Pentagonal quasicrystals and their linear self-similarities

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Abstract. We recall the construction of a cut-and-project scheme providing planar model sets with 5-fold symmetry. The cut-and-project construction consists in projecting points of a lattice \( L \subset \mathbb{R}^4 \) by two orthogonal projections \( \pi_\parallel, \pi_\perp \) to two 2-dimensional subspaces. We describe the set of all linear mappings preserving the \( \mathbb{Z} \)-modules \( \pi_\parallel(L) \) and \( \pi_\perp(L) \) and show that the set of all such mappings is in a correspondence with a 2-dimensional cut-and-project set. Finally, we describe linear self-similarities of a pentagonal cut-and-project set with circular acceptance window.

1. Introduction
We consider the cut-and-project construction of a quasicrystal model with five-fold symmetry. Classical results studying symmetries of such models only focus on rotations and inflations of such sets. For example, the result of Lagarias [1] implies that if a cut-and-project set is closed under a scaling \( \eta > 1 \), then the scaling factor is a Pisot number. In our setting, we examine general linear mappings under which a cut-and-project set is preserved (we call them linear self-similarities of the set). We provide a complete description of possible linear self-similarities of a cut-and-project set with circular acceptance window. We show that there exists a correspondence between a certain two-dimensional cut-and-project set and the set of all linear self-similarities of the quasicrystal. For a specific subclass of self-similarities we provide a full classification of them based on their eigenvalues. The present work extends our study published in [2].

2. Cut-and-project set with five-fold symmetry
Let us recall the construction of a cut-and-project scheme permitting to obtain 5-fold models of quasicrystals. We proceed according to [2], but the construction is equivalent to the classical one, see e.g. [3, 4]. The scheme is derived using the matrix \( C \), the companion matrix of the cyclotomic polynomial \( \Phi_5(X) = X^4 + X^3 + X^2 + X + 1 \). The matrix \( C \) thus satisfies \( C^5 = I \).
Defining the lattice $\mathcal{L}$ as the $\mathbb{Z}$-span of basis vectors $\ell_1, \ldots, \ell_4$, in the standard basis written as

$$
\mathcal{L} = \mathbb{Z} \begin{pmatrix} 1 - \cos \frac{2\pi}{5} \\ \sin \frac{2\pi}{5} \\ 1 - \cos \frac{4\pi}{5} \\ \sin \frac{4\pi}{5} \end{pmatrix} + \mathbb{Z} \begin{pmatrix} 0 \\ 2 \sin \frac{2\pi}{5} \\ 2 \sin \frac{4\pi}{5} \end{pmatrix} + \mathbb{Z} \begin{pmatrix} \cos \frac{4\pi}{5} - \cos \frac{2\pi}{5} \\ \sin \frac{4\pi}{5} + \sin \frac{2\pi}{5} \\ \cos \frac{2\pi}{5} - \cos \frac{4\pi}{5} \end{pmatrix} + \mathbb{Z} \begin{pmatrix} \cos \frac{4\pi}{5} - \cos \frac{2\pi}{5} \\ \sin \frac{2\pi}{5} - \sin \frac{4\pi}{5} \\ \sin \frac{4\pi}{5} + \sin \frac{2\pi}{5} \end{pmatrix} = \sum_{i=1}^{4} \mathbb{Z} \ell_i,
$$

we have that $C\mathcal{L} = \mathcal{L}$, i.e. $C$ gives a lattice transformation of order 5. We consider the projections $\pi_\parallel, \pi_\perp$ on the subspaces generated by the standard vectors $e_1, e_2$, resp. $e_3, e_4$, usually called the physical and internal space. The action of $\pi_\parallel, \pi_\perp$ on a lattice vector $\ell$ is given by

$$
\pi_\parallel(\ell) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \ell, \quad \pi_\perp(\ell) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ell.
$$

Let us depict the projections $\pi_\parallel(\ell_i)$ and $\pi_\perp(\ell_i)$ of the lattice generators in $\mathbb{R}^2$, i.e. in the physical and internal space, in order to underline the naturally arising pentagonal structure.

**Figure 1.** Projections $\pi_\parallel$ and $\pi_\perp$ of lattice generators.

In [2] it is shown that the described cut-and-project scheme is non-degenerate, irreducible and aperiodic, in the sense of [5]. We can thus use it to obtain a cut-and-project set. Let $\Omega$ be a bounded window with non-empty interior. The resulting cut-and-project set $\Sigma(\Omega)$ can be written as

$$
\Sigma(\Omega) = \left\{ \sum_{i=1}^{4} a_i \pi_\parallel(\ell_i) : a_i \in \mathbb{Z}, \sum_{i=1}^{4} a_i \pi_\perp(\ell_i) \in \Omega \right\}.
$$

