Using 3D Stringy Gravity to Understand the Thurston Conjecture

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Abstract

We present a string inspired 3D Euclidean field theory as the starting point for a modified Ricci flow analysis of the Thurston conjecture. In addition to the metric, the theory contains a dilaton, an antisymmetric tensor field and a Maxwell-Chern Simons field. For constant dilaton, the theory appears to obey a Birkhoff theorem which allows only nine possible classes of solutions, depending on the signs of the parameters in the action. Eight of these correspond to the eight Thurston geometries, while the ninth describes the metric of a squashed three sphere. It therefore appears that one can construct modified Ricci flow equations in which the topology of the geometry is encoded in the parameters of an underlying field theory.
1 Introduction: How to Prove Uniformization Theorems

The uniformization theorem in two dimensions \[1\] states that a closed orientable two dimensional manifold with handle number 0,1, > 1 respectively admits uniquely the constant curvature geometry with positive, zero, or negative curvatures. This theorem has proved to be a very powerful tool in two-dimensional physics, with applications in conformal field theories and string theory. Indeed, in the path-integral formalism, one must sum over two-dimensional topologies and geometries. Hence, in the path integral, one can sum over deformations of each of these geometries, then sum over the handle number.

The potential importance of a 3D uniformization theorem is also evident, particularly in the context of (super)membrane physics or three-dimensional quantum gravity where one should be able to perform path-integral quantization via a similar procedure to that in two dimensions. Unfortunately, there is no uniformization theorem in three dimensions, only a conjecture due to W.P. Thurston. \[2\]^1. This conjecture states that a three-manifold with a given topology has a canonical decomposition into ‘simple three-manifolds,’ each of which admits one, and only one, of eight homogeneous geometries: \(H^3\), \(S^3\), \(E^3\), \(S^2 \times S^1\), \(H^2 \times S^1\), Sol, Nil and SL(2,R). The first three geometries are maximally symmetric, and hence isotropic. The remaining five are anisotropic, and hence less symmetric, but all have at least a three parameter group of isometries. \[2\]^2 The conjecture has not been completely proven, but considerable progress has been made by Thurston \[2\] and recently there has been speculation that Perelman \[11\] has overcome some roadblocks in Hamilton’s program to prove the conjecture using the ‘Ricci flow’ \[12, 13\].

To prove a uniformization theorem, one must show that a differentiable manifold with a given topology admits a certain (highly symmetric) metric. This is a formidable task, amounting to showing that structures admitted on small part of the manifold can be extended to cover the entirety of it. An

\[1\] A fairly clear exposition of this can be found in the review article by P. Scott \[3\]. There has been some use made of the Geometrization Conjecture in cosmology, beginning with the paper by Fagundes \[4\]. A non-exhaustive list of other work along these lines is in \[5, 6, 7, 8, 9\]. The role of the conjecture in high-energy physics is explored in \[10\].

\[2\] Relativists are familiar with the classification of homogeneous geometries into the Bianchi models. The relationship of the latter to the Thurston geometries is delineated in \[4\].
alternative is to assume that the manifold admits some metric, then show that the latter can be deformed by some process into the required highly symmetric metric. The obvious choice of the deformation is to treat the initial metric akin to a source of heat, and let the heat flow in ‘time’, via some parabolic system of partial differential equations. The idea is that heat ‘uniformizes’, so that at the end, as $t \to \infty$, the temperature is uniform.

The problem with this strategy is that one must find a parabolic system by which to flow the metric. In other words, the flow should be

$$\dot{g}_{\mu\nu} = \mathcal{O}(g_{\mu\nu}),$$

where the operator $\mathcal{O}$ is elliptic. Furthermore, this operator must transform as a tensor under coordinate transformations. This restricts the operator to be constructed from the curvature tensor. An obvious choice is the Ricci flow

$$\dot{g}_{\mu\nu} = -2R_{\mu\nu}.$$  

The first problem with this flow is that it is not strictly parabolic, though it shares many of the properties of simpler heat equations. In fact, it is not even a linear partial differential system, since the inverse of the metric appears in the Ricci tensor on the right hand side. Thus one must first resolve some hard problems in analysis: the existence of ‘short-time’ flows; extension to finite times; the occurrence of singularities before the flow might otherwise uniformize. Many of these issues have been resolved by Hamilton and others, but not all. In fact, a recent preprint by Perelman claims to have removed one of the major impediments in Hamilton’s program to prove Thurston’s Conjecture via a slightly modified version of the Ricci flow.

