A VARIATIONAL APPROACH TO HOMOGENEOUS SCALAR FIELDS IN
GENERAL RELATIVITY

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ABSTRACT. A result of existence of homogeneous scalar field solutions between pre-
scribed configurations is given, using a modified version [8] of Euler–Maupertuis least
action variational principle. Solutions are obtained as limit of approximating variational
problems, solved using techniques introduced by Rabinowitz in [20].

1. INTRODUCTION

Observational cosmology suggests that our universe has entered a stage of accelerated
expansion (see [21, 22] and references therein). The reason for that is the so called dark
energy, that constitutes the most of the energy density of the whole universe. The existence
of a cosmological constant can be called as a responsible for dark energy, but some
physical problems arise from this interpretation. The most important alternative physical
interpretation for dark energy origin calls into play scalar field models of spacetime (see
[17], and references therein). In the latest years of theoretical physics, the belief for exis-
tence of zero–spin particles – whose description is given in terms of a wave scalar function
– has gained supporters, although there is no observational evidence yet.

The general scalar field spacetime is a Lorentzian manifold \((M, g)\) such that the metric
\(g\) satisfies the Einstein field equation

\[
R - \frac{1}{2} S g = 8\pi T.
\]

On the left hand side, \(R\) and \(S\) are respectively Ricci tensor field and scalar curvature
function of \(g\) (see [16]), and are completely determined by components of \(g\) and their
partial derivatives up to second order. On the righthand side, \(T\) is the energy momentum
tensor field, that in this case is completely determined by a scalar function \(\phi\) on \(M\), and a
potential function \(V(\phi)\). Its expression, evaluated on a couple \((u, v)\) of vectors belonging
to \(T_x M\) (the tangent space to \(M\) in a point \(x \in M\)), reads\(^1\)

\[
4\pi T(x)[u, v] = d\phi(x)[u] \cdot d\phi(x)[v] - \left[ \frac{1}{2} g(\nabla \phi(x), \nabla \phi(x)) + V(\phi(x)) \right] g(u, v),
\]

where \(\nabla \phi\) denotes the Lorentzian gradient vector field of \(\phi\).

In this paper we will make the assumptions that the spacetime is spatially homogeneous
with Bianchi I type symmetry [14]. This allows us to choose a convenient coordinate
system \(x = (x^0 = t, x^1, x^2, x^3)\), such that the metric can be written in the form

\[
g = -dt \otimes dt + a(t)^2 \left[ dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3 \right]
\]

\(^1\)the \(4\pi\) factor on the left hand side of (1.2) is just to simplify the form of Einstein field equations, getting rid
of the \(\pi\) factor in (1.4a)–(1.4b) below.
and the scalar field $\phi$ that determines $T$ is a function $\phi = \phi(t)$ of the variable $t$ only. With the above assumption, a convenient set for Einstein equations is the following:

\begin{align}
(1.4a) \quad (G^0_0 = 8\pi T^0_0) : \quad & -\frac{3\dot{a}^2}{a^2} = - (\dot{\phi}^2 + 2V(\phi)), \\
(1.4b) \quad (G^1_1 = 8\pi T^1_1) : \quad & -\frac{\dot{a}^2 + 2a\ddot{a}}{a^2} = (\dot{\phi}^2 - 2V(\phi)).
\end{align}

where the dot denotes differentiation with respect to $t$.

We will be interested in the problem of determining solutions of (1.4a)–(1.4b) with fixed endpoints. The metric becomes singular when $a(t)$ vanishes in the past or in the future – corresponding to big–bang or big–crash singularity, respectively. In this paper we will want to avoid this situation, so we will consider pieces of evolution where $a(t)$ keeps positive. The central result of the paper is the following theorem:

**Theorem 1.1.** Let $a_0, a_1 \in \mathbb{R}^+$, $\phi_0, \phi_1 \in \mathbb{R}$ be such that

\begin{equation}
3 \min\{a_0, a_1\}(a_1 - a_0)^2 > \max\{a_0, a_1\}(\phi_1 - \phi_0)^2,
\end{equation}

and let $V \in C^1(\mathbb{R}, \mathbb{R})$ such that

\begin{equation}
V(\phi) > 0, \quad \forall \phi \in \mathbb{R}.
\end{equation}

Then, there exists $T > 0$ and $(a(t), \phi(t)) \in C^2([0, T], \mathbb{R}^2)$ solutions of (1.4a)–(1.4b) with the boundary conditions

\begin{equation}
a(0) = a_0, \quad a(T) = a_1, \quad \phi(0) = \phi_0, \quad \phi(T) = \phi_1.
\end{equation}

Note that the above result cannot be seen as a consequence of well-known theory of existence of solutions for Cauchy problems in General Relativity (see pioneering work by Bruhat and following literature). Here, indeed, we do not fix both fundamental forms on a Cauchy surface, and find evolution from it, but we show that, under condition above, the set of initial data can be completed in order to reach a certain configuration from a given one. The solutions under our study stay regular in the interval $[0, T]$, since $a(t) > 0$. Nevertheless, we believe that this approach may be useful to find existence results for scalar field solutions evolving to singularity, a topic which is of relevant interest in the problems related to the Cosmic censorship conjecture. In fact, although existence and causal structure of the singularities in matter-filled spacetimes has been so far widely investigated in the case of fluid-elastic matter (see and references therein) for scalar fields the situation is fully understood only in the very special case of non self-interacting, massless particles. Homogeneous collapse with potential has been treated only in special cases, by Joshi et al (who have also investigated on loop quantum gravity effects in [10]) and in the work [5], where the collapse features are characterized though the dependence of the energy density on the scale factor $a$; an important open question in which models with potentials have been used is also that of cosmic censorship violation in AdS [11],[12].

We will cast the above problem into a suitable variational framework. The use of a variational approach in General Relativity study is not a novelty, of course: for instance, some of the most important results in relativistic gravitational lensing problem are reviewed in [19]. In the present paper, actually, the variational approach is used to determine solutions of Einstein field equations, i.e. spacetimes. Hilbert–Palatini action [15],[23], indeed, provides a functional whose critical points with respect to variation of the metric $g$ yields solutions of (1.1). The particular case under our study produces the functional [22], which is an integral made on the interval $[0, T]$ of definition of the solutions. Of course, since $T$ is let free in principle, it must be treated as an unknown for the system, but this problem
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may be circumvented by using the functional (2.18) in the space of curves reparameterized on the interval $[0, 1]$. Although the functional (2.18) seems in principle more complicated to deal with than (2.5), critical point existence can be obtained as a limit of a sequence of variational problems, for which Rabinowitz’ Saddle Point Theorem [20] techniques apply.

The outline of the paper is the following. Section 2 is devoted to an exposition of the variational formulation for the problem; Section 3 briefly outlines the general theory that applies to the approximating variational problems introduced in Section 4 and studied in Section 5. Final section 6 contains the proof of the main Theorem 1.1.

