ON THE ESTIMATION OF SOME MELLIN TRANSFORMS
CONNECTED WITH THE FOURTH MOMENT OF $|\zeta(\frac{1}{2} + it)|$

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ABSTRACT

Mean square estimates for $\mathcal{Z}_2(s) = \int_1^{\infty} |\zeta(\frac{1}{2} + ix)|^4 x^{-s} \, dx \, (\Re s > 1)$ are discussed, and some related Mellin transforms of quantities connected with the fourth power moment of $|\zeta(\frac{1}{2} + ix)|$.

1. Introduction

Let $\mathcal{Z}_2(s)$ be the analytic continuation of the function

$$\mathcal{Z}_2(s) = \int_1^{\infty} |\zeta(\frac{1}{2} + ix)|^4 x^{-s} \, dx \quad (\Re s > 1),$$

which represents the (modified) Mellin transform of $|\zeta(\frac{1}{2} + ix)|^4$. It was introduced by Y. Motohashi [14] (see also [7], [9], [12] and [15]), who showed that it has meromorphic continuation over $\mathbb{C}$. In the half-plane $\sigma = \Re s > 0$ it has the following singularities: the pole $s = 1$ of order five, simple poles at $s = \frac{1}{2} \pm i\kappa_j \ (\kappa_j = \sqrt{\lambda_j - \frac{1}{4}})$ and poles at $s = \frac{1}{2} \rho$, where $\rho$ denotes complex zeros of $\zeta(s)$. Here as usual $\{\lambda_j = \kappa_j^2 + \frac{1}{4}\} \cup \{0\}$ is the discrete spectrum of the non-Euclidean Laplacian acting on $SL(2,\mathbb{Z})$-automorphic forms (see [15, Chapters 1–3] for a comprehensive account of spectral theory and the Hecke $L$-functions).

The aim of this note is to study the estimation $\mathcal{Z}_2(s)$ in mean square and the (modified) Mellin transforms of certain other quantities related to the fourth power moment of $|\zeta(\frac{1}{2} + it)|$. This research was begun in [12], and continued in [7] and [9]. It was shown there that we have

$$\int_0^T |\mathcal{Z}_2(\sigma + it)|^2 \, dt \ll_{\varepsilon} T^{\varepsilon} \left( T + T^{2 - 2\varepsilon} \right) \quad (\frac{1}{2} < \sigma < 1),$$

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and we also have unconditionally

\begin{equation}
\int_0^T |Z_2(\sigma + it)|^2 \, dt \ll T^{\frac{10-8\sigma}{3}} \log^C T \quad (\frac{1}{2} < \sigma < 1, \ C > 0).
\end{equation}

Here and later \( \varepsilon \) denotes arbitrarily small, positive constants, which are not necessarily the same ones at each occurrence, while \( \sigma \) is assumed to be fixed. The constant \( c \) appearing in (1.1) is defined by

\begin{equation}
E_2(T) \ll \varepsilon \, T^{c+\varepsilon},
\end{equation}

where the function \( E_2(T) \) denotes the error term in the asymptotic formula for the mean fourth power of \( |\zeta(\frac{1}{2} + it)| \). It is customarily defined by the relation

\begin{equation}
\int_0^T |\zeta(\frac{1}{2} + it)|^4 \, dt = TP_4(\log T) + E_2(T),
\end{equation}

with

\begin{equation}
P_4(x) = \sum_{j=0}^4 a_j x^j, \quad a_4 = \frac{1}{2\pi^2}.
\end{equation}

For an explicit evaluation of the \( a_j \)'s in (1.5), see the author's work [3]. The best known value of \( c \) in (1.3) is \( c = 2/3 \) (see e.g., [11] or [15]), and it is conjectured that \( c = 1/2 \) holds, which would be optimal. Namely (see [2], [4], [14] and [15]) one has

\begin{equation}
E_2(T) = \Omega_\pm(T^{1/2}).
\end{equation}

Mean value estimates for \( Z_2(s) \) are a natural tool to investigate the eighth power moment of \( |\zeta(\frac{1}{2} + it)| \). Indeed, one has (see [7, eq. (4.7)])

\begin{equation}
\int_T^{2T} |\zeta(\frac{1}{2} + it)|^8 \, dt \ll \varepsilon \, T^{2\sigma-1} \int_1^{T^{1+\varepsilon}} |Z_2(\sigma + it)|^2 \, dt \quad (\frac{1}{2} < \sigma < 1).
\end{equation}

