Square-root higher-order Weyl semimetals

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The mathematical foundation of quantum mechanics is built on linear algebra, while the application of nonlinear operators can lead to outstanding discoveries under some circumstances. In this Letter, we propose a model of square-root higher-order Weyl semimetal (SHOWS) by inheriting features from its parent Hamiltonians. It is found that the SHOWS hosts both “Fermi-arc” surface and hinge states that connect the projection of the Weyl points. We theoretically construct and experimentally observe the exotic SHOWS state in three-dimensional (3D) stacked electric circuits with honeycomb-kagome hybridizations and double-helix interlayer couplings. Our results open the door for realizing the square-root topology in 3D solid-state platforms.

Nearly all the operators encountered in quantum mechanics are linear (or antilinear) operators, such as the rotation, translation, parity, time reversal, etc, which allows us to construct the mathematical basis of quantum mechanics formulated on linear algebra. Square-root operator is one of the few exceptions. Historically, Paul Dirac derived the Dirac equation through a square-root operation on the Klein-Gordon (KG) equation to describe all spin-$\frac{1}{2}$ massive particles that inherit the Lorentz-covariance of the parent KG equation [1–3]. The approach has inspired Arkinstall et al. [4] to propose the concept of square-root topological insulator (TI) by taking the nontrivial square-root of a tight-binding Hamiltonian in periodic lattices. The most appealing feature of square-root TI is that it inherits the nontrivial nature of Bloch wave function from its parent Hamiltonian. The square-root TI was subsequently observed in a photonic cage [5]. Recently, the square-root operation has been applied to higher-order topological insulators (HOTIs) that allow topologically robust edge states with codimension larger than one [6–15]. Besides the gapped solution, e.g., the electron-positron pair, the Dirac equation allows another crucial gapless or massless solution called Weyl fermion [16] that plays an important role in quantum field theory and the Standard Model. Although not yet observed among elementary particles, Weyl fermions are shown to exist as collective excitations in Weyl semimetals [17–19]. It is thus intriguing to ask if the square-root operation can apply to semimetals [20] or higher-order semimetals [21–25], and particularly how to realize these exotic states in experiments.

In this article, we propose a tight-binding (TB) model of the square-root higher-order Weyl semimetal (SHOWS) by a vertical stacking of two-dimensional (2D) square-root HOTIs with interlayer couplings in a double-helix fashion. It is found that the SHOWS hosts both 2D surface arc states and one-dimensional (1D) hinge states with the topological feature being fully characterized by the quantized bulk polarization. We construct the TB model in honeycomb-kagome (HK) hybridized inductor-capacitor (LC) circuit networks. By performing both the impedance and voltage measurements in the stacked HK circuit, we identify the fingerprint of the SHOWS by directly observing the Weyl points, the “Fermi-arc” surface states, and the hinge states. It is revealed that both the surface states and the hinge states ideally connect the projections of the Weyl points, consistent with theoretical calculations.

Model

Figure 1a shows the lattice structure of the proposed model, the square of which can be viewed as the direct sum of a stacked honeycomb and a breathing kagome lattices (see Fig. 1b and the analysis in Supplementary Information Sec. I [26]). The tight-binding Hamiltonian is given by

$$\mathcal{H} = t_a \sum_{\langle m,n \rangle} (a_{m,n}^\dagger c_n + a_{m,n}^\dagger c_n^\dagger + a_{m,n}^\dagger c_n^\dagger)$$

$$+ t_b \sum_{\langle \langle m,n \rangle \rangle} (b_{m,n}^\dagger d_n + b_{m,n}^\dagger d_n^\dagger + b_{m,n}^\dagger d_n^\dagger)$$

$$+ t_c \sum_{\langle \langle m,n \rangle \rangle} (d_{m,n}^\dagger c_n + d_{m,n}^\dagger c_n^\dagger + d_{m,n}^\dagger c_n^\dagger) + \text{H.c.},$$

