Locally Preserving Projection on Symmetric Positive Definite Matrix Lie Group

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Abstract

Symmetric Positive Definite (SPD) matrices have been widely used as feature descriptors in image recognition. However, the dimension of an SPD matrix built by image feature descriptors is usually high. So SPD matrices oriented dimensionality reduction techniques are needed. The existing manifold learning algorithms only apply to reduce the dimension of high dimensional vector-form data. For high dimensional SPD matrices, it is impossible to directly use manifold learning algorithms to reduce the dimension of matrix-form data, but we need first transform the matrix into a long vector and then reduce the dimension of this vector. This however breaks the spatial structure of the SPD matrix space. To overcome this limitation, we propose a new dimension reduction algorithm on SPD matrix space to transform the high dimensional SPD matrices to lower dimensional SPD matrices. Our work is based on the fact that the set of all SPD matrices with the same size is known to have a Lie group structure and we aims to transform the manifold learning algorithm to SPD matrix Lie group. We make use of the basic idea of manifold learning algorithm LPP (locality preserving projection) to construct the corresponding Laplacian matrix on SPD matrix Lie group. Thus we call our approach Lie-LPP to emphasize its Lie group character. Finally our method gets a lower dimensional and more discriminable SPD matrix Lie group. We also show by experiments that our approach achieves effective results on Human action recognition and Human face recognition.

Keywords: manifold learning, SPD matrix Lie group, Locally preserving

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1. Introduction

Image recognition including dynamic action recognition and static image recognition is a very popular research project in the field of machine vision and pattern recognition [1] [2] [3] [14]. This technique has a wide range of application in many fields, such as intelligent video retrieval, perceived interaction and so on. One key step of image recognition is to construct a high quality image feature descriptor, which determines the accuracy rate of recognition. The original image feature descriptor is a pixel matrix. It is usual to transform a pixel matrix to a high dimensional row feature vector before recognition because it is hard to directly do image recognition on the pixel matrix space. In order to diminish the negative effect of dimension curse, a large number of dimensionality reduction algorithms have been invented to implement image recognition [1] [9]. However the vector-form feature descriptor breaks the geometric structure of pixel matrix space and is highly sensitive to various factors such as illumination intensity, background, object location and so on. To avoid these disadvantages, tensor space dimensionality reduction based on LPP (locality preserving projection) was proposed in [6], which is linear and deals with the image pixel matrix directly. On the other hand, [12] presented a new form of feature descriptor via computing a feature covariance matrix within any size region in an image, which preserves the local geometric structure of the image pixel matrix (see appendix A below). The method of covariance matrix descriptor has mainly been used in static image recognition [10] [12] [14] [11]. For human action recognition, [4] proposed a new method based on spatial-temporal covariance descriptors. [13] applied covariance matrices as descriptors of 3D skeleton location by constructing covariance matrices on 3D joint locations.

Generally the dimensions of covariance matrices calculated from image regions are especially high. Non-singular covariance matrix is an SPD matrix. In addition the space of $D \times D$ SPD matrices forms a Riemannian manifold. If we do image recognition on this high dimensional curved SPD manifold, the recognition quality is very poor and time cost tremendous high. Harandi et al. [18] proposed to learn a kernel function to first map the SPD matrices into a higher dimensional Euclidean space and then use traditional manifold learning algorithm LPP to reduce their dimension. But this method would distort the geometric and algebraic structure of the SPD manifold and would lose a lot of important structure information. To overcome this limitation, Harandi et al. [19] suggested to map the original
SPD manifold to a Grassmann manifold and then solve the optimization problem on the latter. But the time cost for running this method is especially high. Another similar work was done by Huang et al. [7] who proposed transforming SPD matrices to their corresponding tangent space and learning a linear dimensionality reduction map on that tangent space. However, this algorithm needs several parameters which are sensitive factors influencing the algorithm. In addition, this method only preserved the local structure of manifold and did not analysis the global structure. On the whole, all of the three methods [7] [18] [19] need to get help of a linear space and rely on nonlinear dimensionality reduction mappings.

It has already been put forward that the set of SPD matrices with same dimension forms a Riemannian manifold. What’s more, this SPD Riemannian manifold has a group structure formed SPD matrix Lie group $S^D_+$. The group operation in [7] is:

$$S_1, S_2 \in S^D_+, S_1 \odot S_2 = \exp (\log (S_1) + \log (S_2)).$$

Due to the matrix-form data of matrix Lie group, it is difficult to directly reduce the dimension of high dimensional matrix Lie group without breaking the structure of matrix-form.

Recent years manifold learning has become a significant branch of machine learning, which aims to reduce the dimension of high dimensional vector-form data. One popular idea of manifold learning is to construct the corresponding graph Laplacian matrix on data set, such as LPP [5] and LEP [17]. However, this idea only applies to vector-form data. According to the definition of covariance matrix in [8] [12] each covariance matrix can be represented by the product of a set of feature vectors. It summarizes the linear bivariate relationships among a set of variables. Since a covariance matrix is represented by the product of feature vectors and is bilinear, there is an intuitive idea that we can solve the dimensionality reduction problem directly on the SPD matrix Lie group. We extend the idea of LPP [5] to dimensionality reduction learning on SPD matrix Lie group and attempt to construct the corresponding Laplacian matrix which reflects the intrinsic structure of SPD matrix Lie group. We develop the Lie-LPP algorithm to get a dimension reduction algorithm on SPD matrix Lie group, which preserves the intrinsic geometric and algebraic structure of original SPD matrix Lie group.

The main contributions of our work can be summarized as follows:

- Extend the LPP algorithm in [5] to Lie-LPP and apply it on SPD matrix Lie group. Obtain a bi-linear dimensionality reduction mapping on SPD matrix
Lie group, which preserves the intrinsic geometric and algebraic structure of SPD matrix Lie group.

- To overcome the limitation of other methods regarding dimensionality reduction of SPD matrix Lie group, our method solves the dimensionality reduction problem on the SPD matrix Lie group directly without mapping to other spaces and without needing a lot of sensitive parameters, which is more simple and straightforward.

- Construct the graph Laplacian matrix on SPD matrix Lie group to reflect the intrinsic geometric and algebraic structure of the original SPD matrix Lie group.

2. Background

The data points that we consider in this paper are non-singular covariance matrices called SPD matrices. So in this section we first give the definition of covariance matrix and analyze the geometric relationship between covariance matrices and feature vectors. Then we give the definition of Riemannian metric and the corresponding geometric and algebraic properties of SPD matrix Lie group.

2.1. Covariance Matrix

Suppose a set of $n$ feature vectors from an image region is expressed as the following matrix $F$, where the dimensionality of each feature vector is $d$:

$$F = (f_1, f_2, \ldots, f_n),$$

$$f_i = (f_{i1}, f_{i2}, \ldots, f_{id})$$

is the $i^{th}$ feature vector. The corresponding covariance matrix $C$ with respect to these feature vectors is defined as [8]:

$$C = \frac{1}{n} FF^T = \frac{1}{n} \sum_{i=1}^{n} f_i f_i^T.$$

(1)

In this definition we assume the expectation of feature vectors is zero and each term $f_i f_i^T$ in the summation is the outer product of the feature vector $f_i$. Obviously covariance matrix is a symmetric positive semi-definite matrix. In this paper we just consider the non-singular covariance matrix called SPD matrix. When the feature vectors are adjacent, the corresponding covariance matrices are also adjacent. So the covariance matrices preserve the local geometric structure of the corresponding feature vectors. The detailed proof is showed in appendix A.
Covariance matrices (SPD matrices) have several advantages in serving as the feature descriptors of images. First, covariance matrices can fuse all of the features of images. Second, they provide a way of filtering noisy information such as the illumination, the location of the object in the image. In addition, the size of the covariance matrix is dependent on the dimensionality of feature vectors other than the size of image region, so we can construct the covariance matrices with the same size from different regions.

Vector space is a linear space and SPD matrix space is a bilinear space, so it is a natural idea to transform traditional manifold learning algorithms to SPD matrix Lie group. Traditional manifold learning algorithms (LLE [15], Isomap [16], LEP [17], LPP [5]) aim to uncover the lower dimensional manifold structure from high dimensional vector-form data points by analyzing the geometric structures among data points. In our case we need to learn the intrinsic geometric and algebraic structures of SPD matrix Lie group before reducing its dimension.

2.2. Geometric structure of SPD Matrix Lie group

In machine learning we usually have to learn an effective metric for comparing data points. In particular in the image recognition step, metric is required to measure the distance between two different image feature descriptors. In this paper we use the SPD matrices as feature descriptors of images. Thus it is necessary to construct the corresponding Riemannian metric on SPD matrix Lie group to compute the intrinsic geodesic distances between any two SPD matrices.

