The Large $N$ ’t Hooft Limit of Kazama-Suzuki Model

Changhyun Ahn

Department of Physics, Kyungpook National University, Taegu 702-701, Korea

ahn@knu.ac.kr

Abstract

We consider $\mathcal{N} = 2$ Kazama-Suzuki model on $\mathbb{CP}^N = \frac{SU(N+1)}{SU(N) \times U(1)}$. It is known that the $\mathcal{N} = 2$ current algebra for the supersymmetric WZW model, at level $k$, is a nonlinear algebra. The $\mathcal{N} = 2$ $\mathcal{W}_3$ algebra corresponding to $N = 2$ was recovered from the generalized GKO coset construction previously. For $N = 4$, we construct one of the higher spin currents, in $\mathcal{N} = 2 \mathcal{W}_5$ algebra, with spins $(2, \frac{5}{2}, \frac{5}{2}, 3)$. The self-coupling constant in the operator product expansion of this current and itself depends on $N$ as well as $k$ explicitly. We also observe a new higher spin primary current of spins $(3, \frac{7}{2}, \frac{7}{2}, 4)$. From the behaviors of $N = 2, 4$ cases, we expect the operator product expansion of the lowest higher spin current and itself in $\mathcal{N} = 2 \mathcal{W}_{N+1}$ algebra. By taking the large $(N, k)$ limit on the various operator product expansions in components, we reproduce, at the linear order, the corresponding operator product expansions in $\mathcal{N} = 2$ classical $\mathcal{W}_\infty^c[\lambda]$ algebra which is the asymptotic symmetry of the higher spin AdS$_3$ supergravity found recently.
1 Introduction

The duality between the $W_N$ minimal model conformal field theories and the higher spin theory of Vasiliev on the $AdS_3$ has been proposed by Gaberdiel and Gopakumar in [1]. Very recently, in [2], this proposal has been clarified further and they claim that the $W_N$ minimal model conformal field theory is dual, in the 't Hooft $1/N$ expansion, to the higher spin theory coupled to one complex scalar. The duality can hold at finite $N$ because of the nontrivial truncation of the quantum algebra of the higher spin theory.

In [3], the $\mathcal{N} = 2$ supersymmetric extension of [1], the higher spin $AdS_3$ supergravity, has been studied where the dual conformal field theory is given by $\mathcal{N} = 2$ CP$^N$ Kazama-Suzuki(KS) model in two dimensions. The supergravity partition function is computed and agrees with the partition function from the superconformal field theory side. Moreover, this superconformal partition function in the KS model in the 't Hooft limit is described, in detail, in [4]. Recently, in [5], the asymptotic symmetry of the higher spin $AdS_3$ supergravity is obtained, by following the work of [6], and one of the nontrivial checks for the duality [3] is to identify the operator product expansions between the lower higher spin currents in the KS model, in the 't Hooft limit, with the corresponding algebra in the classical $\mathcal{N} = 2\ W^{\infty}_\lambda[\lambda]$ algebra, where $\lambda$ is a free parameter, in higher spin $AdS_3$ supergravity.

Some time ago, Kazama and Suzuki [7, 8] have found a new class of unitary $\mathcal{N} = 2$ superconformal field theories via coset space method. They classified the list of Hermitian symmetric spaces and the Virasoro central charges for the associated $\mathcal{N} = 2$ superconformal field theories. Moreover, Hull and Spence [9] studied the description of $\mathcal{N} = 2$ supersymmetric extension of the Kac-Moody algebra in $\mathcal{N} = 2$ superspace. It turns out that the operator product expansions between the $\mathcal{N} = 2$ currents are nonlinear and this fact produces exactly the same conditions in [7, 8]. Romans [10] has found the $\mathcal{N} = 2\ W_3$ algebra where the higher spin multiplet has spins $(2, \frac{5}{2}, \frac{5}{2}, 3)$ $^1$. One of the discrete series for the central charge matches with the central charge of KS model on CP$^2$ coset model. See also the work of [11]. By applying the $\mathcal{N} = 2$ current algebra in [9, 12] to the supersymmetric WZW conformal field theory, the explicit $\mathcal{N} = 2\ W_3$ current with spins $(2, \frac{5}{2}, \frac{5}{2}, 3)$ in the above CP$^2$ KS model has been found in [13]. The free field realization was discussed in [14]. Moreover, the $\mathcal{N} = 2\ W_4$ algebra was constructed in [15] by adding one more higher spin current with spins $(3, \frac{7}{2}, \frac{7}{2}, 4)$

$^1$We will use this notation for the spins of $\mathcal{N} = 2$ multiplet. The first and last one are bosonic currents while the middle ones are fermionic. The spin contents we are dealing with for the multiplet in this paper take the form $(s, s + \frac{1}{2}, s + \frac{1}{2}, s + 1)$ where the spin $s$ is an integer. That is $s = 1, 2, \cdots$. The lowest case $(1, \frac{1}{2}, \frac{1}{2}, 2)$ corresponds to the usual $\mathcal{N} = 2$ stress energy tensor. For $s \geq 2$, one has higher spin currents. For the $\mathcal{N} = 2\ W_{N+1}$ algebra, the highest spin is $s_{\text{max}} = N$ and there are $N$-multiplets whose first component spins are $s = 1, 2, \cdots, N$. 

1
and they predicted the self-coupling constant, for the lowest higher spin current above, which is valid for any $\mathcal{N} = 2 \mathcal{W}_{N+1}$ algebra.

In this paper, we would like to see the $AdS_3/CFT_2$ correspondence initiated by [3] more detail in the context of supersymmetric WZW model. Contrary to the purely bosonic case where the operator product expansion between the spin 3 and itself does not contain the spin 3 current in the right hand side, the $\mathcal{N} = 2$ supersymmetric model has the operator product expansion between the multiplet with spins $(2, \frac{5}{2}, \frac{5}{2}, 3)$ and itself where the right hand side contains this multiplet itself. For the bosonic case, it is obvious that the spin 3 current can occur in the $\frac{1}{(z-w)^3}$ term. However, due to the symmetry in the operator product expansion of this current and itself, one can also obtain the same operator product expansion by interchanging the arguments between $z$ and $w$ and it turns out in this case, there exists the same spin 3 current in the above singular term but minus sign. Then it automatically becomes zero. This implies that the nontrivial self-coupling in the bosonic case occurs for the next spin 4 where the right hand side has this spin 4 current in the $\frac{1}{(z-w)^4}$ singular term and we do not see any trivial condition like as above because of the even power of this singular term.

What happens if there are $\mathcal{N} = 2$ supersymmetries in two dimensions? One sees the presence of self-coupling even in the operator product expansion for the lowest higher spin current. One simple example is the operator product expansion for the spin 2 current and itself that is the first component of the above multiplet $(2, \frac{5}{2}, \frac{5}{2}, 3)$. One can analyze this situation as above. The spin 2 current occurs in the $\frac{1}{(z-w)^2}$ term in the right hand side. Due to the even power of this singular term, there is a nontrivial spin 2 current in this singular term. It is easy to see that there are no other self-coupling terms except this spin 2-spin 2 operator product expansion. Note that there exists usual spin 2 stress energy tensor and in $\mathcal{N} = 2$ KS $\mathbb{C}P^N$ model, the higher spin current contains other spin 2 current which is contained in the above multiplet. Of course, the spin 3-spin 3 operator product expansion can generate the self-coupling constant term but as we explained in the previous paragraph, this does not give us the self-interacting term. Fortunately, it is known, in [15], that the self-coupling constant for the spin 2 current in $\mathcal{N} = 2 \mathcal{W}_{N+1}$ algebra depends on the $N$ and $k$ explicitly, as in the bosonic case (the footnote 2).

As $N$ increases, one expects that there exist new primary fields in the right hand side of

---

2 Some time ago, the self-coupling constant for the spin 4 current was obtained in [16] for $W_N$ minimal model using the free field realization. It depends on the central charge $c$ and the $N$ explicitly. See also the recent paper by Gaberdiel and Gopakumar [2] where one can find other relevant papers. As far as I know, so far, there is no direct construction for this self-coupling constant from the operator product expansion between the $SU(N)$ Casimir spin 4 operator [17] and itself. It would be interesting to see this feature although there will be lots of work to be done for these computations.
operator product expansion. For example, the spin 3-spin 3 operator product expansion in $SU(N)$ Casimir algebra leads to other spin 4 current and its descendant in the right hand side [18, 19]. The relative coefficient functions appearing in the descendant fields for given primary field are also fixed by conformal invariance.

In section 2, we rewrite the $\mathcal{N} = 2$ current algebra in terms of the currents living in the subgroup $H$ and the currents living in the coset $\frac{G}{H}$ separately. The constraints for the currents are rewritten similarly. The Sugawara stress energy tensor is given in terms of the currents linearly or quadratically. For $\mathcal{N} = 2$, we describe the lowest higher spin current with explicit group index contraction and the corresponding self-coupling constant is given in terms of the central charge or the level. For $\mathcal{N} = 4$, the most of the material is new. We present also the lowest higher spin current in terms of composite Kac-Moody currents and explain the overall normalization constant which depends on either the level $k$ or the central charge. We also observe the presence of a new primary current with spins $(3, \frac{7}{2}, \frac{7}{2}, 4)$ whose structure can be described from the conformal invariance. For the general $\mathcal{N}$, we notice that the self-coupling constant for arbitrary $N$ was determined form unitarity arguments in [15].

In section 3, we take the large $(N, k)$ limit of the operator product expansion between the lowest higher spin current and itself in the context of $\mathcal{N} = 2 \mathcal{W}_{N+1}$ algebra.

In section 4, based on the section 3, we compare the result of section 3 with the classical $\mathcal{N} = 2 \mathcal{W}_{\infty}[\lambda]$ algebra developed in [5]. We will present the three bosonic operator product expansions. At linear order, one sees an agreement between the boundary and bulk theories.

In section 5, We summarize what we have found in this paper and make some comments on the future directions.

In the Appendix, we describe some details discussed in the sections 2, 3, 4.

There exist some related works in [20]-[27], along the line of [1].

## 2 The $\mathcal{N} = 2$ current algebra, Kazama-Suzuki coset model and $\mathcal{W}_{N+1}$ algebra

Let us consider the hermitian symmetric space where the complex structure is preserved $^3$

\[ \mathbb{CP}^N = \frac{SU(N+1)}{SU(N) \times U(1)}. \]  

(2.1)

Let $G = SU(N+1)$ be an even-dimensional Lie group with complex structure and let $H = SU(N) \times U(1)$ be an even-dimensional subgroup. This implies that the $N$ should be even. We

$^3$Following the procedure in [4], the $\mathcal{N} = 1$ supersymmetric coset can be written in terms of the bosonic coset [3, 4, 5] by introducing the $SO(2N)$ factor in the numerator related to the free fermions.
introduce a complex basis for the Lie algebra in which the complex structure is diagonal and let
us label the index of the group generators by \( A \) and \( \bar{A} \) where \( A = 1, 2, \cdots, \frac{1}{2} \text{dim} \, G = \frac{1}{2}[(N + 1)^2 - 1] \)(similarly \( \bar{A} = \bar{1}, \bar{2}, \cdots, \frac{1}{2} \text{dim} \, \bar{G} = \frac{1}{2}[(N + 1)^2 - 1] \). For the Hermitian generators, one has \( T_A = T_{\bar{A}}^d \) and the structure constants appear in the standard commutation relations
\[
[T_A, T_B] = f_{AB}^C T_C, \quad [T_A, T_{\bar{B}}] = f_{AB}^C T_{\bar{C}} + f_{\bar{A}B}^C T_C \quad \text{and} \quad [T_{\bar{A}}, T_{\bar{B}}] = f_{\bar{A}B}^C T_{\bar{C}}.
\]
In other words, the structure constants \( f_{AB}^C \) and \( f_{\bar{A}B}^C \) vanish. Furthermore, there are relations, \( \text{Tr}(T_A T_B) = 0 \), \( \text{Tr}(T_A T_{\bar{B}}) = \delta_{AB} \), and \( \text{Tr}(T_{\bar{A}} T_{\bar{B}}) = 0 \).

Then the \( \mathcal{N} = 2 \) current algebra can be described by the \( \mathcal{N} = 2 \) currents \( Q^A(Z) \) and \( Q^{\bar{A}}(Z) \) with nonlinear constraints where \( Z \) stands for \( \mathcal{N} = 2 \) superspace coordinates, one real bosonic coordinate \( z \), and pair of two conjugate Grassman coordinates \( \theta, \bar{\theta} : Z = (z, \theta, \bar{\theta}) \). We consider the chiral currents where they are annihilated by \( D_- \) and \( \bar{D}_- \) and for simplicity we use \( D \) for \( D_+ \) and \( \bar{D} \) for \( \bar{D}_+ \). We present the \( \mathcal{N} = 2 \) current algebra in the Appendix A.

In order to obtain the generalization of Sugawara construction, it is convenient to decompose the group \( G \) indices into the subgroup \( H \) indices and the coset \( \frac{G}{H} \) indices explicitly. Let lower case middle roman indices \( m, n, p, \cdots \), running from 1 to \( \frac{N^2}{2} \), refer to the Lie algebra of \( H \), lower case top roman indices \( a, b, c, \cdots \), running from \( \frac{N^2}{2} + 1 \) to \( \frac{1}{2}[(N + 1)^2 - 1] \), refer to the remaining Lie algebra generators corresponding to the coset \( \frac{G}{H} \). The complex conjugated indices \( \bar{m}, \bar{n}, \bar{p}, \cdots \) and \( \bar{a}, \bar{b}, \bar{c}, \cdots \) hold similarly. That is,
\[
m, n, p, \cdots = 1, 2, 3, \cdots, \frac{N^2}{2}, \quad a, b, c, \cdots = \frac{N^2}{2} + 1, \cdots, \frac{1}{2}[(N + 1)^2 - 1],
\]
(2.2)
The indices \( A, B, C, \cdots \) corresponding to the group \( G \) are grouped into \( m, n, p, \cdots \) of the subgroup \( H \) and \( a, b, c, \cdots \) corresponding to the coset \( \frac{G}{H} \). For the currents \( Q^A(Z) \) and \( Q^{\bar{A}}(Z) \), one uses \( J^a(Z), J^{\bar{a}}(Z) \) that live in the coset \( \frac{G}{H} \), and \( K^m(Z), K^{\bar{m}}(Z) \) that live in the subgroup \( H \):
\[
Q^A(Z), Q^{\bar{A}}(Z) \rightarrow K^m(Z), \quad K^{\bar{m}}(Z), \quad J^a(Z), \quad J^{\bar{a}}(Z).
\]
(2.3)

Then the original operator product expansions (A.1) can be reexpressed in terms of the currents (2.3) where the subgroup index structure and remaining index structure are manifest. The ten(the all possibility among four currents) operator product expansions between these currents are
\[
K^m(Z_1)K^n(Z_2) = -\frac{\bar{\theta}_{12}}{z_{12}} f^\rho_{\bar{m} \bar{n}} K^\rho(Z_2) - \frac{\theta_{12} \bar{\theta}_{12}}{z_{12}} \frac{1}{(k + N + 1)} f^\rho_{\bar{m} \bar{n}} f_{\bar{m} \bar{n}} f^{\bar{q} \bar{r}} K^\rho K^\bar{q} K^\bar{r}(Z_2) + \cdots,
\]
\[
J^a(Z_1)J^b(Z_2) = -\frac{\theta_{12} \bar{\theta}_{12}}{z_{12}} \frac{1}{(k + N + 1)} f^c_{\bar{m} \bar{n}} f^d_{\bar{m} \bar{n}} f^{\bar{c} \bar{d}} J^c(Z_2) + \cdots,
\]
\[ K^m(Z_1)J^a(Z_2) = -\frac{\theta_{12}}{z_{12}} f_{\dot{a}m} \dot{b} J^b(Z_2) - \frac{\theta_{12}\theta_{12}}{z_{12}} \frac{1}{(k + N + 1)} f_{mn} \dot{p} f_{\bar{a}n} \dot{b} K^p J^b(Z_2) + \cdots, \]
\[ K^m(Z_1)K^n(Z_2) = -\frac{\theta_{12}}{z_{12}} f_{mn} \dot{p} K^p(Z_2) + \frac{\theta_{12}\theta_{12}}{z_{12}} \frac{1}{(k + N + 1)} f_{mp} \dot{q} f_{np} \dot{r} K^q K^r(Z_2) + \cdots, \]
\[ J^a(Z_1)J^\dot{b}(Z_2) = \frac{\theta_{12}\theta_{12}}{z_{12}} \frac{1}{(k + N + 1)} f_{an} \dot{c} f_{bn} \dot{d} J^c J^d(Z_2) + \cdots, \]
\[ K^m(Z_1)J^a(Z_2) = -\frac{\theta_{12}}{z_{12}} f_{ma} \dot{b} J^b(Z_2) + \frac{\theta_{12}\theta_{12}}{z_{12}} \frac{1}{(k + N + 1)} f_{mp} \dot{q} f_{np} \dot{r} K^q J^b(Z_2) + \cdots, \]
\[ K^m(Z_1)K^n(Z_2) = -\frac{\theta_{12}}{z_{12}} f_{mn} \dot{p} K^p(Z_2) - \frac{\theta_{12}\theta_{12}}{z_{12}} \frac{1}{(k + N + 1)} f_{mp} \dot{q} f_{np} \dot{r} K^q K^r(Z_2) + \cdots, \]
\[ \begin{align*}
J^a(Z_1)K^m(Z_2) & = -\frac{\theta_{12}}{z_{12}} f_{\dot{a}m} \dot{b} J^b(Z_2) \\
& - \frac{\theta_{12}\theta_{12}}{z_{12}} \frac{1}{(k + N + 1)} f_{\dot{a}m} \dot{p} f_{\bar{a}n} \dot{p} - \frac{\theta_{12}\theta_{12}}{z_{12}} \frac{1}{(k + N + 1)} f_{\dot{a}m} \dot{q} f_{\bar{a}n} \dot{r} K^q J^b(Z_2) + \cdots, \\
J^a(Z_1)J^\dot{b}(Z_2) & = \frac{\theta_{12}\theta_{12}}{z_{12}} \frac{1}{(k + N + 1)} f_{\dot{a}m} \dot{p} f_{\bar{a}n} \dot{p} - \frac{\theta_{12}\theta_{12}}{z_{12}} \frac{1}{(k + N + 1)} f_{\dot{a}m} \dot{q} f_{\bar{a}n} \dot{r} K^q J^b(Z_2) + \cdots, \\
\end{align*} \]

where the complex spinor covariant derivatives are given by
\[ D = \frac{\partial}{\partial \theta} - \frac{1}{2} \bar{\theta} \frac{\partial}{\partial z}, \quad \bar{D} = \frac{\partial}{\partial \bar{\theta}} - \frac{1}{2} \bar{\theta} \frac{\partial}{\partial \bar{z}}, \]

and they satisfy the algebra
\[ D\bar{D} + \bar{D}D \equiv \{D, \bar{D}\} = -\frac{\partial}{\partial z}. \]

\[ ^4 \text{There is a mathematica package[28] on } \mathcal{N} = 2 \text{ superspace but one cannot use this because in our case the right hand side of the operator product expansion (2.4) has nonlinear structure. This is the limitation of this package. We thank S. Krivonos for pointing out this. However, from time to time, we use this package in order to extract the component approach and are working on [29] mainly for } N = 4 \text{ case.} \]
We also use a simplified notation as
\[ \theta_{12} = \theta_1 - \theta_2, \quad \overline{\theta}_{12} = \overline{\theta}_1 - \overline{\theta}_2, \quad z_{12} = z_1 - z_2 + \frac{1}{2}(\theta_1 \overline{\theta}_2 + \overline{\theta}_1 \theta_2). \] (2.7)

In the first equation of (2.4), the property of \( f_{m \bar{n}}^{\bar{a}} = 0 = f_{ab}^m \) is used. In the second equation, one also uses \( f_{\bar{a} \bar{b}}^m = 0 = f_{\bar{a} \bar{b}}^\bar{c} = f_{m \bar{n}}^{\bar{a}} \). One obtains the third equation after one uses \( f_{m \bar{n}}^{\bar{a}} = 0 = f_{m \bar{n}}^a \). For the fourth-sixth equations, one also uses similar properties of structure constants \( f_{m \bar{n}}^{\bar{a}} = 0 = f_{m \bar{n}}^a = f_{a \bar{b}}^{\bar{c}} \) with above vanishing structure constants. Also the identity \( f_{a \bar{b}}^{\bar{c}} = 0 \) is used in the remaining equations. In the Appendix B, we present the component operator product expansions for (2.4).

