Inverse problems for parabolic equations

A.G. Ramm
Mathematics Department, Kansas State University,
Manhattan, KS 66506-2602, USA
ramm@math.ksu.edu,
fax 785-532-0546, tel. 785-532-0580

Abstract

Let \( u_t - u_{xx} = h(t) \) in \( 0 \leq x \leq \pi, \ t \geq 0 \). Assume that \( u(0, t) = v(t), u(\pi, t) = 0, \) and \( u(x, 0) = g(t) \). The problem is: what extra data determine the three unknown functions \{h, v, g\} uniquely? This question is answered and an analytical method for recovery of the above three functions is proposed.

1 Introduction

Consider the problem

\[
\begin{align*}
    u_t - u_{xx} &= h(t) \quad (x, t) \in [0, \pi] \times [0, \infty), \\
    u(0, t) &= v(t), \quad u(\pi, t) = 0, \quad u(x, 0) = g(x),
\end{align*}
\]

where the three functions \{h, v, g\} are not known.

The Inverse Problem (IP) we are interested in is the following one:

What extra data determine the triple \{h, v, g\} uniquely?

There is an extensive literature on inverse problems for the heat equation (see [1], [2] and references therein), but the above IP has not been studied, as far as the author knows. In [3] the author studied an inverse source problem for multidimensional heat equation in which the source was assumed to be a finite sum of point sources, and the inverse problem was to find the location and the intensity (strength) of these point sources from experimental data. In [4] an inverse problem related to continuation of the solution to heat equation is studied.

Let \( ||f|| := ||f||_{L^2(0, \pi)}\), \( u_m := (u, f_m) = \int_0^\pi u f_m \, dx \),

\[
\begin{align*}
    f''_m + m^2 f_m &= 0, \quad 0 \leq x \leq \pi, \quad f_m(0) = f_m(\pi) = 0, \quad ||f_m|| = 1, \quad m = 1, 2 \ldots,
\end{align*}
\]

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where \( f_m = \sqrt{\frac{2}{\pi}} \sin(mx) \). Let \( y \in (0, \pi) \) be a point such that

\[
f_m(y) \neq 0 \quad \forall m = 1, 2, \ldots
\] (4)

Our result is:

**Theorem 1.** The three functions \( \{u_1(t), u_3(t), u(y, t)\} \), known for all \( t \geq 0 \), determine the triple \( \{h, v, g\} \) uniquely.

We will outline a method for finding \( h, v, \) and \( g \) and discuss the ill-posedness of the IP.

In Section 2 proofs are given.

## 2 Proofs

*Proof of Theorem 1.* Let us look for the solution to problem (1)-(2) of the form

\[
u(x, t) = \sum_{m=1}^{\infty} u_m(t) f_m(x),
\]

(5)

where the functions \( u_m \) are to be found. Multiplying equation (1) by \( f_m(x) \) and integrating over the interval \([0, \pi]\) and then by parts, one gets

\[
\dot{u}_m + m^2 u_m = v(t) f'_m(0) + c_m h(t), \quad u_m(0) = g_m, \quad c_m := (1, f_m) = \sqrt{\frac{2}{\pi}} \frac{1 - \cos(m\pi)}{m},
\]

(6)

where \( m = 1, 2, \ldots \). Thus,

\[
u_m(t) = g_m e^{-m^2 t} + \int_0^t e^{-m^2(t-s)} [v(s) f'_m(0) + c_m h(s)] ds.
\]

(7)

If the data

\[
\{u_1(t), u_3(t), u(y, t)\}
\]

(8)

are known, then one gets

\[
u_1(t) = g_1 e^{-t} + \int_0^t e^{-(t-s)} [v(s) f'_1(0) + c_1 h(s)] ds,
\]

(9)

and

\[
u_3(t) = g_3 e^{-9t} + \int_0^t e^{-9(t-s)} [v(s) f'_3(0) + c_3 h(s)] ds.
\]

(10)

Take \( t = 0 \) in (9) and (10) and get \( g_1 = u_1(0) \) and \( g_3 = u_3(0) \).

