What did Ryser Conjecture?

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Abstract

Two prominent conjectures by Herbert J. Ryser have been falsely attributed to a somewhat obscure conference proceedings that he wrote in German. Here we provide a translation of that paper and try to correct the historical record at least as far as what was conjectured in it.

The two conjectures relate to transversals in Latin squares of odd order and to the relationship between the covering number and the matching number of multipartite hypergraphs.

Introduction

The aim of this document is to make readily available the contents of the following conference proceedings:
H. J. Ryser, Neuere Probleme in der Kombinatorik, Vorträge über Kombinatorik Oberwolfach, 24–29 July (1967), 69–91.

hereafter referred to as [Rys67]. The reason for doing this is that [Rys67] has been attributed as the source of several important conjectures. However, it seems that these attributions are not correct. So this is an attempt to straighten out the historical record.

We should begin with the important caveat that neither of us are fluent in German, the language of the original document. However we did consult native speakers Kevin Leckey and Anita Leibenau on key passages, and rely heavily on google translate elsewhere. We could not have managed without those people/resources, but they should not be blamed for any errors in what is presented below.

The conjectures of present interest are:

Conjecture 1. The number of transversals in a Latin square of order $n$ is congruent to $n \mod 2$.

Conjecture 2. Every Latin square of odd order has at least one transversal.

Conjecture 3. In an $r$-partite hypergraph, $\tau \leq (r−1)\nu$ where $\tau$ is the covering number and $\nu$ is the matching number.

All three have at times been attributed to [Rys67], but only Conjecture 2 is present in that document. The even half of Conjecture 1 was proved by Balasubramanian [3], who attributed the conjecture to [Rys67]. The odd half is false, and there are many counterexamples for orders 7 and above, as has been noted in numerous places including in [4, 7, 8]. The suggestion that Ryser made Conjecture 1 has been repeated many times, including in [2, 7, 8, 9, 10]. It is possible that Ryser did make this conjecture verbally, but we have been unable to find any evidence that he made it in print. The one thing we are certain of is that it does not appear in [Rys67]. If anyone can shed any light on where Balasubramanian obtained the conjecture, or point to an earlier reference to it, we would be very interested to hear from them.

Conjecture 1 does of course imply Conjecture 2, and Conjecture 2 is present in [Rys67], as we will see. In papers on transversals of Latin squares, Conjecture 2 is often referred to as Ryser’s conjecture.

We now turn to Conjecture 3 which is also often referred to as Ryser’s conjecture, and has seen a flurry of activity recently. Conjecture 3 was attributed to [Rys67] in [1], due to a misunderstanding between the authors. This mistake was then copied in [5]. Again we are unaware of any evidence of Ryser making Conjecture 3 in print, although it does appear in an equivalent form in the thesis of his student J. R. Henderson [6, p.26]. It does not appear in [Rys67].
The following pages give a translation of [Rys67]. We took the liberty of correcting a couple of clear and simple typos, but have otherwise tried to stay true to the original. Some commentary has been added in footnotes in blue.

References

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1 Latin Squares

Let $n$ be a natural number, and $S$ be a set of $n$ distinct elements, such as the digits $1, 2, \ldots, n$. A Latin rectangle of order $r \times s$ ($r, s$ natural numbers) is an $r \times s$ matrix $[a_{ij}]_{r \times s}$ of elements of $S$ which has no repetition in any row or column:

$$a_{i,j} = a_{i,k} \implies j = k \quad (i = 1, \ldots, r),$$

$$a_{i,j} = a_{k,j} \implies i = k \quad (j = 1, \ldots, s).$$

Thus, in every Latin rectangle $r \leq n$ and $s \leq n$. Since this definition does not depend on the order of the digits, nor on the distinction between columns and rows, one can restrict oneself to the consideration of Latin rectangles where $s = n$ and the first row is $1, 2, \ldots, n$ in natural order:

$$a_{1,j} = j \quad (j = 1, \ldots, s).$$

These rectangles are called normalized.

If $r = s = n$, then we call it a Latin square of order $n$. The square is called normalized if both the first row and the first column contain the numbers $1, 2, \ldots, n$ in natural order.

$$a_{1,1} = 1 \quad (i = 1, \ldots, n).$$

Some examples of small normalized Latin squares are

$$\begin{pmatrix}
1 & 2 \\
2 & 1
\end{pmatrix}, \begin{pmatrix}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2
\end{pmatrix}.$$

This definition is now followed by the combinatorial question about the number of different objects defined. Obviously, you get only one normalized $1 \times n$ Latin rectangle. The number of different $2 \times n$ Latin rectangles is equal to $D_n$, the number of permutations without any fixed points (“derangements”); this is easy to calculate:

$$D_n = n! \cdot \left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}\right) \approx \frac{n!}{e}.$$

However, an explicit expression for the number of $3 \times n$ Latin rectangles is quite complicated, and for $4 \times n$ Latin rectangles, the solution to the problem is unknown.

The question of the number of normalized Latin squares of order $n$, $\ell(n)$, is interesting. For a given number $n$, however, no explicit expression for $\ell(n)$ is known: Up to $n = 8$, $\ell(n)$ has been calculated ($n = 7$ by A. Sade, and $n = 8$ by Mark Wells with the computer MANIAC in 8 hours of computing time [appears in: J. of Combinatorial Theory]).

$$\begin{array}{cccccccc}
n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\ell(n) & 1 & 1 & 1 & 1 & 56 & 9408 & 16942080 & 535281401856
\end{array}$$

It is noteworthy that in these small known values for $\ell(n)$ for $n \geq 4$, quite high powers of 2 occur.

