Moderate deviations for log-like functions of stationary Gaussian processes

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Abstract
A moderate deviation principle for nonlinear functions of Gaussian processes is established. The nonlinear functions need not be locally bounded. Especially, the logarithm is allowed. (Thus, small deviations of the process are relevant.) Both discrete and continuous time is treated. An integrable power-like decay of the correlation function is assumed.

Introduction

Questions on moderate deviations of random complex zeros \cite{5} lead naturally to questions on moderate deviations of the logarithm of the absolute value of a complex-valued Gaussian random field. Recently, Djellout, Guillin and Wu established a moderate deviation principle for (nonlinear) functions of dependent random variables (Gaussian, and more general) \cite{3} Th. 2.7. However, their result does not answer the questions mentioned above, for several reasons. The most important reason is that the logarithm is not a differentiable function (nor even locally bounded).

The main result of the present work is another moderate deviation principle. Unlike \cite{3}, I restrict myself to Gaussian processes, but admit some non-locally-bounded functions (like the logarithm). My technique is rather far from that of \cite{3}, and includes some arguments about small deviations. Indeed, small values of the process lead to large negative values of the logarithm.

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1 Assumptions on the (nonlinear) function

Let $F : \mathbb{R}^d \to \mathbb{R}$ be a measurable function satisfying

\[(1.1) \quad \int F \, d\gamma^d = 0,\]
\[(1.2) \quad \int e^F \, d\gamma^d < \infty, \quad \int e^{-F} \, d\gamma^d < \infty;\]

here and henceforth $\gamma^d$ is the standard Gaussian measure on $\mathbb{R}^d$,

\[
\gamma^d(dx) = (2\pi)^{-d/2} \exp\left(-\frac{1}{2} |x|^2\right) \, dx.
\]

For any $r \in (0, \infty)$ we define $F_r : \mathbb{R}^d \to (-\infty, +\infty]$ by

\[(1.3) \quad F_r(x) = \sup_{|y-x| \leq r} F(y).\]

Here is the main assumption on $F$: there exists $C < \infty$ such that

\[(1.4) \quad \int \exp(F_r(x+0.5y) - F(x+0.5y)) \, d\gamma^d(dy) \leq e^{Cr}\]

for all $x \in \mathbb{R}^d$ and $r \in (0, \infty)$.

Assumptions (1.1), (1.2), (1.4) will be referred to as ‘the assumptions of Sect. 1’.

Clearly, these assumptions are not invariant under replacement of $F$ with $aF$ for an arbitrary coefficient $a \in (-\infty,0) \cup (0,\infty)$. (Even $a = -1$ is not permitted by (1.4).) However, our main results (Theorems 3.1, 3.2) are evidently invariant under such replacement. Thus, we could assume that $aF$ satisfies (1.1), (1.2), (1.4) for some $a \neq 0$.

1.5 Example. Let $F$ be a Lipschitz function, that is, $|F(x) - F(y)| \leq C|x-y|$ for all $x, y \in \mathbb{R}^d$. Then the function $F(\cdot) - \int F \, d\gamma^d$ satisfies the conditions. Especially, (1.4) holds just because $F_r(x) \leq F(x) + Cr$.

1.6 Example. Let $d \geq 2$. The function

\[
F(x) = \ln |x| - \int \ln |x| \, \gamma^d(dx)
\]

satisfies the conditions. Proof of (1.4): if $|y-x| \leq r$ then $F(y) - F(x) = \ln \frac{|y|}{|x|} \leq \ln \left(1 + \frac{r}{|x|}\right) \leq \ln(1 + \frac{r}{|x|})$, thus, $F_r(x) - F(x) \leq \ln(1 + \frac{r}{|x|})$ and

\[
\int \exp(F_r(0.5y) - F(0.5y)) \gamma^d(dy) \leq \int \left(1 + \frac{2r}{|y|}\right) \gamma^d(dy) = 1 + 2r \int \frac{1}{|y|} \gamma^d(dy) \leq \exp\left(2r \int \frac{1}{|y|} \gamma^d(dy)\right),
\]

2
which is (1.4) for \(x = 0\). For arbitrary \(x\) we have
\[
\int \frac{1}{|x + y|} \gamma^d(dy) \leq \int \frac{1}{|y|} \gamma^d(dy),
\]
which follows from the Anderson inequality, see for instance [2, Th. 1.8.5 and Cor. 1.8.6].

Taking the limit \(r \to 0\) in (1.4) we get
\[
\int |\nabla F(x + 0.5y)| \gamma^d(dy) \leq C
\]
for a smooth \(F\); by approximation, (1.4) implies that the first derivatives of \(F\) are locally finite measures. For the one-dimensional case \((d = 1)\) it means that (1.4) can hold only for locally bounded \(F\). (However, \(F\) need not be continuous.) Especially, the function \(x \mapsto \log |x|\) on \(\mathbb{R}\) violates (1.4).

2 Assumptions on the Gaussian process

Discrete time

Similarly to [3] we consider a process \((X_n)_{n \in \mathbb{Z}}\) that can be written in the form
\[
X_n = \sum_{j \in \mathbb{Z}} a_{j-n} \xi_j = \sum_{j \in \mathbb{Z}} a_j \xi_{n+j}
\]
('moving average process'), where \(\xi_n\) are independent. (About convergence of the series, see below.) Unlike [3] we assume that each \(\xi_n\) is an \(\mathbb{R}^d\)-valued random variable distributed \(\gamma^d\), and each \(a_j\) is a matrix \(d \times d\). We assume that each \(X_n\) is also distributed \(\gamma^d\); it means that \(\sum a_j a_j^*\) is the unit matrix. (Here \(a^*\) is the conjugate matrix.) The main assumption:
\[
a_j = O\left(\frac{1}{|j|^{1.5+\varepsilon}}\right) \quad \text{for some } \varepsilon > 0.
\]

The said above will be referred to as ‘the assumptions of Sect. 2 for discrete time’.

It follows from (2.2) that \(\sum |a_j| < \infty\), which is more than enough for convergence of the series (2.1). The Fourier transform
\[
g(\theta) = \sum_{n \in \mathbb{Z}} a_n e^{in\theta}
\]
is a continuous \(2\pi\)-periodic matrix-valued function. The same holds for the spectral density \(f\),
\[
f(\theta) = \frac{1}{2\pi} g(\theta) g^*(\theta) ; \quad \int_{-\pi}^{\pi} e^{in\theta} f(\theta) d\theta = \mathbb{E} \left( X_n X_0^* \right).
\]
It follows from (2.2) that \( \| g(\theta) - g(\eta) \| = O(|\theta - \eta|^{-0.5+\varepsilon}) \) (see for instance [1 Sect. 11.3]). On the other hand, every twice continuously differentiable (matrix-valued) function \( g \) is Fourier transform of a sequence \((a_j)\) satisfying (2.2) for \( \varepsilon = 0.5 \).

If \( f(\cdot) \) is continuously differentiable twice and \( \det f(\cdot) \) does not vanish then \( f(\cdot) \) is of the form \( f(\theta) = (2\pi)^{-1} g(\theta) g^*(\theta) \) with \( g(\cdot) \) satisfying (2.1). (Just take the positive square root of the positive matrix \( f(\theta) \).) Thus, a process with such a spectral density belongs to our class, provided that \( \int_{-\pi}^{\pi} f(\theta) \, d\theta \) is the unit matrix, and the process is centered (zero-mean).

**Continuous time**

Here we consider a process \((X_t)_{t \in \mathbb{R}}\) that can be written in the form

\[
X_t = \int a_{s-t} \, dw_s = \int a_s \, dw_{t+s},
\]

where \((w_s)_{s \in \mathbb{R}}\) is the standard \( d \)-dimensional Brownian motion (two-sided; the past \( s \leq 0 \) and the future \( s \geq 0 \) are independent, and \( w_0 = 0 \)), and \( s \mapsto a_s \) is a continuous matrix-valued function on \( \mathbb{R} \). (The matrices are of size \( d \times d \).) We assume that each \( X_t \) is distributed \( \gamma^d \); it means that

\[
\int a_s a_s^* \, ds < \infty,
\]

which is more than enough for the linear stochastic integrals (2.3) to be well-defined. The Fourier transform

\[
g(\lambda) = \int_{-\infty}^{\infty} a_s e^{i\lambda s} \, ds
\]

is a continuous matrix-valued function. The same holds for the spectral density \( f \),

\[
f(\lambda) = \frac{1}{2\pi} g(\lambda) g^*(\lambda) ; \quad \int_{-\infty}^{\infty} e^{i\lambda t} f(\lambda) \, d\lambda = \mathbb{E}(X_t X_0^*).
\]

It follows from (2.4), (2.5) that

\[
g(\lambda) = O(|\lambda|^{-\varepsilon}) , \quad g(\lambda) - g(\mu) = O(|\lambda - \mu|^{0.5+\varepsilon}).
\]
On the other hand, every twice continuously differentiable (matrix-valued) function $g$ such that $g(\lambda)$ and $g''(\lambda)$ are $O(|\lambda|^{-1-\varepsilon})$ (as $|\lambda| \to \infty$) is Fourier transform of a function $s \mapsto a_s$ such that the functions $s \mapsto a_s$ and $s \mapsto s^2 a_s$ are bounded and Hölder continuous, thus, (2.4), (2.5) are satisfied.

