(1+2)-DIMENSIONAL BLACK-SCHOLES EQUATIONS WITH MIXED BOUNDARY CONDITIONS

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ABSTRACT. In this paper, we investigate (1+2)-dimensional Black-Scholes partial differential equations (PDE) with mixed boundary conditions. The main idea of our method is to transform the given PDE into the relatively simple ordinary differential equations (ODE) using double Mellin transforms. By using inverse double Mellin transforms, we derive the analytic representation of the solutions for the (1+2)-dimensional Black-Scholes equation with a mixed boundary condition. Moreover, we apply our method to European maximum-quanto lookback options and derive the pricing formula of this options.

1. Introduction. In this paper, we consider the following partial differential equation (PDE) problem:

\[ \mathcal{L} V(t, x, y) = 0, \]
\[ V(T, x, y) = h(x, y), \]

with the mixed boundary condition \( \frac{\partial V}{\partial y}(t, x, 1) = V(t, x, 1) \) on the domain \( \{(t, x, y) \mid 0 \leq t < T, 0 < x < \infty, 0 < y < 1\} \), where the PDE operator \( \mathcal{L} \) is given by

\[ \mathcal{L} = \frac{\partial}{\partial t} + \frac{\sigma_x^2}{2} \frac{x^2}{x^2} \frac{\partial^2}{\partial x^2} + \frac{\sigma_y^2}{2} \frac{y^2}{y^2} \frac{\partial^2}{\partial y^2} + \rho \sigma_x \sigma_y x y \frac{\partial^2}{\partial x \partial y} + r_x \frac{\partial}{\partial x} + r_y \frac{\partial}{\partial y} - r I. \]

Mixed boundary condition usually arises in the option pricing problem related to maximum process of underlying asset. Jeon et al. [5] derived an integral equation satisfying the Russian option using the Mellin transform. The Russian option satisfies (1+1)-dimensional Black-Scholes equations with mixed boundary conditions. Dai et al. [2] obtained a pricing formula for European maximum-rate quanto lookback options by using joint probability density functions of the extreme and terminal values of the prices of the underlying assets. Since the European maximum-rate quanto lookback options can be formulated as (1+2)-dimensional Black-Scholes equation with mixed boundary conditions, the probabilistic method proposed by

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Dai et al. [2] is applicable for solving (1+2)-dimensional Black-Scholes equations with mixed boundary conditions.

Our double Mellin transform technique also gives not only a pricing formula of European maximum-quanto lookback options but also an analytical representation for the solution of general (1+2)-dimensional Black-Scholes PDE with mixed boundary conditions. To the best of our knowledge, there have been no researches which derive an analytic representation of the solution of (1+2)-dimensional Black-Scholes equation with mixed boundary conditions. Since we analytically present the general solution of (1+2)-dimensional Black-Scholes equation with mixed boundary conditions, our approach can be applied to a variety of option pricing problems involving maximum or minimum process of underlying assets.

Since the Mellin transform can be regarded as a general Fourier transform, we can also adapt the double Fourier transform to solve the (1+2)-dimensional Black-Scholes equations with mixed boundary conditions. However, since the (1+2)-dimensional Black-Scholes PDE operator is degenerate, we should consider the change of variables to apply the Fourier transform. Also, after applying the Fourier transform, we have to restore to the original variables. This procedure makes a lot of complexity. In contrast to the Fourier transform, we can directly obtain the simple ordinary differential equation (ODE) without any change of variables when we apply the Mellin transform to the problem. For this reason, when we extend the problem to whole domain, the Mellin transform is simpler and easier than the Fourier method.

We briefly introduce our approach to the proofs of our main results; Theorem 3.2 and Theorem 4.1. In order to extend a solution of the PDE with mixed boundary condition, we shall consider the associated PDE with Dirichlet boundary condition. We then extend the solution of the associated PDE to the domain \{(t, x, y) \mid 0 \leq t < T, 0 < x < \infty, 0 < y < \infty\} by using the double Mellin transform. In the process, assuming a priori that the double Mellin transform of a solution exists, we find a solution and verify that it actually solves the problem. We also derive the extended PDE with Dirichlet boundary condition and deduce the extended PDE with mixed boundary condition. Using Theorem 3.2, we finally provide the pricing formula of European maximum-quanto lookback options in Theorem 4.1.

The rest of the paper is organized as follows. In the next section we review Mellin transform and set up notation. In Section 3 we state and prove the main results. Finally, in the last section we derive the price of the European maximum exchange rate quanto lookback option.

2. Preliminaries.

2.1. The review of Mellin transform. Here, we summarize the definition and basic properties of the double Mellin transform, for readers who are unfamiliar. Most of the properties of the double Mellin transform are similar to those of the single Mellin transform. Readers who are interested in the Mellin transform can refer to Bertrand et al [1] or Sneddon [6] for further details.

**Definition 2.1** (Definition of the (single) Mellin transform and (single) inverse Mellin transform). Let \( f(x) \) be a locally integrable function on \((0, \infty)\). Then, the (single) Mellin transform \( \mathcal{M}_x(f(x))(x^*) \) of \( f(x) \) is defined by

\[
\mathcal{M}_x(f(x))(x^*) := \hat{f}(x^*) = \int_0^\infty f(x)x^{x^*-1}dx, \quad x^* \in \mathbb{C}. \tag{2}
\]
If this integral converges for \( a < \Re(x^*) < b \), then the inverse of the Mellin transform is

\[
f(x) = M^{-1}_x(\hat{f}) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \hat{f}(x^*)x^{-x^*}dx^*, \quad a < c < b.
\]

**Definition 2.2** (Definition of the double Mellin transform and inverse double Mellin transform). Let \( g(x, y) \) be a locally integrable function on \( \mathbb{R}_+ \times \mathbb{R}_+ \). Then, the **double Mellin transform** \( M_{xy}(g(x, y))(x^*, y^*) \) of \( g(x, y) \) is defined by

\[
M_{xy}(g(x, y))(x^*, y^*) := \hat{g}(x^*, y^*) = \int_0^\infty \int_0^\infty g(x, y)x^{x^*-1}y^{y^*-1}dxdy, \quad x^*, y^* \in \mathbb{C}.
\]

If this integral converges for \( a_1 < \Re(x^*) < b_1, \ a_2 < \Re(y^*) < b_2 \), then for \( a_1 < c_1 < b_1, \ a_2 < c_2 < b_2 \) the inverse double Mellin transform is given by

\[
f(x, y) = M^{-1}_{xy}(\hat{g}(x^*, y^*)) = \frac{1}{(2\pi i)^2} \int_{c_2-i\infty}^{c_2+i\infty} \int_{c_1-i\infty}^{c_1+i\infty} \hat{g}(x^*, y^*)x^{-x^*}y^{-y^*}dx^*dy^*.
\]

**Proposition 1** (Convolution property of the double Mellin transform). Let \( f(x, y) \) and \( g(x, y) \) be locally integrable functions on \( \mathbb{R}_+ \times \mathbb{R}_+ \). For \( a_1 < \Re(x^*) < b_1, \ a_2 < \Re(y^*) < b_2 \), suppose that the double Mellin transforms \( \hat{f}(x^*, y^*) \) and \( \hat{g}(x^*, y^*) \) exist. Then the double Mellin convolution is given by the inverse double Mellin transform of \( f(x^*, y^*) \cdot \hat{g}(x^*, y^*) \) as follows:

\[
f(x, y) \ast g(x, y) := \int_0^\infty \int_0^\infty f(u, w) \cdot g\left(\frac{x}{u}, \frac{y}{w}\right) \frac{1}{u \cdot w} dudw.
\]

**Proof.** See Eltayeb and Kilicman [4].

For a positive integer \( n \), the followings hold:

\[
M_{xy}\left(\left(x \frac{\partial}{\partial x}\right)^n f(x, y)\right)(x^*, y^*) = (-x^*)^n \hat{f}(x^*, y^*),
\]

\[
M_{xy}\left(\left(y \frac{\partial}{\partial y}\right)^n f(x, y)\right)(x^*, y^*) = (-y^*)^n \hat{f}(x^*, y^*).
\]

We provide some conditions to guarantee the existence of double Mellin transform in the following proposition. Since the proof is straightforward, we omit it here.

