MONADIC SECOND ORDER FINITE SATISFIABILITY AND
UNBOUNDED TREE-WIDTH

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ABSTRACT. The finite satisfiability problem of monadic second order logic is
decidable only on classes of structures of bounded tree-width by the classic
result of Seese [24]. We prove that the following problem is decidable:

Input: (i) A monadic second order logic sentence \( \alpha \), and (ii) a sen-
tence \( \beta \) in the two-variable fragment of first order logic extended
with counting quantifiers. The vocabularies of \( \alpha \) and \( \beta \) may inter-
sect.

Output: Is there a finite structure which satisfies \( \alpha \land \beta \) such
that the restriction of the structure to the vocabulary of \( \alpha \) has
bounded tree-width? (The tree-width of the desired structure is
not bounded.)

As a consequence, we prove the decidability of the satisfiability problem by a
finite structure of bounded tree-width of a logic MSO\( ^\exists \text{card} \) extending monadic
second order logic with linear cardinality constraints of the form
\(|X_1| + \cdots + |X_r| < |Y_1| + \cdots + |Y_s|\) on the variables \(X_i, Y_j\) of the outer-most quantifier
block. We prove the decidability of a similar extension of WS1S.

1. Introduction

Monadic second order logic (MSO) is among the most expressive logics with
good algorithmic properties. It has found countless applications in computer sci-
ence in diverse areas ranging from verification and automata theory [13, 18, 25] to
combinatorics [16, 17], and parameterized complexity theory [8, 6].

The power of MSO is most visible over graphs of bounded tree-width, and with
second order quantifiers ranging over sets of edges [5]. (1) Courcelle’s famous theorem
shows that MSO model checking is decidable over graphs of bounded tree-width
in linear time [5] [1]. (2) Finite satisfiability by graphs of bounded tree-width is
decidable [5] (with non-elementary complexity) – thus contrasting Trakhtenbrot’s
undecidability result of first order logic. (3) Seese proved [24] that for each class \( \mathcal{K} \)
of graphs with unbounded tree-width, finite satisfiability of MSO by graphs in \( \mathcal{K} \) is
undecidable. Together, (2) and (3) give a fairly clear picture of the decidability of
finite satisfiability of MSO. It appeared that (3) gives a natural limit for decidability
of MSO on graph classes. For instance, finite satisfiability on planar graphs is
undecidable because their tree-width is unbounded.

While Courcelle and Seese circumvent Trakhtenbrot’s undecidability result by
restricting the classes of graphs (or relational structures), several other research
communities have studied syntactic restrictions of first order logic. Modal logic [26],
many temporal logics [21], [23, Chapter 24], the guarded fragment [9], many de-
scription logics [2], and the two-variable fragment [10] are restricted first order

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1The logic we denote by MSO is denoted MS\( _2 \) by Courcelle and Engelfriet [6].
logics with decidable finite satisfiability, and hundreds of papers on these topics have explored the border between decidability and undecidability. While many of the earlier papers exploited variations of the tree model property to show decidability, recent research has also focused on logics such as the two-variable fragment with counting $C^2$ \cite{11,22}, where finite satisfiability is decidable despite the absence of the tree model property. In a recent breakthrough result, Charatonik and Witkowski \cite{4} have extended this result to structures with built-in binary trees. Note that this logic is not a fragment of first order logic, but more naturally understood as a very weak second order logic which can express one specific second order property – the property of being a tree.

Our main result is a powerful generalization of the seminal result on decidability of the satisfiability problem of MSO over bounded tree-width and the recent theorem by \cite{4}: We show decidability of finite satisfiability of conjunctions $\alpha \land \beta$ where $\alpha$ is in MSO and $\beta$ is in $C^2$ by a finite structure $\mathcal{M}$ whose restriction to the vocabulary of $\alpha$ has bounded tree-width. (Theorem 1 in Section 3)

Let us put this result into perspective:

- The MSO decidability problem is a trivial consequence by setting $\beta$ to true; Charatonik and Witkowski’s result follows by choosing $\alpha$ to be an MSO formula which axiomatizes a $d$-ary tree, which is a standard construction \cite{6}.
- The decidability of model checking $\alpha \land \beta$ over a finite structure is a much simpler problem than ours: We just have to model check $\alpha$ and $\beta$ one after the other. In contrast, satisfiability is not obvious because $\alpha$ and $\beta$ can share relational variables. running two finite satisfiability algorithms for the two formulas independently may yield two models which disagree on the shared vocabulary. Thus, the problem we consider is similar in spirit to (but technically very different from) Nelson-Oppen \cite{20} combinations of theories.
- Our result trivially generalizes to Boolean combinations of sentences in the two logics.

**Proof Technique.** We show how to reduce our satisfiability problem for $\alpha \land \beta$ to the finite satisfiability of a $C^2$-sentence with a built-in tree, which is decidable by \cite{4}. The most significant technical challenge is to eliminate shared binary relation symbols between $\alpha'$ and $\beta$. Our Separation Theorem overcomes this challenge by an elegant construction based on local types of universe elements and a coloring argument for directed graphs. The second technical challenge is to replace the MSO-sentence $\alpha$ with an equi-satisfiable $C^2$-sentence $\alpha'$. To do so, we apply tools including the Feferman-Vaught theorem for MSO and translation schemes.

**Monadic Second Order Logic with Cardinalities.** Our main theorem implies new decidability results for monadic second order logic with cardinality constraints, i.e., expressions of the form $|X_1| + \ldots + |X_r| < |Y_1| + \ldots + |Y_t|$ where the $X_i$ and $Y_i$ are monadic second order variables. Klaedtke and Rueß \cite{15} showed that the decision problem for the theory of weak monadic second order logic with cardinality constraints of one successor (WS1S$^{\text{card}}$) is undecidable; they describe a decidable fragment where the second order quantifiers have no alternation and appear after the first order quantifiers in the prefix. Our main theorem implies decidability of a different fragment of WS1S with cardinalities: The fragment MSO$^{\exists \text{card}}$ consists of formulas $\exists X \psi$ where the cardinality constraints in $\psi$ involve only the monadic
second order variables from $\bar{X}$, cf. Theorem 6 in Section 7. Note that in contrast to [15], our fragment is a strict superset of WS1S.

For WS2S, we are not aware of results about decidable fragments with cardinalities. We describe a strict superset of MSO whose satisfiability problem over finite graphs of bounded tree-width is decidable, and which is syntactically similar to the WS1S extension above.

Expressive Power over Structures. Our main result extends the existing body of results on finite satisfiability by structures of bounded tree-width to a significantly richer set of structures. The structures we consider are $C^2$-axiomatizable extensions of structures of bounded tree-width. For instance, we can have interconnected doubly-linked lists as in Fig. 1(a), or a tree whose leaves are connected in a chain and have edges pointing to any of the nodes of a cyclic list as in Fig. 1(b).

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Such structures occur very naturally as shapes of dynamic data structures in programming – where cycles and trees are containers for data, and additional edges express relational information between the data. The analysis of semantic relations between data structures served as a motivation for us to investigate the logics in the current paper [3].

Being a cyclic list or a tree whose leaves are chained can be expressed in MSO and both of these data structures have tree-width at most 3. We can compel the edges between the tree and the cyclic list to obey $C^2$-expressible constraints such as:

- every leaf of the tree has a single edge to the cyclic list;
- every node of the cyclic list has an incoming edge from at least one leaf of the tree; or
- any two leaves pointing to the same node of the cyclic list agree on membership in some unary relation.

Note that while the structures we consider may contain grids of unbounded sizes as subgraphs, the logic cannot axiomatize them.

2. Background

This section introduces basic definitions and results in model theory and graph theory. We follow [14] and [6].

The two-variable fragment with counting $C^2$ is the extension of the two-variable fragment of first order logic with first order counting quantifiers $\exists^{\leq n}$, $\exists^{\geq n}$, $\exists^{= n}$, for every $n \in \mathbb{N}$. Note that $C^2$ remains a fragment of first order logic.
Monadic Second order logic MSO is the extension of first order logic with set quantifiers which can range over elements of the universe or subsets of relations. Throughout the paper all structures consist of unary and binary relations only. Structures are finite unless explicitly stated otherwise (in the discussion of WS1S). Let $C$ be a vocabulary (signature). The arity of a relation symbol $C \in C$ is denoted by $arity(C)$. The set of unary (binary) relation symbols in $C$ are $un(C)$ ($bin(C)$). We write MSO($C$) for the set of MSO-formulas on the vocabulary $C$. The quantifier rank of a formula $\varphi \in MSO$, i.e. the maximal depth of nested quantifiers in $\varphi$ is denoted $qr(\varphi)$. We denote by $A_1 \sqcup A_2$ the disjoint union of two $C$-structures $A_1$ and $A_2$. Given vocabularies $C_1 \subseteq C_2$, a $C_2$-structure $A_2$ is an expansion of a $C_1$-structure $A_1$ if $A_1$ and $A_2$ agree on the symbols in $C_1$; in this case $A_1$ is the reduct of $A_2$ to $C_1$, i.e. $A_1$ is the $C_1$-reduct of $A_2$. We denote the reduct of $A_2$ to $C_1$ by $A_2|_{C_1}$. A $C$-structure $A_0$ with universe $A_0$ is a substructure of a $C$-structure $A_1$ if $A_0 \subseteq A_1$ and for every $R \in C$, $R^{A_0} = R^{A_1} \cap A_0^{arity(R)}$. We say that $A_0$ is the substructure of $A_1$ generated by $A_0$.

Graphs are structures of the vocabulary $C_G = \langle s \rangle$ consisting of a single binary relation symbol $s$. Graphs are simple and undirected unless explicitly stated otherwise. Tree-width $tw(G)$ is a graph parameter indicating how close a simple undirected graph $G$ is to being a tree, cf. [6]. It is well-known that a graph has tree-width at most $k$ iff it is a partial $k$-tree. A partial $k$-tree is a subgraph of a $k$-tree. $k$-trees are built inductively from the $(k + 1)$-clique by repeated addition of vertices, each of which is connected with $k$ edges to a $k$-clique. The Gaifman graph $Gaif(A)$ of a $C$-structure $A$ is the graph whose vertex set is the universe of $A$ and whose edge set is the union of the symmetric closures of $C^A$ for every $C \in bin(C)$. Note the unary relations of $A$ play no role in $Gaif(A)$. The tree-width $tw(A)$ of a $C$-structure $A$ is the tree-width of its Gaifman graph. In this paper, tree-width is a parameter of finite structures only. Fix $k \in \mathbb{N}$ for the rest of the paper. $k$ will denote the tree-width bound we consider.

We introduce the notion of oriented $k$-trees which refines the notion of $k$-trees. Let $R = \{R_1, \ldots, R_k\}$ be a vocabulary consisting of binary relation symbols. An oriented $k$-tree is an $R$-structure $A$ in which all $R^{A}$ are total functions and whose Gaifman graph $Gaif(A)$ is a partial $k$-tree.

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2 On relational structures, MSO is also known as Guards Second Order logic GSO. The results of this paper extend to CMSO, the extension of MSO with modular counting quantifiers.

3 Since we explicitly allowed quantification over subsets of relations for MSO, we do not view graphs and structures as incidence structures, in contrast to [6] Sections 1.8.1 and 1.9.1. 
Lemma 1. Every \( C \)-structure \( \mathcal{M} \) of tree-width \( k \) can be expanded into a \( (C \cup R) \)-structure \( \mathcal{M} \) such that: (i) \( \mathcal{M}|_R \) is an oriented \( k \)-tree, (ii) \( \text{Gaif}(\mathcal{M}) \) is a subgraph of \( \text{Gaif}(\mathcal{M}|_R) \), and (iii) the tree-width of \( \mathcal{M} \) is \( k \).

The oriented 2-tree in Fig. 2(b) is an expansion of the directed graph in Fig. 2(a) as guaranteed in Lemma 1. In Fig. 2(b), \( R_1 \) and \( R_2 \) are denoted by the dashed arrows and the dotted arrows, respectively. There are several other oriented \( k \)-trees which expand Fig. 2(a) and fulfill the requirements in Lemma 1.

To see that Lemma 1 holds, we describe a construction of \( \mathcal{M} \) echoing the process of constructing \( k \)-trees above. For each vertex \( u \) of the initial \((k+1)\)-clique, we can set the values of \( R_1^N(u), \ldots, R_k^N(u) \) to be the other \( k \) vertices of the clique. When a new vertex \( u \) is added to the \( k \)-tree, \( k \) edges incident to it are added. We set \( R_1^N(u), \ldots, R_k^N(u) \) to be the set of vertices incident to \( u \). For oriented \( k \)-trees whose Gaifman graph is not a \( k \)-tree the construction of an oriented \( k \)-tree is augmented by changing the value of \( R_i^N(u) \) to \( R_i^N(u) = u \) whenever \( R_i^N(u) \) is not well-defined. This can happen when the target of \( u \) under \( R_i^N \) is a vertex which was eliminated by taking the subgraph of a \( k \)-tree to obtain the partial \( k \)-tree.

3. Overview of the Main Theorem and its Proof

The precise statement of the main theorem is as follows:

**Theorem 1** (Main Theorem). Let \( C_{\text{bnd}} \) and \( C_{\text{unb}} \) be vocabularies. Let \( s \) be a binary relation symbol not in \( C_{\text{bnd}} \cup C_{\text{unb}} \). Let \( \alpha \in \text{MSO}(C_{\text{bnd}}) \) and \( \beta \in C^2(C_{\text{unb}}) \). There is an effectively computable sentence \( \delta \in C^2(D) \) over a vocabulary \( D \supseteq \{s\} \) such that the following are equivalent:

(i) There is a \((C_{\text{bnd}} \cup C_{\text{unb}})\)-structure \( \mathcal{M} \) such that \( \mathcal{M} \models \alpha \land \beta \) and \( \text{tw}(\mathcal{M}|_{C_{\text{bnd}}}) \leq k \).

(ii) There is a \( D \)-structure \( \mathcal{N} \) such that \( \mathcal{N} \models \delta \) and \( s^{\mathcal{N}} \) is a binary tree.

The first step towards proving Theorem 1 is the Separation Theorem:

**Theorem 2** (Separation Theorem). Let \( C_{\text{bnd}} \) and \( C_{\text{unb}} \) be vocabularies. Let \( \alpha \in \text{MSO}(C_{\text{bnd}}) \) and \( \beta \in C^2(C_{\text{unb}}) \). There are effectively computable sentences \( \alpha' \in \text{MSO}(C_{\text{bnd}}) \) and \( \beta' \in C^2(C_{\text{unb}}) \) over vocabularies \( D_{\text{bnd}} \) and \( D_{\text{unb}} \) such that \( D_{\text{bnd}} \cap D_{\text{unb}} \) only contains unary relation symbols and the following are equivalent:

(i) There is a \((C_{\text{bnd}} \cup C_{\text{unb}})\)-structure \( \mathcal{M} \) with \( \mathcal{M} \models \alpha \land \beta \) and \( \text{tw}(\mathcal{M}|_{C_{\text{bnd}}}) \leq k \).

(ii) There is a \((D_{\text{bnd}} \cup D_{\text{unb}})\)-structure \( \mathcal{N} \) with \( \mathcal{N} \models \alpha' \land \beta' \) and \( \text{tw}(\mathcal{N}|_{D_{\text{bnd}}}) \leq k \).

In conjunction with Theorem 2, we only need to prove Theorem 1 in the case that the MSO-formula \( \alpha \) and the \( C^2 \)-formula \( \beta \) only share unary relation symbols. The significance of Theorem 2 is that it allows us to use tools designed for MSO in our more involved setting. The proof of Theorem 2 uses notions of types for \( C^2 \)-sentences in Scott normal form, oriented \( k \)-trees, coloring arguments, and an induction on ranks of structures. Theorem 2 is discussed in Section 4. The next step is to move from structures whose reducts have bounded tree-width to structures which contain a binary tree.

**Lemma 2.** Let \( C_{\text{bnd}} \) and \( C_{\text{unb}} \) be vocabularies such that \( C_{\text{bnd}} \cap C_{\text{unb}} \) contains only unary relation symbols. Let \( s \) be a binary relation symbol. There is a vocabulary \( D_{\text{bnd}} \) consisting of \( s \) and unary relation symbols only, and, for every \( \alpha \in \text{MSO}(C_{\text{bnd}}) \) and \( \beta \in C^2(C_{\text{unb}}) \), effectively computable sentences \( \alpha' \in \text{MSO}(D_{\text{bnd}}) \) and \( \beta' \in C^2(D_{\text{bnd}} \cup C_{\text{unb}}) \) such that the following are equivalent:
There is a \((C_{\text{bnd}} \cup C_{\text{unb}})\)-structure \(M\) such that \(M|_{C_{\text{unb}}} = \alpha \land \beta\) and such that \(\text{tw}(M|_{C_{\text{bnd}}}) \leq k\).

