RESULTS ON CONTROLLABILITY OF NON-DENSELY CHARACTERIZED NEUTRAL FRACTIONAL DELAY DIFFERENTIAL SYSTEM

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Abstract. This work establishes the controllability of nondense fractional neutral delay differential equation under Hille-Yosida condition in Banach space. The outcomes are derived with the aid of fractional calculus theory, semigroup operator theory and Schauder fixed point theorem. Theoretical results are verified through illustration.

1. Introduction. Classical calculus theory is the core for fractional calculus theory which is related with integrals and derivatives of arbitrary order. Fractional calculus theory has attained more importance and popularity because of its utilization in developments of science and engineering. Due to the colossal opportunity and usage, considerable papers have been concerned to articulate the solutions of fractional systems [1, 2, 4, 6, 7, 9, 10, 11, 13, 15, 17, 19, 20, 24, 26, 28, 29, 33, 35, 36] and the

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sources in that respect. Diversely, in the advancement of present mathematical control theory, controllability acts as a remarkable responsibility. It is an approximate significance in control theory and is of instinctive property of dynamical control system. Controllability permits one to drive the system to an arbitrary final state with the set of permissible controls [3, 5, 8, 12, 14, 18, 23, 25, 27, 30, 31, 32, 37]. One can refer the monographs [16, 21, 22, 38, 34, 39] for further details.

The time delays are periodically experienced in many practical and industrial systems like fuzzy systems, bio engineering, chemical processing, vehicle suspension systems, circuits, automatic control and so on. Neutral delay differential systems are referred that the system not only contain delay in the system also in the derivatives of system states. At(2007), Fu et al. worked on the controllability of abstract nondense neutral fractional differential systems. Recently(2010), Kavitha et al. examined the controllability of abstract nondense neutral fractional differential systems with impulsive condition. Very recently(2017), Gu et al. investigated the integral solution using noncompact measure method for nonlinear fractional order evolution equation.

Stimulated with the above discussion, we start this work to analyze the controllability of delay fractional differential system with nondense linear operator in Banach space of the form

$$C^D_p [z(\omega) - \mathcal{H} (\omega, z_\omega)] = A[z(\omega) - \mathcal{H} (\omega, z_\omega)] + F(\omega, z_\omega) + Bu(\omega), \quad \omega \in J = \{0, b\}$$

$$z(\omega) = \phi(\omega) \in \mathcal{B}_h, \quad \omega \in (-\infty, 0],$$

where $C^D_p$ is the derivative of fractional order with $0 < p < 1$ in Caputo’s sense, the nondense linear operator $A$ from $D(A) \subseteq X$ to $X$ originates a unified semigroup \{\(T(\omega)\)\}_{\omega \geq 0} in \(X\), the Banach space. Also $F : J \times \mathcal{B}_h \rightarrow X$, $\mathcal{H} : J \times \mathcal{B}_h \rightarrow X$ are appropriate functions. The state variable $z(\cdot)$ assigns values in $X$ with $||\cdot||$ and $u(\cdot)$, the control function is defined in a Banach space of permissible control functions $L^2(J, U)$ with a Banach space $U$. The linear operator $B : U \rightarrow X$ is bounded. The histories $z_\omega : (-\infty, 0] \rightarrow X$, defined by $z_\omega(\tau) = z(\omega + \tau)$, $\tau \leq 0$, in $\mathcal{B}_h$, some abstract phase space.

In this work, section II deals with the preliminary results as well as definitions. We discuss some adequate conditions in section III for the presence of control term of fractional neutral delay differential system with nondense linear operator. We provide an illustration in section IV to validate the theoretical outcomes.

2. Primitive results.

$$\mathcal{B}_h = \{ \Omega : (-\infty, 0] \rightarrow X \mbox{ provided for some } c > 0, \Omega |_{[-c, 0]} \in \mathcal{B} \mbox{ and } \int_{-\infty}^{0} h(\alpha) ||\Omega||_{[\alpha, 0]} d\alpha < +\infty \}.$$  

Let $\mathcal{B}_h$ be enriched with

$$||\Omega||_{\mathcal{B}_h} = \int_{-\infty}^{0} h(\alpha) ||\Omega||_{[\alpha, 0]} d\alpha, \quad \forall \Omega \in \mathcal{B}_h,$$

clearly, $\mathcal{B}_h$ is a Banach space which satisfies the norm $||\cdot||_{\mathcal{B}_h}$. Now, define

$$\mathcal{B}_h' = \{ z : (-\infty, b] \rightarrow X \mbox{ provided } z{|}_J \in C(J, X), \quad z_0 = \phi \in \mathcal{B}_h \}.$$
with semi norm
\[ \|z\|_b = \|\phi\|_{\mathcal{B}_h} + \sup\{\|z(\alpha)\| : \alpha \in [0, b]\}, \quad z \in \mathcal{B}'_h. \]

**Lemma 2.1.** [31] Let \( z \in \mathcal{B}'_h \), then for \( \omega \in J \), \( z_\omega \in \mathcal{B}_h \). Also, \( l_1 = \int_0^\infty h(\omega) d\omega < +\infty \).

