Approximate solution for fractional attractor one-dimensional Keller-Segel equations using homotopy perturbation sumudu transform method

Dinkar Sharma*, Gurpinder Singh Samra, and Prince Singh

Abstract: In this paper, homotopy perturbation sumudu transform method (HPSTM) is proposed to solve fractional attractor one-dimensional Keller-Segel equations. The HPSTM is a combined form of homotopy perturbation method (HPM) and sumudu transform using He’s polynomials. The result shows that the HPSTM is very efficient and simple technique for solving nonlinear partial differential equations. Test examples are considered to illustrate the present scheme.

Keywords: homotopy perturbation method; Sumudu transform; Keller-Segel equation; He’s polynomial

1 Introduction

The fractional calculus deal with number of problem arising in the field of fluid mechanics, biology, diffusion, fractional signal, image processing and many other physical process. Fractional differential equations are used to model these types of the problems. In the various field of engineering and science, it is very important to find the approximate or the exact solution of some nonlinear partial differential equations [1]. There are several potent methods such as Homotopy perturbation [2, 3]; homotopy perturbation transformation method (HPTM) [4] and homotopy perturbation sumudu transformation method (HPSTM) have been proposed to obtain the approximate or the exact solutions of nonlinear equations [5–9].

In 1970, Keller and Segel presented a mathematical formulation of cellular slime mold aggregation process [10]. Recently many researchers use different methods to solve Keller-Segel equation [11–13]. Different types of numerical methods are used to solve nonlinear partial differential equations [14–19]. The solution of multidimensional linear and nonlinear partial differential equations are established by using combination of least square approximation and homotopy perturbation approximation [20]. A new semi-analytical method called the homotopy analysis Shehu transform method is used to solve multidimensional fractional diffusion equations and this method is combination of the homotopy analysis method and the Laplace-type integral transform transform [21]. Solution of Reaction-Diffusion-Convection Problem and nonlinear equation is discussed by homotopy perturbation technique [22, 23]. Non-linear Fisher equation is solved with help of homotopy perturbation method then solution is compared with solution from Variational Iteration Method (VIM) and Adomian Decomposition Method (ADM) [24]. In this paper, we propose HPSTM for the solution of fractional attractor one dimensional Keller Siegel equation. The simplified form of the Keller Segel equation in one dimension is given as [25]:

\[
\frac{\partial U(x, t)}{\partial t} = a \frac{\partial^2 U(x, t)}{\partial x^2} - \frac{\partial}{\partial x} \left( U(x, t) \frac{\partial \rho(x)}{\partial x} \right)
\]

\[
\frac{\partial \rho(x, t)}{\partial t} = b \frac{\partial^2 \rho(x, t)}{\partial x^2} + c U(x, t) - d \rho(x, t)
\]

Subject to the boundary conditions

\[
\frac{\partial U(a, t)}{\partial x} = \frac{\partial U(b, t)}{\partial x} = \frac{\partial \rho(a, t)}{\partial x} = \frac{\partial \rho(b, t)}{\partial x} = 0
\]

And the initial conditions

\[
U(x, 0) = U_0(x), \quad \rho(x, 0) = \rho_0(x), \quad x \in I
\]

Where I = (a, b) is a bounded open interval and a, b, c and d are positive constants. The unknown functions \(U(x, t)\) denote the concentration of amoebae where \(\rho(x, t)\) denote the concentration of the chemical substance in \(I \times (0, \infty)\). The chemo tactic term \(\frac{\partial}{\partial x} \left( U(x, t) \frac{\partial \rho(x)}{\partial x} \right)\) indicates the sensitivity of the cells, \(\chi(\rho)\) called the sensitivity function of \(\rho \in (0, \infty)\). Different form of \(\chi(\rho)\) like \(\rho, \rho^2\) and
log \rho have been suggested. But in this paper, we discuss the two cases of the fractional attractor one dimensional Keller- Siegel equations one with chemo tactic sensitivity function \( \chi (p) = 1 \) and other with \( \chi (p) = \rho \).

**Definition:** A real function \( f(t) \) is said to be in space \( C_\mu, \mu \in \mathbb{R} \), if \( \exists \) a real number \( p(> \mu) \) such that \( f(t) = t^p g(t) \), where \( g(t) \in C[0, \infty) \) and it is said to be in the space \( C_\mu^m \) iff \( f^m \in C_\mu, m \in \mathbb{N} \).

**Definition:** The Riemann- Liouville fractional integral operator of order \( a > 0 \) of a function \( f(t) \in C_\mu, \mu \in \mathbb{R} \) is defined as

\[
J^a f(t) = \frac{1}{\Gamma(a)} \int_0^t (t - \tau)^{a-1} f(\tau) \, d\tau,
\]

For \( m - 1 < a \leq m, m \in \mathbb{N}, t > 0 \), where \( D_t^a \) is Caputo derivative operator and \( \Gamma(a) \) is the Gamma function.

**Sumudu Transformation:** The Sumudu transformation over the set of function

\[
A = \{ f(t) \mid M, \tau_1, \tau_2 > 0, |f(t)| < Me^{-\tau_2} \text{ if } t \in (-\infty, 0) \}
\]

is defined by Watugala (1993) as

\[
S[f(t)] = \int_0^\infty u f(t) e^{-\frac{u}{t}} \, dt, \, u \in (-\tau_1, \tau_2)
\]

Some properties of the Sumudu transformations are

\[
S[1] = 1, \quad S \left[ \frac{t^m}{\Gamma(m + 1)} \right] = u^m, \quad m > 0
\]

The Sumudu transformation of the Caputo fractional derivative is defined as

\[
S[D_t^a f(x, t)] = u^{-a} S[f(x, t)] - \sum_{k=0}^{m-1} u^{-a+k} f^{(k)}(0+),
\]

\( m - 1 < a \leq m \).

