FUNDAMENTAL FUNCTION FOR
GRAND LEBESGUE SPACES.

E.Ostrovsky, L.Sirota

Department of Mathematic, Bar-Ilan University, Ramat Gan, 52900, Israel,
e-mails: eugostrovsky@list.ru, sirota3@bezeqint.net

Abstract.

We investigate in this short article the fundamental function for the so-called Grand Lebesgue Spaces (GLS) and show in particular a one-to-one and mutually continuous accordance between its fundamental and generating function.

Key words and phrases: Young-Orlicz function, ordinary and Grand Lebesgue Spaces (GLS); Orlicz, GLS norms, rearrangement invariant spaces, fundamental and generating function, Young-Fenchel, or Legendre transform, theorem of Fenchel-Moraux, inverse function, Exponential Orlicz function (EOF) and Spaces (EOS).

Mathematics Subject Classification (2000): primary 60G17; secondary 60E07; 60G70.

1 Notations. Statement of problem.

A. A triplet $(X, \mathcal{B}, \mu)$, where $X = \{x\}$ is arbitrary set, $\mathcal{B}$ is non-trivial certain sigma-algebra of subsets $X$ and $\mu$ is probabilistic: $\mu(X) = 1$ diffuse non-negative completely additive measure defined on the $\mathcal{B}$.

The non-probabilistic case $\mu(X) = \infty$ will be consider further.

Recall that the measure $\mu$ is said to be diffuse iff for arbitrary measurable set $A_1 \in \mathcal{B}$ with positive measure: $\mu(A_1) > 0$ there exists it subset $A_2 \subset A_1$ such that $\mu(A_2) = \mu(A_1)/2$.

We denote as usually for any arbitrary measurable function $f : X \to R$

$$|f|_p = \left[\int_X f(x)^p \mu(dx)\right]^{1/p}, \quad p \geq 1;$$

$L_p = \{f, \ |f|_p < \infty\}$.

B. The so-called Grand Lebesgue Space (GLS) $G\psi$ with norm $\|\cdot\|_{G\psi}$ is defined (not only in this article) as follows:

$$G\psi = \{f, \ |f|_{G\psi} < \infty\}, \quad |f|_{G\psi} \overset{def}{=} \sup_{p \geq 1} \left[\frac{|f|_p}{\psi(p)}\right].$$ (1.1)

Here $\psi = \psi(p)$, $1 \leq p < \infty$ is some continuous strictly increasing function such that $\lim_{p \to \infty} \psi(p) = \infty$. 




The detail investigation of this spaces (and more general spaces) see in [14], [19]. See also [5], [6], [8], [9], [10] etc.

The case when in (1.1) supremum is calculated over finite interval is investigated in [14], [19], [20]:

\[ G_b \psi = \{ f, \| f \| G_b \psi < \infty \}, \quad \| f \| G_b \psi \overset{\text{def}}{=} \sup_{1 \leq p < b} \left[ \frac{|f|_p}{\psi(p)} \right], \quad b = \text{const} > 1, \quad (1.2) \]

but in (1.2) \( \psi = \psi(p) \) is continuous function in the semi-open interval \( 1 \leq p < b \) such that \( \inf_{p \in (1,b)} \psi(p) > 0 \).

We will denote

\[ [1, b) := \text{supp } \psi(\cdot), \]

or simple \( b = b(\psi) := \text{supp } \psi(\cdot) \), including the case \( b = \infty \).

**Definition 1.1.** The function \( \psi(p) \) which appeared in (1.1) and (1.2), will be named as generating function for the correspondent Banach space \( G \psi \).

An used further example:

\[ \psi^{(\beta,b)}(p) = (b - p)^{-\beta}, \quad 1 \leq p < b, \quad \beta = \text{const} > 0; \quad b = \text{const} > 1, \]

\[ G_{\beta,b}(p) := G_b \psi^{(\beta,b)}(p). \]

C. We denote as ordinary for any measurable set \( A \), \( A \in \mathcal{B} \) its indicator function by \( I(A) = I_A(\omega) \).

D. The Grand Lebesgue Spaces \( \{ G \psi \} \) are rearrangement invariant in the classical definition, see e.g. [2], chapter 1. Therefore, its fundamental function \( \phi_{G(\psi)}(\delta), \delta \geq 0 \) is correctly defined in the considered case as follows:

\[ \phi_{G(\psi)}(\delta) \overset{\text{def}}{=} \sup_{p \in \text{supp } \psi} \left[ \frac{\delta^{1/p}}{\psi(p)} \right], \quad (1.3) \]

see [2], chapters 2 and 5.

