THE BV-ALGEBRA STRUCTURE OF $\mathcal{W}_3$ COHOMOLOGY

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Abstract: We summarize some recent results obtained in collaboration with J. McCarthy on the spectrum of physical states in $\mathcal{W}_3$ gravity coupled to $c = 2$ matter. We show that the space of physical states, defined as a semi-infinite (or BRST) cohomology of the $\mathcal{W}_3$ algebra, carries the structure of a BV-algebra. This BV-algebra has a quotient which is isomorphic to the BV-algebra of polyvector fields on the base affine space of $SL(3, \mathbb{C})$. Details will appear elsewhere.

1. Introduction

Understanding the spectrum of physical states in theories of two-dimensional $\mathcal{W}$-gravity coupled to matter poses an interesting challenge. Unlike in the case of ordinary gravity, the computation of the relevant semi-infinite (or BRST) cohomology of the underlying $\mathcal{W}$-algebra appears to be very difficult, and only a small number of results have been rigorously established. One expects that by studying the structure of this cohomology space it might be possible to achieve a better understanding of (quantum) $\mathcal{W}$-geometry and string field theory. The problem is also mathematically quite interesting as it involves generalizing some of the standard techniques for computing semi-infinite cohomologies to non-linear algebras.

In this paper we summarize some recent work done in collaboration with J. McCarthy on the computation of physical states in $\mathcal{W}_3$-gravity coupled to two scalar fields, as the semi-infinite cohomology of a tensor product of two Fock space modules of the $\mathcal{W}_3$ algebra. A complete result for the cohomology is given in Conjecture 3.1, Theorem 3.2 and Corollary 3.3. We then discuss in some detail the structure of the space of physical states as a Batalin-Vilkovisky (BV) algebra and, in particular, show that it is modelled on the well-known BV-algebra of regular polyvector fields on the base affine space of $SL(3, \mathbb{C})$. The main result here is given in Theorem 4.6. For more details we refer to [1–3] and the forthcoming paper [4].

Throughout this paper we will use the notation $\mathfrak{h}$ for the Cartan subalgebra, $\mathfrak{h}_{\mathbb{Z}}^{\ast}$ for the set of integral weights, $P_+$ for the set of dominant integral weights, $P_{++}$ for the set of strictly dominant integral weights, $\Delta_+$ for the positive roots and $W$ for the Weyl group of some Lie algebra $\mathfrak{g}$. $\mathcal{L}(\Lambda)$ will denote the finite dimensional irreducible representation of $\mathfrak{g}$ with highest weight $\Lambda \in P_+$ and $\ell(w)$ the length of $w \in W$. In the following $\mathfrak{g}$ will always refer to $\mathfrak{sl}_3$.

2. The $\mathcal{W}_3$ algebra and its modules

The $\mathcal{W}_3$ algebra with central charge $c \in \mathbb{C}$ (denoted simply by $\mathcal{W}$ in the sequel) is defined as the quotient of the free Lie algebra generated by $L_m, W_m, m \in \mathbb{Z}$, by the ideal generated by the following commutation

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relations (see e.g. the review on \(\mathcal{W}\)-algebras [5], and references therein).

\[
\begin{align*}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0}, \\
[L_m, W_n] &= (2m-n)W_{m+n}, \\
[W_m, W_n] &= (m-n) \left( \frac{1}{12} (m+n+3)(m+n+2) - \frac{1}{6} (m+2)(n+2) \right) L_{m+n} \\
&\quad + \beta (m-n)\Lambda_{m+n} + \frac{c}{360} m(m^2-1)(m^2-4)\delta_{m+n,0},
\end{align*}
\]

(2.1)

where \(\beta = 16/(22+5c)\) and

\[
\Lambda_m = \sum_{n \leq -2} L_n L_{m-n} + \sum_{n > -2} L_n L_{m-n} - \frac{c}{12} (m+3)(m+2)L_m.
\]

(2.2)