Note that the $\ell(n)$ appear as coefficients of a certain power series related to MacMahon’s “Master Theorem”, which, however, is not suitable for the calculation of $\ell(n)$.

Otherwise, quite little is known about Latin squares. Here are some known results:

1. For this statement to be true the intention must have been for the first row to be in order, but the second row to be unrestricted.
2. To date these numbers have been computed up to $n = 11$. See B. D. McKay and I. M. Wanless, On the number of Latin squares, *Ann. Comb.* 9 (2005), 335–344.
3. D. S. Stones, The many formulae for the number of Latin rectangles, Electron. J. Combin. 17 (2010), #A1.
4. This same observation was independently made by Alter. It was subsequently proved that increasing powers of any prime (not just 2) divide $\ell(n)$, although we are currently unable to explain the apparent rapid growth in the power of 2. For details, see: R. Alter, How many Latin squares are there? *Amer. Math. Monthly*, 82 (1975) 632–634.
5. D. S. Stones and I. M. Wanless, Divisors of the number of Latin rectangles, *J. Combin. Theory Ser. A* 117 (2010), 204–215.
Theorem 1.1 (M. Hall). Every \( r \times n \) Latin rectangle \( r \leq n \) can be completed to a Latin square of order \( n \).

Proof. We assume that the given rectangle as normalized. Let \( S_i \) be the subset of \( S \) that consists of the digits not occurring in the \( i^{th} \) column \( (i = 1, 2, \ldots, n) \). From \( S_1, S_2, \ldots, S_n \), we define \( A \) to be the incidence matrix defined by:

\[
A = [\alpha_{ik}]_{n \times n} \quad \text{where} \quad \alpha_{ik} = \begin{cases} 
1 & \text{if} \ k \in S_i \\
0 & \text{otherwise}.
\end{cases}
\]

Because \( |S_i| = n - r \), each row of \( A \) contains exactly \( n - r \) ones. By the definition of Latin rectangles, any given number occurs in exactly \( r \) of the sets \( S \setminus S_i \) \( (i = 1, \ldots, n) \). Thus, it occurs in exactly \( n - r \) of sets \( S_i \). Therefore, every column of \( A \) contains exactly \( n - r \) ones.

However, according to the Birkhoff-König Theorem (later quoted), \( A \) is the sum of \( n - r \) permutation matrices \( P_j \):

\[
A = P_1 + \cdots + P_{n-r}.
\]

If the \((r + i)^{th}\) row is now set equal to the permutation of the digits \( (1, \ldots, n) \) given by \( P_i \), i.e.,

\[
[\alpha_{r+i,j}] = P_i \begin{bmatrix} 
1 \\
\vdots \\
1 \end{bmatrix}
\]

in the sense of matrix multiplication, one obtains a Latin square. Obviously, there are no repetitions in the rows. Moreover, there are no repetitions in the columns, since the newly added digits of the column \( i \) are exactly the elements of \( S_i \).

The following slightly generalized result also holds (here without a proof).

Theorem 1.2. An \( r \times s \) Latin rectangle can be completed to a Latin square of order \( n \) if and only if

\[
N(i) \geq r + s - n \quad \text{for all} \quad i = 1, \ldots, r,
\]

where \( N(i) \) is the number of times the symbol \( i \) appears in the given \( r \times s \) Latin rectangle.

The above result follows from Theorem 1.1 for \( s = n \).

The problem addressed can be made more general in the following way: Let \( S \) be the set consisting of the numbers \( 1, \ldots, n \) and the symbol \( x \) (“empty cell”). Given an \( n \times n \) matrix of elements of \( S \), what are the necessary and sufficient conditions for this matrix to be completed to a Latin square by substitution of each \( x \) by digits (not necessarily the same digits)? Above, we have dealt with a special case:

\[
\begin{bmatrix}
<\text{Digits}> \\
x \ x \ \cdots \ x \\
\vdots \ \vdots \\
x \ x \ \cdots \ x
\end{bmatrix}
\]

Hereinafter, a line of a matrix refers to either one of its rows or columns.

Given a square matrix \([a_{ik}]\) of order \( n \), a set \( P = \{(i, k)\} \) of size \( n \) is a path if no two of its elements share the same row or column:

\[
(i, k) \in P \implies (i, l) \notin P \text{ if } l \neq k \\
\text{and} \quad (j, k) \notin P \text{ if } i \neq j.
\]

A path \( P \) is called a transversal if all digits from 1 to \( n \) occur among \([a_{ik}]\), for \((i, k) \in P\). For example, the main diagonal of each square matrix is a path, as is the back-diagonal. The main diagonal of the Latin square

\[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2
\end{bmatrix}
\]

is a transversal path, but its back-diagonal is not. The Latin square

\[
\begin{bmatrix}
1 & 2 \\
2 & 1
\end{bmatrix}
\]
does not have any transversal paths. The first example seems to be a special case of a more general fact. We formulate the following conjecture:

A Latin square of odd order always has a transversal path. This assumption was verified in the case of \( n = 5 \) (H. J. Ryser). For general \( n \), it can easily be proved under the extra assumption that it is a symmetric Latin square.

In this case, the main diagonal is always a transversal path. It suffices to show that it contains a 1, but this follows from the fact that 1 occurs exactly \( n \) times in the square, and because of the symmetry, \( \frac{n-1}{2} \) times below the principal diagonal, where \( k \) is the number of the 1s on the main diagonal; \( k \) cannot be zero because \( n \) is odd.

