EXPLICIT SUBCONVEXITY ESTIMATES FOR DIRICHLET $L$-FUNCTIONS

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Abstract. Given a Dirichlet character $\chi$ modulo a prime $q$ and its associated $L$-function, $L(s, \chi)$, we provide an explicit version of Burgess’ estimate for $|L(s, \chi)|$. We use partial summation to provide bounds along the vertical lines $\Re s = 1 - r^{-1}$, where $r$ is a parameter associated with Burgess’ character sum estimate. These bounds are then connected across the critical strip using the Phragmén–Lindelöf principle. In particular, for $\sigma \in [\frac{1}{2}, \frac{9}{10}]$, we establish

$$|L(\sigma + it, \chi)| \leq (1.105)(0.692)^{\sigma}q^{\frac{1}{8} - \frac{3}{8}\sigma}(\log q)^\frac{7}{32} - \frac{2}{5}\sigma|\sigma + it|.$$

1. Introduction

Consider a Dirichlet character $\chi$ modulo $q$ and its associated $L$-function, $L(s, \chi)$. It is regularly of interest for one to know about the size of $L(s, \chi)$ inside the critical strip ($0 < \Re s < 1$). A classical estimate is due to Davenport [3]

$$L(\sigma + it, \chi) \ll (q\tau)^{\frac{1}{2}(1-\sigma)},$$

where $\tau = |t| + 1$. Most authors focus on bounds along the critical line. Hiary [7] shows that if one wishes for an explicit version of the trivial bound,

$$|L(1/2 + it, \chi)| \leq 124.46(q|t|)^{\frac{1}{2}}(q|t| \geq 10^9, |t| \geq \sqrt{q}),$$

suffices for primitive characters $\chi$. However, the trivial bound is not a sub-convexity bound on the half-line, that is one that shows

$$L(1/2 + it, \chi) \ll q^{\theta + \epsilon},$$

with $\theta < \frac{1}{2}$. We do have sub-convexity for Dirichlet $L$-functions, via Burgess [2], who showed that

(1) $$L(1/2 + it, \chi) \ll q^{\frac{1}{4} + \frac{1}{4}}.$$ 

Petrow and Young [10] showed that for $\chi$ of cube-free conductor,

$$L(1/2 + it, \chi) \ll q^{\frac{1}{4} + \epsilon}(1 + |t|)^{\frac{1}{2} + \epsilon}.$$ 

For characters whose modulus is a very large prime power, we have estimates which beat the Weyl exponent of $\frac{1}{6}$, due to [9].

Theorem 1.1 (Theorem 1, [9]). Let $\theta > \theta_0 = 0.1645$. There is an $r > 0$ for which

$$L(1/2, \chi) \ll p^r q^\theta (\log q)^{\frac{1}{2}},$$

holds for every Dirichlet character of modulus $q = p^n$.

The estimate due to Burgess comes to us as a consequence of a character sum estimate known as the Burgess bound, which improves upon the Pólya–Vinogradov inequality for certain lengths of character sums. Recently, in [4], the author improved upon work of Treviño which established explicit Burgess bounds for characters of prime modulus [12]. For characters of composite modulus, further results can be found in [8].
Theorem 1.2 (Theorem 1.8, [4]). Let $p$ be a prime and $2 \leq r \leq 10$ be an integer. Let $\chi$ be a non-principal character modulo $p$. Let $N_0$ and $N_1$ be non-negative integers with the condition that $1 \leq N_1 < 2p^{\frac{2}{3}}$ only when $r = 2$. Then, for $p \geq 10^{10}$, we have a precisely determined constant $B(r)$ for which

$$
\left| \sum_{n=N_0}^{N_0+N_1} \chi(n) \right| < B(r)N_1^{1-r-\frac{4r+1}{p^{2r}}} (\log p)^{\frac{4}{3}}.
$$

For example, when $r = 3$, one can take $B(3) = 2.491$. An extensive table of constants is provided in [4] Table 3.

