ON ALMOST-EQUIDISTANT SETS - II

A. POLYANSKII

Abstract. A set in \(\mathbb{R}^d\) is called almost-equidistant if for any three distinct points in the set, some two are at unit distance apart. We proved that an almost-equidistant set \(V\) in \(\mathbb{R}^d\) has \(O(d)\) points in two cases: if the diameter of \(V\) is at most 1 or if \(V\) is a subset of \((d-1)\)-dimensional sphere of radius at most \(\sqrt{1/2 + O(d^{-2/3})}\). Also, we present a new proof of the result by Kupavskii, Mustafa and Swanepoel [9] that an almost-equidistant set in \(\mathbb{R}^d\) has \(O(d^{4/3})\) elements.

1. Introduction

A set in \(\mathbb{R}^d\) is called almost-equidistant if among any three points in the set, some two are at unit distance apart. The natural conjecture [12, Conjecture 7] claims that an almost-equidistant set in \(\mathbb{R}^d\) has \(O(d)\) points.

Using an elegant linear algebraic argument Rosenfeld [14] proved that an almost-equidistant set on a \((d-1)\)-dimensional sphere of radius \(1/\sqrt{2}\) has at most \(2d\) points. Note that the set of the vertices of two unit \((d-1)\)-simplices lying on the same \((d-1)\)-dimensional sphere of radius \(1/\sqrt{2}\) is almost-equidistant. Bezdek and Lángi [2, Theorem 1] generalized Rosenfeld’s approach and showed that an almost-equidistant set on a \((d-1)\)-dimensional sphere of radius \(<1/\sqrt{2}\) has at most \(2d + 2\) points; this bound is tight because the vertices of two unit \(d\)-simplices inscribed in the same sphere form an almost-equidistant set. Balko, Pór, Scheucher, Swanepoel and Valtr [1] showed that an almost equidistant set has \(O(d^{3/2})\) elements. This bound was improved by the author [12] to \(O(d^{13/9})\). Recently, Kupavskii, Mustafa, Swanepoel [9] further improved to \(O(d^{4/3})\). For more references we refer the interested readers to [1].

The first goal of the current article is to confirm the conjecture in two cases: for almost-equidistant sets of diameter 1 (see Section 3) and for almost-equidistant sets lying on a \((d-1)\)-dimensional sphere of radius \(\leq\sqrt{1/2 + O(d^{-2/3})}\) (see Section 4). The second aim is to give a new proof of the upper bound \(O(d^{4/3})\) for the cardinality of an almost-equidistant set in \(\mathbb{R}^d\) (see Section 5). Also, we will discuss several open problems related to almost-equidistant sets (see Section 6).

2. Preliminaries

Suppose that \(\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset \mathbb{R}^d\) is an almost-equidistant set. Consider the matrix

\[
\mathbf{U} := \|\mathbf{v}_i - \mathbf{v}_j\|^2 + \mathbf{I}_n - \mathbf{J}_n,
\]

where \(\mathbf{J}_n\) is the \(n\)-by-\(n\) matrix of one’s and \(\mathbf{I}_n\) is the identity matrix of size \(n\). We need two simple facts proved in [12, Corollary 4 and Lemma 5]. We join them in the following lemma.

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Lemma 1. 1) \( \text{tr}(U) = \text{tr}(U^3) = 0. \)

2) \( U \) has at most one eigenvalue > 1 and at least \( n - d - 2 \) eigenvalues equal to 1.

Suppose that \( \{v_1, \ldots, v_n\} \) is an almost-equidistant set lying on the \((d - 1)\)-dimensional sphere of radius \( \sqrt{1/(2(1 - \cos \alpha))} \) with center in the origin of \( \mathbb{R}^d \). Consider the matrix
\[
U_\alpha := 2(1 - \cos \alpha)(v_i, v_j) + (\cos \alpha - 1)I_n - \cos \alpha J_n, \tag{2}
\]
where \( 0 < \alpha < \pi \) is a fixed angle.

We omit the proof of the next lemma because it is similar to the proof of Lemma 1.

Lemma 2. 1) \( \text{tr}(U_\alpha) = \text{tr}(U_\alpha^3) = 0. \)

2) \( U_\alpha \) has at most one eigenvalue < \( \cos \alpha - 1 \) and at least \( n - d - 1 \) eigenvalues equal to \( \cos \alpha - 1 \).

We will use the following lemma in next sections.

