On constructing purely affine theories with matter

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Abstract We explore ways to obtain the very existence of a space–time metric from an action principle that does not refer to it a priori. Although there are reasons to believe that only a non–local theory can viably achieve this goal, we investigate here local theories that start with Schrödinger’s purely affine theory [21], where he gave reasons to set the metric proportional to the Ricci curvature aposteriori. When we leave the context of unified field theory, and we couple the non–gravitational matter using some weak equivalence principle, we can show that the propagation of shock waves does not define a lightcone when the purely affine theory is local and avoids the explicit use of the Ricci tensor in realizing the weak equivalence principle. When the Ricci tensor is substituted for the metric, the equations seem to have only a very limited set of solutions. This backs the conviction that viable purely affine theories have to be non–local.

Keywords Affine theories · local gravity theories.

1 Introduction

Purely affine theories became a topic in Relativity through the search for a theory unifying gravitation and electromagnetism. A central question of a purely affine theory is the generation of a metric in the course of the evaluation of the field equations. This property of an a posteriori generation of the space–time metric is a central issue for a relativistic implementation of a Mach principle, e.g. as a Mach-type symmetry breakdown to locally Lorentz invariant theories [123]. The implementations of the Mach principle into a relativistic theory...
of gravity have found different aspects and different directions to explore [1, 5]. Considering a Mach type symmetry breakdown to locally Lorentz invariant theories, the important aspect is that the light-cone is the structure that should be generated through that break-down. This implies that the metric structure itself should not \textit{a priori} enter in the gravitation theory. The metric structure, together with the existence of a light–cone, should be the outcome of the theory. In this context, the distribution of matter in the surrounding universe represents the classical vacuum for the local neighborhood that breaks the \textit{at-least} affine invariance of vector spaces to the Lorentz invariance; for a local breakdown, see Ref. [6]. Although a Mach type symmetry breakdown should require an \textit{a priori} non-local theory [1, 7], it is surely useful to consider the known local pre-metric theories anew. This is the reason why we do not use the in other respect successful path to extend the metric theory to a metric-affine theory [5], but return again to purely affine theories.

In our approach, the metric is only expected to be a second-order tensor that appears in the simplest equations of motion, like the motion of pole particles or the propagation of shock waves of any field. These equations should be compared with the corresponding equations of General Relativity (GR) in order to identify the light-cone structure, or the projective structure used by Ehlers, Pirani, and Schild [9]. In the present work we shall consider shock waves because the appearance of the light-cone structure is our central point. These shockwaves can be matter shocks as well as pure gravitational shocks. In any case, we need only the simplest approximations. Symmetry properties of the tensor that is to be identified as metric are a second-order problem and will not be discussed here.

The first observation is that we cannot avoid the use of a connection $\Gamma^{a}_{bc}$ even when the use of an ordinary metric is avoided: The pure definition of a covariant derivative requires its existence. Then, we have two options. First, we can formulate the problem to find a metric as some solution to the Weyl-Cartan problem: To find a second–order covariant tensor $g_{ik}$ that is covariantly constant with respect to the transport $\Gamma^{a}_{bc}$ [10, 11]. The second option is to find independently this tensor from field equations, and consider its relation to the $\Gamma^{a}_{bc}$ afterwards. We choose this latter option here, because we intend to study the possibility of a dynamical definition of the metric. Its definition through the Weyl–Cartan space problem is a priori to the construction of the coupling to matter and not a posteriori. We are interested in a scheme that constructs the action without the use of the Riemannian metric so that the latter can arise from dynamics, i.e. a posteriori.

In section 2 we construct the general type of theories that we are pursuing to deal with. In section 3 the covariant field equations of our general theory are deduced. Later, in section 4 we explore a way to find the metric tensor as a result of a shock wave. In section 5 we explore local actions and find the necessity to go for nonlocal actions, that are explained in section 6 where also our final remarks are expressed.
2 Constructing a theory

We start from a theory that defines gravitation by a connection field. The question is how to couple external fields and how to get the notion of a metric a posteriori, i.e. to find the equivalent for the metric tensor. We expect that this a posteriori metric tensor is defined only to some approximation, or as a result of some symmetry breaking process, and that its precise definition requires particular configurations of the gravitational field.