2. THE VARIATIONAL PRINCIPLE

In this section we collect some basic facts that lead to the variational formulation of homogeneous scalar field theory. As is well known, solutions of Einstein field equations (1.1) are related to critical points of the Hilbert–Palatini action functional

$$I = \int_M \sqrt{-\det g} (L_g + L_f) \, dV,$$

where $L_g$ and $L_f$ are Lagrangian scalar function related to the contribution of gravitation (the metric $g$) and the external source (the energy momentum tensor) respectively. The Lagrangian $L_g$ is given, in general, by

$$L_g = \frac{3}{8\pi} \frac{\dot{a}^2(t) + a(t) \ddot{a}(t)}{a^2(t)},$$

whereas $L_f$ depends on the source of matter, and in this case takes the form

$$L_f = -\frac{1}{4\pi} \left( \frac{1}{2} g(\nabla \phi, \nabla \phi) + V(\phi) \right) = \frac{1}{8\pi} (\dot{\phi}^2(t) - 2V(\phi(t))).$$

Since all the unknown functions depends on $t$ only, we can reduce the volume integral in (2.1) to an integral made on the interval $[0, T]$ of definition of $a(t)$ and $\phi(t)$. Using (2.2) and (2.3), therefore, the functional we are dealing with is found to be

$$\mathcal{L}(a, \phi) = \int_0^T 3\dot{a}^2(t) + a(t) \ddot{a}(t) a(t) + a^3(t) (\dot{\phi}^2(t) - 2V(\phi(t))) \, dt.$$

We can integrate by parts the term in (2.4) containing $\dddot{a}(t)$. We ignore the contribution of the boundary term $3a^2 \dot{a}^2|_0^T$ coming from the integration. This exactly amounts [23] to modify the functional (2.4), adding the contribution $\frac{1}{16\pi} \int_{\partial M} K$ of the trace of the extrinsic curvature $K$ integrated along the boundary $\partial M$ of the spacetime.

All in all, we obtain

$$\mathcal{L}(a, \phi) = \int_0^T 3a(t) \dot{a}^2(t) - a^3(t) \dot{\phi}^2(t) + 2a^3(t)V(\phi(t)) \, dt,$$

where we have also performed an unsubstantial overall change of sign inside the integral. The following proposition holds.

**Proposition 2.1.** If $(a, \phi) \in C^2(\mathbb{R}^+, \mathbb{R})$ solves Euler–Lagrange equation for $\mathcal{L}$, and

$$3\dot{a}(0)^2 = a_0^2(\dot{\phi}(0)^2 + 2V(\phi_0)),$$

then it is a solution for homogeneous scalar field equation (1.4a)–(1.4b).
Proof. This is a standard result, consequence of Nöther’s Theorem, but an ad–hoc proof can be easily given. Indeed, Euler–Lagrange equations for the functional \( \mathcal{L} \) read
\[
\begin{align*}
\dot{a}^2 + 2a\ddot{a} &= -a^2(\dot{\phi}^2 - 2V(\phi)), \\
\ddot{\phi} + V'(\phi) &= -3\frac{\dot{a}}{a}\phi,
\end{align*}
\]
where also the condition \( a \neq 0 \) has been used. Using these equations, the quantity \( a(3\dot{a}^2 - a^2(\dot{\phi}^2 + 2V)) \) is easily seen to be constant w.r.t. \( t \), and then if \( 1.4a \) holds at initial time – which is just condition \( 2.6 \) – then it vanishes during the whole evolution. Since \( a \neq 0 \) we therefore obtain \( 1.4a \). Equation \( 1.4b \) is equivalent to \( 2.7a \). \( \square \)

Remark 2.2. A partial converse of Proposition above can be of course given, using the following identity:
\[
T_{\alpha\beta} = -2\dot{\phi} \left( \ddot{\phi} + V'(\phi) + 3\frac{\dot{a}}{a}\phi \right).
\]
The left hand side above is the 0-component of the divergence of the Gordon equation, and its intrinsic expression reads \( \nabla g \) is everywhere nonzero, \( 2.7b \) holds. Equation \( 2.7b \) is also known in literature as Klein–Gordon equation, and its intrinsic expression reads \( \Box \phi = V'(\phi) \), where \( \Box \phi = g^{\alpha\beta}\nabla_\alpha \nabla_\beta \phi \) is the d’Alembert operator with respect to the Lorentzian metric \( g \).

Actually, the converse result can be improved further:

**Proposition 2.3.** Se \( (a, \phi) : [0, T] \to \mathbb{R}^+ \times \mathbb{R} \) are \( C^2 \) solutions of \( 1.4a-1.4b \) with \( \phi \neq \phi_0 \) on \( [0, T] \), then solve \( 2.7a-2.7b \). In particular \( 2.6 \) holds.

Proof. Only \( 2.7b \) must be proved. Let \( G(t) := \ddot{\phi}(t) + V'(\phi(t)) + 3\frac{\dot{a}(t)}{a(t)}\phi(t) \), and assume by contradiction the existence of \( t_0 \in [0, T] \) such that \( G(t_0) \neq 0 \). Therefore, considered the (closed) set \( C = \{ t \in [0, T] : \phi(t) = 0 \} \subseteq [0, T] \), Bianchi identity \( 2.8 \) implies the existence of a closed interval \( [\alpha, \beta] \) such that
\[
i_0 \subset [\alpha, \beta] \subseteq C,
\]
and that is maximal with respect to this property \( 2.9 \). We will show \( G \neq 0 \) on \( [\alpha, \beta] \), getting a contradiction.

Since \( [\alpha, \beta] \) is strictly contained in \( [0, T] \) then \( \alpha > 0 \) or \( \beta < T \) (or both). Let us assume for instance \( \alpha > 0 \). Since \( \dot{\phi}(0, \beta) = 0 \), and \( \phi \) is \( C^2 \), then \( \dot{\phi}(0, \beta) = 0 \) and \( \phi(t) = \phi(\alpha), \forall t \in [\alpha, \beta] \). These facts imply
\[
G(t) = V'(\phi(\alpha)), \quad \forall t \in [\alpha, \beta],
\]
Maximality of \( [\alpha, \beta] \) w.r.t. \( 2.9 \) implies the existence of a sequence \( \alpha_n \notin C, \alpha_n \to \alpha^- \). Then \( \phi(\alpha_n) \neq 0 \) and therefore \( G(\alpha_n) = 0 \) by \( 2.8 \), therefore continuity of \( G \) and \( 2.10 \) finally imply
\[
0 = \lim_{n \to \infty} G(\alpha_n) = \lim_{t \to \alpha^-} G(t) = V'(\phi(\alpha)),
\]
obtaining \( G(t) = 0 \) on \( [\alpha, \beta] \). \( \square \)

Remark 2.4. If \( \phi \) is everywhere constant one can easily find counterexamples where \( 1.5a-1.4b \) hold but \( 2.7b \) do not. This is a well known fact in relativistic elasticity theory, where equivalence between Bianchi identity and Euler–Lagrange equations (i.e. Nöther theorem) holds under the assumption that the deformation tensor has maximum rank. We refer the reader to \( 1.3 \) for further details on this topic. In scalar field theory, the gradient
\[ \nabla \phi \text{ plays the role of this deformation tensor. For our purposes here, anyway, we will only use Proposition 2.1.} \]

The problem of finding solutions of (1.4a)–(1.4b) with fixed endpoints is brought to the study of critical points of the functional (2.5) with fixed endpoints and under the initial condition (2.6). Of course, since the arrival time \( T \) is left free so far, one can reparameterize the functions on the interval \([0, 1]\), with the obvious drawback to promote \( T \) as a new unknown of the problem. But this problem can be overcome, applying the following (general) variational principle. \[1\)

**Theorem 2.5.** Let \((\mathbb{M}, g)\) be a semi–Riemannian manifold, \(W\) a \(C^1\) function on \(\mathbb{M}\), \(E \in \mathbb{R}\) and \(p, q \in \mathbb{M}\).

