Modular Solutions to Equations of Generalized Halphen Type

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Abstract

Solutions to a class of differential systems that generalize the Halphen system are determined in terms of automorphic functions whose groups are commensurable with the modular group. These functions all uniformize Riemann surfaces of genus zero and have $q$-series with integral coefficients. Rational maps relating these functions are derived, implying subgroup relations between their automorphism groups, as well as symmetrization maps relating the associated differential systems.

1. Introduction

Differential equations satisfied by modular functions have been studied since the time of Jacobi [J]. Such equations arise naturally in connection with second order Fuchsian differential operators whose monodromy groups coincide with the automorphism group of the function. Recall that a meromorphic function $f(\tau)$ defined on an open, connected domain $\mathcal{D}$ in the Riemann sphere is said to be automorphic with respect to a group $\mathfrak{G}_f$ of linear fractional transformations

$$T : \tau \rightarrow \frac{a\tau + b}{c\tau + d} \equiv T(\tau), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{C}),$$

(1.1)

if the domain $\mathcal{D}$ is $\mathfrak{G}_f$ invariant and

$$f(T(\tau)) = f(\tau), \quad \forall \ T \in \mathfrak{G}_f.$$

(1.2)

In this paper, the term modular will be applied to functions with Fuchsian automorphism groups of the first kind that are commensurable with the modular group $PSL(2, \mathbb{Z})$ (i.e., whose intersection with the latter is of finite index in both).

Using the terminology of [Fo], a “simple” automorphic function is one without essential singularities at ordinary points, whose fundamental region has a finite number of sides and which has a definite (finite or infinite) limiting value at any parabolic point (cusp). The following standard theorem gives the connection between linear second order equations and simple automorphic functions (cf. [Fo], Sec. 44, Theorem 15).
Theorem 1.1:  If \( f(\tau) \) is a nonconstant, simple automorphic function, then the (multi-valued) inverse function \( \tau = \tau(f) \) can be expressed as the quotient of two solutions of a second order linear equation

\[
\frac{d^2 y}{df^2} + R(f)y = 0, \quad (1.3)
\]

where \( R \) is an algebraic function of \( f \). If \( f \) has a single first order pole in the fundamental region, then \( R \) is a rational functions of \( f \).

In fact, the solutions of (1.3) may be expressed, at least locally, as

\[
y = \frac{(A + B\tau(f))}{(\tau')^{1/2}}, \quad (1.4)
\]

Conversely, given any second order linear equation

\[
\frac{d^2 y}{df^2} + P\frac{dy}{df} + Qy = 0 \quad (1.5)
\]

with rational coefficients \( P(f), Q(f), \) singular points \( (a_1, \ldots a_n, \infty) \), a basis of solutions \( (y_1, y_2) \) and a base point \( f_0 \), the image of the monodromy representation

\[
M : \pi_1(\mathbb{P} - \{a_1, \ldots a_n, \infty\}) \to \text{GL}(2, \mathbb{C})
\]

\[
M : \gamma \mapsto M_\gamma =: \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (1.6)
\]

defined up to global conjugation by

\[
\gamma : (y_1, y_2)|_{f_0} = (y_1, y_2)|_{f_0} M_\gamma, \quad (1.7)
\]

determines a subgroup \( \mathfrak{G} \subset \text{GL}(2, \mathbb{C}) \) that acts on the ratio

\[
\tau := \frac{y_1}{y_2} \quad (1.8)
\]

by linear fractional transformations (1.1). By a simple substitution of the type

\[
y \to \prod_{i=1}^{n} (f - a_i)^{\mu_i} y, \quad (1.9)
\]

eq (1.5) can be transformed into the form (1.3) without changing the projective class of the monodromy group; i.e., without changing the \( \mathfrak{G} \)-action (1.1) on \( \tau \). If \( \mathfrak{G} \) is Fuchsian and commensurable with \( \text{SL}(2, \mathbb{Z}) \), the inverse function \( f = f(\tau) \) is a modular function in the above sense.
If $R(f)$ is the resulting rational function in (1.3), then $f = f(\tau)$ satisfies the Schwarzian differential equation
\[ \{f, \tau\} + 2R(f)f'^2 = 0, \quad (1.10) \]
where
\[ \{f, \tau\} := \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2, \quad (f' := \frac{df}{d\tau}) \quad (1.11) \]
denotes the Schwarzian derivative [H1, GS].

Perhaps the oldest example of such an equation involves the square of the elliptic modulus of the associated Jacobi elliptic functions $\lambda(\tau) = k^2(\tau)$, which satisfies the Schwarzian equation [H1]
\[ \{\lambda, \tau\} + \frac{\lambda^2 - \lambda + 1}{2\lambda^2(1-\lambda)^2}\lambda'^2 = 0. \quad (1.12) \]
The automorphism group in this case is the level 2 principal congruence subgroup
\[ \Gamma(2) := \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}), \; g \equiv I \; (\text{mod } 2) \right\}. \quad (1.13) \]
The associated Fuchsian equation is the hypergeometric equation
\[ \lambda(1-\lambda)\frac{d^2y}{d\lambda^2} + (1-2\lambda)\frac{dy}{d\lambda} - \frac{1}{4}y = 0, \quad (1.14) \]
for which a basis of solutions is given by the elliptic $\frac{1}{2}$-period integrals
\[ \kappa = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right) \quad (1.15a) \]
\[ i\kappa' = \int_1^\frac{1}{k} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \frac{i\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; 1-k^2\right) \quad (1.15b) \]
with
\[ \tau = \frac{i\kappa'}{\kappa}. \quad (1.16) \]

An equivalent way of representing the Schwarzian equation (1.12), due to Brioschi [B], is to introduce the functions
\[ w_1 := \frac{1}{2} \frac{d}{d\tau} \ln \frac{\lambda'}{\lambda}, \quad w_2 := \frac{1}{2} \frac{d}{d\tau} \ln \frac{\lambda'}{(\lambda - 1)}, \quad w_3 := \frac{1}{2} \frac{d}{d\tau} \ln \frac{\lambda'}{\lambda(\lambda - 1)}. \quad (1.17) \]
These satisfy the system
\[ w_1' = w_1(w_2 + w_3) - w_2w_3 \]
\[ w_2' = w_2(w_1 + w_3) - w_1w_3 \]
\[ w_3' = w_3(w_1 + w_2) - w_1w_2, \quad (1.18) \]
introduced by Darboux [Da] in his study of orthogonal coordinate systems and solved by Halphen [Ha] with the help of hypergeometric functions. This system is referred to in [T, O1, O2] as the Halphen equations. The general solution [Ha] is obtained by applying an arbitrary \( SL(2, \mathbb{C}) \) transformation (1.1) to the independent variable, while transforming \((2w_1, 2w_2, 2w_3)\) as affine connections

\[
 w_i \rightarrow \frac{1}{(c \tau + d)^2} w_i \circ T - \frac{c}{c\tau + d}.
\]  

(1.19)

The system (1.18) has appeared in a number of recent contexts, including: solutions of reduced self–dual Einstein [GP] and Yang–Mills [CAC, T] equations, the 2–monopole dynamical equations [AH], and the WDVV equations of topological field theory [Du].

A symmetrized version may be obtained by considering the symmetric invariants formed from \((w_1, w_2, w_3)\), such as

\[
 W := 2(w_1 + w_2 + w_3),
\]  

(1.20)

which satisfies the Chazy equation [C1,C2]

\[
 W''' = 2WW'' - 3W'^2.
\]  

(1.21)

The corresponding modular function is obtained by noting that, although the action of the full modular group \( \Gamma := \text{PSL}(2, \mathbb{Z}) \) upon \( \lambda \) does not leave it invariant, the quotient \( \Gamma/\Gamma(2) \) by the normal subgroup \( \Gamma(2) \) is just the symmetric group \( S_3 \), acting on \( \lambda \) as the “group of anharmonic ratios” [Hi2]

\[
 \lambda \mapsto \lambda, \frac{1}{\lambda}, 1 - \lambda, \frac{1}{1 - \lambda}, \frac{\lambda}{\lambda - 1}, \frac{\lambda - 1}{\lambda}.
\]  

(1.22)

These transformations correspond, respectively, to the following modular transformations

\[
 \tau \mapsto \tau, \frac{\tau}{1 + \tau}, -\frac{1}{\tau}, -\frac{1}{1 + \tau}, \tau + 1, \frac{\tau - 1}{\tau}.
\]  

(1.23)

Symmetrization amounts to forming the ring of invariants, which has a single generator that may be taken as Klein’s \( J \)-function

\[
 J := \frac{4(\lambda^2 - \lambda + 1)^3}{27\lambda^2(\lambda - 1)^2},
\]  

(1.24)

whose automorphism group is \( \Gamma \). The corresponding Fuchsian equation is the hypergeometric equation

\[
 J(1 - J) \frac{d^2y}{dJ^2} + \left( \frac{2}{3} - \frac{7}{6}J \right) \frac{dy}{dJ} - \frac{1}{144} y = 0.
\]  

(1.25)
Modular solutions to generalized Halphen equations

The associated Schwarzian equation,

{J, τ} + \frac{36J^2 - 41J + 32}{72J^2(J - 1)^2} J'' = 0, \tag{1.26}

was first studied by Dedekind [De] in relation to the modular properties of J and \lambda. The resulting solution of the Chazy equation (1.21) is given [C1, T] by

\[ W = \frac{1}{2} \frac{d}{d\tau} \ln \frac{J^6}{J^4(J - 1)^3} = \frac{d}{d\tau} \ln \frac{\lambda^3}{\lambda^2(\lambda - 1)^2} = \frac{1}{2} \frac{d}{d\tau} \ln \Delta, \tag{1.27} \]

where \Delta is the modular discriminant.

If we introduce analogous Halphen variables for the J-function

\[ W_1 := \frac{1}{2} \frac{d}{d\tau} \ln \frac{J'}{J}, \quad W_2 := \frac{1}{2} \frac{d}{d\tau} \ln \frac{J'}{J - 1}, \quad W_3 := \frac{1}{2} \frac{d}{d\tau} \ln \frac{J'}{J(J - 1)}, \tag{1.28} \]

these satisfy the system

\[ \begin{align*}
W_1' &= \frac{1}{4} W_1^2 + \frac{1}{9} W_2 + \frac{23}{36} W_1 W_2 + \frac{31}{36} W_1 W_3 - \frac{31}{36} W_2 W_3 \\
W_2' &= \frac{1}{4} W_1^2 + \frac{1}{9} W_2^2 + \frac{23}{36} W_1 W_2 - \frac{41}{36} W_1 W_3 + \frac{41}{36} W_2 W_3 \\
W_3' &= \frac{1}{4} W_1^2 + \frac{1}{9} W_2^2 - \frac{49}{36} W_1 W_2 + \frac{31}{36} W_1 W_3 + \frac{41}{36} W_2 W_3. \tag{1.29} \end{align*} \]

From the transformations (1.22), it follows that the symmetrizing group S3 acts upon the Halphen variables (w1, w2, w3) by permutations. Differentiating the identity (1.24), we deduce that the quantities (W1, W2, W3) are related to (w1, w2, w3) as follows

\[ \begin{align*}
3W_1 + 2W_2 + W_3 &= 2\sigma_1 = W \\
W_1 - W_3 &= -\frac{4(\sigma_1^2 - 3\sigma_2)^2}{2\sigma_1^2 - 9\sigma_1\sigma_2 + 27\sigma_3} \tag{1.30a} \\
W_2 - W_3 &= -\frac{2\sigma_1^3 - 9\sigma_1\sigma_2 + 27\sigma_3}{\sigma_1^2 - 3\sigma_2} \tag{1.30b} \\
\end{align*} \]

where

\[ \sigma_1 := w_1 + w_2 + w_3, \quad \sigma_2 := w_1 w_2 + w_2 w_3 + w_3 w_1, \quad \sigma_3 := w_1 w_2 w_3 \tag{1.31} \]

are the elementary symmetric invariants. In terms of these, the system (1.29) reduces to

\[ \begin{align*}
\sigma_1' &= \sigma_2 \\
\sigma_2' &= 6\sigma_3 \\
\sigma_3' &= 4\sigma_1\sigma_3 - \sigma_2^2 \tag{1.32} \end{align*} \]
(cf. [O1]), which is equivalent to the Chazy equation (1.21).

