Nested algebraic Bethe ansatz for open spin chains with even twisted Yangian symmetry

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Abstract. We present the nested algebraic Bethe ansatz for a one-dimensional open spin chain whose underlying symmetry is Olshanskii twisted Yangian \( Y^{\pm}(\mathfrak{gl}_{2n}) \) [Ol, MNO]. These integrable models have “soliton non-preserving” boundary conditions and are described by a Hamiltonian of an alternating type; they have been intensively studied using techniques of the analytic Bethe ansatz in [ACDFR, ADK, Do]. The full quantum space is a tensor product of irreducible finite-dimensional “bulk” \( \mathfrak{gl}_{2n} \)-modules and a “boundary” \( \mathfrak{so}_{2n-} \) or \( \mathfrak{sp}_{2n-} \)-module. This space is then equipped with the structure of a lowest weight \( Y^{\pm}(\mathfrak{gl}_{2n}) \)-module. We will call this model a \( Y^{\pm}(\mathfrak{gl}_{2n}) \) system.

1. Introduction

The algebraic Bethe ansatz has proven to be a powerful method to study quantum integrable models. It provides an effective approach for determining the spectrum of quantum Hamiltonians by reducing the problem of diagonalizing the Hamiltonian to a set of algebraic equations that in many cases can be solved using numerical methods, see e.g. [BeRa1, BeRa2, KuRs, Sk]. Once these equations are solved, the eigenvectors and their eigenvalues are known exactly. This provides the necessary first step that needs to be taken in the study of scalar products, norms and correlation functions, see e.g. [Bx, KBI, SI].

In this paper we construct a nested algebraic Bethe ansatz for a one-dimensional quantum spin chain with open boundaries, whose underlying symmetry is the Olshanskii twisted Yangian \( Y^{\pm}(\mathfrak{gl}_{2n}) \) [Ol, MNO]. These integrable models have “soliton non-preserving” boundary conditions and are described by a Hamiltonian of an alternating type; they have been intensively studied using techniques of the analytic Bethe ansatz in [ACDFR, ADK, Do]. The full quantum space is a tensor product of irreducible finite-dimensional “bulk” \( \mathfrak{gl}_{2n} \)-modules and a “boundary” \( \mathfrak{so}_{2n-} \) or \( \mathfrak{sp}_{2n-} \)-module. This space is then equipped with the structure of a lowest weight \( Y^{\pm}(\mathfrak{gl}_{2n}) \)-module. We will call this model a \( Y^{\pm}(\mathfrak{gl}_{2n}) \) system.

Studying the spectral problem of spin chains with orthogonal and symplectic symmetries requires elaborate algebraic methods: the usual nesting approach for a \( \mathfrak{gl}_{n} \)-symmetric spin chain, put forward in [KuRs], fails since there are no natural Yangian analogues of the chains of subalgebras \( \mathfrak{so}_{2n} \supset \mathfrak{so}_{2n-2} \supset \ldots \supset \mathfrak{so}_{2} \) and \( \mathfrak{sp}_{2n} \supset \mathfrak{sp}_{2n-2} \supset \ldots \supset \mathfrak{sp}_{2} \). (This problem for the Yangians \( Y(\mathfrak{so}_{N}) \) and \( Y(\mathfrak{sp}_{N}) \) was addressed in [JLM].) It was shown in [DVK, Rs] that the spectral problem of such a system can be addressed using the algebraic Bethe ansatz if the \( R \)-matrix intertwining the monodromy matrices of the model can be written in a six-vertex block-form. This approach has recently been used in [GoPa] to study orthogonal quantum spin chains with open boundary conditions, whose underlying symmetries are provided by the twisted Yangian of type D [GuRe].

Our strategy for solving the spectral problem of the \( Y^{\pm}(\mathfrak{gl}_{2n}) \) system is as follows. We interpret the generating matrix of \( Y^{\pm}(\mathfrak{gl}_{2n}) \) as the monodromy matrix of the model. Inspired by the ideas put forward in [DVK, Rs], we write the defining relations of \( Y^{\pm}(\mathfrak{gl}_{2n}) \) in a block-form, i.e. in terms of the matrix operators \( A, B, C \) and \( D \), that are matrix analogous of the conventional creation, annihilation and number operators of the six-vertex model. Indeed, the exchange relations between these matrix operators turn out to be reminiscent of those of the six-vertex model. We then introduce creation operators that are constructed using matrix entries of the \( B \) operator. We define the vacuum sector of the quantum space as the subspace annihilated by the \( C \) operator. The Bethe vector is constructed by acting with creation operators on vectors in the vacuum sector. Then, following Sklyanin’s pioneering work [Sk], we require the Bethe vector to be an eigenvector of the double-row transfer matrix. Surprisingly, this procedure reduces the initial \( Y^{\pm}(\mathfrak{gl}_{2n}) \) system to a residual \( Y(\mathfrak{gl}_{n}) \) system, the nested algebraic Bethe ansatz for which is well-known [KuRs]. Our main results are the construction of the Bethe vectors and derivation of the nested Bethe equations. The resulting equations are comparable to those obtained in [ACDFR] for an alternating open spin chain, see Remark 4.1.
The plan of the paper is as follows. In Section 2 we provide the necessary preliminaries and definitions. We recall the definition of the Yangian \( Y(\mathfrak{gl}_{2n}) \), the twisted Yangian \( Y^\pm(\mathfrak{gl}_{2n}) \) and relevant details of their representation theory. We then obtain a six-vertex block-form of the Yang \( R \)-matrix and its twisted counterpart, which allows us to write the defining relations of both Yangian and twisted Yangian in terms of the matrix operators \( A, B, C \) and \( D \). In Section 3 we provide the technical details of the main ingredients necessary for the nested Bethe ansatz. We introduce the creation operator of multi-excitations and describe its algebraic properties. We derive the exchange relations for the operators that lead to the so-called wanted and unwanted terms. In Section 4 we present the nested Bethe ansatz, in two steps. First, we demonstrate the method for a single top-level excitation. Then we generalize the method to multi-excitations at the top-level and provide the complete set of Bethe equations. In Appendix A we provide in detail the nested algebraic Bethe ansatz for \( Y(\mathfrak{gl}_n) \) first presented in [KuRs], to which the Bethe ansatz for \( Y^\pm(\mathfrak{gl}_{2n}) \) reduces. Many technical details are omitted in loc. cit. and in other publications where the Bethe ansatz for the Yangian \( Y(\mathfrak{gl}_n) \) is considered, e.g. [TaVa, BeRa1]; our aim is to fill in these gaps and provide the reader with complete details.

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2. Definitions and preliminaries

2.1. Notation. Let \( n \in \mathbb{N} \). The orthogonal Lie algebra \( \mathfrak{so}_{2n} \) or the symplectic Lie algebra \( \mathfrak{sp}_{2n} \) can be realized as a Lie subalgebra of \( \mathfrak{gl}_{2n} \) as follows. For any \( 1 \leq i, j \leq 2n \) set \( \theta_{ij} = \theta_{ji} \) with \( \theta_i = 1 \) in the orthogonal case and \( \theta_i = \delta_{i>n} - \delta_{i<n} \) in the symplectic case. Introduce elements \( F_{ij} = E_{ij} - \theta_{ij} E_{2n-j+1,2n-i+1} \), where \( E_{ij} \) are the standard generators of \( \mathfrak{gl}_{2n} \). These elements satisfy the relations

\[
\begin{align*}
[F_{ij}, F_{kl}] &= \delta_{jk} F_{il} - \delta_{il} F_{kj} + \theta_{ij} \delta_{j,2n-l+1} F_{k,2n-i+1} - \theta_{ij} \delta_{i,2n-k+1} F_{2n-j+l},
F_{ij} + \theta_{ij} F_{2n-j+1,2n-i+1} &= 0,
\end{align*}
\]

which in fact are the defining relations of the Lie algebra \( \mathfrak{so}_{2n} \) or \( \mathfrak{sp}_{2n} \). Namely, we may identify \( \mathfrak{so}_{2n} \) or \( \mathfrak{sp}_{2n} \) with \( \text{span}_\mathbb{C}\{F_{ij} : 1 \leq i, j \leq 2n\} \) and we will use \( \mathfrak{h}_{2n} = \text{span}_\mathbb{C}\{F_{ii} : 1 \leq i \leq n\} \) as a Cartan subalgebra. Given a Lie algebra \( \mathfrak{g} \) its universal enveloping algebra will be denoted by \( U(\mathfrak{g}) \).

Next, we need to introduce some operators on \( \mathbb{C}^N \otimes \mathbb{C}^N \), where the tensor product \( \otimes \) is defined over the field of complex numbers, that is \( \otimes = \otimes_{\mathbb{C}} \), and \( N = n \) or \( N = 2n \) (it will always be clear from the context which \( N \) is used). Let \( e_{ij} \in \text{End}(\mathbb{C}^N) \) be the standard matrix units and let \( e_i \) be the standard basis vectors of \( \mathbb{C}^N \). Then \( P \) will denote the permutation operator on \( \mathbb{C}^N \otimes \mathbb{C}^N \) and we set \( Q = P^{t_1} P^{t_2} \), where the transpose \( t \) is defined by \( (e_{ij})^t = \theta_{ij} e_{ji} \) with \( i = N - i + 1 \) and \( j = N - j + 1 \); explicitly,

\[
\begin{align*}
P &= \sum_{1 \leq i, j \leq N} e_{ij} \otimes e_{ji}, \quad Q &= \sum_{1 \leq i, j \leq N} \theta_{ij} e_{ij} \otimes e_{ji}.
\end{align*}
\]

Let \( I \) denote the identity matrix on \( \mathbb{C}^N \otimes \mathbb{C}^N \) or \( \mathbb{C}^N \) (it will always be clear from the context which \( I \) is used). Then \( P^2 = I \), \( PQ = QP = \pm Q \), \( Q^2 = NQ \), which will be useful below. Here (and henceforth in this paper) the upper sign in \( \pm \) corresponds to the orthogonal case and the lower sign to the symplectic case. Also note that \( P (e_{ij} \otimes I) = (I \otimes e_{ij}) P \). Taking the transpose of this, we obtain a pair of relations for \( Q \) and any \( M \in \text{End}(\mathbb{C}^N) \):

\[
Q (M \otimes I) = Q (I \otimes M^t), \quad (M \otimes I) Q = (I \otimes M^t) Q.
\]

For a matrix \( X \) with entries \( x_{ij} \) in an associative algebra \( A \) we write

\[
X_s = \sum_{1 \leq i, j \leq N} \left( I \otimes \cdots \otimes I \otimes e_{ij} \otimes I \otimes \cdots \otimes I \otimes x_{ij} \right) \in \text{End}(\mathbb{C}^N)^{\otimes k} \otimes A.
\]

Here \( k \geq 2 \) and \( 1 \leq s \leq k \); it will always be clear from the context what \( k \) is.
2.2. The Yangian $Y(\mathfrak{gl}_{2n})$ and twisted Yangian $Y^{\pm}(\mathfrak{gl}_{2n})$. We briefly recall necessary details of the Yangian $Y(\mathfrak{gl}_{2n})$, the twisted Yangian $Y^{\pm}(\mathfrak{gl}_{2n})$ and their representation theory, adhering closely to [Mo3].

Introduce a rational function acting on $\mathbb{C}^{2n} \otimes \mathbb{C}^{2n}$
\begin{equation}
R(u) = I - u^{-1}P
\end{equation}
called the Yang’s $R$-matrix. It satisfies $R(u)R(-u) = (1 - u^{-2})I$ and is a solution of the quantum Yang-Baxter equation,
\begin{equation}
R_{12}(u-v)R_{13}(u-z)R_{23}(v-z) = R_{23}(v-z)R_{13}(u-z)R_{12}(u-v).
\end{equation}

We introduce elements $t^{(r)}_{ij}$ with $1 \leq i, j \leq 2n$ and $r \geq 0$ such that $t^{(0)}_{ij} = \delta_{ij}$. Combining these into formal power series $t_{ij}(u) = \sum_{r \geq 0} t^{(r)}_{ij} u^{-r}$, we can then form the generating matrix $T(u) = \sum_{1 \leq i, j \leq 2n} c_{ij} \otimes t_{ij}(u)$.

\textbf{Definition 2.1.} The Yangian $Y(\mathfrak{gl}_{2n})$ is the unital associative $\mathbb{C}$-algebra generated by elements $t^{(r)}_{ij}$ with $1 \leq i, j \leq 2n$ and $r \in \mathbb{Z}_{\geq 0}$ satisfying the relations
\begin{equation}
R_{12}(u-v)T_{1}(u)T_{2}(v) = T_{2}(v)T_{1}(u)R_{12}(u-v).
\end{equation}

The Hopf algebra structure of $Y(\mathfrak{gl}_{2n})$ is given by
\begin{equation}
\Delta: T(u) \mapsto T(u) \otimes T(u), \quad S: T(u) \mapsto T^{-1}(u), \quad \varepsilon: T(u) \mapsto I.
\end{equation}

We now recall the definition of the lowest weight representation of $Y(\mathfrak{gl}_{2n})$. It is a historic tradition (but merely a convention) to consider lowest weight representations in the algebraic Bethe ansatz instead of highest weight ones.

\textbf{Definition 2.2.} A representation $V$ of $Y(\mathfrak{gl}_{2n})$ is called a lowest weight representation if there exists a non-zero vector $\eta \in V$ such that $V = Y(\mathfrak{gl}_{2n})\eta$ and
\[ t_{ij}(u)\eta = 0 \quad \text{for} \quad 1 \leq j < i \leq 2n \quad \text{and} \quad t_{ii}(u)\eta = \lambda_i(u)\eta \quad \text{for} \quad 1 \leq i \leq 2n, \]
where $\lambda_i(u)$ is a formal power series in $u^{-1}$ with a constant term equal to 1. The vector $\eta$ is called the lowest vector of $V$, and the 2n-tuple $\lambda(u) = (\lambda_1(u), \ldots, \lambda_{2n}(u))$ is called the lowest weight of $V$.

The Yangian $Y(\mathfrak{gl}_{2n})$ contains the universal enveloping algebra $U(\mathfrak{gl}_{2n})$ as a Hopf subalgebra. An embedding $U(\mathfrak{gl}_{2n}) \hookrightarrow Y(\mathfrak{gl}_{2n})$ is given by $E_{ij} \mapsto t^{(1)}_{ij}$ for all $1 \leq i, j \leq 2n$. We will identify $U(\mathfrak{gl}_{2n})$ with its image in $Y(\mathfrak{gl}_{2n})$ under this embedding. Conversely, the map $t^{(1)}_{ij} \mapsto -E_{ji}$ and $t^{(s)}_{ij} \mapsto 0$ for all $s \geq 2$ defines a surjective homomorphism $e\nu: Y(\mathfrak{gl}_{2n}) \rightarrow U(\mathfrak{gl}_{2n})$ called the evaluation homomorphism. By composing the map $e\nu$ with the algebra automorphism called the shift automorphism,
\[ \sigma_c : Y(\mathfrak{gl}_{2n}) \rightarrow Y(\mathfrak{gl}_{2n}), \quad T(u) \mapsto T(u-c) \]
for any $c \in \mathbb{C}$, we obtain the map
\begin{equation}
e\nu_c : t_{ij}(u) \mapsto \delta_{ij} - E_{ji}(u-c)^{-1}.
\end{equation}

Given an 2n-tuple $\lambda = (\lambda_1, \ldots, \lambda_{2n}) \in \mathbb{C}^{2n}$ we will denote by $L(\lambda)$ the irreducible representation of the Lie algebra $\mathfrak{gl}_{2n}$ with the highest weight $\lambda$. In particular, $L(\lambda)$ is a cyclic $\mathfrak{gl}_{2n}$-module generated by a non-zero vector $1_\lambda$ such that
\[ E_{ij} 1_\lambda = 0 \quad \text{for} \quad 1 \leq i < j \leq 2n \quad \text{and} \quad E_{ii} 1_\lambda = \lambda_i 1_\lambda \quad \text{for} \quad 1 \leq i \leq 2n. \]
The representation $L(\lambda)$ is finite-dimensional if and only if $\lambda_i - \lambda_{i+1} \in \mathbb{N}$ for all $1 \leq i \leq 2n - 1$. By the virtue of the map $e\nu$, any $\mathfrak{gl}_{2n}$-representation can be regarded as $Y(\mathfrak{gl}_{2n})$-module. Moreover, any irreducible $\mathfrak{gl}_{2n}$-representation remains irreducible over $Y(\mathfrak{gl}_{2n})$, by surjectivity of $e\nu$. We will denote by $L(\lambda)c$ the $Y(\mathfrak{gl}_{2n})$-module obtained from the irreducible representation $L(\lambda)$ of $\mathfrak{gl}_{2n}$ via the map (2.8). Clearly, $L(\lambda)c$ is a lowest weight $Y(\mathfrak{gl}_{2n})$-module with the components of the lowest weight given by
\[ \lambda_i(u) = 1 - \lambda_i(u-c)^{-1} \quad \text{for} \quad 1 \leq i \leq 2n. \]

Fix $c \in \mathbb{N}$ and consider the tensor product of the $Y(\mathfrak{gl}_{2n})$ evaluation modules
\begin{equation}
L := L(\lambda^{(1)})c_1 \otimes L(\lambda^{(2)})c_2 \otimes \ldots \otimes L(\lambda^{(c)})c_c,
\end{equation}
where $c_i \in \mathbb{C}$ are arbitrary complex numbers and each $\lambda^{(i)}$ is a partition of length not exceeding $\ell$. Set $\Delta^{(1)} := id$ and define recursively $\Delta^{(\ell)} = (id \otimes \cdots \otimes id \otimes \Delta) \circ \Delta^{(\ell-1)}$ with $\Delta^{(2)} := \Delta$. The comultiplication $\Delta^{(\ell)}$ allows us to equip $L$ with the structure of the $Y(\mathfrak{gl}_{2n})$-module using the rule

$$t_{ij}(u) \cdot L = \Delta^{(\ell)}(t_{ij}(u))L.$$  

In particular, the generating matrix $T(u)$ acts on the space $L$ by

$$(2.10) \quad T_u(u) \cdot L = \left( \prod_{1 \leq i \leq \ell} \mathcal{L}_{ai}(u - c_i) \right) L \in \text{End}(\mathbb{C}^{2n}) \otimes L[[u^{-1}]],$$

where

$$\mathcal{L}(u - c) := (id \otimes e_{ij})(T(u)) = \sum_{1 \leq i,j \leq 2n} e_{ij} \otimes (\delta_{ij} - E_{ji}(u - c)^{-1})$$

are the Lax operators. The components of the lowest weight of $L$ are

$$\lambda_i(u) = \prod_{1 \leq j \leq \ell} \left( 1 - \lambda^{(j)}_i(u - c_j)^{-1} \right) \quad \text{for} \quad 1 \leq i \leq 2n.$$  

The binary property of the tensor products of Yangian modules states that, for a suitable choice of weights $\lambda^{(j)}_i$ and parameters $c_j$, the $Y(\mathfrak{gl}_{2n})$-module $L$ is irreducible, see Theorem 6.5.8 in [Mo3].