**Proposition 2.** Suppose that the locally integrable function \( f(x, y) \) on \( \mathbb{R}_+ \times \mathbb{R}_+ \) satisfies the following conditions:

For \( a_1 < b_1 \) and \( a_2 < b_2 \),

\[
f(x, y) = O(x^{-a_1}) \quad \text{for} \quad x \rightarrow 0^+, \quad \text{and} \quad f(x, y) = O(x^{-b_1}), \quad \text{for} \quad x \rightarrow +\infty,
\]

\[
f(x, y) = O(y^{-a_2}) \quad \text{for} \quad y \rightarrow 0^+, \quad \text{and} \quad f(x, y) = O(y^{-b_2}), \quad \text{for} \quad y \rightarrow +\infty.
\]

Then the double Mellin transform \( M_{xy}(f(x, y))(x^*, y^*) \) of \( f(x, y) \) exists for \( a_1 < \Re(x^*) < b_1, \ a_2 < \Re(y^*) < b_2 \).

**2.2. Notation.** In the rest of the paper we shall use the following notations:

- \( \mathcal{R} \equiv \mathcal{R}_{t,x,y} \equiv \{(t, x, y) \mid 0 \leq t < T, \ 0 < x < \infty, \ 0 < y < 1\} \).
- \( \mathcal{R}_{x,y} \equiv \{(x, y) \mid 0 < x < \infty, \ 0 < y < 1\} \).
- \( \mathcal{R}_{t,x} \equiv \{(t, x) \mid 0 \leq t < T, \ 0 < x < \infty\} \).
- \( \overline{\mathcal{R}} \equiv \overline{\mathcal{R}}_{t,x,y} \equiv \{(t, x, y) \mid 0 \leq t < T, \ 0 < x < \infty, \ 0 < y < 1\} \).
- \( \overline{\mathcal{R}}_{x,y} \equiv \{(x, y) \mid 0 < x < \infty, \ 0 < y < \infty\} \).
3. Main results. We extend PDE problem (1) to the domain $\tilde{\mathcal{R}}$ and derive an analytic solution using the double Mellin transform. The following lemma is required to prove the main theorem.

**Lemma 3.1.** Consider the following PDE with Dirichlet boundary condition:

\[
\begin{aligned}
&\mathcal{L} V_D(t, x, y) = 0 \quad \text{in} \quad \mathcal{R}, \\
&V_D(T, x, y) = f(x, y) \quad \text{on} \quad \mathcal{R}_{x,y}, \\
&V_D(t, x, 1) = 0 \quad \text{on} \quad \mathcal{R}_{t,x}.
\end{aligned}
\]  

Then, a solution $V_D(t, x, y)$ of the PDE (7) can be extended to a solution of the following PDE:

\[
\begin{aligned}
&\mathcal{L} \tilde{V}_D(t, x, y) = 0 \quad \text{in} \quad \tilde{\mathcal{R}}, \\
&\tilde{V}_D(T, x, y) = f(x, y)1_{\{y<1\}} - y^{1-k_y}f \left( y^{\frac{-2\rho \sigma x}{\sigma_y}} x, \frac{1}{y^\sigma} \right)1_{\{y>1\}} \quad \text{on} \quad \tilde{\mathcal{R}}_{x,y},
\end{aligned}
\]

where $k_y = \frac{2r_y}{\sigma_y^2}$.

**Proof.** To solve (7), we consider the following PDE on the unrestricted domain $\tilde{\mathcal{R}}$:

\[
\begin{aligned}
&\mathcal{L} \tilde{V}_D(t, x, y) = 0 \quad \text{in} \quad \tilde{\mathcal{R}}, \\
&\tilde{V}_D(T, x, y) = f(x, y)1_{\{y<1\}} \quad \text{on} \quad \tilde{\mathcal{R}}_{x,y}.
\end{aligned}
\]  

Let $\hat{V}_D(t, x^*, y^*)$ denote the double Mellin transform of $\tilde{V}_D(t, x, y)$. Then, for the PDE in (8), we have

\[
\frac{d\hat{V}_D}{dt} + A(x^*, y^*)\hat{V}_D = 0,
\]

where

\[
A(x^*, y^*) = \frac{\sigma_x^2 x^{*2}}{2} + \frac{\sigma_y^2 y^{*2}}{2} + \rho \sigma_x \sigma_y x^* y^* - \left( r_x - \frac{\sigma_x^2}{2} \right) x^* - \left( r_y - \frac{\sigma_y^2}{2} \right) y^* - r.
\]

The solution of (9) is given by

\[
\hat{V}_D(t, x^*, y^*) = \hat{f}(x^*, y^*) e^{A(x^*, y^*)(T-t)}
\]

where $\hat{f}(x^*, y^*)$ is the double Mellin transform of $f(x, y)1_{\{y<1\}}$.

By applying the inverse double Mellin transform, we obtain that

\[
\tilde{V}_D(t, x, y) = \frac{1}{(2\pi i)^2} \int_{c_2-i\infty}^{c_2+i\infty} \int_{c_1-i\infty}^{c_1+i\infty} \hat{f}(x^*, y^*) e^{A(x^*, y^*)(T-t)} x^{-x^*} y^{-y^*} dx^* dy^*.
\]  

(10)

To compute (10), let us consider the following:

\[
G_C(t, x, y) = \frac{1}{(2\pi i)^2} \int_{c_2-i\infty}^{c_2+i\infty} \int_{c_1-i\infty}^{c_1+i\infty} e^{A(x^*, y^*)(T-t)} x^{-x^*} y^{-y^*} dx^* dy^*.
\]  

(11)
From Yoon and Kim [7], we have
\[ G_L(t, x, y) = \exp \left\{ \frac{t}{8(1 - \rho^2)} \left( \sigma_x^2 (k_x - 1)^2 + \sigma_y^2 (k_y - 1)^2 - 2\rho \sigma_x \sigma_y (k_x - 1)(k_y - 1) \right) - rt \right\} \]
where \( k_x = 2r_x/\sigma_x^2, \ k_y = 2r_y/\sigma_y^2 \).

