Conformal Invariance in Inverse Turbulent Cascades

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We study statistical properties of turbulent inverse cascades in a class of nonlinear models describing a scalar field transported by a two-dimensional incompressible flow. The class is characterized by a linear relation between the transported field and the velocity, and include several cases of physical interest, such as Navier-Stokes, surface quasi-geostrophic and Charney-Hasegawa-Mima equations. We find that some statistical properties of the inverse turbulent cascades in such systems are conformal invariant. In particular, the zero-isolines of the scalar field are statistically equivalent to conformal invariant curves within the resolution of our numerics. We show that the choice of the conformal class is determined by the properties of a transporting velocity rather than those of a transported field and discover a phase transition when the velocity turns from a large-scale field to a small-scale one.

Exceptional role of conformal invariance in theoretical physics stems from the fact that most of non-trivial exact solutions of dynamical and statistical models can be traced to the existence of this symmetry. One of the most remarkable recent advances in mathematics was the discovery of Schramm-Loewner Evolution (SLE) and of the bridges it builds between different branches of physics\(^1\)\(^-\)\(^3\). SLE is a class of fractal random curves that can be mapped into a one-dimensional Brownian walk and thus have conformal invariant statistics. SLE curves appear at two-dimensional (2d) critical phenomena as cluster boundaries, thus revealing a statistical geometry of conformal field theories.

In equilibrium, the statistical weight of a state does not depend on how the state was created. If a system is driven away from equilibrium by an external force then the probability of a given configuration depends generally on the history and on the statistics of the driving force. Turbulence statistics are thus generally force-dependent. All the more surprising was then the experimental discovery that the isolines of vorticity in the 2d Navier-Stokes turbulence and of temperature in the Surface Quasi-Geostrophic (SQG) turbulence belong to SLE\(^1\)\(^-\)\(^3\). That means that at least a part of turbulence statistics could be described in terms of a conformal field theory like equilibrium critical phenomena. In particular, nodal vorticity lines happen to be equivalent to the boundaries of percolation clusters\(^4\), while the iso-temperature lines in SQG are equivalent to the domain walls of SO(2) model (that of a 2d gaussian free field)\(^5\). Having only two examples leaves wide open possibilities for different interpretations and hypothesis, particularly trying to relate the scaling and properties of the bulk field to the choice of the curve class for its isolines\(^6\). Here we study additional models from the family and show that the SLE class is actually sensitive to the type of dynamics (i.e. velocity) rather than to the type of a field that is carried; that sensitivity manifested dramatically by the phase transition we discover.

The class of models we investigate has been introduced in\(^5\)\(^-\)\(^8\). It describes the evolution of a scalar field \(\theta\) transported by an incompressible two-dimensional velocity \(\mathbf{u} = (\partial_y\psi, -\partial_x\psi)\), expressed via the stream function \(\psi\). The scalar field \(\theta\) is “active” because it is linearly related to \(\psi\) and \(\mathbf{u}\). In Fourier space the relation reads: \(\theta(k) = |k|^m \psi(k)\). The system is thus governed by the equation

\[
\partial_t \theta + (\mathbf{u} \nabla) \theta = \partial_t \theta + \{\theta, \psi\} = F + D, \tag{1}
\]

where \(\{\theta, \psi\} = \theta_x \psi_y - \theta_y \psi_x\), \(F\) and \(D\) are external forcing and dissipation respectively. Different values of \(m\) give different well-known hydrodynamic equations. For \(m = 2\) one obtains two-dimensional Navier-Stokes (NS) equation, \(\theta\) being the vorticity. For \(m = 1\) the field \(\theta\) represents the temperature in SQG turbulence. Finally, for \(m = -2\) the model corresponds to that derived by Charney and Oboukhov for waves in rotating fluids and by Hasegawa and Mima for drift waves in magnetized plasma in the limit of vanishing Rossby radius (ion Larmor radius for plasma physics).

At all values of \(m\) equation \((1)\) possesses two positive-definite invariants for \(F = D = 0\), namely \(Z = \int \theta^2 d\mathbf{x}\) and \(E = \int \theta \psi d\mathbf{x}/2\). When the system is forced by an external source of scalar fluctuations \(F\), with a correlation length \(\ell_f \sim 1/k_f\), the existence of two conserved quantities causes double turbulent cascade. The sign of \(m\) determines the direction of the cascades. For \(m > 0\) the “energy” \(E\) is transferred toward large scales \(\ell > \ell_f\) giving rise to an inverse cascade, and the “enstrophy” \(Z\) flows toward small scales. The cascades are reversed for \(m < 0\).

