Design-based theory for Lasso adjustment in randomized block experiments with a general blocking scheme

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Abstract

Blocking, a special case of rerandomization, is routinely implemented in the design stage of randomized experiments to balance the baseline covariates. Regression adjustment is highly encouraged in the analysis stage to adjust for the remaining covariate imbalances. This study proposes methods that combine blocking, rerandomization, and regression adjustment techniques in randomized experiments with high-dimensional covariates and a general blocking scheme. In the design stage, we suggest the implementation of blocking and rerandomization to balance the fixed number of covariates most relevant to the outcomes. For the analysis stage, we propose a regression adjustment method based on least absolute shrinkage and selection operator (Lasso) to adjust for the remaining imbalances in additional high-dimensional covariates. We derive the asymptotic properties of the proposed estimator and outline the conditions under which this estimator is more efficient than the unadjusted one. Moreover, we provide a conservative variance estimator to facilitate valid inferences. Our design-based analysis allows for model misspecification and is applicable to heterogeneous block sizes, propensity scores, and treatment effects. Simulation studies and two real-data analyses demonstrate the advantages of the proposed method.

Keywords: causal inference, covariate imbalance, projection estimator, randomization-based inference, stratification

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1 Introduction

Randomized experiments are the basis for evaluating the effect of a treatment on an outcome and are widely used in the industry, social sciences, and biomedical sciences (see, for example, Fisher 1935, Box et al. 2005, Imbens & Rubin 2015, Rosenberger & Lachin 2015). In randomized experiments, complete randomization of treatment assignments can balance the baseline covariates, on average. However, covariate imbalances often occur in a particular treatment assignment (see, for example, Fisher 1926, Morgan & Rubin 2012, Athey & Imbens 2017). To increase the estimation efficiency of the treatment effect, some researchers have recommended balancing the key covariates in the design stage (Fisher 1926, Morgan & Rubin 2012, Krieger et al. 2019), whereas others have emphasized the implementation of adjustments for covariate imbalances in the analysis stage (Fisher 1935, Miratrix et al. 2013, Lin 2013, Bloniarz et al. 2016, Lei & Ding 2021).

Fisher (1926) was the first to recommend the use of blocking or stratification in the design stage to balance a few discrete covariates that were the most relevant to the outcomes. Since then, blocking has been widely used in experimental designs (see, for example, Imai et al. 2008, Higgins et al. 2015, Schochet 2016, Pashley & Miratrix 2021). While blocking can effectively balance discrete covariates, balancing continuous covariates using this approach is less intuitive. Rerandomization is a more general approach for balancing discrete and continuous covariates (Student 1938, Morgan & Rubin 2012, Li et al. 2018, 2020). Recently, scholars have recommended combining blocking and rerandomization techniques (Johansson & Schultzberg 2022, Wang, Wang & Liu 2021). Rubin summarized this design strategy as “Block what you can and rerandomize what you cannot.”

Blocking, rerandomization, or their combination can balance only a fixed number of covariates. However, in modern randomized experiments, a large number of baseline covariates are often collected and the number of covariates can be larger than the sample size. For example, in a randomized controlled trial, the researcher may record the demographic and genetic information of each participant. Bloniarz et al. (2016) highlighted that in such high-dimensional settings, most of the covariates may not be related to the outcomes; thus, the important covariates must be selected to realize efficient treatment effect estimation. In the design stage, when pre-experimental data (outcomes and covariates) were available, Johansson & Schultzberg (2020) used Lasso (Tibshirani 1996) to select important covariates and rerandomization to balance the selected covariates. However, when pre-experimental outcome information is unavailable, it is difficult to perform covariate selection in the design stage. A more realistic approach is to use Lasso in the analysis stage to perform a regression adjustment. Regression adjustment has
been widely used to analyze randomized experiments and increase associated efficiency (see, for example, Fisher 1935, Miratrix et al. 2013, Lin 2013, Bloniarz et al. 2016, Liu & Yang 2020, Lei & Ding 2021). Li & Ding (2020) showed that regression adjustment is equivalent to rerandomization with a threshold tending to zero.

In contrast to regression adjustment based on Lasso, blocking and rerandomization techniques do not use the outcome data and thus can avoid bias due to the specification search of the outcome model. Many scholars have indicated that it is preferable to avoid the occurrence of covariate imbalances before treatment is administered rather than performing post-treatment regression adjustments (Cox 2007, Freedman 2008a,b, Rosenbaum 2010). In this context, a trade-off method is to combine blocking, rerandomization, and high-dimensional regression adjustments. In this method, blocking, rerandomization, or both are implemented in the design stage to balance a few covariates that most significantly affect the outcomes. Next, we perform covariate selection and regression adjustment by using, for example, Lasso in the analysis stage to adjust for the remaining imbalances in the additional high-dimensional covariates.

Recently, researchers have proposed methods that combine blocking, rerandomization, and regression adjustment in randomized experiments under certain constraints. For example, Bugni et al. (2018, 2019) showed that the regression of the outcome of the treatment, block indicators, and their interactions can increase the estimation efficiency of the average treatment effect in covariate-adaptive randomized experiments, including randomized block experiments. Several scholars have discussed regression adjustment using additional covariates beyond block indicators (see, for example, Wang, Susukida, Mojtabai, Amin-Esmaeili & Rosenblum 2021, Ma et al. 2020, Liu et al. 2020). These studies considered a super-population framework under which the potential outcomes, covariates, and treatment indicators are independent and identically distributed. Moreover, they assumed that the number of blocks was fixed, with their sizes tending to infinity. Under a finite-population or design-based framework by conditioning the potential outcomes and covariates, with treatment assignments being the only source of randomness, Liu & Yang (2020) proposed a weighted regression adjustment method for randomized block experiments and demonstrated that this approach could increase the efficiency even when the number of blocks tends to infinity with the block sizes being fixed. However, this method has several limitations. First, it works only for homogeneous propensity scores (proportion of treated units in each block) across blocks and manages only low-dimensional covariates. Second, each block must have at least two treated and two control units, which may be unrealistic in many randomized experiments such as paired experiments and finely stratified experiments or observational studies (Fogarty 2018a, Pashley & Miratrix 2021). For example, in observational studies, full
matching is widely used to balance covariates and create fine blocks with only one treated unit or only one control unit (Rosenbaum 1991, Hansen 2004). After full matching, we can analyze the data as if they come from a finely stratified experiment (Bind & Rubin 2019). Investigations of regression adjustment methods in finely stratified experiments are limited. Third, it does not consider rerandomization in the design stage. Li & Ding (2020) established a unified theory for rerandomization followed by regression adjustment under complete randomization, but did not consider high-dimensional covariates or the combination of blocking and rerandomization in the design stage. To fill the gap, we develop general approaches and theoretical guarantees for the combination of blocking, rerandomization, and regression adjustment with high-dimensional covariates, homogeneous or heterogeneous block sizes, propensity scores, and treatment effects. Importantly, our methods do not require at least two treated and two control units in each block and thus are applicable to paired experiments, finely stratified experiments, and observational studies using full matching.

Specifically, we propose a Lasso-adjusted average treatment effect estimator for randomized block experiments from a projection perspective. We show that under mild conditions, the proposed estimator is consistent, asymptotically normal, and more efficient than, or at least as efficient as, the classic weighted difference-in-means estimator, even when the propensity scores differ across blocks or when only one treated or control unit exists in some blocks. We consider a general asymptotic regime in which both the total number of units and the number of covariates tend to infinity, allowing for a few large blocks, many small blocks, or a combination thereof. As a by-product, we establish novel concentration inequalities for the weighted sample mean and covariance under stratified randomization with possibly heterogeneous block sizes and propensity scores. Moreover, we propose a Neyman-type conservative variance estimator to facilitate valid inferences. Finally, we investigate the asymptotic properties of the proposed Lasso-adjusted estimator under stratified rerandomization (Wang, Wang & Liu 2021), which combines stratification and rerandomization during the design stage. We outline the conditions under which it is more efficient than an unadjusted estimator. As another byproduct, we extend the results of Li & Ding (2020) to high-dimensional settings.

Our asymptotic results were obtained under a finite population and randomization-based inference framework. Under this framework, the potential outcomes and covariates are fixed quantities, and the sole source of randomness is the treatment assignment. Our theory allows the outcome data-generating model to be misspecified.

The remainder of this paper is organized as follows. Section 2 introduces the framework and notation. Section 3 describes the Lasso adjustment method in randomized blocks or stratified
randomized experiments and studies its asymptotic properties. Section 4 describes a method that combines rerandomization and Lasso adjustment. Section 5 describes a novel “blocking + rerandomization + Lasso adjustment” method and establishes its asymptotic theory. The details of the simulation studies and two real data analyses are presented in Sections 6 and 7, respectively. Section 8 presents concluding remarks. All proofs are presented in the Supplementary Material.

2 Framework and notation

Consider a randomized experiment with \( n \) units and binary treatment. For each unit \( i = 1, \ldots, n \), let \( Z_i \) be an indicator of treatment assignment. Specifically, \( Z_i = 1 \) if unit \( i \) is assigned to the treatment group, and \( Z_i = 0 \) otherwise. The treatment assignments are randomized. For example, in a completely randomized experiment, the probability distribution of the treatment assignment vector \( Z = (Z_1, \ldots, Z_n)^T \) is \( P(Z = z) = \frac{n_1!n_0!}{n!}, \sum_{i=1}^n I(z_i = 1) = n_1, \sum_{i=1}^n I(z_i = 0, z_i = 0, 1, \ldots, n \) is the number of control units, and \( I(\cdot) \) is the indicator function. We define treatment effects using the Neyman–Rubin potential outcomes framework (Splawa-Neyman et al. 1990, Rubin 1974). For unit \( i \), let \( Y_i(1) \) and \( Y_i(0) \) be the potential outcomes under treatment and control, respectively. We define the unit-level treatment effect as \( \tau_i = Y_i(1) - Y_i(0) \). As each unit is assigned to either the treatment or control group, but not to both, we cannot simultaneously observe \( Y_i(1) \) and \( Y_i(0) \). Thus, \( \tau_i \) cannot be identified without strong modelling assumptions pertaining to potential outcomes. Under the stable unit treatment value assumption (SUTVA) (Rubin 1980), the average treatment effect is identifiable and is defined as \( \tau = (1/n) \sum_{i=1}^n \tau_i = (1/n) \sum_{i=1}^n \{ Y_i(1) - Y_i(0) \} \).

The observed outcome is \( Y_i = Z_iY_i(1) + (1 - Z_i)Y_i(0) \). For each unit \( i \), we observe a \( p \)-dimensional baseline/pretreatment covariate vector \( x_i = (x_{i1}, \ldots, x_{ip})^T \in \mathbb{R}^p \), where \( p \) is comparable to or even larger than \( n \). According to prior experiments or domain knowledge, certain covariates most relevant to potential outcomes may be preferentially balanced in the design stage. We denote these covariates by \( x_{iK} = (x_{ij}, j \in K)^T \in \mathbb{R}^k \), where \( K \) is the index set, and \( k \) is the dimension of these covariates. For the remaining \( p - k \) covariates, although the designer does not have prior information regarding their importance, we can perform data-driven variable selection and regression adjustment during the analysis stage. Throughout this study, we assume that \( k \) is fixed regardless of \( n \) and \( p \) diverges with \( n \). For notation simplicity, we do not index \( p \) with \( n \). The objective is to make valid and efficient inferences on the average treatment effect \( \tau \) using the observed data \( \{ Y_i, Z_i, x_i \}_{i=1}^n \).
To facilitate the discussion, we use the following notation: For an \( L \)-dimensional column vector \( \mathbf{u} = (u_1, \ldots, u_L)^T \), let \( \| \mathbf{u} \|_0, \| \mathbf{u} \|_1, \| \mathbf{u} \|_2, \) and \( \| \mathbf{u} \|_\infty \) denote the \( \ell_0, \ell_1, \ell_2 \) and \( \ell_\infty \) norms, respectively. For a subset \( S \subset \{1, \ldots, L\} \), \( S^c \) is the complementary set of \( S \), and \( \mathbf{u}_S = (u_j, j \in S)^T \) is the restriction of \( \mathbf{u} \) on \( S \). Let \( |S| \) be the cardinality of \( S \). For matrix \( A \), \( \lambda_{\text{max}}(A) \) indicates the largest eigenvalue of \( A \). Let \( \overset{d}{\to} \) and \( \overset{p}{\to} \) denote the convergence in distribution and probability, respectively. We use \( c, C, \ldots \) to denote universal constants that do not change with \( n \) but whose precise value may change from line to line.

3 Blocking and Lasso adjustment

3.1 Randomized block experiments

Blocking is a traditional approach to balancing discrete covariates in an experimental design. Experiments in which blocking is implemented are known as stratified randomized or randomized block experiments. Blocking can increase the estimation efficiency of the average treatment effect when blocking variables are relevant to the outcomes (Fisher 1926, Imai 2008, Imbens & Rubin 2015). Moreover, Liu & Yang (2020) showed that a weighted regression adjustment can further improve estimation efficiency. However, this method requires that the number of covariates is fixed, that the propensity scores are homogeneous (the same) across blocks, and that there are at least two treated and two control units in each block. We describe a method that combines blocking and high-dimensional regression adjustments to address these issues.

First, we introduce randomized block experiments. Before the physical implementation of the experiment, we stratify the units into \( M \) blocks. Let \( B_i \) denote the block indicator of unit \( i \). The covariates in \( \mathbf{x}_i \) can be continuous or categorical, but we assume that they are not linearly dependent on \( B_i \). Let \( n_{[m]} = \sum_{i=1}^n I(B_i = m) \) denote the number of units in the block \( m \) (\( m = 1, \ldots, M \)). Hereafter, subscript “\( [m] \)” indicates block-specific quantities. Let \( \pi_{[m]} = n_{[m]} / n \) denote the proportion of block size for block \( m \). Within block \( m \), \( n_{[m]} \) units are randomly assigned to the treatment group, and the remaining \( n_{[m]0} \) units are assigned to the control group. The total number of treated units is \( n_1 = \sum_{m=1}^M n_{[m]1} \). We assume that \( 1 \leq n_{[m]1} \leq n_{[m]} - 1 \), which includes both fine- and coarse-stratified experiments (a fine block has one treated or one control unit, whereas a coarse block has at least two treated and two control units). In particular, pair experiments and finely stratified experiments are special cases in our study. Let \( e_{[m]} = n_{[m]1} / n_{[m]} \) denote the propensity score, which may differ across blocks for practical reasons, such as budget restrictions. The treatment assignments are independent across blocks, and thus the probability distribution of \( \mathbf{Z} = (Z_1, \ldots, Z_n)^T \) in randomized block
experiments is $P(Z = z) = \Pi_{i=1}^{M} (n_{[m]l}!n_{[m]0}!)/n_{[m]}!$, $\sum_{i\in[m]} I(z_i = 1) = n_{[m]1}$, $z_i = 0, 1$, where $i \in [m]$ indexes unit $i$ in block $m$.

For potential outcomes or transformed potential outcomes $R_i(z)$ ($z = 0, 1$), the block-specific finite population mean and sample mean are defined as $\bar{R}_{[m]}(z) = n_{[m]}^{-1} \sum_{i\in[m]} R_i(z)$ and $\bar{R}_{[m]}(z) = n_{[m]}^{-1} \sum_{i\in[m]} I(Z_i = z) R_i(z)$, respectively. The overall finite population mean and weighted-sample mean are denoted as $\bar{R}(z) = n^{-1} \sum_{i=1}^{n} R_i(z) = \sum_{m=1}^{M} \pi_{[m]} \bar{R}_{[m]}(z)$ and $\bar{R}_z = \sum_{m=1}^{M} \pi_{[m]} \bar{R}_{[m]} z$, respectively. For finite population quantities $H = (H_1, \ldots, H_n)^T$ and $Q = (Q_1, \ldots, Q_n)^T$, where $H_i$ and $Q_i$ are potential outcomes (scalars) or covariates (column vectors), the block-specific covariance and overall covariance are denoted by $S_{[m]} H = (Q_{[m]})^T$ and $Q = (Q_1, \ldots, Q_n)^T$, where $H_i$ and $Q_i$ are potential outcomes (scalars) or covariates (column vectors), the block-specific covariance and overall covariance are denoted by $S_{[m]} H = (Q_{[m]})^T$ and $S_{[m]} H = (Q_{[m]})^T$. When $H = Q$, the subscript is occasionally simplified to $S_{[m]} H = S_{[m]} H Q$. The corresponding sample quantities are denoted by $s_{[m]} H Q$, $S_{[m]} H Q$, and $s_{[m]} H Q$. These quantities depend on $n$; however, they are not indexed with $n$ to ensure the simplicity of the notation.

The block-specific average treatment effect is $\tau_{[m]} = (1/n_{[m]}) \sum_{i\in[m]} \{ Y_i(1) - Y_i(0) \} = \bar{Y}_{[m]}(1) - \bar{Y}_{[m]}(0)$ and the overall average treatment effect is $\tau = (1/n) \sum_{i=1}^{n} \{ Y_i(1) - Y_i(0) \} = \sum_{m=1}^{M} \pi_{[m]} \tau_{[m]}$. The difference-in-means of the outcomes within block $m$ is an unbiased estimator of $\tau_{[m]}$, $\hat{\tau}_{[m]} = (1/n_{[m]}) \sum_{i\in[m]} Z_i Y_i(1) - (1/n_{[m]}) \sum_{i\in[m]} (1 - Z_i) Y_i(0) = \bar{Y}_{[m]}(1) - \bar{Y}_{[m]}(0)$. Thus, the plug-in estimator of $\tau$ is the weighted difference-in-means, as follows: $\hat{\tau}_{unadj} = \sum_{m=1}^{M} \pi_{[m]} \hat{\tau}_{[m]} = \sum_{m=1}^{M} \pi_{[m]} (\bar{Y}_{[m]}(1) - \bar{Y}_{[m]}(0))$.

Under mild conditions, $\hat{\tau}_{unadj}$ is unbiased, and $\sqrt{n}(\hat{\tau}_{unadj} - \tau)$ is asymptotically normal with a mean of zero and variance of $\sigma^2_{unadj} = \lim_{n \to \infty} \sum_{m=1}^{M} \pi_{[m]} \{ S_{[m]}^2 Y(1)/e_{[m]} + S_{[m]}^2 Y(0)/(1 - e_{[m]}) - S_{[m]}^2 (Y(1) - Y(0)) \}$. When $n_{[m]z} \geq 2$ for all $z$ and $m$, the asymptotic variance can be conservatively estimated by $\sum_{m=1}^{M} \pi_{[m]} \{ s_{[m]}^2 Y(1)/e_{[m]} + s_{[m]}^2 Y(0)/(1 - e_{[m]}) \}$. When there are fine blocks with $n_{[m]z} = 1$, we can use the variance estimator proposed by Pashley & Miratrix (2021).

Specifically, let $A_c = \{ 1 \leq m \leq M : n_{[m]1} > 1, n_{[m]0} > 1 \}$ and $A_f = \{ 1 \leq m \leq M : n_{[m]1} = 1 \text{ or } n_{[m]0} = 1 \}$ denote sets of coarse and fine blocks, respectively. Let $n_f = \sum_{m \in A_f} n_{[m]}$ be the total number of units in fine blocks. For $m \in A_f$, we define a weight $\omega_{[m]} = n_{[m]}^2/(n_f - 2n_{[m]})$ and assume $n_f > 2n_{[m]}$ throughout the paper. Let $\hat{\tau}_f = \sum_{m \in A_f} (n_{[m]}/n_f) \hat{\tau}_{[m]}$. Then, $\sigma^2_{unadj}$ can be estimated using (Pashley & Miratrix 2021)

$$\sigma^2_{unadj} = \sum_{m \in A_c} \pi_{[m]} \{ S_{[m]}^2 Y(1)/e_{[m]} + S_{[m]}^2 Y(0)/(1 - e_{[m]}) \} + \left( \frac{\hat{\tau}_f}{n_f} \right)^2 \frac{n_f}{n_f + \sum_{m \in A_f} \omega_{[m]} \sum_{m \in A_f} \omega_{[m]} (\hat{\tau}_{[m]} - \hat{\tau}_f)^2}.$$
The unadjusted estimator $\hat{\tau}_{\text{unadj}}$ does not incorporate the covariate information. In the following section, we introduce a covariate-adjusted average treatment effect estimator to adjust for covariate imbalance and improve estimation efficiency.

3.2 Regression adjustment from a projection perspective

In this part, we derive a regression adjustment method with low-dimensional covariates ($p \ll n$) based on a projection perspective. We extend this method to high-dimensional settings in the next section.

As the baseline covariates are not affected by the treatment, the average treatment effect of the covariates is $\tau_x = \sum_{m=1}^{M} \pi[m] \{ \bar{x}_m - \bar{x}_m \} = 0$. Let $\hat{\tau}_x = \sum_{m=1}^{M} \pi[m] (\bar{x}_m - \bar{x}_m)$ be the weighted difference-in-means estimator for $\tau_x$. To decrease the variance of $\hat{\tau}_{\text{unadj}}$, we can project it onto $\hat{\tau}_x$. We define the projection coefficient vector $\gamma_{\text{proj}}$ as follows:

$$\gamma_{\text{proj}} = \arg \min_{\gamma} \mathbb{E}(\hat{\tau}_{\text{unadj}} - \tau - \hat{\tau}_x \gamma)^2 = \text{cov}(\hat{\tau}_x)^{-1} \text{cov}(\hat{\tau}_x, \hat{\tau}_{\text{unadj}})$$

$$= \left\{ \sum_{m=1}^{M} \pi[m] \frac{S_{m|X}^2}{e[m](1-e[m])} \right\}^{-1} \left\{ \sum_{m=1}^{M} \pi[m] \frac{S_{m|XY}(1)}{e[m]} + \sum_{m=1}^{M} \pi[m] \frac{S_{m|X}^2}{1-e[m]} \right\}.$$  

The oracle projection estimator $\tilde{\tau}_{\text{proj}} = \hat{\tau}_{\text{unadj}} - \hat{\tau}_x \gamma_{\text{proj}}$ is consistent, asymptotically normal, and has the smallest asymptotic variance among the estimators that are asymptotically equivalent to $\hat{\tau}_{\text{unadj}} - \hat{\tau}_x \gamma$ for some adjusted vector $\gamma \in \mathbb{R}^p$.

