Causal topology in future and past distinguishing spacetimes

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Abstract

The causal structure of a strongly causal spacetime is particularly well endowed. Not only does it determine the conformal spacetime geometry when the spacetime dimension \( n > 2 \), as shown by Malament and Hawking–King–McCarthy (MHKM), but also the manifold dimension. The MHKM result, however, applies more generally to spacetimes satisfying the weaker causality condition of future and past distinguishability (FPD), and it is an important question whether the causal structure of such spacetimes can determine the manifold dimension. In this work, we show that the answer to this question is in the affirmative. We investigate the properties of future or past distinguishing spacetimes and show that their causal structures determine the manifold dimension. This gives a non-trivial generalization of the MHKM theorem and suggests that there is a causal topology for FPD spacetimes which encodes manifold dimension and which is strictly finer than the Alexandrov topology. We show that such a causal topology does exist. We construct it using a convergence criterion based on sequences of ‘chain intervals’ which are the causal analogues of null geodesic segments. We show that when the region of strong causality violation satisfies a local achronality condition, this topology is equivalent to the manifold topology in an FPD spacetime.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

In the Riemannian geometry influenced discourse on Lorentzian geometry, the causal structure \((M, \prec)\) of a spacetime \((M, g)\) is viewed as a derivative construction which relies on the
underlying differentiable structure of $M$, with the causal relation $\prec$ between events on $M$ being obtained from the local light-cone structure provided by $g$. $(M, \prec)$ is, however, itself rich with information about the spacetime geometry, and it has been the focus of several investigations over the years to endow it with a more primitive role in Lorentzian geometry [1–4]. We will concern ourselves in this work only with the causal structure of *causal* spacetimes, i.e. those that harbour no closed causal curves. For such spacetimes, $(M, \prec)$ is a partially ordered set, i.e. $\prec$ is (i) *acyclic*: for $x, y \in M$, $x \prec y \prec x \Rightarrow x = y$, and (ii) *transitive*: for $x, y, z \in M$, $x \prec y$ and $y \prec z \Rightarrow x \prec z$. It is important to note that the set of events $M$ in $(M, \prec)$ does not carry with it the attendant differentiable structure of its spacetime avatar. $(M, \prec)$ has no analogue in Riemannian geometry, and it is therefore of very general interest to understand the role it plays in Lorentzian geometry [1, 2, 5–7].

A set of results due to Malament and Hawking–McCarty–King [3, 4] (MHKM) provides an important relationship between $(M, \prec)$ and $(M, g)$ in spacetimes that are *future and past distinguishing* (FPD). These are spacetimes in which the chronological (time-like) past and future sets are unique for every spacetime event. MHKM address the general question: what aspects of the spacetime geometry are left invariant under a causal structure preserving bijection between two spacetimes? Such a map is called a *causal bijection*.

**Definition 1.** A causal bijection $f : (M_1, g_1) \rightarrow (M_2, g_2)$ is a bijection between the set of events $M_1$ and $M_2$ which, in addition, preserves the causal relations $\prec_1$ and $\prec_2$: for $x_1, y_1 \in M_1$, $x_1 \prec_1 y_1 \Rightarrow f(x_1) \prec_2 f(y_1)$ and for $x_2, y_2 \in M_2$, $x_2 \prec_2 y_2 \Rightarrow f^{-1}(x_2) \prec_1 f^{-1}(y_2)$.

Malament’s original results were restricted to chronological bijections, i.e. those that preserve only the chronological relation, but as shown by Levichev [9], causal bijections themselves imply chronological bijections. We can summarize these results as follows.

**Theorem 2** (Malament–Hawking–King–McCarty–Levichev (MHKML)). If a causal bijection $f$ exists between two $n$-dimensional spacetimes which are both future and past distinguishing, then these spacetimes are conformally isometric when $n > 2$.

Here, a conformal isometry between $(M_1, g_1)$ and $(M_2, g_2)$ is a bijection $f : M_1 \rightarrow M_2$ such that $f$ and $f^{-1}$ are smooth and $g_2 = \Omega^2 g_1$ for some real, smooth, non-vanishing function $\Omega$. Importantly, this implies that $M_1$ and $M_2$ have the same topology.

The power of the MHKML theorem is evident. It tells us that the causal structure $(M, \prec)$ of an $n$-dimensional spacetime which is FPD determines its conformal geometry and topology. The *only* remaining geometric degree of freedom not determined by the causal structure is the conformal factor $\Omega$. This suggests an alternative, non-Riemannian approach to Lorentzian geometry in which a partially ordered set $(M, \prec)$ plays a primitive rather than a derivative role. Indeed, the MHKML theorem provides a strong motivation for the causal set approach to quantum gravity in which a locally finite partially ordered set replaces the spacetime continuum [10].

Can the MHKML theorem be generalized to include spacetimes of different dimensions? Equivalently, are causal bijections rigid enough to constrain the spacetime dimension? Let us consider for the moment a special subclass of FPD spacetimes, namely those that are strongly causal. For such spacetimes, the Alexandrov topology $\mathcal{A}$ is a causal topology which is equivalent to the manifold topology $M$ (Theorem 4.24 in [5]), so that $(M, \prec)$ determines the manifold topology and hence dimension. Since a causal bijection $f$ preserves the chronological relation [9] and hence the topology $\mathcal{A}$, this implies that

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5 The notion of a causal structure preserving map has also been studied from a different perspective in [8], under the name of isocausality class.
**Corollary 3.** If there is a causal bijection between two strongly causal spacetimes, then they both have the same manifold dimension and topology.

This generalizes the MHKML theorem when applied to strongly causal spacetimes. However, since $\mathcal{A}$ is known to be strictly coarser than $\mathcal{M}$ for spacetimes that are not strongly causal, it is an interesting question whether causal bijections impose topological constraints in such spacetimes. In this work, we show that corollary 3 can be generalized to spacetimes satisfying weaker causality conditions, i.e. those that are either past or future distinguishing:

**Proposition 4.** If there is a causal bijection between two FPD spacetimes, then they are of the same dimension.

This gives us a genuine generalization of the MHKML theorem to FPD spacetimes for the spacetime dimension $n > 2$. Thus, for an FPD spacetime, $(M, \prec)$ encodes the manifold topology and hence its dimension. This begs the question: what is the causal topology for FPD spacetimes that corresponds to the manifold topology? We define a new causal topology $\mathcal{N}$ derived from a convergence condition on sequences of *chain intervals* which are order theoretic analogs of null geodesic segments. We show that $\mathcal{N}$ is equivalent to the manifold topology $\mathcal{M}$ for an FPD spacetime in which the regions of strong causality violation satisfy a certain local achronality condition.

In section 2, we briefly review some of the standard material on causal structure, in the process formalizing many of the concepts and definitions given in this section. We also define and discuss the properties of *chain intervals* which we use throughout our paper, not only to construct the new topology $\mathcal{N}$ but also to give a *local* characterization of strong causality and future and past distinguishability. In section 3, we investigate the properties of FPD spacetimes in some detail and find some new and generic features. Proposition 4 follows from this analysis.

In section 4, we define a new convergence criterion on sequences of chain intervals from which we derive the causal topology $\mathcal{N}$. We show that for FPD spacetimes which satisfy an additional local achronality condition for the regions of strong causality violation, $\mathcal{N}$ is equivalent to the manifold topology. This construction is inspired by the work of [4], where null geodesic segments are used instead of chain intervals, albeit with an entirely different motivation. Indeed, a convergence condition using null geodesic segments instead of chain intervals always gives the manifold topology for FPD spacetimes, but since null geodesic segments are not order theoretically defined, this does not give rise to a purely *causal* topology. The construction of $\mathcal{N}$ via a convergence criterion however does not give us a causal basis. Finding such a basis has proved difficult, and we end this section with some speculative remarks.

