Abstract

The purpose of this paper is to study the exceptional eigenvalues of the asymmetric quantum Rabi models (AQRM), specifically, to determine the degeneracy of their eigenstates. Here, the Hamiltonian $H_{\text{Rabi}}^{\varepsilon}$ of the AQRM is defined by adding the fluctuation term $\varepsilon \sigma_x$, with $\sigma_x$ being the Pauli matrix, to the Hamiltonian of the quantum Rabi model, breaking its $\mathbb{Z}_2$-symmetry. The spectrum of $H_{\text{Rabi}}^{\varepsilon}$ contains a set of exceptional eigenvalues, considered to be remainders of the eigenvalues of the uncoupled bosonic mode, which are further classified in two types: Juddian, associated with polynomial eigensolutions, and non-Juddian exceptional. We explicitly describe the constraint relations for allowing the model to have exceptional eigenvalues. By studying these relations we obtain the proof of the conjecture on constraint polynomials previously proposed by the third author. In fact, we prove that the spectrum of the AQRM possesses degeneracies if and only if the parameter $\varepsilon$ is a half-integer. Moreover, we show that non-Juddian exceptional eigenvalues do not contribute any degeneracy and we characterize exceptional eigenvalues by representations of $\mathfrak{sl}_2$. Upon these results, we draw the whole picture of the spectrum of the AQRM. Furthermore, generating functions of constraint polynomials from the viewpoint of confluent Heun equations are also discussed.

2010 Mathematics Subject Classification: Primary 34L40, Secondary 81Q10, 34M05, 81S05.

Keywords and phrases: quantum Rabi models, Bargmann space, degenerate spectrum, constraint polynomials, Lie algebra representations, confluent Heun differential equations, zeta regularized products.

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**1 Introduction and overview**

In quantum optics, the quantum Rabi model (QRM) \[27\] describes the simplest interaction between matter and light, i.e. the one between a two-level atom and photon, a single bosonic mode (see e.g. \[4, 64\]). Actually, it appears ubiquitously in various quantum systems including cavity and circuit quantum electrodynamics, quantum dots and artificial atoms \[65\], with potential applications in quantum information technologies including quantum cryptography, quantum computing, etc. (see e.g \[12, 22\]). In addition, the fact \[61\] that the confluent Heun ODE picture of QRM is derived by coalescing two singularities in the Heun picture of the non-commutative harmonic oscillator (NCHO: \[45, 48\]) strongly suggests the existence of a rich number theoretical structure behind the QRM, including modular forms, elliptic curves \[31\], Apéry-like numbers \[30, 39\] and Eichler cohomology groups \[32\] through the study of the spectral zeta function \[23, 26, 44\] (see also \[56\] for the spectral zeta function for the QRM). For the reasons above and according to recent development of experimental technology (cf. e.g. \[43\]), lately there has been considerable progress in the investigation of the QRM not only in theoretical physics and mathematical analysis (cf. e.g. \[20, 21, 46, 56\]) but also in experimental physics. For instance, there is a proposal to reproduce/realize the quantum Rabi models experimentally \[47\] (see also \[40\]). In practice, in the weak parameter coupling regime the Jaynes-Cummings model, the rotating-wave approximation (RWA) of the QRM \[27\], experimentally meets the QRM. However, this is not the case in the ultra-strong and deep strong coupling regimes, where the RWA, or similar approximations, is no longer suitable and the full Hamiltonian of the QRM has to be considered (for a review of recent developments, see \[64\]). In contrast with the Jaynes-Cummings model, which has a continuous $U(1)$-symmetry, the QRM only has a $\mathbb{Z}_2$-symmetry (parity). In 2011, paying attention to this $\mathbb{Z}_2$-symmetry, Braak found the analytical solutions of eigenstates (for the non-exceptional type) and derived the conditions for determining the energy spectrum of the QRM \[5\] (see also \[8\]). These conditions are described by the so-called $G$-functions in \[5, 6\]. Since then, various aspect of the QRM and its generalizations have been discussed widely and intensively, and developed from the theoretical viewpoint (see \[64\] and references therein).
In the present paper, we study the spectrum of the asymmetric quantum Rabi model (AQRM) [64]. This asymmetric model actually provides a more realistic description of the circuit QED experiments employing flux qubits than the QRM itself [43, 65]. The Hamiltonian $H_{\text{Rabi}}^\varepsilon$ of the AQRM ($\hbar = 1$) is given by

$$H_{\text{Rabi}}^\varepsilon = \omega a^\dagger a + \Delta \sigma_z + g \sigma_x (a^\dagger + a) + \varepsilon \sigma_x,$$

where $a^\dagger$ and $a$ are the creation and annihilation operators of the bosonic mode, i.e. $[a, a^\dagger] = 1$ and $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ are the Pauli matrices, $2\Delta$ is the energy difference between the two levels, $g$ denotes the coupling strength between the two-level system and the bosonic mode with frequency $\omega$ (subsequently, we set $\omega = 1$ without loss of generality) and $\varepsilon$ is a real parameter. The Hamiltonian of the “symmetric” quantum Rabi model (QRM) is then given by $H_{\text{Rabi}}^0$. In this respect, the AQRM has been also referred to as the generalized-, biased- or driven QRM (see, e.g. [5, 37, 64]).

The initial purpose of the present paper is to study the “exceptional” eigenvalues of the AQRM and to determine the degeneracy of its eigenstates. Let us first recall the situation for the QRM. In this case, an eigenvalue $\lambda$ is called exceptional if $\lambda = N - g^2$ for a non-negative integer $N \in \mathbb{Z}_{\geq 0}$. It was shown by Kuś [34] that the degeneracy of an eigenstate (i.e. the energy level crossing at the spectral graph) happens in the QRM if and only if the eigenvalue is exceptional and the corresponding state is essentially described by a polynomial, i.e. a Juddian eigensolution [28]. Non-degenerate exceptional eigenvalues are also present in the spectrum of QRM (cf. [41, 8]), and we call these eigenvalues (and the associated eigensolutions) non-Juddian exceptional. Exceptional eigenvalues, especially Juddian eigenvalues, are considered to be remains of the eigenvalues of the uncoupled bosonic mode (i.e. the quantum harmonic oscillator).

Similarly, in the AQRM case, an eigenvalue $\lambda$ is called exceptional if there is an integer $N \in \mathbb{Z}_{\geq 0}$ such that $\lambda$ is of the form $\lambda = N \pm \varepsilon - g^2$ [37]. An eigenvalue which is not exceptional is called regular and is always non-degenerate [5]. The presence of degeneracy, in other words, a level crossing in the spectral graph, for the asymmetric model is highly non-trivial. This is because the additional term $\varepsilon \sigma_x$ breaks the $\mathbb{Z}_2$-symmetry which couples the bosonic mode and the two-level system by allowing spontaneous tunneling between the two atomic states. The AQRM has been studied, for instance, numerically in the context of the process of physical bodies reaching thermal equilibrium through mutual interaction [36]. In recent works [15, 54] on the AQRM and the quantum Rabi-Stark model (another generalization of the QRM) the respective authors have studied the degeneracies of the spectrum from different points of view than the present work.

Without the $\mathbb{Z}_2$-symmetry, there seems to be no invariant subspaces whose respective spectral graphs intersect to create “accidental” degeneracies in the spectrum for specific values of the coupling. However, the presence of degeneracies (crossings at the spectral graph) was claimed for the case $\varepsilon = \frac{1}{2}$ and supporting numerical evidence was presented for the half-integral parameter $\varepsilon$ in [37], investigating an earlier empirical observation in [5]. Moreover, this numerical verification was proved for $\varepsilon = \frac{1}{2}$ and formulated mathematically as a conjecture (see [4] Conjecture 4.1) for the general half-integer $\varepsilon$ case in [62], hinting at a hidden symmetry present in this case. In this paper we prove the conjecture affirmatively in general for $\varepsilon = \frac{\ell}{2} (\ell \in \mathbb{Z})$ (cf. Theorem 4.12).

Let us now briefly draw the whole picture of the spectrum of the AQRM by restricting ourselves to mention the technical issues that we prove in this paper. The eigenvalues
of the AQRM can be visualized in the spectral graph, that is, the graph of the curves \( \{ \lambda_i(\epsilon) \}_{i=1}^{\infty} \) in the \((g,E)\)-plane (\(E = \text{Energy}\) for fixed \(\epsilon \in \mathbb{R}\) and \(\Delta > 0\)). In this picture, the exceptional eigenvalues are those that lie in the energy curves \(E = N \pm \epsilon - g^2\), as shown conceptually in Figure 1(a).

![Eigenvalue curves](image)

(a) Eigenvalue curves \(\lambda(g), \lambda'(g)\) and exceptional eigenvalues of \(H_{Rabi}^{\epsilon}\) for two integers \(N, M \in \mathbb{Z}_{\geq 0}\). (e.g. see Figure 6(a))

Figure 1: Exceptional eigenvalues of AQRM.

An eigenfunction \(\psi\) corresponding to an exceptional eigenvalue \(\lambda\) is called a Juddian solution if its representation in the Bargmann space \(B\) (cf. \(\S\)) consists of polynomial components. The associated eigenvalue \(\lambda\) is also called Juddian. Juddian solutions are also called quasi-exact and have been investigated by Turbiner \([57]\) with a viewpoint of \(\mathfrak{sl}_2\)-action and Heun operators. Notice that Juddian solutions are not present for arbitrary parameters \(g, \Delta\). In fact, it is known (\([38, 63]\)) that an exceptional eigenvalue \(\lambda = N + \epsilon - g^2\) is present in the spectrum of \(H_{Rabi}^{\epsilon}\) and corresponds to a Juddian solution if and only if the parameters \(g\) and \(\Delta\) satisfy the polynomial equation

\[
P^{(N,\epsilon)}_M((2g)^2, \Delta^2) = 0. \tag{1.2}
\]

The polynomial \(P^{(N,\epsilon)}_M(x, y)\) (cf. \(\S 4.1\)) is called constraint polynomial and \((1.2)\) is called constraint relation. In practice, however, not all exceptional eigenvalues correspond to Juddian (i.e. quasi-exact) solutions and, as in the case of the QRM, we call these eigenvalues and the corresponding eigensolutions non-Juddian exceptional. This situation is illustrated conceptually in Figure 1(a) (see Figure 6(a) for a numerical example). Further, the constraint relation for non-Juddian exceptional eigenvalues (cf. \(\S 6.4\)), which are shown to be non-degenerate when \(\epsilon = 0\) in \([8]\), cannot be obtained in terms of polynomials.

The constraint relation \((1.2)\) determines a curve in the \((g, \Delta)\)-plane consisting of a number of concentric closed curves, shown conceptually in Figure 1(b). In this picture, for fixed \(\Delta = C > 0\) the Juddian eigenvalues \(\lambda = N + \epsilon - g^2\) of \(H_{Rabi}^{\epsilon}\) correspond to points in the intersection of the curve \(P^{(N,\epsilon)}_M((2g)^2, \Delta^2) = 0\) with the horizontal line \(\Delta = C\) in the \(g, \Delta > 0\) quadrant.

Next, in Figure 2 we illustrate conceptually the way degeneracies appear in the exceptional spectrum for the case \(N, \ell \in \mathbb{Z}_{\geq 0}\). When \(\epsilon \in \mathbb{R}\) satisfies \(0 < |\epsilon - \ell/2| < \delta\) for small \(\delta > 0\), there are \(g, g' > 0\) such that \(\lambda = N + \ell - \epsilon - g^2\) and \(\lambda' = N + \epsilon - g'^2\) are non-degenerate eigenvalues corresponding to Juddian solutions (shown with circle marks in Figure 2(a)). In addition, exceptional eigenvalues \(\lambda = N + \ell - \epsilon - g''^2\) with non-Juddian solutions may be present for \(g'' \neq g, g'\) (shown with diamond marks in Figure 2(a)). On the
other hand, the case $\varepsilon = \ell/2$ $(\ell \in \mathbb{Z})$ is illustrated in Figure 2(b). In this case, the energy curves $E = N + \ell - \varepsilon - g^2$ and $E = N + \varepsilon - g^2$ coincide into the curve $E = N + \ell/2 - g^2$. As $\varepsilon \to \ell/2$, the non-degenerate Juddian eigenvalues lying in the disjoint energy curves of Figure 2(a) join into a single degenerate Juddian eigenvalue with multiplicity 2 lying on the resulting energy curve $E = N + \ell/2 - g^2$. However, we remark that for $g' > 0$, with $g \neq g'$ there may be additional non-Juddian solutions with exceptional eigenvalue $\lambda = N + \ell/2 - g^2$, as demonstrated in [37] for the QRM (case $\varepsilon = 0$). In §6.4 we present numerical examples of these graphs, and we direct the reader to [37] for further examples.

In [4] we prove that degenerate exceptional eigenvalues $\lambda$ with Juddian solutions exist in general for any half-integer $\varepsilon$ (Theorem 4.12) by studying certain determinant expressions for the constraint polynomials $P_N^{(N,\varepsilon)}((2g)^2, \Delta^2)$. In particular, if $\lambda = N + \ell/2 - g^2$ $(\ell \in \mathbb{Z})$ is a Juddian eigenvalue (corresponding to a root of the constraint polynomial) then its multiplicity is 2 and the two linearly independent solutions are Juddian. In [4, 5, 5.3] we show that all the Juddian solutions corresponding to exceptional eigenvalues are degenerate. Moreover, in [5] Theorem 5.3 we count the exact number of Juddian solutions relative to the pair $(g, \Delta)$, giving a (complete) generalization of the results given in [37] for the AQRM and in [34] for the QRM.

The situation for the degeneracy of Juddian solutions in the case $N = 5$ and $\varepsilon = \ell/2 = 3/2$ is illustrated in Figure 3 with the graphs of the curves $P_N^{(N,\varepsilon)}((2g)^2, \Delta^2) = 0$ and $P_N^{(N+\ell,-\varepsilon)}((2g)^2, \Delta^2) = 0$ in the $(g, \Delta)$-plane for different choices of $\varepsilon > 0$. As we can see in Figures 3(a) and Figure 3(b), as $\varepsilon$ tends to $\ell/2$ the two curves become coincident until finally, at $\varepsilon = \ell/2$ (Figure 3(c)) the two curves coincide completely. In the case $\varepsilon = \ell/2$ $(\ell \in \mathbb{Z})$, any point $(g, \Delta)$ with $g, \Delta > 0$ in the resulting curve corresponds to a degenerate Juddian solution for the eigenvalue $\lambda = N + \ell/2 - g^2$. The aforementioned Theorem 1.12 (cf. Conjecture 1.1) gives a complete explanation of the coincidence of the two curves. In particular, by Theorem 1.12 we have the divisibility of the constraint polynomial $P_{N+\ell}^{(N+\ell,-\ell/2)}((2g)^2, \Delta^2)$ by $P_{N+\ell}^{(N,\ell/2)}((2g)^2, \Delta^2)$ and positivity of the resulting divisor (a polynomial of degree $\ell$).

We notice that, however, that the crossings between the curves of the constraint relations appearing in Figures 3(a) and 3(b) do not constitute degeneracies as the associated Juddian solutions have different eigenvalues for $\varepsilon \neq \ell/2$ ($\lambda_1 = N + \varepsilon - g^2$ and $\lambda_2 = N + \ell - \varepsilon - g^2$ respectively).

The second purpose of this paper is to complete the whole picture of the spectrum based on the study of the exceptional eigenvalues, particularly the aforementioned Juddian
Determinant expression of constraint polynomials and spectrum of AQRM

Figure 3: Curves $P^{5,\varepsilon}_5((2g)^2, \Delta^2) = 0$ (continuous line) and $P^{8,\varepsilon}_8((2g)^2, \Delta^2) = 0$ (dashed line). The two curves overlap in the case (c) $\varepsilon = 3/2$ (Theorem 4.12).

 eigenvalues.

As in the case of the QRM, eigenvalues other than the exceptional ones are called regular. Equivalently, a regular eigenvalue $\lambda$ is one of the form $\lambda = x \pm \varepsilon - g^2 (x \notin \mathbb{Z}_{\geq 0})$. It is known that regular eigenvalues are non-degenerate. Notice also that regular eigenvalues are always obtained from zeros of the constraint polynomials $G$, $P$, $T$, which correspond to exceptional eigenvalues $\lambda = N \pm \varepsilon - g^2$ with $N \notin \mathbb{Z}_{\geq 0}$. In §6.4 we define the constraint $T$-function $T^{(N)}_\varepsilon (g, \Delta)$ whose zeros correspond to exceptional eigenvalues $\lambda = N \pm \varepsilon - g^2$ with non-Juddian solution. Thus, the transcendental equation $T^{(N)}_\varepsilon (g, \Delta) = 0$ gives the constraint relation for non-Juddian exceptional eigenvalues. In §6.4 we present numerical examples of the curves determined by the constraint relation for non-Juddian exceptional eigenvalues when $\varepsilon = \frac{1}{2}$. Further numerical examples can be found in [41] for the case $\varepsilon = 0$ and in [37] for the non half-integral general case.

We summarize the spectrum of the AQRM using the constraint relations given by the $G$-functions, the constraint polynomials $P^{(N,\varepsilon)}_N(x, y)$ and constraint $T$-functions. Setting $N, \ell \in \mathbb{Z}_{\geq 0}$, we have

- If $G_\varepsilon(x; g, \Delta) = 0$, then $\lambda = x - g^2 (x \pm \varepsilon \notin \mathbb{Z}_{\geq 0})$ is a regular eigenvalue.
- If $P^{(N,\pm \varepsilon)}_N((2g)^2, \Delta^2) = 0 \ (N \in \mathbb{Z}_{\geq 0})$, then $\lambda = N \pm \varepsilon - g^2$ is an exceptional eigenvalue with Juddian solution. Furthermore, if $\varepsilon = \ell/2 \ (\ell \in \mathbb{Z}_{\geq 0})$ and $\lambda = N + \ell/2 - g^2$ is a Juddian eigenvalue for some $N$, then $\lambda$ is degenerate with multiplicity two and the two eigen solutions are Juddian. Moreover, we show that $\lambda = N - \ell/2 - g^2$ with $N \leq \ell$ does not occur as a Juddian eigenvalue (cf. §6.3).
- If $T^{(N)}_\pm(x; g, \Delta) = 0 \ (N \in \mathbb{Z}_{\geq 0})$, then $\lambda = N \pm \varepsilon - g^2$ is an exceptional eigenvalue with non-Juddian exceptional solution (cf. §6.4).
- The spectrum of the AQRM possesses a degenerate eigenvalue if and only if the parameter $\varepsilon$ is a half integer. Furthermore, all degenerate eigenstates consist of Juddian solutions (cf. Theorem 6.3 in §6.4).

We also make an extensive study of the $G$-function $G_\varepsilon(x; g, \Delta)$ and its relation with $T^{(N)}_\varepsilon (g, \Delta)$ and $P^{(N,\varepsilon)}_N((2g)^2, \Delta^2)$ in §6.3, §6.4 and §6.5. Especially, we observe that the meromorphic function $G_\varepsilon(x; g, \Delta)$ actually possesses almost complete information about Juddian and non-Juddian exceptional eigenvalues. In particular, in §6.5 the pole structure of $G_\varepsilon(x; g, \Delta)$ reveals a finer structure for exceptional eigenvalues for $\varepsilon = \ell/2 \ (\ell \in \mathbb{Z}_{\geq 0})$. For instance, we see that $G_{\ell/2}(x; g, \Delta)$ can have, in general, simple poles at $x = N - \ell/2 \ (N < \ell)$ and double poles at $x = N + \ell/2 \ (N \in \mathbb{Z}_{\geq 0})$. When $\lambda = N - \ell/2 - g^2$ with $N < \ell$ is
a non-Juddian exceptional eigenvalue the simple pole at \( x = N - \ell/2 \) disappears (cf. Proposition 6.10). In contrast, when \( \lambda = N + \ell/2 - g^2 \) is a Juddian eigenvalue the double pole of \( G_{\ell/2}(x;g,\Delta) \) at \( x = N \pm \ell/2 \) disappears and if there is a non-Juddian exceptional eigenvalue \( \lambda = N + \ell/2 - g^2 \), then the double pole of \( G_{\ell/2}(x;g,\Delta) \) at \( x = N \pm \ell/2 \) either vanishes or is simple (cf. Proposition 6.11). Moreover, we prove that the meromorphic function \( G_{\ell/2}(x;g,\Delta) \) is essentially, i.e. up to a multiple of two gamma functions, identified with the spectral determinant of the Hamiltonian \( H_{Rabi} \). In other words, \( G_{\ell/2}(x;g,\Delta) \) is expressed by the zeta regularized product (cf. [49]) defined by the Hurwitz-type spectral zeta function of the AQRM, and equivalently this fact confirms the (physically intuitive) experimental numerical observation done in [38].

In §6.6 we make a representation theoretic description of the non-Juddian exceptional eigenvalues. Recall that the eigenstates of the quantum harmonic oscillator are described by certain weight subspaces of the oscillator representation of \( \mathfrak{sl}_2 \) (cf. [23]). Also, the Juddian solutions are known to be captured (i.e. determined) by a pair of irreducible finite dimensional representations of \( \mathfrak{sl}_2 \) [62]. In the same manner, in Theorem 6.15 we show that the non-Juddian exceptional eigenvalues are captured by a pair of lowest weight irreducible representations of \( \mathfrak{sl}_2 \).

Finally, in §7 we study generating functions of constraint polynomials with their defining sequence \( P_k^{(N,\varepsilon)}((2g)^2,\Delta^2) \). In fact, we observe that the generating function of \( P_k^{(N,\varepsilon)}((2g)^2,\Delta^2) \) satisfies a confluent Heun equation (see §7.1), which can be seen as natural by virtue of certain properties of the aforementioned \( G \)-function. As a byproduct of the discussion we have also an alternative proof of the divisibility part of the conjecture.

It is important to notice that, although having analytic solutions, the asymmetric quantum Rabi models, even in the symmetric (QRM) case, are in general known not to be integrable models in the Yang-Baxter sense [11]. However, it is interesting to note that recently the existence of monodromies associated with the singular points of the eigenvalue problem for the quantum Rabi model has been discussed in [13]. Moreover, we note that although there are various coupling regimes of the AQRM given in terms of \( \Delta, \omega (= 1) \) and \( g \) physically, the discussion in this paper is independent of the choice of regimes (cf. [12]).

2 Background from \( \mathfrak{sl}_2 \)-representations

In this section we introduce the necessary background from representation theory of the Lie algebra \( \mathfrak{sl}_2(\mathbb{R}) \) and/or \( \mathfrak{sl}_2(\mathbb{C}) \) to develop the discussion on the spectral theory of AQRM in the following sections. The reader is directed to [62] for an extended discussion and [35, 23] for the general theory of \( \mathfrak{sl}_2 \)-representations.

The standard generators \( H, E \) and \( F \) of \( \mathfrak{sl}_2(\mathbb{R}) \) are given by

\[
H = \begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}, \quad E = \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}, \quad F = \begin{bmatrix}
0 & 0 \\
1 & 0
\end{bmatrix}.
\]

These generators satisfy the commutation relations

\[
[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H.
\]

For \( a \in \mathbb{C} \) define the algebraic action \( \varpi_a \) of \( \mathfrak{sl}_2 \) on the vector spaces \( V_1 := y^{-\frac{1}{4}}\mathbb{C}[y, y^{-1}] \).
and \( V_2 := y^{-\frac{1}{2}} \mathbb{C}[y, y^{-1}] \) given by

\[
\varpi_a(H) := 2y \partial_y + \frac{1}{2}, \quad \varpi_a(E) := y^2 \partial_y + \frac{1}{2}(a + \frac{1}{2})y,
\]

\[
\varpi_a(F) := - \partial_y + \frac{1}{2}(a - \frac{1}{2})y^{-1}
\]

with \( \partial_y := \frac{d}{dy} \). It is not difficult to verify that these operators indeed act on the space \( V_j (j = 1, 2) \), and define infinite dimensional representations of \( \mathfrak{sl}_2 \). Write \( \varpi_{j,a} := \varpi_a|_{V_j} \) and put \( e_{1,n} := y^{n-\frac{1}{4}} \) and \( e_{2,n} := y^{n+\frac{1}{4}} \). Then we have

\[
\begin{align*}
\varpi_{1,a}(H)e_{1,n} &= 2ne_{1,n}, \\
\varpi_{1,a}(E)e_{1,n} &= (n + \frac{a}{2})e_{1,n+1}, \\
\varpi_{1,a}(F)e_{1,n} &= (-n + \frac{a}{2})e_{1,n-1}, \\
\varpi_{2,a}(H)e_{2,n} &= (2n + 1)e_{2,n}, \\
\varpi_{2,a}(E)e_{2,n} &= (n + \frac{a+1}{2})e_{2,n+1}, \\
\varpi_{2,a}(F)e_{2,n} &= (-n + \frac{a-1}{2})e_{2,n-1}.
\end{align*}
\]

Note that \( \varpi_{1,a}, V_1 \) (resp. \( \varpi_{2,a}, V_2 \)) is irreducible when \( a \notin 2\mathbb{Z} \) (resp. \( a \notin 2\mathbb{Z} - 1 \)) and that there is an equivalence between \( \varpi_{j,a} \) and \( \varpi_{j,2-a} \) under the same condition. We call such irreducible representation a principal series. Next, for a non-negative integer \( m \), define subspaces \( D_{2m}^+, F_{2m-1} \) of \( V_{1,2m}(= V_1) \), and \( D_{2m+1}^+, F_{2m} \) of \( V_{2,2m+1}(= V_2) \) respectively by

\[
D_{2m}^+ := \bigoplus_{n \geq m} \mathbb{C} \cdot e_{1,+n}, \quad F_{2m-1} := \bigoplus_{-m+1 \leq n \leq -m} \mathbb{C} \cdot e_{1,n},
\]

\[
D_{2m+1}^- := \bigoplus_{n \geq m+1} \mathbb{C} \cdot e_{2,-n}, \quad D_{2m+1}^+ := \bigoplus_{n \geq m} \mathbb{C} \cdot e_{2,n}, \quad F_{2m} := \bigoplus_{-m \leq n \leq m-1} \mathbb{C} \cdot e_{2,n}.
\]

The spaces \( D_{2m}^+ \) (resp. \( D_{2m+1}^+ \)) are invariant under the action \( \varpi_{1,2m}(X) \), (resp. \( \varpi_{1,2m+1}(X) \)), \( (X \in \mathfrak{sl}_2) \), and define irreducible representations (having the lowest and highest weight vector respectively) known to be equivalent to (holomorphic and anti-holomorphic) discrete series for \( m > 0 \) of \( \mathfrak{sl}_2(\mathbb{R}) \). The irreducible representation \( D_{2m}^+ \) are the (infinitesimal version of) limit of discrete series of \( \mathfrak{sl}_2(\mathbb{R}) \) (see e.g. [23, 33]). Moreover, the finite dimensional space \( F_m \) (dim\( \mathbb{C} F_m = m \)), is invariant and defines irreducible representation of \( \mathfrak{sl}_2 \) for \( a = 2 - 2m \) when \( j = 1 \) and \( a = 1 - 2m \) when \( j = 2 \), respectively.

The following result describes the irreducible decompositions of \( (\varpi_a, V_{j,a}), (a = m \equiv j - 1 \mod 2) \) for \( m \in \mathbb{Z}_{\geq 0} \) and \( j = 1, 2 \).

**Lemma 2.1.** Let \( m \in \mathbb{Z}_{\geq 0} \).

1. The subspaces \( D_{2m}^\pm \) are irreducible submodules of \( V_{1,2m} \) under the action \( \varpi_{1,2m} \) and \( F_{2m-1} \) is an irreducible submodule of \( V_{1,2-2m} \) under \( \varpi_{1,2-2m} \). In the former case, the finite dimensional irreducible representation \( F_{2m-1} \) can be obtained as the subquotient as \( V_{1,2m}/D_{2m}^- \oplus D_{2m}^+ \cong F_{2m-1} \). In the latter case, the discrete series \( D_{2m}^- \) can be realized as the irreducible components of the subquotient representation as \( V_{1,2-2m}/F_{2m-1} \cong D_{2m}^- \oplus D_{2m}^+ \).

2. The subspaces \( D_{2m+1}^- \) are irreducible submodule of \( V_{2,2m+1} \) under the action \( \varpi_{2,2m+1} \) and \( F_{2m} \) is an irreducible submodule of \( V_{2,1-2m} \) under \( \varpi_{2,1-2m} \). In the former case, the finite dimensional irreducible representation \( F_{2m} \) can be obtained as the
Moreover, the spaces of irreducible submodules \( \text{enveloping algebra of } \partial \)
entire functions equipped with the inner product \( \langle f, g \rangle = \int C f(z) g(z) e^{-|z|^2} d(\text{Re}(z)) d(\text{Im}(z)) \).