The modular solutions to these systems may also be represented as logarithmic derivatives of the null theta functions \( \vartheta_2(\tau), \vartheta_3(\tau), \vartheta_4(\tau) \), (cf. Appendix), in terms of which the elliptic modular function \( \lambda(\tau) \) has the well known representation [WW]

\[
\lambda(\tau) = \frac{\vartheta_2^4}{\vartheta_3^4} = 1 - \frac{\vartheta_4^4}{\vartheta_3^4},
\]

(1.33)

From the differential identities satisfied by \( \vartheta_1, \vartheta_2, \vartheta_3 \) (cf. Appendix, eqs. (A.19)–(A.21) and [O1]), it follows that

\[
\frac{\lambda'}{1-\lambda} = i\pi \vartheta_2^4, \quad \frac{\lambda'}{\lambda} = i\pi \vartheta_4^4, \quad \frac{\lambda'}{\lambda(1-\lambda)} = i\pi \vartheta_3^4,
\]

(1.34)

so the solution of the Halphen system may be expressed as

\[
w_1 = 2 \frac{d}{d\tau} \ln \vartheta_4, \quad w_2 = 2 \frac{d}{d\tau} \ln \vartheta_2, \quad w_3 = 2 \frac{d}{d\tau} \ln \vartheta_3.
\]

(1.35)

Using (1.24), the \( J \)-function may also be expressed rationally in terms of the theta functions

\[
J = \frac{(\vartheta_2^8 + \vartheta_3^8 + \vartheta_4^8)^3}{54\vartheta_2^8\vartheta_3^8\vartheta_4^8}.
\]

(1.36)

Taking derivatives and applying the same differential identities, the \( J \) Halphen variables \( (W_1, W_2, W_3) \) may be expressed as logarithmic derivatives of rational expressions in the \( \vartheta \)'s, and the symmetrizing relations (1.30a)–(1.30c) interpreted as differential relations satisfied by the theta functions.

The above examples have been generalized by Ohyama [O2] to other classes of second order equations of type (1.3), both Fuchsian and non–Fuchsian. For the Fuchsian case, the rational functions \( R(f) \) may be expressed as

\[
R(f) = \frac{N(f)}{(D(f))^2},
\]

(1.37)

where

\[
D(f) = \prod_{i=1}^{n} (f - a_i)
\]

(1.38)

and \( N(f) \) is a polynomial of degree \( \leq 2n - 2 \). The generalization of the Halphen variables is given by

\[
X_0 := \frac{1}{2} \frac{d}{d\tau} \ln f', \quad X_i := \frac{1}{2} \frac{d}{d\tau} \ln \frac{f'}{(f - a_i)^2}, \quad i = 1, \ldots n.
\]

(1.39)
For \( n > 2 \), the \( X_i \)'s again satisfy a system of first order equations analogous to (1.18) and (1.29), with suitably defined quadratic forms on the right, but they are also subject to a set of \( n - 2 \) quadratic constraints fixing the anharmonic ratios between any distinct set of four of them. The underlying phase space is therefore still \( 3c \)-dimensional and may, for generic initial conditions, be identified with the \( SL(2, \mathbb{C}) \) group manifold. Besides the above two cases, other explicit solutions were given in terms of modular functions by Ohyama [O3] for a system in which the corresponding Fuchsian operator has four regular singular points and the automorphism group is \( \Gamma(3) \).

For the case of Fuchsian operators with three regular singular points the associated differential systems were already studied by Halphen [Ha]. For the general hypergeometric equation

\[
f(1 - f) \frac{d^2 y}{df^2} + (c - (a + b + 1)f) \frac{dy}{df} - aby = 0,
\]

the corresponding rational function is

\[
R(f) = \frac{1}{4} \left( \frac{1 - \lambda^2}{f^2} + \frac{1 - \mu^2}{(f-1)^2} + \frac{\lambda^2 + \mu^2 - \nu^2 - 1}{f(f-1)} \right),
\]

where

\[
\lambda := 1 - c, \quad \mu := c - a - b, \quad \nu := b - a
\]

are the exponents at the regular singular points \((0,1,\infty)\), which determine the angles \((\lambda \pi, \mu \pi, \nu \pi)\) at the vertices. Introducing the variables

\[
W_1 := \frac{1}{2} \frac{d}{d\tau} \ln \frac{f'}{f}, \quad W_2 := \frac{1}{2} \frac{d}{d\tau} \ln \frac{f'}{(f-1)}, \quad W_3 := \frac{1}{2} \frac{d}{d\tau} \ln \frac{f'}{f(f-1)},
\]

and viewing them as functions of the ratio

\[
\tau = \frac{y_1}{y_2}
\]

of two independent solutions of (1.40), these satisfy the general Halphen system

\[
\begin{align}
W_1'' &= \mu^2 W_1^2 + \lambda^2 W_2^2 + \nu^2 W_3^2 \\
&\quad + (\nu^2 - \lambda^2 - \mu^2 + 1)W_1W_2 + (\lambda^2 - \mu^2 - \nu^2 + 1)W_1W_3 + (\mu^2 - \lambda^2 - \nu^2 - 1)W_2W_3 \\
W_2'' &= \mu^2 W_1^2 + \lambda^2 W_2^2 + \nu^2 W_3^2 \\
&\quad + (\nu^2 - \lambda^2 - \mu^2 + 1)W_1W_2 + (\lambda^2 - \mu^2 - \nu^2 + 1)W_1W_3 + (\mu^2 - \lambda^2 - \nu^2 - 1)W_2W_3 \\
W_3'' &= \mu^2 W_1^2 + \lambda^2 W_2^2 + \nu^2 W_3^2 \\
&\quad + (\nu^2 - \lambda^2 - \mu^2 + 1)W_1W_2 + (\lambda^2 - \mu^2 - \nu^2 + 1)W_1W_3 + (\mu^2 - \lambda^2 - \nu^2 + 1)W_2W_3.
\end{align}
\]
Although this gives a construction, in principle, of the general solution to such systems, the functional inversion involved is not in general globally well defined. Only if the parameters \((\lambda, \mu, \nu)\) are of the form \((\frac{1}{m}, \frac{1}{n}, \frac{1}{p})\), where \(m, n\) and \(p\) are integers or \(\infty\), is there a tessellation of the upper half plane by (triangular) fundamental regions, and even then we cannot in general say much about the explicit form of the inverse function. However, for a small number of special cases which are described below, these again turn out to be modular functions that can be given explicit rational expressions in terms of null \(\vartheta\)–functions or the Dedekind \(\eta\)-function.

Remarkably, the \(J\)-function also appears in connection with “Monstrous Moonshine” [CN], in that the \(q = e^{2\pi i \tau}\)–coefficients of the Fourier expansion of \(j := 12^3 J - 744\) are the dimensions of representations of the Monster simple sporadic group; i.e., the traces on the identity element. Like \(j\), the \(q\)–series with coefficients given by traces on the other conjugacy classes turn out to also be \textit{Hauptmoduls}; i.e., each is the generator of a field of meromorphic functions of genus 0. An alternative characterization of the elliptic modular function \(j\) is the fact that the principal part of its \(q\)–series is \(q^{-1}\), together with its behaviour under the action of the Hecke operator; namely,

\[
T_n(j) = \frac{1}{n} \sum_{0 \leq b < d} j\left(\frac{a\tau + b}{d}\right) = P_{n,j}(j), \quad \forall n \geq 1, \tag{1.45}
\]

where \(P_{n,f}(f)\) is the Faber polynomial [Fa] of degree \(n\). The functions appearing as such character generators are included in a larger class of Hauptmoduls, the \textit{replicable functions} [CN, FMN], which are constructed using a generalization of the Hecke operator. Their automorphism groups all contain a finite index subgroup of the type

\[
\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}), \ c \equiv 0 \bmod N \right\} \tag{1.46}
\]

(and hence their automorphism groups are all commensurable with \(\Gamma\).) The maximal such \(N\) is referred to as the level of the function. Like all Hauptmoduls, they also satisfy Schwarzian equations of the type (1.10) with rational \(R(f)\) of the form (1.37), and hence each has an associated Fuchsian operator of the form (1.3) whose projectivized monodromy group is the automorphism group of the function.

In the present work, we study the differential equations satisfied by such Hauptmoduls and their corresponding generalized Halphen systems. In particular, we consider the Hauptmoduls known as \textit{triangular functions}, for which the tessellation of the upper half-plane is associated with a triangular domain and a corresponding hyperbolic group generated by reflections in its sides. The associated Fuchsian equations are therefore of hypergeometric type. Since the automorphism groups appearing are conjugate to subgroups of \(PGL(2, Q)\), it follows that only
crystallographic angles \((0, \pi, \pi/2, \pi/3, \pi/4, \pi/6)\) occur at the vertices. For the \(j\)-function, the triangle has a cusp at \(i\infty\) and angles of \(\pi/3\) at \(e^{i\pi/3} = J^{-1}(0)\) and \(\pi/2\) at \(i = J^{-1}(1)\), so this is denoted \((\frac{1}{3}, \frac{1}{2}, 0)\). Since the functions considered have \(q\)-expansions with principal part \(q^{-1}\) at \(q = 0\), there must be a cusp at \(\tau = i\infty\), and hence the angle \(\nu\) always vanishes. There are nine such triangular arithmetic groups \([Ta]\) with angles given by: \((0, 0, 0), (1/2, 0, 0), (1/3, 0, 0), (1/3, 1/2, 0), (1/4, 1/2, 0), (1/6, 1/2, 0), (1/3, 1/3, 0), (1/4, 1/4, 0), (1/6, 1/6, 0)\), and these are precisely the ones appearing in the list \([CN]\) of replicable functions with integral \(q\)-series coefficients.

In the following sections, several examples will be given of differential systems of generalized Halphen type whose solutions are completely determined in terms of replicable functions. These will include all the arithmetic triangular groups, and some further cases for which there are four singular points in the associated Fuchsian equation (as well as one in which there are 26). Each such function has a fundamental domain bounded by circular arcs centered on the real axis, with a cusp at \(i\infty\) and one of the crystallographic angles \((0, \pi/2, \pi/3, \pi/4, \pi/6)\) at each remaining vertex. By construction, each has a normalized \(q\)-series of the form \([FMN]\)

\[
F(q) = \frac{1}{q} + \sum_{n=1}^{\infty} a_n q^n, \quad q := e^{2i\pi\tau},
\]

with integer coefficients \(a_n\). In the case of triangular functions, in order to relate these to solutions of the hypergeometric equations, another standard normalization is chosen by applying an affine transformation

\[
F = \alpha f + \beta,
\]

with constants \(\alpha\) and \(\beta\) chosen to assign the values \((0, 1, \infty)\) to \(f(\tau)\) at the vertices of the fundamental triangle. Since all these functions are known explicitly \([FMN]\) in terms of the Dedekind eta function \(\eta(\tau)\) or the null theta functions \(\vartheta_\alpha(\tau)\), we are able to give explicit solutions of the corresponding differential systems in terms of logarithmic derivatives of the \(\eta\)- or \(\vartheta\)-functions.