3 The result

3.1 Theorem. Let a function $F : \mathbb{R}^d \to \mathbb{R}$ satisfy the assumptions of Sect. 1, and a process $(X_n)_{n \in \mathbb{Z}}$ satisfy the assumptions of Sect. 2 for discrete time. Then
(a) the following limit exists:

$$\sigma^2 = \lim_{n \to \infty} \frac{1}{n} \mathbb{E} \left( (F(X_1) + \cdots + F(X_n))^2 \right) ;$$

(b) if $\sigma \neq 0$ then for every $\beta \in (0, 0.5)$ and $\lambda \in \mathbb{R}$,

$$\frac{1}{n^{1-2\beta}} \ln \mathbb{E} \exp \left( \frac{\lambda}{\sigma n^\beta} (F(X_1) + \cdots + F(X_n)) \right) \to \frac{\lambda^2}{2} \text{ as } n \to \infty .$$

3.2 Theorem. Let a function $F : \mathbb{R}^d \to \mathbb{R}$ satisfy the assumptions of Sect. 1, and a process $(X_t)_{t \in \mathbb{R}}$ satisfy the assumptions of Sect. 2 for continuous time. Then
(a) the following limit exists:

$$\sigma^2 = \lim_{t \to \infty} \frac{1}{t} \mathbb{E} \left( \int_0^t F(X_s) \, ds \right)^2 ;$$

(b) if $\sigma \neq 0$ then for every $\beta \in (0, 0.5)$ and $\lambda \in \mathbb{R}$,

$$\frac{1}{t^{1-2\beta}} \ln \mathbb{E} \exp \left( \frac{\lambda}{\sigma t^\beta} \int_0^t F(X_s) \, ds \right) \to \frac{\lambda^2}{2} \text{ as } t \to \infty .$$

It follows by the Gärtner-Ellis theorem (see [4, Sect. 8]) that for every $c \in [0, \infty)$,

$$\frac{1}{n^{1-2\beta}} \ln \mathbb{P} \left( F(X_1) + \cdots + F(X_n) < -c \sigma n^{1-\beta} \right) \to -\frac{c^2}{2} ,$$

$$\frac{1}{n^{1-2\beta}} \ln \mathbb{P} \left( F(X_1) + \cdots + F(X_n) > c \sigma n^{1-\beta} \right) \to -\frac{c^2}{2} .$$
as \( n \to \infty \) (discrete time), and

\[
\frac{1}{t^{1-2\beta}} \ln \mathbb{P}\left( \int_0^t F(X_s) \, ds < -c \sigma t^{1-\beta} \right) \to -\frac{c^2}{2},
\]

\[
\frac{1}{t^{1-2\beta}} \ln \mathbb{P}\left( \int_0^t F(X_s) \, ds > c \sigma t^{1-\beta} \right) \to -\frac{c^2}{2},
\]

as \( t \to \infty \) (continuous time).

\[4\] Splitting the process

\section*{Discrete time}

Given a process \((X_n)_{n \in \mathbb{Z}}\) satisfying the assumptions of Sect. 2 for discrete time, and its representation (2.1), we may split the process in two independent processes,

\[
X_n = X_{\text{past}}^n + X_{\text{future}}^n,
\]

\[
X_{\text{past}}^n = \sum_{j \leq 0} a_{j-n} \xi_j, \quad X_{\text{future}}^n = \sum_{j > 0} a_{j-n} \xi_j.
\]

4.1 Lemma. There exists \( \varepsilon > 0 \) such that

\[
\sup_{n \geq 0} ((n + 1)^{1+\varepsilon} |X_{\text{future}}^n|) < \infty \quad \text{and} \quad \sup_{n > 0} \left( n^{1+\varepsilon} |X_{\text{past}}^n| \right) < \infty \quad \text{a.s.}
\]

Proof. For \( n \geq 0 \), using (2.2),

\[
\mathbb{E} |X_{\text{future}}^n|^2 = \sum_{j=1}^{\infty} \text{trace}(a_{j+n} a_{j+n}^*) \leq \text{const} \cdot \sum_{j=1}^{\infty} \frac{1}{(j+n)^{3+2\varepsilon}} = O(n^{-2-2\varepsilon}).
\]

Thus, \( \sum_{n \geq 0} \mathbb{P}( |X_{\text{future}}^n| > n^{-1-0.5\varepsilon} ) < \infty \) (since \( X_{\text{future}}^n \) is Gaussian); the statement on \( X_{\text{future}} \) follows. The statement on \( X_{\text{past}} \) is similar.

\section*{Continuous time}

Given a process \((X_t)_{t \in \mathbb{R}}\) satisfying the assumptions of Sect. 2 for continuous time, and its representation (2.3), we may split the process in two independent processes,

\[
X_t = X_{\text{past}}^t + X_{\text{future}}^t,
\]

\[
X_{\text{past}}^t = \int_{-\infty}^{0} a_{s-t} \, dw_s, \quad X_{\text{future}}^t = \int_{0}^{\infty} a_{s-t} \, dw_s.
\]
4.2 Lemma. There exists \( \varepsilon > 0 \) such that

\[
\sup_{t \geq 0} \left((t + 1)^{1+\varepsilon}|X_{t+1}^\text{future}|\right) < \infty \quad \text{and} \quad \sup_{t \geq 0} \left((t + 1)^{1+\varepsilon}|X_t^\text{past}|\right) < \infty \quad \text{a.s.}
\]

The proof, given afterwards, uses the following (quite general) lemma.

4.3 Lemma. Let \( \varphi : [0, \infty) \to \mathbb{R} \) and \( \varepsilon > 0 \) be such that the function \( x \mapsto (x + 1)^{0.5+\varepsilon} \varphi(x) \) is bounded and Hölder continuous, that is,

\[
\sup_{x \geq 0} \left|\left((1 + x)^{0.5+\varepsilon} \varphi(x)\right)\right| < \infty, \\
\sup_{x \geq 0, 0 < \delta < 1} \frac{|(1 + x + \delta)^{0.5+\varepsilon} \varphi(x + \delta) - (1 + x)^{0.5+\varepsilon} \varphi(x)|}{\delta^\alpha} < \infty
\]

for a given \( \alpha \in (0, 1] \). Then there exists \( C < \infty \) such that for all \( y \in [0, \infty) \) and \( \delta \in [0, 1] \)

\[
\int_0^\infty |(1 + y + \delta)^\varepsilon \varphi(y + \delta + x) - (1 + y)^\varepsilon \varphi(y + x)|^2 \, dx \leq C \delta^{2\alpha}.
\]

Proof. Denote \( \psi(x) = (1 + x)^{0.5+\varepsilon} \varphi(x) \). Choosing for every \( x \) some \( z_x \in [y, y + \delta] \) we have

\[
(1 + y + \delta)^\varepsilon \varphi(y + \delta + x) - (1 + y)^\varepsilon \varphi(y + x) = \\
\frac{(1 + y + \delta)^\varepsilon}{(1 + y + \delta + x)^{0.5+\varepsilon}} \psi(y + \delta + x) - \frac{(1 + y)^\varepsilon}{(1 + y + x)^{0.5+\varepsilon}} \psi(y + x) = A_x + B_x + C_x,
\]

where

\[
A_x = \frac{(1 + y + \delta)^\varepsilon}{(1 + y + \delta + x)^{0.5+\varepsilon}} \left( \psi(y + \delta + x) - \psi(z_x + x) \right), \\
B_x = \frac{(1 + y)^\varepsilon}{(1 + y + x)^{0.5+\varepsilon}} \psi(z_x + x), \\
C_x = \frac{(1 + y)^\varepsilon}{(1 + y + x)^{0.5+\varepsilon}} \left( \psi(z_x + x) - \psi(y + x) \right).
\]