In this representation, the operators \( a^\dagger \) and \( a \) are realized as the multiplication and differentiation operators over the complex variable: \( a^\dagger = (x - \partial_x) / \sqrt{2} \to z \) and \( a = (x + \partial_x) / \sqrt{2} \to \partial_z \); so that the Hamiltonian \( H^R_{\text{Rabi}} \) is mapped to the operator

\[
\tilde{H}^R_{\text{Rabi}} := \begin{bmatrix}
  z\partial_z + \Delta & g(z + \partial_z) + \varepsilon \\
g(z + \partial_z) + \varepsilon & z\partial_z - \Delta
\end{bmatrix}.
\]

By the standard procedure (cf. [7, 31]), we observe that the Schrödinger equation \( H^R_{\text{Rabi}} \psi = \lambda \psi (\lambda \in \mathbb{R}) \) is equivalent to the system of first order differential equations

\[
\tilde{H}^R_{\text{Rabi}} \psi = \lambda \psi, \quad \psi = \begin{bmatrix} \psi_1(z) \\ \psi_2(z) \end{bmatrix}.
\]

Hence, in order to have an eigenstate of \( H^R_{\text{Rabi}} \), it is sufficient to obtain an eigenstate \( \psi \in \mathcal{B} \), that is, BI: \( \langle \psi_i | \psi_i \rangle < \infty \), and BII: \( \psi_i \) are holomorphic everywhere in the whole complex plane \( \mathbb{C} \) for \( i = 1, 2 \). Actually, it can be shown that any such function satisfying condition BII also satisfies the condition BI (cf. [7]).

Therefore, the eigenvalue problem of the AQRM amounts to finding entire functions \( \psi_1, \psi_2 \in \mathcal{B} \) and real number \( \lambda \) satisfying

\[
\begin{cases}
  (z\partial_z + \Delta) \psi_1 + (g(z + \partial_z) + \varepsilon) \psi_2 = \lambda \psi_1, \\
  (g(z + \partial_z) + \varepsilon) \psi_1 + (z\partial_z - \Delta) \psi_2 = \lambda \psi_2.
\end{cases}
\]

Now, by setting \( f_\pm = \psi_1 \pm \psi_2 \), we get

\[
\begin{cases}
  (z + g) \frac{d}{dz} f_+ + (gz + \varepsilon - \lambda) f_+ + \Delta f_- = 0, \\
  (z - g) \frac{d}{dz} f_- - (gz + \varepsilon + \lambda) f_- + \Delta f_+ = 0,
\end{cases}
\]

(3.1)
where, by using the substitution \( \phi_{1,\pm}(z) := e^{gz} f_{\pm}(z) \) and the change of variable \( y = \frac{a+z}{2g} \), we obtain

\[
\begin{align*}
\left\{ \begin{array}{l}
y \frac{d}{dy} \phi_{1,+}(y) = (\lambda + g^2 - \varepsilon) \phi_{1,+}(y) - \Delta \phi_{1,-}(y), \\
(y - 1) \frac{d}{dy} \phi_{1,-}(y) = (\lambda + g^2 - \varepsilon - 4g^2 + 4g^2y + 2\varepsilon) \phi_{1,-}(y) - \Delta \phi_{1,+}(y).
\end{array} \right.
\end{align*}
\]  

(3.2)

Defining \( a := -(\lambda + g^2 - \varepsilon) \), we get

\[
\begin{align*}
\left\{ \begin{array}{l}
y \frac{d}{dy} \phi_{1,+}(y) = -a \phi_{1,+}(y) - \Delta \phi_{1,-}(y), \\
(y - 1) \frac{d}{dy} \phi_{1,-}(y) = -(4g^2 - 4g^2y + a - 2\varepsilon) \phi_{1,-}(y) - \Delta \phi_{1,+}(y).
\end{array} \right.
\end{align*}
\]  

(3.3)

Similarly, by applying the substitutions \( \phi_{2,\pm}(z) := e^{-gz} f_{\pm}(z) \) and \( \bar{y} = \frac{a-z}{2g} \) to the system (3.1), we get

\[
\begin{align*}
\left\{ \begin{array}{l}
(y - 1) \frac{d}{d\bar{y}} \phi_{2,+}(\bar{y}) = -(4g^2 - 4g^2 \bar{y} + a) \phi_{2,+}(\bar{y}) - \Delta \phi_{2,-}(\bar{y}), \\
\bar{y} \frac{d}{d\bar{y}} \phi_{2,-}(\bar{y}) = -(a - 2\varepsilon) \phi_{2,-}(\bar{y}) - \Delta \phi_{2,+}(\bar{y}).
\end{array} \right.
\end{align*}
\]  

(3.4)

This system gives another (possible) solution \( (\phi_{2,+}(\bar{y}), \phi_{2,-}(\bar{y})) \) to the eigenvalue problem. Note that \( a - 2\varepsilon = -(\lambda + g^2 + \varepsilon) \) and that \( \bar{y} = 1 - y \), where \( y \) is the variable used in (3.3).

The singularities of system (3.3) and (3.4) at \( y = 0 \) and \( y = 1 \) are regular. The exponents of the equation system can be obtained by standard computation, and are shown in Table 1 for reference.

| \( y \) | \( \phi_{1,-}(y) \) | \( \phi_{1,+}(y) \) | \( \phi_{2,-}(1-y) \) | \( \phi_{2,+}(1-y) \) |
|---|---|---|---|---|
| 0 | \( 0, a + 1 \) | \( 0, -a \) | 0 | 0 |
| 1 | \( 0, a + 2\varepsilon \) | \( 0, -a + 2\varepsilon \) | \( 0, -a + 2\varepsilon \) | \( 0, -a + 2\varepsilon + 1 \) |

We remark that Table 1 in particular shows that each regular eigenvalue is not degenerate because one of the exponents is necessarily not an integer.

Next, by using the representation of a particular element of the universal enveloping algebra \( \mathcal{U}(\mathfrak{sl}_2) \) of \( \mathfrak{sl}_2 \) we capture the confluent Heun operators corresponding to the eigenvalue problem of AQRM in the Bargmann space, that is, the second order differential equations corresponding to systems (3.3) and (3.4). Let \( (\alpha, \beta, \gamma; C) \in \mathbb{R}^4 \). Define a second order element \( \mathbb{K} = \mathbb{K}(\alpha, \beta, \gamma; C) \in \mathcal{U}(\mathfrak{sl}_2) \) and a constant \( \lambda_a = \lambda_a(\alpha, \beta, \gamma) \) depending on the representation \( \varpi_a \) as follows:

\[
\mathbb{K}(\alpha, \beta, \gamma; C) := \left[ \frac{1}{2} H - E + \alpha \right] (F + \beta) + \gamma \left[ H - \frac{1}{2} \right] + C,
\]

\[
\lambda_a(\alpha, \beta, \gamma) := \beta \left( \frac{1}{2} a + \alpha \right) + \gamma \left( a - \frac{1}{2} \right).
\]

Noticing \( y^{-\frac{1}{2}(a+\frac{1}{2})} y \partial_y y^{\frac{1}{2}(a+\frac{1}{2})} = y \partial_y + \frac{1}{2}(a - \frac{1}{2}) \), we obtain the following lemma.
Lemma 3.1 ([62, 63]). We have the following expression.
\[
y^{-\frac{1}{2}}(a-\frac{1}{2}) \varpi_a(\mathcal{K}(\alpha, \beta, \gamma; C)) y^{\frac{1}{2}}(a-\frac{1}{2})
\]
\[
y(y - 1) = \frac{d^2}{dy^2} + \left\{ -\beta + \frac{1}{2} a + \alpha + \frac{1}{2} a + 2\gamma - \alpha \right\} \frac{d}{dy} + \frac{-\alpha \beta y + \lambda_a(\alpha, \beta, \gamma) + C}{y(y - 1)}. \]

Now, by choosing suitable parameters \((\alpha, \beta, \gamma; C)\) we define from \(\mathcal{K} = \mathcal{K}(\alpha, \beta, \gamma; C)\) two second order elements \(\mathcal{K}\) and \(\tilde{\mathcal{K}} \in U(\mathfrak{sl}_2)\) that capture the Hamiltonian \(H^{\varepsilon}_{\text{Rabi}}\) of the AQRM. In the following proposition, \(H^{\varepsilon}_1\) is the second order differential operator (confuent Heun ODE [51, 55]) corresponding to the solution \(\phi_{1,+}\) in the system (3.2). Similarly, \(H^{\varepsilon}_2\) is the second order differential operator (confuent Heun ODE) corresponding to \(\phi_{2,+}\) of (3.4).

Proposition 3.2. Let \(\lambda\) be an eigenvalue of \(H^{\varepsilon}_{\text{Rabi}}\). Set \(a = -(\lambda + g^2 - \varepsilon)\), \(a' = a - 2\varepsilon + 1\) and \(\mu = (\lambda + g^2)^2 - 4g^2(\lambda + g^2) - \Delta^2\).

\(1\) Define
\[
\mathcal{K} := \mathcal{K}\left(1 + \frac{a}{2}, 4g^2, \frac{a'}{2}; \mu + 4\varepsilon g^2 - \varepsilon^2\right) \in U(\mathfrak{sl}_2),
\]
\[
\Lambda_a := \lambda_a\left(1 - \frac{a}{2}, 4g^2, \frac{a'}{2}\right).
\]
Then
\[
y(y - 1)H^{\varepsilon}_1(\lambda) = y^{-\frac{1}{2}}(a-\frac{1}{2})(\varpi_a(\mathcal{K}) - \Lambda_a)y^{\frac{1}{2}}(a-\frac{1}{2}). \tag{3.5}\]

\(2\) Define
\[
\tilde{\mathcal{K}} := \mathcal{K}\left(-1 + \frac{a'}{2}, 4g^2, \frac{a}{2}; \mu - 4\varepsilon g^2 - \varepsilon^2\right) \in U(\mathfrak{sl}_2),
\]
\[
\tilde{\Lambda}_{a'} := \lambda_{a'}\left(-1 + \frac{a'}{2}, 4g^2, \frac{a}{2}\right).
\]
Then
\[
y(y - 1)H^{\varepsilon}_2(\lambda) = y^{-\frac{1}{2}}(a'-\frac{1}{2})(\varpi_{a'}(\tilde{\mathcal{K}}) - \tilde{\Lambda}_{a'})y^{\frac{1}{2}}(a'-\frac{1}{2}). \tag{3.6}\]

4 Degeneracies of the spectrum and constraint polynomials

The constraint polynomials for the AQRM were originally defined by Li and Batchelor [37], following the work of Kuś [34] on the (symmetric) quantum Rabi model (QRM). In [63, 62], these polynomials were derived in the framework of finite-dimensional irreducible representations of \(\mathfrak{sl}_2\) in the confluent Heun picture of the AQRM. The zeros of the constraint polynomials give the Juddian, or quasi-exact, solutions of the model. In this section, we give the definition in terms of recurrence relations directly and refer the reader to the cited works for the explicit derivation (see Proposition 6.1 for another derivation).
**Definition 4.1.** Let $N \in \mathbb{Z}_{\geq 0}$. The polynomials $P_k^{(N,\varepsilon)}(x, y)$ of degree $k$ are defined recursively by

\[
P_0^{(N,\varepsilon)}(x, y) = 1, \\
P_1^{(N,\varepsilon)}(x, y) = x + y - 1 - 2\varepsilon, \\
P_k^{(N,\varepsilon)}(x, y) = (kx + y - k(k + 2\varepsilon))P_{k-1}^{(N,\varepsilon)}(x, y) - k(k-1)(N - k + 1)xP_{k-2}^{(N,\varepsilon)}(x, y),
\]

for $k \geq 2$.

**Example 4.1.** For $k = 2, 3$, we have

\[
P_2^{(N,\varepsilon)}(x, y) = 2x^2 + 3xy + y^2 - 2(N + 2(1 + 2\varepsilon))x - (5 + 6\varepsilon)y + 4(1 + 3\varepsilon + 2\varepsilon^2), \\
P_3^{(N,\varepsilon)}(x, y) = 6x^3 + 11x^2y + 6xy^2 + y^3 - 6(2N + 3(1 + 2\varepsilon))x^2 - 2(7 + 6\varepsilon)y^2 \\
- 2(4N + 17 + 22\varepsilon)xy + 6(2N + 3(1 + 2\varepsilon))(2 + 2\varepsilon)x \\
+ (49 + 4\varepsilon(24 + 11\varepsilon))y - 6(1 + 2\varepsilon)(2 + 2\varepsilon)(3 + 2\varepsilon).
\]

When $k = N$, the polynomial $P_k^{(N,\varepsilon)}(x, y)$ is called constraint polynomial. Actually, for a fixed value $y = \Delta^2$, if $x = (2g)^2$ is a root of $P_N^{(N,\varepsilon)}(x, y)$, then $\lambda = N + \varepsilon - g^2$ is an exceptional eigenvalue corresponding to a Juddian solution for $H_{\text{Rabi}}^\varepsilon$. Likewise, if $x = (2g)^2$ is a root of $P_N^{(N,\varepsilon)}(x, y) := P_N^{(N-\varepsilon)}(x, y)$ (62 52), then $\lambda = N - \varepsilon - g^2$ is an exceptional eigenvalue corresponding to a Juddian solution of $H_{\text{Rabi}}^\varepsilon$. Mathematically, the constraint polynomial $P_N^{(N,\varepsilon)}(x, y)$ possesses certain particular properties not shared with $P_k^{(N,\varepsilon)}(x, y)$ with $k \neq N$, these are studied in 4.1.

The main objective is to prove the following conjecture.

**Conjecture 4.1** ([62]). For $\ell, N \in \mathbb{Z}_{\geq 0}$, there exists a polynomial $A_N^\ell(x, y) \in \mathbb{Z}[x, y]$ such that

\[
P_N^{(N+\ell,-\ell/2)}(x, y) = A_N^\ell(x, y)P_N^{(N,\ell/2)}(x, y).
\]

(4.1)

Moreover, the polynomial $A_N^\ell(x, y)$ is positive for any $x, y > 0$. □

If the conjecture holds and $g, \Delta > 0$ satisfy $P_N^{(N,\ell/2)}((2g)^2, \Delta^2) = 0$, the exceptional eigenvalue $\lambda = N + \ell/2 - g^2 = (N + \ell) - \ell/2 - g^2$ of $H_{\text{Rabi}}^\ell$ is degenerate.

Actually, in order to complete the argument, it is necessary to show that the associated Juddian solutions are linearly independent. The outline of the proof is as follows. The main point is that each root $x, y > 0$ of a constraint polynomial $P_N^{(N,\ell/2)}(x, y)$ determines an eigenvector in a finite dimensional representation space ($F_{2m}$ or $F_{2m+1}$ depending on the parity of $N$) associated with the exceptional eigenvalue $\lambda = N + \varepsilon - x$. For instance, suppose $N = 2m \in \mathbb{Z}_{\geq 0}$ and $\varepsilon = \ell \in \mathbb{Z}_{\geq 0}$ and that $x, y > 0$ make $P_{2m}^{(2m,\ell)}(x, y)$ vanish. By Section 5.1 of [62], the eigenvalue $\lambda = 2m + \ell - x$ has a corresponding eigenvector $\nu \in F_{2m+1}$. Moreover, under the assumption of the conjecture, $P_{2m+2\ell-\ell}^{(2m+2\ell,\ell)}(x, y)$ vanishes as well, therefore the eigenvalue $\lambda = 2m + \ell - x$ also has the eigenvector $\tilde{\nu} \in F_{2m+2\ell} = F_{2(m+\ell)}$. For any $\ell \in \mathbb{Z}$, it is clear that $F_{2m+1} \not\cong F_{2(m+\ell)}$ so the eigenvectors $\nu$ and $\tilde{\nu}$ (and hence the associated Juddian solutions) are linearly independent. The linear independence
in the remaining cases is shown in an completely analogous way, with the exception of the case $\varepsilon = 1/2$, where we direct the reader to Proposition 6.6 of [62] for the proof.

The condition $A_N^\ell(x, y) > 0$ of Conjecture 4.1 ensures that for $N \in \mathbb{Z}_{\geq 0}$ there are no non-degenerate exceptional eigenvalues $\lambda = N + \ell/2 - g^2$ corresponding to Juddian solutions (see Corollary 4.15 below).

We prove Conjecture 4.1 in two parts. We show the existence of the polynomial $A_N^\ell(x, y)$ by showing that $P_{N/2}^{(N/2)}(x, y)$ divides $P_{N+\ell/2-\ell/2}^{(N+\ell/2)}(x, y)$ (as polynomials in $\mathbb{Z}[x, y]$) in 4.2. Additionally, this method gives an explicit determinant expression for the polynomial $A_N^\ell(x, y)$. The proof is completed in 4.3 by studying the eigenvalues of the matrices involved in the determinant expressions for $A_N^\ell(x, y)$.

For convenience, we introduce here certain notations used throughout this paper. For a tridiagonal matrix, we put

$$
\text{tridiag} \left[ \begin{array}{ccc}
  a_i & b_i \\
  c_i & & \\
  & & \\
  & & \\
  & & \\
  & & 
\end{array} \right]_{1 \leq i \leq n} :=
\begin{bmatrix}
  a_1 & b_1 & 0 & 0 & \cdots & 0 \\
  c_1 & a_2 & b_2 & 0 & \cdots & 0 \\
  0 & c_2 & a_3 & b_3 & \cdots & 0 \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \cdots \\
  0 & \cdots & 0 & c_{n-2} & a_{n-1} & b_{n-1} \\
  0 & \cdots & 0 & 0 & c_{n-1} & a_n
\end{bmatrix}.
$$

The symbol $(a)_n$ denotes the Pochhammer symbol, or raising factorial, that is,

$$(a)_n := a(a + 1) \cdots (a + n - 1) = \frac{\Gamma(a + n)}{\Gamma(a)}$$

for $a \in \mathbb{C}$ and a non-negative integer $n$.

4.1 Determinant expressions of constraint polynomials

It is well-known that orthogonal polynomials can be expressed as determinants of tridiagonal matrices. Those determinant expressions are derived from the fact that orthogonal polynomials satisfy three-term recurrence relations. It is not difficult to verify that the polynomials $\{P_k^{(N,\varepsilon)}(x, y)\}_{k \geq 0}$ do not constitute families of orthogonal polynomials with respect to either of their variables (in a standard sense). Nevertheless, since they are defined by three-term recurrence relations we can derive determinant expressions using the same methods. We direct the reader to [14] or [29] for the case of orthogonal polynomials.

Let $N \in \mathbb{Z}_{\geq 0}$ and $\varepsilon \in \mathbb{R}$ be fixed, by setting $c^{(\varepsilon)}_k = k(k+2\varepsilon)$ and $\lambda_k = k(k-1)(N-k+1)$ ($k \in \mathbb{Z}$), the family of polynomials $\{P_k^{(N,\varepsilon)}(x, y)\}$ is given by the three-term recurrence relation

$$
P_k^{(N,\varepsilon)}(x, y) = (kx + y - c^{(\varepsilon)}_k)P_{k-1}^{(N,\varepsilon)}(x, y) - \lambda_kP_{k-2}^{(N,\varepsilon)}(x, y),
$$

for $k \geq 2$, with initial conditions $P_1^{(N,\varepsilon)}(x, y) = x + y - c^{(\varepsilon)}_1$ and $P_0^{(N,\varepsilon)}(x, y) = 1$. Hence, the polynomial $P_k^{(N,\varepsilon)}(x, y)$ is the determinant of a $k \times k$ tridiagonal matrix

$$
P_k^{(N,\varepsilon)}(x, y) = \det(I_k y + A_k^{(N)} x + U_k^{(\varepsilon)}) \quad (4.2)
$$

where $I_k$ is the identity matrix of size $k$ and

$$A_k^{(N)} = \text{tridiag} \left[ \begin{array}{ccc}
  i & 0 \\
  \lambda_i & & \\
  & & \\
  & & \\
  & & \\
  & & 
\end{array} \right]_{1 \leq i \leq k}, \quad U_k^{(\varepsilon)} = \text{tridiag} \left[ \begin{array}{ccc}
  -c^{(\varepsilon)}_i & 0 \\
  & 1 \\
  & & \\
  & & \\
  & & \\
  & & 
\end{array} \right]_{1 \leq i \leq k}.$$
Proof. We have to check that \((4.4)\) is equal to zero.

Using the elementary relations
\[
-e_i \cdot e_{i+1} + j(2(N-j) + 1 + 2\varepsilon)e_{ij} - e_{i,j-1} - j(1+1)\varepsilon e_{i,j+1}.
\] (4.4)

Using the elementary relations
\[
j(j+1)e_{N-j}e_{i,j+1} = -(i-j)(N-j+2\varepsilon)e_{ij},
\]
\[
ev_{i+1,j} - e_{i,j-1} = (i^2 + j^2 + ij - j - iN - jN)e_{ij},
\]
we immediately see that \((4.4)\) is equal to zero. \(\square\)
Recall that the determinant $J_n$ of a tridiagonal matrix

$$J_n = \det \text{tridiag} \begin{bmatrix} a_i & b_i \\ c_i & \end{bmatrix}$$

is called **continuant** (see [12]). It satisfies the three-term recurrence relation

$$J_n = a_n J_{n-1} - b_{n-1} c_{n-1} J_{n-2},$$

(4.5)

with initial condition $J_{-1} = 0, J_0 = 1$. As a consequence of this, notice that the continuant equivalence

$$\det \text{tridiag} \begin{bmatrix} a_i & b_i \\ c_i & \end{bmatrix}$$

1 \leq i \leq n

$$= \det \text{tridiag} \begin{bmatrix} a'_i & b'_i \\ c'_i & \end{bmatrix}$$

1 \leq i \leq n

(4.6)

holds whenever $b_i c_i = b'_i c'_i$ for all $i = 1, 2, \cdots, n - 1$, since the continuants on both sides of the equation define the same recurrence relations with the same initial conditions.

**Corollary 4.4.** Let $N \in \mathbb{Z}_{\geq 0}$. We have

$$P_N^{(N,\varepsilon)}(x,y) = \det \left( I_N y + D_N x + S_N^{(N,\varepsilon)} \right),$$

where $D_N$ is the diagonal matrix of Proposition 4.2 and $S_N^{(N,\varepsilon)}$ is the symmetric matrix given by

$$S_N^{(N,\varepsilon)} = \text{tridiag} \begin{bmatrix} -i(2(N-i)+1+2\varepsilon) & \sqrt{i(i+1)c_{N-i}^{(\varepsilon)}} \\ & \sqrt{i(i+1)c_{N-i}^{(\varepsilon)}} & \end{bmatrix}

1 \leq i \leq N.$$

**Proof.** Notice that the matrices $I_N y + D_N x + C_N^{(N,\varepsilon)}$ and $I_N y + D_N x + S_N^{(N,\varepsilon)}$ are tridiagonal. Then, it is clear by the continuant equivalence (4.6) that the determinants of the matrices are equal, establishing the result.

As a corollary to the discussion on the determinant expression (4.2) we have the following result used in §4.3 to prove the positivity of the polynomial $A_N^k(x,y)$.

**Corollary 4.5.** For $x \geq 0, \varepsilon \in \mathbb{R}$ and $N, k \in \mathbb{Z}_{\geq 0}$, all the roots of $P_k^{(N,\varepsilon)}(x,y)$ with respect to $y$ are real.

**Proof.** When $x \geq 0$, using the continuant equivalence (4.6) on the determinant expression (4.2) of $P_k^{(N,\varepsilon)}(x,y)$ we can find an equivalent expression $\det(I_k y - V_k(x))$ for a real symmetric matrix $V_k(x)$. Since the roots of $P_k^{(N,\varepsilon)}(x,y)$ with respect to $y$ are the eigenvalues of the real symmetric matrix $V_k(x)$, the result follows immediately.

In the case of the constraint polynomials $P_N^{(N,\varepsilon)}(x,y)$, the determinant expression of Corollary 4.4 gives the following result of similar type, used for the estimation of positive roots of constraint polynomials in §5.2.

**Theorem 4.6.** Let $N \in \mathbb{Z}_{\geq 0}$ and $\varepsilon > -1/2$. Then, for fixed $x \in \mathbb{R}$ (resp. $y \in \mathbb{R}$), all the roots of $P_N^{(N,\varepsilon)}(x,y)$ with respect to $y$ (resp. $x$) are real.
Proof. Upon setting \( x = \alpha \in \mathbb{R} \), the zeros of \( P_N^{(N,\varepsilon)}(\alpha, y) \) are the eigenvalues of the matrix \(-D_N \alpha + S_N^{(N,\varepsilon)}\). For \( \varepsilon > -\frac{1}{2} \), the matrix is real symmetric, so the eigenvalues, therefore the zeros, are real. The case of \( y = \beta \in \mathbb{R} \) is completely analogous since \( P_N^{(N,\varepsilon)}(x, \beta) = \det D_N \det(I_Nx + D_N^{-1} \beta + D_N^{-1/2} S_N^{(N,\varepsilon)} D_N^{-1/2}) \). \( \square \)

The next example shows that we should not expect a determinant expression of the type of Corollary 4.4 for general \( P_k^{(N,\varepsilon)}(x, y) \) with \( k \neq N \).

**Example 4.2.** For a fixed \( y \), the roots of the polynomial
\[
P_2^{(6,0)}(x, y) = 2x^2 + y^2 - 16x + 3xy - 5y + 4,
\]
are given by
\[
\frac{1}{4} \left( 16 - 3y \pm \sqrt{y^2 - 56y + 224} \right).
\]
Clearly, the roots are not real for every value \( y \in \mathbb{R} \).

### 4.2 Divisibility of constraint polynomials

In this subsection, we study the case where \( \varepsilon \) is a negative half-integer. In this case, the determinant expressions of \([4.4]\) give the proof of the divisibility in Conjecture 4.1.

First, from the determinant expression for \( P_N^{(N,\varepsilon)}(x, y) \) given in Corollary 4.4, by means of the continual equivalence (4.6) and elementary determinant operations it is not difficult to see that
\[
P_N^{(N,\varepsilon)}(x, y) = N! \det \left( I_Nx + D_N^{-1}y + V_N^{(N,\varepsilon)} \right),
\]
where
\[
V_N^{(N,\varepsilon)} = \text{tridiag} \left[ -2(N - i) - 1 - 2\varepsilon \right] \sqrt{c_{N-i}^{(\varepsilon)}} 1 \leq i \leq N.
\]

For \( N, \ell \in \mathbb{Z}_{\geq 0} \), the expression above reads
\[
P_{N+\ell}^{(N+\ell,\varepsilon)}(x, y) = (N + \ell)! \det \left( I_{N+\ell}x + D_{N+\ell}^{-1}y + V_{N+\ell}^{(N+\ell,\varepsilon)} \right). \tag{4.8}
\]
Noting that \( c_{\ell}^{(-\ell/2)} = \ell(-\ell) = 0 \), the matrix \( V_{N+\ell}^{(N+\ell,-\ell/2)} \) has the block-diagonal form
\[
V_{N+\ell}^{(N+\ell,-\ell/2)} = \begin{bmatrix} \mathbf{L}V_N^{(N+\ell,-\ell/2)} & \mathbf{O}^{(N+\ell,-\ell/2)}_{N,\ell} \\ \mathbf{O}_{\ell,N} & \mathbf{R}V_{\ell}^{(N+\ell,-\ell/2)} \end{bmatrix},
\]
where \( \mathbf{O}_{n,m} \) is the \( n \times m \) zero matrix. Next, by setting
\[
D_{\ell}^{(N)} = \text{diag} \left( \frac{1}{N+1}, \frac{1}{N+2}, \ldots, \frac{1}{N+\ell} \right),
\]
immediately it follows that
\[
P_{N+\ell}^{(N+\ell,-\ell/2)}(x, y) = (N + \ell)! \det \left( I_{N+\ell}x + D_N^{-1}y + \mathbf{L}V_N^{(N+\ell,-\ell/2)} \right) \times \det \left( I_{\ell}x + D_{\ell}^{(N)}y + \mathbf{R}V_{\ell}^{(N+\ell,-\ell/2)} \right).
\]
For \(i = 1, 2, \ldots, N\), the \(i\)-th diagonal element of \(\mathbf{LV}^{(N+\ell,-\ell/2)}_N\) is
\[ -(2(N + 1 + \ell - i) - 1 - \ell) = -(2(N + 1 - i) - 1 + \ell) \]
and the off-diagonal elements are \(c_{N+\ell-i}^{(-\ell/2)} = c_{N-i}^{(\ell/2)}\). Therefore,
\[ \mathbf{LV}^{(N+\ell,-\ell/2)}_N = \mathbf{V}^{(N,\ell/2)}_N, \]
and then, from \([4,7]\) we have
\[ P^{(N+\ell,-\ell/2)}_{N+\ell}(x, y) = P^{(N,\ell/2)}_N(x, y) \frac{(N + \ell)!}{N!} \det \left( \mathbf{I}_\ell x + \mathbf{D}^{(N)}_\ell y + \mathbf{R}_\ell^{(N+\ell,-\ell/2)} \right). \]
Let \(A_N^{(\ell)}(x, y) = \frac{(N+\ell)!}{N!} \det \left( \mathbf{I}_\ell x + \mathbf{D}^{(N)}_\ell y + \mathbf{R}_\ell^{(N+\ell,-\ell/2)} \right). \) By expanding the determinant as a recurrence relation (cf. \([4,5]\)) or by appealing to Gauss’ lemma, it is easy to see that \(A_N^{(\ell)}(x, y)\) is a polynomial with integer coefficients. Therefore, the discussion above proves the following theorem.

