STRICHARTZ ESTIMATES FOR N-BODY SCHRODINGER OPERATORS WITH SMALL POTENTIAL INTERACTIONS

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ABSTRACT. In this paper, we prove Strichartz estimates for N-body Schrödinger operators, provided that interaction potentials are small enough. Our tools are new Strichartz estimates with frozen spatial variables, and its improvement in the $V_p^S$-norm of Koch and Tataru [19]. As an application, we prove scattering for N-body Schrödinger operators.

1. Introduction. Let $d \geq 3$. We consider the N-body Schrödinger equation in $\mathbb{R}^d$,

$$i\partial_t u = H_N u,$$

where

$$H_N = -\Delta_x + V_N = -\sum_{\alpha=1}^{N} \Delta_{x_\alpha} + \sum_{1 \leq \alpha < \beta \leq N} V(x_\alpha - x_\beta),$$

and $x = (x_1, \cdots, x_N) \in \mathbb{R}^{dN} = \mathbb{R}^d \times \cdots \times \mathbb{R}^d$. Under suitable assumptions, $H_N$ is a self-adjoint operator, and thus Stone’s theorem implies that the equation is globally well-posed in $L^2_x$, and that the N-body Schrödinger propagator is unitary, i.e.,

$$|e^{-itH_N} u_0|_{L^2_x} = |u_0|_{L^2_x}, \quad \forall t \in \mathbb{R}.$$

The purpose of this article is to investigate time decay properties of solutions to the N-body Schrödinger equation [1]

When there is no potential interaction, using the explicit integral kernel formula for the free Schrödinger propagator $e^{it\Delta_x}$, one can show the dispersive estimate

$$|e^{it\Delta_x} u_0|_{L^2_x} \lesssim \frac{1}{|t|^\frac{d}{2}} |u_0|_{L^2_x}.$$

As a consequence, the space-time norm bounds, namely Strichartz estimates,

$$|e^{it\Delta_x} u_0|_{L^q_{t,x} \cap L^r_x} \lesssim |u_0|_{L^2_x},$$

(2)
hold for all $dN$-dimensional admissible pairs $(q, r)$ \cite{16}. Here, an exponent pair $(q, r)$ is called $D$-dimensional admissible if $2 \leq q, r \leq \infty$, $(q, r, D) \neq (2, \infty, 2)$ and

$$\frac{2}{q} + \frac{D}{r} = \frac{D}{2}.$$ 

In a physical perspective, it is natural to ask whether similar estimates hold in the presence of interaction potentials. In the one-body case, the research on the decay properties of the propagator $e^{it(\Delta_0 - V(x))}$ has a long history \cite{12, 11, 10, 9, 8, 7, 6, 5, 4} (see Section 3 for the definition and the properties). Here, an exponent pair $(q, r)$ is called $D$-dimensional admissible if $2 \leq q, r \leq \infty$, $(q, r, D) \neq (2, \infty, 2)$ and

$$\frac{2}{q} + \frac{D}{r} = \frac{D}{2}.$$ 

Moreover, for any $dN$-dimensional admissible pair $(q, r)$, we have

$$|e^{-itH_N}u_0|_{L^q_{t,x} L^r_{\alpha,x}} \lesssim |u_0|_{L^2_x}.$$ 

Indeed, we will prove an even stronger space-time norm bound in the $V^p_{\Delta_\alpha}$-norm introduced by Koch and Tataru \cite{19} (see Section 4 for the definition and the properties).

**Theorem 1.2** (Strichartz estimates for an $N$-body Schrödinger operator in the $V^p_{\Delta_\alpha}$-norm). Let $d \geq 3$ and $p \in (1, 2)$. There exists a small number $\epsilon > 0$ such that if $|V|_{L_{\frac{d}{2},x}} \leq \frac{\epsilon}{N^2}$, then

$$|1_{[0, +\infty)} e^{-itH_N}u_0|_{V^p_{\Delta_\alpha}} \lesssim |u_0|_{L^2_x}.$$ 

As an application, we prove scattering for an $N$-body Schrödinger operator with rough small interactions.

