COMPLEMENTARITY AND PHASE
DISTRIBUTIONS FOR ANGULAR MOMENTUM
SYSTEMS

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ABSTRACT

Interferences in the distributions of complementary variables for angular momentum - two level systems are discussed. A quantum phase distribution is introduced for angular momentum. Explicit results for the phase distributions and the number distributions for atomic coherent states, squeezed states and superpositions of coherent states are given. These results clearly demonstrate the issue of complementarity and provide us with results analogous to those for the radiation field.

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Complementarity of basic variables in Physics is leading to new types of interference experiments with wide implications [1-5]. For example complementarity of position and momentum variables has been used by Rauch and coworkers [1] to perform new types of neutron interferometric experiments. Wolf and coworkers [2-4] demonstrated the utility of the complementarity of frequency and time variables. An extension of these ideas to other complementary variables would be important. In particular one should examine systems with spins or angular momentum $j$. The excitation in the system is determined by the number distribution $p(m) = \langle j, m|\rho|j, m \rangle$ with $|m| \leq j$. Here $|j, m\rangle$ is the simultaneous eigenstate of $J^2$ and $J_z$ and $\rho$ is the density matrix for the system. Clearly we need to introduce complementary variable. This would be phase variable. However, there are difficulties [6] with the introduction of the phase operator because of the boundedness of angular momentum operator spectrum $|m| \leq j$. We will avoid these difficulties by directly introducing the phase distribution $p(\varphi)$. The interferometric aspects can be discussed in terms of the distributions $p(m)$ and $p(\varphi)$. The analysis that we present is also applicable to a system of identical two level atoms, as such a system is equivalent to an angular momentum system [7]. Here the phase $\varphi$ will refer to the phase of the dipole moment of the system and thus $\varphi$ is an important quantity in the context of atomic coherences and the interferometry based on such coherences.

In this letter we therefore first address the question- what are the phase distributions associated with angular momentum operators. Note that in recent years the phase distributions for the radiation field have been extensively discussed [8-11]. The phase distributions have been defined in a variety of ways as there is no unique way to do so. However most ways of defining are qualitatively equivalent though the different definitions differ in details. Under certain conditions it was demonstrated that the measured phase distributions [12] were related
to the phase space distributions like the Q-function and Wigner function. We introduce $p(\varphi)$ via the phase space distributions [13] for the angular momentum operators. Let $\rho$ represent the density matrix

$$\rho = \sum_{m,m'} \rho_{mm'} |j,m\rangle \langle j,m'|,$$

and let $|\theta,\varphi\rangle$ be the atomic coherent state [14]

$$|\theta,\varphi\rangle = \sum_m \left[ \frac{2j}{j+m} \right]^{1/2} \left( \sin \frac{\theta}{2} \right)^{j+m} \left( \cos \frac{\theta}{2} \right)^{j-m} |j,m\rangle e^{-i(j+m)\varphi}. \quad (2)$$

For this state the mean value of the dipole moment operator $J_-=J_x - iJ_y$ is $j\sin\theta e^{-i\varphi}$. Thus we should look for the distribution of $\varphi$. This is in analogy to the radiation field in the coherent state. A very useful phase space distribution is the Q-function defined by

$$Q(\theta,\varphi) = \langle \theta,\varphi | \rho | \theta,\varphi \rangle, \quad (3)$$

which is normalized according to

$$\left( \frac{2j+1}{4\pi} \right) \int \int Q(\theta,\varphi) \sin \theta d\theta d\varphi = 1. \quad (4)$$

We next define $p(\varphi)$ via

$$p(\varphi) = \left( \frac{2j+1}{4\pi} \right) \int Q(\theta,\varphi) \sin \theta d\theta ; \quad p(\varphi) > 0. \quad (5)$$

On using (1) to (3), we find that

$$p(\varphi) = \sum_{m,m'} \rho_{mm'} B \left( j - \frac{m+m'}{2} + 1, \; j + \frac{m+m'}{2} + 1 \right) e^{i(m-m')\varphi}, \quad (6)$$

where $B(x,y)$ is the Beta function. Note that the number distribution is given by $\rho_{mm}$. If $\rho_{mm'} \propto \delta_{mm'}$, then $p(\varphi)$ is uniform as expected.
We next consider \( p(\varphi) \) and \( p(m) \) for some important states of the angular momentum systems.

