An exact Bianchi V cosmological model in Scale Covariant theory of gravitation: A variable deceleration parameter study

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Abstract. A spatially homogeneous and anisotropic Bianchi type V cosmological model of the universe for perfect fluid within the framework of Scale covariant theory of gravitation proposed by Canuto et al., is studied in view of a variable deceleration parameter which yields the average scale factor $a(t) = \sinh^{1/n}(\beta t)$, where $n$ and $\beta$ are positive constants. The solution represents a singular model of the universe. All physical and geometrical properties of the model are thoroughly studied. The time dependent deceleration parameter supports the recent observation. The model represents an accelerating phase for $0 < n \leq 1$ and for $n > 1$, there is a phase transition from early deceleration to a present accelerating phase.

Keywords: Cosmology. Bianchi type V model. Variable deceleration parameter. Scale Covariant theory

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I. INTRODUCTION

The scalar tensor theories are the generalizations of Einstein’s general theory of relativity in which the metric is generated by a scalar gravitational field together with non-gravitational fields. The scalar gravitational field itself is generated by the non-gravitational fields via a wave equation in curved space-time. Nowadays scalar tensor theories are more in use as the Einstein’s theory of general relativity doesn’t seem to resolve some of the important problems in cosmology such as dark matter or the missing matter[1-13]. Scalar tensor theories provide a convenient set of representations for the observational limits on possible deviations from general relativity. Canuto et al.[14, 15] have formulated Scale covariant scalar tensor theory of gravitation by associating the mathematical operation of scale transformation with the physics of using different dynamical systems to measure space time distances. In this theory the generalized Einstein’s field equations are invariant under scale transformations. This theory also provide a natural interpretation of the variation of the gravitational constant $G$ [16, 17]. Also in this theory, the Einstein’s field equations are valid in gravitational units and the other physical quantities are measured in atomic units. The line element $ds^2 = g_{ij}dx^i dx^j$ in Einstein units corresponds to $ds = \phi^{-1}(x) d\tilde{s}$ in any other units (in atomic units). The metric tensor in the two systems of units are related by a conformal transformation $\tilde{g}_{ij} = \phi^2 g_{ij}$, where the metric $\tilde{g}_{ij}$ giving macroscopic metric properties and $g_{ij}$ giving microscopic metric properties. Here we consider the gauge function $\phi$ as a function of time.

Several workers in the field of general relativity and cosmology have studied the Scale-Covariant theory in different Bianchi space-times. Shri Ram et al. [18] have studied a spatially homogeneous Bianchi type V cosmological model in Scale-Covariant theory of gravitation. Zeyauddin et al. [19] have studied Scale covariant theory for Bianchi type VI space-time and obtained some exact solutions. Reddy et al. [20] have developed a cosmological model with negative constant deceleration parameter in Scale-Covariant theory of gravitation. Reddy et al.[21] have presented the exact Bianchi type II, VIII and IX cosmological models in Scale-Covariant theory of gravitation. Beesham [22] has obtained a solution for Bianchi type I cosmological model in the Scale-Covariant theory. Higher dimensional string cosmologies in Scale-Covariant theory of gravitation have been investigated by Venkateswarlu and Kumar [23].

II. BASIC EQUATIONS

The generalized field equations in Scale covariant theory of gravitation are given as

$$R_{ij} - \frac{1}{2}g_{ij}R + f_{ij}(\phi) = -8\pi GT_{ij} + \Lambda(\phi)g_{ij},$$

(1)

where

$$\phi^2 f_{ij} = 2\phi\phi_{ij} - 4\phi_i \phi_j - g_{ij}(\phi\phi^\lambda_\lambda - \phi^\lambda \phi_\lambda).$$

(2)

for any scalar $\phi, \phi_i = \phi \delta_i$. Here comma denotes ordinary differentiation whereas a semi-colon denotes a covariant differentiation. The line element for the Bianchi V space-time can be written as

$$ds^2 = dt^2 - A^2 dx^2 - e^{2mx}[B^2 dy^2 + C^2 dz^2]$$

(3)

where the scale factors $A, B, C$ are functions of time and $m$ is an arbitrary constant. Here we take the source of gravitational field as a perfect fluid. The energy momentum tensor for a perfect fluid, is given by

$$T_{ij} = (\rho + p) u_i u_j - pg_{ij},$$

(4)

where $\rho$ is the energy-density, $p$ the pressure and $u^i$ is four velocity vector of the fluid following $u^i u_j = 1$.

