Topologically Stratified Energy Minimizers in a Product Abelian Field Theory

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Abstract

The recently developed product Abelian gauge field theory by Tong and Wong hosting magnetic impurities is reformulated into an extended model that allows the coexistence of vortices and anti-vortices. The two Abelian gauge fields in the model induce two species of magnetic vortex-lines resulting from $N_s$ vortices and $P_s$ anti-vortices ($s = 1, 2$) realized as the zeros and poles of two complex-valued Higgs fields, respectively. An existence theorem is established for the governing equations over a compact Riemann surface $S$ which states that a solution with prescribed $N_1, N_2$ vortices and $P_1, P_2$ anti-vortices of two designated species exists if and only if the inequalities

$$|N_1 + N_2 - (P_1 + P_2)| < \frac{|S|}{\pi}, \quad |N_1 + 2N_2 - (P_1 + 2P_2)| < \frac{|S|}{\pi},$$

hold simultaneously, which give bounds for the ‘differences’ of the vortex and anti-vortex numbers in terms of the total surface area of $S$. The minimum energy of these solutions is shown to assume the explicit value

$$E = 4\pi(N_1 + N_2 + P_1 + P_2),$$

given in terms of several topological invariants, measuring the total tension of the vortex-lines.

Key words. Gauge field theory, magnetic vortices, impurities, complex line bundles, connections, topological invariants, nonlinear elliptic equations, Leray–Schauder fixed-point theorem.
1 Introduction

It is well known that the simplest and most important quantum field theory model is the Abelian Higgs model [15,22] which embodies an electromagnetic gauge field and spontaneously broken symmetry and allows mass generation through the Higgs mechanism. In the temporal gauge, its static limit gives rise to the classical Ginzburg–Landau theory for superconductivity [14] so that in its two dimensional setting mixed-state configurations known as the Abrikosov vortices [1] can be rigorously constructed [6,15,27,28,32]. Inspired by the gauged sigma model of Schroers [23,24], the classical Abelian Higgs model is extended in [36,37] to allow the coexistence of vortices and anti-vortices. This extended model is also shown to generate cosmic strings and anti-strings when gravitation is switched on by the Einstein equations which give rise to curvature and mass concentrations essential for matter accretion in the early universe [16,17,30,31,34,35]. In order to understand the topological contents of such an extended Abelian Higgs model, a reformulation of it is carried in [25] in the context of a complex line bundle over a compact Riemann surface $S$ as in [6,12,19,20]. In a sharp and interesting contrast with the Abelian Higgs model where vortices are topologically characterized by the first Chern class, the vortices and anti-vortices in the extended Abelian Higgs model [36,37] are characterized jointly and elegantly [25] by the first Chern class of the line bundle and the Thom class [26] of the associated dual bundle. In the former case, there are only finitely many minimum energy values which can be attained due to the fact that the total number of vortices is confined by the total area $|S|$ of the two-surface $S$ where vortices reside. In the latter case, however, the confinement is made instead to the difference of the numbers of vortices and anti-vortices, but the minimum energy is proportional to the sum of these numbers. Hence the possible minimum energy values becomes an explicitly determined infinite sequence as in the situation of vortices over a non-compact surface in the classical Abelian Higgs theory [15,27,28].

In a recent interesting work of Tong and Wong [29], a product Abelian gauge field theory is formulated to include magnetic impurities in the form of an extra gauge-matter sector. This gauge-matter sector is not treated as a background source but as a fully coupled sector. In other words, this is a product Abelian gauge field theory with two complex Higgs fields. It is shown in [29] that, like in the classical Abelian Higgs model, the new product model allows a BPS (after Bogomol’nyi [5] and Prasad–Sommerfield [21]) reduction, hence a construction of magnetic vortices as in [15]. The present paper aims to enrich our understanding of Abelian (magnetic) vortices by achieving two goals. The first is to extend the product Abelian gauge field theory of Tong and Wong [29] using the ideas in [23,24,36,37] into a new product field theory that allows the coexistence of two species of vortices and anti-vortices. The second is to establish an existence theorem for such vortices of beautiful topological characteristics. For clarity and simplicity, the underlying domain for the vortices to live is assumed to be a compact Riemann surface, as in [25].

An outline of the rest of the paper is as follows. In Section 2, we present our extended product Abelian gauge field theory (in its static limit), describe in detail its field-theoretical properties, and derive the system of the BPS equations. We then state our main existence theorem for coexisting vortices and anti-vortices of two species. In Section 3, we convert the BPS equations into a system of nonlinear elliptic equations, state the main existence theorem in terms of these equations, and carry out some preliminary discussion. In Section 4, we prove the existence theorem by using a Leray–Schauder fixed-point theorem argument [13] under a necessary and sufficient condition. In Section 5, we explicitly compute the (minimum) energy of a vortex and anti-vortex solution and
show that such energy arises topologically and is proportional to the sum of vortex and anti-vortex numbers of two species. In Section 6, we make some concluding remarks.

2 Energy functional, BPS reduction, and existence theorem

Let \( L \) be complex Hermitian line bundle over a Riemann surface \( S \). Use \( q, p \) to denote two sections \( L \to S \) and \( Dq, Dp \) the connections induced from the real-valued connection 1-forms \( \hat{A}, \tilde{A} \), respectively, so that

\[
Dq = dq - i(\hat{A} - \tilde{A})q, \quad Dp = dp - i\hat{A}p. \tag{2.1}
\]

Using \( * \) to denote the usual Hodge dual operating on differential forms, the energy density of the Tong–Wong model \([29]\) for a product Abelian Higgs theory implementing magnetic impurities may be rewritten as

\[
E = \frac{1}{2} * (\hat{F} \wedge \ast \hat{F}) + \frac{1}{2} * (\tilde{F} \wedge \ast \tilde{F}) + \frac{4}{(1 + |q|^2)^2} * (Dq \wedge \ast Dq) + \frac{4}{(1 + |p|^2)^2} * (Dp \wedge \ast Dp)
+ \frac{1}{2} (1 - |q|^2)^2 + \frac{1}{2} (1 - |q|^2 + |p|^2 - 1)^2, \tag{2.2}
\]

where \( \hat{F} = d\hat{A}, \tilde{F} = d\tilde{A} \) are curvature 2-forms, which recovers the classical Ginzburg–Landau model \([14]\) when impurities are switched off by setting

\[
\hat{A} = 0, \quad p = 0. \tag{2.3}
\]

