ZEROES OF THE SPECTRAL DENSITY OF THE
SCHRÖDINGER OPERATOR WITH THE
SLOWLY DECAYING WIGNER–VON NEUMANN POTENTIAL

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Abstract. We consider the Schrödinger operator \( L_\alpha \) on the half-line with a
periodic background potential and a perturbation which consists of two parts:
a summable potential and the slowly decaying Wigner–von Neumann potential
\( \frac{c \sin(2 \omega x + \delta)}{x^\gamma} \), where \( \gamma \in (\frac{1}{2}, 1) \). The continuous spectrum of this operator has
the same band-gap structure as the continuous spectrum of the unperturbed
periodic operator. In every band there exist two points, called critical, where
the eigenfunction equation has square summable solutions. Every critical point \( \nu_{cr} \)
is an eigenvalue of the operator \( L_\alpha \) for some value of the boundary param-
eter \( \alpha = \alpha_{cr} \), specific to that particular point. We prove that for \( \alpha \neq \alpha_{cr} \)
the spectral density of the operator \( L_\alpha \) has a zero of the exponential type at \( \nu_{cr} \).

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Wigner–von Neumann potential \( \frac{c \sin(2\omega x)}{x} \) gives probably the simplest example of an eigenvalue embedded into the continuous spectrum of the Schrödinger operator on the half-line. The eigenvalue can appear at the point \( \omega^2 \) which is called critical or resonance point, and appears for only one boundary condition and only if \(|c| > 2|\omega|\).

At this point a square summable solution of the eigenfunction equation exists, and for one value of the boundary parameter it satisfies the boundary condition. This situation is very unstable: the eigenvalue disappears if one slightly changes the boundary condition or adds a summable perturbation to the potential. This eigenvalue can be related to resonances, which are poles on the unphysical sheet of the analytic continuation of the Green’s function, but may or may not exist. It is meaningful to consider objects that are stable under small perturbations. One can study properties of the Weyl’s \( m \)-function or the spectral density \( \rho' \) of the operator (which is the derivative of the spectral function [24]).

In the present paper we study properties of the spectral density of the Schrödinger operator \( \mathcal{L}_\alpha \) defined by the differential expression

\[
L := -\frac{d^2}{dx^2} + q(x) + q_{WN}(x, \gamma) + q_1(x)
\]

on the positive half-line with the boundary condition

\[
\psi(0) \cos \alpha - \psi'(0) \sin \alpha = 0,
\]

which means that \( \mathcal{L}_\alpha \) acts as

\[
\mathcal{L}_\alpha : \psi \mapsto L\psi
\]

on the domain

\[
\text{dom} \mathcal{L}_\alpha = \{ \psi \in L^2(\mathbb{R}_+) \cap H^2_{\text{loc}}(\mathbb{R}_+) : L\psi \in L^2(\mathbb{R}_+), \psi(0) \cos \alpha - \psi'(0) \sin \alpha = 0 \}
\]

in the Hilbert space \( L^2(\mathbb{R}_+) \). We assume that

\[
q \text{ is periodic with the period } a \quad q \in L^1(0, a),
\]

\[
c, \omega, \delta \in \mathbb{R}, \gamma \in \left( \frac{1}{2}, 1 \right), \quad \frac{2\omega}{\pi} \notin \mathbb{Z},
\]

\[
q_{WN}(x, \gamma) := \begin{cases} 
\frac{c \sin(2\omega x + \delta)}{x}, & \text{if } \gamma \in \left( \frac{1}{2}, 1 \right), \\
\frac{c \sin(2\omega x + \delta)}{x + \frac{1}{2}}, & \text{if } \gamma = 1,
\end{cases}
\]

\[
q_1 \in L^1(\mathbb{R}_+),
\]

\[
\alpha \in [0, \pi).
\]

Under these assumptions the operator \( \mathcal{L}_\alpha \) is self-adjoint [23].

As it was shown by Kurasov and Naboko in [22], the absolutely continuous spectrum of the operator given by the expression \( L \) on the whole real line coincides with the spectrum of the corresponding unperturbed periodic operator on the whole line,

\[
\mathcal{L}_{\text{per}} : \psi \mapsto -\psi'' + q\psi,
\]

\[
\text{dom} \mathcal{L}_{\text{per}} = \{ \psi \in L^2(\mathbb{R}) \cap H^2_{\text{loc}}(\mathbb{R}) : (-\psi'' + q\psi) \in L^2(\mathbb{R}) \},
\]

which means it has a band-gap structure:

\[
\sigma(\mathcal{L}_{\text{per}}) = \bigcup_{j=0}^{\infty} ([\lambda_{2j}, \mu_{2j}] \cup [\mu_{2j+1}, \lambda_{2j+1}]),
\]
where

\[ \lambda_0 < \mu_0 \leq \mu_1 < \lambda_1 \leq \lambda_2 < \mu_2 \leq \mu_3 < \lambda_3 \leq \lambda_4 < \ldots \]

In turn, the absolutely continuous spectrum of \( L_\alpha \) coincides as a set with the spectrum of \( L_{\text{per}} \), although the latter has multiplicity two, whereas the former is simple. Moreover, Kurasov and Naboko in [22] showed that in every band \([\lambda_j, \mu_j]\) or \([\mu_j, \lambda_j]\) there exist two critical points \( \nu_{j+} \) and \( \nu_{j-} \). The type of asymptotics of generalised eigenvectors (solutions of the eigenfunction equation) at these points is different from that type in other points of the absolutely continuous spectrum. At each of the critical points there exists a subordinate solution and hence each of these points can be an eigenvalue of the operator \( L_\alpha \) as long as this solution belongs to \( L^2(\mathbb{R}_+) \) and satisfies the boundary condition. Locations of the points \( \nu_{j\pm} \) are determined by the conditions [22]

\[ k(\nu_{j+}) = \pi \left(j + 1 - \left\{ \frac{a\omega}{\pi} \right\} \right), \quad k(\nu_{j-}) = \pi \left(j + \left\{ \frac{a\omega}{\pi} \right\} \right), \quad j \geq 0, \]

where \( k \) is the quasi-momentum of the periodic operator \( L_{\text{per}} \), and \( \{ \cdot \} \) is the standard fractional part function. The condition \( \frac{a\omega}{\pi} \notin \mathbb{Z} \) ensures that critical points do not coincide with the endpoints of bands and with each other.

In [29] we have studied the asymptotic behaviour of the spectral density of the operator \( L_\alpha \) near critical points for the case \( \gamma = 1 \). In this paper we consider \( \gamma \in (\frac{1}{2}, 1) \), and this case differs significantly. Let us state both results right away.

Let \( \psi_+(x, \lambda) \) and \( \psi_-(x, \lambda) \) be the Bloch solutions of the periodic equation \(-\psi''(x) + q(x)\psi(x) = \lambda\psi(x)\) chosen so that they are complex conjugate to each other whenever \( \lambda \) is inside a spectral band. Let \( \varphi_\alpha(x, \lambda) \) be the solution of the Cauchy problem

\[ L\varphi_\alpha = \lambda \varphi_\alpha, \quad \varphi_\alpha(0, \lambda) = \sin \alpha, \quad \varphi'_\alpha(0, \lambda) = \cos \alpha. \]

Denote by \( W\{\psi_+, \psi_-\}(\lambda) := \psi'_+(x, \lambda)\psi_-(x, \lambda) - \psi_+(x, \lambda)\psi'_-(x, \lambda) \) the Wronskian of two Bloch solutions. In [22] Kurasov and Naboko have found the following asymptotics of generalised eigenvectors at critical points.

**Proposition 1.1** ([22]). Let the operator \( L_\alpha \) be defined by (1.3) and (1.4) where \( L \) is given by (1.1), and let the conditions (1.5) hold. Let \( \nu_{j\pm} \) and \( \varphi_\alpha \) be defined by (1.9) and (1.10), respectively. For every critical point \( \nu_{cr} \in \{\nu_{j+}, \nu_{j-}, j \geq 0\} \) there exist \( \alpha_{cr}(c, \omega, \delta, \gamma) \in [0, \pi) \), \( \beta_{cr}(c, \omega, \delta, \gamma) \geq 0 \), \( \phi_{cr}(c, \omega, \delta, \gamma) \in \mathbb{R} \) and \( d_{cr\pm}(c, \omega, \delta, \gamma) \in \mathbb{R}\setminus\{0\} \) such that, as \( x \to +\infty \),

\[ \varphi_{\alpha_{cr}}(x, \nu_{cr}) = d_{cr-} x^{-\beta_{cr}} \times \left( ie^{\frac{i}{2} \phi_{cr}} \psi_-(x, \nu_{cr}) - ie^{-\frac{i}{2} \phi_{cr}} \psi_+(x, \nu_{cr}) + o(1) \right), \quad \text{if} \; \gamma = 1, \]

\[ \varphi_{\alpha_{cr}}(x, \nu_{cr}) = d_{cr-} \exp \left( -\frac{\beta_{cr} x^{1-\gamma}}{1-\gamma} \right) \times \left( ie^{\frac{i}{2} \phi_{cr}} \psi_-(x, \nu_{cr}) - ie^{-\frac{i}{2} \phi_{cr}} \psi_+(x, \nu_{cr}) + o(1) \right), \quad \text{if} \; \gamma \in \left( \frac{1}{2}, 1 \right), \]

and for every \( \alpha \neq \alpha_{cr} \)

\[ \varphi_{\alpha}(x, \nu_{cr}) = d_{cr+} \sin(\alpha - \alpha_{cr}) x^{\beta_{cr}} \times \left( e^{\frac{i}{2} \phi_{cr}} \psi_-(x, \nu_{cr}) + e^{-\frac{i}{2} \phi_{cr}} \psi_+(x, \nu_{cr}) + o(1) \right), \quad \text{if} \; \gamma = 1, \]
(1.14) \( \varphi_\alpha(x, \nu_{cr}) = d_{cr+} \sin(\alpha - \alpha_{cr}) \exp\left( \frac{\beta_{cr} x^{1-\gamma}}{1-\gamma} \right) \)
\( \times \left( e^{\frac{1}{2} \phi_{cr} x}(x, \nu_{cr}) + e^{-\frac{1}{2} \phi_{cr} x}(x, \nu_{cr}) + o(1) \right), \) if \( \gamma \in \left( \frac{1}{2}, 1 \right) \).

**Remark 1.1.** In fact, formulae for \( \beta_{cr} \) and \( \phi_{cr} \) are known:

(1.15) \( \beta_{j\pm} := \frac{|c|}{2a W\{\psi_+, \psi_-\}(\nu_{j\pm})} \left| \int_0^a \psi_{\pm}^2(t, \nu_{j\pm}) e^{2i\omega t} dt \right|, \)

(1.16) \( \phi_{j\pm} := \pm \left( \delta + \arg \int_0^a \psi_{\pm}^2(t, \nu_{j\pm}) e^{2i\omega t} dt \right). \)

The expression (1.15) was found in [29] and the expression (1.16) for the case \( \gamma \in \left( \frac{1}{2}, 1 \right) \) is given by Lemma 3.1 of the present paper (for the case \( \gamma = 1 \) a slight modification of this lemma is needed).

In [29] we had the following result.

**Proposition 1.2** ([29]). Let the operator \( \mathcal{L}_\alpha \) be defined by (1.3) and (1.4) where \( L \) is given by (1.1), and let the conditions (1.5) hold with \( \gamma = 1 \). Let \( \rho'_\alpha \) be the spectral density of \( \mathcal{L}_\alpha \), let \( \nu_{j\pm} \) and \( \varphi_\alpha \) be defined by (1.9) and (1.10), respectively. Let \( \nu_{cr} \in \{ \nu_{j+}, \nu_{j-}, j \geq 0 \} \) be one of the critical points and \( \alpha_{cr} \) be defined in Proposition 1.1. If \( \alpha \neq \alpha_{cr} \), then there exist two non-zero one-side limits

\[ \lim_{\lambda \to \nu_{cr}^{\pm}} \frac{\rho'_\alpha(\lambda)}{|\lambda - \nu_{cr}|^{2\gamma_{cr}}}, \]

where \( \beta_{cr} \) is given by the expression (1.15).

This means that the spectral density of the operator \( \mathcal{L}_\alpha \) at a critical point in the generic situation \( \alpha \neq \alpha_{cr} \) has zeroes of the power type, and the power is twice the rate of decay of the subordinate solution at this critical point. The main result of the present paper is the following theorem.

**Theorem 1.1.** Let the operator \( \mathcal{L}_\alpha \) be defined by (1.3) and (1.4) where \( L \) is given by (1.1), and let the conditions (1.5) hold with \( \gamma \in \left( \frac{1}{2}, 1 \right) \). Let the potential \( q_1 \) satisfy the estimate

(1.17) \[ |q_1(x)| \leq \frac{c_1}{x^{1+\alpha_1}}, \quad x \in [0, +\infty) \]

with some \( \alpha_1, c_1 > 0 \). Let \( \nu_{j\pm} \) and \( \varphi_\alpha \) be defined by (1.9) and (1.10), respectively. Let \( \nu_{cr} \in \{ \nu_{j+}, \nu_{j-}, j \geq 0 \} \) be one of the critical points and \( \alpha_{cr} \) be defined in Proposition 1.1. If \( \alpha \neq \alpha_{cr} \), then the spectral density \( \rho'_\alpha \) of the operator \( \mathcal{L}_\alpha \) has the following asymptotics:

(1.18) \[ \rho'_\alpha(\lambda) = \frac{a_{cr}}{2c_{cr}^2} \sin^2(\alpha - \alpha_{cr}) \exp\left( -\frac{2c_{cr}}{\gamma} \right) \left( 1 + o(1) \right) \] \( \text{as} \ \lambda \to \nu_{cr}, \)

where

(1.19) \[ c_{cr} := \frac{(2\beta_{cr})^{1/2}}{4\gamma} \int B \left( \frac{3}{2}, \frac{1}{2\gamma} \right) \left( \frac{a_{cr}}{2\pi k'(\nu_{cr})} \right)^{2/\gamma}. \]
and

\begin{equation}
\alpha_{cr} := \frac{1}{\pi |W\{\psi_+ , \psi_- \}(\nu_{cr})|} \exp \left( - \int_{0}^{(2\beta_{cr})^+} \frac{\gamma}{t \left( 1 - \frac{\epsilon^2}{\nu_{cr}^2} \right)} \, dt \right) + \int_{0}^{(2\beta_{cr})^+} \frac{\gamma \, dt}{t \left( 1 - \frac{\epsilon^2}{\nu_{cr}^2} \right)} - \text{v.p.} \int_{2(\beta_{cr})^+}^{+\infty} \frac{\gamma \, dt}{t \left( 1 - \frac{\epsilon^2}{\nu_{cr}^2} \right)},
\end{equation}

\beta_{cr} is given by the expression \((1.13)\), \(a_{cr+}\) is defined in \((1.14)\), \(B\) is the beta function and \(k\) is the quasi-momentum of the unperturbed periodic operator \(\mathcal{L}_{per}\).

Note that the subordinate solution at a critical point decays as \(\exp\left( -\frac{2\beta_{cr}^+ \epsilon}{\nu_{cr}^2} \right)\), while the spectral density vanishes as

\[ \exp \left( - \frac{(2\beta_{cr})^+ B}{2\gamma} \left( \frac{\frac{1}{2} - \frac{1}{\gamma}}{2\pi k'(\nu_{cr}) |\lambda - \nu_{cr}|} \right) \right) \]

Relation between, on the one hand, the behaviour at infinity of the subordinate solution (whenever it exists) compared to the behaviour of a non-subordinate one and, on the other hand, the normal boundary behaviour of the Weyl’s \(m\)-function was, in a general situation, established by Jitomirskaya–Last \([16]\) and Remling \([31]\). However, in our case it yields only a trivial result: that for \(\alpha \neq \alpha_{cr}\) one has \(|m_{\alpha}(\nu_{cr} + i\epsilon)| = O(1)\) and for \(\alpha = \alpha_{cr}\) one has \(|m_{\alpha}(\nu_{cr} + i\epsilon)| \geq \frac{1}{\epsilon} \) as \(\epsilon \rightarrow 0^+\).

Spectral density is related to the boundary behaviour of the Weyl’s \(m\)-function, and one can say in these terms that we study \(m_{\alpha}(\nu_{cr} + \epsilon) = \frac{1}{\pi} \text{Im} m_{\alpha}(\nu_{cr} + \epsilon + i0)\) as \(\epsilon \rightarrow 0\), which, clearly, can behave quite differently to \(\text{Im} m_{\alpha}(\nu_{cr} + i\epsilon)\).

The case \(\gamma = 1\) was earlier considered in detail by Hinton–Klaus–Shaw \([14]\), Klaus \([17]\) and Behncke \([2, 3, 4]\). They included no periodic background potentials. Hinton, Klaus and Shaw considered instead an infinite sum of Wigner–von Neumann terms, while Behncke added Dirac operators into consideration. Their methods are specific to the models they study, while in \([29]\) we proposed an approach which is based on reducing the problem to a certain discrete linear system with small parameter which essentially models the behaviour of solutions of the Schrödinger equation (we call it the “model problem”). This allowed for stating the result in certain generality and using it to study a discrete analogue, a Jacobi matrix \([15, 32]\).

Condition \((1.17)\) in the formulation of the result is of technical nature. We would like to note here that in \([14]\) for the case \(\gamma = 1\) an analogous condition was imposed on the summable part \(q_1\) of the potential, however, with \(\alpha_1 = 1\). It was left as an open question whether this condition could be weakened to include every \(q_1 \in L_1(\mathbb{R}_+)\), see the question \((4)\) in the final section of \([14]\). For the case \(\gamma = 1\) the answer is positive, as our analysis in \([29]\) suggests. For the case \(\gamma \in (\frac{1}{2}, 1)\) we do not have the answer.

In the case \(\gamma \in (\frac{1}{2}, 1)\) we reduce the problem to a model linear differential, rather than discrete, system. In fact, it is possible to do the same in the case \(\gamma = 1\), and then the idea of the method of \([29]\) would work and should lead to the same result as there, but in an easier way (however, on this way we would lose the discrete case). For \(\gamma < 1\) this idea does not produce the result anymore, but can only be used at one of the stages (in Section \([5]\)). On the whole, the method of the present
paper is different from the method of [29], which is insufficient for the situation considered here.

Wigner–von Neumann potentials originally aroused interest as giving an explicit example of an eigenvalue embedded into continuous spectrum [34]. Since then they have been studied by many authors (for example, the papers [2, 3, 4, 5, 6, 9, 14, 17, 20, 21, 22, 25, and many more). Embedded eigenvalues created by such potentials have been observed in experiment, see [7]. Zeroes of density divide the absolutely continuous spectrum into independent parts and for this reason are sometimes called pseudogaps.

A paper by Kreimer, Last and Simon [19] should be also mentioned as an example of the spectral density analysis (in that case of the discrete Schrödinger operator with slowly decaying potential, near the endpoints of the absolutely continuous spectrum). Results on the behaviour of the spectral density have been recently used by Lukic [26] to construct test sequences in the proof of higher-order Szegő theorems for CMV matrices.

The paper is organized as follows. In Section 2 we introduce the Titchmarsh–Weyl formula for the spectral density of the operator $L$ which was proved in [23]. The formula expresses $\rho'(\lambda)$ in terms of asymptotic coefficients of the solution $\varphi(x,\lambda)$ as $x \to +\infty$. In Section 3 we rewrite the eigenfunction equation of $L$ as a model system with a small parameter $\varepsilon$ and express the spectral density in terms of a certain solution of that system. In Section 4 we transform the model system to a new form and determine five regions of the positive coordinate half-line where asymptotic analysis of solutions as $\varepsilon \to 0$ should be carried out in different ways. In Sections 5–9 we consider each of the regions separately and find asymptotics of solutions of the model system. In Section 10 we match the results in order to obtain the double asymptotics, in the coordinate and in the small parameter, of the solution of the model system in terms of which the spectral density of $L$ was earlier expressed. In Section 11 we prove the main result of the paper, Theorem 1.1.

We denote by $M^{2\times 2}(\mathbb{R})$ and $M^{2\times 2}(\mathbb{C})$ matrices of two rows and two columns with real and, respectively, complex entries and use the following notation for two basic vectors in $\mathbb{C}^2$:

$$
e_+ := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_- := \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

### 2. Preliminaries

The spectrum of the periodic Schrödinger operator $L_{\text{per}}$ consists of infinitely many intervals [17], see [11] Theorem 2.3.1], where $\lambda_j$ and $\mu_j$ are the eigenvalues of the Sturm–Liouville problem defined by the differential expression $L$ on the interval $[0, a]$ with, respectively, periodic and antiperiodic boundary conditions. Let us denote by $\vartheta$ the set of the endpoints of the spectral bands of $L_{\text{per}}$,

$$\vartheta := \{\lambda_j, \mu_j, \ j \geq 0\},$$

including the case when the endpoints of the neighbouring bands coincide. Spectral properties of the operator $L_{\text{per}}$ are related to the quasi-momentum

$$k(\lambda) := -i \ln \left( \frac{D(\lambda) + \sqrt{D^2(\lambda) - 4}}{2} \right),$$
where the entire function $D$ is the trace of the monodromy matrix of the periodic equation. We can choose the branch of $k$ so that

$$k(\lambda_0) = 0, k(\mu_0) = k(\mu_1) = \pi, k(\lambda_1) = k(\lambda_2) = 2\pi, \ldots,$$

$$k(\lambda) \in \mathbb{R} \text{ and } k'(\lambda) > 0, \text{ if } \lambda \in \sigma(L_{\text{per}}) \setminus \partial,$$

$$k(\lambda) \in \mathbb{C}_+, \text{ if } \lambda \in \mathbb{C}_+,$$

see [11, Theorem 2.3.1]. The eigenfunction equation of $L_{\text{per}},$

$$-\psi''(x) + q(x)\psi(x) = \lambda\psi(x),$$

has two solutions $\psi_+(x, \lambda)$ and $\psi_-(x, \lambda)$ (Bloch solutions) which satisfy the quasi-periodic conditions:

$$(2.2) \quad \psi_\pm(x + a, \lambda) \equiv e^{\pm ik(\lambda)}\psi_\pm(x, \lambda),$$

Each of them is defined uniquely up to multiplication by a coefficient which depends on $\lambda.$ It is possible to choose these coefficients so that for every $x \geq 0$ the functions $\psi_\pm(x, \cdot)$ and $\psi_\pm'(x, \cdot)$, and hence their Wronskian $W\{\psi_+, \psi_\pm\}(\cdot)$, are analytic in $\mathbb{C}_+$ and continuous up to the set $\sigma(L_{\text{per}}) \setminus \partial.$ Moreover, it is possible for every $\lambda \in \sigma(L_{\text{per}}) \setminus \partial$ to have

$$\psi_+(x, \lambda) \equiv \overline{\psi_-(x, \lambda)} \quad \text{and} \quad iW\{\psi_+, \psi_\pm\}(\lambda) < 0.$$

In what follows we assume that such a choice of the coefficients is made. Bloch solutions can be written in another form:

$$\psi_\pm(x, \lambda) = e^{\pm ik(\lambda)}p_\pm(x, \lambda),$$

where the functions $p_+(x, \lambda)$ and $p_-(x, \lambda)$ are periodic with the period $a$ in the variable $x$ and have the same properties as $\psi_\pm(x, \lambda)$ in the variable $\lambda.$

It is well known that some of spectral properties of one-dimensional Schrödinger operators can be written in terms of the asymptotic behaviour of their generalised eigenvectors (see, for example, [12]). In particular, the spectral density of the operator can be expressed in terms of the Jost function by the Titchmarsh–Weyl formula. The eigenfunction equation $L\psi = \lambda\psi$ is a small perturbation of the periodic equation $-\psi''(x) + q(x)\psi(x) = \lambda\psi(x)$ in the sense that asymptotically $\varphi_\alpha$ is some linear combination of two Bloch solutions.

**Proposition 2.1** ([23]). Let the operator $L_\alpha$ be defined by (1.5) and (1.4) where $L$ is given by (1.1), and let the conditions (1.5) hold. Let $\rho_\alpha'$ be the density of the operator $L_\alpha,$ and $\nu_{j, \pm}, \varphi_\alpha, \sigma(L_{\text{per}}),$ $\partial,$ $\psi_\pm$ be defined by (1.9), (1.10), (1.7), (2.1), (2.2), respectively. For every fixed $\lambda \in \sigma(L_{\text{per}}) \setminus (\partial \cup \{\nu_{j, +}, \nu_{j, -}, j \geq 0\})$ there exists $A_\alpha(\lambda)$ such that, as $x \to +\infty,$

$$(2.3) \quad \varphi_\alpha(x, \lambda) = A_\alpha(\lambda)\overline{\psi_-(x, \lambda)} + A_\alpha(\lambda)\psi_+(x, \lambda) + o(1),$$

$$(2.4) \quad \rho_\alpha'(\lambda) = \frac{1}{2\pi|W\{\psi_+, \psi_\pm\}(\lambda)| |A_\alpha(\lambda)|^2}.$$
follows from results of [28]. In the case of discrete Schrödinger operator with the Wigner–von Neumann potential an analogous formula is also known, see [10, 15].

3. Reduction to the model problem

In this section we reduce the eigenfunction equation \( L\psi = \lambda \psi \) to a specially constructed linear differential system and express the modulus of the coefficient \( A_\alpha \) from the Titchmarsh–Weyl formula [27] in terms of a certain solution of this new system.

**Lemma 3.1.** Let the conditions of Theorem [17] hold and let \( \nu_{cr} \in \{ \nu_{j+}, \nu_{j-}, j \geq 0 \} \) be one of the critical points.

1. There exist the following objects which are determined by the data of the problem \( (q, q_1, c, \omega, \gamma, \delta, \theta, \alpha) \) and have the following properties:
   i. the neighbourhood \( U_{cr} \) of the point \( \nu_{cr} \) such that its closure lies inside the spectral band and does not contain the second critical point of that band,
   ii. the bijective real-valued function \( \varepsilon_{cr}(\lambda) \) such that
   
   \[
   \varepsilon_{cr}(\lambda) = \frac{2\pi k'(\nu_{cr})}{a}(\lambda - \nu_{cr}) + O((\lambda - \nu_{cr})^2) \quad \text{as} \quad \lambda \rightarrow \nu_{cr},
   \]
   \[
   \|R_{cr}(x, \lambda)\| < c_2 \left( |q_1(x)| + \frac{1}{(x^2 + x)x} \right)
   \]
   with some \( c_2 > 0 \),
   iii. the vector \( g_{cr,\alpha} \in \mathbb{R}^2 \backslash \{0\} \),
   iv. the solution \( w_{cr,\alpha}(x, \lambda) \) of the linear differential system
   \[
   w'_{cr}(x) = \frac{\beta_{cr}}{x^\gamma} \begin{pmatrix} \cos(\varepsilon_{cr}(\lambda)x) & \sin(\varepsilon_{cr}(\lambda)x) \\ \sin(\varepsilon_{cr}(\lambda)x) & -\cos(\varepsilon_{cr}(\lambda)x) \end{pmatrix} + R_{cr}(x, \lambda) w_{cr}(x),
   \]
   where \( \beta_{cr} \) is given by (1.15), and such that \( w_{cr,\alpha}(0, \cdot) \) is a continuous function in \( U_{cr} \) and \( w_{cr,\alpha}(0, \nu_{cr}) = g_{cr,\alpha} \).

2. For every \( \lambda \in U_{cr} \backslash \{ \nu_{cr} \} \) the limit \( \lim_{x \to +\infty} w_{cr,\alpha}(x, \lambda) \) exists and

\[
|A_\alpha(\lambda)| = \lim_{x \to +\infty} \|w_{cr,\alpha}(x, \lambda)\|,
\]
where \( A_\alpha \) is defined in Proposition 2.1.

3. With the same \( \alpha_{cr}, d_{cr-} \) and \( d_{cr+} \) as in Proposition 1.7 and with \( \phi_{cr} \) given by (1.16) one has

\[
w_{cr,\alpha_{cr}}(x, \nu_{cr}) = d_{cr-} \exp \left( -\frac{\beta_{cr} x^{1-\gamma}}{1-\gamma} \right) (e_- + o(1))
\]
as \( x \to +\infty \) and the asymptotics [11.12] holds. For every \( \alpha \neq \alpha_{cr} \) one has

\[
w_{cr,\alpha}(x, \nu_{cr}) = d_{cr+} \sin(\alpha - \alpha_{cr}) \exp \left( \frac{\beta_{cr} x^{1-\gamma}}{1-\gamma} \right) (e_+ + o(1))
\]
as \( x \to +\infty \) and the asymptotics [11.14] holds. Vectors \( e_{\pm} \) are defined by (1.21).
Proof. Let us start with the eigenfunction equation $L\psi = \lambda\psi$ and write it in the vector form,
\[
\begin{pmatrix}
\psi(x) \\
\psi'(x)
\end{pmatrix}' = \begin{pmatrix} q(x) + \frac{\cos(2\omega x + \delta)}{x} + q_1(x) - \lambda & 1 \\
q(x) + \frac{\cos(2\omega x + \delta)}{x} & q(x) + \frac{\cos(2\omega x + \delta)}{x} \end{pmatrix} \begin{pmatrix}
\psi(x) \\
\psi'(x)
\end{pmatrix}.
\]

Let us make variation of parameters:
\[
\begin{pmatrix}
\psi(x) \\
\psi'(x)
\end{pmatrix} = \begin{pmatrix} \psi_-(x, \lambda) & \psi_+(x, \lambda) \\
\psi'_-(x, \lambda) & \psi'_+(x, \lambda)
\end{pmatrix} \tilde{w}(x).
\]

This leads to the system
\[
\tilde{w}'(x) = \frac{\cos(2\omega x + \delta) + q_1(x)}{W(\psi_+, \psi_-)} \begin{pmatrix} -\psi_+(x, \lambda)\psi_-(x, \lambda) & -\psi_2^2(x, \lambda) \\
\psi_2^2(x, \lambda) & \psi_+(x, \lambda)\psi_-(x, \lambda)
\end{pmatrix} \tilde{w}(x).
\]

Denote the summable part of its coefficient matrix as
\[
\tilde{R}(x, \lambda) := \frac{q_1(x)}{W(\psi_+, \psi_-)} \begin{pmatrix} -\psi_+(x, \lambda)\psi_-(x, \lambda) & -\psi_2^2(x, \lambda) \\
\psi_2^2(x, \lambda) & \psi_+(x, \lambda)\psi_-(x, \lambda)
\end{pmatrix}.
\]

For the remaining part we use Fourier series decompositions of the periodic functions $p_+, p_-$ and $p_+^2 = p_+^2$ to write the entries of the matrix as follows:
\[
\begin{align*}
\psi_+(x, \lambda)\psi_-(x, \lambda) &= \sum_{n \in \mathbb{Z}} b_n(\lambda) e^{2i\pi n \frac{\omega}{\pi}}, \\
\psi_2^2(x, \lambda) &= \sum_{n \in \mathbb{Z}} b_n^+(\lambda) e^{2i(\pi n + k(\lambda)) \frac{\omega}{\pi}}, \\
\psi_2^2(x, \lambda) &= \sum_{n \in \mathbb{Z}} b_n^+(\lambda) e^{2i(\pi n - k(\lambda)) \frac{\omega}{\pi}}.
\end{align*}
\]

Now let us choose some band of the spectrum with the index $j$ and one of two critical points in it, $\nu_{jr} = \nu_{j+}$. We give the detailed proof for the choice of the sign “+”, and for the sign “−” formulae should be modified in a natural way. Take the neighbourhood of the critical point $U_{j+}$ so that its closure lies inside the band with the index $j$ and does not contain the point $\nu_{j-}$. Let
\[
n_{j+} := -\left( j + 1 + \left\lfloor \frac{a\omega}{\pi} \right\rfloor \right),
\]

so that
\[
2i(\pi n_{j+} + k(\lambda))x + 2iawx = 2i(k(\lambda) + \pi n_{j+} + a\omega)x,
\]

and
\[
(3.10) \quad \pi n_{j+} + a\omega = -\pi \left( j + 1 - \left\{ \frac{a\omega}{\pi} \right\} \right) = -k(\nu_{j+}).
\]

Then we can write
\[
\begin{align*}
&\frac{\cos(2\omega x + \delta)}{x^2W(\psi_+, \psi_-)} \begin{pmatrix} -\psi_+(x, \lambda)\psi_-(x, \lambda) & -\psi_2^2(x, \lambda) \\
\psi_2^2(x, \lambda) & \psi_+(x, \lambda)\psi_-(x, \lambda)
\end{pmatrix} \\
&= \frac{c(e^{2i\omega x + \delta} - e^{-2i\omega x - \delta})}{2i\omega^2W(\psi_+, \psi_-)} (\sum_{n \in \mathbb{Z}} \begin{pmatrix}
-b_n(\lambda)e^{2i\pi n \frac{\omega}{\pi}} & -b_n^+(\lambda)e^{2i(\pi n + k(\lambda)) \frac{\omega}{\pi}} \\
b_n^+(\lambda)e^{-2i(\pi n + k(\lambda)) \frac{\omega}{\pi}} & b_n(\lambda)e^{2i(\pi n - k(\lambda)) \frac{\omega}{\pi}}
\end{pmatrix}) \\
&= \sum_{n \in \mathbb{Z}} S_n(x, \lambda) = S_{j+}^{(1)}(x, \lambda) + S_{j+}^{(2)}(x, \lambda),
\end{align*}
\]
where

\[ S_n(x, \lambda) := \frac{c(e^{2i\omega x + i\delta} - e^{-2i\omega x - i\delta})}{2ix^\gamma W\{\psi_+, \psi_-(\lambda)} \times \left( \begin{array}{cc} -b_n(\lambda)e^{2i\pi n} & -b_n(\lambda)e^{2i(\pi n + k(\lambda))} \\ b_n(\lambda)e^{-2i(\pi n + k(\lambda))} & \end{array} \right), \]

\[ S_{j+}(x, \lambda) := \frac{c}{2ix^\gamma W\{\psi_+, \psi_-(\nu_{j+})} \times \left( \begin{array}{cc} 0 & -b_n(\nu_{j+})e^{i\delta}e^{2i\pi(k(\lambda)-k(\nu_{j+}))} \\ -b_n(\nu_{j+})e^{i\delta}e^{-2i\pi(k(\lambda)-k(\nu_{j+}))} & 0 \end{array} \right). \]

and

\[ S_{j+}(x, \lambda) := \sum_{n \in \mathbb{Z}\{n_{j+}\}} S_n(x, \lambda) + (S_{n_{j+}}(x, \lambda) - S_{j+}^{(1)}(x, \lambda)). \]

In this notation the system (3.7) reads as

\[ \hat{w} = (S_{j+}^{(1)} + S_{j+}^{(2)} + \hat{R}) \hat{w}. \]

To eliminate the non-resonating term \( S_{j+}^{(2)} \) from this system we use the Harris–Lutz transformation based on the matrix

\[ \hat{T}_{j+}(x, \lambda) = -\int_{-\infty}^{\infty} S_{j+}^{(2)}(t, \lambda) dt. \]

First we need to see that this integral is convergent.

**Lemma 3.2.** The integral in the definition (3.15) is convergent, the function \( \hat{T}_{j+}(x, \lambda) \) is continuous in \( \lambda \) for every \( x \in [0, +\infty) \) and satisfies the estimate

\[ \|\hat{T}_{j+}(x, \lambda)\| \leq \frac{c_3}{(x + 1)^\gamma} \]

for every \( x \in [0, +\infty) \) and \( \lambda \in U_{j+} \) with some \( c_3 > 0 \).

**Proof.** Coefficients \( b_n \) and \( b_n^+ \) have the same analyticity properties as the function \( k \), and satisfy the following estimates (see, for example, [23]): there exists \( c_4 > 0 \) such that for every \( \lambda \in U_{j+} \) and \( n \in \mathbb{Z} \)

\[ |b_n(\lambda)|, |b_n^+(\lambda)| < \frac{c_4}{n^\gamma + 1}. \]

One can choose, if necessary, \( c_4 \) large enough to ensure that for every \( \lambda \in U_{j+} \)

\[ \frac{1}{|W\{\psi_+, \psi_-(\lambda)}|} < c_4. \]

We also need the following rough estimate: if \( N_1 < N_2 \), then

\[ \left| \int_{N_1}^{N_2} e^{i\xi t} dt \right| \leq \frac{2^\gamma \left( \frac{1}{N_1} + \frac{1}{N_2} \right)}{(N_1 + 1)^\gamma}. \]
To see this let us consider three cases.

1. If $0 \leq N_1 < N_2 \leq 1$, then

$$\left| \int_{N_1}^{N_2} \frac{e^{i\xi t}}{t^\gamma} dt \right| \leq \int_0^1 \frac{dt}{t^\gamma} = \frac{1}{1-\gamma} \leq \frac{2^\gamma \left( \frac{2}{|\xi|} + \frac{1}{1-\gamma} \right)}{(N_1 + 1)^\gamma}.$$ 

2. If $1 \leq N_1 < N_2$, then

$$\left| \int_{N_1}^{N_2} \frac{e^{i\xi t}}{t^\gamma} dt \right| \leq \frac{2}{|\xi| N_1} \leq \frac{2^\gamma \left( \frac{2}{|\xi|} + \frac{1}{1-\gamma} \right)}{(N_1 + 1)^\gamma}$$

from integrating by parts.

3. If $0 \leq N_1 < 1 < N_2$, then

$$\left| \int_{N_1}^{N_2} \frac{e^{i\xi t}}{t^\gamma} dt \right| \leq \left| \int_{N_1}^{1} \frac{e^{i\xi t}}{t^\gamma} dt \right| + \left| \int_{1}^{N_2} \frac{e^{i\xi t}}{t^\gamma} dt \right| \leq \frac{1}{1-\gamma} + \frac{2}{|\xi|} \leq \frac{2^\gamma \left( \frac{2}{|\xi|} + \frac{1}{1-\gamma} \right)}{(N_1 + 1)^\gamma}$$

using the intermediate estimates from the first case for the first summand and from the second case for the second.

