Convergence Rates of Distributed Nesterov-like Gradient Methods on Random Networks
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Abstract—We consider distributed optimization in random networks where $N$ nodes cooperatively minimize the sum $\sum_{i=1}^{N} f_i(x)$ of their individual convex costs. Existing literature proposes distributed gradient-like methods that are computationally cheap and resilient to link failures, but have slow convergence rates. In this paper, we propose accelerated distributed gradient methods that: 1) are resilient to link failures; 2) computationally cheap; and 3) improve convergence rates over other gradient methods. We model the network by a sequence of independent, identically distributed random matrices $\{W(k)\}$ drawn from the set of symmetric, stochastic matrices with positive diagonals. The network is connected on average and the cost functions are convex, differentiable, with Lipschitz continuous and bounded gradients. We design two distributed Nesterov-like gradient methods that modify the D–NG and D–NC methods we proposed for static networks. We prove their convergence rates in terms of the expected optimality gap at the cost function. Let $k$ and $K$ be the number of per-node gradient evaluations and per-node communications, respectively. Then the modified D–NG achieves rates $O(\log k/k)$ and $O(\log K/K)$, and the modified D–NC rates $O(1/k^2)$ and $O(1/K^{2-\xi})$, where $\xi > 0$ is arbitrarily small. For comparison, the standard distributed gradient method cannot do better than $\Omega(1/k^{2/3})$ and $\Omega(1/K^{2-\xi})$, on the same class of cost functions (even for static networks). Simulation examples illustrate our analytical findings.

Keywords: Distributed optimization, convergence rate, random networks, Nesterov gradient, consensus.

I. INTRODUCTION

We study distributed optimization where $N$ nodes in a (sensor, multi-robot, or cognitive) network minimize the sum $\sum_{i=1}^{N} f_i(x)$ of their individual costs subject to a global optimization variable $x \in \mathbb{R}^d$. Each $f_i : \mathbb{R}^d \to \mathbb{R}$ is convex and known only by node $i$. The goal for each node is to estimate the vector $x^* \in \mathbb{R}^d$ of common interest to all nodes. Each node $i$ acquires locally data $d_i$ that reveals partial knowledge on $x^*$ and forms the cost function $f_i(x; d_i)$ of the global variable $x$. The nodes cooperate to find $x^*$ that minimizes $\sum_{i=1}^{N} f_i(x; d_i)$. This setup has been studied in the context of many signal processing applications, including: 1) distributed estimation in sensor networks, e.g., [1], [2]; 2) acoustic source localization, e.g., [3]; and 3) spectrum sensing for cognitive radio networks, e.g., [4], [5].

For the above problem, reference [6], see also [7], [8], presents two distributed Nesterov-like gradient algorithms for static (non-random) networks, referred to as D–NG (Distributed Nesterov Gradient algorithm) and D–NC (Distributed Nesterov gradient with Consensus iterations). The distributed gradient methods D–NG and D–NC significantly improve the convergence rates over standard distributed gradient methods, e.g., [9], [10].

In this paper, we propose the mD–NG and mD–NC algorithms, which modify the D–NG and D–NC algorithms, and, beyond proving their convergence, we solve the much harder problem of establishing their convergence rate guarantees on random networks. Randomness in networks may arise when inter-node links fail as with random packet dropouts in wireless sensor networks, and when communication protocols are random like with the gossip protocol [11]. We model the network by a sequence of random independent, identically distributed (i.i.d.) weight matrices $W(k)$ drawn from a set of symmetric, stochastic matrices with positive diagonals, and we assume that the network is connected on average (the graph supporting $E[W(k)]$ is connected). We establish the convergence rates of the expected optimality gap in the cost function (at any node $i$) of mD-NG and mD-NC, in terms of the number of per node gradient evaluations $k$ and the number of per-node communications $K$, when the functions $f_i$ are convex and differentiable, with Lipschitz continuous and bounded gradients. We show that the modified methods achieve in expectation the same rates that the methods in [6] achieve on static networks, namely: mD–NG converges at rates $O(\log k/k)$ and $O(\log K/K)$, while mD–NC has rates $O(1/k^2)$ and $O(1/K^{2-\xi})$, where $\xi$ is an arbitrarily small positive number. We explicitly give the convergence rate constants in terms of the number of nodes $N$ and the network statistics, more precisely, in
terms of the quantity \( \eta := (\|\mathbb{E}[W(k)]^2 - J\|)^{1/2} \) (See ahead paragraph with heading Notation.)

We contrast D–NG and D–NC in [6] with their modified variants, mD–NG and mD–NC, respectively. Simulations in Section VIII show that D–NG may diverge when links fail, while mD–NG converges, possibly at a slightly lower rate on static networks and requires an additional (\( d \)-dimensional) vector communication per iteration \( k \). Hence, mD–NG compromises slightly speed of convergence for robustness to link failures.

Algorithm mD–NC has one inner consensus with \( 2d \)-dimensional variables per outer iteration \( k \), while D–NC has two consensus algorithms with \( d \)-dimensional variables. Both D–NC variants converge in our simulations when links fail, showing similar performance.

The analysis here differs from [6], since the dynamics of disagreements are different from the dynamics in [6]. This requires novel bounds on certain products of time-varying matrices. By disagreement, we mean how different the solution estimates of distinct nodes are, say \( x_i(k) \) and \( x_j(k) \) for nodes \( i \) and \( j \).

**Brief comment on the literature.** There is increased interest in distributed optimization and learning. Broadly, the literature considers two types of methods, namely, batch processing, e.g., [9], [12], [13], [14], [15], [4], [5], and online adaptive processing, e.g., [16], [17], [18]. With batch processing, data is acquired beforehand, and hence the \( f_i \)'s are known before the algorithm runs. In contrast, with adaptive online processing, nodes acquire new data at each iteration \( k \) of the distributed algorithm. We consider here batch processing.

Distributed gradient methods are, e.g., in [9], [12], [13], [19], [20], [21], [22], [14], [10], [3], [23], [15]. References [9], [12], [20] proved convergence of their algorithms under deterministically time varying or random networks. Typically, \( f_i \)'s are convex, non-differentiable, and with bounded gradients over the constraint set. Reference [10] establishes \( O(\log k/\sqrt{k}) \) convergence rate (with high probability) of a version of the distributed dual averaging method. We assume a more restricted class \( F \) of cost functions–\( f_i \)'s that are convex and have Lipschitz continuous and bounded gradients, but, in contradistinction, we establish strictly faster convergence rates–at least \( O(\log k/k) \) that are not achievable by standard distributed gradient methods [9] on the same class \( F \). Indeed, [6] shows that the method in [9] cannot achieve a worst-case rate better than \( \Omega(1/k^{2/3}) \) on the same class \( F \), even for static networks. Reference [24] proposes an accelerated distributed proximal gradient method, which resembles our D–NC method for deterministically time varying networks; in contrast, we deal here with randomly varying networks. For a detailed comparison of D–NC with [24], we refer to [6].

Distributed augmented Lagrangian or ordinary Lagrangian dual methods, are e.g., in [4], [25], [26], [27], [28], [29], [1], [2]. They have in general more complex iterations than gradient methods, but may have a lower total communication cost, e.g., [27]. To our best knowledge, the convergence rates of such methods have not been established for random networks.

**Paper organization.** The next paragraph sets notation. Section II introduces the network and optimization models and presents mD–NG and its convergence rate, which is proved in Sections III and IV. Section V presents mD–NC and its convergence rate, proved in Section VI. Section VII discusses extensions to our results. Section VIII illustrates mD–NG and mD–NC on a Huber loss example. We conclude in Section IX. Certain auxiliary arguments are in Appendices IX-A and IX-B.

**Notation.** Denote by: \( \mathbb{R}^d \) the \( d \)-dimensional real space; \( A_{lm} \) or \( [A]_{lm} \) the \((l,m)\)-entry of \( A \); \( A^\top \) the transpose of \( A \); \( [a]_{lm} \) the selection of the \((l-1)\)-th, \( \cdots \), \( m \)-th entries of vector \( a \); \( I, 0, 1 \), and \( e_i \), respectively, the identity matrix, the zero matrix, the column vector with unit entries, and the \( i \)-th column of \( I \); \( J \) the \( N \times N \) ideal consensus matrix \( J := (1/N)11^\top \); \( \mathbb{I} \) the Kroncker product of matrices; \( \| \cdot \| \) the vector (matrix) \( l \)-norm of its argument; \( \| \cdot \|_2 \) the Euclidean (spectral) norm of its vector (matrix) argument \((\| \cdot \| \) also denotes the modulus of a scalar); \( \lambda_i(\cdot) \) the \( i \)-th smallest in modulus eigenvalue; \( A \succ 0 \) a positive definite Hermitian matrix \( A \); \( [a] \) the integer part of a real scalar \( a \); \( \nabla \phi(x) \) and \( \nabla^2 \phi(x) \) the gradient and Hessian at \( x \) of a twice differentiable function \( \phi: \mathbb{R}^d \to \mathbb{R} \), \( d \geq 1 \); \( \mathbb{P}([\cdot]) \) and \( \mathbb{E}([\cdot]) \) the probability and expectation, respectively; and \( \mathcal{I}_A \) the indicator of event \( A \). For two positive sequences \( \eta_n \) and \( \chi_n \), we have: \( \eta_n = O(\chi_n) \) if \( \limsup_{n \to \infty} \frac{\eta_n}{\chi_n} < \infty \); \( \eta_n = \Omega(\chi_n) \) if \( \liminf_{n \to \infty} \frac{\eta_n}{\chi_n} > 0 \); and \( \eta_n = \Theta(\chi_n) \) if \( \eta_n = O(\chi_n) \); and \( \eta_n = \Omega(\chi_n) \).

