POSITIVE SCALAR CURVATURE ON FOLIATIONS: THE NONCOMPACT CASE

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Abstract. Let \((M,g^{TM})\) be a noncompact (not necessarily complete) enlargeable Riemannian manifold in the sense of Gromov-Lawson and \(F\) an integrable subbundle of \(TM\). Let \(k^F\) be the leafwise scalar curvature associated to \(g^F = g^{TM}|_F\). We show that if either \(TM\) or \(F\) is spin, then \(\inf(k^F) \leq 0\). This generalizes the famous result of Gromov-Lawson on enlargeable manifolds to the case of foliations. It also extends an ansatz of Gromov on hyper-Euclidean spaces to general enlargeable Riemannian manifolds, as well as recent results on compact enlargeable foliated manifolds due to Benamer-Heitsch et al to the noncompact situation.

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1. Introduction

A classical result of Lichnerowicz [10] states that if a closed spin manifold carries a Riemannian metric of positive scalar curvature, then its \(\hat{A}\)-genus vanishes. A celebrated result of Connes [5, Theorem 0.2] generalizes the Lichnerowicz vanishing theorem to the case of spin foliations on an oriented closed manifold. In [13], Zhang gives a differential geometric proof of the Connes vanishing theorem, and proves an alternative generalization of the Lichnerowicz vanishing theorem to the case of foliations on a closed spin manifold.

In this paper, we deal with the case of noncompact foliated manifolds. Recall that in his comprehensive paper on positive scalar curvature, Gromov [6, p. 192] has stated an analogue of the Connes vanishing theorem for any hyper-Euclidean space \(\mathbb{R}^n\), which is noncompact.

Proposition 1.1. (Gromov Ansatz) Let \(g^{\mathbb{T}\mathbb{R}^n}\) be any hyper-Euclidean metric on \(T\mathbb{R}^n\) (that is, for any \(X \in T\mathbb{R}^n\), \(|X|_{g^{\mathbb{T}\mathbb{R}^n}} \leq |X|\), where \(|X|\) is a standard Euclidean norm of \(X\)). Let \(F \subseteq T\mathbb{R}^n\) be an integrable subbundle of \(T\mathbb{R}^n\), then the leafwise scalar curvature \(k^F\) associated with \(g^{\mathbb{T}\mathbb{R}^n}|_F\) verifies that \(\inf(k^F) \leq 0\).

In view of the fact that \(\mathbb{R}^n\) is the universal covering space of the torus \(T^n\), Proposition 1.1 implies the fact that there is no foliation admitting metric of positive leafwise scalar curvature on any torus, which is proved in [13, Corollary 0.5]. It generalizes the famous results of Schoen-Yau [12] and Gromov-Lawson [7] stating that there is no Riemannian metric of positive scalar curvature on any torus to the case of foliations on torus.

Gromov [6] indicates implicitly that he would combine Connes’ proof of his celebrated vanishing theorem [5] with the Gromov-Lawson relative index theorem [8] to prove Proposition 1.1. However, as we will see below, this might meet potential problems due to
the noncompactness of $M$. In this paper, we will provide a complete proof of this Gro-mov ansatz, in a more general setting. Our main result generalizes a famous result of Gromov-Lawson to the case of foliations.

In fact, recall that the concept of enlargeability of Gromov-Lawson has played an important role in their classical papers [7], [8] on positive scalar curvature. Following [8, Definition 7.1] (see also [4, Definition 1.1]), a (connected) Riemannian manifold $(M, g^TM)$ is called enlargeable if for any $\varepsilon > 0$, there is a (connected) covering $\widetilde{M} \to M$ (with the lifted Riemannian metric of $g^TM$) and a smooth map $f : \widetilde{M} \to S^\dim M(1)$ of nonzero degree such that $f$ is constant near infinity and that for any $X \in \Gamma(T\widetilde{M})$, one has $|f_*(X)| \leq \varepsilon |X|$. When $M$ is compact, the concept of enlargeability is independent of the Riemannian metric $g^TM$. However, when $M$ is noncompact, this concept does depend on the chosen metric (cf. [4, Theorem B]).

A famous result of Gromov-Lawson [8, Theorem 7.3] states that if $(M, g^TM)$ is a spin enlargeable complete Riemannian manifold and $k^TM$ is the scalar curvature associated to $g^TM$, then $\inf(k^TM) \leq 0$. That is, there is no uniform positive lower bound of $k^TM$.

In this paper, we will generalize this result to the case of foliations. To be more precise, let $F \subseteq TM$ be an integrable subbundle of the tangent bundle $TM$ of $M$. Let $g^F = g^TM|_F$ be the restricted Euclidean metric on $F$, and $k^F \in C^\infty(M)$ be the associated leafwise scalar curvature (cf. [13, (0.1)]).

With the above notation, the main result of this paper can be stated as follows.

**Theorem 1.2.** Let $F \subseteq TM$ be an integrable subbundle of the tangent bundle of a spin enlargeable Riemannian manifold $(M, g^TM)$ and $k^F$ the leafwise scalar curvature associated to $g^F = g^TM|_F$, then one has $\inf(k^F) \leq 0$.

When $F = TM$ and $g^TM$ is complete, one recovers the above Gromov-Lawson result. When $M$ is compact, Theorem 1.2 has been proved (at least in details for compactly enlargeable manifolds) in [14]. So in this paper we will concentrate on the case of noncompact $M$. Clearly, Proposition 1.1 is a corollary of Theorem 1.2.

It might be interesting to note that we do not assume $g^TM$ to be complete in Theorem 1.2.

Recall that in the case of noncompact $M$ and $F = TM$, Gromov and Lawson make use of the relative index theorem in [8, §4] to prove their result. However, in the case of general $F$ here, even if $(M, F, g^TM)$ is a Riemannian foliation, one does not get a uniform positive lower bound of $k^TM$ over $M$ if one only assumes that $\inf(k^F) > 0$ (although one can always obtain positive $k^TM$ on any compact region via an adiabatic process). This indicates that one can not use the relative index theorem directly to prove Theorem 1.2.

To overcome this difficulty, we will use the method developed in [14] where a proof without using the relative index theorem is given for the Gromov-Lawson theorem on spin enlargeable manifolds. This amounts to deform the Dirac operator in question by endomorphisms of the involved (twisted) $\mathbb{Z}_2$-graded vector bundle, which are invertible near infinity (cf. [14, (1.11)]), thus reducing the noncompact problem to a compact one by a gluing process.

To be more precise, we will combine the methods in [13] and [14], in an extended form adapted to the current noncompact situation, to prove Theorem 1.2. In particular,
the sub-Dirac operators constructed in [11] and [13], as well as the Connes fibration introduced in [5] (cf. [13, §2.1]), will play essential roles in our proof. Especially, a key new deformation of the sub-Dirac operator on the Connes fibration is introduced (cf. (2.18)). Note also that we do not assume a priori that \( g^{TM} \) is complete. This is different with respect to what in [8], where the relative index theorem on complete manifolds [8, §4] plays an essential role.

Moreover, just as in [13, §2.5] and [14], our method can also be used to prove the following alternate extension of Gromov-Lawson’s result.

**Theorem 1.3.** Let \( F \subseteq TM \) be a spin integrable subbundle of the tangent bundle of an enlargeable Riemannian manifold \((M, g^{TM})\) and \( k^F \) the leafwise scalar curvature associated to \( g^F = g^{TM}|_F \), then one has \( \inf(k^F) \leq 0 \).

When \( M \) is compact and the homotopy groupoid of \((M, F)\) is Hausdorff, Theorem 1.3 is due to Benameur-Heitsch [2]. In [14], Zhang eliminates this Hausdorff condition. Thus Theorem 1.3 is new in the case of noncompact \( M \).

2. Proof of Theorems 1.2 and 1.3

In this section, we prove Theorems 1.2 and 1.3. In Section 2.1 we recall the basic geometric set up. In Section 2.2 we lift things to the Connes fibration. In Section 2.3 we study the deformed sub-Dirac operators on the Connes fibration. In Section 2.4 we prove Theorem 1.2 in detail. The proof of Theorem 1.3 is similar.

2.1. Foliations on noncompact enlargeable manifolds. Let \( M \) be a noncompact (connected) smooth manifold and \( g^{TM} \) a Euclidean metric on the tangent bundle \( TM \).