Following \([36, 37]\), we show that we may extend the Tong–Wong model \([29]\) to accommodate vortices and anti-vortices by considering the modified energy density

\[
E = \frac{1}{2} * (\hat{F} \wedge \ast \hat{F}) + \frac{1}{2} * (\tilde{F} \wedge \ast \tilde{F}) + \frac{4}{(1 + |q|^2)^2} * (Dq \wedge \ast Dq) + \frac{4}{(1 + |p|^2)^2} * (Dp \wedge \ast Dp)
+ 2 \left( \frac{1 - |q|^2}{1 + |q|^2} \right)^2 + 2 \left( \frac{1 - |q|^2}{1 + |q|^2} + \frac{|p|^2 - 1}{1 + |p|^2} \right)^2. \tag{2.4}
\]

It is interesting to observe that (2.2) is recovered from (2.4) when taking the limit \( |q| \to 1, |p| \to 1 \) in the denominators \( 1 + |q|^2 \) and \( 1 + |p|^2 \) of (2.4). The Euler–Lagrange equations of the energy density are found to be

\[
D * \left( \frac{Dq}{(1 + |q|^2)^2} \right) = \frac{1}{(1 + |q|^2)^3} (Dq \wedge \ast Dq) + 2 * \left( \frac{1 - |q|^2}{1 + |q|^2} \right) q
+ 2 * \left( \frac{1 - |q|^2}{1 + |q|^2} + \frac{|p|^2 - 1}{1 + |p|^2} \right) \left( \frac{1 - |q|^2}{1 + |q|^2} \right) \tag{2.5}
\]

\[
d \ast \hat{F} = 4i \left( \overline{\partial Dq} - q \overline{\partial Dq} \right) \left( \frac{1}{1 + |q|^2} \right), \tag{2.6}
\]

\[
D * \left( \frac{Dp}{(1 + |p|^2)^2} \right) = \frac{1}{(1 + |p|^2)^3} (Dp \wedge \ast Dp)
+ 2 * \left( \frac{1 - |q|^2}{1 + |q|^2} + \frac{|p|^2 - 1}{1 + |p|^2} \right) \left( \frac{|p|^2 - 1}{1 + |p|^2} \right) \tag{2.7}
\]

\[
d \ast \tilde{F} = -4i \left( \overline{\partial Dq} - q \overline{\partial Dq} \right) \left( \frac{1}{1 + |q|^2} \right) + 4i \left( \overline{\partial Dp} - p \overline{\partial Dp} \right) \left( \frac{1}{1 + |p|^2} \right), \tag{2.8}
\]
which appear rather complicated and intractable. In order to obtain interesting solutions of these equations, we follow \[\text{[29, 36]}\] to pursue a BPS reduction.

Introduce the current densities

\[
J(q) = \frac{i}{1 + |q|^2} (q \overline{Dq} - \overline{q} Dq), \quad J(p) = \frac{i}{1 + |p|^2} (p \overline{Dp} - \overline{p} Dp).
\] (2.9)

Then we have

\[
K(q) = i \partial J(q) = \frac{-2|q|^2}{1 + |q|^2} (\hat{F} - \bar{F}) + \left( \frac{Dq \wedge \overline{Dq} - *Dq \wedge *Dq}{(1 + |q|^2)^2} \right),
\]

\[
K(p) = i \partial J(p) = -\frac{2|p|^2}{1 + |p|^2} \bar{F} + i \left( \frac{Dp \wedge \overline{Dp} - *Dp \wedge *Dp}{(1 + |p|^2)^2} \right).
\] (2.10)

Note also that there holds the identity

\[
Dq \wedge *Dq + (*Dq) \wedge Dq = (Dq \pm i * Dq) \wedge (*Dq \pm i * Dq) = \pm i (Dq \wedge Dq - (*Dq) \wedge (*Dq)).
\] (2.12)

So, with \(|Dq|^2 = *Dq \wedge Dq\), etc, we arrive at the decomposition

\[
\mathcal{E} = \frac{1}{2} \left| \hat{F} \mp 2 \left( \frac{1 - |q|^2}{1 + |q|^2} \right) \right|^2 + \frac{1}{2} \left| \bar{F} \mp \left( 2 \left( \frac{1 - |q|^2}{1 + |q|^2} \right) + 2 \left( \frac{|p|^2 - 1}{1 + |p|^2} \right) \right) \right|^2 \\
+ \frac{2}{(1 + |q|^2)^2} |Dq \pm i * Dq|^2 + \frac{2}{(1 + |p|^2)^2} |Dp \pm i * Dp|^2 \\
\pm 2 \left( \hat{F} - \bar{F} \right) \pm 2 \star K(q) \pm 2 \star \bar{F} \pm 2 \star K(p).
\] (2.13)

The quantities \(\frac{1}{2\pi} \int (\hat{F} - \bar{F})\) and \(\frac{1}{4\pi} \int \bar{F}\) are the first Chern numbers induced from the connections \(\hat{A} - \bar{A}\) and \(\hat{A}\) over \(L \rightarrow S\) and \(\frac{1}{4\pi} \int K(q)\) and \(\frac{1}{4\pi} \int K(p)\) the Thom classes over \(L^* \rightarrow S\), respectively \[\text{[25]}\]. Thus, the quantity

\[
\tau = 2\hat{F} + 2K(q) + 2K(p)
\] (2.14)

is a topological density which leads to the topological energy lower bound

\[
E = \int_M \mathcal{E} \star 1 \geq \left| \int_M \tau \right|,
\] (2.15)

measuring the tension \[\text{[7, 11]}\] of the vortex-lines, so that the lower bound is saturated when the quartet \((q, p, \hat{A}, \bar{A})\) satisfies the equations

\[
Dq \pm i * Dq = 0,
\] (2.16)

