PERSISTENCE AND STATIONARY DISTRIBUTION OF A STOCHASTIC PREDATOR-PREY MODEL UNDER REGIME SWITCHING

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(Communicated by Peter Bates)

Abstract. Taking both white noise and colored environment noise into account, a predator-prey model is proposed. In this paper, our main aim is to study the stationary distribution of the solution and obtain the threshold between persistence in mean and the extinction of the stochastic system with regime switching. Some simulation figures are presented to support the analytical findings.

1. Introduction. The dynamic of predator and prey models is a dominant theme in ecology. The classical deterministic predator-prey Lotka-Volterra model with density-dependent logistic growth of the prey takes the form

\[
\begin{align*}
\dot{x}(t) &= x(t)(a - bx(t) - cy(t)), \\
\dot{y}(t) &= y(t)(-h + fcz(t)),
\end{align*}
\]

(1)

where \(x(t)\) and \(y(t)\) are the sizes of prey population and predator population on time \(t\), respectively. \(a, b, c, h\) and \(f\) are positive constants. The prey carrying capacity of the environment is \(\frac{a}{b}\), the feeding efficiency in turning predation into new predators is \(f\). System (1) was formulated by Pielou [18]. Nisbet and Gurney pointed out that if the initial value \(x(0) > 0\) and \(\frac{a}{bc} \leq 1\) or \(x(0) > 0\) and \(y(0) = 0\), then \(x \to \frac{a}{b}\)
and $y \to 0$, as $t \to \infty$. If $x(0) > 0$, $y(0) > 0$ and $\frac{a_f}{b_k} > 1$ hold, system \[ (1) \] has a positive equilibrium $(x^*, y^*) = \left( \frac{b_k}{a_f}, \frac{a_f c - bh}{c^2} \right)$ which is globally asymptotic stable (see e.g. \[ 3, 17 \]). System \[ (1) \] and its deformations have been studied extensively, see e.g. \[ 1, 2, 3, 6, 23 \] and the references therein.

The deterministic systems mentioned above are usually affected by environmental noise. Generally speaking, there are two types of environmental noise. One is white noise, the other is classical colored noise, say telegraph noise \[ 7 \] and \[ 14 \].

The telegraph noise can be described as a random switching between two or more environmental regimes, which differ in terms of factors such as nutrition or rainfall.

Since it introduced his stochastic calculus, many scholars introduced stochastic perturbations to reveal the effect of environmental white noise on the population dynamics (see, e.g. \[ 4, 6, 9, 10, 13, 15, 16, 19, 24, 25 \]).

We focus in this paper on the case when the growth rate $a$ and $-\varepsilon$ in model \[ (1) \] are perturbed with white noise, that is $a \to a + \alpha B_1(t)$, $-h \to -h + \beta B_2(t)$, and we obtain the stochastic system

\[
\begin{align*}
\frac{dx(t)}{dt} &= [x(t)(a - bx(t)) - cx(t)y(t)]dt + \alpha x(t)dB_1(t), \\
\frac{dy(t)}{dt} &= [-hy(t) + f(x(t))y(t)]dt + \beta y(t)dB_2(t),
\end{align*}
\]

where $B_1(t)$ and $B_2(t)$ are one-dimensional standard Brownian motion, $\alpha^2$ and $\beta^2$ stand for the intensities of the white noises. Under the premise of the system \[ (1) \] having a positive equilibrium $(x^*, y^*)$, Li et al. \[ 6 \] gave sufficient conditions for permanence, extinction and the existence of the stationary distribution in \[ (2) \].

We will also consider colored noise in this paper. Let $\{r(t), t \geq 0\}$ be a right-continuous Markov chain on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ with finite-state space $S = \{1, \ldots, m\}$. The generator $\Gamma = (\gamma_{ij})_{1 \leq i, j \leq m}$ is given, for $\delta > 0$, by

\[
P\{r(t + \delta) = j|r(t) = i\} = \begin{cases} 
\gamma_{ij}\delta + o(\delta), & \text{if } i \neq j, \\
1 + \gamma_{ij}\delta + o(\delta), & \text{if } i = j.
\end{cases}
\]

Here $\gamma_{ij}$ is the transition rate from $i$ to $j$ and $\gamma_{ij} \geq 0$ if $i \neq j$, while $\gamma_{ii} = -\sum_{i \neq j} \gamma_{ij}$. In this paper we assume $\gamma_{ij} > 0$ if $i \neq j$. Suppose that the Markov chain $r(t)$ is independent of the Brownian motion $B(\cdot)$ and it is irreducible. Under this condition, the Markov chain has a unique stationary (probability) distribution $\pi = (\pi_1, \ldots, \pi_m) \in \mathbb{R}^1 \times m^*$, which can be determined by solving the linear equation $\pi \Gamma = 0$, $\sum_{k=1}^m \pi_k = 1$ and $\pi_k > 0$, $\forall k \in S$. We refer the reader to \[ 7, 8, 9, 12, 16 \] and \[ 20 \] for details.

The population stochastic system \[ (2) \] with regime switching can be described by the model

\[
\begin{align*}
\frac{dx(t)}{dt} &= [x(t)(a(r(t)) - b(r(t))x(t)) - c(r(t))x(t)y(t)]dt + \alpha(r(t))x(t)dB_1(t), \\
\frac{dy(t)}{dt} &= [-h(r(t))y(t) + f(r(t))c(r(t))x(t)y(t)]dt + \beta(r(t))y(t)dB_2(t).
\end{align*}
\]

Assume, for any $k \in S$, that the coefficients $a(k)$, $b(k)$, $c(k)$, $h(k)$, $f(k)$, $\alpha(k)$ and $\beta(k)$ are positive and bounded above and below by positive constants. $B_i(t)(i = 1, 2)$ are independent standard Brownian motions.

The aim of this paper is to study system \[ (3) \] and find the threshold between persistence in mean and the extinction and obtain sufficient conditions for system \[ (3) \] being positive recurrent (and the existence of a unique ergodic stationary distribution). Settati et al. \[ 19 \] and Liu et al. \[ 11 \], using the theory of \[ 24, 25 \], investigated
a stochastic Lotka-Volterra mutualistic system with regime switching, respectively, and obtained the existence of the stationary distribution. Zu et al. [25] studied a predator-prey model with density dependence of predator and obtained the results about ergodic property. Liu et al. [13] investigated a stochastic logistic model with regime switching and obtained the threshold between weak persistence and extinction. To our knowledge, there are very few results on the model [3] and other stochastic predator-prey systems under regime switching. First we apply the theory and methods of [24], in this paper, to a stochastic Lotka-Volterra predator-prey model with regime switching to study the ergodic stationary distribution by constructing appropriate Lyapunov functions. The persistence in mean-extinction threshold we obtain and sufficient conditions about the stationary distribution fully reflect the effect of white noise and the probability distribution of the Markov chain.

