COUPLED KDV EQUATIONS DERIVED FROM ATMOSPHERICAL DYNAMICS

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ABSTRACT. Some types of coupled Korteweg de-Vries (KdV) equations are derived from an atmospheric dynamical system. In the derivation procedure, an unreasonable \(y\)-average trick (which is usually adopted in literature) is removed. The derived models are classified via Painlevé test. Three types of \(\tau\)-function solutions and multiple soliton solutions of the models are explicitly given by means of the exact solutions of the usual KdV equation. It is also interesting that for a non-Painlevé integrable coupled KdV system there may be multiple soliton solutions.

1. Introduction

The single component KdV equation has been widely used in various natural science fields especially in almost all branches of physics. For instance \(^1\), the KdV equation describes, in a general form, competition between weak nonlinearity and weak dispersion, while the nonlinear Schrödinger (NLS) equation describes the same competition for envelope waves (see, for example, the Introduction in \(^2\)). Some other integrable equations such as the sine-Gordon (SG) equation, the Kadomtsev-Petviashvily equation, the so-called three- and four-wave systems are universal as well.

Some kinds of the coupled KdV equations had also been introduced in literature such as one describing two resonantly interacting normal modes of internal-gravity-wave motion in a shallow stratified liquid \(^3\). In principle, many of other coupled KdV equations are introduced mathematically because of their integrability \(^4\).
In section II of this paper, we derive some new types of coupled KdV equation systems with some arbitrary parameters from a two-layer fluid model which is used to describe the atmospheric phenomena by using long wave approximation.

Whence the coupled KdV systems are derived, an important problem is how to solve them. To get more exact solutions, one hopes to figure out the integrable ones. In section III of the paper, we use the well known Painlevé test classification to find out the Painlevé integrable ones for special types of selections of the parameters.

For some special types of coupled KdV systems, one can find some types of exact solutions by modifying the solutions of the usual KdV equation. Some concrete solution examples, especially, the \( \tau \)-function solutions and the multiple soliton solutions are given in section V. The last section contains a short summary and discussion.

2. Coupled KdV equations derived from atmospheric dynamics

It is known that various integrable models can be derived from fluid dynamics. Similarly, in this section we use a two layer fluid model,

\[
q_{1t} + J\{\psi_1, q_1\} + \beta \psi_{1x} = 0, \quad (1)
\]
\[
q_{2t} + J\{\psi_2, q_2\} + \beta \psi_{2x} = 0, \quad (2)
\]

where

\[
q_1 = \psi_{1xx} + \psi_{1yy} + F(\psi_2 - \psi_1), \quad (3)
\]
\[
q_2 = \psi_{2xx} + \psi_{2yy} + F(\psi_1 - \psi_2), \quad (4)
\]

and \( J\{a, b\} \equiv a_x b_y - b_x a_y \), as a starting point to derive two component KdV equations by using the multiple scale approach and the long wave approximation.

In (1)–(2), \( F \) is the weak coupling constant among two layer fluids and \( \beta = \beta_0(L^2/U) \), \( \beta_0 = (2\omega_0/a_0) \cos \phi_0 \), in which \( a_0 \) is the earth’s radius, \( \omega_0 \) is the angular frequency of the earth’s rotation and \( \phi_0 \) is the latitude, \( U \) is the characteristic velocity scales. The derivation of the dimensionless equation (1) and (2) is based on the characteristic horizontal length scale \( L = 10^6 m \) and the characteristic horizontal velocity scale \( U = 10^{-1} m/s \).
More especially, if we take $\beta = 0$, the system (3) and (4) is reduced to the usual coupled Euler equation which is suitable to be used to describe two-layer inviscid fluid. In other words, all the results obtained in this paper are valid for general two-layer inviscid fluid.

Under the long wave approximation in the $x$ direction, in order to derive the KdV type equations, the stream functions $\psi_1$ and $\psi_2$ should have the form

\[
\psi_i = \phi_{i0}(y) + \phi_i(\epsilon(x - c_0 t), y, \epsilon^3 t) \equiv \phi_{i0} + \phi_i(X, y, T) \equiv \phi_{i0} + \phi_i, \quad i = 1, 2, \quad (5)
\]

where $\epsilon$ is a small parameter. It is reasonably considered that the parameters $F$ and $\beta$ are in order $\epsilon$ and $\epsilon^2$ respectively that means the coupling among two layers is week and the effect of the rotation of the earth is much smaller. Thus we set

\[
F = F_0 \epsilon, \quad \beta = \beta_0 \epsilon^2. \quad (6)
\]

Now, we expand the shift stream functions $\phi_i$, $i = 1, 2$ as

\[
\phi_1 = \epsilon \phi_{11}(X, y, T) + \epsilon^2 \phi_{12}(X, y, T) + \epsilon^3 \phi_{13}(X, y, T) + O(\epsilon^4), \quad (7)
\]

\[
\phi_2 = \epsilon \phi_{21}(X, y, T) + \epsilon^2 \phi_{22}(X, y, T) + \epsilon^3 \phi_{23}(X, y, T) + O(\epsilon^4). \quad (8)
\]