## 2 Homotopy perturbation Sumudu transformation method (HPSTM)

To illustrate the idea of HPSTM technique, we consider the fractional attractor one dimensional Keller-Segel equation

\[
D_t^\mu U(x, t) = a \frac{\partial^2}{\partial x^2} U(x, t) - \frac{\partial}{\partial x} \left( U(x, t) \frac{\partial \chi(p)}{\partial x} \right)
\]

\[
D_t^\mu \rho(x, t) = b \frac{\partial^2}{\partial x^2} \rho(x, t) + c U(x, t) - d \rho(x, t),
\]

\( 0 < \mu \leq 1, 0 \leq \eta \leq 1 \)

Subject to the conditions \( U(x, 0) = f(x) \) and \( \rho(x, 0) = g(x) \),

Where \( D_t^\mu \) and \( D_t^\mu \) is the Caputo fractional derivative of the function \( U(x, t) \) and \( \rho(x, t) \) respectively. Apply sumudu transformation on both sides of the equation (3), we have

\[
S[D_t^\mu U(x, t)] = S \left[ a \frac{\partial^2}{\partial x^2} U(x, t) - \frac{\partial}{\partial x} \left( U(x, t) \frac{\partial \chi(p)}{\partial x} \right) \right]
\]

\[
S[D_t^\mu \rho(x, t)] = S \left[ b \frac{\partial^2}{\partial x^2} \rho(x, t) + c U(x, t) - d \rho(x, t) \right]
\]

Using differentiation property of the sumudu transformation and the initial conditions, we get

\[
S[U(x, t)] = f(x) + u^a S \left[ a \frac{\partial^2}{\partial x^2} U(x, t) - \frac{\partial}{\partial x} \left( U(x, t) \frac{\partial \chi(p)}{\partial x} \right) \right]
\]

\[
S[\rho(x, t)] = g(x) + u^a S \left[ b \frac{\partial^2}{\partial x^2} \rho(x, t) + c U(x, t) - d \rho(x, t) \right]
\]

Operating with inverse sumudu transformation on both sides,

\[
U(x, t) = f(x) + \sum_{n=0}^\infty a^n S^{-1} u^a S \left[ a \frac{\partial^2}{\partial x^2} U(x, t) - \frac{\partial}{\partial x} \left( U(x, t) \frac{\partial \chi(p)}{\partial x} \right) \right]
\]

\[
\rho(x, t) = g(x) + \sum_{n=0}^\infty b^n S^{-1} u^a S \left[ b \frac{\partial^2}{\partial x^2} \rho(x, t) + c U(x, t) - d \rho(x, t) \right]
\]

Now we apply Homotopy perturbation method,

\[
U(x, t) = \sum_{n=0}^\infty U_n(x, t) p^n \quad \text{and} \quad \rho(x, t) = \sum_{n=0}^\infty \rho_n(x, t) p^n
\]

where the nonlinear term can be decomposed as

\[
\frac{\partial}{\partial x} \left( U(x, t) \frac{\partial \chi(p)}{\partial x} \right) = NY(x, t) = \sum_{n=0}^\infty p^n H_n
\]
for some He’s polynomial \( H_n \) given by
\[
H_n = \frac{1}{m!} \frac{\partial^n}{\partial \rho^n} \left[ N \left( \sum_{n=0}^{\infty} Y_n(x, t) \rho^n \right) \right]_{p=0}, \quad n = 0, 1, 2, 3, \ldots.
\]

Substituting these values in (5), we have
\[
\sum_{n=1}^{\infty} p^n U_n(x, t) = U(x, 0) + pS^{-1}
\]
\[
\left[ u^n S \left\{ a \left( \sum_{n=1}^{\infty} p^n U_n(x, t) \right)_{xx} - \sum_{n=1}^{\infty} p^n H_n \right\} \right]
\]
\[
\sum_{n=1}^{\infty} p^n \rho_n(x, t) = \rho(x, 0) + pS^{-1}
\]
\[
\left[ u^n S \left\{ b \left( \sum_{n=1}^{\infty} p^n \rho_n(x, t) \right)_{xx} + c \left( \sum_{n=1}^{\infty} p^n U_n(x, t) \right) \right\} - d \sum_{n=1}^{\infty} p^n \rho_n(x, t) \right] \right)
\]

On comparing the like powers of \( p \) on both sides,
\[
U_0 = U(x, 0) = f(x), \quad U_1 = S^{-1}[u^n S(a U_{0xx} - H_0)],
\]
\[
U_2 = S^{-1}[u^n S(a U_{1xx} - H_1)]
\]
\[
\rho_0 = \rho(x, 0) = g(x), \quad \rho_1 = S^{-1}[u^n S(b \rho_{0xx} + c U_0 - d \rho_0)]
\]
\[
\rho_2 = S^{-1}[u^n S(b \rho_{1xx} + c U_1 - d \rho_1)]
\]
Similarly we can find all the values of \( U_0, U_1, U_2, U_3, \ldots \)
and \( \rho_0, \rho_1, \rho_2, \ldots \).
The approximate solution of equation (3) can be calculated by setting \( p \to 1 \).
\[
U(x, t) = U_0 + U_1 + U_3 + \ldots, \quad \rho(x, 0) = \rho_0 + \rho_1 + \rho_2 + \cdots
\]

### 2.1 Application of HPSTM

In order to understand the solution procedure of the homotopy perturbation sumudu transform method, we consider the following examples:

**Solution of fractional attractor 1-D Keller Segel equation**

**Example:** Consider the following coupled system:
\[
\frac{\partial^\beta v}{\partial t^\beta} = a \frac{\partial^\beta v}{\partial x^\beta} - \frac{\partial}{\partial x} \left( v \frac{\partial \chi (\rho)}{\partial x} \right)
\]
\[
\frac{\partial^\beta \rho}{\partial t^\beta} = b \frac{\partial^\beta \rho}{\partial x^\beta} + cv - dp, \quad 0 < \beta \leq 1
\]
Subject to the boundary conditions
\[
v(x, 0) = m \exp(-x^2), \quad \rho(x, 0) = n \exp(-x^2)
\]

**Case-I:** Consider the sensitivity function \( \chi (\rho) = 1 \), then the chemo-tactic term i.e. \( \frac{\partial}{\partial x} \left( v \frac{\partial \chi (\rho)}{\partial x} \right) = 0 \). Hence Keller Segel equation reduces to:
\[
\frac{\partial^\beta v}{\partial t^\beta} = a \frac{\partial^\beta v}{\partial x^\beta}
\]
\[
\frac{\partial^\beta \rho}{\partial t^\beta} = b \frac{\partial^\beta \rho}{\partial x^\beta} + cv - dp, \quad 0 < \beta \leq 1
\]
By applying HPSTM on Eq. (36), we have
\[
\sum_{n=0}^{\infty} p^n v_n = v(x, 0) + pS^{-1} \left[ u^n S \left\{ a \left( \sum_{n=0}^{\infty} p^n v_n \right) \right\} \right]
\]
\[
\sum_{n=0}^{\infty} p^n \rho_n = \rho(x, 0) + pS^{-1} \left[ u^n S \left\{ b \left( \sum_{n=0}^{\infty} p^n \rho_n \right) \right\} \right]
\]
\[
+ pS^{-1} \left[ u^n S \left\{ c \left( \sum_{n=0}^{\infty} p^n v_n \right) - d \left( \sum_{n=0}^{\infty} p^n \rho_n \right) \right\} \right]
\]
On looking at the like terms of \( p \) of Eq. (37) & (38) and using (35), we have
\[
p^0 : v_0 = me^{-x^2};
\]
\[
p^0 : \rho_0 = ne^{-x^2};
\]
\[
p^1 : v_1 = 2ame^{-x^2} (2x^2 - 1) \frac{\partial^\beta}{\partial t^{\beta}} e^{-x^2};
\]
\[
p^1 : \rho_1 = \frac{t^\beta}{\Gamma(1 + \beta)} \left( (cm - nd) - 2b \left( 2x^2 - 1 \right) e^{-x^2} \right);
\]
\[
p^1 : \rho_1 = \frac{a^2 m t^\beta}{\Gamma(1 + 3\beta)} e^{-x^2} \left[ 12 - 48x^2 + 16x^4 \right];
\]
\[
p^2 : v_2 = \frac{t^\beta}{\Gamma(1 + 2\beta)} e^{-x^2} \left[ d(-cm + dn) + 2acm (-1 + 2x^2) + 2b \left( -1 + 2x^2 \right) \left( cm - 2dn \right) + 4b^2 \left( 3 - 12x^2 + 4x^4 \right) \right];
\]
\[
p^3 : v_3 = \frac{a^3 m t^\beta}{\Gamma(1 + 3\beta)} e^{-x^2} \left[ -120 + 720x^2 - 480x^4 + 64x^6 \right];
\]
\[
p^3 : \rho_3 = \frac{t^\beta}{\Gamma(1 + 3\beta)} e^{-x^2} \left[ d^2 (cm - dn) + b \left( 6 - 24x^2 + 8x^4 \right) + 2acm \left( d - 2dx^2 \right) + 4b^2 \left( cm - 3dn \right) \left( 3 - 12x^2 + 4x^4 \right) + 8b^3 n \left( -15 + 90x^2 - 60x^4 + 8x^6 \right) + 4a^2 cm \left( 3 - 12x^2 + 4x^4 \right) + 2bd \left( -2cm + 3dn \right) (-1 + 2x^2) \right];
\]
The approximation solution of Eq. (36) obtained as $p \to 1$, i.e.

$$
\nu(x, t) = \nu_0 + \nu_1 + \nu_2 + \ldots \\
\rho(x, t) = \rho_0 + \rho_1 + \rho_2 + \ldots 
$$

$$
\nu(x, t) = me^{-x^2} \left( 1 + a \left( -2 + 4x^2 \right) \frac{t^\beta}{\Gamma(1 + \beta)} \right) + a^2 \left( 12 - 48x^2 + 16x^4 \right) \frac{t^{2\beta}}{\Gamma(1 + 2\beta)} \\
+ me^{-x^2} \left( a^3 \left( -120 + 720x^2 - 480x^4 + 64x^6 \right) \frac{t^{3\beta}}{\Gamma(1 + 3\beta)} \right); 
$$

$$
\rho(x, t) = ne^{-x^2} + \frac{t^\beta}{\Gamma(1 + 2\beta)} \left[ (cm - nd) - 2bn \left( 2x^2 - 1 \right) e^{-x^2} \right] \\
+ \frac{t^\beta}{\Gamma(1 + 2\beta)} \left[ (d - cm + dn) + 2acm \left( -1 + 2x^2 \right) \\
+ 2b \left( -1 + 2x^2 \right) (cm - 2dn) + 4b^2(3 - 12x^2 + 4x^4) \right] \\
+ \frac{t^\beta}{\Gamma(1 + 3\beta)} \left[ d^2 (cm - dn) + b \left( 6 - 24x^2 + 8x^4 \right) \right] \\
+ 2acm \left( d - 2dx^2 \right) + 4b^2 (cm - 3dn) \left( 3 - 12x^2 + 4x^4 \right) \\
+ 8b^3n \left( -15 + 90x^2 - 60x^4 + 8x^6 \right) \\
+ 4a^2cm \left( 3 - 12x^2 + 4x^4 \right) + 2bd(-2cm + 3dn)(-1 + 2x^2) \right] 
$$

