Cosmological aspects of $f(R, T)$ gravity in a simple model with a parametrization of $q$

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Abstract

In this paper, we have considered a quadratic variation of the deceleration parameter ($q$) as a function of cosmic time ($t$) which describes a smooth transition from the decelerating phase of the Universe to an accelerating one and also show some distinctive feature from the standard model. The logical move of this article is against the behavior of the future Universe, i.e. whether the Universe expands forever or ends with a Big Rip, and we observe that the outcome of the considered parametrization comes in favor of Big Rip future of the Universe. The whole set up of the parametrization and solution is taken in $f(R, T)$ theory of gravity for a spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) geometry. Furthermore, we have considered the functional form of $f(R, T)$ function as $f(R) + f(T)$, where a quadratic correction of the geometric term $R$ is adopted as the function $f(R)$, and a linear matter term $f(T)$. We have investigated some features of the model by examining the behavior of physical parameters. Our primary goal here is to discuss the physical dynamics of the model in $f(R, T)$ gravity. We have found, the EoS parameter also has the same singularity as that of the Hubble parameter i.e. at the initial phase and at the Big Rip. The EoS parameter is explored in some detail for our choice of $f(R, T)$ function considered here. Different cases for $f(R, T)$ functional form for different values of the coupling parameters are discussed, and the evolution of the physical parameters is shown graphically.

Keywords: $f(R, T)$ gravity, Parametrization, Dark Energy, Late-time acceleration.

1 Introduction

The cosmological data of type Ia supernova (SNIa) independently analyzed by Perlmutter et al. [1], and Riess et al. [2] in different projects have individually disclosed that the Universe is undergoing a phase of cosmic acceleration at present. This late-time acceleration of the Universe is not known exactly, but the inclusion of an extra source in the energy budget can explain the idea of cosmic acceleration well. This additional source is assumed to be weird and has an anti-gravitational effect with highly negative pressure is generally dubbed as dark energy (DE), [3], [4], [5], [6], [7]. The recent results on SNIa research have shown that almost 70% of the energy budget is DE. Though, the right candidate for dark energy is still a point of heavy discussion. Recently, gravitational wave detection and the picture of black hole shadow strengthen Einstein’s general theory of relativity, and any modifications in Einstein’s theory (specifically to the geometry part) is not much appropriate. However, Einstein himself was not convinced with the matter distribution in the Universe i.e. the right-hand side of his field equations (representing matter sector) is considered to be made up of low-grade wood while the geometry part is of solid marble (representing the space-time). Any extra source term such as Einstein’s cosmological constant (representing energy density of vacuum) could be added into the energy-momentum tensor.
and serve as a candidate for dark energy. The most favored candidate of dark energy is the well-known cosmological constant Λ. Also, ΛCDM models have the best fit with many observational datasets. However, with this significant Λ, and due to its non-dynamical character produces a plethora of DE models with the dynamical equation of state explaining the accelerated expansion in the past few decades such as quintessence [8], [9], [10], [11], [12], [13], K-essence [14], [15], [16], [17], [18], [19], tachyons [20], [21], [22], [23], [24], [25], quintom [26], [27], phantom [28], [29], [30]. Although dark energy is not the only possibility, there is another way to explain the late-time acceleration and is modify the gravity [31]. So far, a wide range of modifications in the geometry part of the Einstein field equation has been done. These include f(R) gravity [32], scalar-tensor theories [33], braneworld models [34] etc. Many remarkable studies have already been carried out in the field of alternative gravity theory [35], [36], [37], [38], [39], [40], [41], [42], [43], [44].

We have comprehended that modified gravity theories became the main field of study of modern cosmology, especially due to the motivational search to explore the elusive nature of DE and the reason of late times cosmic acceleration [1], [2]. These theories demand the modification or generalization in the Einstein-Hilbert action and offer gravitational field equations distinct from the field equations of the DE model of cosmic speed up [3], [45]. There are numerous ways to depart from GR, so, therefore, researchers developed several realistic modified gravity theories whose enticing features are recorded in [46], [47], [48], [49]. One of the most simple generalization of f(R) theory of gravity [50], [51], [52], [53], [54], [55], [56], [57], [58], [59], [60], [61] was first proposed in 1984 by Goenner [62]. The consequence of non-minimal coupling between scalar curvature and matter Lagrangian was broadly examined in [63], [64]. Including the above theory, Harko and Lobo extend the Einstein-Hilbert action by expecting an arbitrary coupling of scalar curvature and matter Lagrangian. The straightforward exercise of this arbitrary coupling was initially performed by Poplawski [65] on the basis of the principle of least action. The outcome of this arbitrary matter-geometry coupling leads to the violation of conservation of EMT, which results in the appearance of non-geodesic motion of the massive particle. In 2011, Harko et al. [66] widespread the sphere of this arbitrary coupling of scalar curvature R and trace T of EMT, and this introduces another modification in the GR known as f(R,T) theory of gravity. A lot of notable works have already been carried out in f(R,T) gravity [67], [68], [69], [70], [71], [72], [73], [74], [75], [76], [77], [78], [79], [80], [81].