In this article, we are interested in using the explicit estimates of Theorem 1.2 to obtain explicit bounds on $L(\sigma + it, \chi)$ for any $0 < \sigma < 1$. Hiary [7] uses Pólya–Vinogradov along with partial summation to establish the explicit bound

$$
|L(1/2 + it, \chi)| \leq 4q^{\frac{1}{2}} \sqrt{\tau \log q}.
$$

As Hiary notes, (3) is useful when $\tau$ is small (relative to $q$). On the other hand, one expects explicit bounds for $|L(1/2 + it, \chi)|$ which come from (1) to be useful when $\tau$ is very small, roughly $\tau \ll q^{\frac{2}{3} + \epsilon}$.

We will use the approach of Burgess to obtain bounds of the same type as (1) on the whole critical strip. For each choice of $r$ in the Burgess bound, we obtain a bound for the size of $L(\frac{1}{2} + it)$, and then use the functional equation and the Phragmén–Lindelöf principle to complete the bound for any $\sigma \in \left[\frac{1}{10}, \frac{9}{10}\right]$. This yields the following.

Theorem 1.3. Let $q$ be a prime and $\chi$ be a non-principal, primitive character modulo $q \geq 10^{10}$ and $|t| \geq 1$. Then

$$
|L\left(\frac{1}{2} + it, \chi\right)| \leq 0.918q^{\frac{3}{10}} (\log q)^{\frac{3}{2}} \left|\frac{1}{2} + it\right|.
$$

Moreover, for $\sigma \in \left[\frac{1}{3}, \frac{9}{20}\right],

$$
|L(\sigma + it, \chi)| \leq (1.105)(0.692)^\sigma q^{\frac{1}{2} - \frac{2}{5}\sigma}(\log q)^{\frac{33}{2} - \frac{2}{5}\sigma}|\sigma + it|.
$$

Finally, for $\sigma \in \left[\frac{9}{20}, \frac{1}{2}\right],

$$
|L(\sigma + it, \chi)| \leq (10.094)(0.083)^\sigma q^{\frac{1}{2} - \frac{2}{5}\sigma}(\log q)^{\frac{7}{2} - \frac{2}{5}\sigma}|\sigma + it|.
$$

Additionally, one may use these methods to obtain bounds for $|L(1 + it, \chi)|$, although such bounds are likely only useful for very specific ranges of $|t|$.

Theorem 1.4. Let $q$ be a prime and $\chi$ be a non-principal, primitive character with modulus $q$. Then, for $|t| \geq 1$, $\epsilon > 0$, and $\sigma \in \left[\frac{9}{20}, 1 + \epsilon\right]$, we have

$$
|L(\sigma + it, \chi)| \leq \left(0.792 \left|\frac{9}{10} + it\right| q^{\frac{11}{200}} (\log q)^{\frac{21}{20}}\right)^{\frac{1 + \sigma - \frac{9}{20}}{\frac{9}{20} + \epsilon}} \cdot (1 + \epsilon^{-1})^{\frac{\sigma - \frac{9}{20}}{\frac{9}{20} + \epsilon}}.
$$

Specifically, for $\sigma = 1$, we have

$$
|L(1 + it, \chi)| \leq (1 + \epsilon^{-1})^{\frac{1}{10 + \epsilon}} \left(0.792q^{\frac{11}{200}} (\log q)^{\frac{21}{20}}\right)^{\frac{10}{10 + \epsilon}} \left|\frac{9}{10} + it\right|^{\frac{10}{10 + \epsilon}}.
$$

In the future, it is certainly worth investigating explicit versions of the theorems of Heath–Brown in [3], an example of which includes, for a specific set of moduli $q, L(1/2 + it, \chi) \ll (q\tau)^{\frac{1}{4} + \epsilon}$. Explicitly, Hiary [7] Corollary 1.1 provides

$$
|L(1/2 + it, \chi)| \leq 9.05d(q)(q|t|)^{\frac{1}{2}} \log \frac{q}{|t|}.
$$
with $|t| > 200$, $\chi$ a character modulo a sixth power $q$, and $d(q)$ as the number of divisors function.

However, it appears that making these results explicit would require more than the elementary means of this article.

2. General Results and Useful Lemmas

In the course of the proof of Theorem 1.3, we will obtain bounds for $L(s, \chi)$ depending on the parameter $r$ arising in Burgess’ bound and $\sigma = \Re s$. Roughly speaking, we will follow the approaches of Burgess [1] and Hiary [7, Eq. 4], and “only” use partial summation. As a first step in this direction, we make [1, Lemma 10] explicit. Doing so requires an explicit Pólya–Vinogradov inequality, i.e.,

$$\left| \sum_{n=1}^{N} \chi(n) \right| \leq P \sqrt{q} \log q,$$

where $P$ is an explicit constant.