However, $\tilde{\tau}_{\text{proj}}$ is not feasible in practice, because it depends on the unknown vector $\gamma_{\text{proj}}$. To consistently estimate $\gamma_{\text{proj}}$, we decompose it into two terms: Let $e_{[m]} = ze_{[m]} + (1-z)(1-e_{[m]})$, $z = 0, 1$. Then, $\gamma_{\text{proj}} = \gamma(0) + \gamma(1)$ with

$$\gamma(z) = \left\{ \sum_{m=1}^{M} \pi[m] \frac{S_{m|X}^2}{e[m](1-e[m])} \right\}^{-1} \left\{ \sum_{m=1}^{M} \pi[m] \frac{S_{m|XY}(z)}{e[m]} \right\}, \quad z = 0, 1.$$  

Intuitively, when $n_{[m]} \geq 2$, we can estimate $\gamma(z)$ by replacing the block-specific covariances $S_{m|X}^2$ and $S_{m|XY}(z)$ with the corresponding sample covariances $s_{m|X}^2(X(z)$ and $s_{m|XY}(z)$ in block $m$ under treatment arm $z$. When $n_{[m]} = 1$ for some $m$ and $z$, both $s_{m|X}^2(X(z)$ and $s_{m|XY}(z)$ are not well-defined. To address this issue, we can use the following estimators:

$$s_{m|X}^2(X(z) = \frac{n_{[m]}}{n_{[m]}(n_{[m]} - 1)} \sum_{i \in [m]} I(Z_i = z) (x_i - \bar{x}_m)(x_i - \bar{x}_m)^T,$$

$$s_{m|XY}(z) = \frac{n_{[m]}}{n_{[m]}(n_{[m]} - 1)} \sum_{i \in [m]} I(Z_i = z) (x_i - \bar{x}_m) Y_i,$$
that are unbiased and well-defined when $1 \leq n_{[m]z} \leq n_{[m]} - 1$. Then, we can estimate $\gamma(z)$ by the following plug-in estimator,

$$\hat{\gamma}_{\text{obs},z} = \left\{ \sum_{m=1}^{M} \frac{\tilde{s}_{[m]}^2 x(z)}{e_{[m]z}(1 - e_{[m]z})} \right\}^{-1} \left\{ \sum_{m=1}^{M} \frac{\tilde{s}_{[m]} x Y(z)}{e_{[m]z}} \right\}$$

$$= \arg \min_{\gamma} \sum_{m=1}^{M} \sum_{i \in [m], z_i = z} \frac{n_{[m]}^2}{e_{[m]z} n_{[m]} z (n_{[m]} - 1)} \left\{ \sqrt{1 - e_{[m]z}} Y_i - \frac{1}{\sqrt{1 - e_{[m]z}}} (x_i - \bar{x}_{[m]})^T \gamma \right\}^2.$$ 

To simplify the above expression, we introduce several weights. For $i \in [m]$, let $\omega_i(z) = n_{[m]}^2 / \{e_{[m]z} n_{[m]} z (n_{[m]} - 1)\}$, $\omega_i Y(z) = 1 - e_{[m]z}$, and $\omega_i x(z) = 1 / (1 - e_{[m]z})$. The weighted potential outcomes and covariates are denoted by $Y_i^\omega(z) = \sqrt{\omega_i(z) \omega_i Y(z)} Y_i(z)$ and $x_i^\omega(z) = \sqrt{\omega_i(z) \omega_i x(z)} (x_i - \bar{x}_{[m]})$, respectively. Note that because the weights are different for $z = 0, 1$, we have two groups of weighted covariates $x_i^\omega(z)$. Then, $\hat{\gamma}_{\text{obs},z}$ can be rewritten as

$$\hat{\gamma}_{\text{obs},z} = \arg \min_{\gamma} \sum_{i: z_i = z} \left\{ Y_i^\omega - (x_i^\omega)^T \gamma \right\}^2,$$

where $Y_i^\omega = Z_i Y_i^\omega(1) + (1 - Z_i) Y_i^\omega(0)$ and $x_i^\omega = Z_i x_i^\omega(1) + (1 - Z_i) x_i^\omega(0)$.

The above equation indicates that $\hat{\gamma}_{\text{obs},z}$ is obtained through a regression without an intercept. We show that $\hat{\gamma}_{\text{obs},z}$ has the same asymptotic limit as

$$\hat{\gamma}_{\text{obs},z} = \arg \min_{\gamma} \sum_{i: z_i = z} \left\{ Y_i^\omega - (x_i^\omega - \bar{x}_{i}^\omega)^T \gamma \right\}^2,$$

where $\bar{x}_{i}^\omega = n_{i}^{-1} \sum_{i: z_i = z} x_i^\omega$. The simulation results indicate that $\hat{\gamma}_{\text{obs},z}$ performs better than $\hat{\gamma}_{\text{obs},z}$ in finite samples. Thus, our final recommendation is $\hat{\tau}_{\text{proj}} = \hat{\tau}_{\text{unadj}} - \hat{\tau}_{\text{adj}} \hat{\gamma}_{\text{obs}}$, where $\hat{\gamma}_{\text{obs}} = \hat{\gamma}_{\text{obs},1} + \hat{\gamma}_{\text{obs},0}$. 

**Remark 1.** In the paired experiments, $\hat{\tau}_{\text{proj}}$ is equivalent to the regression-adjusted estimator $\hat{\tau}_{R1}$ proposed by Fogarty (2018b).

### 3.3 Lasso adjustment in randomized block experiments

In a high-dimensional setting, if many of the covariates do not affect the potential outcomes, it is reasonable to assume that the projection coefficient $\gamma_{\text{proj}}$ is sparse. More precisely, let $S \in \{1, \ldots, p\}$ be the set of relevant covariates and let $s = |S|$. We assume that $s \ll n$. We still
use $\gamma_{\text{proj}} = \gamma(0) + \gamma(1)$ to denote the projection coefficient, where $\{\gamma(z)\}_{S^c} = 0$ and

$$\{\gamma(z)\}_S = \left\{ \sum_{m=1}^M \pi_{[m]} \frac{S_{[m]}'X_S}{e_{[m]}(1-e_{[m]})} \right\}^{-1} \left\{ \sum_{m=1}^M \pi_{[m]} \frac{S_{[m]}'X_SY(z)}{e_{[m]z}} \right\}, \quad z = 0, 1.$$  

We can estimate $\gamma(z)$ using Lasso

$$\hat{\gamma}_{\text{lasso},z} = \arg\min_{\gamma} \frac{1}{2n} \sum_{i:Z_i=z} \left\{ Y_i^\omega - (x_i^\omega - \bar{x}_z^\omega)^\top \gamma \right\}^2 + \lambda_z \| \gamma \|_1,$$

where $\lambda_z$ is the tuning parameter, $z = 0, 1$. We can choose $\lambda_z$ by cross validation. We replace $\gamma_{\text{proj}}$ with its estimator $\hat{\gamma}_{\text{lasso}} = \hat{\gamma}_{\text{lasso},1} + \hat{\gamma}_{\text{lasso},0}$ and obtain the projection-originated Lasso-adjusted estimator, $\hat{\tau}_{\text{lasso}} = \hat{\tau}_{\text{unadj}} - \hat{\tau}_{\text{lasso}}^T \hat{\gamma}_{\text{lasso}}$.

**Remark 2.** For the case of equal propensity scores, $e_{[m]} = e^s$, $m = 1, \ldots, M$, Liu & Yang (2020) propose the following weighted regression with low-dimensional covariates:

$$\sqrt{\omega_{C,i}} Y_i \sim \alpha \sqrt{\omega_{C,i}} + \tau \sqrt{\omega_{C,i}} Z_i$$

$$+ \sum_{m=2}^M \zeta_{[m]0} \sqrt{\omega_{C,i}} I(B_i = m) + \sum_{m=2}^M \zeta_{[m]1} \sqrt{\omega_{C,i}} Z_i \{ I(B_i = m) - \pi_{[m]} \}$$

$$+ \sqrt{\omega_{C,i}} x_i^T \theta_0 + \sqrt{\omega_{C,i}} Z_i \left\{ x_i - \sum_{m=1}^M I(B_i = m) \bar{x}_{[m]} \right\}^T \theta_1,$$  

(1)

where $\omega_{C,i} = n_{[m]}/(n_{[m]z} - 1)$ for $i \in [m]$ and $Z_i = z$; $\alpha$, $\tau$, $\zeta_{[m]0}$, $\zeta_{[m]1}$, $\theta_0$, and $\theta_1$ are the parameters being estimated. The regression-adjusted average treatment effect estimator $\hat{\tau}_{\text{ols}}$ is defined as the ordinary least squares (OLS) estimator of $\tau$ (coefficient of $\sqrt{\omega_{C,i}} Z_i$). The Lasso-adjusted estimator $\hat{\tau}_{\text{lasso}}$ is a generalization of the OLS-adjusted estimator to high-dimensional settings, but we use different weights to deal with heterogeneous propensity scores across blocks and remove the requirement of $n_{[m]z} \geq 2$ for all $m = 1, \ldots, M$ and $z = 0, 1$.

To investigate the asymptotic properties of $\hat{\tau}_{\text{lasso}}$, we decompose the original potential outcomes and define the approximation error $\varepsilon_i^*(z)$ as follows:

$$Y_i(z) = Y_{[m]}[z] + (x_i - \bar{x}_{[m]})^T \gamma_{\text{proj}} + \varepsilon_i^*(z), \quad i \in [m], \quad z = 0, 1.$$  

We require the following regularity conditions to guarantee the asymptotic normality of $\hat{\tau}_{\text{lasso}}$.

**Condition 1.** There exists a constant $c \in (0, 0.5)$ independent of $n$ such that $c \leq e_{[m]} \leq 1 - c$ for $m = 1, \ldots, M$.  

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Condition 2. There exists a constant $L < \infty$ independent of $n$ such that for $z = 0, 1$ and $j = 1, \ldots, p$,
\[
\frac{1}{n} \sum_{m=1}^{M} \sum_{i \in [m]} (x_{ij} - \bar{x}_{[m],j})^4 \leq L, \quad \frac{1}{n} \sum_{m=1}^{M} \sum_{i \in [m]} (\varepsilon_i^*(z))^4 \leq L, \quad \frac{1}{n} \sum_{m=1}^{M} \sum_{i \in [m]} (\bar{\varepsilon}_i^*(z))^4 \leq L,
\]
where $\varepsilon_i^*(z) = Y_i^*(z) - \bar{Y}_i^*(z) - (x_i^*(z) - \bar{x}^*(z))^T\gamma(z)$.

Condition 3. The weighted variances $\sum_{m=1}^{M} \pi[m] S^2_{[m]}/e_{[m]}$, $\sum_{m=1}^{M} \pi[m] S^2_{[m]}e_{(0)/(1 - e_{[m]})}$, and $\sum_{m=1}^{M} \pi[m] S^2_{[m]}(\varepsilon(1) - \varepsilon(0))$ tend to finite limits, with positive values for the first two terms. The limit of
\[
\sum_{m=1}^{M} \pi[m] S^2_{[m]}e_{(1)/e_{[m]}} + \sum_{m=1}^{M} \pi[m] S^2_{[m]}e_{(0)/(1 - e_{[m]})} - \sum_{m=1}^{M} \pi[m] S^2_{[m]}(\varepsilon(1) - \varepsilon(0))
\]
is strictly positive.

Condition 4. There exist constants $C > 0$ and $\xi > 1$ independent of $n$ such that $\|h_S\|_1 \leq C\|VXh\|_\infty$, $\forall h \in \{h : \|h_S\|_1 \leq \xi\|h_S\|_1\}$, where $VXX = \sum_{m=1}^{M} \pi[m] S^2_{[m]}X/e_{[m]}(1 - e_{[m]})$.

Condition 5. There exist constants $C > 0$ and $0 < \eta < (\xi - 1)/(\xi + 1)$ such that the tuning parameters of Lasso satisfy
\[
s\sqrt{\log p}\lambda_z \rightarrow 0 \quad \text{and} \quad \lambda_z \geq \frac{1}{\eta} \left\{ C\sqrt{\frac{\log p}{n}} + \delta_n \right\}, \quad z = 0, 1,
\]
where $\delta_n = \max_{z=0,1} \|\sum_{m=1}^{M} (\pi[m] - n^{-1})e_{[m]}S_{[m]}^2X^\varepsilon(0)(z)\|_\infty$.

Remark 3. Conditions 1–3 are required for deriving the asymptotic normality of low-dimensional regression-adjusted treatment effect estimators (Freedman 2008a, Lin 2013). Conditions 4–5 are similar to the typical conditions for deriving the $\ell_1$ convergence rate of Lasso under complete randomization, with treated units being sampled without replacement from the finite population (Bloniarz et al. 2016).

Theorem 1. Under Conditions 1–5, $\sqrt{p}(\hat{\tau}_{lasso} - \tau) \xrightarrow{d} N(0, \sigma^2_{lasso})$, where
\[
\sigma^2_{lasso} = \lim_{n \to \infty} \sum_{m=1}^{M} \pi[m] \left\{ \frac{S^2_{[m]e_{(1)/e_{[m]}}}^2}{\pi[m]} + \frac{S^2_{[m]e_{(0)}}}{1 - e_{[m]}} - \frac{S^2_{[m]}(\varepsilon(1) - \varepsilon(0))}{e_{[m]}(1 - e_{[m]})} \right\}.
\]
Furthermore, $\hat{\tau}_{lasso}$ is asymptotically more efficient than $\hat{\tau}_{unadj}$; that is,
\[
\sigma^2_{lasso} - \sigma^2_{unadj} = -\lim_{n \to \infty} \gamma^T_{\text{proj}} \left\{ \sum_{m=1}^{M} \pi[m] S^2_{[m]}X / e_{[m]}(1 - e_{[m]}) \right\} \gamma_{\text{proj}} \leq 0.
\]
Remark 4. Because \( \sigma^2_{\text{lasso}} = \lim_{n \to \infty} E\{\hat{\tau}_{\text{unadj}} - \tau - \hat{\tau}_x^T \gamma_{\text{proj}}\}^2 \), Theorem 1 indicates that \( \hat{\tau}_{\text{lasso}} \) is not only feasible but also has the same asymptotic distribution as \( \hat{\tau}_{\text{proj}} \) even for unequal propensity scores. In other words, \( \hat{\tau}_{\text{lasso}} \) has the smallest asymptotic variance among the estimators that are asymptotically equivalent to \( \hat{\tau}_{\text{unadj}} - \hat{\tau}_x^T \gamma, \gamma \in \mathbb{R}^p \).

Remark 5. The proof of Theorem 1 relies on novel concentration inequalities for the weighted sample mean and covariance under stratified randomization. These inequalities are crucial for deriving the \( l_1 \) convergence rate of the Lasso estimator in a finite population and randomization-based inference framework. We obtain these inequalities in general asymptotic regimes, including the cases of (1) \( M \) tending to infinity with fixed \( n_{[m]} \) and (2) \( n_{[m]} \) tending to infinity with fixed \( M \). These inequalities are of independent interest in other fields where stratified sampling without replacement is performed. This aspect is discussed extensively in the Supplementary Material.

Similar to \( \hat{\sigma}^2_{\text{unadj}} \), we can conservatively estimate \( \sigma^2_{\text{lasso}} \) by replacing outcomes \( Y_i \) with the adjusted outcomes \( R_i = Y_i - (x_i - \bar{x}[m])^T \gamma_{\text{lasso}} \). Specifically, let \( \hat{\tau}_{R,[m]} = \bar{R}_{[m]} - \bar{R}_{[m]0} \) and \( \hat{\tau}_{R,f} = \sum_{m \in A_f} (n_{[m]}/n_f) \hat{\tau}_{R,[m]} \). Then, \( \sigma^2_{\text{lasso}} \) can be estimated by:

\[
\hat{\sigma}^2_{\text{lasso}} = \frac{n}{n - s} \left[ \sum_{m \in A_c} \pi_{[m]} \left\{ \frac{s^2_{[m] R(1)}}{e_{[m]}} + \frac{s^2_{[m] R(0)}}{1 - e_{[m]}} \right\} + \left( \frac{n_f}{n} \right)^2 \frac{n}{n_f + \sum_{m \in A_f} \omega_{[m]}} \sum_{m \in A_f} \omega_{[m]} (\hat{\tau}_{R,[m]} - \hat{\tau}_{R,f})^2 \right],
\]

where \( s = ||\gamma_{\text{lasso}}||_0 \). The factor \( n/\{n - s\} \) adjusts for the degrees of freedom of the residuals to achieve better finite sample performance.

Condition 6. There exists a constant \( C < \infty \) such that \( \Lambda_{\text{max}}(VXX^T) \leq C \) and \( n^{-1} \sum_{i=1}^n \{Y_i(z) - (x_i - \bar{x}[m])^T \gamma_{\text{proj}}\}^2 \leq C, \ z = 0, 1 \).

Theorem 2. Under Conditions 1–6, \( \hat{\sigma}^2_{\text{lasso}} \) converges in probability to

\[
\sigma^2_{\text{lasso}} + \lim_{n \to \infty} \left\{ \sum_{m \in A_c} \pi_{[m]} S^2_{[m]}(\epsilon^{*}(1) - \epsilon^{*}(0)) + \left( \frac{n_f}{n} \right)^2 \frac{n}{n_f + \sum_{m \in A_f} \omega_{[m]}} \sum_{m \in A_f} \omega_{[m]} (\tau_{[m]} - \tau_f)^2 \right\},
\]

which is no less than \( \sigma^2_{\text{lasso}} \) and no greater than the probability limit of \( \hat{\sigma}^2_{\text{unadj}} \).

If the treatment effects within each coarse block are constant; that is, \( \tau_i = \tau_{[m]} \) for all \( i \in [m] \), then we have \( S^2_{[m]}(\epsilon^{*}(1) - \epsilon^{*}(0)) = 0 \). If further the average treatment effects across fine blocks are constants, i.e., \( \tau_{[m]} = \tau_f \) for all \( m \in A_f \), \( \hat{\sigma}_{\text{lasso}} \) is a consistent estimator of \( \sigma^2_{\text{lasso}} \). In general, \( \hat{\sigma}_{\text{lasso}} \) is a conservative variance estimator. Given \( 0 < \alpha < 1 \), let \( q_{\alpha/2} \) denote the upper \( \alpha/2 \)
Algorithm 1 (Stratified) Rerandomization using the Mahalanobis distance

1. Collect covariate data $X_K, i = 1, \ldots, n$.
2. (Re-)Randomize units into the treatment and control groups by complete randomization (or stratified randomization) and obtain the treatment assignment vector $Z$.
3. If $\text{Ma}(Z, X_K) \leq a$, proceed to Step 4; otherwise, return to Step 2;
4. Conduct the physical experiment using treatment assignment $Z$.

Quantile of the standard normal distribution. Based on Theorems 1 and 2, we can construct an asymptotically valid $1 - \alpha$ confidence interval for $\tau$:

$$\left[ \hat{\tau}_{\text{lasso}} - q_{\alpha/2} \hat{\sigma}_{\text{lasso}} / \sqrt{n}, \hat{\tau}_{\text{lasso}} + q_{\alpha/2} \hat{\sigma}_{\text{lasso}} / \sqrt{n} \right],$$

whose asymptotic coverage rate is greater than or equal to $1 - \alpha$. Moreover, the length of this confidence interval is less than or equal to that constructed by $(\hat{\tau}_{\text{unadj}}, \hat{\sigma}_{\text{unadj}}^2)$. Therefore, $\hat{\tau}_{\text{lasso}}$ improves, or at least does not degrade, the estimation and inference efficiencies, regardless of equal or unequal propensity scores.

4 Rerandomization and Lasso adjustment

4.1 Rerandomization

Although blocking is widely used in practice, it can only balance discrete covariates. Rerandomization is a more general approach for balancing discrete and continuous covariates (Morgan & Rubin 2012). Rerandomization discards the treatment assignments that lead to covariate imbalances and accepts only those assignments that fulfill a pre-specified balance criterion. Morgan & Rubin (2012) proposed the use of the Mahalanobis distance of the covariate means in the treatment and control groups to measure the covariate imbalances. Suppose we want to balance the covariates $X_K$, where $K \subset \{1, \ldots, p\}$. In completely randomized experiments, we define $\hat{\tau}_{x,K} = \bar{x}_{1,K} - \bar{x}_{0,K}$ and the Mahalanobis distance as $\text{Ma}(Z, X_K) = (\hat{\tau}_{x,K})^T \{ \text{cov}(\hat{\tau}_{x,K}) \}^{-1} \hat{\tau}_{x,K}$.

A treatment assignment is acceptable if and only if the corresponding Mahalanobis distance is less than or equal to a pre-specified threshold $a > 0$; that is, $\text{Ma}(Z, X_K) \leq a$. We denote $\mathcal{M}_a = \{ Z : \text{Ma}(Z, X_K) \leq a \}$ the set of acceptable treatment assignments. Li et al. (2018) suggested choosing a suitable $a$ to ensure that the probability of a treatment assignment satisfying the balance criterion equals a certain value, for example, $p_a = P\{\text{Ma}(Z, X_K) \leq a \} = 0.001$.

A general rerandomization procedure is presented in Algorithm 1. Under mild conditions, the average treatment effect estimator $\hat{\tau}_{\text{unadj}}$ (with $M = 1$) subjected to randomization is consistent and asymptotically a convolution of a normal distribution and a truncated normal distribution. Moreover, the asymptotic variance is no greater than that of $\hat{\tau}_{\text{unadj}}$ under complete randomization, as reported by Morgan & Rubin (2012) and Li et al. (2018) and the references therein.
4.2 Rerandomization plus Lasso

Rerandomization is useful for balancing a fixed number of covariates $X_K$ that are most relevant to potential outcomes. However, the other covariates in $X$ may also be predictive of potential outcomes and remain imbalanced under rerandomization.

