Numerical solution of the time-dependent Ginzburg-Landau equation for a superconducting mesoscopic disk: Link variable method

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Abstract. The time dependent Ginzburg-Landau equations (TDGLE) are used to study the superconducting properties of a disk by taking into account the influence of internal defects. The Link variable algorithm is applied to a circular geometry surrounded by an insulator and immersed in an external magnetic field applied perpendicular to its plane. The TDGLE are used upon taking the magnetic field and the order parameter invariant along z-direction. We show that the magnetic response is substantially modified by the competition between the confinement geometry and the geometric position of the defects leads to vortex configurations which are not compatible with the symmetry of the sample geometry.

1. Introduction
Vortex state in conventional mesoscopically tailored superconducting samples has generate much activity in the wide scientific community. Many topologies have been studied experimentally and theoretically, such as simple loops and circular geometry [1, 2, 3, 4], surface roughness and surface defects [5, 6]. All these superconducting structures have attracted attention as potential new components for low-temperature electronics. By solving the non-linear time dependent Ginzburg-Landau (TDGL) equations, using the Link variable technique, we calculate the magnetization and the Coopers pair density for a disk in presence of an external applied magnetic field.

2. Theoretical Formalism
Superconducting state properties are described in the Ginzburg-Landay Theory by the order parameter \( \psi \), and the potential vector \( \mathbf{A} \). The TDGL equations in absence of external currents are given by:

\[
\begin{align*}
\frac{\partial \psi}{\partial t} &= -(i \nabla + \mathbf{A})^2 \psi + (1 - T)\psi(1 - |\psi|^2), \\
\frac{\partial \mathbf{A}}{\partial t} &= (1 - T)\text{Re} \left( \hat{\psi}(-i \nabla - \mathbf{A})\psi \right) - \kappa^2 \nabla \times \mathbf{h} ,
\end{align*}
\]

In the present scenario, \( T \) is temperature in critical temperature units \( T_C \), order parameter in \( \psi_\infty \) units, length in units of coherence length at zero temperature \( \xi(0) \), and fields in \( H_{c2}(0) \).
units. \( \kappa = \lambda(0)/\xi(0) \) is the material-dependent Ginzburg-Landau parameter (for more details, see Ref. [5, 7, 8, 9]). The \( z \)-dependence of the order parameter has been ignored. It is convenient to introduce auxiliary vector field \( \mathcal{U} = (U_\rho, U_\theta) \).

\[
U_\rho(\rho, \theta) = \exp \left( -i \int_{\rho_0}^{\rho} A_\rho(\rho, \xi) \, d\xi \right), \quad U_\theta(\rho, \theta) = \exp \left( -i \int_{\theta_0}^{\theta} A_\theta(\xi, \rho) \, d\xi \right),
\]

where \((\rho_0, \theta_0)\) is an arbitrary reference point. We use \( \mathbf{D} = -i \mathbf{\nabla} - \mathbf{A} \) to write:

\[
D_\rho \psi = -i \frac{\partial (U_\rho \psi)}{\partial \rho}, \quad D_\theta \psi = -i \frac{U_\theta}{\rho} \frac{\partial (U_\theta \psi)}{\partial \theta}.
\]

Upon using these two last equations recursively, we obtain

\[
D_\rho^2 \psi = -\frac{\partial^2 (U_\rho \psi)}{\partial \rho^2}, \quad D_\theta^2 \psi = -\frac{U_\theta}{\rho^2} \frac{\partial^2 (U_\theta \psi)}{\partial \theta^2}.
\]

As a consequence, we obtain for the kinetic term in the first TDGL equation

\[
\mathbf{D} \cdot \mathbf{D} \psi = D_\rho^2 \psi + D_\theta^2 \psi - \frac{i}{\rho} D_\rho \psi = -\frac{U_\rho}{\rho} \frac{\partial (U_\rho \psi)}{\partial \rho} + \frac{U_\theta}{\rho^2} \frac{\partial^2 (U_\theta \psi)}{\partial \theta^2} - (1 - T) \psi (1 - |\psi|^2).
\]