**Lemma 2.1.** Let $\chi$ be a primitive character modulo $q > 1$ and $L(s, \chi)$ be the associated Dirichlet $L$-function. Then, for $s = \sigma + it$ with $\sigma > 0$ fixed, we have

$$|L(\sigma + it, \chi)| \leq \frac{M^{1-\sigma}}{|1-\sigma|} + \frac{1}{M^\sigma} \left| \sum_{n \leq M} \chi(n) \right| + (\sigma + |t|)P \sqrt{q} \log q \frac{N^{-\sigma}}{\sigma}$$

$$+ (\sigma + |t|) \int_{M}^{N} \left| \sum_{1 \leq n \leq u} \chi(n) \right| u^{-\sigma} du,$$

where $M, N$ are arbitrary constants satisfying $0 < M \leq N$.

**Proof.** Since $\chi$ is non-principal, for a fixed $\sigma > 0$, we may write

$$L(\sigma + it, \chi) = \sum_{n=1}^{M} \frac{\chi(n)}{n^{\sigma+it}} + \sum_{n=M+1}^{\infty} \frac{\chi(n)}{n^{\sigma+it}}.$$

Estimate the initial sum trivially. To estimate the tail, cut it at some point $N$ and use partial summation to obtain:

$$\sum_{M < n \leq N} \frac{\chi(n)}{n^{\sigma+it}} = \frac{1}{N^\sigma} \sum_{n \leq N} \chi(n) + \frac{1}{M^\sigma} \sum_{n \leq M} \chi(n) + s \int_{M}^{N} \sum_{1 \leq n \leq u} \chi(n) u^{-1-\sigma} du.$$

On the other hand,

$$\left| \sum_{N < n \leq \infty} \frac{\chi(n)}{n^{\sigma+it}} \right| \leq \frac{1}{N^\sigma} \sum_{n \leq N} \chi(n) + (\sigma + |t|)P \sqrt{q} \log q \frac{N^{-\sigma}}{\sigma}.$$

Combining (6) and (7) with the trivial estimate and taking absolute values gives the result. \(\square\)

Taking Burgess’ character sum estimate with parameter $r$ in Lemma 2.1 offers a bound on $|L(s, \chi)|$ which depends on $r$. The shape of this bound depends on the relationship between $r$ and $\sigma$, as detailed in the following proposition. As detailed in Proposition 2.2, one should observe that when $\sigma > 1 - r^{-1}$, we have $\sigma r - r + 1 > 0$. Therefore, there the second line of (8) may be ignored with regard to asymptotics. A clever choice of parameters may even eliminate these terms entirely. The same is true for the first line of (8) when $\sigma < 1 - r^{-1}$. In this sense, Proposition 2.2 should be compared with Theorem 3 of [2]. Note that the distinction between cases below is entirely due to the behaviour of the integral in Lemma 2.1.
Proposition 2.2. Suppose $\chi$ is a primitive character modulo $q$ for which constants $B, r$ exist such that

$$
\left| \sum_{n=N_1+1}^{N_0+N_1} \chi(n) \right| < BN_1^{1-r-1} q^{\frac{1}{\sqrt{r}}} (\log q)^{\frac{1}{2}},
$$

for $N_1$ at least as large as $q^{\frac{2}{\sqrt{r}}}(\log q)^{\frac{1}{2}}$. Let $\sigma \in (0, 1)$. Then, for $\sigma \neq 1 - \frac{1}{r}$, we have

$$
|L(\sigma + it, \chi)| \leq \left( \frac{1}{1 - \sigma} + B + B(\sigma + |t|) \left( \frac{r}{\sigma r - r + 1} \right) \right) q^{\frac{1}{\sqrt{r}} - \frac{1}{\sigma}} \sigma^{\frac{1}{2 \sigma}} (\log q)^{\frac{1}{2}} \frac{1}{\sigma}
$$

