Double Hopf bifurcation analysis in the memory-based diffusion system∗

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Abstract

In this paper, we derive the algorithm for calculating the normal form of the double Hopf bifurcation that appears in a memory-based diffusion system via taking memory-based diffusion coefficient and the memory delay as the perturbation parameters. Using the obtained theoretical results, we study the dynamical classification near the double Hopf bifurcation point in a predator-prey system with Holling type II functional response. We show the existence of different kinds of stable spatially inhomogeneous periodic solutions, the transition from one kind to the other as well as the coexistence of two types of periodic solutions with different spatial profiles by varying the memory-based diffusion coefficient and the memory delay.

Keywords: Memory-based diffusion; delay; stability; double Hopf bifurcation; normal form

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1 Introduction

The complex internal mechanism of memory-driven movement is still poorly understood although it has been known by many biologists that the spatial memory has an important influence on the animal movement, and results in complex mathematical and computational challenges [10, 11]. In this regard, mathematical models may provide deep insights into the theoretical mechanism behind the biological phenomenon. For example, considering the spatial memories decay over time and the fact that animal movement is affected by the population at the past time [11], Shi et al. [36] introduce a time delay (also known as “memory delay”) into

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the advection term of the classic reaction-diffusion-advection equation and propose a memory-based diffusion equation to model the dynamics of animal movement with memory. From the theoretical analysis in [36] authors found that the stability of a spatially homogeneous steady state depends on the reaction term and the relationship of the coefficients of random diffusion and the directional diffusion, but not the memory delay. Since then many researchers have shown their interests in modeling and investigating the dynamics for the single-species model with the memory [1, 30, 35, 37, 41, 42, 47].

More recently, Song et al. [40] consider a two species model, where authors assume that the prey, such as plants or “drunk” animals are considered as resource so that they have no or negligible memory or cognition. Thus they only introduce a spatial memory into the predator, and propose the following predator-prey system

\[
\begin{align*}
\frac{\partial u(x,t)}{\partial t} &= d_{11} u_{xx}(x,t) + f(u(x,t), v(x,t)), \quad 0 < x < \ell \pi, t > 0, \\
\frac{\partial v(x,t)}{\partial t} &= d_{22} v_{xx}(x,t) - d_{21} (v(x,t)u_x(x,t-\tau))_x + g(u(x,t), v(x,t)), \quad 0 < x < \ell \pi, t > 0, \\
u_x(0,t) = u_x(\ell \pi, t) = v_x(0,t) = v_x(\ell \pi, t) = 0, \quad t \geq 0,
\end{align*}
\]

where \( u(x,t) \) and \( v(x,t) \) are the density of the prey and predator, respectively, at the space \( x \) and time \( t \), \( d_{11} \) and \( d_{22} \) are the random diffusion coefficients of the prey and predator respectively, \( d_{21} \) is the memory-based diffusion coefficient of the predator, \( \tau \) is the time delay representing the averaged memory period of the predator, \( f \) and \( g \) are biological birth/death of prey and predator respectively. The effects of the memory-based diffusion coefficient \( d_{21} \) and time delay \( \tau \) on the stability of the positive constant equilibrium of system (1.1) has been investigated in [40] and it has been shown that unlike the classic prey-taxis model, memory-based prey-taxis destabilizes the positive constant equilibrium, which is a new mechanism for spatiotemporal pattern formation. The spatially inhomogeneous Hopf bifurcation, double Hopf bifurcation and stability switches are also found in [40]. The algorithm for computing the normal form to investigate the spatially inhomogeneous Hopf bifurcation are developed in [39].

In this paper, we are interested in the complex dynamics due to the interaction of two spatially inhomogeneous Hopf bifurcations. For this purpose, we shall first develop the algorithm for computing the normal form of double Hopf bifurcation for system (1.1). Recall that the standard Hopf bifurcation happens when the equilibrium loses its stability with a pair of purely imaginary eigenvalues at the bifurcation value. For this Hopf bifurcation, there exists a family of the periodic solutions with small amplitudes near the neighbourhood of the equilibrium when the bifurcation parameter is taken in the unilateral neighbourhood of the bifurcation value.
However, the interaction of Hopf bifurcations may result in more complex dynamics like quasi-periodic solution or invariant torus [21]. The interaction of two Hopf bifurcations, which have a pair of purely imaginary eigenvalues \( \pm i \omega_1 \) and \( \pm i \omega_2, \omega_j > 0, j = 1, 2 \), respectively, is interpreted in the framework of the double Hopf bifurcation also known as Hopf-Hopf bifurcation. When the ratio of \( \omega_1 / \omega_2 \) is irrational, the bifurcation is said to be non-resonant, otherwise resonant. The resonant double Hopf bifurcation is distinguished into two cases: weakly and strongly resonant double Hopf bifurcations. The bifurcation is said to be strongly resonant if there exit two positive integers \( m_1, m_2 \) so that

\[
m_1 \omega_1 = m_2 \omega_2, \quad m_1 + m_2 \leq 4, \quad (1.2)
\]

and to be weakly resonant if there are no \( m_1, m_2 \) satisfying \( m_1 + m_2 \leq 4 \) such that the condition (1.2) holds. The weakly resonant double Hopf bifurcation is often codimension-two, but the strongly resonant case is often codimension-three and it is more difficult to analyze the related dynamics [25, 26].

The two most popular approaches used to investigate the bifurcations are the rigorous centre manifold reduction and normal form theory [5, 12, 13, 15, 17, 21, 48] and the method of multiple scales [25, 26, 28, 29]. For other methods to investigate the bifurcations such as the method of small parameters or the theory of averaging, please refer to [16, 34] and references therein.

The complex dynamics arising from double Hopf bifurcation has been recently studied by many authors for various dynamical systems, referring to [20, 22, 33, 44, 49] for ordinary differential equations, to [2, 3, 6, 7, 14, 18, 27, 31, 45, 46] for delay differential equations. More recently, based on the theory of normal forms for partial functional differential equations developed by Faria [12], the double Hopf bifurcation in the reaction-diffusion system with delay has attracted the attention of the researchers [4, 8, 9, 24]. The solutions of these systems involve not only the time but also the space, thus the investigation of the bifurcation phenomenon is more difficult [32, 50]. The idea in [12] has also been successfully used to calculate the normal form of the Turing-Hopf bifurcation in the reaction-diffusion system with/without delay [19, 38, 43]. Unfortunately, the procedure of calculating the normal forms of the double Hopf bifurcation for the classical reaction-diffusion system with delay can not apply to the system (1.1), where the delay is involved the diffusion term not in the reaction term and the diffusion terms are not linear.

Motivated by the recent results, particularly the aforementioned, on the double Hopf bifurcation for the reaction-diffusion systems with delays, in this paper, we investigate the dynamics associated with the double Hopf bifurcation arising from (1.1) and our results can be summarized as follows:
1. We derive the algorithm for computing the normal form of the double Hopf bifurcation induced by the memory-based diffusion coefficient and memory delay for the memory-based diffusion system (1.1). The explicit relationship between the second and third terms of the normal form and those in (1.1) near the positive equilibrium are completely established;

2. For system (1.1) with Holling-type II functional response, the dynamical classification near the double Hopf bifurcation point are determined for two cases: (i) the interaction of two Hopf bifurcations with the same spatial mode-2; and (ii) the interaction of two Hopf bifurcations with the spatial mode-1 and mode-2;

3. We find different kinds of stable spatially inhomogeneous periodic solutions and the transition from one to another near the neighbourhood of the double Hopf bifurcation point. Especially, for case 2(i), we find the stable quasi-periodic solution like a “bird” with the spatial mode-1, and for case 2(ii) we find the bistability region of two kinds of stable periodic solutions with spatial mode-1 and mode-2.

The the rest of the paper is organized as follows. In Section 2, we derive the algorithm for computing the normal form associated with the double Hopf bifurcation for (1.1). In Section 3, we study the dynamical classification near the double Hopf bifurcation point for (1.1) with Holling type-II functional response by employing the theoretical results developed in Section 2. We conclude our study with a short discussion in Section 4. Finally, we give all detailed calculations used in Section 2 in the Appendices. Throughout the paper, \( \mathbb{N} \) represents the set of all positive integers, and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \) represents the set of all nonnegative integers.

2 Normal forms for the double Hopf bifurcation and Hopf bifurcation

2.1 Basic assumptions

We assume that system (1.1) has a positive constant equilibrium \( E_*(u_*, v_*) \) and \( f, g \in C^3 \) near the neighbourhood of \( E^* \) for the calculation of the normal form. Then the characteristic equation of the linearized system of (1.1) at the positive equilibrium \( E_*(u_*, v_*) \) is

\[
\prod_{n \in \mathbb{N}_0} \Gamma_n(\lambda) = 0,
\]

(2.1)

where \( \Gamma_n(\lambda) = \det(M_n(\lambda)) \) with characteristic matrix

\[
M_n(\lambda) = \lambda I_2 + (n/\ell)^2 D_1 + (n/\ell)^2 e^{-\lambda \tau} D_2 - A,
\]

(2.2)
where $I_2$ is a $2 \times 2$ identity matrix,

$$
D_1 = \begin{pmatrix} d_{11} & 0 \\ 0 & d_{22} \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 0 \\ -d_{21}v_* & 0 \end{pmatrix}, \quad A = (a_{ij})_{2 \times 2},
$$

with

$$a_{11} = \frac{\partial f(u_*, v_*)}{\partial u}, \quad a_{12} = \frac{\partial f(u_*, v_*)}{\partial v}, \quad a_{21} = \frac{\partial g(u_*, v_*)}{\partial u}, \quad a_{22} = \frac{\partial g(u_*, v_*)}{\partial v}$$

From (2.2), we have

$$
\Gamma_n(\lambda) = \lambda^2 - T_n\lambda + \tilde{J}_n(\tau) = 0,
$$

where

$$
T_n = Tr(A) - Tr(D_1)(n/\ell)^2,
$$

$$
\tilde{J}_n(\tau) = d_{11}d_{22}(n/\ell)^4 - (d_{11}a_{22} + d_{22}a_{11} + d_{21}v_*a_{12}e^{-\lambda\tau})(n/\ell)^2 + Det(A).
$$

Assume that at $(\tau, d_{21}) = (\tau_c, d_{21}^c)$, Eq. (2.1) has two pairs of purely imaginary roots $\pm i\omega_1$ and $\pm i\omega_2$, respectively, for $n = n_1$ and $n = n_2$ with $n_1 \leq n_2$ and the corresponding transversality condition holds, and all other eigenvalues have negative real parts. We are interested in the cases of weak resonance and non-resonance, which are codimension-two bifurcation problem, i.e. the strong resonance condition (1.2) does not hold for these two $\omega_1$ and $\omega_2$.

### 2.2 Normal form for the double Hopf bifurcation

In what follows, we set $\tau = \tau_c + \mu_1, d_{21} = d_{21}^c + \mu_2$ such that $(\mu_1, \mu_2) = (0, 0)$ is the double Hopf bifurcation value for Eq. (1.1). In this section, $D_1$ is defined by (2.3) and

$$
D_2^c = \begin{pmatrix} 0 & 0 \\ -d_{21}^c v_* & 0 \end{pmatrix}.
$$

Define the real-valued Sobolev space

$$
\mathcal{X} = \left\{ U = (U_1, U_2)^T \in (W^{2,2}(0, \ell\pi))^2, \frac{\partial U_1}{\partial x} = 0, \frac{\partial U_2}{\partial x} = 0, x = 0, \ell\pi \right\},
$$

and let $\mathcal{C} := C([-1, 0]; \mathcal{X})$ be the Banach space of continuous mappings from $[-1, 0]$ to $\mathcal{X}$.

Translating $E_*$ to the origin by setting

$$
U(x, t) = (U_1(x, t), U_2(x, t))^T = (u(x, t), v(x, t))^T - (u_*, v_*)^T,
$$

normalizing the delay by the time-scaling $t \rightarrow t/\tau$, and then for simplification of notation, writing $U(t)$ for $U(x, t)$ and $U_t \in \mathcal{C}$ for $U_t(\theta) = U(x, t + \theta), -1 \leq \theta \leq 0$, (1.1) becomes

$$
\frac{dU(t)}{dt} = d(\mu)\Delta(U_t) + L(\mu)(U_t) + F(U_t, \mu),
$$

(2.6)
where for \( \varphi = (\varphi^{(1)}, \varphi^{(2)})^T \in \mathcal{C} \), \( d(\mu)\Delta, L(\mu) : \mathcal{C} \to \mathcal{X} \), \( F : \mathcal{C} \times \mathbb{R}^2 \to \mathcal{X} \) are given, respectively, by

\[
d(\mu)\Delta(\varphi) = d_0\Delta(\varphi) + F^d(\varphi, \mu), \quad L(\mu)(\varphi) = (\tau_c + \mu_1)A\varphi(0),
\]

and

\[
F(\varphi, \mu) = (\tau_c + \mu_1) \begin{pmatrix} f(\varphi^{(1)}(0) + u_s, \varphi^{(2)}(0) + v_s) \\ g(\varphi^{(1)}(0) + u_s, \varphi^{(2)}(0) + v_s) \end{pmatrix} - L(\mu)(\varphi),
\]

where

\[
d_0\Delta(\varphi) = \tau_c D^1_{xx}\varphi(0) + \tau_c D^2_{xx}\varphi(-1),
\]

\[
F^d(\varphi, \mu) = \tau_c \begin{pmatrix} 0 \\ \begin{pmatrix} \varphi_x(1)(-1)\varphi^{(2)}_x(0) + \varphi^{(1)}_{xx}(0) - 1 \varphi^{(2)}(0) \\
\varphi_x(1)(-1)\varphi^{(2)}_x(0) - 1 \varphi^{(1)}_x(0) \\
\varphi_x(1)(-1)\varphi^{(2)}(0) + \varphi^{(1)}_{xx}(0) - 1 \varphi^{(2)}(0) \\
\varphi_x(1)(-1)\varphi^{(2)}(0) + \varphi^{(1)}_{xx}(0) - 1 \varphi^{(2)}(0) \\
\end{pmatrix} + \mu_1 \begin{pmatrix} \varphi_{xx}^{(1)}(0) \\
\varphi_{xx}^{(1)}(0) \\
\varphi_x^{(1)}(-1)\varphi_{xx}^{(2)}(0) + \varphi^{(1)}_{xx}(0) - 1 \varphi^{(2)}(0) \\
\varphi_x^{(1)}(-1)\varphi_{xx}^{(2)}(0) + \varphi^{(1)}_{xx}(0) - 1 \varphi^{(2)}(0) \\
\end{pmatrix} \\
- \tau_c \mu_2 \begin{pmatrix} 0 \\
\varphi_x^{(1)}(-1)\varphi^{(2)}(0) + \varphi^{(1)}_{xx}(0) - 1 \varphi^{(2)}(0) \\
\varphi_x^{(1)}(-1)\varphi^{(2)}(0) + \varphi^{(1)}_{xx}(0) - 1 \varphi^{(2)}(0) \\
\end{pmatrix} - \mu_1 \mu_2 \begin{pmatrix} \varphi^{(1)}_x(-1)\varphi^{(2)}(0) + \varphi^{(1)}_{xx}(0) - 1 \varphi^{(2)}(0) \\
\varphi^{(1)}_x(-1)\varphi^{(2)}(0) + \varphi^{(1)}_{xx}(0) - 1 \varphi^{(2)}(0) \\
\end{pmatrix}.
\]

Noticing that \( \mu_1, \mu_2 \) are perturbation parameters and treated as variables in the calculation of normal forms, we denote \( L_0(\varphi) = \tau_c A \varphi(0) \) and rewrite \( (2.6) \) as the following linear form from nonlinear terms

\[
\frac{dU(t)}{dt} = d_0\Delta(U_t) + L_0(U_t) + \tilde{F}(U_t, \mu_1, \mu_2),
\]

where for \( \varphi = (\varphi_1, \varphi_2)^T \in \mathcal{C} \),

\[
\tilde{F}(\varphi, \mu_1, \mu_2) = \mu_1 A \varphi(0) + F(\varphi, \mu) + F^d(\varphi, \mu).
\]

The characteristic equation for the linearized system

\[
\frac{dU(t)}{dt} = d_0\Delta(U_t) + L_0(U_t)
\]

is

\[
\prod_{n \in \mathbb{N}_0} \bar{\Gamma}_n(\lambda) = 0,
\]
where $\tilde{\Gamma}_n(\lambda) = \det\left(\tilde{M}_n(\lambda)\right)$ with

$$
\tilde{M}_n(\lambda) = \lambda I_2 + \tau_c (n/\ell)^2 D_1 + \tau_c (n/\ell)^2 e^{-\lambda} D_2 - \tau_c A.
$$

(2.13)

Comparing (2.13) with (2.2), we know that Eq. (2.12) has two pairs of purely imaginary roots $\pm i\omega_{c1}$ and $\pm i\omega_{c2}$ for $n = n_1$ and $n = n_2$, respectively, and all other eigenvalues have negative real parts, where $\omega_{jc} = \tau_c \omega_j, j = 1, 2$.

It is well known that the eigenvalue problem

$$
-\gamma'' = \mu \gamma, \quad x \in (0, \ell \pi);
\gamma'(0) = \gamma'(\ell \pi) = 0
$$

has eigenvalues $\mu_n = (n/\ell)^2, n \in \mathbb{N}_0$, with corresponding normalized eigenfunctions

$$
\gamma_n(x) = \frac{\cos \left(\frac{nx}{\ell}\right)}{\| \cos \left(\frac{nx}{\ell}\right) \|_{2,2}} = \begin{cases} 
\frac{1}{\sqrt{\ell \pi}}, & \text{for } n = 0, \\
\frac{\sqrt{2}}{\sqrt{\ell \pi}} \cos \left(\frac{nx}{\ell}\right), & \text{for } n \neq 0,
\end{cases}
$$

where the norm $\| \cdot \|_{2,2}$ is induced by the inner product $\langle \cdot, \cdot \rangle$ as follows

$$
[u, v] = \int_0^{\ell \pi} u^T v dx, \text{ for } u, v \in \mathcal{X}.
$$

Let $\beta^{(j)}_n = \gamma_n(x)e_j, j = 1, 2$, where $e_j$ is the unit coordinate vector of $\mathbb{R}^2$, and $\mathcal{B}_n = \text{span}\left\{ [v(\cdot), \beta^{(j)}_n] | v \in \mathcal{X}, j = 1, 2 \right\}$. Then it is easy to verify that

$$
L_0(\mathcal{B}_n) \subset \text{span}\left\{ \beta^{(1)}_n, \beta^{(2)}_n \right\}, n \in \mathbb{N}_0.
$$

Assume that $z_t(\theta) \in C = C([-1, 0], \mathbb{R}^2)$ and

$$
z_t^T(\theta) \begin{pmatrix} \beta^{(1)}_n \\ \beta^{(2)}_n \end{pmatrix} \in \mathcal{B}_n.
$$

Then, on $\mathcal{B}_n$, the linearized equation (2.11) is equivalent to the following functional differential equation (FDE) in $C:

$$
\dot{z}(t) = L^d_0(z_t(\theta)) + L_0(z_t(\theta)),
$$

(2.14)

where

$$
L^d_0(z_t(\theta)) = \tau_c \begin{pmatrix} -d_{11}(n/\ell)^2 & 0 \\ 0 & -d_{22}(n/\ell)^2 \end{pmatrix} z_t(0) + \tau_c \begin{pmatrix} 0 & 0 \\ d_{21}^c v*(n/\ell)^2 & 0 \end{pmatrix} z_t(-1).
$$

The characteristic equation of linear system (2.14) is the same as given in (2.12).

