Holographic interpretation of two-dimensional O($N$) models coupled to quantum gravity

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Abstract

Various two-dimensional O($N$) models coupled to Euclidean quantum gravity, whose intrinsic dimension is four, are shown to belong to universality classes of nongravitating statistical models in a lower number of dimensions. It is speculated that the matching critical behaviors in the gravitating and dimensionally reduced models may be manifestations of the holographic principle.

Keywords: Euclidean quantum gravity, two-dimensional statistical models, critical behavior, KPZ formula, holographic principle

1. Introduction

The critical behavior of various two-dimensional (2D) statistical models coupled to quantum gravity with an Euclidean signature is known exactly. These exact results are provided in part by the Knizhnik-Polyakov-Zamolodchikov (KPZ) formula [1], which transcribes the critical exponents of 2D statistical models into the exponents characterizing the critical behavior of these models when coupled to gravity. For our purpose it is convenient to regularize these theories of gravity by putting the 2D statistical models on a fluctuating surface constructed through, for example, dynamical triangulation. In its simplest form, the surface is built from equilateral triangles [2, 3, 4]. A canonical ensemble of random surfaces obtains by gluing a given number of (identical) triangles in all possible ways to form a compact surface of fixed (say spherical) topology. Such an ensemble can be sampled in a Monte Carlo simulation by using the (local) Pachner move [5] shown in Fig. 1 as update. This update, which consists of a simple bond flip, is both ergodic and preserves the topology of the surface. On the lattice, the functional integral

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over geometries in the continuum theory is replaced with a sum over all triangulations, each carrying the same weight. Fluctuations in the geometry are in this way properly accounted for in the lattice model. It turns out that the fluctuations are so large that the surface becomes highly irregular and cannot usually be embedded in 3D space. In fact, the fluctuating surface, representing pure quantum gravity, has the large fractal, or Hausdorff dimension \( d = 4 \) \([6]\).

Matter can be included on such an empty fluctuating surface by introducing fields (spin variables) on, for example, the lattice sites. Usually only nearest neighbor interactions are considered for convenience, i.e., only lattice sites connected by a bond are assumed to interact. The matter fields and the lattice must be updated simultaneously, taking into account that a Pachner move changes the nearest neighbor contingency table. The fractal dimension of the decorated fluctuating surface is not known exactly, but numerically found to be consistent with \( d = 4 \) for matter fields with central charge \( 0 \leq c \leq 1 \) \([7]\). Given this observation, it seems somewhat deceptive to refer to these models as “two-dimensional” quantum gravity, as is commonly done. This seems even more so in light of the fact that it is the fractal dimension of the fluctuating surface that features in the scaling laws (see below) and takes the place of the dimensionality of a regular lattice in the absence of gravity. Below, we take the point of view that, at least for \( 0 \leq c \leq 1 \), the gravitating statistical models are in \( d = 4 \). In line with the usual application of the O\((N)\) model to thermal phase transitions in equilibrium, for which time is irrelevant and can be ignored, we take the dimensions to represent space.

The critical exponents of the 2D Ising model on a fluctuating planar surface were first determined exactly by Boulatov and Kazakov \([8]\), who came to the surprising conclusion that these exponents are identical to those of the spherical model on a regular 3D lattice. This match in critical behavior is commonly considered a mere coincidence. By providing additional matches of this type, we in this paper speculate that, instead of being coincidences, they may be manifestations of the \textit{holographic} principle \([9]\).

The paper is organized as follows. We start by recalling some key results of the O\((N)\) spin model on a flat planar surface which will be used in Sec.\([3]\) to study
the model on a fluctuating surface. We show that, in addition to the Ising model ($N = 1$), for two other values of $N$, the critical exponents of the O($N$) model on a fluctuating lattice are mirrored by a statistical model defined on a regular lattice. In Sec. 4, we extend the analysis to the tricritical O($N$) model and uncover an additional match in critical behavior. The nongravitating mirror models are all in 3D or 2D, i.e., they are of lower dimension than their 4D gravitating counterparts.

