Quantum groupoids

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Dedicated to the memory of Moshé Flato

Abstract

We introduce a general notion of quantum universal enveloping algebroids (QUE algebroids), or quantum groupoids, as a unification of quantum groups and star-products. Some basic properties are studied including the twist construction and the classical limits. In particular, we show that a quantum groupoid naturally gives rise to a Lie bialgebroid as a classical limit. Conversely, we formulate a conjecture on the existence of a quantization for any Lie bialgebroid, and prove this conjecture for the special case of regular triangular Lie bialgebroids. As an application of this theory, we study the dynamical quantum groupoid $\mathcal{D} \otimes \hbar U_q G$, which gives an interpretation of the quantum dynamical Yang-Baxter equation in terms of Hopf algebroids.

1 Introduction

Poisson tensors in many aspects resemble classical triangular r-matrices in quantum group theory. A notion unifying both Poisson structures and Lie bialgebras was introduced in [34], called Lie bialgebroids. The integration theorem for Lie bialgebroids encompasses both Drinfeld’s theorem of integration of Lie bialgebras on the one hand, and the Karasev-Weinstein theorem of existence of local symplectic groupoids for Poisson manifolds on the other hand [35]. Quantization of Lie bialgebras leads to quantum groups, while quantizations of Poisson manifolds are the so called star-products. It is therefore natural to expect that there exists some intrinsic connection between these two quantum objects. The purpose of this paper is to fill in this gap by introducing the notion of quantum universal enveloping algebroids (QUE algebroids), or quantum groupoids, as a general framework unifying these two concepts. Part of the results in this paper has been announced in [49] [50].

The general notion of Hopf algebroids was introduced by Lu [32], where the axioms were obtained essentially by translating those of Poisson groupoids to their quantum counterparts. The special case where the base algebras are commutative was studied earlier by Maltsiniotis [36], in a 1992 paper based on the work of Deligne on Tannakian categories [8]. Subsequently, Vainerman [43] found a class of examples of Hopf algebroids arising from a Hopf algebra action on an algebra, *Research partially supported by NSF grants DMS97-04391 and DMS00-72171.
which generalizes that introduced by Maltsiniotis. Recently, Hopf algebroids also appeared in Etingof and Varchenko’s work on dynamical quantum groups \[13\quad16\]. In this paper, we will mainly follow Lu’s definition, but some axioms will be modified. One advantage of our approach is that the tensor product of representations becomes an immediate consequence of the definition.

We refer the interested reader to \[37\quad39\] for other definitions of quantum groupoids, which are originated from different motivations and different from the one we are using here.

As we know, many important examples of Hopf algebras arise as deformations of the universal enveloping algebras of Lie algebras. Given a Lie algebroid $A$, its universal enveloping algebra $UA$ (see the definition in Section 2) carries a natural cocommutative Hopf algebroid structure. For example, when $A$ is the tangent bundle Lie algebroid $TP$, one obtains a cocommutative Hopf algebroid structure on $\mathcal{D}(P)$, the algebra of differential operators on $P$. It is natural to expect that deformations of $UA$, called quantum universal enveloping algebroids or quantum groupoids in this paper, would give us some non-trivial Hopf algebroids. This is the starting point of the present paper. Examples include the usual quantum universal enveloping algebras and the quantum groupoid $\mathcal{D}_h(P)$ corresponding to a star-product on a Poisson manifold $P$.

Another important class of quantum groupoids is connected with the so called quantum dynamical Yang-Baxter equation, also known as the Gervais-Neveu-Felder equation \[4\]:

$$R^{12}(\lambda)R^{13}(\lambda + \hbar h^{(2)})R^{23}(\lambda) = R^{23}(\lambda + \hbar h^{(1)})R^{13}(\lambda)R^{12}(\lambda + \hbar h^{(3)}).$$

Here $R(\lambda)$ is a meromorphic function from $\eta^*$ to $U_h g \otimes U_h g$, $U_h g$ is a quasi-triangular quantum group, and $\eta \subset g$ is an Abelian Lie subalgebra. This equation arises naturally from various contexts in mathematical physics, including quantum Liouville theory, quantum Knizhnik-Zamolodchikov-Bernard equation, and quantum Caloger-Moser model \[2\quad8\quad10\]. One approach to this equation, due to Babelon et al. \[4\], is to use Drinfeld’s theory of quasi-Hopf algebras \[11\]. Consider a meromorphic function $F : \eta^* \longrightarrow U_h g \otimes U_h g$ such that $F(\lambda)$ is invertible for all $\lambda$. Set $R(\lambda) = F^{21}(\lambda)^{-1}RF^{12}(\lambda)$, where $R \in U_h g \otimes U_h g$ is the standard universal $R$-matrix for the quantum group $U_h g$. One can check \[4\] that $R(\lambda)$ satisfies Equation (1) if $F(\lambda)$ is of zero weight, and satisfies the following shifted cocycle condition:

$$(\Delta_0 \otimes id)F(\lambda)F^{12}(\lambda + \hbar h^{(3)}) = (id \otimes \Delta_0)F(\lambda)F^{23}(\lambda),$$

where $\Delta_0$ is the coproduct of $U_h g$. If moreover we assume that

$$(\epsilon_0 \otimes id)F(\lambda) = 1; \quad (id \otimes \epsilon_0)F(\lambda) = 1,$$

where $\epsilon_0$ is the counit map, one can form an elliptic quantum group, which is a family of quasi-Hopf algebras $(U_h g, \Delta_\lambda)$ parameterized by $\lambda \in \eta^*$. $\Delta_\lambda = F(\lambda)^{-1}\Delta_0F(\lambda)$. For $g = sl_2(\mathbb{C})$, a solution to Equations (2) and (3) was obtained by Babelon \[4\] in 1991. For general simple Lie algebras, solutions were recently found independently by Arnaudon et al. \[4\] and Jimbo et al. \[24\] based on the approach of Frønsdal \[23\]. Equivalent solutions are also found by Etingof and Varchenko \[16\] using intertwining operators. Recently, using a method similar to \[1\quad20\quad24\], Etingof et al. found a large class of shifted cocycles \[13\] as quantization of the classical dynamical $r$-matrices of semisimple Lie algebras in Schiffmann’s classification list \[41\].
As we will see in Section 7, Equation (2) arises naturally from the “twistor” equation of a quantum groupoid. This leads to another interpretation of an elliptic quantum group, namely as a quantum groupoid. Roughly speaking, our construction goes as follows. Instead of $U_\hbar g$, we start with the algebra $H = D \otimes U_\hbar g$, where $D$ denotes the algebra of meromorphic differential operators on $\eta^*$. $H$ is no longer a Hopf algebra. Instead it is a QUE algebroid considered as the Hopf algebroid tensor product of $D$ and $U_\hbar g$. Then the shifted cocycle condition is shown to be equivalent to the equation defining a twistor of this Hopf algebroid. Using this twistor, we obtain a new QUE algebroid $D \otimes_\hbar U_\hbar g$ (or a quantum groupoid). We note that $D \otimes_\hbar U_\hbar g$ is co-associative as a Hopf algebroid, although $(U_\hbar g, \Delta_\lambda)$ is not co-associative. The construction of $D \otimes_\hbar U_\hbar g$ is in some sense to restore the co-associativity by enlarging the algebra $U_\hbar g$ by tensoring the dynamical part $D$. The relation between this quantum groupoid and quasi-Hopf algebras $(U_\hbar g, \Delta_\lambda)$ is, in a certain sense, similar to that between a fiber bundle and its fibers. We expect that this quantum groupoid will be useful in understanding elliptic quantum groups, especially their representation theory [19]. The physical meaning of it, however, still needs to be explored.

This paper is organized as follows: In Section 2, we recall some basic definitions and results concerning Lie bialgebroids. Section 3 is devoted to the definition and basic properties of Hopf algebroids. In particular, for Hopf algebroids with anchor, it is proved that the category of left modules is a monoidal category. As a fundamental construction, in Section 4, we study the twist construction of Hopf algebroids, which generalizes the usual twist construction of Hopf algebras. Despite its complexity compared to Hopf algebras, the fundamentals are analogous to those of Hopf algebras. In particular, the monoidal categories of left modules of the twisted and untwisted Hopf algebroids are always equivalent. Section 5 is devoted to the introduction of quantum universal enveloping algebroids. The main part is to show that Lie bialgebroids indeed appear as the classical limit of QUE algebroids, as is expected. However, unlike the quantum group case, the proof is not trivial and is in fact rather involved. On the other hand, the inverse question: the quantization problem, which would encompass both quantization of Lie bialgebras and deformation quantization of Poisson manifolds as special cases, remains widely open. As a very special case, in Section 6, we show that any regular triangular Lie bialgebroid is quantizable. The discussion on quantum groupoids associated to quantum dynamical $R$-matrices (i.e. solutions to the quantum dynamical Yang-Baxter equation) occupies Section 7. The last section, Section 8, consists of an appendix, as well as a list of open questions.

We would like to mention the recent work by Etingof and Varchenko [15] [16], where a different approach to the quantum dynamical Yang-Baxter equation in the framework of Hopf algebroids was given.

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2 Preliminary on Lie bialgebroids

It is well known that the classical objects corresponding to quantum groups are Lie bialgebras. Therefore, it is not surprising to expect that the classical counterparts of quantum groupoids are Lie bialgebroids. However, unlike Lie bialgebras, Lie bialgebroids were introduced and studied before the invention of quantum groupoids. In fact, they were used mainly in the study of Poisson geometry in connection with symplectic and Poisson groupoids (see [44] [47]).

The purpose of this section is to recall some basic facts concerning Lie bialgebroids. We start with recalling some definitions.

Definition 2.1 A Lie algebroid is a vector bundle $A$ over $P$ together with a Lie algebra structure on the space $\Gamma(A)$ of smooth sections of $A$, and a bundle map $\rho: A \to TP$ (called the anchor), extended to a map between sections of these bundles, such that

(i) $\rho([X, Y]) = [\rho(X), \rho(Y)]$; and

(ii) $[X, fY] = f[X, Y] + (\rho(X)f)Y$

for any smooth sections $X$ and $Y$ of $A$ and any smooth function $f$ on $P$.

Among many examples of Lie algebroids are usual Lie algebras, the tangent bundle of a manifold, and an integrable distribution over a manifold (see [33]). Another interesting example is connected with Poisson manifolds. Let $P$ be a Poisson manifold with Poisson tensor $\pi$. Then $T^*P$ carries a natural Lie algebroid structure, called the cotangent bundle Lie algebroid of the Poisson manifold $P$ [7]. The anchor map $\pi^#: T^*P \to TP$ is defined by

$$\pi^#: T^*_pP \to T_pP: \pi^#(\xi)(\eta) = \pi(\xi, \eta), \quad \forall \xi, \eta \in T^*_pP$$

and the Lie bracket of 1-forms $\alpha$ and $\beta$ is given by

$$[\alpha, \beta] = L_{\pi^#(\alpha)}\beta - L_{\pi^#(\beta)}\alpha - d(\pi(\alpha, \beta)).$$

(5)

Given a Lie algebroid $A$, it is known that $\oplus_k \Gamma(\wedge^k A^*)$ admits a differential $d$ that makes it into a differential graded algebra [27]. Here, $d: \Gamma(\wedge^k A^*) \to \Gamma(\wedge^{k+1} A^*)$ is defined by ([33] [34] [46]):

$$d\omega(X_1, \ldots, X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1}\rho(X_i)(\omega(X_1, \ldots, X_{k+1})) + \sum_{i < j} (-1)^{i+j}\omega([X_i, X_j], X_1, \ldots, X_{k+1}),$$

(6)

for $\omega \in \Gamma(\wedge^k A^*), \ X_i \in \Gamma A, \ 1 \leq i \leq k + 1$. Then $d^2 = 0$ and one obtains a cochain complex whose cohomology is called the Lie algebroid cohomology. On the other hand, the Lie bracket on $\Gamma(A)$ extends naturally to a graded Lie bracket on $\oplus_k \Gamma(\wedge^k A)$ called the Schouten bracket, which, together with the usual wedge product $\wedge$, makes it into a Gerstenhaber algebra [51].

As in the case of Lie algebras, associated to any Lie algebroid, there is an associative algebra called the universal enveloping algebra of the Lie algebroid $A$ [23], a concept whose definition we now recall.
Let $A \to P$ be a Lie algebroid with anchor $\rho$. Then the $C^\infty(P)$-module direct sum $C^\infty(P) \oplus \Gamma(A)$ is a Lie algebra over $\mathbb{R}$ with the Lie bracket:

$$[f + X, g + Y] = (\rho(X)g - \rho(Y)f) + [X, Y].$$

Let $U = U(C^\infty(P) \oplus \Gamma(A))$ be its standard universal enveloping algebra. For any $f \in C^\infty(P)$ and $X \in \Gamma(A)$, denote by $f'$ and $X'$ their canonical image in $U$. Denote by $I$ the two-sided ideal of $U$ generated by all elements of the form $(fg)' - f'g'$ and $(fX)' - f'X'$. Define $U(A) = U/I$, which is called the universal enveloping algebra of the Lie algebroid $A$. When $A$ is a Lie algebra, this definition reduces to the definition of usual universal enveloping algebras. On the other hand, for the tangent bundle Lie algebroid $TP$, its universal enveloping algebra is $\mathcal{D}(P)$, the algebra of differential operators over $P$. In between, if $A = TP \times \mathfrak{g}$ as the Lie algebroid direct product, then $U(A)$ is isomorphic to $\mathcal{D}(P) \otimes U\mathfrak{g}$. Note that the maps $f \mapsto f'$ and $X \mapsto X'$ considered above descend to linear embeddings $i_1 : C^\infty(P) \hookrightarrow U(A)$, and $i_2 : \Gamma(A) \hookrightarrow U(A)$; the first map $i_1$ is an algebra morphism. These maps have the following properties:

$$i_1(f)i_2(X) = i_2(fX), \quad [i_2(X), i_1(f)] = i_1(\rho(X)f), \quad [i_2(X), i_2(Y)] = i_2([X, Y]). \quad (7)$$

In fact, $U(A)$ is universal among all triples $(B, \varphi_1, \varphi_2)$ having such properties (see [23] for a proof of this simple fact). Sometimes, it is also useful to think of $U\mathcal{A}$ as the algebra of left invariant differential operators on a local Lie groupoid $G$ which integrates the Lie algebroid $A$.

The notion of Lie bialgebroids is a natural generalization of that of Lie bialgebras. Roughly speaking, a Lie bialgebroid is a pair of Lie algebroids $(A, A^*)$ satisfying a certain compatibility condition. Such a condition, providing a definition of Lie bialgebroids, was given in [34]. We quote here an equivalent formulation from [26].

**Definition 2.2** A Lie bialgebroid is a dual pair $(A, A^*)$ of vector bundles equipped with Lie algebroid structures such that the differential $d_*$ on $\Gamma(\wedge^r A)$ coming from the structure on $A^*$ is a derivation of the Schouten bracket on $\Gamma(\wedge A)$. Equivalently, $d_*$ is a derivation for sections of $A$, i.e.,

$$d_*[X, Y] = [d_*X, Y] + [X, d_*Y], \quad \forall X, Y \in \Gamma(A). \quad (8)$$

In other words, $(\otimes \mathcal{G}(\wedge^k A), \wedge, [\cdot, \cdot], d_*)$ is a strong differential Gerstenhaber algebra [53].

In fact, a Lie bialgebroid is equivalent to a strong differential Gerstenhaber algebra structure on $\otimes \mathcal{G}(\wedge^k A)$ (see Proposition 2.3 in [52]).

For a Lie bialgebroid $(A, A^*)$, the base $P$ inherits a natural Poisson structure:

$$\{f, g\} = \langle df, d_*g \rangle, \quad \forall f, g \in C^\infty(P), \quad (9)$$

which satisfies the identity: $[df, dg] = d\{f, g\}$.

As in the case of Lie bialgebras, a useful method of constructing Lie bialgebroids is via $r$-matrices. More precisely, by an $r$-matrix, we mean a section $\Lambda \in \Gamma(\wedge^2 A)$ satisfying

$$L_X[\Lambda, A] = [X, [\Lambda, A]] = 0, \quad \forall X \in \Gamma(A). \quad (10)$$
An $r$-matrix $\Lambda$ defines a Lie bialgebroid, where the differential $d_* : \Gamma(\wedge^* A) \to \Gamma(\wedge^{*+1} A)$ is simply given by $d_* = [\cdot, \Lambda]$. In this case, the bracket on $\Gamma(A^*)$ is given by

$$[\xi, \eta] = L_{\Lambda^\#} \xi \eta - L_{\Lambda^\#} \eta \xi - d[\Lambda(\xi, \eta)],$$

and the anchor is the composition $\rho \circ \Lambda^\# : A^* \to TP$, where $\Lambda^\#$ denotes the bundle map $A^* \to A$ defined by $\Lambda^\#(\xi)(\eta) = \Lambda(\xi, \eta), \forall \xi, \eta \in \Gamma(A^*)$. Such a Lie bialgebroid is called a coboundary Lie bialgebroid. Here $\Lambda$ is a classical dynamical $r$-matrix, and $\coth(1 \cdot r)$ is a Cartan subalgebra, and $\Delta^+$ is the set of positive roots. Then $\Lambda^\#$ defines a coboundary Lie bialgebroid, in analogy with the Lie algebra case \[14\] \[31\]. It is called a triangular Lie bialgebroid if $[\Lambda, \Lambda] = 0$. In particular it is called a regular triangular Lie bialgebroid if $\Lambda$ is of constant rank.

When $P$ reduces to a point, i.e., $A$ is a Lie algebra, Equation (11) is equivalent to that $[\Lambda, \Lambda]$ is $ad$-invariant, i.e., $\Lambda$ is an $r$-matrix in the ordinary sense. On the other hand, when $A$ is the tangent bundle $TP$ with the standard Lie algebroid structure, Equation (11) is equivalent to that $[\Lambda, \Lambda] = 0$, i.e., $\Lambda$ is a Poisson tensor.

Another interesting class of coboundary Lie bialgebroids is connected with the so called classical dynamical $r$-matrices.

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$ (or $\mathbb{C}$) and $\eta \subset \mathfrak{g}$ an Abelian Lie subalgebra. A classical dynamical $r$-matrix \[2\] \[14\] is a smooth function (or meromorphic function in the complex case) $r : \eta^* \to \wedge^2 \mathfrak{g}$ such that\footnote{Throughout the paper, we follow the sign convention in \[2\] for the definition of a classical dynamical $r$-matrix in order to be consistent with the quantum dynamical Yang-Baxter equation \[1\]. This differs a sign from the one used in \[14\].}

(i). $r(\lambda)$ is $\eta$-invariant, i.e., $[h, r(\lambda)] = 0, \forall h \in \eta$;

(ii). $\mathrm{Alt}(dr) - \frac{1}{2}[r, r]$ is constant over $\eta^*$ with value in $(\wedge^3 \mathfrak{g})^\#$.