\[
Dp \pm i * Dp = 0,
\] (2.17)

\[
\hat{F} = \pm 2 \left( \frac{1 - |q|^2}{1 + |q|^2} \right),
\] (2.18)

\[
\bar{F} = \mp \left( 2 \left( \frac{1 - |q|^2}{1 + |q|^2} \right) + 2 \left( \frac{|p|^2 - 1}{1 + |p|^2} \right) \right).
\] (2.19)
It may directly be checked that (2.16)–(2.19) imply (2.5)–(2.8). In other words, (2.16)–(2.19) may be regarded as a reduction of the system of equations (2.5)–(2.8). These reduced first-order equations are often referred to as the BPS equations after Bogomol’nyi [5] and Prasad–Sommerfield [21] who pioneered the idea of such reduction for the classical Yang–Mills–Higgs equations. When the upper sign is taken, the system is said to be self-dual; the lower, anti-self-dual. It may also be checked that the self-dual and anti-self-dual cases are related to each other through the transformation

\[ \hat{A} \rightarrow -\hat{A}, \quad \tilde{A} \rightarrow -\tilde{A}, \quad q \rightarrow \overline{q}, \quad p \rightarrow \overline{p}. \]  

(2.20)

Hence, in the sequel, we will only consider the self-dual situation.

From (2.16) and (2.17), we know [15, 36, 37] that the zeros and poles of the sections \( q, p \) are isolated and possess integer multiplicities. For simplicity, we may denote the sets of zeros and poles of \( q, p \) by

\[ Z(q) = \{ z_{1,1}', \ldots, z_{1,N_1}' \}, \quad \mathcal{P}(q) = \{ z_{1,1}'', \ldots, z_{1,P_1}' \}, \]  

(2.21)

\[ Z(p) = \{ z_{2,1}', \ldots, z_{2,N_2}' \}, \quad \mathcal{P}(p) = \{ z_{2,1}'', \ldots, z_{2,P_2}' \}, \]  

(2.22)

respectively, so that the associated multiplicities of the zeros and poles are naturally counted by their repeated appearances in the above collections of points.

If we interpret \( *\hat{F} \) as a magnetic or vorticity field, (2.18) indicates that it attains its maximum \( *\hat{F} = 2 \) at the zeros and minimum \( *\hat{F} = -2 \) at the poles of \( q \). Thus, the zeros and poles of \( q \) may be viewed as centers of vortices and anti-vortices. In other words, we may identify the zeros and poles of \( q \) as the locations of vortices and anti-vortices generated from the connection 1-form \( \hat{A} \). Similarly, the zeros and poles of \( p \) may be interpreted as vortices and anti-vortices generated from the connection 1-form \( \tilde{A} + \hat{A} \). Therefore, in what follows, the zeros and poles of \( q, p \) are interchangeably and generically referred to as the vortices and anti-vortices of a solution configuration \((\hat{A}, \tilde{A}, q, p)\).

Here is our main existence theorem.

**Theorem 2.1** Consider the BPS system consisting of equations (2.16)–(2.19) of the energy density (2.4) formulated over a complex Hermitian line bundle \( L \) over a compact Riemann surface \( S \) with canonical total area \( |S| \) governing two connection 1-forms \( \hat{A}, \tilde{A} \) and two cross sections \( q, p \) comprising a reduction of the Euler–Lagrange equations (2.5)–(2.8). For the prescribed sets of zeros and poles for the fields \( q, p \) given respectively in (2.21) and (2.22), the coupled equations (2.16)–(2.19) have a solution to realize these sets of zeros and poles, if and only if the inequalities

\[ |N_1 + N_2 - (P_1 + P_2)| < \frac{|S|}{\pi}, \]  

(2.23)

\[ |N_1 + 2N_2 - (P_1 + 2P_2)| < \frac{|S|}{\pi}, \]  

(2.24)

regarding the total numbers of zeros and poles are fulfilled simultaneously. Moreover, such a solution carries a minimum energy of the form

\[ E = 4\pi(N_1 + N_2 + P_1 + P_2), \]  

(2.25)

which is seen to be stratified topologically by the Chern and Thom classes of the line bundle \( L \) and its dual respectively. In particular, in terms of energy, zeros (vortices) and poles (anti-vortices) of \( q, p \) contribute equally.
It is interesting to note that the inequalities (2.23) and (2.24) imply that the differences of vortices and anti-vortices must stay within suitable ranges to ensure the existence of a solution:

\[
\left| N_1 - P_1 \right| < \frac{3|S|}{\pi}, \quad (2.26)
\]

\[
\left| N_2 - P_2 \right| < \frac{2|S|}{\pi}. \quad (2.27)
\]

However, it may be checked that the conditions (2.26) and (2.27) do not lead to (2.23) and (2.24). The latter may be called the difference of total numbers of vortices and anti-vortices and the difference of ‘weighted total numbers’ of vortices and anti-vortices.

### 3 Governing elliptic equations and basic properties

To proceed, we set

\[
u = \ln |q|^2, \quad v = \ln |p|^2,
\]

in (2.16)–(2.19), which leads us via [15,36,37] to the following equivalent governing elliptic equations

\[
\Delta u = \frac{8(e^u - 1)}{e^u + 1} - \frac{4(e^v - 1)}{e^v + 1} + 4\pi \sum_{z \in Z(q)} \delta_z - 4\pi \sum_{z \in \mathcal{P}(q)} \delta_z, \quad (3.2)
\]

\[
\Delta v = -\frac{4(e^u - 1)}{e^u + 1} + \frac{4(e^v - 1)}{e^v + 1} + 4\pi \sum_{z \in Z(p)} \delta_z - 4\pi \sum_{z \in \mathcal{P}(p)} \delta_z, \quad (3.3)
\]

where \( \Delta \) is the Laplace–Beltrami operator on \((S,g)\) defined by

\[
\Delta u = \frac{1}{\sqrt{g}} \partial_j (g^{jk} \sqrt{g} \partial_k u), \quad (3.4)
\]

and \( \delta_z \) denotes the Dirac measure concentrated at the point \( z \in S \) with respect to the Riemannian metric \( g \) over \( S \).