**Corollary 1** (Scattering). Let $d \geq 3$ and, let $\epsilon > 0$ be a small constant given in Theorem 1.2. If $|V|_{L_{\frac{d}{2},x}} \leq \frac{\epsilon}{N^2}$, then for each $u_0 \in L^2_x$, there exist scattering states $u_{\pm}$ such that

$$\lim_{t \to \pm \infty} |e^{-itH_N}u_0 - e^{it\Delta_\alpha}u_{\pm}|_{L^2_x} = 0.$$ 

We prove Strichartz estimates for an $N$-body Schrödinger operator via the endpoint Strichartz estimates for the free linear propagator. Motivated by the simple proof of Strichartz estimates for $e^{it(\Delta_0 - V(x))}$ in Section 2, we introduce the Strichartz estimates for the $N$-particle free propagator $e^{it\Delta_\alpha}$ freezing $(N - 1)$ spatial variables (Proposition 1). However, as described in Section 3, such estimates are not sufficient due to lack of symmetries of the space-time norms with respect to interchanges of
variables. In Section 4 and 5, we solve this problem using the $V^p_\Delta$-norm, particularly taking advantage of the symmetry of the norm. The novelty of this article is to use of the $U^p$- and $V^p$-spaces for many-body systems.

Our main result is the first step to study dispersive and Strichartz estimates for $N$-body Schrödinger operators with general short range interactions, as well as the related scattering problem \cite{23,12}. The spectral properties of an $N$-body Schrödinger operator are much more complicated without smallness assumption. Nevertheless, our approach might be useful to include large interactions in the future.

We also note that other Strichartz estimates, introduced in this paper, can be employed for general dispersive equations having several spatial variables. For instance, Strichartz estimates with frozen variables (Proposition 1) are used in the study of dispersive equations in the Heisenberg picture \cite{5}. In \cite{7}, X. Chen and Holmer proved the Klainerman-Machedon conjecture on the BBGKY hierarchy, which is a key step for rigorous derivation of the nonlinear Schrödinger equation \cite{18,17,6}. One of their key estimates is the many-particle Strichartz estimates in the so-called Bourgain space $X^{s,b}$ \cite{7, Lemma 4.1}. The Strichartz estimates in the $V^p_\Delta$ (in Section 4) sharpen the bounds in $X^{s,b}$ by $0^+$ in that Strichartz estimates in the $X^{s,b}$ space do not cover the endpoint Strichartz estimates, while those in the $V^p_\Delta$-space do.

2. Single-particle case. The easiest way of proving Strichartz estimates for a linear propagator in the presence of small potentials is to use the endpoint Strichartz estimates for the free linear propagator. In this section, in order to illustrate this strategy, we provide a proof of Strichartz estimates in the single-particle case.

**Theorem 2.1** (Strichartz estimate for $e^{it(\Delta-V)}$). Let $d \geq 3$, and let $c_0 > 0$ be the constant given in Theorem 2.2. If $|V|_{L^{\frac{d}{2},\infty}} < \frac{1}{c_0}$, then

$$|e^{it(\Delta-V)}u_0|_{L^q_tL^r_x} \leq \frac{c_0}{1 - c_0 |V|_{L^{\frac{d}{2},\infty}}} |u_0|_{L^2}$$

for all $d$-admissible pair $(q,r)$.

For the proof, recall the Strichartz estimates for the free linear propagator $e^{it\Delta}$ in \cite{10}.

**Theorem 2.2** (Strichartz estimates). There exists $c_0 > 0$ such that for $d$-admissible pairs $(q,r)$ and $(\tilde{q},\tilde{r})$,

$$|e^{it\Delta}u_0|_{L^q_{t,x}L^r_{x}} \leq c_0 |u_0|_{L^2},$$

$$\left| \int e^{-it\Delta} F(s)ds \right|_{L^2_x} \leq c_0 |F|_{L^q_{t,x}L^{r'}_{x}},$$

$$\left| \int e^{it(\Delta-V)} F(s)ds \right|_{L^q_{t,x}L^{r'}_{x}} \leq c_0 |F|_{L^\tilde{q}_{t}L^{\tilde{r}'}_{x}},$$

where $L^{r,s}$ is the Lorentz norm (see \cite{2} for the definition).

**Proof of Theorem 2.1**. It suffices to prove the theorem in the endpoint case $(q,r) = (2, \frac{2d}{d-2})$, since the full set of Strichartz estimates can be obtained by interpolating with the trivial bound $|e^{it(\Delta-V)}u_0|_{L^q_{t,x}L^2} = |u_0|_{L^2}$. The proof of this case follows the same line of argument as in the free case, where the endpoint Strichartz estimates are used.

