Using $B_0^s$ Decays to Determine the CP Angles $\alpha$ and $\gamma$

C. S. Kim$^a$, D. London$^b$ and T. Yoshikawa$^c$

\begin{itemize}
\item $^a$: Department of Physics, Yonsei University, Seoul 120-749, Korea
\item $^b$: Laboratoire de physique nucléaire, Université de Montréal, C.P. 6128, succ. centre-ville, Montréal, QC, Canada
\item $^c$: Theory Group, KEK, Tsukuba, Ibaraki 305, Japan
\end{itemize}

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Abstract

Dighe, Gronau and Rosner have shown that, by assuming $SU(3)$ flavor symmetry and first-order $SU(3)$ breaking, it is possible to extract the CP angles $\alpha$ and $\gamma$ from measurements of the decay rates of $B_d^0(t) \to \pi^+\pi^-$, $B_d^0 \to \pi^-K^+$ and $B^+ \to \pi^+K^0$, along with their charge-conjugate processes. We extend their analysis to include the $SU(3)$-related decays $B_s^0 \to \pi^+K^-$, $B_s^0(t) \to K^+K^-$ and $B_s^0 \to K^0\bar{K}^0$. There are several advantages to this extension: discrete ambiguities are removed, fewer assumptions are necessary, and the method works even if all strong phases vanish. In addition, we show that $\gamma$ can be obtained cleanly, with no penguin contamination, by using the two decays $B_s^0(t) \to K^+K^-$ and $B_s^0 \to K^0\bar{K}^0$. 

1kim@cskim.yonsei.ac.kr, eskim@kekvax.kek.jp
2london@lps.umontreal.ca
3JSPS Research Fellow, yosikawa@theory.kek.jp
1 Introduction

In the coming years, the CP angles $\alpha$, $\beta$ and $\gamma$, which characterize the unitarity triangle, will be measured at $B$ factories. Through such measurements it will be possible to test the Standard Model (SM) explanation of CP violation, namely that CP violation is due to a complex phase in the Cabibbo-Kobayashi-Maskawa (CKM) matrix. Because of the importance of such tests, many ingenious ways of getting at the CP angles have been devised [1].

All methods involve CP-violating rate asymmetries between the decays $B \to f$ and $\bar{B} \to \bar{f}$. The conventional ways of measuring $\alpha$ and $\beta$ use the CP asymmetries in $B^0_d(t) \to \pi^+\pi^-$ and $B^0_d(t) \to \Psi K_s$, respectively, while the CP angle $\gamma$ can be obtained through the CP asymmetry in the charged $B$ decay $B^\pm \to D K^\pm$ [2]. Alternatively, $\gamma$ can be measured via $B^0_s(t) \to D^\pm K^{\mp}$ [3].

There are many other decay modes which can give information about the CP angles. However, one of the problems which must be addressed is the question of penguin contamination. For example, the penguin contribution to $B^0_d \to \pi^+\pi^-$ is likely to be sizeable, so that $\alpha$ cannot be cleanly extracted from the measurement of the CP asymmetry in $B^0_d(t) \to \pi^+\pi^-$. In this case one can remove the effects of the penguin amplitude by performing an isospin analysis [4], but this will probably not be easy since it requires the measurement of the branching ratio of $B^0_d \to \pi^0\pi^0$, which is expected to be of order $10^{-6}$ or less.

Recently, Dighe, Gronau and Rosner (DGR) proposed an elegant new method [5, 6, 7] of dealing with this penguin contamination. This method requires the measurement of the rates for $B^0_d \to \pi^- K^+$ and $B^+ \to \pi^+ K^0$ ($K_s \to \pi^+\pi^-$), as well as their charge-conjugate processes, along with the CP asymmetry in $B^0_d(t) \to \pi^+\pi^-$. Using $SU(3)$ flavor symmetry [8] and first-order $SU(3)$ breaking [9], these measurements can be used to disentangle the effects of the penguin contribution, and thus obtain $\alpha$ cleanly. In addition, the weak phase $\gamma$ can also be extracted from this set of measurements.

There are a number of advantages to this method. First, it does not suffer from the problems with electroweak penguins [10] and $SU(3)$ breaking that plague other methods. Second, it uses only decays of $B^0_d$ and $B^+$ mesons, which are accessible at asymmetric $e^+e^-$ colliders running on the $\Upsilon(4S)$ ($B$ factories). Finally, the decays involve only charged $\pi$’s or $K$’s, which makes the measurements considerably easier.

However, there are also some problems with this method. First, there is a large number of discrete ambiguities in the extraction of $\alpha$, $\gamma$, and the strong phase difference between the tree and penguin diagrams, $\delta$. Many of these ambiguities can be rejected due to other information that we have about the CKM matrix, but some still remain. This creates some difficulties in identifying the presence of new physics. Second, there are some theoretical
assumptions which, while reasonable, may turn out not to be true. In particular, DGR assume that the strong phase of the penguin diagram in $\Delta S = 0$ transitions is equal to that of the penguin diagram in $\Delta S = 1$ transitions. They also assume that the $b \to d$ penguin is dominated by an internal $t$ quark. If any of these assumptions is relaxed, then there is not enough information from the measurements to determine $\alpha$ and $\gamma$. Finally, the method also breaks down if $\delta$ vanishes. In this case it is necessary to make additional assumptions in order to extract information about the CP angles.

In this paper, we discuss an extension of the DGR method which eliminates many of these problems, or at least improves upon them. In addition to the $B_d^0$ and $B^+$ decays used by DGR, this extension uses their $SU(3)$-counterpart $B_s^0$ decays: $B_s^0 \to \pi^+ K^-$, $B_s^0(t) \to K^+ K^-$, and $B_s^0 \to K^0 \bar{K}^0$. If we make the same assumptions as DGR, we are able to extract $\alpha$ and $\gamma$ with a 2-fold ambiguity, corresponding to the unitarity triangle pointing up or down. If we relax the DGR assumptions, we are still able to obtain the CP angles up to possible discrete ambiguities. Even if $\delta = 0$, we are still able to extract these angles, up to some discrete ambiguities. Thus, this extension allows us to extract the CP angles $\alpha$ and $\gamma$, up to possible discrete ambiguities, with a minimum of theoretical assumptions.

If one relaxes all theoretical assumptions, then we find that it is not possible to extract $\alpha$ from measurements of the six decays. However, perhaps surprisingly, it is still possible to obtain $\gamma$. We show that measurements of the decays $B_s^0(t) \to K^+ K^-$ and $B_s^0 \to K^0 \bar{K}^0$ alone allow the measurement of $\gamma$ with no hadronic uncertainty. This is a new way of obtaining this angle.

The paper is organized as follows. In Section 2 we review the DGR method, followed in Section 3 by a description of our extension of this method. We then examine the effects of relaxing the DGR assumptions. The case $\delta \neq \delta'$ is considered in Section 4, followed by the inclusion of internal $u$ and $c$ quarks in the $b \to d$ penguins in Section 5. Section 5 also includes the description of a new method for obtaining $\gamma$. In Sections 3-5 it is assumed that the strong phases are independent of the spectator quark. Section 6 discusses the case where this assumption is relaxed. In Section 7 we consider the case of vanishing strong phases. We conclude in Section 8.

## 2 The DGR Method

In this section we review the method of Dighe, Gronau and Rosner [5, 6, 7]. Using $SU(3)$ symmetry [8], all $B \to PP$ decays, where $B$ represents $B_d$, $B^+$ or $B_s^0$, and $P$ is a pseudoscalar meson, can be written in terms of five $SU(3)$ amplitudes. These five $SU(3)$ amplitudes can in turn be expressed in terms of six diagrams. Of these six diagrams, three of them — the exchange, annihilation, and penguin annihilation diagrams — can
be neglected, since they are expected to be suppressed by $f_B/m_B \sim 3\text{-}4\%$. The remaining amplitudes are the tree $t\ (t')$, color-suppressed $c\ (c')$, and penguin $p\ (p')$ terms, where the unprimed and primed quantities denote $\Delta S = 0$ and $\Delta S = 1$ processes, respectively. These amplitudes include both the leading-order and electroweak penguin [10] contributions:

$$
\begin{align*}
t & \equiv T + (c_u - c_d)P_{EW}^C \\
c & \equiv C + (c_u - c_d)P_{EW} \\
p & \equiv P + c_dP_{EW}^C,
\end{align*}
$$

(1)

where $P_{EW}$ and $P_{EW}^C$ are the color-favored and color-suppressed electroweak penguin (EWP) diagrams, respectively, and $c_u$ and $c_d$ are the couplings of the $Z$ to $u$ quarks and $d$ quarks, respectively. In fact, although the EWP contributions have been included above, in most processes they are at the level of exchange-type diagrams, and so are negligible. Only the color-favored EWP is non-negligible, and then only in $\Delta S = 1$ transitions.

The DGR method involves the decays $B_d^0 \rightarrow \pi^+\pi^-$, $B_d^0 \rightarrow \pi^-K^+$, and $B^+ \rightarrow \pi^+K^0$. The amplitudes for these decays can be written

$$
\begin{align*}
A_{\pi\pi} & \equiv A\left(B^0 \rightarrow \pi^+\pi^-\right) = -(T + P), \\
A_{\pi K} & \equiv A\left(B^0 \rightarrow \pi^-K^+\right) = -(T' + P'), \\
A_{\pi K}^+ & \equiv A\left(B^+ \rightarrow \pi^+K^0\right) = P'.
\end{align*}
$$

(2)

Note that, since only tree and penguin terms are involved, EWP contributions are negligible. (In fact, DGR include the EWP contributions, but end up effectively setting them to zero by making the approximation $-\frac{1}{3}P_{EW}^C \approx \frac{2}{3}P_{EW}^C$.)