In what follows, we use \( d\Omega_g \) to denote the canonical surface element and \( |S| \) the associated total area of the Riemann surface \((S,g)\).

Regarding the equivalently reduced equations (3.2) and (3.3) from (2.16)–(2.19), we have

**Theorem 3.2** The coupled equations (3.2) and (3.3) admit a solution \((u,v)\) with the prescribed sets \( Z(q), \mathcal{P}(q), Z(p), \mathcal{P}(p) \) in \( S \) specified in (2.21) and (2.22) if and only if the inequalities (2.24) and (2.27) are satisfied simultaneously. Moreover, for the solution to the equations (3.2) and (3.3) obtained above, there hold the quantized integrals

\[
\int_S \frac{1 - e^u}{1 + e^u} d\Omega_g = \pi (N_1 - P_1 + N_2 - P_2), \quad (3.5)
\]

\[
\int_S \frac{1 - e^v}{1 + e^v} d\Omega_g = \pi (N_1 - P_1 + 2(N_2 - P_2)). \quad (3.6)
\]

For convenience, we first need to take care of the Dirac distributions by subtracting suitable background functions. To do so, we let \( u_0^1, u_0^2, v_0^1, v_0^2 \) be the normalized solutions of the equations
that determine the source functions arising from the sets \( \mathcal{Z}(q), \mathcal{P}(q), \mathcal{Z}(p), \mathcal{P}(p) \), respectively. For instance, \( u_0^1 \) is the unique solution \( \ref{eq:3.12} \) to

\[
\Delta u_0^1 = -\frac{4\pi N_1}{|S|} + 4\pi \sum_{z \in \mathcal{Z}(q)} \delta_z, \quad \int_S u_0^1 \, d\Omega_g = 0. \tag{3.7}
\]

Set \( u = u_0^1 - u_0^2 + U, v = v_0^1 - v_0^2 + V \). Then we can rewrite \( \ref{eq:3.2} \) and \( \ref{eq:3.3} \) as

\[
\begin{align*}
\Delta U &= 8f(u_0^1, u_0^2, U) - 4f(v_0^1, v_0^2, V) + \frac{4\pi(N_1 - P_1)}{|S|}, \\
\Delta V &= -4f(u_0^1, u_0^2, U) + 4f(v_0^1, v_0^2, V) + \frac{4\pi(N_2 - P_2)}{|S|},
\end{align*} \tag{3.8}
\]

where and in what follows we use the notation

\[
f(s^1, s^2, t) \equiv \frac{e^{s^1 - s^2 + t} - 1}{e^{s^1 - s^2 + t} + 1} = \frac{e^{s^1 + t} - e^{s^2}}{e^{s^1 + t} + e^{s^2}}, \quad s^1, s^2, t \in \mathbb{R}. \tag{3.10}
\]

For fixed \( s^1, s^2 \in \mathbb{R} \), we have

\[
0 < \frac{d}{dt} f(s^1, s^2, t) = \frac{2e^{s^1 + t}e^{s^2}}{(e^{s^1 + t} + e^{s^2})^2} \leq \frac{1}{2}, \quad \forall t \in \mathbb{R}. \tag{3.11}
\]

We first show that the condition consisting of \( \ref{eq:2.23} \) and \( \ref{eq:2.24} \) is necessary for the existence of solutions for \( \ref{eq:3.8} - \ref{eq:3.9} \). In fact, integrating \( \ref{eq:3.8} - \ref{eq:3.9} \), we find

\[
\begin{align*}
\int_S f(u_0^1, u_0^2, U) \, d\Omega_g &= a|S|, \\
\int_S f(v_0^1, v_0^2, V) \, d\Omega_g &= b|S|,
\end{align*} \tag{3.12}
\]

where \( a, b \) are constants defined by

\[
\begin{align*}
a &\equiv -\frac{\pi}{|S|} (N_1 - P_1 + N_2 - P_2), \\
b &\equiv -\frac{\pi}{|S|} (N_1 - P_1 + 2(N_2 - P_2)).
\end{align*} \tag{3.14}
\]

From \( \ref{eq:3.12} - \ref{eq:3.13} \) we see that the quantized integrals \( \ref{eq:3.5} - \ref{eq:3.6} \) hold. On the other hand, noting

\[
-1 < f(s^1, s^2, t) < 1 \quad \text{for any} \quad s^1, s^2, t \in \mathbb{R}, \tag{3.16}
\]

we arrive at

\[
|a| < 1 \quad \text{and} \quad |b| < 1, \tag{3.17}
\]

which is equivalent to \( \ref{eq:2.23} \) and \( \ref{eq:2.24} \). Thus the inequalities \( \ref{eq:2.23} \) and \( \ref{eq:2.24} \) are necessary for a solution to exist.
4 Proof of existence via a fixed-point argument

In this section, we prove that the condition comprised of (2.23) and (2.24) is also sufficient for
the existence of a solution of the coupled equations (3.2) and (3.3). We will extend a fixed-point

Theorem argument used in [38] when treating a single equa-

tion. We will extend a fixed-point

<ref>
[1]
</ref>

where $a, b$ are defined by (3.14)–(3.15).

We know that the space $W^{1,2}(S)$ can be decomposed as $W^{1,2}(S) = \hat{W}^{1,2}(S) \oplus \mathbb{R}$, where

\[
\hat{W}^{1,2}(S) = \left\{ w \in W^{1,2}(S) : \int_S w \, d\Omega_g = 0 \right\}
\]

is a closed subspace of $W^{1,2}(S)$.

To save notation, in the following we also use $W^{1,2}(S), \hat{W}^{1,2}(S)$ and $L^p(S)$ to denote the spaces
of vector-valued functions.

We begin with the following lemma.

**Lemma 4.1** For any $(U', V') \in \hat{W}^{1,2}(S)$, there exists a unique pair $(c_1(U'), c_2(V')) \in \mathbb{R}^2$ such that

\[
\int_S f(u_0^1, u_0^2, U' + c_1(U')) \, d\Omega_g = a|S|,
\]

\[
\int_S f(v_0^1, v_0^2, V' + c_2(V')) \, d\Omega_g = b|S|,
\]

where $a, b$ are defined by (3.14)–(3.15).