**Proof of Theorem 2.2**. The proof of the general case follows a similar strategy as in the free case. The main difference lies in the treatment of the potential $V$, which requires additional care due to its interaction with the linear propagator. The bound $|V|_{L^{\frac{d}{2},\infty}}$ plays a crucial role in controlling the growth of the solution.
Let \( u(t) = e^{it(\Delta - V)}u_0 \). Then, it solves the integral equation
\[
 u(t) = e^{it\Delta}u_0 - i \int_0^t e^{i(t-s)\Delta}(Vu(s))\,ds.
\]
Applying the endpoint Strichartz estimates (Theorem 2.2) to the Duhamel formula and the Hölder inequality in the Lorentz spaces (see [2]), we get
\[
 |u(t)|_{L_{x,2}^{\infty}L_{t,2}^{\frac{2d}{d-2}}} \leq c_0 |u_0|_{L^2} + c_0 |V u(t)|_{L_{x,2}^{\infty}L_{t,2}^{\frac{2d}{d-2}}} \leq c_0 |u_0|_{L^2} + c_0 |V|_{L_{t,2}^{\frac{4}{d-2}, \infty}} |u(t)|_{L_{x,2}^{\infty}L_{t,2}^{\frac{2d}{d-2}}}.
\]
(3)

Thus, the endpoint Strichartz estimate for \( e^{it(\Delta - V)} \) follows from the smallness assumption and the embedding \( L_r^{r,2} \hookrightarrow L^r \) for \( r \geq 2 \).

3. Strichartz estimates for \( e^{it\Delta x} \) with frozen spatial variables. We observed that the endpoint Strichartz estimates for the free linear propagator imply Strichartz estimates for a linear propagator with a small perturbation. However, it is easy to see from the Duhamel formula (see [4] below) that the \( dN \)-dimensional endpoint Strichartz estimates [2] do not suit well in the \( N \)-body case. Instead, the following Strichartz estimates seem more natural for our purpose.

Proposition 1 (Strichartz estimates for \( e^{it\Delta x} \) with frozen variables). Let \( d \geq 3 \), and let \( c_0 > 0 \) be the constant for Theorem 2.2. Then, for \( 1 < \alpha < N \) and \( d \)-dimensional admissible pairs \((\tilde{q}, r)\) and \((\tilde{q}, \tilde{r})\), we have
\[
 |e^{it\Delta x}u_0|_{L_{x,\tilde{q}}^{\infty}L_{t,\tilde{r}}^{\frac{2d}{d-2}}} \leq c_0 |u_0|_{L^2},
\]
\[
 \left| \int_{\mathbb{R}} e^{-is\Delta x}F(s)ds \right|_{L_{t,\tilde{r}}^{\infty}L_{x,\tilde{r}}^{2}} \leq c_0 |F|_{L_{x,\tilde{r}}^{\infty}L_{t,\tilde{r}}^{\frac{2d}{d-2}}},
\]
\[
 \left| \int_0^t e^{i(t-s)\Delta x}F(s)ds \right|_{L_{x,\tilde{r}}^{\infty}L_{t,\tilde{r}}^{2}} \leq c_0 |F|_{L_{x,\tilde{r}}^{\infty}L_{t,\tilde{r}}^{\frac{2d}{d-2}}},
\]
where \( \tilde{x}_\alpha = (x_1, x_2, \cdots, x_{\alpha-1}, x_{\alpha+1}, \cdots, x_N) \in \mathbb{R}^{d(N-1)} \).

Proof. The proof is identical to that of [5 Theorem 3.1], but we give a proof for completeness of the paper.

We identify a complex-valued function \( f(x) : \mathbb{R}^d \rightarrow \mathbb{C} \) of \( N \)-spatial variables in \( L_{x,\tilde{r}}^{2}\), with the function-valued function \( f(x_\alpha) : \mathbb{R}^{d(\tilde{r})} \rightarrow L_{\tilde{x}_\alpha}^{2} \) of one-spatial variable in \( L_{\tilde{x}_\alpha}^{\frac{2d}{d-2}} \). Let \( r \geq 2 \). Using unitarity of the linear propagator \( e^{it\Delta x} \) in \( L_{\tilde{x}_\alpha}^{2} \),
\[
 |e^{it\Delta x}u_0|_{L_{t,\tilde{r}}^{\infty}(L_{\tilde{x}_\alpha}^{2})} = |e^{it\Delta x u_0}|_{L_{t,\tilde{r}}^{\infty}(L_{\tilde{x}_\alpha}^{2})} \leq |e^{it\Delta x u_0}|_{L_{\tilde{x}_\alpha}^{2}(L_{\tilde{r}}^{\tilde{r}})}.
\]
Then, by the dispersive estimate
\[
 |e^{it\Delta x} f|_{L_{\tilde{x}_\alpha}^{\frac{2d}{d-2}}} \leq \frac{1}{|t|^{\frac{d}{2}-\frac{d}{d-2}}} |f|_{L_{\tilde{x}_\alpha}^{\frac{2d}{d-2}}},
\]
we obtain
\[
 |e^{it\Delta x}u_0|_{L_{t,\tilde{r}}^{\infty}(L_{\tilde{x}_\alpha}^{2})} \leq \frac{1}{|t|^{\frac{d}{2}-\frac{d}{d-2}}} |u_0|_{L_{x,\tilde{r}}^{\infty}L_{x,\tilde{r}}^{\frac{2d}{d-2}}} \leq \frac{1}{|t|^{\frac{d}{2}-\frac{d}{d-2}}} |u_0|_{L_{x,\tilde{r}}^{\infty}(L_{\tilde{x}_\alpha}^{2})}.
\]
The proposition follows from Theorem 10.1 in [16] with \( B_0 = L_{x_\alpha}^{2}(L_{\tilde{x}_\alpha}^{2}) \), \( B_1 = L_{x_\alpha}^{1}(L_{\tilde{x}_\alpha}^{2}) \), \( H = L_{x_\alpha}^{2}(L_{\tilde{x}_\alpha}^{2}) \) and \( \sigma = \frac{d}{2} \).
We denote by $u(t) = e^{it\Delta_s} u_0 - t \int_0^t e^{i(t-s)\Delta_s} V_N u(s) ds$. \hfill (4)