Define $\eta_n(\theta) \in BV([-1, 0], \mathbb{R}^2)$ such that

$$
\int_{-1}^0 d\eta_n(\theta) \varphi(\theta) = L^d_0(\varphi(\theta)) + L_0(\varphi(\theta)), \quad \varphi \in C;
$$

7
and use the adjoint bilinear form on $C^* \times C$, $C^* = C([0, 1], \mathbb{R}^2)$, where $\mathbb{R}^2$ is the 2-dimensional space of row vectors, as follows

$$
\langle \psi(s), \varphi(\theta) \rangle_n = \psi(0)\varphi(0) - \int_{-1}^{0} \int_{0}^{\theta} \psi(\xi - \theta)d\eta(\theta)\varphi(\xi)d\xi, \quad \text{for } \psi \in C^*, \varphi \in C.
$$

Let $\Lambda = \{i\omega_{1c}, -i\omega_{1c}, i\omega_{2c}, -i\omega_{2c}\}$. Denote the generalized eigenspace of (2.14) associated with $\Lambda$, by $P_{n_j}$ and the corresponding adjoint space by $P^*_{n_j}$. Then, by the adjoint theory of functional differential equation [15], $C$ can be decomposed as $C = P_{n_j} \oplus Q_{n_j}$, $j = 1, 2$, where $Q_{n_j} = \{\varphi \in C : \langle \psi, \varphi \rangle = 0, \forall \psi \in P^*_{n_j}\}$. Choose the bases $\Phi_{n_j}(\theta)$ and $\Psi_{n_j}(s)$ of $P_{n_j}$ and $P^*_{n_j}$, respectively, as follows

$$
\Phi_{n_j}(\theta) = \left(\phi_{n_j}(\theta), \overline{\psi}_{n_j}(\theta)\right), \quad \Psi_{n_j}(s) = \text{col}\left(\psi_{n_j}^T(s), \overline{\psi}_{n_j}^T(s)\right),
$$

such that $\langle \Psi_{n_j}, \Phi_{n_j} \rangle_{n_j} = I_2$, where

$$
\phi_{n_j}(\theta) = \left(\phi_{n_j}^{(1)}(\theta) \atop \phi_{n_j}^{(2)}(\theta)\right) = \phi_{n_j}(0)e^{i\omega_{j}\theta}, \quad \psi_{n_j}(s) = \left(\psi_{n_j}^{(1)}(s) \atop \psi_{n_j}^{(2)}(s)\right) = \psi_{n_j}(0)e^{-i\omega_{j}\theta},
$$

and

$$
\phi_{n_j}(0) = \frac{1}{i\omega_j + (n_j/\ell)^2d_{11} - a_{11}}\left(1 \atop \frac{a_{12}}{i\omega_j + (n_j/\ell)^2d_{22} - a_{22}}\right), \quad \psi_{n_j}(0) = \alpha_j \left(1 \atop \frac{a_{12}}{i\omega_j + (n_j/\ell)^2d_{22} - a_{22}}\right),
$$

with

$$
\alpha_j = \frac{i\omega_j + (n_j/\ell)^2d_{22} - a_{22}}{i\omega_j + (n_j/\ell)^2d_{11} - a_{11} + i\omega_j + (n_j/\ell)^2d_{22} - a_{22} + \tau a_{12}d_{21}v_{n_j}(n_j/\ell)^2e^{-i\omega_{j}\theta}}.
$$

Using the decomposition $C = P_{n_j} \oplus Q_{n_j}$, the phase space $\mathcal{C}$ for (2.6) can be decomposed as

$$
\mathcal{C} = \mathcal{P} \oplus \mathcal{Q}, \quad \mathcal{P} = \text{Im} \pi, \quad \mathcal{Q} = \text{Ker} \pi,
$$

where $\pi : \mathcal{C} \to \mathcal{P}$ is the projection operator defined by

$$
\pi(\phi) = \Phi_{n_1}(\theta)\left(\Psi_{n_1}(\theta), \begin{bmatrix} \phi(\cdot), \beta^{(1)}_{n_1} \\ \phi(\cdot), \beta^{(2)}_{n_1} \end{bmatrix}_{n_1} \gamma_{n_1}(x) \right)
+ \Phi_{n_2}(\theta)\left(\Psi_{n_2}(\theta), \begin{bmatrix} \phi(\cdot), \beta^{(1)}_{n_2} \\ \phi(\cdot), \beta^{(2)}_{n_2} \end{bmatrix}_{n_2} \gamma_{n_2}(x) \right).
$$

In the following, for simplification of notations, we use $\phi_{n_j}$, $\phi_{n_j}$, $\psi_{n_j}$ and $\overline{\psi}_{n_j}$ for $\phi_{n_j}(\theta)$, $\overline{\psi}_{n_j}(\theta)$, $\psi_{n_j}(\theta)$ and $\overline{\psi}_{n_j}(\theta)$, respectively. In addition, notice that for $n_1 = n_2$, $\phi_{n_1}(\theta)$ looks like $\phi_{n_2}(\theta)$ but they are actually different since $\omega_{1c} \neq \omega_{2c}$. Thus, in this case, one can not replace $\phi_{n_1}(\theta)$ by $\phi_{n_2}(\theta)$ or conversely.
Following \[12\] and \[43\], we define $\mathcal{C}_0^1 = \left\{ \phi \in \mathcal{C} : \phi \in \mathcal{C}, \phi(0) \in \text{dom}(d\Delta) \right\}$ and let
\[
z_x = (z_1(t)\gamma_{n_1}(x), z_2(t)\gamma_{n_1}(x), z_3(t)\gamma_{n_2}(x), z_4(t)\gamma_{n_2}(x))^T
\]
and
\[
\Phi(\theta) = \left( \phi_{n_1}(\theta), \phi_{n_2}(\theta), \phi_{n_2}(\theta) \right)
\]
For $\varphi(\theta) \in \mathcal{C}_0^1$, we have the following decomposition
\[
\varphi(\theta) = \Phi(\theta)z_x + w, \quad w = (w^{(1)}, w^{(2)})^T \in \mathcal{C}_0^1 \cap \text{Ker} \pi := \mathcal{Z}^1.
\]
For simplicity of notations, we write
\[
\left( \begin{bmatrix} \bar{F}_1, \beta_1^{(1)} \end{bmatrix} \right)_{\nu=n_2} \quad \text{for col} \quad \left( \begin{bmatrix} \bar{F}_1, \beta_1^{(1)} \end{bmatrix}, \left( \begin{bmatrix} \bar{F}_1, \beta_1^{(1)} \end{bmatrix} \right) \right)
\]
and let $z = (z_1(t) z_2(t) z_3(t) z_4(t))^T$. Following the notations in \[12\], we define
\[
X_0(\theta) = \left\{ \begin{array}{cl}
0, & -1 \leq \theta < 0, \\
1, & \theta = 0.
\end{array} \right.
\]
and then
\[
\pi \left( X_0(\theta) \tilde{F}_2 (\Phi(\theta)z_x, 0) \right)
\]
\[
= \Phi_{n_1}(\theta) \Psi_{n_1}(0) \left( \begin{bmatrix} \tilde{F}_2 (\Phi(\theta)z_x, 0), \beta_1^{(1)} \end{bmatrix} \right) \gamma_{n_1}(x)
\]
\[
+ \Phi_{n_2}(\theta) \Psi_{n_2}(0) \left( \begin{bmatrix} \tilde{F}_2 (\Phi(\theta)z_x, 0), \beta_1^{(2)} \end{bmatrix} \right) \gamma_{n_2}(x).
\]
(2.15)
Then system \[2.9\] is decomposed as a system of abstract ODEs on $\mathbb{R}^4 \times \text{Ker} \pi$:
\[
\begin{align*}
\dot{z} &= Bz + \Psi(0) \left( \begin{bmatrix} F (\Phi(\theta)z_x + w, \mu), \beta_1^{(1)} \end{bmatrix} \right)_{\nu=n_2}, \\
\dot{w} &= A_Q w + (I - \pi) X_0(\theta) \tilde{F}(\Phi(\theta)z_x + w, \mu),
\end{align*}
\]
where
\[
\Psi(0) = \text{diag} \left\{ \Psi_{n_1}(0), \Psi_{n_2}(0) \right\}, \quad B = \text{diag} \left\{ i\omega_{1c}, -i\omega_{1c}, i\omega_{2c}, -i\omega_{2c} \right\},
\]
$A_Q : \mathbb{Q}^1 \to \text{Ker} \pi$ is defined by
\[
A_Q w = \dot{w} + X_0(\theta) \left( L_0(w) + L_0^d(w) - \dot{w}(0) \right).
\]
Consider the formal Taylor expansion

\[
\tilde{F}(\varphi, \mu) = \sum_{j \geq 2} \frac{1}{j!} \tilde{F}_j(\varphi, \mu), \quad F(\varphi, \mu) = \sum_{j \geq 2} \frac{1}{j!} F_j(\varphi, \mu)
\]

and

\[
F^d(\varphi, \mu) = \frac{1}{2} F^d_2(\varphi, \mu) + \frac{1}{3!} F^d_3(\varphi, \mu) + \frac{1}{4!} F^d_4(\varphi, \mu).
\]

From (2.8), we have

\[
F^d_2(\varphi, \mu) = F^{d(0,0)}_2(\varphi) + \mu_1 F^{d(1,0)}_2(\varphi) + \mu_2 F^{d(0,1)}_2(\varphi),
\]

(2.17)

\[
F^d_3(\varphi, \mu) = \mu_1 F^{d(1,0)}_3(\varphi) + \mu_2 F^{d(0,1)}_3(\varphi) + \mu_1 \mu_2 F^{d(1,1)}_3(\varphi),
\]

(2.18)

with

\[
\begin{align*}
F^{d(0,0)}_2(\varphi) &= -2d_{21}^e \tau \begin{pmatrix} 0 \\ \varphi_x^{(1)} (-1) \varphi_x^{(2)} (0) + \varphi_{xx}^{(1)} (-1) \varphi^{(2)} (0) \end{pmatrix}, \\
F^{d(1,0)}_2(\varphi) &= 2D_1 \varphi_{xx} (0) + 2D_2 \varphi_{xx} (-1), \\
F^{d(0,1)}_2(\varphi) &= \frac{2\tau}{d_{21}^e} D_2 \varphi_{xx} (-1), \\
F^{d(1,0)}_3(\varphi) &= -6d_{21}^e \begin{pmatrix} 0 \\ \varphi_x^{(1)} (-1) \varphi_x^{(2)} (0) + \varphi_{xx}^{(1)} (-1) \varphi^{(2)} (0) \end{pmatrix}, \\
F^{d(0,1)}_3(\varphi) &= -6\tau c \begin{pmatrix} 0 \\ \varphi_x^{(1)} (-1) \varphi_x^{(2)} (0) + \varphi_{xx}^{(1)} (-1) \varphi^{(2)} (0) \end{pmatrix}, \\
F^{d(1,1)}_3(\varphi) &= -6 \begin{pmatrix} 0 \\ v_* \varphi_{xx}^{(1)} (-1) \end{pmatrix}, \\
\end{align*}
\]

(2.19)

and

\[
F^d_4(\varphi, \mu) = -24 \mu_1 \mu_2 \begin{pmatrix} 0 \\ \varphi_x^{(1)} (-1) \varphi_x^{(2)} (0) + \varphi_{xx}^{(1)} (-1) \varphi^{(2)} (0) \end{pmatrix},
\]

From (2.10), we have

\[
\bar{F}_2(\varphi, \mu) = 2\mu_1 A \varphi (0) + F_2(\varphi, \mu) + F^d_2(\varphi, \mu),
\]

(2.20)

and

\[
\bar{F}_3(\varphi, \mu) = F_3(\varphi, \mu) + F^d_3(\varphi, \mu).
\]

(2.21)
Then (2.16) is written as
\[
\begin{aligned}
\dot{z} &= Bz + \sum_{j \geq 2} \frac{1}{j!} f_j^1(z, w, \mu), \\
\dot{w} &= A \dot{w} + \sum_{j \geq 2} \frac{1}{j!} f_j^2(z, w, \mu),
\end{aligned}
\]
where
\[
f_j^1(z, w, \mu) = \Psi(0) \left( \begin{bmatrix} \tilde{F}_j(\Phi(\theta)z_x + w, \mu), \beta^{(1)}_\nu \\ \tilde{F}_j(\Phi(\theta)z_x + w, \mu), \beta^{(2)}_\nu \end{bmatrix} \right)_{\nu = n_2},
\]
(2.22)
\[
f_j^2(z, w, \mu) = (I - \pi)X_0(\theta)\tilde{F}_j(\Phi(\theta)z_x + w, \mu).
\]
(2.23)

In terms of the normal form theory of partial functional differential equations [12], after a recursive transformation of variables of the form
\[
(z, w) = (\tilde{z}, \tilde{w}) + \frac{1}{j!} (U_j^1(\tilde{z}, \mu), U_j^2(\tilde{z}, \mu)(\theta)), j \geq 2,
\]
(2.24)
where $z, \tilde{z} \in \mathbb{R}^4, w, \tilde{w} \in \mathcal{O}^1$ and $U_j^1 : \mathbb{R}^6 \to \mathbb{R}^4, U_j^2 : \mathbb{R}^6 \to \mathcal{O}^1$ are homogeneous polynomials of degree $j$ in $\tilde{z}$ and $\mu$, the flow on the local center manifold for (2.9) is written as
\[
\dot{z} = Bz + \sum_{j \geq 2} \frac{1}{j!} g_j(z, 0, \mu),
\]
(2.25)
which is the normal form as in the usual sense for ODEs.

Following [23] and [18], we have
\[
g_2^1(z, 0, \mu) = \text{Proj}_{\text{Ker}(M_j)} f_2^1(z, 0, \mu),
\]
and
\[
g_3^1(z, 0, \mu) = \text{Proj}_{\text{Ker}(M_j)} f_3^1(z, 0, \mu) = \text{Proj}_{\text{S}} f_3^1(z, 0, 0) + O(|\mu|^2 |z|),
\]
(2.26)
where $f_3^1(z, 0, \mu)$ is the terms of order 3 in $(z, \mu)$ obtained after performing the change of variables (2.24) of order 2 and is determined by (2.36),
\[
\text{Ker}(M_1^2) = \text{Span} \{\mu_1 z_1 e_1, \mu_i z_2 e_2, \mu_i z_3 e_3, \mu_i z_4 e_4, i = 1, 2\},
\]
\[
\text{Ker}(M_1^1) = \text{Span} \left\{ \mu_1 \mu_2 z_1 e_1, \mu_i^2 z_1 e_1, z_1^2 z_2 e_1, z_1 z_3 z_4 e_1, \mu_1 \mu_2 z_2 e_2, \mu_i^2 z_2 e_2, \mu_1 \mu_2 z_3 e_3, \mu_i^2 z_3 e_3, z_1 z_2 z_3 e_3, \right\},
\]
and
\[
S = \text{Span} \left\{ z_1^2 z_2 e_1, z_1 z_3 z_4 e_1, z_1 z_2^2 e_2, z_2 z_3 z_4 e_2, z_3^2 z_4 e_3, z_1 z_2 z_3 e_3, \right\}.
\]
For convenience, in what follows we set

\[ H(\alpha z_{q_1} z_{q_2} z_{q_3} z_{q_4} \mu_1 \mu_2) = \begin{pmatrix} \alpha z_{q_1} z_{q_2} z_{q_3} z_{q_4} \mu_1 \mu_2 \\ \alpha z_{q_1} z_{q_2} z_{q_3} z_{q_4} \mu_1 \mu_2 \end{pmatrix}, \alpha \in \mathbb{C}. \]

2.2.1 Calculation of \( g^1_2(z, 0, \mu) \)

It follows from (2.22) that

\[ f^1_2(z, 0, \mu) = \Psi(0) \begin{pmatrix} \tilde{F}_2(\Phi(0) z_x, \mu), \beta^{(1)}_\nu \\ \tilde{F}_2(\Phi(0) z_x, \mu), \beta^{(2)}_\nu \end{pmatrix}_{\nu=n_2}. \tag{2.27} \]

From (2.20), we have

\[ \tilde{F}_2(\Phi(0) z_x, \mu) = 2\mu_1 A(\Phi(0) z_x) + F^d_2(\Phi(0) z_x, \mu). \tag{2.28} \]

