FINITENESS PROPERTIES OF FORMAL LIE GROUP ACTIONS

GAL BINYAMINI

Abstract. Following ideas of Arnold and Seigal-Yakovenko, we prove that the space of matrix coefficients of a formal Lie group action belongs to a Noetherian ring. Using this result we extend the uniform intersection multiplicity estimates of these authors from the abelian case to general Lie groups. We also demonstrate a simple new proof for a jet-determination result of Baouendi et al.

In the second part of the paper we use similar ideas to prove a result on embedding formal diffeomorphisms in one-parameter groups extending a result of Takens. In particular this implies that the results of Arnold and Seigal-Yakovenko are formal consequence of our result for Lie groups.

1. Introduction

Let $M$ be a differentiable manifold and $F : M \to M$ a smooth diffeomorphism. The fixed points of $F$ play an important role in determining its dynamical properties. More generally, a point $p \in M$ is said to be periodic of period $n \in \mathbb{Z}$ if $F^n(p) = p$. The asymptotic properties of the set $n$-periodic points are one of the principal dynamical invariants of $F$.

Recall that the index of an isolated fixed point $p$ of $F$ is defined to be the number of fixed points born from $p$ after a generic small perturbation of $F$ (counted with signs according to orientation). The number of $n$-periodic points counted with the corresponding indices is a more natural topological invariant. In particular, it is homotopy invariant and can be computed at the level of homology, for instance using the Lefschetz fixed point formula.

Suppose a map $F$ is given such that the number of $n$-periodic points, counted with indices, tends to infinity. In [8], Shub and Sullivan show that in this case the number of periodic points of the map must be infinite as well. More specifically, they prove the following.

Theorem ([8 Main Proposition]). Suppose that $F : U \to \mathbb{R}^n$ is a $C^1$ map and $0$ is an isolated fixed point of $F^n$ for every $n \in \mathbb{N}$. Then the index of $0$ as a fixed point of $F^n$ is bounded as a function of $n$.

Shub and Sullivan then apply this result to give a homological criterion for the infinitude of the set of periodic points based on the Lefschetz fixed point formula.

The index of an isolated fixed point may be viewed as a topological intersection index. Indeed, if $\Delta \subseteq M \times M$ denotes the diagonal manifold then the index of a fixed point $p$ is equal to the intersection index of $\Delta$ and $(id \times F)(\Delta)$ at $p$. It is natural to ask whether an analog of Shub and Sullivan’s main proposition can be given for general intersection indices.

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An affirmative answer in the holomorphic setting was provided by Arnold in [1]. Consider now $F : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ to be a germ of a biholomorphism and let $V, W \subset (\mathbb{C}^n, 0)$ be the germs of two analytic submanifolds of complementary dimensions. Then $F^n(V)$ is again a germ of an analytic submanifold, and assuming that it has an isolated intersection with $W$ at the origin, a local intersection multiplicity $(F^n(V), W)$ is defined.

**Theorem ([1, Theorem 1]).** The sequence $\left\{ (F^n(V), W) \right\}_{n \in \mathbb{N}}$ is uniformly bounded provided each of its elements is finite.

Arnold’s proof relied on the rather nontrivial Skolem-Mahler-Lech theorem from additive number theory. In [7], Seigal and Yakovenko showed that Arnold’s result could be proved using simpler and much more general Noetherianity arguments. Using this approach, they were able to generalize the result to the case of groups of formal diffeomorphisms generated by finitely many commuting diffeomorphisms and vector fields.

More specifically, a commutative group $G \subset \text{Diff}[[\mathbb{C}^n, 0]]$ is said to be finitely generated if there exist finitely many formal diffeomorphisms $F_1, \ldots, F_p$ and finitely many formal vector fields $V_1, \ldots, V_q$ such that all of these generators commute in the appropriate sense, and every element of $G$ can be written as a finite product of maps $F_i^{\pm 1}$ and $e^{tV_j}$ where $t \in \mathbb{C}$. Alternatively, $G$ is a product of a (discrete) finitely generated abelian subgroup of $\text{Diff}[[\mathbb{C}^n, 0]]$ and a connected (finite-dimensional) abelian Lie subgroup of $\text{Diff}[[\mathbb{C}^n, 0]]$ such that the two subgroups commute.

