \left(\frac{m + \kappa}{m - \kappa}\right)^2 e^{2\kappa(x - y_0)} + \left(\frac{m + \kappa}{m - \kappa}\right)^2 e^{2m(x - x_0) + 2\kappa(x - y_0)}}
\]

\[
= -\kappa \tanh[\kappa(x - y_0 + R)]
\]

\[
+ \frac{\omega}{e^{-\kappa R} \cosh\left[m(x - x_0) + \kappa(x - y_0) + 2\kappa R\right] + \sinh\left[m(x - x_0) - \kappa(x - y_0)\right]} + \frac{\sinh\left[m(x - x_0) - \kappa(x - y_0)\right]}{\cosh\left[m(x - x_0) - \kappa(x - y_0)\right] + e^{\kappa R}}
\]

(4.16)

with \( R = \frac{1}{2\kappa} \log\left(\frac{m + \kappa}{m - \kappa}\right) \) (Eq.(B.67)) and \( \kappa = \sqrt{m^2 - \omega^2} = m \sin \left(\frac{\pi \nu}{2N}\right) \), as usual. The arbitrary real parameters \( x_0 \) and \( y_0 \) are related to the coefficients \( c_1, c_2 \) in (3.4) via (B.65). These are the flat directions, or moduli, which we discussed in subsection 3.1.1. We can invoke global translational invariance to set one of these collective coordinates, say \( x_0 \), to zero. The remaining parameter \( y_0 \) is left arbitrary.

In [23] we have studied the profile of (4.16) in some detail. (See Figures 1-3 in [23].) In general, it has the shape of the profiles of a CCGZ kink and a DHN soliton superimposed on each other, which move relative to each other as we vary the remaining free modulus \( y_0 \). The latter is essentially the separation of the kink and DHN soliton. The mass (4.15) of the HTS remains fixed throughout this variation, as we discussed in subsection 3.1.1.

The fact that \( \partial M_{HTS,\nu}/\partial y_0 = 0 \), just means that the kink and DHN soliton exert no force on each other, whatever their separation is. In other words - they are very loosely bound. Indeed, it is evident from (4.15) that the HTS is degenerate in mass with a pair of non-interacting CCGZ kink and a DHN soliton of rank \( \nu \). If we interpret the latter two as the constituents of the HTS, they are bound at threshold. Thus,
the HTS is marginally stable [23].

This is a somewhat surprising result, since one would normally expect soliton-soliton interactions to be of the order $\frac{1}{g^2} \sim N$ in a weakly interacting field theory. This is consistent, of course, with what one should expect from general $\frac{1}{N}$ counting rules [34], according to which the baryon-baryon interaction is of order $N$. Yet, due to dynamical reasons which elude us at this point, the solitonic constituents of the HTS avoid these general considerations and do not exert force on each other.

The limit $\nu \to N$ is of some interest. Strictly speaking, there is no HTS with $\nu = N$, as was discussed above. However, it is possible to study HTS's with $\nu$ arbitrarily close to $N$. In this case, $\kappa \to m$, with $R \to \infty$ in (4.16). Thus, for $|x|, |x_0|, |y_0| << R$, (4.16) tends in this limit to

$$\sigma(x) = m \frac{1 - e^{-2m(x-y_0)} - e^{-2m(x-x_0)}}{1 + e^{-2m(x-y_0)} + e^{-2m(x-x_0)}}.$$ (4.17)

In the asymptotic region $1 << m|x_0 - y_0|$ ($<< mR$) (4.17) simplifies further, and appears as a kink $m \tanh[m(x - x_{\text{max}})]$, located at $x_{\text{max}} = \max\{x_0, y_0\}$. This clearly has mass $M_{\text{kink}} = \frac{Nm}{\pi}$, but according to (4.15), $M_{\text{HTS},\nu=N}$ should tend to $\frac{3Nm}{\pi} = 3M_{\text{kink}}$. The extra mass $2M_{\text{kink}}$ corresponds, of course, to the kink-anti-kink pair which receded to spatial infinity.
Appendix A: Resolvent of the Dirac Operator With Static Background Fields and Supersymmetry

In this Appendix we recall some useful properties of the resolvent of the Dirac operator \( D = \omega \gamma^0 + i \gamma^1 \partial_x - \sigma(x) \) in a static background. In particular, we make connections with supersymmetric quantum mechanics\(^\text{15}\) and use it to express all four entries of the diagonal resolvent in terms of a single function. This will be useful in evaluating masses of fermion bags.

The presentation in this Appendix is a special case of the recent discussion in [20], which analyzes the more generic Dirac operator \( D \equiv i \partial_t - (\sigma(x) + i \pi(x) \gamma^5) \). The spectral theory of the latter operator gives rise to an interesting generalization of supersymmetric quantum mechanics which was discussed in [20]. We also mention that supersymmetric quantum mechanics was instrumental in the analysis of fermion bags in [16].

As was discussed in Section 2, our task is to invert the Dirac operator (2.10), \( D = \omega \gamma^0 + i \gamma^1 \partial_x - \sigma(x) \), in a static background \( \sigma(x) \) subjected to the boundary conditions (2.8). We emphasize that inverting (2.10) has nothing to do with the large \( N \) approximation, and consequently our results in this section are valid for any value of \( N \).

In the Majorana representation (1.2) we are using in this paper, \( \gamma^0 = \sigma_2, \gamma^1 = i \sigma_3 \) and \( \gamma^5 = -\gamma^0 \gamma^1 = \sigma_1 \), the Dirac operator (2.10) is

\[
D = \begin{pmatrix}
-\partial_x - \sigma & -i\omega \\
i\omega & \partial_x - \sigma
\end{pmatrix} = \begin{pmatrix}
-Q & -i\omega \\
i\omega & -Q^\dagger
\end{pmatrix}, \tag{A.1}
\]

where we introduced the pair of adjoint operators

\[
Q = \sigma(x) + \partial_x, \quad Q^\dagger = \sigma(x) - \partial_x. \tag{A.2}
\]

\(^\text{15}\)For a thorough review of supersymmetric quantum mechanics see [13].
Inverting (A.1) is achieved by solving
\[
\begin{pmatrix}
-Q & -i\omega \\
i\omega & -Q^\dagger
\end{pmatrix}
\begin{pmatrix}
a(x, y) \\
c(x, y)
\end{pmatrix}
\begin{pmatrix}
b(x, y) \\
d(x, y)
\end{pmatrix} = -i\delta(x - y)
\] (A.3)
for the Green’s function of (A.1) in a given background \(\sigma(x)\). By dimensional analysis, we see that the quantities \(a, b, c\) and \(d\) are dimensionless.

### A.1 “Supersymmetry” in a Bag Background

The diagonal elements \(a(x, y), d(x, y)\) in (A.3) may be expressed in terms of the off-diagonal elements as
\[
a(x, y) = -\frac{i}{\omega}Q^\dagger c(x, y), \quad d(x, y) = \frac{i}{\omega}Q b(x, y)
\] (A.4)
which in turn satisfy the second order partial differential equations
\[
\left(Q^\dagger Q - \omega^2\right)\frac{b(x, y)}{\omega} = \left[-\partial_x^2 + \sigma(x)^2 - \sigma'(x) - \omega^2\right]\frac{b(x, y)}{\omega} = \delta(x - y)
\]
\[
\left(QQ^\dagger - \omega^2\right)\frac{c(x, y)}{\omega} = \left[-\partial_x^2 + \sigma(x)^2 + \sigma'(x) - \omega^2\right]\frac{c(x, y)}{\omega} = -\delta(x - y).
\] (A.5)

Thus, \(b(x, y)/\omega\) and \(-c(x, y)/\omega\) are simply the Green’s functions of the corresponding Schrödinger operators
\[
H_b = Q^\dagger Q = \left[-\partial_x^2 + \sigma(x)^2 - \sigma'(x)\right], \quad \text{and} \quad H_c = QQ^\dagger = \left[-\partial_x^2 + \sigma(x)^2 + \sigma'(x)\right]
\] (A.6)
in (A.5), namely,
\[
\frac{b(x, y)}{\omega} = \frac{\theta(x - y) b_2(x) b_1(y) + \theta(y - x) b_2(y) b_1(x)}{W_b}
\]
\[
\frac{c(x, y)}{\omega} = -\frac{\theta(x - y) c_2(x) c_1(y) + \theta(y - x) c_2(y) c_1(x)}{W_c}.
\] (A.7)
Here \(\{b_1(x), b_2(x)\}\) and \(\{c_1(x), c_2(x)\}\) are pairs of independent fundamental solutions of the two equations
\[
H_b b(x) = \omega^2 b(x) \quad \text{and} \quad H_c c(x) = \omega^2 c(x),
\] (A.8)
subjected to the boundary conditions

\[ b_1(x), c_1(x) \xrightarrow{x \to -\infty} A^{(1)}_{b,c} b_1(x), c_1(x) - A^{(2)}_{b,c} e^{ikx} \]

with some possibly \( k \)-dependent coefficients \( A^{(1)}_{b,c}(k) \), \( A^{(2)}_{b,c}(k) \) and with\(^{16}\)

\[ k = \sqrt{\omega^2 - m^2}, \quad \text{Im} k \geq 0. \]  

(A.10)

The purpose of introducing the (yet unspecified) coefficients \( A^{(1)}_{b,c}(k) \), \( A^{(2)}_{b,c}(k) \) will become clear later, following Eqs. (A.15) and (A.16) below. The boundary conditions (A.9) are consistent, of course, with the asymptotic behavior (2.8) on \( \sigma \) due to which both \( H_b \) and \( H_c \) tend to a free particle hamiltonian \(-\partial_x^2 + m^2\) as \( x \to \pm\infty \).

The wronskians of these pairs of solutions are

\[ W_b(k) = b_2(x)b'_1(x) - b_1(x)b'_2(x) \]

\[ W_c(k) = c_2(x)c'_1(x) - c_1(x)c'_2(x) \].

(A.11)

As is well known, \( W_b(k) \) and \( W_c(k) \) are independent of \( x \).

We comment at this point (for later reference) that in the scattering theory of the operators \( H_b \) and \( H_c \) it is more customary to consider pairs of independent fundamental solutions \( \{b_1(x), b_2(x)\} \) and \( \{c_1(x), c_2(x)\} \) for which the coefficients \( A^{(1)}_{b,c}, A^{(2)}_{b,c} \) in (A.9) are \( k \) independent, because, as is well known in the literature [3], with such boundary conditions, the corresponding wronskians are proportional (up to a \( k \)-independent coefficient) to \( k/t(k) \), where \( t(k) \) is the transmission amplitude of the corresponding operator \( H_b \) or \( H_c \). Thus, we will refer to the wronskians of pairs of independent fundamental solutions with asymptotic behavior (A.9) with \( A^{(1)}_{b,c} = A^{(2)}_{b,c} = 1 \) as the standard wronskians \( W^s_b(k) \) and \( W^s_c(k) \). Therefore, we can express the wronskians (A.11) of pairs of fundamental solutions with asymptotic behavior (A.9) as

\[ W_b(k) = A^{(1)}_b(k)A^{(2)}_b(k)W^s_b(k) \quad \text{and} \quad W_c(k) = A^{(1)}_c(k)A^{(2)}_c(k)W^s_c(k) \].

(A.12)

\(^{16}\)We see that if \( \text{Im} k > 0 \), \( b_1 \) and \( c_1 \) decay exponentially to the left, and \( b_2 \) and \( c_2 \) decay to the right. Thus, if \( \text{Im} k > 0 \), both \( b(x, y) \) and \( c(x, y) \) decay as \( |x - y| \) tends to infinity.
We remind the reader that at a bound state energy $\omega = \omega_b$ (or, equivalently, at the corresponding $k_b = i \sqrt{m^2 - \omega_b^2}$), say, of the operator $H_b$, the transmission amplitude $t_b(k)$ of $H_b$ has a simple pole, $t_b(k) \sim 1/(k - k_b)$. Thus, $W_{st}^b(k) \sim k / t_b(k)$ vanishes as $W_{st}^b(k) \sim k_b(k - k_b)$, $k \sim k_b$ near that bound state. This behavior occurs simply because at the bound state $k = k_b$, $b_1(x)$ and $b_2(x)$ are both proportional to the bound state wave function (which decays asymptotically as $\exp(-\sqrt{m^2 - \omega_b^2}|x|)$, and are thus linearly dependent. Thus, the zeros of $W_{st}^b(k)$ determine the spectrum of bound states of $H_b$. The non-standard wronskian $W_b(k)$ will have some extra $k$ dependence resulting from the factors $A^{(1)}(k)A^{(2)}(k)$ in (A.12). Similar assertions hold, of course for $H_c$, $W_{st}^c$ and $W_c$.

Substituting the expressions (A.7) for the off-diagonal entries $b(x, y)$ and $c(x, y)$ into (A.4), we obtain the appropriate expressions for the diagonal entries $a(x, y)$ and $d(x, y)$. We do not bother to write these expressions here. It is useful however to note, that despite the $\partial_x$’s in the $Q$ operators in (A.4), that act on the step functions in (A.7), neither $a(x, y)$ nor $d(x, y)$ contain pieces proportional to $\delta(x - y)$. Such pieces cancel one another due to the symmetry of (A.7) under $x \leftrightarrow y$.

The factorization (A.6) of $H_b = Q_\dagger Q$ and $H_c = QQ_\dagger$ into a product of the two first order differential operators $Q$ and $Q_\dagger$ is the hallmark of supersymmetry. As is well known, this factorization means that $H_b$ and $H_c$ are isospectral, i.e., have the same eigenvalues, save possibly an unmatched zero mode which belongs to the spectrum of only one of the operators.

Let us now recall how this property arises: The factorized equations
\begin{equation}
Q_\dagger Q b = \omega^2 b \quad \text{and} \quad QQ_\dagger c = \omega^2 c
\end{equation}
suggest a map between their solutions. Indeed, given that $H_b b = \omega^2 b$, then clearly
\begin{equation}
c(x) = \frac{1}{\omega} Q b(x)
\end{equation}
is a solution of $H_c c = \omega^2 c$. The factor $\frac{1}{\omega}$ in (A.15) is relevant for the case in which $b(x)$ is an eigenstate of $H_b$ (with eigenvalue $\omega^2$). In such a case, $c(x)$ on the left hand
side of (A.15) is an eigenstate of $H_c$ with the same eigenvalue $\omega^2$, and furthermore, due to the factor $\frac{1}{\omega}$, it also has same norm as $b(x)$. Similarly, if $H_c c = \omega^2 c$, then

$$b(x) = \frac{1}{\omega} Q^\dagger c(x) \quad (A.16)$$

solves $H_b b = \omega^2 b$, and has the same norm as $c(x)$, in case $c(x)$ is an eigenstate of $H_c$.

We remark, that in the more generic case of the Dirac operator $i\partial - \sigma(x) - \gamma_5 \pi(x)$ discussed in [20], the factors analogous to the factors $\frac{1}{\omega}$ in (A.15) and (A.16) (factors of $\frac{1}{\omega \pm \pi(x)}$, see [20], Eqs. (2.12)-(2.14)), arise already at the level of the differential equations analogous to (A.14), and not from considerations of the norm of $b(x)$ and $c(x)$.

One particular useful consequence of the mappings (A.15) and (A.16) is that given a pair \{\(b_1(x), b_2(x)\}\) of independent fundamental solutions of \((H_b - \omega^2) b(x) = 0\), we can obtain from it a pair \{\(c_1(x), c_2(x)\)\} of independent fundamental solutions of \((H_c - \omega^2) c(x) = 0\) by using (A.15), and vice versa. Therefore, with no loss of generality, we henceforth assume, that the two pairs of independent fundamental solutions \{\(b_1(x), b_2(x)\)\} and \{\(c_1(x), c_2(x)\)\}, are related by (A.15) and (A.16).

