Symmetric Mass Generation in Lattice Gauge Theory

Nouman Butt,1 Simon Catterall,2 and Goksu Can Toga2

1Department of Physics, University of Illinois at Urbana-Champaign, 1110 W Green St, Urbana, IL 61801
2Department of Physics, Syracuse University, Syracuse, NY 13244, USA

(Dated: November 2, 2021)

We construct a four dimensional lattice gauge theory in which fermions acquire mass without breaking symmetries as a result of gauge interactions. Our model consists of reduced staggered fermions transforming in the bifundamental representation of a $SU(2) \times SU(2)$ gauge symmetry. This fermion representation ensures that single site bilinear mass terms vanish identically. A symmetric four fermion operator is however allowed and we show numerical results that show that a condensate of this operator develops in the vacuum.

PACS numbers:

I. INTRODUCTION

Can we generate a mass for all physical states in a theory without breaking symmetries? And can we do it using just gauge interactions? In this paper we describe a lattice model which is capable of realizing this scenario.

The model consists of a reduced staggered fermion coupled to a $SU(2) \times SU(2)$ lattice gauge field. The fermion representation is chosen so that single site fermion gauge invariant fermion bilinear terms vanish identically. A symmetric four fermion operator remains invariant under these symmetries and we present evidence that it condenses as a result of the gauge interactions. Since no symmetries are broken by this condensate there are no massless Goldstone bosons and the spectrum consists of color singlet composites of the elementary fermions. This scenario corresponds to symmetric mass generation realized in the context of a confining gauge theory. It gives an explicit and non-supersymmetric realization of a mechanism that has been proposed to gap chiral fermions in the continuum [1,2].

The model can be seen as a generalization of the $SO(4)$ Higgs-Yukawa theory described in [3] which uses strong quartic interactions to gap lattice fermions. This four dimensional model built on earlier work directed at symmetric mass generation with staggered fermions in two, three and four dimensions [4–10].

In the current paper both of the $SU(2)$ subgroups of $SO(4)$ are gauged and confinement rather than strong Yukawa interactions is used to generate the four fermion condensate.

II. STAGGERED FERMION MODEL

We start from a staggered fermion action which takes the form

$$S = \sum_{x,\mu} \eta_\mu(x) \text{Tr} \left( \psi^\dagger \Delta_\mu \psi \right)$$

where $\eta_\mu(x) = (-1)^{\sum_{i=1}^{x_i-1}}$ are the usual staggered phases and $\Delta_\mu$ is the symmetric difference operator whose action on a lattice field $f(x)$ is given by

$$\Delta_\mu f(x) = \frac{1}{2} \left( f(x + \mu) - f(x - \mu) \right)$$

This action has a $U(1)$ staggered symmetry $\psi \to e^{i\epsilon(x) \alpha \hat{x}} \psi$ with $\epsilon(x) = (-1)^{\sum_i x_i}$ the site parity operator. The fermions are additionally taken to transform under a global $G \times H$ symmetry where $G$ and $H$ correspond to independent $SU(2)$ groups:

$$\psi \to \hat{\psi} = G\psi H^\dagger$$

We will also impose the reality condition

$$\psi^\dagger = \sigma_2 \psi^T \sigma_2$$

To see that this reality condition is compatible with the non-abelian symmetry consider the transformed fermion:

$$\hat{\psi}^\dagger = H\psi^\dagger G^\dagger$$

$$= H \sigma_2 \psi^T \sigma_2 G^\dagger$$

$$= \sigma_2 \left( \sigma_2 H \psi^T G \sigma_2 \right) \sigma_2$$

$$= \sigma_2 \left( H^* \psi^T G^T \right) \sigma_2$$

$$= \sigma_2 \hat{\psi}^T \sigma_2$$

The reality condition is automatically satisfied if $\psi = \sum_A \sigma_A \chi_A$ for real $\chi_A$ where $\sigma_A = (I, i\sigma_i)$. Substituting this expression into the kinetic term shows that the action can be written in an explicit $SO(4)$ invariant form

$$S = \sum_{x,\mu} \eta_\mu(x) \chi_A \Delta_\mu \chi^A$$

Indeed in this form one can see that the kinetic term of this model is precisely the same as that considered in previous work with $SO(4)$ invariant staggered fermions [3].