Let us construct the simplest action integral for some fields $\Phi^A$, where $A$ stands for any field components without referring to their quality as scalar, vector, or tensor of any rank. The question of spinors in purely affine theory requires particular attention, see for instance Ref. [6,12]. First, we look for a second-order field theory, i.e. for an action bilinear in the derivatives, $\Phi^A_{,k}$. However, covariance requires the use of covariant derivatives, $\Phi^A_{,k}$, which are defined through some linear connection $\Gamma^m_{nk}$. The correction to the ordinary derivative for obtaining the covariant one is linear in the coefficients of the connection and linear in the field,

$$\Phi^A_{,k} = \Phi^A_{,k} + C^{AB}_{nm} \Gamma^m_{nk} \Phi^B,$$

(1)

where $C^{AB}_{nm}$ are some coefficients to be determined and depend on the nature of the matter fields. For instance, when $A$ stands for indexing the components of a contravariant vector, then $C^{AB}_{nm} = \delta^A_m \delta^n_B$. The interpretation of the torsion and non-metricity part of the $\Gamma^a_{bc}$ is a famous problem [13,14,15]. We will consider here a general connection, not necessarily a symmetric one.

Second, the integrand must be a scalar density, so the indices of derivation have to be compensated by some appropriate construction that provides upper indices. In GR this is done through the metric tensor, more precisely, through its contravariant inverse, and combinations of it. Here, we have two options that are characterized through the use of the Ricci tensor. We shall consider them below.

Third, we need an invariant volume element [30]. When there is no metric, and hence no determinant of the metric tensor, the simplest choice is the determinant of the Ricci tensor, as A.S. Eddington pointed out in the early 1920’s [16]. This is Schrödinger’s choice too [17,18,19,20,21]. With the simple action,

$$S_1 = \int \sqrt{-\det R_{ab}} \, d^4x ,$$

(2)

where $R_{ab}$ denotes the Ricci tensor. This was used by Eddington [16], but with the restriction to a symmetric connection. Schrödinger obtained a theory for the general affine connection that suggested to equate the Ricci tensor with the metric: The Ricci tensor obeys a field equation that tells that it is covariantly constant with respect to the star affinity up to the torsion of the latter. Therefore, Schrödinger postulated that

$$g_{ik} = \frac{1}{\lambda} R_{ik} .$$

(3)
For a unified field theory that does not explicitly contain matter this interpretation might be satisfying, but for a theory with explicit matter terms it is not. Indeed, Schrödinger's original intention was to get a unified field theory with no external matter at all, and the problem was to find equivalents for the conventional matter. In the present work we assume the gravity sector given by that of Schrödinger's and see how and which ordinary matter can be coupled to such a gravitation.

We show that it is not sufficient to purely determine the Ricci tensor to act as metric. The metric that is inferred by observation is that of the motion of matter [15]. This is also the lesson in particular of all theories with more than one metric tensor [2]. Explicit matter defines an effective metric by its motion, either by the motion of test particles or by the motion of shock waves. We have to define test particles of the matter fields that allow to construct an effective metric through the Ehlers–Pirani–Schild procedure [9], or we have to consider the propagation of shock fronts [22,23]. For the electromagnetic field, this has been stated many times [24,25,26,27,28,29]. This is exactly our point of view. We intend here to consider the relation of this construction—generalized to any field—to the Ricci tensor, that was Schrödinger's favorite choice. It is the matter Lagrangian that is important when we intend to define a metric. Because it is quadratic in the derivatives of the fields, we have to use \( \sqrt{-\det R_{ab}} \) itself as the invariant volume element, or alternatively we have to use fields that are densities of weight 1/2 [30].

