This is the accepted version of the journal article:
Marín Pérez, David; Bedrouni, Samir. «Geometry of certain foliations on the complex projective plane». Annali della Scuola normale superiore di Pisa - Classe di scienze, 2022. 27 pàg. Scuola Normale Superiore.

This version is available at https://ddd.uab.cat/record/265181

under the terms of the IN COPYRIGHT license
Abstract. — Let \( d \geq 2 \) be an integer. The set \( \mathcal{F}(d) \) of foliations of degree \( d \) on the complex projective plane can be identified with a Zariski’s open set of a projective space of dimension \( d^2 + 4d + 2 \) on which \( \text{Aut}(\mathbb{P}^2_\mathbb{C}) \) acts. We show that there are exactly two orbits \( O(\mathcal{F}_d^1) \) and \( O(\mathcal{F}_d^2) \) of minimal dimension 6, necessarily closed in \( \mathcal{F}(d) \). This generalizes known results in degrees 2 and 3. We deduce that an orbit \( O(\mathcal{F}) \) of an element \( \mathcal{F} \in \mathcal{F}(d) \) of dimension 7 is closed in \( \mathcal{F}(d) \) if and only if \( \mathcal{F}_d^i \not\in O(\mathcal{F}) \) for \( i = 1, 2 \). This allows us to show that in any degree \( d \geq 3 \) there are closed orbits in \( \mathcal{F}(d) \) other than the orbits \( O(\mathcal{F}_d^1) \) and \( O(\mathcal{F}_d^2) \), unlike the situation in degree 2.

On the other hand, we introduce the notion of the basin of attraction \( B(\mathcal{F}) \) of a foliation \( \mathcal{F} \in \mathcal{F}(d) \) as the set of \( \mathcal{G} \in \mathcal{F}(d) \) such that \( \mathcal{F} \in O(\mathcal{G}) \). We show that the basin of attraction \( B(\mathcal{F}_d^1) \), resp. \( B(\mathcal{F}_d^2) \), contains a quasi-projective subvariety of \( \mathcal{F}(d) \) of dimension greater than or equal to \( \dim \mathcal{F}(d) - (d - 1) \), resp. \( \dim \mathcal{F}(d) - (d - 3) \). In particular, we obtain that the basin \( B(\mathcal{F}_d^2) \) contains a non-empty Zariski open subset of \( \mathcal{F}(3) \). This is an analog in degree 3 of a result on foliations of degree 2 due to Cerveau, Déserti, Garba Belko and Meziani.

2010 Mathematics Subject Classification. — 37F75, 32S65, 32M25, 32M05.

Introduction

The set \( \mathcal{F}(d) \) of holomorphic foliations of degree \( d \) on \( \mathbb{P}^2_\mathbb{C} \) is identified with a Zariski open subset of the projective space \( \mathbb{P}^{d^2 + 4d + 2}_\mathbb{C} \). We are interested here in the action of the group \( \text{Aut}(\mathbb{P}^2_\mathbb{C}) = \text{PGL}_3(\mathbb{C}) \) on \( \mathcal{F}(d) \). We generalize to arbitrary degree some results known in small degrees \([9, 1, 5]\) on this action.

For \( \mathcal{F} \in \mathcal{F}(d) \), we will respectively denote by \( O(\mathcal{F}) \) and \( \text{Iso}(\mathcal{F}) \) the orbit and the isotropy group of \( \mathcal{F} \) under the action of \( \text{Aut}(\mathbb{P}^2_\mathbb{C}) \), i.e.

\[
O(\mathcal{F}) := \{ \varphi^* \mathcal{F} \in \mathcal{F}(d) \mid \varphi \in \text{Aut}(\mathbb{P}^2_\mathbb{C}) \} \quad \text{and} \quad \text{Iso}(\mathcal{F}) := \{ \varphi \in \text{Aut}(\mathbb{P}^2_\mathbb{C}) \mid \varphi^* \mathcal{F} = \mathcal{F} \}.
\]

\( O(\mathcal{F}) \) is a Zariski irreducible subset of \( \mathcal{F}(d) \) and \( \text{Iso}(\mathcal{F}) \) is an algebraic subgroup of \( \text{Aut}(\mathbb{P}^2_\mathbb{C}) \).

Key words and phrases. — foliation, singularity, inflection point, orbit, isotropy group.

