Bose-Einstein Condensation on inhomogeneous complex networks

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The thermodynamic properties of non interacting bosons on a complex network can be strongly affected by topological inhomogeneities. The latter give rise to anomalies in the density of states that can induce Bose-Einstein condensation in low dimensional systems also in absence of external confining potentials. The anomalies consist in energy regions composed of an infinite number of states with vanishing weight in the thermodynamic limit. We present a rigorous result providing the general conditions for the occurrence of Bose-Einstein condensation on complex networks in presence of anomalous spectral regions in the density of states. We present results on spectral properties for a wide class of graphs where the theorem applies. We study in detail an explicit geometrical realization, the comb lattice, which embodies all the relevant features of this effect and which can be experimentally implemented as an array of Josephson Junctions.

I. INTRODUCTION

The impressive experimental demonstration of Bose-Einstein condensation (BEC) has stimulated a wealth of theoretical work aimed at a better understanding of its basic mechanism and of its potential consequences for the engineering of quantum devices [1].

A well known general result [2] is that for an ideal gas of Bose particle, BEC does not occur for homogeneous systems in dimension $d \leq 2$; the same is true for non interacting bosons on regular periodic lattices. The result cannot be extended to more general discrete structures lacking translational invariance, which, in principle, can now be implemented by combining quantum devices in non conventional discrete geometrical settings.

The first important problem one is faced with is the characterization of the geometrical properties of a general network, i.e. a graph. On one hand, one is interested in describing the complex geometry of a sample and identifying its effects on physics. On the other, once the effects of geometry are known, one can use the geometrical setting to tune the physical properties, by a sort of engineering in network design. As we will see, this can lead to effects analogous to the introduction of an external confining potential for regular geometries.

The problem of describing the large scale geometry of graphs by an effective parameter that generalize the Euclidean dimension of regular lattices has been successfully solved by the introduction of the spectral dimension $\bar{d}$ [3,4], which can be experimentally measured [5] and rigorously defined by graph theory [6]. However, one expects the influence of topology to be richer and more complex on discrete structures, due to the possible relevance of local geometrical details, in addition to the large scale structure described by dimensionality.

In this direction, recent works on BEC on inhomogeneous networks [7] have put into evidence that strong inhomogeneities can give rise to condensation at finite temperature for a system of non interacting bosons in absence of an external confining potential even in a low-dimensional structure as a comb lattice, where $\bar{d} = 1$. This phenomenon arises from a peculiar part of the density of states on inhomogeneous networks we shall refer to as hidden spectrum [8]. This hidden spectrum consists of an energy region filled by a finite or infinite number of states which do not contribute to the normalized density in the thermodynamic limit. Hidden spectra do not usually affect bulk thermodynamic quantities but, as we shall show in the following, can have dramatic effects since bosonic statistics allows the macroscopic occupation of a single quantum state.

In this paper we shall determine the conditions under which, even for low dimensional systems, non-interacting bosons on a general discrete network may lead to BEC [8]. We shall see that this is indeed possible if one resorts to

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a suitable discrete inhomogeneous ambient space on which bosons are defined; this is very desirable in view of the engineering of quantum devices.

Bosonic models over generic discrete structures can be made experimentally accessible through the realization of arrays of Josephson junctions (JJA). The latter are devices that can be engineered to realize a variety of non-homogeneous patterns. We shall see indeed that classical JJA JJA arranged in a non-homogeneous network provide a relevant example of the proposed mechanism for BEC, leading to a spatial condensation of bosons in a single state.

The model Hamiltonian describing the low-temperature JJA is the Bose-Hubbard (BH) model. On a generic graph, the Hamiltonian is given by

\[ H^{BH} = U \sum_i n_i^2 + \sum_{ij} A_{ij} \left( V n_i n_j - J (a_i^\dagger a_j + a_j^\dagger a_i) \right) \tag{1} \]

where \( A_{ij} \) is the adjacency matrix of the graph: \( A_{ij} = 1 \) if the sites \( i \) and \( j \) are nearest neighbours and \( A_{ij} = 0 \) otherwise; \( a_i^\dagger \) creates a boson at site \( i \) and \( n_i \equiv a_i^\dagger a_i \). The phase diagram reflects the competition between the boson kinetic energy (hopping term) and the repulsive Coulomb interaction. In a realistic experimental setup, the parameters \( U \) and \( V \) depend on the ratio between the intergrain capacitance \( C \) and the gate capacitance \( C_0 \), while the parameter \( J \) describes Cooper pair hopping. Classical JJA are obtained when \( U, V << J \) (i.e. \( C/C_0 \to 0 \)); in this limit the hopping term dominates the physics of the system.