Following [8, Definition 7.1] (see also [4, Definition 1.1]), we say that the Riemannian metric \( g^{TM} \) is enlargeable if for any \( \varepsilon > 0 \), there is a (connected) covering \( \pi_\varepsilon : M_\varepsilon \to M \) and a smooth map \( f_\varepsilon : M_\varepsilon \to S^{\dim M}(1) \), where \( S^{\dim M}(1) \) is the standard unit sphere of dimension \( \dim M \), such that there exists a compact subset \( K_\varepsilon \subseteq M_\varepsilon \) verifying that \( f_\varepsilon \) is constant on \( M_\varepsilon \setminus K_\varepsilon \) and \( \deg(f_\varepsilon) \neq 0 \). Moreover, for any \( X \in \Gamma(TM_\varepsilon) \), \( f_\varepsilon \) should satisfy

\[
|f_\varepsilon^\ast(X)| \leq \varepsilon |X|_{g^{TM_\varepsilon}},
\]

where \( g^{TM_\varepsilon} \) is the lifted metric \( \pi_\varepsilon^\ast g^{TM} \).

From now on, we assume that \( g^{TM} \) is enlargeable. Without loss of generality, we assume that for any \( \varepsilon > 0 \), \( f_\varepsilon(M_\varepsilon \setminus K_\varepsilon) = x_0 \in S^{\dim M}(1) \), where \( x_0 \) is a fixed point on \( S^{\dim M}(1) \).

Without loss of generality, we assume that \( \dim M \) is even\(^1\)

Let \( F \) be an integrable subbundle of the tangent bundle \( TM \). Let \( g^F = g^{TM}|_F \) be the induced Euclidean metric on \( F \). Let \( k^F \in C^\infty(M) \) be the leafwise scalar curvature associated to \( g^F \) (cf. [13, (0.1)]).

We will prove Theorems 1.2 and 1.3 by contradiction. Thus we assume first that there is \( \delta > 0 \) such that

\[
k^F \geq \delta \quad \text{over} \quad M.
\]

\(^1\)One may consider \( M \times S^1 \) if \( \dim M \) is odd.
Let $F^\perp$ be the orthogonal complement to $F$, i.e., we have the orthogonal splitting
\begin{equation}
TM = F \oplus F^\perp, \quad g^{TM} = g^F \oplus g^{F^\perp}.
\end{equation}

Let $(E_0, g^{E_0})$ be a Hermitian vector bundle on $S^{\dim M}(1)$ verifying
\begin{equation}
\langle \text{ch} (E_0), [S^{\dim M}(1)] \rangle \neq 0
\end{equation}
and carrying a Hermitian connection $\nabla^{E_0}$. Let $(E_1 = C^k|_{S^{\dim M}(1)}, g^{E_1}, \nabla^{E_1})$, with $k = \text{rk}(E_0)$, be the canonical Hermitian trivial vector bundle, with trivial Hermitian connection, on $S^{\dim M}(1)$. Let $w \in \Gamma(\text{Hom}(E_0, E_1))$ be an endomorphism such that $w|_{x_0} : E_0|_{x_0} \to E_1|_{x_0}$ is an isomorphism. Let $w^* : \Gamma(E_1) \to \Gamma(E_0)$ be the adjoint of $w$ with respect to $g^{E_0}$ and $g^{E_1}$. Set
\begin{equation}
W = w + w^*.
\end{equation}
Then the self-adjoint endomorphism $W : \Gamma(E_0 \oplus E_1) \to \Gamma(E_0 \oplus E_1)$ is invertible near $x_0$.

Let $\varepsilon > 0$ be fixed temporarily.

Let $(M_{\varepsilon}, F_{\varepsilon}) = \pi_{\varepsilon}^*(M, F)$ be the lifted foliation, with $g^{F_{\varepsilon}} = \pi_{\varepsilon}^*g^F$ being the lifted Euclidean metric on $F_{\varepsilon}$. The splitting (2.3) lifts canonically to an orthogonal splitting
\begin{equation}
TM_{\varepsilon} = F_{\varepsilon} \oplus F_{\varepsilon}^\perp, \quad g^{TM_{\varepsilon}} = g^{F_{\varepsilon}} \oplus g^{F_{\varepsilon}^\perp}.
\end{equation}

Following [8], we take a compact hypersurface $H_{\varepsilon} \subset M_{\varepsilon} \setminus K_{\varepsilon}$, cutting $M_{\varepsilon}$ into two parts such that the compact part, denoted by $M_{H_{\varepsilon}}$, contains $K_{\varepsilon}$. Then $M_{H_{\varepsilon}}$ is a compact smooth manifold with boundary $H_{\varepsilon}$.

Let $M_{H_{\varepsilon}}$ be another copy of $M_{H_{\varepsilon}}$. We glue $M_{H_{\varepsilon}}$ and $M'_{H_{\varepsilon}}$ along $H_{\varepsilon}$ to get the double, which we denote by $\tilde{M}_{H_{\varepsilon}}$.

2.2. The Connes fibration. Following [5] §5 (cf. [13] §2.1), let $\pi_{\varepsilon} : \mathcal{M}_{\varepsilon} \to M_{\varepsilon}$ be the Connes fibration over $M_{\varepsilon}$ such that for any $x \in M_{\varepsilon}$, $\mathcal{M}_{\varepsilon,x} = \pi_{\varepsilon}^{-1}(x)$ is the space of Euclidean metrics on the linear space $T_xM_{\varepsilon}/F_{\varepsilon,x}$. Let $T^V\mathcal{M}_{\varepsilon}$ denote the vertical tangent bundle of the fibration $\pi_{\varepsilon} : \mathcal{M}_{\varepsilon} \to M_{\varepsilon}$. Then it carries a natural metric $g^{T^V\mathcal{M}_{\varepsilon}}$ such that any two points $p, q \in \mathcal{M}_{\varepsilon,x}$ with $x \in M_{\varepsilon}$ can be joined by a unique geodesic along $\mathcal{M}_{\varepsilon,x}$. Let $d^{\mathcal{M}_{\varepsilon,x}}(p, q)$ denote the length of this geodesic.

By using the Bott connection on $TM_{\varepsilon}/F_{\varepsilon}$ (cf. [13] (1.2)), which is leafwise flat, one lifts $F_{\varepsilon}$ to an integrable subbundle $F_{\varepsilon}$ of $TM_{\varepsilon}$. Then $g^{F_{\varepsilon}}$ lifts to a Euclidean metric $g^{F_{\varepsilon}} = \pi_{\varepsilon}^*g^F_{\varepsilon}$ on $F_{\varepsilon}$.

Let $F_{\varepsilon,1} \subseteq TM_{\varepsilon}$ be a subbundle, which is transversal to $F_{\varepsilon} \oplus T^V\mathcal{M}_{\varepsilon}$, such that we have a splitting $TM_{\varepsilon} = (F_{\varepsilon} \oplus T^V\mathcal{M}_{\varepsilon}) \oplus F_{\varepsilon,1}$. Then $F_{\varepsilon,1}$ can be identified with $TM_{\varepsilon}/(F_{\varepsilon} \oplus T^V\mathcal{M}_{\varepsilon})$ and carries a canonically induced metric $g^{F_{\varepsilon,1}}$. We denote $F_{\varepsilon,2} = T^V\mathcal{M}_{\varepsilon}$.

The metric $g^{F_{\varepsilon,1}}$ in (2.6) determines a canonical embedded section $s : M_{\varepsilon} \hookrightarrow \mathcal{M}_{\varepsilon}$ for any $p \in M_{\varepsilon}$, set $\rho(p) = d^{M_{\varepsilon},\pi_{\varepsilon}(p)}(p, s(\pi_{\varepsilon}(p)))$.\footnote{We may well assume that $TM_{\varepsilon} = F_{\varepsilon} \oplus F_{\varepsilon,1} \oplus F_{\varepsilon,2}$ is lifted from $TM = F \oplus F_1 \oplus F_2$ via $\pi_{\varepsilon}^*$, where $\mathcal{M}$ is the Connes fibration over $M$ as in [13] §2.1.}
For any $\beta, \gamma > 0$, following \[13\] (2.15), let $g^{TM_{\varepsilon}}_{\beta, \gamma}$ be the metric on $TM_{\varepsilon}$ defined by the orthogonal splitting,

$$
(2.7) \quad TM_{\varepsilon} = F_{\varepsilon} \oplus F_{\varepsilon,1}^\perp \oplus F_{\varepsilon,2}^\perp, \quad g^{TM_{\varepsilon}}_{\beta, \gamma} = \beta^2 g_{\varepsilon}^2 \oplus \frac{g_{\varepsilon}^2}{\gamma^2} \oplus g_{\varepsilon}^2.
$$

For any $R > 0$, let $M_{\varepsilon, R}$ be the smooth manifold with boundary defined by

$$
(2.8) \quad M_{\varepsilon, R} = \{ p \in M_{\varepsilon} : \rho(p) \leq R \}.
$$