Notice that, due to the non-linearity of \(\Lambda_m\) in (2.1), \(\mathcal{W}\) is not a Lie algebra. The Cartan subalgebra \(\mathcal{W}_0\) of \(\mathcal{W}\) is spanned by \(L_0\) and \(W_0\), but, because \((\text{ad} W_0)\) is not diagonalizable, \(\mathcal{W}\) does not admit a root space decomposition (a generalized root space decomposition, i.e. a Jordan normal form, does however exist). Nevertheless, it is still convenient to decompose the generators of \(\mathcal{W}\) according to the \((-\text{ad} L_0)\) eigenvalue, and define \(\mathcal{W}_\pm = \{L_n, W_n \mid \pm n > 0\}\). However, this is not a triangular decomposition in the usual sense.

For physical applications the most interesting representations of \(\mathcal{W}\) are the so-called positive energy modules, which are defined by the condition that (the energy operator) \(L_0\) is diagonalizable with finite dimensional eigenspaces, and with the spectrum bounded from below. If the lowest energy eigenspace is one dimensional, we denote the eigenvalues of \(L_0\) and \(W_0\) on the highest weight state by \(h\) and \(w\), respectively.

In particular, the Verma module \(M(h, w, c)\) is defined as the (positive energy) module induced by \(\mathcal{W}_-\) from an 1-dimensional representation of \(\mathcal{W}_0\). By the standard argument, \(M(h, w, c)\) contains a maximal submodule. We denote the corresponding irreducible quotient module by \(L(h, w, c)\). The module contragradient to \(M(h, w, c)\) will be denoted by \(\overline{M}(h, w, c)\).

Another class of positive energy modules of \(\mathcal{W}\) are the Fock space modules \(F(\Lambda, \alpha_0)\), which arise in the free field realization of \(\mathcal{W}\) in terms of two scalar fields (see e.g. [5], and references therein). The modules \(F(\Lambda, \alpha_0)\) are labelled by the background charge \(\alpha_0 \in \mathbb{C}\) and an \(\mathfrak{sl}_3\) weight \(\Lambda\).

The central charge \(c\) and the highest weights \(h\) and \(w\) of \(F(\Lambda, \alpha_0)\) are given by

\[
\begin{align*}
\eta(\alpha_0) &= 2 - 24\alpha_0^2, \\
h(\Lambda) &= -(\theta_1\theta_2 + \theta_1\theta_3 + \theta_2\theta_3) - \alpha_0^2 = \frac{1}{2}(\Lambda, \Lambda + 2\alpha_0), \\
w(\Lambda) &= \sqrt{3} \theta_1\theta_2\theta_3,
\end{align*}
\]

(2.3)

where

\[
\theta_1 = (\Lambda + \alpha_0, \Lambda_1), \quad \theta_2 = (\Lambda + \alpha_0, \Lambda_2 - \Lambda_1), \quad \theta_3 = (\Lambda + \alpha_0, -\Lambda_2).
\]

(2.4)

Here, \(\Lambda_1\) and \(\Lambda_2\) are the fundamental weights of \(\mathfrak{sl}_3\), and \(\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha\) is the Weyl vector. Note that \(h(\Lambda)\) and \(w(\Lambda)\) as in (2.3) determine \(\Lambda\) only up to a Weyl rotation \(\Lambda \mapsto w(\Lambda + \alpha_0) - \alpha_0\), \(w \in \mathcal{W}\).

