I. INTRODUCTION

There are compelling observational evidences that about 25% of today’s energy density of the Universe is made of dark matter (DM). The existence of DM has been probed by temperature anisotropies in the Cosmic Microwave Background (CMB) [1, 2] as well as by the galaxy-clustering surveys ranging from large-scale superclusters down to small-scale dwarf galaxies [3–5]. To reveal the origin of DM is one of the most challenging problems in modern cosmology and particle physics.

From the theoretical viewpoint, ultra-light bosons like axions can be good candidates for DM [6, 7]. The axion is a pseudo-Nambu Goldstone boson originally introduced to address the strong CP problem in quantum chromodynamics (QCD) [8–10]. The mass of QCD axions is in the range $m \gtrsim 10^{-6}$ eV to avoid overclosing the Universe [11]. String theory also gives rise to axions as Kaluza-Klein zero modes of anti-symmetric form fields [12, 13]. Depending on the geometry of string-theory compactification, the mass of axions can span over the light range from $10^{-33}$ eV to $10^{-10}$ eV [14]. In such cases, the axion affects the late-time cosmological dynamics as all or a part of DM [15, 16]. For the mass $m \sim 10^{-33}$ eV, the axion potential energy can even work as dark energy (DE) [17–21].

Cosmologically, the axion field $\phi$ is nearly frozen up to the instant where the Universe expansion rate $H$ drops below its mass $m$ [11, 22–26]. The moment at which the axion starts to oscillate around the minimum of the potential $V(\phi) = m^2 \phi^2/2$ can be quantified by the condition $m \approx 3H$ [16, 26]. After many times of oscillations, the axion field behaves as nonrelativistic matter in the form of a Bose-Einstein condensate (BEC) [27]. In this BEC state, the bosonic particles behave as a classical coherent wave with the Compton wavelength $\lambda_{\phi} \sim m^{-1}$. For the ultra-light axion mass mentioned above, the Compton wavelength can reach the galactic scales and hence there is an intriguing possibility for probing observational signatures of such “fuzzy DM” [28, 29].

Indeed, the BEC has a “quantum pressure” which works against the gravitational clustering below a certain scale $\lambda_J$ [28, 30–33]. This Jeans scale corresponds to the de Broglie wavelength of a particle in the BEC ground state. For scales below $\lambda_J$ the quantum pressure manifests itself by the uncertainty principle, while the BEC density perturbation on scales larger than $\lambda_J$ grows as in the standard Cold-Dark-Matter (CDM). For the mass range around $m = 10^{-22}$ eV $\sim 10^{-21}$ eV, the BEC DM can suppress the small-scale matter power below the 1 Kpc scale [28]. This allows a possibility for alleviating the excess of abundances of dwarf galaxies present in the CDM model. The statistical analysis of Refs. [34, 35] using Lyman-$\alpha$ forest data of the small-scale matter power spectrum placed the 2$\sigma$ bound $m > 2 \times 10^{-21}$ eV. On the other hand, there is a claim that the mass of order $10^{-22}$ eV is still allowed due to uncertainties in a thermal state of the high-redshift intergalactic medium [36].
For the smaller axion mass $m \lesssim 10^{-27}$ eV, the scalar field starts to oscillate after matter-radiation equality [16, 26]. In this case, the Jeans scale $\lambda_J$ can be within the observable range of linear matter power spectrum and CMB temperature anisotropies. Since the quantum pressure suppresses the gravitational instability of the BEC density contrast for scales smaller than $\lambda_J$, the axion field can not be all DM in this ultra-light mass region. Indeed, for the mass $10^{-32}$ eV $< m \lesssim 10^{-25.5}$ eV, today’s axion density parameter is constrained to be less than 5 % of all DM [16]. Even with such a small density parameter, the axion coupling to photons gives rise to an interesting possibility for explaining the isotropic birefringence [37, 38] recently reported by analyzing the Planck2018 data [39] (see also Refs. [40, 41]).

The standard BEC has a self-interaction whose effective potential is related to the s-scattering length. For axions, expanding the periodic potential $V(\phi) = m^2 f^2[1 - \cos(\phi/f)]$ around $\phi = 0$ gives rise to an effective self-coupling energy density $\lambda_\phi \phi^4$ with $\lambda_\phi = -m^2/(24 f^2)$. Since the coupling constant $\lambda_\phi$ is negative for axions, this leads to an attractive self-interaction. In a self-gravitating system of BEC, there is a possibility that this attractive force enhances the gravitational instability of BEC. The effects of self-interactions on the dynamics of BEC perturbations and the formation of boson stars have been studied in Refs. [42–56]. For attractive interactions there are some particular scales in which the effective sound speed squared of linear BEC cosmological perturbations becomes negative, which can induce Laplacian instabilities.

Most of the works about the BEC cosmological perturbations in the literature have been based on the nonrelativistic Gross-Pitaevskii-Poisson (GPP) equation for a single wave function [44, 57–60]. The GPP equation, which follows from a Hamiltonian of interacting condensed bosons, corresponds to the Newtonian limit of the self-gravitating nonrelativistic BEC [61]. With a Madelung representation of the wave function [62], the BEC perturbation equations of motion can be expressed in terms of Newtonian analogue of the continuity and Euler equations. It is not yet clear whether this Newtonian approach remains valid for large-scale perturbations where the general relativistic effect on the dynamics of inhomogeneities comes into play.

In this paper, we take the full general relativistic, covariant approach to the study of cosmological perturbations for nonrelativistic BEC. We begin with an explicit Lagrangian of a complex massive scalar field $\chi$ preserving a $U(1)$ charge, with its self-interaction involved. The similar treatment of BECs with the Lagrangian description was performed in Refs. [63–65], but the full relativistic treatment including the effect on metric perturbations was not fully addressed yet. We also take into account CDM, baryons, and radiation as perfect fluids to accommodate the case in which the BEC is not responsible for all DM. We clarify the difference between BEC and perfect fluids in terms of the covariant description of continuity and Euler equations. We derive the full linear perturbation equations without fixing any particular gauge conditions and express them in terms of gauge-invariant variables. Thus, these master equations can be applied to any convenient gauge choices at hand.

We also study how the perturbation equation of the BEC density contrast can be recovered in the Newtonian limit. Even though the scalar-field oscillation is integrated out at the background level by the Madelung transformation, this oscillating mode appears in the perturbation equations. However, we show that the existence of this mode hardly affects the dynamics of matter perturbations. The approximate second-order equation for the density contrast, which we will derive in the Newtonian limit with the neglect of the oscillating mode, is in good agreement with the full numerical solution except for wavelengths close to the Hubble radius. In addition, we will see that the density contrasts of perfect fluids like CDM and baryons are affected by the BEC sound speed $c_2^2$ through their gravitational interactions with BEC.

We discuss the effect of BEC self-interactions on the dynamics of density perturbations as well. For the mass range $m = 10^{-22}$ eV $\sim 10^{-21}$ eV within which the axion can be the source for all DM, the axion self-interaction is relevant to the growth of linear perturbations only in the deep radiation era [54]. In this case, there are particular scales around 1 Mpc in which the self-interaction dominates over the quantum pressure. For the ultra-light mass range $m \lesssim 10^{-27}$ eV, the self-interaction can be important on larger scales which are in the observational range of CMB and linear matter spectra. In this latter case, we will study its effect on the dynamics of BEC and CDM/baryon density contrasts during matter dominance. We show that, in spite of a negative value of $c_2^2$ for some particular scales, the Laplacian instability of BEC perturbations is suppressed relative to the gravitational instability in the regime where the nonrelativistic BEC description is valid. Since this description amounts to averaging over the field oscillations during the Hubble time scale, it does not accommodate the transient epoch toward the BEC formation during which parametric resonance can potentially enhance the BEC density contrast [51–53, 56].

Throughout the paper, we use the natural unit where the speed of light $c$ and the reduced Planck constant $\hbar$ are equivalent to 1. The reduced Planck mass $M_p$ is related to the gravitational constant $G$, as $M_p = (8\pi G)^{-1/2}$. We take present-day Hubble constant $H_0 = 100 h$ km sec$^{-1}$ Mpc$^{-1}$ = 2.1331 $\times$ 10$^{-33}$ h eV with $h = 0.677$, and choose today’s density parameters of total nonrelativistic matter, baryons, and DE as $\Omega_m = 0.31$, $\Omega_b = 0.05$, and $\Omega_\Lambda = 0.69$, respectively. The present-day BEC and CDM density parameters, which are given by $\Omega_{\chi_0}$ and $\Omega_\phi$, respectively, satisfy the relation $\Omega_{\chi_0} + \Omega_\phi = \Omega_m - \Omega_b = 0.26$. The scale factor at matter-radiation equality is chosen to be $a_{eq} = 1/3400$, with today’s value $a_0 = 1$. 
II. NONRELATIVISTIC BEC AND BACKGROUND COSMOLOGY

We begin with a complex scalar field $\chi$ given by the action

$$S = \int d^4x \sqrt{-g} \left[ \frac{R}{16\pi G} - \nabla^\mu \chi^* \nabla_\mu \chi - m^2 \chi^* \chi - U(\chi^* \chi) \right], \quad (2.1)$$

where $g$ is the determinant of metric tensor $g_{\mu\nu}$, $R$ is the Ricci scalar, and $\nabla^\mu$ is the covariant derivative operator. The scalar field has a constant mass $m$ with self-interactions described by the potential $U$. For an interacting Bose field, the self-coupling potential $U$ depends on the particle probability density given by

$$\rho = \chi^* \chi. \quad (2.2)$$

The two-body interaction is described by the potential $U(\rho) = \lambda \rho^2 / 4$ with a coupling constant $\lambda$. The attractive and repulsive self-interactions correspond to $\lambda < 0$ and $\lambda > 0$, respectively. Many-body interactions contain the terms higher than the order $\rho$ in $U(\rho)$.

In this section we only include the field $\chi$ in the matter sector, but we will take additional matter sources (CDM, baryons, and radiation) into account in Sec. III. Varying the action (2.1) with respect to $\chi^*$, it follows that

$$\Box \chi - m^2 \chi - U_{,\rho} \chi = 0, \quad (2.3)$$

where $U_{,\rho} \equiv dU / d\rho$, and

$$\Box \chi \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu \chi = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \chi \right), \quad (2.4)$$

with the notation $\partial_\nu \chi \equiv \partial \chi / \partial x^\nu$.

Varying the action (2.1) with respect to $g_{\mu\nu}$ leads to the Einstein equation

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \quad (2.7)$$

where $G_{\mu\nu}$ is the Einstein tensor, and $T_{\mu\nu}$ is the energy-momentum tensor given by

$$T_{\mu\nu} = \nabla_\mu \chi^* \nabla_\nu \chi + \nabla_\mu \chi \nabla_\nu \chi^* - g_{\mu\nu} \left[ g^{\alpha\beta} \nabla_\alpha \chi^* \nabla_\beta \chi + m^2 \chi^* \chi + U(\rho) \right]. \quad (2.8)$$

We are interested in the cosmology on the spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) background given by the line element

$$ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j, \quad (2.9)$$

where $a(t)$ is the time-dependent scale factor. A key quantity which determines the transition to the BEC formation is the Hubble expansion rate $H \equiv \dot{a} / a$ in comparison to the mass $m$, where a dot represents the derivative with respect to the cosmic time $t$.

A. Covariant hydrodynamical equations

We derive the hydrodynamical equations of motion in the nonrelativistic regime where the field $\chi$ oscillates with the frequency $m$. Cosmologically, the field oscillation starts when $H$ drops below the order of $m$, more precisely, $H < m/3$ [26]. After averaging over many oscillations, the kinetic term $\nabla^\mu \chi^* \nabla_\mu \chi$ has the same contribution to the energy density as $m^2 \chi^* \chi$. Then, we introduce the energy density associated with the massive $\chi$ field, as

$$\rho_\chi = 2m^2 \chi^* \chi = 2m^2 \rho. \quad (2.10)$$
In the regime \( m \gg H \), the \( \chi \) field behaves as a single classical wave of the nonrelativistic BEC. To describe this condensed stage of bosons, we take the Madelung representation \([62]\) in the form

\[
\chi = \sqrt{\frac{\rho_\chi}{2m^2}} e^{i\theta},
\]

(2.11)

where the phase part \( \theta \) is given by

\[
\theta = -mt - mv_\chi.
\]

(2.12)

The term \(-mt\) in \( \theta \) characterizes the oscillation of \( \chi \) induced by the mass \( m \). The scalar quantity \( v_\chi \), which depends on both time \( t \) and space \( x^i \), corresponds to the velocity potential. This latter contribution is dealt as a perturbation on the background (2.9). We note that including the term \(-mt\) in \( \theta \) allows one to eliminate rapidly oscillating terms in \( \rho_\chi \) at the background level. The self-coupling term \( U_\rho \chi \) in Eq. (2.3) leads to the deviation from the coherent oscillation of \( \chi \) with frequency \( m \). In Sec. II B, we will derive conditions for the validity of the nonrelativistic BEC description in the presence of self-couplings.