The rest of the paper is arranged as follows. In Section 2, we introduce some preliminaries which will be used in the next sections and give two theorems concerning the existence of a global positive solution and we give the $(\theta + 1)$-moment estimation $(0 < \theta < \min_{k \in \mathbb{S}} \{ \frac{2h(k)}{\beta^2(k)} \})$. Some desired population dynamical properties of model [3] are investigated in Section 3 and 4. In Section 3, we establish sufficient conditions for persistence in mean and the extinction of system [3] and we give the threshold between them. We study the stationary distribution and ergodicity in Section 4. We present the figures to illustrate the main results in Section 5 and close the paper with conclusions and future directions in Section 6.

2. Preliminaries. Throughout this paper, unless otherwise specified, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). We denote $R^2_+$ be the positive zone of $R^2$, namely $R^2_+ = \{(x, y) \in R^2 : x > 0, y > 0\}$. For any vector $e = (e(1), \ldots, e(m))^T$, $\lim_{t \to \infty} \frac{1}{t} \int_0^t e(r(s))ds = \sum_{k \in \mathbb{S}} e(k)\pi_k$. Let $\hat{e} = \min_{k \in \mathbb{S}} \{\pi_k\}$ and $\hat{e} = \max_{k \in \mathbb{S}} \{\pi_k\}$.

In investigating the ergodic stationary distribution, we will use the following lemma which gives a criterion for positive recurrence in terms of Lyapunov function (see e.g. Theorem 3.13 in [24], Lemma 2.1 in [11] and Lemma 3.1 in [19]). Let $(z(t), r(t))$ be the diffusion process described by the equation:

\[
\begin{aligned}
dz(t) &= \rho(z(t), r(t))dt + g(z(t), r(t))dB(t), \\
z(0) &= z, \ r(0) = r,
\end{aligned}
\]  

(4)

where $B(\cdot)$ and $r(\cdot)$ are the d-dimensional Brownian motion and the right-continuous Markov chain in the above discussion, respectively, and $\rho(\cdot, \cdot) : R^n \times \mathbb{S} \to R^n$, $g(\cdot, \cdot) : R^n \times \mathbb{S} \to R^{n \times d}$ satisfying $g(z, k)g^T(z, k) = D(z, k)$. For each $k \in \mathbb{S}$, and for any twice continuously differentiable function $V(\cdot, k)$, we define $\mathcal{L}$ by

\[
\mathcal{L}V(z, k) = \sum_{i=1}^n \rho_i(z, k) \frac{\partial V(z, k)}{\partial y_i} + \frac{1}{2} \sum_{i,j=1}^n D_{ij}(z, k) \frac{\partial^2 V(z, k)}{\partial z_i \partial z_j} + \Gamma V(z, \cdot)(k),
\]

where

\[
\Gamma V(z, \cdot)(k) = \sum_{i=1}^n \eta_{ki} V(z, i) = \sum_{i \neq k} \gamma_{ki} (V(z, i) - V(z, k)), \ k \in \mathbb{S}.
\]
Lemma 2.1. If the following conditions are satisfied:

(i). for i ≠ j, γ_{ij} > 0,
(ii). for each k ∈ S, D(z, k) is symmetric and satisfies
\[ q|ξ|^2 \leq ξ^T D(z, k) ξ \leq q^{-1} |ξ|^2 \] for all ξ ∈ R^n,
with some constant q ∈ (0, 1) for all z ∈ R^n,
(iii). there exists a bounded open subset D of R^n with a regular (i.e. smooth) boundary satisfying that, for each k ∈ S there exists a nonnegative function V(·, k) : D ⊆ R^n → R such that V(·, k) is twice continuously differentiable and that
\[ L V(z, k) ≤ −ς, \] for any (z, k) ∈ D × S,

then system (4) is ergodic and positive recurrent. That is to say, there exists a unique stationary density µ(·, ·) and, for any Borel measurable function ρ(·, ·) : R^n × S → R such that
\[ \sum_{k ∈ S} \int_{R^n} |ρ(z, k)| μ(z, k) dy < ∞, \]
we have
\[ \mathbb{P} \left( \lim_{t → ∞} \frac{1}{t} \int_0^t ρ(z(s), r(s)) ds = \sum_{k ∈ S} \int_{R^n} ρ(z, k) μ(z, k) dy \right) = 1. \]

Lemma 2.2. [13] Suppose that z(t) ∈ C[Ω × [0, +∞), R_+]. If there are positive constants λ_0, T and λ ≥ 0 such that
\[ \log z(t) ≥ λt − \lambda_0 \int_0^t z(s) ds + \sum_{i=1}^n β_i B_i(t) \]
for t ≥ T, where β_i is a constant, 1 ≤ i ≤ n, then \( \liminf_{t → ∞} \frac{1}{t} \int_0^t z(s) ds \geq \frac{λ}{λ_0}, \) a.s.

Now we give two theorems concerning the existence and uniqueness of positive solutions and moment boundedness, and based on these we present our main results in the next sections.

Theorem 2.3. For any given initial value \((x(0), y(0), r(0)) \in R_+^2 × S, \) system (3) has a unique positive solution. Moreover, this solution remains in \( R_+^2 × S \) with probability 1.

The proof of Theorem 2.3 is similar to that in Example 4.2 in [22] so we omit it.

Theorem 2.4. For any \( 0 < θ < \min_{k∈S} \left\{ \frac{2h(k)}{T_0(k)} \right\}, \) there exists a positive constant ι(θ), such that for any given initial value \((x(0), y(0), r(0)) ∈ R_+^2 × S, \) the solution \((x(t), y(t), r(t))\) of system (3) has the property
\[ E[x(t) + y(t)]^{θ+1} ≤ ι(θ), \] t ≥ 0.

Proof. Define a C^2-function U : R_+^2 → R_+ by
\[ U(x, y) = (1 + x + py)^{θ+1}, \]
From the condition $0 < \theta < \min_{k \in S} \left\{ \frac{2h(k)}{\beta^2(k)} \right\}$, for sufficiently small positive constant $e$, we can see that the coefficient of $y^2$ is negative. Using the inequality $xy \leq \varepsilon y^2 + \frac{a^2}{4e}$, here $\varepsilon = \frac{2p[(\theta + 1)(h(k) - \frac{\theta \beta^2(k)}{2}) - w]}{(\theta + 1)a(k) + 2w}$, and putting it into (7), yields

$$
\mathcal{L}[e^{wt}U(x, y)] 
\leq e^{wt}(1 + x + py)^{\theta - 1} \left( - (\theta + 1)b(k)x^3 - \frac{p^2}{2} [(\theta + 1)(h(k) - \frac{\theta \beta^2(k)}{2}) - w] y^2 
+ [(\theta + 1)(a(k) + \frac{\theta \alpha^2(k)}{2}) + w + \frac{((\theta + 1)a(k) + 2w)^2}{2p[(\theta + 1)(h(k) - \frac{\theta \beta^2(k)}{2}) - w]}} x^2 
+ [(\theta + 1)a(k) + 2w] x + w \right)
\leq wKe^{wt},
$$

where

$$
K = \frac{1}{w} \max_{(x, y) \in R^+_2} \left\{ (1 + x + py)^{\theta - 1} \left( - (\theta + 1)b(k)x^3 - \frac{p^2}{2} [(\theta + 1)(h(k) - \frac{\theta \beta^2(k)}{2}) - w] y^2 
+ [(\theta + 1)(a(k) + \frac{\theta \alpha^2(k)}{2}) + w + \frac{((\theta + 1)a(k) + 2w)^2}{2p[(\theta + 1)(h(k) - \frac{\theta \beta^2(k)}{2}) - w]}} x^2 
+ [(\theta + 1)a(k) + 2w] x + w \right) \right\}
$$
in which we put 1 in order to make $K$ positive. By applying Theorem 3.1 in [21], we can get