From equations (3), it can also be easily proved that

\[
\text{Re}\left[ \overline{\psi} D_\rho \psi \right] = \text{Im}\left[ \overline{U_\rho \psi} \frac{\partial (U_\rho \psi)}{\partial \rho} \right], \quad \text{Re}\left[ \overline{\psi} D_\theta \psi \right] = \text{Im}\left[ \frac{\overline{U_\theta \psi}}{\rho} \frac{\partial (U_\theta \psi)}{\partial \theta} \right],
\]

where \( \text{Im} \) indicates the imaginary part of a complex variable. Finally, using equations (5) and (6), the TDGL equations of (1) can be rewritten as

\[
\frac{\partial \psi}{\partial t} = \frac{U_\rho}{\rho} \frac{\partial (U_\rho \psi)}{\partial \rho} + \frac{U_\theta}{\rho^2} \frac{\partial^2 (U_\theta \psi)}{\partial \theta^2} + (1 - T) \psi (1 - |\psi|^2),
\]

\[
\beta \frac{\partial A_\rho}{\partial t} = (1 - T) \text{Im}\left[ \overline{U_\rho \psi} \frac{\partial (U_\rho \psi)}{\partial \rho} \right] - \kappa \frac{1}{\rho} \frac{\partial h_z}{\partial \theta},
\]

\[
\beta \frac{\partial A_\theta}{\partial t} = (1 - T) \text{Im}\left[ \frac{\overline{U_\theta \psi}}{\rho} \frac{\partial (U_\theta \psi)}{\partial \theta} \right] - \kappa \frac{\partial h_z}{\partial \rho}.
\]

### 3. Numerical Method

We will discretize the TDGL equations of (7) on a circular sector (for more details see Ref. [1]). Also, Let a superconducting disk domain be given by \( \{(x, y, z) \in R^3, |z| < dg(x, y)\} \), for all \((x, y)\). Here, \(g(x, y)\) is some function which describes the topology of the top surface of the disk and \(d\) is the thickness of the disk. We used the popular link variable formalism to avoid not only the divergence of the derivative of the order parameter at the interface, but also to obtain a discretization of the Ginzburg-Landau equations which preserves the gauge invariance of these equations (see Ref. [7] and [8] for more details). Let us define the following discrete variables.

\[
U_{\rho,i,j} = \overline{U_\rho(\rho_i, \theta_j)} U_\rho(\rho_{i+1}, \theta_j) = \exp (-ia_\rho A_{\rho,i,j}) ,
\]

\[
U_{\theta,i,j} = \overline{U_\theta(\rho_i, \theta_j)} U_\theta(\rho_i, \theta_{j+1}) = \exp (-ia_\theta A_{\theta,i,j}) ,
\]

where \(a_\rho, a_\theta\) are suitable complex numbers. These variables are functions of the order parameter and of its imaginary part. The form of these functions can be made on the basis of the properties of the internal behavior of the order parameter.
Magnetization curves as a function of external magnetic field for a disk with (and without) a linear defects on its surface.

for all \(1 \leq i \leq N_\rho, 1 \leq j \leq N_\theta + 1\), and \(1 \leq i \leq N_\rho + 1, 1 \leq j \leq N_\theta\), respectively. In what follows, it will be important to define the following discrete variable

\[
L_{i,j} = \exp \left( -i \oint_{\partial D} \mathbf{A} \cdot d\mathbf{r} \right) = \exp \left( -i \int_D h_z \rho d\rho d\theta \right) = \exp \left( -ia_{\rho} \rho_{i+1/2} a_{\theta} h_{z,i,j} \right), \tag{9}
\]

for all \(1 \leq i \leq N_\rho, 1 \leq j \leq N_\theta\), where \(D\) is the domain of a unit cell limited by a closed path \(\partial D\). Stoke’s theorem and the midpoint rule for numerical integration have been used. A simple inspection of equation (9) leads to

\[
L_{i,j} = U_{\rho,i,j} U_{\theta,i+1,j} \bar{U}_{\rho,i,j+1} \bar{U}_{\theta,i,j}.
\]

The boundary conditions were as follows. Let \(\mathbf{n}\) be a unit vector normal to the interface. It will be assumed that normal current density not vanishes at the interface (i.e. \(-i\nabla \times \mathbf{A})\psi \cdot \mathbf{n} = 0\).

