STABLE PAIRS WITH A TWIST AND GLUING MORPHISMS FOR MODULI OF SURFACES

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Abstract. We propose a definition for families of KSBA stable pairs $(X, D)$ over an arbitrary (possibly non-reduced) base in the case in which $D$ is reduced. Building on the approach of Abramovich and Hassett in [AH11], we replace the pair $(X, D)$ with a pair of cyclotomic stacks $(\mathcal{X}, \mathcal{D})$. Our definition of a stable family agrees with previous ones over a seminormal base but has the advantage of being more easily amenable to the tools of deformation theory. We observe that adjunction for $(\mathcal{X}, \mathcal{D})$ holds on the nose; there is no correction term coming from the different. As applications, we show the existence of functorial gluing morphisms for families of stable surfaces and functorial morphisms from $(n+1)$ dimensional KSBA stable pairs to $n$ dimensional polarized orbispace.

1. Introduction

Since the introduction of the space of stable curves by Deligne and Mumford [DM69], the theory of modular compactifications of moduli spaces of varieties has held a central role in algebraic geometry. The class of stable pairs $(X, D)$, first introduced in dimension 2 by Kollár and Shepherd-Barron [KSB88], give a natural generalization of stable curves to higher dimensions. Building on significant advances in the minimal model program, boundedness, properness, and projectivity of the coarse moduli space has been proven for the space of stable pairs [KSB88, Kol90, Ale94, Kar00, BCHM10, HX13, HMX18, KP17, Kol17].

One aspect that remains subtle is the correct notion of a flat family of stable pairs over an arbitrary base. Over a semi-normal base, this has been addressed in great detail in [Kol17]. However, to apply the tools of deformation theory and study the infinitesimal structure of the moduli space, one needs to work with families over non-reduced bases. One difficulty is that the natural polarization $K_X + D$ on a stable pair is only a $\mathbb{Q}$-divisor, so the associated sheaf $\mathcal{O}_X(K_X + D)$ is not locally free. A possible solution is to choose an appropriate reflexive power to work with which is a line bundle; but different choices would result in different moduli problems. Therefore it is natural to impose the following, more canonical, condition introduced by Kollár [Kol90]: the sheaves $\omega_{\pi}^m mD$ are flat and commute with base change for every $m$. It turns out that the condition of the fibers of $\pi$ being stable pairs does not guarantee the desired compatibility with base change, even for a single $m$. So contrary to the case of curves, it does not suffice to define the moduli problem by a fiberwise condition.

This is addressed by Kollár using hulls and husks (see for example [Kol17, Chapter 9]). In [AH11], Abramovich and Hassett showed that imposing Kollár’s condition is equivalent to working with an associated orbifold $\mathcal{X} \to B$. When $D = 0$, they constructed a proper Deligne-Mumford stack parametrizing such canonically polarized twisted stable varieties. The goal of this paper is to extend the approach of Abramovich and Hassett to incorporate a non-zero reduced divisor $D$.

**Theorem 1.1** (Theorems 3.9 and 3.12). There exists a proper Deligne-Mumford stack $\mathcal{K}_{n,v}$ parametrizing $n$-dimensional twisted stable pairs $(\mathcal{X}, \mathcal{D})$ of volume $v$ where $\mathcal{X}$ is an orbifold with no stabilizers in codimension one, $\mathcal{D}$ is a reduced divisor, $K_\mathcal{X} + \mathcal{D}$ is a Cartier ample divisor, and $(\mathcal{X}, \mathcal{D})$ has semi-log canonical singularities.
Let us make Theorem 1.1 more precise. The main challenge in generalizing to pairs is to give a suitable notion of the divisor varying in flat families. We incorporate the divisor into the framework of \cite{AH11} using an idea of Kollár \cite[Page 21]{Kol13} (see also \cite[Section 5]{KP17}). Namely, we replace $D$ with the morphism of sheaves $\mathcal{O}_X(K_X) \to \mathcal{O}_X(K_X + D)$. In particular, we consider pairs $(\mathcal{X}, \phi : \omega_\mathcal{X} \to \mathcal{L})$ where $(\mathcal{X}, \mathcal{L})$ is a polarized orbispace (see Definition 2.8) and $\phi$ is a nonzero morphism which cuts out the divisor $D$ as the support of its cokernel (see 3.1.1, 3.7 and 3.10).

The upshot, by taking the coarse moduli space, is that twisted stable pairs $(\mathcal{X} \to B, \phi : \omega_\mathcal{X} \to \mathcal{L})$ satisfying a fiberwise condition are equivalent to flat families $\pi : (X, D) \to B$ with stable fibers that satisfy Kollár’s condition for the log canonical divisor: the sheaves $\omega_{\pi/m}^m(mD)$ are flat and compatible with base change for every $m$. In particular, we obtain:

**Corollary 1.2.** Let $(\mathcal{X}, D)$ be a twisted stable pair with trivial stabilizers in codimension 1 and coarse space $(X, D)$. Then infinitesimal deformations of the associated pair $(\mathcal{X}, \phi : \omega_\mathcal{X} \to \mathcal{L})$ are in bijection with infinitesimal deformations of the stable pair $(X, D)$ that satisfy Kollár’s condition.

Another advantage of working with the formalism of twisted stable pairs is that it simplifies the adjunction formula. There is no longer a correction term coming from the singularities of $X$:

**Theorem 1.3** (Proposition 4.6 and Corollary 4.7). Let $(\mathcal{X}, D)$ be a twisted stable pair. Then $\mathcal{L}|_D \cong \omega_D$. Moreover, let $(\mathcal{X} \to B, \phi : \omega_\mathcal{X} \to \mathcal{L})$ be a twisted stable pair over an arbitrary base and suppose that $D \to B$ is an $S_2$ morphism. Then $(D \to B, \mathcal{L}|_D)$ is a family of canonically polarized orbifolds with semi-log canonical singularities.

The first part of Theorem 1.3 is essentially saying that the different (see Section 4.2) of $(X, D)$ is replaced by the orbifold structure of $(\mathcal{X}, D)$. We study the precise relation between the different on $(X, D)$ and the stabilizers on $(\mathcal{X}, D)$ at the end of Section 4. For surfaces, the second part of Theorem 1.3 gives us a morphism from $\mathcal{M}_{2,v}$ to the moduli space of orbifold stable curves.

As an application, we show that in the case of surfaces, the gluing results of \cite[Theorem 5.13]{Kol13} for slc varieties can be extended to the orbifold setting and can be done functorially for families over an arbitrary base. More precisely, we construct the algebraic stack $\mathcal{G}_{2,v}$ of gluing data consisting of triples $(\mathcal{X}, D, \tau : D^n \to D^n)$ where $(\mathcal{X}, D)$ is an object of $\mathcal{K}_{2,v}$ as in Theorem 1.1, $D^n$ is the normalization of $D$, and $\tau : D^n \to D^n$ is a generically fixed point free involution which preserves the preimage of the nodes of $D$. Denote by $\mathcal{K}_{2,v}^\omega$ the moduli stack of stable surfaces.

**Theorem 1.4** (Theorems 5.14 and 5.17). There is a morphism $\mathcal{G}_{2,v} \to \mathcal{K}_{2,v}^\omega$ which on the level of closed points sends a triple of gluing data as above to the stable surface given by gluing the coarse space $D \subset X$ along the involution $\tau$ as in \cite[Theorem 5.13]{Kol13}.

Theorem 1.4 is an analogue for stable surfaces of the gluing morphisms that describe the boundary of the moduli space of stable curves $\overline{\mathcal{M}}_g$ \cite{Knu83}. In Section 5.3 we show that there exists a finite stratification of $\mathcal{K}_{2,v}^\omega$ such that each boundary stratum is the image of a family of gluing data under the morphism in Theorem 1.4.

### 1.1. Future work.

The initial motivation for writing this paper was to introduce a framework within which one could more easily apply the tools of (derived) deformation theory (e.g. the cotangent complex) to the moduli spaces of stable pairs. We will explore applications in this direction in the forthcoming work \cite{BI19}.

A major question left unanswered is how to define families of stable pairs over an arbitrary base when the coefficients of $D$ are smaller than 1. Here there is a further complication first noted by Hassett: the divisor $D$ may acquire embedded points in a flat limit. Thus the scheme theoretic limit of $D$ may not agree with the divisor theoretic limit. Incorporating embedded points of $D$ into the definition over an arbitrary base has proven difficult (see \cite[KP17] for the base being semi-normal). We hope the formalism introduced here will help in understanding this phenomenon.
Following Theorem 1.4, a natural goal is to compactify the moduli space of gluing data and extend the gluing morphism to a proper map in order to describe the closures of the boundary strata. Finally, it is interesting to ask if the boundary strata of $K_{2,e}$ can be given a functorial interpretation as in the case of curves.

1.2. Outline. In Section 2 we recall the background we need on singularities of the MMP [Kol13, KM98] and polarized orbispaces [AH11]. In Section 3 we give the definition of twisted stable pair (Definition 3.7) and we prove Theorem 1.1. In Section 4 we study the local properties of polarized orbispaces in order to compare our moduli functor with the $\mathbb{Q}$-Gorenstein deformations used by [Hac04] and to prove Theorem 1.3. Finally in Section 5 we prove Theorem 1.4.

Acknowledgements. The authors would like to thank Dan Abramovich, Shamil Asgarli, Brendan Hassett, Davesh Maulik, Jonathan Wise, and Chenyang Xu for many helpful conversations. We thank David Rydh for informing us about Theorem 5.9 and its proof. DB is supported by an NSF Postdoctoral Fellowship DMS-1803124 and was partially supported by NSF grant DMS-1759514. GI would like to thank the UC Berkeley Department of Mathematics for their hospitality while part of this research was conducted and is partially supported by NSF grant DMS-1759514.

Conventions. We work over an algebraically closed field $k$ of characteristic 0. Unless otherwise specified, all the stacks will be of finite type over $k$. When we say that a stack $\mathcal{X}$ has property $\mathcal{P}$ generically, we mean that there is an open embedding $U \to \mathcal{X}$ which intersects all the irreducible components of $\mathcal{X}$, such that the points of $U$ have property $\mathcal{P}$. When algebraic stack $\mathcal{X}$ admits a coarse moduli space, unless otherwise stated we will denote its coarse moduli space by $\mathcal{X}$. When we say that a diagram of stacks commutes, we mean it 2-commutes.

2. Background on the minimal model program and polarized orbispaces

This section is divided into three subsections. In the first one, we begin by recalling the properties we need about the singularities of the MMP. Then we introduce the analogous singularities for a Deligne-Mumford (DM) stack. In the second subsection we recall the definition of the different. Finally, in the last subsection we recall the relevant constructions and definitions from [AH11].

2.1. Singularities of the MMP. In this subsection we recall the properties of the singularities that appear in the MMP, which are relevant for the rest of the paper. For a more detailed exposition, and for the definitions of lc, slc, plt, demi-normal and Du Bois, we refer to [Kol13b] and [KM98].

We begin by introducing the following notation:

**Notation 2.1.** We say that an open subset $U \subset X$ is big if the complement of $U$ has codimension at least 2 in $X$. Furthermore, given $f : X \to B$ a flat morphism of DM stacks, we denote with $U(f)$ the $f$-Gorenstein locus.

We highlight the following useful observation:

**Observation 2.2.** Let $f : \mathcal{X} \to B$ be a flat family of DM stacks and let $B' \to B$ a morphism. Consider the pullback $f' : \mathcal{X}' := \mathcal{X} \times_B B' \to B'$ and let $h : \mathcal{X}' \to \mathcal{X}$ be the first projection. Then $h^{-1}(U(f)) = U(f')$.

The following lemma is known to the experts. For convenience we include a proof.

**Lemma 2.3.** Consider a flat $S_2$ morphism $\pi : \mathcal{X} \to B$ from a DM stack $\mathcal{X}$ to a scheme, and let $U \subseteq \mathcal{X}$ be an open substack which is big on each fiber. Let $\mathcal{E}$ be a reflexive sheaf on $\mathcal{X}$, and let $\mathcal{F}$ be a coherent sheaf on $\mathcal{X}$. Then the restriction map $\text{Hom}_{\mathcal{X}}(\mathcal{F}, \mathcal{E}) \to \text{Hom}_U(\mathcal{F}|_U, \mathcal{E}|_U)$ is an isomorphism.
Proof. Up to replacing $\mathcal{X}$ with an atlas, we can assume it is a scheme, which we denote by $X$. Let $j : U \to X$ be the inclusion of $U$. Since $\mathcal{E}$ is reflexive and $U$ is big along each fiber, from \cite[Proposition 3.6.1]{HK04} the morphism $\mathcal{E} \to j_*(\nu^*\mathcal{E})$ is an isomorphism. Therefore, by the adjunction between $j_*$ and $j^*$, we have $\text{Hom}_X(\mathcal{F}, \mathcal{E}) \cong \text{Hom}_X(\mathcal{F}, j_*(\nu^*\mathcal{E})) \cong \text{Hom}_U(j^*(\mathcal{F}), j^*(\mathcal{E}))$. \hfill \Box

Consider $f : X \to B$ a flat separated morphism of locally Noetherian DM stacks with $S_2$ and pure $d$-dimensional fibers. Let $\omega'_f$ be the relative dualizing complex.

**Definition 2.4.** We define the relative canonical sheaf $\omega_{X/B}$ or $\omega_f$ to be the sheaf $\mathcal{H}^{-d}(\omega'_f)$.

When we assume that the fibers are Gorenstein in codimension 1, then $\omega_{X/B}$ agrees with the pushforward $\iota_*\omega_{U(f)/B}$ where $\iota : U(f) \hookrightarrow X$ the inclusion of the relative Gorenstein locus (see \cite[Section 5]{LN18}). In this case $\omega_{X/B}$ is in fact a reflexive sheaf \cite[Proposition 5.6]{LN18}.

Next we generalize the definitions for singularities of pairs to DM stacks:

**Definition 2.5.** Consider a pair $(\mathcal{X}, \sum a_i D_i)$ consisting of a DM stack $\mathcal{X}$ and reduced equidimensional closed substacks $D_i$ of codimension 1 with $a_i \in \mathbb{Q}_{[0,1]}$. We say that the pair $(\mathcal{X}, \sum a_i D_i)$ is log canonical or lc (resp. semi-log canonical or slc) if there is an étale cover $f : Y \to \mathcal{X}$ by a scheme such that $(Y, \sum a_i f^* D_i)$ is log canonical (resp. semi-log canonical).

The following observation follows from \cite[2.40, 2.41 and Corollary 2.43]{Kol13b} (see also \cite[Proposition 2.15]{Kol13b}).

**Observation 2.6.** Consider a pair $(\mathcal{X}, \sum a_i D_i)$ consisting of a demi-normal scheme $X$ and pure codimension 1 reduced subschemes $D_i$. Let $f : Y \to X$ be an étale surjective morphism, or a finite surjective morphism that is étale in codimension 1, with $Y$ demi-normal. Then the pair $(\mathcal{X}, \sum a_i D_i)$ is lc (resp. slc) if and only if $(Y, \sum a_i f^* D_i)$ is lc (resp. slc).

The main consequences of Observation 2.6 are the following:

- Definition 2.3 does not depend on the choice of the étale cover $Y$;
- Consider a pair $(\mathcal{X}, \sum a_i D_i)$ with $\mathcal{X}$ a demi-normal DM stack that is a scheme in codimension 2. Let $X$ (resp. $D_i$) be the coarse space of $\mathcal{X}$ (resp. $D_i$). Then $(X, \sum a_i D_i)$ is lc (resp. slc) if and only if $(\mathcal{X}, \sum a_i D_i)$ is lc (resp. slc).

