Goldman-type Lie algebras from knots

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Abstract
We define Lie algebras from a class of knots in a homology 3-sphere. Since the definitions in terms of group homology are analogous to Goldman Lie algebra [Gold], we discuss relations among these Lie algebras.

Keywords
Lie algebra, knots, 3-manifold groups, string topology

1 Introduction
Given a group $G$, by $\hat{G}$ we mean the set of representatives of the conjugacy classes of $G$. Let $\Sigma_{g,1}$ be a connected oriented compact genus-$g$ surface with one circle boundary. When $G = \pi_1(\Sigma_{g,1})$, Goldman [Gold] gives a Lie algebra structure on the vector space $\mathbb{Q}[\hat{G}]$ spanned by $\hat{G}$. As shown in [Gold, KK2, MT, Tur1] and references therein, the Goldman Lie algebra is related to topics including (quantizations of) flat moduli spaces, the Teichmüller space, and the Johnson homomorphism; see also [ACC, CS, KK1] for studies from string topology. In low-dimensional topology and hyperbolic geometry, some properties of (hyperbolic) surfaces have been (virtually) extended to those of (hyperbolic) 3-manifolds. Hence, it is reasonable to hope for similar Lie algebras from some 3-manifolds.

In this paper, analogously to the Goldman Lie algebra, we introduce Lie algebras from a class of knots. Let $K \subset \Sigma$ be a knot of an integral homology 3-sphere $\Sigma$. We need terminology of the JSJ-decomposition of the complement $\Sigma \setminus K$; see, e.g., [AFW, Proposition 1.6.2 and Theorem 1.7.5]. More precisely, if $\Sigma \setminus K$ is irreducible, there are disjointly embedded incompressible tori, $T_1, \ldots, T_m$ in $\Sigma \setminus K$ such that the components $M_1, \ldots, M_n$ of $\Sigma \setminus K$ cut along $T_1 \cup \cdots \cup T_m$, are Seifert fibered or hyperbolic. Let $\text{Ab} : \pi_1(\Sigma \setminus K) \to \mathbb{Z}$ be the abelianization. Then, the main result is as follows:

**Theorem 1.1.** Let $K$ be a knot in $\Sigma$ and $G$ be the commutator subgroup of the knot group, that is, $G = [\pi_1(\Sigma \setminus K), \pi_1(\Sigma \setminus K)]$. Assume that the complement $\Sigma \setminus K$ is irreducible, and that the restriction map of $\text{Ab}$ on $\pi_1(T_i)$ is non-trivial for any $i \leq m$.

Then, the $\mathbb{Q}$-vector space, $\mathbb{Q}[\hat{G}]$, spanned by $\hat{G}$ has a Lie algebra structure and is equipped with a $\mathbb{Q}[t^{\pm 1}]$-module, where $t : \mathbb{Q}[\hat{G}] \to \mathbb{Q}[\hat{G}]$ is a Lie algebra isomorphism. Furthermore, if $K$ is fibered of genus $g$, the algebra structure coincides with the Goldman Lie algebra of $\Sigma_{g,1}$.

In Remark 3.2, we discuss a broad class of knots satisfying the assumption. As a corollary, we will introduce Lie algebra structures on $\mathbb{Q}[\Pi_K]$ and $\mathbb{Q}[G/[G,G]]$, where $\Pi_K$ is the set $\{[k] \in \hat{G} | \text{abelianization of } k = 0\}$, and the latter $G/[G,G]$ is the so-called Alexander module of $K$; see §3 for details. In Remark 3.3, we discuss a comparison with the works in [MT, Tur2].

As shown in [Gold], the Goldman Lie algebra is, in some sense, universal among moduli spaces of flat $G$-bundles on $F$, where $G$ is a semi-simple Lie group. Thus, the Lie algebras and
the isomorphism $t$ in Theorem 1.1 would include interesting information on representations varieties $\text{Hom}(\pi_1(\Sigma \setminus K), G)/G$, and would provide topological invariants of knots in terms of Lie algebras. We pose some problems in Section 3.

This paper is organized as follows. After a review of group homology and string products (see Section 2), Section 3 introduces the Lie bracket in Theorem 1.1. Section 4 gives the proofs of our results.

Conventional notation. We denote by $\hat G$ the set of representatives of the conjugacy classes of a group $G$. For $x \in G$, we denote $Z_G(x) \subset G$ by the centralizer subgroup. By $K$ we mean a knot in an integral homology 3-sphere $\Sigma$.