The SPD matrix Lie group that we consider in this paper is represented by $S^+_D$, where every point $S_1 \in S^+_D$ is a $D \times D$ dimension matrix. The tangent space of $S^+_D$ at the identity is $\text{Sym}(D)$, the space of symmetric matrices. The learnt lower dimensional SPD matrix Lie group is represented by $S^+_d$. The family of all scalar products on all tangent spaces of a SPD matrix Lie group is known as Riemannian metric. The geodesic distance $d_G(S_1, S_2)$ between any two points $S_1, S_2 \in S^+_D$ can be computed under this Riemannian metric. In this paper we choose Log-Euclidean metric from [23] as the Riemannian metric of SPD matrix Lie group.

**Definition 2.1. (Log-Euclidean Metric) [23]** The Riemannian metric at a point $S_1 \in S^+_D$ is a scalar product defined in the tangent space $T_{S_1} S^+_D$:

$$\langle T_1, T_2 \rangle = \langle D\text{log}.T_1, D\text{log}.T_2 \rangle,$$

where $T_1, T_2 \in T_{S_1} S^+_D$.

**Definition 2.2. (Bi-invariant Metric) [23]** Any bi-invariant metric on the Lie group of SPD matrices is also called a Log-Euclidean metric because it corre-
sponds to a Euclidean metric in the logarithm domain. The logarithm domain is the tangent space of SPD matrix Lie group.

**Corollary 2.3. (Flat Riemannian Manifold) [23]** Endowed with a bi-invariant metric, the space of SPD matrices is a flat Riemannian space: its sectional curvature is null everywhere.

So under Log-Euclidean metric, $S^+_D$ is a flat manifold and locally isometric to the tangent space $\text{Sym}(D)$. In local neighborhood the mapping between SPD matrix Lie group and the corresponding tangent space is represented by the exponential map and the inverse map is logarithm.

\[
\text{exp}_{S_1}(T_1) = \text{exp}(\log(S_1) + D_{S_1}\log \cdot T_1),
\]

\[
\log_{S_1}(S_2) = D_{\log(S_1)}\exp \cdot (\log(S_2) - \log(S_1)).
\]

where the exponential map is defined at point $S_1 \in S^+_D$, $T_1 \in T_{S_1}S^+_D$ a tangent vector, and the corresponding point of $S_2$ at $T_{S_1}S^+_D$ is $\log_{S_1}(S_2)$.

The geodesic distance between $S_1, S_2$ under log-Euclidean metric is defined as follows:

\[
d_{G}(S_1, S_2) = \langle \log_{S_1}(S_2), \log_{S_1}(S_2) \rangle = \| \log(S_1) - \log(S_2) \|^2_F.
\]

SPD Matrix Lie group is a complete manifold, so every two points on the SPD matrix Lie group are linked by a shortest geodesic line.

3. Algorithm

In this section we first analyze LPP algorithm. Then we give the construction process of graph Laplacian matrix on SPD matrix Lie group. In the last, we describe the process of our proposed dimensionality reduction algorithm Lie-LPP.

3.1. Locally Preserving Projection (LPP)

LPP aims to learn a linear dimensionality reduction map to reduce the dimension of high dimensional vector-form data points, which can be seen as a linear approximation of LEP [17]. This linear dimensionality reduction map optimally preserves local neighborhood geometric structure of data set by building a graph Laplacian matrix on the data set. The graph Laplacian matrix is a discrete approximation to a differential Laplace operator that arises from the manifold. Let $X = [x_1, x_2, \ldots, x_N] \subset \mathbb{R}^D$ is the input data set distributed on a $d$-dimensional manifold $\mathcal{M}$ which is embedded into $\mathbb{R}^D$. $a$ is the linear dimensionality reduction
map. The learned lower dimensional data set is \( Y = a^T X = [y_1, y_2, \cdots, y_D] \subset \mathbb{R}^d \).

The algorithm of LPP is stated as below:

- **Constructing the adjacency graph**: Denote \( G \) a graph with \( N \) nodes. If \( x_i \) and \( x_j \) are ”close”, link a connection between nodes \( i \) and \( j \). The ”closeness” between two nodes is measured by the \( K \)-nearest neighbor method.

- **Choosing the weights**: Here, the authors denote matrix \( W \) as the corresponding weight matrix, where \( W \) is a sparse symmetric \( N \times N \) matrix. The element \( W_{ij} \) is defined as follows:
  \[
  W_{ij} = e^{-\frac{\|x_i - x_j\|_2^2}{\epsilon}}, \text{ if nodes } i \text{ and } j \text{ are connected},
  \]
  \[
  W_{ij} = 0, \text{ if nodes } i \text{ and } j \text{ are not connected}.
  \]

- **Eigenmaps**: Solve the following generalized eigenvector problem to obtain the corresponding dimension reduction map \( a \):
  \[
  XLX^T a = \lambda XD X^T a
  \]  
  where \( D \) is a diagonal matrix, \( D_{ii} = \sum_j W_{ji} \). \( L = D - W \) is the graph Laplacian matrix.

So the critical step of LPP is to construct the graph Laplacian matrix on data points. The global lower dimensional representations are learnt by solving the corresponding generalized eigenfunction.

### 3.2. Laplace operator on SPD matrix Lie group

Laplace operator is a significant operator defined on Riemannian manifold [29]. It measures the intrinsic structure of manifold such as the curvature of manifold, the similarities among different points on Riemannian manifold. The Laplacian matrix on a graph is a discrete analogue of the Laplace operator that we’re familiar with in functional analysis [28]. The critical step of LPP algorithm is to construct a graph Laplacian matrix to represent the intrinsic local geometric structure of data points, which is only applied on vector-form data points. For Lie-LPP, we aim to uncover the intrinsic structure of SPD matrices which are essentially different with vector-form data points in spatial structure. Laplace operator is defined based on Riemannian metric. For vector-form data points, the Laplacian matrix is constructed based on Euclidean metric in each local patch.
For SPD matrices, we need to get help from the Log-Euclidean metric \cite{22} \cite{23} to construct the corresponding Laplacian matrix.

There are several difficulties for the learning of Laplacian matrix on SPD matrices. One critical difficulty is the discrete representations of the first and second order derivatives on SPD matrix space. Another difficulty is to find the representation of Laplace operator on SPD matrix Lie group which is very different from the structure on vector space. For constructing Laplacian matrix on SPD matrices, we need to solve these difficulties during the learning process. In addition the core of Lie-LPP is to construct an accurate Laplacian matrix on SPD matrix Lie group before dimensionality reduction. So it is necessary to state the construction process of Laplacian matrix on SPD matrices and give the descriptions for solving these difficulties in detail in a separate section.

In order to be better understood the construction of Laplacian matrix, we first give an intuitive example of dimension one. Consider a graph $G$ with $n$ nodes. Every node $i$ is adjacent to two nodes $i-1$ and $i+1$. If we assign value $v_i$ to node $i$, the Laplacian is represented as $(v_{i+1} - v_i) - (v_i - v_{i-1})$. Thus, $v_{i+1} - v_i$ is the discrete analogue for a first-order derivative defined over the real number line. $-2\left(v_i - \frac{1}{2}v_{i-1} - \frac{1}{2}v_{i+1}\right)$ is the discrete approximation of the second-order derivative. For higher dimensions the ’normalized’ graph Laplacian $\Delta_{nm}$ about function $f$ is defined as:

$$\left(\Delta_{nm} f\right)(i) = f(i) - \frac{1}{\text{deg}(i)} \sum_{j=1}^{n} W_{ij} f(j) ,$$

where the degree function $\text{deg}$ is defined as $\text{deg}(i) = \sum_{j=1}^{n} W_{ji}$. $W_{ji}$ is the heat kernel weight defined in the same way as in the second algorithmic procedure of LPP \cite{5}.