The following theorem summarizes some of the known results on the structure of Fock space modules \(F(\Lambda, \alpha_0)\):

**Theorem 2.1** [1,2].

(i) Let \(\iota'\) and \(\iota''\) be the canonical (\(\mathcal{W}\)-) homomorphisms

\[
M(h(\Lambda), w(\Lambda), c(\alpha_0)) \xrightarrow{\iota'} F(\Lambda, \alpha_0) \xrightarrow{\iota''} \overline{M}(h(\Lambda), w(\Lambda), c(\alpha_0)).
\]

(2.5)

Then \(\iota'\) (resp. \(\iota''\)) is an isomorphism if \(i(\Lambda + \alpha_0) \in \eta D_+\) (resp. \(-i(\Lambda + \alpha_0) \in \eta D_+\)) and \(\alpha_0^2 \leq -4\). Here \(D_+ = \{\lambda \in \mathfrak{h}^* \mid (\lambda, \alpha) \geq 0 \ \forall \alpha \in \Delta^+\}\) denotes the fundamental Weyl chamber and \(\eta \equiv \text{sign}(-i\alpha_0)\).
(ii) For $c = 2$, the Fock space $F(\lambda, 0)$ is completely reducible. Explicitly, for all $\lambda \in h^*_e$, we have

$$F(\lambda, 0) \cong \bigoplus_{\Lambda \in P_+} m(\Lambda) L(h(\Lambda), w(\Lambda), 2),$$

(2.6)

where $m(\Lambda)$ is equal to the multiplicity of the weight $\lambda$ in the irreducible finite dimensional representation $L(\Lambda)$ of $\mathfrak{sl}_3$ with highest weight $\Lambda$.

3. Fock space cohomology of the $W_3$ algebra

Despite the fact that $W$ is not a Lie algebra, the analog of semi-infinite (or BRST-) cohomology can still be defined [6,7]. As usual, one introduces two sets of ghost operators $(b^i_m, c^i_m)$, $i = 2, 3$ of conformal dimension $(j, -j + 1)$, corresponding to the generators $L_m$ and $W_m$, $m \in \mathbb{Z}$, respectively. These ghost operators satisfy anti-commutation relations $\{b^i_m, c^j_n\} = \delta_{m+n,0}\delta_{i,j}$. Let $F^{gh}$ denote the standard positive energy module. The ghost Fock space $F^{gh} = \bigoplus_{n \in \mathbb{Z}} F^{gh,n}$ is graded by ghost number, where $gh(c^i_m) = -gh(b^i_m) = 1$ and the highest weight state (physical vacuum) is chosen to have ghost number 3 (i.e. such that states and their corresponding operators have identical ghost numbers). For any two positive energy modules $V^M$ and $V^L$, such that $c^M + c^L = 100$, there exists a complex $(V^M \otimes V^L \otimes F^{gh,n}, d)$, graded by ghost number, and with a differential (BRST operator) $d$ of degree 1. For an explicit formula for $d$, which is rather involved, we refer to [7,1,2]. We will denote the cohomology of this complex by $H(W, V^M \otimes V^L)$. The cohomology relative to the Cartan subalgebra $W_0$ will be denoted by $H(W, W_0; V^M \otimes V^L)$.

For $V^L \cong F(\Lambda^L, \alpha^L_0)$ this cohomology is interpreted as the set of physical states in $W$-gravity coupled to some matter theory represented by $V^M$. One is interested mainly in two cases: where $V^M$ is either a so-called minimal model $L(h^M, w^M, c^M)$ or a free field Fock space $F(\Lambda^M, \alpha^M_0)$. The minimal model case was discussed in [1,3]. The analysis of $H(W, F(\Lambda^M, \alpha^M_0) \otimes F(\Lambda^L, \alpha^L_0))$ for generic $\alpha_+$ (i.e. $\alpha_+^2 \notin \mathbb{Q}$ where we have parametrized $\alpha^M_0 = \alpha_+ + \alpha_- = \alpha_+^2 = \alpha_+ + \alpha_- = -1$) was started in [7] and completed in [3]. Here we will complete the analysis, begun in [2], of a non-generic case, namely $\alpha_+ = \pm 1$ (i.e. $\alpha^M_0 = 0$, $-\alpha^L_0 = 2$ or $c^M = 2, c^L = 98$).