This phenomenon has motivated researchers to consider regression adjustment under rerandomization to adjust for the remaining covariate imbalances and further increase efficiency. Li & Ding (2020) showed that rerandomization followed by regression adjustment using the OLS could increase the estimation and inference efficiencies when the analyzer uses both the covariates adopted in the design stage and additional covariates available only in the analysis stage. Nevertheless, this conclusion is only true for low-dimensional covariates and no blocking. The proposed Lasso-adjusted estimator $\hat{\tau}_{lasso}$ can be used to extend this principle to high-dimensional settings with a general blocking scheme. In the next section, we examine its asymptotic properties in the case of rerandomization as a special case of stratified rerandomization with $M = 1$.

5 Blocking, rerandomization, and Lasso adjustment

5.1 Stratified rerandomization (blocking plus rerandomization)

Both blocking and rerandomization are powerful methods for balancing a fixed number of covariates. Scholars have recommended combining these two methods in the design stage. Recently, Johansson & Schultzberg (2022) proposed a stratified rerandomization strategy in which stratified randomization is implemented instead of complete randomization in step 2 of Algorithm 1. However, this method is only applicable in the case of equal propensity scores, mainly because the simple difference-in-means estimator, $\bar{x}_{1,K} - \bar{x}_{0,K}$, may be asymptotically biased for $\tau_{x,K}$ when the propensity scores differ across blocks. To address this issue, Wang, Wang & Liu (2021) modified the definition of $\hat{\tau}_{x,K}$: $\hat{\tau}_{x,K} = \sum_{m=1}^{M} \pi[m] \{ (\bar{x}_{[m]1})_K - (\bar{x}_{[m]0})_K \}$, and showed that the asymptotic distribution of $\hat{\tau}_{unadj}$ in the modified stratified rerandomization strategy is a convolution of a normal distribution and a truncated normal distribution, and its asymptotic variance, denoted as $\sigma^2_{unadj,M_a}$, is less than or equal to that of $\hat{\tau}_{unadj}$ in the case of stratified randomization. Moreover, the asymptotic variance can be estimated using a conservative estimator $\hat{\sigma}^2_{unadj,M_a}$, as indicated in the Supplementary Material. These conclusions hold for cases involving equal or unequal propensity scores.
5.2 Stratified rerandomization plus Lasso

The efficiency of \( \hat{\tau}_{\text{unadj}} \) in the case of Wang et al.’s stratified rerandomization (referred to as stratified rerandomization in the following text) can be further increased by adjusting the remaining imbalances in \( X \). To handle both coarse and fine blocks, homogeneous and heterogeneous block sizes, propensity scores, and treatment effects, we suggest using the projection estimator \( \hat{\tau}_{\text{lasso}} \) introduced in Section 3.2. The asymptotic property of \( \hat{\tau}_{\text{lasso}} \) under stratified rerandomization is presented below. For simplicity, we assume that \( K \subset S \).

**Condition 7.** The weighted covariances 
\[
\sum_{m=1}^{M} \pi_{[m]} S_{[m]}^2 X_K / e_{[m]}, \sum_{m=1}^{M} \pi_{[m]} S_{[m]}^2 X_K / (1 - e_{[m]}), \sum_{m=1}^{M} \pi_{[m]} S_{[m]}^2 X_K e^*(1) / e_{[m]}, \text{ and } \sum_{m=1}^{M} \pi_{[m]} S_{[m]}^2 X_K e^*(0) / (1 - e_{[m]}) \]
tend to finite limits, and the limit of \( V_{X_K X_K} = \sum_{m=1}^{M} \pi_{[m]} S_{[m]}^2 / \{e_{[m]} (1 - e_{[m]})\} \) is strictly positive definite.

**Theorem 3.** Under Conditions 1–5 and 7, for fixed \( a > 0 \), \( \{\sqrt{n}(\hat{\tau}_{\text{lasso}} - \tau) \mid M_a\} \xrightarrow{d} N(0, \sigma_{\text{lasso}}^2) \). Furthermore, \( \hat{\tau}_{\text{lasso}} \) is asymptotically more efficient than \( \hat{\tau}_{\text{unadj}} \) under stratified rerandomization, as implied by \( \sigma_{\text{lasso}}^2 \leq \sigma_{\text{unadj}}^2 | M_a \leq \sigma_{\text{unadj}}^2 \).

**Remark 6.** The asymptotic distribution of the unadjusted estimator under rerandomization is typically a convolution of a normal distribution and a truncated normal distribution (Li et al. 2018). In contrast, the asymptotic distribution of the OLS-adjusted estimator under rerandomization is normal if the covariates used in the design stage are included in the regression adjustment (Li & Ding 2020). Because the Lasso-adjusted estimator uses all covariates in \( K \), the set of covariates used in the rerandomization, the asymptotic distribution of \( \hat{\tau}_{\text{lasso}} \) under stratified rerandomization is normal. Theorem 3 implies that \( \hat{\tau}_{\text{lasso}} \) has the same asymptotic distribution in the stratified randomization and rerandomization scenarios. Thus, the discussion in Remark 4 regarding the optimality of this estimator remains valid.

Theorem 3 shows that the asymptotic distribution of \( \hat{\tau}_{\text{lasso}} \) in the stratified rerandomization case is normal. Moreover, the asymptotic variance of \( \hat{\tau}_{\text{lasso}} \) is no greater than that of \( \hat{\tau}_{\text{unadj}} \) in the stratified randomization and stratified rerandomization scenarios, even for cases involving unequal propensity scores or fine blocks. Thus, the efficiency achieved using \( \hat{\tau}_{\text{lasso}} \) is never lower than that achieved using \( \hat{\tau}_{\text{unadj}} \).

Compared with stratified randomization, the asymptotic efficiency in the stratified rerandomization scenario does not increase when \( \hat{\tau}_{\text{lasso}} \) is used in the analysis stage. Similar conclusions were derived by Li & Ding (2020), who examined the combination of rerandomization and OLS adjustment. However, as discussed by Li & Ding (2020), rerandomization is still preferred because it enables a more transparent analysis and avoids the bias associated with searching for a specific outcome model during the analysis stage (Cox 2007, Rosenbaum 2010, Lin 2013).
Moreover, our simulation results in the next section indicate that stratified rerandomization can decrease the mean squared error of $\hat{\tau}_{\text{lasso}}$ in finite samples and is thus recommended.

**Theorem 4.** Under Conditions 1–7, for fixed $a > 0$, $\sigma^2_{\text{lasso}} |\mathcal{M}_a$ converges in probability to

$$\sigma^2_{\text{lasso}} + \lim_{n \to \infty} \left\{ \sum_{m \in A_c} \pi_{[m]} S^2_{[m]} (e^*(1) - e^*(0)) + \left( \frac{n_f}{n} \right)^2 \frac{n}{n} + \sum_{m \in A_f} \omega_{[m]} (\tau_{[m]} - \tau_f)^2 \right\},$$

which is no less than $\sigma^2_{\text{lasso}}$. Moreover, $\sigma^2_{\text{lasso}} \leq \hat{\sigma}^2_{\text{unadj}}, \mathcal{M}_a \leq \hat{\sigma}^2_{\text{unadj}}$ holds in probability.

In the stratified rerandomization case, the variance estimator $\hat{\sigma}^2_{\text{lasso}}$ is consistent if the unit-level treatment effects are constant within each coarse block and the block-specific average treatment effects are constant across the fine blocks. Generally, the estimator is conservative. Based on Theorems 3 and 4, we can construct an asymptotically valid $1 - \alpha$ confidence interval for $\tau$: $[\hat{\tau}_{\text{lasso}} - q_\alpha/2\hat{\sigma}_{\text{lasso}}/\sqrt{n}, \hat{\tau}_{\text{lasso}} + q_\alpha/2\hat{\sigma}_{\text{lasso}}/\sqrt{n}]$, whose asymptotic coverage rate is greater than or equal to $1 - \alpha$. The length of this confidence interval is less than or equal to that based on the estimated asymptotic distributions of $\hat{\tau}_{\text{unadj}}$ in the stratified randomization and stratified rerandomization cases. Therefore, $\hat{\tau}_{\text{lasso}}$ is the most efficient estimator for all the considered scenarios.

### 6 Simulation

This section describes simulation studies performed to examine the finite-sample performance of the proposed methods. We set the sample size as $n = 300$ and 600. We consider four types of blocks: two large coarse blocks with $n_{[m]} = 150$ or 300 and $M = 2$, many small coarse blocks with $n_{[m]} = 10$ and $M = 30$ or 60, hybrid coarse blocks with $n^S_{[m]} = 10$, $M^S = 10$ or 20, $n^L_{[m]} = 100$ or 200, and $M^L = 2$, where the subscripts “S” and “L” denote small and large blocks, respectively, and many triplet fine blocks with $n_{[m]} = 3$ and $M = 100$ or 200. For the first three types of blocks, we consider equal propensity scores with $e_{[m]}$ equal to 0.5 and unequal propensity scores with $e_{[m]}$ evenly spaced in values between 0.3 and 0.7. For the last type of block, we set $e_{[m]}$ to be 2/3 or evenly spaced in values between 0.3 and 0.7. The number of treated units in each block is equal to $\text{round}(e_{[m]} n_{[m]})$. We also consider a scenario without blocking and set $n_1 = \sum_{i=1}^n Z_i = 0.5n$. The potential outcomes are generated as follows: $Y_i(z) = (B_i/M)^{z+1} + x_i^T \beta(z) - 2x_{bc,i}^T \beta(z) + \varepsilon_i(z)$, $i = 1, ..., n$, $z = 0, 1$, where $x_i$ is generated from a $p$-dimensional multivariate normal distribution $N(0, \Sigma)$ with $\Sigma_{ij} = \rho^{|i-j|}$. $x_{bc,i}$ is generated by centering $x_i$ in each block; the first $s$ elements of $\beta(z)$ are generated from the uniform distribution on $[0, 1]$, the remaining elements are zero, and $\varepsilon_i(z)$ is generated from a
normal distribution with a mean of zero and variance of $\sigma^2$ such that the signal-to-noise ratio is equal to 5. We set $p = 400$, $s = 10$, and $\rho = 0.6$. The potential outcomes and covariates are generated once and then kept fixed.

For each scenario, we consider two designs (with/without rerandomization) and two estimators (unadjusted and Lasso-adjusted estimators). We set $x_{i,k}$ as the first $k = 5$ dimensions of $x_i$ and $p_o = 0.001$ for rerandomization. We use the R package “glmnet” to fit the solution path of Lasso. We chose the tuning parameter in Lasso via 10-fold cross validation. We replicate the randomization/rerandomization 1000 times to evaluate the repeated sampling properties.

Figure 1 shows the distributions (violin plots) of different estimators. All distributions are symmetric around the true value of the average treatment effect. The distributions of the Lasso-adjusted estimator are more concentrated than those of the unadjusted estimator under both randomization and rerandomization. Tables 1 and 2 present several summary statistics for different estimators. First, for all designs, the absolute value of the bias of each estimator is considerably smaller than the standard deviation (SD). Second, compared with the unadjusted estimator without rerandomization, the Lasso-adjusted estimator without rerandomization reduces the standard deviation and root mean squared error (RMSE) by 54%–77%. Third, the empirical coverage probabilities (CP) of all estimators reach the nominal level 95% (in a few cases, the coverage probabilities are less than 95% but very close to 95%). Fourth, compared to the unadjusted estimator without rerandomization, the Lasso-adjusted estimator without reran-
domination decreases the mean confidence interval length (Length) by 37%–58%. Finally, the combination of stratified rerandomization and Lasso adjustment can further decrease the mean squared error. Thus, our final recommendation is to implement stratified rerandomization in the design stage and to use the Lasso-adjusted estimator in the analysis stage.

7 Real data illustration

7.1 “Opportunity knocks” experiment

In this part, we use experimental data to illustrate the merits of the combination of stratified rerandomization and the Lasso adjustment. The “Opportunity Knocks” (OK) randomized experiment aimed at evaluating the effect of academic achievement awards on the academic performance of college students (Angrist et al. 2014). Based on sex and discretized high school grades, second-year college students were stratified into $M = 8$ blocks, with sizes ranging from 42 to 90. In each block, only approximately 25 students were assigned to the treatment group (receiving incentives); thus, the propensity scores were significantly different across blocks.

We consider the grade point average (GPA) at the end of the fall semester as the outcome. There were 23 baseline covariates, such as demographic variables, GPA in the previous year, and whether the students correctly answered tests about the scholarship formula. We adjust for the main effect, quadratic terms of the continuous covariates, and two-way interactions. The design matrix $X$ contains $p = 253$ columns (covariates) and $n = 506$ rows (observations).

Based on the unadjusted estimator, the average treatment effect estimate is 0.032 and the 95% confidence interval is $[-0.099, 0.163]$. Based on the Lasso-adjusted estimator, which selects two covariates (“gpapreviousyear” and “gpapreviousyear:test1correct”) into the model, the average treatment effect estimate is 0.038 and the 95% confidence interval is $[-0.069, 0.146]$. Both confidence intervals contain zero, which means that there is insufficient evidence to support the effectiveness of the scholarship program. The Lasso-adjusted estimator appears to be more efficient than the unadjusted estimator because it shortens the interval length by 18%.

To evaluate the repeated sampling properties of different estimators, which depend on all the potential outcomes, we generate synthetic data based on the experimental data. We use Lasso to fit two sparse linear models for the treatment and control groups, respectively, and impute the unobserved potential outcomes using the fitted models. We use the same blocking and propensity scores for each block as those in the original experiment and implement either stratified randomization or stratified rerandomization in the design stage. For rerandomization, we use the following covariates: age, high school grades, and whether the students correctly answered
Table 1: Simulation results for different block types and equal or unequal propensity scores when \( n = 300 \).

| Scenario | Rerand. | Est. | Bias  | SD    | RMSE  | CP    | Length |
|----------|---------|------|-------|-------|-------|-------|--------|
| No       | no      | unadj| 0.6   | 30.7  | 30.6  | 96.6  | 130.5  |
| block    | yes     | unadj| -0.8  | 22.7  | 22.7  | 97.1  | 98.2   |
|          | no      | lasso| 0.4   | 11.5  | 11.5  | 99.6  | 66.9   |
|          | yes     | lasso| -0.2  | 10.6  | 10.6  | 100.0 | 67.0   |
| Large,   | no      | unadj| 1.2   | 39.4  | 39.4  | 96.0  | 161.2  |
| equal    | yes     | unadj| -1.1  | 22.0  | 22.0  | 97.5  | 99.0   |
|          | no      | lasso| -0.3  | 16.4  | 16.4  | 98.7  | 81.6   |
|          | yes     | lasso| -0.7  | 14.2  | 14.2  | 99.2  | 81.8   |
| Lagre,   | no      | unadj| -1.8  | 37.9  | 37.9  | 95.7  | 154.9  |
| unequal  | yes     | unadj| -1.3  | 20.4  | 20.4  | 97.1  | 90.0   |
|          | no      | lasso| -0.0  | 13.9  | 13.8  | 99.4  | 72.0   |
|          | yes     | lasso| -0.4  | 11.1  | 11.1  | 100.0 | 72.2   |
| Small,   | no      | unadj| 1.0   | 33.8  | 33.8  | 95.3  | 137.0  |
| equal    | yes     | unadj| -0.5  | 23.2  | 23.2  | 94.7  | 93.1   |
|          | no      | lasso| 1.7   | 14.8  | 14.9  | 97.4  | 65.7   |
|          | yes     | lasso| 0.9   | 11.8  | 11.8  | 99.3  | 65.6   |
| Small,   | no      | unadj| -1.1  | 32.6  | 32.6  | 97.2  | 144.5  |
| unequal  | yes     | unadj| -0.1  | 18.6  | 18.6  | 98.7  | 94.3   |
|          | no      | lasso| -0.4  | 13.4  | 13.4  | 99.8  | 82.6   |
|          | yes     | lasso| 0.0   | 11.5  | 11.5  | 99.9  | 82.8   |
| Hybrid,  | no      | unadj| -0.2  | 36.2  | 36.2  | 96.3  | 149.3  |
| equal    | yes     | unadj| -0.2  | 24.8  | 24.8  | 96.6  | 106.0  |
|          | no      | lasso| 1.8   | 17.3  | 17.3  | 98.0  | 80.2   |
|          | yes     | lasso| 1.3   | 14.3  | 14.3  | 99.2  | 80.3   |
| Hybrid,  | no      | unadj| 1.1   | 34.3  | 34.3  | 95.1  | 135.7  |
| unequal  | yes     | unadj| -0.1  | 22.0  | 22.0  | 93.8  | 86.4   |
|          | no      | lasso| 0.2   | 14.5  | 14.5  | 96.2  | 60.0   |
|          | yes     | lasso| -0.2  | 9.4   | 9.4   | 100.0 | 59.5   |
| Triplet, | no      | unadj| -0.9  | 29.7  | 29.7  | 95.1  | 118.9  |
| equal    | yes     | unadj| -0.7  | 17.1  | 17.1  | 93.9  | 67.1   |
|          | no      | lasso| 1.7   | 16.4  | 16.4  | 95.6  | 66.5   |
|          | yes     | lasso| 0.5   | 10.8  | 10.8  | 99.9  | 67.0   |

Note: The numbers in brackets are the corresponding standard errors estimated using the bootstrap with 500 replications. Bias, SD, RMSE, CP, Length, and their standard errors are multiplied by 100.
Table 2: Simulation results for different block types and equal or unequal propensity scores when \( n = 600 \).

| Scenario   | Rerand. | Est.  | Bias   | SD   | RMSE  | CP   | Length  |
|------------|---------|-------|--------|------|-------|------|---------|
| No block   | no      | unadj | -1.0 (0.7) | 21.8 (0.5) | 21.8 (0.5) | 97.0 (0.6) | 93.7 (0.1) |
|            | yes     | unadj | 0.5 (0.5)  | 15.1 (0.3)  | 15.1 (0.3)  | 97.5 (0.5) | 68.4 (0.0)  |
|            | no      | lasso | -0.4 (0.3) | 7.7 (0.2)   | 7.7 (0.2)   | 100.0 (0.0) | 47.6 (0.0)  |
|            | yes     | lasso | 0.2 (0.2)  | 7.2 (0.2)   | 7.2 (0.2)   | 99.6 (0.2) | 47.5 (0.0)  |
| Large, equal | no  | unadj | 0.3 (0.8)  | 25.7 (0.6)  | 25.7 (0.6)  | 96.8 (0.5) | 109.3 (0.0) |
|            | yes     | unadj | 1.4 (0.5)  | 16.2 (0.3)  | 16.2 (0.3)  | 97.7 (0.5) | 70.9 (0.1)  |
|            | no      | lasso | 1.3 (0.4)  | 12.0 (0.3)  | 12.1 (0.3)  | 97.4 (0.5) | 55.6 (0.1)  |
|            | yes     | lasso | 1.6 (0.3)  | 9.6 (0.2)   | 9.7 (0.2)   | 99.2 (0.3) | 55.7 (0.1)  |
| Lagre, unequal | no | unadj | 0.0 (0.7)  | 20.5 (0.4)  | 20.5 (0.4)  | 95.9 (0.6) | 83.1 (0.0)  |
|            | yes     | unadj | 0.2 (0.4)  | 12.3 (0.2)  | 12.3 (0.2)  | 97.6 (0.5) | 53.0 (0.0)  |
|            | no      | lasso | 0.1 (0.2)  | 6.7 (0.1)   | 6.7 (0.1)   | 99.8 (0.1) | 37.1 (0.0)  |
|            | yes     | lasso | -0.2 (0.2) | 6.2 (0.1)   | 6.2 (0.1)   | 99.9 (0.1) | 37.1 (0.0)  |
| Small, equal | no   | unadj | 0.3 (0.8)  | 27.4 (0.6)  | 27.4 (0.6)  | 96.2 (0.7) | 115.5 (0.1) |
|            | yes     | unadj | 1.2 (0.6)  | 18.7 (0.4)  | 18.7 (0.4)  | 97.0 (0.5) | 80.7 (0.1)  |
|            | no      | lasso | 0.8 (0.3)  | 10.7 (0.2)  | 10.8 (0.2)  | 98.8 (0.3) | 55.9 (0.1)  |
|            | yes     | lasso | 0.7 (0.3)  | 8.7 (0.2)   | 8.8 (0.2)   | 99.9 (0.1) | 55.8 (0.1)  |
| Small, unequal | no | unadj | 0.6 (0.9)  | 26.6 (0.6)  | 26.6 (0.6)  | 95.9 (0.6) | 108.6 (0.0) |
|            | yes     | unadj | 0.3 (0.5)  | 16.4 (0.4)  | 16.4 (0.4)  | 96.6 (0.6) | 69.5 (0.1)  |
|            | no      | lasso | 0.3 (0.3)  | 9.3 (0.2)   | 9.3 (0.2)   | 99.5 (0.2) | 48.4 (0.0)  |
|            | yes     | lasso | 0.1 (0.3)  | 8.4 (0.2)   | 8.4 (0.2)   | 99.4 (0.2) | 48.5 (0.0)  |
| Hybrid, equal | no  | unadj | -1.2 (0.9) | 29.0 (0.6)  | 29.0 (0.6)  | 96.2 (0.6) | 117.6 (0.1) |
|            | yes     | unadj | 0.2 (0.6)  | 19.4 (0.4)  | 19.4 (0.4)  | 94.7 (0.7) | 79.9 (0.1)  |
|            | no      | lasso | 0.3 (0.4)  | 11.9 (0.3)  | 11.9 (0.3)  | 97.7 (0.5) | 55.5 (0.1)  |
|            | yes     | lasso | 0.6 (0.3)  | 10.0 (0.2)  | 10.0 (0.2)  | 99.9 (0.1) | 55.6 (0.1)  |
| Hybrid, unequal | no | unadj | 0.3 (0.7)  | 20.6 (0.5)  | 20.5 (0.5)  | 95.7 (0.7) | 82.4 (0.1)  |
|            | yes     | unadj | -0.2 (0.4) | 11.1 (0.2)  | 11.1 (0.2)  | 95.3 (0.7) | 46.8 (0.2)  |
|            | no      | lasso | -0.0 (0.2) | 7.1 (0.2)   | 7.1 (0.2)   | 98.2 (0.4) | 34.4 (0.1)  |
|            | yes     | lasso | -0.2 (0.1) | 4.6 (0.1)   | 4.6 (0.1)   | 100.0 (0.0) | 34.6 (0.1) |
| Triplet, equal | no   | unadj | 0.7 (1.0)  | 30.2 (0.6)  | 30.2 (0.6)  | 95.2 (0.7) | 118.6 (0.1) |
|            | yes     | unadj | -0.4 (0.5) | 15.4 (0.3)  | 15.4 (0.3)  | 93.3 (0.8) | 59.8 (0.3)  |
|            | no      | lasso | 0.6 (0.4)  | 13.0 (0.3)  | 13.0 (0.3)  | 96.5 (0.6) | 53.7 (0.1)  |
|            | yes     | lasso | 0.1 (0.2)  | 8.0 (0.2)   | 8.0 (0.2)   | 100.0 (0.0) | 53.8 (0.1)  |

Note: The numbers in brackets are the corresponding standard errors estimated using the bootstrap with 500 replications. Bias, SD, RMSE, CP, Length, and their standard errors are multiplied by 100.
two tests about the scholarship formula. We replicate the simulation 1000 times on this semi-synthetic dataset. The left panel of Figure 2 and Table 3 show the results. Rerandomization is the preferable strategy, and the Lasso-adjusted estimator is superior to the unadjusted one. Specifically, The combination of rerandomization and Lasso adjustment reduces the standard deviation by 30% and decreases the mean confidence interval lengths by 19%.