It is to be noted that cosmic acceleration is a late-time phenomenon, and the structure formation in the Universe during the matter-dominated era requires a phase of decelerated expansion where gravity must be the dominating force. So, to illustrate the whole evolution of the Universe, one needs a phase of super acceleration in the beginning (inflation) and middle deceleration, and an accelerated phase at late times. This phenomenon is attributed to the cosmological parameter known as the deceleration parameter, and the simplest way to obtain such a scenario is the cosmological parametrization [82], [83]. In literature, there are various schemes of parametrization discussed, suggesting an early deceleration to the present accelerating era together with a cosmological phase transition. Various DE models, as well as modified gravity models, have been explored in the past few years. We are interested here to discuss a simple model of the Universe that describes the observational scenario of the different phases of the Universe in f(R,T) gravity with a parametrization of the deceleration parameter that also helps to find an exact solution of the field equations and try to see the role of the coupling parameters in the quadratic form of f(R,T) function considered here.

The paper is organized in the discussed sequences: The first section is the introduction and describes the present scenario in cosmology. In the second sect., we have discussed the basic formalism of the f(R,T) gravity. The field equations are derived in the third sect. for a homogeneous and isotropic FLRW space-time in the backdrop of f(R,T) gravity. The parametrization scheme and the solution of field equations are discussed in the fourth sect. In the fifth sect., the dynamical behavior of the equation of state parameter is explored, while some special cases for the functional form of the f(R,T) function are discussed in the sixth sect. Finally, we have summarized the physical insights of the results in the seventh sect.
2 Basic formalism in $f(R, T)$ gravity

The general action of $f(R, T) = f(R) + f(T)$ gravity [66]

$$S = \int \left\{ \frac{1}{16\pi G} f(R, T) + L_m \right\} \sqrt{-g} dx^4,$$

(1)

Here, we allow the general form of $f(R, T)$ as $f(R) + f(T)$, where a quadratic correction of the geometric term $R$ is adopted as the function $f(R) = R + \alpha R^2$ [84], and a linear matter term $f(T)$. The term $R^2$ turn up in the general functional form of $f(R)$ indicates the simple corrections to GR. Numerous observations of the Universe are compatible with the Starobinsky model [85], [86], therefore we have extended the form of $f(R, T) = R + f(T)$ to $f(R, T) = f(R) + f(T)$, so that acceleration in the Universe (early and late time acceleration) can be explained by the theories beyond GR [87]. Also, to establish the exotic imperfect fluids and taking quantum effects as well in the above mentioned function $f(R)$, we outset a trace $T$ dependent term which is responsible for matter Lagrangian $L_m$ and exhibit the set of field equations. Here, in this research, we take $f(T)$ as a first degree function of trace $T$ defined as $f(T) = 2\lambda T$, $\lambda$ being a coupling constant. Following from this the concluded form of $f(R, T)$ function is $R + \alpha R^2 + 2\lambda T$.

Energy momentum tensor(EMT) of matter [66] is expressed as

$$T_{ij} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}L_m)}{\delta g^{ij}},$$

(2)

where $T = g^{ij}T_{ij}$ is the trace of EMT. Furthermore, the dependence of $L_m$ is dependent on $g_{ij}$, therefore

$$T_{ij} = g_{ij} L_m - 2 \frac{\delta L_m}{\delta g^{ij}}.$$  

(3)

Varying the action (1) w.r.t. $g_{ij}$, we have

$$f^R(R, T)R_{ij} - \frac{1}{2}g_{ij}f(R, T) + (g_{ij}\Box - \nabla_i \nabla_j) f^R(R, T) = 8\pi G T_{ij} - f^T(R, T)(T_{ij} + \Theta_{ij}),$$

(4)

where $f^R(R, T)$ and $f^T(R, T)$ act as the derivative of $f(R, T)$ w.r.t. $R$ and $T$ respectively, $\Box = g^{ij}\nabla_i \nabla_j$ represents the d’Alembert operator, where $\nabla_i$ shows the covariant derivative w.r.t. $g_{ij}$.