Hence, \( A \) is known as locally Lipschitz continuous.

\[ \|A(z)\|_{\mathcal{B}_h} \leq \|\phi\|_{\mathcal{B}_h} + l_1 \sup_{\alpha \in [0, \omega]} |z(\alpha)|, \]

where \( l_1 = \int_0^\infty h(\omega) d\omega < +\infty \).

Let us get into the basic concepts, few Lemmas and assumptions which will be used further. For more details refer [31]

**Definition 2.2.** [13] Let \( \{T(\omega)\}_{\omega \geq 0} \) be an unified semigroup originated by \( A \) on \( X \) and if there is \( \lambda \in \mathbb{R}^+ \), then \( T(\omega) \) is known as locally Lipschitz continuous.

**Definition 2.3.** [13] Let \( \{T(\omega)\}_{\omega \geq 0} \) be an unified semigroup originated by \( A \) on \( X \) and if there is \( \lambda \in \mathbb{R}^+ \), then \( T(\omega) \) is known as locally Lipschitz continuous.

**Definition 2.4.** If there is a constant \( L \) for all \( \delta > 0 \), provided
\[ |T(\omega) - T(\alpha)| \leq L|\omega - \alpha|, \quad \text{for all } \omega, \alpha \in [0, \delta], \]

then \( \{T(\omega)\}_{\omega \geq 0} \) is known as locally Lipschitz continuous.

Also, \( \{T(\omega)\}_{\omega \geq 0} \) is said to be non-degenerate if \( T(\omega)z = 0 \), for every \( \omega \geq 0 \), gives \( z = 0 \).

**Definition 2.5.** (Hille condition) [34] There is a \( \lambda \) such that \( \omega \in \mathbb{R} \) provided \( (\omega, +\infty) \subset \rho(A) \), the resolvent operator of \( A \) and
\[ \sup\{(\lambda - \omega)^n : n \in \mathbb{N}, \lambda > \omega\} \leq \frac{\lambda}{|\rho(A)|}. \]

**Theorem 2.6.** [13] A fulfills the Hille condition if and only if it originates a non-degenerate locally Lipschitz continuous unified semigroup.

Let \( A_0 \) be the subset of \( A \) on \( \overline{D(A)} \) characterized as
\[ D(A_0) = \{\kappa \in D(A) : A\kappa \in \overline{D(A)}\}, \]
\[ A_0\kappa = A\kappa, \quad \text{for } \kappa \in D(A_0). \]

Hence \( A_0 \) originates a \( C_0 \)-semigroup \( T(\omega) \) on \( \overline{D(A)} \).

To derive the solution of (1)-(2), take the subsidiary problem,
\[ C_0^D z(\omega) = Az(\omega) + f(\omega), \quad \omega \in (0, b], \]

\[ z(0) = z_0. \]

**Theorem 2.7.** [9] \( z(\omega) \) is a solution of (3) if and only if
\[ z(\omega) = S_\omega z_0 + \lim_{\lambda \to +\infty} \int_0^\omega K_\omega(\omega - \alpha)\lambda(\lambda I - A)^{-1} f(\alpha) d\alpha, \]

for \( \omega \in J \) and \( z_0 \in X_0 \).
Definition 2.8. A function \( z : (-\infty, b] \to X \) be a solution of (1)-(2) if \( z(\omega) = \phi(\omega) \in \mathcal{B}_h \) on \( (-\infty, 0] \) and \( \int_0^\omega (\omega - \alpha)^{p-1}[z(\alpha) - \mathcal{H}(\omega, z_\alpha)]d\alpha \in D(A) \), for \( \omega \in J \),
\[
z(\omega) = T(\omega)[\phi(0) - \mathcal{H}(0, \phi)] + \mathcal{H}(\omega, z_\omega)
+ \lim_{\lambda \to +\infty} \int_0^\omega (\omega - \alpha)^{p-1}S(\omega - \alpha)\lambda(\lambda I - A)^{-1}F(\alpha)d\alpha
+ \lim_{\lambda \to +\infty} \int_0^\omega (\omega - \alpha)^{p-1}S(\omega - \alpha)\lambda(\lambda I - A)^{-1}Bu(\alpha)d\alpha, \quad \omega \in J,
\]
where the characteristic operators \( T(\cdot) \) and \( S(\cdot) \) are
\[
T = \int_0^\infty \xi_p(\theta)T(\omega^p\theta)d\theta, \quad S = p \int_0^\infty \theta\xi_p(\theta)T(\omega^p\theta)d\theta,
\]
and for \( \theta \in (0, \infty) \)
\[
\xi_p(\theta) = \frac{1}{p} \theta^{-1-\frac{1}{p}} \overline{\pi}_p(\theta^{-\frac{1}{p}}) \geq 0,
\]
\[
\overline{\pi}_p(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1}\theta^{-n-1}\frac{\Gamma(np+1)}{n!} \sin(n\pi p),
\]
where, the pdf \( \xi_p \) is defined on \((0, \infty)\).

Lemma 2.9. [35] Operators \( T \) and \( S \) satisfies the subsequent claims:
(i) The operators \( T \) and \( S \) are bounded and linear for any fixed \( \omega \geq 0 \), that is,
\[
\|T(\omega)z\| \leq \mathcal{M}\|z\| \quad \text{and} \quad \|S(\omega)z\| \leq \frac{p\mathcal{M}}{\Gamma(1+p)}\|z\|, \quad \text{for any} \quad z \in X.
\]
(ii) \( \{T(\omega), \omega \geq 0\} \) and \( \{S(\omega), \omega \geq 0\} \) are strongly continuous.
(iii) \( T(\omega), S(\omega) \) are compact operators if \( T(\omega) \) is compact for every \( \omega > 0 \).

