Stable limits for Markov chains via the Principle of Conditioning

Mohamed El Machkouri\textsuperscript{1}\textsuperscript{*}, Adam Jakubowski\textsuperscript{2}\textsuperscript{†} and Dalibor Volný\textsuperscript{1}\textsuperscript{‡}

\textsuperscript{1} Université de Rouen Normandie, France
\textsuperscript{2} Nicolaus Copernicus University, Poland

Abstract

We study limit theorems for partial sums of instantaneous functions of a homogeneous Markov chain on a general state space. The summands are heavy-tailed and the limits are stable distributions. The conditions imposed on the transition operator $P$ of the Markov chain ensure that the limit is the same as if the summands were independent. Such a scheme admits a physical interpretation, as given in Jara et al. (Ann. Appl. Probab., 19 (2009), 2270–2300).

We considerably extend the results of Jara et al., ibid. and Cattiaux and Manou-Abi (ESAIM Probab. Stat., 18 (2014), 468–486). We show that the theory holds under the assumption of operator uniform integrability in $L^2$ of $P$ (a notion introduced by Wu (J. Funct. Anal., 172 (2000), 301–376)) plus the $L^2$-spectral gap property. If we strengthen the uniform integrability in $L^2$ to the hyperboundedness, then the $L^2$-spectral gap property can be relaxed to the strong mixing at geometric rate (in practice: to geometric ergodicity).

We provide an example of a Markov chain on a countable space that is uniformly integrable in $L^2$ (and admits an $L^2$-spectral gap), while it is not hyperbounded. Moreover, we show by example that hyperboundedness is still a weaker property than $\phi$-mixing, what enlarges the range of models of interest.

What makes our assumptions working is a new, efficient version of the Principle of Conditioning that operates with conditional characteristic functions rather than predictable characteristics.

Keywords: convergence in distribution, stable laws, Markov chains, transition operators, spectral gap, operator uniform integrability, principle of conditioning, hyperboundedness, ultra-boundedness.

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1 Introduction

Our motivation comes from the paper by Jara, Komorowski and Olla \textsuperscript{34}, where a fractional diffusion was obtained as a scaled limit of functionals of Markov chains forming a probabilistic solution to a linear Boltzmann equation. The main tool used in \textsuperscript{34} was a functional

\textsuperscript{*}E-mail: mohamed.elmachkouri@univ-rouen.fr
\textsuperscript{†}E-mail: adjakubo@mat.umk.pl
\textsuperscript{‡}E-mail: dalibor.volny@univ-rouen.fr
limit theorem on convergence to stable Lévy processes due to Durrett and Resnick [19] and the assumptions that made this functional limit theorem working were $L^2$-spectral gap and strong contractivity properties of the Markov transition operator. In the particular example considered in [34] the ultraboundedness of the transition operator was used, but in the general considerations (Theorem 2.4, ibid.) properties related to a weaker notion of hyperboundedness were assumed (We refer to Section 2 below for formal definitions and discussion of all these notions).

Later Cattiaux and Manou-Abi [11] reexamined the limit theorems from [34] in the context of the general theory of convergence to stable laws for sums of stationary sequences. They considered standard mixing conditions ($\phi$-, $\rho$-, $\alpha$-mixing) and anti-clustering condition $D'$, introduced in [13] and discussed in [17] (see also [37]). While the discussion in [11] was quite extensive, it did not address the question whether the strong assumption of hyperboundedness of the transition operator can be essentially weakened.

In the present paper we suggest replacing the hyperboundedness with the uniform integrability in $L^2$ (2-U.I. in short) of the transition operator, a notion introduced in [46]. We believe that this is the proper minimal form for operator contractivity whenever limit theorems for Markov chains with stable limits are considered. Our main results are formulated in Section 3. In Theorem 3.1 we obtain limit theorems assuming the 2-U.I. condition and the $L^2$-spectral gap property. In Theorem 3.4 we assume the hyperboundedness in place of the 2-U.I. condition, but we weaken the $L^2$-spectral gap property to the geometric ergodicity. The proofs of both main results are deferred to Section 5. In Section 2 we gather all necessary information, notation and comments related to the models considered in the paper.

What allows considerable weakening of the assumptions is a new efficient version of the Principle of Conditioning that operates with conditional characteristic functions rather than predictable characteristics and therefore keeps integrability requirements at the minimal possible level. Recall that the Principle of Conditioning is a heuristic rule that transforms limit theorems for independent random variables into limit theorems for dependent random variables. The mentioned above functional limit theorem by Durrett and Resnick [19] is a particular manifestation of this rule. We state our new result (Theorem A.3) and give more comments and references on the Principle of Conditioning in the Appendix.

In Section 4 we give four examples, each of different nature. First we provide an example of a Markov chain with the transition operator that is uniformly integrable in $L^2$ (and admits an $L^2$-spectral gap) while it is not hypercontractive. This shows that our theory substantially extends that of [34] and [11].

Then we show that the standard stationary AR(1) sequence with Gaussian innovations satisfies the hyperboundedness property and admits an $L^2$-spectral gap. It follows that instantaneous functions of this sequence give stable limit theorems without any need of centering in the whole range $\alpha \in (0, 2)$ (and not only for $\alpha \in (0, 1)$). This partially answers a conjecture formulated in [13]. On the other hand it is well-known that this sequence is not $\phi$-mixing what proves that the hyperboundedness is not as demanding as it looks like.

Finally we study the problem of $m$-skeletons. It is known that the contraction properties may improve after composition of operators. Suppose that some power $P^m$ of the transition operator has the desired (by us) properties while $P^k$, $k = 1, 2, \ldots, m - 1$ not. It follows that our stable limit theorem holds if we sum random variables along $m$-skeleton only and the question is whether the limit theorem can be extended to the whole sequence. The answer is “no” as simple probabilistic examples built upon i.i.d. sequences show. We provide another example, with $m = 3$, that is more oriented towards thinking in terms of operators.
2 Preliminaries

2.1 Transition operator

Let \( \{X_n\}_{n \geq 0} \) be a Markov chain with state space \((S, \mathcal{S})\) and the transition probability \( P(x, dy) \) on \( S \times S \). We will always assume that \( P(x, dy) \) admits a stationary distribution \( \pi \) on \((S, \mathcal{S})\), i.e.

\[
\pi(A) = \int_S \pi(dx) P(x, A), \quad A \in \mathcal{S}.
\] (1)

The transition probability defines the transition operator that acts by the formula

\[
(Pf)(x) = \int_S P(x, dy) f(y)
\] (2)

and is a positive contraction on every space \( L^p(\pi) = L^p(S, \mathcal{S}, \pi) \), \( p \in [1, +\infty] \).

2.2 2-U.I. condition

Following [46] we will say that the transition operator \( P \) is:

- uniformly integrable in \( L^2 \) (or 2-U.I.) if
  \[
  \{ |Pf|^2 ; f \in L^2(\pi), \|f\|_2 \leq 1 \} \text{ is uniformly } \pi\text{-integrable.}
  \] (3)

- hyperbounded if there exists \( q > 2 \) such that \( P : L^2(\pi) \rightarrow L^q(\pi) \) is a bounded linear operator, i.e.
  \[
  \sup \{ \pi(|Pf|^q) ; f \in L^2(\pi), \|f\|_2 \leq 1 \} < +\infty.
  \] (4)

- ultrabounded if
  \[
  \sup \{ \|Pf\|_\infty ; f \in L^1(\pi), \|f\|_1 \leq 1 \} < +\infty.
  \] (5)

The hyperboundedness of the transition operator is, in a sense, independent of the particular choice of \( p < q \), provided \( 1 < p < q < +\infty \). Indeed, by the Riesz-Thorin theorem, if \( P \) is a bounded linear operator from \( L^p \) to \( L^q \), with \( 1 < p < q < +\infty \), then for any other \( 1 < p' < +\infty \) there is \( q' > p' \), \( q' < +\infty \), such that \( P \) is a bounded linear operator from \( L^{p'} \) to \( L^{q'} \). Notice also that if \( P \) is ultrabounded, then for any \( p > 1 \)

\[
\sup \{ \|Pf\|_\infty ; f \in L^p(\pi), \|f\|_p \leq 1 \} < +\infty.
\]

In particular, the ultraboundedness implies the hyperboundedness and the latter implies the uniform integrability in \( L^2 \).

Conditions like (3) - (5) are usually considered in the context of hypercontractivity of Markov semigroups and all examples mentioned in [46] (as well as most of examples in [11]) are related to the continuous time Markov processes analysis.

In the present paper we deal with discrete time Markov chains and show that also in this more elementary setting there are natural examples of Markov chains with contracting properties of the transition operator describable by relations (3) - (5).

For example, suppose that \( P \) is given by a density \( p(x, y) \) with respect to \( \pi \), i.e.

\[
Pf(x) = \int_S \pi(dy)p(x, y)f(y).
\]
Then $P$ is ultrabounded if $p(x, y)$ is a bounded function in $(x, y)$ (as in the main model in [34]), and it is hyperbounded if $p(x, y) \in L^q(\pi \times \pi)$ for some $q > 2$ (see [11, p. 480]). In Section 4.1 we shall provide an example of a countable-space Markov chain with $P$ that is 2-U.I. but not hyperbounded.

2.1 Remark By the linearity of $P$, if any of conditions (3) - (5) holds for real-valued functions $f$, then it is satisfied also for complex-valued functions $f$.

2.3 $L^2$-spectral gap, geometric ergodicity and strong mixing

The transition operator $P$ is said to have an $L^2$-spectral gap if there is a number $a < 1$ such that

$$\sup\{\|Pf\|_{L^2(\pi)} : \int_S f(x) d\pi(x) = 0, \|f\|_{L^2(\pi)} \leq 1\} \leq a.$$ 

By iteration we obtain for $f \in L_0^2(\pi) = \{f \in L^2(\pi) : \pi(f) = \int_S f(x) \pi(dx) = 0\}$

$$\|P^n f\|_{L^2(\pi)} \leq a^n \|f\|_{L^2(\pi)}, \quad n = 1, 2, \ldots.$$ (6)

This means that $\{X_n\}$ satisfies “an $L^2$ norm condition” of [43] and by Theorem 2, p. 217, *ibid.*, a central limit theorem with the standard normalization $\sqrt{n}$ holds for the stationary sequence $\Psi(X_0), \Psi(X_1), \ldots$ whenever $\int \Psi(x) \pi(dx) = 0$ and $\int \Psi^2(x) \pi(dx) < +\infty$. (A proof of this limit theorem that is preferred nowadays can be found e.g. in [22]).

For reversible, $\psi$-irreducible and aperiodic Markov chains the spectral gap property is known to be equivalent to geometric ergodicity, i.e. existence of $0 < \rho < 1$ and $C : S \to \mathbb{R}^+$ such that

$$\|P^n(x, \cdot) - \pi\|_{TV} \leq C(x) \rho^n, \quad \text{for } \pi\text{-a.e. } x \in S,$$

where $\|\cdot\|_{TV}$ is the total variance distance (see [42, Theorem 2.1]). If $\{X_n\}$ is not reversible, then the spectral gap property implies the geometric ergodicity (see [36, Theorem 1.3]), but there are Markov chains that are geometrically ergodic and do not have an $L^2$ spectral gap (see [36, Theorem 1.4]). It is remarkable that the central limit theorem need not hold for such Markov chains (see [6, 23, 24]).

