Generalized Chern-Simons Form and Descent Equation

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Abstract

We present the general method to introduce the generalized Chern-Simons form and the descent equation which contain the scalar field in addition to the gauge fields. It is based on the technique in a noncommutative differential geometry (NCG) which extends the 4-dimensional Minkowski space $M_4$ to the discrete space such as $M_4 \times Z_2$ with two point space $Z_2$. However, the resultant equations do not depend on NCG but are justified by the algebraic rules in the ordinary differential geometry.

1 Introduction

The Chern-Simons theory \cite{1} has been extensively studied so far with great interests both for their theoretical interests as the topological quantum field theories \cite{2} and their practical applications for certain planar condensed matter phenomena such as the fractional quantum Hall effects and high temperature super conductivity \cite{3}, \cite{4}. Especially, three dimensional Chern-Simons theory depending on three dimensional Chern-Simons form \cite{5} provides a field theoretic framework for studying knots and links in three dimension. Furthermore, three dimensional gravity with a negative cosmological constant is described by two Chern-Simons theories \cite{7}. This approach \cite{8} makes it possible to exactly calculate the black hole entropy beyond the semi-classical calculations.

The occurrences of Yang-Mills anomalies and other topological terms such as axial anomaly, Schwinger terms and Chern characters are the important aspect of quantized gauge theories. Thus, the descent equations \cite{6} are very important because a series of these equations prescribe the relations between Yang-Mills anomalies.

Connes proposed the noncommutative geometry on the product space of the 4-dimensional Minkowski space \cite{10} and two point space $Z_2$. The Higgs boson field is regarded as a kind of gauge field on the discrete space $Z_2$ in this formulation. In fact, the Higgs boson has several similarities with the ordinary gauge fields such as the same type couplings with fermions and the trilinear and quartic self-couplings. The Higgs mechanism naturally works without assuming the Higgs potential leading to the spontaneous breakdown of gauge symmetry.

After the original formulation of NCG by Connes \cite{10}, many versions of NCG \cite{11} has appeared and succeeded to reconstruct the spontaneously broken gauge theories. Morita and the present author \cite{12} proposed the generalized differential geometry (GDG) on the discrete space $M_4 \times Z_2$ and reconstructed the Weinberg-Salam model. In this formulation on $M_4 \times Z_2$ the extra differential one-form $\chi$ is introduced in addition to the usual one-form $dx^\mu$ and so our formalism is the generalization of the ordinary differential geometry on the continuous manifold. This formulation was generalized to GDG on the discrete space $M_4 \times Z_N$ \cite{13}, \cite{14} by introducing the extra one-forms $\chi_k (k = 1, 2 \cdots N)$, which generalization enabled us to reconstruct the left-right symmetric gauge theory, SU(5) GUT and SO(10) GUT.

From the standpoint of NCG, the Higgs boson is a gauge field of the principal bundle on the discrete space. Thus, it is expected that there exist the Chern-Simons forms and descent equations including the scalar boson field in addition to the ordinary gauge field. In this letter, we address this problem and present the general method to introduce these generalized Chern-Simons form and descent equations by use of the technique in NCG. It should be noted that we use NCG but the resultant formulas are free from NCG and are justified by the direct calculations in the ordinary differential geometry.

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2 Differential geometry on the product space $M_N \times Z_2$

The generalized gauge field is defined on the product space $M_N \times Z_2$ with $N$-dimensional Minkowski space $M_N$ and the two points space $Z_2$ as

$$A(x) = \begin{pmatrix} A_1(x) & H_{12}(x) \chi_2 \\ H_{21}(x) \chi_1 & A_2(x) \end{pmatrix},$$

(2.1)

where $A_1(x) = -A_1^\dagger(x)$ and $A_2(x) = -A_2^\dagger(x)$ are gauge fields belonging to the self-adjoint representations of unitary gauge groups $G_1$ and $G_2$, respectively and $H_{12}(x) = H_{21}(x)^\dagger$ is a scalar field belonging to the covariant representation of $G_1$ and $G_2$. We do not call $H_{12}$ the Higgs boson field because its vacuum expectation value is irrelevant to our formulation. In addition, $A_1(x) = A_1^\mu(x) dx_\mu$, $A_2(x) = A_2^\mu(x) dx_\mu$ and $\chi_1$ and $\chi_2$ are one-form base on the discrete space $Z_2$ which satisfy the following algebraic rules.

