In this talk, which popularizes some of our recent work, we provide novel insights into the bulk properties of light chiral quarks in a fixed Euclidean volume (e.g. lattice QCD). We show that the spontaneous breakdown of chiral symmetry results into diffusing quarks with a vacuum diffusion constant \( D = \frac{2F_{\pi}^2}{|\langle \bar{q}q \rangle|} \approx 0.22 \text{ fm} \), in striking analogy to diffusing electrons in disordered metals in one-, two- and three-dimensions.

The idea that light quarks diffuse in \( D = 4 \) is key to understand a number of phenomena in QCD in light of results known from disordered electronic systems. We introduce here the concept of the quark return probability in the QCD vacuum as a chirally disordered medium, borrowing from concepts first introduced by Anderson in the context of localization.

The eigenvalue equation for the Euclidean Dirac operator for quarks in the fundamental representation and in the fixed gluon field \( A \)

\[
(i\nabla[A] + im)q_k = \lambda_k[A]q_k
\]

allows us to extend the theory into 4+1 dimensions with proper time \( \tau \), and to define the normalized return probability \( P(\tau) \), for a light quark to start at \( x(0) \) and return back to the same position \( x(\tau) \) after a duration \( \tau \) as,

\[
P(\tau) = \frac{V^2}{N} \left\langle |\langle x(0)|x(\tau)\rangle|^2 \right\rangle = \frac{V^2}{N} \left\langle |\langle x(0)|e^{i(i\nabla[A] + im)\tau} x(0)\rangle|^2 \right\rangle .
\]
Here the Dirac operator acts as a four-dimensional Hamiltonian for the evolution in proper time \( \tau \), and \( N \) is the mean number of quarks states in the four-volume \( V = L^4 \). The ensemble averaging is over the gauge configurations which we note are \( \tau \)-independent (static disorder). Using the arguments developed in [1], we obtain for large proper times

\[
P(\tau) = e^{-2m\tau} \sum_{Q} e^{-DQ^2\tau},
\]

with \( D = 2F_\pi^2/|\langle \bar{q}q \rangle| \approx 0.22 \text{ fm} \) and \( Q_\mu = 2\pi n_\mu / L \) with \( n_\mu \) integers. In Fourier space \( \lambda \) (dual to \( \tau \)),

\[
P(\lambda) = \sum_{Q} \frac{1}{-i\lambda + \gamma + DQ^2}.
\]

The kernel of the sum in (4) corresponds to the kernel of a classically diffusing particle in 4+1 dimensions. The damping factor \( \gamma = 2m \) can be rewritten as \( \gamma \equiv D/L_{coh}^2 \), with the help of the GOR relation [3]. \( L_{coh} \) defines the coherence length for light quarks, which is also the pion Compton wavelength \( L_{coh} = 1/m_\pi \). In the chiral limit and for large times, the quark return probability develops a diffusion pole at \( \lambda = 0 \) and \( Q^2 = 0 \) (diffuson), which is a relic of the massless pion.

It is now easy to identify all the relevant proper-time scales separating different regimes in the QCD vacuum viewed as a disordered medium. Because of the diffusion, we expect the emergence of an ergodic time \( \tau_{\text{erg}} = \sqrt{V/D} = L^2/D \), which is the average time for the diffusing quarks to probe a sample of linear size \( L \), as expected from Einstein’s relation \( \bar{x}^2(\tau) = D\tau \).

For times greater than the ergodic time, but smaller than the Heisenberg time \( \tau_h = 1/\Delta \), where \( \Delta \) is the average spacing between the quantum levels, we are in the ergodic (universal) regime dominated by the diffuson pole. The return probability is then universal and given by \( P(\tau) = \exp(-2m\tau) \). By analogy with disordered electronic systems, this universal regime should be described by random matrix theory. For very large times (very small energies) the classical (ergodic) description breaks down with the onset of a purely quantum regime.

For times smaller than the ergodic time, the diffusing quark probes only the part of the Euclidean volume \( V \). All modes \( Q_\mu \) are equally important, and the return probability explicitly depends on the dimension of the system. This regime, called diffusive, ends at the elastic time \( \tau_{\text{elastic}} = 1/2M_q \), where \( M_q \) is the constituent mass of the quark. For times shorter than the time to cross a mean-free path, the concept of diffusion (and dressing of the quark through multiple scatterings) becomes obsolete. The regime \( \tau < \tau_{\text{elastic}} \) is referred to as ballistic.