Substituting Eq. (2.11) into Eq. (2.6), the current \( j_\mu \) can be expressed as

\[
j_\mu = \frac{\rho_\chi v_\mu}{m} = n_\chi v_\mu,
\]

(2.13)

where \( n_\chi = \rho_\chi/m \) is the particle number density, and \( v_\mu \) is the four vector field defined by

\[
v_\mu \equiv \frac{\partial_\mu \theta}{m} = (-1 - \dot{v}_\chi, -\partial_i v_\chi).
\]

(2.14)

From the definition (2.14), the vector field \( v_\mu \) obeys the irrotational relation

\[
\nabla_\mu v_\nu = \nabla_\nu v_\mu.
\]

(2.15)

Then, the current conservation (2.5) translates to the continuity equation

\[
\nabla_\mu (\rho_\chi v_\mu) = 0.
\]

(2.16)

Substituting Eq. (2.11) into Eq. (2.3) and using Eq. (2.16), it follows that \([63, 64]\)

\[
v^\mu v_\mu = -1 + \beta,
\]

(2.17)

where

\[
\beta \equiv \frac{\Box \sqrt{\rho_\chi}}{m^2 \sqrt{\rho_\chi}} - \frac{U_\rho}{m^2}.
\]

(2.18)

We define the four velocity as

\[
u_\mu \equiv \frac{v_\mu}{\sqrt{-v_\alpha v^\alpha}},
\]

(2.19)

which satisfies the normalization \( u_\mu u^\mu = -1 \). Taking the covariant derivative of Eq. (2.17) and exploiting the irrotational property (2.15) of \( v_\mu \), we obtain

\[
v^\mu \nabla_\mu v_\nu = \frac{1}{2} \nabla_\nu \beta.
\]

(2.20)

This is the analogue of the Euler equation in the hydrodynamical mechanics.

Substituting Eq. (2.11) into Eq. (2.8), the matter energy-momentum tensor can be expressed as

\[
T_{\mu\nu} = \rho_\chi v_\mu v_\nu + \frac{\nabla_\mu \rho_\chi \nabla_\nu \rho_\chi}{4m^2 \rho_\chi} - g_{\mu\nu} \left[ \frac{\rho_\chi}{2} (v^\alpha v_\alpha + 1) + \frac{\nabla^\alpha \rho_\chi \nabla_\alpha \rho_\chi}{8m^2 \rho_\chi} + U(\rho) \right].
\]

(2.21)

The dynamics of nonrelativistic BEC is governed by the continuity Eq. (2.16) and Euler Eq. (2.20) as well as by the Einstein Eq. (2.7) with the energy-momentum tensor (2.21).
B. FLRW background

Let us consider the flat FLRW background given by the line element (2.9). Since \( v_\chi = 0 \) on this background, the vector field \( v_\mu \) in Eq. (2.14) reduces to the four velocity \( u_\mu = (-1, 0, 0, 0) \). Then, the continuity Eq. (2.16) gives

\[
\dot{\rho}_\chi + 3H\rho_\chi = 0 ,
\]

(2.22)

so that the BEC energy density decreases as \( \rho_\chi \propto a^{-3} \). From Eq. (2.17) we have

\[
\beta = 0 ,
\]

(2.23)

where

\[
\beta = \frac{\dot{\rho}_\chi^2 - 2\rho_\chi(\dot{\rho}_\chi + 3H\dot{\rho}_\chi)}{4m^2\rho_\chi^2} - \frac{U_{,\rho}}{m^2} = \frac{3(2\dot{H} + 3H^2)}{4m^2} - \frac{U_{,\rho}}{m^2} .
\]

(2.24)

In the second equality of Eq. (2.24), we used Eq. (2.22) and its time derivative. During exact matter dominance \((a \propto t^{2/3})\) there is the relation \(2\dot{H} + 3H^2 = 0\), in which case the property (2.23) holds in the absence of the self-coupling potential \((U = 0)\). During the radiation era \((a \propto t^{1/2})\), we have \(\dot{H} + 2H^2 = 0\) and hence \(\beta = -3H^2/(4m^2) - U_{,\rho}/m^2\). Then, the relation (2.23) approximately holds for

\[
m^2 \gg H^2 , \quad \text{and} \quad m^2 \gg |U_{,\rho}| .
\]

(2.25)

These conditions ensure the validity of the nonrelativistic BEC description based on the Madelung representation (2.11). We note that the Euler Eq. (2.20) is trivially satisfied on the FLRW spacetime.

On the background (2.9), the scalar-field equation (2.3) yields

\[
\ddot{\chi} + 3H\dot{\chi} + (m^2 + U_{,\rho})\chi = 0 .
\]

(2.26)

Under the conditions (2.25), Eq. (2.26) approximately yields \((a^{3/2}\chi)^{\prime} + m^2(a^{3/2}\chi) \approx 0\). Then, the scalar field exhibits a damped oscillation described by the solution \(\chi = \chi_i (a/a_i)^{-3/2}e^{-int}\), where \(\chi_i\) and \(a_i\) are constants. Comparing the amplitude of this solution with Eq. (2.11), the field energy density evolves as \(\rho_\chi = 2m^2\chi_i^2(a/a_i)^{-3}\), which is consistent with Eq. (2.22). In the early cosmological epoch where the condition \(m^2 \ll H^2\) is satisfied, the field slowly evolves along the potential with the second time derivative \(\ddot{\chi}\) negligible relative to the other terms in Eq. (2.26). After \(H\) drops below the order of \(m\), the scalar field starts to oscillate around the potential minimum. The onset of this oscillation is characterized by the condition \(m \gtrsim 3H\) [16, 26].

Substituting Eq. (2.14) into Eq. (2.21) and using the relation (2.23) with Eq. (2.24), the nonvanishing components of \(T_{\mu\nu}\) are \(T_{00} = \rho_{\text{eff}}\) and \(T_{ij} = a^2 P_{\text{eff}}\delta_{ij}\), where \(\rho_{\text{eff}}\) and \(P_{\text{eff}}\) are the effective field energy density and pressure given by

\[
\rho_{\text{eff}} = \rho_\chi + \frac{\rho_\chi^2}{8m^2\rho_\chi} + U , \quad P_{\text{eff}} = \frac{\rho_\chi^2}{8m^2\rho_\chi} - U .
\]

(2.27)

From the Einstein equation (2.7), we obtain

\[
3H^2 = 8\pi G\rho_{\text{eff}} ,
\]

(2.28)

\[
3H^2 + 2\dot{H} = -8\pi G P_{\text{eff}} .
\]

(2.29)

Under the first condition of Eq. (2.25), the term \(\rho_\chi^2/(8m^2\rho_\chi)\) in Eq. (2.27), which is identical to \(9H^2\rho_\chi/(8m^2)\), is suppressed relative to \(\rho_\chi\). As long as the condition

\[
\rho_\chi \gg |U|
\]

(2.30)

is satisfied, it follows that \(\rho_{\text{eff}} \approx \rho_\chi\) and \(|P_{\text{eff}}/\rho_{\text{eff}}| \ll 1\). In this case, the cosmological dynamics dominated by the rapidly oscillating scalar field over the Hubble time scale \(H^{-1}\) is equivalent to that of the matter era characterized by \(H \approx -3H^2/2\) and \(a \propto t^{2/3}\).

The above result shows that, under the conditions (2.25) and (2.30), the oscillating scalar field can be the source for DM in a state of the nonrelativistic BEC. Using the subscript “osc” at the onset of oscillations, the field energy density today \((a = 1)\) is given by \(\rho_{\chi,0} = (\rho_\chi_{\text{osc}})_{\text{osc}}a_{\text{osc}}^3\). By the end of this section, we consider the case in which the
quadratic potential $m^2|\chi|^2$ dominates over $U$. Then, the initial field density can be estimated as $(\rho_\chi)_{osc} \simeq m^2|\chi_{osc}|^2$. Then, today’s density parameter of the field $\chi$ is given by

$$\Omega_\chi 0 = \frac{\rho_\chi 0}{3M_{pl}^2H_0^2} \simeq \frac{m^2|\chi_{osc}|^2a_{osc}^3}{3M_{pl}^2H_0^2},$$  

(2.31)

where $M_{pl} = (8\pi G)^{-1/2}$ is the reduced Planck mass and $H_0 = 2.1331 \times 10^{-33} h$ eV is the Hubble constant. To estimate the Hubble expansion rate as a function of the scale factor, we take radiation, CDM, baryons, and DE into account. Expressing today’s density parameters of total nonrelativistic matter and DE as $\Omega_{M0}$ and $\Omega_{\Lambda 0}$, respectively, and assuming that the origin of DE is the cosmological constant, the Hubble parameter can be expressed as

$$H(a) = H_0\sqrt{\Omega_{M0}(a + a_{eq})a^{-4} + \Omega_{\Lambda 0}},$$  

(2.32)

where $a_{eq} = \Omega_{r0}/\Omega_{M0} \simeq 1/3400$ is the scale factor at matter-radiation equality ($\Omega_{r0}$ is today’s radiation density parameter). The scale factor $a_*$ at the onset of oscillations can be identified by the condition

$$m = 3H(a_{osc}).$$  

(2.33)

The Hubble parameter at matter-radiation equality is given by $H(a_{eq}) \simeq H_0\sqrt{2\Omega_{M0}a_{eq}^{-3}} \simeq 2.3 \times 10^{-28}$ eV, where we used the values $\Omega_{M0} = 0.31$ and $h = 0.677$. For $m > 7 \times 10^{-28}$ eV the scalar field starts to oscillate during radiation domination, while, for $m < 7 \times 10^{-28}$ eV, the oscillation begins in the matter era.

During radiation dominance ($a \ll a_{eq}$) the Hubble parameter is approximately given by $H(a) \simeq H_0\sqrt{\Omega_{r0}a^{-4}}$, so that $a_{osc} \simeq (9H_0^2\Omega_{r0}/m^2)^{1/4}$. Substituting this relation into Eq. (2.31), it follows that [16, 26]

$$\Omega_\chi 0 \simeq \sqrt{\Omega_{M0}^3/4}(m/\sqrt{H_0})^{1/2}\left|\chi_{osc}/M_{pl}\right|^2 \quad \text{for} \quad m > 7 \times 10^{-28} \text{ eV}.$$  

(2.34)

During matter dominance we have $H(a) \simeq H_0\sqrt{\Omega_{M0}a^{-3}}$ and hence $a_{osc} \simeq (9H_0^2\Omega_{M0}/m^2)^{1/3}$. Then, Eq. (2.31) reduces to

$$\Omega_\chi 0 \simeq 3\Omega_{M0}\left|\chi_{osc}/M_{pl}\right|^2 \quad \text{for} \quad 1 \times 10^{-33} \text{ eV} \ll m < 7 \times 10^{-28} \text{ eV}.$$  

(2.35)

For $m = O(10^{-33})$ eV, the field is nearly frozen until recently, but this is the region in which the field energy density works as DE rather than DM.