From (2.17), (2.19), (2.27) and (2.28), and noticing that

\[ \left[ \gamma_{n_i}(x), \gamma_{n_j}(x) \right] = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \]

then for \( n_1 \neq n_2 \), we have

\[ \begin{pmatrix} 2\mu_1 A(\Phi(0) z_x), \beta^{(1)}_\nu \\ 2\mu_1 A(\Phi(0) z_x), \beta^{(2)}_\nu \end{pmatrix} = \begin{cases} 2\mu_1 A \left( \Phi_{n_1}(0) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right), & \nu = n_1, \\ 2\mu_1 A \left( \Phi_{n_2}(0) \begin{pmatrix} z_3 \\ z_4 \end{pmatrix} \right), & \nu = n_2, \end{cases} \tag{2.29} \]

\[ \begin{pmatrix} \mu_1 F^d_2(\Phi(0) z_x), \beta^{(1)}_\nu \\ \mu_1 F^d_2(\Phi(0) z_x), \beta^{(2)}_\nu \end{pmatrix} = \begin{cases} -2(n_1/\ell)^2 \mu_1 \left( D_1 \left( \Phi_{n_1}(0) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right) + D_2^c \left( \Phi_{n_1}(-1) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right) \right), & \nu = n_1, \\ -2(n_2/\ell)^2 \mu_1 \left( D_1 \left( \Phi_{n_2}(0) \begin{pmatrix} z_3 \\ z_4 \end{pmatrix} \right) + D_5^c \left( \Phi_{n_2}(-1) \begin{pmatrix} z_3 \\ z_4 \end{pmatrix} \right) \right), & \nu = n_2, \end{cases} \tag{2.30} \]
and
\[
\begin{pmatrix}
\mu_2 F_2^{d(0,1)}(\Phi(\theta)z_x), \beta^{(1)}_\nu \\
\mu_2 F_2^{d(0,1)}(\Phi(\theta)z_x), \beta^{(2)}_\nu \\
\end{pmatrix}
\]

(2.31)

By (2.7), it is easy to verify that for all \( \mu \in \mathbb{R}^2 \),
\[
F_2(\Phi(\theta)z_x, \mu) = F_2(\Phi(\theta)z_x, 0).
\] (2.32)

This, together with (2.29), (2.30) and (2.31), yields to
\[
g^1_2(z, 0, \mu) = \text{Proj}_{Ker(M_1)} f^1_2(z, 0, \mu)
\]

(2.33)

where
\[
B^{(1)}_1 = 2 \psi_{n_1}^T(0) \left( A \phi_{n_1}(0) - (n_1/\ell)^2 (D_1 \phi_{n_1}(0) + D_2^c \phi_{n_1}(-1)) \right) = 2i \omega_1 \psi_{n_1}^T(0) \phi_{n_1}(0),
\]
\[
B^{(1)}_3 = 2 \psi_{n_2}^T(0) \left( A \phi_{n_2}(0) - (n_2/\ell)^2 (D_1 \phi_{n_2}(0) + D_2^c \phi_{n_2}(-1)) \right) = 2i \omega_2 \psi_{n_2}^T(0) \phi_{n_2}(0),
\]
\[
B^{(2)}_1 = -2(n_1/\ell)^2 \psi_{n_1}^T(0) (D_2^c \phi_{n_1}(-1)), \quad B^{(2)}_3 = -2(n_2/\ell)^2 \psi_{n_2}^T(0) (D_2^c \phi_{n_2}(-1)).
\]

For \( n_1 = n_2 \), noticing the fact that \( [\Phi(\theta)z_x, \gamma_{n_j}(x)] = \Phi(\theta)z, j = 1, 2 \), and using the similar calculations as above, it is easy to obtain the same \( g^1_2(z, 0, \mu) \) as in (2.33).

**2.2.2 Calculation of \( g^1_3(z, 0, \mu) \)**

In this subsection, we calculate the third term \( g^1_3(z, 0, 0) \) in terms of (2.26). Notice that \( \frac{1}{3} \tilde{f}_3 \) in (2.26) is the term of order 3 obtained after the changes of variables in previous step. Denote
\[
f^{(1,1)}_2(z, w, 0) = \Psi(0) \begin{pmatrix}
F_2(\Phi(\theta)z_x + w, 0), \beta^{(1)}_\nu \\
F_2(\Phi(\theta)z_x + w, 0), \beta^{(2)}_\nu \\
\end{pmatrix}_{\nu = n_2},
\] (2.34)
\[
\]
\[
f^{(1,2)}_2(z, w, 0) = \Psi(0) \begin{pmatrix}
F_2^d(\Phi(\theta)z_x + w, 0), \beta^{(1)}_\nu \\
F_2^d(\Phi(\theta)z_x + w, 0), \beta^{(2)}_\nu \\
\end{pmatrix}_{\nu = n_2}
\] (2.35)
In addition, it follows from (2.33) that \( g_2^1(z, 0, 0) = (0, 0, 0, 0)^T \). Then \( \tilde{f}_1^3(z, 0, 0) \) is determined by

\[
\tilde{f}_1^3(z, 0, 0) = f_3^1(z, 0, 0) + \frac{3}{4} \left[ (D_z f_2^1(z, 0, 0)) U_2^1(z, 0) + \left( D_w f_2^{(1,1)}(z, 0, 0) \right) U_2^2(z, 0)(\theta) \right. \\
+ \left. \left( D_w, w_z, w_x f_2^{(1,2)}(z, 0, 0) \right) U_2^{(2,d)}(z, 0)(\theta) \right],
\]

where \( f_2^1(z, 0, 0) = f_2^{(1,1)}(z, 0, 0) + f_2^{(1,2)}(z, 0, 0) \),

\[
D_w, w_z, w_x f_2^{(1,2)}(z, 0, 0) = \left( D_w f_2^{(1,2)}(z, 0, 0), D_{w_z} f_2^{(1,2)}(z, 0, 0), D_{w_x} f_2^{(1,2)}(z, 0, 0) \right),
\]

\[
U_2^1(z, 0) = (M_2^1)^{-1} \text{Proj}_{\text{Im}(M_2^1)} f_2^1(z, 0, 0), \quad U_2^2(z, 0)(\theta) = (M_2^2)^{-1} f_2^1(z, 0, 0),
\]

and

\[
U_2^{(2,d)}(z, 0)(\theta) = \text{col} \left( U_2^2(z, 0)(\theta), U_2^{2, z_x}(z, 0)(\theta), U_2^{2, z_x}(z, 0)(\theta) \right). \tag{2.37}
\]

Next, we compute \( \text{Proj}_S \tilde{f}_1^3(z, 0, 0) \) step by step according to (2.36). The calculation is divided into the following four steps.

**Step 1: The calculation of \( \text{Proj}_S f_1^3(z, 0, 0) \)**

From (2.18) and (2.21), we have \( \tilde{F}_3(\Phi(\theta)z_x, 0) = F_3(\Phi(\theta)z_x, 0) \), which can be written as

\[
\tilde{F}_3(\Phi(\theta)z_x, 0) = F_3(\Phi(\theta)z_x, 0) = \sum_{q_1 + q_2 + q_3 + q_4 = 3} A_{q_1,q_2,q_3,q_4} \gamma_{n_1}^{q_1+q_2}(x) \gamma_{n_2}^{q_3+q_4}(x) z_1^{q_1} z_2^{q_2} z_3^{q_3} z_4^{q_4}, \tag{2.38}
\]

where \( q_1, q_2, q_3, q_4 \in \mathbb{N}_0 \), and

\[
A_{q_1,q_2,q_3,q_4} = \left( A_{q_1,q_2,q_3,q_4}^{(1)}, A_{q_1,q_2,q_3,q_4}^{(2)} \right)^T. \tag{2.39}
\]

It follows from (2.22) and (2.38) that

\[
f_1^3(z, 0, 0) = \Psi(0) \left( \sum_{q_1 + q_2 + q_3 + q_4 = 3} A_{q_1,q_2,q_3,q_4} \int_0^{\ell \pi} \gamma_{n_1}^{q_1+q_2+1}(x) \gamma_{n_2}^{q_3+q_4}(x) dx z_1^{q_1} z_2^{q_2} z_3^{q_3} z_4^{q_4} \right),
\]

which, together with the fact that

\[
\int_0^{\ell \pi} \gamma_{n_1}^2(x) \gamma_{n_2}^2(x) dx = \begin{cases} 
\frac{3}{2\pi}, & n_2 = n_1, \\
\frac{1}{\ell \pi}, & n_2 \neq n_1,
\end{cases}
\]

implies that

\[
\text{Proj}_S f_1^3(z, 0, 0) = \left( \mathcal{H} \left( C_{11} z_1^2 z_2 + C_{12} z_1 z_3 z_4 \right) \right), \tag{2.40}
\]

\[
\mathcal{H} \left( C_{31} z_2^2 z_4 + C_{32} z_1 z_2 z_3 \right).
\]
where

\[
C_{11} = \frac{3}{2\pi} \psi_{n_1}(0) A_{2100}, \quad C_{31} = \frac{3}{2\pi} \psi_{n_2}(0) A_{0021},
\]

\[
C_{12} = \begin{cases} 
\frac{3}{2\pi} \psi_{n_1}(0) A_{1011}, & n_2 = n_1, \\
\frac{1}{\pi} \psi_{n_1}(0) A_{1011}, & n_2 \neq n_1,
\end{cases} \quad C_{32} = \begin{cases} 
\frac{3}{2\pi} \psi_{n_2}(0) A_{1110}, & n_2 = n_1, \\
\frac{1}{\pi} \psi_{n_2}(0) A_{1110}, & n_2 \neq n_1.
\end{cases}
\]

### Step 2: The calculation of \( \text{Proj}_S ( (D_z f^J_z ) (z,0,0) U_2^3(z,0) ) \)

From (2.20), we have

\[
\tilde{F}_2(\Phi(\theta)z_x,0) = F_2(\Phi(\theta)z_x,0) + F_2^{d(0,0)}(\Phi(\theta)z_x)
\]

By (2.32), we write

\[
F_2(\Phi(\theta)z_x + w, \mu) = F_2(\Phi(\theta)z_x + w, 0)
\]

\[
= \sum_{q_1+q_2+q_3+q_4=2} A_{q_1 q_2 q_3 q_4} \gamma_{n_1}^{q_1+q_2} \gamma_{n_2}^{q_3+q_4} (x) \gamma_{n_1}^{q_1} \gamma_{n_2}^{q_2} \gamma_{n_3}^{q_3} \gamma_{n_4}^{q_4}
\]

\[
+ S_2(\Phi(\theta)z_x, w) + O(|w|^2),
\]

where \( q_1, q_2, q_3, q_4 \in \mathbb{N}_0 \) and \( S_2(\Phi(\theta)z_x, w) \) is the second cross terms of \( \Phi z_x \) and \( w \). In addition, by (2.17), we write

\[
F_2^d(\Phi(\theta)z_x, 0) = F_2^{d(0,0)}(\Phi(\theta)z_x)
\]

\[
= \sum_{q_1+q_2+q_3+q_4=2} A_{1234}^{(d,1)} \gamma_{n_1}^{q_1+q_2} \gamma_{n_2}^{q_3+q_4} \gamma_{n_3}^{q_1} \gamma_{n_4}^{q_2} \gamma_{n_5}^{q_3} \gamma_{n_6}^{q_4}
\]

\[
- (n_1/\ell)^2 A_{0200}^{(d,2)} \gamma_{n_1}^{q_1} \gamma_{n_2}^{q_2} - (n_2/\ell)^2 A_{0002}^{(d,2)} \gamma_{n_1}^{q_1} \gamma_{n_2}^{q_2} - (n_3/\ell)^2 A_{0011}^{(d,2)} \gamma_{n_1}^{q_1} \gamma_{n_2}^{q_2} - (n_4/\ell)^2 A_{0101}^{(d,2)} \gamma_{n_1}^{q_1} \gamma_{n_2}^{q_2}
\]

\[
- \gamma_{n_1}^{q_1} \gamma_{n_2}^{q_2} \left( (n_1/\ell)^2 A_{1010}^{(d,3)} + (n_2/\ell)^2 A_{1001}^{(d,3)} \right) z_1 z_3 + \left( (n_1/\ell)^2 A_{1011}^{(d,3)} + (n_2/\ell)^2 A_{1101}^{(d,3)} \right) z_1 z_4
\]

\[
+ \left( (n_1/\ell)^2 A_{1010}^{(d,3)} + (n_2/\ell)^2 A_{1001}^{(d,3)} \right) z_2 z_3 + \left( (n_1/\ell)^2 A_{1101}^{(d,3)} + (n_2/\ell)^2 A_{1011}^{(d,3)} \right) z_2 z_4,
\]

where \( \xi_{n_j} = \frac{\sqrt{2}}{\sqrt{\ell}} \sin(\frac{n_j x}{\ell}) \) with \( j = 1, 2 \), and \( A_{i_1i_2i_3i_4}^{(d,j)} \) with \( j = 1, 2, 3 \) are given in Appendix A.

It is easy to verify that

\[
\int_0^{\ell\pi} \gamma_{n_1}^3(x) dx = \int_0^{\ell\pi} \gamma_{n_2}^3(x) dx = \int_0^{\ell\pi} \gamma_{n_1}(x) \gamma_{n_2}(x) dx = 0,
\]

\[
\int_0^{\ell\pi} \xi_{n_j}^2(x) \gamma_{n_j}(x) dx = 0, j = 1, 2, \int_0^{\ell\pi} \xi_{n_1}^2(x) \gamma_{n_1}(x) dx = \int_0^{\ell\pi} \xi_{n_1}(x) \xi_{n_2}(x) \gamma_{n_2}(x) dx = 0,
\]

\[
\int_0^{\ell\pi} \gamma_{n_1}(x) \gamma_{n_2}(x) dx = \int_0^{\ell\pi} \xi_{n_1}(x) \xi_{n_2}(x) \gamma_{n_1}(x) dx = \begin{cases} 
\frac{1}{\sqrt{2\pi}}, & n_2 = 2n_1, \\
0, & n_2 \neq 2n_1,
\end{cases}
\]
and

\[
\int_0^{\ell\pi} \xi_{n_1}^2(x) \gamma_{n_2}(x) dx = \begin{cases} \frac{-1}{\sqrt{2\pi}} & n_2 = 2n_1, \\ 0 & n_2 \neq 2n_1. \end{cases}
\]

Then, by (2.41) and a direct calculation, we have

\[
f_{1}^2(z, 0, 0) = \Psi(0) \begin{pmatrix} F_2(\Phi(\theta)z_x, 0), \beta_{\nu}^{(1)} \\ F_2(\Phi(\theta)z_y, 0), \beta_{\nu}^{(2)} \end{pmatrix} \bigg|_{\nu = n_2}
= \begin{cases} \frac{1}{\sqrt{2\pi}} \Psi(0) \begin{pmatrix} A_{1010}z_1z_3 + \bar{A}_{1001}z_1z_4 + \bar{A}_{0110}z_2z_3 + \bar{A}_{0101}z_2z_4 \\ \bar{A}_{0000}z_1^2 + \bar{A}_{0200}z_2^2 + \bar{A}_{1100}z_1z_2 \end{pmatrix}, & n_2 = 2n_1, \\ (0, 0, 0)^T, & n_2 \neq 2n_1, \end{cases}
\]

where

\[
a_{j_1,j_2,j_3,j_4} = A_{j_1,j_2,j_3,j_4} + m_{n_2}^2 A_{j_1,j_2,j_3,j_4} - \frac{n_2^2}{\ell^2} A_{j_1,j_2,j_3,j_4} - \frac{n_2^2}{\ell^2} A_{j_1,j_2,j_3,j_4},
\]

(2.44)

\[
j_1, j_2, j_3, j_4 = 0, 1, \quad j_1 + j_2 = 1, \quad j_3 + j_4 = 1,
\]

(2.45)

Then, for \( n_2 \neq 2n_1, U_{1}^2(z, 0) = (0, 0, 0, 0)^T \), and for \( n_2 = 2n_1, \)

\[
U_{1}^2(z, 0) = (M_1^2)^{-1} \text{Proj}_{\text{Im}M_1^2} f_{1}^2(z, 0, 0)
\]

\[
= \left(2\pi\right)^{-1} \psi_{n_1}^T(0) \left( \begin{array}{c} \frac{\omega_{1c}}{\omega_{2c}} \bar{A}_{1010}z_1z_3 - \frac{\omega_{1c}}{\omega_{2c}} \bar{A}_{1001}z_1z_4 + \frac{\omega_{1c}}{\omega_{2c} - 2\omega_{1c}} \bar{A}_{0110}z_2z_3 - \frac{\omega_{1c}}{\omega_{2c} + 2\omega_{1c}} \bar{A}_{0101}z_2z_4 \\ \frac{\omega_{2c}}{\omega_{3c} + 2\omega_{1c}} \bar{A}_{1010}z_1z_3 - \frac{\omega_{2c}}{\omega_{3c} - 2\omega_{1c}} \bar{A}_{1001}z_1z_4 + \frac{\omega_{2c}}{\omega_{3c} + 2\omega_{1c}} \bar{A}_{0110}z_2z_3 - \frac{\omega_{2c}}{\omega_{3c} - 2\omega_{1c}} \bar{A}_{0101}z_2z_4 \\ \frac{\omega_{1c}}{\omega_{2c} - 2\omega_{1c}} \bar{A}_{2000}z_1^2 - \frac{\omega_{1c}}{\omega_{2c} + 2\omega_{1c}} \bar{A}_{0200}z_2^2 - \frac{\omega_{1c}}{\omega_{2c} - 2\omega_{1c}} \bar{A}_{1100}z_1z_2 \\ \frac{\omega_{1c}}{\omega_{2c} + 2\omega_{1c}} \bar{A}_{2000}z_1^2 + \frac{\omega_{1c}}{\omega_{2c} - 2\omega_{1c}} \bar{A}_{0200}z_2^2 + \frac{\omega_{1c}}{\omega_{2c} - 2\omega_{1c}} \bar{A}_{1100}z_1z_2 \end{array} \right) \psi_{n_2}^T(0)
\]

Hence,

\[
\text{Proj}_S \left( (D_z f_{1}^2)(z, 0, 0)U_{1}^2(z, 0) \right) = \begin{pmatrix} \mathcal{H}(D_{11}z_1^2z_2 + D_{12}z_1z_3z_4) \\ \mathcal{H}(D_{31}z_2^2z_4 + D_{32}z_1z_2z_3) \end{pmatrix},
\]