(ii) There is a \((D_{\text{bnd}} \cup C_{\text{unb}})\)-structure \(N\) such that \(N|_{\alpha}\) and \(s^N\) is a binary tree.

Technically, Lemma 2 is proved using a translation scheme which maps structures with a binary tree into structures whose \(C_{\text{bnd}}\)-reducts have tree-width at most \(k\), and conversely, each of the latter structures is the image of a structure with a binary tree under the translation scheme. Translation schemes capturing the graphs of tree-width at most \(k\) as the image of labeled trees were studied in the context of decidability and model checking of MSO [1]. We need a more refined construction to ensure that the translation scheme also behaves correctly on \(C_2\)-sentences, i.e. that it maps \(C_2\)-sentences to \(C_2\)-sentences, see Lemma 2 in Section 5.

Now that we have reduced our attention to the case that our structures contain a binary tree, we can replace MSO-sentences with equi-satisfiable \(C_2\)-sentences.

Lemma 3. Let \(C\) be a vocabulary which consists only of a binary relation symbol \(s\) and unary relation symbols. Let \(\alpha\) be an MSO\((C)\)-sentence. There is an effectively computable \(C_2(D)\)-sentence \(\gamma\) over a vocabulary \(D \supseteq C\) such that for every \(C\)-structure \(M\) in which \(s^M\) is a binary tree the following are equivalent:

(i) \(M \models \alpha\)  (ii) There is a \(D\)-structure \(N\) expanding \(M\) such that \(N \models \gamma\).

For the proof of Lemma 3 we use a Feferman-Vaught type theorem which states that the Hintikka type (i.e. MSO types) of a binary tree labeled with unary relation symbols depends only on the Hintikka types of its children. We can therefore axiomatize in \(C_2\) that the Hintikka type of the labeled binary tree implies a given MSO-sentence.

Having replaced the MSO-sentence in statement (ii) of Lemma 2 with a \(C_2\)-sentence, we are left with the problem of deciding whether a \(C_2\)-sentence is satisfiable by a structure in which a specified relation is a binary tree, which has recently been shown to be decidable:

Theorem 3 (Charatonik and Witkowski [4]). Let \(C\) be a vocabulary which contains a binary relation symbol \(s\). Given a \(C_2(C)\)-sentence \(\varphi\), it is decidable whether \(\varphi\) is satisfiable by a structure \(M\) in which \(s^M\) is a binary tree.

4. Separation Theorem

4.1. Basic Definitions and Results.

1-types and 2-types. We begin with some notation and definitions in the spirit of the literature on decidability of \(C_2\), cf. e.g. [22, 4]. Let \(A\) be a vocabulary of unary and binary relations.

A 1-type \(\pi\) is a maximal consistent set of atomic \(A\)-formulas or negations of atomic \(A\)-formulas with free variable \(x\), i.e., exactly one of \(A(x)\) and \(\neg A(x)\) belongs to \(\pi\) for every unary relation symbol \(A \in A\), and exactly one of \(B(x, x)\) and \(\neg B(x, x)\) belongs to \(\pi\) for every binary relation symbol \(B \in A\). We denote by \(\text{char}_\pi(x) = \bigwedge_{\varphi \in \pi} \varphi\) the formula that characterizes the 1-type \(\pi\). We denote by 1-Types\((A)\) the set of 1-types over \(A\).

A 2-type \(\lambda\) is a maximal consistent set of atomic \(A\)-formulas or negations of atomic \(A\)-formulas with free variables \(x\) and \(y\), i.e., for every \(z \in \{x, y\}\) and unary
relation symbol \( A \in \mathcal{A} \), exactly one of \( A(z) \) and \( \neg A(z) \) belongs to \( \lambda \), and for every \( z_1, z_2 \in \{x, y\} \) and binary relation symbol \( B \in \mathcal{A} \), exactly one of \( B(z_1, z_2) \) and \( \neg B(z_1, z_2) \) belongs to \( \lambda \). We note that the equality relation \( \approx \) is also part of a 2-type. We write \( \lambda^{-1} \) for the 2-type obtained from \( \lambda \) by substituting all occurrences of \( x \) resp. \( y \) with \( y \) resp. \( x \). We write \( \lambda_x \) for the 1-type obtained from \( \lambda \) by restricting \( \lambda \) to formulas with free variable \( x \). We write \( \lambda_y \) for the 1-type obtained from \( \lambda \) by restricting \( \lambda \) to formulas with free variable \( y \) and substituting \( y \) with \( x \). We denote by \( \text{char}_\lambda(x, y) = \bigwedge_{i \in \lambda} i \) the formula that characterizes the 2-type \( \lambda \). We denote by \( 2\text{-Types}(\mathcal{A}) \) the set of 2-types over \( \mathcal{A} \).

Let \( \mathcal{M} \) be a \( \mathcal{A} \)-structure. We denote by \( 1\text{-tp}^\mathcal{M}(u) \) the unique 1-type \( \pi \) such that \( \mathcal{M} \models \text{char}_\pi(u) \). For elements \( u, v \) of \( \mathcal{M} \), we denote by \( 2\text{-tp}^\mathcal{M}(u, v) \) the unique 2-type \( \lambda \) such that \( \mathcal{M} \models \text{char}_\lambda(u, v) \). We denote by \( 2\text{-tp}(\mathcal{M}) = \{2\text{-tp}^\mathcal{M}(u, v) \mid u, v \text{ elements of } \mathcal{M}\} \) the set of 2-types realized by \( \mathcal{M} \). The following lemma is easy to see:

**Lemma 4.** Let \( \mathcal{M}_1, \mathcal{M}_2 \) be two \( \mathcal{A} \)-structures over the same universe \( M \) and let \( \phi = \forall x, y, \chi \in C^2(\mathcal{A}) \) with \( \chi \) quantifier-free. If \( 2\text{-tp}(\mathcal{M}_1) = 2\text{-tp}(\mathcal{M}_2) \), then \( \mathcal{M}_1 \models \phi \) iff \( \mathcal{M}_2 \models \phi \).

(See proof in Appendix 8.4.)

**Scott Normal Form and \( T \)-functionality.** \( C^2 \)-sentences have a Scott-Normal Form, cf. [12], which can be obtained by iteratively applying Skolemization and introducing new predicates for subformulas, together with predicates ensuring the soundness of this transformation:

**Lemma 5** (Scott Normal Form, [12]). For every \( C^2 \)-sentence \( \beta \) there is a \( C^2 \)-sentence \( \beta' \) of the form

\[
\forall x, y. \chi \land \bigwedge_{i \in [l]} \forall x. \exists y. S_i(x, y),
\]

with \( \chi \) quantifier-free, over an expanded vocabulary such that \( \beta \) and \( \beta' \) are equi-satisfiable. Moreover, \( \beta' \) is computable. The expanded vocabulary contains in particular the fresh binary relation symbols in \( \mathcal{S} = \{S_1, \ldots, S_l\} \).

Let \( T \) be a set of binary relation symbols. We say a structure \( \mathcal{M} \) is \( T \)-functional, if for every \( T \in T \), \( T^\mathcal{M} \) is a total function on the universe of \( \mathcal{M} \). Observe the following are equivalent for every structure \( \mathcal{M} \):

(i) \( \mathcal{M} \) satisfies Eq. (1), and (ii) \( \mathcal{M} \models \forall x, y, \chi \) and \( \mathcal{M} \) is \( \mathcal{S} \)-functional.

**Message Types and Chromaticity.** Let \( T \subseteq \text{bin}(\mathcal{A}) \) be a subset of the binary relation symbols of \( \mathcal{A} \). We write \( \lambda \in T\text{-MsgTypes}(\mathcal{A}) \) and say \( \lambda \) is a \( T \)-message type, if \( \lambda \in 2\text{-Types}(\mathcal{A}) \) and \( T(x, y) \in \lambda \) for some \( T \in T \). Let \( \mathcal{M} \) be a \( \mathcal{A} \)-structure with universe \( M \). We define \( E^\mathcal{M}_T = \{(u, v) \in M^2 \mid \text{there is a } T \in T \text{ with } \mathcal{M} \models T(u, v)\} \). The \( T \)-message-graph is the directed graph \( G^\mathcal{M}_T = (M, E) \), where \( E = \{(u, v) \in M^2 \mid u \neq v \} \cup \{(u, v) \in E^\mathcal{M}_T \circ E^\mathcal{M}_T \} \), where \( R \circ S = \{(a, b) \mid \text{there is a } c \text{ with } (a, c) \in R \text{ and } (c, b) \in S\} \) denotes the usual composition of relations. We say \( \mathcal{M} \) is \( T \)-chromatic, if \( 1\text{-tp}^\mathcal{M}(u) \neq 1\text{-tp}^\mathcal{M}(v) \) for all \( u, v \in E \).

We note that if \( \mathcal{M} \) is \( T \)-functional, then \( G^\mathcal{M}_T \) has out-degree \( \text{deg}^+(u) \leq |T|^2 \) for all \( u \in M \). This allows us to prove Lemma 6 based on Lemma 7; the proofs can be found in Appendices 8.6 and 8.5.
Lemma 6. There is a finite set of unary relations symbols $\text{colors}_T$ such that every $T$-functional $\mathcal{A}$-structure can be expanded to a $T$-chromatic $(\mathcal{A} \cup \text{colors}_T)$-structure.

Lemma 7. Let $G = (V, E)$ be a directed graph with out-degree $\deg^+(v) \leq k$ for all $v \in V$. Then, the underlying undirected graph has a proper $2k + 1$-coloring.

4.2. Separation Theorem. Let $G = (V, E)$ be an (undirected) graph. We say $G$ is $k$-bounded, if the edges of $G$ can be oriented such that every node of $G$ has out-degree less than $k$. We say a structure $\mathcal{M}$ is $k$-bounded if its Gaifman graph is $k$-bounded.

Theorem 2 (Separation Theorem). Let $k$ be a natural number. Let $\mathcal{C}_{\text{bnd}}$ and $\mathcal{C}_{\text{unb}}$ be vocabularies. Let $\alpha \in \text{MSO}(\mathcal{C}_{\text{bnd}})$ and $\beta \in \text{C}^2(\mathcal{C}_{\text{unb}})$. There are effectively computable sentences $\alpha' \in \text{MSO}(\mathcal{D}_{\text{bnd}})$ and $\beta' \in \text{C}^2(\mathcal{D}_{\text{unb}})$ over vocabularies $\mathcal{D}_{\text{bnd}}$ and $\mathcal{D}_{\text{unb}}$ such that $\mathcal{D}_{\text{bnd}} \cap \mathcal{D}_{\text{unb}}$ only contains unary relation symbols such that for every $k$-bounded graph $G$ the following are equivalent:

(i) There is a $(\mathcal{C}_{\text{bnd}} \cup \mathcal{C}_{\text{unb}})$-structure $\mathcal{M}$ with $\mathcal{M} \models \alpha \land \beta$ and $\text{Gaif}(\mathcal{M}_{|\mathcal{C}_{\text{bnd}}}^\text{bnd}) = G$.

(ii) There is a $(\mathcal{D}_{\text{bnd}} \cup \mathcal{D}_{\text{unb}})$-structure $\mathcal{R}$ with $\mathcal{R} \models \alpha' \land \beta'$ and $\text{Gaif}(\mathcal{R}_{|\mathcal{D}_{\text{bnd}}}^\text{bnd}) = G$.

We assume that $\beta$ is in the form given in Eq. (4) for some set of binary relation symbols $S = \{S_1, \ldots, S_l\} \subseteq \mathcal{C}_{\text{unb}}$ and quantifier-free $C^2$-formula $\chi$. Let $\mathcal{R} = \{R_1, \ldots, R_k\}$ be a set of fresh binary relation symbols. We set $T = S \cup \mathcal{R}$. We begin by giving an intuition for the proof of the Separation Theorem in three stages.

1. Syntactic separation coupled with semantic constraints. For a binary relation symbol $B$, we define its copy as the relation symbol $\overline{B}$. For every vocabulary $\mathcal{A}$, we define its copy $\overline{\mathcal{A}} = \text{un}(\mathcal{A}) \cup \{\overline{B} \mid B \in \mathcal{A}\}$ to be the unary relation symbols of $\mathcal{A}$ plus the copies of its binary relations symbols. We assume that copied relation symbols are distinct from non-copied symbols, i.e., $\text{bin}(\mathcal{A}) \cap \text{bin}(\overline{\mathcal{A}}) = \emptyset$. For a formula $\varphi$ over vocabulary $\mathcal{A}$, we define its copy $\overline{\varphi}$ over vocabulary $\overline{\mathcal{A}}$ as the formula obtained from $\varphi$ by substituting every occurrence of a binary relation symbol $B \in \mathcal{A}$ with $\overline{B}$.

The sentences $\overline{\alpha}$ (the copy of $\alpha$) and $\beta$ do not share any binary relation symbols. Clearly, (i) from Theorem 2 holds iff

(1) $\overline{\alpha} \land \beta$ is satisfied by a $(\mathcal{C}_{\text{bnd}} \cup \mathcal{C}_{\text{unb}}) \cup (\mathcal{C}_{\text{bnd}} \cup \mathcal{C}_{\text{unb}}))$-structure $\mathcal{M}$ with $\mathcal{M}^\text{bnd} = \overline{\mathcal{M}}^\text{bnd}$ for all $B \in \text{bin}(\mathcal{C}_{\text{bnd}})$ and $\text{Gaif}(\mathcal{M}_{|\mathcal{C}_{\text{bnd}}}^\text{bnd}) = G$.

In the next two stages we will construct $\alpha'$ and $\beta'$ so that (1) is equivalent to (ii) from Theorem 2. More precisely, we will construct sentences $\mu_{\text{bnd}}, \mu_{\text{unb}} \in \text{C}^2(\mathcal{D}_{\text{unb}})$ with $\mathcal{D}_{\text{unb}} \supseteq \mathcal{C}_{\text{bnd}} \cup \mathcal{C}_{\text{unb}}$ and $\mathcal{D}_{\text{bnd}} = \mathcal{D}_{\text{unb}}$ such that (I) is equivalent to (II):

(II) $(\overline{\alpha} \land \overline{\mu}_{\text{bnd}}) \land (\beta \land \mu_{\text{unb}})$ is satisfied by a $(\mathcal{D}_{\text{bnd}} \cup \mathcal{D}_{\text{unb}})$-structure $\mathcal{R}$ with $\text{Gaif}(\mathcal{R}_{|\mathcal{D}_{\text{bnd}}}^\text{bnd}) = G$.

2. Representation of $k$-bounded structures using functions and unary relations. Theorem 2 as well as (I) and (II) involve reducts which are $k$-bounded structures. $k$-bounded $\mathcal{A}$-structures $\mathcal{A}$ can be represented by introducing new binary relation symbols interpreted as functions and new unary relation symbols as follows.

(a) We add $k$ fresh relation symbols $\mathcal{R} = \{R_1, \ldots, R_k\}$ and axiomatize that these relations are interpreted as total functions.

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The Separation Theorem remains correct if we replace $C^2$ with any logic containing $C^2$ which is closed under conjunction.
(b) We add fresh unary relations \{\rho_\lambda \mid \lambda \in \mathcal{R}\text{-MsgTypes}(A)\} and axiomatize that every element labeled by \rho_\lambda has an outgoing edge with 2-type \lambda. The symbols \rho_\lambda are called unary 2-type annotations.

(c) We axiomatize that \text{Gaif}(\mathfrak{A}|_{\text{bin}(A)}) = \text{Gaif}(\mathfrak{A}|_{\mathcal{R}}).

In other words, the functions interpreting \( R_1, \ldots, R_k \) witness that \( \mathfrak{A} \) can be oriented so that every node in the Gaifman graph of \( \mathfrak{A} \) has outdegree at most \( k \).

The 2-type of each edge \((u, v)\) in \( \mathfrak{A} \) is encode by putting the unary relation symbol \( \rho_\lambda \) of the 2-type of \((u, v)\) on the source \( u \) in the orientation.