Here $dr$ is considered as a $\eta \otimes \wedge^2 \mathfrak{g}$-valued function over $\eta^*$ and $\mathrm{Alt}$ denotes the standard skew-symmetrization operator. In particular, if $\mathrm{Alt}(dr) - \frac{1}{2}[r, r] = 0$, it is called a classical triangular dynamical $r$-matrix. The following is a simple example of a classical dynamical $r$-matrix.

Example 2.1 Let $\mathfrak{g}$ be a simple Lie algebra with root decomposition $\mathfrak{g} = \eta \oplus \sum_{\alpha \in \Delta_+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$, where $\eta$ is a Cartan subalgebra, and $\Delta_+$ is the set of positive roots. Then

$$r(\lambda) = -\frac{1}{2} \sum_{\alpha \in \Delta_+} \coth\left(\frac{1}{2} \begin{smallmatrix} \ll \alpha, \lambda \gg \end{smallmatrix}\right) E_\alpha \wedge E_{-\alpha},$$

is a classical dynamical $r$-matrix, where $\ll, \gg$ is the Killing form of $\mathfrak{g}$, the $E_\alpha$ and $E_{-\alpha}$'s are standard root vectors, and $\coth(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}}$ is the hyperbolic cotangent function.

A classical dynamical $r$-matrix naturally gives rise to a Lie bialgebroid.

Proposition 2.3 \[3\] \[11\] Let $r : \eta^* \to \wedge^2 \mathfrak{g}$ be a classical dynamical $r$-matrix. Then $A = T\eta^* \times \mathfrak{g}$, equipped with the standard Lie algebroid structure, together with $\Lambda = \sum_{i=1}^k \left( \frac{r}{2} h_i \wedge h_i \right) + r(\lambda) \in \Gamma(\wedge^2 A)$ defines a coboundary Lie bialgebroid. Here $\{h_1, h_2, \ldots, h_k\}$ is a basis of $\eta$, and $(\lambda_1, \ldots, \lambda_k)$ is the induced coordinate system on $\eta^*$. 

We end this section by recalling the definition of Hamiltonian operators, which will be needed later on. Given a Lie bialgebroid \((A, A^*)\) with associated strong differential Gerstenhaber algebra \(\oplus_k \Gamma(\Lambda^k A), \wedge, [\cdot, \cdot], d_\ast\), one may construct a new Lie bialgebroid via a twist. For that, simply let \(d_\ast = d_\ast + [\cdot, H]\) for some \(H \in \Gamma(\wedge^2 A)\). It is easy to check that this still defines a strong differential Gerstenhaber algebra (therefore a Lie bialgebroid), if and only if the following Maurer-Cartan type equation holds:

\[
d_\ast H + \frac{1}{2}[H, H] = 0.
\]

(12)

Such an \(H\) is called a Hamiltonian operator of the Lie bialgebroid \((A, A^*)\).

Finally we note that even though we are mainly dealing with real vector bundles and real Lie algebroids in this paper, one may also consider (as already suggested by the early example of classical dynamical \(r\)-matrices) complex Lie algebroids and complex Lie bialgebroids. In that case, one may have to use sheaves of holomorphic sections etc. instead of global ones. Most results in this section still hold after suitable modifications.

3 Hopf algebroids

Definition 3.1 A Hopf algebroid \((H, R, \alpha, \beta, m, \Delta, \epsilon)\) consists of the following data:

1) a total algebra \(H\) with product \(m\), a base algebra \(R\), a source map: an algebra homomorphism \(\alpha : R \rightarrow H\), and a target map: an algebra anti-homomorphism \(\beta : R \rightarrow H\) such that the images of \(\alpha\) and \(\beta\) commute in \(H\), i.e., \(\forall a, b \in R, \alpha(a)\beta(b) = \beta(b)\alpha(a)\). There is then a natural \((R, R)\)-bimodule structure on \(H\) given by \(a \cdot h = \alpha(a)h\) and \(h \cdot a = \beta(a)h\). Thus, we can form the \((R, R)\)-bimodule product \(H \otimes_R H\). It is easy to see that \(H \otimes_R H\) again admits an \((R, R)\)-bimodule structure. This will allow us to form the triple product \(H \otimes_R H \otimes_R H\) and etc.

2) a co-product: an \((R, R)\)-bimodule map \(\Delta : H \rightarrow H \otimes_R H\) with \(\Delta(1) = 1 \otimes 1\) satisfying the co-associativity:

\[
(\Delta \otimes_R id_H)\Delta = (id_H \otimes_R \Delta)\Delta : H \rightarrow H \otimes_R H \otimes_R H;
\]

(13)

3) the product and the co-product are compatible in the following sense:

\[
\Delta(h)(\beta(a) \otimes 1 - 1 \otimes \alpha(a)) = 0, \quad \text{in } H \otimes_R H, \forall a \in R \text{ and } h \in H, \text{ and}
\]

(14)

\[
\Delta(h_1 h_2) = \Delta(h_1)\Delta(h_2), \quad \forall h_1, h_2 \in H, \quad \text{(see the remark below)};
\]

(15)

4) a co-unit map: an \((R, R)\)-bimodule map \(\epsilon : H \rightarrow R\) satisfying \(\epsilon(1_H) = 1_R\) (it follows then that \(\epsilon\beta = \epsilon\alpha = id_R\)) and

\[
(\epsilon \otimes_R id_H)\Delta = (id_H \otimes_R \epsilon)\Delta = id_H : H \rightarrow H.
\]

(16)

Here we have used the identification: \(R \otimes_R H \cong H \otimes_R R \cong H\) (note that both maps on the left hand sides of Equation (14) are well-defined).
Remark. It is clear that any left $H$-module is automatically an $(R, R)$-bimodule. Now given any left $H$-modules $M_1$ and $M_2$, define,

$$h \cdot (m_1 \otimes_R m_2) = \Delta(h)(m_1 \otimes m_2), \quad \forall h \in H, \ m_1 \in M_1, \ m_2 \in M_2.$$  \hfill (17)

The right-hand side is a well-defined element in $M_1 \otimes_R M_2$ due to Equation (14). In particular, when taking $M_1 = M_2 = H$, we see that the right-hand side of Equation (15) makes sense. In fact, Equation (15) implies that $M_1 \otimes_R M_2$ is again a left $H$-module under the action defined by Equation (17). Left $H$-modules are also called representations of the Hopf algebroid $H$ (as an associative algebra). The category of representations of $H$ is denoted by $\text{Rep}H$.

There is an equivalent version for the compatibility condition 3) due to Lu [32]:

**Proposition 3.2** The compatibility condition 3) (Equations (14) and (13)) is equivalent to that the kernel of the map

$$\Psi : H \otimes H \otimes H \longrightarrow H \otimes H : \sum h_1 \otimes h_2 \otimes h_3 \longmapsto \sum (\Delta(h_1)(h_2 \otimes h_3))$$  \hfill (18)

is a left ideal of $H \otimes H^{\text{op}} \otimes H^{\text{op}}$, where $H^{\text{op}}$ denotes $H$ with the opposite product.

**Proof.** Assume that $\text{Ker} \Psi$ is a left ideal. It is clear that for any $a \in H$, $1 \otimes \beta(a) \otimes 1 - 1 \otimes 1 \otimes \alpha(a) \in \text{Ker} \Psi$. Hence $h \otimes \beta(a) \otimes 1 - h \otimes 1 \otimes \alpha(a) = (h \otimes 1 \otimes 1)(1 \otimes \beta(a) \otimes 1 - 1 \otimes 1 \otimes \alpha(a))$ belongs to $\text{Ker} \Psi$. That is, $\Delta(h)(\beta(a) \otimes 1 - 1 \otimes \alpha(a)) = 0$. To prove Equation (15), we assume that $\Delta(h) = \sum_{ij} g_i \otimes_R g_j$ for some $g_i, g_j \in H$. Then $h_2 \otimes 1 - \sum_{ij} 1 \otimes g_i \otimes g_j \in \text{Ker} \Psi$. This implies that $h_1 h_2 \otimes 1 - \sum_{ij} h_1 \otimes g_i \otimes g_j \in \text{Ker} \Psi$ since $\text{Ker} \Psi$ is a left ideal. Hence $\Delta(h_1 h_2) - \sum_{ij} (\Delta(h_1)(g_i \otimes g_j)) = 0$ in $H \otimes_R H$. I.e., $\Delta(h_1 h_2) = \Delta(h_1) \Delta(h_2)$.

Conversely, assume that Equations (14) and (13) hold. Suppose that $\sum h_1 \otimes h_2 \otimes h_3 \in \text{Ker} \Psi$. Then for any $x, y, z \in H$, note that in $H \otimes H^{\text{op}} \otimes H^{\text{op}}$, we have $(x \otimes y \otimes z) \sum (h_1 \otimes h_2 \otimes h_3) = \sum x h_1 \otimes h_2 y \otimes h_3 z$. Then

$$\Psi((x \otimes y \otimes z) \sum (h_1 \otimes h_2 \otimes h_3)) = \sum \Delta(x h_1)(h_2 y \otimes h_3 z) = (\Delta x) \sum (\Delta(h_1)(h_2 \otimes h_3))(y \otimes z) = 0.$$  \hfill (19)

That is, $\text{Ker} \Psi$ is a left ideal.

\[ \square \]

**Remark.** (1) In [32], objects satisfying the above axioms are called bi-algebroids, while Hopf algebroids are referred to those admitting an antipode. However, here we relax the requirement of the existence of an antipode for Hopf algebroids, since many interesting examples, as shown below, often do not admit an antipode.

(2) In the classical case, the compatibility between the Poisson structure and the groupoid structure implies that the base manifold is a coisotropic submanifold of the Poisson groupoid [14].
For a Hopf algebroid, it would be natural to expect that the quantum analogue should hold as well, which means that the kernel of \( \epsilon \) is a left ideal of \( H \). However, we are not able to prove this at the moment (note that this extra condition was required in the definition in [32]).

In most situations, Hopf algebroids are equipped with an additional structure, called an anchor map. Let \((H, R, \alpha, \beta, m, \Delta, \epsilon)\) be a Hopf algebroid (over the ground field \( k \) of characteristic zero). By \( \text{End}_k R \), we denote the algebra of linear endmorphisms of \( R \) over \( k \). It is clear that \( \text{End}_k R \) is an \((R, R)\)-bimodule, where \( R \) acts on it from the left by left multiplication and acts from the right by right multiplication. Assume that \( R \) is a left \( H \)-module and moreover the representation \( \mu : H \rightarrow \text{End}_k R \) is an \((R, R)\)-bimodule map. For any \( x \in H \) and \( a \in R \), we denote by \( x(a) \) the element \( \mu(x)(a) \) in \( R \). Define

\[
\varphi_{\alpha}, \varphi_{\beta} : (H \otimes_R H) \otimes R \rightarrow H, \\
\varphi_{\alpha}(x \otimes_R y \otimes a) = x(a) \cdot y, \quad \text{and} \quad \varphi_{\beta}(x \otimes_R y \otimes a) = x \cdot y(a). 
\]

(19)

Here \( x, y \in H \), \( a \in R \), and the dot \( \cdot \) denotes the \((R, R)\)-bimodule structure on \( H \). Note that \( \varphi_{\alpha} \) and \( \varphi_{\beta} \) are well defined since \( \mu \) is an \((R, R)\)-bimodule map.

**Proposition 3.3** Under the above assumption, and moreover assume that \( \varphi_{\alpha}(\Delta x \otimes a) = x \alpha(a), \) and \( \varphi_{\beta}(\Delta x \otimes a) = x \beta(a), \) \( \forall x \in H, a \in R. \)

(20)

Then the map \( \tilde{\epsilon} : H \rightarrow R, \tilde{\epsilon} x = x(1_R) \), satisfies the co-unit property, i.e., it is an \((R, R)\)-bimodule map, \( \tilde{\epsilon}(1_H) = 1_R \), and satisfies Equation (16).

**Proof.** That \( \tilde{\epsilon} \) is an \((R, R)\)-bimodule map follows from the assumption that the representation \( \mu \) is an \((R, R)\)-bimodule map. It is clear that \( \tilde{\epsilon}(1_H) = 1_R \). To prove Equation (16), assume that \( \Delta x = \sum_i x_i^{(1)} \otimes_R x_i^{(2)} \). Then

\[
(\tilde{\epsilon} \otimes_R \text{id}_R) \Delta x = \sum \tilde{\epsilon}(x_i^{(1)}) \otimes_R x_i^{(2)} = \sum x_i^{(1)}(1_R) \otimes_R x_i^{(2)} = \varphi_{\alpha}(\Delta x \otimes 1_R) = x \alpha(1_R) = x.
\]

Similarly, we have \((\text{id}_R \otimes_R \tilde{\epsilon}) \Delta x = x.\)

\(\square\)

It is thus natural to expect that \( \tilde{\epsilon} \) coincides with the co-unit map.

**Definition 3.4** Given a Hopf algebroid \((H, R, \alpha, \beta, m, \Delta, \epsilon)\), an anchor map is a representation \( \mu : H \rightarrow \text{End}_k R \), which is an \((R, R)\)-bimodule map satisfying.
\( \varphi_\alpha(\Delta \otimes a) = x\alpha(a) \) and \( \varphi_\beta(\Delta \otimes a) = x\beta(a), \ \forall x \in H, a \in R; \)

(ii). \( x(1_R) = \epsilon x, \ \forall x \in H. \)

**Remark.** For a Hopf algebra, since \( R = k \) and \( \text{End}_k R \cong k \), one can simply take the counit as the anchor. In this case, the anchor is in fact equivalent to the counit map. However, for a Hopf algebroid, the existence of an anchor map is a stronger assumption than the existence of a counit. In fact, we can require axioms 1)-3) in Definition (3.1) together with the existence of an anchor map to define a Hopf algebroid with an anchor. Then the existence of the counit would be a direct consequence according to Proposition (3.3).

Given any \( x = \sum x_1 \otimes_R x_2 \cdots \otimes_R x_n \in \otimes^n_R H \) and \( k \) elements \( (k \leq n) a_{i_1}, a_{i_2}, \ldots, a_{i_k} \in R \), we denote by \( x(\cdots, a_{i_1}, \cdots, a_{i_k}, \cdots) \) the element \( \sum x_1 \otimes_R \cdots \otimes_R x_{i_1} \otimes_R \cdots \otimes_R x_{i_k} \otimes_R \cdots \otimes_R x_n \) in \( \otimes^{n-k}_R H \). The anchor map assumption guarantees that this is a well-defined element.

**Proposition 3.5** For any \( a, b \in R, x \in H \),

\[
\alpha(a)(b) = ab, \quad \beta(a)(b) = ba; \tag{21}
\]

\[
\Delta(x)(a, b) = x(ab); \tag{22}
\]

\[
\epsilon(xy) = x(\epsilon(y)). \tag{23}
\]

**Proof.** To prove Equation (21), we note that \( \alpha(a)(b) = (a \cdot 1_H)(b) = a(1_H(b)) = ab \), where we used the fact that \( \mu \) is an \( (R, R) \)-bimodule map. Similarly, we have \( \beta(a)(b) = ba \).

For Equation (22), we have

\[
\Delta(x)(a, b) = \varphi_\alpha(\Delta \otimes a)(b) \\
= (x\alpha(a))(b) \\
= x(\alpha(a)(b)) \\
= x(ab).
\]

Here the last step used Equation (21).

Finally, using (ii) in Definition (3.4), we have

\[
\epsilon(xy) = (xy)(1_R) = x(\epsilon(y)) = x(\epsilon(y)).
\]

\[\Box\]

**Remark.** Equation (21) implies that the induced \( (R, R) \)-bimodule structure on \( R \), where \( R \) is considered as a left \( H \)-module, coincides with the usual one by (left and right) multiplications. In fact, this condition is equivalent to requiring that \( \mu \) is an \( (R, R) \)-bimodule map in Definition (3.4).

Equation (22) simply means that \( R \otimes_R R \cong R \) as left \( H \)-modules. And the last equation, Equation (23), amounts to saying that \( \epsilon : H \to R \) is a module map as both \( H \) and \( R \) are considered as left \( H \)-modules, where \( H \) acts on \( R \) by left multiplication.

The following result follows immediately from the definitions.
Theorem 3.6 Let \((H, R, \alpha, \beta, m, \Delta, \epsilon)\) be a Hopf algebroid with anchor \(\mu\). Then the category \(\text{Rep}_H\) of left \(H\)-modules equipped with the tensor product \(\otimes_R\) as defined by Equation (17), the unit object \((R, \mu)\), and the trivial associativity isomorphisms: \((M_1 \otimes_R M_2) \otimes_R M_3 \rightarrow M_1 \otimes_R (M_2 \otimes_R M_3)\) is a monoidal category.

Example 3.1 Let \(D\) denote the algebra of all differential operators on a smooth manifold \(P\), and \(R\) the algebra of smooth functions on \(P\). Then \(D\) is a Hopf algebroid over \(R\). Here, \(\alpha = \beta\) is the embedding \(R \rightarrow D\), while the coproduct \(\Delta : D \rightarrow D \otimes_R D\) is defined by
\[
\Delta(D)(f, g) = D(fg), \quad \forall D \in D, \text{ and } f, g \in R.
\]
Note that \(D \otimes_R D\) is simply the space of bidifferential operators. Clearly, \(\Delta\) is co-commutative, i.e., \(\Delta^{op} = \Delta\). The usual action of differential operators on \(C^\infty(P)\) defines an anchor \(\mu : D \rightarrow \text{End}_k R\).

In this case, the co-unit \(\epsilon : D \rightarrow R\) is the natural projection to its 0th-order part of a differential operator. It is easy to see that left \(D\)-modules are \(D\)-modules in the usual sense, and the tensor product is the usual tensor product of \(D\)-modules over \(R\). We note, however, that this Hopf algebroid does not admit an antipode in any natural sense. Given a differential operator \(D\), its antipode, if it exists, would be the dual operator \(D^*\). However, the latter is a differential operator on 1-densities, which does not possess any canonical identification with a differential operator on \(R\).

The construction above can be generalized to show that the universal enveloping algebra \(UA\) of a Lie algebroid \(A\) admits a co-commutative Hopf algebroid structure.

Again we take \(R = C^\infty(P)\), and let \(\alpha = \beta : R \rightarrow UA\) be the natural embedding. For the co-product, we set
\[
\Delta(f) = f \otimes_R 1, \quad \forall f \in R;
\]
\[
\Delta(X) = X \otimes_R 1 + 1 \otimes_R X, \quad \forall X \in \Gamma(A).
\]
This formula extends to a co-product \(\Delta : UA \rightarrow UA \otimes_R UA\) by the compatibility condition: Equations (14) and (15). Alternatively, we may identify \(UA\) as the subalgebra of \(D(G)\) consisting of left invariant differential operators on a (local) Lie groupoid \(G\) integrating \(A\), and then restrict the co-product \(\Delta_G\) on \(D(G)\) to this subalgebra. This is well-defined since \(\Delta_G\) maps left invariant differential operators to left invariant bidifferential operators. Finally, the map \((\mu x)(f) = (\rho x)(f), \quad \forall x \in UA, f \in R\) defines an anchor, and the co-unit map is then the projection \(\epsilon : UA \rightarrow R\), where \(\rho : UA \rightarrow D(P)\) denotes the algebra homomorphism extending the anchor of the Lie algebroid (denoted by the same symbol \(\rho\)).

