On Recurrence and Transience of Two-Dimensional Lévy and Lévy-Type Processes

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Abstract

In this paper, we study recurrence and transience of Lévy-type processes, that is, Feller processes associated with pseudo-differential operators. Since the recurrence property of Lévy-type processes in dimensions greater than two is vacuous and the recurrence and transience of one-dimensional Lévy-type processes have been very well investigated, in this paper we are focused on the two-dimensional case only. In particular, we study perturbations of two-dimensional Lévy-type processes which do not affect their recurrence and transience properties, we derive sufficient conditions for their recurrence and transience in terms of the corresponding Lévy measure and we provide some comparison conditions for the recurrence and transience also in terms of the Lévy measures.

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1 Introduction

Let \( \{L_t\}_{t \geq 0}, \{P_x\}_{x \in \mathbb{R}^d} \) be a \( d \)-dimensional, \( d \geq 1 \), Lévy process. The process \( \{L_t\}_{t \geq 0} \) is said to be 

**recurrent** if 
\[
\int_0^\infty P_x (L_t \in B_r(x)) \, dt = \infty \quad \text{for all } x \in \mathbb{R}^d \text{ and all } r > 0,
\]

and **transient** if 
\[
\int_0^\infty P_x (L_t \in B_r(x)) \, dt < \infty \quad \text{for all } x \in \mathbb{R}^d \text{ and all } r > 0.
\]

Here, \( B_r(x) \) denotes the open ball of radius \( r > 0 \) around \( x \in \mathbb{R}^d \). It is well known that every Lévy process is either recurrent or transient (see [Sat99, Theorem 35.3]). However, the above definitions (characterizations) of the recurrence and transience properties are not practical in most cases. Due to the stationarity and independence of the increments, every Lévy process \( \{L_t\}_{t \geq 0} \) can be uniquely
and completely characterized through the characteristic function of a single random variable $L_t$,
$t > 0$, that is, by the famous Lévy-Khintchine formula we have
\[
\mathbb{E}^x[\exp\{iq(\xi, L_t - x)\}] = \exp\{-tq(\xi)\}, \quad t \geq 0, \quad x \in \mathbb{R}^d,
\]
where the function $q : \mathbb{R}^d \to \mathbb{C}$ is called the characteristic exponent of the process $\{L_t\}_{t \geq 0}$ and it enjoys the following Lévy-Khintchine representation
\[
q(\xi) = i\langle \xi, b \rangle + \frac{1}{2}\langle \xi, C\xi \rangle + \int_{\mathbb{R}^d} (1 - \exp\{i\langle \xi, y \rangle\} + i\langle \xi, y \rangle 1_{B_1(0)}(y)) \nu(dy), \quad \xi \in \mathbb{R}^d.
\]
Here, $b \in \mathbb{R}^d$, $C$ is a symmetric non-negative definite $d \times d$ matrix and $\nu(dy)$ is a $\sigma$-finite Borel measure on $\mathbb{R}^d$, the so-called Lévy measure, satisfying $\int_{\mathbb{R}^d} \min\{1, |y|^2\} \nu(y) < \infty$. Now, by having this nice analytical description (characterization) of Lévy processes, a more operable characterization of the recurrence and transience properties has been given by the well-known Chung-Fuchs criterion. A Lévy process $\{L_t\}_{t \geq 0}$ is recurrent if, and only if,
\[
\int_{B_1(0)} \text{Re} \left( \frac{1}{q(\xi)} \right) d\xi = \infty \quad \text{for some (all) } r > 0,
\]
(see [Sat99, Corollary 37.6 and Remark 37.7]). As one of the direct consequences of this criterion we get that in dimensions greater than two every Lévy process is transient (see [Sat99, Theorem 37.8]).

The notion of recurrence and transience can also be defined for a broader class of processes. Let $\{(M_t)_{t \geq 0}, (\mathbb{P}^x)_{x \in \mathbb{R}^d}\}$ be a d-dimensional, $d \geq 1$, Markov process with càdlàg sample paths. The process $\{M_t\}_{t \geq 0}$ is called

(i) $\varphi$-irreducible if there exists a $\sigma$-finite measure $\varphi(dy)$ on $\mathcal{B}(\mathbb{R}^d)$ such that whenever $\varphi(B) > 0$ we have $\int_0^\infty \mathbb{P}^x(M_t \in B)dt > 0$ for all $x \in \mathbb{R}^d$.

(ii) recurrent if it is $\varphi$-irreducible and if $\varphi(B) > 0$ implies $\int_0^\infty \mathbb{P}^x(M_t \in B)dt = \infty$ for all $x \in \mathbb{R}^d$.

(iii) transient if it is $\varphi$-irreducible and if there exists a countable covering of $\mathbb{R}^d$ with sets $\{B_j\}_{j \in \mathbb{N}} \subseteq \mathcal{B}(\mathbb{R}^d)$, such that for each $j \in \mathbb{N}$ there is a finite constant $c_j \geq 0$ such that $\int_0^\infty \mathbb{P}^x(M_t \in B_j)dt \leq c_j$ holds for all $x \in \mathbb{R}^d$.

Recall that, as in the Lévy process case, every $\varphi$-irreducible Markov process is either recurrent or transient (see [Twe94, Theorem 2.3]). Now, let $\{(F_t)_{t \geq 0}, (\mathbb{P}^x)_{x \in \mathbb{R}^d}\}$ be a $d$-dimensional, $d \geq 1$, conservative (that is, $\mathbb{P}^x(F_t \in \mathbb{R}^d) = 1$ for all $t \geq 0$ and all $x \in \mathbb{R}^d$) Feller process with infinitesimal generator $(\mathcal{A}, \mathcal{D}_A)$. If the set of smooth functions with compact support $C^\infty_c(\mathbb{R}^d)$ is contained in $\mathcal{D}_A$, then the operator $\mathcal{A}|_{C^\infty_c(\mathbb{R}^d)}$ has the following representation
\[
\mathcal{A}|_{C^\infty_c(\mathbb{R}^d)} f(x) = -\int_{\mathbb{R}^d} q(x, \xi) \exp\{i\langle \xi, x \rangle\} \hat{f}(\xi) d\xi,
\]
where $\hat{f}(\xi)$ denotes the Fourier transform of $f(x)$ and the function $q : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ is called the symbol of the operator $\mathcal{A}|_{C^\infty_c(\mathbb{R}^d)}$ (process $\{F_t\}_{t \geq 0}$) and, for each fixed $x \in \mathbb{R}^d$, it is the characteristic exponent of some Lévy process. In particular, it enjoys the $(x$-dependent) Lévy-Khintchine representation
\[
q(x, \xi) = i\langle \xi, b(x) \rangle + \frac{1}{2}\langle \xi, C(x)\xi \rangle + \int_{\mathbb{R}^d} (1 - \exp\{i\langle \xi, y \rangle\} + i\langle \xi, y \rangle 1_{B_1(0)}(y)) \nu(x, dy).
\]
Accordingly, a Feller process satisfying the above properties is called a \textit{Lévy-type process} (see Section 2 for details). Observe that this class of processes contains the class of Lévy processes.

In the sequel, we consider only the so-called \textit{open-set irreducible} Lévy-type processes, that is, Lévy-type processes whose irreducibility measure is fully supported. An example of such measure is the Lebesgue measure, which we denote by $\lambda(dy)$. Clearly, a Lévy-type process $\{F_t\}_{t \geq 0}$ will be $\lambda$-irreducible if $\P^x(F_t \in B) > 0$ for all $x \in \mathbb{R}^d$ and all $t > 0$ whenever $\lambda(B) > 0$. In particular, the process $\{F_t\}_{t \geq 0}$ will be $\lambda$-irreducible if the transition kernel $\P^x(F_t \in dy)$, $t > 0$, $x \in \mathbb{R}^d$, possesses a strictly positive transition density function. Let us remark that the $\lambda$-irreducibility of Lévy-type processes is a very well-studied topic in the literature. We refer the readers to [She91] and [ST97] for the case of elliptic diffusion processes, to [Kol00] for the case of a class of pure jump Lévy-type processes (the so-called stable-like processes), to [BC86], [Ish01], [KM14], [Kul07], [KC99] and [Pic96, Pic10] for the case of a class of Lévy-type processes obtained as a solution of certain jump-type stochastic differential equations and [KS12], [KS13] and [PS15] for the case of general Lévy-type processes.

Now, if $\{F_t\}_{t \geq 0}$ is an open-set irreducible Lévy-type process with symbol $q(x, \xi)$, then, under certain additional regularity conditions on the symbol (see conditions (C2) and (C3) in Section 2), in [San14a] and [SW13] it has been proven that $\{F_t\}_{t \geq 0}$ is recurrent if

$$\liminf_{\alpha \to 0} \int_{\mathbb{R}^d} \left( \int_{-\infty}^{\infty} \frac{e^{-\alpha t} \Re \E^0[e^{i(t,F_t)}]}{4 \sup_{x \in \mathbb{R}^d} |q(x, \xi)|} \sum_{i=1}^d \sin^2 \left( \frac{a d_i}{\xi} \right) \right) d\xi > -\infty$$

for all $a > 0$ small enough (see [San15, Proposition 2.5] for further discussion on this condition) and

$$\int_{B_r(0)} \frac{d\xi}{\sup_{x \in \mathbb{R}^d} |q(x, \xi)|} = \infty \quad \text{for some } r > 0, \quad (1.1)$$

and it is transient if $\sup_{x \in \mathbb{R}^d} |\Im q(x, \xi)| \leq c \inf_{x \in \mathbb{R}^d} \Re q(x, \xi)$ for some $0 \leq c < 1$ and all $\xi \in \mathbb{R}^d$ and

$$\int_{B_r(0)} \frac{d\xi}{\inf_{x \in \mathbb{R}^d} \Re q(x, \xi)} < \infty \quad \text{for some } r > 0. \quad (1.2)$$

Again, directly from the above Chung-Fuchs type conditions, in [San14a, Theorem 2.8] it has been shown that in dimensions greater than two every Lévy-type process (satisfying the above mentioned conditions) is transient.

Due to this, the problem of recurrence and transience of Lévy and Lévy-type processes reduces to the dimensions one and two. In the one-dimensional case this problem has been extensively studied in [San14a] and [Sat99]. The main questions regarding the recurrence and transience addressed in these references consider a connection with Prüﬀ indices, problem of perturbations of these processes, conditions for the recurrence and transience in terms of the underlying Lévy measure and problem of comparison for the recurrence and transience also in terms of the Lévy measures.

In this work, we are focused on the same questions as in the one-dimensional case. More precisely, in Theorem 3.2, we prove that if $\{F_t\}_{t \geq 0}$ and $\{\bar{F}_t\}_{t \geq 0}$ are two-dimensional Lévy-type processes with radial (in the co-variable) symbols $q(x, \xi)$ and $\bar{q}(x, \xi)$ and Lévy measures $\nu(x, dy)$ and $\bar{\nu}(x, dy)$, respectively, then $\{F_t\}_{t \geq 0}$ and $\{\bar{F}_t\}_{t \geq 0}$ are recurrent or transient at the same time if there is a rotation of the plane (orthogonal matrix) $R$ such that

$$\sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} |y|^2 |\nu(x, dy) - \bar{\nu}(Rx, dy)| < \infty.$$
Here, $|\mu(dy)|$ denotes the total variation measure of the signed measure $\mu(dy)$. Note that this result automatically implies that the recurrence and transience of Lévy-type processes depend only on the nature of big jumps of the process. Further, since in general it is not always easy to check the Chung-Fuchs type conditions in (1.1) and (1.2), in Theorem 4.4 we give necessary and sufficient conditions for the recurrence and transience in terms of the Lévy measure. More precisely, under the assumption that the corresponding symbol is radial in the co-variable and certain additional regularity conditions, we prove that (1.1) is equivalent to

$$\int_r^\infty \left( \rho \sup_{x \in \mathbb{R}^2} \int_0^\rho u \nu(x,B_u^c(0))du \right)^{-1} d\rho = \infty \quad \text{for some } r > 0,$$

and (1.2) is equivalent to

$$\int_r^\infty \left( \rho \inf_{x \in \mathbb{R}^2} \int_0^\rho u \nu(x,B_u^c(0))du \right)^{-1} d\rho < \infty \quad \text{for some } r > 0.$$

Finally, in Theorem 5.1, we give some comparison conditions for the recurrence and transience by comparing “tails” of the Lévy measures, that is, we prove that the recurrence of the process with the Lévy measure with “bigger tail” implies the recurrence of the one with “smaller tail”. Similarly, we prove that the transience of the process with the Lévy measure with “smaller tail” implies the transience of the process with “bigger tail”.