Let us call \( G_L(t, x, y) \) the Green function of the Black-Scholes PDE operator \( L \). By the double Mellin convolution property, we get
\[
\tilde{V}_D(t, x, y) = \int_0^\infty \int_0^\infty f(u, w) 1_{(w < 1)} \ G_L \left( T - t, \frac{x}{u}, \frac{y}{w} \right) \frac{1}{u} \frac{1}{w} dwdu.
\] (12)

Moreover, since \( \mathcal{A}(x^*, y^*) \) is a quadratic form of \( y^* \), it holds that
\[
\mathcal{A}(x^*, y^*) = \mathcal{A}\left(x^*, -y^* - \frac{2\rho \sigma_x}{\sigma_y} x^* + k_y - 1\right).
\]

Then we obtain from the properties of the Mellin transform that
\[
G_L(t, x, y) = y^{1-k_y} G_L \left( t, y \frac{2\rho \sigma_x}{\sigma_y}, x, \frac{1}{y} \right).
\]

Thus, \( \tilde{V}_D(t, x, y) \) in (12) is expressed by
\[
\tilde{V}_D(t, x, y) = \int_0^\infty \int_0^\infty f(u, w) 1_{(w < 1)} \left( \frac{y}{w} \right)^{1-k_y} G_L \left( T - t, \left( \frac{y}{w} \right) \frac{2\rho \sigma_x}{\sigma_y} x, \frac{w}{u} \right) \frac{1}{u} \frac{1}{w} dwdu,
\]
which implies that
\[
y^{1-k_y} \tilde{V}_D \left( t, y \frac{2\rho \sigma_x}{\sigma_y} x, \frac{1}{y} \right) = y^{1-k_y} \int_0^\infty \int_0^\infty \left( \frac{1}{wy} \right)^{1-k_y} f(u, w) 1_{(w < 1)} G_L \left( T - t, \left( \frac{1}{wy} \right) \frac{2\rho \sigma_x}{\sigma_y} x, wy \right) \frac{1}{u} \frac{1}{w} dwdu
\]
\[
= \int_0^\infty \int_0^\infty \left( \frac{1}{w} \right)^{1-k_y} f(u, w) 1_{(w < 1)} G_L \left( T - t, \frac{x}{w} \frac{2\rho \sigma_x}{\sigma_y} u, wy \right) \frac{1}{u} \frac{1}{w} dwdu
\]
\[
= \int_0^\infty \int_0^\infty \tilde{w}^{1-k_y} f \left( \frac{\tilde{w}}{\tilde{w}} \frac{2\rho \sigma_x}{\sigma_y} \tilde{u}, \frac{1}{\tilde{w}} \right) 1_{(\tilde{w} > 1)} G_L \left( T - t, \frac{x}{\tilde{w}} \frac{2\rho \sigma_x}{\sigma_y} u, \frac{1}{\tilde{w}} \right) \frac{1}{u} \frac{1}{w} d\tilde{u}d\tilde{w}.
\]

where we have used \( \frac{1}{w} dwdu = \frac{1}{w} d\tilde{u}d\tilde{w} \) with \( \tilde{u} := w \frac{2\rho \sigma_x}{\sigma_y} u \) and \( \tilde{w} := \frac{1}{w} \).

From this, we deduce that \( \tilde{V}_D^*(t, x, y) \equiv y^{1-k_y} \tilde{V}_D \left( t, y \frac{2\rho \sigma_x}{\sigma_y} x, \frac{1}{y} \right) \) is the solution of
following PDE:

\[
\begin{cases}
L\tilde{V}_D(t,x,y) = 0 \\
\tilde{V}_D(T,x,y) = y^{1-k_y} f \left(y^{-\frac{2\rho\sigma}{\sigma_y}} x, \frac{1}{y}\right) 1_{\{y>1\}} 
\end{cases}
\text{ in } \tilde{R},
\]

Indeed, a direct calculation shows that

\[
L\tilde{V}_D(t,x,y) = 0
\]

and that

\[
\tilde{V}_D(T,x,y) = y^{1-k_y} \tilde{V}_D \left(T, y^{-\frac{2\rho\sigma}{\sigma_y}} x, \frac{1}{y}\right)
\]

\[
= y^{1-k_y} f \left(y^{-\frac{2\rho\sigma}{\sigma_y}} x, \frac{1}{y}\right) 1_{\{y<1\}} = y^{1-k_y} f \left(y^{-\frac{2\rho\sigma}{\sigma_y}} x, \frac{1}{y}\right) 1_{\{y>1\}}.
\]

We now define \(V_D(t,x,y) = \tilde{V}_D(t,x,y) - \tilde{V}_D(t,x,y)\). Then we have

\[
\begin{cases}
L V_D(t,x,y) = 0 \\
V_D(T,x,y) = f(x,y) 1_{\{y<1\}} - y^{1-k_y} f \left(y^{-\frac{2\rho\sigma}{\sigma_y}} x, \frac{1}{y}\right) 1_{\{y>1\}}
\end{cases}
\text{ in } \tilde{R},
\]

and \(V_D(t,x,1) = \tilde{V}_D(t,x,1) - \tilde{V}_D(t,x,1) = 0\). Hence, \(V_D(t,x,y)\) is the solution of the PDE (7), and is given by

\[
V_D(t,x,y) = \tilde{V}_D(t,x,y) - y^{1-k_y} \tilde{V}_D \left(t, y^{-\frac{2\rho\sigma}{\sigma_y}} x, \frac{1}{y}\right).
\]

Now we can get the following main theorem.

**Theorem 3.2.** The solution \(V(t,x,y)\) of the PDE (1) with mixed boundary condition can be extended to following PDE:

\[
\begin{cases}
L V(t,x,y) = 0 \\
V(T,x,y) = h(x,y) 1_{\{y<1\}} + y h(x,1) 1_{\{y>1\}} \\
+ y 1_{\{y>1\}} \int_1^y u^{-(1+k_y)} h \left(u^{-\frac{2\rho\sigma}{\sigma_y}} x, \frac{1}{u}\right) du \\
- y 1_{\{y>1\}} \int_1^y u^{-(k_y+2)} h_y \left(u^{-\frac{2\rho\sigma}{\sigma_y}} x, \frac{1}{u}\right) du
\end{cases}
\text{ in } \tilde{R},
\]

\[
\text{on } \tilde{R}_{x,y}.
\]
Proof. We define the differential operator \( \mathcal{H}[\cdot] \) as follows:

\[
\mathcal{H}[\cdot] := y \frac{\partial}{\partial y} - \mathcal{I}.
\]

Then we have

\[
\begin{cases}
\mathcal{H}[\mathcal{L}V(t, x, y)] = \mathcal{L} \mathcal{H}[V(t, x, y)] & \text{in } \mathcal{R}, \\
\mathcal{H}[V(t, x, 1)] = 0 & \text{on } \mathcal{R}_{t,x}.
\end{cases}
\]

Let \( P(t, x, y) = \mathcal{H}[V(t, x, y)] \), \( f(x, y) = \mathcal{H}[h(x, y)] \). It follows that

\[
\begin{cases}
\mathcal{L}P(t, x, y) = 0 & \text{in } \mathcal{R}, \\
P(t, x, 1) = 0 & \text{on } \mathcal{R}_{t,x}, \\
P(T, x, y) = f(x, y)1_{(y<1)} & \text{on } \mathcal{R}_{x,y}.
\end{cases}
\] (13)

By Lemma 7, the solution \( P(t, x, y) \) of the PDE (13) can be expressed by

\[
P(t, x, y) = \tilde{P}_D(t, x, y) - y^{1-k_y}\tilde{P}_D \left( t, \frac{2\rho_x}{\sigma_y} x, \frac{1}{y} \right),
\] (14)

where \( \tilde{P}_D(t, x, y) \) satisfies the PDE

\[
\begin{cases}
\mathcal{L}\tilde{P}_D(t, x, y) = 0 & \text{in } \mathcal{R}, \\
\tilde{P}_D(t, x, y) = f(x, y)1_{(y<1)} & \text{on } \mathcal{R}_{x,y}.
\end{cases}
\] (15)

Let us define the right inverse operator \( \mathcal{H}^{-1} \) of the differential operator \( \mathcal{H} \) as follows:

\[
\mathcal{H}^{-1}[g(y)] = y \int_{1}^{y} \frac{g(z)}{z^2} dz.
\]

Then we have \( \mathcal{H} \circ \mathcal{H}^{-1} = g \), that is, \( \mathcal{H} \circ \mathcal{H}^{-1} = \text{id} \).

