The whispering gallery effect in neutron scattering

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Abstract. We present the first experimental evidence for neutron localization (whispering gallery wave) in the quasistationary quantum states near a cylindrical mirror surface. The boundary effective well is formed by the centrifugal effective potential and the mirror neutron–matter optical potential. We present a formalism that describes quantitatively the neutron scattering at a cylindrical mirror surface and compare the experimental results to this model. We discuss further prospects based on this study.

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1. Introduction

Neutron localization in ‘centrifugal states’ near a curved mirror surface [1]–[3] is a quantum analogue of the so-called whispering gallery wave. The whispering gallery effect has been known in acoustics since ancient times and was explained by Lord Rayleigh in his *Theory of Sound* [4, 5]. Whispering gallery electromagnetic waves exist in dielectric spheres [3, 6, 7]. This phenomenon in optics has been the object of growing interest during the last decade due to the waves’ multiple applications [8, 9]. In the following, we will be interested in the matter–wave aspect of the whispering gallery wave phenomenon: namely, large-angle neutron scattering on a curved mirror. Such scattering can be understood in terms of the states of a quantum particle above a mirror in a linear potential—the so-called ‘quantum bouncer’ [10], in analogy to the neutron quantum motion in the Earth’s gravitational field above a flat mirror [11, 12].

The observed phenomenon consists of the localization of cold neutrons near a curved mirror surface due to the superposition of the centrifugal potential and optical potential of the mirror. In this case, the centrifugal states play an essential role in the neutron flux dynamics. Measurement of the gravitationally bound and centrifugal quantum states of neutrons could be considered as a direct confirmation of the Weak Equivalence Principle for a massive particle in a quantum state [13]–[17]. Both problems (the gravitational and centrifugal ones) provide a perfect experimental laboratory for studying neutron quantum optics phenomena, quantum revivals and localization [18]–[25]. Evident advantages of using cold neutrons instead of ultracold neutrons (UCNs) include much higher statistics being attainable, broad accessibility of cold neutron beams as well as crucial reduction of many false effects due to $\sim 10^5$ times higher energies of the quantum states involved. This phenomenon could provide a promising tool for studying fundamental neutron–matter interactions (in analogy to [12, 26]–[29]), as well as quantum neutron optics and surface physics effects.

The neutron whispering gallery phenomenon and the method of its experimental observation are described in sections 2 and 3. In section 4, we develop a formalism, which describes the neutron scattering at a cylinder surface. The results are analyzed within the formalism presented in section 5.
2. Principle of observation

For the first time, the neutron scattering on a curved mirror was studied in [1]. If the neutron energy is much larger than the mirror optical potential, most neutrons scatter to small angles \( \varphi < 2\varphi_c \). Here, the critical angle is \( \varphi_c = \sqrt{U_0/E} \), where \( U_0 \) is the mirror optical potential and \( E \) is the neutron energy. However, some neutrons could be captured into long-living centrifugal quasistationary states localized near the well-polished curved surface of the mirror and thus could deflect to large angles. The curved mirror surface plays the role of a waveguide and the centrifugal states play the role of radial modes in such a waveguide. The spectral dependence of the transmission probability and the neutron angular distribution are determined by the existence of the centrifugal states in such a system.

The quantum well is formed by the effective centrifugal potential and the repulsive optical potential of a curved mirror, as shown in figures 1 and 3. The effective acceleration near the curved mirror surface could be approximated by \( a = v^2/R \), where \( v \) is the neutron velocity and \( R \) is the mirror radius. The classical radial motion of the neutron in this well is limited up to \( z_{class} = RU_0/Mv^2 \) from the surface, where \( M \) is the neutron mass. The condition for the existence of a quasistationary state above a mirror surface can be estimated from the Heisenberg relation,

\[
z_{class} \sqrt{2MU_0} > 2\pi \hbar \quad \text{or} \quad \frac{v^2}{R} < \frac{1}{\pi \hbar \sqrt{2M}} U_0^{3/2}.
\]

(1)

It is natural to use the neutrons with most probable velocities in standard neutron source spectra \( \sim 10^3 \text{ m s}^{-1} \). Then the radius for which just a few quasistationary states exists is a few centimeters. The extension of the states is \( \sim 100 \text{ nm} \), much smaller than the mirror radius.

If we vary continuously the longitudinal velocity \( v \) (i.e. the neutron wavelength \( \lambda \)), we change the width of the triangular barrier, thus changing the number of quasistationary states that can propagate along the mirror. It is important that below some critical wavelength \( \lambda_c \) no

**Figure 1.** The scheme of the neutron centrifugal experiment. 1—The classical trajectories of incoming and outgoing neutrons; 2—the cylindrical mirror; 3—the detector.
quasistationary states with sufficiently long lifetimes can be formed, so the neutron scattering probability to large angles is expected to have a sharp cut-off below $\lambda_c$. With increasing neutron wavelength $\lambda$ the number of quasistationary states increases, resulting in interference maxima and minima in the scattering probability.

In contrast to the gravitationally bound quantum states [11, 12], neutrons tunnel deeply into the mirror through the triangle potential barrier shown in figure 3. The lifetime of deeply bound states is much longer than that for states with energy close to the barrier edge; this phenomenon has to be taken into account. Another essential difference is related to the effects of surface roughness: as the characteristic scale of the centrifugal quantum states is much smaller, the roughness effects are much larger; therefore, the constraints on the roughness of a cylindrical mirror surface are more severe than those for flat mirrors in the gravitational experiments.

In our setup, neutrons pass through the mirror bulk, as shown in figure 1, and arrive at the mirror–vacuum interface. Neutrons arriving at the interface at tangential trajectories might refract into the whispering gallery modes. The whispering gallery wave decays via the neutron tunneling back into the mirror bulk. We detect such neutrons in the position-sensitive detector. In such a setup, the edge effects could be neglected; thus in our theoretical analysis, we consider a non-truncated cylindrical mirror.

3. Experimental evidence for the centrifugal quantum states of neutrons

In the early stages of studying the feasibility of observing the neutron whispering gallery effect, the two most important methodical issues to address were the following [3]: the quality of the surface (roughness at least as low as a few nanometers, waviness at least as low as $\sim 10^{-3}$ rad, reduction of impurities) has to be high enough to provide a sufficiently long neutron quantum state lifetime; the signal-to-noise ratio has to be of the order of unity or better. The latter is particularly difficult due to the extremely small ratio of the deflected to incoming neutron fluxes. We identified two strategies: (i) bending extra-thin perfectly polished crystal plates and (ii) cutting off a sector of a thick-wall polished tube. The first strategy enabled us to meet
Figure 3. A sketch of the potential in the mirror surface vicinity is shown. The potential step at \( z = 0 \) is equal to the mirror optical potential 54 neV. The potential slope at \( z \neq 0 \) is governed by the centrifugal effective acceleration \( a = v^2/R \).

all the mentioned requirements, but it did not provide a perfect cylindrical shape and optimal curvature radius. The results obtained by the second strategy are presented in [3]. The mirror surface quality in this experiment is close to the required one. We succeeded in getting a high signal/background ratio; in identifying neutron tunneling through the triangle potential in figure 3 along all of the mirror surface; and in identifying the neutron intensity spot at the exit mirror edge. However, too large roughness effects mixed considerably the quantum states and did not allow any reliable quantitative analysis.