### III. COSMOLOGICAL PERTURBATIONS

Now, we proceed to the study of linear cosmological perturbations on top of the flat FLRW background. Besides the $\chi$ field described by the action (2.1), we take the perfect fluids of baryons, CDM, DE, and radiation (photons and neutrinos) into account, which are labelled by $b$, $c$, $d$, and $r$ ($\gamma$ and $\nu$) respectively. If the $\chi$ field is responsible for all DM, we do not need to include CDM in the matter action. The source for DE can be the cosmological constant or other dynamical fields [66], but it is also possible to realize the late-time cosmic acceleration in DE models of a perfect fluid [67]. Then, the total system is described by the action

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{16\pi G} - \nabla^\mu \chi^* \nabla_\mu \chi - m^2\chi^* \chi - U(\chi^* \chi)\right] + \int d^4x L_{pf},$$  

(3.1)

where $L_{pf}$ is the perfect-fluid Lagrangian given by [68–72]

$$L_{pf} = - \sum_{I=b,c,d,r} \sqrt{-g} \left[\rho_I(n_I) + j_I^\mu \partial_\mu \ell_I\right].$$  

(3.2)

The Lagrangian (3.2) consists of the energy density $\rho_I$, the current vector $j_I^\mu$, and the Lagrange multiplier $\ell_I$, where $\rho_I$ is a function of the fluid number density $n_I$. Varying the Lagrangian $L_{pf}$ with respect to $\ell_I$, there is the current conservation

$$\nabla_\mu j_I^\mu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} j_I^\mu) = 0,$$  

(3.3)

1 In Refs. [67, 71, 72] the quantity $\tilde{J}_I^\mu = \sqrt{-g} j_I^\mu$ is used instead of $j_I^\mu$, in which case the current conservation (3.3) is expressed as $\partial_\mu \tilde{J}_I^\mu = 0$. 


which is analogous to Eq. (2.5) of the complex scalar field $\chi$. The number density and four velocity of each matter species are given, respectively, by

$$n_I = \sqrt{-g_{\mu\nu} j_I^\mu j_I^\nu}, \quad (3.4)$$

$$u_{I\mu} = \frac{j_{I\mu}}{n_I}. \quad (3.5)$$

From Eqs. (3.4) and (3.5), the four velocity $u_{I\mu}$ satisfies the normalization

$$u_{I\mu}u_I^\mu = -1. \quad (3.6)$$

The variation of $L_{pf}$ with respect to $j_I^\mu$ leads to

$$\partial_\mu \ell_I = \rho_{I,n_I} u_{I\mu}. \quad (3.7)$$

On using the relation (3.5), the current conservation (3.3) translates to

$$\nabla_\mu (n_I u_I^\mu) = 0, \quad (3.8)$$

or equivalently,

$$u_I^\mu \nabla_\mu \rho_I + (\rho_I + P_I) \nabla_\mu u_I^\mu = 0, \quad (3.9)$$

where $P_I$ is the pressure defined by

$$P_I \equiv n_I \rho_{I,n_I} - \rho_I. \quad (3.10)$$

For nonrelativistic matter with $n_I = \rho_I / m_I$ and mass $m_I$, the continuity Eq. (3.8) is analogous to Eq. (2.16) of the BEC. We note, however, that the four vector $u^{\mu}$ does not correspond to the four velocity $u_I^\mu$, so there is the difference between Eqs. (2.16) and (3.8) at the level of perturbations (as we will see in Sec. IIIA).

On using Eq. (3.7) with Eqs. (3.5) and (3.10), the current vector $j_{I\mu}$ is related to the Lagrange multiplier $\ell_I$ as

$$j_{I\mu} = \frac{n_I^2}{\rho_I + P_I} \partial_\mu \ell_I. \quad (3.11)$$

On the other hand, the current (2.13) of the BEC is given by $j_{\nu} = (\rho_\chi / m_\chi^2) \partial_\nu \theta$, where we used Eq. (2.14). Since there is the relation $m = \rho_\chi / n_\chi$ for the nonrelativistic BEC, this current reduces to $j_{I\mu} = (n^2 / \rho_\chi) \partial_\mu \theta$. Comparing it to Eq. (3.11), the quantity $\theta$ in the Madelung representation (2.11) has the correspondence with $\ell_I$ in the perfect-fluid Lagrangian (3.2) with the vanishing pressure ($P_I = 0$).

Taking the covariant derivative of Eq. (3.7), the four velocity $u_{I\mu}$ satisfies the relation

$$\nabla_\nu u_{I\mu} - \nabla_\mu u_{I\nu} = \frac{1}{\rho_{I,n_I}} \left( u_{I\nu} \nabla_\mu \rho_{I,n_I} - u_{I\mu} \nabla_\nu \rho_{I,n_I} \right). \quad (3.12)$$

If we consider nonrelativistic matter with the mass $m_I$ and density $\rho_I = m_I n_I$, we have $\rho_{I,n_I} = m_I$ constant and hence $\nabla_\nu u_{I\mu} = \nabla_\mu u_{I\nu}$. Exerting the operator $\nabla_\nu$ for Eq. (3.6) and using the property (3.12), it follows that

$$u_I^\mu \nabla_\mu u_I^\nu = -\frac{1}{\rho_{I,n_I}} \left( \frac{1}{\rho_{I,n_I}} \left( u_{I\nu} u_I^\mu \nabla_\mu \rho_{I,n_I} + \nabla_\nu \rho_{I,n_I} \right) \right), \quad (3.13)$$

which corresponds to the Euler equation for the perfect fluid. For nonrelativistic matter the right hand-side of Eq. (3.13) vanishes, so that the Euler equation is simplified to $u_I^\mu \nabla_\mu u_I^\nu = 0$. The Euler Eq. (2.20) of the BEC is different from that of the nonrelativistic perfect fluid, in that the right hand-side of Eq. (2.20) contains the derivative term $\nabla_\nu \beta / 2$.

The energy-momentum tensor $T^{(pf)}_{\mu\nu}$ associated with the perfect-fluid Lagrangian (3.2) follows by its variation with respect to $g^{\mu\nu}$. On using the properties $\delta \sqrt{-g} / \delta g^{\mu\nu} = -(1/2) \sqrt{-g} g^{\mu\nu}$, $\delta n_I / \delta g^{\mu\nu} = (n_I^2 / 2) (g^{\mu\nu} - u_{I\mu} u_{I\nu})$, and Eq. (3.7), we have

$$T^{(pf)}_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta L_{pf}}{\delta g^{\mu\nu}} = \sum_I \left[ (\rho_I + P_I) u_{I\mu} u_{I\nu} + P_I g_{\mu\nu} \right]. \quad (3.14)$$

Varying the total action (3.1) with respect to $g^{\mu\nu}$, we obtain the gravitational field equation of motion

$$G_{\mu\nu} = 8\pi G \left[ T_{\mu\nu} + T^{(pf)}_{\mu\nu} \right], \quad (3.15)$$

where $T_{\mu\nu}$ is given by Eq. (2.8). Since we are interested in the regime where the BEC is formed, we will employ the energy-momentum tensor of the form (2.21) in the following.
A. Perturbation equations in a gauge-ready form

The general perturbed line element containing four scalar metric perturbations is given by [73–75]

\[ ds^2 = -(1 + 2a)dt^2 + 2\partial_i Bdx^i + a^2(t) \left[ (1 + 2\zeta)\delta_{ij} + 2\partial_i \partial_j E \right] dx^i dx^j, \]

(3.16)

where \( a, B, \zeta, E \) depend on both cosmic time \( t \) and spatial coordinates \( x^i \). Unlike Refs. [76, 77], we do not take intrinsic vector perturbations into account as they are nondynamical for the theory under consideration. The evolution of tensor perturbations is the same as that in standard general relativity. We first derive the linear perturbation equations of motion without choosing particular gauges and then express them in terms of gauge-invariant quantities.

For the nonrelativistic BEC, the energy density \( \rho_\chi \) and the quantity \( \beta \), which are defined respectively by Eqs. (2.10) and (2.18), are decomposed into the background and perturbed parts, as

\[ \rho_\chi = \bar{\rho}_\chi + \delta\rho_\chi, \quad \beta = \bar{\beta} + \delta\beta, \]

(3.17)

where a bar represents the background values, and \( \bar{\beta} = 0 \). As we will see below, the perturbation equations for matter perturbation \( \delta\rho_\chi \) and velocity potential \( v_\chi \) follow by substituting Eqs. (2.14) and (3.17) into the continuity Eq. (2.16) and the Euler Eq. (2.20).

For the perfect-fluid sector, the fluid number density (3.4) is decomposed into the background and perturbed parts, as \( n_I = \bar{n}_I + \delta n_I \). Since the fluid density \( \rho_I \) depends on its number density \( n_I \), the matter perturbation is given by

\[ \delta\rho_I = \rho_{I,\bar{n}_I}\delta n_I = \frac{\rho_I + P_I}{n_I}\delta n_I. \]

(3.18)

Here and in the following, we omit a bar from the background quantities. For the line element (3.16), the temporal and spatial components of the fluid four velocity \( u_{I\mu} \), up to first order in perturbations, are [71, 72]

\[ u_{I0} = -1 - \alpha, \quad u_{Ii} = -\partial_i v_I, \]

(3.19)

where \( v_I \) is the velocity potential. From Eq. (3.5), the components of \( j_{I\mu} \) are expressed as

\[ j_{I0} = -n_I - n_I\alpha - P_I\delta\rho_I, \quad j_{Ii} = -n_I\partial_i v_I. \]

(3.20)

In the following, we will use \( \delta\rho_I \) and \( v_I \) instead of \( j_{I0} \) and \( j_{Ii} \) for the derivation of perturbation equations of motion. We also introduce the equation of state and adiabatic sound speed squared, as

\[ w_I = \frac{P_I}{\rho_I}, \quad c_I^2 = \frac{\dot{P}_I}{\dot{\rho}_I} = \frac{n_I\rho_{I,\bar{n}_I}n_I}{\rho_{I,n_I}}. \]

(3.21)

Let us first consider the background equations of motion. For the nonrelativistic BEC, the \( \chi \) field obeys the continuity equation (2.22), i.e.,

\[ \dot{\rho}_\chi + 3H\rho_\chi = 0. \]

(3.22)

On the flat FLRW background (2.9), the four velocity of each perfect fluid is given by \( u_{I\mu} = (-1, 0, 0, 0) \), so the continuity Eq. (3.9) leads to

\[ \dot{\rho}_I + 3H(1 + w_I)\rho_I = 0, \quad \text{for} \quad I = b, c, d, r, \]

(3.23)

whereas the Euler Eq. (3.13) trivially holds. From Eq. (3.14), the nonvanishing components of perfect-fluid energy-momentum tensors are given by \( T_{00} = \sum_I \rho_I \) and \( T_{ij} = \sum_I a^2 P_I\delta_{ij} \). Then, the \((00)\) and \((ii)\) components of the Einstein Eq. (3.15) give

\[ 3H^2 = 8\pi G \left( \rho_\chi + \frac{\dot{\rho}_\chi^2}{8m^2\rho_\chi} + U + \sum_I \rho_I \right), \]

(3.24)

\[ 3H^2 + 2\dot{H} = -8\pi G \left( \frac{\dot{\rho}_\chi^2}{8m^2\rho_\chi} - U + \sum_I w_I\rho_I \right), \]

(3.25)

respectively, where we used Eq. (2.27).
Expanding Eqs. (2.16) and (2.20) up to first order in perturbations, the matter perturbation $\delta \rho_x$ and velocity potential $v_x$ obey

$$
\delta \rho_x + 3H \delta \rho_x + \rho_x \left(3 \dot{\zeta} - \frac{\partial^2 v_x}{a^2} - \frac{\partial^2 B}{a^2} + \partial^2 \dot{E}\right) - \frac{1}{2} \rho_x \delta v = 0, \\
\dot{v}_x - \alpha + \frac{1}{2} \delta \beta = 0,
$$

(3.26)

(3.27)

where $\partial^2 = \sum_{i=1}^{3} \partial_i^2$, and

$$
\delta \beta = \frac{1}{2m^2 \rho_x} \left[\frac{1}{a^2} (\partial^2 \delta \rho_x + \rho_x \partial^2 B) + 2 \dot{\rho}_x \alpha + \dot{\rho}_x \left(6H \alpha + \dot{\alpha} - 3 \dot{\zeta} - \partial^2 \dot{E}\right) - \delta \rho_x - 3H \delta \rho_x - \frac{\rho_x}{m^2} \partial^2 \delta \rho_x
\right.