$$dx_\mu \wedge dx_\nu = -dx_\nu \wedge dx_\mu, \quad dx_\mu \wedge \chi_k = -\chi_k \wedge dx_\mu, \quad \chi_k \wedge \chi_l = -\chi_l \wedge \chi_k,$$

(2.2)

with $k, l = 1, 2$. We abbreviate the argument $x$ in the field hereafter except for the case necessary to write. Let us first address the gauge transformation of $A$ with the gauge function

$$g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix},$$

(2.3)

where $g_1 \in G_1$ and $g_2 \in G_2$. It is expressed as

$$A^g = g^{-1} dg + g^{-1} A g,$$

(2.4)

where the operator $d = \partial^\mu dx_\mu$ is the exterior derivative on the space $M_N$. Equation (2.3) brings the gauge transformations of gauge and scalar fields.

$$A_i^g = g_i^{-1} d g_i + g_i^{-1} A_i g_i, \quad H_{12}^g = g_1^{-1} H_{12} g_2.$$

(2.5)

The generalized field strength $F$ is defined as usual and expressed as

$$F = d A + A \wedge A,$$

(2.6)

which is written in components as

$$F = \begin{pmatrix} F_1 + H_{12}H_{21}\chi_2 \wedge \chi_1 & \mathcal{D}H_{12} \chi_2 \\ \mathcal{D}H_{21} \chi_1 & F_2 + H_{21}H_{12}\chi_1 \wedge \chi_2 \end{pmatrix},$$

(2.7)

where $F_1$ and $F_2$ are the field strength of gauge fields $A_1$ and $A_2$, respectively and $\mathcal{D}H_{12}$ and $\mathcal{D}H_{21}$ are the covariant derivatives of the scalar field. Equations of those quantities are written as

$$F_i = d A_i + A_i \wedge A_i, \quad \mathcal{D}H_{12} = d H_{12} + A_1 H_{12} - H_{12} A_2, \quad \mathcal{D}H_{21} = d H_{21} + A_2 H_{21} - H_{21} A_1.$$

(2.8)

According to Eq.(2.4), we can easily find the generalized field strength transformed covariantly under the gauge transformation;

$$F^g = g^{-1} F g,$$

(2.9)

which is written in components as

$$F_i^g = g_i^{-1} F_i g_i, \quad \mathcal{D}H_{12}^g = g_1^{-1} \mathcal{D}H_{12} g_2, \quad \mathcal{D}H_{21}^g = g_2^{-1} \mathcal{D}H_{21} g_1.$$

(2.10)

The generalized field strength defined in Eq.(2.4) satisfies the Bianchi Identity

$$\mathcal{D} F = d F + A F - F A = 0,$$

(2.11)

which is easily proved by use of the algebraic rule in the differential geometry and very important frequently used hereafter. It should be noted that the Bianchi Identity in Eq.(2.10) does not yield any restriction between gauge field $A_i$ and scalar field $H_{12}$.
3 Generalized Chern-Simons form

In order to introduce the generalized Chern-Simons form, we use the Cartan’s homotopy formula

$$P(A, F) = (kd + dk)P(A, F),$$ (3.1)

where $P(A, F)$ is an arbitrary function of $A$ and $F$. The operator $k$ is defined through the equation

$$kP(A, F) = \int_0^1 dt k_t P(A_t, F_t),$$ (3.2)

where $A_t = tA$ and $F_t = tdA + t^2 A \wedge A$. The operator $k_t$ in Eq.(3.2) is an anti-differential operator and is defined as

$$k_t A_t = 0, \quad k_t F_t = tA,$$ (3.3)

through which the identity

$$\frac{\partial}{\partial t} = k_t d + dk_t$$ (3.4)

follows. Equation (3.4) justifies the Cartan’s homotopy formula together with Eq.(3.2).

