

\textbf{T-ADIC EXPONENTIAL SUMS OF POLYNOMIALS IN ONE VARIABLE}

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\textbf{Abstract.} The $T$-adic exponential sum of a polynomial in one variable is studied. An explicit arithmetic polygon in terms of the highest two exponents of the polynomial is proved to be a lower bound of the Newton polygon of the $C$-function of the $T$-adic exponential sum. This bound gives lower bounds for the Newton polygon of the $L$-function of exponential sums of $p$-power order.

\section{Introduction}

Let $p$ be a prime number, $q$ a power of $p$, and $\mathbb{F}_q$ the finite field with $q$ elements. Let $W$ be the Witt ring scheme, $\mathbb{Z}_q = W(\mathbb{F}_q)$, and $\mathbb{Q}_q = \mathbb{Z}_q[[\frac{1}{p}]]$.

Let $\triangle \supseteq \{0\}$ be an integral convex polytope in $\mathbb{R}^n$, and $I$ the set of vertices of $\triangle$ different from the origin. Let

$$f(x) = \sum_{u \in \triangle} (a_u x^u, 0, 0, \ldots) \in W(\mathbb{F}_q[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]) \text{ with } \prod_{u \in I} a_u \neq 0,$$

where $x^u = x_1^{u_1} x_2^{u_2} \cdots x_n^{u_n}$ if $u = (u_1, u_2, \ldots, u_n) \in \mathbb{Z}^n$.

\textbf{Definition 1.1.} For any positive integer $l$, the $T$-adic exponential sum associated to $f$ is the sum:

$$S_f(l, T) = \sum_{x \in (\mathbb{F}_q^*)^n} (1 + T)^{Tr_{\mathbb{Q}_q/\mathbb{Q}_p}(f(x))} \in \mathbb{Z}_p[[T]].$$

The $T$-adic $L$-function $L_f(s, T)$ associated to $f$ is defined by the formula

$$L_f(s, T) = \exp(\sum_{l=1}^{\infty} S_f(l, T) \frac{s^l}{l}) \in 1 + s\mathbb{Z}_p[[T]][[s]].$$

Let $m \geq 1$, $\zeta_p^m$ a primitive $p^m$-th root of unity, and $\pi_m = \zeta_p^m - 1$. Then the specialization $L_f(s, \pi_m)$ is the $L$-function of $p$-power order exponential sums $S_f(l, \pi_m)$. These $p$-power order exponential sums were studied by Adolphson-Sperber \cite{AS} for $m = 1$, and by Liu-Wei \cite{LW} for $m \geq 1$.

We view $L_f(s, T)$ as a power series in the single variable $s$ with coefficients in the $T$-adic complete field $\mathbb{Q}_p((T))$. So we call it a $T$-adic $L$-function.

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**Definition 1.2.** The $T$-adic $C$-function $C_f(s, T)$ associated to $f$ is the generating function

$$C_f(s, T) = \exp(\sum_{l=1}^{\infty} -(q^l - 1)^{-n} S_f(l, T) s^{q^l}).$$

We have

$$L_f(s, T) = \prod_{i=0}^{n} C_f(q^i s, T)^{(-1)^{n+1-i}(i)},$$

and

$$C_f(s, T)^{(-1)^{n-1}} = \prod_{j=0}^{\infty} L_f(q^j s, T)^{\binom{n+j-1}{j}}.$$ 

So the $C$-function $C_f(s, T)$ and the $L$-function $L_f(s, T)$ determine each other. From the last identity, one sees that

$$C_f(s, T) \in 1 + s \mathbb{Z}_p[[T]][[s]].$$

We also view $C_f(s, T)$ as a power series in the single variable $s$ with coefficients in the $T$-adic complete field $\mathbb{Q}_p((T))$. The $C$-function $C_f(s, T)$ was shown to be $T$-adic entire by Liu-Wan [LWn].