7.2 An observational study of fish consumption

The second dataset was obtained from the US National Health and Nutrition Examination Survey (NHANES) 2013–2014. Zhao et al. (2018) investigated the effect of high-level fish consumption on biomarkers. We use this dataset to illustrate the application of the proposed Lasso-adjusted estimator in the observational study. There were 88 people with high fish consumption and 663 with low fish consumption. We use the blood cadmium level on the log scale as the outcome. Many covariates were available, such as demographic variables, disability, and history of drugs, alcohol, and smoking. As in Pashley & Miratrix (2021), we match for age, sex, race, income, education, and smoking. We fit the propensity score model using the R package “brglm” and perform full matching using the R package “optmatch,” which produces 88 fine blocks with one treated unit and one to eight control units.

After full matching, we could analyze the data as a finely stratified experiment (Bind & Rubin 2019, Pashley & Miratrix 2021). In contrast to Pashley & Miratrix (2021), where inference
is based on the weighted difference-in-means estimator, we perform covariate adjustment to improve efficiency. Thirty baseline covariates were included in the initial dataset. We perform feature engineering and include the main effect, quadratic terms of the continuous covariates, and two-way interactions, which generate a design matrix $X$ with $p = 390$ columns (covariates) and $n = 751$ rows (observations). Based on the unadjusted estimator, the average treatment effect estimate is 0.139 and the 95% confidence interval is $[-0.052, 0.330]$. Based on the Lasso-adjusted estimator, which selects 16 covariates into the model, the average treatment effect estimate is 0.129 and the 95% confidence interval is $[-0.046, 0.305]$. Both methods indicate that there is insufficient evidence to support the hypothesis that high fish consumption affects blood cadmium level. Notably, the Lasso-adjusted estimator shortens the confidence interval length by 8% and is thus more efficient than the unadjusted estimator.

To further examine the repeated sampling properties of the Lasso-adjusted estimator, we use the same approach as in the previous section to impute the unobserved potential outcomes. We use the same blocking and propensity scores for each block as those in the matched dataset and implement stratified randomization 1000 times on this semi-synthetic dataset. The right panel of Figure 2 and Table 3 show the results. The distribution of the Lasso-adjusted estimator is more concentrated than that of the unadjusted estimator. Moreover, the Lasso-adjusted estimator reduces the standard deviation by 32% and decreases the mean confidence interval length by 9%. Thus, the Lasso-adjusted estimator improves both the estimation and inference efficiencies.
8 Discussion

This study aimed to enhance the estimation and inference efficiencies of the average treatment effect in randomized experiments when many baseline covariates are available. We propose novel methods that combine blocking, rerandomization, and regression adjustment using Lasso. Under mild conditions, we obtain the asymptotic distribution of the Lasso-adjusted average treatment effect estimator when blocking, rerandomization or both are implemented in the design stage. We demonstrate that the proposed estimator enhances, or at least does not deteriorate, the precision compared with that associated with the unadjusted estimator. Our results are design-based and robust to model misspecification, heterogeneous block sizes, propensity scores, and treatment effects. In addition, we propose a conservative variance estimator to construct asymptotically conservative confidence intervals or tests for the average treatment effect. Our final recommendation is to use blocking and rerandomization in the design stage to balance a subset of covariates that are most relevant to the potential outcomes and then implement regression adjustment using Lasso in the analysis stage to adjust for the remaining covariate imbalances. Similar to the findings reported by Li & Ding (2020), when rerandomization or the combination of blocking and rerandomization is used in the design stage, the Lasso adjustment should consider all of the covariates used in the rerandomization to ensure efficiency gains.

To render the theory and methods more intuitive, we focus on inferring the average treatment effect for a binary treatment. Our analysis can be generalized to multiple value treatments, including factorial experiments (Fisher 1935, Li et al. 2020, Liu et al. 2021). Moreover, it may be interesting to extend our results to other complicated settings, such as binary outcomes based on penalized logistic regression (Freedman 2008b, Zhang et al. 2008) and the use of other machine learning methods such as random forest (Wager et al. 2016), $L_2$-boosting (Kueck et al. 2022), and neural networks (Farrell et al. 2021).

In practice, some experimental units may not comply with their treatment assignment; that is, the actual treatments received by the experimental units may be different from the treatments assigned (Imbens & Angrist 1994, Angrist et al. 1996). When there is noncompliance, investigators often use the two-stage least squares to estimate the complier average treatment effect (Angrist & Pischke 2008). It would be interesting to extend our methods to noncompliance settings.

The asymptotic theory requires that the number of experimental units is large, which may be unrealistic in some randomized experiments. In such cases, we recommend Fisher’s randomization test with estimators studentied by appropriate standard errors. Fisher’s randomization test delivers exact $p$-values under Fisher’s sharp null hypothesis (Young 2019, Bind & Rubin...
and is asymptotically valid under the weak null hypothesis of $\tau = 0$ when using appropriate studentization or prepivoting (Wu & Ding 2021, Zhao & Ding 2021, Cohen & Fogarty 2022). Note that the Fisher randomization test must follow the same treatment assignment rule for stratified rerandomization.

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Supplementary Material

Section A provides bounds for sampling without replacement, concentration inequalities under stratified randomization, and asymptotic results of $\hat{\tau}_{\text{unadj}}$ under stratified randomization and stratified rerandomization. Section B provides proofs of Theorems 1–4.

A Some preliminary results

A.1 Bounds for sampling without replacement

The connection between randomized experiments and survey sampling has been discussed in depth by many scholars (Lin 2013, Li & Ding 2017, Mukerjee et al. 2018, Lei & Ding 2021). Both of them are based on a probability model of sampling without replacement from a finite population. We start by introducing Bobkov’s inequality, which is a powerful tool to prove concentration inequalities for sampling without replacement. In this section, we consider completely
randomized experiments; that is,
\[ P(Z = z) = \frac{n_1!n_0!}{n!}, \quad \sum_{i=1}^{n} I(z_i = 1) = n_1, \quad z_i = 0, 1. \]

We denote the propensity score by \( e_c = n_1/n \). The value space of \( Z \) is defined as
\[ G = \{ z = (z_1, \ldots, z_n) \in \{0, 1\}^n : z_1 + \cdots + z_n = n_1 \}. \]

For every \( z \in G \), we pick a pair of units \((i, j)\) such that \( z_i = 1 \) and \( z_j = 0 \), and switch the value of \( z_i \) and \( z_j \) to obtain a “neighbour” of \( z \), denoted by \( z^{i,j} \). Clearly, for different \((i, j)\), \( z \) has totally \( n_1(n - n_1) \) neighbours. For every real-valued function \( f \) on \( G \), we define the discrete gradient as follows:
\[ \nabla f(z) = (f(z) - f(z^{i,j}))_{i,j}, \]
which is a \( n_1(n - n_1) \) dimensional vector. We define the \( \ell_2 \) norm of \( \nabla f(z) \) as
\[ \|\nabla f(z)\|_2^2 = \sum_{i: z_i = 1} \sum_{j: z_j = 0} |f(z) - f(z^{i,j})|^2. \]

**Lemma S1** (Bobkov (2004)). For every real-valued function \( f \) on \( G \), if \( \|\nabla f(z)\|_2 \leq \sigma \) for all \( z \in G \), then
\[ E \exp \left[ t \left\{ f(Z) - Ef(Z) \right\} \right] \leq \exp \left\{ \frac{\sigma^2 t^2}{n + 2} \right\}, \quad t \in \mathbb{R}. \]

Consider two sequences of real numbers \((a_1, \ldots, a_n)\) and \((b_1, \ldots, b_n)\), we denote
\[ \bar{a} = \frac{1}{n} \sum_{i=1}^{n} a_i, \quad \bar{a}_1 = \bar{a}_1(Z) = \frac{1}{n} \sum_{i=1}^{n} Z_ia_i, \]
\[ S_{ab} = \frac{1}{n-1} \sum_{i=1}^{n} (a_i - \bar{a})(b_i - \bar{b}), \quad s_{ab} = s_{ab}(Z) = \frac{1}{n_1-1} \sum_{i=1}^{n} Z_1(a_i - \bar{a}_1)(b_i - \bar{b}_1). \]

The following result from Zhang et al. (2012) is useful to bound \( \|\nabla f(z)\|_2^2 \).

**Lemma S2** (Zhang et al. (2012)). We have
\[ \sum_{i=1}^{n} (a_i - \bar{a})(b_i - \bar{b}) = \frac{1}{n} \sum_{1 \leq i < j \leq n} (a_i - a_j)(b_i - b_j). \]

Next, we apply Bobkov’s inequality to derive the bounds for the sample mean and sample covariance, respectively.
Lemma S3. For $t \in \mathbb{R}$, we have

$$E \exp \{t \langle \bar{a}_1 - \bar{a} \rangle \} \leq \exp \left\{ \sigma^2_{\text{mean}} t^2 / (n + 2) \right\},$$

where $\sigma^2_{\text{mean}} = e^2 n^{-1} \sum_{i=1}^{n} (a_i - \bar{a})^2$.

Proof. By definition and simple calculation, we have

$$\|\nabla \bar{a}_1(z)\|_2^2 = \sum_{i:z_i = 1} \sum_{j:z_j = 0} |\bar{a}_1(z) - \bar{a}_1(z^{i,j})|^2 = \frac{1}{n^2} \sum_{i:z_i = 1} \sum_{j:z_j = 0} |a_i - a_j|^2. \quad (S1)$$

Then, by Lemma S2, we have

$$\frac{1}{n^2} \sum_{i:z_i = 1} \sum_{j:z_j = 0} |a_i - a_j|^2 \leq \frac{1}{n^2} \sum_{1 \leq i < j \leq n} |a_i - a_j|^2 = \frac{n}{n^2} \sum_{i=1}^{n} (a_i - \bar{a})^2 =: \sigma^2_{\text{mean}}. \quad (S2)$$

Combining (S1) and (S2), Lemma S3 follows from Lemma S1. \qed

Lemma S4. If $n_{[m]} \geq 2$ for all $m$, then, for $t \in \mathbb{R},$

$$E \exp \{t (s_{ab} - S_{ab}) \} \leq \exp \left\{ \sigma^2_{\text{cov}} t^2 / (n + 2) \right\},$$

where

$$\sigma^2_{\text{cov}} = \left\{ \frac{1}{e^2 n} \sum_{i=1}^{n} (a_i - \bar{a})^2 (b_i - \bar{b})^2 + \frac{8}{e^3 n^2} \sum_{i=1}^{n} (a_i - \bar{a})^2 \sum_{i=1}^{n} (b_i - \bar{b})^2 \right\}.$$

Proof. We start by examining $s_{ab}(z) - s_{ab}(z^{i,j})$. By Lemma S2 and some simple calculation, we have

$$s_{ab}(z) - s_{ab}(z^{i,j}) = \frac{1}{n_1(n_1 - 1)} \sum_{1 \leq i' < j' \leq n} \left\{ z_{i'} z_{j'} (a_{i'} - a_{j'}) (b_{i'} - b_{j'}) - z_{i'} z_{j'} z_{i''} z_{j''} (a_{i''} - a_{j''}) (b_{i''} - b_{j''}) \right\}$$

$$= \frac{1}{n_1(n_1 - 1)} \sum_{l \neq i} z_l \{ (a_l - a_i) (b_l - b_i) - (a_l - a_j) (b_l - b_j) \}$$

$$= \frac{1}{n_1(n_1 - 1)} \sum_{l \neq i} z_l \{ a_l b_i - a_j b_l + (a_j - a_i) b_l + a_l (b_j - b_l) \}$$

$$= \frac{1}{n_1(n_1 - 1)} \sum_{l \neq i} z_l \{ (a_l b_i - a_i \bar{b} - \bar{a} b_i + \bar{a} \bar{b}) - (a_j b_j - a_j \bar{b} - a_b j + \bar{a} b) \}$$

$$+ (a_j - a_i) (b_l - \bar{b}) + (a_l - \bar{a}) (b_j - b_l) \}$$

$$= \frac{1}{n_1} (U_{ij} + V_{ij}).$$
where

\[ U_{ij} := (a_i - \bar{a})(b_i - \bar{b}) - (a_j - \bar{a})(b_j - \bar{b}), \]

\[ V_{ij} := \frac{(a_j - a_i)}{n_1 - 1} \sum_{l \neq i} z_l (b_l - \bar{b}) + \frac{(b_j - b_i)}{n_1 - 1} \sum_{l \neq i} z_l (a_l - \bar{a}). \]

By Cauchy–Schwarz inequality, we have

\[
V_{ij} \leq |a_j - a_i| \sqrt{\frac{1}{n_1 - 1} \sum_{l=1}^{n} (b_l - \bar{b})^2 + |b_j - b_i| \sqrt{\frac{1}{n_1 - 1} \sum_{l=1}^{n} (a_l - \bar{a})^2}} \leq |a_j - a_i| \sqrt{\frac{2}{e \epsilon n} \sum_{l=1}^{n} (b_l - \bar{b})^2 + |b_j - b_i| \sqrt{\frac{2}{e \epsilon n} \sum_{l=1}^{n} (a_l - \bar{a})^2}}. \quad \text{(S3)}
\]

Then, we can bound \( \| \nabla s_{ab}(z) \|_2^2 \). By definition and Minkowski’s inequality, we have

\[
\| \nabla s_{ab}(z) \|_2^2 = \sum_{i: z_i = 1} \sum_{j: z_j = 0} |s_{ab}(z) - s_{ab}(z^{i,j})|^2 \\
= \sum_{i: z_i = 1} \sum_{j: z_j = 0} |U_{ij}/n_1 + V_{ij}/n_1|^2 \\
\leq \sum_{1 \leq i < j \leq n} |U_{ij}/n_1 + V_{ij}/n_1|^2 \\
\leq (\sqrt{U} + \sqrt{V})^2, \quad \text{(S4)}
\]

where

\[ U := \frac{1}{n_1^2} \sum_{1 \leq i < j \leq n} U_{ij}^2, \quad V := \frac{1}{n_1^2} \sum_{1 \leq i < j \leq n} V_{ij}^2. \]

We bound \( U \) and \( V \) separately. By Lemma S2, we have

\[
U = \frac{1}{n_1^2} \sum_{1 \leq i < j \leq n} \left\{ (a_i - \bar{a})(b_i - \bar{b}) - (a_j - \bar{a})(b_j - \bar{b}) \right\}^2 \\
= \frac{n}{n_1^2} \sum_{i=1}^{n} \left\{ (a_i - \bar{a})(b_i - \bar{b}) - \frac{1}{n} \sum_{j=1}^{n} (a_j - \bar{a})(b_j - \bar{b}) \right\}^2 \\
\leq \frac{n}{n_1^2} \sum_{i=1}^{n} (a_i - \bar{a})(b_i - \bar{b})^2 \\
= \frac{1}{e^2 \epsilon n} \sum_{i=1}^{n} (a_i - \bar{a})^2 (b_i - \bar{b})^2. \quad \text{(S5)}
\]
By (S3), Minkowski’s inequality, and Lemma S2, we have

\[
V \leq \frac{1}{n^2} \sum_{1 \leq i < j \leq n} (|a_j - a_i| \sqrt{\frac{2}{e \cdot n} \sum_{i=1}^{n} (b_i - \bar{b})^2} + |b_j - b_i| \sqrt{\frac{2}{e \cdot n} \sum_{i=1}^{n} (a_i - \bar{a})^2})^2
\]

\[
\leq \frac{1}{n^2} \left( \frac{2}{e \cdot n} \sum_{i=1}^{n} (b_i - \bar{b})^2 \sum_{1 \leq i < j \leq n} (a_i - a_j)^2 + \frac{2}{e \cdot n} \sum_{i=1}^{n} (a_i - \bar{a})^2 \sum_{1 \leq i < j \leq n} (b_i - b_j)^2 \right)
\]

\[
= 8 e^3 n^2 \sum_{i=1}^{n} (a_i - \bar{a})^2 \sum_{i=1}^{n} (b_i - \bar{b})^2.
\]  
(S6)

Combining (S4), (S5), and (S6), we have

\[
\|\nabla_s a_b(z)\|_2^2 \leq \left\{ \frac{1}{e \cdot n} \sum_{i=1}^{n} (a_i - \bar{a})^2 (b_i - \bar{b})^2 + \frac{8}{e^3 n^2} \sum_{i=1}^{n} (a_i - \bar{a})^2 \sum_{i=1}^{n} (b_i - \bar{b})^2 \right\}^2 =: \sigma^2_{cov}.
\]

Then, the conclusion follows from Lemma S1.

A.2 Concentration inequalities for stratified randomization

Massart (1986), Bloniarz et al. (2016), and Tolstikhin (2017) established concentration inequalities for the sample mean under simple random sampling without replacement. We apply Lemmas S3 and S4 in each block to obtain concentration inequalities for the weighted sample mean and sample covariance under stratified random sampling without replacement. These novel inequalities hold for a wide range of number of blocks, block sizes, and propensity scores.

**Theorem S1.** Consider a sequence of real numbers \(\{a_1, \ldots, a_n\}\). For any \(t > 0\),

\[
P\left( \sum_{m=1}^{M} \pi_{[m]} (\bar{a}_{[m]} - \bar{a}_{[m]}) \geq t \right) \leq \exp \left\{ -\frac{nt^2}{4\sigma_n^2} \right\},
\]

where \(\sigma_n^2 = (1/n) \sum_{m=1}^{M} \sum_{i \in [m]} (a_i - \bar{a}_{[m]})^2 / e_{[m]}^2\).

**Proof of Theorem S1.** For any \(\lambda > 0\) and \(t > 0\), by Markov’s inequality, we have

\[
P\left( \sum_{m=1}^{M} \pi_{[m]} (\bar{a}_{[m]} - \bar{a}_{[m]}) \geq t \right) \leq \exp\{-\lambda t\} \cdot E \exp \left\{ \lambda \sum_{m=1}^{M} \pi_{[m]} (\bar{a}_{[m]} - \bar{a}_{[m]}) \right\}
\]

\[
= \exp\{-\lambda t\} \cdot \prod_{m=1}^{M} E \exp \left\{ \lambda \pi_{[m]} (\bar{a}_{[m]} - \bar{a}_{[m]}) \right\}.
\]
By Lemma S3, we have

\[
\prod_{m=1}^{M} E \exp \left\{ \lambda \pi_{[m]} (a_{[m]} - \bar{a}_{[m]}) \right\} \leq \prod_{m=1}^{M} \exp \left\{ \frac{\lambda^2 \pi_{[m]}^2}{\epsilon_{[m]}^2} \sum_{i \in [m]} (a_i - \bar{a}_{[m]})^2 \right\}
\]

\[= \exp \left\{ \frac{\lambda^2}{n^2} \sum_{m=1}^{M} \sum_{i \in [m]} (a_i - \bar{a}_{[m]})^2 / \epsilon_{[m]}^2 \right\} \]

\[= \exp \left\{ \frac{\lambda^2}{n} \sigma_a^2 \right\} \]

Thus,

\[
P(\sum_{m=1}^{M} \pi_{[m]}(\bar{a}_{[m]} - a_{[m]}) \geq t) \leq \exp \left\{ - \lambda t + \frac{\lambda^2}{n} \sigma_a^2 \right\}.
\]

The conclusion follows by taking \( \lambda = nt/(2\sigma_a^2) \).

\[\square\]

**Theorem S2.** Consider two sequences of real numbers \( \{a_1, \ldots, a_n\} \) and \( \{b_1, \ldots, b_n\} \). If \( n_{[m]} \geq 2 \) for all \( m \), then, for any \( t > 0 \),

\[
P(\sum_{m=1}^{M} \pi_{[m]}(s_{[m]ab} - S_{[m]ab}) \geq t) \leq \exp \left\{ - \frac{nt^2}{60(\kappa_a^4 \kappa_b^4)^{1/2}} \right\},
\]

where \( \kappa_a^4 = (1/n) \sum_{m=1}^{M} \sum_{i \in [m]} (a_i - \bar{a}_{[m]})^4 / \epsilon_{[m]}^3 \) and \( \kappa_b^4 = (1/n) \sum_{m=1}^{M} \sum_{i \in [m]} (b_i - \bar{b}_{[m]})^4 / \epsilon_{[m]}^3 \).