Certain of these functions are expressible in terms of others through a rational map, possibly composed with a transformation of the type \(\tau \mapsto \tau/n\) (where \(n\) has prime factors 2 or 3). This implies that there is a subgroup relation between their respective automorphism groups, and also that the differential system associated to the larger automorphism group is a symmetrization of the one associated to the subgroup. An example of this was seen in the pair of elliptic modular functions \((J, \lambda)\), where the rational expression (1.24) implies the subgroup relation \(\Gamma(2) \subset \Gamma\) and the relations (1.30a)–(1.30c) between the corresponding Halphen variables. In general, if \(f\) satisfies the Schwarzian equation (1.10) and \(g\) satisfies

\[
\{g, \tau\} + 2\tilde{R}(g)g'^2 = 0,
\]

\[
(1.49)
\]
then if \( f \) can be expressed as a function of \( g \)

\[
f = Q(g),
\]

this will satisfy the Schwarzian equation

\[
\{Q, g\} + 2R(Q(g))Q'^2 = 2\tilde{R}(g).
\]

Equivalently, if \( Q(g) \) satisfies (1.51) and \( y(f) \) is a solution to (1.3), the composite function \( \tilde{y}(g) = (Q')^{-\frac{1}{2}}y(Q(g)) \) will satisfy

\[
\frac{d^2\tilde{y}}{dg^2} + \tilde{R}(g)\tilde{y} = 0.
\]

By choosing both \( R(f) \) and \( \tilde{R}(g) \) of the hypergeometric form (1.41), Goursat [Go] found rational transformations of degree \( \leq 4 \) relating various classes of hypergeometric functions. The relation (1.24) may be viewed as a composition of two such transformations, of degrees 2 and 3, connecting the hypergeometric equations of types \( (a, b, c) = \left(\frac{1}{12}, \frac{1}{12}, \frac{2}{3}\right) \) and \( \left(\frac{1}{2}, \frac{1}{2}, 1\right) \). In section 2b, transformations of Goursat’s type are used to relate the associated arithmetic triangular functions, and in section 3b, similar transformations of degree \( \leq 4 \) are given relating these to certain 4–vertex cases. Such relations are determined by finding appropriately normalized rational solutions \( f = Q(g) \) of the Schwarzian equation (1.51) or, equivalently, by equating the \( q \)-series of both sides up to a finite number of terms.

In all the cases treated, we give a tabular summary of the properties of the modular functions \( f \), including explicit rational expressions for them in terms of the \( \eta \)- and \( \vartheta \)-functions. The general solutions to the Halphen systems (1.44) and their multivariable generalizations are thus determined as logarithmic derivatives of \( \eta \)- or \( \vartheta \)-functions. For the hypergeometric cases (section 2a), we list the exponents \( (\lambda, \mu, \nu) \) determining the Halphen type system (1.44), the corresponding hypergeometric parameters \( (a, b, c) \) and the group elements fixing the vertices. For the four vertex cases (section 3a), we give the locations \( (a_1, a_2, a_3) \) of the finite poles of the rational function \( R(f) \); i.e., the values of \( f \) at the vertices, the group elements stabilizing the vertices and a representative of an associated linear class of quadratic forms in the dynamical variables that serves to uniquely define the corresponding constrained 4–variable system, as well as the rational function \( R(f) \) entering eqs. (1.3) and (1.10). Section 4 contains a discussion of the \( n + 1 \) pole case and includes an example with 26 regular singular points, which is the largest number that appears, and automorphism group of level 72. In the general \( n + 1 \) pole case, it is shown that Ohyama’s quadratically constrained \( n + 1 \) variable dynamical system [O2] is equivalent to a 3–variable system defined on the \( SL(2, C) \) group manifold.

For both the triangular cases (section 2b), and certain four vertex cases (section 3b), we examine the various rational maps of degree \( \leq 4 \) relating the different Hauptmoduls, together
with the corresponding relations between the associated automorphism groups. These maps define symmetrizations for the function fields and the corresponding differential systems. Just as in the $J – \lambda$ case (1.30a)–(1.30c), the generalized Halphen variables for the more symmetrical systems consist of rational invariants formed from the less symmetrical one. Not all cases correspond to normal subgroups, however, so the fibre of the quotient map is not necessarily a Galois group, but rather the quotient of two Galois groups, corresponding to a field extension whose automorphism group is the largest subgroup normal in both the groups of the pair of Hauptmoduls.

2. Triangular replicable functions and symmetrization maps

2a. Solutions of Halphen type systems.

In Table 1, we list all cases where the underlying Fuchsian equation is of hypergeometric type (1.40). The Halphen system is always of the form (1.44), and therefore we list only the values of the constants $(a, b, c)$ and $(\lambda, \mu, \nu)$ characterizing these systems. The notation used to designate the corresponding automorphism groups and function fields is that of [FMN]. The triangular cases appearing are denoted: 1A, 2A, 3A, 2B, 3B, 4C, 2a, 4a and 6a, where the integer $N$ denotes the level. (The upper case letters denote functions that are character generators for the Monster.) There are further cases of replicable functions having triangular automorphism groups, but these are all equivalent to one of the above under an affine transformation that just relocates the two finite poles of the Fuchsian equation (1.3), together with a Möbius transformation (1.1) in $\tau$, and therefore they are not listed separately. The $\eta$-function formulae listed for the functions $f(\tau)$ normalized to take values $(0, 1, \infty)$ at the vertices are obtained from those given in [FMN] for the functions $F$ normalized as in (1.47) by applying the appropriate affine transformation (1.48). The $\vartheta$-function formulae are deduced either from standard identities relating the $\eta$- and $\vartheta$-functions (cf. Appendix), or by applying the rational maps relating the different cases listed in sections 2(b) and 3(b). The automorphism group, being the projective image of the monodromy group of the associated hypergeometric equation, has three matrix generators $\rho_0, \rho_1, \rho_\infty$ stabilizing the vertices satisfying the relation

$$\rho_\infty \rho_1 \rho_0 = mI, \quad m \in \mathbb{R} \quad (2.1)$$

for some $m \neq 0$. Representatives of the projective class of these generators may always be chosen to have integer entries, though not necessarily unit determinant, so the constant $m$ in (2.1) is an integer. In all cases the generator $\rho_\infty$ stabilizing the cusp at $i\infty$ is

$$\rho_\infty = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \quad (2.2)$$

and the two remaining elements $\rho_0, \rho_1$ are given in the table.
Table 1. Triangular Replicable Functions

| Name | \((a, b, c)\) | \((\lambda, \mu, \nu)\) | \(\rho_0\) | \(\rho_1\) | \(F\) | \(f(\tau)\) |
|------|-------------|----------------|--------|--------|------|-------|
| \(1A\) \sim \Gamma | \((\frac{1}{12}, \frac{1}{12}, \frac{1}{2})\) | \((\frac{1}{3}, 1)\) | \((-1, 1)\) | \((-1, 0)\) | 1728\(f - 744\) | \(J = \frac{(\phi_1^2 + \phi_2^2 + \phi_3^2)^3}{54\phi_1^2\phi_2^2\phi_3^2}\) |
| \(2A\) \sim \Gamma_0(2) | \((\frac{1}{8}, \frac{3}{4})\) | \((\frac{1}{4}, \frac{1}{2})\) | \((-1, -2)\) | \((-1, 0)\) | 256\(f - 104\) | \(\left(\eta^{12}(\tau) + 27\eta^{12}(3\tau)\right)^2 \over 108\eta^{12}(\tau)\eta^{12}(3\tau)\) |
| \(3A\) \sim \Gamma_0(3) | \((\frac{1}{6}, \frac{5}{6})\) | \((\frac{1}{6}, \frac{1}{2})\) | \((-1, 3)\) | \((-1, 0)\) | 108\(f - 42\) | \(1 + \frac{1}{64}(\eta(\tau)\eta(2\tau))^2 = \frac{\phi_1^2(\tau) + \phi_1^2(4\tau)}{\phi_2^2(\tau)}\) |
| \(2B\) \sim \Gamma_0(2) | \((\frac{1}{4}, \frac{1}{4}, \frac{1}{2})\) | \((\frac{1}{2}, 0, 0)\) | \((-1, -1)\) | \((-1, 0)\) | 64\(f - 40\) | \(1 + \frac{1}{27}(\eta(\tau)\eta(3\tau))^2\) |
| \(3B\) \sim \Gamma_0(3) | \((\frac{1}{3}, \frac{2}{3}, \frac{2}{3})\) | \((\frac{1}{3}, 0, 0)\) | \((-1, 3)\) | \((-1, 0)\) | 27\(f - 15\) | \(1 + \frac{1}{27}(\eta(\tau)\eta(3\tau))^2\) |
| \(4C^*\) \sim \Gamma_0(4) | \((\frac{1}{2}, \frac{1}{2}, 1)\) | \((0, 0, 0)\) | \((-1, 4)\) | \((-1, 0)\) | 16\(f - 8\) | \(\frac{1}{\lambda(2\tau)} \frac{\phi_1^2(2\tau)}{\phi_2^2(2\tau)} = 1 + \frac{1}{18}(\eta(\tau)\eta(4\tau))^2\) |
| \(2a\) | \((\frac{1}{6}, \frac{5}{6}, \frac{1}{3})\) | \((\frac{1}{6}, \frac{1}{3})\) | \((-3, 4)\) | \((-1, 0)\) | 24\(\sqrt{3}i(2f - 1)\) | \(\frac{\sqrt{3}(\phi_1^2(2\tau) - \phi_2^2(2\tau))}{9\phi_1^2(2\tau)\phi_2^2(2\tau)\phi_3^2(2\tau)}\) |
| \(4a\) | \((\frac{1}{4}, \frac{3}{4}, \frac{3}{4})\) | \((\frac{1}{4}, \frac{1}{4})\) | \((-5, 8)\) | \((-1, 0)\) | \(-16i(2f - 1)\) | \(-\frac{i(\phi_1^2(2\tau) + i\phi_2^2(2\tau))^4}{8\phi_1^2(2\tau)\phi_2^2(2\tau)\phi_3^2(2\tau)}\) |
| \(6a\) | \((\frac{1}{3}, \frac{5}{6}, \frac{1}{6})\) | \((\frac{1}{6}, \frac{1}{6})\) | \((-7, 12)\) | \((-1, 0)\) | 6\(\sqrt{3}i(2f - 1)\) | \(-\frac{\sqrt{3}(\phi_1^2(2\tau) + 3\sqrt{3}\eta^6(6\tau))}{36\eta^6(2\tau)\eta^6(6\tau)}\) |

*Remark: Case 4C has the same fundamental domain as the elliptic modular function \(\lambda(\tau)\), but with respect to the variable \(\tau/2\). The automorphism group is \(\Gamma_0(4)\), which is conjugate to \(\Gamma(2)\) under the map \(T \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} T \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\).*

2b. Rational maps between triangular cases.

In the following, we catalogue the rational maps, analogous to (1.24), relating the replicable functions listed above. Each map may be seen as following from an identity relating different hypergeometric functions through a rational change in the independent variable. Such identities were derived systematically by Goursat [Go], and are labelled here according to the numbering scheme of [Go]. For each case, we have a pair \((f, g)\) of function field generators, with \(f\) a rational function of \(g\), normalized consistently with the hypergeometric equations, so that the vertices of the fundamental triangle are mapped to \((0, 1, \infty)\). In order to preserve the standard ordering with angles \((\lambda, \mu, \nu)\) at the vertices with values \((0, 1, \infty)\), we make use
of the fact that $F(a, b, c; z)$ and $F(a, b, 1 - c + a + b; 1 - z)$ satisfy the same equation and compose, in some cases, with the maps $f \mapsto 1 - f$ or $g \mapsto 1 - g$. The corresponding pair of functions, normalized as in (1.47), is denoted $(F, G)$ and the relevant map is denoted $G \mapsto F$.

In order to preserve the form (1.47) of the $q$–series, it is necessary in some cases to compose the rational map with a map $\tau \mapsto \tau^2$ or $\tau \mapsto \tau^3$ on the independent variable. In these cases, the composed map is denoted $G' \mapsto F$. By taking derivatives of the corresponding rational map relating $f$ to $g$, we also obtain relations between the Halphen variables, such as that given in the case of the pair $(J, \lambda)$ by (1.30a)–(1.30c).