\left. + \frac{1}{\rho_x} \left(\dot{\rho}_x \delta \rho_x + \dot{\rho}_x \left(3H \delta \rho_x\right) - \frac{\rho_x}{m^2} \delta \rho_x\right)\right].
$$

(3.28)

The linearly perturbed continuity and Euler Eqs. (3.9) and (3.13) for perfect fluids are given, respectively, by

$$
\delta \rho_l + 3H (1 + c_l^2) \delta \rho_l + \rho_l (1 + w_l) \left(3 \dot{\zeta} - \frac{\partial^2 v_l}{a^2} - \frac{\partial^2 B}{a^2} + \partial^2 \dot{E}\right) = 0,
$$

(3.29)

$$
\dot{v}_l - 3H c_l^2 v_l - \alpha - \frac{c_l^2}{\rho_l (1 + w_l)} \delta \rho_l = 0,
$$

(3.30)

which hold for each $I = b, c, d, r$.

The energy-momentum tensors of the BEC and perfect fluids are given, respectively, by Eqs. (2.21) and (3.14). From the (00), (0t), trace, and traceless components of the perturbed Einstein Eq. (3.15), we obtain

$$
6H \left(H \alpha - \dot{\zeta} \right) + 2 \left(\frac{\partial^2 \zeta}{a^2} + \frac{H}{a^2} \partial^2 B - H \partial^2 \dot{E}\right)
\right.

\left. + 8\pi G \left(\delta \rho_x - \frac{9H^2 - 4U_e}{8m^2} \delta \rho_x - \frac{3H \delta \rho_x}{4m^2} - \frac{9H^2}{4m^2} \rho_x \alpha - \frac{1}{2} \rho_x \delta \rho + \sum_I \delta \rho_I\right) = 0,
$$

(3.31)

$$
H \alpha - \dot{\zeta} - 4\pi G \left[\rho_x v_x - \frac{3H}{4m^2} \delta \rho_x + \sum_I \rho_I (1 + w_I) v_I\right] = 0,
$$

(3.32)

$$
\ddot{\zeta} + 3H \dot{\zeta} - H \dot{\alpha} - \left(3H^2 + 2H\right) \alpha - 4\pi G \left(\frac{9H^2 + 4U_e}{8m^2} \delta \rho_x + \frac{3H \delta \rho_x}{4m^2} + \frac{9H^2}{4m^2} \rho_x \alpha + \frac{1}{2} \rho_x \delta \beta - \sum_I c_l^2 \delta \rho_l\right) = 0,
$$

(3.33)

$$
\alpha + \dot{\zeta} + \dot{B} + H B - a^2 \left(\dot{E} + 3H \dot{E}\right) = 0.
$$

(3.34)

The perturbation Eqs. (3.26)-(3.34) are written in a gauge-ready form [78, 79], i.e., they are ready for fixing any gauge conditions.

B. Gauge-invariant perturbation equations

To study the evolution of cosmological perturbations without worrying about unphysical gauge degrees of freedom, we consider gauge-invariant perturbations invariant under the infinitesimal coordinate transformation $t \to t + \xi^0$ and $x^i \to x^i + \delta^i_j \partial_j \xi$. We introduce the following gauge-invariant combinations [73]

$$
\Psi = \alpha + \frac{d}{dt} \left(B - a^2 \dot{E}\right), \quad \Phi = - \zeta + H \left(B - a^2 \dot{E}\right),
$$

$$
\delta \rho_N = \delta \rho_x + \rho_x \left(B - a^2 \dot{E}\right), \quad \delta \rho_{IN} = \delta \rho_l + \rho_l \left(B - a^2 \dot{E}\right),
$$

$$
v_N = v_x + B - a^2 \dot{E}, \quad v_{lN} = v_l + B - a^2 \dot{E}.
$$

(3.35)

It is possible to express Eqs. (3.26)-(3.27), (3.29)-(3.30), and (3.31)-(3.34) in terms of the above gauge-invariant variables. On using the background equations of motion, all the gauge-dependent quantities such as $B$ and $E$
disappear from the perturbation equations, so that
\[
\dot{\delta \rho}_N + 3H\delta \rho_N - \rho_x \left(3\dot{\Phi} + \frac{\partial^2 v_N}{a^2}\right) - \frac{1}{2}\rho_x \delta \beta_N = 0 ,
\]
(3.36)
\[
\dot{v}_N - \Psi + \frac{1}{2}\dot{\delta \beta}_N = 0 ,
\]
(3.37)
\[
\dot{\delta \rho}_I + 3H (1 + c_I^2) \delta \rho_I - \rho_I (1 + w_I) \left(3\dot{\Phi} + \frac{\partial^2 v_I}{a^2}\right) = 0 ,
\]
(3.38)
\[
\dot{v}_I - 3H c_I^2 v_I - \Psi - c_I^2 \frac{\delta \rho_I}{\rho_I (1 + w_I)} = 0 ,
\]
(3.39)
\[
6H (H\Psi + \dot{\Phi}) - 2\frac{\partial^2 \Phi}{a^2} + 8\pi G \left(\delta \rho_N - \frac{9H^2 - 4U_{\phi}}{8m^2} \delta \rho_N - \frac{3H\delta \rho_N}{4m^2}\right) - \frac{9H^2}{4m^2} \rho_x \Psi - \frac{1}{2} \rho_x \delta \beta_N + \sum_{I} \delta \rho_I ) = 0 ,
\]
(3.40)
\[
H\Psi + \dot{\Phi} - 4\pi G \left[\rho_x v_N - \frac{3H}{4m^2} \delta \rho_N + \sum_{I} \rho_I (1 + w_I) v_I\right] = 0 ,
\]
(3.41)
\[
\Psi = \Phi ,
\]
(3.42)
\[
where \delta \beta_N is the gauge-invariant variable given by
\[
\delta \beta_N = \delta \beta + \beta \left(B - a^2 \dot{E}\right) = \frac{1}{2m^2 \rho_x} \left[\frac{1}{a^2} \partial^2 \rho_N + 2\dot{\rho}_X + \rho \left(6H\Psi + \dot{\Phi} + 3\dot{\Phi}\right) - \delta \rho_N - 3H\delta \rho_N - \frac{\dot{\rho}_N^2}{\rho_x} \right] + \frac{1}{\rho_x} \left[\frac{\dot{\rho}_X}{\rho_x} + \rho \left(\delta \rho_N + 3H\delta \rho_N\right) - \dot{\rho}_N\Psi - \frac{4\rho_{\nu} U_{\nu} \rho_{\nu}}{m^2} \delta \rho_N\right].
\]
(3.44)

Since the background value of \beta vanishes, \delta \beta_N is identical to \delta \beta.

After the CMB recombination epoch, the equations of state and the sound speed squares for CDM and baryons can be taken to be \(w_c = w_b = 0\) and \(c_{\gamma}^2 = c_{\nu}^2 = 0\), while the photons and relativistic neutrinos have the values \(w_{\gamma} = w_{\nu} = 1/3\) and \(c_{\gamma}^2 = c_{\nu}^2 = 1/3\). The DE equation of state \(w_d\) needs to be close to \(-1\) at low redshifts, while the sound speed squared \(c_{\gamma}^2\) should not be much smaller than \(1\) to avoid the clustering of DE perturbations [67].

Prior to the recombination, the baryons and photons are tightly coupled to each other due to the Thomson scattering weighed by the product of cross section \(\sigma_{\gamma}\) and electron number density \(n_e\). Taking into account this coupling, the perturbation Eq. (3.39) of velocity potentials for baryons and photons are modified, respectively, to [80]
\[
\dot{v}_N - \Psi = -\frac{4\rho_{\nu}}{3\rho_b} \sigma_{\gamma} n_e (v_{bN} - v_{\gamma N}) ,
\]
(3.45)
\[
\dot{v}_{\gamma N} - H v_{\gamma N} - \Psi - \frac{1}{4} \frac{\delta \rho_{\gamma N}}{\rho_{\gamma}} = \sigma_{\gamma} n_e (v_{bN} - v_{\gamma N}) .
\]
(3.46)

In the strongly coupled regime, the velocity potentials of baryons and photons are almost equivalent to each other \((v_{bN} \simeq v_{\gamma N})\). After the recombination, it is a good approximation to set the right hand-sides of Eqs. (3.45)-(3.46) to be 0.

**IV. QUASI-STATIC APPROXIMATION FOR SUB-HORZION PERTURBATIONS**

In this section, we derive the second-order equations for the BEC and perfect-fluid density perturbations under a quasi-static approximation for the modes deep inside the Hubble radius. The perturbation \(\delta \beta_N\) consists of a special solution induced by the \(\chi\)-field density contrast and a homogenous solution which oscillates with the approximate frequency \(2m\). The quasi-static approximation amounts to neglecting the oscillating mode of \(\delta \beta_N\) relative to its special solution for the perturbation dynamics over the cosmological time scale \(H^{-1}\). Numerically, we will study the validity of this approximation by numerically solving the perturbation equations including the modes close to the Hubble radius.
A. Analytic estimation

The gauge-invariant density contrasts of the BEC and perfect fluids are defined, respectively, by

\[ \delta_{\chi N} = \frac{\delta \rho_{\chi N}}{\rho_{\chi}}, \quad \delta_{IN} = \frac{\delta \rho_{IN}}{\rho_{I}}. \]  

We study the evolution of perturbations in Fourier space with the comoving wavenumber \( k \). From Eq. (3.36), we have

\[ \delta_{\chi N} - 3\Phi + \frac{k^2}{a^2} v_{\chi N} - \frac{1}{2} \delta_{\beta N} = 0. \]  

(4.2)

Differentiating Eq. (4.2) with respect to \( t \) and using Eqs. (3.37) and (4.2) to eliminate \( \dot{v}_{\chi N} \) and \( v_{\chi N} \), it follows that

\[ \delta_{\beta N} = 2H \delta_{\beta N} + \frac{k^2}{a^2} \Phi - 3\Phi - 6H \dot{\Phi} - \frac{1}{2} \delta_{\beta N} - H \dot{\delta}_{\beta N} - \frac{k^2}{2a^2} \delta_{\beta N} = 0. \]  

(4.3)

In Eq. (3.44), \( \delta_{\beta N} \) contains the second time derivative \( \ddot{\delta}_{\chi N} \). Combining Eq. (3.44) with Eq. (4.3) to eliminate \( \ddot{\delta}_{\chi N} \), we obtain

\[ \delta_{\beta N} + 2H \delta_{\beta N} + \left( \frac{k^2}{a^2} + 4m^2 \right) \delta_{\beta N} + 2 \left( \frac{k^2}{a^2} + \frac{\rho_{\chi} U_{\rho \rho}}{m^2} \right) \delta_{\chi N} - 4H \ddot{\delta}_{\chi N} + 6 \left( \Phi + 6H \dot{\Phi} \right) + 2 \left[ 3(3H^2 + 2\dot{H}) - \frac{k^2}{a^2} \right] \Phi = 0. \]  

(4.4)

Taking the small-scale limit in Eq. (3.44), the perturbation \( \delta_{\beta N} \) has the scale-dependence \( \delta_{\beta N} \simeq -k^2/(2m^2a^2)\delta_{\chi N} \). In the linear regime of perturbation theory (\( |\delta_{\chi N}| \lesssim 0.1 \)), we require the condition \( k/(ma) \lesssim 1 \) to ensure that \( |\delta_{\beta N}| \lesssim 0.1 \). In other words, the nonrelativistic BEC description with the background value \( \beta = 0 \) can be approximately justified for the wavenumber \( k/a \lesssim m \). Physically, this means that the BEC ground state is described by a coherent wave with the length scale \( (k/a)^{-1} \) larger than the Compton wavelength \( m^{-1} \) [28].