There is another problem with the Ricci flow: how do you ‘input’ the topology of the manifold? After all, uniformization really boils down to showing that a given manifold topology implies admissibility of a unique (up to diffeomorphisms) homogeneous metric. For the case of closed 2D manifolds, it is pretty straightforward to modify the Ricci flow so that topology is specified. This is due to the fact these manifolds are characterized as topological spaces by their Euler number $\chi(M_2)$, and the latter is related to the curvature via the Gauss-Bonnet formula:

$$\chi(M_2) = \frac{1}{2\pi} \int_{M_2} d^2x \sqrt{g} R(g).$$

In an earlier work, one of us, (J.G.) in collaboration with S. Braham, considered the Yang-Mills flow as a tool for examining uniformization theorems. Although promising in 2D, the 3D case proved intractable.
The modified Ricci flow \[12\] is then

\[
\dot{g}_{\mu\nu} = -2R_{\mu\nu} + \frac{\chi(M_2)}{V(M_2)}g_{\mu\nu},
\]

(4)

where \(V(M_2) := \int_{M_2} d^2x \sqrt{g}\) is the volume of \(M_2\). This flow has as its fixed points the constant curvature geometries. It was proved by Hamilton \[15\] and Chow \[16\] that this flow exists, and that it converges to its fixed points—thus giving a new proof of the two dimensional uniformization theorem. The important point here is that the topology of \(M_2\) is explicit data in the second term in the flow.

Unfortunately, the 3D case is not nearly so simple. Besides the problem that in general the 3D versions of the Ricci and modified Ricci flows have singularities, the topology of a closed 3D manifold cannot be easily expressed as a data point on the right side of a modified Ricci flow. For example, the Euler number of a closed 3D manifold is zero. Thus a term in the 3D modified Ricci flow proportional to the metric itself could not be obviously specified by topological data.

Our proposal here is that if the flow is broadened to include other fields defined on 3D space, then one can input at least some of the topological data in an explicit way, much as in the 2D case. As will be shown in the next section, such a flow is suggested by the low energy limit of a bosonic string propagating in 3D space. In particular, there exists sectors of the theory, labelled by the values (actually signs) of parameters in the Lagrangian, each of which admit one, and only one, of the eight Thurston geometries, up to coordinate transformations. We are able to prove uniqueness of the solutions for the case when the Chern-Simons term in the theory is turned off; but although plausible, uniqueness in the remaining two sectors has not rigorously been proven. Nonetheless it seems to us that these results are significantly encouraging and suggest that one might be able to use the flow in this model to clarify and understand, and perhaps finally prove, the Thurston Conjecture.

2 Modified 3D Stringy Gravity

Of the Thurston spaces, only \(E^3\), \(S^3\) and \(H^3\) are solutions of Einstein gravity with an appropriate cosmological constant term. In search of a single theory from which all eight of the Thurston geometries arise, we turn to the
low-energy limit of three-dimensional string theory, which has a metric $g_{\mu\nu}$, dilaton $\phi$, Abelian 2-form potential $B_{(2)}$ with field strength $H_{(3)} = dB_{(2)}$ and a ‘constant’ term in the level of the original sigma model [17, 18]. This theory has many more solutions than the constant curvature geometries. In fact, it has propagating modes. If the dilaton is set to a constant value, then for a given sign for the coupling of the $H^2$ term, the only solutions have either constant, non-negative or non-positive metrics. The value of the cosmological constant is given by a constant of integration. But even with the dilaton non-trivial, the remaining five anisotropic Thurston geometries are not solutions.

We therefore modify the above 3D stringy theory by appending to it a $U(1)$ gauge field with potential 1-form $A$ and field strength $F$ which couple as a ‘Maxwell-Chern-Simons theory’. The corresponding action is given by [19]:

$$S = \int d^3x \sqrt{g} e^{-2\phi} \left( -\chi + R + 4|\nabla \phi|^2 - \frac{\epsilon H}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{\epsilon F}{2} F_{\mu\nu} F^{\mu\nu} \right) + \frac{\epsilon}{2} e^{\mu\nu\rho} A_{\mu} F_{\nu\rho},$$

where the last term is the Abelian Chern-Simons term for the one-form potential $A_{(1)}$, and $F_{(2)} = dA_{(1)}$. The Wess-Zumino field $B_{\mu\nu}$ is a 2-form potential whose field strength $H_{\mu\nu\rho} = \partial_{[\mu} B_{\nu\rho]}$. Hence, in 3D, the field strength is proportional to the LeviCivita tensor:

$$H_{\mu\nu\rho} = H(x) \eta_{\mu\nu\rho},$$

where $H(x)$ is a scalar field. The equations of motion for the ‘Wess-Zumino field’ $B_{\mu\nu}$ are

$$H^{\mu\nu} := \nabla_{\mu}(e^{-2\phi} H(x) \eta^{\mu\nu}) = 0.$$

It is easy to see that the latter implies that $H(x) = c = \text{constant}$. Without loss of generality, this result can be substituted into the remaining equations of motion. The result is:

$$E_{\mu\nu} := R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \phi - \frac{\epsilon H}{2} e^{4\phi} g_{\mu\nu} - \epsilon_F F_{\mu}^\rho F_{\nu\rho} = 0;$$

$$J^\mu := \epsilon_F \nabla_\nu \left( e^{-2\phi} F^{\mu\nu} \right) - \frac{\epsilon}{2} \eta^{\mu\nu\rho} F_{\nu\rho} = 0;$$

$$D := -\chi + R(g) + 4\Delta \phi - 4|\nabla \phi|^2 - \frac{\epsilon H}{2} e^{4\phi} - \frac{\epsilon F}{2} F_{\mu\nu} F_{\mu\nu}. $$

4
In this paper we will look for solutions with $\phi = 0$. In this case, by taking appropriate linear combinations of the trace of (8) and (10) one obtains constraints on the Ricci scalar and electromagnetic field strengths. In particular, the Ricci scalar and the square of the field strength must both be constant:

$$R = -\frac{\epsilon_H}{2} c^2 + 2\chi.$$  \hspace{1cm} (11)

$$F_{\mu
u} F^{\mu\nu} = 2\epsilon_F \left( \chi - \epsilon_H c^2 \right)$$  \hspace{1cm} (12)

These constraints will play an important role in what follows. They have the effect of totally eliminating all propagating modes from the theory, leaving a finite dimensional space of solutions. This can be seen heuristically as follows. The full theory has two coupled dynamical modes, one for the scalar and one for the electromagnetic field. The gravitational field does not propagate in three spacetime dimensions. By restricting to constant $\phi$, the coupling between $\phi$ and $F_{\mu\nu}$ gives rise to the constraint (12), which also eliminates the dynamical mode associated with electromagnetic field. Thus, the electromagnetic field cannot fluctuate unless the scalar also fluctuates.

As will be shown below, all of the Thurston geometries are solutions of the equations of motion of this theory for various values of the parameters $\chi$, $\epsilon_H$, $\epsilon_F$, $e$. See Table 1 for a list of the solutions. In particular, the addition of the Maxwell term alone ($e = 0$) yields $S^2 \times E^1$, $H^2 \times E^1$ and Sol as solutions. In this case, there exists a generalized Birkhoff theorem which guarantees that these are the only solutions when $\phi = constant$. With $e \neq 0$, one finds that the remaining Thurston geometries Nil and $SL(2, R)$ are also solutions. Although it seems plausible that these are the only solutions, we have been unable to find a rigorous proof.