Notice that if one is interested in a central limit theorem to hold for particular instantaneous function of the underlying Markov chain, then sufficient conditions weaker than the $L^2$ spectral gap are known (see e.g. [39]).

It is well known that the geometric ergodicity of a Markov chain is equivalent (under natural conditions) to the exponential absolute regularity (see e.g. [8, Theorem 21.19, p. 325]), hence implies also the strong mixing at geometric rate. In this paper we shall use only the following consequence of the last property.

Let $\{X_j\}$ be strongly mixing at geometric rate. Then there exists a number $0 \leq \eta < 1$ such that for any bounded measurable complex-valued function $\chi$ on $(S, S)$

$$\left| \mathbb{E}\left(\chi(X_i) - \mathbb{E}(\chi(X_i))\right)\left(\chi(X_j) - \mathbb{E}(\chi(X_j))\right) \right| \leq 2\pi \eta^{|i-j|}\|\chi\|_{L^\infty}^2, \quad i, j \in \mathbb{N}.$$ (7)

See [7, Theorem 4.5, p. 125].
2.4 Stable limits

In the present paper the limiting distribution $\mu$ will be stable with exponent $\alpha \in (0, 2)$. It is well-known (see e.g. [45] or [26]) that its characteristic function admits the Lévy-Khintchine representation

$$\hat{\mu}(\theta) = \exp \left( i \theta a^h + \int \left( e^{i\theta x} - 1 - i\theta x \mathbb{1}_{|x| \leq h} \right) \nu_{\alpha,c_+,c_-}(dx) \right),$$

where $c_+, c_- \geq 0$, $c_+ + c_- > 0$ and $a^h \in \mathbb{R}^1$, the Lévy measure $\nu_{\alpha,c_+,c_-}$ has the density

$$p_{\alpha,c_+,c_-}(x) = \alpha \left( c_+ x^{-(\alpha+1)} \mathbb{1}_{x>0} + c_- |x|^{-(\alpha+1)} \mathbb{1}_{x<0} \right),$$

and $h > 0$ is a fixed level of truncation. We will denote the stable distribution with characteristic function (8) by $\delta_{a^h \ast \text{Poiss}(\alpha,c_+,c_-)}$.

In the main results of the paper we shall consider somewhat less general limits $\mu_\alpha$ with characteristic function of the form

$$\hat{\mu}_\alpha(\theta) = \begin{cases} 
\exp \left( \int \left( e^{i\theta x} - 1 \right) \nu_{\alpha,c_+,c_-}(dx) \right), & \alpha \in (0, 1); \\
\exp \left( \int \left( e^{i\theta x} - 1 \right) \nu_{1,c,c}(dx) \right), & \alpha = 1; \\
\exp \left( \int \left( e^{i\theta x} - 1 - i\theta x \right) \nu_{\alpha,c_+,c_-}(dx) \right), & \alpha \in (1, 2).
\end{cases}$$

(9)

A reader familiar with the terminology would observe that completing the above list with probability laws of the form $\delta_{a^h \ast \text{Poiss}(\alpha,c_+,c_-)}$, we obtain all strictly stable laws on $\mathbb{R}^1$.

Notice that the integrals under the exponents in (8) or (9) can be evaluated, but obtained this way formulas are usually meaningless within the limit theory.

3 Results

Let $\{X_n\}$ be a Markov chain on the space $(S,S)$ with a stationary distribution $\pi$. Define $F_n = \sigma\{X_j; j \leq n\}$.

We will study distributional limits for suitably normalized and centered partial sums of the form

$$S_n = \sum_{j=1}^{n} \Psi(X_j),$$

where $\Psi : (S,S) \to (\mathbb{R}^1,\mathcal{B}^1)$ is a measurable function.

We will assume that the probability law $\pi \circ \Psi^{-1}$ belongs to the domain of attraction of $\mu_\alpha$, $0 < \alpha < 2$. This means (see e.g. [21], Theorem 1a, p. 313] that

$$\pi(x; |\Psi(x)| > t) = t^{-\alpha} \ell(t),$$

(10)

where $\ell(t)$ is a slowly varying function as $t \to \infty$, and there exist the limits

$$\lim_{t \to \infty} \frac{\pi(x; \Psi(x) > t)}{\pi(x; |\Psi(x)| > t)} = \frac{c_+}{c_+ + c_-}, \quad \lim_{t \to \infty} \frac{\pi(x; \Psi(x) < -t)}{\pi(x; |\Psi(x)| > t)} = \frac{c_-}{c_+ + c_-}.$$
3.1 Theorem Let \( \{X_n\} \) be a Markov chain on the space \((S,S)\), with the transition operator \(P\) and a stationary distribution \(\pi\). We assume that \(P\) has a spectral gap and satisfies the 2-U.I. condition.

Let \(\Psi : (S,S) \rightarrow (\mathbb{R}^1,\mathcal{B}^1)\) be such that \(\pi \circ \Psi^{-1}\) belongs to the domain of attraction of the stable distribution \(\mu_\alpha\), \(\alpha \in (0,2)\) (i.e. both \([\text{III}]\) and \([\text{IV}]\) are fulfilled). Let \(B_n \rightarrow \infty\) satisfies

\[
\frac{n}{B_n^\alpha} \ell(B_n) \to c_+ + c_-.
\]

(i) If \(\alpha \in (0,1)\) or \(\alpha = 1\) and \(c_+ = c_- = c\) then

\[
\frac{\Psi(X_1) + \Psi(X_2) + \ldots + \Psi(X_n)}{B_n} \to^\mathcal{D} \mu_\alpha.
\]

(ii) If \(\alpha \in (1,2)\), then

\[
\frac{\sum_{j=1}^n \Psi(X_j) - \mathbb{E}(\Psi(X_j)|\mathcal{F}_{j-1})}{B_n} \to^\mathcal{D} \mu_\alpha.
\]

3.2 Comments

1. Let us notice that in (ii) the tails of conditional expectations may \textit{a priori} influence the form of the limit. But they do not.

2. It is worth stressing that for \(\alpha = 1\) we need only that \textit{the limit} is symmetric and not \(\pi \circ \Psi^{-1}\) itself.

3.3 Corollary In assumptions of Theorem 3.1, if \(\alpha \in (1,2)\) and

\[
\mathbb{E}(\Psi(X_1)|\mathcal{F}_0) = 0.
\]

i.e. \(\Psi(X_1), \Psi(X_2), \ldots\) form a martingale difference sequence, then

\[
\frac{\Psi(X_1) + \Psi(X_2) + \ldots + \Psi(X_n)}{B_n} \to^\mathcal{D} \mu_\alpha.
\]

There is another important case where we may get rid of centering by conditional expectations. As shown in \([\text{III}]\) and \([\text{IV}]\) such situation takes place when we assume the hyperboundedness in place of condition 2-U.I. We extend considerably the result of \([\text{III}]\) by weakening the \(L^2\)-spectral gap property to the strong mixing at geometric rate.

3.4 Theorem In assumptions of Theorem 3.1 replace the 2-U.I. condition with the hyperboundedness and the \(L^2\)-spectral gap property with the strong mixing at geometric rate (in particular: with the geometric ergodicity).

Then

\[
\frac{\Psi(X_1) + \Psi(X_2) + \ldots + \Psi(X_n)}{B_n} \to^\mathcal{D} \mu_\alpha,
\]

provided

(i) \(\alpha \in (0,1)\);

(ii) \(\alpha = 1\) and \(c_+ = c_- = c\),

(iii) \(\alpha \in (1,2)\) and \(\int \Psi(x)\pi(dx) = 0\).
As a by-product of developed techniques we obtain a weak law of large numbers for geometrically ergodic Markov chains, which might be of independent interest.

3.5 Theorem Let \( \{X_n\} \) be strongly mixing at geometric rate (in particular: geometrically ergodic) Markov chain on \((S,S)\) with a stationary distribution \(\pi\). Suppose \(\Psi : (S,S) \to (\mathbb{R}^1,\mathcal{B}^1)\) is such that for some \(\beta > 1\)

\[
\int \pi(dx)|\Psi(x)|^\beta < +\infty, \quad \int \pi(dx)\Psi(x) = 0.
\]

Then for any \(\alpha \in (0,\beta \wedge 2)\) and any \(1/\alpha\)-regularly varying sequence \(B_n\) we have

\[
\frac{\Psi(X_0) + \Psi(X_1) + \ldots + \Psi(X_{n-1})}{B_n} \xrightarrow{P} 0.
\]

4 Examples

4.1 Example related to the 2-U.I. condition

We are going to construct a discrete in time and space example of the transition operator that exhibits the \(L^2\)-spectral gap property, satisfies the 2-U.I. condition but is not hyperbounded. This will show that our theory essentially extends the results of [34] and [11]. Notice also that all the examples of operators provided in [30] and related to the 2-U.I. condition are taken from the stochastic analysis.

4.1 Example The example is a variant of Rosenblatt’s family of examples [43, pp. 213-214], but it occurs also in many other places, e.g. in [40, p. 54], in the context of the backward recurrence time chain.

Let \(T : (\Omega,\mathcal{F},\mathbb{P}) \to \mathbb{N} = \{0,1,2,\ldots\}\) be an integer valued nonnegative random variable such that

\[
\mathbb{E}T < +\infty, \quad \mathbb{P}(T \geq j) > 0, \quad j \in \mathbb{N}.
\]

(Other requirements imposed on the distribution of \(T\) will be specified later). Let the transition probabilities \(p_{j,k}\) be given by the formula

\[
p_{j,k} = \begin{cases} 
\frac{\mathbb{P}(T = j)}{\mathbb{P}(T \geq j)}, & \text{if } k = 0; \\
\frac{\mathbb{P}(T \geq j + 1)}{\mathbb{P}(T \geq j)}, & \text{if } k = j + 1; \\
0, & \text{otherwise.}
\end{cases}
\]

Then

\[
\pi(j) = \frac{\mathbb{P}(T \geq j)}{1 + \mathbb{E}T}, \quad j = 0,1,2,\ldots,
\]

is the unique stationary distribution for \(P = [p_{j,k}]\) and the transition operator reads

\[
Pf(j) = \frac{\mathbb{P}(T = j)}{\mathbb{P}(T \geq j)}f(0) + \frac{\mathbb{P}(T \geq j + 1)}{\mathbb{P}(T \geq j)}f(j + 1).
\]

Let \(\{X_n\}\) be a Markov chain on \(S = \mathbb{N}\) with the transition probabilities \([p_{j,k}]\).
4.2 Lemma If

\[ 3 \mathbb{E}T < \mathbb{P}(T = 0), \]

and

\[ \mathbb{P}(T \geq 1) \geq \sup_{k \geq 1} \frac{\mathbb{P}(T \geq k + 1)}{\mathbb{P}(T \geq k)}, \tag{13} \]

then the Markov chain \( \{X_n\} \) has the \( L^2 \)-spectral gap property.