Case-II: Consider the sensitivity function $\chi(\rho) = \rho$, then the chemo-tactic term i.e. $\frac{\partial}{\partial x} (\nu \frac{\partial \rho}{\partial x}) = \frac{\partial}{\partial x} \frac{\partial \rho}{\partial x} + \frac{\partial}{\partial x} (\nu \frac{\partial \rho}{\partial x})$; hence Keller–Segel equation reduces to

$$
\frac{\partial^2 \nu}{\partial t^2} = \frac{\partial^2 \nu}{\partial x^2} \frac{\partial \rho}{\partial x} + \frac{\partial}{\partial x} \left( \frac{\partial \rho}{\partial x} + \nu \frac{\partial^2 \rho}{\partial x^2} \right),
$$

$$
\frac{\partial^2 \rho}{\partial t^2} = b \frac{\partial^2 \rho}{\partial x^2} + cv - d \rho, \quad 0 < \beta \leq 1 
$$

(19)

Now, for the solution of Eq. (39), we apply HPSTM on Eq. (37), we have

$$
\sum_{n=0}^{\infty} p^n \nu_n = \nu(x, 0) \\
+ pS^{-1} \left[ \nu^0 S \left\{ \frac{\partial}{\partial x} \left( \sum_{n=0}^{\infty} p^n \nu_n \right) \right\} \right] 
$$

$$
\sum_{n=0}^{\infty} p^n \rho_n = \rho(x, 0) + pS^{-1} \left[ \nu^0 S \left\{ b \left( \sum_{n=0}^{\infty} p^n \rho_n \right) \right\} \right] \\
+ pS^{-1} \left[ \nu^0 S \left\{ \frac{\partial}{\partial x} \left( \sum_{n=0}^{\infty} p^n \nu_n \right) - d \left( \sum_{n=0}^{\infty} p^n \rho_n \right) \right\} \right]. 
$$

(20)

(21)

Where

$$
\sum_{n=0}^{\infty} p^n H_n(x, t) = \frac{\partial \nu}{\partial x} \frac{\partial \rho}{\partial x} + \nu \frac{\partial^2 \rho}{\partial x^2} 
$$

An initial couple of terms of He’s polynomial i.e. $H_n(x, t)$ are given below:

$$
H_0(x, t) = \nu_0 \rho_0 + \nu_0 \rho_0 x; \\
H_1(x, t) = \nu_0 \rho_1 x + \nu_1 \rho_0 x + \nu_0 \rho_1 xx + \nu_1 \rho_0 xx; \\
H_2(x, t) = \nu_0 \rho_2 xx + \nu_1 \rho_1 xx + \nu_0 \rho_2 xx + \nu_1 \rho_1 xx + \nu_2 \rho_0 xx; \\
$$

On looking at the like term of $p\nu$ of Eq. (40) and (41) and using Eq. (35) and He’s polynomial we get

$$
p^0: \nu_0(x, t) = me^{-x^2}; \\
p^0: \rho_0(x, t) = ne^{-x^2}; \\
p^1: \nu_1(x, t) = 2m \frac{t^\beta}{\Gamma(1 + \beta)} \left\{ a \left( 2x^2 - 1 \right) - ne^{-x^2} (4x^2 - 1) \right\}; \\
p^1: \rho_1(x, t) = \frac{t^\beta}{\Gamma(1 + \beta)} e^{-x^2} \left\{ 2bn \left( 2x^2 - 1 \right) + (cm - nd) \right\}; \\
p^2: \nu_2(x, t) = 2m \frac{t^{2\beta} e^{-3x^2}}{\Gamma(1 + 2\beta)} \left\{ -ce^{x^2} m \left( -1 + 4x^2 \right) \right\}; \\
+ 2ae^{x^2} \left( 3 - 12x^2 + 4x^4 \right) - 2ae^{x^2} n \left( 7 - 58x^2 + 40x^4 \right); \\
+ nde^{x^2} \left( -1 + 4x^2 \right) - 2nde^{x^2} \left( 3 - 18x^2 + 8x^4 \right) \\
+ 2n^2 \left( 1 - 18x^2 + 24x^4 \right); \\
p^2: \rho_2(x, t) = \frac{t^{2\beta} e^{-2x^2}}{\Gamma(1 + 2\beta)} \left\{ e^{x^2} d(-cm + nd) \right\}; \\
+ 2cm \left( 1 - 4x^2 \right) + 2e^{x^2} (acm + (cm - 2dn) \left( 1 + 2x^2 \right)) \\
+ 4b^2 ne^{x^2} \left( 3 - 12x^2 + 4x^4 \right) \right\}; \\
p^3: \nu_3(x, t) = 2m \frac{t^{3\beta} e^{-5x^2}}{\Gamma(1 + 3\beta)} \left\{ -cdme^{x^2} \right\}; \\
+ d^2e^{x^2} n + 14ce^{x^2} mn - 2de^{x^2} n^2 + 4n^3 + 4cde^{x^2} mx^2 \\
- 4d^2e^{2x^2} + n^3 e^{x^2} \left( 1056 - 768x^2 \right) - 156cmnx^2 e^{x^2}; \\
+ 36dn^2 e^{x^2} - 248n^2 x^2 + 144cmnx^2 e^{x^2} \\
- 48dn^2 e^{x^2} + 4a^2 e^{3x^2} \left( -15 + 90x^2 - 60x^4 + 8x^6 \right) \\
- 4b^2 e^{x^2} \left( -15 + 120x^2 - 100x^4 + 16x^6 \right) \\
- 4a^2 ne^{x^2} \left( -75 + 924x^2 - 1252x^4 + 336x^6 \right) \\
- 2bcme^{x^2} \left( 3 - 18x^2 + 8x^4 \right) \right\};
Subjected to initial condition:
\[ U(x, 0) = U_0(x) \] (21)