One can rewrite the constraints (A.2), by expanding the \( G \)-indices into \( H \)-indices and \( \overline{G} \)-indices as above,

\[
\begin{align*}
DK^m(Z) &= -\frac{1}{2(k + N + 1)} f_{mn}^{\bar{p}} K^n K^p(Z), \\
DJ^a(Z) &= -\frac{1}{(k + N + 1)} f_{ab}^{\bar{m}} J^b K^m(Z), \\
\overline{D}K^\bar{m}(Z) &= -\frac{1}{2(k + N + 1)} f_{mn}^{\bar{p}} K^n K^p(Z), \\
\overline{D}J^\bar{a}(Z) &= -\frac{1}{(k + N + 1)} f_{ab}^{m} J^b K^m(Z),
\end{align*}
\] (2.8)

where one uses \( f_{a \bar{b}}^{\bar{c}} = 0 = f_{m \bar{n}}^{\bar{a}} = f_{a \bar{b}}^{m} = f_{\bar{a} \bar{b}}^{\bar{m}} \). For example, the \( \theta \) and \( \overline{\theta} \) independent terms in the left hand side can be obtained from the corresponding quantities in the right hand side of (2.8). Note that the unconstrained \( \mathcal{N} = 2 \) currents have too many components and we have to impose constraints in order to preserve the number of the independent \( \mathcal{N} = 1 \) currents [9]. As we will see the component currents explicitly, the unconstrained \( \mathcal{N} = 1 \) affine Kac-Moody currents(or its component currents) are relocated into the component currents in an extended \( \mathcal{N} = 2 \) superspace. One also obtains, from (2.6),

\[
\begin{align*}
\left[D, \overline{D}\right] K^m(Z) &= -\partial K^m(Z) + \frac{1}{(k + N + 1)} f_{mn}^{\bar{p}} \left( \overline{D}K^n K^p - K^n \overline{D}K^p \right)(Z), \\
\left[D, \overline{D}\right] K^\bar{m}(Z) &= \partial K^\bar{m}(Z) - \frac{1}{(k + N + 1)} f_{mn}^{\bar{p}} \left( D K^n K^p - K^n DK^p \right)(Z), \\
\left[D, \overline{D}\right] J^a(Z) &= -\partial J^a(Z) + \frac{2}{(k + N + 1)} f_{ab}^{\bar{m}} \left( \overline{D}J^b K^m - J^b \overline{D}K^m \right)(Z), \\
\left[D, \overline{D}\right] J^\bar{a}(Z) &= \partial J^\bar{a}(Z) - \frac{2}{(k + N + 1)} f_{ab}^{m} \left( D J^b K^m - J^b DK^m \right)(Z).
\end{align*}
\] (2.9)

Also in this case, the quantities in the left hand side are not independent because they can be obtained from the quantities in the right hand side(cubic or linear terms) where all the derivative terms can be written in terms of quadratic currents via (2.8).
The Sugawara stress energy tensor for the group $G = SU(N + 1)$ can be written in terms of
\[
T_G = -\frac{1}{(k + N + 1)} \left[ J^a J^a + K^m K^m - (f_{\bar{m} \bar{n}} + f_{\bar{m} \bar{n}}) DK^m - (f_{\bar{m} \bar{n}} + f_{\bar{m} \bar{n}}) D\bar{K}^m \right].
\] (2.10)

Note that there are linear terms in the currents as well as the quadratic terms. As before, by using the $H$-indices and $G_H$-indices in (A.3) explicitly, the vanishing of structure constants $f_{\bar{a} \bar{b} \bar{c}} = 0 = f_{\bar{a} \bar{m} \bar{n}}$ is used. For the metric, $\delta_{\bar{a} \bar{m}} = 0 = \delta_{\bar{m} \bar{a}}$. Similarly, the stress energy tensor for the subgroup $H = SU(N) \times U(1)$ can be written as
\[
T_H(Z) = -\frac{1}{(k + N + 1)} \left[ K^m K^m(Z) - f_{\bar{m} \bar{n}} DK^m(Z) - f_{\bar{m} \bar{n}} D\bar{K}^m(Z) \right].
\] (2.11)

Then the stress energy tensor $T(Z)$ for the supersymmetric coset model based on $N = 2 \mathbb{CP}^N$ model is obtained, by taking the difference between (2.10) and (2.11),
\[
T(Z) = T_G(Z) - T_H(Z) = -\frac{1}{(k + N + 1)} \left[ J^a J^a - f_{\bar{m} \bar{n}} DK^m - f_{\bar{m} \bar{n}} D\bar{K}^m \right](Z).
\] (2.12)

From the defining operator product expansions (2.4) between the currents, one obtains the standard operator product expansion of $N = 2$ superconformal algebra, together with (2.7),
\[
T(Z_1)T(Z_2) = \frac{1}{z_{12}^2 \theta_{12}} \left[ \frac{\theta_{12} \theta_{12}}{z_{12} - \theta_{12} D + D \bar{\theta}_{12}} \right] T(Z_2).
\] (2.13)

One can easily check that there is no singular term in the operator product expansions between the currents $K^m(Z_1), K^m(Z_1)$ and the stress tensor $T(Z_2)$. The corresponding central charge depends on $N$ and $k$ as follows \footnote{This can be described as $\frac{3k_G}{2(k_G + \tilde{h}_G)} \text{dim} \left( \frac{G}{H} \right)$ \cite{7} where $k_G = k, \tilde{h}_G = N + 1$ is the dual Coxeter number of group $G$ and $\text{dim} \left( \frac{G}{H} \right) = 2N$.}
\[
c(N, k) = c_G - c_H = \frac{3}{2} ((N + 1)^2 - 1) \left[ 1 - \frac{2(N + 1)}{3(k + N + 1)} \right] - \frac{3}{2} (N^2 - 1) \left[ 1 - \frac{2N}{3(k + 1 + N)} \right] - \frac{3}{2}
\] \[
= \frac{3Nk}{N + k + 1}.
\] (2.14)

Note that the coefficients of the stress energy tensors (2.10) and (2.11) are the same. This is different feature from the coset construction for the bosonic theory where the diagonal subalgebra exists and the coefficients of the various stress energy tensors are different. Also, the level of the group $SU(N + 1)$ and the level of $SU(N)$ are same as each other. In other
words, each \( \frac{1}{(z-w)^n} \) term of spin 1-spin 1 operator expansion has the same factor \((k+N+1)\). For the explicit form, see the Appendix B. This shift \((k+N+1)\) rather than \(k\) arises from the \(\mathcal{N} = 1\) supersymmetrization.

There are two requirements on the \(\mathcal{N} = 2\) current \(W(Z)\) of generators with spins \((2, \frac{5}{2}, \frac{5}{2}, 3)\).

1) The operator product expansions between the \(H\)-currents \(K^m(Z), K^m(Z)\) and the \(\bar{G}\)-current \(W(Z)\) should not contain any singular terms:

\[
K^m(Z_1)W(Z_2) = 0, \quad K^m(Z_1)W(Z_2) = 0. \tag{2.15}
\]

In practice, one uses the component approach and due to the constraints (2.8), only after some of the operator product expansions(among 16 operator product expansions for each case in (2.15)) are checked, the coefficient functions appearing in the unknown higher spin current \(W(Z)\) are determined completely except the overall constant.

2) The current \(W(Z)\) with vanishing \(U(1)\) charge is an \(\mathcal{N} = 2\) primary field under the stress energy tensor (2.12)

\[
T(Z_1)W(Z_2) = \left[ \frac{\theta_{12}\theta_{12}}{z_{12}^2} - \frac{\theta_{12}}{z_{12}} D + \frac{\theta_{12}}{z_{12}} D + \frac{\theta_{12}\theta_{12}}{z_{12}} \partial \right] W(Z_2). \tag{2.16}
\]

Here there is no term like \(\frac{1}{z_{12}}\) due to the zero \(U(1)\) charge. The coefficient 2 in \(\frac{\theta_{12}\theta_{12}}{z_{12}^2}\) implies the lowest spin of \(W(Z)\). We present the component results for (2.16) in the Appendix C. In general, there exist \(\frac{1}{z_{12}}, \frac{\theta_{12}\theta_{12}}{z_{12}}, \frac{\theta_{12}}{z_{12}}, \frac{\theta_{12}}{z_{12}}\)-terms with appropriate composite currents. The requirement 2) implies that these extra terms should vanish by choosing the correct coefficient functions.

For the \(\mathcal{N} = 2\) currents, the component currents are given by

\[
K^m(Z) = K^m(z) + \theta D K^m(z) + \theta \overline{D} K^m(z) + \theta \bar{\theta} (-1) \frac{1}{2} [D, \overline{D}] K^m(z),
\]

\[
K^m(Z) = K^m(z) + \theta D K^m(z) + \theta \overline{D} K^m(z) + \theta \bar{\theta} (-1) \frac{1}{2} [D, \overline{D}] K^m(z),
\]

\[
J^a(Z) = J^a(z) + \theta D J^a(z) + \theta \overline{D} J^a(z) + \theta \bar{\theta} (-1) \frac{1}{2} [D, \overline{D}] J^a(z),
\]

\[
J^\bar{a}(Z) = J^\bar{a}(z) + \theta D J^\bar{a}(z) + \theta \overline{D} J^\bar{a}(z) + \theta \bar{\theta} (-1) \frac{1}{2} [D, \overline{D}] J^\bar{a}(z). \tag{2.17}
\]

Due to the constraints (2.8) and (2.9), the \(\theta\)- and \(\theta \bar{\theta}\) components of \(K^m(Z)\) and \(J^a(Z)\) are not independent but they can be written in terms of other independent terms. Similarly, the \(\bar{\theta}\)- and \(\theta \bar{\theta}\) components of \(K^m(Z)\) and \(J^\bar{a}(Z)\) can be written in terms of other independent terms.

Let us emphasize how one applies the above two conditions 1) and 2). For the general \(\mathcal{N} = 2\) \(\mathcal{W}_{N+1}\) algebra, we use them in \(\mathcal{N} = 2\) superspace but for fixed \(N = 4\) case, we
use the package [29] where the component result is necessary to obtain the operator product expansions. Therefore, one should apply the two conditions in the component approach. The component result for (2.16) is summarized in the Appendix C. For the regularity condition, given in 1), among 16 operator product expansions for each case, only the half of them are independent from the arguments in (2.17). Once we have checked the regularity condition for the independent components, then the condition 1) satisfies automatically, by construction. We do not need to check the other remaining half of the equations.

For the stress energy tensor 6, one has

\[ T(Z) = T(z) + \theta DT(z) + \bar{\theta} D\bar{T}(z) + \theta\bar{\theta} (-1)^{\frac{1}{2}}[D, \bar{D}]T(z), \]

(2.18)

where the component fields can be obtained from (2.12) by using the covariant derivatives (2.5) with (2.6) and putting the \( \theta, \bar{\theta} \)'s to vanish. \( T(z) \) is a \( U(1) \) current of spin 1, \( DT(z) \) and \( D\bar{T}(z) \) are fermionic currents of spin \( \frac{3}{2} \) and \( -\frac{1}{2}[D, \bar{D}]T(z) \) is the stress energy tensor of spin 2.

In next subsections, we will construct the higher spin currents explicitly. Starting with \( N = 2 \) case, one considers the \( N = 4 \) case and wants to generalize for arbitrary \( N \) which corresponds to \( \mathcal{N} = 2 \mathcal{W}_{N+1} \) algebra.

\section{2.1 The \( N = 2 \) case: \( \mathbb{CP}^2(=SU(3)/SU(2)\times U(1)) \) coset model}

The \( \mathcal{N} = 2 \mathcal{W}_3 \) algebra has one additional extra higher spin current with spins \( (2, \frac{5}{2}, \frac{5}{2}, 3) \), as well as the \( \mathcal{N} = 2 \) superconformal algebra (2.13), where one can write down the following component currents explicitly 7

\[ W(Z) = W(z) + \theta DW(z) + \bar{\theta} D\bar{W}(z) + \theta\bar{\theta} (-1)^{\frac{1}{2}}[D, \bar{D}]W(z). \]

(2.19)

In this case, the number of independent WZW currents is 8 and it is not so complicated to write down all the possible terms for the current (2.19). However, the explicit form for this current in [13] is not convenient to generalize to the arbitrary \( \mathcal{N} = 2 \mathcal{W}_{N+1} \) algebra because there are no any contractions between the \( SU(3) \) group indices.

For given result for the expression of (2.19) in [13], one can think of the equivalent expression as follows 8:

\[ W(Z) = a_1 f_{pa}^b f_{pm}^n J^a J^b K^m K^{\bar{n}}(Z) + a_2 f_{pa}^b f_{pm}^n J^a J^b K^m K^{\bar{n}}(Z) + a_3 J^a J^b J^{\bar{a}} J^{\bar{b}}(Z) \]

6Our notation corresponds to the one in [10] as follows: \( T(z) \leftrightarrow J_{ro}(z), DT(z) \leftrightarrow G^+_{ro}, D\bar{T}(z) \leftrightarrow G^-_{ro}, \) and \( -\frac{1}{2}[D, \bar{D}]T(z) \leftrightarrow T_{ro}(z). \)

7Similarly, our fields correspond to the ones in [10] as follows: \( W(z) \leftrightarrow V_{ro}(z), DW(z) \leftrightarrow U^+_{ro}, D\bar{W}(z) \leftrightarrow U^-_{ro}, \) and \( -\frac{1}{2}[D, \bar{D}]W(z) \leftrightarrow W_{ro}(z). \)

8There are 48 nonzero structure constants.
\[ + a_4 f_{\bar{m}a} J^a \bar{J}^b DK^{\bar{m}}(Z) + a_5 f_{\bar{m}a} J^a \bar{J}^b DK^{\bar{m}}(Z) + a_6 f_{\bar{m}a} DK^{\bar{m}}.J^a(Z) \\
+ a_7 f_{\bar{m}a} \bar{J}^b DK^{\bar{m}}.J^a(\bar{Z}) + a_8 \bar{J}^a D.J^a(\bar{Z}) + a_9 \bar{D}DK^{\bar{m}}DK^{\bar{m}}(Z) \\
+ a_{10} J^a \bar{J}^a(\bar{Z}) + a_{11} J^a \bar{J}^a(\bar{Z}) + a_{12} K^m \partial K^m(\bar{Z}) + a_{13} \partial K^mK^m(\bar{Z}) \\
+ a_{14} J^a[J, \bar{D}].J^a(\bar{Z}) + a_{15} D.[J, \bar{D}].J^a(\bar{Z}) + a_{16} K^m[D, \bar{D}].K^m(\bar{Z}) \\
+ a_{17} D.[J, \bar{D}]K^mK^m(\bar{Z}) + a_{18} D.J^a[D, \bar{D}].J^a(\bar{Z}) + a_{19} DK^mDK^m(\bar{Z}) \\
+ a_{20} f_{\bar{m}a} \bar{\partial}DK^{\bar{m}}(\bar{Z}) + a_{21} f_{\bar{m}a} \bar{\partial}DK^{\bar{m}}(\bar{Z}) + a_{22} f_{m\bar{p}} f_{m\bar{q}} \bar{D}DK^mDK^{\bar{n}}(Z) \\
+ a_{23} f_{m\bar{p}} f_{m\bar{q}} \bar{D}DK^mDK^{\bar{n}}(Z) + a_{24} f_{m\bar{p}} f_{m\bar{q}} \bar{D}DK^mDK^{\bar{n}}(Z), \quad (2.20) \]

where all the coefficient functions are present in the Appendix (D.1). This explicit structure (2.20) was obtained from the two conditions (2.15) and (2.16). We also present the operator product expansion at the linearized level in the Appendix D where the right hand side contains the central charge

\[ c_{N=2} = \frac{6k}{k+3}, \quad (2.21) \]

and the self-coupling constant

\[
\alpha_{N=2}^2 = \frac{27(2-k)^2(1+k)^2}{(-1+k)(5+k)(3+2k)(-3+5k)} = -\frac{(3+c)^2(-12+5c)^2}{2(-15+c)(1+c)(6+c)(-3+2c)}, \quad (2.22)
\]

where we replace the level \( k \) with the central charge \( c \) (2.21). Compared to the pure bosonic case (for example, the operator product expansion between the spin 3 current and itself in terms of WZW currents), as in introduction, the operator product expansion of \( W(Z) \) and itself in \( \mathcal{N} = 2 \) superspace or in the component approach has a self-coupling constant term in the right hand side. For the bosonic spin 3 case, there is no spin 3 current that will appear in the \( \frac{1}{(z-w)^2} \) term in the right hand side of the operator product expansion. One can easily see this observation by considering the operator product expansion with reversing the arguments and realizing that there will be an inconsistency in the operator product expansion. However, this is not true for the spin 4 case. In general, the operator product expansion between the spin 4 current and itself (in terms of WZW currents) in \( W_{\mathcal{N}} \) algebra generates the spin 4 current in the right hand side. It would be interesting to find out the self-coupling constant for the spin 4 current in the bosonic case.