1. If \(y : [0, 1] \rightarrow \mathbb{M}\) is a critical point for the functional
   \[ \mathcal{F}(y) = \left( \int_0^1 \frac{1}{2} g \left( \frac{d}{ds} y(s), \frac{d}{ds} y(s) \right) ds \right) \cdot \left( \int_0^1 E - W(y(s)) ds \right) \]
   with positive critical value, in the space of \(C^2\) curves defined in \([0, 1]\), such that \(y(0) = p, y(1) = q\), called
   \[ T_0 = \left( \frac{\int_0^1 \frac{1}{2} g \left( \frac{d}{ds} y(s), \frac{d}{ds} y(s) \right) ds}{\int_0^1 E - W(y(s)) ds} \right)^{1/2}, \]
   then the curve \(x : [0, T_0] \rightarrow \mathbb{M}\), \(x(t) = y(T_0 s)\) is a critical point for the functional
   \[ \mathcal{L}(x) = \int_0^T \frac{1}{2} g \left( \frac{d}{dt} x(t), \frac{d}{dt} x(t) \right) - W(x(t)) dt, \]
   with \(T = T_0\), in the space of \(C^2\) curves satisfying the conditions
   \[ x(0) = p, \quad x(T) = q, \quad \frac{1}{2} g \left( \frac{d}{dt} x(t), \frac{d}{dt} x(t) \right) + W(x(t)) = E. \]

2. Viceversa, let us fix \(T > 0\), and let \(x : [0, T] \rightarrow \mathbb{M}\) be a critical point for the functional (2.13) in the space of \(C^2\) curves \(\gamma : [0, T] \rightarrow \mathbb{R}\) satisfying conditions (2.14). If \(\int_0^1 \frac{1}{2} g \left( \frac{d}{ds} x(t), \frac{d}{ds} x(t) \right) dt \neq 0\), then the reparameterization \(y : [0, 1] \rightarrow \mathbb{M}\) of \(x\) on \([0, 1]\), i.e. \(y(s) = x(T s)\), is a critical point for the functional (2.11), with positive critical value, in the space of \(C^2\) curves defined in \([0, 1]\), such that \(y(0) = p, y(1) = q\).

**Proof.** Let \(y(s) : [0, 1] \rightarrow \mathbb{M}\) be a critical point for \(\mathcal{F}\) (2.11). Fixed endpoint first variation of this functional reads

\[ d\mathcal{F}(y)[\eta] = \int_0^1 E - W(y) ds \int_0^1 g \left( \frac{d}{ds} y, \frac{d}{ds} \eta \right) ds - \int_0^1 \frac{1}{2} g \left( \frac{d}{ds} y, \frac{d}{ds} y \right) ds \int_0^1 W'(y)[\eta] ds, \]

and integrating by part we get the following equation

(2.15)

\[ \left( \int_0^1 E - W(y(s)) ds \right) \frac{D^2}{ds^2} y(s) + \left( \int_0^1 \frac{1}{2} g \left( \frac{d}{ds} y(s), \frac{d}{ds} y(s) \right) ds \right) \nabla W(y(s)) = 0. \]
that is, using the value of $T_0$ given by (2.12) — well defined since the critical value is positive

\begin{equation}
\frac{1}{T_0^2} \frac{d^2}{ds^2} y(s) + \nabla W(y(s)) = 0, \quad \forall s \in [0, 1].
\end{equation}

Let $x : [0, T_0] \to \mathfrak{M}$ be the reparameterization of $y$ on the interval $[0, T_0]$ (i.e. $x(t) = y(t/T_0)$). Therefore, equation (2.16) becomes nothing but Euler–Lagrange equation for the functional (2.13) with $T = T_0$. Moreover, contracting the left hand side of (2.16) with $\frac{d}{dt} y(s)$, we obtain the existence of a constant $K$ such that

\begin{equation}
\frac{1}{T_0^2} \frac{d}{ds} [\frac{d}{dt} y(s), \frac{d}{ds} y(s)] + W(y(s)) = K, \quad \forall s \in [0, 1].
\end{equation}

Integrating both sides above between 0 and 1, and using (2.12), we obtain $E = K$, and so

\begin{equation}
\frac{1}{2} \left( \frac{d}{dt} x(t), \frac{d}{dt} x(t) \right) = E, \quad \forall t \in [0, T_0].
\end{equation}

Conversely, let $x(t) : [0, T] \to \mathfrak{M}$ be a critical point for (2.13) in the space of $C^2$ curves defined in $[0, T]$, satisfying (2.14), and $y(s) : [0, 1] \to \mathfrak{M}$ be its reparameterization: $y(s) = x(T s)$. Since $\frac{d}{dt} = T \frac{d}{dt}$, (2.17) holds with $T_0$ and $K$ replaced by $T$ and $E$ respectively. Integrating both sides of (2.17) in $[0, 1]$ implies that $T$ is equal to the value of $T_0$ given by (2.14). Moreover, $\frac{d}{dt} = T \frac{d}{dt}$ also implies (2.16), and substituting the value of $T_0$ given by (2.12) we find that $y(s)$ satisfies (2.13), that is critical point equation for the functional (2.10), and the proof is complete.

**Remark 2.6.** The variational principle given above, is actually a sort of modified version of the classical Euler–Maupertuis least action principle. The above proof is an adaptation of the one given in [7] for the Euclidean case, and may not completely stress the link with the classical principle. For a deeper insight, we refer the reader to the review [8] by the same author.

Applying this variational principle to functional (2.15) (with $E = 0$), the following problem provides solutions of homogeneous scalar field equation with fixed endpoints:

**Problem.** Let $a_0, a_1 \in \mathbb{R}^+$, $\phi_0, \phi_1 \in \mathbb{R}$, and $V \in C^1(\mathbb{R}, \mathbb{R})$.

Find the critical points of the functional

\begin{equation}
F(a, \phi) = \left( \int_0^1 3a(t)\dot{a}(t)^2(t) - a^3(t)\dot{\phi}(t)^2(t) \, dt \right) \cdot \left( \int_0^1 2a^3(t)V(\phi(t)) \, dt \right),
\end{equation}

with positive critical value, in the space of $C^2$ curves $(a, \phi) : [0, 1] \to \mathbb{R}^+ \times \mathbb{R}$ such that

\begin{equation}
a(0) = a_0, \quad a(1) = a_1, \quad \phi(0) = \phi_0, \quad \phi(1) = \phi_1.
\end{equation}

3. THE FUNCTIONAL FRAMEWORK AND THE ABSTRACT CRITICAL POINTS THEORY

We first recall the classical notion of Palais–Smale condition.