Some basic physical parameters for the line element equation (3) are given by,

$$V = ABC,$$

(5)
\[a = (ABC)^{1/3},\]
\[\theta = u^\mu_{\mu} = \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C},\]
\[\sigma^2 = \frac{1}{2} \sigma_{\mu\nu} \sigma^{\mu\nu} = \frac{1}{2} \left( \left( \frac{\dot{A}}{A} \right)^2 + \left( \frac{\dot{B}}{B} \right)^2 + \left( \frac{\dot{C}}{C} \right)^2 \right) - \frac{\theta^2}{6},\]
\[H = \frac{1}{3} (H_1 + H_2 + H_3),\]

where \(V, a, \theta, \sigma^2,\) and \(H\) are volume scalar, scale factor, expansion scalar, shear scalar, and Hubble parameter respectively. Here \(H_1 = \frac{\dot{A}}{A}, H_2 = \frac{\dot{B}}{B}\) and \(H_3 = \frac{\dot{C}}{C}\) are directional Hubble parameters in the directions of \(x, y\) and \(z\) respectively. A dot denotes the derivative with respect to time. A very important relation in terms of the parameters \(H, V\) and \(a\) can be obtained as
\[H = \frac{1}{3} \frac{\dot{V}}{V} = \frac{\dot{a}}{a}.\]

From equations (1)-(4), the field equations can be translated in the following set of non-linear differential equations,
\[\frac{\dot{\dot{B}}}{B} + \frac{\dot{C}}{C} + \frac{\dot{B} \dot{C}}{BC} - \frac{m^2}{A^2} \frac{\dot{A}}{A} - 2 \frac{\dot{A} \dot{\phi}}{A \phi} + \frac{\dot{\phi} \dot{V}}{\phi V} + \frac{\ddot{\phi}}{\phi} - \frac{\dot{\phi}^2}{\phi^2} = -8\pi G p,\]
\[\frac{\dot{A}}{A} + \frac{\dot{\dot{C}}}{C} + \frac{\dot{A} \dot{C}}{AC} - \frac{m^2}{A^2} \frac{\dot{B}}{B} - 2 \frac{\dot{B} \dot{\phi}}{B \phi} + \frac{\dot{\phi} \dot{V}}{\phi V} + \frac{\ddot{\phi}}{\phi} - \frac{\dot{\phi}^2}{\phi^2} = -8\pi G p,\]
\[\frac{\dot{A}}{A} + \frac{\dot{\dot{B}}}{B} + \frac{\dot{A} \dot{B}}{AB} - \frac{m^2}{A^2} \frac{\dot{C}}{C} - 2 \frac{\dot{C} \dot{\phi}}{C \phi} + \frac{\dot{\phi} \dot{V}}{\phi V} + \frac{\ddot{\phi}}{\phi} - \frac{\dot{\phi}^2}{\phi^2} = -8\pi G p,\]
\[\frac{\dot{A} \dot{B}}{AB} + \frac{\dot{A} \dot{C}}{AC} + \frac{\dot{B} \dot{C}}{BC} - \frac{3m^2}{A^2} \frac{\ddot{\phi}}{\phi V} - \frac{\dot{\phi}}{\phi} + 3 \frac{\dot{\phi}^2}{\phi^2} = 8\pi G \rho,\]
\[2 \frac{\dot{A}}{A} - \frac{\dot{B}}{B} - \frac{\dot{C}}{C} = 0.\]

The continuity equation \(T^i_{\ j; i} u^j = 0\) reads,
\[\dot{\rho} + (\rho + p) \frac{\dot{V}}{V} + \rho \left( \frac{\dot{\phi}}{\phi} + \frac{\dot{G}}{G} \right) + 3p \frac{\dot{\phi}}{\phi} = 0.\]

From equations (11)-(14), we obtain the energy density and pressure as follows:
II BASIC EQUATIONS

\[ 8\pi G\rho = 3H^2 - \sigma^2 - \frac{3m^2}{A^2} - \frac{\dot{\phi}}{\dot{\phi}} + 3\left(\frac{\dot{\phi}}{\phi}\right)^2 + 3H\frac{\dot{\phi}}{\phi}. \]  

\[ (17) \]

\[ 8\pi Gp = H^2(2q - 1) - \sigma^2 + \frac{m^2}{A^2} - \frac{\dot{\phi}}{\dot{\phi}} + \left(\frac{\dot{\phi}}{\phi}\right)^2 - H\frac{\dot{\phi}}{\phi}. \]  

\[ (18) \]