We define the rotation operator $R_{\alpha,\beta}$ by

$$R_{\alpha,\beta}(f(x_1, \cdots, x_\alpha, -x_\alpha; x_{\alpha+1}, \cdots; x_{N})) = f(x_1, \cdots, x_N).$$

Then, by Proposition 1 we can estimate one term in the integral term in (4),

$$
\int_0^t e^{[t-s]_{\Delta_s}} V_N u(s) ds = \sum \int_0^t e^{[t-s]_{\Delta_s}} V(x_\alpha - x_\beta) u(s) ds,
$$

with the exponents in (3) as follows:

$$
\begin{align*}
|\tilde{R}_{\alpha_0, \beta_0} & \int_0^t e^{[t-s]_{\Delta_s}} V(x_\alpha - x_\beta) u(s) ds |_{L_{2,1}^{\frac{2d}{\alpha_0}} L_{\infty}^{\frac{2d}{\beta_0}}} \leq c_0 |V|_{L_{\frac{d}{\alpha}}^{\frac{2d}{\alpha}}} |\tilde{R}_{\alpha_0, \beta_0} u(t)|_{L_{2,1}^{\frac{2d}{\alpha_0}} L_{\infty}^{\frac{2d}{\beta_0}}} \\
& \leq \frac{c_0}{2} |V|_{L_{\frac{d}{\alpha}}^{\frac{2d}{\alpha}}} |\tilde{R}_{\alpha_0, \beta_0} u(t)|_{L_{2,1}^{\frac{2d}{\alpha_0}} L_{\infty}^{\frac{2d}{\beta_0}}}.
\end{align*}
$$

where in the first identity, we used that the linear propagator $e^{it\Delta_s}$ commutes with a rotation. However, one cannot estimate other integral terms, with $(\alpha, \beta) \neq (\alpha_0, \beta_0)$, in (3) by the same norm $|\tilde{R}_{\alpha_0, \beta_0} \cdot |_{L_{2,1}^{\frac{2d}{\alpha_0}} L_{\infty}^{\frac{2d}{\beta_0}}}$. \hfill (7)

4. Strichartz estimates for $e^{it\Delta_s}$ in the $V_{\Delta_s}^p$-norm. To overcome the problem mentioned in the previous section, we look for a space-time norm that plays the role of the rotated space-time norm $|\tilde{R}_{\alpha_0, \beta_0} \cdot |_{L_{2,1}^{\frac{2d}{\alpha_0}} L_{\infty}^{\frac{2d}{\beta_0}}}$. \hfill (7)

4.1. A brief review of the $\mathcal{U}_{S_{\alpha}}^p$ and $\mathcal{V}_{S_{\alpha}}^p$-spaces. For readers’ convenience, we give a brief review of the $\mathcal{U}_{S_{\alpha}}^p$ and $\mathcal{V}_{S_{\alpha}}^p$-spaces. We refer the reader to \cite{14} \cite{20} for details.

Let $H$ be a separable Hilbert space, and let $\mathcal{Z}$ be the collection of finite partitions \{tk\}_{k=0}^K of the whole time interval $[\infty, 0]$ with

$$-\infty = t_0 < t_1 < \cdots < t_K = 0.$$ 

We denote by $u(\pm \infty)$ the limit of $u(t)$ as $t \to \pm \infty$ if it exists.

