SCALING LIMIT OF THE STEIN VARIATIONAL GRADIENT DESCENT PART I: THE MEAN FIELD REGIME ∗

JIANFENG LU†, YULONG LU†, AND JAMES NOLEN†

Abstract. We study an interacting particle system in $\mathbb{R}^d$ motivated by Stein variational gradient descent [Q. Liu and D. Wang, NIPS 2016], a deterministic algorithm for sampling from a given probability density with unknown normalization. We prove that in the large particle limit the empirical measure converges to a solution of a non-local and nonlinear PDE. We also prove global well-posedness and uniqueness of the solution to the limiting PDE. Finally, we prove that the solution to the PDE converges to the unique invariant solution in large time limit.

Key words. Stein Variational Gradient Descent, Interacting particle system, Mean field limit, Sampling

AMS subject classifications. 35Q62, 35Q68, 82C22

1. Introduction. In this paper we study the following interacting particle system in $\mathbb{R}^d$:

$$
\dot{x}_i(t) = -\frac{1}{N} \sum_{j=1}^{N} \nabla K(x_i(t) - x_j(t)) - \frac{1}{N} \sum_{j=1}^{N} K(x_i(t) - x_j(t)) \nabla V(x_j(t)),
$$

$$
x_i(0) = x^0_i \in \mathbb{R}^d, \quad i = 1, \ldots, N.
$$

We refer to each of the $N$ functions $x_i(\cdot) \in \mathbb{R}^d$ as a particle. The function $K : \mathbb{R}^d \mapsto \mathbb{R}$ is a smooth, symmetric, and positive definite kernel. The function $V : \mathbb{R}^d \to \mathbb{R}$ is a smooth potential such that $e^{-V(x)}$ is integrable. More specific assumptions about $K$ and $V$ are given below.

We are interested in the macroscopic behavior of the particle system (1.1) as $N \to \infty$ in the framework of mean field limit. Formally this mean field limit is described by the following non-local, nonlinear partial differential equation (PDE):

$$
\partial_t \rho = \nabla \cdot \left( \rho (\nabla K * \rho + K * (\nabla \rho)) \right),
$$

$$
\rho(0, \cdot) = \rho_0(\cdot).
$$

We aim to make a rigorous connection between (1.1) and (1.2). Specifically, we prove global existence and uniqueness of a solution to this initial value problem, for $\rho_0$ in the appropriate regularity class, and we show that the empirical measure

$$
\mu^N_t = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i(t)}
$$

converges as $N \to \infty$ to the solution of (1.2), assuming $\mu_0^N$ converges to $\rho_0(x)dx$ in the appropriate sense. We also want to study the long-time behavior of solutions to the mean field PDE (1.2). The probability density $\rho_\infty = e^{-V(x)}/Z$ is an invariant solution to (1.2); under certain assumptions, we prove that $\rho(t, \cdot)$ converges weakly to $\rho_\infty$ as $t \to +\infty$.

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†Department of Mathematics, Duke University, Durham NC 27708, USA (jianfeng@math.duke.edu, yulongliu@math.duke.edu, nolen@math.duke.edu).
1.1. Motivation. Our interest in the particle system (1.1) is mainly motivated by the recent works by Liu and Wang [23, 22], where a time-discretized form of (1.1) was introduced as an algorithm called Stein Variational Gradient Descent (SVGD). The idea of the algorithm is to transport a set of $N$ particles in $\mathbb{R}^d$ so that their empirical measure $\mu^N$ approximates the target probability measure $\rho_\infty(x)dx = Z^{-1}e^{-V(x)}dx$, with an unknown normalization factor $Z$. At discrete times, the particles are updated via the map

$$T(x) = x + \varepsilon \varphi(x)$$

where $\varepsilon$ is a small time step size and $\varphi$ is a velocity field, which is chosen appropriately so to have a “fastest decay” of the Kullback-Leibler (KL) divergence between the push-forward measure $T#\mu^N$ and the target $\rho_\infty$.

This idea is formalized by choosing the velocity field $\varphi$ to solve the variational problem

$$\sup_{\varphi \in \mathcal{H}} \left\{ -\partial_t KL(T#\mu^N || \rho_\infty)_{\varepsilon=0} \mid \| \varphi \|_{\mathcal{H}} \leq 1 \right\}$$

at each time step, where $\mathcal{H}$ is a suitable space of vector fields. It is not clear that (1.4) is well-defined, because the measure $T#\mu^N$ may be singular with respect to $\rho_\infty$ and $KL(T#\mu^N || \rho_\infty) = +\infty$. However, as shown in [23], (1.4) can be given meaning through the observation that if $\mu$ is absolutely continuous with respect to $\rho$ and $KL(T#\mu || \rho) < \infty$, then

$$-\partial_t KL(T#\mu || \rho)_{\varepsilon=0} = E_\mu[S_\rho \varphi],$$

where $S_\rho$ is the so-called Stein operator defined by

$$S_\rho \varphi := \nabla \log \rho(x) \cdot \varphi(x) + \nabla \cdot \varphi(x).$$

In view of (1.4), this leads to the definition of Stein discrepancy

$$SD(\mu, \rho, \mathcal{H}) := \sup_{\varphi \in \mathcal{H}} \left\{ E_\mu[S_\rho \varphi] \mid \| \varphi \|_{\mathcal{H}} \leq 1 \right\},$$

which has the property that $SD(\mu, \rho, \mathcal{H}) \geq 0$ is equal to zero if and only if $\mu = \rho$ provided that the space $\mathcal{H}$ is sufficiently rich. For the empirical measure $\mu^N$, the objective function $E_{\mu^N}[S_\rho \varphi]$ in (1.5) may be well-defined and finite even though $KL(T#\mu^N || \rho_\infty) = +\infty$. Furthermore, [23] showed that if the space $\mathcal{H}$ is chosen to be a reproducing kernel Hilbert space with a positive definite kernel $K$, then the velocity field optimizing (1.5) can be characterized explicitly and is given by

$$\varphi^*_{\mu,\rho}(\cdot) \propto E_{x \sim \mu}[S_\rho K(x, \cdot)] = \int_{\mathbb{R}^d} (\nabla \log \rho(x)K(x, \cdot) + \nabla_x K(x, \cdot)) \mu(dx).$$

Therefore, interpreting (1.4) by (1.5) and using the fact that $\rho_\infty(x) \propto e^{-V(x)}$, one sees that the optimal solution of (1.4) is given by

$$\varphi^*_{\mu^N,\rho_\infty}(x) = E_{x \sim \mu^N}[S_{\rho_\infty} K(x, \cdot)]$$

$$= -\frac{1}{N} \sum_{j=1}^N \nabla K(x - x_j) + \frac{1}{N} \sum_{j=1}^N K(x - x_j)\nabla V(x_j).$$
Putting this optimal velocity back into (1.3) and letting the step size \( \varepsilon \downarrow 0 \) gives the evolution (1.1).

The variational picture described above about the particle system (1.1) suggests that the mean field limit (1.2) might also admit a variational structure. Indeed, it has been shown heuristically in [22] that equation (1.2) can be viewed formally as a gradient flow for the KL-divergence functional

\[
\rho \mapsto \text{KL}(\rho \| \rho_\infty) = \int_{\mathbb{R}^d} \rho \log \frac{\rho}{\rho_\infty} \, dx,
\]

with respect to a generalized optimal transport metric whose definition involves the reproducing kernel Hilbert space with kernel \( K(x) \). This in particular implies that the KL-divergence functional is a Lyapunov functional for the PDE (1.2), namely

\[
\frac{d}{dt} \text{KL}(\rho_t \| \rho_\infty) \leq 0.
\]

Interpreting an evolutionary PDE as a gradient flow in the space of probability measures with respect to certain Wasserstein metric dates back to the seminar work on Fokker-Planck equation by Jordan, Kinderlehrer and Otto [19]. By now, similar gradient flow structures have been identified for a large family of evolution equations, including porous medium equation [25], McKean-Vlasov equation [5], etc. In the present paper, we will not pursue further the rigorous definition and analysis of the gradient flow structure of (1.2). Instead, we take the system (1.1) as our starting point and prove its connection to the mean field PDE (1.2).

1.2. Relevant Literature. Sampling from a density of the form \( \rho_\infty(x) = e^{-V(x)}/Z \) without knowing the normalization constant \( Z \) is a fundamental problem in Bayesian statistics and machine learning. One generic approach that has been tremendously successful in recent years is the Markov chain Monte Carlo (MCMC) methodology based on Metropolis-Hastings mechanism. The general principle of Metropolis-Hastings algorithms is to build an ergodic Markov chain whose invariant measure is the target measure \( \rho_\infty \) by first making candidate samples (proposals), which are then tuned to ensure stationarity via acceptance/rejection. In practice, one common approach to constructing proposals is by discretizing some stochastic dynamics, such as the following (overdamped) Langevin dynamics:

\[
dX(t) = -\nabla V(X) \, dt + \sqrt{2} \, dB(t),
\]

(1.7)

where \( B \) is a standard Brownian motion in \( \mathbb{R}^d \). A vanilla Euler-Maruyama discretization scheme associated to (1.7) together with Metropolis-Hastings step leads to the famous Metropolis Adjusted Langevin Algorithm (MALA) [27, 3] (whose non-Metropolized version known as unadjusted Langevin algorithm (ULA) [9, 14]).

One advantageous feature of stochastic dynamics-based sampling methods, e.g. MALA or ULA, is that the dynamics tend to explore high probability regions (around the local minima of \( V \)), while the random noise helps the dynamics to escape outside the basin of attraction and thus promotes its exploration of the entire state space. In contrast to this stochastic sampling approach, (1.1) may be viewed as a deterministic system for sampling from \( \rho_\infty \), albeit a coupled system. Qualitatively speaking, the terms in (1.1) which involve \( \nabla V \) tend to drive particles toward local minima of \( V \) (note however the nonlocal interaction due to the presence of \( K \)). On the other hand,
the terms involving $\nabla K$ are repulsive, forcing the particles to disperse; this is seen in the fact that
\[-\frac{1}{N} \sum_{j=1}^{N} \nabla K(x_i - x_j) = -\nabla x_i E(x),\]
where $E(x) = \frac{1}{N} \sum_{i<j} K(x_i - x_j)$ is the interaction energy. Here we assumed that $\nabla K(0) = 0$. This interaction term in SVGD plays a role similar to that of the noisy term in stochastic-dynamics-based sampling methods. Intuitively, one would expect that the empirical measure $\mu_N$ of the particles $\{x_i(t)\}$ tends to be close to $\rho_\infty$ in the limit of both large sample size of large time. One of the contributions of this paper is to prove this convergence rigorously.

To compare these two sampling approaches at the PDE level, observe that the probability density for $X(t)$ defined by (1.7) solves the linear Fokker-Planck equation
\[
\partial_t \rho = \nabla \cdot (\nabla \rho + \rho \nabla V).
\]
(1.8)

It is well known [24] that under some mild assumption on $V$, the solution $\rho$ of (1.8) converges to the equilibrium distribution $\rho_\infty$ exponentially fast. On the other hand, if we formally set $K(x) = \delta_0(x)$, the non-local mean-field equation (1.2) becomes
\[
\partial_t \rho = \nabla \cdot (\rho (\nabla V)),
\]
(1.9)

which is a non-linear porous medium equation with an additional transport due to $\nabla V$. So, compared to (1.8), the mobility term and the transport term in (1.9) are small where the density is small. This suggests that the convergence of the solution of (1.9) towards $\rho_\infty$ may be slower that that of (1.8). In this paper, we consider only a fixed kernel $K$, but if we scale the kernel $K$ as $K_N(\cdot) = N^\beta K(N^\beta \cdot)$, it is natural to expect the large particle limit of (1.1) to be governed by (1.9) instead of (1.2), if $\beta > 0$ is not too large — rigorous justification of such convergence result is still work in process.

One should also compare (1.1) with the following more standard deterministic interacting particle system:
\[
\dot{x}_i = -\frac{1}{N} \sum_{j=1}^{N} \nabla K(x_i - x_j) - \nabla V(x_i), \quad i = 1, 2, \cdots, N.
\]
(1.10)

It is well-known [12] that under suitable assumption on $K$ and $V$, the mean field limit of (1.10) is the following McKean-Vlasov equation
\[
\partial_t \rho = \nabla \cdot (\rho (\nabla K \ast \rho + \nabla V)).
\]
(1.11)

The particle system (1.1) differs from (1.10) in that the external force added to each particle is non-local, and is defined by averaging the individual forces $\nabla V(x_j)$ with weights defined by the kernel $K$. Interestingly, such non-local external force guarantees that $\rho_\infty$ is a stationary solution of (1.2) — this explains the rationale for using the deterministic particle system (1.1) as an approximation algorithm for sampling $\rho_\infty$. On the contrary, $\rho_\infty$ is not a stationary solution of (1.10). We also remark that the nonlocal external force makes the analysis of (1.1) more challenging than that of (1.10).