To construct the matter part of the action within a local theory, we first recall that in GR this is given through

\[
S_2 = \int L_{\text{matter}}[\Phi^A, \Phi^A_{,k}, g^{ik}, g_{ik}] \sqrt{-\det g_{ab}} \, d^4x .
\]  

(4)

Our construction is however performed by using \( R_{ik} \) instead of \( g_{ik} \). Then, we have two options. First, we can try actions with Lagrangians not explicitly containing the Ricci tensor, where the latter enters only the volume element,

\[
S_3 = \int L_{\text{matter}}[\Phi^A, \Phi^A_{,k}] \sqrt{-\det R_{ab}} \, d^4x .
\]

(5)

Alternatively, we may consider a matter action similar to Eq. (4), but not necessarily implying the equality given by Eq. (3),

\[
S_4 = \int L_{\text{matter}}[\Phi^A, \Phi^A_{,k}, R_{ik}] \sqrt{-\det R_{ab}} \, d^4x .
\]

(6)

Jakubiec and Kijowski [31,32] have shown that the latter action can be transformed into GR with a different set of non-gravitational fields. This helps with respect to the dynamical structure, but destroys the interpretation of the deliberately chosen fields.

It is, of course, a drawback in the local action that matter has to exist locally in order to have a geometry defined. In the elementary vacuum, \( \Phi_A \equiv 0 \), the Euler–Lagrange equation might not exist or show singular behavior. Only
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the matter in the surrounding universe, like in Machian approaches, and not
the purely local one, should be as necessary as sufficient to fix a geo-

metry. However, a non–local Lagrangian will be the next step. First, we int-
end to evaluate a local action.

3 Covariant field equations

In this section, we explicitly show the construction of covariant field equa-
tions derived from the action, Eq. (5), in the case where the trans-
formation properties of the field components \( \Phi^A \) are not yet defined. We want to keep
our formalism as general as possible, therefore we consider our ba-

sic matter field of the following form

\[
\Phi^A \equiv [\Phi^{i_1 \cdots i_m}_{k_1 \cdots k_n}, \Phi^{j_1 \cdots j_s}_{l_1 \cdots l_t}, \cdots];
\]

that is, \( A \) represents field components of different fields with different trans-
formation properties.

We assume, as usual, a local variational principle to get the Euler–La grange
field equations in which \( L \) denotes the Lagrangian in the form

\[
L = L[\Phi^A, \Phi_B^A, \Gamma_{ikl}],
\]

to be distinguished from the form

\[
L^* = L^*[\Phi^C, \Phi_B^C, \Gamma_{ikl}].
\]

The change from partial to covariant derivatives implies that

\[
\frac{\partial L^*[\Phi^C, \Phi_B^C, l]}{\partial \Phi^A} = \frac{\partial L[\Phi^C, \Phi_B^C]}{\partial \Phi^A} + \frac{\partial L[\Phi^C, \Phi_B^C]}{\partial \Phi^B} \frac{\partial \Phi^B}{\partial \Phi^A} - \frac{\partial}{\partial x^k} \frac{\partial L}{\partial \Phi^A} + \Gamma^{A}_{ikl} \frac{\partial L}{\partial \Phi^B},
\]

(8)

where

\[
D \equiv \frac{\partial}{\partial x^k}, \quad \text{and}
\]

\[
\frac{\partial L}{\partial \Phi^A} = \frac{\partial}{\partial x^k} \frac{\partial L}{\partial \Phi^A} + \Gamma^{A}_{ikl} \frac{\partial L}{\partial \Phi^B} - C^B_{A} n m \frac{\partial L}{\partial \Phi^B},
\]

(11)

The determinant of the Ricci tensor transforms as follows

\[
\frac{\partial}{\partial x^k} \left( \ln \sqrt{-\det R_{ab}} \right) = \frac{D}{\partial x^k} \left( \ln \sqrt{-\det R_{ab}} \right) + \Gamma_{mk}^m,
\]