This work has been partially funded by the Ministry of Science, Innovation and Universities of Spain through the grant PGC2018-095998-B-I00, by the Agency for Management of University and Research Grants of Catalonia through the grant 2017SGR1725 and by the Spanish State Research Agency, through the Severo Ochoa and María de Maeztu Program for Centers and Units of Excellence in R&D (CEX2020-001084-M).
Following [14] we will say that a foliation of $\mathbf{F}(d)$ is convex if its leaves other than straight lines have no inflection points. We will denote by $\mathbf{FC}(d)$ the subset of $\mathbf{F}(d)$ consisting of convex foliations, which is a ZARISKI closed subset of $\mathbf{F}(d)$.

According to [7, Proposition 2.2] every foliation of degree 0 or 1 is convex. For $d \geq 2$, $\mathbf{FC}(d)$ is a proper closed subset of $\mathbf{F}(d)$ and it contains the foliation $\mathcal{F}_1^d$ defined in the affine chart $(x,y)$ by the 1-form (see [3, page 75])

$$\omega_1^d = y^d dx + x^d (xdy - ydx).$$

We know from [9, Proposition 2.3] that if $\mathcal{F}$ is an element of $\mathbf{F}(d)$ with $d \geq 2$, then the dimension of $O(\mathcal{F})$ is at least 6, or equivalently, the dimension of $\text{Iso}(\mathcal{F})$ is at most 2. In addition these bounds are attained by the convex foliation $\mathcal{F}_1^d$ and the non convex foliation $\mathcal{F}_2^d$ defined by the 1-form (see [3])

$$\omega_2^d = x^d dx + y^d (xdy - ydx).$$

The main result of this paper is the following.

**Theorem A.** — Let $d$ be an integer greater than or equal to 2 and let $\mathcal{F}$ be an element of $\mathbf{F}(d)$. Assume that the isotropy group $\text{Iso}(\mathcal{F})$ of $\mathcal{F}$ has dimension 2. Then $\mathcal{F}$ is linearly conjugated to one of the two foliations $\mathcal{F}_1^d$ and $\mathcal{F}_2^d$ defined respectively by the 1-forms

1. $\omega_1^d = y^d dx + x^d (xdy - ydx)$;
2. $\omega_2^d = x^d dx + y^d (xdy - ydx)$.

In other words, $O(\mathcal{F}_1^d)$ and $O(\mathcal{F}_2^d)$ are the only orbits of dimension 6. They are closed in $\mathbf{F}(d)$. Moreover we have

$$\text{Iso}(\mathcal{F}_1^d) = \left\{ \left( \begin{array}{cc} \alpha^d - 1_x & \alpha^d y \\ 1 + \beta x & 1 + \beta x \end{array} \right) \right| \alpha \in \mathbb{C}^*, \beta \in \mathbb{C} \right\},$$

$$\text{Iso}(\mathcal{F}_2^d) = \left\{ \left( \begin{array}{cc} \alpha^d + 1_x & \alpha^d y \\ 1 + \beta x & 1 + \beta x \end{array} \right) \right| \alpha \in \mathbb{C}^*, \beta \in \mathbb{C} \right\};$$

these two groups are not conjugated.

This theorem is a generalization in arbitrary degree of previous results on foliations of degrees $d = 2$ ([9, Proposition 2.7]) and $d = 3$ ([1, Theorem 10], [5, Corollary B]).

We also obtain the following corollary, which generalizes [5, Corollary 3.9]:

**Corollary B.** — Let $d$ be an integer greater than or equal to 2 and let $\mathcal{F}$ be an element of $\mathbf{F}(d)$. If $\dim O(\mathcal{F}) \leq 7$, then

$$\overline{O(\mathcal{F})} \subset O(\mathcal{F}) \cup O(\mathcal{F}_1^d) \cup O(\mathcal{F}_2^d).$$

In particular, when $\dim O(\mathcal{F}) = 7$, the orbit $O(\mathcal{F})$ of $\mathcal{F}$ is closed in $\mathbf{F}(d)$ if and only if $\mathcal{F}_i^d \notin \overline{O(\mathcal{F})}$ for $i = 1,2$.