**II. ALGORITHM mD–NG**

Subsection II-A introduces the network and optimization models, Subsection II-B the mD–NG algorithm, and Subsection II-C its convergence rate.

**A. Problem model**

**Random network model.** The network is random, due to link failures or communication protocol used (e.g., gossip, [11], [30]). It is defined by a sequence \( \{W(k)\}_{k=1}^\infty \) of \( N \times N \) random weight matrices.

**Assumption 1 (Random network)** We have:

(a) The sequence \( \{W(k)\}_{k=1}^\infty \) is i.i.d.

(b) Almost surely (a.s.), \( W(k) \) are symmetric, stochastic, with strictly positive diagonal entries.

(c) There exists \( \omega > 0 \) such that, for all \( i, j = 1, \cdots, N \), a.s. \( W_{ij}(k) \notin (0, \omega) \).
By Assumptions 1 (b) and (c), $W_{ij}(k) \geq \mu$ a.s., $\forall i$; also, $W_{ij}(k), i \neq j$, may be zero, but if $W_{ij}(k) > 0$ (nodes $i$ and $j$ communicate) it is non-negligible (at least $\mu$).

Let $\tilde{W} := \mathbb{E}[W(k)]$, the supergraph $\mathcal{G} = (N, E), N$ the set of $N$ nodes, and $E = \{\{i,j\} : i < j, W_{ij} > 0\} - \tilde{G}$ collects all realizable links, all pairs $\{i,j\}$ for which $W_{ij}(k) > 0$ with positive probability.

Assumption 1 covers link failures. Here, each link $\{i,j\} \in E$ at time $k$ is Bernoulli: when it is one, $\{i,j\}$ is online (communication), and when it is zero, the link fails (no communication). The Bernoulli links are independent over time, but may be correlated in space. Possible weights are: 1) $i \neq j, \{i,j\} \in E$: $W_{ij}(k) = w_{ij} = 1/N$, when $\{i,j\}$ is online, and $W_{ij}(k) = 0$, else; 2) $i \neq j, \{i,j\} \notin E: W_{ij}(k) = 0$; and 3) $W_{ii}(k) = 1 - \sum_{j \neq i} W_{ij}(k)$. As an alternative, when the link occurrence probabilities and their correlations are known, set the weights $w_{ij}$, $\{i,j\} \in E$, as the minimizers of $\tilde{p}$ (See Section VIII and [31] for details.)

We further make the following Assumption.

**Assumption 2 (Network connectedness)** $\tilde{G}$ is connected.

Denote by $\tilde{W}(k) = W(k) - J = -1(1/N)11^T$, by $\Phi(k, t) = \tilde{W}(k) \cdots \tilde{W}(t+2), t = 0, 1, \ldots, k-2$,

and by $\Phi(k, k-1) = I$. One can show that $\tilde{p} := (\mathbb{E}[W^2(k)] - J)^{1/2}$ is the square root of the second largest eigenvalue of $\mathbb{E}[W^2(k)]$ and that, under Assumptions 1 and 2, $\tilde{p} < 1$. Lemma 1 (proof in Appendix IX-A) shows that $\tilde{p}$ characterizes the geometric decay of the first and second moments of $\Phi(k, t)$.

**Lemma 1** Let Assumptions 1 and 2 hold. Then:

\[
\mathbb{E} \left[ \|\Phi(k,t)\| \right] \leq N \tilde{p}^{k-t-1} \tag{2}
\]

\[
\mathbb{E} \left[ \|\Phi(k,t)^T\|\Phi(k,t)\| \right] \leq N^2 \tilde{p}^{2(k-t-1)} \tag{3}
\]

\[
\mathbb{E} \left[ \|\Phi(k,s)^T\|\Phi(k,t)\| \right] \leq N^3 \tilde{p}^{(k-t-1)+(k-s)} \tag{4}
\]

for all $t, s = 0, \ldots, k-1$, for all $k = 1, 2, \ldots$

The bounds in (2)-(4) may be loose, but are enough to prove the results below and simplify the presentation.

For static networks, $W(k) \equiv W$, $W$ doubly stochastic, deterministic, symmetric, and $\overline{\mu} := \|W - J\|$ equals the spectral gap, i.e., the modulus of the second largest (in modulus) eigenvalue of $W$. For static networks, the constants $N, N^2, N^3$ in (2)-(4) are reduced to one.

**Optimization model.** We now introduce the optimization model. The nodes solve the unconstrained problem:

\[
\text{minimize} \sum_{i=1}^{N} f_i(x) =: f(x). \tag{5}
\]

The function $f_i : \mathbb{R}^d \rightarrow \mathbb{R}$ is known only to node $i$. We impose the following three Assumptions.

**Assumption 3 (Solvability)** There exists a solution $x^* \in \mathbb{R}^d$ such that $f(x^*) = f^* := \inf_{x \in \mathbb{R}^d} f(x)$.

**Assumption 4 (Lipschitz continuous gradient)** For all $i$, $f_i$ is convex and has Lipschitz continuous gradient with constant $L \in [0, \infty)$:

\[
\|\nabla f_i(x) - \nabla f_i(y)\| \leq L \|x - y\|, \forall x, y \in \mathbb{R}^d.
\]

**Assumption 5 (Bounded gradients)** There exists a constant $G \in [0, \infty)$ such that $\forall i$, $\|\nabla f_i(x)\| \leq G, \forall x \in \mathbb{R}^d$.

Assumptions 3 and 4 are standard in gradient methods; in particular, Assumption 4 is precisely the Assumption required by the centralized Nesterov gradient method [32]. Assumption 5 is not required in centralized Nesterov. Reference [6] demonstrates that (even on) static networks and a constant $W(k) \equiv W$, the convergence rates of D–NG or of the standard distributed gradient method in [9] become arbitrarily slow if Assumption 5 is violated.

Examples of functions $f_i$ that obey Assumptions 3–5 include the logistic loss for classification, [33], the Huber loss in robust statistics, [34], or the “fair” function in robust statistics, $\phi : \mathbb{R} \rightarrow \mathbb{R}$, $\phi(x) = b_0^2\left(\frac{|x|}{b_0} - \log\left(1 + \frac{|x|}{b_0}\right)\right)$, where $b_0 > 0$, [34].

**B. Algorithm mD–NG for random networks**

We modify D–NG in [6] to handle random networks. Node $i$ maintains its solution estimate $x_i(k)$ and auxiliary variable $y_i(k), k = 0, 1, \ldots$ It uses arbitrary initialization $x_i(0) = y_i(0) \in \mathbb{R}^d$ and, for $k = 1, 2, \ldots$, performs the updates

\[
x_i(k) = \sum_{j \in O_i(k)} W_{ij}(k) y_j(k-1) - \alpha_{k-1} \nabla f_i(y_i(k-1)) \tag{6}
\]

\[
y_i(k) = (1 + \beta_{k-1}) x_i(k) - \beta_{k-1} \sum_{j \in O_i(k)} W_{ij}(k) x_j(k-1). \tag{7}
\]

In (6)–(7), $O_i(k) = \{j \in \{1, \ldots, N\} : W_{ij}(k) > 0\}$ is the (random) neighborhood of node $i$ (including node $i$) at time $k$. For $k = 0, 1, 2, \ldots$, the step-size $\alpha_k$ is:

\[
\alpha_k = c/(k+1), \ c \leq 1/(2L). \tag{8}
\]

All nodes know $L$ (or its upper bound) beforehand to set $\alpha_k$ in (8). Section VII relaxes this requirement. Let $\beta_k$ be the sequence from centralized Nesterov, [32]:

\[
\beta_k = \frac{k}{k+3}. \tag{9}
\]
The mD–NG algorithm (6)–(7) differs from D–NG in [6] in step (7). With D–NG, nodes communicate only the variables \( y_i(k−1) \)'s; with mD–NG, they also communicate the \( x_j(k−1) \)'s (see sum term in (7)). This modification allows for the robustness to link failures. (See Theorems 2 and 3 and the simulations in Section VIII.) Further, mD–NG does not require the weight matrix to be positive definite, as D–NG in [6] does.