where $a^\dagger (a)$, $b^\dagger (b)$, $c^\dagger (c)$, $d^\dagger (d)$, and $e^\dagger (e)$ are the creation (annihilation) operators on the site 1-5, respectively, $\langle m,n \rangle$ and $\langle \langle m,n \rangle \rangle$ label the nearest-neighbor and next-nearest-neighbor coupling, respectively, and $t_a$, $t_b$, and $t_c$ are the hopping parameters. Without loss of generality, we assume all hopping parameters are positive. In momentum space, the Hamiltonian can be expressed as

$$\mathcal{H} = \begin{pmatrix} O_{2,2} & \Phi_k^\dagger \\ \Phi_k & O_{3,3} \end{pmatrix},$$

where $O_{2,2}$ and $O_{3,3}$ are the $2 \times 2$ and $3 \times 3$ zero matrix,
respectively, and $\Phi_k$ is the $3 \times 2$ matrix

$$
\Phi_k = \begin{pmatrix}
t_a & t_b + 2t_z \cos(k \cdot \mathbf{a}_3) \\
t_a & t_b + 2t_z \cos(k \cdot \mathbf{a}_3) e^{-i k \cdot \mathbf{a}_1} \\
t_a & t_b + 2t_z \cos(k \cdot \mathbf{a}_3) e^{-i k \cdot \mathbf{a}_2}
\end{pmatrix}.
$$

(3)

Here $k = (k_x, k_y, k_z)$ is the wave vector, and $\mathbf{a}_1 = \frac{1}{2} \hat{x} + \frac{\sqrt{3}}{2} \hat{y}$, $\mathbf{a}_2 = -\frac{1}{2} \hat{x} + \frac{\sqrt{3}}{2} \hat{y}$ and $\mathbf{a}_3 = \hat{z}$ are three basic vectors.

By taking the square of the original Hamiltonian (2), we can conveniently obtain the dispersion relation of $|\mathcal{H}|^2$ (see Supplementary Information Sec. I [26])

$$
E_k = 0 \quad \text{and} \quad \frac{3}{2} \left[ t_a^2 + t_b^2 \pm \sqrt{(t_a^2 - t_b^2)^2 + 4t_a^2 t_b^2 |\Delta(k)|^2} \right].
$$

(4)

With $t_b = t_b + 2t_z \cos(k_z)$ and $\Delta(k) = (1 + e^{i k \cdot \mathbf{a}_1} + e^{i k \cdot \mathbf{a}_2})/3$. The band structure of the original Hamiltonian is thus given by $\varepsilon_k = \pm \sqrt{E_k}$. It is found that the band structure closes at the twofold degenerate points $K_{\pm} = (4\pi/3, 0, \pm k_{zw})$, as shown in Fig. 1c, with $k_{zw} = \arccos[(t_a - t_b)/(2t_z)]$ when $|t_a - t_b| < 2t_z$. It is straightforward to verify that their time-reversal counterparts are $G_{\pm} = (-4\pi/3, 0, \pm k_{zw})$, and their equivalence points locate at $G_{\pm}, G_{\mp}, K_{\pm}, K_{\mp}$, as shown in Fig. 1d. By evaluating the topological charge $C_{\text{PS}}$, we find that the hollow and solid circles plotted in Fig. 1d denote the Weyl points with opposite topological charges, i.e., +1 and −1, respectively (see Supplementary Information Sec. II and Fig. S1 [26]). In addition, we derive the low-energy effective Hamiltonian near the degeneracy points, and obtain a linear crossing in the vicinity of the Weyl points (see Supplementary Information Sec. II [26]). The computation of Berry curvatures are plotted in Figs. S1c and S1d, which indeed demonstrates that the Weyl points manifest as singularities (source and drain), a close analogy to the magnetic monopole in momentum space.