Given a \((\bigcup \mathcal{C}_{\text{unb}} \cup \mathcal{C}_{\text{bnd}})\)-structure \( \mathfrak{M} \), we will use the above representation twice, on \( \mathfrak{M}|_{\mathcal{C}_{\text{bnd}}} \) and \( \mathfrak{M}|_{\mathcal{C}_{\text{unb}}} \), by axiomatizing that every element labeled by \( \rho_\lambda \) has an outgoing edge with 2-type \( \lambda \) and an outgoing edges with 2-type \( \overline{\lambda} \). This will allow us to replace the condition from \((I)\) that \( B^\mathfrak{M} = \overline{B}^\mathfrak{M} \) for all \( B \in \text{bin}(\mathcal{C}_{\text{bnd}}) \) with the condition that \( R^\mathfrak{M}_i = \overline{R}^\mathfrak{M}_i \) for all \( R_i \in \mathcal{R} \). We will define a vocabulary \( \mathcal{E} \supseteq \mathcal{C}_{\text{bnd}} \cup \mathcal{C}_{\text{unb}} \cup \mathcal{R} \). Let \( \mathcal{P}_{\mathcal{R}} = \{ \rho_\lambda \mid \lambda \in \mathcal{R}\text{-MsgTypes}(\mathcal{E}) \} \). We will define two vocabularies \( \mathcal{D}_{\text{unb}} \supseteq \mathcal{C}_{\text{bnd}} \cup \mathcal{C}_{\text{unb}} \cup \mathcal{R} \cup \mathcal{P}_{\mathcal{R}} \) and \( \mathcal{D}_{\text{bnd}} = \mathcal{D}_{\text{unb}} \). According to (a), (b), and (c), we will construct \( \nu_{\text{bnd}}, \nu_{\text{unb}} \in \mathcal{C}^2(\mathcal{D}_{\text{unb}}) \) such that \((I)\) is equivalent to the following:

\[(I)’ \quad (\overline{\pi} \land \overline{\mathcal{P}_{\text{bnd}}}) \land (\beta \land \nu_{\text{unb}}) \]

is satisfied by a \( \mathcal{D}_{\text{bnd}} \cup \mathcal{D}_{\text{unb}} \)-structure \( \mathfrak{M} \) with \( R^\mathfrak{M}_i = \overline{R}^\mathfrak{M}_i \) for all \( R_i \in \mathcal{R} \) and \( \text{Gaif}(\mathfrak{M}|_{\mathcal{R}}) = G \).

3. Establishing the semantic condition of \((I’)\) by swapping edges. Here we discuss how to show the implication from \((II)\) to \((I’)\) and make the vocabularies \( \mathcal{D}_{\text{bnd}} \) and \( \mathcal{D}_{\text{unb}} \) precise. Let \( \mathfrak{M} \) be a \((\mathcal{D}_{\text{bnd}} \cup \mathcal{D}_{\text{unb}})\)-structure with \( \mathfrak{M} \models (\overline{\pi} \land \overline{\mathcal{P}_{\text{bnd}}}) \land (\beta \land \nu_{\text{unb}}) \). It simplifies the discussion to split a \((\mathcal{D}_{\text{bnd}} \cup \mathcal{D}_{\text{unb}})\)-structure \( \mathfrak{M} \) into two \( \mathcal{D}_{\text{unb}} \)-structures. The \( \mathcal{D}_{\text{unb}} \)-structure \( \mathfrak{L} \) is \( \mathfrak{M}|_{\mathcal{D}_{\text{unb}}} \). The \( \mathcal{D}_{\text{unb}} \)-structure \( \mathfrak{L’} \) is obtained from \( \mathfrak{M}|_{\mathcal{D}_{\text{unb}}} \) by renaming copies of relation symbols \( \overline{B} \) to \( B \) — i.e., we define the \( \mathcal{D}_{\text{unb}} \)-structure \( \mathfrak{L’} \) by setting \( 1_\mathfrak{L’}^\mathfrak{L’}(u) = 1_\mathfrak{M}^\mathfrak{L}(u) \) for all \( u \in M \) and setting \( \mathfrak{L’} \models B(u, v) \) iff \( \mathfrak{M} \models B(u, v) \) for all \( u, v \in M \) and \( B \in \text{bin}(\mathcal{D}_{\text{bnd}}) \). The interpretations of the relations \( R_i \) might differ in \( \mathfrak{L} \) and \( \mathfrak{L’} \). Observe that we have \( \mathfrak{L’} \models \alpha \) and \( \mathfrak{L} \models \beta \). The key idea of the proof is prove the existence of a sequence of structures \( \mathfrak{L} = \mathfrak{L}_0, \ldots, \mathfrak{L}_p \), where each \( \mathfrak{L}_{i+1} \) is obtained from \( \mathfrak{L}_i \) by swapping edges, until the interpretations of the relations \( R_i \) agree in \( \mathfrak{L}_p \) and \( \mathfrak{L’} \). The edge swapping operation is a local operation which involves changing the 2-types of at most 4 edges.

The edge swapping operation satisfies two crucial preservation requirements: edge swapping preserves \((P_1)\) the truth value of \( \beta \), i.e. \( \mathfrak{L}_p \models \beta \), and \((P_2)\) \( \mathcal{R} \)-functionality. The universal constraint \( \forall x, y, \chi \in \beta \) is maintained under edge swapping because this operation does not change the set of 2-types (see Lemma 3). To satisfy the preservation requirements \((P_1)\) and \((P_2)\), all that remains is to guarantee the existence of a sequence of edge swapping preserving \( \mathcal{T} \)-functionality. Note that \( \mathcal{T} \)-functionality amounts to \( \mathcal{R} \)-functionality and \( \mathcal{S} \)-functionality. We use two main techniques for ensuring the preservation of \( \mathcal{T} \)-functionality: chromaticity and unary 2-type annotations. We will axiomatize that the structures \( \mathfrak{L} \) and \( \mathfrak{L’} \) are chromatic and we will take care that chromaticity is maintained during edge swaps. We will add fresh unary relation symbols \( \mathcal{P}_S = \{ \rho_\lambda \mid \lambda \in \mathcal{S}\text{-MsgTypes}(\mathcal{E}) \} \) and axiomatize that every element of \( \mathfrak{M} \) labeled by \( \rho_\lambda \) has an outgoing edge with 2-type \( \lambda \) and an outgoing edge with 2-type \( \overline{\lambda} \).
Proof of the Separation Theorem. We now start the formal proof of the Separation Theorem. Let colorSet be the vocabulary from Lemma 6. We set $E = C_{bnd} \cup C_{unb} \cup R \cup \text{colorSet}_T$. We set $D_{bnd} = E \cup P_S \cup P_R$ and $D_{unb} = D_{bnd}$. Next we will define formulas $\psi \in \text{MSO}(D_{bnd})$ and $\beta' \in C^2(D_{bnd})$, and set $\alpha' = \overline{\psi} \in \text{MSO}(D_{unb})$. We set $\nu_{bnd} = \delta \land \theta \land \epsilon \land \zeta_{unb}$, $\nu_{unb} = \delta \land \theta \land \zeta_{unb}$, $\mu_{bnd} = \nu_{bnd} \land \eta$, and $\mu_{unb} = \nu_{unb} \land \eta$, $\psi = \alpha \land \mu_{bnd}$, and $\beta' = \beta \land \mu_{unb}$, where:

$$\delta = \bigwedge_{i \in [k]} \forall x. \exists y. R_i(x, y)$$
$$\theta = \forall x, y. x \neq y \rightarrow \left( \bigvee_{i \in [k]} R_i(x, y) \lor R_i(y, x) \leftrightarrow \bigvee_{B \in \text{bin}(C_{bnd})} B(x, y) \lor B(y, x) \right)$$
$$\epsilon = \bigwedge_{B \in \text{bin}(C_{unb})} \forall x, y. x \neq y \rightarrow \left( B(x, y) \rightarrow \bigvee_{i \in [k]} R_i(x, y) \lor R_i(y, x) \right)$$

$$\zeta_{bnd} = \bigwedge_{\lambda \in R - \text{MsgTypes}(E)} \forall x, y. x \neq y \rightarrow \left( \bigwedge_{(\lambda \in T - \text{MsgTypes}(E)) \land \lambda^{-1} \in R - \text{MsgTypes}(E))} P_\lambda(x) \leftrightarrow \exists y. \text{char}_\lambda(x, y) \right)$$

$$\zeta_{unb} = \bigwedge_{\lambda \in T - \text{MsgTypes}(E)} \forall x, y. x \neq y \rightarrow \left( \bigwedge_{\lambda \in \text{MsgTypes}(E)} \forall \lambda. P_\lambda(x) \leftrightarrow \exists y. \text{char}_\lambda(x, y) \right)$$

$$\eta = \bigwedge_{\lambda \in T - \text{MsgTypes}(E)} \forall x, y. T(x, y) \land \neg T'(x, y) \rightarrow \neg \text{char}_\lambda(y)$$

$\delta$ expresses that each $R_i$ is interpreted as a total function, $\theta$ expresses that for every $D_{unb}$-structure $\mathfrak{L}$ with $\mathfrak{L} \models \theta$ we have that $\text{Gaif}(\mathfrak{L}|_R) = \text{Gaif}(\mathfrak{L}|_{C_{bnd}})$, $\epsilon$ expresses that for every $D_{unb}$-structure $\mathfrak{L}$ with $\mathfrak{L} \models \epsilon$ we have that $\text{Gaif}(\mathfrak{L}|_{D_{unb}})$ is a subgraph of $\text{Gaif}(\mathfrak{L}|_R)$, $\zeta$ expresses that for every $\lambda \in T - \text{MsgTypes}(E)$ with $\lambda \in \mathcal{R} - \text{MsgTypes}(E)$ or $\lambda \in \mathcal{R} - \text{MsgTypes}(E)$, $D_{unb}$-structure $\mathfrak{L}$ and element $u$ of $\mathfrak{L}$ we have $\mathfrak{L} \models P_\lambda(u)$ iff there is an element $v$ of $\mathfrak{L}$ such that $2\cdot \text{tp}^{\mathfrak{L}}_\lambda(u, v) = \lambda$, $\zeta$ expresses that for every $\lambda \in T - \text{MsgTypes}(E)$, $D_{unb}$-structure $\mathfrak{L}$ and element $u$ of $\mathfrak{L}$ we have $\mathfrak{L} \models P_\lambda(u)$ iff there is an element $v$ of $\mathfrak{L}$ such that $2\cdot \text{tp}^{\mathfrak{L}}_\lambda(u, v) = \lambda$, $\eta$ expresses that for every $D_{unb}$-structure $\mathfrak{L}$ with $\mathfrak{L} \models \zeta_{unb}$, $\mathfrak{L}$ is chromatic iff $\mathfrak{L} \models \eta$.

The direction “(i) implies (ii)” of the Separation Theorem is not hard:

**Lemma 8.** Let $G$ be a $k$-bounded graph. Let $\mathfrak{M}$ be a $(C_{bnd} \cup C_{unb})$-structure with $\mathfrak{M} \models \alpha \land \beta$ and $\text{Gaif}(\mathfrak{M}|_{C_{bnd}}) = G$. $\mathfrak{M}$ can be expanded to a $(D_{bnd} \cup D_{unb})$-structure $\mathfrak{N}$ with $\mathfrak{N} \models \alpha' \land \beta'$ and $\text{Gaif}(\mathfrak{N}|_{D_{bnd}}) = G$.

**Proof.** Because of $\text{Gaif}(\mathfrak{M}|_{C_{bnd}}) = G$ and $G$ is $k$-bounded, we can expand $\mathfrak{M}$ to a $C_{bnd} \cup C_{unb} \cup R$-structure $\mathfrak{M'}$ such that $\text{Gaif}(\mathfrak{M'}|_R) = \text{Gaif}(\mathfrak{M}|_{C_{bnd}})$ and $R^i_{\mathfrak{M'}}$ is a total function for all $i \in [k]$ (possibly adding self-loops for the relations $R_i$). Thus, $\mathfrak{M'} \models \delta \land \theta$. According to Lemma 6, $\mathfrak{M}$ can be expanded to a chromatic structure $\mathfrak{M}$ over vocabulary $\mathcal{E}$ with $\mathfrak{M} \models \eta$. We expand $\mathfrak{M}$ to a $D_{bnd}$-structure $\mathfrak{L}$ such that for all $u \in M$ and $\lambda \in T - \text{MsgTypes}(\mathcal{E})$ we have $\mathfrak{L} \models P_\lambda(u)$ iff there is an element $v$ of $\mathfrak{L}$ such that $2\cdot \text{tp}^{\mathfrak{L}}_\lambda(u, v) = \lambda$. This definition gives us $\mathfrak{L} \models \zeta_{unb}$, and thus $\mathfrak{L} \models \beta'$. We expand $\mathfrak{L}$ to a $(D_{bnd} \cup D_{unb})$-structure $\mathfrak{N}$ such that for all $u, v \in M$ and $B \in B$ we have $\mathfrak{N} \models B(u, v)$ iff $\mathfrak{N} \models B(u, v)$ and $\mathfrak{N} \models R_i(u, v)$ or $\mathfrak{N} \models R_i(v, u)$ for some $i \in [k]$. We note that $\mathfrak{N} \models \alpha'$.

Now we turn to the direction “(ii) implies (i)”. Let $G$ be a $k$-bounded graph. Let $\mathfrak{N}$ be a $(D_{bnd} \cup D_{unb})$-structure with $\mathfrak{N} \models \alpha' \land \beta'$ and $\text{Gaif}(\mathfrak{N}|_{D_{bnd}}) = G$. Let $M$ be
the universe of $\mathcal{R}$. We define the $\mathcal{D}_{unb}$-structure $\mathcal{L}'$ by setting $1\text{-tp}^{\mathcal{L}'}(u) = 1\text{-tp}^\mathcal{R}(u)$ for all $u \in M$ and setting $\mathcal{L}' \models B(u, v)$ iff $\mathcal{R} \models \mathcal{B}(u, v)$ for all $u, v \in M$ and $B \in \text{bin} (\mathcal{D}_{unb})$. We note that $\mathcal{L}' \models \psi$ and Gaif($\mathcal{L}'$) = $G$. We define the $\mathcal{D}_{unb}$-structure $\mathcal{L}$ by setting $\mathcal{L} = \mathcal{R}|_{\mathcal{D}_{unb}}$. We note that $\mathcal{L} \models \beta'$.

We make the following definition: For $u \in M$ and $i \in [k]$ we set $\text{rank}_u^i(\mathcal{L}, \mathcal{L}') = 1$, if there are $v, w \in M$ with $\mathcal{L} \models R_i(u, v)$, $\mathcal{L}' \models R_i(u, w)$ and $v \neq w$; we set $\text{rank}_u^i(\mathcal{L}, \mathcal{L}') = 0$, otherwise. We set $\text{rank}_u(\mathcal{L}, \mathcal{L}') = \sum_{i\in[k]} \text{rank}_u^i(\mathcal{L}, \mathcal{L}')$ and $\text{rank}(\mathcal{L}, \mathcal{L}') = \sum_{u \in M} \text{rank}_u(\mathcal{L}, \mathcal{L}')$. $\text{rank}$ measures the deviation of the relations $\mathcal{R}$ in $\mathcal{L}$ and $\mathcal{L}'$ (we note that there always are unique $v, w \in M$ for $u \in M$ with $\mathcal{L} \models R_i(u, v)$, $\mathcal{L}' \models R_i(u, w)$ because of $\mathcal{L} \models \delta$ and $\mathcal{L}' \models \delta$) and has the following important property:

**Lemma 9.** If $\text{rank}(\mathcal{L}, \mathcal{L}') = 0$, then $\mathcal{L}|_{\mathcal{C}_{unb} \cup \mathcal{C}_{unb}} \models \alpha \land \beta$ and Gaif($\mathcal{L}|_{\mathcal{C}_{unb} \cup \mathcal{C}_{unb}}$) = $G$.

(See Appendix A.8 for the proof.)

The proof of Lemma 9 uses the following simple but useful property of the rank function:

**Lemma 10.** Let $u \in M$ be an element with $\text{rank}_u^i(\mathcal{L}, \mathcal{L}') = 0$ and let $\lambda \in 2\text{-Types}(\mathcal{E})$ with $R_i \in \lambda$. For all $v \in M$ we have $2\text{-tp}^{\mathcal{E}_\lambda|_v}(u, v) = \lambda$ iff $2\text{-tp}^{\mathcal{E}_\lambda|_v}(u, v) = \lambda$.

(See Appendix A.8 for the proof.)

**Lemma 11.** There is a sequence of $\mathcal{D}_{unb}$-structures $\mathcal{L}_0, \ldots, \mathcal{L}_p$, with universe $M$ and $\mathcal{L} = \mathcal{L}_0$, such that:

1. $1\text{-tp}^{\mathcal{L}_0}(u) = 1\text{-tp}^{\mathcal{L}}(u)$ for all $u \in M$,
2. $2\text{-tp}(\mathcal{L}_i) = 2\text{-tp}(\mathcal{L}_{i+1})$ for all $0 \leq i < p$,
3. $\mathcal{L}_i \models \psi$, (in particular $\mathcal{L}_i$ is $T$-functional and chromatic),
4. rank($\mathcal{L}_i, \mathcal{L}'$) > rank($\mathcal{L}_{i+1}, \mathcal{L}'$) for all $0 \leq i < p$, and rank($\mathcal{L}_p, \mathcal{L}'$) = 0.

