Problems With the Vortex-Boson Mapping in 1+1 Dimensions

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Using the well known boson mapping, we relate the transverse magnetic susceptibility of a system of flux vortices in 1+1 dimensions to an appropriately defined conductivity of a one-dimensional boson system. The tilt response for a system free of disorder is calculated directly, and it is found that a subtle order of limits is required to avoid deceptive results.

1. Introduction

A d + 1 dimensional vortex system with correlated disorder can be modeled as a d dimensional time dependent boson system. This mapping has proved extremely useful for the study of vortex behavior for systems with disorder correlated in one direction. Such pinning may arise from artificially created ion tracks, screw dislocations, and/or twin or grain boundaries. Unfortunately, difficulties in characterizing the disorder lead to uncertainties in the interpretation of experimental measurements.

In contrast, in the 1+1 dimensional system of flux vortices in large Josephson junctions, disorder can be fabricated with great control. In particular, it is easy to create samples with disorder appropriately correlated in one direction. This freedom has encouraged both the experimental and theoretical study of these 1+1 dimensional systems. Researchers have also found 1+1 dimensional vortex models to be a useful simplification for understanding the physics of the more complicated 2+1 dimensional systems.

In this paper, expanding on previous work, we discuss features of the boson mapping for 1+1 dimensional systems. The basic mapping is briefly reviewed in section 2. As might be expected, the response of the vortex system to a transverse magnetic field (the tilt response) is related to a conductivity of the boson system. As is well known, one must be very careful about defining a conductivity in one dimension. The relation between the vortex tilt response and a boson conductivity is derived in section 3. We will see, however, that this result is extremely deceptive due to a subtle order of limits. To illustrate this problem, as well as the method of treating it properly, we explicitly calculate the tilt response for a system with no disorder in section 4.

2. Boson Analogy

We consider a system of vortices confined to the x − y plane and oriented along the y axis, with disorder correlated in the y-direction. In this case, such the pinning potential can be written as \( U(x, y) \) and the free energy \( F_N \) for a system of \( N \) vortices is

\[
F_N = -N\left(\frac{H}{H_{c1}} - 1\right)\epsilon L + \int_0^L dy \left\{ \sum_n \left[ \frac{\dot{r}_n(x_n(y), y)}{2} \right]^2 + U(x_n(y)) \right\} + \sum_{n, n'} V(x_n(y) - x_{n'}(y))
\]

where \( x_i(y) \) describes the path of the \( i^{th} \) vortex, \( H_{c1} \) is the lower critical field, \( H(x, y) = H_x(x, y)/H_y(x, y) \) is the local slope of the applied magnetic field, \( \epsilon \) is the vortex stiffness, and \( V(x) \) is the vortex-vortex interaction.

If periodic boundary conditions are imposed on the system in the y direction, the partition function for the system of flux lines is given by

\[
Z = \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{P} \int D x_1(y) \cdots \int D x_N(y) e^{-F_N/k_B T}
\]

where \( y \) goes from 0 to \( L \). The sum over \( P \) is a sum over all possible permutations of boundary conditions, such that for all \( i \), \( x_i(0) = x_j(L) \) for some \( j \). As discussed in Ref., in the limit of \( h(y) = 0 \), this partition function is identical to an imaginary time Feynman path integral for the grand canonical partition function of a collection of bosons in 1+1 dimensions. The y coordinate for the flux line system is mapped to the imaginary time coordinate of the boson system. The chemical potential of boson system is given by \( \epsilon(H/H_{c1}, 1) \), and the boson mass is given by the fluxline stiffness \( \epsilon \). The temperature \( k_B T \) of the vortex is mapped to Planck’s constant \( \hbar \) for the boson system, and the length \( L \) of the vortex system is \( \beta \hbar \) with \( \beta \) the inverse temperature \( (1/k_B T_{bose}) \) of the boson system. The boson pair potential is given by the flux line pair interaction \( V(|x_i - x_j|) \), and the y-correlated disorder potential \( U(x) \) for the vortex system also maps directly to the boson system.
3. Tilt Response

The above described mapping is particularly useful in allowing one to make contact with the large body of knowledge of interacting boson systems. To make this comparison more fruitful, we now derive a relation between the tilt response of the vortex array and the conductivity of the interacting bosons. As will be discussed in an explicit calculation in section 4, it is necessary to keep careful track of the order of limits defining the dc response to avoid deceptive results.