3. Controllability results.

\( \mathcal{H}_1 \) Let \( \frac{\lambda}{(\lambda I - A)} = \lambda R(A) \). Hence for all \( z \in D(A) \), when \( \lambda \to +\infty, R(A)k \to k \).
Further more by Definition 2.8 (for \( n = 1 \)), then
\[
\lim_{\lambda \to +\infty} |R(A)k| \leq \mathcal{M}|k|, \quad \text{since} \quad |\lambda R(A)| \leq \frac{\mathcal{M}\lambda}{\lambda - \omega}.
\]
Therefore \( \lim_{\lambda \to +\infty} |R(A)| \leq \mathcal{M} \).

\( \mathcal{H}_2 \) \( T(\omega) \) is compact when \( \omega > 0 \) and for a constant \( \mathcal{M} \geq 1 \) provided
\[
\sup_{\omega \in J} \|T(\omega)\| \leq \mathcal{M}.
\]

\( \mathcal{H}_3 \) \( \mathcal{H} : J \times \mathcal{B}_h \) which is continuous and for some constants \( l_{1,\mathcal{H}}, \tilde{l}_{1,\mathcal{H}} > 0 \), provided
\[
\|\mathcal{H}(\omega, z) - \mathcal{H}(\omega, x)\| \leq l_{1,\mathcal{H}}\|z - x\|_{\mathcal{B}_h}, \quad z, x \in \mathcal{B}_h,
\]
\[
\|\mathcal{H}(\omega, z)\| \leq \tilde{l}_{1,\mathcal{H}}(1 + \|z\|_{\mathcal{B}_h}).
\]

\( \mathcal{H}_4 \) \( \mathcal{F} : J \times \mathcal{B}_h \to X; (\omega, \phi) \to \mathcal{F}(\omega, \phi) \) is strongly measurable for every \( \phi \in \mathcal{B}_h \),
continuous with respect to \( \phi \) for a.e. \( \omega \in J \) and is completely continuous.
For every \( \omega \in J \), \( \mathcal{F}(\omega, \cdot) \) is u.s.c; also for every \( z \in \mathcal{B}_h \), \( \mathcal{F}(\cdot, z) \) is measurable.
For every $r > 0$ and $z \in \mathcal{C}$, the space of continuous functions with $\|z\|_{\mathcal{C}} \leq r$, there is a function $\mathcal{L}_r : J \rightarrow \mathbb{R}^+$, such that

$$\sup\left\{ \|\mathcal{F}(\omega, z_\omega)\| \right\} \leq \mathcal{L}_r(\omega),$$

for a.e. $\omega \in J$.

The function $\alpha \rightarrow (\omega - \alpha)^{p-1}\mathcal{L}_r(\alpha) \in L^1(J, \mathbb{R}^+)$ and some $\varrho > 0$ implies

$$\lim_{r \rightarrow \infty} \inf_{\mathcal{M}_2} \int_0^b (\omega - \alpha)^{p-1}\mathcal{L}_r(\alpha)d\alpha = \varrho < +\infty.$$

The operator $W : L^2(J, U) \rightarrow D(A)$ is characterized as

$$W = \lim_{\lambda \rightarrow +\infty} \int_0^b T(b - \alpha)\lambda R(A)Bu(\alpha)d\alpha.$$

Also $W^{-1}$ exists and assumes values in $L^2(J, U) | \ker W$, and for $\mathcal{M}_2, \mathcal{M}_3 \geq 0$, provided $\|B\| \leq \mathcal{M}_2$,

$$\|W^{-1}\| \leq \mathcal{M}_3.$$

**Theorem 3.1.** Let assumptions $H_1 - H_7$ holds, then (1)-(2) has a solution on $J$, provided

$$l_1 \left[ l_2 \left( \frac{b}{\mathcal{M}_2} \mathcal{M}_1 \mathcal{M}_3 \right) \left( 1 + \mathcal{M}_1 \left( \frac{b}{\mathcal{M}_2} \mathcal{M}_3 \right)^{2b^p} \frac{1}{p} \right) \right] < 1,$$

where $\mathcal{M}_2 = \|B\|$.

**Proof of Theorem 3.1.** From hypothesis $H_6$, define the control for an arbitrary function $z(\cdot)$ as

$$u(\omega) = W^{-1} \left[ z_1 - T(b) [\phi(0) - \mathcal{H}(0, \phi)] - \mathcal{H}(b, z_b) \right.\left. - \lim_{\lambda \rightarrow +\infty} \int_0^b (b - \alpha)^{p-1}S(b - \alpha)\lambda R(A)F(\alpha, z_\omega)d\alpha \right] (\omega).$$

Construct the operator $\Gamma$ as

$$\Gamma z(\omega) = \begin{cases} \phi(\omega), & \omega \in (-\infty, 0], \\ T(\omega)[\phi(0) - \mathcal{H}(0, \phi)] + \mathcal{H}(\omega, z_\omega) \end{cases} + \lim_{\lambda \rightarrow +\infty} \int_0^b (\omega - \alpha)^{p-1}S(\omega - \alpha)\lambda R(A)F(\alpha, z_\omega)d\alpha + \lim_{\lambda \rightarrow +\infty} \int_0^b (\omega - \alpha)^{p-1}S(\omega - \alpha)\lambda R(A)Bu(\alpha)d\alpha, \ \omega \in J.$$

Now we prove that $\Gamma$ has fixed point, the solution of (1)-(2). Obviously, $z_b = \Gamma z(b) = z_1$.

