Parity violating elastic electron scattering from $^{27}\text{Al}$ and the Qweak measurement

C. J. Horowitz\textsuperscript{1}\textsuperscript{*}

\textsuperscript{1}Department of Physics and Nuclear Theory Center, Indiana University, Bloomington, IN 47405, USA

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Parity violating elastic electron scattering from a heavy nucleus is interesting for several reasons. First, it is sensitive to neutron distributions, because the weak charge of a neutron is much larger than that of a proton [1, 2]. The PREX experiment at Jefferson Laboratory has pioneered using parity violation to measure the neutron density of $^{208}\text{Pb}$ [3, 4]. This result will be improved with a follow-up Qweak measurement [5], and the approved CREX experiment will measure the neutron radius of $^{48}\text{Ca}$ [6, 7]. Note that neutron densities can also be measured, in principle, with elastic neutrino scattering [8, 9], a process that could be important in astrophysics [10].

Second, parity violation can be used to test the standard model, [11, 12]. For example, a precision electron scattering experiment could measure the weak charge of $^{12}\text{C}$. This is like an atomic parity measurement without many of the atomic structure uncertainties. Electron scattering could also probe radiative corrections, such as coulomb distortions [13], and other nuclear structure effects [14].

Recently, the Qweak experiment is measuring the parity violating asymmetry $A_{pv}$ for electrons scattering from $^{27}\text{Al}$ [15, 16]. This experiment primarily measures $A_{pv}$ from hydrogen to determine the weak charge of the proton and test the standard model. However, the Qweak hydrogen target has $^{27}\text{Al}$ windows. These windows are an important source of background because $A_{pv}$ for $^{27}\text{Al}$ is much larger than that for hydrogen. Therefore a separate precise measurement of $A_{pv}$ for $^{27}\text{Al}$ is being undertaken, in order to subtract the contribution of window scattering from the main hydrogen measurement.

In this paper we calculate $A_{pv}$ for elastic scattering from $^{27}\text{Al}$ in order to compare to the Qweak measurement. While the experimental Qweak background subtraction does not depend directly on theory, our calculations will provide a check of some of the important assumptions involved in the subtraction. For example, if the $^{27}\text{Al}$ measurement disagrees with theory, one could worry that this might be due to a common systematic
correction in the subtraction procedure. For example, if $A_{pv}$ near the diffraction minimum. The Qweak data can be used to measure the neutron radius of $^{27}\text{Al}$, if nuclear structure uncertainties are indeed small, as we suggest, and one can estimate inelastic and impurity contributions. This should provide an important check of the measurement, analysis, and theory.

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In this paper we calculate $A_{pv}$ for elastic scattering from $^{27}\text{Al}$ in order to compare to the Qweak measurement. While the experimental Qweak background subtraction does not depend directly on theory, our calculations will provide a check of some of the important assumptions involved in the subtraction. For example, if the $^{27}\text{Al}$ measurement disagrees with theory, one could worry that this might be due to a common systematic error that could also impact the main hydrogen measurement. Alternatively if the $^{27}\text{Al}$ measurement agrees with theory, this will help support the validity of the background subtraction procedure.

Finally the precision $^{27}\text{Al}$ measurement is interesting in its own right and can be used to measure the neutron radius $R_n$ of $^{27}\text{Al}$. In this paper we calculate the effects of coulomb distortions, discuss the main nuclear structure sensitivities, and present results for the cross section and parity violating asymmetry.