Lemma 3. Suppose that an \( n \)-by-\( n \) matrix \( W \) has \( n - k \) eigenvalues are equal to 1. Also, \( \text{tr}(W) = \text{tr}(W^3) = 0. \)

1. Assume that eigenvalues of \( W \) are \( \leq 1 \). Then \( n \leq 2k \).
2. Assume that \( W \) has eigenvalues \( \lambda > \mu_1 = \cdots = \mu_{n-k} = 1 \geq \lambda_1 \geq \cdots \geq \lambda_{k-2} \geq \lambda' \) such that \( \lambda' + \lambda \leq 0 \). Then \( n \leq 2k - 2 \).
3. Assume that all eigenvalues of \( W \) but \( \lambda \) are \( \leq 1 \). Then
\[
\lambda^3 \geq \frac{(n - k + 1)^3}{(k - 1)^2} - (n - k + 1).
\]

Proof. 1. Denote by \( \lambda_1, \ldots, \lambda_k \) eigenvalues of \( W \) that are \( < 1 \). Then
\[
\sum_{i=1}^{k} (-\lambda_i) = n - k, \quad \sum_{i=1}^{k} (-\lambda_i)^3 = n - k. \tag{3}
\]

To finish the proof, we need Lemma 1 in [2]. For the sake of completeness we provide its proof here.

Lemma 4. Let \( x_1, \ldots, x_m \) be real numbers with the property that \( x_i \geq -2 \) for \( i = 1, \ldots, m \) and \( \sum_{i=1}^{m} x_i = (m + l) \), where \( l \geq 0 \). Then
\[
\sum_{i=1}^{m} x_i^3 \geq \frac{(m + l)^3}{m^2} \geq (m + 3l).
\]

Here the equality is possible if \( x_i \) are equal and therefore, \( x_i \geq 1 \).

Proof. Consider functions \( f, g : [-2, +\infty) \rightarrow \mathbb{R} \) such that
\[
f(x) = x^3 \text{ for any } x \geq -2, \quad g(x) = \begin{cases} 3x - 2, & \text{for any } -2 \leq x \leq 1, \\ x^3, & \text{for any } 1 \leq x. \end{cases}
\]

For \(-2 \leq x \leq 1\) we have \( g(x) \leq f(x) \) because in this range \( g(x) \) has the value of a tangent line to \( f(x) \) at \( x = 1 \) and the second point of the intersection of that tangent line and \( f(x) \) is at \( x = 2 \). Further, \( g(x) \) is a convex function in the range \(-2 \leq x \leq 1\). By Jensen's inequality,
\[
\sum_{i=1}^{m} x_i^3 = \sum_{i=1}^{m} f(x_i) \geq \sum_{i=1}^{m} g(x_i) \geq mg \left( \frac{\sum_{i=1}^{m} x_i}{m} \right) = m \left( \frac{\sum_{i=1}^{m} x_i}{m} \right)^3 \geq \frac{(m + l)^3}{m^2} \geq m + 3l.
\]
The equality case we leave as an exercise.

Assume that \( n > 2k \). Introducing the notation \( l = n - 2k \) we can rewrite the first equality in (3) as:
\[
\sum_{i=1}^{k} (-\lambda_i) = k + l.
\]
Thus Lemma 4 implies that
\[
\sum_{i=1}^{k} (-\lambda_i)^3 \geq k + 3l.
\]
Finally, according to the second equality in (3)
\[
\sum_{i=1}^{k} (-\lambda_i)^3 = k + l,
\]
a contradiction.

2. Since \( \text{tr}(W) = \text{tr}(W^3) = 0 \),
\[
\sum_{i=1}^{k-2} (-\lambda_i) + (-\lambda - \lambda') = n - k, \quad \sum_{i=1}^{k-2} (-\lambda_i)^3 - \lambda^3 - (\lambda')^3 = n - k. \tag{4}
\]
Assume that \( n \geq 2k - 1 \). Introducing the notation \( l = n - 2k - 1 \) we can rewrite the first equality in (4) as:
\[
\sum_{i=1}^{k-2} (-\lambda_i) + (-\lambda - \lambda') = k - 1 + l.
\]
By Lemma 4, we obtain
\[
\sum_{i=1}^{k-2} (-\lambda)^3 + (-\lambda - \lambda')^3 \geq k - 1 + 3l. \tag{5}
\]
Moreover, if \( \lambda + \lambda' = 0 \) then (5) becomes a strict inequality (see Lemma 4). Since the second equality in (4),
\[
k - 1 + l + (-\lambda - \lambda')^3 + \lambda^3 + (\lambda')^3 \geq k - 1 + 3l.
\]
Therefore, \(-3\lambda \lambda' (\lambda + \lambda') \geq 2l\). From \(-3\lambda \lambda' (\lambda + \lambda') \leq 0\) and \( l \geq 0 \) we see \( \lambda + \lambda' = 0 \), but this implies a strict inequality in (5), a contradiction.