Although sampling via a deterministic particle system is less common, the use of deterministic particles is ubiquitous in numerical approximations of partial differential
equations, including for example, the vortex method for equations in fluid mechanics [18, 26], the weighted particle method [10] and the diffusion-velocity method [11] for convection-diffusion and nonlinear-wave equations [7]. For a comprehensive discussion on deterministic particle methods we refer the reader to the recent review paper [6] and references therein. Recently, a blob method was proposed in [8] for an aggregation equation, which is the equation (1.2) with $V = 0$ and with $K$ being attractive rather than repulsive. The same method was generalized by [4] to a more general class of nonlinear diffusion equations, which has a $L^2$-Wasserstein gradient flow structure.

1.3. Plan of The Paper. The rest of the paper is organized as follows. In Section 2, we first make several technical assumptions on $V$ and $K$ and then state our main results under these assumptions. In Section 3, we prove the existence and uniqueness of the mean field equation (1.2). Section 4 devotes to the proof of convergence of the interaction particle system (1.1) to its mean field limit (1.2). Finally, in Section 5 we prove that the solution $\rho_t$ of (1.2) converges to the equilibrium $\rho_\infty$ as $t \to \infty$.

2. Preliminaries and Main Results.

2.1. Assumptions and Notation. Throughout the paper we assume that the kernel $K$ satisfies the following:

**Assumption 2.1.** $K : \mathbb{R}^d \to \mathbb{R}$ is smooth with bounded derivatives of any order. In addition, $K(x - y)$ is symmetric and positive definite, meaning that

$$\sum_{i=1}^{m} \sum_{j=1}^{m} K(x_i - x_j)\xi_i \xi_j \geq 0, \quad \forall \ x_i \in \mathbb{R}^d, \ \xi_i \in \mathbb{R}, \ m \in \mathbb{N}.$$ 

A canonical choice of $K$ satisfying Assumption 2.1 is a Gaussian kernel, e.g. $K(x) = \frac{1}{(4\pi)^{d/2}} \exp(-\frac{|x|^2}{4})$. Most of the results of this paper can be generalized to the case where $K$ has only finite regularity; we choose not to state the most general result for the sake of simplicity.

For the potential function $V : \mathbb{R}^d \to \mathbb{R}$, we will assume the following:

**Assumption 2.2.** (A1) $V \in C^\infty(\mathbb{R}^d), V \geq 0$ and $V(x) \to +\infty$ if $|x| \to +\infty$.

(A2) There exists a constant $C_V > 0$ and some index $q > 1$ such that

$$|\nabla V(x)|^q \leq C_V (1 + V(x))$$

for every $x \in \mathbb{R}^d$ and that

$$\sup_{\theta \in [0,1]} |\nabla^2 V(\theta x + (1 - \theta)y)|^q \leq C_V (1 + V(x) + V(y)).$$

(A3) For any $\alpha, \beta > 0$, there exists a constant $C_{\alpha,\beta} > 0$ such that if $|y| \leq \alpha|x| + \beta$, then

$$(1 + |x|)(|\nabla V(y)| + |\nabla^2 V(y)|) \leq C_{\alpha,\beta}(1 + V(x)).$$

(A4) There exist constants $r < 2$ and $\tilde{C}_V > 0$ such that

$$|\nabla V(x)e^{-|x|^r}| \leq \tilde{C}_V.$$ 

**Remark 2.3.** Setting $\alpha = 1, \beta = 0$ and $y = x$ in (A3), we have that

$$(1 + |x|)(|\nabla V(x)| + |\nabla^2 V(x)|) \leq C_1(1 + V(x))$$

and that

$$(2.1) \quad \sup_{\theta \in [0,1]} |\nabla^2 V(\theta x + (1 - \theta)y)|^q \leq C_V (1 + V(x) + V(y)).$$
for some constant $C_1 > 0$. It is easy to check that Assumption 2.2 is fulfilled by all polynomials of even order.

For $k, p \geq 1$, we denote by $W^{k,p}(\mathbb{R}^d)$ the usual Sobolev space of functions whose weak derivatives up to $k$-th order belong to $L^p(\mathbb{R}^d)$. On the space $W^{k,p}(\mathbb{R}^d)$, we shall write the norm $\| \cdot \|_{k,p}$. In particular, when $p = 2$, we set $H^k(\mathbb{R}^d) = W^{k,2}(\mathbb{R}^d)$. For a function $u \in L^p(\mathbb{R}^d)$, we may write its norm $\| u \|_p$. Given a probability measure $\mu$ and a Borel-measurable map $f$, we denote by $f_#\mu$ the push-forward of the measure $\mu$ under the map $f$. Given a function $u$, we define $\hat{u}$ the Fourier transform of $u$. Let $u * v$ be the convolution of two functions $u$ and $v$. We use $\mathcal{P}_V$ and $\mathcal{P}_p$ denote the set of Borel probability measures $\mu$ on $\mathbb{R}^d$ satisfying

\begin{equation}
\| \mu \|_{\mathcal{P}_V} = \int_{\mathbb{R}^d} (1 + V(x)) \, d\mu < \infty \quad \text{or} \quad \| \mu \|_{\mathcal{P}_p} = \int_{\mathbb{R}^d} |x|^p \, d\mu(x) < \infty,
\end{equation}

respectively. For $\mu, \nu \in \mathcal{P}_p$, $W_p(\mu, \nu)$ denotes the $p$-Wasserstein distance [30].

For our result on well-posedness of the PDE (1.2), we introduce function spaces

\begin{align*}
L^1_V := \{ u \in L^1(\mathbb{R}^d) \mid \int_{\mathbb{R}^d} (1 + V(x)) |u(x)| \, dx < \infty \}, \\
W^{1,1}_V := \{ u \in W^{1,1}(\mathbb{R}^d) \mid \int_{\mathbb{R}^d} (1 + V(x)) (|u(x)| + |\nabla u(x)|) \, dx < \infty \}
\end{align*}

with norms $\| u \|_{L^1_V} := \|(1 + V)u\|_{L^1(\mathbb{R}^d)}$ and $\| u \|_{W^{1,1}_V} := \int_{\mathbb{R}^d} (1 + V(x)) (|u(x)| + |\nabla u(x)|) \, dx$, respectively. We set

$$
\mathcal{B}_{k,V} := H^k(\mathbb{R}^d) \cap L^1_V, \quad \mathcal{B}^{1}_{k,V} = H^k(\mathbb{R}^d) \cap W^{1,1}_V
$$

with the canonical norms

$$
\| u \|_{\mathcal{B}_{k,V}} := \| u \|_{H^k(\mathbb{R}^d)} + \| u \|_{L^1_V}, \quad \| u \|_{\mathcal{B}^{1}_{k,V}} := \| u \|_{H^k(\mathbb{R}^d)} + \| u \|_{W^{1,1}_V}.
$$

We also define the following space-time function spaces

\begin{align}
\mathcal{A}_k(0, T) &= \{ u \in L^\infty((0,T); H^k(\mathbb{R}^d)) \mid \partial_t u \in L^\infty((0,T); H^{k-1}(\mathbb{R}^d)) \}, \\
\mathcal{A}_{k,V}(0, T) &= \{ u \in L^\infty((0,T); \mathcal{B}_{k,V}) \mid \partial_t u \in L^\infty((0,T); H^{k-1}(\mathbb{R}^d)) \}, \\
\mathcal{A}^{1}_{k,V}(0, T) &= \{ u \in L^\infty((0,T); \mathcal{B}^{1}_{k,V}) \mid \partial_t u \in L^\infty((0,T); H^{k-1}(\mathbb{R}^d)) \}.
\end{align}

We will use constant $C(V)$ to denote a generic constant which depends on $V$. Similar rules apply to $C(K), C(V, K)$, etc. We also use constants $C, \tilde{C}, \hat{C}$ to denote generic constants that are independent of quantities of interest. The exact values of these constants may change from line to line.

2.2. Main Results. Our first result is the global well-posedness and regularity of the mean field PDE (1.2):

**Theorem 2.4.** Let $V$ satisfy Assumption 2.2. Let $\rho_0$ be a non-negative function in $\mathcal{B}^{1}_{k,V}$ with some $k \geq 2$. Then for any $T > 0$, there exists a unique non-negative function $\rho \in \mathcal{A}^{1}_{k,V}(0, T)$ which solves

\begin{align}
\partial_t \rho &= \nabla \cdot (\rho (\nabla K * \rho + K * (\nabla V \rho))) \quad \text{in } (0, T) \times \mathbb{R}^d, \\
\rho(0, \cdot) &= \rho_0(\cdot).
\end{align}
Theorem 2.4 is contained in Section 3 below. It is motivated by the paper [20], where the author proved the local and global well-posedness of the PDE (1.2) with $V = 0$ and under different regularity assumptions on $K$. The presence of the potential $V$ makes the problem more difficult since the velocity $\nabla V$ is unbounded at infinity. At the first glance, it is not clear whether the mass of density $\rho$ would accumulate at infinity. However, we will prove below that this can not happen provided that $V$ does not grow too wildly at infinity (see Lemma 3.1).

Our next main result is the following theorem showing that the mean field limit of the particle system (1.1) is given by the PDE (1.2).

**Theorem 2.5.** Let $V$ satisfy Assumption 2.2 with some $q \in (1, \infty)$. Define $p \in (1, \infty)$ by $\frac{1}{p} + \frac{1}{q} = 1$. Fix $R > 0$. Let $\nu_0$ be a probability measure in $\mathcal{P}_V \cap \mathcal{P}_p$ with density $\rho_0 \in Y^1_{\mathcal{P}_V}$ such that $\|\nu_0\|_{\mathcal{P}_V} \leq R$. Let $\rho(t, \cdot)$ be the solution to (1.2) with initial condition $\rho_0$. Let $\{x_i(t)\}^N_{i=1}$ be the solution to the particle system (1.1) and $\mu^N_t = \frac{1}{N} \sum^N_{i=1} \delta_{x_i(t)}$. Suppose also that $\|\mu^N_t\|_{\mathcal{P}_V} \leq R$. Then for any $T > 0$, there exists a constant $C > 0$ depending on $T$, $K$, $V$, and $R$ such that

$$\sup_{t \in [0, T]} W_p(\mu^N_t, \rho(t, \cdot)) \leq C W_p(\mu^N_0, \nu_0).$$

Theorem 2.5 implies that if the initial points $\{x^N_i(0)\}^N_{i=1}$ are such that $W_p(\mu^N_0, \nu_0) \to 0$ as $N \to \infty$, then $W_p(\mu^N_t, \rho(t, \cdot)) \to 0$ as well. The proof of Theorem 2.5 is presented in Section 4.3 below: it is based on the Dobrushin’s coupling argument (see e.g. [12, 17]) for the mean field characteristic flow. The hypothesis of the convergence of the initial empirical measure, i.e., $W_p(\mu^N_0, \nu_0) \to 0$, can be justified rigorously in particular when the initial particles $\{x^N_0\}$ are independent samples drawn from $\nu_0$. For a detailed discussion on the convergence of empirical measures in $\mathcal{W}_p$, we refer the interested readers to the paper [16, 31, 2] in the case that $p \in [1, \infty)$, and to [29, 21] when $p = \infty$.

Our last result pertains to the long time behavior of the solution $\rho_t$ of (1.2). Since the probability density $\rho_{\infty}(x) = e^{-V(x)}/Z$ is an invariant solution to the PDE (1.2), it is natural to ask whether $\rho_{\infty}$ is the unique invariant measure, and whether $\rho_t \to \rho_{\infty}$ as $t \to \infty$. The following theorem provides an affirmative answer to these questions. For technical reasons, we need to restrict our attention to the special case that $K$ is a Gaussian kernel.

**Theorem 2.6.** Assume that the conditions of Theorem 2.4 hold, and for some $\sigma > 0$ let the kernel

$$K(x) = K_{\sigma}(x) := \frac{1}{(4\pi \sigma^2)^d} e^{-x^2 / (4\sigma^2)}.$$

Let $\rho_t$ be the solution to (1.2) with initial condition $\rho_0$ satisfying that $\text{KL}(\rho_0 \| \rho_{\infty}) < \infty$. Then $\rho_t$ converges weakly to $\rho_{\infty}$ as $t \to \infty$.