## 2 Preliminaries

We will describe this product structure directly in terms of the relative homology of groups. For this, we begin by reviewing the (relative) group (co)-homology. Denote the group ring of a group $G$ over $\mathbb{Q}$ by $\mathbb{Q}[G]$. Let $F_n(G)$ be the free $\mathbb{Q}[G]$-module with basis $\{[g_1| \cdots |g_n]; g_i \in G\}$. Let $\bar F_n(G)$ be the quotient of $F_n(G)$ by the $\mathbb{Q}[G]$-submodule generated by $\{[g_1| \cdots |g_n]; g_i = 1 \text{ for some } i\}$. Given a left $\mathbb{Q}[G]$-module $A$, we define $C_n(G; A)$ to be $A \otimes_{\mathbb{Q}[G]} \bar F_n(G)$ and the differential operator $\partial$ by

$$(g_1^{-1}a) \otimes [g_2| \cdots |g_n] + \sum_{i: 1 \leq i \leq n-1} (-1)^i a \otimes [g_1| \cdots |g_{i-1}|g_i|g_{i+1}|g_{i+2}| \cdots |g_n] + (-1)^n a \otimes [g_1| \cdots |g_{n-1}].$$

Then, the group homology is defined to be the homology. Furthermore, for a subgroup $S \subset G$, the relative homology is defined to be the homology of the quotient complex $C_n(G; A)/C_n(S; A)$. In this paper, we often consider the case $A = \mathbb{Q}[G/L]$ for some subgroup $L \subset G$, where $G/L$ is acted on by left multiplication.

Dually, we can define the group cohomology, $H^n(G, S; A)$, the cup product $\smile$, and the cap product $\frown$; see, e.g., [BE] for details.

This paper focuses on rational duality groups: here, a pair of groups $S \subset G$ (possibly $S = \emptyset$) is said to have a $\mathbb{Q}$-relative (Poincaré) duality of dimension $n$ if there is a relative homology $n$-class $\sigma_{G,S} \in H_n(G, S; \mathbb{Q})$ such that the cap product with this class provides an isomorphism

$$D := \bullet \quad \tilde{\sigma}_{G,S} : H^p(G, S; \mathbb{Q}[G/Z_G(x)] \frown) \sim H_{n-p}(G; \mathbb{Q}[G/Z_G(x)])$$

for any $x \in G$. We give a topological example:

**Example 2.1.** Let $M$ be an oriented compact $n$-manifold such that $M$ and the boundary $\partial M$ are aspherical. Let $G := \pi_1(M)$ and $S := \pi_1(\partial M)$. Then, there are isomorphisms $H^p(M, \partial M; \mathbb{Q}[G/Z_G(x)]) \cong H^p(G, S; \mathbb{Q}[G/Z_G(x)])$ and the relative fundamental class $\sigma_{G,S} := [M, \partial M] \in H_n(M, \partial M; \mathbb{Z})$, which ensures relative duality.

Using the map $D$, let us prepare the intersection product in the homology of subgroups of a $\mathbb{Q}$-relative Poincaré duality, as a slight modification of [ACC] §1. For any subgroups...
\(L, H \subset G\), it is known [ACC] Lemma 1 that the correspondence \(g_1L \otimes g_2H \mapsto L(g_1^{-1}g_2)H\) gives rise to a \(\mathbb{Q}[G]\)-module isomorphism,

\[
\psi : \mathbb{Q}[G/L] \otimes \mathbb{Q}[G/H] \cong \bigoplus_{[g] \in I} \mathbb{Q}[G/(L \cap gHg^{-1})],
\]

where \(I\) is a set of representatives for the double coset \(L \backslash G/H\). If \(L = Z_G(x)\) and \(H = Z_G(y)\) for some \(x, y \in G\), then the intersection pairing,

\[
\mathcal{H}_p(G; \mathbb{Q}[G/L]) \otimes \mathcal{H}_q(G; \mathbb{Q}[G/H]) \rightarrow \bigoplus_{[g] \in I} \mathcal{H}_{p+q-n}(G; \mathbb{Q}[G/L \cap gHg^{-1}]),
\]

defined in [ACC] Definition 2 is given by the composition,

\[
\mathcal{H}_p(G; \mathbb{Q}[G/L]) \otimes \mathcal{H}_q(G; \mathbb{Q}[G/H]) \xrightarrow{D^{-1} \otimes D^{-1}} \mathcal{H}^{n-p}(G, S; \mathbb{Q}[G/L]) \otimes \mathcal{H}^{n-q}(G, S; \mathbb{Q}[G/H])
\]

\[
\xrightarrow{\psi_*} \bigoplus_{[g] \in I} \mathcal{H}_{p+q-n}(G; \mathbb{Q}[G/(L \cap gHg^{-1})]).
\]

If \(S\) is the empty set, this definition is exactly Definition 2 in [ACC].