The remaining part of this paper is organized as follows. In Section 2, we give some preliminaries on Lévy and Lévy-type processes and in Section 3 we discuss perturbations of these processes. In Section 4 we derive sufficient conditions for the recurrence and transience in terms of the Lévy measure and, finally, in Section 5 we give some comparison conditions for the recurrence and transience properties also in terms of the Lévy measures.

## 2 Preliminaries on Lévy and Lévy-Type Processes

Let $(\Omega, \mathcal{F}, \{\mathbb{P}^x\}_{x \in \mathbb{R}^d}, \{\mathcal{F}_t\}_{t \geq 0}, \{\theta_t\}_{t \geq 0}, \{M_t\}_{t \geq 0})$, denoted by $\{M_t\}_{t \geq 0}$ in the sequel, be a Markov process with state space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, where $d \geq 1$ and $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel $\sigma$-algebra on $\mathbb{R}^d$. A family of linear operators $\{P_t\}_{t \geq 0}$ on $B_b(\mathbb{R}^d)$ (the space of bounded and Borel measurable functions), defined by

$$P_t f(x) := \mathbb{E}^x[f(M_t)], \quad t \geq 0, \quad x \in \mathbb{R}^d, \quad f \in B_b(\mathbb{R}^d),$$

is associated with the process $\{M_t\}_{t \geq 0}$. Since $\{M_t\}_{t \geq 0}$ is a Markov process, the family $\{P_t\}_{t \geq 0}$ forms a semigroup of linear operators on the Banach space $(B_b(\mathbb{R}^d), || \cdot ||_\infty)$, that is, $P_s \circ P_t = P_{s+t}$ and $P_0 = I$ for all $s, t \geq 0$. Here, $|| \cdot ||_\infty$ denotes the supremum norm on the space $B_b(\mathbb{R}^d)$. Moreover, the semigroup $\{P_t\}_{t \geq 0}$ is contractive, that is, $||P_t f||_\infty \leq ||f||_\infty$ for all $t \geq 0$ and all $f \in B_b(\mathbb{R}^d)$, and positivity preserving, that is, $P_t f \geq 0$ for all $t \geq 0$ and all $f \in B_b(\mathbb{R}^d)$ satisfying $f \geq 0$. The infinitesimal generator $(A_b, \mathcal{D}_{A_b})$ of the semigroup $\{P_t\}_{t \geq 0}$ (or of the process $\{M_t\}_{t \geq 0}$) is a linear operator $A_b : \mathcal{D}_{A_b} \rightarrow B_b(\mathbb{R}^d)$ defined by

$$A_b f := \lim_{t \to 0} \frac{P_t f - f}{t}, \quad f \in \mathcal{D}_{A_b} := \left\{ f \in B_b(\mathbb{R}^d) : \lim_{t \to 0} \frac{P_t f - f}{t} \text{ exists in } || \cdot ||_\infty \right\}.$$ 

We call $(A_b, \mathcal{D}_{A_b})$ the $B_b$-generator for short. A Markov process $\{M_t\}_{t \geq 0}$ is said to be a Feller process if its corresponding semigroup $\{P_t\}_{t \geq 0}$ forms a Feller semigroup. This means that the
family \( \{P_t\}_{t \geq 0} \) is a semigroup of linear operators on the Banach space \( C_\infty(\mathbb{R}^d), || \cdot ||_\infty \) and it is strongly continuous, that is,
\[
\lim_{t \to 0} ||P_t f - f||_\infty = 0, \quad f \in C_\infty(\mathbb{R}^d).
\]

Here, \( C_\infty(\mathbb{R}^d) \) denotes the space of continuous functions vanishing at infinity. Note that every Feller semigroup \( \{P_t\}_{t \geq 0} \) can be uniquely extended to \( B_b(\mathbb{R}^d) \) (see [Sch98a, Section 3]). For notational simplicity, we denote this extension again by \( \{P_t\}_{t \geq 0} \). Also, let us remark that every Feller process possesses the strong Markov property and has càdlàg sample paths (see [Jac05, Theorems 3.4.19 and 3.5.14]). Further, in the case of Feller processes, we call \( (\mathcal{A}, \mathcal{D}_A) := (A_b, \mathcal{D}_{A_b} \cap C_\infty(\mathbb{R}^d)) \) the Feller generator for short. Note that, in this case, \( \mathcal{D}_A \subseteq C_\infty(\mathbb{R}^d) \) and \( \mathcal{A}(\mathcal{D}_A) \subseteq C_\infty(\mathbb{R}^d) \). If the Feller generator \( (\mathcal{A}, \mathcal{D}_A) \) of a Feller process \( \{M_t\}_{t \geq 0} \) satisfies:

\[ (C1) \quad C_\infty^\infty(\mathbb{R}^d) \subseteq \mathcal{D}_A, \]

then, according to [Cou66, Theorem 3.4], \( \mathcal{A}|_{C_\infty^\infty(\mathbb{R}^d)} \) is a pseudo-differential operator, that is, it can be written in the form
\[
\mathcal{A}|_{C_\infty^\infty(\mathbb{R}^d)} f(x) = -\int_{\mathbb{R}^d} q(x, \xi) \exp\{i(\xi, x)\} \hat{f}(\xi) d\xi, \quad (2.1)
\]
where \( \hat{f}(\xi) := (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i(\xi, x)} f(x) dx \) denotes the Fourier transform of the function \( f(x) \). The function \( q : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C} \) is called the symbol of the pseudo-differential operator. It is measurable and locally bounded in \( (x, \xi) \) and continuous and negative definite as a function of \( \xi \). Hence, by [Jac01, Theorem 3.7.7], the function \( \xi \mapsto q(x, \xi) \) has for each \( x \in \mathbb{R}^d \) the Lévy-Khintchine representation
\[
q(x, \xi) = a(x) - i \langle \xi, b(x) \rangle + \frac{1}{2} \langle \xi, C(x) \xi \rangle + \int_{\mathbb{R}^d} (1 - \exp\{i(\xi, y)\} + i \langle \xi, y \rangle 1_{B_1(0)}(y)) \nu(x, dy), \quad (2.2)
\]
where \( a(x) \) is a nonnegative Borel measurable function, \( b(x) \) is an \( \mathbb{R}^d \)-valued Borel measurable function, \( C(x) := (c_{ij}(x))_{1 \leq i, j \leq d} \) is a symmetric non-negative definite \( d \times d \) matrix-valued Borel measurable function and \( \nu(x, dy) \) is a Borel kernel on \( \mathbb{R}^d \times \mathcal{B}(\mathbb{R}^d) \), called the Lévy measure, satisfying
\[
\nu(x, \{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} \min\{1, |y|^2\} \nu(x, dy) < \infty, \quad x \in \mathbb{R}^d.
\]
The quadruple \( (a(x), b(x), c(x), \nu(x, dy)) \) is called the Lévy quadruple of the pseudo-differential operator \( \mathcal{A}|_{C_\infty^\infty(\mathbb{R}^d)} \) (or of the symbol \( q(x, \xi) \)). Let us remark that the local boundedness of \( q(x, \xi) \) implies that for every compact set \( K \subseteq \mathbb{R}^d \) there exists a finite constant \( c_K > 0 \) such that
\[
\sup_{x \in K} |q(x, \xi)| \leq c_K (1 + |\xi|^2), \quad \xi \in \mathbb{R}^d, \quad (2.3)
\]
(see [Jac01, Lemma 3.6.22]). Due to [Sch98b, Lemma 2.1 and Remark 2.2], (2.3) is equivalent to the local boundedness of the Lévy quadruple, that is, for every compact set \( K \subseteq \mathbb{R}^d \) we have
\[
\sup_{x \in K} a(x) + \sup_{x \in K} |b(x)| + \sup_{x \in K} |c(x)| + \sup_{x \in K} \int_{\mathbb{R}^d} \min\{1, y^2\} \nu(x, dy) < \infty.
\]
According to the same reference, the global boundedness of the Lévy quadruple is equivalent to
\[ (C2) \quad ||q(\cdot, \xi)||_\infty \leq c(1 + |\xi|^2) \text{ for some finite } c > 0 \text{ and all } \xi \in \mathbb{R}^d. \]
In the sequel, we also assume the following condition on the symbol \(q(x, \xi)\):

\[ (C3) \quad q(x, 0) = a(x) = 0 \quad \text{for all} \quad x \in \mathbb{R}^d. \]

Let us remark that, according to [BSW13, Theorem 2.34], condition (C3) is closely related to the conservativeness property of \(\{M_t\}_{t \geq 0}\). Namely, \(\{M_t\}_{t \geq 0}\) is conservative, that is, \(\mathbb{P}^x(M_t \in \mathbb{R}^d) = 1\) for all \(t \geq 0\) and all \(x \in \mathbb{R}^d\), if \(q(x, \xi)\) satisfies (C3) and

\[
\liminf_{k \to \infty} \sup_{|y-x| \leq k} \sup_{|\xi| \leq 1/2k} |q(y, \xi)| < \infty, \quad x \in \mathbb{R}^d.
\]  

Moreover, due to [Sch98a, Theorem 5.2], under (C2), (C3) is equivalent to the conservativeness property of the process \(\{M_t\}_{t \geq 0}\). Further, note that in the case when the symbol \(q(x, \xi)\) does not depend on the variable \(x \in \mathbb{R}^d\), \(\{M_t\}_{t \geq 0}\) becomes a Lévy process, that is, a stochastic process with stationary and independent increments and càdlàg sample paths. Moreover, every Lévy process is uniquely and completely characterized through its corresponding symbol (see [Sat99, Theorems 7.10 and 8.1]). According to this, it is not hard to check that every Lévy process satisfies conditions (C1), (C2) and (C3) (see [Sat99, Theorem 31.5]). Thus, the class of processes we consider in this paper contains a subclass of Lévy processes. Let us also remark here that, unlike in the case of Lévy processes, it is not possible to associate a Feller process to every symbol (see [BSW13] for details). Throughout this paper, the symbol \(\{F_t\}_{t \geq 0}\) denotes a Feller process satisfying conditions (C1), (C2) and (C3). Such a process is called a Lévy-type process. Also, a Lévy process is denoted by \(\{L_t\}_{t \geq 0}\). If \(\nu(x, dy) = 0\) for all \(x \in \mathbb{R}^d\), according to [BSW13, Theorem 2.44], \(\{F_t\}_{t \geq 0}\) becomes an elliptic diffusion process. Note that this definition agrees with the standard definition of elliptic diffusion processes (see [RW00]). For more on Lévy and Lévy-type processes we refer the readers to the monographs [Sat99] and [BSW13].