Finally, the results presented here have been obtained using a cylindrical mirror with a radius of \( R = 25.3 \) mm and a height of 40 mm polished from the bulk of a silicon single crystal, as shown in figure 1. The mirror surface roughness is 0.4 nm, much smaller than the characteristic size of the quantum states. The geometric shape is perfect enough to neglect significant false effects at the present level of accuracy. The mirror radius is chosen so that Bragg effects in silicon do not manifest in the wavelength of interest. The use of a crystal reduces the scattering of neutrons passing through the mirror bulk; therefore, the neutron angular distribution at the mirror exit is defined by the neutron interaction with the cylindrical surface alone. We assume that the polished surface is covered with an oxide layer with a thickness of 1.5 nm and the diffuseness on the oxide/silicon interface is 0.7 nm—precise values will be confirmed by further detailed studies.

The present measurements were carried out using the D17 instrument [30] at the ILL. The cylindrical mirror was installed on a translation/rotation table. The neutron beam was shaped using two slits parallel to the cylinder axis. The first slit with a width of 3 mm was installed at a distance of 3.4 m upstream of the neutron beam; it defined the maximum angular beam divergence of 0.2°. Another slit with a width of 0.1 mm was installed in front of the cylindrical mirror; the mirror inclination was adjusted to provide parallelism of the slit to the
cylindrical mirror axis. Due to the chosen geometry, the beam width at the sample was nearly equal to the second slit width. Thus defined, the neutron beam width was much larger than the characteristic quantum state size. The neutron spectrum at the entrance to the mirror is shown in figure 2. The neutron beam entered the cylindrical mirror at a tangent trajectory from the mirror bulk, as shown in figure 1; only a small fraction of neutrons tunneled into the centrifugal quasistationary quantum states. Such neutrons populate mainly short-living high-energy states. Neutrons were counted in a large position-sensitive neutron detector placed at a distance of 3.1 m from the mirror exit. As this distance is much larger than the mirror size, the detection point is related unambiguously to the exit angle. The scattering probability as a function of the neutron wavelength and the scattering angle is presented in figure 7.

The scattering pattern is a set of pronounced rays. Neutrons with a wavelength below the cut-off value do not scatter to measured angles. In the following, we will show that the above mentioned features are essential properties of the neutron whispering gallery phenomenon.

4. Formalism

In this section we develop the formalism describing large angle scattering of a plane neutron wave on a cylindrical mirror. In such an approach, we neglect that the mirror studied experimentally is not a full cylinder; in particular, the effects of the neutron scattering from the edges of the curved mirror are not considered. Such an approximation is justified for the geometry used in our experiment and for the range of measured scattering angles.

The scattering obeys the following Schrödinger equation in the cylindrical coordinates,

\[
-\hbar^2 \left( \frac{1}{2M} \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} \right) - \frac{\hbar^2}{2M \rho^2} \frac{\partial^2}{\partial \varphi^2} + U(\rho) - \frac{p^2}{2M} \right] \Psi(\rho, \varphi) = 0. \tag{2}
\]

Here, \( \Psi(\rho, \varphi) \) is the neutron wavefunction, \( p \) is the neutron momentum, \( \rho \) is the radial distance from the cylinder axis, \( \varphi \) is the angle and \( U(\rho) \) is the mirror optical potential. The neutrons enter from the mirror bulk into vacuum through the curved cylindrical surface. We can choose the energy origin equal to the mirror bulk optical potential. Then the problem is equivalent to the neutron scattering on the cylinder with negative optical potential,

\[
U(\rho) = -U_0 \Theta(R - \rho) = \begin{cases} -U_0, & \rho \leq R, \\ 0, & \rho > R. \end{cases} \tag{3}
\]

Here, \( R \) is the cylinder radius and \( \Theta(R - \rho) \) is the step function. In equation (2), we omit the trivial dependence on the \( z \)-coordinate directed along the cylinder axis. By standard substitution of an analytical form of a wavefunction \( \Psi(\rho, \varphi) = \Phi(\rho, \varphi)/\sqrt{\rho} \), equation (2) is transformed into the following form,

\[
\left[ -\frac{\hbar^2}{2M} \left( \frac{\partial^2}{\partial \rho^2} \right) - \frac{\hbar^2}{2M \rho^2} \left( \frac{\partial^2}{\partial \varphi^2} + \frac{1}{4} \right) - U_0 \Theta(R - \rho) - \frac{p^2}{2M} \right] \Phi(\rho, \varphi) = 0. \tag{4}
\]

The wavefunction \( \Phi(\rho, \varphi) \) asymptotic at large \( \rho \) values is

\[
\Phi(\rho, \varphi) \to \sqrt{\rho} \exp(i \rho \cos(\varphi)) + f(\varphi) \exp(i \rho \varphi + i \pi/4), \tag{5}
\]
where \( f(\varphi) \) is the scattering amplitude. The standard expansion of the two-dimensional (2D) wavefunction \( \Phi(\rho, \varphi) \) in the complete basis of the angular momentum states \( \exp(i\mu \varphi) \) is

\[
\Phi(\rho, \varphi) = \sum_{\mu=-\infty}^{\infty} \chi_{|\mu|}(\rho) \exp(i\mu \varphi),
\]

where \( \chi_{|\mu|}(\rho) \) are the radial wavefunctions.

The scattering amplitude \( f(\varphi) \) expansion in the complete basis of the angular momentum states \( \exp(i\mu \varphi) \) follows from equation (6) and has the following form:

\[
f(\varphi) = \sum_{\mu=-\infty}^{\infty} (S(\mu, p) - 1) \exp(i\mu \varphi) \equiv \sum_{\mu=-\infty}^{\infty} f(\mu, p) \exp(i\mu \varphi).
\]

Here, \( S(\mu, p) = \exp(2i\delta_\mu(p)) \) is the scattering matrix in a partial wave with angular momentum \( \mu \); \( \delta_\mu(p) \) is the scattering phase, which can be found by solving the corresponding radial equation:

\[
\left[ -\frac{\hbar^2}{2M} \frac{\partial^2}{\partial \rho^2} + \frac{\hbar^2}{2MP^2} \left( \mu^2 - \frac{1}{4} - U_0 \Theta(R - \rho) - \frac{p^2}{2M} \right) \right] \chi_\mu(\rho) = 0,
\]

\[
\chi_\mu(\rho \to 0) = 0,
\]

\[
\chi_\mu(\rho \to \infty) = \sqrt{\frac{2}{\pi p}} \sin(\rho \Theta + \delta_\mu - \frac{\pi}{2} (|\mu| - 1/2)).
\]

Thus, we have formulated the general equations describing the scattering of a plane neutron wave on a cylindrical mirror and need to solve them. We will be interested in large-angle neutron scattering \( (\varphi > 2\varphi_c) \) and show that it results from the neutron capture into the quasistationary states, localized near the cylinder surface.