Next, we review string products. Let \(\text{Ad}\) be \(\mathbb{Q}[G]\) as a coefficient module, where the action of \(G\) is given by conjugation. Then,

\[
\mathcal{H}_* (G; \text{Ad}) = \bigoplus_{x \in G} \mathcal{H}_* (G; \mathbb{Q}[G/Z_G(x)]) \cong \bigoplus_{x \in G} \mathcal{H}_* (Z_G(x); \mathbb{Q}),
\]

where the second isomorphism is immediately obtained from the Shapiro Lemma. As mentioned in the introduction of [ACC], \(\mathcal{H}_* (G; \text{Ad})\) is isomorphic to \(\mathcal{H}_* (L(BG); \mathbb{Q})\), where \(L(BG)\) is the free loop space of the Eilenberg-MacLane space of \(G\). Then, as in Definition 4 in [ACC], the string product,

\[
\mu : \mathcal{H}_p(G; \text{Ad}) \otimes \mathcal{H}_q(G; \text{Ad}) \rightarrow \mathcal{H}_{p+q-n}(G; \text{Ad}),
\]

is defined to be the composition,

\[
\mu := j_* \circ \mathcal{H}_p(G; \mathbb{Q}[G/Z_G(x)]) \otimes \mathcal{H}_q(G; \mathbb{Q}[G/Z_G(y)]) \rightarrow \bigoplus_{[g] \in I} \mathcal{H}_{p+q-n}(G; \mathbb{Q}[(Z(x) \cap gZ_G(y))g^{-1}]) \rightarrow \bigoplus_{[g] \in I} \mathcal{H}_{p+q-n}(G; \mathbb{Q}[G/Z_G(xgyg^{-1})]).
\]

Here, \(j\) is induced from the following inclusion of subgroups:

\[
j : Z_G(x) \cap g^{-1}Z_G(y)g = Z_G(x) \cap Z_G(g^{-1}yg) \subset Z_G(xgyg^{-1}).
\]

Then, as in [ACC] Theorem 5, we can easily check that \(\mu\) turns \(\mathcal{H}_{*+n}(G; \text{Ad})\) into a graded, associated, commutative algebra. For instance, it is shown in [ACC] that, if \(S = \emptyset\) and \(N\) is an aspherical closed \(n\)-manifold with \(G = \pi_1(N)\), then the string product coincides with the loop product in the string topology of \(N\) [CS].

Finally, we review the homomorphism \(\lambda\) and define a bracket (Definition 2.2). Let us consider the correspondence \(\lambda : G \rightarrow C_1(G; \text{Ad}); x \mapsto x \otimes [x]\). As can be seen in [KK1] Lemma 3.4.2, we can easily check that \(\lambda(x)\) is a 1-cycle and \(\lambda(y^{-1}xy) = \lambda(x)\) holds in \(\mathcal{H}_1(G; \text{Ad})\). Thus, we have a homomorphism \(\lambda : \mathbb{Q}[\hat{G}] \rightarrow H_1(G; \text{Ad})\).
Definition 2.2. Suppose that \( S \subset G \) are \( Q \)-relative duality groups of dimension 2 and identify \( H_0(G; \text{Ad}) \) with \( Q[\hat{G}] \) as a \( Q \)-vector space. Then, we define a binary bracket,

\[
[,]: Q[\hat{G}] \otimes 2 \longrightarrow H_0(G; \text{Ad}) = Q[\hat{G}]
\]

by setting \([x, y] = \mu(\lambda(x), \lambda(y))\). Here, \( \mu \) is the string product with \( p = q = 1 \).

Example 2.3. Let \( G \) be the surface group \( \pi_1(\Sigma_{g,1}) \) and \( S \) be \( \pi_1(\partial \Sigma_{g,1}) = \mathbb{Z} \). It is shown in [KK1, Proposition 3.4.3] that the bracket is exactly equal to the Goldman Lie bracket \( \text{Gold} \) on \( Q[\hat{G}] \).

In so doing, we hope that this \([,] \) has the anti-commutativity property and satisfies the Jacobi identity for many pairs \( S \subset G \) of dimension two; however, this paper focuses on the fundamental groups from knots.

3 Results and Problems

Here, we state Theorem 3.1; the proof is in §4. We denote the complement \( \Sigma \setminus K \) by \( E_K \).