In the following, we will derive the second-order differential equations of \( \delta_{\chi N} \) and \( \delta_{IN} \) for the sub-horizon modes in the range

\[ H \ll \frac{k}{a} \lesssim m. \]  

(4.5)

We impose the conditions (2.25) to ensure the nonrelativistic BEC description. For the wavenumber in the range (4.5), we exploit the quasi-static approximation under which the time derivatives \( \dot{\Psi}, \dot{\Phi}, \) and \( \ddot{\delta}_{\chi N} \) are at most of the orders \( H \dot{\Psi}, H \Phi, \) and \( H \ddot{\delta}_{\chi N} \), respectively. In this case, Eq. (3.40) approximately reduces to

\[ \frac{k^2}{a^2} \Phi \simeq -4\pi G \left( \rho_{\chi} \delta_{\chi N} - \frac{1}{2} \rho_{\chi} \delta_{\beta N} + \sum I \rho_{I} \delta_{IN} \right). \]  

(4.6)

Substituting Eq. (4.6) into Eq. (4.4) and using the fact that the term \( \pi G \rho_{\chi} \) is at most of the order \( H^2 \), we obtain

\[ \delta_{\beta N} + 2H \delta_{\beta N} + \left( \frac{k^2}{a^2} + 4m^2 \right) \delta_{\beta N} \simeq -2 \left( \frac{k^2}{a^2} + \frac{\rho_{\chi} U_{\rho \rho}}{m^2} \right) \delta_{\chi N} - 8\pi G \sum I \rho_{I} \delta_{IN}. \]  

(4.7)

The general solution to Eq. (4.7) is the sum of a special solution \( \delta_{\beta N}^{(s)} \) and a homogenous solution \( \delta_{\beta N}^{(h)} \), i.e.,

\[ \delta_{\beta N} = \delta_{\beta N}^{(s)} + \delta_{\beta N}^{(h)}. \]  

(4.8)

The special solution can be derived by neglecting the time derivatives on the left hand-side of Eq. (4.7), such that

\[ \delta_{\beta N}^{(s)} = -\frac{2(k^2 + a^2 \rho_{\chi} U_{\rho \rho}/m^2) \delta_{\chi N} + 8\pi G a^2 \sum I \rho_{I} \delta_{IN}}{k^2 + 4m^2a^2}. \]  

(4.9)

Since \( \delta_{\beta N}^{(s)} \) is directly related to the matter density contrasts, the typical time scale for its variation should be of order the Hubble time \( H^{-1} \).
The homogenous solution can be obtained by setting the terms on the right-hand-side of Eq. (4.7) to be 0. Under the condition (4.5), the dominant contribution to the frequency $\omega_k = \sqrt{k^2/a^2 + 4m^2}$ in Eq. (4.7) is the mass term $2m$. This means that $\delta \beta_N^{(h)}$ oscillates with the approximate time period $(2m)^{-1}$, which is much shorter than $H^{-1}$ for $m \gg H$. The WKB solution to this oscillating regime is given by

$$
\delta \beta_N^{(h)} = \frac{1}{a\sqrt{2\omega_k}} \left[ A \cos \left( \int_0^t \omega_k dt' \right) + B \sin \left( \int_0^t \omega_k dt' \right) \right],
$$

(4.10)

where $A$ and $B$ are integration constants. Under the initial condition $\delta \beta_N = \delta \beta_N^{(s)} + \delta \beta_N^{(h)} = 0$ at $t = 0$, the constant $A$ is fixed to be $A = -a_i \sqrt{2\omega_k} \delta \beta_N^{(s)}$, where $a_i$, $\omega_k$, and $\delta \beta_N^{(s)}$ are the initial values of $a$, $\omega_k$, and $\delta \beta_N^{(s)}$ respectively. Imposing the initial condition $\delta \beta_N^{(h)} = 0$ at $t = 0$, we obtain the constant $B$ which is suppressed by the factor $H/m$ in comparison to $A$. Neglecting the term $B \sin(\int_0^t \omega_k dt)$ in Eq. (4.10), the homogenous solution yields

$$
\delta \beta_N^{(h)} \simeq -\frac{a_i}{a} \sqrt{\frac{\omega_k \delta \beta_N^{(s)}}{\omega_k}} \cos \left( \int_0^t \omega_k dt' \right).
$$

(4.11)

Thus, the contributions $\delta \beta_N^{(s)}$ and $\delta \beta_N^{(h)}$ to the total solution (4.8) are given, respectively, by Eqs. (4.9) and (4.11). In the regime $k/a \ll m$, the solution reduces to $\delta \beta_N \simeq \delta \beta_N^{(s)} - \left( a_i/a \right) \delta \beta_N^{(s)} \cos(2mt)$. While $\beta = 0$ at the background level, the perturbation $\delta \beta_N$ contains the matter-induced mode $\delta \beta_N^{(s)}$ slowly varying over the Hubble time scale $H^{-1}$ as well as the oscillating mode $\delta \beta_N^{(h)}$ rapidly changing over the time scale $(2m)^{-1}$.

Applying the quasi-static approximation to Eq. (4.3) for the gravitational potential $\Phi$ deep inside the Hubble radius and using Eq. (4.6), it follows that

$$
\ddot{\delta} \chi_N + 2H \dot{\delta} \chi_N - 4\pi G \left( \rho_N \delta \chi_N + \sum_i \rho_i \delta \chi_i \right) - \frac{1}{2} \delta \beta_N - H \delta \beta_N - \frac{k^2}{2a^2} \delta \beta_N \simeq 0,
$$

(4.12)

where we ignored the second term on the right-hand-side of Eq. (4.6) relative to $k^2/(2a^2)\delta \beta_N$. The time derivatives $\delta \beta_N^{(s)}$ and $H \delta \beta_N^{(s)}$ present in Eq. (4.12) are at most of order $H^2 \delta \beta_N^{(s)}$, so we can neglect these contributions relative to the term $(k^2/2a^2)\delta \beta_N^{(s)}$. For the homogenous solution $\delta \beta_N^{(h)}$, the term $H \dot{\delta} \beta_N^{(h)}$ is suppressed relative to $\delta \beta_N^{(h)}/2$. On using the property $\delta \beta_N^{(h)} \simeq -\omega_k \delta \beta_N^{(h)}$, we have $-\delta \beta_N^{(h)}/2 - k^2/(2a^2)\delta \beta_N^{(h)} \simeq 2m^2 \delta \beta_N^{(h)}$. Then, Eq. (4.12) reduces to

$$
\ddot{\delta} \chi_N + 2H \dot{\delta} \chi_N - 4\pi G \left( \rho_N \delta \chi_N + \sum_i \rho_i \delta \chi_i \right) - \frac{k^2}{2a^2} \delta \beta_N^{(s)} - 2m^2 a_i \sqrt{\frac{\omega_k \delta \beta_N^{(s)}}{\omega_k}} \cos \left( \int_0^t \omega_k dt' \right) \simeq 0.
$$

(4.13)

Initially, the last term on the left-hand-side of Eq. (4.13) is larger than $-k^2/(2a^2)\delta \beta_N^{(s)}$, but the former oscillates between +1 and −1 with the approximate frequency $2m$. Taking the time average over the Hubble time scale in Eq. (4.13) and using the special solution (4.9), we obtain

$$
\ddot{\delta} \chi_N + 2H \dot{\delta} \chi_N + \left( c_s^2 \frac{k^2}{a^2} - 4\pi G \rho_N \right) \delta \chi_N - 4\pi G \left( 1 + \frac{k^2}{4m^2a^2} \right)^{-1} \sum_i \rho_i \delta \chi_i \simeq 0,
$$

(4.14)

where $c_s^2$ is the effective sound speed squared defined by

$$
c_s^2 \equiv \frac{k^2}{4m^2a^2 + k^2} + \frac{a^2 \rho_N U_{,pp}}{m^2(4m^2a^2 + k^2)}.
$$

(4.15)

The formation of a coherent BEC state gives rise to a quantum pressure with the scale-dependent sound speed squared $c_{s1}^2 = k^2/(4m^2a^2 + k^2)$. This value of $c_{s1}^2$ coincides with the one derived in Ref. [32] by considering the perturbation $\delta \phi$ of a massive axion field $\phi$ and taking the time average over the background axion oscillations. This shows that our nonrelativistic BEC description based on the Madelung representation (2.11) is consistent with the approach of scalar-field perturbations in the rapidly oscillating regime with $m \gg H$. In the limit that $k/a \gg m$, $c_s^2$ approaches the value 1, but, for scales much larger than the Compton wavelength $(k/a \ll m)$, it follows that $c_{s1}^2 \simeq k^2/(4m^2a^2) \ll 1$. The fact that this latter sound speed gives rise to a quantum pressure due to the uncertainty principle was originally recognized in Ref. [30]. With the BEC formation, there is a critical Jeans-scale wavenumber
$k_J$ at which the gravitational interaction $4\pi G \rho_\chi$ balances the pressure term $c_{s1}^2 k^2/a^2$. In the regime $k/a \ll m$, we have
\begin{equation}
k_J = a \left(16\pi G m^2 \rho_\chi\right)^{1/4} .
\end{equation}

For $k > k_J$, the quantum pressure suppresses the gravitational instability of $\delta_{\chi N}$.

The self-coupling potential $U(\rho)$ with a repulsive ($U_{,\rho \rho} > 0$) or attractive ($U_{,\rho \rho} < 0$) interaction leads to the suppressed or enhanced growth of $\delta_{\chi N}$ through the second sound speed squared $c_{s2}^2 = a^2 \rho_\chi U_{,\rho \rho}/[m^2(4m^2 a^2 + k^2)]$ in Eq. (4.15). For the self-coupling potential $U(\rho) = \lambda \rho^4/4$ of the two-body interaction, we have $c_{s2}^2 = \lambda \rho_\chi/(8m^4)$ for $k/a \ll m$. This agrees with the expression derived in Refs. [44, 48, 54]. Our result of $c_{s2}^2$ can be applied to the general self-interacting potential $U(\rho)$ as well as to the wavenumber $k/a$ close to $m$. In the regime $k/a \ll m$, the critical wavenumber $k_S$ at which $4\pi G \rho_\chi$ balances $|c_{s2}^2| k^2/a^2$ is given by
\begin{equation}
k_S = a \left(16\pi G m^4 \left|U_{,\rho \rho}\right|\right)^{1/2} .
\end{equation}

For the modes $k > k_S$, the gravitational growth of $\delta_{\chi N}$ is modified by the self-coupling potential.

There is also the critical wavenumber $k_I$ at which $c_{s1}^2$ and $c_{s2}^2$ on the right hand-side of Eq. (4.15) have the same amplitudes, i.e.,
\begin{equation}
k_I = a \left(\rho_\chi \left|U_{,\rho \rho}\right|/m^2\right)^{1/2} .
\end{equation}

For $k > k_I$, the quantum pressure dominates over the self-interaction. In Sec. V, we will study the evolution of $\delta_{\chi N}$ by comparing the three critical wavenumbers $k_J$, $k_S$, and $k_I$.

In the regime $k/a \ll m$, the last term on the left hand-side of Eq. (4.14) reduces to the standard form $-4\pi G \sum_I \rho_I \delta_{IN}$. In the presence of CDM and baryons, their density contrasts $\delta_{cN}$ and $\delta_{bN}$ affect the evolution of $\delta_{\chi N}$ through the gravitational interaction mediated by $\Phi$. Let us derive the second-order differential equations of $\delta_{cN}$ and $\delta_{bN}$ under the same approximation scheme as $\delta_{\chi N}$. Differentiating Eq. (3.38) with respect to $t$ and using Eq. (3.39), we find
\begin{equation}
\ddot{\delta}_{IN} + (2 + 3c_I^2 - 6w_I) H \dot{\delta}_{IN} + \left[c_I^2 k^2/a^2 + 3 \left(5H^2 + \dot{H}\right) (c_I^2 - w_I) + 3H(c_I^2)\right] \delta_{IN} \\
+(1 + w_I) \left[k^2/a^2 \Phi + 3H (3c_I^2 - 2) \Phi - 3\Phi\right] = 0 .
\end{equation}

For the modes deep inside the Hubble radius, the terms $3H(3c_I^2 - 2)\Phi - 3\Phi$ in Eq. (4.19) are negligible relative to $(k^2/a^2)\Phi$. This latter Laplacian term, which is approximately given by Eq. (4.6), contains the contribution $-4\pi G \sum_I \rho_I \delta_{IN}$ responsible for the gravitational clustering of $\delta_{IN}$. In Eq. (4.19), there exists the term $c_I^2(k^2/a^2)\delta_{IN}$ preventing the growth of $\delta_{IN}$ on scales smaller the sound horizon $(c_I k/a > H)$. For $c_I^2 = O(1)$, which is typically the case for DE perturbations, the density contrast does not grow for most of the modes inside the Hubble radius. On the other hand, the sound speeds of CDM and baryons are much smaller than 1 after the recombination epoch, so $\delta_{cN}$ and $\delta_{bN}$ are subject to the gravitational instabilities. In the following, we take the nonrelativistic limits,
\begin{equation}
c_I^2 \rightarrow 0 , \quad w_I \rightarrow 0 \quad \text{for} \quad I = c, b .
\end{equation}