The Laplacian matrix we mentioned above is just defined on vector-form data points. In the following, we construct the ’normalized’ graph Laplacian matrix on SPD matrix Lie group endowed with Log-Euclidean metric. Suppose a parameterized SPD matrix Lie group $\Sigma$ defined as $\Sigma : \mathbb{R}^d \rightarrow \mathbb{S}^D_+$, we call the vector $\overrightarrow{\Sigma(x) \Sigma(x + u)}$ as the standard first-order derivative on $\mathbb{S}^D_+$ \cite{21}:

$$\overrightarrow{\Sigma(x) \Sigma(x + u)} = \Sigma(x)^{\frac{1}{2}} (\log \Sigma(x + u) - \log \Sigma(x)) \Sigma(x)^{\frac{1}{2}} .$$

(8)
The Laplace-Beltrami operator $\Delta$ about function $\Sigma$ is defined as:

$$
\Delta \Sigma = \sum_{i=1}^{d} \Delta_i \Sigma, \quad (9)
$$

$$
\Delta_i \Sigma = \partial_i^2 \Sigma - 2 (\partial_i \Sigma) \Sigma^{-1} (\partial_i \Sigma). \quad (10)
$$

In order to approximate the graph Laplacian matrix on SPD matrices, we need first to approximate the first and second order derivatives on SPD matrix Lie group. Under the approximation of second order derivative, the corresponding approximation of the Laplace-Beltrami operator on $\mathbb{S}^D$ [21] is:

$$
\Delta u \Sigma = \partial_i^2 u \Sigma - 2 (\partial_i u \Sigma) \Sigma^{-1} (\partial_i u \Sigma) \rightarrow \Sigma(x) \Sigma(x + u) + \Sigma(x) \Sigma(x - u) + O(||u||^4) \quad (11)
$$

In order to compute the complete SPD matrix Lie group Laplacian of Eq. 9, we just have to compute the Laplace operator along $d$ orthonormal directions.

In practical application, suppose $\Sigma_1, \Sigma_2, \ldots, \Sigma_N$ are a set of SPD matrices generated from $\Sigma$. The 'normal' graph Laplacian matrix on this data set is:

$$(\Delta_{nm} \Sigma_i) = \Sigma_i^{\frac{1}{2}} \left( \log(\Sigma_i) - \sum_{j=1}^{N} \tilde{W}_{ij} \log(\Sigma_j) \right) \Sigma_i^{\frac{1}{2}} \quad (11)$$

where $\tilde{W}_{ij} = e^{-\frac{||log(\Sigma_i) - log(\Sigma_j)||^2}{2t}}$, if $i$ and $j$ are connected, else $\tilde{W}_{ij} = 0$. The graph Laplacian matrix on SPD matrix Lie group is $\tilde{L} = \tilde{D} - \tilde{W}$, where $\tilde{W}$ is a symmetric matrix, $\tilde{W}_{ij}$ defined as above and $\tilde{D}$ a diagonal matrix, $\tilde{D}_{ii} = \sum_j \tilde{W}_{ji}$.

3.3. Lie-LPP Algorithm

Based on the definition of graph Laplacian matrix on $\mathbb{S}^D$ and the algorithmic procedures of LPP, we describe our Lie-LPP algorithm as follows.

3.3.1. Lie-LPP Algorithm

For SPD matrix Lie group $\mathbb{S}^D$, the SPD matrix logarithms in the tangent space are also symmetric matrices. The linear mapping between tangent spaces is defined as follows:

$$
f (\log (S_1)) = A^T \log (S_1) A. \quad (12)
$$

The corresponding mapping between SPD matrix Lie groups is :

$$
g (S_1) = \exp \circ f (\log (S_1)) = \exp \left( A^T \log (S_1) A \right), \quad (13)
$$
where \( S_1 \in S_D^+ \), \( A \) is a linear map matrix, \( f \) the corresponding map defined on Lie algebras and \( g \) the derived map defined on SPD matrix Lie groups.

It is easy to prove that \( g(S_1) \) is still an SPD matrix. In this paper we attempt to learn a transformation matrix \( A \), where \( A \in \mathbb{R}^{D \times d} \) is a full column rank matrix, \( D \times D \) the dimension of \( S_D^+ \) and \( d \times d \) the mapped dimension of \( \mathbb{S}^d \), \( D \gg d \). It has been proved that the linear mapping matrix \( A \) preserves the algebraic structure of \( S_D^+ \). In order to obtain a more discriminative SPD matrix Lie group \( \mathbb{S}^d \), \( A \) should also inherit and preserve the geometric structure of \( S_D^+ \). According to the idea of LPP, the key step of Lie-LPP algorithm is to construct the Laplacian matrix \( L \) on \( S_D^+ \) which reflects the local geometrical structure of \( S_D^+ \). Under Log-Euclidean metric, \( S_D^+ \) is locally isometric to the tangent space of \( S_D^+ \). So the geodesic distance between two SPD matrices is equal to the Euclidean distance between the corresponding points on tangent space.

Suppose the input data points are \( S_1, S_2, \ldots, S_N \in S_D^+ \), and the output sample points are \( Y_1, Y_2, \ldots, Y_N \in S^d \), where \( N \) is the number of sampled points.

The algorithm steps of Lie-LPP are as follows:

- The first step is to divide the input SPD matrices into a set of local patches. We use the \( K \)-nearest method to find the \( K \)-nearest neighborhoods \( U_i \) of every point \( S_i \), where the distance metric between two points is defined as their geodesic distance on SPD matrix Lie group \( S_D^+ \) using Eq. 5.

- The second step is to construct a weight matrix \( W \) on each local patch \( U_i \) to represent the local intrinsic geometric structure of \( U_i \).

\[
\tilde{W}_{ij} = e^{-\frac{\| \log(S_i) - \log(S_j) \|^2}{t}}, \text{ if } S_j \in U_i,
\]

\[
\tilde{W}_{ij} = 0, \text{ else if } S_j / U_i.
\]

The definition of weight value \( \tilde{W}_{ij} \) is based on the construction of Laplacian matrix on input SPD matrices from Eq. 11.

- The third step is to compute the eigenvectors and eigenvalues for the generalized eigenvector problem:

\[
S^T \tilde{L} S A = \lambda S^T \tilde{D} S A \tag{14}
\]

where \( S = [ \log(S_1), \log(S_2), \ldots, \log(S_N) ]^T \) is a partitioned matrix.

Details will be shown in the next subsection.
3.3.2. **Optimal Embedding**

The optimal dimension reduction map $A$ is obtained by minimizing the following energy function:

$$
\frac{1}{2} \sum_{i,j} d_G(Y_i, Y_j) \tilde{W}_{ij}, \quad (15)
$$

where $d_G(Y_i, Y_j)$ is the geodesic distance between $Y_i$ and $Y_j$ while $W_{ij}$ is the corresponding weight.

According to the definition of geodesic distance and Log-Euclidean metric on SPD matrix Lie group in Eq.5, the energy function Eq.15 can be transformed to the following equation:

$$
\frac{1}{2} \sum_{i,j} \| \log Y_i - \log Y_j \|_F^2 \tilde{W}_{ij}, \quad (16)
$$

According to Eq. 12, the optimization function Eq.16 is represented as follows:

$$
\begin{align*}
\frac{1}{2} \sum_{i,j} \| \log Y_i - \log Y_j \|_F^2 \tilde{W}_{ij} & = \frac{1}{2} \sum_{i,j} \| A^T \log S_i A - A^T \log S_j A \|_F^2 \tilde{W}_{ij} \\
& = \text{tr} \left( P^T S^T \left( \tilde{D} - \tilde{W} \right) S P \right) \quad (17) \\
& = \text{tr} \left( P^T S^T \tilde{L} S P \right),
\end{align*}
$$

$P = AA^T \in \mathbb{R}^{D \times D}$, $P$ and $\tilde{L} = \tilde{D} - \tilde{W}$ are all semi-SPD matrices. So $P^T S^T \left( \tilde{D} - \tilde{W} \right) S P$ is a semi-SPD matrix, the eigenvalues of it are all non-negative. We have $\text{tr} \left( P^T S^T \tilde{L} S P \right) \geq 0$. In order to compute the minimum value of Eq.17, we just need to compute the minimum eigenvalues of matrix $P^T S^T \tilde{L} S P$.