The weak phase of $T$ is $\text{Arg}(V_{ud}V_{ub}^*) = \gamma$, and similarly for $T'$: $\text{Arg}(V_{us}V_{ub}^*) = \gamma$. The $b \rightarrow s$ penguin $P'$ is dominated by the internal $t$-quark, so its weak phase is $\text{Arg}(V_{ts}V_{tb}^*) = \pi$. As for the $b \rightarrow d$ penguin $P$, if it also is dominated by the $t$-quark, its weak phase is $\text{Arg}(V_{td}V_{tb}^*) = -\beta$. This is the assumption made by DGR, but we will relax it in later sections.

If $SU(3)$ were unbroken, the amplitudes $T$ and $T'$ would be related simply by the ratio of their CKM matrix elements: $|T'/T| = |V_{us}/V_{ud}|$. However, if one includes first-order $SU(3)$ breaking [8], there is an additional factor involving the ratio of $K$ and $\pi$ decay constants if factorization is assumed:

$$
\frac{|T'|}{|T|} = \frac{|V_{us}|}{|V_{ud}|} \frac{f_K}{f_\pi} \equiv r_u.
$$

(3)

On the other hand, since factorization is unlikely to hold for penguin amplitudes, $P$ and $P'$ are not related in a simple way. However, DGR do assume that the strong phase of
the penguin diagram, $\delta_e$, is unaffected by $SU(3)$ breaking. This assumption will also be relaxed in later sections.

With these assumptions, the amplitudes in Eqs. (2) can be written

\[ A_{\pi\pi} = |T| e^{i\delta_e} e^{i\gamma} + |P| e^{i\delta_p} e^{-i\beta}, \]
\[ A_{\pi K} = r_u |T| e^{i\delta_e} e^{i\gamma} - |P'| e^{i\delta_p}, \]
\[ A_{\pi K}^+ = |P'| e^{i\delta_p}, \] (4)

where $T \equiv |T|$, $P \equiv |P|$, and $P' \equiv |P'|$.

There are thus six unknown quantities in the above 3 amplitudes: $\alpha \equiv \pi - \beta - \gamma$, $\gamma$, $T$, $P$, $P'$, and $\delta \equiv \delta_e - \delta_p$. These quantities can be extracted as follows. The time-dependent, tagged $B_d^0$ and $\bar{B}_d^0$ decay rates to $\pi^+\pi^-$ are given by

\[ \Gamma \left( B_d^0 (t) \rightarrow \pi^+\pi^- \right) = e^{-\Gamma t} \left[ |A_{\pi\pi}|^2 \cos^2 \left( \frac{\Delta m}{2} t \right) + \left| \bar{A}_{\pi\pi} \right|^2 \sin^2 \left( \frac{\Delta m}{2} t \right) + \text{Im} \left( e^{2i\beta} A_{\pi\pi} \bar{A}_{\pi\pi}^* \right) \sin(\Delta mt) \right], \]
\[ \Gamma \left( \bar{B}_d^0 (t) \rightarrow \pi^+\pi^- \right) = e^{-\Gamma t} \left[ |A_{\pi\pi}|^2 \sin^2 \left( \frac{\Delta m}{2} t \right) + \left| \bar{A}_{\pi\pi} \right|^2 \cos^2 \left( \frac{\Delta m}{2} t \right) - \text{Im} \left( e^{2i\beta} A_{\pi\pi} \bar{A}_{\pi\pi}^* \right) \sin(\Delta mt) \right]. \] (5)

From these measurements one can determine the three quantities $|A_{\pi\pi}|^2$, $|\bar{A}_{\pi\pi}|^2$, and $\text{Im} \left( e^{2i\beta} A_{\pi\pi} \bar{A}_{\pi\pi}^* \right)$. The rates for the self-tagging decays $B_d^0 \rightarrow \pi^- K^+$ and $\bar{B}_d^0 \rightarrow \pi^+ K^-$ are

\[ |A_{\pi K}|^2 = r_u^2 \Gamma^2 + \Gamma^2 - 2r_u \Gamma \Gamma' \cos(\delta + \gamma), \]
\[ |\bar{A}_{\pi K}|^2 = r_u^2 \Gamma^2 + \Gamma^2 - 2r_u \Gamma \Gamma' \cos(\delta - \gamma). \] (6)

Finally, the rates for $B^+ \rightarrow \pi^+ K^0$ and its CP-conjugate decay give

\[ |A_{\pi K}^+|^2 = |\bar{A}_{\pi K}|^2 = \Gamma'^2. \] (7)

Thus, from the above measurements, one can obtain the following six quantities:

\[ A \equiv \frac{1}{2} \left( |A_{\pi\pi}|^2 + |\bar{A}_{\pi\pi}|^2 \right) = \Gamma^2 + \Gamma^2 - 2\Gamma \Gamma \cos \delta \cos \alpha, \] (8)
\[ B \equiv \frac{1}{2} \left( |A_{\pi\pi}|^2 - |\bar{A}_{\pi\pi}|^2 \right) = -2\Gamma \Gamma \sin \delta \sin \alpha, \] (9)
\[ C \equiv \text{Im} \left( e^{2i\beta} A_{\pi\pi} \bar{A}_{\pi\pi}^* \right) = -\Gamma^2 \sin 2\alpha + 2\Gamma \Gamma \cos \delta \sin \alpha, \] (10)
\[ D \equiv \frac{1}{2} \left( |A_{\pi K}|^2 + |\bar{A}_{\pi K}|^2 \right) = r_u^2 \Gamma^2 + \Gamma^2 - 2r_u \Gamma \Gamma' \cos \delta \cos \gamma, \] (11)
\[ E \equiv \frac{1}{2} \left( |A_{\pi K}|^2 - |\bar{A}_{\pi K}|^2 \right) = 2r_u \Gamma \Gamma' \sin \delta \sin \gamma, \] (12)
\[ F \equiv |A_{\pi K}^+|^2 = \Gamma'^2. \] (13)
These give 6 equations in 6 unknowns, so that one can solve for $\alpha$, $\gamma$, $T$, $P$, $P'$, and $\delta$. However, because the equations are nonlinear, there are discrete ambiguities in extracting these quantities. In fact, a detailed study shows that, depending on the actual values of the phases, there can be up to 8 solutions. Many of these can be eliminated due to other information on the CKM phases, but still some ambiguity often remains.

From this brief summary, one can see some of the problems of the method. If $\delta = 0$, the quantities $B$ and $E$ vanish, so that one is left with 4 equations in 5 unknowns. In this case one must use additional assumptions to extract information about the CP phases. Furthermore, even if $\delta \neq 0$, if one relaxes any of the assumptions described above, the method breaks down. For example, if one allows the strong phase of the $P'$ diagram to be different from that of the $P$ diagram, as might be the case in the presence of $SU(3)$ breaking, then one has 6 equations in 7 unknowns. And if one relaxes the assumption that the $b \rightarrow d$ penguin is dominated by the $t$-quark, then once again additional parameters are introduced, and the method breaks down.

All of these potential problems can be dealt with by considering additional $B^0_s$ decays. We discuss this possibility in the following sections.

### 3 Extending the DGR Method with $B^0_s$ Decays

The problems with the DGR method can be resolved by adding amplitudes which depend on the same 6 quantities, thus overconstraining the system. In this case, if one adds a parameter or two, perhaps by relaxing certain assumptions, the method will be less likely to break down.

Within $SU(3)$ symmetry, the obvious decays to consider are the $SU(3)$ counterparts to the DGR decays, namely $B^0_s \rightarrow \pi^+K^-$, $B^0_s(t) \rightarrow K^+K^-$, and $B^0_s \rightarrow K^0\bar{K}^0$. The amplitudes for these decays are completely analogous to those in Eqs. (2):

\[
\begin{align*}
B_{sK} & \equiv A\left(B^0_s \rightarrow \pi^+K^-\right) = - \left(\tilde{T} + \tilde{P}\right), \\
B_{KK} & \equiv A\left(B^0_s \rightarrow K^+K^-\right) = - \left(\tilde{T}' + \tilde{P}'\right), \\
B^s_{KK} & \equiv A\left(B^0_s \rightarrow K^0\bar{K}^0\right) = \tilde{P}'.
\end{align*}
\] (14)

Here we have denoted the tree and penguin diagrams involving a spectator $s$ quark by $\tilde{T}$ and $\tilde{P}$, respectively. As before, the unprimed and primed quantities denote $\Delta S = 0$ and $\Delta S = 1$ processes, respectively.

The weak phase of $\tilde{T}$ and $\tilde{T}'$ is $\gamma$, and that of $\tilde{P}'$ is $\pi$. As for $\tilde{P}$, as a first step we make the same assumptions as DGR, namely that it is dominated by the $t$-quark, so that its weak phase is $-\beta$. Turning to $SU(3)$ breaking, we assume factorization for the tree
amplitudes, so that

$$\frac{|\hat{T}'|}{|\hat{T}|} = r_u .$$

(15)

The magnitudes of the $\hat{P}$ and $\hat{P}'$ amplitudes are unrelated to one another. However, again as a first step, like DGR we assume that they have the same strong phase, $\delta_p$. In subsequent sections, we will examine the consequences of relaxing these assumptions.

The one new assumption that we make is that the relative strong phase between the tree and penguin amplitudes is independent of the flavor of the spectator quark. Thus we have $\delta_s = \delta$, where $\delta_s \equiv \delta_T - \delta_p$ and $\delta \equiv \delta_T - \delta_p$. (The most likely way for this to occur is if $\delta_T = \delta_p$ and $\delta_p = \delta_p$.). This assumption, which is motivated by the spectator model, will also be reexamined in later sections.