**Proof.** Under the condition consisting of (2.23) and (2.24), we easily see that

\[
-1 < a, b < 1.
\]

Noting the expression (3.10), for any $(U', V') \in \hat{W}^{1,2}(S)$, we have

\[
\int_S f(u_0^1, u_0^2, U' + t) \, d\Omega_g, \int_S f(v_0^1, v_0^2, V' + t) \, d\Omega_g \to |S| \quad \text{as} \quad t \to \infty
\]

and

\[
\int_S f(u_0^1, u_0^2, U' + t) \, d\Omega_g, \int_S f(v_0^1, v_0^2, V' + t) \, d\Omega_g \to -|S| \quad \text{as} \quad t \to -\infty.
\]

Then, for any $(U', V') \in \hat{W}^{1,2}(S)$, we conclude from (4.6), (4.7) and (4.8) that there exists a
point $(c_1(U'), c_2(V')) \in \mathbb{R}^2$ such that (4.1) and (4.2) hold.

The uniqueness of $(c_1(U'), c_2(V'))$ follows from the strict monotonicity of $f(s_1, s_2, t)$ with respect to $t$ (see (3.11)).

**Lemma 4.2** For any $(U', V') \in \hat{W}^{1,2}(S)$, let $(c_1(U'), c_2(V'))$ be defined in Lemma 4.1. Then, the
mapping $(c_1(\cdot), c_2(\cdot)) : \hat{W}^{1,2}(S) \to \mathbb{R}^2$, is continuous with respect to the weak topology of $\hat{W}^{1,2}(S)$.
**Proof.** Take a weakly convergent sequence \( \{(U_k', V_k')\} \) in \( \mathring{W}^{1,2}(S) \) such that \( (U_k', V_k') \to (U_0', V_0') \) weakly in \( \mathring{W}^{1,2}(S) \). Then we see that

\[
(U_k', V_k') \to (U_0', V_0') \quad \text{strongly in} \quad L^p(S) \quad \text{for any} \quad p \geq 1,
\]

by the compact embedding \( W^{1,2}(S) \hookrightarrow L^p(S) \) (\( p \geq 1 \)). We aim to prove that \( (c_1(U_k'), c_2(V_k')) \to (c_1(U_0'), c_2(V_0')) \) as \( k \to \infty \).

Claim: The sequence \( \{(c_1(U_k'), c_2(V_k'))\} \) is bounded.

To show this claim we first prove that \( \{(c_1(U_k'), c_2(V_k'))\} \) is bounded from above. We argue by contradiction. Without loss of generality, assume \( c_1(U_k') \to \infty \) as \( k \to \infty \). Noting (4.9) and using the Egorov theorem, we see that for any \( \varepsilon > 0 \), there is a large constant \( K_\varepsilon > 0 \) and a subset \( S_\varepsilon \subset S \) such that

\[
|U_k'| \leq K_\varepsilon, \quad x \in S \setminus S_{\varepsilon}, \quad |S_{\varepsilon}| < \varepsilon, \quad \forall k.
\]

Then by (4.10) and (3.16) we have

\[
|a||S| = \left| \int_{S \setminus S_\varepsilon} f(u_0^1, u_0^2, U_k' + c_1(U_k'))d\omega_g + \int_{S_\varepsilon} f(u_0^1, u_0^2, U_k' + c_1(U_k'))d\omega_g \right| \\
\geq \left| \int_{S \setminus S_\varepsilon} f(u_0^1, u_0^2, U_k' + c_1(U_k'))d\omega_g \right| - \left| \int_{S_\varepsilon} f(u_0^1, u_0^2, U_k' + c_1(U_k'))d\omega_g \right| \\
\geq \int_{S \setminus S_\varepsilon} f(u_0^1, u_0^2, c_1(U_k') - K_\varepsilon)d\omega_g - \varepsilon.
\]

Hence taking \( k \to \infty \) in (4.11) we get

\[
|a||S| \geq |S \setminus S_\varepsilon| - \varepsilon \geq |S| - 2\varepsilon.
\]

Noting that \( \varepsilon \) is arbitrary, we obtain

\[
|a| \geq 1,
\]

which contradicts the condition (2.23) (\( |a| < 1 \)). Hence the sequence \( \{(c_1(U_k'), c_2(V_k'))\} \) is bounded from above.

Now we show that \( \{(c_1(U_k'), c_2(V_k'))\} \) is also bounded from below. In fact, we may suppose \( c_1(U_k') \to -\infty \) as \( k \to \infty \). Using (4.10) and (3.16), we have

\[
a|S| = \left| \int_{S \setminus S_\varepsilon} f(u_0^1, u_0^2, U_k' + c_1(U_k'))d\omega_g + \int_{S_\varepsilon} f(u_0^1, u_0^2, U_k' + c_1(U_k'))d\omega_g \right| \\
\leq \int_{S \setminus S_\varepsilon} f(u_0^1, u_0^2, c_1(U_k') + K_\varepsilon)d\omega_g + \varepsilon.
\]

Then letting \( k \to \infty \) in (4.12), we obtain

\[
a|S| \leq -|S \setminus S_\varepsilon| + \varepsilon \leq -|S| + 2\varepsilon,
\]

which implies \( a \leq -1 \) since \( \varepsilon > 0 \) is arbitrary. Hence we get a contradiction with the condition (2.23) again. So the sequence \( \{(c_1(U_k'), c_2(V_k'))\} \) is bounded from below. Therefore the claim follows.
By the claim above, up to a subsequence, we may assume that
\[
(c_1(U'_k), c_2(V'_k)) \to (c'_1, c'_2) \quad \text{as} \quad k \to \infty \quad \text{for some} \quad (c'_1, c'_2) \in \mathbb{R}^2. \tag{4.13}
\]