In section 5, we examine another candidate for a causal topology inspired by a causal convergence criterion defined in [12]. The convergence criterion used in [12] gives rise to a causal topology $\mathcal{F}$ which is strictly finer than $\mathcal{M}$ and is Hausdorff and equivalent to the path topology constructed by Hawking–McCarthy–King [3]) iff the spacetime is past and future distinguishing. $\mathcal{F}$ can alternatively be defined in terms of its basis elements, obtained from ‘doubled’ chronological intervals, and this makes it attractive to work with. We show that a natural extension of the convergence condition of [12] gives rise to yet another distinct causal topology $\mathcal{P}$ which is strictly coarser than $\mathcal{F}$ and also strictly finer than $\mathcal{M}$ when the spacetime is FPD. It is thus tempting to conclude that the topological content of $(M, \prec)$ is far richer than one might have imagined and that further work could yield valuable insights.
2. Preliminaries

We give below the basic definitions and tools we will need for our paper, and refer the reader to the classic texts on causal structure [5, 6, 11] for a more detailed exposition. A spacetime \((M, g)\) is specified by a smooth, \(n\)-dimensional Hausdorff manifold \(M\) with topology \(\mathcal{M}\) and a smooth Lorentzian geometry \(g\). Every \(x \in M\) lies in a neighbourhood \(N \in \mathcal{M}\) which is *convex and normal*, i.e. where the exponential map \(\exp_p\) is a diffeomorphism from an open neighbourhood of the origin in Minkowski spacetime \(\mathbb{M}^n\) to \(N\) for every \(p \in N\). Thus, locally, the light-cone structure of \((M, g)\) is identical to that of \(\mathbb{M}^n\). Every \(x \in M\) also lies in a *simple region*, i.e. a convex normal open set whose closure lies in a convex normal neighbourhood.

A causal (chronological) curve is a smooth (with respect to \(\mathcal{M}\)) map \(γ : I \to M\) where \(I \subset \mathbb{R}\), such that the tangent to \(γ\) is everywhere non-spacelike (timelike) with respect to the metric \(g\). A causal or chronological curve is future or past directed depending on whether its tangent is everywhere future or past directed. \(x\) is said to causally precede \(y (x \prec y)\) if there is a future-directed causal curve from \(x\) to \(y\). Similarly, \(x\) is said to chronologically precede \(y (x \prec y)\) if there is a future-directed timelike curve from \(x\) to \(y\). The *horismotic relation* \(x \rightarrow y\) is then defined as \(x \prec y, x \nsim y\). We will call a causal curve for which every pair \(x \prec y\) is horismotic a horismotic curve. We will say that two points \(x, y\) are *spacelike related*, or incomparable, if \(x \nsim y\) and \(y \nsim x\).

In the standard usage, \(\prec\) is reflexive, i.e. \(x \prec x\) and \(\nsim\) is irreflexive \(x \nsim x\). The causal future and pasts of an event \(x\) are the sets \(I^+(x) \equiv \{y \in M | y \succ x\}\) and \(I^-(x) \equiv \{y \in M | y \prec x\}\) respectively, and the chronological future and past are the sets \(J^+(x) \equiv \{y \in M | y \gg x\}\) and \(J^-(x) \equiv \{y \in M | y \ll x\}\). Both \(I^\pm(x)\) are open in the manifold topology. Following [1] we write \(x < y\) if \(x \prec y\) and \(x \nsim y\) and \([x, y] = J(x, y) \equiv \{z \in M | x \nsim z \nsim y\}\). The *Alexandrov topology* \(\mathcal{A}\) is generated by the manifold open sets \(x, y\).

\((M, g)\) is said to be *causal* if \(\prec\) is acyclic, i.e. \(x < y\) and \(y < x\) implies that \(x = y\). The causal structure \((M, \prec)\) obtained from such a spacetime is then a (reflexive) partially ordered set, namely \(\prec\) is (i) acyclic and (ii) transitive, i.e. \(x < y\) and \(y < z\) \(\Rightarrow x < z\). Importantly, \(M\) is taken here to be simply the set of events without the additional topological and differentiable structures that one needs in defining the spacetime \((M, g)\).

A spacetime is said to be *FPD* at \(x \in M\), if for all \(y \neq x\), \(I^+(x) \neq I^+(y)\) \((I^-(x) \neq I^-(y))\). An equivalent formulation can be given in terms of *future (past) locality neighbourhoods*.

**Definition 5.** A neighbourhood \(U\) of \(x\) is said to be a future (past) locality neighbourhood if every future (past)-directed causal curve from \(x\) intersects \(U\) in a connected set.

A spacetime is FPD at \(x\) if for every neighbourhood \(V\) of \(x\), there exists a future(past) locality neighbourhood \(U \subset V\) of \(x\). If a spacetime is both future and past distinguishing at \(x\), then every open set \(V \ni x\) contains a future and past locality neighbourhood of \(x\). A future and/or past distinguishing spacetime is one in which every point is future and/or past distinguishing. Such a spacetime is always causal, but the converse is not true.

An open set \(O\) in \((M, g)\) is *causal convex* if every causal curve between events \(x, y \in O\) lies in \(O\). A spacetime is said to be *strongly causal* at \(x\) if \(x\) is contained in a causally convex set whose closure is contained in a simple region. Such a neighbourhood is also referred to as a *local causality neighbourhood*. Equivalently, every open set \(V \ni x\) contains a neighbourhood \(U\) of \(x\) which no causal curve intersects in a disconnected set. A strongly causal spacetime is one in which all events are strongly causal. Such a spacetime is always both future and past distinguishing, but again the converse is not always true. For strongly
causal spacetimes \( \mathcal{A} \) is the manifold topology \( \mathcal{M} \), while it is strictly coarser than \( \mathcal{M} \) when strong causality is violated. Moreover, strong causality is equivalent to \( \mathcal{A} \) being Hausdorff.

In this work we will emphasize the distinction between manifold topology \( \mathcal{M} \) and causal topology, the latter being constructed purely from the order relation \( \prec \) on \( M \). In order to view the Alexandrov topology as a causal topology, we need to be able to obtain the chronological relation \( \preceq \) from \( \prec \). One way of doing this is to first obtain the horismotic relation. A natural candidate definition for this relation involves the use of what we shall refer to as \textit{chain intervals}.

\textbf{Definition 6.} A chain \((C, \prec)\) is a totally ordered subset of \((M, \prec)\), i.e. every pair \( u, v \in C \) is such that either \( u \prec v \) or \( v \prec u \). We say that a causal interval \([x, y]\) is a chain interval if it is a chain.

\textbf{Claim 7.} Let \((M, g)\) be a causal spacetime such that for all \( p \in M \), every neighbourhood of \( p \) contains an incomparable pair of events. Then \( x \rightarrow y \) iff for every \( w \in [x, y] \), distinct from \( x \) and \( y \), \([x, w]\) and \([w, y]\) are both chain intervals.

\textbf{Proof.} If \( x \rightarrow y \), then there is a null geodesic from \( x \) to \( y \). If this null geodesic is unique, then \([x, y]\) itself is a chain interval and we are done. If it is not unique, then (using propositions 2.19 and 2.20 in [5]) for every \( z \) with \( y \rightarrow z \), we see that \( x \preceq z \). For any \( w \in [x, y] \) distinct from \( x \) and \( y \), therefore, since \( x \rightarrow w \) and \( w \rightarrow y \), the null geodesic from \( x \) to \( z \) is unique as is the null geodesic from \( z \) to \( y \). Thus, \([x, w]\) and \([w, y]\) are chain intervals.

Conversely, for every \( w \in [x, y] \), distinct from \( x \) and \( y \), let \([x, w]\) and \([w, y]\) both be chain intervals. Assume that \( x \preceq w \). Then \((x, w) \neq \emptyset \). For every \( z \in (x, w) \), there is an \( O \ni z \) such that \( O \subset (x, y) \). By the assumptions of the claim, there exist \( u, v \in O \) which are incomparable, which is a contradiction. Thus, \( x \rightarrow w \) and similarly \( w \rightarrow y \). If \( x \preceq y \), then \( \exists z \) such that \( x \preceq z \preceq y \). Since \( z \in [x, y] \), this is a contradiction. Thus, \( x \rightarrow y \).