On defining $\Theta_{ij}$, we have

$$\Theta_{ij} \equiv g^{lm}\frac{\delta T_{lm}}{\delta g^{ij}} = -2T_{ij} + g_{ij}S_m - 2g^{lm}\frac{\delta^2 L_m}{\delta g_{ij}\delta g^{lm}}.$$  

(5)

In present paper, we assume perfect fluid matter in the universe, so one can take the form of matter Lagrangian $L_m = -p$.

$$T_{ij} = (\rho + p)u_i u_j - pg_{ij},$$

(6)

where $\rho$ is the energy density and $p$ is the pressure of the fluid present in the Universe. Equation (5) defines the variation of EMT as

$$\Theta_{ij} = -2T_{ij} - pg_{ij}.$$  

(7)

The gravitational field equations can be obtained as using Eq. (7) in Eq. (4)

$$f^R(R, T)R_{ij} - \frac{1}{2}g_{ij}f(R, T) + (g_{ij}\Box - \nabla_i \nabla_j) f^R(R, T) = 8\pi G T_{ij} + f^T(R, T)(T_{ij} + pg_{ij}).$$

(8)

On contracting the above equation (8) w.r.t $g^{ij}$ and reorganize the terms in Eq. (8), one can read the next two equations as

$$R f^R(R, T) - 2f(R, T) + 3\Box f^R(R, T) = 8\pi G T + (T + 4p)f^T(R, T).$$

(9)
and, we have
\[
R_{ij} = \frac{1}{f^R(R,T)} \left( 8\pi G T_{ij} + \frac{1}{2} g_{ij} + (\nabla_i \nabla_j - g_{ij} \Box) f^R(R,T) + f^T(R,T)(T_{ij} + pg_{ij}) \right). \tag{10}
\]

Next, we define a new operator \( \ominus \),
\[
\ominus = \nabla_i \nabla_j - g_{ij} \Box. \tag{11}
\]

Using Eq. (11) in (10),
\[
R_{ij} = \frac{1}{f^R(R,T)} \left( 8\pi G T_{ij} + \frac{1}{2} g_{ij} + \ominus_{ij} f^R(R,T) + f^T(R,T)(T_{ij} + pg_{ij}) \right). \tag{12}
\]

On rewriting the Ricci scalar \( R \) in Eq. (9), we obtain
\[
R = \frac{1}{f^R(R,T)} \left( 8\pi G T + 2f(R,T) - 3\Box f^R(R,T) + (T + 4p)f^T(R,T) \right). \tag{13}
\]

The field equations with the Einstein tensor \( G_{ij} \) on the LHS can be achieved by applying Eqs. (12) and (13) in Eq. (8)
\[
G_{ij} = R_{ij} - \frac{1}{2} R g_{ij} = \frac{8\pi G}{f^R(R,T)} T_{ij} + \frac{1}{f^R(R,T)} \left[ \frac{1}{2} g_{ij} \left( f(R,T) - R f^R(R,T) \right) + \ominus_{ij} f^R(R,T) + (T_{ij} + pg_{ij}) f^T(R,T) \right],
\]
where \( T_{ij}' = \frac{1}{\pi \rho G} \left( \frac{1}{2} g_{ij} \left( f(R,T) - R f^R(R,T) \right) + \ominus_{ij} f^R(R,T) + (T_{ij} + pg_{ij}) f^T(R,T) \right) \). From the above field equations, EFE in GR can be resumed by fixing \( \alpha = 0 \) and \( \lambda = 0 \). Imposing the Bianchi identity on Eq. (14) leads to
\[
\left( 8\pi G + f^T(R,T) \right) \nabla^i T_{ij} + \frac{1}{2} f^T(R,T) \nabla_i T_{ij} + T_{ij} \nabla^i f^T(R,T) + \nabla_j \left( pf^T(R,T) \right) = 0. \tag{15}
\]