$$
EU(x(t), y(t)) \leq EU(x(0), y(0)) e^{-\omega t} + K
$$
and $U(x(t), y(t))$ is continuous. Obviously,

$$
E[x(t) + y(t)]^{\theta + 1} \leq EU(x(0), y(0)) e^{-\omega t} + K,
$$
and then

$$
E[x(t) + y(t)]^{\theta + 1} \leq \frac{EU(x(0), y(0)) + K}{(\min\{1, p\})^{\theta + 1}} \leq \lambda(t) \text{ for all } t \geq 0.
$$
The required assertion (5) follows immediately. \qed
3. The threshold between persistence in mean and the extinction. In the study of population dynamic, extinction and persistence are two important issues. In this section, we will find out the persistence in mean-extinction threshold of system (3).

**Definition 3.1.** System (3) is said to be extinctive almost surely (a.s.), if

\[ \lim_{t \to \infty} y(t) = 0 \text{ a.s.} \]

System (3) is said to be persistent in mean if

\[ \liminf_{t \to \infty} \frac{1}{t} \int_0^t y(s)ds > 0 \text{ a.s.} \]

Now, we consider the system on the boundary

\[ dX(t) = X(t)(a(r(t)) - b(r(t))X(t))dt + \alpha(r(t))X(t)dB_1(t). \]

We can check that \( x(t) \leq X(t) \forall t \geq 0 \) a.s. provided that \( x(0) = X(0) > 0 \) and \( y(0) > 0 \) by a comparison theorem. If \( \sum_{k \in \mathbb{S}} \pi_k [a(k) - \frac{\alpha^2(k)}{2}] < 0 \), we can use Theorem 4.1 in [8] to see that \( \lim_{t \to \infty} X(t) = 0 \) a.s. Hence, \( \lim_{t \to \infty} x(t) = 0 \) a.s., and then \( \lim_{t \to \infty} y(t) = 0 \) a.s. For this reason, we suppose \( \sum_{k \in \mathbb{S}} \pi_k [a(k) - \frac{\alpha^2(k)}{2}] > 0 \) throughout the rest of the paper.

Furthermore, by taking into consideration the results of [19], we can determine that if \( \sum_{k \in \mathbb{S}} \pi_k [a(k) - \frac{\alpha^2(k)}{2}] > 0 \), for any Borel measurable function \( \rho(\cdot, \cdot) : R \times \mathbb{S} \to R \), then system (9) has a unique stationary distribution \( \nu(\cdot, \cdot) \) with ergodic property:

\[ \lim_{T \to \infty} \frac{1}{T} \int_0^T \rho(X(t), r(t))dt = \sum_{k \in \mathbb{S}} \int_{R_+} \rho(x, k)\nu(dx, k) \text{ a.s.} \]

Use the generalized Itô’s formula on (9) and integrate, and one can obtain

\[ \frac{\log X(t) - \log X(0)}{t} = \frac{1}{t} \int_0^t \left[ a(r(s)) - \frac{\alpha^2(r(s))}{2} \right] ds - \frac{1}{t} \int_0^t b(r(s))X(s)ds + \frac{M_1(t)}{t}, \]

where \( M_1(t) = \int_0^t \alpha(r(s))dB_1(s) \) is a real-valued continuous local martingale (see e.g. [16] on page 16) whose quadratic variation is \( \langle M_1, M_1 \rangle_t = \int_0^t \alpha^2(r(s))ds \leq (\bar{a})^2 t \). Making use of the strong law of large numbers for martingales yields \( \lim_{t \to \infty} \frac{M_1(t)}{t} = 0 \) a.s. Taking the superior limit on both sides of inequality (11) leads to

\[ \limsup_{t \to \infty} \frac{\log X(t)}{t} = \lim_{t \to \infty} \frac{1}{t} \int_0^t \left[ a(r(s)) - \frac{\alpha^2(r(s))}{2} \right] ds - \lim_{t \to \infty} \frac{1}{t} \int_0^t b(r(s))X(s)ds \text{ a.s.} \]

By the ergodicity of the Markov chain, we know that

\[ \lim_{t \to \infty} \frac{1}{t} \int_0^t b(r(s))X(s)ds = \sum_{k \in \mathbb{S}} b(k) \int_{R_+} x\nu(dx, k) \text{ a.s.} \]

while applying the fact \( \lim_{t \to \infty} \frac{\log X(t)}{t} = 0 \) by Theorem 5.1 in [8], and then (12) implies that

\[ \sum_{k \in \mathbb{S}} b(k) \int_{R_+} x\nu(dx, k) = \sum_{k \in \mathbb{S}} \pi_k \left[ a(k) - \frac{\alpha^2(k)}{2} \right] \text{ a.s.} \]
Let $\lambda = \sum_{k \in S} f(k)c(k) \int_{R_+} x \nu(dx,k) - \sum_{k \in S} \pi_k \left[ h(k) + \frac{\beta^2(k)}{2} \right]$ which is the threshold we want between persistence in mean and the extinction of the predator population.

**Theorem 3.2.** Let $\sum_{k \in S} \pi_k \left[ a(k) - \frac{\alpha^2(k)}{2} \right] > 0$ hold. If $\lambda < 0$, then for any initial value $(x(0), y(0), r(0)) \in R_+^2 \times S$, the predator populations goes to extinction almost surely.

**Proof.** Applying the generalized Itô’s formula to the second equation of (3) results in

$$\frac{\log y(t) - \log y(0)}{t} = -\frac{1}{t} \int_0^t \left[ h(r(s)) + \frac{\beta^2(r(s))}{2} \right] ds + \frac{1}{t} \int_0^t f(r(s))c(r(s))x(s)ds + \frac{M_2(t)}{t}$$

where $M_2(t) = \int_0^t \beta(r(s))dB_2(s)$. Similar to the previous proof we get

$$\limsup_{t \to \infty} \frac{\log y(t)}{t} \leq \sum_{k \in S} f(k)c(k) \int_{R_+} x \nu(dx,k) - \sum_{k \in S} \pi_k \left[ h(k) + \frac{\beta^2(k)}{2} \right] = \lambda \ a.s. \ (15)$$

In the case when $\lambda < 0$, $y(t)$ tends to zero a.s.. This completes the proof of assertion. 

**Theorem 3.3.** Let $\sum_{k \in S} \pi_k \left[ a(k) - \frac{\alpha^2(k)}{2} \right] > 0$ hold. If $\lambda > 0$, then system (3) will be persistent in mean, for any initial value $(x(0), y(0), r(0)) \in R_+^2 \times S$.