## 3 Perturbations of Lévy and Lévy-Type Processes

In this section, we study perturbations of two-dimensional Lévy and Lévy-type processes which do not affect their recurrence and transience properties. We start with the following proposition which we will need in the sequel.

**Proposition 3.1.** Let \(\{F_t\}_{t \geq 0}\) be a \(d\)-dimensional Lévy-type process with Feller generator \((\mathcal{A}, \mathcal{D}_\mathcal{A})\), symbol \(q(x, \xi)\) and Lévy triplet \((b(x), C(x), \nu(x, dx))\), and let \(M\) be a regular \(d \times d\) matrix. Then, the process \(\{MF_t\}_{t \geq 0}\) is also a \(d\)-dimensional Lévy-type process which is determined by symbol and Lévy triplet of the form

\[
q_M(x, \xi) = q(M^{-1}x, M^T\xi),
\]

\[
b_M(x) = Mb(M^{-1}x) + \int_{\mathbb{R}^d} M y \left(1_{B_1(0)}(y) - 1_{B_1(0)}(My)\right) \nu(x, dy),
\]

\[
C_M(x) = MC(M^{-1}x)MT,
\]

\[
\nu_M(x, dy) = \nu(M^{-1}x, M^{-1}dy),
\]

respectively. Here, \(M^T\) denotes the transpose matrix of the matrix \(M\). Further, if \(\{F_t\}_{t \geq 0}\) is open-set irreducible, then \(\{MF_t\}_{t \geq 0}\) is also open-set irreducible.

**Proof.** First, by a straightforward inspection, it is easy to see that \(\{MF_t\}_{t \geq 0}\) is a Feller process with respect to \(\mathbb{P}_M(MF_t \in dy) := \mathbb{P}^{M^{-1}x}(F_t \in M^{-1}dy), t \geq 0, x \in \mathbb{R}^d\). Next, let us show that
\{MF_t\}_{t \geq 0} satisfies (C1) and that its symbol and Lévy triplet are given by the relations in (3.1). Since, 
\[ \int f(y)^p \mathbb{P}^x(MF_t \in dy) - f(x) = \int f \circ M(y)^p M^{-1}x(F_t \in dy) - f \circ M(M^{-1}x), \quad x \in \mathbb{R}^d, \quad f \in B_b(\mathbb{R}^d), \]
we easily conclude that \( C^{\infty}_c(\mathbb{R}^d) \subseteq D_{A_M} \) and \( A_M f(x) = Af \circ M(M^{-1}x) \) for \( x \in \mathbb{R}^d \) and \( f \in C^{\infty}_c(\mathbb{R}^d) \), where \( (A_M, D_{A_M}) \) denotes the Feller generator of \( \{MF_t\}_{t \geq 0} \). Now, according to (2.1), for \( x \in \mathbb{R}^d \) and \( f \in C^{\infty}_c(\mathbb{R}^d) \),
\[ A_M f(x) = - \int_{\mathbb{R}^d} q_M(x, \xi)e^{i(\xi, x)} \hat{f}(\xi) d\xi \]
\[ = - \int_{\mathbb{R}^d} q(M^{-1}x, \xi)e^{i(\xi, M^{-1}x)} \hat{f}(\xi) d\xi \]
\[ = - \text{det} M^{-1} \int_{\mathbb{R}^d} q(M^{-1}x, \xi)e^{i(\xi, M^{-1}x)} \hat{f}(\xi, (M^{-1}T) \xi) d\xi \]
\[ = - \int_{\mathbb{R}^d} q(M^{-1}x, MT \xi)e^{i(\xi, x)} \hat{f}(\xi) d\xi, \]
which together with (2.2) proves the assertion. Finally, it is straightforward to see that \( \{MF_t\}_{t \geq 0} \) satisfies the conditions in (C2) and (C3) and that it is open-set irreducible. □

Recall that a rotation of the plane for an angle \( \varphi \in [0, 2\pi) \) is a linear mapping \( R_{\varphi} : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) represented by the matrix
\[ R_{\varphi} = \begin{bmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{bmatrix}. \]
Clearly, \( R_{\varphi}^T = R_{\varphi}^{-1} = R_{2\pi - \varphi} \) and \( \det R_{\varphi} = 1 \) for all \( \varphi \in [0, 2\pi) \). A function \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) is said to be radial if, for every \( x \in \mathbb{R}^2 \), \( f(R_{\varphi}x) = f(x) \) for all \( \varphi \in [0, 2\pi) \). In the rest of this section we will always assume that the symbol \( q(x, \xi) \) of a two-dimensional Lévy-type process \( \{F_t\}_{t \geq 0} \) is radial in the co-variable. In particular, if \( (b(x), C(x), \nu(x, dy)) \) is the corresponding Lévy triplet, then, due to [Sat99, Exercise 18.3],
(i) \( b(x) = 0 \) for all \( x \in \mathbb{R}^2 \);
(ii) \( C(x) = c(x)I \) for some Borel measurable function \( c : \mathbb{R}^2 \rightarrow [0, \infty) \), where \( I \) is the \( 2 \times 2 \) identity matrix;
(iii) \( \nu(x, dy) = \nu(x, R_{\varphi}dy) \) for all \( x \in \mathbb{R}^d \) and all \( \varphi \in [0, 2\pi) \).

Also, let us remark that this assumption implies that the condition in (1.1) does not depend on \( r > 0 \) (see [San14a, Proposition 2.4]). On the other hand, note that if (1.2) holds for some \( r_0 > 0 \), then it also holds for all \( 0 < r < r_0 \). According to the same reference, if we need complete independence on \( r > 0 \) in (1.2), it suffices to assume that the function \( \xi \mapsto \inf_{x \in \mathbb{R}^2} \sqrt{q(x, \xi)} \) is subadditive (that is, \( \inf_{x \in \mathbb{R}^2} \sqrt{q(x, \xi_1 + \xi_2)} \leq \inf_{x \in \mathbb{R}^2} \sqrt{q(x, \xi_1)} + \inf_{x \in \mathbb{R}^2} \sqrt{q(x, \xi_2)} \) for all \( \xi_1, \xi_2 \in \mathbb{R}^2 \)). The main result of this section is the following (see also [San14a] and [Sat99, Section 38] for the one-dimensional case).

**Theorem 3.2.** Let \( \{F_t\}_{t \geq 0} \) and \( \{\tilde{F}_t\}_{t \geq 0} \) be two-dimensional Lévy-type processes with symbols \( q(x, \xi) \) and \( \tilde{q}(x, \xi) \) and Lévy triplets \( (0, c(x)I, \nu(x, dy)) \) and \( (0, \tilde{c}(x)I, \nu(x, dy)) \), respectively. If there exists a rotation of the plane \( R_{\varphi} \), \( \varphi \in [0, 2\pi) \), such that
\[ \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^2} |y|^2 |\nu(x, dy) - \tilde{\nu}(R_{\varphi}x, dy)| < \infty, \quad (3.2) \]
then \( q(x, \xi) \) satisfies (1.1) if, and only if, \( \bar{q}(x, \xi) \) satisfies (1.1). Further, denote

\[
c := \frac{1}{2} \sup_{x \in \mathbb{R}^2} |c(x) - \bar{c}(R_x x)| + \sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} |y|^2 |\nu(x, dy) - \nu(R_x x, dy)|.
\]

If

\[
\liminf_{|\xi| \to 0} \frac{\inf_{x \in \mathbb{R}^2} q(x, \xi)}{|\xi|^2} > c,
\]

then, under (3.2), \( q(x, \xi) \) satisfies (1.2) if, and only if, \( \bar{q}(x, \xi) \) satisfies (1.2).

**Proof.** First, observe that, due to Proposition 3.1, \( \nu(R_x x, dy) = \nu(R_{2\pi - \varphi} x, R_{2\pi - \varphi} dy) \) is the Lévy measure of the Lévy-type process \( \{R_{2\pi - \varphi} F_t\}_{t \geq 0} \). Next, note that (3.2) implies

\[
sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} |y|^2 \nu(x, dy) < \infty \quad \text{if, and only if,} \quad sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} |y|^2 \nu(x, dy) < \infty.
\]

Indeed, we have

\[
\int_{\mathbb{R}^2} |y|^2 \nu(R_x x, dy) = \int_{\mathbb{R}^2} |y|^2 |\nu(R_x x, dy) - \nu(x, dy) + \nu(x, dy)|
\]

\[
\leq \int_{\mathbb{R}^2} |y|^2 \nu(x, dy) + \int_{\mathbb{R}^2} |y|^2 |\nu(x, dy) - \nu(R_x x, dy)|,
\]

and similarly

\[
\int_{\mathbb{R}^2} |y|^2 \nu(x, dy) \leq \int_{\mathbb{R}^2} |y|^2 \nu(R_x x, dy) + \int_{\mathbb{R}^2} |y|^2 |\nu(x, dy) - \nu(R_x x, dy)|.
\]

Now, in the case when

\[
\sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} |y|^2 \nu(x, dy) < \infty, \quad \text{or, equivalently,} \quad \sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} |y|^2 \nu(x, dy) < \infty,
\]

the assertion of the theorem easily follows from [San14a, Theorem 2.8]. Next, suppose that

\[
\sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} |y|^2 \nu(x, dy) = \infty \quad \text{or, equivalently,} \quad \sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} |y|^2 \nu(x, dy) = \infty.
\]

Then, by using the radiality of the function \( \xi \mapsto q(x, \xi) \) and Fatou’s lemma, we conclude

\[
\liminf_{|\xi| \to 0} \sup_{x \in \mathbb{R}^2} \frac{q(x, \xi)}{|\xi|^2} \geq \liminf_{|\xi| \to 0} \sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1 - \cos(|\xi| e_1, y)}{|\xi|^2} \nu(x, dy)
\]

\[
\geq \liminf_{|\xi| \to 0} \int_{\mathbb{R}^2} \frac{1 - \cos(|\xi| e_1, y)}{|\xi|^2} \nu(x, dy)
\]

\[
\geq \frac{1}{2} \int_{\mathbb{R}^2} \langle e_1, y \rangle^2 \nu(x, dy)
\]

\[
= \frac{1}{4} \int_{\mathbb{R}^2} \langle e_1, y \rangle^2 \nu(x, dy) + \frac{1}{4} \int_{\mathbb{R}^2} \langle e_2, y \rangle^2 \nu(x, dy)
\]

\[
= \frac{1}{4} \int_{\mathbb{R}^2} |y|^2 \nu(x, dy),
\]
where $e_1$ and $e_2$ denote the coordinate vectors of $\mathbb{R}^2$. Thus,

$$\liminf_{|\xi|\to 0} \frac{\sup_{x \in \mathbb{R}^2} q(x, \xi)}{|\xi|^2} = \infty. \quad (3.4)$$

Next, we have

$$\left| \sup_{x \in \mathbb{R}^2} q(x, \xi) - \sup_{x \in \mathbb{R}^2} \bar{q}(x, \xi) \right|$$

$$= \left| \sup_{x \in \mathbb{R}^2} q(x, \xi) - \sup_{x \in \mathbb{R}^2} \bar{q}(R_\varphi x, \xi) \right|$$

$$\leq \sup_{x \in \mathbb{R}^2} |q(x, \xi) - \bar{q}(R_\varphi x, \xi)|$$

$$\leq \frac{1}{2} |\xi|^2 \sup_{x \in \mathbb{R}^2} |c(x) - c(R_\varphi x)| + \sup_{x \in \mathbb{R}^2} \left| \int_{\mathbb{R}^2} (1 - \cos(\xi, y)) \nu(x, dy) - \int_{\mathbb{R}^2} (1 - \cos(\xi, y)) \bar{\nu}(R_\varphi x, dy) \right|$$

$$\leq \frac{1}{2} |\xi|^2 \sup_{x \in \mathbb{R}^2} |c(x) - c(R_\varphi x)| + \sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} |y|^2 |\nu(x, dy) - \bar{\nu}(R_\varphi x, dy)|$$

$$= c|\xi|^2, \quad (3.5)$$

where in the penultimate step we used the fact that $1 - \cos u \leq u^2$ for all $u \in \mathbb{R}$. Finally, (3.4) and (3.5) imply

$$\lim_{|\xi|\to 0} \frac{\sup_{x \in \mathbb{R}^2} \bar{q}(x, \xi)}{\sup_{x \in \mathbb{R}^2} q(x, \xi)} = 1 + \lim_{|\xi|\to 0} \frac{\sup_{x \in \mathbb{R}^2} \bar{q}(x, \xi) - \sup_{x \in \mathbb{R}^2} q(x, \xi)}{\sup_{x \in \mathbb{R}^2} q(x, \xi)} = 1,$$ 

which together with the radiality of $\xi \mapsto q(x, \xi)$ proves the first assertion.