3. Denote by \( \lambda_1, \ldots, \lambda_{k-1} \) eigenvalues < 1. Since \( \text{tr}(W) = \text{tr}(W^3) = 0 \),
\[
\lambda + \sum_{i=1}^{k-1} \lambda_i + (n - k) = 0, \quad \lambda^3 + \sum_{i=1}^{k-1} \lambda_i^3 + (n - k) = 0.
\]
Therefore, we have
\[
\sum_{i=1}^{k-1} (-\lambda_i) = \lambda + (n - k) > (n - k + 1).
\]
By Lemma 4, we obtain
\[
\lambda^3 = \sum_{i=1}^{k-1} (-\lambda_i)^3 - (n - k) \geq \frac{(-\lambda_1 - \ldots - \lambda_{k-1})^3}{(k-1)^2} - (n - k) \geq \left( \frac{(n - k + 1)^3}{(k-1)^2} - (n - k + 1) \right).
\]
3. Almost-equidistant diameter sets

A subset of $\mathbb{R}^d$ is called an **almost-equidistant diameter set** if it is almost-equidistant and has diameter 1. The next theorem is about the maximal size of such sets.

**Theorem 5.** An almost-equidistant diameter set in $\mathbb{R}^d$ has at most $2d + 4$ points.

**Proof.** Suppose that the matrix $U$ (see (1)) for an almost-equidistant diameter set $V \subseteq \mathbb{R}^d$. Clearly, the entries of $U$ are non-positive.

Assume that $U$ does not have an eigenvalue $> 1$. By Lemmas 1 and 3 (case 1), we have $|V| \leq 2d + 4$.

Assume that $U$ has an eigenvalue $\lambda > 1$. We need the following week form of the Perron–Frobenius theorem; see [5, 11] or [15, Theorem 5.2.1].

**Theorem 6** (Perron–Frobenius theorem). If an $n$-by-$n$ matrix has non-negative entries, then it has a non-negative real eigenvalue, which has maximum absolute value among all eigenvalues.

By the Perron–Frobenius theorem, the matrix $U$ has an eigenvalue $\lambda' < 0$ such that $|\lambda'| \geq \lambda$. Therefore, Lemmas 1 and 3 (case 2) imply $|V| \leq 2d + 2$. \qed

4. Almost-equidistant sets on small spheres

**Theorem 7.** An almost-equidistant set with vertices on a $(d - 1)$-dimensional sphere $S$ of radius $\leq \sqrt{1/(2(1 - (32d^2)^{1/2}))} = \sqrt{1/2 + O(d^{-2/3})}$ has cardinality $O(d)$.

**Proof.** Assume that $\{v_1, \ldots, v_n\} \subseteq S$ is an almost-equidistant set and $S$ has the center in the origin $o$ of $\mathbb{R}^d$. Clearly, that if $|v_i - v_j| = 1$ then $\angle v_i o v_j = \alpha$, where $\cos \alpha \leq 1/(32d^2)^{1/3}$.

Consider the matrix $U_{\alpha}$ (see (2)) for $\{v_1, \ldots, v_n\}$. According to Lemma 2 there are two possible cases.

1. The matrix $U_{\alpha}$ does not have an eigenvalue $< -1 + \cos \alpha$. Then by Lemmas 2 and 3 (case 1) for $U_{\alpha}/(\cos \alpha - 1)$, we get that $n \leq 2d + 2$.

2. The matrix $U_{\alpha}$ has an eigenvalue $\lambda < -1 + \cos \alpha$. Suppose, contrary to our claim, $n > 6d$. Lemmas 2 and 3 (case 3) for $U_{\alpha}/(-1 + \cos \alpha)$ imply that

$$(-\lambda)^3 \geq (1 - \cos \alpha)^3 \frac{(n - d)^3}{d^2} - (n - d) \geq \frac{1}{2} \frac{(n - d)^3}{d^2} - (n - d).$$

We need Weyl’s inequality (see [16, Theorem 1] and [13, Theorem 34.2.1]).

