HILBERT POLYNOMIALS
OF NON-STANDARD BIGRADED ALGEBRAS

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Abstract. This paper investigates Hilbert polynomials of bigraded algebras which are generated by elements of bidegrees (1, 0), (d_1, 1), \ldots, (d_r, 1), where d_1, \ldots, d_r are non-negative integers. The obtained results can be applied to study Rees algebras of homogeneous ideals and their diagonal subalgebras.

Introduction

Let \( R = \bigoplus_{(u,v) \in \mathbb{N}^2} R_{(u,v)} \) be a finitely generated bigraded algebra over a field \( k \). The Hilbert function of \( R \) is the function
\[
H_R(u, v) := \dim_k R_{(u,v)}.
\]
If \( R \) is standard bigraded, i.e. \( R \) is generated by elements of bidegrees (1, 0) and (0, 1), then \( H_R(u, v) \) is equal to a polynomial in \( u, v \) for \( u, v \) large enough [Ba], [KMV], [W]. This fact does not hold if \( R \) is not standard bigraded.

In this paper we will study the case \( R \) is generated by elements of bidegrees (1, 0), (d_1, 1), \ldots, (d_r, 1), where d_1, \ldots, d_r are non-negative integers. This case was considered first by P. Roberts in [Ro1] where it is shown that there exist integers \( c \) and \( v_0 \) such that \( H_R(u, v) \) is equal to a polynomial \( P_R(u, v) \) for \( u \geq cv \) and \( v \geq v_0 \). He calls \( P_R(u, v) \) the Hilbert polynomial of the bigraded algebra \( R \). It is worth to notice that Hilbert polynomials of bigraded algebras of the above type appear in Gabber’s proof of Serre non-negativity conjecture (see e.g. [Ro2]) and that the positivity of certain coefficient of such a Hilbert polynomial is strongly related to Serre’s positivity conjecture on intersection multiplicities [Ro3].

Using a different method we are able to show that for
\[
d := \max\{d_1, \ldots, d_r\},
\]
there exist integers \( u_0, v_0 \) such that \( H_R(u, v) = P_R(u, v) \) for \( u \geq dv + u_0 \) and \( v \geq v_0 \) (Theorem 1.1). Furthermore, the total degree and the degree of \( P_R(u, v) \) in the variable \( u \) can be expressed in terms of the dimension of certain quotient rings of \( R \) (Theorem 2.1 and Theorem 2.3). These results cover recent results of P. Roberts in the case \( R \) is generated by elements of bidegrees (1, 0), (0, 1), (1, 1) [Ro3].

Besides the afore mentioned results we also obtain some results on the leading coefficients of the Hilbert polynomial \( P_R(u, v) \). If \( P_R(u, v) \) is written in the form
\[
P_R(u, v) = \sum_{i=0}^{s} \frac{e_i(R)}{i!(s-i)!} u^i v^{s-i} + \text{lower-degree terms},
\]

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where \( s = \deg P_R(u, u) \), we can show that \( e_i(R) \) is an integer for \( i = 0, \ldots, s \) (Theorem 3.2). The integer \( e_i(R) \) can be negative. That is in sharp contrast to the standard bigraded case where it is always non-negative [W]. We shall see that if \( R \) is a domain or a Cohen-Macaulay ring, then \( e_i(R) > 0 \) for \( i = \deg_n P_R(u, v) \) (Proposition 3.3). Moreover, we will show that the integer \( e_i(R) \) satisfies the associativity formula (Proposition 3.4).

The inspiration for our study also comes from the theory of diagonal subalgebras of bigraded Rees algebras [STV], [CHTV]. Let \( A = k[X_1, \ldots, X_n] \) be a polynomial ring over a field \( k \) and \( I \) a homogeneous ideal of \( A \). One can associate with \( I \) the Rees algebra \( A[I] := \oplus_{n \geq 0} I^n t^n \). Since \( I \) is a homogeneous ideal, \( A[I] \) has a natural bigraded structure by setting \( A[I]_{(u,v)} = (I^v)_u t^v \) for all \( (u,v) \in \mathbb{N}^2 \). It is obvious that \( A[I] \) belongs to the above class of bigraded algebras. Let \( V \) denote the blow-up of the projective space \( \mathbb{P}^{n-1}_k \) along the subvariety defined by \( I \). Then \( V \) can be embedded into a projective space by the linear system \((I^v)_e\) for any pair of positive integers \( e, c \) with \( c > de \) [CH]. Such embeddings often yield interesting rational varieties such as the Bordiga-White surfaces [Gi], the Room surfaces [GG] and the Buchsbaum-Eisenbud varieties [GL]. The homogeneous coordinate ring of the embedded variety is the subalgebra \( k[(I^v)_e] \) of \( A[I] \). It has been observed in [STV] and [CHTV] that \( k[(I^v)_e] \) can be identified as the subalgebra of \( A[I] \) along the diagonal \( \{(cv, ev) | v \in \mathbb{N}\} \) of \( \mathbb{N}^2 \). This idea has been employed successfully to study algebraic properties of the embedded variety. Since \( P_{A[I]}(cv, ev) \) is the Hilbert polynomial of \( k[(I^v)_e] \), we may get uniform information on all such embeddings of \( V \) from \( P_{A[I]}(u, v) \). Furthermore, if we fix \( v \geq v_0 \), then \( P_{A[I]}(u, v) \) is the Hilbert polynomial of the ideal \( I^v \).

If \( R \) is a standard bigraded algebra, the coefficients \( e_i(R) \) of \( P_R(u, v) \) are related to the mixed multiplicities introduced by Teissier in singularity theory [Te]. Furthermore, one can use general reductions to compute the numbers \( e_i(R) \) [Re], [Tr2]. If \( R \) is not standard bigraded, we do not know any general method for the computation of the integers \( e_i(R) \). However, if \( R \) is the bigraded Rees algebra \( A[I] \) of a homogeneous ideal \( I \), we have some preliminary information on the Hilbert polynomial \( P_{A[I]}(u, v) \) (Theorem 4.2) and we will present an effective method for the computation of \( H_{A[I]}(u, v) \) (Proposition 4.5).

We will compute the integers \( e_i(R) \) explicitly in the following cases:

- \( R \) is a bigraded polynomial ring of the above type (Proposition 1.3),
- \( R \) is the Rees algebra of an ideal generated by a homogeneous regular sequence (Corollary 5.3),
- \( R \) is the Rees algebra of the ideal generated by the maximal minors of a generic \((r-1) \times r\) matrix (Corollary 5.5).

The last two cases follow from a more general result on ideals generated by homogeneous \( d \)-sequences (Theorem 5.2). We would like to point out that the computation of the Hilbert polynomial of the Rees algebra \( A[I] \) is usually very difficult (even if \( I \) is generated by a regular sequence) since it amounts to the computation of the Hilbert polynomials of all powers of \( I \). For instance, the computation of these polynomials in the case \( I \) being generated by a regular sequence of two forms in a polynomial ring of three variables was a key result in the study of certain rational surfaces [GGH]. Our method will yield a simple proof of this result (Example 5.4).
The paper is organized as follows. In the first three sections we will study the existence, the degree and the coefficients of Hilbert polynomials of bigraded algebras, respectively. In the fourth section we apply the obtained result to Rees algebras of homogeneous ideals. The last section is devoted to the case the given ideal is generated by a homogeneous \(d\)-sequence.

For unexplained terminologies and notations we refer the reader to [E]. Unless otherwise specified, all summations will be ranged over non-negative indices.

1. Existence of Hilbert polynomial

The aim of this section is to prove the following result on the existence of Hilbert polynomials of bigraded algebras.

**Theorem 1.1.** Let \(R\) be a finitely generated bigraded algebra over a field \(k\). Assume that \(R\) is generated by elements of bidegrees \((1,0),(d_1,1),\ldots,(d_r,1)\), where \(d_1,\ldots,d_r\) are non-negative integers. Set \(d = \max\{d_1,\ldots,d_r\}\). There exist integers \(u_0,v_0\) such that for \(u \geq dv + u_0\) and \(v \geq v_0\), the Hilbert function \(H_R(u,v)\) is equal to a polynomial \(P_R(u,v)\).

If \(R\) is generated by elements of bidegrees \((1,0),(0,1),(1,1)\), P. Roberts has shown that there exist integers \(u_0\) and \(v_0\) such that the function \(\sum_{i=0}^{\infty} H_R(u,v)\) is equal to a polynomial for \(u \geq v + u_0\) and \(v \geq v_0\) [Ro3, Theorem 1]. But this result can be easily deduced from Theorem 1.1.

Unlike the standard bigraded case, we do not have \(H_R(u,v) = P_R(u,v)\) for \(u,v\) large enough. In fact, a bigraded algebra as above may have different Hilbert polynomials, depending on the range of the variables in \(\mathbb{N}^2\).

**Example 1.2.** Let \(S = k[X,Y,Z]\) be a bigraded polynomial ring in three indeterminates \(X, Y, Z\) with bideg \(X = (1,0)\), bideg \(Y = (0,1)\) and bideg \(Z = (1,1)\). It is easy to check that

\[
H_S(u,v) = \begin{cases} 
  v + 1 & \text{if } u \geq v, \\
  u + 1 & \text{if } v \geq u.
\end{cases}
\]

The proof of Theorem 1.1 can be reduced to the case of a bigraded polynomial ring. In this case we have a more precise statement as follows.

**Proposition 1.3.** Let \(S = k[X_1,\ldots,X_n,Y_1,\ldots,Y_r] (n \geq 1, r \geq 1)\) be a bigraded polynomial ring with bideg \(X_i = (1,0), i = 1,\ldots,n,\) and bideg \(Y_j = (d_i,1), j = 1,\ldots,r.\) For \(u \geq dv\), the Hilbert function \(H_S(u,v)\) is equal to a polynomial

\[
P_S(u,v) = \sum_{i+j=n+r-2} \frac{e_{i,j}}{i!j!} u^i v^j + \text{lower-degree terms}
\]

of total degree \(n + r - 2\) with

\[
e_{i,n+r-2-i} = \begin{cases} 
  (-1)^{n-i-1} \sum_{j_1+\ldots+j_i=n-1-i} d_1^{j_1} \cdots d_r^{j_i} & \text{if } i < n, \\
  0 & \text{if } i \geq n.
\end{cases}
\]

Remark. The range \(u \geq dv\) can not be removed. In fact, for \(u < d\) and \(v = 1\), the Hilbert function \(H_S(u,1)\) can be equal to different polynomials than \(P_S(u,1)\), depending on the range of \(u\). For instance, \(H_S(u,1) = 0\) for \(u < \min\{d_1,\ldots,d_r\}\).
Proposition 1.3 follows from the following result which is based on some polynomial identities of [Ve1]. This result will be used also in Section 5.