Taking Laplace transform on both sides of equation (20)
\[ L[U_t(x, t)] = aL[U_{xx}(x, t)] - L[(U(x, t)X_x(\rho))] \] (22)
\[ L[\rho_t(x, t)] = L[b\rho_{xx} + U(x, t) - dp(x, t)] \] (23)

Applying the differentiation property of Laplace transform, we have
\[ U(x, s) = \frac{U(x, 0)}{s} + \frac{1}{s} L[ aU_{xx}(x, t) - (U(x, t)X_x(\rho))] \] (24)
\[ \rho(x, s) = \frac{\rho(x, 0)}{s} + \frac{1}{s} L[ b\rho_{xx} + U(x, t) - dp(x, t)] \] (25)

Taking the inverse Laplace transform on both sides of equation (24) and (25)
\[ U(x, t) = U(x, 0) + L^{-1} \left\{ \frac{1}{s} L[ aU_{xx}(x, t) - (U(x, t)X_x(\rho))] \right\} \] (26)
\[ \rho(x, t) = \rho(x, 0) + L^{-1} \left\{ \frac{1}{s} L[ b\rho_{xx} + U(x, t) - dp(x, t)] \right\} \] (27)

Now, apply homotopy perturbation method, with
\[ U(x, t) = \sum_{n=0}^{\infty} p^n U_n(x, t), \quad NU(x, t) = \sum_{n=0}^{\infty} p^n H_n(U) \] (28)

Where \( H_n(U) \) is He’s polynomial use to decompose the nonlinear terms. This polynomial is of the form:
\[ H_n(U_0, U_1, \ldots, U_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[ N \left( \sum_{i=0}^{n} p^i U_i(x, t) \right) \right], \quad n = 0, 1, 2, \ldots \] (29)

Substituting equation (28) in equation (27) and (26), we get
\[ \sum_{n=0}^{\infty} p^n U_n(x, t) = U(x, 0) + pL^{-1} \left\{ \frac{1}{s} L \left[ a \sum_{n=0}^{\infty} (U_n p^n)_{xx} - \sum_{n=0}^{\infty} (H_n p^n) \right] \right\} \] (30)
\[ \sum_{n=0}^{\infty} p^n \rho_n(x, t) = \rho(x, 0) + pL^{-1} \left\{ \frac{1}{s} \left[ b \sum_{n=1}^{\infty} p^n \rho_n(x, t)_{xx} + c \left( \sum_{n=1}^{\infty} p^n U_n(x, t) \right) \right] - d \sum_{n=0}^{\infty} p^n \rho_n(x, t) \right\} \] (31)

3 Homotopy perturbation transform method (HPTM)

To elucidate the basic idea of this method, we consider coupled attractor for one-dimensional Keller-Segel equation:
\[ U_t(x, t) = aU_{xx}(x, t) - (U(x, t)X_x(\rho))_x \] (20)
\[ \rho_t(x, t) = b\rho_{xx} + cU(x, t) - dp(x, t) \] (20)

Subjected to initial condition:

3.1 To elucidate the basic idea of this method, we consider coupled attractor for one-dimensional Keller-Segel equation:
The Laplace transform and the homotopy perturbation method are coupled here by using He’s polynomials. Comparing the coefficients of like powers of \( p \), the following approximations are obtained

\[
p^0 : U_0 = U(x, 0), \quad \rho_0 = \rho(x, 0)
\]

\[
p^1 : U_1 = L^{-1} \left\{ \frac{1}{s} L [a U_{0xx} - H_0] \right\},
\]

\[
\rho_1 = L^{-1} \left\{ \frac{1}{s} L \left[ b \rho_{0xx} + c U_0 - \rho_{0x} \right] \right\},
\]

\[
p^2 : U_2 = L^{-1} \left\{ \frac{1}{s} L [a U_{1xx} - H_1] \right\},
\]

\[
\rho_2 = L^{-1} \left\{ \frac{1}{s} L \left[ b \rho_{1xx} + c U_1 - \rho_{1x} \right] \right\},
\]

\[
p^3 : U_3 = L^{-1} \left\{ \frac{1}{s} L [a U_{2xx} - H_2] \right\} \tag{32}
\]

And so on. Setting, \( p = 1 \) results the approximate solution of equation (20)

\[
U(x, t) = U_0 + U_1 + U_2 + \ldots, \quad \rho(x, t) = \rho_0 + \rho_1 + \rho_2 + \ldots, \tag{33}
\]

### 3.1 Application of HPTM

In the order to understand solution of the homotopy perturbation transform method, we consider the following example:

**Example**: The simplified form of the Keller Segel equation in one dimension in given as

\[
\frac{\partial^\beta \nu}{\partial t^\beta} = a \frac{\partial^2 \nu}{\partial x^2} + \frac{\partial}{\partial x} \left( \nu \frac{\partial}{\partial x} \chi(\rho) \right)
\]

\[
\frac{\partial^\beta \rho}{\partial t^\beta} = b \frac{\partial^2 \rho}{\partial x^2} + \nu - \rho \tag{34}
\]

Subject to the boundary conditions

\[
\nu(x, 0) = m \exp(-x^2), \quad \rho(x, 0) = n \exp(-x^2) \tag{35}
\]