Theorem 2.6 establishes the convergence of $\rho_t$ to the unique invariant measure $\rho_{\infty}$ when the kernel $K$ is Gaussian. The proof is presented in Section 5. Extension of the result to other smooth reproducing kernels is unclear to us and remains an interesting open problem. Moreover, a quantitative convergence rate is also far from clear. The main obstacle is the lack of a generalized logarithmic Sobolev inequality which could lower bound the Stein discrepancy in terms of the relative entropy. This issue is to be investigated in future works.

### 3. Well-posedness of the mean field PDE

In this section, we prove Theorem 2.4 in three steps: we first prove the uniqueness of the solution of the problem...
(2.5) in Section 3.1, then we prove the local existence of solution in Section 3.2, and finally we show the local solution is a global solution in Section 3.3.

3.1. Uniqueness. Suppose that $\rho_1, \rho_2 \in X_{K,V}(0,T)$ are two solutions to (2.5) with the same initial condition. Then the difference $\omega = \rho_1 - \rho_2$ solves

$$
\frac{\partial \omega}{\partial t} = \nabla \cdot \left( \omega (\nabla K * \rho_1 + K * (\nabla \rho_1)) + \nabla \cdot \left( \rho_2 (\nabla K * \omega + K * (\nabla \omega)) \right) \right),
$$

$$\omega(0, \cdot) = 0.$$

Now given $\delta > 0$, we define the one dimensional function

$$\phi_\delta(x) = \sqrt{|x|^2 + \delta}.$$  

It is clear that $\phi_\delta(x) \to |x|$ as $\delta \to 0$ and that $\sup_{x \in \mathbb{R}} |\phi_\delta'(x)| \leq 1$. In addition, $\phi_\delta(\omega) \in L^\infty((0,T); H_{loc}^{k-1}(\mathbb{R}^d))$ and satisfies the following equation:

$$
\frac{\partial \phi_\delta(\omega)}{\partial t} = \phi_\delta(\omega) \nabla \cdot \left( \omega (\nabla K * \rho_1 + K * (\nabla \rho_1)) + \phi_\delta(\omega) \nabla \cdot \left( \rho_2 (\nabla K * \omega + K * (\nabla \omega)) \right) \right).
$$

Let us also define a smooth cut-off function $\eta_R$ on $\mathbb{R}^d$ by

$$\eta_R(x) = \eta(x/R), \quad \text{where } \eta \in C_c^\infty(\mathbb{R}^d) \text{ and } \eta(x) = 1 \text{ for } |x| \leq 1, \eta(x) = 0 \text{ for } |x| \geq 2.$$  

Multiplying the above equation on $\phi_\delta(\omega)$ with $\eta_R(1 + V)$ and then using integration by parts yields

$$
\frac{\partial}{\partial t} \int_{\mathbb{R}^d} \phi_\delta(\omega)(1 + V) \eta_R dx = - \int_{\mathbb{R}^d} \phi_\delta(\omega) \nabla \cdot \left( (\nabla K * \rho_1 + K * (\nabla \rho_1))(1 + V) \eta_R \right) dx
$$

$$\quad + \int_{\mathbb{R}^d} \phi_\delta(\omega)(1 + V) \eta_R \left( (\nabla K * \rho_1 + K * (\nabla \rho_1)) \right) dx
$$

$$\quad + \int_{\mathbb{R}^d} \phi_\delta(\omega)(1 + V) \eta_R \nabla \cdot \left( \rho_2 (\nabla K * \omega + K * (\nabla \omega)) \right) dx =: \sum_{i=1}^{3} I_i(\delta, R).$$

Since $\rho_1, \rho_2 \in X_{K,V}(0,T) \subset X_{k,V}(0,T)$, an application of Young’s convolution inequality and the assumption (2.2) implies that

$$I_1(\delta, R) \leq C(K, V) \|\rho_1\|_{W^{1,1}_V(1 + O(1/R))} \int_{\mathbb{R}^d} (1 + V) \phi_\delta(\omega) \eta_{2R} dx.$$  

Similarly, one has that

$$I_2(\delta, R) + I_3(\delta, R) \leq C(K, V) \|\rho_1\|_{W^{1,1}_V} \|\omega\|_{L^1_V}.$$  

Consequently, by first letting $\delta \to 0$ and then letting $R \to \infty$, we have

$$\|\omega(t, \cdot)\|_{L^1_V} \leq C(K, V) \|\rho_1\|_{W^{1,1}_V} \int_0^t \|\omega(s, \cdot)\|_{L^1_V} ds.$$  

It follows from the Grönwall’s inequality that $\omega(t, \cdot) = 0$ a.e. $t \in [0, T]$. This finishes the proof of uniqueness.
3.2. Local existence. In this section, we prove the local existence of the solution by the method successive approximation, as in [20]. The idea is to first build a sequence of approximate solutions \( \{ \rho^n \} \) through solving the following linearized equation successively:

\[
\begin{align*}
\partial_t \rho^n &= \nabla \cdot (\rho^n (\nabla K * \rho^{n-1} + K * (\nabla V \rho^{n-1}))) \quad \text{in} \quad (0, T) \times \mathbb{R}^d, \\
\rho^n(0, \cdot) &= \rho_0(\cdot).
\end{align*}
\]

Then we show that limit of the sequence \( \{ \rho^n \} \) is a solution of the problem (2.5). However, unlike [20], we need extra a priori estimates on \( \rho^n \) due to the presence of \( V \) in the second convolution term \( K * (\nabla V \rho^{n-1}) \) on the right side of (3.2). Therefore we first derive a few useful a priori estimates about the solutions of (2.5) and of the iterative equation associated to (3.2) (see (3.4)).

3.2.1. A priori estimates. We start by proving a simple weighted \( L^1 \)-estimate of \( \rho \), which will play an important role in the proof of local and global existence.

**Lemma 3.1.** Let \( V \) satisfy Assumption 2.2. Suppose that \( \rho \in \mathcal{H}^1_{k,V} (0, T) \) is a solution to (2.5) with a non-negative initial data \( \rho_0 \in \mathcal{H}^1_{k,V} \) for some \( k \geq 1 \). Then there exists a constant \( C \) depending on \( V, K \) but not on \( \rho_0 \) or \( T \) such that

\[
\| \rho(t, \cdot) \|_{L^1_v} \leq \| \rho_0 \|_{L^1_v} e^{t C} \| \rho_0 \|_{L^1_v}.
\]

**Proof.** Since the initial condition \( \rho_0 \in \mathcal{H}^1_{k,V} \) is non-negative, and \( \rho \in \mathcal{H}^1_{k,V} (0, T) \) by assumption, we know from the method of characteristics that \( \rho(t, \cdot) \) is also non-negative for every \( t \in [0, T] \). In addition, by integrating both sides of the equation (2.5), one sees that the equation is mass-preserving i.e. \( \| \rho(t, \cdot) \|_{L^1} = \| \rho_0 \|_{L^1} \) for all \( t \in [0, T] \). In order to prove (3.3), similar to the proof of the uniqueness, we first multiply both sides of the equation (2.5) with \( (1 + V) \eta_R \), then use integrating by parts and finally let \( R \to \infty \) to obtain that

\[
\partial_t \int_{\mathbb{R}^d} (1 + V(x)) \rho(t, x) dx = - \int_{\mathbb{R}^d} \rho(t, x) \nabla V(x) \cdot (\nabla K * \rho)(t, x) dx
\]

\[
- \int_{\mathbb{R}^d} \rho(t, x) \nabla V(x) \cdot (K * (\nabla V \rho))(t, x) dx.
\]

Again thanks to the assumption \( K(x - y) \) is positive definite, the second term on the right side of above is non-positive. Hence one obtains from Young's convolution inequality and (2.2) that

\[
\partial_t \| \rho(t, \cdot) \|_{L^1_V} \leq \| \nabla V \rho(t, \cdot) \|_{L^1} \| \nabla K * \rho(t, \cdot) \|_{L^\infty}
\]

\[
\leq C(V) \| K \|_{1, \infty} \| \rho(t, \cdot) \|_{L^1_V} \| \rho(t, \cdot) \|_{L^1_V}
\]

\[
\leq C(V) \| K \|_{1, \infty} \| \rho(t, \cdot) \|_{L^1_V} \| \rho_0 \|_{L^1_V},
\]

where we also used the mass-preserving property of the solution in the last line. This implies that \( \| \rho(t, \cdot) \|_{L^1_V} \leq \exp(t C(V)) \| \rho_0 \|_{L^1_V} \| \rho_0 \|_{L^1_V} \), which proves the lemma.

**Remark 3.2.** Lemma 3.1 shows that \( \| \rho(t, \cdot) \|_{L^1_V} \) is bounded up to any finite time. Because of the assumption (2.2), this in particular implies that \( \nabla V(\cdot) \rho(t, \cdot) \) is bounded in \( L^1 \) up to any finite time, which in turn allows us to control the nonlinear and non-local drift term in (2.5). This uniform-in-time bound is essential in the construction of local solutions in the space \( \mathcal{H}^{-1}_{k,V} (0, T) \) as well as in the proof that the local solution does not blow up in finite time.
The next lemma establishes the well-posedness of an iterative equation associated to (3.2).

**Lemma 3.3.** Let $V$ satisfy Assumption 2.2. Let $\rho_0$ be a non-negative function in $\mathcal{F}^1_{k,V}$ with some $k \geq 2$. Given $\overline{\rho} \in \mathcal{X}^1_{k,V}(0, T)$, there exists a unique function $\rho \in \mathcal{X}^1_{k,V}(0, T)$ which solves the linear transport equation

\[
\begin{align*}
\partial_t \rho &= \nabla \cdot (\rho(\nabla K \ast \overline{\rho} + K \ast (\nabla V \overline{\rho}))) \text{ in } (0, T) \times \mathbb{R}^d, \\
\rho(0, \cdot) &= \rho_0(\cdot).
\end{align*}
\]

In addition, the solution $\rho$ is non-negative and satisfies

\[
||\rho(t, \cdot)||_{\mathcal{F}^1_{k,V}} \leq \exp \left( tC||\overline{\rho}||_{L^\infty((0,T);L^1_{W})} \right) ||\rho_0||_{\mathcal{F}^1_{k,V}},
\]

where the constant $C$ only depends on $V$ and $K$.

**Proof.** The existence of a unique solution $\rho$ which is non-negative follows immediately from the method of characteristics, since the equation has the form $\partial_t \rho = v \cdot \nabla \rho + (\nabla \cdot v)\rho$, where the vector field

\[v(t, x) = \nabla K \ast \overline{\rho} + K \ast (\nabla V \overline{\rho})\]

is Lipschitz and satisfies the bounds

\[
\begin{align*}
|v(t,x)| &\leq (||\nabla K||_\infty + ||\nabla K||_\infty) ||\overline{\rho}||_{L^\infty((0,T);L^1_{W})}, \\
|\nabla v(t,x)| &\leq (||D^2 K||_\infty + ||\nabla K||_\infty) ||\overline{\rho}||_{L^\infty((0,T);L^1_{W})}
\end{align*}
\]

for all $x \in \mathbb{R}^d$ and almost every $t \in [0, T]$. Specifically, the solution is

\[\rho(t, x) = \rho_0(\Phi_t^{-1} x) \exp \left( \int_0^t (\nabla \cdot v)(s, \Phi_s \circ \Phi_t^{-1} x) \, ds \right)\]

where $\Phi_t : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is the flow map, defined by $\frac{d}{dt} \Phi_t(x) = -v(t, \Phi_t(x))$, $\Phi_0(x) = x$.

In fact, since for each $k \geq 1$,

\[|D^k v(t,x)| \leq (||D^{k+1} K||_\infty + ||D^k K||_\infty) ||\overline{\rho}||_{L^\infty((0,T);L^1_{W})}, \quad x \in \mathbb{R}^d \]

holds for almost every $t$, the flow map $\Phi_t$ is smooth in $x$, with bounded derivatives of all order (e.g. see [28, Chapter 2]. Consequently, these estimates imply that if $\rho_0 \in \mathcal{F}^1_{k,V} \subset H^k(\mathbb{R}^d)$, then $\rho \in \mathcal{F}^k(0, T)$ for any $T > 0$. Then we are left to prove that $\rho(t, \cdot) \in W^1_{V_{k,V}}$ for every $t \in [0, T]$ and that it satisfies the a priori estimate (3.5).

