Revivals and and Casimir energy for a free Maxwell field
(spinn–1 singleton) on $R \times S^d$ for odd $d$

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Earlier work on quantum revivals is extended to Maxwell fields (aka spin–one singletons). An evaluation of the Casimir energy on the generalised Einstein universe is also done to illustrate the utility of the Barnes zeta–function and generalised Bernoulli polynomials. Contact is made with some recent calculations in AdS/CFT. In particular, higher order singletons are considered with the Casimir energy being a polynomial in the order.

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1. Introduction and summary.

The following calculation is partly concerned with a topic, originally considered by Cardy, [1], of quantum revivals in higher dimensional free-field CFTs. It simply extends my previous analysis, [2], to the Maxwell (‘spin–one’) field for completeness.

It is not expected that the results will differ qualitatively from the spin–0 ones, both being bosonic. However, the details raise (again) a few small calculational points which might have applications in other situations such as AdS/CFT.

For a particular choice of quenching initial state, the return amplitude is determined by the free energy of a finite temperature free-field theory on the generalised cylinder, $R \times S^d$ (the Einstein universe). The requisite mode information is given in the next section. This is used in section 3, which is the largest one, to compute the spin-1 singleton Casimir energy. The Di and Rac linetion fields are also treated. In section 4, the return amplitude is briefly discussed and plotted.

Nothing is dealt with at very great length since this communication should be regarded, mostly, as an addendum to my earlier work and as a promotion of a particular, Barnsian organisation of the spectral data used before to advantage.

2. Maxwell theory in higher dimensions.

I consider coexact (divergence free) $p$–forms on the Einstein universe. When propagated by the de Rham Laplacian, only those with $p = (d−1)/2$ are conformally invariant and I am, perforce, obliged to take the sphere dimension, $d$, odd in order to have a generalisation of Maxwell theory to higher dimensions, e.g., [3–5].

The eigenproblem has been used in, for example [6], where numerous important earlier references are given.

The eigenvalues are

$$\mu(p, l) = (l + p)^2, \quad p = (d − 1)/2,$$

and the degeneracies, $d(p, l)$, were specifically manipulated in [6] to give the generating function,

$$G_M(p, q) = \sum_{l=1}^{\infty} d(p, l)q^{l+p} = \frac{2}{p!^2} \sum_{l=1}^{\infty} \frac{(l + 2p)!}{(l − 1)!(l + p)}q^{l+p}$$

$$= 2 \sum_{m=p+1}^{2p+1} \binom{m−1}{p}q^{p+1}(1−q)^m,$$  (1)
the last identity following by recursion, [6].

Dolan, [7], has evaluated this generating function from the explicit degeneracies, exactly as here. It was used in [8], App.D, and in [9]. Our earlier, [6], result provides a different, but equivalent, combinatorial form.²

To check this explicitly, rewrite \( G \) as

\[
G_M'(d, q) = \frac{2}{(1-q)^{d+1}} \sum_{m=(d+1)/2}^{d} \left( \frac{m-1}{d-1/2} \right) q^{(d+1)/2} (1-q)^{d+1-m} \]

\[
\equiv 2 \frac{P_{d+1}(q)}{(1-q)^{d+1}}. \tag{2}
\]

Elementary evaluation yields rapid agreement with the polynomials given in [8] footnote 26.

Setting \( q = e^{-\tau} \), \( G(p, q) \) can be interpreted as the cylinder kernel, or the ‘square root’ kernel. Thermally \( \tau = \beta = 1/T \) and \( G \) is the single particle partition function from which the field theory boson free energy can be found from basic statistical physics, e.g. [10],

\[
\beta F = \beta E_0 - \sum_{n=1}^{\infty} \frac{1}{n} G(p, q^n) \]

\[
= \beta E_0 + \Xi' (\beta), \tag{3}
\]

where \( E_0 \) is the zero temperature vacuum energy and \( \Xi' \) is the finite temperature correction to the grand canonical partition function.

Although not required for the computation of the return amplitude, I give the evaluation of \( E_0 \) in the next section in order to show the utility of the present organisation of the spectral data which is one of my aims.

3. The Casimir energy.

Standard theory, [11,12], gives the boson Casimir energy on \( R \times S^d \) in terms of the spectral \( \zeta \)–function on \( S^d \), as,

\[
E_0 = \frac{1}{2} \zeta(-1/2),
\]

when this is finite, as it is here.