Once this reality condition is imposed it is not possible to write down single site mass terms since $\text{Tr} \left( \psi^\dagger \psi \right) = \text{Tr} \left( \sigma_2 \psi^T \sigma_2 \psi \right) = 0$ on account of the Grassmann nature...
of the fields. However a four fermion term invariant under
$SO(4) = SU(2) \times SU(2)$ is possible and takes the form
\[
\text{Tr} \left( \psi \psi^\dagger \right) = \frac{1}{3} \epsilon_{abcd} \chi^a \chi^b \chi^c \chi^d
\]  
(7)
The form of this four fermion term also agrees with the
earlier work [3].
To add such a four fermion term to the action we can use a
Yukawa interaction with an auxiliary scalar field $\phi$ of the
form
\[
\sum_x \text{Tr} \left( \phi \psi \psi^\dagger \right) + \frac{1}{2 \lambda^4} \sum_x \text{Tr}(\phi^2)
\]  
(8)
where $\phi$ transforms in the adjoint representation of $G$ but is
a singlet under $H$:
\[
\phi \rightarrow G \phi G^\dagger
\]  
(9)
After integration over $\phi$ a four fermion term is produced with
coupling $-\lambda^4/2$. The addition of this term breaks the
original $U(1)$ staggered symmetry to a $Z_4$ corresponding to
\[
\psi(x) \rightarrow \omega \epsilon(x) \psi(x)
\]  
(10)
where $\omega$ is an element of $Z_4$.
In [3] we showed that it was possible to achieve symmetric mass
generation in this model for large values of the Yukawa coupling and vanishing
gauge coupling. In this paper we will show that a four fermion condensate
can also be obtained by using strong gauge interactions and small Yukawa coupling. This result is important as it avoids
the problem of relying on perturbatively irrelevant four fermion
operators to induce symmetric mass generation.
To do this we need to generalize eqn. [1] so that it is
invariant under lattice gauge transformations. The
following prescription does the job:
\[
S_F = \sum_{x, \mu} \frac{1}{2} \eta_\mu(x) \text{Tr} \left[ \psi^\dagger(x) \mu(x) \psi(x + \mu) \nu(x) \right]
\]  
(11)
In addition we will add Wilson terms for $G$ and $H$:
\[
S_W = -\frac{\beta_G}{2} \sum_x \sum_{\mu \nu} \left( \mu(x) \nu(x + \mu) \nu(x + \nu) \psi^\dagger(x) \right)
\]  
(12)
\[
-\frac{\beta_H}{2} \sum_x \sum_{\mu \nu} \left( \nu(x) \nu(x + \mu) \nu(x + \nu) \psi^\dagger(x) \right)
\]  
(13)
The resultant action is now invariant under the following
gauge transformations
\[
\psi(x) \rightarrow G(x) \psi(x) H^\dagger(x)
\]  
(14)
\[
\mu(x) \rightarrow G(x) \mu(x) G^\dagger(x + \mu)
\]  
(15)
\[
\nu(x) \rightarrow H(x + \mu) \nu(x) H^\dagger(x)
\]  
(16)
The Yukawa interaction given in eqn. [8] is automatically
invariant under these local symmetries.\footnote{As an aside we remark that four fermion interactions similar to the ones considered here have previously been used to argue for the appearance of Higgs phases in strongly coupled lattice theories [11][12].}
Finally we note that the action of the model is invariant under a $Z_2$ center symmetry transformation
\[
\mu(x) \rightarrow -\mu(x)
\]  
(17)
\[
\nu(x) \rightarrow \epsilon(x) \nu(x)
\]  
(18)
The existence of an exact center symmetry ensures that the
Polyakov line
\[
P(x) = \frac{1}{2} \text{Tr} \prod_{t=1}^L \mu(x + t)
\]  
(19)
is a good order parameter for confinement in this theory.
In the next section we will show numerical results that provide evidence that a four fermion condensate appears in the theory even for small Yukawa coupling. We can think of this condensate as corresponding to the appearance of a bilinear mass term for the color singlet composite scalar $\phi = \psi \psi^\dagger$. This scenario is similar to that advocated for by Tong et al. in [1] as a mechanism for
hopping chiral fermions. It is important to remember though that this model targets a vector-like theory at short distances as $\beta \rightarrow \infty$.

III. NUMERICAL RESULTS

The fermion kinetic term including the Yukawa term takes the form
\[
S_F = \sum_x \text{Tr} \left[ \sigma_2 \psi^\dagger(\Delta_\mu + G\phi) \psi \right]
\]  
(19)
where $\Delta^c$ denotes the covariant difference operator appearing in eqn. [11]. Using the properties
\[
\mu^T = \sigma_2 \mu^\dagger \sigma_2
\]  
(20)
\[
\phi^T = -\sigma_2 \phi \sigma_2
\]  
(21)
it is easy to verify that the fermion operator $M$ is anti-
symmetric and each eigenvector $\varphi_\mu(x)$ with eigenvalue $\lambda_\mu$ is paired with another $\sigma_2 \varphi_\mu(x) \sigma_2$ with eigenvalue $\lambda^*_\mu$. Thus the eigenvalues, which are generically complex, come in quartets $\lambda, \lambda, -\lambda, -\lambda$. This ensures that the
Pfaffian that results after fermion integration is generically real positive definite. We have verified that this is indeed the case by computing the latter for ensembles of
small lattice size \(^2\). Thus the model can be simulated using the RHMC algorithm \([13, 14]\). We now turn to our numerical results.