We present the operator product expansion in (D.3) at linearized level. Note that the coefficient functions in the right hand side are characterized by the central charge \( c \) and self-coupling constant \( \alpha \). One sees that the \( \alpha \) dependence appears in the current \( W(Z_2) \) and its descendants fields and the functions of central charge appear in the other remaining fields.
2.2 The $N = 4$ case: $\text{CP}^4(= SU(5) / SU(4) \times U(1))$ coset model

Let us recall that the field contents of $\mathcal{N} = 2$ $\mathcal{W}_5$ algebra are given by the stress energy tensor with spins $(1, 3/2, 5/2, 2)$ and higher spin currents with spins $(2, 5/2, 3), (3, 7/2, 5), (4, 9/2, 7/2, 5)$. Then how one can determine these currents in terms of the fundamental currents which live in the supersymmetric WZW model? As before, the stress energy tensor is given by (2.12). It is nontrivial to find the extra symmetry currents in the generalization of Sugawara construction (so called Casimir construction) that includes the higher spin generators. Compared to the previous case where there exist only 8 independent fields, there exist 24 independent fundamental WZW currents. One way to write down the lowest higher spin current with spins $(2, 5/2, 3)$ is to take into account of all the possible terms (quartic terms, cubic terms and quadratic terms and linear terms in the WZW currents (2.3)). The other way is to take the $N = 2$ case (2.20) with arbitrary coefficient functions and apply the two conditions (2.15) and (2.16) but did not come out properly. By brute force, one should add other possible terms coming from

$$TT(Z), \partial T(Z), [D, \overline{D}]T(Z), T_H T_H(Z), \partial T_H(Z), [D, \overline{D}]T_H(Z), TT_H(Z), \quad (2.23)$$

where $T(Z)$ is given by (2.12) and $T_H(Z)$ is given by (2.11). In other words, by looking at the explicit expressions (2.23), collecting the independent terms and adding these into the expression (2.20). Finally, it turns out that the correct higher spin current with spins $(2, 5/2, 3)$, satisfying the two conditions (2.15) and (2.16), takes the form

$$W(Z) = b_1 f_{ca}^m f_{cb}^n J^a \overline{J}^b K^m K^n \overline{Z}(Z) + c_2 f_{ca}^n f_{cb}^m J^a \overline{J}^b K^m K^n \overline{Z}(Z) + c_3 J^a \overline{J}^a J^b \overline{J}^b (Z)$$

$$+ b_4 f_{am}^b J^a K^m \overline{J}^b (Z) + b_5 f_{ma}^b K^m \overline{D} J^a \overline{J}^b (Z) + b_6 f_{ma}^b J^a \overline{J}^b D K^m (Z)$$

$$+ b_7 f_{ma}^b J^a \overline{J}^b D K^m (Z) + b_8 f_{mn}^p D K^m K^n \overline{Z}(Z) + b_9 D J^a D \overline{J}^a (Z)$$

$$+ b_{10} D K^m D K^m (Z) + b_{11} J^a \overline{J}^a (Z) + b_{12} \partial J^a \overline{J}^a (Z) + b_{13} K^m \overline{D} K^m (Z)$$

$$+ b_{14} \partial K^m K^m (Z) + b_{15} J^a [D, \overline{D}] J^a (Z) + b_{16} [D, \overline{D}] J^a J^a (Z) + b_{17} K^m [D, \overline{D}] K^m (Z)$$

$$+ b_{18} [D, \overline{D}] K^m K^m (Z) + b_{19} D K^m \overline{D} K^m (Z) + b_{20} f_{mn}^a \overline{D} K^m (Z) + b_{21} f_{mn}^a \overline{D} K^m (Z)$$

$$+ b_{22} f_{mn}^b J^a \overline{J}^b D K^m (Z) + b_{23} f_{mn}^b J^a \overline{J}^b D K^m (Z) + b_{24} f_{mn}^a \overline{J}^b D K^m (Z)$$

$$+ b_{25} f_{mn}^a \overline{J}^b D K^m K^m (Z) + b_{26} f_{mn}^a \overline{J}^b D K^m K^m (Z) + b_{27} f_{mn}^a \overline{D} K^m (Z), \quad (2.24)$$

\(^{9}\)If one considers $\mathcal{N} = 2$ $\mathcal{W}_4$ algebra, then the coset can be described as $\text{CP}^3(= SU(2) / SU(3) \times U(1)) = SU(2) / SU(3) \times U(1)$. By introducing the extra $U(1)$’s in order to have even-dimensional groups $G$ and $H$ from (2.1). In principle, one can find the corresponding $\mathcal{N} = 2$ current algebra with $U(4)$ group in the complex basis. This should correspond to the work of [15].
where all the coefficient functions are given in the Appendix F explicitly\textsuperscript{10}. This is an $\mathcal{N} = 2$ current and one can read off the corresponding component currents with spins $(2, \frac{5}{2}, \frac{5}{2}, 3)$. The spin 2 current $W(z)$ in (2.19) can be obtained by putting all the $\theta$ and $\bar{\theta}$ dependence in the right hand side of (2.24) to zero. For the spin $\frac{5}{2}$ currents $DW(z)$ and $\bar{D}W(z)$ can be obtained also by taking the supercovariant derivatives $D$ and $\bar{D}$ into the right hand side of (2.24) and then putting the $\theta$ and $\bar{\theta}$ to zero at the final expression. For the spin 3 current $-\frac{1}{2}[D, \bar{D}]W(z)$, one can do similar analysis. Due to the constraints (2.8), there are several ways to write down these component currents\textsuperscript{11}.

On the other hands, one can make the explicit operator product expansion between $T(Z_1)$ (2.12) and $W(Z_2)$ (2.24) by using the defining equations (2.4). Since it should satisfy the primary condition (2.16), one can read off the above component currents straightforwardly by looking at the singular terms in the operator product expansion. We list them in the Appendix C. How to determine the spin 3 current, for example? First, we determine the spin $\frac{5}{2}$ current $\bar{D}W(z)$ by using the seventh equation of (C.1) and reading off the $\frac{1}{(z-w)}$ terms where one uses $\bar{D}T(z)$ from (2.18). Next, by using the fifth equation of (C.1) with spin $\frac{3}{2}$ current $DT(z)$ (2.18) and collecting the $\frac{1}{(z-w)}$ terms, one sees the spin 3 current $-\frac{1}{2}[D, \bar{D}]W(w)$ and the descendant field $\partial W(w)$. During this computation, one uses the constraint equations (2.8) all the time. In this way, one obtains all the component fields. For example, the field $DW(w)$ can be obtained from the fourth equation of (C.1).

Let us focus on the $\frac{1}{(z-w)^4}$ terms in the operator product expansion of $W(z)W(w)$ where the spin 2 current $W(z)$ is the first component of $W(Z)$ in (2.19) that has the form in (2.24) together with (F.1). One determines the overall constant $A(k)$

$$A(k)^2 = \frac{-\left(25\sqrt{3} + 135i\sqrt{5} - 23\sqrt{3}k + 15i\sqrt{5}k - 8\sqrt{3}k^2\right)^2}{8(-1+k)(5+k)^4(9+k)(5+2k)(-5+11k)} = \frac{(-12+c)^4 \left(-60\sqrt{3} - 324i\sqrt{5} + 33\sqrt{3}c + 39i\sqrt{5}c + \sqrt{3}c^2 - i\sqrt{5}c^2\right)^2}{207360000(-27+c)(-2+c)(12+c)(12+c)},$$ \hspace{1cm} (2.25)

by requiring that the $\frac{1}{(z-w)^4}$ term should be equal to $\frac{c}{2}$ where the central charge is

$$c_{N=4} = \frac{12k}{k + 5}. \hspace{1cm} (2.26)$$

Let us consider the $\frac{1}{(z-w)^4}$ terms in the operator product expansion of $W(z)W(w)$.

\textsuperscript{10}Totally, there are 249 independent terms if we expand out the structure constants (the number of nonzero structure constants is 492 from the discussion of the Appendix E) and the metric.

\textsuperscript{11}Also note that the previous spin 2 current (2.20) can be written in terms of (2.24) with the coefficients in the Appendix (D.2).
way to determine the self-coupling constant appearing in the coefficient function in front of $W(w)$ (in the right hand side of this operator product expansion $W(z)W(w)$) is to focus on any quartic term which does not appear in the fields $[D,\overline{D}]T(w)$ and $TT(w)$. For example, let us consider the $K^1 K^5 K^3 K^7(w)$ in the $\frac{1}{(z-w)^2}$ term. Definitely, this quartic term does not appear in the $[D,\overline{D}]T(w)$ and $TT(w)$. It turns out that the self-coupling constant is given in terms of either the level $k$ or the central charge $c$:

$$\alpha_{N=4}^2 = \frac{25(-4 + k)^2(1 + k)^2}{(-1 + k)(9 + k)(5 + 2k)(-5 + 11k)} = \frac{(3 + c)^2(-16 + 3c)^2}{2(27 - c)(-2 + c)(-1 + c)(12 + c)}. \quad (2.27)$$

Compared to the previous one for $N = 2$ (2.22), this is different from (2.22). It seems that the factors $(3 + c)^2$ and $(-1 + c)$ are common and they appear as $N$-independent factors but other factors should behave as $N$-dependent factors. Therefore, one should consider the most general self-coupling constant which will depend on $N$ when one describes the Kazama-Suzuki model for the general $N$.

Let us consider the $\frac{1}{(z-w)^2}$ term in the operator product expansion of the spin 2 field and the spin 3 field, $W(z)(-1)_2[D,\overline{D}]W(w)$. We do not present the spin 3 current here because the full expression for this is rather complicated. In general, there exist the seven different spin 3 fields in this singular term:

$$[D,\overline{D}]W(w), \ TW(w), \ \partial[D,\overline{D}]T(w), \ T[D,\overline{D}]T(w), \ TTT(w), \ \overline{T}TDT(w), \ \partial^2T(w). \quad (2.28)$$

As in bosonic case $^{12}$ [17], one expects that there should be the extra higher spin fields because the field contents of $\mathcal{N} = 2 \mathcal{W}_3$ algebra are given by the multiplet $W(Z)$ with spins $(2, \frac{5}{2}, \frac{5}{2}, 3)$, the multiplet $V(Z)$ with spins $(3, \frac{7}{2}, \frac{7}{2}, 4)$, and the multiplet $X(Z)$ with spins $(4, \frac{9}{2}, \frac{9}{2}, 5)$. As before, one has the following component currents for this new primary current with spins $(3, \frac{7}{2}, \frac{7}{2}, 4)$, for example,

$$V(Z) = V(z) + \theta \ DV(z) + \bar{\theta} \overline{D}V(z) + \theta \bar{\theta} \ (-1)_2 \frac{1}{2}[D,\overline{D}]V(z). \quad (2.29)$$

Although it is rather involved procedure to extract the exact form for the new primary field explicitly, one can check the existence of this field by looking at the particular term in the corresponding $\frac{1}{(z-w)^2}$ term. It turns out that, in $\mathcal{N} = 2$ superspace, one should add the following extra singular terms in the operator product expansion $W(Z_1)W(Z_2)$, at the linearized level, compared to the $\mathcal{N} = 2 \mathcal{W}_3$ algebra described in previous subsection,

$$\frac{\theta_{12}\bar{\theta}_{12}}{z_{12}^2} 3V(Z_2) + \frac{\theta_{12}}{z_{12}} \overline{D}V(Z_2) - \frac{\theta_{12}}{z_{12}} DV(Z_2) + \frac{\theta_{12}\bar{\theta}_{12}}{z_{12}} 2\partial V(Z_2). \quad (2.30)$$

$^{12}$Recall that for the bosonic case, the spin 4 current appearing in the operator product expansion between the spin 3 current and itself vanishes for $N = 3$ and one of the levels being 1 in the $W_N$ coset minimal model. However, for general $N$, the spin 4 current occurs naturally.
It is easy to see that all the relative coefficients can be fixed by using the conformal invariance, for given normalized factor 3 in front of $V(Z_2)$ in (2.30). In the operator product expansion of $\phi_m(z)\phi_n(w) \sim \phi_p +$ descendant, the relative coefficient function, for given primary field, can be determined by conformal invariance [30, 31].

For example, let us put the arbitrary coefficients $c_1$ and $c_2$ and $c_3$ in the second, third and fourth term respectively. Then the operator product expansion of $W(z)[D, \overline{D}]W(w)$ (see also (G.4)) can be written in terms of

$$\frac{1}{(z-w)^2}(-1)6V(w) + \frac{1}{(z-w)}[(c_1 - c_2 - 2c_3)\partial V + \cdots] (w) + \cdots.$$  (2.31)

The descendant field of $V(w)$, at $\frac{1}{(z-w)}$ term, is given by $\partial V(w)$. We do not write down other terms which are not relevant to our consideration. One uses the formula for the relative coefficient is given by

$$\frac{h_p + h_m - h_n}{2h_p} = \frac{3 + 2 - 3}{2 \times 3} = \frac{1}{3},$$  (2.32)

where the field $\phi_m$ plays the role of $W(z)$ which has a conformal dimension 2, the field $\phi_n$ corresponds to $[D, \overline{D}]W(w)$ with spin 3 and the field $\phi_p$ corresponds to $V(w)$ with spin 3. Therefore, the coefficient of $\partial V(w)$ should be equal to $-2$ for given the coefficient $-6$ in front of $V(w)$ in (2.31). That is, $\frac{-2}{-6} = \frac{1}{3}$. Then one has

$$c_1 - c_2 - 2c_3 = -2.$$  (2.33)

Similarly, the operator product expansion $DW(z)\overline{D}W(w)$(see also (G.7) and (G.8)) contains

$$\frac{1}{(z-w)^2}3V(w) + \frac{1}{(z-w)}\left[\frac{1}{2}(c_2 + 2c_3)\partial V + \cdots\right] (w) + \cdots.$$  (2.34)

By counting the conformal dimensions and using the above formula, one gets

$$\frac{h_p + h_m - h_n}{2h_p} = \frac{3 + \frac{5}{2} - \frac{5}{2}}{2 \times 3} = \frac{1}{2}.$$

The coefficient of $\partial V(w)$ in (2.34) should be equal to $\frac{3}{2}$. Therefore, the following relation holds

$$\frac{1}{2}(c_2 + 2c_3) = \frac{3}{2}.$$  (2.35)

Due to the structure of the formula, as long as $h_m = h_n$, the relative coefficient becomes $\frac{1}{2}$ [30]. Finally, the operator product expansion $DW(z)[D, \overline{D}]W(w)$ has the singular term

$$\frac{1}{(z-w)^2}(-6 + c_2)DV(w) + \frac{1}{(z-w)}\left[(-c_2 - 2c_3)\partial DV + \cdots\right] (w) + \cdots.$$  (2.36)
Again, in this case, one has
\[
\frac{h_p + h_m - h_n}{2h_p} = \frac{\frac{7}{2} + \frac{5}{2} - 3}{2 \times \frac{1}{2}} = \frac{3}{7}.
\]
The coefficient of \(\partial DV(w)\) in (2.36) should be equal to \(\frac{3}{7}(-6 + c_2)\). The final equation satisfies
\[
(-c_2 - 2c_3) = \frac{3}{7}(-6 + c_2).
\]
By combing these three equations (2.33), (2.35) and (2.37), and solving them, then there exists a unique solution and one has \(c_1 = 1, c_2 = -1,\) and \(c_3 = 2\).

Due to the field contents for \(\mathcal{N} = 2\) \(\mathcal{W}_5\) algebra, there are other five operator product expansions in \(\mathcal{N} = 2\) superspace. In principle, one obtains them by taking the operator product expansions once all the primary fields are determined and expressed in terms of WZW currents, although the computations will be rather complicated.

### 2.3 The general \(\mathcal{N}\) case: \(\mathbb{CP}^N\) (\(= \frac{SU(N+1)}{SU(N) \times U(1)}\)) coset model

The self-coupling constant of the current with spins \((2, \frac{5}{2}, \frac{5}{2}, 3)\) for any \(\mathcal{N} = 2\) \(\mathcal{W}_{N+1}\) algebra is determined from the unitarity arguments in [15]
\[
\alpha(N, k)^2 = \frac{3(1 + k)^2(k - N)^2(1 + N)^2}{(-1 + k)(1 - 1 + N)(1 + 2k + N)(1 + k + 2N)(-1 - N + k(-1 + 3N))}
\]
\[
= \frac{(3 + c)^2(c + 2cN - 3N^2)^2}{(-1 + c)(3 - c + 6N)(-1 + N)(c + 3N)(-3N + c(2 + N))},
\]
where the central charge is given by (2.14). Note that the previous constants (2.22) and (2.27) can be read off from this general expression (2.38) by substituting \(N = 2\) and \(N = 4\) respectively.

This behavior is also observed in the work of [5] by considering the singular terms \(\frac{1}{(z-w)^2}\) and \(\frac{1}{(z-w)^4}\) with spin 2 current in the KS model simultaneously because the normalization in the highest singular term is different from each other. They computed the operator product expansions between the spin 2 field and itself, by following [32] in the context of free field realization, for \(N = 2, 3, 4, 5\) cases and obtained by extrapolating these results.