The coefficients $A_{b,c}^{(1)}(k), A_{b,c}^{(2)}(k)$ in (A.9) are to be adjusted according to (A.15) and (A.16), and this was the purpose of introducing them in the first place.

It is clear (and of course well known from the literature on supersymmetric quantum mechanics), that the mappings $b(x) \leftrightarrow c(x)$ break down if either $H_b$ or $H_c$ has an eigenstate at zero energy. In case that zero mode exists, it must be the ground state of the corresponding operator $H_b$ or $H_c$, since these operators obviously cannot have negative eigenvalues.

### A.1.1 Zero Modes and Topology

Let us determine under which conditions such zero-energy eigenstates exist and what they look like. Assume, for example, that $H_b$ has its ground state $|b_0\rangle$ at zero energy. Our system is defined over the whole real axis, thus bound states are strictly square-integrable. By assumption $\langle b_0 | H_b | b_0 \rangle = \langle b_0 | Q^\dagger Q | b_0 \rangle = \left| \langle Q | b_0 \rangle \right|^2 = 0$. Thus, $H_b | b_0 \rangle = 0$. 

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implies the first-order equation

\[ Q b_0(x) = b'_0(x) + \sigma(x)b_0(x) = 0. \] (A.17)

Thus,

\[ b_0(x) = \exp - \int^x \sigma(y) dy. \] (A.18)

Similarly, if \( H_c \) has its ground state \( |c_0\rangle \) at zero energy, we have

\[ Q^\dagger c_0(x) = -c'_0(x) + \sigma(x)c_0(x) = 0 \] namely \[ c_0(x) = \exp + \int^x \sigma(y) dy. \] (A.19)

For \( \sigma(x) \) subjected to the boundary conditions (2.8), the wave function (A.18) is square-integrable if and only if \( \sigma(\infty) = -\sigma(\infty) = m \), i.e., only when \( \sigma(x) \) carries topological charge \( q = 1 \). Similarly, the wave function (A.19) is square-integrable if and only if \( \sigma(\infty) = -\sigma(\infty) = -m \), i.e., only when \( \sigma(x) \) carries topological charge \( q = -1 \). Thus, \( b_0 \) and \( c_0 \) cannot be simultaneously square-integrable, and correspondingly, only one of the operators \( H_b \) and \( H_c \) can have a zero energy ground state in a given topologically non-trivial background \( \sigma(x) \). Of course, none of these operators has a zero energy state if \( \sigma(x) \) is topologically trivial, i.e., when \( q = 0 \).

To summarize, if \( \sigma(x) \) carries topological charge \( q = 1 \), \( H_c \) is strictly positive and isospectral to \( H_b \), except for the normalizable zero-mode (A.18) of \( H_b \). In the opposite case, where \( \sigma(x) \) carries topological charge \( q = -1 \), \( H_b \) is strictly positive and isospectral to \( H_c \), except for the normalizable zero-mode (A.19) of \( H_c \). In the third case, when \( \sigma(x) \) is topologically trivial, \( q = 0 \), neither \( H_b \) nor \( H_c \) has a zero energy state, and they are strictly isospectral.

Let us reformulate all this in terms of the Dirac equation (1.6). Assuming the existence of a bound state at \( \omega = 0 \), \( \psi(x, t) = u(x) \), we write

\[ \left[ i\gamma^1 \partial_x - \sigma(x) \right] u(x) = 0. \] (A.20)

This equation may be further reduced as follows: The matrix \( i\gamma^1 \) is hermitean with eigenvalues \( \pm 1 \). (This basis independent property is manifest in the particular representation (1.2).) Thus, consider (A.20) with \( i\gamma^1 u_\pm(x) = \pm u_\pm(x) \). The corresponding
solutions are \( u_\pm(x) = u_\pm(0) \exp \pm \frac{x}{\sigma(y)} dy \). If \( q = 1 \), then only \( u_+(x) = u_+(0)b_0(x) \) is normalizable. Similarly, if \( q = -1 \), only \( u_-(x) = u_-(0)c_0(x) \) is normalizable. And if \( q = 0 \), none of these solutions is normalizable. This is a general feature of the Dirac equation (1.6). Thus, as was first noticed in [31], a fermion which is Yukawa-coupled to a topologically non-trivial scalar field (as in (1.1)), will have a single, unpaired, normalizable zero energy mode.

The “Witten index”[35] associated with the pair of isospectral operators \( H_b \) and \( H_c \), which in this context may be defined\(^\text{17}\) as

\[
\text{Witten index} = N_0(H_b) - N_0(H_c),
\]

where \( N_0(H_b,c) \) is the number of normalizable zero modes of \( H_{b,c} \). This index is of course a topological invariant of the space of hamiltonians \( H_b \) and \( H_c \) defined with configurations \( \sigma(x) \) that satisfy the boundary conditions (2.8). Indeed, it is clear from the discussion above that in our context, it coincides with the topological charge (2.9) of the background \( \sigma(x) \):

\[
\text{Witten index} = q. \tag{A.22}
\]

Thus, the Witten index in our system is either \( \pm 1 \) or zero.

If one of the operators \( H_b \) or \( H_c \) supports a normalizable zero mode, we say that supersymmetry is unbroken. When none of the operators has a normalizable zero mode, we say that supersymmetry is broken. Since for our system only one operator can support a normalizable zero mode, we may summarize these definitions by saying that for our system a null Witten index means broken supersymmetry. A non vanishing Witten index, which in our case can take only the values \( \pm 1 \), always means unbroken supersymmetry.

Since the Witten index in our model coincides with the topological charge (2.9) \( q \), we may rephrase the last paragraph by saying that only topologically non-trivial \( \sigma(x) \) backgrounds lead to a Dirac operator with a normalizable bound state at \( \omega = 0 \).

\(^{17}\)With this definition, we are identifying \( H_b \) and \( H_c \), respectively, as the hamiltonians of the bosonic and fermionic sectors of the supersymmetric system.
A.1.2 Wronskians and Isospectrality

An interesting outcome of the isospectrality of $H_b$ and $H_c$ concerns their wronskians. Indeed, from the definition (A.11), it follows for pairs of independent fundamental solutions $\{b_1(x), b_2(x)\}$ and $\{c_1(x), c_2(x)\}$ which are related through (A.15) and (A.16), that

$$\frac{W_c}{\omega} = \frac{c_2 \partial_x c_1 - c_1 \partial_x c_2}{\omega} = c_1 b_2 - c_2 b_1 = \frac{b_2 \partial_x b_1 - b_1 \partial_x b_2}{\omega} = W_b, \quad \omega \neq 0. \quad (A.23)$$

The wronskians of pairs of independent fundamental solutions of $H_b$ and $H_c$ are equal for all $\omega \neq 0$! Eq. (A.23) will play an important role in establishing useful properties of the diagonal resolvent of the Dirac operator in the next subsection.

With no loss of generality, we may choose

$$A_b^{(1)} = A_b^{(2)} = 1 \quad (A.24)$$

in (A.9). The coefficients $A_c^{(1)}, A_c^{(2)}$ are then determined by (A.15):

$$A_c^{(1)} = \frac{\sigma(-\infty) - ik}{\omega}, \quad A_c^{(2)} = \frac{\sigma(\infty) + ik}{\omega}. \quad (A.25)$$

Thus, from (A.12), (A.24), (A.23) and (A.25) we have

$$W_b(k) = W_b^{st}(k) = \frac{(\sigma(-\infty) - ik)(\sigma(\infty) + ik)}{\omega^2} W_c^{st}(k) = W_c(k). \quad (A.26)$$

Using $\omega^2 = m^2 + k^2$, the definition of topological charge $q = (\sigma(\infty) - \sigma(-\infty))/2m$ (2.9), and the boundary condition $\sigma(\pm \infty)^2 = m^2$ (2.8), we may write (A.26) more compactly as

$$W_b^{st}(k) = \left(1 + \frac{2mq}{\sigma(-\infty) + ik}\right) W_c^{st}(k). \quad (A.27)$$

Recall from (A.13) that the standard wronskians $W_b^{st}(k), W_c^{st}(k)$ vanish, respectively, at the bound states of $H_b$ and $H_c$. Thus, contrary to (A.23), which holds for all backgrounds $\sigma(x)$, we cannot have $W_b^{st}(k) = W_c^{st}(k)$ when $q = \pm 1$, i.e., when only
one of the hamiltonians has an unpaired normalizable zero energy ground state. It is
rewarding to verify that (A.27) is consistent with these facts:

For \( q = 0 \), \( H_b \) and \( H_c \) are strictly positive and thus must have the same bound
states. Thus, we expect to find \( W_{st}^b(k) = W_{st}^c(k) \), which is what (A.27) tells us at
\( q = 0 \). For \( q = 1 \), \( H_b \) and \( H_c \) are isospectral, save for the unpaired \( \omega_b^2 = 0 \) ground state
of \( H_b \). \( \omega_b^2 = 0 \) means \( k_b = \text{im} \). Thus, we expect to find \( W_{st}^b(k) \) has an extra zero
at \( k = \text{im} \), relative to \( W_{st}^c(k) \). Indeed, from (A.27) with \( q = 1 \) and \( \sigma(-\infty) = -m \),
we find that \( W_{st}^b(k) = \frac{k - \text{im}}{k + \text{im}} W_{st}^c(k) \). Finally, for \( q = -1 \), the roles of \( H_b \) and \( H_c \) are
reversed relative to the \( q = 1 \) case, and upon substituting \( q = -1 \) and \( \sigma(-\infty) = m \)
in (A.27) we find accordingly that \( W_{st}^b(k) = \frac{k + \text{im}}{k - \text{im}} W_{st}^c(k) \).

We note in passing that isospectrality of \( H_b \) and \( H_c \) is consistent with the \( \gamma_5 \)
symmetry of the system of equations in (A.3), which relates the resolvent of \( D \) with
that of \( \tilde{D} = -\gamma_5 D \gamma_5 \) (recall (2.13)). Due to this symmetry, we can map the pair of
equations \((H_b - \omega^2)b(x,y)/\omega = \delta(x-y)\) and \((H_c - \omega^2)c(x,y)/\omega = -\delta(x-y)\) (Eqs.
(A.5)) on each other by

\[
b(x, y) \leftrightarrow -c(x, y) \quad \text{together with} \quad \sigma \rightarrow -\sigma. \tag{A.28}
\]

(Note that under these reflections we also have \( a(x, y) \leftrightarrow -d(x, y) \), as we can see
from (A.4).) The reflection \( \sigma \rightarrow -\sigma \) flips the signs of both asymptotic values \( \sigma(\pm \infty) \)
and thus flips the sign of the topological charge \( q \). It changes the roles of \( H_b \) and \( H_c \),
but obviously it cannot change physics. Since this reflection interchanges \( b(x, y) \) and
\( c(x, y) \) without affecting the physics, these two objects must have the same singular-
ities as functions of \( \omega \), consistent with isospectrality of \( H_b \) and \( H_c \) (save possibly an
unpaired zero mode).

**A.2 The Diagonal Resolvent**

Following [17, 19, 20] we define the diagonal resolvent \( \langle x | iD^{-1} | x \rangle \) symmetrically as

\[
\langle x | -iD^{-1} | x \rangle \equiv \begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix}
\]
\[
\begin{align*}
= \frac{1}{2} \lim_{\epsilon \to 0^+} \begin{pmatrix}
  a(x, y) + a(y, x) & b(x, y) + b(y, x) \\
  c(x, y) + c(y, x) & d(x, y) + d(y, x)
\end{pmatrix}_{y=x+\epsilon}
\end{align*}
\] (A.29)

Here \(A(x)\) through \(D(x)\) stand for the entries of the diagonal resolvent, which following (A.4) and (A.7) have the compact representation\(^{18}\)

\[
B(x) = \frac{\omega b_1(x)b_2(x)}{W_b}, \quad D(x) = i \frac{[\partial_x + 2\sigma(x)] B(x)}{2\omega},
\]
\[
C(x) = -\frac{\omega c_1(x)c_2(x)}{W_c}, \quad A(x) = i \frac{[\partial_x - 2\sigma(x)] C(x)}{2\omega}. \tag{A.30}
\]

Clearly, from (A.5) and (A.6) we have

\[
\frac{B(x)}{\omega} = \langle x | \frac{1}{H_b - \omega^2} | x \rangle
\]
\[
-\frac{C(x)}{\omega} = \langle x | \frac{1}{H_c - \omega^2} | x \rangle. \tag{A.31}
\]

Thus, comparing with [16], we see that \(B(x)/\omega\) and \(-C(x, \omega)/\omega\) are, respectively, the resolvents \(R_-(x, \omega^2)\) and \(R_+(x, \omega^2)\) defined in Eqs.(9) and (10) of [16].

The expressions for \(A(x)\) and \(D(x)\) in terms of \(B(x)\) and \(C(x)\) in (A.30) have an interesting property concerning zero modes. Consider, for example the case where \(H_b\) has a normalizable zero mode. From (A.31) we expect that as \(\omega^2 \to 0\), \(B(x)/\omega\) will be dominated by the pole at \(\omega_b^2 = 0\) with a residue proportional to the wave function of the zero energy state (A.18) squared,

\[
\lim_{\omega^2 \to 0^+} \frac{B(x)}{\omega} = -\frac{b_0^2(x)}{\omega^2}. \tag{A.32}
\]

How does \(D(x)\) behave near that pole? Note from (A.18) that \((\partial_x + 2\sigma(x))b_0^2(x) = 0.\)

Thus, the operator \((\partial_x + 2\sigma(x))\), which appears in the expression of \(D(x)\) in terms of \(B(x)\) in (A.30), projects the zero mode \(b_0(x)\) out of \(D(x)!\) Similarly, the operator \((\partial_x - 2\sigma(x))\), which appears in the expression of \(A(x)\) in terms of \(C(x)\) in (A.30),

\(^{18}\)\(A, B, C\) and \(D\) are obviously functions of \(\omega\) as well. For notational simplicity we suppress their explicit \(\omega\) dependence.
projects the zero mode $c_0(x)$ out of $A(x)$. $A(x)$ and $D(x)$ do not have a pole at $\omega^2 = 0$, no matter what value the topological charge $q$ takes on.

Since $B(x)/\omega$ and $-C(x,\omega)/\omega$ are, respectively, the diagonal resolvents of the Schrödinger operators $H_b$ and $H_c$, each one of them satisfies its appropriate Gel’fand-Dikii [18] identity:

\[-2B\partial_x^2 B + (\partial_x B)^2 + 4B^2(\sigma^2 - \sigma' - \omega^2) = \omega^2\]
\[-2C\partial_x^2 C + (\partial_x C)^2 + 4C^2(\sigma^2 + \sigma' - \omega^2) = \omega^2.\]  
(A.33)

A linearized form of these identities may be obtained by deriving once and dividing through by $2B$ and $2C$. One obtains

\[\partial_x^3 B - 4\partial_x B(\sigma^2 - \sigma' - \omega^2) - 2B(\sigma^2 - \sigma')' = 0\]
\[\partial_x^3 C - 4\partial_x C(\sigma^2 + \sigma' - \omega^2) - 2C(\sigma^2 + \sigma')' = 0.\]  
(A.34)

(For a simple derivation of the GD identity, see [17, 19].)

### A.2.1 Relations Among A, B, C, and D

We now use the supersymmetry of the Dirac operator, which we discussed in the previous subsection, to deduce some important properties of the functions $A(x)$ through $D(x)$.

