Lee wave breaking region: the map of instability development scenarios

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Abstract. Numerical study of a stably stratified flow above the two-dimensional cosine-shaped obstacle has been performed by DNS and LES. These methods were implemented to solve the three-dimensional Navier–Stokes equations in the Boussinesq approximation, together with by the scalar diffusion equation. The results of scanning in the wide ranges of physical parameters (Reynolds and Prandtl/Schmidt numbers relating to laboratory experiment cases and atmospheric or oceanic situations) are presented for instability and turbulence development scenarios in the overturning internal lee waves. The latter is generated by the obstacle in a flow with the constant inflow values of velocity and stable density gradient. Evolution of lee-wave breaking is explored by visualization of velocity and scalar (density) fields, and the analysis of spectra. Based on the numerical simulation results, the power-law dependence on Reynolds number is demonstrated for the wavelength of the most unstable perturbation.

1. Introduction

The numerical investigations of instability and turbulence development mechanisms in the overturning lee waves were performed in [1, 2] where the DNS data for Re = 4000 and Pr = 1 were presented, with the grid resolution sufficient to capture the fine-scale transition features [1] and the subsequent turbulence behavior [2]. In the preceding and present studies, Reynolds and Prandtl/Schmidt number are $Re = UH/v$ and $Pr = v/κ$ for a stably stratified flow with the constant values of inflow density gradient and velocity $U$ where the 2D cosine-shaped obstacle of height $H$ was inserted. Such a phenomenon has also been explored in laboratory experiments in water tanks with towed body [3-5]. The secondary density-field instability generated after internal lee wave overturning reveals a range of spanwise modes. The smallest mode has the wavelength $λ_y ≈ 0.5H$ for $Re = 4000$ and $Pr = 1$ [1, 2], relating to perturbations of Rayleigh–Taylor instability (RTI) type, growing massively and resulting in convective mushroom-like structures, with associated Kelvin–Helmholtz instability rolls. At late times of transition to turbulence the smaller-scale vortices transform into the bigger vortex structures, and another dominant mode ($λ_H ≈ 2.5H$) arises when the large-scale toroidal vortex structures can be visualized [1, 4, 5, 6]. This larger-scale mode can be associated with the most unstable perturbation of the initial laminar 2D vortex pair in the place of lee wave overturning [1, 2]. Visualization of large structures is complicated by the presence of a cascade of finer-scale eddies, leading to the formation of the inertial “−5/3” range and the dissipative range of spanwise spectra with steeper slopes. For $Pr >> 1$, the earlier and faster growth of RTI with finer-scale structures is seen [7].

The intention of the present paper is to numerically simulate the lee wave overturning region for various Reynolds, Prandtl/Schmidt numbers ($50 ≤ Re ≤ 40000$, $1 ≤ Pr ≤ 700$) and to examine the influence of these parameters on the details of instability development and transition to turbulence. For
example, a decrease in Re allows us to improve the resolution due to smaller grid-size ratios to the Kolmogorov microscale at lower Re, which in turn gives more accurate statistics in the turbulent patch. Calculations for various Re can reveal a critical value (Re_c) for the onset of instability. On the other hand, for large values of Re > Re_c and depending on the grid resolution, the DNS method (which transforms here into the implicit LES approach) will become inadequate, and it is necessary to introduce the models of SGS viscosity and scalar diffusion to parameterize small eddies of sub-grid scales (SGS). Variations of Prandtl number value will also help us to found critical levels, in particular, for determining the DNS applicability limit. It makes sense to consider separately the real situations of non-isothermal flows in the atmosphere (where Pr ~ 1) and water reservoirs (Pr ~ 7), as well as saline water in oceanic estuaries (Pr ~ 700 for diffusion of salted water in still water).

2. Numerical model
Numerical simulation of a stratified flow with the obstacle is based on the continuity, Navier–Stokes and scalar (density) equations with the Boussinesq approximation of buoyancy effects. All details of the governing equations non-dimensionalized by inflow velocity U and obstacle height H, and their numerical realization for DNS cases are given in [1, 2].

To overcome the problems of insufficient resolution, SGS models of Smagorinsky type for velocity and scalar equations are used, with the standard value of Smagorinsky constant C_s = 0.1, and the same boundary conditions, sponge layers, refined uniform grid as in [1, 2]. The SGS Prandtl/Schmidt number is Pr_{sgs} = 0.3 which is enough to efficiently remove the excessive numerical noise distorting the scalar field plots and spectra both in ILES with C_s = 0 and in LES with C_s = 0.1 and Pr_{sgs} = 1.0 [7].