2 Orthogonal Systems of Latin Squares

Let \( A_1, A_2 \) be Latin squares of order \( n \geq 3 \). Form the “superimposition square” of the ordered pairs \( \left( a_{ij}^{(1)}, a_{ij}^{(2)} \right) \) (Euler’s “Graeco-Latin” squares). If the \( n^2 \) pairs in this square are all different, then \( A_1 \) and \( A_2 \) are called orthogonal.

For example,

\[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2 \\
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
1 & 2 & 3 \\
3 & 1 & 2 \\
2 & 3 & 1 \\
\end{bmatrix}
\]

are orthogonal Latin squares. To find the second orthogonal square, the three transversal paths of the first are occupied by the same numbers. A set of Latin squares is called an orthogonal system of Latin squares when its elements are pairwise orthogonal.

The following shows that orthogonal systems cannot be too large.

**Theorem 2.1.** For any orthogonal system \( A_1, \ldots, A_t \) \( (A = [a_{ij}^{(\tau)}], \tau = 1, \ldots, t) \) of \( t \) Latin squares of order \( n \) \( (t \geq 2, n \geq 3) \),

\[ t \leq n - 1. \]

**Proof.** Obviously, permuting the elements of the matrix does not change the orthogonality of squares, so we can assume that \( A_1, \ldots, A_t \) have the same first row consisting of the numbers 1, \ldots, \( n \) in the natural order. The \( a_{21}^{(\tau)}(\tau = 1, \ldots, t) \) must now all be different, and they cannot be 1. From this, the theorem follows. \( \square \)

If \( t = n - 1 \), then we call it a complete system of orthogonal Latin squares

**Theorem 2.2.** For \( n = p^a \geq 3 \) \( (\text{for } p \text{ prime}) \) there exists a complete system of orthogonal Latin squares of order \( n \) (proved by Veblen and Bassey).

**Proof.** Let \( S = GF(p^n) = \{a_0 = 0, a_1 = 1, a_2, \ldots, a_{n-1}\} \). A complete system of orthogonal squares is defined by the fact that \( A_e = [a_{ij}^{(e)}] \) with \( a_{ij}^{(e)} = a_ia_j \). First, we verify that these are Latin squares:

- If elements are in the same row, then
  \[ a_ea_i + a_j = a_ea_i + a_{j'} \]
  from which \( a_j = a_{j'} \) and thus \( j = j' \);

- If elements are in the same column, then
  \[ a_ea_i + a_j = a_ia_j + a_j \]
  and thus, \( a_i = a_{i'} \) and \( i = i' \).

**Proof of Orthogonality**

Assume that

\[
\left( a_{ij}^{(e)}, a_{ij}^{(f)} \right) = \left( a_{i'j'}, a_{i'j'}^{(f)} \right)
\]

\( 4 \)This is Conjecture \( 2 \)
with \( e \neq f \) and \((i, j) \neq (i', j')\). From this would follow that
\[
a_e a_i + a_j = a_e a_i' + a_{j'}
\]
and
\[
a_f a_i + a_j = a_f a_i' + a_{j'}.
\]
Subtraction of these equations give
\[
a_i (a_e - a_f) = (a_e - a_f) a_i',
\]
so \( a_i = a_{i'} \), and consequently also \( a_j = a_{j'} \), which contradicts the assumption. \(\square\)

A difficult question is the uniqueness of complete orthogonal systems. Uniqueness could be demonstrated up to order 8; for \( n = 9 \), it is unknown.\(^5\)

On the other hand, we know more about the existence of orthogonal (i.e., not complete) systems:

**Theorem 2.3** (MacNeish). Let
\[
n = p_1^{\alpha_1} \cdots p_N^{\alpha_N} \,(p_i \text{ prime}), \quad t = \min_{i=1,\ldots,N} (p_i^{\alpha_i} - 1) \geq 2.
\]
Then there are \( t \) pairwise orthogonal Latin squares of order \( n \).

To prove of this theorem, we need the following two lemmas:

**Lemma 2.1.** A system of \( t \) orthogonal Latin squares \( A_1, \ldots, A_t \) corresponds one-to-one to the following schema

\[
\begin{array}{cccccc}
1 & 1 & 1 & \ldots & \ldots & \ldots \\
1 & 2 & 2 & \ldots & \ldots & \ldots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
1 & n & n & \ldots & \ldots & \ldots \\
2 & 1 & 2 & \ldots & \ldots & \ldots \\
2 & 2 & \ldots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
n & 1 & 2 & \ldots & \ldots & \ldots \\
n & n & n & \ldots & \ldots & \ldots \\
\end{array}
\]

which has the property that every \( n^2 \times 2 \) submatrix contains all \( n^2 \) pairs formed from the numbers \( 1, \ldots, n \).

**Proof.** The \( A_\tau (\tau = 1, \ldots, t) \), as indicated in the scheme, form a system of orthogonal Latin squares: Latin because from the form of the 1st and 2nd column of the scheme, it follows that there are no two identical digits in a row or column of \( A_\tau \); orthogonal because the elements of \( A_\tau \) and \( A_{\tau'} \) form an \( n^2 \times 2 \) matrix in the schema, and therefore all \((A_\tau, A_{\tau'})\) pairs are different. This superposition, combined in the manner indicated with the \( n^2 \times (t + 2) \) scheme, has the property in question. \(\square\)

**Lemma 2.2.** If there are \( t \) orthogonal Latin squares of order \( n \) and \( t \) of order \( n' \), then there are also \( t \) orthogonal Latin squares of order \( n \cdot n' \).