For a system with the rotational symmetry (it is $C_3$ in our model), the bulk polarization is the appropriate invariant to characterize the topological features. For the $n$th band, the bulk polarization as a function of $k_z$ is written as

$$
2\pi p_n(k_z) = \arg \theta_n(k) \mod 2\pi,
$$

(5)

where $k = (4\pi/3, 0, \pm k_{zw})$, and $\theta_n(k) = u_n^\dagger(k) U_k u_n(k)$ with $u_n(k)$ the $n$th eigenvector and the $U$-matrix

$$
U_k = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & e^{-i k \cdot \mathbf{a}_2} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
$$

(6)

Here we are particularly interested in the 1st (or 5th) band, because the Weyl points only appear in the intersecting between the first and second energy bands (or
between the fourth and fifth energy bands. As shown in Fig. 1e, \( p_1 \) takes 1/3 for \( |k_z| < |k_{zw}| \) and 0 for \( |k_z| > |k_{zw}| \). The topological phase transition occurs at \( k_z = \pm k_{zw} \). A non-vanishing \( p_1 \) indicates the very presence of the higher-order topological edge states. The present model unambiguously demonstrates the bulk-hinge correspondence and manifests itself as an ideal SHOWS (see Fig. S2c in Supplementary Information Sec. III [26]). It is noted that a pair of Weyl points emerge with opposite wave vectors (see Fig. 1c) because of the inversion-symmetry breaking in our model. It is worth mentioning that the present model also allows a 3D square-root HOTI phase (see Figs. S2d-S2f in Supplementary Information Sec. III [26]). In what follows, we construct the tight-binding SHOWS model in 3D stacked HK LC circuits.

Circuit realization of SHOWS

We consider a stacked 10-layer HK circuit with \( N = 2860 \) nodes, as depicted in Fig. 2a. The circuit dynamics at frequency \( \omega \) obeys Kirchhoff’s law \( I_a(\omega) = \sum_b J_{ab}(\omega)V_b(\omega) \), with \( I_a \) the external current flowing into node \( a \), \( V_b \) the voltage of node \( b \), and \( J_{ab}(\omega) \) being the circuit Laplacian

\[
J(\omega) = \begin{pmatrix}
J_{0A} & -J_B & 0 & 0 & 0 & 0 & \ldots \\
-J_B & J_{0C} & -J_A & 0 & 0 & 0 & \ldots \\
0 & -J_A & J_{0B} & -J_A & 0 & 0 & \ldots \\
0 & 0 & -J_A & J_{0C} & 0 & -J_B & \ldots \\
0 & 0 & -J_A & 0 & J_{0C} & 0 & \ldots \\
0 & 0 & 0 & -J_B & 0 & J_{0B} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}_{N \times N}
\]

(7)

with \( J_{0A} = 3i\omega C_A + 1/(i\omega L_A) \), \( J_{0B} = 3i\omega C_B + 1/(i\omega L_B) \), \( J_{0C} = i\omega (C_A + C_B) + 1/(i\omega L_C) \), \( J_A = i\omega C_A \), and \( J_B = i\omega (C_B + 2C_Z) \). Under the resonance condition \( \omega_0 = 1/\sqrt{3C_A L_A} = 1/\sqrt{(3C_B + 6C_Z)L_B} = 1/(C_A + C_B + 2C_Z)L_C \), the circuit Laplacian (7) exactly recovers the tight-binding Hamiltonian by the following one-to-one correspondence: \( -\omega_0 C_A \leftrightarrow t_a \), \( -\omega_0 C_B \leftrightarrow t_b \), and \( -\omega_0 C_Z \leftrightarrow t_z \). To explore the square-root topological semimetal phase, we set \( C_A = C_B/2 = 0.5 \) nF, \( C_Z = 0.5 \) nF and \( L_A = 30 \) \( \mu \)H, \( L_B = 7.5 \) \( \mu \)H, and \( L_C = 18 \) \( \mu \)H in the following calculations, if not stated otherwise.

To facilitate the detection of the hinge states through a direct two-point impedance measurement [28], we connect a grounded inductor \( L_G = 22 \) \( \mu \)H to all nodes to move the hinge modes to the zero admittance without modifying their wave functions [6]. By measuring the impedance, one can precisely characterize the wave function of the zero-energy hinge states in the circuit [6, 28]. Figure 2b exhibits the corresponding admittance spectrum, where the red, blue, and black dots represent the hinge, surface, and bulk states, respectively. It is obvious to see the three-fold degeneracy of the in-gap hinge states. The spatial distributions of each mode are plotted in Figs. 2c-2e, from which one can straightforwardly distinguish them.

