On the entanglement across a cubic interface in 3 + 1 dimensions

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We calculate the area, edge and corner Renyi entanglement entropies in the ground state of the transverse-field Ising model, on a simple-cubic lattice, by high-field and low-field series expansions. We find that while the area term is positive and the line term is negative as required by strong subadditivity, the corner contributions are positive in 3-dimensions. Analysis of the series suggests that the expansions converge up to the physical critical point from both sides. The leading area-law Renyi entropies match nicely from the high and low field expansions at the critical point, forming a sharp cusp there. We calculate the coefficients of the logarithmic divergence associated with the corner entropy and compare them with conformal field theory results with smooth interfaces and find a striking correspondence.

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FIG. 2: Ratio of successive series coefficients in the variable $\lambda^2 = (J/h)^2$ for the ‘area-law’ coefficient of various Renyi entropies. Only 2nd, sixth and tenth Renyi entropies are shown. The difference between successive Renyi entropies is very tiny. The critical-point value, known from the study of other ground state properties, is shown by a solid line. Assuming standard scaling for the singular part of the ‘area-law’ term, the ratio of coefficients must asymptotically approach the critical-point with the slope given by the dashed line.

FIG. 3: Ratio of successive series coefficients in the variable $x^2 = (h/4J)^2$ for the ‘area-law’ coefficient of various Renyi entropies. Only second, sixth and tenth Renyi entropies are shown. The difference between them is very tiny. The critical-point value, known from the study of other ground state properties, is shown by a solid line. Assuming standard scaling for the singular part of the ‘area-law’ term, the ratio of coefficients must asymptotically approach the critical-point with the slope given by the dashed line.

expected from scaling arguments. For the all important corner term, we simply bias the series to have the expected logarithmic singularity at the critical point and estimate the coefficient of log divergence.

From the point of view of critical phenomena, 3 + 1-D is at the boundary where Gaussian or free fixed point becomes stable. Thus, a correspondence with free field theory is expected, modulo logarithmic corrections. One should note that even in 2 + 1-D, the log coefficients for free field theory and Ising criticality are not far from each other. However, we are not aware of a field theory calculation for a lattice model with a corner. Conformal field theory has been used to study logarithmic singularity associated with smooth interfaces. On a lattice, we can define a closed surface consisting of a cube, with 8 corners. Quite remarkably, we find that the log singularity in the corner entropy for a cube, and its dependence on the Renyi index, is quite close to 1/8th of the result for a sphere. This is potentially a deep result and deserves further theoretical attention.

The transverse-field Ising Hamiltonian is

$$\mathcal{H} = -h \sum_i S_i^x - J \sum_{\langle i,j \rangle} S_i^z S_j^z.$$  \(1\)

We expand around both the $J = 0$ and the $h = 0$ limits of the model. Various ground state properties can be calculated as a power series expansion in the variables $\lambda = J/h$ or $x = h/4J$. When a system is bipartitioned into two subsystems $A$ and $B$, the $\alpha$th Renyi entropy is defined as

$$S_\alpha(A) = \frac{1}{1-\alpha} \ln \text{Tr} (\hat{\rho}_A^\alpha),$$ \(2\)

where $\hat{\rho}_A = \text{Tr}_B |\Psi\rangle \langle \Psi|$. The reduced density matrix for subsystem $A$.

When the infinite system is bipartitioned by a plane (such as the XY plane), the ground state has an entropy per unit area. To define an entropy per unit length, we consider dividing the system by two perpendicular planes. The area contributions can be canceled out analogous to the calculation of corner entropy in 2D, leaving one with an edge-entropy associated with a 90 degree edge. To define a $\pi/2$ solid angle corner entropy, we consider the intersection of three perpendicular planes. Once again, the area and edge entropies can be canceled out by suitable subtraction, leaving one with the corner entropy. If one has a closed cubic interface with large linear dimension, it can be decomposed into planar surfaces, 90 degree edges and $\pi/2$ corners. Within perturbation theory, the total entanglement across the cube is the sum of each contribution.