We first show that $||\rho(t, \cdot)||_{L^1_{W_{k,V}}} \leq \infty$ for every $t \in [0, T]$. As in the proof of Lemma 3.1, by testing the equation (3.4) with $(1 + V)\eta_R$ and then taking the limit $R \rightarrow \infty$, we can obtain that

\[
\begin{align*}
\partial_t \int_{\mathbb{R}^d} (1 + V(x))\rho(t,x)dx &= -\int_{\mathbb{R}^d} \rho(t,x)\nabla V(x) \cdot (\nabla K \ast \overline{\rho})(t,x)dx \\
&\quad - \int_{\mathbb{R}^d} \rho(t,x)\nabla V(x) \cdot (K \ast (\nabla V \overline{\rho}))(t,x)dx \\
&\leq C(V)||K||_{1,\infty}||\rho(t,\cdot)||_{L^1_{W}} ||\overline{\rho}(t,\cdot)||_{L^1_{W}} \\
&= C(V)||K||_{1,\infty}||\rho_0||_{L^1_{W}} ||\overline{\rho}(t,\cdot)||_{L^1_{W}}.
\end{align*}
\]
As a consequence,
\[ \| \rho(t, \cdot) \|_{L^1_V} \leq \exp(\rho(C(V)\|K\|_{\infty} \| \tilde{p} \|_{L^\infty((0,T);L^1_V)})) \| \rho_0 \|_{L^1_V} . \]

Next, we show that \( \rho(t, \cdot) \in W^{1,1}_V \). To see this, we differentiate both sides of (3.4) with respect to \( x_i \) to get the following equation for \( \partial_{x_i} \rho \)
\[
\partial_t(\partial_{x_i} \rho) = \nabla(\partial_{x_i} \rho) \cdot \left( \nabla K * \tilde{p} + K * (\nabla V \tilde{p}) \right) + \nabla \rho \cdot \left( \nabla \partial_{x_i} K * \tilde{p} + \partial_{x_i} K * (\nabla V \tilde{p}) \right) \\
+ \partial_{x_i} \rho \cdot \left( \Delta K * \tilde{p} + \nabla K * (\nabla V \tilde{p}) \right) + \rho \left( \Delta \partial_{x_i} K * \tilde{p} + \nabla \partial_{x_i} K * (\nabla V \tilde{p}) \right).
\]

Notice that since \( \rho \in \mathcal{G}_k(0, T) \) with \( k \geq 2 \) the above equation holds in the space \( L^\infty((0,T);L^2(\mathbb{R}^d)) \). Now recall the function \( \phi_\delta(x) \) defined in (3.1). Then \( \phi_\delta(\partial_{x_i} \rho) \) satisfies the following equation in the weak sense:
\[
\partial_t(\phi_\delta(\partial_{x_i} \rho)) = \phi'_\delta(\partial_{x_i} \rho) \nabla(\partial_{x_i} \rho) \cdot \left( \nabla K * \tilde{p} + K * (\nabla V \tilde{p}) \right) \\
+ \phi'_\delta(\partial_{x_i} \rho) \nabla \rho \cdot \left( \nabla \partial_{x_i} K * \tilde{p} + \partial_{x_i} K * (\nabla V \tilde{p}) \right) + \phi'_\delta(\partial_{x_i} \rho) \partial_{x_i} \rho \cdot \left( \Delta K * \tilde{p} + \nabla K * (\nabla V \tilde{p}) \right) \\
+ \phi'_\delta(\partial_{x_i} \rho) \rho \left( \Delta \partial_{x_i} K * \tilde{p} + \nabla \partial_{x_i} K * (\nabla V \tilde{p}) \right).
\]

Next, by adapting the same arguments as in the proof of the uniqueness, we can multiply the above equation with \((1+V)\eta_R\) and then integrate on the whole space to get
\[(3.8) \quad \partial_t \int_{\mathbb{R}^d} (1+V) \eta_R \phi_\delta(\partial_{x_i} \rho) dx = \int_{\mathbb{R}^d} (1+V) \eta_R \nabla(\phi_\delta(\partial_{x_i} \rho)) \cdot \left( \nabla K * \tilde{p} + K * (\nabla V \tilde{p}) \right) dx \\
+ \int_{\mathbb{R}^d} (1+V) \eta_R \phi'_\delta(\partial_{x_i} \rho) \nabla \rho \cdot \left( \nabla \partial_{x_i} K * \tilde{p} + \partial_{x_i} K * (\nabla V \tilde{p}) \right) dx \\
+ \int_{\mathbb{R}^d} (1+V) \eta_R \phi'_\delta(\partial_{x_i} \rho) \partial_{x_i} \rho \cdot \left( \Delta K * \tilde{p} + \nabla K * (\nabla V \tilde{p}) \right) dx \\
+ \int_{\mathbb{R}^d} (1+V) \eta_R \phi'_\delta(\partial_{x_i} \rho) \rho \left( \Delta \partial_{x_i} K * \tilde{p} + \nabla \partial_{x_i} K * (\nabla V \tilde{p}) \right) dx =: \sum_{j=1}^{4} I_j(\delta, R).
\]

Using the fact that \( |\phi'_\delta| \leq 1 \) and that \( \eta_R \) is uniformly bounded, we have that
\[ I_j(\delta, R) \leq C(V)\|K\|_{2,\infty}\|\rho(t, \cdot)\|_{W^{1,1}_V}\|\tilde{p}(t, \cdot)\|_{L^1_V} \text{ for } j = 2, 3, 4. \]

For \( I_1(\delta, R) \), using integration by parts and the assumption (2.2) one obtains that
\[ I_1(\delta, R) = - \int_{\mathbb{R}^d} \phi_\delta(\partial_{x_i} \rho) \nabla \cdot \left( (1+V) \eta_R \left( \nabla K * \tilde{p} + \nabla K * (\nabla V \tilde{p}) \right) \right) \\
\leq C(V)\|K\|_{2,\infty}\|\tilde{p}(t, \cdot)\|_{L^1_V} \int_{\mathbb{R}^d} (1+V) \phi_\delta(\partial_{x_i} \rho). \]

Consequently, letting \( \delta \to 0 \) and \( R \to \infty \), we obtain from (3.8) that
\[ \partial_t \| (1 + V) \nabla \rho(t, \cdot) \|_{L^1} \leq C(V)\|K\|_{2,\infty}\|\tilde{p}(t, \cdot)\|_{L^1_V} \| (1 + V) \nabla \rho(t, \cdot) \|_{L^1}. \]

This combining with (3.7) yields
\[(3.9) \quad \| \rho(t, \cdot) \|_{W^{1,1}_V} \leq \exp \left( C(V)\|K\|_{2,\infty}\|\tilde{p}\|_{L^\infty((0,T);L^1_V)}^\dagger \right) \| \rho_0 \|_{W^{1,1}_V}. \]
Finally we derive an $H^k$-estimate for the solution. For doing so, let $\alpha$ be an multi-index such that $|\alpha| \leq k$. Taking $\partial^\alpha$ on the both sides of (3.4), multiplying the resulting equation with $\partial^\alpha \rho$ and then integrating gives

$$
\frac{1}{2} \partial_t \|\partial^\alpha \rho\|_{L^2(R^d)}^2 = \int_{R^d} \partial^\alpha (\nabla \cdot (\nabla K * \tilde{\nu})) \partial^\alpha \rho dx + \int_{R^d} \partial^\alpha (\rho \Delta K * \tilde{\nu}) \partial^\alpha \rho dx
$$

$$
+ \int_{R^d} \partial^\alpha (\nabla \rho \cdot K * (\nabla \tilde{\nu})) \partial^\alpha \rho dx
+ \int_{R^d} \partial^\alpha (\rho \nabla K * (\nabla \tilde{\nu})) \partial^\alpha \rho dx =: \sum_{i=1}^4 J_i.
$$

Note that by Leibniz rule and integration by parts,

$$
J_1 = \int_{R^d} \sum_{\beta < \alpha} \partial^\beta (\nabla \rho) \cdot (\nabla (\partial^{\alpha-\beta} K) * \tilde{\nu}) \partial^\alpha \rho dx - \frac{1}{2} \int_{R^d} |\partial^\alpha \rho|^2 \Delta K * \tilde{\nu} dx
$$

$$
\leq \sum_{\beta < \alpha} \|\partial^\beta (\nabla \rho)\|_{L^2} \|\nabla (\partial^{\alpha-\beta} K) * \tilde{\nu}\|_{L^\infty} \|\partial^\alpha \rho\|_{L^2} + \frac{1}{2} \|\partial^\alpha \rho\|_{L^2} \|\Delta K * \tilde{\nu}\|_{L^1}
$$

$$
\leq C(K) \|\rho\|_{H^k(R^d)}^2 \|\nabla \tilde{\nu}\|_{L^1}.
$$

Similarly, we have for $i = 2, 3, 4$,

$$
J_i \leq C(V, K) \|\rho\|_{H^k(R^d)}^2 \|\nabla \tilde{\nu}\|_{L^1}.
$$

Plugging the estimates into (3.10) and then summing over $\alpha$ with $|\alpha| \leq k$ gives

$$
\|\rho(t, \cdot)\|_{H^k(R^d)} \leq \exp \left( C(V, K) \|\nabla \tilde{\nu}\|_{L^\infty((0, T); L^1)} \right) \|\rho_0\|_{H^k(R^d)}.
$$

The estimate follows from (3.5) and (3.11). This finishes the proof of the lemma.

### 3.2.2. Proof of local existence.

We construct an approximating sequence $\{\rho^n\} \subset X_{0, V}(0, T)$ to the solution of (2.5). For doing so, let $\rho^n(t, \cdot) = \rho_0(\cdot)$ for all $t \in [0, T]$. Then for $n \geq 1$ we define $\rho^n$ by

$$
\partial_t \rho^n = \nabla \cdot (\rho^n (\nabla K * \rho^{n-1} + K * (\nabla V \rho^{n-1}))) \text{ in } (0, T) \times R^d,
$$

$$
\rho^n(0, \cdot) = \rho_0(\cdot).
$$

If we choose $T \leq \ln(2) / (C \|\nabla \tilde{\nu}\|_{L^\infty((0, T); L^1)})$ then it follows from estimate (3.5) that

$$
\|\rho^n\|_{L^\infty((0, T); X_{0, V})} \leq 2 \|\rho_0\|_{X_{0, V}}.
$$

In particular, we have

$$
\|\rho^n\|_{L^\infty((0, T); W^{1, 1}_V)} \leq 2 \|\rho_0\|_{X_{0, V}}.
$$

Combing above estimates together with Young’s convolution inequality leads to

$$
\|\partial_t \rho^n\|_{L^\infty((0, T); H^{k-1}(R^d))} \leq C \|\rho^n\|_{L^\infty((0, T); H^k(R^d))} \|K\|_{k+1, \infty} \|\rho^{n-1}\|_{L^\infty((0, T); W^{1, 1}_V)}
$$

$$
\leq C \|\rho^n\|_{L^\infty((0, T); H^k(R^d))} \|\rho_0\|_{X_{0, V}}
$$

$$
\leq C \|\rho_0\|_{X_{0, V}}^2.
$$
The above uniform bounds on $\rho^n$ imply that there exists a subsequence \( \{\rho^{n_m}\} \) and \( \rho \in \mathcal{X}^1_{k,V}(0,T) \) such that as \( m \to \infty \),

\[
\rho^{n_m} \xrightarrow{\ast} \rho \quad \text{in } L^\infty((0,T);\mathcal{Y}^1_V),
\]

\[
\partial_t \rho^{n_m} \xrightarrow{\ast} \partial_t \rho \quad \text{in } L^\infty((0,T);L^2(\mathbb{R}^d)),
\]

\[
\nabla K * \rho^{n_m} \xrightarrow{\ast} \nabla K * \rho \quad \text{in } L^\infty((0,T);W^{1,\infty}(\mathbb{R}^d)),
\]

\[
K * (\nabla V \rho^{n_m}) \xrightarrow{\ast} K * (\nabla V \rho) \quad \text{in } L^\infty((0,T);W^{1,\infty}(\mathbb{R}^d)),
\]

\[
\rho^{n_m}(0) \xrightarrow{\ast} \rho(0).
\]