Substituting (5)–(8) to (1)–(2) yields

\[
[(\phi_{10y} - c)\partial_{yy} - \phi_{10yyy}] \phi_{11x} \epsilon^2 + \{[(\phi_{10y} - c)\partial_{yy} - \phi_{10yyy}] \phi_{12x} + F_0(\phi_{10y} - c) \phi_{21x} + F_0(c_0 - \phi_{20y}) + \phi_{11y} \phi_{11yy}\phi_{11x} - \phi_{11y} \phi_{11yy} \phi_{13} - \phi_{12y} \phi_{11yy} x - \phi_{11y} \phi_{21yy} x) + \phi_{20y} - c_0)(F_0 \phi_{12} + \phi_{21xx}) X + \phi_{11yy} T - F_0 \phi_{21x} \phi_{11y} + \phi_{12yy} + F_0 \phi_{21y} + \beta_0] \phi_{11x} + [F_0(c_0 - \phi_{20y}) + \phi_{11yy}] \phi_{12x} \} \epsilon^4 + O(\epsilon^5) = 0, \quad (9)
\]

and

\[
[(\phi_{20y} - c)\partial_{yy} - v_{0yy}] \phi_{21x} \epsilon^2 + \{[(\phi_{20y} - c)\partial_{yy} - \phi_{20yyy}] \phi_{22x} + F_0(\phi_{20y} - c) \phi_{11x} + F_0(c_0 - \phi_{10y}) + \phi_{21y} \phi_{21yy} \phi_{21x} - \phi_{21y} \phi_{21yy} \phi_{13} - \phi_{22y} \phi_{21yy} x - \phi_{21y} \phi_{22yy} x) + \phi_{20y} - c_0)(F_0 \phi_{12} + \phi_{21xx}) X + \phi_{21yy} T - F_0 \phi_{21x} \phi_{21y} + \phi_{22yy} + F_0 \phi_{21y} + \beta_0] \phi_{21x} + [F_0(c_0 - \phi_{10y}) + \phi_{21yy}] \phi_{22x} \} \epsilon^4 + O(\epsilon^5) = 0. \quad (10)
\]
Vanishing the $\epsilon^2$ terms of (9) and (10), we have a special solution
\[ \phi_{11} = A_1(X, T)B_1(y) \equiv A_1B_1, \quad (11) \]
\[ \phi_{21} = A_2(X, T)B_2(y) \equiv A_2B_2, \quad (12) \]
with $B_1$ and $B_2$ linked to $\phi_{10}$ and $\phi_{20}$ by
\[ U_{0yy}B_1 - B_{1y}\phi_{10y} + C_1 = 0, \quad \phi_{10} = U_0 + c_0y, \quad (13) \]
and
\[ V_{0yy}B_2 - B_{2y}\phi_{20y} + C_2 = 0, \quad \phi_{20} = V_0 + c_0y, \quad (14) \]
respectively with arbitrary constants $C_1$ and $C_2$.

By using the relations (11)–(14), the equations obtained by vanishing the $\epsilon^3$ orders of (9) and (10) and integrating once with respect to $X$ become
\[ 2\phi_{10y}(B_1\partial_{yy} - B_{1yy})\phi_{12} + B_1 [b_{11}A_1^2 - 2F_0(B_1\phi_{20y}A_1 + B_2\phi_{10y}A_2)] = 0, \quad (15) \]
\[ 2\phi_{20y}(B_2\partial_{yy} - B_{2yy})\phi_{22} + B_2 [b_{21}A_2^2 - 2F_0(B_2\phi_{10y}A_2 - B_1\phi_{20y}A_1)] = 0, \quad (16) \]
where the integrating functions have been dropped away and
\[ b_{11} \equiv B_1B_{1yy} - B_{1y}B_{1yy}, \quad b_{21} \equiv B_2B_{2yy} - B_{2y}B_{2yy}. \quad (17) \]

It is readily verified that
\[ \phi_{12} = (B_3A_1^2 + B_4A_2 + B_6A_1)B_1, \quad \phi_{22} = (B_3A_2^2 + B_6A_1 + B_7A_2)B_2, \quad (18) \]
with $B_0$, $B_3$, $B_4$, $B_5$, $B_6$ and $B_7$ being functions of $y$ and determined by
\[ B_{3y} = \frac{b_3}{B_1^2}, \quad b_{3y} = -\frac{B_1b_{11}}{f_1}, \quad B_{5y} = \frac{b_5}{B_2^2}, \quad b_{5y} = -\frac{B_2b_{21}}{g_1}, \quad (19) \]
\[ B_{4y} = \frac{b_4}{B_1^2}, \quad b_{4y} = -F_0B_2B_1, \quad B_{6y} = \frac{b_6}{B_2^2}, \quad b_{6y} = -F_0B_2B_1, \quad (20) \]
\[ B_{0y} = \frac{b_0}{B_1^2}, \quad b_{0y} = F_0B_1^2\frac{g_1}{f_1}, \quad B_{7y} = \frac{b_7}{B_2^2}, \quad b_{7y} = F_0B_2^2\frac{f_1}{g_1}, \quad (21) \]
\[ f_1 = U_{0y}, \quad g_1 = V_{0y}, \quad (22) \]
solves the third order equations (15) and (16).
Because of (11), (12) and (13), the fourth order of (9) and (10) become

\begin{align*}
f_1 \left( \partial_{yy} - B_1^{-1} B_{1yy} \right) \phi_{13X} + B_{1yy} A_{1XT} + f_1 B_1 A_{1XXX} + F_0 (g_1 B_1 B_0 - f_1 B_2 B_7) A_{2X} \\
+ 2 f_1 F_0 B_2 B_3 A_2 A_{2X} - (\beta_0 B_1 - F_0 g_1 B_0 B_1 + F_0 f_1 B_2 B_6) A_{1X} + B_4 b_{11} (A_1 A_2) X \\
+ \left[ 2 b_{11} B_0 - 2 F_0 g_1 B_1 B_3 + \frac{F_0 g_1 B_1}{f_1 B_2} \left( \frac{c_1 B_2}{f_1} - \frac{d_1 B_1}{g_1} - B_2 B_{1y} + B_1 B_{2y} \right) \right] A_{1A_{1X}} \\
+ \frac{1}{2 f_1^2} \left[ b_{11} (6 B_3 f_1^2 + 3 f_1 B_{1y} - c_1) - f_1 B_1 b_{11y} \right] A_1^2 A_{1X} = 0, \tag{23}
\end{align*}