The requirement that every neighbourhood of an event contains an incomparable pair of events seems general enough to apply to any causal spacetime. However, one should be careful in attempting a generalization. The following example\(^6\), though not a counterexample, illustrates the need for care. Consider a two-dimensional spacetime on the cylinder, with the light cones gradually tilting over until there is a single null geodesic which traverses an \( S^1 \). Subsequent to this, the light cones then right themselves (see figure 8.8 of [11]). One can then get a causal spacetime by removing a point from the closed null geodesic. Every \( x \) on this null geodesic is, however, causally related to every other point in the spacetime—in other words, it has no incomparable event. On the other hand, every neighbourhood of \( x \) does indeed contain an incomparable pair. We will be able to provide an explicit proof that this is a feature of future or past distinguishing spacetimes, but we do not know if it is true more generally. Indeed, there may be a broader class of causal spacetimes for which the chain intervals give the horismos relation, but we will not explore this here.

We use the language of causal geodesics to define a future-directed chain interval from \( x \) to \( y \), as the chain interval \([x, y]\), and a past-directed chain interval from \( x \) to \( z \) as the chain interval \([z, x]\). Similarly, an open chain interval is a future and past endless chain \( L \) such that for every \( x, y \in L \), \([x, y] \subset L \).

\(^6\) We thank Fay Dowker for this very clarifying example.
3. Causal but not strongly causal spacetimes

Claim 8. For a future distinguishing spacetime \((M, g)\), every \(x \in M\) is contained in an arbitrarily small future locality neighbourhood \(U \subset M\) such that for every \(y \ni x\) with \(y \ni x\), \(I(x, y)\) is strongly causal. The analogous statement holds in the past distinguishing case.

Proof. Let \((M, g)\) be future distinguishing. For any \(y \ni x\) which lies in a future locality neighbourhood \(U\) of \(x\), \(I(x, y) = I(x, y, U)\) (otherwise there is a future-directed causal curve from \(x\) to \(y\) which leaves \(U\) and re-enters). Let \(N\) be a simple region containing \(x\) and choose the future locality neighbourhood \(U\) of \(x\) such that \(\bar{U} \subset N\). Since \(I(x, y)\) is causally convex and \(I(x, y) \subset N\), any \(z \in I(x, y)\) is strongly causal. Similarly for past distinguishing spacetimes. □

If \(\Delta\) is the set of points in \(M\) at which strong causality is violated, then for either past or future distinguishing spacetimes, the strongly causal region \(M - \Delta \neq \emptyset\) and is moreover open in \(M\) (see proposition 4.13 in [5]). Thus \(\Delta\) is closed in \(M\). We will also employ a minor rephrasing of theorem 4.31 in [5].

Theorem 9 (Penrose). Let \((M, g)\) be a causal spacetime and let strong causality fail at \(p \in \Delta \subset M\). Then, there is a future and past endless null geodesic \(\Gamma_p\) through \(p\) at every point of which strong causality fails, such that if \(u\) and \(z\) are any two points of \(\Gamma_p\) with \(u \prec v\), \(u \neq v\), then \(u \prec x\) and \(y \prec v\) together imply \(y \prec x\).

We will refer to the above null geodesic \(\Gamma_p \subset \Delta\) as a special null geodesic.

Claim 10. Let \((M, g)\) be a causal spacetime. Then the special null geodesic \(\Gamma_p\) through every \(p \in \Delta \subset M\) is horismotic and unique.

Proof. Let \(\Gamma_p\) be a special null geodesic through \(p \in \Delta \subset M\). Assume that there exists a pair \(x \prec y\) on \(\Gamma_p\) which are not horismotic, i.e. \(x \prec y\). Consider a pair of events \(u, v\) such that \(x \prec u \prec v \prec y\). Then by theorem 9, \(v \prec y\) and \(x \prec u\) implies that \(v \prec u\), which violates causality. Thus, \(\Gamma_p\) is horismotic. To show uniqueness, assume that there are two distinct special null geodesics \(\Gamma_p, \Gamma_{p'}\) through \(p \in M\). Let \(x \prec p \prec y\) with \(x, y \in \Gamma_p\) and \(x' \prec p \prec y'\) with \(x', y' \in \Gamma_{p'}\). Then \(x \prec y'\) and \(x' \prec y\) which from theorem 9 implies that \(x' \prec y'\), which is again a contradiction since \(\Gamma_p\) is horismotic. □

Lemma 11. Let \((M, g)\) be a causal spacetime and let \(p\) be a future distinguishing event. Then there exists a neighbourhood \(U\) of \(p\) such that for any future-directed null geodesic segment \(\Omega \subset U\) from \(p\) which is distinct from \(\Gamma_p\), \(\Omega \cap \Delta = \emptyset\), i.e. \(\Omega - \{p\} \subset M - \Delta\). The analogous time-reversed statement holds for \(p\) past distinguishing.

Proof. Assume to the contrary that no such neighbourhood exists. Choose \(U\) to be a future locality neighbourhood of \(p\), which lies in a simple region, and let \(\Omega\) be a future-directed null geodesic segment from \(p\) such that its intersection with \(U\) contains points of \(\Delta\) other than \(p\). For such an \(s \in \Omega \cap U\), there exists a special null geodesic \(\Gamma_s\) through \(s\), which from claim 10 means that \(\Gamma_s\) cannot coincide with \(\Omega\). For any \(r > s\) on \(\Gamma_{p}\), \(p \prec r\). Since \(U\) is open, there exists an \(r' \in U\) with \(r \prec r'\). From claim 8, since \(\langle p, r' \rangle \subset U\) is a strongly causal region, this is a contradiction. Similarly for a past distinguishing event. □

In particular, this means that every neighbourhood of a future distinguishing event \(p\) in a causal spacetime contains a future locality neighbourhood \(Q \ni p\) such that \(J^+(p, Q)\) lies in \(M - \Delta\), where \(\Gamma_p\) is the special null geodesic through \(p\). We illustrate this in figure 1.
Figure 1. The regions $J^\pm(p, Q)$ for $Q$ a future and past locality neighbourhood of $p$ are, excluding $\Gamma_p$, strongly causal. Since the regions of strong causality are open, we depict the strongly causal regions in $Q$ as the interiors of two widened cones from $p$ which contain the sets $J^\pm(p, Q)$ and intersect them only along $\Gamma_p$.

Let us now cast these results into the language of chain intervals as promised. In order to use chain intervals interchangeably with null geodesic segments, we need to satisfy the conditions of claim 7. Equivalently, it suffices to prove that

**Claim 12.** For a future or past distinguishing spacetime, if $[x, y]$ is a chain interval, then $x \rightarrow y$.

**Proof.** Assume otherwise, i.e. let $x \ll y$, and let $O$ be an open set contained in $\langle x, y \rangle \neq \emptyset$. From lemma 11, every future or past locality neighbourhood $U$ of $p \in M$ intersects the strongly causal region $M - \Delta$ non-trivially. For $p \in O$, choose $U \subset O$ to be such a neighbourhood of $p$, and let $r$ be a strongly causal point in $U$. Now, every causally convex neighbourhood $W$ of $r$ contains an $s$ which is incomparable to it. If $W$ is chosen to be a subset of $U$, then $r, s$ are an incomparable pair in $\langle x, y \rangle \subset [x, y]$, which is a contradiction. □

This allows us to give a local characterization of strong causality.

**Lemma 13.** A future or past distinguishing spacetime is strongly causal at $p$ iff every null geodesic segment containing $p$ in its interior contains a chain interval with $p$ in its interior.