(13)
where we assumed that (in contrast to the notation in GR) \( R^{ij} R_{jk} = \delta^i_k \). Note that though the metric tensor also possesses this property, it is not necessarily implied a relation of the type given by Eq. (3). In fact, below we will see that in the presence of matter fields the Ricci and the metric tensors must be different. Combining the above formulas, we obtain the tensorial equation

\[
\frac{\partial L}{\partial \Phi^A} - \frac{D}{\partial x^k} \frac{\partial L}{\partial \Phi_{ik}} = \left[ \frac{D}{\partial x^k} \ln \sqrt{-\det R_{ab}} + 2 I^{m}_{[ik]} \right] \frac{\partial L}{\partial \Phi_{jk}} = 0. \tag{14}
\]

This is the covariant field equation, valid for a general matter field \( \Phi^A \).

Up to this point, an explicit dependence of \( L \) on \( R_{ik} \) was not involved. We now turn to the more general action, Eq. (6). The equation for the affine connection,

\[
\frac{\partial L}{\partial \Phi^C, \Phi^C; \xi, R_{mn}} \sqrt{-\det R_{ab}} - \frac{\partial}{\partial x^k} \frac{\partial L}{\partial \Phi^C, \Phi^C; \xi, R_{mn}} \sqrt{-\det R_{ab}} = 0,
\]

is in this form neither tensorial nor covariant. We start from the definition of the Ricci tensor,

\[
R_{ik} \equiv \Gamma_{il;k} - \Gamma_{ik;l} + \Gamma_{ilm} \Gamma^{lm}_{ik} - \Gamma_{ik} \Gamma^{lm}_{lm}.
\]

A straightforward calculation yields

\[
\frac{\partial R_{ik}}{\partial \Gamma^a_{bc}} = \Gamma^{rs}_{ik} E_{rika}^{stbc} = \delta_{r}^i \delta_{s}^j \delta_{k}^a, \\
E_{rika}^{stbc} = \delta_{r}^i \delta_{s}^j \delta_{k}^a + \delta_{r}^j \delta_{s}^i \delta_{k}^a - \delta_{r}^j \delta_{s}^i \delta_{k}^a - \delta_{r}^j \delta_{s}^i \delta_{k}^a,
\]

Formally, the variational derivatives are

\[
\frac{\delta}{\delta [\Gamma^a_{bc}]} \frac{\partial L}{\sqrt{-\det R_{ab}}} = \frac{\partial}{\partial x^d} \frac{\partial L}{\partial \Gamma^a_{bc,d}} = \frac{\partial}{\partial x^d} \frac{\partial L}{\partial \Gamma^a_{bc,d}} = D_{ika}^{bcd} \frac{\partial}{\partial x^d} G^{ik},
\]

where we used the abbreviations

\[
G^{ik} \equiv \frac{\sqrt{-\det R_{ab}} L}{\partial R_{ik}} \quad \text{and} \quad P_{B}^{C} \equiv \sqrt{-\det R_{ab}} \frac{\partial L}{\partial \Phi_{BC}}.
\]

We now solve the Euler–Lagrange equation for \( G^{ik} \) through use of the relation

\[
D_{ika}^{bcd} \left( \delta_{b}^i \delta_{m}^a \delta_{c}^n - \frac{1}{3} \delta_{c}^i \delta_{b}^m \delta_{a}^n \right) = -\delta_{i}^i \delta_{m}^a \delta_{k}^n
\]

and obtain
\[
\frac{\partial}{\partial x^n} g^{mn} = -\sqrt{-\det R_{ab}} c^{B}_{A} b^{a}_{\Phi A} (\delta^{a}_{c} \delta^{b}_{d} - \frac{1}{3} \delta^{a}_{c} \delta^{b}_{d}) \tag{16}
\]

\[
- E_{vika} = - P_{B}^{m} C^{B}_{A} n^{a}_{\Phi A} + \frac{1}{3} \delta^{a}_{c} P_{B}^{m} C^{B}_{A} n^{a}_{\Phi A} \tag{17}
\]

There are two contractions,

\[
\frac{\partial}{\partial x^n} g^{mn} = - P_{B}^{m} C^{B}_{A} n^{a}_{\Phi A} + \frac{1}{3} \delta^{a}_{c} P_{B}^{m} C^{B}_{A} n^{a}_{\Phi A} \tag{18}
\]