Lemma 1.4. Let 

\[ f(t) = \frac{e}{m!} t^m + \text{lower-degree terms} \]

be a polynomial of degree \( m \). Let \( d_1, \ldots, d_r \) be a sequence of non-negative integers and \( d = \max\{d_1, \ldots, d_r\} \). Set 

\[ H(u, v) := \sum_{\alpha_1 + \ldots + \alpha_r = u} f(u - d_1 \alpha_1 - \ldots - d_r \alpha_r). \]

Then \( H(u, v) \) is equal to a polynomial of degree \( m + r - 1 \). Moreover, if \( H(u, v) \) is written in the form 

\[ H(u, v) = \sum_{i+j=m+r-1} e_{i,m} u^i v^j + \text{lower-degree terms}, \]

then 

\[ e_{i,m+r-1-i} = \begin{cases} (-1)^{m-i} e_{j_1+\ldots+j_r=m-i} d_1^{j_1} \cdots d_r^{j_r} & \text{if } i \leq m, \\ 0 & \text{if } i > m. \end{cases} \]

Proof. For \( r = 1 \) we have 

\[ H(u, v) = \frac{e}{m!} (u - d_1 v)^m + \text{lower-degree terms} = e \sum_{i=0}^m (-1)^{m-i} \frac{d_1^{m-i}}{i!(m-i)!} u^i v^{m-i} + \text{lower-degree terms}. \]

Hence \( e_{i,m-i} = (-1)^{m-i} e_{d_1^{m-i}} \) for \( i = 0, \ldots, m \). Since \( e_{m,0} = e \neq 0 \), we get \( \deg H(u, v) = m \).

For \( r > 1 \) we introduce a new function 

\[ H'(u', v') := \sum_{\alpha_1 + \ldots + \alpha_r = v'} f(u' - d_1 \alpha_1 - \ldots - d_r \alpha_r). \]

By induction on \( r \) we may assume that 

\[ H'(u', v') = \sum_{i+j=m+r-2} e'_{i,j} (u')^i (v')^j + \text{lower-degree terms} \]

is a polynomial of degree \( m + r - 2 \). Then 

\[ H(u, v) = \sum_{\alpha_r = 0}^v H'(u - d_r \alpha_r, v - \alpha_r) \]

\[ = \sum_{\alpha_r = 0}^v \left[ \sum_{i+j=m+r-2} e'_{i,j} (u - d_r \alpha_r)^i (v - \alpha_r)^j + \text{lower-degree terms} \right]. \]

Therefore, \( H(u, v) \) is a polynomial of degree \( m + r - 1 \) if we can show that for any pair of non-negative integers \( i, j \), the function 

\[ G(u, v) = \sum_{\alpha_r = 0}^v (u - d_r \alpha_r)^i (v - \alpha_r)^j \]

is a polynomial of degree \( i + j + 1 \).
We have
\[ G(u, v) = \sum_{\alpha_r=0}^{v} \sum_{p=0}^{i} (-1)^p \binom{i}{p} u^{i-p} (d_r \alpha_r)^p \sum_{q=0}^{j} (-1)^q \binom{j}{q} v^{j-q} \alpha_r^q \]
\[ = \sum_{p=0}^{i} \sum_{q=0}^{j} (-1)^{p+q} \binom{i}{p} \binom{j}{q} p^p u^{i-p} v^{j+1-p+1} \alpha_r^p \alpha_r^q. \]

By [Ve, Lemma 2.8] \[ \sum_{\alpha_r=0}^{v} \alpha_r^{p+q} \] is a polynomial in \( v \) of degree \( p + q + 1 \) with
\[ \sum_{\alpha_r=0}^{v} \alpha_r^{p+q} = \frac{1}{p+q+1} v^{p+q+1} + \text{lower-degree terms}. \]

Therefore, \( G(u, v) \) is a polynomial and we may write
\[ G(u, v) = \sum_{p=0}^{i} \sum_{q=0}^{j} (-1)^{p+q} \binom{i}{p} \binom{j}{q} \frac{1}{p+q+1} p^p u^{i-p} v^{j+1-p+1} + \text{lower-degree terms}. \]

By [Ve, Lemma 2.7] we know that
\[ \sum_{q=0}^{j} (-1)^q \binom{j}{q} \frac{1}{p+q+1} = \frac{1}{(p+j+1)(p+1)}. \]

This implies
\[ G(u, v) = \sum_{p=0}^{i} (-1)^p \binom{i}{p} \frac{1}{(p+j+1)(p+1)} p^p u^{i-p} v^{j+1-p+1} + \text{lower-degree terms} \]
\[ = \sum_{p=0}^{i} (-1)^p \frac{i!j!}{(i-p)!(p+j+1)!} p^p u^{i-p} v^{j+1-p+1} + \text{lower-degree terms} \]

The coefficient of \( u^i v^{j+1} \) is equal to \( 1/(j+1) \). Hence deg \( G(u, v) = i + j + 1 \).

Now we are going to compute the coefficients of the highest degree terms of the polynomial \( H(u, v) \). Using the last formula for \( G(u, v) \) we have
\[ H(u, v) = \sum_{i+j=m+r-2}^{i} \sum_{p=0}^{i} (-1)^p \frac{i!j!}{(i-p)!(p+j+1)!} p^p u^{i-p} v^{j+1-p+1} + \text{lower-degree terms} \]
\[ = \sum_{i+j=m+r-2}^{i} \sum_{p=0}^{i} (-1)^p \frac{i!j!}{(i-p)!(p+j+1)!} d_r^p u^{i-p} v^{j+1-p+1} + \text{lower-degree terms} \]

Putting \( j = m + r - 2 - i \) and \( h = i - p \) we get
\[ H(u, v) = \sum_{i=0}^{m+r-2} \sum_{h=0}^{i} (-1)^{i-h} \frac{e_{i,m+r-2}^{i} h!}{(m+r-1-h)!} d_r^{i-h} u^{h} v^{m+r-1-h} + \text{lower-degree terms} \]
\[ = \sum_{h=0}^{m+r-2} \sum_{i=h}^{m+r-2} (-1)^{i-h} \frac{e_{i,m+r-2}^{i} h!}{(m+r-1-h)!} d_r^{i-h} u^{h} v^{m+r-1-h} + \text{lower-degree terms} \]
Thus, if we write
\[ H(u, v) = \sum_{i+j=m+r-1} \frac{e_{i,j}}{i!j!} u^i v^j + \text{lower-degree terms}, \]
then
\[ e_{i,m+r-1-i} = \begin{cases} 
\sum_{h=i}^{m+r-2} (-1)^{h-i} e_{h,m+r-2-h}^h d_r^{h-i} & \text{if } i \leq m + r - 2, \\
0 & \text{if } i = m + r - 1. 
\end{cases} \]

By the induction hypothesis we have
\[ e_{h,m+r-2-h}^h = \begin{cases} 
(-1)^{m-h} e \sum_{j_1+\ldots+j_{r-1}=m-h} d_1^{j_1} \ldots d_{r-1}^{j_{r-1}} d_r^{h-i} & \text{if } h \leq m, \\
0 & \text{if } h > m. 
\end{cases} \]

It is easy to check that
\[ \sum_{h=i}^{m+r-2} (-1)^{h-i} \left[ (-1)^{m-h} e \sum_{j_1+\ldots+j_{r-1}=m-h} d_1^{j_1} \ldots d_{r-1}^{j_{r-1}} d_r^{h-i} \right] = (-1)^{m-i} e \sum_{j_1+\ldots+j_{r-1}=m-i} d_1^{j_1} \ldots d_r^{j_r}. \]
Therefore,
\[ e_{i,m+r-1-i} = \begin{cases} 
(-1)^{m-i} e \sum_{j_1+\ldots+j_{r-1}=m-i} d_1^{j_1} \ldots d_r^{j_r} & \text{if } i \leq m, \\
0 & \text{if } i > m. 
\end{cases} \]

This completes the proof of Lemma 1.4.

\[ \square \]

**Proof of Proposition 1.3.** Let \( k[X] = k[X_1, \ldots, X_n] \). Then \( k[X] \) is a standard \( \mathbb{N} \)-graded algebra with \( \deg X_i = 1 \). For all \((u, v) \in \mathbb{N}^2 \) we have
\[ S_{(u,v)} = \bigoplus_{\alpha_1+\ldots+\alpha_r=v} k[X]_{u-d_1\alpha_1-\ldots-d_r\alpha_r} Y_1^{\alpha_1} \ldots Y_r^{\alpha_r}. \]

Hence
\[ H_S(u, v) = \sum_{\alpha_1+\ldots+\alpha_r=v} \dim_k k[X]_{u-d_1\alpha_1-\ldots-d_r\alpha_r} = \sum_{\alpha_1+\ldots+\alpha_r=v} \binom{u-d_1\alpha_1-\ldots-d_r\alpha_r+n-1}{n-1}. \]

Let \( f(t) \) denote the polynomial \( \binom{t+n-1}{n-1} \). For \( u \geq dv \) we have
\[ H_S(u, v) = \sum_{\alpha_1+\ldots+\alpha_r=v} f(u-d_1\alpha_1-\ldots-d_r\alpha_r). \]

Hence the conclusion follows from Lemma 1.4.

\[ \square \]

Now we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** First, we represent \( R \) as a bigraded quotient of the bigraded polynomial ring \( S \) defined in Proposition 1.3. For any bigraded \( S \)-module \( M \) we set \( H_M(u, v) := \dim_k M_{(u,v)} \). Let
\[ 0 \rightarrow F_m \rightarrow \cdots \rightarrow F_1 \rightarrow F_1 \rightarrow R \rightarrow 0 \]
be a bigraded minimal free resolution of $R$ over $S$. Then

$$H_R(u, v) = \sum_{i=0}^{m} (-1)^i H_{F_i}(u, v).$$

Every free module $F_i$ is a direct sum of modules of the form $S(-a, -b)$ for some non-negative integers $a, b$, where $S(-a, -b)_{(u,v)} = S_{(u-a, v-b)}$ for all $(u,v) \in \mathbb{Z}^2$.