**Proof.** Let \( f \in L_0^2(\pi) \) and \( \|f\|_2 = 1 \). These relations imply that

\[ |f(0)| = \left| - \sum_{j=1}^{\infty} f(j) \mathbb{P}(T \geq j) \right| \leq \sum_{j=1}^{\infty} (|f(j)| \sqrt{\mathbb{P}(T \geq j)}) \sqrt{\mathbb{P}(T \geq j)} \]

\[ \leq \sqrt{\sum_{j=1}^{\infty} f^2(j) \mathbb{P}(T \geq j)} \sqrt{\sum_{j=1}^{\infty} \mathbb{P}(T \geq j)} = \sqrt{(1 + ET - f^2(0))ET}. \]

Hence

\[ |f(0)| \leq \sqrt{ET}. \tag{14} \]

In a similar way we obtain

\[ \left| \sum_{j=1}^{\infty} f(j) \frac{\mathbb{P}(T = j - 1) \mathbb{P}(T \geq j)}{\mathbb{P}(T \geq j - 1)} \frac{\mathbb{P}(T \geq j)}{1 + ET} \right| \leq \sum_{j=1}^{\infty} \left| f(j) \right| \frac{\mathbb{P}(T \geq j)}{1 + ET} \]

\[ \leq \sqrt{\sum_{j=1}^{\infty} f^2(j) \frac{\mathbb{P}(T \geq j)}{1 + ET} \left| \sum_{j=1}^{\infty} \frac{\mathbb{P}(T \geq j)}{1 + ET} \leq \sqrt{\frac{ET}{1 + ET}}. \tag{15} \]

We are ready for estimates of \( E_\pi(\|Pf\|^2) = (1/(1 + ET)) \sum_{j=0}^{\infty} |Pf(j)|^2 \mathbb{P}(T \geq j). \)

\[ \sum_{j=0}^{\infty} |Pf(j)|^2 \frac{\mathbb{P}(T \geq j)}{1 + ET} = \sum_{j=0}^{\infty} \frac{\mathbb{P}(T = j)}{\mathbb{P}(T \geq j)} f(0) + \frac{\mathbb{P}(T \geq j + 1)}{\mathbb{P}(T \geq j)} f(j + 1) \frac{\mathbb{P}(T \geq j)}{1 + ET} \]

\[ = f^2(0) \frac{1 + ET}{1 + ET} \sum_{j=0}^{\infty} \frac{\mathbb{P}(T = j)}{\mathbb{P}(T \geq j)} + 2f(0) \frac{1 + ET}{1 + ET} \sum_{j=0}^{\infty} f(j + 1) \frac{\mathbb{P}(T = j)}{\mathbb{P}(T \geq j)} \mathbb{P}(T \geq j + 1) \]

\[ + \frac{1 + ET}{1 + ET} \sum_{j=0}^{\infty} f^2(j + 1) \frac{\mathbb{P}(T \geq j + 1)}{\mathbb{P}(T \geq j)} = J_1 + J_2 + J_3. \]

We have by (14)

\[ J_1 \leq \frac{ET}{1 + ET} \sum_{j=0}^{\infty} \frac{\mathbb{P}^2(T = j)}{\mathbb{P}(T \geq j)} \leq ET, \]

while by (14) and (15)

\[ J_2 \leq 2\sqrt{ET} \sqrt{\frac{ET}{1 + ET}} \leq 2ET. \]

Finally, by (13),

\[ J_3 \leq \mathbb{P}(T \geq 1) \sum_{j=1}^{\infty} f^2(j) \frac{\mathbb{P}(T \geq j)}{1 + ET} \leq \mathbb{P}(T \geq 1). \]
Therefore
\[ \mathbb{E}_\pi \left( |Pf|^2 \right) \leq 3ET + \mathbb{P}(T \geq 1) = 1 - (\mathbb{P}(T = 0) - 3ET) = a < 1. \]

The proof of Lemma 4.2 is complete.

It remains to show that for some specific distribution of \( T \) the 2-U.I. condition holds, but there is no hyperboundedness. Choose \( \gamma \in (0, 1) \) and set
\[ \mathbb{P}(T \geq 1) = \gamma, \mathbb{P}(T \geq 2) = \gamma^2, \ldots, \mathbb{P}(T \geq j) = \gamma^{1+2+\ldots+j} = \gamma^{j(j+1)/2}, \ldots. \]

Clearly, \( \mathbb{P}(T \geq j+1)/\mathbb{P}(T \geq j) = \gamma^{j+1}, j = 0, 1, 2, \ldots \) and for \( \gamma < 1/5 \)
\[ ET < \frac{\gamma}{1-\gamma} < (1/3)(1-\gamma) = (1/3)\mathbb{P}(T = 0), \]
so that the assumptions of Lemma 4.2 are satisfied and the corresponding Markov chain \( \{X_n\} \) has the \( L^2 \)-spectral gap property.

In order to prove that the 2-U.I. condition holds, it is enough to show that
\[ \lim_{k \to \infty} \sup_{\|f\|_2 \leq 1} \sum_{j=k}^{\infty} |Pf(j)|^2 \frac{\mathbb{P}(T \geq j)}{1 + ET} = 0. \]

Notice that \( \|f\|_2 \leq 1 \) implies that \( f^2(j) \leq (1 + ET)/\mathbb{P}(T \geq j), j = 0, 1, 2, \ldots \) Keeping this in mind we can proceed as follows.
\[
\sum_{j=k}^{\infty} |Pf(j)|^2 \frac{\mathbb{P}(T \geq j)}{1 + ET} \\
\leq 2\mathbb{P}(0) \sum_{j=k}^{\infty} \frac{\mathbb{P}(T = j)}{\mathbb{P}(T \geq j)} \bigg( \frac{\mathbb{P}(T = j)}{\mathbb{P}(T \geq j)} \bigg)^{2} \frac{\mathbb{P}(T \geq j)}{1 + ET} \\
\leq 2\mathbb{P}(T \geq k) + 2\sum_{j=k}^{\infty} \frac{\mathbb{P}(T \geq j + 1)}{\mathbb{P}(T \geq j)} = 2\mathbb{P}(T \geq k) + 2\sum_{j=k}^{\infty} \gamma^{j+1} \to 0.
\]

Next consider a sequence \( \{f_k\} \) of functions in \( L^2(\pi) \) given by
\[ f_k(j) = \begin{cases} \sqrt{\frac{1 + ET}{\mathbb{P}(T \geq k)}}, & \text{if } j = k; \\ 0, & \text{otherwise}. \end{cases} \]

Take any \( q > 2 \). We have, if \( k \to \infty \),
\[
\|Pf_k\|_q^q = \sum_{j=0}^{\infty} \left| Pf_k(j) \right|^q \frac{\mathbb{P}(T \geq j)}{1 + ET} \\
= \left( \frac{1 + ET}{\mathbb{P}(T \geq k)} \right)^{q/2} \left( \frac{\mathbb{P}(T \geq k)}{\mathbb{P}(T \geq k - 1)} \right)^{q} \frac{\mathbb{P}(T \geq k - 1)}{1 + ET} \\
= \left( 1 + ET \right)^{q/2-1} \left( \frac{\mathbb{P}(T \geq k)}{\mathbb{P}(T \geq k - 1)} \right)^{q/2} = (1 + ET)^{q/2-1} \gamma w(k) \to +\infty,
\]
for \( w(k) = qk(k + 1)/4 - (q - 1)k(k - 1)/2 = (1/4)(k^2(2 - q) + k(3q - 2)) \to -\infty \). It follows that the transition operator \( P \) cannot be a bounded linear map from \( L^2(\pi) \) to \( L^q(\pi) \).
4.2 Gaussian hyperboundedness

Let us examine a standard example, already considered by Doob [18, p.218]. For $0 < |\rho| < 1$ set
\[
P(x, dy) = \frac{1}{\sqrt{2\pi(1 - \rho^2)}} e^{-\frac{(y - \rho x)^2}{2(1 - \rho^2)}} dy,
\]
and notice that for each $x \in \mathbb{R}^1$
\[
P^n(x, dy) = \frac{1}{\sqrt{2\pi(1 - \rho^{2n})}} e^{-\frac{(y - \rho^{2n} x)^2}{2(1 - \rho^{2n})}} dy \xrightarrow{TV} \pi(dy) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy.
\]

Rosenblatt [43, p. 214] proves that the Markov chain \{X_n\} corresponding to $P(x, dy)$ has the $L^2$-spectral gap property. Alternatively, we may observe that \{X_n\} is a Gaussian stationary sequence with the correlation function $\mathbb{E}X_0X_k = |\rho|^{|k|}$, hence an AR(1) process with Gaussian innovations. By [40, p.389] it is a geometrically ergodic Markov chain. Since it is also reversible, it admits an $L^2$-spectral gap.

Using [11, p. 480]), we shall show that \{X_n\} is also hyperbounded. Indeed, $P(x, dy) = p(x, y)\pi(dy)$, where
\[
p(x, y) = \frac{1}{\sqrt{1 - \rho^2}} \exp \left( -\frac{\rho^2}{2(1 - \rho^2)} x^2 + \frac{\rho xy}{1 - \rho^2} - \frac{\rho^2}{2(1 - \rho^2)} y^2 \right).
\]
And we have
\[
\int \pi(dx) \pi(dy) p(x, y)^q < +\infty,
\]
whenever
\[
2 < q < \frac{1 + |\rho|}{|\rho|}.
\]
Hence we can apply Theorem [4,4] and obtain
\[
\frac{\Psi(X_1) + \Psi(X_2) + \ldots + \Psi(X_n)}{B_n} \xrightarrow{D} \mu_\alpha,
\]
for suitably chosen $\Psi$ and $B_n$. Notice that the lack of any necessary centering confirms in this particular case the conjecture of Davis formulated in the frame of Example on p. 267 in [13].

There is another reason for evoking this classic example. It was Doob [18, p.218] who pointed out that this Markov chain does not satisfy Doeblin’s condition (D). And since the work of Davydov [15] we know that Doeblin’s condition means essentially $\phi$-mixing of a Markov chain. It follows that the limit theory developed in our paper is much broader than results depending on uniform ergodicity of Markov chains, as presented e.g. in [12].

4.3 ARCH processes with heavy tails are not hyperbounded

An ARCH(1) process is a Markov chain given by the recurrence formula
\[
X_{j+1} = \sqrt{\beta + \lambda X_j^2} Z_{j+1}, \ j \geq 0,
\]
where $\beta, \lambda > 0$ and $\{Z_n\}_{n \in \mathbb{N}}$ is an i.i.d. sequence, independent of $X_0$. In order to comply with references we shall assume that $Z_n \sim \mathcal{N}(0, 1)$.  

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For basic information on ARCH processes and the properties used below we refer both to the classic book [20] and to the recent source [10].