Let $C(\Delta)$ be the cone generated by $\Delta$, and $M(\Delta) = C(\Delta) \cap \mathbb{Z}^n$. There is a degree function $\deg$ on $C(\Delta)$ which is $\mathbb{R}_{\geq 0}$-linear and takes the values 1 on every co-dimension 1 face not containing 0. For example, if $\Delta \subset \mathbb{Z}$, then the degree function on $C(\Delta)$ is defined by the formula

$$\deg(a) = \begin{cases} 0, & a = 0, \\ \frac{a}{d(\text{sgn}(a))}, & a \neq 0, \end{cases}$$

where $d(\varepsilon)$ is the nonzero endpoint of $\Delta$ with sign $\varepsilon = \pm 1$. For $a \notin C(\Delta)$, we define $\deg(a) = +\infty$.

**Definition 1.3.** A convex function on $\mathbb{R}_{\geq 0}$ which is linear between consecutive integers with initial value 0 is called the Hodge polygon of $\Delta$ if its slopes between consecutive integers are the numbers $\deg(a)$, $a \in M(\Delta)$. We denote it by $H_\infty^\Delta$.

Liu-Wan [LWn] proved the following.

**Theorem 1.4.** We have

$$T - \text{adic NP of } C_f(s, T) \geq \text{ord}_p(q)(p - 1)H_\infty^\Delta,$$

where NP is the short for Newton polygon.

From now on we assume that $\Delta = [0, d]$.

**Definition 1.5.** For $a \in \mathbb{N}$, we define

$$\delta_\varepsilon(a) = \begin{cases} 1, & pi = a(d) \text{ for some } i < d(\frac{a}{d}), \\ 0, & \text{otherwise}, \end{cases}$$

where $\{\cdot\}$ is the fractional part of a real number.
Definition 1.6. A convex function on $\mathbb{R}_{\geq 0}$ which is linear between consecutive integers with initial value 0 is called the arithmetic polygon of $\triangle = [0, d]$ if its slopes between consecutive integers are the numbers

$$\varpi_\triangle(a) = [(p-1)\deg(a)] - \delta_\triangle(a), a \in \mathbb{N},$$

where $[\cdot]$ is the least integer equal or greater than a real number. We denote it by $p_\triangle$.

Liu-Liu-Niu [LLN] proved the following.

Theorem 1.7. If $p > 3d$, then

$$T - \text{adic NP of } C_f(s, T) \geq \ord_p(q)p_\triangle,$$

with equality holding for a generic $f$ of degree $d$.

By a result of Li [Li], $L_f(s, \pi_m)$ is a polynomial with degree $p^{m-1}d$ if $p \nmid d$. Combined this result with the above theorem, one can infer the following.

Theorem 1.8. If $p > 3d$, then

$$\pi_m - \text{adic NP of } L_f(s, \pi_m) \geq \ord_p(q)p_\triangle \text{ on } [0, p^{m-1}d],$$

with equality holding for a generic $f$ of degree $d$.

The Newton polygon of the $L$-function $L_f(s, \pi_m)$ for $m = 1$ was studied by Zhu [Zhu, Zhu2] and Blache-Férard [BF].

We assume that the second highest exponent of $f$ is $k$. So $k \leq d-1$, and

$$f(x) = (a_dx^d, 0, 0, \cdots) + \sum_{i=1}^{k}(a_ix^i, 0, 0, \cdots) \in W(\mathbb{F}_q[x]) \text{ with } a_da_k \neq 0.$$

For $a \in \mathbb{N}$, define

$$\varpi_{d,[0,k]}(a) = [\frac{pa}{d}] - [\frac{a}{d}] + [\frac{ra}{k}] - [\frac{ra}{k}] + \sum_{i=1}^{r_a-1}(1_{\{\frac{ra}{k}\}>(\frac{ra}{k})} - 1_{\{\frac{ra}{k}\}>(\frac{ra}{k})}),$$

where $r_a = d\{\frac{a}{d}\}$ for $a \in \mathbb{N}$.