Under these assumptions, the amplitudes in Eqs. (14) can be written

$$B_{sK} = \hat{T} e^{i\delta_T} e^{i\gamma} + \hat{P} e^{i\delta_p} e^{-i\beta},$$

$$B_{KK} = r_u \hat{T} e^{i\delta_T} e^{i\gamma} - \hat{P}' e^{i\delta_p},$$

$$B^s_{KK} = \hat{P}' e^{i\delta_p},$$

(16)

where $\hat{T} \equiv |\hat{T}|$, $\hat{P} \equiv |\hat{P}|$, and $\hat{P}' \equiv |\hat{P}'|$. The important point here is that three new parameters have been introduced in the above amplitudes: $\hat{T}$, $\hat{P}$, and $\hat{P}'$. However, as in the DGR method, 6 quantities can be extracted from measurements of the rates for these decays. Here, the self-tagging decays are $B_s^0 \to \pi^+ K^-$ and $\bar{B}_s^0 \to \pi^- K^+$, whose rates are

$$|B_{sK}|^2 = \hat{T}^2 + \hat{P}^2 - 2\hat{T}\hat{P}\cos(\delta - \alpha),$$

$$|B_{sK}|^2 = \hat{T}^2 + \hat{P}^2 - 2\hat{T}\hat{P}\cos(\delta + \alpha).$$

(17)

The time-dependent, tagged $B_s^0$ and $\bar{B}_s^0$ decay rates to $K^+ K^-$ are given by

$$\Gamma \left[ B_s^0(t) \to K^+ K^- \right] = e^{-\Gamma t} \left[ |B_{kk}|^2 \cos^2 \left( \frac{\Delta m_s t}{2} \right) + |\bar{B}_{kk}|^2 \sin^2 \left( \frac{\Delta m_s t}{2} \right) \right. $$

$$+ \left. \text{Im} \left( B_{kk} \bar{B}_{kk}^* \sin(\Delta m_s t) \right) \right],$$

$$\Gamma \left[ \bar{B}_s^0(t) \to K^+ K^- \right] = e^{-\Gamma t} \left[ |B_{kk}|^2 \sin^2 \left( \frac{\Delta m_s t}{2} \right) + |\bar{B}_{kk}|^2 \cos^2 \left( \frac{\Delta m_s t}{2} \right) \right. $$

$$- \left. \text{Im} \left( B_{kk} \bar{B}_{kk}^* \sin(\Delta m_s t) \right) \right],$$

(18)

from which the quantities $|B_{kk}|$, $|\bar{B}_{kk}|$, and $\text{Im} \left( B_{kk} \bar{B}_{kk}^* \right)$ can be extracted. Finally, we turn to $B_s^0(t) \to K^0 \bar{K}^0$. In principle there can be indirect CP violation in these decays. However, within the SM, this CP violation is zero to a good approximation, since both $B_s^0$-$\bar{B}_s^0$ mixing and the $b \to s$ penguin diagram, which dominates this decay, are real. Thus, measurements of the rates for these decays yield

$$|B_{kk}|^2 = |\bar{B}_{kk}|^2 = \hat{P}'^2 .$$

(19)
Obviously, any violation of this equality will be clear evidence for new physics.

Therefore the above measurements yield 6 new quantities:

\[ \tilde{A} \equiv \frac{1}{2} \left( |B_{sK}|^2 + |\bar{B}_{sK}|^2 \right) = \tilde{T}^2 + \tilde{P}^2 - 2 \tilde{T} \tilde{P} \cos \delta \cos \alpha, \]  
\[ \tilde{B} \equiv \frac{1}{2} \left( |B_{sK}|^2 - |\bar{B}_{sK}|^2 \right) = -2 \tilde{T} \tilde{P} \sin \delta \sin \alpha, \]  
\[ \tilde{C} \equiv \text{Im} \left( B_{KK} \bar{B}_{KK}^* \right) = r_u^2 \tilde{T}^2 + \tilde{P}^2 - 2 r_u \tilde{T} \tilde{P}' \cos \delta \sin \gamma, \]  
\[ \tilde{D} \equiv \frac{1}{2} \left( |B_{KK}|^2 + |\bar{B}_{KK}|^2 \right) = r_u^2 \tilde{T}^2 + \tilde{P}^2 - 2 r_u \tilde{T} \tilde{P}' \cos \delta \cos \gamma, \]  
\[ \tilde{E} \equiv \frac{1}{2} \left( |B_{sKK}|^2 - |\bar{B}_{sKK}|^2 \right) = 2 r_u \tilde{T} \tilde{P}' \sin \delta \sin \gamma, \]  
\[ \tilde{F} \equiv |B_{sKK}|^2 = \tilde{P}'^2. \]  

Combined with the 6 quantities in Eqs. (8-13), we have 12 equations in 9 unknowns. As shown below, this allows us to solve for the CP angles, as in the DGR method, but greatly reduces the discrete ambiguities.

The CP angles can be obtained as follows. First, one finds the ratios \( \frac{\tilde{T}}{T}, \frac{\tilde{P}}{P}, \) and \( \frac{\tilde{P}'}{P'} \):

\[ a \equiv \frac{\tilde{T}}{T} = \frac{E}{E \sqrt{\frac{F}{F}}}, \quad b \equiv \frac{\tilde{P}}{P} = \frac{BE}{BE \sqrt{\frac{F}{F}}}, \quad c \equiv \frac{\tilde{P}'}{P'} = \sqrt{\frac{F}{F}}. \]  

Using these, we can find the values of all the magnitudes of the amplitudes. The amplitudes \( T \) and \( P \) are obtained from

\[ T^2 = \frac{(acD - \tilde{D}) - c(a - c)F}{a(c - a)r_u^2}, \quad P^2 = \frac{abA - \tilde{A}}{b(a - b)} + \frac{a (acD - \tilde{D}) - c(c - a)F}{a(c - a)r_u^2}, \]  

and the remaining amplitudes can be found using Eq. (26). Note that all magnitudes are positive, by definition.

We now turn to the angles. Using our knowledge of the magnitudes of the amplitudes, we have

\[ \cos(\delta - \alpha) = \frac{T^2 + P^2 - A - B}{2TP}, \]  
\[ \cos(\delta + \alpha) = \frac{T^2 + P^2 - A + B}{2TP}, \]  
\[ \cos(\delta - \gamma) = \frac{r_u^2 T^2 + F - D + E}{2r_u T \sqrt{F}}, \]  
\[ \cos(\delta + \gamma) = \frac{r_u^2 T^2 + F - D - E}{2r_u T \sqrt{F}}. \]  

These equations can be solved to give the phases \( \alpha, \gamma \) and \( \delta \) up to a fourfold ambiguity. That is, if \( \alpha_0, \gamma_0 \) and \( \delta_0 \) are the true values of these phases, then the following four sets of
phases solve the above equations: \( \{ \alpha_0, \gamma_0, \delta_0 \} \), \( \{-\alpha_0, -\gamma_0, -\delta_0 \} \), \( \{\alpha_0 - \pi, \gamma_0 - \pi, \delta_0 - \pi \} \), and \( \{\pi - \alpha_0, \pi - \gamma_0, \pi - \delta_0 \} \). Note, however, that we still haven’t used the \( C \) and \( \tilde{C} \) measurements. Their knowledge eliminates two of the four sets, leaving \( \{\alpha_0, \gamma_0, \delta_0 \} \), \( \{\alpha_0 - \pi, \gamma_0 - \pi, \delta_0 - \pi \} \).

These two solutions correspond to two different orientations of the unitarity triangle, one pointing up, the other down. This final ambiguity cannot be resolved by this method alone. However, within the SM it can be removed by using other measurements such as \( \epsilon \) in the kaon system or the third CP angle \( \beta \).

Thus, for the case \( \delta \neq 0 \), this extension of the DGR method removes almost all of the discrete ambiguities found in the original method.

### 3.1 Special Case: \( a = b = c \)

From Eq. (27), one can see that the above method will not work if \( a = c \) or \( a = b \). We therefore reconsider the analysis in the worst-case scenario of \( a = b = c \). In this case the 12 equations (Eqs. [13], [20]–[25]) reduce to 7:

\[
A = \mathcal{T}^2 + \mathcal{P}^2 - 2\mathcal{T}\mathcal{P}\cos\delta\cos\alpha, \quad (30)
\]

\[
B = -2\mathcal{T}\mathcal{P}\sin\delta\sin\alpha, \quad (31)
\]

\[
C = -\mathcal{T}^2\sin2\alpha + 2\mathcal{T}\mathcal{P}\cos\delta\sin\alpha, \quad (32)
\]

\[
\tilde{C} = a^2\left[r_u^2\mathcal{T}^2\sin2\gamma - 2r_u\mathcal{T}\mathcal{P}'\cos\delta\sin\gamma\right], \quad (33)
\]

\[
D = r_u^2\mathcal{T}^2 + \mathcal{P}'^2 - 2r_u\mathcal{T}\mathcal{P}'\cos\delta\cos\gamma, \quad (34)
\]

\[
E = 2r_u\mathcal{T}\mathcal{P}'\sin\delta\sin\gamma, \quad (35)
\]

\[
F = \mathcal{P}'^2, \quad (36)
\]

with \( \tilde{A} = a^2A, \tilde{B} = B, \tilde{D} = a^2D, \tilde{E} = E, \) and \( \tilde{F} = a^2F \). The quantity \( a \) can therefore be determined by measurements of \( \tilde{A} \) and \( A \), for example. Without loss of generality, we set \( a = 1 \); other values of \( a \) correspond simply to different values of \( \tilde{C} \) above.