In the range of parameters $\beta > 0$ and $\lambda \in (0, 2e\gamma)$ (where $\gamma$ is the Euler constant) the process $\{X_j\}$ admits a stationary distribution given by

$$X_0 \sim r_0 \sqrt{\beta \sum_{m=1}^{\infty} Z_m^2 \prod_{j=1}^{m-1} (\lambda Z_j^2)},$$

where $r_0$ is a Rademacher random variable ($P(r_0 = \pm 1) = 1/2$), independent of $\{Z_n\}$. This stationary distribution exhibits power decay of the tails. Namely, if $\kappa > 0$ is the unique positive solution of the equation

$$E(\lambda Z_1^2)^u = 1,$$

then, as $x \to \infty$,

$$P(X_0 > x) = P(X_0 < -x) \sim \frac{C_{\beta,\lambda}}{x^{2\kappa}},$$

where

$$C_{\beta,\lambda} = \frac{\mathbb{E}\left[ (\beta + \lambda X_0^2)^\kappa - (\lambda X_0^2)^\kappa \right]}{\kappa \lambda^{2\kappa} \mathbb{E}\left[ (Z_1^{2\kappa} \ln(\lambda Z_1^2)) \right]} \in (0, +\infty).$$

It follows that $\lambda > 1$ implies “really” heavy tails and it is likely that the partial sums of $\{X_j\}$ properly normalized converge to stable laws. Indeed, Davis and Mikosch [14] showed that the partial sums under the natural normalization converge to some stable limit and Bartkiewicz et al. [1] identified the parameters of the limit.

For purposes of the present example, let us denote by $\mu_{\alpha,\tau}$ the symmetric $\alpha$-stable distribution given for $\alpha \in (0, 2)$ and $\tau > 0$ by

$$\mu_{\alpha,\tau}(\theta) = \exp\left( \tau\alpha \int_{\mathbb{R}} (e^{i\theta u} - 1)|u|^{-(\alpha+1)} du \right).$$

If our Theorems 3.1 or 3.4 were applicable to $\{X_j\}_{j \geq 0}$, then we would have

$$\frac{X_1 + X_2 + \ldots + X_n}{(nC_{\beta,\lambda})^{\frac{1}{2\kappa}}} \overset{\mathcal{D}}{\to} \mu_{2\kappa,1}.$$

It is, however, proved in [1] that

$$\frac{X_1 + X_2 + \ldots + X_n}{(nC_{\beta,\lambda})^{\frac{1}{2\kappa}}} \overset{\mathcal{D}}{\to} \mu_{2\kappa,\tau},$$

where $\tau = E[|1 + S_\infty|^{2\kappa} - |S_\infty|^{2\kappa}] > 0$ and the series

$$S_\infty = \sum_{j=1}^{\infty} \lambda^{j/2} \prod_{k=1}^{j-1} |Z_k| Z_j$$

converges a.s.

Moreover, ARCH(1) processes are strongly mixing at geometric rate, as is shown in [14] p. 2077.

Therefore the transition operator of an ARCH(1) process is not hyperbounded.
4.3 Remark If $2\kappa \in (1, 2)$, then the corresponding ARCH(1) process $\{X_j\}$ forms a martingale difference sequence, partial sums of which normalized by $n^{1/2\kappa}$ are weakly convergent, but to a different limit than in the independent case. This is in striking contrast to the properties of martingale difference sequences with finite variance!

4.4 $m$-skeletons

It is well known that iterating the transition operator improves its properties from many viewpoints. So it may happen that some power $P^m$ is hyperbounded, for instance, while $P$ itself not. Such situation implies that for $\{\Psi(X_{k,m})\}_{k=0,1,2,...}$ (the $m$-skeleton) some $\alpha$-stable limit theorem holds and one may hope to extend this property to the whole sequence. This is impossible in general, as the simple counterexample provided already by Rosenblatt [13, p. 195] shows. Indeed, take an i.i.d. sequence $\{Y_n\}$ of strictly stable random variables and consider a Markov chain on $S = \mathbb{R}^2$ given by the formula $X_n = (Y_n, Y_{n-1})$. Take $\Psi(x, y) = x - y$. Then $\sum_{j=0}^{n-1} \Psi(X_n)$ remains stochastically bounded while the 1-skeleton consists of independent random variables and therefore satisfies the corresponding limit theorem.

Rosenblatt’s example is of probabilistic provenience. Some people may prefer another example given below that is closer to thinking in terms of dynamical systems.

4.4 Example Set $S = [0, 3)$ and let $Leb$ be the Lebesgue measure restricted to $S$. For $x \in [0, 1)$ and $B \in B_{[0,1]} \cup B_{[2,3]}$ define

$$P(x, \{x + 1\}) = P(x + 1, \{x + 2\}) = 1, \quad P(x + 2, B) = \frac{Leb(B)}{2}.$$ 

The invariant measure $\pi$ is given by the density

$$p(x) = \frac{1}{4} \mathbb{1}_{[0,2)}(x) + \frac{1}{2} \mathbb{1}_{[2,3]}(x).$$

Elementary calculations show that for $f \in L^2_0(\pi)$ we have

$$\mathbb{E}_\pi \left((P^3 f)^2\right) \leq \frac{27}{32} \mathbb{E}_\pi (f^2),$$

i.e. the 3-skeleton has the spectral gap property. Another elementary calculation shows that also

$$\|P^3 f\|_{\infty} \leq 3\|f\|_1,$$

i.e. the 3-skeleton is ultrabounded.

Now take $\psi(\cdot) : [0, 1) \to \mathbb{R}^1$ with a symmetric $\alpha$-stable distribution $\mu$ and define

$$\Psi(x) = \begin{cases} 
\psi(x), & \text{if } x \in [0, 1); \\
-\psi(x - 1), & \text{if } x \in [1, 2); \\
0, & \text{if } x \in [2, 3).
\end{cases}$$

One verifies directly that

$$\pi(\Psi > r) = \frac{1}{2} Leb(\psi > r), \quad \pi(\Psi < -r) = \frac{1}{2} Leb(\psi < -r).$$

Therefore the 3-skeleton $\{\Psi(X_{k,3})\}$ satisfies all assumptions of our Theorem 3.4 while the partial sums of the whole sequence $\{\Psi(X_k)\}$ are bounded in probability.
5 Proofs

5.1 Some auxiliary results

We begin with establishing an important property of conditional distributions \( P(x, dy) \circ \Psi^{-1} \) that is a consequence of solely (10)–(11).

5.1 Proposition Suppose that (10) and (11) hold. Let \( B_n \) be defined by (12). Then

\[
|1 - \mathbb{E}(e^{i\theta \psi(X_1)/B_n} | \mathcal{F}_0)|^2 \to 0, \quad \theta \in \mathbb{R}^1. \tag{18}
\]

Proof. Recall that if \( B_n \) is defined by (12) then \( B_n = n^{1/\alpha}\tilde{f}(n) \), where \( \tilde{f}(t) \) is a slowly varying function. Let \( h > 0 \) be fixed. Using the inequality \(|1 + ix - e^{ix}| \leq \frac{1}{2}|x|^2\), we have

\[
|1 - \mathbb{E}(e^{i\theta \psi(X_1)/B_n} | \mathcal{F}_0)|^2 \leq
\]

\[
2n|1 + i\theta \mathbb{E}(\frac{\Psi(X_1)}{B_n} \mathbb{I}_{\{\psi(X_1) \leq h B_n\}} | \mathcal{F}_0) - \mathbb{E}(e^{i\theta \psi(X_1)/B_n} | \mathcal{F}_0)|^2
\]

\[
+ 2nB_n^{-2}\theta^2|\mathbb{E}(\Psi(X_1) \mathbb{I}_{\{\psi(X_1) \leq h B_n\}} | \mathcal{F}_0)|^2
\]

\[
\leq nB_n^{-2}\theta^2\left(\mathbb{E}(\Psi(X_1)^2 | \mathcal{F}_0\right) + 16n\left(\mathbb{E}(|\Psi(X_1)| > h B_n | \mathcal{F}_0\right)^2
\]

\[
+ 2nB_n^{-2}\theta^2|\mathbb{E}(\Psi(X_1) \mathbb{I}_{\{\psi(X_1) \leq h B_n\}} | \mathcal{F}_0)|^2
\]

\[
= \theta^4 I_{n,1}^h + 16I_{n,2}^h + 2\theta^2 I_{n,3}^h.
\]

At first we shall examine the convergence of \( I_{n,3}^h \). If \( \alpha \in (1, 2) \) then

\[
\left| \mathbb{E}(\Psi(X_1) \mathbb{I}_{\{\psi(X_1) \leq h B_n\}} | \mathcal{F}_0) \right|^2 \to 0 \text{ a.s.},
\]

while \( nB_n^{-2} = n^{1-2/\alpha}(\tilde{f}(n))^{-2} \to 0 \). Consequently, \( I_{n,3}^h \to 0 \text{ a.s.} \).

Now suppose that \( \alpha \in (0, 1) \). Take \( 0 < r < \alpha/2 \). We have

\[
\mathbb{E}(\mathbb{E}(|\Psi(X_1)|^{\alpha-r} | \mathcal{F}_0) = \mathbb{E}|\Psi(X_1)|^{\alpha-r} < +\infty,
\]

and so

\[
(\alpha - r) \int_{0}^{\infty} t^{\alpha-r-1} P(X_0, |\Psi|^{-1}(t, +\infty)) \ dt = \mathbb{E}(|\Psi(X_1)|^{\alpha-r} | \mathcal{F}_0) < +\infty \text{ a.s.}
\]

It follows that

\[
I_{n,3}^h = nB_n^{-2} \mathbb{E}(\Psi(X_1) \mathbb{I}_{\{\psi(X_1) \leq h B_n\}} | \mathcal{F}_0) \]

\[
\leq nB_n^{-2} \mathbb{E}(|\Psi(X_1)| \mathbb{I}_{\{\psi(X_1) \leq h B_n\}} | \mathcal{F}_0) \]

\[
\leq nB_n^{-2} \int_{0}^{hB_n} P(X_0, |\Psi|^{-1}(t, +\infty)) \ dt \]

\[
= nB_n^{-2} \int_{0}^{hB_n} t^{1-\alpha+r} t^{\alpha-r-1} P(X_0, |\Psi|^{-1}(t, +\infty)) \ dt \]

\[
\leq nB_n^{-2} h^{2(1-\alpha+r)} B_n^{2(1-\alpha+r)} \int_{0}^{\infty} t^{\alpha-r-1} P(X_0, |\Psi|^{-1}(t, +\infty)) \ dt \]

\[
= n^{-1+2r/\alpha}(\tilde{f}(n))^{-2(\alpha-r)} h^{2(1-\alpha+r)} \left( \frac{1}{\alpha-r} \mathbb{E}(|\Psi(X_1)|^{\alpha-r} | \mathcal{F}_0) \right)^2 \to 0 \text{ a.s.}
\]
Similarly, if \( \alpha \in (0, 2) \) and \( 0 < r < \alpha/2 \), then we have
\[
I_{n,1}^h = n B_n^{-4} |\mathbb{E}(\Psi(X_1)^2 \mathbb{I}_{\{|\Psi(X_1)| \leq h B_n\}, \mathcal{F}_0})|^2
\leq 4 n B_n^{-4} \left( \int_0^{h B_n} t P(X_0, |\Psi|^{-1}(t, +\infty)) dt \right)^2
= 4 n B_n^{-4} \left( \int_0^{h B_n} t^{2-\alpha+r} t^{\alpha-r-1} P(X_0, |\Psi|^{-1}(t, +\infty)) dt \right)^2
\leq 4 n B_n^{-4} h^{2(2-\alpha+r)} B_n^{2(2-\alpha+r)} \left( \int_0^{\infty} t^{\alpha-r-1} P(X_0, |\Psi|^{-1}(t, +\infty)) dt \right)^2
= 4 n^{-1+2r/\alpha} \left( \ell(n) \right)^{-2(\alpha-r)} h^{2(2-\alpha+r)} \left( \frac{1}{\alpha-r} \mathbb{E}(\Psi(X_1)^{\alpha-r} |\mathcal{F}_0) \right)^2 \to 0, \text{ a.s.}
\]