**Definition 3.1.** Let $\mathfrak{X}$ a Hilbert manifold of class $C^1$ and $f \in C^1(\mathfrak{X}, \mathbb{R})$. We say that $f$ satisfies the Palais–Smale condition at level $c$ (abbrev. $(PS)_c$) if any sequence $\{x_n\}_{n \in \mathbb{N}} \subset \mathfrak{X}$ such that

\[ f(x_n) \to c, \quad \text{and} \quad \nabla f(x_n) \to 0 \]

(where $\nabla f$ represents the gradient of $f$ w.r.t. the Hilbert structure of $\mathfrak{X}$) has a converging subsequence in $\mathfrak{X}$. The sequence $\{x_n\}$ with properties above is called a Palais–Smale sequence for $f$. 

Definition 3.2. We say that \( x \in X \) is a critical point of \( f \) if \( \nabla f(x) = 0 \). A value \( c \in \mathbb{R} \) such that there exists a critical point \( x \) with \( f(x) = c \) is a critical value for \( f \). A value \( c \in \mathbb{R} \) which is not critical will be called regular.

The following lemma is a slight modification of a well known deformation lemma (see e.g. [18]) and it will be used in Theorem 3.4 later.

For any \( d \in \mathbb{R} \) set \( f^d = \{ x \in X : f(x) \leq d \} \).

Lemma 3.3. Let \( X \) be a Hilbert manifold and \( f \in C^1(X, \mathbb{R}) \). Let \( a < b \in \mathbb{R} \). Assume that

1. \( f \) satisfies \((PS)_c\), \( \forall c \in [a, b] \);
2. the strip \( \{ x \in X : a \leq f(x) \leq b \} \subset X \) is complete (w.r.t. the Hilbert structure of \( X \));
3. each value \( c \in [a, b] \) is regular for \( f \).

Then, there exists a homotopy \( \mathbf{h} : [0, 1] \times f^b \to f^a \) such that
- \( \mathbf{h}(0, x) = x \), \( \forall x \in f^b \);
- \( \mathbf{h}(\tau, x) = x \), \( \forall x \in f^a, \forall \tau \in [0, 1] \);
- \( \mathbf{h}(1, x) \in f^a \), \( \forall x \in f^b \).

If the gradient of \( f \) is locally Lipschitz continuous, the above Lemma can be proved using the gradient flow of \( f \); indeed, if \( f \) satisfies \((PS) \), \( \forall c \in [a, b] \), then \( \| \nabla f(x) \|_X \) is uniformly bounded away from 0 when \( x \) satisfies \( a \leq f(x) \leq b \). Thanks also to assumption [18], the solutions of

\[
\begin{align*}
\frac{d}{dt} \eta(\tau) &= \nabla f(\eta(\tau)), \\
\eta(0) &= x,
\end{align*}
\]

with \( a \leq f(x) \leq b \), allow to send \( f^b \) to \( f^a \) in a finite time. If \( \nabla f \) is only continuous, we can use the so-called pseudo gradient vector field, first introduced by Palais [18], which is locally Lipschitz continuous. Also in this case the sublevel \( f^b \) can be deformed on \( f^a \), provided the assumptions [1] and [2] hold.

Using Lemma 3.3, the following result, that is a slight modification of the well known Rabinowits’ Saddle Point Theorem, [20] holds.

Theorem 3.4. Let \( X = \Omega \times Y \), where \( \Omega \) is a Hilbert manifold and \( Y \) is a finite dimensional affine space. Let \( \| \cdot \| \) denote the norm on \( Y \), and let \( f \in C^1(X, \mathbb{R}) \). Assume that

1. there exists \( \omega_0 \in \Omega, e_0 \in Y \), and \( R > 0 \) such that, called \( B_R(e_0) = \{ e \in E : \|e - e_0\| \leq R \} \), it is \( b_0 = \sup_{e \in \partial B_R(e_0)} f(\omega_0, e) < b_1 = \inf_{\omega \in \Omega} f(\omega, e_0) \);
2. if \( b_2 = \sup_{e \in B_{r_1}(e_0)} f(\omega_0, e) \), the strip \( \{ x \in X : b_1 \leq f(x) \leq b_2 \} \subset X \) is complete;
3. \( f \) satisfies \((PS)_c\) at any \( c \in [b_1, b_2] \).

Then, there exists a critical value \( c \) for \( f \) in \([b_1, b_2] \).

The proof can be obtained adapting the scheme developed in [20]. However, the idea behind is quite simple. If, by contradiction, \([b_1, b_2] \) is made by regular value only, by Lemma 3.3, there exists a homotopy sending \( f^{b_2} \) to \( f^{b_1} \), letting \( f^{b_1} \) fixed. Therefore, using the projection on \( Y \) and the retraction of \( Y \) on \( B_R(e_0) \), and observing that \( b_0 < b_1 \), one can define a homotopy that sends \( B_R(e_0) \) to its boundary \( \partial B_R(e_0) \) - recall that \( b_2 = \sup_{e \in B_R(e_0)} f(x_0, e) \) - and that lets \( \partial B_R(e_0) \) fixed, which is impossible in finite dimension.
4. AN APPROXIMATION SCHEME

By assumption (1.5) of Theorem 1.1 we know that \( a_0 \neq a_1 \). To fix ideas, without loss of generality we can assume
\[
a_0 > a_1,
\]
so that (1.5) becomes
\[
3a_1(a_1 - a_0)^2 > a_0^3(\phi_1 - \phi_0)^2.
\]
We can choose constants \( m, M \) such that
\[
0 < m < a_1 < a_0 < M < +\infty
\]
and
\[
3m(a_1 - a_0)^2 - M^3(\phi_1 - \phi_0)^2 > 0.
\]
Consider the Hilbert manifold
\[
\Omega = \{ a \in H^1([0,1], m, M) : a(0) = a_0, a(1) = a_1 \},
\]
where \( H^1([0,1], m, M) \) is the set of absolutely continuous functions defined on \([0,1]\), with values on \([m, M]\), such that \( \int_0^1 \dot{a}^2 \, dt < +\infty \). Let us observe that \( \Omega \) is a not complete Hilbert manifold with Hilbert structure
\[
\langle a_1, a_2 \rangle = \int_0^1 \dot{a}_1(t)\dot{a}_2(t) \, dt.
\]
We denote by \( \|a\|_{\Omega} \) the norm induced by the above inner product:
\[
\|a\|_{\Omega} = \left( \int_0^1 \dot{a}(t) \, dt \right)^{1/2}.
\]
Now set \( \phi_*(t) = (1 - t)\phi_0 + t\phi_1 \). Since \( V > 0 \), we have
\[
\inf_{t \in [0,1]} V(\phi_*(t)) \equiv v_* > 0.
\]
Let us also consider the affine space
\[
\begin{align*}
Y &= \{ \phi = \hat{\phi} + \phi_* : \hat{\phi} \in H^1_0([0,1], R) \}, \\
\text{where } H^1_0([0,1], R) &= \{ \phi \in H^1([0,1], R) : \phi(0) = \phi(1) = 0 \}. \quad \text{Y is a closed affine subspace of } H^1([0,1], R), \text{ with norm}
\end{align*}
\]
\[
\|\phi\|_Y = \left( \int_0^1 \phi^2(t) \, dt \right)^{1/2}.
\]
Since \( \dim Y = +\infty \) we cannot apply Theorem 3.4 as is to our setting, and then we approximate \( Y \) by a sequence \( Y_k \), defined as follows: for any \( k \in \mathbb{N} \) set
\[
W_k = \text{span}\{ \sin(\pi \ell t) : t \in [0,1], \ell = 1, \ldots, k \},
\]
and
\[
Y_k = \{ \phi = \hat{\phi}_k + \phi_* : \hat{\phi}_k \in W_k \}.
\]
Remark 4.1. Since \( \{ \sqrt{2} \sin(\pi \ell t) \}_{\ell \in \mathbb{N}} \) is a complete orthonormal system of \( H^1_0([0,1], R) \), if \( \phi = \hat{\phi} + \phi_* \in Y \) and \( \hat{\phi}_k \) denotes the projection of \( \hat{\phi} \) on \( W_k \), then \( \hat{\phi}_k \rightarrow \hat{\phi} \) in \( H^1_0 \), w.r.t. the norm defined in (4.7).
We shall apply Theorem 3.4 to the space \( X_k = \Omega \times Y_k \). Since \( \Omega \) is not complete, and \( V \) is not bounded in general, we modify the functional \( F(4.10) \) and look for critical points \( x_{\epsilon,\lambda} \), with positive critical value, of a suitable functional \( F_{\epsilon,\lambda} \). Some estimates for the critical points \( x_{\epsilon,\lambda} \) will show that they are critical points for \( F \), whenever \( \epsilon \) is sufficiently small, and \( \lambda \) sufficiently large.