In [9] the pointwise estimate for \( Z_2(s) \) was given by

\begin{equation}
Z_2(\sigma + it) \ll \varepsilon \ t^{\frac{1}{2}(1-\sigma) + \varepsilon},
\end{equation}

for \( \frac{1}{2} < \sigma \leq 1 \) fixed and \( t \geq t_0 > 0 \). This result is still much weaker than the bound conjectured in [7] by the author, namely that for any given \( \varepsilon > 0 \) and fixed \( \sigma \) satisfying \( \frac{1}{2} < \sigma < 1 \), one has

\begin{equation}
Z_2(\sigma + it) \ll \varepsilon \ t^{\frac{1}{2}-\sigma + \varepsilon} \quad (t \geq t_0 > 0).
\end{equation}
To define another Mellin transform related to $\mathcal{Z}_2(s)$, let $P_4(x)$ be defined by (1.4), let
\begin{equation}
Q_4(x) := P_4(x) + P_4'(x)
\end{equation}
and set
\begin{equation}
K(s) := \int_1^\infty (|\zeta(\frac{1}{2} + ix)|^4 - Q_4(\log x)) E_2(x) x^{-s} \, dx.
\end{equation}
The integral on the right-hand side of (1.10) converges absolutely at least for $\sigma > 5/3$, in view of (1.3) with $c = 2/3$ and the bound for the fourth moment of $|\zeta(\frac{1}{2} + ix)|$. However, the interest in $K(s)$ lies in the fact that (1.4) and (1.9) yield
\begin{equation}
E_2'(x) = |\zeta(\frac{1}{2} + ix)|^4 - Q_4(\log x).
\end{equation}
Thus an integration by parts shows that
\begin{equation}
K(s) = -\frac{1}{2} E_2^2(1) - \frac{1}{2} s \int_1^\infty E_2^2(x) x^{-s-1} \, dx.
\end{equation}
In view of the mean square bound (see e.g., [10] and [15])
\begin{equation}
\int_1^T E_2^2(t) \, dt \ll T^2 \log^C T,
\end{equation}
it follows that (1.12) furnishes analytic continuation of $K(s)$ to the region $\sigma > 1$. A true asymptotic formula for the integral in (1.13) would provide further analytic continuation of $K(s)$. For example, a strong conjecture is that
\begin{equation}
\int_1^T E_2^2(t) \, dt = T^2 p(\log T) + R(T), \quad R(T) \ll \varepsilon T^{\rho + \varepsilon}
\end{equation}
with $p(x)$ a suitable polynomial (perhaps of degree zero) and $\frac{3}{2} \leq \rho < 2$. Namely from [2, Theorem 4.1] with $|H| = T^{\rho/3}$ it follows that (1.14) implies (1.3) with $c \leq \rho/3$, hence $\rho \geq \frac{3}{2}$ must hold in view of (1.7). Then the integral in (1.12) becomes
\begin{equation}
\int_1^\infty (2p(\log x) + p'(\log x)) x^{-s} \, dx + O(1) + (s + 1) \int_1^\infty R(x) x^{-s-1} \, dx.
\end{equation}
The first integral above is easily evaluated as $\sum_{j=1}^{m+1} b_j (s - 1)^{-j}$, where $m$ is the degree of $p(x)$. The second integral is regular for $\sigma > \rho - 1$ if $R(x) \ll \varepsilon x^{\rho + \varepsilon}$. Thus,
on (1.14), it is seen that $K(s)$ is regular for $\sigma > \rho - 1$ except for a pole at $s = 1$ of order $1 + \deg p(x)$.

Finally we define a Mellin transform related to the spectral theory of the non-Euclidean Laplacian. Let, as usual, $\alpha_j = |\rho_j(1)|^2 \cosh \pi \kappa_j^{-1}$, where $\rho_j(1)$ is the first Fourier coefficient of the Maass wave form corresponding to the eigenvalue $\lambda_j$ to which the Hecke series $H_j(s) = \sum_{n=1}^{\infty} t_j(n)n^{-s}$ is attached. For $0 < \xi \leq 1$ we define

$$I(t; \xi) = \frac{1}{\sqrt{\pi t \xi}} \int_{-\infty}^{\infty} |\zeta(\frac{1}{2} + it + iu)|^4 \exp(-u/t \xi^2) \, du.$$  

The importance of the function $I(t; \xi)$ in the theory of the fourth power moment of $|\zeta(\frac{1}{2} + it)|$ comes from the fact that, for $\frac{1}{2} \leq \xi < 1$ and suitable $C > 0$, we have (see Y. Motohashi [13] and [15]) the explicit formula

$$I(t; \xi) = \frac{\pi}{\sqrt{2t}} \sum_{j=1}^{\infty} \alpha_j H_j^{3}(\frac{1}{2}) \kappa_j^{-3/2} \sin \left( \kappa_j \log \frac{\kappa_j}{4\xi t} \right) \exp \left( -\frac{1}{4} \xi^2 \kappa_j^2 \right) + O(\log^C t)$$

$$= I(t; \xi) + O(\log^C t),$$

say. Then we let

$$J(s, \xi) = \int_1^\infty I(x; \xi)x^{-s} \, dx$$

denote the (modified) Mellin transform of $I(t; \xi)$. In view of the bound (see e.g. [15])

$$\sum_{\kappa_j \leq T} \alpha_j H_j^{3}(\frac{1}{2}) \ll T^2 \log^C T \quad (C > 0)$$

it easily follows that $J(s, \xi)$ is regular for $\sigma > 2 - \frac{3}{2} \xi$.