Theorem 3.7 \((UA, R, \alpha, \beta, m, \Delta, \epsilon)\) is a co-commutative Hopf algebroid with anchor \(\mu\).

4 Twist construction

As in the Hopf algebra case, the twist construction is an important method of producing examples of Hopf algebroids. This section is devoted to the study on this useful construction. We start with the following
Proposition 4.1 Let \((H, R, \alpha, \beta, m, \Delta, \epsilon)\) be a Hopf algebroid with anchor \(\mu\), and let \(\varphi\alpha\) and \(\varphi\beta\) be the maps defined by Equation (19). Then for any \(x, y, z \in H\) and \(a \in R\),

\[
\begin{align*}
\varphi\alpha((\Delta x)(y \otimes_R z) \otimes a) &= x\alpha(y(a))z; \quad (25) \\
\varphi\beta((\Delta x)(y \otimes_R z) \otimes a) &= x\beta(z(a))y. \quad (26)
\end{align*}
\]

**Proof.** Assume that \(\Delta x = \sum_i x_i^{(1)} \otimes_R x_i^{(2)}\). Then

\[
\begin{align*}
\varphi\alpha((\Delta x)(y \otimes_R z) \otimes a) &= \varphi\alpha(\sum_i (x_i^{(1)} y \otimes_R x_i^{(2)} z) \otimes a) \\
&= \sum_i \alpha((x_i^{(1)} y)(a))x_i^{(2)} z \\
&= \sum_i \alpha(x_i^{(1)} (y(a)))x_i^{(2)} z.
\end{align*}
\]

On the other hand, using (i) in Definition 3.4,

\[
\begin{align*}
x\alpha(y(a))z &= \varphi\alpha(\Delta x \otimes y(a))z \\
&= \varphi\alpha(\sum_i (x_i^{(1)} \otimes_R x_i^{(2)} \otimes y(a))z) \\
&= \sum_i \alpha(x_i^{(1)} (y(a)))x_i^{(2)} z.
\end{align*}
\]

Hence, \(\varphi\alpha((\Delta x)(y \otimes_R z) \otimes a) = x\alpha(y(a))z\). Equation (26) can be proved similarly.

\[\square\]

Now let \(F\) be an element in \(H \otimes_R H\). Define \(\alpha_F, \beta_F : R \rightarrow H\), respectively, by

\[
\alpha_F(a) = \varphi\alpha(F \otimes a), \quad \beta_F(a) = \varphi\beta(F \otimes a), \quad \forall a \in R.
\]

And for any \(a, b \in R\), set

\[
a * F b = \alpha_F(a)(b).
\]

More explicitly, if \(F = \sum_i x_i \otimes_R y_i\) for \(x_i, y_i \in H\), then \(\forall a, b \in R\),

\[
\begin{align*}
\alpha_F(a) &= \sum_i x_i(a) \cdot y_i = \sum_i \alpha(x_i(a))y_i; \quad (29) \\
\beta_F(a) &= \sum_i x_i \cdot y_i(a) = \sum_i \beta(y_i(a))x_i, \quad \text{and} \quad (30) \\
a * F b &= \sum_i x_i(a)y_i(b). \quad (31)
\end{align*}
\]

Proposition 4.2 Assume that \(F \in H \otimes_R H\) satisfies:

\[
(\Delta \otimes_R \text{id})F^{12} = (\text{id} \otimes_R \Delta)F^{23} \quad \text{in} \quad H \otimes_R H \otimes_R H; \quad \text{and} \quad (32)
\]

\[
(\epsilon \otimes_R \text{id})F = 1_H; \quad (\text{id} \otimes_R \epsilon)F = 1_H. \quad (33)
\]

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\[ F^{12} = F \otimes 1 \in (H \otimes_R H) \otimes H, \quad F^{23} = 1 \otimes F \in H \otimes (H \otimes_R H), \]
and in Equation (33) we have used the identification: \( R \otimes_R H \cong H \otimes_R R \cong H \) (note that both maps on the left hand sides of Equation (33) are well-defined). Then

(i). \((R, \ast_F)\) is an associative algebra, and \(1_R \ast_F a = a \ast_F 1_R = a, \ \forall a \in R\).

(ii). \(\alpha_F : R_F \rightarrow H\) is an algebra homomorphism, and \(\beta_F : R_F \rightarrow H\) is an algebra anti-homomorphism. Here \(R_F\) stands for the algebra \((R, \ast_F)\).

(iii). \((\alpha_F a)(\beta_F b) = (\beta_F b)(\alpha_F a), \ \forall a, b \in R\).

**Proof.** As a first step, we prove that for any \(a, b \in R\),

\[
\alpha_F(a \ast_F b) = (\alpha_F a)(\alpha_F b),
\]

\[
\beta_F(a \ast_F b) = (\beta_F b)(\beta_F a).
\]

Assume that \(F = \sum x_i \otimes_R y_i\) for \(x_i, y_i \in H\). Then

\[
(\Delta \otimes_R \text{id})F \cdot F^{12} = \sum_{ij} \Delta x_i(x_j \otimes_R y_j) \otimes_R y_i,
\]

\[
(id \otimes_R \Delta)F \cdot F^{23} = \sum_{ij} x_i \otimes_R \Delta y_i(x_j \otimes_R y_j).
\]

Thus

\[
[(\Delta \otimes_R \text{id})F \cdot F^{12}](a, b, \cdot) = \sum_{ij} \Delta x_i(x_j \otimes y_j)(a, b) \otimes_R y_i
\]

\[
= \sum_{ij} x_i(x_j(a)y_j(b)) \otimes_R y_i
\]

\[
= \sum_{ij} \alpha[x_i(x_j(a)y_j(b))], y_i
\]

\[
= \alpha_F(\sum_j x_j(a))y_j(b))
\]

\[
= \alpha_F(a \ast_F b),
\]

where the second equality used Equation (22).

On the other hand,

\[
[(id \otimes_R \Delta)F \cdot F^{23}](a, b, \cdot) = \sum_{ij} x_i(a) \otimes_R \Delta y_i(x_j \otimes_R y_j) \otimes b
\]

\[
= \sum_{ij} x_i(a) \otimes_R y_i \alpha(x_j(b))y_j
\]

\[
= \sum_{ij} \alpha(x_i(a))y_i \alpha(x_j(b))y_j
\]

\[
= (\alpha_F a)(\alpha_F b).
\]
Thus Equation (34) follows from Equation (32). The equation $\beta_F(a \ast_F b) = (\beta_F b)(\beta_F a)$ can be proved similarly.

Now for any $a, b, c \in R$, $[(\alpha_F a)(\alpha_F b)](c) = (\alpha_F a)((\alpha_F b)(c)) = a \ast_F (b \ast_F c)$. On the other hand, $\alpha_F(a \ast_F b)(c) = (a \ast_F b) \ast_F c$. The associativity of $R_F$ thus follows from Equation (34).

Finally, we have $\alpha_F(1_R) = \sum_i x_i(1_R) \cdot y_i = \sum_i \epsilon(x_i) \cdot y_i = (\epsilon \otimes_R id)F = 1_H$. Similarly, $\beta_F(1_R) = 1_H$. It thus follows that $1_R \ast_F a = \alpha_F(1_R)(a) = 1_H(a) = a$. Similarly, $a \ast_F 1_R = a$.

For the last statement, a similar computation leads to $[(\Delta \otimes_R id)FF^{12}](a, \cdot, b) = (\beta_F b)(\alpha_F a)$, and $[(id \otimes_R \Delta)FF^{23}](a, \cdot, b) = (\alpha_F a)(\beta_F b)$.

Thus (iii) follows immediately. This concludes the proof.

\[\square\]

**Proposition 4.3** Under the same hypotheses as in Proposition 4.2, we have

$$F(\beta_F(a) \otimes 1 - 1 \otimes \alpha_F(a)) = 0 \text{ in } H \otimes_R H, \quad \forall a \in R. \quad (38)$$

\[\text{Proof.} \]

\begin{align*}
F(\beta_F(a) \otimes 1 - 1 \otimes \alpha_F(a)) &= \left(\sum_i x_i \otimes_R y_i\right) \left(\sum_j \beta(y_j(a)) x_j \otimes 1 - \sum_j 1 \otimes \alpha(x_j(a)) y_j\right) \\
&= \sum_{ij} \left[x_i \beta(y_j(a)) x_j \otimes_R y_i - x_i \otimes_R y_i \alpha(x_j(a)) y_j\right].
\end{align*}

Now using Equations (36)-(37) and (25)-(26), we obtain

$$[(\Delta \otimes_R id)FF^{12}](\cdot, a, \cdot) = \sum_{ij} \varphi_{\beta}(\Delta x_i(x_j \otimes_R y_j) \otimes a) \otimes_R y_i$$

and

$$[(id \otimes_R \Delta)FF^{23}](\cdot, a, \cdot) = \sum_{ij} x_i \otimes_R \varphi_{\alpha}(\Delta y_i(x_j \otimes_R y_j) \otimes a)$$

Thus the conclusion follows immediately from Equation (32).
As an immediate consequence, we have

**Corollary 4.4** Let $M_1$ and $M_2$ be any left $H$-modules. Then

$$
F^# : M_1 \otimes_{R_F} M_2 \rightarrow M_1 \otimes_R M_2 \\
(m_1 \otimes_{R_F} m_2) \rightarrow F \cdot (m_1 \otimes m_2), \ m_1 \in M_1, \ and \ m_2 \in M_2,
$$

(39)

is a well defined linear map.

Note that $M_1 \otimes_R M_2$ is automatically an $(R, R)$-bimodule since both $M_1$ and $M_2$ are $(R, R)$-bimodules. Similarly, $M_1 \otimes_{R_F} M_2$ is an $(R_F, R_F)$-bimodule. Besides, $M_1 \otimes_R M_2$ is also a left $H$-module. The next lemma indicates how these module structures are related.

**Lemma 4.5** For any $a \in R$ and $m \in M_1 \otimes_{R_F} M_2$,

$$
F^# (a \cdot_F m) = \alpha_F(a) \cdot F^# (m); \quad (40)
$$

$$
F^# (m \cdot_F a) = \beta_F(a) \cdot F^# (m), \quad (41)
$$

where the dot on the right-hand side means the left $H$-action on $M_1 \otimes_R M_2$, and the dot $\cdot_F$ on the left-hand side refers to both the left and right $R_F$-actions on $M_1 \otimes_{R_F} M_2$.

**Proof.** For simplicity, let us assume that $m = m_1 \otimes_{R_F} m_2$ for $m_1 \in M_1$ and $m_2 \in M_2$. Then

$$
F^# (a \cdot_F (m_1 \otimes_{R_F} m_2)) = F^# ((\alpha_F(a)m_1) \otimes_{R_F} m_2) = F \cdot (\sum_i \alpha(x_i(a))y_i m_1 \otimes_{R_F} m_2) = \sum_{ij} x_j \alpha(x_i(a))y_i m_1 \otimes_{R_F} m_2 = (\Delta \otimes id) F^12 (a, \cdot\cdot) \cdot (m_1 \otimes m_2).
$$

On the other hand,

$$
\alpha_F(a) \cdot F^# (m_1 \otimes_{R_F} m_2) = \alpha_F(a) \cdot (\sum_j x_j m_1 \otimes_{R_Y} m_2) = \sum_j \Delta(\alpha_F(a))(x_j m_1 \otimes_{R_Y} m_2) = \sum_{ij} \Delta(\alpha(x_i(a))y_i)(x_j m_1 \otimes_{R_Y} m_2).
$$

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\[
= \sum_{ij}(x_i(a) \cdot \Delta y_i)(x_j m_1 \otimes_R y_j m_2)
= \sum_{ij}[(x_i(a) \otimes_R \Delta y_i(x_j \otimes_R y_j))] (m_1 \otimes m_2)
= (id \otimes_R \Delta) \mathcal{F} \mathcal{F}^{23}(a, \cdot, \cdot) \cdot (m_1 \otimes m_2).
\]

The conclusion thus follows again from Equation (32).

\[\blacksquare\]

We say that \( \mathcal{F} \) is invertible if \( \mathcal{F}^\# \) defined by Equation (39) is a vector space isomorphism for any left \( H \)-modules \( M_1 \) and \( M_2 \). In this case, in particular we can take \( M_1 = M_2 = H \) so that we have an isomorphism

\[ \mathcal{F}^\# : H \otimes_{R_F} H \rightarrow H \otimes_R H. \]  \hfill (42)

An immediate consequence of Lemma 4.5 is the following

**Corollary 4.6** If \( \mathcal{F} \) is invertible, then for any \( a \in R_F \) and \( n \in M_1 \otimes_R M_2 \),

\[ \mathcal{F}^{-1}(\alpha_F(a) \cdot n) = a \cdot \mathcal{F}^{-1}(n); \]

\[ \mathcal{F}^{-1}(\beta_F(a) \cdot n) = \mathcal{F}^{-1}(n) \cdot \mathcal{F} a. \]

**Definition 4.7** An element \( \mathcal{F} \in H \otimes_R H \) is called a twistor if it is invertible and satisfies Equations (32) and (33).

Now assume that \( \mathcal{F} \) is a twistor. Define a new coproduct \( \Delta_\mathcal{F} : H \rightarrow H \otimes_{R_F} H \) by

\[ \Delta_\mathcal{F} = \mathcal{F}^{-1} \Delta \mathcal{F}, \]  \hfill (43)

where Equation (43) means that \( \Delta_\mathcal{F}(x) = \mathcal{F}^{-1}(\Delta(x)\mathcal{F}), \forall x \in H \). In what follows, we will prove that \( \Delta_\mathcal{F} \) is indeed a Hopf algebroid co-product.

**Lemma 4.8** For any \( x \in H \) and \( a \in R_F \),

\[ \Delta(a \cdot_F x) = \alpha_F(a) \cdot \Delta x; \]  \hfill (44)

\[ \Delta(x \cdot_F a) = \beta_F(a) \cdot \Delta x, \]  \hfill (45)

where \( \cdot_F \) refers to both the left and the right \( R_F \)-actions on \( H \).

**Proof.** We have

\[ \Delta(a \cdot_F x) = \Delta(\alpha_F(a)x) \]
\[ = \Delta(\alpha_F(a)) \Delta x \]
\[ = \alpha_F(a) \cdot \Delta x. \]

Equation (43) can be proved similarly.
Proposition 4.9 \( \Delta_F : H \rightarrow H \otimes_{R_F} H \) is an \((R_F, R_F)\)-bimodule map.

Proof. For any \( a \in R_F \) and \( x \in H \), using Lemma 4.8 and Corollary 4.6, we have

\[
\Delta_F(a \cdot_F x) = F^\#(\Delta(a \cdot_F x)) = F^\#(\alpha_F(a) \cdot \Delta x) = a \cdot_F F^\#(\Delta x) = a \cdot_F \Delta_F(x).
\]

Similarly, we can show that \( \Delta_F(x \cdot_F a) = \Delta_F x \cdot_F a \).

\[\square\]

Proposition 4.10 The comultiplication \( \Delta_F : H \rightarrow H \otimes_{R_F} H \) is compatible with the multiplication in \( H \).

Proof. Consider the maps

\[
\Psi : H \otimes H \otimes H \rightarrow H \otimes_R H : \sum h_1 \otimes h_2 \otimes h_3 \mapsto \sum (\Delta h_1)(h_2 \otimes h_3), \quad (46)
\]

and

\[
\Psi_F : H \otimes H \otimes H \rightarrow H \otimes_{R_F} H : \sum h_1 \otimes h_2 \otimes h_3 \mapsto \sum (\Delta_F h_1)(h_2 \otimes h_3). \quad (47)
\]

We first prove that

\[
F^\# \circ \Psi_F = \Psi \circ F^\#_{23}. \quad (48)
\]

Equation (48) implies that \( \text{Ker} \Psi_F = \text{Ker} \Psi \). To see this, note that \( x \in \text{Ker} \Psi_F \), i.e., \( \Psi_F(x) = 0 \), is equivalent to \( (F^\# \circ \Psi_F)(x) = 0 \), which is equivalent to \( (\Psi \circ F^\#_{23})(x) = 0 \), or \( F^\#_{23}(x) \in \text{Ker} \Psi \). Since \( \text{Ker} \Psi \) is a left ideal in \( H \otimes H^{\text{op}} \otimes H^{\text{op}} \), the latter is equivalent to the fact that \( x \in \text{Ker} \Psi \). The final conclusion thus follows from Proposition 3.2.

\[\square\]
Proposition 4.10 implies that $M_1 \otimes_R F M_2$ is again a left $H$-module for any left $H$-modules $M_1, M_2$. Moreover, it is easy to see that $\mathcal{F}^\#: M_1 \otimes_R M_2 \rightarrow M_1 \otimes_R M_2$ is an isomorphism of left $H$-modules. Now we are ready to prove that $\Delta_\mathcal{F}$ is coassociative.

**Proposition 4.11** $\Delta_\mathcal{F}: H \rightarrow H \otimes_R H$ is coassociative.

**Proof.** Assume that $M_1, M_2, M_3$ are any left $H$-modules. It suffices to prove that the natural identification $\varphi_\mathcal{F}: (M_1 \otimes_R M_2) \otimes_R M_3 \rightarrow M_1 \otimes_R (M_2 \otimes_R M_3)$ is an isomorphism of left $H$-modules.

Consider the following diagram:

\[
\begin{array}{ccc}
(M_1 \otimes_R M_2) \otimes_R M_3 & \xrightarrow{(\mathcal{F}^{1\otimes 2})^\#} & (M_1 \otimes_R M_2) \otimes_R M_3 \\
\varphi_\mathcal{F} & & \downarrow \varphi \\
M_1 \otimes_R (M_2 \otimes_R M_3) & \xrightarrow{(\mathcal{F}^{2\otimes 1})^\#} & M_1 \otimes_R (M_2 \otimes_R M_3)
\end{array}
\]

Equation (32) implies that the above diagram commutes. Since all the other maps involved in the diagram above are isomorphisms of left $H$-modules, $\varphi_\mathcal{F}$ is an $H$-module isomorphism as well. This concludes the proof.

By now, we have actually proved all the Hopf algebroid axioms for $(H, R_\mathcal{F}, \alpha_\mathcal{F}, \beta_\mathcal{F}, m, \Delta_\mathcal{F}, \epsilon)$ except for the condition on counit $\epsilon$. Instead of proving this last condition directly, here we show that $\mu$ is still an anchor after the twist, and therefore Axiom 4) in Definition 3.4 would be a consequence according to the remark following Definition 3.4. Note that $R_\mathcal{F}$ can still be considered as a left $H$-module under the representation $\mu : H \rightarrow \text{End} R_\mathcal{F}$ (here only the underlying vector space structure on $R_\mathcal{F}$ is involved). We prove that $\mu$ still satisfies the anchor axioms.