2. O($N$) model

The O($N$) spin model on a flat planar lattice with intrinsic dimension $d = 2$ undergoes for $-2 \leq N \leq 2$ a continuous phase transition [10]. The model, parametrized as

$$N = 2 \cos \left( \frac{\pi}{m} \right)$$

(1)

with $1 \leq m \leq \infty$, has central charge $c = 1 - 6/m(m + 1)$, and thermal and magnetic exponents [11]

$$\frac{1}{\nu d} = 1 - \Delta_E, \quad \frac{2 - \eta}{d} = 1 - 2\Delta_1.$$  

(2)

The remaining critical exponents, $\beta, \gamma,$ and $\delta$ follow from the two independent exponents $\nu$ and $\eta$ through scaling relations. In Eq. (2), $\Delta_E = \Delta_{1,3}$ and $\Delta_1 = \Delta_{1/2,0}$ denote conformal dimensions expressed in terms of entries in the Kac table [12]

$$\Delta_{p,q} = \frac{[(m + 1)p - mq]^2 - 1}{4m(m + 1)}.$$  

(3)

The conformal dimension $\Delta$ of an operator $\phi$ determines the scaling dimension $x$ of that operator, which in turn specifies the algebraic decay of the correlation function at the critical point $\langle \phi(r)\phi(r') \rangle \sim 1/|r - r'|^{2x}$, through $\Delta = x/d$. The corresponding renormalization-group eigenvalue is

$$y = d(1 - \Delta) = d - x,$$

(4)

with $d = 2$ the dimensionality of a flat surface.

In addition to the standard spin formulation, the O($N$) model also allows for a geometric formulation in terms of high-temperature (HT) graphs [13]. Contributions to the partition function are represented in this equivalent formulation by closed graphs along the links on the underlying lattice. For computational convenience and without changing the universality class, often a truncated O($N$) model
is considered \[14\]. Whereas a link in the standard HT expansion can be multiply occupied, in the truncated model it can at most be occupied once. A spin operator at site \(r\) is pictured in the geometrical approach by putting a bond, or leg on one of the links emanating from that site. As a result, contributions to the spin-spin correlation function \(G_1(r, r') \equiv \langle s(r) \cdot s(r') \rangle\) feature, in addition to possible closed graphs, an open graph, or strand, connecting the legs at \(r\) and \(r'\). The subscript "1" on the conformal dimension in Eq. (2), determining the algebraic behavior of \(G_1(r, r')\), indicates that \(\Delta_1\) denotes the dimension of the one-leg operator. An additional operator of interest is the two-spin, or two-leg operator, whose scaling dimension is \(\Delta_2 = \Delta_{1,0}\). The corresponding renormalization-group eigenvalue \(y_2\) determines the fractal, or Hausdorff dimension \(D_{HT}\) of the HT graphs of the O(\(N\)) model:

\[
\frac{D_{HT}}{d} = \frac{y_2}{d} = 1 - \Delta_2.
\]

A final operator of interest in the geometrical approach is the four-leg operator, which introduces intersections in HT graphs. Its conformal dimension is \[10\] \(\Delta_4 = \Delta_{2,0}\). For all \(1 \leq m < \infty\), the corresponding scaling dimension is larger than two, the dimension of space, \(x_4 = 2\Delta_4 > 2\), and \(x_4 = 2\) in the limit \(m \rightarrow \infty\), which corresponds to the XY model. The four-leg operator is therefore irrelevant at the critical point for \(1 \leq m < \infty\) and becomes marginal in the limit \(m \rightarrow \infty\). Graphs at the O(\(N\)) critical point are for this reason referred to as \textit{dilute} graphs, and the critical properties of the O(\(N\)) model can be studied on a lattice with coordination number \(z = 3\), where closed graphs simply cannot intersect, without changing the universality class. The advantage of such a lattice is that closed graph configurations can be uniquely decomposed into mutually and self-avoiding loops.

The one-, two, and four-leg operators belong to the set of so-called watermelon operators with conformal dimension \[15\]

\[
\Delta_L = \Delta_{L/2,0},
\]

\(L = 1, 2, \ldots\). The corresponding correlation function \(G_L(r, r')\) involves \(L\) HT strands connecting the lattice sites \(r\) and \(r'\).