We begin calculating the cross section, first in Born approximation and then including Coulomb distortions. We describe $^{27}\text{Al}$ as a very simple pure $d_{5/2}$ proton hole in a relativistic mean field model using the FSUgold [17] interaction. Table [4] presents proton $R_p$, neutron $R_n$, and charge $R_{PQ}$ radii of this model. The proton density has a spherically symmetric part $\rho_p^0(r)$ and a part that contains the $d_{5/2}$ hole $\rho_p^2(r)$:

$$\rho_p(r) = \rho_p^0(r) + \rho_p^2(r).$$

Explicitly we write out

$$\rho_p^0(r) = \frac{1}{4\pi r^2} \left\{ 2(G_{s_{1/2}}^2 + F_{s_{1/2}}^2) + 4(G_{p_{1/2}}^2 + F_{p_{1/2}}^2) + 2(G_{p_{1/2}}^2 + F_{p_{1/2}}^2) + 5(G_{d_{5/2}}^2 + F_{d_{5/2}}^2) \right\}.$$

where $G_j$ is the Dirac upper component and $F_j$ is the lower component of the proton wave functions for the occupied states $j = 1s_{1/2}, 1p_{1/2}, 1p_{1/2}, 1d_{5/2}$ [15]. Note that there are only 5 protons in the $d_{5/2}$ level. The contribution of the hole is written,

$$\rho_p^2(r) = \frac{1}{4\pi r^2} \left( G_{d_{5/2}}^2 + F_{d_{5/2}}^2 \right) \left( \frac{1}{4\pi} - |Y_{2M}(\hat{r})|^2 \right),$$

where $Y_{2M}$ is a spherical harmonic. These forms are normalized, $\int d^3r \rho_p^0(r) = Z = 13$, and $\int d^3r \rho_p^2(r) = 0$.

We assume that the neutron density is spherically symmetric, $\rho_n(r) = \rho_n^0(r)$, where $\rho_n^0(r)$ has the same form as Eq. (2), with 5 replaced by 6 and the Dirac wave functions are slightly different for the neutron states. The normalization is $\int d^3r \rho_n^0(r) = N = 14$.

*Electronic address: horowitz@indiana.edu
TABLE I: Proton \( R_p \), neutron \( R_n \), and charge \( R_{e^q} \) radii of \(^{27}\text{Al}\) for FSUgold RMF model.
\[
\begin{array}{ccc}
R_p (\text{fm}) & R_n (\text{fm}) & R_{e^q} (\text{fm}) \\
2.904 & 2.913 & 3.013 \\
\end{array}
\]

The proton form factor \( F_p(q) \) is,
\[
F_p(q) = \frac{1}{Z} \int d^3r \left( \rho_p^0(r) + \rho_p^2(r) \right) e^{iqr},
\]
where \( q \) is the momentum transfer.
\[
F_p(q) = \frac{1}{Z} \int d^3r \left( \rho_p^0(r) + \rho_p^2(r) \right) \{ j_0(qr) - \sqrt{20\pi} j_2(qr)Y_{20} + \sqrt{36\pi} j_4(qr)Y_{40} \}.
\]

Here we have expanded the plane wave and only kept terms that make nonzero contributions. Finally, \( j_L \) are spherical Bessel functions. There are contributions for \( L = 0, 2, 4 \). Squaring \( |F_p|^2 \) and averaging over orbital angular momentum projection \( M \) from -2 to 2 yields,
\[
|F_p|^2 = C_0^2 + C_2^2 + C_4^2,
\]
with
\[
C_0(q) = \frac{1}{Z} \int d^3r \rho_p^0(r) j_0(qr),
\]
\[
C_2(q) = \frac{1}{Z} \sqrt{\frac{10}{7}} \int_0^{\infty} dr j_2(qr) (G_{ds/u^2} + F_{ds/u^2}),
\]
and
\[
C_4(q) = \frac{1}{Z} \sqrt{\frac{2}{7}} \int_0^{\infty} dr j_4(qr) (G_{ds/u^2} + F_{ds/u^2}).
\]

The form factor is normalized \( F_p(0) = C_0(0) = 1 \). The neutron form factor only has an \( L = 0 \) contribution, \( F_n(q) = \frac{1}{Z} \int d^3r \rho_p^0(r) j_0(qr) \), and is also normalized \( F_n(0) = 1 \).