Here we focus on the range of scales corresponding to an inverse cascade. Dimensional argument based on the assumption of scale-independence of the flux of energy in the inverse cascade (for \(m > 0\)) gives the scaling exponent \(h = (2 - 2m)/3\) for the the increments
The first remarkable discovery of conformal invariance in turbulence has been made for the zero-vorticity lines in Navier-Stokes turbulence i.e. for $m = 2$ [4]. Zero-vorticity regions correspond to $\Delta \psi = 0$, i.e. to a harmonic stream-function and are invariant with respect to conformal transformations (which thus map streamlines into themselves). One may think that conformal invariance of zero-vorticity lines is a remnant of the invariance of zero-vorticity domains and is peculiar for $m = 2$. However, conformal invariance of the isolines was then discovered for $m = 1$ [4], where one does not recognize an analogous property of zero-$\theta$ domains. It is then tempting to relate conformal invariance to the properties of $\theta$ which are common for all $m$. The main property seems to be the fact that $\theta$ is a Lagrangian invariant of the flow and determines the symplectic structure. For example, if we denote $\mathbf{R} = (X,Y)$ the initial (Lagrangian) coordinates of the fluid particles then the extremum of the action $I = \int S(t) \, dt$ with

$$S_2(\mathbf{R}) = \int \theta(\mathbf{R}) x(\mathbf{R}, t) \dot{y}(\mathbf{R}, t) \, d\mathbf{R}$$

$$-\frac{1}{2} \int \theta(\mathbf{R}_1) \theta(\mathbf{R}_2) \ln |\mathbf{r}(\mathbf{R}_1) - \mathbf{r}(\mathbf{R}_2)| \, d\mathbf{R}_1 d\mathbf{R}_2$$

(2)

gives $\dot{x} = \partial_y \psi$ and $\dot{y} = -\partial_x \psi$ which is equivalent to [1] for $m = 2$. Generally,

$$S_m(\mathbf{R}) = \int \theta(\mathbf{R}) x(\mathbf{R}, t) \dot{y}(\mathbf{R}, t) \, d\mathbf{R}$$

$$-\frac{1}{2} \int \theta(\mathbf{R}_1) \theta(\mathbf{R}_2) \ln |\mathbf{r}(\mathbf{R}_1) - \mathbf{r}(\mathbf{R}_2)|^{m-2} \, d\mathbf{R}_1 d\mathbf{R}_2.$$  (3)

In other words, the energy $E = \int \psi(\mathbf{r}) x(\mathbf{r}, t) \, d\mathbf{r}/2 = \int \theta(\mathbf{r}_1) \theta(\mathbf{r}_2) |\mathbf{r}(\mathbf{r}_1) - \mathbf{r}(\mathbf{r}_2)|^{m-2} \, d\mathbf{r}_1 d\mathbf{r}_2/2$ is the Hamiltonian. It is tempting to conjecture [4] that zero-$\theta$ lines are special since the Hamiltonian description is singular (non-invertible) there. However, at negative $m$, $\theta$ is a large-scale field and its isolines are not fractal (have dimensionality 1). It is then natural to study the properties of the isolines of $\psi$ which are fractal now. We show below that at $m = -2$ the isolines of $\psi$ seem to have the same statistical properties as the isolines of $\theta$ at $m = 2$, despite the fact that $\psi$ is not a Lagrangian invariant and the equation has no symmetry $m \to -m$.

To investigate the statistical properties of the scalar field $\theta$ we solved numerically eq. [1] on a doubly periodic square domain of size $L = 2\pi$ at different resolution $N^2 = 1024^2, 2048^2$. The scalar fluctuations are sustained by a Gaussian, $\delta$-correlated in time, random forcing, peaked around wavenumber $k_f = 100$. Dealiasing cut-off is set to $k_t = N/3$. Time evolution was computed by means of a second-order Runge-Kutta scheme, with implicit handling of the linear dissipative terms. The direct cascade of enstrophy is halted at wavenumbers $k > k_f$ by means of a hyper-viscous damping $(-1)^{p-1} \nu_{p-1} |\nabla^2 \psi| \, d\mathbf{r}$ of order $p = 8$. Statistically steady state in the inverse cascade is obtained by removing the energy at large scales with a linear friction term $-\eta \theta$. Note that for $m > 0$ the characteristic times of the inverse cascade process scales as $\tau \sim f(4-m)/3$ i.e. the cascade slows down as $m$ goes to zero. This phenomenon limits the resolution achievable in numerical simulations.
The scalar field resulting from numerical simulations with $0 < m \leq 2$ is scale invariant, as confirmed by the perfect collapse of the probability distribution functions (pdfs) of scalar increments $\delta_r \theta$ for different $r$, see e.g. Figure 2. Note that the pdfs are non-Gaussian. The scalar field also has a power-law spectrum for $k < k_f$ in agreement with the prediction:

$$P_\theta(k) = C_m \epsilon^{2/3} k^\zeta$$

where $\zeta = (4m - 7)/3$, and $\epsilon$ is the flux of energy (see e.g. Figure 3).