From (A.30) and from (A.2) we have

\[A(x) = i\frac{\partial_x - 2\sigma(x)}{2\omega} \left(-\frac{\omega c_1 c_2}{W_c}\right) = i\frac{2}{2W_c} \left(c_2 Q^\dagger c_1 + c_1 Q^\dagger c_2\right).\]

Using (A.16) first, and then (A.15), we rewrite this expression as

\[A(x) = \frac{i\omega}{2W_c} (c_2 b_1 + c_1 b_2) = \frac{i}{2W_c} (b_1 Q b_2 + b_2 Q b_1).\]

Then, using the fact that $W_c = W_b$ (Eq. (A.23)) and (A.30), we rewrite the last expression as

\[A(x) = i\frac{\partial_x + 2\sigma(x)}{2\omega} \left(\frac{\omega b_1 b_2}{W_b}\right) = i\frac{(\partial_x + 2\sigma) B(x)}{2\omega}.\]
Thus, finally,
\[ A(x) = D(x). \quad (A.35) \]

Supersymmetry renders the diagonal elements \( A \) and \( D \) equal. Clearly, the projection of zero modes (when they exist) out of \( A(x) \) and \( D(x) \), which we discussed following Eq. (A.32), makes \( A = D \) possible. We certainly cannot have \( B \) and \( C \) equal, since when one of them has a pole at \( \omega^2 = 0 \), the other does not. (See (A.38) and (A.39) below.)

Due to (A.30), \( A = D \) is also a first order differential equation relating \( B \) and \( C \):
\[
(\partial_x + 2\sigma(x)) \frac{B(x)}{\omega} = (\partial_x - 2\sigma(x)) \frac{C(x)}{\omega}. \quad (A.36)
\]

With the identification of the resolvents \( B/\omega \) and \( -C/\omega \), respectively with \( R_-(x, \omega^2) \) and \( R_+(x, \omega^2) \) alluded to above, we can write (A.36) as
\[
(2\sigma - \partial_x)R_+ = (2\sigma + \partial_x)R_-,
\]
which is essentially Eq.(18) of [16].

We can also relate the off diagonal elements \( B \) and \( C \) to each other more directly. From (A.30) and from (A.15) we find
\[
C(x) = -\frac{\omega c_1 c_2}{W_c} = -\frac{(Qb_1)(Qb_2)}{\omega W_c}. \quad (A.37)
\]

After some algebra, and using (A.23), we can rewrite this as
\[
-\omega^2 C = \sigma^2 B + \sigma B' + \frac{\omega b_1' b_2'}{W_b}
\]
The combination \( \omega b_1' b_2'/W_b \) appears in \( B'' = (\omega b_1 b_2/W_b)'' \). After using \( (H_b - \omega^2)b_{1,2} = 0 \) to eliminate \( b_1'' \) and \( b_2'' \) from \( B'' \), we find
\[
\frac{\omega b_1' b_2'}{W_b} = \frac{1}{2} B'' - \left( \sigma^2 - \sigma' - \omega^2 \right) B.
\]
Thus, finally, we have
\[
-\omega^2 C = \frac{1}{2} B'' + \sigma B' + (\sigma' + \omega^2) B. \quad (A.38)
\]
In a similar manner we can prove that
\[ \omega^2 B = -\frac{1}{2} C'' + \sigma C' + (\sigma' - \omega^2) C. \] (A.39)

We can simplify (A.38) and (A.39) further. After some algebra, and using (A.30) we arrive at
\[ C(x) = \frac{i}{\omega} \partial_x D(x) - B(x) = -\frac{1}{2\omega^2} \partial_x [(\partial_x + 2\sigma(x)) B(x)] - B(x) \]
\[ B(x) = \frac{i}{\omega} \partial_x A(x) - C(x) = -\frac{1}{2\omega^2} \partial_x [(\partial_x - 2\sigma(x)) C(x)] - C(x) \] (A.40)

(which are one and the same equation, since \( A(x) = D(x) \)). Supersymmetry, namely, isospectrality of \( H_b \) and \( H_c \), enables us to relate the diagonal resolvents of these operators, \( B \) and \( C \), to each other.

Thus, we can use (A.30), (A.35) and (A.40) to eliminate three of the entries of the diagonal resolvent in (A.30), in terms of the fourth.

The relations (A.40) (or (A.38) and (A.39)) were not discussed in [16], but one can verify them, for example, for the resolvents corresponding to the kink case \( \sigma(x) = m \tanh mx \) (Eq. (29) in [16]), for which
\[ C = -\frac{\omega}{2\sqrt{m^2 - \omega^2}}, \quad B = \left[\left(\frac{m \text{sech} \, mx}{\omega}\right)^2 - 1\right] C. \] (A.41)

Note that the two relations (A.38) and (A.39) transform into each other under
\[ B \leftrightarrow -C \quad \text{simultaneously with} \quad \sigma \rightarrow -\sigma, \] (A.42)
in consistency with (A.28).

From the relations in (A.40), it is clear that away from \( \omega = 0 \), \( B \) and \( C \) must have the same singularities as functions of \( \omega^2 \). However, we see, for example from (A.38), that if \( C \) has a pole \( 1/\omega^2 \) (which corresponds to a normalizable zero mode of \( H_c \)), then \( B \) will not have such a pole, and vice versa, as it should be.
A.3 Bilinear Fermion Condensates and Vanishing of the Spatial Fermion Current

Following basic principles of quantum field theory, we may write the most generic flavor-singlet bilinear fermion condensate in our static background as

$$\langle \bar{\psi}_{a\alpha}(t, x) \Gamma^{\alpha\beta} \psi_{a\beta}(t, x) \rangle_{\text{reg}} = N \int_{C} \frac{d\omega}{2\pi} \text{tr} \left[ \Gamma \frac{-i}{\omega \gamma^0 + i \gamma^1 \partial_x - \sigma} \right]$$

$$= N \int_{C} \frac{d\omega}{2\pi} \text{tr} \left\{ \Gamma \left[ \begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix} - \begin{pmatrix} A & B \\ C & D \end{pmatrix}_{\text{vac}} \right] \right\}, \quad (A.43)$$

where we have used (A.29). The integration contour is specified in Section 3 (see Figure 1). Here $a = 1, \cdots, N$ is a flavor index, and the trace is taken over Dirac indices $\alpha, \beta$. As usual, we regularized this condensate by subtracting from it a short distance divergent piece embodied here by the diagonal resolvent

$$\langle x | - i D^{-1} | x \rangle_{\text{vac}} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}_{\text{vac}} = \frac{1}{2\sqrt{m^2 - \omega^2}} \begin{pmatrix} i\sigma_{\text{vac}} & \omega \\ -\omega & i\sigma_{\text{vac}} \end{pmatrix} \quad (A.44)$$

of the Dirac operator in a vacuum configuration $\sigma_{\text{vac}} = \pm m$.

In our convention for $\gamma$ matrices (1.2) we have

$$\begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix} = \frac{A(x) + D(x)}{2} + \frac{A(x) - D(x)}{2i} \gamma^1 + i \frac{B(x) - C(x)}{2} \gamma^0 + \frac{B(x) + C(x)}{2} \gamma_5. \quad (A.45)$$

An important condensate is the expectation value of the fermion current $\langle j^\mu(x) \rangle$. In particular, consider its spatial component. In our static background $\sigma(x)$, it must, of course, vanish identically

$$\langle j^1(x) \rangle = 0. \quad (A.46)$$

Thus, substituting $\Gamma = \gamma^1$ in (A.43) and using (A.45) we find

$$\langle j^1(x) \rangle = iN \int_{C} \frac{d\omega}{2\pi} \left[ A(x) - D(x) \right]. \quad (A.47)$$
But we have already proved that $A(x) = D(x)$ in any static background $\sigma(x)$ (Eq. (A.35)). Thus, each frequency component of $\langle j^1 \rangle$ vanishes separately, and (A.46) holds identically. It is remarkable that the supersymmetry of the Dirac operator guarantees the consistency of any static $\sigma(x)$ background.

Expressions for other bilinear condensates may be derived in a similar manner (here we write the unsubtracted quantities). Thus, substituting $\Gamma = \gamma^0$ in (A.43) and using (A.45), (A.35) and (A.40), we find that the fermion density is

$$
\langle j^0(x) \rangle = iN \int \frac{d\omega}{2\pi} [B(x) - C(x)] = iN \int \frac{d\omega}{2\pi} \frac{2\omega B(x) - i\partial_x D(x)}{\omega} .
$$

(A.48)

Similarly, the scalar and pseudoscalar condensates are

$$
\langle \bar{\psi}(x) \psi(x) \rangle = N \int \frac{d\omega}{2\pi} [A(x) + D(x)] = 2N \int \frac{d\omega}{2\pi} D(x) ,
$$

(A.49)

and

$$
\langle \bar{\psi}(x) \gamma^5 \psi(x) \rangle = N \int \frac{d\omega}{2\pi} [B(x) + C(x)] = iN \partial_x \int \frac{d\omega}{2\pi} \frac{D(x)}{\omega} .
$$

(A.50)

We can also derive a simple relation between the fermion density (A.48) and the pseudoscalar condensate (A.50). From (A.36) we have $\partial_x (B - C) = -2\sigma(B + C)$. Substituting this result into (A.50) and comparing with (A.48) we deduce that

$$
\partial_x \langle j^0(x) \rangle = -2i\sigma(x) \langle \bar{\psi}(x) \gamma^5 \psi(x) \rangle .
$$

(A.51)

We may verify (A.51) easily for the kink $\sigma(x) = m \tanh mx$ by substituting (A.41) in (A.48) and (A.50).

The result (A.51) has a simple interpretation in terms of bosonization. Since (A.51) stems essentially from the properties of the diagonal resolvant (A.29) of the Dirac operator $i\partial - \sigma(x)$, it is independent of the $O(2N)$ symmetry of the GN model, and also independent of the dynamics of the auxiliary field $\sigma(x)$. Thus, we need to consider only bosonization of the action

$$
S_{\psi} = \int d^2x \bar{\psi} \left( i\partial - \sigma(x) \right) \psi
$$

(A.52)
of a *single* Dirac fermion $\psi$, corresponding to the Abelian case [36]. The relevant bosonization operator identities are

\[ i\bar{\psi}\partial_{\mu}\psi \rightarrow \frac{1}{2} (\partial_{\mu}\phi)^2 \]

\[ \bar{\psi}\gamma^\mu\psi \rightarrow \frac{\epsilon^{\mu\nu}\partial_{\nu}\phi}{\sqrt{\pi}} \]

\[ \bar{\psi}\psi \rightarrow \mu \cos(2\sqrt{\pi}\phi) \quad \text{and} \]

\[ i\bar{\psi}\gamma^5\psi \rightarrow \mu \sin(2\sqrt{\pi}\phi), \quad (A.53) \]

where $\phi$ is the corresponding boson, and $\mu$ is an arbitrary scale parameter.

According to these formulas, the bosonized form of $S_\psi$ is

\[ S_\phi = \int d^2x \left[ \frac{1}{2}(\partial_{\mu}\phi)^2 - \mu\sigma \cos(2\sqrt{\pi}\phi) \right], \quad (A.54) \]

which leads to the equation of motion

\[ \Box\phi - 2\mu\sqrt{\pi}\sigma \sin(2\sqrt{\pi}\phi) = 0. \quad (A.55) \]

In the static limit, the latter equation becomes

\[ \partial_0^2\phi(x) + 2\mu\sqrt{\pi}\sigma(x) \sin(2\sqrt{\pi}\phi(x)) = 0, \quad (A.56) \]

which is easily recognized as the bosonized form of (A.51) (up to a trivial multiplication of (A.56) by a factor $N$). Thus, the *identity* (A.51) is equivalent to the equation of motion of the bosonic field $\phi$ in the static limit. This of course, should be expected on physical grounds. After all, Eq. (A.51) is a consequence of (A.35) (or (A.36)), which implies $\langle j^1(x) \rangle = 0$ in our static background $\sigma(x)$, which leads to $\partial_0\phi = 0$, according to the second equation in (A.53).
Appendix B: Reflectionless Schrödinger Operators and their Resolvents

In this Appendix we gather (without derivation) some useful properties of reflectionless Schrödinger hamiltonians[3] and their resolvents. A particularly useful summary of the theory of inverse scattering in one dimension and of reflectionless Schrödinger operators is given in the first reference in [9] (which we partly follow here).

Following that summary of results from the literature, we use one particular result, namely, the explicit formulas for a pair of fundamental independent solutions of the Schrödinger equation, to derive an explicit simple representation for the diagonal resolvent of a reflectionless Schrödinger operator with a prescribed set of bound states, which we have not encountered in the literature.

B.1 The Potential and Wave Functions

Consider the Schrödinger equation

\[ (-\partial_x^2 + V(x)) b(x) = k^2 b(x) \]  

(B.1)

on the whole real axis, where \( V(x) \) is a reflectionless potential, namely, the reflection amplitude \( r(k) \) of (B.1) vanishes identically. Reflectionless potentials are bounded, and tend to an asymptotic constant value \( V(\pm\infty) = V_0 \) at an exponential rate. With no loss of generality we will set \( V_0 = 0 \). Thus, \( V(x) \) satisfies the boundary conditions

\[ V(x) \xrightarrow{x\to\pm\infty} 0. \]  

(B.2)

Since \( V(x) \) is bounded and tends to its asymptotic value (B.2) at an exponential rate, it can support only a finite number of bound states. Let us assume that the reflectionless \( V(x) \) has \( K \) bound states at energies

\[ -\kappa_1^2 < -\kappa_2^2 < \cdots < -\kappa_K^2 < 0. \]  

(B.3)

Eq.(B.3) is the spectrum of bound states of a one-dimensional Schrödinger operator, which cannot be degenerate. Thus all inequalities in (B.3) are strict.
The reflectionless potential $V(x)$ is uniquely determined [3] by the asymptotic behavior of its $K$ bound state wave functions only. (In order to determine a generic reflectionful potential, one also requires, of course, the reflection amplitude $r(k)$.)

A bound state wave function in a potential which tends to its asymptotic behavior \( B.2 \) at an exponential rate, must also decay exponentially. Thus, from \( B.1 \), the $n$-th bound state wave function $\psi_n(x)$, which corresponds to bound state energy $-\kappa_n^2$, decays asymptotically as

$$
\psi_n(x) \xrightarrow{x \to +\infty} c_n \exp -\kappa_n x ,
$$

with some real coefficient $c_n$. The parameter $c_n$ is determined from the requirement that $\psi_n$ be normalized to unity. With no loss of generality we take $c_n$ to be positive. For convenience, let us name the right hand side of \( B.4 \) as

$$
\lambda_n(x) = c_n \exp -\kappa_n x .
$$

The asymptotic behavior of $\psi_n(x)$ is determined by two positive parameters $\kappa_n$ and $c_n$, and thus, there are $2K$ parameters at our disposal:

$$
\kappa_1 > \kappa_2 > \cdots \kappa_K > 0 \quad \text{and} \quad c_1, c_2, \cdots c_K .
$$

These are the “scattering data” alluded to in Section 3 (see \( 3.4 \)). These parameters determine the reflectionless potential $V(x)$ with $K$ bound states \( B.3 \) neatly as

$$
V(x) = -2 \frac{\partial^2}{\partial x^2} \log \det A ,
$$

where $A(x)$ is the symmetric $K \times K$ matrix

$$
A_{mn} = \delta_{mn} + \frac{\lambda_m \lambda_n}{\kappa_m + \kappa_n} .
$$

It can be shown that the $K$ dimensional vectors $\psi_n$ and $\lambda_n$ are related by

$$
\sum_{n=1}^{K} A_{mn} \psi_n = \lambda_m .
$$

We can solve this linear equation and thus obtain an explicit formula for the $\psi_n(x)$. The desired solution is

$$
\psi_n(x) = - \frac{1}{\lambda_n} \frac{\det A^{(n)}}{\det A} ,
$$
where \( A^{(n)}(x) \) is the \( K \times K \) matrix obtained from (B.8), by replacing the \( n \)-th column of \( A \) by its derivative.