We now focus on the twisted Yangian $Y^{\pm}(\mathfrak{gl}_{2n})$ and its representation theory. Following [ACDFR] we introduce an additional “shift” parameter $\rho \in \mathbb{C}$ in the definition of $Y^{\pm}(\mathfrak{gl}_{2n})$.

**Definition 2.3.** The twisted Yangian $Y^{\pm}(\mathfrak{gl}_{2n})$ is the subalgebra of $Y(\mathfrak{gl}_{2n})$ generated by the coefficients of the entries of the matrix

$$(2.13) \quad S(u) = T(u)T(-u - \rho).$$

The “$\rho$-shifted” twisted Yangian defined above is isomorphic to the usual one studied in [Mo3]. The isomorphism is provided by the map $S(u) \mapsto S(u + \rho/2)$.

The matrix $S(u)$ defined in (2.13) satisfies the reflection equation

$$R(u - v)S_1(u)R^t(-u - v - \rho)S_2(v) = S_2(v)R^t(-u - v - \rho)S_1(u)R(u - v)$$

and the symmetry relation

$$(2.15) \quad S^t(-u - \rho) = S(u) \pm \frac{S(u) - S(-u - \rho)}{2u + \rho}.$$  

The above two relations are in fact the defining relations of $Y^{\pm}(\mathfrak{gl}_{2n})$. Their form in terms of matrix elements $s_{ij}(u)$ of $S(u)$, for $\rho = 0$, can be found in (2.4) and (2.5) of [Ol] (note that indices $i,j,k,l$ are indexed by $-n,-n+1,\ldots,n-1,n$ in loc. cit.).

**Definition 2.4.** A representation $V$ of $Y^{\pm}(\mathfrak{gl}_{2n})$ is called a lowest weight representation if there exists a non-zero vector $\xi \in V$ such that $V = Y^{\pm}(\mathfrak{gl}_{2n})\xi$ and

$$s_{ij}(u)\xi = 0 \quad \text{for} \quad 1 \leq j < i \leq 2n \quad \text{and} \quad s_{ii}(u)\xi = \mu_i(u)\xi \quad \text{for} \quad 1 \leq i \leq n,$$

where $\mu_i(u)$ are formal power series in $u^{-1}$ with constant terms equal to 1. The vector $\xi$ is called the lowest weight vector of $V$, and the $n$-tuple $\mu(u) = (\mu_1(u), \ldots, \mu_n(u))$ is called the lowest weight of $V$.

Note that $\xi$ is also an eigenvector for the action of $s_{ii}(u)$ with $n + 1 \leq i \leq 2n$. Indeed, the symmetry relation (2.15) implies that

$$s_{ii}(u)\xi = \left( \mu_i(-u - \rho) \pm \frac{\mu_i(u) - \mu_i(-u - \rho)}{2u + \rho} \right) \xi \quad \text{for} \quad 1 \leq i \leq n.$$  

Writing $s_{ij}(u) = \sum_{r \geq 0} s^{(r)}_{ij} u^{-r}$, the map $F_{ij} \mapsto -s^{(1)}_{ji}$ defines an embedding $U(\mathfrak{g}_{2n}) \hookrightarrow Y^{\pm}(\mathfrak{gl}_{2n})$, where $\mathfrak{g}_{2n} = \mathfrak{so}_{2n}$ or $\mathfrak{sp}_{2n}$. Conversely, the map $s_{ij}(u) \mapsto \delta_{ij} - F_{ji}(u \pm 1)^{-1}$ defines the evaluation homomorphism $e_{\pm} : Y^{\pm}(\mathfrak{gl}_{2n}) \rightarrow U(\mathfrak{g}_{2n})$. (Note that there is no analogue of the shift automorphism $\sigma_c$ of $Y(\mathfrak{gl}_{2n})$ for the twisted Yangian.)
Given an $n$-tuple $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{C}^n$ we will denote by $M(\mu)$ the irreducible representation of the Lie algebra $\mathfrak{g}_{2n}$ with the highest weight $\mu$. That is, $M(\mu)$ is generated by a non-zero vector $\mu_1$ such that 
\[ F_{ij} \mu_1 = 0 \quad \text{for} \quad 1 \leq i < j \leq 2n \quad \text{and} \quad F_{ii} \mu_1 = \mu_i \mu_1 \quad \text{for} \quad 1 \leq i \leq n. \]

The representation $M(\mu)$ is finite-dimensional if and only if $\mu_i - \mu_{i+1} \in \mathbb{N}$ for $1 \leq i \leq n-1$ and $\mu_{n-1} + \mu_n \in \mathbb{N}$ for $\mathfrak{so}_{2n}$ and $\mu_n \in \mathbb{N}$ for $\mathfrak{sp}_{2n}$. Using the evaluation homomorphism $ev^\pm$ we can extend each representation $M(\mu)$ to a lowest weight $Y^\pm(\mathfrak{gl}_{2n})$-module with the lowest weight given by 
\[ \mu_i(u) = 1 - \mu_i(u + (\rho \pm 1)/2)^{-1} \quad \text{for} \quad 1 \leq i \leq n. \]

The twisted Yangian $Y^\pm(\mathfrak{gl}_{2n})$ is a left coideal subalgebra of $Y(\mathfrak{gl}_{2n})$. In particular, 
\[ \Delta : S(u) \rightarrow (T(u) \otimes 1)(1 \otimes S(u))(T^\pm(\rho - u) \otimes 1) \in \text{End}(\mathbb{C}^{2n}) \otimes Y(\mathfrak{gl}_{2n}) \otimes Y^\pm(\mathfrak{gl}_{2n})[[u^{-1}]]. \]

This allows us to equip the space 
\[ M := L \otimes M(\mu) = L(\lambda^1)_{c_1} \otimes L(\lambda^2)_{c_2} \otimes \cdots \otimes L(\lambda^\ell)_{c_\ell} \otimes M(\mu) \]
with the structure of a lowest weight $Y^\pm(\mathfrak{gl}_{2n})$-module. In particular, $S(u)$ acts on the space $M$ by 
\[ S(u) \cdot M = \left( \prod_{1 \leq i \leq \ell} L_i(u - c_i) \right) \left( \prod_{\ell \leq i \leq 1} L_i'(-u - \rho - c_i) \right) \in \text{End}(\mathbb{C}^{2n}) \otimes M[[u^{-1}]], \]

where 
\[ L^\pm(u) := (id \otimes ev^\pm)(S(u)) = \sum_{1 \leq i \leq 2n} e_{ij} \otimes (\delta_{ij} - F_{ij}(u + (\rho \pm 1)/2)^{-1}) \]
is the “boundary” Lax operator. Let $\xi \in M(\mu)$ be a highest vector. Denote by $\eta_1$ a lowest vector of $L(\lambda^1)_{c_1}$ and set $\zeta = \eta_1 \otimes \cdots \otimes \eta_{\ell} \otimes \xi$. Then the submodule $Y^\pm(\mathfrak{gl}_{2n})\zeta$ of $Y^\pm(\mathfrak{gl}_{2n})$-module $M$ is a lowest weight representation with a lowest vector $\zeta$. It is given by 
\[ \lambda_i(u)\lambda_{n-i+1}(-\rho - u)\mu_i(u) \quad \text{for} \quad 1 \leq i \leq n, \]

with $\lambda_i(u)$ defined in (2.12) and $\mu_i(u)$ defined in (2.16), see Proposition 4.2.11 in [Mo3]. To the best of our knowledge, there are currently no irreducibility criteria known for a tensor product of irreducible representations of $Y(\mathfrak{gl}_{2n})$ and $Y^\pm(\mathfrak{gl}_{2n})$.

**Remark 2.5.** Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\nu = (\nu_1, \ldots, \nu_n)$ be any $\mathfrak{gl}_n$-weights. Set $\lambda' = (\lambda_2, \ldots, \lambda_n)$ and $\nu' = (\nu_2, \ldots, \nu_n)$. The algebraic Bethe ansatz for $Y(\mathfrak{gl}_n)$ relies on the fact that if the $Y(\mathfrak{gl}_n)$-module $L(\lambda) \otimes L(\nu)$ is irreducible, so is the $Y(\mathfrak{gl}_{n-1})$-module $L(\lambda') \otimes L(\nu')$, see Lemma 6.2.2 in [Mo3]. This property combined with the binary property allows one to solve the spectral problem of a $Y(\mathfrak{gl}_n)$ system recursively, via the chain of subalgebras $Y(\mathfrak{gl}_n) \supset Y(\mathfrak{gl}_{n-1}) \supset \cdots \supset Y(\mathfrak{gl}_2)$. In this paper we will use the fact that the restriction of an irreducible $\mathfrak{so}_{2n}$- or $\mathfrak{sp}_{2n}$-representation of weight $\mu = (\mu_1, \ldots, \mu_n)$ to its natural $\mathfrak{gl}_n$ subalgebra is irreducible. Moreover, any irreducible $\mathfrak{gl}_{2n}$-module, upon restriction to the natural $\mathfrak{gl}_n \subset \mathfrak{gl}_{2n}$ subalgebra, factors as a tensor product of its natural irreducible $\mathfrak{gl}_n$-submodules. Hence, starting with the $Y^\pm(\mathfrak{gl}_{2n})$-module $M$ we can restrict to an irreducible $Y(\mathfrak{gl}_n)$-module $M \subset M$, provided the binary property holds. This restriction will allow us to solve the spectral problem for a $Y^\pm(\mathfrak{gl}_{2n})$ system using the chain of subalgebras $Y^\pm(\mathfrak{gl}_{2n}) \supset Y(\mathfrak{gl}_n) \supset Y(\mathfrak{gl}_{n-1}) \supset \cdots \supset Y(\mathfrak{gl}_2)$.

### 2.3. Block decomposition

In this section, inspired by the arguments presented in [Rs, DVK], we demonstrate a block decomposition of the Yangian $Y(\mathfrak{gl}_{2n})$ and the twisted Yangian $Y^\pm(\mathfrak{gl}_{2n})$. We write the matrices $T(u)$ and $S(u)$ in the block form: 
\[ T(u) = \left( \begin{array}{ccc} T\ell(u) & T2(u) \\ C\ell(u) & D\ell(u) \end{array} \right), \quad S(u) = \left( \begin{array}{cc} A(u) & B(u) \\ C(u) & D(u) \end{array} \right). \]

Our goal is to derive the algebraic relations between these smaller matrices, which is the crucial first step of the algebraic Bethe ansatz. We will denote the matrix elements of $A(u)$ by $a_{ij}(u)$ with $1 \leq i, j \leq n$, and similarly for matrices $B(u)$, $C(u)$ and $D(u)$, and their barred counterparts.

The first step is split the vector space $\mathbb{C}^{2n}$ into two even parts, 
\[ \mathbb{C}^{2n} = \mathbb{C}^n \oplus \mathbb{C}^n = \text{span}_\mathbb{C}\{e_1, \ldots, e_n\} \oplus \text{span}_\mathbb{C}\{e_{n+1}, \ldots, e_{2n}\}. \]
Introduce a matrix

\[ C = \sum_{i=1}^{n/2} \sum_{j=1}^{n} \left( (e_{i,2i-1} + e_{i+n,2i-1+n}) \otimes e_{jj} + (e_{i,2i} + e_{i+n,2i+n}) \otimes e_{j+n,j} \right) 
\]

\[ + (e_{i+n,2i-1} + e_{i+n+2i-2i+n}) \otimes e_{j,j+n} + (e_{i+n,2i} + e_{i+n+2i+2i+n}) \otimes e_{j+n,j+n} \]

if \( n \) is even, or

\[ C = \sum_{i=1}^{(n-1)/2} \sum_{j=1}^{n} \left( (e_{i,2i-1} + e_{i+n,2i-1+n}) \otimes e_{jj} + (e_{i,2i} + e_{i+n,2i+n}) \otimes e_{j+n,j} \right) 
\]

\[ + (e_{i+n,2i-1} + e_{i+n+2i-2i+n}) \otimes e_{j,j+n} + (e_{i+n,2i} + e_{i+n+2i+2i+n}) \otimes e_{j+n,j+n} \]

\[ + \sum_{j=1}^{n} \left( (e_{\frac{n+1}{2},n} + e_{\frac{n+1}{2}+n}) \otimes e_{jj} + (e_{\frac{n+1}{2},1} + e_{\frac{n+1}{2}+1}) \otimes e_{j+n,j+n} \right) \]

if \( n \) is odd. This matrix rearranges the standard ordering of the tensor product,

\[ C(e_1, \ldots, e_n) \otimes (e_{n+1}, \ldots, e_{2n}) = (e_{11}, \ldots, e_{1n}, e_{21}, \ldots, e_{2n}, e_{2n-1,n+1}, \ldots, e_{2n-1,2n}, e_{2n,n+1}, \ldots, e_{2n,2n}), \]

where \( e_{ij} = e_i \otimes e_j \). Conjugating the \( R \)-matrix \( R(u) \) defined in (2.4) and its \( t \)-transpose \( R^t(u) \) with \( C \) we obtain

\[ \tilde{R}(u) := CR(u)C^{-1} = \begin{pmatrix} R(u) & I & -u^{-1}P \\ -u^{-1}P & I & R(u) \end{pmatrix}, \]

\[ \tilde{R}^t(u) := CR^t(u)C^{-1} = \begin{pmatrix} I & R^t(u) & \mp u^{-1}Q \\ \mp u^{-1}Q & R^t(u) & I \end{pmatrix}, \]

where the operators inside the matrices are each acting on \( \mathbb{C}^n \otimes \mathbb{C}^n \); note that \( R^t(u) = I - u^{-1}Q \) and \( Q = \sum_{1 \leq i,j \leq n} e_{ij} \otimes e_{ji} \) in both cases above are of the orthogonal type (recall the notation \( \tilde{i} = n - i + 1 \)).

Conjugating the matrices \( T_1(u) = T(u) \otimes I \) and \( T_2(u) = I \otimes T(u) \) of \( Y(\mathfrak{gl}_{2n}) \) with \( C \) we find

\[ \tilde{T}_1(u) := CT_1(u)C^{-1} = \begin{pmatrix} A_1(u) & \overline{A}_1(u) & B_1(u) & \overline{B}_1(u) \\ A_1(u) & \overline{B}_1(u) & B_1(u) & \overline{A}_1(u) \end{pmatrix}, \]

\[ \tilde{T}_2(u) := CT_2(u)C^{-1} = \begin{pmatrix} A_2(u) & \overline{A}_2(u) & B_2(u) & \overline{B}_2(u) \\ A_2(u) & \overline{B}_2(u) & B_2(u) & \overline{A}_2(u) \end{pmatrix}. \]

Consequently, conjugating both sides of the equality (2.6) with \( C \) yields

\[ \tilde{R}(u - v)\tilde{T}_1(u)\tilde{T}_2(v) = \tilde{T}_2(v)\tilde{T}_1(u)\tilde{R}(u - v), \]
allowing us to write the defining relations of $Y(\mathfrak{gl}_{2n})$ in terms of the matrices $\overline{A}(u)$, $\overline{D}(u)$, $\overline{C}(u)$ and $\overline{D}(u)$.

The relations that we will need are:

\begin{align*}
(2.24) \quad & R(u-v)\overline{A}_1(u)\overline{A}_2(v) = \overline{A}_2(v)\overline{A}_1(u)R(u-v), \\
(2.25) \quad & R(u-v)\overline{D}_1(u)\overline{D}_2(v) = \overline{D}_2(v)\overline{D}_1(u)R(u-v), \\
(2.26) \quad & \overline{C}_1(u)\overline{D}_2(v) = \overline{A}_2(v)\overline{C}_1(u)R(u-v) + \frac{P\overline{A}_1(u)\overline{C}_2(v)}{u-v}, \\
(2.27) \quad & \overline{C}_1(u)\overline{D}_2(v) = R(v-u)\overline{D}_2(v)\overline{C}_1(u) - \frac{P\overline{D}_2(u)\overline{C}_1(v)}{u-v}, \\
(2.28) \quad & (u-v)(\overline{D}_1(u)\overline{A}_2(v) - \overline{A}_2(v)\overline{D}_1(u)) = P\overline{B}_1(u)\overline{C}_2(v) - \overline{B}_2(v)\overline{C}_1(u)P.
\end{align*}

In particular, the coefficients of the matrix entries of $\overline{A}(u)$ generate a $Y(\mathfrak{gl}_n)$ subalgebra of $Y(\mathfrak{gl}_{2n})$. The same is true for $\overline{D}(u)$.