Now one can write down the operator product expansion between \(W(Z_1)\) and \(W(Z_2)\) as follows:
\[
W(Z_1)W(Z_2) = \frac{1}{z_{12}^4} c_2 + \frac{\theta_{12}}{z_{12}^4} \frac{\bar{\theta}_{12}}{z_{12}^4} 3T(Z_2) + \frac{\theta_{12}}{z_{12}^3} 3DT(Z_2) - \frac{\bar{\theta}_{12}}{z_{12}^3} 3DT(Z_2) + \theta_{12} \bar{\theta}_{12} \frac{3 \partial T(Z_2)}{z_{12}^3}.
\]
where the central charge is given by (2.14) and the self-coupling constant is given by (2.38). One also has the nonlinear terms given by (2.29). Then one can continue to obtain the operator product expansion between products, one can read off the next higher spin current, for example, the spin 3 current in the right hand side [17] up to the overall normalization constant that can be fixed by the highest singular term of spin 4-spin 4 operator product expansion. Then one can compute the spin 3-spin 4 operator product expansion and determine other higher spin current. For example, the spin 5 current. The \( \mathcal{N} = 2 \mathcal{W}_{N+1} \) algebra should related to this bosonic \( \mathcal{W}_5 \) algebra. It would be interesting to find the structure constant for this particular coefficient in front of spin 5 current in the right hand side and see whether this will coincide with the previous result by using different method.

\[
\begin{align*}
+ \frac{1}{z_{12}^2} & \left[ 2\alpha W + \frac{c}{(1-c)}[D, \overline{D}]T \right] (Z_2) + \frac{\bar{\theta}_{12}}{z_{12}^2} \left[ \alpha D\overline{W} + \frac{(-3+2c)}{(1-c)} \partial D\overline{T} \right] (Z_2) \\
+ \frac{\bar{\theta}_{12}}{z_{12}^2} & \left[ \alpha D\overline{W} - \frac{(-3+2c)}{(1-c)} \partial D\overline{T} \right] (Z_2) + \frac{\theta_{12}\bar{\theta}_{12}}{z_{12}^2} \left[ \frac{3(-8+c)}{2(-12+5c)} \alpha [D, \overline{D}]W \right. \\
& + \frac{9c(-12+5c)}{4(-1+c)(6+c)(3+2c)} \partial [D, \overline{D}]T + \frac{3(18-15c+2c^2+2c^3)}{2(-1+c)(6+c)(3+2c)} \partial^2 T + 3V \right] (Z_2) \\
+ \frac{1}{z_{12}^2} & \left[ \alpha \partial W - \frac{c}{2(-1+c)} \partial [D, \overline{D}]T \right] (Z_2) \\
+ \frac{\bar{\theta}_{12}}{z_{12}^2} & \left[ 3(-6+c)(-1+c) \alpha \overline{D}W + \frac{3c(9+3c+2c^2)}{4(-1+c)(6+c)(3+2c)} \partial^2 \overline{D}T + \overline{D}V \right] (Z_2) \\
+ \frac{\theta_{12}\bar{\theta}_{12}}{z_{12}^2} & \left[ 3(-6+c)(-1+c) \alpha \overline{D}W - \frac{3c(9-3c+c^2)}{2(-1+c)(6+c)(3+2c)} \partial^2 D\overline{T} - D\overline{V} \right] (Z_2) \\
+ \frac{\theta_{12}\bar{\theta}_{12}}{z_{12}^2} & \left[ - \frac{(15+c)c}{(3+c)(-12+5c)} \alpha [D, \overline{D}]W + \frac{(18-3c-2c^2+2c^3)}{2(-1+c)(6+c)(3+2c)} \partial^2 T + 2\partial V \right] (Z_2). \\
& + \text{(Non-linear singular terms)} + \cdots, \\
\end{align*}
\] (2.39)

One should see the linear structure in (2.39) for the operator product expansion of the current with spins \((2, \frac{5}{2}, \frac{5}{2}, 3)\) and itself in \( \mathcal{N} = 2 \mathcal{W}_{N+1} \) algebra. For the nonlinear terms, one has \( TT(Z_2) \) term in the \( \frac{1}{z_{12}^2} \) term and the descendant fields arise in the appropriate singular terms. One also has the nonlinear terms \( TW(Z_2), T[D, \overline{D}]T(Z_2), TTT(Z_2), \overline{D}TDT(Z_2) \) whose component fields appear in (2.28).

In principle, with the two values (2.40), one can find other higher spin currents. For given the higher spin current \( W(Z) \) of spins \((2, \frac{5}{2}, \frac{5}{2}, 3)\) in the \( \mathcal{N} = 2 \mathcal{W}_{N+1} \) algebra, one can construct the operator product expansion of this current and itself. By looking at the singular terms, one can read off the next higher spin current, for example, \( V(Z) \) of spins \((3, \frac{7}{2}, \frac{7}{2}, 4)\) given by (2.29). Then one can continue to obtain the operator product expansion between \( W(Z_1) \) and \( V(Z_2) \) in order to find other higher spin current and so on \(^{13}\).

\(^{13}\)Let us remind that for the bosonic case, the spin 3-spin 3 operator product expansion determines the spin 4 current in the right hand side [17] up to the overall normalization constant that can be fixed by the highest singular term of spin 4-spin 4 operator product expansion. Then one can compute the spin 3-spin 4 operator product expansion and determine other higher spin current. For example, the spin 5 current. The \( \mathcal{N} = 2 \mathcal{W}_5 \) algebra should related to this bosonic \( \mathcal{W}_5 \) algebra. It would be interesting to find the structure constant for this particular coefficient in front of spin 5 current in the right hand side and see whether this will coincide with the previous result by using different method.
According to the observation of [2], the original proposal in [1] should hold at finite \((N,k)\) and one expects that the quantum deformation algebra of \(\mathcal{N} = 2\ \mathcal{W}_\infty^{\text{cl}}[\lambda]\) in [5] should satisfy the algebraic structure in (2.39), at finite \((N,k)\).

3 The large \((N,k)\) ’t Hooft limit of \(\mathcal{N} = 2\ \mathcal{W}_{N+1}\) algebra

We would like to describe the large \((N,k)\) limit for the operator product expansion between the lowest higher spin current \(W(Z_1)\) with spins \((2,\frac{5}{2},\frac{5}{2},3)\) and itself \(W(Z_2)\). The large \((N,k)\) limit for fixed ’t Hooft coupling constant \(\lambda\) is given by

\[
c(N,k) = \frac{3Nk}{N+k+1} \rightarrow 3(1-\lambda)N, \quad \lambda = \frac{N}{N+k}. \tag{3.1}
\]

Similarly, one also has the following limit for the self-coupling constant (2.38)

\[
\alpha(N,k)^2 \rightarrow -\frac{(1+2\lambda)^2}{(2+\lambda)(1+\lambda)}. \tag{3.2}
\]

From the observations for \(N = 2\) and \(N = 4\) cases in previous subsections, one expects that the operator product expansion, in the large \((N,k)\) limit, together with (3.1) and (3.2), takes the form

\[
W(Z_1)W(Z_2) = \frac{1}{z_{12}^{3/12}} \frac{c(N,k)}{2} + \frac{\theta_{12} \bar{\theta}_{12}}{z_{12}^{1/12}} 3T(Z_2) + \frac{\bar{\theta}_{12}}{z_{12}^{3/12}} 3\overline{T}(Z_2) - \frac{\theta_{12}}{z_{12}^{3/12}} 3D(T(Z_2) + \frac{\theta_{12} \bar{\theta}_{12}}{z_{12}^{3/12}} 3\partial T(Z_2)
+ \frac{1}{z_{12}^{3/12}} \left[ 2\alpha(N,k)W + [D,\overline{T}]T \right] (Z_2) + \frac{\bar{\theta}_{12}}{z_{12}^{1/12}} \left[ \alpha(N,k)\overline{D}W + 2\partial \overline{T}T \right] (Z_2)
+ \frac{\theta_{12}}{z_{12}^{1/12}} \left[ \alpha(N,k)DW - 2\partial DT \right] (Z_2) + \frac{\theta_{12} \bar{\theta}_{12}}{z_{12}^{1/12}} \left[ \frac{3}{10} \alpha(N,k)[D,\overline{T}]W + \frac{3}{2} \partial^2 T + 3V \right] (Z_2)
+ \frac{1}{z_{12}^{1/12}} \left[ \alpha(N,k)\partial W - \frac{1}{2} \partial[D,\overline{T}]T \right] (Z_2) + \frac{\bar{\theta}_{12}}{z_{12}^{3/12}} \left[ \frac{3}{5} \alpha(N,k)\overline{T}W + \frac{3}{4} \partial^2 \overline{T} + \overline{D}V \right] (Z_2)
+ \frac{\theta_{12}}{z_{12}^{3/12}} \left[ \frac{3}{5} \alpha(N,k)DW - \frac{3}{4} \partial^2 DT - DV \right] (Z_2)
+ \frac{\theta_{12} \bar{\theta}_{12}}{z_{12}^{3/12}} \left[ -\frac{1}{5} \alpha(N,k)[D,\overline{T}]W + \frac{1}{2} \partial^3 T + 2\partial V \right] (Z_2)
+ \frac{1}{N} \text{(quadratic singular terms)} + \frac{1}{N^2} \text{(cubic singular terms)}
+ \frac{1}{N^3} \text{(quartic singular terms)} + \cdots, \tag{3.3}
\]

where \(c(N,k)\) and \(\alpha(N,k)\) are the values after taking the large \((N,k)\) limit, given in (3.1) and (3.2) respectively. At the linear order in the right hand side, we replace the fixed coupling
constants (2.22) and (2.27) with the general coupling constant (2.38) and allow to include the
new $\mathcal{N} = 2$ primary field $V(Z_2)$ (and its descendant fields) (2.30) in the right hand side of the
operator product expansion (3.3). We list the component results of (3.3) in the Appendix G. Note that the $\partial[D, \bar{T}]T$ term in $\frac{\partial_t \theta_{12}}{z_{12}}$ in (D.3) vanishes in this limit and does not appear in
(3.3) also.

One expects that the extra new composite fields $T^4(Z_2), T^2W(Z_2), W^2(Z_2)$, and $TV(Z_2)$
with spins $(4, \frac{9}{2}, \frac{9}{2}, 5)$ should appear in the lowest singular term $\frac{\partial_t \theta_{12}}{z_{12}}$ in (3.3). We have seen
this feature in the $\mathcal{N} = 2$ $\mathcal{W}_4$ algebra in [15] although the full structure of the algebra is not
given. Also one sees the appearance of these new fields in the $AdS_3$ side. In [5], the nonlinear
terms in (3.46) to (3.53) contain these fields. For example, the $\frac{1}{k_{CS}}$ term corresponds to
$T^4(Z_2)$ term and some of the $\frac{1}{k_{CS}}$ terms contain $T^2W(Z_2)$ term and so on. Note that the
Chern-Simon level $k_{CS}$ behaves as $N$ in the large $N$ 't Hooft limit.

4 Comparison with the $\mathcal{N} = 2$ classical $\mathcal{W}_\infty^\text{cl}[\lambda]$ algebra
of the bulk theory

One identifies the currents in the Kazama-Suzuki model with the higher spin fields in $\mathcal{W}_\infty^\text{cl}[\lambda]$ introduced in [5] as follows:

\[
\begin{align*}
T(z) & \leftrightarrow a_{\frac{3}{2}} \sim W^-_{1,HP}(z), \\
(DT + \bar{T}T)(z) & \leftrightarrow \psi_{\frac{5}{2}} \sim G^+_{2,HP}(z), \\
(DT - \bar{T}T)(z) & \leftrightarrow \psi_{\frac{3}{2}} \sim G^+_{2,HP}(z), \\
-\frac{1}{2}[D, \bar{T}]T(z) & \leftrightarrow a_2 \sim W^+_{2,HP}(z), \\
W(z) & \leftrightarrow a_{\frac{5}{2}} \sim W^-_{2,HP}(z), \\
(DW + \bar{T}W)(z) & \leftrightarrow \psi_{\frac{7}{2}} \sim G^+_{3,HP}(z), \\
(DW - \bar{T}W)(z) & \leftrightarrow \psi_{\frac{3}{2}} \sim G^+_{3,HP}(z), \\
-\frac{1}{2}[D, \bar{T}]W(z) & \leftrightarrow a_3 \sim W^+_{3,HP}(z), \\
V(z) & \leftrightarrow a_{\frac{7}{2}} \sim W^-_{3,HP}(z), \\
(DV + \bar{T}V)(z) & \leftrightarrow \psi_{\frac{9}{2}} \sim G^+_{4,HP}(z), \\
(DV - \bar{T}V)(z) & \leftrightarrow \psi_{\frac{5}{2}} \sim G^+_{4,HP}(z), \\
-\frac{1}{2}[D, \bar{T}]V(z) & \leftrightarrow a_4 \sim W^+_{4,HP}(z).
\end{align*}
\] (4.1)

We also present the CFT fields with $HP$ index in the last entry in order to specify them from
[5].
In order to obtain the $AdS_3$ result, the normalization factor should occur in the operator product expansion of spin 2 current and itself
\[
\beta(N, k)^2 = \frac{(-1 + k)(-1 + N)(1 + 2k + N)(1 + k + 2N)}{3(1 + k + N)^2(-1 - k - N + 3kN)}
\rightarrow -\frac{2}{9}(-1 + \lambda_{HP})(1 + 2\lambda_{HP}) = -\frac{1}{9}(-2 + \lambda)(1 + \lambda), \quad 2\lambda_{HP} = \lambda, \quad (4.2)
\]
where we also present the large $(N, k)$ limit (3.1). This expression is a generalization of [32] where the $\beta(N = 2, k)$ was found for fixed $N = 2$ case. Similarly, one also has the following limit for the self-coupling constant (2.38) as before
\[
\alpha(N, k)^2 \rightarrow -\frac{(-1 + 4\lambda_{HP})^2}{2(-1 + \lambda_{HP})(1 + 2\lambda_{HP})} = \frac{(-1 + 2\lambda)^2}{(-2 + \lambda)(1 + \lambda)}. \quad (4.3)
\]

From the operator product expansion in Appendix $G$ and the following relations between our currents and the field contents in [5]
\[
W(z) \equiv \frac{1}{\beta(N, k)}W_2^-(z), \quad -\frac{1}{2}[D, \overline{D}]T(z) \equiv W_{2, HP}^+(z), \quad (4.4)
\]
one rewrites the equation (G.1), together with (4.1), as
\[
W_{2, HP}^+(z)W_{2, HP}^-(w) = \frac{1}{(z - w)^4} \left[ \frac{1}{2}c(N, k)\beta(N, k)^2 \right.
+ \frac{1}{(z - w)^2}\beta(N, k)^2 \left[ \frac{2\alpha(N, k)}{\beta(N, k)}W_{2, HP}^- + 2W_{2, HP}^+ \right](w)
+ \frac{1}{(z - w)}\beta(N, k)^2 \left[ \frac{\alpha(N, k)}{\beta(N, k)}\partial W_{2, HP}^- + \partial W_{2, HP}^+ \right](w)
+ \frac{1}{N}(\text{Non-linear singular terms}) + \cdots
\rightarrow \frac{1}{(z - w)^4} \left[ -1 - \frac{1}{3}(1 - \lambda_{HP})(2\lambda_{HP} - 1)(2\lambda_{HP} + 1)N \right.
+ \frac{1}{(z - w)^2} \left[ \frac{2}{3}(1 - 4\lambda_{HP})W_{2, HP}^- - \frac{4}{9}(2\lambda_{HP} + 1)(\lambda_{HP} - 1)W_{2, HP}^+ \right](w)
+ \frac{1}{(z - w)} \left[ \frac{1}{3}(1 - 4\lambda_{HP})\partial W_{2, HP}^- - \frac{2}{9}(2\lambda_{HP} + 1)(\lambda_{HP} - 1)\partial W_{2, HP}^+ \right](w)
+ \frac{1}{N}(\text{Non-linear singular terms}) + \cdots, \quad (4.5)
\]
where we use the large $(N, k)$ limits for $\alpha(N, k), \beta(N, k)$ and $c(N, k)$, (4.3), (4.2) and (3.1) respectively.\footnote{A commutator relation for the modes $(W_{2, HP}^\pm)_m$ is as follows: $[(W_{2, HP}^-)_m, (W_{2, HP}^-)_n] = \beta(N, k)^2(m - n)\left[ \frac{\alpha(N, k)}{\beta(N, k)}(W_{2, HP}^-)_{m+n} + (W_{2, HP}^+)_{m+n} \right] + \beta(N, k)^2 c(N, k)(m^2 - 1)\delta_{m+n, 0} + \text{Nonlinear terms}$, where $W_{2, HP}^\pm = \sum_{m \in \mathbb{Z}} (W_{2, HP}^\pm)_m$. One sees the similar structure in [10].} This is exactly the same as the equation (4.8) of [5]. Then it is straightforward
to change the above to the commutator and agree with the AdS \(_3\) result where one can use the identities

\[
\beta(N, k)^2 \rightarrow -N_B^B, \quad N_{3/2}^B = \frac{2}{9}(1 + \lambda_{HP})(1 + 2\lambda_{HP}) = \frac{1}{3}N_3^B, \quad \text{(4.6)}
\]

where \(N_B^B\) and \(N_{3/2}^B\) in [5] are some normalization functions that depend on ’t Hooft coupling constant and they appear in the commutator relations in the AdS \(_3\) side. One might ask whether there exists a possibility for the existence of a new primary field of spin in the \(\frac{1}{(z-w)}\) term (4.5). If there is a new primary field in that singular term, one can change the arguments \(z\) and \(w\) and use the series expansion around \(w\). Then it turns out there is a minus sign for this primary field. This implies that there is no extra new primary field of spin 3.