(2.46)

where for \( n_2 \neq 2n_1, D_{11} = D_{12} = D_{31} = D_{32} = 0 \), and for \( n_2 = 2n_1, D_{31} = 0, \)

\[
D_{11} = \frac{1}{2\ell^2} \left( \frac{1}{\omega_{2c}} \left( \psi_{n_1}^T(0) \bar{A}_{1010} \right) \left( \psi_{n_2}^T(0) \bar{A}_{1100} \right) + \frac{1}{\omega_{2c}} \left( \psi_{n_1}^T(0) \bar{A}_{1001} \right) \left( \psi_{n_2}^T(0) \bar{A}_{1010} \right) \right)
= \frac{1}{\omega_{2c} - 2\omega_{1c}} \left( \psi_{n_1}^T(0) \bar{A}_{0110} \right) \left( \psi_{n_2}^T(0) \bar{A}_{2000} \right) + \frac{1}{\omega_{2c} + 2\omega_{1c}} \left( \psi_{n_1}^T(0) \bar{A}_{0101} \right) \left( \psi_{n_2}^T(0) \bar{A}_{2000} \right),
\]
\[ D_{12} = \frac{1}{2\pi} \left( -\frac{1}{\omega_{2c}} \left( \psi_{n_1}^T(0)A_{1010} \right) \left( \psi_{n_1}^T(0)A_{1001} \right) + \frac{1}{\omega_{2c}} \left( \psi_{n_1}^T(0)A_{1001} \right) \left( \psi_{n_1}^T(0)A_{1010} \right) \right) \]
\[ - \frac{1}{\omega_{2c}+2\omega_{1c}} \left( \psi_{n_1}^T(0)A_{10101} \right) \left( \psi_{n_1}^T(0)A_{10110} \right) + \frac{1}{\omega_{2c}} \left( \psi_{n_2}^T(0)A_{11010} \right) \left( \psi_{n_1}^T(0)A_{10101} \right) \],
\[ D_{32} = \frac{1}{2\pi} \left( \frac{2}{\omega_{2c}-2\omega_{1c}} \left( \psi_{n_2}^T(0)A_{02000} \right) \left( \psi_{n_1}^T(0)A_{01101} \right) + \frac{2}{\omega_{2c}+2\omega_{1c}} \left( \psi_{n_2}^T(0)A_{02000} \right) \left( \psi_{n_1}^T(0)A_{01101} \right) \right). \]

**Step 3:** The calculation of \( \text{Projs} \left( (D_{w} f_2^{(1,1)}(z, 0, 0)) \, U_2^2(z, 0) \right) = \)

Let
\[ U_2^2(z, 0)(\theta) \triangleq h(\theta, z) = \sum_{n \in \mathbb{N}_0} h_n(\theta, z) \gamma_n(x), \tag{2.47} \]
where
\[ h_n(\theta, z) = \sum_{q_1, q_2, q_3, q_4} h_{n, q_1, q_2, q_3, q_4}(\theta) \gamma_1^{q_1} \gamma_2^{q_2} \gamma_3^{q_3} \gamma_4^{q_4} \]
with
\[ h_{n, q_1, q_2, q_3, q_4}(\theta) = \left( h_{n, q_1, q_2, q_3, q_4}(\theta) \, h_{n, q_1, q_2, q_3, q_4}(\theta) \right)^T. \]

By (2.47), we have
\[
\begin{aligned}
U_{2x}^2(z, 0)(\theta) &= h_x(\theta, z) = - \sum_{n \in \mathbb{N}_0} (n/\ell) h_n(\theta, z) \xi_n(x), \\
U_{2xx}^2(z, 0)(\theta) &= h_{xx}(\theta, z) = - \sum_{n \in \mathbb{N}_0} (n/\ell)^2 h_n(\theta, z) \gamma_n(x).
\end{aligned} \tag{2.48}
\]

Then, from (2.34) and (2.47), we obtain
\[
\left( D_{w} f_2^{(1,1)}(z, 0, 0) \right) U_2^2(z, 0) = \Psi(0) \begin{bmatrix}
\left. D_{w} F_2(\Phi(\theta)z_x + w, 0) \right|_{w=0} \left( \sum_{n \in \mathbb{N}_0} h_n(\theta, z) \gamma_n(x) \right), & \beta^{(1)}_\nu \\
\left. D_{w} F_2(\Phi(\theta)z_x + w, 0) \right|_{w=0} \left( \sum_{n \in \mathbb{N}_0} h_n(\theta, z) \gamma_n(x) \right), & \beta^{(2)}_\nu
\end{bmatrix}^{\nu=n_2}.
\]

By (2.42) and a straightforward computation, we obtain
\[
D_{w} F_2(\Phi(\theta)z_x + w, 0)|_{w=0} \left( \sum_{n \in \mathbb{N}_0} h_n(\theta, z) \gamma_n(x) \right) = S_2 \left( \Phi(\theta)z_x, \sum_{n \in \mathbb{N}_0} h_n(\theta, z) \gamma_n(x) \right)
\]
and
\[
\begin{bmatrix}
S_2 \left( \Phi(\theta)z_x, \sum_{n \in \mathbb{N}_0} h_n(\theta, z) \gamma_n(x) \right), & \beta^{(1)}_\nu \\
S_2 \left( \Phi(\theta)z_x, \sum_{n \in \mathbb{N}_0} h_n(\theta, z) \gamma_n(x) \right), & \beta^{(2)}_\nu
\end{bmatrix}
\]
\[
= \sum_{n \in \mathbb{N}_0} b_{n_1, \nu} \left( S_2 \left( \phi_{n_1}(\theta)z_1, h_n(\theta, z) \right) + S_2 \left( \phi_{n_1}(\theta)z_2, h_n(\theta, z) \right) \right) + \sum_{n \in \mathbb{N}_0} b_{n_2, \nu} \left( S_2 \left( \phi_{n_2}(\theta)z_3, h_n(\theta, z) \right) + S_2 \left( \phi_{n_2}(\theta)z_4, h_n(\theta, z) \right) \right),
\]

17
where \( n = 0, 1, 2, \cdots, \nu = n_1, n_2, \)

\[
b_{n_j, n, \nu} = \int_0^{\ell \pi} \gamma_{n_j}(x)\gamma_n(x)dx = \begin{cases} 
\frac{1}{\sqrt{\ell \pi}}, & n = 0, \nu = n_j, \\
\frac{1}{\sqrt{2\ell \pi}}, & n = 2n_j, \nu = n_j, \\
\frac{1}{\sqrt{2\ell \pi}}, & n = n_1 + n_2, \nu = n_j + (-1)^{j+1}, \\
\frac{1}{\sqrt{2\ell \pi}}, & n = n_2 - n_1, \nu = n_j + (-1)^{j+1}, n_1 < n_2, \\
0, & \text{otherwise}.
\end{cases}
\]

Hence,

\[
\left( D_{wJ_2^{(1,1)}}(z, 0, 0) \right) U_2^2(z, 0)(\theta)
= \begin{pmatrix} 
\sum_{n=0,2n_1} b_{n_1,n,n_1} \left( S_2 \left( \phi_{n_1}(\theta)z_1, h_n(\theta, z) \right) + \overline{S_2} \left( \overline{\phi}_{n_1}(\theta)z_2, h_n(\theta, z) \right) \right) \\
+ \sum_{n=n_1+n_2, 2n_1-n_1} b_{n_1,n,n_1} \left( S_2 \left( \phi_{n_2}(\theta)z_3, h_n(\theta, z) \right) + \overline{S_2} \left( \overline{\phi}_{n_2}(\theta)z_4, h_n(\theta, z) \right) \right)
\end{pmatrix}
\Psi(0)
= \begin{pmatrix} 
\sum_{n=0,2n_2} b_{n_2,n,n_2} \left( S_2 \left( \phi_{n_2}(\theta)z_3, h_n(\theta, z) \right) + \overline{S_2} \left( \overline{\phi}_{n_2}(\theta)z_4, h_n(\theta, z) \right) \right)
+ \sum_{n=n_1+n_2, 2n_2-n_1} b_{n_1,n,n_2} \left( S_2 \left( \phi_{n_1}(\theta)z_1, h_n(\theta, z) \right) + \overline{S_2} \left( \overline{\phi}_{n_1}(\theta)z_2, h_n(\theta, z) \right) \right)
\end{pmatrix}.
\]

Then, we have

\[
\text{Proj}_S \left( D_{wJ_2^{(1,1)}}(z, 0, 0)U_2^2(z, 0) \right) = \begin{pmatrix} 
\mathcal{H} \left( E_{11}z_1^2z_2 + E_{12}z_1z_3z_4 \right) \\
\mathcal{H} \left( E_{31}z_2^2z_4 + E_{32}z_1z_2z_3 \right)
\end{pmatrix},
\]

\[
\begin{align*}
E_{11} &= \frac{1}{\sqrt{\ell \pi}} \psi_{n_1}^T(0) \left( S_2 \left( \phi_{n_1}(\theta), h_{0,1100}(\theta) \right) + \overline{S_2} \left( \overline{\phi}_{n_1}(\theta), h_{0,2000}(\theta) \right) \right) \\
&\quad + \frac{1}{\sqrt{2\ell \pi}} \psi_{n_1}^T(0) \left( S_2 \left( \phi_{n_1}(\theta), h_{2n_1,1100}(\theta) \right) + \overline{S_2} \left( \overline{\phi}_{n_1}(\theta), h_{2n_1,2000}(\theta) \right) \right),
\end{align*}
\]

\[
\begin{align*}
E_{12} &= \frac{1}{\sqrt{\ell \pi}} \psi_{n_1}^T(0) S_2 \left( \phi_{n_1}(\theta), h_{0,0011}(\theta) \right) + \frac{1}{\sqrt{2\ell \pi}} \psi_{n_1}^T(0) S_2 \left( \phi_{n_1}(\theta), h_{2n_1,0011}(\theta) \right) \\
&\quad + \frac{1}{\sqrt{2\ell \pi}} \psi_{n_1}^T(0) \left( S_2 \left( \phi_{n_2}(\theta), h_{n+1,1001}(\theta) \right) + \overline{S_2} \left( \overline{\phi}_{n_2}(\theta), h_{n+1,1010}(\theta) \right) \right), \\
&\quad + \delta_{n_1n_2} \psi_{n_1}^T(0) \left( S_2 \left( \phi_{n_2}(\theta), h_{n_1-n_2,1001}(\theta) \right) + \overline{S_2} \left( \overline{\phi}_{n_2}(\theta), h_{n_1-n_2,1010}(\theta) \right) \right),
\end{align*}
\]

\[
\begin{align*}
E_{31} &= \frac{1}{\sqrt{\ell \pi}} \psi_{n_2}^T(0) \left( S_2 \left( \phi_{n_2}(\theta), h_{0,0011}(\theta) \right) + \overline{S_2} \left( \overline{\phi}_{n_2}(\theta), h_{0,0020}(\theta) \right) \right) \\
&\quad + \frac{1}{\sqrt{2\ell \pi}} \psi_{n_2}^T(0) \left( S_2 \left( \phi_{n_2}(\theta), h_{2n_2,0011}(\theta) \right) + \overline{S_2} \left( \overline{\phi}_{n_2}(\theta), h_{2n_2,0020}(\theta) \right) \right),
\end{align*}
\]

\[
\begin{align*}
E_{32} &= \frac{1}{\sqrt{\ell \pi}} \psi_{n_2}^T(0) S_2 \left( \phi_{n_2}(\theta), h_{0,1100}(\theta) \right) + \frac{1}{\sqrt{2\ell \pi}} \psi_{n_2}^T(0) S_2 \left( \phi_{n_2}(\theta), h_{2n_2,1100}(\theta) \right) \\
&\quad + \frac{1}{\sqrt{2\ell \pi}} \psi_{n_2}^T(0) \left( S_2 \left( \phi_{n_1}(\theta), h_{n_1+n_2,0110}(\theta) \right) + \overline{S_2} \left( \overline{\phi}_{n_1}(\theta), h_{n_1+n_2,1010}(\theta) \right) \right), \\
&\quad + \delta_{n_1n_2} \psi_{n_2}^T(0) \left( S_2 \left( \phi_{n_1}(\theta), h_{n_1-n_2,0110}(\theta) \right) + \overline{S_2} \left( \overline{\phi}_{n_1}(\theta), h_{n_1-n_2,1010}(\theta) \right) \right),
\end{align*}
\]
where

\[ \delta_{n_1n_2} = \begin{cases} \frac{1}{\sqrt{2\pi}}, & n_1 < n_2, \\ \frac{1}{\sqrt{\ell}}, & n_1 = n_2. \end{cases} \]

**Step 4: The calculation of \( \text{Proj} \)**

The calculation of \( \text{Proj} \left( \left( D_{w,w_x,w_{xx}} f^{(1,2)}_2 (z, 0, 0) \right) U^{(2,d)}_2 (z, 0)(\theta) \right) \)

The calculation of \( \text{Proj} \left( \left( D_{w,w_x,w_{xx}} f^{(1,2)}_2 (z, 0, 0) \right) U^{(2,d)}_2 (z, 0)(\theta) \right) \) is similar to that in Step 3 but is more tedious. We leave the calculation to Appendix B.

\[
\text{Proj} \left( \left( D_{w,w_x,w_{xx}} f^{(1,2)}_2 (z, 0, 0) \right) U^{(2,d)}_2 (z, 0)(\theta) \right) = \begin{pmatrix} \mathcal{H} (E^{d}_1 z^2_1 z^2_2 + E^{d}_3 z^2_3 + E^{d}_3 z^2_1 z^2_3) \\ \mathcal{H} (E^{d}_3 z^2_2 z^2_3) \end{pmatrix}, \quad (2.52)
\]

where

\[
\begin{align*}
E^{d}_{11} &= -\frac{1}{\sqrt{\ell\pi}} \psi^T_{n_1} (0) \left( S^{(d,1)}_2 (\phi_{n_1} (\theta), h_{0,1100}(\theta)) + S^{(d,1)}_2 (\overline{\phi}_{n_1} (\theta), h_{0,0000}(\theta)) \right) \\
&\quad + \frac{1}{\sqrt{2\pi}} \psi^T_{n_1} (0) \sum_{j=1,2,3} b^{(1,j)}_{2n_1} S^{(d,j)}_2 (\phi_{n_1} (\theta), h_{2n_1,1100}(\theta)) \\
&\quad + \frac{1}{\sqrt{2\pi}} \psi^T_{n_1} (0) \sum_{j=1,2,3} b^{(1,j)}_{2n_1} S^{(d,j)}_2 (\overline{\phi}_{n_1} (\theta), h_{2n_1,0000}(\theta)), \\
E^{d}_{12} &= -\frac{1}{\sqrt{\ell\pi}} \psi^T_{n_1} (0) \left( S^{(d,1)}_2 (\phi_{n_1} (\theta), h_{0,0001}(\theta)) \right) \\
&\quad + \frac{1}{\sqrt{2\pi}} \psi^T_{n_1} (0) \sum_{j=1,2,3} b^{(1,j)}_{2n_1} S^{(d,j)}_2 (\phi_{n_1} (\theta), h_{2n_1,0001}(\theta)) \\
&\quad + \frac{1}{\sqrt{2\pi}} \psi^T_{n_1} (0) \sum_{j=1,2,3} b^{(2,j)}_{n_2+n_1} S^{(d,j)}_2 (\phi_{n_2+n_1} (\theta), h_{(n_2+n_1),1001}(\theta)) \\
&\quad + \frac{1}{\sqrt{2\pi}} \psi^T_{n_1} (0) \sum_{j=1,2,3} b^{(2,j)}_{n_2+n_1} S^{(d,j)}_2 (\overline{\phi}_{n_2+n_1} (\theta), h_{(n_2+n_1),1010}(\theta)) \\
&\quad + \delta_{n_1n_2} \psi^T_{n_1} (0) \sum_{j=1,2,3} b^{(2,j)}_{n_2-n_1} S^{(d,j)}_2 (\phi_{n_2-n_1} (\theta), h_{(n_2-n_1),1001}(\theta)) \\
&\quad + \delta_{n_1n_2} \psi^T_{n_1} (0) \sum_{j=1,2,3} b^{(2,j)}_{n_2-n_1} S^{(d,j)}_2 (\overline{\phi}_{n_2-n_1} (\theta), h_{(n_2-n_1),1010}(\theta)),
\end{align*}
\]

(2.53)
and

\[
\begin{align*}
E_{31}^d &= - \frac{1}{\sqrt{2\pi}} n_2^2 \psi_{n_2}(0) \left( S_2^{(d,1)}(\phi_{n_2}(\theta), h_{0.0011}(\theta)) + S_2^{(d,1)}(\phi_{n_2}(\theta), h_{0.0020}(\theta)) \right) \\
&\quad + \frac{1}{\sqrt{2\pi}} n_2^2 \psi_{n_2}(0) \sum_{j=1,2,3} b_{2n_2}^{(2,j)} S_2^{(d,j)}(\phi_{n_2}(\theta), h_{2n_2,0.0111}(\theta)) \\
&\quad + \frac{1}{\sqrt{2\pi}} n_2^2 \psi_{n_2}(0) \sum_{j=1,2,3} b_{2n_2}^{(2,j)} S_2^{(d,j)}(\phi_{n_2}(\theta), h_{2n_2,0.0020}(\theta)) \,,
\end{align*}
\]

\[
\begin{align*}
E_{32}^d &= - \frac{1}{\sqrt{2\pi}} n_2^2 \psi_{n_2}(0) S_2^{(d,1)}(\phi_{n_2}(\theta), h_{0.1100}(\theta)) \\
&\quad + \frac{1}{\sqrt{2\pi}} n_2^2 \psi_{n_2}(0) \sum_{j=1,2,3} b_{2n_2}^{(2,j)} S_2^{(d,j)}(\phi_{n_2}(\theta), h_{2n_2,1.1100}(\theta)) \\
&\quad + \frac{1}{\sqrt{2\pi}} n_2^2 \psi_{n_2}(0) \sum_{j=1,2,3} b_{2n_2}^{(2,j)} S_2^{(d,j)}(\phi_{n_2}(\theta), h_{(n_2+n_1),0.1110}(\theta)) \\
&\quad + \delta_{n_1n_2} \psi_{n_2}(0) \sum_{j=1,2,3} b_{2n_2-n_1}^{(1,j)} S_2^{(d,j)}(\phi_{n_1}(\theta), h_{(n_2-n_1),0.1110}(\theta)) \\
&\quad + \delta_{n_1n_2} \psi_{n_2}(0) \sum_{j=1,2,3} b_{2n_2-n_1}^{(1,j)} S_2^{(d,j)}(\phi_{n_1}(\theta), h_{(n_2-n_1),1.1010}(\theta)) \,,
\end{align*}
\]

with

\[
b_{(1,1)} = - \frac{n_1^2}{\ell^2}, \quad b_{(1,3)} = - \frac{k^2}{\ell^2}, \quad k = 2n_1, n_2 + n_1, n_2 - n_1, \quad b_{(1,2)} = \begin{cases} \frac{n_1 k}{\ell^2}, & k = 2n_1, n_2 + n_1, \\ - \frac{n_1 k}{\ell^2}, & k = n_2 - n_1, \end{cases}
\]

\[
b_{(2,1)} = - \frac{n_2^2}{\ell^2}, \quad b_{(2,2)} = \frac{n_2 k}{\ell^2}, \quad b_{(2,3)} = - \frac{k^2}{\ell^2}, \quad k = 2n_2, n_2 + n_1, n_2 - n_1.
\]

Clearly, we still need to compute $h_{n,1010}(\theta)$. We leave this tedious calculation to Appendix C.