Using the estimates (3.16), (3.17) and (3.18) we see that there exists $c_5 > 0$ such that for every $\lambda \in U_{j+}, n \in \mathbb{Z}$ and $N_1 < N_2$

$$\left\| \int_{N_1}^{N_2} S_n(t, \lambda) dt \right\| \leq \frac{c_5}{(n^2 + 1)(N_1 + 1)^\gamma} \min_{\lambda \in U_{j+}} \left\{ |\pi n \pm a\omega|, |k(\lambda) + \pi n \pm a\omega| \right\}.$$ 

This estimate works for $n \neq n_{j+}$, because

$$\min_{n \in \mathbb{Z}\setminus\{n_{j+}\}, \lambda \in U_{j+}} \{|\pi n \pm a\omega|, |k(\lambda) + \pi n \pm a\omega| \} > 0,$$

while for $n = n_{j+}$ one has

$$\min_{\lambda \in U_{j+}} \{|k(\lambda) + \pi n_{j+} \pm a\omega| \} = |k(n_{j+}) + \pi n_{j+} \pm a\omega| = 0.$$ 

One can choose $c_5$ so large that for every $\lambda \in U_{j+}$ and $N_1 < N_2$ the estimate

$$\left\| \int_{N_1}^{N_2} S_n(t, \lambda) dt \right\| \leq \frac{c_5}{(n^2 + 1)(N_1 + 1)^\gamma}$$

holds for every $n \in \mathbb{Z}\setminus\{n_{j+}\}$, and so with some $c_6 > 0$ one has

$$(3.19) \quad \left\| \int_{N_1}^{N_2} \sum_{n \in \mathbb{Z}\setminus\{n_{j+}\}} S_n(t, \lambda) dt \right\| \leq \frac{c_6}{(N_1 + 1)^\gamma}.$$
In order to estimate $S_{j+}^{(2)}(x, \lambda)$ consider the difference

$$S_{n,j+}(x, \lambda) - S_{j+}^{(1)}(x, \lambda)$$

\[
\begin{align*}
= & \frac{c e^{i2\pi x \delta}}{2i} \left[ W\{\psi_+, \psi_-\}(\lambda) \left( \begin{array}{cc} -b_{n,j+}^{+}(\lambda)e^{2i\pi n j+} & 0 \\ b_{n,j+}^{-}(\lambda)e^{-2i(\pi n j+ + k(\lambda))} & b_{n,j+}^{+}(\lambda)e^{2i\pi n j+} \end{array} \right) \\
+ & \frac{c e^{-i2\pi x \delta}}{2i} \left[ W\{\psi_+, \psi_-\}(\lambda) \left( \begin{array}{cc} b_{n,j+}^{+}(\lambda)e^{2i\pi n j+} & 0 \\ b_{n,j+}^{-}(\lambda)e^{-2i(\pi n j+ + k(\lambda))} & -b_{n,j+}^{+}(\lambda)e^{2i\pi n j+} \end{array} \right) \right] \\
+ & \frac{c e^{-i\delta}}{2i} \left( \frac{b_{n,j+}^{+}(\nu_j+)}{W\{\psi_+, \psi_-\}(\nu_j+)} - \frac{b_{n,j+}^{+}(\lambda)}{W\{\psi_+, \psi_-\}(\lambda)} \right) e^{2i\pi(\lambda - (\nu_j+))} \begin{array}{c} 0 \\ 0 \end{array} \\
+ & \frac{c e^{i\delta}}{2i} \left( \frac{b_{n,j+}^{+}(\nu_j+)}{W\{\psi_+, \psi_-\}(\nu_j+)} - \frac{b_{n,j+}^{+}(\lambda)}{W\{\psi_+, \psi_-\}(\lambda)} \right) e^{-2i\pi(\lambda - (\nu_j+))} \begin{array}{c} 0 \\ 0 \end{array} \right). 
\end{align*}
\]

The first two summands can be estimated using (3.16)–(3.18) and added into (3.19) under the integral with, possibly, a change of \(c_6\). The integral of the third and the fourth summands from \(N_1\) to \(N_2\) can be estimated using (3.17) and (3.18) by

\[
\begin{align*}
\left| \int_{N_1}^{N_2} S_{j+}^{(2)}(t, \lambda) dt \right| & \leq \frac{c_3}{(N_1 + 1)^\gamma} 
\end{align*}
\]

with some \(c_7 > 0\). This can in turn be estimated by \(\frac{c_3}{(N_1 + 1)^\gamma}\) with some \(c_8 > 0\), since both functions \(\frac{b_{n,j+}^{+}(\nu_j+)}{W\{\psi_+, \psi_-\}(\nu_j+)}\) and \(k\) are differentiable at the point \(\nu_j+\) and since \(k'(\nu_j+) \neq 0\). Combining this with (3.19) we see that there exists \(c_8 > 0\) such that for every \(\lambda \in U_{j+}\) and \(N_1 < N_2\) we have the estimate

\[
(3.20) \quad \int_{N_1}^{N_2} S_{j+}^{(2)}(t, \lambda) dt \leq \frac{c_3}{(N_1 + 1)^\gamma},
\]

with some \(c_3 > 0\). This means that the integral \(\int_{x}^{+\infty} S_{j+}^{(2)}(t, \lambda) dt\) exists for every \(x \in [0, +\infty)\), the definition (3.15) of \(\tilde{T}_{j+}\) is correct and that \(\tilde{T}_{j+}(x, \lambda)\) is continuous in \(\lambda\) and is estimated in norm by \(\frac{c_3}{(x+1)^\gamma}\).

From now on we start writing the index “cr” instead of “\(j+\)”, because all the formulae remain valid if one changes the index to “\(j-\)”, that is, for the second type of critical points. Let us make the Harris-Lutz transformation by substituting

\[
(3.21) \quad \hat{w}(x) = \exp(\tilde{T}_{cr}(x, \lambda))\tilde{w}_{cr}(x)
\]

to the system (3.7). This leads, with the use of the equality \(\tilde{T}_{cr} = S_{cr}^{(2)}\), to the system

\[
\begin{align*}
\tilde{w}_{cr}' &= e^{-\tilde{T}_{cr}}((S_{cr}^{(1)} + S_{cr}^{(2)} + \tilde{R}e^{\tilde{T}_{cr}} - (e^{\tilde{T}_{cr}})'\tilde{w}_{cr} = (S_{cr}^{(1)} + \tilde{R}_{cr})\tilde{w}_{cr} \\
\tilde{R}_{cr} &= e^{-\tilde{T}_{cr}}(S_{cr}^{(1)} + S_{cr}^{(2)})e^{\tilde{T}_{cr}} - (S_{cr}^{(1)} + S_{cr}^{(2)}) + e^{-\tilde{T}_{cr}}\tilde{R}e^{\tilde{T}_{cr}} - e^{-\tilde{T}_{cr}}((e^{\tilde{T}_{cr}})' - \tilde{T}_{cr}').
\end{align*}
\]
Let us estimate this remainder. For every \( \lambda \in U_{cr} \) and \( x \in (0, +\infty) \) we have 
\[
\|S_{cr}^{(1)} + S_{cr}^{(2)}\| < \frac{c_9}{x^2}
\]
with some \( c_9 > 0 \) and \( \|\tilde{T}_{cr}(x, \lambda)\| < \frac{c_{10}}{(x^2 + x)^\gamma} \).
Therefore
\[
\left\|e^{-\tilde{T}_{cr}(x, \lambda)}(S_{cr}^{(1)}(x, \lambda) + S_{cr}^{(2)}(x, \lambda))e^{\tilde{T}_{cr}(x, \lambda)} - (S_{cr}^{(1)}(x, \lambda) + S_{cr}^{(2)}(x, \lambda))\right\| < \frac{c_{11}}{(x^2 + x)^\gamma}
\]
and
\[
\left\|(e^{\tilde{T}_{cr}(x, \lambda)} - \tilde{T}_{cr}(x, \lambda)')\right\| = \left\|\sum_{n=2}^{\infty} \frac{\tilde{T}_{cr}^{(n)}(x, \lambda)}{n!} \right\| \leq \left\|\tilde{T}_{cr}'(x, \lambda)\right\| \left\|\sum_{n=2}^{\infty} \frac{\tilde{T}_{cr}^{(n)}(x, \lambda)}{(n-1)!}\right\|
\]
\[
= \|S_{cr}^{(2)}(x, \lambda)\| \left(\|e^{\tilde{T}_{cr}(x, \lambda)}\| - 1\right) < \frac{c_{11}}{(x^2 + x)^\gamma}.
\]
Combining all this and the estimate of the summable term \((3.8)\) we see that there exists \( c_{12} > 0 \) such that for every \( \lambda \in U_{cr} \) and \( x \in [0, +\infty) \) one has
\[
\|\tilde{R}_{cr}(x, \lambda)\| < c_{12} \left(\frac{1}{(x^2 + x)^\gamma} + |q_1(x)|\right).
\]
Now we can rewrite \( S_{cr}^{(1)} \) so that the system on \( \tilde{w}_{cr} \) reads:
\[
\tilde{w}_{cr}'(x) = \left(\frac{1}{x^\gamma} \begin{pmatrix} 0 & -z_{cr}e^{i\varepsilon_{cr}(\lambda)} \\ -\frac{z_{cr}e^{-i\varepsilon_{cr}(\lambda)}}{2} & \tilde{R}_{cr}(x, \lambda) \end{pmatrix} + \tilde{R}_{cr}(x, \lambda)\right)\tilde{w}(x),
\]
where
\[
\varepsilon_{cr}(\lambda) := \frac{2\pi(k(\lambda) - k(\nu_{cr}))}{a}
\]
and
\[
z_{cr} := -\frac{e^{ib_{\nu_{cr}}(\nu_{cr})}}{2iW(\psi_+, \psi_-)(\nu_{cr})}.
\]
One can check that
\[
|z_{cr}| = \beta_{cr}, \quad \arg z_{cr} = \phi_{cr},
\]
with \( \beta_{cr} \) and \( \phi_{cr} \) given by \((1.15)\) and \((1.16)\), by substituting the expression for the Fourier coefficient
\[
b_n^{\nu_{cr}}(\lambda) = \frac{1}{a} \int_0^a \psi_+^*(t, \lambda)e^{-2i(k(\lambda) + \pi n)\frac{t}{a}} dt
\]
into \((3.24)\) and using the relation \((3.10)\) involving \( n_{cr} \).
Let us substitute \( \tilde{w}_{cr} \) in the form
\[
\tilde{w}_{cr}(x) = \left(\begin{pmatrix} 0 & 1 \\ e^{-\frac{1}{2}\phi_{cr}} & 1 \end{pmatrix} \right)w_{cr}(x),
\]
into the system \((3.23)\). We get for \( w_{cr} \) the system \((3.2)\), and
\[
R_{cr}(x, \lambda) = \left(\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \right)^{-1} \left(\begin{pmatrix} 0 & 1 \\ e^{-\frac{1}{2}\phi_{cr}} & 0 \end{pmatrix} \right) \left(\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \right) \times R_{cr}(x, \lambda) \left(\begin{pmatrix} 0 & 1 \\ e^{\frac{1}{2}\phi_{cr}} & 0 \end{pmatrix} \right) \left(\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \right).
\]
This expression for every \( x \in [0, +\infty) \) is continuous in \( U_{cr} \) as a function of \( \lambda \). For every \( \lambda \in U_{cr} \) and \( x \in [0, +\infty) \) it can be estimated in norm by \( c_2 \left(|q_1(x)| + \frac{1}{(x^2 + x)^\gamma}\right) \)
with some \( c_2 > 0 \).
Let us check that the matrix $R_{cr}$ has real entries. Fix some $x \in [0, \infty)$ and $\lambda \in U_{cr}$. From the expression (3.26) for $\hat{R}$ using that $q_1(x) \in \mathbb{R}$ and $iW\{\psi_+, \psi_-(\lambda)\} < 0$ we see that the matrix $\hat{R}(x, \lambda)$ has the following conjugation property:

$$
\hat{R}_{11}(x, \lambda) = \hat{R}_{22}(x, \lambda), \quad \hat{R}_{12}(x, \lambda) = \hat{R}_{21}(x, \lambda).
$$

This property is preserved for sums or products as well as for real analytic functions of such matrices. As it is clear from the formulae (3.11), (3.12) and (3.13), this property holds for the matrices $S_{cr}(x, \lambda)$, $S_{cr}^{(1)}(x, \lambda)$ and $S_{cr}^{(2)}(x, \lambda)$, and therefore, due to (3.15), for $\tilde{T}_{cr}(x, \lambda) = \tilde{T}_{cr}(x, \lambda)$ as well as for $\exp(\tilde{T}_{cr}(x, \lambda))$ and for $(\exp(\tilde{T}_{cr}(x, \lambda)))'$. Hence it holds for $R_{cr}(x, \lambda)$ given by (3.22) and for

$$
\begin{pmatrix}
\ e^{-\frac{1}{2}\phi_{cr}} & 0 \\
\ 0 & e^{\frac{1}{2}\phi_{cr}}
\end{pmatrix}
\hat{R}_{cr}(x, \lambda)
\begin{pmatrix}
\ e^{\frac{1}{2}\phi_{cr}} & 0 \\
\ 0 & e^{-\frac{1}{2}\phi_{cr}}
\end{pmatrix}.
$$

Since for every $a, b \in \mathbb{C}$

$$
\begin{pmatrix}
\ a & b \\
\ b & \bar{a}
\end{pmatrix}^{-1}
\begin{pmatrix}
\ a & b \\
\ b & \bar{a}
\end{pmatrix}
\begin{pmatrix}
\ a & b \\
\ b & \bar{a}
\end{pmatrix}^{-1} =
\begin{pmatrix}
\ Re\ a + Re\ b & Im\ b - Im\ a \\
\ Im\ a + Im\ b & Re\ a - Re\ b
\end{pmatrix},
$$

we see from the expression (3.26) that the entries of $R_{cr}(x, \lambda)$ are real-valued. As a result $R_{cr}(\cdot, \lambda) \in L_1(\mathbb{R}^+, M^{2 \times 2}(\mathbb{R}))$ for every $\lambda \in U_{cr}$.

Consider the solution $w_{cr,\alpha}$ of the system (3.2) which corresponds to the solution $\varphi_{\alpha}$ of the eigenfunction equation,

$$
w_{cr,\alpha}(x, \lambda) := T_{cr}(x, \lambda)
\begin{pmatrix}
\varphi_{\alpha}(x, \lambda) \\
\varphi'_{\alpha}(x, \lambda)
\end{pmatrix},
$$

where

$$
T_{cr}(x, \lambda) := \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}^{-1}
\begin{pmatrix}
\ e^{-\frac{1}{2}\phi_{cr}} & 0 \\
\ 0 & e^{\frac{1}{2}\phi_{cr}}
\end{pmatrix}
\exp(-\tilde{T}_{cr}(x, \lambda))
\times
\begin{pmatrix}
\ psi_{-}(x, \lambda) & \ psi_{+}(x, \lambda) \\
\ psi'_{-}(x, \lambda) & \ psi'_{+}(x, \lambda)
\end{pmatrix}^{-1}.
$$

The matrix

$$
\begin{pmatrix}
\ e^{-\frac{1}{2}\phi_{cr}} & 0 \\
\ 0 & e^{\frac{1}{2}\phi_{cr}}
\end{pmatrix}
\exp(-\tilde{T}_{cr}(x, \lambda))
$$

has the same conjugation property as the matrix $\hat{R}$. The matrix

$$
\begin{pmatrix}
\ psi_{-}(x, \lambda) & \ psi_{+}(x, \lambda) \\
\ psi'_{-}(x, \lambda) & \ psi'_{+}(x, \lambda)
\end{pmatrix}^{-1}
= \frac{1}{W\{\psi_+, \psi_-(\lambda)\}}
\begin{pmatrix}
\ psi'_{+}(x, \lambda) & -\ psi_{+}(x, \lambda) \\
-\ psi'_{-}(x, \lambda) & \ psi_{-}(x, \lambda)
\end{pmatrix}
$$

has the first row complex conjugate to the second row. Since for every $a, b, c \in \mathbb{C}$

$$
\begin{pmatrix}
\ a & b \\
\ b & \bar{a}
\end{pmatrix}
\begin{pmatrix}
\ c \\
\ \bar{c}
\end{pmatrix} =
\begin{pmatrix}
\ ac + b\bar{c} \\
\ bc + a\bar{c}
\end{pmatrix},
\begin{pmatrix}
\ 1 & i \\ \ 1 & -i \end{pmatrix}^{-1}
\begin{pmatrix}
\ c \\
\ \bar{c}
\end{pmatrix} =
\begin{pmatrix}
\ Re\ c \\
\ Im\ c
\end{pmatrix},
$$

the matrix $T_{cr}(x, \lambda)$ has real entries and therefore $w_{cr,\alpha}(x, \lambda) \in \mathbb{R}^2$ for every $x \in [0, +\infty)$, $\lambda \in U_{cr}$. The initial condition for the solution $w_{cr,\alpha}$ is

$$
w_{cr,\alpha}(0, \lambda) = T_{cr}(0, \lambda)
\begin{pmatrix}
\ sin\ \alpha \\
\ cos\ \alpha
\end{pmatrix},
$$
which is continuous in \( \lambda \). In particular,
\[
    \hat{w}_{cr,\alpha}(0, \nu_{cr}) = T_{cr}(0, \nu_{cr}) \begin{pmatrix} \sin \alpha \\ \cos \alpha \end{pmatrix} =: g_{cr,\alpha}.
\]
The matrix \( T_{cr}(0, \nu_{cr}) \) is non-degenerate, and hence the vector \( g_{cr,\alpha} \) runs over all the directions in \( \mathbb{R}^2 \) as \( \alpha \) runs over the interval \( [0, \pi] \).

From the asymptotics (3.23) due to the relation (3.20) for every \( \lambda \in U_{cr} \setminus \{\nu_{cr}\} \) we have
\[
    \lim_{x \to +\infty} \hat{w}_{cr,\alpha}(x, \lambda) = \left( \frac{A_\alpha(\lambda)}{A_\alpha(\lambda)} \right).
\]
Since \( \hat{T}_{cr}(x, \lambda) \to 0 \) as \( x \to +\infty \),
\[
    \lim_{x \to +\infty} w_{cr,\alpha}(x, \lambda) = \left( \begin{array}{cc} 1 & i \\ 1 & -i \end{array} \right)^{-1} \left( \begin{array}{cc} e^{-i \frac{\phi_{cr}}{2}} & 0 \\ 0 & e^{i \frac{\phi_{cr}}{2}} \end{array} \right) \left( \begin{array}{c} A_\alpha(\lambda) \\ A_\alpha(\lambda) \end{array} \right).
\]
The matrix \( \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & i \\ 1 & -i \end{array} \right) \) is unitary. Therefore
\[
    \left\| \lim_{x \to +\infty} w_{cr,\alpha}(x, \lambda) \right\| = \frac{|A_\alpha(\lambda)|}{\sqrt{2}} \left\| \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \right\| = |A_\alpha(\lambda)|.
\]
In the opposite direction, the relation (3.20) means that
\[
    \varphi_\alpha(x, \lambda) = (T_{cr}^{-1}(x, \lambda)w_{cr,\alpha}(x, \lambda), e_+) \in \mathbb{C}^2
\]
\[
    = \left( w_{cr,\alpha}(x, \lambda), \left( e^{-i \frac{\phi_{cr}}{2}} \psi_-(x, \lambda) + e^{i \frac{\phi_{cr}}{2}} \psi_+(x, \lambda) \right) + o(1) \right) \in \mathbb{C}^2.
\]
3. Consider the system (3.22) for \( \lambda = \nu_{cr} \):
\[
    w'_{cr}(x) = \left( \begin{array}{cc} \beta_{\nu_{cr}} & 0 \\ 0 & -1 \end{array} \right) + R_{cr}(x, \lambda) \left( \begin{array}{c} x \gamma \end{array} \right) w_{cr}(x).
\]
By the asymptotic Levinson theorem [8, Theorem 8.1] this system has two solutions \( w_{cr}^0 \) with the asymptotics
\[
    (3.28) \quad w_{cr}^\pm(x) = \exp \left( \pm \frac{\beta_{\nu_{cr}} x^{1-\gamma}}{1 - \gamma} \right) (c_\pm + o(1)) \quad \text{as} \quad x \to +\infty.
\]
Since the coefficients of the system are real-valued, \( w_{cr}^-(x) \in \mathbb{R}^2 \) for every \( x \in [0, +\infty) \) (since \( w_{cr}^-(x) \) is also a solution which has the same asymptotics and hence is proportional to \( w_{cr}^+(x) \) with the coefficient one). So there exists the unique \( \alpha_{cr} \in [0, \pi) \) such that the vector \( g_{cr,\alpha_{cr}} \) is proportional to \( w_{cr}^-(0) \):
\[
    g_{cr,\alpha_{cr}} = d_{cr} w_{cr}^-(0),
\]
and hence
\[
    w_{cr,\alpha_{cr}}(x, \nu_{cr}) = d_{cr}^- w_{cr}^-(x).
\]
From this and (3.28) the asymptotics (3.24) follows, and from it using the relation (3.27) we get the asymptotics (1.12) of the solution \( \varphi_{\alpha_{cr}}(x, \nu_{cr}) \) as \( x \to +\infty \). For every \( \alpha \neq \alpha_{cr} \) due to (3.28) and since \( w_{cr,\alpha}(0, \nu_{cr}) \parallel w_{cr}^-(0) \) we have:
\[
    w_{cr,\alpha}(x, \nu_{cr}) = d_{cr}(\alpha) \exp \left( \frac{\beta_{\nu_{cr}} x^{1-\gamma}}{1 - \gamma} \right) (c_\pm + o(1)) \quad \text{as} \quad x \to +\infty.
\]
The coefficient \( d_{cr}(\alpha) \) is a linear functional on \( \mathbb{R}^2 \), the space of initial conditions. Hence it can be expressed in terms of the scalar product with some fixed vector, or
in terms of the angle between this vector and the vector \( \left( \frac{\sin \alpha}{\cos \alpha} \right) \). Clearly this functional vanishes for \( \alpha_{cr} \), therefore it should be
\[
d_{cr}(\alpha) = d_{cr+} \sin(\alpha - \alpha_{cr})
\]
with some \( d_{cr+} \in \mathbb{R} \). From this we get the asymptotics (3.4), and using the relation (3.27) the asymptotics (1.14) of \( \varphi_\alpha(x, \nu_{cr}) \) as \( x \to +\infty \). This completes the proof. \( \square \)

4. The model problem

In this section we study the system (3.2) in the general setting and use only the objects and the properties that are listed in Lemma 3.1. We pass from the spectral parameter \( \lambda \) to the small parameter \( \varepsilon_0 \) supposing that the positive constant \( \beta \), the remainder matrix \( R(x, \varepsilon_0) \) with possibly complex entries and the vector of the initial condition \( f \in \mathbb{C}^2 \) are given. In such a setting we are able to establish the asymptotics as \( x \to +\infty \) and then as \( \varepsilon_0 \to 0 \) of solutions of the model system

\[
u'(x) = \left( \frac{\beta}{x^\gamma} \begin{pmatrix} \cos(\varepsilon_0 x) & \sin(\varepsilon_0 x) \\ \sin(\varepsilon_0 x) & -\cos(\varepsilon_0 x) \end{pmatrix} + R(x, \varepsilon_0) \right) \nu(x).
\]

We consider this system for \( \varepsilon_0 \in U_0 \), where \( U_0 \) is some interval with the midpoint zero. Let \( \alpha_r, c_r > 0 \) and
\[
r(x) := \frac{c_r}{(x + 1)^{1 + \alpha_r}}.
\]
We assume the following:

\[
\begin{align*}
\beta > 0, \\
\gamma \in (\frac{1}{2}, 1), \\
R(x, \cdot) \text{ for every } x \in [0, +\infty) \text{ is continuous in } U_0, \\
\|R(x, \varepsilon)\| < r(x) \text{ for every } x \in [0, +\infty) \text{ and } \varepsilon \in U_0.
\end{align*}
\]

Let us define for every \( \varepsilon_0 \in U_0 \) and \( f \in \mathbb{C}^2 \) the solution \( u(x, \varepsilon_0, f) \) of the Cauchy problem for the system (4.1) with the initial condition
\[
u(0, \varepsilon_0, f) = f.
\]
First we need to establish asymptotics of this solution as \( x \to +\infty \) for every fixed \( \varepsilon_0 \in U_0 \). We do this in the same way as in Lemma 3.1.

**Lemma 4.1.** Let the conditions (4.3) and (4.2) hold, let \( f \in \mathbb{C}^2 \) and let \( u(x, \varepsilon_0, f) \) be the solution of the system (4.1) with the initial condition (4.4).

1. For every \( \varepsilon_0 \neq 0 \) and \( f \in \mathbb{C}^2 \) there exists a finite non-zero limit \( \lim_{x \to +\infty} u(x, \varepsilon_0, f) \).

2. For \( \varepsilon_0 = 0 \) and every \( f \in \mathbb{C}^2 \) the following asymptotics holds:
\[
u(x, 0, f) = \exp \left( \frac{\beta x^{1-\gamma}}{1-\gamma} \right) \left( \Phi(f)e_+ + o(1) \right) \text{ as } x \to +\infty,
\]
where \( \Phi \) is a linear functional in \( \mathbb{C}^2 \). This functional has a one-dimensional kernel which consists of the vector \( f_- \) (and its multiples) such that
\[
u(x, 0, f_-) = \exp \left( -\frac{\beta x^{1-\gamma}}{1-\gamma} \right) \left( e_- + o(1) \right) \text{ as } x \to +\infty.
\]
Proof. 1. Just as in the previous section we make a Harris–Lutz transformation

\[ u(x) = \exp(T_u(x, \varepsilon_0))u_1(x) \]

with

\[ T_u(x, \varepsilon_0) := -\int_x^{+\infty} \begin{pmatrix} \cos(\varepsilon_0 x') & \sin(\varepsilon_0 x') \\ \sin(\varepsilon_0 x') & -\cos(\varepsilon_0 x') \end{pmatrix} dx'. \]

For \( \varepsilon_0 \neq 0 \) one has \( T_u(x, \varepsilon_0) = O \left( \frac{1}{x^2} \right) \) as \( x \to +\infty \) and using the same kind of estimates as in the proof of Lemma 3.1 we arrive at the system

\[ u'_1(x) = R_1(x, \varepsilon_0)u_1(x) \]

with

\[ R_1(x, \varepsilon_0) = O \left( \frac{1}{x^{2\gamma} + 1^{1+\alpha_1}} \right) \]

as \( x \to +\infty \),

which means that \( R_1(\cdot, \varepsilon_0) \in L_1(\mathbb{R}_+, M^{2\times2}(\mathbb{C})) \). The asymptotic Levinson theorem [8, Theorem 8.1] is applicable to this system and yields the existence of two solutions which have limits \( e_+ \) and \( e_- \) as \( x \to +\infty \). Hence every solution \( u_1(x) \) has a non-zero limit as \( x \to +\infty \). The same is true for every solution \( u(x) \), because \( T_u(x, \varepsilon_0) \) goes to zero at infinity.

2. If \( \varepsilon_0 = 0 \), then the asymptotic Levinson theorem is directly applicable to the system (4.1). Using it we conclude that there exists a solution with the asymptotics

\[ \exp \left( -\frac{\beta x^{1-\gamma}}{1-\gamma} \right) (e_- + o(1)) \]

as \( x \to +\infty \).

Let us take as \( f_- \) the value of this solution at zero. Again by the Levinson theorem there exists another solution, with the asymptotics

\[ \exp \left( \frac{\beta x^{1-\gamma}}{1-\gamma} \right) (e_+ + o(1)) \]

as \( x \to +\infty \).

Therefore

\[ u(x, \varepsilon_0, f) = \exp \left( \frac{\beta x^{1-\gamma}}{1-\gamma} \right) (\Phi(f)e_+ + o(1)) \]

as \( x \to +\infty \).

with some coefficient \( \Phi(f) \) which depends on \( f \) linearly and which is such that \( \Phi(f_-) = 0 \) and \( \dim \ker \Phi = 1 \). From [23, Lemma 4.2, case (2)] one can write out the formula for \( \Phi(f) \):

\[ \Phi(f) = \left( f + \int_0^{+\infty} \exp \left( -\frac{\beta x^{1-\gamma}}{1-\gamma} \right) R(x, 0)u(x, 0, f)dx \right) e_+ \infty. \]

This completes the proof. \( \square \)

The main result concerning the model problem is given by the following theorem which establishes the behaviour of the limit \( \lim_{x \to +\infty} u(x, \varepsilon_0, f) \) as \( \varepsilon_0 \to 0 \) and which will be proved in Section 10.

**Theorem 4.1.** Let the conditions (1.3) and (1.2) hold, let \( f \in \mathbb{C}^2 \setminus \{0\} \) and let \( u(x, \varepsilon_0, f) \) be the solution of the system (4.1) with the initial condition (4.4). The following asymptotic holds:

\[ \lim_{x \to +\infty} \|u(x, \varepsilon_0, f)\| = C_{mp}(\beta, \gamma) \exp \left( \frac{1}{|\varepsilon_0|^{\frac{1}{2\gamma}}} \int_0^{(2\beta)^\frac{1}{4}} \sqrt{\frac{\beta^2}{t^{2\gamma}} - \frac{1}{4}} dt \right) (|\Phi(f)| + o(1)) \]
as \( \varepsilon_0 \to 0 \), where \( \Phi \) is defined in Lemma 4.1 and

\[
C_{mp}(\beta, \gamma) := \frac{1}{\sqrt{2}} \exp \left( \int_0^{(2\beta)^{\frac{3}{4}}} \frac{\gamma}{2\tau} \left( 1 - \frac{1 - \varepsilon^2}{4\tau^2} \right) d\tau \right.
\]

\[
\left. - \int_{(2\beta)^{\frac{3}{4}}}^{(2\beta)^{\frac{3}{4}}} \frac{\gamma d\tau}{2\tau \sqrt{1 - \frac{\varepsilon^2}{4\tau^2}}} + \text{v.p.} \int_{(2\beta)^{\frac{3}{4}}}^{+\infty} \frac{\gamma d\tau}{2\tau \sqrt{1 - \frac{\varepsilon^2}{4\tau^2}}} \right). 
\]

Remark 4.1. 1. The case when \( f \) is proportional to \( f_- \) is included, and in such a case this is of course an estimate, not an asymptotics.

2. The integral in the exponent in the asymptotics (4.8) can be expressed in terms of the beta function:

\[
\int_0^{(2\beta)^{\frac{3}{4}}} \frac{\beta^2}{t^2 - 4} dt = \frac{(2\beta)^{\frac{3}{2}}}{2} \int_0^1 t^{\frac{1}{2}} \sqrt{1 - t^2} dt_1 \int_0^1 t^{\frac{1}{2}} \sqrt{1 - t^2} dt_2 = \frac{(2\beta)^{\frac{3}{4}}}{4\gamma} B \left( \frac{3}{2}, \frac{1 - \gamma}{2\gamma} \right). 
\]

Note that the asymptotics of \( \lim_{x \to +\infty} u(x, \varepsilon_0, f) \) contains \( \Phi(f) \) which is responsible for the behaviour of solutions for the fixed value of the parameter, \( \varepsilon_0 = 0 \). This is explained by the multiscale nature of the problem: \( \Phi(f) \) may be considered as the result of development of the solution in the “fast” variable \( \varepsilon \) which then enters the initial condition for the scaled system in the “slow” variable \( t \). We consider this scaled system below.

4.1. Reformulation. The system (4.1) was used in [29] for study of the case \( \gamma = 1 \).

Let us see why in the present form it is no longer suitable for the case \( \gamma \in (\frac{1}{2}, 1) \).

Consider \( \varepsilon_0 > 0 \). Scaling of the independent variable \( x = \frac{1}{\varepsilon_0} t \) leads to the system which has in some sense the following limit as \( \varepsilon_0 \to 0^+ \):

\[
v'(t) = \frac{\beta}{t} \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix} v(t),
\]

and the convergence is uniform in \( t \). This system can be further analysed as \( t \to +\infty \). If we did the same for the case \( \gamma \in (\frac{1}{2}, 1) \), we would come the system

\[
\varepsilon_0^{1-\gamma} v'(t) = \frac{\beta}{t^\gamma} \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix} v(t),
\]

which contains a small parameter at the derivative. This is why the difference between the cases \( \gamma = 1 \) and \( \gamma \in (\frac{1}{2}, 1) \) is essential. Moreover, in our case one needs to consider a different scale of the independent variable rather than \( x = \varepsilon_0^{-1} t \), namely \( x = \varepsilon_0^{\frac{1}{\gamma}} t \). This is not immediately clear: such a substitution does not eliminate the small parameter from the derivative and leads to growing oscillations in the coefficient matrix:

\[
\varepsilon_0^{\frac{1-\gamma}{\gamma}} v'(t) = \frac{\beta}{t^\gamma} \begin{pmatrix} \cos \left( \varepsilon_0^{\frac{1-\gamma}{\gamma}} t \right) & \sin \left( \varepsilon_0^{\frac{1-\gamma}{\gamma}} t \right) \\ \sin \left( \varepsilon_0^{\frac{1-\gamma}{\gamma}} t \right) & -\cos \left( \varepsilon_0^{\frac{1-\gamma}{\gamma}} t \right) \end{pmatrix} v(t).
\]
In view of the above let us begin not with scaling, but with getting rid of the oscillations: make the substitution
\[ u(x) = \begin{pmatrix} \cos \left( \frac{\pi x}{\varepsilon_0} \right) \\ \sin \left( \frac{\pi x}{\varepsilon_0} \right) \end{pmatrix} u_1(x), \]
which leads to the system
\[ u'_1(x) = \begin{pmatrix} \beta \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} \varepsilon_0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + R_1(x, \varepsilon_0) \] \[ u_1(x), \]
where
\[ R_1(x, \varepsilon_0) = \begin{pmatrix} \cos \left( \frac{\pi x}{\varepsilon_0} \right) \\ \sin \left( \frac{\pi x}{\varepsilon_0} \right) \end{pmatrix} R(x, \varepsilon_0) \begin{pmatrix} \cos \left( \frac{\pi x}{\varepsilon_0} \right) \\ \sin \left( \frac{\pi x}{\varepsilon_0} \right) \end{pmatrix}. \]
Now let us scale the independent variable so as to make the first and the second terms in the coefficient matrix of the same order,
\[ (4.9) \]
\[ x = \frac{t}{|\varepsilon_0|^{\frac{2\gamma}{\beta}}}, \]
and substitute
\[ u_1(x) = u_2 \left( |\varepsilon_0|^{-\frac{1}{2\gamma}} x \right). \]
This leads to two systems
\[ |\varepsilon_0|^{-\frac{1}{\beta} - \frac{\gamma}{\beta}} u_2(t) = \left( \begin{pmatrix} \beta \\ \varepsilon_0 \end{pmatrix} \begin{pmatrix} 1 & \pm \frac{1}{2} \\ \frac{1}{2} & \mp \beta \end{pmatrix} + \frac{1}{|\varepsilon_0|} R_1 \left( |\varepsilon_0|^{-\frac{1}{2\gamma}} t, \varepsilon_0 \right) \right) u_2(t), \]
for two possible signs of the parameter \( \varepsilon_0 \): one has to consider the cases \( \varepsilon_0 \to 0^+ \) and \( \varepsilon_0 \to 0^- \) separately. Here the signs in \( \pm \) and \( \mp \) correspond to the sign of \( \varepsilon_0 \).
Now we can define a new small positive parameter
\[ (4.10) \]
\[ \varepsilon := |\varepsilon_0|^{\frac{\gamma}{\beta - \frac{\gamma}{\beta}}} \]
from the set
\[ (4.11) \]
\[ U := \{ |\varepsilon_0|^{\frac{\gamma}{\beta - \frac{\gamma}{\beta}}}, \varepsilon_0 \in U_0 \} \backslash \{0\}, \]
and write these systems as
\[ (4.12) \]
\[ \varepsilon u_2^+(t) = (A^+_2(t) + R^+_2(t, \varepsilon))u_2^+(t) \]
and
\[ (4.13) \]
\[ \varepsilon u_2^-(t) = (A^-_2(t) + R^-_2(t, \varepsilon))u_2^-(t), \]
where
\[ A^\pm_2(t) := \left( \begin{pmatrix} \beta \\ \varepsilon_0 \end{pmatrix} \begin{pmatrix} 1 & \pm \frac{1}{2} \\ \frac{1}{2} & \mp \beta \end{pmatrix} \right) \]
and
\[ (4.14) \]
\[ R^\pm_2(t, \varepsilon) := \varepsilon^{-\frac{1}{2\gamma}} \left( \begin{pmatrix} \beta \\ \varepsilon_0 \end{pmatrix} \begin{pmatrix} -\cos \left( \frac{\pi t}{2\varepsilon} \right) \\ \sin \left( \frac{\pi t}{2\varepsilon} \right) \end{pmatrix} \right) R \left( \varepsilon^{-\frac{1}{2\gamma}} t, \pm \varepsilon \right) \]
\[ = \varepsilon^{-\frac{1}{2\gamma}} \begin{pmatrix} \cos \left( \frac{\pi t}{2\varepsilon} \right) \\ \sin \left( \frac{\pi t}{2\varepsilon} \right) \end{pmatrix} R \left( \varepsilon^{-\frac{1}{2\gamma}} t, \pm \varepsilon \right). \]
Let us introduce solutions \( u_2^+(t, \varepsilon, f) \) of the system (4.12) and \( u_2^-(t, \varepsilon, f) \) of the system (4.13), which correspond to \( u(x, \varepsilon_0, f) \), by the formula
\[ (4.15) \]
\[ u^\pm_2(t, \varepsilon, f) := \begin{pmatrix} \cos \left( \frac{\pi t}{2\varepsilon} \right) \\ \sin \left( \frac{\pi t}{2\varepsilon} \right) \end{pmatrix} u \left( \varepsilon^{-\frac{1}{2\gamma}} t, \pm \varepsilon \right). \]
so that they have initial conditions

\[(4.16)\quad u_2^\pm(0, \varepsilon, f) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} f.\]

Each of the systems (4.12) and (4.13) has an analytic part \((A_2^+ + 2)\) or \((A_2^- + 2)\) of the coefficient matrix and a remainder \((R_2^+ + 2)\) or \((R_2^- + 2)\) which is small in some sense. We will show that these remainders can be ignored away from zero. If there were no remainders, the well developed analytic theory would work here, see [35, Chapter VIII]. The eigenvalues of both matrices \(A_2^\pm(t)\) are the same:

\[\sqrt{\beta_2^2 t^2 - \frac{1}{4}}\quad \text{and} \quad -\sqrt{\beta_2^2 t^2 - \frac{1}{4}},\]

and

\[(4.17)\quad t_0 := (2\beta_2)^{\frac{1}{4}}\]

is the turning point for both systems (4.12). At this point eigenvalues of \(A_2^\pm(t)\) coincide, and each of two matrices is similar to a Jordan block. The behaviour as \(\varepsilon \to 0^+\) of solutions of both systems (4.12) has different character in the intervals \((0, t_0)\) and \((t_0, +\infty)\), so one needs to consider these intervals separately. In order to match the results in these two intervals we consider a small neighbourhood of the turning point and introduce a different (now again “fast”) variable \(z\) there.