² In these references the Maxwell form is referred to as a ‘\( d/2 \)–form’, or twice this.
The relation between the spectral $\zeta$–function and the generating function is, trivially, e.g. [13],

$$\zeta(s) = \frac{i}{2\pi} \Gamma(1 - 2s) \int_C d\tau (-\tau)^{2s-1} G(p, e^{-\tau}) ,$$

where $C$ is the Hankel contour. Substituting the expression (1) for $G$, the integral is recognised as a Barnes $\zeta$–function, $\zeta_B$, and so, for the Maxwell $\zeta$–function, [6],

$$\zeta_M(s, p) = 2 \sum_{m=p+1}^{2p+1} \binom{m-1}{p} \zeta_B(2s, p+1 | 1_m) ,$$

which is one of the calculational points I wish to bring out.

For completeness I also give the (known) scalar (S) and spinor (D) $\zeta$–functions for the full $d$–sphere,

$$\zeta_S(s, d) = \zeta_B(2s, (d-1)/2 | 1_d) + \zeta_B(2s, (d+1)/2 | 1_d)$$

$$\zeta_D(s, d) = S \zeta_B(2s, d/2 | 1_d) .$$

Barnes’ result for the $\zeta$–function at a negative integer (essentially just a residue) yields the compact formula for the Maxwell Casimir energy as a sum of generalised Bernoulli polynomials,

$$E_0^M(p) = \sum_{m=p+1}^{2p+1} \frac{(-1)^m}{(m+1)!} \binom{m-1}{p} B_{m+1}^{(m)}(p+1) ,$$

which is rapidly computed and gives agreement with the values listed in [9]. This reference uses Hurwitz $\zeta$–function regularisation. Just to extend the printed values, I find $E_0^M(6) = -36740617/373248000$ in short order.

We have used the method of deriving the relevant $\zeta$–function through the generating function (square root kernel) on several previous occasions, e.g. [4,5,13,10,6]. Many particulars of the spectrum can thereby be bypassed, generally giving a smoother, more efficient analysis.

In the present, rather simple, instance there is actually not much to choose between the two approaches. The direct expression for the Maxwell $\zeta$–function is, [3],

$$\zeta_M(s, p) = 2 \frac{p!^2}{2} \sum_{n=1}^{\infty} \frac{(n^2 - p^2) \ldots (n^2 - 1)}{n^{2s}}$$

$$= 2 \frac{p!^2}{2} \sum_{\nu=0}^{p} A_{\nu}(p) \zeta_R(2s - 2\nu)$$

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and so the Casimir energy takes the form of a sum of Bernoulli numbers,

\[ E = \frac{2}{p!^2} \sum_{\nu=0}^{p} A_\nu(p) \zeta_R(-1-2\nu) \]

where, from the definition, the coefficients have the combinatorial form,

\[
A_\nu(p) = \text{co}_{2\nu} \prod_{i=1}^{p} (n^2 - i^2) \\
= (-1)^\nu \sum_{i_1 < \ldots < i_{2\nu}} i_1^2 i_2^2 \ldots i_{n-\nu}^2
\]

which can be used numerically. Alternatively, recursion can be used. 3

This expansion of the degeneracy, leading to sums of Hurwitz \( \zeta \)-functions, is the traditional approach and frequently employed. The evaluations in [9] derive the generating functions first and from these effectively obtain the degeneracies which are then expanded in the manner just outlined. From our perspective, this is somewhat roundabout.

In complicated situations, use of the Barnes function is a more systematic way of organising the spectral information and means we don’t have to bother with any new expansions, as I now enlarge on.

There are a number of different ways of writing the \( \zeta \)-function, (4), depending on how the \( q \)-series is arranged. In fact, on the sphere, any generating function will give a (non–unique in form) sum of Barnes \( \zeta \)-functions. To illustrate this I assume the generating function takes the form

\[ G'(d, q) = \frac{P(d, q)}{(1 - q)^{d+1}} \]

\( P \) is a polynomial in \( q \) with typical term \( C(d, \Delta) q^\Delta \). I won’t specify the range of the power \( \Delta \). The spectral \( \zeta \)-function is then

\[ \zeta_p(s, d) = \sum_{\Delta} C(d, \Delta) \zeta_B(2s, \Delta \mid 1_{d+1}) , \]

which would yield, for example, a form different (but equivalent) to (4) for the Maxwell field. 4

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3 The coefficients are variously referred to as ‘differentials of nothing’, central factorial numbers or central Stirling numbers.