To avoid getting a singular solution, we impose a constraint as follows:

$$
\text{tr} \left( P^T S^T \tilde{D} S P \right) = 1. \quad (18)
$$

Then the corresponding minimization problem turns to:

$$
\begin{align*}
\min & \quad \text{tr} \left( P^T S^T \tilde{L} S P \right), \\
\text{s.t.} & \quad \text{tr} \left( P^T S^T \tilde{D} S P \right) = 1. \quad (19)
\end{align*}
$$
We use Lagrange multiplier method to solve the minimization problem:

$$L(A, \lambda) = \text{tr} \left( P^T S^T \tilde{L} S P \right) - \lambda \left( \text{tr} \left( P^T S^T \tilde{D} S P \right) - 1 \right)$$

$$= \text{tr} \left( A A^T \tilde{L} S A A^T - \lambda A A^T \tilde{D} S A A^T - 1 \right). \quad (20)$$

As $L$’s derivative to $A$ we obtain the following:

$$\frac{\partial L(A, \lambda)}{\partial A} = 4 \text{tr} \left( A^T S^T \tilde{L} S A A^T - \lambda A^T S^T \tilde{D} S A A^T \right), \quad (21)$$

where $P = A A^T$ and $S^T \tilde{L} S$ are both semi-SPD matrices, so $\text{tr} \left( S^T \tilde{L} S A A^T \right) = \text{tr} \left( A A^T \tilde{L} S \right)$. Eq.21 can be induced to:

$$\frac{\partial L(A, \lambda)}{A} = 4 \text{tr} \left( A^T A A^T S^T \tilde{L} S - \lambda A^T A A^T \tilde{D} S \right)$$

$$= 4 \text{tr} \left( \left( S^T \tilde{L} S A - \lambda S^T \tilde{D} S A \right) A^T A \right). \quad (22)$$

Since $A$ is a $D \times d$ full rank matrix, $A^T A$ is a $d \times d$ SPD matrix. For obtaining $\frac{\partial L(A, \lambda)}{A} = 0$, we just need to minimize the following generalized eigenvector problem:

$$S^T \tilde{L} S A = \lambda S^T \tilde{D} S A. \quad (23)$$

We obtain the bottom smallest $d$ eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_d$, and the corresponding eigenvectors $a_1, a_2, \ldots, a_d \in \mathbb{R}^{D \times 1}$. $A = [a_1, a_2, \ldots, a_d] \in \mathbb{R}^{D \times d}$ is the learnt linear dimensionality reduction map matrix. The corresponding dimension reduction map $g$ between $S^D_+$ and $S^d_+$ is:

$$Y_i = g(S_i) = \exp \left( A^T \log (S_i) A \right). \quad (24)$$

The lower dimensional SPD matrix Lie group $S^d_+$ preserves the local geometric and algebraic structure of $S^D_+$. The local structure between $S^D_+$ and $S^d_+$ is kept by the Laplacian matrix $L$ on $S^D_+$. The more the similarity is between two points, the larger the weight is between them. In addition, by constructing graph Laplacian matrix on SPD matrices, the reduction map is learnt based on global data points. Laplacian matrix can be viewed as an alignment matrix which aligns a set of local patch structures to obtain global lower dimensional representations by solving a generalized eigenfunction. Different from other methods [7] [18] [19], our method uncovers the intrinsic structure of SPD matrices by constructing this discrete Laplacian matrix without the help of other space.
4. Algorithm Analysis

In this section, we mainly analyze the relationships between our Lie-LPP and LPP [5]. In the first subsection, we give the dimension reduction error comparisons between these two algorithms. In the second subsection, we analyze the similarity relation between them.

4.1. Comparison with LPP

The spectral properties of the alignment matrix play an essential role in using local method for manifold learning. We analyze the reconstruction errors during dimension reduction of these two algorithms from three aspects. First, we analyze the local weight matrix construction. The second is to analyze the global alignment matrix and analyze the null space of alignment matrix. The third aspect is to compare the local neighborhood preserving ratios of these two algorithms in experiment. In this section, we mainly compare Lie-LPP and LPP in theory to analyze the improvements of our algorithm.

How well \( \mathcal{N}(\Phi) \) can be determined numerically depends on the magnitude of its smallest nonzero eigenvalue(s). In fact the dimension reduction losses of the two algorithms are determined by the corresponding graph Laplacian matrices defined on data points.

First we analyze the relationship between two graph Laplacian matrices which are defined on vector data points and SPD matrix data points.

The local weight matrix \( W \) of LPP is defined in the second step of LPP algorithm:

\[
W_{ij} = e^{-\frac{\|x_i - x_j\|^2}{t}},
\]

where \( x_i \) and \( x_j \) are in a same neighborhood.

The local weight matrix \( \tilde{W} \) of Lie-LPP is defined in the second step of Lie-LPP algorithm:

\[
\tilde{W}_{ij} = e^{-\frac{\|\log(S_i) - \log(S_j)\|^2_F}{t}},
\]

where \( S_i \) and \( S_j \) are also in a same neighborhood.

The distance \( \|x_i - x_j\| \) between \( x_i \) and \( x_j \) is computed based on Euclidean metric. However, the distance \( \|\log(S_i) - \log(S_j)\|_F \) between \( S_i \) and \( S_j \) is computed based on Log-Euclidean metric. Euclidean metric is not the real Riemannian metric of the embedded manifold \( \mathcal{M} \) defined in LPP algorithm. We have analyzed that the Log-Euclidean metric is the intrinsic Riemannian metric defined on SPD matrix Lie group. So the distance \( \|x_i - x_j\| \) under Euclidean
metric is not the intrinsic geodesic distance \( \text{dist}_M(x_i, x_j) \) on \( M \) and obviously,
\[
\text{dist}_M(x_i, x_j) \geq \|x_i - x_j\|.
\]
Under Log-Euclidean metric, the intrinsic geometric structure of SPD matrix Lie group can be uncovered and \( \| \log(S_i) - \log(S_j) \|_F \) is the real geodesic distance between \( S_i \) and \( S_j \). To our knowledge, \( X_i \) and \( S_i \) are two different feature descriptors of image in computer vision. In appendix we have proved that the SPD matrix descriptors preserve the geometric structures of vector feature descriptors, so
\[
\| \log(S_i) - \log(S_j) \|_F^2 \geq \| x_i - x_j \|_2^2.
\]
Under this analysis, we have:
\[
W_{ij} \geq \tilde{W}_{ij}.
\] (25)

The corresponding graph Laplacian matrices defined on \( \{x_i\} \) and \( \{S_i\}, i = 1, 2, \cdots, N \) are represented as \( L \) and \( \tilde{L} \). Based on the above analysis, we give our first conclusion in Theorem 4.1.

**Theorem 4.1.** If the datasets \( \{x_i\} \) and \( \{S_i\}, (i = 1, 2, \cdots, N) \) are two different feature descriptors of the same \( N \) images, then we have \( L \succeq \tilde{L} \), that is
\[
\lambda_i(L) \geq \lambda_i(\tilde{L}),
\]
for all \( i = 1, 2, \cdots, N \).

**Proof:** According to the definition of weight matrices \( \tilde{W} \) and \( W \), we obtain Eq.25. Then \( D - \tilde{D} \) is a positive diagonal matrix. We have known that \( \tilde{L} = D - \tilde{W} \) and \( L = D - W \), so \( L - \tilde{L} = D - \tilde{D} - (W - \tilde{W}) \) is also a Laplacian matrix. \( L - \tilde{L} \) is a semi-positive definite symmetric matrix. We have
\[
\lambda_i(L - \tilde{L}) \geq 0, \quad L \succeq \tilde{L},
\]
for all \( i = 1, 2, \cdots, N \).

For two symmetric matrices \( L, \tilde{L} \), if \( L - \tilde{L} \succeq 0 \), then we write \( L \succeq \tilde{L} \). And if \( L \succeq \tilde{L} \), then
\[
\lambda_i(L) \geq \lambda_i(\tilde{L}),
\]
for every \( i \).

For special situation, if the original sub-manifold \( M \) in LPP is highly curved and the Riemannian curvature is not zero everywhere, the eigenvalues of \( L \) is strictly greater than those of \( \tilde{L} \). That is \( \lambda_i(L) > \lambda_i(\tilde{L}) \) for every \( i \) under this
situation. So we have proved this theorem.

After analyzing the relationship between two graph Laplacian matrices $L$ and $\tilde{L}$ in Theorem 4.1, we need to analyze the dimension reduction error of the two algorithms Lie-LPP and LPP. In dimension reduction step, all of these two algorithms need to minimize the following generalized eigenvalue function.

$$E = \frac{1}{2} \sum_{i,j} (y_i - y_j)^2 W_{ij} = Y^T LY,$$

where $\{y_1, y_2, \cdots, y_N\}$ are lower dimensional representations of $\{x_i\}$ or $\{S_i\}$, for $i = 1, 2, \cdots, N$.

The dimension reduction errors of Lie-LPP and LPP are measured by the smallest eigenvalues of graph Laplacian matrices. Suppose the dimension reduction error under traditional manifold learning algorithm LPP is represented as $E$ and the error under Lie-LPP is represented as $\tilde{E}$. Based on theorem 4.1, we give our second conclusion in theorem 4.2.