3 Field equations in RW geometry

In modern physical cosmology, the spatial distribution of the matter in the Universe is based on cosmological principle, according to which, the Universe is homogeneous and isotropic on a large scale; therefore, it has no irregularities on the large scale structure over the course of evolution, that was primarily identified by big-bang. In addition to that, present cosmic observations evidence a flat geometry of the Universe; therefore we assume spatially flat FLRW metric of the Universe of the form
\[
ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2), \tag{16}
\]
where \( a(t) \) is the scale factor. The scalar curvature \( R \) and trace \( T \) of EMT (6) are defined as
\[
R = -6(2H^2 + \dot{H}), \tag{17}
\]
where \( H = \frac{\dot{a}}{a} \) represents the Hubble parameter and an overhead dot shows the derivative w.r.t. to \( t \).
\[
T = \rho - 3p, \tag{18}
\]
By considering the form of \( f(R,T) = R + \alpha R^2 + 2\lambda T \) and using Eqs. (6), (18), (17) in Eq. (14), the field equations read as
\[
\left[ \frac{1}{1 + 2\alpha R^2} \right] 3H^2 = 8\pi \rho + \lambda(3\rho - p) + 2\alpha \xi(H,\dot{H},\ddot{H}), \tag{19}
\]
\footnote{Note that this equation has been obtained in [74]. However, because of the metric signature in the present work, the last term in Eq. (15) has obtained the opposite sign.}
\[
\left(\frac{1}{1 + 2\alpha R}\right) 2\dot{H} + 3H^2 = -8\pi p + \lambda (\rho - 3p) + 2\alpha \eta(H, \dot{H}, \ddot{H}),
\]

where \(\xi(H, \dot{H}, \dddot{H}) = -9(4H^4 + \dot{H}^2 - 4H^2\dddot{H} - 2H\dot{H})\) and \(\eta(H, \dot{H}, \dddot{H}) = 3(-12H^4 + 5\dot{H}^2 + 4H^2\dot{H} + 12H\dddot{H} + 2\dot{H})\) are the functions of Hubble parameter \(H\) and its derivatives up to third order respectively. Also, we have set the units so that \(G = 1\).

Applying the functional form of \(f(R, T)\) in Eq. (15), which leads to,

\[
\frac{8\pi + 3\lambda}{8\pi + 2\lambda} \rho - \frac{\lambda}{8\pi + 2\lambda} \dot{\rho} + 3H(p + \dot{p}) = 0,
\]

Eq. (19) reads as

\[
(8\pi + 3\lambda) \rho - \lambda p = \vartheta,
\]

where we have defined,

\[
\vartheta \equiv 3H^2 + 18\alpha \left(\dot{H}^2 - 6H^2\dot{H} - 2H\dddot{H}\right).
\]

Simplifying Eqs. (21) and (22) with the help of (23), we obtain

\[
\rho = \frac{3\vartheta - \lambda (8\pi + 2\lambda)^{-1} H^{-1}\dot{H}}{12(2\pi + \lambda)},
\]

\[
p = \frac{(8\pi + 3\lambda) \rho - \vartheta}{\lambda}.
\]

The above expressions of the density and pressure in equations (24) and (25) consist of terms of Hubble parameter \(H\) and its derivatives. This will provide an exact solution to the field equations with any simple parametrization schemes [82], [83]. There exist various schemes of parametrization available in the literature, and it is generally referred to as a model-independent way to explore various dark energy models. For a huge list of various parametrization schemes, one can refer to the reference [82], [83]. Here, in this paper, we consider a simple algebraic form of the deceleration parameter \(q\) and study the physical properties of the Universe in \(f(R, T)\) gravity to examine the role of the \(f(R, T)\) coupling parameters \(\alpha\) and \(\lambda\) in the evolution of the Universe.

### 4 Parametrization of \(q\) and Solution

An adhoc choice of generalized time-dependent deceleration parameter \(q(t)\) of second degree is discussed in the paper [88], where the deceleration parameter is considered as,

\[
q(t) = (8\gamma^2 - 1) - 12\gamma t + 3t^2,
\]

where \(\gamma > 0\) is an arbitrary constant. For this parametrization of \(q(t)\), our model entirely accelerates when \(\frac{2t}{3} - \frac{1}{2}\sqrt{2 + 3t^2} < \gamma < \frac{2t}{3} + \frac{1}{2}\sqrt{2 + 3t^2}\) and decelerates according as \(\gamma < \frac{2t}{3} - \frac{1}{2}\sqrt{2 + 3t^2}\) or \(\gamma > \frac{2t}{3} + \frac{1}{2}\sqrt{2 + 3t^2}\) and it predicts phase transitions when \(q = 0\) at \(t = 2\gamma \pm \sqrt{\frac{4\gamma^2 + 1}{3}}\). As it is well acknowledged that the Universe experiences an accelerating phase in late time, so the Universe must passed through a phase of slow expansion in the past [1], [2] in such case the parametrization of deceleration parameter is rational.