**Proof of Theorem S2.** We denote

\[
\sigma_{[m]\text{cov}}^2 = \left\{ \frac{1}{\epsilon_{[m]}^2 n_{[m]}} \sum_{i \in [m]} (a_i - \bar{a}_{[m]})^2 (b_i - \bar{b}_{[m]})^2 \right. + \left. \frac{8}{\epsilon_{[m]}^3 n_{[m]}^2} \sum_{i \in [m]} (a_i - \bar{a}_{[m]})^2 \sum_{i \in [m]} (b_i - \bar{b}_{[m]})^2 \right\}^2.
\]
For any $\lambda > 0$ and $t > 0$, by Markov’s inequality and Lemma S4, we have

\[
P\left( \sum_{m=1}^{M} \pi_{[m]}(s_{[m]ab} - S_{[m]ab}) \geq t \right) \leq \exp\{-\lambda t\} \cdot E \exp\left\{ \lambda \sum_{m=1}^{M} \pi_{[m]}(s_{[m]ab} - S_{[m]ab}) \right\}
\]

\[
= \exp\{-\lambda t\} \cdot \prod_{m=1}^{M} E \exp\left\{ \lambda \pi_{[m]}(s_{[m]ab} - S_{[m]ab}) \right\}
\]

\[
\leq \exp\{-\lambda t\} \cdot \prod_{m=1}^{M} \exp\left\{ \frac{\lambda^2 \pi_{[m]}^2}{n_{[m]} \sigma_{[m]cov}^2} \right\}
\]

\[
= \exp\left\{ -\lambda t + \frac{\lambda^2}{n} \sum_{m=1}^{M} \pi_{[m]} \sigma_{[m]cov}^2 \right\}.
\] (S7)

By Minkowski’s inequality, we have

\[
\sum_{m=1}^{M} \pi_{[m]} \sigma_{[m]cov}^2 = \sum_{m=1}^{M} \left\{ \sqrt{\frac{\pi_{[m]}}{e_{[m]}^2 n_{[m]}} \sum_{i \in [m]} (a_i - \bar{a}_{[m]})^2 (b_i - \bar{b}_{[m]})^2} \right\}
\]

\[
+ \sqrt{\frac{8\pi_{[m]}}{e_{[m]}^3 n_{[m]}^2} \sum_{i \in [m]} (a_i - \bar{a}_{[m]})^2 (b_i - \bar{b}_{[m]})^2 / e_{[m]}^2}
\]

\[
\leq \left\{ \frac{1}{n} \sum_{m=1}^{M} \sum_{i \in [m]} (a_i - \bar{a}_{[m]})^2 (b_i - \bar{b}_{[m]})^2 / e_{[m]}^2 \right\}^{1/2}
\]

\[
+ \left\{ \frac{8}{n} \sum_{m=1}^{M} \sum_{i \in [m]} (a_i - \bar{a}_{[m]})^2 (b_i - \bar{b}_{[m]})^2 / (e_{[m]}^3 n_{[m]}) \right\}^{1/2}.
\] (S8)

Then, we deal with the two terms in (S8) separately. By Cauchy–Schwarz inequality,

\[
\frac{1}{n} \sum_{m=1}^{M} \sum_{i \in [m]} (a_i - \bar{a}_{[m]})^2 (b_i - \bar{b}_{[m]})^2 / e_{[m]}^2
\]

\[
\leq \left\{ \frac{1}{n} \sum_{m=1}^{M} \sum_{i \in [m]} (a_i - \bar{a}_{[m]})^4 / e_{[m]}^2 \right\}^{1/2} \left\{ \frac{1}{n} \sum_{m=1}^{M} \sum_{i \in [m]} (b_i - \bar{b}_{[m]})^4 / e_{[m]}^2 \right\}^{1/2}
\]

\[
\leq \left\{ \frac{1}{n} \sum_{m=1}^{M} \sum_{i \in [m]} (a_i - \bar{a}_{[m]})^4 / e_{[m]}^3 \right\}^{1/2} \left\{ \frac{1}{n} \sum_{m=1}^{M} \sum_{i \in [m]} (b_i - \bar{b}_{[m]})^4 / e_{[m]}^3 \right\}^{1/2}.
\] (S9)
Applying Cauchy–Schwarz inequality twice, we have
\[
\frac{8}{n} \sum_{m=1}^{M} \left[ \sum_{i \in [m]} (a_i - \bar{a}_{[m]})^2 \sum_{i \in [m]} (b_i - \bar{b}_{[m]})^2 / (e_{[m]}^3 n_{[m]}) \right] \\
\leq \frac{8}{n} \left\{ \sum_{m=1}^{M} \left[ \sum_{i \in [m]} (a_i - \bar{a}_{[m]})^2 / (e_{[m]}^3 n_{[m]}) \right] \right\}^{1/2} \left\{ \sum_{m=1}^{M} \left[ \sum_{i \in [m]} (b_i - \bar{b}_{[m]})^2 / (e_{[m]}^3 n_{[m]}) \right] \right\}^{1/2} \\
\leq 8 \left\{ \frac{1}{n} \sum_{m=1}^{M} \sum_{i \in [m]} (a_i - \bar{a}_{[m]})^4 / e_{[m]}^3 \right\}^{1/2} \left\{ \frac{1}{n} \sum_{m=1}^{M} \sum_{i \in [m]} (b_i - \bar{b}_{[m]})^4 / e_{[m]}^3 \right\}^{1/2}.
\] (S10)

Recall that
\[
\kappa_a^4 = \frac{1}{n} \sum_{m=1}^{M} \sum_{i \in [m]} (a_i - \bar{a}_{[m]})^4 / e_{[m]}^3, \quad \kappa_b^4 = \frac{1}{n} \sum_{m=1}^{M} \sum_{i \in [m]} (b_i - \bar{b}_{[m]})^4 / e_{[m]}^3.
\]

Combining (S8), (S9), and (S10), we have
\[
\sum_{m=1}^{M} \pi_{[m]}^{2} \sigma_{[m]}^{2}_{\text{cov}} \leq (1 + 2\sqrt{2})^2 (\kappa_a^4 \kappa_b^4)^{1/2} \leq 15(\kappa_a^4 \kappa_b^4)^{1/2}.
\] (S11)

Combining (S7) and (S11), we have
\[
\Pr\left( \sum_{m=1}^{M} \pi_{[m]} (s_{[m]ab} - S_{[m]ab}) \geq t \right) \leq \exp \left\{ -\lambda t + \frac{15\lambda^2}{n} (\kappa_a^4 \kappa_b^4)^{1/2} \right\}.
\]

The conclusion follows by taking \( \lambda = nt/\{30(\kappa_a^4 \kappa_b^4)^{1/2}\} \).

\[\square\]

**Theorem S3.** Consider two sequences of real numbers \( \{a_1, \ldots, a_n\} \) and \( \{b_1, \ldots, b_n\} \). We denote \( U_m = (\bar{a}_{[m]} - \bar{a}_{[m]}) (\bar{b}_{[m]} - \bar{b}_{[m]}) \). For any \( t > 0 \),
\[
\Pr\left( \sum_{m=1}^{M} \pi_{[m]} (U_m - EU_m) \geq t \right) \leq \exp \left\{ -nt^2 / (4\sigma_U^2) \right\},
\]
where \( \sigma_U^2 = \sum_{m=1}^{M} \pi_{[m]} (4/e_{[m]}^3) (S_{[m]a}^2 + S_{[m]b}^2)^2 \).

**Proof of Theorem S3.** Let \( A_m = A_m(z) = \bar{a}_{[m]} - \bar{a}_{[m]} \). By Cauchy–Schwarz inequality, we have
\[
|A_m| \leq \frac{1}{n_{[m]}} \sum_{i \in [m]} Z_i |a_i - \bar{a}_{[m]}| \leq \sqrt{\frac{S_{[m]a}^2}{e_{[m]}^3}}.
\]
Let \( A_m' = A_m(z^{ij}) = \bar{a}_{[m]}' - \bar{a}_{[m]} \). Similarly, we define \( B_m \) and \( B_m' \). We have

\[
|\nabla U_m|^2 = \sum_{i,z_i=1}^{m} \sum_{j,z_j=0}^{m} \{ A_m B_m - A_m' B_m' \}^2 \\
= \sum_{i,z_i=1}^{m} \sum_{j,z_j=0}^{m} \{ A_m (B_m - B_m') - (A_m' - A_m) B_m' \}^2 \\
\leq \frac{S^2_{[m]a} + S^2_{[m]b}}{e_{[m]}^2} \sum_{i,z_i=1}^{m} \sum_{j,z_j=0}^{m} \{ |B_m - B_m'| + |A_m' - A_m| \}^2 \\
= \frac{S^2_{[m]a} + S^2_{[m]b}}{e_{[m]}^2} \sum_{i,z_i=1}^{m} \sum_{j,z_j=0}^{m} (|a_i - a_j| + |b_i - b_j|)^2 \\
\leq \frac{S^2_{[m]a} + S^2_{[m]b}}{e_{[m]}^2} \sum_{1 \leq i < j \leq n_{[m]}} (|a_i - a_j| + |b_i - b_j|)^2 \\
\leq \frac{S^2_{[m]a} + S^2_{[m]b}}{e_{[m]}^2} \left( \sqrt{\sum_{1 \leq i < j \leq n_{[m]}} (a_i - a)^2} + \sqrt{\sum_{1 \leq i < j \leq n_{[m]}} (b_i - b_j)^2} \right)^2 \\
= \frac{S^2_{[m]a} + S^2_{[m]b}}{e_{[m]}^2} \left( \sqrt{n_{[m]}(n_{[m]} - 1)S^2_{[m]a}} + \sqrt{n_{[m]}(n_{[m]} - 1)S^2_{[m]b}} \right)^2 \\
\leq \frac{4}{e_{[m]}^2} (S^2_{[m]a} + S^2_{[m]b})^2 =: \sigma^2_{[m]U}.
\]

For any \( \lambda > 0 \) and \( t > 0 \), by Markov’s inequality and Bobkov’s inequality, we have

\[
P \left( \sum_{m=1}^{M} \pi_{[m]} (U_m - EU_m) \geq t \right) \leq e^{-\lambda t} \cdot \prod_{m=1}^{M} E \exp \left\{ \lambda \pi_{[m]} (U_m - EU_m) \right\} \\
\leq e^{-\lambda t} \cdot \prod_{m=1}^{M} \exp \left\{ \lambda^2 \pi_{[m]} \sigma^2_{[m]U} / n_{[m]} \right\} \\
= \exp \left\{ -\lambda t + \sigma^2_{U} \lambda^2 / n \right\},
\]

where the last equality is due to \( \sigma^2_{U} = \sum_{m=1}^{M} \pi_{[m]} \sigma^2_{[m]U} \). Taking \( \lambda = nt / (2\sigma^2_{U}) \), we have

\[
P \left( \sum_{m=1}^{M} \pi_{[m]} (U_m - EU_m) \geq t \right) \leq \exp \left\{ -nt^2 / (4\sigma^2_{U}) \right\}.
\]

\[\square\]

A.3 Asymptotic theory of \( \hat{\tau}_{\text{unadj}} \) under stratified randomization and stratified rerandomization

In this section, we review some useful results on the asymptotic distributions of \( \hat{\tau}_{\text{unadj}} \) under stratified randomization (Liu & Yang 2020) and stratified rerandomization (Wang, Wang & Liu 2020).
2021), respectively. The maximum second moment condition (Conditions S1 and S3) used in this section is weaker than the bounded fourth moment condition (Condition 2) used in the main text.

**Condition S1.** *The maximum block-specific squared distance of the potential outcomes satisfies* \( n^{-1} \max_{m=1,\ldots,M} \max_{i \in [m]} \{ Y_i(z) - \bar{Y}_{[m]}(z) \}^2 \to 0, \text{ for } z = 0, 1. \)

**Condition S2.** *The weighted variances* \( \sum_{m=1}^M \pi[m] S^2_{[m]} Y(1)/e[m], \sum_{m=1}^M \pi[m] S^2_{[m]} Y(0)/(1 - e[m]), \) *and* \( \sum_{m=1}^M \pi[m] S^2_{[m]} (Y(1) - Y(0)) \) *tend to finite limits, positive for the first two, and the limit of* \( \sum_{m=1}^M \pi[m] \left[ \frac{S^2_{[m]} Y(1)/e[m] + S^2_{[m]} Y(0)/(1 - e[m])}{e[m]} - S^2_{[m]} (Y(1) - Y(0)) \right] \) *is strictly positive.*

**Proposition S1** (Liu & Yang (2020)). *If Conditions 1, S1, and S2 hold, then* \( \sqrt{n} (\hat{\tau}_{\text{unadj}} - \tau) \xrightarrow{d} N(0, \sigma^2_{\text{unadj}}). \) *Moreover, if Conditions 1 and S1 hold and* \( n_{[m]} z \geq 2 \) *for* \( m = 1, \ldots, M \) *and* \( z = 0, 1, \) *then*

\[
\sum_{m=1}^M \pi[m] \left\{ \frac{S^2_{[m]} Y(z)}{e[m]} \right\} - \sum_{m=1}^M \pi[m] \left\{ \frac{S^2_{[m]} Y(z)}{e[m]} \right\} \xrightarrow{p} 0, \quad z = 0, 1.
\]

By Wang, Wang & Liu (2021, Proposition 2), the covariance of \( \sqrt{n} (\hat{\tau}_{\text{unadj}}, \hat{\tau}^T_{x, \mathcal{K}}) \) under stratified randomization is

\[
V\{Y, X_{\mathcal{K}}\} = \begin{pmatrix} V_{YY} & V_{YX_{\mathcal{K}}} \\ V_{X_{\mathcal{K}}Y} & V_{X_{\mathcal{K}}X_{\mathcal{K}}} \end{pmatrix},
\]

where

\[
V_{YY} = \sigma^2_{\text{unadj}}, \quad V_{X_{\mathcal{K}}X_{\mathcal{K}}} = \sum_{m=1}^M \pi[m] \left\{ S^2_{[m]} X_{\mathcal{K}} / e[m] \right\},
\]

\[
V_{X_{\mathcal{K}}Y} = V^T_{X_{\mathcal{K}}} = \sum_{m=1}^M \pi[m] \left\{ S_{[m]} X_{\mathcal{K}} Y(1) / e[m] + S_{[m]} X_{\mathcal{K}} Y(0)/(1 - e[m]) \right\}.
\]

The asymptotic distribution of \( \hat{\tau}_{\text{unadj}} \) under stratified rerandomization depends on the squared multiple correlation between \( \hat{\tau}_{\text{unadj}} \) and \( \hat{\tau}_{x, \mathcal{K}}, \)

\[
R^2_{Y, X_{\mathcal{K}}} = \lim_{n \to \infty} \frac{(V_{YX_{\mathcal{K}}} V_{X_{\mathcal{K}} X_{\mathcal{K}}^{-1}} V_{X_{\mathcal{K}} Y})/V_{YY}}{V_{YY}}.
\]

Since \( V_{YY} = \sigma^2_{\text{unadj}} \), we can conservatively estimate \( V_{YY} \) by \( \hat{\sigma}^2_{\text{unadj}} \). In addition, we can consistently estimate \( V_{X_{\mathcal{K}} Y} \) by \( \hat{V}_{X_{\mathcal{K}} Y} = \sum_{m=1}^M \pi[m] \left\{ s_{[m]} X_{\mathcal{K}} Y(1)/e[m] + s_{[m]} X_{\mathcal{K}} Y(0)/(1 - e[m]) \right\} \) and directly calculate \( V_{X_{\mathcal{K}} X_{\mathcal{K}}} \) based on \( x_i \). Then, we can estimate \( R^2_{Y, X_{\mathcal{K}}} \) by

\[
\hat{R}^2_{Y, X_{\mathcal{K}}} = (\hat{V}^T_{X_{\mathcal{K}} Y} V_{X_{\mathcal{K}} X_{\mathcal{K}}^{-1}} \hat{V}_{X_{\mathcal{K}} Y})/\hat{\sigma}^2_{\text{unadj}}.
\]

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Remark S1. When \( n_{[m]z} = 1 \), we define

\[
 s[m]X_KY(z) = \frac{n_{[m]}}{n_{[m]z}(n_{[m]} - 1)} \sum_{i \in [m]} I(Z_i = z)(x_iK - \bar{x}[m]K)Y_i.
\]

Recall that \( \varepsilon_0, D_1, \ldots, D_k \) are independent standard normal random variables and \( L_{k,a} \sim D_1 \mid \sum_{i=1}^k D_i^2 \leq a \). Let \( v_{k,a} = P(\chi_{k+2}^2 \leq a) / P(\chi_k^2 \leq a) \in (0,1) \). We can conservatively estimate the variance of \( \hat{\tau}_{\text{unadj}} \) under stratified rerandomization by

\[
\hat{\sigma}^2_{\text{unadj} \mid \mathcal{M}_a} = \delta^2_{\text{unadj}} \left\{ 1 - (1 - v_{k,a}) \hat{R}^2_{Y_k, X_K} \right\}.
\]

Condition S3. The maximum block-specific squared distance of the covariates \( x_i \) satisfies

\[
n^{-1} \max_{m=1, \ldots, M} \max_{x_i \in [m]} \|x_i - \bar{x}[m]\|^2 \to 0.
\]

Condition S4. The weighted covariances

\[
\sum_{m=1}^M \pi[m]S[m]X_K/(e[m]), \sum_{m=1}^M \pi[m]S[m]X_k/(1-e[m]), \sum_{m=1}^M \pi[m]S[m]X_kY(1)/e[m], \text{ and } \sum_{m=1}^M \pi[m]S[m]X_kY(0)/(1 - e[m])
\]

tend to finite limits, and the limit of \( V_{X_kX_k} \) is strictly positive definite.

Condition S5. There exists a constant \( C \) such that \( n^{-1} \sum_{i=1}^n Y_i^2(z) \leq C, z = 0,1 \).

Proposition S2 (Wang, Wang & Liu (2021)). If Conditions 1, S1–S4 hold, then, for fixed \( a > 0 \), \( P(\mathcal{M}_a) \to P(\chi_k^2 \leq a) \),

\[
\sqrt{n}(\hat{\tau}_{\text{unadj}} - \tau) \mid \mathcal{M}_a \xrightarrow{d} \sigma_{\text{unadj}} \left\{ \sqrt{1 - \hat{R}^2_{Y_k, X_K}} \varepsilon_0 + \sqrt{\hat{R}^2_{Y_k, X_K}} L_{k,a} \right\},
\]

and the asymptotic variance of \( \sqrt{n}(\hat{\tau}_{\text{unadj}} - \tau) \mid \mathcal{M}_a \) is

\[
\sigma^2_{\text{unadj} \mid \mathcal{M}_a} = \delta^2_{\text{unadj}} \left\{ 1 - (1 - v_{k,a}) \hat{R}^2_{Y_k, X_K} \right\} \leq \sigma^2_{\text{unadj}}.
\]

Furthermore, if Condition S5 holds, then

\[
\hat{\sigma}^2_{\text{unadj} \mid \mathcal{M}_a} \xrightarrow{p} \sigma^2_{\text{unadj} \mid \mathcal{M}_a} + \lim_{n \to \infty} \left\{ \sum_{m \in \mathcal{A}_c} \pi[m]S[m]Y(1) - Y(0) \right\}^2 + \left( \frac{n_f}{n} \right)^2 n_f + \sum_{m \in \mathcal{A}_f} \omega[m] \sum_{m \in \mathcal{A}_f} \omega[m] (\tau[m] - \tau_f)^2 \right\}.
\]

Proposition S2 shows that the asymptotic distribution of \( \hat{\tau}_{\text{unadj}} \) under stratified rerandomization is a convolution of a normal distribution and a truncated normal distribution, and its asymptotic variance is less than or equal to that of \( \hat{\tau}_{\text{unadj}} \) under stratified randomization. Moreover, we can conservatively estimate the asymptotic variance.
B Proofs of Theorems 1–4

B.1 Proof of Theorem 1

Proof. We first prove the asymptotic normality of $\hat{\tau}_{\text{lasso}}$. By definition, we have

$$\hat{\tau}_{\text{lasso}} - \tau = \hat{\tau}_{\text{unadj}} - \tau - x^T \hat{\gamma}_{\text{lasso}}$$

$$= \sum_{m=1}^{M} \pi[m] \left\{ \bar{Y}[m]1 - \bar{Y}[m](1) - (\bar{x}[m]1 - \bar{x}[m])^T \hat{\gamma}_{\text{lasso}} \right\}$$

$$- \left\{ \bar{Y}[m]0 - \bar{Y}[m](0) - (\bar{x}[m]0 - \bar{x}[m])^T \hat{\gamma}_{\text{lasso}} \right\}.$$  

Recall the decomposition of the potential outcomes,

$$Y_i(z) = \bar{Y}[m](z) + (\bar{x}_i - \bar{x}[m])^T \gamma_{\text{proj}} + \varepsilon_i^*(z), \quad i \in [m], \quad z = 0, 1,$$

we have

$$\hat{\tau}_{\text{lasso}} - \tau = \sum_{m=1}^{M} \pi[m] (\varepsilon_i^*[m]1 - \varepsilon_i^*[m]0) + \sum_{m=1}^{M} \pi[m] (\bar{x}[m]1 - \bar{x}[m])^T (\gamma_{\text{proj}} - \hat{\gamma}_{\text{lasso}})$$

$$- \sum_{m=1}^{M} \pi[m] (\bar{x}[m]0 - \bar{x}[m])^T (\gamma_{\text{proj}} - \hat{\gamma}_{\text{lasso}}).$$

Applying Proposition S1 to $\varepsilon_i^*(z)$, we have

$$\sqrt{n} \sum_{m=1}^{M} \pi[m] (\varepsilon_i^*[m]1 - \varepsilon_i^*[m]0) \xrightarrow{d} N(0, \sigma^2_{\text{lasso}}).$$  

(S13)

It suffices for the asymptotic normality of $\hat{\tau}_{\text{lasso}}$ to show that

$$\sqrt{n} \sum_{m=1}^{M} \pi[m] (\bar{x}[m]1 - \bar{x}[m])^T (\gamma_{\text{proj}} - \hat{\gamma}_{\text{lasso}}) \xrightarrow{p} 0,$$

(S14)

$$\sqrt{n} \sum_{m=1}^{M} \pi[m] (\bar{x}[m]0 - \bar{x}[m])^T (\gamma_{\text{proj}} - \hat{\gamma}_{\text{lasso}}) \xrightarrow{p} 0.$$

(S15)

By Hölder inequality,

$$\left| \sum_{m=1}^{M} \pi[m] (\bar{x}[m]1 - \bar{x}[m])^T (\gamma_{\text{proj}} - \hat{\gamma}_{\text{lasso}}) \right| \leq \left\| \sum_{m=1}^{M} \pi[m] (\bar{x}[m]1 - \bar{x}[m]) \right\|_\infty \cdot \left\| \gamma_{\text{proj}} - \hat{\gamma}_{\text{lasso}} \right\|_1.$$

To bound $\left\| \sum_{m=1}^{M} \pi[m] (\bar{x}[m]1 - \bar{x}[m]) \right\|_\infty$ and $\left\| \gamma_{\text{proj}} - \hat{\gamma}_{\text{lasso}} \right\|_1$, we have the following two
lemmas with their proofs given later.