2.2. **The different.** In this subsection we recall the definition of different. We refer the reader to \cite[Definition 2.34 and Chapter 4]{Kol13b} for further details.

Suppose $(X, D + \Delta)$ is an lc pair, where $D$ is a divisor with coefficient 1 and $\Delta$ is a $\mathbb{Q}$-divisor. If $X$ and $D$ are smooth, then the usual adjunction formula gives a canonical isomorphism

\[
\mathcal{O}_X(K_X + D + \Delta)|_D \cong \mathcal{O}_D(K_D + \Delta)|_D.
\]

If either $X$ or $D$ are singular, Equation (1) may no longer hold. However, there is a canonical correction term given by the different.

More precisely, suppose $(X, D + \Delta)$ is lc and let $\nu : D^n \to D$ be the normalization of $D$. Since $K_X + D + \Delta$ is $\mathbb{Q}$-Cartier, for $m$ divisible enough one can compare $(m(K_X + D + \Delta))|_{D^n}$ and $m(K_D + \Delta)|_D$. Using this, one can define an effective divisor $\text{Diff}_{D^n}(\Delta)$, such that

\[
K_{D^n} + \Delta|_D + \text{Diff}_{D^n}(\Delta) \sim_{\mathbb{Q}} (K_X + D + \Delta)|_{D^n}.
\]

We will see that $\text{Diff}_{D^n}(\Delta)$ is an actual effective divisor on $D^n$, not just a divisor class.

There are two equivalent definitions for $\text{Diff}_{D^n}(\Delta)$ and both will be useful in the sequel.
2.2.1. The different, 1st definition. Writing $D$ as a sum of its irreducible components and applying Equation 2 to each component, one can see that it suffices to define $\text{Diff}_{D^n}(\Delta)$ in the case where $D$ is irreducible. Consider then $p : Y \to X$ a log-resolution of $(X, D + \Delta)$, and let $T := p_*^{-1}D$. Then we have the following commutative diagram.

$$
\begin{array}{ccc}
T & \xrightarrow{i} & Y \\
p_T \downarrow & & \downarrow p \\
D^n & \xrightarrow{j} & X
\end{array}
$$

On $Y$, we can write $p^*(K_X + D + \Delta) = K_Y + T + F$ where $F$ is a $\mathbb{Q}$-divisor. Then $i^*p^*(K_X + D + \Delta) = K_T + F|_T$. Now we can define the different as

$$\text{Diff}_{D^n}(\Delta) := (p_T)_*(F|_T).$$

We remark that this definition only depends on the points of codimension at most 2 on $X$, so it suffices to consider the case in which $X$ is a surface. Then from [Kol13b, Proposition 2.35] the different satisfies the following properties:

1. $\text{Diff}_{D^n}(\Delta)$ does not depend on the choice of a log resolution, and
2. $\text{Diff}_{D^n}(\Delta)$ is an effective divisor.

2.2.2. The different, 2nd definition. One can also define the different using the Poincaré residue map. We review the definition below. See [Kol13b, Definition 4.2] for more details. Let $Z \subseteq X$ to be the union of $\text{Supp} \Delta \cap D$ with the closed subset where either $\text{Supp}(D)$ or $X$ are singular. Then on $X \setminus Z$ there exists a canonical isomorphism

$$R : \omega_X(D)|_{(D \setminus Z)} \to \omega_{D \setminus Z}$$

given by the Poincaré residue map. For $m$ divisible enough so that $\omega_X^m(m(D + \Delta))$ is Cartier, consider the $m$th tensor power $R^{\otimes m}$. Since the normalization morphism $\nu : D^n \to D$ is an isomorphism on the locus where $D$ is smooth and since $\Delta|_{D \setminus Z} = 0$, $R^{\otimes m}$ pulls back to a rational section of

$$\mathcal{H}om_{D^n}(\nu^*(\omega_X^m(m(D + \Delta))), \omega_D^m)$$

which we also denote $R^{\otimes m}$. Now $\omega_D^m$ is Cartier in codimension 1 so the rational section $R^{\otimes m}$ defines a divisor $\Delta_m$ in codimension 1 on $D^n$ and we denote its closure by $\Delta_m$. Then we can define the different by

$$\text{Diff}_{D^n}(D') := \frac{1}{m}\Delta_m.$$
Definition 2.8. Let \( f : \mathcal{X} \to B \) be a flat proper equi-dimensional morphism from a cyclotomic stack to a scheme. Assume that each fiber of \( f \) is generically an algebraic space. Let \( \pi : \mathcal{X} \to B \) be the coarse moduli space and \( f_X : X \to B \) the induced induced map. A polarizing line bundle is a line bundle \( L \) on \( \mathcal{X} \) such that:

(i) For every geometric point \( \xi \in \mathcal{X}(\text{Spec}(K)) \), the action of \( \text{Aut}(\xi) \) on the fiber of \( L \) is effective, and

(ii) There is an \( f_X \)-ample line bundle \( M \) on \( X \) and an \( N > 0 \) such that \( L^N \cong \pi^* M \).

A pair \((\mathcal{X} \to B, L)\) as above is a polarized orbispace.

Remark 2.9. We do not require the fibers of \( f \) to be connected.

Note that a weighted projective stack with the line bundle \( O(1) \) is a polarized orbispace and by [AH11 Corollary 2.4.4], any polarized orbispace is a closed substack of a weighted projective stack.

Now one can define a category fibered in groupoids \( \mathcal{O} \) as follows. The objects \( |\mathcal{O}(S)| \) over a scheme \( S \) are polarized orbispace \((\mathcal{X} \to B, L)\). A morphism of \((\pi : \mathcal{X} \to B, L) \to (\pi' : \mathcal{X}' \to B', L')\) over a map \( g : B \to B' \) consists of a morphism \( f : \mathcal{X} \to \mathcal{X}' \) and an isomorphism \( \phi : f^* L' \to L \), such that the following diagram is cartesian:

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{f} & \mathcal{X}' \\
\pi \downarrow & & \downarrow \pi' \\
B & \xrightarrow{g} & B'.
\end{array}
\]

Theorem 2.10 ([AH11 Proposition 4.2.1]). The stack \( \mathcal{O} \) is algebraic and locally of finite type.

Remark 2.11. Contrary to our conventions, we do not claim that \( \mathcal{O} \) is of finite type.

For our purposes the relevance of polarized orbispace lies in their relation with Kollár families of \( \mathbb{Q} \)-line bundles:

Definition 2.12 ([AH11 Definition 5.2.1]). A Kollár family of \( \mathbb{Q} \)-line bundles is the data of a pair \((f : X \to B, F)\) consisting of a morphism of schemes \( f \) and a coherent sheaf \( F \) on \( X \) satisfying the following conditions:

1. \( f \) is flat with fibers that are reduced, \( S_2 \), and of equi-dimension \( n \);
2. For every fiber \( X_b \), the restriction \( F|_{X_b} \) is reflexive of rank \( 1 \);
3. For every \( n \) the formation of \( F^{[n]} \) commutes with base change for maps \( B' \to B \), and
4. For each \( X_b \), there is an \( N_b \) divisible enough such that \( F^{[N_b]} \) is a line bundle.

Points (1), (2) and (4) do not pose any major difficulty as they are fiberwise conditions. However, point (3) is difficult to check and is not automatic (see [AK16]). In [AH11], Abramovich and Hassett give a stack theoretic characterization of families satisfying condition (3) which we now review.

First, observe that given a Kollár family of \( \mathbb{Q} \)-line bundles \((f : X \to B, F)\), one can consider the variety \( \mathcal{P}(F) := \text{Spec}_{\mathcal{X}}(\bigoplus_{n \geq 1} F^{[n]}) \) lying over \( B \). There is a natural action of \( \mathbb{G}_m \) over \( B \) induced by the grading and taking the quotient \( \mathcal{X}_F := [\mathcal{P}(F)/\mathbb{G}_m] \) gives a cyclotomic stack which is flat over \( B \) [AH11 Proposition 5.1.4]. Moreover, the fibers of \( g \) are reduced and \( S_2 \) with trivial stabilizers in codimension one and there is a line bundle \( O(1) \) on \( \mathcal{X}_F \) making \((g : \mathcal{X}_F \to B, O(1))\) into a polarized orbispace. Abramovich and Hassett show that this correspondence can be reversed:

Theorem 2.13 ([AH11 Section 5]). Consider \((f : \mathcal{X} \to B, L)\) a polarized orbispace. Assume that for every \( b \in B \) the fiber \( \mathcal{X}_b \) is reduced and \( S_2 \) with trivial stabilizers in codimension one. Let \( p : \mathcal{X} \to X \) be the coarse moduli space. Then \((X \to B, p_* (L))\) is a Kollár family of \( \mathbb{Q} \)-line bundles.
In particular, consider a Kollár family of \( \mathbb{Q} \)-line bundles \((X \to \text{Spec}(A), F)\) over a local Artin ring \(A\) and let \(A' \to A\) be an extension of local Artin rings. Then the deformations of \((X \to \text{Spec}(A), F)\) along \(\text{Spec}(A) \to \text{Spec}(A')\) which satisfy the condition (4) of Definition 2.12 are identified with the deformations of the polarized orbispace \((g : \mathcal{X}_F \to \text{Spec}(A), \mathcal{O}(1))\).

Definition 2.14. We will say that a polarized orbispace \((\mathcal{X} \to B, \mathcal{L})\) satisfying the assumptions of Theorem 2.13 is an Abramovich-Hassett (or AH) family. Given a Kollár family of \(\mathbb{Q}\)-line bundles \((X \to B, F)\), we will call \((\mathcal{X}_F \to B, \mathcal{O}(1))\) the associated AH family.

Finally, [AH11] also proves the existence of a locally of finite type (but not necessarily of finite type) algebraic stack which parametrizes canonically polarized orbispaces [AH11] Definition 6.1.1].

Definition 2.15. Following [AH11], we define the moduli space parametrizing AH families of canonically polarized orbispaces with at worst slc singularities by \(K^\omega\). Furthermore, we denote by \(\mathcal{K}^\omega_{n,v} \subseteq \mathcal{K}^\omega_{\text{slc}}\) the substack parametrizing those polarized orbispaces of dimension \(n\) and volume \(v\).

3. The moduli space of twisted stable pairs

The goal of this section is to present a definition of a family of stable pairs over an arbitrary base, using polarized orbispaces. We then construct an algebraic stack \(\mathcal{M}_{n,v}\) of these twisted stable pairs.

We start with the usual definition of a stable pair:

Definition 3.1. An slc pair \((X, D)\) is a stable pair if \(K_X + D\) is ample.

There are two obstacles one has to overcome in order to generalize this to a notion of families of stable pairs. The first is that the \(\mathbb{Q}\)-divisor \(K_X + D\) is only defined up to rational equivalence. Moreover, the divisor \(D\) itself is an actual Weil divisor rather than just a divisor class: even when \(D\) is Cartier, the condition of the pair being slc is not invariant under linear equivalence. Thus one needs to find a suitable definition for a family of divisors, over an arbitrary base scheme \(B\).

To address the first point, it is natural to consider Kollár families of \(\mathbb{Q}\)-line bundles \((X \to B, F)\) where \(F\) restricts to the reflexive sheaf \(\mathcal{O}(K_X + D)\) along each fiber. As we saw in the previous section, this is equivalent to considering flat families of polarized orbispaces. To address the second point, we follow an idea originally due to Kollár (see [Koll13] page 21]) and used by Kovács and Patakfalvi in [KP17]: we replace the data of \(D\) with the morphism of sheaves \(\mathcal{O}_X(K_X) \to \mathcal{O}_X(K_X + D)\).

3.1. The stack of pairs \(\mathcal{H}\). We begin by defining a category \(\mathcal{H}\) fibered in groupoids over \(\text{Sch}/k\), consisting of pairs of a polarized orbispace and a morphism of sheaves as above.

3.1.1. The objects of \(\mathcal{H}\). For every scheme \(B\), an object of \(\mathcal{H}(B)\) consists of a pair \((f : \mathcal{X} \to B, \phi : \omega_{\mathcal{X}/B} \to \mathcal{L})\), with \((f : \mathcal{X} \to B, \mathcal{L})\) a polarized orbispace and \(\phi\) a morphism such that:

1. \(f : \mathcal{X} \to B\) is a flat family of equi-dimensional demi-normal stacks, and
2. for every \(b \in B\), \(\phi_b\) is an isomorphism at the generic points and codimension one singular points of \(\mathcal{X}_b\).

Remark 3.2. Note that we do not assume that the fibers of \(f\) are connected. This will simplify the definition of the moduli of gluing data in Section 5.

3.1.2. The arrows of \(\mathcal{H}\). Consider two schemes \(B_1\) and \(B_2\) and a morphism \(a : B_1 \to B_2\). Suppose \(\alpha := (f_1 : \mathcal{X}_1 \to B_1, \phi_1 : \omega_{\mathcal{X}_1/B_1} \to \mathcal{L}_1)\) and \(\beta := (f_2 : \mathcal{X}_2 \to B_2, \phi_2 : \omega_{\mathcal{X}_2/B_2} \to \mathcal{L}_2)\) are objects in \(\mathcal{H}(B_1)\) and \(\mathcal{H}(B_2)\) respectively. An arrow \(\Phi : \alpha \to \beta\) lying over \(a\) is the data of a morphism \((\mu, \nu)\) of the objects \((\mathcal{X}_1 \to B_1, \mathcal{L}_1)\) and \((\mathcal{X}_2 \to B_2, \mathcal{L}_2)\) of \(\text{Orb}^D\) such that following diagram is commutative:
Here $\mu^* \omega_{X_2/B_2} \to \omega_{X_1/B_1}$ is a canonical morphism, defined as follows. By Observation \[\ref{ref:observation}, \text{there is a morphism } U(f_1) \to U(f_2). Since } U(f_2) \to B_i \text{ is Gorenstein, this induces a unique canonical isomorphism (see } \text{[Con00, Theorem 3.6.1]}\]

\[
\begin{array}{ccc}
\mu^* \omega_{X_2/B_2} & \xrightarrow{\mu^*(\phi_2)} & \omega_{X_1/B_1} \\
\mu^* \nu & \downarrow & \phi_1 \\
\mu^* \nu & \downarrow & \nu \\
\end{array}
\]

Letting $\iota_j : U(f_j) \to X_j$ be the inclusion, we have

$\iota_2^*(\omega_{X_2/B_2}) = \iota_2^*(\omega_{U(f_2)/B_2}) \cong \omega_{U(f_2)/B_2},$

and so Equation \[\ref{eq:equation} \text{induces a map } \iota_1^* \mu^*(\omega_{X_2/B_2}) \cong \mu^*|_{U(f_1)}(\iota_2^*(\omega_{X_2/B_2})) \cong \mu^*|_{U(f_1)}(\omega_{U(f_2)/B_2}) \to \omega_{U(f_1)/B_1}.$

Then the push pull adjunction gives the map $\mu^*(\omega_{X_2/B_2}) \to (\iota_1)_*(\omega_{U(f_1)/B_1}) = \omega_{X_1/B_1}.$

**Theorem 3.3.** The category $\mathcal{H}$ is an algebraic stack locally of finite type.

**Proof.** Consider the algebraic stack $\text{Orb}^\mathcal{L}$, let $(X, \mathcal{L}) \to \text{Orb}^\mathcal{L}$ be the universal polarized orbispace, and assume that $X \to \text{Orb}^\mathcal{L}$ has relative dimension $n$. The locus $\mathcal{O}(1) \to \text{Orb}^\mathcal{L}$ where the fibers are $S_2$ and reduced is open by \text{[Gro66, Théorème 12.2.4]}. Furthermore, the condition of having at worst nodal singularities is open, so there is an open substack $\mathcal{O}(2) \hookrightarrow \mathcal{O}(1)$ where the fibers of $X^{(2)} := X \times_{\text{Orb}^\mathcal{L}} \mathcal{O}(2) \to \mathcal{O}(2)$ are demi-normal.