**A. Coherent State \(|\alpha,\beta\rangle\)**

In this case the probability \( p(m) \) of finding the system in the state \(|j,m\rangle\) is given by

\[
p(m) = \binom{2j}{j+m} \left( \sin \frac{\alpha}{2} \right)^{2j+2m} \left( \cos \frac{\alpha}{2} \right)^{2j-2m},
\]

which is just the Binomial distribution in terms of the variable \( (j+m) \) with mean value and variance equal to

\[
\langle j+m \rangle = j(1-\cos \alpha), \quad \langle (j+m)^2 \rangle - \langle j+m \rangle^2 = \frac{j}{2} \sin^2 \alpha.
\]

The corresponding phase distribution is

\[
p(\varphi) = \left( \frac{2j+1}{4\pi} \right) \int \langle \theta,\varphi | \alpha,\beta \rangle \langle \alpha,\beta | \theta,\varphi \rangle \sin \theta \, d\theta
\]

\[
= \left( \frac{2j+1}{4\pi} \right) \sum_{m=-j}^{j} \sum_{m'=-j}^{j} \binom{2j}{j+m} \binom{2j}{j+m'} \left( \sin \frac{\alpha}{2} \right)^{2j+m+m'} \left( \cos \frac{\alpha}{2} \right)^{2j-m-m'} e^{-i(m-m')\beta}
\]

\[
\times \frac{2(j-m+m')!}{(2j+1)!} \frac{(j+m+m')!}{(2j+1)!} e^{i(m-m')\varphi},
\]

which is centered at \( \varphi = \beta \). These distributions \( p(m) \) and \( p(\varphi) \) are shown in the Fig.1. For large \( j \), \( p(\varphi) \) can be approximated by a Gaussian. The width of the distribution \( p(\varphi) \) is proportional to \( 1/\sqrt{j} \) with proportionality factor \( \approx 3.29 \). The width of the distribution \( p(m) \) is proportional to \( \sqrt{j} \). The two widths are thus in agreement with the idea of complementarity.

**B. Atomic Squeezed State \(|\zeta\rangle\)**

The atomic squeezed state \(|\zeta\rangle\) was defined by

\[
|\zeta\rangle = \mathcal{N} \{ \tanh (2 |\zeta|) \}^{J_z/2} e^{-i\frac{\pi}{4} J_y} |j,0\rangle,
\]

\[
|\zeta\rangle = \mathcal{N} \{ \tanh (2 |\zeta|) \}^{J_z/2} e^{-i\frac{\pi}{4} J_y} |j,0\rangle,
\]

\[
(10)
\]
where $\mathcal{N}$ is a normalization constant. The state $| \zeta \rangle$ has a number of very interesting properties. For example the distribution $p(m)$ exhibits very interesting interference effects [15]. This is shown in Fig.2. In terms of the matrix element of the rotation operator $d_{mm'}^j (\pi/2)$ is

$$
d_{mm'}^j \left( \frac{\pi}{2} \right) = \frac{((-j+m)!(j-m)!(j+m')!(j-m')!)}{2^j} \sum_{q=-j}^{j} \frac{(-1)^q}{(j-m'-q)! (j+m'-q)! (j+m-q)!}.$$

The number and the phase distributions are shown to be

$$p(m) = \mathcal{N}^2 \left( d_{m0}^j \left( \frac{\pi}{2} \right) \right)^2 \left\{ \tanh \left( 2 | \zeta \right| \right\}^m,$$

$$p(\varphi) = \mathcal{N}^2 \left( \frac{2j+1}{4\pi} \right) \sum_{m=-j}^{j} \sum_{m'=-j}^{j} \left[ \begin{array}{c} 2j \\ j+m \end{array} \right]^{1/2} \left[ \begin{array}{c} 2j \\ j+m' \end{array} \right]^{1/2} d_{m0}^j \left( \frac{\pi}{2} \right) d_{m'0}^j \left( \frac{\pi}{2} \right) \times \left\{ \tanh \left( 2 | \zeta \right| \right\}^{m+m'} \frac{2 \left( j - \frac{m+m'}{2} \right)! \left( j + \frac{m+m'}{2} \right)!}{(2j+1)!} e^{-i(m-m')\varphi}.$$