Recently, the BH model has been shown to describe the dynamics of ultracold bosonic atoms on a particular regular discrete geometry, the optical lattice. In this context the BH model describes the hopping of bosonic atoms between low vibrational states of the optical lattice; the parameters \( U \) and \( J \) depend on the power of the laser beams used to realize the lattice. The dynamics of a BEC in such a kind of periodic potential can thus be described by a discrete nonlinear Schrödinger equation. Also for atomic Bose systems, the experiments have reached an high level of accuracy with easily tunable external parameters and accurately tailored trapping profiles.

In this paper we will consider a model of pure hopping of non interacting bosons on general non regular geometries, such as complex networks. In particular, we will analyse the possibility of having a Bose-Einstein condensate at finite temperature, providing a general condition for BEC on a general network. We will describe in detail some geometrical settings where inhomogeneities act as a confining potential, giving rise to BEC even in low dimensions.

The extension of this analysis to the full BH model on complex networks is not trivial and it is still the object of on-going investigations.

The paper is organized as follows: in section II we introduce the mathematical notations and some basic concepts in algebraic graph theory; in section III we define the pure hopping model and prove a general theorem giving the most general condition for the existence of BEC at finite temperature for non interacting bosons on a complex network. We discuss in detail the presence of “hidden” regions in the density of states in the thermodynamic limit; in section IV we apply the theorem to the case of the comb graph showing that BEC at finite temperature is present in 1-dimensional structures as a consequence of the effects of the inhomogeneous geometry on the density of states; finally in section V we consider a wide class of networks where the relation between BEC and complex geometry can be analyzed in detail.

II. COMPLEX NETWORKS AND GRAPHS

On complex networks it is not possible to introduce important mathematical tools typical of Euclidean lattices, such as the Fourier transform and the reciprocal lattice. Nevertheless general and effective mathematical techniques are provided by graph theory and the algebraic approach to graph topology. In this section we briefly recall some basic definitions and notations.

A graph \( G \) is a countable set \( V \) of vertices \( i \) connected pairwise by a set \( E \) of unoriented links \((i,j) = (j,i)\). In general sites represent degrees of freedom and links represent interactions between them. In particular, for bosonic models, vertices can be regarded as Josephson junctions or as sites in the optical networks, while edges represent the hopping probabilities between sites.

A subgraph \( S \) of \( G \) is a graph whose set of vertices \( V' \subseteq V \) and whose set of links \( E' \subseteq E \). A path in \( G \) is a sequence of consecutive links \( \{(i,k)(k,h) \ldots (n,m)(m,j)\} \) and a graph is said to be connected, if for any two points \( i,j \in V \) there is always a path joining them. In the following we will consider connected graphs. Every connected graph \( G \) is endowed with an intrinsic metric generated by the chemical distance \( r_{i,j} \), which is defined as the number of links in the shortest path connecting vertices \( i \) and \( j \). On a graph the generalized Van Hove sphere \( S_{o,r} \subseteq G \) of center \( o \) and radius \( r \) is the subgraph of \( G \) containing all \( i \in G \) whose distance from \( o \) is \( \leq r \) and all the links of \( G \) joining them. We will call \( N_{o,r} \) the number of vertices contained in \( S_{o,r} \). The Van Hove spheres play an important role in the study of the thermodynamic limit on infinite graphs. The physical quantities are usually evaluated by first considering the
models restricted to the finite spheres and then taking the thermodynamic limit by letting \( r \to \infty \). The average in the thermodynamic limit \( \bar{\phi} \) of a bounded function defined on the vertices \( \phi(i) \) is:

\[
\bar{\phi} \equiv \lim_{r \to \infty} \frac{\sum_{i \in S_{o,r}} \phi(i)}{N_{o,r}}.
\]  

(2)

With this definition, the measure \( |V'| \) of a subset \( V' \) of \( V \) is the average value of its characteristic function \( \chi_{V'}(i) \), defined by \( \chi_{V'}(i) = 1 \) if \( i \in V' \) and \( \chi_{V'}(i) = 0 \) otherwise. It can be shown [17] that for graphs with polynomial growth, i.e. when \( N_{r,o} \sim r^p \) for \( r \to \infty \), the thermodynamic limit is independent from the choice of the center of the sphere \( o \). Since all realistic networks which can be embedded in a 3-dimensional space have polynomial growth, we restrict to this class graphs and we drop the index \( o \) of the center of the sphere. The graph topology is algebraically described by its adjacency matrix:

\[
A_{ij} = \begin{cases} 
1 & \text{if } (i,j) \in E \\
0 & \text{if } (i,j) \notin E 
end{cases}
\]  

(3)

On disordered structures, a useful generalization of the adjacency matrix can be considered by introducing a bounded distribution of hopping probabilities \( t_{ij} \) which differs from link to link:

\[
t_{ij} = t_{ji} \begin{cases} 
\neq 0 & \text{if } A_{ij} = 1 \\
= 0 & \text{if } A_{ij} = 0 
end{cases}
\]  

(4)

with \( \sup_{(i,j)} |t_{ij}| < \infty \).