Because of Theorem 2.1 (ii) it suffices to compute the cohomology for the $c = 2$ irreducible $W$-modules $L(\Lambda) \equiv L(h(\Lambda), w(\Lambda), 2)$

**Conjecture 3.1** [4]. Let $\Lambda \in P_+$.

(i) The cohomology $H^n(W, W_0; L(\Lambda) \otimes F(\Lambda^L, 2i))$ is nontrivial only if there exist $w \in W$, $\sigma \in W \cup \{0\}$ such that

$$-i\Lambda^L + 2\rho = w^{-1}(\Lambda + \rho - \sigma\rho).$$

(3.1)

(ii) For $w, \sigma, \Lambda$ and $\Lambda^L$ as in (3.1), the cohomology $H^n(W, W_0; L(\Lambda) \otimes F(\Lambda^L, 2i))$ is 1-dimensional in the following cases

$$\begin{align*}
\sigma &\in W, & \Lambda &\in P_+, & w &\in W, & n = \ell(w^{-1}) - \ell(w^{-1}\sigma) + 3, \\
\sigma &\in W, & \Lambda &\in P_+, & w &\in W, & n = \ell(w^{-1}) + 1 \text{ or } n = \ell(-w^{-1}) + 2, \\
\sigma &\in W, & (\Lambda, \alpha) &\neq 0, & w &\in \langle r_1 \rangle \setminus W, & n = \ell(w^{-1}) + 2, \\
\sigma &\in W, & (\Lambda, \alpha) &\neq 0, & w &\in \langle r_1 \rangle \setminus W, & n = \ell(w^{-1}) + 1.
\end{align*}$$

and vanishes otherwise.

In the case that certain weights $(\Lambda, -i\Lambda^L)$ and certain ghost number $n$ satisfy (i) and (ii) for more than one choice of $(w, \sigma)$, the above should be understood in the sense that the corresponding cohomology is nevertheless 1-dimensional.
Let us comment on the status of this conjecture. For $-i\Lambda^L + 2\rho \in P_+$ we have an isomorphism

$$F(\Lambda^L, 2i) \cong \mathcal{M}(h(\Lambda^L), w(\Lambda^L), 2)$$ (see Theorem 2.1 (i)). By taking the (conjectured) resolutions of $L(\Lambda)$ in terms of generalized Verma modules $M(h, w, c = 2)_N$ [2] and using the known result for $H^n(W, W_0; M(h, w, c) \otimes \mathcal{M}(h', w', 100 - c))$, the conjecture follows (see [2] for details). [The resolution of $L(\Lambda)$ for $\Lambda \in P_{++}$ in [2] contains a minor misprint, see [4].]

For the other Weyl chambers, i.e. $w(-i\Lambda^L + 2\rho) \in P_+$, the conjecture is based on an analysis of the cohomology for generic $\alpha_+$ in the limit $\alpha_+ \to 1$ (i.e. $c^M \to 2$) and passes various nontrivial consistency checks. Among others, it is consistent with duality

$$H^{6-n}(W, W_0; L(\Lambda) \otimes F(\Lambda^L, 2i)) \cong H^n(W, W_0; L(\Lambda) \otimes F(\Lambda^L, 2i)),$$

where $F(\Lambda, \alpha_0) \cong F(w_0(\Lambda + \alpha_0\rho) - \alpha_0\rho)$ denotes the module contragradient to $F(\Lambda, \alpha_0)$.

Both the conjectured resolutions of $L(\Lambda)$ as well as the result for the cohomology (Conjecture 3.1) have also been verified by extensive computer calculations using Mathematica™.