### 2.2.3 The normal form of double Hopf bifurcation truncated to third terms

Let

\[
B_{11} = C_{11} + \frac{3}{2} \left( D_{11} + E_{11} + E_{11}^d \right), \quad B_{12} = C_{12} + \frac{3}{2} \left( D_{12} + E_{12} + E_{12}^d \right),
\]

\[
B_{31} = C_{31} + \frac{3}{2} \left( D_{31} + E_{31} + E_{31}^d \right), \quad B_{32} = C_{32} + \frac{3}{2} \left( D_{32} + E_{32} + E_{32}^d \right).
\]

From (2.25), (2.33), (2.40), (2.46), (2.50) and (2.52), we have the following normal form of double Hopf bifurcation truncated to third terms:

\[
\dot{z} = Bz + \frac{1}{2!} \left( \mathcal{H} \left( B_1^{(1)} \mu_1 + B_1^{(2)} \mu_2 \right) z_1 \right) + \frac{1}{3!} \left( \mathcal{H} \left( B_3^{(1)} \mu_1 + B_3^{(2)} \mu_2 \right) z_3 \right).
\]

With the polar coordinates $z_1 = r_1 e^{-i\Theta_1}$, $z_2 = r_1 e^{i\Theta_1}$, $z_3 = r_2 e^{-i\Theta_2}$, $z_4 = r_2 e^{i\Theta_2}$, we have the following amplitude equations for (2.55):

\[
\begin{align*}
\frac{dr_1}{dt} &= (\delta_1 + p_{11}r_1^2 + p_{12}r_2^2) r_1, \\
\frac{dr_2}{dt} &= (\delta_2 + p_{21}r_1^2 + p_{22}r_2^2) r_2,
\end{align*}
\]

(2.56)
where
\[ \delta_1 = \frac{1}{2} \text{Re} \left( B_1^{(1)} \mu_1 + B_1^{(2)} \mu_2 \right), \quad \delta_2 = \frac{1}{2} \text{Re} \left( B_3^{(1)} \mu_1 + B_3^{(2)} \mu_2 \right), \]
\[ p_{11} = \frac{1}{3} \text{Re} (B_{11}), \quad p_{12} = \frac{1}{3} \text{Re} (B_{12}), \quad p_{21} = \frac{1}{3} \text{Re} (B_{32}), \quad p_{22} = \frac{1}{3} \text{Re} (B_{31}). \]

It follows from [21] that depending on whether \( p_{11} \) and \( p_{22} \) have the same or opposite signs, there are two essential bifurcation cases: simple case: \( p_{11}p_{22} > 0 \), and difficult case: \( p_{11}p_{22} < 0 \).

For the simple case or some subcases of the difficult case, it is sufficient to consider the normal form truncated to three-order terms. However, for some subcases of the difficult case, we have to calculate the normal form up to fifth-order terms to determine the dynamics near the bifurcation point.

### 3 Examples

In this section, taking
\[ f(u, v) = u \left( 1 - \frac{u}{a} \right) - \frac{buv}{1 + u}, \quad g(u, v) = \frac{buv}{1 + u} - cv, \]
then \((1.1)\) becomes the following predator-prey model with Holling type II functional response:

\[
\begin{cases}
\frac{\partial u(x, t)}{\partial t} = d_{11} u_{xx}(x, t) + u \left( 1 - \frac{u}{a} \right) - \frac{buv}{1 + u}, & 0 < x < \ell \pi, t > 0, \\
\frac{\partial v(x, t)}{\partial t} = -d_{21} (v(x, t) u_x(x, t - \tau))_x + d_{22} v_{xx}(x, t) - cv + \frac{buv}{1 + u}, & 0 < x < \ell \pi, t > 0, \\
u_x(0, t) = u_x(\ell \pi, t) = v_x(0, t) = v_x(\ell \pi, t) = 0, & t \geq 0.
\end{cases}
\]

System \((3.1)\) has the positive constant steady state \( E_*(\gamma, v_\gamma) \), where
\[ \gamma = \frac{c}{b - c}, \quad v_\gamma = \frac{(a - \gamma)(1 + \gamma)}{ab}, \]
provided that \( b > \frac{c(1+a)}{a} \) (or equivalently, \( 0 < \gamma < a \)) holds. For \( d_{21} = 0 \), \( E_*(\gamma, v_\gamma) \) is asymptotically stable for \( d_{11} \geq 0 \) and \( d_{22} \geq 0 \) provided that \( \frac{a-1}{2} < \gamma < a \). For \( d_{21} > 0 \), the stability and Hopf bifurcation for system \((3.1)\) has been detailedly investigated in [39, 40]. In what follows, we are interested in the double Hopf bifurcation induced by the spatial memory diffusion coefficient \( d_{21} \) and the delay \( \tau \). For this purpose, we need to investigate the critical values of \( d_{21} \) and \( \tau \) at which \((2.4)\) with \((3.2)\) has two pairs of purely imaginary roots. In the following, we use the same notations as in [40] and simply introduce some results from [39, 40] for the analysis.

For \( E_*(\gamma, v_\gamma) \), we have
\[
a_{11} = \frac{\gamma (a - 1 - 2\gamma)}{a(1 + \gamma)} \begin{cases} 
\leq 0, & \frac{a-1}{2} \leq \gamma < a, \\
> 0, & 0 < \gamma < \frac{a-1}{2},
\end{cases} \quad (3.2)
\]
\[ a_{12} = -c < 0, \quad a_{21} = \frac{a-\gamma}{a(1+\gamma)} > 0, \quad a_{22} = 0. \]
Let
\[ \omega^4 + P_n \omega^2 + Q_n = 0, \quad (3.3) \]
where
\[ P_n = (d_{11}^2 + d_{22}^2) (n/\ell)^4 - 2 (d_{11}a_{11} + d_{22}a_{22}) (n/\ell)^2 + a_{11}^2 + a_{22}^2 + 2a_{12}a_{21}, \quad (3.4) \]
and
\[ Q_n = \tilde{J}_n(0) \left( J_n + d_{21}v_*a_{12}(n/\ell)^2 \right). \quad (3.5) \]
It follows from [40] that if Eq. (3.3) has a positive roots \( \omega^+ \) (or \( \omega^- \)), then Eq. (2.4) has a pair of purely imaginary roots \( \pm \omega^+ i \) (or \( \pm \omega^- i \)) at \( \tau = \tau^+_{n,j} \) (or \( \tau = \tau^-_{n,j} \)), where
\[ \omega^\pm = \sqrt{-P_n \pm \sqrt{\Delta_n}}. \quad (3.6) \]
and
\[ \tau^\pm_{n,j} = \frac{1}{\omega^\pm_n} \left\{ \arccos \left\{ \frac{J_n - (\omega^\pm_n)^2}{d_{21}v_*a_{12}(n/\ell)^2} \right\} + 2j\pi \right\}, \quad j \in \mathbb{N}_0, \ n \in \mathbb{N}. \quad (3.7) \]
The number of the positive root of Eq. (3.3) depends on the signs of \( P_n, Q_n \) and \( \Delta_n = P_n^2 - 4Q_n = T^4_n - 4T^2_nJ_n + 4d_{21}^2v_*^2a_{12}^2(n/\ell)^4 \).

Define
\[ d_{21}^{(n)} = \frac{1}{v_*|a_{12}|} \left( d_{11}d_{22}(n/\ell)^2 + \frac{\text{Det}(A)}{(n/\ell)^2} - (d_{11}a_{22} + d_{22}a_{11}) \right), \quad \text{(3.8)} \]
\[ a_{21}^{(n)} = \frac{\sqrt{4T^2_nJ_n - T^4_n}}{2v_*|a_{12}|(n/\ell)^2}. \quad \text{(3.9)} \]
Then, for fixed \( n \), it follows from [40] that \( Q_n > 0 \) if and only if \( 0 < d_{21} < d_{21}^{(n)} \), and \( \Delta_n > 0 \) for \( 4J_n \leq T^2_n \) if and only if \( d_{21} > a_{21}^{(n)} \).

### 3.1 Dynamics near the double Hopf bifurcation point with the same spatial profile

Taking the parameters as follows
\[ a = 1, b = 9, c = 3, d_{11} = 0.6, d_{22} = 0.8, \ \ell = 2, \quad \text{(3.10)} \]
we have \( (u_*, v_*) = (1/2, 1/12) \) and
\[ a_{11} = -\frac{1}{3}, a_{12} = -3, a_{21} = \frac{1}{3}, a_{22} = 0. \]
From (3.4), we have
\[ P_n = \frac{1}{16}n^4 + \frac{1}{10}n^2 - \frac{17}{9} \begin{cases} < 0, & n = 1, 2, \\ > 0, & n = 3, 4, \ldots. \end{cases} \quad \text{(3.11)} \]
By (3.8), we have
\[ d_{21}^{(n)} = \frac{12n^2}{25} + \frac{16}{n^2} + \frac{16}{15}, \]
from which it is easy to verify that
\[ d_{21}^{(2)} = 524/75 < d_{21}^{(3)} = 1612/225 < d_{21}^{(4)} = 731/75 < d_{21}^{(1)} = 1316/75 < d_{21}^{(5)} = 1028/75 < d_{21}^{(6)} = 4228/225, \]
and
\[ \min_{n \in \mathbb{N}} \left\{ d_{21}^{(n)} \right\} = d_{21}^{(2)} = \frac{524}{75} \doteq 6.9867, \quad d_{21}^{(n)} < d_{21}^{(n+1)} \text{ for any } n \geq 5. \tag{3.12} \]

When \(0 \leq d_{21} < d_{21}^{(2)}\), the stability of the positive constant steady state \((u_*, v_*)\) is independent of the delay and we have the following stability result.

**Proposition 3.1.** For system (3.1) with the parameters \(a = 1, b = 9, c = 3, d_{11} = 0.6, d_{22} = 0.8, \ell = 2\), when \(0 \leq d_{21} < d_{21}^{(2)} = \frac{927}{134}\), the positive constant steady state \((u_*, v_*) = (1/2, 1/12)\) is locally asymptotically stable for any \(\tau \geq 0\);

**Proof.** From (3.5) and (3.12), it follows that when \(d_{21} < d_{21}^{(2)}\), \(Q_n > 0\) for any \(n \in \mathbb{N}\). This, together with (3.11), implies that when \(d_{21} < d_{21}^{(2)}\), Eq. (3.3) has no positive root for \(n = 3, 4, \ldots\).

From (3.9), we have
\[ d_{21}^{(2)} = \frac{927}{134} \doteq 6.9179 < d_{21}^{(1)} = \frac{6851}{633}. \]
Notice that
\[ d_{21}^{(2)} = \frac{927}{134} \doteq 6.9179 < d_{21}^{(2)} = \frac{524}{75} \doteq 6.9867, \]
which implies that for \(0 < d_{21} < d_{21}^{(2)} \doteq 6.9179, \Delta_1 < 0 \) and \(\Delta_2 < 0\). Thus, for \(d_{21} < d_{21}^{(2)}\), Eq. (3.3) has no positive root for \(n = 1, 2\).

Combining the above discussion, we can conclude that for \(0 < d_{21} < d_{21}^{(2)}\), Eq. (3.3) has no positive root for any \(n \in \mathbb{N}\) and then the positive constant steady state \((u_*, v_*) = (1/2, 1/12)\) is asymptotically stable for any \(\tau \geq 0\). \(\square\)

When \(d_{21} > d_{21}^{(2)} = 6.9179\), the stability of the positive constant steady state \((u_*, v_*)\) is related to the delay. Fig. 1(a) illustrates the stability region and Hopf bifurcation curves in the \(d_{21} - \tau\) plane for \(6.9 \leq d_{21} \leq 7\) and \(0 \leq \tau \leq 20\).

When \(d_{21}^{(2)} \doteq 6.9179 < d_{21} < d_{21}^{(2)} \doteq 6.9867\), Eq. (3.3) has two positive roots for \(n = 2\) and no positive roots for \(n \in \mathbb{N}\) and \(n \neq 2\). Thus, the characteristic equation (2.4) has two sequences of purely imaginary roots \(\pm i\omega_2^\pm\) at \(\tau = \tau_{2,j}^\pm\). System (3.1) undergoes Hopf bifurcations at \(\tau = \tau_{2,j}^\pm\), as shown in Fig. 1(b). Hopf bifurcation curves \(\tau = \tau_{2,1}^+\) and \(\tau = \tau_{2,0}^+\) intersect at the point \(P_1(6.9618, 13.1290)\), which is the double Hopf bifurcation point with \(\omega_2^+ \approx 0.2222\) and \(\omega_2^+ \approx 0.6629\). This double Hopf bifurcation arises from the interaction of two Hopf bifurcations.
with the same mode-2. For this double Hopf bifurcation point $P_1(6.9618, 13.1290)$, it follows from the normal form theory derived in Section 2.2 with $n_1 = n_2 = 2, d_{21}^c = 6.9618, \tau_c = 13.1290, \omega_1 = \omega_2^−$ and $\omega_2 = \omega_2^+$. The normal form truncated to the third order terms is
\[
\begin{cases}
\dot{r}_1 = r_1 \left( -4.1681 \times 10^{-4} \mu_1 + 0.1332 \mu_2 - 0.1428 r_1^2 + 6.003 r_2^2 \right), \\
\dot{r}_2 = r_2 \left( 0.0036 \mu_1 + 0.1311 \mu_2 - 5.4981 r_1^2 - 2.2507 r_2^2 \right).
\end{cases} 
(3.13)
\]
Since $r_1, r_2 \geq 0$, system (3.13) has a zero equilibrium $E_0(0, 0)$ for any $\mu_1, \mu_2 \in \mathbb{R}$, two boundary equilibria:
\[
E_1 \left( \sqrt{0.9331 \mu_2 - 0.0029 \mu_1}, 0 \right), \quad \mu_1 < 319.6009 \mu_2,
\]
\[
E_2 \left( 0, \sqrt{0.0016 \mu_1 + 0.0583 \mu_2} \right), \quad \mu_1 > -36.4988 \mu_2,
\]
and one interior positive equilibrium:
\[
E_3 \left( \sqrt{6.1905 \times 10^{-4} \mu_1 + 0.0326 \mu_2}, \sqrt{0.8416 \times 10^{-4} \mu_1 - 0.0214 \mu_2} \right)
\]
for
\[
\mu_1 > \max \{ -52.6919 \mu_2, 254.4824 \mu_2 \}.
\]
By analyzing the stability of these equilibria and noticing $\mu_1 = \tau - \tau_c, \mu_2 = d_{21} - d_{21}^c$, it is easy to obtain the phase portrait and their dynamical topologies as shown in Fig 2(a), where the curves $H_1, H_2, L_1, L_2$ are defined by the following:
\[
H_1 : \tau - \tau_c = 319.6 \left( d_{21} - d_{21}^c \right), \quad H_2 : \tau - \tau_c = -36.4988 \left( d_{21} - d_{21}^c \right),
\]
\[
L_1 : \tau - \tau_c = 254.4824 \left( d_{21} - d_{21}^c \right), d_{21} > d_{21}^c, \quad L_2 : \tau - \tau_c = -52.6919 \left( d_{21} - d_{21}^c \right), d_{21} < d_{21}^c.
\]
(3.14)
The straight lines $H_j, L_j, j = 1, 2$, divide the vicinity of the double Hopf bifurcation point $P_1$ into six regions. For each region, the corresponding phase portrait is plotted in the right side of Fig.2.

![Figure 2: The left column: partition of the region near the double Hopf bifurcation point: (a) partition of the region near the point $P_1$ in Fig.1(a) with $H_j, L_j, j = 1, 2$, being defined by (3.14); (b) partition of the region near the point $P_2$ in Fig.1(b) with $H_j, L_j, j = 1, 2$, being defined by (3.17). The right column: phase portraits for different parameter regions in the left figures.]

These phase portraits Fig.2 show that for the zero equilibrium $E_0$ of (3.13), it is stable in Region 1○ and unstable in other regions. Notice that the zero equilibrium $E_0$ of (3.13) corresponds to the positive equilibrium $E_*$ of the original system (1.1). For $(d_{21}, \tau) = (6.96, 12.5)$ in Region 1○, numerical simulations in Fig.3 show the stability of the positive equilibrium $E_*$ of the original system (1.1).

The boundary equilibria $E_1$ and $E_2$ of correspond to the periodic solution of the original system (1.1). From the phase portraits 2○, 3○ and 6○ in Fig.2 it is shown that near the neighbourhood of the point $P_1$, there exist two types of stable periodic solutions, respectively, bifurcating from the Hopf bifurcation at $\tau = \tau_{2,0}^-$ and $\tau = \tau_{2,1}^+$. Figs.4 and 5 numerically illustrate these two types of periodic solutions (1.1) bifurcating from $\tau_{2,0}^-$ and $\tau_{2,1}^+$ for the parameters $(d_{21}, \tau)$ in Regions 2○ and 6○, respectively.

The positive equilibrium $E_3$ correspond to the quasi-periodic solution of the original system (1.1), which exist for $(d_{21}, \tau)$ in Regions 4○ and 5○ of Fig.2(a). Figs.6(a) and 6(d) numerically illustrate this quasi-periodic solution for $(d_{21}, \tau)$ in Region 5○. Figs.6(b) and 6(e) are the truncated curves of Figs.6(a) and 6(d), respectively, for fixed space $x = \pi/5$, and Figs.6(c) and
Figure 3: For \((d_{21}, \tau) = (6.95, 12.5)\) in Region 1\(\circ\), the positive equilibrium \(E_\ast\) of (1.1) is asymptotically stable.

