Amenable groups, topological entropy and Betti numbers

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Abstract. We investigate an analogue of the $L^2$-Betti numbers for amenable linear subshifts. The role of the von Neumann dimension shall be played by the topological entropy.

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1 Introduction

Let $\Gamma$ be a finitely generated group. Then the Hilbert space $l_2(\Gamma)$ has a natural left $\Gamma$-action by translations:

$$L_\gamma(f)(\delta) = f(\gamma^{-1}\delta).$$

Using the so-called von Neumann dimension we can assign a real number to any $\Gamma$-invariant linear subspace of $[l^2(\Gamma)]^n$, $n \in \mathbb{N}$ satisfying the following basic axioms [14].

1. **Positivity:** If $V \subset [l^2(\Gamma)]^n$ $\Gamma$-invariant linear subspace, then $\dim_\Gamma(V) \geq 0$. Also, $\dim_\Gamma(V) = 0$ if and only if $V = 0$.

2. **Invariance:** If $V \subset [l^2(\Gamma)]^n$, $W \subset [l^2(\Gamma)]^m$ and $T$ is a $\Gamma$-equivariant isomorphism from $V$ to a dense subset of $W$, then $\dim_\Gamma(V) = \dim_\Gamma(W)$.

3. **Additivity:** If $Z$ is the orthogonal direct sum of $V$ and $W$, then $\dim_\Gamma(Z) = \dim_\Gamma(V) + \dim_\Gamma(W)$.

4. **Continuity:** If $V_1 \supset V_2 \supset \ldots$ is a decreasing sequence of $\Gamma$-invariant linear subspaces, then:

$$\dim_\Gamma(\cap_{j=1}^{\infty} V_j) = \lim_{j \to \infty} \dim_\Gamma(V_j).$$

5. **Normalization:** $\dim_\Gamma[l^2(\Gamma)] = 1$.

There is an important application of the von Neumann dimension in algebraic topology due to Atiyah [1] (see also [4]). He defined certain invariants of finite simplicial complexes: the $L^2$-Betti numbers. The idea is the following, let $\tilde{K}$ be an infinite, simplicial complex with a free and simplicial $\Gamma$-action as covering transformations such that $\tilde{K}/\Gamma = K$ is finite. Denote by $C_p(\tilde{K})$ the Hilbert space of square-summable, real $p$-cochains of $\tilde{K}$. Then one has the following differential complex of Hilbert spaces,

$$C_0^p(\tilde{K}) \xrightarrow{d_0} C_1^p(\tilde{K}) \xrightarrow{d_1} \ldots \xrightarrow{d_{n-1}} C_n^p(\tilde{K}),$$

where the $d_p$’s are the usual coboundary operators. Note that $C_p(\tilde{K}) \cong [l^2(\Gamma)]^{K_p}$, where $K_p$ denotes the set of $p$-simplices in $K$. Atiyah’s $L^2$-Betti numbers are defined as

$$L_p^2(\tilde{K}) = \dim_\Gamma Ker d_p - \dim_\Gamma Im d_{p-1}.$$

Let us list some basic results on the $L^2$-Betti numbers.

- (Dodziuk, [4]) If $\tilde{K}$ and $\tilde{L}$ are homotopic by a $\Gamma$-invariant homotopy, then the corresponding $L^2$-Betti numbers of $\tilde{K}/\Gamma = K$ and $\tilde{L}/\Gamma = L$ are equal.
\[ \sum_{p=0}^{n} (-1)^p L(2)_p b_p(K) = e(K), \]
the Euler characteristic of \( K \).

- (Cheeger & Gromov, [2]) If \( \tilde{K} \) is contractible and \( \Gamma \) is amenable, then all \( L^2 \)-Betti numbers are vanishing.

- (Linnell, [9]) If \( \Gamma \) is elementary amenable and torsion-free then all \( L^2 \)-Betti numbers are integers.

- (Lück, [10]) Let \( \Gamma \) be residually finite and
\[
\Gamma \supset \Gamma_1 \supset \Gamma_2 \ldots, \quad \cap_{i=1}^{\infty} \Gamma_i = 1_{\Gamma}
\]

normal subgroups of finite index and let \( X_i = \tilde{K}/\Gamma_i \) the corresponding finite coverings of \( K \). Then
\[
L(2)_p b_p(K) = \lim_{i \to \infty} \frac{\dim_{\mathbb{R}} H^p(X_i, \mathbb{R})}{|\Gamma : \Gamma_i|}
\]

- (Dodziuk & Mathai [5]) If \( \{L_n\}_{n=1}^{\infty} \) is an exhaustion of \( \tilde{K} \) by finite simplicial complexes spanned by a \( \{F_n\}_{n=1}^{\infty} \) Følner-exhaustion , then
\[
L(2)_p b_p(K) = \lim_{i \to \infty} \frac{\dim_{\mathbb{R}} H^p(L_n, \mathbb{R})}{|F_n|}
\]

Note that the second and the third results together imply that if \( K \) is an acyclic simplicial complex with amenable fundamental group then its Euler characteristic is zero [2]. Another interesting application is due to Lück: If \( \Gamma \) is amenable, then the group algebra \( \mathbb{C}[\Gamma] \) as a free module over itself generates an infinite cyclic subgroup in the Grothendieck group of \( \mathbb{C}[\Gamma] \) [11].

The analogue setting we are investigating in this paper is the following. Let \( \Gamma \) be a finitely generated amenable group (see [3] why amenability is crucial). We denote by \( \Sigma_{\Gamma} \) the full Bernoulli shift that is the linear space of \( \mathbb{F}_2 \)-valued functions on \( \Gamma \), where \( \mathbb{F}_2 \) is the field of two elements. The space \( \Sigma_{\Gamma} \) is a compact, metrizable space in the pointwise convergence topology equipped with the natural left \( \Gamma \)-action by translations. A space \( V \subset [\Sigma_{\Gamma}]^n \) is a linear subshift if it is linear as a \( \mathbb{F}_2 \)-vector space, closed in the topology and invariant with respect to the \( \Gamma \)-action. The notion of dimension is the topological entropy of the linear subshifts. This is well-known for \( \mathbb{Z} \) and \( \mathbb{Z}^d \)-actions and somehow less-known for general amenable group actions (nevertheless see [12]). We shall observe that our dimension \( h_{\Gamma} \) satisfies similar axioms as \( \dim_{\Gamma} \):

1. **Nonnegativity:** For any \( V \) linear subshift : \( h_{\Gamma}(V) \geq 0 \). But it can be zero even if \( V \) is not zero.
2. **Monotonicity**: If $V \subset W$, then $h_{\Gamma}(V) \leq h_{\Gamma}(W)$.

3. **Invariance**: If $T : V \rightarrow W$ continuous $\Gamma$-equivariant linear isomorphism, then $h_{\Gamma}(V) = h_{\Gamma}(W)$.

4. **Additivity**: If $Z = V \oplus W$, then $h_{\Gamma}(Z) = h_{\Gamma}(V) + h_{\Gamma}(W)$.

5. **Continuity**: If $V_1 \supset V_2 \ldots$ is a decreasing sequence of linear subshifts, then:
   \[ h_{\Gamma}(\cap_{j=1}^{\infty} V_j) = \lim_{j \rightarrow \infty} h_{\Gamma}(V_j). \]

6. **Normalization**: $h_{\Gamma}(\sum_{\Gamma}) = 1$.