The fractal and conformal dimensions characterizing the O(\(N\)) model also appear in the context of the Potts model. That spin model, too, can be equivalently formulated in purely geometrical terms, this time involving clusters. Two types of clusters can be distinguished: plain and so-called Fortuin-Kasteleyn (FK) bond clusters. Plain bond clusters are formed by unconditionally setting bonds between nearest neighbor sites with like spins. FK clusters, which form the basis of the
equivalent representation of the $Q$-state Potts model as a correlated bond percolation model \[16\], are constructed from the plain clusters by erasing bonds between like spins with a prescribed, temperature-dependent probability. In 2D (and in two only), both types of clusters percolate right at the critical point. Moreover, while the FK clusters encode in their fractal structure critical properties of the $Q$-state Potts model, the plain clusters encode in their fractal structure tricritical properties \[17\]. The latter usually obtains when vacant sites are included in the Potts model. If

$$\sqrt{Q} = 2 \cos \left[ \frac{\pi}{(m + 1)} \right]$$

parametrizes the $Q$-state Potts model with central charge $c = 1 - 6/(m(m + 1))$, the tricritical behavior encoded in the plain bond clusters of that model is that of the $Q^t$-state tricritical Potts model with the same central charge, parametrized as

$$\sqrt{Q^t} = N,$$

with $N$, given by Eq. (1), restricted to $0 \leq N \leq 2$, i.e., $2 \leq m \leq \infty$. Note that this parametrization can be obtained from that of the $Q$-state Potts model by applying the map $m \to -m - 1$, which conserves the central charge $c$.

3. Gravitating $O(N)$ model

We next turn to the $O(N)$ model on fluctuating planar lattices. The KPZ formula \[11\],

$$\Delta = \left(1 - \frac{m}{1 + m}\right) \tilde{\Delta} + \frac{m}{1 + m} \tilde{\Delta}^2,$$

relates the conformal dimension $\Delta$ of an operator on a flat planar lattice to its conformal dimension $\tilde{\Delta}$ on a fluctuating lattice, or more precisely, $\tilde{\Delta}$ is the non-negative solution to the above equation. For the Kac table (3), the KPZ formula yields the gravitationally dressed dimensions

$$\tilde{\Delta}_{p,q} = \frac{-1 + |(1 + m)p - mq|}{2m},$$

so that \[8, 18\]

$$\tilde{\Delta}_E = \frac{-1 + m}{m}, \quad \tilde{\Delta}_1 = \frac{-1 + m}{4m}, \quad \tilde{\Delta}_2 = \frac{1}{2}, \quad \tilde{\Delta}_4 = 1 + \frac{1}{2m},$$

and

$$\tilde{\nu} d = m, \quad \frac{2 - \tilde{\eta}}{d} = \frac{1 + m}{2m}$$

(12)
for the O(N) model coupled to quantum gravity. Here and in the following, d denotes the fractal dimension of the fluctuating surface. For matter fields with central charge $0 \leq c \leq 1$, its value was numerically found to be consistent with the value $d = 4$ of an empty fluctuating surface \[7\]. Remarkably, the gravitationally dressed dimensions have a simpler dependence on $m$ than their nongravitating counterparts. This is in particular true for $\tilde{\Delta}_2$, which yields as fractal dimension of the HT graphs \[18\]
\[
\frac{D_{HT}}{d} = 1 - \tilde{\Delta}_2 = \frac{1}{2},
\]
(13)

independent of $m$. Such a universal behavior, where a set of models share the same scaling dimension, is usually reserved for models in their upper critical dimension, such as the O(N) model on a regular 4D lattice for which the ratio $D_{HT}/d$ is also $\frac{1}{2}$. The conformal dimension $\tilde{\Delta}_4$ of the four-leg operator shows that at criticality, intersections are still irrelevant on a fluctuating planar lattice. This implies that the critical properties of the O(N) model can be studied on a fluctuating planar lattice with coordination number $z = 3$, without changing the universality class. Also observe that in the limit $m \to \infty$, corresponding to the gravitating XY model, the correlation length exponent $\tilde{\nu}$ diverges. This characteristic of the XY model on a flat planar surface is therefore preserved when the model is put on a fluctuating surface. Finally note that for self-avoiding walks ($m = 2$) on a fluctuating random planar lattice $\tilde{\nu} = 1/D_{HT}$, while for all other values of $m$, $\tilde{\nu} \neq 1/D_{HT}$. This is similar to that found on a flat lattice \[19\].