For $\phi \in \mathcal{B}_H$, we consider $\hat{\phi}$ as

$$\hat{\phi}(\omega) = \begin{cases} \phi(\omega), & \omega \in (-\infty, 0], \\ T(\omega)\phi(0), & \omega \in J, \end{cases}$$
Therefore \( \hat{\phi} \in \mathcal{B}'_h \). Let \( z(\omega) = x(\omega) + \hat{\phi}(\omega) \), \( -\infty < \omega \leq b \). Clearly \( z \) satisfies Definition 2.8 iff \( x \) satisfies \( x_0 = 0 \) and
\[
x(\omega) = -T(\omega)\mathcal{H}(0,\phi) + \mathcal{H}(\omega,x_0 + \hat{\phi}_0) + \lim_{\lambda \to +\infty} \int_0^\omega (\omega - \alpha)^{p-1} S(\omega - \alpha)\lambda R(A)\mathcal{F}(\alpha,z_\alpha) d\alpha + \lim_{\lambda \to +\infty} \int_0^\omega (\omega - \alpha)^{p-1} S(\omega - \alpha)\lambda R(A)\mathcal{B} \times W^{-1} \left[ z_1 - T(b)[\phi(0) - \mathcal{H}(0,\phi)] - \mathcal{H}(b,z_b) \right] (\alpha) d\alpha, \omega \in J.
\]
Now \( \mathcal{B}'_h = \{ x \in \mathcal{B}'_h : x_0 = 0 \in \mathcal{B}_h \} \). For some \( x \in \mathcal{B}'_h \),
\[
\| x \|_b = \| x_0 \|_{\mathcal{B}_h} + \sup \{ \| x(\alpha) \| : 0 \leq \alpha \leq b \} = \sup \{ \| x(\alpha) \| : 0 \leq \alpha \leq b \},
\]
then \( \mathcal{B}'_h \) is a Banach space. Define \( B_r = \{ x \in \mathcal{B}'_h : \| x \|_b \leq r \} \), for any \( r > 0 \), thus \( B_r \subseteq \mathcal{B}'_h \) is bounded uniformly, also for \( x \in B_r \), by Lemma 2.1,
\[
\| x_0 + \hat{\phi}_0 \|_{\mathcal{B}_h} \leq \| x_0 \|_{\mathcal{B}_h} + \| \hat{\phi}_0 \|_{\mathcal{B}_h} \leq l_1(r + \mathcal{M}(\phi(0))) + \| \phi \|_{\mathcal{B}_h} = r'.
\tag{5}
\]
Construct \( \Psi : \mathcal{B}'_h \to \mathcal{B}'_h \) as \( \Psi x \) the set of \( \tilde{z} \in \mathcal{B}'_h \) provided
\[
\tilde{z}(\omega) = \left\{
0, \quad \omega \in (-\infty,0],
-T(\omega)\mathcal{H}(0,\phi) + \mathcal{H}(\omega,x_0 + \hat{\phi}_0) + \lim_{\lambda \to +\infty} \int_0^\omega (\omega - \alpha)^{p-1} S(\omega - \alpha)\lambda R(A)\mathcal{F}(\alpha,z_\alpha) d\alpha + \lim_{\lambda \to +\infty} \int_0^\omega (\omega - \alpha)^{p-1} S(\omega - \alpha)\lambda R(A)\mathcal{B} \times W^{-1} \left[ z_1 - T(b)[\phi(0) - \mathcal{H}(0,\phi)] - \mathcal{H}(b,z_b) \right] (\alpha) d\alpha, \omega \in J.
\right\}
\]
Obviously, \( \Gamma \) has a fixed point iff \( \Psi \) has a fixed point.