For instance,

\[ \phi_{G(\psi)}(1) = \frac{1}{\inf_{p \in \text{supp } \psi} \psi(p)}. \quad (1.3a) \]

Note also

\[ \phi_{G(C \cdot \psi)} = \phi_{G(\psi)}/C, \quad C = \text{const} > 0. \quad (1.3b) \]

This notion play a very important role in the functional analysis, [2], [22], [23]; in the theory of interpolation of operators, [2], [5], [7], in the theory of probability [13], [15], [16], [17]; in the theory of Partial Differential equations [7], [9]; in the
theory of martingales [20]; in the theory of approximation, in the theory of random processes etc.

E. Let \( g = g(p), \ p \in (a,b), \ 1 \leq a < b \leq \infty \) be some numerical valued continuous strictly increasing (or decreasing) function. The *inverse* function will be denoted by \( g^{-1}(z), \ g(a) \leq z \leq g(b), \) in contradistinction to the usually notation \( g^{-1}(p) = 1/g(p). \)

F. The Young-Fenchel, or Legendre transform \( g^*(q) \) for the function \( g = g(p) \) one can to define
\[
g^*(q) \overset{def}{=} \sup_{p \in \text{supp}g} (p \ |q| - g(p)). \tag{1.4}
\]

Our goal in this short report is to establish a one-to-one and mutually continuous connection between the fundamental and generating functions for the Grand Lebesgue Spaces.

In some previous works: [6], [12], chapter 8; [14], [19], [18], [22] these function was evaluated and applied in many practical cases.

2 Main result.

Problem A. Let the generating function \( \psi \) be a given: \( \psi \in \mathcal{G}(a,b), \ 1 \leq a < b \leq \infty. \) Find the fundamental function for the correspondent Grand Lebesgue Space \( \mathcal{G}_\psi. \)

Suppose the function
\[
p \rightarrow \frac{p}{\psi(p)}, \ p \in (a,b)
\]
is strictly increasing; and define therefore the function
\[
\nu(p) = \nu_\psi(p) = \left[ \frac{p}{\psi(p)} \right]^{(-1)}, \ p \in \text{supp}\psi, \tag{2.1}
\]
and \( \nu(p) = +\infty \) otherwise.

Introduce also the following Young-Orlicz function
\[
N(u) = N_\psi(u) := \exp \left( \nu^*_\psi(u) \right) - \exp \left( \nu^*_\psi(0) \right), \tag{2.2}
\]
and define finally
\[
\theta(\delta) = \theta_\psi(\delta) \overset{def}{=} \frac{1}{N^{(-1)}(1/\delta)}, \ \delta > 0. \tag{2.3}
\]

Proposition 2.1. We propose under formulated above conditions, for instance, \( \mu(X) = 1, \) diffuseness of the measure \( \mu, \) and in the case when \( b = \infty \)
$\phi_{G(\psi)}(\delta) = \theta_{\psi}(\delta), \ \delta > 0. \quad (2.4)$

**Remark 2.1.** The equality (2.4) is more convenient than source definition (1.3). In particular, it allows for a relatively simple inversion.

**Proof** is very simple; it based on the computation of the fundamental function for Orlicz spaces, see the book of Krasnosel’skii M.A. and Rutickii Ya.B. [11], chapter 3; see also the classical monographs [26], [27].

In detail, it is proved in particular in the articles [14], [18], [19] that under our conditions the Grand Lebesgue Space $G\psi$ coincides with certain Orlicz space over source probability triplet $(X, B, \mu)$ relative the Young-Orlicz function $N_{\psi}(u)$.

We deduce reducing considered case to the well-known calculation of fundamental function for Orlicz space, [11], chapter 3

$$\phi_{G(\psi)}(\delta) = \frac{1}{N_{\psi}^{(-1)}(1/\delta)} = \theta_{\psi}(\delta), \ \delta > 0, \quad (2.4)$$

Q.E.D.

**An inverse problem B.** Let the fundamental function $\phi_{G\psi}(\delta) = \phi(\delta)$ be a given. Find the correspondent generating function $\psi(p)$.

A first restrictions: the function $\phi = \phi(\delta)$ is strictly increasing and continuous; in particular $\phi(0+) = \phi(0) = 0$.

We find from the equality (2.4)

$$N_{\psi}(1/\delta) = \left(\frac{1}{\phi(\delta)}\right)^{(-1)}, \quad (2.5)$$

or equivalently

$$N_{\psi}(z) = \left(\frac{1}{\phi(\delta)}\right)^{(-1)} / \delta = 1/z. \quad (2.5a)$$

A second restriction: the function

$$V^*(z) = \ln(C + N_{\psi}(z)), \ z \geq 0, \quad (2.6)$$

where $V^*(0) = \ln C$, is continuous and upward convex.