$$
+ \left( P(\sigma + |t|) - B(\sigma + |t|) \left( \frac{r}{\sigma r - r + 1} \right) \right) q^{\frac{1}{\sqrt{r}} - \frac{1}{\sigma}} \sigma^{\frac{1}{2 \sigma}} (\log q)^{\frac{1}{2}} \frac{1}{\sigma}
$$

whereas, for $\sigma = 1 - \frac{1}{r}$, we have

$$
|L(\sigma + it, \chi)| \leq \left( \frac{1}{1 - \sigma} + B + P(\sigma + |t|) + (\sigma + |t|) B \left( \frac{\log q}{4} + \frac{\log \log q}{2 \sigma} \right) \right)
$$

$$
\cdot q^{\frac{1}{\sqrt{r}} + \frac{1}{2 \sigma}} (\log q)^{\frac{1}{2}} \frac{1}{\sigma}
$$

Proof. For such a $\chi$, when $\sigma + r^{-1} \neq 1$, Lemma 2.1 provides

$$
|L(\sigma + it, \chi)| \leq \frac{M^1 - \sigma}{1 - \sigma} + BM^{1 - \sigma - r^{-1}} q^{\frac{1}{\sqrt{r}} - \frac{1}{\sigma}} \sigma^{\frac{1}{2 \sigma}} (\log q)^{\frac{1}{2}} \frac{1}{\sigma}
$$

$$
+ (\sigma + |t|) B q^{\frac{1}{\sqrt{r}} - \frac{1}{\sigma}} (\log q)^{\frac{1}{2}} \frac{1}{\sigma} \left( \frac{r}{\sigma r - r + 1} \right) \left( M^{1 - \sigma - r^{-1}} - N^{1 - \sigma - r^{-1}} \right).
$$

Making the choices $M = q^{\frac{2}{\sqrt{r}} + \frac{1}{2 \sigma}} (\log q)^{\frac{1}{2}}$ and $N = q^{\frac{2}{\sqrt{r}} + \frac{1}{2 \sigma}} (\log q)^{\frac{1}{2}}$ yields (9). On the other hand, if $\sigma + r^{-1} = 1$, then Lemma 2.1 provides

$$
|L(\sigma + it, \chi)| \leq \frac{M^1 - \sigma}{1 - \sigma} + BM^{1 - \sigma - r^{-1}} q^{\frac{1}{\sqrt{r}} - \frac{1}{\sigma}} \sigma^{\frac{1}{2 \sigma}} (\log q)^{\frac{1}{2}} \frac{1}{\sigma}
$$

$$
+ (\sigma + |t|) B q^{\frac{1}{\sqrt{r}} - \frac{1}{\sigma}} (\log q)^{\frac{1}{2}} \frac{1}{\sigma} \left( \log N - \log M \right).
$$

Here, the choices $M = q^{\frac{2}{\sqrt{r}} + \frac{1}{2 \sigma}} (\log q)^{\frac{1}{2}}$ and $N = q^{\frac{2}{\sqrt{r}} + \frac{1}{2 \sigma}} (\log q)^{\frac{1}{2}}$ yield (11). \qed

While (3) agrees with Theorem 3 of [2] (except in the case $\sigma = \frac{1}{2}$, where (9) fills in), (3) tends to provide better exponents on $q$. We can now easily obtain bounds along the lines $\sigma = \frac{2}{3}, \frac{3}{4}, \ldots, \frac{r-1}{r}, \ldots$, where $r > 2$ is an integer. The bounds we obtain along vertical lines can be connected to bound $L(s, \chi)$ across the entire strip. For this, we use the following version of the Phragmén–Lindelöf principle, which comes from Rademacher [11].