**Theorem ([7, Theorem 1]).** Let $G \subset \text{Diff}[[\mathbb{C}^n, 0]]$ be a finitely generated commutative group, and let $V, W$ be two formal varieties. Then the intersection multiplicities $\mu_g := (gV, W), g \in G$ admit a uniform bound $N \in \mathbb{N}$ whenever they are finite, 

$$\forall g \in G : \quad \mu_g < \infty \implies \mu_g < N \quad (1)$$

Seigal and Yakovenko prove this theorem essentially by showing that the set $G_k = \{ g : \mu_g > k \}$ is given by the zero locus of an ideal $I_k \subset R$, where $R$ is a certain Noetherian ring of continuous functions on $G$. The theorem (and its various generalizations given in [7]) follow by a simple Noetherianity argument.

In this paper we generalize the results of [7] from the commutative case to the action of an arbitrary (finite dimensional) Lie group with finitely many connected components. Moreover, we prove a formal embedding result showing that arbitrary formal diffeomorphisms may be embedded into the action of such groups, thus showing that the results of [1] and [7] follow formally from our result. A synopsis of our results and some applications and corollaries is presented in §2. Proofs of the key results and some further analysis is given in §3.

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2. **Statement of our results**

2.1. **Notation.** Let $\mathbb{C}[[x]]$ denote the ring of formal power series in $n$ variables. Denote by $m \subset \mathbb{C}[[x]]$ the maximal ideal, by $\mathbb{C}_p[[x]] := \mathbb{C}[[x]]/m^{p+1}$ the ring of $p$-jets of formal power series and by $\text{Diff}_p[[\mathbb{C}^n, 0]]$ the corresponding group of $p$-jets of diffeomorphisms.
Let Diff[[C^n, 0]] denote the group of formal diffeomorphisms, i.e. automorphisms of C[[x]]. Let D[[C^n, 0]] denote the Lie algebra of formal singular vector fields, i.e. derivatives of C[[x]] which map m to itself. Since Diff[[C^n, 0]] (resp. D[[C^n, 0]]) maps m to itself, there is an induced group of p-jets Diff_p[[C^n, 0]] (resp. Lie algebra D_p[[C^n, 0]]) and Diff[[C^n, 0]] (resp. D_p[[C^n, 0]]) is the inverse limit of these groups (resp. Lie algebras). When we talk about a morphism of groups, lie algebras or algebraic varieties into Diff[[C^n, 0]] or D[[C^n, 0]] we mean that the induced map into the p-jet spaces are morphisms of the prescribed type, for any p ∈ N.

Recall that the formal exponentiation operator defines a map

\[ \exp : D[[C^n, 0]] \to \text{Diff}[[C^n, 0]], \quad \exp(V) = e^V = I + V + \frac{V^2}{2} + \ldots \] (2)

For any V ∈ D[[C^n, 0]] there is an associated one-parameter group of formal diffeomorphisms τ_V : R → Diff[[C^n, 0]] given by τ_V(t) = e^{tV}. Conversely, any analytic (or even continuous) one parameter group τ : R → Diff[[C^n, 0]] admits an infinitesimal generator V ∈ D[[C^n, 0]], i.e. τ = τ_V (see [11, Equation 3.6]).

For α ∈ N^n, let x^α ∈ C[x] denote the corresponding monomial in the standard multiindex notation. Let \((\cdot, \cdot)\) denote the standard inner product with respect to the monomial basis, i.e. \((x^\alpha, x^\beta) = \delta_{\alpha,\beta}\).

2.2. Noetherianity of Lie group actions. Let G denote a (finite-dimensional) Lie group, and ρ : G → Diff[[C^n, 0]] a homomorphism of Lie groups (by which we mean that ρ induces a homomorphism of finite-dimensional Lie groups ρ_p : G → Diff_p[[C^n, 0]] for every p ∈ N). We say that G acts formally on (C^n, 0).