**Lemma 4.12** For any $x, y \in H$ and $a \in R$,

\[
\begin{align*}
\varphi_\mathcal{F}^\alpha((x \otimes_R y) \otimes a) &= \varphi_\alpha(\mathcal{F}^\#(x \otimes_R y) \otimes a); \\
\varphi_\mathcal{F}^\beta((x \otimes_R y) \otimes a) &= \varphi_\beta(\mathcal{F}^\#(x \otimes_R y) \otimes a).
\end{align*}
\]

**Proof.**

\[
\begin{align*}
\varphi_\mathcal{F}^\alpha((x \otimes_R y) \otimes a) &= x(a) \cdot_\mathcal{F} y \\
&= \alpha_\mathcal{F}(x(a))y \\
&= \sum_i \alpha(x_i(x(a)))y_i y \\
&= \sum_i \alpha(x_i x(a))y_i y.
\end{align*}
\]
On the other hand,

\[
\varphi_{\alpha}(\mathcal{F}^\#(x \otimes_R y) \otimes a) \\
= \varphi_{\alpha}(\sum_i x_i \otimes_R y_i \otimes a) \\
= \sum_i \alpha(x_i(a)) y_i.
\]

Hence, \(\varphi_{\alpha}^\mathcal{F}((x \otimes_R y) \otimes a) = \varphi_{\alpha}(\mathcal{F}^\#(x \otimes_R y) \otimes a)\). Similarly, one can prove that \(\varphi_{\beta}^\mathcal{F}((x \otimes_R y) \otimes a) = \varphi_{\beta}(\mathcal{F}^\#(x \otimes_R y) \otimes a)\).

\[\square\]

**Proposition 4.13** The map \(\mu : H \to \operatorname{End}_R F\) satisfies the anchor axioms in Definition 3.4 for \((H, R, \alpha, \beta, m, \Delta, \epsilon)\).

**Proof.** First we need to show that \(\mu\) is an \((R, R)\)-bimodule map. This can be checked easily since \((a \cdot_F x)(b) = (\alpha_F(a))x(b) = a \ast_F x(b)\). Similarly, \((x \cdot_F a)(b) = x(b) \ast_F a\).

Axiom (ii) in Definition 3.4 holds automatically since \(\mu\) is an anchor for \((H, R, \alpha, \beta, m, \Delta, \epsilon)\).

Now according to Lemma 4.12

\[
\varphi_{\alpha}^\mathcal{F}(\Delta_F x \otimes a) \\
= \varphi_{\alpha}(\Delta x \mathcal{F} \otimes a) \\
= \sum_i \varphi_{\alpha}(\Delta x_i \otimes_R y_i \otimes a) \\
= \sum_i x \alpha(x_i(a)) y_i \\
= x \alpha_F(a).
\]

Here the second from the last equality used Equation (23). Similarly, \(\varphi_{\beta}^\mathcal{F}(\Delta_F x \otimes a) = x \beta_F(a)\). This concludes the proof.

\[\square\]

In summary, we have proved

**Theorem 4.14** Assume that \((H, R, \alpha, \beta, m, \Delta, \epsilon)\) is a Hopf algebroid with anchor \(\mu\), and \(\mathcal{F} \in H \otimes_R H\) a twistor. Then \((H, R, \alpha, \beta, m, \Delta, \epsilon)\) is a Hopf algebroid, which still admits \(\mu\) as an anchor. Moreover, its corresponding monoidal category of left \(H\)-modules is equivalent to that of \((H, R, \alpha, \beta, m, \Delta, \epsilon)\).

We say that \((H, R, \alpha, \beta, m, \Delta, \epsilon)\) is obtained from \((H, R, \alpha, \beta, m, \Delta, \epsilon)\) by twisting via \(\mathcal{F}\).

The following theorem generalizes a standard result in Hopf algebras [3].
Theorem 4.15 If $F_1 \in H \otimes_R H$ is a twistor for the Hopf algebroid $H$, and $F_2 \in H \otimes_{R,F_1} H$ a twistor for the twisted Hopf algebroid $H_{F_1}$, then the Hopf algebroid obtained by twisting $H$ via $F_1$ then via $F_2$ is equivalent to that obtained by twisting via $F_1 F_2$. Here $F_1 F_2 \in H \otimes_R H$ is understood as $F_1^#(F_2)$, where $F_1^#: H \otimes_{R,F_1} H \rightarrow H \otimes_R H$ is the map as defined in Equation (42).

Proof. Clearly, $F = F_1 F_2 = F_1^#(F_2)$ is a well defined element in $H \otimes_R H$. We only need to verify that $F$ is still a twistor. The rest of the theorem follows from a routine verification. For this purpose, it suffices to show that $F$ satisfies both Equation (32) and Equation (33). To check that, in fact we may think of $F_1$ and $F_2$ as elements in $H \otimes_{R,F_1} H$ by taking some representatives. Then

\[
(\Delta \otimes_R \text{id}) FF^{12} = (\Delta \otimes_R \text{id}) F_1 F_2^{12} F_2^{12} = (\Delta \otimes_R \text{id}) F_1 F_2^{12} [(\Delta \otimes_R \text{id}) F_2 F_1^{12}] F_2^{12} = [(\Delta \otimes_R \text{id}) F_1 F_2^{12}] [(\Delta \otimes_R \text{id}) F_2 F_2^{12}] = (id \otimes_R \Delta) F_1 (id \otimes_R \Delta) F_2 F_2^{23} = (id \otimes_R \Delta) F F^{23}.
\]

To prove Equation (33), assume that $F_1 = \sum_i x_i^{(1)} \otimes_R y_i^{(1)}$, and $F_2 = \sum_i x_i^{(2)} \otimes_{R,F_1} y_i^{(2)}$. Then $F_1 F_2 = \sum_{ij} x_i^{(1)} x_j^{(2)} \otimes_R y_i^{(1)} y_j^{(2)}$. And

\[
(\epsilon \otimes_R \text{id}) F = \sum_{ij} \epsilon(x_i^{(1)} x_j^{(2)}) \otimes_R y_i^{(1)} y_j^{(2)} = \sum_{ij} \epsilon(x_i^{(1)} x_j^{(2)}) y_i^{(1)} y_j^{(2)} \quad \text{(using Equation (23))}
\]

\[
= \sum_{ij} x_i^{(1)} (\epsilon x_j^{(2)}) y_i^{(1)} y_j^{(2)} = \sum_{ij} \varphi_a(x_i^{(1)} \otimes_R y_i^{(1)} \otimes \epsilon x_j^{(2)}) y_j^{(2)} = \sum_j \varphi_a(F_1 \otimes x_j^{(2)}) y_j^{(2)} = \sum_j \alpha_{F_1}(x_j^{(2)}) y_j^{(2)} = (\epsilon \otimes_{R,F_1} \text{id}) F_2 = 1_H.
\]

Similarly, we prove that $(id \otimes_R \epsilon) F = 1_H$.

\[\square\]

We end this section by the following:
Example 4.1 Let \( P \) be a smooth manifold, \( D \) the algebra of differential operators on \( P \), and \( R = C^\infty(P) \). Let \( D[[\hbar]] \) denote the space of formal power series in \( \hbar \) with coefficients in \( D \). The Hopf algebroid structure on \( D \) naturally extends to a Hopf algebroid structure on \( D[[\hbar]] \) over the base algebra \( R[[\hbar]] \), which admits a natural anchor map.

Let \( F = 1 \otimes_{R} 1 \) be a formal power series in bidifferential operators. It is easy to see that \( F \) is a twistor iff the multiplication on \( R[[\hbar]] \) defined by:

\[
f \ast \hbar g = F(f, g), \quad \forall f, g \in R[[\hbar]]
\]

is associative with identity being the constant function 1, i.e., \( \ast \hbar \) is a star product on \( P \). In this case, the bracket \( \{ f, g \} = B_1(f, g) - B_1(g, f) \), \( \forall f, g \in C^\infty(P) \), defines a Poisson structure on \( P \), and \( f \ast \hbar g = F(f, g) \) is simply a deformation quantization of this Poisson structure [6].

The twisted Hopf algebroid can be easily described. Here \( D_\hbar = D[[\hbar]] \) is equipped with the usual multiplication, \( R_\hbar = R[[\hbar]] \) is the \( \ast \)-product defined by Equation (51), \( \alpha_\hbar : R_\hbar \to D_\hbar \) and \( \beta_\hbar : R_\hbar \to D_\hbar \) are given, respectively, by

\[
\alpha_\hbar(f)g = f \ast \hbar g, \quad \beta_\hbar(f)g = g \ast \hbar f, \quad \forall f, g \in R.
\]

The co-product \( \Delta_\hbar : D_\hbar \to D_\hbar \otimes_{R_\hbar} D_\hbar \) is

\[
\Delta_\hbar = F^{-1} \Delta F,
\]

and the co-unit \( \epsilon \) remains the same, i.e., the projection \( D_\hbar \to R_\hbar \). This twisted Hopf algebroid \((D_\hbar, R_\hbar, \alpha_\hbar, \beta_\hbar, m_\hbar, \Delta_\hbar, \epsilon_\hbar)\) is called the quantum groupoid associated to the star product \( \ast_\hbar \) [49].

5 Quantum groupoids and their classical limits

The main purpose of this section is to introduce quantum universal enveloping algebroids (QUE algebroids), also called quantum groupoids in the paper, as a deformation of the standard Hopf algebroid \( UA \).

Definition 5.1 A deformation of a Hopf algebroid \((H, R, \alpha, \beta, m, \Delta, \epsilon)\) over a field \( k \) is a topological Hopf algebroid \((H_\hbar, R_\hbar, \alpha_\hbar, \beta_\hbar, m_\hbar, \Delta_\hbar, \epsilon_\hbar)\) over the ring \( k[[\hbar]] \) of formal power series in \( \hbar \) such that

(i). \( H_\hbar \) is isomorphic to \( H[[\hbar]] \) as \( k[[\hbar]] \) module with identity \( 1_H \), and \( R_\hbar \) is isomorphic to \( R[[\hbar]] \) as \( k[[\hbar]] \) module with identity \( 1_R \);

(ii). \( \alpha_\hbar = \alpha(\text{mod } \hbar), \quad \beta_\hbar = \beta(\text{mod } \hbar), \quad m_\hbar = m(\text{mod } \hbar), \quad \epsilon_\hbar = \epsilon(\text{mod } \hbar) \);

(iii). \( \Delta_\hbar = \Delta(\text{mod } \hbar) \).

In this case, we simply say that the quotient \( H_\hbar/\hbar H_\hbar \) is isomorphic to \( H \) as a Hopf algebroid.

Here the meaning of (i) and (ii) is quite clear. However, for Condition (iii), we need the following simple fact:
Lemma 5.2 Under the hypotheses (i) and (ii) as in Definition 5.1, $H \otimes_R H/\hbar (H \otimes_R H) \cong H \otimes_R H$ as a $k$-module.

Proof. Define $\tau : H \otimes_R H \to H \otimes_R H$ by

$$\left( \sum_i x_i \hbar^i \right) \otimes \left( \sum_i y_i \hbar^i \right) \to x_0 \otimes_R y_0.$$  

For any $a \in R$ and $x, y \in H$, since

$$(\beta a \otimes 1 - 1 \otimes \alpha a)(x \otimes y) = (\beta a)x \otimes y - x \otimes (\alpha a)y + O(\hbar),$$

then $\tau[(\beta a \otimes 1 - 1 \otimes \alpha a)(x \otimes y)] = 0$. In other words, $\tau$ descends to a well defined map from $H \otimes_R H$ to $H \otimes_R H$. It is easy to see that $\tau$ is surjective and $\ker \tau = \hbar (H \otimes_R H)$. The conclusion thus follows immediately.

By abuse of notation, we still use $\tau$ to denote the induced map $H \otimes_R H \to H \otimes_R H$. We shall also use the notation $\hbar \mapsto 0$ to denote this map whenever the meaning is clear from the context.

Then, Condition (iii) means that $\lim_{\hbar \mapsto 0} \Delta_1(h(x)) = \Delta(x)$ for any $x \in H$.

Definition 5.3 A quantum universal enveloping algebroid (or QUE algebroid), also called a quantum groupoid, is a deformation of the standard Hopf algebroid $(UA, R, \alpha, \beta, m, \Delta, \epsilon)$ of a Lie algebroid $A$.

Let $U_\hbar A = UA[[\hbar]]$ and $R_\hbar = R[[\hbar]]$. Assume that $(U_\hbar A, R_\hbar, \alpha_\hbar, \beta_\hbar, m_\hbar, \Delta_\hbar, \epsilon_\hbar)$ is a quantum groupoid. Then $R_\hbar$ defines a star product on $P$ so that the equation

$$\{f, g\} = \lim_{\hbar \mapsto 0} \frac{1}{\hbar}(f * \hbar g - g * \hbar f), \quad \forall f, g \in R$$

defines a Poisson structure on the base space $P$.

Now define

$$\delta f = \lim_{\hbar \mapsto 0} \frac{1}{\hbar}(\alpha_\hbar f - \beta_\hbar f) \in UA, \quad \forall f \in R,$$

$$\Delta^1 X = \lim_{\hbar \mapsto 0} \frac{1}{\hbar}(\Delta_\hbar X - (1 \otimes R_\hbar X + X \otimes R_\hbar 1)) \in UA \otimes R UA, \quad \forall X \in \Gamma(A), \quad \text{and}$$

$$\delta X = \Delta^1 X - (\Delta^1 X)_{21} \in UA \otimes R UA.$$  

Here for $T = \sum x \otimes_R y \in UA \otimes R UA$, $T_{21} = \sum y \otimes_R x$. For the convenience of notations, we introduce

$$\text{Alt}T = T - T_{21}, \quad \forall T \in UA \otimes R UA$$

so that $\delta X = \text{Alt}\Delta^1 X.$
Below we will use $\star_h$ to denote both the multiplication in $U_h A$ and that in $R_h$ provided there is no confusion. For any $f, g \in R$, $x, y \in U A$, write

\[
\begin{align*}
\alpha_h f &= f + h \alpha_1 f + h^2 \alpha_2 f + O(h^3); \\
\beta_h f &= f + h \beta_1 f + h^2 \beta_2 f + O(h^3); \\
f \star_h g &= fg + hB_1(f, g) + O(h^2); \\
x \star_h y &= xy + hm_1(x, y) + O(h^2),
\end{align*}
\]

where $\alpha_1 f, \beta_1 f, \alpha_2 f, \beta_2 f$ and $m_1(x, y)$ are elements in $U A$. Hence,

\[
\{f, g\} = B_1(f, g) - B_1(g, f), \quad \text{and} \quad \delta f = \alpha_1 f - \beta_1 f.
\]

**Lemma 5.4** For any $f, g \in R$,

(i). $\alpha_1(fg) = g(\alpha_1 f) + f(\alpha_1 g) + [\alpha_1 f, g] + m_1(f, g) - B_1(f, g);$ 
(ii). $\beta_1(fg) = g(\beta_1 f) + f(\beta_1 g) + [\beta_1 f, g] + m_1(f, g) - B_1(g, f);$ 
(iii). $[\alpha_1 f, g] - [\beta_1 g, f] = m_1(g, f) - m_1(f, g).$

**Proof.** From the identity $\alpha_h(f \star_h g) = \alpha_h f \star_h \alpha_h g$, it follows, by considering the $h^1$-terms, that

\[
\alpha_1(fg) + B_1(f, g) = m_1(f, g) + (\alpha_1 f)g + f(\alpha_1 g).
\]

Thus (i) follows immediately. And (ii) can be proved similarly.

On the other hand, we know, from the definition of Hopf algebroids, that $(\alpha_h f) \star_h (\beta_h g) = (\beta_h g) \star_h (\alpha_h f)$. By considering the $h^1$-terms, we obtain

\[
(\alpha_1 f)g + f(\beta_1 g) + m_1(f, g) = g(\alpha_1 f) + (\beta_1 g)f + m_1(g, f).
\]

This proves (iii).

\[\square\]

**Corollary 5.5** For any $f, g \in R$,

(i). $\delta(fg) = f\delta g + g\delta f;$

(ii). $[\delta f, g] = \{f, g\}.$

**Proof.** By symmetrizing the third identity in Lemma 5.4, we obtain that

\[
[\alpha_1 f - \beta_1 f, g] - [\beta_1 g - \alpha_1 g, f] = 0,
\]

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i.e.,

\[ [\delta f, g] = -[\delta g, f]. \]

Now subtracting Equation (ii) from Equation (i) in Lemma 5.4, one obtains that

\[ \delta(fg) = g\delta f + f\delta g + [\delta f, g] - \{f, g\}. \]

I.e.,

\[ \delta(fg) - (g\delta f + f\delta g) = [\delta f, g] - \{f, g\}. \]

Note that the left-hand side of this equation is symmetric with respect to \( f \) and \( g \), whereas the right-hand side is skew-symmetric, so both sides must vanish. The conclusion thus follows immediately.

\[ \square \]

**Lemma 5.6** For any \( f \in R \),

(i). \( \Delta_h f = f \otimes R_h 1 + h(\alpha_1 f \otimes R_h 1 - \Delta_h \alpha_1 f) + h^2(\alpha_2 f \otimes R_h 1 - \Delta_h \alpha_2 f) + O(h^3); \)

(ii). \( \Delta_h f = 1 \otimes R_h f + h(1 \otimes R_h \beta_1 f - \Delta_h \beta_1 f) + h^2(1 \otimes R_h \beta_2 f - \Delta_h \beta_2 f) + O(h^3); \)

(iii). \( f \otimes R_h 1 - 1 \otimes R_h f = h(1 \otimes R_h \alpha_1 f - \beta_1 f \otimes R_h 1) + h^2(1 \otimes R_h \alpha_2 f - \beta_2 f \otimes R_h 1) + O(h^3). \)

**Proof.** Since \( \Delta_h : U_h A \to U_h A \otimes R_h U_h A \) is an \((R_h, R_h)\)-bimodule map, it follows that

\[ \Delta_h (f \cdot_h 1) = f \cdot_h \Delta_h 1. \]

Here, as well as in the sequel, \( \cdot_h \) denotes both the left and the right \( R_h \)-actions on \( U_h A \), and on its appropriate tensor powers.

Now

\[ f \cdot_h 1 = \alpha_h f = f + h\alpha_1 f + h^2 \alpha_2 f + O(h^3), \]

while

\[ f \cdot_h \Delta_h 1 = f \cdot_h (1 \otimes R_h 1) = \alpha_h f \otimes R_h 1 = (f + h\alpha_1 f + h^2 \alpha_2 f) \otimes R_h 1 + O(h^3). \]

Thus it follows that

\[ \Delta_h f = f \otimes R_h 1 + h(\alpha_1 f \otimes R_h 1 - \Delta_h \alpha_1 f) + h^2(\alpha_2 f \otimes R_h 1 - \Delta_h \alpha_2 f) + O(h^3). \]

Similarly, one can prove (ii).