By using the identification

\[
-\frac{1}{2}[\mathcal{D}, \mathcal{D}] W(z) \equiv \frac{1}{\beta(N, k)} W_{3, HP}^+(z), \quad V(z) \equiv \frac{1}{\beta(N, k)^2} W_{3, HP}^-(z), \quad \text{(4.7)}
\]

and (4.4), one also computes the large \((N, k)\) limit for the operator product expansion between the spin 2 current and the spin 3 current, from (G.4), as follows:

\[
W_{2, HP}^-(z) W_{3, HP}^+(w) = \frac{1}{(z-w)^4} 3\beta(N, k)^2 W_{1, HP}^-(w)
\]

\[
+ \frac{1}{(z-w)^2} \beta(N, k)^2 \left[ \frac{3\alpha(N, k)}{5\beta(N, k)} W_{3, HP}^+ + \frac{3}{\beta(N, k)^2} W_{3, HP}^- \right] (w)
\]

\[
+ \frac{1}{(z-w)} \beta(N, k)^2 \left[ \frac{1}{5\beta(N, k)} \partial W_{3, HP}^+ + \frac{1}{\beta(N, k)^2} \partial W_{3, HP}^- \right] (w)
\]

\[
+ \frac{1}{N} (\text{Non-linear singular terms}) + \cdots
\]

\[
\rightarrow \frac{1}{(z-w)^4} (-1) \frac{2}{3} (2\lambda_{HP} + 1)(\lambda_{HP} - 1) W_{1, HP}^-(w)
\]

\[
+ \frac{1}{(z-w)^2} \left[ \frac{1}{5} (1 - 4\lambda_{HP}) W_{3, HP}^+ + 3W_{3, HP}^- \right] (w)
\]

\[
+ \frac{1}{(z-w)} \left[ \frac{1}{15} (1 - 4\lambda_{HP}) \partial W_{3, HP}^+ + \partial W_{3, HP}^- \right] (w)
\]

\[
+ \frac{1}{N} (\text{Non-linear singular terms}) + \cdots. \quad \text{(4.8)}
\]

One easily sees that this (4.8) agrees with the equation (3.46) in [5] at the linear order \(^{15}\).

For example, the relative coefficient \(\frac{1}{3}\) on the descendant field \(\partial W_{3, HP}^-\) can be obtained from

\(^{15}\)This can be written in terms of modes as follows: \([W_{2, HP}^-(m), W_{3, HP}^+(n)] = \frac{\beta(N, k)^2}{2} m(m+1)(W_{1, HP}^-)_{m+n} - \frac{1}{2\beta(N, k)^2} (2m-n) \frac{\alpha(N, k)}{\beta(N, k)} (W_{3, HP}^+)_{m+n} + (2m-n) (W_{3, HP}^-)_{m+n} + \text{Nonlinear terms, which can be compared to [10].}
\[ h_p + h_m - h_n = \frac{3 + 2 - 3}{2 \times 3} = \frac{1}{3}. \]

It is obvious that this equation (4.9) should correspond to the equation (3.2) of \[ \beta \] because according to the counting of (2.32), the numerator becomes zero \( (h_m = 2, h_n = 3 \text{ and } h_p = 1) \). This implies that the coefficient for the descendant field \( \partial W_{1,HP}^-(w) \) vanishes and there is no such term in the \( \frac{1}{(z-w)^3} \) term in (4.8).

Let us present the final bosonic operator product expansion between the spin 3 current and itself, from (G.10), where we use (4.6)

\[
W_{3,HP}^+(z)W_{3,HP}^+(w) = \frac{1}{(z-w)^6} \frac{5}{2} \alpha(N, k) \beta(N, k)^2 \\
+ \frac{1}{(z-w)^4} \beta(N, k)^2 \left[ \frac{3 \alpha(N, k)}{\beta(N, k)} W_{2,HP}^- + 15 W_{2,HP}^+ \right] (w) \\
+ \frac{1}{(z-w)^3} \beta(N, k)^2 \left[ \frac{3 \alpha(N, k)}{2 \beta(N, k)} \partial W_{2,HP}^- + \frac{15}{2} \partial W_{2,HP}^+ \right] (w) \\
+ \frac{1}{(z-w)^2} \beta(N, k)^2 \left[ \frac{9 \alpha(N, k)}{20 \beta(N, k)} \partial^2 W_{2,HP}^- + \frac{9}{4} \partial^2 W_{2,HP}^+ + \frac{4}{\beta(N, k)^2} W_{4,HP}^+ \right] (w) \\
+ \frac{1}{(z-w)} \beta(N, k)^2 \left[ \frac{1}{10 \beta(N, k)} \partial^3 W_{2,HP}^- + \frac{1}{2} \partial^3 W_{2,HP}^+ + \frac{2}{\beta(N, k)^2} \partial W_{4,HP}^+ \right] (w) \\
+ \frac{1}{N} (\text{Non-linear singular terms}) + \cdots \\
\rightarrow \frac{1}{(z-w)^6} (-1)^5 (1 - \lambda_{HP})(2\lambda_{HP} - 1)(2\lambda_{HP} + 1) N \\
+ \frac{1}{(z-w)^4} \left[ (1 - 4\lambda_{HP}) W_{2,HP}^- - \frac{10}{3}(2\lambda_{HP} + 1)(\lambda_{HP} - 1) W_{2,HP}^+ \right] \\
+ \frac{1}{(z-w)^3} \left[ \frac{1}{2}(1 - 4\lambda_{HP}) \partial W_{2,HP}^- - \frac{5}{3}(2\lambda_{HP} + 1)(\lambda_{HP} - 1) \partial W_{2,HP}^+ \right] \\
+ \frac{1}{(z-w)^2} \left[ \frac{3}{20}(1 - 4\lambda_{HP}) \partial^2 W_{2,HP}^- - \frac{1}{2}(2\lambda_{HP} + 1)(\lambda_{HP} - 1) \partial^2 W_{2,HP}^+ + 4 W_{4,HP}^+ \right] \\
+ \frac{1}{(z-w)} \left[ \frac{1}{30}(1 - 4\lambda_{HP}) \partial^3 W_{2,HP}^- - \frac{1}{9}(2\lambda_{HP} + 1)(\lambda_{HP} - 1) \partial^3 W_{2,HP}^+ + 2 \partial W_{4,HP}^+ \right] \\
+ \frac{1}{N} (\text{Non-linear singular terms}) + \cdots . \tag{4.9}
\]

It is obvious that this equation (4.9) should correspond to the equation (3.47) of [5]. Also

\[ \beta(N, k)^2 \frac{m(m^2 - 1)(m^2 - 4)\delta_{m+n,0} + \beta(N, k)^2 (m - n) \left[ \frac{1}{15}(m + n + 3)(m + n + 2) - \frac{1}{6}(m + n)(n + 2) \right] \left[ \frac{3 \alpha(N, k)}{2 \beta(N, k)} W_{2,HP}^- m + n - \frac{15}{4} W_{2,HP}^+ m + n \right] + 2(m - n) W_{4,HP}^- m + n + \text{Nonlinear terms.} \]

Similarly, one can compare this with the corresponding equation in [10].
note that the relative coefficient function $\frac{1}{2}$ on $\partial W^+_4 H P$ can be obtained from the formula (2.32) by substituting $h_m = 3 = h_n$ and $h_p = 4$. The relative coefficients $1, \frac{1}{2}, \frac{3}{20}$, and $\frac{1}{30}$, for the spin 2 current in the right hand side, are standard values in the well-known $W_3$ algebra. See, for example, the review paper [34]. The coefficient $\frac{3}{20}$ is nothing but $\frac{1}{4} \frac{h_p+1}{2 h_n+1}$ and this becomes $\frac{3}{20}$ at $h_p = 2$ [30].

We also present the remaining 6 operator product expansions, in the large $(N, k)$ limit in (G.11), (G.12), (G.13), (G.14), (G.15), and (G.16). Due to the $\mathcal{N} = 2$ supersymmetry (the current multiplets $T(Z)$ and $W(Z)$ and their operator product expansions can be organized in manifest $\mathcal{N} = 2$ superspace), compared to the bosonic case, one could obtain much informations on the various operator product expansions. In other words, for given operator product expansion of $\mathcal{N} = 2$ currents (only after this is determined by other method, for example, Jacobi identity), there exist 16 component operator product expansions. Without any input for the $\mathcal{N} = 2$ supersymmetry, one should analyze all these operator product expansions separately [10]. For example, for the $\mathcal{N} = 2 \ W_3$ algebra, by exploiting the package of [28] with Jacobi identity, one can easily obtain the operator product expansion for the higher spin current in $\mathcal{N} = 2$ superspace.

5 Conclusions and outlook

We have constructed the $\mathcal{N} = 2$ current with spins $(2, \frac{5}{2}, \frac{5}{2}, 3)$ in (2.24) and the self-coupling constant in (2.27) in $\mathcal{N} = 2 \ W_5$ algebra. We also have found the extra singular terms in the operator product expansion in (2.30) which were not present in $\mathcal{N} = 2 \ W_3$ algebra. By observing the self-coupling constant (2.38) which depends on $(N, k)$ explicitly, the large $(N, k)$ limit of $\mathcal{N} = 2 \ W_{N+1}$ algebra contains the particular operator product expansion given in (3.3). We have identified this with the corresponding $\mathcal{N} = 2$ classical $\mathcal{W}_\infty^{cl}[\lambda]$ algebra in the bulk.

- It is an immediate question to ask how one obtains the higher spin current including (2.20) and (2.24) for $\mathcal{N} = 2 \ W_{N+1}$ algebra. From the structure of (2.24), one can try to write down the correct ansatz for the possible terms (one might add a few extra terms which are not present for $N = 2$ or $N = 4$ case) and then apply to the two conditions 1) and 2) in (2.15) and (2.16). It is nontrivial to find the identities for the multiple products between the structure constants in the complex basis and to collect the independent fields in each singular term. These are necessary to check the right singular structures.

- It would be interesting to obtain the quantum $\mathcal{N} = 2 \ W_{\infty}^{qu}[\lambda]$ algebra which is a deformation of the classical $\mathcal{N} = 2 \ W_{\infty}^{cl}[\lambda]$ algebra. For the KS model side, once we complete
all the operator product expansions at least $\mathcal{N} = 2$ $W_3$ algebra, then this algebra should provide all the informations on the quantum $\mathcal{N} = 2$ $W_3^{\text{su}}[\lambda]$ algebra, along the line of [2]. The $W_3^{\text{su}}[\lambda]$ for the bosonic case was found in [6, 35] and the corresponding quantum algebra has been studied in [2]. See also [18, 17]. We expect that there are extra linear terms for the nonlinear composite currents in (3.3). From the observation [36], as we take $c \to \infty$ in the quantum operator product expansion, any composite field (product of $n$ fields) where the $c$’s power in the denominator is greater than $(n - 1)$ will disappear in the classical limit. For example, the standard spin 3-spin 3 operator product expansion has the nonlinear term $\Lambda(w) \equiv TT(w) - \frac{3}{10} \partial^2 T(w)$ with coefficient function $\frac{32}{22+5c}$ in $\frac{1}{(z-w)^2}$ term as well as $\frac{3}{10} \partial^2 T(w)$ term [34]. In the $c \to \infty$ limit, the $\partial^2 T(w)$ term in the $\Lambda(w)$ vanishes while the $TT(w)$ term survives. In quantum theory, the extra term like as $\partial^2 T(w)$ in the $\Lambda(w)$ exists. On the other hand, it is an open problem to obtain the bosonic subalgebra (how to one gets the bosonic $W_5$ algebra) or $\mathcal{N} = 1$ subalgebra for the algebra we have described, along the line of [10].

• According the classification for the KS model [7], there exists the following coset model also

$$\frac{SO(N+2)}{SO(N) \times SO(2)}, \quad c(N, k) = \frac{3Nk}{N+k}.$$

It would be interesting to find the higher spin currents for this model and see how they arise as an $\mathcal{N} = 2$ nonlinear algebra. Once we construct the complex basis for the group $SO(N+2)$, then the current algebra similar to (2.4) should exist. Only the structure constants and dual Coxeter number can change. Then the standard Sugawara construction can follow similarly, along the line of [37]. See also the relevant works in [38, 39].

• As pointed out in [40], it would be interesting to construct the more supersymmetric higher spin $AdS_3$ supergravity dual to the $\mathcal{N} = 4$ superconformal coset model that can be realized by the $\mathcal{N} = 4$ current algebra for the supersymmetric WZW model. As a first step, one can use the previous work of [12] where the $\mathcal{N} = 4$ superconformal algebra (the spins for all the currents are less than or equal to 2: the spin 2 current, four spin $\frac{3}{2}$ currents, seven spin 1 currents and four spin $\frac{1}{2}$ currents) can be written in terms of the $\mathcal{N} = 2$ affine Kac-Moody currents. It is an open problem to construct the higher spin currents with spin greater than 2. In $\mathcal{N} = 2$ superspace, one should have $T(Z)$ with spins $(1, \frac{3}{2}, \frac{3}{2}, 2)$ and $W(Z)$ with spins $(2, \frac{5}{2}, \frac{5}{2}, 3)$ as well as the extra primary currents. For the minimal extension of $\mathcal{N} = 2$ $W_3$ algebra, the number of these extra currents is equal to 2. One of them corresponds to the $\mathcal{N} = 4$ partner of $T(Z)$ and the other corresponds to the $\mathcal{N} = 4$ partner of $W(Z)$. The spins for the $\mathcal{N} = 2$ multiplets can be either $(\frac{3}{2}, 2, 2, \frac{3}{2})$ or $(\frac{5}{2}, 3, 3, \frac{5}{2})$. The former is more preferable because the extension of $W_3$ current (the last component of $W(Z)$) has its partner of spin $\frac{5}{2}$.
in the context of [41]. Note that the full $\mathcal{N} = 4$ superconformal algebra is generated by the stress energy tensor $T(Z)$ with spins $(1, \frac{3}{2}, \frac{3}{2}, 2)$, two $\mathcal{N} = 2$ currents with spins $(\frac{1}{2}, 1, 1, \frac{3}{2})$ and a $\mathcal{N} = 2$ current with spins $(0, \frac{1}{2}, \frac{1}{2}, 1)$.

- From the result of [6], one expects that the linear structure in (3.3) should have the higher spin algebra in [42, 43], although the explicit relations are not given in this paper. One cannot use their expressions directly because the currents or generators are not primary fields. So in order to compare with our results here, one should obtain the correct primary fields with respect to the stress energy tensor. Of course, the higher spin algebra is not a subalgebra of the ultimate quantum algebra but is a subalgebra in the $c \to \infty$ limit. In general, the ultimate quantum algebra does not contain higher spin algebra as a subalgebra.

- It is an open problem to reconsider the previous analysis in [3], under the large $(N, k)$ limit, along the line of [2]. This can be done only after the $\mathcal{N} = 2$ quantum $\mathcal{V}_{\infty}^\text{qu}[\lambda]$ algebra is found.

**Acknowledgments**

We would like to thank the following people for correspondence on the following topics: R. Gopakumar on the current status of the triality [2], Y. Hikida on the supersymmetric version of higher spin algebra [3], S. Krivonos on his mathematica package for $\mathcal{N} = 2$ operator product expansions [28], S. Odake on his paper [11], C. Peng on the asymptotic symmetry [5] and M. Vasiliev on the super $W_{\infty}(\lambda)$ algebra [42, 43]. This work was supported by the Mid-career Researcher Program through the National Research Foundation of Korea (NRF) grant funded by the Korean government (MEST) (No. 2009-0084601).
Appendix A  The $\mathcal{N} = 2$ current algebra

The on-shell current algebra in $\mathcal{N} = 2$ superspace for the supersymmetric WZW model, with level $k$, on a group $G = SU(N + 1)$ of even-dimension, can be written as [9, 13, 33]

$$Q^A(Z_1)Q^B(Z_2) = -\frac{\tilde{\theta}_{12}}{z_{12}} f_{\bar{A}B}^C Q^C(Z_2) - \frac{1}{z_{12}} \frac{1}{(k + N + 1)} f_{\bar{A}C}^D f_{\bar{B}C}^E Q^D Q^E(Z_2),$$

$$Q^A(Z_1)Q^B(Z_2) = -\frac{\theta_{12}}{z_{12}} f_{AB}^C Q^C(Z_2) + \frac{\theta_{12}}{z_{12}} \frac{1}{(k + N + 1)} f_{AC}^D f_{BC}^E Q^D Q^E(Z_2),$$

$$Q^A(Z_1)Q^B(Z_2) = \frac{\tilde{\theta}_{12}}{z_{12}} \frac{1}{2} \left[ (k + N + 1) \delta^{AB} + f_{\bar{A}C}^D f_{\bar{B}C}^E \right] - \frac{1}{z_{12}} (k + N + 1) \delta^{AB}$$

$$-\frac{\theta_{12}}{z_{12}} f_{AB}^C Q^C(Z_2) - \frac{\theta_{12}}{z_{12}} f_{\bar{A}B}^C Q^C(Z_2)$$

$$-\frac{\theta_{12}}{z_{12}} \left[ f_{AB}^C DQ^C + \frac{1}{(k + N + 1)} f_{AC}^D f_{BC}^E Q^D Q^E \right] (Z_2), \quad (A.1)$$

where the nonlinear constraints are given by [9, 13]

$$DQ^A = -\frac{1}{2(k + N + 1)} f_{\bar{A}B}^C Q^B Q^C, \quad \bar{D}Q^A = -\frac{1}{2(k + N + 1)} f_{AB}^C Q^B Q^C. \quad (A.2)$$

The Jacobi identities of the algebra (A.1) are satisfied under the constraints (A.2). The operator product expansion $Q^\bar{A}(Z_1)Q^B(Z_2)$ can be obtained from the third equation of (A.1). The stress energy tensor can be written as

$$T(Z) = -\frac{1}{(k + N + 1)} \delta_{AB} Q^A Q^B(Z) + \frac{1}{(k + N + 1)} \left[ \delta_{BC} f_{\bar{A}C}^D DQ^A + \delta_{BC} f_{AB}^C \bar{D}Q^A \right]. \quad (A.3)$$

In the section 2, we rewrite the equations (A.1), (A.2) and (A.3) in manifest way of the subgroup $H$ and the coset $G/H$.