The original system of 12 equations in 9 unknowns is therefore reduced to 7 equations in 6 unknowns: \( \mathcal{T}, \mathcal{P}, \mathcal{P}', \alpha, \gamma, \) and \( \delta \). In this case the solution can still be found without discrete ambiguities, although numerical methods are required. The key observation is that 6 of the above 7 equations — \( A, B, C, D, E, F \) — are the same as those of the DGR method in Section 2 (Eqs. [8]–[13]). These 6 equations can be solved for the 6 unknowns, up to discrete ambiguities, by following the method of Ref. [7]. However, the spurious solutions can be eliminated since they do not, in general, satisfy the constraint of the seventh equation, \( \tilde{C} \). Thus, even if \( a = b = c \), we can solve for all the parameters, up to
the discrete ambiguity given by \( \{\alpha, \gamma, \delta\} \rightarrow \{\alpha - \pi, \gamma - \pi, \delta - \pi\} \), corresponding to the unitarity triangle pointing up or down.

4 Unequal Strong Phases in \( \Delta S = 0 \) and \( \Delta S = 1 \) Decays

One of the assumptions made by DGR is that the strong phase difference in the \( \Delta S = 0 \) sector, \( \delta \equiv \delta_T - \delta_P \), is the same as that in the \( \Delta S = 1 \) sector, \( \delta' \equiv \delta_T' - \delta_P' \). This assumption is necessary in the DGR method since otherwise there would be 6 equations in 7 unknowns. However, it is not clear how good this assumption is. \( SU(3) \) breaking could in principle lead to a measurable difference between \( \delta \) and \( \delta' \). DGR argue that, since the phases are expected to be small anyway \[11\], this assumption is unlikely to introduce a significant uncertainty into the method. Still, it does introduce a possible theoretical error into the procedure.

Fortunately, when one considers in addition \( B_s^0 \) decays, this assumption is no longer necessary. We therefore reconsider the analysis of the previous section for the case in which \( \delta \neq \delta' \). (Note that we continue to assume that the relative strong phases are independent of the flavor of the spectator quark, so that there are only two strong phases, and not four.) In this case we have 12 equations in 10 unknowns. The 12 equations are very similar to those shown earlier:

\[
\begin{align*}
A &= T^2 + P^2 - 2TP\cos\delta\cos\alpha, \\
\tilde{A} &= a^2T^2 + b^2P^2 - 2abTP\cos\delta\cos\alpha, \quad (37) \\
B &= -2TP\sin\delta\sin\alpha, \\
\tilde{B} &= -2\tilde{T}\tilde{P}\sin\delta\sin\alpha, \quad (38) \\
C &= -T^2 \sin 2\alpha + 2TP \cos\delta\sin\alpha, \\
\tilde{C} &= a^2r_u^2T^2 \sin 2\gamma - 2r_uaTTP' \cos\delta'\sin\gamma, \quad (39) \\
D &= r_u^2T^2 + P'^2 - 2r_uaTP' \cos\delta'\cos\gamma, \\
\tilde{D} &= a^2r_u^2T^2 + c^2P'^2 - 2r_uaTP' \cos\delta'\cos\gamma, \quad (40) \\
E &= 2r_uTP'\sin\delta'\sin\gamma, \\
\tilde{E} &= 2r_u\tilde{T}\tilde{P}'\sin\delta'\sin\gamma, \quad (41) \\
F &= P'^2, \\
\tilde{F} &= \tilde{P}'^2. \quad (42)
\end{align*}
\]

The magnitudes of the amplitudes are found in exactly the same way as in Section 3 (Eqs. 26-27). Following this analysis, we find

\[
\cos(\delta - \alpha) = \frac{T^2 + P^2 - A - B}{2TP}, \quad (43)
\]

10
\[ \cos(\delta + \alpha) = \frac{T^2 + P^2 - A + B}{2TP}, \]  
(50)  
\[ \cos(\delta' - \gamma) = \frac{r_u^2 T^2 + F - D + E}{2ruT \sqrt{F}}, \]  
(51)  
\[ \cos(\delta' + \gamma) = \frac{r_u^2 T^2 + F - D - E}{2ruT \sqrt{F}}. \]  
(52)  

From Eqs. (49) and (50) we can find 4 sets of solutions for \( \{\alpha, \delta\} \). That is, if \( \{\alpha_0, \delta_0\} \) are the true values of these phases, then the equations are solved by the following sets of phases: \( \{\alpha_0, \delta_0\}, \{-\alpha_0, -\delta_0\}, \{\alpha_0 - \pi, \delta_0 - \pi\}, \) and \( \{\pi - \alpha_0, \pi - \delta_0\} \). However, as in Sec. 3, the measurement of \( C \) eliminates two of the sets, leaving \( \{\alpha_0, \delta_0\} \) and \( \{\alpha_0 - \pi, \delta_0 - \pi\} \).

Similarly, Eqs. (51) and (52) give 4 sets of solutions for \( \{\gamma, \delta'\} \). But the measurement of \( \tilde{C} \) again eliminates two of them, leaving \( \{\gamma_0, \delta_0'\} \) and \( \{\gamma_0 - \pi, \delta_0' - \pi\} \).

Thus, the measurements in Eqs. (37-48) allow the extraction of the angles \( \{\alpha, \delta, \gamma, \delta'\} \) up to a four-fold ambiguity. However, the definition of the unitarity triangle requires us to choose the same sign for \( \alpha \) and \( \gamma \). Thus, once again, the angles can be extracted up to a two-fold ambiguity:

\[ \{\alpha_0, \delta_0, \gamma_0, \delta_0'\}, \]  
\[ \{\alpha_0 - \pi, \delta_0 - \pi, \gamma_0 - \pi, \delta_0' - \pi\}, \]  
(53)

which corresponds to the unitarity triangle pointing up or down.

### 4.1 Special Case: \( a = b = c \)

As in Sec. 3.1, we consider the special case of \( a = b = c \), in which case the above method breaks down. As before, we choose, without loss of generality, \( a = 1 \). The 12 equations of Eqs. (37-48) reduce to 7:

\[ A = T^2 + P^2 - 2TP \cos \delta \cos \alpha, \]  
(54)  
\[ B = -2TP \sin \delta \sin \alpha, \]  
(55)  
\[ C = -T^2 \sin 2\alpha + 2TP \cos \delta \sin \alpha, \]  
(56)  
\[ \tilde{C} = r_u^2 T^2 \sin 2\gamma - 2ruTP' \cos \delta' \sin \gamma, \]  
(57)  
\[ D = r_u^2 T^2 + P^2 - 2ruTP' \cos \delta' \cos \gamma, \]  
(58)  
\[ E = 2ruTP' \sin \delta' \sin \gamma, \]  
(59)  
\[ F = P^2. \]  
(60)  

Since there are 7 unknowns, these 7 equations can be solved, but there will be discrete ambiguities.
The solutions can be obtained as follows. First, from Eqs. (54), (55), (58), and (59), we have

\[ \cos \alpha = \frac{T^2 + P^2 - A}{2TP \cos \delta}, \]  
\[ \sin \alpha = \frac{B}{2TP \sin \delta}, \]  
\[ \cos \gamma = \frac{r_u^2T^2 + F - D}{2r_u \sqrt{F} \cos \delta'}, \]  
\[ \sin \gamma = \frac{E}{2r_u \sqrt{F} \sin \delta'}. \]

The angles \( \alpha \) and \( \gamma \) can be eliminated from the above equations, yielding quadratic equations for \( \cos^2 \delta \) and \( \cos^2 \delta' \):

\begin{align*}
4T^2P^2 \cos^4 \delta - \left\{ 4T^2P^2 + (T^2 + P^2 - A)^2 - B^2 \right\} \cos^2 \delta \\
+ (T^2 + P^2 - A)^2 &= 0, \quad (65) \\
-4r_u^2T^2F \cos^4 \delta' + \left\{ 4r_u^2T^2F - E^2 + (r_u^2T^2 + F - D)^2 \right\} \cos^2 \delta' \\
- (r_u^2T^2 + F - D)^2 &= 0. \quad (66)
\end{align*}

Eqs. (63), (61) and (62) can be combined to yield another quadratic equation in \( \cos^2 \delta \). Similarly, Eqs. (57), (63) and (64) combine to give another quadratic equation in \( \cos^2 \delta' \). These new equations are:

\begin{align*}
4P^4 \left( C^2 + B^2 \right) \cos^4 \delta - \left\{ 4P^4C^2 + 4P^2B^2(T^2 + P^2 - A) \right\} \cos^2 \delta \\
+ B^2(T^2 + P^2 - A)^2 &= 0, \quad (67) \\
4F^2 \left( E^2 + \tilde{C}^2 \right) \cos^4 \delta' - \left\{ 4FE^2(r_u^2T^2 + F - D) + 4F^2\tilde{C}^2 \right\} \cos^2 \delta' \\
+ E^2(r_u^2T^2 + F - D)^2 &= 0. \quad (68)
\end{align*}

Note that Eqs. (67) and (68) depend on \( C^2 \) and \( \tilde{C}^2 \), so that some sign information has been (temporarily) lost. Eqs. (65), (68) can now be solved straightforwardly to give \( T, P, \cos^2 \delta \) and \( \cos^2 \delta' \), and the CP angles \( \alpha \) and \( \gamma \) can be obtained. Of course, since the equations are quadratic, there are multiple solutions for all these quantities. Some of these solutions can be eliminated by now reconsidering Eqs. (56) and (57), i.e. the signs of \( C \) and \( \tilde{C} \). Still, many solutions remain.

In Table 4, we show examples of some of these solutions for various values of the parameters. For the amplitudes, we take \( T = \tilde{T} = 1, P = \tilde{P}' = 1, \) and \( P = \tilde{P} = \frac{|V_{ud}| \sin \gamma}{|V_{us}| \sin \alpha}. \) The values assumed for the weak and strong phases are shown in the table.