**Theorem 4.7.** For \(N, \ell \in \mathbb{Z}_{\geq 0}\), there is a polynomial \(A_N^{(\ell)}(x, y) \in \mathbb{Z}[x, y]\) such that
\[ P^{(N+\ell,-\ell/2)}_{N+\ell}(x, y) = A_N^{(\ell)}(x, y)P^{(N,\ell/2)}_N(x, y). \]
Furthermore, \(A_N^{(\ell)}(x, y)\) is given by
\[
\frac{(N + \ell)!}{N!} \det \text{tridiag} \left[ x + \frac{y}{N+i} - \ell + 2i - 1 \sqrt{c_{N-i}^{(\ell/2)}} \right]_{1 \leq i \leq \ell}.
\]
To complete the proof of Conjecture \([4.1]\) it remains to prove that \(A_N^{(\ell)}(x, y) > 0\) for \(x, y > 0\). This is done in \([4.3]\) below.

**Example 4.3** \([52]\). For small values of \(\ell\), the explicit form of \(A_N^{(\ell)}(x, y)\) is given by
\[
A_N^{(1)}(x, y) = (N + 1)x + y,
\]
\[
A_N^{(2)}(x, y) = (N + 1)2x^2 + \left( \sum_{i=1}^{2} (N + i) \right) xy + y(1 + y),
\]
\[
A_N^{(3)}(x, y) = (N + 1)3x^3 + \left( \sum_{i=1}^{3} (N + i)(N + j) \right) x^2y
\]
\[+ (N + 2)x(3y + 4)y + y(2 + y)^2,
\]
\[
A_N^{(4)}(x, y) = (N + 1)4x^4 + \left( \sum_{i=1}^{4} (N + i)(N + j)(N + j) \right) x^3y
\]
\[+ \left( \sum_{i=1}^{4} (N + i)(N + j) \right) x^2y^2 + 2 \left( \sum_{i=1}^{4} (N + i)(N + j) - (N + 2)(N + 3) \right) x^2y
\]
\[+ \left( \sum_{i=1}^{4} (N + i) \right) xy(y + 2)(y + 3) + y(3 + y)^2(4 + y).
\]
For a fixed degree $\ell \in \mathbb{Z}_{\geq 0}$, the polynomial equation $A_N^\ell(x,y) = 0$ defines certain algebraic curve depending on the parameter $N$: the case $\ell = 2$ is parabolic, $\ell = 3$ gives an elliptic curve and $\ell = 4$ is super elliptic, and so on.

For instance, let us consider the case $\ell = 3$. Here, by using the change of variable $X = -x/y$ and $Y = 1/y$, the equation $A_N^3(x,y) = 0$ turns out to be

$$4Y^2 + 4Y - 4(N + 2)XY = (N + 1)X^3 - (11 + 3N(N + 4))X^2 + 3(N + 2)X - 1,$$

which is easily seen to be (birationally) equivalent to the elliptic curve in Legendre form (cf. [33]).

$$Y_1^2 = X_1(X_1 - 1)\left(1 - \frac{(N + 2)^2}{(N + 1)(N + 3)}\right).$$

with variables $X_1 = \frac{X}{N+2}$ and $Y_1 = \frac{(N+2)}{\sqrt{(N+1)(N+3)}}(2Y - (N + 2)X + 1)$.

4.3 Proof of the positivity of $A_N^\ell(x,y)$

In this subsection we complete the proof of Conjecture 4.1 by proving the positivity of the polynomial $A_N^\ell(x,y)$ for $x,y > 0$. Let $N \in \mathbb{Z}_{\geq 0}$ and $\ell \in \mathbb{Z}_{>0}$ be fixed. From Theorem 4.7 and the continuant equivalence (4.6), we see that the polynomial $A_N^\ell(x,y)$ has the determinant expression

$$\frac{(N + \ell)!}{N!} \det(D_N^{(N)}y + B_\ell(x))$$

where $B_\ell(x)$ is an matrix-valued function given by

$$B_\ell(x) = \text{tridiag} \left[ x - \ell + 2i - 1 \left\| \begin{array}{c} \frac{c^{(\ell/2)}_{-i}}{c^{(\ell/2)}_{i}} \end{array} \right. \right] \quad (4.9)$$

Next, multiplying the $\frac{(N+\ell)!}{N!}$ factor into the determinant in such a way that the $i$-th row is multiplied by $N + i$, we obtain the expression

$$A_N^\ell(x,y) = \det(I_{\ell}y + M_N^{(N)}(x)) = \prod_{\lambda \in \text{Spec}(M_N^{(N)}(x))} (y + \lambda)$$

with

$$M_N^{(N)}(x) = \text{tridiag} \left[ (N + i)(x - \ell + 2i - 1) \left\| \begin{array}{c} N + i \\ (N + i + 1)c_{-i}^{(\ell/2)} \end{array} \right. \right] \quad (4.10)$$

Thus, it suffices to show that all the eigenvalues of $M_N^{(N)}(x)$ are positive for $x > 0$ to prove that $A_N^\ell(x,y) > 0$ when $x, y > 0$.

First, we compute the determinant of the matrix $M_N^{(N)}(x)$, or equivalently, the value of $A_N^\ell(x,0)$.

**Lemma 4.8.** We have

$$\text{det}(M_N^{(N)}(x)) = A_N^\ell(x,0) = \frac{(N + \ell)!}{N!} x^{\ell}.$$
Proof. Consider the recurrence relation

\[ J_i(x) = (x + \ell + 1 - 2i)J_{i-1}(x) + (i - 1)(\ell - 1 - i)J_{i-2}(x), \]

with initial conditions \( J_0(x) = 1 \) and \( J_{-1}(x) = 0 \). Notice that this recurrence relation corresponds to the continuant \( \det B_\ell(x) \) (compare with (4.9) above) and therefore,

\[ \left(\frac{N+\ell}{N}\right)! J_i(x) = \left(\frac{N+\ell}{N}\right)! \det B_\ell(x) = \det(M^{(N)}_\ell(x)). \]

We claim that \( J_i(x) = \sum_{j=0}^i (\ell - i)j!x^{i-j} \). Clearly, the claim holds for \( J_0(x) = 1 \) and \( J_1(x) = x + \ell - 1 \). Assuming it holds for integers up to a fixed \( i \), then \( J_{i+1}(x) \) is given by

\[
(x + \ell - 1 - 2i) \sum_{j=0}^i (\ell - i)j!(\ell - i - j)x^j + i(\ell - i) \sum_{j=0}^{i-1} (\ell - i + 1)j!(\ell - i - 1 - j)x^{i-1-j} + i(\ell - i) \sum_{j=0}^{i-1} (\ell - i + 1)j!(\ell - i - 1 - j)x^{i-1-j},
\]

by grouping the terms in the sums we obtain

\[
x^{i+1} + (\ell - i - 1)x^i + (\ell - i - 1)x^{i+1} + \sum_{j=1}^{i-1} (\ell - i)j! \left( (\ell - i + j) \left( \frac{i}{j+1} \right) + (\ell - 1 - 2i) \left( \frac{i}{k} + j \left( \frac{i}{j} \right) \right) \right)x^{i-j}.
\]

The sum on the right is

\[
\sum_{j=1}^{i-1} (\ell - i)j! \left( (\ell - i + j)(i - j) \right) \frac{i}{j+1} + \ell - 1 - 2i + j \right)x^{i-j}
\]

\[
= \sum_{j=1}^{i-1} (\ell - i)j! \left( (i + 1)(\ell - i - 1) \right) \frac{i}{j+1} \right)x^{i-j}
\]

and the claim follows by joining the remaining terms into the sum. Finally, notice that \( J_\ell(x) = \sum_{j=0}^\ell (0)j!x^{\ell-j} = x^\ell \), as desired. \( \Box \)

Remark 4.1. The lemma above is a generalization of the case \( N = 0 \) studied in [52] (Prop. 4.1) using continued fractions. It would be interesting to study the combinatorial properties of the coefficients of the polynomials \( A^N_{x,y}(x) \) using the determinant expressions given above.

From the lemma above, we immediately obtain the following result.

Corollary 4.9. For \( N \in \mathbb{Z}_{\geq 0} \), the eigenvalue \( \lambda = 0 \) is in \text{Spec}(M^{(N)}_\ell(x)) if and only if \( x = 0 \).

The next result collects some basic properties of the eigenvalues of the matrix \( M^{(N)}_\ell(x) \) that are used in the proof of the positivity of \( A^N_{x,y}(x) \).
Lemma 4.10. Denote the spectrum of the matrix $M^{(N)}(x)$ by $\text{Spec}(M^{(N)}(x))$.

(1) For $x \geq 0$, the eigenvalues $\lambda \in \text{Spec}(M^{(N)}(x))$ are real.

(2) We have $\text{Spec}(M^{(N)}(0)) = \{i(\ell-i) : i = 1, 2, \ldots, \ell\}$. In particular, $0 \in \text{Spec}(M^{(N)}(0))$ is a simple eigenvalue and any eigenvalue $\lambda \in \text{Spec}(M^{(N)}(0))$ satisfies $\lambda \geq 0$.

(3) If $x' > \ell - 1$, all eigenvalues $\lambda \in \text{Spec}(M^{(N)}(x'))$ satisfy $\lambda > 0$.

Proof. Note that by Corollary 4.5 and the divisibility of Theorem 4.7, if $x \geq 0$ all the roots of $A^{(N)}_x(x,y)$ with respect to $y$ are real. By definition, the same holds for the elements of $\text{Spec}(M^{(N)}(x))$, proving the first claim. From the defining recurrence relation, we see that $P^{(N,x)}_N(0,y) = \prod_{i=1}^{N} (y-i(2\pi))$, and by divisibility we have $A^{(N)}_x(0,y) = \prod_{i=1}^{\ell} (y-i(\ell))$ proving the second claim. For the third claim, notice that when $x' > \ell - 1$ all the diagonal elements of $M^{(N)}(x')$ are positive. Therefore, the continuant (4.10) defines a recurrence relation with positive coefficients, so that $A^{(N)}_x(x',y)$ is a polynomial in $y$ with positive coefficients and real roots. Since $y = 0$ is not a root of $A^{(N)}_x(x',y)$ by Corollary 4.9, all of the roots of $A^{(N)}_x(x',y)$ must be negative and the third claim follows.

With these preparations, we come to the proof of the positivity of the polynomial $A^{(N)}_x(x,y)$

Theorem 4.11. With the notation of Theorem 4.7, $A^{(N)}_x(x,y) > 0$ for $x, y > 0$.

Proof. By virtue of (4.10), it is enough to show that all the eigenvalues of $M^{(N)}(x)$ are positive if $x > 0$. Notice that each eigenvalue of $M^{(N)}(x)$ is a real-valued continuous function in $x$. Assume that there is a positive $x'$ such that $M^{(N)}(x')$ has a negative eigenvalue. Then, there also exists $x''$ such that $x' < x'' < \ell$ and $0 \in \text{Spec}(M^{(N)}(x''))$ since all eigenvalues of $M^{(N)}(\ell)$ are positive by Lemma 4.10 (3). This contradicts to Corollary 4.9.

The proof of Conjecture 4.11 which we reformulate as a theorem below, is completed by Theorems 4.7 and 4.11.

Theorem 4.12. For $\ell, N \in \mathbb{Z}_{\geq 0}$, there exists a polynomial $A^{(N)}_x(x,y) \in \mathbb{Z}[x,y]$ such that

$$P^{(N+\ell, -\ell/2)}((2g)^2, \Delta^2) = A^{(N)}_x((2g)^2, \Delta^2)P^{(N, -\ell/2)}((2g)^2, \Delta^2).$$

(4.11)

for $g, \Delta > 0$. Moreover, the polynomial $A^{(N)}_x(x,y)$ is positive for any $x, y > 0$.

A consequence of the positivity of $A^{(N)}_x(x,y)$ in Theorem 4.12 is that all the positive roots of the constraint polynomials $P^{(N+\ell/2)}(x,y)$ and $P^{(N+\ell, -\ell/2)}(x,y)$ ($N, \ell \in \mathbb{Z}_{\geq 0}$) must coincide. This explains the fact that the two curves defined by the constraint polynomials in Figure 3 appear to coincide when $\varepsilon = \ell/2$ ($\ell \in \mathbb{Z}$).

Note that since $A^{(N)}_x(x,y) = P^{(\ell, -\ell/2)}_0(x,y)$ and $P^{(0, -\ell/2)}_0(x,y) = 1 \neq 0$, the positivity of $A^{(N)}_x(x,y)$ also implies the nonexistence of Juddian eigenvalues $\lambda = \ell/2 - y^2$ for $\ell > 0$. In fact, the positivity can be extended to a larger set of constraint polynomials $P^{(k, -\ell/2)}_k(x,y)$.

Proposition 4.13. Let $\ell \in \mathbb{Z}_{\geq 0}$ and $1 \leq k \leq \ell$. Then the constraint polynomial $P^{(k, -\ell/2)}_k(x,y)$ is positive for $x, y > 0$. 


Proof. For $1 \leq k \leq \ell$, define the $k \times k$ matrix

$$M_k(x) = \text{tridiag} \left[ \begin{array}{cccc}
x + \ell - 1 - 2(k - i) & i \\ (i + 1) \epsilon_{k-i}^{(-\ell/2)} & \ddots & i \\ & \ddots & \ddots & \ddots \\ & & \ddots & \ddots & i \\ & & & \ddots & \ddots \\ & & & & x + \ell - 1 - 2(k - i)
\end{array} \right]_{1 \leq i \leq k}$$

then $P_k^{(\ell,-\ell/2)}(x,y) = \det(I_k y + M_k(x))$ and the roots of $P_k^{(\ell,-\ell/2)}(x,y)$ with respect to $y$ are the eigenvalues of the matrix $-M_k(x)$. Thus, as in the case of $A_N^{(\ell)}(x,y)$, it suffices to prove that all the eigenvalues of $M_k(x)$ are positive for $x > 0$.

First, from (4.7), we see that $\det(M_k(x)) = P_k^{(\ell,-\ell/2)}(x,0) = k! \sum_{j=0}^{\ell} (-k-j)(\ell) x^{k-j}$. Indeed, we verify that $\det(M_k(x)) = k! J_k(x)$, where $\{J_i(x)\}_{i \geq 0}$ is the recurrence relation defined in Lemma 4.8. In particular, $\det(M_k(x))$ is a polynomial with positive coefficients and thus it never vanishes for $x > 0$.

Next, we verify that the matrix $M_k(x)$ has the properties of the matrices $M_{\ell}^{(N)}(x)$ given in Lemma 4.10. From Corollary 4.5 it is clear that for $x \geq 0$ the eigenvalues of $M_k(x)$ are real. By the definition of the constraint polynomials, it is obvious that $\text{Spec}(M_k(0)) = \{i(\ell - i) : i = 1, 2, \ldots, k\}$, hence any eigenvalue $\lambda \in \text{Spec}(M_k(0))$ is non-negative. Finally, as in the proof of Lemma 4.10 we see that for $x' > \max(0,2k - \ell - 1)$ all eigenvalues $\lambda \in \text{Spec}(M_k(x'))$ satisfy $\lambda > 0$.

The proof of positivity then follows exactly as in the proof of Theorem 4.11.

Corollary 4.14. For $\ell \in \mathbb{Z}_{>0}$ and $0 \leq k \leq \ell$ there are no Juddian eigenvalues $\lambda = k - \ell/2 - g^2$ in $H_\text{Rabi}^{(\ell/2)}$.

Proof. The case $k = \ell$ was already proved in the discussion above and the case $k = 0$ is trivial since $P_0^{(0,-\ell/2)}((2g)^2, \Delta^2) = 1 \neq 0$. For $1 \leq k < \ell$, if $\lambda = k - \ell/2 - g^2$ is a Juddian eigenvalue then $P_k^{(\ell,-\ell/2)}((2g)^2, \Delta^2) = 0$ for some parameters $g, \Delta > 0$. This is a contradiction to Proposition 4.13. Note that in this case there is no possibility of a contribution of Juddian eigenvalues by roots of constraint polynomials $P_N^{(\ell,\ell/2)}((2g)^2, \Delta^2)$ as this would necessarily require $N = k - \ell < 0$.

Remark 4.2. In Proposition 5.8 of [62], it is shown that the roots of the constraint polynomials $P_N^{(\ell,\ell/2)}(x,y)$ are simple. In particular, this implies that for $\varepsilon \notin \frac{1}{2} \mathbb{Z}$, there are no degenerate exceptional eigenvalues consisting of two Juddian solutions.

Since the multiplicity of the eigenvalues is at most two, as a corollary of the divisibility in Theorem 4.12 and Corollary 4.14 we have the following result.

Corollary 4.15. If $x = (2g)^2$ is a root of the equation $P_N^{(\ell,\ell/2)}(x, \Delta^2) = 0$, then the (Juddian) eigenvalue $\lambda = N + \ell/2 - g^2$ must be a degenerate exceptional eigenvalue. In fact, the multiplicity of the exceptional eigenvalue $\lambda$ is exactly 2 and the two linearly independent solutions are Juddian (see Figure 2(b)).

Remark 4.3. What the corollary means is, although a non-Juddian exceptional eigenvalue may exist on the energy curve $E = N + \ell - \varepsilon - g^2$ (resp. $E = N + \varepsilon - g^2$) (see Figure 2(a)) for $0 < |\varepsilon - \ell/2| < \delta$ for sufficiently small $\delta$, as the numerical result in [38] suggests, the non-Juddian exceptional eigenvalues disappear when $\varepsilon = \ell/2 \in \mathbb{Z}_{>0}$ and the exceptional eigenvalue $\lambda := E$ is Juddian.
Remark 4.4. The mathematical model known as the non-commutative harmonic oscillator (NcHO) [48] (see [45] for a detailed study and information of the NcHO with references therein, and [46] for a recent development) is given by

\[
Q = Q_{\alpha, \beta} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \left( -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} x^2 \right) + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left( x \frac{d}{dx} + \frac{1}{2} \right).
\]

The NcHO is a self-adjoint ordinary differential operator with a \( \mathbb{Z}_2 \) -symmetry that generalizes the quantum harmonic oscillator by introducing an interaction term. When the parameters \( \alpha, \beta > 0 \) satisfy \( \alpha \beta > 1 \), the Hamiltonian \( Q \) is positive definite, whence it has only positive (discrete) eigenvalues. It is known [59] that the multiplicity of the eigenvalues is at most 2. Moreover, the possibilities that an eigenstate of \( Q \) is degenerate (2 dimensional) are the following two cases [61]:

- a quasi-exact (Juddian) solution and a non-Juddian solution with the same parity (in this case the eigenvalue \( \lambda \) is of the form \( \lambda = 2 \sqrt{\frac{\alpha \beta}{\alpha + \beta}} (m + \frac{1}{2}) \) for some \( m \in \mathbb{Z}_{\geq 0} \),
- two non-quasi-exact solutions with different parity.

There is a close connection between the NcHO and the quantum Rabi model [61], arising from their representation theoretical pictures via a confluent process for the Heun ODE (see [55]). It is desirable to clarify the reason concerning the difference of the structure of the degeneracies between the NcHO and the QRM (also AQRM for \( \varepsilon \in \frac{1}{2} \mathbb{Z}_{\geq 0} \); see Theorem 6.3 in [62]). Actually, the degeneracies occur only for quasi-exact solutions in both models and those are considered to be remains of the eigenvalues of the quantum harmonic oscillator. Therefore, it is quite interesting to develop a similar discussion for constraint polynomials in the former “exceptional” case for the NcHO in [61].

Remark 4.5. By Lemma 2.1 when \( m \in \mathbb{Z}_{\geq 0} \) the space \( F_{2m+1} \) is realized as an invariant subspace of \( V = V_{1,2-2m} \) while the space \( D_{2m}^{-} \oplus D_{2m}^{+} \) is just obtained as a subquotient of \( F_{2m+1} \), that is, \( D_{2m}^{\pm} \) do not constitute invariant spaces of \( V_{1,2-2m} \). In other words, \( V_{1,2-2m} \) cannot have the direct sum decomposition \( V_{1,2-2m} = D_{2m}^{-} \oplus F_{2m+1} \oplus D_{2m}^{+} \). It would be interesting to give an explanation in the framework of \( sl_2 \) representations for the absence of non-Juddian exceptional solutions when a Juddian solution exists with the same eigenvalue for \( \varepsilon \in \frac{1}{2} \mathbb{Z}_{\geq 0} \) (Corollary 4.15).

4.4 The degenerate atomic limit

In this subsection we make a brief remark on the case \( \Delta = 0 \) from the orthogonal polynomials viewpoint. Recall that the polynomials \( P^{(N, \varepsilon)}_k(x, y) \) are defined by a three-term recurrence relation. However, it is not possible to set the parameters to define a family of orthogonal polynomials in \( x \) or \( y \).

Consider the determinant expression (4.7) and set \( y = 0 \). The expansion of the continuum from the lower-right corner gives the three-term recurrence relation

\[
\frac{1}{k!} P^{(k, \varepsilon)}_k(x, 0) = (x + 1 - 2k - 2\varepsilon) \frac{1}{(k-1)!} P^{(k-1, \varepsilon)}_{k-1}(x, 0) - (k-1)(k-1+2\varepsilon) \frac{1}{(k-2)!} P^{(k-2, \varepsilon)}_{k-2}(x, 0).
\]
By Favard’s theorem (see, e.g. [14]), when \( \varepsilon > -\frac{1}{2} \) the family of normalized constraint polynomials \( \{ \frac{1}{k!}P^{(k,\varepsilon)}_k(x, 0) \} \) defines a family of orthogonal polynomials. Recall that the generalized Laguerre polynomials [1] are given by

\[
L^{(\alpha)}_k(x) = \frac{x^{-\alpha}e^x}{k!} \frac{d^k}{dx^k}(e^{-x}x^{k+\alpha}).
\]

for \( k \geq 1 \) and \( \alpha > -1 \), and the monic generalized Laguerre polynomials are given by \((-1)^kk!L^{(\alpha)}_k(x)\). Comparing the recurrence relations and the initial conditions we immediately obtain the following result.

**Theorem 4.16.** For \( k \geq 0 \), we have

\[
\frac{1}{k!}P^{(k,\varepsilon)}_k(x, 0) = (-1)^kk!L^{(2\varepsilon)}_k(x).
\]

The case \( y = 0 \) corresponds to the model \( H^\varepsilon_{\text{Rabi}} \) with \( \Delta = 0 \), namely

\[
H^\varepsilon := \omega a^\dagger a + g\sigma_x(a^\dagger + a) + \varepsilon\sigma_x,
\]

called the degenerate atomic limit [37]. The Hamiltonian \( H^\varepsilon \) is a generalization of the displaced harmonic oscillator (corresponding to \( H^0 \)) studied in [33]. For the Hamiltonian \( H^\varepsilon \), the constraint equation for the exceptional eigenvalue parameterized by integer \( N \) is given by

\[
L^{(2\varepsilon)}_N(x) = 0.
\]

The presence of the Laguerre polynomials in the constraint equation is explained in the study of the solutions of the model (4.12). For instance, in [33], the solutions of the displaced harmonic oscillator is given in terms of power series, its coefficients are multiples of associated Laguerre polynomials. The explicit form of the exceptional solutions of (4.12) is obtained in [37] by a related method.

**Remark 4.6.** For the case \( y \neq 0 \), let \( C^{(N,\varepsilon)}_N \) be the matrix in the determinant expression of \( P^{(N,\varepsilon)}_N(x, y) \) of Proposition 4.2, then we have

\[
(C^{(N,\varepsilon)}_N)_{k,k} = -k(2N + 1 - k) - 1 + 2\varepsilon, \quad (C^{(N,\varepsilon)}_N)_{i,i-1} = i(N + 1 - i)(N + 1 - i + 2\varepsilon)
\]

for \( k = 1, 2, \ldots, N \) and \( i = 2, 3, \ldots, N \). By expanding the continuant as a recurrence relation we obtain a family \( \{Q^{(N,\varepsilon)}_k(x, y)\}_{k \geq 0} \) of polynomials in two variables given by

\[
Q^{(N,\varepsilon)}_0(x, y) = 1, \quad Q^{(N,\varepsilon)}_1(x, y) = x + y - (2N + 1 + 2\varepsilon)
\]

\[
Q^{(N,\varepsilon)}_k(x, y) = (kx + y - k(2N + 1 - k) - 1 + 2\varepsilon))Q^{(N,\varepsilon)}_{k-1}(x, y) - k(k-1)(N + 1 - k)(N + 1 - k + 2\varepsilon)Q^{(N,\varepsilon)}_{k-2}(x, y).
\]

By definition \( Q^{(N,\varepsilon)}_k(x, y) = P^{(N,\varepsilon)}_k(x, y) \). In general, for \( k \neq N \), it does not hold that \( Q^{(N,\varepsilon)}_k(x, y) = P^{(N,\varepsilon)}_k(x, y) \). Moreover, in contrast with the case \( y = 0 \), it is not clear how to relate the \( k \)-th polynomial \( Q^{(N,\varepsilon)}_k(x, y) \) with the constraint polynomial \( P^{(k,\varepsilon)}_k(x, y) \).
5 Estimation of positive roots of constraint polynomials

In this section, we study existence of degenerate exceptional eigenvalues corresponding to Juddian solutions by giving an estimate on the number of positive roots $x = (2g)^2$ for the constraint polynomial $P_{N}^{(N,\varepsilon)}(x, y)$ according on the value of $y = \Delta^2$. For our current interest concerning the degeneracy of Juddian solutions, it is sufficient to obtain the estimate when $\varepsilon \geq 0$. However, we give also a conjecture which counts precisely a number of positive roots of $P_{N}^{(N,\varepsilon)}(x, y)$ for negative $\varepsilon$ when $N$ is sufficiently large, i.e. $N \geq -[2\varepsilon]$, $[x]$ being the integer part of $x \in \mathbb{R}$.

5.1 Interlacing of roots for constraint polynomials

When considered as polynomials in $\mathbb{R}[y][x]$, there is non-trivial interlacing among the roots of the coefficients of the constraint polynomials $P_{N}^{(N,\varepsilon)}(x, y)$. This interlacing is essential for the proof of the upper bound on the number of positive roots of the constraint polynomials in the next sections.

For $N \in \mathbb{Z}_{\geq 0}$, let

$$P_{N}^{(N,\varepsilon)}(x, y) = \sum_{i=0}^{N} a_{i}^{(N)}(y)x^{i}.$$ 

Noticing that $\deg(a_{i}^{(N)}(y)) = N - i$, the interlacing property is given in the following lemma.

**Lemma 5.1.** Let $N \in \mathbb{Z}_{\geq 0}$ and $\varepsilon > -1/2$. Then the roots of $a_{j}^{(N)}(y) (0 \leq j \leq N - 1)$ are real. Denote the roots of $a_{j}^{(N)}(y)$ by $\xi_{1}^{(j)} \leq \xi_{2}^{(j)} \leq \cdots \leq \xi_{N-j}^{(j)}$. Then, for $j = 0, 1, \ldots, N - 2$ we have

$$\xi_{i}^{(j)} < \xi_{i+1}^{(j)} < \xi_{i+1}^{(j+1)}$$

for $i = 1, 2, \ldots, N - j - 1$.

The constraint polynomials $P_{N}^{(N,\varepsilon)}(x, y)$, with $\varepsilon > -1/2$, belong to a special class of polynomials in two variables, the class $P_{2}$ (see [16]). The class $P_{2}$ is a generalization of polynomials of one variable with all real roots. A polynomial $p(x, y)$ of degree $n$ belongs to the class $P_{2}$ if it satisfies the following conditions:

- For any $\alpha \in \mathbb{R}$, the polynomials $p(\alpha, y)$ and $p(x, \alpha)$ have all real roots.
- Monomials of degree $n$ in $p(x, y)$ all have positive coefficients.