**Vector form.** Let \( x(k) := (x_1(k)^T, \ldots, x_N(k)^T)^T \), \( y(k) := (y_1(k)^T, \ldots, y_N(k)^T)^T \), and \( F : \mathbb{R}^{Nd} \to \mathbb{R} \), \( F(x_1, \ldots, x_N) := f_1(x_1) + \cdots + f_N(x_N) \). Then, for \( k = 1, 2, \ldots \), with \( x(0) = y(0) \in \mathbb{R}^{Nd} \), \( W(k) \otimes I \) the Kronecker product of \( W(k) \) with the \( d \times d \) identity \( I \), mD–NG in vector form is:

\[
\begin{align*}
\dot{x}(k) &= (W(k) \otimes I) y(k) - \alpha_k - 1 \nabla F(y(k-1)) \\
y(k) &= (1 + \beta_k - 1) x(k) - \beta_k - 1 (W(k) \otimes I) x(k-1).
\end{align*}
\]

**Initialization.** For notation simplicity, without loss of generality (wlog), we assume, with all proposed methods, that nodes initialize their estimates to the same values, i.e., \( x_i(0) = y_i(0) = x_j(0) = y_j(0) \), for all \( i, j \); for example, \( x_1(0) = y_1(0) = x_2(0) = y_2(0) = 0 \).

**C. Convergence rate of mD–NG**

We state our convergence rate for mD–NG operating in random networks. Proofs are in Section IV. We estimate the expected optimality gap in the cost at each node \( i \) normalized by \( N \), e.g., [10], [23]:

\[
\frac{1}{N} E \left[ f(x_i(k)) - f^* \right],
\]

where \( x_i \) is node \( i \)'s solution at a certain stage of the algorithm. We study how node \( i \)'s optimality gap decreases with: 1) the number \( k \) of iterations, or of per-node gradient evaluations; and 2) the total number \( K \) of \( d \)-dimensional vector communications per node. With mD–NG, \( K = \mathcal{K} \) at each \( k \), there is one and only one per-node \( 2d \)-dimensional communication and one per-node gradient evaluation. Not so with mD–NC, as we will see. We establish for both methods the mean square convergence rate on the mean square disagreements of different node estimates in terms of \( k \) and \( \mathcal{K} \), showing that it converges to zero.

Define by: the network-wide global averages of the nodes’ estimates be \( \bar{x}(k) := \frac{1}{N} \sum_{i=1}^{N} x_i(k) \) and \( \bar{y}(k) := \frac{1}{N} \sum_{i=1}^{N} y_i(k) \); the disagreements: \( \bar{x}_i(k) = x_i(k) - \bar{x}(k) \) and \( \bar{y}_i(k) = (\bar{x}_1(k)^T, \ldots, \bar{x}_N(k)^T)^T \), and analogously for \( \bar{y}_i(k) \) and \( \bar{y}(k) \); and \( \bar{z}(k) := (\bar{y}(k)^T, \bar{z}(k)^T)^T \). We have the following Theorem on \( \mathbb{E} \left[ \| \bar{z}(k) \| \right] \) and \( \mathbb{E} \left[ \| \bar{z}(k) \|^2 \right] \). Note \( \| \bar{z}(k) \| \leq \| \bar{z}(k) \| \), and so \( \mathbb{E} \left[ \| \bar{z}(k) \| \right] \leq \mathbb{E} \left[ \| \bar{z}(k) \| \right] \) and \( \mathbb{E} \left[ \| \bar{z}(k) \|^2 \right] \leq \mathbb{E} \left[ \| \bar{z}(k) \|^2 \right] \). (Equivalent inequalities hold for \( \bar{y}(k) \).)

Recall also \( \bar{p} \) in Lemma 1. Theorem 2 states that the mean square disagreement of different nodes’ estimates converges to zero at rate \( 1/k^2 \).

**Theorem 2** Consider mD–NG (6)–(9) under Assumptions 1–5. Then, for all \( k = 1, 2, \ldots \)

\[
\mathbb{E} \left[ \| \bar{z}(k) \| \right] \leq \frac{50 c N^3/2 G}{(1 - \bar{p})^2} \tag{12}
\]

\[
\mathbb{E} \left[ \| \bar{z}(k) \|^2 \right] \leq \frac{50^2 c^2 N^4 G^2}{(1 - \bar{p})^4} + 50 N^2 c G^2 \tag{13}
\]

Theorem 3 establishes the convergence rate of mD–NG as \( O(\log k/k) \) (and \( O(\log K/K) \)).

**Theorem 3** Consider mD–NG (6)–(9) under Assumptions 1–5. Let \( \| \bar{p}(0) - x^* \| \leq R, R \geq 0 \). Then, at any node \( i \), the expected normalized optimality gap \( \frac{1}{N} \mathbb{E} \left[ f(x_i(k)) - f^* \right] \) is \( O(\log k/k) \); more precisely:

\[
\mathbb{E} \left[ f(x_i(k)) - f^* \right] \leq \frac{2 R^2}{N} \left( 1 + \frac{1}{k} \right) \leq \frac{50^2 c^2 N^4 G^2}{(1 - \bar{p})^4} \tag{14}
\]

**III. INTERMEDIATE RESULTS**

We establish intermediate results on certain scalar sums and the products of time-varying \( 2 \times 2 \) matrices that arise in the analysis of mD–NG and mD–NC.

**Scalar sums.** We have the following Lemma.

**Lemma 4** Let \( 0 < r < 1 \). Then, for all \( k = 1, 2, \ldots \)

\[
\sum_{t=1}^{k} t^r \leq \frac{r}{(1 - r)^2} \leq \frac{1}{(1 - r)^2} \tag{15}
\]

\[
\sum_{t=0}^{k-1} \frac{1}{t+1} \leq \frac{1}{(1 - r)^2} \tag{16}
\]

**Proof:** Let \( \frac{d}{dr} h(r) \) be the derivative of \( h(r) \). Then (15) follows from:

\[
\sum_{t=1}^{k} t^r = \sum_{t=1}^{k} \sum_{j=1}^{r} r^{j-1} t = \frac{d}{dr} \left( \sum_{t=1}^{k} r^t \right)
\]

\[
= \frac{d}{dr} \left( \frac{r - r^{k+1}}{1 - r} \right)
\]

\[
= \frac{(1 - (k+1)r)k(1 - r - r^{k+1})}{(1 - r)^2}
\]
We establish bounds on the sums $\sigma_k = \frac{r}{(1-r)^2}$, $\forall k = 1, 2, \cdots$

To obtain (16), use (15) and $k/(t+1) \leq k - t$, $\forall t = 0, 1, \cdots, k - 1$:

$$\sum_{t=0}^{k-1} r^{k-t-1} \frac{1}{t+1} = \frac{1}{k} \sum_{t=0}^{k-1} \frac{r^{k-t-1}}{t+1} \leq \frac{1}{k} \sum_{t=0}^{k-1} \frac{r^{k-t}(k-t)}{k} \leq \frac{1}{k} \frac{1}{(1-r)^2}.$$  

**Products of matrices.** For $k = 1, 2, \cdots$, let $B(k)$ be:

$$B(k) := \begin{bmatrix}
(1 + \beta_{k-1}) & -\beta_{k-1} & 1 \\
0 & 0 & 0
\end{bmatrix},$$  

(17)

with $\beta_{k-1}$ in (9). The proofs of Theorems 2 and 9 rely on products $B(k, t)$. Let $B(k, -1) := I$ and:

$$B(k, t) := B(k) \cdot \cdots \cdot B(k, -t), t = 0, 1, \cdots, k - 2.$$  

(18)

**Lemma 5 (Products $B(k, t)$)** Let $k \geq 3$, $B(k, t)$ in (18), $a_t := 3/(t+3)$, $t = 0, 1, \cdots$, and:

$$B_1 := \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}, B_2 := \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, B_3 := \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.  

(19)

Then, for $t = 1, 2, \cdots, k - 2$:

$$B(k, t) = B_1^{t+1} - \sigma_2(k, t) B_2 - \sigma_3(k, t) B_3,$$

where

$$\sigma_2(k, t) = a_{k-t-1} + a_{k-t} \beta_{k-t-1} (t-1)  
+ a_{k-t+1} \beta_{k-t+1} (t-2) + \cdots + a_{k-2} \beta_{k-3} \cdots \beta_{k-t-1}.$$  

\begin{align*}
\sigma_3(k, t) &= a_{k-t-1} + a_{k-t} \beta_{k-t-1} \\
&+ a_{k-t+1} \beta_{k-t+1} \beta_{k-t-1} + \cdots + a_{k-2} \beta_{k-3} \cdots \beta_{k-t-1}.
\end{align*}  

(20)

(21)

We establish bounds on the sums $\sigma_2(k, t)$ and $\sigma_3(k, t)$.

**Lemma 6** Let $\sigma_2(k, t)$ and $\sigma_3(k, t)$ in (20)-(21), $t = 1, \cdots, k - 2, k \geq 3$. Then:

$$\frac{t^2}{k+2} \leq \sigma_2(k, t) \leq t + 1, \quad 0 \leq \sigma_3(k, t) \leq 1.$$  

(22)

**Proof:** We prove each of the four inequalities above.

**Proof of the right inequality on $\sigma_2(k, t)$**. By induction on $t = 1, \cdots, k - 2$. The claim holds for $t = 1$, since $\sigma_2(k, 1) = a_{k-2} = 3/(k+1) \leq 1 + 1$, $\forall k$. Let it be true for some $t \geq 1$. For $t = 1, \cdots, k - 3$, write $\sigma_2(k, t)$ as:

$$\sigma_2(k, t+1) = a_{k-t-2} (t+1) + \beta_{k-t-2} \sigma_2(k, t).$$  

(23)

Using (23) and the induction hypothesis: $\sigma_2(k, t+1) \leq \frac{(t+1) a_{k-t-2} + \beta_{k-t-2} (t+1) + 1}{t+1} \leq t+1$. Thus, the right inequality on $\sigma_2(k, t)$.