In the spirit of Corollary B we can ask under what condition the closure in $\mathbf{F}(d)$ of the orbit $O(\mathcal{F})$ of an element $\mathcal{F}$ of $\mathbf{F}(d)$ contains the foliations $\mathcal{F}_1^d$ and $\mathcal{F}_2^d$, a question that we have already asked and studied in degree 3 in [5, Section 3]. In Section §3, we extend (Propositions 3.4 and 3.11) in arbitrary degree $d$ our previous results in [5, Propositions 3.10, 3.12, 3.15, 3.17] concerning this question. For $\mathcal{F} \in \mathbf{F}(d)$, we call basin of attraction of $\mathcal{F}$ the subset $\mathbf{B}(\mathcal{F})$ of $\mathbf{F}(d)$ defined by

$$\mathbf{B}(\mathcal{F}) := \{ \mathcal{G} \in \mathbf{F}(d) \mid \mathcal{F} \in \overline{O(\mathcal{G})} \}.$$
It follows from [9, Theorem 2.15] that in degree 2 the basin $B(\mathcal{F}_2^d)$ contains a quasi-projective subvariety of $\mathbb{F}(2)$ of dimension greater than or equal to $\dim \mathbb{F}(2) - 1$. In Section §3, we establish an analogous result in any degree greater than 2.

**Theorem C** (Theorem 3.10). — For any integer $d \geq 2$, the basin of attraction $B(\mathcal{F}_1^d)$ of $\mathcal{F}_1^d$ contains a quasi-projective subvariety of $\mathbb{F}(d)$ of dimension greater than or equal to $\dim \mathbb{F}(d) - (d - 1)$.

Notice that the non-convexity of $\mathcal{F}_2^d$ and the fact that $\mathbb{F}(d)$ is closed in $\mathbb{F}(d)$ imply that

$$B(\mathcal{F}_2^d) \subset \mathbb{F}(d) \setminus \mathbb{F}(d).$$

In degree 2, according to [9, Theorem 3], inclusion (0.1) is an equality:

$$B(\mathcal{F}_2^2) = \mathbb{F}(2) \setminus \mathbb{F}(2).$$

It follows in particular from equality (0.2) that the basin $B(\mathcal{F}_2^2)$ is a ZARISKI open subset of $\mathbb{F}(2)$. For $d \geq 3$ we show the following result.

**Theorem D** (Theorem 3.18). — In any degree $d \geq 3$, the basin of attraction $B(\mathcal{F}_2^d)$ of $\mathcal{F}_2^d$ contains a quasi-projective subvariety of $\mathbb{F}(d)$ of dimension greater than or equal to $\dim \mathbb{F}(d) - (d - 3)$. In particular, the basin $B(\mathcal{F}_2^3)$ contains a non-empty ZARISKI open subset of $\mathbb{F}(3)$.

Along the same order of ideas, we prove the following result.

**Theorem E** (Theorem 3.21). — For any integer $d \geq 2$, the intersection $B(\mathcal{F}_1^d) \cap B(\mathcal{F}_2^d)$ is non-empty and it contains a quasi-projective subvariety of $\mathbb{F}(d)$ of dimension equal to $\dim \mathbb{F}(d) - 3d$.

By combining equality (0.2) with the classification of C. Favre and J. V. Pereira of convex foliations of degree two (cf. [10, Proposition 7.4] or [6, Theorem A]), we see that the only closed orbits in $\mathbb{F}(2)$ under the action of $\text{Aut}(\mathbb{F}_C^2)$ are those of $\mathcal{F}_1^2$ and $\mathcal{F}_2^2$. We show in Section §4 that in any degree $d \geq 3$ there are closed orbits in $\mathbb{F}(d)$ other than the orbits $O(\mathcal{F}_1^d)$ and $O(\mathcal{F}_2^d)$, unlike the situation in degree 2. More precisely, we will consider a family of elements of $\mathbb{F}(d)$ which has been already studied in degree $d = 2$ in [9, page 189], namely the family $(\mathcal{F}_0^d(\lambda))_{\lambda \in \mathbb{C}^*}$ of foliations of degree $d$ on $\mathbb{P}_C^2$ defined by the 1-form

$$\omega_0^d(\lambda) = xdy - \lambda ydx + y^d dy.$$

We will see that, for $\lambda = 1$, $\mathcal{F}_0^d(1)$ is linearly conjugated to the foliation $\mathcal{F}_1^d$ and that, for any $\lambda \neq 1$, $\dim O(\mathcal{F}_0^d(\lambda)) = 7$. Moreover, we will show (Proposition 4.2) that the orbit $O(\mathcal{F}_0^d(\lambda))$ is closed for any $d \geq 3$ and $\lambda = -\frac{1}{d-1}$, resp. for any $d \in \{3,4,5\}$ and any $\lambda \in \mathbb{C}^*$, and we conjecture that it is so for any $d \geq 6$ and any $\lambda \in \mathbb{C}^*$ (see Conjectures 1 and 2).