Let the entropy per unit boundary-area be defined as $a_\alpha = S_\alpha/A$, the entropy per unit length for a 90 degree edge be defined as $s_\alpha = S_\alpha/L$, and the corner entropy...
for a $\pi/2$ solid-angle be defined as $c_\alpha$. Series expansion coefficients for $a_n$, $s_n$, and $c_n$ are calculated complete to order $\lambda^{14}$ in the high-field expansions and to order $x^{22}$ in the low-field expansions.

We first study the critical point of the leading entropy, that is the area-law term, to find where it becomes singular. To locate this critical point, we calculate the ratio of coefficients $r_n = a_n/a_{n-1}$. These coefficients should approach $1/\lambda^2$ or $1/x^2$ as $n \to \infty$. The critical point of the model is well known from previous series expansion study\cite{27} to be at $1/\lambda_c = 5.15$. In Fig. 2 and Fig. 3, we show the ratio of coefficients from the high and low field expansions respectively. The value of the known critical coupling is also shown. Also shown as a dashed-line is the asymptotic slope along which the $r_n$ should approach the critical value as $n \to \infty$, assuming a $1/\xi^2$ singularity with $\nu = 1/2$.\cite{23,28} From the high-field expansion, we find that the numerical values are in very good agreement with the expectations from scaling theory. Note that the data include ratios for second, sixth and tenth Renyi entropies. Evidently there is no discernible dependence of the critical point on the Renyi index. However, we should note that a very small variation in the critical point would probably not be distinguishable in our study.

The low-field expansions show a strong alternation, which is not unusual for series expansions around an ordered phase (also sometimes called a low temperature expansion\cite{20}). However, they also provide strong support for the idea that the singularities happen at the critical point and there is no variation in the critical point with the Renyi index.

The area and line coefficients are analyzed both by simple Pade approximants that lack critical behavior and a biased differential approximant, which builds in a power-law singularity at the critical point. They are shown in Fig. 4 and Fig. 5 respectively. Outside the critical region, where simple Pade approximants and biased differential approximants agree, our results should be highly accurate. One finds that the critical behavior only sets in when one is within a few percent of the critical point. Furthermore, we find that the leading ‘area-law’ entropies nicely meet together from the high and low temperature sides, forming a sharp cusp there. This is further evidence that the only singularity in the system is at the physical critical point. We do not find any evidence for a log singularity in the line term, although, it would be difficult for series analysis to reliably deal with coexisting power-law and logarithmic singularities.

We now turn to the key quantity of interest in the paper, namely, the corner entropy and its logarithmic singularity. We expect that, on approach to the critical point, the corner entropy behaves as

$$c_\alpha = c_\alpha^* \ln \xi = -\nu \ c_\alpha^* \ln (\lambda_c - \lambda).$$

where $c_\alpha^*$ should be universal. By scaling, it should equal the coefficient $c_\alpha^* \ln L$ in the logarithmic size dependence in an $L \times L \times L$ system, when the system is at the critical coupling\cite{4,13}. We are interested in a comparison...
Entropies associated with planes, edges and corners in field Ising model by high and low field series expansions. Tanglement entropies of the three-dimensional transverse would be interesting to study such corner entropies on estimates a corner calculation on a lattice[24]. In future, it s spherical boundary in a continuum theory closely approximates to be 0 high and low field sides respectively. If we shift the critical point value to 1/8-th of the values obtained in conformal field theory for a spherical surface. This suggests a universality between entanglement in lattice statistical models and continuum field theories deserves further attention.

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[29] The differences for $\alpha = 2$ could be due to numerical uncertainties from having a short series. They could also arise from the fact that the Ising criticality in $3 + 1$ D is not completely equivalent to a free-field theory at the upper critical dimension, and logarithmic corrections to critical properties are present.