**Proof of the left inequality on $\sigma_2(k, t)$**. Again, by induction on $t$. The claim holds for $t = 1$, since:

$$\sigma_2(k, 1) = a_{k-2} = \frac{3}{k+1} \geq \frac{1}{k+2}.$$  

Let the claim be true for some $t \in \{1, 2, \cdots, k-3\}$, i.e.:

$$\sigma_2(k, t) \geq \frac{t^2}{k+2}.$$  

(24)

We show that $\sigma_2(k, t+1) \geq \frac{(t+1)^2}{k+2}$ Using (23):

$$\sigma_2(k, t+1) \geq a_{k-t-2} (t+1) + \beta_{k-t-2} \frac{t^2}{k+2}$$

$$= \frac{(t+1)^2}{k+2} + \frac{t (k-t)}{k+2} \frac{(2k+5t+5)}{k+2} \geq \frac{(t+1)^2}{k+2},$$

where the last equality follows after algebraic manipulations. By induction, the last inequality completes the proof of the lower bound on $\sigma_2(k, t)$.

**Proof of bounds on $\sigma_3(k, t)$**. The lower bound is trivial. The upper bound follows by induction. For $t = 1$:

$$\sigma_3(k, 1) = a_{k-2} + a_{k-1} \beta_{k-2} \leq a_{k-2} + \beta_{k-2} = 1.$$  

Let the claim hold for some $t \in \{1, \cdots, k-3\}$, i.e.:

$$\sigma_3(k, t) \leq 1.$$  

From (20):

$$\sigma_3(k, t+1) = \beta_{k-t-2} \sigma_3(k, t) + a_{k-t-2}.$$  

Thus, by the induction hypothesis:

$$\sigma_3(k, t+1) \leq \sigma_3(k, t) + a_{k-t-2} \leq 1,$$

completing the proof of the upper bound on $\sigma_3(k, t)$.

**Lemma 7** Consider $B(k, t)$ in (18). Then, for all $t = 0, \cdots, k-1$, for all $k = 1, 2, \cdots$

$$\|B(k, k-t-2)\| \leq \frac{8}{k+1} \frac{8}{k} + 5.$$  

(25)

**Proof:** Fix some $t \in \{1, \cdots, k-2\}$, $k \geq 3$, and consider $B(k, t)$ in Lemma 5. It follows $B_1^t = t B_2 + I$.

Thus,

$$B(k, t) = (t+1 - \sigma_2(k, t)) B_2 + I - \sigma_3(k, t) B_3.$$  

(26)

By Lemma 6, the term:

$$0 \leq t + 1 - \sigma_2(k, t) \leq t + 1 - \frac{t^2}{k+2}.$$  

Using in (26) this equation, $\sigma_3(k, t) \leq 1$ (by Lemma 6), $\|B_2\| = 2, \text{and} \|B_3\| = \sqrt{2} < 2$, get:

$$\|B(k, t)\| \leq 2(t+1 - \frac{t^2}{k+2}) + 2 \left(1 - \frac{t^2}{k+2}\right) + 5,$$  

(27)
for all \( t = 1, 2, \cdots, k - 2, k \geq 3 \). Next, from (27), for \( t = 0, \cdots, k - 3, k \geq 3, \) get:
\[
\|B(k, k - t - 2)\| \leq 2 \left( k - t - 2 - \frac{(k - t - 2)^2}{k + 2} \right) + 5
\]
\[
= 2(k - t - 2) t + \frac{4}{k + 2} + 5 \leq 8(k - t - 1) \frac{t + 1}{k} + 5,
\]
We used \((t + 4)/(k + 2) \leq 4(t + 1)/k\) and proved (25) for \( t = 0, \cdots, k - 3, k \geq 3 \). To complete the proof, we show that (25) holds also for: 1) \( t = k - 2, k \geq 2; 2) t = k - 1, k \geq 1 \). Consider first case 1 and \( B(k, k - 2) = B(k - 1) = B_1 - a_{k-1}B_3, k \geq 2 \). We have \( \|B(k, k - 2)\| \leq \|B_1\| + \|B_3\| < 5 \), and so (25) holds for \( t = k - 2, k \geq 2 \). Next, consider case 2 and \( B(k, k - 1) = I, k \geq 1 \). We have that \( \|B(k, k - 1)\| = 1 < 5 \), and so (25) also holds for \( t = k - 1, k \geq 1 \). This proves the Lemma.

IV. PROOFS OF THEOREMS 2 AND 3

Subsection IV-A proves Theorem 2, while Subsection IV-B proves Theorem 3.

A. Proof of Theorem 2

Through this proof and the rest of the paper, we establish certain equalities and inequalities on random quantities of interest. These equalities and inequalities further ahead hold either: 1) surely, for any random realization; or: 2) in expectation. From the notation, it is clear which of the two cases is in force. For notational simplicity, we perform the proof of Theorem 2 for the case \( d = 1 \), but the proof extends for generic \( d > 1 \). The proof has three steps. In Step 1, we derive the dynamic equation for the disagreement \( \bar{z}(k) = (\bar{g}(k)^\top, \bar{x}(k)^\top)^\top \).

In Step 2, we unwind the dynamic equation, expressing \( \bar{z}(k) \) in terms of the products \( \bar{\Phi}(k, t) \) in (1) and \( B(k, t) \) in (18). Finally, in Step 3, we apply the already established bounds on the norms of the latter products.

Step 1. Disagreement dynamics. Let \( \bar{z}(k) := (\bar{g}(k)^\top, \bar{x}(k)^\top)^\top \). Multiplying (6)–(7) from the left by \((I - J)\), using \((I - J)W(k) = \bar{W}(k) - J\), obtain:
\[
\bar{z}(k) = (B(k) \otimes \bar{W}(k)) \bar{z}(k - 1) + u(k - 1), \tag{28}
\]
for \( k = 1, 2, \cdots \) and \( \bar{z}(0) = 0 \), where
\[
u(k - 1) = \left[ \alpha_{k-1} (1 + \beta_{k-1}) (I - J) \nabla F(y(k - 1)) \right]. \tag{29}
\]

Step 2. Unwinding recursion (28). Recall \( \bar{\Phi}(k, t) \) in (1), and \( B(k, t) \) in (18). Then, unwinding (28), and using the Kronecker product property \((A \otimes B)(C \otimes D) = (AB) \otimes (CD)\), we obtain for all \( k = 1, 2, \cdots \)
\[
\bar{z}(k) = \sum_{t=0}^{k-1} (B(k, k - t - 2) \otimes \bar{\Phi}(k, t)) u(t), \tag{30}
\]
The quantities \( u(t) \) and \( \bar{\Phi}(k, t) \) in (30) are random, while the \( B(k, k - t - 2) \)'s are deterministic.

Step 3. Finalizing the proof. Consider \( u(t) \) in (29). By Assumption 5, we have \( \|\nabla F(y(t))\| \leq \sqrt{N} \). Using this, the step-size \( \alpha_t = c/(t + 1) \), and \( \|I - J\| = 1 \), get
\[
\|u(t)\| \leq \sqrt{c \sqrt{N}} \frac{k}{t+1}, \tag{31}
\]
for any random realization of \( u(t) \). With this bound, Lemma 7, and the sub-multiplicative and sub-additive properties of norms, obtain from (30):
\[
\mathbb{E}\|\bar{z}(k)\|^2 \leq \left(8 \sqrt{3} c \sqrt{N} G \right) \frac{1}{k} \sum_{t=0}^{k-1} \|\bar{\Phi}(k, t)\| (k - t - 1) \]
\[
+ \left(5 \sqrt{3} c \sqrt{N} G \right) \sum_{t=0}^{k-1} \frac{1}{t + 1} \frac{1}{t} \frac{1}{t+1} \frac{1}{t+1}.
\]
Finally, applying Lemma 4 to the last equation with \( r = 7 \), the result in (12) follows.