Definition 1.9. A convex function on $\mathbb{R}_{\geq 0}$ which is linear between consecutive integers with initial value 0 is called the arithmetic polygon of $\{d\} \cup [0, k]$ if its slopes between consecutive integers are the numbers $\varpi_{d,[0,k]}(a), a \in \mathbb{N}$. We denote it by $p_{d,[0,k]}$.

We can prove the following.

Theorem 1.10. We have $p_{d,[0,k]} \geq p_\triangle$.

The main result of this paper is the following.
**Theorem 1.11.** If \( p > d(2d + 1) \), then
\[
T - \text{adic NP of } C_f(s, T) \geq \text{ord}_p(q)p_{d,[0,k]}.
\]

**Corollary 1.12.** If \( p > d(2d + 1) \), then
\[
\pi_m - \text{adic NP of } L_f(s, \pi_m) \geq \text{ord}_p(q)p_{d,[0,k]} \text{ on } [0, p^{m-1}d].
\]

We put forward the following conjecture.

**Conjecture 1.13.** Let \( k < d \) be fixed positive integers. If \( p \) is sufficiently large, then for a generic \( f \) of the form
\[
f(x) = (a_dx^d, 0, 0, \cdots) + \sum_{i=1}^{k} (a_ix^i, 0, 0, \cdots) \in W(F_q[x]) \text{ with } a_da_k \neq 0,
\]
we have
\[
T - \text{adic NP of } C_f(s, T) = \text{ord}_p(q)p_{d,[0,k]}.
\]

The above conjecture implies the following.

**Conjecture 1.14.** Let \( k < d \) be fixed positive integers. Then for a generic \( f \) of the form
\[
f(x) = (a_dx^d, 0, 0, \cdots) + \sum_{i=1}^{k} (a_ix^i, 0, 0, \cdots) \in W(F_q[x]) \text{ with } a_da_k \neq 0,
\]
the \( T^{p-1} - \text{adic Newton polygon of } C_f(s, T) \) uniformly converges to \( \text{ord}_p(q)H_{\infty,[0,d]} \) as \( p \) goes to infinity.

The above conjecture again implies the following.

**Conjecture 1.15.** Let \( k < d \) be fixed positive integers. Then for a generic \( f \) of the form
\[
f(x) = (a_dx^d, 0, 0, \cdots) + \sum_{i=1}^{k} (a_ix^i, 0, 0, \cdots) \in W(F_q[x]) \text{ with } a_da_k \neq 0,
\]
the \( \pi_m^{p-1} - \text{adic Newton polygon of } C_f(s, \pi_m) \) converges to \( \text{ord}_p(q)H_{\infty,[0,d]} \) as \( p \) goes to infinity.

The last two conjectures are analogues of a conjecture put forward by Wan [Wa, Wa2].

2. **The \( T \)-adic Dwork Theory**

In this section we review the \( T \)-adic analogue of Dwork theory on exponential sums.

Let
\[
E(t) = \exp\left(\sum_{i=0}^{\infty} \frac{tp^i}{p^i} \right) = \sum_{i=0}^{+\infty} \lambda_i t^i \in 1 + t\mathbb{Z}_p[[t]]
\]
be the $p$-adic Artin-Hasse exponential series. Define a new $T$-adic uniformizer $\pi$ of $\mathbb{Q}_p((T))$ by the formula $E(\pi) = 1 + T$. Let $\pi^{1/d}$ be a fixed $d$-th root of $\pi$. Let $a \mapsto \hat{a}$ be the Teichmüller lifting. One can show that the series

$$E_f(x) := E(\pi \hat{a}_d x^d) \prod_{i=1}^{k} E(\pi \hat{a}_i x^i)$$

lies in the $T$-adic Banach module

$$L = \{ \sum_{i \in \mathbb{N}} c_i \pi^{\deg(i)} x^i : c_i \in \mathbb{Z}_q[[\pi^{1/d}]] \}.$$