The photograph of 3D LC electric circuits fabricated on a printed circuit board is displayed in Fig. 5 in Methods. We choose electric elements \( C_A = C_B/2 = 0.5 \) nF, \( C_Z = 0.5 \) nF and \( L_A = 33 \) \( \mu \)H, \( L_B = 7.5 \) \( \mu \)H, \( L_C = 18 \) \( \mu \)H and \( L_G = 22 \) \( \mu \)H, the same as those for theoretical computations above, but with a practical 2% tolerance. The resonant frequency is then \( f_c = 1/(2\pi \sqrt{3C_A L_A}) = 755 \) kHz. We first measure the impedance between three representative nodes and the ground as a function of the exciting frequency with the impedance analyzer (Keysight E4990A). Experimental results are shown in Fig. 3b, which well agree with theoretical calculations plotted in
Fig. 3. a Theoretical impedance versus the driving frequency in a disordered circuit. b Measured impedance as a function of the frequency. Calculated c and measured d impedance distribution of hinge state over the system. Hinge-state voltage distribution in theory e and experiment f.

Fig. 4. a The projected admittances along the k_z direction. b Hinge state dispersion. The red dots and the colour map in a and b represent the theory and experimental hinge spectrum, respectively. c The numerical “Fermi arc” of the surface states at j_n = 0.004082 Ω^{-1}, which connects the projections of the Weyl points (the hollow and solid dots). d “Fermi arc” of the surface states at 860 kHz. The colour map and the white circles represent the experimental and theoretical results, respectively.

Fig. 3a. Here we select the 1432th, 1564th, and 1711th nodes to characterize the properties of the hinge, surface, and bulk states, respectively. We then measure the spatial distribution of the impedance and voltage over the circuit (see Figs. 3d and 3f), which compare reasonably well with the theoretical results plotted in Figs. 3c and 3e.

To characterize the hinge states more carefully, we project the dispersion to the k_z axis, as shown by the color map in Fig. 4a. In numerical simulations, one can conveniently take different j_n and analyze the spectrum subsequently, but we cannot set the specific value of j_n in circuit experiments. Fortunately, by mapping Kirchhoff’s law to the Schrödinger equation in circuit [27, 29], we obtain the frequency dispersion (see Supplementary Information Sec. V [26]) that significantly facilitates our experimental measurements. Experimentally, we impose a voltage source in the middle of one hinge of the circuits, and scan the voltage distribution along the hinge. Specifically, we input a signal $v_s(t) = 5\sin(\omega t)$ V at a hinge node with the arbitrary function generator (GW AFG-3022), and then collect the voltage $v(\omega, z)$ with frequency $f = \omega/(2\pi)$ ranging from 500 kHz to 1600 kHz by using the oscilloscope (Keysight MSOX3024A). We perform the Fourier transformation on the $v(\omega, z)$ and obtain the projected dispersion along the k_z direction, shown by the color map in Fig. 4b. It can be seen that the hinge states connecting two Weyl points at a resonant frequency around 755 kHz, which perfectly agrees with the simulation results marked by the solid red circles.

Furthermore, it is known that the “Fermi arc” surface state is an unique feature of Weyl semimetals. Figure 4c shows the “Fermi arc” surface dispersion at $j_n = 0.004082$ Ω^{-1}. Figure 4d shows the “Fermi arc” surface dispersion at $f = 860$ kHz. The colour map rep-
represents the measured data and the white circles denote
the simulated equal-admittance contour, whereas the hol-
low and solid dots denote the projections of Weyl points
with opposite topological charges +1 and −1, respect-
ively. Our experiment therefore unambiguously supports
the bulk-hinge correspondence and identifies the emer-
gence of SHOWS.