Now prove (13). Consider \( \|\bar{z}(k)\|^2 \). From (30):
\[
\|\bar{z}(k)\|^2 = \sum_{t=0}^{k-1} \sum_{s=0}^{k-1} \|B(k, k - t - 2)\| \|B(k, k - s - 2)\| \]
\[
\left( B(k, k - t - 2) \otimes \bar{\Phi}(k, s) \right) u(s)
\]
\[
= \sum_{t=0}^{k-1} \sum_{s=0}^{k-1} \|u(t)\| \|u(s)\|
\]
\[
\|\bar{\Phi}(k, t) \otimes \bar{\Phi}(k, s)\| 3c^2 N G^2 \frac{1}{(t+1)(s+1)}.
\]
The last inequality uses Lemma 7 and \( \|u(\sqrt{t})\| \leq \sqrt{3c \sqrt{N} G} \). Taking expectation and applying Lemma 1, obtain:
\[
\mathbb{E}\|\bar{z}(k)\|^2 \leq (3c^2 N^2 G^2) \sum_{t=0}^{k-1} \sum_{s=0}^{k-1} \frac{8(k - t - 1)(t + 1)(s + 1) + 5}{k}.
\]
\[
\left( \frac{8(k - s - 1)(s + 1)}{k} + 5 \right) \frac{p^k}{(t + 1)(s + 1)} \frac{(t - 1)}{t + 1}^2 \\
\left( \frac{3c^2 N^4 G^2}{(1 - p)} \right) ^k \sum_{t=0}^{k-1} \left( \frac{8(k - t - 1)(t + 1)}{k} + 5 \right) \frac{p^k}{(t + 1)}^2 \\
\leq 50^2 c^2 N^4 G^2 \frac{k^2}{N^4 k}.
\]

The last inequality applies Lemma 4. Thus, the bound in (13). The proof of Theorem 2 is complete.

**B. Proof of Theorem 3**

The proof parallels that of Theorem 5 (a) in [6]. We outline it and refer to ([6], Lemma 2, Lemma 3, Theorem 5 (a), and their proofs.) It is based on the evolution of the global averages \( \bar{x}_k \). We outline it and refer to ([6], Lemma 2, Lemma 3, and their proofs.)

\[
\begin{align*}
\hat{f}_{k-1} &:= \sum_{i=1}^{N} \left( f_i(y_i(k - 1)) \right) \\
+ \nabla f_i(y_i(k - 1))^\top (\bar{x}_k - y_i(k - 1)) \\
\hat{g}_{k-1} &:= \sum_{i=1}^{N} \nabla f_i(y_i(k - 1)) \\
L_{k-1} := & \frac{N}{\alpha_{k-1}} N k L k \\
\delta_{k-1} := & L \| \bar{x}_k(k - 1) \|^2.
\end{align*}
\]

Then, it is easy to show that \( \bar{x}_k \), \( \bar{y}_k \) evolve as:

\[
\bar{x}_k = \bar{y}_k - \frac{\hat{g}_{k-1}}{L_{k-1}} \\
\bar{y}_k = (1 + \beta_{k-1}) \bar{x}_k - \beta_{k-1} \bar{y}_k(k - 1),
\]

\( k = 1, 2, \ldots \), with \( \bar{x}(0) = \bar{y}(0) \). As shown in [6], \( \hat{f}_{k-1}, \hat{g}_{k-1} \) is a \( (L_{k-1}, \delta_{k-1}) \) inexact oracle, i.e., it holds that for all points \( x \in \mathbb{R}^d \):

\[
\begin{align*}
(\hat{f}_{k-1}, \hat{g}_{k-1}) &\leq f(x) - \bar{f}_{k-1} \\
+ \hat{g}_{k-1}^\top (x - \bar{y}(k - 1)) &\leq f(x) - \bar{f}_{k-1} + L_{k-1} \| x - \bar{y}(k - 1) \|^2 + \delta_{k-1}.
\end{align*}
\]

From (32), \( \hat{f}_{k-1}, \hat{g}_{k-1} \) and \( \delta_{k-1} \) are functions (solely) of \( y(k - 1) \). Inequalities (35) hold for any random realization of \( y(k - 1) \) and any \( x \in \mathbb{R}^d \). We apply now Lemma 2 in [6], with \( \delta_{k-1} \) as in (32). Get:

\[
(k + 1)^2 \left( f(\bar{x}_k(k)) - f^* \right) + \frac{2Nk}{c} \| \bar{x}_k(k) - x^* \|^2
\leq \left( k^2 - 1 \right) \left( f(\bar{x}_k(k)) - f^* \right) + \frac{2Nk}{c} \| \bar{x}_k(k) - x^* \|^2
+ (k + 1)^2 L \| \bar{x}_k(k - 1) \|^2,
\]

where \( \bar{y}_k(k - 1) = \bar{y}_k(k) - \theta \bar{y}_k(k) / \theta \). Dividing (36) by \( k \) and unwinding the resulting inequality, get:

\[
\frac{1}{N} f(\bar{x}_k(k)) - f^* \leq \frac{2}{k c} \| \bar{y}(0) - x^* \|^2
+ \frac{L}{N k} \sum_{i=1}^{k} \frac{(t + 1)^2}{t} \| \bar{y}(t - 1) \|^2.
\]

Next, using Assumption 5, obtain, \( \forall i \):

\[
\frac{1}{N} \left( f(x_i(k)) - f^* \right) \leq \frac{1}{N} \left( f(\bar{x}_k(k)) - f^* \right) + \frac{G}{\sqrt{N}} \| \bar{x}_k(k) \|.
\]

The proof is completed after combining (37) and (38), taking expectation, and using in Theorem 2 the bounds \( E \left[ \| \bar{x}_k(k) \| \right] \leq E \left[ \| \bar{z}(k) \| \right] \) and \( E \left[ \| \bar{y}(k) \| \right] \leq E \left[ \| \bar{z}(k) \| \right] \).

**V. ALGORITHM mD–NC**

We present mD–NC. Subsection V-A defines additional random matrices needed for representation of mD–NC and presents mD–NC. Subsection V-B states our result on its convergence rate.

**A. Model and algorithm**

We consider a sequence of i.i.d. random matrices that obey Assumptions 1 and 2. We index these matrices with two-indices since D–NC operates in two time scales—an inner loop, indexed by \( s \) with \( \tau_k \) iterations, and an outer loop indexed by \( k \), where:

\[
\tau_k = \left[ 3 \log k + \log N \right] / \log 2.
\]

For static networks, the term \( \log N \) can be dropped. At each inner iteration, nodes utilize one communication round—each node broadcasts a \( 2d \times 1 \) vector to all its neighbors. We denote by \( W(k, s) \) the random weight matrix that corresponds to the communication round at the \( s \)-th inner iteration and \( k \)-th outer iteration.

The matrices \( W(k, s) \) are ordered lexicographically as \( W(k = 1, s = 1), W(k = 1, s = 2), \ldots, W(k = 1, s = \tau_1), \ldots, W(k = 2, s = 1), \ldots \). This sequence obeys Assumptions 1 and 2.

It will be useful to define the products of the weight matrices \( W(k, s) \) over each outer iteration \( k \):

\[
W(k) := \prod_{s=1}^{k-1} W(k, \tau(k - s)).
\]

Clearly, \( \{W(k)\}_k^\infty \) is a sequence of independent (but not identically distributed) matrices. We also define \( W(k) := W(k) - J \), and, for \( t = 0, 1, \ldots, k - 1 \):

\[
\bar{W}(k, t) := W(k)W(k-1) \cdots W(t+1).
\]

The Lemma below, proved in Appendix IX-A, follows from Assumptions 1 and 2, the independence of the matrices \( W(k) \), the value of \( \tau_k \) in (39), and Lemma 1.
Lemma 8 Let Assumptions 1 and 2 hold. Then, for all $k = 1, 2, \ldots$, for all $s, t \in \{0, 1, \ldots, k - 1\}$:
\[
E \left[ \left\| \Psi(k, s) - \Psi(k, t) \right\| \right] \leq \frac{1}{k^3} (k-1)^3 \cdots (t+1)^3.
\]
\[
E \left[ \left\| \Psi(k, t) - \Psi(k, t) \right\| \right] \leq \frac{1}{k^3} (k-1)^3 \cdots (t+1)^3.
\]
\[
E \left[ \left\| \Psi(k, t) - \Psi(k, t) \right\| \right] \leq \frac{1}{k^3} (k-1)^3 \cdots (t+1)^3.
\]

The mD–NC algorithm. mD–NC, in Algorithm 1, uses constant step-size $\alpha \leq 1/(2L)$. Each node $i$ maintains over (outer iterations) $k$ the solution estimate $x_i(k)$ and an auxiliary variable $y_i(k)$. Recall $\Pi$ in Lemma 1. Step 3 has $\tau_k$ communication rounds at outer iteration $k$.

Algorithm 1 mD–NC

1. Initialization: Node $i$ sets $x_i(0) = y_i(0) \in \mathbb{R}^d$; and $k = 1$.
2. Node $i$ calculates $x^{(k)}(k) = y_i(k) - \alpha \nabla f_i(y_i(k-1))$.
3. (Consensus) Nodes run average consensus on $\chi_i(s, k)$, initialized by $\chi_i(s = 0, k) = \left( x^{(k)}(k) \right)^{\top}$:
\[
\chi_i(s, k) = \sum_{j \in \Omega_i(k)} W_{ij}(s, k) \chi_j(s = 1, k), \quad s = 1, 2, \ldots, \tau_k
\]
with $\tau_k$ in (39), and set $x_i(k) := [\chi_i(s = \tau_k, k)]_{1:d}$ and $x^{(k)}(k-1) := [\chi_i(s = \tau_k, k)]_{d+1:2d}$. (Here $[a]_{1:m}$ is a selection of $i$-th, $i + 1$-th, $\ldots$, $i$-th entries of vector $a$.)
4. Node $i$ calculates $y_i(k) := (1 - \beta_k) x_i(k) + \beta_k x_i(k-1)$.
5. Set $k \rightarrow k + 1$ and go to step 2.