With the exceptions of Sol and $H^3$, the Thurston spaces can be characterized topologically as Seifert fibre bundles $\eta$ over an orbifold $Y$. The topology of a Seifert fibre bundle is determined by the Euler number $\chi(Y)$ of $Y$ and the Euler number $e(\eta)$ of the bundle $\eta$ \[\text{[2, 3, 10]}\]. Table 2 gives the values of these topological invariants for the 3-manifolds which admit Thurston geometries. It turns out that for all these spaces, the parameter $\chi$ in the action is related to $\chi(Y)$ by $\chi(Y) = \chi$. In addition, the constant $e$ turns out to be related to the Euler number $e(\eta)$. When the three-form field strength $H_{(3)}$ in the string inspired model is real, the parameter $\epsilon_H$ is positive, whereas $\epsilon_H < 0$ corresponds to an imaginary $H_{(3)}$. $\epsilon_H$ is positive only for the Thurston spaces which are Seifert fibre bundles, whereas Sol and $H^3$ require $\epsilon_H < 0$. We shall
| Geometry | $\epsilon_H$ | $\epsilon_F$ | $\chi$ | $e$ | $c$ | $[A_1, A_2, A_3]$ | Metric |
|----------|-------------|-------------|-------|-----|-----|----------------|--------|
| $E^3$    | *           | *           | *     | *   | *   | 0              | $dx_1^2 + dx_2^2 + dy^2$ |
| $S^3$    | 1           | 1           | 1     | 0   | 0   | 0              | $dx_1^2 + sin^2 x_1 dx_2 + (dy + cos x_1 dx_2)^2$ |
| $H^3$    | -1          | *           | -4    | *   | 2   | 0              | $\frac{1}{x_1^2}(dx_1^2 + dx_2^2 + dy^2)$ |
| $S^2 \times E^1$ | * | 1 | 1 | 0 | 0 | 0 | $[0, \cos x_i, 0]$ | $dx_1^2 + sin^2 x_1 dx_2 + dy^2$ |
| $H^2 \times E^1$ | * | -1 | -1 | 0 | 0 | 0 | $[0, \frac{1}{x_1}, 0]$ | $\frac{1}{x_1^2}(dx_1^2 + dx_2^2 + dy^2)$ |
| $Sol$    | -1          | 1           | 2     | 0   | 2   | 0              | $e^{2\nu} dx_1^2 + e^{-2\nu} dx_2^2 + dy^2$ |
| $Nil$    | 1           | -1          | 0     | -1  | 1   | 0              | $dx_1^2 + dx_2^2 + (dy - x_1 dx_2)^2$ |
| $\tilde{SL}(2, R)$ | 1 | -1 | -1 | 1 | 1 | 1 | $[-\frac{\sqrt{2}}{x_2}, 0, 0]$ | $\frac{1}{x_1^2}(dx_1^2 + dx_2^2) + (dy + \frac{1}{x_1} dx_2)^2$ |

Table 1: Thurston Geometries as Solutions

see below that in all cases where both $\epsilon_F$ and $\epsilon_H$ must be specified, they are of opposite sign.

| $c(\eta)$ | $\chi(\eta) > 0$ | $\chi(\eta) = 0$ | $\chi(\eta) < 0$ |
|-----------|------------------|------------------|------------------|
| $c(\eta) = 0$ | $S^2 \times E^1$ | $E^3$ | $H^2 \times E^1$ |
| $c(\eta) \neq 0$ | $S^3$ | $Nil$ | $SL(2, R)$ |

Table 2: Thurston Geometries as Seifert Fibre Spaces

### 3 The Electromagnetic Field

In order to simplify the equations, we define a vector field dual to the Maxwell field strength:

$$ v^\mu := \frac{1}{2} \eta^{\mu \rho} F_{\nu \rho}, \quad (13) $$

where $\eta^{\mu \rho} := \epsilon^{\mu \rho} / \sqrt{g}$ is the completely skewsymmetric Levi-Civita tensor.

Then the Maxwell-Chern-Simons equation (4) becomes

$$ \epsilon_F \eta^{\mu \rho} \nabla_\nu v_\rho = ev^\mu. \quad (14) $$

If we multiply by $\eta_{\mu \alpha \beta}$ (contracting on $\mu$) and use the property

$$ \eta^{\mu \rho} \eta_{\mu \alpha \beta} = \delta^\nu_\alpha \delta^\rho_\beta - \delta^\nu_\beta \delta^\rho_\alpha, \quad (15) $$

$$
we get
\[ 2\epsilon F \nabla_{[\alpha} v_{\beta]} = e \eta_{\mu\alpha\beta} v^\mu = e F_{\alpha\beta}. \] (16)

Now since \( F_{\alpha\beta} = 2\partial_{[\alpha} A_{\beta]} \), it follows that \textit{locally} there exists a (piecewise?) smooth function \( \sigma \) such that
\[ v_\mu = \epsilon_F e A_\mu + \nabla_\mu \sigma. \] (17)

Globally we would have to append to the above a ‘harmonic part’, determined by the topology of the 3-manifold.

Note that \( F_{\mu\pi} F_{\nu\pi} = g_{\mu\nu} v^2 - v_\mu v_\nu \), where \( v^2 := g_{\mu\nu} v_\mu v_\nu \). From this Eq. (12) it follows immediately that:
\[ v^2 = \frac{1}{2} F_{\mu\nu} F_{\mu\nu} = \epsilon_F \left( \chi - \epsilon_H c^2 \right). \] (18)

The gravitational equations now take the simple form:
\[ E_{\mu\nu} = R_{\mu\nu} - \frac{\epsilon H}{2} c^2 g_{\mu\nu} - \epsilon_F (\chi^2 g_{\mu\nu} - v_\mu v_\nu) = 0; \] (19)

From Eq. (19) it is easy to see that on shell, the Ricci scalar has three constant eigenvalues
\[ \lambda_1 = \frac{\epsilon H c^2}{2}, \] (20)
\[ \lambda_2 = \lambda_3 = \chi - \frac{\epsilon H c^2}{2}. \] (21)

with corresponding eigenvectors \( \xi_a^\mu, a = 1, 2, 3 \) satisfy
\[ \xi_1^\mu = \alpha v^\mu, \] (22)
\[ v_\mu \xi_i^\mu = 0, i = 2, 3, \] (23)

where \( \alpha \) is a constant. The second equations tells us that the \( \xi_i \) are a linearly independent set of eigenvectors.