We now repeat the same steps for the twisted Yangian $Y^{\pm}(\mathfrak{gl}_{2n})$. Conjugating both sides of the reflection equation (2.14) with the matrix $C$ yields

\[ R(u-v)\overline{S}_1(u)\overline{R}^{\dagger}(-u-v-\rho)\overline{S}_2(v) = \overline{S}_2(v)\overline{R}^{\dagger}(-u-v-\rho)\overline{S}_1(u)\overline{R}(u-v), \]

where $\overline{S}_1(u)$ and $\overline{S}_2(u)$ have the same form as in (2.22) and (2.23), respectively. The relation above allows us to write the defining relations of $Y^{\pm}(\mathfrak{gl}_{2n})$ in terms of the matrices $A(u)$, $B(u)$, $C(u)$ and $D(u)$. The relations that we will need are:

\begin{align*}
(2.29) \quad & A_2(v)B_1(u) = R(u-v)B_1(u)R^{\dagger}(-u-v-\rho)A_2(v) + \frac{PB_1(v)R^{\dagger}(-u-v-\rho)A_2(u)}{u-v} + \frac{B_2(v)QC_1(u)}{u+v+\rho}, \\
(2.30) \quad & R(u-v)B_1(u)R^{\dagger}(-u-v-\rho)B_2(v) = B_2(v)R^{\dagger}(-u-v-\rho)B_1(u)R(u-v), \\
(2.31) \quad & R(u-v)A_1(u)A_2(v) - A_2(v)A_1(u)R(u-v) = \frac{R(u-v)B_1(u)QC_2(v) - B_2(v)QC_1(u)R(u-v)}{u+v+\rho}, \\
(2.32) \quad & C_1(u)A_2(v) = A_2(v)R^{\dagger}(-u-v-\rho)C_1(u)R(u-v) + \frac{PA_1(u)R^{\dagger}(-u-v-\rho)C_2(v)}{u-v} + \frac{D_1(u)QC_2(v)}{u+v+\rho}.
\end{align*}

It remains to cast the symmetry relation (2.15) in the block form. Observe that

\[ S^{\dagger}(u) = \begin{pmatrix} D^{\dagger}(u) & \pm B^{\dagger}(u) \\ \pm C^{\dagger}(u) & A^{\dagger}(u) \end{pmatrix}. \]

This allows us immediately to extract linear relations between matrices $A(u)$, $D(u)$, $B(u)$ and $D(u)$, of which we will need the following two only:

\begin{align*}
(2.33) \quad & D^{\dagger}(-u-\rho) = A(u) \pm \frac{1}{2u+\rho} (A(u) - A(-u-\rho)), \\
(2.34) \quad & \pm B^{\dagger}(-u-\rho) = B(u) \pm \frac{1}{2u+\rho} (B(u) - B(-u-\rho)).
\end{align*}

3. Technical identities

In this section we derive algebraic relations between certain elements of the twisted Yangian $Y^{\pm}(\mathfrak{gl}_{2n})$ that will be used in the derivation of the nested Bethe equations in the section that follows below. In particular, we recast the exchange relations (2.29-2.32) so that they can be applied directly to the algebraic Bethe ansatz. We then introduce the nested monodromy matrix and show its relevant algebraic properties. We note that all the operators in (2.29-2.32), viewed as matrices, will act on the vector spaces $V_a, V_{a_1}, V_{a_2}, \ldots$, and $V_{\tilde{a}}, V_{\tilde{a}_1}, V_{\tilde{a}_2}, \ldots$, all isomorphic to $\mathbb{C}^n$, which we call the auxiliary spaces.
3.1. Rewriting the AB exchange relation. First, we will use the symmetry relations \((2.33)\) and \((2.34)\) to rewrite the last term in \((2.29)\) in terms of \(A\) and \(B\) operators. Indeed, focussing on this term only,

\[
B_a(v)Q_{ba}D_b(u) = P_{ba}B_b(v)Q_{ba}D_a^\prime(u)
\]

\[
= P_{ba}B_b(v)Q_{ba}\left(A_a(-u - \rho) \pm \frac{1}{2u + \rho} (A_a(u) - A_a(-u - \rho))\right).
\]

Applying this directly to \((2.29)\), we obtain

\[
A_a(v)B_a(u) = R_{a,a}(u-v)B_a(u)R_{a,a}^\prime(-u-v - \rho)A_a(v) \mp p(-u - \rho)P_{a,a}B_a(v)Q_{a,a}\frac{Q_{a,a}}{u + v + \rho}A_a(-u - \rho)
\]

\[
+ P_{a,a}B_a(v)\left(R_{a,a}^\prime(-u-v - \rho)\frac{Q_{a,a}}{u - v} - \frac{Q_{a,a}}{(2u + \rho)(u + v + \rho)}\right)A_a(u),
\]

where we have introduced a rational function

\[
p(v) = 1 \pm \frac{1}{2v + \rho}.
\]

We note that

\[
\frac{R_{a,a}^\prime(-u-v - \rho)}{u - v} - \frac{Q_{a,a}}{(2u + \rho)(u + v + \rho)} = \frac{1}{u - v}\left(I + \left(1 - \frac{u - v}{2u + \rho}\right)\frac{Q_{a,a}}{u + v + \rho}\right) = R_{a,a}^\prime(-2u - \rho).
\]

Following these manipulations, the relation \((2.29)\) takes the form

\[
A_a(v)B_a(u) = R_{a,a}(u-v)B_a(u)R_{a,a}^\prime(-u-v - \rho)A_a(v)
\]

\[
+ \frac{P_{a,a}}{u-v}B_a(v)R_{a,a}^\prime(-2u-v - \rho)A_a(u) \mp p(-u - \rho)\frac{Q_{a,a}}{u + v + \rho}A_a(-u - \rho).
\]

This form is convenient as it does not feature \(D\) operators; however, the third term needs to be manipulated further to arrive at the most convenient expression. Consider the combination \(B_a(v)Q_{a,a}\). Using \((2.3)\) and the symmetry relation \((2.34)\) we write

\[
B_a(v)Q_{a,a} = B_a^\prime(v)Q_{a,a}
\]

\[
= P_{a,a}B_a^\prime(v)Q_{a,a}
\]

\[
= \pm P_{a,a}\left(p(-v - \rho)B_a(-v - \rho) \pm \frac{1}{2v + \rho}B_a(v)\right)Q_{a,a}.
\]

We now reinterpret the matrix \(B_a(u)\) as a row vector in two auxiliary spaces, with components given by the matrix elements of \(B_a(u)\). In particular, we define a creation operator

\[
\beta_{a,a}^\prime(u) := \sum_{1 \leq i,j \leq n} e_i^\dagger \otimes e_j^\dagger \otimes b_{ij}(u) \in V_a^+ \otimes V_a^+ \otimes Y^{\pm}(\mathfrak{gl}_2)\llbracket u^{-1}\rrbracket.
\]

Our goal is to rewrite the exchange relations involving the \(B\) operator using the above notation. For example, for any \(X_a \in \text{End}(V_a)\) and \(Y_a \in \text{End}(V_a)\), we have

\[
\beta_{a,a}(u)X_a^\dagger Y_a = \sum_{1 \leq i,j,k,l \leq n} (e_i \otimes e_j \otimes b_{kl}(u))(e_r \otimes e_s \otimes x_{ir}y_{js}) = \sum_{1 \leq i,j,k,l \leq n} e_i \otimes e_j \otimes b_{kl}(u)x_{ik}y_{lj},
\]

so that \((\beta_{a,a}(u)X_a^\dagger Y_a)_{ij} = \sum_{1 \leq k,l \leq n} b_{kl}(u)x_{ik}y_{lj}\). On the other hand, taking the \((i,j)\)-th matrix element of the expression \(X_aB_a(u)Y_a\) for any \(X_a, Y_a \in \text{End}(V_a)\) we obtain

\[
(X_aB_a(u)Y_a)_{ij} = \sum_{1 \leq k,l \leq n} x_{ik}b_{kl}(u)y_{lj} = (\beta_{a,a}(u)X_a^\dagger Y_a)_{ij}.
\]
This allows us to identify $X_a B_a(u) Y_a$ with $\beta_{a a}(u) X_a Y_a$. Using similar arguments we can rewrite (3.1) and (3.2) as follows:

\[(3.4) \quad A_a(v) \beta_{a, a}(u) = \beta_{a, a}(u) R_{a, a}^t(u-v) R_{a, a}^t(-u-v-\rho) A_a(v) + \frac{1}{u-v} \beta_{a, a}(v) Q_{a, a} R_{a, a}^t(-2u-v) A_a(u) + p(-u-\rho) \frac{u + v + \rho}{u + v} \beta_{a, a}(v) Q_{a, a} Q_{a, a} A_a(-u-\rho),\]

\[(3.5) \quad \beta_{a, a}(v) Q_{a, a} = \pm \left(p(-u-\rho) \beta_{a, a}(-v-\rho) + \frac{1}{2v + \rho} \beta_{a, a}(v)\right) Q_{a, a} Q_{a, a}.
\]

To obtain the most convenient form of (3.4), we consider the action of $p(v) A_a(v) + p(-v-\rho) A_a(-v-\rho)$ on $\beta_{a, a}(u)$ rather than of $A_a(v)$ alone (the motivation for this construction will be explained in Section 4). Introduce the following notation for an even combination of functions or operators,

\[(3.6) \quad \{f(v)\}^v := f(v) + f(-v-\rho).
\]

**Lemma 3.1.** The following identity holds:

\[(3.7) \quad \{p(v) A_a(v)\}^v \beta_{a, a}(u) = \beta_{a, a}(u) \left\{p(v) R_{a, a}^t(u-v) R_{a, a}^t(-u-v-\rho) A_a(v)\right\}^v + \frac{1}{p(u)} \left\{p(v) \beta_{a, a}(v)\right\}^v Q_{a, a} R_{a, a}^t(-2u-v) A_a(u) + p(-u-\rho) \left\{\frac{p(v)}{u-v} \beta_{a, a}(v)\right\}^v Q_{a, a} Q_{a, a} A_a(-u-\rho).
\]

Proof. Starting from (3.4), multiplying by $p(v)$ and symmetrising using (3.6), we obtain

\[(3.8) \quad \{p(v) A_a(v)\}^v \beta_{a, a}(u) = \beta_{a, a}(u) \left\{p(v) R_{a, a}^t(u-v) R_{a, a}^t(-u-v-\rho) p(v) A_a(v)\right\}^v + \frac{1}{p(u)} \left\{p(v) \beta_{a, a}(v)\right\}^v Q_{a, a} R_{a, a}^t(-2u-v) A_a(u) + p(-u-\rho) \left\{\frac{p(v)}{u-v} \beta_{a, a}(v)\right\}^v Q_{a, a} Q_{a, a} A_a(-u-\rho).
\]

We will show that this is equivalent to (3.7) term by term, separating the terms by the parameter carried by $A_a(\cdot)$. Note that the term containing $A_a(v)$ is already the same in both (3.7) and (3.8). For the remaining terms, containing $A_a(u)$ and $A_a(-u-\rho)$, we will work backwards from (3.7). Let

\[U = \frac{1}{p(u)} \left\{p(v) \beta_{a, a}(v)\right\}^v Q_{a, a} R_{a, a}^t(-2u-v) A_a(u)\]

Furthermore, expand the symmetriser inside the residue so that $U = U_+ + U_-$, where

\[U_+ = \frac{1}{p(u)} \left\{p(v) \beta_{a, a}(v)\right\}^v \left[ Q_{a, a} R_{a, a}^t(-2u-v) A_a(u) \right]_{-u-\rho} ;\]

\[U_- = \frac{1}{p(u)} \left\{p(v) \beta_{a, a}(v)\right\}^v \left[ Q_{a, a} R_{a, a}^t(-2u-v) A_a(u) \right]_{u-\rho}.
\]

Focussing first on $U_+$, we evaluate the residue to obtain

\[U_+ = \left\{p(v) \beta_{a, a}(v)\right\}^v Q_{a, a} R_{a, a}^t(-2u-v) A_a(u).
\]

This now matches the term containing $A_a(u)$ in (3.8). It remains to show that $U_-$ is equal to the term containing $A_a(-u-\rho)$ in (3.8). Again evaluating the residue, we obtain

\[U_- = \left\{p(v) \beta_{a, a}(v)\right\}^v Q_{a, a} R_{a, a}^t(-2u-v) A_a(u)\]

\[= \left\{p(v) \beta_{a, a}(v)\right\}^v \left( Q_{a, a} R_{a, a}^t(-2u-v) A_a(u) \right)_{-u-\rho}.
\]

We now apply the symmetry relation (3.5), so

\[U_- = \left\{p(v) \beta_{a, a}(v)\right\}^v \left( Q_{a, a} R_{a, a}^t(-2u-v) A_a(u) \right)_{-u-\rho}.
\]
Since it lies within the symmetriser, the term containing $\beta_{\tilde{a},a,-}(v)$ can be rewritten in terms of $\beta_{\tilde{a},a,-}(v)$ to obtain
\[
U_+ = \frac{p(-u-\rho)}{p(u)} \left( \left( \frac{p(-v-\rho)}{u+v+\rho} + \frac{1}{2u+\rho} \right) p(v) \beta_{\tilde{a},a,-}(v) \right) v Q_{\tilde{a},a} Q_{a,a} A_a(-u-\rho).
\]
All that remains are algebraic manipulations:
\[
U_+ = \frac{p(-u-\rho)}{p(u)} \left( \left( \frac{p(-v-\rho)}{u+v+\rho} + \frac{2}{(2v+\rho)(2u+\rho)} \right) p(v) \beta_{\tilde{a},a,-}(v) \right) v Q_{\tilde{a},a} Q_{a,a} A_a(-u-\rho)
\]
\[
= \frac{p(-u-\rho)}{p(u)} \left( \left( \frac{1}{2v+\rho} \right) p(v) \beta_{\tilde{a},a,-}(v) \right) v Q_{\tilde{a},a} Q_{a,a} A_a(-u-\rho)
\]
\[
= \frac{\pm p(-u-\rho)}{p(u)} \left( \frac{p(v)}{u+v+\rho} \beta_{\tilde{a},a,-}(v) \right) v Q_{\tilde{a},a} Q_{a,a} A_a(-u-\rho).
\]
This matches the term containing $A_a(-u-\rho)$ in (3.8) and completes the proof. 

3.2. Creation operator for multiple excitations. Fix $m \in \mathbb{N}$ and introduce $u = (u_1, u_2, \ldots, u_m) \in \mathbb{C}^m$, an $m$-tuple of distinct complex parameters and an operator acting on the space $V_{\tilde{a}_1} \otimes \cdots \otimes V_{\tilde{a}_m} \otimes V_a \otimes \cdots \otimes V_a$:
\[
(3.9) \quad \beta_{\tilde{a}_1 \tilde{a}_2 \ldots \tilde{a}_m a_m}(u) = \prod_{i=1}^{m} \left( \beta_{\tilde{a}_i a_i}(u_i) \prod_{j=i-1}^{i} R_{a_i \tilde{a}_i}(-u_j - u_i - \rho) \right),
\]
so that
\[
\beta_{\tilde{a}_1 \tilde{a}_2 \ldots \tilde{a}_m a_m}(u) = \beta_{\tilde{a}_1 \tilde{a}_2 \ldots \tilde{a}_m a_m}(u)(u) R_{\tilde{a}_m a_m}(-u_{m-1} - u_m - \rho) \cdots R_{a_1 \tilde{a}_1}(-u_1 - u_m - \rho).
\]
We will refer to this operator as the creation operator for $m$ excitations. Denote
\[
(3.10) \quad u_{i+1} = (u_1, u_2, \ldots, u_{i-1}, u_i, u_{i+1}, u_{i+2}, \ldots, u_m).
\]

Lemma 3.2. The following identity holds:
\[
(3.11) \quad \beta_{\tilde{a}_1 \tilde{a}_2 \ldots \tilde{a}_m a_m}(u) = \beta_{\tilde{a}_1 \tilde{a}_2 \ldots \tilde{a}_m a_m}(u_{i+1}) \tilde{R}_{a_i \tilde{a}_i}(u_i - u_i - \rho) \tilde{R}_{\tilde{a}_i a_i}^{-1}(u_i - u_i - \rho)
\]
for $1 \leq i \leq m - 1$, where $R_{a_i \tilde{a}_i}(u_i - u_i - \rho) = P_{a_i \tilde{a}_i} R_{a_i \tilde{a}_i}(u_i - u_i + 1)$.

Proof. We use induction on $m$. Start from (2.30). Begin by acting from the left with $P_{a_1 \tilde{a}_1}$, then use the defining property of the permutation operator to move it to the right on the r.h.s. of the equation to obtain
\[
\tilde{R}_{a_1 \tilde{a}_2}(u_1 - u_2) B_{a_1}(u_1) R_{a_1 \tilde{a}_2}(-u_1 - u_2 - \rho) B_{a_2}(u_2)
\]
\[
= B_{a_1}(u_2) R_{a_1 \tilde{a}_2}(-u_1 - u_2 - \rho) B_{a_2}(u_1) \tilde{R}_{a_1 \tilde{a}_2}(u_1 - u_2).
\]
We want to rewrite this in terms of the creation operators introduced in (3.3). Choose bases for $V_{a_1}$ and $V_{a_2}$, then denote the matrix components of $R_{a_1 \tilde{a}_2}(-u_1 - u_2 - \rho)$ by $r_{i_1 i_2 j_1 j_2}$, and the matrix components of $\tilde{R}_{a_1 \tilde{a}_2}(u_1 - u_2)$ by $\tilde{r}_{i_1 i_2 j_1 j_2}$. In components, (3.12) becomes
\[
\sum_{1 \leq i_1, j_1, k_1, k_2 \leq n} \tilde{r}_{i_1 i_2 j_1 j_2} b_{i_1 k_1}(u_1) r_{k_1 i_2 j_2} b_{k_2 l_2}(u_2) = \sum_{1 \leq i_1, j_1, k_1, k_2 \leq n} b_{i_1 l_1}(u_2) r_{i_1 k_1 j_2} b_{j_2 k_2}(u_1) \tilde{r}_{k_1 l_1 k_2 l_2}.
\]
In fact, it will be clearer to take the $\tilde{r}_1$ and $\tilde{r}_2$ components instead of $i_1$ and $i_2$. With this in mind, the expression above becomes
\[
\sum_{1 \leq j_1, j_2, k_1, k_2 \leq n} b_{i_1 k_1}(u_1) b_{k_2 l_2}(u_2) r_{k_1 i_2 j_2} \tilde{r}_{i_1 i_2 j_1 j_2} = \sum_{1 \leq j_1, j_2, k_1, k_2 \leq n} b_{i_1 l_1}(u_2) b_{j_2 k_2}(u_1) r_{j_1 k_1 j_2 i_2} \tilde{r}_{k_1 l_1 k_2 l_2}.
\]
Similarly, we may choose to take the sum over the bar components instead of the non-bar components as follows:
\[
\sum_{1 \leq j_1, j_2, k_1, k_2 \leq n} b_{j_1 k_1}(u_1) b_{k_2 l_2}(u_2) r_{k_1 i_2 j_2} \tilde{r}_{i_1 i_2 j_1 j_2} = \sum_{1 \leq j_1, j_2, k_1, k_2 \leq n} b_{j_1 l_1}(u_2) b_{j_2 k_2}(u_1) r_{j_1 k_1 j_2 i_2} \tilde{r}_{k_1 l_1 k_2 l_2}.
\]
Finally, we note that \( \tilde{r}_{ij}^{12} = r_{ij}^{12} \), as \( \tilde{R}_{ab}(u)^{1-1} = R_{ab}(u) \). Regarding the expression as the components of a vector in \( V_{a_1}^* \otimes V_{a_1} \otimes V_{a_2}^* \otimes V_{a_2} \), this is equivalent to

\[
\beta_{\tilde{a}_1 a_1} (u_1) \beta_{\tilde{a}_2 a_2} (u_2) R_{\tilde{a}_1 \tilde{a}_2} (-u_1 - u_2 - \rho) \tilde{R}_{a_1 a_2} (u_1 - u_2) = \beta_{\tilde{a}_1 a_1} (u_1) \beta_{\tilde{a}_2 a_2} (u_2) R_{a_1 a_2} (-u_1 - u_2 - \rho) \tilde{R}_{a_1 a_2} (u_1 - u_2),
\]

yielding

\[
\beta_{\tilde{a}_1 a_1 a_2 a_2} (u_1, u_2) = \beta_{\tilde{a}_1 a_1 a_2 a_2} (u_2, u_1) \tilde{R}_{a_1 a_2} (u_1 - u_2) \tilde{R}_{a_2 a_1}^{-1} (u_1 - u_2).
\]