In Figure 1 we plot the square of the proton form factors, for \( L = 0, 2, 4 \). We see that the \( L = 0 \) contribution \( C_0^2 \) dominates, except near the diffraction minimum around \( q = 1.4 \text{ fm}^{-1} \) where \( C_2 \) is important. The \( L = 4 \) contribution \( C_4 \) is small. These results are similar to much earlier shell model calculations using harmonic oscillator wave functions. 19.

The cross section \( d\sigma/d\Omega \) for elastic electron scattering can be calculated in Born approximation, see for example ref. 19.
\[
\frac{d\sigma}{d\Omega} = \sigma_m \frac{q^2}{2} F_L^2 \approx \sigma_m F_L^2
\]

Here the Mott cross section is, \( \sigma_m = \frac{Z^2 Q^2 \cos^2 \theta/2}{4E_i^2 \sin^2 \theta/2} \), with \( \theta \) the scattering angle and \( E_i \) the incident electron energy.

\textbf{FIG. 1:} Square of the proton form factors \( C_0^2 \) (solid), \( C_2^2 \) (dashed), and \( C_4^2 \) (dotted), Eqs. (7,8,9), versus momentum transfer \( q \).

The recoil factor is \( \eta = (1 + 2E_i/M \sin^2 \theta/2)^{-1} \approx 1 \) with \( M \) the mass of the nucleus. Finally \( q_n^2 = q^2 - (E_i - E_f)^2 \) with \( E_F \) the final electron energy and the three momentum transfer is \( q^2 = 4E_iE_f \sin^2 \theta/2 + (E_i - E_f)^2 \).

The longitudinal form factor \( F_L \) is \( F_p \) folded with the electric form factor of a single proton \( G_E \). \( F_L(q)^2 = G_E(q)^2 F_p^2 \), however see ref. 20 for a discussion of spin-orbit currents. Equation (10) neglects contributions from transverse currents. For \(^{27}\text{Al}\) these have been calculated in ref. 19 and found to be very small, comparable to \( C_4^2 \) in Fig. 1. Note that in general we do not expect large transverse contributions for the forward angle kinematics of Qweak.

Coulomb distortions can be important near diffraction minima. If one neglects the aspherical \( \rho_p^2(r) \), Coulomb distortions have been calculated, in the usual way, by numerically solving the Dirac equation and summing partial waves. 13. Our procedure is to include coulomb distortions for \( C_0 \) contributions exactly 13 and then simply add \( C_2 \) contributions in Born approximation. Our best estimate for the cross section is,
\[
\frac{d\sigma}{d\Omega} \approx \sigma_m F_L^2 + \sigma_m \xi^2 C_2^2.
\]

where we have neglected \( C_4 \) and transverse contributions. We add a parameter \( \xi \) to include a generous estimate for uncertainties in the nuclear structure and for the effects of coulomb distortions on the \( C_2 \) contribution. Note that we do not expect coulomb distortions to be very important for the \( C_2 \) contribution because it is only relevant for \( q \) near \( 1.4 \text{ fm}^{-1} \), which is far away from the diffraction minimum in \( C_2 \). Allowing \( \xi^2 \) to vary between 0.5 and 1.5 provides a generous uncertainty range that likely includes many far more sophisticated nuclear structure models.
Perhaps the simplest approximation is to assume $\rho_p^2 = 0$ and that the proton and neutron distributions have the same shape so that $F_p(q) = F_n(q)$. In Born approximation $A_{pv}$ is then simply proportional to $q^2$,  

$$A_{pv} = A_{pv0} = -\frac{G_F q^2 Q_W}{4\pi\sqrt{2}aZ}. \quad (13)$$

Here $G_F$ is the Fermi constant and the total weak charge of $^{27}$Al, $Q_W$ is,  

$$Q_W = Q_n N + Q_p Z = -12.8919. \quad (14)$$

The weak charge of the neutron $Q_n$ is -1 at tree level. However including radiative corrections \cite{22,23} we use $Q_n = -0.9878$. The weak charge of the proton $Q_p$ is small. It is $1 - 4\sin^2 \theta_W$ at tree level and we use $Q_p = 0.0721$ with radiative corrections.