Now over $\mathcal{O}(2)$, we can consider the relative canonical sheaf $\omega_{X^{(2)}/\mathcal{O}(2)}$ which is reflexive and equal to $\iota_*(\omega_{U/\mathcal{O}(2)})$ where $U$ is the relative Gorenstein locus of $X^{(2)} \to \mathcal{O}(2)$. Denoting by $\mathcal{L}^{(2)}$ the pullback of $\mathcal{L}$ to $X^{(2)}$, we have that the Hom-stack $\mathcal{H} := \text{Hom}_{\mathcal{O}(2)}(\omega_{X^{(2)}/\mathcal{O}(2)}, \mathcal{L}^{(2)})$ is algebraic and locally of finite type by \text{[Lie06, Proposition 2.1.3]}. We will check that $\mathcal{H}$ is a substack of $\mathcal{H}'$ by identifying $\mathcal{H}'$ with the stack of pairs $(f : X \to B, \phi : \omega_X/B \to \mathcal{L})$. That is, for every scheme $B$ there is an equivalence of categories

\[
\text{Hom}(B, \mathcal{H}') \to \{(f : X \to B, \phi : \omega_X/B \to \mathcal{L})\}
\]

where $(X \to B, \mathcal{L})$ is induced by a morphism $B \to \mathcal{O}(2)$.

Indeed given a morphism $B \to \mathcal{H}'$, we can first compose it with the natural map $\mathcal{H}' \to \mathcal{O}(2)$ to get a map $B \to \mathcal{O}(2)$ and thus a family of demi-normal polarized orbispaces $(X \to B, \mathcal{L})$. Then the universal property of the Hom stack identifies the category of liftings as in the dotted arrow below

\[
\begin{array}{ccc}
\mathcal{H} & \xrightarrow{\mathcal{H}} & \mathcal{O}(2) \\
\pi & & \\
B & \xrightarrow{\pi} & \mathcal{O}(2) \\
\end{array}
\]

with maps $\phi_0 : \pi^*(\omega_{X^{(2)}/\mathcal{O}(2)}) \to \mathcal{L}$. Now since $\mathcal{L}$ is a line bundle, $\phi_0$ factors uniquely through the reflexive hull of the source, so we obtain $\phi_0^{(1)} : \phi^*(\omega_{X^{(2)}/\mathcal{O}(2)})^{[1]} \to \mathcal{L}$. Notice that $\phi^*(\omega_{X^{(2)}/\mathcal{O}(2)})^{[1]} \cong \omega_X/B$ so setting $\phi := \phi_0^{[1]}$ gives a pair. We leave it to the reader to check this is an equivalence of categories.

For any $B$ and any object $(f : X \to B, \omega_X/B \to \mathcal{L}) \in \mathcal{H}'(B)$, condition (1) is satisfied by construction since $(f : X \to B, \mathcal{L})$ is an object of $\mathcal{O}(2)(B)$. We must show that (2) imposes
an algebraic condition. In fact, we will show that it cuts out $\mathcal{H}$ as an open substack of $\mathcal{H}'$. Let $(\mathcal{X}^{(3)}, \mathcal{L}^{(3)})$ denote the pullback of the universal polarized orbispace to $\mathcal{H}'$ and let $\Psi : \omega_{\mathcal{X}^{(3)}/\mathcal{H}'} \to \mathcal{L}^{(3)}$ be the universal morphism.

Since $\pi : \mathcal{X}^{(3)} \to \mathcal{H}'$ is Gorenstein in codimension one, $\omega_\pi$ is a line bundle in codimension one and a morphism of line bundles is an isomorphism if and only if it is surjective. Thus we need to show that requiring $\Psi$ to be surjective at generic points and at the codimension one singular points of the fibers of $\pi$ is an open condition.

We begin with the generic points. Consider $\mathcal{C} := \text{Supp}(\text{Coker}(\Psi))$. Since $\mathcal{C} \to \mathcal{X}^{(3)}$ is a closed embedding and $\mathcal{X}^{(3)} \to \mathcal{H}'$ is proper, $\mathcal{C} \to \mathcal{H}'$ is proper. We need to show that the locus where the fibers of $\mathcal{C} \to \mathcal{H}'$ have dimension at most $(n-1)$ is open. Since $\mathcal{C}$ and $\mathcal{H}'$ are tame DM stack, and formation of coarse moduli spaces commutes with base change, it suffices to show that the locus where the fibers of the coarse moduli map $C \to H'$ have dimension at most $(n-1)$ is open. This follows from upper-semicontinuity of fiber dimension.

Finally we consider the codimension one singular points. Let $\mathcal{S} \subset \mathcal{X}^{(3)}$ be the singular locus of $\pi : \mathcal{X}^{(3)} \to \mathcal{H}'$. It is a closed substack of $\mathcal{X}^{(3)}$ so that the map to $\mathcal{H}'$ is proper. Let $\Psi_\mathcal{S}$ be the restriction of $\Psi$ to $\mathcal{S}$ and $\mathcal{C}_\mathcal{S}$ be the support of its cokernel. Then it suffices to show that the locus where the fibers of $\mathcal{C}_\mathcal{S} \to \mathcal{H}'$ have dimension at most $(n-2)$ is open. Since $\mathcal{C}_\mathcal{S}$ is a closed substack of $\mathcal{S}$, it is proper and the result again follows from upper-semicontinuity of fiber dimension applied to the coarse map.

Thus $\mathcal{H}$ is an open substack of the algebraic stack $\mathcal{H}'$. \hfill \Box

3.2. The family of divisors $\mathcal{D}$. Next we produce a family of divisors from the data of a pair $(f : X \to B, \phi : \omega_{X/B} \to \mathcal{L}) \in \mathcal{H}(B)$, and study its properties.

**Lemma 3.4.** Suppose $(f : X \to B, \phi : \omega_{X/B} \to \mathcal{L})$ is an object of $\mathcal{H}(B)$. Then the morphism

$$\phi \otimes \mathcal{L}^{-1} : \omega_{X/B} \otimes \mathcal{L}^{-1} \to \mathcal{O}_X$$

is injective.

**Proof.** The statement is étale local so we can replace $X$ with an étale cover and assume it is a scheme. The sheaf $\omega_{X/B} \otimes \mathcal{L}^{-1}$ is reflexive by [LN18, Proposition 5.6]. Therefore by [HK04, Proposition 3.5], if $j : U(f) \to X$ is the open embedding of the Gorenstein locus,

$$\omega_{X/B} \otimes \mathcal{L}^{-1} \cong j_* j^*(\omega_{X/B} \otimes \mathcal{L}^{-1}) \text{ and } \mathcal{O}_X \cong j_* j^*(\mathcal{O}_X).$$

This means that for every open subset $V \subseteq X$ the restriction morphism $(\omega_{X/B} \otimes \mathcal{L}^{-1})(V) \to (\omega_{X/B} \otimes \mathcal{L}^{-1})(V \cap U(f)) \to \mathcal{O}_X(V \cap U(f))$ is an isomorphism. Therefore, it suffices to show that $(\omega_{X/B} \otimes \mathcal{L}^{-1})(V \cap U(f))$ is injective. Namely, we can assume without loss of generality that $\omega_{X/B}$ is a line bundle.

Now the statement is local so suppose that $B = \text{Spec}(A)$, $X = \text{Spec}(R)$ with $g : \text{Spec}(R) \to \text{Spec}(A)$ where $R$ is a flat $A$-module, and $\omega_{X/B} \otimes \mathcal{L}^{-1} \cong \mathcal{O}_X$ and denote $\phi \otimes \mathcal{L}^{-1}$ by $\psi$. Then $\psi : R \to R$ is an $R$-module homomorphism which is necessarily multiplication by some $a \in R$. Our goal is to show that $a$ is not a zero divisor.

From the commutativity condition on morphisms and point (2) in the definition of objects of $\mathcal{H}$, the element $a$ is not a zero divisor when restricted to each fiber of $X \to B$. For any ring $C$ and $C$-module $M$, we will denote by $\text{Ass}_C(M)$ the associated primes of $M$, and by $\text{Div}_C(M)$ the set of zero divisors for $M$. It is essential now to recall that $\text{Div}_C(M) = \bigcup_{P \in \text{Ass}_C(M)} P$. Then the following chain of implications finishes the proof:
For Definition 3.7.

category fibered in groupoids over $\text{Sch}/k$.

By the commutativity condition in the definition of $H^A.2.1$. See also $[\text{KP}17, \text{Lemma 5.10}]$.

and stable under generalization. Our proof is inspired by $[\text{Kol}17, \text{Lemma 4.48}]$ and $[\text{AH}11, \text{Lemma 3.3}].$

Morphisms are given by morphisms in $\text{Morph}(\mathcal{X}, \mathcal{Y})$.

The stack $\mathcal{M}$ of pairs $\mathcal{H}$ is algebraic by Theorem 3.9. We will call an object of $\mathcal{H}(B)$ an object of $\mathcal{H}(B)$ and suppose that for each $b \in B$, $\mathcal{X}_b$ is $\text{Du Bois}$. Then $\mathcal{D}_b \to B$ is flat and commutes with basechange.

Proof. We will drop the subscript $\phi$ in $\mathcal{D}_b$. Let $B' \to B$ be a morphism and consider the pullback $(\mathcal{X}' \to B', \phi' : \mathcal{X}' \to B')$. Let $g : \mathcal{X}' \to \mathcal{X}$ be the resulting morphism.

By the exact sequence of Tor and using that $\mathcal{X} \to B$ is flat, it suffices to show that the morphism $g^*(\mathcal{O}_X(-\mathcal{D})) \to g^*\mathcal{O}_X \cong \mathcal{O}_{\mathcal{X}'}$ is injective. Since the fibers are $\text{Du Bois}$, the relative canonical sheaf commutes with base change by $[\text{KK}18, \text{Corollary 1.5}]$, so $g^*(\mathcal{O}_X/B) \cong \mathcal{O}_{\mathcal{X}'}$. Thus

\[
g^*(\mathcal{O}_X(-\mathcal{D})) \cong g^*(\mathcal{O}_X/B \otimes \mathcal{L}^{-1}) \cong g^*(\mathcal{O}_X/B) \otimes g^*(\mathcal{L})^{-1} \cong \mathcal{O}_{\mathcal{X}'} \otimes (\mathcal{L}')^{-1}.
\]

By the commutativity condition in the definition of $\mathcal{H}$, the composition $\mathcal{O}_{\mathcal{X}'}(\mathcal{L}')^{-1} \cong g^*(\mathcal{O}_X(-\mathcal{D})) \to \mathcal{O}_{\mathcal{X}'}$ is the map $\phi' \otimes (\mathcal{L}')^{-1}$ which is injective by Lemma 3.4 and identifies $g^*(\mathcal{O}_X(-\mathcal{D}))$ with $\mathcal{O}_{\mathcal{X}'}(-\mathcal{D}')$.

3.3. The stack $\mathcal{M}_{n,v}$. We are now ready to introduce the stack of twisted stable pairs $\mathcal{M}_{n,v}$ as a category fibered in groupoids over $\text{Sch}/k$.

Definition 3.7. For $B$ a scheme, an object of $\mathcal{M}_{n,v}(B)$ consists of a pair $(f : \mathcal{X} \to B, \phi : \mathcal{X}/B \to \mathcal{L})$ such that

1. for every $b \in B$, $(\mathcal{X}_b, \mathcal{D}_b)$ is slc, and
2. for every $b \in B$, the volume of $(\mathcal{X}_b, \mathcal{D}_b)$ is $v$.

Morphisms are given by morphisms in $\mathcal{H}$. We will call an object of $\mathcal{M}_{n,v}$ over $B$, a twisted KSBA-stable pair (or twisted stable pair) over $B$.

Remark 3.8. Despite the fact that $\mathcal{M}_{n,v}$ is defined as a full subcategory of $\mathcal{H}$, it is not immediate that it is a substack. The issue is that condition (1) in Definition 3.7 is not compatible with base change a priori. However, by $[\text{Kol}13b, \text{Theorem 5.14}]$ the fibers $\mathcal{X}_b$ of a twisted stable pair are $\text{Du Bois}$. Therefore, by Corollary 3.6 $\mathcal{D}$ is flat and commutes with base change so $\mathcal{M}_{n,v}$ is a category fibered in groupoids.

Theorem 3.9. The stack $\mathcal{M}_{n,v}$ is algebraic and locally of finite type.

Proof. Since the stack of pairs $\mathcal{H}$ is algebraic by Theorem 3.8, it suffices to show that $\mathcal{M}_{n,v}$ is an open substack. Thus we will check that (1) and (2) in Definition 3.7 are open conditions.

Condition (1). Consider a family of pairs $(f : \mathcal{X} \to B, \mathcal{L}, \phi : \mathcal{X}/B \to \mathcal{L}) \in \mathcal{H}$. We need to show that the locus in $B$ where $(\mathcal{X}_b, \mathcal{D}_b)$ is slc is open. It suffices then to show that it is constructible, and stable under generalization. Our proof is inspired by $[\text{Kol}17, \text{Lemma 4.48}]$ and $[\text{AH}11, \text{Lemma A.2.1}]$. See also $[\text{KP}17, \text{Lemma 5.10}]$.

We begin with some initial reductions. By Corollary 3.6 $\mathcal{D} \to B$ is flat for objects of $\mathcal{M}_{n,v}$, so $\mathcal{M}_{n,v}$ is contained in the substack $\mathcal{H}_1 \subset \mathcal{H}$ where $\mathcal{D} \to B$ is flat. By $[\text{Gro}66, \text{Theorem 11.3.1}]$,
\( \mathcal{H}_1 \subset \mathcal{H} \) is open. Furthermore, \( \mathcal{M}_{n,v} \) is contained in the substack \( \mathcal{H}_2 \subset \mathcal{H}_1 \) where the fibers of \( \mathcal{D} \to B \) are reduced. Since \( \mathcal{D} \to B \) is flat and proper, then \( \mathcal{H}_2 \subset \mathcal{H}_1 \) is open by [Gro66, Theorem 12.2.1]. Thus without loss of generality, we may suppose that \( \mathcal{D} \to B \) is flat with reduced fibers.