Nodes know $L$, $\tau$, and $N$. Section VII relaxes this.

mD–NC in vector form. Let the matrices $W(k)$ in (40). Use the compact notation in mD–NG for $x(k)$, $y(k)$, and $F : \mathbb{R}^{Nd} \rightarrow \mathbb{R}^N$. Then for $k = 1, 2, \ldots$
\[
x(k) = (W(k) \otimes I) \left[ y(k) - 1 \right] - \alpha \nabla F(y(k-1))
\]
\[
y(k) = (1 + \beta_{k-1}) x(k) - \beta_{k-1} (W(k) \otimes I) x(k-1),
\]
with $x(0) = y(0) \in \mathbb{R}^{Nd}$. Note the formal similarity with mD–NG in (10)–(11), except that $W(k)$ is replaced by $W(k)$, and the diminishing step-size $\alpha_k = c/(k+1)$ is replaced with the constant step-size $\alpha_k = \alpha$.

B. Convergence rate

Define, like for mD–NG, the disagreements $\tilde{x}_i(k), \tilde{y}_i(k), \tilde{x}(k)$, and $\tilde{y}(k)$, and $\tilde{z}(k) := \left( \tilde{y}(k)^{\top}, \tilde{x}(k)^{\top} \right)^{\top}$.

Theorem 9 Consider mD–NC in Algorithm 1 under Assumptions 1–5. Then, for all $k = 1, 2, \ldots$
\[
E \left[ \left\| \tilde{z}(k) \right\| \right] \leq \frac{50 \alpha N^{1/2} G}{k^2}.
\]
\[
E \left[ \left\| \tilde{z}(k) \right\| \right] \leq \frac{50 \alpha N^{1/2} G}{k^2}.
\]
\[
E \left[ \left\| \tilde{z}(k) \right\| \right] \leq \frac{50 \alpha N^{1/2} G}{k^2}.
\]

Theorem 10 Consider mD–NC in Algorithm 1 under Assumptions 1–5. Let $\left\| x(0) - x^* \right\| \leq R$, $R \geq 0$. Then, after $K$ communication rounds (after $k$ outer iterations)
\[
K = \sum_{t=1}^{k} \tau_t \leq \frac{3}{\log \bar{r}} \left[ (k+1) \log (N(k+1)) \right],
\]
i.e., $K \sim O(k \log k)$, we have, at any node $i$, $k = 1, 2, \ldots$
\[
E \left[ f(x_i(k)) - f^* \right] \leq \frac{1}{2k} \left( \frac{2}{\alpha} R^2 + 11 \alpha^2 L G^2 + \alpha G^2 \right),
\]

Remark. Theorem 10 implies $\frac{E[f(x_i(k))) - f^*]}{N}$ with mD–NC converges at rate $O(1/K^2-\xi)$ in the number of communications $K$.

VI. PROOFS OF THEOREMS 9 AND 10

We now prove the convergence rate results for mD–NC. Subsection VI-A proves Theorem 9 and Subsection VI-B proves Theorem 10.

A. Proof of Theorem 9

For simplicity, we prove for $d = 1$, but the proof extends to generic $d > 1$. Similarly to Theorem 2, we proceed in three steps. In Step 1, we derive the dynamics for the disagreement $\tilde{z}(k) = \left( \tilde{y}(k)^{\top}, \tilde{x}(k)^{\top} \right)^{\top}$. In Step 2, we unwind the disagreement equation and express $\tilde{z}(k)$ in terms of the $\tilde{y}(k), t$’s in (41) and $B(k, t)$ in (18). Step 3 finalizes the proof using bounds previously established on the norms of $\tilde{y}(k, t)$ and $B(k, t)$.

Step 1. Disagreement dynamics. We write the dynamic equation for $\tilde{z}(k)$. Recall $B(k, t)$ in (17). Multiplying (46)–(47) from the left by $(I - J)$, and using $(I - J)W(k) = \tilde{W}(k)(I - J)$, obtain for $k = 1, 2, \ldots$
\[
\tilde{z}(k) = \left( B(k) \otimes \tilde{W}(k) \right) \left( \tilde{z}(k-1) + u'(k-1) \right),
\]
and $\tilde{z}(0) = 0$, where
\[
u'(k-1) = - \left[ \begin{array}{c} \alpha_{k-1} \nabla F(y(k-1)) \\ 0 \end{array} \right].
\]

Step 2: Unwinding the recursion (50). Recall $B(k, t)$ in (18). Unwinding (50) and using $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$, obtain for $k = 1, 2, \ldots$
\[
\tilde{z}(k-1) = \sum_{t=0}^{k-1} (B(k, k-t-2)B(t+1) \otimes \tilde{W}(k, t)) u'(t),
\]
The quantities $u'(t)$ and $\tilde{W}(k, t)$ in (52) are random, while the $B(k, k-t-2)$’s are deterministic.

Step 3: Finalizing the proof. Consider $u'(t)$ in (51). By Assumption 5, $\left\| \nabla F(y(t)) \right\| \leq \sqrt{NG}$. From this,
obtain $||u'(t)|| \leq \alpha \sqrt{N} G$, for any random realization of $u'(t)$. Using this, Lemma 7, the sub-multiplicative and sub-additive properties of norms, and $||B(t+1)|| \leq 3$, for all $t$, get from (52):

$$||\tilde{z}(k)|| \leq 3 \left(8\alpha \sqrt{N} G \right)^{k-1} \sum_{t=0}^{k-1} \left|\Psi(k, t)\right| \left(\frac{(k-t-1)(t+1)}{k}\right)$$

$$+ 3 \left(5\alpha \sqrt{N} G \right)^{k-1} \sum_{t=0}^{k-1} \left|\Psi(k, t)\right|.$$  

Taking expectation, using $\frac{(k-t-1)(t+1)}{k} \leq t + 1$ and Lemma 8, (48) follows from

$$\mathbb{E}[||\tilde{z}(k)||] \leq 3 \left(8\alpha \sqrt{N} G \right)^{k-1} \sum_{t=0}^{k-1} \left|\Psi(k, t)\right|$$. 

We outline the proof since similar to Theorem 8 in [6] (version v2). Consider the global averages $\tau(k)$ and $\bar{\pi}(k)$ as in mD–NG. Then, $\tau(k)$ and $\bar{\pi}(k)$ follow (33)–(34), with $L_{k-1} := N/\alpha$ and $\hat{g}_{k-1}$ as in (32). Inequalities (35) hold with $L_{k-1} := N/\alpha$ and $\hat{f}_{k-1}$ and $\hat{g}_{k-1}$ as in (32). Applying Lemma 2 in [6] gives:

$$\frac{1}{N} (f(\tau(k)) - f^*) \leq \frac{2}{\alpha k^2} ||\tau(0) - x^*||^2 + \frac{L}{N k^2} \sum_{t=1}^{k} \mathbb{E}[||\bar{y}(t-1)||^2(t+1)^2].$$

(Compare the last equation with (37)). The remainder of the proof proceeds analogously to that of Theorem 3.

**B. Proof outline of Theorem 10**

We outline the proof since similar to Theorem 8 in [6] (version v2). Consider the global averages $\tau(k)$ and $\bar{\pi}(k)$ as in mD–NG. Then, $\tau(k)$ and $\bar{\pi}(k)$ follow (33)–(34), with $L_{k-1} := N/\alpha$ and $\hat{g}_{k-1}$ as in (32). Inequalities (35) hold with $L_{k-1} := N/\alpha$ and $\hat{f}_{k-1}$ and $\hat{g}_{k-1}$ as in (32). Applying Lemma 2 in [6] gives:

$$\frac{1}{N} (f(\tau(k)) - f^*) \leq \frac{2}{\alpha k^2} ||\tau(0) - x^*||^2 + \frac{L}{N k^2} \sum_{t=1}^{k} \mathbb{E}[||\bar{y}(t-1)||^2(t+1)^2].$$

(Compare the last equation with (37)). The remainder of the proof proceeds analogously to that of Theorem 3.

**VII. DISCUSSION AND EXTENSIONS**

We discuss extensions and corollaries: 1) relax the prior knowledge on $L, \tau, \pi$, and $N$ for both mD–NG and mD–NC; 2) establish rates in the convergence in probability of mD–NG and mD–NC; 3) show almost sure convergence with mD–NC; and 4) establish a convergence rate in the second moment with both methods.