**Step 1.** \( \Psi(B_r) \subseteq B_r \) for some \( r > 0 \).
If it fails, there is a function \( z_r \in B_r \), but \( \Psi(z_r) \notin B_r \), that is, \( \| \Psi(z_r)(\omega) \| = \sup \{ \| \phi_r \| : \phi_r \in \Psi(z_r) \} \geq r \). Now
\[
\begin{align*}
r \leq & \| (\Psi x')(\omega) \| \\
\leq & \| -T(\omega)\mathcal{H}(0,\phi) \| + \| \mathcal{H}(\omega,x_0 + \hat{\phi}_0) \| + \lim_{\lambda \to +\infty} \int_0^\omega (\omega - \alpha)^{p-1} S(\omega - \alpha)\lambda R(A)\mathcal{F}(\alpha,x_\alpha + \hat{\phi}_\alpha) d\alpha + \lim_{\lambda \to +\infty} \int_0^\omega (\omega - \alpha)^{p-1} S(\omega - \alpha)\lambda R(A)\mathcal{B}u(\alpha,x + \hat{\phi}) d\alpha \\
= & I_1 + I_2 + I_3 + I_4.
\end{align*}
\]
By $H_1-H_3$ and Lemma 2.1,
\[
I_1 \leq \mathcal{M} \| \mathcal{H}(0, \phi) \| \leq \mathcal{M} \tilde{\tau}_1(1 + \| \phi \|_{\mathcal{B}_h}),
\]
\[
I_2 \leq \| \mathcal{H}(\omega, x_\omega^r + \hat{\theta}_\omega) \| \tilde{\tau}_1(1 + \| x_\omega^r + \hat{\theta}_\omega \|_{\mathcal{B}_h}) \leq \tilde{\tau}_1(1 + r'),
\]
\[
I_3 \leq \lim_{\lambda \to +\infty} \int_0^\omega (\omega - \alpha)^{p-1} |S(\omega - \alpha)\lambda R(A) \mathcal{F}(\alpha, x_\alpha^r + \hat{\theta}_\alpha)\| d\alpha
\leq \frac{pM}{\lambda(1+p)} \int_0^\omega (\omega - \alpha)^{p-1} L_{r'}(\alpha, x_\alpha^r + \hat{\theta}_\alpha) d\alpha.
\]

Also by Lemma 2.9 and $H_4-H_6$,
\[
I_4 \leq \lim_{\lambda \to +\infty} \int_0^\omega \| (\omega - \alpha)^{p-1} S(\omega - \alpha)\lambda R(A)Bu(\alpha, x + \hat{\phi})\| d\alpha
\leq \frac{pM}{\lambda(1+p)} \int_0^\omega (\omega - \alpha)^{p-1} \frac{pM}{\lambda(1+p)} |z_\alpha| + \frac{pM}{\lambda(1+p)} \tilde{\tau}_1(1 + \| \phi \|_{\mathcal{B}_h}) + \tilde{\tau}_1(1 + r')
\leq \frac{pM}{\lambda(1+p)} \int_0^b (b - \eta)^{p-1} L_{r'}(\eta) d\eta
\]
\[
\leq \frac{pM}{\lambda(1+p)} \int_0^b (b - \eta)^{p-1} L_{r'}(\eta) d\eta.
\]

By grouping $I_1-I_4$,
\[
r \leq \mathcal{M} \tilde{\tau}_1(1 + \| \phi \|_{\mathcal{B}_h}) + \tilde{\tau}_1(1 + r') + \frac{pM}{\lambda(1+p)} \int_0^\omega (\omega - \alpha)^{p-1} L_{r'}(\alpha) d\alpha
\leq \frac{pM}{\lambda(1+p)} \int_0^b (b - \eta)^{p-1} L_{r'}(\eta) d\eta.
\]

Moreover
\[
\lim_{r \to \infty} \inf \left( \frac{\int_0^\omega (\omega - \alpha)^{p-1} L_{r'}(\alpha) d\alpha}{r} \right) = \lim_{r \to \infty} \inf \left( \frac{\int_0^\omega (\omega - \alpha)^{p-1} L_{r'}(\alpha) d\alpha}{r} \cdot \frac{r'}{r} \right) = \theta l_1.
\]

Dividing (6) by $r$ and applying $r \to \infty$,
\[
l_1 [\tilde{\tau}_1 + \frac{pM}{\lambda(1+p)}] (1 + \frac{pM}{\lambda(1+p)} \frac{b^p}{p}) \geq 1
\]
which leads contradiction to (4). Hence $\Psi(\mathcal{B}_r) \subseteq \mathcal{B}_r$.

**Step 2.** $\Psi : \mathcal{B}_h^n \to \mathcal{B}_h^n$ is continuous.
Consider \( \{x^{(n)}(\omega)\}_{n=0}^{\infty} \subset \mathcal{B}_h' \) with \( x^{(n)}(\omega) \to x(\omega) \). For \( r > 0 \), \( ||x^{(n)}(\omega)|| \leq r \), for every \( n \) and a.e. \( \omega \in J \). Hence \( x^{(n)}(\omega) \in \mathcal{B}_r \), \( x(\omega) \in \mathcal{B}_r \). We have

\[
||x_\omega^{(n)} + \hat{\phi}_\omega||_{\mathcal{B}_h} \leq r'.
\]

Using \( \mathcal{H}_5 \) we get

1. \( \mathcal{F}(\omega, x_\omega^{(n)} + \hat{\phi}_\omega) \to \mathcal{F}(\omega, x_\omega + \hat{\phi}_\omega) \), for every \( \omega \in J \) and since

\[
||\mathcal{F}^\prime(\omega, x_\omega^{(n)} + \hat{\phi}_\omega) - \mathcal{F}(\omega, x_\omega + \hat{\phi}_\omega)|| \leq 2t_\omega(\omega).
\]

2. \( \mathcal{H}(\omega, x_\omega^{(n)} + \hat{\phi}_\omega) \to \mathcal{H}(\omega, x_\omega + \hat{\phi}_\omega) \), for every \( \omega \in J \) and because

\[
||\mathcal{H}^\prime(\omega, x_\omega^{(n)} + \hat{\phi}_\omega) - \mathcal{H}^\prime(\omega, x_\omega + \hat{\phi}_\omega)|| \leq 2t'_\mathcal{H}(r' + 1).
\]