Note that the Galois group of $\mathbb{Q}_q$ over $\mathbb{Q}_p$ can act on $L$ but keeping $\pi^{1/d}$ as well as the variable $x$ fixed. Let $\sigma$ be the Frobenius element in the Galois group such that $\sigma(\zeta) = \zeta^p$ if $\zeta$ is a $(q - 1)$-th root of unity. Let $\Psi_p$ be the operator on $L$ defined by the formula

$$\Psi_p(\sum_{i \in \mathbb{N}} c_i x^i) = \sum_{i \in \mathbb{N}} c_{pi} x^i.$$

Then $\Psi := \sigma^{-1} \circ \Psi_p \circ E_f$ acts on the $T$-adic Banach module

$$B = \{ \sum_{i \in \mathbb{N}} c_i \pi^{\deg(i)} x^i \in L, \ \text{ord}_T(c_i) \to +\infty \text{ if } \deg(i) \to +\infty \}.$$

We call it Dwork’s $T$-adic semi-linear operator because it is semi-linear over $\mathbb{Z}_q[[\pi]]$. Let $b = \text{ord}_p(q)$, the $b$-iterate $\Psi^b$ is linear over $\mathbb{Z}_q[[\pi^{1/d}]]$, since

$$\Psi^b = \Psi_p^b \circ \prod_{i=0}^{b-1} E_f^{\sigma^i}(x^{p^i}).$$

One can show that $\Psi$ is completely continuous in the sense of Serre [Se]. So $\det(1 - \Psi^b s | \mathbb{Z}_q[[\pi^{1/d}]])$ and $\det(1 - \Psi s | \mathbb{Z}_p[[\pi^{1/d}]]))$ are well-defined.

We now state the $T$-adic Dwork trace formula.

**Theorem 2.1** (T-adic Dwork trace formula [LWn]). We have

$$C_f(s, T) = \det(1 - \Psi^b s | \mathbb{Z}_q[[\pi^{1/d}]])$$

### 3. Key estimate

In order to study

$$C_f(s, T) = \det(1 - \Psi^b s | \mathbb{Z}_q[[\pi^{1/d}]])$$

we first study

$$\det(1 - \Psi s | \mathbb{Z}_p[[\pi^{1/d}]] = \sum_{i=0}^{-\infty} (-1)^i c_i s^i.$$
Theorem 3.1. If \( p > d(2d + 1) \), then we have
\[
\text{ord}_T(c_{bm}) \geq bp_d[0,k](m).
\]

Fix a normal basis \( \xi_1, \ldots, \xi_b \) of \( \mathbb{F}_q \) over \( \mathbb{F}_p \). Let \( \xi_1, \ldots, \xi_b \) be their Teichmüller lifts. Then \( \xi_1, \ldots, \xi_b \) is a normal basis of \( \mathbb{Q}_q \) over \( \mathbb{Q}_p \), and \( \sigma \) acts on \( \xi_1, \ldots, \xi_b \) as a permutation. Let \( (\gamma(i,u),(j,\omega))_{i,j \in \mathbb{N}, 1 \leq u, \omega \leq b} \) be the matrix of \( \Psi \) on \( B \otimes_{\mathbb{Z}_p} \mathbb{Q}_p(\pi^{1/d}) \) with respect to the basis \( \{\xi_u x^i\} \). Then
\[
c_{bm} = \sum_R \det((\gamma(i,u),(j,\omega))(i,u),(j,\omega) \in R),
\]
where \( R \) runs over all subsets of \( \mathbb{N} \times \{1, 2, \ldots, b\} \) with cardinality \( bm \).

So Theorem 3.1 is reduced to the following.

Theorem 3.2. Let \( R \subset \mathbb{N} \times \{1, 2, \ldots, b\} \) be a subset of cardinality \( bm \). If \( p > d(2d + 1) \), then
\[
\text{ord}_T(\det((\gamma(i,u),(j,\omega))(i,u),(j,\omega) \in R)) \geq bp_d[0,k](m).
\]

Let \( O(\pi^\alpha) \) denote any element of \( \pi \)-adic order \( \geq \alpha \).