**Relaxing knowledge of $L, \tau, \pi$, and $N$.** mD–NG requires only knowledge of $L$ to set the step-size $c/L$. We demonstrate that the rate $O(\log k/k)$ (with a deteriorated constant) still holds if nodes use arbitrary $c > 0$. Initialize all nodes to $x_0(1) = y_i(0) = 0$, suppose that $c > 1/(2L)$, and let $k' = 2cL$. Applying Lemma 2 in [6], as in the proof of Theorem 5 (b) in [6], for all $k > k'$, surely:

$$(8(s + 1) + 5) \left(\frac{1}{k^3} \cdot (t+1)^3\right) \left(\frac{1}{k^3} \cdot (s+1)^3\right) \leq 50^{2} \alpha^2 NG^2. $$

Thus, the bound in (49) and Theorem 9 is proved.
resulting inequality, and applying Theorem 2, obtain desired $O(\log k/k)$ rate.

mD–NC uses the constant step-size $\alpha = 1/(2L)$ and $\tau_k$ in (39). To avoid the use of $L, \overline{\tau}$, and $N$, we set in mD–NC: 1) a diminishing step-size $\alpha_k = 1/k^p$, $p \in (0, 1]$; and 2) $\tau_k = k$ (as suggested in [24]). We show the adapted mD–NC achieves rate $O(1/k^{2-p})$. Let $k'' = (2L)^{1/p}$. Then, by Lemma 2 in [6], $\forall k \geq k''$, surely:

$$\frac{(k+1)^2 - 1}{(k+1)^2} \leq (f(\pi(k)) - f^*),$$

$$\leq (k')^{2-p} (f(\pi(k'-1)) - f^*)$$

$$+ 2N(2|\pi(k'-1)|^2 + 2|x|^2) + \sum_{i=1}^{k}(1+1)^2 \leq 2L\|y(t-1)\|^2.$$  \hspace{1cm} (55)

Further, (54) holds here as well (surely.) Modify the argument on the sum in (55). By Lemma 8 and $\tau_k = k$, we have: $E[\|\tilde{\psi}(k)\|^2] \leq N^2\overline{\tau}^2 k$. From this, $\forall k \geq k'' := (6(\log N+1)/\log \overline{\tau})^2$. Next, consider

$$\tilde{\psi}(k, s) = \tilde{\psi}(k, t) = \left(\tilde{\psi}(k) \cdots \tilde{\psi}(s+1)\right) \tilde{\psi}(k) \cdots \tilde{\psi}(t+1),$$

for arbitrary $k \geq k''$, and arbitrary $s, t \in \{0, 1, \ldots, k-1\}$. Clearly, $\|\tilde{\psi}(k, s)\| = \|\tilde{\psi}(k, t)\| \leq \|\tilde{\psi}(k)\|^2$, and hence:

$$E[\|\tilde{\psi}(k, s)\|] \leq \frac{1}{k}, \forall s, t \in \{0, 1, \ldots, k-1\}, k \geq k''.$$  \hspace{1cm} (56)

Now, from step 3 of the proof of Theorem 9, the above implies: $E[\|\tilde{y}(k)\|^2] \leq E[\|\tilde{\psi}(k)\|^2] \leq C/k^2$, for all $k \geq k''$, where $C > 0$ is independent of $k$. Hence, we obtain the desired bound on the sum:

$$\sum_{i=1}^{\infty} \frac{t^2}{t^p} - LE[\|\tilde{y}(t-1)\|^2] = O(1).$$

Using this, (54), multiplying (55) by $\frac{(k+1)^2}{(k+1)^2 - 1}$, and taking expectation in (55), obtains the rate $O(1/k^{2-p})$.

Convergence in probability and almost sure convergence. Through the Markov inequality, Theorems 3 and 10 imply, for any $\epsilon > 0$, $k \to \infty$, $\forall i$:

$$\text{mD – NG: } P\left(k^{1-\xi} \left(f(x_i(k)) - f^*\right) > \epsilon\right) \to 0$$

$$\text{mD – NC: } P\left(k^{2-\xi} \left(f(x_i(k)) - f^*\right) > \epsilon\right) \to 0,$$

where $\xi > 0$ is arbitrarily small. Furthermore, by the arguments in, e.g., ([35], Subsection IV–A), with mD–NC, we have that, $\forall i$, $f(x_i(k)) - f^* \to 0$, almost surely.

Convergence rates in the second moment. Consider a special case of the random network model $G(k)$ in Assumptions 1 and 2 that supports a random instantiation of $W(k)$: $G(k) = (N, E)$, with $E = \{i, j\}$, $W_{ij}(k) > 0$, $i < j$. We assume $G(k)$ is connected with positive probability. This holds with spatio-temporally independent link failures, but not with pairwise gossip, where one edge occurs at a time, hence all realizations of $G(k)$ are disconnected. We establish the bounds on the second moment of the optimality gaps:

$$\text{mD – NG: } E\left[f(x_i(k)) - f^*\right]^2 = \frac{O(\log k/k)}{k^2}, \forall i$$

$$\text{mD – NC: } E\left[f(x_i(k)) - f^*\right]^2 = \frac{1}{k^2}, \forall i, \hspace{1cm} (57)$$

where (57) holds for mD–NC with a modified value of $\tau_k$ (see Appendix IX–B.) We interpret (56), while (57) is similar. Result (56) shows that, not only the mean of the optimality gap decays as $O(\log k/k)$ (by Theorem 3), but also the standard deviation is $O(\log k/k)$.

VIII. SIMULATION EXAMPLE

We compare mD–NG and mD–NC, D–NG and D–NC in [6], and the methods in [9], [19]. We initialize all to $x_i(0) = y_i(0) = 0$, $\forall i$. We generate one sample path (simulation run), and estimate the average normalized optimality gap $\text{err}_\text{f} = 1 \sum_{i=1}^{N} f(x_i(k)) - f^* \text{ versus the total number } K' \text{ of scalar transmissions, across all nodes. We count both the successful and failed transmissions. All our plots are in } \log_{10} - \log_{10} \text{ scales.}$

Setup. Consider a connected geometric supergraph $\mathcal{G} = (N, E)$ generated by placing 10 nodes at random on a unit 2D square and connecting the nodes whose distance is less than a prescribed radius (26 links). We consider random and static networks. With the random graph, nodes fail with probability .9. For online links $\{i, j\} \in E$, the weights $W_{ij}(k) = W_{ji}(k) = 1/N = 1/10$ and $W_{ij}(k) = 1 - \sum_{j \in O(k) - \{i\}} W_{ij}(k), \forall i$. The static network has the same supergraph $\mathcal{G}$, and, $\forall \{i, j\} \in E$, we set $W_{ij} = W_{ji} = 1/N$. With D–NG and mD–NG, the step-size is $\alpha_k = 1/(k + 1)$, while with D–NC and mD–NC, $\alpha_k = 1/(\sqrt{k})$. With random networks, for both variants of D–NC, we set $\tau_k$ as in (39); with static networks, we use $\tau_k = \frac{1}{\log_2 k}$. (As indicated in Section V, the log $N$ term is not needed with static networks.)

We use Huber loss cost functions arising, e.g., in distributed robust estimation in sensor networks [3]; $f_i(x) = ||x - \theta_i|| - 1/2$, else, $\theta_i \in \mathbb{R}$. The $f_i$'s obey Assumptions 3 and 5. We set $\theta_i = \pm 4(1 + \nu_i)$ and $\nu_i$ is generated randomly from the uniform distribution on $[-0.1, 0.1]$. For $i = 1, 2, 3$, we use the + sign and for $i = 4, \ldots, 10$ the − sign.

Results: Link failures. Figure 1 (top) shows that the convergence rates (slopes) of mD–NG, mD–NC, and D–NC, are better than that of the method in [9]. All methods converge, even with severe link failures, while D–NG diverges, see Figure 1 (second from top plot).

Results: Static network. Figure 1 (second from bottom) shows mD–NG, mD–NC, D–NG, D–NC, and
the method in [19] on a static network. As expected with a static network, D-NG performs slightly better than mD-NG, and both converge faster than [19]. D-NC and mD-NC perform similarly on both static and random networks. The bottom plot in Figure 1 shows mD-NG and D-NG when mD-NG is run with Metropolis weights $W$, [36], while D-NG, because it requires positive definite weights, is run with positive definite $W' = \frac{1.01}{2} I + \frac{0.95}{2} W$. The D-NG performs only marginally better than the mD-NG, which has a larger (worse) $\bar{\pi}$.

better convergence constant (agreeing with Theorem 3), reducing the communication cost for the same accuracy.

IX. CONCLUSION

We considered distributed optimization over random networks where $N$ nodes minimize the sum $\sum_{i=1}^{N} f_i(x)$ of their individual convex costs. We model the random network by a sequence $\{W'(k)\}$ of independent, identically distributed random matrices that take values in the set of symmetric, stochastic matrices with positive diagonals. The $f_i$'s are convex and have Lipschitz continuous and bounded gradients. We present mD-NG and mD-NC that are resilient to link failures. We establish their convergence in terms of the expected optimality gap of the cost function at arbitrary node $i$: mD–NG achieves rates $O(\log k/k)$ and $O(\log K/K)$, where $k$ is the number of node communications and $K$ is the number of per-node communications; and mD–NC has rates $O(1/k^2)$ and $O(1/K^2)$, where $\xi > 0$ arbitrarily small. Simulation examples with link failures and Huber loss functions illustrate our findings.