**Proof.** Let $p$ be strongly causal and $U$ a local causality neighbourhood of $p$. Every null geodesic segment in $U$ through $p$ is horismotic in $U$ and is therefore a chain interval. Conversely, if $p$ is not strongly causal, then by theorem 9 there is a special null geodesic $\Gamma_p$ through $p$. Let $\Omega$ be a null geodesic through $p$ distinct from $\Gamma_p$ and let $a, b \in \Gamma_p$ with $a \prec p \prec b$. Then for any $x, y \in \Omega$ such that $x \prec p \prec y, x \ll b$ and $a \ll y$. By theorem 9, this means that $x \ll y$. Since the spacetime is future or past distinguishing, $\langle x, y \rangle$ contains incomparable pairs of events and hence $[x, y]$ is not a chain interval. Since this is true for any pair $x, y$, we see that no null geodesic through $p$ which is distinct from $\Gamma_p$ contains a chain interval with $p$ in its interior. □

Using claim 10, this means that

**Claim 14.** If strong causality fails at a point $p$ in a future or past distinguishing spacetime, then the only chain interval which contains $p$ in its interior lies in $\Gamma_p$.

On the other hand,
Claim 15. If p is a future distinguishing point in a causal spacetime, then any future-directed null geodesic from p contains a chain interval \([p, w]\). The analogous time-reversed statement holds for p past distinguishing.

Proof. Let p be future distinguishing and let U be a future locality neighbourhood of p which lies in a simple region. Let N be a null geodesic from p and \(z \in N \cap U\). Since \(J^+(p, U) = J^+(p) \cap U\), this means that \(p \to z\). Then for any \(w \in [p, z]\), \([p, w]\) is a chain interval by the above claim. □

It is important to know whether causal properties of spacetimes are preserved under a causal bijection. In [9] it was shown that for FPD spacetimes, a causal bijection is also a chronological bijection. Using the results above, we can extend this to the slightly more general statement

Claim 16. If \(f : (M_1, g_1) \to (M_2, g_2)\) is a causal bijection between two spacetimes both of which are either future distinguishing or past distinguishing, then \(f\) is also a chronological bijection.

Proof. First note that a causal bijection preserves chain intervals. Let \(x_1, y_1 \in M_1\) such that \(x_1 \rightarrow_1 y_1\). Then by claims 7 and 12 for every \(w_1 \in [x_1, y_1]\), \([w_1, y_1]\) and \([w_1, x_1]\) are chain intervals. Now, \(f(x_1) \prec\prec f(y_1)\). Assume that \(f(x_1) \prec\prec f(y_1)\). Then by the proof of claim 12, we see that \([f(x_1), f(y_1)]_2\) is not a chain interval. Moreover, there exists a \(w_2 \in (f(x_1), f(y_1))_2 \neq \emptyset\) such that \(f(x_1) \prec\prec w_2 \prec\prec f(y_1)\), so that \([f(x_1), w_2]\) and \([w_2, f(y_1)]_2\) are not chain intervals. Since \(f^{-1}(w_2) \in [x_1, y_1]\), this is a contradiction. Thus, if \(x_1 \rightarrow_1 y_1\), then \(f(x_1) \rightarrow_2 f(y_1)\). Conversely, by the same argument, if \(x_1 \prec\prec y_1\), then \(f(x_1) \prec\prec f(y_1)\). □

Claim 17. Let \(f\) be a causal bijection between two spacetimes \((M_1, g_1)\) and \((M_2, g_2)\) both of which are either future or past distinguishing. Then if \(p \in M_1\) is strongly causal in \((M_1, g_1)\), \(f(p) \in M_2\) is strongly causal in \((M_2, g_2)\).

Proof. Assume otherwise, i.e. that \(f(p)\) is not strongly causal. At \(p\) we can find two distinct chain intervals \([x, y]\) and \([u, v]\) that contain \(p\). This means that \(x \prec v\) and \(u \prec y\). Since \([f(x), f(y)]\) and \([f(u), f(v)]\) are also chain intervals, and they contain \(f(p)\), they must lie in the special null geodesic \(\Gamma_{f(p)}\) (by claim 14). But since \(f\) also preserves chronology by claim 16, \(f(x) \prec\prec f(v)\) which is a contradiction, since \(f(x), f(v) \in \Gamma_p\). □

We are now in a position to prove proposition 4.

Proof of proposition 4. W.l.o.g. let the two spacetimes \((M_1, g_1)\) and \((M_2, g_2)\) both be future distinguishing. Every \(p \in M_1\) is contained in a future locality neighbourhood \(U\) such that \((p, x)\) is strongly causal for every \(x \in U\). Since \((p, x)\) is causally convex and is open in the manifold topology, it can also be topologized with Alexandrov intervals in \((p, x)\), so that \(\mathcal{A}_1|_{(p, x)} = \mathcal{A}_1|_{(p, x)}\) because of strong causality. Similarly, since \(f((p, x)) = (f(p), f(x))\) is also strongly causal \(\mathcal{A}_2|_{(p, x)} = \mathcal{A}_2|_{(f(p), f(x))}\). However, since \(f\) is a homeomorphism from \(\mathcal{A}_1\) to \(\mathcal{A}_2\), it is also a homeomorphism between \(\mathcal{M}_1|_{(p, x)}\) and \(\mathcal{M}_2|_{(f(p), f(x))}\) which implies that \(M_1\) and \(M_2\) have the same dimension. □

A generalization of the MHKML theorem is then immediate

Corollary 18 (Extension of MHKML). If a causal bijection \(f\) exists between two spacetimes of dimensions \(n_1, n_2 \geq 2\) which are both future and past distinguishing, then \(n_1 = n_2\) and the spacetimes are conformally isometric.
It is important to note that while there is no restriction to $n > 2$ in proposition 4, such a restriction is crucial to the Hawking–King–McCarthy result and hence the above corollary.

4. A topology based on convergence of chain intervals

The fact that causal bijections encode manifold dimension and topology so generally is interesting and begs the question of whether there exists a causal topology which is equivalent to the manifold topology when the spacetime fails to be strongly causal. In this section, we present a new topology for future or past distinguishing spacetimes, using an Alexandrov convergence of chain intervals which we term $N$-convergence. In general, this convergence criterion is not equivalent to manifold convergence, but for FPD spacetimes, for which the region of strong causality violation is locally achronal, we can demonstrate that $N$-convergence is the same as manifold convergence.

Just as the convergence of causal curves is defined with respect to the manifold topology, one can also define the convergence of chain intervals with respect to a causal topology like the Alexandrov topology. A sequence of chain intervals $\{\Omega_i\} = \{[p_i, q_i]\}$ will be said to Alexandrov converge to an event $x \in M$ if for every $A$ open neighbourhood of $x$ there exists an $N$ such that for all $i > N$, $\Omega_i \cap A$ is non-empty. In what follows, we use the Alexandrov convergence of chain intervals to define a new convergence condition for the end points $p_i$ or $q_i$ of chain intervals.

**Definition 19.** Let $(M, g)$ be a causal spacetime and let $\Delta$ be the region of strong causality violation. A sequence $\{p_i\}$ is said to future $N$-converge to $p$ if there exists a future-directed chain interval $\Omega^+ = [p, q]$ from $p$ with $\Omega^+ - \{p\} \subset M - \Delta$ and a sequence of future-directed non-intersecting chain intervals $\{\Omega_i^+ = [p_i, q_i]\}$ from $p_i$ such that every point in $\Omega^+ - \{p\}$ is an Alexandrov convergence point of the sequence $\{\Omega_i^+\}$ and, moreover, no subsequence of $\{\Omega_i^+\}$ has any other convergence points off $\Omega^+$ in $M - \Delta$. Past $N$-convergence is similarly defined.

This definition is inspired by a construction used in Malament’s paper [4] which employs null geodesics instead of chain intervals. For completeness and later comparison, we define the null geodesic version of $N$-convergence as follows.