Let $1 \leq p < \infty$. We call $a(t) : \mathbb{R} \to H$ a $U_{S_{\alpha}}^p$-atom if

$$a(t) = \sum_{k=1}^K \phi_k \mathbf{1}_{[t_{k-1}, t_k]},$$

where $\phi_k$ is a measurable function on $[0, 1]$. \hfill (8)
where \( \{t_k\}_{k=0}^K \in \mathbb{Z}, \{\phi_k\}_{k=0}^{K-1} \subset H, \phi_0 = 0 \) and \( \sum_{k=0}^{K-1} |\phi_k|_H^p = 1 \). We define the atomic space by
\[
U^p := \left\{ u = \sum_{j=1}^\infty \lambda_j a_j : a_j \text{ } U^p\text{-atom, } \lambda_j \in \mathbb{C}, \sum_{j=1}^\infty |\lambda_j| < \infty \right\}
\]
with the norm
\[
|u|_{U^p} := \inf \left\{ \sum_{j=1}^\infty |\lambda_j| : u = \sum_{j=1}^\infty \lambda_j a_j, \lambda_j \in \mathbb{C}, a_j \text{ } U^p\text{-atom} \right\}.
\]

**Lemma 4.1** (Proposition 2.2 in [13]). Let \( 1 \leq p < q < \infty \).
(i) \( U^p \) is a Banach space.
(ii) The embeddings \( U^p \hookrightarrow U^q \hookrightarrow L^q(\mathbb{R}; H) \) are continuous.
(iii) Every \( u \in U^p \) is right-continuous, \( u(-\infty) = 0 \) and \( u(\infty) \) exists.

Let \( 1 \leq p < \infty \). We define \( V^p \) by the collection of all functions \( v : \mathbb{R} \to H \) such that \( v(\infty) = 0 \) and \( v(-\infty) \) exists, equipped with the norm
\[
|v|_{V^p} := \sup_{\{t_k\}_{k=0}^K \in \mathbb{Z}} \left\{ \sum_{k=1}^K \left( |v(t_k) - v(t_{k-1})|_H^p \right)^{1/p} \right\}.
\]
We define \( V_0^p \) by the space of all functions \( v : \mathbb{R} \to H \) such that \( v(-\infty) = 0 \) and \( v(\infty) \) exists, equipped with the same norm, and denote by \( V_{0,rc}^p \) the closed subspace of all right continuous functions in \( V_0^p \) with respect to the \( V^p \)-norm.

**Lemma 4.2** (Proposition 2.4 and Corollary 2.6 in [13]). Let \( 1 \leq p < q < \infty \).
(i) \( V^p, V_0^p, V_{0,rc}^p \) are Banach spaces.
(ii) The embedding \( U^p \hookrightarrow V_{0,rc}^p \) is continuous.
(iii) The embedding \( V_{0,rc}^p \hookrightarrow V_{0,rc}^q \) is continuous.
(iv) The embedding \( V_{0,rc}^p \hookrightarrow U^q \) is continuous.

**Lemma 4.3** (Duality; Proposition 2.7 and Theorem 2.8 in [13]). Let \( 1 < p < \infty \).
(i) For \( u \in U^p \) and \( v \in V^p \) and a partition \( t := \{t_k\}_{k=0}^K \in \mathbb{Z} \), we define
\[
B_t(u,v) := \sum_{k=1}^K \langle u(t_{k-1}), v(t_k) - v(t_{k-1}) \rangle_H.
\]
There exists a unique complex number \( B(u,v) \) such that for any \( \epsilon > 0 \), there exists \( t \in \mathbb{Z} \) such that for every \( v' \supset t \),
\[
|B_t(u,v) - B(u,v)| < \epsilon.
\]
The bilinear form \( B : U^p \times V^p \to \mathbb{C} \) satisfies
\[
|B(u,v)| \leq |u|_{U^p} |v|_{V^p}.
\]
(ii) \( (U^p)^* = V^p \) in the sense that \( T : V^p \to (U^p)^* \), defined by \( T(v) = B(\cdot,v) \), is an isometric isomorphism.

Let \( 1 \leq p < \infty \). For a self-adjoint operator \( S \), we define \( U_0^p \) and \( V_{0,rc,S}^p \) respectively by \( e^{itS}U^p \) and \( e^{itS}V_{0,rc}^p \) respectively with the norm
\[
|u|_{U_0^p} := |e^{-itS}u|_{U^p}, \quad |u|_{V_{0,rc,S}^p} := |e^{-itS}u|_{V_{0,rc}^p}, \quad (8)
\]
In this paper, we choose \( H = L^2_x \) and \( S = \Delta_x \). For notational convenience, we denote \( V_{0,rc,S}^p \) and the self-adjoint operator \( S \).
4.2. Strichartz estimates for \( e^{it\Delta x} \) in the \( V_{\Delta x}^p \)-norm. We now establish Strichartz estimates in the \( V_{\Delta x}^p \)-norm, which are the key estimates to prove the main theorem. In the literature, the \( U_{\Delta x}^p \)-space is typically chosen to be the solution space of the given equation. However, we employ the \( V_{\Delta x}^p \)-space for the space of solutions to the N-body Schrödinger equation [1], because it is easier to prove the integral estimate (Proposition 3). Indeed, by Lemma 4.1 and 4.2, one can replace the \( V_{\Delta x}^p \)-norms in the following propositions by \( U_{\Delta x}^p \)-norms with arbitrarily small loss of the exponent \( p \), and such losses are acceptable for our purpose.