However, this is still not enough: we need to consider intermediate regions at both sides of the turning point and use a different method there to treat the remainders \(R_2^\pm\). Only then matching of all the results can be done to trace the behaviour of \(u_2^\pm(t, \varepsilon, f)\) from \(t = 0\) to \(t = +\infty\).

In the statement of Theorem 4.1 one can rewrite the expression (4.6) in the following way:

\[(4.18)\quad \exp\left(-\frac{1}{\varepsilon} \int_0^{t_0} \sqrt{\beta_2^2 t^2 - \frac{1}{4}} \, dt\right) \lim_{t \to +\infty} \|u_2^\pm(t, \varepsilon, f)\| \to C_{mp}(\beta, \gamma) |\Phi(f)|\]

as \(\varepsilon \to 0^+\). Note that due to the oscillations in (4.13) \(u_2^\pm(t, \varepsilon, f)\) cannot have limits as \(t \to +\infty\), and only limits of their norms exist. This can be interpreted in the sense that the vector \(\Phi(f)e_+\) plays the role of the initial condition for the systems (4.12), and the growth of norms of the solutions \(u_2^\pm\) takes place on the interval \((0, t_0)\) at the rate determined by the positive eigenvalue of the matrices \(A_2^\pm\).

4.2. Regions of the half-line. In the next five sections we analyse the asymptotic behaviour of solutions of the system (4.12) in five different regions of the positive half-line. In each of these regions we need to use appropriate transformations in order to simplify the system. Then we combine the results to match asymptotics, and this gives the asymptotic behaviour of the limit of the norm of the solution \(u_2^\pm(t, \varepsilon, f)\) at \(t = +\infty\) as \(\varepsilon \to 0^+\). In Section 10 we will see that there is no need to perform a parallel study for the second system (4.13) and the solution \(u_2^-\). We take points \(t_{I-I} \in (0, t_0)\) and \(t_{IV-V} \in (t_0, +\infty)\) sufficiently close to the turning point \(t_0\). On the interval \([t_{I-I}, t_{IV-V}]\) we use the variable \(z = \varepsilon^{-\frac{1}{4}} \left(1 - \frac{t_{I-I}}{t_0}\right)\) for which this interval corresponds to \([-Z_2(\varepsilon), Z_2(\varepsilon)]\) (travelled in the opposite direction). We further divide this interval into three regions by the points \(\pm Z_0\) which in the variable \(t\) correspond to the points \(t_{II-I} \in (\varepsilon)\) and \(t_{III-I} \in (\varepsilon)\). The regions are
displayed on the following figure:

\[
\begin{array}{cccccc}
I & t_{I-II} & II & III & IV & V \\
\downarrow & Z_2(\varepsilon) & Z_1(\varepsilon) & Z_0 & -Z_0 & -Z_1(\varepsilon) & -Z_2(\varepsilon)
\end{array}
\]

In the region I we obtain the asymptotics of the solution \(u_2^+(t)\), for the regions II and IV we find bases of solutions with known asymptotics, for the region III we find a matrix solution and its asymptotics, and for the region V we find a family of solutions determined by their behaviour as \(t \to +\infty\) and establish their asymptotics as \(\varepsilon \to 0^+\).

In section devoted to the regions II–V we formulate results in a general form, that is for systems of the kind of (4.12), imposing different sufficient conditions on remainders of these systems. We as well check that these conditions are satisfied for the remainder \(R_2^+\) of the system (4.12) itself. In notation for each of these systems and for other related objects we use indices which correspond to the region that is considered.

5. Neighbourhood of the origin (region I): hyperbolic case

We start with the system

\[(5.1)\quad \varepsilon u_2^+(t) = \left( \begin{array}{c}
\frac{\beta}{2} & -\frac{1}{4} \\
-\frac{1}{8} & \frac{1}{2}
\end{array} \right) u_2^+(t) + R_2^+(t, \varepsilon) u_2^+(t),\]

Let us diagonalise the main term of the coefficient matrix with the transformation

\[(5.2)\quad u_2^+(t) = T_I(t)u_{I,1}(t),\]

where

\[(5.3)\quad T_I(t) := \left( \begin{array}{c}
1 \\
\frac{1}{2\beta(1+\sqrt{1-\frac{\varepsilon^2}{4\beta^2}})} \left( 1 + \sqrt{1-\frac{\varepsilon^2}{4\beta^2}} \right)
\end{array} \right)\]

(the eigenvector in the second column is chosen so that it does not have a singularity at \(t = 0\)). The substitution gives:

\[(5.4)\quad u_{I,1}^+(t) = \left( \begin{array}{c}
\lambda_I(t) \\
0
\end{array} \right) \left( \begin{array}{c}
1 \\
0
\end{array} \right) + S_I(t) + R_{I,1}(t, \varepsilon) u_{I,1}(t),\]

where

\[(5.5)\quad \lambda_I(t) := \sqrt{\frac{\beta^2}{t^{2\gamma}} - \frac{1}{4}},\]

\[(5.6)\quad S_I(t) := \frac{\gamma}{8\beta t^{1-\gamma}} \left( \begin{array}{cc}
\frac{1}{2\beta(1+\sqrt{1-\frac{\varepsilon^2}{4\beta^2}})} & -1 - \sqrt{1-\frac{\varepsilon^2}{4\beta^2}} \\
-\frac{1}{\beta(1+\sqrt{1-\frac{\varepsilon^2}{4\beta^2}})} & \frac{1}{2\beta(1+\sqrt{1-\frac{\varepsilon^2}{4\beta^2}})}
\end{array} \right),\]

\[(5.7)\quad R_{I,1}(t, \varepsilon) := \frac{T_I^{-1}(t)R_2^+(t, \varepsilon)T_I(t)}{\varepsilon}.\]

The result for the region I is given by the following lemma.
Lemma 5.1. Let the conditions (1.3) and (1.2) hold and let for $f \in \mathbb{C}^2$ the function $u(x, \varepsilon, f)$ be the solution of the system (4.1) with the initial condition (1.4). Let $u_2^+(t, \varepsilon, f)$ be given by (4.15) with the use of the definitions (1.9) of $t$ and (1.10) of $\varepsilon$, and thus be a solution of the system (5.1) where $R_2^+(t, \varepsilon)$ is given by (4.14). For every $t \in (0, t_0)$ and $\varepsilon \in U$ the following asymptotics holds:

$$u_2^+(t, \varepsilon, f) = T_I(t) \exp \left( \int_0^t \left( \frac{\lambda_I(\tau)}{\varepsilon} + S_{I, +}(\tau) \right) d\tau \right) \left( \Phi(f)e_+ + o(1) \right),$$

as $\varepsilon \to 0^+$, where $T_I, \lambda_I, S_I$ and $e_+$ are given by the expressions (5.3), (5.5), (5.6) and (1.21), respectively, $S_{I, +}$ is the upper-left entry of the matrix $S_I$ and $\Phi$ is defined in Lemma 4.4.

First let us prove an a priori estimate.

Lemma 5.2. Let $t_I \in (0, t_0)$. Under the conditions of Lemma 5.1 there exists $c_{13} > 0$ such that for every $t \in [0, t_I], \varepsilon \in U$ and $f \in \mathbb{C}^2$

$$\|u_2^+(t, \varepsilon, f)\| < c_{13} \exp \left( \frac{1}{\varepsilon} \int_0^t \lambda_I \right) \|f\|,$$

where $\lambda_I$ is given by (5.5).

Proof. Rough estimate of the norm of the coefficient matrix of the system (5.4) immediately gives:

$$\|u_{I, 1}(t, \varepsilon, f)\| < \exp \left( \int_0^t \left( \frac{\lambda_I(\tau)}{\varepsilon} + \|S_I(\tau)\| + \|R_{I, 1}(\tau, \varepsilon)\| \right) d\tau \right) \|f\|.$$

Since $S_I \in L_1((0, t_I), M_{2 \times 2}(\mathbb{C}))$, we need the estimate $\int_0^{t_I} \|R_{I, 1}(\tau, \varepsilon)\| d\tau = O(1)$ as $\varepsilon \to 0^+$. To see this first note that $T_I$ and $T_I^{-1}$ are bounded in $[0, t_I]$, and so, with some $c_{14} > 0$,

$$\int_0^{t_I} \|R_{I, 1}(\tau, \varepsilon)\| d\tau < \frac{c_{14}}{\varepsilon} \int_0^{t_I} \|R_2^+(\tau, \varepsilon)\| d\tau,$$

and so using (4.14), the condition on $R$ from (4.3) and summability of $r$ due to (4.2) we have

$$\int_0^{t_I} \|R_{I, 1}(\tau, \varepsilon)\| d\tau < \frac{c_{14}}{\varepsilon} \int_0^{t_I} \|R_2^+(\tau, \varepsilon)\| d\tau = \frac{c_{14}}{\varepsilon_0} \int_0^{t_I} R \left( \frac{\tau}{\varepsilon_0}, \varepsilon_0 \right) d\tau \leq c_{14} \int_0^{+\infty} r(x) dx,$$

which is finite and does not depend on $\varepsilon$. Using again boundedness of $T_I$ and the relation (5.2) we complete the proof. \qed

Remark 5.1. Note that the estimate (5.9) is not valid for $t \geq t_0$, because $S_I \notin L_1(0, t_0)$. Neighbourhood of the turning point requires special attention.

Let us make the variation of parameters

$$u_{I, 1}(t) = \begin{pmatrix} \exp \left( \frac{1}{\varepsilon} \int_0^t \frac{\tau}{\varepsilon} \lambda_I \right) & 0 \\ 0 & \exp \left( -\frac{1}{\varepsilon} \int_0^t \frac{\tau}{\varepsilon} \lambda_I \right) \end{pmatrix} u_{I, 2}(t),$$

where $u_{I, 2}(t)$ is not valid for $t \geq t_0$, because $S_I \notin L_1(0, t_0)$. Neighbourhood of the turning point requires special attention.
which gives
\[ u'_{I,2}(t) = \begin{pmatrix}
\exp\left(-\frac{1}{\varepsilon} \int_{0}^{t} \lambda_{I}\right) & 0 \\
0 & \exp\left(\frac{1}{\varepsilon} \int_{0}^{t} \lambda_{I}\right)
\end{pmatrix} (S_{I}(t) + R_{I,1}(t,\varepsilon))u_{I,1}(t). \]

Integrating this from 0 to \( t \) and returning to \( u_{I,1} \) we get the integral equation for the solution of the system (5.4),
\[ (5.11) \quad u_{I,1}(t,\varepsilon, f) := T_{I}^{-1}(t)u_{I,2}^{0}(t,\varepsilon, f), \]
\[ (5.12) \quad u_{I,1}(t,\varepsilon, f) = \begin{pmatrix}
\exp\left(\frac{1}{\varepsilon} \int_{0}^{t} \lambda_{I}\right) & 0 \\
0 & -\exp\left(-\frac{1}{\varepsilon} \int_{0}^{t} \lambda_{I}\right)
\end{pmatrix} f + \int_{0}^{t} \begin{pmatrix}
\exp\left(\frac{1}{\varepsilon} \int_{\tau}^{t} \lambda_{I}\right) & 0 \\
0 & \exp\left(-\frac{1}{\varepsilon} \int_{\tau}^{t} \lambda_{I}\right)
\end{pmatrix} (S_{I}(\tau) + R_{I,1}(\tau,\varepsilon))u_{I,1}(\tau,\varepsilon, f) d\tau. \]

Scaling
\[ (5.13) \quad u_{I,3}(t,\varepsilon, f) := \exp\left(-\frac{1}{\varepsilon} \int_{0}^{t} \lambda_{I}\right) u_{I,1}(t,\varepsilon, f) \]
we come to another integral equation,
\[ (5.14) \quad u_{I,3}(t,\varepsilon, f) = \begin{pmatrix}
1 & 0 \\
0 & -\exp\left(-\frac{2}{\varepsilon} \int_{0}^{t} \lambda_{I}\right)
\end{pmatrix} f + \int_{0}^{t} \begin{pmatrix}
1 & 0 \\
0 & \exp\left(-\frac{2}{\varepsilon} \int_{\tau}^{t} \lambda_{I}\right)
\end{pmatrix} (S_{I}(\tau) + R_{I,1}(\tau,\varepsilon))u_{I,3}(\tau,\varepsilon, f) d\tau. \]

Rewrite it as
\[ (5.15) \quad u_{I,3}(t,\varepsilon, f) = h_{I,3}(t,\varepsilon, f) + \int_{0}^{t} K_{I}(t,\tau,\varepsilon)u_{I,3}(\tau,\varepsilon, f) d\tau, \]
where
\[ (5.16) \quad h_{I,3}(t,\varepsilon, f) := \begin{pmatrix}
1 & 0 \\
0 & -\exp\left(-\frac{2}{\varepsilon} \int_{0}^{t} \lambda_{I}\right)
\end{pmatrix} f + \int_{0}^{t} \begin{pmatrix}
1 & 0 \\
0 & \exp\left(-\frac{2}{\varepsilon} \int_{\tau}^{t} \lambda_{I}\right)
\end{pmatrix} R_{I,1}(\tau,\varepsilon)u_{I,3}(\tau,\varepsilon, f) d\tau \]
and
\[ (5.17) \quad K_{I}(t,\tau,\varepsilon) := \begin{pmatrix}
1 & 0 \\
0 & \exp\left(-\frac{2}{\varepsilon} \int_{\tau}^{t} \lambda_{I}\right)
\end{pmatrix} S_{I}(\tau). \]

Define also
\[ (5.18) \quad K_{I}(t,\tau,0) := \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix} S_{I}(\tau). \]

Now fix an arbitrary point \( t_{I} \in (0, t_{0}). \) Consider the equation (5.15) as an equation in the Banach space \( L_{\infty}((0, t_{I}), \mathbb{C}^{2}) \)
\[ (5.19) \quad u_{I,3}(\varepsilon, f) = h_{I,3}(\varepsilon, f) + K_{I}(\varepsilon)u_{I,3}(\varepsilon, f), \]
where $K_I(\varepsilon)$ is the Volterra operator

$$K_I(\varepsilon) : u(t) \mapsto \int_0^t K_I(t, \tau, \varepsilon)u(\tau)d\tau$$

(which makes sense for $\varepsilon = 0$ too).

**Lemma 5.3.** Let the conditions of Lemma [5.1] hold and let $h_{I,3}$ be given by (5.16) with the use of the relations (5.11) and (5.13).

1. For every $t \in (0, t_0)$ and $f \in C^2$ the following limit exists:

$$\lim_{\varepsilon \to 0^+} h_{I,3}(t, \varepsilon, f) = \Phi(f)e_+ =: h_{I,3}(t, 0, f),$$

where $\Phi$ is defined in Lemma [4.7]

2. There exists $c_{15} > 0$ such that for every $t \in (0, t_0)$, $\varepsilon \in U$ and $f \in C^2$ one has $\|h_{I,3}(t, \varepsilon, f)\| < c_{15}\|f\|$.

**Remark 5.2.** Note that there is no convergence $h_{I,3}(\varepsilon, f) \to h_{I,3}(0, f)$ as $\varepsilon \to 0^+$ in the norm of $L_\infty((0, t_1), C^2)$.

**Proof.** Rewrite the second summand in (5.16) in the following way:

$$\int_0^t \left( 1 \begin{bmatrix} 0 & \exp \left( -\frac{2}{\varepsilon} \int_0^l \lambda_I \right) \end{bmatrix} \right) R_{I,1}(\tau, \varepsilon)u_{I,3}(\tau, \varepsilon, f) d\tau$$

$$= \int_0^t \left( 0 \begin{bmatrix} 1 & \exp \left( -\frac{2}{\varepsilon} \int_0^l \lambda_I \right) \end{bmatrix} \right) T_I^{-1}(\tau)R_{I,1}^+(\tau, \varepsilon)T_I(\tau)u_{I,1}(\tau, \varepsilon, f)e^{-\frac{1}{4} \int_0^\tau \lambda_I \frac{d\tau}{\varepsilon}}$$

$$= \int_0^t \left( 0 \begin{bmatrix} 1 & \exp \left( -\frac{2}{\varepsilon} \int_0^l \lambda_I \right) \end{bmatrix} \right) T_I^{-1}(\tau)$$

$$\times \left( \cos \left( \frac{\tau}{2} \right) \sin \left( \frac{\tau}{2} \right) - \cos \left( \frac{\tau}{2} \right) \right) R \left( \varepsilon_0^{-\frac{1}{4}} \tau, \varepsilon_0 \right) u \left( \varepsilon_0^{-\frac{1}{4}} \tau, \varepsilon_0, f \right) e^{-\frac{1}{4} \int_0^\tau \lambda_I \frac{d\tau}{\varepsilon_0}}$$

$$= \int_0^{\varepsilon_0^{-\frac{1}{4}} t} \left( 0 \begin{bmatrix} 1 & \exp \left( -\frac{2}{\varepsilon} \int_0^l \lambda_I \right) \end{bmatrix} \right) T_I^{-1}(\varepsilon_0^x)$$

$$\times \left( \cos \left( \frac{\tau}{2} \right) \sin \left( \frac{\tau}{2} \right) - \cos \left( \frac{\tau}{2} \right) \right) R(x, \varepsilon_0)u(x, \varepsilon_0, f) \exp \left( -\frac{1}{\varepsilon} \int_0^{\varepsilon_0^x} \lambda_I \right) dx.$$

Consider the expression $\exp \left( -\frac{2}{\varepsilon} \int_0^{\frac{x}{\varepsilon_0^x}} \lambda_I \right)$ which is positive and is less than one.

For every fixed $x$ and sufficiently small $\varepsilon$ it is less then $\exp \left( -\frac{2}{\varepsilon} \int_0^{\frac{x}{\varepsilon_0^x}} \lambda_I \right)$ which
converges to zero as $\varepsilon \to 0^+$. Moreover,

$$\exp \left( -\frac{1}{\varepsilon} \int_0^{\varepsilon_0} x \lambda_I \right) = \exp \left( -\frac{1}{\varepsilon} \int_0^{\varepsilon_0} \frac{\sqrt{\beta^2 t^2 - \frac{1}{4} dt}}{\varepsilon_0^2 b^2} \right)$$

$$= \exp \left( -\varepsilon_0 \int_0^x \sqrt{\frac{\beta^2}{\varepsilon_0^2 b^2} - \frac{1}{4} dy} \right) \to \exp \left( -\int_0^x \frac{\beta dy}{y^2} \right) = \exp \left( -\frac{\beta x^{1-\gamma}}{1-\gamma} \right)$$

as $\varepsilon \to 0^+$. Since $T_I(0) = I$ and the functions $R(x, \cdot)$ and $u(x, \cdot, f)$ are continuous in $U_0$, the expression under the integral in the result of the calculation (5.21) for every fixed $x$ converges to

$$\left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) R(x, 0) u(x, 0, f) \exp \left( -\frac{\beta x^{1-\gamma}}{1-\gamma} \right) dx.$$

Since

$$\|u(x, \varepsilon_0, f)\| = \left\| u_\varepsilon \left( \frac{\beta^2}{\varepsilon_0^2 b^2} \right) \left( x, \varepsilon, f \right) \right\| < c_{13} \exp \left( \frac{1}{\varepsilon} \int_0^{\varepsilon_0} x \lambda_I \right) \|f\|,$$

by Lemma 5.2 the estimate $\|R(x, \varepsilon_0)\| < r(x)$ provides a summable majorant for the expression under the integral, and by the Lebesgue’s dominated convergence theorem we get that for every $t \in (0, t_I]$ there exists a limit of $h_{I,3}(t, \varepsilon, f)$ as $\varepsilon \to 0^+$ which equals to

$$\left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \left( f + \int_0^{+\infty} R(x, 0) u(x, 0, f) \exp \left( -\frac{\beta x^{1-\gamma}}{1-\gamma} \right) dx \right) = \Phi(f)e_+,$$

according to the formula (1.5) for $\Phi$. The uniform boundedness of the $L_\infty((0, t_I), C^2)$ norm of $h_{I,3}(t, \varepsilon, f)$ also follows from the existence of a summable majorant. This completes the proof.

We denote by $B(L_\infty((0, t_I), C^2))$ the Banach space of bounded operators in $L_\infty((0, t_I), C^2)$.

**Lemma 5.4.** Let the conditions of Lemma 5.1 hold and $K_I(\varepsilon)$ be given by (5.17) for $\varepsilon \neq 0$ and by (5.13) for $\varepsilon = 0$. Let $h_{I,3}(\varepsilon, f)$ for $\varepsilon \neq 0$ be given by (5.10) with the use of the relations (5.11), (5.13), and for $\varepsilon = 0$ be defined in Lemma 5.3. Let $t_I \in (0, t_0)$.

1. $K_I(\varepsilon) \to K_I(0)$ as $\varepsilon \to 0^+$ in the norm of $B(L_\infty((0, t_I), C^2))$.
2. $K_I(0)h_{I,3}(\varepsilon, f) \to K_I(0)h_{I,3}(0, f)$ as $\varepsilon \to 0^+$ in the norm of $L_\infty((0, t_I), C^2)$.

**Proof.** 1. It suffices to prove that

$$\max_{t \in [0, t_I]} \int_0^t \|K_I(t, \tau, \varepsilon) - K_I(t, \tau, 0)\|d\tau \to 0 \text{ as } \varepsilon \to 0^+,$$

or that

$$\max_{t \in [0, t_I]} \int_0^t \exp \left( -\frac{2}{\varepsilon} \int_\tau^t \lambda_I \right) \|S_I(\tau)\|d\tau \to 0 \text{ as } \varepsilon \to 0^+.$$

On the interval $[0, t_I]$ we can use estimates $\lambda_I(t) > c_{16}$ and $\|S_I(t)\| < \frac{c_{17}}{t^{1-\gamma}}$ with some $c_{16}, c_{17} > 0$ which can be seen directly from the expressions (5.6) and (5.6).
Take an arbitrary small $\Delta > 0$. Firstly, there exists $t(\Delta)$ such that
\[
\int_{0}^{t(\Delta)} \|S_I(\tau)\|d\tau < \frac{\Delta}{2}.
\]
Thus
\[
(5.22) \quad \max_{t \in [0, t(\Delta)]} \int_{0}^{t} \exp \left( -\frac{2}{\varepsilon} \int_{\tau}^{t} \lambda_I(\tau) \right) \|S_I(\tau)\|d\tau < \frac{\Delta}{2}.
\]
Secondly,
\[
(5.23) \quad \max_{t \in [t(\Delta), t_1]} \int_{0}^{t} \exp \left( -\frac{2}{\varepsilon} \int_{\tau}^{t} \lambda_I(\tau) \right) \|S_I(\tau)\|d\tau
\leq \frac{\Delta}{2} + \frac{c_{17}}{(t(\Delta))^{1-\gamma}} \max_{t \in [0, t_1]} \int_{0}^{t} \exp \left( -\frac{2c_{16}}{\varepsilon} (t - \tau) \right) \|S_I(\tau)\|d\tau
\leq \frac{\Delta}{2} + \frac{c_{17}}{(t(\Delta))^{1-\gamma}} \frac{\varepsilon}{2c_{16}}.
\]
One can choose $\varepsilon(\Delta) > 0$ so that for every $\varepsilon \in (0, \varepsilon(\Delta))$ holds $\frac{c_{17}}{(t(\Delta))^{1-\gamma}} \frac{\varepsilon}{2c_{16}} < \frac{\Delta}{2}$, and so, from the estimates (5.22) and (5.23),
\[
\max_{t \in [0, t_1]} \int_{0}^{t} \exp \left( -\frac{2}{\varepsilon} \int_{\tau}^{t} \lambda_I(\tau) \right) \|S_I(\tau)\|d\tau < \Delta.
\]
This proves the convergence.

2. We have:
\[
\|K_I(0)(h_{I,3}(\varepsilon, f) - h_{I,3}(0, f))\|_{L^\infty(0,t_1)}
\leq \max_{t \in [0, t_1]} \left\| \int_{0}^{t} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} S_I(\tau)(h_{I,3}(\tau, \varepsilon, f) - h_{I,3}(\tau, 0, f))d\tau \right\|
\leq \int_{0}^{t_1} \|S_I(\tau)\| \|h_{I,3}(\tau, \varepsilon, f) - h_{I,3}(\tau, 0, f)\|d\tau.
\]
In the expression under the integral $S_I$ is summable and $h_{I,3}$ is point-wise convergent to zero and uniformly bounded by Lemma 5.3 Therefore, by the Lebesgue’s dominated convergence theorem, the integral converges to zero as $\varepsilon \to 0^+$. \(\Box\)

Now we are able to prove convergence of the solution.

**Lemma 5.5.** Let the conditions of Lemma 5.4 hold and let $u_{I,3}$ be defined by (5.18). For every $t \in (0, t_0)$ there exists the limit
\[
\lim_{\varepsilon \to 0^+} u_{I,3}(t, \varepsilon, f) =: u_{I,3}(t, 0, f),
\]
which satisfies the following integral equation on the interval $[0, t_0)$:
\[
(5.24) \quad u_{I,3}(t, 0, f) = h_{I,3}(t, 0, f) + \int_{0}^{t} K_I(t, \tau, 0)u_{I,3}(\tau, 0, f)d\tau,
\]
where $K_I(0)$ is given by (5.18), and $h_{I,3}(0, f)$ is defined in Lemma 5.3.

**Remark 5.3.** Note that again, as with $h_{I,3}(\varepsilon, f)$, there is no convergence in the norm of $L^\infty((0, t_1), C^2)$. However, the difference $u_{I,3}(\varepsilon, f) - h_{I,3}(\varepsilon, f)$ converges in this norm.
Proof. Take some $t_I \in (0, t_0)$. Let us rewrite the equation \((5.19)\) as
\[
u_{1,3}(\varepsilon, f) - h_{1,3}(\varepsilon, f) = K_I(\varepsilon)\nu_{1,3}(\varepsilon, f) + K_I(\varepsilon)(\nu_{1,3}(\varepsilon, f) - h_{1,3}(\varepsilon, f))
\]
and then as
\[
u_{1,3}(\varepsilon, f) - h_{1,3}(\varepsilon, f) = (I - K_I(\varepsilon))^{-1}K_I(\varepsilon)\nu_{1,3}(\varepsilon, f).
\]
By Lemma \([5.3]\) due to the boundedness of $h_{1,3}(\varepsilon, f)$ in the norm of $L_\infty((0, t_I), \mathbb{C}^2)$ provided by Lemma \([5.5]\) we have:
\[
K_I(\varepsilon)h_{1,3}(\varepsilon, f) = (K_I(\varepsilon) - K_I(0))h_{1,3}(\varepsilon, f) + K_I(0)h_{1,3}(\varepsilon, f) \to K_I(0)h_{1,3}(0, f)
\]
as $\varepsilon \to 0^+$. Since $K_I(\varepsilon)$ are Volterra operators, the norms $\| (I - K_I(\varepsilon))^{-1} \| \leq \exp(\| K_I(\varepsilon) \|)$ are bounded as $\varepsilon \to 0^+$. Therefore
\[
(I - K_I(\varepsilon))^{-1} = (I + (I - K_I(\varepsilon))^{-1}(K_I(\varepsilon) - K_I(0))(I - K_I(0))^{-1} \to (I - K_I(0))^{-1}
\]
in the norm of $B(L_\infty((0, t_I), \mathbb{C}^2))$. Hence in the equality \((5.25)\) there exists the limit
\[
\lim_{\varepsilon \to 0^+} (\nu_{1,3}(\varepsilon, f) - h_{1,3}(\varepsilon, f)) = (I - K_I(0))^{-1}K_I(0)h_{1,3}(0, f)
\]
in the norm of $L_\infty((0, t_I), \mathbb{C}^2)$ which means that $\nu_{1,3}(t, \varepsilon, f) - h_{1,3}(t, \varepsilon, f)$ has a limit for every $t \in [0, t_I]$ and uniformly in $t$. Since $t_I$ was chosen arbitrarily, the limit exists for every $t \in (0, t_0)$, however, without uniformity. By Lemma \([5.3]\) for every $t \in (0, t_0)$ the function $h_{1,3}(t, \varepsilon, f)$ has a limit as $\varepsilon \to 0^+$, also not uniform in $t$. On the interval $[0, t_I]$ we have the equality
\[
\nu_{1,3}(0, f) = h_{1,3}(0, f) + (I - K_I(0))^{-1}K_I(0)h_{1,3}(0, f).
\]
Applying $I - K_I(0)$ to both sides we arrive at the formula \((5.24)\). \(\square\)

Now we can prove Lemma \([5.1]\).

Proof of Lemma \([5.7]\). Equation \((5.24)\) is in fact simpler than it looks: it is merely an equation for the upper component of $\nu_{1,3}$ which can be solved explicitly. Indeed, using the formulae \((6.20)\) and \((5.18)\) for the initial condition and the kernel we get the equation
\[
\nu_{1,3}(t, 0, f) = \Phi(f)e_+ + \int_0^t \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} S_I(\tau)\nu_{1,3}(\tau, 0, f)d\tau,
\]
and the solution is given by the expression
\[
(5.26)\quad \nu_{1,3}(t, 0, f) = \Phi(f)\exp\left(\int_0^t S_{I, +}(\tau)d\tau\right)e_+.
\]
Putting this into \((5.13)\) and \((5.2)\) we obtain the asymptotics \((5.8)\), which completes the proof. \(\square\)

6. Neighbourhood of the infinity (region $V$): elliptic case

We start with the same system in different notation:
\[
(6.1)\quad \varepsilon u'_V(t) = \left(\begin{pmatrix} 0 & 1 \\ \frac{1}{\varepsilon} & \frac{1}{\varepsilon} \end{pmatrix} + R_V(t, \varepsilon)\right)u_V(t).
\]
In the same way as in the region $I$ let us diagonalise the main term of the coefficient matrix by the transformation
\[
(6.2)\quad u_V(t) = T_V(t)u_{V, 1}(t),
\]
where

\[(6.3) \quad T_V(t) := \left( \frac{2\beta}{t^\gamma} + i\sqrt{1 - \frac{4\beta^2}{t^{2\gamma}}} \frac{2\beta}{i} - i\sqrt{1 - \frac{4\beta^2}{t^{2\gamma}}} \right). \]

Substitution into the system \((6.1)\) gives:

\[(6.4) \quad u_{V,1}(t) = \left( \frac{\lambda_V(t)}{\epsilon} \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) + S_V(t) + R_{V,1}(t, \epsilon) \right) u_{I,1}(t),\]

where

\[(6.5) \quad \lambda_V(t) := -\frac{i}{2} \sqrt{1 - \frac{4\beta^2}{t^{2\gamma}}},\]

\[(6.6) \quad S_V(t) := \frac{\beta\gamma}{\left(1 - \frac{4\beta^2}{t^{2\gamma}}\right)^{1+\gamma}} \left( \begin{array}{cc} \frac{2\beta}{i} - i\sqrt{1 - \frac{4\beta^2}{t^{2\gamma}}} & \frac{2\beta}{i} - i\sqrt{1 - \frac{4\beta^2}{t^{2\gamma}}} \\ \frac{2\beta}{i} + i\sqrt{1 - \frac{4\beta^2}{t^{2\gamma}}} & \frac{2\beta}{i} + i\sqrt{1 - \frac{4\beta^2}{t^{2\gamma}}} \end{array} \right),\]

\[(6.7) \quad R_{V,1}(t, \epsilon) := \frac{T_V^{-1}(t) R_V(t, \epsilon) T_V(t)}{\epsilon}.\]

Here we consider solutions which are defined not by their values at zero, but rather by their asymptotics at infinity. Asymptotics of solutions in the region \(V\) is given by the following lemma.

**Lemma 6.1.** Let \(\beta > 0, \gamma \in (\frac{1}{2}, 1), t_0 = (2\beta)^{\frac{1}{\gamma}}\) and \(g \in \mathbb{C}^2\). If

\[(6.8) \quad \int_{t_0}^{+\infty} \| R_V(t, \epsilon) \| dt = o(\epsilon) \text{ as } \epsilon \to 0^+,\]

then for every \(\epsilon \in U\) there exists a solution \(u_V(t, \epsilon, g)\) of the system \((6.1)\) on the interval \((t_0, +\infty)\) with the following asymptotics:

\[(6.9) \quad u_V(t, \epsilon, g) = T_V(t) \times \left( \exp \left( \int_{t_0}^{t} \frac{\lambda_V(\tau)}{\epsilon} \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) d\tau - \int_{t}^{+\infty} \text{diag} S_V(\tau) d\tau \right) g + o(1) \right)\]

as \(\epsilon \to 0^+,\) where the convergence of the term \(o(1)\) is uniform with respect to \(t \in [t_V, +\infty)\) for every \(t_V \in (t_0, +\infty)\). Moreover, for every \(\epsilon \in U\)

\[(6.10) \quad \| u_V(t, \epsilon, g) \| \to \sqrt{2} \| g \| \text{ as } t \to +\infty.\]

Let us rewrite the condition \((6.8)\) in terms of the remainder \(R_{V,1}^{-1}\).

**Lemma 6.2.** Under the conditions of Lemma 6.1 if for some \(t_V \in (t_0, +\infty)\)

\[(6.11) \quad \int_{t_V}^{+\infty} \| R_V(t, \epsilon) \| dt = o(\epsilon) \text{ as } \epsilon \to 0^+,\]

then \(\int_{t_V}^{+\infty} \| R_{V,1}(t, \epsilon) \| dt \to 0\) as \(\epsilon \to 0^+\).}

**Proof.** Since \(T_V\) and \(T_V^{-1}\) are bounded in \([t_V, +\infty)\), one has with some \(c_{18} > 0\) using the definition \((6.7)\):

\[(6.12) \quad \int_{t_V}^{+\infty} \| R_{V,1}(\tau, \epsilon) \| d\tau < \frac{c_{18}}{\epsilon} \int_{t_V}^{+\infty} \| R_V(\tau, \epsilon) \| d\tau,\]

which goes to zero as \(\epsilon \to 0^+.\) □
Now let us see that the condition (6.8) is satisfied if we take \( R_V = R_V^+ \).

**Lemma 6.3.** Let \( R_V^+ \) be given by (4.14), \( r \in L_1(\mathbb{R}_+) \) and conditions (4.3) hold. Then

\[
\int_{t_0}^{+\infty} \| R_V^+(t, \varepsilon) \| dt = o(\varepsilon) \quad \text{as} \quad \varepsilon \to 0^+.
\]

**Proof.** Using the definition (4.14), the estimate of \( R \) from (4.3), the relation (4.10) and summability of the function \( r \) we have:

\[
\int_{t_0}^{+\infty} \| R_V^+(\tau, \varepsilon) \| d\tau = \frac{1}{\varepsilon_0} \int_{t_0}^{+\infty} \left\| R \left( \varepsilon_0^{-\frac{1}{2}} \tau, \varepsilon_0 \right) \right\| d\tau
\]

\[
\leq \varepsilon \varepsilon_0^{-\frac{1}{2}} \int_{t_0}^{+\infty} r \left( \varepsilon_0^{-\frac{1}{2}} \tau \right) d\tau = \varepsilon \int_{\varepsilon_0^{-\frac{1}{2}} t_0}^{+\infty} r(x) dx = o(\varepsilon) \quad \text{as} \quad \varepsilon \to 0^+.
\]

This gives the result. \(\square\)

It is more convenient for the later use to reformulate this result in different terms. Making the variation of parameters in the system (6.4),

\[
u_{V,1}(t) = \begin{pmatrix} \exp \left( \frac{1}{\varepsilon} \int_{t_0}^{t} \lambda_V \right) & 0 \\ 0 & \exp \left( -\frac{1}{\varepsilon} \int_{t_0}^{t} \lambda_V \right) \end{pmatrix} \nu_{V,2}(t),
\]

we come to the system

\[
u_{V,2}(t) = \begin{pmatrix} \exp \left( -\frac{1}{\varepsilon} \int_{t_0}^{t} \lambda_V \right) & 0 \\ 0 & \exp \left( \frac{1}{\varepsilon} \int_{t_0}^{t} \lambda_V \right) \end{pmatrix} (S_V(t) + R_{V,1}(t, \varepsilon))
\]

\[
\times \begin{pmatrix} \exp \left( \frac{1}{\varepsilon} \int_{t_0}^{t} \lambda_V \right) \\ 0 \end{pmatrix} \nu_{V,2}(t).
\]

The following lemma will be used as the result for the region \( V \) in Section 10.

**Lemma 6.4.** Let \( \beta > 0, \gamma \in (\frac{1}{2}, 1), t_0 = (2\beta)^{\frac{1}{2}}, g \in \mathbb{C}^2 \) and

\[
\int_{t_0}^{+\infty} \| R_V(t, \varepsilon) \| dt = o(\varepsilon) \quad \text{as} \quad \varepsilon \to 0^+.
\]

There exists the solution \( u_{V,2}(t, \varepsilon, g) \) of the system (6.13) such that the following holds.