**Proof.** Assume we have already defined $\mathcal{L}_i$ and rank($\mathcal{L}_i, \mathcal{L}'$) > 0. In the following we will define $\mathcal{L}_{i+1}$. Because of rank($\mathcal{L}_i, \mathcal{L}'$) > 0 we can choose some elements $u, v, w \in M$ and $j \in [k]$ such that $\mathcal{L}_i \models R_j(u, v)$, $\mathcal{L}' \models R_j(u, w)$ and $v \neq w$. Let $\lambda = 2\text{-tp}^{\mathcal{E}_\lambda|_v}(u, v)$. We have $\lambda \in \mathcal{T}\text{-MsgTypes}(\mathcal{E})$ because of $R_j \in \mathcal{R}$. We have $\mathcal{L}_i \models P_\lambda(u)$ because of $\mathcal{L}_i \models \zeta_{\text{unb}}$. Because $1\text{-tp}(u)^{\mathcal{E}_\lambda} = 1\text{-tp}(u)^{\mathcal{E}_\lambda'}$ we have $\mathcal{L}' \models P_\lambda(u)$. Thus, $\mathcal{L}_i \models \text{char}_\lambda^{\mathcal{L}_i}(v)$ and $\mathcal{L}' \models \text{char}_\lambda^{\mathcal{L}_i}(w)$. With $1\text{-tp}^{\mathcal{E}_\lambda}(w) = 1\text{-tp}^{\mathcal{E}_\lambda'}(w)$ we get $\mathcal{L}_i \models \text{char}_\lambda^{\mathcal{L}_i}(w)$, and thus $1\text{-tp}^{\mathcal{E}_\lambda|_v}(v) = 1\text{-tp}^{\mathcal{E}_\lambda|_v}(w)$. We proceed by a case distinction:

**Case 1:** $\lambda^{-1}$ is a $T$-message type.

We have $\mathcal{L}' \models P_{\lambda^{-1}}(w)$ because of $\mathcal{L}' \models \zeta_{\text{bnd}}$. We get $\mathcal{L}_i \models P_{\lambda^{-1}}(w)$ because of $1\text{-tp}^{\mathcal{L}_i}(w) = 1\text{-tp}^{\mathcal{L}}(w)$. With $\mathcal{L}_i \models \zeta_{\text{unb}}$, there is an element $a \in M$ such that $\lambda = 2\text{-tp}^{\mathcal{E}_\lambda|_v}(a, w)$ and $\mathcal{L}_i \models P_\lambda(a)$. We note that $u \neq a$ because of $\mathcal{L}_i \models R_j(a, w)$, $v \neq w$ and $v$ is the unique element with $\mathcal{L}_i \models R_j(u, v)$ (using $\mathcal{L}_i \models \delta$). Moreover, $\mathcal{L}_i \models \text{char}_\lambda^{\mathcal{L}_i}(a)$ and $\mathcal{L}' \models \text{char}_{\lambda^{-1}}^{\mathcal{L}_i}(u)$. With $1\text{-tp}^{\mathcal{E}_\lambda}(u) = 1\text{-tp}^{\mathcal{E}_\lambda'}(u)$ we get $\mathcal{L}_i \models \text{char}_{\lambda^{-1}}^{\mathcal{L}_i}(u)$, and thus $1\text{-tp}^{\mathcal{E}_\lambda|_v}(u) = 1\text{-tp}^{\mathcal{E}_\lambda|_v}(a)$.

We note that the edges $(u, w)$, $(w, u)$, $(a, v)$ and $(v, a)$ do not have $T$-message types (*) because $\mathcal{L}_i$ is chromatic, $u \neq a$, $v \neq w$, $1\text{-tp}^{\mathcal{E}_\lambda|_v}(v) = 1\text{-tp}^{\mathcal{E}_\lambda|_v}(w)$, $1\text{-tp}^{\mathcal{E}_\lambda|_v}(u) = 1\text{-tp}^{\mathcal{E}_\lambda|_v}(a)$ and the edges $(u, v)$, $(v, u)$, $(a, w)$ and $(w, a)$ have $T$-message types. Similarly, we get $\lambda \neq 2\text{-tp}^{\mathcal{E}_\lambda|_v}(u, v)$ and $\lambda \neq 2\text{-tp}^{\mathcal{E}_\lambda|_v}(a, w)$ (**) because $\mathcal{L}'$ is chromatic, $\lambda = 2\text{-tp}^{\mathcal{E}_\lambda|_v}(v, w)$ and $\lambda$ and $\lambda^{-1}$ are $T$-message types.
We define $\mathcal{L}_{i+1}$ as follows: The unary relations of $\mathcal{L}_{i+1}$ are defined such that we have $1$-$tp(u)_{\mathcal{L}_{i+1}} = 1$-$tp(u)$ for all elements $u \in M$. We obtain the binary relations of $\mathcal{L}_{i+1}$ by swapping the edges $(u, w)$ and $(u, v)$ as well as $(a, w)$ and $(a, v)$ in $\mathcal{L}_i$: we set $2$-$tp(u)_{\mathcal{L}_{i+1}}(u, w) = \lambda$, $2$-$tp(u)_{\mathcal{L}_{i+1}}(a, v) = \lambda$, $2$-$tp(u)_{\mathcal{L}_{i+1}}(a, w) = 2$-$tp(u)_{\mathcal{L}_i}(a, v)$ and $2$-$tp(u)_{\mathcal{L}_{i+1}}(u, v) = 2$-$tp(u)_{\mathcal{L}_i}(u, w)$; these 2-types are well-defined because of $1$-$tp(u)_{\mathcal{L}_i}(v) = 1$-$tp(u)_{\mathcal{L}_i}(v)$. All other 2-types in $\mathcal{L}_{i+1}|_{\mathcal{L}_i}$ are the same as in $\mathcal{L}_i|_{\mathcal{L}_i}$. This completes the definition of $\mathcal{L}_{i+1}$.

We now argue that $\mathcal{L}_{i+1}$ satisfies properties $\ref{1}$-$\ref{4}$. Clearly, $\mathcal{L}_{i+1}$ satisfies $\ref{1}$ by definition. Because we only swapped edges from $\mathcal{L}_i$ to $\mathcal{L}_{i+1}$ we have $\ref{2}$. From $\ref{2}$, Lemma $\ref{4}$ and $\mathcal{L}_i | \chi$, we get $\mathcal{L}_{i+1} | \chi$. Because we only swapped swapped edges from $\mathcal{L}_i$ to $\mathcal{L}_{i+1}$ we get $\mathcal{L}_{i+1} | \theta$ and $\mathcal{L}_{i+1} | \epsilon$ from $\mathcal{L}_i | \theta$ and $\mathcal{L}_i | \epsilon$. Because of $(\ast)$ and $2$-$tp(u)_{\mathcal{L}_i}(v, u) = 2$-$tp(u)_{\mathcal{L}_i}(u, v)$ we get $\mathcal{L}_{i+1} | \zeta_{unb}$ from $\mathcal{L}_i | \zeta_{unb}$, that $\mathcal{L}_{i+1}$ is $T$-functional because $\mathcal{L}_i$ is $T$-functional and $\mathcal{L}_{i+1} | \eta$ from $\mathcal{L}_i | \eta$ (using that $1$-$tp(u)_{\mathcal{L}_i}(v) = 1$-$tp(u)_{\mathcal{L}_i}(v)$ and $1$-$tp(u)_{\mathcal{L}_i}(u) = 1$-$tp(u)_{\mathcal{L}_i}(u)$). Thus, $\mathcal{L}_{i+1}$ satisfies property $\ref{4}$.

It remains to show property $\ref{4}$. Using $(\ast)$, $(\ast\ast)$ and Lemma $\ref{10}$ we get that $\text{rank}_z(\mathcal{L}_{i+1}, x') < \text{rank}_z(\mathcal{L}_i, x')$ and $\text{rank}_z(\mathcal{L}_{i+1}, x') \leq \text{rank}_z(\mathcal{L}_i, x')$ for $z \in \{v, w, a\}$. Moreover, $\text{rank}_z(\mathcal{L}_{i+1}, x') = \text{rank}_z(\mathcal{L}_i, x')$ for $z \in M \setminus \{u, v, w, a\}$. Thus, we have $\text{rank}(\mathcal{L}_i, x') > \text{rank}(\mathcal{L}_{i+1}, x')$.

Case 2: $\lambda^{-1}$ is not a $T$-message type.

We note that the edge $(w, u)$ does not have a $T$-message type because $\mathcal{L}_i$ is chromatic, $v \neq w$, $1$-$tp(u)_{\mathcal{L}_i}(v) = 1$-$tp(u)_{\mathcal{L}_i}(w)$ and the edge $(u, v)$ has a $T$-message type.

We define $\mathcal{L}_{i+1}$ as follows: The unary relations of $\mathcal{L}_{i+1}$ are defined such that we have $1$-$tp(u)_{\mathcal{L}_{i+1}} = 1$-$tp(u)$ for all elements $u \in M$. We obtain the binary relations of $\mathcal{L}_{i+1}$ by swapping edges in $\mathcal{L}_i$: $2$-$tp(u)_{\mathcal{L}_{i+1}}(u, w) = \lambda$, and $2$-$tp(u)_{\mathcal{L}_{i+1}}(u, v) = 2$-$tp(u)_{\mathcal{L}_i}(u, w)$. These 2-types are well-defined because of $1$-$tp(u)_{\mathcal{L}_i}(v) = 1$-$tp(u)_{\mathcal{L}_i}(v)$. All other 2-types in $\mathcal{L}_{i+1}|_{\mathcal{L}_i}$ are the same as in $\mathcal{L}_i|_{\mathcal{L}_i}$. This completes the definition of $\mathcal{L}_{i+1}$.

We now argue that $\mathcal{L}_{i+1}$ satisfies properties $\ref{1}$-$\ref{4}$. As in the previous case, one can argue that $\mathcal{L}_{i+1}$ satisfies $\ref{1}$ and $\ref{2}$, $\mathcal{L}_{i+1} | \chi$, $\mathcal{L}_{i+1} | \theta$ and $\mathcal{L}_{i+1} | \epsilon$. Because $(v, u)$ and $(w, u)$ do not have $T$-message types we get $\mathcal{L}_{i+1} | \zeta_{unb}$ from $\mathcal{L}_i | \zeta_{unb}$ and that $\mathcal{L}_{i+1}$ is $T$-functional because $\mathcal{L}_i$ is $T$-functional. We get $\mathcal{L}_{i+1} | \eta$ from $\mathcal{L}_i | \eta$, $\mathcal{L}_i' | \eta$ and $1$-$tp(w)_{\mathcal{L}_i}(w) = 1$-$tp(w)_{\mathcal{L}_i}(w)$.

It remains to show property $\ref{4}$. From Lemma $\ref{10}$ we get that $2$-$tp(u)_{\mathcal{L}_i}(v, u) \neq \lambda$ and $2$-$tp(u)_{\mathcal{L}_i}(v, u) \neq \lambda$. Again applying Lemma $\ref{10}$ we get $\text{rank}_v(\mathcal{L}_{i+1}, x') < \text{rank}_v(\mathcal{L}_i, x')$. Because $2$-$tp(u)_{\mathcal{L}_i}(v, u)$ and $2$-$tp(u)_{\mathcal{L}_i}(v, u)$ are not $T$-message types, we get $\text{rank}_v(\mathcal{L}_{i+1}, x') = \text{rank}_v(\mathcal{L}_i, x')$ and $\text{rank}_v(\mathcal{L}_{i+1}, x') = \text{rank}_v(\mathcal{L}_i, x')$. Moreover, we have that $\text{rank}_z(\mathcal{L}_{i+1}, x') = \text{rank}_z(\mathcal{L}_i, x')$ for all $z \in M \setminus \{u, v, w\}$. Thus, $\text{rank}(\mathcal{L}_i, x') > \text{rank}(\mathcal{L}_{i+1}, x')$.

\[\square\]

5. From Bounded Tree-width to Binary Trees

This section is devoted to a discussion of the proof of Lemma $\ref{2}$ First we need some background from the literature. A \textit{translation scheme} for $\mathcal{C}_1$ over $\mathcal{C}_2$ is a tuple $\mathcal{T} = \langle \phi, \psi_C : C \in \mathcal{C}_2 \rangle$ of MSO($\mathcal{C}_1$)-formulas such that $\phi$ has exactly one free first order variable and the number of free first order variables in each
The formulas $\phi$ and $\psi_C$, $C \in C_2$, do not have any free second order variables. The quantifier rank $qr(t)$ of $t$ is the maximum of the quantifier ranks of $\phi$ and the $\psi_C$. $t$ is quantifier-free if $qr(t) = 0$. The induced transduction $t^*$ is a partial function from $C_1$-structures to $C_2$-structures which assigns a $C_2$-structure $t^*(\mathfrak{A})$ to a $C_1$-structure $\mathfrak{A}$ as follows. The universe of $t^*(\mathfrak{A})$ is $A_t = \{ a \in A : \mathfrak{A} \models \phi(a) \}$. The interpretation of $C \in C_2$ in $t^*(\mathfrak{A})$ is $C^{t^*(\mathfrak{A})} = \{ \bar{a} \in A_t^{\text{arity}(C) : \mathfrak{A} \models \psi_C(\bar{a})} \}$. Due to the convention that structures do not have an empty universe, $t^*(\mathfrak{A})$ is defined iff $\mathfrak{A} \models \exists x \phi(x)$.

**Lemma 12** (Fundamental property of translation schemes). Let $t$ be a translation scheme for $C_2$ over $C_1$. There is a computable function $t^f$ from $\text{MSO}(C_2)$-sentences to $\text{MSO}(C_1)$-sentences such that for every $C_1$-structure $\mathfrak{A}$ for which $t^*(\mathfrak{A})$ is defined and for every $\text{MSO}(C_2)$-sentence $\theta$, $\mathfrak{A} \models t^f(\theta)$ if and only if $t^*(\mathfrak{A}) \models \theta$. $t^f$ is called the induced translation.

For an $\text{MSO}(C_2)$-sentence $\zeta$, $t^f$ substitutes the relation symbols $C \in C_2$ in $\zeta$ with the formulas $\psi_C$, requires that each of the free variables satisfies $\phi$, and relativizes the quantification to $\phi$. Appendix 8.9 gives an inductive definition of $t^f$ following Definition 3.2 of [19].

From the definition of the induced translation we have:

**Lemma 13** (Quantifier-free translation schemes and $C^2$). Let $t$ be a quantifier-free translation scheme for $C_2$ over $C_1$. The induced translation $t^f$ maps $C^2(C_2)$-formulas to $C^2(C_1)$-formulas.

It is well-known that the class of graphs of tree-width at most $k$ is the image of an induced transduction on a class of labeled trees [1]. For the proof of Lemma 2 we need an analogous translation scheme for $(C_{\text{bnd}} \cup C_{\text{unb}})$-structures $\mathfrak{M}$ whose reducts $\mathfrak{M}|_{C_{\text{bnd}}}$ have tree-width at most $k$. In order that $\alpha$ and $\beta$ be mapped to an $\text{MSO}(D_{\text{bnd}})$-sentence and a $C^2(C_{\text{unb}})$-sentence respectively, we need that the translation scheme satisfy some additional properties.

**Lemma 14.** Let $C_{\text{bnd}}$ and $C_{\text{unb}}$ be vocabularies such that $C_{\text{bnd}} \cap C_{\text{unb}}$ only contains unary relation symbols. There exist the following effectively computable objects: (1) a vocabulary $D_{\text{bnd}}$ consisting of the binary relation symbol $s$ and unary relation symbols only, (2) a translation scheme $\text{tr} = (\phi, \psi_C : C \in C_{\text{bnd}} \cup C_{\text{unb}})$ for $C_{\text{bnd}} \cup C_{\text{unb}}$ over $D_{\text{bnd}} \cup C_{\text{unb}}$, and (3) an $\text{MSO}(D_{\text{bnd}})$-sentence dom, such that:

(a) $\phi$ is quantifier-free over $D_{\text{bnd}}$.

(b) For every relation symbol $C \in C_{\text{unb}}$, $\psi_C$ is quantifier-free.

(c) For every relation symbol $C \in C_{\text{bnd}}$, $\psi_C$ is an $\text{MSO}(D_{\text{bnd}})$-formula.