Conclusions
To summarize, we proposed a TB model of the SHOWS
and constructed it in 3D double-helix stacked LC cir-
cuits. Through the impedance and voltage measurements, we
directly observed both the 1D prismatic states and the
2D “Fermi arc” surface states connecting the projected
Weyl points, the fingerprint of SHOWS. Comparing with
the normal Weyl semimetal [17], the SHOWS supports
robust hinge states, besides the arc surface states. The
emergence of Weyl pairs in SHOWS with both positive
and negative energies marks its difference from the con-
ventional higher-order Weyl semimetals [21, 23, 25]. One
of the parent sublattices, i.e., the honeycomb lattice, ori-
ginally does not support any hinge states or flat-band
states. The square-root operator, however, makes it in-
herit these exotic states from the other parent sublatt-
ique. Our results pave the way to realizing the square-root
higher-order topological states, and may inspire the ex-
ploration in other solid-state systems, such as cold atoms,
photonic crystals, and elastic lattices.

Methods

PCB image in experiments and circuit Laplacians.

![PCB image](image.png)

**Fig. 5.** Side a and top b view of the printed circuit board in experiment.

The circuit dynamics at frequency ω obeys Kirchhoff’s law

\[ I_a(\omega) = \sum_b J_{ab}(\omega)V_b, \]

with \( I_a \) the external current flowing into node \( a \), \( V_b \) the voltage of node \( b \), and \( J_{ab}(\omega) \) being the circuit Laplacian

\[ J_{ab}(\omega) = i\mathcal{H}_{ab}(\omega) = \omega \left[ -C_{ab} + \delta_{ab} \left( \sum_n C_{an} - \frac{1}{\omega^2 L_a} \right) \right], \tag{8} \]

where \( C_{ab} \) is the capacitance between \( a \) and \( b \) nodes, \( L_a \) is
the grounding inductance of node \( a \), and the sum is taken
over all nearest-neighboring nodes. For a finite circuit, the
Laplacian of the circuit can be written as Eq. (7) in the main
text. Considering the resonance condition \( \omega = \omega_0 \), one can
obtain all eigenvalues \( j_n \) (admittances) and eigenfunctions \( \psi_n \),

\[(n = 1, 2, ..., N). \]

We set \( C_a = C_B/2 = 0.5 \text{ nF}, C_Z = 0.5 \text{ nF}, \]
\( L_A = 30 \mu\text{H}, L_B = 7.5 \mu\text{H}, \) and \( L_C = 18 \mu\text{H} \) in the following
calculations. In the calculation of Fig. 2 in the main text, we
consider the ideal situation that all inductors and capacitors
have no loss and disorder. Considering the practical loss and
tolerance of capacitors and inductors, we introduced 2% disor-
ter to each capacitor and inductor in theoretical calculations.

In experiments, we stacked 10 identical 2D printed
circuit boards (PCBs) along the \( z \) direction, as shown in Fig.
**5a.** Figure **5b** shows the top view of the PCB with the inset
zooming in the design details of the electrical circuit.

Data availability

The data that support the plots within this paper and
other findings of this study are available from the corre-
sponding author on reasonable request.

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[26] See Supplementary Information at http://link.aps.org/supplemental/ for solving the squared Hamiltonian (Sec. I), the calculation of the linear admittance spectrum near the Weyl point and the Berry curvature (Sec. II), the 3D square-root HOTI emerging in other parameter spaces (Sec. III), the image of experimental PCB and the circuit Laplacian (Sec. IV), the calculation of the Fermi arcs (Sec. V), and the mapping from Kirchhoff’s law to Schrödinger equation (Sec. VI), which includes Ref. [27].

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Acknowledgement

We acknowledge Z.-X. Li for useful discussions. This work was supported by the National Natural Science Foundation of China (Grants No. 12074057, No. 11704060, and No. 11604041).

Author contributions

P.Y. conceived the idea and contributed to the project design. L.S. and H.Y. designed the circuits and performed the measurements. L.S. developed the theory and wrote the manuscript. All authors discussed the results and revised the manuscript.

Competing interests

The authors declare no competing interests.