### 1. Some definitions and notations

#### 1.1. Singularities and local invariants.

A degree $d$ holomorphic foliation $\mathcal{F}$ on $\mathbb{P}_C^2$ is defined in homogeneous coordinates $[x : y : z]$ by a 1-form

$$\omega = a(x, y, z)dx + b(x, y, z)dy + c(x, y, z)dz,$$

where $a$, $b$ and $c$ are homogeneous polynomials of degree $d + 1$ without common factor and satisfying the EULER condition $i_R \omega = 0$, where $R = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}$ denotes the radial vector field and $i_R$ is the interior product by $R$. 
Dually the foliation $\mathcal{F}$ can also be defined by a homogeneous vector field
\[
Z = U(x,y,z) \frac{\partial}{\partial x} + V(x,y,z) \frac{\partial}{\partial y} + W(x,y,z) \frac{\partial}{\partial z},
\]
the coefficients $U, V$ and $W$ are homogeneous polynomials of degree $d$ without common factor. The relation between $Z$ and $\omega$ is given by
\[
\omega = i_{\mathcal{F}}(dx \wedge dy \wedge dz).
\]

The singular locus $\text{Sing} \mathcal{F}$ of $\mathcal{F}$ is the projectivization of the singular locus of $\omega$
\[
\text{Sing} \omega = \{(x,y,z) \in \mathbb{C}^3 \mid a(x,y,z) = b(x,y,z) = c(x,y,z) = 0\}.
\]

Let $C \subset \mathbb{P}^2$ be an algebraic curve with homogeneous equation $F(x,y,z) = 0$. We say that $C$ is an invariant curve by $\mathcal{F}$ if $C \cap \text{Sing} \mathcal{F}$ is a union of (ordinary) leaves of the regular foliation $\mathcal{F}|_{\mathbb{P}^2 \setminus \text{Sing} \mathcal{F}}$. In algebraic terms, this is equivalent to require that the 2-form $\omega \wedge d\mathcal{F}$ is divisible by $F$, i.e. it vanishes along each irreducible component of $C$.

Let $p$ be an arbitrary point of $C$. When each irreducible component of $C$ passing through $p$ is not $\mathcal{F}$-invariant, we define the tangency order $\text{Tang}(\mathcal{F}, C, p)$ of $\mathcal{F}$ with $C$ at $p$ as follows. We fix a local chart $(u, v)$ such that $p = (0, 0)$; let $f(u, v) = 0$ be a reduced local equation of $C$ in a neighborhood of $p$ and let $X$ be a vector field defining the germ of $\mathcal{F}$ at $p$. We denote by $X(f)$ the Lie derivative of $f$ along $X$ and by $\langle f, X(f) \rangle$ the ideal of $C\{u,v\}$ generated by $f$ and $X(f)$. Then
\[
\text{Tang}(\mathcal{F}, C, p) = \dim_{\mathbb{C}} \frac{\mathbb{C}\{u,v\}}{\langle f, X(f) \rangle}.
\]

Notice that $\text{Tang}(\mathcal{F}, C, p)$ coincides with the intersection multiplicity $(C, C')_p$ at $p$ of the two algebraic curves $C = \{F = 0\}$ and $C' = \{Z(F) = 0\}$. Moreover, $\text{Tang}(\mathcal{F}, C, p) < +\infty$ by the non-invariance of the irreducible components of $C$ passing through $p$. By convention, we put $\text{Tang}(\mathcal{F}, C, p) = +\infty$ if there is at least one irreducible component of $C$ invariant by $\mathcal{F}$ and passing through $p$.