This gives the basis case for (3.11). Assume the result holds for \( m - 1 \) excitations. There are two cases to consider, depending on the spaces \( a_i, a_{i+1} \) on which \( R_{a_i a_{i+1}} (u_i - u_{i+1}) \) acts nontrivially. Consider first the case where \( i < m - 1 \),

\[
\beta_{\tilde{a}_1 a_1 \ldots \tilde{a}_m a_m} (u) = \beta_{\tilde{a}_1 a_1 \ldots \tilde{a}_{m-1} a_{m-1}} (u) \beta_{\tilde{a}_m a_m} (u_m) \prod_{j=m-1}^{1} R_{a_j \tilde{a}_m} (-u_j - u_m - \rho)
\]

\[
= \beta_{\tilde{a}_1 a_1 \ldots \tilde{a}_{m-1} a_{m-1}} (u_{i+1}) \tilde{R}_{a_i a_{i+1}} (u_i - u_{i+1}) \tilde{R}_{a_{i+1} a_i}^{-1} (u_i - u_{i+1})
\]

\[
\times \beta_{\tilde{a}_{m-1} a_m} (u_m) \prod_{j=m-1}^{1} R_{a_j \tilde{a}_m} (-u_j - u_m - \rho).
\]

Notice that the matrix \( \tilde{R}_{a_{i+1} a_i}^{-1} (u_i - u_{i+1}) \) commutes with all matrices to the right of it, so it can be moved to the very right. The matrix \( \tilde{R}_{a_i a_{i+1}} (u_i - u_{i+1}) \) may be moved through the product of \( R \)-matrices using the (braided) Yang-Baxter equation:

\[
\tilde{R}_{a_i a_{i+1}} (u_i - u_{i+1}) R_{a_{i+1} \tilde{a}_m} (-u_{i+1} - u_m - \rho) R_{a_i \tilde{a}_m} (-u_i - u_m - \rho)
\]

\[
= R_{a_i \tilde{a}_m} (-u_i - u_m - \rho) R_{a_i \tilde{a}_m} (-u_{i+1} - u_m - \rho) \tilde{R}_{a_{i+1} a_i} (u_i - u_{i+1}).
\]

This then gives (3.11) for \( i < m - 1 \). For \( i = m - 1 \), we factorise the excitations as follows:

\[
\beta_{\tilde{a}_1 a_1 \ldots \tilde{a}_m a_m} (u) = \beta_{\tilde{a}_1 a_1 \ldots \tilde{a}_{m-2} a_{m-2}} (u) \beta_{\tilde{a}_{m-1} a_{m-1} \tilde{a}_m a_m} (u_{m-1}, u_m)
\]

\[
\times \prod_{j=m-2}^{1} \left( R_{a_j \tilde{a}_{m-1}} (-u_j - u_{m-1} - \rho) R_{a_j \tilde{a}_{m}} (-u_j - u_m - \rho) \right)
\]

\[
= \beta_{\tilde{a}_1 a_1 \ldots \tilde{a}_{m-2} a_{m-2}} (u) \beta_{\tilde{a}_{m-1} a_{m-1} \tilde{a}_m a_m} (u_m, u_{m-1})
\]

\[
\times \tilde{R}_{a_{m-1} a_m} (u_m - u_{m-1}) \tilde{R}_{a_{m-1} \tilde{a}_m}^{-1} (u_m - u_{m-1})
\]

\[
\times \prod_{j=m-2}^{1} \left( R_{a_j \tilde{a}_{m-1}} (-u_j - u_{m-1} - \rho) R_{a_j \tilde{a}_{m}} (-u_j - u_m - \rho) \right).
\]

The matrix \( \tilde{R}_{a_{m-1} \tilde{a}_m}^{-1} (u_m - u_{m-1}) \) may be moved through the product of \( R \)-matrices using another variant of the Yang-Baxter equation,

\[
\tilde{R}_{a_{m-1} \tilde{a}_m}^{-1} (u_m - u_{m-1}) R_{a_j \tilde{a}_{m-1}} (-u_j - u_{m-1} - \rho) R_{a_j \tilde{a}_m} (-u_j - u_m - \rho)
\]

\[
= R_{a_j \tilde{a}_{m-1}} (-u_j - u_{m-1} - \rho) R_{a_j \tilde{a}_m} (-u_j - u_m - \rho) \tilde{R}_{a_{m-1} \tilde{a}_m}^{-1} (u_m - u_{m-1}).
\]

Then, rearranging the commuting matrices in the expression, we reconstruct the full excitation vector and arrive at (3.11) for \( i = m - 1 \). This completes the induction. \( \square \)

**Remark 3.3.** By definition, the operator \( \beta_{\tilde{a}_1 a_1 \ldots \tilde{a}_m a_m} (u) \) in (3.9) is an analogue of the fused boundary operator of Theorem 4.1 in [BaRe] for the twisted reflection equation. More precisely, it is a solution to a fused analogue of twisted reflection equation (2.30) in the sense of *loc. cit.*

### 3.3. The AB exchange relation for multiple excitations.

We want to move \( \{p(v)A_a(v)\}^v \) through the operator \( \beta_{\tilde{a}_1 a_1 \ldots \tilde{a}_m a_m} (u) \). Each time \( \{p(v)A_a(v)\}^v \) is moved through one of the excitations \( \beta_{\tilde{a}_1 a_1} (u_i) \) using (3.4), we obtain a term, where the parameter \( v \) of \( \{p(v)A_a(v)\}^v \) is unchanged. We will call this term the *wanted term*. All the additional terms will be called the *unwanted terms*; we will denote them by \( UWT \). Focussing on the wanted term at each step, \( \{p(v)A_a(v)\}^v \) accures \( R \)-matrices as it moves through
the excitations. In the following lemma, we will show that these $R$-matrices may be moved through those appearing in the operator $\beta_{\tilde{a}_1\ldots\tilde{a}_m}(\mathbf{u})$.

**Lemma 3.4.** The following exchange relation holds

$$
\left(\prod_{k=1}^{i-1} R_{\tilde{a}_k\tilde{a}}^t(u_k - v)\right) \left(\prod_{l=1}^{i-1} R_{\tilde{a}_l\tilde{a}}^t(-u_l - v - \rho)\right) A_\alpha(v) \beta_{\tilde{a}_1\ldots\tilde{a}_i}(u_i) \prod_{j=i-1}^1 R_{a_j\tilde{a}_i}(-u_j - u_i - \rho) \\
= \beta_{\tilde{a}_1\ldots\tilde{a}_i}(u_i) \left(\prod_{j=i-1}^1 R_{a_j\tilde{a}_i}(-u_j - u_i - \rho)\right) \left(\prod_{k=1}^{i-1} R_{\tilde{a}_k\tilde{a}}(u_k - v)\right) \left(\prod_{l=1}^{i-1} R_{\tilde{a}_l\tilde{a}}(-u_l - v - \rho)\right) A_\alpha(v) + UWT.
$$

**Proof.** We begin by using (3.4) and focus on the wanted terms only:

$$
\left(\prod_{k=1}^{i-1} R_{\tilde{a}_k\tilde{a}}(u_k - v)\right) \left(\prod_{l=1}^{i-1} R_{\tilde{a}_l\tilde{a}}(-u_l - v - \rho)\right) A_\alpha(v) \beta_{\tilde{a}_1\ldots\tilde{a}_i}(u_i) \prod_{j=i-1}^1 R_{a_j\tilde{a}_i}(-u_j - u_i - \rho) \\
= \left(\prod_{k=1}^{i-1} R_{\tilde{a}_k\tilde{a}}(u_k - v)\right) \left(\prod_{l=1}^{i-1} R_{\tilde{a}_l\tilde{a}}(-u_l - v - \rho)\right) \\
\times \beta_{\tilde{a}_1\ldots\tilde{a}_i}(u_i) R_{\tilde{a}_1\tilde{a}}(u_i - v) R_{\tilde{a}_1\tilde{a}}(u_i - v) A_\alpha(v) \prod_{j=i-1}^1 R_{a_j\tilde{a}_i}(-u_j - u_i - \rho) + UWT \\
= \beta_{\tilde{a}_1\ldots\tilde{a}_i}(u_i) \left(\prod_{k=1}^{i-1} R_{\tilde{a}_k\tilde{a}}(u_k - v)\right) \left(\prod_{l=1}^{i-1} R_{\tilde{a}_l\tilde{a}}(-u_l - v - \rho)\right) \\
\times R_{\tilde{a}_1\tilde{a}}(u_i - v) \left(\prod_{j=i-1}^1 R_{a_j\tilde{a}_i}(-u_j - u_i - \rho)\right) R_{\tilde{a}_1\tilde{a}}(u_i - v) A_\alpha(v) + UWT.
$$

All that remains is to rearrange the product of $R$-matrices in the centre of the expression. The matrices can be reordered using the Yang-Baxter equation

$$
R_{\tilde{a}_1\tilde{a}}^t(-u_{i-1} - v - \rho) R_{\tilde{a}_1\tilde{a}}^t(u_i - v) R_{\tilde{a}_1\tilde{a}}(-u_{i-1} - u_i - \rho) = R_{\tilde{a}_1\tilde{a}}(-u_{i-1} - u_i - \rho) R_{\tilde{a}_1\tilde{a}}^t(u_i - v) R_{\tilde{a}_1\tilde{a}}^t(-u_{i-1} - v - \rho).
$$

Thus the product of $R$-matrices becomes

$$
\left(\prod_{l=1}^{i-1} R_{\tilde{a}_l\tilde{a}}(-u_l - v - \rho)\right) R_{\tilde{a}_1\tilde{a}}^t(-u_{i-1} - u_i - \rho) R_{\tilde{a}_1\tilde{a}}^t(u_i - v) \\
= \left(\prod_{l=1}^{i-2} R_{\tilde{a}_l\tilde{a}}(-u_l - v - \rho)\right) R_{\tilde{a}_1\tilde{a}}(-u_{i-1} - u_i - \rho) R_{\tilde{a}_1\tilde{a}}^t(u_i - v) \\
\times R_{\tilde{a}_1\tilde{a}}(-u_{i-1} - v - \rho) \left(\prod_{j=i-2}^1 R_{a_j\tilde{a}_i}(-u_j - u_i - \rho)\right) \\
= R_{\tilde{a}_1\tilde{a}}(-u_{i-1} - u_i - \rho) \left(\prod_{l=1}^{i-2} R_{\tilde{a}_l\tilde{a}}(-u_l - v - \rho)\right) R_{\tilde{a}_1\tilde{a}}^t(u_i - v) \\
\times \left(\prod_{j=i-2}^1 R_{a_j\tilde{a}_i}(-u_j - u_i - \rho)\right) R_{\tilde{a}_1\tilde{a}}(-u_{i-1} - v - \rho) \\
= \left(\prod_{j=i-1}^1 R_{a_j\tilde{a}_i}(-u_j - u_i - \rho)\right) R_{\tilde{a}_1\tilde{a}}^t(u_i - v) \left(\prod_{l=1}^{i-1} R_{\tilde{a}_l\tilde{a}}(-u_l - v - \rho)\right),
$$

where the last equality is achieved by inductively applying the same argument. Putting this together, and noting that the $R^t$-matrices all commute with the $R$-matrices, we arrive to
\[
\left( \prod_{k=1}^{i-1} R_{\tilde{a}_k a}^t (u_k - v) \right) \left( \prod_{l=1}^{i-1} R_{a_l \tilde{a}}^t (-u_l - v - \rho) \right) A_a(v) \beta_{\tilde{a}_i a_i}(u_i) \prod_{j=i+1}^1 R_{a_j \tilde{a}_j} (-u_j - u_i - \rho) \\
= \beta_{\tilde{a}_i a_i}(u_i) \left( \prod_{j=i-1}^1 R_{a_j \tilde{a}_j} (-u_j - u_i - \rho) \right) \left( \prod_{k=1}^i R_{\tilde{a}_k a}^t (u_k - v) \right) \left( \prod_{l=1}^i R_{a_l \tilde{a}}^t (-u_l - v - \rho) \right) A_a(v) + UWT
\]
as required.

Applying this result to the product of \( m \) such excitations in (3.9) yields

\[
A_a(v) \beta_{\tilde{a}_1 a_1 \ldots \tilde{a}_m a_m}(u) = \beta_{\tilde{a}_1 a_1 \ldots \tilde{a}_m a_m}(u) \left( \prod_{k=1}^m R_{\tilde{a}_k a}^t (u_k - v) \right) \left( \prod_{l=1}^m R_{a_l \tilde{a}}^t (-u_l - v - \rho) \right) A_a(v) + UWT.
\]

We define the matrix on the right side to be the \textit{nested monodromy matrix},

\[
(3.13) \quad T_a(v; u) := \left( \prod_{k=1}^m R_{\tilde{a}_k a}^t (u_k - v) \right) \left( \prod_{l=1}^m R_{a_l \tilde{a}}^t (-u_l - v - \rho) \right) A_a(v).
\]

Its matrix entries will be denoted by \( t_{ij}(v; u) \). The matrix \( T_a(v; u) \) allows us to write the above identity more compactly,

\[
A_a(v) \beta_{\tilde{a}_1 a_1 \ldots \tilde{a}_m a_m}(u) = \beta_{\tilde{a}_1 a_1 \ldots \tilde{a}_m a_m}(u) T_a(v; u) + UWT,
\]

which leads to the following result.

\textbf{Corollary 3.5.} The \textit{AB exchange relation for multiple excitations} has the form

\[
\{ p(v) A_a(v) \}^v \beta_{\tilde{a}_1 a_1 \ldots \tilde{a}_m a_m}(u) = \beta_{\tilde{a}_1 a_1 \ldots \tilde{a}_m a_m}(u) \{ p(v) T_a(v; u) \}^v + UWT.
\]

\textbf{3.4. Exchange relations for the nested monodromy matrix.} In this section we introduce a vector space \( M' \), called the \textit{nested vacuum sector}, on which the nested monodromy matrix \( T(v; u) \) satisfies the usual RTT relation, \textit{viz.} (2.6). This allows us to identify \( T(v; u) \) as the monodromy matrix for a residual \( Y(\mathfrak{gl}_n) \) system. The space \( M' \) is then interpreted as the full quantum space of this residual system. We start by introducing certain subspaces of the evaluation modules \( M(\mu) \) and \( L(\lambda^{(i)})_{c_i} \), that will be building blocks of the space \( M' \).

Denote by \( M^0(\mu) \) the subspace of the evaluation module \( M(\mu) \) of \( Y^\pm(\mathfrak{gl}_{2n}) \) consisting of vectors annihilated by the operator \( C(u) \) of the matrix \( S(u) \), namely

\[
M^0(\mu) := \{ \zeta \in M(\mu) : e_{ij}(u) \zeta = 0 \quad \text{for} \quad 1 \leq i, j \leq n \}.
\]

The subspace \( M^0(\mu) \) corresponds to the natural embedding \( \mathfrak{gl}_n \subset \mathfrak{so}_{2n} \) or \( \mathfrak{gl}_n \subset \mathfrak{sp}_{2n} \) (generated by \( F_{ij} \) with \( 1 \leq i, j \leq n \), \textit{viz.} (2.1-2.2)) and is an irreducible \( \mathfrak{gl}_n \)-representation of the highest weight \( \mu = (\mu_1, \ldots, \mu_n) \).

The space \( M^0(\mu) \) is stable under the action of the operator \( A(u) \) of the matrix \( S(u) \). Moreover, the operator \( A(u) \) satisfies the usual RTT relation on this space. Indeed, applying equality (2.32) to \( M^0(\mu) \) yields \( C_1(v) A_2(u) M^0(\mu) = 0 \). Applying (2.31) instead we obtain

\[
R(u - v) \quad A_1(u) A_2(v) \zeta = A_2(v) A_1(u) R(u - v) \zeta
\]

for all \( \zeta \in M^0(\mu) \). We thus have the following.

\textbf{Lemma 3.6.} The mapping

\[
Y(\mathfrak{gl}_n) \to Y^\pm(\mathfrak{gl}_{2n}), \quad T(u) \mapsto A(u)
\]
equips the space \( M^0(\mu) \) with a structure of a lowest weight \( Y(\mathfrak{gl}_n) \)-module with the lowest weight given by (2.16).

Note that the operator \( A(u) \) of the matrix \( S(u) \) acts on the space \( M^0(\mu) \) via the Lax operator

\[
(3.14) \quad \mathcal{L}^{\pm,0}(u) := \sum_{i,j=1}^n e_{ij} \otimes (\delta_{ij} - F_{ji} (u + (\rho \pm 1)/2))^{-1},
\]

which is the restriction of \( \mathcal{L}^\pm(u) \) defined in (2.20) to the operator \( A(u) \).
Next, we denote by $L^0(\lambda(i))_{c_i}$ the subspace of the evaluation module $L(\lambda(i))_{c_i}$ of $Y(\mathfrak{gl}_{2n})$ consisting of vectors annihilated by the operator $C(u)$ of the matrix $T(u)$, namely
\begin{equation}
L^0(\lambda(i))_{c_i} := \{ \zeta \in L(\lambda(i))_{c_i} : \tau_{ij}(u)\zeta = 0 \text{ for } 1 \leq i, j \leq n \}.
\end{equation}
The subspace $L^0(\lambda(i))_{c_i}$ corresponds to the natural embedding $\mathfrak{gl}_n \oplus \mathfrak{gl}_n \subset \mathfrak{gl}_{2n}$ (generated by $E_{ij}$ with $1 \leq i, j \leq n$ and $n < i, j \leq 2n$) and is isomorphic to a tensor product of irreducible $\mathfrak{gl}_n$-representations of the highest weights $\lambda(i) = (\lambda_1^{(i)}, \ldots, \lambda_n^{(i)})$ and $\lambda^{(i)} = (\lambda_{n+1}^{(i)}, \ldots, \lambda_{2n}^{(i)})$. Applying equality (2.26) to $L^0(\lambda(i))_{c_i}$ yields $C_1(u)C_2(v)L^0(\lambda(i))_{c_i} = 0$. Applying (2.27) instead we obtain $C_1(u)D_2(v)L^0(\lambda(i))_{c_i} = 0$. Moreover, applying (2.28) to $L^0(\lambda(i))_{c_i}$, we get $[D_1(u), A_2(v)]L^0(\lambda(i))_{c_i} = 0$. This, together with (2.24) and (2.25), implies the following.

**Lemma 3.7.** Each of the mappings
\[
Y(\mathfrak{gl}_n) \to Y(\mathfrak{gl}_{2n}), \quad T(u) \mapsto A(u) \quad \text{and} \quad T(u) \mapsto D(u)
\]
is a homomorphism of algebras. Moreover, these mappings equip the spaces $L^0(\lambda(i))_{c_i}$ and $L^0(\lambda^{(i)}))_{c_i}$ with a structure of a lowest weight $Y(\mathfrak{gl}_n)$-module with the lowest weight given by $\lambda(i)_i(u) = \lambda(i)_i(u)$ and $\lambda^{(i)}_i(u) = \lambda^{(i)}_i(u)$, respectively, for $1 \leq i \leq n$.