From the eigenvalue equation for \( \lambda_1 \), namely,
\[ R_{\mu\nu} v^\mu = \frac{\epsilon H c^2}{2} v_\nu \] (24)

one can show that:
\[ \nabla_\mu v^\mu = 0. \] (25)
Simply take the covariant derivative of both sides of (24), and use the Bianchi
identity and the constancy of the Ricci scalar to obtain (25).

It also follows from the Maxwell-Chern-Simons equations of motion that \( v^\nu \) is constant norm tangent vector to a geodesic, so that

\[
v^\nu \nabla_\nu v^\mu = 0. \tag{26}
\]

Indeed from the first equality in Eq (16), it follows that after contracting with \( v^\alpha \) we get

\[
v^\alpha \nabla_\alpha v_\beta - v^\alpha \nabla_\beta v_\alpha = 0. \tag{27}
\]

Since \( v^\alpha v_\alpha = \text{constant} \), it follows that the second term above vanishes, and
the vanishing of the first term is just the geodesic equation.

To summarize, the above analysis shows that the field equations require
the dual of the electromagnetic field strength to have constant norm, be
divergence free, and obey the geodesic equation. These conditions must hold
for all values of the parameters in the Lagrangian.

### 4 Gravitational Equations

We will now use the vector field \( v^\mu \) to specify a local coordinate system in
which the metric takes a particularly simple form. Choose the coordinate
system \( \{x^1, x^2, y\} \) so that

\[
\left( \frac{\partial}{\partial y} \right)^\mu = v^\mu. \tag{28}
\]

We will denote the dependence of a function \( f \) on the \( x^j \) by \( f(x) \). Then from
the constancy of \( v_\mu \) it follows that

\[
g_{33} = v^2, \tag{29}
\]

where \( v^2 \) is the constant given by (18).

Without loss of generality we can write the metric as

\[
\begin{align*}
\begin{multlined}
 ds^2 = h_{ij}(x, y)dx^i dx^j + v^2(dy + a_i(x)dx^i)^2,
\end{multlined}
\end{align*}
\]

where the ‘2D metric’ \( h_{ij} \) depends on all the coordinates \( x^1, x^2, y \). However,
\( A_i \) depend only on the \( x^j \). This follows from the requirement that \( v^\mu \) is
tangent to a family of geodesics. Indeed, using \( v^\mu \nabla_\mu v^\nu = 0 \), get
\( g^{\mu\nu} \partial_y A_i = 0 \), which gives the desired result due to the invertability of \( h^{ij} \).
It is important to note here that we have used up most of the freedom we have to choose coordinates. Given a fixed vector field $v^\mu$, the form of the metric is preserved by only two restricted types of coordinate transformations. Firstly:

$$y \rightarrow y + f(x)$$

which effects a gauge transformation on the vector field $a_i(x)$:

$$a_i(x) \rightarrow a_i(x) + \partial_i f(x)$$

Secondly, we can do $y$-independent coordinate transformations on the $x^i$. This means that we cannot transform away any $y$-dependence in the metric $h_{ij}$.

Note that the form of the metric (30) suggests that $v^\mu$ is a Killing vector for the full metric. Indeed a straightforward calculation shows that the $i, j$ components of the Killing equation on $v^\mu$

$$\nabla_i v_j + \nabla_j v_i = \dot{h}_{ij}$$

where we have defined the quantity $\dot{h}_{ij} := h_{ij,y}$. The other components vanish identically. This shows as one might expect that $v^\mu$ is a Killing vector if and only if $h$ is independent of $y$.

We have yet to impose the condition $\nabla_\mu v^\mu = 0$ which is equivalent to

$$h^{ij} \dot{h}_{ij} = 0,$$

where $i, j, ...$ indices are lowered and raised by $h_{ij}$ and its inverse matrix $h^{ij}$. This means of course that $h = \det(h_{ij})$ is independent of $y$.