In order to pass to the limit and show that the limit function $\rho$ satisfies (1.2), we also need to prove the following strong convergence results:

\[
\rho^n \to \rho \quad \text{in } L^\infty((0,T);L^2(\mathbb{R}^d)), \quad (3.13)
\]

\[
\nabla K * \rho^n \to \nabla K * \rho \quad \text{in } L^\infty((0,T);L^2(\mathbb{R}^d)), \quad (3.14)
\]

\[
K * (\nabla V \rho^n) \to K * (\nabla V \rho) \quad \text{in } L^\infty((0,T);L^2(\mathbb{R}^d)), \quad (3.15)
\]

To that end, let us define \( \omega^n := \rho^n - \rho^{n-1} \) with \( \omega^0 = 0 \). Then we have

\[
\partial_t(\omega^n) = \nabla \cdot (\omega^n(\nabla K * \rho^{n-1} + K * (\nabla V \rho^{n-1}))) + \nabla \cdot (\rho^n(\nabla K * \omega^{n-1} + K * (\nabla V \omega^{n-1}))). \quad (3.16)
\]

Similar to the proof of uniqueness, we can obtain that

\[
\partial_t \|\omega^n\|_{L^2_V} \leq C(k,V) \left( \|\rho^{n-1}\|_{L^\infty_V} + \|\rho^n\|_{W^{1,1}_V} \right) \|\omega^{n-1}\|_{L^1_V}.
\]

Therefore, thanks to the estimate (3.12), there exists a constant $\tilde{C}$ depending on \( K,V,T \) but independent of \( n \) such that

\[
\|\omega^n(t,\cdot)\|_{L^1_V} \leq \tilde{C} \int_0^t \|\omega^{n-1}(s,\cdot)\|_{L^1_V} ds.
\]

Iterating this estimate yields

\[
\|\omega^n(t,\cdot)\|_{L^1_V} \leq \frac{(\tilde{C}t)^n}{n!} \|\omega^1\|_{L^\infty((0,T);L^1_V)}, \quad \text{for a.e. } t \in (0,T). \quad (3.17)
\]

As a result, the sequence \( \{u^n\} \) is a Cauchy sequence in \( L^\infty((0,T);L^1_V) \). This together with the fact that the convolution $K*$ is a bounded operator from \( L^1(\mathbb{R}^d) \) to \( L^\infty(\mathbb{R}^d) \) as well as the assumption (2.2) proves (3.14)-(3.15). To show (3.13), by multiplying equation (3.16) with \( \omega^n \) and then integrating by parts, we can obtain

\[
\partial_t \|\omega^n\|_{L^2(\mathbb{R}^d)}^2 \leq C(k,V)(\|\omega^n\|_{L^2(\mathbb{R}^d)}^2 + \|\rho^n\|_{W^{1,1}_V} + \|\omega^n\|_{L^2(\mathbb{R}^d)} \|\rho^n\|_{W^{1,1}_V}) \|\omega^{n-1}\|_{L^1_V} \leq \tilde{C}(\|\omega^n\|_{L^2(\mathbb{R}^d)}^2 + \|\omega^{n-1}\|_{L^1_V}^2).
\]

where in reaching the second inequality we have used (3.12) and the Cauchy-Schwarz inequality. An application of Grönwäll’s inequality and the estimate (3.17) leads to

\[
\|\omega^n(t,\cdot)\|_{L^2(\mathbb{R}^d)}^2 \leq \tilde{C} e^{\tilde{C} t} \int_0^t \|\omega^{n-1}(s,\cdot)\|_{L^1_V}^2 ds \leq \frac{\tilde{C} 2n-1 e^{\tilde{C} t} t^{2n-1}}{(n-1)!(2n-1)} \|\omega^1\|_{L^\infty((0,T);L^1_V)}^2.
\]
This implies that the sequence \( \{\rho^n\} \) is also a Cauchy sequence in \( L^\infty((0, T); L^2(\mathbb{R}^d)) \), which proves (3.13). Hence we are eligible to pass to the limit and the limit \( \rho \) solves (2.5).

### 3.3. Global existence
We show that the local solution constructed in the last section is indeed a global solution, namely it does not blow up in finite time. In fact, by going through the proof of Lemma 3.3 with \( \bar{\rho} \) replaced by \( \rho \) in equation (3.4), one can show that the local solution satisfies the a priori estimate

\[
\|\rho(t, \cdot)\|_{\mathcal{X}_{k,V}^1} \leq \exp \left( C\|\rho\|_{L^\infty([0,T]; L^1)} t \right)\|\rho_0\|_{\mathcal{X}_{k,V}^1},
\]

where the constant \( C \) depends on \( V \) and \( K \) but not on \( t \). This together with the estimate (3.3) implies that

\[
\|\rho(t, \cdot)\|_{\mathcal{X}_{k,V}^1} \leq \exp \left( \exp(Ct) t \right)\|\rho_0\|_{\mathcal{X}_{k,V}^1},
\]

where \( C \) depends on \( V, K \) and \( \rho_0 \) but not on \( t \). This concludes the proof of global existence.

### 4. Convergence of the particle system
In this section we prove Theorem 2.5. First, in Section 4.1, we prove a few a priori estimates on the discrete particle system. Then in Section 4.2 we introduce the key ingredient to be used in the proof of Theorem 2.5 — the mean field characteristic flow.

#### 4.1. Estimates on the particle system
In this section, we first show that the ODE system (1.1) is well-defined. It is useful to introduce the Lyapunov function

\[
H_N(x) = \frac{1}{N} \sum_{i=1}^{N} V(x_i),
\]

where \( x = (x_1, x_2, \ldots, x_n) \).

**Lemma 4.1.** Let \( V \) satisfy Assumption 2.2. Then for any initial value \( x^0 = \{x^0_i\}_{i=1}^{N} \in \mathbb{R}^{dN} \) and any \( T > 0 \), the problem (1.1) has a unique solution \( x(t) = \{x_i(t)\}_{i=1}^{N} \in C^1([0, T]; \mathbb{R}^{dN}) \). Moreover, there exists a constant \( C \) depending only on \( V \) and \( K \) such that

\[
H_N(x(t)) \leq H_N(x^0) \cdot e^{Ct}.
\]

**Remark 4.2.** Notice that the estimate (4.1) can be regarded as a discrete analogue of the estimate (3.3) established in Lemma 3.1.

**Proof of Lemma 4.1.** First, since both \( K \) and \( V \) are smooth, the problem (1.1) has a unique solution up to some small time \( T_0 \). Next, we show that the solution does not blow up at finite time. This follows immediately from the estimate (4.1). In fact, if (4.1) is valid, then we know from Assumption 2.2 (A1) that \( x_i(t) \) remains bounded in \([0, T]\) for any \( T > 0 \), whence the solution can be extended up to any finite time. We are left to prove (4.1). In fact, by first differentiating \( V(x_i(t)) \) with respect to \( t \) and then summing over \( i \) gives

\[
\partial_t \left( \frac{1}{N} \sum_{i=1}^{N} V(x_i(t)) \right) = -\frac{1}{N^2} \sum_{i,j=1}^{N} \nabla K(x_i(t) - x_j(t)) \cdot \nabla V(x_i(t))
\]

\[
- \frac{1}{N^2} \sum_{i,j=1}^{N} K(x_i(t) - x_j(t)) \nabla V(x_i(t)) \cdot \nabla V(x_j(t)).
\]
Observe that the second term on the right side of above is non-positive since the matrix \( \{K(x_i - x_j)\}_{i,j=1}^N \) is positive definite by Assumption (2.1). Then it follows from the inequality in Assumptions (2.2) (A-2) and the fact that \( \nabla K \) is uniformly bounded that there exists a constant \( C = C(V,K) > 0 \) such that

\[
\partial_t \left( \frac{1}{N} \sum_{i=1}^N V(x_i(t)) \right) \leq \left| \frac{1}{N^2} \sum_{i,j=1}^N \nabla K(x_i(t) - x_j(t)) \cdot \nabla V(x_i(t)) \right|
\]

\[
\leq \frac{C}{N} \sum_{i=1}^N |\nabla V(x_i(t))|\]

\[
\leq \frac{C}{N} \sum_{i=1}^N V(x_i(t)).
\]

This proves (4.1).

Lemma 4.1 shows that the Lyapunov function \( H_N(x) \) of the finite particle system (1.1) remains finite for all time. We expect that for fixed \( N \), \( H_N(x) \) will remain uniformly bounded in time, although we have been able to prove this only with some further restrictions on \( V \), as the next lemma states.

**Lemma 4.3.** Fix \( N \geq 1 \). Suppose that for some \( p \geq 2 \) and \( m, R > 0 \), \( V(x) = m|x|^p \) if \( |x| > R \). Suppose also that \( K(0) > 0 \) and that \( |x|^{p-1}K(x) \) is bounded. Then \( H_N(x(t)) \) is uniformly bounded for \( t \in [0, \infty) \).

**Proof.** Observe that \( \partial_t H_N(x(t)) = -(S_1 + S_2)/N^2 \), where

\[
S_1 = \sum_{i,j=1}^N \nabla K(x_i(t) - x_j(t)) \cdot \nabla V(x_i(t)),
\]

\[
S_2 = \sum_{i,j=1}^N K(x_i(t) - x_j(t)) \nabla V(x_i(t)) \cdot \nabla V(x_j(t)).
\]

Because \( K \) is positive definite, we know that \( S_2 \geq 0 \). We wish to bound \( S_2 \) from below. For \( y \in \mathbb{R}^d \), let us define

\[
A(y) = \{ j \in \{1, \ldots, N\} \mid \nabla V(y) \cdot \nabla V(x_j) \geq |V(y)|^2/2 \}.
\]

We write \( S_2 \) as

\[
S_2 = \sum_{i=1}^N \sum_{j \in A(x_i)} K(x_i(t) - x_j(t)) |\nabla V(x_i(t))|^2 + \sum_{i=1}^N \sum_{j \notin A(x_i)} K(x_i(t) - x_j(t)) |\nabla V(x_i(t))|^2
\]

\[
\geq \sum_{i=1}^N \frac{1}{2} |\nabla V(x_i(t))|^2 \sum_{j \in A(x_i)} K(x_i(t) - x_j(t))
\]

\[
+ \sum_{i=1}^N \sum_{j \notin A(x_i)} K(x_i(t) - x_j(t)) \nabla V(x_i(t)) \cdot (\nabla V(x_j) - \nabla V(x_i(t))).
\]

Since

\[
|\nabla V(x_j) - \nabla V(x_i)| \leq \int_0^1 \|D^2 V(sx_j + (1-s)x_i)\| ds |x_j - x_i|,
\]
we have
\[ S_2 \geq \frac{1}{2} \sum_{i=1}^{N} \sum_{j \in A(x_i)} |\nabla V(x_i(t))|^2 K(x_i(t) - x_j(t)) \]
\[ - \sum_{i=1}^{N} |\nabla V(x_i(t))| \sum_{j \not\in A(x_i)} |x_j - x_i| K(x_i - x_j) R(x_i, x_j), \]
where
\[ R(a, b) = \int_0^1 \| D^2 V(sa + (1 - s)b) \| \, ds, \quad a, b \in \mathbb{R}^d. \]

By our assumptions on \( V \), we have
\[ R(a, b) \leq C + max(\| D^2 V(a) \|, \| D^2 V(b) \|) \leq C(1 + |a|^{p-2} + |b|^{p-2}). \]

Consequently, there is \( \alpha = (p-2)/(p-1) \in [0, 1) \) such that
\[ K(x_i, x_\ell) |x_\ell - x_i| R(x_\ell, x_i) \leq C(1 + |\nabla V(x_i)|^\alpha), \quad \forall \, \ell \not\in A(x_i). \]

Then
\[ S_2 \geq \frac{1}{2} \sum_{i=1}^{N} |\nabla V(x_i(t))|^2 \sum_{j \in A(x_i)} K(x_i(t) - x_j(t)) \]
\[ - \sum_{i=1}^{N} |\nabla V(x_i(t))|(1 + |\nabla V(x_i(t))|^\alpha)CN. \]

Since \( i \in A(x_i) \), the trivial bound
\[ \sum_{j \in A(x_i)} K(x_i(t) - x_j(t)) \geq K(0) > 0 \]
always holds. Applying Hölder’s inequality with exponents \( p = 2 \) and with \( (p, p^*) = (2/(1 + \alpha), 2/(1 - \alpha)) \) we obtain
\[ S_2 \geq \frac{1}{2} \sum_{i=1}^{N} |\nabla V(x_i(t))|^2 K(0) - \epsilon \sum_i |\nabla V(x_i(t))|^2 - \frac{1}{\epsilon} C^2 N^3 \]
\[ - \delta \sum_{i=1}^{N} |\nabla V(x_i(t))|^2 - \frac{1}{\delta^{(1+\alpha)/(1-\alpha)}} N(CN)^{2/(1-\alpha)}. \]

In particular, we may choose constants \( C_1, C_2 > 0 \) (dependent on \( K(0) \) and \( \alpha \)) so that
\[ S_2 = C_1 \sum_{i=1}^{N} |\nabla V(x_i(t))|^2 - C_2 N^{3/(1-\alpha)}. \]
The sum $S_1$ is bounded by

$$|S_1| = \left| \sum_{i,j=1}^{N} \nabla K(x_i(t) - x_j(t)) \cdot \nabla V(x_i(t)) \right|$$

$$\leq \frac{c}{\epsilon} \sum_{i=1}^{N} |\nabla V(x_i)|^2 + \frac{2}{\epsilon} \sum_{i} | \sum_{j} \nabla K(x_i - x_j)|^2.$$

Combining all these estimates, we obtain

$$\partial_t H_N(x(t)) \leq - \frac{C_1}{N^2} \sum_{i=1}^{N} |\nabla V(x_i(t))|^2 + C_2 N^{3/(1-\alpha)-2}$$

Since $|\nabla V|^2 \geq C V - C'$ holds for all $x$, for some constants $C, C'$, this implies

$$\partial_t H_N(x(t)) \leq - \frac{C_3}{N} H_N(x(t)) + C_4 N^{3/(1-\alpha)-2},$$

which implies that $H_N$ is uniformly bounded in $t$, for $N$ fixed.