**Case-I**: Consider \( \chi(\rho) = 1 \), then

\[
\frac{\partial^\beta \nu}{\partial t^\beta} = a \frac{\partial^2 \nu}{\partial x^2}
\]

\[
\frac{\partial^\beta \rho}{\partial t^\beta} = b \frac{\partial^2 \rho}{\partial x^2} + \nu - \rho \tag{36}
\]

By applying HPTM on equation (16), we have

\[
\sum_{n=0}^{\infty} p^n \nu_n = \nu(x, 0) + pL^{-1} \left\{ \frac{1}{s^\beta} L \left\{ b \left( \sum_{n=0}^{\infty} p^n \rho_n \right) \right\} \right\}
\]

\[
+ pL^{-1} \left\{ \frac{1}{s^\beta} L \left\{ c \left( \sum_{n=0}^{\infty} p^n \nu_n \right) - d \left( \sum_{n=0}^{\infty} p^n \rho_n \right) \right\} \right\} \tag{37}
\]

On looking at the coefficients of like powers of \( p \) of Eq. (17) and (18) and using (15), we have:

\[
p^0 : \nu_0 = me^{-x^2};
\]

\[
p^0 : \rho_0 = ne^{-x^2};
\]

\[
p^1 : \nu_1 = \frac{\nu_{\beta}}{1 + \beta} \left(-2e^{-x^2} + 4x^2e^{-x^2}\right);
\]

\[
p^1 : \rho_1 = \frac{\rho_{\beta}}{1 + \beta} \left[-2b\left(2x^2 - 1\right) + (cm - nd)\right];
\]

\[
p^2 : \nu_2 = \frac{4}{\beta} \frac{e^{-x^2}}{(1 + \beta)^2} \left[ 3 - 12x^2 + 4x^4\right];
\]

\[
p^2 : \rho_2 = e^{-x^2} \frac{\rho_{\beta}}{1 + \beta} \left[d (-cm + dn) + 2ac \left(-1 + 2x^2\right) + 2b \left(-1 + 2x^2\right) (cm - 2dn) + 4b^2n(3 - 12x^2 + 4x^4)\right];
\]

\[
p^3 : \nu_3 = 8a^3me^{-x^2} \left(8x^6 - 60x^4 + 90x^2 - 15\right) \frac{\rho_{\beta}}{1 + \beta};
\]

\[
p^3 : \rho_3 = e^{-x^2} \frac{\rho_{\beta}}{1 + \beta} \left[d^2 \left(cm - dn\right) + b \left(6 - 24x^2 + 8x^4\right) + 2ac \left(d - 2dx^2\right) + 4b^2 \left(cm - 3dn\right) \left(3 - 12x^2 + 4x^4\right) + 8b^3n \left(-15 + 90x^2 - 60x^4 + 8x^6\right) + 4a^2cm \left(3 - 12x^2 + 4x^4\right) + 2bd \left(-2cm + 3dn\right) \left(-1 + 2x^2\right)\right];
\]
The approximate solution is obtained by letting \( p \to 1 \), & \( \beta \to 1 \) i.e.

\[
v(x, t) = v_0 + v_1 + v_2 + v_3 + \ldots
\]

\[
\rho (x, t) = \rho_0 + \rho_1 + \rho_2 + \rho_3 + \ldots
\]

\[
v(x, t) = me^{-x^2} \left( 1 + a \left( -2 + 4x^2 \right) \frac{t}{1} + a^2 \left( 12 - 48x^2 + 16x^4 \right) \frac{t^2}{2} \right) + me^{-x^2} \left( a^3 \left( -120 + 720x^2 - 480x^4 + 64x^6 \right) \frac{t^3}{6} \right) + \ldots
\]

\[
\rho (x, t) = ne^{-x^2} + \frac{te^{-x^2}}{1} \left( cm - dn \right) - 2nb \left( 2x^2 - 1 \right) + \frac{t^2 e^{-x^2}}{2} \left( d \left( -cm + dn \right) + 2acm \left( -1 + 2x^2 \right) \right)
\]

\[+ \frac{t^3 e^{-x^2}}{6} \left( d^2 \left( cm - dn \right) + b \left( 6 - 24x^2 + 8x^4 \right) \right) + 2acm \left( d - 2dx^2 \right) + 4b^2 \left( cm - 3dn \right) \left( 3 - 12x^2 + 4x^4 \right) + 8b^2 n \left( 15 + 90x^2 - 60x^4 + 8x^6 \right) + 4a^2 cm \left( 3 - 12x^2 + 4x^4 \right) + 2bd \left( 2cm + 3dn \right) \left( 1 + 2x^2 \right) \right) + \ldots
\]

**Case-II:** Consider the Keller Siegel equation with sensitivity function \( \chi (\rho) = \rho \). Then

\[
\frac{\partial}{\partial x} \left( v \frac{\partial \rho}{\partial x} \right) = \frac{\partial v}{\partial x} + v \frac{\partial^2 \rho}{\partial x^2};
\]

Hence Keller-segel equation (14) reduces to

\[
\frac{\partial v}{\partial t} = a \frac{\partial^2 v}{\partial x^2} \left( \frac{\partial v}{\partial x} + v \frac{\partial^2 \rho}{\partial x^2} \right),
\]

\[
\frac{\partial \rho}{\partial t} = b \frac{\partial^2 \rho}{\partial x^2} + cv - d \rho
\]