Using the relation of deceleration parameter with Hubble parameter \(H(t)\) [89], Eq. (26) yields the explicit form of the Hubble parameter and the scale factor as,

\[
H(t) = \frac{1}{t(2\gamma - t)(4\gamma - t)},
\]
and

\[ a(t) = \frac{\beta [4\gamma - t]^{\frac{1}{2\gamma}}}{(2\gamma - t)^{\frac{1}{2\gamma}}} \]  

(28)

where \( \beta \) is constant of integration. The explicit form of the expressions for energy density and pressure can be written using the equation (27) in equations (24) and (25).

Now, we have the complete solution to our field equations with explicit forms of all the geometrical and physical parameters to study the dynamics of the obtained model. Bakry and Shafeek [88] have discussed the geometrical behavior of the model obtained with this parametrization of \( q(t) \) in general relativity, wherein they have explored the possibility of \( \gamma = 0.5 \) in the redshift range \(-1 < z < 4\). The analyses show the model have Bang to Rip evolution similar to the one discussed by by Caldwell et al. [90]. Some notable works have also been studied to discuss the finite time singularity [91], [92], [93].

Our presented work is an extended work of the same, where we want to explore the physical dynamics of the Universe in \( f(R,T) \) theory of gravity with a quadratic correction term of the \( f(R,T) \) function i.e. \( f(R,T) = R + \alpha R^2 + 2\lambda T \) with two parameters \( \alpha \) and \( \lambda \) and want to see the role of the correction terms in the evolution of the physical parameters. As, we are more interested to discuss the present phase of the Universe, we formulate these kinematic parameters given in Eqs. (26) and (27) in terms of redshift \( z \) by using the relation of scale factor as \( a(t) = 1/(1 + z) \) (with the normalizing condition \( a_0 = 1 \), \( a_0 \) being the value of the scale factor at \( t = t_0 \)). Now, the expressions for \( q \) and \( H \) are demonstrated in terms of redshift \( z \) as,

\[ H(z) = \frac{H_0 \left( (\beta(z + 1))^{8\gamma^2 + 1} + 1 \right)^{3/2}}{(\beta^{8\gamma^2 + 1} + 1)^{3/2}} \]  

(29)

\[ q(z) = -\frac{(\beta(z + 1))^{8\gamma^2} + \gamma^2 (4 - 8(\beta(z + 1))^{8\gamma^2}) + 1}{(\beta(z + 1))^{8\gamma^2 + 1}} \]  

(30)

For different values of the model parameter \( \gamma \), the evolution of the deceleration parameter \( q \) w.r.t. redshift \( z \) can be plotted as follows,

From the above figure, Fig.1 of deceleration parameter, we can interpret the cosmological phase transition from early deceleration to present acceleration with the present value of the deceleration parameter.
| Substance          | EoS parameter | Observations                                                                 |
|--------------------|---------------|-------------------------------------------------------------------------------|
| Phantom Universe   | $\omega < -1$ | Repulse Weak Energy Condition (WEC) Lead to Big Rip                           |
| Cosmological Constant | $\omega = -1$ | Phantom Universe (Inconsistent with observations)                             |
| Quintessence       | $-\frac{1}{3} < \omega < -1$ | Cosmological Constant (68% of the Universe)                                   |
| Cold matter        | $\omega = 0$  | Pressurless matter (32% of the Universe)                                      |
| Hot matter         | $0 < \omega < \frac{1}{3}$ | Radiation (Insubstantial at present time)                                     |
| Radiation          | $\omega = \frac{1}{3}$ | Hard Universe (Significant in Early Universe)                                 |
| Hard Universe      | $\frac{1}{3} < \omega < 1$ | Exorbitant high densities                                                     |
| Stiff matter Universe | $\omega = 1$  | Ekpyrotic matter                                                              |
| Ekpyrotic matter   | $\omega > 1$ | Repulse Dominant Energy Condition (DEC)(Quintessence)                         |

parameter $q_0 < 0$ and in the far future, the highly negative value of $q$ indicates the super acceleration phase leading to a Big Rip singularity.