In Ref. [20], the fractal structure of plain and FK bond clusters of the $Q$-state Potts model was studied on fluctuating planar lattices with coordination number $z = 3$. The fractal dimension of the hulls of plain bond clusters on a fluctuating lattice was, among others, conjectured on the basis of the KPZ formula, with the result \[13\]. The predictions for the plain and FK cluster dimensions were confirmed numerically in that reference for the Ising model ($Q = 2$) coupled to quantum gravity through Monte Carlo simulations. A rigorous derivation of the geometrical KPZ relation, using a probabilistic approach, was provided in Ref. [21]. For a derivation based on the heat-kernel method, see Ref. [22].

We next show that for various $N$, the O(N) model coupled to quantum gravity belongs to a universality class (as defined by the standard critical exponents) of a statistical model formulated on a regular lattice, i.e., without gravity. Specifically:

- for $m = 1$, the critical exponents \[12\] are the same as on a flat planar lattice, so that putting the Gaussian model ($N = -2$) on a fluctuating lattice has no effect on its critical behavior,
• for \( m = \frac{4}{3} \) \((N = -\sqrt{2})\), the critical exponents \([12]\) are mirrored by the \( Q = 4 \) Potts model on a flat planar lattice,

• for the Ising model \((m = 3)\) coupled to quantum gravity, the critical exponents \([12]\) are known to coincide with those of the spherical model on a cubic lattice \([8]\).

In summary, for \( m = 1, \frac{4}{3}, \) and \( 3 \) (and surely also for other values), the O(\( N \)) model coupled to quantum gravity with intrinsic dimension \( d = 4 \) belongs to universality classes of nongravitating models in dimension \( d = 2, 2, \) and \( 3 \), respectively. Note that these universality classes are all in lower than four dimensions—the dimensionality of the fluctuating surface. That is, we witness a “dimensional reduction in quantum gravity” \([9]\).

Although the critical exponents coincide, the fractal dimensions of the HT graphs featuring in the gravitating and in the dimensionally reduced models without gravity do not. Whereas, for example, the fractal dimension of the gravitating Gaussian model is \( \tilde{D}_{HT}/d = \frac{1}{2} \), that of its nongravitating counterpart on a flat planar lattice is \( D_{HT}/d = \frac{2}{5} \). Both models are nevertheless classified as belonging to the same universality class on the basis that their critical exponents coincide.

### 4. Tricritical O(\( N^t \)) model

We next extend our analysis to the tricritical O(\( N^t \)) model. When vacant sites are included, the O(\( N \)) spin model, too, displays in addition to critical also tricritical behavior. The tricritical point is reached by gradually increasing the activity of the vacancies. The continuous phase transition then eventually becomes discontinuous with the endpoint, marking the change in the order of the transition, being the tricritical point. The tricritical O(\( N^t \)) model with central charge \( c \) is parametrized as \([23]\)

\[
N^t = \sqrt{Q} - 1/\sqrt{Q}
\]

with \( Q \) given by Eq. \( (7) \), denoting the Potts model with the same central charge, and restricted to \( 0 < Q \leq 2 \), i.e., \( 1 < m \leq \infty \). This parametrization constitutes the tricritical counterpart of Eq. \( (8) \), which relates the critical O(\( N \)) model to the tricritical \( Q^t \)-state Potts model.