We substitute Eq. (4.6) into Eq. (4.19) with the approximation $\delta \beta_N \simeq \delta \beta_N^{(s)}$. The oscillating mode $\delta \beta_N^{(h)}$ in Eq. (4.11) should not contribute to the growth of $\delta_{IN}$ over the Hubble time scale. Then, the CDM and baryon density contrasts for the modes deep inside the Hubble radius obey
\begin{equation}
\ddot{\delta}_{IN} + 2H \dot{\delta}_{IN} - 4\pi G \sum_I \rho_I \delta_{IN} - 4\pi G \rho_\chi \left(1 + c_s^2\right) \delta_{\chi N} \simeq 0 , \quad \text{for} \quad I = c, b ,
\end{equation}

where $c_s^2$ is given by Eq. (4.15). To our knowledge, the contribution of $c_s^2$ to $\delta_{IN}$ in the form (4.21) was not recognized in the literature. For the wavenumber $k/a \ll m$, we have $c_s^2 \simeq k^2/(4m^2 a^2) \ll 1$ and hence the last term on the left hand-side of Eq. (4.21) approximately reduces to $-4\pi G \rho_\chi \delta_{\chi N}$. The evolution of $\delta_{\chi N}$ is affected by those of $\delta_{cN}$ and $\delta_{bN}$ through Eqs. (4.14) and (4.21), and vice versa.
B. Numerical solutions

In order to confirm the accuracy of approximations exploited in Sec. IV A, we numerically solve the perturbation equations of motion during the matter era in which the BEC energy density dominates over the other matter densities. In this section we focus on the case without BEC self-interactions ($U = 0$), but in Sec. V we will take into account the self-coupling potential as well as the perturbations of CDM and baryons. We integrate the perturbation Eqs. (3.37), (3.40)-(3.41), (4.2), (4.4) with (3.43), along with the background Eqs. (2.22) and (2.28)-(2.29).

In the left panel of Fig. 1, we plot the evolution of $\delta \beta_N$ for the initial conditions $H = 10^{-2} m$ and $\delta \beta_N = \delta \chi_N = 0$. Since $H$ decreases in time, the nonrelativistic condition $H \ll m$ is always satisfied. We choose the wavenumber $k$ to be $K \equiv k/(aH) = 18$ at $t = 0$, in which case the perturbation is deep inside the Hubble radius during the matter era. Since this mode is in the regime $k/a \ll m$, the special solution (4.9) and the homogenous solution (4.11) reduce, respectively, to

$$\delta \beta_N^{(s)} \simeq -\frac{k^2}{2m^2a^2} \delta \chi_N, \quad \delta \beta_N^{(h)} \simeq -\frac{a_1}{a} \delta \beta_N^{(s)} \cos(2mt), \quad (4.22)$$

where $\delta \beta_N$ is the sum of these two modes. In Fig. 1, we find that $\delta \beta_N$ oscillates with the period $\pi/m$ around the slowly varying central value $\delta \beta_N^{(s)}$ due to the homogenous mode $\delta \beta_N^{(h)}$, with an amplitude related to the initial value $\delta \beta_N^{(s)}$. The special solution $\delta \beta_N^{(s)}$ shown as a thick black line in Fig. 1, which is proportional to $\delta \chi_N$, exhibits only a tiny oscillation with an amplitude much smaller than the time-averaged value $-k^2/(2m^2a^2) \delta \chi_N$. It should be a good approximation to neglect the homogenous oscillating mode $\delta \beta_N^{(h)}$ relative to $\delta \beta_N^{(s)} \simeq -k^2/(2m^2a^2) \delta \chi_N$ for the evolution of $\chi_N$ on cosmological time scales.

![FIG. 1. (Left) The red line shows the evolution of $\delta \beta_N$ versus $mt$ during the matter era dominated by the BEC energy density, without the self-coupling potential $U$. The initial conditions are chosen to be $H = 10^{-2} m$, $\rho_\chi = 3H^2/(8\pi G)$, $\delta \chi_N/\chi = 10^{-3}$, $\delta \beta_N = 0$, $\delta \chi_N = 0$, and $K = k/(aH) = 18$ at $t = 0$. The thick black line is the evolution of the special solution $\delta \beta_N^{(s)}$ given by Eq. (4.9). (Right) The red, blue, and green lines correspond to the evolutions of $\delta \chi_N$ for three different wavenumbers: (a) $K = 3$, (b) $K = 18$, and (c) $K = 30$ at $mt = 0$, respectively. The other initial conditions are the same as those used in the left panel. The thick dashed plots are derived by solving the approximate Eq. (4.14). This approximation is accurate for the perturbations deep inside the Hubble radius ($K \gg 1$).](image)

In the right panel of Fig. 1, we show the evolutions of $\delta \chi_N$ for three different wavenumbers with the same initial conditions as those used in the left. The case (a) corresponds to the wavenumber $K = k/(aH) = 3$ at $t = 0$. The density contrast $\delta \chi_N$ grows by the gravitational source term $-4\pi G\rho_\chi \delta \chi_N$ in Eq. (4.14). For this mode the Laplacian term $c_s^2(k^2/a^2) \delta \chi_N$ is smaller than $4\pi G\rho_\chi \delta \chi_N$, so the growth of $\delta \chi_N$ is hardly prevented by the quantum pressure. The thick dashed line above the red line (a) in Fig. 1 is obtained by integrating the approximate Eq. (4.14) with Eq. (4.15). In case (a), the approximate solution exhibits some difference from the numerical solution. This property...
is mostly attributed to the fact that, for the modes close to the Hubble radius, the terms of order $H^2 \Phi$ cannot be ignored relative to $(k^2/a^2)\Phi$ in Eq. (3.40). Indeed, implementing such contributions to Eq. (3.40) gives rise to the gravitational potential $\Phi$ smaller than that estimated by Eq. (4.6). This is the main reason why the full numerical solution (a) of $\delta\chi_N$ is smaller than the analytic estimation (4.14) derived for the modes $K \gg 1$.

The case (b) in Fig. 1 corresponds to the wavenumber $K = 18$ at $t = 0$. We observe that the growth of $\delta\chi_N$ is suppressed in comparison to case (a). On using the Friedmann equation $3H^2\simeq 8\pi G \rho_\chi$ in Eq. (4.16), the critical wavenumber associated with the Jeans scale can be estimated as

$$K_J \equiv \frac{k_J}{aH} \simeq 1.6 \sqrt{\frac{m}{H}}.$$  (4.23)

With the initial Hubble parameter $H = 10^{-2}m$, we have $K_J \simeq 16$ and hence the mode $K = 18$ is affected by the quantum pressure. The suppressed growth of $\delta\chi_N$ starts to be at work for the initial value of $K$ close to $K_J$. Especially for $K \gg K_J$, the quantum pressure leads to the strong suppression of $\delta\chi_N$. This property can be confirmed in case (c) of Fig. 1, which corresponds to the initial wavenumber $K = 30$.

In cases (b) and (c) the evolutions of $\delta\chi_N$ obtained by solving the approximate Eq. (4.14) show good agreement with the full numerical results, by reflecting the fact that the perturbations are always in the regime $K \gg 1$. Thus, the approximate second-order differential Eq. (4.14) can be trustable for the modes deep inside the Hubble radius. In the right panel of Fig. 1, we also observe that the oscillating mode in $\delta\beta_N$ does not give rise to any large oscillations of $\delta\chi_N$.

V. BEC SELF-INTERACTIONS

In this section, we study the effect of BEC self-interactions on the dynamics of cosmological perturbations. For concreteness, we consider the self-coupling potential

$$U(\rho) = \frac{1}{4}\lambda \rho^2,$$  (5.1)

where $\lambda$ is a coupling constant. In Eq. (2.24), the contribution to $\beta$ arising from $U(\rho)$ is given by

$$\beta_U \equiv -\frac{U_{,\rho}}{m^2} = -\frac{\lambda \rho_\chi}{4m^4} = -\frac{3}{2}\beta_m \Omega_\chi,$$  (5.2)

where

$$\beta_m \equiv \frac{H^2}{m^2}, \quad \mu \equiv \frac{\lambda M^2_{\text{pl}}}{2m^2}, \quad \Omega_\chi \equiv \frac{\rho_\chi}{3M^2_{\text{pl}}H^2}.$$  (5.3)

We require that both $|\beta_U|$ and $\beta_m$ are smaller than the order 1 to ensure the nonrelativistic BEC description. The ratio between $U$ and $\rho_\chi$ is given by

$$\frac{U}{\rho_\chi} = -\frac{1}{4}\beta_U.$$  (5.4)

Under the condition $|\beta_U| \ll 1$, the self-coupling potential is suppressed in comparison to $\rho_\chi = 2m^2\chi^*\chi$.

For a real scalar field $\phi = \sqrt{2} \chi$ with the axion-type potential

$$V = m^2f^2 \left[ 1 - \cos \left( \frac{\phi}{f} \right) \right],$$  (5.5)

the expansion of $V$ around $\phi = 0$ leads to

$$V = m^2\chi^2 - \frac{m^2}{6f^2}\chi^4 + \mathcal{O}(\chi^6).$$  (5.6)

The first contribution to $V$ in Eq. (5.6) is the mass Lagrangian $m^2\chi^*\chi$ in Eq. (3.1), while the second one corresponds to the self-coupling potential (5.1) with $\lambda = -2m^2/(3f^2)$. In this case, the dimensionless constant $\mu$ defined in Eq. (5.3) reads

$$\mu \equiv -\frac{M^2_{\text{pl}}}{3f^2}.$$  (5.7)
Since $\mu$ is negative, the axion potential of the form (5.5) leads to an attractive self-interaction [46, 54]. For $f \ll M_{\text{pl}}$, $|\mu|$ is larger than the order 1.

We study the evolution of density contrasts for the total potential of the form $V = m^2 \chi^2 + \chi (\chi^*)^2 / 4$, but our analysis also covers the axion-type potential (5.5) expanded around its minimum. We will focus on the case $\mu < 0$ in the following discussion.

## A. Scales relevant to the self-coupling

At the onset of BEC formation, which is expressed by the subscript $*$ for time-dependent quantities, the variable (5.2) reads

$$\beta_{U*} = -\frac{3}{2} \beta_{m*} \mu \Omega_{m*},$$  \tag{5.8}$$

where

$$\beta_{m*} = \frac{H_0^2}{m^2} \Omega_{M0} (a_s + a_{eq}) / a_s^2, \quad \Omega_{\chi*} = \frac{\rho_{\chi*}}{3M_{\text{pl}}^2 H_0^2} = \frac{\Omega_{\chi0}}{\Omega_{M0}} \frac{a_s}{a_s + a_{eq}}.$$  \tag{5.9}$$

In Eq. (5.9), we used Eq. (2.32) and neglected the contribution of DE to the Hubble expansion rate. Since $|\beta_U|$ decreases as $|\beta_U| \propto a^{-3}$ for $a > a_s$, the nonrelativistic BEC description can be ensured for $|\beta_{U*}| \ll 1$. This gives an upper limit on the self-coupling strength, as

$$|\mu| \ll \frac{2}{\beta_{m*}} \frac{1}{\Omega_{M0}} \frac{\Omega_{\chi0}}{a_s + a_{eq}}.$$  \tag{5.10}$$

For the modes deep inside the Hubble radius, the density contrast $\delta_{\chi N}$ approximately obeys Eq. (4.14). For the self-coupling potential (5.1), the sound speed squared (4.15) is given by

$$c_s^2 = \left( \frac{k^2}{4m^2 a_s^2} - \frac{1}{2} \beta_U \right) \left( 1 + \frac{k^2}{4m^2 a_s^2} \right)^{-1}.$$  \tag{5.11}$$