The Laplacian matrix \( L_{ij} \) is defined by:

\[
L_{ij} = z_i \delta_{ij} - A_{ij}
\]  

(5)

where \( z_i = \sum_j A_{ij} \), the number of nearest neighbours of \( i \), is called the coordination number. We will consider graph with bounded coordination number, i.e. with \( \max_i z_i < \infty \).

\( L_{ij} \) is the generalization to graphs of the usual Laplacian on a lattice, where \( z_i = z, \forall i \). \( L_{ij} \) has some important spectral properties: its spectrum is real, non-negative and bounded. In particular 0 is a simple eigenvalue of \( L_{ij} \) corresponding to the constant eigenvector. Notice that, while on a regular lattice \( L_{ij} \) is diagonalized by the Fourier transform, this is not the case for a generic graph.

A fundamental problem in graph theory is the characterization of the large scale geometry of the graph by the definition of a dimension which generalizes to infinite graphs the usual Euclidean dimension \( d \) of lattices. The spectrum of the Laplacian operator defines the spectral dimension \( \bar{d} \), which has been shown to be the extension of \( d \) in many physical problems. On a infinite graph, the spectral density of the Laplacian matrix is denoted by \( \rho(l) \). The spectral dimension \( \bar{d} \) is defined by the asymptotic behaviour of \( \rho(l) \) at low eigenvalues [3,4]:

\[
\rho(l) \sim l^{d/2-1} \quad \text{for } l \to 0^+
\]  

(6)

Interestingly, \( \bar{d} \) can be a real number, it can be experimentally measured [3]; it depends only on the large scale topology of the graph [4] and it is the direct extension to graphs of the Euclidean dimension in many physical problems. In particular, the long time asymptotic diffusion of classical particles [6], the low frequencies spectrum of classical harmonic oscillators [3], the zero mass limit of the Gaussian model [4,6] and the existence of spontaneous magnetization in continuous symmetry spin models [18] are determined by the spectral dimension.

### III. BEC ON COMPLEX NETWORKS: THE GENERAL THEOREM

The Hamiltonian for non interacting bosons on a graph \( G \) is:

\[
H = \sum_{i,j \in V} h_{ij} a_i^\dagger a_j
\]  

(7)

where \( a_i^\dagger \) and \( a_i \) are the creation and annihilation operator at site \( i \), with \( [a_i, a_j^\dagger] = \delta_{ij} \). The Hamiltonian matrix \( h_{ij} \) is defined by:
\[ h_{ij} = t_{ij} + \delta_{ij} V_i \]  \hspace{1cm} (8)

The term \( t_{ij} \), defined in (4), describes hopping between nearest neighbours sites. The diagonal term \( V_i \) takes into account a potential at site \( i \) and it satisfies a boundedness condition: \( \sup |V_i| < \infty \). In particular, the theorem holds also when \( V_i = 0 \), i.e. when there is no on-site potential.

The thermodynamic limit on the graph is studied by first considering the finite Van Hove sphere \( S_r \) and then letting \( r \to \infty \). This definition of the thermodynamic limit is very general and it applies to a general discrete structure. On graphs with symmetries (lattices, fractals, bundled structures, etc.) different shapes of Van Hove spheres can be introduced, preserving the symmetry of the network. It can be shown that for regular choices of the spheres the properties of the model for \( r \to \infty \) do not depend on the shape of the sphere. The Hamiltonian (6) restricted to the sphere is:

\[ H^{S_r} = \sum_{i,j \in S_r} h^{S_r}_{ij} a_i^\dagger a_j \]  \hspace{1cm} (9)

where \( h^{S_r}_{ij} = h_{ij} \) if \( i \) and \( j \) belong to \( S_r \) and \( h^{S_r}_{ij} = 0 \) otherwise.

For each finite sphere of radius \( r \) let us consider the eigenvalues equation:

\[ \sum_{j \in S_r} h^{S_r}_{ij} \psi(j) = E \psi(i) \]  \hspace{1cm} (10)

The normalized density of states \( \rho^r(E) \) on \( S_r \) is defined as:

\[ \rho^r(E) = \frac{1}{N^r} \sum_k \delta(E - E^r_k) \]  \hspace{1cm} (11)

where \( E^r_k \) are the \( N^r \) eigenvalues of \( H^{S_r}_{ij} \).

We define \( \rho(E) \) to be the density of states of \( h_{ij} \) in the thermodynamic limit if:

\[ \lim_{r \to \infty} \int |\rho^r(E) - \rho(E)| dE = 0 \]  \hspace{1cm} (12)

Let \( E_m = \text{Inf}(\text{Supp}(\rho(E))) \), where \( \text{Supp}(\rho(E)) \) is the support of the distribution \( \rho(E) \). On a graph, the asymptotic behavior at low energies of the density of states is described by an exponent \( \alpha \):

\[ \rho(E) \sim (E - E_m)^{\frac{\alpha}{2}} \]  \hspace{1cm} (13)

The exponent \( \alpha \) in general depends both on graph topology and on the external potential. Interestingly, when \( h_{ij} = L_{ij} \) and when the external potential \( V_i \) satisfy suitable regularity conditions, \( \alpha \) coincides with the spectral dimension \( d \).