Lemma 3.3. We have
\[
\gamma(i,u),(j,\omega) = O(\pi^{[\frac{p_i-1}{d}]+[\frac{r_{pi-1}}{k}]}).
\]

Proof. Write
\[
E_f(x) = \sum_{i \in \mathbb{N}} \gamma_i x^i.
\]

Then
\[
\gamma_i = \sum_{d \alpha_d + \sum_{j=1}^{k} j n_j = i} \pi^{\sum_{j=1}^{k} n_j + n_d} \prod_{j=1}^{k} \lambda_{n_j} \hat{a}_j^{n_j} \lambda_{n_d} \hat{a}_d^{n_d} = O(\pi^{[\frac{1}{s}]+[\frac{r}{k}]}).
\]

Since
\[
(\xi_u \gamma_{pi-1})^{\sigma^{-1}} = \sum_{u=1}^{b} \gamma(i,u),(j,\omega) \xi_u,
\]
we have
\[
\text{ord}_T(\gamma(i,u),(j,\omega)) = \text{ord}_T(\gamma_{pi-1}).
\]
The lemma now follows. \( \square \)

By the above lemma, Theorem 3.2 is reduced to the following.

Theorem 3.4. Let \( R \subset \mathbb{N} \times \{1, 2, \ldots, b\} \) be a subset of cardinality \( bm \), and \( \tau \) a permutation of \( R \). Suppose that \( p > d(2d + 1) \). Then
\[
\sum_{(i,u) \in R} ([\frac{p_i-\tau(i)}{d}] + [\frac{r_{pi-\tau(i)}}{k}]) \geq bp_d[0,k](m),
\]
where \( \tau(i) \) is defined by \( \tau(i,u) = (\tau(i), \tau(u)) \).
Proof. By definition, we have

\[ p_{d,[0,k]}(m) = \sum_{a=1}^{m-1} \mathbf{w}_{d,[0,k]}(a) \]

\[ = \sum_{a=1}^{m-1} \left( \left\lfloor \frac{pa}{d} \right\rfloor - \left\lfloor \frac{a}{d} \right\rfloor + \left\lfloor \frac{ra}{k} \right\rfloor + 1_{\{ \frac{ra}{k} \}} > \left\{ \frac{r_{m-1}}{k} \right\} - 1_{\{ \frac{ra}{k} \}} > \left\{ \frac{r_{m-1}}{k} \right\} \right). \]

Note that

\[ \sum_{(i,u) \in R} \left( \left\lfloor \frac{pi}{d} \right\rfloor - \frac{i}{d} \right) + \left\lfloor \frac{r_{pi}}{k} \right\rfloor - \frac{r_{pi}}{k} \]

\[ = \sum_{(i,u) \in R} \left( \left\lfloor \frac{pi}{d} \right\rfloor - \frac{i}{d} \right) + \frac{r_{pi}}{k} - \frac{r_{pi}}{k} + \left\lfloor \frac{d-k}{k} \right\rfloor 1_{r_{pi} > r_{pi}} \]

\[ = \sum_{(i,u) \in R} \left( \left\lfloor \frac{pi}{d} \right\rfloor - \frac{i}{d} \right) + \frac{r_{pi}}{k} - \frac{r_{pi}}{k} + \left\lfloor \frac{d-k}{k} \right\rfloor 1_{r_{pi} > r_{pi}} + \left\{ \frac{r_{pi}}{k} \right\} - \frac{r_{pi} - \tau(i)}{k} \right). \]