and

\begin{align*}
g_1 \left( \partial_{yy} - B_2^{-1} B_{2yy} \right) \phi_{23X} + B_{2yy} A_{2XT} + g_1 B_2 A_{2XXX} + F_0 (g_1 B_1 B_0 - f_1 B_2 B_6) A_{1X} \\
+ 2 g_1 F_0 B_1 B_3 A_1 A_{1X} - (\beta_0 B_2 + F_0 f_1 B_7 B_2 - F_0 g_1 B_1 B_4) A_{2X} + B_6 b_{21} (A_1 A_2) X \\
+ \left[ 2 b_{21} B_7 - 2 F_0 f_1 B_2 B_5 + \frac{F_0 f_1 B_2}{g_1 B_1} \left( \frac{d_1 B_1}{g_1} - \frac{c_1 B_2}{f_1} - B_1 B_{2y} + B_2 B_{1y} \right) \right] A_{2A_{2X}} \\
+ \frac{1}{2 g_1^2} \left[ b_{21} (6 B_5 g_1^2 + 3 g_1 B_{2y} - d_1) - g_1 B_2 b_{21y} \right] A_2^2 A_{2X} = 0. \tag{24}
\end{align*}

In the usual studies to solve (23) and (24) type equations, especially in the atmospheric and ocean dynamics, one would take \( \phi_{13} \) and \( \phi_{23} \) as zero. However, whence \( \phi_{13} \) and \( \phi_{23} \) are taken as zero, there may be a non-consistence problem because \( A_1 \) and \( A_2 \) are only the functions of \( X \) and \( T \) while the coefficients of (23) and (24) are explicitly \( y \)-dependent. In generally, equations (23) and (24) are not consistent except that all the \( y \)-dependent coefficients are proportional each other up to constant level. The detailed analysis of the equations (23) and (24) with \( \phi_{13} = \phi_{23} = 0 \) tells that it is impossible to select the functions \( B_0, B_1, ..., B_7, U_0 \) and \( V_0 \) which are proportional each other and satisfy the equations (13), (14), (19) – (21) at the same time. To avoid this kind of inconsistency in the traditional literature, an unreasonable and unclear procedure is usually made, taking a \( y \) average by integrating the inconsistent equations with respect to the variable \( y \) from \( y_1 \) to \( y_2 \).

Nevertheless, it is possible to get some consistent and significant solutions from (23) and (24) by taking nonzero \( \phi_{13} \) and \( \phi_{23} \). In this paper, we only give out a possible selection of \( \phi_{13} \) and \( \phi_{23} \) to derive coupled KdV type equations for the quantities \( A_1 \) and \( A_2 \).
It is straightforward to verify that if we take

$$
\phi_{13} = r_1 \int A_{1X} A_2 dX + r_2 A_1^3 + r_3 A_1^2 + r_4 A_1 + r_5 A_1 A_2 + r_6 A_2^2 + r_7 A_2 + r_8 A_{1XX}, \quad (25)
$$

$$
\phi_{23} = s_1 \int A_{1X} A_2 dX + s_2 A_2^3 + s_3 A_2^2 + s_4 A_2 + s_5 A_1 A_2 + s_6 A_1^2 + s_7 A_1 + s_8 A_{2XX}, \quad (26)
$$

with

$$
r_i = B_1 \int^{y} \frac{1}{B_1(y')^2} \int^{y''} R_i(y') dy' dy'',
$$

$$
s_i = B_2 \int^{y} \frac{1}{B_2(y')^2} \int^{y''} S_i(y') dy' dy'', \quad i = 1, 2, ..., 8,
$$

$$
R_1 = \frac{-\alpha_1 B_1 B_{1yy}}{f_1}, \quad R_2 = \frac{B_1}{6 f_1^3} [B_1 f_1 b_{11y} + b_{11}(c_1 - 3 f_1 B_{1y} - 6 B_3 f_1^3)],
$$

$$
R_3 = \frac{B_2^2 F_0 g_1}{2 f_1} \left( 2 B_3 + \frac{B_2 B_{1y} - B_2 B_{2y}}{B_2 f_1} - \frac{c_1}{f_1^2} \right) - \frac{B_1}{f_1} (\alpha_5 B_{1yy} + b_{11} B_0) + \frac{F_0 d_1 B_1^3}{f_1^2 B_2},
$$

$$
R_4 = -\frac{B_1}{f_1} (\beta_0 B_1 - F_0 B_0 B_1 g_1 + F_0 f_1 B_2 B_6), \quad R_5 = -\frac{B_1}{f_1} (\alpha_3 B_{1yy} + b_{11} B_4),
$$

$$
R_6 = \frac{B_1}{f_1} [(\alpha_2 - \alpha_5) B_{1yy} + F_0 f_1 B_2 B_5], \quad R_7 = \frac{F_0 B_1}{f_1} (g_1 B_1 B_4 - f_1 B_2 B_7),
$$

$$
R_8 = -\frac{B_1}{f_1} (\alpha_4 B_{1yy} + f_1 B_1), \quad S_8 = -\frac{B_2}{g_1} (\delta_4 B_{2yy} + g_1 B_2),
$$

$$
S_1 = \frac{\delta_1 B_2 B_{2yy}}{g_1}, \quad S_2 = \frac{B_2}{6 g_1^3} [B_2 g_1 b_{21y} + b_{21}(d_1 - 3 g_1 B_{2y} - 6 B_5 g_1^3)],
$$

$$
S_3 = \frac{B_2^2 F_0 f_1}{2 g_1} \left( 2 B_5 + \frac{B_1 B_{2y} - B_2 B_{1y}}{g_1 B_1} - \frac{d_1}{g_1^2} \right) + \frac{c_1 F_0 B_2^3}{2 g_1^2 B_2} + \frac{B_2}{g_1} (\delta_5 B_{2yy} - b_{21} B_7),
$$