Inverse scattering theory also provides us with explicit expressions for the pair of fundamental independent solutions of (B.1). Let \( \{b_1(x), b_2(x)\} \) be a pair of independent fundamental solutions of (B.1) subjected to the (standard, or canonical) asymptotic boundary conditions

\[
b_1(x) \xrightarrow{x \to -\infty} e^{-ikx}, \quad b_2(x) \xrightarrow{x \to +\infty} e^{ikx}
\]

(i.e., Eq. (A.9) with \( A_b^{(1)} = A_b^{(2)} = 1 \)). Then, inverse scattering theory tells us that for \( \text{Im} k \geq 0 \), \( b_1 \) and \( b_2 \) are given by

\[
b_1(x) = \frac{1}{t(k)} e^{-ikx} \left[ 1 + i \sum_{n=1}^{K} \frac{\lambda_n(x) \psi_n(x)}{k - i\kappa_n} \right]
\]

\[
b_2(x) = e^{ikx} \left[ 1 - i \sum_{n=1}^{K} \frac{\lambda_n(x) \psi_n(x)}{k + i\kappa_n} \right],
\]

where

\[
\frac{1}{t(k)} = \frac{i}{2k} \left[ b_2(x)b'_1(x) - b_1(x)b'_2(x) \right]
\]

is the inverse of the transmission amplitude \( t(k) \) of (B.1). It is proportional to the wronskian

\[
W = b_2(x)b'_1(x) - b_1(x)b'_2(x)
\]

of \( b_1 \) and \( b_2 \), as was mentioned in Appendix A,

\[
\frac{1}{t(k)} = \frac{iW}{2k}.
\]

According to inverse scattering theory, the transmission amplitude \( t(k) \) of the reflectionless potential with \( K \) bound states which correspond to the data (B.6) is simply

\[
t(k) = \prod_{n=1}^{K} \frac{k + i\kappa_n}{k - i\kappa_n}.
\]

\(^{19}\)The functions \( b_1 \) and \( b_2 \) are, respectively, the functions \( \phi_2 \) and \( \phi_1 \) in the first reference cited in [9].
This transmission amplitude is a pure phase, as it should be, since the potential \( V(x) \) is reflectionless. Note the simple poles of \( t(k) \) at the bound states, or equivalently, the simple zeros of \( 1/t(k) \), which arise, as we have already discussed (in connection with (A.13)), since \( 1/t(k) \) is proportional to the wronskian of the two fundamental solutions, and both these solutions become proportional to the bound state wave function, and thus linear dependent.

Thus, as \( k \) tends to one of its bound state values, say at \( k = i\kappa_m \), the sum in the square brackets in the expression for \( b_1(x) \) in (B.12) has a pole, which dominates the sum. But at the same time, \( 1/t(k) \) vanishes linearly as \( k - i\kappa_m \) (up to some constant one can calculate from (B.16)). This is, of course, a general feature of scattering theory, and is not particular to reflectionless potentials. Thus, at the limit \( k \to i\kappa_m \) the expression for \( b_1(x) \) in (B.12) collapses simply to

\[
b_1(x) = \text{const.} \, \psi_m(x) \tag{B.17}
\]

(where we used \( e^{\kappa_m x} \lambda_m(x) = c_m \)). To see that at \( k = i\kappa_m \), the other fundamental solution \( b_2(x) \) in (B.12) is also proportional to \( \psi_m(x) \) already requires the special properties of reflectionless potentials. Indeed, at \( k = i\kappa_m \), the expression for \( b_2(x) \) may be written

\[
b_2(x) = e^{-\kappa_m x} \left[ 1 - \frac{1}{\lambda_m} \sum_{n=1}^{K} (A_{mn} - \delta_{mn}) \psi_n(x) \right] = \frac{1}{c_m} \psi_m(x), \tag{B.18}
\]

where we have used (B.5) and (B.9). \( b_1(x) \) and \( b_2(x) \) are indeed both proportional to the bound state wave function at \( k = i\kappa_m \).

We can also use the explicit forms of the fundamental solutions (B.12) and of the transmission amplitude (B.16) to derive an explicit formula for the integral over the potential \( V(x) \). According to (B.7), the definition of the matrix \( A^{(n)} \), and (B.10), we have

\[
\int_{-\infty}^{\infty} V(x) \, dx = 2 \left[ \frac{\partial_x \det A(x)}{\det A(x)} \right]_{-\infty}^{\infty} = -2 \lim_{x \to -\infty} \left( \sum_{n=1}^{K} \lambda_n(x) \psi_n(x) \right). \tag{B.19}
\]

Here we used the fact that \( \lim_{x \to -\infty} \lambda_n(x) \psi_n(x) = 0 \), obviously. However, \( \lim_{x \to -\infty} \lambda_n(x) \psi_n(x) \) is a constant, which has to be determined. In fact, in (B.19) we need only the sum
of these constants. This sum can be extracted from (B.12). Indeed, from (B.11) and
from (B.12), we observe that
\[ t(k) = 1 + i \lim_{x \to -\infty} \left( \sum_{n=1}^{K} \frac{\lambda_n(x)\psi_n(x)}{k - i\kappa_n} \right). \]  
(B.20)

Thus, combining (B.20) and (B.19), we finally arrive at the sum-rule
\[ \int_{-\infty}^{\infty} V(x) \, dx = 2i \int_{\mathcal{C}} \frac{dk}{2\pi i} t(k), \]  
(B.21)

where the contour \( \mathcal{C} \) encircles all the poles of \( t(k) \) in (B.16) in the upper half \( k \)-plane.

**The Diagonal Resolvent**

We now use (B.12) to derive a simple formula (Eq. (B.28) below) for the Green’s function of the reflectionless hamiltonian
\[ H = -\partial_x^2 + V(x) \]  
(B.22)
in (B.1). We have not encountered the formula we are about to derive in the literature.

In a similar manner used to derive the Green’s functions in (A.7), we may express the Green’s function of \( H \) in terms of the fundamental solutions (B.12) as
\[ G(x, y; k) = \langle x| \frac{1}{H - k^2} |y \rangle = \frac{\theta (x - y) b_2(x)b_1(y) + \theta (y - x) b_2(y)b_1(x)}{W}. \]  
(B.23)

Then the diagonal resolvent of \( H \),
\[ R(x; k^2) = G(x, x; k) = \lim_{\epsilon \to 0+} \frac{G(x, x + \epsilon; k) + G(x + \epsilon, x; k)}{2} \]
(1 compare with Eq. (A.29)) is
\[ R(x; k) = \frac{b_1(x)b_2(x)}{W} = \frac{i}{2k} \left[ 1 + i \sum_{n=1}^{K} \frac{\lambda_n \psi_n}{k - i\kappa_n} \right] \left[ 1 - i \sum_{m=1}^{K} \frac{\lambda_m \psi_m}{k + i\kappa_m} \right]. \]  
(B.24)

The last expression may be simplified considerably by simple algebraic manipulations. First, let us multiply the factors in (B.24). Thus,
\[ R(x; k) = \frac{i}{2k} \left[ 1 + i \sum_{n=1}^{K} \left( \frac{1}{k - i\kappa_n} - \frac{1}{k + i\kappa_n} \right) \lambda_n \psi_n + \sum_{m,n=1}^{K} \frac{\lambda_m \lambda_n \psi_m \psi_n}{(k - i\kappa_n)(k + i\kappa_m)} \right]. \]  
(B.25)
Then, concentrate on the last term in (B.25). Using (B.8), we can write it as
\[ \frac{\lambda_m \lambda_n \psi_m \psi_n}{(k - i\kappa_n)(k + i\kappa_m)} = -i(A_{mn} - \delta_{mn}) \left( \frac{1}{k - i\kappa_n} - \frac{1}{k + i\kappa_n} \right) \psi_m \psi_n. \] (B.26)

Then, summing the last expression over \( m, n \) and using (B.9) we obtain
\[ \sum_{m,n=1}^{K} \frac{\lambda_m \lambda_n \psi_m \psi_n}{(k - i\kappa_n)(k + i\kappa_m)} = -i \sum_{n=1}^{K} \left( \frac{1}{k - i\kappa_n} - \frac{1}{k + i\kappa_n} \right) \lambda_n \psi_n - 2 \sum_{n=1}^{K} \frac{\kappa_n \psi_n^2}{k^2 + \kappa_n^2}. \] (B.27)

Finally, substituting (B.27) in (B.25) we arrive at the desired simpler formula for the diagonal resolvent:
\[ R(x; k) = \frac{i}{2k} \left( 1 - 2 \sum_{n=1}^{K} \frac{\kappa_n \psi_n^2}{k^2 + \kappa_n^2} \right). \] (B.28)

As one trivial consistency check on (B.28), note from (B.28) that \( R(x; k) \), as a function of energy \( E = k^2 \), has a simple pole at each of the bound state energies \( E_n = -\kappa_n^2 \), with residue \(-\psi_n^2\), as it should be.\(^{20}\)

Also note that (B.28), being the diagonal resolvent of (B.22) with potential (B.7), must satisfy the GD identity
\[ -2RR'' + R'^2 + 4R^2(V(x) - k^2) = 1. \] (B.29)

In fact, in [16], the analysis went the other way around, and used (B.29) to derive (B.28) in the case of a single bound state, \( K = 1 \).

As yet another consistency check on (B.28), note that as \(|x| \to \infty\), it tends to \( R = i/2k \), which is indeed the asymptotic solution of (B.29) in view of (B.2).

### B.2 Application to the Dirac Operator

In order to make contact with the discussion in the text, let us identify the hamiltonian (B.22) with one of the operators \( H_b \) or \( H_c \) in (A.6). With no loss of generality we chose to identify it with \( H_b = -\partial_x^2 + \sigma(x)^2 \sigma'(x) \). Actually, due to the boundary

\(^{20}\)Note that (B.28) is not the most obvious way to write a function of \( x \) and \( k^2 \) with these properties (e.g., the function similar to (B.28), but with all prefactors \( i\kappa_n/\sqrt{k^2} \) removed, has these properties, and also the correct large \( k \) behavior \( i/2k \), and is much simpler). Thus, we really had to go through all the steps of the derivation to obtain (B.28).
conditions (2.8), the potential $\sigma(x)^2 - \sigma'(x)$ tends asymptotically to $m^2$ and not to zero. Thus, in order to comply with (B.2), we consider $H = H_b - m^2$, so that the eigenvalue equation (A.8) $H_b b = \omega^2 b$ turns into $(H_b - m^2)b = (\omega^2 - m^2)b$, which coincides with (B.1), due to $\omega^2 = k^2 + m^2$ (A.10). At the bound state with $k^2 = -\kappa^2_n$ we thus have $\omega^2_n = m^2 - \kappa^2_n$. Thus, from (B.3) the bound state energies of $H_b$ are

$$0 \leq \omega^2_1 < \omega^2_2 < \cdots < \omega^2_K < m^2. \quad (B.30)$$

Since (B.3) is non-degenerate, all the $\omega^2_n$ must be different from each other.

In the topologically trivial sector of the model, $q = 0$, $H_b = -\partial^2_x + \sigma(x)^2 - \sigma'(x)$ and $H_c = -\partial^2_x + \sigma(x)^2 + \sigma'(x)$ are strictly isospectral, and also reflectionless. Thus, we have two different reflectionless potentials, with the same set of bound state energies (B.30). Clearly, it is the additional $K$ parameters $c_1, \cdots, c_K$ that remove the ambiguity between these two isospectral potentials. The parameters $c_1, \cdots, c_K$ parametrize a $K$ dimensional family of reflectionless potentials with given $K$ bound state energies (B.30). As the $c_n$ vary, the system wanders around in that family space without changing its bound state energies. In particular, $H_b$ and $H_c$ should correspond to two points in the $K$ dimensional space of the $c_n$’s. Let us denote these points, respectively, as $\{c_n(H_b)\}$ and $\{c_n(H_c)\}$. It will be interesting to find a representation of the supersymmetric transformation $H_b \leftrightarrow H_c$ as an invertible map on the $K$ dimensional vectors

$$\{c_n(H_b)\} \leftrightarrow \{c_n(H_c)\}. \quad (B.31)$$

Computation of this transformation in the simplest case, i.e., when there is a single bound state, $K = 1$, is given in section B.3 (see (B.47)).

Finally, from (A.31) we have

$$B(x) = \langle x| \frac{\omega}{H_b - \omega^2} |x\rangle = \frac{i\omega}{2k} \left(1 - 2 \sum_{n=1}^K \frac{\kappa_n \psi_n^2}{\omega^2_n - \omega^2_n}\right). \quad (B.32)$$

We can then use (A.38) to determine $C(x)$. (Alternatively, due to (A.31) and isospectrality of $H_b$ and $H_c$, we can just write an expression for $-C(x)$ similar to (B.32) with $\psi_n$ being the bound state wave functions of $H_c$.)
B.2.1 The Condensate $\sigma(x)$

Let us now determine the condensate $\sigma(x)$. We will consider the cases of zero topological charge and non-zero topological charge separately.

(a) topologically trivial condensates, $q = 0$

In this case, none of the operators $H_b$ and $H_c$ has a normalizable zero mode. Thus, none of the $\omega_n$’s in (B.30) vanishes. (Equivalently, none of the $\kappa_n$’s in (B.3) can be equal to $m$.) Since $H_b$ does not have a normalizable zero mode, the fundamental solutions (B.12) at $\omega^2 = 0$, i.e., at $k = im$,

\[
\begin{align*}
    b_1(x)_{k=im} &= \frac{1}{t(im)} e^{mx} \left[ 1 + \sum_{n=1}^{K} \frac{\lambda_n(x)\psi_n(x)}{m - \kappa_n} \right] \\
    b_2(x)_{k=im} &= e^{-mx} \left[ 1 - \sum_{n=1}^{K} \frac{\lambda_n(x)\psi_n(x)}{m + \kappa_n} \right], \quad (B.33)
\end{align*}
\]

are not square-integrable functions. They are the two independent solutions of $Q^\dagger Q b_{1,2} = 0$. According to (A.18), one of these solutions must be $b_0(x) = \exp(-\int \sigma(y) dy)$. Thus, if we knew $b_0$, we could determine

\[
\sigma(x) = -\frac{d}{dx} \log b_0(x). \quad (B.34)
\]

Clearly,

\[
b_0(x) \xrightarrow{x \to +\infty} e^{-\sigma(\infty)x}. \quad (B.35)
\]

On the other hand,

\[
\begin{align*}
    b_1(x)_{k=im} &\xrightarrow{x \to +\infty} \frac{e^{mx}}{t(im)} \quad \text{and} \quad b_2(x)_{k=im} \xrightarrow{x \to +\infty} e^{-mx}, \quad (B.36)
\end{align*}
\]

since in that limit, the terms proportional to $\lambda_n(x)\psi_n(x)$ in (B.33) are negligible compared to 1. Now, identification of $b_0(x)$ with one of the functions in (B.33) is just a matter of comparing (B.35) and (B.36). Thus, if $\sigma(\infty) = -m$, then $b_0(x) = b_1(x)_{k=im}$, and if $\sigma(\infty) = m$, then $b_0(x) = b_2(x)_{k=im}$. Therefore, we conclude that

\[
\begin{align*}
    \sigma(x) &= -m - \frac{d}{dx} \log \left[ 1 + \sum_{n=1}^{K} \frac{\lambda_n(x)\psi_n(x)}{m - \kappa_n} \right], \quad \sigma(\infty) = -m \\
    \sigma(x) &= m - \frac{d}{dx} \log \left[ 1 - \sum_{n=1}^{K} \frac{\lambda_n(x)\psi_n(x)}{m + \kappa_n} \right], \quad \sigma(\infty) = m, \quad (B.37)
\end{align*}
\]

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or more compactly,

\[ \sigma(x) = \sigma(\infty) - \frac{d}{dx} \log \left[ 1 - \sum_{n=1}^{K} \frac{\lambda_n(x)\psi_n(x)}{\kappa_n + \sigma(\infty)} \right]. \]  \quad (B.38)

This is manifestly a configuration belonging to the \( q = 0 \) sector, since the expression on the right hand side of (B.38) tends to \( \sigma(\infty) \) as \( x \to \pm\infty \), i.e., on both sides of the one dimensional world.