Including different neutron and proton distributions, $A_{pv}$ in Born approximation becomes,  

$$A_{pv} = A_{pv0} \frac{C_0^W C_0^W + C_2^W C_2^W + C_4^W C_4^W}{C_0^2 + C_2^2 + C_4^2}. \quad (15)$$

Here the weak Coulomb form factors $C_L^W$ are Fourier transforms of the weak charge density $\rho_W(r)$,  

$$\rho_W(r) \approx Q_n \rho_n(r) + Q_p \rho_p(r). \quad (16)$$

If one defines $C_L^W(q) = \int d^3r \rho_W(r) j_0(qr)/Q_W$ so that $C_0^W(0) = 1$, we have,  

$$C_0^W(q) = \frac{1}{Q_W} \left[ Q_n NF_n(q) + Q_p Z C_0(q) \right]. \quad (17)$$

The weak form factors for $L = 2, 4$ are small because the proton weak charge $Q_p$ is small and we assumed that the neutron density is spherically symmetric. We have $C_2^W(q) = Q_p Z C_2(q)/Q_W$ and $C_4^W = Q_p Z C_4(q)/Q_W$.

It would be very interesting to calculate core polarization corrections to $C_2^W$. The proton hole is expected to polarize the neutron density and this can make a significant contribution to $C_2^W$ because the weak charge of a neutron is much larger than that of a proton. We think the net effect of this core polarization on $A_{pv}$ can be included by reducing the value of $\xi$, see below.

We now include Coulomb distortions for the $C_0$ (and $C_0^W$) contributions as we did for the cross section. We calculate $A_{pv} = A_{DW}(C_0)$ exactly for spherically symmetric weak charge and E+M charge distributions by solving the Dirac equation numerically for an electron moving in an axial vector weak potential (of order a few eV) and the coulomb potential \cite{13}. Then we add the $C_2$ and $C_4^W$ contributions in Born approximation. Our best estimate for $A_{pv}$ is,  

$$A_{pv} \approx \frac{d\sigma}{d\Omega}(C_0)_{|DW} A_{DW}(C_0)_{|DW} + \sigma_m \xi^2 C_2^W A_{pv0}. \quad (18)$$

Note that the second term in the numerator is small because $C_2^W$ and $Q_p$ are small. Therefore the primary impact of the $C_2$ contribution is to increase the denominator, and therefore reduce $A_{pv}$, for $q$ near the diffraction minimum in $C_0$. Equation (18) reproduces the exact distorted wave result if $C_2$ is small and reproduces the full Born approximation result, Eq. (15), when the effects of Coulomb distortions are small.

Figure 5 shows $A_{pv}$ for electrons of energy 1160 MeV (the energy of Qweak) versus scattering angle. Even
for spherically symmetric neutron and proton densities, Coulomb distortions significantly reduce $A_{PV}$, in the diffraction minimum near 13 degrees, compared to the plane wave $A_{PV}^0$ result. Including $C_2$ and $C_2^W$ contributions further reduces $A_{PV}$. Our best estimate for $A_{PV}$ shown by the solid black line, Eq. (18), is only one third of $A_{PV}^0$ in the minimum near 13.5 degrees.

The average momentum transfer of Qweak corresponds to about 7.8 degrees as shown by the red arrow in Fig. 3. At this angle the effects of Coulomb distortions and $C_2$ are small. This suggests that the final uncertainty in acceptance averaged theory results may be small. However the angular acceptance of the Qweak experiment is large, as shown very roughly by the blue arrows in Fig. 3 and includes some acceptance near the diffraction minimum. The cross section falls very rapidly with increasing angle so that only a small fraction of the events may come from angles near the diffraction minimum. Therefore, the acceptance averaged contribution of the large dip in $A_{PV}$ may not be large. Nevertheless it is important to carefully average our $A_{PV}$ predictions, with its complex shape, with the experimental acceptance. Note that our $A_{PV}$ is not proportional to $q^2$. Indeed for angles beyond 11.5 degrees, $A_{PV}$ actually decreases with increasing $q^2$. Therefore one should be somewhat careful in extrapolating a measurement at one $q^2$ to a different $q^2$.