It remains to show that \( I_{n,2}^h \to_{\mathbb{P}} 0 \). This condition is not related to truncated moments and therefore requires a different type argument. Notice that the convergence in probability is metrizable and so it is enough to show that in every subsequence \( n' \) one can find a further subsequence \( n'' \) along which \( I_{n''}^h \to_{\mathbb{P}} 0 \). So choose \( n' \) and consider random variables \( Y_{n'} \) defined on \((\mathcal{S}, \mathcal{S})\) by the formula
\[
Y_{n'}(x) = n' P(x, |\Psi|^{-1}(h B_{n'}, +\infty)).
\]

We know from (10), (11), (12) and the continuity of the stable Lévy measure that
\[
\int_{\mathcal{S}} \pi(dx) Y_{n'}(x) = n'\mathbb{P}(|\Psi(X_1)| > h B_{n'}) \to (c_+ + c_-) h^{-\alpha},
\]
hence, in particular, random variables \( \{Y_{n'}\} \) are uniformly tight. Let \( \{n''\} \) be a subsequence such that \( Y_{n''} \to_{\mathcal{D}} Y_\infty \). By the Skorokhod representation theorem one can construct random variables \( \tilde{Y}_{n''} \) and \( \tilde{Y}_\infty \), defined on the standard probability space \( ([0,1], \mathcal{B}_{[0,1]}, \text{Leb}) \) and such that
\[
\tilde{Y}_{n''} \sim Y_{n''}, \quad \tilde{Y}_\infty \sim Y_\infty,
\]
and
\[
\tilde{Y}_{n''}(\omega) \to \tilde{Y}_\infty(\omega), \quad \text{for almost all } \omega \in [0,1].
\]
This implies that
\[
\frac{1}{n''} \tilde{Y}_{n''}^2(\omega) \to 0, \quad \text{for almost all } \omega \in [0,1].
\]
But under the initial distribution \( \pi \) we have
\[
n'' \left( P(x, |\Psi|^{-1}(h B_{n''}, +\infty)) \right)^2 = \frac{1}{n''} Y_{n''}^2 \sim \frac{1}{n''} \tilde{Y}_{n''}^2.
\]
It follows that
\[
n'' \left( P(x, |\Psi|^{-1}(h B_{n''}, +\infty)) \right)^2 \to_{\mathbb{P}} 0.
\]

**5.2 Remark** It is clear that the convergences \( I_{n,1}^h \to_{\mathbb{P}} 0 \) and \( I_{n,3}^h \to_{\mathbb{P}} 0 \) can be obtained also by the last method. But the proofs given above lead to the a.s. convergence and provide some idea about the rate of convergence.
5.3 Remark It is also clear that relation \([20]\) can be extended to
\[
(n')^{1+\delta} \left( P(x, |\Psi|^{-1}(hB_{n'}, +\infty)) \right)^2 = \frac{1}{(n')^{1-\delta}}Y_{n'}^2 \sim \frac{1}{(n')^{1-\delta}}\tilde{Y}_{n'}^2,
\]
hence, in fact, we have
\[
n^{\delta} I_{n,2}^h \xrightarrow{p} 0,
\]
for every \(\delta \in (0, 1)\). Gathering information on \(I_{n,1}^h, I_{n,2}^h\) and \(I_{n,3}^h\) we obtain existence of some \(\delta > 0\) such that
\[
n^{1+\delta} \left| 1 - \mathbb{E}\left( e^{i\theta X^n/B_0} \right) \right|^2 \xrightarrow{p} 0, \quad \theta \in \mathbb{R}^1.
\]

Now we are ready to prove two universal (i.e. independent of \(\alpha \in (0, 2)\)) limit theorems.

5.4 Proposition Let \(\{X_n\}\) be a Markov chain on the space \((S, S)\), with the transition operator \(P\) and a stationary distribution \(\pi\). We assume that \(P\) has a spectral gap and satisfies the 2-U.I. condition.

Let \(\alpha \in (0, 2)\) and \(h > 0\). Let \(\Psi : (S, S) \rightarrow (\mathbb{R}^1, \mathcal{B})\) be such that \(\pi \circ \Psi^{-1}\) belongs to the domain of attraction of the stable distribution \(\mu_\alpha, \alpha \in (0, 2)\) (i.e. both \((17)\) and \((17)\) are fulfilled). Let \(B_n \rightarrow \infty\) satisfies
\[
\frac{n}{B_n^2} t(B_n) \rightarrow c_+ + c_-.
\]

Set \(S_n^h = \sum_{j=1}^n \Psi(X_j) - \mathbb{E}(\Psi(X_j) | \mathcal{F}_{j-1})\). Then
\[
\frac{S_n^h \mathcal{F}_0}{B_n} \rightarrow \mathcal{D} \text{c}_h\text{-Pois}(\alpha, c_+, c_-). \tag{21}
\]

PROOF. Choose \(\theta \in \mathbb{R}^1\) and notice that by Proposition 5.1 relation \([18]\) holds. We will show that this relation can be strengthened to
\[
n\mathbb{E} \left( 1 - \mathbb{E}\left( e^{i\theta \Psi(X_1)/B_0} \right) \right)^2 \rightarrow 0. \tag{22}
\]

It is enough to show that \(n\left| 1 - \mathbb{E}\left( e^{i\theta \Psi(X_1)/B_0} \right) \right|^2\) is a uniformly integrable sequence. By the 2-U.I. condition we have to prove that the sequence \(\{Z_n = \sqrt{n}(1 - e^{i\theta \Psi(X_1)/B_0})\}\) is bounded in \(L^2\).

\[
\mathbb{E} \left| \sqrt{n}(1 - e^{i\theta \Psi(X_1)/B_0}) \right|^2 = n \mathbb{E} \left( (1 - \cos(\theta \Psi(X_1)/B_0))^2 + (\sin(\theta \Psi(X_1)/B_0))^2 \right) \leq \theta^2 (1 + \theta^2/4) \frac{n}{B_n^2} \mathbb{E} \Psi(X_1)^2 \mathbb{I}_{\{|\Psi(X_1)| \leq B_n\}} + 5n\mathbb{P}(|\Psi(X_1)| > B_n) \leq \theta^2 (1 + \theta^2/4) \frac{n}{B_n^2} \int_0^{B_n} t\mathbb{P}(|\Psi(X_1)| > t) \, dt + 5n\mathbb{P}(|\Psi(X_1)| > B_n).
\]

The last expression converges to \((2\theta^2 (1 + \theta^2/4) + 5)(c_+ + c_-)\) by the definition of \(B_n\) and the direct half of the Karamata theorem (see \([5\) Theorem 1.5.11, p. 28]).
Given (22) we obtain the crucial relation [35]
\[
E\left( \sum_{j=1}^{n} \left| 1 - E\left( e^{i\theta \Psi(X_j)/B_n} \right| \mathcal{F}_{j-1} \right)^2 \right) = nE \left| 1 - E\left( e^{i\theta \Psi(X_1)/B_n} \right| \mathcal{F}_0 \right|^2 \to 0, \quad \theta \in \mathbb{R}.
\]

By Theorem A.3 it is enough to prove (36), i.e.
\[
\Phi^h(\theta) := \sum_{j=1}^{n} E\left( e^{i\theta \Psi(X_j)/B_n} \right| \mathcal{F}_{j-1} \right) - 1 - i\theta B_n^{-1} E\left( \Psi(X_j) \mathbb{1}_{\{\|\Psi(y)\| \leq hB_n\}} \right| \mathcal{F}_{j-1} \right) \quad (24)
\]
\[
\to_p \int \left( e^{i\theta x} - 1 - i\theta x \mathbb{1}_{\{\|x\| \leq h\}} \right) \nu_{\alpha,c_+} (dx) =: \Phi^h(\theta).
\]
Let us notice that by (10) and (11) we have
\[
\left| \sum_{j=1}^{n} \left( \exp \left( i\theta \Psi(y)/B_n \right) - 1 - i\theta \Psi(y)/B_n \mathbb{1}_{\{\|\Psi(y)\| \leq hB_n\}} \right) P(x, dy) \right| \leq 16 \nu_{\alpha,c_+} (dx) \quad (25)
\]
where
\[
\chi^h_{n,\theta}(x) = \int \left( \exp \left( i\theta \Psi(y)/B_n \right) - 1 - i\theta \Psi(y)/B_n \mathbb{1}_{\{\|\Psi(y)\| \leq hB_n\}} \right) P(x, dy)
\]-
\[
- \left( \int \left( \exp \left( i\theta \Psi(y)/B_n \right) - 1 - i\theta \Psi(y)/B_n \mathbb{1}_{\{\|\Psi(y)\| \leq hB_n\}} \right) \pi(dy) \right).
\]
We apply the standard procedure based on the spectral gap property. Let
\[
\delta^h_{n,\theta} = \sum_{m=0}^{\infty} P^n_{m}(\chi^h_{n,\theta}).
\]
\(\delta^h_{n,\theta}\) is a well-defined element of \(L^2_0(\pi)\), because \(\chi^h_{n,\theta} \in L^2_0(\pi)\) and according to [6]
\[
\|\delta^h_{n,\theta}\|_2 \leq \sum_{m=0}^{\infty} \|P^n_{m}(\chi^h_{n,\theta})\|_2 \leq \sum_{m=0}^{\infty} a^m \|\chi^h_{n,\theta}\|_2 = \frac{\|\chi^h_{n,\theta}\|_2}{1 - a}.
\]
Clearly, \(\delta^h_{n,\theta} = (I - P)^{-1} \chi^h_{n,\theta}\), hence \(\chi^h_{n,\theta} = \delta^h_{n,\theta} - P(\delta^h_{n,\theta})\). Consequently,
\[
\sum_{j=1}^{n} \chi^h_{n,\theta}(X_{j-1}) = \delta^h_{n,\theta}(X_0) - \delta^h_{n,\theta}(X_n) + \sum_{j=1}^{n} \left( \delta^h_{n,\theta}(X_j) - P\delta^h_{n,\theta}(X_{j-1}) \right).
\]
The point here is that \(\sum_{j=1}^{k} \left( \delta^h_{n,\theta}(X_j) - P\delta^h_{n,\theta}(X_{j-1}) \right); k = 1, 2, \ldots, n\) is a square integrable martingale. Therefore
\[
E \left| \sum_{j=1}^{n} \chi^h_{n,\theta}(X_{j-1}) \right|^2 \leq 4E \left| \delta^h_{n,\theta}(X_0) \right|^2 + 4E \left| \delta^h_{n,\theta}(X_n) \right|^2 + 4E \sum_{j=1}^{n} \left| \delta^h_{n,\theta}(X_j) - P\delta^h_{n,\theta}(X_{j-1}) \right|^2
\]
\[
= 4\|\delta^h_{n,\theta}(X_0)\|_2^2 + 4\|\delta^h_{n,\theta}(X_n)\|_2^2 + 4n\|\delta^h_{n,\theta}(X_1) - P\delta^h_{n,\theta}(X_0)\|_2^2
\]
\[
\leq 8\|\delta^h_{n,\theta}(X_0)\|_2^2 + 16n\|\delta^h_{n,\theta}(X_1)\|_2^2 \leq \frac{24}{(1 - a)^2} nE \left| \chi^h_{n,\theta}(X_0) \right|^2.
\]
So we have to prove that

\[ n E \left| \chi_{n,\theta}^h(X_0) \right|^2 \to 0. \]  \hspace{1cm} (27)