### A. The Yukawa theory limit \(\beta_H = \beta_G \to \infty\)

In the absence of gauge interactions the model reduces to the \(SO(4)\) Higgs-Yukawa theory examined in \([8]\). In this limit the only way to drive a four fermion condensate is through the use of a large Yukawa coupling \(\lambda\). Fig. 1 shows a picture of the \(\text{Tr} \phi^2\) which serves as a proxy for the four fermion condensate \(\langle \psi \psi \rangle\). The rapid growth near \(\lambda \sim 1.0\) is identical to our earlier results for the pure four fermion model in four dimensions. This conclusion is strengthened in fig. 2 which shows a plot of the associated fermion susceptibility defined by

\[
\chi = \frac{1}{V} \sum_x \langle \psi(0) \psi(0) \psi(x) \psi(x) \rangle
\]

This shows a peak that grows with volume close to the coupling where the enhanced four fermion vev switches on.

![FIG. 1: \(\text{Tr} \phi^2\) vs \(\lambda\) for \(L = 4^4, 6^4, 8^4\)](image1)

We now switch on the gauge interactions, setting \(\beta_H = \beta_G = \beta\) and retaining only a small Yukawa coupling. Fig. 3 shows a plot of \(\text{Tr} (\phi^2)\) vs \(\beta\). The Yukawa coupling is small and fixed to \(\lambda = 0.5\) for lattice sizes \(L = 6^4, 8^4, 10^3 \times 8\). Clearly the condensate grows for \(\beta \leq 2.5\). Notice that the appearance of this four fermion vev is a result of the gauge interactions not the explicit Yukawa coupling since the latter lies well below the threshold to drive the phase transition seen in fig. 1. Indeed the effect of changing the value of the bare Yukawa coupling \(\lambda\) can be seen in fig. 4 which shows the condensate for a range of \(\lambda = 0.25, 0.5, 0.75, 1.0\) on an \(L = 6^4\) lattice. Clearly for all \(\lambda < 0.8\) a condensate develops at small \(\beta\) but is driven to zero in the weak gauge coupling limit \(\beta \to \infty\) consistent with fig. 1.

Notice that the value of the condensate as \(\beta \to 0\) scales according to \(\lambda^2\) as one might expect from perturbation theory. The case where \(\lambda = 1.0\) is close to the threshold required to precipitate a condensate even in the absence of gauge interactions and

\[^2\) Notice though that pure imaginary eigenvalues come only in pairs which allows for a sign change if such an eigenvalue crosses the origin. While this is logically possible we have not seen any sign of this in our simulations.
indeed we see in this case that the condensate survives the $\beta \to \infty$ limit.

![Figure 5: Tr($\phi^2$) vs $\beta$ for $\lambda = 0.25, 0.5, 0.75, 1.0$](image)

The fact that the regime where the four fermion condensate is non-zero corresponds to confinement can be seen in fig. 4 which shows the absolute value of the Polyakov line averaged over the lattice over the same range in $\beta$. It is clear that the Polyakov line vanishes for values of $\beta$ in which the four fermion condensate grows. A vanishing Polyakov line signals a confining phase for the gauge theory. This conclusion can be strengthened by looking at Wilson loops. The Wilson Loops for $L = 8^4$ and $\lambda = 0.5$ are shown in fig. 6 and clearly also decrease rapidly in the small $\beta$ regime. To extract the string tension, we fit the $W(R, R)$ loops to an exponential of form $e^{-(AR^2 + BR + C)}$ corresponding to a combination of area and perimeter laws. For values of $\beta < 1.8$ the fitted values of $B$ and $C$ are consistent with zero and we hence fit only for $A$. However around $\beta = 1.8$ the area term and the perimeter term become comparable so we need to employ the full form of the exponential for couplings $\beta \geq 1.8$. This behavior can be seen in fig. 7 which shows the coefficients $A$ and $B$ versus $\beta$. The plot also shows the pure area law fit as a solid line which yields an estimate of the string tension $\sigma = 0.499(5)$. This agrees well with a strong coupling analysis of the quenched gauge theory and is consistent with the absence of light fermions in this regime due to symmetric mass generation.