We have

\[ \sum_{(i,u) \in R} \left( \left\lfloor \frac{pi}{d} \right\rfloor - \frac{i}{d} \right) + \frac{r_{pi}}{k} - \frac{r_{pi} - \tau(i)}{k} \]

\[ \geq \sum_{(i,u) \in R} 1_{\left\{ \frac{r_{pi}}{k} \right\} \leq \left\{ \frac{r_{m-1}}{k} \right\} < \left\{ \frac{r_{pi}}{k} \right\}} \]

\[ \geq \sum_{(i,u) \in R} 1_{\left\{ \frac{r_{pi}}{k} \right\} \leq \left\{ \frac{r_{m-1}}{k} \right\} } - \sum_{(i,u) \in R} 1_{\left\{ \frac{r_{pi}}{k} \right\} > \left\{ \frac{r_{m-1}}{k} \right\} }. \]

We also have

\[ \sum_{(i,u) \in R} \left( \left\lfloor \frac{pi}{d} \right\rfloor - \frac{i}{d} \right) + \frac{r_{pi}}{k} - \frac{r_{pi}}{k} \]

\[ = b \sum_{i=1}^{m-1} \left( \left\lfloor \frac{pi}{d} \right\rfloor - \frac{i}{d} \right) + \sum_{(i,u) \in R} \left( \left\lfloor \frac{pi}{d} \right\rfloor - \frac{i}{d} \right) - \sum_{i \geq m} \left( \left\lfloor \frac{pi}{d} \right\rfloor - \frac{i}{d} \right) \]

\[ - \sum_{(i,u) \in R} \left( \left\lfloor \frac{pi}{d} \right\rfloor - \frac{i}{d} \right) + \frac{r_{pi}}{k} - \frac{r_{pi}}{k} \]
\[
\geq b \sum_{i=1}^{m-1} \left( \left\lceil \frac{p_i}{d} \right\rceil - \frac{i}{d} + \left\lceil \frac{r_{pi}}{k} \right\rceil - \frac{r_{i}}{k} \right) + N \left( \left\lceil \frac{pm}{d} \right\rceil - \frac{m}{d} - \left\lfloor \frac{d-1}{k} \right\rfloor \right) \\
- N \left( \left\lceil \frac{p(m-1)}{d} \right\rceil - \frac{m-1}{d} + \left\lfloor \frac{d-1}{k} \right\rfloor \right) \\
\geq b \sum_{i=1}^{m-1} \left( \left\lceil \frac{p_i}{d} \right\rceil - \frac{i}{d} + \left\lceil \frac{r_{pi}}{k} \right\rceil - \frac{r_{i}}{k} \right) + N \left( \left\lceil \frac{p}{d} \right\rceil - 1 - 2(d-1) \right).
\]

where \( N = \# \{(i, u) \in R \mid i \geq m \} = \# \{(i, u) \notin R \mid 0 \leq i < m \} \).

Similarly, we have
\[
\sum_{(i, u) \in R} 1_{ \left( \frac{r_{pi}}{k} \right) > \left( \frac{r_{m-1}}{k} \right) } \geq b \sum_{i=1}^{m-1} 1_{ \left( \frac{r_{pi}}{k} \right) > \left( \frac{r_{m-1}}{k} \right) } - N,
\]
and
\[
\sum_{(i, u) \in R} 1_{ \left( \frac{r_{pi}}{k} \right) > \left( \frac{r_{m-1}}{k} \right) } \leq b \sum_{i=1}^{m-1} 1_{ \left( \frac{r_{pi}}{k} \right) > \left( \frac{r_{m-1}}{k} \right) } + N.
\]

Therefore
\[
\sum_{(i, u) \in R} \left( \left\lceil \frac{p_i - \tau(i)}{d} \right\rceil + \left\lceil \frac{r_{pi} - \tau(i)}{k} \right\rceil \right)
\geq b \sum_{i=1}^{m-1} \left( \left\lceil \frac{p_i}{d} \right\rceil - \frac{i}{d} + \left\lceil \frac{r_{pi}}{k} \right\rceil - \frac{r_{i}}{k} \right) + b \sum_{i=1}^{m-1} 1_{ \left( \frac{r_{pi}}{k} \right) > \left( \frac{r_{m-1}}{k} \right) } - b \sum_{i=1}^{m-1} 1_{ \left( \frac{r_{pi}}{k} \right) > \left( \frac{r_{m-1}}{k} \right) } \\
+ N \left( \left\lceil \frac{p}{d} \right\rceil - 1 - 2(d-1) - 2 \right)
= bp_{d, [0, k]}(m) + N \left( \left\lceil \frac{p}{d} \right\rceil - 2d + 1 \right) \geq bp_{d, [0, k]}(m).
\]