\(^5\text{It has since been proved that there are exactly 4 projective planes of order 9, and that these correspond to 19 complete sets of MOLS of order 9. See P. J. Owens and D. A. Preece. Aspects of complete sets of }9 \times 9\text{ pairwise orthogonal Latin squares. Discrete Math. 167/168 (1997), 519–525.}\)
Proof. According to Lemma 2.1, we only need to consider the corresponding \( n^2 \times (t+2) \) and \( n'^2 \times (t+2) \) schema \( B = (b_{ij}) \) and \( B' = (b'_{ij}) \). From these, we form an \( (n \cdot n')^2 \times (t+2) \) scheme \( B' \) by setting

\[
\begin{align*}
b'_{(i,j),j} = (b_{ij}, b'_{ij}) \quad & \text{where} \\
(i, i') = (1,1), \ldots, (n, n'), \\
j = 1, \ldots, t+2,
\end{align*}
\]

according to which the elements of this new scheme are now the pairs formed by the numbers 1, \ldots, \( n \) and 1, \ldots, \( n' \).

The new scheme also has the property that every \( (n \cdot n')^2 \times 2 \) submatrix contains all \( (n \cdot n')^2 \) pairs from \((1,1), \ldots, (n,n')\). So by Lemma 2.1, the result follows. \( \square \)

The theorem follows by the application of the preceding lemma and multiple applications of Lemma 2.2.

Euler even suggested that in this case \( n \equiv 2 \mod 4 \), no pair of orthogonal Latin squares exist. This assumption is trivially correct for \( n = 2 \) and has also been verified for the next case of \( n = 6 \) (Tarry, 1900). All the more surprising is the result of Bose, Shrikhande and Parker (1960), who says that these cases are the only ones in which Euler’s conjecture is true:

**Theorem 2.4.** There are pairs of orthogonal Latin squares of each order \( n \neq 2 \) and \( n \neq 6 \).

We do not give the proof here again because of its complexity, but a refutation of Euler’s assumption is already contained in the more elegant proof for the case \( n \equiv 10 \mod 12 \) [see [1]].

## 3 Projective Planes

Let \( n \geq 3 \). From the standpoint of combinatorics, one can introduce projective planes as follows:

**Definition 1.** A projective plane of order \( n \) is given by a complete system of orthogonal Latin squares of order \( n \).

Equivalent to this is a square zero-one matrix \( A \) of order \( m = n^2 + n + 1 \) with \( AA^T = nI + J \), where

\[
I = (\delta_{ij})_{m \times m} \quad \text{and} \quad J = (1)_{m \times m}.
\]

This is because they give a complete system of orthogonal Latin squares of order \( n \); According to Lemma 2.1, we can assign a \( n^2 \times (n+1) \) scheme to these rows. The rows of this scheme are \( Z_1, \ldots, Z_{n^2} \). Moreover, let \( Z_{n^2+1}, \ldots, Z_{n^2+n+1} \) be additional (pairwise distinct) symbols, and denote by \( G_{ij} \), \((i = 1, \ldots, n; j = 1, \ldots, n+1)\) the set of all \( Z_k \) with \( i \) in the \( j \)th column, combined with \( Z_{n^2+j} \). By assumption, all \( G_{ij} \) are different.

Finally, let \( G \) denote the set of symbols \( Z_{n^2+1}, \ldots, Z_{n^2+n+1} \).

Let \( A \) be the \((n^2 + n + 1) \times (n^2 + n + 1)\) incidence matrix of the \( Z_k \) with respect to the \( G \)'s. Then it is clear that every \( G \) contains exactly \( n + 1 \) \( Z \)'s, and every \( Z \) is contained in precisely \( n + 1 \) \( G \)'s, which follows the matrix equation for \( A \). Geometrically speaking, the \( Z \)'s are the points and the \( G \)'s lines.

Conversely, assume \( A \) is given with the above-mentioned properties. We interpret \( A \) as the incidence matrix of \( n^2 + n + 1 \) “points” \( Z_j \) and the same number of “lines” \( G_k \). For example, for a fixed line, \( G_1 \), the points \( P_1, \ldots, P_{n+1} \) are on it, \( Q_1, \ldots, Q_{n^2} \) being the points which are not on it.

As follows from the matrix equation, exactly \( n + 1 \) lines extend through each point. The \( n \) lines through \( F_j \), different from \( G_1 \), may be assigned the numbers 1, \ldots, \( n \), for every \( j \). Thus, all lines through \( G_1 \) have a number \( \in \{1, \ldots, n\} \).

Let \( C = [C_{ij}] = [\text{number of } Q_{1}P_{j}] \). Exactly one line goes through each pair of points, as follows directly from the matrix equation. It is now easy to verify that \( C \) is a schema of the desired shape from Lemma 2.1. From it, \( n - 1 \) pairs of orthogonal Latin squares can be deduced.

The combinatorial problem that is associated with this is the determination of for which \( n \) projective planes of order \( n \) exist. For \( n = p^m \) with \( p \) prime, the existence of a projective plane of this order follows from Theorem 2.2. The smallest \( n \) for which this problem is not known is \( n = 10 \). Here, even three
pairs of orthogonal squares could not be found (in spite of Parker’s extensive computer calculations). Parker obtained by means of unsystematic searching 84 Latin squares of order 10. Of these, 33 had no orthogonal ‘partners’, 29 had exactly one, 15 had two, 2 had three, 3 had four and 2 had five orthogonal partners. It appears that about 50% of all randomly generated squares have orthogonal partners.