With a choice of base point in \( \partial E_K \), we fix an inclusion \( \mathbb{Z}^2 = \pi_1(\partial E_K) \hookrightarrow \pi_1(E_K) \). Since \( H_1(E_K; \mathbb{Z}) \cong \mathbb{Z} \), we focus on the infinite cyclic cover \( \tilde{E}_K \) of \( E_K \). Let \( \pi_K \) and \( \partial \pi_K \) be \( \pi_1(\tilde{E}_K) \) and \( \pi_1(\partial \tilde{E}_K) \), respectively. In other words, \( \pi_K = \pi_1(\tilde{E}_K) = [\pi_1(E_K), \pi_1(E_K)] \), and \( \partial \pi_K \cong \mathbb{Z} \).

Theorem 3.1. In the above notation, as in Theorem 1.1, assume that the complement \( \Sigma \setminus K \) is irreducible, and that the restriction map of \( \text{Ab} \) on \( \pi_1(T_i) \) is non-trivial for any \( i \leq m \), where \( T_i \) is a JSJ-torus as above.

Then, the pair \( \partial \pi_K \subset \pi_K \) has a \( Q \)-relative duality of dimension two. Furthermore, the bracket \([,]\) in Definition 2.2 admits a Lie algebra structure on \( H_0(\pi; \text{Ad}) = Q[\hat{\pi}_K] \).

Remark 3.2. We mention the assumption. For example, if \( K \) is either hyperbolic or fibered, then \( K \) obviously satisfies the assumption. From a private discussion with T. Ito [Ito], we can verify that \( \Sigma \setminus K \) is irreducible if and only if \( K \) is not contained in a 3-ball in \( \Sigma \setminus K \). Thus, using surgeries around the JSJ-tori \( T_j \), we can construct many examples of knots satisfying the assumption. Conversely, he points out some knots, which satisfy the irreducibility and do not satisfy the assumption. For example, if \( K \) is the Whitehead double of (the Whitehead double of \ldots) a torus knot, then the restriction map is trivial.

Remark 3.3. As seen in Remarks 7.4 and 8.5 in [MT], some bracket on a completed module of \( Q[\hat{\pi}_K] \) is discussed in terms of “Fox pairings” (see also [Tur2, Theorem E]); however, the point of Theorem 3.1 is that the bracket \([,]\) on \( Q[\hat{\pi}_K] \) is homologically defined before any completion and is shown to have the anti-commutativity property and satisfy the Jacobi identity. Furthermore, we conjecture a completion of our bracket \([,]\) is isomorphic to the bracket in [MT].

Now let us discuss the relation to the Goldman Lie algebra. Let \( F \) be a Seifert surface in \( \Sigma \setminus K \), where a Seifert surface is a smooth embedding \( i \) of \( \Sigma_{g,1} \) for some \( g \) to \( \Sigma \) such that \( i(\partial \Sigma_{g,1}) = K \) and the orientations are compatible with the embedding. We have a lift of \( F \) as a surface in \( \tilde{E}_K \), where the lift is unique up to covering transformations. Recall from Example 2.3 that we have the Goldman Lie algebra on \( Q[\pi_1(F)] \).
Proposition 3.4. The linear homomorphism \( \mathbb{Q}[\hat{F}] \to \mathbb{Q}[\pi_K] \) induced by the inclusion \( F \hookrightarrow \hat{E}_K \) preserves the brackets. In particular, if \( K \) is fibered and \( F \) is a fiber surface, the bracket on \( \mathbb{Q}[\pi_K] \) is equal to the Goldman Lie bracket on \( F \).

We defer to the proof in \( \S \). Next, as an application, we introduce other Lie algebras from Theorem 3.1. Consider the covering transformation \( t : \hat{E}_K \to \hat{E}_K \). This induces a group isomorphism \( t_* : \pi_K \to \pi_K \) and a bijection \( t_* : \pi_K \to \pi_K \). Thus, the Lie algebra on \( \mathbb{Q}[\pi_K] \) can be regarded as a \( \mathbb{Q}[t^{\pm 1}] \)-module. Furthermore, as is mentioned in Remark 4.4 later, \( \sigma_{\pi_K, \partial \pi_K} = t_\ast \sigma_{\pi_K, \partial \pi_K} \in H_2(\pi_K, \partial \pi_K; \mathbb{Q}) \). Thus, we have \([tx, ty] = t[x, y] \) by the definitions of the string product and \( \lambda \). Then, by the definition of the semi-direct product \( \pi_1(\hat{E}_K) \cong \pi_K \rtimes \mathbb{Z} \), the quotient set, \( \hat{\pi}_K/ \sim \), of \( \hat{\pi}_K \) determined by the relation \( x \sim t_\ast x \) can be regarded as the subset,

\[
\Pi_K := \{ [x] \in \pi_1(\hat{E}_K) \mid \text{The abelianization of } x = 0 \}.
\]

Thus, the quotient module \( \mathbb{Q}[\hat{\pi}_K] \) subject to the ideal \( (t-1) \) is isomorphic to \( \mathbb{Q}[\Pi_K] \) and is equipped with the induced Lie algebra.