Let $L$ be the lattice

$$L \equiv \{(\lambda, \mu) \in \mathfrak{b}_{\mathbb{R}}^2 \otimes \mathfrak{b}_{\mathbb{R}}^2 \mid \lambda - \mu \in \mathbb{Z} \cdot \Delta_+\}.$$

Note that, in particular,

$$(\lambda, \lambda') - (\mu, \mu') = (\lambda - \mu, \lambda') + (\mu, \lambda' - \mu') \in \mathbb{Z},$$

(3.4)

for all pairs $(\lambda, \mu)$ and $(\lambda', \mu')$ in $L$. We will restrict the momenta $(\Lambda^M, -i\Lambda^L)$ to the lattice $L$. As a consequence, all the vertex operators $V_{(\Lambda^M, -i\Lambda^L)}(z) = \exp(i\Lambda^M \cdot \phi^M + i\Lambda^L \cdot \phi^L)(z)$ will become mutually local because of (3.4) and, moreover, one can find a set of cocycles turning the underlying BRST-complex into a Vertex Operator Algebra (VOA). This will be essential for the construction of the BV-algebra in Section 3. In addition, the most interesting cohomology happens to be situated at $(\Lambda^M, -i\Lambda^L) \in L$.

Now consider the cohomologies

$$\mathcal{H} = \bigoplus_{(\Lambda^M, -i\Lambda^L) \in L} H(W, F(\Lambda^M, 0) \otimes F(\Lambda^L, 2i)),$$

$$\mathcal{H}_{\text{rel}} = \bigoplus_{(\Lambda^M, -i\Lambda^L) \in L} H(W, W_0; F(\Lambda^M, 0) \otimes F(\Lambda^L, 2i)).$$

(3.5)

We recall

**Theorem 3.2 [1,2].**

(i) $\mathcal{H}$ (and $\mathcal{H}_{\text{rel}}$) carries the structure of a $\mathfrak{g} \oplus \mathfrak{h}$ module ($\mathfrak{g} \cong \mathfrak{sl}_3$). The action of $\mathfrak{g}$ is through the zero modes of the Frenkel-Kac-Segal vertex operator construction (in matter fields only), while $\mathfrak{h}$ acts as $-i\Lambda^L$ (with eigenvalues $-i\Lambda^L$). This module is completely reducible under $\mathfrak{g} \oplus \mathfrak{h}$.

(ii) There exists a (non-canonical) isomorphism (as $\mathfrak{g} \oplus \mathfrak{h}$ modules)

$$\mathcal{H}^i \cong \mathcal{H}_{\text{rel}}^i \oplus \mathcal{H}_{\text{rel}}^{i-1} \oplus \mathcal{H}_{\text{rel}}^{i-1} \oplus \mathcal{H}_{\text{rel}}^{i-2}.$$

By combining the results of Theorems 2.1, 3.2 and Conjecture 3.1, we find

**Corollary 3.3.** The cohomology $\mathcal{H}_{\text{rel}}$ is isomorphic (as a $\mathfrak{g} \oplus \mathfrak{h}$ module) to the direct sum of irreducible modules $L(\Lambda) \otimes \mathbb{C}_{\Lambda'}$ with momenta $(\Lambda, \Lambda') \in \mathfrak{b}_{\mathbb{R}}^2 \otimes \mathfrak{b}_{\mathbb{R}}^2$ lying in a set of disjoint cones $\{S^\Lambda_w + (\lambda, w^{-1}\lambda) \mid \lambda \in P_+\}$, i.e.

$$\mathcal{H}_{\text{rel}}^n \cong \bigoplus_{w \in W} \bigoplus_{(\Lambda, \Lambda') \in S^\Lambda_w} \bigoplus_{\lambda \in P_+} (L(\Lambda + \lambda) \otimes \mathbb{C}_{\Lambda'+w^{-1}\lambda}),$$

where the sets $S^\Lambda_w$ (tips of the cones) are given in Table 1.
Table 1. The sets $S^n_w$