$$
S_4 = -\frac{B_2}{g_1} (\beta_0 B_2 - F_0 B_7 B_2 f_1 + F_0 g_1 B_1 B_4), \quad S_5 = \frac{B_2}{g_1} (\delta_3 B_{2yy} - b_{21} B_6),
$$

$$
S_6 = \frac{B_2}{g_1} [(\delta_2 - \delta_5) B_{1yy} - F_0 g_1 B_1 B_3], \quad S_7 = \frac{F_0 B_2}{g_1} (g_1 B_1 B_0 - f_1 B_2 B_6)
$$

for arbitrary $B_1$ and $B_2$, then $A_1$ and $A_2$ satisfy the following coupled KdV system

$$
A_{1T} + \alpha_1 A_2 A_{1X} + (\alpha_2 A_2^2 + \alpha_3 A_1 A_2 + \alpha_4 A_{1XX} + \alpha_5 A_1^2) X = 0, \quad (27)
$$

$$
A_{2T} + \delta_1 A_2 A_{1X} + (\delta_2 A_2^2 + \delta_3 A_1 A_2 + \delta_4 A_{2XX} + \delta_5 A_2^2) X = 0, \quad (28)
$$

where ten constants \(\{\alpha_i, \delta_i, i = 1, 2, 3, 4, 5\}\) are arbitrary.

Now the important question is how to get some types of exact solutions of the coupled KdV system. Before giving out some concrete solutions, we try to make a Painlevé
classification at first. That means we are going to give some constraints on the parameters \(\{\alpha_i, \delta_i, i = 1, 2, 3, 4, 5\}\) such that the solutions of the model are single valued with respect to an arbitrary singular manifold.

3. Painlevé classification of the coupled KdV system

The Painlevé test is one of the best way to study nonlinear systems. In this section, we take a standard Painlevé test by using the Kruskal’s simplification for the coupled KdV system.

To pass the Painlevé test, four steps are required. The leading order analysis, the resonances determination, the test of the primary branch and the test of the secondary branches.

The leading order analysis for the coupled KdV system (27) and (28) around the arbitrary manifold \(\phi\) shows us that there are two possible cases:

Case 1.

\begin{align*}
A_1 \sim \frac{u_0}{\phi^2}, & \quad A_2 \sim \frac{v_0}{\phi^2}.
\end{align*}

In this case the parameters \(\{\alpha_i, \delta_i\}\) and \(\{u_0, v_0\}\) are linked by

\begin{align*}
2\alpha_5u_0^2 + 2\alpha_2v_0^2 + (2\alpha_3 + \alpha_1)u_0v_0 + 12\alpha_4u_0 = 0, \\
2\delta_5v_0^2 + 2\delta_2u_0^2 + (2\delta_3 + \delta_1)u_0v_0 + 12\delta_4v_0 = 0.
\end{align*}

Case 2.

\begin{align*}
A_1 \sim \frac{u_0}{\phi^2}, & \quad A_2 \sim \frac{v_0}{\phi^2}
\end{align*}

or equivalently

\begin{align*}
A_1 \sim \frac{u_0}{\phi}, & \quad A_2 \sim \frac{v_0}{\phi^2}
\end{align*}

which will be not considered since the exchange symmetry \(\{A_1, A_2, \alpha_i, \delta_i\} \leftrightarrow \{A_2, A_1, \delta_i, \alpha_i\}\) for the coupled KdV system (27) and (28).

The case (31) appears only for

\begin{align*}
\delta_2 = 0, & \quad \delta_4 = \frac{\alpha_4}{\alpha_5}(2\delta_1 + 3\delta_3), & \quad u_0 = -6\frac{\alpha_4}{\alpha_5}.
\end{align*}
The resonance analysis for the first case shows us that the resonant points are located at

\[-1, 4, 6, j_1, j_2, j_3 = 9 - j_1 - j_2, \quad (33)\]

where \(j_1, j_2\) and \(j_3\) are three roots of

\[
u_0 v_0 \delta_4 \alpha_4 (j - 9)^2 + [(14 \delta_4 v_0 u_0 - u_0^2 (v_0 \delta_1 + \delta_3 v_0 + 2 \delta_2 u_0)) \alpha_4 - v_0^2 \delta_4 (2 \alpha_2 v_0 + u_0 \alpha_3)] j
\]

\[+(24 \delta_4 v_0 u_0 + 2 u_0^2 (4 \delta_2 v_0 + v_0 \delta_1 + 2 \delta_3 v_0)) \alpha_4 + 2 v_0^2 \delta_4 (u_0 \alpha_1 + 2 u_0 \alpha_3 + 4 \alpha_2 v_0) = 0 \quad (34)\]

for the variable \(j\). Apart from the equivalent decoupled case for both \(A_1\) and \(A_2\) satisfy completely decoupled KdV equations, the positive integer conditions for the resonant points lead to the following only nonequivalent subcases (i) \(j_1 = j_2 = 0, j_3 = 9\), (ii) \(j_1 = 0, j_2 = 1, j_3 = 8\), (iii) \(j_1 = 0, j_2 = 2, j_3 = 7\), (iv) \(j_1 = 0, j_2 = 3, j_3 = 6\), (v) \(j_1 = 0, j_2 = 4, j_3 = 5\), (vi) \(j_1 = j_2 = 1, j_3 = 7\), (vii) \(j_1 = 1, j_2 = 2, j_3 = 6\), (viii) \(j_1 = 1, j_2 = 3, j_3 = 5\), (ix) \(j_1 = j_2 = 2, j_3 = 5\), (x) \(j_1 = 2, j_2 = 3, j_3 = 4\).