3. Computation results
The DNS results at fixed Pr = 1 and lower Re values (Re < 4000) than in [1, 2] demonstrate that, for instance, for Re = 2000 (Figures 1–2) multiple peaks in spanwise spectra (corresponding to the wavelength 0.5H < λ_s ≤ 5H of spanwise periodicity) occur at times 20 < t < 32, during perturbation growth. The wavelength of the most unstable perturbation corresponding to the RTI mode during the exponential growth stage (23 < t < 27) is λ_s ≈ 0.63H according to Figures 1–2. The large-structure mode λ_s ≈ 3.3H arises at t > 40 during the stage the quasi-steady turbulent patch forming due to lee wave breaking. The larger RTI structures (in comparison with those at Re = 4000 in [1]) are evidently occur due to stronger molecular diffusion effects which also lead to instability development delay, as well as smaller density drop between heavier and lighter fluid layers (Figure 1).

The noted diffusion effects are seen for lower Re cases too. In particular, for Re = 1000 (Figure 3a) perturbation growth takes place later (25 < t < 40) and with λ_s ≈ 1.1H as the most unstable perturbation wavelength during the exponential growth stage (30 < t < 35). For Re = 500 the spectra peak with wavelength λ_s ≈ 2H (corresponding to the late RTI growth at 45 ≤ t ≤ 50) dominates almost persistently. For all different Re values studied here (as for the Re = 4000 case tested in [1, 2]), the large-structure enlargement happens with time.

For Re = 200 the spanwise instability magnitude is low and does not lead to the turbulence, having no inertial spectra range (Figure 3b), i.e. the critical value is within 200 < Re_c < 500 at Pr = 1.

In DNS for Pr = 1 and higher Re > 4000 than in [1, 2], the numerical noise becomes evident at large-wavenumber parts of spectra at late times corresponding to turbulent stages where the inertial range (k^{−3/2}) should be followed by larger slopes at higher k corresponding to dissipative ranges for both velocity and scalar spectra. In contrast, after the inertial range at 0.5 < k_s < 2.5 we observe lower spectra slopes at higher wave numbers 2.5 < k_s < 10 (Figure 3c) which indicates to excessive spurious numerical noise appearing because of poor resolution (or absence of SGS models to suppress such a noise). Therefore, the second critical value at Pr = 1 is within 4 000 < Re_c < 10 000, i.e. finer grids or LES tools are needed to predict the small-scale flow features adequately as confirmed by Figure 3(d).
Figure 1. Instantaneous contours of scalar at Re = 2000 and Pr = 1 for t = 25, 27, 40 and x = 2.5.

Figure 2. Spanwise spectra of scalar variance at Re = 2000 and Pr = 1 (averaged at 1.25 ≤ x ≤ 5.00, 1.25 ≤ z ≤ 3.75) for white-noise perturbation decay and RTI growth (left), turbulence stage (right).

In contrast to the case of Re = 200 and Pr = 1, runs with the higher Prandtl number Pr = 700 at both Re = 100 and 200 (Figures 4, 5) do show the RTI growth. For Re = 100 one can see existence of well-organised periodic structures (Figure 5) which may exist during long periods almost without change. This may be similar to measurement [4, 5] at Re = 150 where the spanwise periodicity was associated with toroidal vortices. The spanwise spectra (Figure 4a) do not have convincing inertial ranges, so the case of Pr = 700 and Re = 100 can be considered to have significant instability, but ‘undeveloped’ turbulence. In LES at Re = 200 (Figure 4b) the scalar spectra at the turbulent stage (t > 60) reveal both the inertial range ($k^{-5/3}$) and the dissipative one with higher slopes.

For intermediate value Pr = 7, the instability development at Re = 200 is still seen due to the RTI growth (which occurs quite late), in contrast to that at Pr = 1, with unsteady and intermittent in time of the resulting turbulent state (Figure 6). At Pr = 7 we found 100 < Re$_1$ < 200, and 2000 < Re$_2$ < 4000.

Note, the interesting features of spanwise periodicity at Pr = 7 and Re = 100 where scalar contours illustrate steady triangular cells surviving for long periods (Figure 7), and spanwise spectra do show weak instability without turbulence (Figure 6a). The similar cells appear for Re = 200, but during short periods at $t \sim 70$. Such a feature may demonstrate another scenario of instability growth (rather than RTI with mushroom-like structures), for instance the Rayleigh–Benard convection.
Figure 3. Spanwise spectra (as in Figure 2) for Pr = 1 in DNS at Re = 1000 (a), DNS at Re = 200 (compared to “turbulent stage” curves for other Re) (b), ILES at Re = 10^4 (c), LES at Re = 10^5 (d).