Now let $\widetilde{K}$ be as above. Then we have the ordinary cochain complex of $F_2$-coefficients over $\widetilde{K}$:
\[ C^0(\widetilde{K}, F_2) \xrightarrow{d_0} C^1(\widetilde{K}, F_2) \xrightarrow{d_1} \ldots \xrightarrow{d_{n-1}} C^n(\widetilde{K}, F_2). \]
Then the $p$-cochain space $C^p(\widetilde{K}, F_2)$ is $\Gamma$-isomorphic to $[\sum_{\Gamma}]^{K_p}$, where $K_p$ denotes the set of $p$-simplices in $\widetilde{K}$. We define the $p$-th entropy Betti number $b_{E}^p(K)$ as $h_{\Gamma}(\text{Ker } d_p) - h_{\Gamma}(\text{Im } d_{p-1})$. In this paper we shall prove the following analogues of the $L^2$-results.

- If $\widetilde{K}$ and $\widetilde{L}$ are homotopic by a $\Gamma$-invariant homotopy and $\Gamma$ is poly-cyclic then the corresponding entropy-Betti numbers of $\widetilde{K}/\Gamma = K$ and $\widetilde{L}/\Gamma = L$ are equal.

- $\sum_{p=0}^{n} (-1)^p b_{E}^p(K) = e(K)$,
  the Euler characteristic of $K$.

- If $\widetilde{K}$ is contractible then all entropy-Betti numbers are vanishing. (this is quite obvious, the point is that the corollary on the vanishing Euler-characteristic still follows from this and the previous statement)

- If $\Gamma$ is poly-infinite-cyclic then all entropy-Betti numbers are integers.

- Let $\Gamma$ be free Abelian and
  \[ \Gamma \supset \Gamma_1 \supset \Gamma_2 \ldots, \cap_{i=1}^{\infty} \Gamma_i = 1_{\Gamma} \]
  normal subgroups of finite index and let $X_i = \widetilde{K}/\Gamma_i$ the corresponding finite coverings of $K$. Then
  \[ b_{E}^p(K) = \lim_{i \rightarrow \infty} \frac{\dim_{F_2} H^p(X_i, F_2)}{|\Gamma : \Gamma_i|}, \]
• If \( \{L_n\}_{n=1}^{\infty} \) is an exhaustion of \( \overline{K} \) by finite simplicial complexes spanned by a \( \{F_n\}_{n=1}^{\infty} \) Følner-exhaustion, then
\[
b_E^p(K) = \lim_{n \to \infty} \frac{\dim_{F_2} H^p(L_n, F_2)}{|F_n|}
\]

We shall also prove an analogue of Lück’s result on the Grothendieck-group for the group algebras \( F_2[\Gamma] \).

## 2 Amenable groups and quasi-tiles

Let \( \Gamma \) be a finitely generated group with a symmetric generator set \( \{g_1, g_2, \ldots, g_k\} \). The right Cayley-graph of \( \Gamma \), \( C_\Gamma \) is defined as follows. Let \( V(C_\Gamma) = \Gamma \), \( E(C_\Gamma) = \{(a, b) \in \Gamma \times \Gamma : \text{there exists } g_i : ag_i = b\} \). The shortest path distance \( d \) of \( C_\Gamma \) makes \( \Gamma \) a discrete metric space. We shall use the following notation. If \( H \subset \Gamma \) is a finite set, then \( B_r(H) \) is the set of elements \( a \in \Gamma \) such that there exists \( h \in H, d(a, h) \leq r \). We denote \( B_1(H) \setminus H \) by \( \partial H \) and \( B_r(H) \setminus H \) by \( \partial_r H \). An exhaustion of \( \Gamma \) by finite sets \( 1^n \Gamma \subset F_2 \subset \ldots \subset F_N \) is called a Følner-exhaustion if for any \( r \in \mathbb{N} \): \( \lim_{n \to \infty} \frac{|\partial_r F_n|}{|F_n|} = 0 \). A group \( \Gamma \) is called amenable if it possess a Følner-exhaustion. Some amenable groups have tiling Følner-exhaustion that is any \( F_n \) is a tile: There exists \( C \subset \Gamma \) such that \( \{cF_n\}_{c \in C} \) is a partition of \( \Gamma \). For example \( \mathbb{Z}^n \) has this tiling property. As observed by Ornstein and Weiss \([13]\) any amenable group has quasi-tiling Følner-exhaustion. Let us recall their construction. Let \( \{A_i\}_{i=1}^{\infty} \) be finite sets. Then we call them \( \epsilon \)-disjoint if there exist subsets \( \overline{A_i} \subset A_i \) so that \( \overline{A_i} \cap \overline{A_j} = 0 \) if \( i \neq j \), and \( \frac{|A_i|}{|G|} \geq 1 - \epsilon \) for all \( i \). Now let \( B \) another finite set. We say that \( \{A_i\}_{i=1}^{\infty} (1 - \epsilon) \)-cover \( B \), if
\[
\frac{|B \cap \cup_{i=1}^{\infty} A_i|}{|B|} \geq 1 - \epsilon.
\]

The subsets of \( \Gamma \), \( 1^n \Gamma \subset F_2 \subset \ldots \subset F_N \) form an \( \epsilon \)-quasi-tile system if for any finite subset of \( A \subset \Gamma \), there exists \( C_i \subset \Gamma, i = 1, 2, \ldots, N \) such that
1. \( C_iT_i \cap C_jT_j = \emptyset \) if \( i \neq j \).
2. \( \{cT_i : c \in C_i\} \) are \( \epsilon \)-disjoint sets for any fixed \( i \).
3. \( \{C_iT_i\} \) form a \((1 - \epsilon)\)-cover of \( A \).

The following proposition is Theorem 6. in \([13]\):

**Proposition 2.1** If \( F_1 \subset F_2 \subset \ldots \) is a Følner-exhaustion of an amenable group, then for any \( \epsilon > 0 \) we can choose a finite subset \( F_{n_1} \subset F_{n_2} \subset \ldots \subset F_{n_N} \) such that they form an \( \epsilon \)-quasi-tile system. The number \( N \) may depend on \( \epsilon \).
3 The topological entropy of linear subshifts

First of all we define an averaged dimension $h_\Gamma(W)$ for linear subshifts and then we shall show that it coincides with the topological entropy. Let $\Gamma$ be a finitely generated amenable group with Følner-exhaustion $1_\Gamma \in F_1 \subset F_2 \subset \ldots \cup_{j=1}^{\infty} F_j = \Gamma$. We introduce some notations. If $\Lambda \subset \Gamma$ is a finite set, then $[\sum_{\Lambda}]^r$ be the space of functions in $[\sum_{\Gamma}]^r$ supported on $\Lambda$. Also, $[\sum_{\Gamma}]^r$ denotes the space of finitely supported functions. Now let $W \subset [\sum_{\Gamma}]^r$ be a $\Gamma$-invariant not necessarily closed linear subspace. Then for any finite $\Lambda \subset \Gamma$ let $W_\Lambda \subset [\sum_{\Lambda}]^r$ be the linear space of functions $\eta$ supported on $\Lambda$ such that there exists $\nu \in W : \eta|_\Lambda = \nu|_\Lambda$.

**Definition 3.1** $h_\Gamma(W) = \limsup_{n \to \infty} \frac{\log_2 |W_{F_n}|}{|F_n|}$

Note that $\log_2 |W_{F_n}|$ is just the dimension of the vector space $W_{F_n}$ over the field $F_2$. It will be obvious from the next proposition that $h_\Gamma(W)$ does not depend on the particular choice of the exhaustion.

**Proposition 3.1**

1. $h_\Gamma(W) = h_\Gamma(\overline{W})$, where $\overline{W}$ denotes the closure of $W$ in the point-wise convergence topology.

2. $h_\Gamma(W) = \liminf_{n \to \infty} \frac{\log_2 |W_{F_n}|}{|F_n|}$, hence $\lim_{n \to \infty} \frac{\log_2 |W_{F_n}|}{|F_n|}$ always exists and equals to $h_\Gamma(W)$.