The plan of the paper is as follows. The function $Z_2(s)$ will be studied in Section 2, $K(s)$ in Section 3, while Section 4 is devoted to $J(s, \xi)$.

2. The function $Z_2(s)$

The new result concerning mean square bounds for $Z_2(s)$ is contained in

**Theorem 1.** For $\frac{5}{6} \leq \sigma \leq \frac{5}{4}$ we have

$$\int_1^T |Z_2(\sigma + it)|^2 \, dt \ll_{\varepsilon} T^{\frac{15-12\sigma}{5} + \varepsilon}.$$
Proof. To prove (2.1) we first introduce, as in [7], the function

\[ F_K(s) := \int_{K/2}^{5K'/2} \varphi(x)|\zeta(\frac{1}{2} + ix)|^4 x^{-s} \, dx \quad (K < K' \leq 2K), \]

where \( \varphi(x) \in C^\infty \) is a nonnegative function supported in \([K/2, 5K'/2]\) such that \( \varphi(x) = 1 \) for \( K < K' \leq 2K \), and

\[ \varphi^{(r)}(x) \ll_r K^{-r} \quad (r = 0, 1, 2, \ldots). \]

To connect \( F_K(s) \) and \( Z_2(s) \) note that from the Mellin inversion formula (e.g., [7, eq. (2.6)] we have

\[ |\zeta(\frac{1}{2} + ix)|^4 = \frac{1}{2\pi i} \int_{(1+\varepsilon)} Z_2(s)x^{s-1} \, ds \quad (x > 1), \]

where the \( \int_{(c)} \) denotes integration over the line \( \Re s = c \). Here we replace the line of integration by the contour \( \mathcal{L} \), consisting of the same straight line from which the segment \([1 + \varepsilon - i, 1 + \varepsilon + i]\) is removed and replaced by a circular arc of unit radius, lying to the left of the line, which passes over the pole \( s = 1 \) of the integrand. By the residue theorem we have

\[ |\zeta(\frac{1}{2} + ix)|^4 = \frac{1}{2\pi i} \int_{\mathcal{L}} Z_2(s)x^{s-1} \, ds + Q_4(\log x) \quad (x > 1), \]

where \( Q_4 \) is defined by (1.9). Hence by using (2.4) we obtain

\[ F_K(s) = \frac{1}{2\pi i} \int_{\mathcal{L}} Z_2(w) \left( \int_{K/2}^{5K'/2} \varphi(x)x^{w-s-1} \, dx \right) dw + \int_{K/2}^{5K'/2} \varphi(x)Q_4(\log x)x^{-s} \, dx. \]

In view of (2.3) we infer, by repeated integration by parts, that the last integral in (2.5) is \( \ll T^{-A} \) for any given \( A > 0 \). Similarly we note that

\[ \int_{K/2}^{5K'/2} \varphi(x)x^{w-s-1} \, dx \]

\[ = (-1)^r \int_{K/2}^{5K'/2} \varphi^{(r)}(x)x^{w-s-r-1} \frac{1}{(w-s)\cdots(w-s+r-1)} \, dx \ll T^{-A}. \]
for any given $A > 0$, provided that $|\mathfrak{Im} w - \mathfrak{Im} s| > T^\varepsilon$ and $r = r(A, \varepsilon)$ is sufficiently large. Thus if in the $w$–integral in (2.5) we replace the contour $L$ by the straight line $\Re w = d$ ($\frac{1}{2} < d < 1$), we shall obtain

\begin{equation}
F_K(s) \ll_{\varepsilon} K^{d-\sigma} \int_{t-T^\varepsilon}^{t+T^\varepsilon} |Z_2(d+iv)| \, dv + T^{-2}.
\end{equation}

Squaring (2.6) and integrating it follows that

\begin{equation}
\int_T^{2T} |F_K(s)|^2 \, dt \ll_{\varepsilon} T^{-1} + K^{2d-2\sigma} T^\varepsilon \int_{T-T^\varepsilon}^{2T+T^\varepsilon} |Z_2(d+iv)|^2 \, dv \quad (\sigma > d).
\end{equation}