Equivalently, a polynomial $p(x, y)$ is in the class $P_{2}$ if it has a determinant expression

$$p(x, y) = \det(I_{n}y + D_{n}x + S_{n}),$$

with $D_{n}$ a diagonal matrix with positive entries and $S_{n}$ a real symmetric matrix.

Recall the following property of polynomials of the class $P_{2}$.

**Lemma 5.2** (Lemma 9.63 of [16]). Let $f(x, y) \in P_{2}$ and set

$$f(x, y) = f_{0}(x) + f_{1}(x)y + \cdots + f_{n}(x)y^{n}.$$ 

If $f(x, 0)$ has all distinct roots, then all $f_{i}$ have distinct roots, and the roots of $f_{i}$ and $f_{i+1}$ interlace.
Note that the lemma above tacitly implies that the roots of the polynomials \( f_i \) are real. With these preparations, we prove Lemma 5.1.

**Proof of Lemma 5.1.** By Corollary 4.4, \( P_N^{(N, \varepsilon)}(x, y) \in P_2 \). Since

\[
P_N^{(N, \varepsilon)}(0, y) = \prod_{i=1}^{N} (y - i(2\varepsilon)),
\]

for \( \varepsilon > -1/2 \), the roots are different and the lemma applies, establishing the result.

5.2 Number of positive roots of constraint polynomials

In this section we give an estimation on the number of positive roots of constraint polynomials. In particular, this result proves the existence of exceptional eigenvalues corresponding to Juddian solutions in the spectrum of the AQRM. We note that although there is a description of the statement of Theorem 5.3 for open intervals in [38], the proof provided by the authors only gives a lower bound on the number of positive roots.

**Theorem 5.3.** Let \( \varepsilon > -\frac{1}{2} \). For each \( k (0 \leq k < N) \), there are exactly \( N - k \) positive roots (in the variable \( x \)) of the constraint polynomial \( P_N^{(N, \varepsilon)}(x, y) \) for \( y \) in the range

\[
k(k + 2\varepsilon) \leq y < (k + 1)(k + 1 + 2\varepsilon).
\]

Furthermore, when \( y \geq N(N + 2\varepsilon) \), the polynomial \( P_N^{(N, \varepsilon)}(x, y) \) has no positive roots with respect to \( x \).

We illustrate numerically the proposition for the case \( N = 6 \) and \( \varepsilon = 0.4 \) in Figure 4. For fixed \( \Delta > 0 \) satisfying \( k(k + 2\varepsilon) \leq \Delta^2 < (k + 1)(k + 1 + 2\varepsilon) \) (\( k \in \{1, 2, \ldots, N\} \)), the number of points \((g, \Delta)\) with \( g > 0 \) in the curve \( P_N^{(N, \varepsilon)}((2g)^2, \Delta^2) = 0 \) is exactly \( N - k \). Likewise, as it is clear in the figure, there are no points \((g, \Delta)\) in the curve with \( g > 0 \) and \( \Delta^2 \geq N(N + 2\varepsilon) \).

![Figure 4: Curve \( P_6^{(6, \varepsilon)}((2g)^2, \Delta^2) = 0 \) with \( \varepsilon = 0.4 \) (for \( g, \Delta > 0 \))](image)

First, we establish a lower bound on the number of positive roots for the constraint polynomials. The following Lemma extends Li and Batchelor’s result ([38], Theorem), to the case of semi-closed intervals.
Lemma 5.4. Let \( \varepsilon > -\frac{1}{2} \). For each \( k (0 \leq k < N) \), there are at least \( N - k \) positive roots (in the variable \( x \)) of the constraint polynomial \( P_{N,\varepsilon}^{(N,\varepsilon)}(x, y) \) for \( y \) in the range
\[
k(k + 2\varepsilon) \leq y < (k + 1)(k + 1 + 2\varepsilon).
\]

Remark 5.1. The proof is a modification to the argument given in [38] (Appendix B), which is based on the proof of Kuš for the case of the (symmetric) quantum Rabi model ([34], Section IV, Thm. 3).

Proof. Define the normalized polynomials \( S_{k}^{(N,\varepsilon)}(x, y) \) by
\[
S_{k}^{(N,\varepsilon)}(x, y) = \frac{P_{k}^{(N,\varepsilon)}(x, y)}{k!}.
\]

Fix \( y \) and consider the polynomials \( S_{k}^{(N,\varepsilon)}(x, y) \) as polynomials in the variable \( x \) and write \( S_{k}^{(N,\varepsilon)}(x) \) for simplicity. Set \( \alpha_{i} = (i(i+2\varepsilon) - y)/i \) and \( \beta_{i} = N - i + 1 \), then the recurrence relation becomes
\[
\begin{align*}
S_{0}^{(N,\varepsilon)}(x) &= 1, \\
S_{1}^{(N,\varepsilon)}(x) &= x - \alpha_{1} \\
S_{k}^{(N,\varepsilon)}(x) &= (x - \alpha_{k})S_{k-1}^{(N,\varepsilon)}(x) - \beta_{k}xS_{k-2}^{(N,\varepsilon)}(x). \\
\end{align*}
\tag{5.1}
\]

Let \( k (0 \leq k < N) \) be fixed. If \( k(k + 2\varepsilon) < y < (k + 1)(k + 1 + 2\varepsilon) \), then it is clear that \( \alpha_{i} < 0 \) for \( i < k \), \( \alpha_{i} > 0 \) for \( i > k \) and \( \beta_{i} > 0 \) for \( 0 \leq i < N \). Moreover, when \( y = k(k + 2\varepsilon) \) we have \( \alpha_{k} = 0 \) and, from (5.1), we see that \( x = 0 \) is a root of all polynomials \( S_{k+i}(x) \) for \( i = 1, \ldots, N - k \).

For \( i = 0, 1, \ldots, N - k \), set
\[
S_{k+i}(x) = \begin{cases} 
S_{k+i}(x) & \text{if } y \neq k(k + 2\varepsilon) \\
(1/x)S_{k+i}(x) & \text{if } y = k(k + 2\varepsilon). 
\end{cases}
\]

With this modification, the proof follows as in [34]. First, notice that
\[
\text{sgn}(S_{l}^{(N,\varepsilon)}(0)) = \text{sgn}((-1)^{l}\alpha_{1}\alpha_{2} \cdots \alpha_{l}) = (-1)^{2l} = 1
\]
for \( l < k \). Similarly, \( \text{sgn}(S_{k}^{(N,\varepsilon)}(0)) = 1 \) if \( y \neq k(k + 2\varepsilon) \) and \( \text{sgn}(S_{k}^{(N,\varepsilon)}(0)) = 0 \) if \( y = k(k + 2\varepsilon) \). On the other hand, for \( i = 1, \ldots, N - k \), we have
\[
\text{sgn}(S_{k+i}^{(N,\varepsilon)}(0)) = \begin{cases} 
\text{sgn}((-1)^{k+i+1}a_{1}\alpha_{k-1}a_{k}\alpha_{k+1} \cdots a_{k+i}) & \text{if } y \neq k(k + 2\varepsilon) \\
\text{sgn}((-1)^{k+i-1}a_{1}\alpha_{k-1}a_{k+1} \cdots a_{k+i}) & \text{if } y = k(k + 2\varepsilon)
\end{cases},
\]
and we directly verify that in both cases the expression is equal to \((-1)^{i}\).

In addition, from the recurrence relation (5.1) we easily see the following
- If \( S_{i}^{(N,\varepsilon)}(a) = 0 \) for \( a > 0 \), then \( S_{i+1}^{(N,\varepsilon)}(a) \) and \( S_{i-1}^{(N,\varepsilon)}(a) \) have opposite signs,
- \( S_{i}^{(N,\varepsilon)}(x) \) and \( S_{i-1}^{(N,\varepsilon)}(x) \) cannot have the same positive root.
These remarks are easily seen to hold for the auxiliary polynomials \( \tilde{S}_{k+i}(x) \) as well. Next, denote by \( V(x) \) the number of change of signs of the sequence

\[
\tilde{S}^{(N,\varepsilon)}_N(x), \tilde{S}^{(N,\varepsilon)}_{N-1}(x), \ldots, \tilde{S}^{(N,\varepsilon)}_{k+1}(x), \tilde{S}^{(N,\varepsilon)}_k(x), \tilde{S}^{(N,\varepsilon)}_{k-1}(x), \ldots, \tilde{S}^{(N,\varepsilon)}_0(x).
\]

By the remarks above, variations of \( V(x) \) by \( \pm 1 \) occur only at zeros of \( \tilde{S}^{(N,\varepsilon)}_N(x) \) or \( \tilde{S}^{(N,\varepsilon)}_0(x) \). At \( x = 0 \), the first terms of the sequence are \((-1)^{N-k-i}\) for \( i = 0, \ldots, N - k - 1 \), then \( 0 \) if \( y = k(k+2\varepsilon) \) and all the remaining terms are \( 1 \), hence \( V(0) = N - k \). On the other hand, it is clear that as \( x \) tends to infinity \( \text{sgn}(\tilde{S}^{(N,\varepsilon)}_i(x)) = 1 \) and \( \text{sgn}(\tilde{S}^{(N,\varepsilon)}_{k+i}(x)) = 1 \). This proves that there are at least \( N - k \) positive roots of the polynomial \( \tilde{S}^{(N,\varepsilon)}_N(x) \) and the same holds for \( P^{(N,\varepsilon)}_N(x) \).

To complete the proof we give an upper bound to the number of positive roots using Descartes’ rule of signs (see e.g. [29], Theorem 7.5). This result states that the number of positive roots of a polynomial does not exceed the number of the sign changes in its coefficients.

**Lemma 5.5.** Let \( \varepsilon > -\frac{1}{2} \). When \( y \geq N(N+2\varepsilon) \), the polynomial \( P^{(N,\varepsilon)}_N(x,y) \) has no positive roots with respect to \( x \).

**Proof.** First, using the notation of \([5.1]\) we note that \( y = N(N+2\varepsilon) \) is the largest root of \( a_0^{(N)}(y) = P^{(N,\varepsilon)}_N(0,y) \). Then, by the interlacing of the roots of \( a_i^{(N)}(y) \) (\( i = 0, 1, \ldots, N - 1 \)) of Lemma 5.1, all \( a_i^{(N)}(y) \) must be non-negative. Thus, there are no changes of signs in the coefficients of \( P^{(N,\varepsilon)}_N(x,y) \) (as a polynomial in \( x \)) and the result follows by Descartes’ rule of signs.

**Lemma 5.6.** Let \( \varepsilon > -\frac{1}{2} \). For each \( k \) (\( 0 \leq k < N \)), there are at most \( N - k \) positive roots (in the variable \( x \)) of the constraint polynomial \( P^{(N,\varepsilon)}_N(x,y) \) for \( y \) in the range

\[
k(k+2\varepsilon) \leq y < (k+1)(k+1+2\varepsilon).
\]

**Proof.** First, using the notation of \([5.1]\) note as in Lemma 5.5 that when \( y \geq N(N+2\varepsilon) \), all the coefficients \( a_i^{(N)}(y) \) of the polynomial \( P^{(N,\varepsilon)}_N(x,y) \) are non-negative. By Lemma 5.1, for

\[
(N-1)(N-1+2\varepsilon) < y < N(N+2\varepsilon),
\]

the sign sequence \( (\text{sgn} a_0^{(N)}(y), \text{sgn} a_{N-1}^{(N)}(y), \ldots, \text{sgn} a_0^{(N)}(y)) \) is given by

\[+, +, \cdots, +, -, -, \cdots, -, -,
\]

that is, it consists of a subsequence \(+, +, \ldots, +\) of positive signs followed by a subsequence \(-, -, \ldots, -\) of negative signs. Thus, by Descartes’ rule of signs we have at most \( 1 = N - (N-1) \) positive roots for \( P^{(N,\varepsilon)}_N(x,y) \). When \( y = (N-1)(N-1+2\varepsilon) \), we have \( a_0^{(N)}(y) = 0 \) and the sequence is the same except for a 0 at the end, so the result holds without change. Continuing this process, we see that for \( (N-2)(N-2+2\varepsilon) < y < (N-1)(N-1+2\varepsilon) \), the sign sequence given by

\[+, +, \cdots, +, -, -, \cdots, +, +, \ldots, +,
\]
from where it holds that the polynomial hast at most \( 2 = N - (N - 2) \) roots (with respect to \( x \)). We continue this process until we reach \( 0 < y < 1(1 + 2\varepsilon) \), where we have

\[
+, -, +, -, \ldots, (-1)^{N-1}, (-1)^N
\]
giving \( N = N - 0 \) roots (with respect to \( x \)) by Descartes’ rule of signs. Therefore, to complete the proof it is enough to show that the number of sign changes in the sequence \( (\text{sgn} a_N^{(N)}(y), \text{sgn} a_{N-1}^{(N)}(y), \ldots, \text{sgn} a_0^{(N)}(y)) \) does not vary for \( y \) satisfying \((k - 1)(k - 1 + 2\varepsilon) < y < k(k + 2\varepsilon)\), and that there is exactly an additional sign change when \( y \) crosses \((k - 1)(k - 1 + 2\varepsilon)\). To see this, note that due to the interlacing of roots given in Lemma 5.1, the next sign change in a subsequence \(+, +, \cdots, + \) (or \(-, -, \cdots, -\)) of contiguous coefficients with same sign must happen at right end of the subsequence. When the subsequence \(+, +, \cdots, + \) (or \(-, -, \cdots, -\)) is at the rightmost end of the complete sign sequence \( (\text{sgn} a_N^{(N)}(y), \text{sgn} a_{N-1}^{(N)}(y), \ldots, \text{sgn} a_0^{(N)}(y)) \) there is an additional sign change in the complete sequence and the sign change occurs at roots of \( a_0(y) \), that is, when \( y = k(k + 2\varepsilon) \) for \( k \in \{1, 2, \ldots, N - 1\} \). In any other case there is no additional sign change. This completes the proof.

The combination of Lemmas 5.4 and 5.6 immediately gives Theorem 5.3.

Remark 5.2. It would be interesting to obtain a result where we switch the roles of \( g \) and \( \Delta \) in Theorem 5.3 from both mathematical (e.g. orthogonal polynomials of two variables) and physics (experimental and/or applications) points of view.

5.3 Negative \( \varepsilon \) case

In this subsection we give some remarks on the estimation of positive roots for \( \varepsilon < 0 \). First, we present the generalization of Lemma 5.4. Here, \([x]\) denotes the integer part of \( x \), that is, the unique integer \([x]\) such that \([x]\) \( \leq x < [x] + 1\), and \( \{x\} = x - [x] \) is the fractional part of \( x \).

Lemma 5.7. Let \( \varepsilon < 0 \) and set \( m = -[2\varepsilon] \). For each \( k \) \((0 \leq k < N)\), there are at least \( N - k \) positive roots (in the variable \( x \)) of the constraint polynomial \( P_{N+m}^{(N,m,\varepsilon)}(x,y) \) for \( y \) in the range

\[
(m + k)(m + k + 2\varepsilon) \leq y < (m + k + 1)(m + k + 1 + 2\varepsilon).
\]

Furthermore, let \( \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_m \) be the elements of the multiset \( \{j + 2\varepsilon : 1 \leq j \leq m\} \). Then, for \( k \) \((1 \leq k < m)\) such that \( \alpha_k \neq \alpha_{k+1} \), and \( y \) in the range

\[
\alpha_k \leq y < \alpha_{k+1},
\]

the constraint polynomial \( P_{N+m}^{(N,m,\varepsilon)}(x,y) \) has at least \( N + m - k \) positive roots. If \( y < \alpha_1 \), then the constraint polynomial \( P_{N+m}^{(N,m,\varepsilon)}(x,y) \) has exactly \( N + m \) roots.

Proof. The proof is done in the same manner as in Lemma 5.4 by replacing \( k \) with \( m + k \), and by noticing that for \( i = 1, 2, \ldots, m - 1 \), it holds that \( \alpha_i < 0 \) for any \( y \geq 0 \).

Remark 5.3. The numbers \((m + k)(m + k + 2\varepsilon)\), for \( k \) \((0 \leq k < N)\), in the proposition are also given by \((k - [2\varepsilon])(k + \{2\varepsilon\})\).
The case of negative half-integer $\varepsilon$ of Theorem 5.3 follows directly from Theorem 4.11.

**Corollary 5.8.** Let $N \in \mathbb{Z}_{\geq 0}$ and $\ell \in \mathbb{Z}_{>0}$ satisfying $N - \ell \geq 0$. For each $k(0 \leq k < N)$, there are exactly $N - k$ positive roots (in the variable $x$) of the constraint polynomial $P_{N+\ell}^{(N+\ell,-\ell/2)}(x, y)$ for $y$ in the range

$$k(\ell + k) \leq y < (k + 1)(\ell + k + 1).$$

Furthermore, when $y \geq N(N + \ell)$, the polynomial $P_{N+\ell}^{(N+\ell,-\ell/2)}(x, y)$ has no positive roots with respect to $x$.

**Remark 5.4.** Recall that for the case $N - \ell < 0$ the constraint polynomials $P_{N+\ell}^{(N+\ell,-\ell/2)}(x, y)$ have no positive roots (cf. Proposition 4.13).

**Proof.** Since $P_{N+\ell}^{(N+\ell,-\ell/2)}(x, y) = A_N^\varepsilon(x, y)P_{N}^{(N,\ell/2)}(x, y)$, the result follows immediately from Theorem 5.3 and the fact that $A_N^\varepsilon(x, y)$ has no positive roots for $x, y > 0$. \qed

In the case of non-half integral $\varepsilon < 0$, there is no analog of the polynomial $A_N^\varepsilon(x, y)$. Nevertheless, we expect that the following conjecture holds.

**Conjecture 5.9.** Suppose $\varepsilon < 0$ and set $m = \max(0, -[2\varepsilon])$. For each $k(0 \leq k < N)$, there are exactly $N - k$ positive roots (in the variable $x$) of the constraint polynomial $P_{N+m}^{(N+m,\varepsilon)}(x, y)$ for $y$ in the range

$$(m + k)(m + k + 2\varepsilon) \leq y < (m + k + 1)(m + k + 1 + 2\varepsilon).$$

Furthermore, when $y \geq (N + m)(N + m + 2\varepsilon)$, the polynomial $P_{N+m}^{(N+m,\varepsilon)}(x, y)$ has no positive roots with respect to $x$.

## 6 Further discussion on the spectrum of AQRM

In this section, we further discuss the spectrum of AQRM. Among other things, we show that non-Juddian exceptional solutions do not contribute to any degeneracy for the AQRM and make a study on the non-Juddian exceptional eigenstates with a viewpoint from the lowest weight representations of $\mathfrak{sl}_2$.

### 6.1 Solutions corresponding to the smallest exponent

In the following subsections we return to the study of exceptional eigenvalues from the point of view of the confluent picture of the AQRM started in §3. A part of our discussions follows [7] and [8] for $\varepsilon = 0$. Recall that an eigenvalue $\lambda$ of $H_{\text{Rabi}}$ is exceptional if there is an integer $N \in \mathbb{Z}_{\geq 0}$ such that $\lambda = N \pm \varepsilon - g^2$.

Let us take $a$ as $-a = (\lambda + g^2 - \varepsilon) = N \in \mathbb{Z}_{\geq 0}$. The corresponding system (3.3) of differential equations is then given by

$$\begin{cases}
    y \frac{d}{dy} \phi_{1,+}(y) = N\phi_{1,+}(y) - \Delta \phi_{1,-}(y) \\
    (y - 1) \frac{d}{dy} \phi_{1,-}(y) = (N - 4g^2 + 4g^2y + 2\varepsilon)\phi_{1,-}(y) - \Delta \phi_{1,+}(y).
\end{cases} \tag{6.1}$$
The exponents of \( \phi_{1,-} \) at \( y = 0 \) are \( \rho^{-}_1 = 0, \rho^{-}_2 = N + 1 \). Likewise, the exponents of \( \phi_{1,+} \) at \( y = 0 \) are \( \rho^{+}_1 = 0, \rho^{+}_2 = N \). Since the difference between the exponents is a positive integer, the local analytic solutions will develop a logarithmic branch-cut at \( y = 0 \).

In this subsection, we revisit Juddian solutions. The local Frobenius solution corresponding to the smallest exponent \( \rho^{-}_1 = 0 \) has the form

\[
\phi_{1,-}(y) = \phi_{1,-}(y; \varepsilon) = \sum_{n=0}^{\infty} K^{(N,\varepsilon)}_n y^n, \tag{6.2}
\]

where \( K^{(N,\varepsilon)}_0 \neq 0 \) and \( K^{(N,\varepsilon)}_n = K^{(N,\varepsilon)}_n(g, \Delta) \). Integration of the first equation of (6.1) gives

\[
\phi_{1,+}(y) = \phi_{1,+}(y; \varepsilon) = cy^N - \Delta \sum_{n=0}^{\infty} \frac{K^{(N,\varepsilon)}_n}{n-N} y^n - \Delta K^{(N,\varepsilon)}_N y^N \log y, \tag{6.3}
\]

with constant \( c \in \mathbb{C} \). A necessary condition for \( \phi_{1,+}(y) \) to be an element of the Bargmann space \( \mathcal{B} \) is that \( \phi_{1,+}(y) \) is an entire function, forcing \( K^{(N,\varepsilon)}_N = 0 \) to make the logarithmic term vanish. Suppose \( \phi_{1,+}(y) \in \mathcal{B} \), then by using the second equation of (6.1) we obtain the recurrence relation for the coefficients

\[
(n+1)K^{(N,\varepsilon)}_{n+1} + \left(N-n-(2g)^2 + \frac{\Delta^2}{n-N} + 2\varepsilon\right) K^{(N,\varepsilon)}_n + (2g)^2 K^{(N,\varepsilon)}_{n-1} = 0, \tag{6.4}
\]

valid for \( n \neq N \). This recurrence relation clearly shows the dependence of the coefficients \( K^{(N,\varepsilon)}_n = K^{(N,\varepsilon)}_n(g, \Delta) \) on the parameters of the system. Additionally, for \( n = N \), by the second equation of (6.1), we have

\[
\Delta c = (2g)^2 K^{(N,\varepsilon)}_{N-1} + (N+1)K^{(N,\varepsilon)}_{N+1}. \tag{6.5}
\]

Setting \( c = (2g)^2 K^{(N,\varepsilon)}_{N-1}/\Delta \) makes \( K^{(N,\varepsilon)}_{N+1} \) vanish, and then, by repeated use of the recurrence (6.4), we see that for all positive integers \( k \) the coefficients \( K^{(N,\varepsilon)}_{N+k} \) also vanish. Thus, the solutions of (6.1) given by

\[
\phi_{1,-}(y) = \sum_{n=0}^{N-1} K^{(N,\varepsilon)}_n y^n, \tag{6.6}
\]

\[
\phi_{1,+}(y) = \frac{4g^2K^{(N,\varepsilon)}_{N-1}}{\Delta} y^N - \Delta \sum_{n=0}^{N-1} K^{(N,\varepsilon)}_n y^n \tag{6.7}
\]

are polynomial solutions.

Next, we show the relation between \( K^{(N,\varepsilon)}_N \) and the constraint polynomials.

**Proposition 6.1.** Let \( N \in \mathbb{Z}_{\geq 0} \) and fix \( \Delta > 0 \). Then, the zeros \( g \) of \( K^{(N,\varepsilon)}_N = K^{(N,\varepsilon)}_N(g, \Delta) \) defined by (6.4) and \( P^{(N,\varepsilon)}_N((2g)^2, \Delta^2) \) coincide. In particular, if \( g \) is a zero of \( P^{(N,\varepsilon)}_N((2g)^2, \Delta^2) \), then \( \lambda = N + \varepsilon - g^2 \) is an exceptional eigenvalue with corresponding Juddian solution given by \( \phi_{1,+}(y) \) and \( \phi_{1,-}(y) \) above.
Proof. By multiplying \( K_n^{(N,\varepsilon)} \) by \((K_0^{(N,\varepsilon)})^{-1} \) for all \( n \in \mathbb{Z}_{\geq 0} \), we can assume that \( K_0^{(N,\varepsilon)} = 1 \). Then, we can rewrite the recurrence relation for the coefficients \( K_n^{(N,\varepsilon)} \) as

\[
K_n^{(N,\varepsilon)} = \frac{1}{n} \left( (2g)^2 + \frac{\Delta^2}{N-n+1} + n - 1 - N - 2\varepsilon \right) K_{n-1}^{(N,\varepsilon)} - \frac{1}{n} (2g)^2 K_{n-2}^{(N,\varepsilon)},
\]

for \( n \leq N \). As in [4.1] we easily see that \( K_N^{(N,\varepsilon)} \) has the determinant expression

\[
K_N^{(N,\varepsilon)} = \det \text{tridiag} \left[ \frac{1}{N-i+1} \left( (2g)^2 + \frac{\Delta^2}{i} - i - 2\varepsilon \right) \frac{2g}{N-i+1} \right]_{1 \leq i \leq N}.
\]

Next, for \( i = 1, 2, \ldots, N \), factor \( \frac{1}{(N+1-i)} \) from the \( i \)-th row in the determinant to get the expression of \( K_N^{(N,\varepsilon)} \) as

\[
\frac{1}{(N!)^2} \det \text{tridiag} \left[ \frac{i(2g)^2 + \Delta^2 - i^2 - 2i\varepsilon}{(2N - 1 - i)g} \frac{2ig}{(N+1-i)} \right]_{1 \leq i \leq N}.
\]

The recurrence relation corresponding to this continuant is the same as the recurrence relation of \( P_k^{(N,\varepsilon)}((2g)^2, \Delta^2) \) (cf. Definition 4.1), including the initial conditions. Thus

\[
K_N^{(N,\varepsilon)}(N + \varepsilon; g, \Delta, \varepsilon) = \frac{1}{(N!)^2} P_N^{(N,\varepsilon)}((2g)^2, \Delta^2),
\]

completing the proof. \( \square \)

Remark 6.1. The proposition above gives another explanation of the constraint polynomials for the existence of Juddian solutions which we derived in [62].

6.2 Solutions corresponding to the largest exponent

In this subsection we study the non-Juddian solutions corresponding to exceptional eigenvalues. The largest exponent of \( \phi_{1,-} \) at \( y = 0 \) is \( \rho_2^{-} = N + 1 \), it follows that there is a local Frobenius solution analytic at \( y = 0 \) of the form

\[
\phi_{1,-}(y)(= \phi_{1,-}(y; \varepsilon)) = \sum_{n=N+1}^{\infty} \tilde{K}_n^{(N,\varepsilon)} y^n,
\]

where \( \tilde{K}_{N+1}^{(N,\varepsilon)} \neq 0 \) and \( \tilde{K}_n^{(N,\varepsilon)} = \tilde{K}_n^{(N,\varepsilon)}(g, \Delta) \). Integration of the first equation of (6.1) gives

\[
\phi_{1,+}(y)(= \phi_{1,+}(y; \varepsilon)) = cy^n - \Delta \sum_{n=N+1}^{\infty} \frac{\tilde{K}_n^{(N,\varepsilon)}}{n - N} y^n,
\]

with constant \( c \in \mathbb{C} \). The second equation of (6.1) gives the recurrence relation

\[
(n + 1)\tilde{K}_{n+1}^{(N,\varepsilon)} + (N - n - (2g)^2 + \frac{\Delta^2}{n - N} + 2\varepsilon) \tilde{K}_n^{(N,\varepsilon)} + (2g)^2 \tilde{K}_{n-2}^{(N,\varepsilon)} = 0,
\]

for \( n \geq N + 1 \) with initial conditions \( \tilde{K}_{N+1}^{(N,\varepsilon)} = 1 \) and \( \tilde{K}_N^{(N,\varepsilon)} = 0 \). Furthermore, we also have the condition

\[
(N + 1)\tilde{K}_{N+1}^{(N,\varepsilon)} = (N + 1) = c\Delta,
\]
which determines value of the constant $c = (N + 1)/\Delta$. Notice that the radius of convergence of each series above equals 1 from the defining recurrence relation (6.10).

As a corollary of the discussion on the exceptional eigensolutions, we now present the following result about the general structure of the degeneracy of the spectrum of the AQRM.

**Corollary 6.2.** The degeneracy of the spectrum of $H_{\text{Rabi}}^\varepsilon$ occurs only when $\varepsilon = \ell/2$ for $\ell \in \mathbb{Z}_{\geq 0}$ and $P_N^{(N,\ell/2)}((2g)^2, \Delta^2) = 0$. In particular, any non-Juddian exceptional solution is non-degenerate.