It is easy to note that these equations (8)–(10) coincide with the usual equations used in the Regge formalism [32].

Further on, we will use the corresponding method of complex angular momentum. So far we introduce, following the Regge approach, \( f(\mu, p) \) as a function of complex momentum \( \mu \), which coincides with the scattering matrix for integer values of \( \mu \) and has standard analytical properties in the complex plane of \( \mu \) [32, 33]. The sum (7) over integer \( \mu \) is then transformed to an integral in the complex \( \mu \) plane, which is calculated by using the residue theorem and is replaced by a sum over poles contributions. In 2D problems, an elegant way of performing such a transformation is to use the Poisson sum formula [34],

\[
f(\varphi) = \sum_{\mu=-\infty}^{\infty} \int_{-\infty}^{\infty} (f(\mu, p) \exp(i\mu(\varphi + 2\pi n)))d\mu.
\]

Here, \( f(\mu, p) \) is the analytical function of \( \mu \), as mentioned above. The integer number \( n \) has the sense of the number of neutron wave rotations around the cylinder surface. Using the analytical properties of the amplitude, we transform the above integral to the sum of the amplitude poles contribution,

\[
f(\varphi) = 2\pi i \sum_{n=-\infty}^{\infty} \sum_{j} \text{Res} f(\mu_j, p) \exp(i\mu_j(\varphi + 2\pi n)).
\]
Here, $\mu_j$ is the $j$th pole of the amplitude $f$; $\text{Res}_f(\mu_j, p)$ is a residue of the amplitude in the mentioned pole. In the above expression, the integration contour is chosen such that $\text{Im}\,\mu_j\varphi > 0$ and $\text{Im}\,\mu_j(\pi - \varphi) > 0$. The summation over $n$ can be performed after taking into account that the amplitude in our case is symmetric under the substitution $\mu \to -\mu$. Such an expression takes the form
\[
f(\varphi) = 2\pi i \sum_j \text{Res}_f(\mu_j, p) \frac{\sin(\mu_j(\pi - \varphi))}{\sin(\mu_j\pi)}.
\]

For practical reasons, we will be interested in the scattering on the segment of a cylinder. In such a case, the only contribution comes from $n = 0$ and expression (12) simplifies to
\[
f(\varphi) = 2\pi i \sum_j \text{Res}_f(\mu_j, p) \exp(i\mu_j\varphi)).
\]

The physical sense of the above expressions for the scattering amplitude is transparent. The corresponding amplitude is the sum of the contributions of decaying neutron quasi-stationary states (each corresponds to the $S$-matrix pole), which are formed during the neutron scattering on the cylinder. The neutron states with the longest lifetime determine the neutron scattering to the large angles. In the following, we will show that such long-living states are the states localized near the cylinder surface and corresponding to the whispering gallery waves.

### 4.1. Surface wave solutions

We will be interested in those radial equation solutions with different angular momenta $\mu$ which correspond to the states of neutrons, following the mirror surface. The angular momentum $\mu$ for such neutrons with energy $E = Mv^2/2$ is close to the classical value $\mu_0 = MvR/\hbar$ that is extremely large $\mu_0 \sim 5 \times 10^8$ if $v = 1000 \, \text{m s}^{-1}$ and $R = 2.53 \, \text{cm}$ (the characteristic parameters of the performed experiment). This big value of effective momentum explains naturally the classical character of neutron motion along the angle variable $\varphi$. In particular, we can neglect the quantization of angular momentum and consider it as a continuous variable.

In order to solve the equation given above, we expand the expression for the centrifugal energy in equation (8) in the vicinity of $\rho = R$ introducing the deviation from the cylinder surface $z = \rho - R$. In the first order of small ratio $z/R$, we obtain the following equation,
\[
\left[-\frac{\hbar^2}{2M} \frac{\partial^2}{\partial z^2} - U_0\Theta(-z) + \frac{\hbar^2 \mu^2 - 1/4}{2MR^2} \left(1 - \frac{2z}{R}\right) - E\right] \chi_{\mu}(z) = 0.
\]

Introducing a new variable
\[
\varepsilon_{\mu} = E - \frac{\hbar^2 \mu^2 - 1/4}{2MR^2} \simeq \frac{(MvR)^2 - \hbar^2 \mu^2}{2MR^2},
\]
we obtain the following equation,
\[
\left[-\frac{\hbar^2}{2M} \frac{\partial^2}{\partial z^2} - U_0\Theta(-z) - \frac{Mv^2}{R}z - \varepsilon_{\mu}\right] \chi_{\mu}(z) = 0.
\]

The value $\varepsilon_{\mu}$ can be understood as the radial motion energy within the linear expansion used above with the angular momentum $\mu$ in the vicinity of the curved mirror surface.
Equation (16) describes the neutron motion in a constant effective field \( a = -v^2/R \) superposed with the mirror optical potential \(-U_0\Theta(-z)\). A sketch of the corresponding potential is shown in figure 3.

The regular solution of equation (16) is given by the well-known Airy function \( \text{Ai}(z) \) [35],

\[
\chi_\mu(z) \sim \begin{cases} 
\text{Ai}(-z/l_0 - \varepsilon_\mu/\varepsilon_0) & \text{if } z > 0, \\
\text{Ai}(-u_0 - z/l_0 - \varepsilon_\mu/\varepsilon_0) & \text{if } z \leq 0.
\end{cases} \tag{17}
\]

Here,

\[
l_0 = \sqrt{\frac{\hbar^2 R}{2M^2 v^2}} = \frac{R}{2} \left( \frac{2}{\mu_0} \right)^{2/3}
\]

is the characteristic distance scale of the problem, and

\[
\varepsilon_0 = \sqrt{\frac{\hbar^2 M v^4}{2R^2}} = \frac{M v^2}{2} \left( \frac{2}{\mu_0} \right)^{2/3}
\]

is the characteristic energy scale; \( u_0 = U_0/\varepsilon_0 \) is the optical potential of the mirror in units of the characteristic energy. For the typical experimental setup parameters \( U_0 = 54.1 \text{ neV}, \ v = 1000 \text{ m s}^{-1} \) and \( R = 2.53 \text{ cm} \), the above-mentioned scales are \( l_0 = 37 \text{ nm} \), \( \varepsilon_0 = 15.3 \text{ neV} \) and \( u_0 \simeq 3.5 \).