The resonance analysis for the second case gives that the resonances will appear at

\[-1, 0, 4, 6, j_1, j_2 = 6 - j_1, \quad (35)\]

where \(j_1\) and \(j_2\) are two solutions of

\[j (2 \delta_1 + 3 \delta_3) (j - 6) + 27 \delta_3 + 22 \delta_1 = 0 \quad (36)\]

for the variable \(j\). It is clear that the positive integer conditions for the resonance points lead to the following four nonequivalent subcases (a) \(j_1 = 0, j_2 = 6\), (b) \(j_1 = 1, j_2 = 5\), (c) \(j_1 = 2, j_2 = 4\), and (d) \(j_1 = 3, j_2 = 3\).

To check all the resonances for the subcases (i)–(x) and (a)–(d) yields the possible Painlevé integrable models under some conditions for the model parameters \(\alpha_i\) and \(\delta_i\). Here we just list the final nonequivalent models but omit the detail and tedious analysis because the procedures are standard. The result shows us that there are only six Painlevé integrable subcases of the coupled KdV system and [27] and [28].
P-integrable model 1.

\[ A_{1T} + \left( A_{1XX} + \frac{1}{2}(c_2 - c_1 - c_1 c_2)A_1^2 + c_1 A_1 A_2 - \frac{1}{2} A_2^2 \right)_x = 0, \]
\[ A_{2T} + \left( A_{2XX} + \frac{1}{2}(c_1 - c_2 - 1)A_2^2 + c_2 A_1 A_2 - \frac{1}{2} c_1 c_2 A_1^2 \right)_x = 0, \] (37)

where \(c_1\) and \(c_2\) are arbitrary constants. For this type of coupled KdV system (37), there are three branches with the resonances \(-1, 2, 3, 4, 6\), \(-1, 2, 3, 4, 6\) and \(-1, -1, 4, 4, 6, 6\) respectively and all the resonance conditions satisfied.

P-integrable model 2.

\[ A_{1T} + [A_{1XX} - (c + 3)(c + 6)A_1^2 - c^2 A_2^2]_x + 2c[(c + 6)A_1 A_2 + (c + 3)A_1 A_2X] = 0, \]
\[ A_{2T} + [A_{2XX} - c(c - 3)A_2^2 - (c + 3)^2 A_1^2]_x + 2(c + 3)[c A_2 A_1 X + (c - 3)A_1 A_2X] = 0, \] (38)

where \(c\) is an arbitrary constant. For the model system (38), there is only one branch with the resonances located at \(-1, 1, 2, 4, 6, 6\) and all the resonance conditions satisfied identically.

P-integrable model 3.

\[ A_{1T} + (A_{1XX} + A_1^2 + A_1 A_2)_x = 0, \]
\[ A_{2T} + (A_{2XX} + A_2^2 + A_1 A_2)_x = 0. \] (39)

In this case, the resonance points are \(-1, 0, 4, 4, 5, 6\).

P-integrable model 4.

\[ A_{1T} + [A_{1XX} + (A_1 + A_2)^2]_x = 0, \]
\[ A_{2T} + [A_{2XX} + (A_1 + A_2)^2]_x = 0. \] (40)

This case is corresponding to the resonances are located at \(-1, 2, 3, 4, 4, 6\).

P-integrable model 5.

\[ A_{1T} + (A_{1XX} + A_1^2)_x + 2A_2 A_1 X = 0, \]
\[ A_{2T} + (A_{2XX} + A_2^2)_x + 2A_1 A_2 X = 0. \] (41)
This case is corresponding to the resonances are located at \{-1, 0, 2, 4, 6, 7\}.

**P-integrable model 6.**

\[
A_{1T} + A_{1XXX} + (A_1 + A_2)(3A_1 + A_2)_X = 0,
\]

\[
A_{2T} + A_{2XXX} + (A_1 + A_2)(3A_2 + A_1)_X = 0. \tag{42}
\]

This case is corresponding to the resonances are located at \{-1, 1, 2, 4, 6, 6\}.

### 4. Exact solutions

In this section, we study some types of exact solutions for the general couple KdV system (27) and (28) and some special types of P-integrable models.

#### 4.1. Travelling periodic and solitary wave solutions of the general coupled KdV system (27) and (28)

In [6], it is pointed out that some special types of exact solutions including travelling wave solutions of various nonlinear systems can be obtained via the deformation and mapping approach from the solutions of the cubic nonlinear Klein-Gordon equation (or namely \(\phi^4\) model). It is quite easy to see that for some types of travelling wave solutions of the coupled KdV system (27) and (28) can also be obtained by some suitable deformation relations from the travelling wave solution of the \(\phi^4\) model.

For the travelling wave solution of the coupled KdV system (27) and (28),

\[
A_1 = A_1(\xi) \equiv A_1(kX - kcT), \quad A_2 = A_2(\xi), \tag{43}
\]

we have

\[
\alpha_1 A_{1\xi} A_2 + (\alpha_2 A_2^2 + \alpha_3 A_1 A_2 + \alpha_4 k^2 A_{1\xi} + \alpha_5 A_1^2 - cA_1)\xi = 0, \tag{44}
\]

\[
\delta_1 A_{1\xi} A_2 + (\delta_2 A_2^2 + \delta_3 A_1 A_2 + \delta_4 k^2 A_{2\xi} + \delta_5 A_2^2 - cA_2)\xi = 0. \tag{45}
\]