**Lemma S5.** If Conditions 1 and 2 hold, then

\[
\left\| \sum_{m=1}^{M} \pi[m] (\bar{x}[m]_1 - \bar{x}[m]) \right\|_{\infty} = O_p\left( \sqrt{\frac{\log p}{n}} \right).
\]

**Lemma S6.** If Conditions 1, 2, 4, and 5 hold, then

\[
\| \gamma(z) - \hat{\gamma}_{\text{lasso},z} \|_1 = O_p(s_{\lambda_z}) ,
\]

By Lemma S6, \( \| \gamma_{\text{proj}} - \hat{\gamma}_{\text{lasso}} \|_1 = O_p(s_{\lambda_1} + s_{\lambda_0}) \). Therefore,

\[
\sqrt{n} \left\| \sum_{m=1}^{M} \pi[m] (\bar{x}[m]_1 - \bar{x}[m])^T (\gamma_{\text{proj}} - \hat{\gamma}_{\text{lasso}}) \right\|
\leq \sqrt{n} \left\| \sum_{m=1}^{M} \pi[m] (\bar{x}[m]_1 - \bar{x}[m]) \right\|_{\infty} \cdot \left\| \gamma_{\text{proj}} - \hat{\gamma}_{\text{lasso}} \right\|_1
= O_p \left( \sqrt{n} \cdot \sqrt{\frac{\log p}{n}} \right) \cdot O_p(s_{\lambda_1} + s_{\lambda_0})
= o_p(1),
\]

where the last equality is because of Condition 5. Thus, Statement (S14) holds. Similarly, Statement (S15) holds. Combining (S13)–(S15), we obtain the asymptotic normality of \( \hat{\tau}_{\text{lasso}} \).

Next, we compare the asymptotic variances \( \sigma^2_{\text{unadj}} \) and \( \sigma^2_{\text{lasso}} \). By the decomposition in (S12), we have

\[
S^2_{[m]\epsilon^*(1)} = S^2_{[m]Y(1)} + \gamma_{\text{proj}}^T S^2_{[m]X} \gamma_{\text{proj}} - 2 S_{[m]X}^T Y(1) \gamma_{\text{proj}}
\]

(S16)

Similarly, we have

\[
S^2_{[m]\epsilon^*(0)} = S^2_{[m]Y(0)} + \gamma_{\text{proj}}^T S^2_{[m]X} \gamma_{\text{proj}} - 2 S_{[m]X}^T Y(0) \gamma_{\text{proj}},
\]

(S17)

and

\[
S^2_{[m]\{\epsilon^*(1) - \epsilon^*(0)\}} = S^2_{[m]\{Y(1) - Y(0)\}} + (\gamma_{\text{proj}} - \gamma_{\text{proj}})^T S^2_{[m]X} (\gamma_{\text{proj}} - \gamma_{\text{proj}})
- 2 (S_{[m]X}^T Y(1) - S_{[m]X}^T Y(0))^T (\gamma_{\text{proj}} - \gamma_{\text{proj}})
= S^2_{[m]\{Y(1) - Y(0)\}}.
\]
Thus,

\[ \sigma^2_{\text{unadj}} = \lim_{n \to \infty} \sum_{m=1}^{M} \pi[m] \left( \frac{S^2[m]Y(1)}{e[m]} + \frac{S^2[m]Y(0)}{1 - e[m]} - S^2[m](Y(1) - Y(0)) \right) \]

\[ = \sigma^2_{\text{lasso}} - \lim_{n \to \infty} \gamma_{\text{proj}}^T \left( \sum_{m=1}^{M} \pi[m] \frac{S[m]X}{e[m]} \right) \gamma_{\text{proj}} \]

\[ + 2 \sum_{m=1}^{M} \pi[m] \left( \frac{S[m]XY(1)}{e[m]} + \frac{S[m]XY(0)}{1 - e[m]} \right) \gamma_{\text{proj}} \]

\[ = \sigma^2_{\text{lasso}} - \lim_{n \to \infty} \gamma_{\text{proj}}^T \left( \sum_{m=1}^{M} \pi[m] \frac{S[m]X}{e[m]} \right) \gamma_{\text{proj}}, \]

where the last equality is because of the definition of \( \gamma_{\text{proj}} \):

\[ (\gamma_{\text{proj}})_S = \left\{ \sum_{m=1}^{M} \pi[m] \frac{S[m]XS}{e[m](1 - e[m])} \right\}^{-1} \left\{ \sum_{m=1}^{M} \pi[m] \frac{S[m]XSY(1)}{e[m]} + \sum_{m=1}^{M} \pi[m] \frac{S[m]XSY(0)}{1 - e[m]} \right\}, \]

and \( (\gamma_{\text{proj}})_{Sc} = 0 \).

**B.2 Proof of Theorem 2**

**Proof of Theorem 2.** We first introduce a lemma which bounds the number of the covariates selected by Lasso and will be proved in Section B.7.

**Lemma S7.** If Conditions 1, 2, and 4–6 hold, then there exists a constant \( C \) independent of \( n \), such that the following holds with probability tending to one,

\[ \| \hat{\gamma}_{\text{lasso},1} \|_0 \leq C \hat{s}, \quad \| \hat{\gamma}_{\text{lasso},0} \|_0 \leq C \hat{s}. \]

By Lemma S7, we have

\[ \hat{s} = \| \hat{\gamma}_{\text{lasso},1} + \hat{\gamma}_{\text{lasso},0} \|_0 = O_p(s). \]

Then, \( n/(n - \hat{s}) \xrightarrow{p} 1 \). Thus, we only need to derive the probability limit of

\[ \sum_{m \in \mathcal{A}_c} \pi[m] \left( \frac{S^2[m]R(1)}{e[m]} + \frac{S^2[m]R(0)}{1 - e[m]} \right) + \left( \frac{n_f}{n} \right)^2 n_f + \sum_{m \in \mathcal{A}_f} \sum_{m \in \mathcal{A}_f} \omega[m] (\hat{R}_m - \hat{R}_f)^2. \]  

(S18)

**Step 1 (Coarse blocks):** We derive the probability limit of the first term in (S18). By
the definition of \( R_i = Y_i - (x_i - \bar{x}_{[m]})^T \hat{\gamma}_{lasso} \) and the decomposition \((S12)\), we have

\[
\begin{align*}
S^2_{[m]R(1)} &= \frac{1}{n_{[m]1} - 1} \sum_{i \in [m]} Z_i \{ Y_i(1) - \bar{Y}_{[m]1} - (x_i - \bar{x}_{[m]1})^T \hat{\gamma}_{lasso} \}^2 \\
&= S^2_{[m]\epsilon^*(1)} + (\gamma_{\text{proj}} - \hat{\gamma}_{lasso})^T S^2_{[m]X} (\gamma_{\text{proj}} - \hat{\gamma}_{lasso}) + 2 S^2_{[m]X \epsilon^*(1)} (\gamma_{\text{proj}} - \hat{\gamma}_{lasso}),
\end{align*}
\]

where \( S^2_{[m]X} = S^2_{[m]X(1)} \) stands for the sample covariance of \( X \) under treatment. In the following, we will use this simplified notation if there is no exceptional clarity. Therefore,

\[
\sum_{m \in \mathcal{A}_c} \pi_{[m]} S^2_{[m]\epsilon^*(1)} e_{[m]} = \sum_{m \in \mathcal{A}_c} \pi_{[m]} S^2_{[m]\epsilon^*(1)} e_{[m]} + (\gamma_{\text{proj}} - \hat{\gamma}_{lasso})^T \left( \sum_{m \in \mathcal{A}_c} \pi_{[m]} S^2_{[m]X} e_{[m]} \right) (\gamma_{\text{proj}} - \hat{\gamma}_{lasso})
\]

\[+ 2 \sum_{m \in \mathcal{A}_c} \pi_{[m]} S^2_{[m]X \epsilon^*(1)} e_{[m]} (\gamma_{\text{proj}} - \hat{\gamma}_{lasso}). \tag{S19} \]

For the first term on the right-hand side of \((S19)\), applying Proposition \(S1\) to \( \epsilon^*_1(1) \), we have

\[
\sum_{m \in \mathcal{A}_c} \pi_{[m]} S^2_{[m]\epsilon^*(1)} e_{[m]} - \sum_{m \in \mathcal{A}_c} \pi_{[m]} S^2_{[m]\epsilon^*(1)} e_{[m]} \rightarrow 0.
\]

For the second term on the right-hand side of \((S19)\), by Condition 1, we have

\[
(\gamma_{\text{proj}} - \hat{\gamma}_{lasso})^T \left( \sum_{m \in \mathcal{A}_c} \pi_{[m]} S^2_{[m]X} e_{[m]} \right) (\gamma_{\text{proj}} - \hat{\gamma}_{lasso})
\]

\[\leq \frac{1}{c} (\gamma_{\text{proj}} - \hat{\gamma}_{lasso})^T \left( \sum_{m \in \mathcal{A}_c} \pi_{[m]} S^2_{[m]X} \right) (\gamma_{\text{proj}} - \hat{\gamma}_{lasso})
\]

\[\leq \frac{1}{c} \| \gamma_{\text{proj}} - \hat{\gamma}_{lasso} \|_1^2 \cdot \| \sum_{m \in \mathcal{A}_c} \pi_{[m]} S^2_{[m]X} \|_{\infty}.
\]

By the fourth moment condition of the covariates (see Condition 2), we have \( \| \sum_{m \in \mathcal{A}_c} \pi_{[m]} S^2_{[m]X} \|_{\infty} \leq C \) for some constant \( C \). Similar to the proof of Lemma \(S8\), we can show that

\[
\| \sum_{m \in \mathcal{A}_c} \pi_{[m]} S^2_{[m]X} - \sum_{m \in \mathcal{A}_c} \pi_{[m]} S^2_{[m]X} \|_{\infty} = O_p(\sqrt{\log p/n}).
\]

Thus, \( \| \sum_{m \in \mathcal{A}_c} \pi_{[m]} S^2_{[m]X} \|_{\infty} = O_p(1) \). By Lemma \(S6\) and Condition 5, we have \( \| \gamma_{\text{proj}} - \hat{\gamma}_{lasso} \|_1 = o_p(1) \). Therefore,

\[
(\gamma_{\text{proj}} - \hat{\gamma}_{lasso})^T \left( \sum_{m \in \mathcal{A}_c} \pi_{[m]} S^2_{[m]X} e_{[m]} \right) (\gamma_{\text{proj}} - \hat{\gamma}_{lasso}) \overset{p}{\rightarrow} 0.
\]

The third term on the right-hand side of \((S19)\) tends to zero in probability by Cauchy-
Schwarz inequality. Therefore,

\[
\sum_{m \in A_c} \pi_{[m]} S_{[m]|R(1)}^2 e_{[m]} - \sum_{m \in A_c} \pi_{[m]} S_{[m]|\varepsilon^*(1)}^2 e_{[m]} \xrightarrow{p} 0.
\]

Similarly,

\[
\sum_{m \in A_c} \pi_{[m]} S_{[m]|R(0)}^2 e_{[m]} - \sum_{m \in A_c} \pi_{[m]} S_{[m]|\varepsilon^*(0)}^2 e_{[m]} \xrightarrow{p} 0.
\]

**Step 2 (Fine blocks):** We derive the probability limit of the second term in (S18). By the definition of \( R_i = Y_i - (x_i - \bar{x}_{[m]})^T \hat{\gamma}_{lasso} \) and the decomposition (S12), we have

\[
\frac{1}{n[m]} \sum_{i \in [m], Z_i = z} R_i = \frac{1}{n[m]} \sum_{i \in [m], Z_i = z} \{ Y_i - (x_i - \bar{x}_{[m]})^T \hat{\gamma}_{lasso} \}
= \frac{1}{n[m]} \sum_{i \in [m], Z_i = z} \{ \bar{Y}_i(z) + (x_i - \bar{x}_{[m]})^T (\gamma_{proj} - \hat{\gamma}_{lasso}) + \varepsilon_i^*(z) \}
= \bar{Y}_i(z) + (\bar{x}_{[m]} - \bar{x}_{[m]})^T (\gamma_{proj} - \hat{\gamma}_{lasso}) + \varepsilon_i^*[m]z.
\]

Then, we obtain a key decomposition:

\[
\hat{\tau}_R[m] = \left[ \{ \varepsilon_i^*[m]1 + \bar{Y}_i[m](1) \} - \{ \varepsilon_i^*[m]0 + \bar{Y}_i[m](0) \} \right] + (\bar{x}_{[m]}1 - \bar{x}_{[m]0})^T (\gamma_{proj} - \hat{\gamma}_{lasso})
= (\bar{R}_i^*[m]1 - \bar{R}_i^*[m]0) + (\bar{x}_{[m]}1 - \bar{x}_{[m]0})^T (\gamma_{proj} - \hat{\gamma}_{lasso}),
\]

where \( R_i^*[z] = \varepsilon_i^*[z] + \bar{Y}_i[z] \). We denote

\[
\phi_{[m]} = \frac{1}{\pi_{[m]}} \left( \frac{n_f}{n} \right)^2 \frac{n}{n_f + \sum_{m \in A_f} \omega_{[m]}^2} 
= \frac{1}{\pi_{[m]}} \left( \frac{n_f}{n} \right)^2 \frac{n}{n_f + \sum_{m \in A_f} \{ n_{[m]}^2/(n_f - 2n_{[m]}) \}} \cdot \frac{n_{[m]}^2}{(n_f - 2n_{[m]})}.
\]
By the above decomposition and definition, we have

\[
\left( \frac{n_f}{n} \right)^2 \frac{n}{n_f + \sum_{m \in A_f} \omega[m]} \sum_{m \in A_f} \omega[m] (\hat{\tau}_{R,m} - \hat{r}_{R,f})^2
= \sum_{m \in A_f} \phi[m] \pi[m] (\hat{\tau}_{R,m} - \hat{r}_{R,f})^2
= \sum_{m \in A_f} \phi[m] \pi[m] \{ (\bar{x}_{m} [1] - \bar{x}_{m}[0])^T (\gamma_{\text{proj}} - \hat{\gamma}_{\text{lasso}}) - \sum_{m \in A_f} \frac{n_f[m]}{n_f} (\bar{x}_{m} [1] - \bar{x}_{m}[0])^T (\gamma_{\text{proj}} - \hat{\gamma}_{\text{lasso}}) \}^2
+ \frac{2}{n_f} \sum_{m \in A_f} \phi[m] \pi[m] \{ (\bar{x}_{m} [1] - \bar{x}_{m}[0])^T (\gamma_{\text{proj}} - \hat{\gamma}_{\text{lasso}}) - \sum_{m \in A_f} \frac{n_f[m]}{n_f} (\bar{x}_{m} [1] - \bar{x}_{m}[0])^T (\gamma_{\text{proj}} - \hat{\gamma}_{\text{lasso}}) \} \}
\]

Since \( \bar{x}^*_m(z) = 0 \), we have \( \tau_{R^*_m,m} = \tau_m \) and \( \tau_{R^*_f} = \tau_f \). By replacing \( Y_i(z) \) with \( R_i^*(z) \), we apply Proposition S2 with \( a = \infty \) and obtain

\[
\sum_{m \in A_f} \phi[m] \pi[m] (\hat{\tau}_{R^*_m,m} - \hat{\tau}_{R^*_f})^2 \overset{P}{\longrightarrow} \lim_{n \to \infty} \left[ \sum_{m \in A_f} \pi[m] \left\{ \frac{S^2_{m|e^*(1)}}{e[m]} + \frac{S^2_{m|e^*(0)}}{1 - e[m]} - \frac{S^2_{m|e^*(1)-e^*(0)}}{1} \right\} \right]
+ \left( \frac{n_f}{n} \right)^2 \frac{n}{n_f + \sum_{m \in A_f} \omega[m]} \sum_{m \in A_f} \omega[m] (\tau_{m} - \tau_f)^2
\]

Next, we show that the following term converges to zero in probability:

\[
\sum_{m \in A_f} \phi[m] \pi[m] \left\{ (\bar{x}_{m} [1] - \bar{x}_{m}[0])^T (\gamma_{\text{proj}} - \hat{\gamma}_{\text{lasso}}) - \sum_{m \in A_f} \frac{n_f[m]}{n_f} (\bar{x}_{m} [1] - \bar{x}_{m}[0])^T (\gamma_{\text{proj}} - \hat{\gamma}_{\text{lasso}}) \}^2
= \sum_{m \in A_f} \phi[m] \pi[m] \left\{ (\bar{x}_{m} [1] - \bar{x}_{m}[0])^T (\gamma_{\text{proj}} - \hat{\gamma}_{\text{lasso}}) \}^2
+ \sum_{m \in A_f} \phi[m] \pi[m] \left\{ \sum_{m \in A_f} \frac{n_f[m]}{n_f} (\bar{x}_{m} [1] - \bar{x}_{m}[0])^T (\gamma_{\text{proj}} - \hat{\gamma}_{\text{lasso}}) \}^2
- 2 \sum_{m \in A_f} \phi[m] \pi[m] \left\{ (\bar{x}_{m} [1] - \bar{x}_{m}[0])^T (\gamma_{\text{proj}} - \hat{\gamma}_{\text{lasso}}) \} \right\} \left\{ \sum_{m \in A_f} \frac{n_f[m]}{n_f} (\bar{x}_{m} [1] - \bar{x}_{m}[0])^T (\gamma_{\text{proj}} - \hat{\gamma}_{\text{lasso}}) \} \right\}
\equiv A_1 + A_2 + A_{12}.
\]

By Cauchy–Schwarz inequality, we only need to show that \( A_1 \) and \( A_2 \) converge to zero in probability. Note that \( (\bar{x}_{m} [1] - \bar{x}_{m}[0]) = (\bar{x}_{m} [1] - \bar{x}_{m})/(1 - e[m]) \). For \( m \in A_f \), by Condition 1, \( n[m] \) is bounded, which leads to that \( \phi[m] \) has the same order as a constant. By Lemma S6,
Conditions 1, 2, and 5, we have

\[ A_1 = (\gamma_{\text{proj}} - \hat{\gamma}_{\text{lasso}})^T \left\{ \sum_{m \in A_J} \phi_m [\pi_m] (\bar{x}_m | 1 - \bar{x}_m | 0) (\bar{x}_m | 1 - \bar{x}_m | 0)^T \right\} (\gamma_{\text{proj}} - \hat{\gamma}_{\text{lasso}}) \]

\[ = (\gamma_{\text{proj}} - \hat{\gamma}_{\text{lasso}})^T \left\{ \sum_{m \in A_J} \frac{\phi_m [\pi_m]}{1 - e_m} \bar{x}_m \bar{x}_m^T \right\} (\gamma_{\text{proj}} - \hat{\gamma}_{\text{lasso}}) \]

\[ = o_p(1). \]

By Lemmas S5 and S6, Conditions 1, 2, and 5, we have

\[ A_2 = \left( \sum_{m \in A_J} \phi_m [\pi_m] \right) \cdot \left( \sum_{m \in A_J} \frac{n_m}{n_f (1 - e_m)} (\bar{x}_m | 1 - \bar{x}_m | 0)^T (\gamma_{\text{proj}} - \hat{\gamma}_{\text{lasso}}) \right)^2 = o_p(1). \]

Combining Step 1 and Step 2, (S18) converges in probability to

\[ \sigma_{\text{lasso}}^2 + \lim_{n \to \infty} \left\{ \sum_{m \in A_c} \pi_m S_m^2 | \varepsilon | (1 - \varepsilon)(0) + \frac{n_f}{n} \cdot \frac{n}{n_f + \sum_{m \in A_f} \omega_m} \right\} \left\{ \omega_m (\tau_m - \tau_f)^2 \right\}. \]

**Step 3 (\( \hat{\sigma}_{\text{lasso}}^2 \) is conservative):** We compare the limits of \( \hat{\sigma}_{\text{lasso}}^2 \) and \( \sigma_{\text{lasso}}^2 \). By definition and the above proof, it is easy to see that

\[ \lim_{n \to \infty} (\hat{\sigma}_{\text{lasso}}^2 - \sigma_{\text{lasso}}^2) = \lim_{n \to \infty} \left\{ \sum_{m \in A_c} \pi_m S_m^2 | \varepsilon | (1 - \varepsilon)(0) + \frac{n_f}{n} \cdot \frac{n}{n_f + \sum_{m \in A_f} \omega_m} \right\} \left\{ \omega_m (\tau_m - \tau_f)^2 \right\} \]

\[ \geq 0. \]