4.2. Scattering amplitude and centrifugal states

Let us calculate the scattering amplitude (7). First, we calculate the partial \( S \)-matrix, \( S_\mu = \exp(2i\delta_\mu) \), by matching the scattering state solutions for the radial equation (8) for \( z < 0 \) and \( z \geq 0 \),

\[
\tilde{\chi}_\mu(z) \sim \begin{cases} 
\chi_\mu^-(z) - S_\mu \chi_\mu^+(z) & \text{if } z > 0, \\
\text{Ai}(-u_0 - z/l_0 - \varepsilon_\mu/\varepsilon_0) & \text{if } z \leq 0.
\end{cases} \tag{20}
\]

Here, \( \chi_\mu^-(z) = Bi(-z/l_0 - \varepsilon_m/\varepsilon_0) - i\text{Ai}(-z/l_0 - \varepsilon_\mu/\varepsilon_0) \) and \( \chi_\mu^+(z) = Bi(-z/l_0 - \varepsilon_\mu/\varepsilon_0) + i\text{Ai}(-z/l_0 - \varepsilon_\mu/\varepsilon_0) \).

By matching the logarithmic derivatives of the two solutions at \( z = 0 \), we obtain

\[
S_\mu = \frac{h(\varepsilon_\mu/\varepsilon_0) + ig(\varepsilon_\mu/\varepsilon_0)}{h(\varepsilon_\mu/\varepsilon_0) - ig(\varepsilon_\mu/\varepsilon_0)},
\]

where

\[
h(x) = \text{Ai}(-u_0 - x) \ Bi'(-x) - \text{Bi}(-x) \ \text{Ai}'(-u_0 - x)
\]

and

\[
g(x) = \text{Ai}(-x) \ \text{Ai}'(-u_0 - x) - \text{Ai}(-u_0 - x) \ \text{Ai}'(-x).
\]

The \( S \)-matrix, as a function of the continuous variable \( x \), has poles for those complex values \( x = x_n \) which are the roots of the equation

\[
h(x_n) - ig(x_n) = 0.
\]

One can check that the values \( x_n \) given by the above equation are the complex energies of the quasistationary centrifugal states [36]. Hereafter, we discuss only the physically important
states (so-called class I states [37]) and do not account for unphysical contributions (class II poles [37]).

Such a state is localized within an effective well, formed by the centrifugal potential and a mirror optical potential. The main properties of such states are summarized in appendix A. Here we mention that the quasistationary nature of such a state is explained by the nonvanishing probability of the neutron penetration through the triangular barrier (figure 3) into the mirror bulk. This probability strongly depends on the effective triangular barrier height \( u_0 \), which is a function of neutron velocity

\[
u_0 = \frac{U_0}{\varepsilon_0} = U_0 \sqrt{\frac{2R^2}{\hbar^2 M v^4}}
\]

and determines the width \( \Gamma_n \) of the centrifugal state.

For the following analysis, it would be convenient to write down the expression for the scattering amplitude (14) as follows,

\[
f(\varphi) = f_0 \sum_{n=1}^{\infty} \gamma_n r_n \exp \left[ -i \varphi \left( \frac{\mu_0}{2} \right)^{1/3} (\beta_n - i \gamma_n) \right].
\]

(25)

Here,

\[
f_0 = -2i \left( \frac{\mu_0}{2} \right)^{1/3} \exp(i \mu_0 \varphi) \sqrt{\frac{2\pi \hbar}{p}}.
\]

(26)

\[
r_n = \frac{h'(x_n^*) + ig'(x_n^*)}{h'(x_n) - ig'(x_n)}.
\]

(27)

To obtain the above expression, we took into account that the \( S \)-matrix can be represented in the vicinity of its pole as

\[
S(x \to x_n) = \frac{h'(x_n^*) + ig'(x_n^*)}{h'(x_n) - ig'(x_n)} \left( x - \beta_n - i \gamma_n \right).
\]

(28)

For the following, it would be important to get the asymptotic expression for \( r_n \) for large \( n \). Using the asymptotic expressions for the Airy functions, one can find that

\[
r_n \approx -i \exp \left[ -\frac{4}{3} (\beta_n - u_0)^{3/2} \right].
\]

(29)

The above expression is valid for \( \beta_n \gg u_0 \), i.e. for \( n \gg 1 \). However, it is well justified even for the first \( n \), for which \( \beta_n > u_0 \).

One can see from equation (25) that the scattering amplitude is the sum of the quasistationary state contributions. The contribution of each term is proportional to \( \gamma_n \exp(-\gamma_n (\mu_0/2)^{1/3} \varphi) \). Note that this expression tends to zero both for very small and very big \( \gamma_n \) (\( \gamma_n \to 0 \) and \( \gamma_n \to \infty \)). We demonstrate the position of the Regge poles in the first quadrant of the complex momentum plane in figure 4.

One can see the first few poles with a small imaginary part, corresponding to under-barrier quasistationary states, and a set of poles with large and practically equal imaginary parts, corresponding to above-barrier quasistationary states.
Expression (25) is the final result for the scattering amplitude. Nevertheless, it is interesting for physical understanding to perform a more detailed analysis of this expression by separating contributions from different kinds of states. We will address three types of states:

- The first type is the deeply bound under-barrier states. We will denote the number of such states by $N_I$.
- We attribute two quasistationary states to the second type; their energy is closest to the barrier height (usually this is one under-barrier and one above-barrier state).
- The third type consists of all the rest highly excited above-barrier states. We will show that their contribution could be well approximated by the semiclassical expression.

The above scattering amplitude decomposition is convenient because each type of state, as we will show, dominates for a certain scattering angle range. We represent the scattering amplitude according to this classification,

$$f(\varphi) = f_1(\varphi) + f_{II}(\varphi) + f_{III}(\varphi),$$

where

$$f_1(\varphi) = f_0 \sum_{n=1}^{N_I} \gamma_n r_n \exp \left[ -i \varphi \left( \frac{\mu_0}{2} \right)^{1/3} (\beta_n - i \gamma_n) \right],$$

$$f_{II}(\varphi) = f_0 \sum_{n=N_I+2}^{N_I+2} \gamma_n r_n \exp \left[ -i \varphi \left( \frac{\mu_0}{2} \right)^{1/3} (\beta_n - i \gamma_n) \right],$$

$$f_{III}(\varphi) = f_0 \sum_{n=N_I+3}^{\infty} \gamma_n r_n \exp \left[ -i \varphi \left( \frac{\mu_0}{2} \right)^{1/3} (\beta_n - i \gamma_n) \right].$$

**Figure 4.** The position of the Regge poles in the first quadrant of the complex momentum plane. Neutron wavelength $\lambda = 4$ Å.
The width \( \gamma_1 \) of the deeply bound under-barrier states is exponentially small (see appendix A),

\[
\gamma_n \sim \exp\left(-\frac{4}{3}(u_0 - x_n)^{3/2}\right).
\]

The contribution of \( f_I(\varphi) \) is negligible for

\[
\varphi \ll \ln\left(\frac{\gamma_N}{\gamma_1}\right)\left(\frac{\mu_0}{2}\right)^{1/3},
\]

where \( \gamma_N \) is a typical above-barrier width value.