Pacif et al. [94] have discussed a model in general relativity with this same parametrization of $q$ and constrained the model parameters with some external datasets (Hubble datasets, Supernovae datasets, and Baryonic Acoustic Oscillation datasets) involved in the model i.e., $\gamma$ and $\beta$ and they have found the values of these model parameters as $\gamma = 0.44^{+0.01}_{-0.01}$ and $\beta = 1.09^{+0.11}_{-0.11}$ for which $q_0 = -0.50^{+0.11}_{-0.11}$ and the phase transition occurs at $z_t = 0.67^{+0.26}_{-0.36}$. Since our discussed model is based on the same parametrization scheme $q = (8 \gamma^2 - 1) - 12 \gamma t + 3 t^2$, we can use these constrained values of the model parameters for our subsequent analysis.

5 EoS Parameter & its dynamical behavior

One of the vital parameter of cosmology is EoS parameter which explains the different cosmic regimes in Universe. In a more generic way, $\omega$ can be defined as $p/\rho$ whose diverge values in different ranges discusses the different cosmic matter in the Universe and the same can be observe in the table below.

The cosmic acceleration can be attained with the inequality $1 + 3\omega < 0$ as advised by the Friedmann equations, which can be realized for an exotic matter related to the negative pressure. The existence of different substances in the Universe give rise to various cosmic phases. These different epochs can be observed by varying the EoS parameter $\omega$ (see Table I).

It is worth considering the behavior of EoS parameter in $f(R, T)$ gravity. According to our model in $f(R, T)$ gravity, we have realized that in the initial times, $\omega = \frac{(8 \pi + 3 \lambda) \rho - \theta}{3 \varphi - 4 \pi (8 \pi + 2 \lambda) - 4 \pi - 1} \frac{\rho - \theta}{\beta (8 \pi + 2 \lambda) - 1} \frac{\rho - \theta}{\beta}$ turns to an expression
of coupling constant $\lambda$ (of matter and geometry) and the model parameter $\gamma$ but remains unrelated of the other coupling parameter $\alpha$. In the late times, it can be realized that the EoS parameter $\omega$ depends solely on $f(R,T)$ coupling constant $\lambda$.

Mathematically, we can write,

$$\omega_i \equiv \lim_{t \to 0} \omega = \frac{4\pi(-3 + 32\gamma^2) + 3(-1 + 16\gamma^2)}{12\pi + (3 + 16\gamma^2)\lambda}, \quad (31)$$

and

$$\omega_f \equiv \lim_{t \to \infty} \omega = -3 + \frac{8\pi}{\lambda}. \quad (32)$$

Here, the subscript ‘$i$’ indicates the value of EoS parameter in the early Universe and the subscript ‘$f$’ represents the value of EoS parameter in the late Universe. We can notice that the expression of $\omega_i$ is not tied-up with the parameter $\alpha$ and $\omega_f$ is free from $\alpha$ and $\gamma$.

Some elementary calculations witness that constraining the EoS parameter $\omega$ as $0 < \omega_i < 1$ (which is plausible in cosmology) forces one to prefer

(i) $\lambda > 0$ and $\frac{1}{4}\sqrt{3}\sqrt{\frac{(4\pi+\lambda)}{(8\pi+3\lambda)}} < \gamma < \sqrt{\frac{3}{4}}$.

and the positivity of $\frac{(4\pi+\lambda)}{(8\pi+3\lambda)}$ further leads the condition on the $f(R,T)$ coupling constant $\lambda < -4\pi$ or $\lambda > -\frac{8\pi}{3}$,

(ii) The EoS parameter can behave like ekpyrotic matter ($\omega > 1$) by choosing $\lambda > 0$ and $\gamma > \sqrt{\frac{3}{4}}$ (for positive coupling $f(R,T)$ constant $\lambda$), and

(iii) (a) $\lambda < -4\pi$ and $\gamma > \sqrt{\frac{3}{4}}$ or (b) $-4\pi < \lambda < -2\pi$ and $\frac{1}{4}\sqrt{3}\sqrt{-\frac{4\pi+\lambda}{8\pi+3\lambda}} < \gamma < \sqrt{\frac{3}{4}}$ or (c) $-2\pi < \lambda < 0$ and $\sqrt{\frac{3}{4}} < \gamma < \frac{1}{4}\sqrt{3}\sqrt{-\frac{4\pi+\lambda}{8\pi+3\lambda}}$.

which resists dominant energy condition (DEC) followed by matter-dominated era where energy density is extensively high to the radiation-dominated era in the initial Universe. Some fundamental facts about different eras of cosmic evolution can be comprehended by the results (31) and (32). Also, from (31), one can arbitrate the fundamental gravitational model which portray two different cosmic phase transitions:

(i) for $\gamma = \sqrt{\frac{3}{4}}$, we get $\omega_i = 1$. It is simply observed that the stiff matter epoch of the Universe can be obtained by setting $\gamma = \sqrt{\frac{3}{4}}$ and remain independent of other model parameters. In this case, the model explains a transition from a state in which stiff matter dominates in the early eras to a state which behaves like the DE in the late times.