Set $H_{\varepsilon} = (\pi_{\varepsilon})^{-1}(H_{\varepsilon})$ and

$$
(2.9) \quad M_{H_{\varepsilon}, R} = ((\pi_{\varepsilon})^{-1}(M_{H_{\varepsilon}})) \cap M_{\varepsilon, R}.
$$

Consider another copy $M'_{H_{\varepsilon}, R}$ of $M_{H_{\varepsilon}, R}$. We glue $M_{H_{\varepsilon}, R}$ and $M'_{H_{\varepsilon}, R}$ along $H_{\varepsilon} \cap M_{\varepsilon, R}$ to get the double, denoted by $\tilde{M}_{H_{\varepsilon}, R}$, which is a smooth manifold with boundary. Moreover, $\tilde{M}_{H_{\varepsilon}, R}$ is a disk bundle over $M_{H_{\varepsilon}}$. Without loss of generality, we assume that $\tilde{M}_{H_{\varepsilon}, R}$ is oriented. Let $g^{TM_{H_{\varepsilon}, R}}_{\beta, \gamma}$ be a metric on $TM_{H_{\varepsilon}, R}$ such that $g^{TM_{H_{\varepsilon}, R}}_{\beta, \gamma} |_{M_{H_{\varepsilon}, R}} = g^{TM_{\varepsilon}}_{\beta, \gamma} |_{M_{\varepsilon, R}}$. The existence of $g^{TM_{H_{\varepsilon}, R}}_{\beta, \gamma}$ is clear\(^3\).

Let $\partial \tilde{M}_{H_{\varepsilon}, R}$ bound another oriented manifold $N_{\varepsilon, R}$ so that $\tilde{N}_{\varepsilon, R} = \tilde{M}_{H_{\varepsilon}, R} \cup N_{\varepsilon, R}$ is an oriented closed manifold. Let $g^{TN_{\varepsilon, R}}_{\beta, \gamma}$ be a smooth metric on $TN_{\varepsilon, R}$ so that $g^{TN_{\varepsilon, R}}_{\beta, \gamma} |_{\tilde{M}_{H_{\varepsilon}, R}} = g^{TM_{H_{\varepsilon}, R}}_{\beta, \gamma}$. The existence of $g^{TN_{\varepsilon, R}}_{\beta, \gamma}$ is clear.

We extend $f_{\varepsilon} : M_{H_{\varepsilon}} \to S^{\dim M(1)}$ to $f_{\varepsilon} : \tilde{M}_{H_{\varepsilon}} \to S^{\dim M(1)}$ by setting $f_{\varepsilon}(M'_{H_{\varepsilon}}) = x_0$. Let $\tilde{f}_{\varepsilon, R} : \tilde{M}_{H_{\varepsilon}, R} \to S^{\dim M(1)}$ be the smooth map defined by

$$
(2.10) \quad \tilde{f}_{\varepsilon, R} = f_{\varepsilon} \circ \pi_{\varepsilon} \quad \text{on} \quad M_{H_{\varepsilon}, R}
$$

and $\tilde{f}_{\varepsilon, R}(M'_{H_{\varepsilon}, R}) = x_0$.

For $i = 0, 1$, let $(E_{\varepsilon, R, i}, g_{\varepsilon, R, i}, \nabla_{\varepsilon, R, i}) = \tilde{f}_{\varepsilon, R}^*(E_{\varepsilon, i}, g_{\varepsilon, i}, \nabla_{\varepsilon, i})$ be the induced Hermitian vector bundle with Hermitian connection on $\tilde{M}_{H_{\varepsilon}, R}$. Then $E_{\varepsilon, R} = E_{\varepsilon, R, 0} \oplus E_{\varepsilon, R, 1}$ is a $\mathbb{Z}_2$-graded Hermitian vector bundle over $\tilde{M}_{H_{\varepsilon}, R}$.

### 2.3. Adiabatic limits and deformed sub-Dirac operators on $\tilde{M}_{H_{\varepsilon}, R}$

We assume first that $TM$ is oriented and spin. Then $TM_{\varepsilon} = \pi_{\varepsilon}^*(TM)$ is spin, and thus $F_{\varepsilon} \oplus F_{\varepsilon,1}^\perp = \pi_{\varepsilon}^*(TM_{\varepsilon})$ is spin. Without loss of generality, we assume $F_{\varepsilon}$ is oriented. Then $F_{\varepsilon,1}^\perp$ is also oriented. Without loss of generality, we assume that $\dim M_{\varepsilon}$ is even.

It is clear that $F_{\varepsilon} \oplus F_{\varepsilon,1}^\perp, F_{\varepsilon,2}^\perp$ over $M_{H_{\varepsilon}, R}$ can be extended to $M'_{H_{\varepsilon}, R}$ such that we have the orthogonal splitting

$$
(2.11) \quad TM_{H_{\varepsilon}, R} = (F_{\varepsilon} \oplus F_{\varepsilon,1}^\perp) \oplus F_{\varepsilon,2}^\perp \quad \text{on} \quad \tilde{M}_{H_{\varepsilon}, R}.
$$

Let $S^{TM_{H_{\varepsilon}, R}}_{\beta, \gamma}(F_{\varepsilon} \oplus F_{\varepsilon,1}^\perp)$ denote the spinor bundle over $\tilde{M}_{H_{\varepsilon}, R}$ with respect to the metric $g^{TM_{H_{\varepsilon}, R}}_{\beta, \gamma} |_{F_{\varepsilon} \oplus F_{\varepsilon,1}^\perp}$ (thus with respect to $\beta^2 g_{\varepsilon}^2 \oplus \frac{g_{\varepsilon}^2}{\gamma^2}$ on $M_{H_{\varepsilon}, R}$). Let $\Lambda^*(F_{\varepsilon,2}^\perp)$ denote the exterior algebra bundle of $F_{\varepsilon,2}^\perp$, with the $\mathbb{Z}_2$-grading given by the natural parity (cf. \[14\] (1.15))).

\(^3\)Here we need not assume that $g^{TM_{H_{\varepsilon}, R}}_{\beta, \gamma}$ is of product structure near $H_{\varepsilon}$. 
Let $D_{\mathcal{F}_s \oplus \mathcal{F}_{s,1}, \beta, \gamma} : \Gamma(S_{\beta, \gamma}(\mathcal{F}_s \oplus \mathcal{F}_{s,1}) \otimes \Lambda^*(\mathcal{F}_{s,2}^\perp)) \to \Gamma(S_{\beta, \gamma}(\mathcal{F}_s \oplus \mathcal{F}_{s,1}) \otimes \Lambda^*(\mathcal{F}_{s,2}^\perp))$ be the sub-Dirac operator on $\tilde{\mathcal{M}}_{H,e,R}$ constructed as in [13 (2.16)]. Then it is clear that one can define canonically the twisted sub-Dirac operator (twisted by $\mathcal{E}_{\varepsilon, R}$) on $\tilde{\mathcal{M}}_{H,e,R}$,

\[
\begin{align*}
(2.12) \quad D_{\mathcal{F}_s \oplus \mathcal{F}_{s,1}, \beta, \gamma}^\varepsilon,R & : \Gamma(S_{\beta, \gamma}(\mathcal{F}_s \oplus \mathcal{F}_{s,1}) \otimes \Lambda^*(\mathcal{F}_{s,2}^\perp)) \to \Gamma(S_{\beta, \gamma}(\mathcal{F}_s \oplus \mathcal{F}_{s,1}) \otimes \Lambda^*(\mathcal{F}_{s,2}^\perp)) \otimes \mathcal{E}_{\varepsilon, R}\n
\end{align*}
\]

Moreover, by [13 (2.28)], one sees that the following identity holds on $\mathcal{M}_{H,e,R}$, using the same notation for Clifford actions as in [13],

\[
(2.13) \quad \left(D_{\mathcal{F}_s \oplus \mathcal{F}_{s,1}, \beta, \gamma}^\varepsilon,R\right)^2 = -\Delta_{\varepsilon, \beta, \gamma} + \frac{k_{\mathcal{F}_s}}{4\beta^2} + \frac{1}{2\beta^2} \sum_{i,j=1}^{\text{rk}(\mathcal{F})} R_{\mathcal{F}_s,R}(f_i, f_j) c_{\beta, \gamma} (\beta^{-1} f_i) c_{\beta, \gamma} (\beta^{-1} f_j) + O_{\varepsilon,R} \left( \frac{1}{\beta} + \frac{\gamma^2}{\beta^2} \right),
\]

where $-\Delta_{\varepsilon, \beta, \gamma} \geq 0$ is the corresponding Bochner Laplacian,

\[
(2.14) \quad k_{\mathcal{F}_s} = \pi^*_s \left( \pi^*_s \left( k_{\mathcal{F}} \right) \right) \geq \delta,
\]

$R_{\mathcal{F}_s,R} = (\nabla \mathcal{E}_{\varepsilon,R})^2 + (\nabla \mathcal{E}_{\varepsilon,R,1})^2$ and $f_1, \ldots, f_{\text{rk}(\mathcal{F})}$ is an orthonormal basis of $(\mathcal{F}_s, g_{\mathcal{F}_s})$. The subscripts in $O_{\varepsilon,R}(\cdot)$ mean that the estimating constant may depend on $\varepsilon$ and $R$.