Denoting for simplicity $u = \pi_\parallel(\ell_1)$, $v = \pi_\parallel(\ell_4)$, $u^* = \pi_\perp(\ell_1)$, $v^* = \pi_\perp(\ell_4)$, we can rewrite $\Sigma(\Omega)$ into

$$
\Sigma(\Omega) = \left\{ (a + b\tau)u + (c + d\tau)v : a, b, c, d \in \mathbb{Z}, (a + b\tau^*)u^* + (c + d\tau^*)v^* \in \Omega \right\},
$$

where $\tau = \frac{1}{2}(1 + \sqrt{5}) = 2 \cos \pi/5$ is the golden ratio and $\tau' = \frac{1}{2}(1 - \sqrt{5})$ is its algebraic conjugate, see [2].
3. Self-similarities of $\Lambda_5$

Any self-similarity of the cut-and-project set $\Sigma(\Omega)$, i.e. a linear map $A$ such that $A\Sigma(\Omega) \subset \Sigma(\Omega)$ is a self-similarity of the $\mathbb{Z}$-module $\pi_\parallel(L)$, i.e. $A\pi_\parallel(L) \subset \pi_\parallel(L)$. Since $\pi_\parallel$ restricted to $L$ is an injection, it induces a linear lattice transformation $Z$ such that $ZL \subset L$, and a linear map $B$ satisfying $B\pi_\perp(L) \subset \pi_\perp(L)$. In [2] we have described transformations $Z$ of the lattice $L$ as well as the mappings $A,B$ preserving the $\mathbb{Z}$-modules $\pi_\parallel(L)$ and $\pi_\perp(L)$ induced by $Z$. Note that transformations $Z \in \mathbb{Z}^{4\times4}$ form an 8-dimensional $\mathbb{Z}$-algebra with a 4-dimensional commutative subalgebra of mappings that commute with $C$. The elements of the commutative subalgebra induce transformations $A,B$ of $\pi_\parallel(L)$ and $\pi_\perp(L)$ that are scaled rotations. Note that the matrices of the mappings $A,B$ in standard basis have components in the extension field $\mathbb{Q}(\cos 2\pi/5, \sin 2\pi/5)$. In particular, $A = (A_{ij})$ with

$$
A_{11} = a + b \cos \frac{2\pi}{5} + (c + d) \cos \frac{4\pi}{5},
A_{12} = b \sin \frac{2\pi}{5} + (c - d) \sin \frac{4\pi}{5},
A_{21} = \frac{1}{2} ((2a - 3b + 2c + 2d + 8e - 2f - 2g - 2h) \sin \frac{2\pi}{5} +
\qquad + (-6a + 4b - c - d - 4e + 6f - 4g - 4h) \sin \frac{4\pi}{5}),
A_{22} = -c + d + f + h + (2g - b) \cos \frac{2\pi}{5} + (-c + d + 2h) \cos \frac{4\pi}{5}.
$$

$a,b,c,d,e,f,g,h \in \mathbb{Z}$. The components of the matrix $B = (B_{ij})$ can be obtained from $A_{ij}$ by application of the automorphism $\psi$ of the field $\mathbb{Q}(\cos 2\pi/5, \sin 2\pi/5)$, defined by

$$
\psi: \cos \frac{2\pi}{5} \mapsto \cos \frac{4\pi}{5}, \quad \cos \frac{4\pi}{5} \mapsto \cos \frac{2\pi}{5}, \quad \sin \frac{2\pi}{5} \mapsto \sin \frac{4\pi}{5}, \quad \sin \frac{4\pi}{5} \mapsto -\sin \frac{2\pi}{5}.
$$

In order to handle the self-similarities more easily, let us give another description. Let us rewrite the condition on the mapping $Z$ (i.e. it has to preserve the two modules) in terms of its characteristic polynomial. Then we show that any such mapping $Z$ corresponds to an element of a cut-and-project set with a triangular window. It is not difficult to show that $Z \in \mathbb{Z}^{4\times4}$ is a self-similarity of $L$ inducing linear mappings $A,B \in \mathbb{R}^{2\times2}$, which preserve modules $\pi_\parallel(L)$ and $\pi_\perp(L)$, respectively, if and only if the characteristic polynomial $\chi_Z \in \mathbb{Z}[X]$ of $Z$ satisfies $\chi_Z(X) = \chi(X) \cdot \chi'(X)$, where $\chi \in \mathbb{Z}[\tau][X]$ and $\tau$ is the non-trivial field automorphism of $\mathbb{Q}(\tau)$, i.e. $\chi(X) = X^2 + pX + q$, $p,q \in \mathbb{Z}[\tau]$ and $\chi'(X) = X^2 + p'X + q'$. A necessary condition for existence of a bounded window $\Omega$ so that $A\Sigma(\Omega) \subset \Sigma(\Omega)$ is that the eigenvalues of the corresponding matrix $B$ are in modulus smaller or equal to 1, see [2]. With this in mind, one can prove the following statement.