**(b) the \( q = 1 \) sector**

According to the discussion following Eqs. (A.18), (A.19), in the \( q = 1 \) sector, only \( H_b \) has a normalizable zero mode. According to (A.18) this normalizable ground state is \( b_0(x) = \exp - \int^x \sigma(y)dy \), and thus must be just the lowest energy state \( \psi_1 \) in (B.10), corresponding to \( \omega_1^2 = 0 \), i.e., \( \kappa_1 = m \). Thus, from (B.10) and (B.34) we find

\[ \sigma(x) = -\frac{d}{dx} \log \psi_1(x) = -\frac{d}{dx} \log \left( -\frac{1}{\lambda_1} \frac{\det A(1)}{\det A} \right). \]  \quad (B.39)

Let us make the side remark, that due to the discussion which led to (B.17) and (B.18), the present case can be considered as a singular limit of the previous case (a) (with \( \sigma(\infty) = m \), as appropriate to the sector \( q = 1 \)), simply by taking the second equation in (B.37) at \( \kappa_1 \to m \). Following the same steps as in the derivation of (B.18) with \( \kappa_m = \kappa_1 = m \) we end up with

\[ \sigma(x) = m - \frac{d}{dx} \log \frac{\psi_1}{\lambda_1} = -\frac{d}{dx} \log \psi_1(x) \]  \quad (B.40)

i.e., back to (B.39).

**(c) the \( q = -1 \) sector**

According to the discussion following Eqs. (A.18), (A.19), in the \( q = -1 \) sector, only \( H_c \) has a normalizable zero mode. If it has \( K - 1 \) additional bound states at positive energies, they will be isospectral with those of \( H_b \). Thus, in that case \( H_c \) has \( K \) bound states at energies

\[ 0 = \omega_1^2 < \omega_2^2 \cdots < \omega_K^2 < m^2. \]  \quad (B.41)
Determination of a $\sigma(x)$ configuration that carries topological charge $q = -1$ can be reduced to the previous case, of determination of the configuration $-\sigma(x)$ that carries $q = 1$. A sign flip $\sigma(x) \to -\sigma(x)$ obviously interchanges $H_b$ and $H_c$. Thus, $-\sigma(x)$ will give rise to an $H_b$ with bound state spectrum (B.41), and we will be able to determine this $-\sigma(x)$ by the method of case (b), from (B.39).

So far, we have assumed $K \geq 2$ in (B.41). If $K = 1$, the zero mode is the only bound state of $H_c$, and $H_b$ cannot have bound states at all. In that case, it is therefore the free particle hamiltonian $H_b = -\partial_x^2 + m^2$. Thus, $\sigma^2 - \sigma' = m^2$ and $\sigma(x)$ is the CCGZ antikink $\sigma(x) = -mtanh mx$.

**B.3 Reflectionless Potentials with One and Two Bound States**

We end this appendix by working out the cases of reflectionless potentials with one and two bound states, which are the two cases relevant to our discussion of stable static bags in the GN model.

**B.3.1 A Single Bound State, $K = 1$**

Let us set the bound state energy at $\omega_b^2 < m^2$ (with the corresponding $\kappa^2 = m^2 - \omega_b^2$). We also have a single parameter $c > 0$. Then, from (B.5), (B.8), (B.10) and (B.7) we find

$$\lambda(x) = c e^{-\kappa x}, \quad A(x) = 1 + \frac{c^2 e^{-2\kappa x}}{2\kappa} \quad (B.42)$$

and

$$\psi_b(x) = -\frac{1}{\lambda} \frac{A'(x)}{A(x)} = \sqrt{\frac{\kappa}{2}} \text{sech} (\kappa(x - x_0)) \quad (B.43)$$

where $x_0$ is determined from

$$e^{-\kappa x_0} = \sqrt{2\kappa/c}. \quad (B.44)$$

$\psi_b$ is of course normalized to 1. From (B.7) we find the potential $\sigma^2 - \sigma' = m^2 + V(x) = m^2 - 2\partial_x^2 \log A(x) = m^2 - 2\kappa^2 \text{sech}^2 (\kappa(x - x_0))$, which is a potential of the Pöschl-Teller type.

The role of the parameter $c$ is explicit in these formulas: it just shifts the center of the potential, and thus cannot affect the energy of the bound state. Thus, $\log c$ is
essentially the translational collective mode of the static soliton $\sigma(x)$, which is obvious from the explicit expression (B.46) below.

The diagonal resolvent is found from (B.32) and is given by

$$B(x) = \frac{i\omega}{2k} \left( 1 - 2\frac{\kappa \psi_b^2}{\omega^2 - \omega_b^2} \right) = \frac{i\omega}{2k} \left[ 1 - \kappa^2 \frac{\text{sech}^2(\kappa(x - x_0))}{\omega^2 - \omega_b^2} \right],$$  \hspace{1cm} (B.45)

in agreement with the expression found in [16].

Let us determine the corresponding $\sigma(x)$. Since $\omega_b > 0$, we have to use (B.38). We find

$$\sigma(x) = \sigma(\infty) - \frac{d}{dx} \log \left[ 1 - \frac{\lambda(x)\psi_b(x)}{\kappa + \sigma(\infty)} \right]$$

$$= \sigma(\infty) + \kappa \tanh [\kappa(x - x_0)] - \kappa \tanh \left[ \kappa(x - x_0) + \frac{1}{2} \log \left( \frac{m + \kappa}{m - \kappa} \right) \right],$$  \hspace{1cm} (B.46)

in agreement with the result quoted in Eq. (3.28) of [1]. It has the profile of a bound state of a kink and an antikink, with interkink distance $\frac{1}{2\kappa} \log \left( \frac{m + \kappa}{m - \kappa} \right)$. It thus carries topological charge $q = 0$.

Let us now find the representation of the supersymmetric transformation $H_b \leftrightarrow H_c$, alluded to in the previous subsection, as a map (B.31) between the two normalization constants $c(H_b)$ and $c(H_c)$ in (B.31). Since the transformation $H_b \leftrightarrow H_c$ is achieved simply by $\sigma(x) \leftrightarrow -\sigma(x)$, we have to find a pair of normalization constants $c(H_b)$ and $c(H_c)$, which will yield, upon substitution into (B.38) and (B.42) (with boundary conditions $\sigma(\infty;b) = m = -\sigma(\infty;c)$ in the latter equation), two $\sigma(x)$ configurations of opposite signs. The calculation is straightforward, and we find

$$c^2(H_c) = c^2(H_b) \frac{m - \kappa}{m + \kappa},$$  \hspace{1cm} (B.47)

or equivalently, from (B.44),

$$\kappa x_0(H_c) = \kappa x_0(H_b) + \frac{1}{2} \log \frac{m - \kappa}{m + \kappa}.$$  \hspace{1cm} (B.48)

It is easy to verify, by substituting (B.48) in (B.46), followed by $\sigma(\infty) = m \to -m$, that $\sigma(x - x_0(H_b)) = -\sigma(x - x_0(H_c))$, as required. Furthermore, note that if $x_0 =$
\( x_0(H_b) \) in the first hyperbolic tangent in (B.46), the second hyperbolic tangent there is shifted by \( x_0(H_c) \).

As \( \kappa \) tends to \( m \), \( \omega_b^2 \) tends to zero. An eigenvalue \( \omega_b^2 = 0 \) cannot occur in the topologically trivial sector. Thus, the limit \( \kappa \rightarrow m \) in (B.46) is singular, and the kink and antikink become infinitely separated, as we discussed following (3.36).

In the sector with topological charge \( q = 1 \), we have \( \omega_b^2 = 0 \) in the spectrum of \( H_b \), and we should use (B.39) for \( \sigma(x) \). Thus, from (B.39) and (B.43) with \( \kappa = m \) we find

\[
\sigma(x) = -\frac{d}{dx} \log \psi_1(x) = m \tanh \left( m(x-x_0) \right),
\]

which is of course the CCGZ kink. In the CCGZ kink background, \( H_b \) has a single bound state at \( \omega_b = 0 \) and no other bound states. Thus, \( H_c \) has no bound states at all, and is thus the free particle Hamiltonian \( H_c = -\partial_x^2 + m^2 \). Indeed, with (B.49), one has \( \sigma^2 + \sigma' = m^2 \) for the potential of \( H_c \). The CCGZ antikink was already mentioned at the end of case (c) in the previous subsection.

### B.3.2 Two Bound States, \( K = 2 \)

We now determine the reflectionless potentials with two bound states. Let us set the bound state energies (B.30) at \( 0 \leq \omega_1^2 < \omega_2^2 < m^2 \), with the corresponding \( 0 < \kappa_2 < \kappa_1 \leq m \). We also need \( c_1, c_2 \), the two positive parameters in (B.4). The case relevant for our discussion of stable static fermion bags with two bound states is, of course, the case \( 0 = \omega_1 < \omega_2 \). We will discuss it at the end of this subsection.

Then, from (B.5) and (B.8) we construct

\[
A = \begin{pmatrix}
1 + \frac{c_1^2}{2\kappa_1} e^{-2\kappa_1 x} & \frac{c_1 c_2}{\kappa_1 + \kappa_2} e^{-(\kappa_1 + \kappa_2) x} \\
\frac{c_1 c_2}{\kappa_1 + \kappa_2} e^{-(\kappa_1 + \kappa_2) x} & 1 + \frac{c_2^2}{2\kappa_2} e^{-2\kappa_2 x}
\end{pmatrix}.
\]

We will also need

\[
A^{(1)} = \begin{pmatrix}
-c_1^2 e^{-2\kappa_1 x} & \frac{c_1 c_2}{\kappa_1 + \kappa_2} e^{-(\kappa_1 + \kappa_2) x} \\
-c_1 c_2 e^{-(\kappa_1 + \kappa_2) x} & 1 + \frac{c_2^2}{2\kappa_2} e^{-2\kappa_2 x}
\end{pmatrix},
A^{(2)} = \begin{pmatrix}
1 + \frac{c_1^2}{2\kappa_1} e^{-2\kappa_1 x} & -c_1 c_2 e^{-(\kappa_1 + \kappa_2) x} \\
\frac{c_1 c_2}{\kappa_1 + \kappa_2} e^{-(\kappa_1 + \kappa_2) x} & -c_2^2 e^{-2\kappa_2 x}
\end{pmatrix}.
\]

(B.51)
The determinants of these matrices are

\[
F(x) = \det A = 1 + \frac{c_1^2}{2\kappa_1} e^{-2\kappa_1 x} + \frac{c_2^2}{2\kappa_2} e^{-2\kappa_2 x} + \frac{c_1^2 c_2^2}{4\kappa_1 \kappa_2} \left( \frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2} \right)^2 e^{-2(\kappa_1 + \kappa_2) x}
\]

\[
F_1(x) = \det A^{(1)} = -c_1^2 e^{-2\kappa_1 x} \left( 1 + \frac{c_2^2}{2\kappa_2} \frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2} e^{-2\kappa_2 x} \right)
\]

\[
F_2(x) = \det A^{(2)} = -c_2^2 e^{-2\kappa_2 x} \left( 1 + \frac{c_1^2}{2\kappa_1} \frac{\kappa_2 - \kappa_1}{\kappa_2 + \kappa_1} e^{-2\kappa_1 x} \right).
\]

(B.52)

The potential \(\sigma^2 - \sigma'\) can be found from (B.7) and (B.52) as

\[
\sigma^2 - \sigma' = m^2 + V(x) = m^2 - 2 \frac{d^2}{dx^2} \log F(x).
\]

(B.53)

We will not need an explicit expression for \(V(x)\).

The bound state wave functions may be found from (B.10)

\[
\psi_1(x) = -\frac{1}{\lambda_1(x)} F_1(x)
\]

\[
= \frac{c_1 e^{-\kappa_1 x} \left( 1 + \frac{c_2^2}{2\kappa_2} \frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2} e^{-2\kappa_2 x} \right)}{1 + \frac{c_1^2}{2\kappa_1} e^{-2\kappa_1 x} + \frac{c_2^2}{2\kappa_2} e^{-2\kappa_2 x} + \frac{c_1^2 c_2^2}{4\kappa_1 \kappa_2} \left( \frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2} \right)^2 e^{-2(\kappa_1 + \kappa_2) x}}
\]

\[
\psi_2(x) = -\frac{1}{\lambda_2(x)} F_2(x)
\]

\[
= \frac{c_2 e^{-\kappa_2 x} \left( 1 + \frac{c_1^2}{2\kappa_1} \frac{\kappa_2 - \kappa_1}{\kappa_2 + \kappa_1} e^{-2\kappa_1 x} \right)}{1 + \frac{c_1^2}{2\kappa_1} e^{-2\kappa_1 x} + \frac{c_2^2}{2\kappa_2} e^{-2\kappa_2 x} + \frac{c_1^2 c_2^2}{4\kappa_1 \kappa_2} \left( \frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2} \right)^2 e^{-2(\kappa_1 + \kappa_2) x}}.
\]

(B.54)

The parameters \(c_1, c_2\) appear in the potential and wave functions (B.53) and (B.54) explicitly. They parametrize a 2-dimensional family of reflectionless potentials with given two bound state energies \(\omega_1^2 < \omega_2^2\). In other words, as \(c_1\) and \(c_2\) vary, the profiles of \(V(x),\psi_1(x)\) and \(\psi_2(x)\) (and in particular, their “center of gravity”) change accordingly, with the energies fixed. Thus, as in the single bound state case, \(c_1\) and \(c_2\) represent collective coordinates of the soliton \(\sigma(x)\). This is manifest, for example, in the explicit expression (B.64) below, for the \(\sigma(x)\) corresponding to \(\kappa_1 = m > \kappa_2 = \kappa\).
One particular choice of $c_1$ and $c_2$ leads to a symmetric potential $V(-x) = V(x)$. From (B.7) and (B.52) we see that this will occur when $\frac{d^2}{dx^2} \log \frac{F(-x)}{F(x)} = 0$, i.e., at $F(-x) = e^{ax+b}F(x)$ for some constants $a, b$. Setting $x = 0$ in the last equation tells us that $b = 0$. Thus, an even potential occurs when

$$F(-x) = e^{ax}F(x). \quad (B.55)$$

Making this demand on $F(x)$ in (B.52), leads, after some algebra, to the condition

$$\frac{c_1^2}{2\kappa_1} = \frac{c_2^2}{2\kappa_2} = \frac{\kappa_1 + \kappa_2}{\kappa_1 - \kappa_2}. \quad (B.56)$$

For these values of $c_1$ and $c_2$ we obtain from (B.52)

$$F(x) = 1 + \frac{\kappa_1 + \kappa_2}{\kappa_1 - \kappa_2} \left( e^{-2\kappa_1x} + e^{-2\kappa_2x} \right) + e^{-2(\kappa_1 + \kappa_2)x}. \quad (B.57)$$

It is straightforward to check that (B.57) satisfies (B.55) with $a = 2(\kappa_1 + \kappa_2)$,

$$F(-x) = e^{2(\kappa_1 + \kappa_2)x}F(x). \quad (B.58)$$

In this case, we can read the ground state wave function off (B.54). It is

$$\psi_1(x) = \frac{c_1 e^{-\kappa_1x}}{F(x)} \left( 1 + e^{-2\kappa_2x} \right). \quad (B.59)$$

Similarly, the excited state is

$$\psi_2(x) = \frac{c_2 e^{-\kappa_2x}}{F(x)} \left( 1 - e^{-2\kappa_1x} \right). \quad (B.60)$$

Thanks to (B.58), $\psi_1(-x) = \psi_1(x)$ is symmetric, and $\psi_2(-x) = -\psi_2(x)$ is antisymmetric, as it should be.