To fix notation, the Halphen variables for the two cases will be denoted

$$W_1 := \frac{1}{2} \frac{d}{d\tau} \ln \frac{f'}{f}, \quad W_2 := \frac{1}{2} \frac{d}{d\tau} \ln \frac{f'}{(f - 1)}, \quad W_3 := \frac{1}{2} \frac{d}{d\tau} \ln \frac{f'}{f(f - 1)}$$

for the first element of the pair $(f, g)$, and

$$w_1 := \frac{1}{2} \frac{d}{d\tau} \ln \frac{g'}{g}, \quad w_2 := \frac{1}{2} \frac{d}{d\tau} \ln \frac{g'}{(g - 1)}, \quad w_3 := \frac{1}{2} \frac{d}{d\tau} \ln \frac{g'}{g(g - 1)}$$

for the second. Since $f$ is in the function field generated by $g$, it is $\mathfrak{G}_g$–invariant; i.e., $\mathfrak{G}_g \subset \mathfrak{G}_f$. When $\mathfrak{G}_g$ is a normal subgroup, $\mathfrak{G}_f$ acts on $g$ by rational transformations, and on the Halphen variables $(w_1, w_2, w_3)$ linearly. Since this defines a finite group action, the ring of invariants in the $(w_1, w_2, w_3)$ variables is generated by 3 elements, and the $(W_1, W_2, W_3)$ are uniquely determined as rational expressions of these. In all the cases listed below we give the hypergeometric identity underlying the rational maps between the modular functions and the maps, expressed in terms of both the $(f, g)$ normalizations and the $(F, G)$ ones. When $\mathfrak{G}_g \subset \mathfrak{G}_f$ is a normal subgroup, we give the symmetrizing group $\mathfrak{S}_f^g := \mathfrak{G}_f/\mathfrak{G}_g$ (i.e., the Galois group of the field extension), the action of $\mathfrak{G}_f$ upon $g$ (analogous to the $\mathfrak{S}_3$ action (1.22) on the elliptic modular function $\lambda(\tau)$), the elementary symmetric invariants under this action, and the linear relations determining $(W_1, W_2, W_3)$ in terms of the latter following from differentiation of the rational map relating $f$ to $g$. When $\mathfrak{G}_g \subset \mathfrak{G}_f$ is not normal, we list a symmetrization quotient $\mathfrak{S}_f^h/\mathfrak{S}_g^h$, where $\mathfrak{S}_f^h$ and $\mathfrak{S}_g^h$ are the symmetrizing groups of the smallest field extension, with generator $h$, that is Galois for both. The symmetrizing groups all turn out to be either a symmetric group $\mathfrak{S}_n$, an alternating group $\mathfrak{A}_n$ or a cyclic group $\mathfrak{Z}_n$.

We note that in each case, there exists at least one relation involving a linear symmetric invariant of the form

$$PW_1 + QW_2 + RW_3 = pw_1 + qw_2 + rw_3$$

where $(P, Q, R, p, q, r)$ are all integers, and

$$P + Q + R = p + q + r =: k.$$
This is deduced from a relation of the form

\[
\frac{f^{p+q+r}}{f^{p+q}} = M \frac{g^{p+q+r}}{g^{p+q+r}},
\]

(2.7)

where \((P, Q, p, q, r)\) are integers, which may always be found by simply choosing \((P, Q, R)\) so that the left hand side of (2.7) has no singularities at the vertices. The resulting quantity is an analytic form of weight \(2k\), and its logarithmic derivative, given by (2.5), satisfies a third order equation analogous to the Chazy equation (1.21).

\(2a' \mapsto 1A:\)

Hypergeometric identity (Goursat (138)):

\[
F \left( \frac{1}{12}, \frac{1}{12}; \frac{2}{3}; 4x(1-x) \right) = F \left( \frac{1}{6}, \frac{1}{6}; \frac{2}{3}; x \right)
\]

(2.8)

Rational map:

\[
F = 984 + G^2 \circ (\tau \mapsto \tau/2),
\]

\[
f = 4g(1-g) \circ (\tau \mapsto \tau/2)
\]

(2.9)

Symmetrizing group: \(S_{1A}^{2a'} = \mathbb{Z}_2\)

\[
\tau \mapsto -\frac{1}{\tau}, \quad g(\tau/2) \mapsto 1 - g(\tau/2), \quad (w_1, w_2, w_3) \mapsto (w_2, w_1, w_3)
\]

(2.10)

Polynomial invariants:

\[
\sigma_1 := w_1 + w_2, \quad \sigma_2 := w_1w_2, \quad \Sigma_1 := w_3
\]

(2.11)

Powers and coefficient in (2.7):

\[
(P, Q, R; p, q, r; M) = (3, 2, 1; 2, 2, 2; -\frac{1}{4})
\]

(2.12)

Relation between the Halphen variables:

\[
W_1 + W_3 = \Sigma_1 \circ (\tau \mapsto \tau/2)
\]

\[
W_1 - W_3 = \frac{2(\sigma_2 - \Sigma_1 \sigma_1 + \Sigma_1^2)}{\sigma_1 - 2\Sigma_1} \circ (\tau \mapsto \tau/2)
\]

(2.13)

\[
W_2 - W_3 = \frac{1}{2} \sigma_1 - \Sigma_1 \circ (\tau \mapsto \tau/2)
\]

\(4a' \mapsto 2A:\)

Hypergeometric identity (Goursat (138)):

\[
F \left( \frac{1}{8}, \frac{1}{8}; \frac{3}{4}; 4x(1-x) \right) = F \left( \frac{1}{4}, \frac{1}{4}; \frac{3}{4}; x \right)
\]

(2.14)
Modular solutions to generalized Halphen equations

Rational map:
\[
F = 152 + G^2 \circ (\tau \mapsto \tau/2),
\]
\[
f = 4g(1-g) \circ (\tau \mapsto \tau/2)
\]  
(2.15)

Symmetrizing group: \(S_{2A}^{4a'} = \mathbb{Z}_2:\)
\[
\tau \mapsto -\frac{1}{2\tau}, \quad g(\tau/2) \mapsto 1 - g(\tau/2), \quad (w_1, w_2, w_3) \mapsto (w_2, w_1, w_3)
\]  
(2.16)

Polynomial invariants:
\[
\sigma_1 := w_1 + w_2, \quad \sigma_2 := w_1w_2, \quad \Sigma_1 := w_3
\]  
(2.17)

Powers and coefficient in (2.7):
\[
(P, Q, R; p, q, r; M) = (2, 1, 1; 1, 1, 2; -\frac{1}{4})
\]  
(2.18)

Relation between the Halphen variables:
\[
W_1 + W_3 = \Sigma_1 \circ (\tau \mapsto \tau/2)
\]
\[
W_1 - W_3 = \frac{2(\sigma_2 - \Sigma_1\sigma_1 + \Sigma_1^2)}{\sigma_1 - 2\Sigma_1} \circ (\tau \mapsto \tau/2)
\]
\[
W_2 - W_3 = \frac{1}{2}\sigma_1 - \Sigma_1 \circ (\tau \mapsto \tau/2)
\]  
(2.19)

6a' \mapsto 3A:
Hypergeometric identity (Goursat (138)):
\[
F\left(\frac{1}{6}; \frac{1}{6}, \frac{5}{6}; 4x(1-x)\right) = F\left(\frac{1}{3}; \frac{1}{3}, \frac{5}{6}; x\right)
\]  
(2.20)

Rational map:
\[
F = 66 + G^2 \circ (\tau \mapsto \tau/2),
\]
\[
f = 4g(1-g) \circ (\tau \mapsto \tau/2)
\]  
(2.21)

Symmetrizing group: \(S_{3A}^{6a'} = \mathbb{Z}_2:\)
\[
\tau \mapsto -\frac{1}{3\tau}, \quad g(\tau/2) \mapsto 1 - g(\tau/2), \quad (w_1, w_2, w_3) \mapsto (w_2, w_1, w_3)
\]  
(2.22)

Polynomial invariants:
\[
\sigma_1 := w_1 + w_2, \quad \sigma_2 := w_1w_2, \quad \Sigma_1 := w_3
\]  
(2.23)
Powers and coefficient in (2.7):

\[
(P, Q, R; p, q, r; M) = \left(3, 1, 2; 1, 1, 4; \frac{1}{16}\right) \tag{2.24}
\]

Relation between the Halphen variables:

\[
W_1 + W_3 = \Sigma_1 \circ (\tau \mapsto \tau/2)
\]

\[
W_1 - W_3 = \frac{2(\sigma_2 - \Sigma_1 \sigma_1 + \Sigma_1^2)}{\sigma_1 - 2\Sigma_1} \circ (\tau \mapsto \tau/2) \tag{2.25}
\]

\[
W_2 - W_3 = \frac{1}{2} \sigma_1 - \Sigma_1 \circ (\tau \mapsto \tau/2)
\]

4\(C \mapsto 2\B):  
Hypergeometric identity (Goursat (44)):

\[
(1 - z)^{-\frac{1}{4}} F\left(\frac{1}{4}, \frac{1}{4}, 1; \frac{x^2}{4(x - 1)}\right) = F\left(\frac{1}{2}, \frac{1}{2}, 1; x\right) \tag{2.26}
\]

Rational map:

\[
F = G + \frac{2^8}{G + 8}, \quad f = \frac{(g + 1)^2}{4g} \tag{2.27}
\]

Symmetrizing group:  
\(S_{2B}^{4C} = \Gamma_0(2)/\Gamma_0(4) = \mathbb{Z}_2: \)

\[
\tau \mapsto \frac{\tau}{2\tau + 1}, \quad g \mapsto \frac{1}{g}, \quad (w_1, w_2, w_3) \mapsto (w_1, w_3, w_2) \tag{2.28}
\]

Polynomial invariants:

\[
\sigma_1 := w_2 + w_3, \quad \sigma_2 := w_2w_3, \quad \Sigma_1 := w_1 \tag{2.29}
\]

Powers and coefficient in (2.7):

\[
(P, Q, R; p, q, r; M) = \left(2, 1, -1; 4, -1, -1; \frac{1}{4}\right) \tag{2.30}
\]

Relation between the Halphen variables:

\[
W_1 + W_2 = 2\Sigma_1
\]

\[
W_1 - W_3 = 2\Sigma_1 - \sigma_1 \tag{2.31}
\]

\[
W_2 - W_3 = \frac{\sigma_1^2 - 4\sigma_2}{2\Sigma_1 - \sigma_1}
\]
Modular solutions to generalized Halphen equations

\(4C' \mapsto 2B:\)

Hypergeometric identity (Goursat (138)):

\[
F\left(\frac{1}{4}, \frac{1}{4}; 1; 4x(1-x)\right) = F\left(\frac{1}{2}, \frac{1}{2}; 1; x\right) \tag{2.32}
\]

Rational map:

\[
F = -40 + G^2 \circ (\tau \mapsto \tau/2), \quad f = (2g - 1)^2 \circ (\tau \mapsto \tau/2) \tag{2.33}
\]

Symmetrizing group: \(S^4C'_{2B} = \Gamma_0(2)/\Gamma(2) = \mathbb{Z}_2:\)

\[
\tau \mapsto \tau + 1, \quad g(\tau/2) \mapsto 1 - g(\tau/2), \quad (w_1, w_2, w_3) \mapsto (w_2, w_1, w_3) \tag{2.34}
\]

Polynomial invariants:

\[
\sigma_1 := w_1 + w_2, \quad \sigma_2 := w_1w_2, \quad \Sigma_1 := w_3 \tag{2.35}
\]

Powers and coefficient in (2.7):

\[(P, Q, R; p, q, r; M) = (2, 1, -1; 2, 2, -2; 4) \tag{2.36}
\]