**Theorem 4.2.** The dimension reduction error $\tilde{E}$ under Lie-LPP is less than the dimension reduction error $E$ under LPP. That is, we have

$$\|\tilde{E}\|_F \leq \|E\|_F.$$

**Proof:** According to the algorithm procedures of Laplacian eigenmap, The dimension reduction errors $E$ of LPP and Lie-LPP are mainly determined by the smallest eigenvalues of graph Laplacian matrix. Thus the norm of general reconstruction error $E$ is measured as:

$$\|E\|_F = \sum_{i=1}^{d} \lambda_i,$$

where $d$ is the intrinsic dimension of lower dimensional representations.

From Theorem 4.1 we can obtain that for the same image database, the graph Laplacian matrix $\tilde{L}$ constructed on SPD matrices is lower than that on vector form descriptor. So we have $\lambda_i (L) \geq \lambda_i (\tilde{L})$, for all $i = 1, 2, \cdots, N$. Then based on the definition of dimension reduction error norm of Lie-LPP and LPP, we have

$$\|\tilde{E}\|_F \leq \|E\|_F.$$
Under the same special situation in Theorem 4.1, if the embedded manifold \( M \) in LPP is highly curved, the reconstruction error \( E \) is strictly greater than \( \tilde{E} \).

\[
\| \tilde{E} \|_F < \| E \|_F.
\]

The key reason is that in LPP the authors used Euclidean metric to uncover the local intrinsic geometric structure of \( M \), which is not the real local Riemannian metric of \( M \).

\[\blacksquare\]

By theoretical analysis, we have compared the reconstruction errors between Lie-LPP and LPP. In practice, the reconstruction error is measured by the neighborhood preserving ratios between high dimensional space and lower dimensional space of different algorithms. So, in order to better analyze the advantages of our algorithm Lie-LPP, we give the experimental comparisons between Lie-LPP and LPP in subsection 5.3.

4.2. Connection to LPP

Besides the reconstruction error comparisons between these two algorithms, we give a theoretical analysis of the similarity relation between Lie-LPP and LPP in the following. By analyzing, we can see that under the following special situation Lie-LPP is equivalent to LPP by defining a new weight matrix. Suppose the vector form descriptor of an object is represented as \( x_i \), the corresponding SPD matrix descriptor of this object is shown as \( x_i^T x_i \). SPD matrix Lie group is a flat Riemannian manifold, where the local neighborhood of Lie group is locally isometric to the corresponding tangent space. So the local tangent space can be approximately represented by the local neighborhood of SPD matrix Lie group. Under this special situation and theoretical analysis, Lie-LPP can be transformed into LPP with a special weight matrix. The theoretical analysis is stated in proposition 4.1.

**Proposition 4.1.** Let the vector form descriptor of an object is shown as \( x_i \), the corresponding SPD matrix descriptor is shown as \( x_i^T x_i \). Under this special situation, Lie-LPP can be transformed to LPP by defining a new weight matrix.

**Proof:** First we give the representation of generalized eigenvalue function of Lie-LPP shown as follows:

\[
S^T \tilde{L} S A = \lambda S^T \tilde{D} S A,
\]

\[
S^T \tilde{W} S A = (1 - \lambda) S^T \tilde{D} S A,
\]

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where $\tilde{L} = \tilde{D} - \tilde{W}$ is the corresponding graph Laplacian matrix defined on a set of SPD matrices $S$ represented as $S^T = [x_1^Tx_1, x_2^Tx_2, \cdots, x_N^Tx_N]$ and $S^T$ is the transpose matrix of $S$.

Rewrite $S^T$ as a matrix representation, we obtain the following

$$S^T = [x_1^T, x_2^T, \cdots, x_N^T] \begin{pmatrix} x_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x_N \end{pmatrix},$$

Under this representation, we obtain

$$S^T\tilde{W}S^TA = [x_1^T, x_2^T, \cdots, x_N^T] \begin{pmatrix} x_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x_N \end{pmatrix} \tilde{W} \begin{pmatrix} x_1^T & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x_N^T \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} A.$$

Define

$$W_V = \begin{pmatrix} x_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x_N \end{pmatrix} \tilde{W} \begin{pmatrix} x_1^T & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x_N^T \end{pmatrix},$$

$$D_V = \begin{pmatrix} x_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x_N \end{pmatrix} \tilde{D} \begin{pmatrix} x_1^T & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x_N^T \end{pmatrix}.$$

Under this new weight matrix $W_V$, we rewrite $S^T\tilde{W}S^A$ as follows:

$$S^T\tilde{W}S^A = [x_1^T, x_2^T, \cdots, x_N^T]W_V \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} A.$$

Suppose $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$, $S^T\tilde{W}S^A = X^T W_V X A$. 

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So under this new weight matrix $W$, Lie-LPP is equivalent to LPP. By this analysis, we can see that Lie-LPP algorithm can be transformed into LPP by defining a new weight matrix $W$ if the feature descriptors of vector form and SPD matrix form are respectively \( \{x_i\} \) and \( \{x_i^T x_i\}, i = 1, 2, \cdots, N \).

\[ \Box \]

5. Experiments

In this section, we first report the results of running our algorithm Lie-LPP on two Human action databases: Motion Capture HDM05 [26] and CMU Motion Graph. For all these two experiments, we compare our algorithm with traditional manifold learning algorithms. In the second part, we test our algorithms on static face database e.g. Yale Face Database and YouTube Celebrities Database. Correspondingly we use LEML algorithm [7] and SPD-ML algorithm [19] to compare with our algorithm. Finally, we give the experimental comparison between Lie-LPP and LPP [5].

5.1. Human Action Recognition

In this subsection we test our algorithm Lie-LPP on two Human action databases. Each action segment trajectory can be seen as a curve traversing a manifold. Action recognition is to classify the different action curves. In the recognition step, we use the nearest neighborhood framework to carry out the classification of Human action sequences. After feature extraction, the embedded covariance feature descriptors form a matrix which belongs to a low dimensional SPD matrix Lie group. The similarity measure of action sequences we use is the median Hausdorff geodesic distance. The median Hausdorff geodesic distance $d_2 (A_1, A_2)$ between two action sequences $A_1$ and $A_2$ is defined as follows:

\[
H (A_1, A_2) = \text{median}_i \left( \min_j (d_G (A_1 (i), A_2 (j))) \right),
\]

\[
d_2 (A_1, A_2) = H (A_1, A_2) + H (A_2, A_1).
\]

(26)

where $d_G (A_1 (i), A_2 (j))$ represents the geodesic distance between two SPD matrices $A_1 (i)$ and $A_2 (j)$.

5.1.1. Motion Capture HDM05 Database

HDM05 database [26] contains more than 70 motion classes in 10 to 50 realizations executed by various actors. The actions are performed by five subjects whose names are 'bd’, 'bk’, 'dg’, 'mm’, 'tr’. For each subject we choose the
Table 1: Time comparison, Accuracy Rates of HDM05 Database under different reduced dimensions and different Riemannian metrics, i.e. Euclidean metric (EM) and Log-Euclidean metric (LEM).

| Methods               | Dimension $d$ | time       | Accuracy Rate |
|-----------------------|---------------|------------|---------------|
| Lie-LPP-EM            | $10 \times 10$ | 16.1961s   | 0.5905        |
| Lie-LPP-EM            | $15 \times 15$ | 21.0645s   | 0.6095        |
| Lie-LPP-EM            | $20 \times 20$ | 30.2843s   | 0.6095        |
| Lie-LPP-LEM           | $10 \times 10$ | 12175.1s   | 0.7619        |
| Lie-LPP-LEM           | $15 \times 15$ | 16822.2s   | 0.7857        |
| Lie-LPP-LEM           | $20 \times 20$ | 19503.1s   | 0.7905        |
| EM without Reduce Dim | $93 \times 93$ | 438.777s   | 0.5857        |
| LEM without Reduce Dim | $93 \times 93$ | 21133.9s   | 0.6000        |

following 14 actions: 'clap above head', 'deposit floor', 'elbow to knee', 'grab high', 'hop both legs', 'jog', 'kick forward', 'lie down floor', 'rotate both arms backward', 'sit down chair', 'sneak', 'stand up lie' and 'throw basketball'. We choose three motion fragments per subject per action. So the total number of motion fragments in this experiment is 210. The dataset provides the 3D locations of 31 joints over time acquired at the speed of 120 frames per second. In our experiment we describe each action observed over $T$ frames [3] by its joint covariance descriptor, which is an SPD matrix of size $93 \times 93$. $T$ is a parameter. In this experiment, we choose $T = 20$.