For any pair α, β ∈ N^n we define the matrix coefficient

\[ \rho_{\alpha,\beta} : G \to \mathbb{C}, \quad \rho_{\alpha,\beta}(g) = (\rho(g)(x^\alpha), x^\beta). \] (3)

The space of matrix coefficients of ρ agrees with the union of spaces of matrix coefficients of ρ_p for p ∈ N. Since the space of matrix coefficients of a finite-dimensional group representation is independent of the choice of basis elements, it follows that the space of matrix coefficients of ρ is invariant under a formal change of coordinates (since it induces an isomorphism at the level of jets).

We are now ready to state our main result.

**Theorem 1.** Let G be a Lie group with finitely many connected components and ρ : G → Diff[[C^n, 0]] a Lie group homomorphism. Then all matrix coefficients ρ_{α,β} are contained in a Noetherian ring R_p ⊂ C(G) of continuous functions on G.

Moreover,

\[ \dim R_p \leq \dim G + n(\dim G - \dim K) \] (4)

where K denotes a maximal compact subgroup of G.

Theorem [11] generalizes the Noetherianity result of [7] to the case of arbitrary Lie groups (with finitely many connected components) acting formally on (C^n, 0).

2.3. Local intersection dynamics. By a germ of a formal scheme we shall mean a closed subset V ⊂ Spec C[[x]]. In other words, V will be identified with an ideal I_V of the ring C[[x]]. Two germs of formal schemes V, W of complementary dimension are said to intersect properly if I_V + I_W is an ideal of finite codimension in C[[x]].

In this case, we define their intersection multiplicity at the origin to be

\[ (V, W) := \dim C[[x]]/(I_V + I_W) \] (5)
Remark 1. Note that this definition of multiplicity gives the correct notion when \( V, W \) are complete intersections. In the general case a more delicate definition is needed to provide the proper notion of intersection multiplicities, cf. [3] Section 8. However, for our purposes this naive definition will suffice, since it provides an upper bound for the true intersection multiplicity [3, Proposition 8.2, (a)].

Since the ring \( \mathbb{C}[[x]] \) is Noetherian, its ideals are finitely generated. The following lemma is standard.

**Lemma** ([7, Lemma 3]). For any ideal \( I \subset \mathbb{C}[[x]] \) and any \( m \in \mathbb{N} \), the condition \( \dim_{\mathbb{C}} \mathbb{C}[[x]]/I > m \) is equivalent to a finite number of algebraic conditions imposed on the \( m \)-jets of the generators of \( I \).

Let \( G \) be a Lie group and \( \rho : G \to \text{Diff}[[\mathbb{C}^n, 0]] \) a Lie group homomorphism. Then \( G \) acts on germs of formal schemes: \( g^*V \) is defined to be the germ associated to the ideal \( \rho(g)^*I_V \) where \( f^* \) denotes pullback of ideals with respect to \( f \).

We can now state and prove our result on dynamics of intersections for formal Lie group actions.

**Theorem 2.** Let \( G \) be a Lie group with finitely many connected components and \( \rho : G \to \text{Diff}[[\mathbb{C}^n, 0]] \) a Lie group homomorphism. Let \( V, W \) be two germs of formal schemes. Then the intersection multiplicities \( \mu_g := (g^*V, W), g \in G \) admit a uniform bound \( N \in \mathbb{N} \) whenever they are finite,

\[
\forall g \in G : \quad \mu_g < \infty \implies \mu_g < N
\]  

**Proof.** Let \( v_1, \ldots, v_s \) and \( w_1, \ldots, w_t \) denote generators for the ideals \( I_V \) and \( I_W \) respectively. Let \( R_\rho \) denote the ring of Theorem 1. Then the functions mapping \( g \) to the Taylor coefficients of \( g^*v_i \) and \( g^*w_i \) are matrix coefficients of \( \rho \), hence elements of \( R_\rho \). Then, by Lemma 2.3 the set \( \{ g : (g^*V, W) > k \} \) is defined by the common zeros of an ideal \( I_k \subset R_\rho \). Moreover, one can assume that \( I_k \subset I_{k+1} \) (otherwise simply add to each ideal \( I_k \) the union of all \( I_l \) for \( l < k \)).

Since the ring \( R_\rho \) is Noetherian, the chain \( I_k \) stabilizes at some finite index \( k = N - 1 \). Then the set of common zeros of the ideals also stabilizes, i.e. for any \( g \in G \) we have \( \mu_g \geq N \implies \mu_g = \infty \), as claimed. \( \square \)

2.4. **Finite jet determination for Lie group actions.** As another application of Theorem 1, we give an immediate proof of a result of Baouendi et al. concerning finite jet determination in [2, Theorem 2.10 and Proposition 5.1].