To probe the linear response of the flux line system, a small field in the x direction can be applied. To avoid problems with flux lines piling up at the x boundaries of the system, the field is applied over a finite width 0 < x < W, and we will take the limit W → ∞ at the end. For simplicity, we will choose $H_x$ to be given in the form

$$ h(x, y) = \begin{cases} h(y) & \text{for } 0 < x < W \\ 0 & \text{otherwise} \end{cases} \quad (3) $$

although it is a trivial generalization to include more complicated variations of $h(x, y)$ in the x-direction. The Fourier components of the function $h(y)$ are defined by

$$ \hat{h}(q) = \frac{1}{L} \int_{0}^{L} dy e^{i q y} h(y) \quad (4) $$

where $q$ must be of the form $2\pi n/L$ with $n$ an integer. It should be noted that since $h(y)$ is real, $\hat{h}(q) = \hat{h}(-q)^*$. The tilt modulus $t(q)$ can now be defined as

$$ t(q; W) = \left. \frac{d^2 \ln Z}{d h(q)} \right|_{h=0} \quad (5) $$

where $\hat{\theta}(q; W)$ is the Fourier component of the local slope

$$ \hat{\theta}(q; W) = \frac{1}{L} \int_{0}^{L} dy e^{i q y} \theta(y; W) \quad (6) $$

and $\theta(y; W)$ is the spatially averaged slope of the flux vortices in the range $0 < x < W$. More formally,

$$ \theta(y; W) = \frac{1}{W} \int_{0}^{W} dx s(x, y) \quad (7) $$

where $s(x, y)$ is the local slope density

$$ s(x, y) = \sum_{i} \frac{dx_i}{dy} \delta(x - x_i(y)). \quad (8) $$

The expectation of the slope $\hat{\theta}(q; W)$ is given by differentiating the log of the partition function

$$ \langle \hat{\theta}(q; W) \rangle = \frac{k_B T}{W L \epsilon} \int dh(-q) \quad (9) $$

Thus, the tilt response is the second derivative, which can be written as

$$ t(q; W) = \frac{k_B T}{W L \epsilon} \frac{d^2 \ln Z}{d h(q) dh(-q)} \quad (10) $$

$$ = \frac{W \epsilon}{k_B T} \left( \int_{0}^{L} dy e^{-iqy} \langle \theta(0; W) \theta(y; W) \rangle \right) \quad (11) $$

where we have used the fact that $\langle \hat{\theta}(q; W) \rangle$ is zero in the limit of zero field, as well as the $y$-translational invariance of the system. In the limit $L \to \infty$ this expression becomes analogous to the expression for the conductivity of the bose system at zero temperature and finite frequency $\omega$ where the external field is applied over the range $0 < x < W$:

$$ \sigma(\omega; W) = \frac{e^2}{\pi \hbar \omega} \int_{-\infty}^{\infty} d\epsilon e^{-i\epsilon\omega} \langle T_\tau J(\tau) J(0) \rangle \quad (12) $$

where $T_\tau$ is the time ordering operator, and $J(\tau)$ is the current at imaginary time $\tau$ averaged over $0 < x < W$,

$$ J(\tau) = \frac{1}{W} \int_{0}^{W} dx j(x, \tau) \quad (13) $$

with the local current density

$$ j(x, \tau) = \sum_{i} \frac{dx_i}{d\tau} \delta(x - x_i(y)). \quad (14) $$

Thus with $q$ taking the place of $\omega$, the formal relation is

$$ t(q; W) = q \sigma(q; W) \left( \frac{W \epsilon}{k_B T} \frac{\pi \hbar}{e^2} \right). \quad (15) $$

4. Calculation of the Tilt Response: Clean Limit — Luttinger Liquid

Eq. (3) is appealing, but can lead to drastically incorrect results if applied naively. In an interacting one-dimensional quantum system, we expect a finite dc conductivity. Taking the limit $q \to 0$, Eq. (3) seems to imply a vanishing tilt response to a uniform field. It is intuitively clear, however, that the vortices should rotate uniformly to accommodate the transverse field if there are no pinning defects present. To resolve this apparent paradox, we now calculate the conductivity more carefully for general $q$ and $W$ (with $W$ much larger than the mean vortex/boson spacing but still finite).

If there is no disorder in the system, the ground state is a periodic array of vortices. Fluctuations of this array are limited by the mean vortex/boson spacing but still finite). The free energy can then be written as

$$ F = \int_{-\infty}^{\infty} dx \int_{-L/2}^{L/2} dy \left[ \frac{K_x}{2} \left( \frac{du}{dx} \right)^2 + \frac{K_y}{2} \left( \frac{du}{dy} \right)^2 \right] \quad (16) $$
Here we have shifted the zero value of $y$.