By Proposition 1.3, $H_{S(-a, -b)}(u,v) = H_S(u-a, v-b)$ is a polynomial in $u, v$ for $u \geq a + \max\{0, d(v-b)\}$ and $v \geq b$. Thus, if we define $u_0$ to be the maximum of all numbers $a$ and $v_0$ to be the maximum of all numbers $b$, where $(a, b)$ runs all shifting degrees occurring in the bigraded minimal free resolution of $R$ over $S$, then the functions $H_{F_i}(u,v)$ and therefore $H_R(u,v)$ are equal to polynomials in $u, v$ for $u \geq dv + u_0$ and $v \geq v_0$.

\[\square\]

2. Degree of Hilbert Polynomial

Throughout this section, we assume that the bigraded algebra $R$ is generated by $n$ homogeneous elements $x_1, \ldots, x_n$ of bidegree $(1,0)$ and $r$ elements $y_1, \ldots, y_r$ of bidegree $(d_1,1), \ldots, (d_r,1)$, where $d_1, \ldots, d_r$ are non-negative integers.

As we have seen in Theorem 1.1, we can associate with $R$ a Hilbert polynomial $P_R(u,v)$. The aim of this section is to compute the total degree $\deg P_R(u,v)$ and the degree $\deg_u P_R(u,v)$ of $P_R(u,v)$ in the variable $u$.

Let $R_{++}$ denote the ideal of $R$ generated by the elements $x_i y_j$, $i = 1, \ldots, n$, $j = 1, \ldots, r$. Let $\text{Proj} R$ denote the set of the bigraded prime ideals $P$ of $R$ with $P \not\supseteq R_{++}$. Let

$$\text{rdim } R := \begin{cases} 1 & \text{if } \text{Proj } R = \emptyset, \\ \max\{\dim R/P \mid P \in \text{Proj } R\} & \text{if } \text{Proj } R \neq \emptyset. \end{cases}$$

Following [KMV] we call $\text{rdim } R$ the relevant dimension of the bigraded algebra $R$.

The total degree $\deg P_R(u,v)$ can be expressed in terms of the relevant dimension of $R$ as follows.

**Theorem 2.1.** $\deg P_R(u,v) = \text{rdim } R - 2$.

This theorem covers results of Katz, Mandal and Verma for standard bigraded algebras [KMV, Theorem 2.2]. There is another formula for $\deg P_R(u,v)$ given by P. Roberts in [Ro1, Theorem 8.3.4], namely,

$$\deg P_R(u,v) = \max\{\dim R(P) \mid P \in \text{Proj } R\},$$

where $R(P)$ denotes the degree zero part of the bigraded localization of $R$ at $P$ (see [Ro3, Theorem 2(1)]) for the case $R$ is generated by elements of bidegree $(1,0), (0,1), (1,1)$. Since the relationship between these two formulas is not trivial, we shall present below a direct and short proof for Theorem 2.1. For that we shall need the following notion.

Let $\Delta = \{(cv, ev) \mid v \in \mathbb{N}\}$, where $c, e$ are two given positive integers. For any bigraded $R$-module $M$ we define

$$M_\Delta := \oplus_{v \in \mathbb{Z}} M_{(cv, ev)}.$$

It is clear that $R_\Delta$ is a $\mathbb{N}$-graded algebra and $M$ a graded $R_\Delta$-module. We call $R_\Delta$ a diagonal subalgebra of $R$. Diagonal subalgebras were introduced in [STV], [CHTV].
They have been studied mainly in the case $R$ is the Rees algebra of a homogeneous ideal.

**Lemma 2.2.** Let $d = \max\{d_1, \ldots, d_r\}$. Then

(a) $R_\Delta$ is a standard graded algebra if $c \geq de$,

(b) $\dim R_\Delta = \operatorname{rdim} R - 1$ if $c > de$.

**Proof.** (a) By definition, $(R_\Delta)_v = R_{(cv, ev)}$. Therefore, $(R_\Delta)_v$ is the vector space spanned by the products $x_1^{a_1} \cdots x_n^{a_n} y_1^{b_1} \cdots y_r^{b_n}$ of bidegree $(cv, ev)$, which means

$$a_1 + \ldots + a_n + b_1d_1 + \ldots + b_r d_r = cv,$$

$$b_1 + \ldots + b_r = ev.$$

If $v \geq 2$, a product $x_1^{a_1} \cdots x_n^{a_n} y_1^{b_1} \cdots y_r^{b_n}$ of bidegree $(cv, ev)$ is always divisible by a product $x_1^{a_1'} \cdots x_n^{a_n'} y_1^{b_1'} \cdots y_r^{b_n'}$ of bidegree $(c(v - 1), ev(v - 1))$. Indeed, we first choose non-negative integers $b_1' \leq b_1, \ldots, b_r' \leq b_r$ such that

$$b_1' + \ldots + b_r' = ev.$$

Since $(b_1 - b_1') + \ldots + (b_r - b_r') = e$, we have

$$c \geq de \geq (b_1 - b_1')d_1 + \ldots + (b_r - b_r')d_r.$$

From this it follows that

$$c(v - 1) - (b_1'd_1 + \ldots + b_r'd_r) \leq cv - (b_1d_1 + \ldots + b_r d_r) = a_1 + \ldots + a_n.$$

Hence there exist non-negative integers $a_1' \leq a_1, \ldots, a_r' \leq a_r$ such that

$$a_1' + \ldots + a_r' = c(v - 1) - b_1'd_1 + \ldots + b_r'd_r.$$

Clearly, the chosen integers $a_1', \ldots, a_r', b_1', \ldots, b_r'$ yield a product $x_1^{a_1'} \cdots x_n^{a_n'} y_1^{b_1'} \cdots y_r^{b_n'}$ as required. So we can conclude that the $\mathbb{N}$-graded algebra $R_\Delta$ is generated by elements of degree 1.

(b) It is easy to check that if $P$ is a bigraded prime ideal of $R$ and $I$ is a bigraded $P$-primary ideal, then $P_\Delta$ is a prime ideal of $R_\Delta$ and $I_\Delta$ is a graded $P_\Delta$-primary ideal. Further, if $0_R = \cap I_j$ is a graded primary decomposition of the zero ideal $0_R$ of $R$, then $0_{R_\Delta} = \cap (I_j)_\Delta$ is a graded primary decomposition of the zero ideal $0_{R_\Delta}$ of $R_\Delta$. Thus, to show that $\dim R_\Delta = \operatorname{rdim} R - 1$ we may assume that $R$ is a domain.

If $\operatorname{Proj} R = \emptyset$, we have $\operatorname{rdim} R = 1$ and $(R_{++})^v = 0$ for some positive integer $v$. Since from the assumption $c > de$ we can deduce that $R_{(cv, ev)} \subseteq (R_{++})^v$, we get $(R_\Delta)_v = 0$. Therefore, $\dim R_\Delta = 0$.

If $\operatorname{Proj} R \neq \emptyset$, the zero ideal belongs to $\operatorname{Proj} R$. This implies $R_{++} \neq 0$ and $\operatorname{rdim} R = \dim R$. In this case, we will use the formulas

$$\dim R = \operatorname{tr. deg}_k Q(R),$$

$$\dim R_\Delta = \operatorname{tr. deg}_k Q(R_\Delta),$$

where $Q(R)$ and $Q(R_\Delta)$ denote the quotient fields of $R_\Delta$ and $R$, respectively. Since $R_{++} \neq 0$, we can find a homogeneous element $x \in R$ of bidegree $(1, 0)$. Since $e \geq 1$, $x \not\in R_\Delta$. Because of its degree, the element $x$ can not be a root of any algebraic equation over $Q(R_\Delta)$. Hence $x$ is a transcendent element over $Q(R_\Delta)$. It suffices now to show that $Q(R)$ is an algebraic extension of $Q(R_\Delta)(x)$. For this will imply

$$\operatorname{tr. deg}_k Q(R) = \operatorname{tr. deg}_k Q(R_\Delta) + 1.$$
Since \( R_{++} \neq 0 \), there exists an element \( y \in R \) of bidegree \((d, 1)\). For any generator \( x_i \) of \( R \) we can write \( x_i = x_i x^{c-de} y^e / x^{c-de} y^e \). Since \( \text{bideg } x_i x^{c-de} y^e / x^{c-de} y^e = \text{bideg } x^{c-de} y^e = (c, e) \), we get \( x_i \in Q(R_\Delta)(x) \). For any generator \( y_j \) of \( R \) we have \( \text{bideg } x^{c-de} y^e_j = (c, e) \). As \( c > d \), we get \( x^{c-de} y^e_j \in R_\Delta \). Hence \( y_j \) is an algebraic element over \( Q(R_\Delta)(x) \). Since \( R \) is generated by the elements \( x_i \) and \( y_j \), we can conclude that \( Q(R) \) is an algebraic extension of \( Q(R_\Delta)(x) \). \( \square \)

**Proof of Theorem 2.1.** Let \( s \) be the degree of the Hilbert polynomial \( P_R(u, v) \) and write

\[
P_R(u, v) = \sum_{i+j=s} \frac{e_{i,j}}{i!j!} u^i v^j + \text{lower-degree terms.}
\]

We consider the diagonal subalgebras \( R_\Delta \) for any \( c > d \) and \( e = 1 \). Let

\[
H_{R_\Delta}(v) := H_R(cv, v) = \dim_k R_{(cv, v)}
\]

for all \( v \in \mathbb{N} \). Then \( H_{R_\Delta}(v) \) is the Hilbert function of \( R_\Delta \). By Lemma 2.2, \( H_{R_\Delta}(v) \) is equal to a polynomial \( P_{R_\Delta}(v) \) of degree \( \text{rdim } R - 2 \) for \( v \) large enough. Since \( cv \geq dv + v \), by Theorem 1.1 we also have \( H_R(cv, v) = P_R(cv, v) \) for \( v \) large enough. Therefore,

\[
P_{R_\Delta}(v) = P_R(cv, v) = \sum_{i=0}^{s} \frac{e_{i,s-i}}{i!(s-i)!} c^i v^s + \text{lower-degree terms.}
\]

If we choose \( c \) large enough then \( \sum_{i=0}^{s} \frac{e_{i,s-i}}{i!(s-i)!} c^i \neq 0 \). So we can conclude that \( s = \text{rdim } R - 2 \). \( \square \)

Now we shall see that \( \text{deg } P_R(u, v) \) can be expressed as the dimension of a quotient ring of \( R \).