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Supplementary Information

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I. THE SQUARED HAMILTONIAN

It is noted that $\mathcal{H}$ [Eq. (2) in the main text] is chiral-symmetric, because it meets the condition $\mathcal{H} = -\gamma \mathcal{H} \gamma$ with

$$\gamma = \begin{pmatrix} I_{2,2} & O_{2,3} \\ O_{3,2} & -I_{3,3} \end{pmatrix},$$

where $O_{2,3} (I_{2,2})$ and $O_{3,2} (I_{3,3})$ are the $2 \times 2$ $(2 \times 2)$ and $3 \times 2$ $(3 \times 3)$ zero (identity) matrices, respectively. The Hamiltonian $\mathcal{H}$ with chiral symmetry indicates the existence of a parent Hamiltonian $\mathcal{H}^2$. With the help of parent Hamiltonian, one can obtain the eigenvalues of $\mathcal{H}$ by taking its square

$$\mathcal{H}^2 = \begin{pmatrix} h_k^H & O_{2,3} \\ O_{3,2} & h_k^K \end{pmatrix},$$

where $h_k^H = \Phi_k^\dagger \Phi_k$ and $h_k^K = \Phi_k \Phi_k^\dagger$ represent the Hamiltonian of a stacked honeycomb sublattice and breathing kagome sublattice with on-site potentials, respectively. Their explicit expressions are

$$h_k^H = \begin{pmatrix} h_{11} & h_{12} \\ h_{12}^\dagger & h_{22} \end{pmatrix},$$

with

$$h_{11} = 3t_z^2,$$

$$h_{12} = t_a t_b + 2t_a t_z \cos k_z + (t_a t_b + 2t_a t_z \cos k_z) e^{-ik_a_1} + (t_a t_b + 2t_a t_z \cos k_z) e^{-ik_a_2},$$

$$h_{22} = 3t_z^2 + 12t_b t_z \cos k_z + 6t_z^2 + 6t_z^2 \cos (2k_z),$$

and

$$h_k^K = \begin{pmatrix} h_{33} & h_{34} & h_{35} \\ h_{34}^\dagger & h_{44} & h_{45} \\ h_{35}^\dagger & h_{45}^\dagger & h_{55} \end{pmatrix},$$

with

$$h_{33} = h_{44} = h_{55} = t_{a_1}^2 + t_{b_1}^2 + 2t_z^2 + 4t_b t_z \cos k_z + 2t_z^2 \cos (2k_z),$$

$$h_{34} = t_a^2 + [t_b^2 + 2t_z^2 + 4t_b t_z \cos k_z + 2t_z^2 \cos (2k_z)] e^{ik_a_1},$$

$$h_{35} = t_a^2 + [t_b^2 + 2t_z^2 + 4t_b t_z \cos k_z + 2t_z^2 \cos (2k_z)] e^{ik_a_2},$$

$$h_{45} = t_a^2 + [t_b^2 + 2t_z^2 + 4t_b t_z \cos k_z + 2t_z^2 \cos (2k_z)] e^{-ik_a_1 - i k_a_2}.$$

We note that $h_k^H$ and $h_k^K$ have the same energy band solution, except that $h_k^K$ has an additional flat band pinned to zero energy. The energy band solution of the $h_k^K$ is

$$E_k = 0 \quad \text{and} \quad \frac{3}{2} \left[ t_a^2 + t_b^2 \pm \sqrt{(t_a^2 - t_b^2)^2 + 4t_a^2t_b^2|\Delta(k)|^2} \right],$$

with $t_b = t_b + 2t_z \cos (k_z)$ and $\Delta(k) = (1 + e^{ik_a_1} + e^{ik_a_2})/3$. The band structure of the original Hamiltonian is therefore given by $\varepsilon_k = \pm \sqrt{E_k}$.
II. THE LINEAR ADMITTANCE SPECTRUM NEAR THE WEYL POINT AND THE BERRY CURVATURE