Let us notice that

\[ \left\| \chi_{n,\theta}^h(X_0) \right\|_2^2 = E \left| \chi_{n,\theta}^h(X_0) \right|^2 = \text{Var}(W_{n,\theta}^h) \leq E \left| W_{n,\theta}^h \right|^2, \]

where

\[ W_{n,\theta}^h = 1 + i\theta E \left( \frac{\Psi(X_1)}{B_n} \mathbb{I}(\left| \Psi(X_1) \right| \leq h B_n) \mid \mathcal{F}_0 \right) - E(e^{i\theta \Psi(X_1)/B_n} \mid \mathcal{F}_0). \]

By inspection of (19) we see that

\[ \text{By (7) there is a number} \ 0 \leq \eta < 1 \ \text{such that} \]

\[ \left| E \left( \chi_{n,\theta}^h(X_i) \chi_{n,\theta}^h(X_j) \right) \right| \leq 2\pi \eta^{i-j} (2 + \theta h)^2, \quad i, j = 0, 1, \ldots, n - 1. \]

Set \( m_n = [n^\delta] \), for some \( \delta > 0 \), and note that \( n\eta^{mn} \to 0 \). It follows that

\[ \leq 2\pi (2 + \theta h)^2 \sum_{0 \leq i, j \leq n - 1 \atop |i-j| > m_n} \eta^{i-j} \leq \frac{4\pi (2 + \theta h)^2}{1 - \eta} (n - m_n)\eta^{mn}. \]
So we have to consider the remaining covariances only.

\[
\sup_{0 \leq i,j \leq n-1, |i-j| \leq \lfloor n/2 \rfloor} E\left| \chi_{n,\theta}^h(X_i)\chi_{n,\theta}^h(X_j) \right|^2 \\
= \sum_{i=0}^{n-1} E\left| \chi_{n,\theta}^h(X_i) \right|^2 \\
+ \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} \left( E\left| \chi_{n,\theta}^h(X_i)\chi_{n,\theta}^h(X_j) \right| + E\left| \chi_{n,\theta}^h(X_i) \right|^2 + E\left| \chi_{n,\theta}^h(X_j) \right|^2 \right) \\
\leq n E\left| \chi_{n,\theta}^h(X_0) \right|^2 + \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} \left( E\left| \chi_{n,\theta}^h(X_i) \right|^2 + E\left| \chi_{n,\theta}^h(X_j) \right|^2 \right) \\
\leq (n + 2(n-1)[n]) E\left| \chi_{n,\theta}^h(X_0) \right|^2 \leq 3n^{1+\delta} E\left| \chi_{n,\theta}^h(X_0) \right|^2,
\]

where the last inequality holds for \( n \) large enough. It follows that the proof of Proposition 5.5 will be complete if we are able to prove that for some \( \delta > 0 \)

\[
n^{1+\delta} E\left| \chi_{n,\theta}^h(X_0) \right|^2 \to 0. \tag{29}
\]

Using the notation introduced in (19) we have the following estimate.

\[
n^{1+\delta} E\left| \chi_{n,\theta}^h(X_0) \right|^2 \leq n^\delta \left( \frac{1}{2} \theta^4 E I_{n,1}^h + 8 E I_{n,2}^h \right).
\]

Notice that by Remark 5.3 one can always find \( \delta > 0 \) such that \( n^\delta (I_{n,1}^h + I_{n,2}^h) \longrightarrow P 0 \).

We shall strengthen this convergence using the hyperboundedness. By this assumption there exists \( q > 2 \) such that the transition operator \( P \) is a bounded linear map from \( L^2(\pi) \) to \( L^q(\pi) \):

\[
\sup \left\{ \int (dx) |Pf(x)|^q : \|f\|_2 \leq 1 \right\} = \|P\|_{2\rightarrow q}^q < +\infty.
\]

We have

\[
\sup_n \left\| \sqrt{n} \frac{\Psi(X_1)^2}{B_n^2} \mathbb{I}_{\{\Psi(X_1) \leq hB_n\}} \right\|_2^2 = \\
= \sup_n E \left( \frac{\Psi(X_1)^4}{B_n^4} \mathbb{I}_{\{\Psi(X_1) \leq hB_n\}} \right) \\
\leq h^2 \sup_n E \left( \frac{\Psi(X_1)^2}{B_n^2} \mathbb{I}_{\{\Psi(X_1) \leq hB_n\}} \right) \leq K_1 < +\infty.
\]

Therefore

\[
\sup_n E \left| \sqrt{n} \frac{\Psi(X_1)^2}{B_n^2} \mathbb{I}_{\{\Psi(X_1) \leq hB_n\}} \right|_F^q \leq \|P\|_{2\rightarrow q}^q K_1^{q/2} < +\infty.
\]
Next we apply the Hölder inequality.

\[
n^\delta \mathbb{E} I_{n,1}^h = n^{1+\delta} \mathbb{E} \left( \mathbb{E} \left( \frac{\Psi(X_1)^2}{B_n^2} \mathbb{I}_{\{|\Psi(X_1)| \leq hB_n\}} \bigg| \mathcal{F}_0 \right) \right)^2
\]

\[
= \mathbb{E} \left( \sqrt{n} \mathbb{E} \left( \frac{\Psi(X_1)^2}{B_n^2} \mathbb{I}_{\{|\Psi(X_1)| \leq hB_n\}} \bigg| \mathcal{F}_0 \right) \right) \times \left( n^{1/2+\delta} \frac{\Psi(X_1)^2}{B_n^2} \mathbb{I}_{\{|\Psi(X_1)| \leq hB_n\}} \right)
\]

\[
\leq \left( \mathbb{E} \left( \sqrt{n} \mathbb{E} \left( \frac{\Psi(X_1)^2}{B_n^2} \mathbb{I}_{\{|\Psi(X_1)| \leq hB_n\}} \bigg| \mathcal{F}_0 \right) \right)^q \right)^{1/q} \times \left( \mathbb{E} \left( n^{1/2+\delta} \frac{\Psi(X_1)^2}{B_n^2} \mathbb{I}_{\{|\Psi(X_1)| \leq hB_n\}} \right)^{q-1} \right)^{\frac{1}{q}}
\]

\[
\leq \|P\|_{2\to q} K_1^{1/2} \left( n^{1/2+\delta} \mathbb{E} \left( \frac{\Psi(X_1)^2}{B_n^2} \mathbb{I}_{\{|\Psi(X_1)| \leq hB_n\}} \bigg| \mathcal{F}_0 \right) \right)^{q-1} \to 0,
\]

if \(0 < \delta < \frac{1}{2} - \frac{1}{q}\).

Similarly we handle the other convergence \(n^\delta \mathbb{E} I_{n,2} \to 0\). First, we see that

\[
\sup_n \left\| \sqrt{n} \mathbb{I}_{\{|\Psi(X_1)| > hB_n\}} \right\|_2^2 = \sup_n \mathbb{P} \left( |\Psi(X_1)| > hB_n \right) \leq K_2 < +\infty.
\]

This implies that

\[
\sup_n \mathbb{E} \left( \sqrt{n} \mathbb{P}( |\Psi(X_1)| > hB_n | \mathcal{F}_0 ) \right)^q \leq \|P\|_{2\to q} K_2^{q/2} < +\infty.
\]

Finally we obtain by the Hölder inequality and for \(0 < \delta < \frac{1}{2} - \frac{1}{q}\) that

\[
n^\delta \mathbb{E} I_{n,2}^h = n^{1+\delta} \mathbb{E} \left( \mathbb{P}( |\Psi(X_1)| > hB_n | \mathcal{F}_0 ) \right)^2
\]

\[
= \mathbb{E} \left( \sqrt{n} \mathbb{P}( |\Psi(X_1)| > hB_n | \mathcal{F}_0 ) \right) \left( n^{1/2+\delta} \mathbb{I}_{\{|\Psi(X_1)| > hB_n\}} \right)
\]

\[
\leq \left( \mathbb{E} \sqrt{n} \mathbb{P}( |\Psi(X_1)| > hB_n | \mathcal{F}_0 ) \right)^{q/4} \left( \mathbb{E} \left( n^{1/2+\delta} \mathbb{I}_{\{|\Psi(X_1)| > hB_n\}} \right)^{q-1} \right)^{\frac{1}{4}}
\]

\[
\leq \|P\|_{2\to q} K_2^{1/2} \left( n^{1/2+\delta} \mathbb{E} \left( \mathbb{I}_{\{|\Psi(X_1)| > hB_n\}} \bigg| \mathcal{F}_0 \right) \right)^{q-1} \to 0.
\]

The proof of Proposition 5.5 is complete.

### 5.2 Proof of Theorem 3.1

Given (21), i.e.

\[
\frac{\sum_{j=1}^n \Psi(X_j) - \mathbb{E} (\Psi(X_j) \mathbb{I}_{\{|\Psi(X_j)| \leq hB_n\}} | \mathcal{F}_{j-1}) \bigg/ B_n \to_{D} \mathcal{C}_h \text{-Poiss}(\alpha, c_+, c_-),
\]

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we shall apply classic Theorem 4.2 from [4] in a way suitable for each case \( \alpha \in (0, 1) \), \( \alpha = 1 \) or \( \alpha \in (1, 2) \).

The reasoning is equally simple for \( \alpha \neq 1 \) and is based on the direct half of Karamata’s theorem [5, Theorem 1.5.11, p. 28]).