Moreover, let us consider the abelianization, \( H_1(\pi_K; \mathbb{Z}) \), of \( \pi_K \), which is called the Alexander module (of \( K \)) in knot theory; see [Lic]. This can be identified with the set \( \hat{\pi}_K \setminus \pi_1(\pi_K) \setminus \{1\} \).

Therefore, the group ring \( \mathbb{Q}[H_1(\pi_K; \mathbb{Z})] \) has the quotient Lie algebra from Theorem 3.1 where \([tx, ty] = t[x, y] \) also holds. By construction, if \( K \) is fibered, the Lie algebra is isomorphic to the homological Goldman Lie algebra in [Gold p.295–p.297]. To summarize,

Corollary 3.5. Let \( \Pi_K \) be the above set and \( H \) be the abelianization \( H_1(\pi_K; \mathbb{Z}) \). Then, the Lie algebra structure in Theorem 3.1 induces those on \( \mathbb{Q}[\Pi_K] \) and \( \mathbb{Q}[H] \). The covering transformation induces Lie algebra isomorphisms \( t_* : \mathbb{Q}[\Pi_K] \to \mathbb{Q}[\Pi_K] \) and \( t_* : \mathbb{Q}[H] \to \mathbb{Q}[H] \).

Finally, we will discuss some problems. In hyperbolic geometry, some properties of hyperbolic surfaces can be (virtually) extended to those of hyperbolic 3-manifolds or hyperbolic groups. Thus, it is sensible to consider the following questions:

Problem 3.6. Define a Lie algebra structure on the \( \mathbb{Q} \)-vector space, \( \mathbb{Q}[\pi_1(\hat{E}_K)] \) spanned by the set of conjugacy classes of \( \pi_1(\hat{E}_K) \), which contains the Lie algebra in Corollary 3.5 as a subalgebra.

Problem 3.7. For a (relative) hyperbolic duality group \( G \) of dimension 2, define a Lie algebra structure on \( \mathbb{Q}[\hat{G}] \). For example, consider hyperbolic knots in a rational homology sphere.

Furthermore, let us discuss dual brackets. Turaev [Tur] defines a cobracket on \( \mathbb{Q}[\pi_1(\Sigma_{g,1})] \), which turns the Goldman Lie algebra into a Lie bialgebra (see, e.g., [KK2; MT] for further studies). Likewise, we may hope for a similar cobracket compatible with Theorem 3.1.

Problem 3.8. Define a cobracket \( \delta : \mathbb{Q}[\hat{\pi}_K] \to \mathbb{Q}[\hat{\pi}_K]^{\otimes 2} \), which turns the Lie algebra in Theorem 3.1 into a Lie bialgebra.

4 The proofs of the theorems

In what follows, we establish terminology. Let \( \hat{E}_K \) be the complement \( \Sigma \setminus K \), and \( \hat{E}_K \) be the infinite cyclic cover. As above, let \( \pi_K \) and \( \partial \pi_K \) be \( \pi_1(\hat{E}_K) \) and \( \pi_1(\partial \hat{E}_K) \), respectively. We fix
the JSJ-decomposition $E_K = M_1 \cup \cdots \cup M_n$. We also suppose that $\Sigma \setminus K$ is irreducible, and the restriction maps $\pi_1(T_j) \to \mathbb{Z}$ are non-trivial, as in the assumption of Theorem 1.1.

Then, the following four lemmas play key roles in the proofs.

**Lemma 4.1** (cf. [AFW, Theorem 2.5.2]). Let $g \in \pi_K$ with $g \not= 1$. Then, the centralizer subgroup $Z_{\pi_K}(g)$ is one among of $\mathbb{Z}$, $\mathbb{Z}^2$, the semi-direct product $\mathbb{Z} \rtimes \mathbb{Z}/2$, and $\pi_1(M_j) \cap \text{Ker}(\text{Ab})$ for some $j$, where $M_j$ is a Seifert piece of the JSJ-decomposition.

**Proof.** It directly follows from [AFW, Theorem 2.5.1] that, if $Z_{\pi_K}(g)$ is neither isomorphic to $\mathbb{Z}$ nor $\mathbb{Z}^2$, the equality $Z_{\pi_1(E_K)}(g) = Z_{\pi_1(M_j)}(g)$ holds for some $j$. In this case, $Z_{\pi_K}(g) = Z_{\pi_1(M_j)}(g) \cap \text{Ker}(\text{Ab})$ since $Z_{\pi_K}(g) = Z_{\pi_1(E_K)}(g) \cap \text{Ker}(\text{Ab})$. [AFW, Theorem 2.5.2] says that $Z_{\pi_1(M_j)}(g)$ is one among of $\mathbb{Z}$, $\mathbb{Z}^2$, $\mathbb{Z} \rtimes \mathbb{Z}/2$, or $\pi_1(M_j)$. Hence, the required statement is obvious.