Due to the triangle inequality in \( L_2(0, \infty) \) it is sufficient to choose \( z_x \) (measurable in \( x \)) such that \( \int_0^\infty A_x^2 \, dx = O(\delta^{2\alpha}), \int_0^\infty B_x^2 \, dx = O(\delta^{2\alpha}) \) and \( \int_0^\infty C_x^2 \, dx = O(\delta^{2\alpha}) \). We have

\[
\int_0^\infty A_x^2 \, dx = (1 + y + \delta)^{2\varepsilon} \int_0^\infty \frac{O((y + \delta - z_x)^{2\alpha}) \, dx}{(1 + y + \delta + x)^{1+2\varepsilon}} = O(\delta^{2\alpha}).
\]
Similarly, \( \int_0^\infty C_x^2 \, dx = O(\delta^{2\alpha}) \). These two statements hold irrespective of the choice of \( z_x \in [y, y + \delta] \). Now we choose \( z_x \) such that
\[
(1 + y + \delta)^{\varepsilon} \frac{d}{dz} \bigg|_{z = z_x} \left( \frac{(1 + y + x)^{0.5 + \varepsilon}}{(1 + z + x)^{0.5 + \varepsilon}} \right) = \delta \frac{(1 + z_x)^{\varepsilon}}{(1 + z_x + x)^{0.5 + \varepsilon}} \left( \frac{\varepsilon}{1 + z_x} - \frac{0.5 + \varepsilon}{1 + z_x + x} \right).
\]
The bracketed difference is evidently bounded; \( \psi(z_x + x) \) is also bounded, thus,
\[
\begin{align*}
|B_x| \leq \text{const} \cdot \frac{(1 + z_x)^{\varepsilon}}{(1 + z_x + x)^{0.5 + \varepsilon}} \leq \text{const} \cdot \frac{(1 + y + \delta)^{\varepsilon}}{(1 + y + x)^{0.5 + \varepsilon}};
\int_0^\infty B_x^2 \, dx = O(\delta^2) \cdot (1 + y + \delta)^{2\varepsilon} \int_0^\infty \frac{dx}{(1 + y + x)^{1 + 2\varepsilon}} = O(\delta^2) = O(\delta^{2\alpha}).
\end{align*}
\]

**Proof of Lemma 4.2** First, for every \( t > 0 \), using (2.4),
\[
\begin{align*}
\mathbb{E} |X_{t + 1}^{\text{future}}|^2 = & \int_0^\infty \text{trace}(a_{s+t}a_{s+t}^*) \, ds \leq \text{const} \cdot \int_0^\infty \frac{ds}{(s + t + 1)^{3 + 2\varepsilon}} = O((t + 1)^{-2 - 2\varepsilon}).
\end{align*}
\]
Second, we note that Lemma 4.3 holds also for vector-valued (and matrix-valued) functions, and apply it to the function \( \varphi(t) = a_t \ (t \geq 0), \varepsilon \) in place of \( \alpha \) and \( 1 + \varepsilon \) in pace of \( \varepsilon \) (recall (2.5)). We get
\[
\begin{align*}
\sup_{t \geq 0} \int_0^\infty \| (t + 1 + \delta)^{1 + \varepsilon} a_{t+\delta+s} - (t + 1)^{1 + \varepsilon} a_{t+s} \|^2_{\text{HS}} \, ds = O(\delta^{2\varepsilon})
\end{align*}
\]
for \( \delta \leq 1 \); here \( \| \cdot \|_{\text{HS}} \) is the Hilbert-Schmidt norm, \( \| a \|^2_{\text{HS}} = \text{trace}(aa^*) \). Thus,
\[
\begin{align*}
\sup_{t \geq 0} \mathbb{E} \| (t + 1 + \delta)^{1 + \varepsilon} X_{t+\delta}^{\text{future}} - (t + 1)^{1 + \varepsilon} X_t^{\text{future}} \|^2 = O(\delta^{2\varepsilon}),
\sup_{t \geq 0} \mathbb{E} \| (t + 1)^{1 + \varepsilon} X_t^{\text{future}} \|^2 < \infty.
\end{align*}
\]
It follows that the sample paths of the Gaussian process \( (t + 1)^{1 + \varepsilon} X_t^{\text{future}} \) are locally bounded (in fact, continuous). The corresponding estimations are uniform, thus (see for instance [2, Th. 7.1.2])
\[
\begin{align*}
\sup_{n \geq 0} \mathbb{E} \sup_{t \in [n, n+1]} \| (t + 1)^{1 + \varepsilon} X_t^{\text{future}} \| < \infty.
\end{align*}
\]
and moreover,

$$\sup_{n \geq 0} \mathbb{P} \left( \sup_{t \in [n,n+1]} |(t + 1)^{1+\varepsilon} X_{-t}^{\text{future}}| > C \right)$$

decays rapidly as $C \to \infty$, namely, it is $O(e^{-\delta C^2})$ for some $\delta > 0$ (which can be obtained by Fernique’s theorem, see for instance [2 Th. 2.8.5]). By the Borel-Cantelli lemma,

$$\sum_{n=0}^{\infty} \mathbb{P} \left( \max_{t \in [n,n+1]} ((1 + t)^{1+\varepsilon}|X_{-t}^{\text{future}}|) > n^{\varepsilon/2} \right) < \infty .$$

Therefore

$$\sup_{t \geq 0} ((t + 1)^{1+0.5\varepsilon}|X_{-t}^{\text{future}}|) < \infty \quad \text{a.s.}$$

The statement on $X_{\text{past}}$ is similar. \hfill \qed

5 A small deviation argument

Discrete time

Let $(X_n)_{n \in \mathbb{Z}}$ be a process satisfying the assumptions of Sect. 2 for discrete time.

5.1 Lemma. For every $\sigma \in (0, 1)$ there exist $m \in \{1, 2, \ldots \}$ and an $\mathbb{R}^d$-valued stationary Gaussian process $(Y_k)_{k \in \mathbb{Z}}$ such that the two processes

$$(X_{mk})_{k \in \mathbb{Z}} \quad \text{and} \quad (Y_k + \sigma \xi_k)_{k \in \mathbb{Z}}$$

are identically distributed; here $\xi_k$ are independent $\mathbb{R}^d$-valued random variables, each distributed $\gamma^d$, and the process $(\xi_k)_{k \in \mathbb{Z}}$ is independent of the process $(Y_k)_{k \in \mathbb{Z}}$.

\textbf{Proof.} Here is a condition sufficient (and necessary, in fact) for existence of such $(Y_k)_k$ (for given $m$ and $\sigma$): the spectral density $f_m$ of the process $(X_{mk})_k$ should exceed the spectral density of the process $(\sigma \xi_k)_k$. That is, we need

$$f_m(\theta) \geq \frac{1}{2\pi} \sigma^2 I ;$$

here $I$ is the unit matrix, and the inequality means that all the eigenvalues of the Hermitian matrix $f_m(\theta)$ lie on $[\sigma^2, \infty)$.

\footnote{\text{in $X_{mk}$ is just the product of $m$ and $k$.}}
We have

\[ f_m(\theta) = \frac{1}{m} \left( f\left(\frac{\theta}{m}\right) + f\left(\frac{\theta + 2\pi}{m}\right) + \cdots + f\left(\frac{\theta + 2\pi(m-1)}{m}\right) \right), \]

the spectral density \( f \) of the given process \((X_n)_n\) being a continuous \(2\pi\)-periodic matrix-valued function on \( \mathbb{R} \) such that \( \int_{-\pi}^{\pi} f(\theta) \, d\theta = I \) (recall Sect. 2). Therefore \( f_m(\theta) \to (2\pi)^{-1}I \) (as \( m \to \infty \)) uniformly in \( \theta \). It follows that \( f_m(\theta) \geq (2\pi)^{-1}\sigma^2I \) for all \( \theta \), if \( m \) is large enough.

\[ \square \]

**Continuous time**

Let \((X_t)_{t \in \mathbb{R}}\) be a process satisfying the assumptions of Sect. 2 for continuous time.

**5.2 Lemma.** For every \( \sigma \in (0, 1) \) there exist \( t \in (0, \infty) \) and a discrete-time \( \mathbb{R}^d \)-valued stationary Gaussian process \((Y_k)_{k \in \mathbb{Z}}\) such that the two discrete-time processes

\[ (X_{tk})_{k \in \mathbb{Z}} \quad \text{and} \quad (Y_k + \sigma \xi_k)_{k \in \mathbb{Z}} \]

are identically distributed; here \( \xi_k \) are independent \( \mathbb{R}^d \)-valued random variables, each distributed \( \gamma^d \), and the process \((\xi_k)_{k \in \mathbb{Z}}\) is independent of the process \((Y_k)_{k \in \mathbb{Z}}\).

**Proof.** Similarly to the proof of Lemma 5.1 we consider the spectral density \( f_t(\cdot) \) of the process \((X_{tk})_k\) and prove the inequality \( f_t(\theta) \geq (2\pi)^{-1}\sigma^2I \) (which is sufficient).