For \( \alpha \in (0, 1) \) we shall show that

\[
\lim_{h \to 0} \lim_{n \to \infty} \mathbb{E} \left[ \sum_{j=1}^{n} \mathbb{E} \left( \frac{\Psi(X_j)}{B_n} \mathbb{1}_{\{ |\Psi(X_j)| \leq hB_n \}} \right) \mathcal{F}_{j-1} \right] = 0, \tag{30}
\]

and that

\[
c_h \text{-Poiss}(\alpha, c_+, c_-) \Rightarrow \mu_\alpha, \quad \text{as } h \to 0.
\]

The latter relation holds because \( \int |x| \mathbb{1}_{\{|x| \leq 1\}} \nu_{\alpha, c_+, c_-}(dx) < +\infty \) for \( \alpha \in (0, 1) \). In order to prove (30) we proceed also in the standard way.

\[
\mathbb{E} \left[ \sum_{j=1}^{n} \mathbb{E} \left( \frac{\Psi(X_j)}{B_n} \mathbb{1}_{\{ |\Psi(X_j)| \leq hB_n \}} \right) \mathcal{F}_{j-1} \right] \leq n \mathbb{E} \left( \frac{\Psi(X_1)}{B_n} \right) \mathbb{1}_{\{ |\Psi(X_1)| \leq hB_n \}} \leq \frac{n B_n}{B_n} \int_{0}^{hB_n} \mathbb{P}(|\Psi(X_1)| > t) \, dt \to n \to \infty (1 - \alpha)^{-\frac{1}{\alpha}} h^{1-\alpha} (c_+ + c_-) \to h \to 0.
\]

For \( \alpha \in (1, 2) \) we shall show that

\[
\lim_{h \to \infty} \lim_{n \to \infty} \mathbb{E} \left[ \sum_{j=1}^{n} \mathbb{E} \left( \frac{\Psi(X_j)}{B_n} \mathbb{1}_{\{ |\Psi(X_j)| \geq hB_n \}} \right) \mathcal{F}_{j-1} \right] = 0, \tag{31}
\]

and that

\[
c_h \text{-Poiss}(\alpha, c_+, c_-) \Rightarrow \mu_\alpha, \quad \text{as } h \to \infty.
\]

Here again the latter relation holds due to the fact that

\[
\int |x| \mathbb{1}_{\{|x| \geq 1\}} \nu_{\alpha, c_+, c_-}(dx) < +\infty,
\]

if \( \alpha \in (1, 2) \). And we have

\[
\mathbb{E} \left[ \sum_{j=1}^{n} \mathbb{E} \left( \frac{\Psi(X_j)}{B_n} \mathbb{1}_{\{ |\Psi(X_j)| \geq hB_n \}} \right) \mathcal{F}_{j-1} \right] \leq n \mathbb{E} \left( \frac{\Psi(X_1)}{B_n} \right) \mathbb{1}_{\{ |\Psi(X_1)| \geq hB_n \}} \leq \frac{n B_n}{B_n} \int_{hB_n}^{\infty} \mathbb{P}(|\Psi(X_1)| > t) \, dt + hB_n \mathbb{P}(\{|\Psi(X_1)| \geq hB_n \})
\]

\[
\quad \quad \to n \to \infty \frac{\alpha}{\alpha - 1} h^{1-\alpha} (c_+ + c_-) \to h \to \infty.
\]

The proof for \( \alpha = 1 \) is somewhat different. Let us notice first that due to the symmetry of \( \nu_{1,c,c} \) we have the equality

\[
c_h \text{-Poiss}(\alpha, c, c) = \mu_1, \quad h \in \mathbb{R}^1,
\]

hence

\[
c_h \text{-Poiss}(\alpha, c, c) \Rightarrow h \to 0 \mu_1.
\]
Let \( h > h' > 0 \). By (24) we have also
\[
\Phi_n^h(\theta) - \Phi_n^{h'}(\theta) = -i\theta \sum_{j=1}^n B_n^{-1} E(\Psi(X_j) \mathbb{I}_{\{h'B_n < |\Psi(X_j)| \leq hB_n\}} | F_{j-1}) \overset{p}{\to} 0, \quad \theta \in \mathbb{R}^1.
\]

Therefore
\[
\lim_{h' \to 0} \limsup_{n \to \infty} \mathbb{P}\left( \left| \sum_{j=1}^n E\left( \frac{\Psi(X_j)}{B_n} \mathbb{I}_{\{h'B_n < |\Psi(X_j)| \leq hB_n\}} \right) \right| > \varepsilon \right) = 0, \quad \varepsilon > 0.
\]

### 5.3 Proof of Theorem 3.4

The starting point is the same as in the proof of Theorem 3.1: by Proposition 5.5 convergence (21) holds and we have to reduce it to the desired form. The reduction for the cases \( \alpha \in (0, 1) \) and \( \alpha = 1 \) is identical. So assume that \( \alpha \in (1, 2) \). Take \( 0 < r < \alpha - 1 \). We know that
\[
\int \pi(dx)|\Psi(x)|^{\alpha - r} < +\infty, \quad 0 < r < \alpha - 1.
\]
We shall show that the hyperboundedness gives some \( \beta > \alpha \) such that
\[
\int \pi(dx)|P\Psi(x)|^\beta < +\infty.
\]
By the Riesz-Thorin interpolation theorem (see e.g. [2, Theorem 1.1.1]) applied to the transition operator \( P \) considered as a bounded linear map from \( L^1(\pi) \) to \( L^1(\pi) \) and from \( L^2(\pi) \) to \( L^q(\pi) \) we have
\[
\|P\|_{(\alpha-r) \to \beta} \leq \|P\|_{2 \to q}^{2(\alpha-r-1)/(\alpha-r)} < +\infty,
\]
where
\[
\beta = \frac{q(\alpha-r)}{2(q-1) - (q-2)(\alpha-r)} > \alpha.
\]
Therefore
\[
\mathbb{E}\left| \mathbb{E}(\Psi(X_1) | F_0) \right|^\beta < +\infty,
\]
and we may apply Theorem 3.5 to the conditional expectations \( \mathbb{E}(\Psi(X_{j+1}) | F_j) \), \( j = 0, 1, 2, \ldots \) which are instantaneous functions of the strongly mixing at geometric rate Markov chain \( \{X_j\} \).

### 5.6 Remark

In [34] a similar result on negligibility of sums of conditional expectations was obtained, but assuming the \( L^2 \)-spectral gap property, which is stronger than the geometric ergodicity (and the strong mixing at geometric rate) and leads to a martingale decomposition (like in the proof of Theorem 3.1).

### 5.4 Proof of Theorem 3.5

Let \( \{X_n\} \) be strongly mixing at geometric rate and suppose that for some \( \beta > 1 \)
\[
\int \pi(dx)|\Psi(x)|^\beta < +\infty, \quad \int \pi(dx)\Psi(x) = 0,
\]
where \( \pi \) is the stationary distribution. Let \( B_n = n^{1/\alpha} \tilde{\ell}(n) \), where \( \tilde{\ell}(x) \) is a slowly varying function and \( \alpha \in (0, \beta \land 2) \). Without loss of generality we may assume that \( \beta < 2 \).

Suppose we are able to prove that

\[
\lim_{n \to \infty} n \mathbb{E} \left| 1 - \mathbb{E} \left( e^{i \theta \Psi(X_1)/B_n} \big| \mathcal{F}_0 \right) \right|^2 = 0, \quad \theta \in \mathbb{R}.
\]

Then by Theorem [A.3] it is enough to prove that

\[
\Phi_n(\theta) := \sum_{j=1}^{n} \mathbb{E} \left( e^{i \theta \Psi(X_j)/B_n} \big| \mathcal{F}_{j-1} \right) - 1 \xrightarrow{p} 0, \quad \theta \in \mathbb{R}.
\]

Moreover, by the Marcinkiewicz-Zygmunt law of large numbers, we have \( \mathbb{E} \Phi_n(\theta) \to 0 \) and so it is enough to prove that

\[
\mathbb{E} \left| \sum_{j=0}^{n-1} (\chi_{n,\theta}(X_j) - \mathbb{E}\chi_{n,\theta}(X_j)) \right|^2 \to 0,
\]

where

\[
\chi_{n,\theta}(x) = \int \left( \exp \left( i \theta \Psi(y)/B_n \right) - 1 \right) P(x, dy) - \left( \int \left( \exp \left( i \theta \Psi(y)/B_n \right) - 1 \right) \pi(dy) \right).
\]

Similarly as in the proof of relation (28), in presence of strong mixing at geometric rate it suffices to show that for some \( \delta > 0 \)

\[
n^{1+\delta} \mathbb{E} \left| \chi_{n,\theta}(X_0) \right|^2 \leq n^{1+\delta} \mathbb{E} \left| 1 - \mathbb{E} \left( e^{i \theta \Psi(X_1)/B_n} \big| \mathcal{F}_0 \right) \right|^2 \to 0.
\]  \((32)\)

Notice that this convergence gives us also the crucial condition of Theorem [A.3]. So proving \((32)\) will complete the proof of Theorem 3.5.

We have

\[
\left| 1 - \mathbb{E} \left( e^{i \theta \Psi(X_1)/B_n} \big| \mathcal{F}_0 \right) \right| \\
\leq |\theta| \mathbb{E} \left( \left| \frac{\Psi(X_1)}{B_n} \right| \mathbb{I}_{\{\Psi(X_1) \leq B_n\}} \big| \mathcal{F}_0 \right) + 2 \mathbb{P}(\{\Psi(X_1) > B_n\} | \mathcal{F}_0),
\]

and so, for \( 0 < \delta < \beta/\alpha - 1 \) and \( \beta < 2, \)

\[
n^{1+\delta} \mathbb{E} \left| 1 - \mathbb{E} \left( e^{i \theta \Psi(X_1)/B_n} \big| \mathcal{F}_0 \right) \right|^2 \\
\leq 2 |\theta|^2 n^{1+\delta} \mathbb{E} \left( \left| \frac{\Psi(X_1)}{B_n} \right| \mathbb{I}_{\{\Psi(X_1) \leq B_n\}} \big| \mathcal{F}_0 \right)^2 \\
+ 4 n^{1+\delta} \mathbb{P}(\{\Psi(X_1) > B_n\} | \mathcal{F}_0)^2) \\
\leq 2 |\theta|^2 n^{1+\delta} \mathbb{E} \left( \left| \frac{\Psi(X_1)}{B_n} \right| \mathbb{I}_{\{\Psi(X_1) \leq B_n\}} \big| \mathcal{F}_0 \right)^2 \\
+ 4 n^{1+\delta} \mathbb{P}(\{\Psi(X_1) > B_n\}) \\
\leq (2 |\theta|^2 + 4) n^{1+\delta} \left( \frac{\Psi(X_1)}{B_n} \right)^\beta \\
= (2 |\theta|^2 + 4) n^{1+\delta - \beta/\alpha} \tilde{\ell}(n) - \beta \mathbb{E} |\Psi(X_1)|^\beta \to 0.
\]
Appendix: Complements on the Principle of Conditioning

As mentioned in Introduction, the Principle of Conditioning (PoC) is a heuristic rule that allows producing limit theorems for dependent random variables given limit theorems for independent random variables. For example, applying the PoC one obtains the following theorem on convergence to stable laws.

A.1 Theorem Let \( \{X_{n,j} ; j \in \mathbb{N}, n \in \mathbb{N}\} \) be an array of random variables, which are row-wise adapted to a sequence of filtrations \( \{\mathcal{F}_{n,j} ; j = 0, 1, \ldots \ ; n \in \mathbb{N}\} \). Let \( h > 0 \) and let \( k_n \to \infty \) be a sequence of numbers.