4. Results and Discussion

The parameters used in our numerical simulations were: \(\kappa = 2.17\), which is a value for a thin film of Nb with thickness \(d\) (assuming \(\xi(0) = 380\) nm, \(T_c = 3.7K\), \(d \approx 60\) nm, \(T = 0\), the internal and external radius of the disk were \(r_i = 0.3\xi(0)\) and \(r_e = 6.5\xi(0)\). Largest unit cell length was taken as being no larger than \(0.2 \times 0.2\), the applied magnetic field was ramped in steps of \(\Delta H = 10^{-4}\). \(g(x, y) = 0.8\) for \(r_i < r < r_e\) and \(\theta = 0\). Figures 1 show the magnetization as a function of increasing and decreasing applied magnetic field for a superconducting disk with (and without) a lineal defect. From these figures, we can see that the both the surface barrier field and the third critical thermodynamic fields are independent of the nature of defects. In the Figure 2 we shown the contour plot of the order parameter for several vorticities. Following the panels from the left to the right, in this order, we can see initially the sample with the defect at zero magnetic field, later three vortices nucleate asymmetrically, one of them at the defect position. Increasing the magnetic field (Figure 3 (up)) four vortices are trying to nucleated, they will form an unusual vortex shape for a circle geometry, it is, the geometry of the defects is imposing the vortex configuration. However, for higher magnetic fields (Figure 3 (down)), we observe a chain of twelve vortices with equal distance from the center but with a space in the equator and poles of the disk due to the defect.

References

[1] E. Sardella and E. H. Brandt, Supercond. Sci. Technol. 23, 025015 (2010).
Figure 2. (Color online) Contour plot of the order parameter for a disk with radius $R = 6.5\xi(T)$ and one defect in $r_i < r < r_e$ and $\theta = 0$. Dark and bright regions represent values of the modulus of the order parameter from 0 to 1.

Figure 3. (Color online) Contour plot of the order parameter for a disk with radius $R = 6.5\xi(T)$ and one defect in $r_i < r < r_e$ and $\theta = 0$. Dark and bright regions represent values of the modulus of the order parameter from 0 to 1.

[2] J. Barba-Ortega, Miryam R. Joya, J. Albino Aguiar, J. Sup. Nov. Magn, In press (2012).
[3] J. Barba-Ortega, Edson Sardella, J. Albino Aguiar, Supercond. Sci. Technol.24, 015001 (2011).
[4] F. M Peeters, V.A Schweigert, B.J Baelus, P.S Deo, Physica C 332, 255 (2000).
[5] J.R. Clem, in: K.D. Timmerhaus, W.J. O Sullivan, E.F. Hammel (Eds.), Low Temperature Physics, vol. 3, Plenum, New York, 1974, p. 102.
[6] J. Barba-Ortega, Edson Sardella, J. Albino Aguiar, E. H. Brandt, Physica C, 479, 49 (2012).
[7] W. D. Gropp, H. G. Kaper, G. K. Leaf, D. M. Levine, M. Palumbo, and V. M. Vinokur, J. Comput. Phys. 123, 254 (1996).
[8] J. Berger and J. Rubinstein, Phys. Rev. B. 59, 8896 (1999).
[9] G. C. Buscaglia, C. Bolech and A. López, Connectivity and Superconductivity, J. Berger, and J. Rubinstein (Eds.), Springer, (2000).
[10] V. V. Moshchalkov et al. in Connectivity and Superconductivity, edit by J. Berger and J. Rubinstein (Springer, Heidelberg, 2000).
[11] V. A. Schweigert and F. M. Peeters, Phys. Rev. B 60, 3084 (1999).
[12] A. K. Geim, S. V. Dubonos, I. V. Grigorieva, K. S. Novoselov, F. M. Peeters, and V. A. Schweigert, Nature 407, 55 (2000).