**Proof.** First, by the generalized Itô’s formula, we get from the first equation of (3) that

$$\frac{\log x(t) - \log x(0)}{t} = \frac{1}{t} \int_0^t \left[ a(r(s)) - \frac{\alpha^2(r(s))}{2} \right] ds - \frac{1}{t} \int_0^t b(r(s))x(s)ds$$

$$- \frac{1}{t} \int_0^t c(r(s))y(s)ds + \frac{M_1(t)}{t}.$$ 

Combining (11) and (16) yields

$$0 \geq \frac{\log x(t) - \log X(t)}{t} = \frac{1}{t} \int_0^t b(r(s))[X(s) - x(s)]ds - \frac{1}{t} \int_0^t c(r(s))y(s)ds$$

$$\geq \frac{b}{t} \int_0^t [X(s) - x(s)]ds - \frac{c}{t} \int_0^t y(s)ds,$$

which implies

$$\frac{1}{t} \int_0^t [X(s) - x(s)]ds \leq \frac{\tilde{c}}{b} \frac{1}{t} \int_0^t y(s)ds. \quad (17)$$
Now, we will focus on finding \( \frac{1}{t} \int_0^t [X(s) - x(s)] \mathrm{d}s \) from system (3). Let us first calculate, by (14) and (17), that
\[
\log y(t) - \log y(0) = -\frac{1}{t} \int_0^t \left[ h(r(s)) + \frac{\beta^2(r(s))}{2} \right] \mathrm{d}s + \frac{1}{t} \int_0^t f(r(s))c(r(s))x(s) \mathrm{d}s + \frac{M_2(t)}{t}
\]
\[
\geq -\frac{1}{t} \int_0^t \left[ h(r(s)) + \frac{\beta^2(r(s))}{2} \right] \mathrm{d}s + \frac{1}{t} \int_0^t f(r(s))c(r(s))X(s) \mathrm{d}s
\]
\[
- \frac{\epsilon}{b} \frac{1}{t} \int_0^t y(s) \mathrm{d}s + \frac{M_2(t)}{t}.
\]
(18)

It follows from the property of inferior limits and \( \lim_{t \to \infty} \frac{M_2(t)}{t} = 0 \) a.s., that for arbitrary \( \epsilon > 0 \), there exists \( t > 0 \) such that
\[
-\frac{1}{t} \int_0^t \left[ h(r(s)) + \frac{\beta^2(r(s))}{2} \right] \mathrm{d}s + \frac{\epsilon}{b} \frac{1}{t} \int_0^t y(s) \mathrm{d}s,
\]
\[
\geq \lambda - \frac{\epsilon}{b} \frac{1}{t} \int_0^t y(s) \mathrm{d}s,
\]
(19)

and by virtue of the arbitrariness of \( \epsilon \) and Lemma 2.2 we have
\[
\lim_{t \to \infty} \inf \frac{1}{t} \int_0^t y(s) \mathrm{d}s \geq \frac{\lambda b}{\epsilon^2} > 0 \text{ a.s.}
\]
(20)

We therefore obtain the desired assertion from (20).

4. Ergodic property of positive recurrence. In this section, we will investigate the ergodic property of system (3) by using the Lyapunov function method. Now we impose the condition

\[
(\text{H}) \quad \bar{\lambda} = \bar{\beta} \sum_{k \in S} \pi_k \left[ a(k) - \frac{\alpha^2(k)}{2} \right] - \bar{b} \sum_{k \in S} \pi_k \left[ h(k) + \frac{\beta^2(k)}{2} \right] > 0.
\]

and \( \min \{ h(k) - \frac{\beta^2(k)}{2} \} > 0 \).

For the purpose of proving our theorem, we will first introduce a transformation of system (3). Let \( u(t) = \log x(t) \) and \( v(t) = \log y(t) \), for \( t \geq 0 \). Applying the generalized Itô’s formula, yields
\[
\begin{aligned}
\left\{ \begin{array}{l}
\mathrm{d}u(t) = \left[ a(r(t)) - \frac{\alpha^2(r(t))}{2} - b(r(t))e^u(t) - c(r(t))e^v(t) \right] \mathrm{d}t + \alpha(r(t))dB_1(t),
\end{array} \right.
\end{aligned}
\]
\[
\begin{aligned}
\left\{ \begin{array}{l}
\mathrm{d}v(t) = \left[ -h(r(t)) - \frac{\beta^2(r(t))}{2} + f(r(t))c(r(t))e^u(t) \right] \mathrm{d}t + \beta(r(t))dB_2(t).
\end{array} \right.
\end{aligned}
\]
(21)
By Lemma 3.2 in [15], we know that the ergodic property and positive recurrence of system (3) are equivalent to those of system [21]. The proof of the following theorem will be to verify that system [21] satisfy the three conditions of Lemma 2.1.

**Theorem 4.1.** Let us assume that hypothesis 2.1. theorem will be to verify that system (21) satisfy the three conditions of Lemma 2.1 is ergodic and has a unique stationary distribution in $R^+_2 \times S$.

**Proof.** In Section 1, we assume $\gamma_{ij} > 0$, $i \neq j$ and thus condition (i) in Lemma 2.1 is satisfied. To verify condition (ii), consider the bounded open subset $D = \{(u, v) : |u| \leq \log \epsilon^{-1}, |v| \leq \log \epsilon^{-1}, (u, v) \in R^2\}$, where $0 < \epsilon < 1$ is a sufficiently small number. Let $g(z, k) = \text{diag}(\alpha(k), \beta(k))(k \in S)$, and we have $D(z, k) = g(z, k)g^T(z, k) = \text{diag}(\alpha^2(k), \beta^2(k))$ which is positive definite. Therefore, condition (ii) in Lemma 2.1 is satisfied. It therefore remains for us to verify condition (iii) in Lemma 2.1 is satisfied. Define a $C^2$-function

$$\phi(u, v) = M[-\hat{f}\hat{c}u - \hat{b}v + \frac{\hat{f}\hat{c}}{h}e^v] + \frac{[e^u + pe^v]^2}{2},$$

(22)

where $p = \frac{\hat{c}}{\hat{f}}$ and $M = (2/\hat{b}) \max \{2, \sup_{(u, v) \in R^2} \{\frac{\hat{b}}{2}e^{3u} - \frac{\hat{b}^2}{4} \min_{k \in S} \{h(k) - \beta(k)\} e^v + qe^{2u}\}\}$ and $q = \hat{a} + \frac{\hat{a}^2}{2} + \frac{\hat{a}^2}{2 \min_{k \in S} \{h(k) - \beta(k)\}}$. By calculating the equation set of partial derivative functions of $\phi(u, v)$, we know that

$$-\frac{M\hat{f}\hat{c}}{e^u} + e^u + \frac{pe^v}{\hat{f}\hat{c}} = 0$$

has a unique solution $u_0$, which can be seen from the monotonically property of the left function. The unique solution $(u_0, v_0)$ of the equation

$$\frac{p\hat{f}\hat{c}}{e^u} = \frac{\hat{b}}{e^v} + \frac{\hat{f}\hat{c}}{h}$$

is the minimum point of $\phi(u, v)$, here $v_0 = \log \left(\frac{\hat{b}}{\hat{f}\hat{c}(h + pe^{-u_0})}\right)$. We assert that $\phi(u, v) - \phi(u_0, v_0) \geq 0$.