**Proof:** The first part is obvious from the definition, for the second part we argue by contradiction. Suppose that

$$h_\Gamma(W) - \liminf_{n \to \infty} \frac{\log_2 |W_{F_n}|}{|F_n|} = \delta > 0.$$

Consider a subsequence $F_{n_1} \subset F_{n_2} \subset \ldots$ such that

$$\sup_{i \to \infty} \frac{\log_2 |W_{F_{n_i}}|}{|F_{n_i}|} < \liminf_{n \to \infty} \frac{\log_2 |W_{F_n}|}{|F_n|} + \epsilon,$$

where the explicit value of $\epsilon$ shall be chosen later accordingly. Then pick an $\epsilon$-quasi-tile system from our subsequence: $F_{m_1} \subset F_{m_2} \subset \ldots \subset F_{m_N}$. Now we take an arbitrary $F_n$ from the original Følner-exhaustion. By Proposition 2.1 we have an $\epsilon$-disjoint, $(1 - \epsilon)$-covering of $F_n$ by translates of the quasi-tile system. Denote by $R_1, R_2, \ldots R_k$ those tiles which are properly contained in $F_n$. Then we have the following estimate.

$$|W_{F_n}| \leq 2^{ \left( r \left( |F_n| + r |F_n \setminus B_{D+1}(\partial F_n)| \right) \right)} \prod_{i=1}^{k} |W_{R_i}|,$$

where $D$ is the diameter of the largest tile $F_{m_N}$. The inequality (1) follows from the fact that a function $\xi \in W_{F_n}$ is uniquely determined by its restrictions on the covering tiles and its restriction on the uncovered elements. The later one consists of two parts; the elements which are not covered by the original covering and the elements which are
covered by tiles intersecting the complement of $F_n$. These “badly” covered elements are in a $D + 1$-neighbourhood of the boundary of $F_n$. Also, by $\epsilon$-disjointness we have the estimate

$$\sum_{i=1}^{k} |R_i| \leq \frac{1}{1 - \epsilon} |F_n|.$$  

(2)

Therefore,

$$|W_{F_n}| \leq 2^{(r|F_n| + r|F_n \setminus B_{D+1}(\partial F_n)|)} 2^{\frac{1}{1 - \epsilon} (h_\Gamma - \delta + \epsilon)}$$

Hence,

$$\log_2 \frac{|W_{F_n}|}{F_n} \leq r \epsilon + \frac{r|F_n \setminus B_{D+1}(\partial F_n)|}{|F_n|} + \frac{1}{1 - \epsilon} (h_\Gamma - \delta + \epsilon)$$

(3)

Consequently if we choose $\epsilon$ small enough, then for large $n$, $\log_2 \frac{|W_{F_n}|}{F_n} \leq h_\Gamma - \frac{\delta}{2}$, leading to a contradiction.  

Now we recall the notion of topological entropy. Let $\Gamma$ be an amenable group as above and let $X$ be a compact metric space equipped with a continuous $\Gamma$-action; $\alpha : \Gamma \to \text{Homeo}(X)$. Instead of the original definition of Moulin-Ollagnier [12] we use the equivalent “spanning-separating” definition, that is a direct generalization of the Abelian case [13]. We call a finite set $S \subset X$ $(n, \epsilon)$-separated if for any distinct points $s, t \in S$ there exists $\gamma \in F_n$ such that $d(\alpha(\gamma^{-1})(s), \alpha(\gamma^{-1})(t)) > \epsilon$. We denote by $s(n, \epsilon)$ the maximal cardinality of such sets. We call a finite set $R \subset X$ $(n, \epsilon)$-spanning if for any $x \in X$ there exists $y \in R$ such that $d(\alpha(\gamma^{-1})(x), \alpha(\gamma^{-1})(y)) \leq \epsilon$, for all $\gamma \in F_n$. We denote by $r(n, \epsilon)$ the minimal cardinality of such sets. Obviously if $\epsilon' < \epsilon$ then $s(n\epsilon') \geq s(n\epsilon), r(n\epsilon') \geq r(n\epsilon)$. Also, we have the inequalities:

$$r(n, \epsilon) \leq s(n, \epsilon) \leq r(n, \frac{\epsilon}{2}).$$

Indeed, any $(n, \epsilon)$-separating set is $(n, \epsilon)$-spanning. On the other hand if $R = x_1, x_2, \ldots, x_k$ is a $(n, \frac{\epsilon}{2})$-spanning set then $X = \bigcup_{i=1}^{k} D(x_i, n, \frac{\epsilon}{2})$, where

$$D(x_i, n, \frac{\epsilon}{2}) = \{ y \in X : d(\alpha(\gamma^{-1})(x), \alpha(\gamma^{-1})(y)) \leq \frac{\epsilon}{2}, \text{for all } \gamma \in F_n \}.$$ 

Any $D(x_i, n, \frac{\epsilon}{2})$ can contain at most one element of a $(n, \epsilon)$-separating set, hence $s(n, \epsilon) \leq r(n, \frac{\epsilon}{2})$. Consequently, $\lim_{\epsilon \to 0} \limsup_{n \to \infty} \log_2 r(n, \epsilon) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \log_2 s(n, \epsilon)$. This joint limit is called the topological entropy of the $\Gamma$-action and denoted by $h^{\text{top}}(X)$. Note that it follows from the definition that $h^{\text{top}}(X)$ depends only on the topology and not the particular choice of the metric on $X$.

**Proposition 3.2** Let $V \subset \{ \Sigma_{\Gamma} \}^r$ be a linear subshift. Then $h^{\text{top}}_E(V) = h_\Gamma(V)$.

**Proof:** First we fix a metric on $V$ that defines the pointwise convergence topology. If $v, w \in V$, then let $d_V(v, w) = 2^{-(n-1)}$, where $n$ is the infimum of $k$’s such that $v | F_k \neq w | F_k$. First note that $|V_{F_n}| \leq s(n, 1)$. Indeed if $v_1, v_2, \ldots, v_s$ is a subset of $V$ such that $v_i | F_n \neq v_j | F_n$ when $i \neq j$, then there exists $\gamma \in F_n$ such that $L_{\gamma^{-1}} v_i(1 \Gamma) \neq L_{\gamma^{-1}} v_j(1 \Gamma)$. Fix an $\epsilon$ and choose $K_\epsilon, M_\epsilon \in \mathbb{N}$ such that $\epsilon > 2^{-K_\epsilon}$ and $F_{K_\epsilon} \subset B_{M_\epsilon}(1 \Gamma)$. Then we
claim that \( s(n, \epsilon) \leq |V_{B_M(F_n)}| \). Indeed, if \( x \mid B_M(F_n) = y \mid B_M(F_n) \), then for any \( \gamma \in F_n \),
\[ d_V(L_{\gamma^{-1}}(x), L_{\gamma^{-1}}(y)) \leq 2^{-K_i} \epsilon. \]
Hence if \( \epsilon < 1 \),
\[ h_\Gamma(V) = \lim_{n \to \infty} \frac{\log_2 |V_{F_n}|}{|F_n|} \leq \log_2 (s(n, \epsilon)) \leq \lim_{n \to \infty} \frac{\log_2 |V_{B_M(F_n)}|}{|F_n|} = h_\Gamma(V) \]

Note that the previous proposition immediately shows that \( h_\Gamma(V) \) does not depend on how \( V \) is imbedded in a full shift.