On the other hand, since $g_{\varepsilon,R} = \pi^*_s g_{\mathcal{F}_s}$, one has via (2.11) and (2.10) that

\[
(2.15) \quad R_{\mathcal{F}_s,R}(f_i, f_j) = \sum_{k=0}^{1} \delta_{\varepsilon,R}(\left( (\nabla \mathcal{E}_{\varepsilon,R})^2 (\hat{f}_{\varepsilon,R,s}(f_i), \hat{f}_{\varepsilon,R,s}(f_j)) \right) = O(\varepsilon^2),
\]

where the estimating constant does not depend on $\varepsilon$ and $R$.

Let $f : [0, 1] \to [0, 1]$ be a smooth function such that $f(t) = 0$ for $0 \leq t \leq \frac{1}{4}$, while $f(t) = 1$ for $\frac{1}{2} \leq t \leq 1$. Let $h : [0, 1] \to [0, 1]$ be a smooth function such that $h(t) = 1$ for $0 \leq t \leq \frac{3}{4}$, while $h(t) = 0$ for $\frac{7}{8} \leq t \leq 1$.

For any $p \in \mathcal{M}_{H,e,R}$, we connect $p$ and $s(\pi_s(p))$ by the unique geodesic in $\mathcal{M}_{e,\pi_s(p)}$. Let $\sigma(p) \in \mathcal{F}_{s,2,1}^\perp$ denote the unit vector tangent to this geodesic. Then

\[
(2.16) \quad \hat{\sigma} = f \left( \frac{\rho}{R} \right) \sigma
\]

is a smooth section of $\mathcal{F}_{s,2}^\perp|_{\tilde{\mathcal{M}}_{H,e,R}}$. It extends to a smooth section of $\mathcal{F}_{s,2}^\perp|_{\mathcal{M}_{H,e,R}}$, which we still denote by $\hat{\sigma}$. It is easy to see that we may and we will assume that $\hat{\sigma}$ is transversal to (and thus no where zero on) $\partial \tilde{\mathcal{M}}_{H,e,R}$.

The Clifford action $\tilde{\sigma}(\hat{\sigma})$ (cf. [13 (1.47)]) now acts on $S_{\beta, \gamma}(\mathcal{F}_s \oplus \mathcal{F}_{s,1}) \otimes \Lambda^*(\mathcal{F}_{s,2}^\perp) \otimes \mathcal{E}_{\varepsilon, R}$ over $\tilde{\mathcal{M}}_{H,e,R}$.

We also set

\[
(2.17) \quad W_{\mathcal{F}_s,R} = \hat{f}_{\varepsilon,R}^*(W),
\]

where $W$ is defined in (2.3). Then $W_{\mathcal{F}_s,R}$ is an odd endomorphism of $\mathcal{E}_{\varepsilon,R}$ and thus also acts on $S_{\beta, \gamma}(\mathcal{F}_s \oplus \mathcal{F}_{s,1}) \otimes \Lambda^*(\mathcal{F}_{s,2}^\perp) \otimes \mathcal{E}_{\varepsilon, R}$ in an obvious way.
Inspired by [13] (2.21) and [14] (1.11), we introduce the following deformation of $D^{\varepsilon,R}_{\mathcal{F}_{\varepsilon} \oplus \mathcal{F}_{\varepsilon,1},\beta,\gamma}$ on $\widehat{\mathcal{M}}_{\mathcal{H}_\varepsilon,R}$:

\begin{equation}
D^{\varepsilon,R}_{\mathcal{F}_{\varepsilon} \oplus \mathcal{F}_{\varepsilon,1},\beta,\gamma} + \frac{\widehat{c}(\sigma)}{\beta} + \frac{W_{\varepsilon,R}}{\beta}.
\end{equation}

For this deformed sub-Dirac operator, we have the following analogue of [13] Lemma 2.4.

**Lemma 2.1.** There exist $c_0 > 0$, $\varepsilon > 0$ and $R > 0$ such that when $\beta, \gamma > 0$ (which may depend on $\varepsilon$ and $R$) are small enough,

(i) for any $s \in \Gamma(S_{\beta,\gamma}(\mathcal{F}_\varepsilon \oplus \mathcal{F}_{\varepsilon,1}) \widehat{\otimes} \Lambda^*(\mathcal{F}_{\varepsilon,2}) \widehat{\otimes} \mathcal{E}_{\varepsilon,R})$ supported in the interior of $\widehat{\mathcal{M}}_{\mathcal{H}_\varepsilon,R}$, one has

\begin{equation}
\left\| \left( D^{\varepsilon,R}_{\mathcal{F}_{\varepsilon} \oplus \mathcal{F}_{\varepsilon,1},\beta,\gamma} + \frac{\widehat{c}(\sigma)}{\beta} + \frac{W_{\varepsilon,R}}{\beta} \right) s \right\| \geq \frac{c_0}{\beta} \| s \|.
\end{equation}

(ii) for any $s \in \Gamma(S_{\beta,\gamma}(\mathcal{F}_\varepsilon \oplus \mathcal{F}_{\varepsilon,1}) \widehat{\otimes} \Lambda^*(\mathcal{F}_{\varepsilon,2}) \widehat{\otimes} \mathcal{E}_{\varepsilon,R})$ supported in the interior of $\widehat{\mathcal{M}}_{\mathcal{H}_\varepsilon,R} \setminus \mathcal{M}_{\mathcal{H}_\varepsilon,\frac{R}{2}}$, one has

\begin{equation}
\left\| \left( h \left( \frac{\rho}{R} \right) D^{\varepsilon,R}_{\mathcal{F}_{\varepsilon} \oplus \mathcal{F}_{\varepsilon,1},\beta,\gamma} h \left( \frac{\rho}{R} \right) + \frac{\widehat{c}(\sigma)}{\beta} + \frac{W_{\varepsilon,R}}{\beta} \right) s \right\| \geq \frac{c_0}{\beta} \| s \|.
\end{equation}

**Proof.** Recall that $x_0 \in S^{\dim M}(1)$ is fixed and $W|_{x_0}$ is invertible. Let $U_{x_0} \subset S^{\dim M}(1)$ be a (fixed) sufficiently small open neighborhood of $x_0$ such that the following inequality holds on $U_{x_0}$, where $\delta_1 > 0$ is a fixed constant,

\begin{equation}
W^2 \geq \delta_1.
\end{equation}

Following [14], let $\psi : S^{\dim M}(1) \to [0,1]$ be a smooth function such that $\psi = 1$ near $x_0$ and Supp($\psi$) $\subset U_{x_0}$. Then

\begin{equation}
\varphi_{\varepsilon,R} = 1 - \widehat{f}_{\varepsilon,R}^* \psi
\end{equation}

is a smooth function on $\widehat{\mathcal{M}}_{\mathcal{H}_\varepsilon,R}$ such that $\varphi_{\varepsilon,R} = 0$ on $\mathcal{M}_{\mathcal{H}_\varepsilon,R}$.