In particular this happens when \( V_i = 0 \). A hidden region of the spectrum is an energy interval \([E_1,E_2]\) such that \([E_1,E_2]\) \( \cap \text{Supp}(\rho(E)) = \emptyset \) and \( \lim_{r \to \infty} N^r_{[E_1,E_2]} > 0 \), where \( N^r_{[E_1,E_2]} \) is the number of eigenvalues of \( H^{S_r}_{ij} \) in the interval \([E_1,E_2]\). Notice that in general \( N^r_{[E_1,E_2]} \) can diverge for \( r \to \infty \) and the eigenvalues can become dense in \([E_1,E_2]\) in the thermodynamic limit. Therefore this condition not only includes the trivial case of discrete spectrum but is far more general; an interesting example of this behaviour is found in the comb lattice without external potential \( \mathbb{F} \) which will be studied in detail in the next section.

We now define the lowest energy level for the sequence of densities \( \rho_r(E) \), setting \( E^r_0 = \text{Inf}_k(E_k) \) and \( E_0 = \lim_{r \to \infty} E^r_0 \). In general, \( E_0 \leq E_m \). If \( E_0 < E_m \), then \([E_0,E_m]\) is a hidden region of the spectrum, which will be called hidden low-energy spectrum.

In the following we will consider models at fixed filling \( f = N/N_r \) (\( N \) is the number of particles in the system). In the grand canonical ensemble the equation that determines the fugacity \( z \) in the thermodynamic limit is:

\[ f = \lim_{r \to \infty} \int \frac{\rho^r(E) dE}{z^{-1} e^{\beta E} - 1} \]  \hspace{1cm} (14)

Setting \( E_0 = 0 \) we have that \( 0 \leq z \leq 1 \).

The integral in the equation (14) can be divided into two sums, the first one considering the energies smaller than an arbitrary constant \( \epsilon \) and the second the energies larger than \( \epsilon \):

\[ \int \frac{\rho^r(E) dE}{z^{-1} e^{\beta E} - 1} = \sum_{k=0}^{E_k \leq \epsilon} \frac{1}{z^{-1} e^{\beta E_k} - 1} + \int_{E_{\epsilon}} \frac{\rho^r(E) dE}{z^{-1} e^{\beta E} - 1} \]  \hspace{1cm} (15)
We define:
\[ n^r_\epsilon \equiv \sum_{k=0}^{E_k \leq \epsilon} \frac{1}{z^{-1}e^{\beta E_k} - 1} \]  
(16)

as the fraction of particles with energy smaller than \( \epsilon \). Bose-Einstein condensation occurs in this systems if it exists a critical temperature \( T_c > 0 \) such that, for any \( T < T_c \), \( n^r_\epsilon \equiv \lim_{r \to \infty} n^r_\epsilon > k > 0 \), for all \( \epsilon > 0 \); i.e. \( n_0 \equiv \lim_{r \to \infty} n_\epsilon = k > 0 \).

From definition (16), \( n_0 \) can be strictly positive only if \( \lim_{r \to \infty} z(r) = 1 \). Indeed, if this limit is smaller than 1 it follows that \( n_\epsilon \leq (1 - z)^{-1} \lim_{r \to \infty} N_r/N^r \), where \( N^r \) is the number of state with energy smaller than \( \epsilon \). If the ground state is not infinitely degenerate, \( \lim_{r \to \infty} (N_r/N^r) = 0 \) and then \( n_0 = 0 \).

Taking first the limit \( r \to \infty \) and then \( \epsilon \to 0 \) in equation (17) we obtain:
\[ f = n_0 + \lim_{r \to \infty} \int_{E > \epsilon} \frac{(\rho^r(E) - \rho(E))dE}{z^{-1}e^{\beta E} - 1} + \lim_{r \to \infty} \int_{E > \epsilon} \frac{\rho(E)dE}{z^{-1}e^{\beta E} - 1} \]  
(17)

Now, from the boundedness of \((z^{-1}e^{\beta E} - 1)^{-1}\) for \( E > \epsilon \) and from the definition (12), the first of the two limits in the right hand side of (17) vanishes:
\[ f = n_0 + \lim_{r \to \infty} \int_{E > \epsilon} \frac{\rho(E)dE}{z^{-1}e^{\beta E} - 1} = n_0 + \int_{E > \epsilon} \frac{\rho(E)dE}{z^{-1}e^{\beta E} - 1} \]  
(18)

where, again, \( n_0 \) can be different from 0 only if \( z = 1 \).