### 4 Extended Configurations

The notion of extended configuration is due to Ruelle in a slightly different form. Again we start with a linear subshift \( V \subset [\Sigma_\gamma]^\mathbb{N} \). For any \( \Lambda \subset \Gamma \) finite set let \( V_\Lambda^\Omega \subset [\Sigma_\gamma]^\mathbb{N} \) be a finite dimensional linear subspace satisfying the following axioms:

- **Extension:** \( V_\Lambda \subset V_\Lambda^\Omega \).
- **Invariance:** \( V_\Lambda^\Omega = L_\gamma(V_\Lambda^\Omega) \).
- **Transitivity:** If \( \Lambda \subset M \), then for any \( \xi \in V_M^\Omega \) there exists \( \mu \in V_\Lambda^\Omega \) such that \( \xi \mid_\Lambda = \mu \mid_\Lambda \).
- **Determination:** If \( \xi \in [\Sigma_\gamma]^\mathbb{N} \) and for any \( \Lambda \subset \Gamma \) finite there exists \( \xi_\Lambda \in V_\Lambda^\Omega \) such that \( \xi \mid_\Lambda = \xi_\Lambda \).

We call such a system an extended configuration of \( V \). Its topological entropy is defined as \( h_\Gamma^\Omega(V) = \limsup_{n \to \infty} \frac{|V_{F_n}|}{|F_n|} \).

**Proposition 4.1** \( h_\Gamma^\Omega(V) = h_\Gamma(V) \) (compare to Theorem 3.6 [17])

**Proof:** Let us suppose that \( h_\Gamma^\Omega(V) = h_\Gamma(V) = \delta > 0 \). Again choose an \( \epsilon \)-quasi-tile system \( F_{n_1}, F_{n_2}, \ldots, F_{n_N} \) such that \( |V_{F_{n_i}}| < 2^{(h_\Gamma^\Omega + \epsilon)|F_{n_i}|} \), where the explicit value of \( \epsilon \) will be given later. Denote by \( V_{F_{n_i}}^k \) the space of functions \( \mu \) in \( V_{F_{n_i}} \) such that there exists \( \xi \in V_{B_{n_i}(F_{n_i})}^\Omega \) with \( \xi \mid_{F_{n_i}} = \mu \mid_{F_{n_i}} \). By Extension and Determination properties it is easy to see that for large \( p \):
\[ V_{F_{n_i}}^p = V_{F_{n_i}} \]
where \( 1 \leq i \leq N \). Let us pick a large \( p \). Then we can proceed almost the same way as in the previous section. Denote by \( R_i \), \( 1 \leq i \leq k \) those translates in the \( \epsilon \)-disjoint \((1 - \epsilon)\)-covering of a Følner-set \( F_n \) such that not only the \( R_i \)'s but even the \( B_p(R_i) \) balls are contained in \( F_n \). Then by Transitivity we have the following estimate
\[ |V_{F_n}^\Omega| \leq 2^{r\epsilon|F_n| + r|F_n \setminus B_{p+D+1}(\partial F_n)|} \prod_{i=1}^{k} |V_{R_i}^p| \]
That is by Invariance, (2) and (3):
\[ \frac{\log_2 |V_{F_n}^\Omega|}{|F_n|} \leq r\epsilon + \frac{r|F_n \setminus B_{p+D+1}(\partial F_n)|}{|F_n|} + \frac{1}{1 - \epsilon} (h_\Gamma - \delta + \epsilon) \]
which leads to a contradiction provided that we choose \( \epsilon \) small enough.
5 Basic Properties

Now we are in the position to prove the basic properties of $h_\Gamma$ as stated in the Introduction. The Monotonicity, Normalization and Positivity axioms are obviously satisfied.

**Lemma 5.1** Let $V, W \subset [\Sigma]\,^r$ be linear subshifts such that $V \cap W = 0$, then $h_\Gamma(V) + h_\Gamma(W) = h_\Gamma(V \oplus W)$.

**Proof:** First note that just because $V \cap W$ is the zero subspace it is not necessarily true that $V_\Lambda \cap W_\Lambda = 0$ as well. However, we can prove that $N_\Lambda^\Omega = V_\Lambda \cap W_\Lambda$ is an extended configuration of the zero subspace. We only need to show that the Determination axiom is satisfied. Suppose that $\xi \in [\Sigma]\,^r$ such that $\xi \mid_{F_n} \in V_{F_n} \cap W_{F_n}$. Therefore there exists $v_n \in V, w_n \in W$ such that $\xi \mid_{F_n} = v_n \mid_{F_n} = w_n \mid_{F_n}$. Hence $v_n \to \xi, w_n \to \xi$ in the topology of $[\Sigma]\,^r$. The spaces $V$ and $W$ are closed, thus $\xi \in V \cap W$, hence $\xi = 0$. By elementary linear algebra,

$$\dim_{F_2}(V \oplus W)_\Lambda + \dim_{F_2}N_\Lambda^\Omega = \dim_{F_2}V_\Lambda + \dim_{F_2}W_\Lambda.$$

Hence by our Proposition our Lemma follows.

Now we prove a property of the entropy that is slightly more general then invariance.

**Proposition 5.1** Let $T : V \to W$ be a continuous $\Gamma$-equivariant linear map between linear subshifts $V \subset [\Sigma]\,^r, W \subset [\Sigma]\,^r$. Then $h_\Gamma(Ker T) + h_\Gamma(Im T) = h_\Gamma(V)$.

**Proof:** First of all let us note that by the compactness of $V$ and the continuity of $T$ both $Ker T$ and $Im T$ are linear subshifts. Now let us consider the natural right action of $F_2(\Gamma)$ on $[\Sigma]\,^r, R_\gamma f(x) = f(x\gamma)$. This action obviously commutes with our previously defined left $\Gamma$-action. The right action can be extended to $s \times r$-matrices with coefficients in $F_2(\Gamma)$ acting on the column vectors $[\Sigma]\,^r$. Obviously, any such matrix $M$ defines a $\Gamma$-equivariant map; $T_M : [\Sigma]\,^r \to [\Sigma]\,^s$.

**Lemma 5.2** Any continuous $\Gamma$-equivariant linear map $T : V \to W$ can be given via multiplication by some $s \times r$-matrix $T_M$ with coefficients in $F_2(\Gamma)$.

**Proof:** Since $T$ is uniformly continuous the value of $T(v)(1_\Gamma)$ is determined by the value of $v$ on a finite ball $B$, where $B$ does not depend on $v$. Hence for any $1 \leq i \leq s$:

$$T(v)(1_\Gamma) = \sum_{\gamma \in B} \sum_{j=1}^r c_{ij}^\gamma \cdot v_j(\gamma),$$

where $c_{ij}^\gamma \in F_2$. By $\{T_M\}_{ij} = \{\sum_{\gamma \in B} c_{ij}^\gamma \}$ define a $s \times r$-matrix. Then for any $v \in V, T_M(v)(1_\Gamma) = T(v)(1_\Gamma)$. Hence by the $\Gamma$-equivariance of the matrix multiplication:

$$T_M(v)(\gamma) = L_{\gamma^{-1}}(T_M(v))(1_\Gamma) = T_M(L_{\gamma^{-1}}(v))(1_\Gamma) = T(L_{\gamma^{-1}}(v))(1_\Gamma) = T(v)(\gamma)$$
Now we return to the proof of our Proposition. We denote by $T_M$ the matrix and by $k$ the
diameter of the ball $B$ defined in our Lemma. Let

$$N^\Omega_\Lambda = \{v \in [\sum\Lambda]^r: \text{there exists } z \in V_{B_k(\Lambda)}, \text{ such that } z |_{\Lambda} = v \text{ and } T(z) |_\Lambda = 0\}$$

$$M^\Omega_\Lambda = \{w \in [\sum\Lambda]^s: \text{there exists } z \in V_{B_k(\Lambda)}, \text{ such that } T(z) |_\Lambda = w\}$$