Define a $C^2$-function $V : R^2 \times S \rightarrow R$, by

$$V(u, v, k) = M[-\hat{f}\hat{c}u - \hat{b}v + \frac{\hat{f}\hat{c}}{h}e^v] + \frac{[e^u + pe^v]^2}{2} - \phi(u_0, v_0) + M(\varpi_k + |\varpi|)$$

$$= V_1(u, v) + V_2(u, v) - \phi(u_0, v_0) + V_3(k),$$

(23)

where $\varpi = (\varpi_1, \ldots, \varpi_m)^T$, $|\varpi| = \sqrt{\sum_{k=1}^{m} \varpi_k^2}$ and $\varpi_k (k \in S)$ will be determined in the rest of the proof. Note that we put $|\varpi|$ in order to make $\varpi_k + |\varpi|$ non-negative. By Itô’s formula, we have, respectively

$$\mathcal{L}V_1(u, v) \leq M\left(-\hat{f}\hat{c}[a(k) - \frac{\alpha^2(k)}{2}] + \hat{b}[h(k) + \frac{\beta^2(k)}{2}] + \frac{\hat{f}\hat{c}}{h}e^{u+v}\right)$$

(24)

and

$$\mathcal{L}V_3(k) = M \sum_{i \neq k, i \in S} \gamma_{kl}(\varpi_i - \varpi_k).$$

(25)
Let us define the vector $\Lambda = (\Lambda_1, \ldots, \Lambda_m)^T$ with $\Lambda_k = \hat{f}\hat{c}[a(k) - \frac{\alpha^2(k)}{2}] - \hat{b}[h(k) + \frac{\beta^2(k)}{2}]$. Since the generator matrix $\Gamma$ is irreducible, then for $\Lambda_k$, there exists $\varpi = (\varpi_1, \ldots, \varpi_m)^T$ a solution of the Poisson system (see Lemma 2.3 in [5] and Remark 2 for detailed description), such that

$$\Gamma \varpi - \Lambda = -(\sum_{j=1}^{m} \pi_j \Lambda_j)^T,$$

(26)

where $\overline{1}$ denotes the column vector with all its entries equal to 1. Thus, we have

$$\sum_{l \neq k, \ l \in S} \gamma_{kl}(\varpi_l - \varpi_k) - \left(\hat{f}\hat{c}[a(k) - \frac{\alpha^2(k)}{2}] - \hat{b}[h(k) + \frac{\beta^2(k)}{2}]\right)$$

$$= -\left(\hat{f}\hat{c}\sum_{k \in S} \pi_k[a(k) - \frac{\alpha^2(k)}{2}] - \hat{b}\sum_{k \in S} \pi_k[h(k) + \frac{\beta^2(k)}{2}]\right)$$

$$= -\lambda.$$

Thereby, combining (24), (25) and (27), yields $\mathcal{L}(V_1 + V_3) \leq M(-\lambda + \frac{\hat{f}\hat{c}e^2}{h} e^{u+v})$.

We note

$$\mathcal{L}V_2(u, v)$$

$$\leq (e^u + pe^v)[e^u(a(k) - h(k)e^u) - \hat{c}e^{u+v} - p\hat{h}e^{u+v} + p\hat{f}\hat{c}e^{u+v}]$$

$$+ \frac{\alpha^2(k)}{2}e^{2u} + p^2 \frac{\beta^2(k)}{2}e^{2v}$$

$$\leq -\hat{c}e^{3u} - p^2 \left(\min_{k \in S}\{h(k) - \frac{\beta^2(k)}{2}\}\right)e^{2v} + (\hat{a} + \frac{\hat{c}^2}{2})e^{2u} + p\hat{a}e^{u+v}$$

$$\leq -\hat{c}e^{3u} + p^2 \left(\min_{k \in S}\{h(k) - \frac{\beta^2(k)}{2}\}\right)e^{2v} + \left(\hat{a} + \frac{\hat{c}^2}{2} + \frac{\hat{a}^2}{2} + \frac{\hat{c}^2}{2}\right)e^{2u}\right)$$

$$= -\hat{c}e^{3u} + p^2 \left(\min_{k \in S}\{h(k) - \frac{\beta^2(k)}{2}\}\right)e^{2v} + qe^{2u}.$$
where \( K_1 \) and \( K_2 \) are positive constants which can be found in (33) and (35). Denote that
\[
\mathcal{D}_e^1 = \{(u, v) \in \mathbb{R}^2 : -\infty < u \leq \log \epsilon \}, \quad \mathcal{D}_e^2 = \{(u, v) \in \mathbb{R}^2 : -\infty < v \leq \log \epsilon \},
\]
\[
\mathcal{D}_e^3 = \{(u, v) \in \mathbb{R}^2 : u \geq \log \epsilon^{-1} \}, \quad \mathcal{D}_e^4 = \{(u, v) \in \mathbb{R}^2 : v \geq \log \epsilon^{-1} \}.
\]
Obviously, \( \mathcal{D}_e^c = \mathcal{D}_e^1 \cup \mathcal{D}_e^2 \cup \mathcal{D}_e^3 \cup \mathcal{D}_e^4 \). In the following we prove \( \mathcal{L}V(u, v, k) \leq -1 \) on \( \mathcal{D}_e^c \), which is equivalent to proving it on the above four domains, respectively.

**Case 1.** On \( \mathcal{D}_e^1 \times S \), owing to \( e^{u+v} \leq ce^v \leq \epsilon(1 + e^{2v}) \), we have
\[
\mathcal{L}V(u, v, k) 
\leq -\frac{M\bar{\lambda}}{4} + \left( -\frac{M\bar{\lambda}}{4} + \frac{M\int \hat{f} \hat{c}^2 \hat{\epsilon}}{h} \right) \hat{b} + \frac{p^2}{4} \min_{k \in S} \left\{ h(k) - \frac{\beta^2(k)}{2} \right\} + \frac{M\int \hat{f} \hat{c}^2 \hat{\epsilon}}{h} e^{2v} + \left[ -\frac{M\bar{\lambda}}{4} + \frac{M\int \hat{f} \hat{c}^2 \hat{\epsilon}}{h} - \frac{\hat{b}}{2} + \frac{p^2}{4} \min_{k \in S} \left\{ h(k) - \frac{\beta^2(k)}{2} \right\} e^{2v} + qe^{2u} \right].
\]
Since \( M = (2/\bar{\lambda}) \max \{ 2, \sup_{(u, v) \in \mathbb{R}^2} \left\{ -\frac{\hat{b}}{2} e^{3u} - \frac{p^2}{4} \min_{k \in S} \left\{ h(k) - \frac{\beta^2(k)}{2} \right\} e^{2v} + qe^{2u} \right\} \} \), then we have \( \frac{M\bar{\lambda}}{4} \geq 1 \). Combining (28) and (29), we obtain
\[
\mathcal{L}V(u, v, k) \leq -\frac{M\bar{\lambda}}{4} - \frac{\hat{b}}{2} e^{3u} \leq -\frac{M\bar{\lambda}}{4} \leq -1.
\]
Thus \( \mathcal{L}V \leq -1 \) for all \( (u, v, k) \in \mathcal{D}_e^1 \times S \).