The integral in equation (18) is an increasing continuous function of \( z \) with \( 0 \leq z < 1 \). If the limit:
\[ f_c(\beta) = \lim_{z \to 1} \int \frac{\rho(E)dE}{z^{-1}e^{\beta E} - 1} \]  
(19)

is equal to \( \infty \), then \( z < 1 \) and \( n_0 = 0 \). If the limit is finite, \( f_c(\beta) \) is a decreasing function of \( \beta \) with \( \lim_{\beta \to \infty} f_c(\beta) = 0 \) and \( \lim_{\beta \to 0} f_c(\beta) = \infty \). Then, for a suitable \( \beta_c, f_c(\beta_c) = f \). For \( \beta > \beta_c \), (i.e. \( T < T_c \)) \( z = 1 \) and \( n_0 = f - f_c(\beta) > 0 \) while for \( \beta < \beta_c \), (i.e. \( T > T_c \)) \( z < 1 \) and \( n_0 = 0 \).

From the divergence or finiteness of the limit (19) one obtains the most general condition for the occurrence of Bose-Einstein condensation on a graph.

First, if \( 0 = E_0 < E_m \) (i.e. the system presents a low-energy hidden spectrum) the limit (13) is finite and there is Bose-Einstein condensation at finite temperature:
\[ f_c(\beta) \leq \frac{1}{e^{\beta E_m} - 1} \int \rho(E)dE = \frac{1}{e^{\beta E_m} - 1}. \]  
(20)

On the other hand, when \( E_0 = E_m \) the value of the limit (19) is determined by the exponent \( \alpha \). Indeed if \( \alpha > 2 \):
\[ f_c(\beta) \leq \int_0^\delta c_1 E^{\frac{\alpha}{2} - 1}dE + \int_{E > \delta} \frac{\rho(E)dE}{e^{\beta E} - 1} < \infty \]  
(21)

where \( \delta \) and \( c_1 \) are suitable constants. Therefore in this case Bose-Einstein condensation occurs at finite temperature. For \( \alpha < 2 \) we have:
\[ f_c(\beta) \geq \lim_{z \to 1} \int_0^\delta \frac{z c_1 E^{\frac{\alpha}{2} - 1}dE}{\beta E + 1 - z} = \infty \]  
(22)

and there is no Bose-Einstein condensation. When \( \alpha = 2 \) we have to consider the logarithmic correction to (13) and it is possible to show that the limit (19) diverges.

The Hamiltonian (1) describes different models of non interacting bosons on graphs. The simplest example is the discretization of the usual Schrödinger equation. In this case the Hamiltonian is: \( h_{ij} = \frac{\hbar^2}{2m} L_{ij} \), where \( L_{ij} \) is the Laplacian operator. This is a pure topological model, completely defined by the graph geometry and it can be shown that \( E_0 = E_m \), i.e. there are no hidden states in the low energy region and \( \alpha = d \). Therefore the occurrence of BEC is determined by the spectral dimension (3), which has been calculated for a wide class of discrete networks (13,20).

On real condensed matter structures an interesting case is a pure hopping of non interacting bosons on graphs. This has been considered in (9) for the description of the Josephson junction arrays in the weak coupling limit with
Hamiltonian matrix given by: \( h_{ij} = -tA_{ij} \). In this case the behaviour of the model is much more complex and the presence of hidden spectra can give rise to Bose-Einstein condensation also in low dimensional structures [3].

Finally, the most general application of our result concerns particles interacting with an external potential, described by the Hamiltonian: \( h_{ij} = -tA_{ij} + V_i \delta_{ij} \). Indeed, the introduction of a local potential strongly enhances the possibility of modifying the energy spectra in order to induce BEC in non regular geometries.

In the next section, we will study in detail the pure hopping model defined on the comb graph, where the inhomogeneous topology produces low energy hidden states which give rise to Bose-Einstein condensation in a 1-dimensional structure \( (\alpha = \bar{d} = 1) \).

**IV. PURE HOPPING MODELS AND BEC ON THE COMB GRAPH**

The comb lattice is an infinite graph, which can be obtained connecting to each site of a linear chain, called backbone, a 1-dimensional chain called finger (see figure 1). The sites of the comb can be naturally labelled introducing two integer indices \( (x, y) \) with \( x, y \in \mathbb{Z} \), where \( x \) labels the different fingers and \( y \) represents the distance from the backbone.

A suitable definition of the finite Van Hove sphere for the comb lattice is the square periodic comb of \( \bar{L} \times \bar{L} \) sites. In this case the backbone and each finger are rings of \( \bar{L} \) sites. Each vertex can be labelled by \( (x, y) \), \( x, y \in \{0, 1, \ldots, \bar{L} - 1, \bar{L}\} \), with the conditions \( (0, 0) \equiv (\bar{L}, 0) \) and \( (x, 0) \equiv (x, \bar{L}) \) in order to guarantee periodic boundaries.