Then $N^\Omega_\Lambda$ is an extended configuration of $Ker T_M$, $M^\Omega_\Lambda$ is an extended configuration of
$Im T_M$. Let $\hat{T}_\Lambda: V_{B_k(\Lambda)} \to [\sum\Lambda]^s$ be the restriction of $T$ onto $\Lambda$. Then $Im \hat{T}_\Lambda = M^\Omega_\Lambda$. We
have the usual pigeon-hole estimate:

$$|N^\Omega_{F_n}| \leq |Ker \hat{T}_{F_n}| \leq |N^\Omega_{F_n}|2^{|B_k(\partial F_n)|}$$

(5)

Also, by linear algebra we obtain

$$\dim F_2 Ker \hat{T}_{F_n} + \dim F_2 Im \hat{T}_{F_n} = \dim F_2 V_{B_k(F_n)}$$

That is

$$\log_2 |Ker \hat{T}_{F_n}| + \log_2 |Im \hat{T}_{F_n}| = \log_2 |V_{B_k(F_n)}|$$

(6)

It is easy to see that (5) and (6) together implies the statement of our Proposition. □

Now we prove the continuity property.

**Proposition 5.2** If $V^1 \supset V^2 \supset \ldots$ is a decreasing sequence of linear subshifts then

$$h_\Gamma(\bigcap_{j=1}^\infty V^j) = \lim_{j \to \infty} h_\Gamma(V^j)$$

**Proof:** For any $k \in N, V^k \supset \bigcap_{j=1}^\infty V^j$. Hence $h_\Gamma(\bigcap_{j=1}^\infty V^j) \leq \lim_{j \to \infty} h_\Gamma(V^j)$. We need to
prove the converse inequality. Suppose that for all $j$, $h_\Gamma(V^j) \geq h_\Gamma(\bigcap_{j=1}^\infty V^j) + 2\epsilon$. Let $T(j)$
be a monotone increasing function such that

$$\frac{\log_2 |V^j_j|}{|F^j_s|} \geq h_\Gamma(\bigcap_{j=1}^\infty V^j) + \epsilon$$

when $s \geq T(j)$. We define a monotone non-increasing function $S: N \to N$ such that

$S(n) = 1$ if there is no such $j$ so that $T(j) \leq n$ and $S(n) = \inf \{j: T(j) \leq n\}$ otherwise.

Then $S(n) \to \infty$ as $n \to \infty$. Define $V^\Omega_\Lambda = V^S(n)$, where $n$ is the smallest integer such that

$|F^\Omega_n| \geq |\Lambda|$. It is easy to see that $V^\Omega_\Lambda$ is an extended configuration of $\bigcap_{j=1}^\infty V^j$. Therefore,

$$\frac{\log_2 |V^\Omega_\Lambda|}{|F^\Omega_n|} \to h_\Gamma(\bigcap_{j=1}^\infty V^j).$$

On the other hand, by our construction:

$$\frac{\log_2 |V^\Omega_\Lambda|}{|F^\Omega_n|} \geq h_\Gamma(\bigcap_{j=1}^\infty V^j) + \epsilon,$$

leading to a contradiction. □

6 **Pontryagin Duality**

In this section we recall the Pontryagin Duality theory [8]. Let $A$ be a locally compact Abelian group and let $\hat{\Lambda}$ be its dual. That is the group of continuous homomorphisms
$\chi: A \to S^1 = \{z \in C : |z| = 1\}$. According to the duality theorem $A$ is naturally isomorphic to its double dual. The relevant example for us is $A = [\Sigma_\Gamma]^r$, its dual is $[\Sigma_\Gamma]^\ast_r$ the group of finitely supported elements, where $\langle \chi, f \rangle = \sum_{\gamma \in \Gamma} \chi(\gamma) f(\gamma)$ for $\chi \in [\Sigma_\Gamma]^\ast_r$ and $f \in [\Sigma_\Gamma]^r$. Here $(a, b)$ is defined as $\sum_{i=1}^r a_i b_i$. The additive group of $F_2$ is viewed as the subgroup $\{ -1, 1 \} \in S^1$. If $H \subset [\Sigma_\Gamma]^\ast_r$ is a compact subgroup then

$$H^\perp = \{ \chi \in [\Sigma_\Gamma]^\ast_r : \langle \chi, h \rangle = 1 \text{ for any } h \in H \}$$

Conversely if $B \subset [\Sigma_\Gamma]^\ast_r$ is a subgroup then $B^\perp = \{ f \in [\Sigma_\Gamma]^r : \langle f, \chi \rangle = 1 \text{ for any } \chi \in B \}$.

Then $(H^\perp)^\perp = H$, $(B^\perp)^\perp = B$. If $A, B$ locally compact groups and $\psi: A \to B$ continuous homomorphisms, then its dual $\hat{\psi}: \hat{B} \to \hat{A}$ is defined by $\langle \hat{\psi}(\chi), a \rangle = \langle \chi, \psi(a) \rangle$. Again the double dual of $\psi$ is itself if $A$ and $B$ are both compact or both discrete. Then $\psi$ is injective resp. surjective if and only if $\hat{\psi}$ is surjective (resp. injective). Moreover, if we have a short exact sequence of compact or discrete groups

$$1 \to A_1 \to A_2 \to \ldots \to A_n \to 1$$

Then its dual sequence

$$0 \to \hat{A}_n \to \ldots \to \hat{A}_2 \to \hat{A}_1 \to 0$$

is also exact. The next proposition is a version of a result of Schmidt \[18\].

**Proposition 6.1** The Pontryagin duality provides a one-to-one correspondence between linear subshifts and finitely generated left $F_2[\Gamma]$-modules.

**Proof:** First note that if $L_\gamma$ is the left multiplication by $\gamma$ on $[\Sigma_\Gamma]^r$ then $\hat{L}_\gamma$ is the left multiplication by $\gamma^{-1}$ on $[\Sigma_\Gamma]^\ast_r$. Hence if $V \xrightarrow{\iota} [\Sigma_\Gamma]^r$ is the natural embedding of a linear subshift, then $(F_2[\Gamma])^r \cong [\Sigma_\Gamma]^r \xrightarrow{\iota} \hat{V}$ is a surjective $F_2[\Gamma]$-module homomorphism that is $\hat{V}$ is a finitely generated left $F_2[\Gamma]$-module. Conversely, the dual of a finitely generated $F_2[\Gamma]$-module is a linear subshift. It is important to note that if $V \subset [\Sigma_\Gamma]^r$, $W \subset [\Sigma_\Gamma]^a$ are isomorphic linear subshifts then the dual of this isomorphism provides a module- isomorphism between $\hat{W}$ and $\hat{V}$. Conversely, the duals of isomorphic modules are isomorphic linear subshifts. \quad \blacksquare

### 7 The Noether property of group algebras

Let $V \subset [\Sigma_\Gamma]^r$ be a linear subshift. We denote by $V^0$ the subspace of finitely supported elements.