**Proof.** We first consider the case $N \neq 0$. When $P_N^{(N,\varepsilon)}((2g)^2, \Delta^2) \neq 0$ if we look at the local Frobenius solutions at $y = 0$, then there is always a local solution containing a log-term as seen in (6.1) (see Proposition 6.1), so the solutions corresponding to the smaller exponent cannot be components of the eigenfunction. Then, the solution corresponds to the largest exponent (i.e. non-Juddian exceptional) and this implies that the dimension of the corresponding eigenspace is at most one (cf. [5, 67] and also [6, 4]). We note that in the case $\varepsilon = \ell/2 (\ell \in \mathbb{Z})$ there is no chance of a contribution of Juddian solution (i.e. $P_N^{(N+\ell,-\ell/2)}((2g)^2, \Delta^2) = 0$) by Theorem 4.12. Suppose next that $P_N^{(N,\varepsilon)}((2g)^2, \Delta^2) = 0$ for $\varepsilon \notin \frac{1}{2}\mathbb{Z}_{\geq 0}$. Looking at the local Frobenius solutions at $y = 1$, since the exponent different from 0 is not a non-negative integer (see Table 1), we observe that only the solution corresponding to the exponent 0 can give a eigensolution of $H_{\text{Rabi}}^\varepsilon$ so that the dimension of the eigenspace is also at most one. By Corollary 6.15, there is no non-Juddian exceptional eigensolution when $P_N^{(N,\ell/2)}((2g)^2, \Delta^2) = 0$ for $\ell \in \mathbb{Z}_{\geq 0}$.

On the other hand, if $N = 0$, the exponents of the system (6.1) are $\rho_1^-=0$ and $\rho_2^- = 1$, therefore there is one holomorphic Frobenius solution and a local solution with a log-term. This implies that the corresponding eigenstate cannot be degenerate. In addition, note that if $K_0^{(N,\varepsilon)}(g, \Delta) = 0$, the log-term in the Frobenius solution with smaller exponent (6.2) vanishes making it identical to the solution (6.8) (corresponding to the larger exponent). Hence, the exceptional eigenvalue $\lambda = \pm \varepsilon - g^2$ must be non-Juddian exceptional, and thus, non-degenerate. Since $P_0^{(0, \pm \varepsilon)}((2g)^2, \Delta^2) = 1 \neq 0$ and $P_0^{(\ell, -\ell/2)}((2g)^2, \Delta^2) \neq 0$ for $g, \Delta > 0$ and $\ell > 0$ (cf. Proposition 4.13), the desired claim follows.

**Remark 6.2.** The non-degeneracy of the ground state for the QRM is shown in [20].

Thus, summarizing the results so far obtained in Theorem 4.7 with Theorem 4.11 and Corollary 6.2, we have the following result.

**Theorem 6.3.** The spectrum of the AQRM possesses a degenerate eigenvalue if and only if the parameter $\varepsilon$ is a half integer. Furthermore, all degenerate eigenvalues of the AQRM are Juddian.

We conclude this subsection by illustrating numerically the degeneracy structure of the spectrum of the AQRM described in Theorem 6.3 (for the numerical computation of spectral curves see Theorem 6.13). For half-integer $\varepsilon$, Figure 5 shows the spectral graphs for fixed $\Delta = 1$ and $\varepsilon = 0, 1/2, 3/2$. In the graphs, the dashed lines represent the exceptional energy curves $y = i + \ell/2 - g^2$ for $i \in \mathbb{Z}_{\geq 0}$, any crossings of these curves with the spectral curves correspond to exceptional eigenvalues. The crossings of the eigenvalue curves in the exceptional points correspond to Juddian degenerate solutions, given by Theorem 4.11. Notice also the non-degenerate exceptional points in the curves, these points correspond to the non-Juddian exceptional eigenvalues.
The case of $\varepsilon \notin \frac{1}{2}\mathbb{Z}$ is shown in Figure 6. In these graphs, for $i \in \mathbb{Z}_{\geq 0}$ the dashed lines represent the exceptional energy curves $y = i \pm \varepsilon - g^2$. Notice that we have the situation of the conceptual graphs of Figure 2 in the introduction. In particular, note that due to the bounds on positive solutions of constraint polynomials of §5.1, not all exceptional eigenvalues $\lambda = N \pm \varepsilon - g^2$ with the same $N \in \mathbb{Z}_{\geq 0}$ can be Juddian (see also the discussion on Figure 9 below).

Figure 5: Spectral curves for the case of $\Delta = 1$ for the cases $\varepsilon \in \{0, 0.5, 1.5\}$ for $0 \leq g \leq 2.7$ and energy $(E) -1.5 \leq E \leq 5.5$.

Figure 6: Spectral curves for the case of $\Delta = 1$ for the cases $\varepsilon \in \{0.2, 1.4\}$ for $0 \leq g \leq 2.7$ and energy $(E) -1.5 \leq E \leq 5.5$. In (a), circle marks denote points corresponding to Juddian solutions and diamond marks denote non-Juddian exceptional solutions (cf. Figure 1(a)).

Remark 6.3. Notice that $\Delta c = (2g)^2 K_{N-1}^{(N,\varepsilon)} + (N + 1)K_{N+1}^{(N,\varepsilon)}$ in (6.5). This shows that there are two possibilities for the choice of solutions of (6.1). In other words, the choice of $c = (2g)^2 K_{N-1}/\Delta$ (for the smallest exponent) provides a polynomial solution, i.e. the Juddian solution while the choice $c = (N + 1)K_{N+1}/\Delta$ (for the largest exponent) provides a non-degenerate exceptional solution when the $g$ satisfies $T_1^{(N)}(g, \Delta) = 0$ (cf. §6.3). However, there is no chance to have contributions from both Juddian and non-Juddian exceptional eigenvalues (cf. Remark 4.3).
6.3 Remarks on $G$-function for the AQRM

The $G$-function was originally introduced by Braak [5] in the study of the integrability of the (symmetric) quantum Rabi model. The main point is that its zeros give the so called regular spectrum. The $G$-function for the Hamiltonian $H^\varepsilon_{\text{Rabi}}$ is defined as

$$G_\varepsilon(x; g, \Delta) := \Delta^2 \tilde{R}^+(x; g, \Delta, \varepsilon) \tilde{R}^-(x; g, \Delta, \varepsilon) - R^+(x; g, \Delta, \varepsilon) R^-(x; g, \Delta, \varepsilon)$$

where

$$R^\pm(x; g, \Delta, \varepsilon) = \sum_{n=0}^{\infty} K_n^\pm(x) g^n, \quad \tilde{R}^\pm(x; g, \Delta, \varepsilon) = \sum_{n=0}^{\infty} \frac{K_n^\pm(x)}{x - n \pm \varepsilon} g^n, \quad \text{(6.11)}$$

whenever $x \mp \varepsilon \notin \mathbb{Z}_{\geq 0}$, respectively. For $n \in \mathbb{Z}_{\geq 0}$, define the functions $f_n^\pm(x, g, \Delta, \varepsilon)$ by

$$f_n^\pm(x, g, \Delta, \varepsilon) = 2g + \frac{1}{2g} \left( n - x \pm \varepsilon + \frac{\Delta^2}{x - n \pm \varepsilon} \right), \quad \text{(6.12)}$$

then, the coefficients $K_n^\pm(x) = K_n^\pm(x, g, \Delta, \varepsilon)$ are given by the recurrence relation

$$nK_n^\pm(x) = f_{n-1}^\pm(x, g, \Delta, \varepsilon)K_{n-1}^\pm(x) - K_{n-2}^\pm(x) \quad (n \geq 1) \quad \text{(6.13)}$$

with initial condition $K_0^\pm = 0$ and $K_1^\pm = 1$, whence $K_1^\pm = f_0^\pm(x, g, \Delta, \varepsilon)$. It is also clear from the definitions that the equality

$$K_n^\pm(x, g, \Delta, -\varepsilon) = K_n^\mp(x, g, \Delta, \varepsilon), \quad \text{(6.14)}$$

holds.

It is well-known (e.g. [5, 7, 64]) that for fixed parameters $\{g, \Delta, \varepsilon\}$ the zeros $x_n$ of $G_\varepsilon(x; g, \Delta)$ correspond to eigenvalues $\lambda_n = x_n - g^2$ of $H^\varepsilon_{\text{Rabi}}$. These eigenvalues are called regular and always non-degenerate as in the case of regular eigenvalues of the quantum Rabi model.

**Remark 6.4.** We remark that in the case of the QRM (i.e. $\varepsilon = 0$), we have $G_0(x; g, \Delta) = G_+(x) \cdot G_-(x)$, $G_\pm(x)$ being the $G$-functions corresponding to the parity defined as

$$G_\pm(x) = \sum_{n=0}^{\infty} K_n(x) \left( 1 \mp \frac{\Delta}{x - n} \right) g^n,$$

where $K_n(x) = K_n^\pm(x, g, \Delta, 0)$ [5]. It is also known that these functions can be written in terms of confluent Heun functions (cf. [8]). Note also that there are no degeneracies within each parity subspace.

The following result is obvious from the definition and the property (6.14) above.

**Lemma 6.4.** The $G$-functions of $H^\varepsilon_{\text{Rabi}}$ coincides with that of $H^{-\varepsilon}_{\text{Rabi}}$:

$$G_\varepsilon(x; g, \Delta) = G_{-\varepsilon}(x; g, \Delta). \quad \text{(6.15)}$$

In other words, the regular spectrum of $H^\varepsilon_{\text{Rabi}}$ depends only on $|\varepsilon|$.
Lemma 6.5. Let $N \in \mathbb{Z}_{\geq 0}$. Then the following relation hold for $g > 0$.

$$
(N!)^2 (2g)^N K_N^-(N + \varepsilon; g, \Delta, \varepsilon) = P^{(N, \varepsilon)}_N((2g)^2, \Delta^2),
$$

(6.16)

In addition, if $\varepsilon = \ell/2 (\ell \in \mathbb{Z})$, it also holds that

$$
((N + \ell)!)^2 (2g)^{N+\ell} K^{+}_{N+\ell}(N + \ell/2; g, \Delta, \ell/2) = P^{(N+\ell, -\ell/2)}_{N+\ell}((2g)^2, \Delta^2).
$$

Now, let us consider the case $x = N + \varepsilon$. Observe that $\tilde{f}_N^-(x, g, \Delta, \varepsilon)$, as a function in $x$, has a simple pole at $x = N + \varepsilon$, and thus, each of the rational functions $K_n^-(x) = K_n^-(x, g, \Delta, \varepsilon)$, for $n \geq N + 1$, also has a simple pole at $x = N + \varepsilon$. If $P^{(N, \varepsilon)}_N((2g)^2, \Delta^2) = 0$, then $K_N^-(N + \varepsilon) = 0$ by the lemma above and the coefficients $K_n^-(N + \varepsilon)$ are finite for $n \geq N + 1$. Therefore, the functions $R^-(x; g, \Delta, \varepsilon)$ and $R^+(x; g, \Delta, \varepsilon)$ converge to a finite value at $x = N + \varepsilon$.

In the case $P^{(N, \varepsilon)}_N((2g)^2, \Delta^2) \neq 0$ the G-function may or may not have a pole depending on the value of the residue (as a function of $g$ and $\Delta$) at $x = N + \varepsilon$. We leave the detailed discussion to the subsection §6.5 after we have introduced the constraint $T$-function for non-Juddian exceptional eigenvalues.

To illustrate the discussion we show in Figure 7 the plot of the G-function $G_{\varepsilon}(x; g, \Delta)$ for fixed $g, \Delta > 0$ corresponding to roots of the constraint polynomials $P^{(N, \varepsilon)}_N((2g)^2, \Delta^2)$. In Figure 7(a) we show the case of $\varepsilon = 0.3, g \approx 0.5809, \Delta = 1$ and $N = 1$, observe the finite value of the G-function $G_{\varepsilon}(x; g, \Delta)$ at $x = 1.3$ and the poles at $x = N \pm 0.3 (N \in \mathbb{Z}_{\geq 0}, N \neq 1)$. In the Figures 7(b)-(c) we show the half-integer case. Concretely, in Figure 7(b) we show the case $\varepsilon = 1/2, g = 1/2, \Delta = 1$ and $N = 1$ and in Figure 7(c) the case $\varepsilon = 1, g \approx 1.01229, \Delta = 1.5$ and $N = 2$. As expected from the discussion above, the function $G_{\varepsilon}(x; g, \Delta)$ has a finite value at $x = 1.5$ (for Figure 7(b)) and $x = 3$ (for Figure 7(c)), while other values of $x = N \pm \varepsilon$ are poles.

To conclude this subsection, we remark that there is a non-trivial relation between the parameters $g, \Delta$ and the pole structure of $G_{\varepsilon}(x; g, \Delta)$, that is, the lateral limits at the poles for $x \in \mathbb{R}$.

For instance, in Figure 8 we show the plots of $G_{\varepsilon}(x; g, \Delta)$ for fixed $\Delta = 3/2, \varepsilon = 2$ and $g = 1$ (Figure 8(a)), a root $g \approx 1.283$ of $P^{(2,2)}_2((2g)^2, (3/2)^2)$ (Figure 8(b)) and $g = 2$ (Figure 8(c)). Note that in all cases the lateral limits of the G-function $G_{\varepsilon}(x; g, \Delta)$ at the pole $x = 1$ are the same, while at the poles $x = 2, 3, 4$ the limits have different signs in Figures 8(a) and (c). In addition, in Figure 8(b) the pole at $x = 4$ actually vanishes. A deep understanding of this relation is crucial for the study of the distribution of eigenvalues of the AQRM, for instance, the conjecture of Braak for the QRM [5] (see also Remark 6.14 below) and its possible generalizations to the AQRM.

Remark 6.5. The constraint $T$-function $T^{(N)}_\varepsilon(g, \Delta)$ to be introduced in §6.4 below, is defined in a similar manner to the G-function $G_{\varepsilon}(x; g, \Delta)$. Thus, in addition to the references already given, we direct the reader to [6.4] for the derivation and properties of the $G$-function.
Determinant expression of constraint polynomials and spectrum of AQRM

Figure 7: Plot of $G_\varepsilon(x; g, \Delta)$ for fixed $g$ and $\Delta$, corresponding to roots of constraint polynomials $P^{(N,\varepsilon)}_N((2g)^2, \Delta^2)$. Notice the vanishing of the poles (indicated with dashed circles) at $x = N + \varepsilon$ for $N = 1$ in (a) and (b), and $N = 2$ in (c).

6.4 Non-Juddian exceptional solutions

In this subsection, we study the constraint relation for non-Juddian exceptional eigenvalues. Recall from §6.3 that the zeros of the $G$-function $G_\varepsilon(x; g, \Delta)$ corresponds to points of the regular spectrum $\lambda = x - g^2$. Similarly, zeros of the constraint polynomial $P^{(N,\varepsilon)}_N(x, y)$ correspond to exceptional eigenvalues $\lambda = N + \varepsilon - g^2$ with Juddian solutions.

For non-Juddian exceptional eigenvalues, we define a constraint $T$-function $T^{(N)}_\varepsilon(g, \Delta)$ that vanishes for parameters $g$ and $\Delta$ for which $H_{Rabi}$ has the exceptional eigenvalue $\lambda = N + \varepsilon - g^2$ with non-Juddian solution (see [7] for the case of the quantum Rabi model).

In order to define the function $T^{(N)}_\varepsilon(g, \Delta)$, we first describe the local Frobenius solutions of system of differential equations (3.3) and (3.4) at the regular singular points $y = 0, 1$ (cf. [6.2]).

Define the functions $\phi_{1,\pm}(y; \varepsilon)$ as follows:

$$
\phi_{1,+}(y; \varepsilon) = \frac{(N + 1)}{\Delta} y^N - \Delta \sum_{n=N+1}^{\infty} \frac{\tilde{K}^{-}_n(N + \varepsilon; g, \Delta, \varepsilon)}{n - N} y^n, 
$$

$$
\phi_{1,-}(y; \varepsilon) = \sum_{n=N+1}^{\infty} \tilde{K}^{-}_n(N + \varepsilon; g, \Delta, \varepsilon) y^n,
$$

with initial conditions $\tilde{K}^{-}_n(N + \varepsilon; g, \Delta, \varepsilon) = 0 (n \leq N)$, $\tilde{K}^{-}_N(N + \varepsilon; g, \Delta, \varepsilon) = 1$ and

$$(n + 1)\tilde{K}^{-}_{N+1}(N + \varepsilon; g, \Delta, \varepsilon) = \left(n - N + (2g)^2 - 2\varepsilon + \frac{\Delta^2}{N - n}\right)\tilde{K}^{-}_n(N + \varepsilon; g, \Delta, \varepsilon) - (2g)^2 \tilde{K}^{-}_{n-1}(N + \varepsilon; g, \Delta, \varepsilon),$$

where $\tilde{K}^{-}_n(N + \varepsilon; g, \Delta, \varepsilon)$ is the $n$-th value of the $\tilde{K}^{-}_n(N + \varepsilon; g, \Delta, \varepsilon)$ sequence.
Figure 8: Plot of $G_{\varepsilon}(x; g, \Delta)$ for fixed $\Delta = 1.5, \varepsilon = 2$ and different values of $g$.

for $n \geq N + 1$. Then, $(\phi_{1,+}(y; \varepsilon), \phi_{1,-}(y; \varepsilon))$ is the local Frobenius solution corresponding to the largest exponent of the system (3.3) at $y = 0$.

Next, consider the solutions at $y = 1$. For the case $N + 2\varepsilon \notin \mathbb{Z}_{\geq 0}$ (i.e. $\varepsilon \notin \frac{1}{2}\mathbb{Z}$ or $\varepsilon = -\ell/2 (\ell \in \mathbb{Z}_{\geq 0})$ and $N - \ell < 0$ ) we define

$$
\phi_{2,+}(\bar{y}; -\varepsilon) = \Delta \sum_{n=0}^{\infty} \frac{\bar{K}^+_n(N + \varepsilon; g, \Delta, \varepsilon)}{N + 2\varepsilon - n} \bar{y}^n,
$$

with initial conditions $\bar{K}^+_n(N + \varepsilon; g, \Delta, \varepsilon) = 0 (n < 0)$, $\bar{K}^+_0(N + \varepsilon; g, \Delta, \varepsilon) = 1$, while for the case $N + 2\varepsilon \in \mathbb{Z}_{\geq 0}$ (i.e. $\varepsilon = \ell/2 (\ell \in \mathbb{Z}_{\geq 0})$ or $\varepsilon = -\ell/2 (\ell \in \mathbb{Z}_{\geq 0})$ and $N - \ell \geq 0$) we define

$$
\phi_{2,+}(\bar{y}; -\ell/2) = \left(\frac{N + \ell + 1}{\Delta}\right)\bar{y}^{N + \ell} - \Delta \sum_{n=N+\ell+1}^{\infty} \frac{\bar{K}^+_n(N + \ell/2; g, \Delta, \ell/2)}{n - N - \ell} \bar{y}^n,
$$

with initial conditions $\bar{K}^+_n(N + \ell/2; g, \Delta, \ell/2) = 0 (n < N + \ell)$, $\bar{K}^+_0(N + \ell/2; g, \Delta, \ell/2) = 1$ and in both cases the coefficients satisfy

$$
\left(n - N + (2g)^2 + \frac{\Delta^2}{N + 2\varepsilon - n}\right)\bar{K}^+_n(N + \varepsilon; g, \Delta, \varepsilon) - (2g)^2\bar{K}^+_n(N + \varepsilon; g, \Delta, \varepsilon)
\right) = (n + 1)\bar{K}^+_n(N + \varepsilon; g, \Delta, \varepsilon)
$$

Then $(\phi_{2,+}(\bar{y}; -\varepsilon), \phi_{2,-}(\bar{y}; -\varepsilon))$ is the local Frobenius solution of the system (3.4) at $\bar{y} = 0$, where $\bar{y} = 1 - y$. 


Note also that the radius of convergence of each series appearing above equals 1. Moreover, the solutions can be expressed in terms of the confluent Heun functions (see e.g. [41, 63, 67]).

A similar discussion to [7] (see also [6]) leads to the following set of equations to assure the existence of the non-Juddian exceptional solutions. Actually, the eigenvalue equation for $H_{Rab}$, that is (3.1), is equivalent via embedding to the system of differential equations given by

\[
\frac{d}{dz} \Psi(z) = A(z) \Psi(z),
\]

(6.23)

where

\[
A(z) = \begin{bmatrix}
\frac{\lambda - \varepsilon - gz}{z+g} & 0 & 0 & -\frac{\Delta}{z+g} \\
0 & \frac{\lambda + \varepsilon - gz}{z+g} & \frac{-\Delta}{z+g} & 0 \\
0 & \frac{\lambda - \varepsilon - gz}{z-g} & \frac{\lambda + \varepsilon - gz}{z-g} & 0 \\
-\frac{\Delta}{z-g} & 0 & 0 & \frac{\lambda + \varepsilon - gz}{z-g}
\end{bmatrix},
\]

(6.24)

for the vector valued function

\[
\Psi(z) := \begin{pmatrix} e^{-g_2 \phi_{2,-}}(\frac{g + z}{2g}; -\varepsilon), e^{g_2 \phi_{2,+}}(\frac{g + z}{2g}; -\varepsilon), e^{-g_2 \phi_{2,-}}(\frac{g + z}{2g}; -\varepsilon), e^{g_2 \phi_{2,+}}(\frac{g + z}{2g}; -\varepsilon) \end{pmatrix}.
\]

It is not difficult to see that the function

\[
\Phi(z) := \begin{pmatrix} e^{-g_0 \phi_{1,-}}(\frac{g + z}{2g}; -\varepsilon), e^{g_0 \phi_{1,+}}(\frac{g + z}{2g}; -\varepsilon), e^{-g_0 \phi_{1,-}}(\frac{g + z}{2g}; -\varepsilon), e^{g_0 \phi_{1,+}}(\frac{g + z}{2g}; -\varepsilon) \end{pmatrix}
\]

also satisfies (6.23). Hence, in order for a non-Juddian exceptional solution to exist it is necessary and sufficient that for some $z_0 (-g < z_0 < g)$ (an ordinary point of the system), there exists a non-zero constant $c = c_N(g, \Delta, \varepsilon)$ and such that

\[
\begin{align*}
e^{-g_0 \phi_{1,+}}(\frac{g + z_0}{2g}; \varepsilon) &= ce^{g_0 \phi_{2,-}}(\frac{g - z_0}{2g}; -\varepsilon), \\
e^{g_0 \phi_{1,-}}(\frac{g - z_0}{2g}; \varepsilon) &= ce^{-g_0 \phi_{2,+}}(\frac{g + z_0}{2g}; -\varepsilon), \\
e^{g_0 \phi_{1,+}}(\frac{g - z_0}{2g}; \varepsilon) &= ce^{-g_0 \phi_{2,-}}(\frac{g + z_0}{2g}; -\varepsilon), \\
e^{-g_0 \phi_{1,-}}(\frac{g + z_0}{2g}; \varepsilon) &= ce^{g_0 \phi_{2,+}}(\frac{g - z_0}{2g}; -\varepsilon).
\end{align*}
\]

(6.25)

For $z_0 = 0$, it is obvious that the first and third, and the second and forth equations are equivalent respectively. Namely, the four equations reduce to the following two equations when $y = \bar{y} = \frac{1}{2}$.

\[
\begin{align*}
\phi_{1,-}(y; \varepsilon) &= c \phi_{2,+}(\bar{y}; -\varepsilon) = c \phi_{2,+}(1 - y; -\varepsilon), \\
\phi_{1,+}(y; \varepsilon) &= c \phi_{2,-}(\bar{y}; -\varepsilon) = c \phi_{2,-}(1 - y; -\varepsilon).
\end{align*}
\]

(6.26)

for some non-zero constant $c$ (as can be seen by applying the substitutions $y \rightarrow \bar{y} = 1 - y$ and $\varepsilon \rightarrow -\varepsilon$ to the system [6.1]). Therefore, by setting $y = 1/2$ ($z = 0$ in the original variable, an ordinary point of the system) and eliminating the constant $c$ in these linear relations gives the following constraint $T$-function

\[
T_{\varepsilon}^{(N)}(g, \Delta) = \left( R_{(N,+)}^{(N,-)} - R_{(N,+)}^{(N,-)} \right) (g, \Delta; \varepsilon),
\]

(6.27)
where \( \cdot \) denotes the usual multiplication of functions and

\[
\begin{align*}
R^{(N,-)}(g, \Delta; \varepsilon) &= \phi_{1,+}(\frac{1}{2}; \varepsilon), \\
R^{(N,+)}(g, \Delta; \varepsilon) &= \phi_{2,+}(\frac{1}{2}; -\varepsilon), \\
\tilde{R}^{(N,-)}(g, \Delta; \varepsilon) &= \phi_{1,-}(\frac{1}{2}; \varepsilon), \\
\tilde{R}^{(N,+)}(g, \Delta; \varepsilon) &= \phi_{2,-}(\frac{1}{2}; -\varepsilon).
\end{align*}
\]

(6.28)

Conversely, if there exists such \( c = c_N(g, \Delta, \varepsilon)(\neq 0) \), \( \lambda = N + \varepsilon - g^2 \) is a non-Juddian exceptional eigenvalue and the corresponding functions \( (\phi_{j,+}, \phi_{j,-}, j = 1, 2) \) satisfy \( (6.26) \) and \( (6.25) \) (cf. [24]).

**Remark 6.6.** When \( \varepsilon = 0 \) we observe that

\[
T_0^{(N)}(g, \Delta) = \left( \tilde{R}^{(N,+)} - R^{(N,+)} \right) \cdot \left( \tilde{R}^{(N,+)} + R^{(N,+)} \right) (g, \Delta, 0)
\]

since \( R^{(N,+)}(g,\Delta,0) = \tilde{R}^{(N,-)}(g,\Delta,0) \) and \( \tilde{R}^{(N,+)}(g,\Delta,0) = \tilde{R}^{(N,-)}(g,\Delta,0) \).

**Remark 6.7.** By Corollary [62] for any fixed \( \Delta > 0 \), there are no common zeros between the constraint polynomial \( P_N^{(N,\varepsilon)}(2g, \Delta^2) \) and the \( T \)-function \( T_\varepsilon^{(N)}(g, \Delta) \).

In the same manner, we can define a \( T \)-function \( T_\varepsilon^{(N)}(g, \Delta) \) that vanishes for values \( g, \Delta \) corresponding to the non-Juddian exceptional eigenvalue \( \lambda = N - \varepsilon - g^2 \). Clearly, we have \( \tilde{T}_0^{(N)}(g, \Delta) = T_0^{(N)}(g, \Delta) \), and in general it is straightforward to verify that the identity

\[
T_\varepsilon^{(N)}(g, \Delta) = T_{-\varepsilon}^{(N)}(g, \Delta)
\]

(6.30)

holds (up to a constant) as in the case of \( \tilde{T}_\varepsilon^{(N,\varepsilon)}(2g, \Delta^2) \) (see [62] and also [37]).

We consider the particular case of \( \varepsilon = \ell/2 \) (\( \ell \in \mathbb{Z}_{\geq 0} \)). Then, from \( (6.26) \) we have \( \phi_{1,-}(y; \ell/2) = c\phi_{2,+}(1 - y; -\ell/2) \) and \( \phi_{1,+}(y; \ell/2) = c\phi_{2,-}(1 - y; -\ell/2) \). This shows that the non-Juddian exceptional solution corresponding to \( \lambda = (N + \ell) - \ell/2 - g^2 = N + \ell/2 - g^2 \) whose existence is guaranteed by the constraint equation \( T_{\ell/2}^{(N)}(g, \Delta) = 0 \) (resp. \( T_{\ell/2}^{(N+\ell)}(g, \Delta) = 0 \)) are identical up to a scalar multiple. Since the non-Juddian exceptional solution is non-degenerate, the compatibility of this fact, that is, that \( T_{\ell/2}^{(N)}(g, \Delta) \) and \( T_{\ell/2}^{(N+\ell)}(g, \Delta) \) have the same zero with respect to \( g \) for a fixed \( \Delta \), is confirmed by the lemma below.