Now, for the solution of Eq. (19), we apply HPTM on Eq. (19), we have

\[
\sum_{n=0}^{\infty} p^n v_n = v(x, 0) + pL^{-1} \left[ \frac{1}{s} L \left\{ a \frac{\partial^2 v}{\partial x^2} \left( \sum_{n=0}^{\infty} p^n U_n \right) - \sum_{n=0}^{\infty} p^n H_n \right\} \right]
\]

\[
\sum_{n=0}^{\infty} p^n \rho_n = \rho(x, 0) + pL^{-1} \left[ \frac{1}{s} L \left\{ b \frac{\partial^2 \rho}{\partial x^2} \left( \sum_{n=0}^{\infty} p^n \rho_n \right) \right\} \right] + pL^{-1} \left[ \frac{1}{s} L \left\{ c \frac{\partial^2 \rho}{\partial x^2} \left( \sum_{n=0}^{\infty} p^n \rho_n \right) - d \left( \sum_{n=0}^{\infty} p^n \rho_n \right) \right\} \right]
\]

Where

\[
\sum_{n=0}^{\infty} p^n H_n (x, t) = \left\{ \frac{\partial v}{\partial x} + v \frac{\partial^2 \rho}{\partial x^2} \right\}
\]

An initial couple of terms of He’s polynomial i.e. \( H_0 \) \( (x, t) \) are given below:

\[
H_0 (x, t) = v_0 \rho_{0x} + v_0 \rho_{0xx};
\]

\[
H_1 (x, t) = v_0 \rho_{1x} + v_1 \rho_{0x} + v_0 \rho_{1xx} + v_1 \rho_{0xx};
\]

\[
H_2 (x, t) = v_0 \rho_{2x} + v_1 \rho_{1x} + v_2 \rho_{0x} + v_0 \rho_{2xx} + v_1 \rho_{1xx} + v_2 \rho_{0xx};
\]

On looking at the like terms of \( p \) of Eq. (20) & (21) and using Eq. (15) and He’s polynomial, we get

\[
p_0^0 : v_0 (x, t) = me^{-x^2};
\]

\[
p_0^0 : \rho_0 (x, t) = ne^{-x^2};
\]

\[
p_1^1 : v_1 (x, t) = 2mte^{-2x^2} \left\{ n - 4nx^2 + ae^{-x^2} (-1 + 2x^2) \right\};
\]

\[
p_1^1 : \rho_1 (x, t) = te^{-x^2} \left[ 2bn \left( 2x^2 - 1 \right) + \left( cm - nd \right) \right];
\]

\[
p_2^2 : v_2 (x, t) = mt^2 \left( -cme^{-2x^2} (-1 + 4x^2) + 2a^2 e^{-x^2} \left( 3 - 12x^2 + 4x^4 \right) \right)
\]

\[
- mt^2 \left( 2ane^{-2x^2} \left( 7 - 58x^2 + 40x^4 \right) + dne^{-2x^2} (-1 + 4x^2) \right) - mt^2 \left( 2be^{-2x^2} \left( 3 - 18x^2 + 8x^4 \right) + 2ne^{-3x^2} \left( 1 - 18x^2 + 24x^4 \right) \right);
\]
The solution of Eq. (19) obtained as \( p \to 1 \), i.e.

\[
\begin{align*}
    v(x, t) &= v_0 + v_1 + v_2 + \ldots \\
    \rho(x, t) &= \rho_0 + \rho_1 + \rho_2 + \ldots
\end{align*}
\]

The solution of Eq. (19) obtained as \( p \to 1 \), i.e.

\[
\begin{align*}
    \rho(x, t) &= \rho_0 + \rho_1 + \rho_2 + \ldots
\end{align*}
\]

Figure 1: The surface graph of approximate solution \( v(x, t) \) for case-

(a) \( v(x, t) \) for \( \beta = 0.4 \) (b) \( v(x, t) \) for \( \beta = 0.6 \) (c) \( v(x, t) \) for \( \beta = 0.8 \) (d) \( v(x, t) \) for \( \beta = 1 \)
Figure 2: The surface graph of approximate solution $\rho(x, t)$ for case-I: (a) $\rho(x, t)$ for $\beta = 0.4$ (b) $\rho(x, t)$ for $\beta = 0.6$ (c) $\rho(x, t)$ for $\beta = 0.8$ (d) $\rho(x, t)$ for $\beta = 1$

Figure 3: The surface graph of approximate solution $v(x, t)$ for case-II: (a) $v(x, t)$ for $\beta = 1$ (b) $v(x, t)$ for $\beta = 0.8$ (c) $v(x, t)$ for $\beta = 0.6$ (d) $v(x, t)$ for $\beta = 0.4$
Figure 4: The surface graph of approximate solution $\rho(x, t)$ for case-II: (a) $\rho(x, t)$ for $\beta = 1$ (b) $\rho(x, t)$ for $\beta = 0.8$ (c) $\rho(x, t)$ for $\beta = 0.6$ (d) $\rho(x, t)$ for $\beta = 0.4$

Figure 5: The surface graph of approximate solution $v(x, t)$ and $p(x, t)$ for $\beta = 1$: (a) $v(x, t)$ for case-I (b) $p(x, t)$ for case-I (c) $v(x, t)$ for case-II (d) $p(x, t)$ for case-II
4 Results and discussion

In this section, the numerical solution of examples obtained by HPSTM and HPTM through a graphical representation are studied. The surface graphs of Keller-Segel equation for respective cases (I & II) at different values of $\beta$ are represented in Figures 1-4. For graphical representation of solution we take $m = 0.000012$, $n = 0.000016$, $a = 0.5$, $b = 3$, $c = 1$, $d = 2$. Figure 1 represents solution $v(x, t)$ at $\beta = 0.4$, $\beta = 0.6$, $\beta = 0.8$, $\beta = 1$, respectively, whereas Figure 2 indicates $\rho(x, t)$ corresponding to different values of $\beta$ for Case-I.