**Case 2.** For any \( (u, v, k) \in \mathcal{D}_e^2 \times S \), owing to \( e^{u+v} \leq ce^u \leq \epsilon(1 + e^{3u}) \), we have
\[
\mathcal{L}V(u, v, k) 
\leq -\frac{M\bar{\lambda}}{4} + \left( -\frac{M\bar{\lambda}}{4} + \frac{M\int \hat{f} \hat{c}^2 \hat{\epsilon}}{h} \right) \hat{b} + \frac{p^2}{4} \min_{k \in S} \left\{ h(k) - \frac{\beta^2(k)}{2} \right\} e^{2v} + \left[ -\frac{M\bar{\lambda}}{4} + \frac{M\int \hat{f} \hat{c}^2 \hat{\epsilon}}{h} - \frac{\hat{b}}{2} + \frac{p^2}{4} \min_{k \in S} \left\{ h(k) - \frac{\beta^2(k)}{2} \right\} e^{2v} + qe^{2u} \right].
\]
From (28) and (30), yields
\[
\mathcal{L}V(u, v, k) \leq -\frac{M\bar{\lambda}}{4} - \frac{p^2}{4} \min_{k \in S} \left\{ h(k) - \frac{\beta^2(k)}{2} \right\} e^{2v} \leq -\frac{M\bar{\lambda}}{4} \leq -1.
\]
Therefore, \( \mathcal{L}V \leq -1 \) on \( \mathcal{D}_e^2 \times S \).
Case 3. When \((u, v, k) \in \mathcal{D}_t^3 \times \mathcal{S}\),
\[
\mathcal{L}V(u, v, k) \leq -M \lambda - \frac{b}{2} e^{3u} + \left( - \frac{\hat{b}}{2} e^{3u} - \frac{p^2}{4} \min_{k \in \mathcal{S}} \{h(k) - \frac{\beta^2(k)}{2}\}\right) e^{2v} + \left[q + \frac{(Mf \hat{\delta} \hat{\epsilon})^2}{p^2 h^2 \min_{k \in \mathcal{S}} \{h(k) - \frac{\beta^2(k)}{2}\}}\right] e^{2u}
\]
\[
\leq -M \lambda - \frac{\hat{b}}{2} e^{3u} + K_1,
\]
where
\[
K_1 = \sup_{(u, v) \in \hat{R}^2} \left\{ - \frac{\hat{b}}{2} e^{3u} - \frac{p^2}{4} \min_{k \in \mathcal{S}} \{h(k) - \frac{\beta^2(k)}{2}\}\right\} e^{2v} + \left[q + \frac{(Mf \hat{\delta} \hat{\epsilon})^2}{p^2 h^2 \min_{k \in \mathcal{S}} \{h(k) - \frac{\beta^2(k)}{2}\}}\right] e^{2u},
\]
which implies \(\mathcal{L}V \leq -1\) in this domain, in view of (31).

Case 4. On \(\mathcal{D}_t^3 \times \mathcal{S}\),
\[
\mathcal{L}V(u, v, k) \leq -M \lambda - \frac{p^2}{4} \min_{k \in \mathcal{S}} \{h(k) - \frac{\beta^2(k)}{2}\} e^{2v} + \left(- \frac{\hat{b}}{2} e^{3u} + \left[q + \frac{(Mf \hat{\delta} \hat{\epsilon})^2}{p^2 h^2 \min_{k \in \mathcal{S}} \{h(k) - \frac{\beta^2(k)}{2}\}}\right] e^{2u}\right)\]
\[
\leq -M \lambda - \frac{p^2}{4} \min_{k \in \mathcal{S}} \{h(k) - \frac{\beta^2(k)}{2}\} \frac{1}{e^2} + K_2,
\]
where \(K_2 = \sup_{u \in \hat{R}} \left\{ - \frac{\hat{b}}{2} e^{3u} + \left[q + \frac{(Mf \hat{\delta} \hat{\epsilon})^2}{p^2 h^2 \min_{k \in \mathcal{S}} \{h(k) - \frac{\beta^2(k)}{2}\}}\right] e^{2u}\right\}\). Noticing (32), we have \(\mathcal{L}V \leq -1\) on \(\mathcal{D}_t^3 \times \mathcal{S}\).

Therefore according to Lemma 2.1 (\((u(t), v(t), r(t))\) is ergodic and positive recurrent, and so the system \([3]\) is positive recurrent and admits a unique stationary distribution, in addition, the stationary density is in \(R_+^2 \times \mathcal{S}\) which can be testified by a similar proof to Lemma 3.1 of [11].

Remark 1. (a) The purpose of analyzing \(\phi(u, v)\) appearing in [22] is to construct a nonnegative Lyapunov function \(V(u, v, k)\) which consists of three parts. Constructing this \(V(u, v, k)\) reflect in ensuring \(\mathcal{L}V \leq -1\) on all four domains \(\mathcal{D}_t^1, \ldots, \mathcal{D}_t^4\). In addition, with the help of \(\omega(k)\), we can get the \(\lambda\) in \(\mathcal{L}V\) and \(\lambda\) reflects the impact of environmental white noise and the probability distribution of the Markov Chain.

(b) We divided \(\mathcal{D}_t^i\) into four domains, so that, the trends of functions in each domain will be seen clearly.

(c) By comparing \(\lambda\) and \(\tilde{\lambda}\), we find, in this paper, that the condition \(\tilde{\lambda} > 0\) is stronger than \(\lambda > 0\) which also reveals the biological significance, that is, stationary distribution is weak stability, and it is more difficult to achieve than persistence in mean.
Remark 2. We introduce Lemma 2.3 in [5]: Suppose that $\Gamma$, an $m \times m$ constant matrix, is the generator of a continuous-time Markov chain $r(t)$ and that $\Gamma$ is irreducible. Then $\pi\tau = \eta$ has a solution if and only if $\pi\eta = 0$, where $\eta \in R^m$ and $\pi \in R^1 \times m$ denotes the associated stationary distribution.

We will verify $\pi\eta = 0$ and choose the vector $\Lambda = (\Lambda_1, \ldots, \Lambda_m)^T$ with $\Lambda_k = \hat{f}_c[a(k) - \frac{\alpha^2(k)}{2}] - \hat{b}[h(k) + \frac{\beta^2(k)}{2}]$ and $\eta = \Lambda - (\sum_{j=1}^{m} \pi_j \Lambda_j)^T$. Obviously, $\pi\eta = (\pi_1, \pi_2, \ldots, \pi_m)(\Lambda_1, \ldots, \Lambda_m)^T - (\pi_1, \pi_2, \ldots, \pi_m)(\sum_{j=1}^{m} \pi_j \Lambda_j)^T = \sum_{j=1}^{m} \pi_j \Lambda_j - (\sum_{j=1}^{m} \pi_j \Lambda_j) \cdot (\sum_{j=1}^{m} \pi_j) = 0$. Therefore, (26) has a solution $\pi$ in Theorem 4.1.

Theorem 4.1 provides us with the good property of system (3) having a unique stationary distribution in $R^2_+ \times \mathbb{S}$, expressed as $\mu(\cdot, \cdot)$. Thus, we can get the following theorem.