**Theorem 8** (Weyl’s inequality). Let $A$ and $B$ are Hermitian matrices of size $n$. Suppose that $\alpha_1 \leq \cdots \leq \alpha_n$ are eigenvalues of $A$, $\beta_1 \leq \cdots \leq \beta_n$ are eigenvalues of $B$, $\gamma_1 \leq \cdots \leq \gamma_n$ are eigenvalues of $A + B$. Then

$$\gamma_i \geq \alpha_j + \beta_{i-j+1} \text{ for } i \geq j \text{ and } \gamma_i \leq \alpha_j + \beta_{i-j+n} \text{ for } i \leq j.$$ 

In particular, $\gamma_1 \geq \alpha_1 + \beta_1$.

From (2), Weyl’s inequality and positivity of the Gram matrix $\langle v_i, v_j \rangle$ we conclude that $\lambda \geq (\cos \alpha - 1) - n \cos \alpha$. Therefore, $\lambda \geq -2 \max\{n \cos \alpha, 1\}$. This forces

$$\max\left\{\frac{n^3}{4d^2}, 8\right\} \geq (2 \max\{n \cos \alpha, 1\})^3 \geq (\lambda)^3 \geq \frac{1}{2} \frac{(n - d)^3}{d^2} - (n - d),$$

so

$$\max\left\{\frac{x^3}{2}, \frac{16}{d}\right\} \geq (x - 1)^3 - (x - 1),$$

where $x := n/d$. Clearly, the last inequality contradicts our assumption that $x \geq 6$. \qed
5. Almost-equidistant sets: general case

**Theorem 9** (Kupavskii, Mustafa, Swanepoel, 2018). *The cardinality of an almost-equidistant set in \( \mathbb{R}^d \) is \( \leq 38d^{4/3} \).

*Proof.* Assume that there is an almost equidistant set \( V := \{v_1, \ldots, v_n\} \subset \mathbb{R}^d \) such that \( n > 38d^{4/3} \). Clearly, if the matrix \( U \) (see (1)) for \( V \) does not have eigenvalue \( > 1 \) then Lemmas 1 and 3 (case 1) imply \( n \leq 2d + 4 \). Assume that \( U \) has an eigenvalue \( \lambda > 1 \). By Lemmas 1 and 3 (case 3) for \( U \), we have

\[
\lambda^3 \geq \left( \frac{(n - d - 1)^3}{(d + 1)^2} - (n - d - 1) \right) \geq \frac{n^3}{8d^2},
\]

and so \( \lambda \geq \frac{n}{2d^{2/3}}. \) \( \underline{(6)} \)

Assume that the following inequality holds:

\[
f := \max_{i=1,\ldots,n} \left\{ \sum_{j=1}^{n} \|v_i - v_j\|^2 - 1 \right\} = \sum_{j=1}^{n} \|v_1 - v_j\|^2 - 1 \leq 19 \max \left\{ 1, \sqrt{d}, 19d^2/n \right\}. \] \( \underline{(7)} \)

Now we need to apply the Gershgorin circle theorem [6] (or [13, Problem 34.1]) for \( U \).

**Theorem 10** (Gershgorin circle theorem). *Every eigenvalue of an \( n \)-by-\( n \) matrix \( (a_{ij}) \) over \( \mathbb{C} \) belong to one of the disks

\[
\left\{ z \in \mathbb{C} : |a_{kk} - z| \leq \sum_{1 \leq j \leq n, j \neq k} |a_{kj}| \right\} \text{ for } k = 1, \ldots, n.
\]

From the Gershgorin circle theorem, (6) and (7) we conclude that

\[
\frac{n}{2d^{2/3}} \leq \lambda \leq f \leq 19 \max \left\{ 1, \sqrt{d}, 19d^2/n \right\},
\]

and so \( n \leq 38d^{4/3} \).

Therefore, it is enough to show that (7) holds.

Without loss of generality we can assume that \( \sum_{i=1}^{n} v_i = o \), where \( o \) is the origin. By the definition of \( f \) (see (7)) we have

\[
\left| \sum_{j=1}^{n} \|v_i - v_j\|^2 - 1 \right| \leq f \text{ or } \|v_i\|^2 + \frac{\sum_{j=1}^{n} \|v_j\|^2}{n} - 1 \leq \frac{f}{n} \text{ for } i = 1, \ldots, n.
\]

Summing up the last inequality for \( i = 1, \ldots, n \) we get

\[
\left| \sum_{i=1}^{n} \|v_i\|^2 - \frac{1}{2} \right| \leq \frac{f}{n}.
\]

Thus

\[
\left| \|v_i\|^2 - \frac{1}{2} \right| \leq \frac{2f}{n}. \] \( \underline{(8)} \)

We will use the following theorem (see [4, Theorem 1]) several times.