Figure 4: For \((d_{21}, \tau) = (6.96, 12.5)\) in Region 2\(\circ\), the positive equilibrium \(E_\ast\) is unstable and the periodic solution bifurcating from the Hopf bifurcating at \(\tau_{2,0}\) appears. The initial value is chosen as \(u(x, 0) = u_\ast + 0.005\cos(x), v(x, 0) = v_\ast - 0.005\cos(x)\).
Figure 5: For \((d_{21}, \tau) = (6.945, 13.9)\) in Region \(\mathcal{O}\), the positive equilibrium \(E_*\) is unstable and the periodic solution bifurcating from the Hopf bifurcating at \(\tau_{2,1}^+\) appears. The initial value is chosen as \(u(x, 0) = u_* + 0.01 \cos(x), v(x, 0) = v_* - 0.01 \cos(x)\).

\(\mathcal{O}(f)\) are the truncated curves of Figs.6(a) and 6(d), respectively, for fixed time \(t = 3000\). Fig.7 illustrates the phase portrait of \(u(x, t)\) and \(v(x, t)\) in the \(u-v\) plane for fixed space \(x = \pi/5\), which looks like a “bird”.

3.2 Dynamics near the double Hopf bifurcation point with different spatial modes

Taking the same parameters as used in [39]:
\[
a = 1, \quad b = \frac{3}{10}, \quad c = \frac{1}{10}, \quad d_{11} = \frac{6}{10}, \quad d_{22} = \frac{8}{10}, \quad \ell = 2,
\]
we have \((u_*, v_*) = (1/5, 5/2)\). For the dynamical classification near the double Hopf bifurcation point, we introduce the results of linear analysis from [39], as shown in Fig.1(b). In Fig.1(b), Hopf bifurcation curves \(\tau = \tau_{1,0}^+\) and \(\tau = \tau_{2,0}^+\) intersect at the point \(P_2(4.1350, 4.0276)\), which is the double Hopf bifurcation point. This double Hopf bifurcation arises from the interaction of spatially inhomogeneous Hopf bifurcations with mode-1 and mode-2. For this point \(P_2\), we can obtain \(\omega_{1}^+ \approx 0.2671, \omega_{2}^+ = 0.3666\).

To investigate the dynamics near this point \(P_2\), we need to calculate the corresponding normal form. Setting \((d_{21}, \tau_c) = (13.4531, 0.6811)\), \(\omega_1 = \omega_1^+\) and \(\omega_2 = \omega_2^+\), and then employing the algorithm developed in Section 2, we have the following normal form for this double Hopf bifurcation point \(P_2\)
\[
\begin{align*}
\dot{r}_1 &= r_1 \left(0.0781\mu_1 + 0.1224\mu_2 - 1.4203r_1^2 - 4.2174r_2^2\right), \\
\dot{r}_2 &= r_2 \left(0.0595\mu_1 + 0.1653\mu_2 - 1.8176r_1^2 - 2.3315r_2^2\right).
\end{align*}
\]

Figure 6: For \((d_{21}, \tau) = (6.95, 14)\) in Region 5, the positive equilibrium \(E_*\) is unstable and there exists the stable quasi-periodic solution with the spatial profile like \(\cos(x)\). (a) and (d): the spatiotemporal dynamics of the prey \(u\) and predator \(v\); (b) and (e): the truncated curves of (a) and (d) for fixed space \(x = \pi/5\) showing the evolution of \(u\) and \(v\) in time; (c) and (f): the truncated curves of (a) and (d) for fixed time \(t = 3000\) showing the spatial profiles of \(u\) and \(v\). The initial value is chosen as \(u(x, 0) = u_* + 0.02 \cos(x), v(x, 0) = v_* + 0.01 \cos(x)\).
Figure 7: For \((d_{21}, \tau) = (6.95, 14)\) in Region 5 and fixed space \(x = \pi/5\), the evolution of the dynamics of the prey \(u\) and predator \(v\) in the \(u-v\) plane.

System (3.16) has the similar dynamics to (3.13). The dynamical classification of (3.16) is plotted in Fig.1(b), where

\[
H_1 : \tau - \tau_c = -2.7794 (d_{21} - d_{c21}), \quad H_2 : \tau - \tau_c = -1.5672 (d_{21} - d_{c21}),
\]

\[
L_1 : \tau - \tau_c = 0.2140 (d_{21} - d_{c21}), d_{21} > d_{c21}, \quad L_2 : \tau - \tau_c = -5.991 (d_{21} - d_{c21}), d_{21} < d_{c21}.
\]

(3.17)

It follows from Fig.2(b) and the corresponding phase portraits that there exist stable periodic solutions with spatial profile like \(\cos(x)\) and \(\cos(x/2)\), respectively, for \((d_{21}, \tau)\) in Region 2 and Region 6. For \((d_{21}, \tau)\) in Region 3, there exist a connection orbit from the periodic solutions with spatial profile like \(\cos(x)\) to the one spatial profile like \(\cos(x/2)\), but in the reverse direction for \((d_{21}, \tau)\) in Region 8.

However, for \((d_{21}, \tau)\) in Region 7 of Fig.2(b), there exists a bistability phenomenon, i.e., the coexistence of two types of stable periodic solutions with spatial profile like \(\cos(x)\) and \(\cos(x/2)\). Taking \((d_{21}, \tau) = (4.4, 4.3)\) in Region 4, Figs 8 and 9 numerically illustrate this bistability phenomenon. With the same parameter \((d_{21}, \tau) = (4.4, 4.3)\) and different initial values, Figs 8(a) and 8(b) show that the solution \((u, v)\) finally converges to the periodic solutions with spatial profile like \(\cos(x/2)\) for the initial values \(u(x, 0) = u_* + 0.1 \cos(x/2), v(x, 0) = v_* + 0.1 \cos(x/2)\), but Figs 8(c) and 8(d) show that the solution \((u, v)\) finally converges to the periodic solutions with spatial profile like \(\cos(x)\) for the initial values \(u(x, 0) = u_* + 0.1 \cos(x), v(x, 0) = v_* + 0.1 \cos(x)\).
For fixed space $x = \pi/5$, Fig. 9 numerically illustrate the orbit of $(u, v)$ in the $u$-$v$ plane.

Figure 8: For $(d_{21}, \tau) = (4.4, 4.3)$ in Region 7 of Fig. 2(b), two types of stable spatially inhomogeneous periodic solutions with different spatial profiles coexist. (a)-(b) The initial value is chosen as $u(x, 0) = u_* + 0.1 \cos(x/2), v(x, 0) = v_* + 0.1 \cos(x/2)$; (c)-(d) The initial value is chosen as $u(x, 0) = u_* + 0.1 \cos(x), v(x, 0) = v_* + 0.1 \cos(x)$.

4 Discussion

In this paper, we have paid our attention on the development of an algorithm for computing the normal form of the double Hopf bifurcation induced by the memory-based diffusion coefficient and memory delay for the memory-based diffusion system. The calculating formulae of the second and third terms in the normal form are explicitly derived from those in the original
system. This algorithm, which is for the memory-based diffusion system, where the delay appears in the directional diffusion terms and the diffusion terms are nonlinear, is the counterpart of the existing algorithm for the related theory of the classical reaction-diffusion system. By analysing the corresponding normal form, we can determine the dynamical classification near the double Hopf bifurcation point.

Employing the obtained theoretical results to the predator-prey system with Holling-II functional response, we investigate the spatio-temporal dynamics due to the interaction of double Hopf bifurcation for two cases: (i) double Hopf bifurcation with the same spatial mode; (ii) double Hopf bifurcation with the different spatial modes. For the former, we find two types of stable spatially inhomogeneous periodic solutions with the same spatial mode and similar oscillatory frequency, and the quasi-periodic solution in some region near the double Hopf bifurcation point. For latter, we find two types of stable spatially inhomogeneous periodic solutions with different spatial mode and oscillatory frequency, and the coexistence of two stable periodic solutions with different spatial mode in some region near the double Hopf bifurcation point. In both cases, the pattern transitions from an unstable periodic solution to a stable periodic solution are found.

We would also like to iterate that the theoretical results in Section 2 are derived for the non-resonant and weakly resonant double Hopf bifurcations. Thus it is not applicable for the strongly resonant double Hopf bifurcation. The algorithm for computing the normal form for

Figure 9: For \((d_{21}, \tau) = (4.4, 4.3)\) in Region \(\tilde{T}\) of Fig.2(b) and fixed space \(x = \pi/5\), two stable periodic orbits in \(u-v\) plane coexist.
the case of strongly double Hopf bifurcation for the memory-based diffusion system (1.1) still remains open and we leave this for the further investigation.

Appendices

A  Expressions of $A^{(d,j)}_{i_1i_2i_3i_4}$, $j = 1, 2, 3$. 

\[
A^{(d,1)}_{1010} = -2d_{21}^c \tau_c \begin{pmatrix}
0 \\
\phi_{n_1}^{(1)}(-1)\phi_{n_2}^{(2)}(0) + \phi_{n_2}^{(1)}(-1)\phi_{n_1}^{(2)}(0)
\end{pmatrix}, \\
A^{(d,1)}_{1001} = -2d_{21}^c \tau_c \begin{pmatrix}
0 \\
\phi_{n_1}^{(1)}(-1)\phi_{n_2}^{(2)}(0) + \phi_{n_2}^{(1)}(-1)\phi_{n_1}^{(2)}(0)
\end{pmatrix}, \\
A^{(d,1)}_{0110} = \overline{A^{(d,1)}_{1001}}, \\
A^{(d,1)}_{0101} = \overline{A^{(d,1)}_{1010}}, \\
A^{(d,1)}_{2000} = A^{(d,2)}_{2000} = -2d_{21}^c \tau_c \begin{pmatrix}
0 \\
\phi_{n_1}^{(1)}(-1)\phi_{n_1}^{(2)}(0)
\end{pmatrix}, \\
A^{(d,1)}_{0020} = A^{(d,2)}_{0020} = -2d_{21}^c \tau_c \begin{pmatrix}
0 \\
\phi_{n_2}^{(1)}(-1)\phi_{n_2}^{(2)}(0)
\end{pmatrix}, \\
A^{(d,1)}_{1100} = A^{(d,2)}_{1100} = -2d_{21}^c \tau_c \begin{pmatrix}
0 \\
2\text{Re}\left\{\phi_{n_1}^{(1)}(-1)\phi_{n_1}^{(2)}(0)\right\}
\end{pmatrix}, \\
A^{(d,1)}_{0011} = A^{(d,2)}_{0011} = -2d_{21}^c \tau_c \begin{pmatrix}
0 \\
2\text{Re}\left\{\phi_{n_2}^{(1)}(-1)\phi_{n_2}^{(2)}(0)\right\}
\end{pmatrix}, \\
A^{(d,2)}_{1010} = -2d_{21}^c \tau_c \begin{pmatrix}
0 \\
\phi_{n_1}^{(1)}(-1)\phi_{n_2}^{(2)}(0)
\end{pmatrix}, \\
A^{(d,3)}_{1010} = -2d_{21}^c \tau_c \begin{pmatrix}
0 \\
\phi_{n_2}^{(1)}(-1)\phi_{n_1}^{(2)}(0)
\end{pmatrix}, \\
A^{(d,2)}_{1001} = -2d_{21}^c \tau_c \begin{pmatrix}
0 \\
\phi_{n_1}^{(1)}(-1)\phi_{n_2}^{(2)}(0)
\end{pmatrix}, \\
A^{(d,3)}_{1001} = -2d_{21}^c \tau_c \begin{pmatrix}
0 \\
\phi_{n_2}^{(1)}(-1)\phi_{n_1}^{(2)}(0)
\end{pmatrix}, \\
A^{(d,1)}_{0200} = A^{(d,2)}_{0200} = \overline{A^{(d,1)}_{2000}}, \\
A^{(d,1)}_{0002} = A^{(d,2)}_{0002} = \overline{A^{(d,1)}_{0020}}, \\
A^{(d,1)}_{0200} = A^{(d,2)}_{0200} = \overline{A^{(d,1)}_{2000}}, \\
A^{(d,1)}_{0002} = A^{(d,2)}_{0002} = \overline{A^{(d,1)}_{0020}}, \\
A^{(d,1)}_{0200} = A^{(d,2)}_{0200} = \overline{A^{(d,1)}_{2000}}, \\
A^{(d,1)}_{0002} = A^{(d,2)}_{0002} = \overline{A^{(d,1)}_{0020}},
\]
\[
A^{(d,2)}_{0110} = A^{(d,2)}_{1001}, \quad A^{(d,3)}_{0110} = A^{(d,3)}_{1001}, \quad A^{(d,2)}_{0101} = A^{(d,2)}_{1010}, \quad A^{(d,3)}_{0101} = A^{(d,3)}_{1010}.
\]

**B The calculation of Proj\(s\left(\left(D_{w, w_x, w_{xx}} f_{2}^{(1,2)}(z, 0, 0)\right) U_{2}^{(2,d)}(z, 0)(\theta)\right)\)**

Denote \(\varphi(\theta) = \Phi(\theta)z_{x}\) and

\[
F_{2}^{d}(\varphi(\theta), w, w_{x}, w_{xx}) = F_{2}^{d}(\Phi(\theta)z_{x} + w, 0)
\]

\[
=-2d_{21}^{c}\tau_{c}\left(\begin{array}{c}
0 \\
(\varphi_{xx}(1) + w_{xx}(1)) (\varphi(2)(0) + \omega(2)(0))
\end{array}\right)
\]

\[
-2d_{21}^{c}\tau_{c}\left(\begin{array}{c}
0 \\
(\varphi_{x}(1) + w_{x}(1)) (\varphi(2)(0) + w_{x}(2)(0))
\end{array}\right),
\]

\[
S_{2}^{(d,1)}(\varphi(\theta), w) = -2d_{21}^{c}\tau_{c}\left(\begin{array}{c}
0 \\
\varphi_{xx}(1)w_{x}(2)(0)
\end{array}\right),
\]

\[
S_{2}^{(d,2)}(\varphi(\theta), w_{x}) = -2d_{21}^{c}\tau_{c}\left(\begin{array}{c}
0 \\
\varphi_{x}(1)w_{x}(2)(0)
\end{array}\right) - 2d_{21}^{c}\tau_{c}\left(\begin{array}{c}
0 \\
\varphi_{x}(2)(0)w_{x}(1)(-1)
\end{array}\right),
\]

\[
S_{2}^{(d,3)}(\varphi(\theta), w_{xx}) = -2d_{21}^{c}\tau_{c}\left(\begin{array}{c}
0 \\
\varphi_{x}(2)(0)w_{x}(1)(-1)
\end{array}\right).
\]

Then, we have

\[
D_{w, w_{x}, w_{xx}} F_{2}^{d}(\varphi(\theta), w, w_{x}, w_{xx})|_{w, w_{x}, w_{xx} = 0} U_{2}^{(2,d)}(z, 0)(\theta)
\]

\[
= S_{2}^{(d,1)}(\varphi(\theta), h(\theta, z)) + S_{2}^{(d,2)}(\varphi(\theta), h_{x}(\theta, z))
\]

\[
+ S_{2}^{(d,3)}(\varphi(\theta), h_{xx}(\theta, z))
\]

and

\[
\left[\begin{array}{c}
\tilde{S}_{2}^{(d,1)}(\varphi(\theta), h(\theta, z)), \\
\tilde{S}_{2}^{(d,1)}(\varphi(\theta), h(\theta, z)), \beta_{\nu}^{(1)}
\end{array}\right] = -\left(n_{1}/\ell\right)^{2} \sum_{n \in \mathbb{N}_{0}} b_{n, n, \nu} \left(\tilde{S}_{2}^{(d,1)}(\phi_{n_{1}}(\theta)z_{1}(\theta), h_{n}(\theta, z)) + S_{2}^{(d,1)}(\tilde{\phi}_{n_{1}}(\theta)z_{2}, h_{n}(\theta, z))\right)
\]

\[
-\left(n_{2}/\ell\right)^{2} \sum_{n \in \mathbb{N}_{0}} b_{n, n, \nu} \left(\tilde{S}_{2}^{(d,1)}(\phi_{n_{2}}(\theta)z_{3}, h_{n}(\theta, z)) + S_{2}^{(d,1)}(\tilde{\phi}_{n_{2}}(\theta)z_{4}, h_{n}(\theta, z))\right),
\]

33
\[
\left( \begin{bmatrix} S_2^{(d,2)}(\phi(\theta), h_x(\theta, z)), \beta^{(1)}_\nu \\ S_2^{(d,2)}(\phi(\theta), h_x(\theta, z)), \beta^{(2)}_\nu \end{bmatrix} \right) \\
= \left( n_1/\ell \right) \sum_{n \in \mathbb{N}_0} (n/\ell) b^s_{n_1,n,\nu} \left( S_2^{(d,2)}(\phi_1(\theta)z_1, h_n(\theta, z)) + S_2^{(d,2)}(\phi_2(\theta)z_2, h_n(\theta, z)) \right) \\
+ \left( n_2/\ell \right) \sum_{n \in \mathbb{N}_0} (n/\ell) b^s_{n_2,n,\nu} \left( S_2^{(d,2)}(\phi_3(\theta)z_3, h_n(\theta, z)) + S_2^{(d,2)}(\phi_4(\theta)z_4, h_n(\theta, z)) \right) ,
\]

where, for \( \nu = n_1, n_2, b_{n_1,n,\nu} \) is defined as in \( (2.49) \) and

\[
b_{n_1,n,\nu} = \int_{0}^{\ell \pi} \xi_{n_1}(x) \xi_n(x) \gamma_{\nu}(x) dx = \begin{cases} 
\frac{1}{\sqrt{2\pi}}, & n = 2n_1, \nu = n_1, \\
\frac{1}{\sqrt{2\pi}}, & n = n_1 + n_2, \nu = n_j + (-1)^{j+1}, \\
\frac{1}{\sqrt{2\pi}}, & n = n_2 - n_1, n_j = n_2, \nu = n_1, \\
\frac{1}{\sqrt{2\pi}}, & n = n_2 - n_1, n_j = n_1, \nu = n_2, \\
0, & \text{otherwise},
\end{cases}
\]

and for \( \phi(\theta) = (\phi^{(1)}(\theta), \phi^{(2)}(\theta))^T, y(\theta) = (y^{(1)}(\theta), y^{(2)}(\theta))^T \in C([-1, 0], \mathbb{R}^2) \),

\[
S_2^{(d,1)}(\phi(\theta), y(\theta)) = -2d_2^{\tau_c} \begin{pmatrix} 0 \\ \phi^{(1)}(-1) y^{(2)}(0) \end{pmatrix} ,
\]

\[
S_2^{(d,2)}(\phi(\theta), y(\theta)) = -2d_2^{\tau_c} \begin{pmatrix} 0 \\ \phi^{(1)}(-1) y^{(2)}(0) \end{pmatrix} - 2d_2^{\tau_c} \begin{pmatrix} 0 \\ \phi^{(2)}(0) y^{(1)}(-1) \end{pmatrix} ,
\]

Then, from \( (2.35), (2.37), (2.47) \) and \( (2.48) \), we have

\[
\left( D_{w,w_x,w_{xx}} F_2^{(1,2)}(z_1, 0, 0) \right) U_2^{(2,d)}(z, 0)(\theta) = \Psi(0) \left[ \begin{array}{c} D_{w,w_x,w_{xx}} F_2^{d}(\phi(\theta), w, w_x, w_{xx}) \\ D_{w,w_x,w_{xx}} F_2^{d}(\phi(\theta), w, w_x, w_{xx}) \end{array} \right]_{w,w_x,w_{xx}=0} U_2^{(2,d)}(z, 0)(\theta), \beta^{(1)}_\nu \\
\left[ \begin{array}{c} U_2^{(2,d)}(z, 0)(\theta), \beta^{(2)}_\nu \end{array} \right]_{w,w_x,w_{xx}=0} \right)_{\nu=n_1}^{\nu=n_2}.
\]
and then we obtain (2.52).