Next we show that the locus in \( B \) where \( (X_b, D_b) \) is slc is constructible. For that, we can assume that \( B \) is irreducible. Let \( \eta \) be the generic point of \( B \) and consider the generic fiber \( (X_\eta, D_\eta) \). Let \( \mathcal{Y}_\eta \to X_\eta \) of \( (X_\eta, D_\eta) \) be a log-resolution, which exists for DM stacks by functorial resolution of singularities (see for example [W05]). Since \( B \) is locally of finite type over a field, this resolution can be spread out to give a simultaneous log-resolution \( \pi : \mathcal{Y} \to X \) of \( (X_{U|_U}, D_{U|_U}) \) for a suitable open subset \( U \subseteq B \) using the method of [Kol17, Lemma 4.48]. Then we have \( \omega_{\mathcal{Y}_{U|_U}} \otimes \mathcal{O}_Y(\pi^{-1}(D_{U|_U})) \cong \pi^*(\omega_{X_{U|_U}}) \otimes \mathcal{O}_Y(\sum d_i \Delta_i) \) for appropriate irreducible \( \pi \)-exceptional divisors \( \Delta_i \) and coefficients \( d_i \). Then for every \( b \in U \), the pair \( (X_b, D_b) \) is slc if and only if \( d_i \geq -1 \) for every \( i \) which is a constructible condition by Chevalley’s Theorem.

To show that being slc is stable under generalization, we proceed as in [KP17, Lemma 5.10]. Consider a family of demi-normal orbifolds \( \mathcal{X} \to \text{Spec}(R) \) over a DVR \( R \) and assume we have a divisor \( \mathcal{D} \) such that \( K_\mathcal{X} + \mathcal{D} = \mathbb{Q} \)-Cartier. After passing to an étale cover, we may assume \( (\mathcal{X}, \mathcal{D}) \) are schemes. If the closed fiber \( (X_0, D_0) \) is slc, then by inversion of adjunction ([Pat16, Corollary 2.11]) we conclude that the pair \( (\mathcal{X}, \mathcal{D}) \) is slc so the condition is stable under generalization.

**Condition (2).** We need to show that the locus in \( \mathcal{H} \) where the fibers have volume \( v \) is open for every integer \( v \). Note that since \( \mathcal{L} \) is a line bundle, the volume must be integral. Since \( \mathcal{H} \) is locally of finite type, we can restrict to an open connected substack \( \mathcal{H}_0 \) of finite type. Let \( (X_0, \mathcal{L}_0) \to \mathcal{H}_0 \) be the pullback of the universal polarized orbispace and let \( \pi : X_0 \to X_0 \) be the relative coarse moduli space of \( X_0 \to \mathcal{H}_0 \). Since \( X_0 \) is of finite type, there exists an \( m \) such that \( \mathcal{L}_0^{\otimes m} = \pi^*L \) for some line bundle \( L \) on \( X_0 \). It suffices to know that the self intersection \( (L|_{(X_0)_U})^n \) is locally constant, which follows from the flatness of \( X_0 \to \mathcal{H}_0 \).

Observe that the stack \( \mathcal{M}_{n,v} \) comes with a morphism to \( \mathit{Orb^E} \). From [AH11, Corollary 5.3.7], there is an open embedding \( \mathcal{K}^E \to \mathit{Orb^E} \) parametrizing cyclotomic stacks which are \( S_2 \) and have no stabilizers in codimension one.

**Definition 3.10.** Let \( \mathcal{K}_{n,v} := \mathcal{K}^E \times \mathit{Orb^E} \mathcal{M}_{n,v} \to \mathcal{M}_{n,v} \) be the open substack parametrizing orbispaces which have no stabilizers in codimension one. We call \( \mathcal{K}_{n,v} \) the Kollár component of \( \mathcal{M}_{n,v} \) and define the twisted stable pairs parametrized by \( \mathcal{K}_{n,v} \) the AH twisted stable pairs.

By Theorem 2.13 the objects of \( \mathcal{K}_{n,v} \) are the twisted stable pairs \( (f : X \to B, \phi : \omega_X \to \mathcal{L}) \) such that \( (f : X \to B, \mathcal{L}) \) is the associated AH family of a Kollár family of \( \mathbb{Q} \)-line bundles. That is, they are twisted stable pairs that are also AH families as in Definition 2.14.

**Notation 3.11.** Let \( \mathcal{K}^c_{n,v} \) denote the open substack of \( \mathcal{K}_{n,v} \) parametrizing connected twisted stable pairs.

**Theorem 3.12.** The stacks \( \mathcal{K}^c_{n,v} \) and \( \mathcal{K}_{n,v} \) are proper and DM.

**Proof.** It suffices to prove the desired claim for \( \mathcal{K}^c_{n,v} \). We divide the proof into three steps.

**\( \mathcal{K}^c_{n,v} \) is of finite type:** This follows from the results [HMX18] and [Kol08] (see also [Kol11]). By [HMX18, Theorem 1.1], there is a projective morphism of quasi-projective varieties \( X \to B \), with a divisor \( D \subseteq X \) such that every stable pair of dimension \( n \) and volume \( v \) appears as \( (X_b, D_b) \) for a certain \( b \in B \). Up to replacing \( B \) with a resolution, we may assume that \( B \) is smooth. By taking the flattening stratification, we can assume that \( X \to B \) is flat. Using boundedness and proceeding as in [Kol08, Corollary 25], we can stratify further so that \( (K_{X_b} + D_b)^{[m]} \) commutes with base change for every \( m \in \mathbb{Z} \). In particular, we can consider the associated AH family \( (f : X \to B, \mathcal{L}) \). On the locus \( \mathcal{U} \) where \( f \) is Gorenstein and has trivial stabilizers, we have a morphism \( \omega_{f|_U} \to \mathcal{L}|_U \) which
extends uniquely to a morphism \( \omega_f \to \mathcal{L} \) by Lemma\[2.3\]. Since every point of \( K_{n,v}^c \) corresponds to the AH stack of a fiber of \( X \to B \) and \( B \) is of finite type, it follows that \( K_{n,v}^c \) is as well.

\( K_{n,v}^c \) is proper: The valuative criterion for properness now follows from \cite[Koll17, Theorem 5.20 and Proposition 2.76 (2)]{Koll17}.

\( K_{n,v}^c \) is DM: Consider an AH twisted stable pair \((X \to \text{Spec}(k), \omega_X \to \mathcal{L})\), and let \( X \) be the coarse space of \( X \). Since \( X \) has no generic stabilizers, an automorphism of \( X \) which induces the identity on \( X \) is the identity. So it suffices to know that the pair \((X, D)\) has finite automorphisms where \( D = D_\phi^c \). This is \cite[KP17, Proposition 5.5]{KP17}.

\section{Local and global structure of twisted stable pairs}

The goal of this section is to study the twisted stable pairs of Definition\[3.7\] The section is divided into two subsection. In the first one we focus on the properties of an AH family, and we give conditions that are equivalent to having a morphism to an AH family. In the second section we study the local structure of a twisted stable pair in a neighbourhood of the divisor \( D \). By a careful analysis of the case that \( X \) is a surface, we show that the the only contribution to the different on a twisted stable pair comes from the double locus of \( D \) (see Proposition\[4.6\] and Corollary\[4.7\]).

\subsection{Global structure of AH families.}

In this subsection, we study the following questions. Given an AH family \( X \to B \) over a scheme \( B \) with coarse space \( X \) and a morphism \( f : Y \to X \) from a DM stack \( Y \), then:

1. when can we lift \( f \) to a map \( Y \to X \);
2. when is such a lift to be an isomorphism?

Among other things, this will allow us to relate our definition of a family of twisted stable pairs over the spectrum of an Artin ring with the \( \mathbb{Q} \)-Gorenstein deformations of Hacking in \cite{Hac04} (see Corollary\[4.4\]).

\begin{proposition}
Suppose \((X \to B, F)\) is a Kollár family of \( \mathbb{Q} \)-line bundles with associated AH family \((X \to B, \mathcal{L})\). Let \( p : X \to X \) be the coarse moduli space map and consider a morphism \( f : Y \to X \) from a scheme \( Y \). Then the groupoid \( \text{Hom}_X(Y, X) \) of maps lifting \( f \) is equivalent to the following groupoid:

- objects are given by pairs \((G, \phi)\) where \( G \) is a line bundle on \( Y \), and \( \phi : \bigoplus_{n \in \mathbb{Z}} f^*(F[n]) \to \bigoplus_{n \in \mathbb{Z}} G^{\otimes n} \) is a homomorphism of graded \( \mathcal{O}_Y \)-algebras, and
- morphisms between \((G_1, \phi_1) \to (G_2, \phi_2)\) are given by an isomorphism of line bundles \( \psi : G_1 \to G_2 \) such that \( \phi_2 = (\bigoplus_n \psi^{\otimes n}) \circ \phi_1 \).
\end{proposition}

\begin{proof}
Recall that, as a stack over \( X \), we have \( X = [\mathcal{P}(F) / \mathbb{G}_m] = [\text{Spec}_{\mathcal{O}_X}(\bigoplus_{n \in \mathbb{Z}} F[n]) / \mathbb{G}_m] \). Therefore for a scheme \( Y \) over \( X \), the groupoid \( \mathcal{X}(Y) \) is equivalent to the groupoid of \( \mathbb{G}_m \)-torsors \( E \to Y \), with a \( \mathbb{G}_m \)-equivariant morphism \( E \to \mathcal{P}(F) \) over \( X \). This is equivalent to a \( \mathbb{G}_m \)-equivariant morphism \( E \to \mathcal{P}(F) \times_X Y \cong \text{Spec}(\bigoplus_{n \in \mathbb{Z}} f^*(F[n])) \). There exists a unique line bundle \( G \) on \( Y \) such that the \( \mathbb{G}_m \)-torsor \( E \to Y \) may be written as \( \text{Spec}(\bigoplus_{n \in \mathbb{Z}} G^{\otimes n}) \). To conclude the proof it suffices to notice that \( \mathbb{G}_m \)-equivariant morphisms

\[ \text{Spec}(\bigoplus_{n \in \mathbb{Z}} G^{\otimes n}) \to \text{Spec}(\bigoplus_{n \in \mathbb{Z}} f^*(F[n])) \]

correspond to graded \( \mathcal{O}_Y \)-algebra homomorphisms \( \bigoplus_{n \in \mathbb{Z}} f^*(F[n]) \to \bigoplus_{n \in \mathbb{Z}} G^{\otimes n} \).
\end{proof}

\begin{corollary}
Let \((p : X \to B, F)\) a Kollár family of \( \mathbb{Q} \)-line bundles with associated AH family \((X \to B, \mathcal{L})\). Consider a DM stack \( Y \) with a morphism \( f : Y \to X \). Assume that there is a line
bundle \( \mathcal{G} \) on \( \mathcal{Y} \) and an isomorphism of graded \( \mathcal{O}_X \)-algebras \( \bigoplus_{n \in \mathbb{Z}} f_*(\mathcal{G}^\otimes n) \cong \bigoplus_{n \in \mathbb{Z}} F^{[n]} \). Then there is a morphism \( g : \mathcal{Y} \to \mathcal{X} \) over \( X \) such that the following diagram is cartesian.

\[
\begin{array}{ccc}
P(\mathcal{G}) & \longrightarrow & P(F) \\
\downarrow & & \downarrow \\
\mathcal{Y} & \longrightarrow & \mathcal{X}
\end{array}
\]

**Proof.** First observe that

\[
\text{Hom} \left( \bigoplus_n f^* F^{[n]}, \bigoplus_n \mathcal{G}^\otimes n \right) \cong \text{Hom} \left( \bigoplus_n f_* F^{[n]}, \bigoplus_n f_* \mathcal{G}^\otimes n \right) \cong \text{Hom} \left( \bigoplus_n F^{[n]}, \bigoplus_n f_*(\mathcal{G}^\otimes n) \right).
\]

The graded isomorphism in the hypothesis gives an isomorphism of \( \mathbb{G}_m \)-torsors \( P(\mathcal{G}) \to \text{Spec}(f^*(\bigoplus_n F^{[n]})) \cong P(F) \times_X \mathcal{Y} \). The resulting morphism \( \mathbb{G}_m \)-equivariant map \( P(\mathcal{G}) \to P(F) \) induces \( \mathcal{Y} \cong [P(\mathcal{G})/\mathbb{G}_m] \to [P(F)/\mathbb{G}_m] \cong \mathcal{X} \) and the diagram is cartesian by [Ols16, Exercise 10.F]. \( \square \)

Note that in Corollary 4.2 if we assume that \( \mathcal{Y} \to X \) is the coarse space, we cannot conclude that the morphism \( \mathcal{Y} \to \mathcal{X} \) is an isomorphism. For example, consider \( \mathcal{Y} = B\mu_2 \) with \( \mathcal{G} = \mathcal{O}_Y \) and \( \mathcal{X} = X = \text{Spec}(k) \) with \( L = \mathcal{O}_X \). The problem is that \( \mathcal{Y} \) might have more stabilizers than \( \mathcal{X} \). The following lemma shows that this is the only reason for the failure of \( \mathcal{Y} \to \mathcal{X} \) to be an isomorphism.

**Lemma 4.3.** In the situation of Corollary 4.2, suppose \( \mathcal{Y} \) is separated with coarse moduli space \( X \). Then the morphism \( g : \mathcal{Y} \to \mathcal{X} \) is the relative coarse moduli space of the map \( \mathcal{Y} \to B\mathbb{G}_m \) induced by \( \mathcal{G} \).

**Proof.** Let \( \mathcal{Y} \to \mathcal{Y}' \to B\mathbb{G}_m \) be the relative coarse moduli space of the map \( \mathcal{Y} \to B\mathbb{G}_m \). Then \( \mathcal{Y}' \to B\mathbb{G}_m \) is representable and equips \( \mathcal{Y}' \) with a line bundle \( \mathcal{G}' \) that pulls back to \( \mathcal{G} \). Furthermore, \( f \) factors through \( f' : \mathcal{Y}' \to \mathcal{X} \) and we have \( f'_*(\mathcal{G}')^\otimes m = f_*(\mathcal{G})^\otimes m = F^{[m]} \). Thus we may apply Corollary 4.2 to obtain a morphism \( g' : \mathcal{Y}' \to \mathcal{X} \) and it suffices to prove \( g' \) is an isomorphism. In particular, without loss of generality we may assume that \( \mathcal{Y} \to B\mathbb{G}_m \) is representable and we wish to prove that \( g : \mathcal{Y} \to \mathcal{X} \) is an isomorphism.

To prove this claim, note first that the question is local over \( X \), so from [Ols16, Theorem 11.3.1] we can assume that \( X = V/G \) and \( \mathcal{Y} = [V/G] \) for a finite \( G \). Moreover, we can assume that \( V = \text{Spec}(A) \) is affine. Then \( \mathcal{G} \) corresponds to a locally free \( A \)-module \( M \) with a \( G \)-action and the coarse space of \( \mathcal{G} \cong [\text{Spec}(\bigoplus_{n \in \mathbb{Z}} M^\otimes n)/G] \) is given by

\[
\text{Spec} \left( \bigoplus_{n \in \mathbb{Z}} (M^\otimes n)^G \right) = \text{Spec} \left( \bigoplus_{n \in \mathbb{Z}} f_*(\mathcal{G}^\otimes n) \right) \cong \text{Spec} \left( \bigoplus_{n \in \mathbb{Z}} F^{[n]} \right) = P(F).
\]

Now, by [AH11, Proposition 2.3.10] the representability of \( \mathcal{Y} \to B\mathbb{G}_m \) implies that for every \( p \in \mathcal{Y} \), the action of \( \text{Aut}_\mathcal{Y}(p) \) on \( \mathcal{G}_p \) is faithful. Then \( P(\mathcal{G}) = \text{Spec}_\mathcal{Y}(\bigoplus_{n \in \mathbb{Z}} f_*(\mathcal{G}^\otimes n)) \) is already an algebraic space so it is isomorphic to its coarse space: \( P(F) \cong P(\mathcal{G}) \). Thus, since \( P(F) \to \mathcal{X} \) is a smooth atlas and the diagram of Lemma 4.2 is cartesian, \( g \) is an isomorphism. \( \square \)

Lemma 4.3 allows us to compare our definition of twisted stable family with \( \mathbb{Q} \)-Gorenstein deformations in the case where the divisor \( D \) is Cartier.