Figures 3 and 4 show surface graphs of solution $v(x, t)$ and $\rho(x, t)$ for Case-II at different values of $\beta$. Figure 5 represents solution $v(x, t)$ and $\rho(x, t)$ obtained from HPTM for both cases. It is clear from the graphs that results of HPSTM and HPTM are in good harmony with each other for $\beta = 1$.

5 Conclusion

In this work, homotopy perturbation transform method (HPTM) combined with sumudu transform has been successfully applied to approximate solution for a system of nonlinear partial differential equations derived from an attractor for a one-dimensional Keller-Segel dynamics system. On comparing the results of this method with HPTM, it is observed HPSTM is extremely simple, straightforward and easy to handle the nonlinear terms. Maple 13 package is used to calculate series obtained from iteration. Further, the method needs much less computational work which shows fast convergent for solving nonlinear system of partial differential equations.

Acknowledgement: Authors wish to acknowledge DST-FIST sponsored research computational laboratory of Lyallpur Khalsa College, Jalandhar for providing necessary assistance.

References

[1] Deb Nath L., Recent applications of fractional calculus to science and engineering, Int J Math Math Sci, 2003, 54, 3413-3442.
[2] Grover D., Kumar V., Sharma D., A comparative study of numerical techniques and homotopy perturbation method for solving parabolic equations and nonlinear equations, Int. J. Comput. Method Eng. Sci. Mech., 2012, 13, 403-407.
[3] Sharma D., Kumar S., Homotopy perturbation method for Korteweg and de Vries Equation, Int. J. Nonlinear Sci., 2013, 1, 173-177.
[4] Singh P., Sharma D., Comparative study of Homotopy perturbation transformation with Homotopy Perturbation elzaki transform method for solving nonlinear fractional PDE, Nonlin. Eng., 2019, 9, 60-71.
[5] Singh P., Sharma D., Convergence and error analysis of series solution of nonlinear partial differential equation, Nonlin. Eng., 2018, 7, 303-308.
[6] Sharma D., Singh P., Chauhan S., Homotopy perturbation transform method with He’s polynomial for solution of coupled nonlinear partial differential equations, Nonlin. Eng., 2016, 5, 17-23.
[7] Rathore S., Kumar D., Singh J., Gupta S., Homotopy analysis sumudu transform method for nonlinear equations, Int. J. Ind. Math., 2012, 4, 301-304.
[8] Sundas R., Jamshaid A., Muhammad N., Exact solution of Klein Gordon equation via homotopy perturbation sumudu transform method, Int. J. Hybrid Inf. Technol., 2014, 7, 445-452.
[9] Singh J., Kumar D., Sushila, Homotopy perturbation sumudu transform method for non-linear equations, Adv. Theor. Appl. Mech., 2011, 4, 165-175.
[10] Yousif E.A., Solution of nonlinear fractional differential equations using the homotopy perturbation sumudu transform method, Appl. Math. Sci., 2014, 8, 2195-2210.
[11] Watugala G.K., Sumudu transform: a new integral transform to solve differential equations and control engineering problems, Int. J. Math. Educ. Sci. Technol., 1993, 24, 35-43.
[12] Keller E.F., Segel L.A., Initiation of slime mold aggregation viewed as instability, J. Theor. Biol., 1970, 26, 399-415.
[13] Atangana A., Alabaraoyez E., Solving a system of fractional partial differential equations arising in the model of HIV infection of CD4+ cells and attractor one-dimensional Keller-Segel equations, Adv. Differ. Equ., 2013, 94, 1-14.
[14] Atangana A., Vermeulen P.D., Modelling the aggregation process of cellular slime mold by the chemical attraction, Biomed Res. Int., 2014, 815690.
[15] Atangana A., Extension of the Sumudu homotopy perturbation method to an attractor for one-dimensional Keller-Siegel equations. Appl. Math. Model., 2015, 39, 2909-2916.
[16] Jiwari R., Kumar S., Mittal R.C., Meshfree algorithms based on radial basis functions for numerical simulation and to capture shocks behavior of Burgers’ type problems, Eng. Computation., 2019, 36, 1142-1168.
[17] Jiwari R., A hybrid numerical scheme for the numerical solution of the Burgers’ equation, Comput. Phys. Commun., 2015, 188, 59-67.
[18] Jiwari R., A Haar wavelet quasilinearization approach for numerical simulation of Burgers’ equation, Comput. Phys. Commun., 2012, 183, 2413-2423.
[19] Mittal R.C., Jiwari R., A differential quadrature method for numerical solutions of Burgers’-type equations, Int. J. Number. Method H., 2012, 22, 880-885.
[20] Kumar R., Koundal R., Shehzad S.A., Least Square Homotopy Solution to Hyperbolic Telegraph Equations: Multi-dimension Analysis, Int. J. Appl. Comput. Math., 2020, 6.
[21] Maitama S., Zhao W., New homotopy analysis transform method for solving multidimensional fractional diffusion equations, Arab. J. Basic Appl. Sci., 2020, 27, 503-518.
[22] Akter M.T., Mansur Chowdhury M.A., 2019. Homotopy Perturbation Method for Solving Highly Nonlinear Reaction-Diffusion-Convection Problem, Amer. J. Math., 9:136-141.

[23] Pasha S.A., Nawaz Y., Arif M.S., The modified homotopy perturbation method with an auxiliary term for the nonlinear oscillator with discontinuity, J. Low FrEq. Noise V. A., 2019, 38, 1363-1373.

[24] Singh R., Maurya D.K., Rajoria Y.K., A Novel Approach of Homotopy Perturbation Technique to Solution of Non-Linear Fisher Equation, Int. J. Appl. Eng. Res., 2019, 14, 957-964.

[25] Kumar S., Kumar A., Argyros I.K., A New Analysis for the Keller-Segel Model of Fractional Order, Numer. Algor., 2016, 75, 213-228.