Let \( \bar{R} := R/0 : (R_{++})^\infty \), where \( 0 : (R_{++})^\infty \) denotes the ideal of the elements of \( R \) annihilated by some power of \( R_{++} \). Note that \( 0 : (R_{++})^\infty \) can be obtained from any decomposition of the zero ideal of \( R \) by deleting those primary components whose associated primes contain \( R_{++} \). Then \( \text{rdim } R = \dim \bar{R} \) if \( \text{Proj } R \neq \emptyset \), and \( \bar{R} = 0 \) if \( \text{Proj } R = \emptyset \). Thus, we always have

\[
\text{deg } P_R(u, v) = \dim \bar{R} - 2.
\]

The degree \( \text{deg}_u P_R(u, v) \) can be also expressed in terms of \( \bar{R} \) by relating \( P_R(u, v) \) to the Hilbert polynomials of the graded modules

\[
R_v := \oplus_{u \geq 0} R_{(u,v)}.
\]

Note that \( R_0 \) is a finitely generated standard \( \mathbb{N} \)-graded algebra and \( R_v \) is a finitely generated graded \( R_0 \)-module. We set

\[
x-\dim R := \dim R_0
\]

and call it the \textit{x-dimension} of \( R \). The name stems from [Ro3] where \( x-\dim R \) is defined to be the dimension of \( R_v \) for \( v \) large enough. But the following formulas for \( \text{deg}_u P_R(u, v) \) show that both definitions are equal. These formulas were already proved for standard bigraded algebras in [Tr2, Theorem 1.7] and implicitly for bigraded algebras generated by elements of bidegree \((1, 0), (0, 1), (1, 1)\) in [Ro3, Theorem 2(2)].

**Theorem 2.3.** \( \text{deg}_u P_R(u, v) = \dim R_v - 1 = x-\dim R - 1 \) for \( v \) large enough.
Proof. Let \( H_{R_v}(u) \) and \( P_{R_v}(u) \) denote the Hilbert function and the Hilbert polynomial of \( R_v \). Then \( H_{R_v}(u) = \dim_k R_{(u,v)} \) for all \( u \in \mathbb{N} \). By Theorem 1.1, there exist integers \( u_0, v_0 \) such that \( H_{R_v}(u) = P_{R_v}(u,v) \) for \( u \geq cv + u_0 \) and \( v \geq v_0 \). Therefore \( P_{R_v}(u,v) = P_{R_v}(u,v) \) for \( v \geq v_0 \). For \( v \) large enough, the leading coefficient of \( P_{R_v}(u,v) \) as a polynomial in \( u \) does not vanish. From this we deduce that \( \deg_u P_{R_v}(u,v) = \deg P_{R_v}(u,v) \). Since \( \deg P_{R_v}(u,v) = \dim R_v - 1 \), that implies

\[
\deg_u P_{R_v}(u,v) = \dim R_v - 1.
\]

Choose a positive integer \( m \) such that \( 0 : (R_{++})^\infty = 0 : (R_{++})^m \). For \( u \geq cv + m \) and \( v \geq m \), we have \( R_{(u,v)} \subset (R_{++})^m \); hence

\[
\left( 0 : (R_{++})^\infty \right)_{(u,v)} \subseteq (R_{++})^m \left( 0 : (R_{++})^\infty \right) = 0,
\]

which implies \( H_{R_v}(u,v) = H_{R_v}(u,v) \). Therefore, \( P_{R_v}(u,v) = P_{R_v}(u,v) \). As we have seen above, \( \deg_u P_{R_v}(u,v) = \dim R_v - 1 \). But \( \dim \bar{R} \leq x \dim R \). So we get

\[
\deg_u P_{R_v}(u,v) = \deg_u P_{\bar{R}_v}(u,v) \leq x \dim R - 1.
\]

It remains to show that \( \dim R_v \geq x \dim R \) for \( v \) large enough. For this will imply the converse inequality \( \deg_u P_{R_v}(u,v) \geq x \dim R - 1 \); hence \( \deg_u P_{R_v}(u,v) = x \dim R - 1 \). Let \( R_+ \) denote the ideal \( \oplus_{v>0} R_v \) of \( R \). Then \( R_v \cong (R_+)^v/(R_+)^{v+1} \) and \( \bar{R}_0 = R/(R_+ + 0 : (R_{++})^\infty) \). Let \( P \) be an associated prime of \( R_+ + 0 : (R_{++})^\infty \) such that \( \dim R/P = \dim \bar{R}_0 = x \dim R \). Then \( 0 : (R_+)^v \subseteq 0 : (R_{++})^\infty \subset P \). Therefore, \( (0 : (R_+)^v)_P \) is a proper ideal in the local ring \( R_P \) so that \( (R_+)^v_P \neq 0 \). By Nakayama’s lemma, this implies \( (R_+)^v_P/(R_+)^{v+1}_P \neq 0 \). Hence

\[
\dim R_v = \dim (R_+)^v/(R_+)^{v+1} \geq \dim R/P = \dim \bar{R}_0.
\]

\[\square\]

It is easy to find examples with \( \dim R_0 > \dim \bar{R}_0 = x \dim R \).

**Example 2.4.** Let \( R = k[X_1, X_2, Y_1]/(X_1 Y_1) \) with \( \text{bideg } X_1 = \text{bideg } X_2 = (1,0) \) and \( \text{bideg } Y_1 = (1,1) \). Then \( \bar{R} = k[X_1, X_2, Y_1]/(X_1 Y_1) = k[X_2, Y_1] \). Since \( R_0 = k[X_1, X_2] \) and \( \bar{R}_0 = k[X_2] \), we have \( \dim R_0 = 2 > 1 = \dim \bar{R}_0 \).

By Theorem 2.1 and Theorem 2.3 we always have

\[
r \dim R = \deg P_{R_v}(u,v) \geq \deg_u P_{R_v}(u,v) = x \dim R + 1.
\]

The following example shows that the inequality may be strict.

**Example 2.5.** Let \( R = k[X_1, X_2, Y_1, Y_2, Y_3]/(X_1 Y_1) \) with \( \text{bideg } X_i = (1,0) \) and \( \text{bideg } Y_j = (d_j,1) \) for any sequence of non-negative integers \( d_1, d_2, d_3 \). Then \( \bar{R} = R \) and \( R_0 = k[X_1, X_2] \). Hence \( r \dim R = 4 > 2 + 1 = x \dim R + 1 \).

We are not able to find a formula for \( \deg_u P_{R_v}(u,v) \) (the degree of \( P_{R_v}(u,v) \) in \( v \)). The method of Theorem 2.3 does not work in this case since by the range of the equality \( H_{R_v}(u,v) = P_{R_v}(u,v) \) we can not fix \( u \) to estimate \( \deg_u P_{R_v}(u,v) \).
3. Leading coefficients of Hilbert polynomial

Let $R$ be a bigraded algebra $R$ generated by $n$ homogeneous elements $x_1, \ldots, x_n$ of bidegree $(1, 0)$ and $r$ elements $y_1, \ldots, y_r$ of bidegree $(d_1, 1), \ldots, (d_r, 1)$, where $d_1, \ldots, d_r$ are non-negative integers.

We write the Hilbert polynomial $P_R(u, v)$ in the form

$$P_R(u, v) = \sum_{i=0}^{s} \frac{e_i(R)}{i!(s-i)!} u^i v^{s-i} + \text{lower-degree terms},$$

where $s = \deg P_R(u, v)$. Note that $P_R(u, v) = 0$ if $s = -1$. Following [Te] we call the numbers $e_i(R)$ the mixed multiplicities of $R$. Moreover, we set

$$\rho_R := \max\{i | e_i(R) \neq 0\}.$$

The aim of this section is to establish some basic properties of mixed multiplicities.

We shall need the following technical notion. Let $x$ be a homogeneous element of $R$. We call $x$ a filter-regular element of degree $(1, 0)$ if $x \not\in P$ for any associated prime ideal $P \in \text{Proj } R$ of $R$. This notion has its origin in the theory of Buchsbaum rings (see e.g. [SV]). It has been already used to study Hilbert polynomials of standard bigraded algebras in [Tr2]. It is obvious that if the base field $k$ is infinite and $R/(1,0) \neq 0$ then we can always find a filter-regular element of degree $(1, 0)$.

**Lemma 3.1.** Let $x$ be a filter-regular element of degree $(1, 0)$ in $R$. Then

(a) $\deg P_{R/xR}(u, v) \leq \deg P_R(u, v) - 1$,
(b) $\deg_u P_{R/xR}(u, v) = \deg_u P_R(u, v) - 1$,
(c) $e_{i-1}(R/xR) = e_i(R)$ for $i \geq 1$,
(d) $\rho_{R/xR} = \rho_R - 1$ if $\deg P_{R/xR}(u, v) = \deg P_R(u, v) - 1$.