4 The Barnes function can be written as a sum of Hurwitz \( \zeta \)-functions, but this is not necessary as its properties can be developed independently.
In the simplest case of just one term, $q^\Delta$, the Casimir energy is

$$E_0(d, \Delta) = \frac{(-1)^{d-1}B_{d+2}^{(d+1)}(\Delta)}{2(d + 2)!}$$

which is that for a massive scalar (primary) field of weight $\Delta$. This quickly and efficiently reproduces the list in [14], App.B, obtained there using a regulating exponential and the discarding of poles.

Very basic properties of the Bernoulli polynomials, outlined in the Appendix, transcribe immediately into known results for the Casimir energy,

$$E_0(d, (d + 1)/2) = 0, \quad \text{even } d$$

$$\frac{\partial}{\partial \Delta}E_0(d, \Delta)|_{\Delta=(d+1)/2} = 0, \quad \text{odd } d$$

$$\frac{\partial^2}{\partial \Delta^2}E_0(d, \Delta) = \frac{(-1)^{d+1}}{2d!} (\Delta - 1)(\Delta - 2) \ldots (\Delta - d).$$

As is well known, the generating functions of the SO$(d + 2, 2)$ representations, $D_i$ and $Rac$, are identical, respectively, to those of spinors and conformal scalars on $R \times S^d$,

$$G'_{(Rac)}(d, q) = \frac{q^{(d-1)/2}}{(1 - q)^d} + \frac{q^{(d+1)/2}}{(1 - q)^d}$$

$$G'_{(Di)}(d, q) = 2^{[(d+1)/2]} \frac{q^{d/2}}{(1 - q)^d}.$$  \hfill (9)

The spinor expression was given in [15,16]. A useful review, with later references, is contained in [17].

These forms are equivalent to (5) and lead to Casimir energies in the compact forms,

$$E_0(d, Rac) = 1 - \frac{(-1)^d}{2(d + 1)!} B_{d+1}^{(d)}((d - 1)/2)$$

$$E_0(d, Di) = \frac{2^{[(d+1)/2]}}{(d + 1)!} B_{d+1}^{(d)}(d/2),$$

$$\equiv \frac{2^{-(d+1)/2}}{(d + 1)!} D_{d+1}^{(d)},$$

which agree, numerically, with the historic values, frequently reobtained in more recent works.$^5$

$^5$ These have been known since the early 1980s and are available in a number of places. The earliest known to me is [18].
The Rac (scalar) expression is given in [10] and I note that the terms in (9) (cf (5)) arise from considering the sphere spectrum as the union of \textit{hemisphere} spectra, with conditions, on the rims, of Neumann and Dirichlet for the scalar, and local for the spinor (for which the two sets give the same value).

The (anti–) symmetry of the Bernoulli polynomials has been used to obtain these expressions and is very convenient for showing any vanishing of the Casimir energy, otherwise complicated sums of Hurwitz \(\zeta\)–functions can arise. A typical case is equn.(5.13) in [19]. Equivalently, the parity properties of the generating function under \(\tau \rightarrow -\tau\) can be employed, as first described some time ago in [10], [4,5], and used more recently in e.g. [20], [21].

A simple example that generalises the above is the higher derivative Rac \(l\)--lineton with generating function,\(^6\)

\[
G'_{(\text{Rac})}(d,q,l) = \frac{q^{(d+1)/2+l} - q^{(d+1)/2-l}}{(1-q)^{d+1}}.
\]

(10)

This gives a vacuum energy of,

\[
E_0^{(\text{Rac})}(d,l) = \frac{(-1)^{d+1}}{2(d+2)!} \left( B_{d+2}^{(d+1)} ((d+1)/2 + l) - B_{d+2}^{(d+1)} ((d+1)/2 - l) \right)
\]

\[
= \frac{(-1)^{d+1}}{2(d+2)!} (1 - (-1)^d) B_{d+2}^{(d+1)} ((d+1)/2 + l),
\]

which again is zero for even \(d\).