The second term \( f_{II} \) is a superposition of two states with moderate width. Their contribution would be significant for angles \( \varphi < 1/\gamma_{II}(\mu_0/2)^{1/3} \), where \( \gamma_{II} \) is a typical width value for the second-type states.

The interference between the two states results in maxima and minima in the amplitude. From equation (32), one can find that maxima positions are

\[
\varphi_{\text{max}} = \frac{2\pi k - (\theta_2 - \theta_1)}{(\beta_2 - \beta_1) \left(\frac{\mu_0}{2}\right)^{1/3}} \approx 2 \left( k - \frac{\theta_2 - \theta_1}{2\pi}\right) \varphi_c.
\]

Here, \( \theta_{1,2} = -\text{Im} \ln(r_{1,2}) \), and \( k \) is an integer. Due to exponential decay, the maxima values decrease with increasing \( k \).

The third term is a sum over the higher part of the spectrum of quasistationary states. These are above-barrier states, i.e. the states with \( \text{Re} x_n > u_0 \).

Their energy and width \( x_n = \beta_n - i\gamma_n \) in the case of the step-like optical potential can be well approximated by the following expression (see appendix A),

\[
\beta_n = \left(\frac{3}{2}\pi (n - \frac{3}{4})\right)^{2/3},
\]

\[
\gamma_n = \left(\ln(2\sqrt{\beta_n/u_0}) - \frac{u_0}{4\beta_n}\right)/\sqrt{\beta_n}.
\]

These states are described within semiclassical approximation. Taking into account equation (29), we represent equation (33) as

\[
f_{III}(\varphi) = f_0 \sum_{n=N_1+3}^{\infty} \gamma_n \exp\left(-i\delta_n - \gamma_n \left(\frac{\mu_0}{2}\right)^{1/3} \varphi\right),
\]

\[
\delta_n = \frac{4}{3}(\beta_n - u_0)^{3/2} + \beta_n \left(\frac{\mu_0}{2}\right)^{1/3} \varphi.
\]

The detailed properties of these states are described in appendix A. We establish there that the mentioned states’ width changes as a very slow function of the state number \( n \). This justifies the application of the stationary phase method (see appendix B). The important result of the stationary phase method consists of the statement that for a given angle \( \varphi \), only a limited number of states from the sum in equation (38) give a dominant contribution to the scattering amplitude. The quantum number \( n \) of such states is given by the stationary phase equation (B.2) and it turns out to be an increasing function of angle \( \varphi \). Within this approach, the contribution \( f_{III} \) can be written in a closed form and is a decreasing function for large angles \( \varphi \) as given by equation (B.9).

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The above mentioned behavior of the amplitude as a function of scattering angle changes radically if the finite diffuseness of the optical potential is taken into account. In such a case, the width of a highly excited above-barrier state with the wavelength comparable to the optical potential diffuseness tends to a constant value $\gamma_{sc} \rightarrow \pi \alpha \varepsilon_0 \bar{\hbar}$ (see appendix B). The energy of such states is $\beta_n \geq 1/(4\pi^2 \alpha^2)$. This means that the contribution of such states to the scattering amplitude is proportional to $\exp(-\pi \alpha (\mu_0/2)^{1/3} \varphi)$. The amplitude $f_{III}$ at angles $\varphi > 1/(\pi \alpha (\mu_0/2)^{1/3})$ can be neglected if $\gamma_{sc} > \gamma_{II}$; only the $f_I$ and $f_{II}$ amplitudes are important. We come to the conclusion that the contribution of $f_{III}$ amplitude depends strongly on the optical potential diffuseness for $\varphi < 1/(\pi \alpha (\mu_0/2)^{1/3})$. At larger angles the contribution of $f_I$ and $f_{II}$ is dominant, which weakly depends on the diffuseness of the optical potential.

The total scattering amplitude is a result of the interference of the three amplitudes studied above. The corresponding differential cross-section is shown in figure 5 as a density plot function of neutron wavelength $\lambda$ and scattering angle $\varphi$ for the case of a sharp-edge optical potential. The positions of maxima (bright rays) coincide with those of $|f_{II}(\varphi)|^2$, given by equation (35). One can express equation (35) in the form

$$\varphi_{max} \simeq 2 \left( k - \frac{\theta_2 - \theta_1}{2\pi} \right) \sqrt{2mU_0 \lambda / \bar{\hbar}}.$$  

The above expression clarifies the approximate linear dependence of scattering maxima on the neutron wavelength. These maxima are substantially softened by the interference with $f_{III}$ amplitude. The contribution of $f_{III}$ is rejected in the case of a diffuse optical potential; the maxima of the cross-section are much more pronounced in such a case. We show in figure 6 the differential cross-section for the case of the optical potential effective diffuseness $a = 7$ nm.

5. Analysis of experimental results

We present the experimental data on neutron scattering by a curved mirror in figure 7 as a function of neutron wavelength $\lambda$ and scattering angle $\varphi$. The positions of scattering maxima (rays) and their contrast are in good agreement with the results presented in figure 6.

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Another important observation consists of a sharp cut-off of the neutron count rate at angles \( \varphi > 1^\circ \) when the neutron wavelength is below 3 Å. This feature is clearly reproduced by the theory. The explanation for the cut-off is clear: the effective triangular potential does not hold any long-living quantum state for \( \lambda < \lambda_c \) following from equation (1). The mentioned cut-off is even more spectacular if one takes into account that the neutron spectrum at the entrance to the mirror, shown in figure 2, has a wide maximum around this wavelength.

We compare in detail the experimental and theoretical results on the neutron count rate at a fixed scattering angle of 1\(^{\circ}\) in figure 8. A theoretical curve assuming an optical potential
Figure 8. The neutron count rate as a function of the neutron wavelength: experimental data (red circles) versus theory (black straight and blue dash-dot curves). The scattering angle is 1°; the effective diffuseness of optical potential is 7 nm for curve a and 0 nm for curve b.

with an effective diffuseness of 7 nm (denoted as ‘a’) approximates well the general behavior of the experimental data. The systematic shift of the observed maximum in the experimental data towards shorter wavelengths might be due to the presence of an oxide layer on the silicon surface. The curve ‘b’, corresponding to the sharp optical potential, gives a sufficiently less prominent maximum than that observed experimentally.