(ii) Also, it is feasible to obtain the following relation on model parameter $\gamma$ that shows a transition in the initial Universe $\omega^p_i = 0$ (a pressure-less matter dominated era) to a Universe with DE in the late times.

$$\gamma = \frac{\sqrt{3}}{4} \sqrt{\frac{4\pi+\lambda}{8\pi+3\lambda}}, \quad (33)$$

(iii) From Eq. (32) one more analysis can be performed on EoS parameter $\omega$. As we all know that, the standard $\Lambda$CDM model fits well with various measurements and cosmic observational data [95], [96] and is based on the most consistent and prevailing big bang scenario. In accord with many observations [95], [96], $\Lambda$ CDM model is considered as one of the candidates to describe the accelerated expansion in the Universe. Therefore, it is worthwhile to discuss the conditions from
in our model, which acts like cosmological constant $\omega = -1$ in the late time. The negative coupling between matter and geometry i.e. $\lambda = -2\pi$ is required to get the stated condition. Again, it is require to remark that the behavior of our model with standard $\Lambda$CDM depends only on the model constant $\lambda$.

From the above computations, it has been inferred that the parameter $\alpha$ does not play any role in deciding the initial and final stages of cosmic evolution, which simply imply that the only changes in the matter part of the Lagrangian $\lambda$ is responsible for different states of the cosmic evolution. Moreover, it is always worthwhile to study the behavior of matter density and pressure in the limit large times. Straightforward calculations reveal that both terms $\rho$ and $p$ tend to 0 provided that the model parameter $\gamma > 0$ and is free from the model constants $\lambda$ and $\alpha$. Mathematically, the limit of large times gives

$$\lim_{t\to\infty} \rho = \lim_{t\to\infty} p = 0. \quad (34)$$

In the next section, we shall discuss the behavior of the physical parameters e.g. energy density, pressure and EoS parameter in $f(R,T)$ gravity with the quadratic form of the $f(R,T)$ function with the two coupling parameters $\alpha$ and $\lambda$. The following special cases arise for which we can discuss the physical evolution of the energy density, pressure, and also the EoS parameter through some graphical representations.

### 6 Special cases for $f(R,T)$ function

The following four cases arise for different values of the coupling parameters are showing the role of the correction terms in $f(R,T)$ gravity.

#### 6.1 case I: $\lambda = 0$, $\alpha = 0$ i.e. $f(R,T) = R$

For these vanishing values of the coupling parameters ($\lambda = 0$ & $\alpha = 0$), the case reduces to standard general relativity. The evolution of the physical parameters for this case is described in some detail in the ref. [88].

To understand the recent past, present, and future evolution of the physical parameters $\rho$, $p$ and $\omega$, we plot them with respect to redshift $z$, which are shown in the following graphical representations in the following figure, Fig.2.

#### 6.2 case II: $\alpha = 0$, $\lambda \neq 0$ i.e. $f(R,T) = R + 2\lambda T$

For this case also, to understand the recent past, present, and future evolution of the physical parameters $\rho$, $p$ and $\omega$, we plot them with respect to redshift $z$, which are shown in the following graphical representations in the following figure, Fig.3.

From the above plotted Fig. 3, we can interpret that the negative value of the coupling parameter $\lambda$ is incompatible with the present scenario of the late-time Universe as expected. Furthermore, it is found that other two cases for the values of the coupling parameters with $\alpha \neq 0$, $\lambda = 0$ i.e. $f(R,T) = R + \alpha R^2$ and $\alpha \neq 0$, $\lambda \neq 0$ i.e. $f(R,T) = R + \alpha R^2 + 2\lambda T$ are incompatible with the considered parametrization of $q$.

### 7 Discussion and conclusion

In this work, we have examined the cosmological model in the framework of FLRW space-time using the non-linear alternative theory of gravity, namely $f(R,T)$ gravity. The dynamics of the model using the specific form of $f(R,T) = R + \alpha R^2 + 2\lambda T$ is investigated in section (??). Thus, the behavior of the Universe is based on the preferred choice of $f(R,T)$ function, which pulls out an explicit set of field equations. Additionally, this article is an attempt to design a cosmic model by taking a suitable
Figure 2: (a) The plot of energy density $\rho \sim z$, (b) The plot of cosmic pressure $p \sim z$ and (c) The plot of EoS $\omega \sim z$ for $\gamma = 0.44$ and $\beta = 1.09$. In this plot the energy scale is $(8\pi G)^{-1/2} = M_{\text{pl}}$ ($G = 1$ here).