It is easily seen that the quantity $\text{Tr} F^n$ is invariant under the gauge transformation and satisfies the equation

$$d \text{Tr} F^n = \text{Tr} (dF^n + A F^n - F^n A) = 0,$$ (3.5)

which is proved from the Bianchi Identity Eq. (2.11). Putting $P(A, F) = \text{Tr} F^{n+1}$ in Eq.(3.1), we find the transgression formula $[5]$ $\omega_{2n+1}(A, F)$ is the generalized Chern-Simons form and written as

$$\omega_{2n+1}(A, F) = k \text{Tr} F^{n+1} = (n + 1) \int_0^1 dt A F^n_t.$$ (3.6)

Paying attention on $F_t = tF + (t^2 - t)A^2$, we obtain, after calculations of integral over $t$ for $n = 1$ and $n = 2$,

$$\omega_3(A, F) = \text{Tr} \left( A F - \frac{1}{3} A^3 \right),$$ (3.8)

$$\omega_5(A, F) = \text{Tr} \left( A F^2 - \frac{1}{2} A^3 F + \frac{1}{10} A^5 \right).$$ (3.9)

Equations (3.8) and (3.9) have the same forms as the ordinary Chern-Simons forms for $n = 2$ and $n = 3$, respectively but it should be noted that the generalized gauge field $A$ and field strength $F$ are expressed in matrix forms and contain the scalar field as in Eqs.(2.1) and (2.7).

Let us investigate in more detail the generalized Chern-Simons form $\omega_{2n+1}$ for $n = 1$ and $n = 2$. Inserting $A$ and $F$ in Eqs.(2.1) and (2.7) into Eq.(3.8), we find

$$\omega_3 = \omega_3^1 + \omega_3^2 + \omega_3^3 \chi_2 \wedge \chi_1,$$ (3.10)

where

$$\omega_3^1 = \text{Tr} \left( A_i F_i - \frac{1}{3} A_i^3 \right),$$

$$\omega_3^2 = \text{Tr} \left( DH_{12} \cdot H_{21} - H_{12} \cdot DH_{21} \right).$$ (3.11)

$\omega_3^i (i = 1, 2)$ is the Chern-Simons form for the ordinary gauge field. $\omega_3^3$ is the new type Chern-Simons form containing the scalar field $H_{12}$. The transgression formula for this Chern-Simons form $\omega_3^3$ is written as

$$d \omega_3^3 = 2 \text{Tr} \left( F_1 H_{12} H_{21} - H_{12} F_2 H_{21} - DH_{12} \cdot DH_{21} \right).$$ (3.12)

When $A_2 = 0$, Eq.(3.12) leads to

$$d \text{Tr} \left( DH_{12} \cdot H_{21} - H_{12} \cdot DH_{21} \right) = 2 \text{Tr} \left( F_1 H_{12} H_{21} - DH_{12} \cdot DH_{21} \right),$$ (3.13)
where $DH_{12} = dH_{12} + A_1 H_{12}$ and $DH_{21} = dH_{21} - H_{21} A_1$. In this case, $H_{12}$ belongs to the fundamental representation of the group $G_1$. Inserting $A$ and $F$ in Eqs. (2.1) and (2.7) into Eq. (2.8), we find
\[
\omega_5 = \omega_5^1 + \omega_5^2 + \omega_5' \chi_2 \land \chi_1, \tag{3.14}
\]
where
\[
\omega_5^i = \text{Tr} \left( A_1 F_i^2 - \frac{1}{2} A_i F_i + \frac{1}{10} A_i^7 \right),
\]
\[
\omega_5' = 3 \text{Tr} (F_1 DH_{12} \cdot H_{21}) - 3 \text{Tr} (F_2 DH_{21} \cdot H_{12})
\]
\[
- \frac{3}{2} d \text{Tr} (F_1 H_{12} H_{21}) + \frac{3}{2} d \text{Tr} (F_2 H_{21} H_{12}) - \frac{1}{2} d \text{Tr} (A_1 H_{12} A_2 H_{21})
\]
\[
- \frac{1}{2} d \text{Tr} \{ A_1 (DH_{12} \cdot H_{21} - H_{21} \cdot DH_{21}) \} + \frac{1}{2} d \text{Tr} \{ A_2 (DH_{21} \cdot H_{12} - H_{12} \cdot DH_{12}) \}. \tag{3.15}
\]
\(
\omega_5^i (i = 1, 2)
\)
is the Chern-Simons form for the ordinary gauge field. $\omega_5'$ is the new type Chern-Simons form containing the scalar field $H_{12}$. The transgression formula for this Chern-Simons form $\omega_5'$ is written as
\[
d \omega_5' = 3 \text{Tr} \left( F_1^2 H_{12} H_{21} - F_1 DH_{12} \cdot DH_{21} \right) - 3 \text{Tr} \left( F_2^2 H_{21} H_{12} - F_2 DH_{21} \cdot DH_{12} \right). \tag{3.16}
\]
We can find from this equation much more compact equation that
\[
d \text{Tr} F_1 DH_{12} \cdot H_{21} = \text{Tr} \left( F_1^2 H_{12} H_{21} - F_1 H_{12} F_2 H_{21} - F_1 DH_{12} \cdot DH_{21} \right), \tag{3.17}
\]
and the same equation replacing 1 by 2 and 2 by 1 also follows. When $A_2 = 0$, Eq. (3.16) leads to
\[
d \text{Tr} \left( F_1 DH_{12} \cdot H_{21} \right) = \text{Tr} \left( F_1^2 H_{12} H_{21} - F_1 DH_{12} \cdot DH_{21} \right). \tag{3.18}
\]
These transgression formulas are easily justified by the direct calculations according to algebraic rules in the differential geometry.