**Definition 20.** Let $(M, g)$ be a causal spacetime and let $\Delta$ be the region of strong causality violation. A sequence $\{p_i\}$ is said to future geodesic-$N$-converge to $p$ if there exists a future-directed null geodesic segment $\gamma^+$ from $p$ with $\gamma^+ - \{p\} \subset M - \Delta$ and a sequence of future-directed non-intersecting null geodesic segments $\{\gamma_i^+\}$ from $p_i$ such that every point in $\gamma^+ - \{p\}$ is an Alexandrov convergence point of the sequence $\{\gamma_i^+\}$ and, moreover, no subsequence of $\{\gamma_i^+\}$ has any other convergence points off $\gamma^+$ in $M - \Delta$. Past geodesic $N$-convergence is similarly defined.

We will find it useful to first show the following, straightforward, result and then employ it to deal with the more complicated case of FPD spacetimes.

**Lemma 21.** Future or past $N$-convergence is equivalent to manifold convergence in Minkowski spacetime.

**Proof.** Let $\{p_i\} \xrightarrow{M} p$. Any future- or past-directed null geodesic segment from $p$ is a chain interval, and since $\Delta = \emptyset$, it lies in $M - \Delta$. Let $O$ be an open neighbourhood of $p$ and let $\Omega^+$ be a future-directed null geodesic segment from $p$ which is future inextendible in $O$. Then there exists a non-contracting or expanding future-directed null geodesic congruence $\Omega^+_\alpha$ in $O$
which contains $\Omega^+$, generated by a null-vector field (strictly an equivalence class of null-vector fields) $\xi^a$. If $\Omega$ is the (unique) past completion of $\Omega^+$ in $O$, then $\Omega_o$ is the corresponding null geodesic congruence that continues $\Omega^+$ in the past. For the sub-sequence of the $\{p_i\}$’s which lie in $O$, let $\Omega_i^+$ be the future-directed null geodesic in this congruence from $p_i$ which is future inextendible in $O$. $\Omega_o$ are chain intervals since the spacetime is Minkowski.

Let $q \in \Omega^+$ such that $q$ is not an Alexandrov (and hence manifold) limit point of the sequence $\{\Omega^+_i\}$. Then there exists a neighbourhood $U \subset O$ of $q$ which does not intersect any of the $\{\Omega^+_i\}$. Now, $\xi^a$ generates a one parameter family of diffeomorphisms, and hence $U$ defines a collar neighbourhood $T_\varepsilon(U)$ of $\Omega$. Since $U \cap \Omega_i^+ = \emptyset$ for all $i$, this means that none of the $\Omega_i^+$ can enter $T_\varepsilon(U)$ which means that there is a neighbourhood $O' \subset T_\varepsilon(U)$ of $p$ which intersects none of the $\Omega_i^+$ which is a contradiction. Therefore, every $q \in \Omega^+$ is a limit point of $\{\Omega^+_i\}$. Moreover, since the $\Omega_i^+$ and $\Omega^+$ are future inextendible in $O$, all convergence points of $\{\Omega_o\}$ lie on $\Omega^+$. Therefore, the $\{p_i\}$ future $\mathcal{N}$-converge to $p$. A similar argument shows that the $\{p_i\}$ also past $\mathcal{N}$-converge to $p$.

Conversely, let $\{p_i\}$ future $\mathcal{N}$-converge to $p$. Then there exists a future-directed chain interval $\Omega_i$ from $p$ and a sequence of future chain intervals $\Omega_i^+$ from $p_i$ such that every point on $\Omega^+ - \{p\}$ is an Alexandrov convergence point of $\{\Omega_i^+\}$ with no other convergence points besides $p$. If $\{p_i\}$ does not converge in $M$ to $p$, then there exists a neighbourhood $U$ of $p$ which does not contain any of the $p_i$. On the other hand, since every point on $\Omega^+ \cap U - \{p\}$ is an Alexandrov and hence a manifold convergence point for $\{\Omega_i^+\}$, there exists an $N$ such that for all $i > N$, $\Omega^+_i \cap U \neq \emptyset$. Thus each $\Omega_i^+$ must enter and then leave $U$ to reach $p_i$, to the past. However, in $U$ since each $\Omega_i^+$ is also a null geodesic segment, it is uniquely defined and hence it must also have points of Alexandrov convergence on the past-extension $\Omega^-$ of $\Omega^+$ in $U$, which is a contradiction since $\Delta = \emptyset$. \hfill \Box

For a generic spacetime, some elements of the above proof are still valid, as long as we restrict to a simple region $O$ around $p$. However, since the Alexandrov topology is used in defining $\mathcal{N}$-convergence, rather than the manifold topology, care has to be exercised in the generalization. In a strongly causal spacetime, for example, since $\mathcal{A} \sim M$, this distinction is no longer important and the existence of arbitrarily small causally convex neighbourhoods of $p$ implies that the entire proof of lemma 21 can be reproduced for this case.

When strong causality is violated, however, much more caution is required. From lemma 11, we see that if the spacetime is future distinguishing at $p$, it admits a future-directed chain interval $\Omega^+$ from $p$ with $\Omega^+ - \{p\}$ in $M - \Delta$, and similarly if it is past distinguishing at $p$, it admits a past-directed chain interval $\Omega^-$ from $p$ with $\Omega^- - \{p\}$ in $M - \Delta$. However, the converses are not always possible. Thus, in order to be able to equate manifold convergence to either future or past $\mathcal{N}$-convergence, one needs the spacetime to be both future and past distinguishing. However, even this is not quite enough. Although a null geodesic congruence can be constructed through the $p_i$’s which Alexandrov converge to all points on $\Omega^+ - \{p\}$, these do not necessarily give rise to a sequence of chain intervals that Alexandrov converge to all points on $\Omega^+ - \{p\}$. This is because even though $\Delta$ is locally achronal with respect to $p$, it need not itself be locally achronal, i.e. there may exist no open neighbourhood $U \ni p$ in the manifold topology such that $\Delta \cap U$ is achronal. Hence, the null geodesic congruence through $p_i$ could, for all $i$, intersect $\Delta$ both in the future and the past. This means that the corresponding chain intervals that one can construct are trapped between different ‘leaves’ of $\Delta$, and hence manifold convergence would not imply $\mathcal{N}$-convergence for such ‘trapped’ sequences. Figure 2 illustrates the problem.

**Definition 22.** The region of strong causality violation $\Delta$ in a causal spacetime $(M, g)$ is said to be locally manifold achronal if for every $p \in \Delta$ there exists an open neighbourhood
Figure 2. The strong causality-violating region around $p \in \Delta$ can have a complicated structure. Here, the regions $\Delta_1$ and $\Delta_2$ intersect only on $\Gamma_p$. They are each spacelike with respect to $\Gamma_p$, but there is no neighbourhood of $p$ in which $\Delta$ is achronal. A sequence $\{p_i\}$ which manifold converges to $p$ can be trapped in between $\Delta_1$ and $\Delta_2$ as shown. Even though the $p_i$ can eventually lie off $\Delta$, chain intervals from and to $p_i$ can manifold converge only to $p$ since they are also trapped in this region.

$U \ni p$ in the manifold topology such that $\Delta \cap U$ is achronal. It is said to be locally Alexandrov achronal if $U$ is required instead to be open in the Alexandrov topology.

Since $\mathcal{A}$ is strictly coarser than $\mathcal{M}$ for spacetimes that are not strongly causal, and in particular for those Alexandrov sets which contain events in $\Delta$, the requirement of local Alexandrov achronality is stronger than the manifold version. Thus, although one might prefer using the former because it is intrinsically causal, it is from the spacetime perspective more restrictive than necessary to establish the equivalence of $\mathcal{N}$-convergence to manifold convergence. For this purpose, we will use only the manifold version of local achronality for $\Delta$, which is then also true for Alexandrov local achronality.

In order to deal with all manner of manifold converging sequences, either past or future $\mathcal{N}$-convergence is required, depending on whether the sequence lies in the causal future of a local patch of $\Delta$ or to its causal past.