At this stage we have effectively solved the Maxwell-Chern-Simons equations. The only remaining field equations are the Einstein equations (19), which in terms of the $h$ ‘metric’ reduce to

$$E_{yy} := -\frac{1}{4} \dot{h}^{ij} \dot{h}_{ij} + \frac{v^2}{2} \left(-\epsilon_H c^2 + e^2 v^4 \right) = 0;$$

$$E_{iy} := \frac{1}{2} \left[ \nabla_j \dot{h}_i^j - a_j \partial_y \dot{h}_i^j + \frac{1}{2} a_i \dot{h}^{jk} \dot{h}_{jk} \right]$$

$$+ \frac{v^2}{2} \left(-\epsilon_H c^2 + e^2 v^4 \right) a_i = 0;$$

9
\[ h^{ij} E_{ij} := \mathcal{R}(h) + \nabla_k (\dot{h}^{ki} a_i) + a_k \nabla_i \dot{h}^{ki} - a_k a_i \ddot{h}^{ki} - 2 \epsilon_F v^2 + \left( 1 - \frac{\gamma^2}{2} h^{ij} a_i a_j \right) \left( -\epsilon_H c^2 + \epsilon_F v^4 \right) = 0. \]

In the above, the covariant derivative \( \nabla_j \) is with respect to the ‘metric’ \( h_{ij} \) and \( \dot{h}_{ij} := h_{jk} \dot{h}^{ki} \), etc. Note that the first term in (35) is positive definite, which in turn requires the second term to be negative definite. Moreover, if \( v^\mu \) is a Killing vector, so that \( h_{ij} = 0 \), \( \epsilon_H \) and \( \epsilon_F \) must be of opposite sign.

The considerations so far are completely general. In the next section show how the Thurston geometries emerge for restricted values of the parameters in the Lagrangian.

5 Existence of three Killing Vector fields

So far, for the case of constant dilaton field, we have not been able to prove that the only solutions to the equations of motion have metrics that are diffeomorphic to the eight Thurston geometries. However, in the case \( e = 0 \), i.e. no Chern-Simons contribution, one can show that the solutions all admit at least three algebraically independent Killing vector fields. This is sufficient to show that the metrics are homogeneous, and hence diffeomorphic to one of the Thurston geometries. To demonstrate this, we will use a procedure for examining the integrability of the Killing equations developed by Eisenhart [20], [21] and Yano [22].

We begin with the set of twelve variables \( S := \{ \xi^\mu, \xi_{\mu \nu} \} \). We impose on these variables the set of nine constraints

\[ F_0 := \{ \xi_{\mu \nu} = \xi_{\nu \mu}, \partial_{\mu} (v \cdot \xi) = 0 \} \]

where \( v \cdot \xi := g_{\mu \nu} v^\mu \xi^\nu \).

Now the system of partial differential equations that are to be satisfied by the variables \( S \), subject to the constraints \( F_0 \), are

\[ \nabla_\mu \xi_\nu = \xi_{\mu \nu}; \]
\[ \nabla_\mu \xi_{\nu \rho} = R_{\nu \rho \mu \pi} \xi^\pi. \]

The latter set is obtained by differentiating the former and using the first set of constraints along with the Ricci and Bianchi identities, as shown in [22], and also in [23], for the case when the constraint set includes only the first, enforcing the symmetry of \( \xi_{\mu \nu} \). The other constraint in \( F_0 \) is already
integrable. Now it is shown in [22] that the integrability of Eq. (39) is the set of equations

\[ F_1 := \{ \mathcal{L}_\xi R_{\mu\nu} = 0 \}. \tag{40} \]

We now show that these integrability conditions are identically satisfied as long as \( e = 0 \).

We write the field equations for the metric as

\[ R_{\mu\nu} = a g_{\mu\nu} - \epsilon F_{\nu} v^\mu, \tag{41} \]

where \( a := \epsilon_H c^2 / 2 + \epsilon F v^2 \), with \( v^2 = \text{constant} \). We also use the Maxwell-Chern-Simons equations (14). Then it is straightforward to show that

\[ \mathcal{L}_\xi R_{\mu\nu} = a \mathcal{L}_\xi g_{\mu\nu} + v_\mu L_\nu + v_\nu L_\mu, \tag{42} \]

where the vector field \( L_\mu \) is defined as

\[ L_\mu := \nabla_\mu (g_{\nu\rho} \xi^\nu v^\rho) + e \epsilon_F \eta_{\mu\nu\rho} \xi^\nu v^\rho. \tag{43} \]

So if \( e = 0 \), both terms in Eq. (42) vanish by the constraints \( F_0 \).