We have

\[ f_t(\theta) = \frac{1}{t} \sum_{k \in \mathbb{Z}} f\left(\frac{\theta + 2\pi k}{t}\right), \]

the spectral density \( f \) of the given process \((X_t)_t\) being an integrable continuous matrix-valued function on \( \mathbb{R} \) such that \( f(\lambda) \geq 0 \) for all \( \lambda \), and \( \int_{-\infty}^{\infty} f(\lambda) \, d\lambda = I \) (recall Sect. 2).

For \( M \) large enough,

\[ \int_{-M}^{M} f(\lambda) \, d\lambda \geq (\sigma^2 + \varepsilon)I \quad \text{for some} \ \varepsilon > 0. \]

For \( t \) large enough,

\[ f_t(\theta) \geq \frac{1}{2\pi} \int_{-M}^{M} f(\lambda) \, d\lambda - \frac{1}{2\pi} \varepsilon I \geq \frac{1}{2\pi} \sigma^2I \quad \text{for all} \ \theta. \]

\[ \square \]
6 Surgery

Discrete time

Let \((X_n)_{n \in \mathbb{Z}}\) be a process satisfying the assumptions of Sect. 2 for discrete time, and \((Y_n)_{n \in \mathbb{Z}}\) its independent copy. We apply the split of Sect. 4 to both:

\[ X_n = X_n^{\text{past}} + X_n^{\text{future}}, \quad Y_n = Y_n^{\text{past}} + Y_n^{\text{future}}. \]

The four processes \(X_n^{\text{past}}, X_n^{\text{future}}, Y_n^{\text{past}}, Y_n^{\text{future}}\) are independent. The processes \(X_n^{\text{past}}\) and \(Y_n^{\text{past}}\) are identically distributed; symbolically, \(X_n^{\text{past}} \sim Y_n^{\text{past}}\). Also \(X_n^{\text{future}} \sim Y_n^{\text{future}}\). Thus, we have four identically distributed stationary processes:

\[ X = X^{\text{past}} + X^{\text{future}} \sim Y^{\text{past}} + Y^{\text{future}} \sim Y^{\text{past}} + X^{\text{future}} \sim Y^{\text{past}} + Y^{\text{future}} = Y. \]

Let \(F\) be a function satisfying the assumptions of Sect. 1. We introduce

\[ S_n^{\text{future}}(X) = \sum_{k=1}^{n} F(X_k), \]
\[ S_n^{\text{past}}(X) = \sum_{k=0}^{n-1} F(X_{-k}) = \sum_{k=1}^{n} F(X_{-n+k}), \]

Denote by \(\mu_n\) the distribution of \(S_n^{\text{future}}(X)\), symbolically \(S_n^{\text{future}}(X) \sim \mu_n\), and observe that \(S_n^{\text{past}}(X) \sim \mu_n\) and \(S_m^{\text{past}}(X) + S_n^{\text{future}}(X) \sim \mu_{m+n}\). Further, we introduce

\[ D_n^{\text{past}} = S_n^{\text{past}}(X^{\text{past}} + Y^{\text{future}}) - S_n^{\text{past}}(X^{\text{past}} + X^{\text{future}}), \]
\[ D_n^{\text{future}} = S_n^{\text{future}}(Y^{\text{past}} + X^{\text{future}}) - S_n^{\text{future}}(X^{\text{past}} + X^{\text{future}}) \]

and observe that

\[ S_m^{\text{past}}(X) + S_n^{\text{future}}(X) = S_m^{\text{past}}(X^{\text{past}} + Y^{\text{future}}) + S_n^{\text{future}}(Y^{\text{past}} + X^{\text{future}}) - D_m^{\text{past}} - D_n^{\text{future}}. \]

This fact is instrumental to our purpose, since the two random variables \(S_m^{\text{past}}(X^{\text{past}} + Y^{\text{future}}), S_n^{\text{future}}(Y^{\text{past}} + X^{\text{future}})\) are independent, distributed \(\mu_m, \mu_n\) respectively, and the distribution of their sum is close to \(\mu_{m+n}\) as far as \(D_m^{\text{past}} + D_n^{\text{future}}\) is relatively small.

6.2 Lemma. There exists \(\varepsilon > 0\) such that

\[ \sup_{n>0} \mathbb{E} \exp(\varepsilon |D_n^{\text{past}}|) < \infty, \quad \sup_{n>0} \mathbb{E} \exp(\varepsilon |D_n^{\text{future}}|) < \infty. \]
Proof. It is sufficient to prove that
\[ \sup_{n > 0} \mathbb{E} \exp(\varepsilon D_n^\text{future}) < \infty, \]
since the distribution of \( D_n^\text{future} \) is symmetric (around 0), and the assumptions of Sect. 2 are invariant under time reversal. Equivalently, we may prove that
\[ (6.3) \sup_{n > 0} \mathbb{P}(D_n^\text{future} > C) = O(e^{-\varepsilon C}) \]
for some \( \varepsilon > 0 \) and all \( C > 0 \).

By Lemma 4.1, \( \sup_{k > 0} (k^{1+\varepsilon}|X_k^\text{past}|) < \infty \) a.s. The same holds for \( Y^\text{past} \).

We consider events \( A_u : \sup_{k > 0} (k^{1+\varepsilon}|X_k^\text{past} - Y_k^\text{past}|) \leq u \).

Fernique’s theorem (mentioned in Sect. 4) gives us \( \delta > 0 \) such that
\[ (6.4) \mathbb{P}(A_u) \geq 1 - 2e^{-\delta u^2} \quad \text{for } u \in (0, \infty). \]

We introduce \( Z_{u,k} = F_{uk^{-1-\varepsilon}}(X_k) - F(X_k) \geq 0 \) (where \( F_{uk^{-1-\varepsilon}} \) means \( F_r \) of (1.3) for \( r = uk^{-1-\varepsilon} \)) and \( Z_u = \sum_{k > 0} Z_{u,k} \in [0, \infty] \). The intersection of \( A_u \) and the event \( D_n^\text{future} > C \) is contained in the event \( Z_u > C \), since \( |X_k^\text{past} - Y_k^\text{past}| \leq uk^{-1-\varepsilon} \) implies
\[ F(Y_k^\text{past} + X_k^\text{future}) - F(X_k^\text{past} + X_k^\text{future}) \leq F_{uk^{-1-\varepsilon}}(X_k) - F(X_k). \]

Thus,
\[ (6.5) \sup_{n > 0} \mathbb{P}(D_n^\text{future} > C) \leq \mathbb{P}(Z_u > C) + 1 - \mathbb{P}(A_u). \]

Lemma 5.1 for \( \sigma = 0.5 \) gives us \( m \in \{1, 2, \ldots\} \) and \( (Y_k)_{k \in \mathbb{Z}} \) such that \( (X_m)_k \sim (Y_k + 0.5\xi_k)_k \). We have
\[ Z_u = \sum_{k > 0} Z_{u,k} = \sum_{j=1}^m \sum_{k > 0} Z_{u, mk+j}; \]
\[ \exp(m^{-1}Z_u) \leq \frac{1}{m} \sum_{j=1}^m \exp \left( \sum_{k > 0} Z_{u, mk+j} \right). \]

For each \( j \in \{1, \ldots, m\} \), \( \sum_{k > 0} Z_{u, mk+j} = \sum_{k > 0} (F_{u(mk+j)^{-1-\varepsilon}}(X_{mk+j}) - F(X_{mk+j})) \) is distributed like \( \sum_{k > 0} (F_{u(mk+j)^{-1-\varepsilon}}(Y_k + 0.5\xi_k) - F(Y_k + 0.5\xi_k)) \).
For every (nonrandom) sequence \((y_k)_k\), using (1.4) and denoting the constant \(C\) of (1.4) by \(C_F\),
\[
\mathbb{E} \exp \left( \sum_{k \geq 0} \left( F_{u(mk+j)-1-\varepsilon} (y_k + 0.5\xi_k) - F(y_k + 0.5\xi_k) \right) \right) = \\
= \prod_{k \geq 0} \mathbb{E} \exp \left( F_{u(mk+j)-1-\varepsilon} (y_k + 0.5\xi_k) - F(y_k + 0.5\xi_k) \right) \leq \\
\leq \prod_{k \geq 0} \exp \left( C_F u(mk + j)^{1-\varepsilon} \right) = \exp \left( C_F u \sum_{k \geq 0} (mk + j)^{1-\varepsilon} \right) \leq \exp(Bu),
\]
where \(B = C_F \sum_{k \geq 0} (mk + 1)^{1-\varepsilon} < \infty\). Therefore
\[
\mathbb{E} \exp \left( \sum_{k \geq 0} Z_{u,mk+j} \right) = \\
= \mathbb{E} \exp \left( \sum_{k \geq 0} \left( F_{u(mk+j)-1-\varepsilon} (Y_k + 0.5\xi_k) - F(Y_k + 0.5\xi_k) \right) \right) \leq \exp(Bu)
\]
and \(\mathbb{E} (m^{-1}Z_u) \leq e^{Bu}\), which implies
\[
\mathbb{P} (Z_u > C) \leq \exp(Bu - m^{-1}C) .
\]
We return to (6.5) and (6.4):
\[
\sup_{n>0} \mathbb{P} \left( D_{n}^{\text{future}} > C \right) \leq \exp(Bu - m^{-1}C) + 2e^{-\delta u^2}
\]
for every \(u \in (0, \infty)\). Taking \(u = \sqrt{C}\) we get (6.3).