It follows immediately from (2.6) by virtue of theorem of Fenchel-Moraux

$$V(z) = \{\ln(C + N_{\psi}(z))\}^*, \ z \geq 0. \quad (2.7)$$

Since

$$V(p) = \left[\frac{p}{\psi(p)}\right]^{(-1)},$$

we derive finally

**Proposition 2.2.** We conclude under formulated in this pilcrow conditions
\[ \psi(p) = \frac{p}{V(-1)(p)}. \quad (2.8) \]

3 The case of infinite measure.

The case when \( \mu(X) = \infty \) is more complicated.

Recall first of all definition and some facts about the so-called Exponential Orlicz Spaces (EOS), see for example [21].

Let \( N = N(u) \) be an \( N - \) Young-Orlicz’s function, i.e. downward convex, even, continuous function differentiable for all sufficiently great values \( u, u \geq u_0, u_0 = \text{const} > 0 \), strongly increasing along the right semi-axis and such that \( N(u) = 0 \iff u = 0 \); \( u \to \infty \implies dN(u)/du \to \infty \). We can say that \( N(\cdot) \)

\( N(\cdot) \) is an exponential Orlicz function, briefly, \( N(\cdot) \in EOS \), if \( N(u) \) has a form of a continuous differentiable strongly increasing downward convex function \( W = W(u) \)

in the domain \([2, \infty]\) such that \( u \to \infty \implies W'(u) \to \infty \) and

\[ N(u) = N(W, u) = \exp(W(\log |u|)), \ |u| \geq e^2. \]

For the values \( u \in [-e^2, e^2] \) we define \( N(W, u) \) arbitrarily, but so that the function \( N(W, u) \) is even, continuous, convex, strictly increasing along the right semi-axis and so that \( N(u) = 0 \iff u = 0 \). We denote the correspondent Orlicz space on \((X, \mu)\) with a measure \( \mu \) and with \( N - \) function of the form \( N(W, u) \) as \( L(N) = EOS(W); EOS = \bigcup W \{EOS(W)\} \) (exponential Orlicz’s space).

For example, let \( m = \text{const} > 0 \), \( r = \text{const} \in \mathbb{R}^1 \),

\[ N_{m,r}(u) = \exp\left[|u|^m \left(\log^{-m/r}(C_1(r) + |u|)\right)\right] - 1, \]

\( C_1(r) = e, r \leq 0; C_1(r) = \exp(r), r > 0 \). Then \( N_{m,r}(\cdot) \in EOS \). In the case \( r = 0 \) we can write \( N_m = N_{m,0} \).

Recall that the Orlicz’s norm on the arbitrarily measurable space \((X, A, \mu)\)

\[ ||f||L(N) = ||f||L(N, X, \mu) \]

can be calculated by the following formula (see, for example, [11], p. 66; [26], p. 73)

\[ ||f||L(N) = \inf_{v > 0} \left\{ v^{-1} \left(1 + \int_X N(v|f(x)|) \mu(dx)\right) \right\}. \]

Let \( \alpha \) be arbitrary number, \( \alpha = \text{const} \geq 1 \), and \( N(\cdot) \in EOS(W) \) for some \( W = W(\cdot) \). For such a function \( N = N(W, u) \) we denote by \( N^{(\alpha)}(W; u) = N^{(\alpha)}(u) \)

a new Young-Orlicz’s function \( N^{(\alpha)}(u) \) such that

\[ N^{(\alpha)}(u) = C_1 |u|^{\alpha}, \ |u| \in [0, C_2]; \]

\[ N^{(\alpha)}(u) = C_3 + C_4 |u|, \ |u| \in (C_2, C_5]; \]

\[ N^{(\alpha)}(u) = N(u), \ |u| > C_5, \ 0 < C_2 < C_5 < \infty, \]

\[ C_{1,2,3,4,5} = C_{1,2,3,4,5}(\alpha, N(\cdot)). \]

5
In the case of $\alpha = m(j+1)$, $m > 0$, $j = 0, 1, 2, \ldots$ the function $N_{m}^{(\alpha)}(u)$ is equivalent to the following Trudinger’s function:

$$N_{m}^{(\alpha)}(u) \sim N_{[m]}^{(\alpha)}(u) = \exp (|u|^m) - \sum_{l=0}^{j} u^{ml}/l!.$$  

This method is described in [28], p. 42-47. These Orlicz spaces are applicable to the theory of non-linear partial differential equations.