However, from (1.1) and (1.3) with $c = 2/3$ it follows that

\begin{equation}
\int_T^{2T} |Z_2(\frac{5}{6} + it)|^2 \, dt \ll_{\varepsilon} T^{1+\varepsilon},
\end{equation}

so that (2.7) yields

\begin{equation}
\int_T^{2T} |F_K(s)|^2 \, dt \ll_{\varepsilon} T^{-1} + K^{5/3-2\sigma} T^{1+\varepsilon} \quad (\frac{5}{6} \leq \sigma \leq 1).
\end{equation}

Now we write

\[ Z_2(s) = \int_X \left| \zeta \left( \frac{1}{2} + ix \right) \right|^4 x^{-s} \, dx + \int_X^\infty \left| \zeta \left( \frac{1}{2} + ix \right) \right|^4 x^{-s} \, dx = I_1 + I_2, \]

say, which initially holds for $\sigma > 1$. To estimate the mean square of $I_1$, we use the bound (which, up to `$\varepsilon$', is the strongest one known; see e.g., [1])

\begin{equation}
\int_X \left| \zeta \left( \frac{1}{2} + ix \right) \right|^8 \, dx \ll_{\varepsilon} X^{3/2+\varepsilon},
\end{equation}

and the following lemma, whose proof can be found in [7].

**LEMMA 1.** Suppose that $g(x)$ is a real-valued, integrable function on $[a, b]$, a subinterval of $[2, \infty)$, which is not necessarily finite. Then

\begin{equation}
\left| \int_0^T \left( \int_a^b g(x)x^{-s} \, dx \right)^2 \, dt \right| \leq 2\pi \int_a^b g^2(x)x^{1-2\sigma} \, dx \quad (s = \sigma + it, \, T > 0, \, a < b).
\end{equation}
Then, from (2.10) and (2.11), we find that

\[(2.12) \quad \int_{T}^{2T} I_1^2 \, dt \ll_{\varepsilon} X^{5/2 - 2\sigma + \varepsilon}.\]

From (2.6) and (2.7) with \(d = 5/6\) we obtain the analytic continuation of \(I_2 = I_2(s)\) to the region \(\sigma > 5/6\), taking first \(K = 2X\), writing 1 = \(\varphi(x) + (1 - \varphi(x)\) in the integral over \([\frac{1}{2}X, X]\), and estimating the mean square of

\[\int_{X/2}^{X} (1 - \varphi(x))|\zeta(\frac{1}{2} + ix)|^4 x^{-s} \, dx\]

by the bound in (2.11). For the remaining integrals we use, after the integrals are split into subintegrals of the type \(F_K(s)\), the bound given by (2.9). We obtain

\[(2.13) \quad \int_{T}^{2T} |Z_2(\sigma + it)|^2 \, dt \ll_{\varepsilon} X^{5/2 - 2\sigma + \varepsilon} + T^{1+\varepsilon} X^{5/3 - 2\sigma} \ll_{\varepsilon} T^{(15-12\sigma)/5+\varepsilon}\]

with the choice \(X = T^{6/5}\). Replacing \(T\) by \(T2^{-j}\) and adding up all the results, we obtain (2.1) in the range \(\frac{5}{6} \leq \sigma \leq 1\).

To obtain (2.1) in the remaining range \(1 < \sigma \leq \frac{5}{4}\), first we note that by a slight change of proof we see that (2.7) holds for \(d \geq 1\). Thus invoking (2.1) with \(\sigma = 1\) it is seen that for \(1 < \sigma \leq \frac{5}{4}\) (when the exponent in (2.12) is non-negative) we obtain

\[(2.14) \quad \int_{T}^{2T} |Z_2(\sigma + it)|^2 \, dt \ll_{\varepsilon} X^{5/2 - 2\sigma + \varepsilon} + T^{3/5+\varepsilon} X^{2-2\sigma} \ll_{\varepsilon} T^{(15-12\sigma)/5+\varepsilon}\]

again with the choice \(X = T^{6/5}\). The proof of Theorem 1 is complete.