**Definition 23.** $\{p_i\}$ is said to $\mathcal{N}$ converge to $p$ if it either future or past $\mathcal{N}$-converges to $p$.

**Proposition 24.** Let $(M, g)$ be a FPD spacetime. Then $\mathcal{N}$-convergence is equivalent to manifold convergence everywhere in $M$ if either $\Delta = \emptyset$ or it is locally manifold achronal.

**Proof.** If the limit point $p \in M - \Delta$, then the proof is similar to that of lemma 5; so in what follows, we will assume $p \in \Delta$. Let $\{p_i\} \overset{\mathcal{M}}{\rightarrow} p$. Let $U$ be a future and past locality neighbourhood of $p$ which is contained in a simple region $O$. Let $\Omega$ be a null geodesic through $p$ with $\Omega - \{p\} \subset M - \Delta$ which is both past and future inextendible in $O$. If $\Omega^+$ and $\Omega^-$ are the segments of $\Omega$ that are to the causal future and the causal past of $p$, respectively, then for any $q \in \Omega^+$ and $r \in \Omega^-$, $L_q^+ \equiv [p, q]$ and $L_r^- \equiv [r, p]$ are both chain intervals. In particular, $L^+ \equiv \Omega^+$ and $L^- \equiv \Omega^-$ are both ‘open’ chain intervals in $O$ (i.e. future and past endless in $O$, respectively, as defined in section 2).

We further assume that $O$ is contained in a neighbourhood $Q$ of $p$ in which $\Delta$ is achronal. Let $\Omega_i$ be null geodesics through the $p_i$ which belong to a locally non-singular null geodesic congruence constructed as in lemma 21. Let $\Omega^+_i$ and $\Omega^-_i$ be the future and past segments of $\Omega_i$, respectively. Using the exponential map, we see that every $q \in \Omega^+ - \{p\}$ is a manifold convergence point of $\{\Omega^+_i\}$ and since $\Omega^+ - p \subset M - \Delta$, it is also an Alexandrov convergence...
point. In particular, there is no other manifold and hence Alexandrov convergence point of \( \Omega^* \).

In order to construct chain intervals from \( \Omega^\pm \), one has to see whether they intersect \( \Delta \) or not. Since \( \Omega - \{ p \} \subset M - \Delta \) and every \( q \in \Omega - \{ p \} \) is an Alexandrov convergence point of \( \{ \Omega_i \} \), eventually \( \Omega_i \nsubseteq \Delta \). If \( \Omega_i \cap \Delta \) is either empty or just \( p_i \), then both \( \Omega_i^+ \) and \( \Omega_i^- \) are open chain intervals. Since \( O \) is not a subset of \( \Delta \), \( \Omega \) can at worst intersect \( \Delta \) at two disjoint points \( p_1 \) and \( p_2 \). However, since \( \Delta \) is achronal in \( O \), this leads to a contradiction: since \( p_1, 2 \in \Delta \) \( \exists r_1 \in \Gamma_{p_1} \), and since \( p_2 \notin \Gamma_{p_1}, r_1 \ll p_2 \) which is a contradiction.

Let each \( \Omega_i \) intersect \( \Delta \cap O \) at \( q_i \) or nowhere at all. If \( p_i > q_i \), then \( L_1^+ \equiv \Omega_i^+ \) is an open chain interval, and if \( p_i < q_i \), then \( L_1^- \equiv \Omega_i^- \) is an open chain interval. Now, for every sequence \( \{ p_i \} \), one can extract a subsequence \( \{ p_i' \} \) such that either all the \( \{ \Omega_i^+ \} \) or all the \( \{ \Omega_i^- \} \) are open chain intervals. Thus, \( \{ p_i \} \overset{\mathcal{N}}{\rightarrow} p \).

The converse argument is similar to that in lemma 21. Let \( \{ p_i \} \overset{\mathcal{N}}{\rightarrow} p \) and assume wlog that this is future \( \mathcal{N} \)-convergent. Namely, there exists a sequence of future chain intervals \( \{ L_i^+ \} \) from \( p_i \) which Alexandrov converges off \( p \) to a chain interval \( L^+ \) from \( p \) where \( L^+ - \{ p \} \subset M - \Delta \), with no other convergence points in \( M - \Delta \). Now assume, contrary to the assertion, that \( \{ p_i \} \) does not \( M \)-converge to \( p \). Then there exists a simple region \( O \ni p \) which eventually contains none of the \( p_i \). Then \( L^+ \) and \( L_i^+ \) correspond to future-directed null geodesic segments from \( p \) and \( p_i \), respectively, and have unique completions in \( O \) both to the past and the future. If \( L^- \) is the unique null geodesic past extension of \( L^+ \) in \( O \), then it cannot lie in \( \Delta \) by claim 10. The arguments of lemma 21 can then be reproduced to show that \( \{ L_i^+ \} \) also Alexandrov converges to \( L^- \) which is a contradiction. Thus, \( \{ p_i \} \overset{\mathcal{M}}{\rightarrow} p \).

From the above proofs it is clear that future or past geodesic \( \mathcal{N} \)-convergence, which uses null geodesic segments instead of chain intervals, is equivalent to manifold convergence for all FPD spacetimes without the further assumption that \( \Delta \) be locally achronal.

**Definition 25.** A set \( O \) is \( \mathcal{N} \)-open if every sequence of points in its complement \( O^c \), which \( \mathcal{N} \)-converges, has its limit point in \( O^c \).

Let \( \mathcal{N} \) denote the collection of \( \mathcal{N} \)-open sets.

**Lemma 26.** The collection of sets \( \mathcal{N} \) forms a topology on \( M \).

**Proof.** To show that \( \mathcal{N} \) forms a topology on \( M \), we need to prove that it satisfies the three properties of a topology. (i) The proof of \( M, \emptyset \in \mathcal{N} \) is trivial. (ii) Next let \( U_{\alpha} \) be sets in \( \mathcal{N} \), where \( \alpha \) belongs to some indexing set. To show that \( \bigcup_{\alpha} U_{\alpha} \in \mathcal{N} \), note that if an \( \mathcal{N} \)-convergent sequence lies outside \( \bigcup_{\alpha} U_{\alpha} \), then it must lie outside each of the \( U_{\alpha} \). Since \( U_{\alpha} \in \mathcal{N} \) for all \( \alpha \), the limit point of the sequence must also lie outside \( U_{\alpha} \) for all \( \alpha \). Thus, the limit point is not contained in \( \bigcup_{\alpha} U_{\alpha} \), which proves that \( \bigcup_{\alpha} U_{\alpha} \in \mathcal{N} \). (iii) To show that \( \bigcap_{i=1}^n U_i \in \mathcal{N} \), let \( \{ x_i \} \) be an \( \mathcal{N} \)-convergent sequence in \( M = \bigcap_{i=1}^n U_i = \bigcup_{i=1}^n (M - U_i) \). Since the \( (M - U_i) \)'s are finite in number, at least one of them, say \( (M - U_i) \), should contain a subsequence \( \{ y_j \} \) of \( \{ x_i \} \). It is clear from the definition of \( \mathcal{N} \) convergence that every subsequence of an \( \mathcal{N} \)-convergent sequence also \( \mathcal{N} \)-converges to the same limit point. But since \( U_i \in \mathcal{N} \), the limit point of \( \{ y_j \} \) must lie in \( (M - U_i) \). Thus, the limit point of \( \{ x_i \} \) is in \( \bigcup_{i=1}^n (M - U_i) = M - \bigcap_{i=1}^n U_i \), and \( \bigcap_{i=1}^n U_i \in \mathcal{N} \).
O in M. If O were not M open, then from the first countability of M, there exists a sequence \{x_i\} entirely contained in O which M-converges to a point \(x \in O\). But from lemma 24, this would mean that \(x_i\) converges to \(x\), contradicting the fact that \(O\) is N open. Similarly, if \(U\) is an M-open set, then it follows that \(U\) is also N open, because otherwise the definition of N would imply the existence of an N-convergent, and thus M-convergent sequence contained entirely in \(U^\circ\), with its limit point in \(U\), a contradiction.