**Theorem** ([2, Proposition 5.1]). Let \( G \) be a Lie group with finitely many connected components and \( \rho : G \to \text{Diff}[[\mathbb{C}^n, 0]] \) a continuous injective homomorphism. Then there exists \( p \in \mathbb{N} \) such that, for any \( g_1, g_2 \in G \),

\[
\begin{align*}
\mathcal{J}^p \rho(g_1) = \mathcal{J}^p \rho(g_2) \quad &\text{if and only if} \quad g_1 = g_2
\end{align*}
\]  

We note that the statement of [2, Proposition 5.1] refers to analytic rather than formal diffeomorphisms, and is implied by the statement above. We also note that the original formulation works in the real (rather than complex) category, but the proof is not affected by this difference.

**Proof.** By considering \( g = g_1g_2^{-1} \), it is clearly enough to prove the existence of \( p \in \mathbb{N} \) such that for any \( g \in G \),

\[
\mathcal{J}^p \rho(g) = \text{id} \quad \text{if and only if} \quad g = e.
\]  

(8)
Let \( R_\rho \) denote the ring of Theorem 1. Then the Taylor coefficients of \( \rho(g) \) are elements of \( R_\rho \), and in particular the condition \( j^k \rho(g) = \text{id} \) can be expressed as the common zero locus of an ideal \( I_k \subset R_\rho \), where the chain \( I_k \) is increasing.

Since the ring \( R_\rho \) is Noetherian, the chain \( I_k \) stabilizes at some finite index \( k = p \). Then the set of common zeros of the ideals also stabilizes, i.e. the condition \( j^p \rho(g) = \text{id} \) is equivalent to the condition \( \rho(g) = \text{id} \). Since \( \rho \) is an injection, the claim is proved. □

2.5. Embedding in abelian Lie groups and formal flows. In this subsection we state a theorem demonstrating that the results of Arnold [1, Theorem 1] and Seigal-Yakovenko [7, Theorem 1] are formal consequences of Theorem 2. We note, to be clear, that in the cases considered in these papers our proof essentially follows the local computation of Arnold and the Noetherianity argument of Seigal-Yakovenko. However, the embedding results given in this subsection appear to be new and may be of some independent interest.

Theorem 3. Let \( \rho : \mathbb{Z}^p \times \mathbb{R}^q \to \text{Diff}(\mathbb{C}^n, 0) \) be a homomorphism of Lie groups defining a finitely-generated commutative group of formal diffeomorphisms. Then there exists an abelian Lie group \( G \) with finitely many connected components (in fact, a linear algebraic group) and a homomorphism of Lie groups \( \rho^* : G \to \text{Diff}(\mathbb{C}^n, 0) \) such that \( \text{Im} \rho \subset \text{Im} \rho^* \).

We record a simple corollary about embeddings of a diffeomorphism in the flow of a formal vector field.

Corollary 2. Let \( F \in \text{Diff}(\mathbb{C}^n, 0) \) be a formal diffeomorphism. Then there exists some \( k \in \mathbb{N}^+ \) such that \( F^k \) can be embedded in a formal flow, i.e. there exists a formal vector field \( V \in \mathbb{D}(\mathbb{C}^n, 0) \) such that \( e^V = F^k \).

Proof. Let \( \rho : \mathbb{Z} \to \text{Diff}(\mathbb{C}^n, 0) \) be given by \( \rho(1) = F \) and let \( G \) and \( \rho^* \) be as provided by Theorem 3. Let \( G_0 \) denote the connected component of the identity.