**Proposition 3.1.** Let $\Lambda_5 = (L \subset \mathbb{R}^4, \mathbb{R}^2)$ be a CPS with five-fold symmetry. Let $Z$ be a self-similarity of $L$ inducing linear mappings $A,B$, which preserve the modules $\pi_\parallel(L)$ and $\pi_\perp(L)$ respectively. Let us denote $\chi_Z = \chi_A \cdot \chi'_A$ with $\chi_A(X) = X^2 + pX + q$. Then there is a window $\Omega \subset \mathbb{R}^2$ such that $Z$ induces a mapping $A$ that preserves $\Sigma(\Omega)$ if and only if

$$(p,q) \in \left\{ (x,y) \in (\mathbb{Z}[\tau])^2 : (x', y') = \Delta \right\}$$

where $\Delta$ denotes the convex hull of vectors $(0, -1)$, $(2, 1)$, $(-2, 1)$.

Let us focus on the subclass of mappings $Z$ that commute with $C$. This subclass contains all matrices of the form $Z = \varphi(C)$ where $\varphi \in \mathbb{Z}[X]$. The induced mappings $A,B$ in the physical and inner spaces, respectively, are then scaled rotations. We can classify the mappings $A$ for which there exists a cut-and-project set $\Sigma(\Omega)$ with $A\Sigma(\Omega) \subset \Sigma(\Omega)$ according to factorization of the characteristic polynomial $\chi_Z$ of $Z$ into irreducible factors, or, equivalently, according to the degree of eigenvalues of $A$ over the rationals. The proof of the following statement can be found in [2].
Proposition 3.2. Let \( A = \mu R = \begin{pmatrix} s & -t \\ t & s \end{pmatrix} \) with \( \mu > 0 \), \( R \in O(2, \mathbb{R}) \), be such that \( A \Sigma(\Omega) \subset \Sigma(\Omega) \) for a cut-and-project set of the form (2). Then one of the following happens.

- \( A \in D_5 \) is an element of the dihedral group of order 10.
- \( A = \pm \mu I \) is a scaling by a factor \( \pm \mu \in \mathbb{Q}(\tau) \) where \( \mu \) is a Pisot number, i.e. an algebraic number of degree 2, \( \mu > 1 \), with conjugate in modulus smaller than 1.
- \( s + ti = \mu e^{i \phi} \in \mathbb{Q}(\omega) \) is a complex Pisot number, i.e. an algebraic number in \( \mathbb{Q}(\omega) \) of degree 4, \( |s + ti| = \mu > 1 \), whose algebraic conjugates different from \( s \pm ti \) are in modulus smaller than 1.

Besides the above mappings in the form of scaled rotations, a cut-and-project set (2) may have other linear self-similarities. Their eigenvalues belong to some quadratic extension \( K \) of \( \mathbb{Q}(\tau) \). The opposite statement holds as well. Let \( K \) be a quadratic extension of \( \mathbb{Q}(\tau) \), and \( \psi \) the non-trivial automorphism in \( \text{Aut}(K/\mathbb{Q}(\tau)) \). Denote by \( O_K \) the ring of integers in \( K \).

Then for all \( \lambda \in O_K \) there exists a transformation \( Z \in Z^{4 \times 4} \) inducing a self-similarity \( A \) of a cut-and-project set \( \Sigma(\Omega) \), such that \( \lambda, \psi(\lambda) \) are eigenvalues of \( A \).

4. General self-similarity preserving circular window

Let us focus on the cut-and-project set (2) where \( \Omega \) is fixed to be a disk. It is important to note that not all the mappings \( A \) described in Proposition 3.1 are self-similarities of this fixed cut-and-project set. A counterexample can be found in [2]. The condition \( \rho(B) \leq 1 \) on the spectral radius \( \rho \) of the corresponding mapping \( B \) is necessary but not sufficient. In order to describe all linear self-similarities of a given cut-and-project set, we have to verify whether \( B \) preserves the chosen window \( \Omega \).

Let \( \Omega \) be a disk of radius \( r \) centered in the origin. Such \( \Omega \) is closed under the action of \( B \) if every \( x \in \mathbb{R}^2 \) of norm \( \|x\| \leq r \) satisfies \( \|Bx\| \leq r \), where \( \|\cdot\| \) stands for the standard Euclidean norm on \( \mathbb{R}^2 \). Defining the operator norm \( \|B\| := \sup_{\|x\|=1} \|Bx\| \), we translate the condition to the requirement that \( \|B\| \leq 1 \). This matrix norm is usually called the spectral norm.

It can be shown (see e.g. [6]) that the spectral norm can be computed as

\[
\|B\| = \sqrt{\rho(B^*B)},
\]

where \( B^* \) is Hermitian conjugate of \( B \). Simple algebraic manipulation yields the following statement.

Proposition 4.1. Let \( \Sigma(\Omega) \) be as in (2) with \( \Omega \) being a disc centered at the origin. Let \( A \) be a linear mapping such that \( A \pi(L) \subset \pi(L) \). Then \( A \Sigma(\Omega) \subset \Sigma(\Omega) \) if and only if the mapping \( B \) corresponding to \( A \) satisfies

\[
\text{tr} B^*B \leq 1 + \det B^*B.
\]

Let us mention that all mappings \( A \) from Proposition 3.2 satisfy the condition of Proposition 4.1. They are therefore among the self-similarities of cut-and-project sets with circular acceptance window.

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