**Proposition 7.1** If $V^0$ contains a non-zero element then $h_\Gamma(V^0) > 0$.

**Proof:** Let us suppose that a ball $B_n(1_\Gamma)$ contains the support of a non-zero element in $V^0$. We claim that there exists an $\epsilon > 0$ such that if $n$ large enough then $F_n$ contains at least $\epsilon |F_n|$ disjoint translates of $B_n(1_\Gamma)$. First note that the claim implies our Proposition. If we have $M_n$ translates of $B_n(1_\Gamma)$ in $F_n$ then we can find $2^{M_n}$ different elements of $V^0$ which are all supported in $F_n$. Therefore

$$\frac{\log_2 |V_{F_n}|}{|F_n|} \geq \frac{M_n}{|F_n|} \geq \frac{\epsilon |F_n|}{|F_n|} = \epsilon.$$
Hence \( h_{\Gamma}(V^0) \geq \epsilon \). Let us prove the claim. Pick a maximal \( 2r \)-net \( a_1, a_2, \ldots, a_{M_n} \), that is maximal set of points in \( F_n \) such that any two has distance greater or equal than \( 2r \). Then the \( 4r \)-balls around the points \( a_i \) are covering \( F_n \). Hence \( M_n \geq \frac{|F_n|}{B_r(r)} \). Then at least half of the \( a_i \)'s are not in \( F_n \setminus B_{r+1}(\partial F_n) \). The balls around these elements far being from the boundary are completely in \( F_n \). Hence we have at least \( \frac{1}{2} \frac{|F_n|}{2 B_r(r)} \) disjoint translates of \( B_r(1r) \) in \( F_n \).

In the rest of this section we shall have an extra assumption on the amenable group \( \Gamma \). We call an amenable group Noether if \( F_2[\Gamma] \) is a Noether ring. That is any left submodule of \( (F_2[\Gamma])^r \) is finitely generated. According to Hall’s theorem \([15]\) if \( \Gamma \) is polycyclic-by-finite then \( \Gamma \) is Noether.

**Proposition 7.2** If \( V \subset [\Sigma_\Gamma]^r \) is a linear subshift and \( \Gamma \) is Noether, then \( h_{\Gamma}(V) + h_{\Gamma}(V^\perp) = r \).

First of all \( V^\perp \subset [\Sigma_\Gamma]^r \subset [\Sigma_\Gamma]^{r+1} \) hence the expression \( h_{\Gamma}(V^\perp) \) is meaningful. By our assumption \( V^\perp \) is a finitely generated module, so let us choose a \( r_1, r_2, \ldots, r_k \) finite generator set. We need to prove that \( \lim_{n \to \infty} \frac{\log_2 |V_{F_n}^\perp|}{|F_n|} = r - h_{\Gamma}(V) \). In order to do so in it is enough to see that

\[
\lim_{n \to \infty} \frac{\log_2 |V_{F_n}^\perp| - \log_2 |V_{n}^\perp|}{|F_n|} = 0 ,
\]

(7)

where \( V_{F_n}^\perp \) denote the set of elements in \( V^\perp \), supported in \( F_n \). Remember that \( V_{F_n}^\perp \) denotes the restrictions of the elements of \( V^\perp \), therefore \( V_{F_n}^\perp \subset V_{n}^\perp \). By linear algebra,

\[
\dim F_2(V_{F_n}) + \dim F_2(V_{n}^\perp) = r|F_n|,
\]

that is \( \lim_{n \to \infty} \frac{\log_2 |V_{F_n}^\perp|}{|F_n|} = r - h_{\Gamma}(V) \). Let us prove (7). Any element of \( V^\perp \) can be written (not in a unique way!) in the form of \( \sum_{i=1}^k a_i r_i \), where \( a_i \in F_2[\Gamma] \). Denote by \( D \) the supremum of the diameters of the \( r_i \)'s. If \( \text{supp}(a_i) \subset F_n \setminus B_{D+1}(\partial F_n) \), for all \( i \), then \( \sum_{i=1}^k a_i r_i \in V_n^\perp \). On the other hand if \( \text{supp}(a_i) \cap B_{D+1}(F_n) = 0 \) for all \( i \), then \( \sum_{i=1}^k a_i r_i |F_n| = 0 \). Therefore we have the pigeon-hole estimate

\[
|V_{F_n}^\perp| \leq |V_{n}^\perp| 2^{kr B_{D+1}(\partial F_n)}
\]

that is

\[
\frac{\log_2 |V_{F_n}^\perp| - \log_2 |V_{n}^\perp|}{|F_n|} \leq \frac{kr B_{D+1}(\partial F_n)}{|F_n|}
\]

and the right hand side tends to zero. \( \blacksquare \)

Now we prove the density property.

**Proposition 7.3** If \( \Gamma \) is Noether and \( V \subset [\Sigma_\Gamma]^r \) is a linear subshift, then \( h_{\Gamma}(V) = h_{\Gamma}(V^0) \).

**Proof:** By our previous Proposition, \( h_{\Gamma}(V^\perp) = r - h_{\Gamma}(V) \). Therefore \( h_{\Gamma}(\overline{V^\perp}) = r - h_{\Gamma}(V) \), where \( \overline{V^\perp} \) is the closure of \( V^\perp \) as \( [\Sigma_\Gamma]^r \) imbeds into \( [\Sigma_\Gamma^0]^r \). Using our previous Proposition again,

\[
h_{\Gamma}(\overline{V^\perp})^\perp = h_{\Gamma}(V).
\]
If $\xi \in (V^\perp)^\perp$ then $\xi$ is finitely supported and $\xi \in (V^\perp)^\perp = V$, that is $\xi \in V^0$. Therefore,

$$h_\Gamma(V) = h_\Gamma((V^\perp)^\perp) \leq h_\Gamma(V^0) \leq h_\Gamma(V).$$

Actually our proofs of the last two Propositions gives a little bit stronger result:

**Proposition 7.4** Let $\Gamma$ be Noether and let $V$ be a linear subshift such that $V^0$ is generated by $r_1, r_2, \ldots, r_k$ as left $F_2[\Gamma]$-module. Denote by $V_n$ the set of those elements in $V^0$ which can be written in the form of $\sum_{i=1}^k a_i r_i$, where all the $a_i$’s are supported in $F_n$. Then

$$\lim_{n \to \infty} \frac{\log_2 |V_n|}{|F_n|} = h_\Gamma(V).$$

**8 The Yuzvinskii formula**

Recall Yuzvinskii’s additivity formula for Abelian groups [18]. Let $\Gamma \cong \mathbb{Z}^d$ and $\alpha$ be a $\Gamma$-action of continuous automorphisms on a compact metric group $X$. Suppose that $Y$ is a compact $\alpha$-invariant subgroup then

$$h_{\alpha}^{\text{top}}(X) = h_{\alpha}^{\text{top}}(Y) + h_{\alpha}^{\text{top}}(X/Y) \quad (8)$$

The results of Ward and Zhang [19] suggest that a similar statement might be true for general amenable actions. In our paper we prove only a very special case.

**Proposition 8.1** Let $Y \subset X \subset [\Sigma^r_\Gamma]^r$ be linear subshifts where $\Gamma$ is Noether. Then

$$h_\Gamma(Y) = h^{\text{top}}_L(Y) + h^{\text{top}}_L(X/Y) = h^{\text{top}}_L(X) = h_\Gamma(X),$$

where $L$ is the usual left $\Gamma$-action.

**Proof:** The key observation is the following lemma.

**Lemma 8.1** Let $V \subset [\Sigma^r_\Gamma]^r$ be a linear subshift where $\Gamma$ is Noether. Then there exists a constant $D$ such that if for some $\xi \in [\Sigma^r_\Gamma]^r$ with $\xi |_{B_D(\gamma)} \in V_{B_D(\gamma)}$ for all $\gamma \in \Gamma$, then $\xi \in V$. That is for Noether groups linear subshifts are of finite type.

**Proof:** Let $V^\perp \subset [\Sigma^r_\Gamma]^r$ be the orthogonal ideal of $V$. It is generated by $r_1, r_2, \ldots, r_N$, where all $r_i$’s are supported in $B_D(1_\Gamma)$. then $\xi \notin V$ if and only if $\langle \xi, L_\gamma(r_i) \rangle \neq 1$ for some $i$ and $\gamma \in \Gamma$. It means that $\xi |_{B_D(\gamma)} \notin V_{B_D(\gamma)}$. Now we define a metric on $X/Y$ if $v, w \in X$ let $d([v], [w]) = 2^{-(n-1)}$, where $n$ is the smallest integer such that $(v - w) |_{B_D(\gamma)} \notin Y_{B_D(\gamma)}$ for all $\gamma \in F_n$. here $D$ denotes the diameter of the joint support of a generator system $s_1, s_2, \ldots, s_M$ of the ideal $Y^\perp$.