Finally, since \( 1 \otimes R_h \alpha_h f = \beta_h f \otimes R_h 1 \), we have

\[ 1 \otimes R_h (f + h\alpha_1 f + h^2 \alpha_2 f + O(h^3)) = (f + h\beta_1 f + h^2 \beta_2 f + O(h^3)) \otimes R_h 1. \]

This implies (iii).

\[ \square \]
Corollary 5.7 For any \( f, g \in R \),

(i) \( \Delta(\delta f) = \delta f \otimes R 1 + 1 \otimes R \delta f \); 
(ii) \( \Delta^1(\delta f) = \nu_2 f \otimes R 1 + 1 \otimes_R \nu_2 f - \Delta \nu_2 f \), where \( \nu_2 f = \alpha_2 f - \beta_2 f \).

Proof. Combining the three identities in Lemma 5.6 ((i)-(ii)+(iii)), we obtain that

\[
\delta f \otimes R 1 + 1 \otimes_R \delta f - \Delta \nu_2 f + \nu_2 f \otimes R 1 + 1 \otimes_R \nu_2 f - \Delta \nu_2 f + O(h^2) = 0.
\]

I.e.,

\[
\delta f \otimes R 1 + 1 \otimes_R \delta f - \Delta \nu_2 f + \nu_2 f \otimes R 1 + 1 \otimes_R \nu_2 f - \Delta \nu_2 f + O(h^2) = 0. 
\] (52)

By letting \( h \to 0 \), this implies that \( \delta f \otimes R 1 + 1 \otimes_R \delta f - \Delta \nu_2 f \) equals zero. This concludes the proof of (i).

Now writing \( \Delta_1 \nu_2 f = \nu_2 f \otimes R 1 + 1 \otimes_R \nu_2 f - \Delta \nu_2 f \), and substituting it back into Equation (52), we obtain that

\[
\Delta^1_1 \nu_2 f = \nu_2 f \otimes R 1 + 1 \otimes_R \nu_2 f - \Delta \nu_2 f + O(h).
\]

(ii) thus follows immediately by letting \( h \to 0 \). □

An immediate consequence is

Corollary 5.8 For any \( f \in R \), \( \delta f \in \Gamma(A) \) and \( \delta^2 f = 0 \).

Proof. From Corollary 5.7 (i), it follows that \( \delta f \) is primitive, i.e., \( \delta f \in \Gamma(A) \). According to Corollary 5.7 (ii), \( \Delta^1 \nu_2 f \) is symmetric. Therefore, \( \delta^2 f = \delta(\delta f) \), being the skew-symmetric part of \( \Delta^1 \nu_2 f \), equals zero. □

Lemma 5.9 For any \( X \in \Gamma(A) \),

\[
\Delta^1 X \otimes R 1 + (\Delta \otimes_R id) \Delta^1 X = 1 \otimes_R \Delta^1 X + (id \otimes_R \Delta) \Delta^1 X. 
\] (53)

Proof. For any \( X \in \Gamma(A) \), denote

\[
\Delta^1_h X = \frac{1}{h} [\Delta_h X - (1 \otimes_R X + X \otimes_R 1)].
\]

Thus \( \Delta^1 X = \lim_{h \to 0} \Delta^1_h X \) and

\[
\Delta_h X = 1 \otimes_R X + X \otimes_R 1 + h \Delta^1_h X.
\]
Then
\[(\Delta_h \otimes_{R_h} \text{id}) \Delta_h X = 1 \otimes_{R_h} 1 \otimes_{R_h} X + \Delta_h X \otimes_{R_h} 1 + \hbar (\Delta_h \otimes_{R_h} \text{id}) \Delta_h^1 X\]
and
\[(id \otimes_{R_h} \Delta_h) \Delta_h X = 1 \otimes_{R_h} \Delta_h X \otimes_{R_h} 1 + \hbar (id \otimes_{R_h} \Delta_h) \Delta_h^1 X\]

It thus follows that
\[\Delta_h^1 X \otimes_{R_h} 1 + (\Delta_h \otimes_{R_h} \text{id}) \Delta_h^1 X = 1 \otimes_{R_h} \Delta_h^1 X + (id \otimes_{R_h} \Delta_h) \Delta_h^1 X.\]  
(54)

The conclusion thus follows immediately by letting \(\hbar \to 0\).
\[\square\]

According to Proposition 8.1 in the Appendix, we immediately have the following

Corollary 5.10 For any \(X \in \Gamma(A)\), \(\delta X \in \Gamma(\land^2 A)\).

Lemma 5.11 For any \(f \in R\) and \(X \in \Gamma(A)\),
\[\delta(fX) = f \delta X + \delta f \land X.\]

Proof. For any \(X \in \Gamma(A)\), again we let
\[\Delta_h^1 X = \frac{1}{\hbar} \left[ \Delta_h X - (1 \otimes_{R_h} X + X \otimes_{R_h} 1) \right].\]

Now
\[f \cdot_h X = \alpha_h f \ast_h X = fX + \hbar [(\alpha_1 f)X + m_1(f, X)] + O(\hbar^2).\]

Hence
\[\Delta_h(f \cdot_h X) = \Delta_h(fX) + \hbar [\Delta_h((\alpha_1 f)X) + \Delta_h m_1(f, X)] + O(\hbar^2) = 1 \otimes_{R_h} fX + fX \otimes_{R_h} 1 + \hbar [\Delta_h^1(fX) + \Delta_h((\alpha_1 f)X) + \Delta_h m_1(f, X)] + O(\hbar^2).\]  
(55)

On the other hand,
\[f \cdot_h \Delta_h X = f \cdot_h (X \otimes_{R_h} 1 + 1 \otimes_{R_h} X + \hbar \Delta_h^1 X) = [fX + \hbar((\alpha_1 f)X + m_1(f, X))] \otimes_{R_h} 1 + (f + \hbar \alpha_1 f) \otimes_{R_h} X + \hbar f \cdot_h \Delta_h^1 X + O(\hbar^2).\]  
(56)
From Equations (55) and (56), it follows that

\[
1 \otimes_R \hbar f X - f \otimes_R \hbar X = \hbar [(\alpha_1 f) X \otimes_R 1 + m_1(f, X) \otimes_R 1 + \alpha_1 f \otimes_R X + f \cdot_\hbar \Delta^1_{\hbar} X - \Delta^1_{\hbar}(f X) - \Delta_{\hbar}((\alpha_1 f) X) - \Delta_{\hbar} m_1(f, X)] + O(\hbar^2).
\]  

From the identity \(1 \otimes_R \hbar f \cdot_\hbar X = (1 \cdot_\hbar f) \otimes_R \hbar X\), it follows that

\[
1 \otimes_R (f X + \hbar ((\alpha_1 f) X + m_1(f, X))) = (f + \hbar \beta_1 f) \otimes_R \hbar X + O(\hbar^2).
\]

That is,

\[
1 \otimes_R f X - f \otimes_R X = \hbar [\beta_1 f \otimes_R X - 1 \otimes_R (\alpha_1 f) X - 1 \otimes_R m_1(f, X)] + O(\hbar^2).
\]  

By comparing Equations (57) and (58), one obtains that

\[
\Delta^1_{\hbar}(f X) = f \cdot_\hbar \Delta^1_{\hbar} X + (\alpha_1 f - \beta_1 f) \otimes_R X + (\alpha_1 f) X \otimes_R 1 + 1 \otimes_R (\alpha_1 f) X - \Delta_{\hbar}((\alpha_1 f) X) + m_1(f, X) \otimes_R 1 + 1 \otimes_R m_1(f, X) - \Delta_{\hbar} m_1(f, X) + O(\hbar).
\]

Taking the limit by letting \(\hbar \to 0\), we obtain that

\[
\Delta^1(f X) = f \Delta^1 X + \delta f \otimes_R X + (\alpha_1 f) X \otimes_R 1 + 1 \otimes_R (\alpha_1 f) X - \Delta((\alpha_1 f) X) + m_1(f, X) \otimes_R 1 + 1 \otimes_R m_1(f, X) - \Delta m_1(f, X).
\]

The conclusion thus follows by taking the skew-symmetrization.

\[\square\]

In summary, we have proved the following

**Proposition 5.12** For any \(f, g \in R\) and \(X \in \Gamma(A)\),

(i). \(\delta f \in \Gamma(A)\) and \(\delta X \in \Gamma(\wedge^2 A)\);

(ii). \(\delta(fg) = f \delta g + g \delta f\);

(iii). \(\delta(fX) = f \delta X + \delta f \wedge X\);

(iv). \([\delta f, g] = \{f, g\}\);

(v). \(\delta^2 f = 0\).

Properties (i)-(iii) above allow us to extend \(\delta\) to a well-defined degree 1 derivation \(\delta : \Gamma(\wedge^* A) \to \Gamma(\wedge^{*+1} A)\). Below we will show that \((\oplus \Gamma(\wedge^* A), \wedge, [\cdot, \cdot], \delta)\) is a strong differential Gerstenhaber algebra. For this purpose, it suffices to show that \(\delta\) is a derivation with respect to \([\cdot, \cdot]\), and \(\delta^2 = 0\). We will prove these facts in two separate propositions below.
Proposition 5.13  For any $X, Y \in \Gamma(A)$,
\[ \delta[X, Y] = [\delta X, Y] + [X, \delta Y]. \]  \hfill (59)

Proof.  \( \forall X, Y \in \Gamma(A) \),
\[ \Delta_h(X \ast_h Y) = \Delta_h X \ast_h \Delta_h Y = (1 \otimes_{R_h} X + X \otimes_{R_h} 1 + \hbar \Delta^1_h X) \ast_h (1 \otimes_{R_h} Y + Y \otimes_{R_h} 1 + \hbar \Delta^1_h Y). \]
It thus follows that
\[ \Delta_h[X, Y]_h = 1 \otimes_{R_h}[X, Y]_h + [X, Y]_h \otimes_{R_h} 1 + \hbar(\Delta^1_h X)(1 \otimes Y + Y \otimes 1) + (1 \otimes X + X \otimes 1)\Delta^1_h Y \\
- (\Delta^1_h Y)(1 \otimes X + X \otimes 1) - (1 \otimes Y + Y \otimes 1)\Delta^1_h X + O(\hbar^2). \]  \hfill (60)

Here \( [X, Y]_h = X \ast_h Y - Y \ast_h X \). Then \( [X, Y]_h = [X, Y] + \hbar l_1(X, Y) + O(\hbar^2) \), where \( l_1(X, Y) = m_1(X, Y) - m_1(Y, X) \). Hence
\[ \Delta_h[X, Y]_h = \Delta_h[X, Y] + \hbar \Delta_h l_1(X, Y) + O(\hbar^2) \\
= 1 \otimes_{R_h}[X, Y] + [X, Y] \otimes_{R_h} 1 + \hbar(\Delta^1_h[X, Y] + \Delta_h l_1(X, Y)) + O(\hbar^2). \]  \hfill (61)

Comparing Equation (61) with Equation (60), we obtain that
\[ \Delta^1_h[X, Y] = -\Delta l_1(X, Y) + 1 \otimes_{R_h} l_1(X, Y) + l_1(X, Y) \otimes_{R_h} 1 \\
+ (\Delta^1_h X)(1 \otimes Y + Y \otimes 1) + (1 \otimes X + X \otimes 1)\Delta^1_h Y \\
- (\Delta^1_h Y)(1 \otimes X + X \otimes 1) - (1 \otimes Y + Y \otimes 1)\Delta^1_h X + O(\hbar). \]

This implies, by letting \( \hbar \mapsto 0 \), that
\[ \Delta^1[X, Y] = -\Delta l_1(X, Y) + 1 \otimes_{R_h} l_1(X, Y) + l_1(X, Y) \otimes_{R_h} 1 \\
+ (\Delta^1 X)(1 \otimes Y + Y \otimes 1) + (1 \otimes X + X \otimes 1)\Delta^1 Y \\
- (\Delta^1 Y)(1 \otimes X + X \otimes 1) - (1 \otimes Y + Y \otimes 1)\Delta^1 X. \]

Equation (61) thus follows immediately by taking the skew-symmetrization.

\[ \square \]

Proposition 5.14  For any $X \in \Gamma(A)$,
\[ \delta^2 X = 0. \]

Proof. Let
\[ J_h = (\Delta_h \otimes_{R_h} id)\Delta^1_h X - (id \otimes_{R_h} \Delta_h)\Delta^1_h X - 1 \otimes_{R_h} \Delta^1_h X + \Delta^1_h X \otimes_{R_h} 1. \]  \hfill (62)

From Equation (62), we know that \( J_h = 0 \). Let \( \{e_i \in UA\} \) \( (e_0 = 1) \) be a local basis of \( UA \) over the left module \( R \). Assume that \( \delta X = \sum Y_i \wedge Z_i \) with \( Y_i, Z_i \in \Gamma(A) \), and
\[ \Delta^1 X = \sum f^{ij} e_i \otimes_{R} e_j + \sum (Y_i \otimes R Z_i - Z_i \otimes R Y_i). \]
where \( f^{ij} \in R \) are symmetric: \( f^{ij} = f^{ji} \). We may also assume that \( \Delta e_i = \sum g_i^{kl} e_k \otimes_R e_l \) with \( g_i^{kl} = g_i^{lk} \in R \) since \( \Delta \) is co-commutative. Let us write

\[
\Delta^1_h X = \sum f^{ij} \cdot h e_i \otimes_R e_j + \sum (Y_i \otimes_R Z_i - Z_i \otimes_R Y_i) + h\Delta^2_h X; \quad \text{and} \qquad (63)
\]

\[
\Delta_h e_i = \sum g_i^{kl} \cdot h e_k \otimes_R e_l + h\Delta^1_h e_i, \quad \text{and} \qquad (64)
\]

for some \( \Delta^2_h X \) and \( \Delta^1_h e_i \in U_h A \otimes_R U_h A \).

Then

\[
(id \otimes_R \Delta_h)\Delta^1_h X = \sum (\alpha_h f^{ij} \cdot h \beta_h g_j^{kl} \cdot h e_i) \otimes_R e_k \otimes_R e_l + \sum (Y_i \otimes_R Z_i \otimes_R R_{h} 1 + Y_i \otimes_R 1 \otimes_R Z_i - Z_i \otimes_R Y_i) + h[\sum f^{ij} \cdot h e_i \otimes_R \Delta^1_h e_j + \sum Y_i \otimes_R \Delta^1_h Z_i - \sum Z_i \otimes_R \Delta^1_h Y_i + (id \otimes_R \Delta_h)\Delta^2_h X],
\]

and

\[
1 \otimes_R \Delta^1_h X = \sum \beta_h f^{ij} \cdot h e_i \otimes_R e_j + \sum (1 \otimes_R Y_i \otimes_R Z_i - 1 \otimes_R Z_i \otimes_R Y_i) + h(1 \otimes_R \Delta^2_h X).
\]

Similarly, we may write

\[
\Delta^1_h X = \sum e_i \otimes_R e_j \cdot h f^{ij} + \sum (Y_i \otimes_R Z_i - Z_i \otimes_R Y_i) + h\tilde{\Delta}^2_h X; \quad \text{and} \qquad (65)
\]

\[
\Delta_h e_i = \sum e_k \otimes_R e_l \cdot h g_i^{kl} + h\Delta^1_h e_i, \quad \text{and} \qquad (66)
\]

for some \( \tilde{\Delta}^2_h X \) and \( \tilde{\Delta}^1_h e_i \in U_h A \otimes_R U_h A \).

Hence,

\[
(D_h \otimes_R id)\Delta^1_h X = \sum e_k \otimes_R e_l \cdot h f^{ij} + \sum (Y_i \otimes_R Z_i \otimes_R R_{h} 1 + 1 \otimes_R Y_i \otimes_R Z_i - Z_i \otimes_R Y_i \otimes_R R_{h} 1) + h[\sum \tilde{\Delta}^1_h e_i \otimes_R e_j \cdot h f^{ij} + \sum \Delta^1_h Y_i \otimes_R Z_i - \sum \Delta^1_h Z_i \otimes_R Y_i + (D_h \otimes_R id)\Delta^2_h X],
\]

and

\[
\Delta^1_h X \otimes_R 1 = \sum e_i \otimes_R e_j \cdot h \alpha_h f^{ij} + \sum (Y_i \otimes_R Z_i \otimes_R 1 - Z_i \otimes_R Y_i \otimes_R 1) + h\tilde{\Delta}^2_h X \otimes_R 1.
\]

Thus we have \( J_h = I_h + hK_h \), where

\[
I_h = \sum e_i \otimes_R e_j \cdot h \alpha_h f^{ij} + \sum (\alpha_h f^{ij} \cdot h \beta_h g_j^{kl} \cdot h e_i) \otimes_R e_k - \sum \beta_h f^{ij} \cdot h e_i \cdot h e_j \otimes_R e_k + \sum e_i \otimes_R e_j \cdot h \alpha_h f^{ij},
\]

and

\[
K_h = \sum \Delta^1_h e_i \otimes_R e_j \cdot h f^{ij} + \sum \Delta^1_h Y_i \otimes_R Z_i - \sum \Delta^1_h Z_i \otimes_R Y_i + (D_h \otimes_R id)\tilde{\Delta}^2_h X + \tilde{\Delta}^2_h X \otimes_R 1 - [\sum f^{ij} \cdot h e_i \otimes_R \Delta^1_h e_j + \sum Y_i \otimes_R \Delta^1_h Z_i - \sum Z_i \otimes_R \Delta^1_h Y_i + (id \otimes_R \Delta_h)\Delta^2_h X + 1 \otimes_R \Delta^2_h X].
\]
By Alt, we denote the standard skew-symmetrization operator on $UA \otimes_R UA \otimes_R UA$:

$$\text{Alt}(x_1 \otimes_R x_2 \otimes_R x_3) = \sum_{\sigma \in S_3} \frac{1}{3!} (-1)^{|\sigma|} x_{\sigma(1)} \otimes_R x_{\sigma(2)} \otimes_R x_{\sigma(3)},$$

where $x_1, x_2, x_3 \in UA$. It is tedious but straightforward to verify that $\text{Alt}(\lim_{h \to 0} \frac{1}{h} I_h) = 0$. Therefore, $\text{Alt}(\lim_{h \to 0} K_h) = 0$, i.e.,

$$\text{Alt} \left[ \sum f_{ij} \tilde{\Delta}^1 e_i \otimes_R e_j + \sum \Delta^1 Y_i \otimes_R Z_i - \sum \Delta^1 Z_i \otimes_R Y_i + (\Delta \otimes_R \text{id}) \tilde{\Delta}^2 X + \tilde{\Delta}^2 X \otimes_R 1 \right.$$  
$$- (\sum f_{ij} e_i \otimes_R \Delta^1 e_j + \sum Y_i \otimes_R \Delta^1 Z_i - \sum Z_i \otimes_R \Delta^1 Y_i + (\text{id} \otimes_R \Delta) \Delta^2 X + 1 \otimes_R \Delta^2 X) \right] = 0.$$

The final conclusion thus follows immediately by applying the skew-symmetrization operator Alt to the equation above and using the following simple facts:

**Lemma 5.15**

(i). $\text{Alt}(\tilde{\Delta}^2 X \otimes_R 1 - 1 \otimes_R \Delta^2 X) = 0$.