Let \( \chi : \mathbb{R}^+ \to \mathbb{R}^+ \) of class \( C^1 \), such that \( \chi(s) = 0 \) if \( s \leq 0 \) and \( \chi(s) = s^2 \) if \( s > 0 \). Fix \( \epsilon \in [0,1] \), and define

\[
U_{\epsilon}(a) = \chi \left( \frac{1}{a - m} - \frac{1}{\epsilon} \right) + \chi \left( \frac{1}{M - a} - \frac{1}{\epsilon} \right).
\]

Moreover, consider \( \psi : \mathbb{R} \to \mathbb{R} \) of class \( C^1 \) such that \( \psi(s) = s \) if \( s \leq 0 \), \( \psi(s) = 1 \) if \( s \geq 1 \), and \( \psi \) is strictly increasing on the interval \([0,1]\). Fix \( \lambda > 0 \) and define

\[
V_\lambda(\phi) = \psi(V(\phi) - \lambda) + \lambda.
\]

Observe that \( V_\lambda = V \) whenever \( V(\phi) \leq \lambda \). Finally, define

\[
F_{\epsilon,\lambda}(a,\phi) = \int_0^1 \left[ (3a + U_{\epsilon}(a)) \dot{a}^2 - a^3 \dot{\phi}^2 \right] dt + \int_0^1 2a^3 V_\lambda(\phi) dt.
\]

It is a straightforward computation to show that the above functional is \( C^1 \). In the next section we shall show how to apply Theorem 3.4 to \( F_{\epsilon,\lambda} \) on the space \( \Omega \times Y_k \).

5. Critical points for the functional \( F_{\epsilon,\lambda} \)

The aim of this section is to prove the following result.

**Proposition 5.1.** For any \( k \in \mathbb{N} \) there exists \( x_k = (a_k, \phi_k) \) critical point of \( F_{\epsilon,\lambda} \) on \( X_k \) such that

\[
F_{\epsilon,\lambda}(x_k) \in [b_1, b_2],
\]

where \( b_1, b_2 \) are positive, and independent of \( k \).

We must first show that hypotheses \( 1 \) and \( 2 \) of Theorem 3.4 hold for \( F_{\epsilon,\lambda} \) on \( X_k = \Omega \times Y_k \). The key point will be to show \( 3 \), namely Palais–Smale condition (Definition 3.1). This will be done in Lemma 5.2. First, let us prove the validity of hypotheses \( 2 \).

**Lemma 5.2.** Denoted by \( F^c \) the sublevel \( F^c = \{ x \in \Omega \times E : F(x) \leq c \} \), then the set \( F^c \cap X_k \) is complete in \( X_k \).

**Proof.** Take a Cauchy sequence \( x_n = (a_n, \phi_n) \in \Omega \times E_k \). Since the closure of \( \Omega \) is complete, and so is \( E_k \), then there exists \((a, \phi) \in \overline{\Omega} \times E_k \) such that \( a_n \to a \) in \( H^1 \) and \( \phi_n \to \phi \) in \( H^1 \). Now \( \int_0^1 a_n^3 V_\lambda(\phi_n) dt \to \int_0^1 a^3 V_\lambda(\phi) dt \) which is strictly positive, and

\[
\int_0^1 \left( 3a_n \dot{a}_n^2 - a_n^3 \dot{\phi}_n^2 \right) dt \to \int_0^1 3a\dot{a}^2 - a^3 \dot{\phi}^2 dt.
\]

If \( a(t) \in [m, M] \) \( \forall t \), then \((a_n, \phi_n)\) converges in \( \Omega \times E_k \). The proof is complete as one shows that there is no possibility that some \( \tilde{t} \in [0,1] \) exists such that either \( a(\tilde{t}) = m \) or \( a(\tilde{t}) = M \). By contradiction, suppose for instance

\[
\exists \tilde{t} : a(\tilde{t}) = m, \quad a(t) > m \quad \forall t \in [0,\tilde{t}].
\]

Observe that \( F_{\epsilon,\lambda}(a_n, \phi_n) \leq c, \int_0^1 a^3 V_\lambda(\phi) dt > 0, \) and \( \int_0^1 \left( 3a\dot{a}^2 - a^3 \dot{\phi}^2 \right) dt \) is finite.

We will show that the hypothesis \( 5.1 \) implies

\[
\int_0^1 U_{\epsilon}(a_n) \dot{a}_n^2 dt \to +\infty,
\]

\[
\int_0^1 \left( 3a\dot{a}^2 - a^3 \dot{\phi}^2 \right) dt\]
obtaining a contradiction.

Since \( a_n \to a \) uniformly, \( \forall \epsilon > 0 \) there exists \( \bar{s} < \bar{r} \) and \( n_0 \in \mathbb{N} \) such that
\[
|a_n(t) - m| \leq \epsilon, \quad \forall t \in [\bar{s}, \bar{r}], \forall n \geq n_0.
\]

Fix \( \epsilon \) such that \( m + \epsilon < M \). Then, recalling the definition of \( U_\epsilon \), it will suffice to show that

\[
\lim_{n \to \infty} \int_{\bar{s}}^{\bar{r}} \frac{\dot{a}_n^2}{(a_n - m)^2} \, dt = +\infty,
\]

which easily follows from the following estimate:

\[
\left( \int_{\bar{s}}^{\bar{r}} \frac{\dot{a}_n^2}{(a_n - m)^2} \, dt \right)^{1/2} \geq \frac{1}{\sqrt{\bar{r} - \bar{s}}} \int_{\bar{s}}^{\bar{r}} \frac{|\dot{a}_n|}{a_n - m} \, dt \geq \frac{1}{\sqrt{\bar{r} - \bar{s}}} \left| \int_{\bar{s}}^{\bar{r}} \dot{a}_n \, dt \right| = \frac{1}{\sqrt{\bar{r} - \bar{s}}} \log \left( \frac{a_n(\bar{r}) - m}{a_n(\bar{s}) - m} \right),
\]

that diverges because \( a_n(\bar{r}) \to a(\bar{r}) = m \) whereas \( a_n(\bar{s}) \to a(\bar{s}) < m \).