This table shows that there are indeed many solutions for these equations. (And note that there are additional solutions which we have not listed, in which the phases \( \{ \alpha_{out}, \gamma_{out}, \delta_{out}, \delta'_{out} \} \) are changed to \( \{ \alpha_{out} - \pi, \gamma_{out} - \pi, \delta_{out} - \pi, \delta'_{out} - \pi \} \).) However, not
### Table 1: Output values of the strong and weak phases, as well as the amplitudes, for given values of the input strong phases, and weak phases, $\alpha_{\text{in}}$ and $\gamma_{\text{in}}$. All phase angles are given in degrees. In the ‘Notes’ column, ‘a’, ‘b’ and ‘c’ indicate respectively the correct solution, a solution inconsistent with other experimental constraints, and a potential ambiguity.
all solutions are allowed within the context of the SM. For example, present experimental information constrains $20^\circ \lesssim \alpha \lesssim 120^\circ$, $30^\circ \lesssim \gamma \lesssim 150^\circ$, and $10^\circ \lesssim \beta \lesssim 45^\circ$ [12]. Following Refs. [6, 7], in the table we have labeled the solutions as follows: (a) correct solution, (b) one or more of the CP angles outside of the SM domain, and (c) potential ambiguity. As is clear from the table, most of the spurious solutions disappear when one imposes the SM constraints. Of course, one might be overlooking the presence of new physics in this way, but that is one of the problems caused by the presence of discrete ambiguities.

5 Non-negligible $u$- and $c$-quark Penguin Contributions

Another assumption made by DGR is that the $b \to d$ penguin is dominated by internal $t$-quarks. This has the effect that the weak phase of the penguin is simply $-\beta$. However, Buras and Fleischer have argued that the contributions of internal $u$- and $c$-quarks to $b \to d$ penguins are not negligible [13, 14]. They estimate that these additional diagrams can be between 20% and 50% of the leading $t$-quark contribution. By their own admission, this is a very rough estimate, but it indicates a potential complication to the method of DGR, as well as its extension.

If their estimate is correct, these additional contributions must be taken into account. The simplest way to do this is to write:

$$P = \sum_{q=u,c,t} V^*_{qd} V_{qb} P_q = V^*_{ud} V_{ub} (P_u - P_c) + V^*_{td} V_{tb} (P_t - P_c),$$

![Equation 69](image)

where the unitarity of the CKM matrix has been used. The key point here is that the combination of CKM matrix elements $V^*_{ud} V_{ub}$, which multiplies the new contributions, is the same as that which appears in the tree diagram. Thus, the amplitude for $B^0 \to \pi\pi$ can be written

$$A_{\pi\pi} = T e^{i\delta_T} e^{i\gamma} + (P_{ue} e^{i\delta_u} - P_{ce} e^{i\delta_c}) e^{i\gamma} + (P_{te} e^{i\delta_t} - P_{ce} e^{i\delta_c}) e^{-i\beta}$$

$$\equiv T e^{i\delta_T} e^{i\gamma} + P_{pe} e^{i\delta_P} e^{-i\beta},$$

![Equation 70](image)

where $T$ and $\delta_T$ include the tree and $u$- and $c$-quark penguin pieces, and similarly for $P$ and $\delta_P$. Likewise, the amplitude for $B^0_s \to \pi^+ K^-$ can be written

$$B_{\pi K} = \bar{T} e^{i\delta_T} e^{i\gamma} + (\bar{P}_{ue} e^{i\delta_u} - \bar{P}_{ce} e^{i\delta_c}) e^{i\gamma} + (\bar{P}_{te} e^{i\delta_t} - \bar{P}_{ce} e^{i\delta_c}) e^{-i\beta}$$

$$\equiv \bar{T} e^{i\delta_T} e^{i\gamma} + \bar{P}_{pe} e^{i\delta_P} e^{-i\beta}.$$  

![Equation 71](image)

Note that the new penguin contributions modify the sizes and phases of the tree and penguin amplitudes, but leave the forms of $A_{\pi\pi}$ and $B_{\pi K}$ unchanged.
The $b \to s$ penguin is not affected in the same way. For this penguin, one can perform a similar decomposition as in Eq. (59). However, in this case, the contribution proportional to $P_t - P_c$ dominates, since $|V_{us}^* V_{ub}| \ll |V_{ts}^* V_{tb}|$. Thus, the remaining amplitudes can be written

$$A_{sK} = T'e^{i\delta_T} e^{i\gamma} - P'e^{i\delta_P},$$

(72)

$$B_{KK} = \tilde{T}'e^{i\delta_T} e^{i\gamma} - \tilde{P}'e^{i\delta_P},$$

(73)

$$A^+_{sK} = P'e^{i\delta_P},$$

(74)

$$B^s_{K} = \tilde{P}'e^{i\delta_P}.$$  

(75)

Looking at the above 6 amplitudes, there are two points to be noted. First, the inclusion of the $u$- and $c$-quark penguins modifies the strong phase appearing in the $\Delta S = 0$ processes: $\delta = \delta_{TP} - \delta_{PP}$. This is clearly not the same as that appearing in $\Delta S = 1$ processes: $\delta' = \delta_T - \delta_{PP}$. Thus, consideration of $u$- and $c$-quark penguins requires us to take $\delta \neq \delta'$. Second, the presence of these additional penguin contributions destroys the relation between the tree contributions of $\Delta S = 0$ and $\Delta S = 1$ processes: although $T'/T = r_u$, $T'/T_p \neq r_u$. Therefore $T'$ must be left as an independent parameter. We will return to this point below.

With the above modified amplitudes, the 12 measurements are

$$A = T^2 + P^2 - 2T P \cos \delta \cos \alpha,$$

(76)

$$\tilde{A} = \tilde{T}^2 + \tilde{P}^2 - 2\tilde{T} \tilde{P} \cos \delta \cos \alpha,$$

(77)

$$B = -2T P \sin \delta \sin \alpha,$$

(78)

$$\tilde{B} = -2\tilde{T} \tilde{P} \sin \delta \sin \alpha,$$

(79)

$$C = -T^2 \sin 2\alpha + 2T P \cos \delta \sin \alpha,$$

(80)

$$\tilde{C} = \tilde{T}^2 \sin 2\gamma - 2\tilde{T} \tilde{P} \cos \delta' \sin \gamma,$$

(81)

$$D = T^2 + P^2 - 2T' P' \cos \delta' \cos \gamma,$$

(82)

$$\tilde{D} = \tilde{T}^2 + \tilde{P}^2 - 2\tilde{T}' \tilde{P}' \cos \delta' \cos \gamma,$$

(83)

$$E = 2T P' \sin \delta' \sin \gamma,$$

(84)

$$\tilde{E} = 2\tilde{T}' \tilde{P}' \sin \delta' \sin \gamma,$$

(85)

$$F = P^2,$$

(86)

$$\tilde{F} = \tilde{P}^2$$

(87)

where $\delta = \delta_{TP} - \delta_{PP}$ and $\delta' = \delta_T - \delta_{PP}$. Thus we end up with 12 equations in 12 unknowns.

However, there is a problem. An examination of the above equations reveals that the 12 measurements separate into two independent categories, those for $\Delta S = 0$ processes, and those for $\Delta S = 1$. The 5 measurements in the $\Delta S = 0$ sector, Eqs. (76-80), depend only on the 6 parameters $T_p$, $\tilde{T}_p$, $P_p$, $\tilde{P}_p$, $\alpha$, and $\delta$. Since the $\Delta S = 0$ sector has 5
equations in 6 unknowns, it is therefore impossible to extract the CP angle $\alpha$. Thus, the DGR method, as well as its extension, breaks down when the $u$- and $c$-quark penguin contributions are included.

Before discussing the $\Delta S = 1$ processes, let us examine why the method breaks down in this case. The crucial problem is that, in the presence of the additional penguin contributions, the relation $T'/\bar{T} = r_u$, which takes into account $SU(3)$ breaking, is apparently no longer valid. However, it is not clear how badly this relation is violated. Including the $u$- and $c$-quark penguin contributions, we have

$$T'/\bar{T} = \frac{T'e^{i\delta_T}}{\bar{T}e^{i\delta_T} + \bar{P}_ue^{i\delta_u} - \bar{P}_ce^{i\delta_c}} \simeq r_u \left[ 1 - \frac{P_ue^{i\delta_u} - P_ce^{i\delta_c}}{\bar{T}e^{i\delta_T}} \right].$$

(88)

Buras and Fleischer found the ratio $|\frac{(P_c - P_u)}{(P_t - P_u)}|$ to be between 20% and 50% [13, 14]. However, this does not give us any information about the ratio $|\frac{(P_u - P_c)}{T}|$. Even if the $u$- and $c$-quark penguin contributions are sizeable when compared to the $t$-quark penguin, it may still be that they are quite a bit smaller than the tree diagram. In this case, we would still have $T'/\bar{T} \simeq r_u$, and the situation would reduce to that of the previous section, with 12 equations in 10 unknowns. As shown in that section, the CP angles $\alpha$ and $\gamma$ can both be found, up to a 2-fold ambiguity. Thus, even taking into account the $u$- and $c$-quark penguins, it may be possible to extract $\alpha$. However, it is difficult to know for sure, and this will introduce some theoretical uncertainty into the method.

We now turn to the 7 measurements in $\Delta S = 1$ processes, Eqs. (81-87), and assume that the $u$- and $c$-quark penguins are sizeable (since otherwise the method of the previous section holds). In this case these measurements depend only on the remaining 6 parameters $T', \bar{T}', P'_u, \bar{P}'_u, \gamma$, and $\delta'$. As we have shown in previous sections, this implies that $\gamma$ can be extracted up to a 2-fold ambiguity.