To map the travelling waves of the cubic nonlinear Klein-Gordon equation to those of the coupled KdV system, one may use different mapping relations such as the polynomial forms [6], rational forms [7] and may be more complicated and/or derive dependent forms [8]. However, here we just give a simple polynomial deformation relation

\[
A_1 = a_0 + a_1 \Phi(\xi) + a_2 \Phi(\xi)^2, \quad A_2 = b_0 + b_1 \Phi(\xi) + b_2 \Phi(\xi)^2, \tag{46}
\]
where $\phi(\xi)$ is a travelling wave solution of the cubic nonlinear Klein-Gordon equation, i.e., $\phi$ satisfies

$$\phi^2 = \mu \phi^2 + \frac{1}{2} \lambda \phi^4 + C. \quad (47)$$

It is not very difficult to find that $\{A_1, A_2\}$ given by (46) with (47) is a solution of the coupled KdV system (27) and (28) if and only if the eleven solution parameters $a_0, a_1, a_2, b_0, b_1, b, \mu, \lambda, C, k, c$ and ten model parameters $\alpha_i, \delta_i, i = 1, 2, ..., 8$ satisfy the following eight constraints

$$
(2a_1 + 3a_3 + 6a_2 b)a_2 b_1 + [(3a_3 b + 6a_5 + a_1 b)a_2 + 3k^2 \alpha_4 \lambda]a_1 = 0,
$$
$$
(2a_3 b + 4a_5) a_2 a_0 + 2a_2 b_1^2 + (a_1 + 2a_3) a_1 b_1 + 2a_5 a_1^2
$$
$$
+ [8k^2 \alpha_4 \mu - 2c + (2a_3 + 4a_2 b + 2a_1) b_0] a_2 = 0,
$$
$$
(2a_5 a_1 + b_1 a_3) a_0 + 2a_2 b_0 b_1 + [k^2 \alpha_4 \mu - c + (a_1 + a_3) b_0] a_1 = 0,
$$
$$
(b_1 \delta_3 + 2\delta_2 a_1) a_0 + (k^2 \delta_4 \mu + 2b_0 \delta_5 - c) b_1 + (\delta_1 + \delta_0) b_0 a_1 = 0,
$$
$$
(4\delta_5 b^2 + 2\delta_1 b + 4\delta_2 + 4\delta_3 b)a_2 + 12k^2 \delta_4 b \lambda = 0,
$$
$$
(4\delta_2 + 2 \delta_3) a_2 a_0 + 2\delta_3 b_1^2 + (2\delta_3 + \delta_1) a_1 b_1 + 2\delta_2 a_1^2 + [8k^2 \delta_4 b \mu
$$
$$
- 2cb + (4\delta_5 b + 2\delta_1 + 2\delta_3) b_0] a_2 = 0, \quad (48)
$$
$$
(4a_3 b + 2a_1 b + 4a_2 b^2 + 4a_5) a_2 + 12k^2 \alpha_4 \lambda = 0,
$$
$$
[(6\delta_5 b + 2\delta_1 + 3\delta_3) a_2 + 3k^2 \delta_4 \lambda] b_1 + (3\delta_3 b + \delta_1 b + 6\delta_2) a_2 a_1 = 0. \quad (49)
$$

It is obvious that the algebraic equation system (48) possesses many kinds of solutions. Here we just write down a most important and simplest solution when $\delta_4 = \alpha_4$:

$$a_0 = a_1 = b_0 = b_1 = 0, \quad c = 4k^2 \mu \alpha_4, \quad a_2 = -\frac{6k^2 \lambda \alpha_4}{2 \alpha_5 + 2b \alpha_3 + b \alpha_1 + 2b^2 \alpha_2}, \quad (50)$$

while $b$ is linked the model parameters by a cubic algebraic equation

$$\delta_2 + \left(\delta_3 - \alpha_5 + \frac{1}{2} \delta_1\right) b + \left(\delta_5 - \alpha_3 - \frac{1}{2} \alpha_1\right) b - \alpha_2 b^3 = 0. \quad (51)$$

More concretely, if we take $\phi(\xi)$ as the Jacobi elliptic conoid function

$$\phi = cn(\xi, m)$$
which is a special solution of the $\phi^4$ model with the parameters

$$\mu = 2m^2 - 1, \ \lambda = -2m^2, \ C = 1 - m^2;$$

then we have a simple periodic wave solution for the coupled KdV equation (27) and (28) with $\delta_4 = \alpha_4$

$$A_1 = \frac{12k^2m^2\alpha_4}{2\alpha_5 + 2b\alpha_3 + b\alpha_1 + 2b^2\alpha_2} \text{cn}^2 \left( kX - 4k^3(2m^2 - 1)\alpha_4 T, \ m \right),$$

$$A_2 = \frac{12k^2m^2\alpha_4 b}{2\alpha_5 + 2b\alpha_3 + b\alpha_1 + 2b^2\alpha_2} \text{cn}^2 \left( kX - 4k^3(2m^2 - 1)\alpha_4 T, \ m \right),$$

where $b$ is a solution of (51). Furthermore, when $m = 1$, the periodic solution (52) becomes a simple solitary wave solution

$$A_1 = \frac{12k^2\alpha_4}{2\alpha_5 + 2b\alpha_3 + b\alpha_1 + 2b^2\alpha_2} \text{sech}^2 \left( kX - 4k^3\alpha_4 T \right),$$

$$A_2 = \frac{12k^2\alpha_4 b}{2\alpha_5 + 2b\alpha_3 + b\alpha_1 + 2b^2\alpha_2} \text{sech}^2 \left( kX - 4k^3\alpha_4 T \right).$$

4.2. $\tau$-function solutions and Multi-soliton solutions of the coupled KdV system.