Following [3] p. 115], let $\varphi_{\varepsilon,R,1}, \varphi_{\varepsilon,R,2} : \widehat{\mathcal{M}}_{\mathcal{H}_\varepsilon,R} \to [0,1]$ be defined by

\begin{equation}
\varphi_{\varepsilon,R,1} = \frac{\varphi_{\varepsilon,R}}{(\varphi_{\varepsilon,R}^2 + (1 - \varphi_{\varepsilon,R})^2)^{\frac{1}{2}}}, \quad \varphi_{\varepsilon,R,2} = \frac{1 - \varphi_{\varepsilon,R}}{(\varphi_{\varepsilon,R}^2 + (1 - \varphi_{\varepsilon,R})^2)^{\frac{1}{2}}}.
\end{equation}

Then $\varphi_{\varepsilon,R,1}^2 + \varphi_{\varepsilon,R,2}^2 = 1$. Thus, for any $s \in \Gamma(S_{\beta,\gamma}(\mathcal{F}_\varepsilon \oplus \mathcal{F}_{\varepsilon,1}) \widehat{\otimes} \Lambda^*(\mathcal{F}_{\varepsilon,2}) \widehat{\otimes} \mathcal{E}_{\varepsilon,R})$ supported in the interior of $\widehat{\mathcal{M}}_{\mathcal{H}_\varepsilon,R}$, one has

\begin{equation}
\left\| \left( D^{\varepsilon,R}_{\mathcal{F}_{\varepsilon} \oplus \mathcal{F}_{\varepsilon,1},\beta,\gamma} + \frac{\widehat{c}(\sigma)}{\beta} + \frac{W_{\varepsilon,R}}{\beta} \right) s \right\|^2 \leq \varphi_{\varepsilon,R,1} \left( D^{\varepsilon,R}_{\mathcal{F}_{\varepsilon} \oplus \mathcal{F}_{\varepsilon,1},\beta,\gamma} + \frac{\widehat{c}(\sigma)}{\beta} + \frac{W_{\varepsilon,R}}{\beta} \right) s \right\|^2 + \varphi_{\varepsilon,R,2} \left( D^{\varepsilon,R}_{\mathcal{F}_{\varepsilon} \oplus \mathcal{F}_{\varepsilon,1},\beta,\gamma} + \frac{\widehat{c}(\sigma)}{\beta} + \frac{W_{\varepsilon,R}}{\beta} \right) s \right\|^2,
\end{equation}

The norms below depend on $\beta$ and $\gamma$. In case of no confusion, we omit the subscripts for simplicity.
from which one gets,

\begin{equation}
(2.25) \quad \sqrt{2} \left\| \left( D_{F,\epsilon}^{\epsilon,R} + \frac{\overline{\mathcal{C}}(\overline{\sigma})}{\beta} + \frac{W_{f,R}}{\beta} \right) s \right\| \geq \left\| \varphi_{\epsilon,R,1} \left( D_{F,\epsilon}^{\epsilon,R} + \frac{\overline{\mathcal{C}}(\overline{\sigma})}{\beta} + \frac{W_{f,R}}{\beta} \right) s \right\| + \left\| \varphi_{\epsilon,R,2} \left( D_{F,\epsilon}^{\epsilon,R} + \frac{\overline{\mathcal{C}}(\overline{\sigma})}{\beta} + \frac{W_{f,R}}{\beta} \right) s \right\|
\end{equation}

where we identify a one form with its gradient.

Let \( f_1, \ldots, f_q \) (resp. \( h_1, \ldots, h_q \); resp. \( e_1, \ldots, e_{q_2} \)) be an orthonormal basis of \( (\mathcal{F}_\epsilon, g^{F_\epsilon}) \) (resp. \( (\mathcal{F}_\epsilon^{\perp}, g^{F_\epsilon^{\perp}}) \); resp. \( (\mathcal{F}_\epsilon^{\perp,1}, g^{F_\epsilon^{\perp,1}}) \)). Then by \([13, (2.17)]\) one has

\begin{equation}
(2.26) \quad \left( D_{\mathcal{F}_{\epsilon}^{\perp}}^{\epsilon,R} + \frac{\overline{\mathcal{C}}(\overline{\sigma})}{\beta} + \frac{W_{f,R}}{\beta} \right)^2 = \left( D_{\mathcal{F}_{\epsilon}^{\perp}}^{\epsilon,R} + \frac{\overline{\mathcal{C}}(\overline{\sigma})}{\beta} \right)^2 + \sum_{i=1}^{q} \beta^{-1} c_{\beta,\gamma} (\beta^{-1} f_i) \left[ \nabla_{f_i}^{\mathcal{F}_{\epsilon},R} \cdot \frac{W_{f,R}}{\beta} \right] + \sum_{s=1}^{q_1} \gamma c_{\beta,\gamma} (\gamma h_s) \left[ \nabla_{h_s}^{\mathcal{F}_{\epsilon},R} \cdot \frac{W_{f,R}}{\beta} \right] + \sum_{j=1}^{q_2} c(e_j) \left[ \nabla_{e_j}^{\mathcal{F}_{\epsilon},R} \cdot \frac{W_{f,R}}{\beta} \right] + \frac{W_{f,R}^2}{\beta^2}.
\end{equation}

From \((2.5), (2.10)\) and \((2.17)\), one has

\begin{equation}
(2.27) \quad \left[ \nabla^{\mathcal{F}_{\epsilon},R} \cdot W_{f,R} \right] = \tilde{f}_{\epsilon,R} \left( \left[ \nabla^{E_0} + \nabla^{E_1} \right] \right) = 0 \quad \text{on} \quad \mathcal{M}'_{\mathcal{H}_{\epsilon,R}},
\end{equation}

while for any \( X \in \mathcal{F}_{\epsilon}^{\perp,2} \), one has

\begin{equation}
(2.28) \quad \left[ \nabla_{X}^{\mathcal{F}_{\epsilon},R} \cdot W_{f,R} \right] = \tilde{f}_{\epsilon,R} \left( \left[ \nabla^{E_0}_{f,R+\epsilon(X)} + \nabla^{E_1}_{f,R+\epsilon(X)} \right] \right) = 0 \quad \text{on} \quad \mathcal{M}_{\mathcal{H}_{\epsilon,R}}.
\end{equation}

Also, since \( g^{F_\epsilon} = \pi_\epsilon^* g^{F_\epsilon} \), one has via \((2.1)\) and the first equality in \((2.28)\) that for any \( X \in \mathcal{F}_{\epsilon}\),

\begin{equation}
(2.29) \quad \left[ \nabla_{X}^{\mathcal{F}_{\epsilon},R} \cdot W_{f,R} \right] = O(\epsilon |X|) \quad \text{on} \quad \mathcal{M}_{\mathcal{H}_{\epsilon,R}},
\end{equation}

and that for any \( X \in \mathcal{F}_{\epsilon}^{\perp,1}\),

\begin{equation}
(2.30) \quad \left[ \nabla_{X}^{\mathcal{F}_{\epsilon},R} \cdot W_{f,R} \right] = O_{\epsilon,R}(|X|) \quad \text{on} \quad \mathcal{M}_{\mathcal{H}_{\epsilon,R}}.
\end{equation}
From (2.27)-(2.30), one gets\(^5\)

\[
\sum_{i=1}^{q} \beta^{-1} c_{\beta, \gamma} (\beta^{-1} f_i) \left[ \nabla_{\frac{\varepsilon}{\beta}}^{\varepsilon, R} - \frac{W_{\hat{f}_i}}{\beta} \right] + \sum_{s=1}^{q_2} \gamma c_{\beta, \gamma} (\beta h_s) \left[ \nabla_{\frac{h_s}{\beta}}^{\varepsilon, R} - \frac{W_{\hat{f}_s}}{\beta} \right] + \sum_{j=1}^{q_1} c(e_j) \left[ \nabla_{\frac{e_j}{\beta}}^{\varepsilon, R} - \frac{W_{\hat{f}_j}}{\beta} \right] = O \left( \frac{\varepsilon}{\beta^2} \right) + O_{\varepsilon, R} \left( \frac{\gamma}{\beta} \right).
\]

Similarly, by proceeding as in (2.31) and \cite[(1.21)]{14}, one has for \(j = 1, 2\) that

\[
|c_{\beta, \gamma} (d\varphi_{\varepsilon, R,j})| = O \left( \frac{\varepsilon}{\beta} \right) + O_{\varepsilon, R}(\gamma).
\]

From (2.26) and (2.31), one has

\[
\sum_{j=1}^{2} \left\| D^{\varepsilon, R} - \frac{\tilde{c}(\sigma)}{\beta} + \frac{W_{\hat{f}_j}}{\beta} \right\|^2 \left( \varphi_{\varepsilon, R, j}s \right) = \sum_{j=1}^{2} \left\| D^{\varepsilon, R} - \frac{\tilde{c}(\sigma)}{\beta} + \frac{W_{\hat{f}_j}}{\beta} \right\|^2 \left( \varphi_{\varepsilon, R, j}s \right) + \frac{1}{\beta^2} \left\| W_{\hat{f}_j} \right\|^2 + O \left( \frac{\varepsilon}{\beta} \right) \left\| s \right\|^2 + O_{\varepsilon, R} \left( \frac{\gamma}{\beta} \right) \left\| s \right\|^2.
\]

By (2.13)-(2.15) and proceeding as in \cite[p. 1058-1059]{13}, one gets

\[
\left\| D^{\varepsilon, R} - \frac{\tilde{c}(\sigma)}{\beta} + \frac{W_{\hat{f}_j}}{\beta} \right\|^2 \left( \varphi_{\varepsilon, R, 1}s \right) \geq \frac{\delta}{4\beta^2} \left\| \varphi_{\varepsilon, R, 1}s \right\|^2 + O_{\varepsilon, R} \left( \frac{\gamma^2}{\beta^2} \right) \left\| \varphi_{\varepsilon, R, 1}s \right\|^2.
\]

From (2.21), we know that

\[
\varphi_{\varepsilon, R, 2} W_{\hat{f}_j} s \geq \delta_1 \left\| \varphi_{\varepsilon, R, 2}s \right\|^2.
\]

From (2.33)-(2.35), one finds

\[
\sum_{j=1}^{2} \left\| D^{\varepsilon, R} - \frac{\tilde{c}(\sigma)}{\beta} + \frac{W_{\hat{f}_j}}{\beta} \right\|^2 \left( \varphi_{\varepsilon, R, j}s \right) \geq \frac{\delta}{4\beta^2} \left\| s \right\|^2 + O_{\varepsilon, R} \left( \frac{\gamma^2}{\beta^2} \right) \left\| s \right\|^2.
\]

From (2.25), (2.32) and (2.36), one gets (2.19) easily.