The solution can be explicitly constructed as follows. As before, we write $\bar{T}' = aT'$, where $a$ is defined in Eq. (26). Using Eqs. (82) and (83), we can then solve for $T'$:

$$T' = \left[ \frac{\sqrt{F}(D - F) - \sqrt{\bar{F}}(\bar{D} - \bar{F})}{\sqrt{\bar{F}} - a^2\sqrt{F}} \right].$$

(89)

Given values for all the amplitudes, Eqs. (82) and (83) can then be used to obtain the phases $\gamma$ and $\delta'$, up to a 4-fold ambiguity. Finally, Eq. (81) can be used to eliminate two of these solutions.

This is interesting in its own right, as it is a new method for extracting $\gamma$. Note that the $\Delta S = 0$ processes are not needed at all. The four decays which need to be measured here are $B_d^0 \rightarrow \pi^- K^+$, $B^+ \rightarrow \pi^+ K^0$, $B^0_s(t) \rightarrow K^+ K^-$, and $B^0_s \rightarrow K^0 K^0$. The assumption
of \( SU(3) \) symmetry is rather minimal here – one assumes only that the strong phases are independent of the spectator quarks. (Even this assumption is relaxed in the next section.) In essence, this method removes the penguin contribution from the \( \mathrm{CP} \)-violating asymmetry in \( B^0_s(t) \to K^+K^- \). We will have more to say about this method in the next section.

6 Different Strong Phases for Different Spectator Quarks

The one assumption which we have continued to make throughout the previous sections is that the strong dynamics (\textit{i.e.} the strong phases) is independent of the flavor of the spectator quark. Given the success of the spectator model in \( B \) decays, this is probably justified. Nevertheless, in this section we explore the consequences of relaxing this assumption.

If the flavor \( SU(3) \) symmetry were unbroken, there would be a single strong phase, \( \delta \). In the decays considered in this paper, there are two distinct ways in which \( SU(3) \) can be broken: (i) \( \Delta S = 0 \) processes vs. \( \Delta S = 1 \) processes, and (ii) \( B^0_d \) decays \textit{vs.} \( B^0_s \) decays. Therefore in the general case one must consider four different strong phases:

\[
\begin{align*}
\delta & : (B^0_d \text{ or } B^+ \text{ decays, } \Delta S = 0), \\
\delta' & : (B^0_d \text{ or } B^+ \text{ decays, } \Delta S = 1), \\
\delta_s & : (B^0_s \text{ decays, } \Delta S = 0), \\
\delta'_s & : (B^0_s \text{ decays, } \Delta S = 1).
\end{align*}
\]

(90)

We will assume, however, that the \( u \)- and \( c \)-quark penguins are unimportant. In Section 5, it was shown that if these contributions are sizeable when compared to the tree diagram the angle \( \alpha \) cannot be extracted. Since in this section we are adding more parameters, the situation is even worse, and the only way that any information can be obtained is if the \( u \)- and \( c \)-quark penguins are in fact negligible compared to the tree diagram.

In this case, measurements of the various processes yield the following 12 quantities:

\[
\begin{align*}
A & = T^2 + P^2 - 2TP \cos \delta \cos \alpha, \\
\tilde{A} & = \tilde{T}^2 + \tilde{P}^2 - 2\tilde{T}\tilde{P} \cos \delta_s \cos \alpha, \\
B & = -2TP \sin \delta \sin \alpha, \\
\tilde{B} & = -2\tilde{T}\tilde{P} \sin \delta_s \sin \alpha, \\
C & = -T^2 \sin 2\alpha + 2TP \cos \delta \sin \alpha, \\
\tilde{C} & = r_u^2 \tilde{T}^2 \sin 2\gamma - 2r_u \tilde{T}\tilde{P}' \cos \delta'_s \sin \gamma, \\
D & = r_u^2 T^2 + P^2 - 2r_u T\mathcal{P}' \cos \delta' \cos \gamma, \\
\tilde{D} & = r_u^2 \tilde{T}^2 + \tilde{P}^2 - 2r_u \tilde{T}\tilde{P}' \cos \delta'_s \cos \gamma,
\end{align*}
\]

(91-98)


\[ E = 2r_u \mathcal{T} \mathcal{P}' \sin \delta' \sin \gamma, \quad (99) \]
\[ \tilde{E} = 2r_u \tilde{\mathcal{T}} \tilde{\mathcal{P}}' \sin \delta' \sin \gamma, \quad (100) \]
\[ F = \mathcal{P}'^2 \quad (101) \]
\[ \tilde{F} = \tilde{\mathcal{P}}'^2 \quad (102) \]

Since there are 12 equations in 12 unknowns, this system of equations can be solved, but there will be discrete ambiguities.

The solutions can be obtained as follows. First, from Eqs. (91–94), we have

\[
\cos \alpha = \frac{\mathcal{T}^2 + \mathcal{P}^2 - A}{2\mathcal{T} \mathcal{P} \cos \delta}, \quad (103)
\]
\[
\sin \alpha = -\frac{B}{2\mathcal{T} \mathcal{P} \sin \delta} = -\frac{\tilde{B}}{2\mathcal{T} \mathcal{P} \sin \delta_s}, \quad (104)
\]

and from Eqs. (97–100), we have

\[
\cos \gamma = \frac{r_u^2 \mathcal{T}^2 + F - D}{2r_u \mathcal{T} \sqrt{F} \cos \delta'} = \frac{r_u^2 \tilde{\mathcal{T}}^2 + \tilde{F} - \tilde{D}}{2r_u \tilde{\mathcal{T}} \sqrt{F} \cos \delta'_s}, \quad (105)
\]
\[
\sin \gamma = \frac{E}{2r_u \mathcal{T} \sqrt{F} \sin \delta'} = \frac{\tilde{E}}{2r_u \tilde{\mathcal{T}} \sqrt{F} \sin \delta'_s}. \quad (106)
\]

The angles \( \alpha \) and \( \gamma \) can be eliminated from the above equations to obtain the quadratic equations for \( \cos \delta \), \( \cos \delta' \), \( \cos \delta_s \) and \( \cos \delta'_s \):

\[
-4\mathcal{T}^2 \mathcal{P}^2 \cos^4 \delta + \{4\mathcal{T}^2 \mathcal{P}^2 - B^2 + (\mathcal{T}^2 + \mathcal{P}^2 - A)^2\} \cos^2 \delta
\]
\[
= -(\mathcal{T}^2 + \mathcal{P}^2 - A)^2 = 0, \quad (107)
\]
\[
-4r_u^2 \mathcal{T}^2 F \cos^4 \delta' + \{4r_u^2 \mathcal{T}^2 F - E^2 + (r_u^2 \mathcal{T}^2 + F - D)^2\} \cos^2 \delta'
\]
\[
= -(r_u^2 \mathcal{T}^2 + F - D)^2 = 0, \quad (108)
\]
\[
-4\tilde{\mathcal{T}}^2 \tilde{\mathcal{P}}^2 \cos^4 \delta_s + \{4\tilde{\mathcal{T}}^2 \tilde{\mathcal{P}}^2 - \tilde{B}^2 + (\tilde{\mathcal{T}}^2 + \tilde{\mathcal{P}}^2 - \tilde{A})^2\} \cos^2 \delta_s
\]
\[
= -(\tilde{\mathcal{T}}^2 + \tilde{\mathcal{P}}^2 - \tilde{A})^2 = 0, \quad (109)
\]
\[
-4r_u^2 \mathcal{T}^2 \tilde{F} \cos^4 \delta'_s + \{4r_u^2 \mathcal{T}^2 \tilde{F} - \tilde{E}^2 + (r_u^2 \mathcal{T}^2 + \tilde{F} - \tilde{D})^2\} \cos^2 \delta'_s
\]
\[
= -(r_u^2 \mathcal{T}^2 + \tilde{F} - \tilde{D})^2 = 0. \quad (110)
\]

Eliminating \( \alpha \) and \( \gamma \) in Eqs. (105) and (106), we also obtain the following equations:

\[
4\mathcal{P}^4 \left( B^2 + C^2 \right) \cos^4 \delta - \{4\mathcal{P}^2 B^2 (\mathcal{T}^2 + \mathcal{P}^2 - A) + 4\mathcal{P}^4 C^2 \} \cos^2 \delta
\]
\[
= +(\mathcal{T}^2 + \mathcal{P}^2 - A)^2 B^2 = 0, \quad (111)
\]
\[
4\tilde{\mathcal{F}}^2 \left( \tilde{E}^2 + \tilde{C}^2 \right) \cos^4 \delta'_s - \{4\tilde{\mathcal{F}} \tilde{E}^2 (r_u^2 \mathcal{T}^2 + \tilde{F} - \tilde{D}) + 4\tilde{\mathcal{F}}^2 \tilde{C}^2 \} \cos^2 \delta'_s
\]
\[
= +(r_u^2 \mathcal{T}^2 + \tilde{F} - \tilde{D})^2 \tilde{E}^2 = 0. \quad (112)
\]
Table 2: Output values of the strong and weak phases, as well as the amplitudes, for values of the input parameters given in the text. All phase angles are given in degrees.

In the ‘Notes’ column, ‘a’, ‘b’, ‘c’ and ‘a’ indicate respectively the correct solution, a solution inconsistent with other experimental constraints, a potential ambiguity, and a solution with the correct values for the CP angles, but different values for some of the other input parameters.
We can therefore determine $\tilde{T}$ and $\cos \delta'$ from Eqs. (110) and (112). Then the angle $\gamma$ can be obtained from Eq. (106), and $\mathcal{T}$ and $\cos \delta'$ from Eqs. (103) and (108). For each $\mathcal{T}$, we can determine $\mathcal{P}$ and $\cos \delta'$ from Eqs. (105) and (109). Evidently there are numerous solutions, some of which can be eliminated by now reconsidering Eqs. (95) and (96), i.e. the signs of $C$ and $\tilde{C}$. Nevertheless, we are left with many possible solutions.