4 **Generalized descent equation**

In order to introduce the generalized descent equation [1], we incorporate the ghost field [15] in the generalized gauge field $A$.

\[
A^C(x, \theta) = A(x, \theta) + C(x, \theta) = \left( \begin{array}{c}
A_1(x, \theta) + C_1(x, \theta) d\theta \\
H_{21}(x, \theta) \chi_1 \\
A_2(x, \theta) + C_2(x, \theta) d\theta
\end{array} \right), \tag{4.1}
\]

where $C_1(x, \theta)$ and $C_2(x, \theta)$ are ghost fields belonging to the adjoint representation of the groups $G_1$ and $G_2$, respectively and $\theta$ is an argument of Grassmann number in ghost space. $C_i(x, \theta) (i = 1, 2)$ is anti-Hermitian. The generalized field strength for $A^C$ is given as
\[
F^C = dA^C + A^C \land A^C, \tag{4.2}
\]
where
\[
\begin{align*}
d &= d + d\theta, \\
d &= \partial^\mu dx_\mu, \\
d\theta &= \partial_\theta d\theta, \\
dx_\mu \land d\theta &= -d\theta \land dx_\mu, \\
\chi_i \land d\theta &= -d\theta \land \chi_i, \\
d\theta \land d\theta &\neq 0, \quad \partial_\theta^2 = 0. \tag{4.3}
\end{align*}
\]
Therefore, it is easy to see that the exterior derivative $d$ satisfies the nilpotency $d^2 = 0$. According to the nilpotency of $d$ and Eq. (4.3), the Bianchi Identity for $F^C$
\[
dF^C + [A^C, F^C] = 0 \tag{4.4}
\]
follows. By applying the horizontality condition [17] to $F^C$,
\[
F^C(x, \theta)|_{\theta=0} = F(x) \tag{4.5}
\]
we find the BRST transformations for fields involved.

\[
\delta \theta A_i = dC_i + A_i C_i - C_i A_i = D C_i,
\]
\[
\delta \theta H_{12} = H_{12} C_2 - C_i H_{12},
\]
\[
\delta \theta C_i = -C_i^2,
\]
(4.6)

where the operator \( \delta \) stands for the BRST transformation.