By construction there exists \( g \in G \) such that \( \rho^*(g) = F \). Letting \( k = |G/G_0| \) we have \( g^k \in G_0 \). Since \( G_0 \) is an abelian connected Lie group, its exponential map is surjective. Thus there exists an analytic one-parameter group \( \eta^* : \mathbb{R} \to G_0 \) such that \( \eta^*(1) = g^k \). Composing with \( \rho^* \) we obtain an analytic one parameter group \( \eta = \rho^* \circ \eta^* : \mathbb{R} \to \text{Diff}(\mathbb{C}^n, 0) \) with \( \eta(1) = F^k \). Then the derivative of \( \eta \) at the origin gives a formal vector field \( V \) satisfying \( e^V = F^k \) as claimed. □

In §3.2 a more accurate description of the group \( G \) is provided. In particular we show how to compute \( k \) above in terms of the spectrum of the linear part of \( F \), and use this to give a new proof of a result of Takens [9] about embeddings of formal diffeomorphisms with a unipotent linear part.

3. Proofs

In this section we prove the two theorems stated without proof in [2] namely Theorems 1 and 3.

\footnote{We remark that [7] uses a homomorphism of the complex Lie group \( \mathbb{Z}^p \times \mathbb{C}^q \); but the resulting notion is equivalent.}
3.1. **Proof of Theorem** \([\text{I}]\). Before presenting the proof we require two results from the theory of Lie groups. The first is a decomposition theorem due independently to Iwasawa [\text{5}] and Malcev [\text{6}].

**Theorem** ([\text{5}, Theorem 6]). Let \(G\) be a connected Lie group, and \(K\) a maximal compact subgroup. Then there exist one parameter subgroups \(H_1, \ldots, H_r\) isomorphic to \(\mathbb{R}\) such that any element \(g \in G\) can be decomposed uniquely and continuously in the form

\[
g = h_1 \cdots h_r k, \quad h_i \in H_i, \quad k \in K.
\] (9)

The following linearization theorem is due to Bochner. Since we are not aware of a reference for the formal case, we present the proof below.

**Theorem.** Let \(K\) be a compact Lie group and \(\rho : K \to \text{Diff}[[\mathbb{C}^n, 0]]\) a homomorphism of Lie groups. Then after a formal change of coordinate \(s\), \(\rho\) is equivalent to its linear representation (mapping \(g\) to the linear part \(d\rho(g)\)).

**Proof.** Let \(\mu\) denote the normalized left-invariant Haar measure on \(K\). We define a formal map \(U \in \mathbb{C}^n[[x]]\) by averaging,

\[
U(x) = \int_K (d\rho(g))^{-1} \rho(g) d\mu(g).
\] (10)

Taking linear parts we see that

\[
dU = \int_K (d\rho(g))^{-1} d\rho(g) = \text{id},
\] (11)

so \(U \in \text{Diff}[[\mathbb{C}^n, 0]]\). Moreover, by invariance of the Haar measure,

\[
\rho(h)^*U = \int_K (d\rho(g))^{-1} \rho(gh) d\mu(g)
\]

\[
= \int_K (d\rho(gh^{-1}))^{-1} \rho(g) d\mu(g) = d\rho(h)U
\] (12)

so \(U\) conjugates \(\rho\) to its linear part \(d\rho\) as claimed. \(\square\)

We require one more lemma. While simple, this lemma is in fact the heart of the proof. The lemma is due to Arnold [\text{I}, Lemma 2] (see also [\text{II}, Lemma 5]).

**Lemma 3.** Let \(\rho : \mathbb{R} \to \text{Diff}[[\mathbb{C}^n, 0]]\) be given by \(\rho(t) = e^{tV}\) where \(V \in \mathcal{D}[[\mathbb{C}^n, 0]]\). Denote by \(\lambda_1, \ldots, \lambda_n\) the spectrum of the linear part of \(V\). Denote by \(t\) the coordinate on \(\mathbb{R}\).

Then the matrix coefficients of \(\rho\) are contained in the ring \(\mathbb{C}[e^{\lambda_1 t}, \ldots, e^{\lambda_n t}, t]\).

**Proof.** The matrix coefficients of \(\rho\) are the union of the matrix coefficients of the jets \(j^p \rho\) for all \(p \in \mathbb{N}\). It is a classical result, following easily from the Jordan decomposition, that the matrix coefficients of the matrix exponential \(e^{tL}\) of a finite dimensional operator \(L\) are contained in the ring \(\mathbb{C}[e^{\mu t}, t]\) where \(\mu\) are the elements of the spectrum of \(L\). Thus the claim will be proved if we show that the spectrum of the jet \(j^p \rho\) is contained in the semigroup generated by \(\lambda_1, \ldots, \lambda_n\) for every \(p\).