In the thermodynamic limit the spectrum of the infinite comb is evaluated as the limit for \( L \to \infty \) of the spectrum of the hopping model on the finite \( L \times L \) combs. The eigenvalue equation \( (14) \) on the finite comb with \( h_{ij} = -tA_{ij} \) with the previous labeling of sites reads:

\[
-t \sum_{x',y'=0}^{L-1} \left[ (\delta_{x,x'+1} + \delta_{x,x'-1})\delta_{y,y'} + (\delta_{y,y'+1} + \delta_{y,y'-1})\delta_{x,x'} \right] \psi(x', y') = E\psi(x, y)
\]  

(23)

By exploiting the translation invariance in the direction of the backbone, a Fourier transform in the variable \( x \) reduces \( (23) \) to a 1-dimensional eigenvalue problem. Let us define:

\[
\psi(k, y) = \sum_x e^{ikx} \psi(x, y)
\]  

(24)

with \( k = 2\pi n/\bar{L}, n = 1 \ldots \bar{L} \). The eigenvalues equation \( (23) \) becomes:

\[
-t \sum_{k',y'} \left[ -2 \cos(k)\delta_{y,0}\delta_{k,k'} + (\delta_{y,y'+1} + \delta_{y,y'-1})\delta_{k,k'} \right] \psi(k', y') = E\psi(k, y)
\]  

(25)

Now \( (25) \) is diagonal in the variable \( k \) and can be written as:

\[
-t \sum_{y'} \left[ -2 \cos(k_0)\delta_{y,0}\delta_{0,y'} + (\delta_{y,y'+1} + \delta_{y,y'-1}) \right] \psi(y') = E\psi(y)
\]  

(26)

where \( \psi(k, y) = \delta(k - k_0)\psi(y) \) with \( k_0 = 2\pi n/\bar{L}, n = 0 \ldots \bar{L} \). Equation \( (26) \) can be regarded as a 1-dimensional problem of a quantum particle interacting with a potential in the origin, \( V(k_0) = -2t \cos(k_0) \). The eigenvalues and eigenvectors of \( (26) \) are obtained by imposing the matching condition in \( y = 0 \) for the free particle solutions on the chain. Indeed, for \( y \neq 0 \), \( (24) \) the wave function satisfying \( (26) \) in \( y \neq 0, \bar{L} - 1 \) are:

\[
\psi(y) = \cos(hy + \alpha) \quad \text{for} \quad -2t \leq E = -2t \cos(h) \leq 2t;
\]  

(27)

\[
\psi(y) = Ae^{hy} + Be^{-hy} \quad \text{for} \quad E = -t(e^h + e^{-h}) < 2t;
\]  

(28)

\[
\psi(y) = A(-1)^y e^{hy} + B(-1)^y e^{-hy} \quad \text{for} \quad E = t(e^h + e^{-h}) > 2t.
\]  

(29)

Imposing \( \psi(y) \) to be a solution in \( y = 0 \) and \( y = \bar{N} - 1 \) introduces restrictions on the parameters \( h, \alpha, A \) and \( B \), and on the eigenvalues \( E \).

Let us first consider the conditions for \( (27) \), in \( y = \bar{L} - 1 \) and \( y = 0 \):

\[
-t \cos[h(\bar{L} - 2) + \alpha] - t \cos(\alpha) = -2t \cos(h) \cos[h(\bar{L} - 1) + \alpha]
\]  

(30)

\[
-t \cos[h(\bar{L} - 1) + \alpha] - t \cos(h + \alpha) - 2t \cos(k_0) \cos(\alpha) = -2t \cos(h) \cos[h(\bar{L} - 1) + \alpha]
\]  

(31)

The system has odd solutions with \( \alpha = \pi/2 \) (i.e. \( \psi(i) = \sin(hi) \)), \( h = 2\pi n/\bar{L} \) and \( n = 1, \ldots, \bar{L}/2 - 1 \) and even solutions, obtained by solving:
in the thermodynamic limit for $E$ solutions fill densely the interval $[E, E_D]$. Equation (36) represents the normalized density of states leading to:

$$\cos(k_0) \cot(hL/2) = \sinh(h)$$

(32)

Equation (32) can be solved graphically (see figure 2) obtaining $L/2$ solutions. In the large $L$ limit the allowed values for $h$ are: $h \approx \pi(2n-1)/L$ with $n = 1, \ldots, L/2$. Each value of $k_0$ corresponds to $L-1$ solutions with energy between $-2t$ and $2t$.