**Corollary 4.4.** Let \( (X, D) \) be an slc pair where \( D \) is a Cartier divisor. Then \( K_X \) is \( \mathbb{Q} \)-Cartier and the canonical covering stack \( \mathcal{X}' \to X \) is isomorphic to the AH stack \( \mathcal{X} \) for the \( \mathbb{Q} \)-line bundle \( L = K_X + D \).

**Proof.** Consider the coarse space map \( p : \mathcal{X}' \to X \) and let \( \mathcal{G} := K_{\mathcal{X}'} + p^*D \). Then:

1. \( \mathcal{G} \) is a line bundle with \( p_*(\mathcal{G}^\otimes n) = \mathcal{O}_X(K_X + D)^{[n]} \), and
(2) for every point \( q \in X' \), the action of \( \text{Aut}_{X'}(q) \) on \( G_q \) is faithful.

Point (1) follows since both sides of the equality are reflexive and they agree on the big open set where \( p \) is an isomorphism. Point (2) holds because the action of \( \text{Aut}_{X'}(q) \) on \( (K_{X'})_q \) is faithful by definition, the action on \( (p^*D)_q \) is trivial, and the tensor product of a trivial and a faithful representation is a faithful representation. Thus we may apply Lemma 4.3 to conclude that \( X \cong X' \). \qed

4.2. Local structure of twisted stable pairs. The goal of this subsection is to study the local structure of \( X \) along the divisor \( D \). First, we prove Theorem 1.3 (see Proposition 4.6 and Corollary 4.7) which says that on a twisted stable pair the singularities of \( X \) do not contribute to the different. Then we explore the relationship between the stack structure on \( D \) and the different on the coarse space of the pair (Lemmas 4.11 and 4.10).

We start with the local notion of a twisted pair:

Definition 4.5. A twisted pair is a pair \( (\mathcal{X}, \phi : \omega_X \rightarrow \mathcal{L}) \) where \( \mathcal{X} \) is an open substack of an \( \mathcal{X}' \) with \( (\mathcal{X}' \rightarrow \text{Spec}(k), \phi' : \omega_{\mathcal{X}'} \rightarrow \mathcal{L}) \) a twisted stable pair and \( \phi \) pulled back from \( \phi' \).

Proposition 4.6. Let \( (f : \mathcal{X} \rightarrow B, \phi : \omega_f \rightarrow \mathcal{L}) \) be a twisted surface pair. Then \( \mathcal{L}|_{D_\phi} \cong \omega_{D_\phi/B} \).

Proof. We drop the subscript \( \phi \) in \( D_\phi \). The proof follows closely [Kol13b, Page 152].

Let \( \iota : D \rightarrow \mathcal{X} \) denote the natural closed embedding. From Grothendieck duality, since \( f \) is Cohen-Macaulay and \( D \rightarrow B \) is Gorenstein, we have \( \iota_* \omega_{D/B} \cong \text{Ext}^1(\iota_* \mathcal{O}_D, \omega_f) \). Consider the exact sequence

\[
0 \rightarrow \mathcal{O}_X(-D) \rightarrow \mathcal{O}_X \rightarrow \iota_* \mathcal{O}_D \rightarrow 0.
\]

Taking \( \text{Hom}_X(\mathcal{O}_X, \omega_f) \) induces a morphism

\[
\text{Hom}(\mathcal{O}_X, \omega_f) \rightarrow \text{Hom}(\mathcal{O}_X(-D), \omega_f) \rightarrow \text{Ext}^1(\iota_* \mathcal{O}_D, \omega_f) \rightarrow \text{Ext}^1(\mathcal{O}_X, \omega_f) = 0
\]

where the last term is 0 since \( \mathcal{O}_X \) is locally free. But from Lemma 3.4 the ideal \( \mathcal{O}_X(-D) \) is defined as \( \omega_f \otimes \mathcal{L}^{-1} \), therefore

\[
\text{Hom}(\mathcal{O}_X(-D), \omega_f) \cong \text{Hom}(\omega_f \otimes \mathcal{L}^{-1}, \omega_f) \cong \text{Hom}(\omega_f, \omega_f) \otimes \mathcal{L}.
\]

Now from Lemma 2.3 taking \( U \) the \( f \)-Gorenstein locus, we have \( \text{Hom}(\omega_f, \omega_f) \cong \mathcal{O}_X \). Therefore the sequence above gives a surjective morphism \( \mathcal{L} \rightarrow \iota_* (\omega_{D/B}) \). This induces a surjective morphism \( \iota^*(\mathcal{L}) \rightarrow \omega_{D/B} \), but a surjective morphism of line bundles is an isomorphism. \qed

Corollary 4.7. Let \( (\mathcal{X} \rightarrow \text{Spec}(k), \omega_X \rightarrow \mathcal{L}) \) be a twisted stable pair of any dimension. Then \( \mathcal{L}|_D \cong \omega_D \). Moreover, when \( D \) is \( S_2 \), it is an slc canonically polarized orbispace.

Proof. By [Kov17, Lemma 3.7.5], the canonical sheaf \( \omega_Z \) is \( S_2 \) on any excellent scheme \( Z \) admitting a dualizing complex. Furthermore, by [BH93, Proposition 1.2.16], the property of being \( S_2 \) can be checked after finite étale base change. Therefore \( \omega_D \) is an \( S_2 \) sheaf on \( D \). Now, we may apply Proposition 4.6 on the codimension 2 points of \( \mathcal{X} \) to see that \( \omega_D \) agrees with \( \mathcal{L}|_D \) in codimension 1 on \( D \). Since both sheaves are \( S_2 \), we conclude that \( \mathcal{L}|_D \cong \omega_D \).

If \( D \) is \( S_2 \), since it is nodal in codimension 1, it is demi-normal. In particular by inversion of adjunction ([Pat16, Lemma 2.11]) \( D \) is slc. Finally, \( \omega_D \) is a polarizing line bundle since it is the pullback of one by a closed embedding. \qed

Remark 4.8. Usually when one considers questions about adjunction for stable pairs, one works with the normalization of the divisor \( n : D^n \rightarrow D \). By Corollary 4.7 we have that \( (\iota \circ n)^* \mathcal{L} = n^* \omega_D \). Since \( D \) is nodal in codimension 1, this latter sheaf is isomorphic to \( \omega_D(G) \) where \( G \) is the double locus of \( D \). Therefore, we may interpret the corollary as stating that the only contribution to the different on \( D^n \) comes from the double locus of \( D \).
Note that in general, we have an adjunction morphism $\mathcal{M}_{n,v} \to \mathcal{O}rb^\mathcal{L}$ given by

\[(\mathcal{X} \to B, \phi : \omega_\mathcal{X} \to \mathcal{L}) \mapsto (D_\phi, \mathcal{L}|_{D_\phi}).\]

Indeed $D_\phi \to B$ is flat with reduced fibers (Corollary 3.9) and $\mathcal{L}|_D$ is a polarizing line bundle. Consider the locus $\mathcal{U} \subseteq \mathcal{M}_{n,v}$ where $D_\phi \to \mathcal{M}_{n,v}$ is $S_2$ which is open by [Gro66, Théorème 12.1.6] and the fact that the morphism is closed. Over $\mathcal{U}$, Corollary 4.11 gives us that $(D_\phi \to B, \mathcal{L}|_{D_\phi})$ is a family of canonically polarized slc orbispaces of dimension $(n - 1)$ and so over this locus, the morphism above lands in the open substack of $\mathcal{O}rb^\mathcal{L}$ of those orbispaces with slc singularities.

The following corollary follows from Proposition 4.6 and [Kol13b, Proposition 2.35]:

**Corollary 4.9.** Let $(\mathcal{X}, D)$ be a twisted surface pair over $\text{Spec}(k)$ and $p \in D$ such that $D$ is smooth at $p$. Then $\mathcal{X}$ is smooth at $p$.

Proposition 4.10 tells us that in replacing a surface pair $(X, D)$ with its associated AH pair $(\mathcal{X}, D)$, the different on $D$ gets replaced by the stack structure on $D$. Our goal now is to make explicit the relation between the stack structure on $D$ and the different on $D$.

**Lemma 4.10.** Consider $(X, D)$ an lc surface and assume that $p \in D$ is a nodal point of $D$. Then $K_X + D$ is Cartier at $p$ and $X$ agrees with its AH stack in a neighbourhood of $p$.

**Proof.** From Step 2 of the proof of [Kol13b, Corollary 2.32], it follows that the singularity of $X$ at $p$ is a cyclic quotient singularity. In particular, denoting the completed local ring $\tilde{O}_{X,p}$ by $R$, there is a morphism $\pi : \text{Spec}(\mathbb{k}\llbracket x, y \rrbracket) \to \text{Spec}(R)$ which is the quotient by a cyclic group $G$. The action of $G$ is étale in codimension one so $\pi^*K_{\text{Spec}(R)} = K_{\text{Spec}(\mathbb{k}\llbracket x, y \rrbracket)}$ and the pair $(\text{Spec}(\mathbb{k}\llbracket x, y \rrbracket), \pi^*D)$ is lc from [Kol13b, Corollary 2.43]. But then $\pi^*D$ is nodal, and the group $G$ preserves the two branches, so up to an analytic change of coordinates we may suppose that $D = (xy)$ and $x, y$ are eigenvectors for $G$. Furthermore, the log canonical divisor is Cartier generated by $\frac{dx \wedge dy}{xy}$. Then the action of $G$ on the log canonical divisor is trivial so the log canonical divisor descends to a Cartier divisor on $\text{Spec}(R)$.

We are left with understanding $D$ along the points where $D$ is smooth:

**Lemma 4.11.** Let $(X, D)$ be an lc surface and $q \in D$ a closed point at which $D$ is smooth. Assume that $q$ appears in the different of $(X, D)$ with multiplicity $m$. Let $(\mathcal{X}, D)$ be the twisted surface pair associated to $(X, D)$ and let $p \in D$ be the point lying over $q$ with stabilizer group $G := \text{Aut}_X(p)$. Then we have the following.

1. If $p$ is a smooth point of $D$, then $m = 1 - \frac{1}{|G|}$.
2. If $p$ is a node, then $m = 1$, $G = \mathbb{Z}_2$ and $G$ acts by swapping the two branches of $D$.

**Remark 4.12.** For case (1), $m$ determines the stabilizer group since the stack is cyclotomic.

**Proof.** Since the question is local, we may pass to an open subset and assume that $\mathcal{X}$ has no stabilizers outside of $p$ and $D$ has no different outside of $q$.

For case (1), consider the following commutative diagram.

\[
\begin{array}{ccc}
\mathcal{D} & \xrightarrow{\alpha} & \mathcal{X} \\
\pi_D \downarrow & & \downarrow \pi \\
D & \xrightarrow{\alpha} & X
\end{array}
\]

Letting $n$ be the index of $K_X + D$, we have the following equalities:

- $\pi^*(n(K_X + D)) = n(K_X + D)$ since $\pi$ is an isomorphism in codimension one,
- $\alpha^*(n(K_X + D)) = n(K_D + mq)$ by assumption on the different, and
\[ \cdot \alpha^*(n(K_X + D)) = nK_D \] by Proposition 4.6.

Putting this together along with commutativity of the above diagram, we obtain

\[ \pi_D^*(n(K_D + mq)) = \pi_D^*\alpha^*(n(K_X + D)) = \alpha^*\pi^*(n(K_X + D)) = nK_D, \]

or simply \( \pi_D^*(K_D + mq) = K_D \). Up to completing around \( p \), we may assume that coarse map \( \mathcal{D} \to D \) is given by \( \text{Spec}(k[x]) \to \text{Spec}(k[x^r]) \) where \( r = |G| \). In this case, one readily sees that \( \pi_D^*(q) = rp \) and \( \pi_D^*K_D = K_D - (r - 1)p \). Putting this together with the previous equality, we have then \( K_D = K_D - (r - 1)p + rmp \) which gives \( m = \frac{r - 1}{r} \).

For point (2), we may again replace \((X, \mathcal{D})\) with an étale neighbourhood of \( p \) and assume the following:

1. there is an lc surface pair \((X', D')\) and a distinguished point \( p' \in D' \);
2. there is an action of \( G \) on \( X' \) preserving \( D' \) and fixing only \( q \), and
3. \( X = X'/G, D = D'/G \) and the quotient map sends \( p' \) to \( p \).

Since \( D \) is smooth, we know that \( G \) swaps the two branches of \( D' \). Let \( H \) be the normal subgroup of \( G \) which preserves the branches of \( D' \). Then the pair \((X'/H, D'/H)\) is lc, the map \( \pi : X' \to X'/H \) is étale away from \( q \), and \( D'/H \) is a nodal curve. But then from Lemma 4.10 the log canonical divisor \( L \) of \((X'/H, D'/H)\) is Cartier. Since \( \pi \) is étale in codimension one, \( K_{X'} + D' \cong \pi^*L \). In particular, \( H \) acts trivially on \((K_{X'} + D')_p \). Then \( H = \{1\} \) since the action of \( G \) on \((K_{X'} + D')_p \) is faithful (recall that \( X \to BG_m \) is representable). Therefore, any non-zero element of \( G \) swaps the branches of \( D' \) and \( G \cong \mu_2 \). \( \square \)

Remark 4.13. For other relations between the singularities of \( X \) along \( D \) and the different on \( D \), one can consult [Kol13b] Theorem 3.36.

Example 4.14. Consider \((\mathbb{A}^2_{u,v}, D' := V(uv))\), with the action of \( \mathbb{Z}/2\mathbb{Z} \) that sends \((u, v) \mapsto (-u, -v) \). Then \( D' \) is invariant and the quotient is the surface pair \((X := \text{Spec}(k[x, y, z]/xy - z^2), D := V(z))\). The divisor \( D \) is still a nodal curve and the pair is lc since it is the quotient of an lc pair (see [Kol13b] Corollary 2.43]). Consider now the action of \( G := \mu_2 \) on \( X \), which sends \((x, y, z) \mapsto (y, x, -z) \). The action is free in codimension one and preserves \( D \) so \((X/G, D/G)\) is again lc.

The log-canonical divisor on \( X/G \) is not Cartier. Indeed, it suffices to show that the generator of \( G \) acts nontrivially on a section of the log canonical divisor on \( X \). By adjunction, \( \omega_X = (\omega_{X/G})_X \) so a section of \( \omega_X \) is given by \( \left( \frac{dx \wedge dy \wedge dz}{xy - z^2} \right)_X \) which is invariant under \( G \). It follows that \( G \) acts on a generator for \( \omega_X(D) \) by

\[ \frac{dx \wedge dy \wedge dz}{xy - z^2} \mapsto -\frac{dx \wedge dy \wedge dz}{zxy - z^3} \]

so the log canonical divisor is not Cartier on the quotient. Then the twisted pair associated to \((X/G, D/G) = ((X/G), [D/G])\), the divisor \( D/G \) is smooth, but \([D/G]\) is not. By Lemma 4.11 we may conclude that the fixed point \( p \) has different 1 on \( D/G \).