C Calculation of $h_{n,q_1q_2q_3q_4}(\theta)$.

From [12], we have

$$M_2^2 (h_n(\theta, z) \gamma_n(x)) = D_z (h_n(\theta, z) \gamma_n(x)) Bz - A_{\Omega} (h_n(\theta, z) \gamma_n(x)),$$

which leads to

$$\left( \begin{bmatrix} M_2^2 (h_n(\theta, z) \gamma_n(x)) , \beta_n^{(1)} \\ M_2^2 (h_n(\theta, z) \gamma_n(x)) , \beta_n^{(2)} \end{bmatrix} \right)$$

$$= 2 i \omega_1 c (h_{n,0002}(\theta) z_1^2 - h_{n,0002}(\theta) z_2^2) + 2 i \omega_2 c (h_{n,0020}(\theta) z_2^2 - h_{n,0002}(\theta) z_4^2)$$

$$+ i (\omega_1c + \omega_2c) h_{n,1001}(\theta) z_1 z_3 + i (\omega_1c - \omega_2c) h_{n,1001}(\theta) z_1 z_4$$

$$- i (\omega_1c - \omega_2c) h_{n,0110}(\theta) z_2 z_3 - i (\omega_1c + \omega_2c) h_{n,0110}(\theta) z_2 z_4$$

$$- \left( \dot{h}_n(\theta, z) + X_0(\theta) \left( \mathcal{L}_0 (h_n(\theta, z)) - \dot{h}_n(0, z) \right) \right),$$

where

$$\mathcal{L}_0 (h_n(\theta, z)) = -\tau_c (n/\ell)^2 (D_1 h_n(0, z) + D_2 h_n(-1, z) + \tau_c A h_n(\theta, z)).$$

By (2.23), we get

$$f_2^2(z, 0, 0) = X_0(\theta) \bar{F}_2 (\Phi(\theta) z_x, 0) - \pi \left( X_0(\theta) \bar{F}_2 (\Phi(\theta) z_x, 0) \right).$$
C.1 Case 1: $n_1 \neq n_2$.

By (2.15), (2.41), (2.42) and (2.43), we have, for $n_2 \neq n_1$,

$$
\begin{align*}
\left( \begin{array}{c}
\tilde{f}_2^2(z, 0, 0), \beta_n^{(1)} \\
\tilde{f}_2^2(z, 0, 0), \beta_n^{(2)} 
\end{array} \right) \\
\left\{ \begin{array}{l}
\frac{1}{\sqrt{2\pi}} X_0(\theta) \left( A_{2000} z_1^2 + A_{0200} z_2^2 + A_{0020} z_3^2 ight. \\
+ A_{0002} z_1^2 + A_{1100} z_1 z_2 + A_{0011} z_1 z_4 
\end{array} \right) \\
\left. \frac{1}{\sqrt{2\pi}} X_0(\theta) \left( \tilde{A}_{2000} z_1^2 + \tilde{A}_{0200} z_2^2 + \tilde{A}_{1100} z_1 z_2 ight) \right), \\
\left\{ \begin{array}{l}
\frac{1}{\sqrt{2\pi}} \left( X_0(\theta) I_2 - \Phi_{n_2}(\theta) \Psi_{n_2}(0) \right) \left( \tilde{A}_{2000} z_1^2 \\
+ \tilde{A}_{0200} z_2^2 + \tilde{A}_{1100} z_1 z_2 \right) \\
\frac{1}{\sqrt{2\pi}} X_0(\theta) \left( \tilde{A}_{1001} z_1 z_3 + \tilde{A}_{1001} z_1 z_4 + \tilde{A}_{0110} z_2 z_3 + \tilde{A}_{0101} z_2 z_4 \right) \\
\frac{1}{\sqrt{2\pi}} X_0(\theta) \left( \tilde{A}_{0020} z_3^2 + \tilde{A}_{0002} z_4^2 + \tilde{A}_{0011} z_3 z_4 \right) \\
\left\{ \begin{array}{l}
\frac{1}{\sqrt{2\pi}} X_0(\theta) \left( \tilde{A}_{0101} z_2 z_4 \right) \\
\frac{1}{\sqrt{2\pi}} X_0(\theta) \left( A_{1010} z_1 z_3 + A_{1001} z_1 z_4 + A_{0110} z_2 z_3 \\
+ A_{0101} z_2 z_4 \right) \\
\left\{ \begin{array}{l}
\frac{1}{\sqrt{2\pi}} X_0(\theta) \left( A_{1010} z_1 z_3 + A_{1001} z_1 z_4 \right. \\
+ A_{0110} z_2 z_3 + A_{0101} z_2 z_4 \n\end{array} \right) \\
\left( X_0(\theta) I_2 - \Phi_{n_1}(\theta) \Psi_{n_1}(0) \right) \left( A_{1010} z_1 z_3 \\
+ A_{1001} z_1 z_4 + A_{0110} z_2 z_3 + A_{0101} z_2 z_4 \right) \\
\frac{1}{\sqrt{2\pi}} X_0(\theta) \left( \tilde{A}_{1010} z_1 z_3 + \tilde{A}_{1001} z_1 z_4 + \tilde{A}_{0110} z_2 z_3 \\
+ \tilde{A}_{0101} z_2 z_4 + \tilde{A}_{2000} z_1^2 + \tilde{A}_{0200} z_3^2 + \tilde{A}_{1100} z_1 z_2 \right) \\
\left\{ \begin{array}{l}
\frac{1}{\sqrt{2\pi}} X_0(\theta) \left( \tilde{A}_{0101} z_2 z_4 \right) \\
\right. \right) \\
\right. \right) \\
\right. \right) \\
\right. \right) \\
\right. \right) \\
\right. \right) \\
\right. \right) \\
\right. \right) \\
\right. \right),
\end{align*}
$$

where $\tilde{A}_{j_1 j_2 j_3 j_4}$ is defined by (2.44), (2.45) and the following (C.3)

$$
\begin{align*}
\left\{ \begin{array}{l}
\tilde{A}_{j_1 j_2 j_3 j_4} = A_{j_1 j_2 j_3 j_4} - \frac{n_2}{2\pi} \left( A_{j_1 j_2 j_3 j_4}^{(d,1)} + A_{j_1 j_2 j_3 j_4}^{(d,2)} \right), \\
j_3, j_4 = 0, 1, 2, \quad j_3 + j_4 = 2, \quad j_1 = j_2 = 0,
\end{array} \right.
\end{align*}
$$

(C.2)
and
\[
\left\{ \begin{array}{l}
\tilde{A}_{j_1j_2j_3j_4} = A_{j_1j_2j_3j_4} - \frac{n_1n_2}{\ell^2} A^{(d,1)}_{j_1j_2j_3j_4} - \frac{n_3^2}{\ell^2} A^{(d,2)}_{j_1j_2j_3j_4} - \frac{n_4^2}{\ell^2} A^{(d,3)}_{j_1j_2j_3j_4}, \\
\end{array} \right.
\]  
\quad (C.4)

Hence, from (C.1), (C.2) and matching the coefficients of $z_1^2$, $z_2z_2$, $z_1z_3$, $z_2z_3$, $z_3^2$, we have
\[
\begin{align*}
&n = 0, \quad \begin{cases}
&z_1^2 : \quad \begin{cases}
&\hat{h}_{0,2000}(\theta) - 2i\omega_1c h_{0,2000}(\theta) = (0 \ 0)^T, \\
&\hat{h}_{0,0002}(0) - L_0(\hat{h}_{0,0002}(\theta)) = \frac{1}{\sqrt{\ell^n}} A_{0002}, \\
&\hat{h}_{0,1100}(\theta) = (0 \ 0)^T, \\
&L_0(\hat{h}_{0,1100}(\theta)) = \frac{1}{\sqrt{\ell^n}} A_{1100}, \\
&\hat{h}_{0,0011}(\theta) = (0 \ 0)^T, \\
&L_0(\hat{h}_{0,0011}(\theta)) = \frac{1}{\sqrt{\ell^n}} A_{0011},
\end{cases} \\
&z_2^2 : \quad \begin{cases}
&\hat{h}_{2n_1,2000}(\theta) - 2i\omega_1c h_{2n_1,2000}(\theta) = (0 \ 0)^T, \\
&\hat{h}_{2n_1,2000}(0) - L_0(\hat{h}_{2n_1,2000}(\theta)) = \frac{1}{\sqrt{2\ell^n}} \tilde{A}_{2000}, \\
&\hat{h}_{2n_1,1100}(\theta) = (0 \ 0)^T, \\
&\hat{L}_0(\hat{h}_{2n_1,1100}(\theta)) = \frac{1}{\sqrt{2\ell^n}} \tilde{A}_{1100}, \\
&\hat{h}_{2n_1,0011}(\theta) = (0 \ 0)^T, \\
&\hat{L}_0(\hat{h}_{2n_1,0011}(\theta)) = (0 \ 0)^T,
\end{cases} \\
&z_3^2 : \quad \begin{cases}
&\hat{h}_{0,0011}(\theta) - 2i\omega_2 c h_{0,0011}(\theta) = (0 \ 0)^T, \\
&\hat{h}_{0,0011}(0) - L_0(\hat{h}_{0,0011}(\theta)) = \frac{1}{\sqrt{\ell^n}} A_{0011}, \\
&\hat{L}_0(\hat{h}_{0,0011}(\theta)) = (0 \ 0)^T, \\
&\hat{h}_{2n_1,0011}(\theta) = (0 \ 0)^T, \\
&\hat{L}_0(\hat{h}_{2n_1,0011}(\theta)) = (0 \ 0)^T,
\end{cases}
\end{align*}
\]  
\quad (C.5)

\[
\begin{align*}
&n = 2n_1, \quad \begin{cases}
&z_1^2 : \quad \begin{cases}
&\hat{h}_{2n_1,2000}(\theta) - 2i\omega_1c h_{2n_1,2000}(\theta) = (0 \ 0)^T, \\
&\hat{h}_{2n_1,2000}(0) - L_0(\hat{h}_{2n_1,2000}(\theta)) = \frac{1}{\sqrt{2\ell^n}} \tilde{A}_{2000}, \\
&\hat{h}_{2n_1,1100}(\theta) = (0 \ 0)^T, \\
&\hat{L}_0(\hat{h}_{2n_1,1100}(\theta)) = \frac{1}{\sqrt{2\ell^n}} \tilde{A}_{1100}, \\
&\hat{h}_{2n_1,0011}(\theta) = (0 \ 0)^T, \\
&\hat{L}_0(\hat{h}_{2n_1,0011}(\theta)) = (0 \ 0)^T,
\end{cases} \\
&z_2^2 : \quad \begin{cases}
&\hat{h}_{2n_1,2000}(\theta) - 2i\omega_1c h_{2n_1,2000}(\theta) = \frac{1}{\sqrt{2\ell^n}} \Phi_{n_2}(\theta) \Psi_{n_2}(0) \tilde{A}_{2000}, \\
&\hat{h}_{2n_1,2000}(0) - L_0(\hat{h}_{2n_1,2000}(\theta)) = \frac{1}{\sqrt{2\ell^n}} \tilde{A}_{2000}, \\
&\hat{h}_{2n_1,1100}(\theta) = \frac{1}{\sqrt{2\ell^n}} \Phi_{n_2}(\theta) \Psi_{n_2}(0) \tilde{A}_{1100}, \\
&\hat{L}_0(\hat{h}_{2n_1,1100}(\theta)) = \frac{1}{\sqrt{2\ell^n}} \tilde{A}_{1100}, \\
&\hat{h}_{2n_1,0011}(\theta) = (0 \ 0)^T, \\
&\hat{L}_0(\hat{h}_{2n_1,0011}(\theta)) = (0 \ 0)^T,
\end{cases} \\
&z_3^2 : \quad \begin{cases}
&\hat{h}_{2n_1,0011}(\theta) = (0, 0)^T, \\
&\hat{L}_0(\hat{h}_{2n_1,0011}(\theta)) = (0 \ 0)^T,
\end{cases}
\end{align*}
\]  
\quad (C.6)

\[
\begin{align*}
&n = 2n_1, \quad \begin{cases}
&z_1^2 : \quad \begin{cases}
&\hat{h}_{2n_1,2000}(\theta) - 2i\omega_1c h_{2n_1,2000}(\theta) = \frac{1}{\sqrt{2\ell^n}} \Phi_{n_2}(\theta) \Psi_{n_2}(0) \tilde{A}_{2000}, \\
&\hat{h}_{2n_1,2000}(0) - L_0(\hat{h}_{2n_1,2000}(\theta)) = \frac{1}{\sqrt{2\ell^n}} \tilde{A}_{2000}, \\
&\hat{h}_{2n_1,1100}(\theta) = \frac{1}{\sqrt{2\ell^n}} \Phi_{n_2}(\theta) \Psi_{n_2}(0) \tilde{A}_{1100}, \\
&\hat{L}_0(\hat{h}_{2n_1,1100}(\theta)) = \frac{1}{\sqrt{2\ell^n}} \tilde{A}_{1100}, \\
&\hat{h}_{2n_1,0011}(\theta) = (0 \ 0)^T, \\
&\hat{L}_0(\hat{h}_{2n_1,0011}(\theta)) = (0 \ 0)^T,
\end{cases} \\
&z_2^2 : \quad \begin{cases}
&\hat{h}_{2n_1,1100}(\theta) = \frac{1}{\sqrt{2\ell^n}} \Phi_{n_2}(\theta) \Psi_{n_2}(0) \tilde{A}_{1100}, \\
&\hat{h}_{2n_1,1100}(0) - L_0(\hat{h}_{2n_1,1100}(\theta)) = \frac{1}{\sqrt{2\ell^n}} \tilde{A}_{1100}, \\
&\hat{h}_{2n_1,0011}(\theta) = (0, 0)^T, \\
&\hat{L}_0(\hat{h}_{2n_1,0011}(\theta)) = (0 \ 0)^T,
\end{cases} \\
&z_3^2 : \quad \begin{cases}
&\hat{h}_{2n_1,0011}(\theta) = (0 \ 0)^T,
\end{cases}
\end{align*}
\]  
\quad (C.7)
\[
\begin{align*}
\text{\(n = 2n_2\),} & \quad \begin{cases} 
\dot{h}_{2n_2,0020}(\theta) - 2i\omega_{2c}h_{2n_2,0020}(\theta) = (0 \ 0)^T, \\
\dot{h}_{2n_2,0020}(0) - \mathcal{L}_0(h_{2n_2,0020}(\theta)) = \frac{1}{\sqrt{2\pi}} \tilde{A}_{0020}, \\
\dot{h}_{2n_2,1100}(\theta) = (0,0)^T, \\
\dot{h}_{2n_2,1100}(0) - \mathcal{L}_0(h_{2n_2,1100}(\theta)) = (0 \ 0)^T, \\
\dot{h}_{2n_2,0011}(\theta) = (0 \ 0)^T, \\
\dot{h}_{2n_2,0011}(0) - \mathcal{L}_0(h_{2n_2,0011}(\theta)) = \frac{1}{\sqrt{2\pi}} \tilde{A}_{0011}, 
\end{cases} \\
\text{(C.8)} \end{align*}
\]

\[
\begin{align*}
\text{\(n = n_1 + n_2\),} & \quad \begin{cases} 
\dot{h}_{n_1+n_2,1010}(\theta) - i(\omega_{1c} + \omega_{2c}) h_{n_1+n_2,1010}(\theta) = (0 \ 0)^T, \\
\dot{h}_{n_1+n_2,1010}(0) - \mathcal{L}_0(h_{n_1+n_2,1010}(\theta)) = \frac{1}{\sqrt{2\pi}} \tilde{A}_{1010}, \\
\dot{h}_{n_1+n_2,1001}(\theta) - i(\omega_{1c} - \omega_{2c}) h_{n_1+n_2,1001}(\theta) = (0 \ 0)^T, \\
\dot{h}_{n_1+n_2,1001}(0) - \mathcal{L}_0(h_{n_1+n_2,1001}(\theta)) = \frac{1}{\sqrt{2\pi}} \tilde{A}_{1001}, \\
\dot{h}_{n_1+n_2,0110}(\theta) - i(\omega_{2c} - \omega_{1c}) h_{n_1+n_2,0110}(\theta) = (0 \ 0)^T, \\
\dot{h}_{n_1+n_2,0110}(0) - \mathcal{L}_0(h_{n_1+n_2,0110}(\theta)) = \frac{1}{\sqrt{2\pi}} \tilde{A}_{0110}, 
\end{cases} \\
\text{(C.9)} \end{align*}
\]