**Avoiding Degeneracy** It is amusing to observe how this one-dimensional quantum system avoids degeneracy in its bound state spectrum, as it must. (From now on, we consider generic backgrounds and do not assume a symmetric $V(x)$.) If we attempt to construct a reflectionless potential with two degenerate bound states, i.e., with $\kappa_1 = \kappa_2 = \kappa$, we observe from (B.54) that

$$\frac{\psi_1(x)}{c_1} = \frac{\psi_2(x)}{c_2} = \frac{e^{-\kappa x}}{1 + \frac{c_1^2 + c_2^2}{2\kappa}e^{-2\kappa x}}. \quad (B.61)$$
The two wave functions become linearly dependent, and thus, there is no degeneracy. Rather, the system loses one bound state, since now

\[ F(x) = 1 + \frac{c_1^2 + c_2^2}{2\kappa} e^{-2\kappa x} \]  

(B.62)

coincides with \( A(x) = 1 + \frac{c_2 e^{-2\kappa x}}{2\kappa} \) in (B.42) of the single bound state case, with \( c^2 = c_1^2 + c_2^2 \).

The Case \( 0 = \omega_1^2 < \omega_2^2 \)

Rather than continuing the general discussion, let us specialize at this point in the reflectionless potential which is relevant for our discussion of stable fermion bags with two bound states, namely, the case with \( 0 = \omega_1^2 < \omega_2^2 \). This corresponds to setting \( \kappa_1 = m > \kappa_2 \). Let us also denote \( \kappa_2 = \kappa \) in (B.50) through (B.54) above. In particular, the ground state wave function is

\[ \psi_1(x) = \frac{c_1 e^{-mx}}{1 + \frac{c_1^2}{2m} e^{-2mx} + \frac{c_2^2}{2\kappa} e^{-2\kappa x} + \frac{c_1 c_2}{4\kappa m} (\frac{m-\kappa}{m+\kappa})^2 e^{-2(m+\kappa)x}}. \]  

(B.63)

The wave function \( \psi_1(x) \) is the normalizable zero mode of \( H_b \). Thus, we are in the \( q = 1 \) topological sector. In this sector we can determine \( \sigma(x) \) from (B.39). Thus, substituting (B.63) in (B.39), we obtain

\[ \sigma(x) = -\frac{d}{dx} \log \psi_1(x) = m + \frac{2\kappa}{1 + \frac{2m}{c_1^2} \frac{m+\kappa}{m-\kappa} e^{2\kappa x}} \]

\[ - 2(m + \kappa) \frac{1 + \frac{2m\kappa}{c_1^2} \frac{m+\kappa}{m-\kappa} e^{2m x} + \frac{2m\kappa}{c_2^2} \frac{m+\kappa}{m-\kappa} e^{2\kappa x} + \frac{4m\kappa}{c_1^2 c_2^2} \frac{m+\kappa}{m-\kappa} e^{2(m+\kappa)x}}{1 + \frac{2m}{c_1^2} \frac{m+\kappa}{m-\kappa} e^{2m x} + \frac{2m}{c_2^2} \frac{m+\kappa}{m-\kappa} e^{2\kappa x} + \frac{4m}{c_1^2 c_2^2} \frac{m+\kappa}{m-\kappa} e^{2(m+\kappa)x}}. \]  

(B.64)

One can readily check that the asymptotic values of \( \sigma(x) \) are \( \sigma(\infty) = m \) and \( \sigma(-\infty) = m + 2\kappa - 2(\kappa + m) = -m \). Thus, \( \sigma(x) \) indeed carries topological charge \( q = 1 \).

By varying the parameters \( c_1, c_2 \) (and keeping \( \kappa \) fixed), we can modify its shape (e.g., translate its center of gravity), without affecting the bound state energies. Eq. (B.64) represents a two parameter family of isospectral kink backgrounds. Thus, as we have asserted earlier, \( c_1 \) and \( c_2 \) are related to the collective coordinates of the
soliton $\sigma(x)$. In this case, we have two such translational collective coordinates, $x_0$ and $y_0$, given by
\[ e^{2m x_0} = \frac{c_1^2}{2m} \quad \text{and} \quad e^{2\kappa y_0} = \frac{c_2^2}{2\kappa}, \]
in complete analogy with (B.44). In terms of $x_0$ and $y_0$, we can write (B.64) in a slightly less cluttered form as
\[
\sigma(x) = m + \frac{2\kappa}{1 + \frac{m + \kappa}{m - \kappa} e^{2\kappa(x-y_0)}}
- 2(m + \kappa) \frac{1 + \left(\frac{m + \kappa}{m - \kappa}\right)^2 e^{2m(x-x_0)} + \left(\frac{m + \kappa}{m - \kappa}\right)^2 e^{2\kappa(x-y_0)} + \left(\frac{m + \kappa}{m - \kappa}\right)^2 e^{2m(x-x_0)+2\kappa(x-y_0)}}{1 + \left(\frac{m + \kappa}{m - \kappa}\right)^2 e^{2m(x-x_0)} + \left(\frac{m + \kappa}{m - \kappa}\right)^2 e^{2\kappa(x-y_0)} + \left(\frac{m + \kappa}{m - \kappa}\right)^2 e^{2m(x-x_0)+2\kappa(x-y_0)}}.
\]
(B.65)

We may simplify (B.63) and (B.66) further as follows. In terms of the distance scale
\[ R = \frac{1}{2\kappa} \log \left(\frac{m + \kappa}{m - \kappa}\right) \]
we rewrite (B.63) as
\[
\psi_1(x) = \sqrt{2m} \frac{\cosh [\kappa(x - y_0 + R)]}{e^{-\kappa R} \cosh [m(x - x_0) + \kappa(x - y_0) + 2\kappa R] + e^{\kappa R} \cosh [m(x - x_0) - \kappa(x - y_0)]}.
\]
(B.66)

Thus, we obtain the somewhat more transparent expression
\[
\sigma(x) = -\frac{d}{dx} \log \psi_1(x) = -\kappa \tanh [\kappa(x - y_0 + R)]
+ \frac{\omega_b}{e^{-\kappa R} \cosh [m(x - x_0) + \kappa(x - y_0) + 2\kappa R] + e^{\kappa R} \cosh [m(x - x_0) - \kappa(x - y_0)]} \sinh [m(x - x_0) + \kappa(x - y_0) + 2\kappa R] + \sinh [m(x - x_0) - \kappa(x - y_0)]
\]
(B.67)

for $\sigma(x)$, where we used $(m + \kappa)e^{-\kappa R} = (m - \kappa)e^{\kappa R} = \sqrt{m^2 - \kappa^2} = \omega_b$.

With no loss of generality, we can set, of course, one of the collective coordinates to zero. In [23] we have studied the profile of (B.66) in some detail. (See Figures 1-3 in [23].)

\[ \text{Note that (B.67) coincides with the interkink distance in (B.46).} \]

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We can also tune the parameters $c_1$ and $c_2$ in (B.64) (for purely mathematical reasons and without apparent physical motivation), such that $V_b(x) = \sigma^2(x) - \sigma'(x)$ be an even function (and $\sigma(x)$ be an odd function). Thus, using (B.56) and (B.65), we set $2mx_0 = 2\kappa y_0 = 2\kappa R$ in (B.69), and obtain

$$\sigma(x) = -\kappa \tanh(\kappa x) + \omega_b \frac{\sinh[(m+\kappa)x] + \sinh[(m-\kappa)x]}{e^{-\kappa R} \cosh[(m+\kappa)x] + e^{\kappa R} \cosh[(m-\kappa)x]}.$$  \hspace{1cm} (B.70)

Then, if for example, we set $\kappa = m/2$ in (B.70) we see that it simplifies considerably, and we obtain

$$\sigma(x) = m \tanh(mx/2), \hspace{1cm} (B.71)$$

i.e., a deformation of the CCGZ kink $m \tanh mx$. It is clearly spread out in space as twice as much as the CCGZ kink.
Appendix C: Reflectionless $\sigma(x)$’s Are Extremal

As a consistency check of our calculations in Section 3, we will verify now that vanishing of the right hand side of (3.19) guarantees that the corresponding $\sigma(x)$ is indeed extremal among all possible admissible static configurations (i.e., a solution of $\frac{\delta E[\sigma]}{\delta \sigma(x)} = 0$), and not just an extremum over the space of reflectionless configurations:

Substituting $V_b$ and $V_c$ from (3.9) into (3.10), and integrating by parts over $\delta \sigma'(x)$, we obtain

$$\delta E[\sigma] = \int dx \delta \sigma(x) \left\{ \frac{\sigma(x)}{g^2} + N \int \frac{d\omega}{2\pi} \left[ \frac{i(2\sigma(x) + \partial_x)B(x)}{2\omega} + \frac{i(-2\sigma(x) + \partial_x)C(x)}{2\omega} \right] \right\}.$$  \hfill (C.1)

With the help of (A.30) we recognize the two expressions in the square brackets in (C.1), respectively, as $D(x)$ and $A(x)$, the two diagonal entries of the diagonal resolvent (2.11). Thus, from (C.1) we conclude that

$$\frac{\delta E[\sigma]}{\delta \sigma(x)} = \frac{\sigma(x)}{g^2} + N \int \frac{d\omega}{2\pi} (A(x) + D(x)). \hfill (C.2)$$

Finally, we may simplify (C.2) further, by invoking (A.35), which tells us that $A(x) = D(x)$. Thus, we may write the variational derivative as

$$\frac{\delta E[\sigma]}{\delta \sigma(x)} = \frac{\sigma(x)}{g^2} + 2N \int \frac{d\omega}{2\pi} D(x). \hfill (C.3)$$

As a consistency check, we obtain (C.3) in the next subsection as the static limit of the general extremum condition (2.3).

For later use, let us record here (C.3), evaluated at the vacuum condensate $\sigma = \sigma_{VAC} = \pm m$. Thus,

$$\frac{\sigma_{VAC}}{g^2} + 2N \int \frac{d\omega}{2\pi} D_{VAC} = 0, \hfill (C.4)$$

where $D_{VAC} = \frac{i\sigma_{VAC}}{2\sqrt{m^2 - \omega^2}}$ is given in (A.44) in Appendix A. It is a trivial matter to verify from (A.44) and (2.3) that (C.4) is equivalent to the bare gap equation (2.4).

Equations (C.1) - (C.3) are general identities, valid for any static $\sigma(x)$. We will now evaluate (C.3) at a reflectionless $\sigma(x)$ and show that its right hand side vanishes.
We start by substituting the relation $D(x) = i\frac{\partial_x + 2\sigma(x)}{2\omega}B(x)$ (from (A.30)), and the expression (B.32) for $B(x)$ into (C.3), and obtain

$$
\frac{\delta E[\sigma]}{\delta \sigma(x)} = \left( \frac{1}{g^2} + iN \int \frac{d\omega}{2\pi} \frac{1}{\sqrt{m^2 - \omega^2}} \right) \sigma(x)
$$

$$
- iN \sum_{n=1}^{K} \kappa_n \left( (\partial_x + 2\sigma)\psi_n^2 \right) \int \frac{d\omega}{2\pi} \frac{1}{\sqrt{m^2 - \omega^2}} \cdot \frac{1}{\omega^2 - \omega_n^2}
$$

(C.5)

The first term on the right hand side of this equation vanishes due to the gap equation (C.4) (compare with (3.16)). As for the sum in (C.5), we now show that it vanishes term by term. One has to be a little bit more careful in case $H_b$ has its ground state at zero energy. Consider first all terms in the sum which correspond to positive bound state energies $\omega_n^2 > 0$. Due to (3.19), each one of these terms is proportional to the corresponding $\frac{\partial M}{\partial \omega_n}$, which vanishes at an extremal $\omega_n$. If $H_b$ has a zero energy ground state, i.e., if $\omega_1^2 = 0$, its contribution to the sum in (C.5) vanishes also, but due to the fact that it is proportional to $(\partial_x + 2\sigma(x))\psi_1^2$. The latter expression vanishes, since in this case $\psi_1(x)$ is proportional to $b_0(x) = \exp - \int^x \sigma(y)dy$ in (A.18). In any case, the right hand side of (C.5) vanishes. Thus, we have verified that a reflectionless $\sigma(x; \omega_n, c_n)$, with parameters $\omega_n$ that satisfy (3.19) (and any set of $c_n$’s), is an extremal configuration of $E[\sigma]$.

C.1 A Consistency Check of the Static Saddle Point Equation (C.3)

As a consistency check on (C.3), let us verify that the saddle point equation $\frac{\delta E[\sigma]}{\delta \sigma(x)} = 0$, with the variational derivative given by (C.3), is equivalent to the extremum condition (2.3), evaluated at a static configuration\(^{22}\).

To this end, observe that for static $\sigma(x)$ backgrounds, we have the (divergent)

\(^{22}\)Note that (2.3) is obtained by taking a generic space-time dependent variation $\delta \sigma(x,t)$ of the effective action $S_{eff}$ around a static $\sigma(x)$, whereas (C.3) is obtained from varying the energy functional, which is defined, of course, only over the space of static $\sigma(x)$ configurations, and thus allows only static variations $\delta \sigma(x)$.

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formal relation

\[ \langle x, t \mid \frac{1}{i\partial - \sigma} \mid x, t \rangle = \int \frac{d\omega}{2\pi} \langle x \mid \frac{1}{\omega\gamma^0 + i\gamma^1\partial_x - \sigma} \mid x \rangle \]

\[ = i \int \frac{d\omega}{2\pi} \begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix} \]  

(C.6)

(see Eq. (A.43)). Thus, we may write the (bare) saddle point equation (2.3) for static bags as

\[ \frac{\delta S_{\text{eff}}}{\delta \sigma(x, t)} \bigg|_{\text{static}} = -\frac{\sigma(x)}{g^2} - N \int \frac{d\omega}{2\pi} (A(x) + D(x)) = 0 \]  

(C.7)

which is manifestly equivalent to the saddle point condition obtained by equating the right hand side of (C.2) to zero.

From (A.43) (with \( \Gamma = 1 \)), we observe that

\[ N \int \frac{d\omega}{2\pi} (A(x) + D(x)) = N \int \frac{d\omega}{2\pi} \text{tr} \begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix} = \langle \bar{\psi}_a(t, x) \psi_a(t, x) \rangle_{\sigma(x)} \]  

(C.8)

Thus, (C.7) is just the statement that

\[ \langle \bar{\psi}_a(t, x) \psi_a(t, x) \rangle_{\sigma(x)} = -\frac{\sigma(x)}{g^2} \]  

(C.9)

i.e., that the auxiliary field \( \sigma \) in (1.1) is proportional to the condensate \( \langle \bar{\psi}\psi \rangle \), which is nothing but the equation of motion of \( \sigma \) with respect to the action \( S \) in (1.1).
Appendix D: The $\text{O}(2N)$ Energy Multiplets

D.1 Quantization of Majorana Fields in a Background $\sigma(x)$

The $\text{O}(2N)$ quantum numbers of the bound state multiplets associated with the static solitons discussed in Sections 3 and 4 can be determined by considering the action\textsuperscript{23}

$$S = \int d^2x \left\{ \sum_{a=1}^{N} \bar{\psi}_a \left( i\frac{\partial}{\partial x} - \sigma \right) \psi_a \right\}$$

for $N$ non-interacting Dirac fermions in a static background $\sigma(x)$. Following (1.4), we write (D.1) in the explicit $\text{O}(2N)$ invariant form

$$S = \int d^2x \sum_{i=1}^{2N} \left[ i\dot{\xi}_i^T \xi_i - i\xi_i^T \sigma_1 \xi_i - (\xi_i^T \sigma_2 \xi_i) \sigma(x) \right],$$

in terms of the $2N$ Majorana fermions $\xi_i$, where we used the obvious notations

$$\phi_a = \xi_{2a-1}, \quad \chi_a = \xi_{2a}$$

in (1.3).