Lemma 4.1 or Lemma 4.3 allows us to define the characteristic flow map

$$T_t : x^0 = \{x_i^0\}_{i=1}^{N} \in \mathbb{R}^{dN} \mapsto x(t) = \{x_i(t)\}_{i=1}^{N} \in \mathbb{R}^{dN}.$$

If $\mu_t^N$ is the empirical measure

$$\mu_t^N(dx) = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i(t)}, \quad t \geq 0,$$

then $\mu_t^N = (T_t)_# \mu_0^N$ is a weak solution of the Cauchy problem for the mean field PDE (1.2), namely

$$\partial_t \mu_t^N = \nabla \cdot (\mu_t^N (\nabla K * \mu_t^N + K * (\nabla V \mu_t^N))).$$

4.2. Mean field characteristic flow. Here we define the so-called the mean field characteristic flow for the PDE (1.2) (c.f [17]), which will play an essential role in the proof of the large particle limit of (1.1).

**Definition 4.4.** Given a probability measure $\nu$, we say that the map

$$X(t, x, \nu) : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$$

is a mean field characteristic flow associated to the particle system (1.1) or to the mean field PDE (1.2) if $X$ is $C^1$ in time and solves the following problem

$$\partial_t X(t, x, \nu) = -(\nabla K * \mu_t)(X(t, x, \nu)) - (K * (\nabla V \mu_t))(X(t, x, \nu)),$$

$$\mu_t = X(t, \cdot, \nu)_# \nu,$$

$$X(0, x, \nu) = x.$$
The expression $\mu_t = X(t, \cdot, \nu)_{#} \nu$ means that the measure $\mu_t$ is the push-forward of $\nu$ under the map $x \mapsto X(t, \cdot, \nu)$. We think of $\{X(t, \cdot, \nu)\}_{t \geq 0, \nu}$ as a family of maps from $\mathbb{R}^d$ to $\mathbb{R}^d$, parameterized by $t$ and $\nu$. For every $x^0 = \{x^0_i\}_{i=1}^N \in \mathbb{R}^{dN}$, the solution $x(t) = \{x_i(t)\}_{i=1}^N$ of the particle system (1.1) is linked to the mean field characteristic flow through

$$x_i(t) = X(t, x^0_i, \mu^0 N) \text{ or equivalently } \mu^N_t(dx) = (X(t, \cdot, \mu^0 N))_{#} \mu^0 N,$$

where $\mu^N_t$ is defined by (4.4).

We first prove in the theorem below that the mean field characteristic flow (4.5) is well-defined. To this end, let us define the function space

$$Y := \{u \in C(\mathbb{R}^d; \mathbb{R}^d) \mid \sup_{x \in \mathbb{R}^d} \frac{|u(x)|}{1 + |x|} < \infty \}.$$ 

Recall the space of measures $\mathcal{P}_V$ and the space of functions $\mathcal{D}^{1} k, V$ defined in (2.3) and (2.4).

**Theorem 4.5.** Assume the conditions of Theorem 2.5 hold. Let $x \in \mathbb{R}^d$ and $\nu \in \mathcal{P}_V$. For any $T > 0$, there exists a unique solution $X(t, x, \nu) \in C^1([0, T]; Y)$ to the problem (4.5).

**Proof.** We follow the proof of Theorem 1.3.2 in [17]. The proof of the theorem consists of two steps.

**Step 1 (local well-posedness):** We prove that given a small $r > 0$, there exists $T_0 > 0$ such that the problem (4.5) has a unique solution $X(t, x)$ in the following set

$$S = \{u \in C([0, T_0]; Y) \mid \sup_{t \in [0, T_0]} \|u(t, \cdot) - \cdot\|_V \leq r \}.$$ 

To see this, consider the integral formulation of (4.5) given by

$$X(t, x, \nu) = x - \int_0^t \int_{\mathbb{R}^d} \nabla K(X(s, x, \nu) - X(s, x', \nu)) \nu(dx') ds$$

$$- \int_0^t \int_{\mathbb{R}^d} K(X(s, x, \nu) - X(s, x', \nu)) \nabla V(X(s, x', \nu)) \nu(dx') ds.$$

Let us define the operator $\mathcal{F}: u(t, \cdot) \mapsto \mathcal{F}(u)(t, \cdot)$ by

$$\mathcal{F}(u)(t, x) := x - \int_0^t \int_{\mathbb{R}^d} \nabla K(u(s, x) - u(s, x')) \nu(dx') ds$$

$$- \int_0^t \int_{\mathbb{R}^d} K(u(s, x) - u(s, x')) \nabla V(u(s, x')) \nu(dx') ds.$$

Our goal is to show that $\mathcal{F}$ is a contraction in $S$. We first check that $\mathcal{F}$ maps $S$ into $S$. In fact, if $u \in S$, then for any $s \in [0, T_0]$ and $x' \in \mathbb{R}$,

$$|u(s, x')| \leq |x'| + |u(s, x') - x'| \leq |x'| + r(1 + |x'|) \leq r + (1 + r)|x'|.$$

Then according to Assumptions (2.2) (A3), there exists a positive constant $C = C(r)$ such that

$$|\nabla V(u(s, x'))| \leq C(1 + V(x')).$$
As a consequence, we have $F(u)(t, \cdot) \in Y$ since

$$|F(u)(t, x)| \leq |x| + tC\|K\|_{1,\infty} \int_{\mathbb{R}^d} (1 + V(x'))\nu(dx') \leq |x| + \tilde{C}t,$$

where we used the assumption that $\nu \in \mathcal{P}_V$. In addition, from the derivations above, one sees that

$$\sup_{t \in [0, T_0]} \|F(u)(t, \cdot) - \cdot\|_Y \leq \sup_{t \in [0, T_0]} \|F(u)(t, \cdot) - \cdot\|_\infty \leq \tilde{C}T_0 \leq r$$

if $T_0 \leq r/\tilde{C}$. Therefore $F$ maps from $S$ to $S$. Next, we show that $F$ is indeed a contraction. To see this, if $u, v \in S$, then

$$\begin{align*}
\sup_{t \in [0, T_0]} \|F(u)(t, \cdot) - F(v)(t, \cdot)\|_Y &\leq \int_0^{T_0} \left\| \int_{\mathbb{R}^d} \nabla K(u(s, x) - u(s, x')) - \nabla K(v(s, x) - v(s, x'))\nu(dx') \right\|_Y ds \\
&+ \int_0^{T_0} \left\| \int_{\mathbb{R}^d} \left[ K(u(s, x) - u(s, x')) - K(v(s, x) - v(s, x')) \right] \nabla V(u(s, x'))\nu(dx') \right\|_Y ds \\
&+ \int_0^{T_0} \left\| \int_{\mathbb{R}^d} K(v(s, x) - v(s, x')) \left[ \nabla V(u(s, x')) - \nabla V(v(s, x')) \right]\nu(dx') \right\|_Y ds.
\end{align*}$$

Note that the first term on the right side of above can be bounded from above by

$$T_0\|K\|_{2,\infty} \left(1 + \int_{\mathbb{R}^d} |x'|\nu(dx') \right) \sup_{t \in [0, T_0]} \|u(s, \cdot) - v(s, \cdot)\|_Y.$$

Thanks to (4.8) and (2.2), the second term can be bounded from above by

$$T_0C\|K\|_{1,\infty} \left(1 + \int_{\mathbb{R}^d} (1 + |x'|)\nabla V(u(s, x'))\nu(dx') \right) \sup_{t \in [0, T_0]} \|u(s, \cdot) - v(s, \cdot)\|_Y \leq T_0C\|K\|_{1,\infty} \left(1 + V(x') \right) \sup_{t \in [0, T_0]} \|u(s, \cdot) - v(s, \cdot)\|_Y.$$

To bound the last term on the right side of (4.10), using Assumption 2.2 (A2) one obtains that

$$\left| \nabla V(u(s, x')) - \nabla V(v(s, x')) \right| \leq \max_{\theta \in [0, 1]} |\nabla^2 V(\theta u(s, x') + (1 - \theta)v(s, x'))|(1 + |x'|) \sup_{t \in [0, T_0]} \|u(s, \cdot) - v(s, \cdot)\|_Y \leq C(1 + V(x')) \sup_{t \in [0, T_0]} \|u(s, \cdot) - v(s, \cdot)\|_Y,$$

where in the last inequality we have used the fact that $u, v \in S$ so that $u, v$ and $\theta u + (1 - \theta)v$ satisfy the inequality (4.8), which enables us to apply (A3) of Assumption 2.2. Plugging (4.11) into the integral of the last term on the right side of (4.10), we can bound the last term by

$$T_0C\|K\|_{\infty} \int_{\mathbb{R}^d} (1 + V(x'))\nu(dx') \sup_{t \in [0, T_0]} \|u(s, \cdot) - v(s, \cdot)\|_Y.$$

Combining the estimates above leads to
\[ \sup_{t \in [0,T_0]} \| \mathcal{F}(u)(t, \cdot) - \mathcal{F}(v)(t, \cdot) \|_Y \leq T_0 C \sup_{t \in [0,T_0]} \| u(t, \cdot) - v(t, \cdot) \|_Y. \]
which implies that \( \mathcal{F} \) is a contraction map in \( S \) when \( T_0 \) is small. By the contraction mapping theorem, \( \mathcal{F} \) has a unique fixed point \( X(t, x, \nu) \), that is \( X \) solves (4.7). After defining \( \mu_t = X(t, \cdot, \nu) \# \nu \), one sees that \( X(t, x, \nu) \) solves (4.5) in the small time interval \([0, T_0]\).

**Step 2 (Extension of local solution):** We show that the local solution can be extended up to any finite time. We prove this by establishing an a priori bound on the solution. In fact, by considering the evolution of the quantity \( 1 + V(X(t,x,\nu)) \), one has that
\[
\partial_t \int_{\mathbb{R}^d} \left( 1 + V(X(t,x,\nu)) \right) \nu(dx)
= - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \nabla V(X(t,x,\nu)) \cdot \nabla K(X(t,x,\nu) - X(t,x',\nu)) \nu(dx') \nu(dx)
- \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(X(t,x,\nu) - X(t,x',\nu)) \nabla V(X(t,x,\nu)) \cdot \nabla V(X(t,x',\nu)) \nu(dx') \nu(dx)
\leq C \|K\|_{1,\infty} \int_{\mathbb{R}^d} \left( 1 + V(X(t,x,\nu)) \right) \nu(dx).
\]
The last inequality follows from Assumption 2.2 (A3) and the fact that \( K \) is a reproducing kernel so that the third line above is non-positive. As a consequence,
\[ (4.12) \quad \int_{\mathbb{R}^d} \left( 1 + V(X(t,x,\nu)) \right) \nu(dx) \leq e^{C \|K\|_{1,\infty} t} \int_{\mathbb{R}^d} \left( 1 + V(x) \right) \nu(dx) \]
holds for all \( t > 0 \). With this a priori bound, one can iterate this argument to extend the local solution defined on \([0,T_0] \times \mathbb{R}^d \) to all of \([0,\infty) \times \mathbb{R}^d \). Finally, thanks to the integral formulation (4.7) \( \partial_t X \) is continuous on \([0,\infty) \times \mathbb{R}^d \). The proof is complete.

