A Conjecture Connected with Units of Quadratic Fields

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Abstract

In this article, we consider the order $O_f = \{x + yf\sqrt{d} : x, y \in \mathbb{Z}\}$ with conductor $f \in \mathbb{N}$ in a real quadratic field $K = \mathbb{Q}(\sqrt{d})$ where $d > 0$ is square-free and $d \equiv 2, 3 \pmod{4}$. We obtain numerical information about $n(f) = n(p) = \min\{\nu \in \mathbb{N} : \nu^e \in \mathcal{O}_p\}$ where $e > 1$ is the fundamental unit of $K$ and $p$ is an odd prime. Our numerical results suggest that the frequencies of $\frac{e+1}{2n(p)}$ or $\frac{e-1}{2n(p)}$ should have a limit as the ranges of $d$ and $p$ go to infinity.

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1 Introduction

Throughout the paper, we consider the real quadratic field $K = \mathbb{Q}(\sqrt{d})$ where $d > 0$ is square-free and $d \equiv 2, 3 \pmod{4}$. An order $\mathcal{O}$ in the field $K$ is a subset $\mathcal{O} \subset K$ such that

i) $\mathcal{O}$ is a subring of $K$ containing 1.
ii) $\mathcal{O}$ is a finitely generated $\mathbb{Z}$-module.
iii) $\mathcal{O}$ contains a $\mathbb{Q}$-basis of $K$.

Maximal order of an algebraic number field $K$ is the integral closure of the ring $\mathbb{Z}$ of rational numbers in the field $K$. So, among various orders of the field $K$ there is one maximal one which contains all the other orders which we call $\mathcal{O}_K$. Any order $\mathcal{O}$ of the field $\mathbb{Q}(\sqrt{d})$ is of the form $\mathcal{O}_f = \{1, f\omega\}$ where $f$ is the index $[\mathcal{O}_K : \mathcal{O}]$ namely, the conductor of the order. One can represent any number of
the order \( \mathcal{O}_f \) in the form \( \mathcal{O}_f = \{x + yf\sqrt{d} : x, y \in \mathbb{Z}\} \). For \( d \equiv 2, 3 \pmod{4} \) a basis \( \{1, \omega\} = \{1, \sqrt{d}\} \).

For \( d \equiv 2, 3 \pmod{4} \) one can write very well known Pell’s equation as

\[
(x + y\sqrt{d})(x - y\sqrt{d}) = \pm 1
\]

The \( s \text{-th} \) solution \( x_s, y_s \) can be expressed in terms of the first one \( x_1, y_1 \) by

\[
x_s + y_s\sqrt{d} = (x_1 + y_1\sqrt{d})^s \quad (s \in \mathbb{N})
\]  

(1.1)

The first solution \( (x_1, y_1) \in \mathbb{N}^2 \) is called fundamental solution to the Pell equation. Fundamental units \( \varepsilon \) for real quadratic fields \( \mathbb{Q}(\sqrt{d}) \) can be computed from the fundamental solution of Pell’s equation. So, fundamental units for the maximal order \( \mathcal{O}_K \) of the field \( K = \mathbb{Q}(\sqrt{d}) \) are also called fundamental units for the algebraic number field \( K = \mathbb{Q}(\sqrt{d}) \). For more information on units, orders and Pell’s equation see [Cox, p.133], [JW, p.80], [Neu, p.72] and other many algebraic number theory books. For the efficient computation of units and solution of Pell’s equation see [Len]. One can also refer to [JLW] to see about the question how large the fundamental unit \( \varepsilon \) can be.