As a corollary of (2.1) we can obtain (2.10), although this is somewhat going round in a circle, since we actually used (2.10) in the course of proof of (2.1). Recall that we have (1.7), but the analysis of its proof clearly shows that it remains valid for \(\sigma \geq 1\) as well. If we use (2.1) with \(\sigma = 5/4\) in (1.7), then (2.10) immediately follows. The essentialy new result provided by Theorem 1 is the bound

\[(2.15) \quad \int_{1}^{T} |Z_2(1 + it)|^2 \, dt \ll_{\varepsilon} T^{3/5+\varepsilon},\]

and it would be of great interest to decrease the exponent of \(T\) on the right-hand side of (2.15). In fact, the hypothetical estimate

\[(2.16) \quad \int_{1}^{X} |\zeta(\frac{1}{2} + ix)|^8 \, dx \ll_{\varepsilon} X^{1+\varepsilon}\]
is equivalent to

\[(2.17) \quad \int_1^T |Z_2(1+it)|^2 \, dt \ll T^\varepsilon.\]

From (1.7) with \(\sigma = 1\) it follows at once that (2.17) implies (2.16), and the other implication follows by the method of proof of (1.7) in [7]. This fact stresses out once again the importance of mean square bounds for \(Z_2(s)\).

3. The function \(K(s)\)

In this section we shall deal with the function \(K(s)\), defined by (1.10) or (1.12). Our result is the following

**THEOREM 2.** The function \(K(s)\), defined by (1.10), admits analytic continuation which is regular for \(\Re s > 1\). It satisfies

\[(3.1) \quad K(\sigma + it) \ll |t|^\varepsilon (|t|^{3-2\sigma} + 1) \quad (\sigma > 1)\]

and

\[(3.2) \quad \int_0^T |K(\sigma + it)|^2 \, dt \ll T^{13/6 + \varepsilon} \quad \left(\frac{7}{6} \leq \sigma \leq \frac{13}{6}\right).\]

**Proof.** To prove (3.1) note first that, by the Cauchy-Schwarz inequality for integrals, we have \((C > 0)\)

\[(3.3) \quad \int_Y^{2Y} \left( |\zeta(\frac{1}{2} + ix)|^4 - Q_4(\log x) \right) |E_2(x)| x^{-\sigma} \, dx \ll Y^{3/2 - \sigma} \log^C Y +
\]

\[Y^{-\sigma} \left( \int_Y^{2Y} |\zeta(\frac{1}{2} + ix)|^4 \, dx \right)^{1/2} \left( \int_Y^{2Y} (|\zeta(\frac{1}{2} + ix)|^4 - Q_4(\log x)) E_2^2(x) \, dx \right)^{1/2} \ll Y^{3/2 - \sigma} \log^C Y.\]

In the last integral we integrated by parts, recalling that that (1.11) holds, as well as (1.13) and (1.3) with \(c = 2/3\). The above bound shows then that \(K(s) \ll 1\) for \(\sigma > 3/2\). Suppose now that \(1 < \sigma \leq 3/2\). Similarly to (1.12) we have

\[(3.4) \quad K(s) = \int_1^X (|\zeta(\frac{1}{2} + ix)|^4 - Q_4(\log x)) E_2(x) x^{-s} \, dx
\]

\[- \frac{1}{2} E_2^2(X) X^{-s} + \frac{1}{2} s \int_X^\infty E_2^2(x) x^{-s-1} \, dx.\]
From (3.3) it follows that the first integral above is \( \ll X^{3/2-\sigma} \log^C X \), and the second (by (1.13)) is \( \ll t^2 \log^C X \). The choice \( X = t^2 \) easily leads then to (3.1).

To prove (3.2) we start from (3.4) and use Lemma 1. We obtain

\[
\int_T^{2T} \int_1^X \left| \left| \frac{1}{2} + ix \right|^4 - Q_4(\log x) \right| E_2(x) x^{-s} \, dx \, dt \\
\ll \int_1^X \left| \left| \frac{1}{2} + ix \right|^4 - Q_4(\log x) \right| E_2(x) x^{1-2\sigma} \, dx.
\]

(3.5)

Defining the Lindelöf function

\[
\mu(\sigma) = \limsup_{t \to \infty} \frac{\log |\zeta(\sigma + it)|}{\log t} \quad (\sigma \in \mathbb{R})
\]

in the customary way and letting \( \varphi(x) \) be as in (2.2), we see that

\[
\int_{K/2}^{5K'/2} \varphi(x) \left| \left| \frac{1}{2} + ix \right|^4 - Q_4(\log x) \right| E_2(x) x^{1-2\sigma} \, dx \\
\ll \varepsilon K^{-2\sigma + 4 \mu(\frac{1}{2})} + \int_{K/2}^{5K'/2} \varphi(x) \left| \left| \frac{1}{2} + ix \right|^4 + \log^8 x \right| E_2(x) \, dx \\
= K^{-2\sigma + 4 \mu(\frac{1}{2})} + \int_{K/2}^{5K'/2} \varphi(x) (E'_2(x) + Q_4(\log x) + \log^8 x) E_2(x) \, dx \\
= K^{-2\sigma + 4 \mu(\frac{1}{2})} + \int_{K/2}^{5K'/2} \varphi(x) \left( \frac{1}{2} E'_2(x) \right) = O_{\varepsilon}(K^{3-2\sigma + 4 \mu(\frac{1}{2})} + \varepsilon) \\
\ll \varepsilon K^{3-2\sigma + 4 \mu(\frac{1}{2})} + \varepsilon.
\]

Therefore the expression on the right-hand side of (3.5) is, for \( X = T^C, C > 0 \),

(3.6)

\( \ll \varepsilon T^\varepsilon (1 + X^{3-2\sigma + 4 \mu(\frac{1}{2})}) \).