Now, we prove the second assertion. First, as above,

$$\left| \inf_{x \in \mathbb{R}^2} q(x, \xi) - \inf_{x \in \mathbb{R}^2} \bar{q}(x, \xi) \right|$$

$$\leq \sup_{x \in \mathbb{R}^2} |q(x, \xi) - \bar{q}(R_\varphi x, \xi)|$$

$$\leq c|\xi|^2. \quad (3.6)$$

Hence, by (3.3) and (3.6),

$$\liminf_{\xi \to 0} \frac{\inf_{x \in \mathbb{R}^2} \bar{q}(x, \xi)}{\inf_{x \in \mathbb{R}^2} q(x, \xi)} = 1 + \liminf_{\xi \to 0} \frac{\inf_{x \in \mathbb{R}^2} \bar{q}(x, \xi) - \inf_{x \in \mathbb{R}^2} q(x, \xi)}{\inf_{x \in \mathbb{R}^2} q(x, \xi)}$$

$$\geq 1 - \frac{c}{\liminf_{|\xi|\to 0} \frac{\inf_{x \in \mathbb{R}^2} q(x, \xi)}{|\xi|^2}}$$

$$> 0,$$

and

$$\limsup_{\xi \to 0} \frac{\inf_{x \in \mathbb{R}^2} \bar{q}(x, \xi)}{\inf_{x \in \mathbb{R}^2} q(x, \xi)} = 1 + \limsup_{\xi \to 0} \frac{\inf_{x \in \mathbb{R}^2} \bar{q}(x, \xi) - \inf_{x \in \mathbb{R}^2} q(x, \xi)}{\inf_{x \in \mathbb{R}^2} q(x, \xi)}$$

$$\leq 1 + \frac{c}{\liminf_{|\xi|\to 0} \frac{\inf_{x \in \mathbb{R}^2} q(x, \xi)}{|\xi|^2}}$$

$$\leq 2,$$

which proves the desired result.  \(\square\)
Note that every two-dimensional Lévy process with radial symbol automatically satisfies the relation in (3.3). A situation where the perturbation condition in (3.2) trivially holds true is given in the following proposition.

**Proposition 3.3.** Let \( \{F_t\}_{t \geq 0} \) and \( \{\tilde{F}_t\}_{t \geq 0} \) be two-dimensional Lévy-type processes with Lévy measures \( \nu(x, dy) \) and \( \tilde{\nu}(x, dy) \), respectively. If there exist a rotation of the plane \( R_\varphi, \varphi \in [0, 2\pi) \), and \( r > 0 \) such that \( \nu(x, B) = \tilde{\nu}(R_\varphi x, B) \) for all \( x \in \mathbb{R}^2 \) and all \( B \in B(\mathbb{R}^2) \), \( B \subseteq B_c^c(0) \), then the condition in (3.2) holds true.

Observe that Proposition 3.3 implies that the recurrence and transience properties of two-dimensional Lévy-type processes, satisfying the conditions from Theorem 3.2, depend only on big jumps, that is, they do not depend on the continuous part of the process and small jumps. In the following theorem we slightly generalize Proposition 3.3.

**Theorem 3.4.** Let \( \{F_t\}_{t \geq 0} \) and \( \{\tilde{F}_t\}_{t \geq 0} \) be two-dimensional Lévy-type processes with symbols \( q(x, \xi) \) and \( \tilde{q}(x, \xi) \) and Lévy triplets \( (0, c(x)I, \nu(x, dy)) \) and \( (0, \tilde{c}(x)I, \tilde{\nu}(x, dy)) \), respectively. Assume that there exist a rotation of the plane \( R_\varphi, \varphi \in [0, 2\pi) \), and compact set \( C \subseteq \mathbb{R}^2 \), such that \( \nu(x, B) \geq \tilde{\nu}(R_\varphi x, B) \) for all \( x \in \mathbb{R}^2 \) and all \( B \in B(\mathbb{R}^2) \), \( B \subseteq C^c \). Then,

(i) if \( q(x, \xi) \) satisfies (1.1), \( \tilde{q}(x, \xi) \) also satisfies (1.1).

(ii) if \( \tilde{q}(x, \xi) \) satisfies (1.2) and \( q(x, \xi) \) satisfies (3.3), \( q(x, \xi) \) also satisfies (1.2).

**Proof.** First, fix \( r > 0 \) large enough such that \( C \subseteq B_r(0) \). Then, by the same reasoning as in the proof of Theorem 3.2, we conclude

\[
\sup_{x \in \mathbb{R}^2} \tilde{q}(x, \xi) = \sup_{x \in \mathbb{R}^2} \tilde{q}(R_\varphi x, \xi) \leq \tilde{c} |\xi|^2 + \sup_{x \in \mathbb{R}^2} \int_{B_r(0)} (1 - \cos(\xi, y)) \nu(x, dy),
\]

where

\[
\tilde{c} = \frac{1}{2} \sup_{x \in \mathbb{R}^2} \tilde{c}(x) + \sup_{x \in \mathbb{R}^2} \int_{B_r(0)} |y|^2 \tilde{\nu}(x, dy).
\]

Finally, (3.4) implies

\[
\limsup_{|\xi| \to 0} \frac{\sup_{x \in \mathbb{R}^2} \tilde{q}(x, \xi)}{\sup_{x \in \mathbb{R}^2} q(x, \xi)} \leq \limsup_{|\xi| \to 0} \frac{\tilde{c} |\xi|^2 + \sup_{x \in \mathbb{R}^2} \int_{B_r(0)} (1 - \cos(\xi, y)) \nu(x, dy)}{\sup_{x \in \mathbb{R}^2} q(x, \xi)}
\]

\[
\leq 1 + \limsup_{|\xi| \to 0} \frac{\tilde{c} |\xi|^2}{\sup_{x \in \mathbb{R}^2} q(x, \xi)}
\]

\[
= 1,
\]

which together with the radiality of \( \xi \to q(x, \xi) \) proves the first assertion.

Now, we prove the second assertion. Again, fix \( r > 0 \) large enough such that \( C \subseteq B_r(0) \). By the same reasoning as above, we have

\[
\inf_{x \in \mathbb{R}^2} \tilde{q}(x, \xi) = \inf_{x \in \mathbb{R}^2} \tilde{q}(R_\varphi x, \xi) \leq \tilde{c} |\xi|^2 + \inf_{x \in \mathbb{R}^2} \int_{B_r(0)} (1 - \cos(\xi, y)) \nu(x, dy).
\]
Thus,

\[
\limsup_{|\xi| \to 0} \frac{\inf_{x \in \mathbb{R}^2} q(x, \xi)}{\inf_{x \in \mathbb{R}^2} q(x, \xi)} \leq \limsup_{|\xi| \to 0} \frac{\bar{c} |\xi|^2 + \inf_{x \in \mathbb{R}^2} \int_{B_1(0)} (1 - \cos(\xi, y)) \nu(x, dy)}{\inf_{x \in \mathbb{R}^2} q(x, \xi)} \\
\leq 1 + \limsup_{|\xi| \to 0} \frac{\bar{c} |\xi|^2}{\inf_{x \in \mathbb{R}^2} q(x, \xi)} \\
\leq 1 + \frac{\bar{c}}{c},
\]

where the constant \(c\) is defined in Theorem 3.2. \(\square\)

### 4 Conditions for Recurrence and Transience

In many cases the Chung-Fuchs type conditions are not practical, that is, it is not always easy to compute (appropriately estimate) the integrals appearing in (1.1) and (1.2). According to this, in the sequel we derive necessary and sufficient conditions for the recurrence and transience of two-dimensional Lévy-type processes in terms of the Lévy measures. Throughout this section we again assume that the symbol \(q(x, \xi)\) of a two-dimensional Lévy-type process \(\{F_t\}_{t \geq 0}\) is radial in the co-variable. We start this section with the following auxiliary result (see also [San14a] and [Sat99, Section 38] for the one-dimensional case).

**Proposition 4.1.** Let \(\{F_t\}_{t \geq 0}\) be a two-dimensional Lévy-type process with symbol \(q(x, \xi)\) and Lévy triplet \((0, c(x)I, \nu(x, dy))\). Define

\[
M(x, \rho, u) := \nu \left(x, \bigcup_{n=0}^{\infty} (2n\rho + u, 2(n+1)\rho - u] \times \mathbb{R} \cap B_1(0) \right), \quad x \in \mathbb{R}^2, \ \rho \geq 0, \ 0 \leq u \leq \rho.
\]

Then,

\[
\int_r^\infty \left( \rho \sup_{x \in \mathbb{R}^2} \int_0^\rho u M(x, \rho, u) du \right)^{-1} d\rho = \infty \quad \text{for some (all)} \ r > 0 \quad (4.1)
\]

if, and only if, (1.1) holds true. Further, if

\[
\liminf_{|\xi| \to 0} \inf_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1 - \cos(\xi, y)}{|\xi|^2} \nu(x, dy) = \infty, \quad (4.2)
\]

then (1.2) holds true if, and only if,

\[
\int_r^\infty \left( \rho \inf_{x \in \mathbb{R}^2} \int_0^\rho u M(x, \rho, u) du \right)^{-1} d\rho < \infty \quad \text{for some} \ r > 0. \quad (4.3)
\]