**Proof.** Since $x \not\in P$ for any associated prime ideal $P \not\supseteq R_{++}$ of $R$, $0 : x \subseteq 0 : (R_{++})^\infty$. As we have seen in the proof of Theorem 2.3, there is an integer $m$ such that $(0 : x)_{(u,v)} \subseteq (0 : (R_{++})^\infty)_{(u,v)} = 0$ for $u > dv + m$ and $v \geq m$. By Theorem 1.1, this implies $P_R(u, v) = P_{R/(0:x)}(u, v)$. Rewrite $P_R(u, v)$ in the form

$$P_R(u, v) = \sum_{i+j \leq s} e_{i,j} \binom{u}{i} \binom{v}{j},$$

and consider the exact sequence of bigraded algebras

$$0 \rightarrow R/0 : x \xrightarrow{x} R \rightarrow R/xR \rightarrow 0.$$

Since $x$ is a homogeneous element of degree $(1, 0)$,

$$P_{R/xR}(u, v) = P_R(u, v) - P_{R/(0:x)}(u-1, v) = P_R(u, v) - P_R(u-1, v)$$

$$= \sum_{i+j \leq s} e_{i,j} \binom{u-1}{i} \binom{v}{j} - \sum_{i+j \leq s} e_{i,j} \binom{u}{i} \binom{v}{j}$$

$$= \sum_{i+j \leq s} \frac{e_{i,j}}{(i-1)! j!} u^{i-1} v^j + \text{terms of degree } i+j-1 \text{ with } \deg_u < i - 1.$$

Now we can easily verify (a), (b) and (c). Moreover, we have $\rho_{R/xR} = \rho_R - 1$ if $\rho_R > 0$ and $\deg P_{R/xR}(u, v) < \deg P_R(u, v) - 1$ if $\rho_R = 0$. Hence (d) is obvious. \qed
Theorem 3.2. The mixed multiplicities $e_i(R)$ are integers with $e_{\rho_R}(R) > 0$.

Remark. As shown in Proposition 1.3, a mixed multiplicity can be a negative number. However, if $R$ is a standard bigraded algebra, all mixed multiplicities are non-negative [W]. Moreover, $e_0(R), \ldots, e_n(R)$ can be any sequence of non-negative integers with at least one positive entry [KMV, Example 5.2].

Proof. We will first show that $e_{\rho_R}(R) > 0$ is an integer. Consider the diagonal subalgebra $R_\Delta := \oplus_{v \geq 0} R_{(c v, v)}$ for any integer $c > \max\{d_1, \ldots, d_r\}$. By Lemma 2.2(a), $R_\Delta$ is a standard graded algebra. By the proof of Theorem 2.1 we have

$$P_{R_\Delta}(v) = \sum_{i=0}^{s} \frac{e_i(R)}{i!(s-i)!} c^i v^s + \text{lower-degree terms}$$

with $\deg P_{R_\Delta}(v) = s$. From this it follows that

$$\sum_{i=0}^{s} \frac{e_i(R)}{i!(s-i)!} c^i = \frac{e(R_\Delta)}{s!} > 0.$$ 

The leading coefficient of the polynomial in $c$ on the left side must be positive. Hence $e_{\rho_R}(R) > 0$.

From the above formula we also get

$$e_0(R) = e(R_\Delta) - \sum_{i=1}^{s} \binom{s}{i} e_i(R).$$

Hence $e_0(R)$ is an integer. Since $\deg_u P_R(u, v) \geq \rho_R$, to show that $e_1(R), \ldots, e_{\rho_R}(R)$ are integers we may assume that $\deg_u P_R(u, v) > 0$. Extending the base field $k$ by an indeterminate, we may also assume that $k$ is an infinite field. Then we can find a filter-regular element $x \in R$ of degree $(1, 0)$. By Lemma 3.1(b), $\deg_u P_{R/\mathfrak{p}}(u, v) = \deg_u P_R(u, v) - 1$. Using induction on $\deg_u P_R(u, v)$ we may assume that $e_i(R/\mathfrak{p})$ is an integer for $i \geq 0$. By Lemma 3.1(c), this implies that $e_i(R)$ is an integer for $i \geq 1$.

Now we will give a condition for $\rho_R = \deg_u P_R(u, v)$ (see Example 3.5 for an example with $\rho_R < \deg_u P_R(u, v)$). Note that this condition is satisfied if $R$ is a domain or a Cohen-Macaulay ring.

Proposition 3.3. Assume that $\dim R/\mathfrak{p} = \cdim R$ for all minimal prime ideals of $\text{Proj } R$. Then $e_i(R) > 0$ for $i = \deg_u P_R(u, v)$.

Proof. By Theorem 3.2 we only need to show that $\rho_R = \deg_u P_R(u, v)$. For this we may assume that $\deg_u P_R(u, v) > 0$. By Theorem 2.3 we have $R_{(1,0)} \neq 0$. Extending the base field $k$ by an indeterminate we may assume that $k$ is an infinite field. Then we can find a filter-regular element $x \in R$ of degree $(1, 0)$. By Lemma 3.1(b),

$$\deg P_{R/\mathfrak{p}}(u, v) \geq \deg_u P_{R/\mathfrak{p}}(u, v) = \deg_u P_R(u, v) - 1 \geq 0.$$ 

By Theorem 2.1, this implies $\cdim R/\mathfrak{p} \geq 2$. Hence $\text{Proj } R/\mathfrak{p} \neq \emptyset$. Let $P$ be an arbitrary minimal associated prime ideal of $xR$ such that $P \not\ni xR + R_{++}$. Then $P \in \text{Proj } R$. Let $P' \subseteq P$ be a minimal prime ideal of $\text{Proj } R$. Then $P'$ is an associated prime ideal of $R$. Hence $x \notin P'$. From this it follows that $\dim(R/P')_P = 1$. Using the assumption we obtain

$$\dim R/\mathfrak{p} = \dim R/P' - \text{ht } P/P' = \cdim R - \dim(R/P')_P = \cdim R - 1.$$ 

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So we have proved that \( \text{rdim } R / xR = \text{rdim } R - 1 \) and that \( R / xR \) satisfies the assumption of the theorem. By Theorem 2.1 we have
\[
\deg P_{R / xR}(u, v) = \text{rdim } R / xR - 2 = \text{rdim } R - 3 = \deg P_R(u, v) - 1.
\]
By Lemma 3.1(d), this implies \( \rho_{R / xR} = \rho_R - 1 \). By induction we may assume that
\[
\rho_{R / xR} = \deg_u P_{R / xR}(u, v) = \deg_u P_R(u, v) - 1.
\]
Hence we can conclude that \( \rho_R = \deg_u P_R(u, v) \). 

Like for the usual multiplicity, we have an associativity formula for mixed multiplicities of bigraded algebras.

**Proposition 3.4.** Let \( \mathcal{A}(R) \) be the set of the associated prime ideals \( P \in \text{Proj } R \) with \( \dim R / P = \text{rdim } R \). For any \( P \in \mathcal{A}(R) \) let \( \ell(R_P) \) denote the length of the local ring \( R_P \). Then
\[
e_i(R) = \sum_{P \in \mathcal{A}(R)} \ell(R_P) e_i(R / P).
\]

**Proof.** By [Ma, Theorem 6.4] there exists a filtration
\[
R = Q_1 \supset Q_2 \supset \ldots \supset Q_m \supset Q_{m+1} = 0
\]
of bigraded ideals of \( R \) such that \( Q_j / Q_{j+1} \cong (R / P_j)(a_j, b_j) \) for some associated prime ideal \( P_j \) of \( R \) and integers \( a_j, b_j, j = 1, \ldots, m \). From this it follows that \( H_R(u, v) = \sum_{j=1}^m H_{R/P_j}(u + a_j, v + b_j) \). Hence \( P_R(u, v) = \sum_{j=1}^m P_{R/P_j}(u + a_j, v + b_j) \). By Theorem 2.1 this implies
\[
e_i(R) = \sum_{P_j \in \mathcal{A}(R)} e_i(R / P_j).
\]
For every prime ideal \( P \in \mathcal{A}(R) \) we consider the induced filtration
\[
R_P = (Q_1)_P \supset (Q_2)_P \supset \ldots \supset (Q_m)_P \supset (Q_{m+1})_P = 0.
\]
We have \((Q_j / Q_{j+1})_P = 0\) if \( P_j \neq P \) and \((Q_j / Q_{j+1})_P \cong k(P)\) if \( P_j = P \), where \( k(P) \) denotes the residue field of \( R_P \). Therefore, \( \ell(R_P) \) is the number of indices \( j \) for which \( P_j = P \). So we can conclude that
\[
e_i(R) = \sum_{P \in \mathcal{A}(R)} \ell(R_P) e_i(R / P).
\]

By Proposition 3.4, the mixed multiplicities of \( R \) depend only on the prime ideals of \( \text{Proj } R \) with the highest dimension. Using this fact we can easily construct examples with \( \rho_R < \deg_u P_R(u, v) \).

**Example 3.5.** Let \( R = k[X_1, X_2, Y_1, Y_2, Y_3] / (X_1) \cap (Y_1, Y_2) \) with bideg \( X_i = (1, 0) \) and bideg \( Y_j = (d_j, 1) \), where \( d_1, d_2, d_3 \) can be any sequence of non-negative integers. Then
\[
\deg_u P_R(u, v) = x \text{-dim } R - 1 = \dim k[X_1, X_2] - 1 = 1.
\]
By Proposition 3.4, the mixed multiplicities of \( R \) are equal to those of the quotient ring \( R / X_1R \). Hence
\[
\rho_R = \rho_{R / X_1R} \leq \deg_u P_{R / X_1R}(u, v) = x \text{-dim } R / X_1R - 1 = \dim k[X_2] - 1 = 0.
\]
Combining Proposition 3.3 with Proposition 3.4 we obtain the following generalization of a result of P. Roberts in the case $R$ is generated by elements of degree $(1,0),(0,1),(1,1)$ [Ro3, Theorem 2(3)]. This result was used to give a criterion for the positivity of Serre’s intersection multiplicity [Ro3, Proposition 6].

**Corollary 3.6.** Assume that there exists a prime ideal $P \in \mathcal{A}(R)$ with $x\dim R/P = x\dim R$. Then $e_i(R) > 0$ for $i = x\dim R - 1$.