**Lemma 4.2.** Suppose that $M_j$ is a Seifert piece. Then, the rational homology of the group $\pi_1(M_j) \cap \text{Ker}(\text{Ab})$ is of finite dimension.

**Proof.** Denote by $\tilde{M}_j$, $\tilde{T}_j$ the infinite cyclic covering space of $M_j$, $T_j$ associated the abelianization $\text{Ab} : \pi_1(E_K) \to \mathbb{Z}$. Since the restriction maps $\pi_1(T_i) \to \mathbb{Z}$ and $\pi_1(M_j) \to \mathbb{Z}$ are not trivial by assumption, the connected components of $\tilde{M}_j$ and $\tilde{T}_i$ are of finite order. In particular, $H^*(\tilde{T}_j; \mathbb{Q})$ is of finite dimension. Furthermore, notice that each $M_j$ is an Eilenberg-MacLane space by irreducibility and infinite order of $\pi_1(M_j)$, so is each connected component of $\tilde{M}_j$. Thus, $\pi_1(M_j) \cap \text{Ker}(\text{Ab}) \cong \pi_1(\tilde{M}_j, \ast)$ and $H_*(\tilde{M}_j; \mathbb{Q}) \cong H_*(\pi_1(\tilde{M}_j, \ast); \mathbb{Q})$.

Finally, let us consider the Mayer-Vietoris sequence

$$
\cdots \to \bigoplus_i H_*(\tilde{T}_i; \mathbb{Q}) \to \bigoplus_j H_*(\tilde{M}_j; \mathbb{Q}) \to H_*(\tilde{E}_K; \mathbb{Q}) \to \cdots.
$$

Since $H_*(\tilde{E}_K)$ is known to be of finite dimension (see [Mil, Assertion 5]), so is each $H_*(\tilde{M}_j; \mathbb{Q})$ as desired.

**Lemma 4.3.** The pair $\pi_K \subset \pi_K$ has a $\mathbb{Q}$-relative duality of dimension two.

**Proof.** The proof is a priori due to the Milnor pairing [Mil]. Take a Seifert surface $F \subset E_K$, and fix a lift of $F$ as a surface in $\tilde{E}_K$ up to covering transformations. Then, as a modification of the Milnor paring with local coefficient $A$ over $\mathbb{Q}$, if $H_*(\tilde{E}_K; A)$ is of finite dimension, there is an isomorphism,

$$
\delta : H^i(\tilde{E}_K, \partial \tilde{E}_K; A) \cong H_{i+1}^{\text{compact support}}(\tilde{E}_K, \partial \tilde{E}_K; A),
$$

such that the composite,

$$
Poincare dual \circ \delta : H^{2-i}(\tilde{E}_K, \partial \tilde{E}_K; A) \xrightarrow{\sim} H_{3-i}^{\text{compact support}}(\tilde{E}_K, \partial \tilde{E}_K; A) \xrightarrow{\sim} H_i(\tilde{E}_K; A),
$$

is equal to the cap product with $F$; see [Mil, Assertion 9 and Remark 3] or [Yana] for the proof.

Since $\tilde{E}_K$ and $\partial \tilde{E}_K$ are Eilenberg-MacLane spaces of type $(\pi_1(\tilde{E}_K), 1)$ and $(\mathbb{Z}^2, 1)$, respectively, we have an isomorphism $H_*(\tilde{E}_K; A) \cong H_*(\pi_K; A)$, which preserves the cup and cap products. In addition, for any $x \in \pi_K$, the homology $H_*(\tilde{E}_K; \mathbb{Q}[\pi_K / Z_{\pi_K}(x)]) \cong H_*(Z_{\pi_K}(x); \mathbb{Q})$. 


is of finite dimension by Lemmas 4.1 and 4.2 (here, we should notice that $H_1(\pi_K;\mathbb{Q})$ is also finite dimensional by [Mil, Assertion 5]). Hence, the pair $\partial \pi_K \subset \pi_K$ is a $\mathbb{Q}$-relative duality group pair of dimension 2.

**Remark 4.4.** Since $t(F)$ is bordant to $F$ in $E_K$ from the construction of $E_K$ using $F$ (see [Li, Chapters 6–7]) and the relative homology 2-class $\sigma_{\pi_K,\partial \pi_K}$ is represented by $F$, we have $t_*(\sigma_{\pi_K,\partial \pi_K}) = \sigma_{\pi_K,\partial \pi_K}$.