In this article, \( \varepsilon \) is the fundamental unit of the real quadratic field \( \mathbb{Q}(\sqrt{d}) \) and \( N(\varepsilon) = \pm 1 \) is its norm. We consider the case that the conductor \( f \) is an odd prime \( p \) with \( p \nmid d \). The aim of this article is to get numerical information about

\[
n(f) = n(p) = \min\{ \nu \in \mathbb{N} : \varepsilon^\nu \in \mathcal{O}_p \}\.
\]

In [BP], we compute very good upper bounds for \( n(f) \) for the case \( N(\varepsilon) = +1 \) and in [Bii] for the case in particular that \( N(\varepsilon) = -1 \). It is known that (see for instance [BP]) for \( \left( \frac{d}{p} \right) = \mp 1 \), \( n(f) = n(p) \) is always a divisor of \( \frac{d + 1}{2n(p)} \) if \( N(\varepsilon) = +1 \) and of \( p \pm 1 \) if \( N(\varepsilon) = -1 \). Our results become easier to state if we consider the integers, quotients \( q \) defined by

\[
q = \begin{cases} 
\frac{p + 1}{2n(p)} & \text{if } N(\varepsilon) = +1 \quad \text{where } \left( \frac{d}{p} \right) = \mp 1 \\
\frac{p + 1}{n(p)} & \text{if } N(\varepsilon) = -1 \quad \text{where } \left( \frac{d}{p} \right) = \mp 1.
\end{cases}
\]

(1.2)

We compute the frequencies of \( q \). Our numerical results suggest that the frequencies may have a limit as the ranges of \( p \) and \( d \) go to infinity. We prove a theorem suggested by the numerical results and study the expectation of the frequency.

2 Computations and Conjecture

We recall that we always assume that \( d \equiv 2, 3 \pmod{4} \) is positive, square-free and \( p \) is an odd prime. We have \( \mathcal{O}_f = \{x + yf\sqrt{d} : x, y \in \mathbb{Z}\} \). Every unit of \( \mathcal{O} \) that belongs to \( \mathcal{O}_f \) is also a unit of \( \mathcal{O}_f \). 
We compute \( n(f) = n(p) = \min\{\nu \in \mathbb{N} : \varepsilon^\nu \in O_f\} \) for the case that \( f \) is an odd prime \( p \). For \( N(\varepsilon) = +1 \), the possible maximal values of \( n(f) \) are \( \frac{p+1}{2} \) for \( \left( \frac{q}{p} \right) = -1 \), \( \frac{p-1}{2} \) for \( \left( \frac{q}{p} \right) = +1 \); These give \( q = 1 \) defined in (1.2).

The computer we use has 2x DualCore-Opteron 2218 (2.6 GHz) with 16 GB(1 Node with 32 GB) RAM. One can also refer to the appendix to see the algorithm.

Our results depend on two arguments, the range and the quotient \( q \). It would be difficult to give a plot. For reasons of space we only present the first 20 frequencies in our tables as percentages. We take ranges \( d = [3, \ldots, d_{\text{max}}] \) and \( p = [3, \ldots, p_{\text{max}}] \) for \( N(\varepsilon) = +1 \). For \( N(\varepsilon) = -1 \), we take \( d = [2, \ldots, d_{\text{max}}] \) and \( p = [3, \ldots, p_{\text{max}}] \). For \( N(\varepsilon) = -1 \), the computations run faster because there are few occurrences so this enables us to implement larger and different ranges. To save space and to be neat, we only write \( d_{\text{max}} = p_{\text{max}} \) and some of the ranges we calculate in the following tables. The Legendre symbol differs according to the elements \( q \) defined in (1.2). For \( m \in \mathbb{N} \) we define four sequences according to the formula (1.2);

\[
F_1(q; m) = \text{card}\{q = (p - 1)/2n(p) : d, p \leq m, \left( \frac{q}{p} \right) = +1, N(\varepsilon) = +1\}.
\]

\[
F_2(q; m) = \text{card}\{q = (p + 1)/2n(p) : d, p \leq m, \left( \frac{q}{p} \right) = -1, N(\varepsilon) = +1\}.
\]

\[
F_3(q; m) = \text{card}\{q = (p - 1)/n(p) : d, p \leq m, \left( \frac{q}{p} \right) = +1, N(\varepsilon) = -1\}.
\]

\[
F_4(q; m) = \text{card}\{q = (p + 1)/n(p) : d, p \leq m, \left( \frac{q}{p} \right) = -1, N(\varepsilon) = -1\}.
\]

For instance, for the case \( N(\varepsilon) = +1 \), by the definition (1.2) of \( q \), we have \( q \leq \frac{p - 1}{2} \). Hence \( F_1 = 0 \) for \( q > \frac{p - 1}{2} \). This means that, when the ranges become larger, more and more frequencies will become positive. This suggests that for a fixed \( q \) the frequencies will slowly fluctuate.