Using dominated convergence theorem,

\[
||\Psi x^{(n)} - \Psi x|| \\
\leq \left\| - T(\omega) \mathcal{H}(0, \phi) \right\| + ||\mathcal{H}(\omega, x_\omega^{(n)} + \hat{\phi}_\omega) - \mathcal{H}(\omega, x_\omega + \hat{\phi}_\omega)|| \\
+ \left\| \lim_{\lambda \to +\infty} \int_{0}^{\omega} (\omega - \alpha)^{p-1} S(\omega - \alpha) \lambda R(A) \right\| \\
\times \left[ F(\alpha, x_\alpha^{(n)} + \hat{\phi}_\alpha) - F(\alpha, x_\alpha + \hat{\phi}_\alpha) \right] d\alpha \\
+ \left\| \lim_{\lambda \to +\infty} \int_{0}^{\omega} (\omega - \alpha)^{p-1} S(\omega - \alpha) \lambda R(A) \right\| \\
\times \left[ u(\alpha, x^{(n)} + \hat{\phi}) - u(\alpha, x + \hat{\phi}) \right] d\alpha \\
\leq \mathcal{M}_1 \mathcal{H}(||\phi||_{\mathcal{B}_h} + 1) + ||\mathcal{H}(\omega, x_\omega^{(n)} + \hat{\phi}_\omega) - \mathcal{H}(\omega, x_\omega + \hat{\phi}_\omega)|| \\
+ \frac{p \mathcal{M} \mathcal{H}}{1 + p} \int_{0}^{\omega} (\omega - \alpha)^{p-1} \left| F(\alpha, x_\alpha^{(n)} + \hat{\phi}_\alpha) - F(\alpha, x_\alpha + \hat{\phi}_\alpha) \right| d\alpha \\
+ \frac{p \mathcal{M} \mathcal{H}_2}{1 + p} \int_{0}^{\omega} (\omega - \alpha)^{p-1} \frac{p \mathcal{M} \mathcal{H}_2}{1 + p} \left[ \mathcal{H}(b, y_b^{(n)} + \hat{\phi}_b) - \mathcal{H}(b, y_b + \hat{\phi}_b) \right] \\
\to 0 \text{ as } n \to \infty.
\]

**Step 3.** \( \Psi \) maps \( \mathcal{B}_r \) into a precompact set in \( X \).
Since \( T(\omega) \) is strongly continuous and \( \mathcal{H}_4, \mathcal{H}_5, \{T(\omega - \alpha)(F(\alpha, x_\alpha + \hat{\phi}_\alpha)) : \omega, \alpha \in J, x \in \mathcal{B}_r \} \) is relatively compact in \( X \). Also for \( x \in \mathcal{B}_r \), from Bochner integral, we get

\[
\Psi x(\omega) \in \omega \text{compact} \{T(\omega - \alpha)(F(\alpha, x_\alpha + \hat{\phi}_\alpha)) : \omega, \alpha \in J, x \in \mathcal{B}_r \},
\]

for all \( \omega \in J \). Hence \( \{\Psi x(\omega) : x \in \mathcal{B}_r \} \) is precompact in \( X \), for every \( \omega \in J \).

**Step 4.** \( \Psi(\mathcal{B}_r) \) is equicontinuous.
Let \( \varepsilon > 0 \) be small, \( 0 < \omega_1 < \omega_2 < b \). For every \( x \in \mathcal{B}_r \) and \( \hat{\xi} \) belong to \( \Psi_1 x \), and
for each \( \omega \in J \),

\[
\| \tilde{z}(\omega_2) - \tilde{z}(\omega_1) \| = \| (T(\omega_2) - T(\omega_1)) \mathcal{H}(0, \phi) \|
\]

\[
+ \| \mathcal{H}(\omega_2, x_{\omega_2} + \hat{\phi}_{\omega_2}) - \mathcal{H}(\omega_1, x_{\omega_1} + \hat{\phi}_{\omega_1}) \|
\]

\[
+ \lim_{\lambda \to +\infty} \int_{\omega_1}^{\omega_2} (\omega_2 - \alpha)^{p-1} S(\omega_2 - \alpha) \lambda R(A) \mathcal{F}(\alpha, x_\alpha + \hat{\phi}_\alpha) d\alpha
\]

\[
+ \lim_{\lambda \to +\infty} \int_{\omega_1}^{\omega_2} (\omega_2 - \alpha)^{p-1} [S(\omega_2 - \alpha) - S(\omega_1 - \alpha)] \lambda R(A) \mathcal{F}(\alpha, x_\alpha + \hat{\phi}_\alpha) d\alpha
\]

\[
+ \lim_{\lambda \to +\infty} \int_{\omega_1}^{\omega_2} [S(\omega_2 - \alpha)(\omega_1 - \alpha)^{p-1}] \lambda R(A) \mathcal{F}(\alpha, x_\alpha + \hat{\phi}_\alpha) d\alpha
\]

\[
+ \lim_{\lambda \to +\infty} \int_{\omega_1}^{\omega_2} \lambda R(A) B u(\alpha, x + \hat{\phi}) d\alpha
\]

\[
+ \lim_{\lambda \to +\infty} \int_{\omega_1}^{\omega_2} [S(\omega_2 - \alpha)(\omega_1 - \alpha)^{p-1}] \lambda R(A) B u(\alpha, x + \hat{\phi}) d\alpha
\]

\[
+ \lim_{\lambda \to +\infty} \int_{\omega_1}^{\omega_2} [S(\omega_2 - \alpha)(\omega_1 - \alpha)^{p-1}] \lambda R(A) B u(\alpha, x + \hat{\phi}) d\alpha
\]

\[
+ \lim_{\lambda \to +\infty} \int_{\omega_1}^{\omega_2} [S(\omega_2 - \alpha)(\omega_1 - \alpha)^{p-1}] \lambda R(A) B u(\alpha, x + \hat{\phi}) d\alpha
\]