Although N is a causal topology, we do not yet have a useful basis representation unlike the Alexandrov intervals \(\langle x, y \rangle\) for A and the Fullwood ‘double’ intervals \(\langle x, y, z \rangle\) for \(F\) described in the following section. We end this section with a discussion on the type of difficulties one encounters in trying to construct a manifold local causal basis. We first explore in more detail the role played by chain intervals in a future or past distinguishing spacetime.

Let \((M, g)\) be strongly causal at \(p\). If L is the set of all chain intervals that contain \(p\) in its interior, define the equivalence relation \(\sim\) as follows. For every \(l_1, l_2 \in L\), \(l_1 \sim l_2\) if \(l_1 \cap l_2 - \{p\} \neq \emptyset\). Since every null geodesic with \(p\) in its interior contains a chain interval with \(p\) in its interior (lemma 13), there exists a bijection \(\mu : [L] \rightarrow S^{n-2}\), where \([L]\) are the equivalence classes of chains under \(\sim\) and \(S^{n-2}\) represents the set of either future or past null directions from \(p\). Similarly, if the spacetime is future distinguishing at \(p\) and if \(L^+\) is the set of chain intervals of the form \([p, q]\), then again \(\sim\) can be used to determine an equivalence between chains. Using claim 14, we see that \(\mu_+ : [L^+] \rightarrow S^{n-2}\) is a bijection where \(S^{n-2}\) is the set of future null directions from \(p\). \(L^-\) is similarly defined for a past distinguishing point \(p\) with \(\mu_- : [L^-] \rightarrow S^{n-2}\) and \(S^{n-2}\) is now the set of past null directions from \(p\). Requiring a spacetime to be both future and past distinguishing means that one has a purely local causal definition of the future and past light cones emanating from each point in the spacetime. In particular, for every \(p \in \Delta\) which is both future and past distinguishing there is an \(S^{n-2} - \{\text{point}\}\) worth of (equivalence classes of) chain intervals from \(p\) which lie in the strongly causal region \(M - \Delta\) (see figure 1). Conversely, without future or past distinguishability, there seems to be no local definition of future or past light cones, respectively.

Lemma 13 moreover allows us to characterize the null geodesics in a strongly causal spacetime in a natural way. Let \((C, \prec)\) be a chain or totally ordered subset of \((M, \prec)\). If every \(x \in C\) lies in the interior of a chain interval which itself lies in \(C\), then we say that \((C, \prec)\) is a locally causally rigid chain (LCRC). Such a chain is also suitably dense and corresponds to a null geodesic in the spacetime. In a strongly causal spacetime, this provides a purely causal description of an arbitrary null geodesic. If on the other hand a chain contains an event \(x\) at which strong causality is violated, then it cannot be locally causally rigid past this point, unless it is itself the special null geodesic containing \(x\). Thus, if an LCRC contains an \(x \in M - \Delta\), then it must lie in \(M - \Delta\). We can at best attach end points to an LCRC which are not required to be locally rigid themselves and hence can lie on \(\Delta\). Let \(C\) be an LCRC with a future end point \(x \in \Delta\), such that \(C \not\subseteq \Gamma_p\). In a FPD spacetime, it is possible to begin a new future-directed LCRC \(C'\) from \(x\), but there is no natural causal choice of \(C'\) that ensures that \(C \cup C'\) is a null geodesic. This directional ‘floppiness’ at \(x\) is what makes the construction of a basis for the topology \(\mathcal{N}\) particularly difficult.

It is useful to examine the structure of Alexandrov intervals in spacetimes in which strong causality is violated. We notice that

**Claim 27.** For a causal spacetime \((M, g)\), if \((x, y)\) contains a \(p \in \Delta\), then \(\Gamma_p \subset (x, y)\).

**Proof.** For any \(q > p, q \in \Gamma_p, z \ll q \Rightarrow z \ll y\). Moreover, since \(x \ll p < q \Rightarrow x \ll q\), every \(z \in (x, q)\) also lies in \((x, y)\). Similarly, for \(r < p, r \in \Gamma_p, w > r \Rightarrow w > x\), and since \(r \ll y\), every \(w \in (r, y)\) belongs to \((x, y)\). Since this is true for every \(q > p\) on \(\Gamma_p\) and every \(r < p\) on \(\Gamma_p\), we have the desired result. \(\square\)

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At this point, it is useful to make use of the specific two-dimensional example of figure 3. This spacetime has a single special null geodesic \( \Gamma \), and for pairs \( x \ll y \) which straddle \( \Gamma \), the Alexandrov interval \( \langle x, y \rangle \) ‘spreads’ across the spacetime. This example makes explicit the fact that \( A \) is strictly coarser than \( M \) in such spacetimes. Thus, the Alexandrov interval is not sufficiently local to be useful. On the other hand, if \( O \) is manifold open, then the restricted Alexandrov interval \( \langle x, y \rangle \mid O \subset O \) if \( x, y \in O \). This restricted interval is appropriately manifold local and one might hope to find a purely causal way of defining such a set. In our two-dimensional example, we can define this set via its boundary. For every \( x \in M \), there are exactly two forward-directed and two backward-directed null geodesics and hence (classes of) chain intervals. One of the future-directed pairs of chains starting from \( x, f^+_x \) hits \( \Gamma \) at some \( z \) before it can hit a past-directed chain from \( y \). Across \( z \), it is no longer a chain interval. Thus, at \( z \), we have two future directions again to take, \( f^+_z \) and \( f^-_z \). The latter choice lies along \( \Gamma \), and we may reject it and instead take the union of chains \( f^+_x \cup f^+_z \). This intersects a past-directed chain from \( y, p^-_y \) at some \( r \). A similar construction from the past-directed chains from \( y \) carves out the boundary of a region with the desired local properties.

Even if such a construction were universally possible, how would one define the interior region purely causally? This is surprisingly difficult even for the simple spacetime under consideration, without the added complication of higher dimensions. However, by suitably ‘carving out’ sufficiently manifold local sets, using locally defined sequences of chain intervals, it may be possible to make further progress on this question.

5. Other causal topologies

In [12] a new topology \( F \) was constructed from a basis of sets obtained by taking the union of two Alexandrov intervals \( \langle x, y, z \rangle \equiv \langle x, y \rangle \cup \langle y, z \rangle \cup y \). These sets are not open in the manifold topology since they include the intermediate point \( y \). For a \( y \in \Delta \), \( \langle x, y, z \rangle \) is an \( F \) open neighbourhood of \( y \), and for appropriate choices of \( x \) and \( z \), either \( \langle x, y \rangle \subset M - \Delta \) or \( \langle y, z \rangle \subset M - \Delta \) depending on whether the spacetime is future or past distinguishing. Thus, by the arguments in section 3, \( F \) also contains information about the spacetime dimension. \( F \) is Hausdorff iff the spacetime is FPD and is, moreover, strictly finer than \( M \) [12].

It was shown in [12] that \( F \) can also be obtained via a causal convergence criterion on timelike sequences of events. As we will show below, a slight generalization of this definition to include all monotonic causal sequences gives rise to yet another causal topology which we call \( P \). We will show, however, that \( P \) is strictly coarser than \( F \) and also strictly finer than
\( \mathcal{M} \). Some of the following definitions and results have also been considered in a somewhat different framework in [13, 14].

**Definition 28.** A sequence \( \{ p_i \} \) is said to be future directed monotonic timelike if for \( i < j \), \( p_i \ll p_j \), and is past directed monotonic timelike if for \( i < j \), \( p_i \gg p_j \). We can similarly define future- and past-directed monotonic causal sequences.

In [12] causal convergence is defined as follows.