For odd \(d\), \(E_0\) is a polynomial in \(l\). I list a few,

\[
-\frac{l \left( 6 l^4 - 20 l^2 + 11 \right)}{720}, \quad d = 3
\]

\[
-\frac{l \left( 12 l^6 - 126 l^4 + 336 l^2 - 191 \right)}{60480}, \quad d = 5
\]

\[
-\frac{l \left( 10 l^8 - 240 l^6 + 1764 l^4 - 4320 l^2 + 2497 \right)}{3628800}, \quad d = 7.
\]

(11)

The Di \(l\)--lineton is also easily treated without further work, its generating function being, [20], (3.13),

\[
G'_{(Di)}(d,q,l) = 2^{(d+1)/2} \frac{q^{d/2-l+1} - q^{d/2+l}}{(1-q)^{d+1}}
\]

\[
= 2^{(d+1)/2} G'_{(\text{Rac})}(d,q,l - 1/2),
\]

(12)

\(^6\) This is the partition function of a GJMS scalar on \(S^1 \times S^d\). See [22] equn.C8.
where I am now continuing $l$ into the reals. Field–theoretic and thermodynamical quantities will likewise be formally related. The Casimir energy is a simple, explicit example,

$$E_0^{(Di)}(d, l) = -2^{(d+1)/2} E_0^{(Rac)}(d, l - 1/2).$$

The left–hand side can be calculated at spinor physical values (integers) by evaluating the analytic polynomials (11) at scalar unphysical ones (half–integers) i.e. at values meaningless in terms of Young diagrams.

The relation (12) reflects the spectral fact that, on the sphere, the square root eigenvalues for the Dirac field differ from those for scalar fields by $\pm 1/2$. More precisely, $+1/2$ holds for N scalar conditions on the hemi–sphere and $-1/2$ for Dirichlet. 7

For amusement, Fig.1 shows continuous plots of some Rac polynomials for low $l$. The Di curves are obtained from these by changing the sign and normalisation, then translating the origin by $1/2$.

Fig.1. l–Rac Casimir. $d=3,5,7$

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7 This shift can be transferred to the GJMS order, $l$, and then the two types – Di and Rac – correspond to the two possible factorisations of the Gamma function ratio form of the GJMS operators, (cf [23]).
Physical $l$–Rac numerical values for $l$ from 1 to 6, are,

\[
\begin{array}{ccccccc}
  l & 1 & 3 & 4 & 9 & 1087 & 7067 \\
  d = 3 & & & & & & \\
  & \frac{1}{240} & -\frac{3}{40} & -\frac{317}{240} & -\frac{409}{60} & -\frac{120}{48} & -\frac{1087}{120} \\
  d = 5 & & & & & & \\
  & -\frac{31}{60480} & -\frac{19}{6048} & -\frac{24032}{4032} & -\frac{15120}{8183} & -\frac{12096}{1153247} & -\frac{10080}{7731841} \\
  d = 7 & & & & & & \\
  & \frac{289}{3628800} & -\frac{1814400}{172800} & -\frac{129600}{725760} & -\frac{604800}{604800} & -\frac{116959}{12096} & -\frac{408481}{10080} \\
\end{array}
\]

and those for $l$–Di,

\[
\begin{array}{ccccccc}
  l & 1 & 3 & 4 & 9 & 1087 & 7067 \\
  d = 3 & & & & & & \\
  & \frac{17}{960} & -\frac{29}{960} & -\frac{107}{960} & -\frac{12439}{960} & -\frac{16531}{960} & -\frac{143627}{960} \\
  d = 5 & & & & & & \\
  & -\frac{367}{48384} & -\frac{1021}{80640} & -\frac{1331}{48384} & -\frac{113221}{34560} & -\frac{174689}{5376} & -\frac{7984867}{48384} \\
  d = 7 & & & & & & \\
  & \frac{27859}{8294400} & -\frac{98587}{19353600} & -\frac{136741}{11612160} & -\frac{218747}{8294400} & -\frac{41852933}{4528166423} & -\frac{58060800}{58060800} \\
\end{array}
\]

4. **The Maxwell return amplitude**

Finally, I turn, somewhat briefly, to an application of the partition functions *viz.* the quantum return amplitude. Details of the analysis have been described by Cardy, [1], and also in [2]. Hence I proceed immediately to the results.

Figs. 1 and 2 illustrate the typical behaviours of the logs of the return amplitudes, $A$, for $d = 3$ and $d = 5$, respectively. Fig.3 shows the maximum at $s = 0$ in finer detail for $d = 5$.