By contracting the indices \(a\) and \(c\), and substituting that equation again into Eq. (18) implies that

\[
0 = \frac{\partial}{\partial x^{n}} (G^{mn} - G^{nm}) = P_{B}^{m} C^{B}_{A} n^{a}_{\Phi A} \tag{19}
\]

Schrödinger discovered that by defining a new affinity, \(\Gamma_{abc} \equiv \Gamma_{abc} + \frac{2}{3} \delta^{a}_{c} R^{bc} \), equation (19) with \(L = \text{const.} \) reduces to \(R^{ab}_{\Gamma_{abc}} \equiv R^{ab}_{\Gamma_{abc}} + \frac{1}{2} \delta^{b}_{c} R^{bc} \Gamma_{abc} + \Gamma^{a}_{bc} R^{bc} = 0.\) In our case, these definitions imply that

\[
\frac{\partial}{\partial x^{n}} (G^{mn} - G^{nm}) = P_{B}^{m} C^{B}_{A} n^{a}_{\Phi A}. \tag{20}
\]
\[ R_{cb}^{ab} = -2 \left[ \frac{\partial \ln L}{\partial \Gamma^a_{bc}} - \frac{1}{3} \left( \frac{\partial \ln L}{\partial \Gamma^b_{kc}} \delta^b_a + \frac{\partial \ln L}{\partial \Gamma^b_{bk}} \delta^c_a \right) \right] \]

\[ + \left[ \frac{\partial \ln L}{\partial \Gamma^a_{kl}} - \frac{1}{3} \left( \frac{\partial \ln L}{\partial \Gamma^b_{kl}} \delta^m_a + \frac{\partial \ln L}{\partial \Gamma^b_{mk}} \delta^l_a \right) \right] R_{ml} R^{cb}. \] (21)

The introduction of matter fields \((L \neq \text{const.})\) avoids \(R_{cb}\) being parallel transported into itself by the star affinity; the same holds for the Einstein affinity, see Ref. [33]. Then, the presence of matter fields preclude us to interpret the Ricci tensor as being the metric, see Eq. (20).

4 The metric of space–time in the shock-wave picture

We identify the metric through the propagation of shock waves. The observation of the propagation of (shock) waves defines the metric of the wave in question. In ordinary wave mechanics, the wave operator determines the shocks to propagate along its bisectrices. Each wave equation has its own causal cone when the wave operators differ in the highest order of derivatives. The principle of relativity requires that the propagation is the same for the different fields that one intends to include as fundamental, but this is a second question. In a construction like the action given by Eq. (5), the propagation of shock waves is given through substitution of

\[ \Phi_{\text{shock}} = \Phi_0 + \theta |z| z^2 \phi \]

for the fields \(\Phi\), where \(z = z[x^k] = 0\) defines the shock hypersurface. \(\Phi_0\) and \(\phi\) are at least \(C_2\) in a neighborhood of the shock. The difference in the second-order derivatives of the two sides of the hypersurface is

\[ \Delta(\Phi_{ik}) = \phi \ z_i z_k \]

On the shock front, the Euler-Lagrange equation requires \[35\]:

\[ \frac{\partial^2 L}{\partial \phi^A \partial \phi^B} \phi^B z_i z_k = 0. \]

In the case of only one scalar field, the result is trivially

\[ g^{ik} \propto \frac{\partial^2 L}{\partial \phi^i \partial \phi^k} \]

In the case of more than one field component, we obtain a component-dependent propagation of the form

\[ K_{AB}^{ik} \phi^B z_i z_k = 0, \]
where $K_{AB}^{ik} \equiv \frac{\partial^2 L}{\partial \Phi^A_i \partial \Phi^B_k}$. Local Lorentz invariance requires that the light-cones at least for the fundamental free fields coincide. Therefore, GR implies the separability

$$K_{AB}^{ik} = a_{AB} g^{ik}$$

in order to obtain equal propagation cones for all field components $[34,35]$. Note that the coefficients $K_{AB}^{ik}$ depend on the construction of $L$, and not on the volume element $\sqrt{-\det R_{ab}} d^4 x$. The space–time Ricci curvature is irrelevant for the propagation of the shocks as long as it is not explicitly used in forming $L_{\text{matter}}[\Phi^A, \Phi^A_i, R_{ik}]$. However, explicit use implies higher order non-linearity, again.