**Proof.** Let $m := x\dim R$. Then $x\dim R/P \leq m$ for every prime ideal $P \in \mathcal{A}(R)$. If $x\dim R/P < m$, then $\deg u P_{R/P} (u,v) < m - 1$. Hence $e_{m-1}(R/P) = 0$. If $x\dim R/P = m$, then $\deg u P_{R/P} (u,v) = m - 1$. Hence $e_{m-1}(R/P) > 0$ by Proposition 3.3. Applying Proposition 3.4 we get

$$e_{m-1}(R) = \sum_{P \in \mathcal{A}(R)} \ell(R_P) e_{m-1}(R/P) > 0.$$  

$\square$

4. **Rees algebras of homogeneous ideals**

Let $A$ be a standard graded algebra over a field $k$. Let $I$ be a homogeneous ideal of $A$. The **Rees algebra** $A[It]$ of $I$ is the subring $\bigoplus_{v \geq 0} I^v t^v$ of $A[t]$. It has a natural bigraded structure by setting

$$A[It]_{(u,v)} := (I^v)_u t^v$$

for all $(u,v) \in \mathbb{N}^2$.

Let $\mathfrak{m} = (x_1, \ldots, x_n)$ be the maximal graded ideal of $A$, where $x_1, \ldots, x_n$ are homogeneous elements with $\deg x_i = 1$. Let $I = (f_1, \ldots, f_r)$, where $f_1, \ldots, f_r$ are homogeneous elements with $\deg f_j = d_j$. Put $y_j = f_j t$. Then $A[It]$ is generated by the elements $x_1, \ldots, x_n$ and $y_1, \ldots, y_r$ with $\text{bideg } x_i = (1,0)$ and $\text{bideg } y_j = (d_j, 1)$. Hence $A[It]$ belongs to the class of bigraded algebras considered in the preceding section.

Let $0 : I^\infty$ denote the ideal of the elements of $A$ which are annihilated by some power of $I$. Note that $\dim A/0 : I^\infty = \dim A$ if $I$ has positive height.

**Lemma 4.1.** Let $A[It]$ be the above bigraded Rees algebra. Then

1. $\text{rdim } A[It] = \dim A/0 : I^\infty + 1$,
2. $x\dim A[It] = A/0 : I^\infty$.

**Proof.** (a) For every primary ideal $Q$ of $A$ let $Q^* := \bigoplus_{v \geq 0} (Q \cap I^v) t^v$. It is easy to see that if $P$ is a prime ideal of $A$ and $Q$ a $P$-primary ideal, then $P^*$ is a prime ideal of $A[It]$ and $Q^*$ a $P^*$-primary ideal. Let $A[It]_{++}$ be the ideal of $A[It]$ generated by the elements $x_i y_j$, $i = 1, \ldots, n$, $j = 1, \ldots, r$. Then $A[It]_{++} = \bigoplus_{v \geq 1} m^{v} t^v$. Therefore, $P^* \not\supset A[It]_{++}$ if and only if $P \not\supset I$. By [Va, Section 1] we have

$$\dim A[It]/P^* = \dim A/P + 1$$

if $P \not\supset I$. Let $0_A = \cap Q_i$ be a primary decomposition of the zero ideal of $A$. Then $0_{A[It]} = \cap Q_i^*$ is a primary decomposition of the zero ideal of $A[It]$. Therefore, $P^*$ is an associated prime of $A[It]$ with $P^* \not\supset A[It]_{++}$ if and only if $P$ is an associated prime ideal of $A$ with $P \not\supset I$, which is equivalent to the condition that $P$ is an associated prime of $A/0 : I^\infty$.
Theorem 4.2. \( \text{Set } A \text{ algebra of integers } P \) is equal to a polynomial \( \text{By Theorem 2.3, this implies } x \in A \). We have \( \text{ann } s \). Moreover, if \( \text{Proj } A \) then the coefficients \( e_i \) for \( s \) in \( A \). For any finitely generated graded \( A \)-module \( M \) we will denote by \( e(M) \), \( H_M(u) \) and \( P_M(u) \) the multiplicity, the Hilbert function and the Hilbert polynomial of \( M \).

We can summarize the results of the preceding sections for the bigraded Rees algebra \( A[I] \) as follows.

**Theorem 4.2.** Set \( d = \max\{d_1, \ldots, d_r\} \) and \( s = \dim A/0 : I^\infty - 1 \). There exist integers \( u_0, v_0 \) such that for \( u \geq dv + u_0 \) and \( v \geq v_0 \), the Hilbert function \( H_{A[I]}(u, v) \) is equal to a polynomial \( P_{A[I]}(u, v) \) with

\[
\deg P_{A[I]}(u, v) = \deg u P_{A[I]}(u, v) = s.
\]

Moreover, if \( s \geq 0 \) and \( P_{A[I]}(u, v) \) is written in the form

\[
P_{A[I]}(u, v) = \sum_{i=0}^{s} e_i(A[I]) u^i v^{s-i} + \text{lower-degree terms},
\]

then the coefficients \( e_i(A[I]) \) are integers for all \( i \) with \( e_s(A[I]) = e(A/0 : I^\infty) \).

**Proof.** Except the last formula for \( e_s(A[I]) \), all statements of Theorem 4.2 follow from Theorem 1.1, Theorem 2.1, Theorem 2.3, and Theorem 3.2 by taking Lemma 4.1 into consideration.

To prove the formula for \( e_s(A[I]) \) we first note that \( H_{A[I]}(u, v) = H_{I^v}(u) \) for all \( u \). By Theorem 1.1, this implies

\[
P_{A[I]}(u, v) = P_{I^v}(u)
\]

for \( v \) large enough. Since \( \deg P_{A[I]}(u, v) = \deg u P_{A[I]}(u, v) = s \), the term of \( u^s \) in the polynomial \( P_{A[I]}(u, v) \) must be non-zero. The coefficient of this term is equal to \( \frac{e_s(A[I])}{s!} \). Therefore, for \( v \) large enough, we may write

\[
P_{I^v}(u) = \frac{e_s(A[I])}{s!} u^s + \text{lower-degree terms}.
\]

Since \( \dim I^v = s + 1 \), this implies \( e_s(A[I]) = e(I^v) \). Note that \( I^v \cap (0 : I^\infty) = 0 \). Then \( I^v \cong I^v + (0 : I^\infty)/0 : I^\infty \). Hence there is an exact sequence of the form

\[
0 \rightarrow I^v \rightarrow A/0 : I^\infty \rightarrow A/I^v + (0 : I^\infty) \rightarrow 0.
\]
Since \((0 : I^\infty) : I = 0 : I^\infty\), we have \(\dim A/0 : I^\infty > \dim A/I^v + (0 : I^\infty)\). Therefore, the above sequence implies \(e(I^v) = e(A/0 : I^\infty)\). So we can conclude that \(e_s(A[It]) = e(A/0 : I^\infty)\). \(\square\)

**Remark.** There is another kind of mixed multiplicities associated with \(I\). Let \(M := (m, It)\). The associated graded ring \(R := \oplus_{n \geq 0} M^n/M^{n+1}\) has a natural standard bigrading with \(R_{(u,v)} := m^u I^v / m^{u+1} I^v\). Teissier [Te] called \(e_i(R)\) a mixed multiplicity of the pair \((m, I)\) and denoted it by \(e_i(m|I)\). The multiplicity of the Rees algebra \(A[It]\) and of the extended Rees algebra \(A[It, t^{-1}]\) can be expressed in terms of the mixed multiplicities \(e_i(m|I)\) (see [Ve1], [Ve2], [KV]). One may use \(e_i(A[It])\) to compute \(e_i(m|I)\) if \(I\) is generated by forms of the same degree. In this case, there is a bigraded isomorphism \(R \cong A[It]\) with a linear transformation of the bidegree.

The Hilbert polynomial \(P_{A[It]}(u, v)\) can be used to compute the Hilbert polynomial of the quotient ring \(A/I^v\) for \(v\) large enough.

**Corollary 4.3.** For \(v\) large enough,

\[
P_{A/I^v}(u) = P_A(u) - P_{A[It]}(u, v).
\]

**Proof.** By the proof of Theorem 4.2 we have \(P_{I^v}(u) = P_{A[It]}(u, v)\) for \(v\) large enough. Hence the conclusion is immediate. \(\square\)

Let \(V\) denote the blow-up of the subscheme of \(\text{Proj } A\) defined by \(I\). It is known that \(V\) can be embedded into a projective space by the linear system \((I^e)_c\) for any pair of positive integers \(e, c\) with \(c > de\) [CH, Lemma 1.1], and that the embeddings often give concrete varieties with interesting algebraic properties (see e.g. [Gi], [GG], [GHH], [GL]). Let \(V_{c,e}\) denote the embedded variety. We can use the Hilbert polynomial \(P_{A[It]}(u, v)\) to describe the degree of \(V_{c,e}\) as follows.

**Corollary 4.4.** Let \(s = \dim A/0 : I^\infty - 1\). Assume that \(c > de\). Then

\[
\deg V_{c,e} = \sum_{i=0}^s \binom{s}{i} e_i(A[It]) c^i e^{s-i}.
\]

**Proof.** The homogeneous coordinate ring of the embedded variety \(V_{c,e}\) is the subalgebra \(k[[I^e]]_c\) of \(A\) generated by the elements of \((I^e)_c\). As observed in [STV] and [CHVT], we may identify \(k[[I^e]]_c\) with the diagonal subalgebra \(A[It]_\Delta\) of the bigraded Rees algebra \(A[It]\), where \(\Delta = \{(cv, ev) | v \in \mathbb{N}\}\). Hence

\[
\deg V_{c,e} = e(A[It]_\Delta).
\]

Let \(P_{A[It]_\Delta}(v)\) denote the Hilbert polynomial of \(A[It]_\Delta\). As we have seen in the proof of Theorem 2.1,

\[
P_{A[It]_\Delta}(v) = P_{A[It]}(cv, v).
\]

By Theorem 4.2 we may write

\[
P_{A[It]}(cv, v) = \sum_{i=0}^s \frac{e_i(A[It])}{i!(s-i)!} c^i e^{s-i} v^s + \text{lower-degree terms}.
\]

By Lemma 2.2(b) and Lemma 4.1(a),

\[
\deg P_{A[It]_\Delta}(u) = \dim A[It]_\Delta - 1 = r \dim A[It] - 2 = s.
\]
Hence we can conclude that
\[ e(A[It]_\Delta) = \sum_{i=0}^{s} \binom{s}{i} e_i(A[It]) c^i e^{s-i}. \]

Now we will present a method for the computation of \( H_{A[It]}(u, v) \). This method was introduced in [HeTU] in order to compute the multiplicity of \( A[It] \) (see also [RaS], [Tr1]).