---

\(^5\)This is the formula which depends essentially on the enlargeability condition (2.1), where the so-called area enlargeability is not enough.
To prove (2.20), for any smooth section \( s \) in question, one has as in (2.25) that

\[
\begin{align*}
(2.37) \quad \sqrt{2} \left\| \left( h \left( \frac{\rho}{R} \right) D^{\varepsilon R}_{\mathcal{F}_\varepsilon \oplus \mathcal{F}^{1}_{\varepsilon,1,\beta,\gamma}} h \left( \frac{\rho}{R} \right) + \frac{\hat{c}(\sigma)}{\beta} + \frac{W_{\hat{f},R}}{\beta} \right) s \right\| \\
\geq \left\| \left( h \left( \frac{\rho}{R} \right) D^{\varepsilon R}_{\mathcal{F}_\varepsilon \oplus \mathcal{F}^{1}_{\varepsilon,1,\beta,\gamma}} h \left( \frac{\rho}{R} \right) + \frac{\hat{c}(\sigma)}{\beta} + \frac{W_{\hat{f},R}}{\beta} \right) (\varphi_{\varepsilon,R,1}s) \right\| \\
+ \left\| \left( h \left( \frac{\rho}{R} \right) D^{\varepsilon R}_{\mathcal{F}_\varepsilon \oplus \mathcal{F}^{1}_{\varepsilon,1,\beta,\gamma}} h \left( \frac{\rho}{R} \right) + \frac{\hat{c}(\sigma)}{\beta} + \frac{W_{\hat{f},R}}{\beta} \right) (\varphi_{\varepsilon,R,2}s) \right\| - \left\| c_{\beta,\gamma} (d\varphi_{\varepsilon,R,1}) s \right\| \\
- \left\| c_{\beta,\gamma} (d\varphi_{\varepsilon,R,2}) s \right\|.
\end{align*}
\]

Clearly (cf. \[13\] (2.29)),

\[
\begin{align*}
(2.38) \quad \left( h \left( \frac{\rho}{R} \right) D^{\varepsilon R}_{\mathcal{F}_\varepsilon \oplus \mathcal{F}^{1}_{\varepsilon,1,\beta,\gamma}} h \left( \frac{\rho}{R} \right) + \frac{\hat{c}(\sigma)}{\beta} + \frac{W_{\hat{f},R}}{\beta} \right)^2 \\
= \left( h \left( \frac{\rho}{R} \right) D^{\varepsilon R}_{\mathcal{F}_\varepsilon \oplus \mathcal{F}^{1}_{\varepsilon,1,\beta,\gamma}} h \left( \frac{\rho}{R} \right) + \frac{\hat{c}(\sigma)}{\beta} \right)^2 + h \left( \frac{\rho}{R} \right)^2 \left[ D^{\varepsilon R}_{\mathcal{F}_\varepsilon \oplus \mathcal{F}^{1}_{\varepsilon,1,\beta,\gamma}} \frac{W_{\hat{f},R}}{\beta} \right] + \frac{\beta^2}{2} \\
= \left( h \left( \frac{\rho}{R} \right) D^{\varepsilon R}_{\mathcal{F}_\varepsilon \oplus \mathcal{F}^{1}_{\varepsilon,1,\beta,\gamma}} h \left( \frac{\rho}{R} \right) \right)^2 + \frac{h \left( \frac{\rho}{R} \right)^2}{\beta^2} \left[ D^{\varepsilon R}_{\mathcal{F}_\varepsilon \oplus \mathcal{F}^{1}_{\varepsilon,1,\beta,\gamma}} \hat{c}(\sigma) \right] + \frac{\beta^2}{2} \\
+ \frac{h \left( \frac{\rho}{R} \right)^2}{\beta^2} \left[ D^{\varepsilon R}_{\mathcal{F}_\varepsilon \oplus \mathcal{F}^{1}_{\varepsilon,1,\beta,\gamma}} W_{\hat{f},R} \right] + \frac{W_{\hat{f},R}^2}{\beta^2}.
\end{align*}
\]

From (2.31) and the first equality in (2.38), one has

\[
(2.39) \quad \left\| \left( h \left( \frac{\rho}{R} \right) D^{\varepsilon R}_{\mathcal{F}_\varepsilon \oplus \mathcal{F}^{1}_{\varepsilon,1,\beta,\gamma}} h \left( \frac{\rho}{R} \right) + \frac{\hat{c}(\sigma)}{\beta} + \frac{W_{\hat{f},R}}{\beta} \right) (\varphi_{\varepsilon,R,2}s) \right\|^2 \\
\geq \frac{1}{\beta^2} \left\| \varphi_{\varepsilon,R,2} W_{\hat{f},R} s \right\|^2 + O \left( \frac{\varepsilon}{\beta^2} \right) \left\| s \right\|^2 + O_{\varepsilon,R} \left( \frac{1}{\beta R} \right) \left\| s \right\|^2.
\]

By proceeding as in \[13\] (2.27), one has on \( \mathcal{M}_{\varepsilon,R} \setminus s(\mathcal{M}_{\varepsilon}) \) that

\[
(2.40) \quad \left[ D^{\varepsilon R}_{\mathcal{F}_\varepsilon \oplus \mathcal{F}^{1}_{\varepsilon,1,\beta,\gamma}} \hat{c}(\sigma) \right] = O_{\varepsilon} \left( \frac{1}{\beta R} \right) + O_{\varepsilon,R}(1).
\]

From (2.16), (2.31), the second equality in (2.38) and (2.40), one gets

\[
(2.41) \quad \left\| \left( h \left( \frac{\rho}{R} \right) D^{\varepsilon R}_{\mathcal{F}_\varepsilon \oplus \mathcal{F}^{1}_{\varepsilon,1,\beta,\gamma}} h \left( \frac{\rho}{R} \right) + \frac{\hat{c}(\sigma)}{\beta} + \frac{W_{\hat{f},R}}{\beta} \right) (\varphi_{\varepsilon,R,1}s) \right\|^2 \geq \frac{1}{\beta^2} \left\| \varphi_{\varepsilon,R,1}s \right\|^2 \\
+ \frac{1}{\beta^2} \left\| \varphi_{\varepsilon,R,1} W_{\hat{f},R} s \right\|^2 + O \left( \frac{\varepsilon}{\beta^2} \right) \left\| s \right\|^2 + O_{\varepsilon} \left( \frac{1}{\beta^2 R} \right) \left\| s \right\|^2 + O_{\varepsilon,R} \left( \frac{1}{\beta} \right) \left\| s \right\|^2.
\]
From (2.35), (2.39) and (2.41), one gets

\begin{equation}
\sum_{j=1}^{2} \left\| \left( h \left( \frac{\rho}{R} \right) D_{F_\epsilon \oplus F_{\epsilon,1}^{\perp},\beta,\gamma}^{\epsilon,R} h \left( \frac{\rho}{R} \right) + \frac{\widehat{c}(\sigma)}{\beta} + \frac{W_{\hat{f}_{\epsilon,R}}}{\beta} \right) (\varphi_{\epsilon,R,j}s) \right\|^2 \geq \frac{1}{\beta^2} \| \varphi_{\epsilon,R,1}s \|^2 + \frac{1}{\beta^2} \| W_{\hat{f}_{\epsilon,R}}s \|^2 + O \left( \frac{\varepsilon}{\beta^2} \right) \| s \|^2 + O_{\epsilon,R} \left( \frac{1}{\beta^2 R} \right) \| s \|^2 + O_{\epsilon,R} \left( \frac{1}{\beta} \right) \| s \|^2.
\end{equation}

From (2.32) and (2.42), one gets (2.20) easily. \qed

2.4. Elliptic operators on \( \tilde{N}_{\epsilon,R} \). Let \( Q \) be a Hermitian vector bundle over \( \tilde{N}_{\epsilon,R} \) such that \( (S_{\beta,\gamma} (F_\epsilon \oplus F_{\epsilon,1}^{\perp}) \otimes \Lambda^* (F_{\epsilon,2}^{\perp}) \otimes \mathcal{E}_{\epsilon,R})_+ \oplus Q \) is a trivial vector bundle over \( \tilde{N}_{\epsilon,R} \). Then \( (S_{\beta,\gamma} (F_\epsilon \oplus F_{\epsilon,1}^{\perp}) \otimes \Lambda^* (F_{\epsilon,2}^{\perp}) \otimes \mathcal{E}_{\epsilon,R})_+ \oplus Q \) is a trivial vector bundle near \( \partial \tilde{N}_{\epsilon,R} \) under the identification \( \widehat{c}(\sigma) + \hat{f}_{\epsilon,R}(w) + \text{Id}_Q \).