1. For every \( \varepsilon \in U \)

\[ u_{V,2}(t, \varepsilon, g) \to g \text{ as } t \to +\infty. \]

2. For every \( t \in (t_0, +\infty) \)

\[ u_{V,2}(t, \varepsilon, g) \to \exp \left( -\int_{t}^{+\infty} \text{diag} S_V(\tau) d\tau \right) g \text{ as } \varepsilon \to 0^+, \]

and the limit is uniform with respect to \( t \in [t_V, +\infty) \) for every \( t_V \in (t_0, +\infty) \).

**Proof of the item 1.** We can use directly the asymptotic Levinson theorem. The coefficient matrix of the system (6.13) is summable near infinity: from (6.6) it is clear that \( S_V \in L_1((t_V, +\infty), M^{2 \times 2}(\mathbb{C})) \), and \( R_{V,1} \) is summable due to Lemma 6.2. Conditions of the asymptotic Levinson theorem [5, Theorem 8.1] are satisfied which gives the result. We will prove the item 2. later. \(\square\)
Let us integrate both sides of the equation \(6.13\) from \(t\) to \(+\infty\):

\[
(6.16) \quad u_{V,2}(t,\varepsilon, g) = g - \int_{t}^{+\infty} \left( \begin{array}{cc} \exp\left(\frac{-1}{\varepsilon} \int_{t_0}^{\tau} \lambda_V\right) & 0 \\ 0 & \exp\left(\frac{1}{\varepsilon} \int_{t_0}^{\tau} \lambda_V\right) \end{array} \right) \\
\times (S_V(\tau) + R_{V,1}(\tau,\varepsilon)) \left( \begin{array}{cc} \exp\left(\frac{1}{\varepsilon} \int_{t_0}^{\tau} \lambda_V\right) & 0 \\ 0 & \exp\left(-\frac{1}{\varepsilon} \int_{t_0}^{\tau} \lambda_V\right) \end{array} \right) u_{V,2}(\tau,\varepsilon, g) d\tau.
\]

Rewrite this as

\[
(6.17) \quad u_{V,2}(t,\varepsilon, g) = g + \int_{t}^{+\infty} K_V(\tau,\varepsilon) u_{V,2}(\tau,\varepsilon, g) d\tau,
\]

where

\[
(6.18) \quad K_V(\tau,\varepsilon) := - \left( \begin{array}{cc} \exp\left(\frac{-1}{\varepsilon} \int_{t_0}^{\tau} \lambda_V\right) & 0 \\ 0 & \exp\left(\frac{1}{\varepsilon} \int_{t_0}^{\tau} \lambda_V\right) \end{array} \right) \\
\times (S_V(\tau) + R_{V,1}(\tau,\varepsilon)) \left( \begin{array}{cc} \exp\left(\frac{1}{\varepsilon} \int_{t_0}^{\tau} \lambda_V\right) & 0 \\ 0 & \exp\left(-\frac{1}{\varepsilon} \int_{t_0}^{\tau} \lambda_V\right) \end{array} \right).
\]

Define also

\[
(6.19) \quad K_V(\tau, 0) := - \text{diag} S_V(\tau).
\]

Take some \(t_V \in (t_0, +\infty)\) and consider \((6.17)\) as an equation in the Banach space
\(L_\infty((t_V, +\infty), \mathbb{C}^2)\)

\[
(6.20) \quad u_{V,2}(\varepsilon, g) = g + K_V(\varepsilon) u_{V,2}(\varepsilon, g),
\]

where \(K_V(\varepsilon)\) is the Volterra operator

\[
(6.21) \quad K_V(\varepsilon) : u(t) \mapsto \int_{0}^{t} K_V(\tau, \varepsilon) u(\tau) d\tau,
\]

which is also defined by this rule for \(\varepsilon = 0\).

**Remark 6.1.** Until this moment we were following the scheme which was already used for the region \(I\). This analogy does not go further: the operators \(K_V(\varepsilon)\) do not converge in the norm of \(B(L_\infty((t_V, +\infty), \mathbb{C}^2))\), while the solutions \(u_{V,2}(\varepsilon, g)\) do converge in the norm of \(L_\infty((t_V, +\infty), \mathbb{C}^2)\).

**Lemma 6.5.** Let the conditions of Lemma \(6.4\) hold, \(t_V \in (t_0, +\infty)\) and \(K_V(\varepsilon)\) be the operator in the space \(L_\infty((t_V, +\infty), \mathbb{C}^2)\) defined by \(6.21\) with the use of expressions \(6.18\) and \(6.19\). Operator \(K_V(\varepsilon)\) converges in the strong sense to the operator \(K_V(0)\) as \(\varepsilon \to 0^+\), and \((I - K_V(\varepsilon))^{-1}\) converges in the strong sense to \((I - K_V(0))^{-1}\).
Proof. Take some arbitrary \( u \in L_\infty((t_\nu, +\infty), \mathbb{C}^2) \). We have:

\[
\| (K_V(\varepsilon) - K_V(0))u \|_{L_\infty((t_\nu, +\infty)} \leq \left\| \int_{t_\nu}^{+\infty} (K_V(\tau, \varepsilon) - K_V(\tau, 0))u(\tau)d\tau \right\|
\]

\[
\leq \left| \int_{t_\nu}^{+\infty} S_{V,12}(\tau)u_2(\tau) \exp \left( -\frac{2}{\varepsilon} \int_{t_\nu}^{\tau} \lambda_V(\tau')d\tau' \right) d\tau \right|
\]

\[
+ \left| \int_{t_\nu}^{+\infty} S_{V,21}(\tau)u_1(\tau) \exp \left( \frac{2}{\varepsilon} \int_{t_\nu}^{\tau} \lambda_V(\tau')d\tau' \right) d\tau \right|
\]

\[
+ \| u \|_{L_\infty((t_\nu, +\infty)} \int_{t_\nu}^{+\infty} \| R_{V,1}(\tau, \varepsilon) \|d\tau,
\]

where \( S_{V,12} \) and \( S_{V,21} \) are the upper-right and the lower-left entries of the matrix \( S_V \), and \( u_1 \) and \( u_2 \) are the upper and the lower components of the vector \( u \). From the estimate (6.11) it follows that the third term in the last formula goes to zero as \( \varepsilon \to 0^+ \). For the first term after the change of the variable of integration to

\[
s(\tau) = \int_{t_\nu}^{\tau} \sqrt{1 - \frac{4\beta^2}{\tau^2}}d\tau'
\]

we have

\[
\int_{t_\nu}^{+\infty} S_{V,12}(\tau)u_2(\tau) \exp \left( -\frac{2}{\varepsilon} \int_{t_\nu}^{\tau} \lambda_V(\tau')d\tau' \right) d\tau
\]

\[
= \int_{t_\nu}^{+\infty} S_{V,12}(\tau)u_2(\tau) \exp \left( \frac{i}{\varepsilon} \int_{t_\nu}^{\tau} \sqrt{1 - \frac{4\beta^2}{\tau^2}}d\tau' \right) d\tau
\]

\[
= \int_{s(t_\nu)}^{+\infty} S_{V,12}(\tau(s))u_2(\tau(s)) \exp \left( \frac{is}{\varepsilon} \right) ds \to 0 \text{ as } \varepsilon \to 0^+,
\]

by the Riemann-Lebesgue lemma, because \( S_Vu \in L_1((t_\nu, +\infty), M^{2\times 2}(\mathbb{C})) \) and

\[
\int_{s(t_\nu)}^{+\infty} |S_{V,12}(\tau(s))u_2(\tau(s))| \frac{ds}{\sqrt{1 - \frac{4\beta^2}{\tau(s)^2}}} = \int_{t_\nu}^{+\infty} |S_{V,12}(\tau)u_2(\tau)|d\tau < \infty.
\]

The second term goes to zero for the analogous reason and the third term by Lemma 6.2. This proves strong convergence of operators \( K_V(\varepsilon) \).

It remains to prove strong convergence of \( (I - K_V(\varepsilon))^{-1} = \sum_{n=0}^{\infty} K^n_V(\varepsilon) \). This series converges in the norm of \( B(L_\infty((t_\nu, +\infty), \mathbb{C}^2)) \) uniformly in \( \varepsilon \in U \cup \{0\} \), because for every \( \varepsilon \in U \cup \{0\} \) and \( n \in \mathbb{N} \) one has

\[
\| K^n_V(\varepsilon) \|_{B(L_\infty)} \leq \left( \frac{\| K_V(\varepsilon) \|_{B(L_\infty)}}{n!} \right)^n \text{ and } \| K_V(\varepsilon) \|_{B(L_\infty)} \leq \int_{t_\nu}^{+\infty} \| S_V(\tau) \|d\tau.
\]

Take \( u \in L_\infty((t_\nu, +\infty), \mathbb{C}^2) \). By induction one proves that for every \( n \in \mathbb{N} \) it holds that \( K^n_V(\varepsilon)u \to K^n_V(0)u \) as \( \varepsilon \to 0^+ \):

\[
K^n_V(\varepsilon)u = K_V(\varepsilon)(K^n_V(1) - K^{n-1}_V(0))u + K_V(\varepsilon)K^{n-1}_V(0)u \to K^n_V(0)u,
\]

where the first term goes to zero by the induction hypothesis and the second converges to the result due to the strong convergence of \( K_V(\varepsilon) \). Take arbitrarily small \( \Delta > 0 \). There exists \( N(\Delta) \) such that for every \( \varepsilon \in U \cup \{0\} \) and \( N > N(\Delta) \) one has

\[
\left\| \sum_{n=N}^{\infty} K^n_V(\varepsilon)u \right\| < \frac{\Delta}{3}.
\]
There also exists $\varepsilon(\Delta) > 0$ such that for every $\varepsilon < \varepsilon(\Delta)$ it holds that
\[
\left| \sum_{n=0}^{\infty} (\mathcal{K}_V^n(\varepsilon) - \mathcal{K}_V^n(0))u \right| < \frac{\Delta}{3}.
\]
Therefore for every $\varepsilon < \varepsilon(\Delta)$ we have
\[
\left| \sum_{n=0}^{\infty} \mathcal{K}_V^n(\varepsilon)u - \sum_{n=0}^{\infty} \mathcal{K}_V^n(0)u \right| < \frac{2\Delta}{3} + \sum_{n=0}^{N(\Delta)} (\mathcal{K}_V^n(\varepsilon) - \mathcal{K}_V^n(0))u < \Delta,
\]
which proves that
\[(I - \mathcal{K}_V(\varepsilon))^{-1}u \to (I - \mathcal{K}_V(0))^{-1}u \text{ as } \varepsilon \to 0^+.
\]
Since $u \in L_\infty((t_V, +\infty), \mathbb{C}^2)$ was arbitrary, this proves the strong convergence and thus completes the proof of the lemma. \(\square\)

Now we are able to prove the remaining part of Lemma 6.4.

Proof of Lemma 6.4, item 2. Rewriting the equation (6.20) as
\[u_{V,2}(\varepsilon, g) = (I - \mathcal{K}_V(\varepsilon))^{-1}g\]
we use Lemma 6.3 to conclude that $u_{V,2}(\varepsilon, g) \to u_{V,2}(0, g)$ as $\varepsilon \to 0^+$ in the norm of $L_\infty((t_V, +\infty), \mathbb{C}^2)$, where
\[(6.22) \quad u_{V,2}(0, g) := (I - \mathcal{K}_V(0))^{-1}g,\]
which means that for every $g \in \mathbb{C}^2$ and $t \in [t_V, +\infty)$, $u_{V,2}(t, \varepsilon, g)$ converges as $\varepsilon \to 0^+$ to $u_{V,2}(t, 0, g)$ uniformly with respect to $t$ in this interval. Applying the operator $I - \mathcal{K}_V(0)$ to both sides of the equality (6.22) we get:
\[u_{V,2}(t, 0, g) = g - \int_{t}^{+\infty} \text{diag} \, S_V(\tau) u_{V,2}(\tau, 0, g) d\tau.\]
Solution of this equation is
\[u_{V,2}(t, 0, g) = \exp \left( - \int_{t}^{+\infty} \text{diag} \, S_V(\tau) d\tau \right) g,\]
which coincides with the expression in (6.15) and thus completes the proof. \(\square\)

The proof of Lemma 6.1 follows.

Proof of Lemma 6.1. Convergence in (6.9) follows from Lemma 6.4 and substitution of the limit (6.10) to the relations (6.12) and (6.2). To prove convergence in (6.10) first note that $u_V(t, \varepsilon, g)$ is bounded as $t \to +\infty$, because
\[\|u_V(t, \varepsilon, g)\| \leq \|T_V(t)\|\|u_{V,2}(t, \varepsilon, g)\|\]
and $u_{V,2}(t, \varepsilon, g)$ is bounded by Lemma 6.3. Then, since
\[T_V(t) \to \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \text{ as } t \to +\infty\]
and the matrix $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$ is unitary, by Lemma 6.4 we have the convergence of the norm: $\|u_V(t, \varepsilon, g)\| \to \sqrt{2}\|g\|$ as $\varepsilon \to +\infty$. This completes the proof. \(\square\)
7. Intermediate region II: hyperbolic case

Consider the system (4.12)
\[
\varepsilon u''_{II}(t) = \left( \begin{array}{c} \frac{\beta t}{2} - \frac{1}{2t^2} \\ \frac{\beta}{4} \end{array} \right) + R_{II}(t, \varepsilon) u_{II}(t).
\]

Let us again, like in the region I, diagonalise the main term of the coefficient matrix, this time with the transformation
\[
(7.1) \quad u_{II}(t) = T_{II}(t) u_{II,1}(t),
\]
where
\[
(7.2) \quad T_{II}(t) := \begin{pmatrix} \frac{1}{2\beta (1 + \sqrt{1 - \frac{t^2}{4\beta^2}})} & \frac{1}{2\gamma} \left( 1 + \sqrt{1 - \frac{t^2}{4\beta^2}} \right) \\ \frac{1}{2\gamma} \left( 1 - \sqrt{1 - \frac{t^2}{4\beta^2}} \right) & \frac{1}{2\beta (1 - \sqrt{1 - \frac{t^2}{4\beta^2}})} \end{pmatrix}.
\]

Note that the first column is the same as of the matrix \(T_I(t)\) in (5.3). The substitution gives:
\[
(7.3) \quad u'_{II,1}(t) = \left( \frac{\lambda_{II}(t)}{\varepsilon} \right) \left( \begin{array}{c} 1 \\ 0 \end{array} \right) + S_{II}(t) + R_{II,1}(t, \varepsilon) u_{II,1}(t),
\]
where
\[
(7.4) \quad \lambda_{II}(t) := \sqrt{\frac{\beta^2}{2\gamma} - \frac{1}{4}} = \lambda_I(t),
\]
\[
(7.5) \quad S_{II}(t) := \frac{\gamma}{2t \left( 1 - \frac{t^2}{4\beta^2} \right)} \left( \begin{array}{cc} 1 & -1 - \frac{\sqrt{1 - \frac{t^2}{4\beta^2}}}{\sqrt{1 - \frac{t^2}{4\beta^2}}} \\ -1 + \frac{\sqrt{1 - \frac{t^2}{4\beta^2}}}{\sqrt{1 - \frac{t^2}{4\beta^2}}} & 1 + \frac{\sqrt{1 - \frac{t^2}{4\beta^2}}}{\sqrt{1 - \frac{t^2}{4\beta^2}}} \end{array} \right),
\]
\[
R_{II,1}(t, \varepsilon) := \frac{T_{II}^{-1}(t) R_{II}(t, \varepsilon) T_{II}(t)}{\varepsilon}.
\]

In the region I our analysis was based on the fact that the term \(S_I\) is summable over the whole interval. Since \(S_I\) is not summable up to the turning point \(t_0 = (2\beta)^{\frac{1}{2}}\), the integral containing \(S_I\) diverges as \(t\) approaches \(t_0\), so Lemma 5.1 does not work for the interval \((0, t_0]\). In the same way and for the same reason Lemma 6.1 does not work for the interval \([t_0, +\infty)\). This can be seen as an effect of the interplay between the first and the second terms in the coefficient matrix of the system (7.3) which should be taken into account. We do this by scaling the independent variable near the turning point. To choose the new scale observe that near the turning point the first term has the order \(\sqrt{\frac{t - t_0}{\varepsilon}}\), while the second term has the order \(\frac{1}{\varepsilon - t_0}\). To match these orders we have to consider the values of \(t\) such that \(t - t_0 \approx \varepsilon \frac{t_0}{4}\).

Therefore let us take
\[
(7.6) \quad z := \frac{1 - \frac{t^2}{4\beta^2}}{\varepsilon^{\frac{3}{2}}},
\]
Since
\[
1 - \frac{t^2}{4\beta^2} = 2\gamma (t_0 - t) + O((t - t_0)^2) \quad \text{as} \quad t \to t_0,
\]
this is almost the same as taking \(z' = \frac{2\gamma}{(4\beta)^{\frac{3}{2}}} \frac{t_0 - t}{\varepsilon^{\frac{1}{2}}}\). These two substitutions lead to different systems, but essentially they are equivalent, because in the limit as \(\varepsilon \to 0^+\), asymptotically, solutions of these systems behave similarly. The first choice leads to
simpler formulae, so we use the variable $z$ defined by (7.6). One could also choose the sign differently, $\tilde{z} := \frac{c_0^2 - 1}{\varepsilon} = -z$, but our choice of sign will be more natural for the region $III$ where we use the same variable $z$.

Let us take (7.7) $u_{II,2}(t) = u_{II,2}(z(t, \varepsilon))$ and substitute this into the system (7.3). We come to the system

$$u''_{II,2}(z) = (\Lambda_{II,2}(z, \varepsilon) + S_{II,2}(z, \varepsilon) + R_{II,2}(z, \varepsilon))u_{II,2}(z)$$

with

$$\Lambda_{II,2}(z, \varepsilon) := \lambda_{II,2}(z, \varepsilon) \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right),$$

$$\lambda_{II,2}(z, \varepsilon) := \frac{\lambda_{II}(t(z, \varepsilon))t'(z, \varepsilon)}{\varepsilon} = -\frac{c_0\sqrt{z}}{(1 - \varepsilon^4 z)^{\frac{3}{2}}},$$

$$c_0 := \frac{t_0}{4\gamma}, \quad \kappa := \frac{3}{2} - \frac{1}{2\gamma},$$

$$S_{II,2}(z, \varepsilon) := S_{II}(t(z, \varepsilon))t'(z, \varepsilon) = -\frac{1}{4\varepsilon(1 - \varepsilon^4 z)} \left( \begin{array}{cc} 1 - \varepsilon^4 \sqrt{z} & -1 - \varepsilon^4 \sqrt{z} \\ -1 + \varepsilon^4 \sqrt{z} & 1 + \varepsilon^4 \sqrt{z} \end{array} \right)$$

and

$$R_{II,2}(z, \varepsilon) := -\frac{t_0}{2\gamma \varepsilon^4} \frac{T_{II}^{-1}(t(z, \varepsilon))R_{II}(t(z, \varepsilon), \varepsilon)T_{II}(t(z, \varepsilon))}{(1 - \varepsilon^4 z)^{1 - \frac{3}{2}\varepsilon}}.$$ We consider the system (7.8) on the interval $[Z_0, Z_2(\varepsilon)]$ with

$$Z_2(\varepsilon) = \frac{1}{\varepsilon^{\frac{3}{2}}} \left( 1 - \frac{t_{II-1II}^2}{4\beta^2} \right)$$

(so that $t_{II-II}(\varepsilon) = t_0(1 - \varepsilon^4 Z_0^{\frac{3}{2}})$), where the constant $Z_0$ should be large enough and the constant $t_{II-II}$ such that the distance $(t_0 - t_{II-II})$ is small enough. We will choose these constants later.

Consider for a moment the free system. We need the next lemma to be formulated here, although its statement will automatically follow from our analysis in the proof of Lemma 7.6, see Remark 7.2.

**Lemma 7.1.** The system

$$v''_{II,2}(z) = \left(-c_0\sqrt{z}\left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) - \frac{1}{4\varepsilon} \left( \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right) \right) v_{II,2}(z)$$

has for $z \in [1, +\infty)$ solutions $v_{II,2}^{\pm}(z)$ such that

$$v_{II,2}^{\pm}(z) = \frac{\exp\left( \mp \frac{2c_0}{3} z^{\frac{3}{2}} \right)}{z^{\frac{3}{4}}}(e_{\pm} + o(1))$$

as $z \to +\infty$.

The main result for the region $II$ is the following lemma.
Lemma 7.2. Let \( c_0, \kappa > 0 \) and let \( R_{II,2}(z, \varepsilon) \) be given by (7.13) with the use of the expression (7.2) and the definition (7.6) of \( z \). There exist \( Z_0 > 0 \) and \( t_{I-1I} \in (0, t_0) \) such that, with \( Z_2(\varepsilon) \) given by (7.14), if

\[
\int_{Z_0}^{Z_2(\varepsilon)} \frac{\| R_{II}(t(s, \varepsilon), \varepsilon) \|}{\sqrt{s}} ds = o \left( \varepsilon^{\frac{4}{3}} \right) \quad \text{as} \ \varepsilon \to 0^+,
\]

then for every sufficiently small \( \varepsilon > 0 \) the system (7.3) on the interval \([Z_0, Z_2(\varepsilon)]\) has two solutions \( u_{II,2}^\pm(z, \varepsilon) \) such that, as \( \varepsilon \to 0^+ \),

\[
u_{II,2}(z, \varepsilon) = a_{II}^\pm \exp \left( \int_{Z_0}^{Z_2(z)} \left( \frac{c_0 \sqrt{s}}{(1 - \varepsilon \mp s)^\kappa} - \frac{1}{4s(1 \pm \varepsilon \mp \sqrt{s})} \right) ds \right) (e_\pm + o(1)),
\]

where \( a_{II}^\pm \) are positive constants and the vectors \( e_\pm \) are given by (1.21). Moreover, \( u_{II,2}^\pm(z, \varepsilon) \to v_{II,2}^\pm(z) \) as \( \varepsilon \to 0^+ \) for every fixed \( z \geq Z_0 \), where \( v_{II,2}^\pm \) are defined in Lemma 7.1.

Let us rewrite the condition (7.17) in terms of the remainder \( R_{II,2} \).

Lemma 7.3. Under the conditions of Lemma 7.2 for every \( z_0 > 0 \) and \( \nu \in (0, 1) \) with \( z_2(\varepsilon) = \frac{\nu \varepsilon}{\sqrt{s}} \), if

\[
\int_{z_0}^{z_2(\varepsilon)} \frac{\| R_{II}(t(s, \varepsilon), \varepsilon) \|}{\sqrt{s}} ds = o \left( \varepsilon^{\frac{4}{3}} \right), \quad \text{then} \quad \int_{z_0}^{z_2(\varepsilon)} \| R_{II,2}(s, \varepsilon) \| ds \to 0
\]
as \( \varepsilon \to 0^+ \).

Proof. Let

\[
t_\nu := t_0 (1 - \nu)^{\frac{1}{4}}.
\]

From the expression (7.2) for \( T_{II} \) it is clear that there exists \( c_{19} > 0 \) such that \( \| T_{II}(t) \| < c_{19} \) for every \( t \in [t_\nu, t_0] \) and that

\[
det T_{II}(t) \sim 2 \sqrt{1 - \frac{t^{2\gamma}}{4\beta^2}} = 2\varepsilon^{\frac{4}{3}} \sqrt{z(t, \varepsilon)} \quad \text{as} \ t \to t_0,
\]

therefore there exists \( c_{20} > 0 \) such that

\[
\| T_{II}^{-1}(t(z, \varepsilon)) \| < \frac{c_{20}}{\varepsilon^{\frac{4}{3}} \sqrt{z}} \quad \text{for every} \ \varepsilon \in U \ \text{and} \ z \in [z_0, z_2(\varepsilon)].
\]

Hence from the definition (7.13) of \( R_{II,2} \) one has that with some \( c_{21} > 0 \)

\[
\int_{z_0}^{z_2(\varepsilon)} \| R_{II,2}(s, \varepsilon) \| ds \leq \frac{c_{21}}{\varepsilon^{\frac{4}{3}}} \int_{z_0}^{z_2(\varepsilon)} \frac{\| R_{II}^+ (t(s, \varepsilon), \varepsilon) \| ds}{\sqrt{s}}
\]

which goes to zero as \( \varepsilon \to 0^+ \) by the hypothesis. \( \square \)

Now let us see that the condition (7.17) is satisfied, if \( R_{II} = R_{II}^+ \).

Lemma 7.4. Let \( R_{II}^+(t, \varepsilon) \) be given by (1.14) and \( t(z, \varepsilon) \) be defined by (4.9). Let the conditions (4.3) and (4.2) hold. Then for every \( z_0 > 0 \) and \( \nu \in (0, 1) \) with \( z_2(\varepsilon) = \frac{\nu \varepsilon}{\sqrt{s}} \), the following estimate holds:

\[
\int_{z_0}^{z_2(\varepsilon)} \frac{\| R_{II}^+(t(s, \varepsilon), \varepsilon) \|}{\sqrt{s}} ds = O \left( \varepsilon^{\frac{4}{3}} \varepsilon_0^{2\nu} \right) \quad \text{as} \ \varepsilon \to 0^+.
\]
Proof. Using the equalities (4.14) and the estimate of the norm of \( R \) from the conditions (4.4) we have
\[
\int_{z_0}^{z_2(\varepsilon)} \frac{\| R_{t_0}^+ (t(s, \varepsilon), \varepsilon) \| ds}{\sqrt{s}} < \frac{1}{\varepsilon_0} \int_{z_0}^{z_2(\varepsilon)} \left( \varepsilon_0^{-\frac{1}{2}} t(s, \varepsilon) \right) ds \frac{\sqrt{s}}{s}
\]
Since the derivative of \( \varepsilon \) is \( \frac{d \varepsilon(t, \varepsilon)}{dt} = -\frac{2t^{\gamma-1}}{2\beta^2 s^3} \), one has
\[
\frac{1}{\varepsilon_0} \int_{z_0}^{z_2(\varepsilon)} r \left( \varepsilon_0^{-\frac{1}{2}} t(s, \varepsilon) \right) ds \frac{\sqrt{s}}{s} < -\frac{\gamma}{2\beta^2 \varepsilon_0^3} \int_{t_\nu}^{t_0} r \left( \varepsilon_0^{-\frac{1}{2}} t \right) \frac{t^{2\gamma-1} dt}{\sqrt{1 - \frac{t^2}{t_0^2}}},
\]
where \( t_\nu \) is given by (7.19). It follows that there exists \( c_{22} > 0 \) such that
\[
\frac{\gamma}{2\beta^2 \varepsilon_0^3} \int_{t_\nu}^{t_0} r \left( \varepsilon_0^{-\frac{1}{2}} t \right) \frac{t^{2\gamma-1} dt}{\sqrt{1 - \frac{t^2}{t_0^2}}} < c_{22} \varepsilon_0^3 \int_{t_\nu}^{t_0} r \left( \varepsilon_0^{-\frac{1}{2}} t \right) \frac{dt}{t_0^2 - t}.
\]
Using the expression (4.13) for \( r \) we arrive at the estimate
\[
\int_{z_0}^{z_2(\varepsilon)} \frac{\| R_{t_0}^+ (t(s, \varepsilon), \varepsilon) \| ds}{\sqrt{s}} < c_{22} \varepsilon_0^3 \int_{t_\nu}^{t_0} \frac{dt}{t_0^2 - t} = O \left( \varepsilon_0^3 \right)
\]
as \( \varepsilon \to 0^+ \). This completes the proof. \( \square \)

To prove Lemma 7.2 we need to further divide the interval \([Z_0, Z_2(\varepsilon)]\) into two parts by the point \( Z_1(\varepsilon) \) such that \( Z_1(\varepsilon) \to +\infty \) and \( Z_1(\varepsilon) = o(\varepsilon^{-\frac{1}{2}}) \) as \( \varepsilon \to 0^+ \) which we will choose later, in (7.25). On the first subinterval \([Z_0, Z_1(\varepsilon)]\) solutions of the system (7.8) behave like solutions of the free system (7.15) which does not contain \( \varepsilon \). On the second subinterval \([Z_1(\varepsilon), Z_2(\varepsilon)]\) one cannot neglect the terms \( \varepsilon \varepsilon_0^{\frac{1}{2}} \sqrt{s} \) and \( \varepsilon \varepsilon_0^{\frac{1}{2}} z \), so the free system should depend on \( \varepsilon \). Nevertheless, the answer in the second subinterval is even simpler, because \( Z_1(\varepsilon) \to +\infty \) and solutions are already in their asymptotic regime as \( \varepsilon \to +\infty \).

Remark 7.1. Since we are interested only in the behaviour as \( \varepsilon \to 0^+ \) we do not need to ensure that the inequality \( Z_0 < Z_1(\varepsilon) < Z_2(\varepsilon) \) holds for every \( \varepsilon \in U \). It is enough if it holds for sufficiently small values of \( \varepsilon \).

For both subintervals one has to prove that the remainder \( R_{II,2} \) does not affect the asymptotics. Some difficulty in showing this is that the second term of the coefficient matrix has off-diagonal entries. To get rid of them we use the Harris–Lutz transformation, although slightly different for subintervals \([Z_0, Z_1(\varepsilon)]\) and \([Z_1(\varepsilon), Z_2(\varepsilon)]\). Let us see how it works generally in the hyperbolic case [13].

7.1. Formulae for the Harris–Lutz transformation. Suppose that we start with the linear differential system in \( \mathbb{C}^2 \)
\[
u' = (\Lambda + S + R)u,
\]
where \( \Lambda = \text{diag}\{\lambda_1, \lambda_2\} \) and \( \text{Re} \lambda_1 > \text{Re} \lambda_2 \). The aim is to transform it to the system
\[
u'_1 = (\Lambda + \text{diag} S + R_1)u_1
\]
by the substitution \( u = (I + \widehat{T})u_1 \). We suppose that \( (I + \widehat{T}) \) is invertible. This gives
\[
(I + \widehat{T}) \nu' = (\Lambda + \Lambda \widehat{T} - \widehat{T}' + S + (S \widehat{T} + R(I + \widehat{T})))u_1
\]
and 

\[ u'_1 = (\Lambda + [\Lambda, \hat{T}] - \hat{T}' + S + R_1)u_1, \]

where \([\cdot, \cdot]\) is the commutator of matrices and 

\[ R_1 = (I + \hat{T})^{-1}((R(I + \hat{T}) + S\hat{T}) - \hat{T}(\Lambda\hat{T} - \hat{T}' + S) + \hat{T}^2\Lambda). \]

Let \( S_d := \text{diag} S, \ S_{ad} := S - S_d. \) We need the following equality:

\[ [\Lambda, \hat{T}] - \hat{T}' = -S_{ad}. \]

This, in particular, implies

(7.20) \[ R_1 = (I + \hat{T})^{-1}R(I + \hat{T}) + (I + \hat{T})^{-1}(S\hat{T} - \hat{T}S_d). \]

If one represents \( \hat{T} \) as

\[ \hat{T}(x) = \exp \left( \int_0^x \Lambda \right) \hat{T}(x) \exp \left( - \int_0^x \Lambda \right), \]

differentiation gives

\[ \hat{T}' = [\Lambda, \hat{T}] + \exp \left( \int_0^x \Lambda \right) \hat{T}' \exp \left( - \int_0^x \Lambda \right), \]

and so one needs to solve the equation

\[ \exp \left( \int_0^x \Lambda \right) \hat{T}' \exp \left( - \int_0^x \Lambda \right) = S_{ad}. \]

Its solution is

\[ \hat{T}(x) = \begin{pmatrix} 0 & -\int_x^\infty S_{12}(x') \exp(\int_x^{x'} (\lambda_2 - \lambda_1)) dx' \\ \int_0^x S_{21}(x') \exp(\int_0^{x'} (\lambda_1 - \lambda_2)) dx' & 0 \end{pmatrix}, \]

where \( S_{12} \) and \( S_{21} \) are the off-diagonal entries of the matrix \( S \) and the choice of the domain of integration is determined by the sign of the real part of the exponent. This eventually gives

(7.21) \[ \hat{T}(x) = \begin{pmatrix} 0 & -\int_x^\infty S_{12}(x') \exp(\int_x^{x'} (\lambda_2 - \lambda_1)) dx' \\ \int_0^x S_{21}(x') \exp(\int_0^{x'} (\lambda_1 - \lambda_2)) dx' & 0 \end{pmatrix}, \]

where both exponential terms are less than one in modulus. If \( R \) is summable and \( \Lambda, V \) are such that \( \hat{T} \) goes to zero at infinity, then \( R_1 \) can be also summable, which means that the transformation effectively eliminates the off-diagonal terms of \( V \).

According to the argument above, for the system (7.8) we take for \( z \in [Z_0, Z_2(\varepsilon)] \)

(7.22) \[ \hat{T}_{11}(z, \varepsilon) = \begin{pmatrix} 0 & \hat{T}_{112}(z, \varepsilon) \\ \hat{T}_{112}(z, \varepsilon) & 0 \end{pmatrix} \]

with

\[ \hat{T}_{112}(z, \varepsilon) := \int_z^\infty ds \frac{ds}{4s(1 - \varepsilon^2 \sqrt{s})} \exp \left( - \int_s^\infty \frac{2c_0 \sqrt{\sigma} d\sigma}{(1 - \varepsilon^2 \sigma)^{\infty}} \right), \]

\[ \hat{T}_{1121}(z, \varepsilon) := -\int_z^{Z_2(\varepsilon)} ds \frac{ds}{4s(1 + \varepsilon^2 \sqrt{s})} \exp \left( - \int_z^s \frac{2c_0 \sqrt{\sigma} d\sigma}{(1 - \varepsilon^2 \sigma)^{\infty}} \right). \]
Note that we have interchanged the domains of integration according to the signs of the entries of $\Lambda_{II}$. Note also that the argument works with $Z_2(\varepsilon)$ in the upper limit instead of $+\infty$ for the lower-left entry.

**Lemma 7.5.** There exist $t_{I-II} \in (0, t_0)$ and $c_{I-II} > 0$ such that for every $\varepsilon \in U \cup \{0\}$ and $z \in [1, Z_2(\varepsilon)]$ with $Z_2(\varepsilon)$ given by (7.11), the matrix $\widehat{T}_{I-II}(z, \varepsilon)$ given by (7.22) satisfies the following estimate:

$$\|\widehat{T}_{I-II}(z, \varepsilon)\| < \frac{c_{I-II}}{z^2}. \tag{7.23}$$

**Proof.** Let us choose the point $t_{I-II}$ so close to $t_0$ that for every $z \in [0, Z_2(\varepsilon)]$ it holds that $\varepsilon \sqrt{z} < \frac{3}{4}$, which is equivalent to

$$t_0 \left(\frac{3}{4}\right)^2 < t_{I-II} < t_0.$$  

So let us take

$$t_{I-II} := t_0 \left(\frac{4}{5}\right)^3. \tag{7.24}$$

Then

$$|\widehat{T}_{I-II}(z, \varepsilon)| < \int_{z_0}^z \exp \left(-\int_{s_0}^s 2c_0 \sqrt{\sigma} d\sigma\right) ds = \exp \left(-\frac{4c_0}{3} \frac{z^{3/2}}{2}\right) \int_{z_0}^z \exp \left(\frac{4c_0}{3} \frac{s^{3/2}}{2}\right) ds.$$  

The last expression does not depend on $\varepsilon$ and can be estimated for large $z$ using integration by parts. Indeed, for $z > 2Z_0$ one has:

$$\exp \left(-\frac{4c_0}{3} \frac{z^{3/2}}{2}\right) \int_{z_0}^z \exp \left(\frac{4c_0}{3} \frac{s^{3/2}}{2}\right) ds = \frac{1}{4c_0 z^{3/2}} \exp \left(-\frac{4c_0}{3} \frac{z^{3/2}}{2}\right) \int_{z_0}^z \exp \left(\frac{4c_0}{3} \frac{s^{3/2}}{2}\right) ds$$

$$= \frac{1}{4c_0 z^{3/2}} \exp \left(-\frac{4c_0}{3} \frac{z^{3/2}}{2}\right) \left(\int_{z_0}^{\frac{z}{2}} + \int_{\frac{z}{2}}^z\right) \exp \left(\frac{4c_0}{3} \frac{s^{3/2}}{2}\right) ds$$

$$= \frac{1}{4c_0 z^{3/2}} \left(1 - \frac{1}{2} \frac{1}{2}\right) = O \left(\frac{1}{z^{3/2}}\right)$$

as $z \to +\infty$, and hence we have a uniform estimate for the upper-right entry. For the lower-left entry one has:

$$|\widehat{T}_{I-II}(z, \varepsilon)| < \int_{z}^{Z_2(\varepsilon)} \frac{ds}{4s} \exp \left(-\int_{z}^{s} 2c_0 \sqrt{\sigma} d\sigma\right)$$

$$< \int_{z}^{+\infty} \frac{ds}{4s} \exp \left(-\int_{z}^{s} 2c_0 \sqrt{\sigma} d\sigma\right) = \exp \left(\frac{4c_0}{3} \frac{z^{3/2}}{2}\right) \int_{z}^{+\infty} \exp \left(-\frac{4c_0}{3} \frac{s^{3/2}}{4s}\right) ds$$

$$= \frac{1}{8c_0 z^{3/2}} - \frac{3}{16c_0} \int_{z}^{+\infty} \exp \left(-\frac{4c_0}{3} \frac{s^{3/2}}{4s}\right) ds < \frac{1}{8c_0 z^{3/2}}$$

for every $z \in [Z_0, Z_2(\varepsilon)]$ and $\varepsilon \in U \cup \{0\}$. Therefore we have the estimate (7.23). \( \square \)
On the first part \([Z_0, Z_1(\varepsilon)]\) of the region \(I\) we treat the system (7.8) as a perturbation of the free system (7.15). Let us see how to choose \(Z_1(\varepsilon)\) in order to make this possible. Rewrite (7.8) as
\[
u'_{I1,2}(z) = \left(-c_0\sqrt{\varepsilon} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{1}{4\varepsilon} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \tilde{R}_{I1,2}(z, \varepsilon)\right) u_{I1,2}(z),
\]
where
\[
\tilde{R}_{I1,2}(z, \varepsilon) = R_{I1,2}(z, \varepsilon)
\]
\[-c_0\sqrt{\varepsilon} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left(\frac{1}{1-\varepsilon^2z^2} - 1\right) + \frac{\varepsilon^{\frac{1}{2}}}{4\sqrt{\varepsilon}} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \left(\frac{1}{1+\varepsilon^2\sqrt{\varepsilon}} - \frac{1}{1-\varepsilon^2\sqrt{\varepsilon}}\right)\].