Relation between the Halphen variables:

\[
W_2 + W_3 = \Sigma_1 \circ (\tau \mapsto \tau/2) \\
W_1 - W_3 = \frac{1}{2} \sigma_1 - \Sigma_1 \circ (\tau \mapsto \tau/2) \tag{2.37}
\]

\[
W_2 - W_3 = \frac{2(\sigma_2 - \Sigma_1 \sigma_1 + \Sigma_1^2)}{\sigma_1 - 2\Sigma_1} \circ (\tau \mapsto \tau/2)
\]

\(2B \mapsto 2A:\)

Hypergeometric identity (Goursat (44)):

\[
(1 - z)^{-\frac{3}{4}} F\left(\frac{1}{8}, \frac{1}{8}; \frac{3}{4}; z := \frac{x^2}{4(x-1)}\right) = F\left(\frac{1}{4}, \frac{1}{4}; \frac{1}{2}; x\right) \tag{2.38}
\]

Rational map:

\[
F = G + \frac{2^{12}}{G - 24}, \quad f = \frac{g^2}{4(g - 1)} \tag{2.39}
\]

Symmetrizing group: \(S^2B_{2A} = \Gamma_0^{(2)}(2)/\Gamma_0(2) = \mathbb{Z}_2:\)

\[
\tau \mapsto -\frac{1}{2\tau}, \quad g \mapsto \frac{g}{g - 1}, \quad (w_1, w_2, w_3) \mapsto (w_3, w_2, w_1) \tag{2.40}
\]

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Polynomial invariants:

\[ \sigma_1 := w_1 + w_3, \quad \sigma_2 := w_1 w_3, \quad \Sigma_1 := w_2 \]  \hspace{1cm} (2.41)

Powers and coefficient in (2.7):

\[ (P, Q, R; p, q, r; M) = (2, 1, 1; 1, 2, 1; 4) \]  \hspace{1cm} (2.42)

Relation between the Halphen variables:

\[ W_1 + W_2 = 2\Sigma_1 \]
\[ W_1 - W_3 = \frac{\sigma_1^2 - 4\sigma_2}{2\Sigma_1 - \sigma_1} \]  \hspace{1cm} (2.43)
\[ W_2 - W_3 = -\sigma_1 + 2\Sigma_1 \]

3B \mapsto 3A:

Hypergeometric identity (Goursat (44)):

\[ (1 - z)^{-\frac{8}{3}} F\left(\frac{1}{6}, \frac{1}{6}; \frac{5}{6}; z := \frac{x^2}{4(x - 1)}\right) = F\left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}; x\right) \]  \hspace{1cm} (2.44)

Rational map:

\[ F = G + \frac{3^6}{G - 12}, \quad f = \frac{g^2}{4(g - 1)} \]  \hspace{1cm} (2.45)

Symmetrizing group: \( S_{3B}^{3A} = \mathbb{Z}_2 \):

\[ \tau \mapsto -\frac{1}{3\tau}, \quad g \mapsto \frac{g}{g - 1}, \quad (w_1, w_2, w_3) \mapsto (w_3, w_2, w_1) \]  \hspace{1cm} (2.46)

Polynomial invariants:

\[ \sigma_1 := w_1 + w_3, \quad \sigma_2 := w_1 w_3, \quad \Sigma_1 := w_2 \]  \hspace{1cm} (2.47)

Powers and coefficient in (2.7):

\[ (P, Q, R; p, q, r; M) = (3, 1, 2; 2, 2; 2; 16) \]  \hspace{1cm} (2.48)

Relation between the Halphen variables:

\[ W_1 + W_2 = 2\Sigma_1 \]
\[ W_1 - W_3 = \frac{\sigma_1^2 - 4\sigma_2}{2\Sigma_1 - \sigma_1} \]  \hspace{1cm} (2.49)
\[ W_2 - W_3 = -\sigma_1 + 2\Sigma_1 \]

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2B ↦ 1A:
Hypergeometric identity (Goursat (123)):

\[
F \left( \frac{1}{12}, \frac{1}{12}; \frac{1}{2}; -\frac{x(x-9)^2}{27(x-1)^2} \right) = (1-x)^{\frac{1}{2}} F \left( \frac{1}{4}, \frac{1}{4}; \frac{1}{2}; x \right)
\] (2.50)

Rational map:

\[
F = G + \frac{2^{16}(3G + 184)}{(G - 24)^2},
\]
\[
f = \frac{(g + 3)^3}{27(g - 1)^2}
\] (2.51)

Symmetrization quotient: \( \Gamma/\Gamma_0(2) = S_{1A}^{4C}/S_{2B}^{4C} = S_3/Z_2 \)

Polynomial invariants:

\[
\Sigma_1 := 3w_1 + 2w_2 + w_3, \quad \Sigma_2 := (w_1 - w_3)(4w_1 - 3w_2 - w_3),
\]
\[
\Sigma_3 := (w_1 - w_3)^2(4w_1 - 3w_2 - w_3)
\] (2.52)

Powers and coefficient in (2.7):

\[(P, Q, R; p, q, r; M) = (3, 2, 1; 3, 2, 1; -27)\] (2.53)

Relation between the Halphen variables:

\[
2W_1 + 3W_2 + W_3 = \Sigma_1
\]
\[
W_1 - W_3 = \frac{-\Sigma_2}{\Sigma_3}
\]
\[
W_2 - W_3 = \frac{-\Sigma_3}{\Sigma_2}
\] (2.54)

2B’ ↦ 1A:
Hypergeometric identity (Goursat (122)):

\[
F \left( \frac{1}{12}, \frac{1}{12}; \frac{1}{2}; \frac{x(9-8x)^2}{27(1-x)} \right) = (1-x)^{\frac{1}{2}} F \left( \frac{1}{4}, \frac{1}{4}; \frac{1}{2}; x \right)
\] (2.55)

Rational map:

\[
F = G^2 - 552 + \frac{2^{12}}{G - 24} \circ (\tau \mapsto \tau/2),
\]
\[
f = \frac{(3 - 4g)^3}{27(1 - g)} \circ (\tau \mapsto \tau/2)
\] (2.56)

Symmetrization quotient: \( \Gamma/\Gamma_0(2) = S_{1A}^{4C’}/S_{2B}^{4C’} = S_3/Z_2 \)
Polynomial invariants:

\[ \Sigma_1 := 3w_1 + w_2 + 2w_3, \quad \Sigma_2 := (w_1 - w_3)(w_1 + 3w_2 - 4w_3), \]
\[ \Sigma_3 := (w_1 - w_3)^2(w_1 - 9w_2 + 8w_3) \quad (2.57) \]

Powers and coefficient in (2.7):

\[ (P, Q, R; p, q, r; M) = (3, 2, 1; 3, 1, 2; -\frac{27}{64}) \quad (2.58) \]

Relation between the Halphen variables:

\[ \begin{align*}
3W_1 + 2W_2 + W_3 &= \frac{1}{2} \Sigma_1 \circ (\tau \mapsto \tau/2) \\
W_1 - W_3 &= -\frac{\Sigma_2}{2\Sigma_3} \circ (\tau \mapsto \tau/2) \\
W_2 - W_3 &= -\frac{\Sigma_3}{2\Sigma_2} \circ (\tau \mapsto \tau/2)
\end{align*} \quad (2.59) \]

3B \mapsto 1A:

Hypergeometric identity (Goursat (131)):

\[ F \left( \frac{1}{12}, \frac{1}{12}; \frac{2}{3}; \frac{x(x+8)^3}{64(x-1)^3} \right) = (1-x)^{\frac{1}{3}} F \left( \frac{1}{3}, \frac{1}{3}; \frac{2}{3}; x \right) \quad (2.60) \]

Rational map:

\[ F = G + \frac{3^9(10G^2 + 732G + 9459)}{(G-12)^3}, \quad f = \frac{g(g+8)^3}{64(g-1)^3} \quad (2.61) \]

Symmetrization quotient: \( \Gamma/\Gamma_0(3) = S_{3B}^{9B} / S_{1A}^{9B} = A_4 / \mathbb{Z}_3 \)

Polynomial invariants:

\[ \begin{align*}
\Sigma_1 := 3w_1 + 2w_2 + w_3, \quad \Sigma_2 := (w_1 - w_3)(9w_1 - 8w_2 - w_3), \\
\Sigma_3 := (w_1 - w_3)^2(27w_1^2 - 36w_1w_2 + 8w_2^2 - 18w_1w_3 + 2w_2w_3 - w_3^2)
\end{align*} \quad (2.62) \]

Powers and coefficient in (2.7):

\[ (P, Q, R; p, q, r; M) = (3, 2, 1; 3, 2, 1; 64) \quad (2.63) \]

Relation between the Halphen variables:

\[ \begin{align*}
3W_1 + 2W_2 + W_3 &= \Sigma_1 \\
W_1 - W_3 &= -\frac{\Sigma_2}{\Sigma_3} \\
W_2 - W_3 &= -\frac{\Sigma_3}{\Sigma_2}
\end{align*} \quad (2.64) \]
Modular solutions to generalized Halphen equations

3B’ \mapsto 1A:

Hypergeometric identity (Goursat (130)):

\[
F\left(\frac{1}{12}, \frac{1}{6}; \frac{2}{3}; \frac{x(9x - 8)^3}{64(x - 1)}\right) = (1 - x)^\frac{1}{3} F\left(\frac{1}{3}; \frac{1}{3}; \frac{2}{3}; x\right)
\] (2.65)

Rational map:

\[
F = G^3 - 162G + 228 + \frac{3^6}{(G - 12)} - 744 \circ (\tau \mapsto \tau/3), \quad f = \frac{g(9g - 8)^3}{64(g - 1)} \circ (\tau \mapsto \tau/3)
\] (2.66)

Symmetrization quotient: \( \Gamma/\Gamma_0(3) = S_{1A}^{3B’}/S_{3B’}^{9B} = A_4/Z_3 \)

Polynomial invariants:

\[
\Sigma_1 := w_1 + 2w_2 + 3w_3, \quad \Sigma_2 := (w_1 - w_3)(w_1 + 8w_2 - 9w_3), \\
\Sigma_3 := (w_1 - w_3)(w_1^2 - 20w_1w_2 - 8w_2^2 + 18w_1w_3 + 36w_2w_3 - 27w_3^2)
\] (2.67)

Powers and coefficient in (2.7):

\[
(P, Q, R; p, q, r; M) = (3, 2, 1; 2, 3, 1; \frac{64}{729})
\] (2.68)

Relation between the Halphen variables:

\[
3W_1 + 2W_2 + W_3 = \frac{1}{3} \Sigma_1 \circ (\tau \mapsto \tau/3) \\
W_1 - W_3 = -\frac{\Sigma_2}{3\Sigma_3} \circ (\tau \mapsto \tau/3) \\
W_2 - W_3 = -\frac{\Sigma_3}{3\Sigma_2} \circ (\tau \mapsto \tau/3)
\] (2.69)

4C \mapsto 2a:

Hypergeometric identity:

\[
F\left(\frac{1}{6}; \frac{1}{6}; \frac{2}{3}; \frac{i(x + \omega)^3}{3\sqrt[3]{3x(1 - x)}}\right) = (x(1 - x))^{\frac{1}{3}} F\left(\frac{1}{2}; \frac{1}{2}; 1; x\right), \quad \omega := e^{\frac{2\pi i}{3}}
\] (2.70)

Rational map:

\[
F = G + \frac{2^9G}{G^2 - 64}, \quad f = \frac{i(g + \omega)^3}{3\sqrt[3]{3g(1 - g)}}
\] (2.71)

Symmetrizing group: \( S_{2a}^4 = Z_3 \):

\[
\tau \mapsto -\frac{1}{4\tau - 2}, \quad g \mapsto 1 - \frac{1}{g}, \quad (w_1, w_2, w_3) \mapsto (w_3, w_1, w_2)
\] (2.72)
Polynomial invariants:
\[ \sigma_1 := w_1 + w_2 + w_3, \]
\[ \sigma_2 := w_1w_2 + w_2w_3 + w_1w_3, \]
\[ \Sigma_3 := (w_1 + \omega w_2 + \omega^2 w_3)^3. \]

Powers and coefficient in (2.7):
\[ (P, Q, R; p, q, r; M) = (1, 1, 1; 1, 1, 1; 3\sqrt{3}i) \]

Relation between the Halphen variables:
\[ W_1 + W_2 + W_3 = \sigma_1 \]
\[ W_1 - W_3 = -\frac{(\sigma_1^2 - 3\sigma_2)^2}{\Sigma_3} \]
\[ W_2 - W_3 = -\frac{\Sigma_3}{\sigma_1^2 - 3\sigma_2} \]

3. Four vertex systems

3a. Solutions of Generalized Halphen systems.