Then, using \((3.11)\), the Schwartz inequality, \((4.9)\) and \((4.13)\) we have
\[
\left| \int_S f(u^1_0, u^2_0, U'_k + c_1(U'_k)) \, d\Omega_g - \int_S f(u^1_0, u^2_0, c'_1) \, d\Omega_g \right|
\leq \frac{1}{2} \int_S | U'_k - U'_0 + c_1(U'_k) - c'_1 | \, d\Omega_g
\leq \frac{1}{2} \left( |S| \frac{1}{2} \| U'_k - U'_0 \|_2 + |S| | c_1(U'_k) - c'_1 | \right) \to 0 \quad \text{as} \quad k \to \infty, \tag{4.14}
\]
where \(\theta \in (0, 1)\). Noting \((4.14)\) and
\[
\int_S f(u^1_0, u^2_0, U'_k + c_1(U'_k)) \, d\Omega_g = a|S|, \tag{4.15}
\]
we have
\[
\int_S f(u^1_0, u^2_0, c'_1) \, d\Omega_g = a|S|. \tag{4.16}
\]

Similarly, we get
\[
\int_S f(v^1_0, v^2_0, V'_0 + c'_2) \, d\Omega_g = b|S|. \tag{4.17}
\]

Hence from Lemma \((4.1)\) we see that \((c'_1, c'_2) = (c_1(U'_0), c_2(V'_0))\). Then Lemma \((4.2)\) follows.

At this point we can define an operator
\[
T : \dot{W}^{1,2}(S) \to \dot{W}^{1,2}(S)
\]
as follows. For \((U', V') \in \dot{W}^{1,2}(S)\), let \((c_1(U'), c_2(V'))\) be defined by Lemma \((4.1)\) Define \((\tilde{U}', \tilde{V}') = T(U', V')\) where \(\tilde{U}'\) and \(\tilde{V}'\) are the unique solutions of
\[
\Delta \tilde{U}' = 8 \left( f(u^1_0, u^2_0, U' + c_1(U')) - a \right) - 4 \left( f(v^1_0, v^2_0, V' + c_2(V')) - b \right), \tag{4.18}
\]
\[
\Delta \tilde{V}' = -4 \left( f(u^1_0, u^2_0, U' + c_1(U')) - a \right) + 4 \left( f(v^1_0, v^2_0, V' + c_2(V')) - b \right), \tag{4.19}
\]
respectively. In fact, for any \((U', V') \in \dot{W}^{1,2}(S)\), since the right-hand sides of \((4.18)\) and \((4.19)\) have zero averages, the solutions \(\tilde{U}'\) and \(\tilde{V}'\) of \((4.18)\) and \((4.19)\), respectively, are unique (cf. \([4]\)).

Next we show that the operator \(T\) admits a fixed point in \(\dot{W}^{1,2}(S)\). To this end, we first establish the following lemma.

**Lemma 4.3** The above operator \(T : \dot{W}^{1,2}(S) \to \dot{W}^{1,2}(S)\) is completely continuous.

**Proof.** Assume \((U'_k, V'_k) \to (U'_0, V'_0)\) weakly in \(\dot{W}^{1,2}(S)\). Hence by the compact embedding theorem we see that \((4.9)\) holds.

Denote
\[
(\tilde{U}'_k, \tilde{V}'_k) = T(U'_k, V'_k) \quad \text{and} \quad (\tilde{U}'_0, \tilde{V}'_0) = T(U'_0, V'_0). \tag{4.20}
\]
Therefore we have

\[
\Delta(\tilde{U}'_0 - \tilde{U}'_0) = 8\left(f(u_0^1, u_0^2, \tilde{U}'_k + c_1(U'_0)) - f(u_0^1, u_0^2, U'_0 + c_1(U'_0))\right) \\
-4\left(f(v_0^1, v_0^2, V'_k + c_2(V'_0)) - f(v_0^1, v_0^2, V'_0 + c_2(V'_0))\right) \\
= 8f(t_0^1, t_0^2, \tilde{U}' + \tilde{c}_1)(U'_k - U'_0 + c_1(U'_k) - c_1(U'_0)) \\
-4f(v_0^1, v_0^2, \tilde{V}' + \tilde{c}_2)(V'_k - V'_0 + c_2(V'_k) - c_2(V'_0)), \quad (4.21)
\]

\[
\Delta(\tilde{V}'_0 - \tilde{V}'_0) = -4\left(f(u_0^1, u_0^2, \tilde{U}'_k + c_1(U'_k)) - f(u_0^1, u_0^2, U'_0 + c_1(U'_0))\right) \\
+4\left(f(v_0^1, v_0^2, V'_k + c_2(V'_0)) - f(v_0^1, v_0^2, V'_0 + c_2(V'_0))\right) \\
= -4f(v_0^1, v_0^2, \tilde{V}' + \tilde{c}_1)(U'_k - U'_0 + c_1(U'_k) - c_1(U'_0)) \\
+4f(v_0^1, v_0^2, \tilde{V}' + \tilde{c}_2)(V'_k - V'_0 + c_2(V'_k) - c_2(V'_0)), \quad (4.22)
\]

where \(\tilde{U}'_k\) lies between \(U'_k\) and \(U'_0\), \(\tilde{V}'_k\) between \(V'_k\) and \(V'_0\), \(\tilde{c}_1\) between \(c_1(U'_k)\) and \(c_1(U'_0)\), and \(\tilde{c}_2\) between \(c_2(V'_k)\) and \(c_2(V'_0)\).

Multiplying both sides of (4.21) and (4.22) by \(\tilde{U}'_0 - \tilde{U}'_0\) and \(\tilde{V}'_0 - \tilde{V}'_0\), respectively, and integrating by parts, we obtain

\[
\|\nabla(\tilde{U}'_0 - \tilde{U}'_0)\|^2 \leq \int_S \left\{4(|U'_k - U'_0| + |c_1(U'_k) - c_1(U'_0)|) \\
+2(|V'_k - V'_0| + |c_2(V'_k) - c_2(V'_0)|)\right\}|\tilde{U}'_k - \tilde{U}'_0| d\Omega_g, \quad (4.23)
\]

\[
\|\nabla(\tilde{V}'_0 - \tilde{V}'_0)\|^2 \leq 2\int_S \left(|U'_k - U'_0| + |c_1(U'_k) - c_1(U'_0)| \\
+|V'_k - V'_0| + |c_2(V'_k) - c_2(V'_0)|\right)|\tilde{V}'_k - \tilde{V}'_0| d\Omega_g, \quad (4.24)
\]

where the property (3.11) is used.