**Continuous time**

Let \((X_t)_{t \in \mathbb{R}}\) be a process satisfying the assumptions of Sect. 2 for continuous time, and \(F\) a function satisfying the assumptions of Sect. 1. We proceed similarly to the discrete-time case: \(X = X^{\text{past}} + X^{\text{future}}, Y = Y^{\text{past}} + Y^{\text{future}}\) etc.; \(S_t^{\text{future}}(X) = \int_0^t F(X_s) \, ds, \ S_t^{\text{past}}(X) = \int_{-t}^0 F(X_s) \, ds;\) \(D_t^{\text{future}} = S_t^{\text{future}}(Y^{\text{past}} + X^{\text{future}}) - S_t^{\text{future}}(X^{\text{past}} + X^{\text{future}})\), and similarly \(D_t^{\text{past}}\). We get
\[
(6.6) \quad S_s^{\text{past}}(X) + S_t^{\text{future}}(X) = \\
= S_s^{\text{past}}(X^{\text{past}} + Y^{\text{future}}) + S_t^{\text{future}}(Y^{\text{past}} + X^{\text{future}}) - D_s^{\text{past}} - D_t^{\text{future}} .
\]

**6.7 Lemma.** There exists \(\varepsilon > 0\) such that
\[
\sup_{t>0} \mathbb{E} \exp(\varepsilon|D_t^{\text{past}}|) < \infty , \quad \sup_{t>0} \mathbb{E} \exp(\varepsilon|D_t^{\text{future}}|) < \infty .
\]
The proof being quite similar to that of Lemma 6.2, I give a sketch. Events

$$A_u : \sup_{t>0}((t+1)^{1+\varepsilon}|X_t^{\text{past}} - Y_t^{\text{past}}|) \leq u$$

satisfy $\mathbb{P}(A_u) \geq 1 - 2e^{-\delta u^2}$ by Lemma 4.2 and Fernique’s theorem. We introduce $Z_{u,t} = F_{u(t+1)^{-1-\varepsilon}}(X_t) - F(X_t) \geq 0$, $Z_u = \int_0^\infty Z_{u,t} \, dt \in [0, \infty]$ and get

$$\sup_{t>0} \mathbb{P}(D_{\text{future}} > C) \leq \mathbb{P}(Z_u > C) + 2e^{-\delta u^2}.$$ 

Lemma 5.2 for $\sigma = 0.5$ gives $T$ and $(Y_k)_k$ such that $(X_{T_k})_k \sim (Y_k + 0.5\xi_k)_k$. We have

$$Z_u = \int_0^\infty Z_{u,t} \, dt = \int_0^T ds \sum_{k\geq 0} Z_{u,T_k+s};$$

$$\exp(T^{-1}Z_u) \leq \frac{1}{T} \int_0^T ds \exp \left( \sum_{k\geq 0} Z_{u,T_k+s} \right).$$

We use (1.4) as before:

$$\mathbb{E} \exp \left( \sum_{k\geq 0} (F_{u(T_k+s+1)^{-1-\varepsilon}}(y_k + 0.5\xi_k) - F(y_k + 0.5\xi_k)) \right) \leq \exp(Bu),$$

where $B = C_F \sum_{k\geq 0} (Tk + 1)^{-1-\varepsilon} < \infty$. Thus, $\mathbb{E} \exp \left( \sum_{k\geq 0} Z_{u,T_k+s} \right) \leq \exp(Bu)$; $\mathbb{E} \exp(T^{-1}Z_u) \leq e^{Bu}$; $\mathbb{P}(Z_u > C) \leq \exp(Bu - T^{-1}C)$. Finally, $u = \sqrt{C}$ leads to $\sup_{t>0} \mathbb{P}(D_{\text{future}} > C) = O(e^{-\varepsilon C}).$

## 7 Asymptotic variance

### Discrete time

Here we prove Item (a) of Theorem 3.1. Let $\mu_n$ be the distribution of $F(X_1) + \cdots + F(X_n)$. According to Sect. 6, $\mu_{m+n}$ is close to the convolution $\mu_m * \mu_n$ in the following sense. There exist random variables $S_{m,n}, S'_m, S''_n$ such that

$$S_{m,n} \sim \mu_{m+n}, \quad S'_m \sim \mu_m, \quad S''_n \sim \mu_n,$$

$$S'_m \text{ and } S''_n \text{ are independent},$$

$$\mathbb{E} \exp(\varepsilon|S_{m,n} - S'_m - S''_n|) \leq C$$

14
for some $\varepsilon > 0$, $C < \infty$ not depending on $m, n$. Namely, we may take $S_{m,n} = S_{m}^{\text{past}}(X) + S_{n}^{\text{future}}(X)$, $S_{m}' = S_{m}^{\text{past}}(X^{\text{past}} + Y^{\text{future}})$, $S_{n}'' = S_{n}^{\text{future}}(Y^{\text{past}} + X^{\text{future}})$ and note that
\[
\mathbb{E} \exp(\varepsilon |S_{m,n} - S_{m}' - S_{n}''|) \leq \mathbb{E} \exp(2\varepsilon |D_{m}^{\text{past}}| + \varepsilon |D_{n}^{\text{future}}|) \leq \left(\mathbb{E} \exp(2\varepsilon |D_{m}^{\text{past}}|)\right)^{1/2} \left(\mathbb{E} \exp(\varepsilon |D_{n}^{\text{future}}|)\right)^{1/2} \leq C
\]
by (6.1) and Lemma 6.2, if $\varepsilon$ is small enough and $C$ is large enough.

Also,
\[
\int e^{\varepsilon |x|} \mu_1(dx) < \infty
\]
by (1.2). In this section we need only second moments: $\int x^2 \mu_1(dx) < \infty$ and
\[
\sup_{m,n} \mathbb{E} |S_{m,n} - S_{m}' - S_{n}''|^2 < \infty.
\]
Taking into account that the expectations vanish by (1.1), we use orthogonality and the triangle inequality in the space $L_2$ of random variables:
\[
\sup_{m,n} \|S_{m,n}\| - \sqrt{\|S_{m}'\|^2 + \|S_{n}''\|^2} < \infty.
\]
Thus, the numbers
\[
\sigma^2_n = \int x^2 \mu_n(dx) = \mathbb{E} (F(X_1) + \cdots + F(X_n))^2
\]
satisfy
\[
\sup_{m,n} |\sigma_{m+n} - \sqrt{\sigma^2_m + \sigma^2_n}| < \infty.
\]
Existence of $\lim_k (2^{-k/2}\sigma_{2^k})$ could be deduced readily, but existence of $\lim_n (n^{-1/2}\sigma_n)$ needs more effort. Here are two quite general lemmas.

7.3 Lemma. Let numbers $a_1, a_2, \ldots \in [0, \infty)$ and $\varepsilon > 0$ satisfy
\[
a_{m+n} \leq \sqrt{a_m^2 + a_n^2} + \varepsilon
\]
for all $m, n \in \{1, 2, \ldots\}$. Then
\[
a_n \leq \left(a_1 + \frac{\sqrt{2}}{\sqrt{2} - 1} \varepsilon\right) \sqrt{n}
\]
for all $n$. 

15
Proof. For \( k = 0, 1, 2, \ldots \) consider

\[
 b_k = \max_{n \leq 2^k} \frac{a_n}{\sqrt{n}}.
\]

For each \( n \in \{1, 2, 3, \ldots, 2^k\} \)

\[
 \frac{a_{2^k+n}}{\sqrt{2^k+n}} \leq \frac{a_{2^k}^2 + a_n^2 + \varepsilon}{\sqrt{2^k+n}} \leq \frac{2^{2k}b_k^2 + nb_k^2 + \varepsilon}{\sqrt{2^k+n}} \leq b_k + \frac{\varepsilon}{\sqrt{2^k}},
\]

therefore

\[
 b_{k+1} \leq b_k + 2^{-k/2} \varepsilon; \quad b_k \leq b_0 + \frac{\sqrt{2}}{\sqrt{2} - 1} \varepsilon.
\]

However, \( b_0 = a_1 \).