We are still giving an idea that is useful in this context. In the Latin square

\[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2 \\
\end{bmatrix},
\]

one finds three transversal paths, which are symbolically added by the three matrices

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}, \quad \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0 \\
\end{bmatrix}, \quad \begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
\end{bmatrix}
\]

which add up to \( J = (1)^{3\times3} \): the transversal paths are pairwise disjoint. If the first transversal path is occupied by 1, the second by 2, etc., one obtains an orthogonal partner to the given square.

It holds true that any Latin square of order \( n \) has an orthogonal partner if it has \( n \) pairwise disjoint transversal paths.

**Example:**

\[
\begin{bmatrix}
5 & 1 & 7 & 3 & 4 & 0 & 6 & 2 & 8 & 9 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 0 \\
7 & 3 & 4 & 5 & 6 & 2 & 8 & 9 & 0 & 1 \\
3 & 4 & 5 & 6 & 7 & 8 & 9 & 0 & 1 & 2 \\
4 & 5 & 6 & 7 & 8 & 9 & 0 & 1 & 2 & 3 \\
0 & 6 & 2 & 8 & 9 & 3 & 1 & 7 & 3 & 4 \\
6 & 7 & 8 & 9 & 0 & 1 & 2 & 3 & 4 & 5 \\
2 & 8 & 9 & 0 & 1 & 7 & 3 & 4 & 5 & 6 \\
8 & 9 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
9 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{bmatrix}
\]

is a Latin square of order 10 (found by Parker). If you swap all 0 and 5 which are highlighted, as well as all highlighted 2 and 7, a cyclic Latin square (with even order) is obtained, which, as is easily verified, cannot possess a transversal path, and consequently has no orthogonal partner. On the other hand, the given square has 5504 transversal paths and approximately \( 10^6 \) orthogonal partners.

The above representation of Latin squares mean that \( n \) zero-one matrices which indicate the position of the transversal paths can be further refined. To form a given Latin square \([a_{ij}]\) form the \( n \) matrices

\[
[d_{ijk}] \text{ with } d_{ijk} = \begin{cases} 
1 & \text{for } a_{ij} = k \\
0 & \text{otherwise.}
\end{cases}
\]

These form a so-called Latin cube, a three-dimensional zero-one matrix, which has exactly one 1 in every (axis-parallel) line. This output square has an orthogonal partner exactly if the cube can be decomposed into other cubes with a 1 in each axis-parallel plane. (See articles by Ryser and Jurkat, published in the J. of Algebra).

### 4 Two combinatorial results for zero-one matrices

On the basis of the following two problems, two characteristic conclusions of combinatorics are to be presented.

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7 This comment applies to order 10 only. The proportion was estimated at a touch over 60%, in B. D. McKay, A. Meynert and W. Myrvold, Small latin squares, quasigroups and loops, *J. Combin. Des.* 15 (2007), 98–119.

8 Ryser is quoting Parker’s estimate here, but the true figure is 12 265 168. See B. M. Maenhaut and I. M. Wanless, Atomic Latin squares of order eleven, *J. Combin. Designs*, 12 (2004), 12–34.

9 In particular, see “Extremal configurations and decomposition theorems I”, *J. Algebra* 8 (1968) 194–222.
Problem 1. Let $A = (a_{ij})$ be a zero-one matrix of order $m \times n$ such that every element of $A^T A$ is positive and $A$ has no sub-matrix of the form

$$
\begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{bmatrix}
$$

after permuting rows and columns. Show that $A$ has a row of all ones.

Problem 2. Let $A$ be a symmetric zero-one matrix of order $v$ such that

$$
AA^T = (k - \lambda)I + \lambda J,
$$

where $I = (\delta_{ij})_{v \times v}$, $J = (1)_{v \times v}$ and $0 < \lambda < k < v - 1$.

Assuming that $k - \lambda$ is not a square, show that $A$ has exactly $k$ ones on the main diagonal. (This problem occurs with $(b, k, \lambda)$ configurations, see [1]).

Proof of Problem 1: We introduce the proof by induction over the number of columns. For $n = 1$ and $n = 2$, the assertion is true. So let $n \geq 3$.

Let $B$ denote the matrix which is formed by removing the first column from $A$. Obviously, $B^T B$ is a submatrix of $A^T A$ and thus, has only positive elements. This implies, according to the induction requirement, that $B$ has a row of ones; For example, the first line of $B$.

If $a_{11} = 1$, the theorem is proved. Therefore, we consider only the non-trivial case $a_{11} = 0$. Let us repeat the statements which have just been applied, omitting the second column instead of the first column of $A$, the row of all ones must be in the last $m - 1$ lines. Thus, we obtain that $A$ has the following form (up to permutation of rows and columns):

$$
\begin{bmatrix}
0 & 1 & 1 & 1 & \ldots & 0 \\
1 & 0 & 1 & 1 & \ldots & 0 \\
1 & 1 & 0 & 1 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{bmatrix}
$$

This matrix has a submatrix of the forbidden form, and consequently, the unfavourable case could never occur.

Proof of Problem 2: We examine the spectrum $\{\lambda_1, \ldots, \lambda_v\}$ of $A$.