Let \( \tilde{V}_D(t, x, y) = \mathcal{H}^{-1}[\tilde{P}_D(t, x, y)] \) and \( \tilde{V}_D^*(t, x, y) = \mathcal{H}^{-1}[\tilde{P}_D^*(t, x, y)] \), where

\[
\tilde{P}_D^*(t, x, y) = y^{1-k_y}\tilde{P}_D \left( t, \frac{2\rho_x}{\sigma_y} x, \frac{1}{y} \right).
\]

Then we obtain \( \mathcal{H}[\tilde{V}_D(t, x, y)] = \mathcal{H} \circ \mathcal{H}^{-1}[\tilde{P}_D(t, x, y)] = \tilde{P}_D(t, x, y) \).

By applying the double Mellin transform, we obtain

\[
\begin{align*}
\hat{P}(t, x^*, y^*) &= -(1 + y^*)\hat{V}(t, x^*, y^*), \\
\hat{P}_D(t, x^*, y^*) &= -(1 + y^*)\hat{V}_D(t, x^*, y^*), \\
\hat{P}_D^*(t, x^*, y^*) &= -(1 + y^*)\hat{V}_D^*(t, x^*, y^*),
\end{align*}
\] (16)

where \( \hat{V}, \hat{P}, \hat{V}_D, \hat{P}_D, \hat{V}_D^*, \) and \( \hat{P}_D^* \) represent the double Mellin transforms of \( V, P, \hat{V}_D, \hat{P}_D, \hat{V}_D^* \), and \( \hat{P}_D^* \), respectively.

Especially, we see that

\[
\hat{P}_D(t, x^*, y^*)
\]

\[
= \hat{P}_D(t, x^*, y^* - \frac{2\rho_x}{\sigma_y} x^* + k_y - 1)
\]

\[
= -(1 - y^* - \frac{2\rho_x}{\sigma_y} x^* + k_y - 1) \hat{V}_D(t, x^*, y^* - \frac{2\rho_x}{\sigma_y} x^* + k_y - 1).
\]

Applying double Mellin transform to (14) and using the relations (16), we get

\[
\hat{P}(t, x^*, y^*) = \hat{P}_D(t, x^*, y^*) - \hat{P}_D^*(t, x^*, y^*)
\]

\[
= P_D(t, x^*, y^*) - P_D(t, x^*, y^* - \frac{2\rho_x}{\sigma_y} x^* + k_y - 1),
\] (17)
We obtain from the properties of the Mellin transform that

\[ \hat{V}(t, x^*, y^*) = \hat{V}_D(t, x^*, y^*) + \hat{V}_D \left( t, x^*, -y^* - \frac{2\rho \sigma_x}{\sigma_y} x^* + k_y - 1 \right) - \frac{k_y + 1}{1 + y^*} \hat{V}_D \left( t, x^*, -y^* - \frac{2\rho \sigma_x}{\sigma_y} x^* + k_y - 1 \right) + 2 \frac{2\rho \sigma_x}{\sigma_y} x^* \hat{V}_D \left( t, x^*, -y^* - \frac{2\rho \sigma_x}{\sigma_y} x^* + k_y - 1 \right). \]  

(18)

We observe that

\[ \mathcal{M}^{-1}_{xy} \left( \frac{1}{1 + y^*} \hat{V}_D \left( t, x^*, -y^* - \frac{2\rho \sigma_x}{\sigma_y} x^* + k_y - 1 \right) \right) = \frac{1}{2\pi i} \int_{c_2-i\infty}^{c_2+i\infty} \hat{V}_D \left( t, x^*, -y^* - \frac{2\rho \sigma_x}{\sigma_y} x^* + k_y - 1 \right) \frac{1}{1 + y^*} x^{-z} y^{-y} dx^* dy^*. \]  

(19)

Furthermore, we note that

\[ \mathcal{M}^{-1}_y \left( \frac{1}{1 + y^*} \right) = -y 1_{\{y > 1\}} \]

\[ \mathcal{M}^{-1}_y \left( \frac{1}{1 + y^*} \right) \int_{c_1-i\infty}^{c_1+i\infty} \hat{V}_D \left( t, x^*, -y^* - \frac{2\rho \sigma_x}{\sigma_y} x^* + k_y - 1 \right) x^{-z} dz^* \]  

(20)

By the Mellin convolution property of a single variable, we have

\[ \mathcal{M}^{-1}_{xy} \left( \frac{1}{1 + y^*} \hat{V}_D \left( t, x^*, -y^* - \frac{2\rho \sigma_x}{\sigma_y} x^* + k_y - 1 \right) \right) \]

\[ = -\int_0^\infty \left( \frac{y}{u} \right) 1_{\{y \geq u\}} u^{1-k_y} \hat{V}_D \left( t, u - \frac{2\rho \sigma_x}{\sigma_y} x, 1 \right) \frac{1}{u} du \]  

(21)

We obtain from the properties of the Mellin transform that

\[ \mathcal{M}^{-1}_{xy} \left( \frac{x^*}{1 + y^*} \hat{V}_D \left( t, x^*, -y^* - \frac{2\rho \sigma_x}{\sigma_y} x^* + k_y - 1 \right) \right) \]

\[ = -x \frac{\partial}{\partial x} \mathcal{M}^{-1}_{xy} \left( \frac{1}{1 + y^*} \hat{V}_D \left( t, x^*, -y^* - \frac{2\rho \sigma_x}{\sigma_y} x^* + k_y - 1 \right) \right) \]  

(22)
By the definition of \( \mathcal{H} \) and its linearity, we see that if \( \mathcal{H}[g_1(y)] = \mathcal{H}[g_2(y)] \), then 
\[ g_1(y) = g_2(y) + cy \]
for some constant \( c \). It follows from (18), (21) and (22) that

\[
V(t, x, y) = V_D(t, x, y) + y^{1-k_y} \int_0^y u^{-(1+k_y)} e^u \left( t, u^{-\frac{2\rho \sigma_x}{\sigma_y}} x, \frac{1}{u} \right) du + \frac{2\rho \sigma_x}{\sigma_y} x y \int_0^y u^{-(1+k_y+2\rho \sigma_x/\sigma_y)} \partial_x V_D \left( t, u^{-\frac{2\rho \sigma_x}{\sigma_y}} x, \frac{1}{u} \right) du + c(t, x) y. \tag{23}
\]

for some function \( c(t, x) \) being independent of \( y \). Indeed, since \( \mathcal{L} \mathcal{H}[V_D] = \mathcal{L} \mathcal{P}_D = 0 \),
a direct computation shows that \( \mathcal{H}[\mathcal{L}V] = \mathcal{L} \mathcal{H}[V] = 0 \).