First, we prove homogeneous Strichartz estimate in \( V_{\Delta x}^p \).

**Proposition 2** (Homogeneous Strichartz estimate in \( V_{\Delta x}^p \)). Let \( d \geq 1 \) and \( 1 \leq p < \infty \). Then,

\[
|1_{[0, +\infty]} e^{it\Delta x} u_0|_{V_{\Delta x}^p} = |u_0|_{L_x^2}.
\]

**Proof.** The total variation of \( e^{-it\Delta x}(1_{[0, +\infty]} e^{it\Delta x} u_0) = 1_{[0, +\infty]} u_0 \) is simply \( u_0 \). Hence, by definition, we have \(|1_{[0, +\infty]} e^{it\Delta x} u_0|_{V_{\Delta x}^p} = |u_0|_{L_x^2} \). \( \square \)

Next, we prove the transference principle.

**Proposition 3** (Transference principle). Let \( d \geq 1 \), \( p \in (1, 2) \), \( q \geq 2 \) and \( X \) be a Banach space. If a function \( u : \mathbb{R} \to X \) satisfies the bound

\[
|e^{it\Delta x} u_0|_{L^{q,\infty}_x} \lesssim |u_0|_{L_x^2},
\]

then

\[
|u|_{L^{q,\infty}_x} \lesssim |u|_{V_{\Delta x}^p}.
\]

**Proof.** By Lemma 4.2 (ii) and density, it suffices to show that

\[
|u|_{L^{q,\infty}_x} \lesssim |u|_{L_x^2}
\]

for an \( U_{\Delta x}^p \)-atom \( a(t) = \sum_{k=1}^K 1_{[t_k-\epsilon, t_k]} e^{it\Delta x} \phi_{k-1} \). Using that the interval \([t_{k-1}, t_k)\)'s are mutually disjoint, we write

\[
|a|_{L^{q,\infty}_x} \lesssim \left\{ \sum_{k=1}^K |e^{it\Delta x} \phi_{k-1}|_{L_x^q} \right\}^{1/q}.
\]

Then, by the assumption and the embedding \( \ell^2 \hookrightarrow \ell^q \) for \( q \geq 2 \), it follows that

\[
|a|_{L^{q,\infty}_x} \lesssim \left\{ \sum_{k=1}^K |\phi_{k-1}|_{L_x^2}^q \right\}^{1/q} \lesssim \left\{ \sum_{k=1}^K |\phi_{k-1}|_{L_x^2}^2 \right\}^{1/2} = 1.
\]

\( \square \)

As a consequence, we show that the \( V_{\Delta x}^p \)-norm dominates the space-time norms in Proposition 1 and 2.

**Corollary 2.** Let \( d \geq 1 \), \( p \in (1, 2) \), and \( R \) be a rotation in \( \mathbb{R}^{3N}_x \). Then, we have

\[
|R u|_{L^{q,\infty}_x L^{r,2}_t L_{\Delta x}^2} \leq c_1 |u|_{V_{\Delta x}^p}
\]

for a \( d \)-dimensional admissible pair \((q, r)\) and \( 1 \leq \alpha \leq N \). Moreover,

\[
|R u|_{L^{q,\infty}_x L^{r,2}_t} \leq c_1 |u|_{V_{\Delta x}^p}
\]

for a \( dN \)-dimensional admissible pair \((q, r)\).

Finally, we prove the integral estimate analogous to (7).
Proposition 4 (Integral estimate in $V^p_{\Delta s}$). Let $d \geq 3$ and $p \in (1, 2)$. Then, we have
\[
\left| 1_{[0,+\infty)} \int_0^t e^{i(t-s)\Delta s}(V(x_\alpha - x_\beta)u(s))ds \right|_{V^p_{\Delta s}} \leq \frac{c_0c_1}{2} |V|_{L^\infty} |u|_{V^p_{\Delta s}}.
\]

Proof. For notational convenience, we denote
\[
w(t) := 1_{[0, +\infty)} \int_0^t e^{-is\Delta s}(V(x_\alpha - x_\beta)u(s))ds.
\]

We will estimate $w$ by duality. Since we only expect $w \in V^p_{\Delta s}$, not $w \in V^p$, we consider $\tilde{w}(t) = w(-t)$. Note that $w \in V^p_{\Delta s}$ if and only if $\tilde{w}(t) \in V^p$, and that by Lemma 4.3 (ii),
\[
|w|_{V^p} = |\tilde{w}|_{V^p} = \sup_{|s|_{\mu, \beta} \leq 1} B(u, w).
\]