The examples listed in Table 2 below all involve Fuchsian operators (1.3) with four regular singular points (including \( \infty \)). We use the following linear combination of Ohyama’s dynamical variables [O2] as our phase space coordinates
\[ u := X_0 = \frac{1}{2} \frac{f''}{f'}, \quad v_i := \frac{1}{2} (X_0 - X_i) = \frac{1}{2} \frac{f'}{f - a_i} \quad i = 1, 2, 3, \]
where \((a_1, a_2, a_3)\) are the locations of the finite poles of the rational function \( R(f) \) in the Fuchsian equation (1.3) and the Schwarzian equation (1.10). The cases considered here involve the replicable functions denoted: 6C, 6D, 6E, 6c, 6E, 9B in [F MN]. These include all cases with integer \( q \)-series coefficients that, when composed with a suitably defined rational map of degree \( \leq 4 \), give one of the triangular functions listed in Table 1. Each provides modular solutions to a system of the type formulated by Ohyama [O2], generalizing the equations of Halphen type. Further cases of such replicable functions do exist, but they may all be related to one of the above through an affine transformation that relocates the finite poles of the Fuchsian operator (1.3), composed with a M"{o}bius transformation of the modular variable \( \tau \), and hence they are not listed separately. As in the previous section, the function \( F(\tau) \) denotes the normalized \( q \)-series (1.47) as in [FMN], which is related by an affine transformation (1.48) to a function \( f(\tau) \) normalized, if possible, to take values \((a, 0, 1, \infty)\) at the vertices, for some real \( a < 0 \). In two cases: 6D and 9B, this is not possible, because the three finite singular points are not collinear. In case 9B, they are normalized instead to the cube roots of 1, as in
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[O3], while in case 6D they are normalized so that the single real pole is located at 1. We also list, for each case, expressions for $f(\tau)$ in terms of the $\eta$-function or null $\vartheta$-functions and the generators $\{\rho_1, \rho_2, \rho_3\}$ of the automorphism group corresponding to the finite vertices of the fundamental region normalized to have integer entries. These satisfy

$$\rho_\infty \rho_3 \rho_2 \rho_1 = m I$$

where $\rho_\infty$ is given by (2.2) and $m \neq 0$ is an integer.

The differential equations satisfied by the quantities $(u, v_1, v_2, v_3)$ are determined by a quadratic vector field together with a quadratic constraint. The equations for $v'_1, v'_2, v'_3$ following from (3.1) are of the form

$$v'_i = -2v^2 + 2uv_i, \quad i = 1, 2, 3,$$

while the constraint equation is

$$(a_1 - a_2)v_1 v_2 + (a_2 - a_3)v_2 v_3 + (a_3 - a_1)v_1 v_3 = 0.$$  

When the rational function $R(f)$ appearing in eqs. (1.3) and (1.10) is expressible in the form

$$R(f) = \frac{1}{4} \sum_{i,j=1}^{3} \frac{r_{ij}}{(f - a_i)(f - a_j)},$$

the remaining equation, for $u'$, is

$$u' = u^2 - \sum_{i,j=1}^{n} r_{ij} v_i v_j.$$  

In the table, rather than listing $R(f)$ directly, we give the quadratic form in $(v_1, v_2, v_3)$ appearing on the RHS of eq. (3.6). (Note that this quadratic form, and hence also the coefficients $r_{ij}$ appearing in (3.5), is arbitrary up to the addition of any multiple of the constraint (3.4), but such a change leaves $R(f)$ invariant.) The general solution is again obtained by composing the function $f$ with a Möbius transformation (1.1). This amounts to transforming the function $2u$ as an affine connection

$$u \rightarrow \frac{1}{(c\tau + d)^2} u \circ T - \frac{c}{c\tau + d},$$

and the functions $v_i$ as 1–forms

$$v_i \rightarrow \frac{1}{(c\tau + d)^2} v_i \circ T.$$
Table 2. Four Vertex Replicable Functions
(admitting a rational map of degree ≤ 4 to a triangular replicable function)

| Name | \((a_1, a_2, a_3)\) | \(\rho_1\) | \(\rho_2\) | \(\rho_3\) | \(\sum_{i,j=1}^3 r_{ij}v_iv_j\) | \(F\) | \(f(\tau)\) |
|------|-------------------|--------|--------|--------|-----------------|------|----------|
| 6C   | \((-3, 0, 1)\)    | \((3 - 2)\) | \((3 - 1)\) | \((-1)\) | \(\frac{3}{4}v_1^2 + \frac{3}{4}v_2^2 + v_3^2 - \frac{1}{2}v_2v_3 - v_1v_3\) | \(4f + 2\) | \(1 + \frac{1}{4}\eta^2(\tau)\eta^2(3\tau)\eta^4(6\tau)\) |
| 6D   | \((\beta, \bar{\beta}, 1)\) | \((4 - 3)\) | \((2 - 1)\) | \((-1)\) | \(\frac{3}{4}v_1^2 + \frac{3}{4}v_2^2 + v_3^2 + \frac{131}{81}v_1v_2 - \frac{28+16\sqrt{2}i}{81}v_2v_3\) | \(4f\) | \(1 + \frac{1}{4}\eta^2(\tau)\eta^4(3\tau)\eta^4(6\tau)\) |
| 6E \(\sim \Gamma_0(6)\) | \((-\frac{1}{8}, 0, 1)\) | \((5 - 3)\) | \((5 - 2)\) | \((-1)\) | \(-\frac{10}{9}v_2v_3 - \frac{8}{9}v_1v_3\) | \(8f - 3\) | \(1 + \frac{1}{8}\eta^2(\tau)\eta^2(3\tau)\eta^4(6\tau)\) |
| 6c   | \((-1, 1, 0)\)    | \((8 - 7)\) | \((2 - 1)\) | \((-1)\) | \(\frac{8}{9}v_1^2 + \frac{8}{9}v_2^2 + v_3^2 + \frac{16}{9}v_1v_2\) | \(i3\sqrt{3}f\) | \(-\frac{i}{3}\eta^2(\tau)\eta^4(6\tau)\) |
| 8E \(\sim \Gamma_0(8)\) | \((-1, 0, 1)\)    | \((3 - 2)\) | \((3 - 1)\) | \((-1)\) | \(v_1^2 + v_2^2 + v_3^2 - 2v_1v_3\) | \(4f\) | \(1 + \frac{1}{4}\eta^2(\tau)\eta^2(4\tau)\eta^4(8\tau)\eta^4(16\tau)\) |
| 9B \(\sim \Gamma_0(9)\) | \((\omega, \bar{\omega}, 1)\) | \((5 - 4)\) | \((2 - 1)\) | \((-1)\) | \(v_1^2 + v_2^2 + v_3^2 - v_1v_2 - (1 - \omega)v_1v_3 - (1 - \bar{\omega})v_2v_3\) | \(3f\) | \(1 + \frac{1}{3}\eta^3(\tau)\eta^3(9\tau)\) |

3. Rational maps to triangular cases.

In the following, we catalogue the irreducible rational maps of degree \(\leq 4\) which, like those in Section 2b, relate the modular functions listed above to the triangular cases of Section 2a. The same notation is used as in Section 2b, \((F, G)\) denoting a pair of replicable functions, with \(F\) given as a rational function of \(G\), where \(F\) is one of the triangular functions of Section 2a, with normalized \(q\)-series given by (1.47), and the four vertex function \(G\) is one of the cases listed in Section 3a, similarly normalized. The corresponding pair, with values normalized to \((0, 1, \infty)\) at the vertices in the triangular case and to the values \((a_1, a_2, a_3)\) listed in Table 2.
for the four vertex case, is denoted \((f, g)\). The generalized Halphen variables for \(f\) are again as defined in (2.3), while the corresponding ones for \(g\) are defined by replacing \(f\) in (3.1) by \(g\). Taking derivatives of the rational map relating \(f\) to \(g\), we obtain relations between the associated generalized Halphen variables. Analogously to relation (2.7), there is always a relation of the form

\[
\frac{f'^P Q + R}{f^P R (f - 1)^Q + R} = M \frac{g^k}{(g - a_1)^p (g - a_2)^q (g - a_3)^r},
\]

with integer powers \((P, Q, R, p, q, r, k)\) satisfying

\[
P + Q + R = k,
\]

and \((P, Q, R)\) again chosen so that the LHS of (3.9) has no singularities at the vertices. The resulting quantity is again an analytic form of weight \(2k\) and, taking logarithmic derivatives of both sides of (3.9), we again obtain a linear relation between the generalized Halphen variables for the pair \((f, g)\)

\[
P W_1 + Q W_2 + R W_3 = ku - pv_1 - qv_2 - rv_3.
\]

Two other relations also follow from the definitions of these variables, allowing us to determine the quantities \((W_1, W_2, W_3)\) for the triangular case as simple rational symmetric functions of the variables \((u, v_1, v_2, v_3)\) for the four vertex case, invariant under the larger automorphism group \(\mathfrak{G}_f\). In those cases where \(\mathfrak{G}_g \subset \mathfrak{G}_f\) is a normal subgroup we list the linear, quadratic and cubic polynomial invariants, in terms of which \((W_1, W_2, W_3)\) are expressed. In all cases a fourth invariant, which we do not list separately, is provided by the vanishing quadratic form (3.4). (A notational convention that is slightly different from that of Table 2 is used below; the subscripts on the \(v\)-variable designate the location of the poles rather than their order in the sequence \((1, 2, 3)\) indicated in the second column of the table; i.e., what appears in the table as \(v_i\) is here denoted \(v_{a_i}\).)