Since \( \tilde{V}_D(t, x, y) = \mathcal{H}^{-1} [\mathcal{P}_D(t, x, y)] \) and \( f(x, y) = y h_y(x, y) - h(x, y) \), we get

\[
\tilde{V}_D(T, x, y) = \mathcal{H}^{-1} [\mathcal{P}_D(T, x, y)] = \mathcal{H}^{-1} [f(x, y) 1_{\{y < 1\}}] = y \int_1^y \left( zh_y(x, z) - h(x, z) \right) 1_{\{z < 1\}} dz \tag{24}
\]

This and (23) yield

\[
V(T, x, y) = (h(x, y) - h(x, 1) y) 1_{\{y < 1\}} + y^{1-k_y} \left( h \left( y^{-\frac{2\rho \sigma_x}{\sigma_y}} x, \frac{1}{y} \right) - h \left( y^{-\frac{2\rho \sigma_x}{\sigma_y}} x, 1 \right) \frac{1}{y} \right) 1_{\{y > 1\}} \\
+ (k_y + 1) y 1_{\{y > 1\}} \int_1^y u^{-(1+k_y)} h \left( u^{-\frac{2\rho \sigma_x}{\sigma_y}} x, \frac{1}{u} \right) - h \left( u^{-\frac{2\rho \sigma_x}{\sigma_y}} x, 1 \right) \frac{1}{u} du \\
+ \frac{2\rho \sigma_x}{\sigma_y} x y 1_{\{y > 1\}} \int_1^y u^{-(1+k_y+2\rho \sigma_x/\sigma_y)} h_x \left( u^{-\frac{2\rho \sigma_x}{\sigma_y}} x, \frac{1}{u} \right) - \partial_u h \left( u^{-\frac{2\rho \sigma_x}{\sigma_y}} x, 1 \right) \frac{1}{u} du \\
+ \tilde{c}(x) y,
\]

where \( \tilde{c}(x) := c(T, x) \) is a function depending only on \( x \). An integration by parts gives

\[
k_y \int_1^y u^{-(1+k_y)} h \left( u^{-\frac{2\rho \sigma_x}{\sigma_y}} x, \frac{1}{u} \right) du = - y^{-k_y} h \left( y^{-\frac{2\rho \sigma_x}{\sigma_y}} x, \frac{1}{y} \right) + h(x, 1) + \int_1^y u^{-k_y} \partial_u \left( h \left( u^{-\frac{2\rho \sigma_x}{\sigma_y}} x, \frac{1}{u} \right) \right) du \\
= - y^{-k_y} h \left( y^{-\frac{2\rho \sigma_x}{\sigma_y}} x, \frac{1}{y} \right) + h(x, 1) \\
- \frac{2\rho \sigma_x}{\sigma_y} x \int_1^y u^{-(1+k_y+2\rho \sigma_x/\sigma_y)} h_x \left( u^{-\frac{2\rho \sigma_x}{\sigma_y}} x, \frac{1}{u} \right) du \\
+ \int_1^y u^{-k_y} \left( - \frac{1}{u^2} \right) h_y \left( u^{-\frac{2\rho \sigma_x}{\sigma_y}} x, \frac{1}{u} \right) du.
\]
Similarly, we have
\[
(k_y + 1) \int_1^y u^{-(2+k_y)} h \left( u - \frac{2\rho\sigma_x}{\sigma_y}, 1 \right) du \\
= -y^{-(1+k_y)} h \left( y - \frac{2\rho\sigma_x}{\sigma_y}, 1 \right) + h(x, 1) + \int_1^y u^{-(2+k_y)} \frac{\partial}{\partial u} \left( h \left( u - \frac{2\rho\sigma_x}{\sigma_y}, 1 \right) \right) du \\
= -y^{-(1+k_y)} h \left( y - \frac{2\rho\sigma_x}{\sigma_y}, 1 \right) + h(x, 1) \\
- \frac{2\rho\sigma_x}{\sigma_y} \int_1^y u^{-(2+k_y)+2\rho\sigma_x/\sigma_y} h_x \left( u - \frac{2\rho\sigma_x}{\sigma_y}, \frac{1}{u} \right) du.
\]
Therefore, we finally obtain
\[
V(T, x, y) = h(x, y)1_{\{y<1\}} + y^{1-k_y} h \left( y - \frac{2\rho\sigma_x}{\sigma_y}, \frac{1}{y} \right) 1_{\{y>1\}} \\
+ y1_{\{y>1\}} \int_1^y u^{-(1+k_y)} h \left( u - \frac{2\rho\sigma_x}{\sigma_y}, \frac{1}{u} \right) du \\
+ y1_{\{y>1\}} k_y \int_1^y u^{-(1+k_y)} h \left( u - \frac{2\rho\sigma_x}{\sigma_y}, \frac{1}{u} \right) du \\
+ \frac{2\rho\sigma_x}{\sigma_y} y1_{\{y>1\}} \int_1^y u^{-(1+k_y)+2\rho\sigma_x/\sigma_y} h_x \left( u - \frac{2\rho\sigma_x}{\sigma_y}, \frac{1}{u} \right) du \\
- h(x, 1)y + \tilde{c}(x)y \\
= h(x, y)1_{\{y<1\}} + yh(x, 1)1_{\{y>1\}} \\
+ y1_{\{y>1\}} \int_1^y u^{-(1+k_y)} h \left( u - \frac{2\rho\sigma_x}{\sigma_y}, \frac{1}{u} \right) du \\
- y1_{\{y>1\}} \int_1^y u^{-(k_y+2)} h_y \left( u - \frac{2\rho\sigma_x}{\sigma_y}, \frac{1}{u} \right) du \\
+ (\tilde{c}(x) - h(x, 1)) y.
\]
Since \( V(T, x, y) = h(x, y) \) in \( \mathcal{R}_{x,y} \), we deduce that \( \tilde{c}(x) \equiv h(x, 1) \). This gives
\[
V(T, x, y) = h(x, y)1_{\{y<1\}} + yh(x, 1)1_{\{y>1\}} \\
+ y1_{\{y>1\}} \int_1^y u^{-(1+k_y)} h \left( u - \frac{2\rho\sigma_x}{\sigma_y}, \frac{1}{u} \right) du \\
- y1_{\{y>1\}} \int_1^y u^{-(k_y+2)} h_y \left( u - \frac{2\rho\sigma_x}{\sigma_y}, \frac{1}{u} \right) du,
\]
which is the desired result. \( \square \)

The following lemma is useful for deriving the analytic representation of the solution.

**Lemma 3.3.** Consider the following PDE problem:
\[
\begin{aligned}
&\mathcal{L}V(t, x, y) = 0 \quad \text{in } \overline{\mathcal{R}}, \\
&V(T, x, y) = y1_{\{y>1\}} \int_1^y u^{-(1+k_y)} h \left( u - \frac{2\rho\sigma_x}{\sigma_y}, \frac{1}{u} \right) du \quad \text{on } \overline{\mathcal{R}}_{x,y}.
\end{aligned}
\]  
(25)

Then, the solution \( V(t, x, y) \) of the PDE (25) is given by
\[
V(t, x, y) = y \int_0^y u^{-(1+k_y)} V_D \left( t, u - \frac{2\rho\sigma_x}{\sigma_y}, \frac{1}{u} \right) du,
\]
where \( V_D(t, x, y) \) satisfies \( \mathcal{L}V_D(t, x, y) = 0 \) and \( V_D(T, x, y) = h(x, y)1_{\{y<1\}} \).
Proof. Let us define
\[
\xi(x,y) \equiv y1_{\{y>1\}} \int_{1}^{y} u^{-(1+k_y)} h \left( u - \frac{2\rho_\sigma \xi}{\sigma_y} x, \frac{1}{u} \right) du. \tag{26}
\]
Then,
\[
\xi(x,y) = y1_{\{y>1\}} \int_{1}^{y} u^{-(1+k_y)} h \left( u - \frac{2\rho_\sigma \xi}{\sigma_y} x, \frac{1}{u} \right) du
= \int_{0}^{\infty} \left( \frac{y}{u} \right) \left( \frac{y}{u} > 1 \right) u^{1-k_y} h \left( u - \frac{2\rho_\sigma \xi}{\sigma_y} x, \frac{1}{u} \right) 1_{\{\frac{y}{u} < 1\}} \, du. \tag{27}
\]
By (19), (20) and (21) in Theorem 3.2, we deduce that
\[
\hat{\xi}(x^*, y^*) = -\frac{1}{1+y^*} \hat{h} \left( t, x^*, -y^* - \frac{2\rho_\sigma \xi}{\sigma_y} x^* + k_y - 1 \right),
\]
where \(\hat{\xi}\) and \(\hat{h}\) are the double Mellin transform of \(\xi(x,y)\) and \(h(x, y)1_{\{y<1\}}\), respectively.