One can argue indirectly that \( p \) must have different 1. Indeed the different is an effective divisor with coefficient at most 1. If it was not 1, then \((D/G, \text{Diff}_{D/G}(0))\) would be klt and \((X/G, D/G)\) would be plt by inversion of adjunction. Then from [Kol13b] Corollary 2.43], the pairs \((X, D)\) and \((\mathbb{A}^2_{u,v}, V(uv))\) would be plt but this is not the case.

Finally we study how the normalization map \( \mathcal{D}^n \to \mathcal{D} \) behaves around the node point \( p \in \mathcal{D} \) in the situation of point (2) of Lemma 4.11.

Corollary 4.15. Suppose that \((X, \mathcal{D})\) is an lc twisted surface pair and \( p \in \mathcal{D} \) is a node lying over a smooth point \( q \) of \( D \). Let \( \mathcal{D}^n \) be the normalization of \( \mathcal{D} \). Then the composition \( \mathcal{D}^n \to \mathcal{D} \to \mathcal{D} \) is an isomorphism in a neighbourhood of each \( q \).
Proof. It suffices to check the required map is an isomorphism after taking étale charts so without loss of generality, we may suppose that $D$ is isomorphic to $[\text{Spec}(k[x,y]/xy)/\mu_2]$ with the action that sends $x \mapsto y$ and $y \mapsto x$. Since normalization commutes with étale base change, we have that the normalization $D^n$ is the quotient

$[\text{Spec}(k[x]) \sqcup \text{Spec}(k[y])]/\mu_2$

with the action $(x, y) \mapsto (y, x)$. Then $D^n$ is just isomorphic to the scheme $\text{Spec}(k[z])$ by the map $x, y \mapsto z$, and we can write the morphisms $D^n \to D \to D$ as

$\text{Spec}(k[x]) \cong [\text{Spec}(k[x]) \sqcup \text{Spec}(k[y])]/\mu_2 \to [\text{Spec}(k[x,y]/xy)/\mu_2] \to \text{Spec}(k[x+y]) \cong \text{Spec}(k[t]).$

The composition is the morphism induced by $t \to x + y \to x$ which is an isomorphism.  

5. Gluing morphisms for families of twisted stable surfaces

In this section we produce gluing morphisms which describe the boundary of the moduli of twisted stable surfaces in terms of moduli of twisted stable surface pairs (see Theorem 1.4). More precisely, in Subsection 5.1, we prove that one can functorially glue a family of twisted stable surface pairs over an arbitrary base to obtain a family of twisted stable surfaces. In Subsection 5.2, we define an algebraic stack $G_{2,b}$ of gluing data. Finally, in Subsection 5.3 we show that the image of $G_{2,v} \to K_{2,v}$ stratifies $K_{2,v}$ into finitely many boundary components.

5.1. Gluing morphisms for twisted stable surfaces. We begin this subsection by recalling some of the results in [Kol13b, Chapter 5].

Consider an slc stable surface $X'$ and let $X$ be its normalization with conductor $D \subseteq X$. Since $X'$ is nodal in codimension 1, the normalization $X \to X'$ induces a rational and generically fixed point free involution $D \dashrightarrow D$. By [Kol13b Proposition 5.12], this gives a generically fixed point free involution on the normalization $\tau : D^n \to D^n$ which preserves the different. It follows from [Kol13b Theorem 5.13] that the converse also holds. Namely, the data of a stable lc pair $(X, D)$ and a generically fixed point free involution $D^n \to D^n$ which preserves the different determine a unique stable surface $X'$. The goal of this subsection is to understand [Kol13b Theorem 5.13] in terms of twisted stable surfaces, which will give us a generalization to families over arbitrary bases.

We begin with two auxiliary results:

**Lemma 5.1.** Let $(X, D)$ be a normal AH twisted stable surface pair, and let $D^n$ be the normalization of $D$. Denote the coarse space of $D^n$ by $D^n$ and let $\tau : D^n \to D^n$ be a different preserving involution on $D^n$. Then there is a unique involution $\sigma : D^n \to D^n$ lifting $\tau$:

$$
\begin{array}{ccc}
D^n & \xrightarrow{\sigma} & D^n \\
\downarrow{\text{coarse space}} & & \downarrow{\text{coarse space}} \\
D^n & \xrightarrow{\tau} & D^n
\end{array}
$$

**Proof.** It follows from Corollary 4.15 and Lemma 4.11 that the coarse space morphism $D^n \to D^n$ is an isomorphism away from the points where the different has a coefficient $0 < c < 1$. These correspond to smooth points of $D$ with nontrivial stabilizer and it suffices check that we can extend the morphism along such points.

From Lemma 4.11 two points with stabilizers of different orders cannot be swapped. Locally around such points on the coarse space, we have an involution $\tau$ of $Y := \text{Spec}(k[s])$ that we wish to extend uniquely to an involution of the $m^{th}$ root stack $\hat{Y} := [\text{Spec}(k[t])/(t \mapsto \zeta_m t)]$ for some $m$. Here $\zeta_m$ is a primitive $m^{th}$ root of unity and $\zeta_m^m = s$. The required assertion follows from the universal property of root stacks (see [Ols16, 10.3]). Such a root stack is the universal objects for
triples \((Y, L, v)\) where \(Y\) has a morphism \(f : Y \to \text{Spec}(k[s])\), \(L\) is a line bundle on \(Y\), and \(v\) is a section of \(L\). Moreover, we require that there is an isomorphism \(L^\otimes_m \to f^*\mathcal{O}_{\text{Spec}(k[s])}\) sending \(v^\otimes_m \to s\).

**Notation 5.2.** Let \(\nu : \mathcal{D}^n \to \mathcal{D}\) be the normalization morphism. We will denote by \(\mathcal{N}\) by \(\mathcal{N}\) and set \(\Delta := \nu^{-1}(\mathcal{N})\). Then \(\Delta\) is a scheme from Lemma [4.10] and Corollary [4.15].

**Lemma 5.3.** Consider a normal twisted stable surface pair \((\mathcal{X}, \mathcal{D})\). Let \(\mathcal{D}^n\) be the normalization of \(\mathcal{D}\) and let \(\tau : \mathcal{D}^n \to \mathcal{D}^n\) be an involution on \(\mathcal{D}^n\) which is generically fixed point free and preserves \(\Delta\). Let \(X\) and \(D\) be the coarse spaces of \(\mathcal{X}\) and \(\mathcal{D}\) respectively. Consider the stable surface \(X'\) obtained from \(X, D\) and \(\tau\) using [Kol13b] Theorem 5.13. Let \(\mathcal{X}'\) be the AH stack associated to \((X', \omega_{X'})\). Then \(\mathcal{X}\) is the normalization of \(\mathcal{X}'\) and there is a map \(h : [\mathcal{D}^n/\tau] \to \mathcal{X}'\) which makes the following diagram commute:

\[
\begin{array}{ccc}
\mathcal{D}^n & \xrightarrow{\mathcal{X}} & \mathcal{X}' \\
\downarrow & & \downarrow h \\
[\mathcal{D}^n/\tau] & \xrightarrow{h} & \mathcal{X}'
\end{array}
\]

**Proof.** Recall that \(\mathcal{X}'\) is a scheme away from a finite set of points. Therefore the normalization \(\nu : \mathcal{Y} \to \mathcal{X}'\) has \(X\) as coarse space, and if we denote by \(p : \mathcal{Y} \to X\) the coarse space map, \(p_*((\nu^*\omega_{\mathcal{X}'})^n) \cong (\omega_X(D))^n\) for every \(n\) by [Kol13b] 5.7.1. But \(\omega_{\mathcal{X}'}\) is a uniformizing line bundle for \(\mathcal{X}'\), and \(\nu\) is representable, so the morphism \(\mathcal{Y} \to B\mathbb{G}_m\) induced by \(\nu^*(\omega_{\mathcal{X}'})\) is also representable. It follows from Lemma [1.3] that \(\mathcal{Y} \cong \mathcal{X}'\). In particular we have a map \(\mathcal{X} \to \mathcal{X}'\) which induces \(f : \mathcal{D}^n \to \mathcal{X}'\). There is an open dense subset \(U\) of \(\mathcal{D}^n\) where \(\mathcal{D}^n\) is a scheme and the two maps \(f|_U, (f \circ \tau)|_U : U \to \mathcal{X}'\) agree. By [DH18] Lemma 7.3, the arrows \(f\) and \(f \circ \tau\) agree and so they induce a map \(h : [\mathcal{D}^n/\tau] \to \mathcal{Y}\) making the diagram commute. \(\square\)

The canonical line bundle \(\omega_{\mathcal{X}'}\) induces a map \(\mathcal{X}' \to B\mathbb{G}_m\) and we obtain a map \([\mathcal{D}^n/\tau] \to B\mathbb{G}_m\) given by composition with the map \(h\) from Lemma 5.3. We wish to understand this map \([\mathcal{D}^n/\tau] \to B\mathbb{G}_m\) in terms of \(\mathcal{X}', \mathcal{D}^n\) and \(\tau\). First, observe that \(\tau\) induces a map \(\psi : [\mathcal{D}^n/\tau] \to B\mu_2\). We can identify \(\text{Pic}(B\mu_2)\) with the characters of \(\mu_2\) and we denote by \(\chi_{B\mu_2}\) the line bundle corresponding to the sign representation. Given a morphism \(\mathcal{Y} \to B\mu_2\), we will denote by \(\chi_\mathcal{Y}\) the pull back of \(\chi_{B\mu_2}\) to \(\mathcal{Y}\).

Now, the involution \(\tau\) naturally acts on \(\omega_{\mathcal{D}^n}(\Delta)\) since \(\tau\) preserves the different \(\Delta\). Thus \(\omega_{\mathcal{D}^n}(\Delta)\) descends to a line bundle \(\mathcal{G}\) on \([\mathcal{D}^n/\tau]\).

**Proposition 5.4.** In the situation above, the line bundle corresponding to \([\mathcal{D}^n/\tau] \to B\mathbb{G}_m\) is \(\mathcal{G} \otimes \chi_{[\mathcal{D}^n/\tau]}\).

**Proof.** Let \(\pi : \mathcal{D}^n \to [\mathcal{D}^n/\tau]\) be the quotient map, let \(\mathcal{H} := h^*\omega_{\mathcal{X}'},\) and let \(\sigma : \mu_2 \times \mathcal{D}^n \to \mathcal{D}^n\) be the action induced by \(\tau\). The key ingredient is [Kol13b] Proposition 5.8.

From the diagram of Lemma 5.3 and Proposition 4.6, we have \(\pi^*\mathcal{H} \cong \omega_{\mathcal{D}^n}(\Delta)\). The line bundle \(\mathcal{H}\) is uniquely determined by the isomorphism \(\alpha : \sigma^*(\omega_{\mathcal{D}^n}(\Delta)) \to p_2^*(\omega_{\mathcal{D}^n}(\Delta))\) (see [Rom05] Example 4.3). To determine \(\alpha\), it suffices to restrict \(\mathcal{H}\) to a dense open subset of \(\mathcal{D}^n\); we can assume that \(\mathcal{X}'\) and \([\mathcal{D}^n/\tau]\) are schemes. Now from [Kol13b] Proposition 5.8, the sections of \(\mathcal{H}\) and those of \(\mathcal{G} \otimes \chi_{B\mu_2}\) agree so \(\mathcal{H} \cong \mathcal{G} \otimes \chi_{B\mu_2}\). \(\square\)

**Notation 5.5.** We will denote by \(\mathcal{C}\) the relative coarse moduli space of the map \([\mathcal{D}^n/\tau] \to B\mathbb{G}_m\) induced by \(\mathcal{G} \otimes \chi_{[\mathcal{D}^n/\tau]}\).
Now, since \( \tau \) sends \( \Delta \) to itself, we can factor \( \Delta \to C \) as \( \Delta \to [\Delta/\tau] \to [D^n/\tau] \to C \) and we have the following commutative diagram.

\[
\begin{array}{ccc}
\Delta & \xrightarrow{f} & N \\
\downarrow & & \downarrow \\
[\Delta/\tau] & \xrightarrow{\nu} & D \\
\end{array}
\]

\[
\begin{array}{ccc}
D^n & \xrightarrow{\nu} & D \\
\downarrow & & \downarrow \\
N & \xrightarrow{\psi} & C \\
\end{array}
\]

**Lemma 5.6.** The map \( f : [\Delta/\tau] \to C \) is a closed embedding. In particular, the composition \([\Delta/\tau] \to C \to B\mathbb{G}_m \) is representable.

**Proof.** Since \( \Delta \to D^n \) is a closed embedding, then \([\Delta/\tau] \to [D^n/\tau] \) is also a closed embedding. Therefore the claim follows if we can prove that \( \psi : [D^n/\tau] \to C \) is an isomorphism along \([\Delta/\tau] \). Let \( \pi : D^n \to [D^n/\tau] \) be the quotient map.

If \( q \in \Delta \) such that \( \tau(q) \neq q \), then \( \pi(q) \) has no stabilizer so \( \psi \) is necessarily an isomorphism in a neighborhood of \( \pi(q) \). Assume then that \( \tau(q) = q \). Formally locally around \( q \), we can replace \( D^n \) with \( \Spec(k[z]) \) and assume the involution \( \tau \) sends \( z \mapsto -z \) where \( z \) is a uniformizer for \( q \). Since \( dz/z \) is a generator of \( \omega_{D^n}(\Delta) \), \( \tau \) acts trivially on the fiber of \( \omega_{D^n}(\Delta) \) at \( q \). Therefore, \( \tau \) acts faithfully on the fiber of \( G \otimes X_{[D^n/\tau]} \) at \( q \) and so by [AH11 Proposition 2.3.10], \( \psi \) is an isomorphism at \( \pi(q) \).

We now define a DM stack \( S \) with two representable morphisms \( g_1 : N \to S \) and \( g_2 : [\Delta/\tau] \to S \), such that the two morphisms \( \Delta \to [\Delta/\tau] \to S \) and \( \Delta \to N \to S \) are isomorphic. We will denote with roman letters the coarse space of a DM stack, as in the conventions. We define \( S \) to be the pushout in algebraic spaces of the diagram below:

\[
\begin{array}{ccc}
\Delta & \xrightarrow{f} & N \\
\downarrow & & \downarrow \\
[\Delta/\tau] & \xrightarrow{g} & S \\
\end{array}
\]

Thus \( S \) is a disjoint union of points. We determine a stack \( S \) with coarse space \( S \) by requiring that for every point \( q \in S \), the group \( \text{Aut}_q(S) \) is \( \mu_2 \) if there is a \( \mu_2 \) stabilizer on a point of \( g_1^{-1}(q) \cup g_2^{-1}(q) \) and trivial otherwise. Recall these are the only options for the stabilizers of \([\Delta/\tau]\) and \( N \). Now by Lemma 5.7, the morphisms \( g_i \) are uniquely determined.

**Lemma 5.7.** Let \( a, b \) be positive integers with \( a \) even. Then, up to isomorphism, there are unique representable morphisms \( \Spec(k) \to B\mu_b \) and \( B\mu_2 \to B\mu_a \).