**Lemma 4.5.** Take duality group pairs $S \subset G$ and $S' \subset G'$ of dimension two and a group homomorphism $f : G \to G'$ satisfying $f(S) \subset S'$ and $f_*(\sigma_{G,S}) = N_f \sigma_{G',S'}$ for some $N_f \in \mathbb{Q}$. Assume that, for any $g \in G \setminus \{1\}$, the centralizers $Z_G(g)$ is infinite cyclic.

Then, the string product has the naturality $f_*(\mu(x,y)) = N_f \mu'(f_*(x), f_*(y))$.

**Proof.** We claim that, for any $g \in G \setminus \{1\}$, the image $f(Z_G(g))$ is contained in the centralizer $Z_{G'}(f(g))$. By assumption, any $h \in Z_G(g)$ admits $m \in \mathbb{Z}$ with $h^m = g$. In general, if $g', h' \in G'$ satisfy $(h')^m = g'$, then the roof $h'$ of $g'$ necessarily lies in $Z_{G'}(g')$; see [AFW, Page 37]. Therefore, $f(h)^m = f(g)$ implies $f(h) \in Z_{G'}(f(g))$. Then, the required naturality follows from the claim and the definition of the string product.

Using the above lemmas, we will show Proposition 3.4.

**Proof of Proposition 3.4.** Let $G = \pi_1(\Sigma_{g,1})$ and $G'$ be $\pi_K$, where the Seifert surface of genus $g$ is regarded as $\Sigma_{g,1}$. Similarly, $Z(h)$ for any $h \in G \setminus \{1\}$ is widely known to be infinite cyclic (see, e.g., [KKL, Proposition 3.4.3]). By Lemma 4.5 with $N_f = 1$, the linear homomorphism preserves the Lie bracket.

It remains to show Theorems 3.1 and 1.1. For the proofs, we review the Heegaard decomposition of $E_K$ and group presentation of $\pi$; see [3] below. It is known (see, e.g., [Sav, Lemma 17.2]) that we can choose a Seifert surface $F$ of genus $g$ such that the complement $\Sigma \setminus F$ is homeomorphic to a handlebody of genus $g$. Fix a meridian $m \in \pi_1(E_K)$. Take a bicollar $F \times [-1,1]$ of $F$ such that $S \times \{0\} = F$. Let $\iota_\pm : F \to \Sigma \setminus F$ be embeddings whose images are $F \times \{\pm1\}$.

Take generating sets $W := \{u_1, \ldots, u_{2g}\}$ of $\pi_1$ and $X := \{x_1, \ldots, x_{2g}\}$ of $\pi_1(\Sigma \setminus F)$. Set $y_i := (\iota_+)_*(u_i)$ and $z_i := (\iota_-)_*(u_i)$; a van Kampen argument yields a presentation of $\pi_1(\Sigma \setminus K)$:

\[ \langle m, x_1, \ldots, x_{2g}, \mid m^{-1}y_imz_i^{-1} \quad (1 \leq i \leq 2g) \rangle. \quad (2) \]

Then, by using the Reidemeister-Schreier method, $\pi_1(\widetilde{E}_K)$ is presented by

\[ \langle x_1^{(n)}, \ldots, x_{2g}^{(n)} \mid (n \in \mathbb{Z}) \mid y_i^{(k)}(z_i^{(k+1)})^{-1} \quad (1 \leq i \leq 2g, \ n \in \mathbb{Z}) \rangle. \quad (3) \]