**Theorem 11.** Let \( X = \{x_1, \ldots, x_n\} \) and \( Y = \{y_1, \ldots, y_n\} \) be two point-sets in \( \mathbb{R}^d \). Then

\[
\sum_{1 \leq i, j \leq n} \|x_i - y_j\|^2 = \sum_{1 \leq i, j \leq n} \|x_i - x_j\|^2 + \sum_{1 \leq i, j \leq n} \|y_i - y_j\|^2 + n^2 \|x - y\|^2,
\]

where \( x \) and \( y \) are barycenters of \( X \) and \( Y \), respectively, that is,

\[
x = (x_1 + \cdots + x_n)/n, \quad y = (y_1 + \cdots + y_n)/n.
\]
Suppose that \( w_1, \ldots, w_k, w \in V \) are such that \( \| w - w_i \|^2 \neq 1 \). Since \( V \) is an almost-equidistant set, \( w_1, \ldots, w_k \) form a unit simplex. Hence, by Theorem 11 for \( \{ w, \ldots, w \} \) and \( \{ w_1, \ldots, w_k \} \), we have
\[
\sum_{1 \leq i \leq k} \| w - w_i \|^2 = \frac{k - 1}{2} + k\| w - o' \|^2 \leq \frac{k - 1}{2} + k(\| w \|^2 + \| o' \|^2 + 2\| w \|\| o' \|), \tag{9}
\]
where \( o' \) is the center of the simplex \( w_1 \ldots w_k \). Also, by Theorem 11 for \( \{ o, \ldots, o \} \) and \( \{ w_1, \ldots, w_k \} \), we have
\[
k\| o' \|^2 = \sum_{1 \leq i \leq k} \| w_i \|^2 - \frac{k - 1}{2} \leq \frac{2fk}{n} + \frac{k}{2} - \frac{k - 1}{2} = \frac{2fk}{n} + \frac{1}{2}. \tag{10}
\]
By (8) and (10), we obtain
\[
(9) \leq \frac{k - 1}{2} + \frac{2fk}{n} + \frac{k}{2} + \frac{2fk}{n} + \frac{1}{2} + 2k \sqrt{\frac{2f}{n + 2} \cdot \sqrt{\frac{2f}{n + 2k}}} \leq k + 9k \max \left\{ \sqrt{f/n, 1} \right\} \cdot \max \left\{ \sqrt{f/n, \sqrt{1/k}} \right\}.
\]
Likewise, we get
\[
\left| \sum_{1 \leq i \leq k} \| w - w_i \|^2 - 1 \right| \leq 9k \max \left\{ \sqrt{f/n, 1} \right\} \cdot \max \left\{ \sqrt{f/n, \sqrt{1/k}} \right\}. \tag{11}
\]
Without loss of generality we assume \( \| v_1 - v_j \|^2 > 1 \) for \( j = 2, \ldots, l+1, \| v_1 - v_j \|^2 < 1 \) for \( j = l + 2, \ldots, l + m + 1 \) and \( \| v_1 - v_j \|^2 = 1 \) for \( j = l + m + 2, \ldots, n \). Since \( V \) is an almost-equidistant set and \( \| v_1 - v_j \| \neq 1 \) for \( j = 2, \ldots, l + m + 1 \), the simplex \( v_2 \ldots v_{l + m + 1} \) is regular. Hence \( l + m \leq d + 1 \). By (7), we thus get
\[
f = 1 + \left( \sum_{j=2}^{l+1} \| v_1 - v_j \|^2 - 1 \right) - \left( \sum_{j=l+2}^{l+m+1} \| v_1 - v_j \|^2 - 1 \right). \tag{12}
\]
Also, we can assume that \( l \geq m \). There are several possible cases.