**Lemma 8.2** The metric $d$ defines the pointwise convergence topology.

**Proof:** We need to prove that $d([v_n], 0) \to 0$ implies that $[v_n] \to Y$ in the factor topology of $X/Y$. (the converse is obvious). Suppose that $\{[v_n]\}$ does not converge to $Y$ in the factor topology. Then there exists a subsequence $v_{n_k}$ such that $v_{n_k} \to v \notin Y$ in the convergence
topology of $[\sum_x]^r$. But then there exists a ball $B_D(\gamma)$ such that $v_{n_k} |B_D(\gamma) \notin Y_{B_D(\gamma)}$ for large $k$. This contradicts to the assumption that $d([v_{n_k}], 0) \to 0$.

Now let us turn back to the proof of our Proposition. Similarly to Proposition 3.2 we have a lower estimate for $s_{X/Y}(n, 1)$ in the $d$-metric. Let us denote by $G_n$ the set of elements in $Y^\perp$ which can be written in the form $\sum_{i=1}^m c_i s_i$ such that all the $c_i$’s are supported in $F_n$. Denote by $H_n$ the set of those elements of $[\sum_x]^r$ which are supported on $B_D(F_n)$ and orthogonal to $G_n$. Then by Propositions 2.4 and 2.6 $\frac{\log_2 |G_n|}{|F_n|} \to h_\Gamma(Y)$. We have the following inequality:

$$k_n = |X_{B_D(F_n)}|/|G_n \cap X_{B_D(F_n)}| \leq s_{X/Y}(n, 1).$$

Indeed, there exists $k_n$ elements of $X_{B_D(F_n)}$ such that their pairwise differences $x_i - x_j \notin G_n$ thus $\langle x_i - x_j, L_\gamma s_k \rangle \neq 1$ for some $s_k$ and $\gamma \in F_n$. Hence $d(L_{\gamma^{-1}}([x_i]), L_{\gamma^{-1}}([x_j])) = 1$. Consequently, $h_\Gamma(X) - h_\Gamma(Y) \leq h_{L^p}^\top(X/Y)$. Now fix an $\epsilon$ and let $B_R(1_\Gamma) \supset F_m$, where $2^{-m} < \epsilon$. Then obviously,

$$s_{X/Y}(n, \epsilon) \leq \frac{|X_{B_D(F_n)}|}{Y_{B_D(F_n)}},$$

which implies the converse inequality: $h_\Gamma(X) - h_\Gamma(Y) \geq h_{L^p}^\top(X/Y)$.

### 9 Betti numbers

In this section we define an analogue of the $L^2$-Betti numbers. Let $\tilde{K}$ be a regular, normal $\Gamma$-covering of a finite simplicial complex $K$, where $\Gamma$ is an amenable group that acts freely and simplicially on $\tilde{K}$ and $\tilde{K}/\Gamma = K$. We have the ordinary cochain complex of $F_2$-coefficients over $\tilde{K}$:

$$C^0(\tilde{K}, F_2) \xrightarrow{d_0} C^1(\tilde{K}, F_2) \xrightarrow{d_1} \ldots \xrightarrow{d_{n-1}} C^n(\tilde{K}, F_2),$$

Then the $p$-cochain space $C^p(\tilde{K}, F_2)$ is $\Gamma$-isomorphic to $[\sum_x]|K_p|$, where $K_p$ denotes the set of $p$-simplices in $K$. We define the $p$-th entropy Betti number $b_E^p(K)$ as $h_\Gamma(Ker d_p) - h_\Gamma(Im d_{p-1})$. The following theorem is the analogue of Cohen’s theorem [3]

**Proposition 9.1** $\sum_{p=0}^n (-1)^p b_E^p(K)$ equals to the Euler-characteristics of $K$.

**Proof:** By Proposition 5.1,

$$b_E^p(K) = h_\Gamma(Ker d_p) + h_\Gamma(Ker d_{p-1}) - h_\Gamma(C^{p-1}(\tilde{K}, F_2)).$$

Summing up these equations for all $p$ with alternating signs we obtain that

$$\sum_{p=0}^n (-1)^p b_E^p(K) = \sum_{p=0}^n (-1)^p|K_p| = e(K).$$

Now let us see the analogue of the result of Cheeger & Gromov.

**Proposition 9.2** If $\tilde{K}$ is contractible, then all entropy Betti numbers are vanishing.
The proof is much easier than for the $L^2$-Betti numbers. If $p > 0$, then $F_2$-cohomologies are vanishing therefore $b_p^E(K) = 0$ if $p > 0$. If $p = 0$, then the cocycle space is finite so the entropy Betti number must be zero.

**Corollary 9.1** If $K$ is a finite acyclic simplicial complex with an amenable fundamental group then its Euler characteristics is zero.

Now we prove the analogue of the result of Dodziuk and Mathai.

**Proposition 9.3**

$$b_p^E(K) = \lim_{n \to \infty} \frac{\dim_{F_2} H^p(L_n, F_2)}{|F_n|},$$

where $\{L_n\}$ is an exhaustion of $\tilde{K}$ spanned by the Følner-sets.

**Remark:** Since $\dim_{f_2} H^p(L_n, F_2) \geq \dim_{R} H^p(L_n, R)$ the entropy Betti numbers are at least as large as the corresponding $L^2$-Betti numbers. It is easy to construct examples, where some entropy Betti numbers are strictly larger than the corresponding $L^2$-Betti number (cf. the remark after Proposition 10.1.)

**Proof:** (of Proposition 9.3) First note again that $C^0(\tilde{K}, F_2) \cong [\sum \Gamma] |K_p|$. Denote by $R$ a constant such that for any $[(\gamma, p), (\delta, q)]$ 1-simplex of $\tilde{K}$, $d_\Gamma(\gamma, \delta) \leq R$. Now we can build an extended configuration for $Ker d_p$ and $Im d_p$ the following way. Let $S(\Lambda)$ be the simplicial complex spanned by vertices of the form $(\gamma, p)$, where $\gamma \in B_{2r}(\Lambda)$ and $p \in K_0$. Also, let $L_n$ be the simplicial complex spanned by the vertices with first coordinate in $F_n$. Consider the coboundary operator as $[\sum \Gamma] |K_p| \xrightarrow{d_\Gamma} [\sum \Gamma] |K_{p+1}|$. Let $A_p(\Lambda)$ be the space of those functions in $[\sum \Gamma] |K_p|$ which are supported on $\Lambda$ and are the restriction of a cocycle of $S(\Lambda)$ respectively let $B_p(\Lambda)$ be the space of restrictions of coboundaries of $S(\Lambda)$. Obviously, $A_p(\Lambda)$ is an extended configuration of $Ker d_p$ and $B_p(\Lambda)$ is an extended configuration of $Im d_p$. Then the usual pigeon-hole argument and Proposition 10.1 implies that

$$h_\Gamma(Ker d_p) = \lim_{n \to \infty} \frac{\dim_{F_2}(Z^p(S(F_n)))}{|F_n|},$$

$$h_\Gamma(Im d_p) = \lim_{n \to \infty} \frac{\dim_{F_2}(B^p(S(F_n)))}{|F_n|},$$

where $Z^p$ resp. $B^p$ denote the space of cocycles resp. coboundaries. Therefore

$$b^E_p(K) = \lim_{n \to \infty} \frac{\dim_{F_2}(H^p(S(F_n), F_2))}{|F_n|}.$$ 

Finally we must prove that

$$\lim_{n \to \infty} \frac{\dim_{F_2}(H^p(S(F_n), F_2)) - \dim_{F_2}(H^p(L_n, F_2))}{|F_n|} = 0.$$

Note that it follows from the long exact cohomology sequence induced by the inclusion $L_n \to S(F_n)$ and the obvious fact that $\frac{\dim_{F_2}(H^p(S(F_n), L_n, F_2))}{|F_n|}$ tends to zero as $n \to \infty$. 