In Ref. \([19]\), we conjectured that the fractal dimension \( D_{HT}^t \) of the HT graphs at the tricritical point on a flat planar lattice, where they collapse, follow from those \([5]\) at the critical point by applying the central-charge conserving map \( m \rightarrow \)
−m − 1, so that \( \Delta_2 = \Delta_{1,0} \rightarrow \Delta_{0,1} = \Delta_2^t \). This yields

\[
\frac{D_{HT}^t}{d} = 1 - \Delta_2^t = \frac{1 + 3m}{4m}.
\] (15)

In the limit \( m \rightarrow \infty \) (\( N \rightarrow 2 \)), the fractal dimensions \( D_{HT} \) and \( D_{HT}^t \) approach the limiting value \( \frac{3}{2} \) from opposite directions, with the collapsing graphs being, of course, more crumpled than the dilute graphs. In that limit, corresponding to \( c = 1 \), the critical and tricritical fixed points merge. In the opposite limit, \( m \rightarrow 1 \), the collapsing graphs even start to fill the entire available space, \( D_{HT}^t / d \rightarrow 1 \).

Equation (15) reduces to the known results \( D_{HT}^t = \frac{7}{4} \) for polymers (\( m = 2, N^t = 0 \)) at the theta point [24], and \( D_{HT}^t = \frac{13}{8} \) for the tricritical Ising model (\( m = 4, N^t = 1 \)), which coincides with the tricritical \( Q^t = 2 \) Potts model. We further conjectured in Ref. [19] that also the dimensions of the leading magnetic operators of the critical and tricritical models are related by the dual map \( m \rightarrow -m - 1 \). Specifically, \( \Delta_1 = \Delta_{1/2,0} \rightarrow \Delta_{0,1/2} = \Delta_1^t \). This prediction has been confirmed by high-precision numerical transfer-matrix calculations in Ref. [23].

Transcribing the conjecture (15) to the tricritical \( O(N^t) \) model on a fluctuating planar lattice, we obtain the fractal dimension

\[
\tilde{D}_{HT}^t = 1 - \tilde{\Delta}_2^t = \frac{1 + m}{2m},
\] (16)

which, in contrast to the fractal dimension of dilute graphs on a fluctuating lattice, does depend on \( m \). As expected, the collapsing graphs are more crumpled than the dilute graphs for all \( 1 < m < \infty \), see Eq. (5). In the limit \( m \rightarrow 1 \), the collapsing graphs fill the entire available space, \( \tilde{D}_{HT}^t / d \rightarrow 1 \), as on a flat planar lattice. The two fractal dimensions \( \tilde{D}_{HT}^t / d \) and \( \tilde{D}_{HT}^t / d \) coincide in the limit \( m \rightarrow \infty \). The KPZ formula yields as magnetic dimension for the gravitating tricritical \( O(N^t) \) model

\[
\tilde{\Delta}_1^t = \frac{-2 + m}{4m},
\] (17)

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1Earlier numerical work by Guo et al. [23] already showed agreement with our prediction for \( 0 \leq c \lesssim 0.7 \), but the two data points reported in the interval \( 0.7 \lesssim c \lesssim 1.0 \) deviated significantly from our prediction (see Fig. 1 of Ref. [19]). In Ref. [23], the data point \( \{c, x_h\} = \{0.860(1), 0.0923(2)\} \) of the earlier work shifted upwards to \( \{0.860(1), 0.0950(2)\} \), while the old data point \( \{0.998(2), 0.111(1)\} \) received a much larger error bar \( \{1.001(2), 0.12(1)\} \), so that the revised numerical results now fit our prediction perfectly over the entire range \( 0 \leq c \leq 1 \). Moreover, it was shown that the transition of the tricritical model ceases to be continuous for \( c > 1 \), in accordance with our prediction.
which, as on a flat planar lattice, is physical only for \( m \geq 2 \). With the exception of collapsing self-avoiding walks and of the tricritical Ising model, the leading tricritical thermal exponent on a flat planar lattice does not correspond to an entry in the Kac table \([23]\). When conformal dimensions are not rational numbers, it is not clear whether these can be fed into the KPZ formula. In the following, we, therefore, restrict ourselves to the gravitating \( O(N^t = 0) \) and \( O(N^t = 1) \) models.