From this we see that with some \(c_{23} > 0\) for every \(z \in [Z_0, Z_2(\varepsilon)]\)
\[
\|\tilde{R}_{I1,2}(z, \varepsilon)\| \leq \|R_{I1,2}(z, \varepsilon)\| + c_{23} \left(\varepsilon^{\frac{1}{2}}z^2 + \frac{\varepsilon^{\frac{1}{2}}}{\sqrt{\varepsilon}}\right).
\]
Since by Lemma 7.3 one has
\[
\int_{Z_0}^{Z_2(\varepsilon)} \|R_{I1,2}(s, \varepsilon)\| ds \to 0 \text{ as } \varepsilon \to 0^+,
\]
the condition on \(Z_1(\varepsilon)\) is the following:
\[
\int_{Z_0}^{Z_1(\varepsilon)} \left(\varepsilon^{\frac{1}{2}}s^{\frac{1}{2}} + \frac{\varepsilon^{\frac{1}{2}}}{\sqrt{s}}\right) ds \to 0 \text{ as } \varepsilon \to 0^+.
\]
This is equivalent to the condition
\[
\varepsilon^{\frac{1}{2}}(Z_1(\varepsilon))^{\frac{1}{2}} + \frac{\varepsilon^{\frac{1}{2}}}{\sqrt{s}}(Z_1(\varepsilon))^{\frac{1}{2}} \to 0,
\]
which in turn is equivalent to \(Z_1(\varepsilon) = o \left(\varepsilon^{-\frac{1}{2}}\right)\). Therefore let us take
\[
Z_1(\varepsilon) := \frac{1}{\varepsilon^{\frac{1}{2}}},
\]
This ensures that
\[
\int_{Z_0}^{Z_1(\varepsilon)} \|	ilde{R}_{I1,2}(s, \varepsilon)\| ds \to 0 \text{ as } \varepsilon \to 0^+.
\]
Now we can obtain results for the subintervals \([Z_0, Z_1(\varepsilon)]\) and \([Z_1(\varepsilon), Z_2(\varepsilon)]\).

**Lemma 7.6.** Let the conditions of Lemma 7.3 hold and let \(Z_1(\varepsilon)\) be given by (1.21). There exists \(Z_0 > 0\) such that if
\[
\int_{Z_0}^{Z_1(\varepsilon)} \left|\frac{R_{I1}(s, \varepsilon)}{\sqrt{s}}\right| ds = o \left(\varepsilon^{\frac{1}{2}}\right) \text{ as } \varepsilon \to 0^+,
\]
then for every sufficiently small \(\varepsilon > 0\) the system (7.8) on the interval \([Z_0, Z_1(\varepsilon)]\)
has two solutions \(\tilde{u}_{I1,2}^\pm(z, \varepsilon)\) such that, as \(\varepsilon \to 0^+\),
\[
\tilde{u}_{I1,2}^\pm(Z_1(\varepsilon), \varepsilon) = \exp\left(\frac{\mp 2m(\varepsilon)}{3} (Z_1(\varepsilon))^{\frac{1}{2}}\right) (e_\pm + o(1)),
\]
where the vectors \(e_{\pm}\) are given by (1.21). Moreover, \(\tilde{u}_{I1,2}^\pm(z, \varepsilon) \to \tilde{v}_{I1,2}^\pm(z)\) as \(\varepsilon \to 0^+\) for every fixed \(z \geq Z_0\), where \(v_{I1,2}^\pm\) are defined in Lemma 7.1.
Proof. Let us choose
\begin{align}
Z_0 := \max\{(2c_{II})^{\frac{2}{3}}, (2c_{IV})^{\frac{2}{3}}\}
\end{align}
where $c_{II}$ is defined in Lemma 7.3 and $c_{IV}$ will be defined in Lemma 8.5 independently. We now only need to know that $Z_0 \geq (2c_{II})^{\frac{2}{3}}$: this ensures that $\|\tilde{T}_{II}(z)\| < \frac{1}{2}$ for every $z \in [Z_0, Z_2(\varepsilon)]$ by Lemma 7.3 and hence $(I + \tilde{T}_{II}(z,0))$ is invertible. Take
\begin{align}
(7.29) 
\quad u_{II,2}(z) = (I + \tilde{T}_{II}(z,0))u_{II,3}(z).
\end{align}
According to the argument for the Harris–Lutz transformation and due to the formula (7.20) this leads to the system
\begin{align}
(7.30) 
\quad u'_{II,3}(z) = \left(-c_0\sqrt{z}\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right) - \frac{1}{4z}I + Q_{II,3}(z) + R_{II,3}(z,\varepsilon) u_{II,3}(z)
\end{align}
with
\begin{align}
Q_{II,3}(z) := (I + \tilde{T}_{II}(z,0))^{-1}(S_{II,3}(z,0)\tilde{T}_{II}(z,0) - \tilde{T}_{II}(z,0)\text{diag} S_{II,2}(z,0))
\end{align}
and
\begin{align}
(7.31) 
\quad R_{II,3}(z,\varepsilon) := (I + \tilde{T}_{II}(z,0))^{-1}R_{II,2}(z,\varepsilon)(I + \tilde{T}_{II}(z,0)).
\end{align}
From the expression (7.12) for $S_{II,3}(z,0)$ and the estimate (7.23) for $T_{II}(z,0)$ we have
\begin{align}
(7.32) 
\quad Q_{II,3}(z) = O\left(\frac{1}{z^2}\right) \text{ as } z \to +\infty.
\end{align}
Consider the free system
\begin{align}
(7.33) 
\quad v'_{II,3}(z) = \left(-c_0\sqrt{z}\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right) - \frac{1}{4z}I + Q_{II,3}(z) v_{II,3}(z).
\end{align}
Since $\int_1^{\infty} \|Q_{II,3}(s)\| ds < \infty$, the asymptotic Levinson theorem is applicable and yields the existence of two solutions $v_{II,3}^\pm$ of the system (7.33) with the asymptotics
\begin{align}
(7.34) 
\quad v_{II,3}^+(z) = \frac{\exp\left(\mp \frac{2\pi}{3} z^{\frac{2}{3}}\right)}{z^{\mp}} (e_\pm + o(1)) \text{ as } z \to +\infty.
\end{align}

**Remark 7.2.** Now we can easily obtain the proof of Lemma 7.1.

**Proof of Lemma 7.1.** The following definition of solutions $v_{II,3}^\pm$ of the system (7.15) immediately gives the asymptotics (7.16) as stated in the lemma:
\begin{align}
(7.35) 
\quad v_{II,3}^\pm(z) := (I + \tilde{T}_{II}(z,0))v_{II,3}^\pm(z) = \frac{\exp\left(\mp \frac{2\pi}{3} z^{\frac{2}{3}}\right)}{z^{\mp}} (e_\pm + o(1)) \text{ as } z \to +\infty.
\end{align}

From the integral estimate of the remainder (7.20) using Lemma 7.5 we have the following estimate of $R_{II,3}$ defined in (7.31)
\begin{align}
(7.36) 
\quad \int_{Z_0}^{Z_1(\varepsilon)} \|R_{II,3}(s,\varepsilon)\| ds \to 0 \text{ as } \varepsilon \to 0^+.
\end{align}
Let us prove that solutions $u_{II,3}$ of the system (7.30) behave on the interval $[Z_0, Z_1(\varepsilon)]$ similarly to solutions $v_{II,3}$ of the free system (7.33). This will imply
that solutions \(u_{II,2}\) of the system (7.8) behave similarly to solutions \(v_{II,2}\) of the free system (7.15). To do this make variation of parameters

\[
(7.37) \quad u_{II,3}(z) = \frac{1}{z^4} \begin{pmatrix} \exp \left( -\frac{2\alpha_0}{3} z^\frac{3}{2} \right) & 0 \\ 0 & \exp \left( \frac{2\alpha_0}{3} z^\frac{3}{2} \right) \end{pmatrix} u_{II,4}(z),
\]

which turns the system (7.38) into the system

\[
(7.38) \quad u'_{II,4}(z) = \begin{pmatrix} \exp \left( \frac{2\alpha_0}{3} z^\frac{3}{2} \right) & 0 \\ 0 & \exp \left( -\frac{2\alpha_0}{3} z^\frac{3}{2} \right) \end{pmatrix} \times (Q_{II,3}(z) + R_{II,3}(z, \varepsilon)) \begin{pmatrix} \exp \left( -\frac{2\alpha_0}{3} z^\frac{3}{2} \right) & 0 \\ 0 & \exp \left( \frac{2\alpha_0}{3} z^\frac{3}{2} \right) \end{pmatrix} u_{II,4}(z).
\]

At this point we need to consider separately the “small” and the “large” solutions.

"Small" solution. Consider the following particular solution of the system (7.38):

\[
(7.39) \quad u_{II,4}^+(z, \varepsilon) = e^+ - \int_{z}^{Z_{1}(\varepsilon)} \begin{pmatrix} \exp \left( \frac{2\alpha_0}{3} s^\frac{3}{2} \right) & 0 \\ 0 & \exp \left( -\frac{2\alpha_0}{3} s^\frac{3}{2} \right) \end{pmatrix} \times (Q_{II,3}(s) + R_{II,3}(s, \varepsilon)) \begin{pmatrix} \exp \left( -\frac{2\alpha_0}{3} s^\frac{3}{2} \right) & 0 \\ 0 & \exp \left( \frac{2\alpha_0}{3} s^\frac{3}{2} \right) \end{pmatrix} u_{II,4}^+(s, \varepsilon) ds.
\]

Returning to \(u_{II,3}\) we have:

\[
(7.39) \quad u_{II,3}^+(z, \varepsilon) = \frac{\exp \left( -\frac{2\alpha_0}{3} z^\frac{3}{2} \right)}{z^4} e^+ - \int_{z}^{Z_{1}(\varepsilon)} \begin{pmatrix} \exp \left( \frac{2\alpha_0}{3} (s^\frac{3}{2} - z^\frac{3}{2}) \right) & 0 \\ 0 & \exp \left( -\frac{2\alpha_0}{3} (s^\frac{3}{2} - z^\frac{3}{2}) \right) \end{pmatrix} \times (Q_{II,3}(s) + R_{II,3}(s, \varepsilon)) u_{II,4}^+(s, \varepsilon) ds.
\]

Normalising the solution

\[
(7.39) \quad u_{II,3}^+(z, \varepsilon) = \frac{\exp \left( -\frac{2\alpha_0}{3} z^\frac{3}{2} \right)}{z^4} u_{II,5}^+(z, \varepsilon)
\]

we come to the integral equation

\[
(7.40) \quad u_{II,5}^+(z, \varepsilon) = e^+ - \int_{z}^{Z_{1}(\varepsilon)} \begin{pmatrix} 1 & 0 \\ 0 & \exp \left( -\frac{4\alpha_0}{3} (s^\frac{3}{2} - z^\frac{3}{2}) \right) \end{pmatrix} \times (Q_{II,3}(s) + R_{II,3}(s, \varepsilon)) u_{II,5}^+(s, \varepsilon) ds.
\]

Repeating the same manipulations with the system (7.33) we get

\[
(7.41) \quad v_{II,5}^+(z) = e^+ - \int_{z}^{+\infty} \begin{pmatrix} 1 & 0 \\ 0 & \exp \left( -\frac{4\alpha_0}{3} (s^\frac{3}{2} - z^\frac{3}{2}) \right) \end{pmatrix} Q_{II,3}(s) v_{II,5}^+(s) ds.
\]
with
\[ \tilde{v}_{1,3}^{+}(z) := \exp\left(-\frac{2\alpha z^{2}}{3}\right) v_{1,3}^{+}(z) \]
in place of (7.39). Note that we need to formally distinguish this \( \tilde{v}_{1,3}^{+} \) from \( v_{1,3}^{+} \) of (7.34), because we have not yet shown that they are the same. Subtracting (7.41) from (7.40) we obtain the equality
\[ u_{1,3}^{+}(z,\varepsilon) - v_{1,3}^{+}(z) = -\int_{z}^{Z_{1}(\varepsilon)} \begin{pmatrix} 1 & 0 \\ 0 & \exp\left(-\frac{4\alpha}{3}(s^{2} - z^{2})\right) \end{pmatrix} Q_{1,3}(s) v_{1,3}^{+}(s) ds \]
\[ \times R_{1,3}(s,\varepsilon) u_{1,3}^{+}(s,\varepsilon) ds + \int_{Z_{1}(\varepsilon)}^{+\infty} \begin{pmatrix} 1 & 0 \\ 0 & \exp\left(-\frac{4\alpha}{3}(s^{2} - z^{2})\right) \end{pmatrix} Q_{1,3}(s) u_{1,3}^{+}(s,\varepsilon) ds \]
\[ - \int_{z}^{Z_{1}(\varepsilon)} \begin{pmatrix} 1 & 0 \\ 0 & \exp\left(-\frac{4\alpha}{3}(s^{2} - z^{2})\right) \end{pmatrix} Q_{1,3}(s) (u_{1,3}^{+}(s,\varepsilon) - v_{1,3}^{+}(z)) ds. \]
The following variant of Gronwall lemma helps to obtain an estimate on \( u_{1,3}^{+} - v_{1,3}^{+} \).

**Lemma 7.7.** Let \( -\infty \leq N_{1} < N_{2} \leq +\infty \), \( v \in L_{\infty}(N_{1}, N_{2}) \), let \( K(z,s) \) be a measurable matrix-valued function defined for \( N_{1} < z < s < N_{2} \) (or \( N_{1} < s < z < N_{2} \)) and such that for every \( z, s \) it satisfies the estimate \( \|K(z,s)\| < k(s) \) with some \( k \in L_{1}(N_{1}, N_{2}) \). Then the equation
\[ u(z) = v(z) - \int_{z}^{N_{2}} K(z,s) u(s) ds \]
(or the equation
\[ u(z) = v(z) + \int_{N_{1}}^{z} K(z,s) u(s) ds, \]
respectively) has the unique solution in \( L_{\infty}(N_{1}, N_{2}) \) which satisfies the estimate
\[ \|u\|_{L_{\infty}(N_{1}, N_{2})} \leq \exp\left(\int_{N_{1}}^{N_{2}} k(s) ds\right) \|v\|_{L_{\infty}(N_{1}, N_{2})}. \]

**Proof.** One has to consider the operator \( \mathcal{K} \) in \( L_{\infty}(N_{1}, N_{2}) \) defined as
\[ \mathcal{K} : u \mapsto -\int_{z}^{N_{2}} K(z,s) u(s) ds \quad \text{(or } u \mapsto \int_{N_{1}}^{z} K(z,s) u(s) ds, \text{ respectively) } \]
and to check that \( \|\mathcal{K}\|_{\mathcal{B}(L_{\infty}(N_{1}, N_{2}))} \leq \|k\|_{L_{1}(N_{1}, N_{2})} \), that
\[ \|\mathcal{K}^{n}\|_{\mathcal{B}(L_{\infty}(N_{1}, N_{2}))} \leq \frac{\|\mathcal{K}\|_{\mathcal{B}(L_{\infty}(N_{1}, N_{2}))}^{n}}{n!} \]
and that hence
\[ \|(I - \mathcal{K}^{n})^{-1}\|_{\mathcal{B}(L_{\infty}(N_{1}, N_{2}))} \leq \exp(\|k\|_{L_{1}(N_{1}, N_{2})}), \]
which completes the proof. \( \square \)

The lemma immediately yields for the equation (7.40):
\[ \sup_{z \in [Z_{0}, Z_{1}(\varepsilon)]} \|u_{1,3}^{+}(z,\varepsilon)\| \leq \exp\left(\int_{Z_{0}}^{Z_{1}(\varepsilon)} \|Q_{1,3}(s) + R_{1,3}(s,\varepsilon)\| ds\right), \]

\[ (7.43) \]
for the equation (7.41):

\[ \sup_{z \in [Z_1(\epsilon), +\infty)} \| u_{1,3}(z) \| \leq \exp \left( \int_{Z_1(\epsilon)}^{+\infty} \| Q_{1,3}(s) \| ds \right), \]

and finally for the equality (7.42):

\[ \sup_{z \in [Z_0, Z_1(\epsilon)]} \| u_{1,5}(z, \epsilon) - v_{1,3}(z) \| \leq \exp \left( \int_{Z_0}^{Z_1(\epsilon)} \| Q_{1,3}(s) \| ds \right) \]

\[ \times \left( \sup_{z \in [Z_0, Z_1(\epsilon)]} \| u_{1,5}(z, \epsilon) \| \int_{Z_0}^{Z_1(\epsilon)} \| R_{1,3}(s, \epsilon) \| ds \right) \]

\[ + \sup_{z \in [Z_1(\epsilon), +\infty)} \| v_{1,3}(z) \| \int_{Z_1(\epsilon)}^{+\infty} \| Q_{1,3}(s) \| ds \right) \]

\[ \leq \exp \left( 2 \int_{Z_0}^{+\infty} \| Q_{1,3}(s) \| ds \right) \exp \left( \int_{Z_0}^{Z_1(\epsilon)} \| R_{1,3}(s, \epsilon) \| ds \right) \]

\[ \times \int_{Z_0}^{Z_1(\epsilon)} \| R_{1,3}(s, \epsilon) \| ds + \int_{Z_1(\epsilon)}^{+\infty} \| Q_{1,3}(s) \| ds \right) \rightarrow 0 \]

as \( \epsilon \rightarrow 0^+ \) due to the estimate (7.32) for \( Q_{1,3} \) and the integral estimate (7.36) for \( R_{1,3} \). Using again (7.32) with (7.41) to estimate the integral in the equation (7.41) we conclude that \( u_{1,5}(z) \rightarrow e_+ \) as \( z \rightarrow +\infty \), and \( v_{1,3}(z) \) has the same exponentially vanishing asymptotics as \( v_{1,3}(z) \). Therefore \( \tilde{v}_{1,3}(z) = v_{1,3}(z) \) and

\[ v_{1,3}(z) = (I + \hat{T}_{1,3}(z, 0)) \frac{\exp \left( -\frac{2\alpha_0 z^+}{3} \right)}{z^+} \tilde{v}_{1,3}(z). \]

Now, using the formulae (7.39) and (7.29) which establish the relation between \( u_{1,2}, u_{1,3} \) and \( u_{1,5} \) we define:

\[ \tilde{u}_{1,2}(z, \epsilon) = (I + \hat{T}_{1,1}(z, 0)) \frac{\exp \left( -\frac{2\alpha_0 z^+}{3} \right)}{z^+} u_{1,5}(z, \epsilon), \]

and this is a solution of the system (7.8). From the convergence in (7.35) and the equality (7.46) we conclude that

\[ \tilde{u}_{1,2}(z, \epsilon) = u_{1,2}(z) + o(1) \]

as \( \epsilon \rightarrow 0^+ \) for every fixed \( z \geq Z_0 \). For \( z = Z_1(\epsilon) \) we use the estimate for \( \hat{T}_{1,1}(z, 0) \) by Lemma 7.5 and the fact that \( u_{1,5}(Z_1(\epsilon), \epsilon) = e_+ \) which follows from (7.40). Thus we have

\[ \tilde{u}_{1,2}(Z_1(\epsilon), \epsilon) = \exp \frac{-2\alpha_0 (Z_1(\epsilon))^+}{3} (e_+ + o(1)) \]

as \( \epsilon \rightarrow 0^+ \). This proves the part of Lemma 7.6 concerning the “small” solution. “Large” solution. Take

\[ v^- := Z_0^+ \begin{pmatrix} \exp \frac{2\alpha_0 (Z_0)^+}{3} & 0 \\ 0 & \exp \frac{-2\alpha_0 (Z_0)^+}{3} \end{pmatrix} v_{1,3}(Z_0), \]

for the equation (7.41):
where \(v_{II,3}^{-}\) is defined in (7.34). Consider the following solution of the system (7.38):

\[
\begin{aligned}
&u_{II,4}(z, \varepsilon) = v^{-} + \int_{z_0}^{z} \left( \begin{array}{cc}
\exp \left( \frac{2c_0 s^2}{3} \right) & 0 \\
0 & \exp \left( -\frac{2c_0 s^2}{3} \right)
\end{array} \right) \\
&\times (Q_{II,3}(s) + R_{II,3}(s, \varepsilon)) \left( \begin{array}{cc}
\exp \left( -\frac{2c_0 s^2}{3} \right) & 0 \\
0 & \exp \left( \frac{2c_0 s^2}{3} \right)
\end{array} \right) u_{II,4}^{-}(s, \varepsilon) ds.
\end{aligned}
\]

Returning to \(u_{II,3}^{-}\) by the relation (7.38) we have

\[
\begin{aligned}
&u_{II,3}^{-}(z, \varepsilon) = \frac{1}{z^\frac{1}{4}} \left( \begin{array}{cc}
\exp \left( -\frac{2c_0 z^2}{3} \right) & 0 \\
0 & \exp \left( \frac{2c_0 z^2}{3} \right)
\end{array} \right) v^{-} \\
&+ \int_{z_0}^{z} \left( \begin{array}{cc}
\exp \left( -\frac{2c_0 (s^2 - z^2)}{3} \right) & 0 \\
0 & \exp \left( \frac{2c_0 (s^2 - z^2)}{3} \right)
\end{array} \right) \\
&\times (Q_{II,3}(s) + R_{II,3}(s, \varepsilon)) u_{II,4}^{-}(s, \varepsilon) ds.
\end{aligned}
\]

Normalising the solution

\[
(7.48) \quad u_{II,3}^{-}(z, \varepsilon) = \frac{\exp \left( \frac{2c_0 z^2}{3} \right)}{z^\frac{1}{4}} u_{II,6}^{-}(z, \varepsilon)
\]

we come to the integral equation

\[
(7.49) \quad u_{II,6}^{-}(z, \varepsilon) = \left( \begin{array}{cc}
\exp \left( -\frac{4c_0 z^2}{3} \right) & 0 \\
0 & 1
\end{array} \right) v^{-} \\
+ \int_{z_0}^{z} \left( \begin{array}{cc}
\exp \left( -\frac{4c_0 (s^2 - z^2)}{3} \right) & 0 \\
0 & 1
\end{array} \right) (Q_{II,3}(s) + R_{II,3}(s, \varepsilon)) u_{II,6}^{-}(s, \varepsilon) ds.
\]

Doing the same transformations with the free system (7.33) we get the equation

\[
(7.50) \quad v_{II,6}^{-}(z) = \left( \begin{array}{cc}
\exp \left( -\frac{4c_0 z^2}{3} \right) & 0 \\
0 & 1
\end{array} \right) v^{-} \\
+ \int_{z_0}^{z} \left( \begin{array}{cc}
\exp \left( -\frac{4c_0 (z^2 - s^2)}{3} \right) & 0 \\
0 & 1
\end{array} \right) Q_{II,3}(s) v_{II,6}^{-}(s) ds.
\]

and define \(\tilde{v}_{II,3}^{-}(z) := \frac{\exp \left( \frac{2c_0 z^2}{3} \right)}{z^\frac{1}{4}} v_{II,6}^{-}(z)\). Since

\[
\tilde{v}_{II,3}^{-}(Z_0) = \frac{1}{(Z_0)^{\frac{1}{4}}} \left( \begin{array}{cc}
\exp \left( -\frac{2c_0 (Z_0)^{\frac{3}{2}}}{3} \right) & 0 \\
0 & \exp \left( \frac{2c_0 (Z_0)^{\frac{3}{2}}}{3} \right)
\end{array} \right) v^{-} = v_{II,3}^{-}(Z_0)
\]
and both $u_{1,3}$ and $\tilde{v}_{1,3}$ satisfy the same system (7.33), they coincide. From the asymptotics (7.31) of $v_{1,3}$ we conclude that $v_{1,6}(z) \to e_-$ as $z \to +\infty$. Subtracting (7.50) from (7.49) we get the equality

$$(7.51)$$

$$u_{1,6}(z, \varepsilon) - v_{1,6}(z) = \int_{Z_0}^z \left( \exp \left( -\frac{4\alpha}{3} (\frac{z}{2} - \frac{s}{2}) \right) \right) R_{1,3}(s, \varepsilon) u_{1,6}(s, \varepsilon) ds$$

$$+ \int_{Z_0}^z \left( \exp \left( -\frac{4\alpha}{3} (\frac{z}{2} - \frac{s}{2}) \right) \right) Q_{1,3}(s) (u_{1,6}(s, \varepsilon) - v_{1,6}(s)) ds.$$  

Applying Lemma 7.7 to the equation (7.49) and the equality (7.51) we have:

$$\sup_{z \in [Z_0, Z_1(\varepsilon)]} \| u_{1,6}(z, \varepsilon) - v_{1,6}(z) \| \leq \| v_- \| \exp \left( \int_{Z_0}^{Z_1(\varepsilon)} \| Q_{1,3}(s) + R_{1,3}(s, \varepsilon) \| ds \right),$$

and

$$(7.52)$$

$$\sup_{z \in [Z_0, Z_1(\varepsilon)]} \| u_{1,6}(z, \varepsilon) - v_{1,6}(\varepsilon) \| \leq \exp \left( \int_{Z_0}^{Z_1(\varepsilon)} \| Q_{1,3}(s) \| ds \right)$$

$$\times \sup_{z \in [Z_0, Z_1(\varepsilon)]} \| u_{1,6}(z, \varepsilon) \| \int_{Z_0}^{Z_1(\varepsilon)} \| R_{1,3}(s, \varepsilon) \| ds \leq \| v_- \| \int_{Z_0}^{Z_1(\varepsilon)} \| R_{1,3}(s, \varepsilon) \| ds$$

$$\times \exp \left( 2 \int_{Z_0}^{+\infty} \| Q_{1,3}(s) \| ds + \int_{Z_0}^{Z_1(\varepsilon)} \| R_{1,3}(s, \varepsilon) \| ds \right) \to 0$$

as $\varepsilon \to 0^+$ due to the estimate (7.32) of $Q_{1,3}$ and the integral estimate (7.36) of $R_{1,3}$. Define $u_{1,2}$ using the formulae (7.29) and (7.48) which relate $u_{1,2}, u_{1,3}$ and $u_{1,6}$ as

$$u_{1,2}(z, \varepsilon) := (I + \tilde{T}_{11}(z, 0)) \exp \left( \frac{2\alpha z}{3} \frac{1}{z^\frac{1}{2}} \right) u_{1,6}(z, \varepsilon),$$

and this is a solution of the system (7.33). For $z = Z_1(\varepsilon)$ from the convergence (7.52), asymptotics $u_{1,6}(z) \to e_-$ as $z \to +\infty$ and the estimate for $\tilde{T}_{11}(z, 0)$ by Lemma 7.9 we have:

$$\tilde{u}_{1,2}(Z_1(\varepsilon), \varepsilon) = \frac{\exp \left( \frac{2\alpha(2\alpha Z_1(\varepsilon))}{3} \right)}{(Z_1(\varepsilon))^\frac{1}{2}} (e_- + o(1))$$

as $\varepsilon \to 0^+$. For every fixed $z \geq Z_0$ convergence in (7.52) implies that

$$\tilde{u}_{1,2}(z, \varepsilon) \to (I + \tilde{T}_{11}(z, 0)) \exp \left( \frac{2\alpha z}{3} \frac{1}{z^\frac{1}{2}} \right) v_{1,6}(z) = v_{1,2}(z)$$

as $\varepsilon \to 0^+$. This completes the proof of Lemma 7.6.

The result for the interval $[Z_1(\varepsilon), Z_2(\varepsilon)]$ is given by the following lemma.
Lemma 7.8. Let the conditions of Lemma 7.2 hold, $Z_1(\varepsilon)$ be given by (7.25) and
$Z_2(\varepsilon)$ by (7.44) with $t_{II}$ given by (7.24). If
\[
\int_{Z_1(\varepsilon)}^{Z_2(\varepsilon)} \frac{\|R_{II}(t(s, \varepsilon), \varepsilon)\|}{\sqrt{s}} ds = o\left(\varepsilon^{\frac{5}{2}}\right) \quad \text{as} \quad \varepsilon \to 0^+,
\]
then for every sufficiently small $\varepsilon > 0$ the system (7.28) on the interval $[Z_1(\varepsilon), Z_2(\varepsilon)]$
has two solutions $\hat{u}_{II,2}(z, \varepsilon)$ such that, as $\varepsilon \to 0^+$,
\[
(7.53) \quad \hat{u}_{II,2}(z, \varepsilon) = \exp \left( \int_{Z_1(\varepsilon)}^{z} \left( \mp \frac{c_0 \sqrt{s}}{1 - \varepsilon^{2}\varepsilon s} - \frac{1}{4s(1 \pm \varepsilon^{2}\varepsilon s)} \right) ds \right) (e_\pm + o(1)),
\]
where the vectors $e_\pm$ are given by (1.21) and the remainder $o(1)$ converges uniformly
with respect to $z \in [Z_1(\varepsilon), Z_2(\varepsilon)]$.

Proof. First let us eliminate the off-diagonal entries of the matrix $V_{II}(z, \varepsilon)$ with the Harris–Lutz transformation
\[
(7.54) \quad u_{II,2}(z) = (I + \hat{T}_{II}(z, \varepsilon))u_{II,1}(z)
\]
where $\hat{T}_{II}$ is given by (7.22). On the one hand, in contrast with the interval
$[Z_0, Z_1(\varepsilon)]$, the transformation depends on $\varepsilon$, because one cannot ignore the difference between $V_{II}(z, \varepsilon)$ and $V_{II}(z, 0)$ anymore. On the other hand, we do not need to prove convergence to solutions of some other system independent of $\varepsilon$ (it is no longer true that there is convergence to solutions of (7.15)). With the substitution (7.54) we come to the system
\[
(7.55) \quad u'_{II,1}(z) = (\Lambda_{II,1}(z, \varepsilon) + R_{II,1}(z, \varepsilon))u_{II,1}(z),
\]
where, according to (7.9), (7.12) and (7.20),
\[
(7.56) \quad \Lambda_{II,1}(z, \varepsilon) := \Lambda_{II,2}(z, \varepsilon) + \text{diag} S_{II,2}(z, \varepsilon) = \begin{pmatrix} \lambda_{II,1}(z, \varepsilon) & 0 \\ 0 & \lambda_{II,1}(z, \varepsilon) \end{pmatrix},
\]
\[
(7.57) \quad \lambda_{II,1}(z, \varepsilon) := \mp \frac{c_0 \sqrt{z}}{1 - \varepsilon^{2}z} - \frac{1}{4z(1 \pm \varepsilon^{2}z)},
\]
\[
(7.58) \quad R_{II,1}(z, \varepsilon) := (I + \hat{T}_{II}(z, \varepsilon))^{-1}R_{II,2}(z, \varepsilon)(I + \hat{T}_{II}(z, \varepsilon))
+ (I + \hat{T}_{II}(z, \varepsilon))^{-1}(S_{II,2}(z, \varepsilon)\hat{T}_{II}(z, \varepsilon) - \hat{T}_{II}(z, \varepsilon)\text{diag} S_{II,2}(z, \varepsilon)).
\]
Making variation of parameters
\[
(7.59) \quad u_{II,1}(z) = \exp \left( \int_{Z_1(\varepsilon)}^{z} \Lambda_{II,1}(\sigma, \varepsilon) d\sigma \right) u_{II,8}(z),
\]
and substituting to the system (7.55) we have:
\[
(7.60) \quad u'_{II,8}(z) = \exp \left( - \int_{Z_1(\varepsilon)}^{z} \Lambda_{II,1}(\sigma, \varepsilon) d\sigma \right)
\times R_{II,1}(z, \varepsilon) \exp \left( \int_{Z_1(\varepsilon)}^{z} \Lambda_{II,1}(\sigma, \varepsilon) d\sigma \right) u_{II,8}(z).
\]
Let us now introduce two solutions $u_{I1,s}^{\pm}$ of this system which satisfy the following equations:

$$u_{I1,s}^{+}(z,\varepsilon) = e_{+} - \int_{z}^{Z_{2}(\varepsilon)} \exp \left( - \int_{Z_{1}(\varepsilon)}^{s} \Lambda_{I1,7}(\sigma,\varepsilon) d\sigma \right) u_{I1,s}^{+}(s,\varepsilon) ds$$

$$\times R_{I1,7}(s,\varepsilon) \exp \left( \int_{Z_{1}(\varepsilon)}^{s} \Lambda_{I1,7}(\sigma,\varepsilon) d\sigma \right)$$

and

$$u_{I1,s}^{-}(z,\varepsilon) = e_{-} + \int_{Z_{1}(\varepsilon)}^{s} \exp \left( - \int_{Z_{1}(\varepsilon)}^{\sigma} \Lambda_{I1,7}(\sigma,\varepsilon) d\sigma \right) u_{I1,s}^{-}(s,\varepsilon) ds$$

$$\times R_{I1,7}(s,\varepsilon) \exp \left( \int_{Z_{1}(\varepsilon)}^{s} \Lambda_{I1,7}(\sigma,\varepsilon) d\sigma \right) u_{I1,s}^{-}(s,\varepsilon) ds.$$

For solutions $u_{I1,7}^{\pm}(z,\varepsilon) := \exp \left( \int_{Z_{1}(\varepsilon)}^{s} \Lambda_{I1,7}(\sigma,\varepsilon) d\sigma \right) u_{I1,s}^{\pm}(z,\varepsilon)$ these equations read as follows:

$$u_{I1,7}^{+}(z,\varepsilon) = \exp \left( \int_{Z_{1}(\varepsilon)}^{s} \lambda_{I1,7}^{+}(\sigma,\varepsilon) d\sigma \right) e_{+}$$

$$- \int_{z}^{Z_{2}(\varepsilon)} \exp \left( - \int_{Z_{1}(\varepsilon)}^{s} \Lambda_{I1,7}(\sigma,\varepsilon) d\sigma \right) R_{I1,7}(s,\varepsilon) u_{I1,7}^{+}(s,\varepsilon) ds$$

and

$$u_{I1,7}^{-}(z,\varepsilon) = \exp \left( \int_{Z_{1}(\varepsilon)}^{s} \lambda_{I1,7}^{-}(\sigma,\varepsilon) d\sigma \right) e_{-}$$

$$+ \int_{Z_{1}(\varepsilon)}^{s} \exp \left( \int_{s}^{\sigma} \Lambda_{I1,7}(\sigma,\varepsilon) d\sigma \right) R_{I1,7}(s,\varepsilon) u_{I1,7}^{-}(s,\varepsilon) ds.$$

Normalising these solutions by the substitution

$$u_{I1,7}^{\pm}(z,\varepsilon) = \exp \left( \int_{Z_{1}(\varepsilon)}^{s} \lambda_{I1,7}^{\pm}(\sigma,\varepsilon) d\sigma \right) u_{I1,9}^{\pm}(z,\varepsilon)$$

we come to the following equations:

$$u_{I1,9}(z,\varepsilon) = e_{+} - \int_{z}^{Z_{2}(\varepsilon)} \begin{pmatrix} 1 & 0 \\ \exp \left( \int_{z}^{s} (\lambda_{I1,7}^{+}(\sigma,\varepsilon) - \lambda_{I1,7}^{-}(\sigma,\varepsilon)) d\sigma \right) & 0 \end{pmatrix}$$

$$\times R_{I1,7}(s,\varepsilon) u_{I1,9}^{+}(s,\varepsilon) ds$$

and

$$u_{I1,9}(z,\varepsilon) = e_{-} + \int_{Z_{1}(\varepsilon)}^{s} \begin{pmatrix} \exp \left( \int_{s}^{Z_{1}(\varepsilon)} (\lambda_{I1,7}^{+}(\sigma,\varepsilon) - \lambda_{I1,7}^{-}(\sigma,\varepsilon)) d\sigma \right) & 0 \\ 0 & 1 \end{pmatrix}$$

$$\times R_{I1,7}(s,\varepsilon) u_{I1,9}^{-}(s,\varepsilon) ds.$$
These can be rewritten as

\begin{equation}
\label{7.62}
u_{II,9}^+(z, \varepsilon) - e_+ = - \int_z^{Z_2(\varepsilon)} \left( \begin{array}{cc} 1 & 0 \\ 0 & \exp \left( \int_z^s (\lambda_{1,7}(s, \varepsilon) - \lambda_{1,7}^{-}(s, \varepsilon))d\sigma \right) \end{array} \right) R_{II,7}(s, \varepsilon)e_+ds \\
- \int_z^{Z_2(\varepsilon)} \left( \begin{array}{cc} 1 & 0 \\ 0 & \exp \left( \int_z^s (\lambda_{1,7}^{-}(s, \varepsilon) - \lambda_{1,7}(s, \varepsilon))d\sigma \right) \end{array} \right) R_{II,7}(s, \varepsilon) \times (u_{II,9}^+(s, \varepsilon) - e_+)ds
\end{equation}

and

\begin{equation}
\label{7.63}
u_{II,9}^-(z, \varepsilon) - e_- = \int_z^{Z_1(\varepsilon)} \left( \begin{array}{cc} \exp \left( \int_z^s (\lambda_{1,7}(s, \varepsilon) - \lambda_{1,7}^{-}(s, \varepsilon))d\sigma \right) & 0 \\ 0 & 1 \end{array} \right) R_{II,7}(s, \varepsilon)e_-ds \\
+ \int_z^{Z_1(\varepsilon)} \left( \begin{array}{cc} \exp \left( \int_z^s (\lambda_{1,7}^{-}(s, \varepsilon) - \lambda_{1,7}(s, \varepsilon))d\sigma \right) & 0 \\ 0 & 1 \end{array} \right) R_{II,7}(s, \varepsilon) \times (u_{II,9}^-(s, \varepsilon) - e_-)ds.
\end{equation}

From the expression \eqref{7.12} for $S_{II,2}$, the estimate for $T_{II}$ by Lemmas \ref{7.3} and \ref{7.3} we get:

\begin{equation}
\int_{Z_1(\varepsilon)}^{Z_2(\varepsilon)} \| R_{II,7}(s, \varepsilon) \| ds \to 0 \text{ as } \varepsilon \to 0^+.
\end{equation}

We also have

\[
\lambda_{1,7}^+(z, \varepsilon) - \lambda_{1,7}^{-}(z, \varepsilon) = - \frac{2\sqrt{z}}{1 - \varepsilon^2 z} \left( c_0 (1 - \varepsilon^2 z)^{1 - \varepsilon} - \frac{\varepsilon^2}{4z} \right).
\]

Since $1 - \varepsilon = \frac{1 - 2\varepsilon}{2\varepsilon}$ and the values $1 - \varepsilon^2 z$ and $z$ are separated from zero for $z \in [Z_1(\varepsilon), Z_2(\varepsilon)]$, the above expression is strictly negative for all $z$ from the interval considered, if $\varepsilon$ is small enough. Hence Lemma \ref{7.7} yields for both \eqref{7.62} and \eqref{7.63}:

\begin{equation}
\label{7.64}
sup_{z \in [Z_1(\varepsilon), Z_2(\varepsilon)]} \| u_{II,9}^\pm(z, \varepsilon) - e_\pm \| \\
\leq \exp \left( \int_{Z_1(\varepsilon)}^{Z_2(\varepsilon)} \| R_{II,7}(s, \varepsilon) \| ds \right) \int_{Z_1(\varepsilon)}^{Z_2(\varepsilon)} \| R_{II,7}(s, \varepsilon) \| ds \to 0
\end{equation}

as $\varepsilon \to 0^+$. Now using the relations \eqref{7.54} and \eqref{7.61} we define functions

\[
\hat{u}_{II,2}^\pm(z, \varepsilon) := \exp \left( \int_{Z_1(\varepsilon)}^z \lambda_{1,7}^\pm(s, \varepsilon)d\sigma \right) (I + \hat{T}_{II}(z, \varepsilon))u_{II,9}^\pm(z, \varepsilon)
\]

and they are solutions of the system \eqref{7.64}. Lemma \ref{7.7} expressions \eqref{7.57} and convergence in \eqref{7.65} imply that these solutions have asymptotics \eqref{7.53}. This completes the proof. $\square$

The following trivial lemma helps to match the results in different intervals.
Lemma 7.9. Let \( g(\varepsilon), f_+(\varepsilon), f_-\varepsilon \) be functions of \( \varepsilon \) defined in some neighbourhood of the point \( \varepsilon = 0 \) with values in \( \mathbb{C}^2 \) and such that \( g(\varepsilon) \to g, f_\pm(\varepsilon) \to f_\pm \) as \( \varepsilon \to 0 \), where the vectors \( f_+ \) and \( f_- \) are linearly independent and \( g = c_+ f_+ + c_- f_- \). Then in the decomposition \( g(\varepsilon) = c_+(\varepsilon) f_+(\varepsilon) + c_-(\varepsilon) f_-(\varepsilon) \) the coefficients converge: \( c_\pm(\varepsilon) \to c_\pm \) as \( \varepsilon \to 0 \).