4.2.1. The first type of $\tau$-function and multi-soliton solutions related to the KdV reductions. It is straightforward to verify that for the coupled KdV equation system (27) and (28) with

$$\delta_4 = \alpha_4,$$

one can find at least one type of multiple soliton solutions because there is a simple KdV reduction

$$A_{1T} + \alpha_4 A_{1XX} + (a\alpha_1 + 2\alpha_2 a^2 + 2a\alpha_3 + 2a\alpha_5) A_1A_{1X} = 0, \ A_2 = aA_1,$$

where $a$ is a solution of the algebraic cubic equation

$$2a^3\alpha_2 + (\alpha_1 + 2\alpha_3 - 2\delta_5)a^2 + (2\alpha_5 - \delta_1 - 2\delta_3)a - 2\delta_2 = 0.$$
Then the coupled KdV equation system (27) and (28) with (54) possesses the following \( \tau \) function and multiple soliton solutions

\[
A_1 = \frac{A_2}{a} = \frac{12\alpha_4}{a\alpha_1 + 2\alpha_2a^2 + 2\alpha_3 + 2\alpha_5}(\ln \tau)_{XX},
\]

where \( \tau \) is just the usual \( \tau \) function. For the multi-soliton solution, the \( \tau \) function reads

\[
\tau = 1 + \sum_{i=1}^{N} \pi_i + \sum_{i_1 < i_2}^{N} A_{i_1 i_2} \pi_{i_1} \pi_{i_2} + \sum_{i_1 < i_2 < i_3}^{N} A_{i_1 i_2 i_3} \pi_{i_1} \pi_{i_2} \pi_{i_3} + \ldots + A_{i_1 i_2 \ldots i_N} \pi_{i_1} \pi_{i_2} \ldots \pi_{i_N},
\]

\[
\pi_i \equiv \exp(k_i X - \alpha_4k_i^3T), \quad A_{i_1 i_2 \ldots i_k} \equiv \prod_{i_a < i_b, a, b = 1, 2, \ldots k} A_{i_a i_b}.
\]

(58)

It is interesting that there is only one parameter condition (54) to get multiple soliton solution (55) while it has been proved that the model is non-Painlevé integrable. In other words, the existence condition for multiple soliton solutions is not a sufficient condition of the integrability.

Especially, because of there are three real solutions of (56) for the special coupled KdV equation (37), we can obtain three types of multiple soliton solutions \( \{u_1, v_1\}, \{u_2, v_2\} \) and \( \{u_3, v_3\} \),

\[
v_1 = u_1 = \frac{12}{(c_1 - 1)(c_2 - 1)}(\ln \tau)_{XX},
\]

(59)

\[
u_2 = \frac{12}{(c_1 - 1)(c_1 - c_2)}(\ln \tau)_{XX}, \quad v_2 = c_1 u_2,
\]

and

\[
u_3 = \frac{12}{(c_2 - 1)(c_1 - c_2)}(\ln \tau)_{XX}, \quad v_3 = c_2 u_2
\]

(61)

with \( \tau \) being given by (58).

4.2.2. The second type of \( \tau \)-function and multi-soliton solutions of the coupled KdV system. The multi-solitons of the coupled KdV system listed in the last subsection are obtained from its special KdV reduction. In [9] it has been found that even if for the non-integrable cases, the coupled nonlinear system may have much more abundant solitary wave structure. So we believe that for the coupled KdV system there may be other types of multiple soliton solutions.
For instance, if the model parameters have the following conditions

$$\alpha_1 \alpha_2 (\alpha_1 \delta_3 - \delta_1 \alpha_3) \neq 0, \; \delta_4 = \alpha_4,$$

$$\delta_5 = \frac{1}{2} \alpha_3 + \frac{2 \alpha_2 \delta_1}{\alpha_1} - \frac{\alpha_1 \delta_3}{2 \delta_1}, \; \alpha_5 = -\frac{2 \delta_1 \alpha_1 \alpha_3 + \alpha_1 \delta_2}{\delta_1} \frac{1}{\alpha_1},$$ (62)

$$\alpha_3 = -\frac{2 \delta_2 \alpha_1^2}{\delta_1^2} - \frac{\alpha_1 (\delta_1 + \delta_3)}{\delta_1} - \frac{2 \delta_1 \alpha_2}{\alpha_1},$$ (63)

then we have a new type of multiple soliton solution

$$A_1 = \frac{12 \alpha_1 \alpha_4}{\alpha_1 \delta_3 - \delta_1 \alpha_3} \ln(\tau)_{XX} + \frac{\alpha_1}{\delta_1} A_2,$$ (64)

where $\tau$ is still the $\tau$ function of the usual KdV equation and for the multi-soliton solution it is given by (58) while $A_2$ is linked to the $\tau$-function by a linear equation

$$A_{2T} + \frac{12 \alpha_1 \alpha_4 (\delta_1 \delta_3 + 2 \delta_2 \alpha_1)}{\delta_1 \alpha_3 - \alpha_1 \delta_3} A_{2X} \ln(\tau)_{XX} + \frac{144 \delta_2 \alpha_1^2 \alpha_4^2}{(\delta_1 \alpha_3 - \alpha_1 \delta_3)^2} [\ln(\tau)^2_{XX}]_X$$

$$- \frac{12 \alpha_1 \alpha_4 (\delta_1 \delta_3 + \delta_2^2 + 2 \delta_2 \alpha_1)}{\delta_1 \alpha_3 - \alpha_1 \delta_3} A_2 \ln(\tau)_{XXX} + \alpha_4 A_2_{XXX} = 0,$$ (65)

If the third condition (63) is not satisfied, then a nonlinear term

$$\left(\delta_1 + \delta_3 + \frac{\delta_1 \alpha_3}{\alpha_1} + \frac{2 \delta_2 \alpha_1}{\delta_1} + \frac{2 \delta_2 \alpha_2}{\alpha_1^2}\right) A_2 A_{2X}$$

has to be added to the left hand side of (65).