The asymmetry $A_{PV}$ is somewhat sensitive to nuclear structure uncertainties, for scattering angles beyond about 11 degrees. This is shown in Fig. 3 by the dotted error bands which correspond to different $\xi^2$ values. However, this nuclear structure uncertainty is very small at the average $q^2$ near 7.8 degrees. Therefore the remaining nuclear structure uncertainty, by the time one averages over the acceptance, may be small. This should be carefully checked.

If the remaining nuclear structure uncertainty is in fact small, one can use the Qweak $A_{PV}$ data to measure the neutron radius $R_n$ of $^{27}\text{Al}$. This is one of our main results. To determine the sensitivity to $R_n$, we uniformly stretch our FSUgold neutron density, so that $R_n$ increases by 1%, while keeping the proton density unchanged, see also [24]. We then calculate the log derivative of $A_{DW}(C_0)$ with respect to $R_n$, evaluated at 7.8 degrees. We find,

$$\frac{d \ln A_{DW}(C_0)}{d \ln R_n} \approx 2.$$  

This shows that a 4% measurement of $A_{PV}$ is sensitive to 2% changes in $R_n$. Therefore, in principle, one can use $A_{PV}$, good to 4%, to measure the neutron radius of $^{27}\text{Al}$ to about 2%. For comparison, the PREX experiment measured $R_n$ of $^{208}\text{Pb}$ to 3% [3]. Note that the follow up experiment PREX-II aims to improve this to 1% [5].

However, there are important backgrounds that need to be estimated before one can determine $R_n$. The Qweak spectrometer has only modest energy resolution and also accepts inelastically scattered electrons with energy losses up to about 100 MeV. Therefore one will also have contributions from discrete excited states, collective giant resonances, and quasielastic scattering. For the forward angle Qweak kinematics we expect the discrete excited states to be dominated by Coulomb multipoles. For these one can easily make an estimate of $A_{PV}$, see also ref. [2]. The most important property is the isospin of the excitation. Isoscalar excitations, where neutrons move in phase with protons, should have $A_{PV} \approx A_{PV}^0$, see Eq. (13). For isovector excitations, where the neutrons move out of phase with the protons, one has $A_{PV} \approx -A_{PV}^0$. We would expect excitations of mixed isospin to be in-between. These estimates should also hold for giant resonances where for example the isovector giant dipole resonance should have $A_{PV} \approx -A_{PV}^0$. We have calculated $A_{PV}$ for quasielastic scattering in some detail, see ref. [25].

Finally there are impurities in the Qweak target. An alloy is used that is about 90% $^{27}\text{Al}$ but also contains some Cu, Mg, and Zn and other trace elements. In future work we will calculate $A_{PV}$ for elastic scattering from these nuclei using relativistic mean field densities.

If one estimates contributions from inelastic excitations and impurity scattering, and the nuclear structure uncertainties for $^{27}\text{Al}$ are indeed small, one can measure $R_n$. What is the physics content of this measurement? For $^{208}\text{Pb}$, $R_n$ determines the density dependence of the symmetry energy and the pressure of pure neutron mat-
ter with important applications to astrophysics \[2, 26-28\]. However, for \(^{27}\text{Al}\) we have \(N \approx Z\) and many relativistic mean field models have \(R_n \approx R_p\), where \(R_p\) is the proton radius. There may only be a small range of \(R_n\) values predicted by all reasonable nuclear structure models. This should be explicitly checked by looking at a large number of nuclear structure models. Very likely theory makes a sharp prediction for \(R_n - R_p\) for \(^{27}\text{Al}\) that is essentially independent of the density dependence of the symmetry energy. Therefore the measurement of \(R_n\) provides a sharp test of theory and experiment. A disagreement would suggest an important problem either in the measurement or in the theory. While agreement of the Qweak \(R_n\) measurement with theory would support the validity of many aspects of the measurement, analysis, and theory.

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