![Figure 7: A and B vs $\beta$ for $L = 8^4$](image)

Of course while the single site fermion bilinear is forced to vanish by symmetry in this model it is possible to construct other gauge invariant and $Z_4$ symmetric fermion mass terms that involve coupling different fermion fields within the hypercube. It is logically possible that the model would choose to condense these other fermion bilinear operators rather than the four fermion operator we have considered so far. To check for this we have added the simplest of these operators, the one link term, to the action with coupling $m_l$.

$$O_1 = \frac{1}{8} \sum_{x, \mu} \epsilon(x) \xi_\mu(x) \text{Tr}[\psi^\dagger(x) (U_\mu(x) \psi(x + \mu) V_\mu^\dagger(x) + U_\mu^\dagger(x - \mu) V_\mu(x - \mu))]$$

where the phase $\xi_\mu(x) = (-1)^{x_{\mu+1}}$. Notice though that a vev for this operator as $m_l \to 0$ will necessarily break a set of discrete shift symmetries given by

$$\psi(x) \to \xi_\mu(x) \psi(x + \rho)$$

$$V_\mu(x) \to V_\mu^\dagger(x + \rho)$$

$$U_\mu(x) \to U_\mu^\dagger(x + \rho)$$

In fig. 8 we show a plot of the vev of this operator for several lattice sizes as a function of $m_l$ on a $6^4$ lattice for $\lambda = 0.5$ and $\beta = 2.0$. Notice that the measured vev is small in comparison with the four fermion condensate and decreases smoothly to zero as $m_l \to 0$ with no significant dependence on lattice volume. This result argues against the condensation of such a link term and a corresponding spontaneous breaking of these shift symmetries in the thermodynamic limit.

---

3 We use the absolute value of the line in our measurements since the Polyakov line itself vanishes for all $\beta$ at finite volume as a consequence of the exact center symmetry.
FIG. 8: Link vev vs $m_l$ for $L = 64, 84$ at $\beta = 2.0$.

IV. CONCLUSIONS

In this paper we have argued that a particular lattice gauge theory composed of massless reduced staggered fermions transforming under a local $SU(2) \times SU(2)$ symmetry develops a four fermion rather than bilinear fermion condensate due to confinement. Furthermore, since this four fermion condensate breaks no symmetries there are no Goldstone bosons in the spectrum of the theory. This gives an explicit realization of symmetric mass generation in a lattice model which describes sixteen Majorana fermions at high energies. This number of fermion flavors is precisely what is needed to cancel certain discrete anomalies of Weyl fermions in the continuum\cite{17}. Our work furnishes the first example of of a lattice theory capable of supporting symmetric mass generation using just gauge interactions.

[1] S. S. Razamat and D. Tong, Phys. Rev. X 11, 011063 (2021), 2009.05037.
[2] D. Tong (2021), 2104.03907.
[3] N. Butt, S. Catterall, and D. Schaich, Phys. Rev. D 98, 114514 (2018), 1810.06117.
[4] V. Ayyar and S. Chandrasekharan, Phys. Rev. D 91, 065035 (2015), 1410.6474.
[5] S. Catterall, JHEP 01, 121 (2016), 1510.04153.
[6] V. Ayyar and S. Chandrasekharan, Phys. Rev. D 93, 081701 (2016), 1511.09071.
[7] V. Ayyar and S. Chandrasekharan, JHEP 10, 058 (2016), 1606.06312.
[8] V. Ayyar and S. Chandrasekharan, Phys. Rev. D 96, 114506 (2017), 1709.06048.
[9] S. Catterall, Phys. Rev. D 104, 014503 (2021), 2010.02290.
[10] N. Butt, S. Catterall, A. Pradhan, and G. C. Toga (2021), 2101.01026.
[11] S. Catterall and A. Veernala, Phys. Rev. D 88, 114510 (2013), 1306.5668.
[12] S. Catterall and A. Veernala, Int. J. Mod. Phys. A 29, 1445002 (2014), 1401.0457.
[13] S. Catterall and N. Butt, Phys. Rev. D 99, 014505 (2019), 1810.00853.
[14] N. Butt and S. Catterall, PoS LATTICE2018, 294 (2019), 1811.01015.
[15] C. van den Doel and J. Smit, Nucl. Phys. B 228, 122 (1983).
[16] M. F. Golterman and J. Smit, Nucl. Phys. B 245, 61 (1984).
[17] I. n. García-Etxebarria and M. Montero, JHEP 08, 003 (2019), 1808.00009.