For the wavenumber in the range $k/a \ll m$, Eq. (5.11) reduces to $c_s^2 \simeq k^2 / (4m^2 a_s^2) - \beta_U / 2$. For $k < k_I$, where $k_I$ is given by Eq. (4.18), the self-coupling term $-\beta_U / 2$ is the dominant contribution to $c_s^2$. Since we are considering the case $\mu < 0$, the term $-\beta_U / 2$ is negative. This can induce the Laplacian instability of $\delta_{\chi N}$ for some particular values of $k$. As we estimated in Sec. IV A, whether the Laplacian instability is present or not is the comparison of the $c_s^2 k^2 / a_s^2$ term with those responsible for the gravitational instability. On using the property $\rho_{\chi} = \rho_{00} a^{-3}$, where $ho_{00} = 3 M_{\text{pl}}^2 H_0^2 \Omega_{\chi0}$ is today’s energy density of the BEC, the three critical wavenumbers (4.16), (4.17), and (4.18) can be expressed, respectively, as

$$k_J = 5.22 \times 10^{-4} h \Omega_{\chi0}^{1/4} \left( \frac{m}{H_0} \right)^{1/2} a^{-1/4} \text{ Mpc}^{-1},$$  \tag{5.12}$$

$$k_S = 4.72 \times 10^{-4} h |\mu|^{-1/2} \frac{m}{H_0} a \text{ Mpc}^{-1}. \tag{5.13}$$

$$k_I = 5.78 \times 10^{-4} h \Omega_{\chi0}^{1/2} |\mu|^{1/2} a^{-1/2} \text{ Mpc}^{-1}.$$  \tag{5.14}$$

In the asymptotic past ($a \to 0$), the largest wavenumber is $k_I$, while the smallest one is $k_S$. Let us consider the case in which the inequality $k_I > k_J > k_S$ is satisfied at $a = a_s$. As we see in Fig. 2, there is a moment at which $k_I$, $k_J$, and $k_S$ become equivalent to each other. This corresponds to the instant $\mu^2 \rho_{\chi} = 2m^2 M_{\text{pl}}^2$, which translates to the scale factor

$$a_E = \left( \frac{3 \mu^2 H_0^2 \Omega_{\chi0}}{2m^2} \right)^{1/3},$$  \tag{5.15}$$

with the wavenumber

$$k_E = 5.40 \times 10^{-4} h \Omega_{\chi0}^{1/3} |\mu|^{1/6} \left( \frac{m}{H_0} \right)^{1/3} \text{ Mpc}^{-1}.$$  \tag{5.16}$$
The necessary condition for the Laplacian instability to occur is given by

$$a_E > a_0 \quad \rightarrow \quad |\mu| > \frac{m}{H_0} \sqrt{\frac{2a_0^3}{3\Omega_0}}. \quad (5.17)$$

Under this condition, for the modes in the range $k_S < k < k_I$, the negative Laplacian term $c_s^2 k^2 / a^2$ associated with the self-coupling dominates over the term $-4\pi G \rho_\chi$ during the time interval $a_* < a < a_E$. This is shown as a yellow shaded region in Fig. 2. On using Eq. (5.9) and combining the bound (5.17) with (5.10), it follows that

$$\sqrt{\frac{2}{3} \frac{\Omega_{M0} a_+ + a_{eq}}{\Omega_0 a_*}} < |\mu| \ll \sqrt{\frac{2}{3} \frac{\Omega_{M0} a_+ + a_{eq}}{\Omega_0 a_*}}. \quad (5.18)$$

For the existence of $|\mu|$ in this range, we require the condition

$$\frac{\Omega_0}{\Omega_{M0}} \ll \frac{2}{3} \frac{\beta_{m*}^2}{\Omega_0 a_* + a_{eq}}. \quad (5.19)$$

Since $\Omega_0 \leq \Omega_{M0}$ and $(a_+ + a_{eq}) / a_* > 1$, the inequality (5.19) holds for $\beta_{m*} \ll 1$.

Expressing the coefficient in front of $\delta_\chi$ in Eq. (4.14) as

$$\omega_\chi^2 = c_s^2 \frac{k^2}{a^2} - 4\pi G \rho_\chi = \frac{2m^2k^2 + \lambda a^2 \rho_\chi}{2m^3(4m^2a^2 + k^2) a^2} - 4\pi G \rho_\chi, \quad (5.20)$$

the behavior of $\omega_\chi^2$ is different depending on $k$. During the time interval $a_* < a < a_E$, its scale dependence is given by

$$\omega_\chi^2 \simeq \begin{cases} 
\frac{k^4}{a^2(4m^2a^2 + k^2)}, & (k > k_I), \\
\frac{\lambda a^2 \rho_\chi k^2}{2m^3(4m^2a^2 + k^2)}, & (k_S < k < k_I), \\
-4\pi G \rho_\chi, & (k < k_S).
\end{cases} \quad (5.21)$$

For $k > k_I$ the quantum pressure suppresses the growth of $\delta_\chi$, while, for $k < k_S$, $\delta_\chi$ grows in the standard manner by the gravitational instability. In the intermediate wavenumber $k_S < k < k_I$, the self-coupling contribution to $\omega_\chi^2$...
dominates over both quantum pressure and gravitational instability. Taking into account the contribution of quantum pressure to $c_s^2$ in the regime $k_S < k < k_I$ with $k/a < m$, we find that $\omega^2_\chi$ takes a minimum value

$$\omega^2_{\chi, \text{min}} = -4\pi G\rho_\chi - \frac{\lambda^2 \rho^2_\chi}{64 m^6} = -4\pi G\rho_\chi \left(1 - \frac{1}{4} \beta_U \mu \right), \quad (5.22)$$

at

$$k_{\text{min}} = \frac{a\sqrt{|\lambda|\rho_\chi}}{2m} = \frac{k_I}{\sqrt{2}}. \quad (5.23)$$

Even if $|\beta_U| \ll 1$, the term $-\beta_U \mu/4$ can be of order 1, so that it is compatible with the gravitational instability term in $\omega^2_{\chi, \text{min}}$. In the presence of other perturbations like $\delta_N$, the last term on the left hand-side of Eq. (4.14) also works as a source term for the gravitational instability of $\delta_N$.

For $a > a_E$, the quantity $\omega^2_{\chi}$ has the following scale dependence

$$\omega^2_{\chi} \simeq \begin{cases} 
\frac{k^4}{a^2(4m^2a^2 + k^2)}, & (k > k_J), \\
-4\pi G\rho_\chi, & (k < k_J). 
\end{cases} \quad (5.24)$$

In this regime, there is no range of $k$ in which the self-coupling potential affects the growth of $\delta_N$.

### B. Evolution of density contrasts for ultra-light BEC

For the mass range $m < 7 \times 10^{-28}$ eV, the scalar field starts to oscillate after matter-radiation equality. In this case, the scales of Laplacian instabilities discussed above can be as large as those relevant to the observations of CMB and matter power spectra in the linear regime. In the following, we study the evolution of perturbations in this ultra-light mass range. On using the approximation $a_e \gg a_{eq}$ for $\beta_{m*}$ in Eq. (5.9), we have

$$a_* \simeq \left(\frac{H_0^2 \Omega_{M0}}{m^2 \beta_{m*}}\right)^{1/3}. \quad (5.25)$$

The inequality (5.18) yields

$$\sqrt{\frac{2}{3}} \frac{\Omega_{M0}}{\beta_{m*} \Omega_{\chi0}} \frac{\Omega_{\chi0}}{\beta_{m*} \Omega_{\chi0}} < |\mu| \ll 2 \frac{1}{3} \frac{\Omega_{M0}}{\beta_{m*} \Omega_{\chi0}}. \quad (5.26)$$

Under this upper bound of $|\mu|$, the quantity $-\beta_U \mu/4$ in Eq. (5.22) is in the range

$$-\frac{1}{4} \beta_U \mu \ll \frac{\beta_U \Omega_{M0}}{6 \beta_{m*} \Omega_{\chi0}}. \quad (5.27)$$

If the $\chi$ field is responsible for all DM ($\Omega_{\chi0} = \Omega_{M0}$), the right hand-side of Eq. (5.27) is given by $\beta_U/(6 \beta_{m*})$. At $a = a_e$, it is natural to consider the situation in which $\beta_{U*}$ is of the similar order to $\beta_{m*}$ and hence the quantity $\beta_{U*}/(6 \beta_{m*})$ is at most of order 1. For $a > a_e$, $\beta_U$ decreases in proportion to $a^{-3}$, so the term $-\beta_U \mu/4$ is constrained to be much smaller than 1. This means that, for $\Omega_{\chi0} = \Omega_{M0}$, the self-coupling term in $\omega^2_{\chi, \text{min}}$ does not significantly dominate over the gravitational instability term $-4\pi G\rho_\chi$.

If $\Omega_{\chi0} \ll \Omega_{M0}$, then the term $-\beta_U \mu/4$ can be larger than 1 due to the large upper limit on the right hand-side of Eq. (5.27). For $\mu = 0$, the likelihood analysis using the CMB and galaxy clustering data in the ultra-light axion mass range $10^{-32}$ eV $\leq m \leq 10^{-25.5}$ eV showed that the $\chi$-field density parameter is constrained to be $\Omega_{\chi0}/\Omega_{M0} \leq 0.05$ [16]. This small value of $\Omega_{\chi0}$ relative to the total DM density is attributed to the fact that the quantum pressure suppresses the growth of the BEC density contrast for $k > k_J$. Even if the self-interaction were to enhance $\delta_N$ at some particular scales, the quantum pressure still leads to the suppression of $\delta_N$ on scales relevant to the observational range of CMB and linear matter power spectrum. Hence the fact that $\Omega_{\chi0}$ needs to be much smaller than $\Omega_{M0}$ for the above ultra-light mass region should not be modified.

Let us consider the mass range $m < 7 \times 10^{-28}$ eV together with the condition $\Omega_{\chi0} \ll \Omega_{M0}$ in the following discussion. In this case, we need to take CDM and baryons into account to study the evolution of perturbations after matter dominance. For $a > a_e \gg a_{eq}$, it is unnecessary to distinguish between CDM and baryons, in that
both perturbations satisfy Eqs. (3.38) and (3.39) with Eq. (4.20). We define the CDM-baryon density contrast \( \delta_{\text{N}} = (\Omega_c \delta_c + \Omega_b \delta_b) / (\Omega_c + \Omega_b) \), with today’s density parameter \( \Omega_{m0} = \Omega_{c0} + \Omega_{b0} = \Omega_{m0} - \Omega_{\Lambda 0} \). In Eq. (4.14), there exists the contribution of order \(-4\pi G\rho_m \delta_{mN} \) (where \( \rho_m = \rho_c + \rho_b \)), besides the term proportional to \( \delta_{\text{N}} \). We recall that the coefficient of \( \delta_{\text{N}} \) takes the minimum value (5.22) at the wavenumber (5.23). The ratio between \( \omega_{\text{min}}^2 \delta_{\text{N}} \) and \(-4\pi G\rho_m \delta_{mN} \) is given by

\[
\frac{\omega_{\text{min}}^2 \delta_{\text{N}}}{-4\pi G\rho_m \delta_{mN}} = \left( 1 - \frac{1}{4} \frac{\beta_U}{\beta_{\text{m*}}} \right) \Omega_{\Lambda 0} \frac{\delta_{\text{N}}}{\Omega_{m0} \delta_{mN}}. \tag{5.28}
\]

Applying the upper limit of \( |\mu| \) in Eq. (5.26) to the second term on the right-hand-side of Eq. (5.28), it follows that

\[
r_U \equiv \frac{1}{4} \frac{\beta_U}{\beta_{\text{m*}}} \Omega_{\Lambda 0} \frac{\delta_{\text{N}}}{\Omega_{m0} \delta_{mN}} \ll \frac{\beta_U}{6\beta_{\text{m*}}} \Omega_{\Lambda 0} \frac{\delta_{\text{N}}}{\Omega_{m0} \delta_{mN}} \equiv r_{U\text{max}}. \tag{5.29}
\]