**Step 4 (Improved efficiency):** We compare the limits of \( \hat{\sigma}_{\text{lasso}}^2 \) and \( \sigma_{\text{unadj}}^2 \). By Proposition S2 with \( a = \infty \),

\[ \sigma_{\text{unadj}}^2 \xrightarrow{p} \sigma_{\text{unadj}}^2 + \lim_{n \to \infty} \left\{ \sum_{m \in A_c} \pi_m S_m^2 | \varepsilon | (1 - \varepsilon)(0) \right\} + \left( \frac{n_f}{n} \right) \cdot \left( \frac{n}{n_f + \sum_{m \in A_f} \omega_m} \right) \sum_{m \in A_f} \omega_m (\tau_m - \tau_f)^2 \}

Sine \( S_m^2 | \varepsilon | (1 - \varepsilon)(0) = S_m^2 | \varepsilon | (1 - \varepsilon)(0) \), then

\[ \lim_{n \to \infty} (\hat{\sigma}_{\text{lasso}}^2 - \sigma_{\text{unadj}}^2) = \lim_{n \to \infty} \left\{ \sum_{m = 1}^{M} S_m^2 | \varepsilon | (1 - \varepsilon)(0) - \sum_{m = 1}^{M} S_m^2 | \varepsilon | (1 - \varepsilon)(0) \right\} + \sum_{m = 1}^{M} \pi_m S_m^2 | \varepsilon | (1 - \varepsilon)(0) - \sum_{m = 1}^{M} S_m^2 | \varepsilon | (1 - \varepsilon)(0) \right\}. \]
We have shown in (S16) and (S17) that

\[ S^2_{[m]}(z) = S^2_{[m]}(z) + \gamma^T_{\text{proj}} S^2_{[m]}X \gamma_{\text{proj}} - 2S^T_{[m]}XY(z) \gamma_{\text{proj}}, \quad z = 0, 1. \]

Therefore,

\[
\begin{align*}
\lim_{n \to \infty} (\hat{\sigma}^2_{\text{lasso}} - \hat{\sigma}^2_{\text{unadj}}) &= \lim_{n \to \infty} \left[ \gamma^T_{\text{proj}} \sum_{m=1}^{M} \frac{S^2_{[m]}X}{\epsilon_{[m]}(1 - \epsilon_{[m]})} \gamma_{\text{proj}} - 2 \sum_{m=1}^{M} \pi_{[m]} \left\{ \frac{S_{[m]}X Y(1)}{\epsilon_{[m]}} + \frac{S_{[m]}X Y(0)}{1 - \epsilon_{[m]}} \right\} \gamma_{\text{proj}} \right] \\
&= \lim_{n \to \infty} \gamma^T_{\text{proj}} \left\{ \sum_{m=1}^{M} \pi_{[m]} \frac{S^2_{[m]}X}{\epsilon_{[m]}(1 - \epsilon_{[m]})} \right\} \gamma_{\text{proj}} \leq 0,
\end{align*}
\]

where the second equality is because of the definition of \( \gamma_{\text{proj}} \).

\[ \Box \]

### B.3 Proof of Theorem 3

**Proof.** First, we prove the result on the asymptotic distribution of \( \hat{\gamma}_{\text{lasso}} \) under stratified rerandomization. In the proof of Theorem 1, we have shown that

\[
\hat{\gamma}_{\text{lasso}} - \tau = \sum_{m=1}^{M} \pi_{[m]} (\bar{\varepsilon}_{[m]} - \tilde{\varepsilon}_{[m]}) + \sum_{m=1}^{M} \pi_{[m]} (\bar{\varepsilon}_{[m]} - \tilde{\varepsilon}_{[m]})^T (\gamma_{\text{proj}} - \hat{\gamma}_{\text{lasso}}) - \sum_{m=1}^{M} \pi_{[m]} (\bar{\varepsilon}_{[m]} - \tilde{\varepsilon}_{[m]})^T (\gamma_{\text{proj}} - \hat{\gamma}_{\text{lasso}}).
\]

Applying Proposition S2 to \( \varepsilon^*_i(z) \), we have

\[
\left\{ \sqrt{n} \sum_{m=1}^{M} \pi_{[m]} (\varepsilon^*_{[m]} - \tilde{\varepsilon}^*_{[m]}) \mid \mathcal{M}_a \right\} \xrightarrow{d} \sigma_{\text{lasso}} \sqrt{1 - R^2_{\varepsilon^*,X_K}} \tilde{\varepsilon}_0 + \sqrt{R^2_{\varepsilon^*,X_K}} L_{k,a},
\]

where

\[
R^2_{\varepsilon^*,X_K} = \lim_{n \to \infty} \left( V_{\varepsilon^*X_K} V^{-1}_{\varepsilon^*X_K} V_{X_K} \right) / V_{\varepsilon^*},
\]

\[
V_{X_K \varepsilon^*} = \sum_{m=1}^{M} \pi_{[m]} \left( \frac{S_{[m]}X_K \varepsilon^*_{[m]}}{\epsilon_{[m]}} + \frac{S_{[m]}X_K \varepsilon^*_{[m]}}{1 - \epsilon_{[m]}} \right).
\]

Recall the definition of \( \gamma_{\text{proj}} \):

\[
(\gamma_{\text{proj}})_S = \left\{ \sum_{m=1}^{M} \pi_{[m]} \frac{S^2_{[m]}X_S}{\epsilon_{[m]}(1 - \epsilon_{[m]})} \right\}^{-1} \left\{ \sum_{m=1}^{M} \pi_{[m]} \frac{S_{[m]}X_S Y(1)}{\epsilon_{[m]}} + \sum_{m=1}^{M} \pi_{[m]} \frac{S_{[m]}X_S Y(0)}{1 - \epsilon_{[m]}} \right\},
\]

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\((\gamma_{\text{proj}})_{\mathcal{S}} = 0\), and the decomposition \(Y_i(z) = \bar{Y}_{[m]}(z) + (\bar{x}_i - \bar{x}_{[m]})^T \gamma_{\text{proj}} + \varepsilon_i^*(z)\), we have

\[
\sum_{m=1}^{M} \pi_{[m]} \left\{ \frac{S_{[m]} X \varepsilon^*(1)}{e_{[m]}} + \frac{S_{[m]} X \varepsilon^*(0)}{1 - e_{[m]}} \right\} = 0.
\]

Since \(\mathcal{K} \subset \mathcal{S}\), we have

\[
\sum_{m=1}^{M} \pi_{[m]} \left\{ \frac{S_{[m]} X \varepsilon^*(1)}{e_{[m]}} + \frac{S_{[m]} X \varepsilon^*(0)}{1 - e_{[m]}} \right\} = 0.
\]

Therefore, \(V_{X \varepsilon^*} = 0\). Then, \(R_{\varepsilon^*, X_\mathcal{K}}^2 = 0\) and

\[
\sqrt{n} \sum_{m=1}^{M} \pi_{[m]} (\bar{x}_{[m]} z - \bar{x}_{[m]})^T (\gamma_{\text{proj}} - \hat{\gamma}_{\text{lasso}}) \xrightarrow{\mathcal{M}_a} N(0, \sigma^2_{\text{lasso}}).
\]

It suffices for the asymptotic normality of \(\hat{\tau}_{\text{lasso}}\) to show that,

\[
\sqrt{n} \sum_{m=1}^{M} \pi_{[m]} (\bar{x}_{[m]} z - \bar{x}_{[m]})^T (\gamma_{\text{proj}} - \hat{\gamma}_{\text{lasso}}) \xrightarrow{\mathcal{M}_a} N(0, \sigma^2_{\text{lasso}}), \quad z = 0, 1.
\]  
\((S20)\)

By Proposition \(S2\), \(P(\mathcal{M}_a) \rightarrow P(\chi^2_k < a) > 0\). Thus, it suffices for \((S20)\) to show that, under stratified randomization,

\[
\sum_{m=1}^{M} \pi_{[m]} (\bar{x}_{[m]} z - \bar{x}_{[m]})^T (\gamma_{\text{proj}} - \hat{\gamma}_{\text{lasso}}) \xrightarrow{\mathcal{P}} 0, \quad z = 0, 1,
\]

which hold as shown in the proof of Theorem 1.

Next, we compare the asymptotic variances. By Proposition \(S2\), \(\sigma^2_{\text{unadj}|\mathcal{M}_a} \leq \sigma^2_{\text{unadj}}\). Thus, it suffices to show that \(\sigma^2_{\text{lasso}} \leq \sigma^2_{\text{unadj}|\mathcal{M}_a}\). By Proposition \(S2\), we have

\[
\sigma^2_{\text{unadj}|\mathcal{M}_a} = \sigma^2_{\text{unadj}} \left[ 1 - \{1 - v_{k,a}\} R^2_{X_\mathcal{K}} \right] \\
\geq \sigma^2_{\text{unadj}} (1 - R^2_{X_\mathcal{K}}) \\
= \lim_{n \to \infty} \left[ \text{var}(\hat{\tau}_{\text{unadj}}) - \text{var}\{\text{proj}(\hat{\tau}_{\text{unadj}} | \hat{\tau}_{X,\mathcal{K}})\} \right] \\
= \lim_{n \to \infty} \text{var}\{\hat{\tau}_{\text{unadj}} - \text{proj}(\hat{\tau}_{\text{unadj}} | \hat{\tau}_{x,\mathcal{K}})\},
\]

where \(\text{proj}(\hat{\tau}_{\text{unadj}} | \hat{\tau}_{x,\mathcal{K}})\) denote the projection (minimizing the variance) of \(\hat{\tau}_{\text{unadj}}\) onto \(\hat{\tau}_{x,\mathcal{K}}\), and the last but one equality holds because (see Li & Ding (2020)):

\[
R^2_{X_\mathcal{K}} = \lim_{n \to \infty} \frac{\text{var}\{\text{proj}(\hat{\tau}_{\text{unadj}} | \hat{\tau}_{x,\mathcal{K}})\}}{\text{var}(\hat{\tau}_{\text{unadj}})} = \lim_{n \to \infty} \frac{\text{var}\{\text{proj}(\hat{\tau}_{\text{unadj}} | \hat{\tau}_{x,\mathcal{K}})\}}{\sigma^2_{\text{unadj}}}.
\]
By definition and the above proof for the asymptotic normality, we have

\[
\sigma^2_{\text{lasso}} = \lim_{n \to \infty} \sum_{m=1}^{M} \pi[m] \left\{ \frac{S[m] e^*(1)}{e[m]} + \frac{S[m] e^*(0)}{1 - e[m]} - S[m] \{e^*(1) - e^*(0)\} \right\} \\
= \lim_{n \to \infty} E \left\{ \hat{\tau}_{\text{unadj}} - \tau - \hat{\tau}_{\text{proj}} \right\}^2 \\
= \lim_{n \to \infty} \text{var} \left\{ \hat{\tau}_{\text{unadj}} - \text{proj} \left( \hat{\tau}_{\text{unadj}} \mid \hat{\tau}_{x, S} \right) \right\} \\
\leq \lim_{n \to \infty} \text{var} \left\{ \hat{\tau}_{\text{unadj}} - \text{proj} \left( \hat{\tau}_{\text{unadj}} \mid \hat{\tau}_{x, K} \right) \right\} \\
\leq \sigma^2_{\text{unadj}|M_a},
\]

where the first inequality is due to $K \subset S$.

\[\square\]

### B.4 Proof of Theorem 4

**Proof.** By Theorem 2, under stratified randomization,

\[
\hat{\sigma}^2_{\text{lasso}} \xrightarrow{p} \sigma^2_{\text{lasso}} + \lim_{n \to \infty} \left\{ \sum_{m \in A_c} \pi[m] S[m] \{e^*(1) - e^*(0)\} + \left( \frac{n_f}{n} \right)^2 \frac{n}{n_f + \sum_{m \in A_f} \omega[m]} \sum_{m \in A_f} \omega[m] (\tau[m] - \tau_f)^2 \right\}.
\]

Since $P(M_a) \to P(\chi^2_k \leq a) > 0$, then the above statement also holds under stratified rerandomization, i.e., conditional on $M_a$.

Next, we show that $\sigma^2_{\text{unadj}|M_a} \leq \sigma^2_{\text{unadj}}$ holds in probability under stratified rerandomization.

By Proposition S2,

\[
\hat{\sigma}^2_{\text{unadj}} \xrightarrow{p} \sigma^2_{\text{unadj}} + \lim_{n \to \infty} \left\{ \sum_{m \in A_c} \pi[m] S[m] \{Y(1) - Y(0)\} + \left( \frac{n_f}{n} \right)^2 \frac{n}{n_f + \sum_{m \in A_f} \omega[m]} \sum_{m \in A_f} \omega[m] (\tau[m] - \tau_f)^2 \right\},
\]

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Therefore, with probability tending to one, $\hat{\sigma}_{\text{unadj}|M_a}^2 \leq \sigma_{\text{unadj}|M_a}^2$.

Finally, we show that $\hat{\sigma}_{\text{lasso}}^2 \leq \sigma_{\text{unadj}|M_a}^2$ holds in probability under stratified rerandomization. Under stratified rerandomization, the probability limit of $\hat{\sigma}_{\text{unadj}|M_a}^2$ satisfies:

$$
\lim_{n \to \infty} \hat{\sigma}_{\text{unadj}|M_a}^2 = \sigma_{\text{unadj}|M_a}^2 + \lim_{n \to \infty} \left\{ \sum_{m \in A_c} \pi[m] S^2_{[m]}(Y(1) - Y(0)) + \left( \frac{n_f}{n} \right)^2 \frac{n}{n_f + \sum_{m \in A_f} \omega[m]} \sum_{m \in A_f} \omega[m](\tau[m] - \tau_f)^2 \right\}.
$$

In the proof of Theorem 1, we have shown that $S^2_{[m]}(\epsilon^*(1) - \epsilon^*(0)) = S^2_{[m]}(Y(1) - Y(0))$. Therefore, $\hat{\sigma}_{\text{lasso}}^2 \leq \sigma_{\text{unadj}|M_a}^2$ holds in probability under stratified rerandomization.

\[ \square \]

### B.5 Proof of Lemma S5

**Proof.** For any $t > 0$, we have

$$
P\left( \left\| \sum_{m=1}^{M} \pi[m] (x[m]_1 - \bar{x}[m]) \right\|_{\infty} \geq t \right) \leq p \cdot \max_{1 \leq j \leq p} P\left( \left\| \sum_{m=1}^{M} \pi[m] (x[m]_{1,j} - \bar{x}[m]_{1,j}) \right\|_{\infty} \geq t \right).
$$

Applying Theorem S1 to the $j$th covariate $X_j$ (and $-X_j$), we have

$$
P\left( \left\| \sum_{m=1}^{M} \pi[m] (x[m]_1 - \bar{x}[m]) \right\|_{\infty} \geq t \right) \leq 2p \cdot \max_{1 \leq j \leq p} \left\{ - \frac{nt^2}{4\hat{\sigma}_{x,j}^2} \right\},
$$

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where $\sigma^2_{x,j} = (1/n) \sum_{m=1}^{M} \sum_{i \in [m]} (x_{ij} - \bar{x}_{[m],j})^2/e^2_{[m]}$. By Conditions 1–2 and Cauchy-Schwarz inequality, we have,

$$
\sigma^2_{x,j} \leq \frac{1}{c^2} \cdot \frac{1}{n} \sum_{m=1}^{M} \sum_{i \in [m]} (x_{ij} - \bar{x}_{[m],j})^2 \leq \frac{1}{c^2} \cdot \left\{ \frac{1}{n} \sum_{m=1}^{M} \sum_{i \in [m]} (x_{ij} - \bar{x}_{[m],j})^4 \right\}^{1/2} \leq \frac{L^{1/2}}{c^2}.
$$

Therefore,

$$
P\left( \left\| \sum_{m=1}^{M} \pi_{[m]} (\bar{x}_{[m]} - \bar{x}_{1}) \right\|_\infty \geq t \right) \leq 2 \exp \left( \log p - \frac{c^2 n t^2}{4L^{1/2}} \right).
$$

Taking $t = \sqrt{8L^{1/2}/c^2} \cdot \sqrt{\log p}/n$ gives the result.

### B.6 Proof of Lemma S6

Before proving Lemma S6, we first prove two useful lemmas. Define

$$
V_{XX} = \sum_{m=1}^{M} \pi_{[m]} \frac{S_{[m]} X}{e_{[m]} (1 - e_{[m]})}, \quad \hat{\Sigma}_{X}^{\omega, (1)} = \sum_{m=1}^{M} (\pi_{[m]} - n^{-1} e_{[m]} S_{[m]} X - \epsilon_{\omega, (1)}).
$$

Then

$$
\hat{\Sigma}_{XX} = \frac{1}{n} \sum_{m=1}^{M} \sum_{i \in [m]} Z_i (\bar{x}_{i}^{\omega} - \bar{x}_{1}^{\omega}) (\bar{x}_{i}^{\omega} - \bar{x}_{1}^{\omega})^T, \quad \hat{\Sigma}_{XX}^{\omega, (1)} = \frac{1}{n} \sum_{m=1}^{M} \sum_{i \in [m]} \epsilon_{i} (\bar{x}_{i}^{\omega} - \bar{x}_{1}^{\omega}) \{ \epsilon_{\omega, (1)} - \epsilon_{1}^{\omega} \}.
$$

**Lemma S8.** Suppose that Conditions 1 and 2 hold, then there exists a constant $C$, such that for the event

$$
L_{XX}^\omega = \left\{ \left\| \hat{\Sigma}_{XX}^\omega - V_{XX} \right\|_\infty \leq C \sqrt{\log p} / n \right\},
$$

we have $P(L_{XX}^\omega) \rightarrow 1$.

**Proof.** Since

$$
\hat{\Sigma}_{XX} = \frac{1}{n} \sum_{m=1}^{M} \sum_{i \in [m]} Z_i (\bar{x}_{i}^{\omega} - \bar{x}_{1}^{\omega}) (\bar{x}_{i}^{\omega} - \bar{x}_{1}^{\omega})^T = \frac{1}{n} \sum_{m=1}^{M} \sum_{i \in [m]} Z_i (\bar{x}_{i}^{\omega}) (\bar{x}_{i}^{\omega})^T - \frac{n_1}{n} (\bar{x}_{1}^{\omega}) (\bar{x}_{1}^{\omega})^T,
$$

we have

$$
\left\| \hat{\Sigma}_{XX}^\omega - V_{XX} \right\|_\infty \leq \left\| \frac{1}{n} \sum_{m=1}^{M} \sum_{i \in [m]} Z_i (\bar{x}_{i}^{\omega}) (\bar{x}_{i}^{\omega})^T - V_{XX} \right\|_\infty + \left\| \frac{n_1}{n} (\bar{x}_{1}^{\omega}) (\bar{x}_{1}^{\omega})^T \right\|_\infty.
$$

Thus, we only need to bound the above two terms.
To bound the first term in (S21), we have

\[
E \left\{ \frac{1}{n} \sum_{m=1}^{M} \sum_{\ell \in [m]} Z_i(x_i^{\omega}) (x_i^{\omega})^T \right\} = E \left\{ \frac{1}{n} \sum_{m=1}^{M} \sum_{\ell \in [m]} Z_i(\omega_i(1)\omega_i(1) \{ x_i - x_i[m] \}) (x_i - x_i[m])^T \right\}
\]

\[
= \frac{1}{n} \sum_{m=1}^{M} \sum_{\ell \in [m]} \frac{n_{[m]}^1}{n_{[m]}} \frac{n_{[m]}^2}{n_{[m]}^1 (n_{[m]}^1 - 1)} \frac{1}{e_{[m]}} \{ x_i - x_i[m] \} (x_i - x_i[m])^T
\]

\[
= \sum_{m=1}^{M} \pi_{[m]} \frac{S_{[m]}^2 x}{e_{[m]} (1 - e_{[m]})} = V_{xx}.
\]

For any \( t > 0 \), applying Theorem S1 to \( a_i = (n_{[m]}^1/n_{[m]}) x_{i,j} x_{i,l} \), we have

\[
P \left( \left\| \frac{1}{n} \sum_{m=1}^{M} \sum_{\ell \in [m]} Z_i(x_i^{\omega}) (x_i^{\omega})^T - V_{xx} \right\|_\infty \geq t \right) \]

\[
P \left( \left\| \sum_{m=1}^{M} \pi_{[m]} \frac{1}{n_{[m]}} \sum_{\ell \in [m]} Z_i \left( \frac{n_{[m]}^1}{n_{[m]}} x_{i,j} x_{i,l} \right) - \sum_{m=1}^{M} \pi_{[m]} \frac{S_{[m]} x_{i,j} x_{i,l}}{e_{[m]} (1 - e_{[m]})} \right\|_\infty \geq t \right) \]

\[
\leq p^2 \max_{1 \leq j, l \leq p} P \left( \left\| \sum_{m=1}^{M} \pi_{[m]} \frac{1}{n_{[m]}} \sum_{\ell \in [m]} Z_i \left( \frac{n_{[m]}^1}{n_{[m]}} x_{i,j} x_{i,l} \right) - \sum_{m=1}^{M} \pi_{[m]} \frac{S_{[m]} x_{i,j} x_{i,l}}{e_{[m]} (1 - e_{[m]})} \right\| \geq t \right) \]

\[
\leq 2p^2 \exp \left\{ - \frac{c \rho^2 L^2}{16} \right\},
\]

where the last inequality is due to Condition 2.

To bound the second term in (S21), we have

\[
E \{ \bar{x}_i^{\omega} \} = E \left\{ \sum_{m=1}^{M} \pi_{[m]} \frac{1}{n_{[m]}} \sum_{\ell \in [m]} Z_i \left( \frac{n}{n_{1, [m]}} x_{i,j} \right) \right\} = \sum_{m=1}^{M} \pi_{[m]} \frac{1}{n_{1, [m]}} \sum_{\ell \in [m]} \left( \frac{n}{n_{1, [m]}} \right) x_{i,j} = 0,
\]

where the last equality is due to \( \sum_{\ell \in [m]} x_{i,j} = 0 \). Applying Theorem S1 to \( a_i = (n/n_1)(n_{[m]}^1/n_{[m]}) x_{i,j}^{\omega} \), we have \( \| \bar{x}_i^{\omega} \|_\infty = O_p(\sqrt{\log p}/n) \). Thus, \( \| (n_1/n) (\bar{x}_i^{(1)}) (\bar{x}_i^{(1)})^T \|_\infty = o_p(\sqrt{\log p}/n) \).