If we do not reduce the dimension before recognition, the time cost of this experiment would be as high as up to 21133.9s. In practice each joint action is controlled only by a few features much less than 93. Therefore dimension reduction is possible before recognition. In this experiment, we perform eight groups of experiments. Each experiment procedure is divided into two parts. The first part is to use Lie-LPP algorithm to reduce the dimension of SPD matrices while the second part is to use the leave-one-out method to compute the recognition rate where we measure the similarity between two SPD matrices under two Riemannian metrics, the Euclidean metric (EM) and the Log-Euclidean metric (LEM), for comparison. The final comparison results are shown in Table 1. The purpose of this experiment is to analyze the time complexity of our method Lie-LPP, so we just give the comparisons of our method under different Riemannian metrics and different reduced dimensions. It turns out that under the same reduced dimensions, the accuracy rates under LEM are totally higher than the accuracy rates under EM, but the time cost under LEM is much higher than under EM. That is because the
time cost of Log operation on SPD matrix is especially high. On the other hand, under the same Riemannian metric, the time cost increases along with the growth of the reduced dimension. In addition, if we do not reduce the dimensionality of SPD matrices before recognition, the recognition accuracies under both Riemannian metrics are respectively 0.5857 and 0.6000. However, the recognition rates under low dimensional SPD matrix Lie group with LEM Riemannian metric are respectively 0.7619, 0.7857, and 0.7905 with regard to three different reduced dimensions $10 \times 10$, $15 \times 15$ and $20 \times 20$, see Table 1, which are much higher than it would be when without reducing dimension. In conclusion, it is a necessary step to reduce the dimension of original SPD matrices before recognition. On the one hand, our method reduces the time cost of experiment. On the other hand, it improves the recognition rate. In addition, based on Log-Euclidean metric, the recognition rates are relatively higher. So the intrinsic structure of SPD matrices is more accurately uncovered based on Log-Euclidean metric.

5.1.2. CMU Motion Graph Database

We consider four different action classes, i.e. 'walking', 'running', 'jumping' and 'swing'. Each class contains ten sequences, and 40 sequences in total for our experiment. There are 31 joints marked in the skeletal joint sequences. Only the root joint is represented by a $6D$ vector, the rest of the other 30 joints are represented by $3D$ rotation angle vectors. Each frame of an action is represented by a $96D$ vector. The SPD matrix which is a $96 \times 96$ matrix is constructed by computing the covariance of $T$ frames sub-windows, where we choose $T = 20$. In order to guarantee the connectivity between sub-windows, we take a $T/2$ frames overlapping between adjacent sub-windows, as mentioned in [3].

In this experiment we use leave-one out method to compute the recognition rate. Every time we choose one sequence of each class as the test set and the other sequences as the training set. The comparison results in Table 2 show the recognition accuracy for different action classes, where the recognition accuracy of our method is highly 0.975 which is higher than the other three methods. It is worth noting that SPD matrix descriptor under joint locations in our method gets a very surprising result.

5.2. Human Face Recognition

In this subsection we test our algorithm on two static human face databases. The above of this paper has said that there are only two different algorithms LEML and SPD-ML that are similar with our method to reduce the dimensionality of
Table 2: Classification performance of CMU Motion Graph database, together with the comparison results for Lie-LPP and traditional manifold learning algorithms.

| Methods | CMU Dimension | $d$  |
|---------|---------------|------|
| Lie-LPP | 0.9750        | $3 \times 3$ |
| LPP [5] | 0.8500        | 9    |
| LEP [17]| 0.9000        | 9    |
| PCA [20]| 0.2500        | 9    |

SPD matrix manifold. So in this subsection we report the results of comparing our method with the two algorithms LEML [7] and SPD-ML [19].

5.2.1. Extended Yale Face Database B

The extended YaleFace B database [27], or YFB DB for short, contains 2414 single light source images of 38 individuals each seen under about 64 near frontal images under different illuminations per individual. For every subject in a particular pose, an image with ambient illumination was also captured. The face region in each image is resized into $32 \times 32$. We use the raw intensity feature to construct the corresponding SPD matrix, in which we follow [7] to construct the SPD matrix for each image. On this dataset since the image size is $32 \times 32$, the size of the corresponding SPD matrix is $1024 \times 1024$.

5.2.2. YouTube Celebrities (YTC) Database

YTC database [25] collects 1910 video sequences of 47 subjects from YouTube. The face regions of the original images from the video sequences are not all the same and especially high. In this experiment, we resize the face region into $20 \times 20$ intensity image as the same in [7]. Each video clip generates an image set of faces. In this experiment, we choose 497 video sequences of 10 different people who are: 'Adam Aandler', 'Al Gore', 'Al Pacino', 'Alanis Morissette', 'Anderson Cooper', 'Andy Garcia', 'Angelina Jolie', 'Ashley Judd', 'Bill Clinton', 'Bill Gates'. As the same in Yale Face Database, in this experiment, we also use the $20 \times 20$ intensity feature to construct the corresponding SPD matrix descriptor. So the size of the corresponding SPD matrix is $400 \times 400$.

5.2.3. Recognition

In the recognition step, SPD-ML [19] used Affine-Invariant metric and Stein divergence metric to measure the similarities among SPD matrices. LEML [7] used Log-Euclidean metric under which the SPD matrix Lie group is locally isometric to Lie algebra. For our algorithm Lie-LPP, we also use Log-Euclidean
Table 3: Classification performance of YFB DB, YTC DB, together with the comparison results for Lie-LPP and traditional manifold learning algorithms PCA, LPP as well as LEML, SPD-ML-Stain, SPD-ML-Airm.

|         | YFB DB     |                 |                 |                 |                 |
|---------|------------|----------------|----------------|----------------|----------------|
|         | YFD-trn2/tst9 | YFD-trn3/tst8   | YFD-trn4/tst7   | YFD-trn5/tst6   |                 |
| PCA [20]| 46.2 ± 3.2  | 48.3 ± 2.8     | 52.7 ± 1.9     | 54.3 ± 2.7     |                 |
| LPP [5] | 44.5 ± 3.5  | **56.4 ± 4.5** | 62.8 ± 4.8     | 67.2 ± 2.4     |                 |
| LEML [7]| 43.8 ± 2.6  | 50.8 ± 1.2     | 51.2 ± 1.7     | 73.6 ± 4.6     |                 |
| SPD-Stain [19] | 43.0 ± 1.9  | 51.7 ± 1.8     | 50.5 ± 2.3     | 70.4 ± 3.7     |                 |
| SPD-Airm [19] | 44.1 ± 1.9  | 52.6 ± 2.7     | 51.2 ± 3.6     | 69.3 ± 1.9     |                 |
| Lie-LPP | 45.3 ± 2.4  | 55.2 ± 3.2     | **63.4 ± 2.7** | **73.9 ± 1.8** |                 |

|         | YTC DB     |                 |                 |                 |                 |
|---------|------------|----------------|----------------|----------------|----------------|
|         | YTC-trn3/tst7 | YTC-trn4/tst6   | YTC-trn5/tst5   | YTC-trn6/tst4   |                 |
| PCA [20]| 44.2 ± 2.2  | 47.3 ± 2.2     | 49.6 ± 1.5     | 50.1 ± 1.7     |                 |
| LPP [5] | 56.5 ± 1.4  | 58.6 ± 2.3     | 59.8 ± 2.1     | 61.2 ± 1.7     |                 |
| LEML [7]| **67.5 ± 2.2** | 68.2 ± 1.4     | 70.3 ± 1.7     | 73.6 ± 2.6     |                 |
| SPD-Stain [19] | 57.3 ± 1.9  | 59.7 ± 1.8     | 60.5 ± 2.3     | 61.6 ± 1.7     |                 |
| SPD-Airm [19] | 58.5 ± 2.3  | 59.6 ± 1.7     | 62.2 ± 1.6     | 64.6 ± 1.9     |                 |
| Lie-LPP | 66.3 ± 2.1  | **70.5 ± 2.6** | **76.1 ± 1.6** | **78.5 ± 2.2** |                 |

Metric to compute the geodesic distances between two SPD matrices. Different from the other two algorithms, our algorithm aims to construct Laplace-Beltrami operator on SPD matrix Lie group, and then learns a more discriminable Lie group which preserves the geometric and algebra structure of the original one.

For YFB DB, we totally do the experiment four times by each algorithm. In each experiment, we randomly choose \( p (p = 2, 3, 4, 5) \) image sets per subject as the training dataset, the rest \( 11 - p \) image sets per subject as the test dataset respectively. The recognition accuracy results of different algorithms are reported in Table 3. In these experiments we choose the same classification methods with LEML and SPD-ML in the recognition step. From Table 3 we can see that the recognition result of our proposed algorithm is especially similar to the results of LEML and SPD-ML only when we choose \( p = 5 \) images per subject as the training dataset. The accuracy recognition rates of Lie-LPP are slightly higher than the accuracies of LEML and SPD-ML when we choose \( p = 2, 3, 4 \). The results of these experiments mean that the effect of dimensionality reduction for SPD matrices by Lie-LPP is better than the effects of LEML and SPD-ML. In addition, LEML and SPD-ML need several parameters when performing their algorithms. Our algorithm just needs to analyze the Laplacian operator of SPD matrix Lie group and solves the dimensionality reduction problem directly on the
SPD matrix Lie group.