**Proof.** First, due to Theorem 3.2 and Proposition 3.3, without loss of generality, we can assume that \(\sup_{x \in \mathbb{R}^2} \nu(x, B_1(0)) = 0\). Next, note that, because of the radiality of the function \(\xi \mapsto q(x, \xi)\), the conditions in (1.1) and (1.2) are equivalent to

\[
\int_0^r \frac{\rho d\rho}{q(\rho)} = \infty \quad \text{for some (all)} \ r > 0 \quad (4.4)
\]

and

\[
\int_0^r \frac{\rho d\rho}{q(\rho)} < \infty \quad \text{for some} \ r > 0, \quad (4.5)
\]
respectively, where \( \overline{q}(\rho) := \sup_{x \in \mathbb{R}^2} q(x, \xi) \) and \( q(\rho) := \inf_{x \in \mathbb{R}^2} q(x, \xi) \) for \( \rho \in [0, \infty) \) and \( \xi \in \mathbb{R}^2 \), \( \rho = |\xi| \). Denote by \( N(x, u) := \nu(x, (u, \infty) \times \mathbb{R}) \) for \( x \in \mathbb{R}^2 \) and \( u \geq 0 \). Then, by following [San14a, Theorem 3.7], we have

\[
q(x, \xi) - \frac{1}{2} c(x)|\xi|^2 = \int_{\mathbb{R}^2} (1 - \cos(\xi, y))\nu(x, dy)
= \int_{\mathbb{R}^2} (1 - \cos(|\xi|e_i, y))\nu(x, dy)
= 2 \int_{(0, \infty) \times \mathbb{R}} (1 - \cos(|\xi|e_i, y))\nu(x, dy)
= 2 \int_{0}^{\infty} (1 - \cos |\xi|u)d(-N(x, u))
= 2|\xi| \int_{0}^{\infty} N(x, u) \sin |\xi|u du
= 2|\xi| \sum_{n=0}^{\infty} \int_{0}^{\frac{2\pi n}{|\xi|}} N \left( x, \frac{2\pi n}{|\xi|} + u \right) \sin |\xi|u du
= 2|\xi| \sum_{n=0}^{\infty} (I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}), \quad i = 1, 2,
\]

where in the second step we employed the fact that \( \nu(x, dy) \) is rotationally invariant, in the fifth step we used the integration by parts formula and

\[
I_{n,1} := \int_{0}^{\frac{2\pi n}{|\xi|}} N \left( x, \frac{2\pi n}{|\xi|} + u \right) \sin |\xi|u du,
I_{n,2} := \int_{\frac{2\pi n}{|\xi|}}^{\frac{2\pi n}{|\xi|} + \pi} N \left( x, \frac{2\pi n}{|\xi|} + u \right) \sin |\xi|u du = \int_{0}^{\frac{2\pi n}{|\xi|}} N \left( x, \frac{2\pi n + \pi}{|\xi|} - u \right) \sin |\xi|u du,
I_{n,3} := \int_{\frac{2\pi n}{|\xi|} + \pi}^{\frac{2\pi n}{|\xi|} + \pi + \frac{\pi}{|\xi|}} N \left( x, \frac{2\pi n}{|\xi|} + u \right) \sin |\xi|u du = -\int_{0}^{\frac{2\pi n}{|\xi|}} N \left( x, \frac{2\pi n + \pi}{|\xi|} + u \right) \sin |\xi|u du,
I_{n,4} := \int_{\frac{2\pi n}{|\xi|} + \pi + \frac{\pi}{|\xi|}}^{\frac{2\pi n}{|\xi|} + \pi + \frac{2\pi}{|\xi|}} N \left( x, \frac{2\pi n}{|\xi|} + u \right) \sin |\xi|u du = -\int_{0}^{\frac{2\pi n}{|\xi|}} N \left( x, \frac{2\pi n + \pi}{|\xi|} + \frac{2\pi}{|\xi|} - u \right) \sin |\xi|u du.
\]

Thus,

\[
I_{n,1} + I_{n,4} = \int_{\frac{2\pi n}{|\xi|}}^{\frac{2\pi n}{|\xi|} + \pi} \nu \left( x, \left( \frac{2\pi n + \pi}{|\xi|}, u, \frac{2\pi(n + 1)}{|\xi|} - u \right) \times \mathbb{R} \right) \sin |\xi|u du,
I_{n,2} + I_{n,3} = \int_{0}^{\frac{\pi}{|\xi|}} \nu \left( x, \left( \pi, \frac{\pi}{|\xi|} + u, \frac{\pi(n + 1)}{|\xi|} - u \right) \times \mathbb{R} \right) \sin |\xi|u du.
\]

Now, by defining

\[
\tilde{M}(x, \rho, u) := \nu \left( x, \bigcup_{n=0}^{\infty} ((2n + 1)\rho - u, (2n + 1)\rho + u] \times \mathbb{R} \right), \quad x \in \mathbb{R}^2, \quad \rho \geq 0, \quad 0 \leq u \leq \rho,
\]

\[
q(x, \xi) - \frac{1}{2} c(x)|\xi|^2 = 2|\xi| \left( \int_{0}^{\frac{\pi}{|\xi|}} \tilde{M} \left( x, \frac{\pi}{|\xi|}, u \right) \sin |\xi|u du + \int_{0}^{\frac{2\pi}{|\xi|}} \tilde{M} \left( x, \frac{2\pi}{|\xi|}, u \right) \sin |\xi|u du \right).
\]

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Further, note that
\[
M \left( x, \frac{\pi}{|\xi|}, u \right) \geq \bar{M} \left( x, \frac{\pi}{|\xi|}, u \right) \geq 0, \quad u \in \left( 0, \frac{\pi}{2|\xi|} \right],
\]
and
\[
\frac{2u}{\pi} \leq \sin u \leq u, \quad u \in \left( 0, \frac{\pi}{2} \right].
\]
Thus
\[
\frac{4}{\pi^2} |\xi| \int_0^\pi uM \left( x, \frac{\pi}{|\xi|}, u \right) du \leq \frac{q(x, \xi) - \frac{1}{2}c(x)|\xi|^2}{|\xi|} \leq 4 |\xi| \int_0^\pi uM \left( x, \frac{\pi}{|\xi|}, u \right) du.
\] (4.6)
Next, we have
\[
\int_0^\pi \left( \rho \sup_{x \in \mathbb{R}^2} \int_0^\pi uM \left( x, \frac{\pi}{\rho}, u \right) du \right)^{-1} d\rho = \int_r^\infty \left( \rho \sup_{x \in \mathbb{R}^2} \int_0^\rho uM \left( x, \rho, u \right) du \right)^{-1} d\rho
\]
and
\[
\int_0^\pi \left( \rho \inf_{x \in \mathbb{R}^2} \int_0^\pi uM \left( x, \frac{\pi}{\rho}, u \right) du \right)^{-1} d\rho = \int_r^\infty \left( \rho \inf_{x \in \mathbb{R}^2} \int_0^\rho uM \left( x, \rho, u \right) du \right)^{-1} d\rho,
\]
where we use the change of variables \( \rho \mapsto \pi/\rho \). Thus, (4.1) implies
\[
\int_0^\pi \frac{\rho \ d\rho}{\sup_{x \in \mathbb{R}^2} \left( q(x, (\rho, 0)) - \frac{1}{2}c(x)\rho^2 \right)} = \infty
\]
and
\[
\int_0^\pi \frac{\rho \ d\rho}{\inf_{x \in \mathbb{R}^2} \left( q(x, (\rho, 0)) - \frac{1}{2}c(x)\rho^2 \right)} < \infty
\]
implies (4.3). Finally,
\[
1 \leq \liminf_{\rho \to 0} \frac{\sqrt{q(\rho)}}{\sup_{x \in \mathbb{R}^2} \left( q(x, (\rho, 0)) - \frac{1}{2}c(x)\rho^2 \right)}
\]
\[
\leq \limsup_{\rho \to 0} \frac{\sqrt{q(\rho)}}{\sup_{x \in \mathbb{R}^2} \left( q(x, (\rho, 0)) - \frac{1}{2}c(x)\rho^2 \right)}
\]
\[
\leq \limsup_{\rho \to 0} \frac{\frac{1}{2}\rho^2 \sup_{x \in \mathbb{R}^2} c(x) + \sup_{x \in \mathbb{R}^2} \left( q(x, (\rho, 0)) - \frac{1}{2}c(x)\rho^2 \right)}{\sup_{x \in \mathbb{R}^2} \left( q(x, (\rho, 0)) - \frac{1}{2}c(x)\rho^2 \right)}
\]
\[
\leq 1 + \frac{1}{2} \sup_{x \in \mathbb{R}^2} c(x)
\]
\[
\leq 1 + \frac{1}{2} \sup_{x \in \mathbb{R}^2} \frac{c(x)}{\int_{\mathbb{R}^2} \frac{1 - \cos((\rho, 0), y)}{\rho^2} \nu(x, dy)}
\]
\[
\leq 1 + 2 \sup_{x \in \mathbb{R}^2} \frac{c(x)}{\int_{\mathbb{R}^2} \nu(x, dy)}
\] (4.7)
where in the final step we employed Fatou’s lemma and the fact that \( \nu(x, dy) \) is rotationally invariant. Analogously,
\[
1 \leq \liminf_{\rho \to 0} \frac{q(\rho)}{(q(x, (\rho, 0)) - \frac{1}{2}c(x)\rho^2)}
\]
\[
\leq \limsup_{\rho \to 0} \frac{q(\rho)}{\inf_{x \in \mathbb{R}^2} \left( q(x, (\rho, 0)) - \frac{1}{2}c(x)\rho^2 \right)}
\]
\[
\leq 1 + \frac{1}{2} \sup_{x \in \mathbb{R}^2} c(x)
\]
\[
\leq 1 + \frac{1}{2} \liminf_{\rho \to 0} \inf_{x \in \mathbb{R}^2} \frac{1 - \cos((\rho, 0), y)}{\rho^2} \nu(x, dy)
\] (4.8)
Now, the assertion follows from (4.2).

To prove the converse, first note that

\[
\int_0^{\infty} u \nu \left( x, \frac{2n \pi}{|\xi|} + u, \frac{2(n+1)\pi}{|\xi|} - u \right) \times \mathbb{R} \, du
\]

\[
= \int_0^{\infty} u \nu \left( x, \frac{2n \pi}{|\xi|} + u, \frac{2(n+1)\pi}{|\xi|} - u \right) \times \mathbb{R} \, du
\]

\[
+ \int_0^{\infty} u \nu \left( x, \frac{2n \pi}{|\xi|} + u, \frac{2(n+1)\pi}{|\xi|} - u \right) \times \mathbb{R} \, du
\]

\[
= \int_0^{\infty} u \nu \left( x, \frac{2n \pi}{|\xi|} + u, \frac{2(n+1)\pi}{|\xi|} - u \right) \times \mathbb{R} \, du
\]

\[
+ 4 \int_0^{\infty} u \nu \left( x, \frac{2n \pi}{|\xi|} + u, \frac{2(n+1)\pi}{|\xi|} - 2u \right) \times \mathbb{R} \, du
\]

\[
\leq \int_0^{\infty} u \nu \left( x, \frac{2n \pi}{|\xi|} + u, \frac{2(n+1)\pi}{|\xi|} - u \right) \times \mathbb{R} \, du
\]

\[
+ 4 \int_0^{\infty} u \nu \left( x, \frac{2n \pi}{|\xi|} + u, \frac{2(n+1)\pi}{|\xi|} - u \right) \times \mathbb{R} \, du
\]

\[
\leq 5 \int_0^{\infty} u \nu \left( x, \frac{2n \pi}{|\xi|} + u, \frac{2(n+1)\pi}{|\xi|} - u \right) \times \mathbb{R} \, du.
\]

Hence,

\[
\int_0^{\infty} u M \left( x, \frac{\pi}{|\xi|}, u \right) \, du \leq 5 \int_0^{\infty} u M \left( x, \frac{\pi}{|\xi|}, u \right) \, du,
\]

that is,

\[
\int_r^{\infty} \left( \rho \sup_{x \in \mathbb{R}^2} \int_0^{\rho} u M \left( x, \rho, u \right) \, du \right)^{-1} \, d\rho = \int_0^{\frac{\pi}{\rho}} \left( \rho \sup_{x \in \mathbb{R}^2} \int_0^{\rho} u M \left( x, \frac{\pi}{\rho}, u \right) \, du \right)^{-1} \, d\rho
\]

\[
\geq \frac{1}{5} \int_0^{\frac{\pi}{\rho}} \left( \rho \sup_{x \in \mathbb{R}^2} \int_0^{\rho} u M \left( x, \frac{\pi}{\rho}, u \right) \, du \right)^{-1} \, d\rho
\]

and

\[
\int_r^{\infty} \left( \rho \inf_{x \in \mathbb{R}^2} \int_0^{\rho} u M \left( x, \rho, u \right) \, du \right)^{-1} \, d\rho = \int_0^{\frac{\pi}{\rho}} \left( \rho \inf_{x \in \mathbb{R}^2} \int_0^{\rho} u M \left( x, \frac{\pi}{\rho}, u \right) \, du \right)^{-1} \, d\rho
\]

\[
\geq \frac{1}{5} \int_0^{\frac{\pi}{\rho}} \left( \rho \inf_{x \in \mathbb{R}^2} \int_0^{\rho} u M \left( x, \frac{\pi}{\rho}, u \right) \, du \right)^{-1} \, d\rho,
\]

where in the first steps we again used the change of variables \( \rho \to \pi / \rho \). Thus, according to (4.2), (4.6), (4.7) and (4.8), (4.4) and (4.3) imply (4.1) and (4.5), respectively.