The lemma below show that Palais–Smale condition actually holds in every closed interval of \( \mathbb{R} \).

**Lemma 5.3.** Fixed two values \( c_1, c_2 \) such that \( 0 < c_1 < c_2 < +\infty \), the functional \( F_{\epsilon, \lambda} \) satisfies \((PS)_c\), \( \forall c \in [c_1, c_2] \).

**Proof.** Let \((a_n, \phi_n)\) be a Palais–Smale sequence for \( F_{\epsilon, \lambda} \) such that \( F_{\epsilon, \lambda}(a_n, \phi_n) \in [c_1, c_2], \forall n \in \mathbb{N} \). Since \( \nabla F_{\epsilon, \lambda}(a_n, \phi_n) \) is infinitesimal we have, \( \forall \theta \in W_k \) (recall (4.38))

\[
|\nabla F_{\epsilon, \lambda}(a_n, \phi_n)[0, \theta]| \leq \delta_n \|\theta\|_{H_k^1}, \quad \text{with } \delta_n \to 0.
\]

Therefore

\[
\left| \left( \int_0^1 2a_n^3 V_\lambda(\phi_n) \, dt \right) \int_0^1 \left( -2a_n^3 \dot{\phi}_n \theta \right) \, dt + \int_0^1 \left( 3a_n + U_\epsilon(a_n) \right) \ddot{a}_n^2 - a_n^3 \ddot{\phi}_n^2 \, dt \right| \int_0^1 2a_n^3 V_\lambda'(\phi_n) |\theta| \, dt \leq \delta_n \|\theta\|_{H_k^1}.
\]

Multiplying end terms of the inequality chain above by the bounded and strictly positive quantity \( \int_0^1 2a_n^3 V_\lambda(\phi_n) \, dt \) we have

\[
\left( \int_0^1 2a_n^3 V_\lambda(\phi_n) \, dt \right)^2 \int_0^1 \left( -2a_n^3 \dot{\phi}_n \theta \right) \, dt + F_{\epsilon, \lambda}(a_n, \phi_n) \int_0^1 2a_n^3 V_\lambda'(\phi_n) \theta \, dt \leq \tilde{\delta}_n \|\theta\|_{H_k^1},
\]

where \( \tilde{\delta}_n := \delta_n \int_0^1 2a_n^3 V_\lambda(\phi_n) \, dt \to 0 \). Since \( \phi_n - \phi_* \in W_k \) we can choose \( \theta = \phi_n - \phi_* \), and observe the following facts:

- \( \int_0^1 2a_n^3 V_\lambda(\phi_n) \, dt \) is bounded away from 0, independently of \( n \),
- \( F_{\epsilon, \lambda}(a_n, \phi_n) \) is bounded,
- \( a_n^3 V_\lambda(\phi_n) \) is uniformly bounded, and
- \( \|\phi_n - \phi_*\|_{L^\infty} \leq \|\dot{\phi} - (\phi_1 - \phi_0)\|_{L^1} \leq \|\dot{\phi} - (\phi_1 - \phi_0)\|_{L^2} \).
Therefore, (5.8)–(5.9) together gives
\[ \|\theta\|_{H^1_0} = \|\dot{\theta}\|_{L^2}. \]
Then, by (5.3) we deduce the existence of a positive constant \( D_0 \) such that
\[ \int_0^1 \phi_n^2 \, dt \leq D_0, \quad \forall n \in \mathbb{N}. \]  
Since \( m \leq a_n \leq M, \) \( F_{\epsilon,\delta}(a_n, \phi_n) \leq c, \) and \( \int_0^1 2a_n^3 V_k(\phi_n) \, dt \) is bounded away from zero we deduce, by (5.4), that \( \int_0^1 (3a_n + U_\epsilon(a_n)) \dot{\phi}_n^2 \, dt \) is bounded; moreover, since \( 3a_n + U_\epsilon(a_n) \geq 3a_1 > 0 \) \( \forall n \in \mathbb{N}, \) there exists a positive constant \( D_1 \) such that
\[ \int_0^1 \dot{\phi}_n^2 \, dt \leq D_1, \quad \forall n \in \mathbb{N}. \]
By (5.4) and (5.5), there exists \( a \) and \( \phi \) of class \( H^1 \) such that, up to subsequences,
\[ a_n \rightarrow a, \quad \phi_n \rightarrow \phi \]
weakly in \( H^1 \) and uniformly. Since \( F(a_n, \phi_n) \) is bounded from above, arguing as in Lemma 5.1, one proves that \( a(t) \in [m, M], \) \( \forall t \in [0, 1]. \) Therefore it remains to show
\[ \dot{a}_n \rightarrow \dot{a} \quad \text{in } L^2([0, 1], \mathbb{R}), \]
and
\[ \phi_n \rightarrow \phi \quad \text{in } L^2([0, 1], \mathbb{R}). \]
Observing that \( \phi_n - \phi \in W_k, \) we can choose \( \theta = \phi_n - \phi \) in (5.3). Now \( F_{\epsilon,\lambda}(a_n, \phi_n) \) is bounded and
\[ \int_0^1 2a_n^3 V_k(\phi_n)|\phi_n - \phi| \, dt \rightarrow 0 \]
because \( 2a_n^3 V_k(\phi_n) \) is uniformly bounded and \( \sup_{t \in [0, 1]} |\phi_n(t) - \phi(t)| \rightarrow 0, \) while \( ||\phi_n - \phi||_{H^1_0} \) is bounded and \( \int_0^1 2a_n^3 V_k(\phi_n) \, dt \) is bounded away from zero. Then, by (5.3) we deduce
\[ \int_0^1 \dot{a}_n^2 \phi_n(\dot{\phi}_n - \dot{\phi}) \, dt \rightarrow 0. \]
Moreover, \( \int_0^1 a_n^3 \phi_n(\dot{\phi}_n - \dot{\phi}) \, dt \rightarrow \int_0^1 a_\lambda^3 \phi_n(\dot{\phi}_n - \dot{\phi}) \, dt \rightarrow 0, \) since \( \int_0^1 |\phi_n(\dot{\phi}_n - \dot{\phi})| \, dt \) is bounded and \( a_n^3 \rightarrow a_\lambda^3 \) uniformly. Then
\[ \int_0^1 a_n^3 \phi_n(\dot{\phi}_n - \dot{\phi}) \, dt \rightarrow 0, \]
and since \( \dot{\phi}_n \rightarrow \dot{\phi} \) weakly in \( L^2 \) we have
\[ \int_0^1 a_\lambda^3 \dot{\phi}(\dot{\phi}_n - \dot{\phi}) \, dt \rightarrow 0. \]
Therefore, (5.8)–(5.9) together gives
\[ \int_0^1 a_\lambda^3(\dot{\phi}_n - \dot{\phi})^2 \, dt \rightarrow 0, \]
and since \( a_\lambda^3 \geq a_\lambda^3 > 0, \) \( \forall t \in [0, 1] \) we obtain (5.7).