Furthermore, let \( S_j(m) = \sum_q F_j(q; m) \). In the tables values shows the number of occurrences of \( \left( \frac{d}{p} \right) = +1 \) and \( \left( \frac{d}{p} \right) = -1 \) respectively. The tables suggest the following conjecture:

**Conjecture 2.1.** Let \( j = 1, 2, 3, 4 \). There is a probability distribution \( P_j(q) \) such that, for all \( q \),

\[
F_j(q; m)/S_j(m) \to P_j(q) \text{ as } m \to \infty.
\]

More precisely: For \( j = 1, \ldots, 4 \) there is a function

\[
P_j : \mathbb{N} \to [0, 1] \text{ with } \sum_q P_j(q) = 1
\]

such that for every \( \delta > 0 \) there exists \( m_0 \) with the property that

\[
|F_j(q; m)/S_j(m) - P_j(q)| < \delta
\]

for all \( q \) and \( m \geq m_0 \).
\( N(\varepsilon) = +1, \left( \frac{\varepsilon}{2} \right) = +1 \)

| \( q \) | \( d_{\text{max}} \) | 10000 | 20000 | 40000 | 60000 | 80000 | 100000 |
|-----|----------------|------|-------|-------|-------|-------|--------|
| 1   | 57.3           | 56.9 | 56.9  | 56.7  | 56.7  | 56.6  |
| 2   | 11.8           | 12.0 | 12.0  | 12.1  | 12.2  | 12.2  |
| 3   | 9.91           | 9.89 | 9.89  | 9.88  | 9.86  | 9.88  |
| 4   | 4.66           | 4.72 | 4.64  | 4.64  | 4.65  | 4.67  |
| 5   | 2.86           | 2.84 | 2.84  | 2.84  | 2.82  | 2.82  |
| 6   | 1.98           | 2.05 | 2.10  | 2.13  | 2.14  | 2.16  |
| 7   | 1.34           | 1.35 | 1.34  | 1.35  | 1.34  | 1.34  |
| 8   | 1.12           | 1.16 | 1.16  | 1.16  | 1.16  | 1.16  |
| 9   | 1.09           | 1.08 | 1.09  | 1.08  | 1.09  | 1.09  |
| 10  | 0.604          | 0.610| 0.609 | 0.612 | 0.618 | 0.619 |
| 11  | 0.530          | 0.515| 0.521 | 0.518 | 0.517 | 0.510 |
| 12  | 0.817          | 0.823| 0.841 | 0.854 | 0.866 | 0.874 |
| 13  | 0.349          | 0.365| 0.363 | 0.362 | 0.363 | 0.363 |
| 14  | 0.394          | 0.292| 0.289 | 0.292 | 0.291 | 0.295 |
| 15  | 0.497          | 0.489| 0.494 | 0.492 | 0.491 | 0.495 |
| 16  | 0.280          | 0.306| 0.302 | 0.296 | 0.293 | 0.293 |
| 17  | 0.216          | 0.214| 0.210 | 0.214 | 0.213 | 0.210 |
| 18  | 0.221          | 0.229| 0.237 | 0.239 | 0.238 | 0.240 |
| 19  | 0.161          | 0.164| 0.159 | 0.164 | 0.163 | 0.162 |
| 20  | 0.233          | 0.238| 0.239 | 0.236 | 0.238 | 0.240 |
| ... | ...            | ...  | ...   | ...   | ...   | ...   |
| Values | 2240624 | 8330759 | 31140682 | 67496484 | 116576513 | 178609439 |