In particular we see that, as an $\mathfrak{sl}_3$ module, the ‘ground ring’ $\mathcal{H}^0$ decomposes as $\mathcal{H}^0 \cong \bigoplus_{\lambda \in \mathbb{P}^+} \mathcal{L}(\lambda)$ and is therefore a so-called ‘model space’ for $\mathfrak{sl}_3$. It is well-known that this model space can be realized as the space $\mathcal{P}^0(A)$ of polynomial functions on the so-called ‘base-affine space’ $A \equiv \mathbb{N}^+ \setminus G$ [8]. For $\mathfrak{sl}_3$ this model space is given by $\mathbb{C} [x^i, y_i]/(x^i y_i)$ ($i = 1, 2, 3$), i.e. polynomials in 6 variables $x^i, y_i$ transforming in the $\mathbf{3}$ and $\bar{\mathbf{3}}$ of $\mathfrak{sl}_3$ respectively, with a single relation $x^i y_i = 0$ [9]. In fact, one can show that $\mathcal{H}^0 \cong \mathcal{P}^0(A)$ as algebras [4]. One might think that, just as in the Virasoro case (corresponding to $\mathfrak{g} \cong \mathfrak{sl}_2$) [10–12], part of the rest of $\mathcal{H}$ allow an interpretation in terms of polyvector fields on this base affine space. This turns out to be true and will be elaborated on in the next section.

| $n$ | $w$ | $S^n_w$ |
|-----|-----|---------|
| 0   | 1   | (0, 0)  |
| 1   | 1   | (A_1, -A_1 + A_2), (A_1 + A_2, 0), (A_2, A_1 - A_2) |
|     | $r_1$ | (0, -2A_1 + A_2) |
|     | $r_2$ | (0, A_1 - 2A_2) |
| 2   | 1   | (2A_1, -A_1), (0, -A_1 - A_2), (2A_2, -A_2) |
|     | $r_1$ | (A_1, -2A_1), (A_2, -3A_1 + A_2), (0, -4A_1 + 2A_2) |
|     | $r_2$ | (A_2, -2A_2), (A_1, A_1 - 3A_2), (0, 2A_1 - 4A_2) |
|     | $r_{1r_2}$ | (0, -3A_1) |
|     | $r_{1r_2}$ | (0, -3A_2) |
| 3   | 1   | (A_1 + A_2, -A_1 - A_2) |
|     | $r_1$ | (A_2, -2A_1 - A_2), (A_1, -4A_1 + A_2), (A_2, -5A_1 + 2A_2) |
|     | $r_2$ | (A_1, -A_1 - 2A_2), (A_2, A_1 - 4A_2), (A_1, 2A_1 - 5A_2) |
|     | $r_{2r_1}$ | (A_1, -3A_1 - A_2), (0, -5A_1 + A_2), (A_1, -5A_1) |
|     | $r_{1r_2}$ | (A_2, -A_1 - 3A_2), (0, A_1 - 5A_2), (A_2, -5A_2) |
|     | $r_{1r_2r_1}$ | (0, -2A_1 - 2A_2) |
| 4   | $r_1$ | (0, -4A_1 - A_2) |
|     | $r_2$ | (0, -A_1 - 4A_2) |
|     | $r_{2r_1}$ | (A_1, -4A_1 - 2A_2), (A_2, -5A_1 - A_2), (0, -6A_1) |
|     | $r_{1r_2}$ | (A_2, -2A_1 - 4A_2), (A_1, -A_1 - 5A_2), (0, -6A_2) |
|     | $r_{1r_2r_1}$ | (0, -3A_1 - 3A_2), (2A_1, -4A_1 - 3A_2), (2A_2, -3A_1 - 4A_2) |
| 5   | $r_{2r_1}$ | (0, -5A_1 - 2A_2) |
|     | $r_{1r_2}$ | (0, -2A_1 - 5A_2) |
|     | $r_{1r_2r_1}$ | (A_1, -5A_1 - 3A_2), (A_1 + A_2, -4A_1 - 4A_2), (A_2, -3A_1 - 5A_2) |
| 6   | $r_{1r_2r_1}$ | (0, -4A_1 - 4A_2) |
4. The BV-structure of $\mathcal{H}$

To explain the algebraic structure of the cohomology $\mathcal{H}$ of Section 2 we will first need to recall the definition of a Gerstenhaber algebra (or G-algebra, for short) [13] and a BV-algebra (or coboundary G-algebra) [14–16,12] as well as some basic facts.