**Theorem 4.2.** Let hypothesis $(H)$ holds. For any $k \in \mathbb{S}$, we have

$$\sum_{k \in \mathbb{S}} b(k) \int_{R^+} x\mu(dx, k) + \sum_{k \in \mathbb{S}} c(k) \int_{R^+} y\mu(dy, k) = \sum_{k \in \mathbb{S}} \pi_k [a(k) - \frac{\alpha^2(k)}{2}] \quad a.s. \quad (37)$$

and

$$\sum_{k \in \mathbb{S}} f(k)c(k) \int_{R^+} x\mu(dx, k) = \sum_{k \in \mathbb{S}} \pi_k [h(k) + \frac{\beta^2(k)}{2}] \quad a.s. \quad (38)$$

**Proof.** By [5] of Theorem 2.4, for any $\theta > \min_{k \in \mathbb{S}} \left( \frac{2b(k)}{\alpha^2(k)} \right)$, there exists a positive constant $\iota(\theta)$, such that $E[x(t)+y(t)]^{\theta+1} \leq \iota(\theta)$, additionally, $E[x(t)]^{\theta+1} \geq (E[x(t)])^{\theta+1}$, and we have $E[x(t)] \leq [\iota(\theta)]^{\frac{1}{\theta+1}}$, and similarly, $E[y(t)] \leq [\iota(\theta)]^{\frac{1}{\theta+1}}$. Then, by the ergodic property of $(x(t), y(t), r(t))$ and for any $n > 0$, we have

$$\mathbb{P} \left( \lim_{t \to \infty} \frac{1}{t} \int_{0}^{t} (x(s) \wedge n) ds = \sum_{k \in \mathbb{S}} \int_{R^+} (x \wedge n) \mu(x, k) dx \right) = 1.$$ 

Thus, by the dominated convergence theorem, we get

$$E \left( \lim_{t \to \infty} \frac{1}{t} \int_{0}^{t} (x(s) \wedge n) ds \right) = \lim_{t \to \infty} \frac{1}{t} \int_{0}^{t} E(x(s) \wedge n) ds = \sum_{k \in \mathbb{S}} \int_{R^+} (x \wedge n) \mu(x, k) dx.$$ 

Therefore,

$$\sum_{k \in \mathbb{S}} \int_{R^+} x\mu(x, k) dx = \lim_{n \to \infty} \sum_{k \in \mathbb{S}} \int_{R^+} (x \wedge n) \mu(x, k) dx \leq \limsup_{t \to \infty} E[x(t)] < +\infty, \quad (39)$$

which implies the function $f(x) = x$ is integrable with respect to the measure $\mu$. Similarly, we can get also

$$\sum_{k \in \mathbb{S}} \int_{R^+} y\mu(y, k) dy \leq \limsup_{t \to \infty} E[y(t)] < +\infty.$$ 

Return to (16), from the ergodic property of $(x(t), y(t), r(t))$, we know that

$$\lim_{t \to \infty} \frac{\log x(t)}{t} = 0 \quad a.s.$$ 

Otherwise, if $\lim_{t \to \infty} \frac{\log x(t)}{t} = 0 < 0$, we know that, when $t \to \infty$, $x(t) \to 0$ or $0 < \lim_{t \to \infty} \frac{\log x(t)}{t} = 0$, which contradicts the fact that its stationary density lies in $R^+$. Therefore, we have

$$\sum_{k \in \mathbb{S}} \pi_k [a(k) - \frac{\alpha^2(k)}{2}] - \sum_{k \in \mathbb{S}} b(k) \int_{R^+} x\mu(dx, k) - \sum_{k \in \mathbb{S}} c(k) \int_{R^+} y\mu(dy, k) = 0, \quad a.s. \quad (40)$$
which implies the first result (37). The second result also holds by the same way.

5. Computer simulations.

![Figure 1](image1.png)

Figure 1. The curves on the subgraphs (a) and (b) are the density functions of $x(t)$ and $y(t)$ in $k = 1, k = 2$, respectively. The white noise intensity on $x(t)$ and $y(t)$ are all relatively small.

Assume that the system (3) switches from one to the other according to the movement of the Markov chain $r(t)$ on the state space $S = \{1, 2\}$ with the generator

$$
\Gamma = \begin{pmatrix}
-1 & 1 \\
2 & -2
\end{pmatrix}
$$

The stationary distribution is given by $\pi = (\pi_1, \pi_2) = \left(\frac{2}{3}, \frac{1}{3}\right)$. Let us choose $(a(1), a(2)) = (5, 6), (b(1), b(2)) = (6, 4), (c(1), c(2)) = (5, 6), (e(1), e(2)) = (2, 4), (f(1), f(2)) = (0.9, 0.9)$. Here we will discuss two cases according to the white noise intensity.

**Case 1.**

Assume that $(\alpha(1), \alpha(2)) = (0.15, 0.09), (\beta(1), \beta(2)) = (0.21, 0.05)$. Predator and prey are affected by smaller white noise.

This gives

$$
\sum_{k=1}^{2} \pi_k \left[ a(k) - \frac{\alpha^2(k)}{2} \right] = 5.3245 > 0, \quad \bar{\lambda} = \hat{\lambda} = \hat{\beta} \sum_{k=1}^{2} \pi_k \left[ h(k) + \frac{\beta^2(k)}{2} \right] = 7.8695 > 0 \quad \text{and} \quad \min_{k \in S} \{ h(k) - \frac{\beta^2(k)}{2} \} = 1.9780 > 0.
$$

Then, the conditions of Theorem 4.1 hold, and system (3) has a unique stationary distribution. Here we will use Fig.1-2 to illustrate the results.

The density function images of stationary distribution of $x(t)$ and $y(t)$ in $k = 1$ and $k = 2$ will be seen in Fig.1. In Fig.2 (a), the red □, blue ◦ and black + represent the phase portrait of $x(t)$ and $y(t)$ when there is only one state $k = 1, k = 2$ and switching back and forth from one state $k = 1$ to another state $k = 2$ according to the movement of $r(t)$, respectively. Clearly, the black area is located between the red region and the blue region, and the red area and the blue area is similar to the two limit state of the black region. Correspondingly, the subgraph Fig.2 (b) describes the state with no random disturbance. From Fig.2, we can clearly see the impact of white noise and colored environment noise on populations.

**Case 2.**

Assume that $(\alpha(1), \alpha(2)) = (1.2, 1.7), (\beta(1), \beta(2)) = (1.5, 1.4)$. Predator and prey are affected by bigger white noise.
Figure 2. The subgraphs (a) and (b) denote sample phase portrait of stochastic system and the corresponding deterministic system. The red, blue and black area denote $x(t)$, $y(t)$ in $k = 1$, $k = 2$ and converting between the above two states, respectively. The white noise intensity on $x(t)$ and $y(t)$ are all relatively small.

Figure 3. The subgraphs (a) and (b) have the same definitions as in Fig.2. The white noise intensity on $x(t)$ and $y(t)$ are all relatively big.