Proof. Consider scalar products with the vectors orthogonal to \( f_\pm \) to immediately see the result. □

Combining the results of Lemmas [7.6] and [7.8] we can now prove Lemma [7.2]

Proof of Lemma [7.2]. Let us first rewrite the formula (7.27) for the asymptotics from Lemma [7.6] using that

\[
\exp \left( \pm \frac{2a(\varepsilon)}{s}(Z_1(\varepsilon)) \frac{1}{s} \right) = a_{II}^\pm \exp \left( \int_{Z_0}^{Z_1(\varepsilon)} \lambda_{II,\varepsilon}^\pm (s,\varepsilon) ds \right) (1 + o(1))
\]

as \( \varepsilon \to 0^+ \), where \( a_{II}^\pm := \frac{\exp \left( \frac{2a(\varepsilon)}{s}(Z_0^\varepsilon) \right)}{Z_0^\varepsilon} \), which we can do because

\[
\left| \lambda_{II,\varepsilon}^\pm (s,\varepsilon) - \left( \mp c_0 \sqrt{s} - \frac{1}{4s} \right) \right| < c_{24} \left( \varepsilon^2 s^{\frac{3}{2}} + \frac{\varepsilon^4}{\sqrt{s}} \right)
\]

with some \( c_{24} > 0 \) and

\[
\int_{Z_0}^{Z_1(\varepsilon)} \left( \varepsilon^2 s^{\frac{3}{2}} + \frac{\varepsilon^4}{\sqrt{s}} \right) ds \to 0 \quad \text{as} \quad \varepsilon \to 0^+
\]
due to the choice of \( Z_1(\varepsilon) \) in (7.23). Let us define for \( z \in [Z_1(\varepsilon), Z_2(\varepsilon)] \)

\[
(7.66) \quad u_{II,2}(z,\varepsilon) := a_{II}^\pm \exp \left( \int_{Z_0}^{Z_1(\varepsilon)} \lambda_{II,\varepsilon}^\pm (s,\varepsilon) ds \right) \tilde{u}_{II,2}(z,\varepsilon).
\]

On the interval \( [Z_0, Z_1(\varepsilon)] \) the continuation of the solution \( \tilde{u}_{II,2}(z,\varepsilon) \) has a decomposition with some coefficients in terms of the basis of solutions \( \tilde{u}_{II,2}^\pm(z,\varepsilon) \), and at the point \( Z_1(\varepsilon) \) one has:

\[
\hat{u}_{II,2}(Z_1(\varepsilon),\varepsilon) = c_+ + o(1),
\]

\[
\hat{u}_{II,2}(Z_1(\varepsilon),\varepsilon) = c_- + o(1).
\]

By Lemma [7.9] we conclude that for \( z \in [Z_0, Z_1(\varepsilon)] \)

\[
\hat{u}_{II,2}(z,\varepsilon) = (1 + o(1)) \frac{\tilde{u}_{II,2}^+(z,\varepsilon)}{a_{II}^+} \exp \left( - \int_{Z_0}^{Z_1(\varepsilon)} \lambda_{II,\varepsilon}^+(s,\varepsilon) ds \right)
\]

\[
+ o \left( \tilde{u}_{II,2}(z,\varepsilon) \exp \left( - \int_{Z_0}^{Z_1(\varepsilon)} \lambda_{II,\varepsilon}^+(s,\varepsilon) ds \right) \right)
\]
and therefore by (7.66)
\[ u_{II,2}^+(z, \varepsilon) = (1 + o(1)) \tilde{u}_{II,2}^+(z, \varepsilon) + o \left( \int_{Z_0}^{Z_1(\varepsilon)} (\lambda_{II,7}(s, \varepsilon) - \lambda_{II,7}^-)(s, \varepsilon) ds \right). \]

For every fixed \( z \geq Z_0 \) this means that
\[ u_{II,2}^+(z, \varepsilon) \to v_{II,2}^+(z) \text{ as } \varepsilon \to 0^+. \]

Asymptotics of \( u_{II,2}^+(z, \varepsilon) \) at \( z = Z_2(\varepsilon) \) is due to Lemma 7.8.

For the second solution define for \( z \in [Z_0, Z_1(\varepsilon)] \)
\[ u_{II,2}^-(z, \varepsilon) := \tilde{u}_{II,2}^-(z, \varepsilon) \]
Analogously, we have
\[ \tilde{u}_{II,2}^-(Z_1(\varepsilon), \varepsilon) \exp \left( - \int_{Z_0}^{Z_1(\varepsilon)} \lambda_{II,7}^- (s, \varepsilon) ds \right) = e_- + o(1), \]
\[ \tilde{u}_{II,2}^+(Z_1(\varepsilon), \varepsilon) = e_+ + o(1) \]
as \( \varepsilon \to 0^+ \). Therefore for the continuation of \( u_{II,2}^- = \tilde{u}_{II,2}^- \) to the interval \([Z_1(\varepsilon), Z_2(\varepsilon)]\)
one has
\[ u_{II,2}^-(z, \varepsilon) = \tilde{u}_{II,2}^-(z, \varepsilon) \]
\[ = a_{II} \exp \left( \int_{Z_0}^{Z_1(\varepsilon)} \lambda_{II,7}^- (s, \varepsilon) ds \right) ((1 + o(1)) \tilde{u}_{II,2}^-(z, \varepsilon) + o(\tilde{u}_{II,2}^+(z, \varepsilon))). \]

For \( z = Z_2(\varepsilon) \) this means that
\[ u_{II,2}^-(Z_2(\varepsilon), \varepsilon) = a_{II} \exp \left( \int_{Z_0}^{Z_2(\varepsilon)} \lambda_{II,7}^- (s, \varepsilon) ds \right) \times \left( e_- + o(1) + o \left( \exp \left( \int_{Z_1(\varepsilon)}^{Z_2(\varepsilon)} (\lambda_{II,7}^+ (s, \varepsilon) - \lambda_{II,7}^- (s, \varepsilon)) ds \right) \right) \right) \]
\[ = a_{II} \exp \left( \int_{Z_0}^{Z_2(\varepsilon)} \lambda_{II,7}^- (s, \varepsilon) ds \right) (e_- + o(1)) \]
as \( \varepsilon \to 0^+ \), which is the desired asymptotics (7.15). Convergence
\[ u_{II,2}^-(z, \varepsilon) \to v_{II,2}^-(z) \text{ as } \varepsilon \to 0^+ \]
for every fixed \( z \geq Z_0 \) is due to (7.67) and Lemma 7.6. □

8. INTERMEDIATE REGION IV: ELLIPTIC CASE

Once again we start with the system (4.12) written in the form
\[ \varepsilon u_{IV}'(t) = \left( \begin{pmatrix} \frac{\alpha}{2} & -1 \frac{1}{2} \\ \frac{1}{2} & \frac{\beta}{2} \end{pmatrix} + R_{IV}(t, \varepsilon) \right) u_{IV}(t) \]
We diagonalise the main term of the coefficient matrix with the transformation
(8.1)
\[ u_{IV}(t) = T_{IV}(t) u_{IV,1}(t), \]
where

\[
T_{IV}(t) := \begin{pmatrix}
\frac{1}{t^\gamma (1 - i\sqrt{\frac{t^2\gamma}{4\beta^2}} - 1)} & \frac{1}{t^\gamma (1 + i\sqrt{\frac{t^2\gamma}{4\beta^2}} - 1)} \\
\end{pmatrix}.
\]

Note that this matrix coincides with \(T_V(t)\) given by (7.2) and with \(T_{II}(t)\) given by (7.2) with one of the possible choices of the branch of the square root. The substitution gives:

\[
u'_{IV,1}(t) = \left( \frac{\lambda_{IV}(t)}{\varepsilon} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + S_{IV}(t) + R_{IV,1}(t, \varepsilon) \right) u_{IV,1}(t),
\]

where

\[
\lambda_{IV}(t) := -i \frac{\beta}{t^\gamma} \sqrt{\frac{t^2\gamma}{4\beta^2} - 1} = \lambda_V(t),
\]

\[
S_{IV}(t) := \frac{\gamma}{2t \left( \frac{t^2\gamma}{4\beta^2} - 1 \right)} \begin{pmatrix} -1 - i \sqrt{\frac{t^2\gamma}{4\beta^2} - 1} & 1 - i \sqrt{\frac{t^2\gamma}{4\beta^2} - 1} \\
1 + i \sqrt{\frac{t^2\gamma}{4\beta^2} - 1} & -1 + i \sqrt{\frac{t^2\gamma}{4\beta^2} - 1} \end{pmatrix} = S_V(t),
\]

\[
R_{IV,1}(t, \varepsilon) := \frac{T_{IV}^{-1}(t)R_{IV}(t, \varepsilon)T_{IV}(t)}{\varepsilon} = R_{IV,1}(t, \varepsilon).
\]

In the region \(IV\) analysis goes along the same lines as in the region \(II\). However, there are slight changes and formulae should be written in a different way. To avoid confusion we repeat the argument highlighting differences and omitting details which are the same. Let us take

\[
u_{IV,1}(t) = u_{IV,2}(z(t, \varepsilon))
\]

with

\[z(t, \varepsilon) = \frac{1}{\varepsilon^{\frac{1}{2}}} \left( 1 - \frac{t^2\gamma}{4\beta^2} \right),\]

as in (7.6), and substitute this into the system (8.3). This gives the system

\[
u'_{IV,2}(z) = (\Lambda_{IV,2}(z, \varepsilon) + S_{IV,2}(z, \varepsilon) + R_{IV,2}(z, \varepsilon))u_{IV,2}(z)
\]

with

\[
\Lambda_{IV,2}(z, \varepsilon) := \lambda_{IV,2}(z, \varepsilon) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

\[
\lambda_{IV,2}(z, \varepsilon) := \frac{\lambda_{IV}(t(z, \varepsilon))t'(z, \varepsilon)}{\varepsilon} = \frac{i\epsilon_0 \sqrt{-z}}{(1 - \varepsilon^{\frac{1}{2}}z)^k},
\]

\[
S_{IV,2}(z, \varepsilon) := S_{IV}(t(z, \varepsilon))t'(z, \varepsilon)
\]

\[= -\frac{1}{4\varepsilon(1 - \varepsilon^{\frac{1}{2}}z)} \begin{pmatrix} 1 + i\varepsilon^{\frac{1}{2}}\sqrt{-z} & -1 + i\varepsilon^{\frac{1}{2}}\sqrt{-z} \\
-1 - i\varepsilon^{\frac{1}{2}}\sqrt{-z} & 1 - i\varepsilon^{\frac{1}{2}}\sqrt{-z} \end{pmatrix},
\]

and

\[
R_{IV,2}(z, \varepsilon) := \frac{t_0}{2}\frac{T_{IV}^{-1}(t(z, \varepsilon))R_{IV}(t(z, \varepsilon), \varepsilon)T_{IV}(t(z, \varepsilon))}{(1 - \varepsilon^{\frac{1}{2}}z)^{1 - \frac{1}{2}\varepsilon}}.
\]
where \(c_0\) and \(\nu\) are given by (7.11). We consider the system (8.7) on the interval \([-Z_2(\varepsilon), -Z_0]\) where \(Z_0\) is given by (7.28) and \(Z_2(\varepsilon)\) by (7.14) with \(t_{1-11}\) given by (7.24). The point \(z = -Z_0\) corresponds to the point \(t = t_{111-IV}(\varepsilon)\),

\[
t_{111-IV}(\varepsilon) := (2\beta)^{\frac{1}{2}}(1 + \varepsilon^2 Z_0)^{\frac{1}{2}},
\]

and the point \(z = -Z_2(\varepsilon)\) corresponds to the point \(t = t_{IV-V}\),

\[
t_{IV-V} := (2\beta)^{\frac{1}{2}}(1 + \varepsilon^2 Z_2(\varepsilon))^{\frac{1}{2}} = (8\beta^2 - t_{111-III}^2)^{\frac{1}{2}}.
\]

Consider the free system.

**Lemma 8.1.** The system

\[
v'_{IV,2}(z) = \left(i c_0 \sqrt{-z} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{1}{4\varepsilon} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right) v_{IV,2}(z)
\]

has for \(z \in (-\infty, 1]\) solutions \(v_{IV,2}^\pm(z)\) such that

\[
v_{IV,2}^\pm(z) = \frac{\exp \left( \pm \frac{2ic_0}{3} (-z)^{\frac{3}{2}} \right)}{(-z)^{\frac{1}{2}}} (e_\pm + o(1))
\]
as \(z \to -\infty\).

As in the previous section, the proof of this lemma will be given later, see Remark 8.1. The main result for the region IV is the following.

**Lemma 8.2.** Let \(c_0, \nu > 0\) and \(R_{IV,2}(z, \varepsilon)\) be given by (8.11) with the use of the expression (8.2) and the definition (7.4). Let \(Z_0\) be given by (7.28) and \(Z_2(\varepsilon)\) by (7.14) with \(t_{1-11}\) given by (7.24). If

\[
\int_{-Z_2(\varepsilon)}^{-Z_0} \frac{\|R_{IV}(t(s, \varepsilon), \varepsilon)\|}{\sqrt{-s}} ds = o \left( \varepsilon^{\frac{1}{2}} \right) \text{ as } \varepsilon \to 0^+,
\]

then for every sufficiently small \(\varepsilon > 0\) the system (8.11) on the interval \([-Z_2(\varepsilon), -Z_0]\) has two solutions \(v_{IV,2}^\pm(z, \varepsilon)\) such that, as \(\varepsilon \to 0^+\),

\[
v_{IV,2}^\pm(-Z_2(\varepsilon), \varepsilon) = a_{IV}^\pm \exp \left( - \int_{-Z_2(\varepsilon)}^{-Z_0} \left( \pm \frac{ic_0 \sqrt{-s}}{1 - \varepsilon^2 s} - \frac{1}{4s(1 + \varepsilon^2 \sqrt{-s})} \right) ds \right) (e_\pm + o(1)),
\]

where \(a_{IV}^\pm\) are complex constants which are conjugate to each other and the vectors \(e_\pm\) are given by (1.21). Moreover, \(v_{IV,2}^\pm(z, \varepsilon) \to v_{IV,2}^\pm(z)\) as \(\varepsilon \to 0^+\) for every fixed \(z \leq -Z_0\), where \(v_{IV,2}^\pm\) are defined in Lemma 8.1.

The following lemma shows that the condition (8.16) is satisfied, if \(R_{IV} = R_2^+\).

**Lemma 8.3.** Let \(R_2^+(t, \varepsilon)\) be given by (1.14) and \(t(z, \varepsilon)\) be defined by (7.6). Let the conditions (7.3) and (7.2) hold. Then for every \(z_0 > 0\) and \(z_2(\varepsilon) = -\frac{\nu}{\varepsilon^2}\) with \(\nu \in (-\infty, 0)\) the following estimate holds:

\[
\int_{-Z_2}^{-z_0} \frac{\|R_2^+(t(s, \varepsilon), \varepsilon)\|}{\sqrt{-s}} ds = O \left( \varepsilon^{\frac{1}{2}} \varepsilon_0^{\frac{\nu}{2}} \right) \text{ as } \varepsilon \to 0^+.
\]
Proof. Using the same argument as in the proof of Lemma 8.3 we have with some $c_{25} > 0$ and $t_1$ given by (7.19):

$$
\int_{-z_2(\varepsilon)}^{-z_0} \frac{\|R_{IV}^1(t(s, \varepsilon), \varepsilon)\|}{\sqrt{-s}} \, ds < c_{25} \varepsilon^{\frac{2}{3}} \bar{z}_0 \int_{t_0}^{t_1} \frac{dt}{t^{1+\alpha/2}}.
$$

Now let us rewrite the estimate (8.16) in terms of $R_{IV,2}$.

**Lemma 8.4.** Under the conditions of Lemma 8.3 for every $z_0 > 0$ and $z_2(\varepsilon) = -\frac{\nu}{\varepsilon^2}$ with $\nu \in (\infty, 0)$, if

$$
\int_{-z_2(\varepsilon)}^{-z_0} \frac{\|R_{IV,2}(t(s, \varepsilon), \varepsilon)\|}{\sqrt{-s}} \, ds = o\left(\varepsilon^{-\frac{3}{2}}\right), \text{ then } \int_{-z_2(\varepsilon)}^{-z_0} \frac{\|R_{IV,2}(s, \varepsilon)\|}{\varepsilon^{\frac{3}{2}} \sqrt{-s}} \, ds \to 0
$$

as $\varepsilon \to 0^+$.

**Proof.** Following the proof of Lemma 7.3 we have with some $c_{26} > 0$:

$$
\int_{-z_2(\varepsilon)}^{-z_0} \frac{\|R_{IV,2}(s, \varepsilon)\|}{\varepsilon^{\frac{3}{2}} \sqrt{-s}} \, ds < c_{26} \int_{-z_2(\varepsilon)}^{-z_0} \frac{\|R_{IV}(t(s, \varepsilon), \varepsilon)\|}{\varepsilon^{\frac{3}{2}} \sqrt{-s}} \, ds \to 0 \text{ as } \varepsilon \to 0^+.
$$

In the elliptic case the formula for the Harris–Lutz transformation can be simplified. It is not difficult to check that one can take

$$
\hat{T}(x) = \left(\begin{array}{cc}
\int_\infty^- S_2(x') \exp(f_2(x')(\lambda_2 - \lambda_1)) dx' & \int_\infty^- S_1(x') \exp(f_2(x')(\lambda_1 - \lambda_2)) dx' \\
0 & 0
\end{array}\right)
$$

instead of (7.21). Therefore let us take for $z \in [-Z_2(\varepsilon), -z_0]$

$$(8.18) \quad \hat{T}_{IV}(z, \varepsilon) = \left(\begin{array}{cc}
0 & \hat{T}_{IV2}(z, \varepsilon) \\
\hat{T}_{IV1}(z, \varepsilon) & 0
\end{array}\right)
$$

with

$$
\hat{T}_{IV1}(z, \varepsilon) := \int_{-Z_2(\varepsilon)}^{z} \frac{ds}{4s(1 + i\varepsilon^2 \sqrt{-s})} \exp\left(\int_{s}^{z} \frac{2ic_0 \sqrt{-\sigma} d\sigma}{(1 - \varepsilon^2 \sigma)^\alpha}\right),
$$

$$
\hat{T}_{IV2}(z, \varepsilon) := \int_{-Z_2(\varepsilon)}^{z} \frac{ds}{4s(1 - i\varepsilon^2 \sqrt{-s})} \exp\left(-\int_{s}^{z} \frac{2ic_0 \sqrt{-\sigma} d\sigma}{(1 - \varepsilon^2 \sigma)^\alpha}\right) = \hat{T}_{IV1}(z, \varepsilon).
$$

**Lemma 8.5.** There exists $c_{1IV} > 0$ such that for every $\varepsilon \in U \cup \{0\}$ and $z \in [-Z_2(\varepsilon), -1]$, where $Z_2(\varepsilon)$ is given by (6.14) with $t_{1-t_1}$ given by (7.21), the matrix $\hat{T}_{IV}(z, \varepsilon)$ given by (8.18) satisfies the following estimate:

$$(8.19) \quad \|\hat{T}_{IV}(z, \varepsilon)\| < \frac{c_{1IV}}{|z|^{\alpha}}.$$

**Proof.** This is the place where the proof for the elliptic case differs from the corresponding proof in the hyperbolic case not only in notation. It is impossible to
effectively estimate oscillating exponentials by something which is independent of $\varepsilon$, so one has to pay more attention. Denote $\nu_2 := \varepsilon \hat{T}_2(\varepsilon)$. For $\varepsilon \in U$ we have:

$$
\|\hat{T}_{IV}(z,\varepsilon)\| = |\hat{T}_{IV,1}(z,\varepsilon)| = |\hat{T}_{IV,2}(z,\varepsilon)| = \left| \int_{-Z_2(\varepsilon)}^{z} \frac{\exp \left( \int_{s}^{\nu_2} \frac{2i\varepsilon \sqrt{\sigma_1 \delta s}}{(1+\varepsilon \frac{\kappa}{2})^{\alpha}} \right) ds}{4s(1+i\varepsilon \frac{\kappa}{2} s^{\alpha})} \right|
$$

$$
= \left| \int_{|z|}^{Z_2(\varepsilon)} \frac{\exp \left( \int_{|z|}^{\nu_2} \frac{2i\varepsilon \sqrt{\sigma_1 \delta s}}{(1+\varepsilon \frac{\kappa}{2})^{\alpha}} \right) ds}{4s(1+i\varepsilon \frac{\kappa}{2} s^{\alpha})} \right|
$$

$$
= \frac{\varepsilon}{8c_0} \left| \int_{\varepsilon |z|^2}^{\nu_2} \frac{(1+\varepsilon |z|^2)^{\alpha}}{(1+\varepsilon |z|^2)^{\alpha}} \left( \int_{\varepsilon |z|^2}^{\nu_2} \exp \left( \int_{\varepsilon |z|^2}^{\nu_2} \frac{2i\varepsilon \sqrt{\sigma_1 \delta s}}{\varepsilon (1+\sigma_1)^{\alpha}} \right) d \left( \frac{(1+\sigma_1)^{\alpha}}{\varepsilon |z|^2} \right) \right) \right|
$$

The first two terms can be estimated as follows:

$$
\left| \frac{(1+\nu_2)^{\alpha}}{\nu_2 (1+i\nu_2)} \exp \left( \int_{\varepsilon |z|^2}^{\nu_2} \frac{2i\varepsilon \sqrt{\sigma_1 \delta s}}{\varepsilon (1+\sigma_1)^{\alpha}} \right) \right| \leq \frac{(1+\nu_2)^{\alpha}}{\varepsilon |z|^2} \leq \frac{(1+\nu_2)^{\alpha}}{\varepsilon |z|^2},
$$

$$
\left| \frac{(1+\varepsilon |z|^2)^{\alpha}}{\varepsilon |z|^2 (1+i\varepsilon |z|^2)^{\alpha}} \leq \frac{(1+\nu_2)^{\alpha}}{\varepsilon |z|^2}.\right|
$$

Furthermore,

$$
\left| \left( \frac{(1+\nu_2)^{\alpha}}{s_1^2 (1+i\sqrt{s_1})} \right) \left( \frac{(1+\nu_2)^{\alpha}}{s_1^2 (1+i\sqrt{s_1})} \right) \left( \frac{1}{1+s_1} - \frac{3}{2s_1} - \frac{i}{2\sqrt{s_1}(1+i\sqrt{s_1})} \right) \right| \leq \frac{(1+\nu_2)^{\alpha}}{s_1^2} \left( \frac{\kappa+3}{2} + \frac{\sqrt{2\pi}}{2} \right),
$$

and thus

$$
\left| \int_{\varepsilon |z|^2}^{\nu_2} \exp \left( \int_{\varepsilon |z|^2}^{\nu_2} \frac{2i\varepsilon \sqrt{\sigma_1 \delta s}}{\varepsilon (1+\sigma_1)^{\alpha}} \right) d \left( \frac{(1+\nu_2)^{\alpha}}{s_1^2 (1+i\sqrt{s_1})} \right) \right| \leq (1+\nu_2)^{\alpha} \left( \frac{\kappa+3}{2} + \frac{\sqrt{2\pi}}{2} \right) \int_{\varepsilon |z|^2}^{\nu_2} \frac{ds_1}{s_1^2} \leq \frac{2(1+\nu_2)^{\alpha} \left( \frac{\kappa+3}{2} + \frac{\sqrt{2\pi}}{2} \right)}{3\varepsilon |z|^2}.\right|
$$

Combining everything we get:

$$
\|\hat{T}_{IV}(z,\varepsilon)\| \leq \frac{(1+\nu_2)^{\alpha}}{8c_0|z|^2} \left( 2 + \frac{2}{3} \left( \frac{\kappa+3}{2} + \frac{\sqrt{2\pi}}{2} \right) \right).
$$
The case \( \varepsilon = 0 \) should be considered separately:

\[
\|\tilde{T}_{IV}(z, 0)\| = \left| \int_{-\infty}^{z} \exp \left( i \int_{s}^{z} 2i \sqrt{-s} \right) ds \right| = \left| \int_{|z|}^{\infty} \exp \left( \frac{i \varepsilon}{s^3} \right) ds \right|
\]

\[
= \left| \frac{i}{8c_0|z|^2} \exp \left( \frac{4i \varepsilon}{3} \right) - \frac{3i}{16c_0} \int_{|z|}^{\infty} \exp \left( \frac{4i \varepsilon}{s^3} \right) ds \right|
\]

\[
\leq \frac{1}{8c_0|z|^2} + \frac{3}{16c_0} \int_{|z|}^{\infty} \frac{ds}{s^7} = \frac{1}{4c_0|z|^2}.
\]

Thus the estimate is proved for both cases. \( \square \)

On the first part \([-Z_1(\varepsilon), -Z_0]\) of the region \(IV\) we treat the system \((8.7)\) as a perturbation of the free system \((8.14)\).

\[u'_{IV,2}(z) = \left( i \varepsilon \sqrt{-z} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{1}{4z} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \tilde{R}_{IV,2}(z, \varepsilon) \right) u_{IV,2}(z),\]

where

\[\tilde{R}_{IV,2}(z, \varepsilon) = R_{IV,2}(z, \varepsilon) + \varepsilon \rho \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} v_{IV,2}(z),\]

From this we see that with some \(c_{27} > 0\) for every \(z \in [-Z_2(\varepsilon), -Z_0]\)

\[\|\tilde{R}_{IV,2}(z, \varepsilon)\| \leq \|R_{IV,2}(z, \varepsilon)\| + c_{27} \left( \varepsilon \sqrt{|z|} + \frac{\varepsilon}{\sqrt{|z|}} \right).\]

In the same way as in the region \(II\) due to Lemma \(8.4\) and the choice of \(Z_1(\varepsilon) = \frac{1}{\varepsilon^2}\), one has:

\[\int_{-Z_0}^{-Z_1(\varepsilon)} \|\tilde{R}_{IV,2}(s, \varepsilon)\| ds \to 0 \text{ as } \varepsilon \to 0^+.\]

Now we can obtain results for the subintervals \([-Z_1(\varepsilon), -Z_0]\) and \([-Z_2(\varepsilon), -Z_1(\varepsilon)]\).

**Lemma 8.6.** Let the conditions of Lemma \(8.4\) hold and let \(Z_1(\varepsilon)\) be given by \((7.25)\). If

\[
\int_{-Z_1(\varepsilon)}^{-Z_0} \frac{\|R_{IV}(t(s, \varepsilon), \varepsilon)\|}{\sqrt{-s}} ds = o \left( \varepsilon \sqrt{|z|} \right) \text{ as } \varepsilon \to 0^+,
\]

then for every sufficiently small \(\varepsilon > 0\) the system \((8.7)\) on the interval \([-Z_1(\varepsilon), -Z_0]\) has two solutions \(\tilde{u}_{IV,2}^\pm(z, \varepsilon)\) such that, as \(\varepsilon \to 0^+\),

\[\tilde{u}_{IV,2}^\pm(-Z_1(\varepsilon), \varepsilon) = \frac{\exp \left( \frac{2i\varepsilon}{(Z_1(\varepsilon))^2} \right)}{(Z_1(\varepsilon))^2} (e_\pm + o(1)),\]

where the vectors \(e_\pm\) are given by \((7.21)\). Moreover, \(\tilde{u}_{IV,2}^\pm(z, \varepsilon) \to v_{IV,2}^\pm(z)\) as \(\varepsilon \to 0^+\) for every fixed \(z \leq -Z_0\), where \(v_{IV,2}^\pm\) are defined in Lemma \(8.4\).
Proof. Take
\begin{equation}
(8.22) \quad u_{IV,2}(z) = (I + \widehat{T}_{IV}(z,0))u_{IV,3}(z)
\end{equation}
with $\widehat{T}_{IV}$ given by (8.18). According to the argument of the Subsection 7.1 and due to the formula (7.20) this leads to the system
\begin{equation}
(8.23) \quad u'_{IV,3}(z) = \left( ic_0 \sqrt{-z} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{1}{4z} I + Q_{IV,3}(z) + R_{IV,3}(z,\varepsilon) \right) u_{IV,3}(z)
\end{equation}
with
\[ Q_{IV,3}(z) := (I + \widehat{T}_{IV}(z,0))^{-1}(S_{IV,2}(z,0)\widehat{T}_{IV}(z,0) - \widehat{T}_{IV}(z,0) \text{ diag } S_{IV,2}(z,0) ) \]
and
\[ R_{IV,3}(z,\varepsilon) := (I + \widehat{T}_{IV}(z,0))^{-1}R_{IV,2}(z,\varepsilon)(I + \widehat{T}_{IV}(z,0)). \]
From the expression (8.10) for $S_{IV,2}(z)$ and the estimate (8.19) for $T_{IV}$ we have
\begin{equation}
(8.24) \quad Q_{IV,3}(z) = O \left( \frac{1}{\sqrt{|z|}} \right) \quad \text{as} \quad z \to -\infty.
\end{equation}
Consider the free system
\begin{equation}
(8.25) \quad v'_{IV,3}(z) = \left( ic_0 \sqrt{-z} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{1}{4z} I + Q_{IV,3}(z) \right) v_{IV,3}(z).
\end{equation}
Since \( \int_{-\infty}^{1} \|Q_{IV,3}(s)\| ds < \infty \), the asymptotic Levinson theorem [8, Theorem 8.1] is applicable and yields the existence of two solutions $v^{\pm}_{IV,3}$ of the system (8.25) with asymptotics
\begin{equation}
(8.26) \quad v^{\pm}_{IV,3}(z) = \exp \left( \mp \frac{2ic_0}{3} \right) \left( -z \right)^{\frac{1}{4}} (e_{\pm} + o(1)) \quad \text{as} \quad z \to -\infty.
\end{equation}
Remark 8.1. Now the proof of Lemma 8.1 follows.

Proof of Lemma 8.1. Define solutions $v^{\pm}_{IV,2}$ of the system (8.14) as
\begin{equation}
(8.27) \quad v^{\pm}_{IV,2}(z) := (I + \widehat{T}_{IV}(z,0))v^{\pm}_{IV,3}(z).
\end{equation}
Due to (8.26) and Lemma 8.5 they have asymptotics (8.15)
\[ v^{\pm}_{IV,2}(z) = \exp \left( \mp \frac{2ic_0}{3} \right) \left( -z \right)^{\frac{1}{4}} (e_{\pm} + o(1)) \quad \text{as} \quad z \to -\infty. \]
From (8.19) and the integral estimate of the remainder (8.20) we also have
\begin{equation}
(8.28) \quad \int_{-Z_0}^{1} \|R_{IV,3}(s,\varepsilon)\| ds \to 0 \quad \text{as} \quad \varepsilon \to 0^+.
\end{equation}
Let us make a variation of parameters transformation: denote
\[ E_1(z) := \frac{2ic_0}{3} \left( -z \right)^{\frac{1}{4}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]
and take
\begin{equation}
(8.29) \quad u_{IV,3}(z) = e^{-iE_1(z)} \left( -z \right)^{\frac{1}{4}} u_{IV,4}(z),
\end{equation}
which turns the system (8.23) into the system
\begin{equation}
(8.30) \quad u'_{IV,4}(z) = e^{iE_1(z)}(Q_{IV,3}(z) + R_{IV,3}(z, \varepsilon))e^{-iE_1(z)}u_{IV,4}(z).
\end{equation}
Consider the following particular solutions of the system (8.30):
\begin{equation}
(8.31) \quad u_{IV,4}^\pm(z, \varepsilon) = e_{\pm} + \int_{-Z_1(\varepsilon)}^z e^{iE_1(s)}(Q_{IV,3}(s) + R_{IV,3}(s, \varepsilon))e^{-iE_1(s)}u_{IV,4}^\pm(s, \varepsilon)ds.
\end{equation}
Repeating the same manipulations with the system (8.25) we get
\begin{equation}
(8.32) \quad v_{IV,4}^\pm(z) = e_{\pm} + \int_{-\infty}^z e^{iE_1(s)}Q_{IV,3}(s)e^{-iE_1(s)}v_{IV,4}^\pm(s)ds.
\end{equation}
with
\[
\overline{v}_{IV,3}^\pm := \frac{e^{-iE_1(z)}}{(z)'} v_{IV,4}^\pm(z),
\]
instead of (8.29). As in the previous section we need to show that \(\overline{v}_{IV,3}^\pm = v_{IV,3}^\pm\).
Subtracting (8.32) from (8.31) we obtain the equality
\begin{equation}
(8.33) \quad u_{IV,4}^\pm(z, \varepsilon) - v_{IV,4}^\pm(z, \varepsilon) = \int_{-Z_1(\varepsilon)}^z e^{iE_1(s)}(Q_{IV,3}(s) + R_{IV,3}(s, \varepsilon))e^{-iE_1(s)}u_{IV,4}^\pm(s, \varepsilon)ds
\end{equation}
\[\quad - \int_{-\infty}^{-Z_1(\varepsilon)} e^{iE_1(s)}Q_{IV,3}(s)e^{-iE_1(s)}v_{IV,4}^\pm(s)ds
\end{equation}
\[\quad + \int_{Z_1(\varepsilon)}^z e^{iE_1(s)}Q_{IV,3}(s)e^{-iE_1(s)}(u_{IV,4}^\pm(s, \varepsilon) - v_{IV,4}(s))ds.
\end{equation}
Lemma 7.7 yields for the equation (8.31):
\begin{equation}
(8.34) \quad \sup_{z \in [-Z_1(\varepsilon), -Z_0]} \|u_{IV,4}^\pm(z, \varepsilon)\| \leq \exp \left( \int_{-Z_1(\varepsilon)}^{-Z_0} \|Q_{IV,3}(s) + R_{IV,3}(s, \varepsilon)\| ds \right),
\end{equation}
for the equation (8.32):
\begin{equation}
(8.35) \quad \sup_{z \in (-\infty, -Z_1(\varepsilon))] \|v_{IV,4}^\pm(z)\| \leq \exp \left( \int_{-\infty}^{-Z_1(\varepsilon)} \|Q_{IV,3}(s)\| ds \right),
\end{equation}
and finally for the equality (8.33):
\begin{equation}
(8.36) \quad \sup_{z \in [-Z_1(\varepsilon), -Z_0]} \|u_{IV,4}^\pm(z, \varepsilon) - v_{IV,4}(z)\| \leq \exp \left( \int_{-Z_1(\varepsilon)}^{-Z_0} \|Q_{IV,3}(s)\| ds \right)
\end{equation}
\[\times \left( \sup_{z \in [-Z_1(\varepsilon), -Z_0]} \|u_{IV,4}^\pm(z, \varepsilon)\| \int_{-Z_1(\varepsilon)}^{-Z_0} \|R_{IV,3}(s, \varepsilon)\| ds \right)
\end{equation}
\[+ \sup_{z \in (-\infty, -Z_1(\varepsilon))] \|v_{IV,4}(z)\| \int_{-\infty}^{-Z_1(\varepsilon)} \|Q_{IV,3}(s)\| ds \right)
\end{equation}
\[\times \leq \exp \left( 2 \int_{-\infty}^{-Z_0} \|Q_{IV,3}(s)\| ds \right) \left( \exp \left( \int_{-Z_1(\varepsilon)}^{-Z_0} \|R_{IV,3}(s, \varepsilon)\| ds \right) \right)
\end{equation}
\[\times \int_{-Z_1(\varepsilon)}^{-Z_0} \|R_{IV,3}(s, \varepsilon)\| ds + \int_{-\infty}^{-Z_1(\varepsilon)} \|Q_{IV,3}(s)\| ds \right) \to 0
\end{equation}
as \( \varepsilon \to 0^+ \) due to the estimate (8.24) for \( Q_{IV,3} \) and the integral estimate (8.28) for \( R_{IV,3} \). Using (8.24) with (8.35) to estimate the integral in the equation (8.32) we conclude that

\[
v_{IV,4}^\pm(z) \to e_\pm \text{ as } z \to -\infty,
\]

and \( \tilde{v}_{IV,3}(z) \) have the same asymptotics as \( v_{IV,3}^\pm(z) \). Therefore \( \tilde{v}_{IV,3}(z) = v_{IV,3}^\pm(z) \) and

\[
v_{IV,2}^\pm(z) = (I + \hat{T}_{IV}(z,0)) \frac{e^{-iE_1(z)}}{(-z)^{1/4}} v_{IV,4}^\pm(z).
\]

Using (8.29) and (8.22) we define functions

\[
\tilde{u}_{IV,2}^\pm(z,\varepsilon) := (I + \hat{T}_{IV}(z,0)) \frac{e^{-iE_1(z)}}{(-z)^{1/4}} u_{IV,4}^\pm(z,\varepsilon),
\]

and they are solutions of the system (8.7). From the convergence in (8.36) and the equality (8.37) we conclude that

\[
\tilde{u}_{IV,2}^\pm(z,\varepsilon) \to v_{IV,2}^\pm(z)
\]

as \( \varepsilon \to 0^+ \) for every fixed \( z \leq -Z_0 \). For \( z = -Z_1(\varepsilon) \) we use the estimate for \( \hat{T}_{IV}(z,0) \) by Lemma 8.5 and the fact that \( u_{IV,4}^\pm(-Z_1(\varepsilon),\varepsilon) = e_\pm \) which follows from the equation (8.31) to get

\[
\tilde{u}_{IV,2}^\pm(-Z_1(\varepsilon),\varepsilon) = (I + o(1)) \frac{e^{-iE_1(Z_1(\varepsilon))}}{(Z_1(\varepsilon))^{1/4}} e_\pm = \frac{\exp \left( \pm \frac{2i\varepsilon}{3} (Z_1(\varepsilon))^{3/2} \right)}{(Z_1(\varepsilon))^{1/4}} (e_\pm + o(1))
\]

as \( \varepsilon \to 0^+ \). This completes the proof of Lemma 8.6 \( \Box \)

The result for the interval \([-Z_2(\varepsilon), -Z_1(\varepsilon)]\) is given by the following lemma.