1. If \( f/n \geq 1 \) then (11) and (12) imply
\[
f \leq 1 + 9(d + 1)f/n \leq 19df/n, \quad \text{so } n \leq 19d.
\]

2. If \( 1/m \leq f/n < 1 \) then (11) and (12) imply
\[
f \leq 1 + 9(d + 1)\sqrt{f/n} \leq 19 \max \{1, d\sqrt{f/n}\}, \quad \text{and so } f \leq 19 \max \{1, 19d^2/n\}.
\]

3. If \( 1/l \leq f/n < 1/m \leq 1 \) or \( m = 0 \) and \( 1/l \leq f/n < 1 \) then (11) and (12) imply
\[
f \leq 1 + 9l\sqrt{f/n} + 9\sqrt{m} \leq 19 \max \{1, d\sqrt{f/n}, \sqrt{d}\}, \quad \text{and so } f \leq 19 \max \left\{1, \sqrt{d}, 19d^2/n\right\}.
\]

4. If \( f/n < 1/l \) then (11) and (12) imply
\[
f \leq 1 + 9\sqrt{l} + 9\sqrt{m} \leq 19\sqrt{d}.
\]
This completes the proof. \( \square \)
6. Discussion

6.1. Almost-equidistant diameter sets. A graph \((V, E)\) is called a diameter graph if its vertex set \(V \subseteq \mathbb{R}^d\) is a set of points of diameter 1 and a pair of vertices forms an edge if they are at unit distance apart. Of course, the set of vertices of two cliques in a diameter graph is an almost-equidistant diameter set. For instance, in [10, the last paragraph of Section 3] there is given an example of diameter graph in \(\mathbb{R}^d\) consisting of two cliques without common vertices such that they have \(d + 1\) and \(\left\lfloor \frac{d+1}{2} \right\rfloor\) vertices respectively. We believe that the vertex set of this diameter graph has the maximal size among almost-equidistant diameter sets in \(\mathbb{R}^d\).

**Conjecture 12.** An almost-equidistant diameter set in \(\mathbb{R}^d\) has at most \(\left\lfloor \frac{3(d+1)}{2} \right\rfloor\) points.

There is the following conjecture [7, Conjecture 5.5] that arose in the context of study of cliques in diameter graphs.

**Conjecture 13** (Schur). Let \(S_1\) and \(S_2\) be two unit simplices in \(\mathbb{R}^d\) forming a set of diameter 1 such that they have \(k\) and \(m\) vertices respectively. Then \(S_1\) and \(S_2\) share at least \(\min\{0, k + 2m - 2d - 2\}\) vertices for \(k \geq m\).

Clearly, this conjecture is closely related to Conjecture 12. Note that Conjecture 13 was confirmed in two special (but not trivial!) cases: for \(k = m = d\) (see [10]) and \(k = 5, m = 3, d = 4\) (see [8]).

6.2. Two-distant almost-equidistant sets. A subset of \(\mathbb{R}^d\) is called a two-distant set if there are only two distances formed by any two distinct points of the set. The following question seems to be interesting.

**Problem 14.** What is the largest cardinality of a set in \(\mathbb{R}^d\) that is two-distant and almost-equidistant at the same time?

Let us consider a two-distant almost-equidistant set \(V \subseteq \mathbb{R}^d\) with distances 1 and \(a > 1\) between points of \(V\) (the case \(a < 1\) is not interesting because of Theorem 5). If the matrix \(U\) (see (1)) for \(V\) does not have an eigenvalue \(> 1\) then Lemmas 1 and 3 imply \(|V| \leq 2d + 4\). Thus we can assume that \(U\) has an eigenvalue \(> 1\). Note that the matrix \(U/(a^2 - 1)\) is an adjacency matrix of some triangle-free graph. By Lemma 1, Problem 14 is reduced to following question.

**Problem 15.** What is the minimal rank of the matrix \(A - \lambda_2 I_n\), where \(A\) is the adjacency matrix of some triangle-free graph on \(n\) vertices and \(\lambda_2 > 0\) is its second largest eigenvalue?

It is worth pointing out that there are infinitely many triangle-free graphs on \(n\) vertices such that \(\text{rank}(A - \lambda I) = O(n^{3/4})\), where \(A\) is the adjacency matrix of the graph and \(\lambda\) is some real number; see [3, the proof of Theorem 5].

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ALEXANDR POLYANSKII,
Moscow Institute of Physics and Technology
Institutskiy per. 9
Dolgoprudny, Russia 141700
Institute for Information Transmission Problems RAS
Bolshoy Karetny per. 19
Moscow, Russia 127994
E-mail address: alexander.polyanskii@yandex.ru
URL: http://polyanskii.com