**4.3. Convergence of particle system.** We prove theorem 2.5 by following Dobrushin’s coupling argument [12, 17]. The crucial step is establish the stability of the mean field characteristic flow with respect to the initial probability measure, as shown in the theorem below.

**Theorem 4.6.** Let \( V \) satisfy Assumption 2.2 with \( q \in (1, \infty) \) in (A2). Let \( p \) be the conjugate index of \( q \), i.e. \( \frac{1}{p} + \frac{1}{q} = 1 \). Let \( R > 0 \). Assume that \( \nu_1, \nu_2 \) are two initial probability measures in \( \mathcal{P}_V \cap \mathcal{P}_p \), satisfying \( \| \nu_i \|_{\mathcal{P}_V} < R, \ i = 1,2 \). Let \( \mu_{i,t} = (Z_{i,\cdot,\nu_i})_{\cdot,\nu_i}. \) Then given any \( T > 0 \), there exists a constant \( C > 0 \) depending on \( K, V, R, p \) and \( T \) such that
\[ (4.13) \quad W_p(\mu_{1,t}, \mu_{2,t}) \leq C W_p(\nu_1, \nu_2) \ \forall t \in [0, T]. \]

The proof of Theorem 4.6 follows closely the proof of Theorem 1.4.1 of [17], which dealt with the flow (4.5) with \( V = 0 \). The stability estimate there was stated in terms of 1-Wasserstein distance, and mainly resulted from the Lipschitz condition of \( \nabla K \). However, we are only be able to prove the stability of mean field characteristic flow (4.5) in \( p \)-Wasserstein distance with \( p \) strictly larger than one. This is again due to the presence of the nonlinear drift term \( K * (\nabla V \mu_t) \) on the right side of (4.5).
Proof. Let \( \pi^0 \) be a coupling measure between the probability measures \( \nu_1 \) and \( \nu_2 \). Define for \( \delta > 0 \), \( \phi_{\delta}(x) = \frac{1}{p}|x|^2 + \delta^{p/2} \) to be an approximation to \( \frac{1}{p}|x|^p \). Given any two points \( x_1, x_2 \in \mathbb{R}^d \), we have from (4.5) that

\[
\partial_t \phi_{\delta} \left( X(t, x_1, \nu_1) - X(t, x_2, \nu_2) \right) = -\nabla \phi_{\delta} \left( X(t, x_1, \nu_1) - X(t, x_2, \nu_2) \right) \times \\
\times \left\{ \int_{\mathbb{R}^d} \nabla K(X(t, x_1, \nu_1) - X(t, x_1', \nu_1)) \nu_1(dx_1') \\
- \int_{\mathbb{R}^d} \nabla K(X(t, x_2, \nu_2) - X(t, x_2', \nu_2)) \nu_2(dx_2') \right\} \\
- \left\{ \int_{\mathbb{R}^d} K(X(t, x_1, \nu_1) - X(t, x_1', \nu_1)) \nabla V(X(t, x_1', \nu_1)) \nu_1(dx_1') \\
- \int_{\mathbb{R}^d} K(X(t, x_2, \nu_2) - X(t, x_2', \nu_2)) \nabla V(X(t, x_2', \nu_2)) \nu_2(dx_2') \right\} \\
= -\nabla \phi_{\delta} \left( X(t, x_1, \nu_1) - X(t, x_2, \nu_2) \right) \times \\
\times \left\{ \int_{\mathbb{R}^d} \left( \nabla K(X(t, x_1, \nu_1) - X(t, x_1', \nu_1)) \\
- \nabla K(X(t, x_2, \nu_2) - X(t, x_2', \nu_2)) \right) \pi^0(dx_1' dx_2') \right\} \\
+ \left\{ \int_{\mathbb{R}^d} \left( K(X(t, x_1, \nu_1) - X(t, x_1', \nu_1)) - K(X(t, x_2, \nu_2) - X(t, x_2', \nu_2)) \right) \right. \\
\times \nabla V(X(t, x_1', \nu_1)) \pi^0(dx_1' dx_2') \\
\left. + \int_{\mathbb{R}^d} K(X(t, x_2, \nu_2) - X(t, x_2', \nu_2)) \right) \right. \\
\left. \times \left( \nabla V(X(t, x_1', \nu_1)) - \nabla V(X(t, x_2', \nu_2)) \right) \pi^0(dx_1' dx_2') \right\} \\
=: I_1 + I_2 + I_3.
\]

Below we bound \( I_i \) individually. First, it is important to notice that

\[
|\nabla \phi_{\delta}(x)| = |(|x|^2 + \delta)^{p/2-1} x| \leq |x|^{p-1}.
\]

Then thanks to Assumption (2.1) on \( K \) and the fact that the inclusion \( L^p \rightarrow L^1 \) is bounded for \( p > 1 \), we have

\[
I_1 \leq \| K \|_{2, \infty} \left| X(t, x_1, \nu_1) - X(t, x_2, \nu_2) \right|^p \\
+ \| K \|_{2, \infty} \left| X(t, x_1, \nu_1) - X(t, x_2, \nu_2) \right|^{p-1} \\
\times \left( \int_{\mathbb{R}^d} \left| X(s, x_1', \nu_1) - X(s, x_2', \nu_2) \right|^p \pi(dx_1' dx_2') \right)^{1/p}.
\]
For $I_2$, it follows from Assumption 2.2 (A2)-(A3) and Hölder’s inequality that
\[
I_2 \leq \|K\|_{1,\infty} \left| X(t, x_1, \nu_1) - X(t, x_2, \nu_2) \right|^p \cdot \int_{\mathbb{R}^d} \left| \nabla V(X(t, x'_1, \nu_1)) \right| \nu_1(dx'_1) \\
+ \|K\|_{1,\infty} \left| X(t, x_1, \nu_1) - X(t, x_2, \nu_2) \right|^{p-1} \cdot \int_{\mathbb{R}^d} \left| X(t, x'_1, \nu_1) - X(t, x'_2, \nu_2) \right| \times \left| \nabla V(X(t, x'_1, \nu_1)) \right| \pi^0(dx'_1 dx'_2) \\
\leq \|K\|_{1,\infty} \left| X(t, x_1, \nu_1) - X(t, x_2, \nu_2) \right|^p \int_{\mathbb{R}^d} (1 + V(X(t, x'_1, \nu_1)))\nu_1(dx'_1) \\
+ \|K\|_{1,\infty} \left| X(t, x_1, \nu_1) - X(t, x_2, \nu_2) \right|^{p-1} \cdot \left( \int_{\mathbb{R}^d} \left| X(t, x'_1, \nu_1) - X(t, x'_2, \nu_2) \right| \pi^0(dx'_1 dx'_2) \right)^{1/p} \\
\times \left( \int_{\mathbb{R}^d} \left| \nabla V(X(t, x'_1, \nu_1)) \right| \mu^0(dx'_1) \right)^{1/q}.
\]
Observe that the integrals involving $V$ on the right side of above can be bounded in exactly the same way as (4.12). Hence we can obtain
\[
I_2 \leq e^{Ct} \|K\|_{1,\infty} \|\mu^0\|_{\varphi_V} \left( \left| X(t, x_1, \nu_1) - X(t, x_2, \nu_2) \right|^p \right) \\
+ \left| X(t, x_1, \nu_1) - X(t, x_2, \nu_2) \right|^{p-1} \cdot \left( \int_{\mathbb{R}^d} \left| X(t, x'_1, \nu_1) - X(t, x'_2, \nu_2) \right| \pi^0(dx'_1 dx'_2) \right)^{1/p}
\]
with the constant $C$ depending only on $V$. Finally, we find an upper bound for $I_3$. In fact, an application of the intermediate value theorem to the difference of $\nabla V$ and the inequality (2.1) of Assumption 2.2 (A-2) yields that
\[
I_3 \leq \|K\|_{\infty} \left| X(t, x_1, \nu_1) - X(t, x_2, \nu_2) \right|^{p-1} \int_{\mathbb{R}^d} \left| X(t, x'_1, \nu_1) - X(t, x'_2, \nu_2) \right| \\
\times \sup_{\theta \in [0,1]} \left| \nabla^2 V(\theta X(t, x'_1, \nu_1) + (1-\theta) X(t, x'_2, \nu_2)) \right| \pi^0(dx'_1 dx'_2) \\
\leq C_V \|K\|_{\infty} \left| X(t, x_1, \nu_1) - X(t, x_2, \nu_2) \right|^{p-1} \\
\times \left( \int_{\mathbb{R}^d} \left| X(t, x'_1, \nu_1) - X(t, x'_2, \nu_2) \right| \pi^0(dx'_1 dx'_2) \right)^{1/p} \\
\times \left( \int_{\mathbb{R}^d} \left| 1 + V(X(t, x'_1, \nu_1)) + V(X(t, x'_2, \nu_2)) \right| \pi^0(dx'_1 dx'_2) \right)^{1/q} \\
\leq e^{Ct} C_V \|K\|_{\infty} \left( \left| \mu^0_1 \right|_{\varphi_V} + \left| \mu^0_2 \right|_{\varphi_V} \right) \left| X(t, x_1, \nu_1) - X(t, x_2, \nu_2) \right|^{p-1} \\
\times \left( \int_{\mathbb{R}^d} \left| X(t, x'_1, \nu_1) - X(t, x'_2, \nu_2) \right| \pi^0(dx'_1 dx'_2) \right)^{1/p}.
\]
If we define
\[
D_p(\pi)(s) := \left( \int_{\mathbb{R}^d} \left| X(s, x'_1, \nu_1) - X(s, x'_2, \nu_2) \right| \pi(dx'_1 dx'_2) \right)^{1/p},
\]
then
then by combing the estimates above, we obtain that for any \( t \in [0, T] \),

\[
\phi_\delta(X(t, x_1, \nu_1) - X(t, x_2, \nu_2)) = \phi_\delta(x_1 - x_2) + \int_0^t \partial_s \phi_\delta\left(X(s, x_1, \nu_1) - X(s, x_2, \nu_2)\right) ds \\
\leq \phi_\delta(x_1 - x_2) + C(K, V)e^{CT}\left(\|\nu_1\|_{\mathcal{P}} + \|\nu_2\|_{\mathcal{P}}\right) \int_0^t \left(\|X(s, x_1, \nu_1) - X(s, x_2, \nu_2)\|_P^p + \|X(s, x_1, \nu_1) - X(s, x_2, \nu_2)\|_P^{-1} \cdot D_p(\pi^0)(s)\right) ds.
\]

Now integrating the above inequality with respect to the coupling \( \pi^0(dx_1dx_2) \), using the fact that

\[
\int_{\mathbb{R}^d} \|X(s, x_1, \nu_1) - X(s, x_2, \nu_2)\|_P^{-1} \pi^0(dx_1dx_2) \leq D_p^{-1}(\pi^0)(s)
\]

and finally letting \( \delta \to 0 \) yields

\[
D_p^p(\pi^0)(t) \leq D_p^p(\pi^0)(0) + C(K, V)e^{CT}\left(\|\nu_1\|_{\mathcal{P}} + \|\nu_2\|_{\mathcal{P}}\right) \int_0^t D_p^p(\pi^0)(s) ds.
\]

By the Grönwall’s inequality we obtain that

\[
D_p^p(\pi^0)(t) \leq D_p^p(\pi^0)(0) \exp\left(C(K, V)e^{CT}\left(\|\nu_1\|_{\mathcal{P}} + \|\nu_2\|_{\mathcal{P}}\right) t\right).
\]

Now since \( \pi^0 \in \Gamma(\nu_1, \nu_2) \) and \( \mu_{t, t} = (X(t, \cdot, \nu_t)\#\nu_t) \), the mapping

\[
\Xi_t : (x_1, x_2) \in \mathbb{R}^{2d} \mapsto (X(t, x_1, \nu_1), X(t, x_2, \nu_2)) \in \mathbb{R}^{2d}
\]

satisfies that \( (\Xi_t)\#\pi^0 \in \Gamma(\mu_{t, t}, \mu_{t, t}) \). As a consequence, we have that

\[
W_p^p(\mu_{t, t}, \mu_{t, t}) = \inf_{\pi \in \Gamma(\mu_{t, t}, \mu_{t, t})} \int_{\mathbb{R}^{2d}} |x_1 - x_2|^p \pi(dx_1dx_2) \\
\leq \inf_{\pi^0 \in \Gamma(\nu_1, \nu_2)} D_p^p(\pi^0)(t) \\
\leq \exp\left(C(K, V)e^{CT}\left(\|\nu_1\|_{\mathcal{P}} + \|\nu_2\|_{\mathcal{P}}\right) t\right) \cdot \inf_{\pi^0 \in \Gamma(\nu_1, \nu_2)} D_p^p(\pi^0)(0) \\
= \exp\left(C(K, V)e^{CT}\left(\|\nu_1\|_{\mathcal{P}} + \|\nu_2\|_{\mathcal{P}}\right) t\right) \cdot W_p^p(\nu_1, \nu_2).
\]

This finishes the proof. \( \square \)

With the stability result in Theorem 4.6, we are ready to prove Theorem 2.5.