5 Local action integrals

Let us assume a contravariant vector field $\Phi^k$. When the Ricci tensor enters the action through the volume element only, we can construct actions such as

$$S_3 = \int (\alpha \Phi^k_\ell \Phi^{\ell k} + \beta \Phi^k_\ell \Phi^\ell_k) \sqrt{-\det R_{ab}} d^4 x$$

and we obtain

$$K_{ab}^{ik} = \frac{\partial^2 L}{\partial \Phi^a_i \partial \Phi^b_k} \propto 2 \alpha \delta^k_a \delta^i_b + 2 \beta \delta^i_a \delta^k_b$$

and, therefore for arbitrary $\alpha$ and $\beta$ such that $\alpha + \beta \neq 0$, one has that $\delta^k z_\ell = 0$.

This is a limitation only for the amplitude of the shock, and no limitation for its front. Again, the form of the volume element does not enter the shock condition. In a local theory, its construction cannot yield the metric of space–time.

It is not difficult to see that in a local theory any propagation depends on the local amplitudes of the interacting fields and not on the geometry of the shock fronts as long as the lightcones are not deliberately constructed through use of some second–order contravariant tensor field, i.e. an a priori metric. Such an a priori metric however destroys our program, and cannot be its solution.

Let us now take Schrödinger’s choice to go around the fatal result that the matter fields itself cannot locally determine a viable light cone. We still stick to a local construction and replace the ordinary metric with the Ricci tensor. This might not be the final construction because one expects, at least approximately, metricity of the connection $[10]$, but we need only the shock approximation and the qualitative features of the field equations for our argument. We consider an action of the type given by Eq. (6) that is constructed using the methods of GR or of the metric-affine theory followed by a substitution of
$R_{ik}$ for $g_{ik}$ (the undifferentiated $g_{ik}$, not in the connection $\Gamma^a_{bc}$). The propagation of matter fields, of course, follows now the cone that is determined by $R_{ik}$ as constructed. However, the field equation for the connection now yields a restricting condition for the decisive part of the energy-momentum tensor density,

$$T_{ik} = \frac{\delta[L[g_,\Phi^A_m + C^A B_k \Gamma^a_{bm} g \cdots] \Phi^B] \sqrt{-\det g_{ab}}}{\delta[g_{ik}]} ,$$

namely,

$$G^{ik} = \left. \frac{\partial[L[g_,\Phi^A_m] \sqrt{-\det g_{ab}}]}{\partial g_{ik}} \right|_{g=R_..} ,$$

where the implicit dependence of $\Phi^A_m$ on $g_{ik}$ and its derivatives does not enter. We arrive at field equations for the connection, Eq. (16), that restrict the energy-momentum tensor density to kind of constant values, i.e., to peculiar, and not general, physical cases.

Summarizing: Local theories of the type given by Eqs. (5) and (6) do not achieve a viable causal structure. In the former case, when the Ricci tensor only enters the volume element, the shock waves of the matter fields do not feel that metric and are not null surfaces as expected. In the latter case, when the Ricci tensor is deliberately substituted for the metric, the Schrödinger result of a covariantly constant Ricci tensor turns into a correspondingly constant matter tensor and excludes nearly all physical cases. It was our intention to show this in due generality.

We may use matter fields to construct a volume element in order to get field equations in space–times without curvature, too [36,30]. In doing that, it is difficult not to introduce an a priori metric. Akama and Terazawa [36] hide it in the summation of their scalar fields, Gronwald et al [30] have it explicitly in their Lagrangians (see their section IV).