In order to prove (5.6) note that there exist an infinitesimal sequence \( \epsilon_n \)
\[ |\nabla F_{\epsilon,\lambda}(\alpha, 0)| \leq \epsilon_n \|\alpha\|_{H^1_0}, \quad \forall \alpha \in H^1_0, \]
and then
\[
\int_0^1 \left[ (3a + U_\varepsilon(a_n))\dot{\phi}_n^2 + 2(3a_n + U_\varepsilon(a_n))\dot{a}_n^2 - 3a_n^2\alpha^2 \right] dt + \int_0^1 \left[ (3a_n + U_\varepsilon(a_n))\dot{a}_n^2 - a_n^2\rho_n^2 \right] dt \geq \int_0^1 6a_n^2\alpha V_\lambda(\phi_n) dt \leq \epsilon_n\|\alpha\|_{H^1_0}.
\]

Taking \(\alpha := a_n - a\), and recalling that
- \(\int_0^1 \dot{a}_n^2 dt\) is bounded,
- \(\|U_\varepsilon'(a_n)\|_{L^\infty}\) is bounded (since \(a_n\) doesn’t approach either \(m\) or \(M\)),
- \(a_n \to a\) uniformly, and \(a_n\) is bounded,
- \(\int_0^1 2a_n^2 V_\lambda(\phi_n) dt\) is bounded away from zero, and
- \(V_\lambda(\phi_n)\) is bounded,

we obtain
\[
\int_0^1 (3a_n + U_\varepsilon(a_n))\dot{a}_n(\dot{a}_n - a) dt \to 0.
\]

But \(3a_n + U_\varepsilon(a_n) \to 3a + U_\varepsilon(a)\) uniformly, and \(3a + U_\varepsilon(a) \geq 3a \geq 3a_1 > 0, \forall t \in [0, 1]\). Then, arguing as before, we also obtain (5.6), and the proof is complete. \(\square\)

**Proof of Proposition 5.7.** As already outlined, the aim is to apply Theorem 3.4 to the functional \(F_{\varepsilon, \lambda}\). In view of lemmas 5.2 and 5.3 it still remains to show that hypothesis (1) holds.

Take \(a_*(t) = (1 - t)a_0 + t a_1\), and choose \(\omega_0 = a_*, \epsilon_0 = \phi_*\). Note that, by (4.5) and the choice of \(m\), we have
\[
\int_0^1 a_3 V_\lambda(\phi_*) dt \geq m^3 v_* > 0, \quad \forall a \in \Omega, \forall \lambda \geq \sup_{t \in [0, 1]} V(\phi_*).
\]

Moreover by assumption (4.2) we have, for any \(a \in \Omega\),
\[
\int_0^1 \left[ (3a + U_\varepsilon(a)) \dot{a}^2 - a^3 \rho^2 \right] dt \geq \int_0^1 3m \dot{a}^2 - M^3 (\dot{\phi}_1 - \dot{\phi}_0)^2 dt \geq 3m(a_1 - a_0)^2 - M^3 (\dot{\phi}_1 - \dot{\phi}_0)^2 > 0,
\]
therefore, by (5.10),
\[
b_1 := \inf_{a, \lambda} F_{\varepsilon, \lambda}(a, \phi_*) > 0.
\]

Clearly, \(b_1 = b_1(m, M)\) and is independent of \(k\). Actually, \(b_1\) is independent of \(\lambda\) too, but this property won’t be used here.

Since \(V_\lambda\) is bounded,
\[
\sup_{\lambda} V_\lambda \equiv B_\lambda < +\infty,
\]
and
\[
\int_0^1 a_3^2 V_\lambda(\phi) dt \leq a_0^3 B_\lambda.
\]

Moreover, if \(\epsilon\) is sufficiently small, \(U_\varepsilon(a_*(t)) = 0 \forall t\), so
\[
F_{\varepsilon, \lambda}(a_*, \phi) = \int_0^1 \left( 3a_\ast \dot{a}_\ast^2 - a_\ast^3 \dot{\phi}_\ast^2 \right) dt \cdot \int_0^1 2a_3^2 V_\lambda(\phi) dt,
\]
but \[ \int_0^1 \left(3a_e \dot{a}_e^2 - a_e^3 \dot{\phi}_e^2 \right) \, dt \leq \int_0^1 \left(3a_e \dot{a}_e^2 - m^3 \dot{\phi}^2 \right) \, dt, \] and then there exists \( R > 0 \) independent of \( k \) such that
\[
\sup_{\|\phi\|_{\mathcal{K}_k} = R} F_{e,\lambda}(a_e, \dot{\phi} + \phi_e) =: b_0 < b_1, \tag{5.12}
\]
(note that \( b_0 \) depends on \( \lambda \) and then hypothesis \( \text{(1)} \) of Theorem 5.4 also holds. This implies that there exists a critical value for the functional \( F_{e,\lambda} \) on \( \mathcal{X}_k \) in the interval \([b_1, b_2]\), where
\[
b_2 := \sup_{\|\phi\|_{\mathcal{K}_k} \leq R} F_{e,\lambda}(a_e, \dot{\phi} + \phi_e) \tag{5.13}
\]
and since \( b_2 \), though depending on \( \lambda \), is independent of \( k \), the proof is complete. \(
\square \)

6. PROOF OF THE MAIN RESULT

To complete the proof of Theorem 1.1 using Proposition 5.1, we first need the following lemma. Recall the definitions of \( b_1, b_2 \) given in \( \text{(5.11)} \) and \( \text{(5.13)} \).

**Lemma 6.1.** There exists a critical value of \( F_{e,\lambda} \) on \( \mathcal{X} = \Omega \times E \) in \([b_1, b_2]\).

**Proof.** Let \( x_k = (a_k, \phi_k) \) the critical point given by Proposition 5.1. Arguing as in the proof of Lemma 5.3, we deduce the existence of \( x = (a, \phi) \in \mathcal{X} \) such that, up to subsequence,
\[
x_k \to x \quad \text{in } H^1.
\]
We will show that \( x \) is a critical point of \( F_{e,\lambda} \). Since \( F_{e,\lambda}(x_k) \in \] \([b_1, b_2]\) and \( F_{e,\lambda} \) is continuous, it immediately will follow that \( F_{e,\lambda}(x) \in \] \([b_1, b_2]\).

Take \((\alpha, \theta) \in H^1_0\), and consider \( \theta_k \), the orthogonal projection of \( \theta \) on \( W_k \). As observed in remark \( \text{(4.1)} \), \( \theta_k \to \theta \) in \( H^1 \). Since \( \nabla F_{e,\lambda}(x_k)[\alpha, \theta_k] = 0 \) for any \( k \), we have
\[
\nabla F_{e,\lambda}(x)[\alpha, \theta] = \nabla F_{e,\lambda}(x)[\alpha, \theta - \theta_k] + (\nabla F_{e,\lambda}(x) - \nabla F_{e,\lambda}(x_k))[\alpha, \theta_k].
\]
Let us observe that, since \([\alpha, \theta_k]\) is bounded in \( H^1 \), and recalling that \( F_{e,\lambda} \) is \( C^1 \) on \( \mathcal{X} \), then \( (\nabla F_{e,\lambda}(x) - \nabla F_{e,\lambda}(x_k))[\alpha, \theta_k] \to 0 \). Moreover
\[
\nabla F_{e,\lambda}(x)[\alpha, \theta - \theta_k] \to 0
\]
since \( \theta_k \to \theta \) in \( H^1 \). Then \( \nabla F_{e,\lambda}(x)[\alpha, \theta] = 0 \), \( \forall \alpha, \theta \in H^1_0 \), and the proof is complete. \(
\square \)

**Proof of Theorem 1.1.** Recall first that, looking for solutions of \( \text{(1.4a)} \)–\( \text{(1.4b)} \) in the space of curves \((a, \phi)\) defined on \([0, T]\) and with boundary conditions \( \text{(1.7)} \), amounts to find critical points for the functional \( \text{(2.13)} \) in the space of curves \((a, \phi)\) defined in \([0, 1]\) with boundary conditions \( \text{(2.19)} \).