**Lemma 6.6.** For \( \ell, N \in \mathbb{Z}_{\geq 0} \) we have

\[
T_{\ell/2}^{(N+\ell)}(g, \Delta) = T_{\ell/2}^{(N)}(g, \Delta).
\]

**Proof.** From the definitions, we have \( K_\pm^{\Delta}(N - \ell/2; g, \Delta, -\ell/2) = K_\Delta^{\ell/2}(N + \ell/2; g, \Delta, \ell/2) \), therefore

\[
R^{(N+\ell,\pm)}(g, \Delta, -\ell/2) = \bar{R}^{(N,\mp)}(g, \Delta, \ell/2),
\]

\[
R^{(N+\ell,\pm)}(g, \Delta, -\ell/2) = R^{(N,\mp)}(g, \Delta, \ell/2).
\]

Hence, it follows that

\[
\begin{align*}
\tilde{T}_{\ell/2}^{(N+\ell)}(g, \Delta) &= T_{-\ell/2}^{(N+\ell)}(g, \Delta) \\
&= \left( \bar{R}^{(N,\mp)} - R^{(N,\pm)} \right) (g, \Delta; -\ell/2) \\
&= \left( \bar{R}^{(N,\mp)} - R^{(N,\pm)} \right) (g, \Delta; \ell/2) \\
&= T_{\ell/2}^{(N)}(g, \Delta).
\end{align*}
\]
By the discussion above, the condition \( T_\varepsilon^{(N)}(g, \Delta) = 0 \) (resp. \( T_\varepsilon^{(N)}(g, \Delta) = 0 \)) can be indeed be regarded as the constraint equation for the exceptional eigenvalues \( \lambda = N + \varepsilon - g^2 \) (resp. \( \lambda = N - \varepsilon - g^2 \)) with non-Juddian exceptional solutions.

We illustrate numerically the constraint relations \( P_N^{(N,\varepsilon)}((2g)^2, \Delta^2) = 0 \) (for Juddian eigenvalues) and \( T_\varepsilon^{(N)}(g, \Delta) = 0 \) (for non-Juddian exceptional eigenvalues) in Figure 9 showing the curves in the \((g, \Delta)\)-plane defined by these relations for \( \varepsilon = 0.45 \) and \( N = 3 \). Concretely, Figures 9(a) and 9(b) depict the graph of the curve \( G_\varepsilon(x, g, \Delta) = 0 \) for the values \( x = 3.2 \) and \( x = 3.4 \), while Figure 9(c) shows the graph of the curve \( T_\varepsilon^{(3)}(g, \Delta) = 0 \) in continuous line and \( P_3^{(3,\varepsilon)}((2g)^2, \Delta^2) = 0 \) in dashed line. Notice that as \( x \to 3.45 \) adjacent closed curves near the origin in the graph of \( G_\varepsilon(x, g, \Delta) = 0 \) approach each other. Some of these curves join to form the closed curves (ovals) of \( P_N^{(N,\varepsilon)}((2g)^2, \Delta^2) = 0 \), corresponding to Juddian eigenvalues, while others form curves in the graph of \( T_\varepsilon^{(N)}(g, \Delta) = 0 \), corresponding to non-Juddian exceptional eigenvalues. Also observe that we have ovals (corresponding to non-Juddian solutions) near the origin of the graph in Figure 9(c), some of them very close to dashed ovals (corresponding to Juddian eigenvalues).

Figure 9: Curves of constraint relations for fixed regular eigenvalues ((a),(b)) and exceptional eigenvalues ((c)) for \( \varepsilon = 0.45 \) for \(-3 \leq g \leq 3 \) and \(-10 \leq \Delta \leq 10 \).

On the other hand, the case \( \varepsilon \in \frac{1}{2}\mathbb{Z} \geq 0 \) is illustrated in Figure 10. As in the case above, Figures 10(a) and 10(b) depict the curves given by the relation \( G_\varepsilon(x, g, \Delta) = 0 \) for the values \( x = 3.2 \) and \( x = 3.4 \), while Figure 10(c) shows the graph of the curve \( T_\varepsilon^{(3)}(g, \Delta) = 0 \) in continuous line and \( P_3^{(3,\varepsilon)}((2g)^2, \Delta^2) = 0 \) in dashed line. Different from the case \( \varepsilon \notin \frac{1}{2}\mathbb{Z} \geq 0 \) above, there are no continuous ovals (non-Juddian) near the origin in Figure 10(c). It is worth mentioning that Figure 9(c) and Figure 10(c) support the conceptual graphs of Figure 2(a) and Figure 2(b) presented in the Introduction. Actually, we can observe there are both dashed (Juddian) and continuous (non-Juddian) ovals when \( \varepsilon = 0.45 \) in Figure 9(c), while the continuous ovals disappear when \( \varepsilon = \frac{1}{2}(\varepsilon \in \frac{1}{2}\mathbb{Z}) \) in Figure 10(c) (see Corollary 4.15 and its subsequent Remark 4.3).

We next generalize Lemma 6.4, by including the exceptional eigenvalues. As a result, we see that the spectrum of \( H_\varepsilon^{\text{Rabi}} \) does not depend on the sign of \( \varepsilon \).

**Proposition 6.7.** The spectrum of the Hamiltonian \( H_\varepsilon^{\text{Rabi}} \) of AQRM depends only on \( |\varepsilon| \). In other words, the spectrum of Hamiltonian \( H_{\varepsilon}^{\text{Rabi}} \) coincides with that of \( H_{\varepsilon}^{\text{Rabi}} \).
Proof. For the regular spectrum, since $G_{-\varepsilon}(x,g,\Delta) = G_{\varepsilon}(x,g,\Delta)$ by Lemma 6.4 the result follows immediately. Moreover, since the constraint polynomials $P_{N,\varepsilon}(x,y)$ of $H_{\text{Rabi}}^\varepsilon$ are also constraint polynomials $P_{N,\varepsilon}^{\Omega}(x,y)$ of $H_{\text{Rabi}}^{-\varepsilon}$, the result holds for Juddian eigenvalues as well. Finally, if $g$ is a positive zero of $T_{\varepsilon}^N(g,\Delta)$, that is, $\lambda = N + \varepsilon - g^2$ is a non-Juddian exceptional eigenvalue of $H_{\text{Rabi}}^\varepsilon$, then, $\lambda = N + \varepsilon - g^2 = N - (-\varepsilon) - g^2$ is also a non-Juddian exceptional eigenvalue of $H_{\text{Rabi}}^{-\varepsilon}$ since $g$ is actually a zero of $\tilde{T}_{\varepsilon}^N(g,\Delta)(= T_{\varepsilon}^N(g,\Delta))$. Hence the assertion follows.

Remark 6.8. The graphs in Figures 9 and 10 might explain the conical structure discussed in [10]. It is therefore important to understand why ovals corresponding to the non-Juddian exceptional eigenvalues appear in Figure 9(c).

Remark 6.9. Proposition 6.7 follows also from the comparison of the two systems of differential equations (3.3) and (3.4). It should be also noted that the proposition does not imply that $N - \varepsilon - g^2$ gives an eigenvalue of $H_{\text{Rabi}}^{-\varepsilon}$ even if $N + \varepsilon - g^2$ is an eigenvalue of $H_{\text{Rabi}}^\varepsilon$.

Remark 6.10. For a fixed $\Delta > 0$, define the involution $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$

$$\sigma : (y,\varepsilon) \to (y + 2\varepsilon, -\varepsilon).$$

Associating a tuple $(y,\varepsilon) \in \mathbb{R}^2$ to each eigenvalue $\lambda = y + \varepsilon - g^2$ of $H_{\text{Rabi}}^\varepsilon$, we easily see that the eigenvalues of $H_{\text{Rabi}}^\varepsilon$ are invariant under $\sigma$. For instance, if the eigenvalue $\lambda$ is a regular, we have $\lambda = x - g^2$ for some $x \in \mathbb{R} (x \neq N \pm \varepsilon)$, thus $y = x - \varepsilon$ and the image $\sigma(x - \varepsilon,\varepsilon) = (x + \varepsilon, -\varepsilon)$ corresponds to same eigenvalue $\lambda = (x + \varepsilon) - \varepsilon - g^2$ under this interpretation. The case of exceptional eigenvalues follows in a similar manner. It is an interesting problem to relate the involution $\sigma$ with the identities (4.1), (6.15) and (6.31) when $\varepsilon$ is a half integer. Actually, it is widely believed among physicists (cf. [2]) that there must be a symmetry if there exist energy level crossings (i.e. spectral degeneration) like we have Juddian eigenvalues for a half-integral $\varepsilon$. See also e.g. [18] for further discussion on the symmetry for the $\varepsilon = 0$ case.
6.5 Residues of $G$-function and spectral determinants of $H^\varepsilon_{Rabi}$

In this subsection, we return to the discussion of the structure of poles of the $G$-function started in §6.3. This enables us to deepen the understanding of non-Juddian exceptional eigenvalues. Also, we establish the relation between the $G$-function and the spectral determinant (i.e., the zeta regularized product of the spectrum [38][39]) of the Hamiltonian $H^\varepsilon_{Rabi}$. This can be regarded as a mathematical refinement of the discussion partially made in [38].

Formally, to study the behavior of the $G$-function $G_\varepsilon(x; g, \Delta)$ at a point $x = N \pm \varepsilon$ ($N \in \mathbb{Z}_{\geq 0}$) we consider a sufficiently small punctured disc centered at a fixed point $x = N \pm \varepsilon$ and compute the residue of $G_\varepsilon(x; g, \Delta)$ as a function of the parameters $g$ and $\Delta$. According to the value of the residue for the parameters $g$ and $\Delta$ we classify the singularity as a removable singularity or a pole. In the case of a removable singularity we consider the $G$-function $G_\varepsilon(x; g, \Delta)$ as a function defined at $x = N \pm \varepsilon$ for the particular parameters $g$ and $\Delta$. It is clear from the definition that the only singularities of $G_\varepsilon(x; g, \Delta)$ (as a function of $x$) appear at the points $x = N \pm \varepsilon$ ($N \in \mathbb{Z}_{\geq 0}$) and that all singularities are either removable singularities or poles. To simplify the notation, we say that a function has a pole of order $\leq N$ when it has a removable singularity or a pole of order at most $N$.

We consider the case $\varepsilon \not\in \frac{1}{2}\mathbb{Z}$ and $\varepsilon \not\in \frac{1}{2}\mathbb{Z}$ by separate. For the case of $\varepsilon \not\in \frac{1}{2}\mathbb{Z}$, by the defining recurrence formula (6.13), we observe that the rational functions $K_1^\varepsilon(x)$, for $n \geq N + 1$, have poles of order $\leq 1$ at $x = N \pm \varepsilon$. Hence, $G_\varepsilon(x; g, \Delta)$ has a pole of order $\leq 1$ at $x = N \pm \varepsilon$. The residue of the $G$-function at a point $x = N \pm \varepsilon$ is given in the following result.

**Proposition 6.8.** Let $\varepsilon \not\in \frac{1}{2}\mathbb{Z}$. Then any pole of the $G$-function $G_\varepsilon(x; g, \Delta)$ is simple. If $N \in \mathbb{Z}_{\geq 0}$, the residue of $G_\varepsilon(x; g, \Delta)$ at the points $x = N \pm \varepsilon$ is given by

$$\text{Res}_{x=N\pm\varepsilon} G_\varepsilon(x; g, \Delta) = C(N)\Delta^2 P_{\varepsilon}(N; x; g, \Delta),$$

where $C(N) = \frac{1}{N(N+1)!}$.

**Proof.** We give the proof for the case $x = N + \varepsilon$, for the case $x = N - \varepsilon$ is completely analogous. From the definition of $f^-_N(x, g, \Delta, \varepsilon)$, it is clear that $\text{Res}_{x=N\pm\varepsilon} f^-_N(x, g, \Delta, \varepsilon) = \frac{\Delta^2}{2g(N+1)}\delta_N(n)$, where $\delta_N(n)$ is the Kronecker delta function. Likewise, for $0 \leq n \leq N$ it is clear that $\text{Res}_{x=N\pm\varepsilon} K^-_N(x; g, \Delta, \varepsilon) = 0$, and for $n = N + 1$ we have

$$\text{Res}_{x=N\pm\varepsilon} K^-_{N+1}(x) = \lim_{x\to N\mp\varepsilon} (x - N - \varepsilon) \frac{1}{N+1} \left( f^-_N(x)K^-_N(x) - K^-_{N-1}(x) \right)$$

$$= \frac{1}{N+1} K^-_{N}(N + \varepsilon) \text{Res}_{x\to N\pm\varepsilon} f^-_N(x) = \frac{\Delta^2}{2g(N+1)} K^-_{N}(N + \varepsilon).$$

Setting $a_0 = 0$, $a_1 = 1$, and

$$a_k = \frac{1}{N+k} \left( f^-_{N+k-1}(N + \varepsilon)a_{k-1} - a_{k-2} \right),$$

for $k \geq 2$, it is easy to see that $\text{Res}_{x=N\pm\varepsilon} K^-_{N+k}(x) = \frac{\Delta^2}{2g(N+1)} K^-_{N}(N + \varepsilon)a_k$. Furthermore, by the same method of the proof of Proposition 6.1, we observe that

$$(2g)^{k-1}a_k = K^-_{N+k}(N + \varepsilon; g, \Delta, \varepsilon),$$
for \( k \geq 1 \), where \( \tilde{K}_{N+k}^- (N + \varepsilon; g, \Delta, \varepsilon) \) are the coefficients of \( \phi_{1,-}(y; \varepsilon) \) in (6.18). Then, from the definition, \( R^{(N,-)}(g, \Delta; \varepsilon) \) is given by

\[
\phi_{1,-}\left(\frac{1}{2}; \varepsilon\right) = \sum_{n=N+1}^{\infty} \tilde{K}_n^- (N + \varepsilon; g, \Delta, \varepsilon) \left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^{N+1} \sum_{n=N+1}^{\infty} \tilde{K}_n^- (N + \varepsilon; g, \Delta, \varepsilon) \left(\frac{1}{2}\right)^{n-N-1} = \left(\frac{1}{2}\right)^{N+1} \sum_{n=N+1}^{\infty} a_{n-N} g^{n-N-1},
\]

and, similarly, \( \tilde{R}^{(N,-)}(g, \Delta; \varepsilon) \) is given by

\[
\phi_{1,+}\left(\frac{1}{2}; \varepsilon\right) = \frac{(N+1)}{\Delta} \left(\frac{1}{2}\right)^N - \Delta \sum_{n=N+1}^{\infty} \tilde{K}_n^- (N + \varepsilon; g, \Delta, \varepsilon) \left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^{N+1} \left(\frac{2(N+1)}{\Delta}\right) - \Delta \sum_{n=N+1}^{\infty} \tilde{K}_n^- (N + \varepsilon; g, \Delta, \varepsilon) \left(\frac{1}{2}\right)^{n-N-1} = \left(\frac{1}{2}\right)^{N+1} \left(2(N+1)\Delta\right) - \Delta \sum_{n=N+1}^{\infty} a_{n-N} g^{n-N-1}.\]

Moreover, we recall that for \( \varepsilon \notin \frac{1}{2} \mathbb{Z} \), both functions \( R^+(x) \) and \( \tilde{R}^+(x) \) are analytic at \( x = N + \varepsilon \) and

\[
R^+(N + \varepsilon) = R^{(N,+)}(g, \Delta; \varepsilon), \quad \Delta \tilde{R}^+(N + \varepsilon) = \tilde{R}^{(N,+)}(g, \Delta; \varepsilon).
\]

With these preparations, we compute \( \text{Res}_{x=N+\varepsilon} R^+(x) \tilde{R}^-(x) \) as

\[
R^+(N + \varepsilon) \ \text{Res}_{x=N+\varepsilon} \sum_{n=0}^{\infty} K^-_N(x) g^n = \frac{\Delta^2}{2g(N+1)} K^-_N(N + \varepsilon) R^+(N + \varepsilon) \sum_{n=N+1}^{\infty} a_{n-N} g^n = \frac{(2g)^N \Delta^2}{(N+1)} K^-_N(N + \varepsilon) R^{(N,+)}(g, \Delta; \varepsilon) \tilde{R}^{(N,-)}(g, \Delta; \varepsilon).
\]

and, since \( \text{Res}_{x=N+\varepsilon} K^-_N(x) = K^-_N(N + \varepsilon) \) holds trivially, we also obtain

\[
\text{Res}_{x=N+\varepsilon} \frac{\Delta^2 \tilde{R}^+(x) \tilde{R}^-(x)}{x - n - \varepsilon} = \Delta^2 \tilde{R}^+(N + \varepsilon) \ \text{Res}_{x=N+\varepsilon} \sum_{n=0}^{\infty} \frac{K^-_N(x)}{x - n - \varepsilon} g^n = \Delta^2 \tilde{R}^+(N + \varepsilon) \left( K^-_N(N + \varepsilon) g^N - \frac{\Delta^2}{2g(N+1)} K^-_N(N + \varepsilon) \sum_{n=N+1}^{\infty} \frac{a_{n-N} g^n}{n-N} \right) = \frac{g^N \Delta^2}{2(N+1)} K^-_N(N + \varepsilon) \left( \Delta \tilde{R}^+(N + \varepsilon) \right) \left(\frac{2(N+1)}{\Delta}\right) - \Delta \sum_{n=N+1}^{\infty} \frac{a_{n-N} g^{n-N-1}}{n-N} = \frac{(2g)^N \Delta^2}{(N+1)} K^-_N(N + \varepsilon) \tilde{R}^{(N,+)}(g, \Delta; \varepsilon) \tilde{R}^{(N,-)}(g, \Delta; \varepsilon).
\]
Finally, using Lemma 6.5 we have

\[
\text{Res}_{x=N+\varepsilon} G_\varepsilon(x; g, \Delta) = \frac{(2g)^N \Delta^2}{(N+1)!} K_{N+\varepsilon}(N+\varepsilon) T_{\varepsilon}^{(N)}(g, \Delta)
\]

\[
= \frac{\Delta^2}{N!(N+1)!} p_{N}^{(N,\varepsilon)}((2g)^2, \Delta^2) T_{\varepsilon}^{(N)}(g, \Delta),
\]

which is the desired result.

Remark 6.11. We make a remark on the relation \((2g)^{k-1} a_k = K_{N+1}(N+\varepsilon; g, \Delta, \varepsilon)\) appearing in the proof of the proposition above. The coefficients \(K_{N+1}(x; g, \Delta, \varepsilon)\) (and thus the numbers \(a_k\)) in the definition of the \(G\)-function \(G_\varepsilon(x; g, \Delta)\) arise from the solution of the system of differential equations (3.1) by using the change of variable \(y = g + z\) (instead of \(y = \frac{g + z}{2g}\)). The use of the change of variable \(y = \frac{g + z}{2g}\) results on the system (3.3) compatible with the representation theoretical description of Proposition 3.2 and therefore we use the solutions arising from this system for the definition of the \(T\)-function \(T_{\varepsilon}^{(N)}(g, \Delta)\) (see § 6.2). We also note that it is possible to equivalently redefine the \(G\)-function \(G_\varepsilon(x; g, \Delta)\) using the solutions of the system (3.3) (i.e. with the change of variable \(y = \frac{g + z}{2g}\)), however, we use the definition given in § 6.3 since it is standard in the literature, including e.g., 3 37 63.

The proposition above completely characterizes the poles of the \(G\)-function for the case \(\varepsilon \notin \frac{1}{2}\mathbb{Z}\) in terms of the exceptional spectrum of \(H_{\text{Rabi}}\). In particular, it shows that the function \(G_\varepsilon(x; g, \Delta)\) is finite at points \(x = N \pm \varepsilon (N \in \mathbb{Z}_{\geq 0})\) corresponding to non-Juddian exceptional eigenvalues \(\lambda = N \pm \varepsilon - g^2\) (i.e. the parameters \(g\) and \(\Delta\) are positive zeros of \(T_{\varepsilon}^{(N)}(g, \Delta)\)). This situation is illustrated in Figure 11(a) for the parameters \(\varepsilon = 0.3, g = 0.8695, \Delta = 1/2\), showing the finite value of \(G_{0.3}(x; g, 1/2)\) at \(x = 1.3\). Here, \(g \approx 0.8695\) is a root (computed numerically) of \(T_{0.3}^{(1)}(g, 1/2)\). By Corollary 6.2, this value of \(g\) must be different to the value \(g' \approx 0.5809\) in the Juddian case, shown in Figure 7(a), which also has a finite value of \(G_{0.3}(x; g', 1/2)\) at \(x = 1.3\).

The following corollary justifies the claim that the exceptional eigenvalues \(\lambda = N \pm \varepsilon - g^2\) vanish (or “kill”) the poles of the \(G\)-function.

Corollary 6.9. Suppose \(\varepsilon \notin \frac{1}{2}\mathbb{Z}, N \in \mathbb{Z}_{\geq 0}\) and \(\Delta > 0\). Then, \(H_{\text{Rabi}}'\) has the exceptional eigenvalue \(\lambda = N \pm \varepsilon - g^2\) if and only if the \(G\)-function \(G_\varepsilon(x; g, \Delta)\) does not have a pole at \(x = N \pm \varepsilon\).

Next, we consider the case \(\varepsilon = \ell/2 (\ell \in \mathbb{Z}_{\geq 0})\). On the one hand, the functions \(R^+(x; g, \Delta, \varepsilon)\) and \(R^-(x; g, \Delta, \varepsilon)\) (resp. \(R^+_{\ell/2}(x; g, \Delta, \varepsilon)\) and \(R^-_{\ell/2}(x; g, \Delta, \varepsilon)\)) have poles of order \(\leq 1\) at points \(x = N + \ell/2 (N \in \mathbb{Z}_{\geq 0})\). On the other hand, at points \(x = N - \ell/2\) with \(0 \leq N \leq \ell - 1\) only the functions \(R^+(x; g, \Delta, \varepsilon)\) and \(R^+_{\ell/2}(x; g, \Delta, \varepsilon)\) have poles of order \(\leq 1\). Consequently, the \(G\)-function \(G_{\ell/2}(x; g, \Delta)\) has poles of order \(\leq 1\) at the points \(x = N - \ell/2 (0 \leq N \leq \ell - 1)\) and poles of order \(\leq 2\) at points \(x = N + \ell/2 (N \in \mathbb{Z}_{\geq 0})\). Note that all possible poles of the \(G\)-function are accounted since \(N = \ell + i (i \in \mathbb{Z}_{\geq 0})\) yields \(x = N - \ell/2 = \ell + i - \ell/2 = i + \ell/2 (i \in \mathbb{Z}_{\geq 0})\). The residue at the poles of order \(\leq 1\) are given in the following proposition. The proof is identical to Proposition 6.8 and is therefore omitted.
Determinant expression of constraint polynomials and spectrum of AQRM

Figure 11: Plot of $G_\varepsilon(x; g, \Delta)$ for parameters $g$ and $\Delta$ corresponding to a non-Juddian eigenvalue $\lambda = 1 \pm \varepsilon - g^2$ (i.e. $T_\pm^{(1)}(g, \Delta) = 0$). A finite value of $G_\varepsilon(x; g, \Delta)$ at $x = 1 \pm \varepsilon$ is indicated by a dashed circle, while a simple pole at $x = 1 + \varepsilon$ is indicated with a dashed vertical line.

Proposition 6.10. Suppose $\ell > 1$ and let $1 \leq N < \ell$. Then any pole of the $G$-function $G_{\ell/2}(x; g, \Delta)$ at a point $x = N - \ell/2$ is simple. The residues of $G_{\ell/2}(x; g, \Delta)$ at the point $x = N - \ell/2$ is given by

$$\text{Res}_{x=N-\ell/2} G_{\ell/2}(x; g, \Delta) = C(N)\Delta^2 P_{N-\ell/2}((2g)^2, \Delta^2)T_{\ell/2}((N+\ell, -\ell/2)\Delta_{N+\ell})$$

with $C(N) = \frac{1}{N(N+1)!}$.

Similar to the non half-integer case, the residues of $G_{\ell/2}(x; g, \Delta)$ at the points $x = N - \ell/2$ with $1 \leq N < \ell$ depend on the constraint polynomial $P_{N-\ell/2}((2g)^2, \Delta^2)$ and $T$-function $T_{\ell/2}((g, \Delta))$ for $1 \leq N < \ell$. However, by Proposition 4.13 $P_{N-\ell/2}((2g)^2, \Delta^2)$ is positive for $g, \Delta > 0$, in other words, the pole vanishes (i.e. it is a removable singularity) if and only if $T_{\ell/2}((g, \Delta)) = 0$, as illustrated in Figure 11(b).

In the following proposition we consider the remaining poles of $G_{\ell/2}(x; g, \Delta)$.

Proposition 6.11. Suppose $\varepsilon = \ell/2$ ($\ell \in \mathbb{Z}_{\geq 0}$) and let $N \in \mathbb{Z}_{\geq 0}$. Let

$$G_{\ell/2}(x; g, \Delta) = \frac{A}{(x-N-\ell/2)^2} + \frac{B}{x-N-\ell/2} + H_{\ell/2}(x; \Delta, g)$$

for a function $H_{\ell/2}(x; \Delta, g)$ analytic at $x = N + \ell/2$. We have

$$A = C(N)C(N+\ell)\Delta^4 P_{N-\ell/2}^{(N+\ell, -\ell/2)}((2g)^2, \Delta^2)P_{N+\ell}^{(N, \ell/2)}((2g)^2, \Delta^2)T_{\ell/2}((N, -\ell/2)\Delta_{N+\ell})$$

where $C(N)$ is defined as in Proposition 6.8 and
Similarly, setting

\[ Q + (x; g, \Delta) = C(N)C(N + \ell)\Delta^2 P_N^{(N, \ell/2)}((2g)^2, \Delta^2) \]

\[ \times \left[ \frac{1}{C(N + \ell)} \left( \tilde{R}^{(N,-)}(g, \Delta, \frac{\ell}{2})(\Delta \tilde{Q}^+(N + \frac{\ell}{2}; g, \Delta)) \right.ight.

\[ \left. - \tilde{R}^{(N,-)}(g, \Delta)Q^+(N + \frac{\ell}{2}; g, \Delta) \right) + A_N((2g)^2, \Delta^2) \frac{1}{C(N)} \left( \tilde{R}^{(N,+)}(g, \Delta, \frac{\ell}{2})(\Delta \tilde{Q}^-(N + \frac{\ell}{2}; g, \Delta)) \right.

\[ \left. - \tilde{R}^{(N,+)}(g, \Delta, \frac{\ell}{2})Q^-(N + \frac{\ell}{2}; g, \Delta) \right) \right], \]

where \( Q^+(x; g, \Delta) \) is defined by \( Q^+(x; g, \Delta) = R^+(x; g, \Delta) - \frac{\text{Res}_{x=N+\ell/2} R^+(x; g, \Delta)}{x-N-\ell/2} \). The functions \( Q^+(x; g, \Delta) \), \( Q^-(x; g, \Delta) \) and \( \tilde{Q}^-(x; g, \Delta) \) are defined similarly.