For collapsing self-avoiding walks \([26]\) \((m = 2, \ N^t = 0)\) the conformal dimension of the leading thermal operator is \( \Delta^t_E = \Delta_{2,2} = \frac{1}{8} \), while for the tricritical Ising model \([27]\) \((m = 4, \ N^t = 1)\) it is \( \Delta^t_E = \Delta_{1,2} = \frac{1}{10} \). Both dimensions happen to yield \( \tilde{\nu} d = \frac{4}{3} \) for the tricritical correlation length exponent on a fluctuating planar lattice. Together with the magnetic dimension \([17]\), this gives for the exponents of collapsing self-avoiding walks coupled to quantum gravity:

\[
\frac{1}{\tilde{\nu} d} = 1 - \tilde{\Delta}_E = \frac{3}{4}, \quad \frac{2 - \tilde{\eta}^t}{d} = 1 - 2\tilde{\Delta}_E = 1,
\]

and for the gravitating tricritical Ising model:

\[
\frac{1}{\tilde{\nu} d} = \frac{3}{4}, \quad \frac{2 - \tilde{\eta}^t}{d} = \frac{3}{4}.
\]

The remaining tricritical exponents \( \tilde{\beta}^t, \tilde{\gamma}^t, \) and \( \tilde{\delta}^t \) follow from \( \tilde{\nu}^t \) and \( \tilde{\eta}^t \) through the usual scaling relations. Note that for collapsing self-avoiding walks coupled to gravity, \( \tilde{\nu}^t = 1/\tilde{D}_{\text{HT}}^t \), so that the characteristic relation between correlation length exponent and fractal dimension for (collapsing) self-avoiding walks on flat planar lattices persists on fluctuating lattices.

The gravitating tricritical Ising model also belongs to a universality class defined by a nongravitating model in lower dimensions. Specifically, it shares the same critical exponents as the fourth-order critical point of the Ising model on a cubic lattice with antiferromagnetic nearest neighbor and ferromagnetic next-nearest neighbor interactions in an external magnetic field \([28]\). Although the exponents determined in Ref. \([28]\) are mean-field exponents, we expect them to be exact in \( d = 3 \), for this is above the upper critical dimension

\[
d_u = \frac{2k}{k-1} = \frac{8}{3}
\]

of a fourth-order critical point \((k = 4)\). That is, we again witness “dimensional reduction in quantum gravity” \([9]\), with the gravitating tricritical Ising model in \( d = 4 \) sharing the same critical exponents as a nongravitating model in \( d = 3 \).
5. Discussion

In this paper, the critical properties of the 2D O($N$) model coupled to Euclidean quantum gravity, whose intrinsic dimension is four, were studied. It was pointed out that for various $N$, the critical exponents of these models are mirrored by nongravitating models in dimensions lower than four. Rather than consider this to be mere coincidence, we submit that these matches may, in fact, be manifestations of the holographic principle. This principle, put forward by 't Hooft [9], asserts that all of the information about a system with gravity in some region of space, which can be very large, or even infinite, is coded in degrees of freedom that are confined to a boundary to that region and are not subject to gravity. At the least, the principle mandates that the critical properties of a statistical model coupled to gravity be mirrored by one without gravity in one dimension less, representing the gravity-free degrees of freedom on a boundary to space. Since such matches in critical behavior exist for the gravitating critical and tricritical Ising models, it is tempting to speculate that these models are holographic projections of the corresponding nongravitating models defined on a 3D boundary to 4D space.

For the gravitating O($N = -2$) and O($N = -\sqrt{2}$) models, whose matter fields have a negative central charge, the nongravitating counterparts are in 2D. If the dimensionality of the fluctuating surface with matter fields of negative central charge is still four, and if the holographic principle applies, it would imply that, in these cases, a 2D model suffices to reconstruct the critical behavior of a 4D gravitating world.

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We disagree with the conclusion these authors draw from their numerical results. In our view, their results support the value $d = 4$, certainly for matter fields of central charge $0 \leq c \leq 1$. 

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\[11\]

\[12\]

\[13\]

\[14\]

\[15\]

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