The above lemma ensures the existence of \((a_{e,\lambda}, \phi_{e,\lambda})\), critical point of \( F_{e,\lambda} \) in \( \Omega \times E \), with critical value in \([b_1, b_2]\). First, let us observe that a simple bootstrap argument shows that both \( a_{e,\lambda} \) and \( \phi_{e,\lambda} \) are \( C^2 \). Then the following conservation law follows from the variational principle:
\[
(3a_{e,\lambda} + U_e(a_{e,\lambda})) \dot{a}_{e,\lambda}^2 - a_{e,\lambda}^3 \dot{\phi}_{e,\lambda}^2 - 2a_{e,\lambda}^3 V_e(\phi_{e,\lambda}) = 0.
\]
Since \( a_{e,\lambda}^3 (\dot{\phi}_{e,\lambda}^2 + 2V_e(\phi_{e,\lambda})) > 0 \), \( \forall t \), then \( \dot{a}_{e,\lambda}(t) \neq 0, \forall t \), and since we have supposed (see \( \text{(4.1)} \)) \( a_0 > a_1 \) it is
\[
\dot{a}_{e,\lambda}(t) < 0, \quad \forall t \in [0, 1]
\]
and
\[
a_1 \leq a_{e,\lambda}(t) \leq a_0, \quad \forall t \in [0, 1].
\]
Then $a_{\epsilon,\lambda}(t)$ is bounded away from $m$ and $M$. Taking a sufficiently small $\epsilon$ we have

$$m + \epsilon < a_{\epsilon,\lambda}(t) < M - \epsilon$$

so that $U_\epsilon'(a_{\epsilon,\lambda}) = 0$. Furthermore, $a_{\epsilon,\lambda}$ is a critical point of

$$F_\lambda(a, \phi) := \int_0^1 (3a^2 - a^3 \phi^2) \, dt \cdot \int_0^1 2a^3 V_\lambda(\phi) \, dt,$$

with critical value in $[b_1, b_2]$. Moreover, the conservation law (6.1) takes the form

$$(6.2) \quad 3a_\lambda \dot{a}_\lambda^2 - a_\lambda^3 \dot{\phi}_\lambda^2 - 2a_\lambda^3 V_\lambda(\phi_\lambda) = 0.$$ 

Recalling that $\dot{a}_\lambda$ is negative, that implies

$$\dot{a}_\lambda = -\sqrt{\frac{\dot{\phi}_\lambda^2 + 2V_\lambda(\phi_\lambda)}{3}}.$$ 

Integrating the above relation in $[0, 1]$ we obtain

$$a_\lambda(t) = a_0 e^{-\frac{1}{3} \int_0^1 \sqrt{\dot{\phi}_\lambda^2 + 2V_\lambda(\phi_\lambda)} \, ds},$$

and in particular

$$a_1 = a_0 e^{-\frac{1}{3} \int_0^1 \sqrt{\dot{\phi}_\lambda^2 + 2V_\lambda(\phi_\lambda)} \, ds}.$$ 

But $a_1 > 0$ is of course fixed and therefore independent of $\lambda$, then there exists a positive constant $D$ independent of $\lambda$ such that

$$\int_0^1 |\phi_\lambda| \, dt \leq \int_0^1 \sqrt{\dot{\phi}_\lambda^2 + 2V_\lambda(\phi_\lambda)} \, ds \leq D.$$ 

Since $\phi_0$ is fixed, we see that, for any $\lambda$, the function $\phi_\lambda$ satisfies

$$|\phi_\lambda(t)| \leq |\phi_0| + D,$$

and therefore choosing $\lambda \geq \sup \{ V(\phi) : |\phi| \leq |\phi_0| + D \}$ we have

$$V(\phi_\lambda) = V_\lambda(\phi_\lambda), \quad V'(\phi_\lambda) = V'_\lambda(\phi_\lambda).$$

This means that $(a_\lambda, \phi_\lambda)$ is also a critical point of $F(2.18)$ with positive critical value. □

REFERENCES

[1] A. Ambrosetti, Mem. S. M. F., 49 (1992), 1–139
[2] Bruhat, Y., Acta Math., 88 1952, 141–225
[3] D. Christodoulou, Ann. Math. 140 607 (1994).
[4] D. Christodoulou, Ann. Math. 149 183 (1999)
[5] R. Giambò, Class. Quantum Grav. 22 (2005) 2295
[6] R. Giambò, F. Giannoni, G. Magli, P. Piccione, Commun. Math. Phys. 235(3), 545-563 (2003).
[7] E. van Groesen, Journ. Math. Anal. Appl. 132 (1988) 1-12
[8] E. van Groesen, Hamiltonian Flow on an Energy Surface: 240 Years After the Euler-Maupertuis Principle, on “Geometric Aspects of the Einstein Equations and Integrable Systems” (R. Martini ed.), Lecture Notes in Physics, Vol. 239, Springer-Verlag, 1985.
[9] R. Goswami, P. S. Joshi, gr-qc/0401044
[10] R. Goswami, P. S. Joshi and P. Singh, gr-qc/0506129
[11] T. Hertog, G.T. Horowitz, K. Maeda, Phys.Rev.Lett. 92 (2004) 131101
[12] T. Hertog, G. T. Horowitz, K. Maeda, gr-qc/0404050
[13] J. Kijowski, D. Bambusi, and G. Magli, Elasticità finita e relativistica: introduzione ai metodi geometrici della teoria dei campi, Pitagora, 1991 (in Italian)
[14] D. Kramer, H. Stephani, E. Herlt, M. MacCallum, Exact Solutions of the Einstein’s Field Equations, (Cambridge University Press, Cambridge, 1980)
[15] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*, W. H. Freeman and Company, 1973
[16] B. O’Neill, *Semi–Riemannian Geometry with Applications to Relativity*, Academic Press, New York, 1983.
[17] N. J. Nunes, J. E. Lidsey, Phys.Rev. D 69 (2004) 123511
[18] R. Palais, Topology 5, (1966), 115–132.
[19] V. Perlick, *Gravitational lensing from a spacetime perspective*, Living Rev. Relativity 7 (2004)
[20] P.H. Rabinowitz, *Minimax Method in Critical Point Theory with Applications to Differential Equations*, CBMS Regional Conf. Series in Math., No. 65, AMS, Providence, 1986
[21] A. G. Riess et al. [Supernova Search Team Collaboration], Astron. J. 116 (1998) 1009
[22] S. Tsujikawa, Phys. Rev. D 72 (2005) 083512
[23] R. M. Wald, *General Relativity*, University of Chicago Press, 1984.

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