**Definition 4.1.** A G-algebra $(\mathcal{A}, \cdot, [\cdot, \cdot])$ is a Z-graded, supercommutative, associative algebra $\mathcal{A} = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}^i$ (under $\cdot$) as well as a Z-graded Lie superalgebra (under $[\cdot, \cdot]$), such that the (odd) bracket acts as a superderivation of the algebra, i.e.

$$[x, y \cdot z] = [x, y] \cdot z + (-1)^{(|x|-1)(|y|+1)} y \cdot [x, z], \quad x, y, z \in \mathcal{A}. \quad (4.1)$$

For any commutative algebra $\mathcal{A}$ and $\mathcal{A}$-module $\mathcal{M}$, one defines the set $D(\mathcal{A}, \mathcal{M})$ of derivations of $\mathcal{A}$ with coefficients in $\mathcal{M}$ as the set of elements $D \in \text{Hom}(\mathcal{A}, \mathcal{M})$ that satisfy the Leibniz rule

$$D(x \cdot y) = y(Dx) + x(Dy). \quad (4.2)$$

The set $D^n(\mathcal{A})$ of polyderivations of order $n$ is defined by induction as $D^n = \text{Hom}(\mathcal{A}, D^{n-1}(\mathcal{A}))$ satisfying the Leibniz rule (4.2) and being completely antisymmetric when considered as elements of $\text{Hom}(\mathcal{A}^n, \mathcal{A})$. We recall

**Theorem 4.2** [17]. Let $\mathcal{A}$ be a commutative algebra. The set of polyderivations $D(\mathcal{A})$ carries the structure of a G-algebra, with the bracket given by the Schouten bracket.

Another example of a G-algebra is the Hochschild cohomology $H(\mathcal{A}, \mathcal{A})$ of an associative algebra $\mathcal{A}$ [13].

**Definition 4.3.** A BV-algebra $(\mathcal{A}, \cdot, \Delta)$ is a Z-graded, supercommutative, associative algebra $\mathcal{A}$ with a second order derivation $\Delta$ (BV-operator) of degree $-1$ satisfying $\Delta^2 = 0$.

**Lemma 4.4** [18,12,16]. For any BV-algebra $(\mathcal{A}, \cdot, \Delta)$ we may define an odd bracket by

$$[x, y] = (-1)^{|x||y|} \left( \Delta(x \cdot y) - (\Delta x) \cdot y - (-1)^{|x|} x \cdot (\Delta y) \right), \quad x, y \in \mathcal{A}. \quad (4.3)$$

This will equip $\mathcal{A}$ with the structure of a G-algebra. Moreover, the BV-operator acts as a superderivation of the bracket

$$\Delta[x, y] = [\Delta x, y] + (-1)^{|x|-1}[x, \Delta y]. \quad (4.4)$$

In general, given a commutative algebra $\mathcal{A}$, the G-algebra $D(\mathcal{A})$ of polyderivations of $\mathcal{A}$ will not carry the structure of a BV-algebra. However, if $\mathcal{A}$ is the algebra of (smooth or polynomial) functions on some smooth manifold $M$, then $D(\mathcal{A})$ is isomorphic to the set of polyvector fields $\mathcal{P}(M)$ on $M$ [17]. If, moreover, $M$ possesses a volume form, then we can in fact equip $D(\mathcal{A}) (= \mathcal{P}(M))$ with the structure of a BV-algebra [18,12]. Another example of a BV-algebra is the Grassmann algebra $\wedge^* \mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ [12].