In a straightforward way we can translate these scales onto the spectral properties of the Euclidean Dirac operator for finite volume QCD, e.g. lattice calculations. Using Heisenberg’s uncertainty relation \( E \sim 1/t \) we define the spectral scale \( \lambda \sim 1/\tau \). The inverse of the ergodic time defines the Thouless virtuality \( \Lambda_c = D/L^2 = D/\sqrt{V} \), in analogy to the Thouless energy in condensed matter systems. The inverse of the Heisenberg time is just the mean quantum spacing between the Dirac eigenvalues \( \Delta \). Eigenvalues greater than \( 2M_q \) are part of the ballistic regime.

We depict various regimes of disorder in the spectrum of the Dirac operator in Fig. 1.

![Figure 1: Disorder regimes in the eigenvalue spectrum of the Dirac operator.](image-url)

quantum ergodic diffusive ballistic
\[\Delta \quad \Lambda_c \quad 2M_q \quad \lambda\]
The first milestone in this diagram is the average level spacing $\Delta$ at small virtualities, which corresponds to the inverse of the spectral density at zero virtuality. It is directly related to the value of the quark condensate, via the Banks-Casher relation:

$$|\langle \bar{q}q \rangle| = \frac{\pi}{(V\Delta)}.$$  

The second milestone is played by the Thouless virtuality. Here the crucial physical ingredient is the occurrence of the pion decay constant through the diffusion constant.

The third milestone is the breakdown of the diffusive picture, corresponding to distance scales where the concept of dressing a quark is meaningless.

All these regimes are amenable to lattice verifications, as we explain below.

In the ergodic regime, the quark return probability is given solely by the diffuson pole. Hence, any model with the same global symmetries as QCD, leading to $P(\tau) = e^{-2m\tau}$ would do for describing the small quark eigenvalues. The simplest realizations are chiral random matrix models in the microscopic limit, an outgrowth of the matrix models used in the macroscopic limit. This regime is well accounted for by detailed lattice simulations. However, this agreement holds naturally only in the ergodic regime, where a direct relation to most of the dynamical observables in QCD is lost.

In the diffusive regime, QCD is characterized by a complex dynamics in D-dimensions. Nevertheless, there is still a possibility for a systematic study of the spectral properties, e.g. by semi-classical arguments borrowing on arguments from mesoscopic systems. Indeed, the spectral form factor

$$K(\tau) \approx \frac{2\tau \Delta^2}{4\pi^2 \beta} P(\tau), \quad (5)$$

is directly related to the average two-level spectral correlation function, for different quark representations $\beta = 1, 2, 4$, for pseudo-real, complex and quaternion, respectively. In the diffusive regime, the spectral properties are reflecting on two-pion exchange, again in close analogy to the spectral properties of the two-diffuson and/or cooperon exchange in disordered electron systems.

Soon after our theoretical analysis, various numerical studies appeared suggesting the occurrence of the Thouless virtuality. A numerical study in the instanton model and lattice QCD have confirmed some of our predictions, although the diffusion scenario we have unraveled is yet to be explored. The first results reflecting on the spectral properties mentioned above are by now also available.

The ballistic regime is probably the most difficult one to investigate on the lattice, since the virtualities are large, and may interfere with the lattice cut-off. We note, that a recent multi-matrix study has confirmed that the ballistic regime is delineated by the constituent quark mass.

Last but not least, let us ask the crucial question about the microscopic source of disorder in Euclidean QCD. For disordered electron systems, the diffusion is triggered by the elastic scattering of electrons on external, static defects in ‘dirty’ wires, plaquettes or grains, for $D = 1, 2, 3$ respectively. QCD is a fundamental theory, so where does the ‘dirt’ come from? One possibility are instantons. They are static quasi-particles in 4+1 dimensions, that cause a net chirality flip on light quarks (index theorem), with usually a random distribution in color and position space. However, other lumps of gauge-fields can act similarly as well. In many ways, the renewal interest in random instanton systems can be regarded as the longstanding motivation for the ideas developed in this note. We are pleased that early suggestions that such systems are amenable to generic results from matrix models and Anderson’s ideas have come full circle, with potentially novel applications, some of which we have addressed recently.
Acknowledgments

M.A.N. is grateful to the organizers for the invitation to such an enjoyable and productive meeting. This work was supported in part by the US DOE grants DE-FG-88ER40388 and DE-FG02-86ER40251, by the Polish Government Project (KBN) grants 2P03B04412 and 2P03B00814 and by the Hungarian grant OTKA-F026622.

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