Observe that in the Lévy process case the condition in (4.2) is trivially satisfied. As a direct consequence of Proposition 4.1, we get the following characterization of the recurrence and transience in terms of the tail behavior of the Lévy measures.
Theorem 4.2. Let \( \{F_1\}_{t \geq 0} \) be a two-dimensional Lévy-type process with symbol \( q(x, \xi) \) and Lévy measure \( \nu(x, dy) \). Define

\[
N(x, u) := \nu(x, (u, \infty) \times \mathbb{R}), \quad x \in \mathbb{R}^2, \ u \geq 0.
\]

Then,

\[
\int_r^\infty \left( \rho \sup_{x \in \mathbb{R}^2} \int_0^\rho uN(x, u)du \right)^{-1} d\rho = \infty \quad \text{for some } r > 0 \tag{4.9}
\]

implies (4.1), and (4.3) implies

\[
\int_r^\infty \left( \rho \inf_{x \in \mathbb{R}^2} \int_0^\rho uN(x, u)du \right)^{-1} d\rho < \infty \quad \text{for some } r > 0. \tag{4.10}
\]

Proof. The assertion follows directly from the fact \( N(x, u) \geq M(x, \rho, u) \) for all \( x \in \mathbb{R}^2 \), all \( \rho \geq 0 \) and all \( 0 \leq u \leq \rho \). \( \square \)

In general, we cannot conclude the equivalence in Theorem 4.2 (see [Sat99, Theorem 38.4]). However, if, in addition, we assume the quasi-unimodality of the measure \( \nu(x, du \times \mathbb{R}) \), \( x \in \mathbb{R}^2 \), then (4.1) will be equivalent to (4.9) and (4.3) will be equivalent to (4.10). Recall that a symmetric Borel measure \( \mu(dx) \) on \( \mathcal{B}(\mathbb{R}) \) is quasi-unimodal if there exists \( x_0 \geq 0 \) such that \( x \mapsto \mu(x, \infty) \) is a convex function on \( (x_0, \infty) \). Equivalently, a symmetric Borel measure \( \mu(dx) \) on \( \mathcal{B}(\mathbb{R}) \) is quasi-unimodal if it is of the form \( \mu(dx) = \mu_0(dx) + f(x) dx \), where the measure \( \mu_0(dx) \) is supported on \( [-x_0, 0) \), for some \( x_0 \geq 0 \), and the density function \( f(x) \) is supported on \( [0, x_0] \), it is symmetric and decreasing on \( (x_0, \infty) \) and \( \int_{x_0}^\infty f(x) dx < \infty \) for every \( \varepsilon > 0 \) (see [Sat99, Chapters 5 and 7]). When \( x_0 = 0 \), then \( \mu(dx) \) is said to be unimodal.

Theorem 4.3. Let \( \{F_1\}_{t \geq 0} \) be a two-dimensional Lévy-type process with symbol \( q(x, \xi) \) and Lévy triplet \( (0, c(x)I, \nu(x, dy)) \). Assume that there exists \( u_0 > 1 \) such that the measure \( \nu(x, du \times \mathbb{R}) \) is quasi-unimodal with respect to \( u_0 \) for all \( x \in \mathbb{R}^2 \). Then, (1.1) holds true if, and only if, (4.9) holds true. Further, if, in addition, \( \nu(x, dy) \) satisfies (4.2), then (1.2) holds true if, and only if, (4.10) holds true.

Proof. According to Theorem 4.2, we only have to prove that (1.1) implies (4.9) and that (4.10) implies (1.2). First, we prove that (1.1) implies (4.9). Due to the radiality of the function \( \xi \mapsto q(x, \xi) \), (1.1) is equivalent to

\[
\int_0^r \frac{\rho d\rho}{\sup_{x \in \mathbb{R}^2} q(x, (\rho, 0))} = \infty \quad \text{for some (all) } r > 0. \tag{4.11}
\]

Further, we have

\[
1 \leq \liminf_{\rho \to 0} \frac{\sup_{x \in \mathbb{R}^2} q(x, (\rho, 0))}{\sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} (1 - \cos((\rho, 0), y)) \nu(x, dy)}
\leq \limsup_{\rho \to 0} \frac{\sup_{x \in \mathbb{R}^2} q(x, (\rho, 0))}{\sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} (1 - \cos((\rho, 0), y)) \nu(x, dy)}
\leq 1 + \limsup_{\rho \to 0} \frac{\sup_{x \in \mathbb{R}^2} c(x)}{\sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1 - \cos((\rho, 0), y)}{\rho^2} \nu(x, dy)}
\leq 1 + 2 \frac{\sup_{x \in \mathbb{R}^2} c(x)}{\sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} |y|^2 \nu(x, dy)},
\]
which, together with [San14a, Proposition 2.4], yields that (4.11) is equivalent to
\[
\int_0^r \sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} (1 - \cos((\rho, 0), y)) \nu(x, dy) = \infty \quad \text{for some (all) } r > 0.
\]
(4.12)

Now, define \( \bar{\nu}(x, dy) := \nu(x, dy \cap B_{u_0}^c(0)) \), \( x \in \mathbb{R}^2 \). Obviously, \( \bar{\nu}(x, dy) \) is rotationally invariant, \( \sup_{x \in \mathbb{R}^2} \bar{\nu}(x, \mathbb{R}^2) < \infty \) and
\[
\sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} |y|^2 |\nu(x, dy) - \bar{\nu}(x, dy)| \leq \sup_{x \in \mathbb{R}^2} \int_{B_{u_0}(0)} |y|^2 \nu(x, dy) < \infty.
\]
Thus, by analogues arguments as in the proof of Theorem 3.2, it is easy to see that (4.12) is equivalent to
\[
\int_0^r \sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}} (1 - \cos((\rho, 0), y)) \bar{\nu}(x, dy) = \infty \quad \text{for some (all) } r > 0,
\]
that is,
\[
\int_0^r \frac{\rho \, d\rho}{\sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}} (1 - \cos \rho u) \bar{\nu}(x, du \times \mathbb{R})} = \infty \quad \text{for some (all) } r > 0.
\]
(4.13)

Next, due to [San14a, the proof of Theorem 3.9], for every \( x \in \mathbb{R}^2 \) there exists an unimodal probability measure \( \eta_U(x, du) \) on \( \mathcal{B}(\mathbb{R}) \) such that \( \bar{\nu}(x, (u, \infty) \times \mathbb{R}) = c\eta_U(x, (u, \infty)) \) for all \( x \in \mathbb{R}^2 \) and all \( u \geq u_0 + 1 \), where \( c := c(u_0) \) is an appropriate norming constant. Note that
\[
\sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}} u^2 |\bar{\nu}(x, du \times \mathbb{R}) - c\eta_U(x, du)| < \infty.
\]
According to this, [San14a, Theorem 3.1] implies that (4.13) is equivalent to
\[
\int_0^r \frac{\rho \, d\rho}{\sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}} (1 - \cos \rho u) \eta_U(x, du)} = \infty \quad \text{for some (all) } r > 0.
\]
(4.14)

In the sequel we prove that (4.14) implies (4.9). First, since \( \eta_U(x, du) \) is unimodal, by [Sat99, Exercise 29.21], there exists a random variable \( X_x \) such that \( \eta_U(x, du) \) is the distribution of the random variable \( U_x X_x \), where \( U_x \) is uniformly distributed random variable on [0, 1] independent of \( X_x \). Further, let \( \eta(x, du) \) be the distribution of the random variable \( X_x. \) By [Sat99, Lemma 38.6], \( \eta(x, (u, \infty)) \geq \eta_U(x, (u, \infty)) \) for all \( x \in \mathbb{R}^2 \) and all \( u \geq 0 \). Now, we have
\[
\int_{\mathbb{R}} (1 - \cos \rho u) \eta_U(x, du) = \int_0^1 \int_{\mathbb{R}} (1 - \cos \rho uv) \eta(x, dv) = \int_{\mathbb{R}} \left( 1 - \frac{\sin \rho u}{\rho u} \right) \eta(x, dv).
\]
Next, since
\[
1 - \frac{\sin \rho u}{\rho u} \geq \bar{c} \min\{1, u^2\}
\]
for all \( u \in \mathbb{R} \) and all \( 0 < \bar{c} < \frac{1}{6} \),
\[
\int_{\mathbb{R}} (1 - \cos \rho u) \eta_U(x, du) \geq \bar{c} \int_{\mathbb{R}} \min\{1, \rho^2 u^2\} \eta(x, dv) = 4\bar{c}^2 \int_0^\infty u E(x, u) du,
\]

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where \( E(x, u) := \eta(x, (u, \infty)) \) for \( x \in \mathbb{R}^2 \) and \( u \geq 0 \). Set \( E_U(x, u) := \eta_U(x, (u, \infty)) \) for \( x \in \mathbb{R}^2 \) and \( u \geq 0 \). Then, for any \( r > 0 \), we have

\[
\int_r^\infty \left( \rho \sup_{x \in \mathbb{R}^2} \int_0^\rho u E_U(x, u) du \right)^{-1} d\rho \geq \int_r^\infty \left( \rho \sup_{x \in \mathbb{R}^2} \int_0^\rho u E(x, u) du \right)^{-1} d\rho \\
= \int_0^\frac{1}{r} \left( \rho \sup_{x \in \mathbb{R}^2} \int_0^\rho u E(x, u) du \right)^{-1} d\rho \\
\geq 4\epsilon \int_0^\frac{1}{r} \frac{\rho d\rho}{\sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}} (1 - \cos \rho u) \eta_U(x, du)}. \tag{4.15}
\]

Further,

\[
\lim_{\rho \to \infty} \frac{\sup_{x \in \mathbb{R}^2} \int_{u_0}^\rho u N(x, u) du}{\sup_{x \in \mathbb{R}^2} \int_{u_0}^\rho u E_U(x, u) du + \frac{1}{\epsilon} \sup_{x \in \mathbb{R}^2} \int_{u_0}^\rho u N(x, u) du}
\leq \lim_{\rho \to \infty} \frac{\sup_{x \in \mathbb{R}^2} \int_{u_0}^\rho u N(x, u) du}{\sup_{x \in \mathbb{R}^2} \int_{u_0}^\rho u E_U(x, u) du}
\leq \lim_{\rho \to \infty} \frac{\sup_{x \in \mathbb{R}^2} \int_{u_0}^\rho u N(x, u) du}{\sup_{x \in \mathbb{R}^2} \int_{u_0}^\rho u E_U(x, u) du + \sup_{x \in \mathbb{R}^2} \int_{u_0}^\rho u N(x, u) du}
\leq \lim_{\rho \to \infty} \frac{\sup_{x \in \mathbb{R}^2} \int_{u_0}^\rho u N(x, u) du}{\int_{\mathbb{R}} (1 - \cos \rho u) \eta(x, du)}.
\]

Now, if \( \sup_{x \in \mathbb{R}^2} \int_{u_0}^\rho u N(x, u) du = 0 \) the desired result trivially follows. On the other hand, if \( \sup_{x \in \mathbb{R}^2} \int_{u_0}^\rho u N(x, u) du > 0 \), we have

\[
0 < \lim_{\rho \to \infty} \frac{\sup_{x \in \mathbb{R}^2} \int_{u_0}^\rho u N(x, u) du}{\sup_{x \in \mathbb{R}^2} \int_{u_0}^\rho u E_U(x, u) du} \leq \lim_{\rho \to \infty} \frac{\sup_{x \in \mathbb{R}^2} \int_{u_0}^\rho u N(x, u) du}{\sup_{x \in \mathbb{R}^2} \int_{u_0}^\rho u E_U(x, u) du} < \infty, \tag{4.16}
\]

which together with (4.15) proves the assertion.