APPENDIX

A. Proofs of Lemmas 1 and 8

Proof of Lemma 1: We prove (3). For $t = k - 1$, $\bar{\Phi}(k, t) = I$ and (3) holds. Fix $t$, $0 \leq t \leq k - 2$. For $N \times N$ matrix $A$: $\|\bar{\Phi}(k, t)\|^2 \leq N \sum_{i=1}^{N} \left\| \bar{\Phi}(k, t) e_i \right\|^2$, $e_i$ is the $i$-th canonical vector. Using this, taking expectation:

$$
\mathbb{E} \left[ \left\| \bar{\Phi}(k, t)^T \bar{\Phi}(k, t) \right\|^2 \right] = \mathbb{E} \left[ \left\| \bar{\Phi}(k, t) \right\|^2 \right] \leq N \sum_{i=1}^{N} \mathbb{E} \left[ \left\| \bar{\Phi}(k, t) e_i \right\|^2 \right].
$$

Let $\chi_i(s + 1) := \bar{\Phi}(s, t) e_i = \bar{W}(s + t + 2) \cdots \bar{W}(t + 2) e_i$, $s = 0, \cdots, k - t - 2$, $\chi_i(0) = e_i$. Get the recursion:

$$
\chi_i(s + 1) = \bar{W}(s + t + 2) \chi_i(s), s = 0, \cdots, k - t - 2.
$$
Independence of the $\tilde{W}(k)$’s and nesting expectations:
\[
\mathbb{E}[|x_i(k-t-1)|^2] = \mathbb{E}[\mathbb{E}[x_i(k-t-2)\tilde{W}(k)^2x_i(k-t-2)]]
\]
\[
= \mathbb{E}[|x_i(k-t-2)|^2] \mathbb{E}[|x_i(k-t-2)|^2]
\]
\[
\leq \mathbb{E}\left[\mathbb{E}\left[|\tilde{W}(k)|^2\right]\right] \leq \tau \mathbb{E}[|\tilde{W}(k)|^2].
\]

Repeating for $\mathbb{E}[|x_i(k-t-2)|^2]$, obtain for all $i$:
\[
\mathbb{E}\left[\left\langle \tilde{\Phi}(k,t) e_i, e_i \right\rangle^2 \right] = \mathbb{E}\left[\left\langle x_i(k-t-1) \right\rangle^2\right]
\]
\[
\leq (\tau^2)^k t^{-1} \mathbb{E}[|\tilde{W}(k)|^2] = (\tau^2)^k t^{-1}.
\]

Plugging this in (58), (3) follows. Next, (2) follows from (3) and Jensen’s inequality. To prove (4), consider $0 \leq s < t \leq k-2$ ($t < s$ by symmetry). By the independence of the $\tilde{W}(k)$’s, the sub-multiplicative property of norms, and taking expectation, obtain:
\[
E\left[\left\langle \tilde{\Phi}(k,s)^T \tilde{\Phi}(k,t) \right\rangle^2 \right] \leq E\left[\left\langle \tilde{W}(t+1) \cdots \tilde{W}(s+2) + 1 \right\rangle^2 \right] \\
\leq (N\tau^2)(N^2\tau^2)^{k-t-2} = N^{k-1}(k-s-1). (59)
\]

We applied (2) and (3) to get (59); thus, (4) for $s, t \in \{0, \ldots, k-2\}$. If $s = k-1, t < k-1, \tilde{\Phi}(k,s)^T \tilde{\Phi}(k,t) = \tilde{\Phi}(k,t)$ and the result reduces to (3). The case $s < k-1, t = k-1$ is symmetric. Finally, if $s = k-1, t = k-1$, the result is trivial. The proof is complete. ■

Proof of Lemma 8: We prove (42). By (41), $W(k)$ is the product of $\tau_k$ i.i.d. matrices $W(t)$ that obey Assumptions 1 and 2. Hence, by (3), obtain (42):
\[
E\left[|W(k)|^2\right] \leq (\tau^2)^k N^2 e^{2\tau k log(\tau)} = N^2 e^{-2(3\log k + \log N)} = \frac{1}{k^6}.
\]

We prove (44). Let $\tilde{W}(k,t) = \tilde{W}(k,t) \cdots, \tilde{W}(t+1,k) \geq t+1$.

For square matrices $A, B$: $|B|A A B |B| \leq A A |B| A B |B| A$.

Applying it $k$ times, obtain:
\[
\left\langle \tilde{\Phi}(k,t)^T \tilde{\Phi}(k,t) \right\rangle \leq \left\langle \tilde{W}(k)^2 \cdots \tilde{W}(t+1)\right\rangle^2.
\]

Using independence, taking expectation, and applying (42), obtain (44). By Jensen’s inequality, (43) follows from (44); relation (45) is proved similarly. ■

B. Proof of (56)–(57)

Recall the random graph $G(k)$. For a certain connected graph $G_0, G(k) = G_0$ with probability $p_G > 0$.

This, together with $w > 0$ by Assumption 1, implies that there exists $\mu_4 \in [0, 1)$, such that $E\left[|W(k)|^4\right] \leq (\mu_4)^4$. In particular, $\mu_4$ can be taken as:
\[
\mu_4 = (1 - p_G) + p_G \left(1 - w^2 \lambda_F(G_0)^2\right)^2 < 1,
\]
where $\lambda_F(G_0)$ is the second largest eigenvalue of the unweighted Laplacian of $G_0$. Now, consider (37). Let $f_k := \frac{1}{k}(f(x) - f^*)$. Squaring (37), taking expectation, and by the Cauchy-Schwarz inequality, get:
\[
\mathbb{E}\left[\tilde{f}_k^2\right] \leq \frac{4R^2}{c^2 k^2} + \frac{4RL}{c^2 k^2} \sum_{t=1}^k \frac{t(1+t)}{t} \mathbb{E}[|\tilde{g}(t-1)|^2] \\
+ \frac{L^2}{c^2 N^2 k^2} \sum_{t=1}^k \sum_{s=1}^k \frac{(t+s)^2}{t} \mathbb{E}[|\tilde{g}(t-1)|^2] \\
\times \sqrt{\mathbb{E}[|\tilde{g}(s-1)|^2]}, (60)
\]

where recall $R := \|\mathbb{E}[x]\|$ and the first term in (60) is $O(1/k^2)$; by Theorem 2, the second is $O(\log k/k^2)$. We upper bound the third term. Recall (31), let by $U_k := \|W(k)\|$. Fix $s < t, t \in \{0, 1, \ldots, k-1\}$. By the sub-multiplicative property of norms:
\[
\|\tilde{\Phi}(k,t)^T \tilde{\Phi}(k,s)\| \leq \|U_k^2 U_{k-1} \cdots U_{t+2}(U_{t+1} \cdots U_{s+2})
\]

For $t = s, \|\tilde{\Phi}(k,t)^T \tilde{\Phi}(k,t)\| \leq U_k^2 U_{k-1} \cdots U_{t+2}$. Let $\tilde{U}(t, s) := \tilde{U}(U_{k-1} \cdots U_{t+1}, t > s, and $\tilde{U}(t, t) = I$. Further, let $\tilde{b}(k, t) := \tilde{b}(k, t) \cdots (k-t-1)/k + 1$. Squaring (31):
\[
\|\tilde{U}(k, t)\|^4 \leq (9c^4 N^2 G^4)^{\sum_{t=1}^k \sum_{s=1}^k \tilde{b}(k,t)\tilde{U}(k,t)} \\
\times \tilde{b}(k,t)\tilde{U}(k,t) (61)
\]

Fix $t_i \in \{0, \ldots, k-1\}, i = 1, \ldots, 4, t_i \geq t_2 \geq t_3 \geq t_4$.

By independence of $U_k$’s, $\{\tilde{U}(U_k^2)^4 \leq \tilde{U}(U_k^2)^4, j = 1, 2, 3,$

\[
\mathbb{E}\left[\tilde{U}(k, t_1 + 1)\right]^4 (\tilde{U}(k, t_1 + 1) t_2 + 1)^3 \\
(\tilde{U}(k, t_2 + 1) t_3 + 1)^2 (\tilde{U}(k, t_3 + 1) t_4 + 1)^2 \\
(\tilde{U}(k, t_4 + 1) t_4 + 1)^2 (\tilde{U}(k, t_4 + 1) t_4 + 1)^2
\]

Taking expectation in (61), and applying (62), we get:
\[
\mathbb{E}[|\tilde{z}(k)|^4] \leq (9c^4 N^2 G^4)^{\left(\sum_{t=0}^{k-1} \tilde{b}(k,t)\mu_4^{k-t-1}(t+1)^{-1}\right)^4} \\
= O(1/k^4), (63)
\]

where the last equality uses Lemma 4. Applying (63) to (60), the third term in (60) is $O(\log^2 k/k^2)$. Thus,
\[ \mathbb{E} \left[ \left( f_k \right)^2 \right] = O(\log^2 k/k^2). \]

Express \( f(x_i(k)) - f^* = N f_k + (f(x_i(k)) - f(\tilde{x}(k))) \), and use \( (f(x_i(k)) - f^*)^2 \leq 2(N f_k)^2 + 2(f(x_i(k)) - f^*)^2 \leq 2(N f_k)^2 + 2GN \| \tilde{x}(k) \|^2 \), where the last inequality follows by Assumption 5. Taking expectation, applying Theorem 2, and using (63), the result (56) follows. For mD-NC, prove (57) like (56) by letting \( \tau_k = \frac{3 \log k}{\log \mu_k^2} \).

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