According to the nilpotency of \( d \) and the Bianchi Identity for \( F^C \) in Eq.(4.4), we obtain the transgression formula for \( A^C \) and \( F^C \) same as in Eq.(3.6).

\[
\text{Tr} F^{C,n+1} = d \omega_{n+1} (A^C, F^C),
\]
(4.7)

If we consider the horizontarity condition \([17]\) in Eq.(4.5), the equation

\[
d \omega_{2n+1}(A^C, F^C)|_{\theta = 0} = d \omega_{2n+1}(A, F)
\]
(4.8)

follows. Here,

\[
\omega_{2n+1}(A^C, F^C) = (n + 1) \int_0^1 dt A^C F^{C,n}_t,
\]
(4.9)

where

\[
F^C_t = t d A^C + i^2 A^C \wedge A^C.
\]
(4.10)

By use of Eq.(4.1), we expand \( \omega_{2n+1}(A^C, F^C) \) in power of the ghost field \( C \) as

\[
\omega_{2n+1} = \omega_{2n+1}^0 + \omega_{2n+1}^1 + \omega_{2n+1}^2 + \cdots + \omega_{2n+1}^{2n+1},
\]
(4.11)

where the superscript of \( \omega \) in the right hand side stands for the power of the ghost field \( C \) and the subscript stands for the degree of the form \( dx_\mu \). Here, \( \omega_{2n+1}^0 \) is the Chern-Simons form. From Eq.(4.8), we find the generalized descent equation

\[
d \theta \omega_{2n+1}^0 + d \omega_{2n+1}^1 = 0,
\]
\[
d \theta \omega_{2n+1}^1 + d \omega_{2n-1}^2 = 0,
\]
\[
d \theta \omega_{2n-1}^2 + d \omega_{2n-2}^3 = 0,
\]
\[
\vdots
\]
\[
d \theta \omega_{1}^{2n} + d \omega_{0}^{2n+1} = 0,
\]
\[
d \theta \omega_{0}^{2n+1} = 0.
\]
(4.12)

\( \omega_{2n}^1 \) is a solution of the Wess-Zumino consistency condition \([3]\)

\[
\Delta = \int \Omega, \quad d \theta \Omega = d \mathcal{B},
\]
(4.13)

where \( \Delta \) is anomaly term, \( \Omega \) is a 4-form with ghost number 1 and \( \mathcal{B} \) is a 3-form. \( \omega_{2n}^1 \) is written as

\[
\omega_{2n}^1 = n(n + 1) \int_0^1 dt(1 - t) \text{STr}(C d A F^{n-1}_t),
\]
(4.14)

where \( \text{STr} \) stands for the symmetrized trace. We obtain for \( n = 1, 2, 3 \)

\[
\omega_2 = \text{Tr} (C d A),
\]
\[
\omega_4 = \text{Tr} \left\{ C d \left( A d A + \frac{1}{2} A^3 \right) \right\},
\]
\[
\omega_6 = \text{Tr} \left\{ C d \left( A d A \cdot d A + \frac{3}{5} A d A \cdot A^2 + \frac{3}{5} A^3 d A \cdot A^2 + \frac{2}{5} A^5 \right) \right\},
\]
(4.15)

where the notation of the wedge product is abbreviated. It should be noted that these equations are same in form as in the ordinary case without the scalar field. However, the gauge field \( A \) written in Eq.(2.1)
includes the scalar field $H_{12}$. We extract the term containing the scalar field from Eq. (4.15). $\omega_1'$ does not include the scalar field.

$$\omega_1' = \text{Tr} \, C_1 \, d \left\{ -H_{12} \, d \, H_{21} + \frac{1}{2} \left( A_1 \, H_{12} \, H_{21} - H_{12} \, A_2 \, H_{21} + H_{12} \, H_{21} \, A_1 \right) \right\} - \text{Tr} \, C_2 \, d \left\{ -H_{21} \, d \, H_{12} + \frac{1}{2} \left( A_2 \, H_{21} \, H_{12} - H_{21} \, A_1 \, H_{12} + H_{21} \, H_{12} \, A_2 \right) \right\}.$$  
(4.16)

$\omega_6'$ has the complicated terms containing the scalar field and thus, we write it in the case of $A_2 = C_2 = 0$.

$$\omega_6' = \text{Tr} \, C_1 \, d \left\{ -A_1 \, d \, H_{12} \cdot d \, H_{21} - H_{12} \cdot d \, H_{21} \, d \, A_1 \
+ \frac{3}{5} \left( A_1 \, d \, A_1 \cdot H_{12} \, H_{21} + A_1 \, d \, H_{12} \cdot H_{21} \, A_1 + H_{12} \, d \, H_{21} \, A_1 \right) \
+ \frac{3}{5} \left( A_1 \, H_{12} \, H_{21} \, d \, A_1 + H_{12} \cdot H_{21} \, A_1 \, d \, A_1 - A_1^2 \, H_{12} \, d \, H_{21} \right) \
+ \frac{2}{5} \left( A_1^3 \, H_{12} \, H_{21} + A_1^2 \, H_{12} \, H_{21} \, A_1 + H_{12} \, H_{21} \, A_1^3 + A_1 \, H_{12} \, H_{21} \, A_1^2 \right) \right\}.$$  
(4.17)