We may, after a linear change of coordinates, assume that the linear part of \(V\) is a lower-triangular matrix. Put the corresponding degree-lexicographic ordering on all monomials in \(\mathbb{C}[x]\). Then the matrix representing \(j^p V\) is again lower-triangular, and the diagonal entry corresponding to the monomial \(x^\alpha\) is precisely \(\sum_{i} \alpha_i \lambda_i\). Thus the spectrum is contained in the semigroup generated by the \(\lambda_i\), as claimed. \(\square\)
We are now ready to present the proof of Theorem 1.

Proof of Theorem 1. Recall that $K$ denotes a maximal compact subgroup of $G$. By Theorem 3.1, making a formal change of coordinates we may assume that the action of $K$ is linear (note that a formal change of variables does not change the space of matrix coefficients).

Assume first that $G$ is connected. By Theorem 3.1 we have for every $g \in G$ a decomposition
$$g = h_1 \cdots h_r k \quad h_i \in H_i \quad k \in K$$
where we may think of $h_i, k$ as continuous functions of $g$.

We first note that since $K$ acts linearly by assumption, the matrix coefficients of $\rho(k)$ belong to the ring of $\mathbb{R}[K]$ of regular functions on the linear algebraic group $K$. This is a finitely-generated $\mathbb{C}$-algebra of dimension $\dim K$.

Secondly, $H_i$ is an analytic one-parameteric group isomorphic to $\mathbb{R}$, and its action $\rho_i : H_i \to \text{Diff}[[\mathbb{C}^n, 0]]$ is therefore given by $\rho_i(t) = e^{tV_i}$ for some infinitesimal generator $V_i \in \mathbb{D}[[\mathbb{C}^n, 0]]$. By Lemma 3 the matrix coefficients of $\rho_i$ belong to a ring $R_i$ generated by the spectrum of the linear part of $V_i$ and $t$. In particular it is a finitely-generated $\mathbb{C}$-algebra of dimension at most $n + 1$.

Now,
$$\rho(g) = \rho(h_1) \cdots \rho(h_r) \rho(k)$$
so the matrix coefficients of $\rho(g)$ are multilinear combinations of the matrix coefficients of $\rho(h_i), \rho(k)$. In particular they belong to the ring
$$R_\rho = R_1 \otimes_\mathbb{C} \cdots \otimes_\mathbb{C} R_r \otimes_\mathbb{C} R[K] (15)$$
which, by the above, is a Noetherian ring (in fact a finitely generated $\mathbb{C}$-algebra) of dimension bounded by $\dim G + nr$ as claimed.

For the non-connected case, one only needs to add to the decomposition (13) another factor $s \in S$, where $S \subset G$ is a (finite) subset of representatives of the cosets of $G/G_0$. Since $S$ is a finite set, the matrix coefficients of $\rho(s)$ belong to the zero-dimensional finitely-generated algebra of functions on $S$, and the preceding argument can be carried out verbatim.

3.2. Proof of Theorem 3 and further results. In this subsection we prove Theorem 3 and give some more detailed results about embeddings of formal diffeomorphisms in formal flows.

We begin with a slightly more detailed analog of Lemma 3 for the discrete case.

Lemma 4. Let $F \in \text{Diff}[[\mathbb{C}^n, 0]]$ and denote by $\lambda_1, \ldots, \lambda_n$ the spectrum of the linear part of $F$. Let $\tau : \mathbb{Z} \to \text{Diff}[[\mathbb{C}^n, 0]]$ be given by $\tau(1) = F$. Denote by $t$ the coordinate on $\mathbb{Z}$.

Then one of the following is true:

1. (Semisimple case) The space of matrix coefficients of $\tau$ is equal to $\mathbb{C}[\lambda_1^t, \ldots, \lambda_n^t]$.
2. (Non-semisimple case) The space of matrix coefficients of $\tau$ is contained in $\mathbb{C}[\lambda_1^t, \ldots, \lambda_n^t, t]$ and contains a function of the form $\lambda_\alpha t^\beta = \lambda_1^{\alpha_1} t^{\beta_1} \cdots \lambda_n^{\alpha_n} t^{\beta_n}$ for some $\alpha, \beta \in \mathbb{N}^n$.