Appendix B  The $\mathcal{N} = 2$ current algebra (2.4) in the component approach

The 22 operator product expansions, by expanding the above operator product expansions (A.1) or (2.4) into the component, can be summarized by

$$K^m(z)\bar{D}K^n(w) = -\frac{1}{(z - w)} f_{m\bar{n}} \bar{p} K^p(w) + \cdots,$$

$$\bar{D}K^m(z)\bar{D}K^n(w) = -\frac{1}{(z - w)} f_{m\bar{n}} \bar{p} \bar{D}K^p(w) + \cdots,$$

$$K^m(z)\bar{D}J^a(w) = -\frac{1}{(z - w)} f_{m\bar{n}} \bar{b} J^b(w) + \cdots,$$
\[
\mathcal{D} K^m(z) J^a(w) = -\frac{1}{(z-w)} f_{\hat{m} \hat{a}} J^b(w) + \cdots,
\]
\[
\mathcal{D} K^m(z) \mathcal{D} J^a(w) = -\frac{1}{(z-w)} f_{\hat{m} \hat{a}} \mathcal{D} J^b(w) + \cdots,
\]
\[
K^m(z) D K^\bar{a}(w) = -\frac{1}{(z-w)} f_{mn} \bar{p} K^p(w) + \cdots,
\]
\[
D K^m(z) D K^\bar{a}(w) = -\frac{1}{(z-w)} f_{mn} \bar{p} D K^p(w) + \cdots,
\]
\[
K^m(z) D J^\bar{a}(w) = -\frac{1}{(z-w)} f_{\hat{m} \hat{a}} J^\bar{b}(w) + \cdots,
\]
\[
D K^m(z) J^\bar{a}(w) = -\frac{1}{(z-w)} f_{\hat{m} \hat{a}} J^\bar{b}(w) + \cdots,
\]
\[
D K^m(z) D J^\bar{a}(w) = -\frac{1}{(z-w)} f_{\hat{m} \hat{a}} D J^\bar{b}(w) + \cdots,
\]
\[
K^m(z) K^\bar{a}(w) = -\frac{1}{(z-w)} (k+N) \delta^m \bar{a} + \cdots,
\]
\[
K^m(z) D K^\bar{a}(w) = -\frac{1}{(z-w)} f_{mn} \bar{p} K^p(w) + \cdots,
\]
\[
\mathcal{D} K^m(z) K^\bar{a}(w) = -\frac{1}{(z-w)} f_{mn} \bar{p} K^p(w) + \cdots,
\]
\[
\mathcal{D} K^m(z) D K^\bar{a}(w) = \frac{1}{(z-w)^2} \left[ 2(k+N+1) \delta^m \bar{a} + f_{\hat{m} \hat{p}} f_{\hat{n} \hat{p}} \right]
- \frac{1}{(z-w)} \left[ f_{mn} D K^\bar{p} + f_{mn} \mathcal{D} K^p + \frac{1}{k+N+1} f_{\hat{m} \hat{p}} f_{\hat{n} \hat{p}} K^q K^r \right] (w) + \cdots,
\]
\[
\mathcal{D} J^a(z) D J^\bar{b}(w) = \frac{1}{(z-w)^2} \left[ 2(k+N+1) \delta^{ab} + f_{\hat{a} \hat{m}} f_{\hat{b} \hat{n}} + f_{\hat{a} \hat{c}} \bar{m} f_{\hat{b} \hat{c}} \bar{m} \right]
- \frac{1}{(z-w)} \left[ f_{\hat{a} \hat{b}} \mathcal{D} K^m + f_{\hat{a} \hat{b}} \mathcal{D} K^\bar{m} \right.
+ \frac{1}{k+N+1} \left( f_{\hat{a} \hat{m}} \bar{c} f_{\hat{b} \hat{n}} \bar{d} J^c J^\bar{d} + f_{\hat{a} \hat{c}} \bar{m} f_{\hat{b} \hat{c}} \bar{m} K^m K^\bar{a} \right) \left( w \right) + \cdots,
\]
\[
J^a(z) J^\bar{b}(w) = -\frac{1}{(z-w)} (k+N+1) \delta^{ab} + \cdots,
\]
\[
J^a(z) D J^\bar{b}(w) = -\frac{1}{(z-w)} f_{\hat{a} \hat{b}} \bar{m} K^m(w) + \cdots,
\]
\[
\mathcal{D} J^a(z) J^\bar{b}(w) = -\frac{1}{(z-w)} f_{\hat{a} \hat{b}} \bar{m} K^m(w) + \cdots,
\]
\[
\mathcal{D} K^m(z) J^a(w) = -\frac{1}{(z-w)} f_{\hat{m} \hat{a}} J^\bar{b}(w) + \cdots,
\]
\[
\mathcal{D} K^m(z) D J^a(w) = \frac{1}{(z-w)^2} \frac{1}{2} f_{\hat{m} \hat{n}} f_{\hat{a} \hat{n}} \bar{p}
\]

\(- \frac{1}{(z-w)} \left[ f_{m a}^b D J^b + \frac{1}{k+N+1} \sum_{\mu} f_{m \mu}^a f_{\mu \rho}^b K^q J^b \right] (w) + \cdots, \)

\(J^a(z) D K^m(w) = - \frac{1}{(z-w)} f_{\bar{a} m}^b J^b(w) + \cdots, \)

\(\overline{\partial} J^a(z) D K^m(w) = - \frac{1}{(z-w)} \left[ f_{\bar{a} m}^b \overline{\partial} J^b + \frac{1}{k+N+1} \sum_{\mu} f_{\mu \rho}^{\bar{a}} f_{\mu \rho}^n J^b K^n \right] (w) + \cdots \) \hspace{1cm} (B.1)

Here we use the component currents (2.17) in order not to introduce many different notations for various fields with different spins. Of course, the remaining 14 (= 36 - 22) operator product expansions do not have any singular terms.

Due to the limitation of [28], we compute the operator product expansions for \(N = 4\) via the component approach given in [29]. Note that the \(k\)-dependent terms in \(\overline{\partial} K^m(z) D K^a(w)\) and \(\overline{\partial} J^a(z) D J^b(w)\) (B.1) are the same as \((k + N + 1)\). This was used in (2.14).

**Appendix C**  The operator product expansion (2.16) in the component approach

One writes the operator product expansion (2.16), from (2.18) and (2.19), in terms of components as follows:

\(T(z) D W(w) = \frac{1}{(z-w)} D W(w) + \cdots, \)

\(T(z) \overline{\partial} W(w) = - \frac{1}{(z-w)} \overline{\partial} W(w) + \cdots, \)

\(T(z) [D, \overline{\partial}] W(w) = - \frac{1}{(z-w)^2} 4 W(w) + \cdots, \)

\(D T(z) W(w) = - \frac{1}{(z-w)} D W(w) + \cdots, \)

\(D T(z) \overline{\partial} W(w) = \frac{1}{(z-w)^2} 2 W(w) + \frac{1}{(z-w)} \frac{1}{2} \left[ - [D, \overline{\partial}] W + \partial W \right] (w) + \cdots, \)

\(D T(z) [D, \overline{\partial}] W(w) = - \frac{1}{(z-w)^2} 5 D W(w) - \frac{1}{(z-w)} \partial D W(w) + \cdots, \)

\(\overline{\partial} T(z) W(w) = \frac{1}{(z-w)} \overline{\partial} W(w) + \cdots, \)

27
Appendix D

The coefficient functions in the primary field

in \( \mathcal{N} = 2 \) \( \mathcal{W}_3 \) algebra

In (2.20), we write down the spin 2 current with various contracted terms. The coefficient functions are given by

\[
\begin{align*}
\mathcal{D} T(z)DW(w) &= -\frac{1}{(z-w)^2}2W(w) - \frac{1}{(z-w)^2} \frac{1}{2} [D, \mathcal{D}]W + \partial W \right) (w) + \cdots, \\
\mathcal{D} T(z)[D, \mathcal{D}]W(w) &= -\frac{1}{(z-w)^2}5DW(w) - \frac{1}{(z-w)^2} \partial \mathcal{D}W(w) + \cdots, \\
\left(-\frac{1}{2}\right) [D, \mathcal{D}] T(z)W(w) &= \frac{1}{(z-w)^2}2W(w) + \frac{1}{(z-w)^2} \partial W(w) + \cdots, \\
\left(-\frac{1}{2}\right) [D, \mathcal{D}] T(z)DW(w) &= \frac{1}{(z-w)^2}2DW(w) + \frac{1}{(z-w)^2} \partial DW(w) + \cdots, \\
\left(-\frac{1}{2}\right) [D, \mathcal{D}] T(z)[D, \mathcal{D}]W(w) &= \frac{1}{(z-w)^2}3[D, \mathcal{D}]W(w) + \frac{1}{(z-w)^2} \partial [D, \mathcal{D}]W(w) + \cdots. \quad \text{(C.1)} \\
\end{align*}
\]

Note that the remaining three operator product expansions \( T(z)W(w), DT(z)DW(w), \) and \( \mathcal{D} T(z)\mathcal{D}W(z) \) do not have any singular terms. It is obvious, from the last four equations of (C.1), that the component fields (2.19) are primary with respect to the stress energy tensor.

\[
\begin{align*}
a_1 &= -1, \\ a_2 &= -1, \\ a_3 &= \frac{(15 - 7k - 2k^2)}{(-6 + 10k)}, \\ a_4 &= -3 - k, \\ a_5 &= -3 - k, \\ a_6 &= -\frac{2i\sqrt{3}(-6 + k + k^2)}{(-3 + 5k)}, \\ a_7 &= \frac{2i\sqrt{3}(-6 + k + k^2)}{(-3 + 5k)}, \\ a_8 &= \frac{k(15 + 8k + k^2)}{(-3 + 5k)}, \\ a_9 &= -3 - k, \\ a_{10} &= \frac{3(-6 - 5k + 2k^2 + k^3)}{(6 - 10k)}, \\ a_{11} &= \frac{3(-6 - 5k + 2k^2 + k^3)}{(6 - 10k)}, \\ a_{12} &= \frac{1}{2}(3 + k), \\ a_{13} &= -\frac{1}{2}(3 + k), \\ a_{14} &= \frac{k(15 + 8k + k^2)}{(-6 + 10k)}, \\ a_{15} &= \frac{k(15 + 8k + k^2)}{(-6 + 10k)}, \\ a_{16} &= -\frac{1}{2}(3 + k), \\ a_{17} &= -\frac{1}{2}(3 + k), \\ a_{18} &= -\frac{k(15 + 8k + k^2)}{(-3 + 5k)}, 
\end{align*}
\]
The other way to express the spin 2 current with spins \((2, \frac{5}{2}, \frac{5}{2}, 3)\) is to take the ansatz for \(N = 4\) choice given in \((2.24)\). By requiring the two conditions \((2.15)\) and \((2.16)\), one gets the following coefficient functions which depend on the level \(k\) explicitly as follows:

\[
\begin{align*}
a_{19} &= 3 + k, \\
a_{20} &= \frac{(3 + k) \left( -3 + \left( 5 + 5i\sqrt{3} \right) k + i\sqrt{3}k^2 \right)}{(-6 + 10k)}, \\
a_{21} &= \frac{(3 + k) \left( 3 + 5i \left( i + \sqrt{3} \right) k + i\sqrt{3}k^2 \right)}{(-6 + 10k)}, \\
a_{22} &= \frac{(27 + 6k - k^2)}{(-6 + 10k)}, \\
a_{23} &= \frac{(27 + 6k - k^2)}{(-6 + 10k)}, \\
a_{24} &= \frac{(-27 - 6k + k^2)}{(-3 + 5k)}, \\
a_{25} &= \frac{(3 + 5k)}{(-3 + 5k)}. \\
\end{align*}
\tag{D.1}
\]

Now we present the operator product expansion of spin 2 current and itself \([13]\), at the
linearized level, as follows:

\[
W(Z_1)W(Z_2) = \frac{1}{z_{12}^2} \left[ \frac{c}{2} + \frac{\theta_{12} \theta_{12}}{z_{12}^4} 3T(Z_2) + \frac{\theta_{12}}{z_{12}^4} 3\overline{D}T(Z_2) - \frac{\theta_{12}}{z_{12}^4} 3\overline{D}T(Z_2) + \frac{\theta_{12} \theta_{12}}{z_{12}^4} 3\partial T(Z_2) \right] (Z_2)
\]

\[
\frac{1}{z_{12}^2} \left[ 2\alpha W + \frac{c}{-1+c} [D, D]T \right] (Z_2) + \frac{\theta_{12}}{z_{12}^2} \left[ \alpha \overline{D}W + \frac{(-3+2c)}{(-1+c)} \partial \overline{D}T \right] (Z_2)
\]

\[
\frac{\theta_{12}}{z_{12}^2} \left[ \alpha DW - \frac{(-3+2c)}{(-1+c)} \partial DT \right] (Z_2) + \frac{\theta_{12} \theta_{12}}{z_{12}^2} \left[ \frac{3(-8+c)}{2(-12+5c)} \alpha [D, D]W \right]
\]

\[
\frac{9c(-12+5c)}{4(-1+c)(6+c)(-3+2c)} \partial [D, \overline{D}]T + \frac{3(18-15c+2c^2+2c^3)}{2(-1+c)(6+c)(-3+2c)} \partial^2 T \right] (Z_2)
\]

\[
\frac{1}{z_{12}^2} \left[ \alpha \partial \overline{W} - \frac{c}{2(-1+c)} \partial [D, D]T \right] (Z_2)
\]

\[
\frac{\theta_{12}}{z_{12}^2} \left[ \frac{3(-6+c)(-1+c)}{(3+c)(-12+5c)} \alpha \overline{D}W + \frac{3c(9+3c+2c^2)}{4(-1+c)(6+c)(-3+2c)} \partial^2 \overline{D}T \right] (Z_2)
\]

\[
\frac{\theta_{12}}{z_{12}^2} \left[ \frac{3(-6+c)(-1+c)}{(3+c)(-12+5c)} \alpha DW - \frac{3c(9-3c+c^2)}{2(-1+c)(6+c)(-3+2c)} \partial^2 DT \right] (Z_2)
\]

\[
\frac{\theta_{12} \theta_{12}}{z_{12}^2} \left[ \frac{(-15+c)c}{(3+c)(-12+5c)} \alpha [D, \overline{D}]W + \frac{(-18-3c-2c^2+2c^3)}{2(-1+c)(6+c)(-3+2c)} \partial^3 T \right] (Z_2)
\]

\[
+ \text{(Non-linear singular terms)} + \cdots , \tag{D.3}
\]

where the nonlinear terms are given in the original paper [13]. As we take the large \( c \) limit blindly, the quadratic term has \( \frac{1}{c} \)-behavior and the cubic term has \( \frac{1}{c^2} \)-behavior. For the linear term in (D.3), all the fields in the right hand side behave like as \( c \) independent term except the \([D, \overline{D}]T(w)\) which goes to \( \frac{1}{c} \). In particular, the self-coupling constant appearing in the right hand side of (D.3) can be written as

\[
\alpha^2_{\text{N=2}} = \frac{(c+3)^2(5c-12)^2}{2(15-c)(-1+c)(6+c)(-3+2c)},
\]

which can be generalized to the \( N = 4 \) (2.27) and the arbitrary \( N \) (2.38). It is also useful to compare the above expression (D.3) with the previous results in the component approach [10].

**Appendix E**  The generators and structure constants of \( SU(5) \) in complex basis in the subsection 2.2

One can choose 4 diagonal Cartan generators in the normalization of [44]. Let us describe the 8 generators \( T_m \) where \( m = 1, 2, \cdots, 8 \) and the remaining 4 generators \( T_a \) where \( a = 9, 10, 11, 12 \).
(2.2) as follows:

\[
T_1 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
T_2 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
T_3 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix},
T_4 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix},
T_5 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix},
T_6 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
T_7 = \begin{pmatrix}
\frac{i}{2} + \frac{1}{\sqrt{12}} & 0 & 0 & 0 \\
0 & -\frac{i}{2} + \frac{1}{\sqrt{12}} & 0 & 0 \\
0 & 0 & -\frac{2}{\sqrt{12}} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
T_8 = \begin{pmatrix}
\sqrt{\frac{1}{24} + \frac{1}{\sqrt{40}}} & 0 & 0 & 0 \\
0 & \sqrt{\frac{1}{24} + \frac{1}{\sqrt{40}}} & 0 & 0 \\
0 & 0 & \sqrt{\frac{1}{24} + \frac{1}{\sqrt{40}}} & 0 \\
0 & 0 & 0 & -\frac{3i}{\sqrt{24}} + \frac{1}{\sqrt{40}}
\end{pmatrix},
T_9 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix},
T_{10} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix},
T_{11} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
T_{12} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

Note that \( T_7 = iH_1 + H_2 \) and \( T_8 = iH_3 + H_4 \) where the four Cartan generators are given \( H_1 = \frac{1}{2} \text{diag}(1, -1, 0, 0, 0), H_2 = \frac{1}{\sqrt{12}} \text{diag}(1, 1, -2, 0, 0), H_3 = \frac{1}{\sqrt{24}} \text{diag}(1, 1, 1, -3, 0) \) and \( H_4 = \frac{1}{\sqrt{40}} \text{diag}(1, 1, 1, 1, -4) \) [44]. The conjugated generators \( T_m \) and \( T_{\bar{m}} \) can be obtained from the relations from the Hermiticity \( T_m = T_m^\dagger \) and \( T_{\bar{m}} = T_{\bar{m}}^\dagger \) as we mentioned before. One can easily see that the 14 generators \( T_m, T_{\bar{m}} \) where \( m = 1, 2, \cdots, 7 \) and the \( (T_8 - T_\bar{8}) \) consist of the \( SU(4) \) subgroup generators. The remaining diagonal generator \( (T_8 + T_{\bar{8}}) \) corresponds to the \( U(1) \) in the subgroup \( H \) of the coset model.
Moreover, the 492 nonzero structure constants can be obtained from the commutator relations

\[
[T_m, T_n] = f_{mn}^p T_p + f_{mn}^p T_p, \quad [T_m, T_{\bar{n}}] = f_{mn}^p T_p + f_{mn}^p T_{\bar{p}}, \\
[T_m, T_{\bar{n}}] = f_{mn}^p T_p + f_{mn}^p T_{\bar{p}}, \quad [T_m, T_a] = f_{ma}^b T_b, \\
[T_m, T_{\bar{a}}] = f_{ma}^b T_b, \quad [T_{\bar{m}}, T_a] = f_{\bar{m}a}^b T_b, \quad [T_{\bar{m}}, T_{\bar{a}}] = f_{\bar{m}a}^b T_b.
\]

It is straightforward to obtain the complex basis for general \( N \). This will be necessary to analyze the \( \mathcal{N} = 2 \) \( \mathcal{W}_{N+1} \) algebra. It is nontrivial to write down two \( U(1) \) generators in the subgroup \( H \) explicitly.