We denote hereinafter generally by $C_k = C_k(\cdot)$, $k = 1, 2, \ldots$ some positive finite essentially constructive constants, and by $C, C_0$ non-essentially constants, also constructive. We proved the existence of constants $C_{1,2,3,4,5} = C_{1,2,3,4,5}(\alpha, N(\cdot))$ such that $N^{(\alpha)}$ is a new exponential $N$ Orlicz’s function in [21]. We denote classical absolute constants by the symbols $K_j$.

Now we introduce some new Grand Lebesgue Spaces. Let $\psi = \psi(p)$, $p \geq \alpha, \alpha = \text{const} \geq 1$ be a continuous positive $\psi(\alpha) > 0$ finite strictly increasing function such that the function $p \rightarrow p \log \psi(p)$ is downward convex, and

$$\lim_{p \rightarrow \infty} \psi(p) = \infty.$$  

We denote the set of all these functions by $\Psi$; $\Psi = \{\psi\}$. A particular case

$$\psi(p) = \psi(W; p) = \exp(W^*(p)/p),$$

where

$$W^*(p) = \sup_{z \geq \alpha}(pz - W(z))$$

is so-called Young-Fenchel, or Legendre transform of $W(\cdot)$. It follows from the theorem of Fenchel-Moraux that in this case

$$W(p) = [p \log \psi(W;p)]^*, \quad p \geq p_0 = \text{const} \geq 2,$$

and, consequently, for all $\psi(\cdot) \in \Psi$ we introduce a correspondent Young-Orlicz $N$ – function by the equality:

$$N([\psi]) = N([\psi], u) = \exp \{[p \log \psi(p)]^* (\log u)\}, \quad u \geq e^2.$$  

**Definition 3.1.** We introduce for such arbitrary function $\psi(\cdot) \in \Psi$ the so-called $G(\alpha; \psi)$ norms and correspondent Banach GLS space $G(\alpha; \psi)$ as a set of all measurable (complex) functions with finite norms:

$$||f||_G(\alpha; \psi) = \sup_{p \geq \alpha}(|f|_p/\psi(p)). \tag{3.2}$$

For instance, $\psi(p)$ may be $\psi(p) = \psi_m(p) = p^{1/m}$, $m = \text{const} > 0$; in this case, we can write $G(\alpha, \psi_m) = G(\alpha, m)$ and

$$||f||_G(\alpha, m) = \sup_{p \geq \alpha} \left(|f|_p \cdot p^{-1/m}\right).$$
Theorem A, see [21]. Let the measure $\mu$ be diffuse, $\mu(X) = \infty$, $\alpha = \text{const} \geq 1$ and $\psi \in \Psi$. We assert that the norms Orlicz-Luxemburg norm $\| \cdot \|_{L(N^{(\alpha)}, [\psi])}$ and Grand Lebesgue Space norm $\| \cdot \|_{G(\alpha, \psi)}$, $\alpha \geq 1$ are equivalent.

Arguing similarly to the second section, we obtain the following result.

Proposition 3.1. We propose under conditions of theorem A

$$
\phi_{G(\psi)}(\delta) \asymp \theta_{\psi}(\delta), \; \delta > 0.
$$

Remark 3.1. Note that this case $\delta \in (0, \infty)$, in contradiction to the proposition 2.1, where it in naturally to take $\delta \in (0, 1)$.

4 Concluding remarks. Open problems.

It is interest by our opinion to investigate the notion of fundamental function and also its relation with generating function for the so-called mixed, or equally anisotropic Grand Lebesgue Spaces.

Recall that the definition of mixed, or equivalently anisotropic ordinary Lebesgue Spaces appeared at first in the article [1] and was investigated in detail in the classical books [3], [4].

The anisotropic Grand Lebesgue Spaces as a slight generalization of $L_p$ spaces arises in turn in [23] with the correspondent fundamental function; in the preprint [24] both this notions was applied in the operator’s theory.

References

[1] Benedek A. and Panzone R. The space $L(p)$ with mixed norm. Duke Math.

[2] Bennet C., Sharpley R. Interpolation of operators. Orlando, Academic Press Inc., (1988).

[3] Besov O.V., Il’in V.P., Nikolskii S.M. Integral representation of functions and imbedding theorems. Vol.1; Scripta Series in Math., V.H.Winston and Sons, (1979), New York, Toronto, Ontario, London.

[4] Besov O.V., Il’in V.P., Nikolskii S.M. Integral representation of functions and imbedding theorems. Vol.2; Scripta Series in Math., V.H.Winston and Sons, (1980), New York, Toronto, Ontario, London.