Figure 4. Spanwise spectra (as in Figures 2, 3) for Pr = 700: (a) ILES, Re = 100, (b) LES at Re = 200.
Figure 5. Instantaneous contours of scalar at $Pr = 700$ and $Re = 100$ for $t = 47, 85, 110$ and $x = 2.5$.

Figure 6. The same spectra as in Fig. 4, 5 for $Pr = 7$ in DNS at $Re = 100$ (a), $Re = 200$ (b).

Figure 7. Scalar contours in DNS at $Re = 100$ and $Pr = 7$ for $t = 67, 78, 100$ and $x = 2.5$. 
Figure 8. Scalar contours at $y = 0$ in DNS: (a) $Re = 4000$, $Pr = 1$, $t = 21$; (b) $Re = 4000$, $Pr = 7$, $t = 19$.

Figure 9. Contours of scalar in spanwise sections for different times in LES at $Re = 40\,000$, $Pr = 700$.

It should be noted that at for $Re \geq 4000$ and $Pr \geq 7$ the RTI periodicity is seen not only along the span, but also in the streamwise direction as shown by visualization of scalar contours at $y = 0$ (Figure 8). The similar effects are observed for $Re \geq 10^3$ and $Pr = 1$. Although, in both cases we have the under-resolved density field resulting in excessive numerical noise, this evidence may indicate to another route of instability growth than that discussed in [1] with quasi-two-dimensional structures.

For high $Re$ cases one can also see the multiple rows of the small-scale perturbations of RTI type (Figure 9) instead of the single row for lower Re cases (Figures 1, 5, 7). The similar picture is given in [8] where several unstable layers in overturning wave region have been observed, with the convective elements appearing and growing in the spanwise sections, creating a multilayered structure of vortices.

The LES results at high $Re$ and $Pr$ show that in all cases a SGS model (used in [7]) efficiently suppresses numerical noise in spectra and scalar contours, allowing us to view very fine convective structures along the span (Figure 9). Although, the SGS models lead to slight delay in the wave breaking development, working in fact together with molecular viscosity and diffusion.

From LES with higher $Re$, one can also see that the results become almost independent on $Pr$ or $Re$. This means that the SGS diffusion coefficients in fact are much larger than the molecular diffusion ones. Note, at finer grids (when SGS viscosity and diffusion coefficients will be smaller), the $Pr$ and $Re$ dependence will be more visible. Nevertheless, at very high $Re$ levels corresponding to geophysical flows, it is natural to omit the molecular viscosity/diffusion coefficients as done e.g. in [8].

Finally, the results for instability development scenarios can be summarized in Figure 10 where the most unstable perturbation wavelengths deduced from density snapshot visualizations and spectra peak analysis are given by symbols (each relates to one computation at different $Re$ or/and $Pr$). One can see (red dashed line in Figure 10) that the analytical expression [9]
The range of \( \Delta \neq 0 \) is seen in growing mushroom 

\[ \lambda' = 4\pi H(2\Delta \rho / \Delta \rho)(F_H)^2 \Re^{-2/3} \]  

written in dimensionless form [1] at \( \Re = 1 \) with the typical density drop, \( \Delta \rho \approx 10^{-3} \Delta \rho_H \), between the layers of heavier and lighter fluids related to earlier RTI stages is satisfied for dependence on \( \Re \).

Note, results at \( \Re = 700 \) are affected by the density field under-resolution, even at \( \Re = 200 \). As the result, the numerical wiggles can be seen in growing mushroom-like structures (Figures 1, 5, 8), and the numerical dissipation effect is a possible reason of obvious over-prediction of the most unstable perturbation wavelength (Figure 10), as well as the SGS model effect in the case of LES computations.

4. Conclusions

The results for scenarios of instability development in overturning lee waves have been presented for \( 50 \leq \Re \leq 40000 \) and \( 1 \leq \Pr \leq 700 \). The range of \( \Pr \) relates to density stratification in atmospheric and oceanic flows, whereas the range of \( \Re \) corresponds to that in measurements [3-5] and concurrent simulations [6, 10]. The typical phenomena in nature have much higher levels of \( \Re \). However, the preceding and present studies replicate well the cases in natural conditions, except for the instability development stage which can reveal various routes with different structures at different \( \Re \) and \( \Pr \). The scenarios viewed here allow us to extrapolate results to geophysical cases with much higher \( \Re \) values.

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