We present the theory versus the experimental data for the neutron count rate at a fixed scattering angle of 3° in figure 9. The best agreement is again obtained for the diffused optical potential with the diffuseness of 7 nm.

We give in figure 10 the neutron count rate as a function of scattering angle for a fixed wavelength of 5 Å compared to the theoretical calculations with diffused optical potential with a diffuseness of 7 nm. The theoretical curve reproduces well the positions of minima and maxima of the scattering probability.

The theoretical calculations represent the main features of the experimental results. Residual deviations between theory and experimental data might be explained if the precise shape of the optical surface potential is taken into account; in particular, the presence of a thin oxide layer on the mirror surface. We will analyze this problem in future publications.

6. Conclusions

We have measured for the first time the quasistationary quantum states of cold neutrons localized near a cylindrical mirror surface. The effective potential well is formed by the centrifugal effective potential and the mirror optical potential. The experiment consisted in the scattering of cold neutrons at the surface of a well-polished cylindrical mirror made of single-crystal silicon. The probability of neutron scattering was measured as a function of neutron wavelength and deviation angle. We present a theoretical formalism describing this process. The measured
data are explained within the formalism presented. We conclude that deeply bound centrifugal quantum states might present a kind of precision clock with a long lifetime; highly excited states are very sensitive to the precise wall potential shape. Therefore, they are a promising tool for studying fundamental neutron–matter interaction, quantum neutron optics and surface physics effects. Various possible applications of the described centrifugal quantum states have to be explored in further studies.

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Figure A.1. The real part of the two lowest eigenvalues $x_{1,2}$ as a function of $u_0$ obtained by numerical integration of equation (A.2).

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Appendix A. Properties of the centrifugal quantum states

The effective triangle potential in figure 3 supports the existence of the quasistationary states. They correspond to the solution of equation (16) with the outgoing wave boundary condition,

$$\tilde{\chi}_n(z) \sim \begin{cases} B_i(-z/l_0 - \epsilon_n/\epsilon_0) + i A_i(-z/l_0 - \epsilon_n/\epsilon_0) & \text{if } z > 0, \\ A_i(-u_0 - z/l_0 - \epsilon_n/\epsilon_0) & \text{if } z \leq 0. \end{cases} \tag{A.1}$$

Here, $n = 1, 2, \ldots$ is an integer number that plays the role of the quasistationary states quantum number. The complex energies of such quasistationary states can be found by matching the logarithmic derivative at $z = 0$,

$$\epsilon_n \equiv \epsilon_0 x_n, \tag{A.2}$$

$$\frac{A_i(-u_0 - x_n)}{A_i(-u_0 - x_n)} = \frac{B_i(-x_n) + i A_i(-x_n)}{B_i(-x_n) + i A_i(-x_n)}. \tag{A.3}$$

The real and imaginary parts of the eigenvalue $x_n$, obtained by numerical solution of equation (A.2) for the two lowest states, are shown in figures A.1 and A.2 as a function of a dimensionless variable $u_0$, depending on the neutron velocity, and in figures A.3 and A.4 as a function of quantum number $n$.

One can calculate the widths of the centrifugal quasistationary states in a semiclassical approximation, which is quite precise for linear potentials even for the lowest state [38].
Figure A.2. The imaginary part of the two lowest eigenvalues $x_{1,2}$ as a function of $u_0$ obtained by numerical integration of equation (A.2).

Figure A.3. The real part of the eigenvalues $x_n$ as a function of quantum number $n$, $u_0 = 3.5$.

an asymptotic expression for the Airy functions of a large argument, one obtains the following equation,

$$\sqrt{\frac{x_n}{u_0 - x_n}} \tan\left(\frac{2}{3} x_n^{3/2} - \pi/4\right) = 1 - i \exp\left[-\frac{4}{3} (u_0 - x_n)^{3/2}\right]. \quad (A.4)$$

The approximate solution of the above equation for deeply bound states $x_n \ll u_0$ gives

$$x_n \simeq \left(\frac{3}{2} \pi (n - 1/4)\right)^{2/3} - \sqrt{\frac{1}{u_0 - (\frac{3}{2} \pi (n - 1/4))^{2/3}}}. \quad (A.5)$$
Figure A.4. The absolute value of the imaginary part of the eigenvalues \( x_n \) as a function of the quantum number \( n \). Neutron wavelength \( \lambda = 4 \) Å. Circles correspond to a smooth optical potential in equation (A.21) with a diffuseness parameter \( a = 8 \text{ nm} \). Squares indicate results for a sharp step-like optical potential.

\[
\Gamma_n \simeq 2\varepsilon_0 \sqrt{\frac{1}{u_0 - x_n}} \exp \left[ -\frac{4}{3}(u_0 - x_n)^{3/2} \right].
\]

(A.6)

The approximate solution for the states close to the barrier \( 1 \ll u_0 - x_n \ll u_0 \) gives

\[
x_n \simeq \left( \frac{3}{2\pi} (n - 3/4) \right)^{2/3} \sqrt{\frac{u_0 - (\frac{3}{2}\pi (n - 3/4))^{2/3}}{(\frac{3}{2}\pi (n - 3/4))^{2/3}}}.
\]

(A.7)

\[
\Gamma_n \simeq 2\varepsilon_0 \frac{\sqrt{u_0 - x_n}}{x_n} \exp \left[ -\frac{4}{3}(u_0 - x_n)^{3/2} \right].
\]

(A.8)

The energy and width of the quasistationary states depend strongly on the centrifugal acceleration \( |\alpha| = v^2/R \). A small acceleration \( \alpha \) results in a broad barrier, which separates the states in the effective well from continuum. Indeed, \( u_0 = U_0/\varepsilon_0 = U_0\left((2R^2)/(\hbar^2 M v^4)\right)^{1/3} \) increases if \( v \) decreases. The width of a quasistationary state decreases exponentially, as is clear from equation (A.4). Equation (A.4) estimates the critical values of the neutron velocity \( v_n^c \), which correspond to the appearance of the \( n \)th state in the effective well or in other terms to the equality of the \( n \)th energy eigenvalue to the mirror optical potential in the characteristic energy units,

\[
u_n^c \approx \sqrt{\frac{U_0^3}{(3/2\pi (n - 3/4))^2 \hbar^2 M}}.
\]

(A.10)

The accuracy of the above approximation increases with \( n \).
Solutions of equation (A.3) include the ‘above-barrier’ quasistationary states as well, i.e. the states with \( \text{Re} x_n > u_0 \). The asymptotic form of equation (A.3) for \( x_n \gg u_0 \) is

\[
\sqrt{x_n} \cos(2/3x_n^{3/2} + \pi/4) - i\sqrt{x_n} - u_0 \sin(2/3x_n^{3/2} + \pi/4) = 0.
\]