## 5 Concluding remarks

From the standpoint of NCG that the Higgs field is a kind of gauge field on the discrete space, we incorporated the scalar field $H_{12}$ into the generalized gauge field so as to generalize the Chern-Simons form and descent equations. We obtained the generalized Chern-Simons form and its transgression formula which include the scalar field $H_{12}$, for example, as in Eqs. (3.15) and (3.16). The more compact transgression formula follows in Eq. (3.17). We also introduced the generalized descent equations in Eq. (4.13). The physical implications of these formulas will be explored in future work.

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**References**

[1] Recent reviews: G. V. Dunne, Lectures at the 1998 Les Houches Summer School, *Aspects of Chern-Simons Theory*, [hep-th/9902115](http://arxiv.org/abs/hep-th/9902115).

  R. K. Kaul, *Topological Quantum Field Theories - A Meeting Ground for Physicists and Mathematicians*, [hep-th/9907119](http://arxiv.org/abs/hep-th/9907119).

  J. M. F. Labastida, *Chern-Simons Gauge Theory: Ten Years After*, [hep-th/9905057](http://arxiv.org/abs/hep-th/9905057).

[2] E. Witten, Comm. Math. Phys. **121**, 351 (1989). **37B**, 95 (1971).

[3] F. Wilczek, *Fractional Statistics and Anyon Superconductivity*, World Scientific, Singapore, 1990.

[4] A. Lerda, *Anyons: Quantum Mechanics of Particles with Fractional Statistics*, Lecture Notes in Physics Vol. 14, Springer, Berlin 1992.

[5] S. S. Chern and J. Simons, Ann. Math. **99**, 48 (1974).

[6] B. Zumino, Nucl. Phys. **B253**, 477 (1985).

[7] A. Achucarro and P. K. Townsend, Phys. Lett. **B180**, 89 (1986). E. Witten, Nucl. Phys. **B311**, 46 (1989).

[8] A. Strominger, High Energy Phys. **02**, 009 (1998).

[9] J. Wess and B. Zumino, Phys. Lett. **37B**, 95 (1971).
[10] A. Connes, p.9 in *The Interface of Mathematics and Particle Physics*, ed. D. G. Quillen, G. B. Segal, and Tsou. S. T., Clarendon Press, Oxford, 1990. See also, Alain Connes and J. Lott, Nucl. Phys. B (Proc. Suppl.) 18B, 57 (1990).

[11] Recent review papers of NCG: C. P. Martín, J. M. Gracia-Bondia, J. S. Várilly, Phys. Rep. 294, 363 (1998).
J. Madore and J. Mourad, “Noncommutative Kaluza-Klein Theory”, hep-th/9601169.

[12] K. Morita and Y. Okumura, Prog. Theor. Phys. 91, 959 (1994)
K. Morita and Y. Okumura, Phys. Rev. D 50, 1016 (1994).

[13] Y. Okumura, Phys. Rev. D 50, 1026 (1994).

[14] Y. Okumura, Nuovo Cim. 110A, 267 (1997).

[15] S. Ferrara, O. Piguet and S. Schweda, Nucl. Phys. 119, 493 (1977).

[16] D. Quillen, Topology 24, 89 (1985).
Y. Ne'eman and S. Sternberg, Proc. Natl. Acad. Sci. U. S. A. 87, 7875 (1990).

[17] Y. Ne'eman and J. Thierry-Mieg, in *Proceedings of the 1979 Salamanca International Conference on Differential Geometric Methods in Mathematical Physics* (1979), edited by A. Perez-Rendon, Lecture Notes in Mathematics No. 836 (Springer, Berlin, 1980).
J. Thierry-Mieg, J. Math. Phys. 21, 2834 (1980).