In this section, we demonstrate that the Weyl semimetal in our system hosts linear dispersion in all three dimensions in the vicinity of the Weyl points which act like monopoles of Berry curvature. To this end, we expand $h_k^H$ in terms of Pauli matrix $h_k^H = \lambda_0 \sigma_0 + \lambda_x \sigma_x + \lambda_y \sigma_y + \lambda_z \sigma_z$ with $\sigma_0$ the identity matrix, $\sigma_x$, $\sigma_y$ and $\sigma_z$ being the Pauli matrices. The parameters $\lambda_i$ ($i = 0, x, y, z$) are explicitly expressed as

$$\lambda_0 = \frac{3}{2} t_a^2 + \frac{3}{2} t_b^2 + 6 t_b t_z \cos k_z + 3 t_z^2 + 3 t_z^2 \cos(2k_z),$$

$$\lambda_x = t_a t_b + 2 t_a t_z \cos k_z + 2(t_a t_b + 2 t_a t_z \cos k_z) \cos(\frac{1}{2} k_x) \cos(\frac{\sqrt{3}}{2} k_y),$$

$$\lambda_y = 2 t_a t_b \cos\left(\frac{1}{2} k_x\right) \sin\left(\frac{\sqrt{3}}{2} k_y\right) + 4 t_a t_z \sin\left(\frac{1}{2} k_x\right) \sin\left(\frac{\sqrt{3}}{2} k_y\right) \cos k_z,$$

$$\lambda_z = \frac{3}{2} t_a^2 - \frac{3}{2} t_b^2 - 6 t_b t_z \cos k_z - 3 t_z^2 - 3 t_z^2 \cos(2k_z).$$

(8)

Fig. 1. The admittance dispersion around the Weyl point $K_+$ in the a $q_x - q_y$ and b $q_x - q_z$ planes. The spatial distribution of the Berry curvature around c $K_+$ and d $K_-$. Open and solid circles represent the Weyl points with opposite topological charges +1 and −1.
Near the point $K_+ = (4\pi/3, 0, kzw)$, using the Taylor expansion, the parameters $\lambda_i \ (i = 0, x, y, z)$ of the effective Hamiltonian can be written as:

$$
\begin{align*}
\lambda_0 &= \frac{3}{2} t_a^2 + \frac{3}{2} t_a^2 + \frac{3}{2} t_z^2 + \frac{3}{2} t_b t_z - 3\sqrt{3}(t_b t_z - t_a^2) q_z, \\
\lambda_x &= t_a t_z - \frac{\sqrt{3}}{2} (t_a t_b - t_a t_z) q_x - \sqrt{3} t_a t_z q_z, \\
\lambda_y &= -\frac{\sqrt{3}}{2} (t_a t_b - t_a t_z) q_y, \\
\lambda_z &= \frac{3}{2} t_a^2 + \frac{3}{2} t_b^2 - \frac{3}{2} t_z^2 - \frac{3}{2} t_b t_z + 3\sqrt{3}(t_b t_z - t_a^2) q_z,
\end{align*}
$$

with $q = k - K_+$. From Eqs. (9), one can clearly see that the band linearly touches at $K_+$, which is a typical feature of band crossing of Weyl semimetals. The energy bands around the Weyl point of SHOWS inherited from (9) are also linear. We then investigate the distribution of Berry curvature in momentum based on the low-energy effective Hamiltonian expanding around the Weyl points. It will be demonstrated that the Weyl points will generate Fermi arc states on the surface. We first consider the degenerate point at $K_+$. Here, we plot the 3D band dispersion around $K_+$ in Figs. 1a and 1b. The band dispersion around $K_- \triangleleft$ is similar to the case around $K_+$. Obviously, the band dispersion around the degenerate points along any direction is linear. Furthermore, the Berry curvature is expressed as

$$
F_x = \frac{\partial A_x}{\partial q_y} - \frac{\partial A_y}{\partial q_z}, \quad F_y = \frac{\partial A_y}{\partial q_z} - \frac{\partial A_z}{\partial q_x}, \quad F_z = \frac{\partial A_z}{\partial q_x} - \frac{\partial A_x}{\partial q_y},
$$

where $A_\mu = -i(\phi|\nabla_\mu|\phi)$ is the berry connection, with $\mu = x, y, z$ and $\phi(q)$ being its wave function. Figures 1c and 1d show that the flux of the Berry curvature flowing from $K_+$ to $K_-$, which is similar to the magnetic monopole in momentum space. The monopole charge is defined as

$$
C_{FS} = \frac{1}{2\pi} \int_{FS} \mathbf{F}(k) \cdot d\mathbf{S},
$$

where $FS$ is the curved surface surrounding the Weyl point. By evaluating $C_{FS}$, we find that $K_+$ and $K_-$ are a pair of Weyl points with opposite charge +1 and −1, denoted by the open and solid circles respectively. This means that this 3D circuit system hosts four Weyl points that reside at the same admittance and is thus a Weyl semimetal.