Note that the operator $A(u)$ of the matrix $T(u)$ of $Y(\mathfrak{gl}_{2n})$ acts on the space $L^0(\lambda(i))_{c_i}$ via the restriction of the Lax operator (2.11),
\begin{equation}
L^0(u - c_i) := \sum_{i,j=1}^n e_{ij} \otimes (\delta_{ij} - E_{ji}(u - c_i))^{-1},
\end{equation}
and the operator $A(u)$ of the transposed matrix $T^t(u)$ acts on $L^0(\lambda^{(i)})_{c_i}$ via the Lax operator $(L^0(u - c_i))^t$.

We are now ready to introduce the vacuum sector $M^0 \subset M$ by
\begin{equation}
M^0 := L^0(\lambda(1))_{c_1} \otimes \cdots \otimes L^0(\lambda(\ell))_{c_{\ell}} \otimes M^0(\mu).
\end{equation}

**Lemma 3.8.** The space $M^0$ is stable under the action of the operator $A(u)$ of the matrix $S(u)$.

**Proof.** We start by showing that operator $C(u)$ of the matrix $S(u)$ annihilates the space $M^0$: $e_{ij}(v) \cdot M^0 = 0$. We use induction on $\ell$. For $\ell = 0$ we have $M^0 = M^0(\mu)$ and $e_{ij}(v)M^0(\mu) = 0$, by definition (3.17). For $\ell \geq 1$ let $\zeta \in M^0$ and $\zeta' \in L^0(\lambda^{(\ell+1)})_{c_{\ell+1}}$ be any non-zero vectors. Using (2.17) and the notation (2.21) we find
\[
e_{ij}(u) \cdot (\zeta' \otimes \zeta) = \sum_{k,l=1}^n \left( \tau_{ik}(u) \overline{d}_{jl}(-u - \rho) \zeta' \otimes a_{kl}(u) \cdot \zeta \pm \tau_{ik}(u) \overline{d}_{ji}(-u - \rho) \zeta' \otimes b_{kl}(u) \cdot \zeta \right)
\]
\[
+ \overline{d}_{ik}(u) \overline{d}_{jl}(-u - \rho) \zeta' \otimes e_{kl}(u) \cdot \zeta \pm \overline{d}_{ik}(u) \overline{d}_{ji}(-u - \rho) \zeta' \otimes d_{kl}(u) \cdot \zeta
\]
\[
= \sum_{k,l=1}^n \left( \tau_{ik}(u) \overline{d}_{jl}(-u - \rho) \zeta' \otimes e_{kl}(u) \cdot \zeta \pm \overline{d}_{ik}(u) \overline{d}_{ji}(-u - \rho) \zeta' \otimes e_{kl}(u) \cdot \zeta \right),
\]
by definition (3.15); here we used the notation $\overline{i} = n - i + 1$. Assuming induction, $e_{kl}(u) \zeta = 0$. Finally, by (2.27) and (2.15), we have that
\[
\tau_{ik}(u) \overline{d}_{jl}(-u - \rho) \zeta = \overline{d}_{jl}(-u - \rho) \tau_{ik}(u) \zeta' = 0.
\]
Hence $e_{ij}(u) \cdot (\zeta' \otimes \zeta) = 0$, as required. Next, we need to show that $e_{ij}(u) \cdot M^0 \subseteq M^0[[u^{-1}]]$. The base case is given by Lemma 3.6. For $\ell \geq 1$ we have
\[
e_{ij}(u) \cdot (\zeta' \otimes \zeta) = \sum_{k,l=1}^n \left( \tau_{ik}(u) \overline{d}_{jl}(-u - \rho) \zeta' \otimes a_{kl}(u) \cdot \zeta \pm \tau_{ik}(u) \overline{d}_{ji}(-u - \rho) \zeta' \otimes b_{kl}(u) \cdot \zeta \right)
\]
\[
+ \tau_{ik}(u) \overline{d}_{jl}(-u - \rho) \zeta' \otimes e_{kl}(u) \cdot \zeta \pm \tau_{ik}(u) \overline{d}_{ji}(-u - \rho) \zeta' \otimes d_{kl}(u) \cdot \zeta
\]
\[
= \sum_{k,l=1}^n \tau_{ik}(u) \overline{d}_{jl}(-u - \rho) \zeta' \otimes a_{kl}(u) \cdot \zeta,
\]
by definition (3.15) and the result above. Assuming induction, \(a_{kl}(u) \cdot \zeta \in M^0[[u^{-1}]]\) and, by Lemma 3.7, 
\[ a_{kl}(u) \overline{f}_{ij}(u-\rho) \zeta' \in L^0(\lambda^{(\ell+1)}_{c+1})[[u^{-1}]]. \]
Hence \(a_{ij}(u) \cdot (\zeta' \otimes \zeta) \in L^0(\lambda^{(\ell+1)}_{c+1}) \otimes M^0[[u^{-1}]]. \) This proves the claim.

The last ingredients we will need are the auxiliary spaces \(V_{\tilde{a}}\) and \(V_a\). They are vector representations of \(\mathfrak{gl}_n\) of weight \(\lambda^{(\tilde{a})}_i = \lambda^{(a)}_i = (1, 0, \ldots, 0)\). Denote by \(L^t(\lambda)\) the evaluation module of \(Y(\mathfrak{gl}_n)\) obtained from \(\mathfrak{gl}_n\)-representation \(L(\lambda)\) by composing the evaluation map \(ev_c\) in (2.8) with the algebra automorphism \(T(u) \to T(-u)\). The spaces \(V_{\tilde{a}}\) and \(V_a\) can thus be viewed as evaluation modules \(L^t(\lambda^{(\tilde{a})})_{-u_i}\) and \(L^t(\lambda^{(a)})_{-u_i}\) of \(Y(\mathfrak{gl}_n)\), respectively, with the highest weights given by

\[
\lambda^{(\tilde{a})}_i(v) = \frac{v-u_i+1}{v-u_i}, \quad \lambda^{(a)}_i(v) = \frac{v+u_i+1}{v+u_i} \quad \text{and} \quad \lambda^{(\tilde{a})}_j(u) = \lambda^{(a)}_j(u) = 1 \quad \text{for} \quad 1 \leq j \leq n-1.
\]

In particular, the matrix \(T_a(v)\) of \(Y(\mathfrak{gl}_n)\) acts on the space \(L(\lambda^{(\tilde{a})})_{-u_i}\) as \(R^t_{\tilde{a}a}(u_i - v)\) and on the space \(L(\lambda^{(a)})_{-u_i}\) as \(R^t_{aa}(u_i - v)\) and on the space \(L(\lambda^{(a)})_{-u_i}\). Finally, acting with \(t_{ii}(v; u)\) on \(\eta\) for \(1 \leq i \leq n\) and using (2.12), (2.16) and (3.18) yields (3.20).

**Remark 3.10.** (i) By Lemma 3.7, we may write the space \(M^0\) as

\[
M^0 \cong L^0(\lambda^{(1)})_{c_1} \otimes \cdots \otimes L^0(\lambda^{(t)})_{c_t} \otimes M^0(\mu) \otimes L^0(\lambda^{(t)}_{c_1}) \otimes \cdots \otimes L^0(\lambda^{(t)}_{c_t}).
\]
By Proposition 3.9, we may view this space as a lowest weight $Y(\mathfrak{g}_n)$-module. Provided the binary property holds, it is an irreducible $Y(\mathfrak{g}_n)$-module, see Theorem 6.5.8 in [Mo3]. (ii) Enumerate the tensorands of $M^0$ above by $1, 2, \ldots, 2\ell, 2\ell + 1$. Then the matrix $T_a(v; u)$ acts on the space $M'$ via the operator

$$\left( \prod_{k=1}^m R_{a_k}^{-1}(u_k - v) \right) \left( \prod_{i=1}^{\ell} L_a^0(u_i - c_i) \right) \left( \prod_{i=1}^{2\ell + 1} L_a^0(-u - \rho_i) \right),$$

where the Lax operators are those defined in (3.16) and (3.14).

We end this section with a lemma which will assist us in finding the explicit expressions of the unwanted terms in Section 4.3.

**Lemma 3.11.** The following identities hold:

$$\tilde{R}_{a_i,\tilde{a}_{i+1}}(u_i - u_{i+1}) \tilde{R}_{\tilde{a}_i,a_{i+1}}^{-1}(u_i - u_{i+1}) t_{kl}(w; v) = t_{kl}(w; u_{i+1}) \tilde{R}_{a_i,\tilde{a}_{i+1}}(u_i - u_{i+1}) \tilde{R}_{\tilde{a}_i,a_{i+1}}^{-1}(u_i - u_{i+1}).$$

**Proof.** The first identity is achieved by moving the $\tilde{R}$-matrices through each matrix in the definition of the nested monodromy matrix. Indeed, the $\tilde{R}$-matrices each commute with all but a pair of adjacent $R$-matrices in (3.13), for which we use the Yang Baxter equations,

$$\tilde{R}_{a_i,\tilde{a}_{i+1}}(u_i - u_{i+1}) t_{kl}(w; v) = t_{kl}(w; u_{i+1}) \tilde{R}_{a_i,\tilde{a}_{i+1}}(u_i - u_{i+1}) \tilde{R}_{\tilde{a}_i,a_{i+1}}^{-1}(u_i - u_{i+1}),$$

and the result follows.

To see why the second identity is true, notice that the lowest weight vector $\eta$ (3.21) is an eigenvector of $P_{a_i,\tilde{a}_{i+1}}(u_i - u_{i+1})$. This is true also for $P_{\tilde{a}_i,a_{i+1}}$. Thus, acting with both $\tilde{R}_{a_i,\tilde{a}_{i+1}}(u_i - u_{i+1})$ and $\tilde{R}_{\tilde{a}_i,a_{i+1}}^{-1}(u_i - u_{i+1})$, the eigenvalues cancel, which gives the result. \qed

### 4. Nested algebraic Bethe ansatz for $Y^\pm(\mathfrak{gl}_{2n})$

We are now ready to consider the nested algebraic Bethe ansatz for a one-dimensional spin chain with open boundary conditions and having twisted Yangian $Y^\pm(\mathfrak{gl}_{2n})$ as its underlying symmetry. The full quantum space is the lowest weight $Y^\pm(\mathfrak{gl}_{2n})$-module defined in (2.18):

$$M = L(\lambda^{(1)})_{c_1} \otimes L(\lambda^{(2)})_{c_2} \otimes \ldots \otimes L(\lambda^{(l)})_{c_l} \otimes M(\mu),$$

The generating matrix $S(u)$ of $Y^\pm(\mathfrak{gl}_{2n})$ acts on this space via a product of Lax operators (2.19):

$$S_a(v) : M = \left( \prod_{i=1}^l L_{ai}(v - c_i) \right) L_{a_{i+1}}^\pm(v) \left( \prod_{i=1}^l L_{ai}^\mp(-v - \rho_i - c_i) \right) M.$$

Taking the trace of the generating matrix we obtain a double-row transfer matrix

$$\tau(v) := \text{tr} S(v) = \text{tr} A(v) + \text{tr} D(v) = \text{tr} A(v) + \text{tr} D(v).$$

One can show using the usual methods that $[\tau(u), \tau(v)] = 0$; see Section 2 in [ACDFR], also [Sk]. We seek an eigenvector of $\Psi \in M$ of $\tau(v)$, which we will refer to as the Bethe vector. The problem of finding an eigenvector of the transfer matrix (4.1) can be substantially simplified with the help of the symmetry relation (2.33) which allows us to write the transfer matrix $\tau(v)$ in a symmetric form

$$\tau(v) = p(v) \text{tr} A(v) + p(-v - \rho) \text{tr} A(-v - \rho) = \left( p(v) \text{tr} A(v) \right)^v \text{ where } p(v) = 1 \pm \frac{1}{2v + \rho}.$$

Here we used the notation introduced in (3.6). It will therefore be sufficient to focus on the action of $A(v)$, without needing to consider $D(v)$.

The last ingredient we will need is the nested transfer matrix, see (3.13):

$$t(v; u) := \text{tr} T(v; u).$$

It will play the role of $\tau(v)$ at the nested level of the ansatz.
4.1. Bethe vector for a single excitation. To introduce the nesting technique, we start by constructing the Bethe vector with a single excitation, i.e., \( m = 1 \), as this case allows us to expose the main idea of our approach while keeping the technical difficulties to the minimum; for example, in this case we find the unwanted terms without additional computations. Recall the definition of the vacuum sector \( M^0 \) (3.17) and the nested vacuum sector \( M' \) (3.19). For \( m = 1 \) we have \( M' = V_{a_1} \otimes V_{a_1} \otimes M^0 \). Let \( \Phi \in M' \), which we will refer to as the nested Bethe vector. The vector \( \Phi \) may depend on \( u \in \mathbb{C} \), hence we will write \( \Phi = \Phi(u) \).

Using (3.3) we write an ansatz for the Bethe vector with a single excitation

\[
\Psi(u) := \beta_{\bar{a}_1 a_1}(u) \cdot \Phi(u) \in M.
\]

We now compute the action of the transfer matrix \( \tau(v) \) on this Bethe vector. Using Lemma 3.1 we have

\[
\tau(v) \cdot \Psi(u) = \{ p(v) \text{ tr}(A(v)) \}^v \beta_{\bar{a}_1 a_1}(u) \cdot \Phi(u)
\]

\[
= \beta_{\bar{a}_1 a_1}(u) \text{ tr}_a \left( \left\{ p(v) R^t_{\bar{a}_1 a}(u-v) R^1_{a_1 a}(-u-v) A_a(v) \right\}^v \right) \cdot \Phi(u)
\]

\[
+ \frac{1}{p(u)} \left\{ p(v) \beta_{\bar{a}_1 a_1}(v) \right\}^v \text{ Res}_{w \to u} \left( \left\{ p(w) R^t_{\bar{a}_1 a}(u-w) R^1_{a_1 a}(-u-w) A_a(w) \right\}^w \right) \cdot \Phi(u)
\]

\[
= \beta_{\bar{a}_1 a_1}(u) \{ p(v) t(v; u) \}^v \cdot \Phi(u) + \frac{1}{p(u)} \left\{ p(v) \beta_{\bar{a}_1 a_1}(v) \right\}^v \text{ Res}_{w \to u} \{ p(w) t(w; u) \}^w \cdot \Phi(u).
\]

The first term in the r.h.s. of the equality above is the wanted term, as the parameter carried by \( \beta_{\bar{a}_1 a_1}(u) \) is unchanged. The second term has \( \beta_{\bar{a}_1 a_1}(v) \) and is the unwanted term, which we will require to vanish.

Let us now make the additional requirement, which we will justify later, that vector \( \Phi(u) \) is an eigenvector of the nested transfer matrix \( t(v; u) \) with an eigenvalue \( \Gamma(v; u) \):

\[
t(v; u) \cdot \Phi(u) = \Gamma(v; u) \Phi(u).
\]

This allows us to rewrite (4.3) as

\[
\tau(v) \cdot \Psi(u) = \{ p(v) \Gamma(v; u) \}^v \Psi(u) + \frac{1}{p(u)} \text{ Res}_{w \to u} \{ p(w) \Gamma(w; u) \}^w \left\{ \frac{p(v)}{u-v} \beta_{\bar{a}_1 a_1}(v) \right\}^v \cdot \Phi(u)
\]

\[
= \Lambda(v; u) \Psi(u) + \text{ Res}_{w \to u} \Lambda(w; u) \left\{ \frac{p(v)}{u-v} \beta_{\bar{a}_1 a_1}(v) \right\}^v \cdot \Phi(u),
\]

where \( \Lambda(v; u) := \{ p(v) \Gamma(v; u) \}^v \). We thus conclude that \( \Phi(u) \) is an eigenvector of \( \tau(v) \) with eigenvalue \( \Lambda(v; u) \) if

\[
\text{Res}_{w \to u} \Lambda(w; u) = 0.
\]

This is the Bethe equation for \( u \), solutions of which, by (4.2), give a set of possible eigenvectors of \( \tau(v) \).

It remains to find a nested Bethe vector \( \Phi(u) \) satisfying (4.4): we seek an eigenvector \( \Phi(u) \in M' \) of \( t(v; u) \). By Proposition 3.9, the nested monodromy matrix \( T_u(v; u) \) and the nested vacuum sector \( M' \) form a \( Y(\mathfrak{gl}_n) \) system. The spectral problem of this system can be solved by means of the usual nested algebraic Bethe ansatz presented in [KuRs], which we have recalled in detail in Appendix A. For example, the ansatz for \( \Phi(u) \) has the form

\[
\Phi(u) = \Phi(u'; u) := B'_{u_1}(u_1') \cdots B'_{u_{m'}}(u_{m'}') \cdot \Phi'(u'; u),
\]

where \( u' = (u_1', \ldots, u_{m'}') \in \mathbb{C}^{m'} \) and \( B'_{u_i}(u_j') \) are creation operators taken from the \( T_u(v; u) \). Continuing this nesting procedure, we obtain an eigenvector \( \Phi(u; u') \) with eigenvalue, see (A.22),

\[
\Gamma(v; u) = \lambda_1(v; u) \prod_{i=1}^{m'} \frac{v - u_1^{(1)}}{v - u_i^{(1)}} + \lambda_n(v; u) \prod_{i=1}^{m^{(n-1)}} \frac{v - u_i^{(n-1)}}{v - u_i^{(n-1)}} - 1
\]

\[
+ \sum_{k=2}^{n-1} \lambda_k(v; u) \prod_{i=1}^{m_{(k-1)}} \frac{v - u_i^{(k-1)}}{v - u_i^{(k-1)}} - 1 \prod_{j=1}^{m_{(k)}} \frac{v - u_j^{(k)}}{v - u_j^{(k)}} - 1,
\]

where \( \lambda_k(v; u) \) are given by (3.9) and the \( u_i^{(k)} \) with \( 1 \leq k \leq n - 1 \) are parameters introduced at level \( k \) of nesting when diagonalizing the \( \mathfrak{gl}_n \)-symmetric periodic spin chain. These parameters are fixed to be solutions of their respective Bethe equations, given in (A.23).
The boundary eigenvalue $\lambda(v; u)$ and Bethe equation for $u$ can then be found by substituting the values for $\lambda_k(v; u)$ from (3.20) into the above expression. These are given explicitly for multiple excitations in (4.6).