Hence, by Lemma 4.3 (i) and density, it suffices to show that
\[
B_t(a, \tilde{w}) = \sum_{j=1}^k \langle a(t_{j-1}), \tilde{w}(t_j) - \tilde{w}(t_{j-1}) \rangle_{L^2} \leq \frac{c_0c_1}{2} |V|_{L^\infty} |u|_{V^p_{\Delta s}}
\]
for any fine partition of unity $t = \{t_j\}_{j=0}^K \in \mathbb{Z}$ and any $U^d$-atom $a(t) = \sum_{k=1}^K 1_{[s_k, s_{k+1})} \phi_{k-1}$. Here, including $s_0, \cdots, s_K$ if necessary, we may assume that $t = \{t_j\}_{j=0}^K$ contains $s_0, \cdots, s_K$. Moreover, due to the cut-off $1_{[0, +\infty)}$ in (9), we also assume that $t_0 = -\infty$, $t_J = \infty$ and $t_1, \cdots, t_{j-1} \in (-\infty, 0)$.

By the assumptions, for each $k$, there exist $t_k, t_m \in t$ such that
\[
\cdots < t_{k-1} < t_k < t_{k+1} < t_{k+2} < \cdots < t_{m-1} < t_m = s_k < t_{m+1} < \cdots
\]
and $a(t_k) = a(t_{k+1}) = \cdots = a(t_{m-1}) = \phi_{k-1}$. Hence, we have
\[
\sum_{j=t_k+1}^m \langle a(t_{j-1}), \tilde{w}(t_j) - \tilde{w}(t_{j-1}) \rangle_{L^2} = \langle \phi_{k-1}, \tilde{w}(s_k) - \tilde{w}(s_{k-1}) \rangle_{L^2}.
\]

Inserting this sum into $B_t(a, \tilde{w})$ in (10), we get a simpler sum
\[
B_t(a, \tilde{w}) = \sum_{k=1}^K \langle \phi_{k-1}, \tilde{w}(s_k) - \tilde{w}(s_{k-1}) \rangle_{L^2}.
\]

By (9) with $\tilde{w}(t) = w(-t)$, unitarity of the linear propagator $e^{is\Delta s}$ and change of the variables by the rotation $R_{\alpha \beta}$, we write
\[
B_t(a, \tilde{w}) = -\sum_{k=1}^K \int_{s_k}^{s_{k-1}} \langle \phi_{k-1}, e^{-is\Delta s}(V(x_\alpha - x_\beta)u(s)) \rangle_{L^2} ds
\]
\[
= -\sum_{k=1}^K \int_{s_k}^{s_{k-1}} \langle e^{is\Delta s} \phi_{k-1}, V(x_\alpha - x_\beta)u(s) \rangle_{L^2} ds
\]
\[
= -\sum_{k=1}^K \int_{s_k}^{s_{k-1}} \langle e^{is\Delta s} R_{\alpha \beta} \phi_{k-1}, V(\sqrt{2}x_\alpha) R_{\alpha \beta} u(s) \rangle_{L^2} ds
\]
\[
= -\sum_{k=1}^K \int_{s_k}^{s_{k-1}} \langle e^{is\Delta s} R_{\alpha \beta} \phi_{k-1}, V(\sqrt{2}x_\alpha) 1_{[-s_k, -s_{k-1})} R_{\alpha \beta} u(s) \rangle_{L^2} ds,
\]
where in the last step, we used that \( u \in V^p_{\Delta x} \) is right-continuous. Then, applying the Hörder inequality, Proposition 1 and Corollary 2, we obtain

\[
\begin{align*}
B_t(a, \tilde{w}) & \leq \sum_{k=1}^{K} |V(\sqrt{2})|_{L^\infty} \cdot |e^{it\Delta x} R_{\alpha \beta} \phi_{k-1}|_{L^2_{\alpha \rho} L^2_{\xi \rho}} \cdot |1_{[-s_k, s_k]} \cdot | R_{\alpha \beta} u|_{L^2_{\alpha \rho} L^2_{\xi \rho}} \\
& \leq \frac{1}{2} \sum_{k=1}^{K} |V|_{L^2_{\alpha \rho}} \cdot c_0 |\phi_{k-1}|_{L^2_{\xi \rho}} \cdot |1_{[-s_k, s_k]} u|_{V^p_{\Delta x}} \\
& \leq \frac{c_0 c_1}{2} |V|_{L^2_{\alpha \rho}} \left\{ \sum_{k=1}^{K} |\phi_{k-1}|_{L^2_{\xi \rho}} \right\}^{1/p} \left\{ \sum_{k=1}^{K} |1_{[-s_k, s_k]} u|_{V^p_{\Delta x}} \right\}^{1/p} \\
& = \frac{c_0 c_1}{2} |V|_{L^2_{\alpha \rho}} \left\{ \sum_{k=1}^{K} |1_{[-s_k, s_k]} u|_{V^p_{\Delta x}} \right\}^{1/p}.
\end{align*}
\]