(A.11)

Its approximative solutions are given by the following expressions

\[
x_n = \beta_n - i\gamma_n,
\]

\[
\beta_n = (3/2\pi(n - 3/4))^{2/3},
\]

(A.12)

\[
\gamma_n = \left( \ln(2\sqrt{x_n/u_0}) - u_0/4\beta_n \right)/\sqrt{\beta_n}.
\]

(A.13)

The real part of the above-barrier quasistationary energies is close to the energy levels of a quantum bouncer. The level spacing (the quasiclassical frequency) of such a spectrum is

\[
\omega_n = \frac{\varepsilon_0}{\hbar} \partial \beta_n/\partial n = \frac{\varepsilon_0}{\hbar} \pi / \sqrt{\beta_n}.
\]

(A.14)

One can find the value \( \beta_n \) and the corresponding quantum number \( n \), above which the level spacing is smaller than the state width,

\[
\beta_{\text{max}} \approx \frac{u_0}{4}(e^{2\pi} + 2) \approx 134.4u_0,
\]

(A.15)

\[
n_{\text{max}} = \frac{2}{3\pi} \beta_{n}^{3/2} + 3/4 \approx 330u_0^{3/2}.
\]

(A.16)

The spectrum of such states is ‘discrete’ (i.e. the level spacing is larger than the level width) for rather large number \( n \). This spectrum property is related to the phenomenon of the sharp edge reflection. To demonstrate this statement, let us mention that the semiclassical expression for the norm \( |C(t)|^2 \) of the above-barrier quasistationary state is

\[
|C(t)|^2 = \exp(-\Gamma t) \approx R(E)^{t/T} = \exp(-|\ln R(E)|t/T),
\]

(A.17)

where \( R(E) \) is the energy-dependent coefficient of the above-barrier reflection \( R(E) \ll 1 \), and \( T \) is the classical period. So far the semiclassical expression for the width is

\[
\Gamma/2 = \hbar|\ln R(E)|/(2T).
\]

(A.18)

The coefficient of the above-barrier reflection from the triangular barrier with a height \( U_0 \) in case \( E \gg U_0 \) is [31]

\[
R(E) \approx \frac{U_0^2}{4E_0^2}.
\]

(A.19)

while the period of a bouncer is \( T \sim 2\sqrt{E} \). One can easily check that a corresponding expression for the width \( \gamma_{qs}/2 \) coincides with equation (A.13) within the accuracy of semiclassical approximation. The classical period \( T \) equals the lifetime \( \tau = \hbar/\Gamma = T/|\ln R(E)| \) when

\[
1 = |\ln R(E)|.
\]

(A.20)

We see that the width of a highly excited state is small compared to the level spacing due to the slow (inverse square in the case of a triangle barrier) decrease in the reflection coefficient as a function of energy. Such an asymptotic dependence \( R(E) \) is a consequence of a sharp change in the potential barrier. One can show that the relation between the energy and the width of the above-barrier states (A.13) is determined by the energy dependence of the above-barrier reflection coefficient \( R(E) \) and is sensitive to the shape of the optical potential.
As demonstrated in section 3, such a relation is exposed in the scattering amplitude dependence on the scattering angle. So far it is useful to establish how the diffuseness of the optical potential modifies equation (A.13) in the case of the following optical potential,

$$U(x) = U_0/(1 + \exp(-x/b)).$$  \hspace{1cm} (A.21)

The reflection coefficient $R(E)$ (expressed in dimensionless variables $\varepsilon = E/\varepsilon_0$, $u_0$ and $\alpha = b/l_0$) is [31]

$$R(E) = \left(\frac{\sinh(\pi\alpha(\sqrt{\varepsilon} - \sqrt{\varepsilon - u_0}))}{\sinh(\pi\alpha(\sqrt{\varepsilon} + \sqrt{\varepsilon - u_0}))}\right)^2. \hspace{1cm} (A.22)$$

The corresponding width of highly excited over-barrier states in the case of an optical potential (A.21) is

$$\gamma_{sc} \approx \varepsilon_0/\hbar \left|\ln \left(\frac{(\pi\alpha u_0)^2}{\varepsilon}\right)\right|/(4\sqrt{\varepsilon}) + \pi\alpha. \hspace{1cm} (A.23)$$

The effect of diffuseness is essential when the neutron wavelength is smaller than the diffusion radius: $\alpha\sqrt{\varepsilon} \gg 1$; in such a case, the width $\gamma_{sc} \rightarrow \pi\alpha\varepsilon_0/\hbar$. In the opposite limit $\alpha\sqrt{\varepsilon} \ll 1$, we return to the case of the step-like barrier.

Appendix B. The stationary phase method

We use the stationary phase method to perform summation in equation (38). The sum (38) can be substituted by the contribution of a few terms around the stationary phase points. Such points can be found from the equation

$$\frac{d\delta}{dn} = (2\sqrt{\beta_n - u_0} + (\mu_0/2)^{1/3} \varphi) \frac{d\beta_n}{dn} = 0. \hspace{1cm} (B.1)$$

Using an explicit form of phase (39) and the semiclassical expression for $\beta_n$ (A.12), we obtain an equation for the stationary points,

$$(2\sqrt{\beta_n - u_0} + (\mu_0/2)^{1/3} \varphi - kT) \frac{\pi}{\sqrt{\beta_n}} = 0. \hspace{1cm} (B.2)$$

Here, $k$ is an integer and $T$ is a ‘classical period’, $T = 2\pi dn/d\beta_n = 2\sqrt{\beta_n}$. The term $-kT$ is an important consequence of the discrete character of the quasistationary states spectrum. The integer number $k - 1$ can be understood as a classical number of reflections. When the summation is substituted by the integration in equation (38), the contribution of the stationary phase points to the sum (38) is given by

$$f_{III}(\varphi) \approx f_0 \sum_k \gamma_k \sqrt{\frac{2\pi}{|d^2\delta_n/dn^2|}} \exp\left(-\frac{i}{3}(\beta_k - u_0)^{3/2} - (i\beta_k + \gamma_k) \left(\frac{\mu_0}{2}\right)^{1/3} \varphi\right). \hspace{1cm} (B.3)$$

Here, $\beta_k$ and $\gamma_k$ are the values taken at stationary phase points, determined by equation (B.2).