### III. THE 3D SQUARE-ROOT HOTI

The non-zero bulk polarization (in Fig. 2b) gives rise to the hinge states in a triangular prism sample with the dispersion connecting the projections of the Weyl points along the $k_z$ direction, as shown by the hinge state distribution in Fig. 2c. It is worth mentioning that a 3D square-root HOTI can also emerge in our system for other parameters (see Figs. 2d-2f). Comparing the bulk band structures in Fig. 2a and Fig. 2d), one can see that the band gap of high-order topological insulators always exists from $K$ to $\bar{K}$. In this region, the bulk polarization is always non-zero in Fig. 2b, but not the case for Fig. 2e.

### IV. CALCULATIONS OF THE FERMI ARCS

The Fermi arc is the equi-energy contour of the surface states at a fixed $j_n = 0.004082 \Omega^{-1}$. Figure 3a shows the Fermi arcs with the same energy of Weyl points. Because all the four Weyl points are at the same energy, the Fermi arcs connect two Weyl points with opposite charges. These surface states are clearly gapped, as shown in Figs. 3b-3f.

### V. MAPPING FROM KIRCHHOFF’S LAW TO SCHRODINGER EQUATION

We derive the relation between Kirchhoff’s laws and Schroedinger equation, which enables us to calculate the frequency spectrum.

In electric circuits, the equation of motion is given by

$$
\frac{dI(t)}{dt} = C\frac{d^2V(t)}{dt^2} + L V(t),
$$
Fig. 2. The parameters in a, b, and c were chosen as with $t_a = 0.5$, $t_b = 1$, and $t_z = 0.1$, corresponding to SHOWS. As a comparison, the parameters in d, e, and f were chosen as with $t_a = 0.5$, $t_b = 1$, and $t_z = 0.5$, corresponding to square-root HOTI. ad Bulk band structures. be Bulk polarization $p_1$ as a function of $k_z$, with the subscript 1 indicating the 1th band. cf The projected dispersion of a triangular prism, i.e., admittance along the $k_z$ direction. The yellow line indicates the hinge state dispersion.

Fig. 3. Fermi arc and surface state dispersions. a The contour of the surface states at the admittance of the Weyl points ($\chi_n = 0.004082 \, \Omega^{-1}$). The open and solid circles denote the Weyl points with opposite topological charges. b-f The surface state dispersions along the $k_x$ direction for different $k_z$ (1.66, 1.78, 1.975, 2.03, and 2.283). The solid red line denotes the surface state dispersion and the dashed line shows the position of $\chi_n = 0.004082 \, \Omega^{-1}$. 
where $V$ is the $N$-component voltage measured at each node against the ground and $I$ is the $N$-component input current at each node.

The homogeneous equations of motion ($I = 0$) can be rewritten as $2N$ differential equations of first order [1]:

$$-i \frac{d}{dt} \psi(t) = \mathcal{H}_S \psi(t),$$

(13)

with $\psi = (\dot{V}(t), V(t))^T$ and the Hamiltonian block matrix being $\mathcal{H}_S = i \begin{pmatrix} 0 & C^{-1}L \\ -1 & 0 \end{pmatrix}$. By diagonalizing $\mathcal{H}_S$, we can obtain the frequency dispersion $\omega(k_x)$.

[1] Hofmann, T., Helbig, T., Lee, C. H., Greiter, M. & Thomale, R. Chiral Voltage Propagation and Calibration in a Topolectrical Chern Circuit, Phys. Rev. Lett. 122, 247702 (2019).