Figure 3: (a) The plot of energy density $\rho \sim z$, (b) The plot of cosmic pressure $p \sim z$ and (c) The plot of EoS $\omega \sim z$ for $\gamma = 0.44$ and $\beta = 1.09$. 
parametrization of the deceleration parameter, which was first considered in the paper [88] wherein two phenomena Big Rip and Big Bang of the Universe were discussed together with the cosmic evolution in the general theory of relativity. Later on, the same model is studied in the paper [94] where the authors found some observational constraints using some external datasets and also discussed the statefinder diagnostics. Here, in our study, we have taken the motivation from [88] and [94] and extended the study in $f(R,T)$ gravity wherein our main intention is to study the physical parameters (especially EoS $\omega$) in $f(R,T)$ gravity with quadratic correction terms i.e. $f(R,T) = R + \alpha R^2 + 2\lambda T$. Some distinctive features of the model are recorded and discussed as follows.

- The parametrization of second-degree time-dependent deceleration parameter $q(t)$ has been chosen such that cosmos passes through different phases depending on the value of model parameter $\gamma$. The functional form of $q(t)$ exhibits different regimes of the Universe as the model demonstrates the bouncing criteria depending on the values of $\gamma$. The model begins with Big bang at time $t = 0$ and ends at $4\gamma$ with $q = 8\gamma^2 - 1$ in both scenarios. The model completes one cycle in the time range $t \in (0,4\gamma)$, i.e. Universe is in the stage of Big Rip at $t = 2\gamma$ while elapses through $t \in (0,2\gamma)$ and recurring at the stage of Big Bang at $t = 4\gamma$ while passes through $t \in (0,2\gamma)$. The flipping behavior of the Universe in two different time ranges is the remarkable feature of the parametrization (26).

- As we have mentioned that the motive of our study is to examine the dynamics of physical parameters in the framework of $f(R,T)$ gravity, thereby it is worthwhile to understand the working of EoS parameter $\omega$. In accord with our findings, we have noted that our cosmic model in the early times remains unaffected by the quadratic correction term $\alpha$, wherein the effect of $\lambda$ and model parameter $\gamma$ is significant. Furthermore, $\omega$ is only $\lambda$ dependent in the late time. Consequently, the role of $\gamma$, $\lambda$, and $\alpha$ make our results utterly different from the findings of [88].

- A comprehensive analysis of $\omega$ has been performed in section (5) with the view to understand the existence of various substances and their dynamic behavior in the Universe, which is classified as EoS parameter (see Table I).

- The analyses in section (5) demonstrates that the parameter $\alpha$ does not play any vital role in deciding the initial and final stages of cosmic evolution. The only variations in the matter part of the Lagrangian coupling constant $\lambda$ are responsible for various states of the cosmological evolution. Straightforward calculations reveal that both terms $\rho$ and $p$ tend to 0 (34) provided that the model parameter $\gamma > 0$ and is free from the model constants $\lambda$ and $\alpha$.

- To see the evolution of the physical parameters energy density ($\rho$), pressure ($p$) and equation of state parameter ($\omega$) in the recent past, present and future evolution, we have plotted them w.r.t. the redshift ($z$) for two cases $\alpha = 0$, $\lambda = 0$ (case-I corresponding to GR) and $\alpha = 0$, $\lambda \neq 0$ (case-II). The other two cases $\alpha \neq 0$, $\lambda = 0$ and $\alpha \neq 0$, $\lambda \neq 0$ are incompatible with the discussed scenario.

By inspecting all the above points, one can interpret that this cosmological model describes a cyclic Universe scenario with the considered scheme of parametrization of deceleration parameter and reconstructing some physical parameters in $f(R,T)$ theory of gravity. The above study imparts a reason to understand several cosmic scenarios right from the evolution of the Universe (Big Bang) to its end (Big Rip). Undoubtedly, the integration of observational cosmology in this study provides a more precise range to model parameters so that the behavior of geometrical and physical parameters could be investigated in a more suitable way. However, the existing study is only an attempt to figure out the dynamics of the physical parameters of the Universe.

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