The limit $m \to 0$ of the active scalar model is singular. Indeed for $m = 0$ the two fields $\theta$ and $\psi$ coincide, and the advection term $\{\theta, \psi\}$ in eq. (1) vanishes. Therefore no turbulent state can be produced and the field $\theta$ is simply determined by local balance between forcing and the dissipation at exactly $m = 0$. Conversely, for arbitrary small values of $m$ we find a turbulent cascade with power law spectrum in agreement with eq. (4) (see Figure 4). As the parameter $m$ goes to zero, the amplitude of the scalar field diverges, to compensate for the less efficient transfer of energy in the cascade. This is signalled by the power law behavior of the analogous of Kolmogorov’s constant for the spectrum $C(m) \sim m^{-2/3}$ (see inset of Figure 4).

To study the limit $m \to 0$ let us write the l.h.s. of eq. (1) in $k$-space, and use the symmetry $j \leftrightarrow k-j$:

$$\frac{\partial \theta_k}{\partial t} = \frac{1}{m} \sum_j [\theta_j] j^{-m} \theta_j \theta_{k-j}$$

$$= \frac{1}{2m} \sum_j \{ [k,j] j^{-m} + [k, k-j] |k-j|^{-m} \} \theta_j \theta_{k-j}$$

$$= \frac{1}{2m} \sum_j [k,j] \{ j^{-m} - |k-j|^{-m} \} \theta_j \theta_{k-j}$$

(5)

where $[k,j] = k_1 j_2 - k_2 j_1$, $k = |k|$ and $j = |j|$. In the limit $m \to 0$, equation (5) has still the form of a transport equation with the link between $\theta$ and the stream function being $\psi(\theta) = -\ln |k| \theta(k)$. Renormalizing $t \to mt$ one gets

$$\frac{\partial \theta_k}{\partial t} = \frac{1}{2} \sum_j [k,j] \ln(j/|k-j|) \theta_j \theta_{k-j} + F + D$$

(6)

Numerical integration of eq. (6) produces an inverse turbulent cascade with power law spectrum $P_\theta(k) \sim k^{-7/3}$. (see Figure 5) In the range of scales of the inverse cascade the field $\theta$ is self similar with scaling exponent $h = 2/3$, as confirmed by the re-scaling of the pdfs of scalar increments (see Figure 2).

Numerical investigation of Navier-Stokes (NS) equation [4] and Surface Quasi Geostrophy (SQG) model [5]
have shown that for two peculiar cases, namely for $m = 2, 1$ the zero-isolines of the scalar field are statistically equivalent to SLE i.e. could be mapped to 1d Brownian walk. The class SLE is characterized by the respective dimensionless diffusivity $\kappa$ [1, 2]. In particular for NS the zero-vorticity isolines belong to the same universality class of critical percolation and are equivalent to SLE curves with $\kappa = 6$. For SQG the zero-temperature isolines are SLE curves with $\kappa = 4$. It is therefore natural to ask if the properties of conformal invariance for the zero-isolines is a general property that holds for arbitrary values of $m$.

To investigate this issue we consider the connected regions of positive/negative sign of $\theta$. The boundaries of these clusters are closed loops formed by the zero-$\theta$ isolines.

For $0 < m \leq 1$ the scalar $\theta$ is a self-similar rough field with scaling exponent $0 < h < 1$. The relation between the scaling exponent $h$ of a height function and the fractal dimension $D$ of its isolines was suggested in [10]:

$$\frac{3 - h}{2} = \frac{7 + 2m}{6}$$

for $0 < h < 1$ and $0 < m < 1$. Let us stress that this is not the fractal dimension of the iso-set (known to be equal to $2 - h$ for $h > 0$ and to 2 for $h < 0$) but that of a single long isoline. One can thus conjecture the relation $\kappa = 4(1 + 2m)/3$.

Indeed for $m = 1$ it was found [5] that the zero-isolines are SLE curves with $\kappa = 4$, in agreement with the above prediction. Nevertheless, in Figures 6 and 7 we show the fractal dimension of the isolines for $m = 1/2$ and for the asymptotic model $m \to 0$. It both cases the fractal dimension measured is not agreement with the prediction $D = (7 + 2m)/6$ but is compatible with $D = 3/2$. 