Let \( S = A[Y_1, \ldots, Y_r] \) be a polynomial ring over \( A \). Mapping \( Y_j \) to \( f_j t, j = 1, \ldots, r \) we obtain a representation of the Rees algebra:
\[ A[It] \cong S/J, \]
where \( J \) is the ideal of \( A \) generated by the forms vanishing at \( f_1, \ldots, f_r \). If we set \( \text{bideg} x_i = (1, 0) \) and \( \text{bideg} Y_j = (d_j, 1) \), then \( S \) is a bigraded algebra and the above isomorphism is a bigraded isomorphism.

For every \( h = (\alpha_0, \ldots, \alpha_r) \in \mathbb{N}^{r+1} \) put \( S_h := A_{\alpha_0} Y_1^{\alpha_1} \cdots Y_r^{\alpha_r} \). Then \( S = \bigoplus_{h \in \mathbb{N}^{r+1}} S_h \) is an \( \mathbb{N}^{r+1} \)-graded algebra. This \( \mathbb{N}^{r+1} \)-grading is finer than the above bigrading because
\[ S_{(u,v)} = \bigoplus_{\alpha_0 + a_1 \alpha_1 + \cdots + a_r \alpha_r = u} S_{(\alpha_0, \alpha_1, \ldots, \alpha_r)}, \]
for all \( (u, v) \in \mathbb{N}^2 \). We order \( \mathbb{N}^{r+1} \) as follows: \( (\alpha_0, \ldots, \alpha_r) < (\beta_0, \ldots, \beta_r) \) if the first non-zero component from the left side of
\[ (\alpha_0 + \sum_{j=1}^{r} \alpha_j d_j - \beta_0 - \sum_{j=1}^{r} \beta_j d_j, \sum_{j=1}^{r} \alpha_j - \sum_{j=1}^{r} \beta_j, \alpha_0 - \beta_0, \ldots, \alpha_r - \beta_r) \]
is negative. Then \(< \) is a term order on \( \mathbb{N}^{r+1} \), i.e., \( h < h' \) implies \( h + g < h' + g \) for any \( g \in \mathbb{N}^{r+1} \). Note that this term order is different from that of [HeTU].

For every polynomial \( f \in S \) let \( f^* \) denote the initial term of \( f \), i.e. \( f^* := f_h \) if \( f = \sum_{h' \in \mathbb{N}^{r+1}} f_{h'} \) and \( h = \min \{h' \mid f_{h'} \neq 0 \} \). Let \( J^* \) denote the ideal of \( S \) generated by the elements \( f^*, f \in J \). Then \( S/J^* \) is a bigraded algebra. This algebra has a simpler structure than that of \( S/J \). We can use \( S/J^* \) to compute the Hilbert function \( H_{A[It]}(u, v) \) by the following lemma.

**Proposition 4.5.** \( H_{A[It]}(u, v) = H_{S/J^*}(u, v) \) for all \( (u, v) \in \mathbb{N}^2 \).

**Proof.** Fix \( (u, v) \in \mathbb{N}^{r+1} \). Let
\[ D := \{ (\alpha_0, \ldots, \alpha_r) \in \mathbb{N}^{r+1} \mid 0 + a_1 \alpha_1 + \cdots + a_r \alpha_r = u, \alpha_1 + \cdots + \alpha_r = v \}. \]
Then \( S_{(u,v)} = \bigoplus_{h \in D} S_h \). Hence \( S_{(u,v)} = 0 \) if \( D = \emptyset \). If \( D \neq \emptyset \), we set
\[ h_m := \min \{h \mid h \in D\}, \]
\[ h_M := \max \{h \mid h \in D\}. \]
By the definition of the order \(< \) we have \( D = \{ h \in \mathbb{N}^{r+1} \mid h_m \leq h \leq h_M \} \). For every \( h \in \mathbb{N}^{r+1} \) let
\[ F_h := \bigoplus_{h' \geq h} S_{h'}, \]
\[ h^* := \min \{h' \in \mathbb{N}^{r+1} \mid h' > h \}. \]
Hence \( \dim_k J_{(u,v)} = \sum \dim_k(J \cap F_h/J \cap F_{h^*}) \)
\[ = \sum_{h \in D} \dim_k J^*_h = \dim_k \bigoplus_{h \in D} J^*_h \]
\[ = \dim_k J^*_{(u,v)}. \]
Hence \( \dim_k(S/J)_{(u,v)} = \dim_k(S/J^*)_{(u,v)} \). So we get \( H_{A[I^q]}(u,v) = H_{S/J^*}(u,v). \)

5. Rees algebras of homogeneous d-sequences

Let \( A = \oplus_{u \geq 0} A_u \) be a standard graded algebra over a field \( k \). Let \( f_1, \ldots, f_r \) be a sequence of homogeneous elements of \( A \) and \( I = (f_1, \ldots, f_r) \).

We call \( f_1, \ldots, f_r \) a \( d \)-sequence if the following conditions are satisfied:

1. \( f_i \notin (f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_r) \),
2. \((f_1, \ldots, f_i) : f_{i+1} f_j = (f_1, \ldots, f_i) : f_j \) for all \( j \geq i + 1 \) and all \( i \geq 0 \).

This notion was introduced by Huneke in [Hu], where one can find basic properties and abundant examples of \( d \)-sequences. In this section we will estimate the Hilbert polynomial of the bigraded Rees algebra \( A[I^q] \) when \( I \) is generated by a \( d \)-sequence of homogeneous elements of increasing degrees.

By Proposition 4.5 we only need to compute the Hilbert polynomial of \( S/J^* \), where \( S = A[Y_1, \ldots, Y_r] \) and \( J^* \) is the initial ideal of the defining ideal of \( A[I^q] \) with respect to the term order introduced there. The following result on the initial ideal \( J^* \) is similar to [HeTU, Lemma 1.2] (which uses a different term order). It can be proved by the same proof. We leave the reader to check the details.

**Lemma 5.1.** Let \( f_1, \ldots, f_r \) be a homogeneous \( d \)-sequence with \( \deg f_1 \leq \ldots \leq \deg f_r \). Then
\[
J^* = (I_1 Y_1, \ldots, I_r Y_r),
\]
where \( I_q = (f_1, \ldots, f_{q-1}) : f_q \) for \( q = 1, \ldots, r \).

Using Lemma 5.1 we can compute the Hilbert polynomial of \( S/J^* \) and get the following result on the Hilbert polynomial of \( A[I^q] \).

**Theorem 5.2.** Let \( I \) be an ideal generated by a homogeneous \( d \)-sequence \( f_1, \ldots, f_r \) with \( \deg f_j = d_j \) and \( d_1 \leq \ldots \leq d_r \). Let \( I_q = (f_1, \ldots, f_{q-1}) : f_q \) for \( q = 1, \ldots, r \). Set \( s := \dim A/I_1 - 1 \) and \( m := \max\{|q| : \dim A/I_q + q - 2 = s\} \). Then \( \deg P_{A[I^q]}(u,v) = s \) and
\[
e_i(A[I^q]) = \min\{m, s-i+1\} \sum_{q=1}^{\min\{m, s-i+1\}} (-1)^{s-q-i+1} e(A/I_q) \sum_{j_1+\ldots+j_q=s-q-i+1} d_{j_1}^{f_1} \ldots d_{j_q}^{f_q}
\]
for \( i = 0, \ldots, s \).

**Proof.** As observed above, we may replace \( A[I^q] \) by \( S/J^* \). We have
\[
(S/J^*)_{(u,v)} = \bigoplus_{\alpha_0 + d_1 \alpha_1 + \ldots + d_r \alpha_r = u, \alpha_1 + \ldots + \alpha_r = v} (S/J^*)_{(\alpha_0, \alpha_1, \ldots, \alpha_r)}. \]
Every element of $J^*(\alpha_0, \alpha_1, \ldots, \alpha_r)$ has the form $fY_1^{\alpha_1} \cdots Y_r^{\alpha_r}$ with $f \in (I_j)_{\alpha_0}$ for some $j = 1, \ldots, r$ with $\alpha_j \neq 0$. Since $I_1 \subset \ldots \subset I_r$ [Hu, Remarks (2)], this implies

$$J^*(\alpha_0, \alpha_1, \ldots, \alpha_r) = (I_{m(\alpha_1, \ldots, \alpha_r)})_{\alpha_0}Y_1^{\alpha_1} \cdots Y_r^{\alpha_r},$$

where $m(\alpha_1, \ldots, \alpha_r) := \max \{q| \alpha_q \neq 0\}$. Hence

$$(S/J^*)_{(\alpha_0, \alpha_1, \ldots, \alpha_r)} \cong (A/I_{m(\alpha_1, \ldots, \alpha_r)})_{\alpha_0}.$$ 

So we obtain

$$H_{S/J^*}(u, v) = \sum_{\alpha_0 + d_1 \alpha_1 + \ldots + d_r \alpha_r = u} \frac{H_{A/I_{m(\alpha_1, \ldots, \alpha_r)}}(\alpha_0)}{\alpha_1 + \ldots + \alpha_r = v}$$

$$= \sum_{\alpha_1 + \ldots + \alpha_r = v} H_{A/I_{m(\alpha_1, \ldots, \alpha_r)}}(u - d_1 \alpha_1 - \ldots - d_r \alpha_r)$$

$$= \sum_{q=1}^{r} \sum_{\alpha_1 + \ldots + \alpha_q = v} H_{A/I_q}(u - d_1 \alpha_1 - \ldots - d_q \alpha_q)$$

$$= \sum_{q=1}^{r} \sum_{\alpha_1 + \ldots + \alpha_q = v} H_{A/I_q}(u - d_1 \alpha_1 - \ldots - d_q \alpha_q) - \sum_{\alpha_1 + \ldots + \alpha_{q-1} = v} H_{A/I_q}(u - d_1 \alpha_1 - \ldots - d_{q-1} \alpha_{q-1}).$$