Thus, \( g_1 \) and \( g_3 \) are determined uniquely by the data.
Define \( u_1(t) - g_1 e^{-t} := F_1(t), \ u_3(t) - g_3 e^{-9t} := F_3(t), \) and rewrite (9) and (10) as

\[
F_1(t) = \int_0^t e^{-(t-s)} [v(s)f_1'(0) + c_1 h(s)] ds,
\]

(11)

and

\[
F_3(t) = \int_0^t e^{-9(t-s)} [v(s)f_3'(0) + c_3 h(s)] ds.
\]

(12)

Differentiate (11) and (12) and get

\[
v(t)f_1'(0) + c_1 h(t) = e^{-t} \frac{d}{dt}[e^t F_1(t)]
\]

(13)

\[
v(t)f_3'(0) + c_3 h(t) = e^{-9t} \frac{d}{dt}[e^{9t} F_3(t)]
\]

(14)

This is a linear system for finding \( v \) and \( h \). The determinant of this system is

\[
\begin{vmatrix}
  f_1'(0) & c_1 \\
  f_3'(0) & c_3
\end{vmatrix} = -\frac{32}{3 \pi} \neq 0,
\]

(15)

so \( v \) and \( h \) are uniquely, explicitly and analytically determined by the data.

If \( v(t) \) and \( h(t) \) are found, then one has

\[
u(y, t) = \sum_{m=1}^{\infty} e^{-m^2 t} g_m f_m(y) + w(y, t),
\]

(16)

where \( w(y, t) \) is known:

\[
w(y, t) = \sum_{m=1}^{\infty} f_m(y) \int_0^t e^{-m^2(t-s)} [v(s)f_m'(0) + c_m h(s)] ds.
\]

(17)

Denote \( q(y, t) := u(y, t) - w(y, t). \) Then \( q(y, t) \) is known and

\[
\sum_{m=1}^{\infty} e^{-m^2 t} g_m f_m(y) = q(y, t).
\]

(18)

This relation allows one to determine the numbers \( g_m f_m(y) \) uniquely for all \( m = 1, 2, \ldots \), by the formulas:

\[
g_1 f_1(y) = \lim_{t \to \infty} e^t q(y, t), \quad g_2 f_2(y) = \lim_{t \to \infty} e^{4t} [q(y, t) - e^{-t} g_1 f_1(y)],
\]

(19)

and so on. Thus, consequitively one finds all the numbers \( b_m := g_m f_m(y). \)

If the numbers \( b_m \) are found for all \( m = 1, 2, \ldots \), then the numbers \( g_m \) are uniquely determined by the formulas:

\[
g_m = \frac{b_m}{f_m(y)}.
\]

(20)
Formulas (20) make sense because of the assumption (4). If all the coefficients $g_m$ are found, then the function $g$ is calculated by the formula:

$$g(x) = \sum_{m=1}^{\infty} g_m f_m(x).$$

(21)

Thus, the triple $\{h, v, g\}$ is uniquely and analytically found from the data $\{u_1(t), u_3(t), u(y, t)\}$, known for all $t > 0$. Theorem 1 is proved.

In the proof of Theorem 1 we assume that the data are exact. The inverse problem under discussion is ill-posed: small perturbations of the data may throw the data out of the set of admissible data. For example, the solution $u(x, t)$ is infinitely differentiable (even analytic) with respect to $t$ in the region $t > 0$, so $u(y, t)$ cannot be an arbitrary function. Also, calculation by formulas (19) is an ill-posed problem: small errors in calculation of $g_m f_m(y)$ lead to large errors in calculation of $g_{m+1} f_{m+1}(y)$ because of the exponential factor $e^{-(m+1)^2 t}$. A detailed study of a similar problem, arising in the singularity expansion method (SEM), developed in scattering theory, is presented in [5], pp.365-393. Formula (20) also leads to ill-posedness, because the denominator in this formula is small for large $m$. Therefore the IP is severely ill-posed.

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