Finally, we prove that (4.10) implies (1.2). Due to the radiality of \( \xi \mapsto q(x, \xi) \) and (4.2), by completely the same arguments as above,

\[
4\epsilon \int_0^{\frac{1}{r}} \frac{\rho d\rho}{\inf_{x \in \mathbb{R}^2} \int_{\mathbb{R}} (1 - \cos \rho u) \eta_U(x, du)} \leq \int_r^\infty \left( \rho \sup_{x \in \mathbb{R}^2} \int_0^\rho u E_U(x, u) du \right)^{-1} d\rho.
\]

Now, the desired result follows by a similar argumentation as in (4.16) and employing (4.2). \( \square \)

Note that the measure \( \nu(x, du \times \mathbb{R}) \) will be quasi-unimodal uniformly in \( x \in \mathbb{R}^2 \) if there exists \( u_0 \geq 0 \) such that \( \nu(x, dy) = n(x, |y|) dy \) on \( B(B^c_0(0)) \) for some Borel function \( n : \mathbb{R}^2 \times (0, \infty) \to (0, \infty) \) which is decreasing on \((u_0, \infty)\) for all \( x \in \mathbb{R}^2 \). Also, let us remark that in the Lévy process case the condition in (4.2) will be satisfied if, and only if, \( \int_{\mathbb{R}^2} |y|^2 \nu(dy) = \infty \). Recall that \( \int_{\mathbb{R}^2} |y|^2 \nu(dy) < \infty \) implies that the underlying Lévy process is recurrent (see [San14a, Theorem 2.8]).

Finally, as a direct consequence of Theorem 4.3, we get the following characterization of the recurrence and transience in terms of the tail behavior of the Lévy measures.

**Theorem 4.4.** Let \( \{F_t\}_{t \geq 0} \) be a two-dimensional Lévy-type process with Lévy measure \( \nu(x, dy) \), satisfying the assumptions from Theorem 4.3. Then, (1.1) holds true if, and only if,

\[
\int_r^\infty \left( \rho \sup_{x \in \mathbb{R}^2} \int_0^\rho u \nu(x, B^c_0(0)) du \right)^{-1} d\rho = \infty \quad \text{for some (all) } r > 0, \tag{4.17}
\]
and (1.2) holds true if, and only if,
\[ \int_r^\infty \left( \rho \inf_{x \in \mathbb{R}^2} \int_0^\rho u \nu(x, B_u^c(0))du \right)^{-1} d\rho < \infty \quad \text{for some } r > 0. \quad (4.18) \]

**Proof.** The assertion is a direct consequence of the following simple fact
\[ \frac{1}{4} \nu(x, B_{1/2}^c(0)) \leq N(x, u) \leq \nu(x, B_u^c(0)), \quad x \in \mathbb{R}^2, \ u > 0. \]

Proposition 4.5. Let \( \{F_t\}_{t \geq 0} \) be a two-dimensional Lévy-type process with Lévy measure \( \nu(x, dy) \).
Then, (4.17) holds if
\[ \int_r^\infty \left( \rho^3 \sup_{x \in \mathbb{R}^2} \nu(x, B_\rho^c(0)) + \rho \sup_{x \in \mathbb{R}^2} \int_{B_\rho(0)} |y|^2 \nu(x, dy) \right)^{-1} d\rho = \infty \quad \text{for some } r > 0, \]
and (4.18) holds if
\[ \int_r^\infty \left( \rho^3 \inf_{x \in \mathbb{R}^2} \nu(x, B_\rho^c(0)) + \rho \inf_{x \in \mathbb{R}^2} \int_{B_\rho(0)} |y|^2 \nu(x, dy) \right)^{-1} d\rho < \infty \quad \text{for some } r > 0. \]

In particular, (4.18) holds if either one of the following conditions holds
\[ \int_r^\infty \left( \rho^3 \inf_{x \in \mathbb{R}^2} \nu(x, B_\rho^c(0)) \right)^{-1} d\rho < \infty \quad \text{for some } r > 0 \]
or
\[ \int_r^\infty \left( \rho \inf_{x \in \mathbb{R}^2} \int_{B_\rho(0)} |y|^2 \nu(x, dy) \right)^{-1} d\rho < \infty \quad \text{for some } r > 0. \]

In addition, if \( \nu(x, dy) \) is of the form \( \nu(x, dy) = n(x, |y|)dy \), where \( n : \mathbb{R}^2 \times (0, \infty) \rightarrow (0, \infty) \) is a Borel function, and there exists \( u_0 \geq 0 \) such that \( n(x, u) \) is decreasing on \((u_0, \infty)\) for all \( x \in \mathbb{R}^2 \), then (4.18) holds if
\[ \int_r^\infty \frac{du}{u^3 \inf_{x \in \mathbb{R}^2} n(x, u)} < \infty \quad \text{for some } r \geq u_0. \]

**Proof.** By employing the integration by parts formula, for any \( x \in \mathbb{R}^2 \), any \( \rho > 0 \) and any \( 0 < \varepsilon < \rho \), we have
\[ \frac{\rho^2}{2} \nu(x, B_{\rho}^c(0)) + \frac{1}{2} \int_{B_{\rho}(0)} |y|^2 \nu(x, dy) \]
\[ \geq \int_0^\rho u \nu(x, B_u^c(0))du \]
\[ = \int_0^\rho u \nu(x, B_u^c(0))du + \int_0^\varepsilon u \nu(x, B_u^c(0))du \]
\[ \geq \frac{\varepsilon^2}{2} \nu(x, B_{\varepsilon}^c(0)) + \frac{\rho^2}{2} \nu(x, B_{\rho}^c(0)) - \frac{\varepsilon^2}{2} \nu(x, B_{\varepsilon}^c(0)) + \frac{1}{2} \int_{B_{\rho}(0) \cap B_{\varepsilon}(0)} |y|^2 \nu(x, dy) \]
\[ = \frac{\rho^2}{2} \nu(x, B_{\rho}^c(0)) + \frac{1}{2} \int_{B_{\rho}(0) \cap B_{\varepsilon}(0)} |y|^2 \nu(x, dy). \]
Now, by letting $\varepsilon \to 0$, Fatou’s lemma yields
\[
\int_0^\rho u \nu(x, B_u^c(0))du = \frac{\rho^2}{2} \nu(x, B_\rho^c(0)) + \frac{1}{2} \int_{B_\rho(0)} |y|^2 \nu(x, dy).
\]

In order to prove the last assertion, observe that for any $r > u_0$,
\[
\int_r^\infty \left( \rho \inf_{x\in\mathbb{R}^2} \int_{B_\rho(0)} |y|^2 \nu(x, dy) \right)^{-1} d\rho \leq \frac{1}{2\pi} \int_r^\infty \left( \rho \inf_{x\in\mathbb{R}^2} \int_{u_0}^\rho u^3 n(x, u)du \right)^{-1} d\rho \\
\leq \frac{2}{\pi} \int_r^\infty \frac{d\rho}{\rho(\rho^4 - u_0^4) \inf_{x\in\mathbb{R}^2} n(x, \rho)} \\
\leq c \int_r^\infty \frac{\rho d\rho}{\rho^5 \inf_{x\in\mathbb{R}^2} n(x, \rho)},
\]
where in the second step we employed the fact that $n(x, u)$ is decreasing on $(u_0, \infty)$ and $c > \frac{2\rho^4}{\pi(r^2 - u_0^2)}$ is arbitrary.

Observe that from the previous proposition we again conclude that if $\sup_{x\in\mathbb{R}^2} \int_{\mathbb{R}^2} |y|^2 \nu(x, dy) < \infty$, then $\nu(x, dy)$ automatically satisfies (4.17). Let us now give some applications of the results presented above.

**Example 4.6.** Let $\alpha : \mathbb{R}^2 \to (0, 2)$ and $\beta : \mathbb{R}^2 \to (0, \infty)$ be arbitrary bounded and continuously differentiable functions with bounded derivatives, such that $0 < \inf_{x\in\mathbb{R}^2} \alpha(x) \leq \sup_{x\in\mathbb{R}^2} \alpha(x) < 2$ and $\inf_{x\in\mathbb{R}^2} \beta(x) > 0$. Under this assumptions, in [Bas88], [Kol00, Theorem 5.1] and [SW13, Theorem 3.3] it has been shown that there exists a unique open-set irreducible Lévy-type process $\{F_t\}_{t \geq 0}$, called a (two-dimensional) stable-like process, determined by a Lévy triplet and symbol of the form $(0, 0, \beta(x)|y|^{-2-\alpha(x)}dy)$ and $q(x, \xi) = \gamma(x)|\xi|^{\alpha(x)}$, respectively, where
\[
\gamma(x) := \beta(x) \frac{\pi^{1/2} \Gamma(1 - \alpha(x)/2)}{\alpha(x) \Gamma((\alpha(x) + 1)/2)}, \quad x \in \mathbb{R}^2.
\]
Here, $\Gamma(x)$ denotes the Gamma function. Note that when $\alpha(x)$ and $\beta(x)$ are constant functions, then we deal with a rotationally invariant two-dimensional stable Lévy process. Now, by a direct application of the Chung-Fuchs type condition in (1.2) we easily see that $\{F_t\}_{t \geq 0}$ is transient. On the other hand, the corresponding Lévy measure also satisfies all the assumptions from Theorem 4.4, which again implies the transience of $\{F_t\}_{t \geq 0}$. For more on stable-like processes and their recurrence and transience properties we refer the readers to [Bas88], [Böt11], [Fra06, Fra07], [San13], [San14a] and [SW13].

**Example 4.7.** Let $\alpha, \beta : \mathbb{R}^2 \to (0, \infty)$ and $\gamma : \mathbb{R}^2 \to \mathbb{R}$ be arbitrary bounded and continuous functions such that $\inf_{x\in\mathbb{R}^2} \alpha(x) > 0$ and $\inf_{x\in\mathbb{R}^2} \beta(x) > 0$. Define $n : \mathbb{R}^2 \times (0, \infty) \to (0, \infty)$ by
\[
n(x, u) := \frac{\beta(x) \ln \gamma(x) u}{u^{2+\alpha(x)}} 1_{\{v : v \geq \gamma\}}(u).
\]
Because of the continuity of $\alpha(x)$, $\beta(x)$ and $\gamma(x)$, without loss of generality, we can assume that $\int_{\mathbb{R}^2} n(x, |y|)dy = 1$ for all $x \in \mathbb{R}^2$. Now, by (a straightforward adaptation of) [San14b, Proposition 2.9], there exists a unique open-set irreducible Lévy-type process $\{F_t\}_{t \geq 0}$ determined by a Lévy measure and symbol of the form $\nu(x, dy) := n(x, |y|)dy$ and $q(x, \xi) := \int_{\mathbb{R}^2} (1 - \cos(\xi, y))\nu(x, dy)$, respectively. Put $\alpha := \inf_{x\in\mathbb{R}^2} \alpha(x) \leq \sup_{x\in\mathbb{R}^2} \alpha(x) =: \overline{\alpha}$, $\beta := \inf_{x\in\mathbb{R}^2} \beta(x) \leq \sup_{x\in\mathbb{R}^2} \beta(x) =: \overline{\beta}$ and $\gamma := \inf_{x\in\mathbb{R}^2} \gamma(x) \leq \sup_{x\in\mathbb{R}^2} \gamma(x) =: \overline{\gamma}$. Then,
(i) if $\alpha > 2$ (which automatically implies that $\sup_{x \in \mathbb{R}^2} \int_{\mathbb{R}^2} |y|^2 \nu(x, dy) < \infty$), a direct application of the Chung-Fuchs type condition in (1.1) (or Proposition 4.5) entails the recurrence of $\{F_t\}_{t \geq 0}$.