The unnormalized phase distribution $p(\varphi)$ is shown in the Fig.2. The phase distribution has a doublet structure which arises from the fact that $p(m)$ is zero for odd values of $m$. The peaks are at $\pm \pi/2$. This is because of the rotation by $\pi/2$ in the definition (10). This bifurcation in phase distribution is similar to the one for squeezed states of the radiation field [10]. The width of each peak is proportional to $1/\sqrt{j}$ for large $j$ with proportionality factor $\approx 2.12$. The numerical factor is less than that for the coherent state.

**C. Superposition of Atomic Coherent States**

Finally we consider a state which is a superposition of two atomic coherent states. Extensive literature exists [16] on superpositions of coherent states of the radiation field. For illustration we consider superposition of two coherent states i.e.,

$$ | \psi \rangle = \mathcal{N} \left( | \pi/4, \pi/4 \rangle + | \pi/4, \pi/4 + \pi/8 \rangle \right),$$

(14)
where $N$ is the normalization factor. It has been shown by Agarwal and Puri [17] that such superpositions can be produced by considering the interaction of a set of atoms with a field in a cavity with large detuning. This dispersive interaction which is proportional to $J_+ J_-$, is like the nonlinear phase shift term. The number and the phase distributions for this state are found to be

$$p(m) = 2N^2 \begin{pmatrix} \frac{2j}{j + m} \end{pmatrix} \left( \sin \frac{\pi}{8} \right)^{2j+2m} \left( \cos \frac{\pi}{8} \right)^{2j-2m} \left( 1 + \cos (j + m) \frac{\pi}{8} \right),$$

$$p(\varphi) = N^2 \left( \frac{2j+1}{4\pi} \right) \sum_{m=-j}^{j} \sum_{m'=-j}^{j} \left[ \frac{2j}{j + m} \right] \left[ \frac{2j}{j + m'} \right] \left( \sin \frac{\pi}{8} \right)^{2j+m+m'} \left( \cos \frac{\pi}{8} \right)^{2j-m-m'}$$

$$\times e^{-i(m-m')\pi/4} \left( 1 + e^{-i(m-m')\pi/8} + e^{-i(j+m)\pi/8} + e^{i(j+m')\pi/8} \right)$$

$$\times \frac{2 \left( j - \frac{m+m'}{2} \right)! \left( j + \frac{m+m'}{2} \right)!}{(2j+1)!} e^{i(m-m')\varphi}. \tag{16}$$

These distributions (unnormalized) are shown in Fig.3. For large values of $j$, the number distribution $p(m)$ shows the interference minimum as given by (15) i.e., at $(j+m)\pi/8 = \pi$ and the two peaks in $p(\varphi)$ become visible. The situation is similar [16] to CAT states for harmonic oscillator.

Thus in conclusion we have introduced a phase distribution for angular momentum systems. This enables us to discuss interferences in complementary spaces for angular momentum systems and for two level systems.

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FIGURE CAPTIONS

1. Phase distribution $p(\varphi)$ and number distribution $p(m)$ for a system in the atomic coherent state $|\pi/4, \pi/4\rangle$ for $j=10$ (dotted), 20 (dashed), and 30 (solid), the phase angle is in units of $\pi$.

2. Phase distribution $p(\varphi)$ and number distribution $p(m)$ for a system in atomic squeezed state $|\zeta\rangle$, where $\zeta$ (squeezing parameter) is equal to 2.6892, for $j=10$ (dotted), 20 (dashed). For illustration $p(\varphi)$ for $j=2$ (long dash) is also drawn.

3. Phase distribution $p(\varphi)$ and number distribution $p(m)$ for superposition of two coherent states i.e., $|\pi/4, \pi/4\rangle + |\pi/4, \pi/4 + \pi/8\rangle$ for $j=10$ (dotted), 20 (dashed), and 30 (solid).
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