By obviously extending the above trivial vector bundles to \( N_{\epsilon,R} \), we get a \( \mathbb{Z}_2 \)-graded Hermitian vector bundle \( \xi = \xi_+ \oplus \xi_- \) over \( \tilde{N}_{\epsilon,R} \) and an odd self-adjoint endomorphism \( V = v + v^* \in \Gamma(\text{End}(\xi)) \) (with \( v : \Gamma(\xi_+) \to \Gamma(\xi_-) \), \( v^* \) being the adjoint of \( v \)) such that

\begin{equation}
\xi_\pm = (S_{\beta,\gamma} (F_\epsilon \oplus F_{\epsilon,1}^{\perp}) \otimes \Lambda^* (F_{\epsilon,2}^{\perp}) \otimes \mathcal{E}_{\epsilon,R})_\pm \oplus Q
\end{equation}

over \( \tilde{N}_{\epsilon,R} \), \( V \) is invertible on \( N_{\epsilon,R} \) and

\begin{equation}
V = \widehat{c}(\sigma) + W_{\hat{f}_{\epsilon,R}} + \begin{pmatrix}
0 & \text{Id}_Q \\
\text{Id}_Q & 0
\end{pmatrix}
\end{equation}
on \( \tilde{N}_{\epsilon,R} \), which is invertible on \( \tilde{N}_{\epsilon,R} \setminus M_{\epsilon,R}^{\frac{\varepsilon}{\beta^2}} \).

Recall that \( h(\frac{\rho}{R}) \) vanishes near \( M_{\epsilon,R} \cap \partial M_{\epsilon,R} \). We extend it to a function on \( \tilde{N}_{\epsilon,R} \) which equals to zero on \( N_{\epsilon,R} \) and an open neighborhood of \( \partial \tilde{N}_{\epsilon,R} \) in \( \tilde{N}_{\epsilon,R} \), and we denote the resulting function on \( \tilde{N}_{\epsilon,R} \) by \( \tilde{h}_R \).

Let \( \pi_{\tilde{N}_{\epsilon,R}} : T\tilde{N}_{\epsilon,R} \to \tilde{N}_{\epsilon,R} \) be the projection of the tangent bundle of \( \tilde{N}_{\epsilon,R} \). Let \( \gamma^\tilde{N}_{\epsilon,R} : \text{Hom}(\pi_{\tilde{N}_{\epsilon,R}}^*, \xi_+^*, \pi_{\tilde{N}_{\epsilon,R}}^*, \xi_-) \) be the symbol defined by

\begin{equation}
\gamma^\tilde{N}_{\epsilon,R}(p,u) = \pi_{\tilde{N}_{\epsilon,R}}^* \left( \sqrt{-1} \tilde{h}_R^2 c_{\beta,\gamma}(u) + v(p) \right) \quad \text{for} \quad p \in \tilde{N}_{\epsilon,R}, \quad u \in T_p\tilde{N}_{\epsilon,R}.
\end{equation}

By (2.44) and (2.45), \( \gamma^\tilde{N}_{\epsilon,R} \) is singular only if \( u = 0 \) and \( p \in M_{\epsilon,R}^{\frac{\varepsilon}{\beta^2}} \). Thus \( \gamma^\tilde{N}_{\epsilon,R} \) is an elliptic symbol.

On the other hand, it is clear that \( \tilde{h}_R D_{F_\epsilon \oplus F_{\epsilon,1}^{\perp},\beta,\gamma}^{\epsilon,R} \tilde{h}_R \) is well defined on \( \tilde{N}_{\epsilon,R} \) if we define it to equal to zero on \( \tilde{N}_{\epsilon,R} \setminus \tilde{N}_{\epsilon,R} \).

Let \( A : L^2(\xi) \to L^2(\xi) \) be a second order positive elliptic differential operator on \( \tilde{N}_{\epsilon,R} \) preserving the \( \mathbb{Z}_2 \)-grading of \( \xi = \xi_+ \oplus \xi_- \), such that its symbol equals to \( |\eta|^2 \) at \( \eta \in T\tilde{N}_{\epsilon,R} \). As in [13] (2.33), let \( P_{\epsilon,R,\beta,\gamma}^{\tilde{N}_{\epsilon,R}} : L^2(\xi) \to L^2(\xi) \) be the zeroth order symbol.
pseudodifferential operator on \( \tilde{\mathcal{N}}_{\varepsilon,R} \) defined by

\begin{equation}
(2.46) \quad \tilde{P}^{\varepsilon,R}_{\varepsilon,R,\beta,\gamma} = A^{-\frac{1}{4}} \tilde{h}_R D^{\varepsilon,R}_{F_{\varepsilon} \oplus F_{\varepsilon},\beta,\gamma} \tilde{h}_R A^{-\frac{1}{4}} + \frac{V}{\beta},
\end{equation}

Let \( \tilde{P}^{\varepsilon,R}_{\varepsilon,R,\beta,\gamma,+} : L^2(\xi^+) \to L^2(\xi^-) \) be the obvious restriction. Then the principal symbol of \( \tilde{P}^{\varepsilon,R}_{\varepsilon,R,\beta,\gamma,+} \), which we denote by \( \gamma(\tilde{P}^{\varepsilon,R}_{\varepsilon,R,\beta,\gamma,+}) \), is homotopic through elliptic symbols to \( \gamma(\tilde{N}_{\varepsilon,R}) \). Thus \( \tilde{P}^{\varepsilon,R}_{\varepsilon,R,\beta,\gamma,+} \) is a Fredholm operator. Moreover, by the Atiyah-Singer index theorem [1] (cf. [9, Th. 13.8 of Ch. III]) and the computation in [8, §5], one finds

\begin{equation}
(2.47) \quad \text{ind} \left( \tilde{P}^{\varepsilon,R}_{\varepsilon,R,\beta,\gamma,+} \right) = \text{ind} \left( \gamma(\tilde{P}^{\varepsilon,R}_{\varepsilon,R,\beta,\gamma,+}) \right) = \text{ind} \left( \gamma(\tilde{N}_{\varepsilon,R}) \right) = \langle \tilde{A} \left( T \tilde{M}_{H_{\varepsilon}} \right) (\text{ch}(f_{\varepsilon}E_0) - \text{ch}(f_{\varepsilon}E_1)), \left[ \tilde{M}_{H_{\varepsilon}} \right] \rangle = (\deg(f_{\varepsilon})) \langle \text{ch}(E_0), [S^{\dim M}(1)] \rangle \neq 0,
\end{equation}

where the inequality comes from (2.41).

For any \( 0 \leq t \leq 1 \), set

\begin{equation}
(2.48) \quad \tilde{P}^{\varepsilon,R}_{\varepsilon,R,\beta,\gamma,+}(t) = \tilde{P}^{\varepsilon,R}_{\varepsilon,R,\beta,\gamma,+} + \frac{(t - 1)v}{\beta} + A^{-\frac{1}{4}} \frac{(1 - t)v}{\beta} A^{-\frac{1}{4}}.
\end{equation}

Then \( \tilde{P}^{\varepsilon,R}_{\varepsilon,R,\beta,\gamma,+}(t) \) is a smooth family of zeroth order pseudodifferential operators such that the corresponding symbol \( \gamma(\tilde{P}^{\varepsilon,R}_{\varepsilon,R,\beta,\gamma,+}(t)) \) is elliptic for \( 0 < t \leq 1 \). Thus \( \tilde{P}^{\varepsilon,R}_{\varepsilon,R,\beta,\gamma,+}(t) \) is a continuous family of Fredholm operators for \( 0 < t \leq 1 \) with \( \tilde{P}^{\varepsilon,R}_{\varepsilon,R,\beta,\gamma,+}(1) = \tilde{P}^{\varepsilon,R}_{\varepsilon,R,\beta,\gamma,+} \).