From the matrix equations, it follows that $A$ has exactly $k$ ones in each row; Hence, $\lambda_1 = k$ is an eigenvalue of $A$ (with the eigenvector $(1 \ldots 1)^T$).

Because $AA^T = A^2$, $(k - \lambda)I + \lambda J$ has eigenvalues $\lambda_1^2, \lambda_2^2, \ldots, \lambda_v^2$. Now, however, $J$ has only the eigenvalue $v$ which is different from 0, since it has rank 1. Therefore, the remaining eigenvalues of $AA^T$ are all $k - \lambda$. I.e., the spectrum of $A$ is $\{k, \sqrt{k - \lambda}, -\sqrt{k - \lambda}\}$.

And so applies to the trace:

$$
\text{tr}(A) = k + a\sqrt{k - \lambda} - b\sqrt{k - \lambda} \quad \text{with integers } a, b.
$$

But since $\text{tr}(A)$ must be an integer and $k - \lambda$ is not a square, $a = b$. The trace of $A$ is thus $k$.

5 The Marriage theorem

In this section, we prove two related propositions.

Theorem 5.1 (König-Egerváry). Let $A$ be a zero-one matrix of order $m \times n$. The minimum number of the lines containing all of the ones in $A$ is equal to the maximum number of ones of which no two share a line.

Example:

$$
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}
$$
Proof. Let $r$ be the minimum number of the lines of $A$ that contain all of the ones of $A$, and let $e$ be the maximum number of the ones that do not share a line.

Note that $r \geq e$, since none of the $r$ elements of a minimum set of lines contains two ones of a maximal system of ones. We will show that $r \leq e$.

For this purpose, we fix a minimal set of lines consisting of $z$ rows and $s$ columns ($z + s = r$). Without loss of generality, we may assume that they are the first $z$ rows and $s$ columns of the matrix:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{bmatrix}$$

where $A_{11}$ is a $z \times s$ matrix. First, let $0 < z < m$.

Now $A_{12}$ has a maximal system of $z$ ones, since otherwise — we may assume that the theorem is already proven for $A_{12} - A_{12}$ would have a minimal system of size less than $z$, which in combination with the first $s$ columns of $A$ would yield a system of less than $r$ lines that cover all ones of $A$, contradicting the minimality of $r$. Similarly, $A_{21}$ has a maximal system of $s$ ones. Thus, in $A$, we have a maximal system of at least $s + z = r$ ones, so $e \geq r$. In the cases $z = m$ and $z = 0$, one can consider $A$ and/or $A^T$ as the incidence matrix of a system of $n$ or $m$ sets for which the assertion is a consequence of the following theorem. \hfill $\square$

Now let’s discuss the marriage theorem of P. Hall: First, a definition.

**Definition 2.** Let $S_1, \ldots, S_m$ be subsets of $S$. An $m$-tuple $(a_1, \ldots, a_m)$ of distinct elements of $S$ with $a_i \in S_i$ ($i = 1, \ldots, m$) is called an individual representative system.

The marriage theorem provides a necessary and sufficient condition for a system $S_1, \ldots, S_m$ of subsets to have an individual representative system.

**Theorem 5.2.** $S_1, \ldots, S_m$ have an individual representative system if and only if $|S_{i_1} \cup \cdots \cup S_{i_k}| \geq k$ for all $k = 1, \ldots, m$ for all $k$-subsets $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, m\}$.

**Proof.** It is only necessary to prove the sufficiency of the condition. For $m = 1$ it is trivial. To prove it for $m$, if it is true for all $m' < m$, we break into two cases.

**Case 1:** For all $k = 1, \ldots, m - 1$ and all $k$-subsets $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, m\}$, $|S_{i_1} \cup \cdots \cup S_{i_k}| \geq k + 1$.

**Case 2:** For some $k \in \{1, \ldots, m - 1\}$, there exists a $k$-subset $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, m\}$, $|S_{i_1} \cup \cdots \cup S_{i_k}| = k$.

In the first case, fix an $a_1 \in S_1$ and set $S'_j = S_j - \{a_1\}$ ($j = 2, \ldots, m$). This set system then satisfies the condition of the theorem and thus has, by induction, an individual representative system for $S_1, \ldots, S_n$.

In the second case, the exceptional indices are, without loss of generality, $\{i_1, \ldots, i_k\} = \{1, \ldots, k\}$. By induction, $S_1, \ldots, S_k$ have an individual representative system $(a_1, \ldots, a_k)$. To the sets $S_{i(k)} := S_j - \{a_1, \ldots, a_k\}$ ($j = k+1, \ldots, m$), we may apply the induction hypothesis to find an individual representative system $(a_{k+1}, \ldots, a_m)$ since when

$$|S_{k+1}^* \cup \cdots \cup S_{k+m}^*| < k^*$$

then

$$|S_1 \cup \cdots \cup S_k \cup S_{k+1}^* \cup \cdots \cup S_{k+m}^*| < k + k^*$$

a contradiction. Thus, $(a_1, \ldots, a_m)$ is an individual representative system of $S_1, \ldots, S_m$. \hfill $\square$

This completes the proof of Theorem 5.2; A sharper statement is proved in [1].

## 6 Permanents

Let $A = [a_{ij}]$ be an $m \times n$ matrix ($m \leq n$). The permanent of $A$ is defined by

$$\text{per}(A) = \sum A_{i_1i_2} \cdots A_{i_{m-1}i_m},$$

where $(i_1, \ldots, i_m)$ runs over all permutations of $\{1, \ldots, m\}$.