**Proof.** To compute the term \( A \) we notice that since each of the factors \( R^+ (x) \) and \( R^- (x) \) (resp. \( \tilde{R}^+ (x) \) and \( \tilde{R}^- (x) \)) can have a pole of order exactly one (simple pole) we have

\[ \lim_{x \to N+\ell/2} (x - N - \ell/2)^2 R^+(x)R^-(x) = \frac{\text{Res}_{x=N+\ell/2} R^+(x; g, \Delta)}{x-N-\ell/2} \frac{\text{Res}_{x=N+\ell/2} R^-(x; g, \Delta)}{x-N-\ell/2}, \]

and then the proof follows as in Proposition 6.8. The second claim follows from the basic identity

\[ \text{Res}_{x=a} \left( \frac{R_1}{x-a} + A_1(x) \right) \left( \frac{R_2}{x-a} + A_2(x) \right) = R_1A_2(a) + R_2A_1(a), \]

valid for functions \( A_1(x), A_2(x) \) analytic at \( x = a \) and \( R_1, R_2 \in \mathbb{C} \). □

By comparing the recurrence relations of \( R^\pm (x; g, \Delta) \) and \( \tilde{R}^\pm (x; g, \Delta) \) with the residues (cf. the proof of Proposition 6.8) the functions \( Q^- (x; g, \Delta) \), \( \tilde{Q}^+(x; g, \Delta) \) and \( Q^+(x; g, \Delta) \) can also be expressed by recurrence relations. Namely, if we set \( s_{-1}(x; g, \Delta) = 0, s_0(x; g, \Delta) = 1, \)

\[ s_{N+1}(x; g, \Delta) = \frac{1}{N+1} \left( \left( 2g + \frac{N-x-\ell/2}{2g} \right) s_N(x; g, \Delta) - s_{N-1}(x; g, \Delta) \right), \]

and

\[ s_k(x; g, \Delta) = \frac{1}{k} \left( f_k^{-1}(x; g, \Delta) s_{k-1}(x; g, \Delta) - s_{k-2}(x; g, \Delta) \right), \]

for positive integer \( k \neq N + 1 \), we have

\[ Q^-(x; g, \Delta) = \sum_{n=0}^{\infty} s_n(x; g, \Delta)g^n, \quad \tilde{Q}^+(x; g, \Delta) = \sum_{n=0}^{\infty} s_n(x; g, \Delta) \frac{x-n-\ell/2}{g^n}. \quad (6.32) \]

Similarly, setting \( r_{-1}(x; g, \Delta) = 0, r_0(x; g, \Delta) = 1, \) and

\[ r_{N+\ell+1}(x; g, \Delta) = \frac{1}{N+\ell+1} \left( \left( 2g + \frac{N-x+\ell/2}{2g} \right) r_{N+\ell}(x; g, \Delta) - r_{N+\ell-1}(x; g, \Delta) \right), \]

and

\[ r_k(x; g, \Delta) = \frac{1}{k} \left( f_k^{-1}(x; g, \Delta) r_{k-1}(x; g, \Delta) - r_{k-2}(x; g, \Delta) \right), \]
for positive integer \( k \neq N + \ell + 1 \), we have

\[
Q^+(x; g, \Delta) = \sum_{n=0}^{\infty} r_n(x; g, \Delta) g^n, \quad \bar{Q}^+(x; g, \Delta) = \sum_{n \geq 0, n \neq N + \ell} \frac{r_n(x; g, \Delta)}{x - n + \ell/2} g^n.
\] (6.33)

Note that by Theorem 4.12, when \( \lambda = N + \ell/2 - g^2 \) is a Juddian eigenvalue, the coefficients of the poles of \( G_{\ell/2}(x; g, \Delta) \) vanish and the function \( G_{\ell/2}(x; g, \Delta) \) has a finite value at \( x = N + \ell/2 \). However, it is possible to find numerically examples of parameters such that \( G_{\ell/2}(x; g, \Delta) \) has a pole at \( x = N + \ell/2 \) yet \( \lambda = N + \ell/2 - g^2 \) is a non-Juddian exceptional eigenvalue. One such example is shown in Figure 11(c) for the parameters \( \epsilon = 1/2, g \approx 1.3903, \Delta = 1 \). In this case, there is a pole of \( G_{1/2}(x; g, 1) \) even though the parameters \( g \) and \( \Delta \) correspond (numerically) to a zero of \( T_{1/2}^{(1)}(g, 1) \) at \( x = 1.5 \). We remark that the pole \( x = 1.5 \) must be simple. Indeed, in the notation of Proposition 6.11 since \( T_{1/2}^{(1)}(g, 1) = 0 \) the second order term \( A \) vanishes while the residue term \( B \) is non-vanishing. This is also apparent in the graph of \( G_{1/2}(x; g, 1) \) in Figure 11(c), since the lateral limits at \( x = 1.5 \) have different signs the pole must be simple and the term \( B \) must be non-zero in a neighborhood of \( x = 1.5 \).

The situation for the poles of the \( G_{\ell/2}(x; g, \Delta) \) is summarized in the following result.

**Corollary 6.12.** Suppose \( \ell \in \mathbb{Z}_{\geq 0} \) and \( \Delta > 0 \). The \( G \)-function \( G_{\ell/2}(x; g, \Delta) \) has \( \ell \) poles of order \( \leq 1 \) at \( x = N - \ell/2 \) for \( 0 \leq N < \ell \) and poles of order \( \leq 2 \) at \( x = N + \ell/2 \) for \( N \in \mathbb{Z}_{\geq 0} \). Moreover, for \( N \in \mathbb{Z}_{\geq 0} \), we have:

- **If** \( \lambda = N \pm \ell/2 - g^2 \) is a Juddian eigenvalue of \( H_{\text{Rabi}}^{\ell/2} \), then \( x = N \pm \ell/2 \) is not a pole of \( G_{\ell/2}(x; g, \Delta) \).
- **For** \( 0 \leq N < \ell \), the function \( G_{\ell/2}(x; g, \Delta) \) does not have a pole at \( x = N - \ell/2 \) if and only if \( \lambda = N - \ell/2 - g^2 \) is a non-Juddian exceptional eigenvalue of \( H_{\text{Rabi}}^{\ell/2} \).
- **If** \( G_{\ell/2}(x; g, \Delta) \) has a simple pole at \( x = N + \ell/2 \), then \( \lambda = N + \ell/2 - g^2 \) is a non-Juddian exceptional eigenvalue of \( H_{\text{Rabi}}^{\ell/2} \).
- **If** \( G_{\ell/2}(x; g, \Delta) \) has a double pole at \( x = N + \ell/2 \), then there is no exceptional eigenvalue \( \lambda = N \pm \ell/2 - g^2 \) of \( H_{\text{Rabi}}^{\ell/2} \).

**Remark 6.12.** In the case of the QRM (i.e. \( \epsilon = 0 \)), all the singularities of the \( G \)-function \( G_0(x; g, \Delta) \) are of the type described in Proposition 6.11 (i.e. poles of order \( \leq 2 \)).

Note that it is possible that non-Juddian exceptional eigenvalues corresponding to finite values of \( G_{\ell/2}(x; g, \Delta) \) at points \( x = N \pm \epsilon \) are present in the spectrum. If such eigenvalues were to exist then the structure of the poles of the \( G \)-function alone would not be sufficient to completely discriminate the structure of the exceptional spectrum.

To get a better understanding of the vanishing of the residues \( \text{Res}_{x=N+\ell/2} G_{\ell/2}(x; g, \Delta) \), we define the function

\[
B_{\ell}^N(g, \Delta) := \frac{1}{C(N)} \left( \bar{R}^{(N,+)}(g, \Delta, \ell/2)(\Delta Q^{-}(N + \ell/2; g, \Delta)) - R^{(N,+)}(g, \Delta, \ell/2)Q^{-}(N + \ell/2; g, \Delta) \right).
\]
This function may be thought of a “regularized” $T$-function. In the case $T_{\ell/2}^{(N)}(g, \Delta) = 0$ (thus $P_N^{(N,\ell/2)}((2g)^2, \Delta^2) \neq 0$), by Proposition 6.11 and the proof of Lemma 6.6 the vanishing of the residue $\text{Res}_{x=N+\ell/2} G_{\ell/2}(x; g, \Delta)$ is equivalent to the equation
\[
B_{N+\ell}^0(g, \Delta) + A_N^0((2g)^2, \Delta^2) B_{N}^0(g, \Delta) = 0,
\]
(6.34)

resembling the divisibility problem of constraint polynomials. Actually, this equation distinguishes the cases where $x = N+\ell/2$ is a pole or not and the polynomial $A_N^\ell((2g)^2, \Delta^2)$ again may play a particular role for its determination. Also, in terms of the constraint polynomials, we notice that $B = 0$, i.e. (6.34), is equivalent to
\[
\begin{align*}
\det \begin{bmatrix}
P_N^{(N,\ell/2)}((2g)^2, \Delta^2) & -B_{N}^0(g, \Delta) \\
P_{N+\ell}^{(N+\ell,-\ell/2)}((2g)^2, \Delta^2) & B_{N+\ell}^0(g, \Delta)
\end{bmatrix} \\
= P_N^{(N,\ell/2)}((2g)^2, \Delta^2) \det \begin{bmatrix}
1 & -B_{N}^0(g, \Delta) \\
A_N^\ell((2g)^2, \Delta^2) & B_{N+\ell}^0(g, \Delta)
\end{bmatrix} = 0.
\end{align*}
\]

**Problem 6.1.** With the notation of Proposition 6.11

- Are there non-Juddian exceptional eigenvalues $\lambda = N \pm \ell/2 - g^2$ corresponding to finite values of $G$-function $G_{\ell/2}(x; g, \Delta)$ at the point $x = N \pm \ell/2$? If the answer is affirmative, what are the properties of these non-Juddian exceptional eigenvalues?

- More concretely, can we characterize the vanishing of $B$ (equivalently (6.34)) in terms of the function $T_{\ell/2}^{(N)}(g, \Delta)$? It would be quite interesting if the vanishing of $B$ can be formulated as a sort of duality of the equation $P_{N+\ell}^{(N+\ell,-\ell/2)} = A_N^\ell \cdot P_N^{(N,\ell/2)}$ in Theorem 4.12. We actually notice that $B = 0$ is equivalent to the fact that the vector $t[B_{N+\ell} B_N]$ is perpendicular to the vector $t[P_{N+\ell}^{(N,\ell/2)} P_{N+\ell}^{(N+\ell,-\ell/2)}]$.

As a first step for the understanding of this problem, we present the graphs in the $(g, \Delta)$-plane of the curves defined by the residue vanishing condition (6.34) and the constraint conditions for exceptional eigenvalues in Figure 12. In the graphs, we show the curve described by $T_{\ell/2}^{(N)}(g, \Delta) = 0$ in continuous gray lines, the curve given by $P_N^{(N,\ell/2)}((2g)^2, \Delta^2) = 0$ in dashed gray lines and the residue vanishing condition (6.34) in black lines. Figure 12(a) shows the case $N = 1$ and $\ell = 2$ while Figure 12(b) depicts the case $N = 3$ and $\ell = 1$. Notice that in both cases there appears to be intersections in the vanishing condition (6.34) and the constraint relation $T_{\ell/2}^{(N)}(g, \Delta) = 0$, in other words, there are non-Juddian eigenvalues which kill the corresponding (double) poles of the $G$-function $G_{\ell/2}(x; g, \Delta)$. While further investigation including numerical experiments is needed, the observations made on the numerical graphs shown in Figure 12 provide actually an evidence for the affirmative answer of the problem above. In addition, from Figure 12 we notice there are apparently no intersections between the curves of the Juddian constraint conditions and the curves of the vanishing condition (6.34), which may be related to the perpendicularity described in the problem above. Actually, further numerical experimentations we have done so far support that this observation can be true in general.
Hence, if there exists a constant $\beta$ then the equation (6.34) holds.

**Remark 6.13.** We make a small remark about Problem 6.1. It is obvious that

$$B_{\ell}^N(g, \Delta) = \frac{1}{C(N)} \det \begin{bmatrix} \bar{R}^{(N, +)}(g, \Delta, \ell/2) & Q^{-}(N + \ell/2; g, \Delta) \\ \bar{R}^{(N, +)}(g, \Delta, \ell/2) & \Delta \bar{Q}^{-}(N + \ell/2; g, \Delta) \end{bmatrix}. \quad (6.35)$$

It follows also that

$$B_{-\ell}^{N+\ell}(g, \Delta) = \frac{1}{C(N + \ell)} \det \begin{bmatrix} \bar{R}^{(N+\ell, +)}(g, \Delta, -\ell/2) & Q^{+}(N + \ell/2; g, \Delta) \\ \bar{R}^{(N+\ell, +)}(g, \Delta, -\ell/2) & \Delta \bar{Q}^{+}(N + \ell/2; g, \Delta) \end{bmatrix} = \frac{1}{C(N + \ell)} \det \begin{bmatrix} \bar{R}^{(N, -)}(g, \Delta, \ell/2) & Q^{+}(N + \ell/2; g, \Delta) \\ \bar{R}^{(N, -)}(g, \Delta, \ell/2) & \Delta \bar{Q}^{+}(N + \ell/2; g, \Delta) \end{bmatrix}. \quad (6.36)$$

Since

$$T_{\ell/2}^{(N)}(g, \Delta) = \det \begin{bmatrix} \bar{R}^{(N, -)}(g, \Delta, \ell/2) & R^{(N, +)}(g, \Delta, \ell/2) \\ \bar{R}^{(N, -)}(g, \Delta, \ell/2) & R^{(N, +)}(g, \Delta, \ell/2) \end{bmatrix} = \epsilon_N \begin{bmatrix} \bar{R}^{(N, +)}(g, \Delta, \ell/2) \\ \bar{R}^{(N, -)}(g, \Delta, \ell/2) \end{bmatrix}. \quad (6.37)$$

It follows that

$$B_{-\ell}^{N+\ell}(g, \Delta) = -\frac{\epsilon_N}{C(N + \ell)} \det \begin{bmatrix} \bar{R}^{(N+\ell, +)}(g, \Delta, -\ell/2) & \bar{Q}^{+}(N + \ell/2; g, \Delta) \\ \bar{R}^{(N+\ell, +)}(g, \Delta, -\ell/2) & \Delta \bar{Q}^{+}(N + \ell/2; g, \Delta) \end{bmatrix}. \quad (6.38)$$

Hence, if there exists a constant $\beta_N = \beta_N(g, \Delta, \ell/2)$ such that

$$\begin{bmatrix} \bar{R}^{(N, +)}(g, \Delta, \ell/2) \\ \bar{R}^{(N, +)}(g, \Delta, \ell/2) \end{bmatrix} = \beta_N \left\{ \frac{\epsilon_N}{C(N + \ell)} \begin{bmatrix} \Delta \bar{Q}^{+}(N + \ell/2; g, \Delta) \\ \bar{Q}^{+}(N + \ell/2; g, \Delta) \end{bmatrix} + \frac{A^2_N((2g)^2, \Delta^2)}{C(N)} \begin{bmatrix} Q^{-}(N + \ell/2; g, \Delta) \\ \Delta \bar{Q}^{-}(N + \ell/2; g, \Delta) \end{bmatrix} \right\},$$

then the equation (6.34) holds.
In the paper of Li and Batchelor [38] (p. 4), the authors define a new $G$-function $G_\varepsilon(x; g, \Delta)$ for numerical computation of the spectrum of the AQRM. The new definition uses a divergent product to make the function $G_\varepsilon(x; g, \Delta)$ vanish for all eigenvalues of AQRM, including the exceptional ones (i.e. at $x = N \pm \varepsilon$). We note that, however, it is not well-defined theoretically due to the use of the divergent product. Nevertheless, according to the following theorem, the numerical observation in [38] by taking a certain truncation of the divergent product does seem to work properly. To obtain a correct understanding, we use the gamma function $\Gamma(x)$ to alternatively define the new $G$-function $G_\varepsilon(x; g, \Delta)$ as

$$G_\varepsilon(x; g, \Delta) := G_\varepsilon(x; g, \Delta)\Gamma(\varepsilon - x)^{-1}\Gamma(-\varepsilon - x)^{-1}. \quad (6.39)$$

As a consequence of our discussion above on the poles of the $G$-function, we can establish the claim made in [38].

**Theorem 6.13.** For fixed $g, \Delta > 0$, $x$ is a zero of $G_\varepsilon(x; g, \Delta)$ if and only if $\lambda = x - g^2$ is an eigenvalue of $H_{\text{Rabi}}^\varepsilon$.

**Proof.** The statement for regular eigenvalues is clear since the factor $\Gamma(\varepsilon - x)^{-1}\Gamma(-\varepsilon - x)^{-1}$ does not contribute any further zeros in this case. Next, suppose $\varepsilon \notin \frac{1}{2} \mathbb{Z}$. Then, the point $x = N + \varepsilon$ is a simple zero of $\Gamma(\varepsilon - x)^{-1}$, therefore $G_\varepsilon(N + \varepsilon; g, \Delta) = C \text{Res}_{x=N+\varepsilon} G_\varepsilon(x; g, \Delta)$ for a nonzero constant $C \in \mathbb{C}$ and the result follows from Proposition 6.8. In the case of $\varepsilon = \ell/2 (\ell \in \mathbb{Z}_{\geq 0})$, the result for $x = N - \ell/2$ with $0 \leq N < \ell$ follows by Proposition 6.10 in the same way as the case $\varepsilon \notin \frac{1}{2} \mathbb{Z}$. Similarly, notice that the double zero of $\Gamma(\ell/2 - x)^{-1}\Gamma(-\ell/2 - x)^{-1}$ at $x = N + \ell/2 (N \in \mathbb{Z}_{\geq 0})$ makes $G_\varepsilon(x; g, \Delta)$ equal (up to a nonzero constant) to the coefficient $A$ of $(x - N - \ell/2)^{-2}$ in the Laurent expansion of $G_\varepsilon(x; g, \Delta)$ at $x = N + \ell/2$ given in the Proposition 6.11. Hence the theorem follows.

We now recall the so-called spectral determinant of the Hamiltonian $H_{\text{Rabi}}^\varepsilon$ of the AQRM. Let $\lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq \ldots$ be the set of all eigenvalues of the Hamiltonian $H_{\text{Rabi}}^\varepsilon$ of the AQRM. Here note that the first eigenvalue $\lambda_0$ is always simple (see the proof of Corollary 6.2). Then the Hurwitz-type spectral zeta function of the AQRM is defined by

$$\zeta_{H_{\text{Rabi}}^\varepsilon}(s, \tau) = \sum_{i=0}^{\infty} (\tau - \lambda_i)^{-s}, \quad \text{Re}(s) > 1. \quad (6.40)$$

Here we fix the log-branch by $-\pi \leq \arctan(\tau - \lambda_i) < \pi$. We then define the zeta regularized product (cf. [39]) over the spectrum of the AQRM as

$$\prod_{i=0}^{\infty} (\tau - \lambda_i) := \exp \left( - \frac{d}{ds} \zeta_{H_{\text{Rabi}}^\varepsilon}(s, \tau) \right)_{s=0}. \quad (6.41)$$

We can prove that $\zeta_{H_{\text{Rabi}}^\varepsilon}(s, \tau)$ is holomorphic at $s = 0$ by the same way as in the case of the QRM [56]. Actually, the meromorphy of $\zeta_{H_{\text{Rabi}}^\varepsilon}(s, \tau)$ in the whole plane $\mathbb{C}$ follows in a similar way to the case $H_{\text{Rabi}}^0$. As the notation may indicate that this regularized product is an entire function possessing its zeros exactly at the eigenvalues of $H_{\text{Rabi}}^\varepsilon$. Now define the spectral determinant of the AQRM $\det H_{\text{Rabi}}^\varepsilon$ as

$$\det(\tau - H_{\text{Rabi}}^\varepsilon) := \prod_{i=0}^{\infty} (\tau - \lambda_i). \quad (6.42)$$

The following result follows immediately from Theorem 6.13.
Corollary 6.14. There exists an entire non-vanishing function $c_\varepsilon(\tau; g, \Delta)$ such that
\[
\det(\tau - g^2 - H_{\text{Rabi}}^\varepsilon) = c_\varepsilon(\tau; g, \Delta) G_\varepsilon(\tau; g, \Delta).
\] (6.43)

Remark 6.14. The product of the two gamma functions in (6.39) may be interpreted as a sort of “gamma factor” in a sense of the zeta function theory (see e.g. [19]) but it is hard to expect any functional equation satisfied by this spectral determinant. On the other hand, the study of the special values $\zeta_{H_{\text{Rabi}}}^0(n, 0) (n = 2, 3, \ldots)$ of the spectral zeta function as in [31] is awaited in the future research. This study may build a bridge between the arithmetics (particularly, modular forms and Apéry-like numbers, e.g. [66]) and the spectrum of the AQRM. Actually, the value $\frac{d}{ds}\zeta_{H_{\text{Rabi}}}^0(0, 0)$ (see (6.41)) may be thought to giving an analogue of the Euler constant (the constant term of the Riemann zeta function $\zeta(s)$ at $s = 1$) and hence, considered also as the “special value” (constant term) at $s = 1$ which describes the Weyl law [56] (notice that since $s = 1$ is a pole of the zeta functions, $\frac{d}{ds}\zeta(0)$ can be considered as the special value of $\zeta(s)$ at $s = 1$ by the functional equation). This result is important because, among other reasons, the Weyl law is relevant to the conjecture on the distribution of eigenvalues for $H_{\text{Rabi}}^0$ by Braak [5]. The conjecture by Braak on the distribution of eigenvalues of the QRM (i.e. $\varepsilon = 0$ case) can be summarized [9] as follows: The number of eigenvalues $H_{\text{Rabi}}^0$ in each interval $[n, n + 1)$ ($n \in \mathbb{Z}$) is restricted to 0, 1 or 2 for a given parity (see Remark 6.4), and moreover, two intervals $[n, n + 1)$ containing two (resp. no) eigenvalues (i.e. two (resp. no) roots of $G_{\pm}(x)$) cannot be contiguous.

Remark 6.15. We may define also the following two functions by zeta regularized products over Juddian and non-Juddian exceptional eigenvalues for given $g$ and $\Delta$:
\[
\Gamma_{\varepsilon, \text{Judd}}(\tau)^{-1} := \prod_{N \in \mathbb{Z}_{\geq 0} : P_{\Lambda}^{(N, \varepsilon)}((2g)^2) \Delta^2 = 0} (\tau - (N + \varepsilon)),
\] (6.44)
\[
\Gamma_{\varepsilon, \text{n-Judd}}(\tau)^{-1} := \prod_{N \in \mathbb{Z}_{\geq 0} : T_{\Lambda}^{N}(g, \Delta) = 0} (\tau - (N + \varepsilon)).
\] (6.45)

Note that if the number of Juddian (resp. non-Juddian exceptional) eigenvalues is finite, then the regularized product $\Gamma_{\varepsilon, \text{Judd}}(\tau)^{-1}$ (resp. $\Gamma_{\varepsilon, \text{n-Judd}}(\tau)^{-1}$) is actually a standard product. In fact, as widely believed among physicists, it can be strongly conjectured that the product
\[
\prod_{a = \pm \varepsilon} \prod_{b = \text{Judd}, \text{n-Judd}} \Gamma_{a, b}(\tau)^{-1}
\] is a polynomial, that is, almost all the eigenvalues of $H_{\text{Rabi}}^\varepsilon$ are regular for fixed parameters $g, \Delta > 0$. If it is true, it is also quite interesting to determine also the degree of this polynomial, more precisely the degrees of $\Gamma_{\varepsilon, \text{Judd}}(y)^{-1}$ and $\Gamma_{\varepsilon, \text{n-Judd}}(y)^{-1}$ in terms of the parameters $g$ and $\Delta$, and to study any symmetry appearing in them (see also Remark 6.9). In order to study these functions and possible symmetry, it is necessary to analyze $\det(\tau - H_{\text{Rabi}}^\varepsilon)$ more deeply than in [56]. Actually, it seems difficult to obtain a sufficient knowledge from the current method (the study of coefficients of the asymptotic expansion of the trace of the heat kernel of $H_{\text{Rabi}}^\varepsilon$) of meromorphic extension of $\zeta_{H_{\text{Rabi}}}^0(s, \tau)$ investigated in [56] (see also [23, 45, 51]). It might also be useful to consider the spectrum from the viewpoints of dynamics as in [46].
We conclude this remark by discussing the finiteness of the product in \( \Gamma_{\epsilon, \text{Judd}}(\tau)^{-1} \) for the degenerate cases \( g = 0 \) and \( \Delta = 0 \). First, suppose that \( g = 0 \). Directly from the definition it is clear that \( P_N^{(N,\epsilon)}(0, i(i + 2\epsilon)) = 0 \) for any \( i \in \mathbb{Z}_{>0} \) and \( N \geq i \), thus for \( \Delta = \sqrt{i(i + 2\epsilon)} \) the gamma factor \( \Gamma_{\epsilon, \text{Judd}}(\tau)^{-1} \) is an infinite product, essentially given by \( \Gamma(\epsilon - \tau)^{-1} \prod_{N=0}^{\epsilon-1} (\tau - N - \epsilon)^{-1} \).

On the other hand, suppose that \( \Delta = 0 \) and \( \epsilon \geq 0 \). In this case, by Theorem 4.16 the constraint polynomials \( P_k^{(k,\epsilon)}(x, 0) \) are given by \( (-1)^k k!^2 L_k^{(2\epsilon)}(x) \), where \( L_k^{(2\epsilon)}(x) \) are the generalized Laguerre polynomials. It is well-known that orthogonal polynomials interlace zeros strictly (see e.g. [1, 14]) and therefore \( \Gamma_{\epsilon, \text{Judd}}(\tau)^{-1} \) is a polynomial of degree 1 or 0 depending on whether \( L_k^{(2\epsilon)}(y^2) = 0 \) for some \( k \in \mathbb{Z}_{\geq0} \) or not. Furthermore, when \( \epsilon = \ell/2 \) (\( \ell \in \mathbb{Z}_{\geq0} \)), we have shown in Proposition 4.14 that for \( 0 \leq k < \ell \) the constraint polynomials \( P_k^{(k,\ell/2)}(x, 0) \) have no positive roots. Moreover, by the divisibility of Theorem 4.12 we have \( P_{N+\ell}^{(N+\ell,\ell/2)}(x, 0) = A_N^{\ell}(x, 0)P_N^{(N,\ell/2)}(x, 0) \) for \( N \in \mathbb{Z}_{\geq0} \). It follows that \( \Gamma_{\ell/2, \text{Judd}}(\tau)^{-1} \Gamma_{\ell/2, \text{Judd}}(\tau)^{-1} \) is a polynomial of degree 2 or 0, depending on whether \( L_k^{(\ell)}(y^2) = 0 \) for some \( k \in \mathbb{Z}_{\geq0} \) or not.

6.6 Exceptional solutions and irreducible representations of \( \mathfrak{sl}_2 \)

In this subsection, we describe the way the exceptional solutions of the AQRM can be captured in the representation theoretical picture of the AQRM (cf. 3). In fact, we show that the non-Juddian exceptional eigenstates are captured by a pair of irreducible lowest weight representations of \( \mathfrak{sl}_2 \).

We begin with the solution corresponding to the larger exponent from 6.2. For \( N = 2m \), define

\[
v_1^{\phi_1} := y^{\frac{1}{2}(a - \frac{1}{2})} \phi_1^{\phi_1}(y) = y^{-m - \frac{1}{2}} \phi_1^{\phi_1}(y) = \frac{2m + 1}{\Delta} K_{2m+1}^{(N,\epsilon)} e_{1,m} - \Delta \sum_{n=2m+1}^{\infty} \frac{K_n^{(N,\epsilon)}}{n - 2m} e_{1,n-m},
\]

(6.46)

where \( \phi_1^{\phi_1} \) is the solution (6.9).

Next, we continue the discussion following Proposition 7.2 of [62]. Namely, we claim that the vector \( v_1^{\phi_1} \) is a non-zero eigenvector corresponding to the eigenvalue problem

\[
\varpi_{1,-2m}(\mathcal{K})v_1^{\phi_1} = \Lambda_{-2m}v_1^{\phi_1}.
\]

(6.47)

To see this, it is enough to compute the recurrence relation satisfied by the solution of the eigenvalue problem (the computation follows like in Section 5.1 of [62]). Concretely, let \( v = \sum_{n \in \mathbb{Z}} a_n e_{1,n} \) be a solution of \( (\varpi_{1,-2m}(\mathcal{K}) - \Lambda_{-2m}) v = 0 \), then from the definition of the representation \( \varpi_{1,-2m} \), the coefficients \( \{a_n\}_{n \in \mathbb{Z}} \) must satisfy

\[
(m + n + 1)(m - n - 1)a_{n+1} + ((m - n)^2 - 4g^2(m - n) + 2\epsilon(m - n) - \Delta^2) a_n + 4g^2(m - n + 1)a_{n-1} = 0.
\]

By shifting the index \( n \) by \( m \), and relabeling the equation becomes

\[
(2m + n + 1)(n + 1)a_{n+1} + (-n^2 - 4g^2n + 2\epsilon n + \Delta^2) a_n - 4g^2(n - 1)a_{n-1} = 0.
\]

(6.48)
Note that the coefficient of $v_{\phi_1}^+$ corresponding to the basis vector $e_{1,m+n}$ ($n \in \mathbb{Z}_{\geq 0}$) is $-\Delta(K_{2m+n}^{(N,\varepsilon)}/n$, therefore by plugging these coefficients into (6.48) we get
\[(2m + n + 1)K_{2m+n+1}^{(N,\varepsilon)} + \left(-n - 4g^2 + 2\varepsilon + \frac{\Delta^2}{n}\right)K_{2m+n}^{(N,\varepsilon)} - 4g^2 K_{2m+n-1}^{(N,\varepsilon)} = 0,
\]
which is equivalent to recurrence (6.10), thus proving the claim.