In Table 2, we give an example of these solutions. For the amplitudes, we take $T = 1$, $\tilde{T} = 0.8$, $P' = 1$, $\tilde{P}' = 0.9$, $\mathcal{P} = \frac{|V_{us}|}{|V_{ud}|} \sin \alpha \mathcal{P}'$ and $\tilde{\mathcal{P}} = 1.1 \mathcal{P}$. The values assumed for the weak and strong angles are $\alpha_{\text{in}} = 45.0^\circ$, $\gamma_{\text{in}} = 120.0^\circ$, $\delta_{\text{in}} = 84.3^\circ$, $\delta'_{\text{in}} = 5.7^\circ$, $\delta_{s,\text{in}} = 174.3^\circ$ and $\delta'_{s,\text{in}} = 36.9^\circ$.

This table shows that there are 60 solutions for these equations (and there are 60 others, not listed, in which $\pi$ is subtracted from all 6 output angles). However, as before, not all solutions are allowed within the context of the SM. In the table we have labeled the solutions as follows: (a) correct solution, (b) one or more of the CP angles outside of the SM domain, (c) potential ambiguity, and (a') correct solution for CP angles but some of the other parameters are different from the inputs. As is clear from the table, most of the spurious solutions disappear when one imposes the SM constraints. However there are still some discrete ambiguities which can not be eliminated.

There is one more point to make here. As explained previously, if the $u$- and $c$-quark penguins are sizeable, then the CP angle $\alpha$ cannot be obtained via this method. However, the new method for obtaining $\gamma$, described in the previous section, is still viable, even when all four strong phases are included. In this case the four measurements $\tilde{C}$, $\tilde{D}$, $\tilde{E}$, and $\tilde{F}$, as obtained from the decays $B^0_s(t) \rightarrow K^+K^-$ and $B^0_s \rightarrow K^0\bar{K}^0$, become:

$$\tilde{C} = \tilde{T}^2 \sin 2\gamma - 2\tilde{T}'\tilde{P}'_\rho \cos \delta'_s \sin \gamma,$$

$$\tilde{D} = \tilde{T}^2 + \tilde{P}'^2 - 2\tilde{T}'\tilde{P}'_\rho \cos \delta'_s \cos \gamma,$$

$$\tilde{E} = 2\tilde{T}'\tilde{P}'_\rho \sin \delta'_s \sin \gamma,$$

$$\tilde{F} = \tilde{P}'^2.$$

Note that $\tilde{C}-\tilde{F}$ depend only on four parameters: $\tilde{T}'$, $\tilde{P}'_\rho$, $\delta'_s$, and $\gamma$. Thus we have 4 equations in 4 unknowns, which is soluble. Thus, even in this worst-case scenario, where all corrections are important, it is still possible to extract $\gamma$ from measurements of the two processes $B^0_s(t) \rightarrow K^+K^-$, and $B^0_s \rightarrow K^0\bar{K}^0$. In this case, there will be discrete ambiguities.

The solutions are obtained as in the previous cases. First, from Eqs. (114) and (115), we have

$$\cos \gamma = \frac{\tilde{T}^2 + \tilde{F} - \tilde{D}}{2\tilde{T}' \sqrt{\tilde{F}} \cos \delta'_s},$$

(117)
| $\gamma_{in}$ | $\delta_{s,in}$ | $\gamma_{out}$ | $\delta_{s,out}$ | $\mathcal{T}_{out}$ | Notes |
|---|---|---|---|---|---|
| 60.0 | 36.9 | 60.0 | 36.9 | 0.80 | a |
| | | 47.5 | 8.3 | 3.93 | c |
| | | 141.9 | 171.7 | 4.69 | c |
| | | 171.7 | 143.1 | 4.78 | b |
| 60.0 | 84.3 | 60.0 | 84.3 | 0.80 | a |
| | | 108.9 | 95.7 | 0.73 | c |
| | | 82.1 | 90.0 | 0.62 | c |
| | | 134.3 | 157.5 | 4.46 | c |
| | | 45.4 | 12.5 | 4.78 | c |
| 60.0 | 174.3 | 60.0 | 174.3 | 0.80 | a |
| | | 124.9 | 178.9 | 4.40 | c |
| | | 42.8 | 1.1 | 5.31 | c |
| | | 5.84 | 5.7 | 6.82 | c |
| 90.0 | 36.9 | 90.0 | 36.9 | 0.80 | a |
| | | 51.7 | 8.6 | 4.98 | c |
| | | 139.9 | 171.4 | 4.98 | c |
| | | 171.1 | 143.1 | 5.19 | b |
| 90.0 | 84.3 | 90.0 | 84.3 | 0.80 | a |
| | | 46.7 | 14.4 | 4.41 | c |
| | | 134.8 | 165.6 | 4.52 | c |
| | | 128.5 | 95.7 | 1.02 | c |
| 90.0 | 174.3 | 90.0 | 174.3 | 0.80 | a |
| | | 126.9 | 178.6 | 4.01 | c |
| | | 38.7 | 1.42 | 5.13 | c |
| | | 7.1 | 5.7 | 6.43 | b |
| 120.0 | 36.9 | 120.0 | 36.9 | 0.80 | a |
| | | 53.6 | 6.8 | 4.39 | c |
| | | 136.8 | 173.2 | 5.17 | c |
| | | 172.9 | 143.1 | 5.57 | b |
| 120.0 | 84.3 | 120.0 | 84.3 | 0.80 | a |
| | | 47.6 | 12.2 | 4.42 | c |
| | | 135.1 | 167.8 | 4.64 | c |
| | | 133.7 | 90.0 | 0.95 | c |
| | | 146.3 | 95.7 | 1.25 | c |
| 120.0 | 174.3 | 120.0 | 174.3 | 0.80 | a |
| | | 131.9 | 178.6 | 3.77 | c |
| | | 35.8 | 1.4 | 4.80 | c |
| | | 6.6 | 5.7 | 6.02 | b |

Table 3: Output values of $\gamma$, $\delta_s'$ and $\mathcal{T}$, for given values of the input phases. All phase angles are given in degrees. In the ‘Notes’ column, ‘a’, ‘b’ and ‘c’ indicate respectively the correct solution, a solution inconsistent with other experimental constraints, and a potential ambiguity.
\[
\sin \gamma = \frac{\tilde{E}}{2\tilde{T}'\sqrt{\tilde{F}} \sin \delta_s'}. \tag{118}
\]

Eliminating \( \gamma \) from Eqs. (117) and (118) and in Eq. (113), we obtain the following quadratic equations for \( \cos \delta'_s \):

\[
-4\tilde{T}'^2 \tilde{F} \cos^4 \delta'_s + \left\{ 4\tilde{T}'^2 \tilde{F} - \tilde{E}^2 + (\tilde{T}'^2 + \tilde{F} - \tilde{D})^2 \right\} \cos^2 \delta'_s
- (\tilde{T}'^2 + \tilde{F} - \tilde{D})^2 = 0, \tag{119}
\]

\[
4\tilde{F}^2 \left( \tilde{E}^2 + \tilde{C}^2 \right) \cos^4 \delta'_s - \left\{ 4\tilde{F} \tilde{E}^2 (\tilde{T}'^2 + \tilde{F} - \tilde{D}) + 4\tilde{F}^2 \tilde{C}^2 \right\} \cos^2 \delta'_s
+ (\tilde{T}'^2 + \tilde{F} - \tilde{D})^2 \tilde{E}^2 = 0. \tag{120}
\]

Eqs. (117)-(120) can now be solved straightforwardly to give \( \tilde{T}' \) and \( \cos^2 \delta'_s \), and the CP angle \( \gamma \) can be obtained. As usual, there are multiple solutions, some of which can be eliminated by now reconsidering Eq. (113).

In Table 3, we show examples of some of these solutions. For the amplitudes, we take \( \tilde{T} = 0.8 \) and \( \tilde{P}' = 0.9 \), and assume various values for the weak and strong phases, shown in the table. This table shows that, for the values of the phases we have chosen, there are always at least four solutions for these equations, some of which are inconsistent with present experimental constraints (30° ≤ \( \gamma \) ≤ 150°). (And there are other solutions, not listed, in which \( \{ \gamma_{\text{out}}, \delta'_{s,\text{out}} \} \rightarrow \{ \gamma_{\text{out}} - \pi, \delta'_{s,\text{out}} + \pi \} \).) In the table we have labeled the solutions as follows: (a) correct solution, (b) the CP angles outside of the SM domain, and (c) potential ambiguity. As is clear from the table, there are some discrete ambiguities which can not be eliminated.

This method can also be applied to the \( \Delta S = 0 \) sector [13]. The analogue of \( B_d^0(t) \rightarrow K^+K^- \) is \( B_d^0(t) \rightarrow \pi^+\pi^- \), so that this technique might be a way of eliminating the troublesome penguin contribution. However, things do not work quite as well for \( \Delta S = 0 \) decays. The main problem is that the only pure penguin decays are \( B^+ \rightarrow K^+\bar{K}^0 \) or \( B_d^0 \rightarrow K^0\bar{K}^0 \), which at the quark level are \( \bar{b} \rightarrow \bar{d}s\bar{s} \). Within flavor \( SU(3) \) symmetry, this is the same amplitude as \( \bar{b} \rightarrow \bar{d}u\bar{u} \), which contributes to \( B_d^0 \rightarrow \pi^+\pi^- \). However, \( SU(3) \)-breaking effects are likely to ruin this equality, and it is very difficult to get an accurate estimate of such effects. The analysis is also more complicated in the \( \Delta S = 0 \) sector. Since the \( u- \) and \( c- \)quark contributions to \( b \rightarrow d \) penguins may be significant, this would lead to direct CP violation in pure penguin decays in the \( \Delta S = 0 \) sector [13]. It is therefore necessary to perform a time-dependent measurement of both \( B_d^0(t) \rightarrow \pi^+\pi^- \) and \( B_d^0(t) \rightarrow K^0\bar{K}^0 \) to disentangle all the parameters.