Within the boson mapping, Eq. 14 can be interpreted as the bosonized action for the quantum particles. Generic values of $K_x$ and $K_y$ actually correspond to the non-Fermi-liquid “Luttinger liquid” state induced by interactions in one dimension. It is satisfying that this highly non-trivial behavior of the quantum system is encoded in our simple displacement field description. The values of the coefficients $K_x$ and $K_y$ are dependent on the precise form of the interaction $V$ as well as on the average spacing $\ell$. Standard methods can be used to evaluate these constants in various limits.

By transforming into Fourier space, the free energy becomes a sum (ie, an integral) of uncoupled harmonic oscillator modes. The equipartition theorem then yields

$$\langle \tilde{u}_{q_x, q_y} \tilde{u}_{q_x', q_y'} \rangle = \frac{(2\pi)^2 \delta(q_x + q_x') \delta(q_y + q_y')}{K_x q_x^2 + K_y q_y^2},$$  
(17)

where $\tilde{u}_{q_x, q_y}$ is the Fourier transform of $u(x, y)$. With the current $(j(x, \tau))$ or local slope density given in terms of the displacement field as $s(x, y) = du/dy$, the slope averaged from 0 to $W$ (see Eq. 18) can be written as

$$\theta(y; W) = \int_0^W dx \int \frac{dq_x dq_y}{(2\pi)^2} e^{i(q_x y + q_y x)} \tilde{u}(x, y; W)$$  
(18)

Performing the $x$ integration and substituting into Eq. 17 yields the tilt response

$$t(q; W) = \frac{2W q^2}{k_B T (2\pi)} \int dq_x \frac{(1 - \cos(q_x W))}{(q_x W)^2 [K_y q_y^2 + K_x q_x^2]},$$  
(19)

By adding a small piece $\delta^2$ to the $(q_x W)^2$ term of the denominator, the integral can be performed by contour integration and the limit $\delta \to 0$ can be taken at the end to give the result

$$t(q; W) = \frac{2\delta}{K_y k_B T} f \left( qW \sqrt{K_y/K_x} \right),$$  
(20)

with

$$f(x) = \left[ 1 - \frac{1 - e^{-x}}{x} \right] \cdot$$  
(21)

As noted above, this calculation can easily be generalized to the case where we are concerned with the response to a field $b(x, y)$ that is an arbitrary function of $x$.

Eq. 20 is an appealing physical result, and resolves the “paradox” described at the beginning of this section. To define the conductivity of the quantum system, we must take $q \to 0$ for finite $W$, to prevent difficulties with equilibration and transport across the ends of the sample. Then, using $\lim_{x \to 0} f(x) = x/2$ and Eq. 13, the dc conductivity is finite. In the vortex system, however, it is clear that for finite $W$ the fluxons cannot tilt, since they would have to pile up at the boundaries to match the untitled flux lines outside the field region. Taking therefore $W \to \infty$ first, the behavior $f(x) \to 1$ in this limit gives a finite $t(q = 0)$.

5. Conclusion

We have shown that a surprising subtlety in the order of limits arises in the calculation of tilt response and conductivity even for the simple pure problem discussed above. The origin of the difficulty lies in boundary effects, which are generally ignored in applications of the vortex–boson analogy. It is natural that transport properties such as the conductivity are sensitive to changes at the boundary, due to the constraint of charge conservation. One might conjecture that in the localized Bose glass phase, the order of limits may become unimportant. This possibility, and the extent to which such problems manifest themselves in other types of correlation functions remain interesting open questions.

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For the physically relevant case of the Josephson junctions described in Ref. 7, near the lower critical field ($H/H_{c1} < 1$), where the fluxon density is low, standard methods give (with notation from Ref. 7) $K_x = c k_B T / (4\pi \tilde{\varepsilon} \ell)$ and $K_y = \tilde{\varepsilon} \ell / (4\pi k_B T c)$, with $c$ a non-universal order one constant. In the high magnetic field limit ($H \gg \phi_0 / d \lambda$), these methods are not appropriate since the vortices overlap and become poorly defined. Nonetheless, it can be shown that the free energy defined in Eq. 16 still applies, with $K_x \approx K_y \approx \varepsilon_J / k_B T$.

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