**Lemma 8.7.** Let the conditions of Lemma 8.2 hold and let \( Z_1(\varepsilon) \) be given by (7.25). If

\[
\int_{-Z_1(\varepsilon)}^{-Z_2(\varepsilon)} \frac{\|R_{IV}(t(s,\varepsilon),\varepsilon)\|}{\sqrt{s}} ds = o \left( \varepsilon^{1/2} \right) \text{ as } \varepsilon \to 0^+,
\]

then for every sufficiently small \( \varepsilon > 0 \) the system (8.7) on the interval \([-Z_2(\varepsilon), -Z_1(\varepsilon)]\) has two solutions \( \tilde{u}_{IV,2}^\pm(z,\varepsilon) \) such that, as \( \varepsilon \to 0^+ \),

\[
\tilde{u}_{IV,2}^\pm(z,\varepsilon) = \exp \left( - \int_z^{-Z_1(\varepsilon)} \left( \pm \frac{i\varepsilon \sqrt{-s}}{(1 - \varepsilon^{1/4} s)^{1/4}} - \frac{1}{4\varepsilon(1 + \varepsilon^{1/4} \sqrt{-s})} \right) ds \right) (e_\pm + o(1)),
\]

where the vectors \( e_\pm \) are given by (1.21) and the remainder \( o(1) \) converges uniformly with respect to \( z \in [-Z_2(\varepsilon), -Z_1(\varepsilon)] \).

**Proof.** Let us make the Harris–Lutz transformation

\[
u_{IV,2}(z) = (I + \hat{T}_{IV}(z,\varepsilon)) u_{IV,2}(z)
\]

where \( \hat{T}_{IV} \) is given by formula (8.18). The substitution gives

\[
u_{IV,2}(z) = (A_{IV,2}(z,\varepsilon) + R_{IV,2}(z,\varepsilon)) u_{IV,2}(z),
\]
Let us denote
\[(8.41)\]
\[
\Lambda_{IV,7}(z, \varepsilon) := i \left( \frac{c_0 \sqrt{-z}}{(1 - \varepsilon z)^2} + \frac{\varepsilon^\frac{1}{2}}{4\sqrt{-z}(1 - \varepsilon z)} \right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} - \frac{1}{4z(1 - \varepsilon z)} I,
\]
\[
\Lambda_{IV,7}(z, \varepsilon) = \Lambda_{IV,2}(z, \varepsilon) + \text{diag} S_{IV,2}(z, \varepsilon) = \begin{pmatrix} \lambda_{IV,7}^+(z, \varepsilon) & 0 \\ 0 & \lambda_{IV,7}^-(z, \varepsilon) \end{pmatrix},
\]
\[
\lambda_{IV,7}^\pm(z, \varepsilon) := \pm \frac{ic_0 \sqrt{-z}}{(1 - \varepsilon^\frac{1}{2} z)^2} - \frac{1}{4z(1 + i\varepsilon^\frac{1}{2} \sqrt{-z})}
\]
and, from \[(7.20),\]
\[(8.42)\]
\[
R_{IV,7}(z, \varepsilon) := (I + \hat{T}_{IV}(z, \varepsilon))^{-1} R_{IV,2}(z, \varepsilon)(I + \hat{T}_{IV}(z, \varepsilon))
\]
\[
+ (I + \hat{T}_{IV}(z, \varepsilon))^{-1} (S_{IV,2}(z, \varepsilon) \hat{T}_{IV}(z, \varepsilon) - \hat{T}_{IV}(z, \varepsilon) \text{diag} S_{IV,2}(z, \varepsilon))
\]
with \(S_{IV,2}\) given by \[(8.10).\] From Lemma \[8.4\] and the estimate of \(T_{IV}(z, \varepsilon)\) by Lemma \[8.5\] we get:
\[(8.43)\]
\[
\int_{-Z_2(\varepsilon)}^{Z_1(\varepsilon)} \|R_{IV,7}(s, \varepsilon)\| ds \to 0 \text{ as } \varepsilon \to 0^+.
\]
Variation of parameters
\[(8.44)\]
\[
u_{IV,7}(z) = \exp \left( - \int_{z}^{-Z_1(\varepsilon)} \Lambda_{IV,7}(\sigma, \varepsilon) d\sigma \right) u_{IV,8}(z),
\]
leads to the system
\[(8.45)\]
\[
u'_{IV,8}(z) = \exp \left( \int_{z}^{-Z_1(\varepsilon)} \Lambda_{IV,7}(\sigma, \varepsilon) d\sigma \right)
\]
\[
\times R_{IV,7}(z, \varepsilon) \exp \left( - \int_{z}^{-Z_1(\varepsilon)} \Lambda_{IV,7}(\sigma, \varepsilon) d\sigma \right) u_{IV,8}(z).
\]
Let us denote
\[
E_2(z, \varepsilon) := \int_{z}^{-Z_1(\varepsilon)} \left( \frac{c_0 \sqrt{-\sigma}}{(1 - \varepsilon^\frac{1}{2} \sigma)^2} + \frac{\varepsilon^\frac{1}{2}}{4\sqrt{-\sigma}(1 - \varepsilon^\frac{1}{2} \sigma)} \right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} d\sigma,
\]
and then
\[
\exp \left( - \int_{z}^{-Z_1(\varepsilon)} \Lambda_{IV,7}(\sigma, \varepsilon) d\sigma \right) = e^{-iE_2(z, \varepsilon)} \exp \left( \int_{z}^{-Z_1(\varepsilon)} \frac{d\sigma}{4\sigma(1 - \varepsilon^\frac{1}{2} \sigma)} \right),
\]
so that the equation reads
\[
u_{IV,8}'(z) = e^{iE_2(z, \varepsilon)} R_{IV,7}(z, \varepsilon) e^{-iE_2(z, \varepsilon)} u_{IV,8}(z).
\]
Let us now introduce two solutions \(u^\pm_{IV,8}\) of this system which satisfy the following equations:
\[
u^\pm_{IV,8}(z, \varepsilon) = e^\pm + \int_{-Z_2(\varepsilon)}^{Z_1(\varepsilon)} e^{iE_2(s, \varepsilon)} R_{IV,7}(s, \varepsilon) e^{-iE_2(s, \varepsilon)} u^\pm_{IV,8}(s, \varepsilon) ds.
\]
They can be rewritten as

\[(8.46) \quad u_{IV,8}^\pm(z, \varepsilon) - e_\pm = \int_{-Z_2(\varepsilon)}^{z} e^{iE_2(s, \varepsilon)} R_{IV,7}(s, \varepsilon)e^{-iE_2(s, \varepsilon)}e_\pm ds + \int_{-Z_2(\varepsilon)}^{z} e^{iE_2(s, \varepsilon)} R_{IV,7}(s, \varepsilon)e^{-iE_2(s, \varepsilon)}(u_{IV,8}^\pm(s, \varepsilon) - e_\pm) ds.\]

For these equalities Lemma 7.7 yields:

\[(8.47) \quad \sup_{z\in[-Z_2(\varepsilon), -Z_1(\varepsilon)]} \left\| u_{IV,8}^\pm(z, \varepsilon) - e_\pm \right\| \leq \exp \left( \int_{-Z_2(\varepsilon)}^{-Z_1(\varepsilon)} \| R_{IV,7}(s, \varepsilon) \| ds \right) \int_{-Z_2(\varepsilon)}^{-Z_1(\varepsilon)} \| R_{IV,7}(s, \varepsilon) \| ds \to 0\]

as \(\varepsilon \to 0^+\). Now using the relations (8.39) and (8.44) we define

\[\tilde{u}_{IV,2}^\pm(z, \varepsilon) := (I + \tilde{T}_{IV}(z, \varepsilon)) \exp \left( -\int_{-Z_2(\varepsilon)}^{-Z_1(\varepsilon)} \Lambda_{IV,7}^\pm(\sigma, \varepsilon) d\sigma \right) u_{IV,8}^\pm(z, \varepsilon)\]

which are solutions of the system (8.6). Lemma 8.5, the expression (8.41) and convergence in (8.47) imply that these solutions have asymptotics (8.38). This completes the proof. \(\square\)

Combining the results of Lemmas 8.6 and 8.7 we now come to the proof of Lemma 8.2.

**Proof of Lemma 8.2** First let us rewrite the formula (8.24) for the asymptotics from Lemma 8.6 using that

\[\frac{\exp \left( \mp \frac{2i\alpha_0}{(Z_1(\varepsilon))^\frac{3}{2}} \right)}{(Z_1(\varepsilon))^\frac{3}{2}} = a_{IV}^\pm \exp \left( -\int_{-Z_2(\varepsilon)}^{-Z_1(\varepsilon)} \lambda_{IV,7}^\pm(s, \varepsilon) ds \right) (1 + o(1))\]

as \(\varepsilon \to 0^+\), where \(a_{IV}^\pm := \frac{\exp \left( \mp \frac{2i\alpha_0 Z_0^\pm}{Z_0^\pm} \right)}{Z_0^\pm}\), which is true because

\[\left| \lambda_{IV,7}^\pm(s, \varepsilon) - \left( \pm i\alpha_0 \sqrt{-s} - \frac{1}{4s} \right) \right| < c_{28} \left( \varepsilon^\frac{5}{2} |s|^{\frac{3}{2}} + \frac{\varepsilon^{\frac{7}{2}}}{\sqrt{|s|}} \right)\]

with some \(c_{28} > 0\) and

\[\int_{Z_0}^{Z_1(\varepsilon)} \left( \varepsilon^\frac{5}{2} s^{\frac{3}{2}} + \varepsilon \frac{1}{\sqrt{s}} \right) ds \to 0\]

due to the choice of \(Z_1(\varepsilon)\) in (7.23). Let us define for \(z \in [-Z_2(\varepsilon), -Z_1(\varepsilon)]\)

\[(8.48) \quad u_{IV,2}^\pm(z, \varepsilon) := a_{IV}^\pm \exp \left( -\int_{-Z_2(\varepsilon)}^{-Z_1(\varepsilon)} \lambda_{IV,7}^\pm(s, \varepsilon) ds \right) \tilde{u}_{IV,2}^\pm(z, \varepsilon).\]

Continuations of the solutions \(\tilde{u}_{IV,2}^\pm(z, \varepsilon)\) to the interval \([-Z_1(\varepsilon), -Z_0]\) have decompositions with some coefficients in terms of the basis of solutions \(\tilde{u}_{IV,2}^\pm(z, \varepsilon)\), and at
the point $-Z_1(\varepsilon)$ one has:

$$\hat{u}_{IV,2}^\pm(\varepsilon) = e_\pm + o(1),$$

By Lemma 7.9 we conclude that for $z \in [-Z_1(\varepsilon), -Z_0]$

$$\hat{u}_{IV,2}(z, \varepsilon) = (1 + o(1)) \frac{u_{IV,2}(z, \varepsilon)}{a_{IV}} \exp \left( \int_{-Z_1(\varepsilon)}^{-Z_0} \lambda_{IV,\gamma}(s, \varepsilon) ds \right) + o \left( \int_{-Z_1(\varepsilon)}^{-Z_0} \lambda_{IV,\gamma}(s, \varepsilon) ds \right)$$

and, since $\lambda_{IV,\gamma}(s, \varepsilon) = \lambda_{IV}(s, \varepsilon)$, by (9.3) we have

$$u_{IV,2}(z, \varepsilon) = (1 + o(1)) \hat{u}_{IV,2}(z, \varepsilon) + o \left( \hat{u}_{IV,2}(z, \varepsilon) \right).$$

For every fixed $z \leq -Z_0$ this means that

$$u_{IV,2}(z, \varepsilon) \to v_{IV,2}(z) \text{ as } \varepsilon \to 0^+.$$  

Asymptotics of $u_{IV,2}(z, \varepsilon)$ at $z = -Z_2(\varepsilon)$ are due to Lemma 8.7 \hfill \square

9. Neighbourhood of the turning point (region III)

Consider the system

$$\varepsilon u_{III}'(t) = \left( \begin{pmatrix} \gamma & -1 \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} + R_{III}(t, \varepsilon) \right) u_{III}(t). \tag{9.1}$$

The main term of its coefficient matrix is analytic near the turning point $t_0 = (2\beta_t)^{\frac{1}{2}}$ and degenerates as a matrix at $t_0$. Analytic theory for the case $R_{III} \equiv 0$ is well known (see, for example, [35, Chapter VIII]). It suggests the transformation

$$u_{III}(t) = T_{III}(t) u_{III,1}(t) \tag{9.2}$$

with

$$T_{III}(t) := \begin{pmatrix} 1 & \frac{\beta}{\gamma} + \frac{1}{2} \\ 1 & -\frac{\beta}{\gamma} - \frac{1}{2} \end{pmatrix} \tag{9.3}$$

which makes the structure of the main term simpler:

$$\varepsilon u_{III,1}'(t) = \left( \begin{pmatrix} 0 & 1 \\ \frac{\beta^2}{\gamma^2} - \frac{1}{4} & 0 \end{pmatrix} + R_{III,1}(t, \varepsilon) \right) u_{III,1}(t), \tag{9.4}$$

where

$$R_{III,1}(t, \varepsilon) := -\frac{\varepsilon \beta \gamma}{t^{1+\gamma} \left( \frac{\beta}{\gamma} + \frac{1}{2} \right)} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + T_{III}^{-1}(t) R_{III}(t, \varepsilon) T_{III}(t). \tag{9.5}$$

The problem comes from the remainder $R_{III,1}$ which is by no means analytic (we know that it wildly oscillates), but is small in the integral sense. Analytic theory would proceed with making the change of the variable $\tau(t) \sim \text{const} \cdot (t - t_0)$ as $t \to t_0$ and considering

$$u_a(\tau) = P(\tau, \varepsilon) u_{III,1}(t(\tau))$$
with the matrix-valued function \( P \) analytic in both variables such that the system \((9.4)\) is transformed into

\[
\varepsilon u'_a(\tau) = \begin{pmatrix} 0 & 1 \\ \tau & 0 \end{pmatrix} u_a(\tau).
\]

Using the variable

\[
\zeta = \frac{\tau}{\varepsilon^{3/4}}
\]

and the function \( u_{a,1} \) defined by the equality

\[
u_a(\tau) = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon^{1/4} \end{pmatrix} u_{a,1} \left( \frac{\tau}{\varepsilon^{3/4}} \right),
\]

this system can be further transformed into

\[
u'_{a,1}(\zeta) = \begin{pmatrix} 0 & 1 \\ \zeta & 0 \end{pmatrix} u_{a,1}(\zeta).
\]

Solutions of the latter are expressed in terms of Airy functions: the system has the following matrix solution:

\[
U_{a,1}(\zeta) = \begin{pmatrix} \text{Ai}(\zeta) & \text{Bi}(\zeta) \\ \text{Ai}'(\zeta) & \text{Bi}'(\zeta) \end{pmatrix}.
\]

In our case, due to presence of the remainder \( R_{III,1} \), such transformations are not possible. Instead, in this section we show that in the scale of the variable \( z \) which was used in the regions \( II \) and \( IV \), on any fixed interval \([-z_0, z_0]\), the presence of the remainder \( R_{III} \) does not affect the asymptotics of solutions of the system \((9.1)\).

In terms of the variable \( t \) the interval \([-Z_0, Z_0]\) of the region \( III \) corresponds to the interval \([t_{III-III}(\varepsilon), t_{III-IV}(\varepsilon)]\) which shrinks to the turning point \( t_0 \). Analytic method of [35] gives the result for a fixed neighbourhood of the turning point in the scale of the variable \( t \), the result which we do not have here. However, we already know what happens in the regions \( II \) and \( IV \).

Let consider the variable \( z = \frac{1 - \varepsilon^{3/2}}{\varepsilon^{3/4}} \) and take

\[
u_{III,1}(t) = \begin{pmatrix} 2 & 0 \\ 0 & -\varepsilon^{1/4} \end{pmatrix} u_{III,2}(z(t, \varepsilon)).
\]

Substituting this to \((9.4)\) and simplifying the result we have:

\[
u'_{III,2}(z) = \frac{c_0}{(1 - \varepsilon^{3/2} z)^{1 - \delta/2}} \begin{pmatrix} 0 & 1 \\ \frac{z}{1 - \varepsilon^{3/2} z} & 0 \end{pmatrix} u_{III,1}(t(z, \varepsilon), \varepsilon) \begin{pmatrix} 2 & 0 \\ 0 & -\varepsilon^{1/4} \end{pmatrix} R_{III,1}(t(z, \varepsilon), \varepsilon) \left( \frac{\varepsilon^{1/4}}{0 - 2} \right) u_{III,2}(z)
\]

with \( c_0 \) given by \((7.11)\). This system can be written in the form

\[
u'_{III,2}(z) = \left( c_0 \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} + R_{III,2}(z, \varepsilon) \right) u_{III,2}(z),
\]
Lemma 9.1. Let \( (9.7) \) then the system \( (9.10) \) Under the conditions of Lemma 9.1, if

\[
(9.9) \quad U_{III}(z, \varepsilon) = \begin{pmatrix} A_0(c_0 z) & B_0(c_0 z) \\ c_0^{-\frac{1}{2}} A_1'(c_0^2 z) & c_0^{-\frac{1}{2}} B_1'(c_0^2 z) \end{pmatrix} \quad \text{as } \varepsilon \to 0^+.
\]

The following lemma is the main result for the region III.

Lemma 9.2. Under the conditions of Lemma 9.1, if

\[
(9.10) \quad \int_{-\varepsilon_0}^{\varepsilon_0} \| R_{III}(t(s), \varepsilon) \| ds = o\left(\varepsilon^{-\frac{5}{2}}\right), \quad \text{then } \int_{-\varepsilon_0}^{\varepsilon_0} \| R_{III,2}(s, \varepsilon) \| ds \to 0
\]
as \( \varepsilon \to 0^+ \).

Proof. From the expression \( (9.8) \) one immediately has: there exists \( c_{29} > 0 \) such that for every sufficiently small \( \varepsilon \) and every \( z \in [-\varepsilon_0, \varepsilon_0] \)

\[
\| R_{III,2}(z, \varepsilon) \| \leq c_{29} \left(\varepsilon^{-\frac{5}{2}} + \varepsilon^{-\frac{5}{2}} \| R_{III,1}(t(z), \varepsilon) \|\right).
\]

Furthermore, due to \( (9.5) \), boundedness and bounded invertibility of \( T_{III} \) in the neighbourhood of the point \( t_0 \), one has:

\[
\| R_{III,2}(z, \varepsilon) \| \leq c_{30} \left(\varepsilon^{-\frac{5}{2}} + \varepsilon^{-\frac{5}{2}} \| R_{III}(t(z), \varepsilon) \|\right)
\]
with some \( c_{30} > 0 \), which converges to zero as \( \varepsilon \to 0^+ \). \( \square \)

Now let us see that conditions of Lemma 9.1 are satisfied, if \( R_{III} = R_{III}^\ast \).

Lemma 9.3. Let \( z_0 > 0 \), let \( R_{III}^\ast(t, \varepsilon) \) be given by \( (4.14) \), \( t(z, \varepsilon) \) be defined by \( (7.6) \) and conditions \( (4.3) \) and \( (4.2) \) hold. Then

\[
\int_{-\varepsilon_0}^{\varepsilon_0} \| R_{III}^\ast(s, \varepsilon) \| ds = O\left(\varepsilon \varepsilon_0^{-\frac{5}{4}}\right) \quad \text{as } \varepsilon \to 0^+.
\]

Proof. Since \( \frac{dz}{dt}(t) = -\frac{t^{2z-1}}{2\beta z^2} \), we have with some \( c_{31} > 0 \):

\[
\int_{-\varepsilon_0}^{\varepsilon_0} \| R_{III}^\ast(s, \varepsilon) \| ds < c_{31} \int_{t_0(1+\varepsilon^{-\frac{5}{2}}z_0)}^{t_0(1+\varepsilon^{-\frac{5}{2}}z_0)} \| R_{III}^\ast(t, \varepsilon) \| dt.
\]
By equalities (4.14) and the estimate of the norm of $R$ from conditions (13) we have:

$$\frac{1}{\varepsilon^3} \int_{t_0(1-\varepsilon^2 z_0)}^{t_1} \left\| R_2^+(t, \varepsilon) \right\| dt = \frac{1}{\varepsilon^3} \int_{t_0(1-\varepsilon^2 z_0)}^{t_1} r \left( \varepsilon_0^{-\frac{1}{4}} t \right) dt.$$  

Using the expression (4.12) for $r$ we have:

$$\int_{t_0(1-\varepsilon^2 z_0)}^{t_1} r \left( \varepsilon_0^{-\frac{1}{4}} t \right) dt < c \varepsilon_{t_0}^\varepsilon \int_{t_0(1-\varepsilon^2 z_0)}^{t_1} \frac{dt}{t^{1+\alpha}},$$

as $\varepsilon \to 0^+$. Putting everything together we come to the following:

$$\int_{-\varepsilon_0^\varepsilon}^{\varepsilon_0^\varepsilon} \left\| R_2^+(t(s, \varepsilon), \varepsilon) \right\| ds = O \left( \frac{1}{\varepsilon_0^{-\frac{1}{4}}} \right) = O \left( \frac{\sqrt{\varepsilon}}{\varepsilon_0^{-\frac{1}{4}}} \right)$$

as $\varepsilon \to 0^+$. This completes the proof.

We need the following technical lemma, which is simple and standard.

**Lemma 9.4.** Let $A(x, \varepsilon)$ be a $n \times n$ matrix-valued function defined for $x \in [a, b]$ and $\varepsilon$ from some neighbourhood of zero. Let $A(\cdot, \varepsilon) \in L_1([a, b], M^{n \times n}(\mathbb{C}))$ for every $\varepsilon$ and $A(x, \varepsilon) \to A(x, 0)$ in the norm of $L_1([a, b], M^{n \times n}(\mathbb{C}))$. If $U(x)$ is a non-degenerate matrix solution of the system

$$(9.11) \quad u'(x) = A(x, \varepsilon)u(x),$$

for $\varepsilon = 0$, then there exist non-degenerate solutions $U(x, \varepsilon)$ of this system for all $\varepsilon \neq 0$ such that $U(x, \varepsilon) \to U(x)$ as $\varepsilon \to 0$ uniformly in $x \in [a, b]$.

**Proof.** Let us look for $U(x, \varepsilon) = U(x)Y(x, \varepsilon)$. The system (9.11) is equivalent to

$$Y'(x) = U^{-1}(x)(A(x, \varepsilon) - A(x, 0))U(x)Y(x).$$

Function $U$ is bounded on the interval $[a, b]$ and its determinant,

$$\det U(x) = \det U(a) \exp \left( \int_a^x \text{tr} A(t, 0) dt \right),$$

is separated from zero, since $A(\cdot, 0) \in L_1([a, b], M^{n \times n}(\mathbb{C}))$. Therefore the function $U^{-1}$ is also bounded on $[a, b]$. Take $Y$ as the solution of the following Volterra equation:

$$Y(x, \varepsilon) = I + \int_a^x U^{-1}(t)(A(t, \varepsilon) - A(t, 0))U(t)Y(t, \varepsilon) dt.$$

Repeating the standard argument on inverting $I - K_{III}(\varepsilon)$ where

$$K_{III}(\varepsilon) : Y(x) \mapsto \int_a^x U^{-1}(t)(A(t, \varepsilon) - A(t, 0))U(t)Y(t, \varepsilon) dt$$

is a Volterra operator in the space $L_\infty((a, b), M^{n \times n}(\mathbb{C}))$ and estimating the norm of the inverse, we finally come to the estimate

$$\sup_{x \in [a, b]} \left\| Y(x, \varepsilon) - I \right\| \leq \left\| K_{III}(\varepsilon) \right\|_{B(L_\infty([a, b]))} \exp \left( \left\| K_{III}(\varepsilon) \right\|_{B(L_\infty([a, b]))} \right)$$

and

$$\left\| K_{III}(\varepsilon) \right\|_{B(L_\infty([a, b]))} \leq \int_a^b \left\| U^{-1}(t)(A(t, \varepsilon) - A(t, 0))U(t) \right\| dt \to 0$$

as $\varepsilon \to 0^+$. From this the statement of the lemma follows. □
Lemma 10.1. Let solutions $\epsilon \to III$ the results for the region with some vector coefficients $\xi$ on $\epsilon$.

Proof. Linear dependence of solutions can be written in the following form:

$$v'_{III,2}(z) = c_0 \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} v_{III,2}(z)$$

has the solution

$$V_{III,2}(z) = \begin{pmatrix} \text{Ai}(c_0^\frac{2}{3} z) & \text{Bi}(c_0^\frac{2}{3} z) \\ c_0^{-\frac{2}{3}} \text{Ai}'(c_0^\frac{2}{3} z) & c_0^{-\frac{2}{3}} \text{Bi}'(c_0^\frac{2}{3} z) \end{pmatrix}.$$  

This can be verified by the entry-wise direct substitution using the Airy equation $u''(x) = xu(x)$.

10. Matching of the results in regions

In this section we prove Theorem 4.1 by putting together results of the previous five sections considering $R_{II} = R_{III} = R_{IV} = R_{V} = R_{V}^2$.

For the regions $II$, $III$ and $IV$ let us, according to Lemmas 7.2, 8.2 and 9.1, define solutions $u_{II}^\pm$, $u_{IV}^\pm$ and $U_{III}$ (matrix-valued) of the system (4.12) in the following way:

$$u_{II}^\pm(t,\varepsilon) := T_{II}(t)u_{III,2}^\pm(z(t,\varepsilon),\varepsilon),$$

$$u_{IV}^\pm(t,\varepsilon) := T_{IV}(t)u_{IV,2}^\pm(z(t,\varepsilon),\varepsilon)$$

and

$$U_{III}(t,\varepsilon) := T_{II}(t) \begin{pmatrix} 2 & 0 \\ 0 & -\varepsilon^\frac{1}{3} \end{pmatrix} U_{III,2}(z(t,\varepsilon),\varepsilon)$$

with $T_{II}$, $T_{III}$ and $T_{IV}$ given by (7.2), (9.3) and (8.2). These all are solutions of the same system and hence are linearly dependent with coefficients which depend on $\varepsilon$. By matching we mean finding asymptotic behaviour of these coefficients as $\varepsilon \to 0^+$. The following lemma matches solutions in the regions $II$ and $IV$ using the results for the region $III$.

Lemma 10.1. Let solutions $u_{II}^\pm(t,\varepsilon)$ and $u_{IV}^\pm(t,\varepsilon)$ of the system (4.12) be defined by (10.1) and (10.2). One has:

$$u_{II}^\pm(t,\varepsilon) = \left( \frac{i}{\sqrt{2}} + \delta_1(\varepsilon) \right) u_{IV}^\pm(t,\varepsilon) + \left( -\frac{i}{\sqrt{2}} + \delta_2(\varepsilon) \right) \bar{u}_{IV}^\pm(t,\varepsilon),$$

$$\bar{u}_{II}(t,\varepsilon) = \left( 1 + \frac{i\alpha_m}{2\sqrt{2}} + \delta_3(\varepsilon) \right) u_{IV}^\pm(t,\varepsilon) + \left( 1 - \frac{i\alpha_m}{2\sqrt{2}} + \delta_4(\varepsilon) \right) \bar{u}_{IV}^\pm(t,\varepsilon),$$

with some $\alpha_m \in \mathbb{C}$ and $\delta_1(\varepsilon), \delta_2(\varepsilon), \delta_3(\varepsilon), \delta_4(\varepsilon) \to 0$ as $\varepsilon \to 0^+$.

Proof. Linear dependence of solutions can be written in the following form:

$$u_{II}^\pm(t,\varepsilon) = U_{III}(t,\varepsilon) \xi_{II}^\pm(\varepsilon),$$

$$u_{IV}^\pm(t,\varepsilon) = U_{III}(t,\varepsilon) \xi_{IV}^\pm(\varepsilon)$$

with some vector coefficients $\xi_{II}^\pm(\varepsilon)$ and $\xi_{IV}^\pm(\varepsilon)$, and also in the following form:

$$u_{II}^\pm(t,\varepsilon) = d_{\varepsilon}^+ u_{IV}^\pm(t,\varepsilon) + d_\varepsilon^+(\varepsilon) \bar{u}_{IV}^\pm(t,\varepsilon),$$

$$\bar{u}_{II}(t,\varepsilon) = d_{\varepsilon}^- u_{IV}^\pm(t,\varepsilon) + d_\varepsilon^-(\varepsilon) \bar{u}_{IV}^\pm(t,\varepsilon)$$
with some coefficients $d^+_i(\varepsilon), d^+_i(\varepsilon), d^-_i(\varepsilon), d^-_i(\varepsilon)$ which we need to determine. These coefficients should be related as

$$\xi_{II}(\varepsilon) = d^+_i(\varepsilon)\xi_{IV}^+(\varepsilon) + d^+_i(\varepsilon)\xi_{IV}^-(\varepsilon),$$

$$\xi_{II}(\varepsilon) = d^-_i(\varepsilon)\xi_{IV}^+(\varepsilon) + d^-_i(\varepsilon)\xi_{IV}^-(\varepsilon).$$

This can be rewritten in matrix notation as $C_{II}(\varepsilon) = C_{IV}(\varepsilon)D(\varepsilon)$, where

$$C_{II} := (\xi_{II}^+|\xi_{II}^-), C_{IV} := (\xi_{IV}^+|\xi_{IV}^-), D := \begin{pmatrix} d^+_i & d^-_i \\ d^+_i & d^-_i \end{pmatrix}.$$ 

Therefore if we know $\xi_{II}^\pm$ and $\xi_{IV}^\pm$, we can calculate $d^+_i, d^+_i, d^-_i, d^-_i$ by the formula

$$(10.6) \quad D(\varepsilon) = C_{IV}(\varepsilon)^{-1}C_{II}(\varepsilon).$$

From the expressions (10.1), (10.2) and (10.3) we can write for any solutions $u_{II,2}, u_{IV,2}$ and $u_{III,2}$ of the systems (7.8), (8.7) and (9.7):

$$(10.7) \quad u_{III,2}(z) = P_{III}(z, \varepsilon)u_{III,2}(z) = P_{IV}(z, \varepsilon)u_{IV,2}(z)$$

with

$$(10.8) \quad P_{II}(z, \varepsilon) := \begin{pmatrix} 2 & 0 \\ 0 & -e^{\frac{\varepsilon}{2}} \end{pmatrix}^{-1} T_{III}(t(z, \varepsilon))T_{II}(t(z, \varepsilon)),$$

$$(10.9) \quad P_{IV}(z, \varepsilon) := \begin{pmatrix} 2 & 0 \\ 0 & -e^{\frac{\varepsilon}{2}} \end{pmatrix}^{-1} T_{III}(t(z, \varepsilon))T_{IV}(t(z, \varepsilon)).$$

Using the expressions (7.2) and (9.3) for $T_{II}$ and $T_{III}$ we have:

$$P_{II}(z, \varepsilon) = \frac{1}{2e^{\frac{\varepsilon}{2}}} \begin{pmatrix} e^{\frac{\varepsilon}{2}} & 0 \\ 0 & -2 \end{pmatrix} \times \left( \frac{1}{1 - \left(\frac{\beta + i}{t_0} + \frac{1}{2} \right)^2} \right)^{-1} \left( \frac{2\beta}{t_0} - \sqrt{\frac{4\beta^2}{t_0^2} - 1} \frac{2\beta}{t_0} + \sqrt{\frac{4\beta^2}{t_0^2} - 1} \right)$$

with $t = t_0(1 - e^{\frac{\varepsilon}{2}}z)^{\frac{1}{2}}$. Since

$$\frac{2\beta}{t_0} = 1 + o(1) \quad \text{and} \quad \sqrt{\frac{4\beta^2}{t_0^2} - 1} = e^{\frac{\varepsilon}{2}}(\sqrt{z} + o(1)) \quad \text{as} \quad \varepsilon \to 0^+,$$

we have

$$(10.10) \quad P_{II}(z, \varepsilon) \to \frac{1}{2} \begin{pmatrix} 1 & 1 \\ \sqrt{z} & -1 \sqrt{z} \end{pmatrix} =: P_{II}(z) \quad \text{as} \quad \varepsilon \to 0^+.$$ 

Analogously,

$$P_{IV}(z, \varepsilon) = \frac{1}{2e^{\frac{\varepsilon}{2}}} \begin{pmatrix} e^{\frac{\varepsilon}{2}} & 0 \\ 0 & -2 \end{pmatrix} \times \left( \frac{1}{1 - \left(\frac{\beta + i}{t_0} + \frac{1}{2} \right)^2} \right)^{-1} \left( \frac{2\beta}{t_0} + i\sqrt{1 - \frac{4\beta^2}{t_0^2}} - \frac{2\beta}{t_0} - i\sqrt{1 - \frac{4\beta^2}{t_0^2}} \right) \to \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i\sqrt{-z} & -i\sqrt{-z} \end{pmatrix} =: P_{IV}(z)$$

as $\varepsilon \to 0^+$. For the solutions $u_{III,2}^\pm, u_{IV,2}^\pm$ and $U_{III,2}$ this means:

$$P_{II}(z, \varepsilon)u_{III,2}^\pm(z, \varepsilon) = U_{III,2}(z, \varepsilon)\xi_{II}^\pm(\varepsilon)$$
Due to the asymptotics as $z \to -\infty$, we have

$$\xi_{II}(\varepsilon) = U_{III,2}(z,\varepsilon) P_{II}(z,\varepsilon) u_{II,2}^\pm(z,\varepsilon) \to V_{III,2}^{-1}(z) P_{II}(z) v_{II,2}^\pm(z) =: \xi_{II}^\pm$$

and

$$\xi_{IV}^\pm(\varepsilon) = U_{III,2}^{-1}(z,\varepsilon) P_{IV}(z,\varepsilon) u_{IV,2}^\pm(z,\varepsilon) \to V_{III,2}^{-1}(z) P_{IV}(z) v_{IV,2}^\pm(z) =: \xi_{IV}^\pm,$$

as $\varepsilon \to 0^+$, with $V_{III,2}$ given by (9.12), $v_{II,2}^\pm$ defined in Lemma 7.1 and $v_{IV,2}^\pm$ defined in Lemma 8.1.