(ii). $\text{Alt} \left[ \sum f_{ij} \tilde{\Delta}^1 e_i \otimes_R e_j - f_{ij} e_i \otimes_R \Delta^1 e_j \right] = 0$.

(iii). $\text{Alt}((\text{id} \otimes_R \Delta) \Delta^2 X) = \text{Alt}((\Delta \otimes_R \text{id}) \tilde{\Delta}^2 X) = 0$.

**Proof.** It follows from Equations (63) and (65) that

$$\hbar \tilde{\Delta}^2 X - \hbar \tilde{\Delta}^2 X = \sum (f_{ij} \cdot \hbar e_i \otimes_R e_j - e_i \otimes_R \hbar e_j \cdot f_{ij})$$
$$= \sum (f_{ij} \cdot \hbar e_i \otimes_R e_j - e_i \otimes_R \hbar f_{ij} \cdot \hbar e_j + e_i \otimes_R \hbar f_{ij} \cdot \hbar e_j - e_i \otimes_R \hbar e_j \cdot f_{ij})$$
$$= \hbar \sum (\delta f_{ij} \cdot \hbar e_i) \otimes_R e_j + \hbar \sum e_i \otimes_R e_i (\delta f_{ij} \cdot \hbar e_j) + O(h^2).$$

Hence,

$$\Delta^2 X - \tilde{\Delta}^2 X = \sum (\delta f_{ij}) e_i \otimes_R e_j + \sum e_i \otimes_R (\delta f_{ij}) e_j$$
$$= \sum \Delta(\delta f_{ij}) e_i \otimes_R e_j,$$

which is symmetric. It thus follows that

$$\text{Alt}(\tilde{\Delta}^2 X \otimes_R 1 - 1 \otimes_R \Delta^2 X) = \text{Alt}(\Delta^2 X \otimes_R 1 - 1 \otimes_R \Delta^2 X - \sum \Delta(\delta f_{ij}) e_i \otimes_R e_j \otimes_R 1)$$
$$= 0.$$

Similarly, one can show that

$$\Delta^1 e_i - \tilde{\Delta}^1 e_i = \sum_{kl} \Delta(\delta g_{ik}^{kl}) (e_k \otimes_R e_l).$$

Thus, (ii) follows immediately. Finally, (iii) is obvious since $\Delta$ is co-commutative.

$\square$
Combining Propositions 5.12-5.14, we conclude that \((\oplus \Gamma(\wedge^i A), \wedge, [\cdot, \cdot], \delta)\) is indeed a strong differential Gerstenhaber algebra. Hence \((A, A^*)\) is a Lie bialgebroid, which is called the classical limit of the quantum groupoid \(U_\hbar A\). In summary, we have proved

**Theorem 5.16** A quantum groupoid \((U_\hbar A, R_\hbar, \alpha_\hbar, \beta_\hbar, m_\hbar, \Delta_\hbar, \epsilon_\hbar)\) naturally induces a Lie bialgebroid \((A, A^*)\) as a classical limit. The induced Poisson structure of this Lie bialgebroid on the base manifold \(P\) coincides with the one obtained as the classical limit of the base \(*\)-algebra \(R_\hbar\).

As an example, in what follows, we will examine the case where the quantum groupoids are obtained from the standard Hopf algebroid \(UA[[\hbar]]\) by twists. Consider \((UA[[\hbar]], R[[\hbar]], \alpha, \beta, m, \Delta, \epsilon)\) equipped with the standard Hopf algebroid structure induced from that on \(UA\). Assume that

\[
F_\hbar = 1 \otimes_R 1 + \hbar \bar{\Lambda} + O(\hbar^2) \in UA \otimes_R UA[[\hbar]],
\]

where \(\bar{\Lambda} \in UA \otimes_R UA\), is a twistor, and let \((U_\hbar A, R_\hbar, \alpha_\hbar, \beta_\hbar, m_\hbar, \Delta_\hbar, \epsilon_\hbar)\) be the resulting twisted QUE algebroid.

**Lemma 5.17** Assume that \(F_\hbar \in UA \otimes_R UA[[\hbar]]\) given by Equation (67) is a twistor. Then

\[
\bar{\Lambda} \otimes_R 1 + (\Delta \otimes_R id) \bar{\Lambda} = 1 \otimes_R \bar{\Lambda} + (id \otimes_R \Delta) \bar{\Lambda}.
\]

**Proof.** This follows immediately from computing the \(\hbar^1\)-term in the \(\hbar\)-expansion of Equation (32). □

Using Proposition 8.1 in the Appendix, we have

**Corollary 5.18** Under the same hypothesis as in Lemma 5.17, then \(\Lambda = \text{Alt}\bar{\Lambda}\) is a section of \(\wedge^2 A\).

Now it is natural to expect the following:

**Theorem 5.19** Let \((U_\hbar A, R_\hbar, \alpha_\hbar, \beta_\hbar, m_\hbar, \Delta_\hbar, \epsilon_\hbar)\) be the quantum groupoid obtained from \((UA[[\hbar]], R[[\hbar]], \alpha, \beta, m, \Delta, \epsilon)\) by twisting via \(F_\hbar\), where

\[
F_\hbar = 1 \otimes_R 1 + \hbar \bar{\Lambda} + O(\hbar^2) \in UA \otimes_R UA[[\hbar]].
\]

Then its classical limit is a coboundary Lie bialgebroid \((A, A^*, \Lambda)\), where \(\Lambda = \text{Alt}\bar{\Lambda}\). In particular, its induced Poisson structure on the base manifold is \(\rho \Lambda\), which admits \(R_\hbar\) as a deformation quantization.

**Proof.** It suffices to prove that \(\delta f = [f, \Lambda]\) and \(\delta X = [X, \Lambda]\), \(\forall f \in R\) and \(X \in \Gamma(A)\).

Write

\[
\bar{\Lambda} = \sum_i d_i \otimes_R e_i \in UA \otimes_R UA.
\]
Using Equations (29)-(30), it is easy to see that for any $f \in R$,
\[
\alpha_h f = f + \hbar \sum_i ((pd_i)f)e_i + O(h^2), \quad \text{and} \\
\beta_h f = f + \hbar \sum_i ((pe_i)f)d_i + O(h^2).
\]

It thus follows that
\[
\delta f = \sum_i [((pd_i)f)e_i - ((pe_i)f)d_i] = [f, \Lambda].
\]

Now we will prove the second identity $\delta X = [X, \Lambda]$. From the definition of $\mathcal{F}_h^\#$, it follows that
\[
\mathcal{F}_h^\#[1 \otimes R_h X + X \otimes R_h 1 + \hbar \sum_i (Xd_i \otimes R_h e_i + d_i \otimes R_h X e_i - d_i X \otimes R_h e_i)] = \mathcal{F}_h [1 \otimes X + X \otimes 1 + \hbar \sum_i (Xd_i \otimes e_i + d_i \otimes X e_i - d_i X \otimes e_i)]
\]
\[
= 1 \otimes R_h X + X \otimes R_h 1 + \hbar \sum_i (Xd_i \otimes R_h e_i + d_i \otimes R_h e_i X) + O(h^2)
\]
\[
= (\Delta X) \mathcal{F}_h + O(h^2).
\]

It thus follows that
\[
\Delta_h X = \mathcal{F}_h^\#^{-1}((\Delta X) \mathcal{F}_h)
\]
\[
= 1 \otimes R_h X + X \otimes R_h 1 + \hbar \sum_i (Xd_i \otimes R_h e_i + d_i \otimes R_h X e_i - d_i X \otimes R_h e_i) + O(h^2).
\]

Therefore
\[
\Delta^1 X = (1 \otimes X + X \otimes 1) \bar{\Lambda} - \bar{\Lambda}(1 \otimes X + X \otimes 1).
\]

This immediately implies that
\[
\delta X = \Delta^1 X - (\Delta^1 X)_{21}
\]
\[
= [(1 \otimes X + X \otimes 1) \bar{\Lambda} - \bar{\Lambda}(1 \otimes X + X \otimes 1)] - [(1 \otimes X + X \otimes 1) \bar{\Lambda}_{21} - \bar{\Lambda}_{21}(1 \otimes X + X \otimes 1)]
\]
\[
= (1 \otimes X + X \otimes 1) \bar{\Lambda} - \bar{\Lambda}(1 \otimes X + X \otimes 1)
\]
\[
= [X, \Lambda].
\]

This concludes the proof.

\[\square\]

More generally, we have
Theorem 5.20 Assume that \((U_\hbar A, R_\hbar, \alpha_\hbar, \beta_\hbar, m_\hbar, \Delta_\hbar, \epsilon_\hbar)\) is a quantum groupoid with classical limit \((A, A^*)\). Let \(\mathcal{F}_\hbar \in U_\hbar A \otimes R_\hbar U_\hbar A\) be a twistor such that \(\mathcal{F}_\hbar = 1 \otimes R_\hbar 1(\text{mod} \hbar)\). Then \(\Lambda = \text{Alt}(\lim_{\hbar \to 0} \hbar^{-1}(\mathcal{F}_\hbar - 1 \otimes R_\hbar 1))\) is a section of \(\wedge^2 A\), and is a Hamiltonian operator of the Lie bialgebroid \((A, A^*)\). If \((U_\hbar A, \tilde{R}_\hbar, \tilde{\alpha}_\hbar, \tilde{\beta}_\hbar, m_\hbar, \tilde{\Delta}_\hbar, \epsilon_\hbar)\) is obtained from \((U_\hbar A, R_\hbar, \alpha_\hbar, \beta_\hbar, m_\hbar, \Delta_\hbar, \epsilon_\hbar)\) by twisting via \(\mathcal{F}_\hbar\), its corresponding Lie bialgebroid is obtained from \((A, A^*)\) by twisting via \(\Lambda\). In particular, if \((A, A^*)\) is a coboundary Lie bialgebroid with \(r\)-matrix \(\Lambda_0\), the latter is still a coboundary Lie bialgebroid and its \(r\)-matrix is \(\Lambda_0 + \Lambda\).

Proof. The proof is similar to that of Theorem 5.19, even though it is a little bit more complicated. We omit it here.

Remark. It is easy to see that the classical limit of the quantum groupoid \((\mathcal{D}_\hbar, R_\hbar, \alpha_\hbar, \beta_\hbar, m, \Delta_\hbar, \epsilon)\) in Example 4.1 is the standard Lie bialgebroid \((TP, T^*P)\) associated to a Poisson manifold \(P\). It would be interesting to explore its “dual” quantum groupoid, namely the one with the Lie bialgebroid \((T^*P, TP)\) as its classical limit.

6 Quantization of Lie bialgebroids

Definition 6.1 A quantization of a Lie bialgebroid \((A, A^*)\) is a quantum groupoid \((U_\hbar A, R_\hbar, \alpha_\hbar, \beta_\hbar, m_\hbar, \Delta_\hbar, \epsilon_\hbar)\) whose classical limit is \((A, A^*)\).

It is a deep theorem of Etingof and Kazhdan that every Lie bialgebra is quantizable. On the other hand, the existence of \(*\)-products for an arbitrary Poisson manifold was recently proved by Kontsevich. In terms of Hopf algebroids, this amounts to saying that the Lie bialgebroid \((TP, T^*P)\) associated to a Poisson manifold \(P\) is always quantizable. It is therefore natural to expect:

Conjecture Every Lie bialgebroid is quantizable.

Below we will prove a very special case of this conjecture by using Fedosov quantization method.

Theorem 6.2 Any regular triangular Lie bialgebroid is quantizable.

We need some preparation first. Recall that given a Lie algebroid \(A \to P\) with anchor \(\rho\), an \(A\)-connection on a vector bundle \(E \to P\) is an \(\mathbb{R}\)-linear map:

\[
\Gamma(A) \otimes \Gamma(E) \to \Gamma(E)
\]

\[X \otimes s \to \nabla_X s,
\]
satisfying the axioms resembling those of usual linear connections, i.e., \( \forall f \in C^\infty(P), \ X \in \Gamma(A), s \in \Gamma(E) \),

\[
\nabla_{fX}s = f\nabla_Xs;
\n\nabla_X(fs) = (\rho(X)f)s + f\nabla_Xs.
\]

In particular, if \( E = A \), an A-connection is called torsion-free if

\[
\nabla_XY - \nabla_YX = [X, Y], \ \forall X, Y \in \Gamma(A).
\]

A torsion-free connection always exists for any Lie algebroid.

Let \( \omega \in \Gamma(\wedge^2 A^*) \) be a closed two-form, i.e., \( d\omega = 0 \). An A-connection on A is said to be compatible with \( \omega \) if \( \nabla_X\omega = 0, \forall X \in \Gamma(A) \). If \( \omega \) is non-degenerate, a compatible torsion-free connection always exists.

**Lemma 6.3** If \( \omega \in \Gamma(\wedge^2 A^*) \) is a closed non-degenerate two-form, there exists a compatible torsion-free A-connection on A.

**Proof.** This result is standard (see [38] [45]). The proof is simply a repetition of that of the existence of a symplectic connection for a symplectic manifold. For completeness, we sketch a proof here.

First, take any torsion-free A-connection \( \tilde{\nabla} \). Then any other A-connection can be written as

\[
\tilde{\nabla}_XY = \nabla_XY + S(X, Y), \ \forall X, Y \in \Gamma(A),
\]

where \( S \) is a (2,1)-tensor. Clearly, \( \tilde{\nabla} \) is torsion-free if and only if \( S \) is symmetric, i.e., \( S(X, Y) = S(Y, X) \) for any \( X, Y \in \Gamma(A) \).

\( \tilde{\nabla} \) is compatible with \( \omega \in \Gamma(\wedge^2 A^*) \) if and only if \( \nabla_X\omega = 0 \). The latter is equivalent to

\[
\omega(S(X, Y), Z) - \omega(S(X, Z), Y) = (\nabla_X\omega)(Y, Z).
\]

Let \( S \) be the (2,1)-tensor defined by the equation:

\[
\omega(S(X, Y), Z) = \frac{1}{3}[(\nabla_X\omega)(Y, Z) + (\nabla_Y\omega)(Z, X)].
\]

Clearly, \( S(X, Y) \), defined in this way, is symmetric with respect to \( X \) and \( Y \). Now

\[
\omega(S(X, Y), Z) - \omega(S(X, Z), Y)
= \frac{1}{3}[(\nabla_X\omega)(Y, Z) + (\nabla_Y\omega)(Z, X)] - \frac{1}{3}[(\nabla_X\omega)(Z, Y) + (\nabla_Z\omega)(X, Y)]
= \frac{1}{3}[(\nabla_X\omega)(Y, Z) + (\nabla_Y\omega)(Z, X) + (\nabla_X\omega)(Y, Z) + (\nabla_Z\omega)(Y, X)]
= (\nabla_X\omega)(Y, Z).
\]

Here the last step follows from the identity:

\[
(\nabla_X\omega)(Y, Z) + (\nabla_Y\omega)(Z, X) + (\nabla_Z\omega)(X, Y) = 0,
\]

which is equivalent to \( d\omega = 0 \). This implies that \( \tilde{\nabla} \) is a torsion-free symplectic connection.
Proof of Theorem 6.2 Let \((A, A^*, \Lambda)\) be a regular triangular Lie bialgebroid. Then \(\Lambda^\#: A^* \to A\) is a Lie algebroid morphism \([34]\). Therefore its image \(\Lambda^\# A^*\) is a Lie subalgebroid of \(A\), and \(\Lambda\) can be considered as a section of \(\wedge^2(\Lambda^# A^*)\). Hence, by restricting to \(\Lambda^# A^*\) if necessary, one may always assume that \(\Lambda\) is nondegenerate. Let \(\omega = \Lambda^{-1} \in \Gamma(\wedge^2 A^*)\). Then \(\omega\) is closed: \(d\omega = 0\). Let \(\nabla\) be a compatible torsion-free \(A\)-connection on \(A\), which always exists according to Lemma 6.3. Let \((G \to P, \alpha, \beta)\) be a local Lie groupoid corresponding to the Lie algebroid \(A\). Let \(\Lambda^l\) denote the left translation of \(\Lambda\), so \(\Lambda^l\) defines a left invariant Poisson structure on \(G\). This is a regular Poisson structure, whose symplectic leaves are simply \(\alpha\)-fibers. The \(A\)-connection \(\nabla\) induces a fiberwise linear connection \(\hat{\nabla}\) for the \(\alpha\)-fibrations. To see this, simply define for any \(X, Y \in \Gamma(A)\),

\[
\hat{\nabla}_X Y^l = (\nabla_X Y)^l,
\]

where \(X^l, Y^l\) and \((\nabla_X Y)^l\) denote their corresponding left invariant vector fields on \(G\). Since left invariant vector fields span the tangent space of \(\alpha\)-fibers, this indeed defines a linear connection on each \(\alpha\)-fiber \(\alpha^{-1}(u), \forall u \in P\), which is denoted by \(\hat{\nabla}_u\). Clearly, \(\hat{\nabla}_u\) is torsion-free since \(\nabla\) is torsion-free. Moreover, \(\hat{\nabla}\) preserves the Poisson structure \(\Lambda^l\), and is left-invariant in the sense that

\[
L_x^u \hat{\nabla}_u = \hat{\nabla}_v, \quad \forall x \in G \text{ such that } \beta(x) = u, \alpha(x) = v.
\]

Applying Fedosov quantization to this situation, one obtains a \(*\)-product on \(G\):

\[
f \ast_h g = fg + \frac{1}{2} h \Lambda^l(f, g) + \cdots + h^k B_k(f, g) + \cdots
\]

quantizing the Poisson structure \(\Lambda^l\). In fact, this \(*\)-product is given by a family of leafwise \(*\)-products indexed by \(u \in P\) quantizing the leafwise symplectic structures on \(\alpha\)-fibers. The Poisson structure \(\Lambda^l\) is left invariant, so the leafwise symplectic structures are invariant under left translations. Moreover, since the symplectic connections \(\hat{\nabla}_u\) are left-invariant, the resulting Fedosov \(*\)-products are invariant under left translations. In other words, the bidifferential operators \(B_k(\cdot, \cdot)\) are all left invariant, and therefore can be considered as elements in \(UA \otimes_R UA\). In this way, we obtain a formal power series \(F_h = 1 + \frac{1}{2} h \Lambda + O(h^2) \in UA \otimes_R UA[[h]]\) so that the \(*\)-product on \(G\) is

\[
f \ast_h g = F_h(f, g), \quad \forall f, g \in C^\infty(G).
\]

The associativity of \(*_h\) implies that \(F_h\) satisfies Equation (32):

\[
(\Delta \otimes_R id) F_h F_h^{12} = (id \otimes_R \Delta) F_h F_h^{23}.
\]

The identity \(1 \ast_h f = f \ast_h 1 = f\) implies that

\[
(\epsilon \otimes_R id) F_h = 1_H; \quad (id \otimes_R \epsilon) F_h = 1_H.
\]

Thus \(F_h \in UA \otimes_R UA[[h]]\) is a twistor, and the resulting twisted Hopf algebroid \((U_h A, R_h, \alpha_h, \beta_h, m_h, \Delta_h, \epsilon_h)\) is a quantization of the triangular Lie bialgebroid \((A, A^*, \Lambda)\) according to Theorem 5.19. This concludes the proof of the theorem.

\[\square\]
In particular, when the base \( P \) reduces to a point, Theorem 6.2 implies that every finite dimensional triangular \( r \)-matrix is quantizable. Of course, there is no need to use Fedosov method in this case. There is a very nice short proof due to Drinfel’d [10].