4.2. **Bethe vector for multiple excitations.** For multiple excitations the argument proceeds similarly to the previous section. Recall that $m \in \mathbb{N}$ is the excitation number and $u \in \mathbb{C}^m$ is an $m$-tuple of distinct complex parameters. Let $\Phi$, the nested Bethe vector, be a vector from the lowest weight $Y(gl_m)$-module $M'$ defined in (3.19):

$$\Phi \in M' = \mathcal{V}_\alpha \mathcal{V}_\beta \cdots \mathcal{V}_\alpha \mathcal{V}_\beta \cdots \mathcal{V}_\alpha \mathcal{V}_\beta \cdots \mathcal{V}_\alpha \mathcal{V}_\beta \cdots \mathcal{V}_\alpha \mathcal{V}_\beta \mathcal{M}^0.$$ 

The vector $\Phi$ may also depend on the parameters $u$, and we will write $\Phi = \Phi(u)$. From the nested Bethe vector, we make the following ansatz for the full Bethe vector:

$$\Psi(u) := \beta_\alpha \beta_\gamma \cdots \beta_\alpha \beta_\gamma \left( u \right) \cdot \Phi(u) \in M.$$

We now act with the transfer matrix $\tau(v)$ on this Bethe vector. Using Corollary 3.5 we find

$$\tau(v) \cdot \Psi(u) = \beta_\alpha \beta_\gamma \cdots \beta_\alpha \beta_\gamma \left( u \right) \left( p(v) \tau(v; u) \right)^v \cdot \Phi(u) + UWT.$$

The unwanted terms $UWT$ are less simple than in the $m = 1$ case, and will be discussed in detail in the section below. With the expectation that the $u$ may be chosen such that the unwanted terms vanish, the Bethe vector $\Phi(u)$ will be an eigenvector of $\tau(v)$ if we take the additional requirement, as for $m = 1$, that

$$\tau(v) \cdot \Phi(u) = \Gamma(v; u) \cdot \Phi(u).$$

We therefore seek an eigenvector $\Phi(u) \in M'$ of $\tau(v; u)$. This is found again by the algebraic Bethe ansatz for $Y(gl_n)$ with the full quantum space $M'$ and monodromy matrix $\Gamma(v; u)$.

From here, proceeding as we did in the $m = 1$ case,

$$\tau(v) \cdot \Psi(u) = \Lambda(v; u) \Psi(u) + UWT, \quad \text{where} \quad \Lambda(v; u) = \left( p(v) \Gamma(v; u) \right)^v.$$

4.3. **Dealing with unwanted terms.** In this section, we will find an exact expression for the unwanted terms from the action of $\tau(v)$ on the Bethe vector and, by setting these terms to zero, we will obtain the Bethe equations.

We begin by introducing some notation for the unwanted terms. Let $U(v; u)$ denote the terms initially excluded from the expression in

$$\tau(v) \beta_\alpha \beta_\gamma \cdots \beta_\alpha \beta_\gamma (u) = \beta_\alpha \beta_\gamma \cdots \beta_\alpha \beta_\gamma \left( u \right) \left( p(v) \tau(v; u) \right)^v + U(v; u).$$

To find an expression for $U(v; u)$, begin by acting on $\beta_\alpha \beta_\gamma \cdots \beta_\alpha \beta_\gamma (u)$. By repeated applications of (3.7), as in Lemma 3.4, we may move $A_u(\cdot)$ through each of the remaining creation operators in $\beta_\alpha \beta_\gamma \cdots \beta_\alpha \beta_\gamma (u)$, generating a sum of terms in which the rightmost operator is a matrix element of $A_u(\cdot)$ for $u \in \{ v, u_1, \ldots, u_m, -v - \rho, -u_1 - \rho, \ldots, -u_m - \rho \}$. To find a more concise expression for $U(v; u)$, it will be useful to partition the terms by the parameter appearing in $A_u(\cdot)$. Let $E$ denote the subalgebra of $Y^{(+)}(gl_n)$ generated by coefficients of $b_{ij}(u)$ for $1 \leq i, j \leq n$, the closure of which is guaranteed by (2.30). Then

$$U(v; u) = \sum_{j=1}^{m} \left( U_{+,j}(v; u) + U_{-,j}(v; u) \right),$$

where

$$U_{+,j}(v; u) = \sum_{k,l=1}^{n} B_{+,j,kl} a_{kl}(u_j), \quad U_{-,j}(v; u) = \sum_{k,l=1}^{n} B_{-,j,kl} a_{kl}(-u_j - \rho)$$

for some $B_{+,j,kl} \in \mathcal{B} \otimes (\mathbb{C}^n)^{\otimes 2m}$. Additionally, define $U_{1}(v; u) := U_{+,1}(v; u) + U_{-,1}(v; u)$. We will now proceed to find $U_{1}(v; u)$ using the standard techniques. Indeed, consider moving $\tau(v)$ through only the first
\( \tau(v) \beta_{\tilde{a}_1 a_1 \ldots \tilde{a}_m a_m}(u) = \left( \beta_{\tilde{a}_1 a_1}(u_1) \text{tr}_a \left\{ p(v) R^t_{\tilde{a}_1 a_1}(u_1 - v) R^t_{a_1 a_1}(-u_1 - v - \rho) A_a(v) \right\} \right)^v 
+ \frac{1}{p(u_1)} \left\{ \frac{p(v)}{u_1 - v} \beta_{\tilde{a}_1 a_1}(v) \right\}^v 
\cdot \text{Res}_{w \rightarrow u_1} \left\{ p(w) R^t_{\tilde{a}_1 a_1}(u_1 - w) R^t_{a_1 a_1}(-u_1 - w - \rho) A_a(w) \right\}^w 
\times \prod_{i=2}^m \left( \beta_{\tilde{a}_i a_i}(u_i) \prod_{j=i-1}^1 R_{a_j a_i}(-u_j - u_i - \rho) \right). \)

We focus on the second term here, which, upon taking the residue, contains \( A_a(u_1) \) and \( A_a(-u_1 - \rho) \). As all the entries of the \( m \)-tuple \( u \) are distinct, all contributions to \( U_1(v; u) \) must originate from moving \( A_a(u_1) \) and \( A_a(-u_1 - \rho) \) through the remaining creation operators without any further parameter exchanges. Therefore, by repeated applications of Lemma 3.4,

\( U_1(v; u) = \frac{1}{p(u_1)} \left\{ \frac{p(v)}{u_1 - v} \beta_{\tilde{a}_1 a_1}(v) \right\}^v \prod_{i=2}^m \left( \beta_{\tilde{a}_i a_i}(u_i) \prod_{j=i-1}^1 R_{a_j a_i}(-u_j - u_i - \rho) \right) \text{Res}_{w \rightarrow u_1} \left\{ p(w) t(w; u) \right\}^w. \)

It now remains to find similar expressions for \( U_j(v; u) \) for \( 2 \leq j \leq m \). Recall Lemma 3.2. By repeatedly applying such transpositions, we may apply an arbitrary permutation to the parameters \( u \) in the \( m \)-excitation creation operator. For \( \sigma \in S_m \), let \( u_\sigma \) denote the ordered set \( (u_{\sigma(1)}, u_{\sigma(2)}, \ldots, u_{\sigma(m)}) \). Additionally, let \( \sigma_j \) denote the cyclic permutation \( \sigma_j : (1, 2, \ldots, m) \rightarrow (j, j+1, \ldots, 1) \). We have

\[ \beta_{\tilde{a}_1 a_1 \ldots \tilde{a}_m a_m}(u) = \beta_{\tilde{a}_1 a_1 \ldots \tilde{a}_m a_m}(u_\sigma) \tilde{R}_{a_1 \ldots a_m}[\sigma_j](u) \tilde{R}_{\tilde{a}_1 \ldots \tilde{a}_m}[\sigma_j](u) \]

where \( \tilde{R}[\sigma_j](u) \) is the product of \( \tilde{R} \)-matrices that generates this permutation. Using this expression for \( \beta_{\tilde{a}_1 a_1 \ldots \tilde{a}_m a_m}(u) \) and following the argument above, we obtain an expression for \( U_k(v; u) \):

\[ U_k(v; u) = \frac{1}{p(u_k)} \left\{ \frac{p(v)}{u_k - v} \beta_{\tilde{a}_1 a_1}(v) \right\}^v \prod_{i=2}^m \left( \beta_{\tilde{a}_i a_i}(u_{\sigma_k(i)}) \prod_{j=i-1}^1 R_{a_j a_i}(-u_{\sigma_k(j)} - u_{\sigma_k(i)} - \rho) \right) \text{Res}_{w \rightarrow u_k} \left\{ p(w) t(w; u_{\sigma_k}) \right\}^w \tilde{R}_{a_1 \ldots a_m}[\sigma_k](u) \tilde{R}_{\tilde{a}_1 \ldots \tilde{a}_m}[\sigma_k](u). \]

By applying this to the nested Bethe vector, we obtain an expression for all the unwanted terms from the action of \( \tau(v) \) on \( \Psi(u) \). However, in order to obtain the Bethe equations, we must assume one additional property of the nested Bethe vector \( \Phi \). We require \( \Phi = \Phi(u) \) such that:

\[ \tilde{R}_{a_i a_{i+1}}(u_i - u_{i+1}) \tilde{R}_{a_{i+1} a_i}^{-1}(u_i - u_{i+1}) \Phi(u) = \Phi(u_{i+1}) \quad \text{for} \quad 1 \leq i \leq n - 1. \]

By Proposition 3.9 and Lemma 3.11, this is true for a vector generated by acting with the nested monodromy matrix on the lowest vector

\[ \Phi(u) \in \text{span}\{t_{i_1 j_1}(w_1; u) \cdots t_{i_K j_K}(w_K; u) : \eta : K \geq 0, 1 \leq i_1, j_1, \ldots, i_K, j_K \leq n - 1, w \in \mathbb{C}^K \}. \]

The \( \Phi(u) \) constructed using the nested Bethe ansatz for \( Y(\mathfrak{gl}_n) \) will have this form, so we may simultaneously assume (4.5). Note that the action of \( \tilde{R} \)-matrices may be combined with (3.11) in such a way that transpositions of the parameters \( u \) leave the full Bethe vector \( \Psi(u) \) unchanged. Therefore,

\[ \Psi(u) = \Psi(u_\sigma) \quad \text{for all} \quad \sigma \in S_m. \]
Acting on $\Phi(u)$ with the expression for $U_k(v; u)$ above, and summing over $k$, we obtain

$$U(v; u) \cdot \Phi(u) = \sum_{k=1}^{m} \frac{1}{p(u_k)} \left\{ \frac{p(v)}{u_k - v} \beta_{\bar{a}_i a_i}(v) \right\} v \prod_{i=2}^{m} \left( \beta_{\bar{a}_i a_i}(u_{\sigma_k(i)}) \prod_{j=1}^{i-1} R_{a_j \bar{a}_j}(-u_{\sigma_k(j)} - u_{\sigma_k(i)} - \rho) \right) \times \text{Res}_{w \to u_k} \{p(w) t(w; u_{\sigma_k})\}^v \cdot \Phi(u_{\sigma_k})$$

$$= \sum_{k=1}^{m} \frac{1}{p(u_k)} \text{Res}_{w \to u_k} \{p(w) \Gamma(w; u_{\sigma_k})\}^v \times \left\{ \frac{p(v)}{u_k - v} \beta_{\bar{a}_i a_i}(v) \right\} v \prod_{i=2}^{m} \left( \beta_{\bar{a}_i a_i}(u_{\sigma_k(i)}) \prod_{j=1}^{i-1} R_{a_j \bar{a}_j}(-u_{\sigma_k(j)} - u_{\sigma_k(i)} - \rho) \right) \cdot \Phi(u_{\sigma_k}).$$

Note that we may use $\bar{R}$-matrices to permute the parameters in (4.5) to show that $\Gamma(v; u_{\sigma}) = \Gamma(v; u)$ for all $\sigma \in S_m$. The Bethe equations are then extracted by demanding $U(v; u) \cdot \Phi(u) = 0$. Since each summand is independent, we obtain

$$\text{Res}_{w \to u_k} \{p(w) \Gamma(w; u)\}^v = 0 \quad \text{for} \quad 1 \leq k \leq m$$

or, more concisely

$$\text{Res}_{w \to u_k} \Lambda(w; u) = 0 \quad \text{for} \quad 1 \leq k \leq m.$$  \hspace{1cm} (4.7)

4.4. Boundary eigenvalues and Bethe equations. From the algebraic Bethe ansatz for $Y(\mathfrak{gl}_n)$, we have explicit values for the eigenvalues of the nested system, see (A.22),

$$\Gamma(v; u) = \lambda_1(v; u) \prod_{i=1}^{m(n)} \frac{v - u_i^{(1)}}{v - u_i^{(1)} + 1} + \lambda_n(v; u) \prod_{i=1}^{m(n-1)} \frac{v - u_i^{(n-1)}}{v - u_i^{(n-1)} - 1} + \sum_{k=2}^{n-1} \lambda_k(v; u) \prod_{i=1}^{m(k-1)} \frac{v - u_i^{(k-1)}}{v - u_i^{(k-1)} - 1} \prod_{i=1}^{m(k)} \frac{v - u_i^{(k)}}{v - u_i^{(k)} + 1},$$

where $\lambda_k(v; u)$ are given by Proposition 3.9. (Note that the $(i+1)$-th level of nesting for $Y^{\pm}(\mathfrak{gl}_n)$ corresponds to $i$-th level for $Y(\mathfrak{gl}_n)$.) The parameters $u_i^{(k)}$ satisfy the appropriate Bethe equations of $Y(\mathfrak{gl}_n)$ given in (A.23). Substituting our values for $\lambda_k(v; u)$ from (3.20) yields

$$\Gamma(v; u) = \left( \prod_{j=1}^{\ell} \frac{v - c_j - \lambda_j^{(1)}}{v - c_j}, v + \rho - c_j + \lambda_j^{(1)} \right) \left( \prod_{i=1}^{m(n)} \frac{v - u_i^{(1)} + 1}{v - u_i^{(1)}} \right) \left( \frac{v + (\rho \pm 1)/2 - \mu_1}{v + (\rho \pm 1)/2} \right)$$

$$+ \left( \prod_{j=1}^{\ell} \frac{v - c_j - \lambda_n^{(n)}}{v - c_j}, v + \rho + c_j + \lambda_n^{(n)} \right) \left( \prod_{i=1}^{m(n-1)} \frac{v - u_i^{(n-1)} + 1}{v - u_i^{(n-1)}} \right) \left( \frac{v + (\rho \pm 1)/2 - \mu_{n-1}}{v + (\rho \pm 1)/2} \right)$$

$$+ \sum_{k=2}^{n-1} \left( \prod_{j=1}^{\ell} \frac{v - c_j - \lambda_j^{(j)}}{v - c_j}, v + \rho + c_j + \lambda_j^{(j)} \right) \left( \prod_{i=1}^{m(k-1)} \frac{v - u_i^{(k-1)} + 1}{v - u_i^{(k-1)}} \right) \left( \frac{v + (\rho \pm 1)/2 - \mu_{k-1}}{v + (\rho \pm 1)/2} \right).$$

The full eigenvalues $\Lambda(v; u)$ are obtained from these nested ones using (4.6):

$$\Lambda(v; u) = \{p(v) \Gamma(v; u)\}^v = \left( 1 \pm \frac{1}{2v + \rho} \right) \Gamma(v; u) + \left( 1 \mp \frac{1}{2v + \rho} \right) \Gamma(-v - \rho; u).$$
By (4.7), the Bethe equations for \( u \) are found by demanding that the residue of the above eigenvalue vanishes at each of the \( u_k \). Evaluating this residue, we obtain

\[
\frac{u_k + (\rho - 1)/2}{u_k + (\rho + 1)/2} \cdot \frac{\prod_{i \neq k} u_k - u_i - 1}{u_k + u_i + \rho - 1} = \left( \prod_{j=1}^{\ell} \frac{u_k - c_j - \lambda_n^{(j)}}{u_k - c_j - \lambda_{n+1}^{(j)}} \cdot \frac{u_k + \rho + c_j + \lambda_n^{(j)}}{u_k + \rho + c_j + \lambda_{n+1}^{(j)}} \right) \left( \prod_{i=1}^{m(n-1)} \frac{u_k - u_i^{(n-1)} - 1}{u_k - u_i^{(n-1)} + 1} \right),
\]

which, together with (A.23), are the Bethe equations for a \( Y^\pm(\mathfrak{gl}_{2n}) \) system. Note that the condition (A.21) is equivalent to the vanishing of the residue of \( \Lambda(v; u) \) at each of the \( u_i^{(k)} \), which is the expected Bethe equation for a system of equations.

**Remark 4.1.** The eigenvalue \( \Lambda(v; u) \) for a \( Y^\pm(\mathfrak{gl}_{2n}) \) system, when the \( Y^\pm(\mathfrak{gl}_{2n}) \)-evaluation module \( M(\mu) \) in (2.18) is a one-dimensional, was calculated in [ACDFR] by means of the analytical Bethe ansatz. By shifting the roots of the equations and introducing the assumption that the roots come in pairs, one can recover the eigenvalue found in [ACDFR] from (4.8) and (4.9). However, this assumption appears to be unnecessary for the algebraic Bethe ansatz.

**Appendix A. Nested algebraic Bethe ansatz for \( Y(\mathfrak{gl}_n) \)**

In this appendix we give, in full detail, the nested algebraic Bethe ansatz for the Yangian \( Y(\mathfrak{gl}_n) \), first constructed by Kulish and Reshetikhin in [KuRs], to which the algebraic Bethe ansatz for the twisted Yangian \( Y^\pm(\mathfrak{gl}_{2n}) \) reduces. We assume that the full quantum space \( L \) of the system is a tensor product of the evaluation modules \( Y(\mathfrak{gl}_n) \), as defined in (2.9),

\[
L := L(\lambda^{(1)})_{c_1} \otimes L(\lambda^{(2)})_{c_2} \otimes \ldots \otimes L(\lambda^{(\ell)})_{c_\ell},
\]

with the lowest weight \( \lambda(u) \) given by (2.12).

**A.1. Exchange relations.** Consider the Yangian \( Y(\mathfrak{gl}_n) \), as defined in Section 2.2. The \( R \)-matrix is \( R(u) = I - u^{-1} P \in \text{End}(\mathbb{C}^n \otimes \mathbb{C}^n)[[u^{-1}]] \), and the generating matrix \( T_a(u) \in \text{End}(V_a) \otimes Y(\mathfrak{gl}_n)[[u^{-1}]] \); here \( V_a = \mathbb{C}^n \) is an auxiliary space. We will refer to \( T_a(u) \) as the monodromy matrix.