### Appendix F  The coefficient functions in the primary field \( (2.24) \) in \( \mathcal{N} = 2 \) \( \mathcal{W}_5 \) algebra

For the spin 2 current \( (2.24) \), the coefficient functions can be obtained from the two conditions (2.15) and (2.16):

\[
\begin{align*}
   b_1 & = \frac{i(-25 + k(23 + 8k))}{6\sqrt{2}(5 + k)} A(k), & b_2 & = \frac{i(5 - 11k)}{6\sqrt{2}} A(k), \\
   b_3 & = \frac{i(9 + k)(-5 + 2k)}{12\sqrt{2}} A(k), & b_4 & = -\frac{ik(9 + k)}{4\sqrt{2}} A(k), \\
   b_5 & = \frac{ik(9 + k)}{4\sqrt{2}} A(k), & b_6 & = \frac{i(-50 + k(73 + 19k))}{12\sqrt{2}} A(k), \\
   b_7 & = \frac{i(-50 + k(73 + 19k))}{12\sqrt{2}} A(k), & b_8 & = \frac{ik(9 + k)}{4\sqrt{2}} A(k), \\
   b_9 & = -\frac{ik(5 + k)(9 + k)}{2\sqrt{2}} A(k), & b_{10} & = \frac{i(5 + k)(-5 + 11k)}{6\sqrt{2}} A(k), \\
   b_{11} & = \frac{5i(-20 + k(-1 + k(4 + k)))}{8\sqrt{2}} A(k), & b_{12} & = -\frac{5i(-20 + k(-1 + k(4 + k)))}{8\sqrt{2}} A(k), \\
   b_{13} & = -\frac{i(5 + k)(-5 + 11k)}{12\sqrt{2}} A(k), & b_{14} & = -\frac{i(50 + k(305 + k(50 + 3k)))}{24\sqrt{2}} A(k), \\
   b_{15} & = -\frac{ik(5 + k)(9 + k)}{8\sqrt{2}} A(k), & b_{16} & = -\frac{ik(5 + k)(9 + k)}{8\sqrt{2}} A(k), \\
   b_{17} & = \frac{i(5 + k)(5 + 11k)}{12\sqrt{2}} A(k), & b_{18} & = -\frac{i(5 + k)(10 + k(5 + 3k))}{24\sqrt{2}} A(k), \\
   b_{19} & = -\frac{i(5 + k)(5 + 11k)}{6\sqrt{2}} A(k), & b_{20} & = \frac{i(5 + k)^2(1 + 3k)}{12\sqrt{2}} A(k), \\
   b_{21} & = \frac{i(5 + k)(-5 + 11k)}{12\sqrt{2}} A(k), & b_{22} & = \frac{i(-20 + k + k^2)}{3\sqrt{2}} A(k),
\end{align*}
\]
\[ b_{23} = \frac{i(-20+k+k^2)}{3\sqrt{2}}A(k), \quad b_{24} = \frac{i(5+k)(35+k)}{60\sqrt{2}}A(k), \]
\[ b_{25} = \frac{i(5+k)(35+k)}{30\sqrt{2}}A(k), \quad b_{26} = \frac{i(5+k)(35+k)}{60\sqrt{2}}A(k), \]
\[ b_{27} = \frac{ik(5+k)(9+k)}{4\sqrt{2}}A(k), \quad \] (F.1)

where the overall coefficient function \( A(k) \) is determined and is given by (2.25). One can also analyze the large \( k \) limit where the \( N \) is fixed along the line of [45]. Then some of the coefficients in (F.1) can survive.

**Appendix G  The operator product expansions in the component approach for (3.3) in \( \mathcal{N} = 2 \mathcal{W}_{N+1} \) algebra**

In practice, it is better to compute the operator product expansions in the component approach (there are some problems in using the package [28] directly) and they are given by

\[
W(z)W(w) = \frac{1}{(z-w)^4} c(N,k) + \frac{1}{(z-w)^2} \left[ 2\alpha(N,k)W - \mathcal{D}[D,\mathcal{D}]T \right](w) + \cdots,
\]
\[
W(z)(DW + \mathcal{D}W)(w) = \frac{1}{(z-w)^3} \left[ DT - \mathcal{D}T \right](w) + \frac{1}{(z-w)^2} \left[ \alpha(N,k)(DW + \mathcal{D}W) + \partial(DT - \mathcal{D}T) \right](w) + \cdots, \quad \] (G.1)

\[
W(z)(DW - \mathcal{D}W)(w) = \frac{1}{(z-w)^3} \left[ DT + \mathcal{D}T \right](w) + \frac{1}{(z-w)^2} \left[ \alpha(N,k)(DW - \mathcal{D}W) + \partial(DT + \mathcal{D}T) \right](w) + \cdots, \quad \] (G.2)

\[
W(z)(-1)^{1/2}[D,\mathcal{D}]W(w) = \frac{1}{(z-w)^4} \left[ \mathcal{D}T \right](w) + \frac{1}{(z-w)^2} \left[ \frac{3}{10} \alpha(N,k)[D,\mathcal{D}]W + 3V \right](w) + \cdots, \quad \] (G.3)
\begin{align}
&+ \frac{1}{(z-w)} \left[ -\frac{1}{10} \alpha(N,k) \partial [D, \overline{D}] W + \partial V \right] (w) + \frac{1}{N} \text{(Non-linear singular terms)} + \cdots, \\
&(DW + \overline{DW})(z) W(w) = \\
&\quad - \frac{1}{(z-w)^3} \left[ DT - \overline{DT} \right] (w) + \frac{1}{(z-w)^2} \left[ \alpha(N,k)(DW + \overline{DW}) - 2\partial(DT - \overline{DT}) \right] (w) \\
&\quad + \frac{1}{(z-w)} \left[ -(DV - \overline{DV}) + \frac{3}{5} \alpha(N,k) \partial(DW + \overline{DW}) - \frac{3}{4} \partial^2(DT - \overline{DT}) \right] (w) \\
&\quad + \frac{1}{N} \text{(Non-linear singular terms)} + \cdots, \\
&(DW - \overline{DW})(z) W(w) = \\
&\quad - \frac{1}{(z-w)^3} \left[ DT + \overline{DT} \right] (w) + \frac{1}{(z-w)^2} \left[ \alpha(N,k)(DW - \overline{DW}) - 2\partial(DT + \overline{DT}) \right] (w) \\
&\quad + \frac{1}{(z-w)} \left[ -(DV + \overline{DV}) + \frac{3}{5} \alpha(N,k) \partial(DW - \overline{DW}) - \frac{3}{4} \partial^2(DT + \overline{DT}) \right] (w) \\
&\quad + \frac{1}{N} \text{(Non-linear singular terms)} + \cdots, \\
&DW(z) DW(w) = + \cdots, \quad \text{(G.4)} \\
&\overline{DW}(z) \overline{DW}(w) = + \cdots, \quad \text{(G.5)}
\end{align}

\begin{align}
&DW(z) \overline{DW}(w) + \overline{DW}(z) DW(w) = \\
&\quad \frac{1}{(z-w)^2} \frac{2c(N,k)}{5} + \frac{1}{(z-w)^3} \left[ 4\alpha(N,k)W - 5|D, \overline{D}|T \right] (w) \\
&\quad + \frac{1}{(z-w)^2} \left[ 2\alpha(N,k) \partial W - \frac{5}{2} \partial[D, \overline{D}]T \right] (w) \\
&\quad + \frac{1}{(z-w)} \left[ \frac{3}{5} \alpha(N,k) \partial^2 W - \frac{3}{4} \partial^2[D, \overline{D}]T - [D, \overline{D}]V \right] (w) \\
&\quad + \frac{1}{N} \text{(Non-linear singular terms)} + \cdots, \quad \text{(G.6)}
\end{align}

\begin{align}
&DW(z) \overline{DW}(w) - \overline{DW}(z) DW(w) = \\
&\quad \frac{1}{(z-w)^4} \left[ 6T(w) + \frac{1}{(z-w)^3} 3\partial T(w) + \frac{1}{(z-w)^2} \left[ \frac{2}{5} \alpha(N,k)|D, \overline{D}|W + \partial^2 T + 6V \right] (w) \\
&\quad + \frac{1}{(z-w)} \left[ \frac{1}{5} \alpha(N,k) \partial[D, \overline{D}]W + \frac{1}{4} \partial^3 T + 3\partial V \right] (w) \\
&\quad + \frac{1}{N} \text{(Non-linear singular terms)} + \cdots, \quad \text{(G.7)}
\end{align}

\begin{align}
&(DW + \overline{DW})(z) (-1) \frac{1}{2} |D, \overline{D}| W(w) = \\
&\quad \frac{1}{(z-w)^4} \left[ \frac{15}{2} DT + \overline{DT} \right] (w) + \frac{1}{(z-w)^3} \left[ \alpha(N,k)(DW - \overline{DW}) + \frac{5}{2} \partial(DT + \overline{DT}) \right] (w) \\
&\quad + \frac{1}{(z-w)^2} \left[ \frac{2}{5} \alpha(N,k) \partial(DW - \overline{DW}) + \frac{5}{8} \partial^2(DT + \overline{DT}) + \frac{7}{2} (DV + \overline{DV}) \right] (w)
\end{align}
\[ + \frac{1}{(z - w)} \left[ \frac{1}{10} \alpha(N, k) \partial^2 (DW - \overline{DW}) + \frac{1}{8} \partial^3 (DT + \overline{DT}) + \frac{3}{2} \partial (DV + \overline{DV}) \right] (w) \]

\[ + \frac{1}{N} (\text{Non-linear singular terms}) + \cdots, \]

\[ (DW - \overline{DW}) (z) \left( -1 \right) \frac{1}{2} [D, \overline{D}] W (w) = \]

\[ \frac{1}{(z - w)^4} \left[ \frac{15}{2} DT - \partial DT \right] (w) + \frac{1}{(z - w)^3} \left[ \alpha(N, k)(DW + \overline{DW}) + \frac{5}{2} \partial (DT - \overline{DT}) \right] (w) \]

\[ + \frac{1}{(z - w)^2} \left[ \frac{2}{5} \alpha(N, k) \partial (DW + \overline{DW}) + \frac{5}{8} \partial^2 (DT - \overline{DT}) + \frac{7}{2} (DV - \overline{DV}) \right] (w) \]

\[ + \frac{1}{(z - w)} \left[ \frac{1}{10} \alpha(N, k) \partial^2 (DW + \overline{DW}) + \frac{1}{8} \partial^3 (DT - \overline{DT}) + \frac{3}{2} \partial (DV - \overline{DV}) \right] (w) \]

\[ + \frac{1}{N} (\text{Non-linear singular terms}) + \cdots, \]

\[ (-1) \frac{1}{2} [D, \overline{D}] W (z) W (w) = \]

\[ \frac{1}{(z - w)^4} \left[ \frac{3}{2} T (w) + \frac{1}{(z - w)^3} 3 \partial T (w) + \frac{1}{(z - w)^2} \left[ \frac{3}{10} \alpha(N, k) [D, \overline{D}] W + 3 V + \frac{3}{2} \partial^2 T \right] \right] (w) \]

\[ + \frac{1}{(z - w)} \left[ - \frac{1}{5} \alpha(N, k) \partial [D, \overline{D}] W + 2 \partial V + \frac{1}{2} \partial^3 T \right] (w) \]

\[ + \frac{1}{N} (\text{Non-linear singular terms}) + \cdots, \]

\[ (-1) \frac{1}{2} [D, \overline{D}] W (z) (DW + \overline{DW}) (w) = \]

\[ \frac{1}{(z - w)^4} \left[ \frac{15}{2} DT + \partial DT \right] (w) + \frac{1}{(z - w)^3} \left[ - \alpha(N, k)(DW - \overline{DW}) + 5 \partial (DT + \overline{DT}) \right] (w) \]

\[ + \frac{1}{(z - w)^2} \left[ - \frac{3}{5} \alpha(N, k) \partial (DW - \overline{DW}) + \frac{15}{8} \partial^2 (DT + \overline{DT}) + \frac{7}{2} (DV + \overline{DV}) \right] (w) \]

\[ + \frac{1}{(z - w)} \left[ - \frac{1}{5} \alpha(N, k) \partial^2 (DW - \overline{DW}) + \frac{1}{2} \partial^3 (DT + \overline{DT}) + 2 \partial (DV + \overline{DV}) \right] (w) \]

\[ + \frac{1}{N} (\text{Non-linear singular terms}) + \cdots, \]

\[ (-1) \frac{1}{2} [D, \overline{D}] W (z) (DW - \overline{DW}) (w) = \]

\[ \frac{1}{(z - w)^4} \left[ \frac{15}{2} DT - \partial DT \right] (w) + \frac{1}{(z - w)^3} \left[ - \alpha(N, k)(DW + \overline{DW}) + 5 \partial (DT - \overline{DT}) \right] (w) \]

\[ + \frac{1}{(z - w)^2} \left[ - \frac{3}{5} \alpha(N, k) \partial (DW + \overline{DW}) + \frac{15}{8} \partial^2 (DT - \overline{DT}) + \frac{7}{2} (DV - \overline{DV}) \right] (w) \]

\[ + \frac{1}{(z - w)} \left[ - \frac{1}{5} \alpha(N, k) \partial^2 (DW + \overline{DW}) + \frac{1}{2} \partial^3 (DT - \overline{DT}) + 2 \partial (DV - \overline{DV}) \right] (w) \]

\[ + \frac{1}{N} (\text{Non-linear singular terms}) + \cdots, \]  

(G.9)
\((-1)\frac{1}{2} [D, \overline{D}] W(z) (-1)\frac{1}{2} [D, \overline{D}] W(w) =
\frac{1}{(z-w)^3} 5c(N,k) + \frac{1}{(z-w)^4} \left[ 3\alpha(N,k) W - \frac{15}{2} [D, \overline{D}] T \right] (w)
+ \frac{1}{(z-w)^3} \left[ \frac{9}{20} \alpha(N,k) \partial W - \frac{9}{8} \partial^2 [D, \overline{D}] T - 2[D, \overline{D}] V \right] (w)
+ \frac{1}{(z-w)} \left[ \frac{1}{10} \alpha(N,k) \partial^3 W - \frac{1}{4} \partial^3 [D, \overline{D}] T - \partial [D, \overline{D}] V \right] (w)
+ \frac{1}{N} \text{(Non-linear singular terms)} + \cdots , \tag{G.10}

where we write down the operator product expansions for the $DW(z)$ and $\overline{D}W(z)$ by taking the sum or difference between them because we want to compare them with the asymptotic symmetry algebra in the $AdS_3$ bulk theory directly. One can also obtain the separate operator product expansions by adding or subtracting the corresponding equations without any difficulty.

In section 4, we only presented some of the large $(N,k)$ limit for the full algebra. Here we complete them by repeating those analysis as follows. For the operator product expansion between the spin 2 and spin $\frac{5}{2}$, one has, from (G.2),

\[
W_{2,HP}^{-}(z)G_{3,HP}^{-}(w) = \frac{1}{(z-w)^3} 3\beta(N,k)^2 G_{2,HP}^{-}(w)
+ \frac{1}{(z-w)^2} \beta(N,k)^2 \left[ \frac{\alpha(N,k)}{\beta(N,k)} G_{3,HP}^{-} + \partial G_{2,HP}^{+} \right] (w)
+ \frac{1}{(z-w)^2} \beta(N,k)^2 \left[ \frac{2}{5} \frac{\alpha(N,k)}{\beta(N,k)} \partial G_{3,HP}^{-} + \frac{1}{4} \partial^2 G_{2,HP}^{+} + \frac{1}{\beta(N,k)} G_{4,HP}^{+} \right] (w)
+ \frac{1}{N} \text{(Non-linear singular terms)} + \cdots 
\]

\[
\rightarrow \frac{1}{(z-w)^3} (-1)\frac{2}{3} (2\lambda_{HP} + 1)(\lambda_{HP} - 1) G_{2,HP}^{+}(w)
+ \frac{1}{(z-2)^2} \left[ \frac{1}{3} (1 - 4\lambda_{HP}) G_{3,HP}^{-} - \frac{2}{9} (2\lambda_{HP} + 1)(\lambda_{HP} - 1) \partial G_{2,HP}^{+} \right] (w)
+ \frac{1}{(z-w)} \left[ \frac{2}{15} (1 - 4\lambda_{HP}) \partial G_{3,HP}^{-} - \frac{1}{18} (2\lambda_{HP} + 1)(\lambda_{HP} - 1) \partial^2 G_{2,HP}^{+} + G_{4,HP}^{+} \right] (w)
+ \frac{1}{N} \text{(Non-linear singular terms)} + \cdots . \tag{G.11}
\]

This corresponds to the equation of (3.48) of [5].\footnote{The commutator is given by $[W_{2,HP}^{-},G_{3,HP}^{-}] = \beta(N,k)^2 \frac{3}{16} (-9 + 12m^2 - 8mn + 4n^2) (G_{2,HP}^{+})_{m+n} + \beta(N,k)^2 \frac{3m(12m^2 - 8mn + 4n^2)}{16} (3m - 2n)(G_{2,HP}^{+})_{m+n} + (G_{4,HP}^{+})_{m+n} + \text{Nonlinear terms.}$} We also use the normalizations like as
Similarly, one has the following operator product expansion between the spin 2 current and other spin $\frac{5}{2}$ current, from (G.3),

$$W_{2,HP}^{+}(z)G_{3,HP}^{-}(w) = \frac{1}{(z-w)^{3}} 3\beta(N,k)^{2}G_{2,HP}^{-}(w)$$

$$+ \frac{1}{(z-w)^{2}} \beta(N,k)^{2} \left[ \frac{\alpha(N,k)}{\beta(N,k)}G_{3,HP}^{+} + \partial G_{2,HP}^{-} \right] (w)$$

$$+ \frac{1}{(z-w)} \beta(N,k)^{2} \left[ \frac{2\alpha(N,k)}{5\beta(N,k)} \partial G_{3,HP}^{+} + \frac{1}{4} \partial^{2} G_{2,HP}^{-} + \frac{1}{\beta(N,k)^{2}} G_{4,HP}^{-} \right] (w)$$

$$+ \frac{1}{N} \text{(Non-linear singular terms)} + \cdots$$

$$\rightarrow \frac{1}{(z-w)^{3}} (-1)^{2} \frac{2}{3} (2\lambda_{HP} + 1)(\lambda_{HP} - 1)G_{2,HP}^{-}(w)$$

$$+ \frac{1}{(z-w)^{2}} \beta(N,k)^{2} \left[ \frac{1}{12} (1 - 4\lambda_{HP})G_{3,HP}^{+} - \frac{2}{9} (2\lambda_{HP} + 1)(\lambda_{HP} - 1)\partial G_{2,HP}^{-} \right] (w)$$

$$+ \frac{1}{(z-w)} \beta(N,k)^{2} \left[ \frac{2}{15} (1 - 4\lambda_{HP})\partial G_{3,HP}^{+} - \frac{1}{18} (2\lambda_{HP} + 1)(\lambda_{HP} - 1)\partial^{2} G_{2,HP}^{-} + G_{4,HP}^{-} \right] (w)$$

$$+ \frac{1}{N} \text{(Non-linear singular terms)} + \cdots,$$  \hspace{1cm} \text{(G.12)}

which agrees with the equation (3.49) of [5].