Similarly,

(7.4) if \( a_{m+n} \geq \sqrt{a_m^2 + a_n^2} - \varepsilon \) then \( a_n \geq \left(a_1 - \frac{\sqrt{2}}{\sqrt{2} - 1} \varepsilon\right)\sqrt{n} \).

7.5 Lemma. Let numbers \( a_1, a_2, \ldots \in [0, \infty) \) satisfy

\[
 \sup_{m,n} \left| a_{m+n} - \sqrt{a_m^2 + a_n^2} \right| < \infty.
\]

Then there exists \( \lim_n (a_n/\sqrt{n}) \in [0, \infty) \).

Proof. Denote the given supremum by \( C \). For any \( k \in \{1, 2, \ldots\} \) we may apply Lemma 7.3 (together with (7.4)) to the sequence \((a_k, a_{2k}, \ldots)\), obtaining

\[
 \left| a_{kn} - a_k \sqrt{n} \right| \leq \frac{\sqrt{2}}{\sqrt{2} - 1} C \sqrt{n}; \quad \left| \frac{a_{kn}}{\sqrt{nk}} - \frac{a_k}{\sqrt{k}} \right| \leq \frac{C}{\sqrt{2} - 1} \sqrt{\frac{k}{n}}.
\]

All limiting points of the sequence \((a_{nk}/\sqrt{nk})_n\) belong to the \( O(1/\sqrt{k}) \)-neighborhood of the number \( a_k/\sqrt{k} \). The same holds for all limiting points of the sequence \((a_n/\sqrt{n})_n\), since for \( \theta \in \{0, 1, \ldots, k-1\} \)

\[
 \left| a_{kn+\theta} - \sqrt{a_{kn}^2 + a_\theta^2} \right| \leq C; \quad \left| \frac{a_{kn+\theta}}{\sqrt{kn+\theta}} - \sqrt{\frac{kn}{kn+\theta}} \frac{a_{kn}^2}{kn} + \frac{a_\theta^2}{kn+\theta} \right| \leq \frac{C}{\sqrt{kn+\theta}}; \quad \left| \frac{a_{kn+\theta}}{\sqrt{kn+\theta}} - \sqrt{(1+o(1))\frac{a_{kn}^2}{kn} + o(1)} \right| \leq o(1).
\]
Let \( x \) be a limiting point of \((a_n/\sqrt{n})_n, \) then

\[
\left| x - \frac{a_k}{\sqrt{k}} \right| \leq \frac{\sqrt{2}}{\sqrt{2} - 1} \frac{C}{\sqrt{k}}
\]

for all \( k. \) Thus, \( a_k/\sqrt{k} \to x. \)

It remains to apply Lemma \( 7.5 \) to the sequence \((\sigma_n)_n. \)

**Continuous time**

Item (a) of Theorem \( 3.2 \) is verified similarly to that of Theorem \( 3.1. \) We consider the distribution \( \mu_t \) of \( \int_0^t F(X_s) \, ds \) and note that \( \mu_s \ast \mu_t \) is close to \( \mu_s \ast \mu_t \) similarly to (7.1). Also, (7.6)

\[
E \exp \left( \frac{1}{t} \left| \int_0^t F(X_s) \, ds \right| \right) \leq E \exp |F(X_0)| < \infty.
\]

The numbers

\[
\sigma^2_t = \int x^2 \mu_t(dx) = E \left( \int_0^t F(X_s) \, ds \right)^2
\]

defined for \( t \in [0, \infty) \) satisfy

\[
\sup_{s \in (0,t]} \sigma_s < \infty \quad \text{for} \quad t < \infty,
\]

\[
\sup_{s,t} \left| \sigma_{s+t} - \sqrt{\sigma^2_s + \sigma^2_t} \right| < \infty.
\]

**7.7 Lemma.** Let a function \( a : [0, \infty) \to [0, \infty) \) satisfy

\[
\sup_{s,t} \left| a(s + t) - \sqrt{a^2(s) + a^2(t)} \right| < \infty
\]

and be bounded on \([0, t]\) for some (therefore, every) \( t > 0. \) Then there exists \( \lim_{t \to \infty} (a(t)/\sqrt{t}) \in [0, \infty). \)

The proof is similar to that of Lemma \( 7.5. \) For any \( t \in (0, \infty) \) we apply Lemma \( 7.3 \) (and (7.4)) to the sequence \((a(nt))_n. \) A limiting point \( x \) of the function \( s \mapsto a(s)/\sqrt{s} \) is also a limiting point of the sequence \((a(nt)/\sqrt{nt})_n \)
(boundedness of \( a(\cdot) \) on \([0, t]\) is used here), and we get \( \left| x - \frac{a(t)}{\sqrt{t}} \right| \leq \frac{\sqrt{2}}{\sqrt{2} - 1} \frac{C}{\sqrt{t}}. \)

It remains to apply Lemma \( 7.4 \) to the function \( t \mapsto \sigma_t. \)

17
8 Asymptotic exponential moments

Discrete time

Here we prove Item (b) of Theorem 3.1. Recall the numbers \( \sigma \in (0, \infty) \) from Item (a), \( \beta \in (0, 0.5) \) from Item (b), \( \varepsilon > 0 \) from (7.1). Denote \( S_{m,n} - S'_m - S''_n \) from (7.1) by \( R_{m,n} \). Taking into account that \( \mathbb{E} R_{m,n} = 0 \) we get

\[
\mathbb{E} \exp(uR_{m,n}) \leq \exp(C\sigma^2u^2)
\]

for some \( C < \infty \) and all \( u \in [-\varepsilon, \varepsilon] \). We define functions \( \varphi_n \) by

\[
\varphi_n(\lambda) = \int \exp\left( \frac{\lambda}{\sigma n^\beta} x \right) \mu_n(dx) = \mathbb{E} \exp\left( \frac{\lambda}{\sigma n^\beta}(F(X_1) + \cdots + F(X_n)) \right) \in (0, \infty].
\]

8.1 Lemma. For all \( m, n \) and \( \lambda \) such that \( |\lambda| \leq \varepsilon\sigma(m+n)^{\beta/2} \),

\[
\varphi_m^p\left( \frac{1}{p} \left( \frac{m}{m+n} \right)^\beta \lambda \right) \varphi_n^p\left( \frac{1}{p} \left( \frac{n}{m+n} \right)^\beta \lambda \right) \exp\left( -C\frac{1}{p} \left( \frac{\lambda^2}{(m+n)^{3\beta/2}} \right) \right) \leq \varphi_{m+n}(\lambda) \leq \varphi_m^{1/p}\left( \frac{m}{m+n} \right)^\beta \lambda \varphi_n^{1/p}\left( \frac{n}{m+n} \right)^\beta \lambda \exp\left( C\frac{\lambda^2}{(m+n)^{3\beta/2}} \right),
\]

where

\[
p = \frac{(m+n)^{\beta/2}}{(m+n)^{\beta/2} - 1}.
\]

Proof. The upper bound for \( \varphi_{m+n}(\lambda) \): first,

\[
\varphi_{m+n}(\lambda) = \mathbb{E} \exp\left( \frac{\lambda}{\sigma(m+n)^\beta} S_{m,n} \right) = \mathbb{E} \exp\left( \frac{\lambda}{\sigma(m+n)^\beta}(S'_m + S''_n + R_{m,n}) \right) \leq \left\| \exp\left( \frac{\lambda}{\sigma(m+n)^\beta}(S'_m + S''_n) \right) \right\|_{L_p} \cdot \left\| \exp\left( \frac{\lambda}{\sigma(m+n)^\beta} R_{m,n} \right) \right\|_{L_q},
\]

where \( q = (m+n)^{\beta/2} \). Second,

\[
\left\| \exp\left( \frac{\lambda}{\sigma(m+n)^\beta} R_{m,n} \right) \right\|_{L_q} = \left( \mathbb{E} \exp\left( q\frac{\lambda}{\sigma(m+n)^\beta} R_{m,n} \right) \right)^{1/q} \leq \exp\left( \frac{1}{(m+n)^{3\beta/2}} C\sigma^2 \left( \frac{\lambda}{\sigma(m+n)^{3\beta/2}} \right)^2 \right) = \exp\left( C\frac{\lambda^2}{(m+n)^{3\beta/2}} \right).
\]
Third,

\[
\left\| \exp \left( \frac{\lambda}{\sigma(m+n)^\beta} (S'_m + S''_m) \right) \right\|_{L_p} = \left( \mathbb{E} \exp \left( \frac{\lambda}{\sigma(m+n)^\beta} (S'_m + S''_m) \right) \right)^{1/p} = \\
= \left( \mathbb{E} \exp \left( \frac{\lambda}{\sigma(m+n)^\beta} S'_m \right) \right)^{1/p} \left( \mathbb{E} \exp \left( \frac{\lambda}{\sigma(m+n)^\beta} S''_m \right) \right)^{1/p},
\]

and

\[
\mathbb{E} \exp \left( \frac{\lambda}{\sigma(m+n)^\beta} S'_m \right) = \mathbb{E} \exp \left( \frac{1}{\sigma m^\beta} \left( \frac{m}{m+n} \right)^\beta \lambda S'_m \right) = \varphi_m \left( \frac{m}{m+n} \right)^\beta \lambda ;
\]

the same for \( S''_m \).