Combining (4.23) with (4.24), and using the Poincaré inequality, we arrive at

\[
\|\nabla(\tilde{U}'_k - \tilde{U}'_0)\|^2 + \|\nabla(\tilde{V}'_k - \tilde{V}'_0)\|^2 \leq C \left(\|U'_k - U'_0\|^2_2 + \|V'_k - V'_0\|^2_2 \\
+|c_1(U'_k) - c_1(U'_0)|^2 + |c_2(V'_k) - c_2(V'_0)|^2\right) \quad (4.25)
\]

for some \(C > 0\). Then, from (4.9), Lemma 4.2 and (4.25), we see that

\[
(\nabla \tilde{U}'_k, \nabla \tilde{V}'_k) \to (\nabla \tilde{U}'_0, \nabla \tilde{V}'_0) \quad \text{strongly in} \quad L^2(S) \quad \text{as} \quad k \to \infty,
\]

which, with (4.9), yields

\[
(\tilde{U}'_k, \tilde{V}'_k) \to (\tilde{U}'_0, \tilde{V}'_0) \quad \text{strongly in} \quad \tilde{W}^{1,2}(S) \quad \text{as} \quad k \to \infty.
\]

Then the proof of Lemma 4.3 is complete.

Before applying the Leray–Schauder fixed-point theory, we need to estimate the solution of the fixed-point equation,

\[
(U'_t, V'_t) = tT(U'_t, V'_t), \quad 0 \leq t \leq 1. \quad (4.26)
\]
Lemma 4.4 For any \((U'_i, V'_i)\) satisfying (4.26), there exists a constant \(C > 0\) independent of \(t \in [0, 1]\) such that
\[
\|U'_i\|_{W^{1,2}(S)} + \|V'_i\|_{W^{1,2}(S)} \leq C.
\] (4.27)

Proof. From (4.26) we have
\[
\Delta U'_i = 8t \left( f(u^1_0, u^2_0, U'_i + c_1(U'_i)) - a \right) - 4t \left( f(v^1_0, v^2_0, V'_i + c_2(V'_i)) - b \right),
\]
(4.28)
\[
\Delta V'_i = -4t \left( f(u^1_0, u^2_0, U'_i + c_1(U'_i)) - a \right) + 4t \left( f(v^1_0, v^2_0, V'_i + c_2(V'_i)) - b \right).
\]
(4.29)

Multiplying both sides of (4.28) and (4.29) by \(U'_i\) and \(V'_i\), respectively, and integrating by parts, we see that
\[
\|\nabla U'_i\|^2 \leq \int_S \left( 8|f(u^1_0, u^2_0, U'_i + c_1(U'_i))| + 4|f(v^1_0, v^2_0, V'_i + c_2(V'_i))| \right) |U'_i| d\Omega_g
\]
\[
\|\nabla V'_i\|^2 \leq \int_S \left( 4|f(u^1_0, u^2_0, U'_i + c_1(U'_i))| + 4|f(v^1_0, v^2_0, V'_i + c_2(V'_i))| \right) |V'_i| d\Omega_g
\]
where we have used (3.16). Then by the Poincaré inequality, we get the desired estimate (4.27).

Now using Lemmas 4.3, 4.4 and the Leray–Schauder fixed-point theorem (cf. [13]), we see that the operator \(T\) admits a fixed point, say \((U', V')\), in \(W^{1,2}(S)\). Thus \((U' + c_1(U'), V' + c_2(V'))\) is a solution of (4.1) and (4.2), i.e., a solution of (5.8) and (5.9).

Hence we have completed the proof of Theorem 3.2.

## 5 Explicit calculation of minimum energy

In this section we establish the minimum energy formula (2.25) and show how it is stratified topologically.

By the equations (4.16)–(4.19), the fact \(*1 = d\Omega_g\), and (3.5)–(3.6), we see that
\[
\int_S (\tilde{F} - \tilde{F}) = 4 \int_S \frac{1 - e^u}{e^u + 1} - 2 \int_S \frac{1 - e^v}{e^v + 1} = 2\pi(N_1 - P_1),
\]
(5.1)
\[
\int_S \tilde{F} = -2 \int_S \frac{1 - e^u}{e^u + 1} + 2 \int_S \frac{1 - e^v}{e^v + 1} = 2\pi(N_2 - P_2),
\]
(5.2)
are valid, which give us
\[
\int_S \tilde{F} = 2\pi(N_1 - P_1 + N_2 - P_2).
\]
(5.3)

To calculate the lower bound of the energy, we need to compute the fluxes contributed by the current densities \(K(q)\) and \(K(p)\).

Take a coordinate chart \(\{U_j\}\) of \(S\). Assume \(z''_{1,j} \in U_j, j = 1, \ldots, P_1\). In local coordinates, we have \(D_i q = \partial_i q - i(\dot{A}_i - \dot{\bar{A}}_i)q, i = 1, 2\) and the density \(K(q)\) in \(U_j\) can be written as
\[
K(q) = -\frac{2|q|^2}{1 + |q|^2} (\tilde{F} - \tilde{F}) + i \frac{D_i q\overline{D_j q} - \overline{D_i q}D_j q}{(1 + |q|^2)^2} dx^i \wedge dx^j.
\]
(5.4)
Besides, in $K(q) = dJ(q)$, we have

$$J(q) = \frac{i}{1 + |q|^2} (q\overline{D_i q} - \overline{q} D_i q) dx^i. \quad (5.5)$$

Then it follows from the Stokes formula that

$$\int_S K(q) = \int_S dJ(q) = \sum_{j=1}^{P_1} \lim_{r \to 0} \oint_{\partial B(z''_{1,j}, r)} J(q), \quad (5.6)$$

where $B(z, r)$ denotes a disc centered at $z$ with radius $r > 0$ and all the line integrals are taken counterclockwise.