The relationship with the individual representative systems is the following: Let $S_1, \ldots, S_m$ be subsets of an $n$-element set $S$ with $m \leq n$, and let $A$ be the incidence matrix of the system. The number of all individual representative systems is $\text{per}(A)$. 

10
The permanent of a matrix is often very difficult to calculate. The following formula, which we cite without proof from [1], reduces the computation a bit with \( n \times n \) matrices:

\[
\text{per}(A) = S(A) - \sum_{A_1} S(A_1) + \sum_{A_2} S(A_2) - \cdots + (-1)^{n-1} \sum_{A_{n-1}} S(A_{n-1})
\]  \hspace{1cm} (6.1)

where \( A_r \) is an \( n \times (n - r) \) submatrix of \( A \) and \( S(A_r) \) is the product over all row sums of \( A_r \).

For example, if we compute the permanent of \( J - I \) according to this formula, we get the following expression:

\[
\text{per}(J - I) = \sum_{r=0}^{n-1} (-1)^r \binom{n}{r} (n - r)^r (n - r - 1)^r.
\]

On the other hand, of course, \( \text{per}(J - I) \) is equal to the number of derangements \( D_n \) (‘derangement number’) of \( n \) objects, thus \( = n! \left( 1 - \frac{1}{1!} + \cdots + (-1)^n \frac{1}{n!} \right) \), so

\[
n! \left( 1 - \frac{1}{1!} + \cdots + (-1)^n \frac{1}{n!} \right) = \sum_{r=0}^{n-1} (-1)^r \binom{n}{r} (n - r)^r (n - r - 1)^r.
\]

\[
X = \begin{bmatrix}
    x & y & 0 & \cdots & \cdots & 0 \\
    y & x & 0 & \cdots & \cdots & 0 \\
    0 & x & y & \cdots & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\
    0 & 0 & 0 & \cdots & \cdots & x \end{bmatrix}
\]

We will compute \( \text{per}(X) \) in two different ways.

On the one hand, \( \text{per}(X) = x^n + y^n \). On the other hand, by applying (6.1),

\[
\text{per}(A) = \sum_{r=0}^{n-1} (-1)^r S(X_r)
\]

\[
= \sum_{r=0}^{n-1} (-1)^r a_r (xy)^r (x + y)^{n-2r},
\]

where \( a_r \) is the number of possibilities to select \( r \) non-neighbouring elements from a circle of \( n \) objects. According to a lemma from Kaplanski (see [1] p. 34):

\[
a_r = \frac{n}{n - r} \binom{n - r}{r} \quad (\text{obviously } a_r = 0 \text{ for } r > \left\lfloor \frac{n}{2} \right\rfloor).
\]

Similar to the case of determinants, the question arises for which matrices \( A \) is \( \text{per}(A) \neq 0 \).

A square matrix of order \( n \) is called doubly stochastic if its elements are nonnegative and all row and column sums are equal to 1.

For doubly stochastic matrices \( A \), \( \text{per}(A) \) is positive. This follows immediately from the Birkhoff-König Theorem:

**Theorem 6.1.** Let \( A \) be doubly stochastic of order \( n \). Then \( A = c_1 P_1 + \cdots + c_t P_t \), where each \( P_t \) is a permutation matrix and \( c_1, \ldots, c_t \) are nonnegative real numbers with \( \sum c_i = 1 \).

**Proof.** \( A \) has \( n \) positive elements which occur in different rows and columns; Otherwise, Theorem 5.1 applied with the positive elements of \( A \) with \( z \) rows and \( s \) columns where \( z + s < n \), we have that \( A \) would not be doubly stochastic. Now let \( P_1 \) be the permutation matrix, which has ones at the corresponding entries; Let \( c_1 \) be the smallest of the selected \( n \) positive elements. Then \( A - c_1 P_1 \) is a matrix of nonnegative elements whose row and column sums are all equal to \( 1 - c_1 \). Repeated application of the preceding conclusion to \( A - c_1 P_1 \) instead of to \( A \) and so on finally yields the desired representation. \( \square \)

**Note:** If \( A \) is a zero-one matrix, with identical row and column sums, then one finds this in the same way, of course, a representation with \( c_i = 1 \) for all \( i \).

A sharp lower estimate of \( \text{per}(A) \) for doubly stochastic matrices \( A \) is not yet known. The conjecture of van der Waerden, which states that

\[
\text{per}(A) \geq \frac{n!}{n^n}
\]
holds, with equality only if $A = \frac{1}{n} J$, has only been verified for small values of $n$ ($n \leq 4$).

We briefly check for $n = 2$: Obviously a doubly stochastic matrix has the form

$$\begin{bmatrix} x & 1-x \\ 1-x & x \end{bmatrix}$$

with $0 \leq x \leq 1$, and therefore

$$\text{per} \left[ \begin{bmatrix} x & 1-x \\ 1-x & x \end{bmatrix} \right] = x^2 + (1-x)^2 \geq \frac{1}{2} = 2!/2^2.$$  

Marcus and Minc (in: Permanents, Am. Math. Monthly, 72, (1965), 577-591) were able to prove the weaker result:

**Theorem 6.2.** For a doubly stochastic matrix $A = [a_{ij}]$, there exists a permutation $\sigma$ of $\{1, \ldots, n\}$ such that

$$\prod_{j=1}^{n} a_{j\sigma(j)} \geq \frac{1}{n^n}.$$  

One may wish to generalize van der Waerden’s conjecture by the following:

$$\text{per}(AB) \leq \min(\text{per}(A), \text{per}(B)).$$  

This inequality was, however, rejected by Jurkat by the counter-example

$$A = \frac{1}{24} \begin{bmatrix} 11 & 5 & 8 \\ 13 & 11 & 0 \\ 0 & 8 & 16 \end{bmatrix}, \quad B = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

In this case, $\text{per}(A) = \frac{3808}{13824} < \text{per}(AB) = \frac{3840}{13824}$. A further conjecture has already been expressed, namely by $\text{per}(AA^T) < \text{per}(A)$. But it too could be disproved, by M. Newman with the counter-example

$$A = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

for which $\text{per}(AA^T) = \frac{9}{64} > \text{per}(A) = \frac{8}{64}$.