From the double Mellin transform approach, we get
\[
\hat{\tilde{V}}(t, x^*, y^*) = \hat{\xi}(x^*, y^*) e^{A(x^*, y^*)(T-t)}
= -\frac{1}{1+y^*} \hat{h} \left( t, x^*, -y^* - \frac{2\rho_\sigma \xi}{\sigma_y} x^* + k_y - 1 \right) e^{A(x^*, y^*)(T-t)}
= -\frac{1}{1+y^*} \mathcal{M}_{xy} \left( y^{1-k_y} \mathcal{V}_D \left( t, y - \frac{2\rho_\sigma \xi}{\sigma_y} x, \frac{1}{y} \right) \right)
= -\frac{1}{1+y^*} \hat{\mathcal{V}}_D \left( t, x^*, -y^* - \frac{2\rho_\sigma \xi}{\sigma_y} x^* + k_y - 1 \right),
\]
where \(\hat{\mathcal{V}}_D\) is the double Mellin transform of \(\mathcal{V}_D\). Thus, we can deduce from (21) in Theorem 3.2 that
\[
\mathcal{V}(t, x, y) = y \int_{0}^{y} u^{-(1+k_y)} \mathcal{V}_D \left( t, u - \frac{2\rho_\sigma \xi}{\sigma_y} x, \frac{1}{u} \right) du.
\]

\[\square\]

**Theorem 3.4.** The solution \(\mathcal{V}(t, x, y)\) of the PDE (1) with mixed boundary condition has the following analytic representation:
\[
\mathcal{V}(t, x, y) = \int_{0}^{\infty} \int_{0}^{\infty} h(u, w)1_{\{w<1\}} \mathcal{G}_L \left( T-t, x, \frac{y}{w} \right) \frac{du \, dw}{u \, w} \\
+ \int_{0}^{\infty} \int_{0}^{\infty} h(u, 1)1_{\{w>1\}} \mathcal{G}_L \left( T-t, x, \frac{y}{w} \right) \frac{du \, dw}{u \, w} \\
+ \int_{1}^{\infty} \int_{0}^{w} w \eta^{-(1+k_y)} h \left( \eta - \frac{2\rho_\sigma \xi}{\sigma_y} u, \frac{1}{\eta} \right) \mathcal{G}_L \left( T-t, x, \frac{y}{w} \right) \frac{du \, dw}{u \, w} \\
- \int_{1}^{\infty} \int_{0}^{w} w \eta^{-(2+k_y)} h \left( \eta - \frac{2\rho_\sigma \xi}{\sigma_y} u, \frac{1}{\eta} \right) \mathcal{G}_L \left( T-t, x, \frac{y}{w} \right) \frac{du \, dw}{u \, w}.
\]

**Proof.** By Theorem 3.2, the PDE (1) with mixed boundary condition is extended to following PDE:
\[
\begin{align*}
\mathcal{L} \mathcal{V}(t, x, y) &= 0, \quad \text{in} \ \mathcal{R}, \\
\mathcal{V}(T, x, y) &= h_{\text{ext}}(x, y), \quad \text{on} \ \mathcal{R}_{x,y},
\end{align*}
\]

\[\square\]
where
\[
 h_{\text{ext}}(x, y) = h(x, y)1_{\{y < 1\}} + y h(x, 1)1_{\{y > 1\}} \\
+ y 1_{\{y > 1\}} \int_1^y u^{-(1+k_y)} h \left( u - \frac{2\sigma_x}{\sigma_y} x, \frac{1}{u} \right) du \\
- y 1_{\{y > 1\}} \int_1^y u^{-(k_y+2)} h_y \left( u - \frac{2\sigma_x}{\sigma_y} x, \frac{1}{u} \right) du.
\]

By the Mellin transform approaches in Lemma 7, the value function \(V(t, x, y)\) is given by
\[
 V(t, x, y) = \int_0^\infty \int_0^\infty h_{\text{ext}}(u, w) G_L \left( T - t, \frac{x}{u}, \frac{y}{w} \right) \frac{du \, dw}{u \, w} \\
+ \int_0^\infty \int_0^\infty h(u, 1)1_{\{w > 1\}} G_L \left( T - t, \frac{x}{u}, \frac{y}{w} \right) \frac{du \, dw}{u \, w} \\
+ \int_0^\infty \int_0^\infty (w 1_{\{w > 1\}}) \int_1^w \eta^{-(1+k_y)} h \left( \eta - \frac{2\sigma_x}{\sigma_y} u, \frac{1}{\eta} \right) \eta \, d\eta \ G_L \left( T - t, \frac{x}{u}, \frac{y}{w} \right) \frac{du \, dw}{u \, w} \\
- \int_0^\infty \int_0^\infty (w 1_{\{w > 1\}}) \int_1^w \eta^{-(2+k_y)} h_y \left( \eta - \frac{2\sigma_x}{\sigma_y} u, \frac{1}{\eta} \right) \eta \, d\eta \ G_L \left( T - t, \frac{x}{u}, \frac{y}{w} \right) \frac{du \, dw}{u \, w}.
\]

Thus, we have just proved the desired result.

4. Application to option pricing. In this section, we consider the European maximum-exchange quato lookback option defined in Dai et al. [2]. Let \(F_t\) denote the exchange rate at time \(t\), which means that \(F_t\) represents the domestic price at time \(t\) of one unit of foreign currency. Let \(S_t\) be the foreign currency price at time \(t\), and let \(r_d\) and \(r_f\) be the constant domestic and foreign riskless interest rates, respectively. Then, the stochastic dynamics of \(S_t\) and \(F_t\) are described by
\[
\begin{aligned}
    dS_t &= \delta_d S_t dt + \sigma_s S_t dW^d_s \\
    dF_t &= (r_d - r_f) F_t dt + \sigma_f F_t dW^d_f,
\end{aligned}
\]
where \(\sigma_s\) and \(\sigma_f\) are the volatility of \(S_t\) and \(F_t\), respectively. Here, \(W^d_s\) and \(W^d_f\) represent standard Brownian motion under the domestic risk neutral measure \(Q^d\), and \(dW^d_s dW^d_f = \rho dt\). By using the quanto prewashing technique introduced by Dravid et al. [3], we obtain that
\[
\delta^d_s = r_f - q - \rho \sigma_f \sigma_s,
\]
where \(q\) is the dividend yield of the foreign asset \(S_t\) in a foreign country.