Let \( V'_a = \mathbb{C}^{n-1} \) and \( V^{(k)}_a = \mathbb{C}^{n-k} \) for any \( 0 \leq k < n \), so that \( V_a = V_a^{(0)} \) and \( V'_a = V_a^{(1)} \). We begin by splitting the auxiliary space \( V_a = \mathbb{C} + V'_a \). Accordingly, the monodromy matrix \( T_a(u) \) splits as follows:

\[
T_a(u) = \begin{pmatrix}
\alpha_a(u) & B_a(u) \\
C_a(u) & D_a(u)
\end{pmatrix},
\]

where \( \alpha_a(u) = t_{11}(u) \) and

\[
B_a(u) = (t_{12}(u), \ldots, t_{1n}(u)) \quad \in (V'_a)^* \otimes Y(\mathfrak{gl}_n)[[u^{-1}]],
\]

\[
C_a(u) = (t_{21}(u), \ldots, t_{n1}(u))^T \quad \in V'_a \otimes Y(\mathfrak{gl}_n)[[u^{-1}]],
\]

\[
D_a(u) = \begin{pmatrix}
t_{22}(u) & \ldots & t_{2n}(u) \\
\vdots & \ddots & \vdots \\
t_{n2}(u) & \ldots & t_{nn}(u)
\end{pmatrix} \quad \in \text{End}(V'_a) \otimes Y(\mathfrak{gl}_n)[[u^{-1}]].
\]

In particular, \( B_a(u) \) is a row-vector and \( C_a(u) \) is a column-vector. It will be convenient to denote the matrix entries of \( B_a(u) \) by \( b_{ij}(u) \) with \( 1 \leq i \leq n-1 \), and similarly for \( C_a(u) \) and \( D_a(u) \). Additionally, we introduce a reduced \( R \)-matrix \( R'(u) \) acting on \( \mathbb{C}^{n-1} \otimes \mathbb{C}^{n-1} \),

\[
R'(u) := I - u^{-1} \sum_{i,j=1}^{n-1} e'_{ij} \otimes e'_{ji} = I - u^{-1} P'.
\]
The defining relations of \( Y(\mathfrak{gl}_n) \) imply the following exchange relations for \( a(v), B_a(v) \) and \( D_a(v) \):

\[
\begin{align*}
A.2 & \quad a(v)B_a(u) = \frac{v-u+1}{v-u}B_a(u)a(v) - \frac{1}{v-u}B_a(v)a(u), \\
A.3 & \quad D_a(v)B_a(u) = B_a(u)D_a(v)R'_{a1}(v-u) + \frac{1}{v-u}B_a(v)D_a(u)P'_{a1}, \\
A.4 & \quad B_a(v)B_a(u) = \frac{v-u}{v-u-1}B_a(u)B_a(v)R'_{a2}(v-u),
\end{align*}
\]

along with an RTT relation

\[
A.5 \quad R'_{a1a2}(u-v)D_a(u)D_a(v) = D_a(v)D_a(u)R'_{a1a2}(u-v).
\]

In particular, the coefficients of the matrix entries of \( D_a(v) \) generate a subalgebra \( Y(\mathfrak{gl}_{n-1}) \subset Y(\mathfrak{gl}_n) \) (note that this is not a Hopf subalgebra). Two additional relations will be used, which can be stated more clearly in terms of individual matrix entries of \( T_a(u) \). For any \( 1 \leq i,j,k \leq n-1 \),

\[
A.6 \quad c_k(u) d_{ij}(v) = d_{ij}(v) c_k(u) - \frac{1}{u-v}(d_{kj}(u)c_i(v) - d_{kj}(v)c_i(u)),
\]

\[
A.7 \quad [a(v), d_{ij}(u)] = \frac{1}{v-u}(\delta_{ij}(u)c_i(v) - \delta_{ij}(v)c_i(u)).
\]

A.2. **Technical identities.** We now use the exchange relations stated above to establish algebraic relations that will be important in the nested algebraic Bethe ansatz.

Let \( m \in \mathbb{N} \). Introduce \( u := (u_1, \ldots, u_m) \), an \( m \)-tuple of distinct complex parameters, and \( V_{a_1'}, \ldots, V_{a_m'} \), copies of \( V_{a'} \). The creation operator for \( m \) excitations is

\[
\mathcal{B}_{a_1 \ldots a_m}(u) := B_{a_1}(u_1) \cdots B_{a_m}(u_m).
\]

The operator \( \mathcal{B}_{a_1 \ldots a_m}(u) \) is a row-vector in \( (V'_{a_1})^* \otimes \cdots \otimes (V'_{a_m})^* \) with entries in \( Y(\mathfrak{gl}_n) \). The parameters carried by \( \mathcal{B}_{a_1 \ldots a_m}(u) \) may be exchanged by the braided \( R \)-matrix defined by

\[
A.8 \quad \tilde{R}'(u) := \frac{u}{u-1}R'(u)P'.
\]

This \( R \)-matrix allows us to rewrite (A.4) in a more elegant form,

\[
B_{a_1}(u_1)B_{a_2}(u_2) = B_{a_1}(u_2)B_{a_2}(u_1)\tilde{R}'_{a_1a_2}(u_1-u_2).
\]

Consequently, for \( m \) excitations, we have that

\[
A.9 \quad \mathcal{B}_{a_1 \ldots a_m}(u) = \mathcal{B}_{a_1 \ldots a_{m-i}}(u_{i+1})\tilde{R}'_{a_i,a_{i+1}}(u_i-u_{i+1}) \quad \text{for} \quad 1 \leq i \leq m-1,
\]

where we used the notation (3.10).

We now move the \( a(v) \) and \( D_a(v) \) operators through the \( m \)-excitation creation operator. Consider first the action of \( a(v) \) on \( \mathcal{B}_{a_1 \ldots a_m}(u) \), for \( v \notin u \),

\[
a(v)\mathcal{B}_{a_1 \ldots a_m}(u) = \left( \frac{v-u+1}{v-u_1}B_{a_1}(u_1)a(v) - \frac{1}{v-u_1}B_{a_1}(v)a(u_1) \right)B_{a_2}(u_2) \cdots B_{a_m}(u_m).
\]

Note that we may repeatedly apply this relation to move \( a(v) \) through each of the excitations, resulting in a sum of \( 2^m \) terms in which \( a(\cdot) \) is the rightmost operator. From this sum we note that there is a unique term in which \( a(v) \) retains its parameter each time we apply (A.2). We will refer to this term as the wanted term, and the other terms as unwanted terms (UWT). Then,

\[
A.10 \quad a(v)\mathcal{B}_{a_1 \ldots a_m}(u) = \prod_{i=1}^m \frac{v-u_i+1}{v-u_i}\mathcal{B}_{a_1 \ldots a_m}(u)a(v) + UWT.
\]

The unwanted terms will be discussed in detail in Section A.4.

The wanted term for the action of \( D_a(v) \) on \( \mathcal{B}_{a_1 \ldots a_m}(u) \) is found similarly. From repeated applications of (A.3),

\[
D_a(v)\mathcal{B}_{a_1 \ldots a_m}(u) = \mathcal{B}_{a_1 \ldots a_m}(u)D_a(v)R'_{aa_m}(v-u_m) \cdots R'_{aa_1}(v-u_1) + UWT.
\]
Here note that the rightmost matrix acting on the auxiliary space $V'_a$ is
\[(A.11) \quad T'_a|_{a_1\ldots a_m} (v; u) := D_a(v) R'_{u a_m} (v - u_m) \cdots R'_{u a_1} (v - u_1).\]
We will refer to this matrix as the *nested monodromy matrix*. The nontrivial action on the auxiliary spaces $V'_1, \ldots, V'_m$ will often be omitted for clarity, and we will write instead simply $T'_a (v; u)$. Using this notation we get
\[(A.12) \quad D_a(v) B_{a_1 \ldots a_m} (u) = B_{a_1 \ldots a_m} (u) T'_a(v; u) + U W T.\]

The nested monodromy matrix satisfies the following properties.

**Lemma A.1.** The matrix $T'_a(v; u)$ satisfies the RTT relation,
\[R'_{ab} (v - w) T'_a(v; u) T'_b(w; u) = T'_a(w; u) T'_b(v; u) R'_{ab} (v - w).\]

**Proof.** Starting from the l.h.s. of the equation and using the definition (A.11) of $T'_a(v; u)$,
\[l.h.s. = R'_{ab} (v - w) D_a(v) R'_{u a_m} (v - u_m) \cdots R'_{u a_1} (v - u_1) D_b(w) R'_{b a_m} (w - u_m) \cdots R'_{b a_1} (w - u_1)\]
\[= R'_{ab} (v - w) D_a(v) D_b(w) R'_{u a_m} (v - u_m) R'_{b a_m} (w - u_m) \cdots R'_{u a_1} (v - u_1) R'_{b a_1} (w - u_1)\]
\[= D_b(w) D_a(v) R'_{u a_m} (v - u_m) R'_{b a_m} (w - u_m) \cdots R'_{u a_1} (v - u_1) R'_{b a_1} (w - u_1) \text{ by (A.5)}\]
\[= D_b(w) D_a(v) R'_{b a_m} (w - u_m) R'_{a a_m} (v - u_m) \cdots R'_{b a_1} (w - u_1) R'_{a a_1} (v - u_1) R'_{a b} (v - w) \text{ by YBE}\]
\[= T'_a(w; u) T'_b(v; u) R'_{ab} (v - w).\]

By the above Lemma, the matrix $T'_a(v; u)$ is a homomorphic image of the generating matrix $T'_a(v)$ of $Y(\mathfrak{gl}_{n-1})$. We may use $R$-matrices $R'$ to exchange the ordering of complex parameters in $u$.

**Lemma A.2.** Matrix elements $t'_{ij}(v; u)$ of $T'_a(v; u)$ satisfy the relation:
\[R'_{a_i a_{i+1}} (u_i - u_{i+1}) t'_{jk}(v; u) = t'_{jk}(v; u_i u_{i+1}) R'_{a_i a_{i+1}} (u_i - u_{i+1}).\]

**Proof.** Moving $R'_{a_i a_{i+1}} (u_i - u_{i+1})$ from left to right through each of the $R$-matrices in the definition (A.11), the $R$-matrices with which it does not commute will undergo parameter exchange $u_i \leftrightarrow u_{i+1}$ due to the (braided) Yang-Baxter equation:
\[R'_{a_i a_{i+1}} (u_i - u_{i+1}) R'_{a_{i+1} a_i} (v - u_i) = R'_{a_{i+1} a_i} (v - u_i) R'_{a_i a_{i+1}} (u_i - u_{i+1}).\]

The required identity is now immediate.

We now construct a finite-dimensional vector space, called the *nested vacuum sector*, which the matrix $T'_a(v; u)$ will act on. Denote by $L(\lambda^{(i)})^{0}_{c_i}$ the subspace of the $Y(\mathfrak{gl}_n)$-evaluation module $L(\lambda^{(i)})_{c_i}$ consisting of vectors annihilated by all operators $e_j(u)$, namely
\[L(\lambda^{(i)})^{0}_{c_i} := \{ \zeta \in L(\lambda^{(i)})_{c_i} : e_j(u) \zeta = 0 \text{ for } 1 \leq j \leq n - 1 \}.\]
This subspace corresponds to the natural embedding $\mathfrak{gl}_{n-1} \subset \mathfrak{gl}_n$ and is an irreducible lowest weight $Y(\mathfrak{gl}_{n-1})$-module with the lowest weight given by
\[(A.13) \quad \lambda_i(u)^0 = \lambda_{i+1}(u) \text{ for } 1 \leq i \leq n - 1\]
and $\lambda_i(u)$ defined in (2.12).

We define the *vacuum sector* $L^{0} \subset L$ by
\[L^{0} = L(\lambda^{(1)})^{0}_{c_1} \otimes L(\lambda^{(2)})^{0}_{c_2} \otimes \ldots \otimes L(\lambda^{(n-1)})^{0}_{c_{n-1}}.\]
By the initial assumption, the space $L$ is an irreducible $Y(\mathfrak{gl}_n)$-module. Then, by Lemma 6.2.2 and Theorem 6.5.8 in [Mo3], the space $L^{0}$ is an irreducible $Y(\mathfrak{gl}_{n-1})$-module. In particular, the space $L^{0}$ is annihilated by all operators $e_i(u)$,
\[L^{0} = \{ \zeta \in L : e_i(u) \cdot \zeta = 0 \text{ for } 1 \leq i \leq n - 1 \},\]
and is stable under the action of the operators $d_{ij}(u)$ for $1 \leq i, j \leq n - 1$, see (A.6).
Each auxiliary space $V_{a_i}'$ is a vector representation of the Lie algebra $\mathfrak{gl}_{n-1}$ of weight $\lambda' = (1, 0, \ldots, 0)$ and may be viewed as an evaluation module $L(\lambda')_{a_i}$ of $Y(\mathfrak{gl}_{n-1})$ with the lowest weight given by

$$\lambda'_i(u) = \frac{u - u_i - 1}{u - u_i} \quad \text{and} \quad \lambda'_j(u) = 1 \quad \text{for} \quad 2 \leq j \leq n - 1.$$  

In particular, the generating matrix $T_a'(u)$ of $Y(\mathfrak{gl}_{n-1})$ acts on $L(\lambda')_{a_i}$ as $R'_{a_i}(u - u_i)$.

We have now all the necessary ingredients to define the nested vacuum sector

$$L' := L^0 \otimes V_{a_m} \otimes \cdots \otimes V_{a_1}.$$

**Proposition A.3.** Let $T'(v)$ denote the generating matrix of $Y(\mathfrak{gl}_{n-1})$. Then the map

$$Y(\mathfrak{gl}_{n-1}) \rightarrow Y(\mathfrak{gl}_n) \otimes \text{End}(V_{a_m} \otimes \cdots \otimes V_{a_1}), \quad T'(v) \mapsto T'(v; u)$$

is a homomorphism of algebras. Moreover, it equips the space $L'$ with a structure of a lowest weight $Y(\mathfrak{gl}_{n-1})$-module with the lowest weight given by

$$\lambda'_1(v; u) = \prod_{j=1}^\ell \frac{v - \lambda_j^{(j)} - c_j - 1}{v - \lambda_j^{(j)} - c_j} \prod_{k=1}^m \frac{v - u_k - 1}{v - u_k} \quad \text{and}$$

$$\lambda'_i(v; u) = \prod_{j=1}^\ell \frac{v - \lambda_{i-1}^{(j)} - c_j - 1}{v - \lambda_{i-1}^{(j)} - c_j} \quad \text{for} \quad 2 \leq i \leq n - 1.$$  

**Proof.** The homomorphism property follows from Lemma A.1. We already know that $L^0$ is an irreducible $Y(\mathfrak{gl}_{n-1})$-module. It follows from (A.11) and (A.15) that the space $L'$ is stable under the action of $T_a'(v; u)$. Thus the map (A.16) equips the space $L'$ with a structure of a lowest weight $Y(\mathfrak{gl}_{n-1})$-module. The lowest vector is

$$\eta = \eta_1 \otimes \cdots \otimes \eta_\ell \otimes e_1' \otimes \cdots \otimes e_1',$$

where each $\eta_i$ is a lowest vector of $L(\lambda_i^{(i)})_{c_i}$ for $1 \leq i \leq \ell$ and each $e_1'$ is a lowest vector of $V_{a_i}$ for $1 \leq i \leq m$ (viewed as an evaluation module $L(\lambda')_{a_i}$). Finally, acting with $t_{t_i}'; v; u)$ on $\eta$ for $1 \leq i \leq n$ and using (A.13) and (A.14) yields (A.17).  

**Lemma A.4.** For any vector $\zeta \in L'$ we have that

$$a_i(u) \cdot \zeta = \lambda_1(u) \zeta,$$

where $\lambda_1(u)$ is defined by (2.12).

**Proof.** By Proposition A.3 we know that $L' = Y(\mathfrak{gl}_{n-1})\eta$ for $\eta$ defined in (A.18) and $c_i(u) \cdot L' = 0$. Using (A.7) and definition of $t_{t_i}'(v; u)$, we find that $[a_i(u), t_{t_i}'(v; u)] \cdot \zeta = 0$ for any $1 \leq i, j \leq n$. Hence it is enough to act with $a_i(u)$ on the lowest vector $\eta$, which yields the required result.

### A.3. Nested algebraic Bethe ansatz.

We are now ready to consider the nested algebraic Bethe ansatz for a $Y(\mathfrak{gl}_n)$ system. The monodromy matrix $T_a(v)$ acts on the space $L$ in (A.1) by

$$T_a(v) \cdot L = \left( \prod_{i=1}^\ell L_{a_i}(v - c_i) \right) L \in \text{End}(\mathbb{C}^n) \otimes L[[v^{-1}]].$$

The transfer matrix is defined as

$$t(v) := \text{tr}_a T_a(v) \in Y(\mathfrak{gl}_n)[[v^{-1}]].$$

Taking the trace of the RTT relation reveals that $[t(v), t(u)] = 0$, and so $t(v)$ is a generating series for conserved quantities. We diagonalise $t(v)$ by means of the nested algebraic Bethe ansatz, adhering closely to [KuRs]. In particular, we construct an ansatz for our eigenvector recursively, at each stage reducing the diagonalisation problem to a similar problem with a smaller symmetry algebra, relying on the chain of subalgebras $Y(\mathfrak{g}_n) \supset Y(\mathfrak{g}_{n-1}) \supset \cdots \supset Y(\mathfrak{g}_2)$ and the irreducibility criterion of a tensor product of evaluation modules.
Recall the definition of the full quantum space \((A.1)\) and the nested vacuum sector \((A.15)\). Let \(\Phi' \in L'\).
We will refer to this as the \textit{nested Bethe vector}, imposing additional properties in a later section.  The ansatz for the eigenvector of the transfer matrix, the \textit{Bethe vector}, is given by
\[
\Phi(u) := \mathcal{B}_{a_1 \ldots a_m}(u) \cdot \Phi' \in L.
\]
To find the eigenvalues and Bethe equations, we act on the Bethe vector with \(t(v) = \phi(v) + \text{tr}_u D_u(v)\). Using \((A.10)\) and \((A.12)\) we write
\[
t(v) \cdot \Phi(u) = \mathcal{B}_{a_1 \ldots a_m}(u) \left( \prod_{i=1}^m \frac{v - u_i + 1}{v - u_i} \phi(v) + \text{tr}_u T_u'(v; u) \right) \cdot \Phi' + UWT.
\]
By Lemma A.4, \(\phi(v) \cdot \Phi' = \lambda_1(v) \Phi'\). We now impose that \(\Phi'\) is an eigenvector of the nested transfer matrix \(t'(v; u) := \text{tr}_u T_u'(v; u)\), with the eigenvalue \(\Gamma'(v; u)\), namely
\[
t'(v; u) \cdot \Phi' = \Gamma'(v; u) \Phi'.
\]
With this condition, the full action of the transfer matrix is diagonal, plus unwanted terms,
\[
t(v) \cdot \Phi(u) = \Gamma(v; u) \Phi(u) + UWT, \quad \text{where} \quad \Gamma(v; u) = \lambda_1(v) \prod_{i=1}^m \frac{v - u_i + 1}{v - u_i} + \Gamma'(v; u).
\]
Finding \(\Phi'\) satisfying \((A.19)\) defines another transfer matrix diagonalisation problem, namely for the Yangian \(Y(\mathfrak{gl}_{n-1})\). The monodromy matrix in this case is given by \(T_u'(v; u)\) and the full quantum space is the lowest weight \(Y(\mathfrak{gl}_{n-1})\)-module \(L'\) defined in \((A.15)\), so the problem may again be reduced by means of the nested algebraic Bethe ansatz; this is ensured by Proposition A.3. For example, constructing the ansatz for the nested Bethe vector, we fix \(m' \in \mathbb{N}\) and introduce an \(m'\)-tuple \(u' = (u_1', \ldots, u_{m'}')\) of distinct complex parameters, so that
\[
\Phi' = \Phi'(u'; u) = B_{a_1'}(u_1'; u) \cdots B_{a_{m'}}(u_{m'}'; u) \cdot \Phi''
\]
where, upon decomposing the nested transfer matrix \(T'(v, u)\) in the same way as we did for \(T(v)\),
\[
\Phi'' \in L'^0 \otimes V''_{a_{m'}} \otimes \cdots \otimes V''_{a_1'}.
\]
Here \(L'^0\) is the vacuum sector of \(L'\) defined analogously to that of \(L\), and each \(V''_{a_i'}\) is a \(\mathfrak{gl}_{n-2}\)-module of weight \(\lambda'' = (1, 0, \ldots, 0)\). Repeating this process, we reduce the problem to a \(Y(\mathfrak{gl}_2)\) system, the solution of which is well known, see e.g. [FdTk] or [Bx].