The following conditions

\[
\begin{aligned}
\max_{1 \leq j \leq k_n} P \left( |X_{n,j}| > \varepsilon \left| \mathcal{F}_{n,j-1} \right. \right) & \xrightarrow{p} 0, \quad \varepsilon > 0; \\
\sum_{j=1}^{k_n} P \left( X_{n,j} > x \left| \mathcal{F}_{n,j-1} \right. \right) & \xrightarrow{p} c_+ x^{-\alpha}, \quad x > 0; \\
\sum_{j=1}^{k_n} P \left( X_{n,j} < x \left| \mathcal{F}_{n,j-1} \right. \right) & \xrightarrow{p} c_- |x|^{-\alpha}, \quad x < 0; \\
\sum_{j=1}^{k_n} \mathbb{E} \left( X_{n,j} \mathbb{1}_{\{ |X_{n,j}| \leq h \}} \right| \mathcal{F}_{n,j-1} \right) & \xrightarrow{p} a^h; \\
\sum_{j=1}^{k_n} \text{Var} \left( X_{n,j} \mathbb{1}_{\{ |X_{n,j}| \leq h \}} \right| \mathcal{F}_{n,j-1} \right) & \xrightarrow{p} \int_{\{|x| \leq h\}} x^2 \nu_{\alpha,c_+,c_-} (dx); \\
\end{aligned}
\]

imply that

\[
\sum_{j=1}^{k_n} X_{n,j} \xrightarrow{D} \delta_{a^h} * c_h \text{-Pois}(\alpha,c_+,c_-),
\]

where \( \delta_{a^h} * c_h \text{-Pois}(\alpha,c_+,c_-) \) is the stable distribution with the characteristic function \( \phi \).

In other words the PoC says that if we replace in a limit theorem for row-wise independent summands:

- the expectations by conditional expectations with respect to the past,
- the convergence of numbers by convergence in probability of random variables appearing in the conditions,

then still the conclusion (in our case: (33)) will hold. In fact, one can also replace the summation to constants by summation to stopping times.

We refer to [28] for exposition of results related to various versions of the PoC, beginning with the Brown-Eagleson martingale CLT [9], through multidimensional [35, 3] and functional [19, 25] limit theorems, up to the PoC in infinite dimensional Hilbert [27, 29] and Banach spaces. The ideas standing behind the PoC motivated further research devoted to so called decoupling inequalities, described in detail in the well-known books by Kwapień and Wojczyński [38] and de la Peña and Giné [16]. It might be interesting to realize that the tools developed to cope with the PoC find unexpected applications even today [41, 32].

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Behind the verbal form of the PoC there is a result on convergence of conditional characteristic functions (see [27]).

**A.2 Theorem** Let the system \( \{X_{n,j}, F_{n,j}\} \) be as in Theorem A.1. If for some \( z \in \mathbb{C}, z \neq 0 \), we have

\[
\phi_n(\theta) = \prod_{j=1}^{k_n} \mathbb{E}\left( e^{i\theta X_{n,j}} \mid F_{n,j-1} \right) \xrightarrow{p} z,
\]

then also

\[
\mathbb{E}\exp(i\theta \sum_{j=1}^{k_n} X_{n,j}) \longrightarrow z.
\]

In particular, if for some probability measure \( \mu \) on \( \mathbb{R}^1 \) we have

\[
\phi_n(\theta) \xrightarrow{p} \hat{\mu}(\theta), \quad \theta \in \mathbb{R}^1, \tag{34}
\]

then

\[
\sum_{j=1}^{k_n} X_{n,j} \xrightarrow{D} \mu.
\]

Mimicking the case of independent random variables one can prove that conditions obtained by the PoC imply (34). But in many cases this is not the most efficient way of applying the PoC. It was observed in [30] that for highly structured models we can often check directly and that going this way we can keep integrability requirements at the minimal possible level.

We extend the results of [27], [31] and [30] in the following theorem that provides a convenient tool in many cases of interest.

**A.3 Theorem** Let \( \{X_{n,j}; j \in \mathbb{N}, n \in \mathbb{N}\} \) be an array of random variables, which are row-wise adapted to a sequence of filtrations \( \{\{F_{n,j}; j = 0, 1, \ldots\}; n \in \mathbb{N}\} \).

Suppose that the following condition holds.

\[
\sum_{j=1}^{k_n} \left| 1 - \mathbb{E}\left( e^{i\theta X_{n,j}} \mid F_{n,j-1} \right) \right|^2 \xrightarrow{p} 0, \quad \theta \in \mathbb{R}^1. \tag{35}
\]

Let \( A_n \) be arbitrary random variables and \( \Phi(\theta) \in \mathbb{C} \) be a constant for each \( \theta \in \mathbb{R}^1 \). The following conditions are equivalent:

\[
\left( \sum_{j=1}^{k_n} \left( \mathbb{E}\left( e^{i\theta X_{n,j}} \mid F_{n,j-1} \right) - 1 \right) - i\theta A_n \right) \xrightarrow{p} \Phi(\theta). \tag{36}
\]

\[
\left( \prod_{j=1}^{k_n} \mathbb{E}\left( e^{i\theta X_{n,j}} \mid F_{n,j-1} \right) \right) e^{-i\theta A_n} \xrightarrow{p} e^{\Phi(\theta)}. \tag{37}
\]

In either case we have also

\[
\mathbb{E}\exp(i\theta \left( \sum_{j=1}^{k_n} X_{n,j} - A_n \right)) \longrightarrow e^{\Phi(\theta)}. \tag{38}
\]
In particular, if \( e^{\Phi(\theta)} = \hat{\mu}(\theta) \), \( \theta \in \mathbb{R}^1 \), for some probability measure \( \mu \), then either of conditions (36) or (37) imply
\[
\sum_{j=1}^{k_n} X_{n,j} - A_n \xrightarrow{D} \mu.
\]

**Proof.** Set
\[
\phi_n(\theta) = \prod_{j=1}^{k_n} \mathbb{E}\left(e^{i\theta X_{n,j}|F_{n,j-1}}\right);
\]
\[
\Phi_n(\theta) = \sum_{j=1}^{k_n} \left( \mathbb{E}\left(e^{i\theta X_{n,j}|F_{n,j-1}}\right) - 1 \right).
\]

If \( z \in \mathbb{C} \) satisfies \(|z| \leq 1\), then \(|z - e^{z-1}| \leq 5|z - 1|^2\). Hence we have
\[
|\phi_n(\theta)e^{-i\theta A_n} - \exp\left(\Phi_n(\theta) - i\theta A_n\right)| = |\phi_n(\theta) - \exp\left(\Phi_n(\theta)\right)|
\leq \sum_{j=1}^{k_n} \mathbb{E}\left|e^{i\theta X_{n,j}|F_{n,j-1}} - \exp\left(\mathbb{E}\left(e^{i\theta X_{n,j}|F_{n,j-1}}\right) - 1\right)\right|
\leq 5 \sum_{j=1}^{k_n} \mathbb{E}\left|e^{i\theta X_{n,j}|F_{n,j-1}} - 1\right|^2 \xrightarrow{p} 0 \quad \text{by (35)}.
\]

We have thus established the equivalence of (36) and (37). To prove that (37) implies (38) we need a suitable version of Lemma 2 in [33].

**A.4 Lemma** For every \( \varepsilon > 0 \)
\[
|\mathbb{E}\exp(i\theta(\sum_{j=1}^{k_n} X_{n,j} - A_n)) - \mathbb{E}(\phi_n(\theta)e^{-i\theta A_n})| \leq 2(1 + \frac{1}{\varepsilon})\mathbb{P}(|\phi_n(\theta)| < \varepsilon) + \frac{1}{\varepsilon}\mathbb{E}|\phi_n(\theta)e^{-i\theta A_n} - \mathbb{E}(\phi_n(\theta)e^{-i\theta A_n})|.
\]

**Proof.** We follow the idea of the proof of Theorem A in [27]. Define
\[
\phi_{n,k}(\theta) = \prod_{j=1}^{k} \mathbb{E}\left(e^{i\theta X_{n,j}|F_{n,j-1}}\right).
\]

Fix \( \theta \in \mathbb{R}^1 \) and \( \varepsilon > 0 \) and consider random variables
\[
X_{n,k}^* = X_{n,k} \mathbb{I}\{|\phi_{n,k}(\theta)| \geq \varepsilon\}.
\]

Then we have both
\[
|\mathbb{E}\exp(i\theta(\sum_{j=1}^{k_n} X_{n,j} - A_n)) - \mathbb{E}\exp(i\theta(\sum_{j=1}^{k_n} X_{n,j}^* - A_n))| \leq 2\mathbb{P}(|\phi_n(\theta)| < \varepsilon),
\]

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and, if we set $\phi_n^*(\theta) = \prod_{j=1}^{k_n} E\left(e^{i\theta X_{n,j}^*}|F_{n,j-1}\right)$, 
\[ E|e^{-i\theta A_n}\phi_n(\theta) - e^{-i\theta A_n}\phi_n^*(\theta)| \leq 2\mathbb{P}(|\phi_n(\theta)| < \varepsilon). \]
The advantage of random variables $\{X_{n,j}^*\}$ consists in the fact that 
\[ |\phi_n^*(\theta)| = \prod_{j=1}^{k_n} E\left(e^{i\theta X_{n,j}^*}|F_{n,j-1}\right) \geq \varepsilon, \]
and so, by the backward induction (or the martingale property) 
\[ \mathbb{E}\frac{\exp(i\theta\left(\sum_{j=1}^{k_n} X_{n,j}^* - A_n\right))}{e^{-i\theta A_n}\phi_n^*(\theta)} = \mathbb{E}\frac{\exp(i\theta\left(\sum_{j=1}^{k_n} X_{n,j}^*\right))}{\prod_{j=1}^{k_n} E\left(e^{i\theta X_{n,j}^*}|F_{n,j-1}\right)} = 1. \]
Therefore, 
\[ |\mathbb{E}\exp(i\theta\left(\sum_{j=1}^{k_n} X_{n,j}^* - A_n\right)) - \mathbb{E}(\phi_n(\theta)e^{-i\theta A_n})| \]
\[ = \left| \mathbb{E}\frac{\exp(i\theta\left(\sum_{j=1}^{k_n} X_{n,j}^*\right))}{\phi_n^*(\theta)}\phi_n^*(\theta)e^{-i\theta A_n} - \mathbb{E}(\phi_n(\theta)e^{-i\theta A_n})\mathbb{E}\frac{\exp(i\theta\left(\sum_{j=1}^{k_n} X_{n,j}^*\right))}{\phi_n^*(\theta)} \right| \]
\[ \leq \frac{1}{\varepsilon}\mathbb{E}|\phi_n^*(\theta)e^{-i\theta A_n} - \mathbb{E}\phi_n(\theta)e^{-i\theta A_n}| \]
\[ \leq \frac{1}{\varepsilon}\left(2\mathbb{P}(|\phi_n(\theta)| < \varepsilon) + \mathbb{E}|\phi_n(\theta)e^{-i\theta A_n} - \mathbb{E}(\phi_n(\theta)e^{-i\theta A_n})| \right). \]

**Proof of Theorem A.3 (continued).** Now assume that (37) holds. Let $\varepsilon = 1/2|e^{\Phi(\theta)}|$. Then $\mathbb{P}(\phi_n(\theta) < \varepsilon) = \mathbb{P}(\phi_n(\theta)e^{-i\theta A_n} < \varepsilon) \to 0$ and by the dominated convergence 
\[ E\phi_n(\theta)e^{-i\theta A_n} - \mathbb{E}(\phi_n(\theta)e^{-i\theta A_n}) \to 0. \]

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