One has the following operator product expansion between the spin 3 current and the spin $\frac{5}{2}$ current:

$$W_{3,HP}^{+}(z)G_{3,HP}^{-}(w) = \frac{1}{(z-w)^{4}} \frac{15}{2} \beta(N,k)^{2} G_{2,HP}^{-}(w)$$

$$+ \frac{1}{(z-w)^{3}} \beta(N,k)^{2} \left[ - \frac{\alpha(N,k)}{\beta(N,k)} G_{3,HP}^{+} + 5\partial G_{2,HP}^{-} \right] (w)$$

$$+ \frac{1}{(z-w)^{2}} \beta(N,k)^{2} \left[ - \frac{3\alpha(N,k)}{5\beta(N,k)} \partial G_{3,HP}^{+} + \frac{15}{8} \partial^{2} G_{2,HP}^{-} + \frac{7}{2\beta(N,k)^{2}} G_{4,HP}^{-} \right] (w)$$

$$+ \frac{1}{(z-w)} \beta(N,k)^{2} \left[ - \frac{1}{5} \alpha(N,k) \partial^{2} G_{3,HP}^{+} + \frac{1}{2} \partial^{3} G_{2,HP}^{-} + \frac{1}{2\beta(N,k)^{2}} \partial G_{4,HP}^{-} \right] (w)$$

$$+ \frac{1}{N} \text{(Non-linear singular terms)} + \cdots$$

$$\rightarrow \frac{1}{(z-w)^{4}} (-1)^{2} \frac{5}{3} (2\lambda_{HP} + 1)(\lambda_{HP} - 1)G_{2,HP}^{-}(w)$$

$$+ \frac{1}{(z-w)^{3}} \beta(N,k)^{2} \left[ - \frac{1}{3} (1 - 4\lambda_{HP}) G_{3,HP}^{+} - \frac{10}{9} (2\lambda_{HP} + 1)(\lambda_{HP} - 1) \partial G_{2,HP}^{-} \right]$$

$$+ \frac{1}{(z-w)^{2}} \beta(N,k)^{2} \left[ - \frac{1}{5} (1 - 4\lambda_{HP}) \partial G_{3,HP}^{+} - \frac{5}{12} (2\lambda_{HP} + 1)(\lambda_{HP} - 1) \partial^{2} G_{2,HP}^{-} + \frac{7}{2} G_{4,HP}^{-} \right]$$

\footnote{This is equivalent to \((W_{2,HP}^{-})_{m,n}(G_{3,HP}^{+})_{m,n}\) = \(\beta(N,k)^{2} \frac{1}{10} \left( -9 + 12m^{2} - 8mn + 4n^{2} \right) (G_{2,HP}^{-})_{m+n} + \beta(N,k)^{2} \frac{\alpha(N,k)}{3m} (3m - 2n)(G_{4,HP}^{+})_{m+n} + (G_{4,HP}^{-})_{m+n} + \text{Nonlinear terms.}}
leads to
which will lead to the equation (3.50) of [5].

Further, one obtains, from (G.9) and the normalization (4.7),

\[ W_{3,HP}^+(z) G_{3,HP}^+(w) = \frac{1}{(z-w)^4} \left[ \frac{15}{2} \beta(N,k)^2 G_{2,HP}^+(w) \right. \]
\[ + \frac{1}{(z-w)^3} \beta(N,k)^2 \left[ \frac{\alpha(N,k)}{\beta(N,k)} G_{3,HP}^- + 5 \partial G_{2,HP}^+ \right] (w) \]
\[ + \frac{1}{(z-w)^2} \beta(N,k)^2 \left[ \frac{3 \alpha(N,k)}{5 \beta(N,k)} \partial G_{3,HP}^- + \frac{15}{8} \partial^2 G_{2,HP}^+ + \frac{7}{2} \frac{1}{\beta(N,k)^2} G_{4,HP}^+ \right] (w) \]
\[ + \frac{1}{(z-w)} \beta(N,k)^2 \left[ \frac{1}{5 \beta(N,k)} \partial^2 G_{3,HP}^- + \frac{2}{3} \partial^3 G_{2,HP}^+ + 2 \frac{1}{\beta(N,k)^2} \partial G_{4,HP}^+ \right] (w) \]
\[ + \frac{1}{N} \text{(Non-linear singular terms)} + \cdots \]

\[ \rightarrow \frac{1}{(z-w)^4} (-1) \frac{5}{3} (2\lambda_{HP} + 1) (\lambda_{HP} - 1) G_{2,HP}^+(w) \]
\[ + \frac{1}{(z-w)^3} \left[ - \frac{1}{3} (1 - 4\lambda_{HP}) G_{3,HP}^- + \frac{10}{9} (2\lambda_{HP} + 1) (\lambda_{HP} - 1) \partial G_{2,HP}^+ \right] \]
\[ + \frac{1}{(z-w)^2} \left[ - \frac{1}{5} (1 - 4\lambda_{HP}) \partial G_{3,HP}^- + \frac{5}{12} (2\lambda_{HP} + 1) (\lambda_{HP} - 1) \partial^2 G_{2,HP}^+ + \frac{7}{2} G_{4,HP}^+ \right] \]
\[ + \frac{1}{(z-w)} \left[ - \frac{1}{15} (1 - 4\lambda_{HP}) \partial^2 G_{3,HP}^- + \frac{1}{9} (2\lambda_{HP} + 1) (\lambda_{HP} - 1) \partial^3 G_{2,HP}^+ + 2 \partial G_{4,HP}^+ \right] \]
\[ + \frac{1}{N} \text{(Non-linear singular terms)} + \cdots \]

which will lead to the equation (3.50) of [5].

The operator product expansion of spin \( \frac{5}{2} \) and itself, from (G.7) with (G.5) and (G.6), leads to

\[ G_{3,HP}^-(z) G_{3,HP}^-(w) = \frac{1}{(z-w)^5} 2c(N,k) \beta(N,k)^2 \]
\[ + \frac{1}{(z-w)^4} \beta(N,k)^2 \left[ \frac{2}{3} \alpha(N,k) W_{2,HP}^- + 10 W_{2,HP}^+ \right] (w) \]
\[ + \frac{1}{(z-w)^3} \beta(N,k)^2 \left[ \frac{2}{3} \alpha(N,k) \partial W_{2,HP}^- + 5 \partial W_{2,HP}^+ \right] (w) \]

\(^{19}\)That is, \( [W_{3,HP}^+, G_{3,HP}^-] = \frac{1}{32} \left( -19m + 4n^3 + 36n - 8m^2n + 12mn^2 - 16n^3 \right) \beta(N,k)^2 (G_{3,HP}^-)_{m+n} \]
\[ - \frac{1}{32} \left( -5 + 2m^2 - 4mn + 4n^2 \right) \alpha(N,k) \beta(N,k) (G_{3,HP}^+)_{m+n} + \frac{3}{2} (m - 2n) \beta(N,k)^2 (G_{4,HP}^-)_{m+n} + \text{Nonlinear terms} \]

\(^{20}\)That is, \( [W_{3,HP}^+, G_{3,HP}^+] = \frac{1}{32} \left( -19m + 4n^3 + 36n - 8m^2n + 12mn^2 - 16n^3 \right) \beta(N,k)^2 (G_{3,HP}^+)_{m+n} \]
\[ - \frac{1}{32} \left( -5 + 2m^2 - 4mn + 4n^2 \right) \alpha(N,k) \beta(N,k) (G_{3,HP}^-)_{m+n} + \frac{3}{2} (m - 2n) \beta(N,k)^2 (G_{4,HP}^+)_{m+n} + \text{Nonlinear terms} \]
which agrees with the equation (3.51) of [5]. We also use the similar normalization for [D, ¯D]V(w) as (4.7). Furthermore, one has, from (G.8) with (G.5) and (G.6),

\[
G_{3, HP}^-(z)G_{3, HP}^+(w) = -\frac{1}{(z-w)^4}6\beta(N, k)^2W_{1, HP}^- (w) - \frac{1}{(z-w)^3}3\beta(N, k)^2\partial W_{1, HP}^- (w) \\
-\frac{1}{(z-w)^2}\beta(N, k)^2 \left[ -\frac{4\alpha(N, k)}{5\beta(N, k)}W_{3, HP}^+ + \partial^2 W_{1, HP}^- + \frac{6}{\beta(N, k)^2}W_{3, HP}^- \right] (w) \\
-\frac{1}{(z-w)^2}\beta(N, k)^2 \left[ -\frac{2\alpha(N, k)}{5\beta(N, k)}\partial W_{3, HP}^+ + \frac{1}{4}\partial^3 W_{1, HP}^- + \frac{3}{\beta(N, k)^2}\partial W_{3, HP}^- \right] (w) \\
+\frac{1}{N}(\text{Non-linear singular terms}) + \cdots, 
\]

which coincides with the equation (3.52) of [5] and we also use the normalization like (4.7).
Similar analysis can be done for the operator product expansions (G.1)-(G.10) we did not consider.

References

[1] M. R. Gaberdiel and R. Gopakumar, “An $AdS_3$ Dual for Minimal Model CFTs,” Phys. Rev. D 83, 066007 (2011) [arXiv:1011.2986 [hep-th]].

[2] M. R. Gaberdiel and R. Gopakumar, “Triality in Minimal Model Holography,” arXiv:1205.2472 [hep-th].

[3] T. Creutzig, Y. Hikida and P. B. Ronne, “Higher spin $AdS_3$ supergravity and its dual CFT,” JHEP 1202, 109 (2012) [arXiv:1111.2139 [hep-th]].

[4] C. Candu and M. R. Gaberdiel, “Supersymmetric holography on $AdS_3$,” arXiv:1203.1939 [hep-th].

[5] K. Hanaki and C. Peng, “Symmetries of Holographic Super-Minimal Models,” arXiv:1203.5768 [hep-th].

[6] M. R. Gaberdiel and T. Hartman, “Symmetries of Holographic Minimal Models,” JHEP 1105, 031 (2011) [arXiv:1101.2910 [hep-th]].

[7] Y. Kazama and H. Suzuki, “New N=2 Superconformal Field Theories and Superstring Compactification,” Nucl. Phys. B 321, 232 (1989).

[8] Y. Kazama and H. Suzuki, “Characterization of N=2 Superconformal Models Generated by Coset Space Method,” Phys. Lett. B 216, 112 (1989).

[9] C. M. Hull and B. J. Spence, “N=2 Current Algebra And Coset Models,” Phys. Lett. B 241, 357 (1990).

[10] L. J. Romans, “The N=2 super W(3) algebra,” Nucl. Phys. B 369, 403 (1992).

[11] S. Odake, “Superconformal Algebras and Their Extensions,” Soryushiron Kenkyu(Kyoto) 78 (1989) 201 (in Japanese).

\[22\] There is also an anticommutator relation for the fermionic modes as follows: \( \{\langle G_{3,HP}^- \rangle_m, \langle G_{3,HP}^+ \rangle_n \} = -\frac{1}{8}(24 + 39m + 24m^2 + 6m^3 + 49n + 48mn + 10m^2n + 24n^2 + 14mn^2 + 2n^3)\beta(N,k)^2(W_{1,HP}^-)_{m+n} \). Nonlinear terms.
[12] M. Rocek, C. Ahn, K. Schoutens and A. Sevrin, “Superspace WZW models and black holes,” hep-th/9110035.

[13] C. Ahn, “Explicit construction of N=2 W(3) current in the N=2 coset SU(3) / SU(2) x U(1) model,” Phys. Lett. B 348, 77 (1995) [hep-th/9410170].

[14] C. Ahn, “Free superfield realization of N=2 quantum super W(3) algebra,” Mod. Phys. Lett. A 9, 271 (1994) [hep-th/9304038].

[15] R. Blumenhagen and A. Wisskirchen, “Extension of the N=2 virasoro algebra by two primary fields of dimension 2 and 3,” Phys. Lett. B 343, 168 (1995) [hep-th/9408082].

[16] K. Hornfeck, “The Minimal supersymmetric extension of WA(n-1),” Phys. Lett. B 275, 355 (1992).

[17] C. Ahn, “The Coset Spin-4 Casimir Operator and Its Three-Point Functions with Scalars,” JHEP 1202, 027 (2012) [arXiv:1111.0091 [hep-th]].

[18] F. A. Bais, P. Bouwknegt, M. Surridge and K. Schoutens, “Extensions of the Virasoro Algebra Constructed from Kac-Moody Algebras Using Higher Order Casimir Invariants,” Nucl. Phys. B 304, 348 (1988).

[19] F. A. Bais, P. Bouwknegt, M. Surridge and K. Schoutens, “Coset Construction for Extended Virasoro Algebras,” Nucl. Phys. B 304, 371 (1988).

[20] M. A. Vasiliev, “Holography, Unfolding and Higher-Spin Theory,” arXiv:1203.5554 [hep-th].

[21] C. -M. Chang and X. Yin, “Correlators in WN Minimal Model Revisited,” arXiv:1112.5459 [hep-th].

[22] M. Ammon, P. Kraus and E. Perlmutter, “Scalar fields and three-point functions in D=3 higher spin gravity,” arXiv:1111.3926 [hep-th].

[23] A. Castro, R. Gopakumar, M. Gutperle and J. Raeymaekers, “Conical Defects in Higher Spin Theories,” JHEP 1202, 096 (2012) [arXiv:1111.3381 [hep-th]].

[24] K. Papadodimas and S. Raju, “Correlation Functions in Holographic Minimal Models,” Nucl. Phys. B 856, 607 (2012) [arXiv:1108.3077 [hep-th]].
[25] A. Campoleoni, S. Fredenhagen and S. Pfenninger, “Asymptotic W-symmetries in three-dimensional higher-spin gauge theories,” JHEP 1109, 113 (2011) [arXiv:1107.0290 [hep-th]].

[26] C. -M. Chang and X. Yin, “Higher Spin Gravity with Matter in $AdS_3$ and Its CFT Dual,” arXiv:1106.2580 [hep-th].

[27] M. R. Gaberdiel, R. Gopakumar, T. Hartman and S. Raju, “Partition Functions of Holographic Minimal Models,” JHEP 1108, 077 (2011) [arXiv:1106.1897 [hep-th]].

[28] S. Krivonos and K. Thielemans, “A Mathematica package for computing N=2 superfield operator product expansions,” Class. Quant. Grav. 13, 2899 (1996) [hep-th/9512029].

[29] K. Thielemans, “A Mathematica package for computing operator product expansions,” Int. J. Mod. Phys. C 2, 787 (1991).

[30] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, “Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory,” Nucl. Phys. B 241, 333 (1984).

[31] V. S. Dotsenko, “Lectures On Conformal Field Theory;” IN *KYOTO 1986, PROCEEDINGS, CONFORMAL FIELD THEORY AND SOLVABLE LATTICE MODELS* 123-170. KYOTO UNIV. - RIMS-559 (86,REC.JAN.87) 84 P. (SEE CONFERENCE INDEX)

[32] K. Ito, “Free field realization of N=2 superW(3) algebra,” Phys. Lett. B 304, 271 (1993) [hep-th/9302039].

[33] C. Ahn, “Extended conformal symmetry in two-dimensional quantum field theory,” UMI-93-10057.

[34] P. Bouwknegt and K. Schoutens, “W symmetry in conformal field theory,” Phys. Rept. 223, 183 (1993) [hep-th/9210010].

[35] M. R. Gaberdiel, T. Hartman and K. Jin, “Higher Spin Black Holes from CFT,” JHEP 1204, 103 (2012) [arXiv:1203.0015 [hep-th]].

[36] P. Bowcock and G. M. T. Watts, “On the classification of quantum W algebras,” Nucl. Phys. B 379, 63 (1992) [hep-th/9111062].

[37] C. Ahn, “The Primary Spin-4 Casimir Operators in the Holographic SO(N) Coset Minimal Models,” JHEP 1205, 040 (2012) [arXiv:1202.0074 [hep-th]].
[38] C. Ahn, “The Large N ’t Hooft Limit of Coset Minimal Models,” JHEP 1110, 125 (2011) [arXiv:1106.0351 [hep-th]].

[39] M. R. Gaberdiel and C. Vollenweider, “Minimal Model Holography for SO(2N),” JHEP 1108, 104 (2011) [arXiv:1106.2634 [hep-th]].

[40] M. Henneaux, G. Lucena Gomez, J. Park and S. -J. Rey, “Super- W(infinity) Asymptotic Symmetry of Higher-Spin AdS$_3$ Supergravity,” arXiv:1203.5152 [hep-th].

[41] C. Ahn, K. Schoutens and A. Sevrin, “The full structure of the super W(3) algebra,” Int. J. Mod. Phys. A 6, 3467 (1991).

[42] E. Bergshoeff, M. A. Vasiliev and B. de Wit, “The SuperW(infinity) (lambda) algebra,” Phys. Lett. B 256, 199 (1991).

[43] E. Bergshoeff, B. de Wit and M. A. Vasiliev, “The Structure of the superW(infinity) (lambda) algebra,” Nucl. Phys. B 366, 315 (1991).

[44] H. Georgi, “Lie Algebras In Particle Physics. From Isospin To Unified Theories,”, 2nd Edition(1999), Front. Phys. 54, 1 (1982).

[45] M. R. Gaberdiel and P. Suchanek, “Limits of Minimal Models and Continuous Orbifolds,” JHEP 1203, 104 (2012) [arXiv:1112.1708 [hep-th]].