\[
\begin{align*}
\text{\(n = n_2 - n_1\),} & \quad \begin{cases} 
\dot{h}_{n_2-n_1,1010}(\theta) - i(\omega_{1c} + \omega_{2c}) h_{n_2-n_1,1010}(\theta) = (0 \ 0)^T, \\
\dot{h}_{n_2-n_1,1010}(0) - \mathcal{L}_0(h_{n_2-n_1,1010}(\theta)) = \frac{1}{\sqrt{2\pi}} \tilde{A}_{1010}, \\
\dot{h}_{n_2-n_1,1001}(\theta) - i(\omega_{1c} - \omega_{2c}) h_{n_2-n_1,1001}(\theta) = (0 \ 0)^T, \\
\dot{h}_{n_2-n_1,1001}(0) - \mathcal{L}_0(h_{n_2-n_1,1001}(\theta)) = \frac{1}{\sqrt{2\pi}} \tilde{A}_{1001}, \\
\dot{h}_{n_2-n_1,0110}(\theta) - i(\omega_{2c} - \omega_{1c}) h_{n_2-n_1,0110}(\theta) = (0 \ 0)^T, \\
\dot{h}_{n_2-n_1,0110}(0) - \mathcal{L}_0(h_{n_2-n_1,0110}(\theta)) = \frac{1}{\sqrt{2\pi}} \tilde{A}_{0110}, 
\end{cases} \\
\text{(C.10)} \end{align*}
\]

\[
\begin{align*}
\text{\(n = n_2 - n_1\),} & \quad \begin{cases} 
\dot{h}_{n_2-n_1,1010}(\theta) - i(\omega_{1c} + \omega_{2c}) h_{n_2-n_1,1010}(\theta) = (0 \ 0)^T, \\
\dot{h}_{n_2-n_1,1010}(0) - \mathcal{L}_0(h_{n_2-n_1,1010}(\theta)) = \frac{1}{\sqrt{2\pi}} \Phi_{n_1}(\theta) \Psi_{n_1}(0) \tilde{A}_{1010}, \\
\dot{h}_{n_2-n_1,1001}(\theta) - i(\omega_{1c} - \omega_{2c}) h_{n_2-n_1,1001}(\theta) = (0 \ 0)^T, \\
\dot{h}_{n_2-n_1,1001}(0) - \mathcal{L}_0(h_{n_2-n_1,1001}(\theta)) = \frac{1}{\sqrt{2\pi}} \Phi_{n_1}(\theta) \Psi_{n_1}(0) \tilde{A}_{1001}, \\
\dot{h}_{n_2-n_1,0110}(\theta) - i(\omega_{2c} - \omega_{1c}) h_{n_2-n_1,0110}(\theta) = (0 \ 0)^T, \\
\dot{h}_{n_2-n_1,0110}(0) - \mathcal{L}_0(h_{n_2-n_1,0110}(\theta)) = \frac{1}{\sqrt{2\pi}} \Phi_{n_1}(\theta) \Psi_{n_1}(0) \tilde{A}_{0110}, 
\end{cases} \\
\text{(C.11)} \end{align*}
\]
Solving (C.5), (C.6), (C.7), (C.8), (C.9), (C.10) and (C.11), we obtain

\[
\begin{align*}
\begin{cases}
  h_{0,0000}(	heta) &= \frac{1}{\sqrt{2\pi}} \left( \mathcal{M}_0 (2i\omega_{1c}) \right)^{-1} A_{0000} e^{2i\omega_{1c}\theta}, \\
  h_{0,0020}(	heta) &= \frac{1}{\sqrt{2\pi}} \left( \mathcal{M}_0 (2i\omega_{2c}) \right)^{-1} A_{0020} e^{2i\omega_{2c}\theta}, \\
  h_{0,1100}(	heta) &= \frac{1}{\sqrt{2\pi}} \left( \mathcal{M}_0 (0) \right)^{-1} A_{1100}, \\
  h_{0,0011}(	heta) &= \frac{1}{\sqrt{2\pi}} \left( \mathcal{M}_0 (0) \right)^{-1} A_{0011}, \\
\end{cases}
\end{align*}
\]  

\[
\begin{align*}
\begin{cases}
  h_{2n_1,0000}(	heta) &= \frac{1}{\sqrt{2\pi}} \left( \mathcal{M}_{2n_1} (2i\omega_{1c}) \right)^{-1} \tilde{A}_{0000} e^{2i\omega_{1c}\theta}, \\
  h_{2n_1,1100}(	heta) &= \frac{1}{\sqrt{2\pi}} \left( \mathcal{M}_{2n_1} (0) \right)^{-1} \tilde{A}_{1100}, \\
  h_{2n_1,0011}(	heta) &= (0,0)^T, \\
\end{cases}
\end{align*}
\]  

\[
\begin{align*}
\begin{cases}
  h_{2n_2,0020}(	heta) &= \frac{1}{\sqrt{2\pi}} \left( \mathcal{M}_{2n_2} (2i\omega_{2c}) \right)^{-1} \tilde{A}_{0020} e^{2i\omega_{2c}\theta}, \\
  h_{2n_2,1100}(	heta) &= (0,0)^T, \\
  h_{2n_2,0011}(	heta) &= \frac{1}{\sqrt{2\pi}} \left( \mathcal{M}_{2n_2} (0) \right)^{-1} \tilde{A}_{0011}, \\
\end{cases}
\end{align*}
\]  

\[
\begin{align*}
\begin{cases}
  h_{n_1+n_2,1010}(	heta) &= \frac{1}{\sqrt{2\pi}} \left( \mathcal{M}_{n_1+n_2} (i(\omega_{1c} + \omega_{2c})) \right)^{-1} \tilde{A}_{1010} e^{i(\omega_{1c} + \omega_{2c})\theta}, \\
  h_{n_1+n_2,1001}(	heta) &= \frac{1}{\sqrt{2\pi}} \left( \mathcal{M}_{n_1+n_2} (i(\omega_{1c} - \omega_{2c})) \right)^{-1} \tilde{A}_{1001} e^{i(\omega_{1c} - \omega_{2c})\theta}, \\
  h_{n_1+n_2,0110}(	heta) &= \frac{1}{\sqrt{2\pi}} \left( \mathcal{M}_{n_1+n_2} (i(\omega_{2c} - \omega_{1c})) \right)^{-1} \tilde{A}_{0110} e^{i(\omega_{2c} - \omega_{1c})\theta}, \\
\end{cases}
\end{align*}
\]  

\[
\begin{align*}
\begin{cases}
  h_{2n_1,0000}(	heta) &= \frac{1}{\sqrt{2\pi}} \left( \mathcal{M}_{2n_1} (2i\omega_{1c}) \right)^{-1} \left( A_{2000} - C_1 \mathcal{M}_{2n_1} (i\omega_{2c}) \phi_{n_2}(0) ight) e^{2i\omega_{1c}\theta} \\
  &\quad - C_2 \mathcal{M}_{2n_1} (-i\omega_{2c}) \overline{\phi}_{n_2}(0) \right) e^{2i\omega_{1c}\theta}, \\
  &\quad + \frac{1}{\sqrt{2\pi}} C_1 \phi_{n_2}(\theta) + \frac{1}{\sqrt{2\pi}} C_2 \overline{\phi}_{n_2}(\theta), \\
  h_{2n_1,1100}(	heta) &= (0,0)^T, \\
\end{cases}
\end{align*}
\]  

\[
\begin{align*}
\begin{cases}
  h_{2n_1,0000}(	heta) &= \frac{1}{\sqrt{2\pi}} \left( \mathcal{M}_{2n_1} (0) \right)^{-1} \left( \tilde{A}_{1100} - C_3 \mathcal{M}_{2n_1} (i\omega_{2c}) \phi_{n_2}(0) ight) \\
  &\quad - C_4 \mathcal{M}_{2n_1} (-i\omega_{2c}) \overline{\phi}_{n_2}(0) \right) + \frac{1}{\sqrt{2\pi}} C_3 \phi_{n_2}(\theta) + \frac{1}{\sqrt{2\pi}} C_4 \overline{\phi}_{n_2}(\theta), \\
  h_{2n_1,0011}(	heta) &= (0,0)^T, \\
\end{cases}
\end{align*}
\]

where

\[
\begin{align*}
C_1 &= \frac{1}{i(\omega_{2c} - 2\omega_{1c})} \psi_{n_2}(0) \tilde{A}_{2000}, \\
C_2 &= -\frac{1}{i(\omega_{2c} + 2\omega_{1c})} \overline{\psi}_{n_2}(0) \tilde{A}_{2000}, \\
C_3 &= \frac{1}{i\omega_{2c}} \psi_{n_2}(0) \tilde{A}_{1100}, \\
C_4 &= -\frac{1}{i\omega_{2c}} \overline{\psi}_{n_2}(0) \tilde{A}_{1100}, \\
\end{align*}
\]
\[
\begin{align*}
\begin{cases}
    h_{n_2-n_1,1010}(\theta) = & \frac{1}{\sqrt{2\pi}} \left( \tilde{M}_{n_2-n_1} (i(\omega_{1c} + \omega_{2c})) \right)^{-1} \tilde{A}_{1010} e^{i(\omega_{1c} + \omega_{2c})\theta}, \\
    h_{n_2-n_1,1001}(\theta) = & \frac{1}{\sqrt{2\pi}} \left( \tilde{M}_{n_2-n_1} (i(\omega_{1c} - \omega_{2c})) \right)^{-1} \tilde{A}_{1001} e^{i(\omega_{1c} - \omega_{2c})\theta}, \text{ for } n_2 \neq 2n_1, \\
    h_{n_2-n_1,0110}(\theta) = & \frac{1}{\sqrt{2\pi}} \left( \tilde{M}_{n_2-n_1} (i(\omega_{2c} - \omega_{1c})) \right)^{-1} \tilde{A}_{0110} e^{i(\omega_{2c} - \omega_{1c})\theta},
\end{cases}
\end{align*}
\]
and for \( n_2 = 2n_1 \), we have
\[
\begin{align*}
\begin{cases}
    h_{n_2-n_1,1010}(\theta) = & \frac{1}{\sqrt{2\pi}} \left( \tilde{M}_{n_2-n_1} (i(\omega_{1c} + \omega_{2c})) \right)^{-1} \left( \tilde{A}_{1010} - C_5 \tilde{M}_{n_2-n_1} (i\omega_{1c}) \phi_{n_1}(0) \right) \\
    & - C_6 \tilde{M}_{n_2-n_1} (-i\omega_{1c}) \phi_{n_1}(0) \right) e^{i(\omega_{1c} + \omega_{2c})\theta} \\
    & + \frac{1}{\sqrt{2\pi}} C_5 \phi_{n_1}(\theta) + \frac{1}{\sqrt{2\pi}} C_6 \tilde{\phi}_{n_1}(\theta), \\
    h_{n_2-n_1,1001}(\theta) = & \frac{1}{\sqrt{2\pi}} \left( \tilde{M}_{n_2-n_1} (i(\omega_{1c} - \omega_{2c})) \right)^{-1} \left( \tilde{A}_{1001} - C_7 \tilde{M}_{n_2-n_1} (i\omega_{1c}) \phi_{n_1}(0) \right) \\
    & - C_8 \tilde{M}_{n_2-n_1} (-i\omega_{1c}) \phi_{n_1}(0) \right) e^{i(\omega_{1c} - \omega_{2c})\theta} \\
    & + \frac{1}{\sqrt{2\pi}} C_7 \phi_{n_1}(\theta) + \frac{1}{\sqrt{2\pi}} C_8 \tilde{\phi}_{n_1}(\theta), \\
    h_{n_2-n_1,0110}(\theta) = & \frac{1}{\sqrt{2\pi}} \left( \tilde{M}_{n_2-n_1} (i(\omega_{2c} - \omega_{1c})) \right)^{-1} \left( \tilde{A}_{0110} - C_9 \tilde{M}_{n_2-n_1} (i\omega_{1c}) \phi_{n_1}(0) \right) \\
    & - C_{10} \tilde{M}_{n_2-n_1} (-i\omega_{1c}) \phi_{n_1}(0) \right) e^{i(\omega_{2c} - \omega_{1c})\theta} \\
    & + \frac{1}{\sqrt{2\pi}} C_9 \phi_{n_1}(\theta) + \frac{1}{\sqrt{2\pi}} C_{10} \tilde{\phi}_{n_1}(\theta),
\end{cases}
\end{align*}
\]
where
\[
\begin{align*}
    C_5 &= -\frac{1}{\omega_{2c}} \psi_{n_1}^T(0) \tilde{A}_{1010}, & C_6 &= -\frac{1}{i(2\omega_{1c} + \omega_{2c})} \psi_{n_1}^T(0) \tilde{A}_{1010}, \\
    C_7 &= \frac{1}{\omega_{2c}} \psi_{n_1}^T(0) \tilde{A}_{1001}, & C_8 &= -\frac{1}{i(2\omega_{1c} - \omega_{2c})} \psi_{n_1}^T(0) \tilde{A}_{1001}, \\
    C_9 &= \frac{1}{i(2\omega_{1c} - \omega_{2c})} \psi_{n_1}^T(0) \tilde{A}_{0110}, & C_{10} &= -\frac{1}{\omega_{2c}} \psi_{n_1}^T(0) \tilde{A}_{0110}.
\end{align*}
\]
C.2 Case 2: $n_1 = n_2$.

By (2.15), (2.41), (2.42) and (2.43), for $n_2 = n_1$, we have

$$
\begin{align*}
&\begin{bmatrix}
  f_2^0(z, 0, 0, \beta_n^{(1)}) \\
  f_2^0(z, 0, 0, \beta_n^{(2)})
\end{bmatrix} \\
= &\begin{cases}
\frac{1}{\sqrt{\pi}} X_0(\theta) \left( A_{2000} z_2^2 + A_{0200} z_2^2 + A_{0020} z_2^2 + A_{0002} z_4^2 \\
+ A_{1100} z_1 z_2 + A_{0011} z_3 z_4 + A_{1010} z_1 z_3 + A_{1100} z_1 z_4 \
+ \tilde{A}_{0110} z_2 z_3 + \tilde{A}_{0101} z_2 z_4 \\
\right), \\
&\left\{ \begin{array}{l}
n = 0, \\
n = 2n_1, \\
\end{array} \right.
\end{cases}
\end{align*}
$$

(C.16)

where $\tilde{A}_{j_1j_2j_3j_4}$ and $\tilde{A}_{j_1j_2j_3j_4}$ are defined by (2.44), (2.45), (C.3) and (C.4), and $\tilde{A}_{j_1j_2j_3j_4}$ is defined by the following

$$
\begin{align*}
\tilde{A}_{j_1j_2j_3j_4} &= A_{j_1j_2j_3j_4} + \frac{n_1^2}{\pi^2} \left( A_{j_1j_2j_3j_4}^{(d,1)} - A_{j_1j_2j_3j_4}^{(d,2)} - A_{j_1j_2j_3j_4}^{(d,3)} \right), \\
j_1, j_2, j_3, j_4 &= 0, 1, \quad j_1 + j_2 = 1, \quad j_3 + j_4 = 1.
\end{align*}
$$

From (C.1) and (C.16), we have

$$
\begin{align*}
&h_{0,2000}(\theta) = \frac{1}{\sqrt{\pi}} \left( \tilde{M}_0(2i\omega_1c) \right)^{-1} A_{2000} e^{2i\omega_1 c \theta}, \\
h_{0,0020}(\theta) = \frac{1}{\sqrt{\pi}} \left( \tilde{M}_0(2i\omega_2 c) \right)^{-1} A_{0020} e^{2i\omega_2 c \theta}, \\
h_{0,1100}(\theta) = \frac{1}{\sqrt{\pi}} \left( \tilde{M}_0(0) \right)^{-1} A_{1100}, \\
h_{0,0011}(\theta) = \frac{1}{\sqrt{\pi}} \left( \tilde{M}_0(0) \right)^{-1} A_{0011}, \\
h_{0,1010}(\theta) = \frac{1}{\sqrt{\pi}} \left( \tilde{M}_0(i(\omega_1 c + \omega_2 c)) \right)^{-1} A_{1010} e^{i(\omega_1 c + \omega_2 c) \theta}, \\
h_{0,1001}(\theta) = \frac{1}{\sqrt{\pi}} \left( \tilde{M}_0(i(\omega_1 c - \omega_2 c)) \right)^{-1} A_{1001} e^{i(\omega_1 c - \omega_2 c) \theta}, \\
h_{0,0110}(\theta) = \frac{1}{\sqrt{\pi}} \left( \tilde{M}_0(i(\omega_2 c - \omega_1 c)) \right)^{-1} A_{0110} e^{i(\omega_2 c - \omega_1 c) \theta}.
\end{align*}
$$

(C.17)
and

\[
\begin{align*}
  h_{2n_1,0000}(\theta) &= \frac{1}{\sqrt{2\pi}} \left( \tilde{M}_{2n_1}(2i\omega_1c) \right)^{-1} \tilde{A}_{0000}e^{2i\omega_1c\theta}, \\
  h_{2n_1,0020}(\theta) &= \frac{1}{\sqrt{2\pi}} \left( \tilde{M}_{2n_1}(2i\omega_2c) \right)^{-1} \tilde{A}_{0020}e^{2i\omega_2c\theta}, \\
  h_{2n_1,1100}(\theta) &= \frac{1}{\sqrt{2\pi}} \left( \tilde{M}_{2n_1}(0) \right)^{-1} \tilde{A}_{1100}, \\
  h_{2n_1,0011}(\theta) &= \frac{1}{\sqrt{2\pi}} \left( \tilde{M}_{2n_1}(0) \right)^{-1} \tilde{A}_{0011}, \\
  h_{2n_1,1010}(\theta) &= \frac{1}{\sqrt{2\pi}} \left( \tilde{M}_{2n_1}(i(\omega_1c + \omega_2c)) \right)^{-1} \tilde{A}_{1010}e^{i(\omega_1c + \omega_2c)\theta}, \\
  h_{2n_1,1001}(\theta) &= \frac{1}{\sqrt{2\pi}} \left( \tilde{M}_{2n_1}(i(\omega_1c - \omega_2c)) \right)^{-1} \tilde{A}_{1001}e^{i(\omega_1c - \omega_2c)\theta}, \\
  h_{2n_1,0110}(\theta) &= \frac{1}{\sqrt{2\pi}} \left( \tilde{M}_{2n_1}(i(\omega_2c - \omega_1c)) \right)^{-1} \tilde{A}_{0110}e^{i(\omega_2c - \omega_1c)\theta}.
\end{align*}
\]

(C.18)

Notice that for \( n_1 = n_2 \), \( 2n_1 = 2n_2 = n_1 + n_2 \) and \( n_2 - n_1 = 0 \). Thus, substituting (C.17) and (C.18) into (2.51), (2.53) and (2.54), we can calculate \( E_{ij} \) and \( E^d_{ij} \).

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