Given a BV-algebra $(\mathcal{A}, \cdot, \Delta)$, let $\mathcal{A}^0$ be its ‘ground ring.’ It follows from equations (4.1), (4.3) and (4.4) that there exists a natural way to embed $\mathcal{A}$ into the G-algebra of polyderivations of $\mathcal{A}^0$, i.e. $D(\mathcal{A}^0)$, namely

**Theorem 4.5.** Let $(\mathcal{A}, \cdot, \Delta)$ be a BV-algebra. Suppose $\mathcal{A}^n = 0$ for all $n < 0$.

(i) There exists a homomorphism of G-algebras $\pi : \mathcal{A} \to D(\mathcal{A}^0)$ defined by

$$\pi(y)(x_1, x_2, \ldots, x_n) = \left[ \ldots \left[ y, x_1 \right], x_2, \ldots, x_n \right], \quad y \in \mathcal{A}^n, x_1, x_2, \ldots, x_n \in \mathcal{A}^0. \quad (4.5)$$

(ii) Suppose that the G-algebra $D(\mathcal{A}^0)$ admits a BV-structure $(D(\mathcal{A}^0), \cdot, \Delta')$ and that $\pi \Delta(x) = \Delta' \pi(x)$ for all $x \in \mathcal{A}^1$, then $\pi$ is a BV-homomorphism and $\mathcal{I} \equiv \text{Ker} \pi$ is a BV-ideal of $\mathcal{A}$.

We are now ready to state the main result of this paper
THEOREM 4.6. Let $\mathcal{H}$ be the cohomology defined in (3.5). Then
(i) $\mathcal{H}$ can be equipped with the structure of a BV-algebra.
(ii) There exists an ideal $\mathcal{I} \subset \mathcal{H}$ such that we have an exact sequence of BV-algebras
\begin{equation}
0 \rightarrow \mathcal{I} \rightarrow \mathcal{H} \xrightarrow{\pi} D(\mathcal{H}^0) \rightarrow 0,
\end{equation}
where $D(\mathcal{H}^0)$ is isomorphic to the BV-algebra $\mathcal{P}(\mathcal{A})$ of polyvector fields on the base affine space $\mathcal{A} = N_+ \backslash G$.

Let us make some comments on the proof. Quite generally, as has been shown in [10–12,16], BRST cohomologies of VOA’s carry the structure of a BV-algebra. The product in this BV-algebra is given by the normal ordered product of the VOA while $\Delta = \delta_0^{[2]}$. The crucial part of the proof of (i) is therefore to show that the complex carries the structure of a VOA. This amounts to showing that one can find an appropriate set of cocycles for the lattice $L$. This is a straightforward exercise. [One might wonder whether there exists additional structure in $\mathcal{H}$ beyond that of a BV-algebra, in particular whether $\delta_0^{[3]}$ gives rise to a second BV-operator. It turns out however that, due to the non-diagonalizability of $W_0$, $\delta_0^{[3]}$ does not act on $\mathcal{H}_0$.] As we have seen in Section 2, there exists a canonical isomorphism of algebras $\mathcal{H}^0 \cong \mathcal{P}(\mathcal{A})$, where $\mathcal{P}(\mathcal{A})$ denotes the (commutative) algebra of polynomials on $\mathcal{A}$. This implies $D(\mathcal{H}^0) \cong \mathcal{P}(\mathcal{A})$ as algebras. That $\pi$ is in fact a BV-epimorphism follows from Theorem 4.5 by explicitly checking that $\pi$ intertwines the BV-operators on $\mathcal{H}^1$ and $\mathcal{P}^1(\mathcal{A})$ and that it acts onto.

We would like to remark here that, contrary to the Virasoro case [12], both the dot product and the bracket in $\mathcal{I}$ are not identically zero. Also, the exact sequence (4.6) splits both as an exact sequence of $\mathcal{H}^0$ and $\mathfrak{g} \oplus \mathfrak{h}$ modules, but not as an exact sequence of BV-algebras.

Details of this paper as well as a more detailed analysis of the BV-algebra structure of the entire $\mathcal{H}$ will appear elsewhere [4].

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