By standard theorems in partial differential equations [20] we can express the variables in the set \( S \) as a convergent multivariable Taylor series containing twelve parameters determined from the values of the fields at the centre of the Taylor expansion. But the nine constraints then reduce the number of independent parameters to three. Hence, our fields \( \xi_\mu \) are a three parameter family of Killing vector fields. This is the result we were after.

For the \( e \neq 0 \) case, it is straightforward to show that if \( v_\mu \) is a Killing vector field, then the metric admits a four parameter family of isometries. This follows from a result in Bona and Coll [24] that if the Ricci tensor satisfies the condition given by the field equation Eq. (11) and in addition \( v_\mu \) is shear-free, then the metric necessarily admits a four-parameter isometry group. It is easy to see that the condition that \( v_\mu \) is Killing is equivalent to it being shear-free. There are in fact three solutions with \( e \neq 0 \) and \( v_\mu \) a Killing vector field. Two of these are precisely the Thurston geometries Nil and \( SL(2, R) \). The third is a twisted line or circle bundle over \( S^2 \), with isometry group the product of \( SO(3) \) with the group of translations on the fibre. So locally these geometries are Hopf fibrations of the squashed (topological) three-sphere. This geometry, which forms the boundary of Taub-Nut-AdS and Taub-Bolt-AdS space, has recently been studied in the context of the holographic principle [25].
So if the theory admits any solutions which are not homogeneous geometries, then those geometries have \( e \neq 0 \) and \( \nu_{\mu} \) is not a Killing vector field. In the context of the field equations \([32, 35, 37]\), this requires solutions with both \( h_{ij} \) and \( a_{i} \) non-vanishing. The system appears to be overconstrained, and the only such solution we have found corresponds to \( Sol \) for which \( e = 0 \).

6 Letting it Flow

We will describe here the flow suggested by our three dimensional gravity theory.\(^4\) The idea is that the right side of the flow has as its zeroes the solutions of the equations of motion Eqs.\(^3\) to Eqs.\(^10\):

\[
\begin{align*}
\dot{g}_{\mu\nu} &= -2 \left[ R_{\mu\nu} + 2 \nabla_\mu \nabla_\nu \phi + \left( -\frac{\epsilon_H}{4} H_{\mu\rho\tau} H_{\nu}^{\rho\tau} - \epsilon_F F_{\mu}^{\rho} F_{\nu\rho} \right) \right]; \\
\dot{B}_{\mu\nu} &= \nabla_\rho \left( e^{-2\phi} H^{\rho}_{\mu\nu} \right); \\
\dot{A}_\mu &= \nabla_\nu \left( e^{-2\phi} F^\nu_{\mu} \right) + \frac{\epsilon_F}{2} \eta_{\mu \nu} F_{\nu\rho}; \\
\dot{\phi} &= -\chi + R(g) + 4\Delta \phi - 4|\nabla \phi|^2 - \frac{\epsilon_H}{12} H^2 - \frac{\epsilon_F}{2} F^2.
\end{align*}
\]

One needs to first specify the values of the parameters in the set \{\( \chi, e, \epsilon_H \)\}, according to the topology of the 3-manifold. If the manifold has the topology of a Seifert bundle \( \eta \) over an orbifold \( Y \), we specify \( \epsilon_H = +1, \chi = \chi(Y), e = e(\eta) \). If it is not a Seifert bundle then \( \epsilon_H = -1 \). It is not clear what the topological significance is of the parameters \( \chi, e \) for this case. So far, we have not found a topological interpretation for the parameter \( \epsilon_F \). However, we note in Table 1, that \( \epsilon_F \) always has the opposite sign of either \( \epsilon_H \) or \( \chi \). Hence we tentatively suggest that \( \epsilon_F \) has the opposite sign of \( \chi(Y) \), if the manifold is a Seifert bundle with \( \chi(Y) \neq 0 \); otherwise choose its sign to be opposite that of \( \epsilon_H \).

Once the parameters (and hence topology) are specified, one begins with an arbitrary configuration of metric, dilaton field, 2-form potential \( B_{\mu\nu} \) and \( U(1) \) potential \( A_{\mu} \) as initial conditions for the flow equations \([12, 37]\). If the

\(^4\) The flow equations below, when the field \( A_{\mu} \) is turned off, are the RG flow for a non-linear sigma model coupled to the target space metric \( g_{\mu\nu} \) and two-form potential \( B_{\mu\nu} \). We are currently exploring the incorporation of the target space Maxwell-CS field in the sigma model.
flow is to be useful then in the case where the flow is non-singular, the metric should reach the appropriate Thurston geometry.