4. Proof of the main result

In this section we prove Theorem \ref{main result}, which says that, if \( p > d(2d+1) \), then
\[
T - \text{adic NP of } C_f(s, T) \geq \text{ord}_p(q)p_{d, [0, k]}.
\]

Lemma 4.1. The Newton polygon of \( \det(1 - \Psi^{b} s^{b} \mid B/\mathbb{Z}_q[[\pi^{\frac{1}{d}}]]) \) coincides with that of \( \det(1 - \Psi s \mid B/\mathbb{Z}_p[[\pi^{\frac{1}{d}}]]) \).

Proof. Note that
\[
\det(1 - \Psi^{b} s \mid B/\mathbb{Z}_p[[\pi^{\frac{1}{d}}]]) = \text{Norm}(\det(1 - \Psi^{b} s \mid B/\mathbb{Z}_q[[\pi^{\frac{1}{d}}]])).
\]
where \( \text{Norm} \) is the norm map from \( \mathbb{Z}_q[[\pi^{\frac{1}{d}}]] \) to \( \mathbb{Z}_p[[\pi^{\frac{1}{d}}]] \). The lemma now follows from the equality
\[
\prod_{\zeta^{b} = 1} \det(1 - \Psi \zeta s \mid B/\mathbb{Z}_p[[\pi^{\frac{1}{d}}]]) = \det(1 - \Psi^{b} s^{b} \mid B/\mathbb{Z}_p[[\pi^{\frac{1}{d}}]])
\]
Corollary 4.2. The $T$-adic Newton polygon of $\det(1 - \Psi^b s \mid B/\mathbb{Z}_q[[\pi^T]])$ is the lower convex closure of the points

$$(m, \text{ord}_T(c_{bm})), \ m = 0, 1, \cdots.$$ 

Proof. By Lemma 4.1, the $T$-adic Newton polygon of $\det(1 - \Psi^b s \mid B/\mathbb{Z}_q[[\pi^T]])$ is the lower convex closure of the points

$$(i, \text{ord}_T(c_i)), \ i = 0, 1, \cdots.$$ 

It is clear that $(i, \text{ord}_T(c_i))$ is not a vertex of that polygon if $b \nmid i$. So that Newton polygon is the lower convex closure of the points

$$(bm, \text{ord}_T(c_{bm})), \ m = 0, 1, \cdots.$$ 

It follows that the $T$-adic Newton polygon of $\det(1 - \Psi^b s \mid B/\mathbb{Z}_q[[\pi^T]])$ is the lower convex closure of the points

$$(m, \text{ord}_T(c_{bm})), \ m = 0, 1, \cdots.$$ 

We now prove Theorem 1.11.

Proof of Theorem 1.11. By Theorem 2.1, we have

$$C_f(s, T) = \det(1 - \Psi^b s \mid B/\mathbb{Z}_q[[\pi^T]])..$$

Then by Corollary 4.2, the $T$-adic Newton polygon of $C_f(s, T)$ is the lower convex closure of the points

$$(m, \text{ord}_T(c_{bm})), \ m = 0, 1, \cdots.$$ 

Therefore the result follows from Theorem 3.1, which says that, if $p > d(2d + 1)$, then we have

$$\text{ord}_\pi(c_{bm}) \geq bp_{d, [0, k]}(m).$$

We conclude this section by proving Corollary 4.12.