The conclusion follows from (S21) and the above bounds for the two terms in (S21). \( \square \)

**Lemma S9.** Suppose that Conditions 1, 2, 4, and 5 hold, then for the event

\[
\mathcal{L}_{X\varepsilon}^{\omega} = \left\{ \| \Sigma_{X\varepsilon}^{\omega} (1) \|_\infty \leq \eta \lambda_1 \right\},
\]

we have \( P(\mathcal{L}_{X\varepsilon}^{\omega}) \rightarrow 1 \).
Proof. Since
\[ \hat{\Sigma}_{X}^{\omega}(1) = \frac{1}{n} \sum_{m=1}^{M} \sum_{i \in [m]} Z_i(x_i^{\omega} - \bar{x}_i^{\omega}) \{ \varepsilon_i^{\omega}(1) - \bar{\varepsilon}_i^{\omega} \} \]
\[ = \frac{1}{n} \sum_{m=1}^{M} \sum_{i \in [m]} Z_i x_i^{\omega} \varepsilon_i^{\omega}(1) - \frac{n_1}{n} \bar{x}_i^{\omega} \bar{\varepsilon}_i^{\omega}, \]
we have
\[ ||\hat{\Sigma}_{X}^{\omega}(1)||_{\infty} \leq \frac{1}{n} \sum_{m=1}^{M} \sum_{i \in [m]} Z_i x_i^{\omega} \varepsilon_i^{\omega}(1) + \frac{n_1}{n} ||\bar{x}_i^{\omega} \bar{\varepsilon}_i^{\omega}||_{\infty}. \tag{S22} \]
We bound the above two terms separately.

To bound the first term in (S22), we have
\[ E \left\{ \frac{1}{n} \sum_{m=1}^{M} \sum_{i \in [m]} Z_i x_i^{\omega} \varepsilon_i^{\omega}(1) \right\} = \frac{1}{n} \sum_{m=1}^{M} \sum_{i \in [m]} c_i^{[m]} x_i^{\omega} \varepsilon_i^{\omega}(1) = \sum_{m=1}^{M} \left( \eta_{[m]} - n^{-1} \right) c_i^{[m]} S_{[m]} X \varepsilon^{\omega}(1) = \hat{\Sigma}_{X}^{\omega}(1). \]

For any \( t > 0 \), by triangle inequality, we have
\[ P(\frac{1}{n} \sum_{m=1}^{M} \sum_{i \in [m]} Z_i x_i^{\omega} \varepsilon_i^{\omega}(1) \geq t) \]
\[ = P(\frac{1}{n} \sum_{m=1}^{M} \sum_{i \in [m]} Z_i x_i^{\omega} \varepsilon_i^{\omega}(1) - \hat{\Sigma}_{X}^{\omega}(1) + \hat{\Sigma}_{X}^{\omega}(1) \geq t) \]
\[ \leq P(\frac{1}{n} \sum_{m=1}^{M} \sum_{i \in [m]} Z_i x_i^{\omega} \varepsilon_i^{\omega}(1) - \hat{\Sigma}_{X}^{\omega}(1) \geq t - \hat{\Sigma}_{X}^{\omega}(1) ||_{\infty} \geq t) \]
\[ \leq P(\frac{1}{n} \sum_{m=1}^{M} \sum_{i \in [m]} Z_i x_i^{\omega} \varepsilon_i^{\omega}(1) - \hat{\Sigma}_{X}^{\omega}(1) \geq t - \delta_n) \]
\[ \leq p \cdot \max_{1 \leq j \leq p} P(\frac{1}{n} \sum_{m=1}^{M} \sum_{i \in [m]} Z_i x_{ij}^{\omega} \varepsilon_i^{\omega}(1) - \hat{\Sigma}_{X}^{\omega}(1) \geq t - \delta_n). \]

Applying Theorem S1 to \( a_i = (n_{[m]} / n_{[m]}) x_{ij}^{\omega} \varepsilon_i^{\omega}(1) \), we have
\[ P\left( \frac{1}{n} \sum_{m=1}^{M} \sum_{i \in [m]} Z_i x_{ij}^{\omega} \varepsilon_i^{\omega}(1) - \hat{\Sigma}_{X}^{\omega}(1) \right) \geq t - \delta_n) \leq 2 \exp \left\{ -\frac{c^3 n (t - \delta_n)^2}{8L} \right\}, \]
where the inequality is due to Condition 2. Let \( t = 4 \sqrt{L (\log p) / (c^3 n)} + \delta_n \), we have
\[ P(\frac{1}{n} \sum_{m=1}^{M} \sum_{i \in [m]} Z_i x_i^{\omega} \varepsilon_i^{\omega}(1) ||_{\infty} \geq 4 \sqrt{L (\log p) / (c^3 n)} + \delta_n) \leq \frac{2}{p} \rightarrow 0. \tag{S23} \]
To bound the second term in (S22), we have

\[ E \{ \bar{x}_1^\omega \} = E \left\{ \sum_{m=1}^{M} \frac{1}{M} \sum_{i \in [m]} Z_i \left( \sum_{m=1}^{M} \frac{1}{M} \sum_{i \in [m]} \left( n \frac{n[m]^1}{n_1} x_i^\omega \right) \right) \right\} = \sum_{m=1}^{M} \frac{1}{M} \sum_{i \in [m]} \left( n \frac{n[m]^1}{n_1} x_i^\omega \right) = 0, \]

where the last equality is due to \( \sum_{i \in [m]} x_i^\omega = 0 \). Applying Theorem S1 to \( a_i = (n/n_1)(n[m]/n[m])x_{i,j}^\omega \), we have

\[ P(\| \bar{x}_1^\omega \|_\infty \leq C_1 \sqrt{(\log p)/n}) \to 1 \]

for some constant \( C_1 > 0 \). Moreover, by Condition 2,

\[ |\bar{\varepsilon}_1^\omega| \leq \frac{1}{n_1} \sum_{i=1}^{n} |\varepsilon_i^\omega(1)| \leq \frac{n}{n_1} L^{1/4}. \]

Therefore,

\[ P \left( \frac{n_1}{n} \| \frac{n_1}{n} \bar{x}_1^\omega \|_\infty \leq \frac{L^{1/4}}{n_1} C_1 \sqrt{(\log p)/n} \right) \to 1. \]  

(S24)

Combining (S22), (S23), and (S24), there exists a constant \( C > 0 \), such that for \( \eta \lambda_1 \geq C \sqrt{(\log p)/n + \delta_n} \), we have \( P(\| \Sigma_{X,\varepsilon(1)} \|_\infty \leq \eta \lambda_1) \to 1. \)

Now, we can prove Lemma S6.

**Proof of Lemma S6.** We will only prove the result for \( z = 1 \), as the proof for \( z = 0 \) is similar.

Define \( h = \hat{\gamma}_{\text{lasso},1} - \gamma(1) \). By KKT condition, we have

\[ \frac{1}{n_1} \sum_{i: Z_i = 1} (x_i^\omega - \bar{x}_1^\omega) |Y_i^\omega - \bar{Y}_1^\omega - (x_i^\omega - \bar{x}_1^\omega)^T \hat{\gamma}_{\text{lasso},1} = \lambda_1 \kappa. \]  

(S25)

where \( \kappa \) is the sub-gradient of \( \| \gamma \|_1 \) taking value at \( \gamma = \hat{\gamma}_{\text{lasso},1} \), i.e.,

\[ \kappa \in \partial \| \gamma \|_1 \bigg|_{\gamma = \hat{\gamma}_{\text{lasso},1}} \text{ with } \begin{cases} \kappa_j = \text{sign}((\hat{\gamma}_{\text{lasso},1})_j) \text{ for } j \text{ such that } (\hat{\gamma}_{\text{lasso},1})_j \neq 0 \\ \kappa_j \in [-1, 1], \text{ otherwise.} \end{cases} \]

Define

\[ \varepsilon_i^\omega(1) = Y_i^\omega(1) - \bar{Y}_1^\omega(1) - (x_i^\omega - \bar{x}_1^\omega)^T \gamma(1). \]

Then,

\[ Y_i^\omega(1) - \bar{Y}_1^\omega = \{ x_i^\omega - \bar{x}_1^\omega \}^T \gamma(1) + \{ \varepsilon_i^\omega(1) - \varepsilon_1^\omega \}. \]

Plugging in to (S25), we have

\[ \Sigma_{XX} \{ \gamma(1) - \hat{\gamma}_{\text{lasso},1} \} + \Sigma_{XX,\varepsilon(1)} = \lambda_1 \kappa. \]  

(S26)

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Multiplying both sides of (S26) by $-\mathbf{h}^T = \{c(1) - \hat{y}_{\text{lasso},1}\}^T$, we have

$$h^T \hat{\Sigma}_{XX}^c \mathbf{h} - h^T \hat{\Sigma}_{XX}^c \mathbf{c} = \lambda_1(-\mathbf{h})^T \kappa \leq \lambda_1(\|c(1)\|_1 - \|\hat{y}_{\text{lasso},1}\|_1),$$

where the last inequality holds because

$$\{c(1)\}^T \kappa \leq \|c(1)\|_1 \|\kappa\|_\infty \leq \|c(1)\|_1 \text{ and } \hat{\gamma}_{\text{lasso},1}^T \kappa = \|\hat{\gamma}_{\text{lasso},1}\|_1.$$ 

Rearranging and by Hölder inequality, we have

$$h^T \hat{\Sigma}_{XX}^c \mathbf{h} \leq \lambda_1 \left\{\|c(1)\|_1 - \|\hat{y}_{\text{lasso},1}\|_1 \right\} + \|\mathbf{h}\|_1 \cdot \|\hat{\Sigma}_{XX}^c \mathbf{c}\|_\infty.$$ 

(S27)

By the definition of $\mathbf{h}$ and several applications of the triangle inequality, we have

$$\|c(1)\|_1 - \|\hat{y}_{\text{lasso},1}\|_1 = \{c(1)\} \|s\|_1 - \{\hat{y}_{\text{lasso},1}\} \|s\|_1 + \{c(1)\} \|s^c\|_1 - \{\hat{y}_{\text{lasso},1}\} \|s^c\|_1$$

$$\leq \|h \|_1 + \|\{c(1)\} \|s^c\|_1 - \{\hat{y}_{\text{lasso},1}\} \|s^c\|_1$$

$$= \|h \|_1 - \|h \|_1 + 2\|\{c(1)\} \|s^c\|_1.$$ 

Therefore, conditional on $L_{XX}^c$ with $P(L_{XX}^c) \rightarrow 1$ (Lemma S9), we have

$$0 \leq h^T \hat{\Sigma}_{XX}^c \mathbf{h} \leq \lambda_1 \left(\|h \|_1 - \|h \|_1 + 2\|\{c(1)\} \|s^c\|_1 + \eta\|h\|_1\right)$$

$$\leq \lambda_1 \left\{(\eta - 1)\|h \|_1 + (1 + \eta)\|h \|_1 + 2\|\{c(1)\} \|s^c\|_1\right\}.$$ 

Then,

$$(1 - \eta)\|h \|_1 \leq (1 + \eta)\|h \|_1 + 2\|\{c(1)\} \|s^c\|_1 \leq (1 + \eta)\|h \|_1 + 2s\lambda_1.$$ 

(S28)

Recall that $\xi > 1$ and $0 < \eta < (\xi - 1)/(\xi + 1) < 1$, thus $(1 - \eta)\xi - (1 + \eta) > 0$. To proceed, considering the following two cases:

(1) If $\|h \|_1 \leq 2s\lambda_1/\{(1 - \eta)\xi - (1 + \eta)\}$, then by (S28):

$$\|h\|_1 = \|h \|_1 + \|h \|_1$$

$$\leq \|h \|_1 + \frac{1 + \eta}{1 - \eta}\|h \|_1 + 2s\lambda_1$$

$$\leq \frac{1}{1 - \eta}\left\{2\|h \|_1 + 2s\lambda_1\right\}$$

$$\leq \frac{2s\lambda_1}{1 - \eta}\left\{(1 - \eta)\xi - (1 + \eta) + 1\right\}.$$ 

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Then,
\[ \|h\|_1 = \|\hat{\gamma}_{lasso, 1} - \gamma(1)\|_1 = O_p(s\lambda_1). \]

(2) If \(2s\lambda_1 < \{(1 - \eta)\xi - (1 + \eta)\} \|h_S\|_1\), then by (S28), we have
\[ \|h_{S^c}\|_1 \leq \frac{1 + \eta}{1 - \eta} \|h_S\|_1 + \frac{(1 - \eta)\xi - (1 + \eta)}{1 - \eta} \|h_S\|_1 = \xi \|h_S\|_1. \]

By Condition 4, we have
\[ \|h\|_1 = \|h_S\|_1 + \|h_{S^c}\|_1 \leq (1 + \xi)\|h_S\|_1 \leq (1 + \xi)Cs\|V_{XX}h\|_{\infty}. \]  

(S29)

Taking the \(l_\infty\)-norm on both sides of the KKT condition (S26), we have, conditional on the event \(L_{\omega X} \cap L_{\omega XX}^c\),
\[ \|\hat{\Sigma}_{XX} h\|_{\infty} \leq \lambda_1 + \|\hat{\Sigma}_{XX} - \gamma(1)\|_{\infty} \leq (1 + \eta)\lambda_1. \]  

(S30)

By triangle inequality and H\"{o}lder inequality, and conditional on the event \(L_{\omega X} \cap L_{\omega XX}^c\), we have
\[ s\|V_{XX}h\|_{\infty} \leq s\|\hat{\Sigma}_{XX} - V_{XX}\|_{\infty}\|h\|_1 + s\|\hat{\Sigma}_{XX}h\|_{\infty} \leq sC\sqrt{(\log p)/n}\|h\|_1 + s\|\hat{\Sigma}_{XX}h\|_{\infty} \leq sC\sqrt{(\log p)/n}\|h\|_1 + s(1 + \eta)\lambda_1 \]
\[ = o(1)\|h\|_1 + s(1 + \eta)\lambda_1, \]
where the third inequality is because of inequality (S30) and the last equality is because of Condition 5. Combining above inequality and (S29), we obtain
\[ \|h\|_1 = \|\hat{\gamma}_{lasso, 1} - \gamma(1)\|_1 = O_p(s\lambda_1). \]

B.7 Proof of Lemma S7

Proof. We will only prove that \(\|\hat{\gamma}_{lasso, 1}\|_0 \leq Cs\) holds in probability for some constant \(C\); the proof for \(\|\hat{\gamma}_{lasso, 0}\|_0 \leq Cs\) is similar. First, we bound \(\|\hat{\Sigma}_{XX} \{\gamma(1) - \hat{\gamma}_{lasso, 1}\}\|_2^2\) from below. Specifically, we consider the \(j\)th element of \(\hat{\Sigma}_{XX} \{\gamma(1) - \hat{\gamma}_{lasso, 1}\}\); that is, \(e_j^T\hat{\Sigma}_{XX} \{\gamma(1) - \hat{\gamma}_{lasso, 1}\}\), where \(e_j\) is a \(p\)-dimensional vector with one on its \(j\)th entry and zero on other entries.

By KKT condition, we have shown that (see (S26))
\[ \hat{\Sigma}_{XX} \{\gamma(1) - \hat{\gamma}_{lasso, 1}\} + \hat{\Sigma}_{XX} \{\gamma(1) - \hat{\gamma}_{lasso, 1}\} = \lambda_1 \kappa. \]
For any \( j \in \{1, 2, \ldots, p\} \) such that \( (\hat{\gamma}_{\text{lasso},1})_j \neq 0 \), we have

\[
|e_j^T \hat{\Sigma}_{XX} \{ \gamma(1) - \hat{\gamma}_{\text{lasso},1} \} + e_j^T \hat{\Sigma}_{X \omega}(1) | = \lambda_1.
\]

Conditional on \( L_{\hat{X}_\varepsilon} = \{ \| \hat{\Sigma}_{X \omega}(1) \|_\infty \leq \eta \lambda_1 \} \) with \( P(L_{\hat{X}_\varepsilon}) \to 1 \) (Lemma S9), and by triangle inequality, we have,

\[
|e_j^T \hat{\Sigma}_{XX} \{ \gamma(1) - \hat{\gamma}_{\text{lasso},1} \} | \geq \lambda_1 - |e_j^T \hat{\Sigma}_{X \omega}(1) | \geq \lambda_1 - \| \hat{\Sigma}_{XX}(1) \|_\infty \geq (1 - \eta) \lambda_1.
\]

Then, summing up the square of the elements of \( \hat{\Sigma}_{XX} \{ \gamma(1) - \hat{\gamma}_{\text{lasso},1} \} \), we have

\[
\| \hat{\Sigma}_{XX} \{ \gamma(1) - \hat{\gamma}_{\text{lasso},1} \} \|_2^2 = \sum_{j=1}^p |e_j^T \hat{\Sigma}_{XX} \{ \gamma(1) - \hat{\gamma}_{\text{lasso},1} \} |^2 \geq \sum_{j \neq (\hat{\gamma}_{\text{lasso},1})_j \neq 0} |e_j^T \hat{\Sigma}_{XX} \{ \gamma(1) - \hat{\gamma}_{\text{lasso},1} \} |^2 \geq (1 - \eta)^2 \lambda_1^2 \| \hat{\gamma}_{\text{lasso},1} \|_0. \quad (S31)
\]

Second, we bound \( \| \hat{\Sigma}_{XX} \{ \gamma(1) - \hat{\gamma}_{\text{lasso},1} \} \|_2^2 \) from above:

\[
\| \hat{\Sigma}_{XX} \{ \gamma(1) - \hat{\gamma}_{\text{lasso},1} \} \|_2 \leq \Lambda_{\text{max}}(\hat{\Sigma}_{XX}) \cdot \| (\hat{\Sigma}_{XX})^{1/2} \{ \gamma(1) - \hat{\gamma}_{\text{lasso},1} \} \|_2 = \Lambda_{\text{max}}(\hat{\Sigma}_{XX}) \cdot \gamma(1) - \hat{\gamma}_{\text{lasso},1} \}^T \hat{\Sigma}_{XX} \{ \gamma(1) - \hat{\gamma}_{\text{lasso},1} \} = \Lambda_{\text{max}}(\hat{\Sigma}_{XX}) \cdot h^T \hat{\Sigma}_{XX} h. \quad (S32)
\]

We deal with \( h^T \hat{\Sigma}_{XX} h \) and \( \Lambda_{\text{max}}(\hat{\Sigma}_{XX}) \) separately. To bound \( h^T \hat{\Sigma}_{XX} h \), we have shown that (see (S27)),

\[
h^T \hat{\Sigma}_{XX} h \leq \lambda_1 \{ \| \gamma(1) \|_1 - \| \hat{\gamma}_{\text{lasso},1} \|_1 \} + \| h \|_1 \| \hat{\Sigma}_{XX}(1) \|_\infty.
\]

Conditional on \( L_{\hat{X}_\varepsilon} = \{ \| \hat{\Sigma}_{XX}(1) \|_\infty \leq \eta \lambda_1 \} \) with \( P(L_{\hat{X}_\varepsilon}) \to 1 \) (Lemma S9), and by triangle inequality, we have

\[
h^T \hat{\Sigma}_{XX} h \leq \lambda_1 (1 + \eta) \| h \|_1.
\]

According to the proof of Lemma S6, with probability tending to one, there exists a constant \( C \), such that

\[
\| h \|_1 = \| \gamma(1) - \hat{\gamma}_{\text{lasso},1} \|_1 \leq C \lambda_1 s.
\]

Therefore, we have

\[
h^T \hat{\Sigma}_{XX} h \leq C (1 + \eta) \lambda_1^2 s. \quad (S33)
\]
To bound \( \Lambda_{\text{max}}(\hat{\Sigma}_{XX}^\omega) \), we expand its expression as follows:

\[
\Lambda_{\text{max}}(\hat{\Sigma}_{XX}^\omega) = \max_{\|u\|_2=1} u^T \hat{\Sigma}_{XX}^\omega u. \tag{S34}
\]

By expanding \( \hat{\Sigma}_{XX}^\omega \), we have

\[
u^T \hat{\Sigma}_{XX}^\omega u \leq \frac{1}{n} \sum_{m=1}^{M} \sum_{i \in [m]} Z_i u^T (x_i^\omega)(x_i^\omega)^T u
\leq \frac{1}{n} \sum_{m=1}^{M} u^T (x_i^\omega)(x_i^\omega)^T u
= u^T \left\{ \sum_{m=1}^{M} \pi_{[m]} \frac{S_{[m]}^2 X}{e_{[m]}^2 (1 - e_{[m]})} \right\} u
\leq \frac{1}{c^3} u^T \left\{ \sum_{m=1}^{M} \pi_{[m]} S_{[m]}^2 X \right\} u,
\]

where the last inequality is due to Condition 1. Plugging above inequality into (S34), and by Condition 6, we have

\[
\Lambda_{\text{max}}(\hat{\Sigma}_{XX}^\omega) \leq \frac{1}{c^3} \max_{\|u\|_2=1} u^T \sum_{m=1}^{M} \pi_{[m]} S_{[m]}^2 X u \leq \frac{1}{c^3} \Lambda_{\text{max}}(\Sigma_{XX}) \leq \frac{C}{c^3}. \tag{S35}
\]

Combining (S32), (S33), and (S35), the following inequality holds in probability:

\[
\|\hat{\Sigma}_{XX}^\omega \{ \gamma(1) - \hat{\gamma}_{\text{lasso}},1 \} \|^2 \leq C^2 (1 + \eta) \lambda_1^2 s / c^3. \tag{S36}
\]

Finally, combining (S31) and (S36), we have, with probability tending to one,

\[
\|\hat{\gamma}_{\text{lasso}},1 \|_0 \leq \frac{C^2 (1 + \eta)}{c^3 (1 - \eta)^2} s =: \tilde{C} s.
\]

\[\square\]