Taking into account equation (B.1), we obtain for the amplitude $f_{III}$,

$$f_{III}(\varphi) \approx \sqrt{\frac{2\pi}{\beta_k}} \sum_k \gamma_k \exp\left(-\frac{i}{3}(\beta_k - u_0)^{3/2} - (i\beta_k + \gamma_k) \left(\frac{\mu_0}{2}\right)^{1/3} \varphi\right) \sigma_k. \hspace{1cm} (B.4)$$

$$\sigma_k = \frac{\sqrt{\beta_k\sqrt{\beta_k(\beta_k - u_0)}}}{\sqrt{k\sqrt{\beta_k - u_0} - \sqrt{\beta_k}}}. \hspace{1cm} (B.5)$$
Figure B.1. The scattering angle, given by equation (B.2). Neutron wavelength \( \lambda = 4 \, \text{Å} \). Solid line is given by \( k = 1 \), dashed line by \( k = 2 \) and dash-dotted line by \( k = 3 \).

To analyze the stationary points values, we write down equation (B.2) in the form

\[
\frac{1}{(\mu_0/2)^{1/3}} \left( 2k \sqrt{\beta_k} - 2 \sqrt{\beta_k - u_0} \right) = \varphi. \tag{B.6}
\]

We plot the left-hand side of this equation in figure B.1 for different values of the integer number \( k \). Each branch of this function, corresponding to a given number \( k \), has a minimum value,

\[
\varphi_k^{\text{min}} = 2(2/\mu_0)^{1/3} \sqrt{(k^2 - 1)u_0}. \tag{B.7}
\]

So far the branch with a given \( k \) value contributes to the scattering angles greater than \( \varphi_k^{\text{min}} \). The exception is a branch \( k = 1 \) with a maximum value \( \varphi_1^{\text{max}} = 2 \sqrt{u_0} \). An important consequence is the absence of the stationary points for angles in between \( 2 \sqrt{z_0/(\mu_0/2)^{1/3}} < \varphi < 2 \sqrt{3z_0/(\mu_0/2)^{1/3}} \). Thus the amplitude \( f_{\text{III}}(\varphi) \) is negligible for angles in between these two values.

One can see that the number of stationary points which contribute to some scattering angle is \( k - 1 \) for the angles lying between the minimum of the \( k \)th and \((k + 1)\)th branches.

For large scattering angles such that \( \beta_k \gg u_0 \), the solution of equation (B.2) can be obtained in a closed form,

\[
\beta_k = \frac{(\mu_0/2)^{4/3} \varphi^2}{4(k-1)^2} \equiv \xi^2 \tag{B.8}
\]

The corresponding expression for amplitude \( f_{\text{III}} \) for such large angles is

\[
 f_{\text{III}}(\varphi) \approx \sqrt{\frac{2}{\pi}} f_0 \sum_{k=2}^{k_{\text{max}}} \ln \frac{\xi}{\sqrt{u_0}} \exp \left( -i \xi^3 \frac{3k - 1}{12} \right) \frac{u_0^{k-1}}{\xi^{2k-5/2}}. \tag{B.9}
\]

The summation in the above formula is performed over the ‘number of neutron classical reflections from the mirror’, as explained in appendix B, and starts from \( k = 2 \), as far as \( k = 1 \) contributes to small angles only. \( k_{\text{max}} - 1 \) is the maximum number of reflections for
which the condition $\beta_k \gg u_0$ is valid, $k_{\text{max}} \leq \sqrt{(\mu_0/2)^{2/3}\varphi^2/(4u_0) + 1}$. The main contribution for sufficiently large $\varphi$ comes from $k = 2$; the scattering amplitude behaves like $\ln(\varphi)/\varphi^{3/2}$.

References

[1] Nesvizhevsky V V, Voronin A Yu, Cubitt R and Protasov K V 2010 Nat. Phys. 6 114
[2] Nesvizhevsky V V, Petukhov A K, Protasov K V and Voronin A Y 2008 Phys. Rev. A 78 033616
[3] Cubitt R et al 2009 Nucl. Instrum. Methods A 611 322
[4] Strutt Baron Rayleigh J W 1878 The Theory of Sound vol 2 (London: Macmillan)
[5] Rayleigh L 1914 Phil. Mag. 27 100
[6] Mic G 1908 Ann. Phys. 25 371
[7] Debye P 1909 Ann. Phys. 30 57
[8] Oraevsky A N 2002 Quantum Electron. 32 377
[9] Vahala K J 2003 Nature 424 839
[10] Flugge V S 1974 Practical Quantum Mechanics I (Berlin: Springer)
[11] Nesvizhevsky V V et al 2002 Nature 415 297
[12] Nesvizhevsky V V et al 2003 Phys. Rev. D 67 102002
[13] Onofrio R and Viola L 1996 Phys. Rev. A 53 3773
[14] Viola L and Onofrio R 1997 Phys. Rev. D 55 455
[15] Herdegen A and Wawrzycki J 2002 Phys. Rev. D 66 044007
[16] Wawrzycki J 2004 Acta Phys. Pol. B 35 613
[17] Chryssomalakos C and Sudarsky D 2003 Gen. Relativ. Gravit. 35 605
[18] Kalbermann G 2002 J. Phys. A: Math. Gen. 35 9829
[19] Robinett R W 2004 Phys. Rep. 392 1
[20] Berberan-Santos M et al 2005 J. Math. Chem. 37 101
[21] Belloni M et al 2005 Phys. Scr. 72 122
[22] Mather W H and Fox R F 2006 Phys. Rev. A 73 032109
[23] Withhaut D and Korsch H J 2006 J. Phys. A: Math. Gen. 39 14687
[24] Romero E and de los Santos F 2007 Phys. Rev. Lett. 99 263601
[25] Gonzalez G 2008 Rev. Mex. Fis. 54 5
[26] Bertolami O and Nunes O 2005 Class. Quantum Gravity 20 2005
[27] Nesvizhevsky V V and Protasov K V 2004 Class. Quantum Gravity 21 4557
[28] Nesvizhevsky V V, Pignol G and Protasov K V 2008 Phys. Rev. D 77 034020
[29] Baeßler S et al 2008 Phys. Rev. D 75 075006
[30] Cubitt R and Fragnetto G 2002 Appl. Phys. A 74 2002
[31] Landau L D and Lifshitz E M 1965 Quantum Mechanics. Nonrelativistic Theory (London: Pergamon)
[32] Nussenzweig H M 1972 Causality and Dispersion Relations (New York: Academic)
[33] De Alfaro V and Regge T 1965 Potential Scattering (Amsterdam: North-Holland)
[34] Decanini Antoine Folacci Y 2003 Phys. Rev. A 67 042704
[35] Abramowitz M and Stegun I E 1965 Handbook of Mathematical Functions (New York: Dover)
[36] Baz A I, Zeldovich Ya B and Perelomov A M 1969 Scattering, Reactions and Decays in the Nonrelativistic Quantum Mechanics (Jerusalem: Israel Program for Scientific Translations)
[37] Nussenzweig H M 1992 Diffraction Effects in Semiclassical Scattering (Cambridge: Cambridge University Press)
[38] Voronin A Yu et al 2006 Phys. Rev. D 73 044029