**Proof of Theorem 2.5.** By the definition of the mean field characteristic flow, we know that \( \rho(t, dx) = (Z(t, \cdot, \nu_0)\#\nu_0) \), where \( \nu_0 = \rho_0(x) dx \). According to (4.6),

\[
\mu^N_t = (X(t, \cdot, \mu^N_0))\#\mu^N_0.
\]

Then the theorem follows directly from the Dobrushin’s stability estimate (4.13). \( \square \)

**5. Long time behavior of the solution of the mean field PDE.** In this section we prove Theorem 2.6. Recall that we assume the kernel is Gaussian:

\[
K(x) = K_1(x) := (4\pi\sigma)^{d/2}e^{-\frac{|x|^2}{4\sigma}}.
\]

For the sake of notational simplicity, from now on we assume that \( K(x) = K_1(x) \).
Proof of Theorem 2.6. To prove \( \rho_t \to \rho_\infty \) as \( t \to \infty \), we only need to prove that \( \rho_{t_k} \to \rho_\infty \) for any sequence \( t_k \nearrow \infty \). Indeed, suppose that the later is true and that \( \rho_t \) does not converge weakly to \( \rho_\infty \). Then there exists a constant \( \varepsilon > 0 \) and a bounded continuous function \( \varphi \), such that there exists a sequence \( t_k \nearrow \infty \) such that

\[
\left| \int_{\mathbb{R}^d} \rho_{t_k} \varphi dx - \int_{\mathbb{R}^d} \rho_\infty \varphi dx \right| \geq \varepsilon,
\]

which contradicts with the assumption. To prove \( \rho_{t_k} \to \rho_\infty \) for any sequence \( t_k \nearrow \infty \), according to [1, Theorem 2.6], it suffices to show that each subsequence of \( \{\rho_{t_k}\}_{k \in \mathbb{N}} \), still denoted by \( \{\rho_{t_k}\}_{k \in \mathbb{N}} \), has a further subsequence \( \{\rho_{t_{k_m}}\}_{m \in \mathbb{N}} \) converging weakly to \( \rho_\infty \). Below we divide our proof into three steps.

**Step 1:** Tightness of \( \{\rho_{t_k}\}_{k \in \mathbb{N}} \). In fact, since \( \rho_t \) solves (1.2), it is straightforward to check that

\[
\partial_t \text{KL}(\rho_t \parallel \rho_\infty) = -\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho_t(x) \rho_t(y) \nabla \log \left( \frac{\rho_t}{\rho_\infty} \right) \cdot \nabla \left( K(x-y) \cdot \nabla \log \left( \frac{\rho_t}{\rho_\infty} \right) \right) dxdy
\]

\[
= -\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( \nabla \rho_t + \nabla V \rho_t \right)(x) \cdot K(x-y) \cdot \left( \nabla \rho_t + \nabla V \rho_t \right)(y) dxdy
\]

\[
\leq 0,
\]

where the inequality follows from the fact that \( K(x-y) \) is positive definite. Furthermore, noticing that

\[
-\int_0^t \partial_s \text{KL}(\rho_s \parallel \rho_\infty) ds = \text{KL}(\rho_0 \parallel \rho_\infty) - \text{KL}(\rho_t \parallel \rho_\infty) < \infty,
\]

one can obtain that \( \partial_t \text{KL}(\rho_t \parallel \rho_\infty) \to 0 \) as \( t \to \infty \). As a result of (5.1), we have

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( \nabla \rho_t + \nabla V \rho_t \right)(x) \cdot K(x-y) \cdot \left( \nabla \rho_t + \nabla V \rho_t \right)(y) dxdy \to 0 \text{ as } t \to \infty.
\]

Since the relative entropy functional \( \rho \to \text{KL}(\rho \parallel \rho_\infty) \) has compact sub-level sets in the weak topology (see e.g. [13, Lemma 1.4.3]), it follows from \( \text{KL}(\rho_{t_k} \parallel \rho_\infty) \leq \text{KL}(\rho_t \parallel \rho_\infty) < \infty \) that \( \{\rho_{t_k}\}_{k \in \mathbb{N}} \) is tight. Consequently there exists a subsequence \( t_{k_m} \uparrow \infty \) and \( \bar{\rho} \in \mathcal{P}(\mathbb{R}^d) \) such that \( \text{KL}(\bar{\rho} \parallel \rho_\infty) < \infty \) and \( \rho_{t_{k_m}} \rightharpoonup \bar{\rho} \).

**Step 2:** We show that \( \bar{\rho} \) satisfies

\[
K_2 \ast (\nabla \bar{\rho} + \nabla V \bar{\rho}) = 0
\]

in the sense of tempered distribution. To this end, using Fourier transform and the fact that \( \tilde{K}_1 = K_2^2 \) we can write

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\nabla \rho_{t_{k_m}} + \nabla V \rho_{t_{k_m}})(x) \cdot K(x-y) \cdot (\nabla \rho_{t_{k_m}} + \nabla V \rho_{t_{k_m}})(y) dxdy
\]

\[
= \int_{\mathbb{R}^d} \tilde{K}_1(\xi) \left( \nabla \rho_{t_{k_m}} - \nabla V \rho_{t_{k_m}} \right)(\xi) d\xi
\]

\[
= \int_{\mathbb{R}^d} \left| \tilde{K}_2(\xi) (\nabla \rho_{t_{k_m}} + \nabla V \rho_{t_{k_m}})(\xi) \right| d\xi
\]

\[
= \|K_2 \ast (\nabla \rho_{t_{k_m}} + \nabla V \rho_{t_{k_m}})\|_{L^2(\mathbb{R}^d)}^2.
\]

Note that we are allowed to take the Fourier transform since \( \rho_{t_{k_m}} \in \mathcal{B}_{k,V} \) (c.f. Theorem 2.4). This together with (5.2) implies that \( K_2 \ast (\nabla \rho_{t_{k_m}} + \nabla V \rho_{t_{k_m}}) \to 0 \) in...
$L^2(\mathbb{R}^d)$. On the other hand, using $\rho_{tkm} \to \bar{\rho}$ with $\bar{\rho} \in \mathcal{P}(\mathbb{R}^d)$ and integration by parts, one sees that

$$K_2 * (\nabla \rho_{tkm} + \nabla V \rho_{tkm}) = \int_{\mathbb{R}^d} K_2(x-y)(\nabla \rho_{tkm} + \nabla V \rho_{tkm})(y)dy$$

$$= \int_{\mathbb{R}^d} \nabla K_2(x-y)\rho_{tkm}(y) + K_2(x-y)\nabla V (\rho_{tkm}(y)dy)$$

$$\to \int_{\mathbb{R}^d} \nabla K_2(x-y)\bar{\rho}(y) + K_2(x-y)\nabla V (\bar{\rho}(y)dy).$$

Therefore we have that $\int_{\mathbb{R}^d} \nabla K_2(x-y)\bar{\rho}(y) + K_2(x-y)\nabla V (\bar{\rho}(y)dy = 0 \ a.e. \ x \in \mathbb{R}^d.$

This in particular, implies that

$$f(x, \tau) = K_\tau * (\nabla \bar{\rho} + \nabla V \bar{\rho})(x).$$

Then from (5.3) we know that $f(\cdot, 2) = 0.\ \text{Moreover, for any} \ \tau \in (0, 2], \ f_\tau \in C^\infty(\mathbb{R}^d)$ and satisfies

$$|f_\tau(x)| \leq M_\tau e^{M_\tau |x|^2}.$$ 

Indeed, since $\bar{\rho} \in \mathcal{P}(\mathbb{R}^d)$,

$$|K_\tau * (\nabla \bar{\rho})(x)| \leq \|\nabla K_\tau * \bar{\rho}\|_{L^\infty(\mathbb{R}^d)} \leq \|\nabla K_\tau\|_{L^\infty(\mathbb{R}^d)} \|\bar{\rho}\|_{L^\infty(\mathbb{R}^d)} = \|\nabla K_\tau\|_{L^\infty(\mathbb{R}^d)}. \ (5.5)$$

Using the elementary inequality $|x-y|^2 \geq |y|^2/2 - |x|^2$ and Assumption 2.2 (A4) on $V,$ we obtain that

$$|K_\tau * (\nabla \bar{\rho})(x)| = \left| \int_{\mathbb{R}^d} K_\tau(x-y)\nabla V(y)\bar{\rho}(y)dy \right|$$

$$\leq \frac{1}{(4\pi \tau)^{d/2}} \left| \int_{\mathbb{R}^d} e^{-|\frac{y-x}{4\tau}|^2} \nabla V(y)\bar{\rho}(y)dy \right|$$

$$\leq \frac{1}{(4\pi \tau)^{d/2}} e^{\frac{|x|^2}{4\tau}} \left| \int_{\mathbb{R}^d} e^{-|\frac{y}{8\pi \tau}|^2} \nabla V(y)\bar{\rho}(y)dy \right|$$

$$\leq \frac{1}{(4\pi \tau)^{d/2}} e^{\frac{|x|^2}{4\tau}} \cdot \|e^{-|\frac{x}{4\tau}|^2} \nabla V(\cdot)\|_{L^\infty(\mathbb{R}^d)}.$$ 

Hence (5.4) follows from (5.5) and (5.6) with

$$M_\tau = \max \left( \|e^{-|\frac{x}{4\tau}|^2} \nabla V(\cdot)\|_{L^\infty(\mathbb{R}^d)}, \|\nabla K_\tau\|_{L^\infty(\mathbb{R}^d)}, \frac{1}{4\tau} \right).$$

The smoothness of $f_\tau$ follows from the smoothness and the decaying property of $K_\tau.$

Now if we define $u(\cdot, \tau) = f(\cdot, 2 - \tau),$ then it is clear that for any $\tau \in [0, 2 - \varepsilon]$ with $\varepsilon > 0,$ $u$ solves the backward heat equation $\partial_\tau u + \Delta u = 0$ with initial condition $u(\cdot, 0) = f(\cdot, 2 - \varepsilon) = 0,$ and $u$ satisfies the growth bound (5.4). An application of Lemma 5.1 yields $u(\cdot, \tau) \equiv 0$ for all $\tau \in [0, 2 - \varepsilon].$ In particular, we have for any $\varepsilon > 0,$

$$K_\varepsilon * (\nabla \bar{\rho} + \nabla V \bar{\rho}) = f(\cdot, \varepsilon) = u(\cdot, 2 - \varepsilon) = 0.$$
Now letting $\varepsilon \to 0$, we obtain that for any $\varphi \in C_0^\infty (\mathbb{R}^d)$,
\[
\langle (\nabla \bar{\rho} + \nabla V \bar{\rho}), \varphi \rangle = \lim_{\varepsilon \to 0} \langle (\nabla \rho^\varepsilon + \nabla V \rho^\varepsilon), K_\varepsilon \ast \varphi \rangle
= \lim_{\varepsilon \to 0} (K_\varepsilon \ast (\nabla \rho + \nabla V \bar{\rho}), \varphi)
= 0, 
\]
which implies that $\nabla (e^V \bar{\rho}) = 0$ in the sense of distribution. Therefore $\bar{\rho} = C \rho_\infty$ a.e. for some constant $C$. Finally since both $\bar{\rho}$ and $\rho_\infty$ are probability density, $C = 1$ and $\bar{\rho} = \rho_\infty$ a.e. This finishes the proof. \[\square\]

**Lemma 5.1.** \cite{15} A function $u$ satisfying $|\partial_t u + \Delta u| \leq M(|u| + |\nabla u|), |u(x, t)| \leq Me^{M|x|^2}$ in $(\mathbb{R}^d \setminus B_R) \times [0, T]$ and $u(x, 0) = 0$ for $x \in \mathbb{R}^d \subset B_R$ must vanishes identically in $(\mathbb{R}^d \setminus B_R) \times [0, T]$.

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