**Proof.** Since \( k \) is algebraically closed, there is a unique \( \mu_b \)-torsor over \( \Spec(k) \), so there is a unique morphism \( \Spec(k) \to B\mu_b \). For the claim about \( B\mu_2 \), consider the presentation of \( B\mu_2 \) given by \( \mu_2 \Rightarrow \Spec(k) \), where one arrow \( \sigma \) is the action of \( \mu_2 \) on the point and the other one \( \pi_2 \) is the structure map. Note however, since \( \mu_2 \) acts trivially on \( \Spec(k) \) these are both just the structure map. Then a morphism \( B\mu_2 \to B\mu_a \) is equivalent to the data of:

1. A morphism \( f : \Spec(k) \to B\mu_a \), and
2. An isomorphism \( \psi \) in \( B\mu_a(\mu_2) \cong B\mu_a(\Spec(k)) \times B\mu_a(\Spec(k)) \) between \( f \circ \sigma \) and \( f \circ \pi_2 \).
Moreover, \( \psi \) must satisfy the cocycle condition. More precisely, let \( m : \mu_2 \times \mu_2 \to \mu_2 \) be the multiplication and \( p_{2,3} \) the projection onto the second factor \( \mu_2 \times \mu_2 \to \mu_2 \). Then we require that 
\[
p_{2,3}^* \psi \circ (\text{Id}_{\mu_2} \times \sigma)^* \psi = m^* \psi.
\]

Now, there is a unique morphism \( \text{Spec}(k) \to B\mu_a \) from the previous point. An isomorphism \( \psi \) as above is the data of two automorphisms \((\alpha, \beta)\) in \( B\mu_a(\text{Spec}(k)) \). We will denote by \( \{\pm 1\} \) the two points of \( \mu_2 \) and by \((1, 1), (1, -1), (-1, 1), (-1, -1)\) the four points of \( \mu_2 \times \mu_2 \). To fix the notation, \( \alpha \) will be the automorphism over 1 and \( \beta \) the one over \(-1\). Then the arrows \( p_{2,3}, \text{Id}_{\mu_2} \times \sigma \) and \( m \) behave as follows:

- \( p_{2,3}(1, 1) = p_{2,3}(-1, 1) = 1 \) and \( p_{2,3}(1, -1) = p_{2,3}(-1, -1) = -1 \);
- \( \text{Id}_{\mu_2} \times \sigma(1, 1) = \text{Id}_{\mu_2} \times \sigma(1, -1) = 1 \) and \( \text{Id}_{\mu_2} \times \sigma(-1, 1) = \text{Id}_{\mu_2} \times \sigma(-1, -1) = -1 \);
- \( m(1, 1) = m(-1, -1) = 1 \), whereas \( m(1, -1) = m(-1, 1) = -1 \).

In particular, the cocycle condition is the following equality of automorphisms over \( \mu_2 \times \mu_2 \):
\[
(\alpha, \alpha, \beta, \beta) \circ (\alpha, \alpha, \beta, \beta) = (\alpha, \beta, \beta, \alpha).
\]

Therefore we must have \( \alpha \circ \beta = \beta \) which implies \( \alpha = \text{Id} \) and that \( \beta \circ \beta = \text{Id} \). Thus the only \( \psi \) which satisfy the cocycle condition are \((\text{Id}, \text{Id})\) and \((\text{Id}, \beta)\) where \( \beta \) is the unique element of \( \mu_a \) with \( \beta^2 = \text{Id} \) and \( \beta \neq \text{Id} \). This means that there are exactly two morphisms \( B\mu_2 \to B\mu_a \). The first one is the composition \( B\mu_2 \to \text{Spec}(k) \to B\mu_a \) and the second one is induced by the morphism \( \mu_2 \hookrightarrow \mu_a \) as in [Ols16 Exercise 10.F]. Only the second is representable. \( \square \)

**Proposition 5.8.** With notation as above, let \( X' \) be the AH stack associated to \( (X' \to \text{Spec}(k), \omega_{X'}) \). Then there is a unique representable morphism \( S \to X' \) which makes the diagram below commutative.

\[
\begin{array}{ccc}
\Delta & \to & N \\
\downarrow & & \downarrow \\
D^n & \to & D \\
\downarrow \left[ \Delta/\tau \right] & & \downarrow f \\
S & \to & X \\
\downarrow & & \downarrow \\
[\Delta/\tau] & \to & X'
\end{array}
\]

Proof. It suffices to consider the following subdiagram:

\[
\begin{array}{ccc}
N' & \to & X' \\
\downarrow & & \downarrow \\
[\Delta/\tau] & \to & S & \to & X'.
\end{array}
\]

Up to considering one connected component of \( S \) at a time, we can assume that \( S \cong \text{Spec}(k) \). Now, recall that the diagram on coarse spaces is a pushout and the diagram of solid arrows commutes. Then if we replace all the stacks above with their coarse spaces, we have an arrow \( q : S = \text{Spec}(k) \to X' \). In particular, if we denote \( \mu_a := \text{Aut}_{X'}(q) \), we have a closed embedding \( B\mu_a \to X' \) (this follows from [Ols16 Theorem 11.3.1]). The arrows \[ \Delta/\tau \to X' \] and \( N \to X' \) factor through \( B\mu_a \to X' \), so in the diagram above we can replace \( X' \) with \( B\mu_a \). Now the claim follows from the description of \( S \) and Lemma [5.7]. \( \square \)

We recall a theorem due to David Rydh on the existence of pinchings in algebraic stacks:
Theorem 5.9 ([Ryd14, Theorem A.4]). Consider a diagram of solid arrows between algebraic stacks over $B$ as below, with $i$ a closed embedding and $f$ a finite morphism.

\[
\begin{array}{ccc}
\mathcal{X} & \overset{i}{\to} & \mathcal{Y} \\
\downarrow f & & \downarrow f' \\
\mathcal{X}' & \overset{i'}{\to} & \mathcal{Y}'
\end{array}
\]

Then there are arrows $i'$ and $f'$ over $S$, such that the resulting diagram is a pushout in algebraic stacks. Moreover, $i'$ is a closed embedding and $f'$ is integral and an isomorphism away from $\mathcal{X}$. If we take the topological spaces of the algebraic stacks above, the corresponding square is a pushout in topological spaces. Finally, if $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{X}'$ are proper over $B$, then also $\mathcal{Y}'$ is proper over $B$ (recall that for us all the algebraic stacks are of finite type).

Remark 5.10. The construction of $\mathcal{Y}'$ can be performed smooth locally. In particular, if we take $\text{Spec}(A') \to \mathcal{Y}'$ a smooth morphism, and we pullback $i'$, $f'$, $i$ and $f$ through it, the corresponding diagram will be a pushout in schemes.

The following Lemma is the last technical result we need before proving Theorem 5.14.

Lemma 5.11. The following diagram is a pushout in algebraic stacks:

\[
\begin{array}{ccc}
\Delta & \longrightarrow & \mathcal{N} \\
\downarrow & & \downarrow \\
\mathcal{D}^n & \longrightarrow & \mathcal{D}
\end{array}
\]

Proof. From [Ryd14, Theorem A.4], the pushout $\mathcal{D}'$ of the diagram without $\mathcal{D}$ exists; we need to show that $\mathcal{D}' \cong \mathcal{D}$. Since $\mathcal{D}'$ can be constructed étale locally, we can assume that $\mathcal{D}$, $\mathcal{D}^n$, $\mathcal{N}$ and $\Delta$ are schemes. Then the conclusion follows from [Sch].

Now we aim at constructing $\mathcal{X}'$ from the data of $\mathcal{D}$, $\tau$ and $\mathcal{X}$. The situation is summarized in the following diagram, where the red arrows are pushouts, the blue ones come from the universal property of a pushout, and all the arrows are representable. From the definition of $S$, the morphism $[\Delta/\tau] \to S$ is finite. Therefore from [Ryd14, Theorem A.4] the two maps $[\Delta/\tau] \to S$ and $[\Delta/\tau] \to C$ have a pushout $C \to Q$ and $S \to Q$. Then we have a morphism $\alpha : Q \to \mathcal{X}'$. To check that $\alpha$ is representable, it suffices to check that it is injective on automorphisms of closed points. But from the description of $Q$ given in [Ryd14, Theorem A.4], it suffices that $C \to \mathcal{X}'$ and $S \to \mathcal{X}'$ are representable, which is true.

Then by Lemma 5.11 we have a morphism $\mathcal{D} \to Q$ which is representable by [Ryd14, Theorem A.4]. One can check that it is also quasifinite, and $\mathcal{D}$ is proper, so $\mathcal{D} \to Q$ is finite. Finally, again from [Ryd14, Theorem A.4], the two maps $\mathcal{D} \to \mathcal{X}$ and $\mathcal{D} \to Q$ admit a pushout $Z$. As before, we have a representable morphism $\Phi : Z \to \mathcal{X}'$. 

(4)
Proposition 5.12. The morphism $\Phi$ is an isomorphism. In particular, $Z$ is Gorenstein.

Proof. From the universal property of the pushouts and coarse spaces, one can check that taking the coarse space commutes with pushouts. In particular, by [Kol13b, Chapter 5], the morphism $\Phi$ is an isomorphism on coarse spaces. Furthermore, $Z$ is seminormal since it is the pushout of seminormal stacks.

Observe that $\Phi$ is an isomorphism away from a finite set of points $\{p_1, \ldots, p_n\}$. Let $p := q_i$ and let $m := |\text{Aut}_{\mathcal{X}}(p)|$ be the index of $K\mathcal{X}$ at $p$. If $n := |\text{Aut}_{\mathcal{Z}}(p)|$, the line bundle $\Phi^* (\omega_{\mathcal{X}})^{\otimes m}$ descends to a line bundle in a neighborhood of $p \in \mathcal{X}$ and agrees with $\omega_{\mathcal{X}}^{[n]}$ in codimension one so the index $m$ divides $n$. On the other hand, by representability, $n$ divides $m$. Thus $m = n$ and the morphism $\Phi_p : \text{Aut}_{\mathcal{Z}}(p) \to \text{Aut}_{\mathcal{X}}(p)$ is bijective. We conclude that $\Phi$ is a proper morphism of seminormal DM stacks such that $Z(\text{Spec}(k)) \to X'(\text{Spec}(k))$ is an equivalence so $\Phi$ is an isomorphism.

Example 5.13. Consider $(X, D) = (\mathbb{A}^2_{x,y}, V(xy))$, so $D$ is a nodal curve and consider the involution $\tau : D \to D$ which sends $(x,y) \mapsto (-x,-y)$. Call the resulting slc surface $X'$.

Observe that $(\omega_{\mathcal{X}}(D))|_D = \omega_D$ is generated by the section $\frac{dx}{y}$ which is fixed by $\tau$. Thus the action on $\omega_D \otimes \chi$ is non-trivial and $\mathcal{C} \cong [D^n/\tau]$. Then $N = p$ is a node, $\Delta = q_1 \sqcup q_2$ lying above the node which are both fixed points of $\tau$, and $S = [p/\mu_2] = B\mu_2$. It follows that $Q = [D/\tau]$ and the AH stack $X'$ of $X'$ is the pushout in stacks of $D \to \mathbb{A}^2$ and $D \to [D/\tau]$. In particular it has a $\mu_2$ stabilizer at the origin. Therefore $\omega_{\mathcal{X'}}$ has index 2 at the origin.

Using a computer algebra system, one can compute equations for $X'$ directly from the presentation above. They are given by the following ideal in $k[a, b, c, d, e]$:

$$(cd - ae, bd - ce, ab - c^2, de - e^2, bc^2 - e^2, ac^2 - d^2).$$

We are now ready to prove the main theorem of this subsection, which states that the construction of $Z$ above can be carried out in families over an arbitrary base.

Theorem 5.14. Consider a twisted stable family of normal surface pairs $(f : \mathcal{X} \to B, \phi : \omega_f \to \mathcal{L})$. Assume that $\mathcal{D}_{\phi} \to B$ has a simultaneous normalization, namely that there is a flat proper morphism $D^n \to B$ and a morphism $\nu : D^n \to \mathcal{D}_{\phi}$ such that for every $b \in B$, the map $\nu_b : D^n_b \to (\mathcal{D}_{\phi})_b$ is a normalization. Let $\{\sigma_i : B \to D^n\}$ be disjoint sections which surject onto the locus where $D^n \to \mathcal{D}_{\phi}$ is not an isomorphism. Finally let $\tau : D^n \to D^n$ be an involution over $B$ which preserves the closed substack $\bigsqcup \sigma_i(B)$ and which is fiberwise generically fixed point free.

Then there exists a DM stack $\mathcal{Z}$ which fits in the commutative diagram below.

$$
\begin{array}{ccc}
\mathcal{D^n} & \xrightarrow{\phi} & \mathcal{X} \\
\downarrow & & \downarrow \\
[\mathcal{D^n}/\tau] & \xrightarrow{\tau} & \mathcal{Z}
\end{array}
$$

Moreover, $\mathcal{Z} \to B$ is flat and proper, and for every $p \in B$, the fiber $\mathcal{Z}_p$ is the AH stack of the stable surface obtained from gluing data $D^n_p$, $\tau|_{D^n_p}$ and $X_p$ by [Kol13b, Theorem 5.13].

Proof. Throughout the proof, we will drop the subscript $\phi$ on $\mathcal{D}_{\phi}$. We will denote by $\Delta$ the closed substack, which is actually a scheme, given by the sections $\sigma_i$. Furthermore, we can assume that $S$ is connected.

From Proposition 4.6 we have that $\mathcal{L}|_{D^n} \cong \omega_{\mathcal{D}^n/B}(\Delta)$. It descends to a line bundle $\mathcal{G}$ on $[\mathcal{D}^n/\tau]$, and we let $[\mathcal{D}^n/\tau] \to \mathcal{C}$ be the relative coarse moduli space of the map $[\mathcal{D}^n/\tau] \to B\mathbb{G}_m,B$ induced by $\mathcal{G} \otimes \chi^{[\mathcal{D}^n/\tau]}$. The induced morphism $\mathcal{C} \to B$ is flat and the construction of $\mathcal{C}$ commutes with arbitrary base change $B' \to B$ by [AOV11, Proposition 3.4].

Consider the quotient stack $[\Delta/\tau]$. Now, $\Delta \subseteq D^n$ is a disjoint union of sections of $\mathcal{D}^n \to B$, so $\Delta \cong \bigsqcup B$. Since $\tau$ acts fiberwise and preserves $\Delta$, and since $B$ is connected, we have that
\[ \frac{\Delta}{\tau} \cong \frac{\Delta_b}{\tau} \times B \] for any \( b \in B \). In other words, \( \frac{\Delta}{\tau} \) is a constant family over \( B \). Similarly, if we denote by \( N \) the singular locus of \( D \to B \), then \( N \cong N_b \times B \) and \( \Delta \cong \Delta_b \times B \). For any fixed \( b \), considering \( \frac{\Delta_b}{\tau} \) and \( N_b \), we can construct \( S_b \) as in Proposition 5.8. Now we can define \( S := S_b \times B \). Then \( S \) fits into a commutative diagram as below and by construction its formation commutes with base change.

\[
\begin{array}{ccc}
\Delta & \longrightarrow & N \\
\downarrow & & \downarrow \\
\frac{\Delta}{\tau} & \longrightarrow & S
\end{array}
\]

Next we proceed by constructing the analogous pushouts from Diagram 4 and the discussion before it. Along the way, we will need to check that our pushouts, which are pinchings as in Theorem 5.9 commute with arbitrary basechange \( B' \to B \). This will be checked étale locally (see Remark 5.10) using Lemma 5.15.

First we have the following diagram which, as in Lemma 5.11, we claim that is a pushout:

\[
\begin{array}{ccc}
\Delta & \longrightarrow & N \\
\downarrow & & \downarrow \\
& & \\
& & \\
D^\alpha & \longrightarrow & D.
\end{array}
\]

Indeed, \( \Delta, N, D^\alpha \) and \( D \) are flat over \( B \), so we can check that the diagram above is a pushout after pulling back along Spec\((k) \to B \), but this is the content of Lemma 5.11.