For the recognition of YTC Database, we also do the experiments four times by each algorithm respectively. As the same method with YFB DB, we randomly choose \( p (p = 3, 4, 5, 6) \) respectively image sets per subject for training, the rest for test. The results of all these experiments are presented in Table 3 lower part. In Table 3, we can see that the recognition of Lie-LPP is higher than the other four algorithms, with more than 20% average improvement. For PCA and LPP algorithms the recognition rates are especially lower than our method. That is because PCA and LPP algorithms reduce the dimension of long image pixel vectors from original pixel matrix, which is sensitive to a lot of noises such as illumination intensity, background, object location and so on. What's more, the long pixel vector breaks the matrix structure of original image pixel matrix. Compared with SPDML and LEML algorithms, our algorithm achieves comparable performance with them for \( p = 3 \). In addition, for \( p = 3, 4, 5, 6 \), our proposed method outperforms them. Compared with the other two Riemannian metrics AIM and Stein, Log-Euclidean metric obtains higher recognition rates in all cases. The results show that constructing the corresponding graph Laplacian matrix on SPD matrix Lie group based on Log-Euclidean metric is an effective step to uncover the intrinsic structure of SPD matrix Lie group.

5.3. Neighborhood Preserving Ratio Comparison

In this subsection, we mainly give the experimental reconstruction error comparison between Lie-LPP and LPP [5] to analyze the improvement of Lie-LPP during dimension reduction. We use neighborhood preserving ratio (NPR) [24] to measure the dimension reduction results of Lie-LPP and LPP. In theory, we have analyzed that the reconstruction errors of Lie-LPP and LPP during dimension reduction are fully determined by the smallest eigenvalues of graph Laplacian matrices \( \tilde{L} \) and \( L \). In practice, the dimension reduction results are influenced by several parameters: the curvature of sub-manifold, the density of data points, the noise level of data points and lower dimension of sub-manifold. Thus, in order to completely analyze the reconstruction errors of Lie-LPP and LPP, we need to compare them both in theory and experiment.

The neighborhood preserving ratio [24] is defined as follows:

\[
NB = \frac{1}{KN} \sum_{i=1}^{N} |\mathcal{N}(x_i) \cap \mathcal{N}(y_i)|,
\]

where \( \mathcal{N}(x_i) \) is the set of \( K \)-nearest sample subscripts of \( x_i \), and \( \mathcal{N}(y_i) \) is the set of \( K \)-nearest sample subscripts of \( y_i \). \( | \cdot | \) represents the number of intersection.
Table 4: Neighborhood preserving ratios (NPRs) comparisons between Lie-LPP and LPP about YTC Database under different embedding dimensions $d$ and neighbor size parameters $K$.

|       | Lie-LPP |       |       |       |       |       |       |
|-------|---------|-------|-------|-------|-------|-------|-------|
| $K = 10$ | $K = 20$ | $K = 30$ | $K = 40$ | $K = 50$ | $K = 60$ | $K = 70$ |
| $d = 2 \times 2$ | 0.2778 | 0.2823 | 0.2958 | 0.2981 | 0.3144 | 0.3314 | 0.3461 |
| $d = 3 \times 3$ | 0.3211 | 0.3101 | 0.3373 | 0.3448 | 0.3603 | 0.3814 | 0.4132 |
| $d = 4 \times 4$ | 0.3808 | 0.3753 | 0.3916 | 0.3942 | 0.4113 | 0.4554 | 0.4613 |
| $d = 5 \times 5$ | 0.3951 | 0.4156 | 0.4250 | 0.4475 | 0.4777 | 0.5219 | 0.5302 |
| $d = 6 \times 6$ | 0.4345 | 0.4535 | 0.4828 | 0.5055 | 0.5212 | 0.5505 | 0.5837 |
| $d = 7 \times 7$ | 0.4939 | 0.5066 | 0.5264 | 0.5511 | 0.5775 | 0.6066 | 0.6349 |
| $d = 8 \times 8$ | 0.5987 | 0.6093 | 0.6267 | 0.6544 | 0.7136 | 0.7312 | 0.7418 |

|       | LPP [5] |       |       |       |       |       |       |
|-------|---------|-------|-------|-------|-------|-------|-------|
| $K = 10$ | $K = 20$ | $K = 30$ | $K = 40$ | $K = 50$ | $K = 60$ | $K = 70$ |
| $d = 2 \times 2$ | 0.1392 | 0.1347 | 0.1347 | 0.1408 | 0.1484 | 0.1558 | 0.1664 |
| $d = 3 \times 3$ | 0.2288 | 0.2099 | 0.1983 | 0.1972 | 0.2104 | 0.2259 | 0.2434 |
| $d = 4 \times 4$ | 0.3166 | 0.2726 | 0.2607 | 0.2752 | 0.2843 | 0.3116 | 0.3403 |
| $d = 5 \times 5$ | 0.4001 | 0.3460 | 0.3367 | 0.3537 | 0.3747 | 0.4090 | 0.4444 |
| $d = 6 \times 6$ | 0.4816 | 0.4246 | 0.4158 | 0.4396 | 0.4669 | 0.5062 | 0.5371 |
| $d = 7 \times 7$ | 0.5381 | 0.5043 | 0.5141 | 0.5401 | 0.5590 | 0.5900 | 0.6097 |
| $d = 8 \times 8$ | 0.5971 | 0.5829 | 0.5993 | 0.6210 | 0.6350 | 0.6611 | 0.6754 |

In this experiment, we compare the NPRs of Lie-LPP and LPP under two databases (YTC database [25] and YFB database [27]) which have been described in subsection 5.2 in detail. The neighbor size parameter $K$ is chosen 10, 20, 30, 40, 50, 60, 70 respectively. And we respectively use Lie-LPP and LPP to reduce the dimension of the same database and then use NPR equation in Eq.27 to measure the neighborhood preserving ratios of these two algorithms. In dimension reduction step, we respectively choose, $d = 2 \times 2, 3 \times 3, 4 \times 4, 5 \times 5, 6 \times 6, 7 \times 7, 8 \times 8$, as the final dimensions of lower dimensional representations. The comparison results are shown in Table 4 and Table 5. From Table 4, we give the comparisons between Lie-LPP and LPP under YTC database. From this table we can see that the neighborhood preserving ratio of Lie-LPP is especially higher than that of LPP with the same neighbor size parameter $K$ as well as the same low dimension $d$. From Table 5, we give the comparisons between these two algorithms under YFB database. From these comparisons we can see that our Lie-LPP outperforms LPP with different neighbor size parameter $K$ and low dimension $d$. In addition, the NPRs of these two algorithms increase gradually as the growth of neighbor size parameter $K$. Both of these two algorithms have this same property. As a whole,
Table 5: Neighborhood preserving ratios (NPRs) comparisons between Lie-LPP and LPP about YFB Database under different embedding dimensions $d$ and neighbor size parameters $K$.