To find $\xi_{II}^\pm$ consider the equation $P_{II}(z) v_{II,2}^\pm(z) = V_{III,2}(z) \xi_{II}^\pm$ which, due to the expressions (9.12) for $V_{III,2}$, (10.10) for $P_{II}$ and Lemma 7.1 reads as

$$\frac{1}{2} \begin{pmatrix} 1 & 1 \\ -\sqrt{z} & \sqrt{z} \end{pmatrix} \exp \left( \frac{-2\alpha \xi^2}{3z} \right) (1 + o(1)) \begin{pmatrix} \text{Ai}(c_0^2 z) & \text{Bi}(c_0^2 z) \\ -\frac{1}{2} \text{Ai}'(c_0^2 z) & c_0^2 \text{Bi}'(c_0^2 z) \end{pmatrix} \xi_{II}^\pm.$$

Using the asymptotics of the Airy functions as $z \to +\infty$, see [1],

$$\text{Ai}(c_0^2 z) = \exp \left( \frac{-2\alpha \xi^2}{3z} \right) (1 + o(1)), \quad \text{Ai}'(c_0^2 z) = -\frac{1}{2} \text{Ai}'(c_0^2 z) \exp \left( \frac{-2\alpha \xi^2}{3z} \right) (1 + o(1)),$$

$$\text{Bi}(c_0^2 z) = \frac{1}{2\sqrt{\pi} c_0 z} \exp \left( \frac{-2\alpha \xi^2}{3z} \right) (1 + o(1)), \quad \text{Bi}'(c_0^2 z) = -\frac{1}{2} \text{Bi}'(c_0^2 z) \exp \left( \frac{-2\alpha \xi^2}{3z} \right) (1 + o(1)),$$

we conclude that

$$\xi_{II}^+ = c_0^2 \sqrt{\pi} e_+, \quad \xi_{II}^- = c_0^2 \sqrt{\pi} \left( e_- + \alpha_m e_+ \right)$$

and

$$(10.11) \quad C_{II} = \frac{c_0^2 \sqrt{\pi}}{2} \begin{pmatrix} 2 & \alpha_m \\ 0 & 1 \end{pmatrix}.$$

with some $\alpha_m$.

To find $\xi_{IV}^\pm = V_{III,2}^{-1}(z) P_{IV}(z) v_{IV,2}^\pm(z)$ recall that

$$P_{IV}(z) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i\sqrt{-z} & -i\sqrt{-z} \end{pmatrix}$$

and, since $W\{\text{Ai}, \text{Bi}\} = -\frac{1}{z}$ (11),

$$V_{III,2}^{-1}(z) = \pi \begin{pmatrix} \text{Bi}'(c_0^2 z) - c_0^2 \text{Bi}(c_0^2 z) \\ -\text{Ai}'(c_0^2 z) + c_0^2 \text{Ai}(c_0^2 z) \end{pmatrix},$$

therefore

$$\xi_{IV}^\pm = \frac{\pi}{2} \begin{pmatrix} \text{Bi}'(c_0^2 z) - i\sqrt{-c_0^2 z} \text{Bi}(c_0^2 z) & \text{Bi}'(c_0^2 z) + i\sqrt{-c_0^2 z} \text{Bi}(c_0^2 z) \\ -\text{Ai}'(c_0^2 z) - i\sqrt{-c_0^2 z} \text{Ai}(c_0^2 z) & -\text{Ai}'(c_0^2 z) + i\sqrt{-c_0^2 z} \text{Ai}(c_0^2 z) \end{pmatrix} \times v_{IV,2}^\pm(z).$$

Due to the asymptotics as $z \to -\infty$ (see 11)

$$\text{Ai}(c_0^2 z) = \frac{1}{\sqrt{\pi} c_0^2 (-z)^{\frac{3}{4}}} \left( \sin \left( \frac{2c_0^2}{3} (-z)^{\frac{3}{2}} + \frac{\pi}{4} \right) - \cos \left( \frac{2c_0^2}{3} (-z)^{\frac{3}{2}} + \frac{\pi}{4} \right) + o(1) \right),$$
Ai’(c_0^{-1}z) = -\frac{1}{\sqrt{\pi}}c_0^{\frac{1}{2}}(-z)^{\frac{1}{2}}\left(\cos\left(\frac{2c_0}{3}(-z)^{\frac{3}{2}} + \frac{\pi}{4}\right) + \sin\left(\frac{2c_0}{3}(-z)^{\frac{3}{2}} + \frac{\pi}{4}\right) + o(1)\right),

Bi(c_0^{-1}z) = \frac{1}{\sqrt{\pi}}c_0^{\frac{1}{2}}(-z)^{\frac{1}{2}}\left(\cos\left(\frac{2c_0}{3}(-z)^{\frac{3}{2}} + \frac{\pi}{4}\right) + \sin\left(\frac{2c_0}{3}(-z)^{\frac{3}{2}} + \frac{\pi}{4}\right) + o(1)\right),

Bi’(c_0^{-1}z) = -\frac{1}{\sqrt{\pi}}c_0^{\frac{1}{2}}(-z)^{\frac{1}{2}}\left(-\sin\left(\frac{2c_0}{3}(-z)^{\frac{3}{2}} + \frac{\pi}{4}\right) + \cos\left(\frac{2c_0}{3}(-z)^{\frac{3}{2}} + \frac{\pi}{4}\right) + o(1)\right)

and by Lemma 8.1 we have:

\xi_{IV}^\pm = c_0^{-1}\sqrt{\frac{\pi}{2}}(-z)^{\frac{1}{2}}\left(\begin{array}{c}-i \exp\left(\frac{2ic_0}{3}(-z)^{\frac{1}{2}}\right) \\ \exp\left(\frac{2ic_0}{3}(-z)^{\frac{1}{2}}\right) \end{array}\right) \times \exp\left(-\frac{2ic_0}{3}(-z)^{\frac{1}{2}}\right) c_{0}^{\frac{1}{2}} \sqrt{\frac{\pi}{2}}\begin{array}{c} \mp i \\ 1 \end{array}.

This means that

C_{IV} = c_0^{\frac{1}{2}}\sqrt{\frac{\pi}{2}}\begin{array}{c} -i \\ 1 \end{array}.

Using the formula (10.11) for C_{I1} and the relation (10.6) we come to the following convergence as \varepsilon \to 0^+:

D(\varepsilon) \to C_{IV}^{-1}C_{I1} = \frac{i}{2\sqrt{2}}\begin{array}{cc} 2 & \alpha_m - i \\ -2 & -\alpha_m - i \end{array},

which gives (10.4) and (10.5). This completes the proof. \square

Now we have everything to prove Theorem 4.1.

Proof of Theorem 4.1. Since

\Lambda_{I1,2}(z, \varepsilon) + \text{diag} S_{I1,2}(z, \varepsilon) = \left(\frac{\lambda_{I1}(t(z, \varepsilon))}{\varepsilon} \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} + \text{diag} S_{I1}(t(z, \varepsilon))\right) t'(z, \varepsilon),

and

\Lambda_{IV,2}(z, \varepsilon) + \text{diag} S_{IV,2}(z, \varepsilon) = \left(\frac{\lambda_{IV}(t(z, \varepsilon))}{\varepsilon} \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} + \text{diag} S_{IV}(t(z, \varepsilon))\right) t'(z, \varepsilon),

see the formulae (7.4), (7.6) and (8.8), (8.10), we can, using Lemmas 7.2 and 8.2, rewrite the asymptotics (7.18) and (8.17) at the points t_{I-11} and t_{IV-V} as follows:

(10.12) \quad u_{I1}^{\pm}(t_{I-11}, \varepsilon)

= u_{I1}^{\pm} T_{I1}(t_{I-11}) \exp\left(\int_{t_{I1-11}(\varepsilon)}^{t_{I-11}} \left(\pm \frac{\lambda_{I1}(\tau)}{\varepsilon} + S_{I1,\pm}(\tau)\right) d\tau\right) (e_\pm + o(1))

and

(10.13) \quad u_{IV}^{\pm}(t_{IV-V}, \varepsilon)

= u_{IV}^{\pm} T_{IV}(t_{IV-V}) \exp\left(\int_{t_{IV-V}(\varepsilon)}^{t_{IV-V}} \left(\pm \frac{\lambda_{IV}(\tau)}{\varepsilon} + S_{IV,\pm}(\tau)\right) d\tau\right) (e_\pm + o(1))
as $\varepsilon \to 0^+$, where $S_{II,+}$ and $S_{IV,+}$ are upper-left and $S_{II,-}$ and $S_{IV,-}$ lower-right entries of the matrices $S_{II}$ and $S_{IV}$. Rewrite the last formula as

$$\exp\left(\int_{t_{IV-V}}^{t_{IV-V}(\varepsilon)} \left(\pm \frac{\lambda_{IV}(\tau)}{\varepsilon} + S_{IV,\pm}(\tau)\right) d\tau\right) u_{IV}^+(t_{IV-V}, \varepsilon) \to a_{IV}^+ T_{IV}(t_{IV-V}) e_\pm.$$ 

By Lemma 6.1 we also have:

$$\exp\left(- \int_{t_{IV-V}}^{t_{IV-V}(\varepsilon)} \frac{\lambda_{IV}(\tau)}{\varepsilon} d\tau + \int_{t_{IV-V}}^{\infty} S_{IV,\pm}(\tau) d\tau\right) u_{IV}(t_{IV-V}, \varepsilon, e_\pm) \to T_{IV}(t_{IV-V}) e_\pm.$$ 

Since $T_{IV} = T_V$, Lemma 7.9 yields:

(10.14) \[ u_{IV}^+(t, \varepsilon) = \exp\left(\int_{t_{II-IV}(\varepsilon)}^{t_{IV-V}} \left(\frac{\lambda_{IV}(\tau)}{\varepsilon} + S_{IV,+}(\tau)\right) d\tau\right) \times \left((a_{IV}^+ + \delta_5(\varepsilon)) \exp\left(- \int_{t_{IV-V}}^{t_{IV-V}(\varepsilon)} \frac{\lambda_{IV}(\tau)}{\varepsilon} d\tau + \int_{t_{IV-V}}^{\infty} S_{IV,+}(\tau) d\tau\right) u_{IV}(t, \varepsilon, e_+) + \delta_6(\varepsilon) \exp\left(\int_{t_{IV-V}}^{t_{IV-V}(\varepsilon)} \frac{\lambda_{IV}(\tau)}{\varepsilon} d\tau + \int_{t_{IV-V}}^{\infty} S_{IV,-}(\tau) d\tau\right) u_{IV}(t, \varepsilon, e_-) \right) \]

and

(10.15) \[ u_{IV}^-(t, \varepsilon) = \exp\left(\int_{t_{II-IV}(\varepsilon)}^{t_{IV-V}} \left(-\frac{\lambda_{IV}(\tau)}{\varepsilon} + S_{IV,-}(\tau)\right) d\tau\right) \times \left((a_{IV}^- + \delta_7(\varepsilon)) \exp\left(- \int_{t_{IV-V}}^{t_{IV-V}(\varepsilon)} \frac{\lambda_{IV}(\tau)}{\varepsilon} d\tau + \int_{t_{IV-V}}^{\infty} S_{IV,-}(\tau) d\tau\right) u_{IV}(t, \varepsilon, e_-) + \delta_8(\varepsilon) \exp\left(\int_{t_{IV-V}}^{t_{IV-V}(\varepsilon)} \frac{\lambda_{IV}(\tau)}{\varepsilon} d\tau + \int_{t_{IV-V}}^{\infty} S_{IV,+}(\tau) d\tau\right) u_{IV}(t, \varepsilon, e_+) \right) \]

with some $\delta_5(\varepsilon), \delta_6(\varepsilon), \delta_7(\varepsilon), \delta_8(\varepsilon) \to 0$ as $\varepsilon \to 0^+$.

Let us now consider the solution $u_{IV}^\pm(t, \varepsilon, f)$ for the case $f \parallel f_-$ and prove the asymptotics (4.6). By Lemma (5.1),

$$u_{IV}^\pm(t_{I-II}, \varepsilon, f) = T_I(t_{I-II}) \exp\left(\int_0^{t_{I-II}} \left(\frac{\lambda_I(\tau)}{\varepsilon} + S_{I,+}(\tau)\right) d\tau\right) (\Phi(f) e_+ + o(1)).$$

Rewrite this as

$$\exp\left(- \int_0^{t_{I-II}} \left(\frac{\lambda_I(\tau)}{\varepsilon} + S_{I,+}(\tau)\right) d\tau\right) u_{IV}^\pm(t_{I-II}, \varepsilon, f) \to T_I(t_{I-II}) \Phi(f) e_+.$$ 

Rewrite also the asymptotics (10.12) as

$$\exp\left(\int_{t_{II-II}}^{t_{II-II}(\varepsilon)} \left(\pm \frac{\lambda_{II}(\tau)}{\varepsilon} + S_{II,\pm}(\tau)\right) d\tau\right) u_{II}^\pm(t_{II-II}, \varepsilon) \to a_{II}^\pm T_{II}(t_{I-II}) e_\pm.$$
Since \( T_1 e_+ = T_1 e_+ \), by Lemma 7.9 we get that

\[
u^+_2 (t, \varepsilon, f) = \exp \left( \int_0^{t_{111-111}} \left( \frac{\lambda_I(\tau)}{\varepsilon} + S_{I,+}(\tau) \right) d\tau \right) \times \left( \frac{\Phi(f)}{a^+_{11}} + \delta_9(\varepsilon) \right) \exp \left( \int_0^{t_{111-111}} \left( \frac{\lambda_{IV}(\tau)}{\varepsilon} + S_{IV,+}(\tau) \right) d\tau \right) u^+_I(t, \varepsilon) \]

\[
+ \delta_{10}(\varepsilon) \exp \left( \int_0^{t_{111-111}} \left( - \frac{\lambda_{IV}(\tau)}{\varepsilon} + S_{IV,-}(\tau) \right) d\tau \right) u^-_I(t, \varepsilon) \]

with some \( \delta_9(\varepsilon), \delta_{10}(\varepsilon) \to 0 \) as \( \varepsilon \to 0^+ \). Using Lemma 10.1 and the fact that

\[
\exp \left( \int_{t_{111-111}}^{t_{111-111}} \left( - \frac{\lambda_{IV}(\tau)}{\varepsilon} + S_{IV,-}(\tau) \right) d\tau \right) = O \left( \exp \left( \int_{t_{111-111}}^{t_{111-111}} \left( \frac{\lambda_{IV}(\tau)}{\varepsilon} + S_{IV,+}(\tau) \right) d\tau \right) \right) \quad \text{as} \quad \varepsilon \to 0^+, \]

as well as the identities \( \lambda_I = \lambda_{IV} \) and \( S_{I,+} = S_{IV,+} \), we have:

\[
u^+_2 (t, \varepsilon, f) = \frac{i\Phi(f)}{2a^+_{11}} \exp \left( \int_0^{t_{111-111}} \left( \frac{\lambda_I(\tau)}{\varepsilon} + S_{I,+}(\tau) \right) d\tau \right) \times \left( (1 + \delta_{11}(\varepsilon)) u^+_I(t, \varepsilon) - (1 + \delta_{12}(\varepsilon)) u^-_I(t, \varepsilon) \right) \]

with some \( \delta_{11}(\varepsilon), \delta_{12}(\varepsilon) \to 0 \) as \( \varepsilon \to 0^+ \). Since

\[
\left| \exp \left( \int_{t_{111-111}}^{t_{111-111}} \pm \frac{\lambda_{IV}(\tau)}{\varepsilon} d\tau \right) \right| = 1, \]

the relations (10.14) and (10.15) together with the identities \( \lambda_{IV} = \lambda_V \) and \( S_{IV} = S_V \) imply that

\[
u^+_2 (t, \varepsilon, f) = \frac{i\Phi(f)}{\sqrt{2a^+_{11}}} \exp \left( \int_0^{t_{111-111}} \left( \frac{\lambda_I(\tau)}{\varepsilon} + S_{I,+}(\tau) \right) d\tau \right) \times \left( (a^+_{IV} + \delta_{13}(\varepsilon)) \exp \left( - \int_{t_{0}}^{t_{111-111}} \frac{\lambda_V(\tau)}{\varepsilon} d\tau + \int_{t_{111-111}}^{+\infty} S_{V,+}(\tau) d\tau \right) u_V(t, \varepsilon, e_+) \right)

\[
- (a^-_{IV} + \delta_{14}(\varepsilon)) \exp \left( \int_{t_{0}}^{t_{111-111}} \frac{\lambda_V(\tau)}{\varepsilon} d\tau + \int_{t_{111-111}}^{+\infty} S_{V,-}(\tau) d\tau \right) u_V(t, \varepsilon, e_-) \right) \]}
with some $\delta_{13}(\varepsilon), \delta_{14}(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. By Lemma 6.4 this means that

\begin{equation}
(10.16) \quad \exp \left( - \int_{t_0}^t \frac{\lambda_V(\tau)}{\varepsilon} \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) d\tau \right) T_V^{-1}(t)u_{-+}^+(t, \varepsilon, f)
\end{equation}

\[ \to \frac{i\Phi(f)}{\sqrt{2a_{II}^+}} \exp \left( \int_0^{t_{II-II}(\varepsilon)} \left( \frac{\lambda_I(\tau)}{\varepsilon} + S_{I,+}(\tau) \right) d\tau \right) \]

\[ \times \left( a_{IV}^+ + \delta_{13}(\varepsilon) \right) \exp \left( - \int_{t_0}^{t_{IV-IV}(\varepsilon)} \frac{\lambda_V(\tau)}{\varepsilon} d\tau + \int_{t_{II-II}(\varepsilon)}^{+\infty} S_{V,+}(\tau) d\tau \right) e_{++} \]

\[ - \left( a_{IV}^- + \delta_{14}(\varepsilon) \right) \exp \left( \int_{t_0}^{t_{IV-IV}(\varepsilon)} \frac{\lambda_V(\tau)}{\varepsilon} d\tau + \int_{t_{II-II}(\varepsilon)}^{+\infty} S_{V,-}(\tau) d\tau \right) e_{--} \]

as $t \to +\infty$. Let us now simplify the right-hand side. As

\[ a_{II}^+ = \exp \left( -\frac{2\varepsilon a_{II}^+}{Z_0^+} \right), \quad a_{IV}^+ = \exp \left( \frac{Z_0^+}{2} \right) \]

we have

\[ Z_0^+ a_{II}^+ \exp \left( - \int_{t_{II-II}(\varepsilon)}^{t_0} \frac{\lambda_I(\tau)}{\varepsilon} d\tau \right) = Z_0^+ a_{II}^+ \exp \left( \int_0^{Z_0} \lambda_{II,2}(s, \varepsilon) ds \right) \]

\[ = Z_0^+ a_{II}^+ \exp \left( \int_0^{Z_0} (c_0 \sqrt{s} + o(1)) ds \right) = 1 + o(1) \text{ as } \varepsilon \to 0^+ \]

and

\[ Z_0^+ a_{IV}^+ \exp \left( \pm \int_{t_0}^{t_{IV-IV}(\varepsilon)} \frac{\lambda_V(\tau)}{\varepsilon} d\tau \right) = Z_0^+ a_{IV}^+ \exp \left( \pm \int_0^{Z_0} \lambda_{IV,2}(s, \varepsilon) ds \right) \]

\[ = Z_0^+ a_{IV}^+ \exp \left( \int_0^{Z_0} (\pm i c_0 \sqrt{-s} + o(1)) ds \right) = 1 + o(1) \text{ as } \varepsilon \to 0^+ \]

Furthermore,

\[ S_{I,+}(t) = \frac{\gamma t\gamma}{8\beta^2 t^{1-\gamma} \left( 1 - \frac{e^{2\pi}}{4\pi^2} \right) \left( 1 + \sqrt{1 - \frac{e^{2\pi}}{4\pi^2}} \right)} = \frac{\gamma \left( 1 - \sqrt{1 - \frac{e^{2\pi}}{4\pi^2}} \right)}{2t \left( 1 - \frac{e^{2\pi}}{4\pi^2} \right)} \]

\[ S_{V,\pm}(t) = -\frac{\beta \gamma \left( \frac{2\gamma}{t^{\gamma+1}} \pm i \sqrt{1 - \frac{4\gamma^2}{\pi^2}} \right)}{1 - \frac{4\gamma^2}{\pi^2}} = \frac{\gamma \left( 1 \pm i \sqrt{\frac{4\gamma^2}{\pi^2} - 1} \right)}{2t \left( 1 - \frac{4\gamma^2}{\pi^2} \right)} \]
Therefore there exists the limit

\[(10.17) \quad c_{\text{op}}(\beta, \gamma) := \lim_{\varepsilon \to 0^+} \left( \int_0^{t_{11} - t_{11}(\varepsilon)} S_{t,+}(\tau) d\tau + \int_{t_{11} - t_{11}(\varepsilon)}^{+\infty} \text{Re } S_{V,\pm}(\tau) d\tau \right) \]

\[= \lim_{\Delta \to 0^+} \left( \int_0^{t_0 - \Delta} \frac{\gamma (1 - \sqrt{1 - \frac{\varepsilon^2}{4\lambda^2}})}{2\tau (1 - \frac{\varepsilon^2}{4\lambda^2})} d\tau + \int_{t_0 + \Delta}^{+\infty} \frac{\gamma d\tau}{2\tau (1 - \frac{\varepsilon^2}{4\lambda^2})} \right) \]

\[= \int_0^{t_0} \frac{\gamma (1 - \sqrt{1 - \frac{\varepsilon^2}{4\lambda^2}})}{2\tau (1 - \frac{\varepsilon^2}{4\lambda^2})} d\tau - \int_{t_0}^{\infty} \frac{\gamma d\tau}{2\tau (1 - \frac{\varepsilon^2}{4\lambda^2})} + \text{v.p.} \int_{t_0}^{+\infty} \frac{\gamma d\tau}{2\tau (1 - \frac{\varepsilon^2}{4\lambda^2})}. \]

Then the right-hand side of (10.16) can be rewritten as

\[
\frac{i e^{c_{\text{op}}(\beta, \gamma)} \Phi(f)}{\sqrt{2}} \exp \left( \int_0^{t_0} \frac{\lambda_I(\tau)}{\varepsilon} d\tau \right) 
\times \left( 1 + \delta_{15}(\varepsilon) \right) \exp \left( i \int_{t_{11} - t_{11}(\varepsilon)}^{+\infty} \text{Im } S_{V,+}(\tau) d\tau \right) e_+ 
- \left( 1 + \delta_{16}(\varepsilon) \right) \exp \left( i \int_{t_{11} - t_{11}(\varepsilon)}^{+\infty} \text{Im } S_{V,-}(\tau) d\tau \right) e_- \]

with some \( \delta_{15}(\varepsilon) , \delta_{16}(\varepsilon) \to 0 \) as \( \varepsilon \to 0^+ \). It follows that

\[
\| T_V^{-1}(t) u^+_2(t, \varepsilon, f) \| \to \frac{e^{c_{\text{op}}(\beta, \gamma)}|\Phi(f)|}{\sqrt{2}} \exp \left( \int_0^{t_0} \frac{\lambda_I(\tau) d\tau}{\varepsilon} \right) \left\| \begin{pmatrix} 1 + \delta_{15}(\varepsilon) \\ 1 + \delta_{16}(\varepsilon) \end{pmatrix} \right\| \]

as \( t \to +\infty \). To get rid of \( T_V^{-1} \) we recall that

\[ T_V(t) \to \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \text{ as } t \to +\infty \]

and that the matrix \( \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \) is unitary. Besides that, for every \( \varepsilon \in U \) the solution \( u^+_2(t, \varepsilon, f) \) is bounded as \( t \to +\infty \), which follows from Lemma 4.1 and the relation (4.1a). Thus we conclude that

\[
\| u^+_2(t, \varepsilon, f) \| \to \frac{e^{c_{\text{op}}(\beta, \gamma)}|\Phi(f)|}{2} \exp \left( \int_0^{t_0} \frac{\lambda_I(\tau) d\tau}{\varepsilon} \right) \left\| \begin{pmatrix} 1 + \delta_{15}(\varepsilon) \\ 1 + \delta_{16}(\varepsilon) \end{pmatrix} \right\| \text{ as } t \to +\infty \]

for every fixed sufficiently small \( \varepsilon > 0 \). Now we can take \( \varepsilon \to 0^+ \) to obtain the asymptotics. With the expressions (5.5) for \( \lambda_I \) and (4.7) for \( C_{\text{mp}}(\beta, \gamma) \) this proves the formula (4.18), and the assertion of Theorem 4.1 for \( \varepsilon \to 0^+ \) follows.

Consider now the solution \( u^+_2(t, \varepsilon, f_-) \). Let us prove the estimate (4.9). First use the relations (10.14) and (10.13) together with the facts that \( \lambda_{IV} = \lambda_V \) are purely imaginary, \( S_V \) is summable at infinity and that \( \| u_V(t, \varepsilon, e_{\pm}) \| \to \sqrt{2} \) as \( t \to +\infty \) by Lemma 6.4 to conclude that

\[
\lim_{t \to +\infty} \| u_{IV}^\pm(t, \varepsilon) \| = O \left( \exp \left( \int_{t_{11} - t_{11}(\varepsilon)}^{t_{11} - t_{11}(\varepsilon)} S_{IV,+}(\tau) d\tau \right) \right) \text{ as } \varepsilon \to 0^+. \]
By Lemma 10.1 this means that

$$\lim_{t \to +\infty} \|u_2^+(t, \varepsilon)\| = O \left( \exp \left( \int_{t_1}^{t_2} \lambda_1(\tau) d\tau \right) \right) \text{ as } \varepsilon \to 0^+. $$

Let us define two solutions $u_{I-1}^\pm$ of the system (4.12) as

$$u_{I-1}^\pm(t, \varepsilon) := \frac{1}{a_{II}} \exp \left( \int_{t_1}^{t_2} \left( \pm \lambda_{I-1}(\tau) + S_{II, \pm}(\tau) \right) d\tau \right) u_{II}^\pm(t, \varepsilon)$$

On the one hand, from (10.19), the equalities $\lambda_I = \lambda_{II}, S_I = S_{II}, S_{IV} = S_{V}$ and finiteness of the limit in (10.17), we have

$$\lim_{t \to +\infty} \|u_{II}^\pm(t, \varepsilon)\| = O \left( \exp \left( \int_{t_1}^{t_2} \frac{\lambda_I(\tau)}{\varepsilon} d\tau \right) \right) \text{ as } \varepsilon \to 0^+.$$ 

On the other hand, from the asymptotics (10.12)

$$u_{II}^\pm(t_{II-1}, \varepsilon) \to T_I(t_{II-1})\varepsilon^\pm \text{ as } \varepsilon \to 0^+.$$

From this we will now estimate the growth of the norm of the fundamental solution. Define for every $h \in \mathbb{C}^2$ the solution $u_{I-1}(t, \varepsilon, h)$ of the system (4.12) on the interval $[t_{I-1}, +\infty)$ with the initial condition

$$u_{I-1}(t_{I-1}, \varepsilon, h) = T_{II}(t_{II-1})h.$$

The fundamental solution $F$ of the system (4.12) can be written as follows:

$$F(t, t_{I-1}, \varepsilon) = (u_{I-1}(t, \varepsilon, e_+))^{-1} u_{II}^\pm(t, \varepsilon) T_{II}^{-1}(t_{II-1}),$$

therefore its norm can be estimated by the norms of solutions $u_{I-1}(t, \varepsilon, e_\pm)$. For them we have from (10.20) by Lemma 5.1

$$u_{I-1}(t, \varepsilon, e_+) = (1 + \delta_{17}(\varepsilon)) u_{I-1}^+(t, \varepsilon) + \delta_{18}(\varepsilon) u_{I-1}^-(t, \varepsilon),$$

$$u_{I-1}(t, \varepsilon, e_-) = \delta_{19}(\varepsilon) u_{II-1}^+(t, \varepsilon) + (1 + \delta_{20}(\varepsilon)) u_{II-1}^-(t, \varepsilon)$$

with some $\delta_{17}(\varepsilon), \delta_{18}(\varepsilon), \delta_{19}(\varepsilon), \delta_{20}(\varepsilon) \to 0 \text{ as } \varepsilon \to 0^+$. Therefore (10.19) means that

$$\lim_{t \to +\infty} \|F(t, t_{I-1}, \varepsilon)\| = O \left( \exp \left( \int_{t_1}^{t_2} \frac{\lambda_I(\tau)}{\varepsilon} d\tau \right) \right) \text{ as } \varepsilon \to 0^+.$$ 

This estimate together with the estimate of $u_2^+(t_{II-1}, \varepsilon, f_-)$ by Lemma 5.1 imply that

$$\lim_{t \to +\infty} \|u_2^+(t, \varepsilon, f_-)\| \leq \frac{1}{\varepsilon} \frac{1}{\varepsilon^2} + R_2^-(t, \varepsilon) u_2^-(t).$$

To prove (4.12) for $\varepsilon_0 \to 0^-$ consider the solution $u_2^-(t, \varepsilon)$ of the system

$$u_2^-(t) = \left( \begin{array}{cc} \frac{1}{\varepsilon^2} & -\frac{1}{\varepsilon} \\ -\frac{1}{\varepsilon^2} & \frac{1}{\varepsilon} \end{array} \right) + R_2^-(t, \varepsilon) u_2^-(t).$$
If one takes \( u^+_2(t) = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) u_2^+(t) \), this system turns into the following:

\[
(10.21) \quad u_2^+ (t) = \left( \begin{array}{cc} \frac{\beta}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{array} \right) + \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) R_2^-(t, \varepsilon) \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) u_2^+(t).
\]

If one proves that the asymptotics as \( \varepsilon \to 0^+ \) of the solution

\[
\left( \begin{array}{c} 1 \\ 0 \end{array} \right) u_2^-(t, \varepsilon, f)
\]

in the region \( I \), namely at the point \( t = t_{II} \), coincides with the asymptotics of the solution \( u_2^+(t, \varepsilon, f) \) at the same point (which is given by Lemma 5.1), then the rest follows automatically. This is because Lemmas 6.1, 7.2, 8.2 and 9.1 are directly applicable to the system \( (10.21) \) and the matching procedure (or estimating, in the case \( f = f_- \)) works for the solution \( \left( \begin{array}{c} 1 \\ 0 \end{array} \right) u_2^- \) literally as above. Hence the asymptotic behaviour of the norm is the same. In the argument of Section 5 every estimate and convergence remain the same as for the case of \( u_2^+ \), except the calculation \( (5.21) \). In an analogue of that calculation one finally arrives at

\[
\int_{\varepsilon_0}^{\frac{1}{\varepsilon} t} \frac{1}{2} \exp \left( -\frac{1}{2} \int_{\varepsilon_0}^{x} \lambda_I \right) T_I^{-1} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \times \left( \frac{\varepsilon_0 \lambda_I}{x} \right) R(x, \varepsilon_0) u(x, \varepsilon_0, f) \exp \left( -\frac{1}{\varepsilon} \int_{\varepsilon_0}^{x} \lambda_I \right) dx,
\]

which differs from the result of the calculation \( (5.21) \) by presence of the matrix \( \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \). Since \( T_I(t) \to I \) as \( t \to 0 \), in the limit as \( \varepsilon \to 0^+ \) this term affects the lower component which goes to zero for the same reason as in the case of \( u_2^- \), so the result is still the same. This proves \( (4.6) \) for \( \varepsilon_0 \to 0^- \) and completes the proof of Theorem 4.1.

\[ \square \]

11. PROOF OF THE MAIN RESULT

In this section we prove Theorem 1.1 putting together everything that was obtained in the previous sections.

**Proof of Theorem 1.1** Consider the critical point \( \nu_{cr} \) and let \( \alpha \neq \alpha_{cr} \). By Lemma 3.1 there exists the neighbourhood \( U_{cr} \), and there the eigenfunction equation for the operator \( \mathcal{L}_\alpha \) is equivalent to the system

\[
w'_{cr}(x) = \left( \frac{\beta_{cr}}{x^\gamma} \right) \left( \frac{\cos(\varepsilon_{cr}(\lambda)x)}{\sin(\varepsilon_{cr}(\lambda)x)} \sin(\varepsilon_{cr}(\lambda)x) - \frac{\sin(\varepsilon_{cr}(\lambda)x)}{\cos(\varepsilon_{cr}(\lambda)x)} \right) w_{cr}(x).
\]

Rewrite this system as

\[
w'(x) = \left( \frac{\beta_{cr}}{x^\gamma} \right) \left( \frac{\cos(\varepsilon_0 x)}{\sin(\varepsilon_0 x)} \sin(\varepsilon_0 x) - \frac{\sin(\varepsilon_0 x)}{\cos(\varepsilon_0 x)} \right) u(x)
\]

with \( \lambda(\varepsilon_0) = \varepsilon_{cr}^{-1}(\varepsilon_0) \), which means that \( \lambda \) is parametrised by \( \varepsilon_0 \) so that \( \varepsilon_{cr}(\lambda) = \varepsilon_0 \). Properties of the remainder provided by Lemma 3.1 are such that the conditions
\[ \text{(1.2)} \text{ and } \text{(1.3)} \text{ are satisfied. By Lemma 4.1} \text{ there exists } u_{cr}(x,0) \text{ such that} \]

the solution \( u_{cr}(x,0) \) has the asymptotics

\[
u_{cr}(x,0, f_{cr-}) = \exp \left( -\frac{\beta_{cr}x^{1-\gamma}}{1-\gamma} \right) (e^{-} + o(1)) \text{ as } x \to +\infty \]

and for \( f \parallel f_{cr-} \)

\[
u_{cr}(x,0, f) = \exp \left( \frac{\beta_{cr}x^{1-\gamma}}{1-\gamma} \right) (\Phi_{cr}(f)e^{+} + o(1)) \text{ as } x \to +\infty. \]

In fact, the matrix \( R_{cr} \) has real entries, so \( f_{cr-} \in \mathbb{R}^2 \setminus \{0\} \). Moreover, comparing these asymptotics with \( \text{(3.4)} \) and \( \text{(3.5)} \) provided by Lemma 3.1, we see that

\[ f_{cr-} = \frac{g_{cr,\alpha_{cr}}}{d_{cr-}} \]

and

\[ \Phi_{cr}(f_{cr,\alpha}) = d_{cr+} \sin(\alpha - \alpha_{cr}) \].

Since \( \alpha \neq \alpha_{cr} \), the vectors \( g_{cr,\alpha} \) and \( f_{cr-} \) are linearly independent and so, by Lemma 7.9,

\[ w_{cr,\alpha}(0,\lambda(\varepsilon_0)) = (1 + \delta_{21}^{(\varepsilon_0)})g_{cr,\alpha} + \delta_{22}^{(\varepsilon_0)}f_{cr-} \]

with some \( \delta_{21}(\varepsilon), \delta_{22}(\varepsilon) \to 0 \text{ as } \varepsilon \to 0 \). Therefore

\[ w_{cr,\alpha}(x,\lambda(\varepsilon_0)) = (1 + \delta_{21}^{(\varepsilon_0)})u(x,\varepsilon_0, g_{cr,\alpha}) + \delta_{22}^{(\varepsilon_0)}u(x,\varepsilon_0, f_{cr-}). \]

By Theorem 4.1 using the relation (11.1) we have

\[
\lim_{x \to +\infty} \| u_{cr}(x,\varepsilon_0, g_{cr,\alpha}) \| = C_{mp}(\beta_{cr}, \gamma) \exp \left( \frac{1}{|\varepsilon_0|^{\frac{1}{1-\gamma}}} \int_{0}^{1} \frac{(2\beta_{cr})^{\frac{1}{4}}}{\sqrt{\frac{\beta_{cr}}{t^{2\gamma}} - \frac{1}{4}}} dt \right) \times (|d_{cr+} \sin(\alpha - \alpha_{cr})| + o(1)),
\]

and

\[
\lim_{x \to +\infty} \| u_{cr}(x,\varepsilon_0, f_{cr-}) \| = o \left( \exp \left( \frac{1}{|\varepsilon_0|^{\frac{1}{1-\gamma}}} \int_{0}^{1} \frac{(2\beta_{cr})^{\frac{1}{4}}}{\sqrt{\frac{\beta_{cr}}{t^{2\gamma}} - \frac{1}{4}}} dt \right) \right)
\]

as \( \varepsilon_0 \to 0 \). With these asymptotics it follows from (11.2) that

\[
\lim_{x \to +\infty} \| w_{cr,\alpha}(x,\lambda) \| = C_{mp}(\beta_{cr}, \gamma) \exp \left( \frac{1}{|\varepsilon_{cr}(\lambda)|^{\frac{1}{1-\gamma}}} \int_{0}^{1} \frac{(2\beta_{cr})^{\frac{1}{4}}}{\sqrt{\frac{\beta_{cr}}{t^{2\gamma}} - \frac{1}{4}}} dt \right) \times (|d_{cr+} \sin(\alpha - \alpha_{cr})| + o(1)) \text{ as } \lambda \to \nu_{cr}.
\]

Since, by Proposition 2.1 and the equality 3.3, \( \rho_{\alpha}'(\lambda) = \frac{1}{2\pi |W\{\psi_+, \psi_-\}(\lambda)| |A_{\alpha}(\lambda)|^{2}} = \frac{1}{2\pi |W\{\psi_+, \psi_-\}(\lambda)| \lim_{x \to +\infty} \| w_{cr,\alpha}(x,\lambda) \|^{2}}, \)

using continuity of \( W\{\psi_+, \psi_-\}(\cdot) \) we have

\[ (11.3) \rho_{\alpha}'(\lambda) = \frac{a_{cr}}{d_{cr+}^{2} \sin^{2}(\alpha - \alpha_{cr})} \exp \left( -\frac{2}{|\varepsilon_{cr}(\lambda)|^{\frac{1}{1-\gamma}}} \int_{0}^{1} \frac{(2\beta_{cr})^{\frac{1}{4}}}{\sqrt{\frac{\beta_{cr}}{t^{2\gamma}} - \frac{1}{4}}} dt \right) (1 + o(1)) \],
where
\[ a_{cr} = \frac{1}{2\pi |W\{\psi_+\psi_-\}(\nu_{cr})|C_{mp}(\beta_{cr}, \gamma)} \]
which coincides with (1.20) after substitution of \( C_{mp} \) from (4.7). Using the property (3.1) we have
\[ \frac{1}{|\varepsilon_{cr}(\lambda)|_{\pm}^{\frac{1}{2}}} = \frac{1}{|\lambda - \nu_{cr}|_{\pm}^{\frac{1}{2}}} \left( \frac{a}{2\pi k'(\nu_{cr})} \right)^{\frac{1}{2}} (1 + O(|\lambda - \nu_{cr}|^{\frac{1}{2}})) \] as \( \lambda \to \nu_{cr} \).
Substituting this and the result of the calculation (4.8) into (11.3) we finally arrive at the asymptotics (1.18) with \( c_{cr} \) given by (1.19). This completes the proof. □

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