We compute $\sum_{k=1}^{2} \pi_k \left[ a(k) - \frac{\sigma^2(k)}{2} \right] = 4.3717 > 0$ and $\bar{\lambda} = -2.7875 < 0$. The hypothesis ($H$) does not hold. From Remark 1(c), we know that $\lambda < \bar{\lambda} = -2.7875 < 0$, that is, the predator will become extinct by Theorem 3.2 which can be seen from Fig.3 (a). Large white noise will lead to population extinction, even though the corresponding deterministic model is persistent (see Fig.3 (b)).

6. Conclusions and future directions. This paper studies a stochastic predator-prey Lotka-Volterra model with density-dependent logistic growth of the prey under regime switching. Sufficient conditions for persistence in mean and the extinction of system (3) are established and we obtain the threshold between them. We prove the stochastic system with regime switching has a unique stationary distribution which is the first attempt to solve this problem by using the Lyapunov function method.
The conclusions we obtained on system (3) in this paper can be a good description of the biological significance. From $\lambda < 0$, we can see that, the predator can die out when the parameters $c(k)$ and feeding efficiency $f(k)$ or the intrinsic growth rate $-h(k)$ is too small or the white noise intensity $\alpha^2(k)$ is too big, which takes place also in the case without noise. If the predator-prey model wants to have a stationary distribution, that is $\bar{\lambda} = \hat{\lambda}$, then $\lambda = \hat{\lambda} - \frac{\sigma^2(k)}{2} > 0$, the intrinsic growth rate $a(k)$ of prey, $c(k)$ and $f(k)$ will not be too small, and the density-dependent $b(k)$ and $h(k)$ will not be too big. Of course, the white noise intensity $\alpha^2(k)$ and $\beta^2(k)$ can not be too big. Therefore, the main difference between the deterministic and stochastic model is that large noise can also cause extinction. Obviously, these important factors play a key role in the predator extinction or not, that gives its threshold $\lambda$ between persistence in mean and extinction. Furthermore, the role of the colored noise is clear in the examples. The presence of the Markov chain in this stochastic system can contribute to the survival of the system, and reduce the risk of extinction.

In future studies, we will try to promote this approach to other predator-prey models.

Acknowledgments. We would like to thank the editor and referee for their very helpful comments and suggestions. We also thank Programs for the NSFC of China (No: 11601038, 11371085), and the Fundamental Research Funds for the Central Universities (No: 15CX08011A) for their financial support.

REFERENCES

[1] G. Q. Cai and Y. K. Lin, Stochastic analysis of predator-prey type ecosystems, *Ecol. Complex.*, 4 (2007), 242–249.
[2] L. S. Chen and Z. J. Jing, The existence and uniqueness of limit cycles for the differential equations of predator-prey interactions, *Chinese Sci. Bull.*, 9 (1984), 521–523.
[3] H. W. Hethcote, W. Wang, L. T. Han and Z. E. Ma, A predator-prey model with infected prey, *Theor. Popul. Biol.*, 66 (2004), 259–268.
[4] C. Y. Ji, D. Q. Jiang and N. Z. Shi, A note on a predator-prey model with modified Leslie-Gower and Holling-type II schemes with stochastic perturbation, *J. Math. Anal. Appl.*, 377 (2011), 435–440.
[5] R. Z. Khasminskii, C. Zhu and G. Yin, Stability of regime-switching diffusions, *Stochastic Process. Appl.*, 117 (2007), 1037–1051.
[6] H. H. Li, D. Q. Jiang, F. Z. Cong and H. X. Li, Persistence and non-persistence of a predator-prey system with stochastic perturbation, *Abstr. Appl. Anal.*, 2014 (2014), Article ID 720283, 10 pages.
[7] X. Y. Li, D. Q. Jiang and X. R. Mao, Population dynamical behavior of Lotka-Volterra system under regime switching, *J. Comput. Appl. Math.*, 232 (2009), 427–448.
[8] X. Y. Li, A. Gray, D. Q. Jiang and X. R. Mao, Sufficient and necessary conditions of stochastic permanence and extinction for stochastic logistic populations under regime switching, *J. Math. Anal. Appl.*, 376 (2011), 11–28.
[9] X. Y. Li and X. R. Mao, A note on almost sure asymptotic stability of neutral stochastic delay differential equations with Markovian switching, *Automatica*, 48 (2012), 2329–2334.
[10] H. Liu, Q. S. Yang and D. Q. Jiang, The asymptotic behavior of stochastically perturbed DII SIR epidemic models with saturated incidences, *Automatica*, 48 (2012), 820–825.
[11] H. Liu, X. X. Li and Q. S. Yang, The ergodic property and positive recurrence of a multi-group Lotka-Volterra mutualistic system with regime switching, *Syst. Control Lett.*, 62 (2013), 805–810.
[12] M. Liu and K. Wang, Asymptotic properties and simulations of a stochastic logistic model under regime switching II, *Math. Comput. Model.*, 55 (2012), 405–418.
[13] M. Liu and K. Wang, Asymptotic properties and simulations of a stochastic logistic model under regime switching. *Math. Comput. Model.*, 54 (2011), 2139–2154.

[14] X. R. Mao, S. Sabanis and E. Renshaw, Asymptotic behaviour of the stochastic Lotka-Volterra model. *J. Math. Anal. Appl.*, 287 (2003), 141–156.

[15] X. R. Mao, Stationary distribution of stochastic population systems. *Syst. Control Lett.*, 60 (2011), 398–405.

[16] X. R. Mao and C. G. Yuan, *Stochastic Differential Equations with Markovian Switching*, Imperial College Press, London, 2006.

[17] R. M. Nisbet and W. S. C. Gurney, *Modelling Fluctuating Populations*, Wiley-Interscience, New York, 1982.

[18] E. C. Pielou, *Introduction to Mathematical Ecology*, Wiley-Interscience, New York, 1969.

[19] A. Settati and A. Lahrouz, Stationary distribution of stochastic population systems under regime switching. *Appl. Math. Comput.*, 244 (2014), 235–243.

[20] Y. Takeuchi, N. H. Dub, N. T. Hieu and K. Satoa, Evolution of predator-prey systems described by a Lotka-Volterra equation under random environment. *J. Math. Anal. Appl.*, 323 (2006), 938–957.

[21] D. Y. Xu, B. Li, S. J. Long and L. Y. Teng, Moment estimate and existence for solutions of stochastic functional differential equations. *Nonlinear Anal.*, 108 (2014), 128–143.

[22] D. Y. Xu, X. H. Wang and Z. G. Yang, Further results on existence-uniqueness for stochastic functional differential equations. *Sci. China Math.*, 56 (2013), 1169–1180.

[23] T. Zhao, Y. Kung and H. L. Smith, Global existence of periodic solutions in a class of delayed Gause-type predator-prey systems. *Nonlinear Anal. Theor.*, 28 (1997), 1373–1394.

[24] C. Zhu and G. Yin, Asymptotic properties of hybrid diffusion systems. *Siam. J. Control Optim.*, 46 (2007), 1155–1179.

[25] L. Zu, D. Q. Jiang and D. O’Regan, Conditions for persistence and ergodicity of a stochastic Lotka-Volterra predator-prey model with regimeswitching. *Commun. Nonlinear Sci. Numer. Simul.*, 29 (2015), 1–11.

Received October 2014; revised January 2017.

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