\( N(\varepsilon) = +1, \left( \frac{\varepsilon}{2} \right) = -1 \)

| \( q \) | \( d_{\text{max}} \) | 10000 | 20000 | 40000 | 60000 | 80000 | 100000 |
|-----|----------------|------|-------|-------|-------|-------|--------|
| 1   | 56.3           | 56.2 | 56.0  | 56.1  | 56.1  | 56.1  |
| 2   | 14.5           | 14.4 | 14.4  | 14.3  | 14.3  | 14.4  |
| 3   | 9.94           | 9.97 | 9.98  | 9.97  | 10.0  | 9.98  |
| 4   | 3.32           | 3.31 | 3.36  | 3.37  | 3.37  | 3.34  |
| 5   | 2.77           | 2.76 | 2.80  | 2.80  | 2.80  | 2.81  |
| 6   | 2.45           | 2.48 | 2.49  | 2.49  | 2.49  | 2.50  |
| 7   | 1.32           | 1.36 | 1.35  | 1.36  | 1.36  | 1.35  |
| 8   | 0.868          | 0.840| 0.849 | 0.836 | 0.839 | 0.837 |
| 9   | 1.07           | 1.10 | 1.10  | 1.11  | 1.12  | 1.12  |
| 10  | 0.718          | 0.728| 0.721 | 0.725 | 0.736 | 0.729 |
| 11  | 0.512          | 0.502| 0.518 | 0.518 | 0.519 | 0.514 |
| 12  | 0.561          | 0.566| 0.583 | 0.581 | 0.578 | 0.571 |
| 13  | 0.352          | 0.363| 0.352 | 0.355 | 0.353 | 0.354 |
| 14  | 0.341          | 0.339| 0.328 | 0.339 | 0.333 | 0.332 |
| 15  | 0.480          | 0.487| 0.508 | 0.507 | 0.508 | 0.506 |
| 16  | 0.202          | 0.200| 0.206 | 0.206 | 0.210 | 0.209 |
| 17  | 0.190          | 0.199| 0.201 | 0.201 | 0.200 | 0.204 |
| 18  | 0.274          | 0.268| 0.274 | 0.274 | 0.272 | 0.273 |
| 19  | 0.165          | 0.168| 0.169 | 0.168 | 0.168 | 0.165 |
| 20  | 0.177          | 0.176| 0.175 | 0.175 | 0.174 | 0.171 |
| ... | ...            | ...  | ...   | ...   | ...   | ...   |
| Values | 2272057 | 8390244 | 31303879 | 67789746 | 117013651 | 179198341 |
\[ N(\varepsilon) = -1, \left( \frac{\varepsilon}{2} \right) = +1 \]

| \( q \) | \( d_{\text{max}} \) |
|--------|------------------|
|       | 40000 | 60000 | 80000 | 100000 | 200000 | 400000 |
| 1     | 37.8  | 37.6  | 37.6  | 37.5   | 37.4   | 37.5   |
| 2     | 18.7  | 18.8  | 18.8  | 18.8   | 18.7   | 18.7   |
| 3     | 6.58  | 6.56  | 6.60  | 6.62   | 6.64   | 6.62   |
| 4     | 14.0  | 14.0  | 14.0  | 14.0   | 14.0   | 14.0   |
| 5     | 1.90  | 1.91  | 1.89  | 1.90   | 1.89   | 1.90   |
| 6     | 3.28  | 3.30  | 3.29  | 3.29   | 3.31   | 3.31   |
| 7     | 0.891 | 0.897 | 0.899 | 0.898  | 0.894  | 0.892  |
| 8     | 3.46  | 3.48  | 3.49  | 3.51   | 3.50   | 3.49   |
| 9     | 0.724 | 0.716 | 0.725 | 0.728  | 0.733  | 0.735  |
| 10    | 0.930 | 0.928 | 0.936 | 0.937  | 0.943  | 0.938  |
| 11    | 0.352 | 0.351 | 0.348 | 0.340  | 0.344  | 0.339  |
| 12    | 2.49  | 2.50  | 2.48  | 2.48   | 2.49   | 2.49   |
| 13    | 0.237 | 0.239 | 0.240 | 0.243  | 0.239  | 0.245  |
| 14    | 0.440 | 0.440 | 0.438 | 0.442  | 0.444  | 0.447  |
| 15    | 0.329 | 0.329 | 0.330 | 0.333  | 0.333  | 0.334  |
| 16    | 0.866 | 0.868 | 0.863 | 0.866  | 0.872  | 0.874  |
| 17    | 0.140 | 0.143 | 0.143 | 0.140  | 0.138  | 0.140  |
| 18    | 0.371 | 0.371 | 0.365 | 0.366  | 0.369  | 0.368  |
| 19    | 0.105 | 0.109 | 0.108 | 0.107  | 0.109  | 0.109  |
| 20    | 0.705 | 0.706 | 0.705 | 0.702  | 0.706  | 0.703  |