| Lie-LPP | $K = 10$ | $K = 20$ | $K = 30$ | $K = 40$ | $K = 50$ | $K = 60$ | $K = 70$ |
|---------|---------|---------|---------|---------|---------|---------|---------|
| $d = 2 \times 2$ | 0.3295  | 0.4362  | 0.4944  | 0.5348  | 0.5645  | 0.5843  | 0.6009  |
| $d = 3 \times 3$ | 0.4344  | 0.5139  | 0.5626  | 0.5960  | 0.6230  | 0.6434  | 0.6589  |
| $d = 4 \times 4$ | 0.5042  | 0.5758  | 0.6207  | 0.6527  | 0.6771  | 0.6964  | 0.7115  |
| $d = 5 \times 5$ | 0.5651  | 0.6207  | 0.6599  | 0.6896  | 0.7120  | 0.7293  | 0.7424  |
| $d = 6 \times 6$ | 0.6042  | 0.6577  | 0.6924  | 0.7192  | 0.7394  | 0.7542  | 0.7663  |
| $d = 7 \times 7$ | 0.6529  | 0.7025  | 0.7366  | 0.7585  | 0.7744  | 0.7855  | 0.7939  |
| $d = 8 \times 8$ | 0.6896  | 0.7335  | 0.7580  | 0.7781  | 0.7928  | 0.8031  | 0.8101  |

| LPP [5] | $K = 10$ | $K = 20$ | $K = 30$ | $K = 40$ | $K = 50$ | $K = 60$ | $K = 70$ |
|---------|---------|---------|---------|---------|---------|---------|---------|
| $d = 2 \times 2$ | 0.2686  | 0.3095  | 0.3696  | 0.3980  | 0.4183  | 0.4317  | 0.4445  |
| $d = 3 \times 3$ | 0.4002  | 0.4229  | 0.4380  | 0.4771  | 0.5076  | 0.5348  | 0.5564  |
| $d = 4 \times 4$ | 0.4742  | 0.4853  | 0.5123  | 0.5386  | 0.5684  | 0.5912  | 0.6098  |
| $d = 5 \times 5$ | 0.5309  | 0.5460  | 0.5715  | 0.5939  | 0.6171  | 0.6330  | 0.6498  |
| $d = 6 \times 6$ | 0.5714  | 0.5865  | 0.6123  | 0.6363  | 0.6589  | 0.6793  | 0.6979  |
| $d = 7 \times 7$ | 0.6044  | 0.6168  | 0.6446  | 0.6689  | 0.6919  | 0.7087  | 0.7232  |
| $d = 8 \times 8$ | 0.6350  | 0.6418  | 0.6715  | 0.6927  | 0.7118  | 0.7264  | 0.7371  |

our algorithm preserves the local geometric structure better than LPP during the dimension reduction.

Under Log-Euclidean metric, SPD matrix Lie group is a flat manifold, where the Riemannian curvature is zero everywhere. The embedded Riemannian manifold $\mathcal{M}$ in LPP is curved in high dimensional feature space. However, under LPP, the local Riemannian metric of $\mathcal{M}$ is Euclidean metric. The Riemannian curvature tensor under local Euclidean metric is zero. Actually, the Riemannian curvature of embedded manifold $\mathcal{M}$ may be not zero everywhere, so Euclidean metric may be not the intrinsic Riemannian metric of $\mathcal{M}$. Under Euclidean metric, LPP cannot uncover the local intrinsic geometric structure of $\mathcal{M}$.

6. Conclusions and Future directions

In summary, the main conclusions of this paper are:

1. We construct Laplace-Beltrami operator on SPD matrix Lie group and give the corresponding discrete Laplacian matrix.
2. We extend manifold learning algorithm LPP to Lie-LPP performing on SPD Matrix Lie group. And we have shown how it can be successfully applied to Human action recognition and Human face recognition;

3. According to analyze the geometric and algebraic structure of SPD matrix Lie group, we obtain that SPD matrix Lie group is a complete and compact manifold. And the sectional curvature on every point is zero, so SPD matrix Lie group is a flat manifold and locally isometric to Lie algebra;

4. However this is not a simple application of the idea in paper [5]. We have developed a new algorithm Lie-LPP which made a substantial extension of the LPP algorithm;

5. We analyze the relationships between Lie-LPP and LPP in theory and obtain three theoretical conclusions;

6. We have made practical experiments whose results show that our Lie-LPP algorithm outperforms the existing ones significantly.

In the future, we will go a further step forwards to improve this algorithm. In practice, we will try to add a time dimension to enhance the human action recognition. Furthermore, we will try to introduce manifold learning algorithms to other types of Lie groups. Also we will try to introduce new manifold learning algorithms on higher dimensional tensor space.

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Appendix

In this section, we give a detailed proof about the local geometrical structure relationship between covariance matrices and feature vectors which are two different feature descriptors of the same images.
Theorem: Covariance matrix used as the feature descriptor of images preserves the spatial geometrical structure of image pixels, that is in local neighborhoods there are two positive constants $c_1 < c_2$ satisfying the following inequality:

$$c_1 \cdot d(I_i, I_j) \leq D_{te}(S_i, S_j) \leq c_2 \cdot d(I_i, I_j)$$

Proof:
Let $I$ be the original image pixel matrix, where $p_j = (p_{jx}, p_{jy})$ is the location expressed w.r.t the origin of this patch and is the same for all patches. In this proof, we just consider the gray scale image pixels. In image recognition, one usually considers image pixel matrix as a high dimensional row vector where sample images are the corresponding points in the high dimensional vector space. According to the properties of a manifold, the distance between two adjacent image points can be approximately viewed as Euclidean distance. Suppose $I_i, I_j$ are two adjacent images, the corresponding pixel dimension is $m \times m$. The Euclidean distance can be written as:

$$d(I_i, I_j) = \sum_{k=1}^{m \times m} \| I_i(p_k) - I_j(p_k) \|^2. \quad (28)$$

Suppose the corresponding covariance matrix descriptors of $I_i, I_j$ are $S_i, S_j$ which lie on a SPD matrix Lie group. The geodesic distance $D_{te}(S_i, S_j)$ between $S_i$ and $S_j$ is shown as:

$$D_{te}(S_i, S_j) = \| \text{log}(S_i) - \text{log}(S_j) \|^2_F. \quad (29)$$

According to the definition of logarithmic map on matrix Lie group, we can write the geodesic distance $D_{te}(S_i, S_j)$ as:

$$\| \sum_{k=1}^{\infty} \frac{(I - S_j)^k}{k} - \sum_{k=1}^{\infty} \frac{(I - S_i)^k}{k} \| \leq \| (S_i - S_j) M \|^2_F \leq \| S_i - S_j \|^2_F \| M \|^2_F,$$

where we define:

$$M := \left( 3I - \frac{3}{2} (S_i + S_j) + \frac{1}{3} (S_i^2 + S_i S_j + S_j^2) + \cdots \right)$$

As the definition of covariance matrix:

$$S_i (f_k, f_l) - S_j (f_k, f_l) = \frac{1}{n} \sum_{h=1}^{n} \left( I_i(t_h + p_k) - I_j(t_h + p_k) \right) \left( I_i(t_h + p_l) - I_j(t_h + p_l) \right),$$

$$\quad (31)$$
where $t_h$ is the first point location of the $h^{th}$ patch, and $f_k$ is the feature vector defined as in [8].

The Frobenius norm of covariance matrix is defined as follows:

$$\|S_i - S_j\|_F^2 = \sum_{k=1}^{d} \sum_{l=1}^{d} \| (S_i - S_j) (k, l) \|^2,$$

where $d$ is the number of patches of each image.

$$\| (S_i - S_j) (k, l) \|^2 = \|S_i (f_k, f_l) - S_j (f_k, f_l) \|^2.$$  (33)

From Eq.32 and Eq.33, we give the Frobenius norm of covariance matrix:

$$\|S_i - S_j\|_F^2$$

$$= \frac{1}{n} \| f_i \cdot f_i^T - f_j \cdot f_j^T \|_F^2$$

$$\leq \frac{1}{n^2} \| (f_i - f_j) \cdot (f_i^T + f_j^T) \|^2 \|_F^2 + \frac{1}{n^2} \| f_j f_i^T - f_i f_j^T \|^2 \|_F^2$$

$$\leq \frac{1}{n^2} \| f_i - f_j \|^2 \cdot (\| f_i^T + f_j^T \|^2 \|_F^2 + 2 \| f_i^T \|^2 \|_F^2 ).$$  (34)

where $f_i = (f_{i1}, f_{i2}, \cdots, f_{in})$, $f_j = (f_{j1}, f_{j2}, \cdots, f_{jn})$, and $f_{ik}$ represent the $k^{th}$ block subimage feature vector of the $i^{th}$ image.

Then

$$\| f_i - f_j \|^2 = \sum_{k=1}^{n} \| f_{ik} - f_{jk} \|^2$$

$$= \sum_{k=1}^{n} \sum_{l=1}^{d} \| I_i (p_{kl}) - I_j (p_{kl}) \|^2$$

$$\leq d \cdot \sum_{k=1}^{m \times m} \| I_i (p_k) - I_j (p_k) \|^2 = d \cdot d (I_i, I_j).$$  (35)

From Eq.34 and Eq.35, we obtain:

$$D_{le} (S_i, S_j) \leq c_2 \cdot d (I_i, I_j),$$

where $c_2$ is a positive constant. And simultaneously $D_{le} (S_i, S_j) \geq 0$, so there are two constants $0 \leq c_1 \leq c_2$ satisfied:

$$c_1 \cdot d (I_i, I_j) \leq D_{le} (S_i, S_j) \leq c_2 \cdot d (I_i, I_j).$$

In summary, we can prove that covariance descriptor preserves the geometrical structure of images.
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