Quanta created by the Dirac-Majorana fields $\xi_i$ may form non-interacting, many-fermion bound states in the external background $\sigma(x)$, which fall into multiplets of the $\text{O}(2N)$ symmetry group. In this appendix we study these multiplets.

In the context of the large-$N$ limit of the GN model, $\sigma(x)$ is determined self-consistently from the static saddle point equation (C.7). The $\text{O}(2N)$ multiplet content associated with that soliton then follows from the simple group theoretic considerations presented here.

The Dirac-Majorana equation associated with (D.2) is

$$i\dot{\xi} - i\sigma_1 \xi' - (\sigma_2 \xi) \sigma(x) = 0.$$ \hspace{1cm} (D.4)

Stationary solutions $f_n(x,t) = e^{-i\omega_n t}f_n(x)$ of (D.4) with $\omega_n \neq 0$, are complex. However, since (D.4) has purely imaginary coefficients, the spinors $\text{Re}f_n(x,t)$ and

\textsuperscript{23}The presentation in subsections D.1 and D.2.1 follows, in part, portions of section 3 of [30]. For more details on the representation theory of orthogonal groups see, e.g., [37].
Im$f_n(x,t)$ comprise a pair of independent solutions of (D.4). This is just the statement, in terms of Majorana spinors, that the stationary solutions of the Dirac equation (1.6), $[i\partial - \sigma(x)]\psi = 0$, come in charge-conjugate pairs $e^{-i\omega nt}f_n(x)$ and $e^{i\omega nt}f^*_n(x)$, as was discussed in the Introduction. In addition, if $\sigma(x)$ is topologically non-trivial, as we discussed in subsection A.1.1, (D.4) has a non-degenerate real zero energy eigenstate $f_0(x)$ [31]. Thus, we need only consider stationary solutions of (D.4) with $\omega \geq 0$.

From (D.2) we see that the quantum Dirac-Majorana field operators $\xi_i(x,t), i = 1, \ldots, 2N$ satisfy the equal time canonical anticommutation relations

$$\{\xi_{i\alpha}(x,t), \xi_{j\beta}(x',t)\} = \frac{1}{2}\delta_{ij}\delta_{\alpha\beta}\delta(x-x'), \quad (D.5)$$

where $\alpha, \beta$ are spinor indices. As usual, these (hermitean) operators are to be expanded [30] in normal modes of (D.4). This expansion, which is exact for the simple field theory (D.1), is the semiclassical expansion of the fermion field in a self-consistent static background in the GN model. For example, in the presence of a topologically non-trivial $\sigma(x)$, with its unpaired real zero energy bound state spinor $f_0(x)$, we have the expansion \(^{24}\)

$$\xi_{i\alpha}(x,t) = f_{0\alpha}(x)b_i + \sum_{\omega_n>0} \left\{ f_{n\alpha}(x,t)a_{n,i} + f^*_{n\alpha}(x,t)a_{n,i}^\dagger \right\}. \quad (D.6)$$

(For topologically trivial backgrounds, we will obtain an expansion similar to (D.6), with the first term $f_{0\alpha}(x)b_i$ excluded.)

In (D.6) $a_{n,i}$ and $a_{n,i}^\dagger$ are, respectively, annihilation and creation operators for a particle of type $i$ in the state corresponding to $f_n(x,t)$. Indeed, from (D.5) (with the appropriate normalization of the complete set of stationary states $f_n$) one obtains the anticommutation relations

$$\{a_{n,i}, a_{m,j}^\dagger\} = \delta_{nm}\delta_{ij}, \quad (D.7)$$

and all other anticommutators involving $a$’s or $a^\dagger$’s vanish.

\(^{24}\)For simplicity, we assume here that the system lives in a big spatial box, rendering its spectrum discrete.
The operators $b_i$ are hermitean and transform according to the vector representation of $O(2N)$, as they inherit these properties from the $\xi_i$. Moreover, the canonical anticommutation relations (D.5) (with the appropriate normalization of $f_0$) imply that the $b_i$ satisfy the Clifford algebra

$$\{b_i, b_j\} = 2\delta_{i,j}.$$  \hspace{1cm} (D.8)

In other words, the $b_i$ have the transformation properties and the anticommutation relations of the gamma matrices of $O(2N)$.

In terms of the creation and annihilation operators discussed above, the (normal ordered) Hamiltonian corresponding to (D.2) is

$$H = \sum_{\omega_n > 0} \omega_n a_{n,i}^\dagger a_{n,i}.$$  \hspace{1cm} (D.9)

## D.2 The $O(2N)$ Multiplets

### D.2.1 The Spinor Representation

The multiplet of states associated with topologically non-trivial solitons, on which the Clifford generators $b_i$ act, must contain a factor transforming according to the spinor representation of $O(2N)$. Since according to our discussion in subsection A.1.1 (and according to [31]), the Dirac-Majorana equation (D.4) with a topologically non-trivial $\sigma(x)$ has only a single zero energy bound state $f_0(x)$, there is only one such factor of the spinor representation associated with a given topologically non-trivial $\sigma(x)$. For example, CCGZ kinks, for which (D.4) has only a zero energy bound state, are pure isospinors[30].

It will be useful at this point to list a few mathematical facts concerning this $O(2N)$ spinor representation factor: In any representation of (D.8) we may define the antihermitean $O(2N)$ generators as

$$M_{ij} = b_i b_j - b_j b_i.$$  \hspace{1cm} (D.10)

In addition, let us define the hermitean operator $\Gamma_5 = i b_1 b_2 \ldots b_{2N}$, which commutes with all group generators $M_{ij}$, but anticommutes with all generators of the Clifford
algebra $b_i$. Clearly, $\Gamma_5^2 = 1$, and thus the possible eigenvalues of $\Gamma_5$ are $\pm 1$. There is only a single irreducible representation of the Clifford algebra (D.8), since $\Gamma_5$ does not commute with the $b_i$, and any representation of the Clifford algebra (D.8) must contain both “left-handed” isospinors with $\Gamma_5 = +1$, and “right-handed” isospinors with $\Gamma_5 = -1$.

We would like to construct a unitary irreducible representation of this Clifford algebra on a positive norm Fock space. Thus, we have to group the Clifford generators $b_i$ into $N$ pairs of Grassmannian creation and annihilation operators which act on that space. This we achieve e.g., by pairing the $b_i$ according to (D.3), namely

$$B_a = \frac{1}{2}(b_{2a-1} + ib_{2a}), \quad a = 1, \ldots, N,$$

which satisfy the anticommutation relations

$$\{B_a, B_a^\dagger\} = \delta_{ab},$$

with all other anticommutators vanishing. An orthonormal basis for this space is then obtained by applying the creation operators $B_a^\dagger$ repeatedly on the Fock vacuum:

$$|0\rangle, B^\dagger_a|0\rangle, B^\dagger_a B^\dagger_b|0\rangle, \ldots, B_N^\dagger B_2^\dagger \cdots B_N^\dagger|0\rangle.$$

Thus, this irreducible representation of the Clifford algebra has dimension $2^N$.

However, since $\Gamma_5$ commutes with all the $O(2N)$ generators $M_{ij}$, these $2^N$ states transform irreducibly under the $O(2N)$ group; the left-handed and right-handed states transform independently. For example, if we choose the Fock vacuum such that $\Gamma_5|0\rangle = +|0\rangle$, then the right-handed states are those with an even number of $B_a^\dagger$’s acting on $|0\rangle$, and the left-handed states are those with an odd number of $B_a^\dagger$’s acting on $|0\rangle$. For later reference, let us demonstrate this reducibility in a concrete basis of the algebra, in which the $O(2N)$ Cartan subalgebra is generated by the $N$ generators

$$M_{12}, M_{34}, \ldots, M_{(2N-1)(2N)}.$$

From (D.11) we see that $M_{(2a-1)(2a)} = 2b_{2a-1}b_{2a} = -2i[B_a^\dagger, B_a]$. Thus, the states (D.13) are simultaneous eigenstates of all the generators (D.14), and the corresponding eigenvalues are essentially the components of the weight vectors of the representation. The step operators, on the other hand, when acting on any of these states,
change the number of Fock quanta of that state by 0 or \( \pm 2 \) units (consider, e.g., \( M_{13} = 2b_1b_3 = 2(B_1 + B_1^\dagger)(B_2 + B_2^\dagger) \)). In other words, the states with an even number of \( B_a^\dagger \)'s acting on \(|0\rangle\) and the states with an odd number of \( B_a^\dagger \)'s acting on \(|0\rangle\) form two disjoint (and irreducible) invariant subspaces under the action of the \( O(2N) \) generators.

**D.2.2 The Antisymmetric Tensor Representations**

Suppose now that (D.4), in a given solitonic background \( \sigma(x) \), has a pair of bound states at \( \pm \omega_n \). What are the irreducible \( O(2N) \) factors in the multiplet of states born by the soliton \( \sigma(x) \), which are acted upon by the operators \( a_{n,i} \) and \( a_{n,i}^\dagger \)? To answer this question consider the following situation: Assume for simplicity that \( \pm \omega_n \) are the only bound states in (D.4). Then choose \( k \) distinct operators among the \( 2N \) \( \xi_i \)'s, and apply them to the vacuum state \(|0\rangle\). The resulting state \( \prod_{j=1}^{k} \xi_{ij} |0\rangle \) will contain bound as well as scattering states. Since we are interested only in static stable soliton states, we have to consider only those \( a_{n,i} \)'s which correspond to bound states of (D.4). Thus, projecting out all scattering states we are left with the state \( (f_n^*)^k a_{n,1i}^\dagger \ldots a_{n,k1}^\dagger |0\rangle \) (where \( (f_n^*)^k \) stands for the direct product of the \( k \) spinors).

Clearly, we can form \( (2N)!/k!(2N-k)! \) bound states in this way, all of which have common energy \( k\omega_n \), and thus form an energy multiplet. We readily identify this multiplet as the antisymmetric tensor of \( O(2N) \) of rank \( k \), since all \( a_{n,i}^\dagger \) anticommute. In this construction, we can obviously take \( 1 \leq k \leq 2N \), and thus obtain all \( O(2N) \) antisymmetric tensors. However, in the context of studying extremal static \( \sigma(x) \) configurations in the GN model, the number \( \nu_n \) of quanta trapped in the bound states \( \pm \omega_n \), is part of the definition of \( \sigma(x) \), in which case one must take \( k = \nu_n \).

For pedagogical clarity, let us derive this conclusion in a more pedestrian way. To this end we have to invoke the notions of particles and antiparticles. In order to differentiate “particles” from “antiparticles” in our model we have to pair the \( 2N \) Majorana fields in (D.6) into \( N \) Dirac fields \( \psi_a \). (The fermion number operator is of course \( \int dx : \sum_a \psi_a^\dagger \psi_a : \)). One possible pairing is according to (D.3), namely
\( \psi_a = \xi_{2a-1} + i\xi_{2a}, a = 1, \ldots N \)\(^{25} \), which gives rise to the Dirac field operators

\[
\psi_a = \xi_{2a-1} + i\xi_{2a} = f_0(x)(b_{2a-1} + ib_{2a}) + \sum_{\omega_n > 0} \left\{ f_n(x,t)(a_{n,2a-1} + ia_{n,2a}) + f_n^*(x,t)(a_{n,2a-1} - ia_{n,2a})^\dagger \right\}.
\] (D.15)

Suppose that the Dirac-Majorana equation (D.4) has a pair of charge conjugate bound states at \( \pm \omega_n \). Consider the \((p_n, h_n)\) many-body configuration of \( p_n \) particles and \( h_n \) holes defined following (3.20). The energy of this state is simply \((p_n + h_n - N)\omega_n\). It depends on the sum \( \nu_n = p_n + h_n \) of numbers of particles and holes, and not on each one of them separately. Evidently, Fermi statistics implies that \( 0 \leq \nu_n \leq 2N \). Thus, considering the set of all \((p_n, h_n)\) configurations with \( \nu_n = p_n + h_n \) fixed, we obtain a multiplet of \( \sum_{k=0}^{\nu_n} C_N^k \sum_{k=0}^{\nu_n-k} C_N^{\nu_n-k} = C_{2N}^{\nu_n} = (2N)!/\nu_n!(2N - \nu_n)! \) states with common energy \((\nu_n - N)\omega_n\). This is, of course, precisely the result we obtained earlier directly from the \( O(2N) \) symmetry, without distinguishing between particles and antiparticles. The many-body wave functions of states in this multiplet are Slater determinants of \( \nu_n \) single particle orbitals, with \( 2N \) possible orbitals (i.e., with a flavor index running through 1, \ldots 2N). Thus, we readily identify this multiplet as the antisymmetric tensor representation of rank \( \nu_n \) of \( O(2N) \). Since \( 0 \leq \nu_n \leq 2N \), all possible \( O(2N) \) antisymmetric tensor multiplets may occur. Thus, in particular, both the antisymmetric tensors of rank \( \nu_n \), and its dual tensor, of rank \( 2N - \nu_n \), appear in the spectrum. These two tensor multiplets have the same dimension and opposite energies \( \pm(\nu_n - N)\omega_n \). Note, however, that the extremal fermion bags, discussed in subsection 3.1, realize only half the antisymmetric tensors allowed by the \( O(2N) \) symmetry, namely, only tensors of ranks in the range \( 0 \leq \nu_n \leq N \) (recall (3.26)).

\(^{25}\)This is by no means the only possible pairing. There is clearly a one to one correspondence between permutations of the 2N fields \( \xi_i \) and their pairings into N Dirac fermions, and thus \( (2N)! \) ways to pair the 2N Majorana fields into N Dirac fields, namely, \( \psi_a = \xi_{P(2a-1)} + i\xi_{P(2a)} = f_0(b_{P(2a-1)} + ib_{P(2a)}) + \sum_{\omega_n > 0} \left\{ f_n(a_{n,P(2a-1)} + ia_{n,P(2a)}) + f_n^*(a_{n,P(2a-1)} - ia_{n,P(2a)})^\dagger \right\} \), where \( P \) is a permutation of 2N objects. Then clearly, the operator \( a_{n,P(2a-1)} + ia_{n,P(2a)} \) may be thought of as annihilating a particle in the state \( f_n(x,t) \), and the operator \( (a_{n,P(2a-1)} - ia_{n,P(2a)})^\dagger \) may be thought of as creating an antiparticle in the state \( f_n^*(x,t) \). However, since the symmetry of the system in question is \( O(2N) \) (rather than \( U(N) \)), all these pairings are physically equivalent. They simply correspond to different orientations of the corresponding \( U(N) \) subgroup in \( O(2N) \). Thus, with no loss of generality, we choose to pair according to (D.15), and define the notions of particles and antiparticles (or holes) accordingly.
Referring, as usual, to the Dirac particles occupying the state at $+\omega_n$ as fermions and to the Dirac holes in the state at $-\omega_n$ as antifermions, then the $(p_n, h_n)$ configuration would carry fermion number $N_f^{(n)} = p_n - h_n$. This is what we referred to as the *valence* fermion number $N_{f,\text{val}}^{(n)}$ in (3.43). Thus, states in our antisymmetric tensor multiplet carry different fermion numbers $N_f^{(n)} = p_n - h_n = \nu_n - 2h_n$, which vary in the range $-\nu_n \leq N_f^{(n)} \leq \nu_n$ (if $\nu_n \leq N$), or $-(2N - \nu_n) \leq N_f^{(n)} \leq 2N - \nu_n$ (if $\nu_n > N$). Thus, the $N_f^{(n)}$ spectrum in the antisymmetric multiplet of rank $\nu_n$ coincides with the $N_f^{(n)}$ spectrum in the dual antisymmetric multiplet of rank $2N - \nu_n$.