(d) Let $\mathcal{K}$ be the class of $(D_{\text{bnd}} \cup C_{\text{unb}})$-structures in which $s$ is interpreted as a binary tree and which satisfy dom. The image of $\mathcal{K}$ under $\text{tr}^*$ is exactly the class of $(C_{\text{bnd}} \cup C_{\text{unb}})$-structures $\mathfrak{M}$ such that $\mathfrak{M}|_{C_{\text{bnd}}}$ has tree-width at most $k$.

The proof of Lemma 14 is technically similar to the discussion in [7]. We include the proof in the Appendix 8.2 for completeness.

We are now ready to prove Lemma 2. By Lemma 14(d) and Lemma 12, statement (i) in Lemma 2 holds iff there is a $(D_{\text{bnd}} \cup C_{\text{unb}})$-structure $\mathfrak{M}$ such that $s^{\mathfrak{M}}$ is a

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5 All translation schemes in this paper are scalar (i.e. non-vectorized). In the notation of [6], a translation scheme is a parameterless non-copying MSO-definition scheme with precondition formula ($x \approx x$).
binary tree and $\mathfrak{M} \models \text{dom} \land \text{tr}^4(\alpha) \land \text{tr}^2(\beta)$. Let $\alpha' = \text{dom} \land \text{tr}^4(\alpha)$ and $\beta' = \text{tr}^2(\beta)$. By Lemma 14(a) and the definition of $\text{tr}^4$, $\alpha' \in \text{MSO}(D_{\text{bnd}})$. Let $\text{tr}|_{\mathfrak{C}_{\text{unk}}}$ be the translation scheme for $\mathfrak{C}_{\text{unk}}$ over $D_{\text{bnd}} \cup \mathfrak{C}_{\text{unk}}$ which agrees with $\text{tr}$ on all formulas, i.e. $\text{tr}|_{\mathfrak{C}_{\text{unk}}} = (\phi, \psi_C : C \in \mathfrak{C}_{\text{unk}})$. The image of $\text{tr}|_{\mathfrak{C}_{\text{unk}}}^*$ on a structure $\mathfrak{M}$ is the $\mathfrak{C}_{\text{unk}}$-reduct of the image of $\text{tr}^*$ on $\mathfrak{M}$. Since $\beta \in C^2(\mathfrak{C}_{\text{unk}})$, $\text{tr}|_{\mathfrak{C}_{\text{unk}}}^*(\beta)$ is well-defined and $\text{tr}|_{\mathfrak{C}_{\text{unk}}}^*(\beta) = \beta'$. By Lemma 14(b), $\text{tr}|_{\mathfrak{C}_{\text{unk}}}$ is a quantifier-free translation scheme, implying that $\beta' \in C^2(D_{\text{bnd}} \cup \mathfrak{C}_{\text{unk}})$ by Lemma 13.

6. From MSO to $C^2$ on Binary Trees

The purpose of this section is to show that, on structures consisting only of a binary tree and additional unary relations, every MSO-sentence can be rewritten to a $C^2$-sentence which is equi-satisfiable and whose length is linear in the length of the input MSO-sentence. We start by introducing some tools for the literature.

**Theorem 4 (Hintikka sentences).** Let $C$ be a vocabulary. For every $q \in \mathbb{N}$ there is a finite set $\mathcal{H}.s.\mathcal{K}_{C,q}$ of $\text{MSO}(C)$ of quantifier rank $q$ such that:

1. every $\epsilon \in \mathcal{H}.s.\mathcal{K}_{C,q}$ has a model;
2. the conjunction of any two distinct sentences $\epsilon_1, \epsilon_2 \in \mathcal{H}.s.\mathcal{K}_{C,q}$ is not satisfiable;
3. every $\text{MSO}(C)$-sentence $\alpha$ of quantifier rank at most $q$ is equivalent to exactly one finite disjunction of sentences $\mathcal{H}.s.\mathcal{K}_{C,q}$;
4. every $C$-structure $\mathfrak{A}$ satisfies exactly one sentence $\text{hin}_{C,q}(\mathfrak{A})$ of $\mathcal{H}.s.\mathcal{K}_{C,q}$.

We may omit $C$ or $q$ from the subscript when they are clear from the context.

For a class of $C$-structures $\mathcal{H}$ an $n$-ary operation $\text{Op}$ over $C$-structures is called smooth over $\mathcal{H}$, if for all $\mathfrak{A}_1, \ldots, \mathfrak{A}_n \in \mathcal{H}$, $\text{hin}_{C,q}(\text{Op}(\mathfrak{A}_1, \ldots, \mathfrak{A}_n))$ depends only on $\text{hin}_{C,q}(\mathfrak{A}_i)$: $i \in [n]$ and this dependence is computable. We omit “over $\mathcal{H}$” when $\mathcal{H}$ consists of all $C$-structures.

**Theorem 5 (Smoothness).**

1. The disjoint union is smooth.
2. For every quantifier-free translation scheme $\mathfrak{t}$, the operation $\mathfrak{t}^*$ is smooth.
3. Let $\mathfrak{T}_1 \supset \mathfrak{T}_2$ denote the operation which augments the disjoint union of $\mathfrak{T}_1$ and $\mathfrak{T}_2$ by adding an edge from the root of tree $\mathfrak{T}_1$ to the root of tree $\mathfrak{T}_2$. This operation is smooth over labeled trees.

For an in-depth introduction to Hintikka sentences and smoothness and references to proofs see [14, Chapter 3, Theorem 3.3.2] and [19] respectively.

We are now ready for the main lemma of this section.

Let $\text{rt}(x)$ be the sentence $\forall y \neg s(y, x)$. This sentence defines the root of the binary tree $s$.

**Lemma 15.** Let $q \in \mathbb{N}$ and let $C$ be a vocabulary which consists only of the binary relation symbol $s$ and (possibly) additional unary relation symbols. Let $\mathcal{C}_\epsilon : \epsilon \in \mathcal{H}.s.\mathcal{K}_{C,q}$ be new unary relation symbols. Let $\text{hin}(C, q)$ be the vocabulary which extends $C$ with $\{ \mathcal{C}_\epsilon : \epsilon \in \mathcal{H}.s.\mathcal{K}_{C,q} \}$. There is a computable $C^2(\text{hin}(C, q))$-sentence $\Theta_{\mathcal{C}_q}^{\text{hin}}$ such that if $\mathfrak{T}_0$ is a $C$-structure such that $s^{\mathfrak{T}_0}$ is a binary tree:

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6. Smooth operations here are called *effectively smooth* in [19].

7. We use the smoothness of the disjoint union and quantifier-free transductions. The transduction adds the edge between the roots. Technically, in order for the transduction to be quantifier-free, we need that our trees have a special unary relation symbol root with the natural interpretation as the root of the tree.
There is an expansion $T_1$ of $T_0$ to $\text{hin}(C, q)$ such that $T_1 \models \Theta_{\text{hin}}^{C,q}$ and $qr(\omega) = q$, the following holds: there exists a $C^2$-sentence $\omega_{\text{hin}}$ such that $T_0 \models \omega$ iff $T_1 \models \omega_{\text{hin}}$. The $C^2$-sentence $\omega_{\text{hin}}$ is
\[
\forall x \left( \text{rt}(x) \rightarrow \bigvee_{\epsilon \in \mathcal{H}_{\text{F}} C,q; \epsilon = \omega} C_\epsilon(x) \right)
\]

The sentence $\Theta_{\text{hin}}^{C,q}$ is defined so that for every $T_0$ there is a unique expansion $T_1$ such that $T_1 \models \Theta_{\text{hin}}^{C,q}$. For every $u$ in the universe of $T_1$, we will have $u \in C_{T_1}$ iff the subtree $T_u$ of $T_0$ whose root is $u$ satisfies $T_u \models \epsilon$. Using the smoothness of $\rightarrow$, whether an element of $T_1$ belongs to $C_{T_1}$ depends only on its children. This can be axiomatized in $C^2$. Lemma 3 follows from Lemma 15 with $q = qr(\alpha)$, $D = \text{hin}(C, q)$, and $\omega_{\text{hin}} = \gamma$.

Appendix 8.3 spells out the proof of Lemma 15.

7. MSO with Cardinality Constraints

MSO$^{\text{card}}$ denotes the extension of MSO with atomic formulas called cardinality constraints $\sum_{i=1}^m |X_i| < \sum_{i=1}^n |Y_i|$, where the $X_i$ and $Y_i$ are MSO variables, and $|X|$ denotes the cardinality of $X$. Let WS1S (WS1S$^{\text{card}}$) be the weak monadic second order theory (with cardinality constraints) of the structure $\langle \mathbb{N}, +1, < \rangle$. Let MSO$^{\text{card}}$ be the set of sentences $\rho$ such that (1) $\rho$ is of the form $\rho = \exists X_1 \cdots \exists X_m \omega$, and (2) only the $X_1, \ldots, X_m$ participate in cardinality constraints.

**Theorem 6.** Given a sentence $\rho \in \text{MSO}^{\text{card}}$, it is decidable (A) whether $\langle N, +1, < \rangle \models \rho$, and (B) whether $\rho$ is satisfiable by a finite structure of bounded tree-width.

This theorem follows from Theorem 1. The main observation needed for (B) is that cardinality constraints can be expressed in terms of injective functions, which are axiomatizable in $C^2$. (A) is reducible to (B). The main observations for (A) are:

1. that $X_1, \ldots, X_m$ are contained in the substructure $\mathcal{A}_1$ of $\langle N, +1, < \rangle$ generated by $\{0, \ldots, \ell\}$ for some $\ell \in \mathbb{N}$,
2. that substructure $\mathcal{A}_2$ of $\langle N, +1, < \rangle$ generated by $N - \{0, \ldots, \ell\}$ is isomorphic to $\langle N, +1, < \rangle$, and therefore $\mathcal{A}_2$ and $\langle N, +1, < \rangle$ have the same weak monadic second order theory,
3. that the weak monadic second order theory of $\langle N, +1, < \rangle$ is decidable,
4. and that $\langle N, +1, < \rangle$ is a transduction $t$ of $\mathcal{A}_1 \sqcup \mathcal{A}_2$.

It remains to use that $\sqcup$ is smooth, cf. Appendix 8.1.
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8. Appendix

8.1. Proof of Theorem 6. We start with (B).

Let \( \rho \) be a MSO_{\text{card}} sentence, i.e. the outermost block of quantifiers in \( \rho \) is existential and only variables from the outermost blocks may appear in cardinality constraints. For simplicity we consider \( \rho = \exists X_1 \exists X_2 \omega \) with only two quantifiers in the outermost block. W.l.o.g. the only cardinality constraint in \( \omega \) is \( |X_1| < |X_2| \).

By a slight abuse of notation, we sometimes treat finite satisfiability of \( \omega \) to finite satisfiability of a sentence \( X \) symbols. Let \( C_{\text{bnd}} \) exist. and only variables from the outermost blocks may appear in cardinality constraints in the outermost block. W.l.o.g. the only cardinality constraint in \( \omega \) is \( |X_1| < |X_2| \).

Let \( C_{\text{bnd}} \) extend the vocabulary of \( \rho \) with new unary relation symbols \( X_1, X_2, W_{\text{img}}, W_{\text{dom}} \). Let \( C_{\text{unb}} \) extend \( C_{\text{bnd}} \) with a new binary relation symbol \( B \). Finite satisfiability of \( \omega \) by a structure \( \mathfrak{M} \) such that \( \text{tw}(\mathfrak{M}) \leq k \) can be reduced to finite satisfiability of a sentence \( \alpha \land \beta \), \( \alpha \in \text{MSO}(C_{\text{bnd}}) \) and \( \beta \in C^2(C_{\text{unb}}) \), by a structure \( \mathfrak{A}_1 \) such that \( \text{tw}(\mathfrak{A}_1|_{C_{\text{bnd}}}) \leq k \). Let \( \beta \) be the \( C^2 \)-sentence

\[
\beta = (\text{inj}_{12} \lor \text{inj}_{21}) \land \text{dom} \land \text{img}
\]

where

- \( \text{inj}_{12} \) expresses that \( B \) is an injective function from \( X_1 \) to \( X_2 \),
- \( \text{inj}_{21} \) expresses that \( B \) is an injective function from \( X_2 \) to \( X_1 \),
- \( \text{dom} \) expresses that the domain of \( B \) is \( W_{\text{dom}} \),
- \( \text{img} \) expresses that the image of \( B \) is \( W_{\text{img}} \).

For every \( C_{\text{unb}} \)-structure \( \mathfrak{A}_1 \), \( |X_1^{\mathfrak{A}_1}| < |X_2^{\mathfrak{A}_1}| \) iff \( W_{\text{dom}}^{\mathfrak{A}_1} = X_1^{\mathfrak{A}_1} \) and \( X_2^{\mathfrak{A}_1} \setminus W_{\text{img}}^{\mathfrak{A}_1} \neq \emptyset \).

Let \( \alpha \) be obtained from \( \omega \) by substituting every \( |X_1| < |X_2| \) by

\[
\forall x (X_1(x) \leftrightarrow W_{\text{dom}}(x)) \land \exists x (\neg W_{\text{img}}(x) \land X_2(x))
\]

For any \( C_{\text{bnd}} \)-structure \( \mathfrak{A}_0 \) with \( \text{tw}(\mathfrak{A}_0) \leq k \), \( \mathfrak{A}_0 \models \omega \) iff there is an expansion \( \mathfrak{A}_1 \) of \( \mathfrak{A}_0 \) such that \( \mathfrak{A}_1 \models \alpha \land \beta \). The treatment of other cardinality constraints \( \sum_{i=1}^t |X_i| < \sum_{i=1}^t |Y_i| \) is similar; it is helpful to assume w.l.o.g. that the \( X_i \) and the \( Y_i \) are disjoint.

Now we turn to (A). The unary function \(+1\) is the successor relation of \( \mathbb{N} \) and interprets the binary relation symbol \( \text{suc} \). The binary relation \( < \) is the natural order relation on \( \mathbb{N} \). In the proof of (A) we will use the theory of Hintikka sentences as presented in Section \( \S \) with one caveat, namely that instead of restricting to finite structures, we allow arbitrary structures. Theorems \( \S \) and \( \S \) hold for arbitrary structures. Theorem \( \S \) guarantees the existence of a set \( \mathcal{H}_{\mathcal{X}, \mathcal{X}_{\text{arb}}} \) analogous to \( \mathcal{H}_{\mathcal{X}, \mathcal{X}_{\text{arb}, \mathcal{C}_q}} \) for arbitrary structures. For every \( C \)-structure \( \mathfrak{A} \), Theorem \( \S \) guarantees the existence of a sentence \( \text{hin}_{\text{arb}}^{\mathfrak{A}}(\mathfrak{A}) \) of \( \mathcal{H}_{\mathcal{X}, \mathcal{X}_{\text{arb}}} \) analogous to \( \text{hin}_{\mathcal{C}_q}(\mathfrak{A}) \). For the rest of the section, we omit the superscript \( \text{arb} \) to simplify notation.

Consider \( \rho = \exists X_1 \exists X_2 \omega \) for the vocabulary \( C_{\mathbb{N}} \) of \( \langle \mathbb{N}, +1, \langle \rangle \rangle \). Let \( \alpha \in \text{MSO}(C_{\text{bnd}}) \) and \( \beta \in C^2(C_{\text{unb}}) \) be as discussed in the proof of (B) above. The following are equivalent:

1. \( \langle \mathbb{N}, +1, \langle \rangle \rangle \models \rho \)
2. There are finite unary relations \( U_1 \) and \( U_2 \) such that \( \langle \mathbb{N}, +1, <, U_1, U_2 \rangle \models \omega \). \( U_1 \) and \( U_2 \) interpret \( X_1 \) and \( X_2 \) respectively.
3. There are finite unary relations \( U_1 \) and \( U_2 \) and an expansion \( \mathfrak{A} \) of \( \langle \mathbb{N}, +1, <, U_1, U_2 \rangle \) such that \( \mathfrak{A} \models \alpha \land \beta \). \( \mathfrak{A} \) expands \( \langle \mathbb{N}, +1, <, U_1, U_2 \rangle \) with interpretations of the symbols in \( D_{\text{unb}} = \{ B, W_{\text{dom}}, W_{\text{img}} \} \).
4. \( \rho' = \exists X_1 \exists X_2 (\alpha \land \beta) \) is satisfiable by an expansion of \( \langle \mathbb{N}, +1, < \rangle \) with interpretations for the symbols of \( D_{\text{unb}} \).
We have 1. iff 2. and 3. iff 4. by the semantics of $\exists X_1 \exists X_2$ in weak monadic second order logic. We have 2. iff 3. similarly to the discussion of $\alpha$ and $\beta$ in the proof of (A) above. The rest of the proof is devoted to proving that 4. is decidable.