The lower bound for \( \varphi_{m+n}(\lambda) \): first,

\[
\varphi_m \left( \frac{1}{p} \left( \frac{m}{m+n} \right)^\beta \lambda \right) \cdot \varphi_n \left( \frac{1}{p} \left( \frac{n}{m+n} \right)^\beta \lambda \right) = \\
= \mathbb{E} \exp \left( \frac{1}{\sigma m^\beta} p \left( \frac{m}{m+n} \right)^\beta \lambda S'_m \right) \cdot \mathbb{E} \exp \left( \frac{1}{\sigma n^\beta} p \left( \frac{n}{m+n} \right)^\beta \lambda S''_m \right) = \\
= \mathbb{E} \exp \left( \frac{1}{p \sigma (m+n)^\beta} (S'_m + S''_m) \right) \leq \left\| \exp \left( \frac{1}{p \sigma (m+n)^\beta} S_{m,n} \right) \right\|_{L_p} \cdot \left\| \exp \left( - \frac{1}{p \sigma (m+n)^\beta} R_{m,n} \right) \right\|_{L_q}.
\]

Second, the \( L_q \) norm is estimated by \( \exp \left( C \frac{1}{(m+n)^{\beta/2}} \right) \) in the same way as before. Third, the \( L_p \) norm is \( \left( \mathbb{E} \exp \left( \frac{\lambda}{\sigma(m+n)^\beta} S_{m,n} \right) \right)^{1/p} = \varphi_{m+n}^{1/p}(\lambda) \). It remains to raise all that to the power \( p \).

Given a number \( \varepsilon_1 \in (0, \varepsilon] \), we consider (for every \( n \)) the smallest \( b_n \) and the largest \( a_n \) such that the inequalities

\[
\exp \left( 0.5n^{1-2\beta} a_n \lambda^2 \right) \leq \varphi_n(\lambda) \leq \exp \left( 0.5n^{1-2\beta} b_n \lambda^2 \right)
\]

hold for all \( \lambda \) satisfying \( |\lambda| \leq \varepsilon_1 \sigma n^{\beta/2} \).

\begin{equation}
(8.3)
\end{equation}

8.4 Lemma. There exists \( N \) such that for all \( m, n \) satisfying \( N \leq m \leq n \leq 2m, \)

\[
a_{m+n} \geq \frac{(m+n)^{\beta/2} - 1}{(m+n)^{\beta/2}} \left( \frac{ma_m + na_n}{m+n} - \frac{2C}{(m+n)^{1-0.5\beta}} \right),
\]

\[
b_{m+n} \leq \frac{(m+n)^{\beta/2}}{(m+n)^{\beta/2} - 1} \left( \frac{mb_m + nb_n}{m+n} + \frac{2C}{(m+n)^{1-0.5\beta}} \right).
\]
Lemma 8.5. Let numbers $B_1, B_2, \cdots \in [0, \infty)$, $\alpha \in (0, \infty)$ and $r \in [0, 1]$ satisfy

$$B_{m+n} \leq \frac{(m+n)^\alpha}{(m+n)^\alpha - r} \frac{mB_m + nB_n}{m+n} + \frac{r}{(m+n)^\alpha}$$

for all $m, n$ such that $m \leq n \leq 2m$. Then

$$\sup_B \leq (1 + C_\alpha r)B_1 + C_\alpha r$$

for some $C_\alpha$ that depends on $\alpha$ only.
Proof. We choose integers \( n_0 < n_1 < \ldots \) such that \( \frac{4}{3} \leq \frac{n_{k+1}}{n_k} \leq \frac{3}{2} \) for all \( k \), and \( n_0 = 2 \). We consider
\[
M_k = \max(B_1, B_2, \ldots, B_{2n_k}).
\]
For every integer \( n \in (2n_k, 3n_k] \)
\[
B_n = B_{n_k + (n - n_k)} \leq \frac{n}{n^\alpha - r} M_k + \frac{r}{n^\alpha}.
\]
Taking into account that \( 2n_{k+1} \leq 3n_k \) we get
\[
M_{k+1} \leq \frac{(2n_{k+1})^\alpha}{(2n_{k+1})^\alpha - r} M_k + \frac{r}{(2n_{k+1})^\alpha}.
\]
Introducing
\[
P_k = \prod_{j=0}^{k-1} \frac{(2n_j + 1)^\alpha}{(2n_j + 1)^\alpha - r}, \quad P_0 = 1,
\]
we have \( P_k \geq 1 \), \( P_k \to P_\infty < \infty \), and \( M_{k+1} \leq \frac{P_{k+1}}{P_k} M_k + \frac{r}{(2n_{k+1})^\alpha} \), \( \frac{M_{k+1}}{P_{k+1}} \leq \frac{M_k}{P_k} + \frac{r}{(2n_{k+1})^\alpha} \), therefore
\[
\sup_k \frac{M_k}{P_k} \leq \frac{M_0}{P_0} + r \sum_{k=0}^\infty (2n_k + 1)^{-\alpha};
\]
\[
\sup_n B_n = \sup_k M_k \leq P_\infty M_0 + r P_\infty \sum_{k=0}^\infty (2n_k + 1)^{-\alpha}.
\]

We note that
\[
B_2 \leq \frac{2^\alpha}{2^\alpha - r} B_1 + \frac{r}{2^\alpha} \leq (1 + C_\alpha r) B_1 + C_\alpha r;
\]
here and henceforth \( C_\alpha \) is some constant that depends on \( \alpha \) only, not necessarily the same in all occurrences. Similarly,
\[
B_3 \leq (1 + C_\alpha r) \max(B_1, B_2) + C_\alpha r \leq (1 + C_\alpha r) B_1 + C_\alpha r,
\]
the same for \( B_4 \), and we get
\[
M_0 = \max(B_1, B_2, B_3, B_4) \leq (1 + C_\alpha r) B_1 + C_\alpha r.
\]
Finally, \( n_k \geq 2(4/3)^k \), thus
\[
\ln P_\infty = - \sum_{k=0}^\infty \ln (1 - r(2n_k + 1)^{-\alpha}) \leq - \sum_{k=0}^\infty \ln (1 - r(4(4/3)^k + 1)^{-\alpha}) \leq C_\alpha r;
\]
\[
P_\infty \leq \exp(C_\alpha r) \leq 1 + (e^{C_\alpha} - 1)r.
\]
\qed
8.6 Lemma. Let numbers $A_1, A_2, \ldots \in [0, \infty)$, $\alpha \in (0, \infty)$ and $r \in [0, 1]$ satisfy
\[ A_{m+n} \geq \frac{(m+n)^\alpha - r}{(m+n)^\alpha} \cdot mA_m + nA_n - \frac{r}{(m+n)^\alpha}, \]
for all $m, n$ such that $m \leq n \leq 2m$. Then
\[ \inf_n A_n \geq (1 - C_\alpha r)A_1 - C_\alpha r \]
for some $C_\alpha$ that depends on $\alpha$ only.

**Proof.** Let $n_k$ and $P_k$ be as in the proof of Lemma 8.5. We introduce $M_k = \min(A_1, \ldots, A_{2n_k})$ and get
\[
M_{k+1} \geq \frac{(2n_k + 1)^\alpha - r}{(2n_k + 1)^\alpha} \cdot M_k - \frac{r}{(2n_k + 1)^\alpha};
\]
\[
P_{k+1}M_{k+1} \geq P_kM_k - P_{k+1}r(2n_k + 1)^{-\alpha};
\]
\[
\inf_k P_k M_k \geq P_0 M_0 - P_\infty \sum_k (2n_k + 1)^{-\alpha};
\]
\[
\inf_n A_n = \inf_k M_k \geq \frac{M_0}{P_\infty} - r \sum_k (2n_k + 1)^{-\alpha} \geq (1 - C_\alpha r)A_1 - C_\alpha r
\]
similarly to the proof of Lemma 8.5. \qed

Recall that $a_n, b_n$ defined by (8.3) depend implicitly on $\varepsilon_1$.