For the exchange rate process, define the maximum process of \(F_t\) as
\[
M_t = \max_{0 \leq \gamma \leq t} F_\gamma, \quad t \geq 0.
\]

Consider the European maximum exchange-rate quanto lookback call option, whose terminal payoff function in the domestic currency world is given by
\[
M_T \cdot (S_T - K)^+,
\]
where $K$ is the strike price in foreign currency, $T$ is the expiration date of call options, and $(x)^+ := \max(x, 0)$. In the absence of arbitrage opportunities, the value $C(t, s, f, m)$ is given by

$$C(t, s, f, m) = \mathbb{E} \left[ e^{-r_d(T-t)} M_T (S_T - K)^+ \mid S_t = s, F_t = f, M_t = m \right].$$

By Feynman-Kac formula, $C(t, s, f, m)$ is the solution of

$$\begin{cases}
\mathcal{L}^* C(t, s, f, m) = 0, \\
\frac{\partial C}{\partial m}(t, s, m, m) = 0, \\
C(T, s, f, m) = m \cdot (s - K)^+,
\end{cases}$$

on the domain $\{(t, s, f, m) \mid 0 \leq t < T, 0 < s < \infty, 0 < f < m, 0 < m < \infty\}$. Here the operator $\mathcal{L}^*$ is defined by

$$\mathcal{L}^* = \frac{\partial}{\partial t} + \frac{\sigma_f^2}{2} \frac{\partial^2}{\partial s^2} + \frac{\sigma_f^2}{2} f^2 \frac{\partial^2}{\partial f^2} + \rho \sigma_s \sigma_f s f \frac{\partial^2}{\partial s \partial f} + \delta_f \sigma_s \frac{\partial}{\partial s} + (r_d - r_f) f \frac{\partial}{\partial f} - r_d I.$$

We introduce a new change of variables

$$z = \frac{f}{m}, \quad V(t, s, z) = \frac{C(t, s, f, m)}{m}$$

Then, we can rewrite the problem (32) as follows:

$$\begin{cases}
\mathcal{L}^* V(t, s, z) = 0, \\
\frac{\partial V}{\partial z}(t, s, 1) = V(t, s, 1), \\
V(T, s, z) = (s - K)^+,
\end{cases}$$

on the domain $\{(t, s, z) \mid 0 \leq t < T, 0 < s < \infty, 0 < z < 1\}$.

Since $\partial_z h(s) = 0$, Theorem 3.2 implies that the PDE (33) can be extended to Theorem 4.1. The price of the European maximum exchange rate quanto lookback option, $C(t, s, f, m)$, is given by

$$C(t, s, f, m) = m se^{-(r_d - \delta^f + r_f)\tau} N_2 \left( d_1 \left( \tau, \frac{s}{K} \right) + \sigma_s \sqrt{\tau}, -d_2 \left( \tau, \frac{f}{m} \right) - \rho \sigma_s \sqrt{\tau}, -\rho \right)$$

$$- K me^{-\alpha^f + \gamma} N_2 \left( d_1 \left( \tau, \frac{s}{K} \right), -d_2 \left( \tau, \frac{f}{m} \right), \alpha \right)$$

$$+ sf e^{-\gamma T} N_2 \left( d_1 \left( \tau, \frac{s}{K} \right) + \left( \sigma_s + \rho \sigma_f \right) \sqrt{\tau}, d_2 \left( \tau, \frac{f}{m} \right) + \left( \rho \sigma_s + \sigma_f \right) \sqrt{\tau}, -\rho \right)$$

$$- K f e^{-\gamma T} N_2 \left( d_1 \left( \tau, \frac{s}{K} \right) + \sigma_f \sqrt{\tau}, d_2 \left( \tau, \frac{f}{m} \right) + \sigma_f \sqrt{\tau}, -\rho \right)$$

$$+ f \int_0^{1/m} u^{-\left(1 + k_f\right)} \left[ se^{-(r_d - \delta^f + r_f)\tau} N_2 \left( d_1 \left( \tau, \frac{u \frac{2 \rho \sigma_s}{K} \sqrt{\tau}}{K} \right) + \sigma_s \sqrt{\tau}, -d_2 \left( \tau, \frac{1}{u} \right) - \rho \sigma_s \sqrt{\tau}, -\rho \right) - K e^{-\gamma T} N_2 \left( d_1 \left( \tau, \frac{u \frac{2 \rho \sigma_s}{K} \sqrt{\tau}}{K} \right), -d_2 \left( \tau, \frac{1}{u} \right), -\rho \right) \right] du,$$
Proof. In (34), let \( V(t, s, z) = V_1(t, s, z) + V_2(t, s, z) + V_3(t, s, z) \) where \( V_1, V_2, \) and \( V_3 \) satisfy

\[
\mathcal{L} V_1(t, s, z) = 0, \quad V_1(T, s, z) = h(s)1_{\{z < 1\}},
\]

\[
\mathcal{L} V_2(t, s, z) = 0, \quad V_2(T, s, z) = \zeta h 1_{\{z > 1\}},
\]

\[
\mathcal{L} V_3(t, s, z) = 0, \quad V_3(T, s, z) = z 1_{\{z > 1\}} \int_1^z u^{-(1+k)} h\left(u - \frac{2\rho\sigma_s}{\sigma_f}\right) \frac{du}{u}.
\]

Then, by applying the double Mellin transform approach described in Section 3, we obtain

\[
V_1(t, s, z) = \int_0^\infty \int_0^\infty h(u)1_{\{u < 1\}} \mathcal{G}_L \cdot \left( T - t, \frac{s}{u}, \frac{z}{w} \right) \frac{1}{u} \frac{1}{w} \frac{du}{dw},
\]

\[
= \int_0^1 \int_0^\infty u \cdot \mathcal{G}_L \cdot \left( T - t, \frac{s}{u}, \frac{z}{w} \right) \frac{1}{u} \frac{1}{w} \frac{du}{dw} - K \int_0^1 \int_K^\infty \mathcal{G}_L \cdot \left( T - t, \frac{s}{u}, \frac{z}{w} \right) \frac{1}{u} \frac{1}{w} \frac{du}{dw},
\]

where \( \mathcal{G}_L \cdot \) is the Green function for the Black-Scholes PDE operator \( \mathcal{L}^* \).

From Section 3, we know that the Green function \( \mathcal{G}_L^* \) for the PDE operator \( \mathcal{L}^* \) is given by

\[
\mathcal{G}_L^*(t, x, y) = \exp\left\{ \frac{t}{8(1-\rho^2)} \right\} \left( \sigma_s^2(k_s - 1)^2 + \sigma_f^2(k_f - 1)^2 - 2\rho\sigma_s\sigma_f(k_s - 1)(k_f - 1) \right) - r_d t
\]

\[
\times \exp\left\{ -\frac{\rho\sigma_s\sigma_f}{(1-\rho^2)^{3/2}} \right\} \frac{x}{y} \int_0^\infty \frac{e^{-\frac{x}{(1-\rho^2)^{3/2}}} \frac{1}{\sigma_f \sqrt{2\pi(1-\rho^2)t}}}{e^{-\frac{y}{(1-\rho^2)^{3/2}}} \frac{1}{\sigma_s \sqrt{2\pi(1-\rho^2)t}}},
\]

where \( k_s := \frac{2\delta_s}{\sigma_s^2} \).