Next we have, by Lemma 1, (1.3) with \( c = 2/3 \) and \( (1.13) \),

\[
\int_T^{2T} \left| s \int_X^\infty E_2^4(x) x^{-s-1} \, dx \right|^2 \, dt \\
\ll T^2 \int_X^\infty E_2^4(x) x^{-2\sigma-1} \, dx \\
\ll \varepsilon T^2 \int_X^\infty E_2^2(x) x^{1/3-2\sigma+\varepsilon} \, dx \\
\ll \varepsilon T^2 X^{7/3-2\sigma+\varepsilon},
\]

(3.7)

provided that \( \sigma > \frac{7}{6} \). From (3.4), (3.6) and (3.7) we infer that

\[
\int_T^{2T} |\mathcal{K}(\sigma + it)|^2 \, dt \\
\ll \varepsilon T^\varepsilon (1 + X^{3-2\sigma + 4 \mu(\frac{1}{2})} + T^2 X^{7/3-2\sigma}) \\
\ll \varepsilon T^{\frac{13+6\sigma}{3} + \varepsilon} \quad (\frac{7}{6} \leq \sigma \leq \frac{13}{6})
\]
with the trivial bound \( \mu(\frac{1}{2}) < \frac{1}{6} \) and \( X = T \). This easily gives (3.2), and slight improvements are possible with a better value of \( \mu(\frac{1}{2}) \). A mean square bound can also be obtained for the whole range \( \sigma > 1 \), by using the trivial bound \( tX^{1-\sigma+\varepsilon} \) for the second integral in (3.4). This will lead to

\[
\int_1^T |\mathcal{K}(\sigma + it)|^2 \, dt \ll \varepsilon \begin{cases} 
T^{1+\varepsilon} & (\sigma > 3/2), \\
T^{\frac{23-18\sigma+\varepsilon}{2}} & (1 < \sigma \leq 3/2).
\end{cases}
\]

Mean square estimates for \( \mathcal{K}(s) \) can be used to bound the fourth moment of \( E_2(t) \), much in the same way that mean square estimates for \( Z_2(s) \) can be used (cf. (1.7)) to bound the eighth moment of \( |\zeta(\frac{1}{2} + it)| \). We have

**THEOREM 3.** For \( \sigma > 1 \) fixed

\[
\int_T^{2T} E_2^4(t) \, dt \ll \varepsilon T^{2\sigma+1} \left( 1 + \int_0^{T^{1+\varepsilon}} \frac{|\mathcal{K}(\sigma + it)|^2}{1 + t^2} \, dt \right).
\]

**Proof.** Write (1.12) as

\[
k(s) := \int_1^{\infty} E_2^2(x)x^{-s-1} \, dx = \frac{2}{s}(\mathcal{K}(s) + \frac{1}{2}E_2^2(1)),
\]

so that \( k(s) \) is regular for \( \sigma > 1 \). From the Mellin inversion formula for the (modified) Mellin transform (see [7, Lemma 1]) we have

\[
E_2^2(x) = \frac{1}{2\pi i} \int_{(c)} k(s)x^s \, ds \quad (x > 1, c > 1).
\]

If \( \psi(t) \) is a smooth, nonnegative function supported in \([T/2, 5T/2]\) such that \( \psi(t) = 1 \) for \( T \leq t \leq 2T \), then

\[
\int_T^{2T} E_2^4(x) \, dx \leq \int_{T/2}^{5T/2} \psi(t)E_2^4(x) \, dx = \frac{1}{2\pi i} \int_{(c)} \left( \int_{T/2}^{5T/2} \psi(x)E_2^2(x)x^s \, dx \right) \, ds.
\]

In the last integral over \( x \) we perform a large number of integrations by parts, keeping in mind that \( \psi^{(j)}(x) \ll_j T^{-j} \) \((j = 0, 1, \ldots)\). It transpires that only the
values of $|t| \leq T^{1+\varepsilon}$ in the integral over $s = \sigma + it$ will make a non-negligible contribution. Hence (3.10) (with $c = \sigma > 1$) and Lemma 1 yield

$$I := \int_{T/2}^{T/2} \psi(x)E_2^4(x) \, dx \ll \varepsilon \left( 1 + \int_{-T^{1+\varepsilon}}^{T^{1+\varepsilon}} |k(\sigma + it)|^2 \, dt \right)^{1/2} \left( \int_{T/2}^{5T/2} \psi^2(x)E_2^4(x)x^{2\sigma+1} \, dx \right)^{1/2} \ll \varepsilon \left( 1 + \int_{0}^{T^{1+\varepsilon}} |k(\sigma + it)|^2 \, dt \right)^{1/2} T^{\sigma + \frac{1}{2}} I^{1/2}.$$ 

Simplifying the above expression and using (3.9) we arrive at (3.8).