(ii) if $\overline{\alpha} < 2$, since
\[
n(x, u) \geq \frac{\beta}{u^{2 + \alpha + \varepsilon}} 1_{\{v : v \geq \varepsilon\}}(u)
\]
for some $0 < \varepsilon < 2 - \overline{\alpha}$, all $x \in \mathbb{R}^2$ and all $u > 0$ large enough, Theorem 3.2 and Example 4.6 (or Proposition 4.5) imply that $\{F_t\}_{t \geq 0}$ is transient.

On the other hand, in order to conclude the recurrence or transience of $\{F_t\}_{t \geq 0}$ in the cases when $\overline{\alpha} = 2$ or $\overline{\alpha} = 2$, it is not immediately clear how to (explicitly) compute or (appropriately) bound its symbol and apply the Chung-Fuchs type conditions. However, since $\nu(x, dy)$ obviously satisfies all the assumptions of Theorem 4.4, we conclude that

(iii) if $\alpha \geq 2$ and $\gamma \leq 0$, then
\[
\int_r^\infty \left( \rho \sup_{x \in \mathbb{R}^2} \int_0^\rho u \nu(x, B_{\rho}^c(0)) \, du \right)^{-1} \frac{d\rho}{\rho \ln \rho}, \quad r \geq e,
\]
which entails the recurrence of $\{F_t\}_{t \geq 0}$.

(iv) if $\overline{\alpha} \leq 2$ and $\gamma > 0$, then
\[
\int_r^\infty \left( \rho \inf_{x \in \mathbb{R}^2} \int_0^\rho u \nu(x, B_{\rho}^c(0)) \, du \right)^{-1} \frac{d\rho}{\rho \ln \rho + r}, \quad r \geq e,
\]
which implies that $\{F_t\}_{t \geq 0}$ is transient.

**Example 4.8.** Let $\{L_t\}_{t \geq 0}$ be a Lévy process with Lévy measure of the form $\nu(dy) = n(|y|)dy$, where $n : (0, \infty) \rightarrow (0, \infty)$ is a decreasing (on $[u_0, \infty)$ for some $u_0 \geq 0$) and regularly varying function with index $\delta \leq -2$ (that is, $\lim_{u \rightarrow \infty} n(\lambda u)/n(u) = \lambda^\delta$ for all $\lambda > 0$). Observe that, due to [BGT87, Theorem 1.5.11], for any $-4 \leq \delta \leq -2$,
\[
\lim_{\rho \rightarrow \infty} \frac{\nu(B_{\rho}^c(0))}{\rho^2 n(\rho)} = \frac{1}{2\pi(-2 - \delta)} \quad \text{and} \quad \lim_{\rho \rightarrow \infty} \frac{\int_{B_{\rho}^c(0)} |y|^2 \nu(dy)}{\rho^4 n(\rho)} = \frac{1}{2\pi(4 + \delta)}.
\]
Consequently, Proposition 4.5 and [BGT87, Proposition 1.3.6] yield that

(i) if $\delta < -4$, then $\{L_t\}_{t \geq 0}$ is recurrent.

(ii) if $-4 < \delta \leq -2$, then $\{L_t\}_{t \geq 0}$ is transient.

(iii) if $\delta = -4$, then $\{L_t\}_{t \geq 0}$ is transient if
\[
\int_r^\infty \frac{d\rho}{\rho^3 n(\rho)} < \infty \quad \text{for some } r > 0.
\]
5 Comparison of Lévy and Lévy-Type Processes

In this section, we provide some comparison conditions for the recurrence and transience in terms of the Lévy measures. Again, we assume that the symbol \( q(x, \xi) \) of a two-dimensional Lévy-type process \( \{ F_t \}_{t \geq 0} \) is radial in the co-variable.

**Theorem 5.1.** Let \( \{ F_t \}_{t \geq 0} \) and \( \{ \bar{F}_t \}_{t \geq 0} \) be two-dimensional Lévy-type processes with symbols \( q(x, \xi) \) and \( \bar{q}(x, \xi) \) and Lévy measures \( \nu(x, dy) \) and \( \nu(x, dy) \), respectively. Assume that

1. \( \nu(x, du \times \sigma) \) is quasi-unimodal uniformly in \( x \in \mathbb{R}^2 \);
2. there exists \( u_0 \geq 0 \) such that \( \nu(x, B_u^c(0)) \geq \bar{\nu}(x, B_u^c(0)) \) (or \( \nu(x, (u, \infty) \times \sigma) \geq \bar{\nu}(x, (u, \infty) \times \sigma) \)) for all \( x \in \mathbb{R}^2 \) and all \( u \geq u_0 \).

Then,
\[
\int_{B_r(0)} \frac{d\xi}{\sup_{\xi \in \mathbb{R}^2} q(x, \xi)} = \infty \quad \text{for all} \ r > 0
\]
implies
\[
\int_{B_r(0)} \frac{d\xi}{\sup_{\xi \in \mathbb{R}^2} \bar{q}(x, \xi)} = \infty \quad \text{for all} \ r > 0.
\]

In addition, if \( q(x, \xi) \) satisfies (4.2), then
\[
\int_{B_r(0)} \frac{d\xi}{\inf_{\xi \in \mathbb{R}^2} q(x, \xi)} < \infty \quad \text{for some} \ r > 0
\]
implies
\[
\int_{B_r(0)} \frac{d\xi}{\inf_{\xi \in \mathbb{R}^2} \bar{q}(x, \xi)} < \infty \quad \text{for some} \ r > 0.
\]

**Proof.** The assertion of the theorem is a direct consequence of Theorems 4.3 and 4.4. \( \square \)

**Corollary 5.2.** Let \( \{ F_t \}_{t \geq 0} \) be a two-dimensional Lévy-type process with symbol \( q(x, \xi) \) and Lévy measure \( \nu(x, dy) \). Assume that there exists \( x_0 \in \mathbb{R}^2 \) such that

1. \( \sup_{\xi \in \mathbb{R}^2} q(x, \xi) = q(x_0, \xi) \) for all \( |\xi| \) small enough;
2. there exists a two-dimensional Lévy process \( \{ L_t \}_{t \geq 0} \) with symbol \( q(\xi) \) and Lévy measure \( \nu(dy) \), such that \( q(\xi) \) is radial, \( \nu(du \times \sigma) \) is quasi-unimodal and \( \nu(x, B_u^c(0)) \geq \nu(x, B_u^c(0)) \) (or \( \nu((u, \infty) \times \sigma) \geq \nu((u, \infty) \times \sigma) \)) for all \( u \geq u_0 \), for some \( u_0 \geq 0 \).

Then, the recurrence property of \( \{ L_t \}_{t \geq 0} \) implies (1.1). Further, if there exists \( x_0 \in \mathbb{R}^2 \) such that

1. \( \inf_{\xi \in \mathbb{R}^2} q(x, \xi) = q(x_0, \xi) \) for all \( |\xi| \) small enough;
2. the measure \( \nu(x_0, du \times \sigma) \) is quasi-unimodal and \( \int_{\mathbb{R}^2} |y|^2 \nu(x_0, dy) = \infty \);
3. there exists a two-dimensional Lévy process \( \{ L_t \}_{t \geq 0} \) with symbol \( q(\xi) \) and Lévy measure \( \nu(dy) \), such that \( q(\xi) \) is radial and \( \nu(x_0, B_u^c(0)) \geq \nu(x, B_u^c(0)) \) (or \( \nu((u, \infty) \times \sigma) \geq \nu((u, \infty) \times \sigma) \)) for all \( u \geq u_0 \), for some \( u_0 \geq 0 \),

then the transience property of \( \{ L_t \}_{t \geq 0} \) implies (1.2).

**Proof.** The claims of the corollary follow directly from [Sat99, Corollary 37.6] and Theorem 5.1. \( \square \)
Examples of Lévy-type processes which satisfy the conditions in Corollary 5.2 can be found in the class of Feller processes obtained by variable order subordination. More precisely, let $q(\xi)$ be the symbol of some $d$-dimensional symmetric Lévy process (that is, $q(\xi) = q(-\xi)$ for all $\xi \in \mathbb{R}^d$), and let $f : \mathbb{R}^d \times [0, \infty) \rightarrow [0, \infty)$ be a measurable function such that $\sup_{x \in \mathbb{R}^d} f(x, t) \leq c(1 + t)$ for some $c \geq 0$ and all $t \in [0, \infty)$ and for fixed $x \in \mathbb{R}^d$ the function $t \mapsto f(x, t)$ is a Bernstein function with $f(x, 0) = 0$. Bernstein functions are the characteristic Laplace exponents of subordinators (Lévy processes with nondecreasing sample paths). For more on Bernstein functions we refer the readers to the monograph [SSV12]. Now, since $q(\xi) \geq 0$ for all $\xi \in \mathbb{R}^d$, the function $\bar{q}(x, \xi) := f(x, q(\xi))$, $x, \xi \in \mathbb{R}^d$, is well defined and, according to [SSV12, Theorem 5.2] and [Sat99, Theorem 30.1], $\xi \mapsto \bar{q}(x, \xi)$ is a continuous and negative definite function satisfying conditions (C2) and (C3). Hence, $\bar{q}(x, \xi)$ is possibly the symbol of some Lévy-type process. Of special interest is the case when $\bar{q}(x, \xi)$ describes variable order subordination. For sufficient conditions on the symbol $q(\xi)$ and function $\alpha(x)$ such that $\bar{q}(x, \xi)$ is the symbol of some Lévy-type process see [EJ07] and [Hoh00] and the references therein. Now, assume that $\alpha(x) = \inf_{x \in \mathbb{R}^d} \alpha(x) \leq \sup_{x \in \mathbb{R}^d} \alpha(x) = \alpha(\pi)$, for some $\pi, \underline{x} \in \mathbb{R}^d$. Then, since the symbol $q(\xi)$ is continuous and $q(0) = 0$, there exists $r > 0$ small enough such that $q(\xi) \leq 1$ for all $|\xi| < r$. In particular, $q^\alpha(\pi)(\xi) = \inf_{x \in \mathbb{R}^d} q^\alpha(x)(\xi) = \inf_{x \in \mathbb{R}^d} \bar{q}(x, \xi) \leq \sup_{x \in \mathbb{R}^d} \bar{q}(x, \xi) = \sup_{x \in \mathbb{R}^d} q^\alpha(x)(\xi) = q^\alpha(\underline{x})(\xi), \quad |\xi| < r.$

Let us remark that when $q(\xi)$ is the symbol of a standard Brownian motion, then by variable order subordination we get a stable-like process.

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