Values: 2812887 5965306 10168304 15404441 56642335 205444859

\[ N(\varepsilon) = -1, \left( \frac{\varepsilon}{2} \right) = -1 \]

| \( q \) | \( d_{\text{max}} \) |
|--------|------------------|
|       | 40000 | 60000 | 80000 | 100000 | 200000 | 400000 |
| 1     | 37.9  | 37.6  | 37.6  | 37.6   | 37.6   | 37.5   |
| 2     | 37.1  | 37.2  | 37.2  | 37.2   | 37.3   | 37.3   |
| 3     | 6.66  | 6.65  | 6.64  | 6.62   | 6.61   | 6.65   |
| 4     | 0.000 | 0.000 | 0.000 | 0.000  | 0.000  | 0.000  |
| 5     | 1.90  | 1.90  | 1.92  | 1.92   | 1.90   | 1.89   |
| 6     | 6.59  | 6.60  | 6.64  | 6.63   | 6.64   | 6.64   |
| 7     | 0.889 | 0.910 | 0.897 | 0.891  | 0.891  | 0.899  |
| 8     | 0.000 | 0.000 | 0.000 | 0.000  | 0.000  | 0.000  |
| 9     | 0.727 | 0.732 | 0.729 | 0.723  | 0.731  | 0.733  |
| 10    | 1.85  | 1.85  | 1.84  | 1.87   | 1.87   | 1.88   |
| 11    | 0.344 | 0.345 | 0.347 | 0.351  | 0.338  | 0.343  |
| 12    | 0.000 | 0.000 | 0.000 | 0.000  | 0.000  | 0.000  |
| 13    | 0.233 | 0.236 | 0.238 | 0.240  | 0.241  | 0.239  |
| 14    | 0.895 | 0.907 | 0.907 | 0.900  | 0.899  | 0.895  |
| 15    | 0.342 | 0.336 | 0.336 | 0.332  | 0.334  | 0.333  |
| 16    | 0.000 | 0.000 | 0.000 | 0.000  | 0.000  | 0.000  |
| 17    | 0.135 | 0.138 | 0.139 | 0.140  | 0.138  | 0.138  |
| 18    | 0.734 | 0.741 | 0.750 | 0.754  | 0.742  | 0.742  |
| 19    | 0.112 | 0.112 | 0.113 | 0.111  | 0.110  | 0.110  |
| 20    | 0.000 | 0.000 | 0.000 | 0.000  | 0.000  | 0.000  |

Values: 2828439 5992331 10206629 15404072 56197833 205855014
Comments: The first two tables appear to be rather similar. In the first two tables \( q = 1 \) predominates. In the third table the sum of frequencies for \( q = 1 \) and \( q = 2 \) is about the frequency for \( q = 1 \) of the first two tables.

But the last table is quite different. The first two frequencies are almost equal and we notice that when \( q = 4k \) for \( k \in \mathbb{N} \), \( F_4(4;m) = 0 \). Now, we give a proof that this is indeed true.

**Theorem 2.1.** Let \( N(\varepsilon) = -1, \left( \frac{d}{p} \right) = -1 \). Then the case \( q = 4 \) does not appear.

**Proof.** Suppose that \( q = 4 \) does occur. Then \( F_4(4;m) \neq 0 \) and thus \( \nu = \frac{p+1}{4} \).

In [Bir] we wrote \( \varepsilon^n = A\nu + B\nu - 1 v\sqrt{d} \).