8E \(\mapsto\) 4C:

Rational map:

\[
F = G + \frac{2^4}{G},
\]

\[
f = \frac{(g + 1)^2}{4g}
\]

Powers and coefficient in (3.9):

\[
(P, Q, R; p, q, r, k; M) = (1, 0, 0; -1, 1, 1, 1; 1)
\]
Symmetrizing group: \( S^{SE}_{4/1} = \Gamma_0(8)/\Gamma_0(4) = \mathbb{Z}_2 \):

\[
\tau \mapsto \frac{-\tau}{4\tau - 1}, \quad g \mapsto \frac{1}{g},
\]

\[(u, v_{-1}, v_0, v_1) \mapsto (u - 2v_0, v_1 - v_0, -v_0, v_1 - v_0) \]  \hspace{1cm} (3.14)

Polynomial invariants:

\[
\Sigma_1 := u + v_{-1} - v_0 - v_1, \quad \Sigma'_1 := 4v_1 - 2v_0, \quad \Sigma_2 := 4v_0^2
\]  \hspace{1cm} (3.15)

Relation between the Halphen variables:

\[
W_1 = \Sigma_1
\]

\[
W_1 - W_3 = \frac{\Sigma_2}{\Sigma'_1}.
\]  \hspace{1cm} (3.16)

\[
W_2 - W_3 = \Sigma'_1
\]

6c \mapsto 6a:

Rational map:

\[
F = G - \frac{3^3}{G},
\]

\[
f = \frac{(g + 1)^2}{4g}
\]  \hspace{1cm} (3.17)

Powers and coefficient in (3.9):

\[
(P, Q, R; p, q, r, k; M) = (1, 1, 4; 4, 2, 4, 6; 256)
\]  \hspace{1cm} (3.18)

Symmetrizing group: \( S^{6c}_{6a} = \mathbb{Z}_2 \):

\[
\tau \mapsto \frac{-1}{12\tau - 6}, \quad g \mapsto \frac{1}{g},
\]

\[(u, v_{-1}, v_0, v_1) \mapsto (u - 2v_0, v_{-1} - v_0, -v_0, v_1 - v_0) \]  \hspace{1cm} (3.19)

Polynomial invariants:

\[
\Sigma_1 := 6u - 4v_{-1} - 2v_0 - 4v_1, \quad \Sigma'_1 := 4v_1 - 2v_0, \quad \Sigma_2 := 4v_0^2
\]  \hspace{1cm} (3.20)

Relation between the Halphen variables:

\[
W_1 = \Sigma_1
\]

\[
W_1 - W_3 = \Sigma'_1.
\]  \hspace{1cm} (3.21)

\[
W_2 - W_3 = \frac{\Sigma_2}{\Sigma'_1}
\]
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6C $\mapsto$ 3A:
Rational map:

$$F = G + \frac{2^8(3G - 2)}{(G - 6)^2},$$

$$f = \frac{(g + 3)^3}{27(g - 1)^2}$$

Powers and coefficient in (3.9):

$$(P, Q, R; p, q, r, k; M) = (3, 1, 2; 2, 3, 2, 6; 3^6)$$

Symmetrization quotient: $S_{6C}^{12E'}/S_{3A}^{12E'} = S_3/Z_2$

Polynomial Invariants:

$$\Sigma_1 := 6u - 2v_3 - 3v_0 - 2v_1, \quad \Sigma_2 := v_1(4v_1 - 3v_0), \quad \Sigma_3 := v_1^2(9v_0 - 8v_1)$$

Relation between the Halphen variables:

$$3W_1 + W_2 + 2W_3 = \Sigma_1$$

$$W_1 - W_3 = \frac{2\Sigma_2}{\Sigma_3}$$

$$W_2 - W_3 = \frac{2\Sigma_3}{\Sigma_2}.$$  

6E $\mapsto$ 3B:
Rational map:

$$F = G + \frac{2^4}{(3G + 13)^2},$$

$$f = \frac{(1 + 2g)^3}{27g^2}$$

Powers and coefficient in (3.9):

$$(P, Q, R; p, q, r, k; M) = (3, 1, -1; 0, 5, -3, 3; \frac{8}{27})$$

Symmetrization quotient: $\Gamma_0(3)/\Gamma_0(6) = S_{3B}^{12f'}/S_{6E}^{12f'} = S_3/Z_2$

Relation between the Halphen variables:

$$3W_1 + W_2 - W_3 = 3u - 5v_0 + 3v_1$$

$$W_1 - W_3 = \frac{4(v_0 - 3v_1)^2}{9v_1 - v_0}$$

$$W_2 - W_3 = \frac{4v_0^2}{3v_1 - v_0}.$$
$9B \mapsto 3B$: 

Rational map:

$$F = G + \frac{3^3(2G + 3)}{G^2 + 3G + 9},$$

$$f = \frac{(2 + g)^3}{9(1 + g + g^2)}$$

(3.29)

Powers and coefficient in (3.9):

$$(P, Q, R; p, q, r, k; M) = (3, 1, -1; 4, 4, -6, 3; \frac{1}{9})$$

(3.30)

Symmetrizing group: $S^B_{3B} = \Gamma_0(3)/\Gamma_0(9) = \mathbb{Z}_3$

$$\tau \mapsto \frac{\tau}{3\tau - 1}, \quad g \mapsto \frac{\omega g + 2}{g - \bar{\omega}},$$

$$(u, v_\omega, v_{\bar{\omega}}, v_1) \mapsto (u - 2v_{\bar{\omega}}, -v_{\bar{\omega}}, v_\omega - v_{\bar{\omega}}, v_1 - v_{\bar{\omega}})$$

(3.31)

Polynomial invariants:

$$\Sigma_1 := 3u + 6v_1 - 4v_\omega - 4v_{\bar{\omega}}, \quad \Sigma_2 := 3(v_\omega^2 - v_\omega v_{\bar{\omega}} + v_{\bar{\omega}}^2)$$

$$\Sigma_3 := ((1 - \omega)v_\omega - (1 - \bar{\omega})v_{\bar{\omega}})^3$$

(3.32)

Relation between the Halphen variables:

$$2W_1 + W_2 = \Sigma_1$$

$$W_1 - W_3 = \frac{2\sqrt{3}i\Sigma_2}{3\Sigma_3}$$

(3.33)

$$W_2 - W_3 = -\frac{2\sqrt{3}i\Sigma_3}{3\Sigma_2}.$$  

$6D \mapsto 2A$: 

Rational map:

$$F = G + \frac{3^7(2G^2 + 20G + 131)}{(G - 4)^3},$$

$$f = \frac{(4g + 23)^4}{4^7(g - 1)^3}$$

(3.34)

Powers and coefficient in (3.9):

$$(P, Q, R; p, q, r, k; M) = (2, 1, 1; 2, 2, 1, 4; 4^7)$$

(3.35)

Symmetrization quotient: $S^D_{2A}/S^D_{6D} = S_4/S_3$
Relation between the Halphen variables:

\[ W_1 + W_2 + W_3 = 4u - 2v_\beta - 2v_\overline{\beta} - v_1 \]
\[ W_1 - W_3 = \frac{2\sqrt{2}((-5i + \sqrt{2})v_\beta + (5i + \sqrt{2})v_\overline{\beta})^3}{((22i + \sqrt{2})v_\beta + (-22i + \sqrt{2})v_\overline{\beta})((7i + 4\sqrt{2})v_\beta + (-7i + 4\sqrt{2})v_\overline{\beta})} \]
\[ W_2 - W_3 = \frac{16\sqrt{2}v_\beta v_\overline{\beta}((22i + \sqrt{2})v_\beta + (-22i + \sqrt{2})v_\overline{\beta})}{((-5i + \sqrt{2})v_\beta + (5i + \sqrt{2})v_\overline{\beta})((7i + 4\sqrt{2})v_\beta + (-7i + 4\sqrt{2})v_\overline{\beta})}. \] (3.36)

6c \mapsto 2a:
Rational map:
\[ F = G - \frac{3^5(2G^2 + 81)}{G^3}, \]
\[ f = \frac{(g + 3)^3(g - 1)}{16g^3} \] (3.37)

Powers and coefficient in (3.9):
\[ (P, Q, R; p, q, r, k; M) = (1, 1, 1; 2, 0, 2, 3; 16) \] (3.38)

Symmetrization quotient: \( S_{6c}^{18|2}/S_{2a}^{18|2} = A_4/Z_3 \)
(Remark: The group 18|2 has a function field that is not of genus 0, and hence it does not appear in the list of replicable functions [FMN].)

Relation between the Halphen variables:
\[ W_1 + W_2 + W_3 = 3u - 2v_{-1} - 2v_1 \]
\[ W_1 - W_3 = -\frac{2v_0(3v_0 - 4v_1)^2}{(v_0 - 2v_1)(3v_0 - 2v_1)} \] (3.39)
\[ W_2 - W_3 = -\frac{2(3v_0 - 2v_1)^2}{3v_0 - 4v_1}. \]

6E \mapsto 2B:
Rational map:
\[ F = G + \frac{3^3(10G^2 + 44G + 43)}{(G + 4)^3}, \]
\[ f = \frac{(8g^2 + 20g - 1)^2}{(8g + 1)^3} \] (3.40)

Powers and coefficient in (3.9):
\[ (P, Q, R; p, q, r, k; M) = (2, 1, -1; 5, 0, -4, 2; 2^{12}) \] (3.41)
Symmetrization quotient: \( \mathbf{S}_{6E}^{18D'} / \mathbf{S}_{2B}^{18D'} \)
Relation between the Halphen variables:

\[
\begin{align*}
W_1 + W_2 + W_3 &= 2u - 5v - \frac{1}{18} + 4v_1 \\
W_1 - W_3 &= -\frac{2(v_0^2 + 18v_0v_1 - 27v_1^2)}{v_0 - 9v_1} \\
W_2 - W_3 &= \frac{128v_0^2v_1}{(v_0 - 9v_1)(v_0^2 + 18v_0v_1 - 27v_1^2)}. \\
\end{align*}
\]

(3.42)

8E \mapsto 4a:
Rational map:

\[
F = G - \frac{2^4(5G^2 - 16)}{G(G^2 - 16)}, \quad f = \frac{i(g - i)^4}{8g(g^2 - 1)}
\]

(3.43)
Powers and coefficient in (3.9):

\[
(P, Q, R; p, q, r, k; M) = (1, 1, 2; 2, 2, 4; 64)
\]

(3.44)
Symmetrizing group: \( \mathbf{S}_{4a}^{8E} = \mathbf{Z}_4 \)

\[
\tau \mapsto \frac{-1}{8\tau - 4}, \quad g \mapsto \frac{g - 1}{g + 1}
\]

(3.45)

(\(u, v_1, v_0, v_1\)) \mapsto (\(u - 2v - 1, v_0 - v_1, v_1 - v_1, -v_1\))

Polynomial invariants:

\[
\begin{align*}
\Sigma_1 &:= 4u - 2v_1 - 2v - 2v_1, & \Sigma_2 &:= (v_0 - v_1 - v_1)^2 \\
\Sigma_3 &:= (v_0 - v_1 - v_1)(v_0 + iv_1 - iv_1)^2
\end{align*}
\]

(3.46)
Relation between the Halphen variables:

\[
\begin{align*}
W_1 + W_2 + W_3 &= \Sigma_1 \\
W_1 - W_3 &= -\frac{\Sigma_2}{\Sigma_3} \\
W_2 - W_3 &= -\frac{\Sigma_3}{\Sigma_2}.
\end{align*}
\]

(3.47)

4. \( n + 1 \)-Vertex Systems. Further Remarks.

The systems listed in the preceding sections comprise a small subclass of the set of Hauptmoduls in [FMN]. These functions all satisfy Schwarzian equations of type (1.10) with \( R(f) \)
Modular solutions to generalized Halphen equations

of the form (1.37), (1.38) for \(1 \leq n \leq 25\) and determine solutions of generalized Halphen equations of the type introduced in [O2]. In the case of \(n\) finite poles in the rational function \(R(f)\), at points \(\{a_i\}_{i=1,\ldots,n}\), the quantities \(\{v_i\}_{i=1,\ldots,n}\) defined, as in eq. (3.1), by

\[
\begin{align*}
u := X_0 = 1 & = \frac{1}{2} \frac{f''}{f'}, & v_i := \frac{1}{2}(X_0 - X_i) = \frac{1}{2} \frac{f'}{f - a_i} & \quad i = 1, \ldots, n,
\end{align*}
\]

satisfy the set of quadratic constraints

\[
(a_i - a_j)v_i v_j + (a_j - a_k)v_j v_k + (a_k - a_i)v_k v_i = 0. \quad i, j, k = 1, \ldots, n,
\]

which span an \(n - 2\) dimensional space of quadratic forms vanishing on the quantities \(\{v_i\}\). They satisfy the differential equations

\[
v_i' = -2v_i^2 + 2uv_i, \quad i = 1, \ldots, n,
\]

while the remaining phase space variable \(u\) satisfies

\[
u' = u^2 - \sum_{i,j=1}^n r_{ij} v_i v_j
\]

when \(R(f)\) is of the form

\[
R(f) = \frac{1}{4} \sum_{i,j=1}^n \frac{r_{ij}}{(f - a_i)(f - a_j)}.
\]

In (4.5) we have the freedom of adding any linear combination of the vanishing expressions

\[
\frac{a_i - a_j}{(f - a_i)(f - a_j)} + \frac{a_j - a_k}{(f - a_j)(f - a_k)} + \frac{a_k - a_i}{(f - a_k)(f - a_i)} = 0,
\]

which amounts to adding linear combinations of the quadratic forms of eqs. (4.2) to the RHS of (4.3) and (4.4). Using this freedom, we can always choose the decomposition in (4.4) defined by the coefficients \(r_{ij}\) be tridiagonal. Choosing the ordering \((a_1, \ldots, a_n, \infty)\) corresponding to a positively oriented path on the boundary of the fundamental region makes this unique. The differential systems are defined on the 3-dimensional subvariety of the space with linear coordinates \((u, v_1, \ldots, v_n)\) cut out by the quadrics in (4.2), and are determined by the set of \(n+1\) quadratic forms appearing in eqs. (4.3), (4.4), defined modulo those in eq. (4.2). We note that the \(n\) quadratic forms appearing in eq. (4.3) are independent of the values of the parameters, while those defining the constraints (4.2) depend only on the locations \(\{a_i\}_{i=1,\ldots,n}\) of the poles. The diagonal coefficients of the quadratic form in (4.4) are related to the angles \(\{\alpha_i\}_{i=1,\ldots,n}\) at the \(n\) finite vertices of the fundamental polygon by

\[
r_{ii} = 1 - \alpha_i^2.
\]
In the case of real poles the off diagonal coefficients are determined, modulo the vanishing quadratic forms (4.2), by the values of the accessory parameters \([\text{GS}]\) of the corresponding mapping of the fundamental polygon to the upper half plane.