Theorem 2.3 (Theorem 2, [11]). Let $f(s)$ be regular analytic in the strip $a \leq \Re s \leq b$. Suppose for some positive constants $c, C$, we have

$$
|f(s)| < Ce^{\|s\|^c}.
$$

Additionally, suppose we have

$$
|f(a + it)| \leq A|Q + a + it|^\alpha,
$$

and

$$
|f(b + it)| \leq B|Q + b + it|^\beta,
$$

where

$$
Q + a > 0 \text{ and } \alpha \geq \beta.
$$

Then, in the strip $a \leq \Re s \leq b$, we have

$$
|f(s)| \leq (A|Q + s|^\alpha)^{\frac{2 - \sigma}{2}} \cdot (B|Q + s|^\beta)^{\frac{\sigma}{2 - \sigma}}.
$$
3. Proof of Theorem 1.3

First, let us consider \( \sigma = \frac{1}{2} \) and \( r = 2 \). We do not have access to a bound of the form (9), since Theorem 1.2 is restricted when \( r = 2 \). We can get around this by sacrificing the exponent on the logarithm in (9) and using a Burgess bound in the form

\[
(12) \quad \left| \sum_{n=N_0+1}^{N_0+N_1} \chi(n) \right| < B(2)N_1^\frac{1}{2}q^{\frac{1}{3}}(\log q)^{\frac{1}{3}}.
\]

From Table 6 in [4], the above bound holds with \( B(2) = 1.520 \) when we have a prime \( q > 10^{10} \). Arguing as in Proposition 2.2 we may replace (9) with

\[
L \left( \frac{1}{2} + it, \chi \right) \leq \left( \frac{1}{2} + |t| \right) q^{\frac{1}{4}}(\log q)^{\frac{3}{4}}.
\]

(13)

\[
\cdot \left( B(2) \left( \frac{1}{4} + \frac{2\log \log q}{\log q} \right) + \frac{2 + B(2)}{(\sigma + |t|)\log q}^{\frac{1}{2}} + \frac{2P}{(\log q)^{2}} \right)
\]

\[
\leq 0.918 \left( \frac{1}{2} + |t| \right) q^{\frac{1}{4}}(\log q)^{\frac{3}{4}}.
\]

To obtain the constant, we have assumed that \( q \geq 10^{10} \) and \( t \geq 1 \) and taken \( P = 2\pi^{-2} + (\log 10^{10})^{-1} \), as in [5]. For other integers \( r \), we may appeal to Proposition 2.2 via Theorem 1.2 directly, obtaining bounds of the form

\[
(14) \quad \left| L \left( 1 - r^{-1} + it, \chi \right) \right| \leq C_{1-r^{-1}} \left( 1 - r^{-1} + |t| \right) q^{\beta_r}(\log q)^{\gamma_r}.
\]

Table 1 records specific values for each \( r \) from 2 to 10. Here, for \( r \geq 3 \), we have used the constants \( B(r) \) from Table 3 of [4]. For \( r \) larger than 10, one should in principle be able to take \( B(10) \) in place of determining the explicit Burgess constant directly. The general form of \( \beta_r \) is \( \frac{r+1}{4r^2} \), while \( \gamma_r \) is \( \frac{2r+1}{2r} \) except when \( r = 2 \).

### Table 1. Acceptable values in (14) for several \( r \).

| \( r \) | \( \sigma \) | \( B(r) \) | \( C_{1-r^{-1}} \) | \( \beta_r \) | \( \gamma_r \) |
|-----|-----|-----|-----|-----|-----|
| 2   | 1/2 | 1.5197 | 0.918 | 3/16 | 3/2 |
| 3   | 2/3 | 2.4910 | 1.036 | 1/9  | 7/6 |
| 4   | 3/4 | 2.1551 | 0.902 | 5/64 | 9/8 |
| 5   | 5/6 | 1.9688 | 0.842 | 3/50 | 13/10 |
| 6   | 5/7 | 1.8476 | 0.812 | 7/144 | 13/12 |
| 7   | 5/8 | 1.7596 | 0.798 | 2/49 | 15/14 |
| 8   | 7/8 | 1.6875 | 0.791 | 9/256 | 17/16 |
| 9   | 8/9 | 1.6292 | 0.789 | 5/162 | 19/18 |
| 10  | 9/10 | 1.5810 | 0.792 | 11/400 | 21/20 |

For \( \sigma < \frac{1}{2} \), observe that Proposition 2.2 taken with \( r = 2 \) (adjusting the exponent on the logarithms as necessary) yields

\[
|L(\sigma + it, \chi)| \leq (\sigma + |t|)q^{\frac{4-5\sigma}{3}}(\log q)^{2-3\sigma}
\]

\[
\cdot \left( \left( \frac{1}{1-\sigma} + B(2) \right) + \frac{2B(2)}{2\sigma - 1} \right) q^{\frac{2\sigma - 1}{3}}(\log q)^{2\sigma - 1} + \left( \frac{P}{\sigma} - \frac{2B(2)}{2\sigma - 1} \right).
\]

(15)

We may bound the above by again assuming that \( |t| \geq 1 \). This gives us bounds of the form

\[
|L(\sigma + it, \chi)| \leq C_\sigma(\sigma + |t|)q^{\beta_\sigma}(\log q)^{\gamma_\sigma}.
\]

We have recorded the value of \( C_\sigma \) for several \( \sigma \) in Table 2.