The transfer matrix will therefore act diagonally on \(\Phi(u)\) if all the unwanted terms vanish. We will show in the next section how this requirement leads to a set of Bethe equations for \(u\).

A.4. \textit{Dealing with unwanted terms}. Recall \((A.10)\) and \((A.12)\). Introduce the following notation for the unwanted terms:
\[
\phi(v) \mathcal{B}_{a_1 \ldots a_m}(u) = \prod_{i=1}^m \frac{v - u_i + 1}{v - u_i} \mathcal{B}_{a_1 \ldots a_m}(u) \phi(v) + U_1(v; u),
\]
\[
\text{tr}_u D_u(v) \mathcal{B}_{a_1 \ldots a_m}(u) = \mathcal{B}_{a_1 \ldots a_m}(u) \text{tr}_u T_u'(v; u) + U_2(v; u).
\]
By applying \((A.2)\) repeatedly, we obtain an expression for \(U_1(v; u)\) as a sum of \(2^m\) terms, in each of which \(\phi(\cdot)\) is the rightmost operator. This inspires a further partition of the unwanted terms. Let \(\mathcal{B}\) denote the subalgebra of \(Y(\mathfrak{gl}_n)\) generated by the coefficients of the series \(b_i(u)\) for \(1 \leq i \leq n - 1\), whose closure is guaranteed by \((A.4)\). We decompose the unwanted terms \(U_1(v; u)\) and \(U_2(v; u)\) by
\[
U_1(v; u) = \sum_{j=1}^m U_{1,j}(v; u) \quad \text{such that} \quad U_{1,j}(v; u) = B_{1,j} \phi(u_j) \quad \text{and}
\]
\[
U_2(v; u) = \sum_{j=1}^m U_{2,j}(v; u) \quad \text{such that} \quad U_{2,j}(v; u) = \sum_{k,l=1}^{n-1} B_{2,j;kl} d_{kl}(u_j)
\]
for some \(B_{1,j}, B_{2,j;kl} \in \mathcal{B} \otimes (\mathbb{C}^{n-1})^\otimes m[[v^{-1}]]\).
To find $U_{1,1}(v; u)$, we begin by acting on $\mathfrak{B}_{a_1...a_m}(u)$ with $\phi(v)$. From (A.2), we have

$$\phi(v)\mathfrak{B}_{a_1...a_m}(u) = \left(\frac{v-u_1+1}{v-u_1}B_{a_1}(u_1)\phi(v) - \frac{1}{v-u_1}B_{a_1}(v)\phi(u_1)\right)B_{a_2}(u_2)\cdots B_{a_m}(u_m).$$

Now, moving $\phi(v)$ through the remaining creation operators, we note that the only contribution to $U_{1,1}(v; u)$ will be from the second term in the above expression, in the instance when there are no further parameter swaps in the remaining commutators. Therefore,

$$U_{1,1}(v; u) = -\frac{1}{v-u_1} \prod_{j=2}^{m} \frac{v-u_j+1}{v-u_j} B_{a_1}(v)B_{a_2}(u_2)\cdots B_{a_m}(u_m)\phi(u_1).$$

We can find $U_{2,1}(v; u)$ in a similar way. Acting with $\text{tr}_a D_a(v)$ on $\mathfrak{B}_{a_1...a_m}(u)$,

$$\text{tr}_a D_a(v)\mathfrak{B}_{a_1...a_m}(u) = \left(B_{a_1}(u_1)\text{tr}_a D_a(v)R'_{a_1a_1}(v-u_1) - \frac{1}{v-u_1}B_{a_1}(v)\text{tr}_a D_a(u_1)P'_{a_1a_1}\right)$$

$$\times B_{a_2}(u_2)\cdots B_{a_m}(u_m).$$

As above, we move the $D_a(u_1)$ operator through the remaining creation operators, and the only contribution to $U_{2,1}(v; u)$ is

$$U_{2,1}(v; u) = -\frac{1}{v-u_1} B_{a_1}(v)B_{a_2}(u_2)\cdots B_{a_m}(u_m) \text{Res} \text{tr}_a D_a(w)R'_{a_1a_1}(w-u_1)R'_{a_2a_2}(w-u_2)R'_{a_3a_3}(w-u_3)\cdots R'_{a_ma_m}(w-u_m).$$

Note that the operators to the right of $D_a(u_1)$ act trivially on $L$, so this is indeed in the correct form. It will be useful to rewrite this in terms of a residue as follows

$$U_{2,1}(v; u) = \frac{1}{v-u_1} B_{a_1}(v)B_{a_2}(u_2)\cdots B_{a_m}(u_m) \text{Res} \text{tr}_a D_a(w)R'_{a_1a_1}(w-u_1)R'_{a_2a_2}(w-u_2)R'_{a_3a_3}(w-u_3)\cdots R'_{a_ma_m}(w-u_m).$$

Proceeding this way we can find the remaining unwanted terms. Consider the relation (A.9). We may use this relation to transpose the parameters $u$ and, since the transpositions generate the symmetric group $\mathfrak{S}_m$, we can apply an arbitrary permutation to the parameters prior to acting with $\phi(v)$. Indeed for $\sigma \in \mathfrak{S}_m$, let $u_\sigma$ denote the ordered set $(u_{\sigma(1)}, u_{\sigma(2)}, \ldots, u_{\sigma(m)})$. Additionally, let $\sigma_j$ denote the cyclic permutation $\sigma_j : (1, 2, \ldots, m) \mapsto (j, j+1, \ldots, 1).$ We have

$$\mathfrak{B}_{a_1...a_m}(u) = \mathfrak{B}_{a_1...a_m}(u_{\sigma_j})R'_{a_1...a_m}[\sigma_j](u),$$

where $R'_{a_1...a_m}[\sigma_j](u)$ is the product of $R'$-matrices required to realise this permutation. Acting now with $\phi(v)$ on the r.h.s. and following the argument above, we obtain an exact expression for $U_{1,j}(v; u)$, namely

$$U_{1,j}(v; u) = -\frac{1}{v-u_1} \prod_{k \neq j} \frac{u_j-u_k+1}{u_j-u_k} \mathfrak{B}_{a_1...a_m}(u_{\sigma_j,u_j \mapsto v})\phi(u_j)R'_{a_1...a_m}[\sigma_j](u),$$

where $\mathfrak{B}_{a_1...a_m}(u_{\sigma_j,u_j \mapsto v}) = B_{a_1}(v)B_{a_2}(u_2+1)\cdots B_{a_m}(u_m-1)$. The expression for $U_{2,j}(v; u)$ obtained by the same method. Indeed,

$$U_{2,j}(v; u) = -\frac{1}{v-u_j} \mathfrak{B}_{a_1...a_m}(u_{\sigma_j,u_j \mapsto v}) \text{Res} \text{tr}'_{w-u_1} t'(w; u_{\sigma_j})R'_{a_1...a_m}[\sigma_j](u).$$

Having found these expressions, the next step is to act with them on the nested Bethe vector to find the full expression for the action of the transfer matrix on the Bethe vector. However, we will first make an assumption about the form of $\Phi'(u'; u)$, namely

$$R'_{a_1a_1}(u_i - u_{i+1}) \Phi'(u'; u) = \Phi'(u'; u_{i+i+1}) \quad \text{for} \quad 1 \leq i \leq m - 1.$$  

This may be achieved if the nested Bethe vector is of the form

$$\Phi'(u'; u) \in \text{span}_{\mathcal{C}} \left\{ t_{i_1j_1}(w_1; u) \cdots t_{i_Kj_K}(w_K; u) \cdot \eta' : K \geq 0, 1 \leq i_1, j_1, \ldots, i_K, j_K \leq n - 1, w \in \mathbb{C}_K \right\},$$

where $\eta'$ is a lowest weight vector of $L'$. Indeed, for any such vector we may use Lemma A.2 to move $R'_{a_1a_1}(u_i - u_{i+1})$ through the $t_{i_kj_k}(w_k; u)$, exchanging $u_i$ with $u_{i+1}$. Then, by definition and (A.8), $\eta'$ is an
eigenvector of $\hat{R}_{a_i,a_{i+1}}(u_i - u_{i+1})$ with eigenvalue 1. The nested Bethe vector constructed using the nested algebraic Bethe ansatz is exactly of this form.

Note that this, combined with the relation (A.9), gives the parameter symmetry of the full Bethe vector

$$\Phi(u) = \Phi(u_\sigma)$$

for all $\sigma \in S_m$.

Applying the expressions for the unwanted terms to the nested Bethe vector, noting (A.19) and Lemma A.4,

$$U_1(v; u) \cdot \Phi'(u'; u) = -\frac{1}{v - u_j} \lambda_j(u_j) \prod_{k \neq j} \frac{u_j - u_k + 1}{u_j - u_k} \mathcal{R}_{a_1 \ldots a_m}(u_{\sigma_j,u_j+v}) \cdot \Phi'(u'; u_{\sigma_j});$$

$$U_2(v; u) \cdot \Phi'(u'; u) = -\frac{1}{v - u_j} \operatorname{Res}_{w \to u_j} \Gamma'(w; u_{\sigma_j}) \mathcal{R}_{a_1 \ldots a_m}(u_{\sigma_j,u_j+v}) \cdot \Phi'(u'; u_{\sigma_j}).$$

In fact, by acting with $\hat{R}'$-matrices on $\Gamma'(v; u) \cdot \Phi'(u'; u)$, we obtain $\Gamma'(v; u) = \Gamma'(v; u_\sigma)$. Putting everything together, we have

$$U_1(v; u) + U_2(v; u) = -\frac{1}{v - u_j} \lambda_j(u_j) \prod_{k \neq j} \frac{u_j - u_k + 1}{u_j - u_k} + \operatorname{Res}_{w \to u_j} \Gamma'(w; u) \mathcal{R}_{a_1 \ldots a_m}(u_{\sigma_j,u_j+v}) \cdot \Phi'(u'; u_{\sigma_j}).$$

From (A.20), $\Phi(u)$ is an eigenvector of the transfer matrix $t(v)$ if the parameters $u$ are chosen such that the above expression vanishes. Since each summand is independent, we require

$$\operatorname{Res}_{w \to u_j} \Gamma(w; u) = 0 \quad \text{for} \quad 1 \leq j \leq m.$$

These are the Bethe equations for $u$.

A.5. End of recursion. Upon reducing to the residual $Y(g\mathfrak{gl}_2)$ system, we have the familiar $2 \times 2$ monodromy matrix

$$T_a^{(n-2)}(v) = \begin{pmatrix} a^{(n-2)}(v) & b^{(n-2)}(v) \\ c^{(n-2)}(v) & d^{(n-2)}(v) \end{pmatrix}.$$ Dependence on parameters $u, u', \ldots, u^{(n-3)}$ has been suppressed. The RTT relation yields the relations

$$a^{(n-2)}(v) b^{(n-2)}(u) = \frac{v - u + 1}{v - u} b^{(n-2)}(u) a^{(n-2)}(v) - \frac{1}{v - u} b^{(n-2)}(v) a^{(n-2)}(u),$$

$$d^{(n-2)}(v) b^{(n-2)}(u) = \frac{v - u - 1}{v - u} b^{(n-2)}(u) d^{(n-2)}(v) + \frac{1}{v - u} b^{(n-2)}(v) d^{(n-2)}(u),$$

$$[b^{(n-2)}(v), b^{(n-2)}(u)] = 0.$$ The Bethe vector with $m^{(n-2)}$ excitations is

$$\Phi^{(n-2)}(u) = b^{(n-2)}(u_1^{(n-2)}) \cdots b^{(n-2)}(u_{m^{(n-2)}}^{(n-2)}) \cdot \eta^{(n-2)},$$

where $\eta^{(n-2)}$ is a lowest vector of the nested vacuum sector $L^{(n-2)}$. The associated eigenvalue of the transfer matrix $t^{(n-2)}(v)$ is

$$\Gamma^{(n-2)}(v; u, \ldots, u^{(n-2)}) = \lambda_1^{(n-2)}(v; u, \ldots, u^{(n-2)}) \prod_{i=1}^{m^{(n-2)}} \frac{v - u^{(n-2)} + 1}{v - u^{(n-2)}} + \lambda_2^{(n-2)}(v; u, \ldots, u^{(n-2)}) \prod_{i=1}^{m^{(n-2)}} \frac{v - u^{(n-2)} - 1}{v - u^{(n-2)}},$$

provided the $u^{(n-2)}$ satisfy the Bethe equations

$$\operatorname{Res}_{w \to u_j^{(n-2)}} \Gamma^{(n-2)}(w; u, \ldots, u^{(n-2)}) = 0 \quad \text{for} \quad 1 \leq j \leq m^{(n-2)}.$$
A.6. Full expressions for eigenvalues and Bethe equations. In this section, we unpack the recursion steps to give the explicit expressions for the eigenvalues of the transfer matrix in terms of the parameters of the $Y(gl_n)$ system.

In order to match the notation used in the Bethe ansatz for the $Y^\pm(gl_{2n})$ chain, we begin by relabelling the spectral parameters as follows. For the initial step, relabel parameters $u_j \rightarrow u_j^{(1)}$ and excitation number $m \rightarrow m^{(1)}$, and for subsequent levels of nesting $u_i^{(k)} \rightarrow u_i^{(k+1)}$ and $m^{(k)} \rightarrow m^{(k+1)}$.

Begin by using Proposition A.3 to rewrite the weights $\lambda_i^{(k)}(v; u, \ldots, u^{(k-1)})$ of the nested system in terms of the weights of the initial $Y(gl_n)$ system,

$$\lambda_i^{(k)}(v; u^{(1)}, \ldots, u^{(k)}) = \lambda_{k+1}(v) \prod_{i=1}^{m^{(k)}} \frac{v - u_j^{(k)} - 1}{v - u_j^{(k)}} \quad \text{for} \quad 1 \leq k \leq n - 1,$$

$$\lambda_i^{(k)}(v; u^{(1)}, \ldots, u^{(k)}) = \lambda_{k+l}(v) \quad \text{for} \quad l > 1, \quad 1 \leq k \leq n - l.$$

From the recursion relation in (A.20), a general expression can be found for $\Gamma^{(k)}(v; u^{(1)}, \ldots, u^{(n-1)})$, for $1 \leq k \leq n - 2$:

$$\Gamma^{(k)}(v; u^{(1)}, \ldots, u^{(n-1)}) = \lambda_2^{(n-2)}(v; u^{(1)}, \ldots, u^{(n-2)}) \prod_{i=1}^{m^{(n-1)}} \frac{v - u_i^{(n-1)} - 1}{v - u_i^{(n-1)}} + \sum_{l=k}^{n-2} \lambda_1^{(l)}(v; u^{(1)}, \ldots, u^{(l)}) \prod_{i=1}^{m^{(l)}} \frac{v - u_i^{(l)} - 1}{v - u_i^{(l)}} \prod_{i=1}^{m^{(l+1)}} \frac{v - u_i^{(l+1)} + 1}{v - u_i^{(l+1)}}.$$

Additionally, the eigenvalue for the transfer matrix is given by

$$\Gamma(v) = \lambda_1(v) \prod_{i=1}^{m^{(1)}} \frac{v - u_i^{(1)} + 1}{v - u_i^{(1)}} + \lambda_n(v) \prod_{i=1}^{m^{(n-1)}} \frac{v - u_i^{(n-1)} - 1}{v - u_i^{(n-1)}}$$

$$+ \sum_{i=1}^{n-2} \lambda_{i+1}(v) \prod_{i=1}^{m^{(i)}} \frac{v - u_i^{(i)} - 1}{v - u_i^{(i)}} \prod_{i=1}^{m^{(i+1)}} \frac{v - u_i^{(i+1)} + 1}{v - u_i^{(i+1)}}.$$

Recall also the Bethe equations (A.21) satisfied by parameters $u_j^{(k)}$. In fact, comparing the above two expressions, we note that equivalent Bethe equations can be obtained by demanding instead that the residue of the full eigenvalue $\Gamma(v)$ vanishes at each $u_j^{(k)}$ for $1 \leq k \leq n - 1, 1 \leq j \leq m^{(k)}$. This is exactly the condition that the eigenvalue of the transfer matrix is analytic. We may now evaluate the residue to obtain the Bethe equations in terms of $\lambda_i(v)$ with $1 \leq k \leq n$:

$$\frac{\lambda_k(u_j^{(k)})}{\lambda_{k+1}(u_j^{(k)})} = \prod_{i=1}^{m^{(k-1)}} \frac{u_j^{(k)} - u_i^{(k-1)} - 1}{u_j^{(k)} - u_i^{(k-1)}} \prod_{i \neq j} \frac{u_j^{(k)} - u_i^{(k)} - 1}{u_j^{(k)} - u_i^{(k)} + 1},$$

$$\frac{\lambda_1(u_j^{(1)})}{\lambda_2(u_j^{(1)})} = \prod_{i \neq j} \frac{u_j^{(1)} - u_i^{(1)} - 1}{u_j^{(1)} - u_i^{(1)} + 1} \cdot \prod_{i=1}^{m^{(2)}} \frac{u_j^{(1)} - u_i^{(2)} + 1}{u_j^{(1)} - u_i^{(2)}},$$

$$\frac{\lambda_{n-1}(u_j^{(n-1)})}{\lambda_n(u_j^{(n-1)})} = \prod_{i=1}^{m^{(n-2)}} \frac{u_j^{(n-1)} - u_i^{(n-2)} - 1}{u_j^{(n-1)} - u_i^{(n-2)}} \prod_{i \neq j} \frac{u_j^{(n-1)} - u_i^{(n-1)} + 1}{u_j^{(n-1)} - u_i^{(n-1)} + 1}.$$
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