Let us recall some local notions attached to the pair $(\mathcal{F}, s)$, where $s \in \text{Sing} \mathcal{F}$. The germ of $\mathcal{F}$ at $s$ is defined, up to multiplication by a unity in the local ring $O_s$ at $s$, by a vector field $X = A(u,v) \frac{\partial}{\partial u} + B(u,v) \frac{\partial}{\partial v}$. The algebraic multiplicity $\nu(\mathcal{F}, s)$ of $\mathcal{F}$ at $s$ is given by
\[
\nu(\mathcal{F}, s) = \min \{\nu(A,s), \nu(B,s)\},
\]
where $\nu(g,s)$ denotes the algebraic multiplicity of the function $g$ at $s$. Let us denote by $\Sigma_s(\mathcal{F})$ the family of straight lines through $s$ which are not invariant by $\mathcal{F}$. For any line $\ell$ of $\Sigma_s(\mathcal{F})$, we have the inequalities
\[1 \leq \text{Tang}(\mathcal{F}, \ell, s) \leq d .\]
This allows us to associate to the pair $(\mathcal{F}, s)$ the following (invariant) integers
\[\tau(\mathcal{F}, s) = \min \{\text{Tang}(\mathcal{F}, \ell, s) \mid \ell \in \Sigma_s(\mathcal{F})\}, \quad \kappa(\mathcal{F}, s) = \max \{\text{Tang}(\mathcal{F}, \ell, s) \mid \ell \in \Sigma_s(\mathcal{F})\} .\]

The invariant $\tau(\mathcal{F}, s)$ represents the tangency order of $\mathcal{F}$ with a generic line passing through $s$. It is easy to see that
\[\tau(\mathcal{F}, s) = \min \{k \geq 1 \mid \det(J^k_s X, R_s) \neq 0\} \geq \nu(\mathcal{F}, s),\]
where $J^k_s X$ denotes the $k$-jet of $X$ at $s$ and $R_s$ is the radial vector field centered at $s$. The Milnor number of $\mathcal{F}$ at $s$ is the integer
\[\mu(\mathcal{F}, s) = \dim_{\mathbb{C}} O_s / \langle A, B \rangle ,\]
where $\langle A, B \rangle$ denotes the ideal of $O_s$ generated by $A$ and $B$.

The singularity $s$ is called radial of order $n - 1$, with $n \in \{2, \ldots, d\}$, if $\nu(\mathcal{F}, s) = 1$ and $\tau(\mathcal{F}, s) = n$. 

The singularity \( s \) is called non-degenerate if \( \mu(\mathcal{F}, s) = 1 \), or equivalently if the Jacobian matrix of \( X \) at \( s \), denoted by \( \text{Jac}X(s) \), possesses two nonzero eigenvalues \( \lambda, \mu \). In this case, the quantity

\[
BB(\mathcal{F}, s) = \frac{\text{tr}^2(\text{Jac}X(s))}{\text{det}(\text{Jac}X(s))} = \frac{\lambda}{\mu} + \frac{\mu}{\lambda} + 2
\]

is called the Baum-Bott index of \( \mathcal{F} \) at \( s \), see \([2]\).

We will say that the singularity \( s \) is quasi-radial of order \( n \) if \( \mu(\mathcal{F}, s) = 1 \), \( BB(\mathcal{F}, s) = 4 \) and \( \kappa(\mathcal{F}, s) = n \).

In the sequel we will denote by \( \text{QRad}(\mathcal{F}, n) \) the set of quasi-radial singularities of \( \mathcal{F} \) of order \( n \) and by \( \text{QRad}(\mathcal{F}, n - 1) \) the subset of \( \text{Sing}(\mathcal{F}) \times \mathcal{L}_s(\mathcal{F}) \) defined by

\[
\text{QRad}(\mathcal{F}, n - 1) := \left\{ (s, \ell) \in \text{Sing}(\mathcal{F}) \times \mathcal{L}_s(\mathcal{F}) \mid \mu(\mathcal{F}, s) = 1, \ BB(\mathcal{F}, s) = 4, \text{Tang}(\mathcal{F}, \ell, s) = n \right\}.
\]

**Remark 1.1.** — Every radial singularity \( s \) of order \( n - 1 \) of a foliation \( \mathcal{F} \) of degree \( d \geq 2 \) on \( \mathbb{P}^2 \) is quasi-radial of order \( \geq n - 1 \), because \( \kappa(\mathcal{F}, s) \geq \tau(\mathcal{F}, s) \). The converse is false: for instance, for the foliation defined in the affine chart \( z = 1 \) by the 1-form \((x+y)\,dy - y\,dx + (x^n + y^d)\,dx\), with \( n \in \{2, 3, \ldots, d\} \), the point \([0 : 0 : 1]\) is a quasi-radial singularity of order \( n - 1 \), but it is not radial.