We note that the induced Poisson structure on the base manifold of a non-degenerate triangular Lie bialgebroid, also called a symplectic Lie algebroid, was studied by Nest-Tsygan [38] and Weinstein [45], for which a \( \ast \)-product was constructed. Indeed, our algebra \( R_{\hbar} \) provides a \( \ast \)-product for such a Poisson structure, where the multiplication is simply defined by the push forward of \( F_{\hbar} \) under the anchor \( \rho \):

\[
a \ast_{\hbar} b = (\rho F_{\hbar})(a, b), \quad \forall a, b \in C^\infty(P)[[\hbar]].
\]

So here we obtain an alternative proof of (a slightly more general version of) their quantization result.

**Corollary 6.4** The induced Poisson structure on the base manifold of a regular (in particular non-degenerate) triangular Lie bialgebroid is quantizable.

### 7 Dynamical quantum groupoids

This section is devoted to the study of an important example of quantum groupoids, which are connected with the so called quantum dynamical \( R \)-matrices. Let \( U_{\hbar}\mathfrak{g} \) be a quasi-triangular quantum universal enveloping algebra over \( \mathbb{C} \) with \( R \)-matrix \( R \in U_{\hbar}\mathfrak{g} \otimes U_{\hbar}\mathfrak{g} \), \( \eta \subset \mathfrak{g} \) a finite dimensional Abelian Lie subalgebra such that \( U\eta[[\hbar]] \) is a commutative subalgebra of \( U_{\hbar}\mathfrak{g} \). By \( \mathcal{M}(\eta^*) \), we denote the algebra of meromorphic functions on \( \eta^* \), and by \( \mathcal{D} \) the algebra of meromorphic differential operators on \( \eta^* \). Consider \( H = \mathcal{D} \otimes U_{\hbar}\mathfrak{g} \). Then \( H \) is a Hopf algebroid over \( \mathbb{C} \) with base algebra \( R = \mathcal{M}(\eta^*)[[\hbar]] \), whose coproduct and counit are denoted, respectively, by \( \Delta \) and \( \epsilon \). Moreover the map

\[
\mu(D \otimes u)(f) = (\epsilon_0 u)D(f), \quad \forall D \in \mathcal{D}, \ u \in U_{\hbar}\mathfrak{g}, \ f \in \mathcal{M}(\eta^*)[[\hbar]],
\]

is an anchor map. Here \( \epsilon_0 \) is the counit of the Hopf algebra \( U_{\hbar}\mathfrak{g} \). Let us fix a basis in \( \eta \), say \( \{h_1, \ldots, h_k\} \), and let \( \{\xi_1, \ldots, \xi_k\} \) be its dual basis, which in turn defines a coordinate system \( (\lambda_1, \ldots, \lambda_k) \) on \( \eta^* \).

Set

\[
\theta = \sum_{i=1}^k \left( \frac{\partial}{\partial \lambda_i} \otimes h_i \right) \in H \otimes H, \quad \text{and } \Theta = \exp \hbar \theta \in H \otimes H.
\]

Note that \( \theta \), and hence \( \Theta \), is independent of the choice of bases in \( \eta \). The following fact can be easily verified.

**Lemma 7.1** \( \Theta \) satisfies Equations (33) and (33).

**Proof.** Consider \( H_0 = \mathcal{D}^{inv} \otimes U\eta[[\hbar]] \), where \( \mathcal{D}^{inv} \) consists of holomorphic differential operators on \( \eta^* \) invariant under the translations. Then \( H_0 \) is a Hopf subalgebroid of \( H \), which is in fact a Hopf algebra. Clearly, \( \theta \in H_0 \otimes H_0 \), so \( \Theta \in H_0 \otimes H_0 \). It thus suffices to prove that
\((\Delta \otimes \text{id}) \Theta \Theta^{12} = (\text{id} \otimes \Delta) \Theta \Theta^{23}\) in \(H_0 \otimes H_0 \otimes H_0\).

Now

\[
(\Delta \otimes \text{id}) \Theta \Theta^{12} = ((\Delta \otimes \text{id}) \exp \hbar \theta) \exp \hbar \Theta^{12} = \exp \hbar ((\Delta \otimes \text{id}) \theta + \Theta^{12}) = \exp \hbar \sum_{i=1}^{k} \left( \frac{\partial}{\partial \lambda_i} \otimes 1 \otimes h_i + 1 \otimes \frac{\partial}{\partial \lambda_i} \otimes h_i + \frac{\partial}{\partial \lambda_i} \otimes h_i \otimes 1 \right).
\]

Here in the second equality we used the fact that \((\Delta \otimes \text{id}) \theta\) and \(\Theta^{12}\) commute in \(H_0 \otimes H_0 \otimes H_0\).

Similarly, we have

\[
(\text{id} \otimes \Delta) \Theta \Theta^{23} = \exp \hbar \sum_{i=1}^{k} \left( \frac{\partial}{\partial \lambda_i} \otimes 1 \otimes h_i + 1 \otimes \frac{\partial}{\partial \lambda_i} \otimes h_i + \frac{\partial}{\partial \lambda_i} \otimes h_i \otimes 1 \right).
\]

This proves Equation (32). Finally, Equation (33) follows from a straightforward verification.

Remark. There is a more intrinsic way of proving this fact in terms of deformation quantization. Consider \(T^* \eta^*\) equipped with the standard cotangent bundle symplectic structure \(\sum_{i=1}^{k} d\lambda_i \wedge dp_i\). It is well-known that, \(\forall f, g \in C^\infty(T^* \eta^*)[[\hbar]],\)

\[
f \ast_h g = f e^{\hbar \sum_{i=1}^{k} \left( \frac{\partial}{\partial \lambda_i} \otimes \frac{\partial}{\partial p_i} \right)} g \tag{79}
\]

defines a \(*\)-product on \(T^* \eta^*\), called the Wick type \(*\)-product corresponding to the normal ordering quantization. Hence \(\exp \hbar (\sum_{i=1}^{k} \left( \frac{\partial}{\partial \lambda_i} \otimes \frac{\partial}{\partial p_i} \right))\), as a (formal) bidifferential operator on \(T^* \eta^*\) (i.e. as an element in \(\mathcal{D}(T^* \eta^*) \otimes \mathcal{M}(T^* \eta^*) \mathcal{D}(T^* \eta^*)[[\hbar]]\)) satisfies Equations (32)-(33) according to Example 4.1. Note that elements in \((\mathcal{D} \otimes \mathcal{U} \eta) \otimes \mathcal{M}(\eta^*) \mathcal{D}(\mathcal{U} \eta)[[\hbar]]\) can be considered as (formal) bidifferential operators on \(T^* \eta^*\) invariant under the \(p\)-translations, so \((\mathcal{D} \otimes \mathcal{U} \eta) \otimes \mathcal{M}(\eta^*) \mathcal{D}(\mathcal{U} \eta)[[\hbar]]\) is naturally a subspace of \(\mathcal{D}(T^* \eta^*) \otimes \mathcal{M}(T^* \eta^*) \mathcal{D}(T^* \eta^*)[[\hbar]]\). Clearly \(\exp \hbar (\sum_{i=1}^{k} \left( \frac{\partial}{\partial \lambda_i} \otimes \frac{\partial}{\partial p_i} \right))\) is a \(p\)-invariant bidifferential operator on \(T^* \eta^*\), and is equal to \(\Theta\) under the above identification. Equations (32) and (33) thus follow immediately.

In other words, \(\Theta\) is a twistor of the Hopf algebroid \(H\). As we see below, it is this \(\Theta\) that links a shifted cocycle \(F(\lambda)\) and a Hopf algebroid twistor.
Given $F \in \mathcal{M}(\eta^*, U_{h\mathfrak{g}} \otimes U_{h\mathfrak{g}})$, define $F^{12}(\lambda + \hbar h^{(3)}) \in \mathcal{M}(\eta^*, U_{h\mathfrak{g}} \otimes U_{h\mathfrak{g}})$ by

$$F^{12}(\lambda + \hbar h^{(3)}) = F(\lambda) \otimes 1 + h \sum_i \frac{\partial F}{\partial \lambda_i} \otimes h_i + \frac{1}{2!} h^2 \sum_{i_1 i_2} \frac{\partial^2 F}{\partial \lambda_{i_1} \partial \lambda_{i_2}} \otimes h_{i_1} h_{i_2}$$

$$+ \cdots + \frac{h^k}{k!} \sum \frac{\partial^k F}{\partial \lambda_{i_1} \cdots \partial \lambda_{i_k}} \otimes h_{i_1} \cdots h_{i_k} + \cdots,$$

(80)
similarly for $F^{23}(\lambda + \hbar h^{(1)})$ etc.

**Lemma 7.2** Let $X$ be a meromorphic vector field on $\eta^*$, and $F \in \mathcal{M}(\eta^*, U_{h\mathfrak{g}} \otimes U_{h\mathfrak{g}})$. Then

$$\Delta X F = F(\Delta X) + X(F), \quad \text{in} \quad H \otimes_R H.$$

**Proof.** Note that $F$, being considered as an element in $H \otimes_R H$, clearly satisfies the condition that

$$[F(\lambda), 1 \otimes h + h \otimes 1] = 0, \quad \forall \lambda \in \eta^*, \ h \in \eta.$$

(82)

Thus Equation (81) follows.

An element $F \in \mathcal{M}(\eta^*, U_{h\mathfrak{g}} \otimes U_{h\mathfrak{g}})$ is said to be of zero weight if

$$[F(\lambda), 1 \otimes h + h \otimes 1] = 0, \quad \forall \lambda \in \eta^*, \ h \in \eta.$$

(82)

**Lemma 7.3** Assume that $F \in \mathcal{M}(\eta^*, U_{h\mathfrak{g}} \otimes U_{h\mathfrak{g}})$ is of zero weight. Then $\forall n \in \mathbb{N}$,

$$[(\Delta \otimes_R id)^n] F^{12} = \sum_{k=0}^n \sum_{0 \leq i_1, \cdots, i_k \leq n} C_k^n \left( \frac{\partial^k F}{\partial \lambda_{i_1} \cdots \partial \lambda_{i_k}} \otimes h_{i_1} \cdots h_{i_k} \right) (\Delta \otimes_R id)^{n-k};$$

(83)

$$[(id \otimes_R \Delta) \theta] F^{23} = F^{23} (id \otimes_R \Delta) \theta.$$

(84)

**Proof.** To prove Equation (83), let us first consider $n = 1$. Then

$$[(\Delta \otimes_R id) \theta] F^{12} = \sum_i (\Delta \otimes_R id h_i) F^{12}.$$
\[ \sum_i (\Delta \frac{\partial}{\partial \lambda_i}) F \otimes R h_i \]
\[ = \sum_i (F \Delta \frac{\partial}{\partial \lambda_i} + \frac{\partial F}{\partial \lambda_i}) \otimes R h_i \]
\[ = F^{12}(\Delta \otimes R id) \theta + \sum_i \frac{\partial F}{\partial \lambda_i} \otimes R h_i. \]

The general case follows from induction, using the above equation repeatedly.

For Equation (84), we have
\[ [(id \otimes R \Delta) \theta] F^{23} = \sum_i \frac{\partial}{\partial \lambda_i} \otimes R \Delta h_i F^{23} = \sum_i \frac{\partial}{\partial \lambda_i} \otimes R (\Delta h_i) F = \sum_i \frac{\partial}{\partial \lambda_i} \otimes R (F \Delta h_i) = F^{23} (id \otimes R \Delta) \theta, \]
where the second from the last equality follows from the fact that \( F \) is of zero weight. Equation (84) thus follows.

\[ \square \]

**Proposition 7.4** Assume that \( F \in \mathcal{M}(\eta^*, U_h \otimes U_h) \) is of zero weight. Then
\[ [(\Delta \otimes R id) \Theta] F^{12}(\lambda) = F^{12}(\lambda + \hbar h(3))(\Delta \otimes R id) \Theta; \]
\[ [(id \otimes R \Delta) \Theta] F^{23}(\lambda) = F^{23}(\lambda)(id \otimes R \Delta) \Theta. \]

**Proof.** Note that \((\Delta \otimes R id) \Theta = \exp \hbar ((\Delta \otimes R id) \theta)\). Equations (85) and (86) thus follow immediately from Lemma 7.3.

\[ \square \]

**Remark.** One may rewrite Equation (85) as
\[ F^{12}(\lambda + \hbar h(3)) = [(\Delta \otimes R id) \Theta] F^{12}(\lambda) [(\Delta \otimes R id) \Theta]^{-1} = e^{\hbar \sum_{i=1}^k (\Delta \frac{\partial}{\partial \lambda_i} \otimes h_i)} F^{12}(\lambda) e^{-\hbar \sum_{i=1}^k (\Delta \frac{\partial}{\partial \lambda_i} \otimes h_i)}. \]
This is essentially the definition of \( F^{12}(\lambda + \hbar h(3)) \) used in [2], where the operator \( \sum_{i=1}^k (\Delta \frac{\partial}{\partial \lambda_i} \otimes h_i) \) was denoted by \( \sum_{i=1}^k \frac{\partial}{\partial \lambda_i} h_i^{(3)}. \)

Now set
\[ \mathcal{F} = F(\lambda) \Theta \in H \otimes R H. \]
Theorem 7.5 Assume that $F \in M(\eta^*, U_{R}\otimes U_{H\mathfrak{g}})$ is of zero weight. Then $F$ is a twistor (i.e. satisfies Equations (32)-(33)) if and only if

$$\left(\Delta_0 \otimes \text{id}\right) F(\lambda) F^{12}(\lambda + h h^{(3)}) = \left(\text{id} \otimes \Delta_0\right) F(\lambda) F^{23}(\lambda),$$

(88)

$$\left(\epsilon_0 \otimes \text{id}\right) F(\lambda) = 1; \quad \left(\text{id} \otimes \epsilon_0\right) F(\lambda) = 1,$$

(89)

where $\Delta_0$ is the coproduct of $U_{H\mathfrak{g}}$, and $\epsilon_0$ is the counit map.

**Proof.** Using Proposition 7.4, we have

$$(\Delta \otimes \text{id}) F F^{12} = \left(\Delta \otimes \text{id}\right) F(\lambda) \Theta F^{12}(\lambda) \Theta^{12} = \left(\text{id} \otimes \Delta_0\right) F(\lambda) \Theta F^{12}(\lambda) \Theta^{12} = \left(\text{id} \otimes \Delta_0\right) F(\lambda) F^{12}(\lambda + h h^{(3)}) \left(\Delta \otimes \text{id}\right) \Theta \Theta^{12},$$

and

$$(\text{id} \otimes \Delta) F F^{23} = \left(\Delta \otimes \text{id}\right) F(\lambda) \Theta F^{23}(\lambda) \Theta^{23} = \left(\text{id} \otimes \Delta_0\right) F(\lambda) \Theta F^{23}(\lambda) \Theta^{23} = \left(\text{id} \otimes \Delta_0\right) F(\lambda) F^{23}(\lambda + h h^{(3)}) \Theta \Theta^{23}.$$}

Thus it follows from Lemma 7.1 that Equation (32) and Equation (88) are equivalent.

For Equation (33), we note that $F = F(\lambda) \Theta = \sum_{k=0}^{\infty} \frac{h^k}{k!} F(\lambda) \theta^k$. It is easy to see that for $k \geq 1$, $(\text{id} \otimes \epsilon_0)(F(\lambda) \theta^k) = (\Delta \otimes \text{id})(F(\lambda) \theta^k) = 0$ since $\epsilon_0(\frac{\partial}{\partial \theta_i}) = 0$ and $h_1 \cdots h_k = 0$. Thus it is immediate that Equations (33) and (89) are equivalent. This concludes the proof of the theorem.

A solution to Equations (88)-(89) is often called a shifted cocycle [1] [4] [24]. Moreover, if $U_{R\mathfrak{g}}$ is a quasi-triangular Hopf algebra with a quantum $R$-matrix $R$ satisfying the quantum Yang-Baxter equation, then $R(\lambda) = F^{21}(\lambda)^{-1} RF^{12}(\lambda)$ is a solution of the quantum dynamical Yang-Baxter equation [4]:

$$R^{12}(\lambda) R^{13}(\lambda + h h^{(2)}) R^{23}(\lambda) = R^{23}(\lambda + h h^{(1)}) R^{13}(\lambda) R^{12}(\lambda + h h^{(3)}).$$

(90)

Now assume that $F(\lambda)$ is a solution to Equations (88)-(89) so that we can form a quantum groupoid by twisting $\mathcal{D} \otimes U_{R\mathfrak{g}}$ via $F$. The resulting quantum groupoid is denoted by $\mathcal{D} \otimes U_{H\mathfrak{g}}$, and is called a dynamical quantum groupoid.

As an immediate consequence of Theorem 4.14, we have

**Theorem 7.6** As a monoidal category, the category of $\mathcal{D} \otimes U_{R\mathfrak{g}}$-modules is equivalent to that of $\mathcal{D} \otimes U_{H\mathfrak{g}}$-modules, and therefore is a braided monoidal category.
Remark. It is expected that representations of a quantum dynamical $R$-matrix [19] can be understood using this monoidal category of $\mathcal{D} \otimes \mathcal{U}_\mathfrak{g}$-modules. The further relations between these two objects will be investigated elsewhere.

In what follows, we describe various structures of $\mathcal{D} \otimes \mathcal{U}_\mathfrak{g}$ more explicitly.

**Proposition 7.7**

(i). $f \ast_f g = fg, \ \forall f, g \in \mathcal{M}(\eta^*)[[\hbar]], \ i.e., \ R_F$ is the usual algebra of functions.

(ii). $\alpha f = \exp (\hbar \sum_{i=1}^k h_i \frac{\partial}{\partial \lambda_i}) f = \sum_{1 \leq i_1, \ldots, i_n \leq k} \frac{\hbar^n}{n!} \frac{\partial^n}{\partial \lambda_{i_1} \cdots \partial \lambda_{i_n}} h_{i_1} \cdots h_{i_n}, \ \forall f \in \mathcal{M}(\eta^*)[[\hbar]]$.

(iii). $\beta f = f, \ \forall f \in \mathcal{M}(\eta^*)[[\hbar]]$.