As a consistency check for the program, we consider the case of locally homogeneous initial conditions for the flow. It should be noted that the Ricci-Hamilton flow of locally homogeneous geometries only converges to the fixed points for the case of locally homogeneous and isotropic geometries [27]. The anisotropic geometries become degenerate or singular [27].

We shall consider a few of the details for the flow of an initial geometry which is locally $H^2 \times E^1$. Thus the metric, $U(1)$ gauge field and dilaton are of the form:

$$ds^2 = \frac{\ell^2}{x_1^2} \left( D_1(t)dx_1^2 + D_2(t)dx_2^2 \right) + E(t)dy^2;$$

$$A_\mu = [0, A(t)\frac{\ell}{x_1}, 0];$$

$$\phi = \phi(t).$$  \hspace{1cm} (48)

From the flow of the metric, we find first that the factor $E(t)$ must be constant, and hence can be absorbed by rescaling the $y-$ coordinate. Second, it turns out that for any value of the flow parameter $t$, there must be a constant $\alpha$ such that $D_2(t) = \alpha D_1(t)$. The constant $\alpha$ can be be absorbed by rescaling $x_2$. Third, the function $A(t)$ in the gauge potential is frozen by its flow to be a constant $A(t) = a$. Finally, we calculate

$$\frac{dD_1}{d\phi} := \frac{\dot{D}_1(t)}{\phi(t)} = -2 \frac{D_1(t)(D_1(t) - a^2)}{(D_1^2(t) + 2D_1(t) - a^2)}. \hspace{1cm} (49)$$

The solution is

$$\phi(D_1) = \phi_0 - \frac{1}{2} \left\{ D_1 + \log \left[ D_1 \left( -D_1 + a^2 \right)^{1+a^2} \right] \right\}. \hspace{1cm} (50)$$

Hence we find that $D_1 \rightarrow a^2$, in the limit $\phi \rightarrow \infty$. Similar behaviour occurs for the case of the locally homogeneous flow of $S^2 \times E^1$. This is already an improvement over the normalized Ricci flow (see below), since in the latter, the locally homogeneous flow is singular in these cases.

The above calculation suggests that the dilaton field $\phi$, ‘normalizes’ the flow and can in some sense be considered as the physical flow parameter. If we had solved the locally homogeneous flows for $D_1, D_2, \phi$ in terms of $t$, we would have found that the first two do not converge to a finite value as
Instead, the fields flow to their fixed points as \( t \to -\infty \). In the usual Ricci flow, the locally homogeneous and isotropic geometries do not converge to their global ‘round’ form in the limit \( t \to \infty \). To accomplish this, the flow is normalized by adding to it a term \( 2/3 r g_{\mu\nu} \), where \( r \) is the average value of the Ricci scalar over the manifold\(^{12, 27} \). These considerations suggest the idea that occurrence of singularities in the flow of the metric is tracked by the flow of the dilaton field, instead of the rather arbitrary parameter \( t \).

## 7 Conclusion

The stringy gravity flow described in the last section is a promising approach to proving the Thurston Geometrization Conjecture. We support this claim by the following observations:

- It is quite closely related to the Ricci flow and its various modifications considered by Hamilton, Perelman and others. Hence the recent progress made by Perelman in resolving the analytical properties of these flows will almost certainly apply to the flow described here.

- The parameters that appear in the flow are determined by the topology of the 3-manifold. This makes it easier to ‘input’ the 3-manifold into the flow at the beginning.

- All the Thurston geometries are fixed points of the flow. Hence we can follow non-singular flows directly to the Thurston geometries.

- The dilaton field in the flow seems to track the singularities in the flow. This should streamline the procedure of performing surgery on the manifolds in regions where these singularities occur.

Finally, and more speculatively, we believe that underlying the stringy flow is a quantum field theoretic understanding of the Thurston Geometrization Conjecture. In particular, if the stringy gravity on the 3-manifolds are the bulk theory, then the sigma model whose RG flow is the stringy gravity equations of motion is its holographical dual theory. Work along these lines is in progress.

**Acknowledgments** We wish to acknowledge useful discussions with S. Das, V. Husain, R. B. Mann, C. Martinez, R. Troncoso, S. Vaidya, J. Vazquez-Poritz and J. Zanelli. We thank NSERC for partial funding.
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