### 1.2. Inflection points.

Let us consider a foliation \( \mathcal{F} \) of degree \( d \) on \( \mathbb{P}^2 \) and let \( p \) be a regular point of \( \mathcal{F} \). Let us denote by \( T^p_{\mathcal{F}} \) the tangent line to the leaf of \( \mathcal{F} \) passing through \( p \); it is the straight line of \( \mathbb{P}^2 \) passing through \( p \) with direction \( T_p\mathcal{F} \). If \( k \in \{2, \ldots, d\} \), we will say that \( p \) is a (transverse) inflection point of order \( k - 1 \) of \( \mathcal{F} \) if \( \text{Tang}(\mathcal{F}, T^p_{\mathcal{F}}, p) = k \), in which case the line \( T^p_{\mathcal{F}} \) is not invariant by \( \mathcal{F} \). When \( T^p_{\mathcal{F}} \) is \( \mathcal{F} \)-invariant, the point \( p \) will be called a trivial inflection point of \( \mathcal{F} \). If we denote by \( \text{Inv}(\mathcal{F}) \) the set of invariant lines of \( \mathcal{F} \), then the set of trivial inflection points of \( \mathcal{F} \) is precisely \( \text{Inv}(\mathcal{F}) \cap \text{Sing}(\mathcal{F}) \). In the sequel, we will denote by \( \text{Flex}(\mathcal{F}) \) the set of inflection points of \( \mathcal{F} \) and by \( \text{Flex}(\mathcal{F}, k - 1) \) the subset of \( \text{Flex}(\mathcal{F}) \) consisting of transverse inflection points of \( \mathcal{F} \) of order \( k - 1 \), i.e.

\[
\text{Flex}(\mathcal{F}, k - 1) := \left\{ p \in \text{Flex}(\mathcal{F}) \mid p \notin \text{Sing}(\mathcal{F}), \text{Tang}(\mathcal{F}, T^p_{\mathcal{F}}, p) = k \right\}.
\]

Let us recall the notion of inflection divisor of \( \mathcal{F} \), introduced by PEREIRA \([16]\), which allows to determine the set \( \text{Flex}(\mathcal{F}) \). Let \( Z \) be a homogeneous vector field of degree \( d \) on \( \mathbb{C}^3 \) defining \( \mathcal{F} \). The inflection divisor of \( \mathcal{F} \), denoted by \( I_\mathcal{F} \), is the divisor of \( \mathbb{P}^2 \) defined by the homogeneous equation

\[
\begin{vmatrix}
1 & Z(x) & Z^2(x) \\
1 & Z(y) & Z^2(y) \\
1 & Z(z) & Z^2(z)
\end{vmatrix} = 0.
\]

According to \([16]\), \( I_\mathcal{F} \) satisfies the following properties:

1. The support of \( I_\mathcal{F} \) is exactly the closure of the set \( \text{Flex}(\mathcal{F}) \) of inflection points of \( \mathcal{F} \). More precisely, \( I_\mathcal{F} \) can be decomposed as \( I_\mathcal{F} = I_{\mathcal{F}^{\text{inv}}} + I_{\mathcal{F}^{\text{tr}}} \), where the support of \( I_{\mathcal{F}^{\text{inv}}} \) is the set \( \text{Inv}(\mathcal{F}) \) of \( \mathcal{F} \)-invariant lines and the support of \( I_{\mathcal{F}^{\text{tr}}} \) is the closure of the set of transverse inflection points of \( \mathcal{F} \).

2. If \( \mathcal{C} \) is an algebraic curve invariant by \( \mathcal{F} \), then \( \mathcal{C} \subset I_\mathcal{F} \) if and only if \( \mathcal{C} \subset \text{Inv}(\mathcal{F}) \).