Observe that by the definition of $\beta$, and in weak monadic second order $X_1$ and $X_2$ are quantified to be finite sets, $B$ is axiomatized to be a function with finite domain, and $W_{\text{dom}}$ and $W_{\text{img}}$ are finite. Hence models $\mathfrak{A}$ of $\rho'$ can be decomposed into a finite part containing $B^A$, and an infinite part isomorphic to an expansion of $\langle N, +1, \rangle$ in which the symbols of $P_{\text{unb}}$ are interpreted as empty sets. We will use a similar decomposition, but first we want to move from the structure $\langle A, < \rangle$ and its expansions to $\langle B, < \rangle$ into a finite part containing $1$ and its expansions. There is a translation scheme $\iota_<$ such that for every structure $\mathfrak{A} = \langle N, +1, B^A, W_{\text{dom}}^A, W_{\text{img}}^A \rangle$, $\iota_<(\mathfrak{A}) = \langle \mathfrak{B}, < \rangle$, where $\langle \mathfrak{B}, < \rangle$ is the expansion of $\mathfrak{A}$ with $\langle$. This is true since $\langle$ is MSO definable from $+1$. We have that $\rho'$ is satisfied by an expansion of $\langle N, +1, \rangle$ if $\iota_<(\rho')$ is satisfied by the same expansion of $\langle N, +1, \rangle$.

Now we turn to the decomposition of models of $\iota_<(\rho')$ into a finite part containing $B^A$, and an infinite part isomorphic to a $P_{\text{unb}}$-expansion of $\langle N, +1, \rangle$. Let $D_2 \supseteq D_{\text{unb}}$ be the vocabulary of $\iota_<(\rho')$. For every $D_2$-expansion $\mathfrak{P}$ of $\langle N, +1, \rangle$ and every $n \in \mathbb{N}$, let $\mathfrak{P}_{1,n}$ and $\mathfrak{P}_{n,\infty}$ be the substructures of $\mathfrak{P}$ generated by $[n]$ and $\mathbb{N} \setminus [n-1]$ respectively. There is a translation scheme $u$ such that $u^\ast (\mathfrak{P}_{1,n} \sqcup \mathfrak{P}_{n,\infty}) = \mathfrak{P}$ if $B^\mathfrak{P} \subseteq [n] \times [n]$. $u$ existentially quantifies the set $[n]$ (which is the only non-empty finite set closed under suc and its inverse), $1$ and $n$ (as the first and last elements of $[n]$) and $n+1$ (as the only element without a suc-predecessor except for $1$) and adds the edge $(n, n+1)$ to suc. We have $\mathfrak{P} \models \iota_<(\rho')$ iff $\mathfrak{P}_{1,n} \sqcup \mathfrak{P}_{n,\infty} \models u^\ast (\iota_<(\rho'))$. Note that the vocabulary of $u^\ast (\iota_<(\rho'))$ is $D_2$. Let $q$ be the quantifier rank of $u^\ast (\iota_<(\rho'))$.

The Hintikka sentence $\text{hin}(\mathfrak{P}_{n,\infty})$ of quantifier rank $q$ of $\mathfrak{P}_{n,\infty}$ is uniquely defined since $\mathfrak{P}_{n,\infty}$ is isomorphic to the expansion of $\langle N, +1, \rangle$ with empty sets. Moreover, $\text{hin}(\mathfrak{P}_{n,\infty})$ is computable using that the theory of $\langle N, +1, \rangle$ is decidable. Hence, by the smoothness of the disjoint union, for every Hintikka sentence $\epsilon \in \mathcal{H}\mathcal{F}\mathcal{N}_{D_2,q}$ there is a computable set $E_\epsilon \subseteq \mathcal{H}\mathcal{F}\mathcal{N}_{D_2,q}$ such that $\text{hin}(\mathfrak{P}_{1,n} \sqcup \mathfrak{P}_{n,\infty}) = \epsilon$ iff $\text{hin}(\mathfrak{P}_{1,n}) \in E_\epsilon$. Then $\mathfrak{P}_{1,n} \sqcup \mathfrak{P}_{n,\infty} \models u^\ast (\iota_<(\rho'))$ iff $\mathfrak{P}_{1,n}$ satisfies the sentence $\mathcal{V}_{(\epsilon, \epsilon')} \epsilon'$, where the $\mathcal{V}_{(\epsilon, \epsilon')} \epsilon'$ ranges over pairs $(\epsilon, \epsilon')$ such that

1. $\epsilon \in \mathcal{H}\mathcal{F}\mathcal{N}_{D_2,q}$,
2. $\epsilon \models u^\ast (\iota_<(\rho'))$, and
3. $\epsilon' \in E_\epsilon$.

Hence, $\rho'$ is satisfiable by an expansion of $\langle N, +1, \rangle$ iff $\mathcal{V}_{(\epsilon, \epsilon')} \epsilon'$ is satisfiable by a finite structure in which suc is interpreted as a successor relation (i.e., as a simple directed path on the whole universe). Let suc-rel be the weak MSO-sentence such that the interpretation of suc is a successor relation. By Theorem [B], it is decidable whether $\mathcal{V}_{(\epsilon, \epsilon')} \epsilon' / \text{suc-rel}$ is finitely satisfiable using that the class of simple directed paths annotated with unary relations has tree-width 1.

**Remark 1.** While we assumed for simplicity in (B) that $X_1$ and $X_2$ range over subsets of the universe, it is not hard to extend the proof to the case that $X_1$ and $X_2$ are guarded second order variables which range over subsets of any relation in the structure. This is true since we can use the translation scheme $\text{tr}$ from Lemma [14] to obtain the structures of tree-width at most $k$ as the image of $\text{tr}^\ast$ of labeled trees; $X_1$ and $X_2$ then translate naturally to monadic second order variables.
8.2. Proof of Lemma 14. This appendix is devoted to the proof of Lemma 14. Appendix 8.2 introduces a tree encoding of structures of tree-width at most \( k \) based on \textit{aligned} \( k \)-trees. Appendix 8.2 introduces a translation scheme such that the image of the induced transduction on an MSO-definable class of labeled binary trees is the class of structures of tree-width at most \( k \). This is done based on translation schemes introduced in Appendix 8.2. Finally, the proof of Lemma 14 is given in Subsection 8.2.

Note that the notion of \textit{aligned} \( k \)-trees is quite similar to the notion of \textit{oriented} \( k \)-trees defined in Section 2. In fact, we could have replaced \textit{aligned} \( k \)-trees with \textit{oriented} \( k \)-trees here, or replaced \textit{oriented} \( k \)-trees with \textit{aligned} \( k \)-trees throughout the paper. However, it simplifies the proofs to have two notions.

**Aligned \( k \)-tree encodings and (decorated) aligned \( k \)-trees.** An aligned \( k \)-tree encoding \( \mathcal{T} \) is a binary tree whose vertices are labeled by certain unary relations. All structures of bounded tree-width can be obtained by applying transductions to aligned \( k \)-tree encodings with additional vertex annotations in three steps:

- **Step A** Aligned \( k \)-tree encoding
- **Step B** Decorated aligned \( k \)-tree
- **Step C** Aligned \( k \)-tree

**interpret, undecorate** and **structurize** (shorthand \( \text{i} \), \( \text{u} \) and \( \text{s} \)) are unary operations on structures given in terms of translation schemes (see Section 8.2).

Two key properties of our encoding:

(a) Using that the number of edges in a \( k \)-tree is at most \( k \cdot \) (number of vertices), our (decorated) aligned \( k \)-trees have functions \( R_1, \ldots, R_k \) rather than an edge relation.

(b) The elements of the universe of a structure appear as elements in the aligned \( k \)-tree, in the decorated aligned \( k \)-tree, and in the aligned \( k \)-tree encoding.

Aligned \( k \)-tree encodings and decorated aligned \( k \)-trees have additional auxiliary vertices which are eliminated by **undecorate**.

Let \( \mathcal{V} \) be the vocabulary containing the binary relation symbol \( s \), and the unary relation symbols \( \text{root}, \text{Label}_1, \ldots, \text{Label}_k, \text{Label}_{\text{blank}} \).

An **aligned \( k \)-tree encoding** is a \( \mathcal{V} \)-structure \( \mathcal{T} \) with universe \( T \) such that

(i) \((T, s^T)\) is a directed tree (i.e., an acyclic directed graph in which all vertices have in-degree 1 except exactly one.), (ii) every vertex in \((T, s^T)\) has out-degree at most 2, (iii) \( \text{Label}_1^T, \ldots, \text{Label}_k^T, \text{Label}_{\text{blank}}^T \) form a partition of \( T \), (iv) \( \text{root}^T \) denotes the unary relation containing only the unique vertex with in-degree 0 in \((T, s^T)\), and (v) all the children of vertices in \( T \backslash \text{Label}_{\text{blank}}^T \) belong to \( \text{Label}_{\text{blank}}^T \). Fig. 3(a) shows an aligned 3-tree encoding. The edges represent \( s \). The small black circles represent \( \text{Label}_{\text{blank}} \), the larger red circles represent \( \text{Label}_1 \), the squares represent \( \text{Label}_2 \), and the diamonds represent \( \text{Label}_3 \).

**Lemma 16.** There is a \( C^2 \)-sentence \( \text{enc} \) such that for every \( \mathcal{V} \)-structure \( \mathcal{M} \) in which \( s^\mathcal{M} \) is a binary tree, \( \mathcal{M} \models \text{enc} \) iff \( \mathcal{M} \) is an aligned \( k \)-tree encoding.

**Proof.** Being an aligned \( k \)-tree encoding is expressible in \( C^2 \) for structures in which \( s \) is interpreted as a binary tree. Below \( \xi_X \) refers to requirement \( X \) in the definition of an aligned \( k \)-tree encoding. \( \xi_i \) and \( \xi_{ii} \) are already taken care of since \( s \) is a binary
A decorated aligned \( k \)-tree \( \mathfrak{A} \) is an expansion of an aligned \( k \)-tree encoding \( \mathfrak{T} \) to \( \mathcal{R} \cup \mathcal{R} \). For every \( j \in [k] \), \( R^T_j \) contains all pairs \((v, u)\) in \((T \setminus (\text{Label}_T^T \cup \text{Label}^{\text{blank}}_T)) \times \text{Label}^T_j \) such that there is a directed path \( P \) from \( u \) to \( v \) in \((T, s^T)\) which does not intersect with \( \text{Label}^T_j \) except on \( u \), i.e., \( \text{Label}^T_j \cap (P \setminus \{u\}) = \emptyset \); moreover, for every \( u \), if \( R^T_j \) does not contain any pair \((u, v)\), then \( R^T_j \) contains additionally the self loop \((u, u)\). Observe that the relation \( R^T_j \) is a total function. Fig. 3(b) shows the decorated aligned 3-tree obtained from (a). The new edges represent \( R^T_1 \), \( R^T_2 \), and \( R^T_3 \); dashed edges represent \( R^T_1 \), solid edges represent \( R^T_2 \), and the single thick edge (from a square to a diamond) represents \( R^T_3 \).
A **aligned** $k$-**tree** is the $\mathcal{R}$-reduct of the substructure of a decorated aligned $k$-tree $\mathfrak{A}_0$ generated by $\text{Label}^0_1 \cup \cdots \cup \text{Label}^0_k$. Fig. 2(c) shows the aligned 3-tree obtained from (b). The vertices of (c) are not labeled by any unary relation.

**Lemma 17.**

1. For every aligned $k$-tree $\mathfrak{A}$, $\text{Gaif}(\mathfrak{A})$ is a partial $(k-1)$-tree.
2. For every partial $(k-1)$-tree $G$ there is an aligned $k$-tree $\mathfrak{A}$ such that $G$ is a subgraph of $\text{Gaif}(\mathfrak{A})$.

**Proof.** For the proof of Lemma 17(1) we define an aligned $k$-tree encoding $\Xi$ to be **perfect** if (1) there are $w_1, p_1, w_2, \ldots, p_{k-1}, w_k \in T$ such that $w_1, p_1, w_2, \ldots, p_{k-1}, w_k$ is a directed path, (2) $\text{root}(\Xi) = \{w_1\}$, (3) $w_i \in \text{Label}^0_i$ of all $i \in [k]$, (4) $p_i \in \text{Label}_{\text{blank}}$ for all $i \in [k-1]$, and (5) the out-degree of each vertex in the path is exactly 1. If $\Xi$ is a perfect aligned $k$-tree encoding, then $\mathbf{u}^*(\mathbf{i}^*(\Xi))$ is a perfect aligned $k$-tree. We have:

   1. For every perfect aligned $k$-tree $\mathfrak{A}$, $\text{Gaif}(\mathfrak{A})$ is a $(k-1)$-tree.

(1) is proved by induction on the number of vertices in a perfect aligned $k$-tree encoding $\Xi$ such that $\mathbf{u}^*(\mathbf{i}^*(\Xi)) = \mathfrak{A}$. Lemma 17(1) follows from (1) by building a perfect aligned $k$-tree such that $\mathfrak{A}$ is its substructure.

First we prove:

   1. For every perfect aligned $k$-tree $\mathfrak{A}$, $\text{Gaif}(\mathfrak{A})$ is a $(k-1)$-tree.

Let $\Xi$ be a perfect aligned $k$-tree encoding such that $\mathbf{u}^*(\mathbf{i}^*(\Xi)) = \mathfrak{A}$. We prove the claim by induction on the number of vertices in $\Xi$.

   - In the base case, $\Xi$ consists only of the vertices $w_1, p_1, \ldots, p_{k-1}, w_k$, and $\mathbf{u}^*(\mathbf{i}^*(\Xi))$ is a $k$-clique, which is a $(k-1)$-tree.

   - Let $\Xi$ be a perfect aligned $(k-1)$-tree encoding. Let $\Xi_0$ be obtained from $\Xi$ by removing a leaf $v$. By the induction hypothesis, $\text{Gaif}(\mathbf{i}^*(\Xi_0))$ is a $(k-1)$-tree. If $v \in \text{Label}^\Xi_0$, then $\mathbf{u}^*(\mathbf{i}^*(\Xi_0)) = \mathbf{u}^*(\mathbf{i}^*(\Xi))$. Otherwise, let $v \in \text{Label}^\Xi_1$. We denote by $B = \{u_i : i \in [k] \setminus \{j\}\}$ the set of elements such that $u_i \in \text{Label}$, and there is a path $P_i$ from $u_i$ to $v$ in $\Xi$ satisfying the following property: $P_i$ does not visit any vertex in $\text{Label}$ except $u_i$. Note that for all $i \in [k] \setminus \{j\}$ $u_i$ exists, and hence $|B| = k - 1$, using that $\Xi_0$ is perfect. By the choice of $B$ and since $v$ is a leaf, $B$ is the set of neighbors of $v$ in $\text{Gaif}(\mathbf{u}^*(\mathbf{i}^*(\Xi)))$. It remains to show that $B$ is a $(k-1)$-clique to get that $\text{Gaif}(\mathbf{u}^*(\mathbf{i}^*(\Xi)))$ is a $(k-1)$-tree. Let $i_1, i_2 \in [k] \setminus \{j\}$ with $i_1 \neq i_2$. Since $(T, s^\Xi)$ is a directed tree, either $P_{i_1} \subseteq P_{i_2}$ or $P_{i_2} \subseteq P_{i_1}$. W.l.o.g. let $P_{i_1} \subseteq P_{i_2}$, so $P_{i_2}$ is a path from $u_{i_1}$ to $v$ which visits $u_{i_1}$. Let $P_{21} \subseteq P_{i_2}$ be the path from $u_{i_2}$ to $u_{i_1}$. By the choice of $B$, there is no vertex in $P_{21}$ which belongs to $\text{Label}_{i_2}$ except $u_{i_2}$. Hence, $(u_{i_1}, u_{i_2}) \in R_{i_2}^{\mathbf{u}^*(\mathbf{i}^*(\Xi))}$, i.e. there is an edge between $u_{i_1}$ and $u_{i_2}$ in $\text{Gaif}(\mathbf{u}^*(\mathbf{i}^*(\Xi)))$. We get that $B$ is a $(k-1)$-clique and $\text{Gaif}(\mathbf{u}^*(\mathbf{i}^*(\Xi)))$ is a $(k-1)$-tree.

Now we are ready to prove Lemma 17(1) and Lemma 17(2).