Recall that there is an intertwining operator $A_a$ between the representations $(\varpi_{1,a}, V_{1,a})$ and $(\varpi_{1,2-a}, V_{1,2-a})$ for $a \not\in 2\mathbb{Z}$ (see [62]). The isomorphism $A_a : V_{1,a} \rightarrow V_{1,2-a}$ \((V_{1,a} = V_{1,2-a} = V_1)\) is explicitly given with respect to the basis \(\{e_{1,n}\}_{n \in \mathbb{Z}}\) by the diagonal matrix
\[A_a = \text{Diag}(\cdots, c_{-n}, \cdots, c_0, \cdots, c_n, \cdots),\]
with $c_0 \neq 0$ and
\[c_n = \left(A_n\right)_n = c_0 \prod_{k=1}^{[n]} \frac{k - \frac{a}{2}}{k - 1 + \frac{a}{2}}.
\]
Recall from Lemma 2.3 for $a = -2m$ ($m \in \mathbb{Z}_{>0}$), there is an isomorphism
\[V_{1,-2m}/F_{2m+1} \simeq D^{-}_{2(m+1)} \oplus D^{+}_{2(m+1)} \subset V_{1,2(m+1)}.
\]
In fact, from the expression of the intertwiner $A_a$ ($a \not\in 2\mathbb{Z}$), we can construct the linear isomorphism $\tilde{A}_{-2m}$ of (6.49) by defining
\[\tilde{A}_{-2m} := \frac{1}{4\pi} \lim_{a \rightarrow -2m} \sin(2\pi a) A_a,
\]
multiplication being elementwise. Then, as we may take $c_0 = 1$, we have
\[(\tilde{A}_{-2m})_n = \begin{cases} (2m + 1) \prod_{k=1, k \neq m+1}^{[n]} \frac{k + m}{k - m - 1} & \text{if } |n| > m \\ 0 & \text{if } |n| \leq m. \end{cases}
\]
Since $e_{1,m} \in \text{Ker} \tilde{A}_{-2m}$, we have $\tilde{A}_{-2m} v_{\phi_1}^+ \in D^{-}_{2(m+1)} \oplus D^{+}_{2(m+1)}$. Hence, it follows from the formula
\[\tilde{A}_{-2m} v_{\phi_1}^+ = -\Delta \sum_{n=2m+1}^{\infty} \frac{K_{n}^{(N,\varepsilon)}}{n-2m} \tilde{A}_{-2m} e_{1,n-m}
\[= -\Delta(2m + 1) \sum_{n=m+1}^{\infty} \frac{K_{n+m}^{(N,\varepsilon)}}{n-m} \prod_{k=1, k \neq m+1}^{n} \frac{k + m}{k - m - 1} e_{1,n}
\]
that $\tilde{A}_{-2m} v_{\phi_1}^+ \in D^{+}_{2(m+1)}$.

By definition, if $v \in V_{1,-2m}$ is a solution of $(\varpi_{1,-2m}(\mathcal{K}) - \Lambda_{-2m}) v = 0$, then $\tilde{A}_{-2m} v \in V_{1,2(m+1)}$ satisfies $(\varpi_{1,2(m+1)}(\mathcal{K}) - \Lambda_{2m}) \tilde{A}_{-2m} v = 0$.

The discussion above is summarized in the following theorem.

**Theorem 6.15.** Let $N \in \mathbb{Z}_{\geq 0}$, $\Delta > 0$ and $T_{\varepsilon}^{(N)}(g, \Delta)$ the constraint $T$-function defined in §6.4. If $g$ is a positive zero of $T_{\varepsilon}^{(N)}(g, \Delta)$, we have a non-degenerate non-Juddian exceptional eigenvalue $\lambda = N + \varepsilon - g^2$. Furthermore:
(1) If $N = 2m$, let $v^+_\phi_1 \in \mathbf{V}_{1,-2m}$ be as in \((6.46)\). Then $w := \tilde{A}_{-2m}v^+_\phi_1$ is a solution to the eigenproblem $(\varpi_{1,2(m+1)}(K) - \Lambda_{-2m})w = 0$ and $w \in \mathbf{D}_{2(m+1)}^+$.

(2) If $N = 2m - 1$, let $v^+_\phi_1 \in \mathbf{V}_{2,1-2m}$ be as described above. Then $w := \tilde{A}_{-2m}v^+_\phi_1$ is a solution of the eigenproblem $(\varpi_{2,2m+1}(K) - \Lambda_{-2m})w = 0$ and $w \in \mathbf{D}_{2m+1}^+$.

Let $N \in \mathbb{Z}_{\geq 0}$, $\Delta > 0$ and $\tilde{T}_\varepsilon(N)(g, \Delta)$ the constraint $T$-function defined in \((6.4)\). If $g$ is a zero of $\tilde{T}_\varepsilon(N)(g, \Delta)$, we have a non-degenerate non-Juddian exceptional eigenvalue $\lambda = N - \varepsilon - g^2$. Furthermore:

(1) If $N = 2m$, let $v^+_\phi_2 \in \mathbf{V}_{2,1-2m}$ be as described above. Then $w := \tilde{A}_{-2m}v^+_\phi_2$ is a solution to the eigenproblem $(\varpi_{2,2m+1}(K) - \Lambda_{-2m})w = 0$ and $w \in \mathbf{D}_{2m+1}^+$.

(2) If $N = 2m + 1$, let $v^+_\phi_2 \in \mathbf{V}_{1,-2m}$ be as described above. Then $w := \tilde{A}_{-2m}v^+_\phi_2$ is a solution of the eigenproblem $(\varpi_{1,2(m+1)}(K) - \Lambda_{-2m})w = 0$ and $w \in \mathbf{D}_{2(m+1)}^+$.

Proof. In the foregoing discussion we proved the case for $\lambda = N + \varepsilon - g^2$ and $N = 2m$. The remaining cases are proved in a similar manner, so we leave the proof of those cases to the reader (for the computations, see Section 5 of \([62]\)).

Remark 6.16. When $\varepsilon = \ell/2$ ($\ell \in \mathbb{Z}_{\geq 0}$), the relation $\tilde{T}_{\ell/2}(N+\ell)(g, \Delta) = T_{\ell/2}(N)(g, \Delta)$ in Lemma \((6.6)\) guarantees the compatibility of the former and latter assertions in Theorem 6.15. For instance, let us observe the case when $N = 2m$ and $\ell = 2\ell'$ ($\ell' \in \mathbb{Z}$), the remaining cases are verified in the same manner. If there exists a non-Juddian exceptional solution $(e^{-g^2\phi_1,+(\frac{2\ell}{2\ell}; \ell/2)}, e^{-g^2\phi_1,-(\frac{2\ell}{2\ell}; \ell/2)})$ for $\lambda = N + \ell/2 - g^2$ then the theorem asserts $T_{\ell/2}(N)(g, \Delta) = 0$ must hold and $w := \tilde{A}_{-2m}v^+_\phi_1 \in \mathbf{D}_{2m+1}^+$. Then, since $\tilde{T}_{\ell/2}(N+\ell)(g, \Delta) = 0$, the latter assertion of the theorem shows that there exists a non-Juddian exceptional solution $(e^{g^2\phi_2,+(\frac{2\ell}{2\ell}; -\ell/2)}, e^{g^2\phi_2,-(\frac{2\ell}{2\ell}; -\ell/2)})$ (corresponding to eigenvalue $\lambda = (N + \ell) - \ell/2 - g^2$) and $w' := \tilde{A}_{-2m+1(\ell'+\frac{2\ell}{2\ell})}v^+_\phi_2 \in \mathbf{D}_{2(\ell+\frac{2\ell}{2\ell})+1}^+$, we can verify that $w'$ corresponds to the first and $w'$ (up to a constant) to the second component of the solution $(e^{-g^2\phi_1,+(\frac{2\ell}{2\ell}; \ell/2)}, e^{-g^2\phi_1,-(\frac{2\ell}{2\ell}; \ell/2)})$, and thus, there is no contradiction with the non-degeneracy of the non-Juddian exceptional solution even for the case $\varepsilon = \ell/2$.

Remark 6.17. From the Juddian solutions of \((6.1)\) we can similarly construct the corresponding eigenvectors $v$ captured in the finite dimensional irreducible submodules $\mathbf{F}_{2m}$ (or $\mathbf{F}_{2m+1}$) of $\varpi_{j,a}$. For instance, defining $v^+ := y^{m-\frac{1}{2}}\phi_{1,+}(y)$, where

$$\phi_{1,+}(y) = \frac{4g^2K^{(N,e)}_{2m-1}}{\Delta} y^{2m} - \Delta \sum_{n=0}^{2m-1} \frac{K^{(N,e)}_n}{n-2m} y^n,$$

we observe directly that $v^+ \in \mathbf{F}_{2m+1}$ as an irreducible submodule of $\mathbf{V}_{1,-2m}$ \([62]\).

Remark 6.18. The eigenvector corresponding to the non-Juddian exceptional solutions corresponding to $N = 0$ in the proof of Corollary \((6.2)\) is captured in the limit of discrete series $\mathbf{D}_{1}^+$.

Remark 6.19. For non-Juddian exceptional eigenvalues $\lambda = N \pm \varepsilon - g^2$, the proposition above describes how the eigenvectors $w \in (\varpi_{j,a}, \mathbf{V}_j)$ ($j \in \{1, 2\}$) in the corresponding...
For the case of the Juddian solutions, we can capture any corresponding eigenvector in a (finite dimensional) irreducible subspace of $V_j$ and such irreducible representation is uniquely determined by the eigenvector [62]. Unlike the case of Juddian solutions, any non-Juddian exceptional solution cannot be captured in an invariant subspace $V_j$ in an appropriate manner but it is required to consider the subquotient of $V_j$. Theorem 6.15 shows that through the isomorphism obtained by the operator $\tilde{V}$, an appropriate manner but it is required to consider the subquotient of $V_j$, the non-Juddian exceptional solution cannot be captured in an invariant subspace $V_j$. Unlike the case of Juddian solutions, any non-Juddian exceptional solution determines uniquely an (infinite dimensional) irreducible subspace of $V_j$ which contains the eigenvector. Since each of the following short exact sequences

$$0 \rightarrow \mathbf{D}_m^+ \oplus \mathbf{D}_m^- \rightarrow V_{1,2m} \rightarrow F_{2m-1} \rightarrow 0,$$

$$0 \rightarrow \mathbf{D}_m^{2m+1} \oplus \mathbf{D}_m^{-2m+1} \rightarrow V_{2,2m+1} \rightarrow F_{2m} \rightarrow 0$$

of $\mathfrak{sl}_2$-modules for $m > 0$ are not split, we might give a representation theoretic explanation of Corollary 4.15 for $\varepsilon \in \frac{1}{2}\mathbb{Z}_{>0}$.

Moreover, the eigenvector corresponding to the regular spectrum determines the irreducible principal series representation and vice-versa. We summarize the relation of eigenvalue type, constraint relation and related irreducible representations in Table 2.

| Type                | Eigenvalue$^d$       | Rep. of $\mathfrak{sl}_2$                 | Constraint relation     |
|---------------------|----------------------|--------------------------------------------|-------------------------|
| Juddian$^c$         | $N \pm \varepsilon - g^2$ | $\mathbf{F}_m$: finite dim. irred. rep.$^a$ | $P^{(N, \pm \varepsilon)}_N((2g)^2, \Delta^2) = 0$ |
| Non-Juddian         | $N \pm \varepsilon - g^2$ | $\mathbf{D}_m^\pm$: irred. lowest weight rep.$^b$ | $T^{(N, \pm \varepsilon)}_\varepsilon(g, \Delta) = 0$ |
| Regular             | $x \pm \varepsilon - g^2$ | $\varpi_{j,a}$: irred. principal series     | $G_\varepsilon(x; g, \Delta) = 0$ |

$^a$ Determination of $m$ in $\mathbf{F}_m$. Case $N + \varepsilon$: $m = N + 1$. Case $N - \varepsilon$: $m = N$.

$^b$ Determination of $m$ in $\mathbf{D}_m^\pm$ for the first component of the solution (see Theorem 6.15). Case $N + \varepsilon$: $m = N + 1$. Case $N - \varepsilon$: $m = N - 1$.

$^c$ Case $\varepsilon \in \frac{1}{2}\mathbb{Z}$. Non-Juddian exceptional solutions are non-degenerate. Juddian solutions are always degenerate ($\varepsilon$ corresponding to the space $\mathbf{F}_m \oplus \mathbf{F}_{m+1}$ when $\varepsilon = 0$. See [62] for general $\varepsilon = \ell/2$.

$^d$ Constants: $N \in \mathbb{Z}_{>0}$, $x \notin \mathbb{Z}_{>0}$.

Remark 6.20. We may ask if there is a geometric (group theoretic) interpretation of the spectrum of the AQRM. In fact, since the Lie algebra of any covering group of $SL_2(\mathbb{R})$ is $\mathfrak{sl}_2(\mathbb{R})$ (and $\mathfrak{sl}_2(\mathbb{R})_C = \mathfrak{sl}_2(\mathbb{C})$), relating to the spectral zeta function the following question comes up naturally (cf. Table 2). Is there any covering group $G := G(g, \Delta)$ of $SL_2(\mathbb{R})$ (or $G := SL_2(\mathbb{C})$) with a discrete subgroup $\Gamma := \Gamma(g, \Delta)$ of $G$ such that the regular spectrum of $H^\infty_{Rabi}$ can be captured in $L^2(\Gamma \backslash G)$? (see Problem 6.1 in [62] for a relevant question).
7 Generating functions of the constraint polynomials

In this section, we study the generating functions of constraint polynomials $P_{N,(2g)^2,\Delta^2}^{(N,\varepsilon)}$ with their defining sequence $P_{N,\Delta^2}^{(N,\varepsilon)}((2g)^2)$ from the viewpoints of confluent Heun equations. As we have already observed (cf. Proposition 6.1), $P_{N,(2g)^2,\Delta^2}^{(N,\varepsilon)}$ is essentially identified with the coefficient of the log-term of the local Frobenius solution of the smallest exponent $\rho^\pm = 0$ for the confluent Heun equation corresponding to the eigenvalue problem of the AQRM. As a byproduct of this discussion, we obtain an alternative proof of the divisibility part of Theorem 4.12.

7.1 Constraint polynomials and confluent Heun differential equations

In this subsection we study certain confluent Heun differential equations satisfied by the generating function of the polynomials $P_{N,\Delta^2}^{(N,\varepsilon)}((x,y))$ and obtain combinatorial relations between the constraint polynomials.

For convenience, define the normalized polynomials $P_{N,\Delta^2}^{(N,\varepsilon)}((x,y))$ by

$$P_{N,\Delta^2}^{(N,\varepsilon)}((x,y)) := \frac{P_{k}^{(N,\varepsilon)}((x,y))}{k!(k+1)!}$$

and their generating function

$$\mathcal{P}^{(N,\varepsilon)}(x,y,t) := \sum_{k=0}^{\infty} P_{k}^{(N,\varepsilon)}((x,y)) t^k.$$  

Clearly, the normalized polynomials $\mathcal{P}_{k}^{(N,\varepsilon)}((x,y))$ satisfy the recurrence relation

$$P_{0}^{(N,\varepsilon)}((x,y)) = 1, \quad P_{1}^{(N,\varepsilon)}((x,y)) = \frac{x + y - 1 - 2\varepsilon}{2},$$

$$P_{k}^{(N,\varepsilon)}((x,y)) = \frac{kx + y - k^2 - 2k\varepsilon}{k(k+1)} P_{k-1}^{(N,\varepsilon)}((x,y)) - \frac{(N - k + 1)x}{k(k+1)} P_{k-2}^{(N,\varepsilon)}((x,y)),$$  

for $k \geq 2$. With these preparations, we study the differential equation satisfied by the generating function $\mathcal{P}^{(N,\varepsilon)}((x,y,t))$.

**Proposition 7.1.** As a function in the variable $t$, the generating function $z = \mathcal{P}^{(N,\varepsilon)}((x,y,t))$ satisfies the differential equation

$$\left\{ t(1 + t) \frac{\partial^2}{\partial t^2} + (2 - (x - 3 - 2\varepsilon)t - xt^2) \frac{\partial}{\partial t} - (x + y - 1 - 2\varepsilon - (N - 1)xt) \right\} z = 0.$$  

(7.2)
Proof. We observe that
\[
\frac{\partial^2}{\partial t^2} = \sum_{k=1}^{\infty} k(k+1)p_k^{(N,\varepsilon)}(x,y)k^{-1}
\]
\[
= 2p_1^{(N,\varepsilon)}(x,y) + \sum_{k=2}^{\infty} (kx + y - k^2 - 2k\varepsilon)p_{k-1}^{(N,\varepsilon)}(x,y) - (N - k + 1)x\frac{\partial p_{k-1}^{(N,\varepsilon)}}{\partial x}(x,y)k^{-1}
= 2p_1^{(N,\varepsilon)}(x,y) + y\sum_{k=2}^{\infty} p_{k-1}^{(N,\varepsilon)}(x,y)k^{-1} + (x + 1 - 2\varepsilon)\sum_{k=2}^{\infty} kp_{k-1}^{(N,\varepsilon)}(x,y)k^{-1}
\]
\[
- \sum_{k=2}^{\infty} k(k+1)p_{k-1}^{(N,\varepsilon)}(x,y)k^{-1} - (N + 1)x\sum_{k=2}^{\infty} p_{k-2}^{(N,\varepsilon)}(x,y)k^{-1} + x\sum_{k=2}^{\infty} kp_{k-2}^{(N,\varepsilon)}(x,y)k^{-1}
\]
\[
= 2p_1^{(N,\varepsilon)}(x,y) + y(z - 1) + (x + 1 - 2\varepsilon)\frac{\partial}{\partial t} (tz - t) - \frac{\partial^2}{\partial tz^2}(t^2z - t^2) - (N + 1)x\frac{\partial}{\partial t}(t^2z)
\]
\[
yz + (x + 1 - 2\varepsilon)(z + t\frac{\partial z}{\partial t}) - (2z + 4t\frac{\partial z}{\partial t} + t^2\frac{\partial^2 z}{\partial t^2}) - (N + 1)x\frac{\partial}{\partial t}(tz).
\]
Hence, it follows that
\[
t\frac{\partial^2 z}{\partial t^2} + 2\frac{\partial z}{\partial t} = yz + (x + 1 - 2\varepsilon)(z + t\frac{\partial z}{\partial t}) - (2z + 4t\frac{\partial z}{\partial t} + t^2\frac{\partial^2 z}{\partial t^2}) - (N + 1)x\frac{\partial}{\partial t}(tz).
\]
Since \(\frac{\partial^2}{\partial t^2}(tz) = t\frac{\partial^2 z}{\partial t^2} + 2\frac{\partial z}{\partial t}\), we have the desired conclusion immediately. \(\square\)

Remark 7.1. Note that equation (7.2) is a confluent Heun differential equation. By (6.16), we know that the constraint polynomials \(P_k^{(N,\varepsilon)}(x,y)\) are the constant multiples of the coefficients \(K_n^\pm(N \pm \varepsilon; g, \Delta, \varepsilon)\) of solutions of the confluent Heun picture of the spectral problem of AQRM. Therefore, it is not surprising that \(P^{(N,\varepsilon)}(x,y,t)\) satisfies the confluent Heun differential equation (7.2).

Our main application is the following combinatorial relation between the polynomials \(P_k^{(N,\varepsilon)}(x,y)\).

Theorem 7.2. For \(N, \ell, k \in \mathbb{Z}_{\geq 0}\), the following equation holds.
\[
P_k^{(N+\ell, -\ell/2)}(x,y) = \sum_{i=0}^{\ell} \binom{\ell}{k-i} p_i^{(N,\ell/2)}(x,y).
\]
(7.3)

We illustrate Theorem 7.2 with an example, which is used later in the proof of the theorem.

Example 7.1. When \(k = 0\), both sides of (7.3) are equal to 1. When \(k = 1\), the left-hand side of (7.3) is \(p_1^{(N+\ell, -\ell/2)}(x,y) = \frac{x+y-1+\ell}{2}\), and the right-hand side of (7.3) is
\[
\left(\begin{array}{c} \ell \\ 1 \end{array}\right) p_0^{(N,\ell/2)}(x,y) + \left(\begin{array}{c} \ell \\ 0 \end{array}\right) p_1^{(N,\ell, -\ell/2)}(x,y) = \ell + \frac{x+y-1-\ell}{2} = \frac{x+y-1+\ell}{2}.
\]
Hence (7.3) is valid when \(k = 0, 1\).

Instead of proving Theorem 7.2 directly, we use the following equivalent formulation in terms of generating functions.
Lemma 7.3. The equation (7.3) is equivalent to the equation
\[ \mathcal{P}^{(N+\ell,-\ell/2)}(x,y,t) = (1 + t)^{\ell} \mathcal{P}^{(N,\ell/2)}(x,y,t). \] (7.4)

Proof. We see that

\[ (1 + t)^{\ell} \mathcal{P}^{(N,\ell/2)}(x,y,t) = \sum_{j=0}^{\infty} \binom{\ell}{j} t^j \sum_{k=0}^{\infty} \mathcal{P}_i^{(N,\ell/2)}(x,y)t^k = \sum_{k=0}^{\infty} \sum_{i+j=k} \binom{\ell}{i} \mathcal{P}_i^{(N,\ell/2)}(x,y)t^k \]

from which the lemma follows.

The proof of the equivalent statement is done using the differential equation (7.2) satisfied by the generating function \( \mathcal{P}^{(N,\ell)}(x,y,t) \). We need the followings lemmas.

Lemma 7.4. Let \( p(t), q(t) \) be polynomials in \( t \) and \( \alpha \in \mathbb{R} \). Suppose that a function \( z = z(t) \) satisfies the differential equation

\[ t(1 + t) \frac{\partial^2 z}{\partial t^2} + p(t) \frac{\partial z}{\partial t} + q(t)z = 0. \]

Then \( w = (1 + t)^{\alpha}z \) satisfies

\[ t(1 + t) \frac{\partial^2 w}{\partial t^2} + (p(t) - 2\alpha t) \frac{\partial w}{\partial t} + (q(t) - \alpha u(t))w = 0, \]

where \( u(t) = \frac{p(t)-(\alpha+1)t}{1+t} \).

Proof. Since

\[ \frac{\partial w}{\partial t} = \alpha(1 + t)^{\alpha-1}z + (1 + t)^{\alpha} \frac{\partial z}{\partial t}, \]

\[ \frac{\partial^2 w}{\partial t^2} = \alpha(\alpha-1)(1 + t)^{\alpha-2}z + 2\alpha(1 + t)^{\alpha-1} \frac{\partial z}{\partial t} + (1 + t)^{\alpha} \frac{\partial^2 z}{\partial t^2}, \]

we have

\[ t(1 + t) \frac{\partial^2 w}{\partial t^2} + p(t) \frac{\partial w}{\partial t} = (1 + t)^{\alpha}t(1 + t) \frac{\partial^2 z}{\partial t^2} + 2\alpha(1 + t)^{\alpha} \frac{\partial z}{\partial t} + (1 + t)^{\alpha-1}p(t)z + (1 + t)^{\alpha-1}p(t)z + (1 + t)^{\alpha-1}p(t)z \]

\[ = -q(t)w + 2\alpha t \left( \frac{\partial w}{\partial t} - \alpha(1 + t)^{\alpha-1}z \right) + (1 + t)^{\alpha-1}t(z) \]

\[ = -q(t)w + 2\alpha t \left( \frac{\partial w}{\partial t} - \alpha(1 + t)^{\alpha-1}z \right) + \alpha(\alpha-1)(1 + t)^{\alpha-1}t \]

from which the conclusion follows.

Lemma 7.5. The function \( w = (1 + t)^{2\varepsilon}z \), where \( z = \mathcal{P}^{(N,\varepsilon)}(x,y,t) \), satisfies

\[ \left\{ t(1 + t) \frac{\partial^2}{\partial t^2} + (2 + (x - 3 + 2\varepsilon)t - x^2) \frac{\partial}{\partial t} - (x + y - 1 + 2\varepsilon - (N + 2\varepsilon - 1)t) \right\} w = 0. \]
Proof. Let
\[ p(t) = 2 - (x - 3 - 2\varepsilon)t - xt^2, \quad q(t) = -(x + y - 1 - 2\varepsilon) + (N - 1)xt, \quad \alpha = 2\varepsilon. \]
Then \( z \) satisfies the equation
\[ t(1 + t) \frac{\partial^2 z}{\partial t^2} + p(t) \frac{\partial z}{\partial t} + q(t)z = 0. \]
We see that
\[ u(t) = \frac{p(t) - (\alpha + 1)t}{1 + t} = \frac{2 - (x - 3 - 2\varepsilon)t - xt^2 - (2\varepsilon + 1)t}{t + 1} = 2 - xt, \]
and
\[ p(t) - 2\alpha t = 2 - (x - 3 + 2\varepsilon)t - xt^2, \quad q(t) - \alpha u(t) = -(x + y - 1 + 2\varepsilon) + (N + 2\varepsilon - 1)xt. \]
Thus, by Lemma 7.4 \( w \) satisfies the equation
\[ t(1 + t) \frac{\partial^2 w}{\partial t^2} + (2 - (x - 3 + 2\varepsilon)t - xt^2) \frac{\partial w}{\partial t} + ((x + y - 1 + 2\varepsilon) + (N + 2\varepsilon - 1)xt)w = 0 \]
as desired. \( \square \)

**Proof of Theorem 7.2.** By Proposition 7.1 and Lemma 7.5, the functions \( \mathcal{P}^{(N+\ell,-\ell/2)}(x,y,t) \) and \((1 + t)^{\ell} \mathcal{P}^{(N,\ell/2)}(x,y,t) \) satisfy the same second order linear differential equation
\[ \left\{ t(1 + t) \frac{\partial^2}{\partial t^2} + (2 + (x - 3 + \ell)t - xt^2) \frac{\partial}{\partial t} - (x + y - 1 + \ell - (N + \ell - 1)xt) \right\} z = 0. \]
By Example 7.1, the constant and linear terms of the power series expansion of these two functions at \( t = 0 \) are equal, and hence they are equal. By Lemma 7.3 this implies (7.3). \( \square \)

### 7.2 Revisiting divisibility of constraint polynomials

Using the identity of Theorem 7.2, we give another proof of the divisibility property of constraint polynomials stated in Theorem 4.7.

**Proof of Theorem 4.7.** From the defining recurrence relations (7.1) of the polynomials \( \mathcal{P}^{(N,e)}_k(x,y) \) we notice that
\[ \mathcal{P}^{(N,e)}_{N+1}(x,y) = \frac{(N + 1)x + y - (N + 1)^2 - 2(N + 1)e}{(N + 1)(N + 2)} \mathcal{P}^{(N,e)}_N(x,y), \]
that is, \( \mathcal{P}^{(N,e)}_{N+1}(x,y) \) is divisible by \( \mathcal{P}^{(N,e)}_N(x,y) \). Hence, using the recurrence (7.1) again, we see by induction on \( k \) that \( \mathcal{P}^{(N,e)}_k(x,y) \) is divisible by \( \mathcal{P}^{(N,e)}_N(x,y) \) if \( k \geq N \).

Next, putting \( k = N + \ell \) and \( e = \frac{\ell}{2} \) in (7.3), we have
\[ \mathcal{P}^{(N+\ell,-\ell/2)}_{N+\ell}(x,y) = \sum_{i=N}^{N+\ell} \binom{\ell}{N + \ell - i} \mathcal{P}^{(N,\ell/2)}_i(x,y) = \sum_{j=0}^{\ell} \binom{\ell}{j} \mathcal{P}^{(N,\ell/2)}_{N+j}(x,y). \tag{7.5} \]
Since each of the terms in the right-hand side are divisible by \( \mathcal{P}^{(N,\ell/2)}_N(x,y) \), this completes the proof. \( \square \)
Remark 7.2. Define the polynomials by \( Q_k^{(N,\varepsilon)}(x,y) = \frac{P_{N+k}^{(N,\varepsilon)}(x,y)}{P_N^{(N,\varepsilon)}(x,y)} \). Then we observe that \((7.5)\) is equivalent to

\[
A_N^\ell(x,y) = \sum_{j=0}^\ell \binom{\ell}{j} \frac{(N + \ell)! (N + \ell + 1)!}{(N + j)! (N + j + 1)!} Q_j^{(N,\ell/2)}(x,y). \quad (7.6)
\]

We notice that for \( x < 0 \) the family of polynomials \( Q_k^{(N,\ell/2)}(x,y) \) actually forms an orthogonal system of polynomials (see [14]) and might have a representation theoretic interpretation. It is therefore desirable to obtain a straightforward proof of the positivity for the polynomial

\[
A_N^\ell(x,y) = \frac{P_{N+\ell}^{(N+\ell,\ell/2)}(x,y)}{P_N^{(N,\ell/2)}(x,y)}
\]

for \( x,y > 0 \) in a reasonable framework (e.g. [17] for the harmonic analysis on symmetric cones) of orthogonal polynomials in two variables.

Acknowledgements

This work was supported by JST CREST Grant Number JPMJCR14D6, Japan, and by Grand-in-Aid for Scientific Research (C) JP16K05063 of JSPS, Japan. CRB was supported during the duration of the research by the Japanese Government (MONBUKAGAKUSHO: MEXT) scholarship.

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