Now since \( \tilde{P}^{\varepsilon,R}_{\varepsilon,R,\beta,\gamma,+}(t) \) is continuous on the whole \( [0,1] \), if \( \tilde{P}^{\varepsilon,R}_{\varepsilon,R,\beta,\gamma,+}(0) \) is Fredholm and has vanishing index, then we would reach a contradiction with respect to (2.47), and then complete the proof of Theorem 1.2.

Thus we need only to prove the following analogue of [13, Proposition 2.5].

**Proposition 2.2.** There exist \( \varepsilon, R, \beta, \gamma > 0 \) such that the following identity holds:

\begin{equation}
(2.49) \quad \dim \left( \ker \left( \tilde{P}^{\varepsilon,R}_{\varepsilon,R,\beta,\gamma,+}(0) \right) \right) = \dim \left( \ker \left( \tilde{P}^{\varepsilon,R}_{\varepsilon,R,\beta,\gamma,+}(0)^* \right) \right) = 0.
\end{equation}

**Proof.** Let \( \tilde{P}^{\varepsilon,R}_{\varepsilon,R,\beta,\gamma}(0) : L^2(\xi) \to L^2(\xi) \) be given by

\begin{equation}
(2.50) \quad \tilde{P}^{\varepsilon,R}_{\varepsilon,R,\beta,\gamma}(0) = A^{-\frac{1}{4}} \tilde{h}_R D^{\varepsilon,R}_{F_{\varepsilon},\beta,\gamma} \tilde{h}_R A^{-\frac{1}{4}} + A^{-\frac{1}{4}} \frac{V}{\beta} A^{-\frac{1}{4}}.
\end{equation}

Since \( \tilde{P}^{\varepsilon,R}_{\varepsilon,R,\beta,\gamma}(0) \) is formally self-adjoint, by (2.46) and (2.48) we need only to show that

\begin{equation}
(2.51) \quad \dim \left( \ker \left( \tilde{P}^{\varepsilon,R}_{\varepsilon,R,\beta,\gamma}(0) \right) \right) = 0
\end{equation}

for certain \( \varepsilon, R, \beta, \gamma > 0 \).

Let \( s \in \ker(\tilde{P}^{\varepsilon,R}_{\varepsilon,R,\beta,\gamma}(0)) \). By (2.50) one has

\begin{equation}
(2.52) \quad \left( \tilde{h}_R D^{\varepsilon,R}_{F_{\varepsilon},\beta,\gamma} \tilde{h}_R + \left( \frac{V}{\beta} \right) \right) A^{-\frac{1}{4}} s = 0.
\end{equation}

Since \( \tilde{h}_R = 0 \) on \( \tilde{\mathcal{N}}_{\varepsilon,R} \setminus \tilde{\mathcal{M}}_{\mathcal{H}_{\varepsilon,R}} \), while \( V \) is invertible on \( \tilde{\mathcal{N}}_{\varepsilon,R} \setminus \tilde{\mathcal{M}}_{\mathcal{H}_{\varepsilon,R}} \), by (2.52) one has

\begin{equation}
(2.53) \quad A^{-\frac{1}{4}} s = 0 \text{ on } \tilde{\mathcal{N}}_{\varepsilon,R} \setminus \tilde{\mathcal{M}}_{\mathcal{H}_{\varepsilon,R}}.
\end{equation}
Write on $\mathcal{K}_{h_\epsilon,R}$ that

\begin{equation}
A^{-\frac{1}{2}}s = s_1 + s_2,
\end{equation}

with $s_1 \in L^2(S_{\beta,\gamma} (F_\epsilon \oplus F_{\epsilon,1}^\perp) \otimes \Lambda^* (F_{\epsilon,2}^\perp) \otimes E_{\epsilon,R})$ and $s_2 \in L^2(Q \oplus Q)$.

By (2.44), (2.52) and (2.54), one has

\begin{equation}
s_2 = 0,
\end{equation}

while

\begin{equation}
\left( \tilde{h}_R D_{F_\epsilon \oplus F_{\epsilon,1}^\perp}^{\beta,\gamma} \tilde{h}_R + \frac{\tilde{c}(\tilde{\sigma})}{\beta} + \frac{W_{f_\epsilon,R}}{\beta} \right) s_1 = 0.
\end{equation}

We need to show that (2.56) implies $s_1 = 0$.

As in (2.37), one has

\begin{equation}
\sqrt{2} \left\| \left( \tilde{h}_R D_{F_\epsilon \oplus F_{\epsilon,1}^\perp}^{\beta,\gamma} \tilde{h}_R + \frac{\tilde{c}(\tilde{\sigma})}{\beta} + \frac{W_{f_\epsilon,R}}{\beta} \right) s_1 \right\|
\geq \left\| \left( \tilde{h}_R D_{F_\epsilon \oplus F_{\epsilon,1}^\perp}^{\beta,\gamma} \tilde{h}_R + \frac{\tilde{c}(\tilde{\sigma})}{\beta} + \frac{W_{f_\epsilon,R}}{\beta} \right) (\varphi_{\epsilon,R,1}s_1) \right\|
\end{equation}

\begin{equation}
+ \left\| \left( \tilde{h}_R D_{F_\epsilon \oplus F_{\epsilon,1}^\perp}^{\beta,\gamma} \tilde{h}_R + \frac{\tilde{c}(\tilde{\sigma})}{\beta} + \frac{W_{f_\epsilon,R}}{\beta} \right) (\varphi_{\epsilon,R,2}s_1) \right\| - \| c_{\beta,\gamma} (d\varphi_{\epsilon,R,1}) s_1 \|
- \| c_{\beta,\gamma} (d\varphi_{\epsilon,R,2}) s_1 \|.\end{equation}

By proceeding as in the proof of (2.39), one gets

\begin{equation}
\left\| \left( \tilde{h}_R D_{F_\epsilon \oplus F_{\epsilon,1}^\perp}^{\beta,\gamma} \tilde{h}_R + \frac{\tilde{c}(\tilde{\sigma})}{\beta} + \frac{W_{f_\epsilon,R}}{\beta} \right) (\varphi_{\epsilon,R,2}s_1) \right\|^2
\geq \frac{1}{\beta^2} \left\| \varphi_{\epsilon,R,2} W_{f_\epsilon,R} s_1 \right\|^2 + O \left( \frac{\epsilon}{\beta^2} \right) \| s_1 \|^2 + O_{\epsilon,R} \left( \frac{\gamma}{\beta} \right) \| s_1 \|^2.
\end{equation}

On the other hand, by using Lemma 2.1 and proceeding as in [13, p. 1062], one finds that there exist $c_1 > 0$, $\epsilon > 0$ and $R > 0$ such that when $\beta, \gamma > 0$ are sufficiently small, one has

\begin{equation}
\left\| \left( \tilde{h}_R D_{F_\epsilon \oplus F_{\epsilon,1}^\perp}^{\beta,\gamma} \tilde{h}_R + \frac{\tilde{c}(\tilde{\sigma})}{\beta} + \frac{W_{f_\epsilon,R}}{\beta} \right) (\varphi_{\epsilon,R,1}s_1) \right\| \geq \frac{c_1}{\beta} \| \varphi_{\epsilon,R,1}s_1 \|.
\end{equation}

From (2.32), (2.35) and (2.57)-(2.59), one finds that there exist $c_2 > 0$, $\epsilon > 0$ and $R > 0$ such that when $\beta, \gamma > 0$ are sufficiently small, one has

\begin{equation}
\left\| \left( \tilde{h}_R D_{F_\epsilon \oplus F_{\epsilon,1}^\perp}^{\beta,\gamma} \tilde{h}_R + \frac{\tilde{c}(\tilde{\sigma})}{\beta} + \frac{W_{f_\epsilon,R}}{\beta} \right) s_1 \right\| \geq \frac{c_2}{\beta} \| s_1 \|,
\end{equation}

which implies, via (2.56), $s_1 = 0$.  \hfill \Box

**Remark 2.3.** By combining the above method with what in [13, §2.5], one gets a proof of Theorem 1.3. We leave the details to the interested reader.
Remark 2.4. From the above proof, one sees that for Theorems 1.2 and 1.3 to hold, one need only to assume that (2.1) holds for \( X \in \Gamma(F_\varepsilon) \). Moreover, when \( M \) is compact and \( M_\varepsilon \) might be noncompact, the above proof can also be seen as to complete in details the proof of the main results in [14] for non-compactly enlargeable foliations.

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