[5] Capone C., Fiorenza A., Krbec M. On the Extrapolation Blowups in the $L_p$ Scale. Collectanea Mathematica, 48, 2, (1998), 71-88.

[6] Fiorenza A. Duality and reflexivity in grand Lebesgue spaces. Collectanea Mathematica (electronic version), 51, 2, (2000), 131-148.
[7] Fiorenza A., and Karadzhov G.E. *Grand and small Lebesgue spaces and their analogs*. Consiglio Nationale Delle Ricerche, Instituto per le Applicazioni del Calcoto Mauro Picine, Sezione di Napoli, Rapporto tecnico n. 272/03, (2005).

[8] Iwaniec T., and Sbordone C. *On the integrability of the Jacobian under minimal hypotheses*. Arch. Rat.Mech. Anal., 119, (1992), 129-143.

[9] Iwaniec T., P. Koskela P., and Onninen J. *Mapping of finite distortion: Monotonicity and Continuity*. Invent. Math. 144 (2001), 507-531.

[10] Kozachenko Yu. V., Ostrovsky E.I. (1985). *The Banach Spaces of random Variables of subgaussian type*. Theory of Probab. and Math. Stat. (in Russian). Kiev, KSU, 32, 43-57.

[11] Krasnosel’skii, M.A.; Rutickii, Ya.B. (1961). *Convex Functions and Orlicz Spaces*. Groningen: P.Noordhoff Ltd

[12] A. Kufner, O. John and S. Fucik. *Function Spaces*. Noordhoff International Publishingr, Leyden, 1977.

[13] Ledoux M., Talagrand M. (1991) *Probability in Banach Spaces*. Springer, Berlin, MR 1102015.

[14] Liflyand E., Ostrovsky E., Sirota L. *Structural Properties of Bilateral Grand Lebesgue Spaces*. Turk. Journal of Math., 34, (2010), 207-219.

[15] Ostrovsky E., Rogover E. *Exact exponential Bounds for the random field Maximum Distribution via the majorizing Measures (Generic Chaining)*. arXiv:0802.0349v1 [math.PR] 4 Feb 2008.

[16] Ostrovsky E.I. (1999). *Exponential estimations for random Fields and its applications*. (in Russian). Moscow-Obninsk, OINPE.

[17] Ostrovsky E.I. (1994.) *Exponential Bounds in the Law of Iterated Logarithm in Banach Space*. Math. Notes, 56, 5, p. 98-107.

[18] Ostrovsky E., Sirota L. *Fourier Transforms in exponential rearrangement invariant Spaces*. arXiv:040639v1 [math.FA], 20 Jun 2004.

[19] Ostrovsky E. and Sirota L. *Moment Banach spaces: theory and applications*. HIAT Journal of Science and Engineering, C, Volume 4, Issues 1-2, pp. 233-262, (2007).

[20] Ostrovsky E. and Sirota L. *Moment and Tail Inequalities for polynomial Martingales. The case of heavy tails*. arXiv: 1112.2768v1 [math.PR] 13 Dez 2011.

[21] Ostrovsky E. and Sirota L. *Fourier transforms in exponential rearrangement invariant spaces*. arXiv:math/0406391v1 [math.FA] 20 Jun 2004
[22] Ostrovsky E. *Exponential Orlicz Spaces: New Norms and Applications.* arXiv:math/0406534v1 [math.FA] 25 Jul 2004

[23] Ostrovsky E. and Sirota L. *Exact norm estimates for multivariate dilation operators between two Bilateral Weight Grand Lebesgue Spaces.* arXiv:1503.05235v1 [math.FA] 17 Mar 2015

[24] Ostrovsky E. and Sirota L. *Central Limit Theorem and exponential tail estimation in mixed (anisotropic) Grand Lebesgue Spaces.* arXiv:1308.5606v1 [math.PR] 26 Aug 2013

[25] Pizier G. *Condition d’entropic assupant la continuite de certains processus et applications a lanalyse harmonique.* Seminaire d’analyse fonctionelle. (1980), Exp.13, p. 23-34.

[26] Rao M.M., Ren Z.D. *Theory of Orlicz Spaces.* Marcel Dekker Inc., 1991. New York, Basel, Hong Kong.

[27] Rao M.M., Ren Z.D. *Applications of Orlicz Spaces.* Marcel Dekker Inc., 2002. New York, Basel, Hong Kong.

[28] Taylor M.E. *Partial Differential Equations.* v.3, Nonlinear Equations. Springer Verlag, 1991. Berlin, Heidelberg, New York.

[29] Talagrand M. (1996). *Majorizing measure: The generic chaining.* Ann. Probab., 24 1049-1103. MR1825156