15
10 Towers and fixed points

In this section we recall some ideas of Farber [7]. Let $\Gamma$ be a finitely generated residually-$p$ group. That is, there exists a chain of normal subgroups of prime power index, $\Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \ldots$, where $\bigcap_{j=1}^{\infty} \Gamma_j = \{1\}$. Let $\bar{K}/\Gamma = K$ be as in the previous section. Then one can consider the tower of finite simplicial complexes $X_i = \bar{K}/\Gamma_i$. Note that $X_i$ is a simplicial $(\Gamma : \Gamma_i)$-covering of $K$. Farber proved ([7], Theorem 11.1) that $\lim_{j \to \infty} \frac{\dim F_2 H^i(X_j, F_2)}{|\Gamma : \Gamma_i|}$ always exists. The following conjecture is the analogue of Lück’s theorem on approximating the $L^2$-Betti numbers [10]:

**Conjecture 10.1** If $\Gamma$ is as above a residually-$2$ group, then

$$\lim_{j \to \infty} \frac{\dim F_2 H^i(X_j, F_2)}{|\Gamma : \Gamma_i|} = b_i^E(K).$$

**Proposition 10.1** The conjecture is true if $\Gamma$ is free Abelian.

**Proof:** First of all note that if $\Gamma$ is Noether, then any $V \subset [\Sigma_\Gamma]^\infty$ linear subshift is expansive. That is there exists $\epsilon > 0$ such that if $x \neq y \in V$, then for some $\gamma \in \Gamma : d(L_\gamma(x), L_\gamma(y)) \geq \epsilon$. This is just a reformulation of Lemma [5.1]. The following result is due to Schmidt ([18], Theorem 21.1).

**Proposition 10.2** If $\alpha$ is an expansive $\mathbb{Z}d$-action by automorphisms of a compact Abelian group $X$, then

$$\lim_{|\mathbb{Z}^d : \Lambda| \to \infty} \frac{|\text{Fix } \Lambda|}{|\mathbb{Z}^d : \Lambda|} = h^\text{top}_{\alpha}(X)$$

where $\text{Fix } \Lambda$ denotes the set of fixed points of the subgroup $\Lambda$.

Now let us turn to the proof of Proposition 10.1. Let $Z^i_j$ be the space of $i$-cocycles on $X_j$ and $Z^i$ be the space of $i$-cocycles on $\bar{K}$. Then $Z^i_j$ is exactly the set of fixed points of the subgroup $\Gamma_j$ on $Z^i_j$. (Note that the similar statement on coboundaries would not be necessarily true).

By Proposition 10.1, $\lim_{j \to \infty} \frac{\dim F_2 Z^i_j}{|\Gamma : \Gamma_j|} = h_\Gamma(Z^i)$. If $C^i_j$ denotes the space of $i$-cochains on $X_j$ and $C^i$ denotes the space of $i$-cochains on $\bar{K}$, then $\frac{\dim F_2 C^i_j}{|\Gamma : \Gamma_j|} = |K_i| = h_\Gamma(C^i)$, for all $j$.

By our Proposition 5.1,

$$b_i^E(K) = h_\Gamma(Z^i) + h_\Gamma(Z^{i-1}) - h_\Gamma(C^{i-1}).$$

Also,

$$\dim F_2 H^i(X_j, F_2) = \dim F_2 (Z^i_j) + \dim F_2 (Z^{i-1}_j) - \dim F_2 (C^{i-1}_j).$$

Hence our Proposition follows. □
It is not hard to construct a $\tilde{K}$, where for some $p$ the entropy and $L^2$-Betti numbers differ. Simply consider the Cayley graph of $\mathbb{Z}^d$ and then just stick a $\mathbb{R}P^4$ on each vertex. Then if $L_n$ denote the approximative complexes for some Følner-exhaustion:

$$b^4_{\mathcal{E}}(K) = \lim_{n \to \infty} \dim_{\mathbb{F}_2} \frac{H^4(L_n, \mathbb{F}_2)}{|F_n|} = 1$$
and

$$L_{(2)}b^4_{\mathcal{E}}(K) = \lim_{n \to \infty} \dim_{\mathbb{R}} \frac{H^4(L_n, \mathbb{R})}{|F_n|} = 0.$$

11 The Grothendieck group and the integrality of the Betti numbers

First we recall the notion of the Grothendieck group of a non-commutative ring $R$ [16]. Let $G(R)$ be the Abelian group, defined by generators $\{[M]\}$, where the $M$’s are the finitely generated left $R$-modules up to isomorphism. The relations are in the form $[M] + [N] = [L]$, for any exact sequence $0 \to M \to L \to N \to 0$. Lück [11] proved that if $R = \mathbb{C}[\Gamma]$, where $\Gamma$ is amenable, then $[\mathbb{C}[\Gamma]]$ generates an infinite cyclic subgroup in $G(R)$.

**Proposition 11.1** $[\mathbb{C}[\Gamma]]$ generates an infinite cyclic subgroup in $G(\mathbb{C}[\Gamma])$ for any finitely generated amenable group $\Gamma$.

**Proof:** It is enough to define a rank on finitely generated $\mathbb{C}[\Gamma]$-modules, such that $rk([\mathbb{C}[\Gamma]]) = 1$ and $rk([M]) + rk([N]) = rk([L])$ if

$$0 \to M \stackrel{j}{\to} L \stackrel{p}{\to} N \to 0.$$

Let $rk(M) = h_{\Gamma}(\hat{M})$. Now apply Proposition 5.1 for the subshifts

$$0 \to \hat{N} \stackrel{p}{\to} \hat{L} \stackrel{i}{\to} \hat{M} \to 0$$

and the additivity follows. 

Linnell [9] proved that all $L^2$-Betti numbers are integers for torsion-free elementary amenable group $\Gamma$. We can prove the following proposition.

**Proposition 11.2** If $\Gamma$ is poly-infinite-cyclic, then $h_{\Gamma}(V)$ is an integer for any linear subshift $V$.

**Proof:** Let $M = \hat{V}$ be the dual $F_2[\Gamma]$-module of our subshift. Then, by Theorem 3.13 [15] $M$ has a finite resolution by finitely generated projective modules:

$$0 \to M_n \to \ldots \to M_2 \to M_1 \to M \to 0$$

Then, as we pointed out earlier the dual sequence

$$0 \to V \to V_1 \to V_2 \to \ldots \to V_n \to 0$$


is an exact sequence of linear subshifts and continuous homomorphisms, where $V_i = \tilde{M}_i$. By Proposition 5.1 it is enough to show that all the $h_\Gamma(V_i)$'s are integers. By a result of Grothendieck & Serre (Theorem 4.13 [15]) if $\Gamma$ is poly-infinite-cyclic, then all finitely generated, projective $F_2[\Gamma]$-module is stably free. Hence, using the notation of the previous section:

$$h_\Gamma(V_i) = rk(\tilde{V}_i) = rk((F_2[\Gamma])^n) - rk((F_2[\Gamma])^m) = n - m$$

is an integer. ■

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