The proof, which we omit, proceeds in the same way as the proof of Lemma 3. The first case is obtained if $j^pF$ is a semisimple operator for every $p \in \mathbb{N}$, and the second case is obtained otherwise (in which case the statement follows by a well-known formula for the powers of a matrix in Jordan form).
Denote by $\text{TDiff}[[\mathbb{C}^n, 0]]$ the group of formal diffeomorphisms which are lower-triangular with respect to the degree-lexicographic ordering on the standard monomial basis, and by $\text{TDiff}_p[[\mathbb{C}^n, 0]]$ the corresponding group of jets. Recall from the proof of Lemma 3 that any formal diffeomorphism belongs to $\text{TDiff}[[\mathbb{C}^n, 0]]$ after a linear change of coordinates.

We now state the key proposition on the embedding of a formal diffeomorphism in an abelian linear algebraic group.

**Proposition 5.** Let $F \in \text{Diff}[[\mathbb{C}^n, 0]]$. Then there exists an abelian linear algebraic group $G$ and a homomorphism of Lie groups $\rho : G \to \text{Diff}[[\mathbb{C}^n, 0]]$ such that $F \in \text{Im} \rho$.

Additionally, if a formal diffeomorphism $F' \in \text{Diff}[[\mathbb{C}^n, 0]]$ commutes with $F$ then it also commutes with any element of $\text{Im} \rho$.

**Proof.** As indicated above, we may assume that $F \in \text{TDiff}[[\mathbb{C}^n, 0]]$. Let $\lambda_1, \ldots, \lambda_n$ denote the spectrum of the linear part of $F$. Suppose that we are in case (2) of Lemma 4. Case (1) is similar but simpler.

Denote by $p$ the degree of the first jet for which the function $\lambda^{ai}t$ appears in the space of matrix coefficients of $\tau : Z \to \text{Diff}[[\mathbb{C}^n, 0]]$ defined by $t \to F^t$ (in fact, this is the first jet for which $j^pF$ is not a semisimple operator).

We claim that there exist regular functions $L_1, \ldots, L_n, T \in R[\text{TDiff}_p[[\mathbb{C}^n, 0]]$ such that, for every $t \in \mathbb{Z}$,

$$L_i(j^pF^t) = \lambda^t_i, \quad T(j^pF^t) = t$$

(16)

Indeed, $L_i$ is given simply by the $i$-th diagonal matrix element on $\text{TDiff}_p[[\mathbb{C}^n, 0]]$. To define $T$, we note first that by assumption $\lambda^{ai}t$ appears in the space of matrix coefficients. It is therefore given by a linear combination of matrix coefficients. That is, there exists a regular function $T^* \in R[\text{TDiff}_p[[\mathbb{C}^n, 0]]$ such that $T^*(j^pF^t) = \lambda^{ai}t$. It remains to define $T := T^*/(L_1^{n_1} \cdots L_n^{n_n})$, noting that each $L_i$, being a diagonal element, is invertible as a regular function on the group of lower triangular matrices.

Let $G_* \subset \text{TDiff}_p[[\mathbb{C}^n, 0]]$ denote the subgroup generated by $j^pF$, and let $G$ denote the Zariski closure of $G_*$. As the closure of an abelian group, $G$ itself is also abelian.

Recall that the ring of matrix coefficients of $\tau$ is contained in $\mathbb{C}[\lambda_1^t, \ldots, \lambda_n^t]$. Thus for every pair $\alpha, \beta \in \mathbb{N}^n$ there exist polynomials $P_{\alpha, \beta}$ satisfying

$$(F^t, x^\alpha, x^\beta) = \tau_{\alpha, \beta} = P_{\alpha, \beta}(\lambda_1^t, \ldots, \lambda_n^t, t)$$

(17)

We define a regular homomorphism $\rho : G \to \text{Diff}[[\mathbb{C}^n, 0]]$ coordinate-wise by the following identity

$$\rho(g)x^\alpha, x^\beta = P_{\alpha, \beta}(L_1(g), \ldots, L_n(g), T(g)).$$

(18)