FIG. 6. Perimeter $P$ of zero-isolines versus gyration radius $L$ for $m = 1/2$. Forcing length scale is $r_f \sim 0.06$. In the inset we show the local slope of perimeter $P$ before and after randomization of the phases of the scalar field $\theta$ (solid and dashed line respectively).

FIG. 7. Perimeter $P$ of zero-isolines versus gyration radius $L$ for $m \to 0$. Forcing length scale is $r_f \sim 0.06$.

FIG. 8. Statistics of the driving $\xi(t)$ for $m = 1/2$

FIG. 9. Statistics of the driving $\xi(t)$ for $m \to 0$
Following the procedure described in [5] from the zero-field lines we obtain an ensemble of curves in the half plane which are expected to converge in the scaling limit to chordal SLE. Then we extract the driving $\xi(t)$ of the corresponding Loewner equation. As shown in Figures 8 and 9 the driving has Gaussian statistics with variance $\langle \xi^2(t) \rangle \sim \kappa t$. For all the cases considered with $0 < m < 1$ we found $\kappa = 4$, which is in agreement with the fractal dimension observed.

As a further test we study the statistics of the winding angle of the zero-isolines. The winding angle $\phi$ is defined as the degree with which the curve wind in the complex plane about a point $w \phi(z) = \arg(z - w)$ [11, 12]. The asymptotic distribution for the winding angle at long distance $\ell$ along the curve is Gaussian, with variance $\kappa/(4 + \kappa/2) \log(\ell)$. As shown in Figures 10 and 11 we found that its variance grows like $2/3 \log(\ell)$, thus supporting the conjecture $\kappa = 4$, and is not compatible with the prediction $\kappa = (4 + 8m)/3$.

This model provides an example of non-trivial relation between the scaling exponent of the field $\theta$ and the fractal dimension of its isolines. For $0 < m < 1$ the scaling exponent varies in the range $0 < h < 2/3$, but the fractal dimension remains constant $D = 3/2$, at variance with what one would expect from the relation $D = (3 - h)/2$ which holds for Gaussian random field. The crucial difference is that the scalar field $\theta$ is not random, but is the result of turbulent dynamics.

Our findings can be understood in Lagrangian terms. The scaling exponent of the velocity field is $h_v = (m - 1)/3$. For $m > 1$ the scaling exponent is positive, and therefore velocity difference scales as $\delta v(t) \propto t^{(m-1)/3}$. Two Lagrangian trajectories moving in such velocity field separate according to the Richardson law $\ell(t) \sim t^3/(4-m)$. Conversely for $0 < m < 1$, the exponent is negative (that is the velocity is a small-scale field like vorticity in Navier-Stokes), and velocity differences are independent of the separation $\delta v(t) \approx v_{rms}$. Lagrangian trajectories will separate as $\ell(t) \propto t$. Perimeter $P$ and gyration radius $L$ of clusters can be related by assuming that their ratio $P/L$, which is proportional to the number of folds, grows as a random walk, i.e. as $t^{1/2}$. Gyration radius grows as two-point distance $L(t) \propto \ell(t)$, which gives $P \propto P t^{1/2} \propto L^{(10-m)/6}$ for $m \geq 1$ and $P \propto L^{3/2} \propto 0 < m < 1$.

The property of conformal invariance of the isolines is therefore determined by the underlying dynamics of the field. As a test we took the field $\theta$ and randomize its phases in Fourier space. This procedure does not change the scaling exponent of the field, but destroys all the correlations generated by the turbulent dynamics. The isolines of this randomized field are no more conformal invariant, but their fractal dimension recover the predic-
FIG. 13. Statistics of the driving \( \xi(t) \) for the stream-function isolines for \( m = -2 \)

Under the hypothesis that the isoline of the scalar field \( \theta \) are SLE curves one can obtain a conjecture for their universality class \( \kappa \) from the formula for the fractal dimension \( D_\ast = 1 + 2/\kappa \) of the outer perimeter \( P \), which holds for \( \kappa \geq 4 \) \[13\]. One obtains \( \kappa = 4 \) for \( 0 < m < 1 \) and \( \kappa = 12/(4 - m) \) for \( m > 1 \), which are in agreement with our findings \( (m = 3/2, 1/2 \text{ and } m \to 0) \) and with previous results \( (m = 1, 2) \).

Approach based on Schramm-Loewner Evolution provides a refreshingly novel geometric insight into the statistics of turbulence and hints at deep symmetry aspects of 2d flows which we are yet far from understanding.

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