Lemma 4.2. For constant \( A, B > 0 \) and \( \tau = T - t \), we have

\[
\int_0^B \int_A^\infty \mathcal{G}_L \left( \tau, \frac{s}{u}, \frac{z}{w} \right) \frac{1}{u} \frac{1}{w} \frac{du}{dw} = e^{-r_d \tau} \mathcal{N}_2 \left( d_1 \left( \tau, \frac{B}{A} \right), -d_2 \left( \tau, \frac{z}{B} \right), -\rho \right),
\]

\[
\int_0^B \int_A^\infty u \cdot \mathcal{G}_L \left( \tau, \frac{s}{u}, \frac{z}{w} \right) \frac{1}{u} \frac{1}{w} \frac{du}{dw}
\]

\[
= se^{-r_d \tau} \mathcal{N}_2 \left( d_1 \left( \tau, \frac{B}{A} \right) + \sigma_s \sqrt{\tau}, -d_2 \left( \tau, \frac{z}{B} \right) - \rho \sigma_s \sqrt{\tau}, -\rho \right),
\]

\[
\int_0^\infty \int_A^\infty w \cdot \mathcal{G}_L \left( \tau, \frac{s}{u}, \frac{z}{w} \right) \frac{1}{u} \frac{1}{w} \frac{du}{dw}
\]
By Lemma 3.3 in Section 3, it holds that
\[
\int_{B} \int_{A} \Phi \cdot G \cdot \left( \begin{array}{c} \frac{x}{u} \\ \frac{y}{w} \end{array} \right) \frac{1}{u \cdot v} \, dv \, dw
= sze^{-qt}N_{2} \left( d_{1} \left( \tau, \frac{s}{A} \right) + (\sigma_{f} + \rho \sigma_{f}) \sqrt{\tau}, d_{2} \left( \tau, \frac{z}{B} \right) + (\rho \sigma_{s} + \sigma_{f}) \sqrt{\tau}, \rho \right).
\]

Proof. The computations are almost the same as in Theorem 1 in Yoon and Kim [7].

By using changes of variables and the methods of undetermined coefficients, we derive the desired results. \(\square\)

It follows from Lemma 4.2 that
\[
\int_{0}^{1} \int_{K}^{\infty} u \cdot G \cdot \left( T - t, \frac{s}{u} \right) \frac{1}{u \cdot w} \, dv \, dw
= se^{-(r-a-\delta_{s})\tau}N_{2} \left( d_{1} \left( \tau, \frac{s}{K} \right) + \sigma_{s} \sqrt{\tau}, -d_{2}(\tau, z) - \rho \sigma_{s} \sqrt{\tau}, -\rho \right)
\]
\[
\int_{0}^{1} \int_{K}^{\infty} u \cdot G \cdot \left( T - t, \frac{s}{u} \right) \frac{1}{u \cdot w} \, dv \, dw = e^{-r_{a}\tau}N_{2} \left( d_{1} \left( \tau, \frac{s}{K} \right), -d_{2}(\tau, z), -\rho \right).
\]

Hence, we have
\[
V_{1}(t, s, z) = se^{-(r-a-\delta_{s})\tau}N_{2} \left( d_{1} \left( \tau, \frac{s}{K} \right) + \sigma_{s} \sqrt{\tau}, -d_{2}(\tau, z) - \rho \sigma_{s} \sqrt{\tau}, -\rho \right)
\]
\[
K \cdot e^{-r_{a}\tau}N_{2} \left( d_{1} \left( \tau, \frac{s}{K} \right), -d_{2}(\tau, z), -\rho \right).
\]

Similarly, we obtain
\[
V_{2}(t, s, z)
= \int_{0}^{\infty} \int_{0}^{\infty} v \cdot \Phi \cdot \left( \begin{array}{c} \frac{u}{w} \\\ \frac{v}{w} \end{array} \right) \frac{1}{u \cdot w} \, dv \, dw
= 1 \int_{0}^{\infty} \int_{K}^{\infty} \Phi \cdot \left( \begin{array}{c} \frac{u}{w} \\\ \frac{v}{w} \end{array} \right) \frac{1}{u \cdot w} \, dv \, dw
- \int_{0}^{\infty} \int_{K}^{\infty} v \cdot \Phi \cdot \left( \begin{array}{c} \frac{u}{w} \\\ \frac{v}{w} \end{array} \right) \frac{1}{u \cdot w} \, dv \, dw
\]
\[
= sze^{-qt}N_{2} \left( d_{1} \left( \tau, \frac{s}{K} \right) + \sigma_{s} \sqrt{\tau}, d_{2}(\tau, z) + (\rho \sigma_{s} + \sigma_{f}) \sqrt{\tau}, -\rho \right)
\]
\[
- K \cdot e^{-r_{a}\tau}N_{2} \left( d_{1} \left( \tau, \frac{s}{K} \right), + \rho \sigma_{f} \sqrt{\tau}, d_{2}(\tau, z) + \sigma_{f} \sqrt{\tau}, \rho \right).
\]

By Lemma 3.3 in Section 3, it holds that
\[
V_{3}(t, s, z)
= z \int_{0}^{z} u^{-(1+k_{l})} V_{1} \left( t, u \cdot \frac{-2\rho \sigma_{s}}{\sigma_{f}} s, \frac{1}{u} \right) \, du
= zse^{-(r-a-\delta_{s})\tau} \int_{0}^{z} u^{-(1+k_{l})} \frac{2\rho \sigma_{s}}{\sigma_{f}}
\]
\[
\cdot N_{2} \left( d_{1} \left( \tau, \frac{u}{K} \right) + \sigma_{s} \sqrt{\tau}, -d_{2}(\tau, \frac{1}{u}) - \rho \sigma_{s} \sqrt{\tau}, -\rho \right) \, du
\]
\[
- K \cdot e^{-r_{a}\tau} z \int_{0}^{z} u^{-(1+k_{l})} N_{2} \left( d_{1} \left( \tau, \frac{u}{K} \right), -d_{2}(\tau, \frac{1}{u}), -\rho \right) \, du.
\]
Therefore, it follows from (35), (36) and (37) that
\[ V(t, s, z) = V_1(t, s, z) + V_2(t, s, z) + V_3(t, s, z) \]
\[ = se^{-(r_d-s)^2}N_2 \left( d_1 \left( \frac{\tau}{K}, \frac{\sigma_s}{s} \right) + \sigma_s \sqrt{\tau}, -d_2(\tau, z) - \rho \sigma_s \sqrt{\tau}, -\rho \right) \]
\[ - K e^{-(r_d-s)^2}N_2 \left( d_1 \left( \frac{\tau}{K}, \frac{\sigma_s}{s} \right), -d_2(\tau, z), -\rho \right) \]
\[ + sze^{-q_T}N_2 \left( d_1 \left( \frac{\tau}{K}, \frac{\sigma_s}{s} \right) + (\sigma_s + \rho \sigma_f) \sqrt{\tau}, d_2(\tau, z) + (\rho \sigma_s + \sigma_f) \sqrt{\tau}, \rho \right) \]
\[ - K ze^{-r_f}N_2 \left( d_1 \left( \frac{\tau}{K}, \frac{\sigma_s}{s} \right) + \rho \sigma_f \sqrt{\tau}, d_2(\tau, z) + \sigma_f \sqrt{\tau}, \rho \right) \]
\[ + ze^{-(r_d-s)^2} \int_0^s \left( 1 + k_f + \frac{2\rho \sigma_s}{s} \right) N_2 \left( d_1 \left( \frac{\tau}{K}, \frac{u \sigma_s}{s} \right), -d_2(\tau, \frac{1}{u}) - \rho \sigma_s \sqrt{\tau}, -\rho \right) du \]
\[ - K e^{-(r_d-s)^2} \int_0^s \left( 1 + k_f \right) N_2 \left( d_1 \left( \frac{\tau}{K}, \frac{u \sigma_s}{s} \right), -d_2(\tau, \frac{1}{u}), -\rho \right) du. \]

Since \( C(t, s, f, m) = m \cdot V \left( t, s, \frac{f}{m} \right) \), we have demonstrated the desired result. \( \square \)

We notice that the formula we have derived for European maximum exchange rate quanto options in Theorem 4.1 is exactly the same as the one derived by Dai et al. [2], which utilized the joint density function of the extreme values and the terminal values of the stock price and exchange rate.

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