(1) Let $\Xi$ be an aligned $k$-tree encoding such that $\mathbf{u}^*(\mathbf{i}^*(\Xi)) = \mathfrak{A}$. Let $\Xi_0$ be obtained from the directed path $w_1, p_1, \ldots, p_{k-1}, w_k$ with $w_i \in \text{Label}^\Xi_i$ for

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8 A small but important note is that the definition of aligned $k$-trees used throughout this appendix deviates slightly from the one given in Section 2 and used elsewhere in the paper.
each \(i \in [k]\), \(p_i \in \text{Label}^\text{\text{blank}}_i\) for each \(i \in [k-1]\), and root \(\underline{\Sigma}\) = \{\(w_1\)\} by attaching \(\underline{\Sigma}\) as a child of \(w_k\). By construction, \(\Sigma_0\) is perfect. By (\(c\)) we have that 
\(\text{Gaif}(\text{u}^*(\text{i}^*(\Sigma_0)))\) is a \((k-1)\)-tree. Deleting the vertices \(w_1, p_1, \ldots, p_{k-1}, w_k\) and the edges incident to them from \(\text{Gaif}(\text{u}^*(\text{i}^*(\Sigma_0)))\) we get \(\text{Gaif}(\text{u}^*(\text{i}^*(\underline{\Sigma})))\), implying that \(\text{Gaif}(\text{u}^*(\text{i}^*(\underline{\Sigma})))\) is a partial \(k\)-tree.

(2) We prove this by induction on the construction of \((k-1)\)-trees.
(a) Let \(G\) be a \((k-1)\)-clique with vertices \(w_1, \ldots, w_{k-1}\). Let \(\Sigma_0\) be the directed path on the vertices \(w_1, p_1, \ldots, p_{k-2}, w_{k-1}\) such that \(w_j \in \text{Label}^\text{\text{blank}}_j\) for all \(i \in [k-1]\) and \(p_i \in \text{Label}^\text{\text{blank}}_i\) for all \(i \in [k-2]\). We have that 
\(\text{Gaif}(\text{u}^*(\text{i}^*(\Sigma_0)))\) = \(G\).
(b) Let \(G_0\) be a \((k-1)\)-tree and let \(\Sigma_0\) be an aligned \(k\)-tree encoding such that 
\(\text{Gaif}(\text{u}^*(\text{i}^*(\Sigma_0)))\) = \(G_0\). Let \(C = \{u_1, \ldots, u_{k-1}\}\) be a \((k-1)\)-clique in \(G_0\). Let \(G\) be the graph obtained from \(G_0\) by adding a new vertex \(v\) as well as edges between \(v\) and each of the vertices in \(C\). Since \(C\) is a \((k-1)\)-clique in \(G_0\), for every distinct \(t_1, t_2 \in [k-1]\) there is a directed path in \(\Sigma_0\) between from \(u_{t_1}\) to \(u_{t_2}\) or vice versa. As a consequence, there is a path \(P\) in \(\Sigma_0\) from the root to some \(u_t\) such that \(u_1, \ldots, u_{k-1}\) all occur on \(P\). For every \(t \in [k-1]\) and \(i \in [k]\) such that \(u_t \in \text{Label}^\text{\text{blank}}_i\), \(u_t\) is the last vertex in \(P\) to belong to \(\text{Label}^\text{\text{base}}_i\). Since \(|C| = k-1\), there is \(j \in [k]\) such that \(\text{Label}^\text{\text{base}}_j \cap C = \emptyset\). Since \(C\) is a clique in \(\text{Gaif}(\text{u}^*(\text{i}^*(\Sigma_0)))\), it must be the case that \(\text{Label}^\text{\text{base}}_i \cap C = 1\) for all \(i \in [k]\) \(\setminus\{j\}\). If \(u_t\) has two children, let \(\Sigma\) be a subtree of \(\Sigma_0\) whose root is a child of \(u_t\). Let \(\Sigma\) be obtained from \(\Sigma_0\) by attaching a new child vertex \(v_{\text{blank}} \in \text{Label}^\text{\text{blank}}_i\) to \(u_t\), moving \(\Sigma\) to be a child of \(v_{\text{blank}}\), and adding \(v\) as a child of \(v_{\text{blank}}\). If \(u_t\) is a leaf or has one child in \(\Sigma_0\), let \(\Sigma\) be obtained from \(\Sigma_0\) by attaching \(v\) as a child of \(u_t\). Setting \(v\) to belong to \(\text{Label}^\text{\text{base}}_j\), we get \(\text{Gaif}(\text{u}^*(\text{i}^*(\Sigma)))\) = \(G\).

\(\square\)

The translation schemes interpret, undecorate and structurize\(_C\). There are translation schemes \text{interpret} (shorthand \(i\)) for \(R \cup V\) over \(V\) and \text{undecorate} (shorthand \(u\)) for \(R\) over \(R \cup V\) which take an aligned \(k\)-tree encoding to its decorated aligned \(k\)-tree respectively a decorated aligned \(k\)-tree to its aligned \(k\)-tree. The induced transductions \(i^*\) and \(u^*\) are surjective with respect to the classes of decorated aligned \(k\)-trees resp. aligned \(k\)-trees.

For every vocabulary \(C\) there is a translation scheme \text{structurize} \(_C\) (shorthand \(s_C\)) which takes an aligned \(k\)-tree whose elements are annotated with unary relations to a \(C\)-structure. The induced transduction \(s_C\) is surjective with respect to the class of \(C\)-structures of tree-width less than \(k\). The binary relations of such structures are encoded inside aligned \(k\)-trees by unary relations on the sources of \(R_j\) edges, using that the \(R_j\) are functions. The vocabulary consisting of these new unary relation symbols as well as the unary relation symbols of \(C\) is denoted by \(N_C\). \(s_C\) is a translation scheme for \(C\) over \(R \cup N_C\). Fig. 3(d) shows an aligned \(k\)-tree annotated with the unary relations of \(N_C\) for \(C = \langle E \rangle\), where \(E\) is a binary relation symbol. Fig. 3(e) shows the directed graph obtained from (d) by applying \(s_C\).

Lemma 18.
A.: For every decorated aligned \(k\)-tree \(\mathfrak{A}\), \(i^*(\mathfrak{A}|_V) = \mathfrak{A}\).
B.: For every aligned \(k\)-tree \(\mathfrak{A}\) obtained from a decorated aligned \(k\)-tree \(\mathfrak{A}_0\), \(u^*(\mathfrak{A}_0) = \mathfrak{A}\).
C.: There is a vocabulary $\mathcal{N}_{C}$ consisting of unary relation symbols only, such that for every $\mathcal{C}$-structure $\mathfrak{M}$, $\mathfrak{M}$ has tree-width at most $k$ if there is an aligned $k$-tree $\mathfrak{A}_{ori}$ and an expansion $\mathfrak{A}$ of $\mathfrak{A}_{ori}$ to $\mathcal{N}_{C} \cup \mathcal{R}$ such that $s_{C}^{k}(\mathfrak{A}) = \mathfrak{M}$.

Lemma 2 follows from Lemma 18, Lemma 16 and Lemma 12 with $\mathcal{D}_{bnd} = \mathcal{N}_{C} \cup \mathcal{V}$, $\alpha' = \text{enc}(u^{T}(s_{C}^{k}(\alpha)))$, and $\beta' = i^{T}(u^{T}(s_{C}^{k}(\beta)))$.

Proof of Lemma 18(A). Let $\mathfrak{A} = \langle \varphi, \psi_{C} : C \in \mathcal{R} \rangle$ be as follows: $\varphi(x) = \neg \text{Labelblank}(x)$ and for every $i \in [k]$, $\psi_{C}(x, y) = R_{i}(x, y)$.

Proof of Lemma 18(C). We prove the following lemma which spells out the two directions of Lemma 18(C):

Lemma 19. For every vocabulary $\mathcal{C}$, there is a set $\mathcal{N}_{C}$ of unary relation symbols and a translation scheme $\text{structurize}_{C}$ (shorthand $s_{C}$) for $\mathcal{C}$ over $\mathcal{N}_{C} \cup \mathcal{R}$ such that:

1. If $\mathfrak{M}$ is a $\mathcal{C}$-structure whose Gaifman graph is a partial $(k - 1)$-tree, then there is an expansion $\mathfrak{A}$ of an aligned $k$-tree to $\mathcal{N}_{C} \cup \mathcal{R}$ for which $s_{C}^{k}(\mathfrak{A}) = \mathfrak{M}$.
2. If $\mathfrak{A}$ is a $(\mathcal{N}_{C} \cup \mathcal{R})$-structure and $\mathfrak{A}_{red} = \mathfrak{A} | \mathcal{R}$ is an aligned $k$-tree, then $\text{Gaif}(s_{C}^{k}(\mathfrak{A}))$ is a partial $(k - 1)$-tree.

For every vocabulary $\mathcal{C}$, let $\mathcal{N}_{C}$ be the vocabulary extending the set $\text{un}(\mathcal{C})$ of unary relation symbols in $\mathcal{C}$ by fresh unary relation symbols $U_{B, \text{self}}$, $U_{B, j}$, and $U_{B, \text{inv}, j}$ for every binary relation symbol $B \in \mathcal{C}$ and $j \in [k]$. Recall $\mathcal{V}_{C} = \mathcal{V} \cup \mathcal{N}_{C}$ and $\mathcal{R}_{C} = \mathcal{V}_{C} \cup \mathcal{R} \cup \mathcal{C}$.

Since the $R_{i}$ are functions, we use a fixed number of unary relation symbols to encode, for every universe element $m_{1}$ such that $(m_{1}, m_{2}) \in R_{i}^{\text{bnd}}$, what other
relations of \( \mathcal{M} \) do \((m_1, m_2)\) and \((m_2, m_1)\) belong to. The unary relation symbols are of the form \( U_{B,j} \) and \( U_{B,\text{inv},j} \), where \( j \in [k] \) and \( B \) is a binary relation symbol. We use that tuples \((m_1, m_2)\) occur in any relation of \( \mathcal{M} \), occur also in \( R^2_j \) or in \( (R^2_j)^{-1} \). Additional unary relation symbols \( U_{B,\text{self}} \) encode self loops \((m, m)\) in \( B^\mathcal{M} \).

Let \( s_C = \langle \varphi, \psi_C : C \in \mathcal{C} \rangle \) is the translation scheme given as follows.

- \( \varphi(x) = (x \approx x) \).
- For every unary relation symbol \( U \in \mathcal{C} \), \( \psi_U(x) = U(x) \).
- For every binary relation symbol \( B \in \mathcal{C} \), \( B_{\mathcal{E}}(\mathcal{F}) \) consists of all pairs \((v, u)\) such that \( (a) (v, u) \in R^2_j \) and \( v \in U^\mathcal{M}_{B,j} \) or \( (b) (u, v) \in R^2_j \) and \( u \in U^\mathcal{M}_{B,\text{inv},j} \).

This is given by

\[
\psi_B(x, y) = \neg(x \approx y) \land \bigvee_{j \in [k]} \left( (R_j(x, y) \land U_{B,j}(x)) \lor (R_j(y, x) \land U_{B,\text{inv},j}(y)) \right)
\]

\[
\lor (x \approx y) \land U_{B,\text{self}}(x)
\]

Let \( \mathcal{M} \) be a \( C \)-structure with universe \( M \). Let \( \mathcal{A}_0 \) be an aligned \( k \)-tree such that \( \text{Gaif}(\mathcal{M}) = \text{Gaif}(\mathcal{A}_0) \) guaranteed by Lemma 17. Let \( \mathcal{A}_1 \) be the expansion of \( \mathcal{A}_0 \) such that, for every \( B \in \mathcal{C} \) and \( j \in [k] \),

\[
U^\mathcal{A}_1_{B,j} = \left\{ m_1 \mid (m_1, m_2) \in R^2_j \cap B^\mathcal{M} \right\}
\]

\[
U^\mathcal{A}_1_{B,\text{inv},j} = \left\{ m_1 \mid (m_2, m_1) \in (R^2_j)^{-1} \cap B^\mathcal{M} \right\}
\]

\[
U^\mathcal{A}_1_{B,\text{self}} = \left\{ m \mid (m, m) \in B^\mathcal{M} \right\}
\]

Self loops are encoded by the relations of the form \( U^\mathcal{A}_1_{B,\text{self}} \). Consider \( a, b \in M \) which are distinct. We divide into cases.

- Assume \((a, b)\) is not an edge of \( \text{Gaif}(\mathcal{M}) \). By \( \text{Gaif}(\mathcal{M}) = \text{Gaif}(\mathcal{A}_1) \), neither \((a, b)\) nor \((b, a)\) belong to any \( R^2_j \in \mathcal{F}_j \). Hence, \( \mathcal{A}_1 \not\models \psi_B(a, b) \), i.e. \((a, b) \notin B_{\mathcal{E}}(\mathcal{F}) \), for any binary relation symbol \( B \in \mathcal{C} \).

- Assume \((a, b)\) is an edge of \( \text{Gaif}(\mathcal{M}) \). By \( \text{Gaif}(\mathcal{M}) = \text{Gaif}(\mathcal{A}_0) \), either \((a, b)\) or \((b, a)\) belong to some \( R^2_j = R^1_j \). We have: \((a, b) \in B^\mathcal{M} \) iff exists \( j \in [k] \) such that \( (a, b) \in R^1_j \) and \( a \in U^\mathcal{M}_{B,j} \) or \( (b, a) \in R^1_j \). Then \( b \in U^\mathcal{M}_{B,\text{inv},j} \) iff \( \mathcal{A}_1 \models \psi_B(a, b) \) iff \((a, b) \in B_{\mathcal{E}}(\mathcal{F}) \).

In both cases \((a, b) \in B^\mathcal{M} \) iff \((a, b) \in B_{\mathcal{E}}(\mathcal{F}) \). We get that \( \mathcal{M} = \mathcal{A}_1 \) and (1) follows.

Now we turn to (2). By Lemma 17, the Gaifman graph of \( \mathcal{A} \) is a partial \((k-1)\)-tree. By definition of \( \psi_B \), for every binary \( B \in \mathcal{C} \), \( B_{\mathcal{E}}(\mathcal{F}) \subseteq \bigcup_{j \in [k]} R^2_j \cup (R^2_j)^{-1} \), implying that \( \text{Gaif}(s_C(\mathcal{A})) \) is a partial \((k-1)\)-tree.

**The translation scheme \( \text{tr}_C \): from structures of bounded tree-width to labeled trees.** Let \( i_C \) and \( u_C \) be the translation schemes which extend \( i \) and \( u \) respectively as follows. The unary relations \( C \) of \( \mathcal{N}_C \) are additionally defined under \( i_C \) and \( u_C \) to be \( \psi_C(x) = C(x) \). Hence, \( i_C \) is a translation scheme for \( \mathcal{R} \cup \mathcal{V} \cup \mathcal{N}_C \) over \( \mathcal{V} \cup \mathcal{N}_C \), \( u_C \) is a translation scheme for \( \mathcal{R} \cup \mathcal{N}_C \) over \( \mathcal{R} \cup \mathcal{V} \cup \mathcal{N}_C \), and \( s_C \) is a translation scheme for \( C \) over \( \mathcal{R} \cup \mathcal{N}_C \).
Before we define \( \text{tr}_C \), we need to introduce the notion of composition of translation schemes. Let \( C_0, C_1 \) and \( C_2 \) be vocabularies. Let \( t_i = (\phi^i, \psi^i : C_i \in \mathcal{C}_i) \) be a translation scheme for \( \mathcal{C}_i \) over \( \mathcal{C}_{i-1} \). The composition of \( t_1 \) and \( t_2 \), denoted \( t_1 \circ t_2 \), is the translation scheme given by \( t = (\phi, \psi_C : C \in \mathcal{C}_i) \) such that:

1. \( \phi(x) = (t^1)^2((t^2)^2(x \approx y)) \)
2. For every \( C \in \text{un}(\mathcal{C}), \psi_C(x) = (t^1)^2((t^2)^2(C(x))) \)
3. For every \( C \in \text{bin}(\mathcal{C}), \psi_C = (t^1)^2((t^2)^2(C(x), y))) \)

The translation scheme \( t_C \) is the composition of \( i_C, u_C \) and \( s_C \), i.e. \( t_C = i_C \circ (u_C \circ s_C) \). \( t_C \) is a translation scheme for \( C \) over \( \mathcal{V} \cup \mathcal{N}_C \). As a consequence of Lemma 18, we have:

**Lemma 20.** There is a vocabulary \( \mathcal{N}_C \) consisting of unary relation symbols only, such that for every \( C \)-structure \( \mathfrak{M} \), \( \mathfrak{M} \) has tree-width at most \( k \) iff there is an \( \mathcal{V} \cup \mathcal{N}_C \)-structure \( \mathfrak{A} \) such that \( t^*_C(\mathfrak{A}) = \mathfrak{M} \) and \( \mathfrak{A} \) is an aligned \( k \)-tree encoding.