Using Holder’s inequality and Lemma 2.9,

\[
\| \tilde{z}(\omega_2) - \tilde{z}(\omega_1) \| \leq \|(T(\omega_2) - T(\omega_1))\| \mathcal{H}(0, \phi) + \| \mathcal{H}(\omega_2, x_{\omega_2} + \hat{\phi}_{\omega_2}) \|
\]

\[
- \mathcal{H}(\omega_1, x_{\omega_1} + \hat{\phi}_{\omega_1}) + \frac{p \mathcal{M} \mathcal{H}}{\Gamma(1 + p)} \int_{\omega_1}^{\omega_2} (\omega_2 - \alpha)^{p-1} \mathcal{L}_\nu(\alpha) d\alpha
\]

\[
+ \mathcal{H} \int_{\omega_1}^{\omega_1-\varepsilon} (\omega_2 - \alpha)^{p-1} \| S(\omega_2 - \alpha) - S(\omega_1 - \alpha) \| \mathcal{L}_\nu(\alpha) d\alpha + \frac{p \mathcal{M} \mathcal{H}}{\Gamma(1 + p)}
\]

\[
\times \int_{\omega_1-\varepsilon}^{\omega_1} [(\omega_2 - \alpha)^{p-1} - (\omega_1 - \alpha)^{p-1}] \mathcal{L}_\nu(\alpha) d\alpha
\]

\[
+ \mathcal{H} \int_{\omega_1-\varepsilon}^{\omega_1-\varepsilon} (\omega_2 - \alpha)^{p-1} \| S(\omega_2 - \alpha) - S(\omega_1 - \alpha) \| \mathcal{L}_\nu(\alpha) d\alpha
\]

\[
+ \frac{p \mathcal{M} \mathcal{H}}{\Gamma(1 + p)} \int_{\omega_1-\varepsilon}^{\omega_1-\varepsilon} [(\omega_2 - \alpha)^{p-1} - (\omega_1 - \alpha)^{p-1}] \mathcal{L}_\nu(\alpha) d\alpha
\]

\[
+ \mathcal{H} \left( \frac{p \mathcal{M} \mathcal{H}}{\Gamma(1 + p)} \right)^2 \int_{\omega_1}^{\omega_2} (\omega_2 - \alpha)^{p-1} \left( \| z_1 \| + \mathcal{H}_1 \mathcal{H} (1 + \| \phi \|_{\mathcal{B}_1}) \right)
\]
\[ I_{1,p}(1 + r^p) + \frac{p \mathcal{M}}{\Gamma(1 + p)} \int_0^b (b - \nu)^{p-1} \mathcal{L}_{r^p}(\nu) d\nu + \mathcal{M} \left( \frac{p \mathcal{M}}{\Gamma(1 + p)} \right)^2 \times \int_{\omega_{1-\varepsilon}}^{\omega_1} (\omega_2 - \alpha)^{p-1} ||S(\omega_2 - \alpha) - S(\omega_1 - \alpha)|| \left[ ||z_1|| + \mathcal{M} I_{1,p}(1 + ||\phi||_{\mathcal{M}}) \right] \]

\[ + I_{1,p}(1 + r^p) \frac{p \mathcal{M}}{\Gamma(1 + p)} \int_0^b (b - \nu)^{p-1} \mathcal{L}_{r^p}(\nu) d\nu d\alpha + \mathcal{M} \left( \frac{p \mathcal{M}}{\Gamma(1 + p)} \right)^2 \times \int_{\omega_{1-\varepsilon}}^{\omega_1} (\omega_2 - \alpha)^{p-1} ||S(\omega_2 - \alpha) - S(\omega_1 - \alpha)|| \left[ ||z_1|| + \mathcal{M} I_{1,p}(1 + ||\phi||_{\mathcal{M}}) \right] \]

\[ + I_{1,p}(1 + r^p) + \frac{p \mathcal{M}}{\Gamma(1 + p)} \int_0^b (b - \nu)^{p-1} \mathcal{L}_{r^p}(\nu) d\nu d\alpha \]

for \( \varepsilon \) sufficiently small. This leads to the continuity in the uniform operator topology using the compactness of \( \{S(\omega)\}_{\omega > 0} \). Hence \( \Psi \) maps \( B_r \) into an equicontinuous family of functions. By Arzela theorem, we conclude that \( \Psi \) is completely continuous. Also by Schauder fixed point theorem, the system (1)-(2) is controllable such that \( z(b) = z_h \).