The van der Waerden conjecture also makes sense for matrices other than doubly stochastic. Consider, e.g., the class $U(R, S)$ of all $m \times n$ matrices of nonnegative elements with given vectors $R = (r_i)$ and $S = (s_k)$ for the row and column sums. Substituting $R = S = (1, \ldots, 1)_n$, we again have the class of doubly stochastic matrices; Here at least the maximum of the permanent function is known; it is equal to $1$ and is attained precisely for the permutation matrices.

Now let $m = n$ and $R = (1, n+1, \ldots, n+1)$, $S = (n+1, \ldots, n+1, 1)$. The class $U(R, S)$ contains, for example,

$$\begin{bmatrix} 1 \\ n \\ \cdot \cdot \cdot \\ 0 \end{bmatrix}$$

The permanent of such matrices is always positive, and it is useful to ask for the maximum and minimum of the permanent function. Is the minimum of $\text{per}(A)$ over this class $= 1$ if only integer elements are allowed? A sharp estimate for the upper bound was shown recently (see [8]):

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10 The van der Waerden conjecture has since been proved. See
J. H. van Lint, The van der Waerden conjecture: two proofs in one year, Math. Intelligencer 4 (1982), 72–77.
G. P. Egorychev, Solution of the van der Waerden problem for permanents, Soviet Math. Dokl. 23 (1981), 619–622.
D. I. Falikman, Proof of the van der Waerden conjecture regarding the permanent of a doubly stochastic matrix, Math. Notes 29 (1981), 475–479.
Theorem 6.3. Let \( A \in U(R, S) \), where \( R = (r_j) \), \( S = (s_j) \), and \( r_1 \leq r_2 \leq \cdots \leq r_n \), \( s_1 \leq s_2 \leq \cdots \leq s_n \). Then

\[
\text{per}(A) \leq \prod_{j=1}^{n} \min(r_j, s_j).
\]

In view of the difficulties of these problems, it is useful first to restrict to the class \( R \) of all zero-one matrices of order \( n \) with row and column sums \( = k \) (\( 1 \leq k \leq n \)) and determine the extremes of the permanent over \( R \).

For this, there is initially a lower estimate of M. Hall: For \( A \in R \), \( \text{per}(A) > k! \). And for the more general case of row sums \( r_1, r_2, \ldots, r_n \), an inequality of Minc

\[
\text{per}(A) \leq \frac{\prod_{j=1}^{n} r_j + 1}{2}.
\]

In addition, Minc conjectured\[1^\text{\textsuperscript{11}}\]

\[
\text{per}(A) \leq \prod_{j=1}^{n} (r_j !)^{1/r_j}
\]

and

\[
\text{per}(A) \leq \prod_{j=1}^{n} (r_j !)^{1/n} \cdot \left( \frac{r_j + 1}{2} \right)^{\frac{n-r_j}{n}}.
\]

Combining van der Waerden’s conjecture of the first bound of Minc’s, one obtains the estimate

\[
\left( \frac{n!}{n^n} \right)^r \cdot \prod_{j=1}^{r} (n+1-j)^n \leq L(r, n) \leq \prod_{j=1}^{r} (n+1-j)^{\frac{n-n^r}{n}}
\]

where \( L(r, n) \) is the number of (non-normalized) \( r \times n \) Latin rectangles; The first inequality is based on van der Waerden’s conjecture and the second of Minc’s conjectures.

References

Basic facts dealt with in this paper can be found in detail in the literature:

[1] Ryser, H. J., Combinatorial Mathematics, Carus Mathematical Monographs 14, 1963.

[2] Hall, M., Combinatorial Theory, Ginn & Co., 1967.

Many things that could only be hinted at here without proof are contained in recent journal articles by H.J. Ryser and co-authors:

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[4] Ryser, H. J., Inequalities of compound and induced matrices with applications to combinatorial analysis, Illinois J. Math. 2, 240–253 (1958).

[5] Fulkerson, D. R., and Ryser, H. J., Multiplicities and Minimal widths for (0, 1)-matrices, Can. J. Math. 14, 498–508 (1962).

[6] Fulkerson, D. R., and Ryser, H. J., Width sequences for special classes of (0, 1)-matrices, Can. J. Math. 15, 371–396.

[7] Jurkat, W. B., and Ryser, H. J., Matrix factorizations of determinants and permanents, J. Algebra 3, 1–27.

[8] Jurkat, W. B., Term ranks and permanents of nonnegative matrices, J. Algebra 5, 342–357 (1967).

\[11^\text{The first part of this conjecture has also been proved. See L. M. Brégman, Some properties of nonnegative matrices and their permanents, Soviet Math. Dokl. 14 (1973), 945–949. A. Schrijver, A short proof of Minc’s conjecture, J. Combinatorial Theory Ser. A 25 (1978), 80–83.}\]