**Definition 29.** A monotonic causal sequence \( \{ p_j \} \) is said to causally converge to \( p \) as \( j \to \infty \) if either (a) \( I^-(p) = \bigcup_j I^-(p_j) \) or (b) \( I^+(p) = \bigcup_j I^+(p_j) \).

Before discussing the relation between these various topologies, we pause to outline some properties of monotonic causal sequences and the \( \mathcal{P} \)-topology.

**Lemma 30.** A future or past monotonic causal sequence \( \{ x_i \} \) causally converges to \( x \), iff every (infinite) subsequence of \( \{ x_i \} \) also causally converges to \( x \).

**Proof.** First, let \( \{ x_i \} \) be a future-directed monotonic causal sequence which causally converges to \( x \) and let \( \{ y_j \} \) be a subsequence of \( \{ x_i \} \). Thus, \( \bigcup_j I^-(y_j) \subseteq \bigcup_j I^-(x_i) = I^-(x) \). For any \( z \in I^-(x) \), there exists a \( k \) such that for all \( n \geq k \), \( z \in I^-(x_n) \). Since \( \{ x_i \} \) is future directed monotonic and \( \{ y_j \} \) is an infinite subsequence, there exists \( l \) such that \( y_l = x_n \) for some \( n \geq k \) so that \( I^-(x) \subseteq \bigcup_j I^-(y_j) \).

Conversely, let \( \{ x_i \} \) be a future-directed monotonic causal sequence such that every infinite subsequence \( \{ y_j \} \) converges to \( x \). Then since \( \bigcup_j I^-(y_j) \subseteq \bigcup_j I^-(x_i) \), \( I^-(x) \subseteq \bigcup_j I^-(x_i) \). On the other hand, for every \( x_k \), there exists an \( l \) such that \( x_k \ll y_l \), so that \( I^-(x_k) \subset I^-(y_l \subset I^-(x) \) which implies that \( \bigcup_j I^-(x_i) \subset I^-(x) \), so that \( \{ x_i \} \) also causally converges to \( x \). The proof proceeds in an analogous manner for past-directed monotonic causal sequences.

**Lemma 31.** A causal spacetime \((M, g)\) is FPD iff causal convergence of every monotonic causal sequence \( \{ x_i \} \) to \( x \) implies its manifold converge to \( x \), for all \( x \in \mathcal{M} \).

**Proof.** Let \((M, g)\) be past and future distinguishing. Let \( \{ x_i \} \) be a future-directed monotonic causal sequence which causally converges to \( x \). If \( \{ x_i \} \) does not manifold converge to \( x \), then there exists a neighbourhood \( O \) of \( x \) which contains none of the \( x_i \). Let \( \bar{O} \) be a future and past locality neighbourhood of \( x \) which lies in \( O \) such that \( \bar{Q} \subset O \). For any \( y \in I^-(x, \bar{Q}) \), there exists an \( N \) such that for all \( i > N \), \( y \in I^-(x_i) \). (Note that \( x_i \) itself does not have to lie in \( I^-(x) \).) Since \( x_i \notin O \), there exists \( O' \ni x_i \) such that \( O' \cap O = \emptyset \), and such that \( O' \in I^-(y) \). If \( z \in I^-(x_i, O') \), then \( z \in I^-(x) \) so that \( y \ll z \ll x \). Thus, there exists a future-directed timelike curve from \( y \) to \( x \) via \( z \notin Q \) which is a contradiction. Similarly for a past-directed monotonic causal sequence. Thus for a FPD spacetime, future or past causal convergence implies manifold convergence.

Conversely, assume that every future and past causally convergent sequence also converges in \( \mathcal{M} \) to the same point. Assume wlog that the future distinguishing condition fails at some \( x \in \mathcal{M} \), so that there exists a \( y \in \mathcal{M} \) with \( x \neq y \), such that \( I^+(x) = I^+(y) \). Let \( \{ x_i \} \) be a past-directed monotonic causal sequence which causally converges to \( x \), i.e., \( I^+(x) = \bigcup_j I^+(x_j) = I^+(y) \) so that \( \{ x_i \} \) also causally converges to \( y \). Since \( \{ x_i \} \) causally converges to \( x \) and \( y \), it also converges in \( \mathcal{M} \) to \( x \) and \( y \). But this immediately leads to a contradiction since \( \mathcal{M} \) is Hausdorff.

As in [12], we can use causal convergence of monotonic sequences to construct a causal topology \( \mathcal{P} \), namely
Definition 32. A set $O$ is open in $\mathcal{P}$ if every monotonic causal sequence in $O^*$ which causally converges, also has its limit point in $O^*$.

It is easy to establish that $\mathcal{P}$ is indeed a topology on $M$.

Lemma 33. The collection of sets $\mathcal{P}$ forms a topology on $M$.

Proof. To show that $\mathcal{P}$ forms a topology on $M$, we need to prove that it satisfies the three properties of a topology. (i) The proof of $M, \emptyset \in \mathcal{P}$ is trivial. (ii) Next let $U_\alpha$ be sets in $\mathcal{P}$, where $\alpha$ belongs to some indexing set. To show that $\bigcup_{\alpha} U_\alpha \in \mathcal{P}$, note that if a monotonic causal, causally convergent sequence lies outside $\bigcup_{\alpha} U_\alpha$, then it must lie outside each of the $U_\alpha$. Since $U_\alpha \in \mathcal{P}$ for all $\alpha$, the limit point of the sequence must also lie outside $U_\alpha$ for all $\alpha$. Thus, the limit point is not contained in $\bigcup_{\alpha} U_\alpha$, which proves that $\bigcup_{\alpha} U_\alpha \in \mathcal{P}$. (iii) To show that $\bigcap_{n=1}^{m} U_i \in \mathcal{P}$, let $\{x_i\}$ be a monotonic causal, causally convergent sequence in $M - \bigcup_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (M - U_i)$. Since the $(M - U_i)$'s are finite in number, at least one of them, say $(M - U_k)$, should contain a subsequence $\{y_j\}$ of $\{x_i\}$. But since $U_k \in \mathcal{P}$, the limit point of $\{y_j\}$ (which is also the limit point of $\{x_i\}$ from lemma 30) must lie in $(M - U_k)$. Thus, the limit point of $\{x_i\}$ is in $\bigcup_{n=1}^{m} (M - U_i) = M - \bigcap_{i=1}^{n} U_i$ and $\bigcap_{n=1}^{m} U_i \in \mathcal{P}$. □

Lemma 34. If $(M, g)$ is FPD, then $M \subset \mathcal{P} \subset \mathcal{F}$.

Proof. (a) First we show that $M \subset \mathcal{P}$. Let $O$ be an $M$-open set. Consider a monotonic causal sequence $\{x_i\}$ in $O^*$ which future or past causally converges to $x$. Since $M$ is FPD, $\{x_i\}$ must also converge to $x$ in $M$ by lemma 31 and hence $x \in O^*$. Since this is true for all future or past causally converging monotonic causal sequences in $O^*$, $O$ is also $\mathcal{P}$-open. Figure 4 shows an example of a set whose complement is closed in $\mathcal{P}$, but it is not, in an obvious way, closed in $M$. Thus, $M \subset \mathcal{P}$. (b) To show that $\mathcal{P} \subset \mathcal{F}$, consider a $V$ which is $\mathcal{P}$-open. Every monotonic causal sequence, and therefore every monotonic timelike sequence in $V^*$ which causally converges to the past or the future, has its limit point in $V^*$. Thus, $V$ is $\mathcal{F}$-open. Conversely, consider the $\mathcal{F}$-open set $U = (x, p, y)$, and let $\gamma$ be a past-directed null geodesic from $p$. Choose a monotonically increasing causal $\{p_i\}$ on $\gamma$ such that $\{p_i\}$ causally converges to $p$. Clearly $\{p_i\} \in U^*$ but $p \in U$ which means that $U$ is not $\mathcal{P}$-open. □

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