**Proof.** Assume that $F(\lambda) = \sum_i F_i(\lambda) u_i \otimes v_i$, with $u_i, v_i \in \mathcal{U}_\mathfrak{g}$. Let

$$F_n = F(\lambda) \theta^n = \sum_i \sum_{1 \leq i_1, \ldots, i_n \leq k} F_i(\lambda) u_{i_1} \otimes \cdots \otimes R v_i h_{i_1} \cdots h_{i_n}.$$ 

Then

$$\alpha_{F_n} f = \sum_i \sum_{1 \leq i_1, \ldots, i_n \leq k} F_i(\lambda) (\epsilon_0 u_i) \frac{\partial^n f}{\partial \lambda_{i_1} \cdots \partial \lambda_{i_n}} v_i h_{i_1} \cdots h_{i_n}$$

$$\begin{align*}
&= \sum_{1 \leq i_1, \ldots, i_n \leq k} \frac{\partial^n f}{\partial \lambda_{i_1} \cdots \partial \lambda_{i_n}} [\sum_i F_i(\lambda) (\epsilon_0 u_i) v_i] h_{i_1} \cdots h_{i_n} \\
&= \sum_{1 \leq i_1, \ldots, i_n \leq k} \frac{\partial^n f}{\partial \lambda_{i_1} \cdots \partial \lambda_{i_n}} h_{i_1} \cdots h_{i_n},
\end{align*}$$

where the last equality used the fact that $\sum_i F_i(\lambda) (\epsilon_0 u_i) v_i = (\epsilon_0 \otimes \text{id}) F(\lambda) = 1$.

Similarly, we have $\beta_{F_n} f = f$ if $n = 0$, and otherwise $\beta_{F_n} f = 0$.

Combining these equations, one immediately obtains that

$$\alpha f = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \alpha_{F_n} f = \sum_{1 \leq i_1, \ldots, i_n \leq k} \frac{\hbar^n}{n!} \frac{\partial^n f}{\partial \lambda_{i_1} \cdots \partial \lambda_{i_n}} h_{i_1} \cdots h_{i_n},$$

$$\beta f = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \beta_{F_n} f = f,$$

and

$$f \ast_f g = (\alpha f)(g) = fg.$$

As it is standard [3] [24], using $F(\lambda)$, one may form a family of quasi-Hopf algebras $(U_h \mathfrak{g}, \Delta_\lambda)$, where the coproducts are given by $\Delta_\lambda = F(\lambda)^{-1} \Delta_0 F(\lambda)$. To describe the relation between $\Delta_F$ and these quasi-Hopf coproducts $\Delta_\lambda$, we need to introduce a “projection” map from $H \otimes \mathcal{D}$ to $\mathcal{M}(\eta^*, U_h \mathfrak{g} \otimes U_h \mathfrak{g})$. This can be defined as follows. Let $Ad_\Theta : H \otimes H \to H \otimes H$ be the adjoint operator: $Ad_\Theta w = \Theta w \Theta^{-1}, \ \forall w \in H \otimes H$. Composing with the natural projection, one obtains a map, denoted by the same symbol $Ad_\Theta$, from $H \otimes H$ to $H \otimes_R H$. Since $\alpha_\Theta = \alpha_F, \ \beta_\Theta = \beta_F$ and
\[ \Theta(\beta_\Theta f \otimes 1 - 1 \otimes \alpha_\Theta f) = 0, \forall f \in R, \text{ in } H \otimes_R H, \text{ then } \Theta(\beta_f f \otimes 1 - 1 \otimes \alpha_f f) = 0 \text{ in } H \otimes_R H. \] This implies that \( Ad_\Theta \) descends to a map from \( H \otimes_R H \) to \( H \otimes_R H \). On the other hand, there exists an obvious projection map \( Pr \) from \( H \otimes R H \) to \( M(\eta^*, U_h \otimes U_h \mathfrak{g}) \), which is just taking the 0th-order component. Now composing with this projection, one obtains a map from \( H \otimes_R H \) to \( M(\eta^*, U_h \otimes U_h \mathfrak{g}) \), which is denoted by \( T \). The following proposition gives an explicit description of this map \( T \).

An element \( x = D \otimes h u \in H \), where \( D \in \mathcal{D}[[h]] \) and \( u \in U_h \mathfrak{g} \), is said to be of order \( k \) if \( D \) is a homogeneous differential operator of order \( k \).

**Proposition 7.8**

(i) \( T(x \otimes_R y) = 0 \) if either \( x \) or \( y \) is of order greater than zero.

(ii) \( T(fu \otimes_R gv) = \sum_{1 \leq i_1, \ldots, i_n \leq k} \frac{h^n}{n!} \frac{\partial^n f}{\partial \lambda_{i_1} \cdots \partial \lambda_{i_n}} g(u \otimes h_{i_1} \cdots h_{i_n} v) = gu \otimes_R (\alpha_f f) v, \forall f, g \in M(\eta^*) \) and \( u, v \in U_h \mathfrak{g} \).

**Proof.** (i) is obvious. We prove (ii) below.

\[
T(fu \otimes_R gv) = Pr(e^h \sum_{i=1}^n (\frac{\partial}{\partial h_i}) (fu \otimes gv) e^{-h} \sum_{i=1}^n \frac{\partial}{\partial h_i} (fu \otimes gv))
\]
\[
= Pr(e^h \sum_{i=1}^n (\frac{\partial}{\partial h_i}) (fu \otimes gv))
\]
\[
= Pr(\sum_{1 \leq i_1, \ldots, i_n \leq k} \frac{h^n}{n!} \frac{\partial^n f}{\partial \lambda_{i_1} \cdots \partial \lambda_{i_n}} g(u \otimes h_{i_1} \cdots h_{i_n} v))
\]
\[
= \sum_{1 \leq i_1, \ldots, i_n \leq k} \frac{h^n}{n!} \frac{\partial^n f}{\partial \lambda_{i_1} \cdots \partial \lambda_{i_n}} g(u \otimes h_{i_1} \cdots h_{i_n} v)
\]
\[
= gu \otimes_R (\alpha_f f) v.
\]

\[ \square \]

**Proposition 7.9** The following diagram:

\[
\begin{array}{ccc}
H & \xrightarrow{\Delta_f} & H \otimes_R H \\
| & \searrow & \downarrow T \\
U_h \mathfrak{g} & \xrightarrow{\Delta^\lambda} & M(\eta^*, U_h \otimes U_h \mathfrak{g})
\end{array}
\]

commutes, where \( i : U_h \mathfrak{g} \to H \) is the natural embedding. I.e., \( \Delta^\lambda = T \circ \Delta_{f \circ i} \).

**Proof.** For any \( u \in U_h \mathfrak{g} \), \( (\Delta_f \circ i)(u) = \Delta_f(u) = F^{-1}(\Delta_0 u) F = \Theta^{-1} F(\lambda)^{-1}(\Delta_0 u) F(\lambda) \Theta \). Then \( (T \circ \Delta_{f \circ i})(u) = T[\Delta_f \circ i](u) = Pr[F(\lambda)^{-1}(\Delta_0 u) F(\lambda)] = F(\lambda)^{-1}(\Delta_0 u) F(\lambda) = \Delta^\lambda u \). The conclusion thus follows.
The following theorem describes the classical limit of the dynamical quantum groupoid $\mathcal{D} \otimes \hbar U_\hbar g$.

**Theorem 7.10** Let $(U_\hbar g, R)$ be a quasitriangular quantum universal enveloping algebra, and $R = 1 + \hbar r_0 \pmod{\hbar}$. Assume that $F(\lambda) \in U_\hbar g \otimes U_\hbar g$ is a shifted cocycle and that $F(\lambda) = 1 + \hbar f(\lambda) \pmod{\hbar}$. Then the classical limit of the corresponding dynamical quantum groupoid $\mathcal{D} \otimes \hbar U_\hbar g$ is a coboundary Lie bialgebroid $(\Lambda, \Lambda^*, \Lambda)$, where $\Lambda = T^\theta \otimes g$ and $\Lambda = \sum_{i=1}^k \partial_{\lambda i} \wedge h_i + \text{Alt}(\frac{1}{2} r_0 + f(\lambda))$.

**Proof.** It is well known \[9\] that $r_0 \in g \otimes g$, and the operator $\delta : g \rightarrow \wedge^2 g$, $\delta a = [1 \otimes a + a \otimes 1, r_0]$, $\forall a \in g$, defines the cobracket of the corresponding Lie bialgebra of $(U_\hbar g, R)$. Thus the Lie bialgebroid corresponding to $\mathcal{D} \otimes U_\hbar g$ is a coboundary Lie bialgebroid $(T^\theta \otimes g, T^* \otimes g^*)$ with $r$-matrix $\frac{1}{2} \text{Alt}(r_0)$. On the other hand, it is obvious that $\text{Alt} \lim_{\hbar \to 0} \hbar^{-1}(F - 1) = \text{Alt} f(\lambda) + \sum_{i=1}^k \partial_{\lambda i} \wedge h_i$. The conclusion thus follows from Theorem 5.20.

□

As a consequence, we have

**Corollary 7.11** Under the same hypotheses as in Theorem 7.10, $r(\lambda) = \text{Alt}(\frac{1}{2} r_0 + f(\lambda))$ is a classical dynamical $r$-matrix.

We refer the interested reader to $[1]$, $[13]$, $[20]$, $[24]$ for an explicit construction of shifted cocycles $F(\lambda)$ for semisimple Lie algebras.

We end this section by the following:

**Remark.** We may replace $\theta$ in Equation (78) by $\tilde{\theta} = \sum_{i=1}^k \frac{h}{2} (\frac{\partial}{\partial \lambda_i} \otimes h_i - h_i \otimes \frac{\partial}{\partial \lambda_i}) \in H \otimes H$ and set $\tilde{\Theta} = \exp \hbar \tilde{\theta} \in H \otimes H$. It is easy to show that $\tilde{F} = F(\lambda) \tilde{\Theta}$ satisfies Equation (92) is equivalent to the following condition for $F(\lambda)$:

$$[(\Delta_0 \otimes \text{id})F(\lambda)]F^{12}(\lambda + \frac{1}{2} \hbar h(3)) = [(id \otimes \Delta_0)F(\lambda)]F^{23}(\lambda - \frac{1}{2} \hbar h(1)).$$

In this case, $R(\lambda) = F^{21}(\lambda)^{-1}RF^{12}(\lambda)$ satisfies the symmetrized quantum dynamical Yang-Baxter equation:

$$R^{12}(\lambda - \frac{1}{2} \hbar h(3))R^{13}(\lambda + \frac{1}{2} \hbar h(2))R^{23}(\lambda - \frac{1}{2} \hbar h(1)) = R^{23}(\lambda + \frac{1}{2} \hbar h(1))R^{13}(\lambda - \frac{1}{2} \hbar h(2))R^{12}(\lambda + \frac{1}{2} \hbar h(3)).$$

In fact, both $\Theta$ and $\tilde{\Theta}$ can be obtained from the quantization of the cotangent bundle symplectic structure $T^* \eta^*$, using the normal ordering and the Weyl ordering respectively, so they are equivalent. This indicates that solutions to Equation (78) and Equation (92) are equivalent as well.
8 Appendix and open questions

Given any element \((i_1i_2i_3)\) in the symmetric group \(S_3\), by \(\sigma_{i_1i_2i_3}\) we denote the permutation operator on \(UA \otimes_R UA \otimes_R UA\) given by

\[
\sigma_{i_1i_2i_3}(x_1 \otimes_R x_2 \otimes_R x_3) = x_{i_1} \otimes_R x_{i_2} \otimes_R x_{i_3}.
\]

**Proposition 8.1** Assume that \(T \in UA \otimes_R UA\) satisfies

\[
T \otimes_R 1 + (\Delta \otimes_R id)T = 1 \otimes_R T + (id \otimes_R \Delta)T.
\]  

(93)

Then \(\text{Alt} \overset{\text{def}}{=} T - T_{21}\) is a section of \(\wedge^2 A\).

**Proof.** First we show that

\[
(\Delta \otimes_R id)\text{Alt} = 1 \otimes_R \text{Alt} + \sigma_{132}(\text{Alt} \otimes_R 1).
\]  

(94)

To prove this, write \(T = \sum_i u_i \otimes_R v_i\), where \(u_i, v_i \in UA\). Then Equation (93) becomes

\[
\sum_i u_i \otimes_R v_i \otimes_R 1 + \sum_i \Delta u_i \otimes_R v_i = \sum_i 1 \otimes_R u_i \otimes_R v_i + \sum_i u_i \otimes_R \Delta v_i.
\]  

(95)

Applying the permutation operators \(\sigma_{231}\) and \(\sigma_{132}\), respectively, on both sides of the above equation, one leads to

\[
\sum_i v_i \otimes_R 1 \otimes_R u_i + \sum_i \sigma_{231}(\Delta u_i \otimes_R v_i) = \sum_i u_i \otimes_R v_i \otimes_R 1 + \sum_i \Delta v_i \otimes_R u_i;
\]  

(96)

\[
\sum_i u_i \otimes_R 1 \otimes_R v_i + \sum_i \sigma_{132}(\Delta u_i \otimes_R v_i) = \sum_i 1 \otimes_R v_i \otimes_R u_i + \sum_i u_i \otimes_R \Delta v_i.
\]  

(97)

Combining Equations (95)-(97), \((95)+ (96)-(97))\), we obtain:

\[
\sum_i (\Delta u_i \otimes_R v_i - \Delta v_i \otimes_R u_i) = \sum_i (1 \otimes_R u_i \otimes_R v_i - 1 \otimes_R v_i \otimes_R u_i + u_i \otimes_R 1 \otimes_R v_i - v_i \otimes_R 1 \otimes_R u_i),
\]

which is equivalent to Equation (94). Here we used the identity: \(\sigma_{231}(\Delta u_i \otimes_R v_i) = \sigma_{132}(\Delta u_i \otimes_R v_i)\), which can be easily verified using the fact that \(\Delta u_i\) is symmetric.

The final conclusion essentially follows from Equation (94). To see this, let us write \(\text{Alt} T = \sum_i u_i \otimes_R v_i\), where \(\{v_i \in UA\}\) are assumed to be \(R\)-linearly independent.

From Equation (94), it follows that

\[
\sum_i \Delta u_i \otimes_R v_i = \sum_i (1 \otimes_R u_i \otimes_R v_i + u_i \otimes_R 1 \otimes_R v_i).
\]

I.e., \(\sum_i (\Delta u_i - 1 \otimes_R u_i - u_i \otimes_R 1) \otimes_R v_i = 0\). Hence \(\Delta u_i = 1 \otimes_R u_i + u_i \otimes_R 1\), which implies that \(u_i \in \Gamma (A)\). Since \(\text{Alt} T\) is skew symmetric, we conclude that \(\text{Alt} T \in \Gamma (\wedge^2 A)\).
Remark. It might be useful to consider the following cochain complex:

\[
0 \to R \xrightarrow{\partial} UA \xrightarrow{\partial} UA \otimes_R UA \xrightarrow{\partial} UA \otimes_R UA \otimes_R UA \xrightarrow{\partial} \]

where \( \partial : \otimes^n_RUA \to \otimes^{n+1}_RUA \), \( \partial = \partial^0 - \partial^1 + \cdots + (-1)^{n+1}\partial^n \), \( \partial^i(x_1 \otimes_R \cdots \otimes_R x_i) = x_1 \otimes_R \cdots \otimes_R x_i \Delta x_i \otimes_R x_i, \partial^{n+1}x = x \otimes_R 1 \). It is simple to check that \( \partial^2 = 0 \). In fact, this is the subcomplex of the Hochschild cochain complex of the algebra \( C^\infty(G) \ (G \text{ is a local Lie groupoid integrating the Lie algebroid } A) \) by restricting to the space of left invariant multi-differential operators. It is natural to expect that the cohomology of this complex is isomorphic to \( \Gamma(\wedge^*A) \), where the isomorphism from the cohomology group (more precisely, cocycles) to \( \Gamma(\wedge^*A) \) is the usual skew-symmetrization map. This is known to be true for Lie algebras [11] and the tangent bundle Lie algebroid [25]. However we could not find such a general result in the literature. In terms of this cochain complex, it is simple to describe what we have proved in Proposition 8.1. It simply means that \( \text{Alt} : UA \otimes_R UA \to UA \otimes_R UA \) maps 2-cocycles into \( \Gamma(\wedge^2A) \).

We end this paper by a list of open questions.

**Question 1:** We believe that techniques in [12] would be useful to prove the conjecture in Section 6. While the proof of Etingof and Kazhdan relies heavily on the double of a Lie bialgebra, the double of a Lie bialgebroid is no longer a Lie algebroid. Instead it is a Courant algebroid [28], where certain anomalies are inevitable. As a first step, it is natural to ask: what is the universal enveloping algebra of a Courant algebroid? Roytenberg and Weinstein proved that Courant algebroids give rise to homotopy Lie algebras [40]. It is expected that these homotopy Lie algebras are useful to understand this question as well as the quantization problem.

**Question 2:** One can form a Kontsevich’s formality-type conjecture for Lie algebroids, where one simply replaces in Kontsevich formality theorem [25] polyvector fields by sections of \( \wedge^*A \) and multi-differential operators by \( UA \otimes_R \cdots \otimes_R UA \) for a Lie algebroid \( A \). Does this conjecture hold? It is not clear if the method in [25] can be generalized to the context of general Lie algebroids. If this conjecture holds, it would imply that any triangular Lie bialgebroid is quantizable.

**Question 3:** Given a solution \( r : \eta^* \to \mathfrak{g} \otimes \mathfrak{g} \) of the classical dynamical Yang-Baxter equation:

\[
\text{Alt}dr - [r^{12}, r^{13}] - [r^{12}, r^{23}] - [r^{13}, r^{23}] = 0
\]

(in this case \( \text{Alt}(r) \) satisfies Condition (ii) of a dynamical \( r \)-matrix as in Section 2, if \( r + r^{21} \) is ad-invariant), a quantization of \( r \) is a quantum dynamical \( R \)-matrix \( R : \eta^* \to U_h \mathfrak{g} \otimes U_h \mathfrak{g} \) such that \( R(\lambda) = 1 + hr(\text{mod } h^2) \), where \( U_h \mathfrak{g} \) is a quantum universal enveloping algebra. Is every classical dynamical \( R \)-matrix quantizable? Many examples are known to be quantizable (e.g., see [13] for the quantization of classical dynamical \( r \)-matrices in Schiffmann’s classification list, and [52] for the quantization of classical triangular dynamical \( r \)-matrices). However, this problem still remains open for a general dynamical \( R \)-matrix.

**Question 4:** According to the general principle of deformation theory, any deformation corresponds to a certain cohomology. In particular, the deformation of a Hopf algebra is controlled by the cohomology of a certain double complex [21] [22]. It is natural to ask what is the proper
cohomology theory controlling the deformation of a Hopf algebroid, and in particular what is the premier obstruction to the quantization problem.

**Question 5:** What is the connection between dynamical quantum groupoids and quantum Virasoro algebra or quantum W-algebras [18] [12]?

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