From equation (15), we have

\[ A^2 = BC. \]  

\[ (19) \]

Following the approach of Saha and Rikhvitsky [24], Zeyauddin and Saha [25], Shri Ram et al.[26-28], and Singh et al.[29], we solve the equations from (11) to (14) and (19) to get the quadrature solution for the metric functions \( A, B \) and \( C \) as

\[ A(t) = a, \]  

\[ (20) \]

\[ B(t) = B_0a \exp\left(M \int \frac{dt}{a^3\phi^2}\right), \]  

\[ (21) \]

\[ C(t) = C_0a \exp\left(-M \int \frac{dt}{a^3\phi^2}\right), \]  

\[ (22) \]

where \( B_0, C_0 \) and \( M \) are arbitrary constants. In this paper, we obtain exact solution to the field equations of Scale-Covariant theory for Bianchi type V space-time metric, using the concept of variable deceleration parameter. For that we follow the assumptions made by Pradhan[30], N. Ahmad et al.[31], Pradhan and Otarod[32], Akarsu and Dereli[33], Pradhan et al.[34], Chawla et al.[35], Chawla and Mishra[36], Mishra et al.[37], Pradhan [38], Pradhan et al. [39], Pradhan et al. [40], Amirhashchi et al.[41]. The deceleration parameter can be time dependent and for that the average scale factor is given by the following relationship as

\[ a(t) = \left[\sinh (\beta t)\right]^{\frac{1}{n}}. \]  

\[ (23) \]

The time varying deceleration parameter can be obtained as

\[ q = \frac{\ddot{a}}{a^2} = n \left[1 - (\tanh(\beta t))^2\right] - 1. \]  

\[ (24) \]

The time dependent deceleration parameter \( (q) \) suggests the accelerating nature of the universe at present and decelerating in the past. This argument is recently supported by the observations of Type Ia supernova (Riess et al. [42,43]; Perlmutter et al.[44,45,46]; Tonry et al.[47]; Clochietti et al. [48]) and CMB anisotropies (Bennett et al.[49]; de Bernardis et al.[50]; Hanany et al.[51]). The value of the transition red-shift from decelerated expansion to accelerated expansion is about 0.5. At present, the concept of constant deceleration is not acceptable as the deceleration parameter must show signature flipping [52-54]. The gauge function \( \phi \) [14, 15, 18] can be considered directly proportional to some constant power of average scale factor as,

\[ \phi = \phi_0a^\alpha = \phi_0\sinh^{\alpha/n}(\beta t), \]  

\[ (25) \]

where \( \alpha \)=any constant and \( \phi_0 \)=arbitrary constant.
FIG. 1. Variation of average scale factor $a$ for the parameter $n = 0.5$ with time.

FIG. 2. Variation of the average scale factor $a$ for the parameter $n = 2$ with time.

FIG. 3. Variation of the gauge parameter $\phi$ for $n = 0.5$ with time.

FIG. 4. Variation of the gauge parameter $\phi$ for $n = 2$ with time.

III. EXACT SOLUTIONS

We solve the quadrature solution equations (20) to (22) by substituting in these equations the values of the average scale factor $a$ and gauge function $\phi$ from equations (23) and (25) and integrating these equations to get the exact solutions for scale factors $A$, $B$ and $C$ as,

$$A = [\sinh(\beta t)]^{1/n},$$

(26)

$$B = B_0 [\sinh(\beta t)]^{1/n} \exp \left[ \frac{\sinh(2\beta t) - 2\beta t}{4\beta} \right],$$

(27)

$$C = C_0 [\sinh(\beta t)]^{1/n} \exp \left[ -\frac{\sinh(2\beta t) - 2\beta t}{4\beta} \right],$$

(28)

provided $\beta \neq 0$. Here $M = 1$ and $2\alpha + 2n + 3 = 0$. The expansion scaler $\theta$ is given by

$$\theta = \frac{3\beta}{n} \coth(\beta t).$$

(29)
The shear scalar $\sigma^2$ can be obtained as

$$\sigma^2 = \frac{1}{8} [\cosh(2\beta t) - 1]^2 - \frac{\beta^2}{2n^2} \coth^2(\beta t).$$