Let us now consider cases (28-29). For $E < 2t$ the wave function in $y = L - 1$ and $y = 0$ satisfies the conditions:

$$-t(Ae^{-h(L-2)} + Be^{h(L-2)}) - t(A + B) = -t(e^h + e^{-h})(Ae^{-h(L-1)} + Be^{h(L-1)})$$

$$-t(Ae^{-h(L-1)} + Be^{h(L-1)}) - t(Ae^h + Be^h) - 2\cos(k_0)(A + B) = -t(e^h + e^{-h})(A + B)$$

(33, 34)

leading to:

$$\cos(k_0) \coth(hL/2) = \sinh(h)$$

(35)

Equation (35) can be solved graphically. For $\cos(k_0) > 0$, (35) has a real solution. In the limit $L \to \infty$ this is $\sinh(h) = \cos(k_0)$, corresponding to energy is $E = -2t \sqrt{1 + \cos^2(k_0)}$. If $\cos(k_0) < 0$, only the solution with $E > 2t$ is present and for $\cos(k_0) = 0$ one has the constant solution $\psi(y) = 1$ with energy $E = 0$.

The complete spectrum of the comb is now obtained by considering the eigenvalues of (26) for the $L$ values of $k_0$. For each $k_0$ there are $L-1$ eigenvalues of type (27) with energy $E = -2t \cos(h)$ and wave functions $\psi(x, y) = e^{ik_0x} \sin(hx)$ and $\psi(x, y) = e^{ik_0x} \cos(hy + \alpha)$, with $h$ and $\alpha$ satisfying conditions (30-31). The fraction of states in this spectral region is $f = L(L-1)/L^2$. Notice that $f$ tends to 1 in the thermodynamic limit. The normalized density of states in this spectral region is:

$$\frac{dn}{L^2} = \frac{ld(Lh/2\pi)}{L^2} = \frac{dE}{\pi \sqrt{4t^2 - E^2}}$$

(36)

with $E \in [-2t, 2t]$. Since the fraction of states with energies $E < -2t$ or $E > 2t$ vanishes in the thermodynamic limit, (36) represents the normalized density of states $\rho(E)$ of the pure hopping model on the comb graph.

However, the lowest energy state is obtained from a solution of type (28). These solutions appear only for $\cos(k_0) > 0$ and they satisfy relation (30). Since the energy is a decreasing function of $h$, the lowest energy level is obtained for $\cos(k_0) = 1$. In this case we have $\coth(hL/2) = \sinh(h)$, and in the thermodynamic limit this corresponds to $\sinh(h) = 1$. From (28) we get $E_0 = -t \sqrt{8}$. The lowest energy in the normalized density of states is $E_m = -2t$ and therefore $E_0 < E_m$. Then the eigenvalue equation (23) presents a hidden spectrum in the thermodynamic limit, and the pure hopping boson model on the comb lattice exhibits BEC at low enough temperatures. From equations (28-30) it is also possible to show that on the infinite graph the eigenvector corresponding to the lowest energy eigenvalue is $\psi(x, y) = e^{-h}|y|$. The wave function for the condensate is localized along the backbone and it decreases exponentially along the fingers (see figure 3).

This model does not present an energy gap between $E_0$ and $E_m$, contrary to the typical instance of non interacting bosons trapped in a harmonic well. Indeed for each value of $k_0$ ($\cos(k_0) > 0$) there is a solution of (35) with a different energy in the interval $[E_0, E_m]$. In a finite comb of $L^2$ sites there are $L/2$ solution of this type and for $L \to \infty$ these solutions fill densely the interval $[E_0, E_m]$. Normalizing the density of states to $L$, we can obtain the spectral density in the thermodynamic limit for $E \in [E_0, E_m]$:

$$\frac{dn}{L} = \frac{d(Lk_0/2\pi)}{L} = \frac{|E|dE}{2\pi \sqrt{8t^2 - E^2} \sqrt{E^2 - 4t^2}}$$

(37)

An analogous equation holds for the spectral region at high energy ($E = [2t, \sqrt{8t}]$), where the other hidden states appear. The density of states can then be represented as in figure 4, where we used the two different normalizations for the hidden spectra and for $\rho(E)$.

From the knowledge of the complete spectral density of the comb, the thermodynamic quantities relative to BEC such as the filling of the ground state as a function of $T$, the specific heat and the critical temperature as a function of $f$ can be analytically calculated (36).

V. BOSE-EINSTEIN CONDENSATION ON GRAPHS: FURTHER EXAMPLES

The spectrum of the pure hopping model can be studied in detail in a wide class of fractals and inhomogeneous discrete structures. In particular, the behaviour observed in the comb lattice is typical of bundle structures (20). This
class of graphs are obtained by a “fibering” procedure, i.e. attaching a fiber graph to every point of a “base” graph. They are characterized by a hidden low-energy region, and from the general result of section III they exhibit BEC at finite temperature. Moreover, as in the comb graph, the wave function of the condensate is localized along the base and presents a fast decay along the fibers. An example of this behaviour is found in the brush graph, which is illustrated in figure 5.