By Theorem 1.1 there exist integers $u_0$ and $v_0$ such that $H_{S/J^*}(u, v) = P_{S/J^*}(u, v)$ for $u \geq dv + u_0$ and $v \geq v_0$. If $\alpha_1 + \ldots + \alpha_j = v$, then $u - d_1 \alpha_1 - \ldots - d_j \alpha_j \geq u - dv$. Hence we may choose $u_0$ such that

$$H_{A/I_q}(u - d_1 \alpha_1 - \ldots - d_j \alpha_j) = P_{A/I_q}(u - d_1 \alpha_1 - \ldots - d_j \alpha_j)$$

for $u \geq dv + u_0$. Put $H_{q,j}(u, v) := \sum_{\alpha_1 + \ldots + \alpha_j = v} P_{A/I_q}(u - d_1 \alpha_1 - \ldots - d_j \alpha_j).$ Then

$$P_{S/J^*}(u, v) = \sum_{q=1}^{r} [H_{q,j}(u, v) - H_{q,q-1}(u, v)].$$

For $q = 1, \ldots, r$ let $s_q := \dim A/I_q - 1$. By Lemma 1.4, $H_{q,j}(u, v)$ is a polynomial of degree $s_q + j - 1$. Therefore,

$$\deg P_{S/J^*}(u, v) \leq \max \{s_q + q - 1| q = 1, \ldots, r\}.$$ 

Since $I_q \supseteq (I_{q-1}, f_{q-1})$ and $f_q$ is a non-zerodivisor modulo $I_{q-1}$, we have

$$s_q < \dim A/(I_{q-1}, f_{q-1}) = \dim A/I_{q-1} - 1 = s_{q-1}$$

for $q \geq 2$. From this it follows that $s_q + q - 1 \leq s_1 = s$. Hence $\deg P_{S/J^*}(u, v) \leq s$.

Since $s = s_1 > \ldots > s$, and $m = \max \{q| s_q + q - 1 = s\}$, we have $s_q + q - 1 = s$ if $q \leq m$ and $s_q + q - 1 < s$ if $q > m$. Therefore, $P_{S/J^*}(u, v)$ and $\sum_{q=1}^{m} H_{q,j}(u, v)$ share the same terms of degree $s$. Using Lemma 1.4 we can compute these terms and we may write

$$P_{S/J^*}(u, v) = \sum_{q=1}^{m} \sum_{i=0}^{s_q - i} \frac{(-1)^{s_q - i} e(A/I_q)}{i! (s - i)!} u^i v^{s - i} +$$

+ lower-degree terms.
For \( q = 1, \ldots, m \) we have \( s_q = s - q + 1 \). Therefore, \( P(u, v) \) can be rewritten as

\[
P_{S/J^*}(u, v) = \sum_{i=0}^{\min\{m, s-i+1\}} \left( \sum_{q=1}^{\min\{m, s-i+1\}} (-1)^{s-q-i+1} e(A/I_q) \sum_{j_1+\ldots+j_q=s-q-i+1} d_1^{j_1} \ldots d_q^{j_q} \right) i!(s-i)! u^i v^{s-i} + \]

+ lower-degree terms.

The coefficient of \( u^s \) in \( P(u, v) \) is equal to \( e(A/I_1) \). So we can conclude that \( \deg P_{S/J^*}(u, v) = s \) and

\[
e_i(S/J^*) = \sum_{q=1}^{\min\{m, s-i+1\}} (-1)^{s-q-i+1} e(A/I_q) \sum_{j_1+\ldots+j_q=s-q-i+1} d_1^{j_1} \ldots d_q^{j_q}.
\]

\( \square \)

**Remark.** The above method was already used in [RaV] (see also [Ho]) to establish the following formula for \( e_i(m/I) \) when \( I \) is generated by a d-sequence:

\[
e_i(m/I) = \begin{cases} 0 & \text{if } 0 \leq i \leq s - m, \\ e(R/I_{s-i+1}) & \text{if } s - m < i \leq s. \end{cases}
\]

This displays a completely different behavior than that of \( e_i(A[I^t]) \).

Now we will apply Theorem 5.2 to some special class of d-sequences.

**Corollary 5.3.** Let \( f_1, \ldots, f_r \) be a homogeneous regular sequence with \( \deg f_1 = d_1 \leq \ldots \leq \deg f_r = d_r \) and \( I = (f_1, \ldots, f_r) \). Set \( s = \dim A - 1 \). Then \( \deg P_{A[I^t]}(u, v) = s \) and

\[
e_i(A[I^t]) = \sum_{q=1}^{\min\{r, s-i+1\}} (-1)^{s-q-i+1} e(A) \sum_{j_1+\ldots+j_q=s-q-i+1} d_1^{j_1+1} \ldots d_q^{j_q+1} d_q^{j_q}
\]

for \( i = 0, \ldots, s \).

**Proof.** Since \( I_q = (f_1, \ldots, f_q-1) : f_q = (f_1, \ldots, f_q-1) \), we have \( \dim A/I_q = \dim A - q + 1 \) and \( e(A/I_q) = d_1 \ldots d_{q-1} e(A), q = 1, \ldots, r \). It follows that \( s = \dim A/I_1 - 1 \) and \( r = \max\{q \mid \dim A/I_q + q - 2 = s\} \). Hence the conclusion follows from Theorem 5.2.

\( \square \)

If \( A \) is a polynomial ring and \( I \) a completion ideal in \( A \), the Hilbert function of \( A/I_q \) can be expressed in terms of \( d_1, \ldots, d_q, q = 1, \ldots, r \). Therefore, we can use the formula for \( H_{S/J^*}(u, v) \) in the proof of Theorem 5.2 to give an explicit formula for \( H_{A[I^t]}(u, v) \) in terms of \( d_1, \ldots, d_r \). We will demonstrate this method by the following example which covers a key result of Geramita, Gimigliano and Harbourne in [GGH].

**Example 5.4.** Let \( f_1, f_2 \) be a homogeneous regular sequence in \( A = k[x_1, x_2, x_3] \) with \( \deg f_1 = d_1 \leq \deg f_2 = d_2 \) and \( I = (f_1, f_2) \). The rational surface \( V \) obtained by blowing up the intersection of the curves \( f_1 \) and \( f_2 \) was the main focus of study in [GGH], where it was assumed that \( f_1 \) and \( f_2 \) meet transversally in \( d_1d_2 \) points. The key result there is an explicit formula for the cohomological function \( h^0(V, D_{u', v}) \) in terms of \( d_1, d_2 \), where \( D_{u', v} \) denotes the divisor class of \( V \) associated to \( (I^t)_{u'+d_2v} \).
[GGH, Proposition III.1(a)]. This formula can be easily derived by our method. We first note that
\[ h^0(V, D_{u'v}) = \dim_k(I^u_{u'+d_2v}) = H_{A[I]}(u' + d_2v, v). \]
Since \( I_1 = 0, I_2 = (f_1) \) and \( H_A(u) = \left( \frac{u+2}{2} \right), H_{A/(f_1)}(u) = \left( \frac{u+2}{2} \right) - \left( \frac{u-d_1+2}{2} \right) \), we have
\[
H_{A[I]}(u, v) = H_A(u - d_1v) + \sum_{j=0}^v H_{A/(f_1)}(u - d_1j - d_2v + d_2j) - H_{A/(f_1)}(u - d_1v) \\
= \sum_{j=0}^v \left[ \left( \frac{u - d_1j - d_2v + d_2j + 2}{2} \right) - \left( \frac{u - d_1j - d_2v + d_2j - d_1 + 2}{2} \right) \right] + \\
\left( \frac{u - d_1v - d_1 + 2}{2} \right)
\]
for \( u > d_2v \). Putting \( u = u' + d_2v \) and \( \delta = d_2 - d_1 \) we get
\[
H_{A[I]}(u' + d_2v, v) = \sum_{j=0}^{v-1} \left[ \left( \frac{u' + \delta j + 2}{2} \right) - \left( \frac{u' + \delta j - d_1 + 2}{2} \right) \right] + \left( \frac{u' + \delta v + 2}{2} \right).
\]
This is precisely the formula of [GGH, Proposition III.1(a)] for \( h^0(V, D_{u'v}) \). Moreover, using Corollary 5.3 we can easily compute the leading coefficients of the above polynomial:
\[ e_0(A[I]) = -d_1d_2, \quad e_1(A[I]) = 0, \quad e_2(A[I]) = 1. \]
By Corollary 4.4, this implies the following formula for the degree of the embedded variety \( V_{c,e} \) of \( V \):
\[ \deg V_{c,e} = c^2 - d_1d_2e. \]

**Corollary 5.5.** Let \( A = k[X] \), where \( X \) is a \( (r - 1) \times r \) matrix of indeterminates. Let \( I \) be the ideal of the maximal minors of \( X \) in \( A \). Set \( s = (r - 1) \times r - 1 \). Then \( \deg P_{A[I]}(u, v) = s \) and
\[
e_i(A[I]) = \sum_{q=1}^{\min\{r, s-i+1\}} (-1)^s-q-i+1 \binom{r-1}{q-1} \binom{s-i}{q-1} r^{s-q-i+1}
\]
for \( i = 0, \ldots, s \).

**Proof.** Let \( f_q \) be the determinant of the submatrix of \( X \) obtained by deleting the \((r - q + 1)\)th column, \( q = 1, \ldots, r \). Then \( \deg f_1 = \ldots = \deg f_r = r \) and \( f_1, \ldots, f_r \) is a \( d \)-sequence by [Hu, Example 1.1]. Let \( I_q = (f_1, \ldots, f_{q-1} : f_q) \). Then \( I_q \) is the ideal generated by the maximal minors of the matrix of the first \( r - q + 1 \) columns of \( X \). Hence \( \dim R/I_q = (r - 1)r - q + 1 \) and \( e(A/I_q) = \binom{r-1}{q-1} \) [HeT, Theorem 3.5]. It follows that \( s = \dim R/I_1 - 1 \) and \( r = \max\{q| \dim A/I_q + q - 2 = s\} \). Since
\[
\sum_{j_1 + \ldots + j_q = s-q-i+1} r^{j_1} \ldots r^{j_q} = \binom{s-i}{q-1} r^{s-q-i+1},
\]
the conclusion follows from Theorem 5.2. \( \square \)
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