7 Vanishing Strong Phases

In the original DGR method, if \( \delta = 0 \), the method breaks down, and additional information is required to extract the CP angles. In this section we examine what happens to
the extended DGR method with $B_s^0$ decays if all strong phases vanish.

If $\delta = 0$, Eqs. (8-13) and (20-25) reduce to 8 equations in 8 unknowns:

\begin{align*}
n A &= T^2 + \mathcal{P}^2 - 2T \mathcal{P} \cos \alpha, \\
\tilde{A} &= \tilde{T}^2 + \tilde{\mathcal{P}}^2 - 2\tilde{T} \tilde{\mathcal{P}} \cos \alpha, \\
C &= -T^2 \sin 2\alpha + 2T \mathcal{P} \sin \alpha, \\
\tilde{C} &= r_u^2 \tilde{T}^2 \sin 2\gamma - 2r_u \tilde{T} \tilde{\mathcal{P}} \sin \gamma, \\
D &= r_u^2 T^2 + \mathcal{P}'^2 - 2r_u T \mathcal{P}' \cos \gamma, \\
\tilde{D} &= r_u^2 \tilde{T}^2 + \tilde{\mathcal{P}}'^2 - 2r_u \tilde{T} \tilde{\mathcal{P}}' \cos \gamma, \\
F &= \mathcal{P}'^2, \\
\tilde{F} &= \tilde{\mathcal{P}}'^2.
\end{align*}

In this case, the equations can be solved for the 8 parameters. However, numerical methods are required, and, as in the original DGR method with $\delta \neq 0$, discrete ambiguities appear.

The solutions can be obtained as follows. First, from Eqs. (121) and (122), we have

\begin{equation}
\cos \alpha = \frac{T^2 + \mathcal{P}^2 - A}{2T \mathcal{P}} = \frac{\tilde{T}^2 + \tilde{\mathcal{P}}^2 - \tilde{A}}{2\tilde{T} \tilde{\mathcal{P}}},
\end{equation}
and from Eqs. (125-128),
\[
\cos \gamma = \frac{r_u^2 \bar{T}^2 + F - D}{2r_u \bar{T} \sqrt{\bar{F}}} = \frac{r_u^2 \tilde{T}^2 + \tilde{F} - \tilde{D}}{2r_u \tilde{T} \sqrt{\tilde{F}}}. \tag{130}
\]
Eliminating \(\alpha\) and \(\gamma\) in Eqs. (123) and (124), we obtain the following equations:
\[
C^2 = \left(-2\bar{P}^2 + \bar{P}^2 - \bar{A}^2 + 2\bar{T} \bar{P} + 2\bar{P} \bar{T}^2 - A^2 + 2\bar{T} \bar{P}^2\right)^2 \left(1 - \frac{(\bar{T}^2 + \bar{P}^2 - \bar{A}^2)^2}{4\bar{T}^2 \bar{P}^2}\right), \tag{131}
\]
\[
\tilde{C}^2 = \left(2r_u^2 \tilde{T}^2 + \tilde{F} - \tilde{D} - 2r_u \tilde{T} \sqrt{\tilde{F}}\right)^2 \left(1 - \frac{(r_u^2 \tilde{T}^2 + \tilde{F} - \tilde{D})^2}{4r_u^2 \tilde{T}^2 \tilde{F}}\right). \tag{132}
\]
We can therefore determine \(\tilde{T}\) from Eq. (132). This allows us to get \(T\) from Eq. (130), and then \(P\) from Eq. (131). The angles \(\alpha\) and \(\gamma\) can then be obtained from Eqs. (129) and (130). There are, of course, many solutions. Some of these can be eliminated by reconsidering Eqs. (123) and (124), i.e. the signs of \(C\) and \(\tilde{C}\), but multiple solutions still remain.

To illustrate this, we take \(T = 1, \tilde{T} = 0.8, P' = 1, \tilde{P}' = 0.9, P = \frac{|V_{us}| \sin \gamma}{|V_{ud}| \sin \alpha} P', \tilde{P} = 1.1P\). We choose three representative sets of values of the CP angles \(\alpha\) and \(\gamma\), and solve the equations as described above. The solutions are shown in Table 4.

As is clear from the table, there are many ambiguities (and there are additional solutions in which \(\{\alpha_{out}, \gamma_{out}\}\) are replaced by \(\{\alpha_{out} - \pi, \gamma_{out} - \pi\}\)). As before, not all solutions are allowed within the context of the SM. We have labeled the solutions as (a) correct solution, (b) one or more of the CP angles outside of the SM domain, and (c) potential ambiguity. For the particular values of \(\alpha\) and \(\gamma\) that we have chosen, there are a few solutions consistent with the SM.

Finally, we note that, for the special case in which \(\tilde{T} = T, \tilde{P} = P, \text{ and } \tilde{P}' = P'\), the system reduces to 5 equations in 5 unknowns, which can be solved just as above. One can still extract the parameters up to discrete ambiguities.

8 Conclusions

The method proposed by Dighe, Gronau and Rosner (DGR) for obtaining the CP angles \(\alpha\) and \(\gamma\) involves the measurement of the decays
\(B^0_L(t) \rightarrow \pi^+ \pi^-, B^0_d \rightarrow \pi^- K^+, B^+ \rightarrow \pi^+ K^0,\)
and their charge-conjugate processes, and assumes \(SU(3)\) flavor symmetry and first-order \(SU(3)\) breaking. This method has a number of advantages: there are no problems with electroweak penguins, all decays are accessible at asymmetric \(e^+ e^- B\) factories, and the decays involve only charged \(\pi\)'s or \(K\)'s, which are easy to detect experimentally. Even so, there are some problems as well. First, there are a large number of discrete ambiguities in the extraction of the CP angles. Second, some theoretical assumptions are required:
the strong phases in $\Delta S = 0$ transitions are assumed to be equal to their counterparts in the $\Delta S = 1$ transitions, even in the presence of $SU(3)$ breaking, and the $b \to d$ penguin is assumed to be dominated by an internal $t$ quark. If either of these assumptions is relaxed, then there is not enough information to determine the CP angles. Finally, if all strong phases vanish, the method again breaks down.

In this paper, we have proposed an extension of this method which avoids most of the problems with the DGR method. In addition to the $B_0^d$ and $B^+$ decays used by DGR, it requires the measurement of their $SU(3)$-counterpart $B_s^0$ decays: $B_s^0 \to \pi^+K^-$, $B_s^0(t) \to K^+K^-$, and $B_s^0 \to K^0\bar{K}^0$. This overconstrains the system, which eliminates most discrete ambiguities in the extraction of $\alpha$ and $\gamma$. Furthermore, if DGR’s assumptions are relaxed, there is still enough information in most cases to obtain the CP angles.

We have found the following results:

1. If we make the same assumptions as DGR, we are able to extract $\alpha$ and $\gamma$ up to a 2-fold ambiguity, corresponding to the unitarity triangle pointing up or down. In the special case where the magnitudes of the amplitudes are independent of the spectator quark, it is still possible to extract the CP angles up to the same 2-fold ambiguity.

2. If we allow the strong phases in the $\Delta S = 1$ transitions to be different from those in the $\Delta S = 0$ transitions, we can still obtain $\alpha$ and $\gamma$ with a 2-fold ambiguity. If the magnitudes of the amplitudes are independent of the spectator quark, then it is still possible to extract the CP angles, but there are multiple discrete ambiguities. Many of these can be eliminated by imposing the present experimental constraints on the angles, but of course one might be overlooking the presence of new physics by doing so.

3. If we consider nonzer ou- and $c$-quark contributions to the $b \to d$ penguin, then, strictly speaking, the method partially breaks down. The angle $\alpha$ cannot be extracted, but $\gamma$ can still be obtained up to a 2-fold ambiguity. However, if the $u$- and $c$-quark penguins are much smaller than the tree diagram (even if they are not negligible compared to the $t$-quark penguin) then this situation reduces to the case of different phases in the $\Delta S = 0$ and $\Delta S = 1$ transitions, described above. So it may still be possible to obtain $\alpha$ in this case, but some theoretical uncertainty may be introduced.

4. When one includes $B_s^0$ decays, there are two distinct ways in which $SU(3)$ can be broken: (i) $\Delta S = 0$ processes vs. $\Delta S = 1$ processes, and (ii) $B_s^0$ decays vs. $B_0^d$ decays. Therefore, if one includes all types of first-order $SU(3)$ breaking, four different strong
phases must be considered. In this case, we find that the CP angles $\alpha$ and $\gamma$ can be extracted, but with multiple discrete ambiguities.

5. If all strong phases vanish, then one can still obtain $\alpha$ and $\gamma$, up to multiple discrete ambiguities.

Finally, we have found a new method of measuring $\gamma$. By measuring $B^0_s(t) \to K^+K^-$ and $B^0_s \to K^0\bar{K}^0$, it is possible to obtain $\gamma$, up to discrete ambiguities, with no hadronic uncertainties. Experimentally this will be difficult, as it requires isolating the tree contribution to $B^0_s \to K^+K^-$ by “subtracting off” the penguin contribution. However, this penguin contribution is much larger than the tree, which means that one is essentially subtracting two big numbers to get a small number. Still, B-physics experiments at hadron colliders may have the precision to carry out such measurements.

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