For \( \sigma \) between the vertical lines \( \Re s = 1 - \frac{1}{r} \) and \( \Re s = 1 - \frac{1}{R} \) \( (r < R) \), we can apply Theorem 2.3. Unless \( r = 2 \), this gives us the bound
Table 2. The value $C_{\sigma}$ in (15) for $\sigma = \frac{1}{r}$.

| $\sigma$ | $C_{\sigma}$ | $\beta_{\sigma}$ | $\gamma_{\sigma}$ |
|----------|-------------|-----------------|-----------------|
| $\frac{1}{3}$ | 8.934 | $\frac{7}{24}$ | 1 |
| $\frac{1}{4}$ | 6.877 | $\frac{11}{32}$ | $\frac{5}{4}$ |
| $\frac{1}{5}$ | 6.222 | $\frac{3}{8}$ | $\frac{7}{5}$ |
| $\frac{1}{6}$ | 5.996 | $\frac{19}{48}$ | $\frac{3}{2}$ |
| $\frac{1}{7}$ | 5.953 | $\frac{23}{56}$ | $\frac{11}{7}$ |
| $\frac{1}{8}$ | 6.003 | $\frac{27}{64}$ | $\frac{13}{8}$ |
| $\frac{1}{9}$ | 6.109 | $\frac{31}{72}$ | $\frac{5}{3}$ |
| $\frac{1}{10}$ | 6.249 | $\frac{7}{16}$ | $\frac{17}{10}$ |

$|L(\sigma + it, \chi)| \leq \left( C_{1-R^{-1}} q^{\frac{\sigma+1}{2R}} (\log q)^{\frac{2R+1}{2R}} |\sigma + it| \right)^{\frac{1-R^{-1}-\sigma}{R^{-1}-R^{-1}}}$

(16)

$\cdot \left( C_{1-R^{-1}} q^{\frac{\sigma+1}{2R}} (\log q)^{\frac{2R+1}{2R}} |\sigma + it| \right)^{\sigma -(1-R^{-1})}$

When $r = 2$, the only change to (16) we should make is that the exponent on the first log $q$ will be replaced by $\frac{3}{2}$. Taking $r = 2$ and $R = 10$, we obtain the second claim of Theorem 1.3. A similar approach using (15) with the line $\Re s = \frac{1}{10}$ and (13) completes Theorem 1.3.

These bounds may be extended to include $\sigma = 1$.

Proof of Theorem 1.4. We require an estimate for the size of $L(s, \chi)$ for $s = 1 + \epsilon + it$. One simple estimate is

(17) $|L(1 + \epsilon + it, \chi)| \leq \zeta(1 + \epsilon) < 1 + \frac{1}{\epsilon}$.

Now, take Theorem 2.3 using (14) (with $r = 10$) and (17). This provides

(18) $|L(\sigma + it, \chi)| \leq \left( 0.792 \left| \frac{9}{10} + it \right| q^{\frac{11}{10} \log(q)^{\frac{21}{20}}} \right)^{\frac{1+\epsilon-\sigma}{\frac{9}{10} + it}}$

for $\frac{9}{10} \leq \sigma \leq 1 + \epsilon$. For $\sigma = 1$, this bound becomes

$|L(\sigma + it, \chi)| \leq \left( 1 + \epsilon^{-1} \right)^{\frac{10}{1+10\epsilon}} \left( 0.792 q^{\frac{11}{10} \log(q)^{\frac{21}{20}}} \right)^{\frac{10}{1+10\epsilon}} \left| \frac{9}{10} + it \right|^{\frac{10}{1+10\epsilon}}$.

\[ \square \]

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