Note that near $z''_{1,j} \in \mathcal{P}(q)$, the section $q$ has the representation

$$q(z) = z^{-1} h_j(z, \overline{z}), \quad z = x^1 + ix^2, \quad x^1(z''_{1,j}) = x^2(z''_{1,j}) = 0, \quad (5.7)$$

where $h_j$ is a non-vanishing function defined near $z''_{1,j}$.

From the equation (2.16) we see that

$$\hat{A}_1 - \tilde{A}_1 = -2\Re(i\overline{\partial} \ln u), \quad \hat{A}_2 - \tilde{A}_2 = -2\Im(i\overline{\partial} \ln u), \quad (5.8)$$

which, with $u = \ln |q|^2$, implies

$$D_1 q = (\partial + \overline{\partial}) q + \left(\frac{\partial q}{q} - \frac{\overline{q} \partial q}{q}\right) q = q \partial u, \quad (5.9)$$

$$D_2 q = i(\partial - \overline{\partial}) q + i \left(\frac{\overline{q} \partial q}{q} + \frac{\partial q}{q}\right) q = iq \partial u. \quad (5.10)$$

Then, by (5.6), (5.9), and (5.10), we have

$$\oint_{\partial B(z''_{1,j}, r)} J(q) = \int_{\partial B(z''_{1,j}, r)} \frac{|q|^2}{1 + |q|^2} (|\partial - \overline{\partial}| u dx^1 - i|\overline{\partial} + \partial| u dx^2)$$

$$= \int_{\partial B(z''_{1,j}, r)} e^u (\partial_2 u dx^1 - \partial_1 u dx^2). \quad (5.11)$$

Noting (5.7), near $z''_{1,j} \in \mathcal{P}(q)$, we see that

$$u = -2 \ln |z| + w_j, \quad (5.12)$$

where $w_j$ is a smooth function. Thus we obtain

$$\lim_{r \to 0} \oint_{\partial B(z''_{1,j}, r)} J(q) = 4\pi, \quad (5.13)$$

which, with (5.6), gives

$$\int_S K(q) = 4\pi P_1. \quad (5.14)$$

Following a similar procedure, we have

$$\int_S K(p) = 4\pi P_2. \quad (5.15)$$
As described in [25], the normalized integrals $\frac{1}{4\pi} \int K(q)$ and $\frac{1}{4\pi} \int K(p)$, counting the numbers $P_1, P_2$ of anti-vortices of the two species, are the Thom classes of the dual bundle $L^* \rightarrow S$, of two respective classification (Chern) classes, $\frac{1}{2\pi} \int (\hat{F} - \tilde{F})$ and $\frac{1}{2\pi} \int \tilde{F}$.

Hence, by (2.13)–(2.15), (5.3), (5.14), and (5.15), we obtain the following topologically stratified minimum energy

$$E = \int_S 2(|\hat{F} - \tilde{F}| + \tilde{F} + K(q) + K(p)) = 4\pi(N_1 + P_1 + N_2 + P_2),$$

(5.16)

as stated in Theorem 2.1.

6 Conclusions and remarks

In this work we have extended the formalism of Tong and Wong [29] of a product Abelian Higgs theory describing a coupled vortex system with magnetic impurities to accommodate coexisting vortices and anti-vortices of two species realized as topological solitons governed by a BPS system of equations. In addition to the usual first Chern classes suited over a complex Hermitian line bundle, the presence of anti-vortices switches on the Thom classes over the dual bundle, as in [25]. When the underlying Riemann surface $S$ where vortices and anti-vortices reside is compact, we have established a theorem which spells out a necessary and sufficient condition, consisting of two inequalities, (2.23) and (2.24), for prescribed $N_1, N_2$ vortices and $P_1, P_2$ anti-vortices, of two respective species, to exist.

This necessary and sufficient condition contains a few special situations worthy of mentioning.

(i) When $N_2 = P_2 = 0$ (only vortices and anti-vortices of the first species are present), the condition becomes

$$|N_1 - P_1| < \frac{|S|}{\pi}. \quad (6.1)$$

(ii) When $N_1 = P_1 = 0$ (only vortices and anti-vortices of the second species are present), the condition reads

$$|N_2 - P_2| < \frac{|S|}{2\pi}. \quad (6.2)$$

(iii) When $N_1 = N_2 = N$ and $P_1 = P_2 = P$ (there are equal numbers of vortices and anti-vortices, respectively, of two species), the condition is

$$|N - P| < \frac{|S|}{3\pi}. \quad (6.3)$$

In all these situations, the numbers of vortices and anti-vortices may be arbitrarily large, provided that the differences of these numbers are kept in suitable ranges as given.

Although the vortices and anti-vortices of the two species do not appear in the model in a symmetric manner as seen in the field-theoretical Lagrangian density and the governing equations, they make equal contributions to the total topologically stratified minimum energy as stated in (2.25) of an elegant form.

Let $M(N_1, P_1, N_2, P_2)$ denote the moduli space of solutions of the BPS equations (2.16)–(2.19) with $N_1 + N_2$ and $P_1 + P_2$ prescribed vortices and anti-vortices, of two respective species. Since...
these solutions depend on at least $2(N_1 + N_2 + P_1 + P_2)$ continuous parameters which specify the locations of zeros and poles of the two sections $q, p$, respectively, we obtain the following upper bound for the dimensionality of $\mathcal{M}(N_1, P_1, N_2, P_2)$:

$$\dim(\mathcal{M}(N_1, P_1, N_2, P_2)) \geq 2(N_1 + N_2 + P_1 + P_2).$$  \hspace{1cm} (6.4)

Since we have not established the uniqueness of a solution with $N_1 + N_2$ and $P_1 + P_2$ prescribed vortices and anti-vortices of the two species yet, we do not know whether the inequality (6.4) is actually an equality. In this regard, it will be interesting to carry out an investigation along the (well-known classical) index theory work of Atiyah, Hitchin, and Singer [2,3] on the Yang–Mills instantons, of Weinberg [33] on the BPS system of the Abelian Higgs model, and of Lee [18] on supersymmetric domain walls, for our new system of equations, (2.16)–(2.19).

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