One expects, in conjunction with the conjecture $E_2(x) \ll_{\varepsilon} x^{1/2+\varepsilon}$, the bound

$$\int_{0}^{T} E_2^4(t) \, dt \ll \varepsilon T^{3+\varepsilon}$$

(3.11)

to hold as well. In fact, from the author’s work [5, Theorem 2] with $a = 4$, one sees that the lower bound

$$\int_{0}^{T} E_2^4(t) \, dt \gg T^3$$

does indeed hold. The upper bound in (3.11) nevertheless seems unattainable at present. If true, it implies (by e.g., [7, eq. (4.4)] and Hölder’s inequality) the hitherto unproved bounds $E_2(T) \ll_{\varepsilon} T^{3/5+\varepsilon}$ and ([2, Lemma 4.1]) $\zeta(\frac{1}{2} + it) \ll_{\varepsilon} |t|^{3/20+\varepsilon}$. From (3.2) of Theorem 2 with $\sigma = 7/6$ we obtain

$$\int_{0}^{T} E_2^4(t) \, dt \ll_{\varepsilon} T^{10/3+\varepsilon}.$$ 

(3.12)

However, the bound (3.12) was already used in proving Theorem 3 via (3.7). It is (up to ‘$\varepsilon$’) the strongest known bound for the integral in question. From (3.8) it is seen that the conjectural bound (3.11) holds if

$$\int_{1}^{T} |K(\sigma + it)|^2 \, dt \ll_{\varepsilon} T^{2+\varepsilon} \quad (\sigma > 1)$$

(3.13)

holds. Conversely, if (3.11) holds, then the bound in (3.7) is to be replaced by $\ll_{\varepsilon} T^2 X^{2-2\sigma+\varepsilon} \quad (\sigma > 1)$, and (3.13) follows from this bound and (3.6) (with $X = T^{6/5}$). Therefore (3.11) is equivalent to the mean square bound (3.13).
4. The function \( J(s, \xi) \)

The result on the function \( J(s, \xi) \) \((0 \leq \xi < 1)\) is contained in

**THEOREM 4.** The function \( J(s, \xi) \) admits analytic continuation to the region \( \Re s > \frac{1}{2} \), where it represents a regular function. Moreover

\[
J(\sigma + it, \xi) \ll t^{-1} + t^{1 - \frac{1}{2} - \frac{\xi - \sigma}{\frac{1}{2} - \xi} + \varepsilon} \quad (\sigma > \frac{1}{2}, \ t \geq t_0, \ 0 \leq \xi < 1).
\]

**Proof.** Let \( X = t^{1/(1-\xi)-\delta} \) for a small, fixed \( \delta > 0 \). We define a sequence of non-negative, smooth functions \( \rho_j(x) \) \((j \in \mathbb{N})\) in the following way. Let \( \rho_1(x) \geq 0 \) be a smooth function supported in \([1, 2X]\) such that \( \rho_1(x) = 1 \) for \( 1 \leq x \leq X \), and \( \rho_1(x) \) monotonically decreases from 1 to 0 in \([X, 2X]\). The function \( \rho_2(x) \) is supported in \([X, 6X]\), where \( \rho_2(x) = 1 - \rho_1(x) \) for \( X \leq x \leq 2X \), \( \rho_2(x) = 1 \) for \( 2X \leq x \leq 4X \) and \( \rho_2(x) \) monotonically decreases from 1 to 0 in \([4X, 6X]\). In general, the function \( \rho_j(x) \), supported in \([2^{j-1}X, 3 \cdot 2^jX]\), satisfies

\[
\rho_j(x) = 1 - \rho_{j-1}(x) \quad \text{for} \quad 2^{j-1}X \leq x \leq 3 \cdot 2^{j-1}X, \ \rho_j(x) = 1 \quad \text{for} \quad [2^{j-1}X, 2^jX] \quad \text{and then decreases monotonically from 1 to 0 in} \ [2^jX, 3 \cdot 2^jX].
\]

In this way we obtain that

\[
\rho_j^{(r)}(x) \ll_{j,r} (2^j X)^{-r} \quad (j, r \in \mathbb{N}).
\]