Proof of Corollary 4.12. Assume that $L_f(s, \pi_m) = \prod_{i=1}^{p^{m-1}d} (1 - \beta_i s)$. Then

$$C_f(s, \pi_m) = \prod_{j=0}^{\infty} L_f(q^j s, \pi_m) = \prod_{j=0}^{\infty} \prod_{i=1}^{p^{m-1}d} (1 - \beta_i q^j s).$$

Therefore the slopes of the $q$-adic Newton polygon of $C_f(s, \pi_m)$ are the numbers

$$j + \text{ord}_q(\beta_i), \ 1 \leq i \leq p^{m-1}d, j = 0, 1, \cdots.$$ 

It is well-known that $\text{ord}_q(\beta_i) \leq 1$ for all $i$. Therefore,

$q$-adic NP of $L_f(s, \pi_m) = q$-adic NP of $C_f(s, \pi_m)$ on $[0, p^{m-1}d]$. 

It follows that
\[ \pi_m - \text{adic NP of } L_f(s, \pi_m) = \pi_m - \text{adic NP of } C_f(s, \pi_m) \text{ on } [0, p^{m-1}d]. \]

By the integrality of \( C_f(s, T) \) and Theorem 1.11, we have
\[ \pi_m - \text{adic NP of } C_f(s, \pi_m) \geq T - \text{adic NP of } C_f(s, T) \geq \text{ord}_p(q)p_{d, [0, k]}. \]

Therefore,
\[ \pi_m - \text{adic NP of } L_f(s, \pi_m) \geq \text{ord}_p(q)p_{d, [0, k]} \text{ on } [0, p^{m-1}d]. \] □

5. Comparison between arithmetic polygons

In this section we prove Theorem 1.10, which says that
\[ p_{d, [0, k]} \geq p_\Delta. \]

Proof of Theorem 1.10 It is clear that \( p_{d, [0, k]}(0) = p_\Delta(0) \). It suffices to show that, for \( m \in \mathbb{N} \), we have \( p_{d, [0, k]}(m + 1) \geq p_\Delta(m + 1) \). By a result in Liu-Liu-Niu [LLN], we have
\[ p_\Delta(m + 1) = \sum_{a=1}^{m} \varpi_\Delta(a) = \sum_{a=1}^{m} \left( \left\lfloor \frac{pa}{d} \right\rfloor - \left\lfloor \frac{a}{d} \right\rfloor \right) + \sum_{a=1}^{r_m} 1_{r_{pa} > r_m}. \]

By definition, we have
\[ p_{d, [0, k]}(m + 1) = \sum_{a=1}^{m} \varpi_{d, [0, k]}(a) \]
\[ = \sum_{a=1}^{m} \left( \left\lfloor \frac{pa}{d} \right\rfloor - \left\lfloor \frac{a}{d} \right\rfloor \right) + \sum_{a=1}^{r_m} \left( \left\lfloor \frac{r_{pa}}{k} \right\rfloor - \left\lfloor \frac{a}{k} \right\rfloor + 1_{r_{pa} > r_m} \right) \left( \frac{r_{pa}}{k} - \frac{a}{k} \right) - 1_{r_{pa} > \frac{r_m}{k}}. \]

For \( m \geq 0 \),
\[ \sum_{a=1}^{m} \left( \left\lfloor \frac{pa}{d} \right\rfloor - \left\lfloor \frac{a}{d} \right\rfloor \right) = \sum_{a=1}^{m} \left( \left\lfloor \frac{pa}{d} \right\rfloor - \left\lfloor \frac{a}{d} \right\rfloor \right). \]

Let \( A_1 = \{ 1 \leq a \leq r_m | a \neq r_{pi} \text{ for some } 1 \leq i \leq r_m \} \). Note that
\[ \{ a : 1 \leq a \leq r_m \} = A_1 \cup A_2, \]
where \( A_2 = \{ r_{pa} | 1 \leq a, r_{pa} \leq r_m \} \). And
\[ \{ r_{pa} | 1 \leq a \leq r_m \} = A_2 \cup A_3, \]
where \( A_3 = \{ r_{pa} > r_m | 1 \leq a \leq r_m \} \), so we have \( |A_1| = |A_3| \). Then
The theorem now follows. 

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