Since \( \Omega_{m0} \approx \Omega_{M0} \) for \( \Omega_{\Lambda 0} \ll \Omega_{M0} \), the upper limit of Eq. (5.29) reduces to \( r_{U\text{max}} \approx \beta_U/(6\beta_{\text{m*}}) \delta_{\text{N}}/\delta_{mN} \). Provided that \( \beta_{U*} \) is the similar order to \( \beta_{\text{m*}} \) and that \( \delta_{\text{N}} \) is initially of the same order as \( \delta_{mN} \) (which is the case for adiabatic initial conditions discussed below), the term \( r_U \) is suppressed to be smaller than 1. Moreover, the first term \( (\Omega_{\Lambda 0}/\Omega_{m0}) (\delta_{\text{N}}/\delta_{mN}) \) on the right-hand-side of Eq. (5.28) is much less than 1. Then the evolution of \( \delta_{\text{N}} \) is dominated by the gravitational instability term \(-4\pi G\rho_m \delta_{mN} \) rather than the negative Laplacian term \( c_2^2 (k^2/a^2) \delta_{\text{N}} \). This suggests that the self-coupling would not lead to the strong Laplacian instability of \( \beta_{\text{m}} \) inside the Hubble radius, the CDM-baryon density contrast \( \beta_{\text{m}} \) be larger. In particular the wavenumbers in cases (a) and (b) are initially close to the Hubble radius \( \langle k^2 \rangle \sim 10^{-3} \, \text{Mpc}^{-1} \), and \( \langle k \rangle \approx 30 \). In case (c) of Fig. 3 the wavenumber is larger than \( \langle k \rangle \approx 10^{-2} \, \text{Mpc}^{-1} \). For the density contrasts and velocity potentials, we choose the adiabatic initial conditions

\[
\delta_{\text{N}}(a_s) = \delta_{\text{N}}(a_s) = \delta_{\text{m}}(a_s), \quad v_{\text{N}}(a_s) = v_{\text{N}}(a_s) = v_{\text{m}}(a_s), \tag{5.30}
\]

which are given, respectively, by

\[
\delta_{\text{N}} = \delta_{\text{m}} = \delta_{\text{m}}(a_s), \quad v_{\text{N}} = v_{\text{m}} = v_{\text{m}}(a_s), \tag{5.31}
\]

For a given mode \( k \), this critical scale factor is determined by the condition \( k_I = k \), such that

\[
\frac{a_1}{10^{-7}h^2} = \frac{a_1}{10^{-7}h^2} = \frac{a_1}{10^{-7}h^2} \Bigg|_{a_s} = \frac{a_1}{10^{-7}h^2} = \frac{a_1}{10^{-7}h^2}. \tag{5.32}
\]

In cases (a) and (b) we have \( a_1 = 9.10 \times 10^{-2} \) and \( a_1 = 1.74 \times 10^{-2} \), respectively, which are in agreement with the numerical results of Fig. 3. We note that the wavenumber (5.16) at \( k_J = k_S = k_I \) is \( k_E = 3.77 \times 10^{-3} \, \text{Mpc}^{-1} \) with \( a_E = 2.69 \times 10^{-2} \). In case (a), which corresponds to \( k < k_E \), \( a_1 \) is larger than \( a_E \). However, the parameter space in which \( c_2^2 k^2/a^2 \) dominates over \(-4\pi G\rho_k \) is limited to the yellow shaded region in Fig. 2. For \( k < k_E \), this dominance occurs for \( a_s < a < a_S \), where \( a_S \) is determined by the condition \( k_S = k \), i.e.,

\[
\frac{a_S}{10^{-7}h^2} = \frac{a_S}{10^{-7}h^2} = \frac{a_S}{10^{-7}h^2} = \frac{a_S}{10^{-7}h^2} \Bigg|_{a_s} = \frac{a_S}{10^{-7}h^2} = \frac{a_S}{10^{-7}h^2}. \tag{5.33}
\]

For \( k_E < k < k_{I*} \), the region of self-coupling dominance is in the interval \( a_s < a < a_I \), where \( a_I \) is smaller than \( a_E \). The wavenumber in case (b) belongs to this range. In case (c) of Fig. 3 the wavenumber is larger than \( k_{I*} \), so \( c_2^2 \) is always positive for \( a > a_s \).
Density contrasts $c_s^2$ gives rise to some difference in comparison to the full numerical results. Indeed, using the approximate Eqs. (4.14) and (4.21) for these modes gives rise to some difference in comparison to the full numerical results.

In Fig. 3, we observe that the evolutions of $\delta_{\chi N}$ in cases (a) and (b) are practically identical to that of $\delta_{mN}$. This means that the negative value of $c_s^2$ induced by the self-coupling does not lead to the additional enhancement of $\delta_{\chi N}$ besides the gravitational instability. In case (b), the wavenumber corresponds to $k = k_{\text{max}} = k_{1s}/\sqrt{2}$, so $\omega_s^2$ takes the minimum value (5.22). At $a = a_*$, the quantities in Eq. (5.29) are given by $r_{U*} = 0.25$ and $r_{U_{\text{max}*}} = 2.1$. Since $r_{U*}$ is less than the order 1, the self-coupling effect on the growth of $\delta_{\chi N}$ is suppressed relative to the gravitational instability term $-4\pi G \rho_m \delta_{mN}$ arising from CDM and baryons. Due to the decrease of $r_{U*}$ in time, the contribution of $r_{U*}$ to the ratio (5.28) tends to be weaker at late times. Not only in cases (a) and (b), but also for the wavenumbers in the range $k_{2s} < k < k_{1s}$, we find that $\delta_{\chi N}$ is hardly subject to the Laplacian instability. This property is mostly attributed to the upper limit of $|\mu|$ given in Eq. (5.26). We recall that the baryon-CDM density contrast $\delta_{mN}$ acquires the BEC sound speed squared $c_s^2$ in Eq. (4.21). Since $|c_s^2|$ decreases in time with the initial value smaller than the order 1, the effect of $c_s^2$ on the evolution of $\delta_{mN}$ can be negligible especially for $\Omega_{\chi 0} \ll \Omega_{m0}$.

In case (c) of Fig. 3, the wavenumber is in the range $k > k_{1s}$ and hence the quantum pressure suppresses the growth of $\delta_{\chi N}$ at early times. In the late epoch, however, the term $-4\pi G \rho_m \delta_{mN}$ dominates over the positive Laplacian term $c_s^2(k^2/a^2)\delta_{\chi N}$ in Eq. (4.14). Then, $\delta_{\chi N}$ starts to grow at some point to catch up with $\delta_{mN}$. For increasing $k$, the initial epoch during which $\delta_{\chi N}$ does not grow by the quantum pressure tends to be longer, so today’s value of $\delta_{\chi N}$ is more significantly suppressed in comparison to $\delta_{mN}$. Thus, for the modes $k > k_{1s}$, the self-coupling does not affect the dynamics of perturbations, but the quantum pressure plays an important role to modify the gravitational clustering of $\delta_{\chi N}$.

Finally, we comment on the analysis of Refs. [51–53, 56], in which the authors studied the evolution of an axion density contrast for the potential (5.5). They considered the axion mass in the range $m = 10^{-22}$ eV $\sim 10^{-21}$ eV and showed that the self-coupling with a large field misalignment can induce instabilities of the axion density contrast for particular wavenumbers around $k = 10$ Mpc$^{-1}$. In this case, the axion starts to oscillate long before matter-radiation equality, so it is necessary to take the radiation perturbation into account. For the large misalignment, the axion density contrast can be enhanced by parametric resonance between the onset of field oscillation and the formation of BEC ($a_{\text{osc}} < a < a_*$). Since our nonrelativistic BEC description amounts to averaging over oscillations in the regime $a > a_*$, it does not accommodate the phenomenon of parametric resonance during such a transient epoch. What we
showed in this paper is that, after the BEC formation, the Laplacian instability associated with negative values of $c_s^2$ induced by the attractive self-interaction is no longer effective for the mass range $m < 7 \times 10^{-28}$ eV. It will be of interest to explore whether the resonant instability is also present for such a ultra-light mass range to probe signatures of axion perturbations on scales larger than 10 Mpc.

VI. CONCLUSIONS

In this paper, we provided a general framework for studying the evolution of cosmological perturbations for an ultra-light complex scalar field $\chi$ in a state of the nonrelativistic BEC. Using the Madelung representation (2.11), we expressed the continuity and Euler equations as well as the gravitational field equations in general relativistic, covariant forms. We also included other matter sources like CDM as perfect fluids and explicitly showed their difference from the BEC in a covariant manner. On the FLRW background, the regime in which the nonrelativistic BEC is formed is characterized by the conditions (2.25).

In Sec. III, we derived the full linear perturbation equations of motion for the line element (3.16) containing four scalar perturbed variables $\alpha, B, \zeta, E$. The BEC matter perturbation $\delta \rho_\chi$ and its velocity potential $c_\chi$ obey Eqs. (3.26) and (3.27), respectively. In comparison to the continuity and Euler Eqs. (3.29) and (3.30) of nonrelativistic perfect fluids with $w_\gamma = c_s^2 = 0$, there exists the perturbation $\delta \beta$ given by Eq. (3.28) which contains the effects of quantum pressure and BEC self-interactions. Metric perturbations are coupled to the energy-momentum tensors of both BEC and perfect fluids through the Einstein Eq. (3.15). Our perturbation equations can be applied to any choice of gauges depending on the problem at hand. We showed that all the perturbed equations can be expressed in terms of the gauge-invariant variables introduced in Eq. (3.35).

In Sec. IV, we used the quasi-static approximation for perturbations deep inside the Hubble radius. Even though the background value of $\beta$ vanishes, its gauge-invariant perturbation $\delta \beta_N$ contains an oscillating mode $\delta \beta_N^{(b)}$ with the frequency associated with the field mass $m$. There is also the special solution $\delta \beta_N^{(s)}$ sourced by the density contrasts $\delta \chi_N$ and $\delta \rho_N$, see Eq. (4.9). Taking the average of Eq. (4.13) over the Hubble time scale amounts to ignoring $\delta \beta_N^{(b)}$ relative to $\delta \beta_N^{(s)}$. Then, the BEC density contrast $\delta \chi_N$ obeys the approximate equation of the form (4.14), with the effective sound speed squared (4.15). This value of $c_s^2$, which contains the contributions of quantum pressure and BEC self-interactions, reproduces those known in the literature in the regime $k/a \ll m$ for the two-body self-interacting potential $U(\rho) = \lambda \rho^2/4$. We also showed that the density contrasts $\delta \rho_N$ in the perfect-fluid sector are affected by the BEC sound speed in the form (4.21) through the gravitational interaction. We numerically solved the perturbation equations of $\delta \rho_N$ during the matter era dominated by the BEC energy density without self-interactions and found that the approximate Eq. (4.14) can be trustable except for the wavenumbers initially around the Hubble radius.

In Sec. V, we investigated the effect of the BEC two-body self-interacting potential $U(\rho) = \lambda \rho^2/4$ on the dynamics of linear perturbations. We also included CDM, baryons, and the cosmological constant to discuss the dynamics after the onset of matter dominance. We studied the evolutions of $\delta \chi_N$ and the CDM-baryon density contrast $\delta m_N$ in the ultra-light mass range $m < 7 \times 10^{-28}$ eV, which was unexplored before in the presence of self-interactions. The BEC sound speed squared $c_s^2$ can be negative by its self-interactions for some particular range of scales relevant to the CMB and large-scale structure measurements. In this ultra-light mass range, the BEC cannot be all DM due to the suppression of the matter power spectrum for $k > k_f$. Under the constraint $\Omega_{\chi,0} \ll \Omega_{m,0}$, the negative Laplacian term $c_s^2 k^2/a^2$ induced by the self-coupling can dominate over the other term $-4\pi G \rho_\chi$ appearing as the coefficient of $\delta \chi_N$ in Eq. (4.14). However, the gravitational instability term $-4\pi G \rho_m \delta m$ arising from the CDM-baryon density contrast $\delta m$ in Eq. (4.14) overwhelms $c_s^2 k^2/a^2$ for the adiabatic initial conditions (5.30). Numerically, we confirmed that, even when $c_s^2$ is negative, the self-coupling hardly induces the Laplacian instabilities of $\delta \chi_N$ and $\delta m_N$ besides their gravitational instabilities.

We have thus shown that the BEC self-coupling is almost ineffective to modify the dynamics of linear density perturbations at least in the regime where the nonrelativistic BEC description is valid. In the context of axions, there is a transient epoch between the moments at which the axion starts to oscillate ($m \approx 3H$) and when the BEC is formed ($m \gg H$). To deal with such a transition including the phenomenon of parametric resonance, we need to consider perturbations of the bosonic field itself and relate them with its density contrast and velocity potential. After the BEC formation, those perturbations should be matched with the solutions derived in this paper. The detailed study about the evolution of inhomogeneities including such a transient epoch with the various axion mass range deserves for a future separate work.
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