(30)

The directional Hubble parameters $H_1$, $H_2$ and $H_3$ in the directions of $x-$, $y-$ and $z-$ coordinate axes respectively are calculated as follows

$$H_1 = \frac{\beta}{n} \coth(\beta t),$$

(31)

$$H_2 = \frac{1}{2n} [2\beta \coth(\beta t) + n(\cosh(2\beta t) - 1)],$$

(32)

$$H_3 = \frac{1}{2n} [2\beta \coth(\beta t) - n(\cosh(2\beta t) - 1)].$$

(33)
The Hubble parameter $H$ can be obtained as

$$H = \frac{\beta}{n} \coth(\beta t).$$  \hfill (34)

The volume scalar $V$ in this model is given by

$$V = \sinh^{3/n}(\beta t).$$  \hfill (35)

The energy density $\rho$ and the pressure $p$ can be obtained as

$$\rho = \frac{2\beta^2}{n^2} (1 + 2\phi_0 \alpha) \coth^2(\beta t) - \frac{1}{2} \sinh^4(\beta t) - 3m^2[\sinh^{-2/n}(\beta t)] - \phi_0 \frac{\alpha \beta^2}{n},$$  \hfill (36)

$$p = \frac{\beta^2}{n^2} \left[ \phi_0 \alpha (n - 1) - \frac{5}{2} \right] \coth^2(\beta t) + \frac{2\beta}{n} \csc h^2(\beta t) - \frac{1}{2} \sinh^4(\beta t) + m^2 \sinh^{-2/n}(\beta t) - \phi_0 \frac{\alpha \beta^2}{n}. \hfill (37)$$
The behaviour of the above parameters can be thoroughly discussed. The average scale factor $a$ varies in such a manner that it starts with zero time ($t = 0$) and becomes infinity for the large time as $t \to \infty$. Also the accelerating and decelerating nature of the average scale can clearly be seen in Figures 1 and 2 for $0 < n < 1$ and $n > 1$ respectively. Due to this value of average scale factor, the time varying deceleration parameter $q$ can be analyzed for its positive and negative values, i.e., for $q > 0$, $t < \frac{1}{\beta} \tanh^{-1}(1 - \frac{1}{n})^{1/2}$ and $q < 0$, $t > \frac{1}{\beta} \tanh^{-1}(1 - \frac{1}{n})^{1/2}$. The volume scalar $V$ is zero at $t = 0$ and it tends to infinity as $t \to \infty$. These behaviours of $V$ can be observed in figures 11 and 12. The accelerating and decelerating nature of the gauge function $\phi$ can be seen in figures 3 and 4. It has the same behaviour as that of average scale factor. The figures 5 and 6 show the variation of the expansion scalar. It will become infinity at $t = 0$ but as $t \to \infty$, this parameter vanish. The behaviour of the shear scalar $\sigma^2$ can be observed from figures 7 and 8. The graphs in figures 9 and 10 give the variation of the Hubble parameter. The energy density $\rho$ and isotropic pressure $p$ will also diverge at $t = 0$ and approach to zero for an infinite time. Here we conclude from the above results that the universe starts with zero volume and it has transition from initial anisotropy to isotropy at present epoch. So we can say that the model indicates a shearing, non rotating, expanding and accelerating universe with big bang singularity at its start. Here we observe that the model is accelerating for $0 < n \leq 1$ and for $n > 1$, the model evolves from a decelerating phase to an accelerating one. Recent observation SNe Ia supports the accelerating nature of the universe.

IV. CONCLUSION

In this paper, we have obtained exact solutions for Scale Covariant theory of gravitation in Bianchi type V space-time. Here, we have used a time dependent deceleration parameter which yields the average scale factor $a(t) = \sinh^{1/n}(\beta t)$, where $n$ and $\beta$ are positive constants. The time dependent deceleration parameter supports the recent observation. The model represents an accelerating phase for $0 < n \leq 1$ and for $n > 1$, there is a phase transition from early deceleration to a present accelerating phase. Following the concept of variable deceleration parameter, we get a singular cosmological model. Different physical and kinematical parameters have been obtained and thoroughly studied in all cases geometrically.

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