It remains to show that

\[
\left\{ \sum_{k=1}^{K} |1_{[-s_k, s_k]} u|_{V^p_{\Delta x}} \right\}^{1/p} \leq |u|_{V^p_{\Delta x}}.
\]

Indeed, by the definition of the norm, given \( \epsilon > 0 \) and \( k \), there exists a partition \( t^k = \{ t_{j_k} \}_{j_k=1}^{J_k} \) in the interval \( [-s_k, s_{k-1}] \) such that

\[
|1_{[t_{j-1}, t_j]} u|^P_{V^p_{\Delta x}} = \sum_{j_k=1}^{J_k} \left| e^{it_{j-1} \Delta x} u(t_{j_k}) - e^{it_{j-1} \Delta x} u(t_{j_k-1}) \right|^p_{L^2} + \epsilon.
\]

Since \( \bigcup_{k=1}^{K} t^k \) is also a partition of unity of \( \mathbb{R} \), by the definition of the norm, we have

\[
\sum_{k=1}^{K} |1_{[t_{j-1}, t_j]} u|^P_{V^p_{\Delta x}} = \sum_{k=1}^{K} \sum_{j_k=1}^{J_k} \left| e^{it_{j-1} \Delta x} u(t_{j_k}) - e^{it_{j-1} \Delta x} u(t_{j_k-1}) \right|^p_{L^2} + \epsilon
\]
\[
\leq |u|^p_{V^p_{\Delta x}} + \epsilon.
\]

Since \( \epsilon \) is arbitrary, this completes the proof.

5. Proof of Theorem 1.2 and Corollary 1. Now, we are ready to prove our main results.

Proof of Theorem 1.1 and 1.2. Applying Proposition 2 and 3 to the Duhamel formula (1) for \( u(t) = e^{-itH_N} u_0 \), we get

\[
|1_{[0, + \infty)} u(t)|_{V^p_{\Delta x}} \leq |u_0|_{L^2} + \frac{N(N-1)}{2} \cdot \frac{c_0 c_1}{2} |V|_{L^2_{\alpha \rho}} |1_{[0, + \infty)} u(t)|_{V^p_{\Delta x}}.
\]

Theorem 1.2 follows from the smallness assumption. Then, Theorem 1.1 follows from Corollary 2.

Proof of Corollary 1. We consider only for the positive time. It suffices to show that

\[
u_+ := \lim_{t \to +\infty} e^{-it\Delta x} e^{-itH_N} u_0
\]
exists in $L^2_x$ as $t \to +\infty$. Indeed, by the Duhamel formula \[4\],
\[
|e^{-it_2\Delta_x}e^{-it_2H_Nu_0} - e^{-it_1\Delta_x}e^{-it_1H_Nu_0}|_{L^2_x} \leq \sum_{1 \leq \alpha, \beta \leq N} \left| \int_{t_1}^{t_2} e^{-is\Delta_x} (V(x_{\alpha} - x_{\beta})e^{-isH_Nu_0}) ds \right|_{L^2_x}.
\]
By the rotation $R_{\alpha\beta}$, Proposition \[1\] and Corollary 1.2, we prove that
\[
\left| \int_{t_1}^{t_2} e^{-is\Delta_x} (V(x_{\alpha} - x_{\beta})e^{-isH_Nu_0}) ds \right|_{L^2_x} = \left| R_{\alpha\beta} \int_{t_1}^{t_2} e^{-is\Delta_x} (V(x_{\alpha} - x_{\beta})e^{-isH_Nu_0}) ds \right|_{L^2_x} = \left| \int_{t_1}^{t_2} e^{-is\Delta_x} (V(\sqrt{2}x_{\alpha})(R_{\alpha\beta}e^{-isH_Nu_0})) ds \right|_{L^2_x} \leq c_0 \left| V(\sqrt{2}x_{\alpha})R_{\alpha\beta}e^{-isH_Nu_0} \right|_{L^2_{e^{|x_{\alpha}|^2 |t_1-t_2|^{2}}}e^{x_{\alpha}^2 |t_2-t_1|^2}} \rightarrow 0
\]
as $t_1, t_2 \to \infty$. Thus, we conclude that the limit exists. \hfill \square

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