Now we write (cf. (1.16))

\[
J(s, \xi) = \int_1^{2X} \rho_1(x)I(x; \xi)x^{-s} \, dx + \sum_{j \geq 2} \int_{2^{j-1}X}^{3 \cdot 2^j X} \rho_j(x)I(x; \xi)x^{-s} \, dx.
\]

In the first integral in (4.3) we insert the expression (1.16) for \( I(x; \xi) \) and integrate repeatedly by parts the factor \( x^{-1/2 - i\kappa j} \). The integrated terms, after \( r \) integrations by parts, will be

\[
\sum_{j=1}^{r} \frac{A_j}{(s - \frac{1}{2})^j}
\]

for suitable constants \( A_j \). The remaining integral, in view of (4.2), will be \( \ll t^{-B} \) for any given \( B > 0 \), provided that \( r = r(B) \) is sufficiently large. There remain the integrals

\[
I(K) := \int_{K/2}^{3K} \rho(x)I(x; \xi)x^{-s} \, dx \quad (\rho(x) = \rho_j(x), K = 2^jX).
\]
Writing the sine in (1.16) as a sum of exponentials, it follows that \( I(K) \) is a linear combination of expressions of the type

\[
J_\pm(K) := \sum_{j=1}^{\infty} \alpha_j H_j^3(\frac{1}{2}) \kappa_j^{-1/2} e^{\pm \kappa_j \log \frac{\kappa_j}{4x}} \int_{K/2}^{3K} \rho(x) x^{-\frac{1}{2} \pm i \kappa_j - s} \exp\left(-\frac{1}{4} (x^{\xi-1} \kappa_j)^2\right) \, dx,
\]

and we may consider only the case of the `+' sign, since the other case is treated analogously. The above series may be truncated at \( \kappa_j = K^{1-\xi} \log K \) with a negligible error. After an integration by parts the integral in \( J_\pm(K) \) becomes

\[
\frac{1}{s - i \kappa_j + \frac{i}{2}} \int_{K/2}^{3K} x^{\frac{1}{2} + i \kappa_j - s} \exp\left(-\frac{1}{4} (x^{\xi-1} \kappa_j)^2\right) \times \left( \rho'(x) + \frac{1}{2} (1 - \xi) \rho(x) x^{2 \xi - 3 \kappa_j^2} \right) \, dx.
\]

In the range \( \kappa_j \leq K^{1-\xi} \log K \) the above expression in parentheses is

\[
\ll K^{-1} + K^{2 \xi - 3} K^{2-2\xi} \log^2 K \ll K^{-1} \log^2 K.
\]

It transpires that, performing sufficiently many integrations by parts, only the values of \( \kappa_j \) for which \( |\kappa_j - t| \leq K^\xi \) will make a non-negligible contribution. For the estimation of \( \alpha_j H_j^3(\frac{1}{2}) \) in short intervals we shall need (see the author’s work [6])

**Lemma 2.** We have

\[
(4.4) \quad \sum_{K-G \leq \kappa_j \leq K+G} \alpha_j H_j^3(\frac{1}{2}) \ll \varepsilon \quad GK^{1+\varepsilon} \quad (K^\varepsilon \leq G \leq K).
\]

Note that (4.4) implies the bound

\[
H_j(\frac{1}{2}) \ll \varepsilon \quad \kappa_j^{1/3+\varepsilon},
\]

which breaks the convexity bound \( H_j(\frac{1}{2}) \ll \varepsilon \quad \kappa_j^{1/2+\varepsilon} \), but is still far away from the conjectural bound

\[
H_j(\frac{1}{2} + it) \ll \varepsilon \quad (\kappa_j + |t|)^{\xi},
\]

which may be thought of as the analogue of the classical Lindelöf hypothesis \((\zeta(\frac{1}{2} + it) \ll \varepsilon \quad |t|^\xi)\) for the Hecke series.

To complete the proof of Theorem 4, note that with the use of Lemma 2 we obtain

\[
J_\pm(K) \ll \varepsilon \quad K^{-1/2-\sigma} K \quad \sum_{|\kappa_j - t| \leq K^\varepsilon} \alpha_j H_j^3(\frac{1}{2}) \kappa_j^{-1/2}
\]

\[
\ll \varepsilon \quad K^{1/2-\sigma t^{1/2+\varepsilon}} \ll \varepsilon \quad t^{\frac{1}{2} \frac{1 - \xi - \sigma}{1 - \xi + \varepsilon} + \varepsilon}
\]

since \( K \gg X(= t^{1/(1-\xi)-\delta}) \). This leads to (4.1) in view of (4.3) and the preceding discussion.
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