**Proof of Theorem 3.1.** As a preparation, for $k, \ell \in \mathbb{Z}$ with $k < \ell$, consider the copies $t^k(F), t^{k-1}(F), \ldots, t^\ell(F)$, and the connected sum $F_{k,\ell} := t^k(F) \# t^{k-1}(F) \# \cdots \# t^\ell(F)$ in $\widetilde{E}_K$, where the connected sum is obtained by adding half-tubes around $\partial K = \partial F \times (-\infty, \infty)$. Then, by Remark 4.4, $[F_{k,\ell}, \partial F_{k,\ell}] = (\ell - k + 1)[F, \partial F]$ in $H_2(\widetilde{E}_K;\mathbb{Q})$. By the construction of [3], the image of $\pi_1(F_{k,\ell})$ by inclusion $F_{k,\ell} \subset \widetilde{E}_K$ is the subgroup of $\pi_1(\widetilde{E}_K)$ generated by the set $\{x_1^{(n)}, \ldots, x_{2g}^{(n)} \mid k \leq n \leq \ell\}$. Let us denote the subgroup by $G_{k,\ell}$. 
Next, we show the Jacobi identity of $[,]$. For $[a_1], [a_2], [a_3] \in \hat{\pi}_K$, take respective representatives $a_1, a_2, a_3 \in \hat{\pi}_K$. By (3), there are $k, \ell \in \mathbb{Z}$ such that $a_1, a_2, a_3 \in \hat{G}_{k, \ell}$. By Lemma 4.5 with $N_f = \ell - k + 1$, the inclusion $i : F_{k, \ell} \subset \tilde{E}_K$ induces $i_* : Q[\hat{G}_{k, \ell}] \to Q[\hat{\pi}_K]$, which preserves the bracket up to constant multiples and admits $b_j \in \pi_1(F_{k, \ell})$ with $i_*([b_j]) = [a_j] \in \hat{\pi}_K$. The Goldman Lie algebra on $Q[\hat{G}_{k, \ell}]$ in Example 4.4 inherits the required Jacobi identity. Precisely,
\[
[[a_1, a_2], a_3] + [[a_2, a_3], a_1] + [[a_3, a_1], a_2] = (\ell - k + 1)i_*([[b_1, b_2], b_3] + [[b_2, b_3], b_1] + [[b_3, b_1], b_2]) = 0.
\]
Finally, the anti-commutativity of $[,]$ directly comes from that of the intersection pairing.

(Proof of the main theorem 1.1). The former claim readily follows from Theorem 3.1. Recall from §3 that the covering transformation $t$ induces an isomorphism $t : Q[\pi_K] \to Q[\pi_K]$ preserving the bracket $[,]$; the latter part follows from Proposition 3.1 since $\pi_K$ of a fibered knot $K$ of genus $g$ is isomorphic to $\pi_1(\Sigma_{g,1})$.

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References

[AFW] Matthias Aschenbrenner, Stefan Friedl, and Henry Wilton. 3-manifold groups, EMS Series of Lectures in Mathematics. European Mathematical Society, Zürich, 2015.

[ACC] Abbaspour, Hossein; Cohen, Ralph; Gruher, Kate. String topology of Poincaré duality groups. Groups, 1–10, Geom. Topol. Monogr., 13, Geom. Topol. Publ., Coventry, 2008. MR2508199

[BE] Bieri, Robert; Eckmann, Beno. Relative homology and Poincaré duality for group pairs. J. Pure Appl. Algebra 13 (1978), no. 3, 277–319. MR0509165

[CS] Chas. M, Sullivan Dennis, String topology, Preprint, math.GT/9911159.

[Gold] Goldman, William M. Invariant functions on Lie groups and Hamiltonian flows of surface group representations. Invent. Math. 85 (1986), no. 2, 263–302. MR0846929

[Lic] Lickorish, W. B. Raymond. An introduction to knot theory. Graduate Texts in Mathematics, 175. Springer-Verlag, New York, 1997. x+201 pp. ISBN: 0-387-98254-X MR1472978

[Ito] Ito, Tetsuya. private communications.

[KK1] Kawazumi, Nariya; Kuno, Yusuke. The logarithms of Dehn twists, Quantum Topol. 5 (2014), no. 3, 347–423. MR3283405

[KK2] Kawazumi, Nariya; Kuno, Yusuke. The Goldman-Turaev Lie bialgebra and the Johnson homomorphisms. Handbook of Teichmüller theory. Vol. V, 97–165, IRMA Lect. Math. Theor. Phys., 26, Eur. Math. Soc., Zürich, 2016. MR3497295

[Mil] Milnor, John W. Infinite cyclic coverings. 1968 Conference on the Topology of Manifolds (Michigan State Univ., E. Lansing, Mich., 1967) pp. 115–133 Prindle, Weber & Schmidt, Boston, Mass. MR0242163

[MT] Massuyeau, Gwénaël; Turaev, Vladimir. Fox pairings and generalized Dehn twists. Ann. Inst. Fourier (Grenoble) 63 (2013), no. 6, 2403–2456. MR3237452

[Sav] Saveliev, Nikolai. Lectures on the topology of 3-manifolds. An introduction to the Casson invariant. Second revised edition. De Gruyter Textbook. Walter de Gruyter & Co., Berlin, 2012. xii+207 pp. ISBN: 978-3-11-025035-0 MR2893651
[Tur1] Turaev, Vladimir. *Skein quantization of Poisson algebras of loops on surfaces*. Ann. Sci. École Norm. Sup. (4) 24 (1991), no. 6, 635–704. MR1142906

[Tur2] Turaev, Vladimir. *Multiplace generalizations of the Seifert form of a classical knot*. (Russian) Mat. Sb. (N.S.) 116(158) (1981), no. 3, 370–397, 463–464. MR0665689

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