The formula plotted for log $A$ is,

\[
\log A(s) = \text{Re} \Xi'(\beta + 2it) \quad s = t/\pi
\]

where $\beta$ is a chosen (usually small) inverse ‘temperature’ and $t$ is the quantum mechanical propagation time from the initial quenched state, [1]. $\Xi'$ is obtained from (3), with (1) or (2).
Fig. 2. Spin-1 log return amplitude, $d=3$

Fig. 3. Spin-1 log return amplitude, $d=5$
The graphs show the expected full revivals when $s$ is an integer. (Fig.2 should be reflected in the $s = 1/2$ line to get the full period.) They also exhibit partial revivals at rational $s$ which are explained in exactly the same way, via modular invariance, as for the scalar field. This is because, in both cases, the degeneracies are polynomials of the same degree in the mode label ($cf$ (7)), and, for small $\beta$, only the highest power is relevant. Normalisations (Stefan’s constant) will, however, differ.

For information, and possible interest, I also present in figs.5 and 6, the results for some scalar GJMS fields. The Paneitz one has a period of $2$. 

Fig.4. Spin-1 log return amplitude, $d=5$

Fig.5. Paneitz GJMS return amplitude, $d=3$, $l=2$
5. Conclusion

The results for the quantum return amplitude are, as expected, qualitatively the same as for spin–0.

The spectral data pertaining to spheres is again compendiously organised into Barnes $\zeta$–functions leading to generalised Bernoulli polynomials allowing systematic evaluation. This also permits factored spheres, e.g. $S^d/\mathbb{Z}_m$, to be treated without too much difficulty, e.g. [4,5] [24].

Appendix

In view of the expression (8) for the basic Casimir energy, It might be useful to outline some relevant properties of the generalised Bernoulli polynomials, $B_{\nu}^{(n)}(x \mid \omega)$, where $\omega$ stands for a set of $n$ reals. The essential reference is Nörlund, [25]. Some basic facts are in [26].

The most frequently occurring, and the simplest, case is when all the $\omega$ are unity\(^8\) $\omega = 1_n$. It is then conventional to drop reference to these parameters. I have not done so in the previous discussion but I will from now on.

Using the theory of ordinary Bernoulli polynomials as a guide, the generalised variety can be defined by the difference equation

$$\Delta_1 B_{\nu}^{(n)}(x) \equiv B_{\nu}^{(n)}(x + 1) - B_{\nu}^{(n)}(x) = \nu B_{\nu-1}^{(n-1)} \quad (13)$$

\(^8\) See [27].
together with the initial condition,
\[ B^{(n)}_{\nu}(0) = B^{(n)}_{\nu}. \]  
(14)

The \( B^{(n)}_{\nu} \) are generalised Bernoulli numbers defined, by analogy with the standard ones, \( B_{\nu} \), by
\[ \sum_{s=1}^{\nu} \binom{\nu}{s} B^{(n)}_{\nu-s} = \nu B^{(n-1)}_{\nu-1}, \]
with the starting value,
\[ B^{(1)}_{\nu} = B_{\nu}. \]

Then, for example, from (13),
\[ B^{(0)}_{\nu}(x) = x^\nu \]
\[ B^{(1)}_{\nu}(x) = B_{\nu}(x), \]
where
\[ B_{\nu}(x) = \sum_{s=0}^{\nu} \binom{\nu}{s} x^s B_{\nu-s}, \]
are the usual Bernoulli polynomials.

Then by induction in general,
\[ B^{(n)}_{\nu}(x) = \sum_{s=0}^{\nu} \binom{\nu}{s} x^s B^{(n)}_{\nu-s}, \]
from which one concludes the useful differential recursion,
\[ D_x B^{(n)}_{\nu}(x) = \nu B^{(n)}_{\nu-1}(x). \]  
(15)

As well as the particular values at \( x = 0 \), (14), those at \( x = n/2 \) are singled out,
\[ B^{(n)}_{\nu}(n/2) \equiv 2^{-\nu} D^{(n)}_{\nu}, \]
the ‘Nörlund \( D \)–numbers’. It is then shown that
\[ D^{(n)}_{2\nu+1} = 0, \]
which, using (15), means that the \( B(x) \)s, have the zeros,
\[ B^{(n)}_{2\nu+1}(n/2) = 0 \]
\[ D_x B^{(n)}_{2\nu}(x) \big|_{x=n/2} = 0. \]  
(16)

When \( \nu = n-1 \) simplifications occur and recursion leads to the explicit formula,
\[ B^{(n+1)}_{n}(x) = (x - 1)(x - 2) \ldots (x - n). \]
The right–hand side can be written in several forms.

Now, in particular, set \( \nu = n+1 \) in the recursion (15) and iterate once to give,
\[ D_x^2 B^{(n)}_{n+1}(x) = n(n+1) B^{(n)}_{n-1}(x) = n(n+1) (x-1)(x-2) \ldots (x-n+1). \]
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