The construction of a metric through local non-gravitational fields has the consequence that a strong dependence of the metric on local perturbation must be expected. For instance, the metric components should be expected to be proportional to the local mass already at zeroth order. Therefore, we conclude that:

1. an a posteriori observation-based definition of a metric must rely on non-gravitational fields even in presence of a curved affine connection, and
2. its definition requires an explicitly non-local action for the non-gravitational fields.

### 6 Non-local action integrals and final remarks

The concluding remark shall discuss non-local action integrals. When we construct action integrals with fields and connections alone we find field equations
that exist only in the case when both $\Phi^A$ and $R_{ik}$ are non-trivial. If there is no matter, the geometry cannot be measured and is free. If there is no curvature, the motion of matter is not defined. It is, of course, a drawback in the local action that matter has to exist locally in order to have a geometry defined. We think that matter somehow should be enough to fix a geometry, like in Machian approaches. A non-local Lagrangian will be the next step.

A non-local interaction is constructed through at least a twofold integration over space–time, such as

$$S_5 = \int d^4 x \int d^4 y \sqrt{-\det R_{ik}[x]} \sqrt{-\det R_{jk}[y]}$$

$$\times L[\Phi^A[x], \Phi^A[x], R_{ij}[x], \Phi^B[y], \Phi^R_{ik}[y], R_{kl}[y]]$$

where, for instance, the $x$–coordinate can be used to label a local integration and the $y$–coordinate to refer the rest of the world. The non–local interaction, however, is a delicate point to be constructed properly. As long as the Lagrangian $L$ can be expanded in a series of scalar functions at $x$ with coefficients that are scalar functions at $y$, for instance,

$$S_5 = \int d^4 x \int d^4 y \sqrt{-\det R_{ik}[x]} \sqrt{-\det R_{jk}[y]} \sum_\alpha \eta_\alpha X_\alpha[x] Y_\alpha[y], \quad (23)$$

one of the integrations can formally performed. The result is a local action,

$$S_5 = \int d^4 x \sqrt{-\det R_{ik}[x]} \sum_\alpha \eta_\alpha X_\alpha[x]$$

with coefficients

$$\eta_\alpha = \int d^4 y \sqrt{-\det R_{jk}[y]} Y_\alpha[y].$$

No new physics is found.

When we now try to implement terms like $\Psi^k[x] \Phi^k[y]$, we have to see that they are not scalars at all: $\Psi^k[x]$ is a vector for substitutions of $x$, and a scalar for substitutions of $y$. On the opposite, $\Phi^k[y]$ is a scalar for substitutions of $x$, and a vector for substitutions of $y$. The product can be made a scalar for both substitutions only when there exist a bi-tensor $\gamma^m[x, y]$ that depends on the two points, and transforms as a covariant vector when the $x$ are substituted, and as a contravariant vector when $y$ is substituted [37]. In this case, $\Psi^k[x] \gamma^m[x, y] \Phi^k[y]$ is a scalar and may be used in constructing Lagrangians.

With the bi–tensor $\gamma$, however, we introduced an additional teleparallel connection [37]:

$$**\Gamma^a_{bc} = - \frac{\partial}{\partial y^c} \gamma^a[x, y] \mid_{y=x}$$

This makes the connection $\Gamma^a_{bc}$ superfluous. In addition, the connection $T^a_{bc}$ is now equivalent to a tensor field, $T^a_{bc} = \Gamma^a_{bc} - **\Gamma^a_{bc}$, of third order. Using $**\Gamma^a_{bc}$, we lose the Ricci tensor as equivalent of some metric: The curvature of a teleparallel connection vanishes, and the Schrödinger choice must be replaced
by some other construction. In addition, it is important to note that the bi-tensor $\gamma$ is mixed-variant. A covariant bi-tensor $\gamma_{ik}[x, y] = \lim_{y \to x} \gamma_{ik}[x, y]$. Teleparallel theories are discussed in connection with e.g. string theory or with rotation in the universe \cite{38,39}; but these are different approaches that are out of the scope of the present work.

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