Similarly, according to our explicit construction of the spinorial representation in the previous subsection, we see that the states in that representation carry different fermion numbers in the range $0 \leq N_f^{(\text{spinor})} \leq N$; just count the number of $B_a^\dagger$’s in a given state. (In subsection 3.2 we referred to this quantity as the *valence* fermion number $n_0$.) However, this is an intolerable assignment of fermion numbers in a multiplet appearing in the spectrum of a quantum field theory with charge conjugation invariance, such as the GN model. One may say that the theory preserves its invariance under charge conjugation through the phenomenon of fermion number fractionalization [31, 33], which shifts, as we discussed in subsection 3.2 (recall (3.45)), the valence fermion numbers in the spinorial representation by $-N/2$, rendering the spectrum of fermion numbers in that representation symmetric: $-N/2 \leq N_f^{(\text{spinor})} \leq N/2$. More formally, we can identify the fermion number operator $N_f^{(\text{spinor})}$ as the linear symmetric combination [37]

$$N_f^{(\text{spinor})} = \frac{i}{4} \sum_{a=1}^{N} M_{(2a-1)(2a)} = \sum_{a=1}^{N} B_a^\dagger B_a - \frac{N}{2} \quad \text{(D.16)}$$

of the generators (D.14) of the Cartan subalgebra of $O(2N)$, where we used (D.12). The splitting of $N_f^{(\text{spinor})}$ into valence and fractional parts is manifest.

**D.3 Multiplet Dimensions from the Partition Function**

The dimensions of degenerate energy multiplets occurring in the quantization of (D.2) can be read off the partition function $Z(\beta) = \text{Tr} \exp -\beta H$ in a straightforward manner. We consider here a topologically non-trivial $\sigma(x)$, so that all possible
multiplets contribute. The Hamiltonian (D.9) is the sum of contributions of non-interacting Grassmannian oscillators, and the partition function is a product over the contributions of individual modes. The modes at frequency $\omega_n$ contribute the factor
$$\prod_i \text{Tr} \exp -\beta \omega_n a_n^\dagger a_n = (1 + \exp -\beta \omega_n)^{2N}$$
to the partition function. The Fock space associated with the zero mode is $2^N$ dimensional, and thus contributes a factor $2^N$ to the partition function.

Gathering all these facts together, we thus obtain the formal expression

$$Z(\beta) = 2^N \prod_{\omega_n > 0} (1 + e^{-\beta \omega_n})^{2N}$$

$$= 2^N \prod_{\omega_n > 0} \left( \sum_{\nu_n=0}^{2N} \frac{(2N)!}{\nu_n!(2N - \nu_n)!} e^{-\nu_n \beta \omega_n} \right).$$

(D.17)

Thus, a state in which there are $\nu_n$ quanta at frequency $\omega_n$ contributes a factor
$$C_{2N}^{\nu_n} = (2N)!/\nu_n!(2N - \nu_n)!$$
to the overall degeneracy. This is, of course, the dimension of the antisymmetric tensor multiplet of rank $\nu_n$. From the expansion of (D.17),

$$Z(\beta) = \sum_{\{\nu_k\}} \left( 2^N \prod_{\omega_n > 0} C_{2N}^{\nu_n} \right) e^{-\beta \sum_{\nu_l > 0} \nu_l \omega_l},$$

(D.18)

where the summation runs over sets of integers $0 \leq \nu_k \leq 2N$, we see that the eigenstates of the Hamiltonian (D.9) fall into all possible direct products of the $O(2N)$ antisymmetric tensors and the spinorial representation, which are built on the normal modes in the expansion (D.6) of $\xi$.

It is somewhat amusing to obtain these results in the language of Dirac fields and the $U(N)$ group. In this language it is more convenient to consider the Hamiltonian without normal ordering, i.e., to include the zero point energy (which we can subtract in the end). Thus, consider one of the $N$-fold degenerate normal modes at frequency $+\omega_n$. If that state is empty the energy is $-\frac{\omega_n}{2}$ (i.e., the zero point energy), and if it is occupied the energy is $+\frac{\omega_n}{2}$. Thus, these modes contribute a factor $(\exp \beta \omega_n + \exp -\beta \omega_n)^N = (2 \cosh \frac{\beta \omega_n}{2})^N$ to the partition function. The modes at $-\omega_n$ obviously contribute the same factor. Thus, the states at frequency $\pm \omega_n$ contribute altogether a factor $(2 \cosh \frac{\beta \omega_n}{2})^{2N} = (1 + \exp -\beta \omega_n)^{2N} \exp N \beta \omega_n$. The $N$-fold degenerate zero
mode contributes a factor \((2 \cosh \beta \cdot 0)^N = 2^N\), which is, of course, the dimension of the spinor representation. Multiplying all these factors together, and subtracting the overall divergent contribution of the zero point energies, we obtain (D.17).
Appendix E: Derivation of the Mass Formula for the CCGZ Kink

In this Appendix we provide our own derivation of the mass formula $M_{kink} = \frac{Nm}{\pi}$ (Eq. (3.33)) of the CCGZ kink. The kink mass, as all other soliton masses in the GN model, ought to be proportional to $N$. In addition, due to dimensional considerations, it has to be proportional to the dynamical mass $m$. Thus, on very general grounds,

$$M_{kink} = Nmc,$$

(E.1)

where $c$ is a dimensionless pure number to be determined. In particular, $c$ cannot depend on $m$. Therefore,

$$c = \frac{1}{N} \frac{\partial M_{kink}}{\partial m}.$$  

(E.2)

$M_{kink}$ is given by the renormalized energy functional (2.19), evaluated at $\sigma_{kink}(x) = m \tanh mx$. Note that due to the subtracted vacuum term in (2.19), that functional is also a function of $\sigma_{VAC} = m$, and of course, also of the bare coupling $g^2$. Thus,

$$\frac{\partial M_{kink}}{\partial m} = \int_{-\infty}^{\infty} dx \frac{\delta \mathcal{E}}{\delta \sigma(x)} \frac{\partial \sigma_{kink}(x)}{\partial m} + \frac{\partial \mathcal{E}}{\partial \sigma_{VAC}} \frac{\partial \sigma_{VAC}}{\partial m} + \frac{\partial \mathcal{E}}{\partial g^2} \frac{\partial g^2}{\partial m},$$

(E.3)

where all derivatives of $\mathcal{E}$ are evaluated at $\sigma(x) = \sigma_{kink}(x)$. Now, $\sigma_{kink}(x)$ and $\sigma_{VAC}$ are extremal configurations. Thus, by definition\textsuperscript{26}, $(\delta \mathcal{E}/\delta \sigma(x))_{\sigma_{kink}(x)} = \partial \mathcal{E}/\partial \sigma_{VAC} = 0$. It follows from this and from (2.19) that

$$\frac{\partial M_{kink}}{\partial m} = \frac{m \partial g^2}{g^4 \partial m} = \frac{1}{g^2} \frac{\partial \log g^2}{\partial \log m},$$

(E.4)

where we used $\int_{-\infty}^{\infty} dx (\sigma_{kink}^2 - m^2) = -2m$.

What is the meaning of taking the derivative of the bare coupling constant $g^2$ with respect to the physically observable dynamical fermion mass $m$? The answer

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\textsuperscript{26}This simple argument can be verified by substituting the kink resolvents $B$ and $C$ (A.41) into (3.10), and by using the fact that $V_c = \sigma^2 + \sigma' = m^2 = \sigma_{VAC}^2$ for the kink. Then, the sum of the first two terms on the right hand side of (E.3) produces a convergent integral in $x$, which is proportional to the bare gap equation (C.4), and thus vanishes.
is that in order to produce the RG invariant mass $m$ at the large distance scale, we have to tune the bare coupling $g^2$ according to (2.5), namely, $\Lambda e^{-\frac{g^2}{\alpha_0(\Lambda)}} = m$. The RG invariant mass $m$ thus parametrizes the RG trajectory of the bare coupling $g^2$ as a function of the cutoff $\Lambda$. Therefore, $(\partial g^2/\partial m) \delta m$ measures the change in $g^2$ as we move to a neighboring trajectory, keeping $\Lambda$ fixed. Thus, substituting $\partial \log g^2/\partial \log m$ from (2.5) into (E.4), followed by substituting the resulting $\partial M_{kink}/\partial m$ into (E.2), we finally obtain

$$c = \frac{1}{\pi},$$

(E.5)

thereby proving (3.33).
Appendix F: An Alternative Proof of Eq. (4.12)

In this Appendix we present an alternative proof of (4.12). The idea is to consider the behavior of

\[ f(\Theta_1, \ldots, \Theta_K) = \sum_{n=1}^{K} \sin \Theta_n \]  

over the hyperplane

\[ \tilde{\Sigma}_{r\alpha} : \quad \Theta_1 + \cdots + \Theta_K = \alpha + \frac{\pi}{2} r, \]  

where \( 0 \leq r \leq K \) is an integer, and \( 0 \leq \alpha < \frac{\pi}{2} \). This hyperplane corresponds, of course, to solitons with a fixed value of the maximal fermion number \( N_f^{\text{max}}(D_{\text{parent}}) \).

We will now prove that \( f(\Theta_1, \ldots, \Theta_K) \) attains its absolute minimum on \( \tilde{\Sigma}_{r\alpha} \) in the positive orthant, at the vertices of the intersection of \( \tilde{\Sigma}_{r\alpha} \) and the hypercube \([0, \frac{\pi}{2}]^K\), namely, the points

\[ \Theta^{(n)} = \frac{\pi}{2} (\delta_{n1} + \delta_{n2} + \cdots + \delta_{nr}) + \alpha\delta_{nr+1}, \quad (n = 1, \ldots, K), \]  

with all possible choices of \( r + 1 \) coordinates \( i_1, \ldots, i_{r+1} \) out of \( K \). Once we have established that, the fact that the potentially stable solitons are given by (4.12) follows in a straightforward manner.

By standard constrained extremum analysis one can show that the local extremum of \( f(\Theta_1, \ldots, \Theta_K) \) occurs at the symmetric point \( \Theta_n = \frac{1}{K}(\alpha + \frac{\pi}{2} r), \forall n \), and that it is a maximum. Thus the minimum occurs at the boundary of the domain under consideration. By the same argument, any local extremum on the boundary is again a maximum. Thus, repeating this analysis, we conclude that the absolute minimum occurs at the vertices (F.3) of the intersection of \( \tilde{\Sigma}_{r\alpha} \) and the hypercube \([0, \frac{\pi}{2}]^K\), where\(^{27}\)

\[ \min f|_{\tilde{\Sigma}_{r\alpha}} = r + \sin \alpha. \]  

An alternative elegant proof of this minimum behavior of (F.1), due to Raphael Yuster, goes as follows: Consider a sequence \( 0 \leq \Theta_1 \leq \cdots \leq \Theta_K \leq \frac{\pi}{2} \), subjected

\(^{27}\)For \( r = 0 \), (F.4) reduces to the well known inequality \( \sin(\sum_{n=1}^{K} \Theta_n) \leq \sum_{n=1}^{K} \sin \Theta_n \) where \( \sum_{n=1}^{K} \Theta_n = \alpha < \frac{\pi}{2} \).
to (F.2). Assume that for some $i$, $0 < \Theta_i \leq \Theta_{i+1} < \frac{\pi}{2}$. We will show that there exists another sequence of $\Theta$'s, with the same sum, but with a lower sum of the sines. Thus, let $\delta > 0$ be chosen such that $0 < \Theta_i - \delta > 0$ and $\Theta_i + 1 + \delta < \frac{\pi}{2}$, i.e., $0 \leq \delta \leq \min\{\Theta_i, \frac{\pi}{2} - \Theta_{i+1}\}$. Modify the sequence under consideration by replacing $\Theta_i$ by $\Theta_i - \delta$ and $\Theta_{i+1}$ by $\Theta_{i+1} + \delta$, keeping the other $K - 2$ terms unaltered. The new sequence thus obtained has the same sum as the original sequence, and thus defines another point on $\tilde{\Sigma}_r$. We must show that $D(\delta) = f(\text{original sequence}) - f(\text{new sequence}) > 0$. Indeed, $D(\delta) = \sin \Theta_i - \sin(\Theta_i - \delta) + \sin \Theta_{i+1} - \sin(\Theta_{i+1} + \delta)$. Clearly, $D(0) = 0$, and also $D'(\delta) > 0$ in the relevant range of $\delta$. Thus, $D(\delta)$ increases monotonically with $\delta$, and reaches its maximum at $\delta_{\text{max}} = \min\{\Theta_i, \frac{\pi}{2} - \Theta_{i+1}\}$, where, depending on the initial condition at $\delta = 0$, either $\Theta_i - \delta_{\text{max}} = 0$ or $\Theta_{i+1} + \delta_{\text{max}} = \frac{\pi}{2}$. Thus, the sequence of $\Theta$'s constrained to $\tilde{\Sigma}_r$, which minimizes (F.1), cannot have more than one element in the interior of $[0, \frac{\pi}{2}]$. Thus, due to (F.2), the absolute minimum is the sequence in which the $r$ largest $\Theta$'s are $\frac{\pi}{2}$, one $\Theta$ is $\alpha$ and the rest are zero, namely, the vertices (F.3).

Thus, a parent soliton corresponding to a point in the interior of the intersection of $\tilde{\Sigma}_r$ and the hypercube $[0, \frac{\pi}{2}]^K$, can decay into a final state with quantum numbers corresponding to the points (F.3), i.e., $L = K$ and $\theta_n = \Theta_n^{(v)}$ in (4.3). In fact, such a parent soliton can also decay at least into the set of final states contained in small pockets above the vertices (F.3), which correspond to $L = K$ and $\theta_n = \Theta_n^{(v)} + \epsilon$ in (4.3), with

$$\epsilon << \frac{\sum_{n=1}^{K} \sin \Theta_n - (r + \sin \alpha)}{K - r - 1 + \cos \alpha},$$

or into final states corresponding to $L > K$ in (4.3), with $\theta_i = \Theta_i^{(v)}$ for $1 \leq i \leq K$ and $\theta_i = \epsilon$ for $K + 1 \leq i \leq L$, where

$$\sin \epsilon < \frac{\sum_{n=1}^{K} \sin \Theta_n - (r + \sin \alpha)}{L - K}.$$ 

On the other hand, the parent soliton which corresponds to the vertices (F.3) has no open channel to decay through. Hence it is potentially stable. Indeed, if it could decay through a channel corresponding to $\theta_1, \ldots, \theta_L$, then, from the requirement that
these parameters satisfy (4.3), we would have

\[ \sum_{i=1}^{L} \theta_i > \alpha + \frac{\pi}{2} r \]

\[ \sum_{i=1}^{L} \sin \theta_i < r + \sin \alpha. \]  

(F.5)

Define the hyperplane

\[ \Sigma_{r\alpha} : \quad \theta_1 + \cdots + \theta_L = \alpha + \frac{\pi}{2} r. \]  

(F.6)

From the analysis in this Appendix we know that the absolute minimum of \( \sum_{i=1}^{L} \sin \theta_i \) over the intersection of \( \Sigma_{r\alpha} \) and the hypercube \([0, \frac{\pi}{2}]^L\) is \( r + \sin \alpha \). Thus, the points which satisfy the second inequality in (F.5) are bounded by \( \sum_{i=1}^{L} \theta_i < \alpha + \frac{\pi}{2} r \), in contradiction with the first inequality in (F.5). This completes our alternative proof of (4.12).
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