As an illustration of a system associated to one of the higher level functions, consider the case 72e which, taken in the normalization of \([\text{FMN}]\), may be expressed as a ratio of \(\eta\)-functions

\[
f = \frac{\eta(24\tau)\eta(36\tau)}{\eta(12\tau)\eta(72\tau)}. \tag{4.8}
\]

The fundamental region has 25 finite vertices, the largest number amongst the replicable functions. These are mapped in the \(f\)-plane to the origin \(f = 0\) and to the twelfth roots of unity times the two reciprocal radii \((\sqrt{2} + 1)^\frac{1}{12}\) and \((\sqrt{2} - 1)^\frac{1}{12}\). The rational function \(R(f)\) entering in eqs. (1.3) and (1.10) is given by

\[
R(f) = \frac{1}{4f^2} + \frac{864f^{10}(f^4 + 1)^2(f^8 - f^4 + 1)^2}{(f^6 + 2f^3 - 1)^2(f^6 - 2f^3 - 1)^2(f^{12} + 6f^6 + 1)^2}. \tag{4.9}
\]

We denote the finite poles as

\[
a_0 := 0, \quad a_m := e^{\frac{(m-1)\pi i}{6}}(\sqrt{2} + 1)^\frac{1}{12}, \quad a_{12+m} := e^{\frac{(m-1)\pi i}{6}}(\sqrt{2} - 1)^\frac{1}{12}, \quad m = 1, \ldots 12. \tag{4.10}
\]

In the notation defined above, the 25 functions \(\{v_0, v_m\}_{m=1, \ldots 24}\) satisfy the usual constraint equations (4.2), and equation (4.3) for the derivatives of the \(v\)-variables. The quadratic form entering in the eq. (4.4) for \(u'\) is

\[
\sum_{i,j=1}^{n} r_{ij} v_i v_j = v_0^2 + \frac{3}{4} \sum_{m=1}^{24} v_m^2 - \frac{3}{8} \sum_{m=1}^{11} (1 - e^{\frac{(m-1)\pi i}{6}}) \left( (2 + \sqrt{2})v_m v_{m+1} + (2 - \sqrt{2})v_{12+m} v_{13+m} \right). \tag{4.11}
\]

The two angles in the fundamental polygon at the vertices mapping to 0 and \(\infty\) therefore vanish, while the others are \(\pi/2\). The simplicity of this expression is due to the invariance of the Schwarzian derivative under the transformation \(\tau \mapsto \tau + \frac{1}{12}\), which generates the cyclic group action

\[
f \mapsto e^{\frac{\pi i}{12}} f, \quad R(f) \mapsto e^{-\frac{\pi i}{12}} R(f), \tag{4.12}
\]

and to the inversion symmetry

\[
R(1/f) = f^4 R(f). \tag{4.13}
\]

As a final remark we note that, for the general \(n\) finite pole case, when \(R(f)\) is of the form (4.5) there is an equivalent way of expressing the Schwarzian equation (1.10) and the associated system (4.2)–(4.4) in terms of an unconstrained dynamical system on the \(SL(2, \mathbb{C})\) group manifold. To do this, let

\[
g(\tau) := \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL(2, \mathbb{C}), \quad AD - BC = 1 \tag{4.14}
\]
denote an integral curve in $SL(2, \mathbb{C})$ for the equation

$$g' = \begin{pmatrix} 0 & \gamma \\ -1 & 0 \end{pmatrix} g, \quad (4.15)$$

where

$$\gamma := -\frac{1}{4C^2} \sum_{i,j=1}^{n} \frac{b_{ij}}{(Ca_i + D)(Ca_j + D)} = -\frac{1}{C^2} R \left( -\frac{D}{C} \right). \quad (4.16)$$

Defining $f(\tau)$ to be

$$f := -\frac{D}{C}, \quad (4.17)$$

it follows that this satisfies (1.10), and that

$$u = \frac{A}{C}, \quad v_{a_i} = \frac{1}{2C(Ca_i + D)}, \quad i = 1, \ldots, n \quad (4.18)$$

satisfy the system (4.2)–(4.4). Equivalently, Ohyama’s variables (1.39) are obtained by applying $g$ as a linear fractional transformation to $\{\infty, a_1, \ldots, a_m\}$

$$X_0 = \frac{A}{C}, \quad X_i = \frac{Aa_i + B}{Ca_i + D}, \quad i = 1, \ldots, n. \quad (4.19)$$

Conversely, up to a choice of branch in $(f')^{\frac{1}{2}}$ (which does not affect the projective class of $g$), we may always express a solution $f$ of (1.10) in this way by defining $g$ to be

$$g = \frac{i}{2(f')^{\frac{1}{2}}} \begin{pmatrix} f'' & 2f'^2 - f f'' \\ 2f' & -2f f' \end{pmatrix}. \quad (4.20)$$

A number of further questions suggest themselves in relation to this work. The first concerns the origin of the Fuchsian equation associated to each of these Hauptmoduls. For the case $\lambda(\tau)$ discussed in the introduction, the elliptic integral formulae (1.15a), (1.15b) show that the associated hypergeometric equations are Picard–Fuchs equations corresponding to the family of elliptic curves parametrized, in Jacobi’s form, by $\lambda(\tau)$. A similar interpretation was found by Ohyama [O3] for the 4–vertex example denoted here as 9B, for which the underlying Fuchsian equation is a Picard–Fuchs equation for the Hesse pencil of elliptic curves. All the hypergeometric functions have Euler integral representations, but it is not clear whether their equations might similarly be interpreted in terms of parametric families of elliptic curves. The same question may be asked for the other Hauptmoduls considered here. Another question that naturally arises is: what are the analogues of the Chazy equation that are satisfied by the logarithmic derivatives of the analytic forms entering in eqs. (2.7) and (3.9), and how may these forms be expressed in terms of some standard set such as, e.g., the modular discriminant appearing in eq. (1.27), Eisenstein series, or simply as ratios of $\vartheta$–functions?
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A. Appendix

We recall here the standard definitions [A, F, WW] of the functions \( \vartheta_2, \vartheta_3, \vartheta_4 \) and \( \eta \), and a number of properties and relations between them that are helpful in verifying some of the formulae of Sections 2 and 3.

Dedekind \( \eta \)-function:

\[
\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n), \quad (q := e^{2\pi i \tau})
\]

(A.1)

Null \( \vartheta \)-functions:

\[
\vartheta_2(\tau) := \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(n+\frac{1}{2})^2} = 2q^{\frac{1}{8}} \prod_{n=1}^{\infty} (1 - q^n)(1 + q^n)^2
\]

(A.2)

\[
\vartheta_3(\tau) := \sum_{n=-\infty}^{\infty} q^{\frac{3}{2}n^2} = \prod_{n=1}^{\infty} (1 - q^n)(1 + q^{n-\frac{1}{2}})^2
\]

(A.3)

\[
\vartheta_4(\tau) := \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{3}{2}n^2} = \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{n-\frac{1}{2}})^2.
\]

(A.4)

Relations between \( \vartheta_2, \vartheta_3, \vartheta_4, \) and \( \eta \):

\[
\vartheta_2(\tau) = 2 \frac{\eta^2(2\tau)}{\eta(\tau)}
\]

(A.5)

\[
\vartheta_3(\tau) = \frac{\eta^5(\tau)}{\eta^2(2\tau)\eta^2(\tau/2)}
\]

(A.6)

\[
\vartheta_4(\tau) = \frac{\eta^2(\tau/2)}{\eta(\tau)}
\]

(A.7)

\[
\eta^3(\tau) = \frac{1}{2} \vartheta_2(\tau) \vartheta_3(\tau) \vartheta_4(\tau)
\]

(A.8)

Modular transformations of \( \vartheta_2, \vartheta_3, \vartheta_4, \) and \( \eta \):

\[
\vartheta_2(\tau + 1) = e^{\frac{1+i}{4}} \vartheta_2(\tau), \quad \vartheta_2(-1/\tau) = (-i\tau)^{\frac{1}{2}} \vartheta_4(\tau)
\]

(A.9)

\[
\vartheta_3(\tau + 1) = \vartheta_4(\tau), \quad \vartheta_3(-1/\tau) = (-i\tau)^{\frac{1}{2}} \vartheta_3(\tau)
\]

(A.10)

\[
\vartheta_4(\tau + 1) = \vartheta_3(\tau), \quad \vartheta_4(-1/\tau) = (-i\tau)^{\frac{1}{2}} \vartheta_2(\tau)
\]

(A.11)

\[
\eta(\tau + 1) = e^{\frac{1+i}{4}} \eta(\tau), \quad \eta(-1/\tau) = (-i\tau)^{\frac{1}{2}} \eta(\tau)
\]

(A.12)
Identities satisfied by \( \vartheta_2, \vartheta_3, \vartheta_4, \eta \):

\[
\vartheta_3^4(\tau) = \vartheta_2^4(\tau) + \vartheta_4^4(\tau) \quad (A.13)
\]

\[
2\vartheta_3^2(2\tau) = \vartheta_3^2(\tau) + \vartheta_4^2(\tau) \quad (A.14)
\]

\[
\vartheta_2^2(\tau) = 2\vartheta_2(2\tau)\vartheta_3(2\tau) \quad (A.15)
\]

\[
\vartheta_4^2(2\tau) = \vartheta_3(\tau)\vartheta_4(\tau) \quad (A.16)
\]

\[
\eta(\tau + \frac{1}{2}) = \frac{e^{\pi i/12} \eta^3(2\tau)}{\eta(\tau)\eta(4\tau)} \quad (A.17)
\]

\[
\eta^3(\tau + \frac{1}{3}) = e^{\pi i/12} \eta^3(\tau) - 3\sqrt{3}e^{-\pi i/12} \eta^3(9\tau) \quad (A.18)
\]

Differential relations satisfied by \( \vartheta_2, \vartheta_3, \vartheta_4 \):

\[
\frac{\vartheta_2'}{\vartheta_2} - \frac{\vartheta_3'}{\vartheta_3} = \frac{i\pi}{4} \vartheta_4^4 \quad (A.19)
\]

\[
\frac{\vartheta_3'}{\vartheta_3} - \frac{\vartheta_4'}{\vartheta_4} = \frac{i\pi}{4} \vartheta_2^4 \quad (A.20)
\]

\[
\frac{\vartheta_2'}{\vartheta_2} - \frac{\vartheta_4'}{\vartheta_4} = \frac{i\pi}{4} \vartheta_3^4. \quad (A.21)
\]

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