Next we construct the analogous pushouts to the ones in red in Diagram 4. This produces the desired proper morphism \( Z \to B \) which is flat and commutes with base change by the above discussion. So \( Z \) satisfies the claimed properties.

In the proof of Theorem 5.14, we needed the following technical result to check that the gluing construction commutes with base change and produces a flat family.

**Lemma 5.15.** Let \( R \) be a ring, and consider two homomorphisms of \( R \)-algebras \( f : A' \to B' \) and \( g : B \to B' \) with \( A := A' \times_{B'} B \) their fiber product. Assume that \( B' \) is flat over \( R \) and that \((g, f) : B \times A' \to B' \) is surjective. Then the square

\[
\begin{array}{ccc}
A' \otimes_R S & \longrightarrow & B' \otimes_R S \\
\downarrow & & \downarrow \\
A \otimes_R S & \longrightarrow & B \otimes_R S
\end{array}
\]

is cartesian for any ring homomorphism \( R \to S \). Moreover, if \( A' \) and \( B \) are flat over \( R \), so is \( A \).

**Proof.** First note that the following sequence is exact:

\[
0 \to A \to B \times A' \xrightarrow{(g, f)} B' \to 0.
\]

Since \( B' \) is flat over \( R \), we also have that

\[
0 \to A \otimes_R S \to (B \times A') \otimes_R S \to B' \otimes_R S \to 0
\]

is exact. On the other hand, \((B \times A') \otimes_R S \cong (B \otimes_R S) \times (A' \otimes_R S)\), so \( A \otimes_R S \) is the pullback of \( B \otimes_R S \to B' \otimes_R S \) and \( A' \otimes_R S \to B' \otimes_R S \), proving the first claim. The second claim follows by applying the long exact sequence of Tor\(_i\) to the first exact sequence in the proof above. \( \square \)
5.2. Gluing data. In this subsection we package the information of a gluing data into an algebraic stack \( \mathcal{G}_{2,v} \). Therefore, Theorem \([\text{[Kol13]} \text{, Theorem 5.13]}\) produces a gluing morphism \( \mathcal{G}_{2,v} \to \mathcal{K}_{2,n}^{2} \), which on the level of points agrees with the gluing morphism of \([\text{Kol13]} \text{, Theorem 5.13]}\).

**Proposition 5.16.** There is an algebraic stack \( \mathcal{G}_{2,v} \) which parametrizes the following objects. Over a scheme \( B \), the objects of \( \mathcal{G}_{2,v}(B) \) are quadruples
\[
(f : \mathcal{X} \to B, \phi : \omega_f \to \mathcal{L}); g : \mathcal{D}^n \to \mathcal{D}_\phi; \{\sigma_i : B \to \mathcal{D}^n\}; \tau : \mathcal{D}^n \to \mathcal{D}^n)
\]
where:

1. \((f : \mathcal{X} \to B, \phi : \omega_f \to \mathcal{L})\) is an object of \( \mathcal{K}_{2,v}(B)\);
2. \(\mathcal{D}^n \to B\) is a flat family of orbifold smooth curves and \( g \) is a simultaneous normalization;
3. For a certain \( n \), we have \( n \) disjoint sections \( \sigma_i \) of \( \mathcal{D}^n \to B \), such that \( g \circ \sigma_i(b) \in (D^\text{sing})_b \) and such that \( g \) is an isomorphism away from \( \bigcup \sigma_i(B) \), and
4. \( \tau \) is a generically fixed point free involution which preserves \( \bigcup \sigma_i(B) \).

The morphisms are pullback diagrams which satisfy the obvious commutativity conditions.

**Proof.** We will construct \( \mathcal{G}_{2,v} \) one condition at the time. It suffices to construct \( \mathcal{G}_{2,v} \) for a fixed choice of \( n \) of point \( (3) \), and then to construct \( \mathcal{G}_{2,v} \) by taking an union over \( n \in \mathbb{N} \). Therefore, from now on we consider the number of sections \( n \) as part of the data.

First, consider \( \mathcal{D} \to \mathcal{K}_{2,v} \) the universal divisor. Consider the stack \( \mathcal{H} := \mathcal{O}_B \times \mathcal{C} \) parametrizing polarized orbifold curves, and let \( \mathcal{C} \to \mathcal{O} \) be the universal curve. Consider now \( \mathcal{H}_1 := \mathcal{Hom}_{\mathcal{O}_B \times \mathcal{K}_{2,v}}(\mathcal{C} \times \mathcal{K}_{2,v}, \mathcal{O} \times \mathcal{D}) \), where for the definition of this Hom stack we refer to \([\text{AOV11]} \text{, Appendix C]}\). Over \( \mathcal{H}_1 \) we have an universal curve \( \mathcal{C}_1 \), obtained from the morphism \( \mathcal{H}_1 \to \mathcal{O} \). Consider then \( \mathcal{H}_2 := \mathcal{Hom}_{\mathcal{H}_1}(\mathcal{H}_1, \mathcal{C}_1)^\times_{\mathcal{H}_1} \), which is the stack that parametrizes \( n \) sections of \( \mathcal{C}_1 \to \mathcal{H}_1 \). Let \( \mathcal{C}_2 \) be the universal curve of \( \mathcal{H}_2 \), and finally consider \( \mathcal{H}_3 := \mathcal{Hom}_{\mathcal{H}_2}(\mathcal{C}_2, \mathcal{C}_2) \). Over a base \( B \), the objects of \( \mathcal{H}_3(B) \) are the following:

1. \((f : \mathcal{X} \to B, \phi : \omega_f \to \mathcal{L})\), an object of \( \mathcal{K}_{2,v}(B)\);
2. \(\mathcal{C} \to B\) is a flat family of orbifold nodal curves and a map \( \mathcal{g} : \mathcal{C} \to \mathcal{D}\);
3. \( n \) sections \( \sigma_i : B \to \mathcal{C} \), and
4. \( \tau : \mathcal{C} \to \mathcal{C} \) a morphism.

Let \( \mathcal{C} \) be the universal curve over \( \mathcal{H}_3 \), and let \( \mathcal{D}_3 \to \mathcal{H}_3 \) be the pull-back of \( \mathcal{D} \).

Now, according to \([\text{AOV11]} \text{, Appendix C]}\), there is an open substack of \( \mathcal{H}_1 \) which parametrizes representable morphisms \( g \). So up to replacing \( \mathcal{H}_1 \) with this open substack, we can assume \( g \) to be representable. We will denote with \( \pi : \mathcal{C} \to \mathcal{C}^{\text{r.c.}} \) the relative coarse space of \( \mathcal{C} \to \mathcal{H}_3 \), and let \( U \subseteq \mathcal{C} \) to be the locus where \( \pi \) is an isomorphism, and \( \mathcal{C} \to \mathcal{H}_3 \) is smooth. Having the sections \( \sigma_i \) to be disjoint, and to map to \( U \), is an open condition. Then up to shrinking \( \mathcal{H}_3 \), we can assume \( \sigma_i \) to be disjoint, and to map to \( U \).

From the upper semicontinuity of the dimension of the fibers, we can also assume that the morphism \( \mathcal{C} \to \mathcal{D} \) is quasi-finite. But then it is finite since it is also proper and representable. Consider now the morphism \( \mathcal{O}_D \to \mathcal{g}_*\mathcal{O}_C \), let \( \mathcal{K}_1 \) be its kernel, and let \( \mathcal{S}_1 \) be the support of \( \mathcal{K}_1 \). Then \( \mathcal{S}_1 \to \mathcal{H}_3 \) is proper. Thus from the upper-semicontinuity of the dimension of the fiber, applied to \( \mathcal{S}_1 \to \mathcal{H}_3 \), there is an open substack where \( \mathcal{S}_1 \) is empty so \( g \) is dominant. Since \( g \) is proper it must also be surjective on this subset. Next let \( \mathcal{K}_2 \) be the cokernel of \( \mathcal{O}_D \to \mathcal{g}_*\mathcal{O}_C \) and \( \mathcal{S}_2 \) its support. As before, by upper-semicontinuity there is an open substack where \( \mathcal{S}_2 \) is 0-dimensional and so \( g \) is generically an isomorphism. Since it is also a representable morphism of nodal curves, the locus where it is not an isomorphism must be contained in the nodes.

To recap, we have now cut out an algebraic stack where \( g \) is a simultaneous normalization and \( \sigma_i(B) \) are disjoint sections. We need to identify the locus where:

- \((g \circ \sigma_i)(b) \in \mathcal{D}_b^{\text{sing}}\) for every \( b \in B \);
• Fiber by fiber, \( \tau \) is a generically fixed point free involution.

To address the first bullet point, consider \( S' \to D \) the inclusion of the \( g \)-singular locus. This is a closed embedding, and consider the following fibred diagram:

\[
\begin{array}{ccc}
F & \xrightarrow{h} & S' \\
\downarrow & & \downarrow \\
\bigsqcup_{i=1}^n \sigma_i(H_3) & \to & D
\end{array}
\]

Then \( h \) will be a closed embedding. Requiring \( h \) to be an isomorphism is equivalent to the first bullet point. Then the flattening stratification guarantees that there is a well-defined closed substack where \( h \) is an isomorphism. In other terms, up to replacing \( H_3 \) with a locally closed substack, we can assume that the first bullet point is satisfied.

Finally, being an isomorphism is an open condition, so there is an open substack of \( H_3 \) where \( \tau \) is an isomorphism. Observe now that if \( \sigma \) is an isomorphism of an orbifold nodal curve, such that it agrees with the identity on an open dense subset, then \( \sigma = \text{Id} \). Thus if \( \tau \) fixes a generic point of \( C \), then it fixes the irreducible component that is its closure.

Consider then the following fiber diagram:

\[
\begin{array}{ccc}
F & \xrightarrow{\psi} & C \\
\downarrow & & \downarrow \text{Diag} \\
\bar{C} & \to & C \times_{H_3} \bar{C}
\end{array}
\]

\( F \) is the fixed locus of \( \tau \). We need to cut out the locus where \( F \) contains no irreducible components of \( C \). This is equivalent to \( F \to H_3 \) being finite so by semi-continuity of fiber dimension, there is an open subset where \( \tau \) fixes no generic points. Similarly, to ensure that \( \tau^2 = \text{Id} \), we can replace in the diagram above \( \tau \) with \( \tau^2 \). Let \( F' \) be the new fiber product we obtain. Then the locus where \( \tau^2 = \text{Id} \) is the locus where \( \pi_1 : F' \to \bar{C} \) is surjective. Or in other terms, where the kernel of the map \( \mathcal{O}_{\bar{C}} \to (\pi_1)_*\mathcal{O}_{F'} \) is zero. This is the locus where the support of \( \text{Coker}(\mathcal{O}_{\bar{C}} \to (\pi_1)_*\mathcal{O}_{F'}) \) is empty: it is an open substack of \( H_3 \).

Putting this together with Theorem 5.14 we obtain:

**Theorem 5.17.** There is a functorial gluing morphism \( \mathcal{G}_{2,v} \to \mathcal{K}'_{2,v} \) from the stack of gluing data to the stack of twisted stable surfaces.

5.3. **The boundary strata of \( \mathcal{K}'_{2,v} \).** In this subsection, we show that there is a locally closed stratification of \( \mathcal{K}'_{2,v} \) of equinormalizable surfaces with equinormalizable double locus which are the images of the gluing morphisms above.

**Lemma 5.18.** Let \( f : \mathcal{Y} \to B \) be a proper family of generically reduced DM stacks over a base scheme \( B \). Then there exists a locally closed stratification of \( B \) over which \( f \) is simultaneously normalizable.

**Proof.** Using Noetherian induction, it suffices to prove that there is a nonempty open subset of \( B \) where \( f \) is simultaneously normalizable. Up to replacing \( B \) with its reduced structure, we can assume that \( B \) is reduced. Then it is generically smooth, so up to further shrinking \( B \) we can assume that it is smooth and connected. Consider \( \nu : \mathcal{Y}^n \to \mathcal{Y} \) the normalization. Up to shrinking \( B \) we can assume that \( \mathcal{Y}^n \to B \) is flat. The generic geometric fiber is normal and the locus \( U \) in \( \mathcal{Y} \) where the fibers are normal is open from [Gro66, Théorème 12.2.6]. So its complement \( Z := \mathcal{Y} \setminus U \) is closed, and since \( f \) is proper, \( f(Z) \) is closed too. Then up to shrinking \( B \) we can assume that \( \mathcal{Y}^n \to B \) has normal fibers.

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But $\nu$ is an isomorphism on the smooth locus of the morphism $Y \to B$, since $B$ is normal. In particular, for every $b \in B$, the map $\nu_b^* \to \nu_b$ is an isomorphism at the generic points of $\nu_b$, and it is finite. So it is a simultaneous normalization.

**Proposition 5.19.** There is a finite, locally closed stratification of $K^2_{2,v}$ such that each stratum is the image of a family of gluing data under the gluing morphism $G_{2,v} \to K^2_{2,v}$. In particular, taking the scheme theoretic images of components under the gluing morphism stratifies $K^2_{2,v}$ into a finite, locally closed union of boundary components.

**Proof.** Let $X \to K^\omega_{n,v}$ be the universal twisted stable surface. From Lemma 5.18 there is a locally closed embedding $S \to K^\omega_{n,v}$ where the fibers of $X \to K^\omega_{n,v}$ admit a simultaneous normalization. Let $X' := X \times_{K^\omega_{n,v}} S$ be the pull back, and let $D$ be the closed substack which cuts the fiberwise double locus. Again from Lemma 5.18 there is a locally closed embedding where the fibers of $D \to S$ are simultaneously normalizable. So up to further stratifying $S$ we can assume that both $X'$ and $D$ admit a simultaneous normalization. Let $\nu : \mathcal{D}^n \to D$ be the simultaneous normalization of $D \to S$.

Let $Z$ be the singular locus of $D \to S$, and $F$ its pull back through $\nu$. Up to stratifying further $S$, we can assume that $F \to S$ is étale, and up to replacing $S$ with an étale cover, we can assume that there are disjoint sections sections $\sigma_i : S \to \mathcal{D}^n$ which surject to $F$.

The composition $\mathcal{D}^n \to X'$ is generically a 2 to 1 cover onto its image so $\mathcal{D}^n$ is equipped with a rational involution $\tau$. After further stratifying and applying Lemma 5.1 we can assume that $\tau$ is a morphism. One can check fiber by fiber to see that $\tau$ preserves $\lim\im(\sigma_i)$. This gives a family of gluing data whose image in $K^2_{2,v}$ is this stratum. Finally, by [Kol13b, Theorem 5.13], every point of $K^2_{2,v}$ corresponding to a non-normal twisted stable surface lies in such a stratum.

**Remark 5.20.** Proposition 5.19 only posits the existence of some finite stratification and says nothing about how to enumerate the strata, nor the components of $G_{2,v}$. We include it to rule out pathological behaviour like the image of $G_{2,v}$ being an infinite disjoint union of points. One hopes for a more functorial stratification described in terms of combinatorial and numerical data of the surfaces, as well as a generalization of functorial gluing morphisms to higher dimensions. Doing this will require a generalization of Kollár’s theory of hulls and husks, e.g. [Kol11], to cyclotomic stacks, which we will pursue in a future article.

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