4. Illustration. Consider the fractional neutral differential equation

\[ C D^p_0 [z(\omega, \eta) - \mathcal{H}(\omega, z_\omega)] = \frac{\partial^2}{\partial \eta^2} [z(\omega, \eta) - \mathcal{H}(\omega, z_\omega)] + \gamma(\eta) u(\omega) + \mathcal{F}(\omega, z_\omega), \quad \omega \in (0, 1), \]

\[ z(\omega, \eta) = \phi(\omega, \eta), \quad \omega \in (-\infty, 0], \quad \eta \in [0, \pi], \]

\[ z(\omega, 0) = z(\omega, \pi) = 0, \quad \omega \geq 0, \]

where \( 0 < p < 1, z_\omega = z(\omega + \theta, \eta), \quad \theta \leq 0 \) and \( \phi(\omega, \eta) \in \mathcal{B}_h \). Define \( \mathcal{H} : [0, 1] \times \mathcal{B}_h \to X, \quad \gamma : [0, 1] \times [0, \pi] \to [0, \pi] \).

Let \( X = L^2(0, \pi) \) with \( ||z||_{L^2} = \left( \int_0^\pi |z(s)|^2 ds \right)^{\frac{1}{2}} \), \( X \times X \times X \) with the norm \( ||z||_2 = ||z||_{L^2} \). For convenience choose \( h = e^{4s}, s > 0 \), then

\[ l_1 = \int_{-\infty}^0 h(s) ds = \frac{1}{4} < \infty \] for \( \omega \geq 0 \) and \( ||\phi||_{\mathcal{M}} = \int_{-\infty}^0 h(s) \sup_{\theta} ||\phi(\theta)||_{L^2} ds \).

The closed linear operator \( A : D(A) \subset X \to X \) by \( Az = z'' \) with

\[ D(A) = \{ z \in X, z, z' \text{ are absolutely continuous}; z'' \in X, z(0) = z(\pi) = 0 \} \]
Then
\[ Az = \sum_{n=1}^{\infty} n^2 (z(z_n) z_n, z \in D(A), \text{ together with } z_n(\eta) = \sqrt{\frac{2}{\pi}} \sin(n\eta), [n = 1, 2, ...] \]

the orthogonal set of eigen vectors of \( A \).

Also \( A \) fulfills the Hille condition with \((0, \infty) \subset \rho(A), \quad ||(\lambda I - A)^{-1}|| \leq \frac{1}{\lambda}, \text{ for } \lambda > 0, \)
\[ X_0 = \{ z \in X : z(0) = z(\pi) = 0 \} \neq X. \]

Now the semigroup \( \{ T_p(\omega) \}_{\omega \geq 0} \) in \( X \) provided as
\[ T_p(\omega)z = \sum_{n=1}^{\infty} e^{\omega n^2} (z(z_n) w_n, \forall X, \omega \geq 0 \text{ which is uniformly bounded.} \]

To show the validity, further we assume \( \mathcal{H}(\omega, z_\omega) = \frac{\sin(\omega z_\omega^2)}{100} \) and \( \mathcal{F}(\omega, z_\omega) = \frac{\omega^2 + \cos(z_\omega)}{100} \) with
\[
\left\| \mathcal{H}(\omega, z_\omega) - \mathcal{H}(\omega, z_\omega) \right\| = \frac{\sqrt{\pi}}{50} \left\| z_\omega - z_\omega \right\|_{\mathcal{B}_h},
\]
\[
\left\| \mathcal{F}(\omega, z_\omega) - \mathcal{F}(\omega, z_\omega) \right\| = \frac{\sqrt{\pi}}{100} \left\| z_\omega - z_\omega \right\|_{\mathcal{B}_h}.
\]

The control term \( W : L^2(J, U) \rightarrow D(\mathcal{A}) \) is characterized as
\[ W = \lim_{\lambda \rightarrow +\infty} \int_0^1 u(\omega, s) ds \text{ with } ||B|| = M_2. \]

From the above results \( l_1 = \frac{1}{4}, b = 1, l_{1,\mathcal{H}} = \frac{\sqrt{\pi}}{50}, \varrho = \frac{\sqrt{\pi}}{100}. \) Take \( \mathcal{M} = \mathcal{H} = \frac{1}{5}, \mathcal{M}_2 = \frac{1}{2}, p = \frac{1}{2} \) we derive
\[
l_1 \left[ l_{1,\mathcal{H}} + \frac{p, \mathcal{M}}{\Gamma(1 + p)} \varrho \right] \left( 1 + \frac{\mathcal{M}}{p} \left( \frac{p, \mathcal{M}_2}{\Gamma(1 + p)} \right)^2 \right)
\]
\[
= \frac{1}{4} \left[ \frac{\sqrt{\pi}}{50} + \frac{\frac{1}{2} \sqrt{\frac{1}{2}} \sqrt{\frac{1}{2}}}{100} \left( 1 + \frac{1}{5} \left( \frac{\frac{1}{2} \sqrt{\frac{1}{2}}}{\Gamma(\frac{3}{2})} \right)^2 \frac{1}{2} \right) \right]
\]
\[
= \frac{1}{4} [0.06828 + 0.025][1 + 0.00018746] = 0.0233 < 1.
\]

Thus conditions \( H_1-H_7 \) holds. By Theorem 3.1, the system 7 is controllable.

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