In our case \( \nu = \frac{p+1}{4} \) and thus \( B_{\frac{p+1}{4}-1} \). Since \( \nu \in \mathbb{N} \) it follows that \( p \equiv 3 \pmod{4} \). In [Bir] Theorem 3.1 we proved that, for \( \left( \frac{d}{p} \right) = -1, A_{\frac{p+1}{4}} \equiv 0 \pmod{p} \) if \( \left( \frac{\nu}{p} \right) = -1 \) and for \( N \in \mathbb{Z}, N \neq 0 \) by using the following formula

\[
2(x^2 - N)B_{\nu-1}(x)^2 = A_{2\nu}(x) - N^\nu
\]

we obtain

\[
2(x^2 + 1)B_{\frac{p+1}{4}-1}(x)^2 \equiv A_{\frac{p+1}{4}}(x) - (-1)^{\frac{p+1}{4}} \\
\equiv 1 \pmod{p}
\]

So, \( B_{\frac{p+1}{4}-1} \neq 0 \pmod{p} \) which is a contradiction. \( \square \)

We compute the expectation of the probability distribution for \( N(\varepsilon) = +1, \left( \frac{d}{p} \right) = +1 \) which are listed below.

| Ranges= m | 40000 | 50000 | 60000 | 70000 | 80000 | 100000 |
|-----------|-------|-------|-------|-------|-------|--------|
| Expectation= E | 7.300 | 7.742 | 7.681 | 7.786 | 7.922 | 8.152 |
| \log(m) \times \log(\log m) | 25.012 | 25.763 | 26.383 | 26.536 | 27.362 | 28.129 |
| \frac{E}{(\log(m) \times \log(\log m))} | 0.291 | 0.290 | 0.291 | 0.293 | 0.289 | 0.289 |

| Ranges= m | 10000 | 10000 | 15000 | 20000 | 25000 | 30000 |
|------------|-------|-------|-------|-------|-------|--------|
| Expectation= E | 3.921 | 6.086 | 6.297 | 6.759 | 6.946 | 7.098 |
| \log(m) \times \log(\log m) | 13.344 | 20.448 | 21.758 | 22.697 | 23.441 | 24.047 |
| \frac{E}{(\log(m) \times \log(\log m))} | 0.293 | 0.297 | 0.294 | 0.297 | 0.296 | 0.296 |

This table suggests that the expectation has the order of \( \log(m) \times \log(\log m) \) for large \( m \) where \( m \) is the range of \( d \) and \( p \).
Appendix

We use the following procedure to obtain the values. The computation is similar for \( N(\varepsilon) = -1 \). At each step of the computation the numbers are modulo \( p \) so that they are 32 bit numbers. To speed things up, one should not calculate \( \varepsilon^i \) as an element of the number field. It is faster to just calculate the coefficients modulo \( p \) with respect to the basis \( \{1, \sqrt{d}\} \). They can be calculated with a fast exponentiation algorithm. Computations are made by using [BCP].

Algorithm 1 Finding \( n(f) = n(p) \)

1: Compute the Norm of \( \varepsilon \) and the Legendre Symbol \( ls \leftarrow \left( \frac{d}{p} \right) \)
2: if \( \text{Norm}(\varepsilon) = 1 \) then
3: \( q \leftarrow (p - ls)/2 \).
4: else
5: \( q \leftarrow p - ls \)
6: end if
7: Calculate factorization of \( q = p_1^{r_1} \ldots p_t^{r_t} \) and set \( \nu \leftarrow 1 \)
8: for \( i \) in \([1..t]\) do
9: Calculate \( F[i] \leftarrow \max\{b \in \{0, \ldots, r_i\} | \text{for which the second coefficient } y_{q/p_i^b} \text{ (see (1.1)) of } \varepsilon q/p_i^b \text{ is divisible by } p\} \)
10: \( \nu \leftarrow \nu \cdot p_i^{F[i]} \)
11: end for
12: Compute \( \frac{p+1}{2n(p)} \) for \( N(\varepsilon) = 1 \) or \( \frac{p+1}{n(p)} \) for \( N(\varepsilon) = -1 \).

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