8.7 Lemma. For every $\delta > 0$ there exist $\varepsilon_1 \in (0, \varepsilon]$ and $k \in \{1, 2, \ldots \}$ such that
\[
\inf_n a_{kn} \geq 1 - \delta \quad \text{and} \quad \sup_n b_{kn} \leq 1 + \delta.
\]

**Proof.** Lemma 8.4 shows that Lemma 8.5 may be applied to $B_n = b_{kn}$, $\alpha = \beta/2$ and $r = k^{-\beta/2} \max(1, 2C)$ provided that $k$ exceeds the number $N$ of Lemma 8.4. Therefore $\sup_n b_{kn} \leq (1+\delta)b_k+\delta$ for all $k$ large enough. Similarly (using Lemma 8.6), $\inf_n a_{kn} \geq (1 - \delta)a_k - \delta$ for all $k$ large enough. Also,
\[
1 - \delta \leq \frac{\mathbb{E}(F(X_1) + \cdots + F(X_k))^2}{k\sigma^2} \leq 1 + \delta
\]
for all $k$ large enough. After choosing such $k$ we choose $\varepsilon_1$ such that
\[
\exp((1 - \delta) \cdot 0.5\varphi''_k(0)\lambda^2) \leq \varphi_k(\lambda) \leq \exp((1 + \delta) \cdot 0.5\varphi''_k(0)\lambda^2)
\]
for all $\lambda$ satisfying $|\lambda| \leq \varepsilon_1 \sigma k^{3/2}$; this is possible, since $\varphi_k(0) = 1$ and

$$
\varphi_k'(0) = \frac{1}{\sigma k^3} \mathbb{E} (F(X_1) + \cdots + F(X_k)) = 0.
$$

Taking into account that

$$
\varphi_k''(0) = \frac{1}{\sigma^2 k^{3.5}} \mathbb{E} (F(X_1) + \cdots + F(X_k))^2 \in [(1 - \delta) k^{1.2}, (1 + \delta) k^{1.2}]
$$

we get

$$
\exp((1 - \delta)^2 \cdot 0.5 k^{1.2} \lambda^2) \leq \varphi_k(\lambda) \leq \exp((1 + \delta)^2 \cdot 0.5 k^{1.2} \lambda^2),
$$

which means that $b_k \leq (1 + \delta)^2$ and $a_k \geq (1 - \delta)^2$. Finally, $\inf_n a_{kn} \geq (1 - \delta)^3 - \delta$ and $\sup_n b_{kn} \geq (1 + \delta)^3 + \delta$.

We see that

$$
(1 - \delta)^2 \lambda^2 \leq \frac{1}{(kn)^{1.2}} \ln \varphi_{kn}(\lambda) \leq (1 + \delta)^2 \lambda^2 \frac{2}{2}
$$

for $|\lambda| \leq \varepsilon_1 \sigma (kn)^{3/2}$. Now we consider $\varphi_{kn+\theta}(\lambda)$ for $\theta \in \{0, 1, \ldots, k - 1\}$ and $|\lambda| \leq 0.5 \varepsilon_1 \sigma (kn+\theta)^{3/2}$, assuming that $kn+\theta$ is large enough (namely, exceeds $2^{2/\beta}$). Similarly to the proof of Lemma 8.4 we use Lemma 8.1. Taking into account that $p = (kn+\theta)^{3/2} \leq 2$ and

$$
|p\left(\frac{kn}{kn+\theta}\right)^{\beta} \lambda| \leq p\left(\frac{kn}{kn+\theta}\right)^{\beta} \cdot 0.5 \varepsilon_1 \sigma (kn+\theta)^{3/2} \leq \varepsilon_1 \sigma (kn)^{3/2}
$$

we get

$$
\varphi_{kn+\theta}(\lambda) \leq \varphi_{\theta}^{1/p}\left(p\left(\frac{\theta}{kn+\theta}\right)^{\beta}\lambda\right) \varphi_{kn}^{1/p}\left(p\left(\frac{kn}{kn+\theta}\right)^{\beta}\lambda\right) \exp\left(C\frac{\lambda^2}{(kn+\theta)^{3\beta/2}}\right)
$$

$$
= \varphi_{\theta}^{1/p}(o(1)) \cdot \exp\left(\frac{1}{p}(kn)^{1-2\beta}(1 + \delta)\frac{1}{2}p^2(1 + o(1))\lambda^2\right) \exp(o(1)) =
$$

$$
= \exp\left(\frac{1}{2}(kn+\theta)^{1-2\beta} \lambda^2(1 + \delta + o(1)) + o(1)\right)
$$

for large $kn+\theta$. Similarly,

$$
\varphi_{kn+\theta}(\lambda) \geq \exp\left(\frac{1}{2}(kn+\theta)^{1-2\beta} \lambda^2(1 - \delta + o(1)) + o(1)\right),
$$

which completes the proof of Theorem 3.1(b).
Proof.  
By (8.8), Lemma 8.5 may be applied to  
similar to the proof of Lemma 8.7.  

\[ \varphi_t(\lambda) = \int \exp \left( \frac{\lambda}{\sigma t^\beta} x \right) \mu_t(dx) = \mathbb{E} \exp \left( \frac{\lambda}{\sigma t^\beta} \int_0^t F(X_s) \, ds \right) \in (0, \infty] ; \]

similarly to Lemma 8.1,

\[ \varphi_p^{s} \left( \frac{1}{p} \left( \frac{s}{s+t} \right) \lambda \right) \varphi_t^{p} \left( \frac{1}{p} \left( \frac{t}{s+t} \right) \lambda \right) \exp \left( -C \frac{1}{p} \frac{\lambda^2}{(s+t)^{3\beta/2}} \right) \leq \varphi_{s+t}(\lambda) \leq \varphi_s^{1/p} \left( p \left( \frac{s}{s+t} \right) \lambda \right) \varphi_t^{1/p} \left( p \left( \frac{t}{s+t} \right) \lambda \right) \exp \left( C \frac{\lambda^2}{(s+t)^{3\beta/2}} \right) , \]

where \( p = \frac{(s+t)^{\beta/2}}{(s+t)^{\beta/2} - 1} \). Given \( \varepsilon_1 \in (0, \varepsilon] \), we consider (for every \( t \)) the smallest \( b_t \) and the largest \( a_t \) such that the inequalities

\[ \exp(0.5t^{1-2\beta}a_t^{2}) \leq \varphi_t(\lambda) \leq \exp(0.5t^{1-2\beta}b_t^{2}) \]

hold for all \( \lambda \) satisfying \( |\lambda| \leq \varepsilon_1\sigma^{\beta/\gamma} \). Similarly to Lemma 8.3, there exists \( T \) such that for all \( s, t \) satisfying \( T \leq s \leq t \leq 2s \),

\[ a_{s+t} \geq \frac{(s+t)^{\beta/2} - 1}{(s+t)^{\beta/2}} \left( \frac{s a_s + t a_t}{s + t} - \frac{2C}{(s+t)^{1-0.5\beta}} \right) , \]

\[ b_{s+t} \leq \frac{(s+t)^{\beta/2}}{(s+t)^{\beta/2} - 1} \left( \frac{s b_s + t b_t}{s + t} + \frac{2C}{(s+t)^{1-0.5\beta}} \right) . \]

8.9 Lemma. For every \( \delta > 0 \) there exist \( \varepsilon_1 \in (0, \varepsilon] \) and \( t \in (0, \infty) \) such that

\[ \inf_n a_{tn} \geq 1 - \delta \quad \text{and} \quad \sup_n b_{tn} \leq 1 + \delta . \]

Proof. By (8.8), Lemma 8.5 may be applied to \( B_n = b_{tn} \), \( \alpha = \beta/2 \) and \( r = t^{-\beta/2} \max(1, 2C) \) provided that \( t \) exceeds \( T \) of (8.8). The rest is completely similar to the proof of Lemma 8.7.

We see that

\[ (1 - \delta) \frac{\lambda^2}{2} \leq \frac{1}{(tn)^{1-2\beta}} \ln \varphi_{tn}(\lambda) \leq (1 + \delta) \frac{\lambda^2}{2} \]

for \( |\lambda| \leq \varepsilon_1\sigma(tn)^{\beta/\gamma} \). Now we consider \( \varphi_{tn+\theta}(\lambda) \) for \( \theta \in [0, t] \) and \( |\lambda| \leq 0.5\varepsilon_1\sigma(tn + \theta)^{\beta/\gamma} \), assuming that \( tn + \theta \) is large enough (namely, exceeds \( 2^{2/\beta} \)). We proceed similarly to the discrete case, taking into account that the functions \( \varphi_\theta(\cdot) \) are continuous at 0 uniformly in \( \theta \in [0, t] \), which follows from (7.6) and convexity of these functions.

24
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