Each coordinate is given by a regular function on $G$, and by construction we have

$$\rho(j^pF^t) = F^t \quad \text{for every} \; t \in \mathbb{Z}$$

(19)

To show that $\text{Im} \rho \subset \text{Diff}[[\mathbb{C}^n, 0]]$ it suffices to note that:

- $\rho(g)$ preserves the $m$-filtration for every $g \in G$: for $g \in G_*$ this follows from (19), and this extends to all $g \in G$ by density.
- $\text{Diff}[[\mathbb{C}^n, 0]]$ is a closed subvariety of the space of all linear endomorphisms of $\mathbb{C}[[x]]$ which preserve the $m$-filtration.
- $\rho$ maps the group $G_*$ into $\text{Diff}[[\mathbb{C}^n, 0]]$ by (19), and this extends to $G$ by density.
Formally one should carry out the argument above at the level of truncations to any finite jet, but we leave these details to the reader. Similarly, to show that $\rho$ is a homomorphism it suffices to note that it satisfies the homomorphism property on the dense subgroup $G_s \times G_s$ of $G \times G$, again by (19). Finally, by (19) we have $F = \rho(j^p F) \in \text{Im } \rho$ as claimed.

For the second part of the proposition, note that commutation with a formal diffeomorphism $F'$ is a closed algebraic condition (at the level of any finite jet). Since $F'$ commutes with $F$ it commutes with any integer power of $F$, and by (19) we see that $F'$ commutes with a dense subset of $\text{Im } \rho$, and thus conclude that $F'$ commutes with any element of $\text{Im } \rho$.

**Remark 6.** From the proof above it is easy to find the structure of the group $G$. It is either $T$ or $T \times \mathbb{C}^a$ (for the semisimple and non-semisimple cases, respectively) where $T$ is the algebraic group generated by the semisimple part of the linear part of $F$. If $T_0$ denotes the connected component of the identity in $T$, then $T_0$ is a torus group $\mathbb{C}^a$, and $T$ is an extension of $T_0$ by a finite torsion group.

A torsion element must be mapped by some multiplicative character to a (non-trivial) root of unity. It follows that the exponent of torsion is equal to the size of the group of roots of unity inside the multiplicative group generated by $\lambda_1, \ldots, \lambda_n$.

Theorem 3 is now an easy corollary.

**Proof of Theorem 3.** We prove the claim by induction on $p$. Let $\rho : \mathbb{Z}^p \times \mathbb{R}^q \to \text{Diff}[[\mathbb{C}^n, 0]]$ be a homomorphism of Lie groups. By the inductive hypothesis, there exists an abelian linear algebraic group $G$ and a homomorphism of Lie groups $\rho^* : G \to \text{Diff}[[\mathbb{C}^n, 0]]$ such that $\rho(\{0\} \times \mathbb{Z}^{q-1} \times \mathbb{R}^q) \subset \text{Im } \rho^*$, and $\rho(1, 0, \ldots, 0)$ commutes with $\text{Im } \rho^*$.

By Proposition 5 there exists an abelian linear algebraic group $G'$ and a homomorphism of Lie algebras $\rho' : G' \to \text{Diff}[[\mathbb{C}^n, 0]]$ such that $\rho(1, 0, \ldots, 0) \in \text{Im } \rho'$ and $\text{Im } \rho^*$ commutes with $\text{Im } \rho'$. Then the group $G \times G'$ with the homomorphism $\rho^* \times \rho'$ satisfies the required conditions.

From Remark 6 and the proof of Corollary 2 we have the following corollary. In particular, this extends a result of Takens [9] stating the a formal diffeomorphism with a unipotent linear part can always be embedded in the flow of a formal vector field.

**Corollary 7.** Let $F \in \text{Diff}[[\mathbb{C}^n, 0]]$ be a formal diffeomorphism. Let $\lambda_1, \ldots, \lambda_n$ denote the spectrum of the linear part of $F$, and let $k$ denote the size of the group of roots of unity in the multiplicative group generated by $\lambda_1, \ldots, \lambda_n$.

Then $F^k$ can be embedded in a formal flow, i.e. there exists a formal vector field $V \in \mathcal{D}[[\mathbb{C}^n, 0]]$ such that $e^V = F^k$.

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