An interesting example of a graph where the geometrical setting gives rise to an isolated eigenstate in the low-energy region of the pure hopping model is the star graph. The effect, which is analogous to that caused by an impurity on a line, is not produced by the presence of an external potential but simply by the topology of the system. Interestingly, the wave function is here completely localized in the center of the star, with an exponential decay along the arms. In figure 6 the star graph with the ground state and the spectrum of the pure hopping model are shown.

Graphs with constant coordination number are typical examples in which the pure hopping model do not present hidden low energy regions. This is due to the fact that for this class of graphs the spectrum of the model can be obtained from that of the Laplacian matrix by a shift of the zero in the energy. The existence of BEC is therefore determined by the spectral dimension $d = \alpha$ of the graph, describing the density of eigenvalues of the Laplacian at low energy. This parameter can be exactly calculated for a wide class of discrete structures. On lattices, $d$ coincides with the usual Euclidean dimension and one recovers the classical result for BEC on translation invariant structures. For exactly decimable graphs (e.g. the Sierpinski gasket and the T-fractal, see figure 7) where one always has $d < 2$ there is no Bose-Einstein condensation.

A fundamental property of the spectral dimension is its independence from the local details of the graph, i.e. universality. The value of $d$ and the behaviour of the density of eigenvalues in the low energy region is not changed under a wide class of transformations, called isospectralities, which can strongly modify the geometry of the graph. A simple consequence of this property is that if we consider the pure hopping model on a graph with constant coordination number, which differs from a graph of known dimension $d$ by an isospectrality, BEC occurs only if $d > 2$. An example of this behaviour is given by the ladder graph (figure 8) which can be obtained from the linear chain by the addition of finite-range links. On this structure then the pure hopping model does not exhibit BEC.

An important properties of $d$ is that the dimension of the graph obtained as a direct product of two graphs is the sum of the dimensions of the original structures. An example is illustrated in figure 9, where we show the direct product of a linear chain and a well known fractal, a Sierpinski gasket. In this the coordination number case $z_i$ is constant, and applying the previous properties it is easy to show that $d = 1 + 2 \ln(3)/\ln(4) > 2$, and therefore, applying the previous results, one immediately infers that BEC occurs on this graph.

VI. CONCLUDING REMARKS

We studied in detail BEC of non-interacting bosons on general discrete networks, i.e. on graphs. We evidenced that these systems support spatial condensation of bosons into a single state even if $d < 2$.

Due to recent advances in engineering technology, Josephson junction networks can be build in a variety of controllable, non-conventional architectures. The theoretical interest for these systems stems naturally from the ability of experimentalists in varying the properties of the networks by acting directly on their geometry. We expect that non universal quantities - such as the critical temperature for condensation - will depend crucially on the geometry of a graph, thus leading to the possibility of engineering network geometries suitable to observe spatial BEC of Cooper pairs.

Our analysis shows that classical inhomogeneous Josephson junction networks build on graphs may support a spatial BEC of Cooper pairs induced only by the geometry of the ambient space on which the junctions lie. We feel our prediction of the existence of spatial BEC in Josephson junction networks may be brought soon to direct experimental testing.

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FIG. 1. The comb lattice.

FIG. 2. The numerical solution of equation (32) for $k_0 = 0$ and $L = 16$. We represent the function $\cot(Lh/2)$ with a continuous line and $\sin(h)$ with a dashed line. The intersections of the two curves are the $L/2$ solutions of (32).
FIG. 3. The wave function of the low energy eigenstate in the comb lattice.

FIG. 4. The spectrum of the pure hopping model on the comb graph. $\rho(E)$ (normalized to $L^2$) is plotted with a continuous line, while for the hidden spectra (normalized to $L$) we used the dashed line.

FIG. 5. The brush graph and the spectrum of the pure hopping model. As in the comb graph, different normalizations are used for the hidden spectra (dotted line) and $\rho(E)$ (continuous line).

FIG. 6. The star graph with the wave function of the lowest energy state and the spectrum of the pure hopping model on this structure.
FIG. 7. Two classical examples of exactly decimable graph: the 3d Sierpinski gasket (\( \bar{d} = \frac{2 \ln(4)}{\ln(6)} < 2 \)) and the t-fractal (\( \bar{d} = \frac{2 \ln(3)}{\ln(6)} < 2 \)).

FIG. 8. The ladder graph and the spectrum of the pure hopping model on this structure. Here we do not have any hidden spectrum and \( \rho(E) \to \infty \) when \( E \to E_m = E_0 \) (\( \bar{d} = 1 \)).

FIG. 9. The graph obtained as a product of the Sierpinski gasket and a linear chain. Since the coordination number is constant, there is not hidden spectra in the low energy region. However \( \bar{d} = 1 + \frac{\ln(3)}{\ln(4)} > 2 \) (sum of the dimension of the original graphs) and therefore BEC occurs on this structure.