**Proof of Lemma 14.** Let \( \mathcal{C}_{\text{bnd}} \) and \( \mathcal{C}_{\text{unb}} \) be vocabularies such that \( \mathcal{C}_{\text{bnd}} \cap \mathcal{C}_{\text{unb}} \) only contains unary relation symbols. Let \( \mathcal{D}_{\text{bnd}} = \mathcal{V} \cup \mathcal{N}_{\mathcal{C}_{\text{bnd}}} \). We define \( \text{tr} \) which extends \( t_{\mathcal{C}_{\text{unb}}} \) as follows: For every \( C \in \mathcal{C}_{\text{unb}} \setminus \mathcal{C}_{\text{bnd}} \), let \( \psi_C(x) = C(x) \) or \( \psi_C(x, y) = C(x, y) \). I.e. \( \text{tr} \) copies the relations of \( \mathcal{C}_{\text{unb}} \setminus \mathcal{C}_{\text{bnd}} \) from the source structure to the target structure without change (except taking the projection to the universe of the target structure imposed by the formula \( \phi(x) \)). Let \( \text{dom} = \text{enc} \) from Lemma 16.

(a) \( \phi \) is quantifier-free since \( t^*_C \) is obtained from the composition of translation schemes in which \( \phi \) is quantifier-free.

(b) \( \psi_C \) is quantifier-free for every \( C \in \mathcal{C}_{\text{unb}} \setminus \mathcal{C}_{\text{bnd}} \) by the definition in this subsection. \( \psi_C \) is quantifier-free for every \( C \in \text{un}(\mathcal{C}_{\text{bnd}}) \) since we have in fact \( \psi_C(x) = C(x) \); this is true because it is true for each of \( i_C, u_C \) and \( s_C \).

(c) For every \( C \in \mathcal{C}_{\text{bnd}} \), \( \psi_C \) is defined in the translation scheme \( t_{\mathcal{C}_{\text{bnd}}} \). Since \( t_{\mathcal{C}_{\text{bnd}}} \) is a translation scheme for \( \mathcal{C}_{\text{bnd}} \) over \( \mathcal{V} \cup \mathcal{N}_{\mathcal{C}_{\text{bnd}}} = \mathcal{D}_{\text{bnd}} \), \( \psi_C \in \text{MSO}(\mathcal{D}_{\text{bnd}}) \).

(d) This item follows from Lemma 20 and using that \( \text{dom} = \text{enc} \) defines the class of aligned \( k \)-tree encodings on structures \( \mathfrak{A} \) in which \( s^\mathfrak{A} \) is a binary tree.

8.3. **Proof of Lemma 15.** For a leaf \( b \), \( \text{hin}_q(\Sigma_b) \) depends only on the unary relations which \( b \) satisfies. By Theorem 3 for a vertex \( b \) with one child \( b_0 \) (two children \( b_0, b_1 \), \( \text{hin}_q(\Sigma_b) \) depends only on the unary relations which \( b \) satisfies and on \( \text{hin}_q(\Sigma_{b_0}) \) (on \( \text{hin}_q(\Sigma_{b_0}) \) and \( \text{hin}_q(\Sigma_{b_1}) \)).

Let

\[
\Theta^\text{hin}_{\mathcal{C}, q} = \text{part} \land \text{leaves} \land \text{ints}_1 \land \text{ints}_2
\]

where part says that \( \{C_e : e \in \mathcal{E} \setminus \mathcal{C}_{\mathcal{F}, q}\} \) partition the universe, and leaves, \( \text{ints}_1 \), and \( \text{ints}_2 \) define the \( C_e \) for the leaves respectively the internal vertices of \( \Sigma_{b_0} \) with one or two children.

We give part, leaves, \( \text{ints}_1 \) and \( \text{ints}_2 \) below. There are \( C^2 \) formulas \( \text{leaf}(x) \), \( \text{int}_1(x) \), and \( \text{int}_2(x) \), which express that \( x \) is a leaf, has one child, or has two children, respectively. Let

\[
\text{part} = \forall x \left( \bigvee_{e \in \mathcal{E} \setminus \mathcal{C}_{\mathcal{F}, q}} C_e(x) \right) \land \left( \bigwedge_{e_1 \neq e_2 \in \mathcal{E} \setminus \mathcal{C}_{\mathcal{F}, q}} (\neg C_{e_1}(x) \lor \neg C_{e_2}(x)) \right).
\]
For \( U \subseteq \text{un}(C) \), let \( \mathcal{D}_U \) be a \( C \)-structure with universe \( O \) of size 1 satisfying that \( U^{\mathcal{D}_U} = \emptyset \) iff \( U \in \mathcal{U} \), and \( \text{root}^{\mathcal{D}_U} = O \). Let

\[
\begin{align*}
\text{ints}_1 &= \forall x \left( \text{int}(x) \rightarrow \bigwedge_{U \subseteq \text{un}(C)} ((\text{this}_{\mathcal{U}}(x) \land \text{child}_{\epsilon_2}(x)) \rightarrow C_{\epsilon_1}(x)) \right) \\
\text{this}_{\mathcal{U}}(x) &= \bigwedge_{U \subseteq \text{un}(C)} - (U \in \mathcal{U}) U(x) \\
\text{child}_{\epsilon_2}(y) &= \exists y \ (s(x, y) \land C_{\epsilon_2}(y)) \\
\bigwedge_{U \subseteq \text{un}(C)} \text{int}(x) &\text{ ranges over } \epsilon_1, \epsilon_2 \in \mathcal{H}_{\mathcal{A}, \mathcal{N}, C, q} \text{ and } U \subseteq \text{un}(C) \text{ such that } \epsilon_2 \neq \epsilon_3, \text{ and for every two structure } \mathcal{A}, \mathcal{B} \in \mathcal{K}_{\text{root}} \text{ with } \text{hin}(\mathcal{A}) = \epsilon_2 \text{ and } \text{hin}(\mathcal{B}) = \epsilon_3, \text{ the notation } -^X Y \text{ stands for } Y \text{ if } X \text{ is false, and } -Y \text{ otherwise.}
\end{align*}
\]

We define

\[
\text{leaves} = \forall x \left( \text{leaf}(x) \rightarrow \bigwedge_{U \subseteq \text{un}(C)} \text{this}_{\mathcal{U}}(x) \rightarrow C_{\epsilon_1}(x) \right)
\]

\( \bigwedge_{U \subseteq \text{un}(C)} \text{int}(x) \) ranges over \( \epsilon_1 \in \mathcal{H}_{\mathcal{A}, \mathcal{N}, C, q} \) and \( U \subseteq \text{un}(C) \), where \( \text{hin}(\mathcal{D}_U) = \epsilon_1 \).

Finally, \( \text{ints}_2 \) is defined as follows:

\[
\begin{align*}
\text{ints}_2 &= \forall x (\text{int}(x) \rightarrow (\text{int}(x) \land \text{int}(x))) \\
\text{int}(x) &= \bigwedge_{U \subseteq \text{un}(C)} ((\text{this}_{\mathcal{U}}(x) \land \text{child}_{\epsilon_2}(x) \land \text{child}_{\epsilon_3}(x)) \rightarrow C_{\epsilon_1}(x)) \\
\text{int}(x) &= \bigwedge_{U \subseteq \text{un}(C)} ((\text{this}_{\mathcal{U}}(x) \land \text{child}_{\epsilon_2}(x) \land \text{child}_{\epsilon_3}(x)) \rightarrow C_{\epsilon_1}(x)) \\
\text{children}_{\epsilon_2}(x) &= \exists y \ (s(x, y) \land C_{\epsilon_2}(y)) \\
\bigwedge_{U \subseteq \text{un}(C)} \text{int}(x) &\text{ ranges over } \epsilon_1, \epsilon_2, \epsilon_3 \in \mathcal{H}_{\mathcal{A}, \mathcal{N}, C, q} \text{ and } U \subseteq \text{un}(C) \text{ such that for every structure } \mathcal{A}, \mathcal{B} \in \mathcal{K}_{\text{root}} \text{ with } \text{hin}(\mathcal{A}) = \epsilon_2 \text{ and } \text{hin}(\mathcal{B}) = \epsilon_3, \text{ the notation } -^X Y \text{ stands for } Y \text{ if } X \text{ is false, and } -Y \text{ otherwise.}
\end{align*}
\]

By definition, \( \Theta^{\text{hin}}_{\mathcal{C}, q} \in C^2 \). \( \mathcal{T}_0 \) is a \( C \)-structure in which \( s \) is interpreted as a binary tree. Let \( \mathcal{T}_1 \) be the expansion of \( \mathcal{T}_0 \) such that, for every element \( b \) of the universe \( T_1 = T_0 \cup \mathcal{T}_0 \), \( b \in C^{\text{hin}(\mathcal{T}_0)} \), where \( \mathcal{T}_0 \) is the subtree of \( \mathcal{T}_0 \) rooted at \( b \). In particular, for the root \( r \) of \( \mathcal{T}_0 \), \( r \in C^{\text{hin}(\mathcal{T}_0)} \). We have \( \mathcal{T}_1 \models \Theta^{\text{hin}}_{\mathcal{C}, q} \), and hence (i) holds. Since \( \mathcal{T}_1 \) is the only expansion of \( \mathcal{T}_0 \) which satisfies \( \Theta^{\text{hin}}_{\mathcal{C}, q} \), (ii) holds.

### 8.4. Proof of Lemma 4

We assume \( 2 \text{-tp}(\mathcal{M}_1) = 2 \text{-tp}(\mathcal{M}_2) \). Let \( u_1, v_1 \in M \) be two elements and let \( \lambda = 2 \text{-tp}^{\mathcal{M}_1}(u_1, v_1) \). Then there are two elements \( u_2, v_2 \in M \) such that \( \lambda = 2 \text{-tp}^{\mathcal{M}_2}(u_2, v_2) \) because of \( 2 \text{-tp}(\mathcal{M}_1) = 2 \text{-tp}(\mathcal{M}_2) \). Thus, \( \mathcal{M}_1 \models \chi(u_1, v_1) \) implies \( \mathcal{M}_2 \models \chi(u_2, v_2) \). Because \( u_1, v_1 \) have been chosen arbitrarily, we get that \( \mathcal{M}_1 \models \phi \) implies \( \mathcal{M}_2 \models \phi \). The other direction is symmetric.

### 8.5. Proof of Lemma 6

**Proof.** Let \( z = 2|\mathcal{T}|^2 + 1 \). We set \( \text{colors}(\mathcal{T}) = \{ A_i \mid 1 \leq i \leq z \} \), where the \( A_i \) are fresh unary relation symbols.
Let $\mathfrak{M}$ be a $C_{\text{unb}}$-structure with universe $M$. By Lemma 7, there is a proper $z$-coloring of $\mathcal{G}_T^{\mathfrak{M}}$. We expand $\mathfrak{M}$ to a $(C_{\text{unb}} \cup \text{colors}(T))$-structure $\mathfrak{N}$ as follows: for all elements $u \in M$ we have $A^i_c(u)$ iff $u$ has color $i$ in the proper $z$-coloring of $\mathcal{G}_T^{\mathfrak{M}}$. Clearly, $\mathfrak{N}$ is a chromatic $(C_{\text{unb}} \cup \text{colors}(T))$-structure. □

8.6. Proof of Lemma 7. We proceed by induction on the number of vertices $n = |V|$. The case $n = 1$ is trivial. Assume that $n > 1$ and that the lemma holds for $n - 1$. We have $\sum_{v \in V} deg(v) = 2 \cdot \sum_{v \in V} deg^+(v) \leq 2nk$. Thus there is a vertex $v \in V$ with $deg(v) \leq 2k$. By the induction hypothesis, the graph $G - v$ has a proper $2k + 1$-coloring. Because of $deg(v) \leq 2k$ the proper $2k + 1$-coloring of $G - v$ can be extended to a proper $2k + 1$-coloring of $G$.

8.7. Proof of Lemma 10. Let $v \in M$ be an element with $2\text{-}\text{tp}^{\mathfrak{L}_k}(u, v) = \lambda$. We get $\mathfrak{L} \models P_\lambda(u)$ because of $\mathfrak{L} \models \zeta_{\text{unb}}$. Because of $1\text{-}\text{tp}^{\mathfrak{L}_k}(u) = 1\text{-}\text{tp}^{\mathfrak{L}'}(u)$ we have $\mathfrak{L}' \models P_\lambda(u)$. Because of $\mathfrak{L}' \models \zeta_{\text{bnd}}$ we have $\lambda = 2\text{-}\text{tp}^{\mathfrak{L}'}(u, w)$ for some $w \in M$. By Lemma 10 we get $\mathfrak{L}' \models R_\lambda(u, v)$. Because of $\mathfrak{L}' \models \theta$ we get $v = w$. In the same way one can show that $2\text{-}\text{tp}^{\mathfrak{L}_k}(u, v) = \lambda$ implies $2\text{-}\text{tp}^{\mathfrak{L}'}(u, v) = \lambda$.

8.8. Proof of Lemma 8. We show $\mathfrak{L}|_{\text{unb}} = \mathfrak{L}'|_{\text{unb}}$. Let $u, v \in M$ with $\mathfrak{L} \models B(u, v)$ for some $B \in \text{bin}(\mathfrak{L}_{\text{unb}})$. Because of $\mathfrak{L} \models \theta$ we have $\mathfrak{L} \models R_\lambda(u, v)$ or $\mathfrak{L} \models R_\lambda(v, u)$ for some $i \in [k]$. Because of $\text{rank}(\mathfrak{L}, \mathfrak{L}') = 0$ we have $\text{rank}_{\text{unb}}(\mathfrak{L}, \mathfrak{L}') = 0$. By Lemma 10 we have $2\text{-}\text{tp}^{\mathfrak{L}_k}(u, v) = 2\text{-}\text{tp}^{\mathfrak{L}'}(u, v)$. Thus, $\mathfrak{L}' \models B(u, v)$. In the same way one can show that $\mathfrak{L}' \models B(u, v)$ implies $\mathfrak{L} \models B(u, v)$. Thus, $\mathfrak{L}|_{\text{unb}} = \mathfrak{L}'|_{\text{unb}}$.

We show $\text{Gaif}(\mathfrak{L}|_{\text{unb}}) = \text{Gaif}(\mathfrak{L}')$: We have that $\text{Gaif}(\mathfrak{L}') = \text{Gaif}(\mathfrak{L}'|_{\text{bnd}})$ because of $\mathfrak{L}' \models \theta$ and $\mathfrak{L}' \models \epsilon$. We have $\text{Gaif}(\mathfrak{L}|_{\text{unb}}) = \text{Gaif}(\mathfrak{L}|_{\text{bnd}})$ because of $\mathfrak{L} \models \theta$. We have $\text{Gaif}(\mathfrak{L}'|_{\text{bnd}}) = \text{Gaif}(\mathfrak{L}|_{\text{bnd}})$ because of $\text{rank}(\mathfrak{L}, \mathfrak{L}') = 0$. Thus, the claim follows.

8.9. The induced translation $t^\sharp$. We spell out the inductive definition of the induced translation.

Let $C_0$ and $C_1$ be vocabularies. Given a translation scheme $t = \langle \phi, \psi_C : C \in C_0 \rangle$ for $C_0$ over $C_1$ we define the induced translation $t^\sharp$ to be a function from $\text{MSO}(C_0)$-formulas to $C_1$-formulas inductively as follows:

1. For $C \in \text{un}(C_0)$ or for monadic second order variables $C$, and for $\theta = C(x)$, we put $t^\sharp(\theta) = \psi_C(x) \land \phi(x)$

2. For $C \in \text{bin}(C_0)$ and $\theta = C(x, y)$, we put $t^\sharp(\theta) = \psi_C(x, y) \land \phi(x) \land \phi(y)$

3. For $x \approx y$, we put $t^\sharp(\theta) = x \approx \phi(x) \land \phi(y)$

4. For the Boolean connectives the translation distributes, i.e.

   - if $\theta = \theta_1 \lor \theta_2$ then $t^\sharp(\theta) = (t^\sharp(\theta_1) \lor t^\sharp(\theta_2))$

   - if $\theta = \neg \theta_1$ then $t^\sharp(\theta) = \neg t^\sharp(\theta_1)$
(5) For the existential quantifiers, we relativize to $\phi$:

If $\theta = \exists y \theta_1$, we put

$$t^x(\theta) = \exists y (\phi(y) \land t^x(\theta_1))$$

If $\theta = \exists U \theta_1$, we put

$$t^x(\theta) = \exists U (t^x(\theta_1) \land \forall y U(y) \rightarrow \phi(y))$$

We have somewhat simplified the presentation in [19, Definition 2.3] to fit our setting.

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