Similarly, under the conditions

$$\alpha_3 \neq 0, \; \alpha_1 = \delta_1 = 0, \; \delta_4 = \alpha_4, \; \delta_5 = \frac{\alpha_2 (\delta_3 - c_1)^2}{c_1 \alpha_3^3} - \frac{1}{2},$$ (66)

$$\alpha_2 = -\frac{\alpha_3^2 (2 c_1 \delta_3 + 2 \delta_2 - c_1^2)}{2 c_1 (c_1 - \delta_3)^2},$$ (67)

where the constant $c_1$ is determined by

$$c_1 = \alpha_5 \pm \sqrt{\alpha_3^2 - 2 \delta_2},$$ (68)

we have the following new type of multiple soliton solutions

$$A_1 = \frac{12 \alpha_4}{c_1} \ln(\tau)_{XX} + \frac{\alpha_3}{\delta_3 - c_1} A_2,$$ (69)

where $\tau$ is same as given by (58) while $A_2$ is also linked to the $\tau$ function by a linear equation

$$A_{2T} + \alpha_4 A_{2XXX} + \frac{12 \alpha_4 (c_1 \delta_3 + 2 \delta_2)}{c_1^2} A_2 \ln(\tau)_{XXX} + \alpha_3 \alpha_1^2 [(\ln(\tau)^2_{XX})_X = 0.$$ (70)
In the same way, if the parameter condition (67) is not satisfied, then we have to add a nonlinear term
\[
\frac{2}{\alpha_3} \frac{2(\delta_3 - c_1)}{\alpha_3} + \frac{\alpha_3(2c_1\delta_3 + 2\delta_2 - c_1^2)}{\alpha_3} c_1(\delta_3 - c_1)
\] to the left hand side of (70).

4.2.3. *The third type of \(\tau\)-function and multi-soliton solutions of the coupled KdV system.*

Actually, in addition to the above types of multiple soliton solutions there may be other types of soliton solutions. Here is a further simple example for the following more specific model

\[
A_1T + aA_1XXX + bA_1A_2X + bcA_2A_2X = 0, \\
A_2T + aA_2XXX + bA_1A_2X + bA_2A_1X = 0. 
\]

(71)

For this special model, its first type of multiple soliton solutions has the form

\[
A_1 = \frac{6}{b} (\ln \tau)_{XX}, \quad A_2 = \pm \frac{1}{\sqrt{c}} A_1 
\]

(72)

and the second type of multiple soliton solutions is given by

\[
A_1 = \frac{12a}{b} (\ln \tau)_{XX} \pm \sqrt{c} A_2, 
\]

(73)

while \(A_2\) satisfies

\[
A_2T + aA_2XXX + 12a[A_2(\ln \tau)_{XX}]X \pm 2b\sqrt{c}A_2A_2X = 0. 
\]

(74)

For the special coupled KdV system (71), there is the following third type of multiple soliton solutions,

\[
A_1 = \frac{6}{b} [\ln(\tau_1^2 + \tau_2^2)]_{XX}, \\
A_2 = \pm \frac{12a}{b\sqrt{c}} \left( \arctan \frac{\tau_2}{\tau_1} \right)_{XX}, 
\]

(75)

(76)

where

\[
\tau \equiv \tau_1 + i\tau_2
\]

is just the usual \(\tau\) function of the KdV equation but with *complex* parameters, \(\tau_1\) and \(\tau_2\) are the real and imaginary parts of \(\tau\) respectively.
Fig. 1 and Fig. 2 are the special interaction plots of two soliton solution for the coupled KdV system (71) of the field $A_1$ and $A_2$ expressed by (75) and (76) respectively with

$$
\tau = 1 + (1 + i)e^{k_1X-k_1^3T} + (1 + 3i)e^{k_2X-k_2^3T} + \frac{(k_1-k_2)^2}{(k_1+k_2)^2}(4i-2)e^{(k_1+k_2)X-(k_1^3+k_2^3)T},
$$

$$k_1 = 1, \quad k_2 = \frac{3}{2}
$$

at times $T = -10, -5, 0, 5$ and 10 respectively.
Figure 2. The interaction plots of the two-soliton solution for the field $A_2 \equiv A_2$ expressed by (76) and (77) at times (a) $T = -10$, (b) $T = -5$, (c) $T = -0$, (d) $T = 5$ and (e) $T = 10$ respectively.

5. Summary and discussions

In summary, a general type of coupled KdV system is derived from the coupled Euler equation system (1) and (2) with ($\beta \neq 0$) and/or without ($\beta \neq 0$) the consideration of the earth rotation effects. In the derivation procedure, the frequently used inconsistent $y$-average trick in the past literature is removed.
The integrability of the derived KdV system is checked by means of the well known Weiβ-Tabor-Carnevale’s Painlevé test procedure. It is found that there are six types of Painlevé integrable subcases for the derived coupled KdV system.

The deformation and mapping method are used to get some types of travelling wave solutions including the conoidal periodic waves and single solitary waves for the general derived coupled KdV system with (54).

It is found that the coupled nonlinear systems may possess much more abundant solution structures. This phenomena is found before for the coupled non-integrable high-dimensional Klein-Gordon equation [9]. In this paper we found that whence some kinds of model parameter conditions are satisfied, then there may be some different kinds of multiple soliton solutions and $\tau$ function solutions. On the other hand, for the coupled KdV equation obtained here

The dynamics of atmospheric blockings has been one of the central and important problem since they are the main representations of the general circulation anomaly in the areas of mid-high latitudes. Atmospheric blocking events have a strong influence on regional weather and climate. The observations have shown atmospheric blockings may locate in the mid-high latitudes, usually over the ocean, as the dipole pattern which was first discovered by Rex [11].

For one layer atmospheric model, the single soliton solution of the constant coefficient KdV equation is responsible for the dipole pattern of the atmospheric blockings. To explain the blocking life cycle, one has to use the soliton solutions of the variable coefficient KdV equation [12]. Using the similar analysis as the single layer atmospheric model, the soliton solution of the coupled KdV equation can also be used to explain the blockings under the two layer atmospheric description. In the same way, the soliton solutions of the constant coefficient coupled KdV equation proposed here can not be used to describe the blocking life cycle. To describe the blocking life cycle one has to extended the coupled KdV system (1) and (2) to variable coefficient case and the problem will be detailed studied in the near future study.
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