3. The degree of the divisor \( I_\mathcal{F} \) is \( 3d \).

The foliation \( \mathcal{F} \) will be called convex if its inflection divisor \( I_\mathcal{F} \) is totally invariant by \( \mathcal{F} \), i.e. if \( I_\mathcal{F} \) is a product of invariant lines.
2. Description of the foliations $\mathcal{F}$ of degree greater than or equal to 2 such that $\dim O(\mathcal{F}) = 6$

Recall that the foliations $\mathcal{F}_1^d$ and $\mathcal{F}_2^d$ are respectively defined in the affine chart $z = 1$ by the 1-forms

$$\omega_1^d = y^d dx + x^d(\text{d}y - y \text{d}x) \quad \text{and} \quad \omega_2^d = x^d dx + y^d(\text{d}x - x \text{d}y).$$

The foliation $\mathcal{F}_1^d$ is convex with inflection divisor $I_{\mathcal{F}_1^d} = \text{inv}_{\mathcal{F}_1^d} = x^{d+1}y^{2d-1}$ and it has two singular points $s_1 = [0 : 0 : 1]$ and $s_2 = [0 : 1 : 0]$; the singularity $s_1$ has maximal algebraic multiplicity $d$ and $s_2$ is radial of maximal order $d - 1$. The foliation $\mathcal{F}_2^d$ is not convex with invariant inflection divisor $\text{inv}_{\mathcal{F}_2^d} = x^{2d+1}$ and transverse inflection divisor $I_{\mathcal{F}_2^d}^{\text{inv}} = y^{d-1}$. The singular locus $\text{Sing}(\mathcal{F}_2^d)$ is reduced to the point $s_1 = [0 : 0 : 1]$; moreover $\nu(\mathcal{F}_2^d, s_1) = d$. We note that the 1-forms $\omega_1^d/x^2y^d$ and $\omega_2^d/x^{d+2}$ are closed and they respectively admit as first integrals

$$\frac{1}{d-1} \left( \frac{x}{y} \right)^{d-1} + \frac{1}{x} \quad \text{and} \quad \frac{1}{d+1} \left( \frac{y}{x} \right)^{d+1} - \frac{1}{x};$$

this allows to check that

$$\text{Iso}(\mathcal{F}_1^d) = \left\{ \left( \frac{\alpha x^{d-1} y}{1 + \beta x}, \frac{\alpha x^d y}{1 + \beta x} \right) \mid \alpha \in \mathbb{C}^*, \beta \in \mathbb{C} \right\} \quad \text{and} \quad \text{Iso}(\mathcal{F}_2^d) = \left\{ \left( \frac{\alpha x^{d+1} + x}{1 + \beta x}, \frac{\alpha x^d y}{1 + \beta x} \right) \right\} \quad \text{where} \quad \alpha \in \mathbb{C}^*, \beta \in \mathbb{C}. $$

In particular, $\dim \text{Iso}(\mathcal{F}_i^d) = 2$ for $i = 1, 2$. Thus the orbits $O(\mathcal{F}_1^d)$ and $O(\mathcal{F}_2^d)$ are both of dimension 6, which is the minimal dimension possible in any degree $d \geq 2$ ([9, Proposition 2.3]). Theorem A announced in the Introduction shows that the orbits $O(\mathcal{F}_1^d)$ and $O(\mathcal{F}_2^d)$ are the only orbits having minimal dimension 6. The goal of this section is to prove this theorem.

Let us denote by $\chi(\mathbb{P}^2_\mathbb{C})$ the Lie algebra of holomorphic vector fields on $\mathbb{P}^2_\mathbb{C}$; $\chi(\mathbb{P}^2_\mathbb{C})$ is of course the Lie algebra of the automorphism group of $\mathbb{P}^2_\mathbb{C}$. Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2_\mathbb{C}$ and let $X$ be an element of $\chi(\mathbb{P}^2_\mathbb{C})$. Following [9] we will say that $X$ is a symmetry of the foliation $\mathcal{F}$ if the flow $\exp(tX)$ is, for each $t$, in the isotropy group $\text{Iso}(\mathcal{F})$ of $\mathcal{F}$. If $\omega$ defines $\mathcal{F}$ in an affine chart, $X$ is a symmetry of $\mathcal{F}$ if and only if $L_X \omega \wedge \omega = 0$, where $L_X \omega$ denotes the Lie derivative of $\omega$ along $X$.

**Lemma 2.1.** — Let $\mathcal{F}$ be a foliation of degree $d$ on $\mathbb{P}^2_\mathbb{C}$ and let $X$ be a symmetry of $\mathcal{F}$. Assume that there is an affine chart $\mathbb{C}^2 \subset \mathbb{P}^2_\mathbb{C}$ such that the vector field $X$ is affine (i.e. $\deg X \leq 1$) and let $\omega$ be a 1-form defining $\mathcal{F}$ in this chart. Then there is a constant $\lambda \in \mathbb{C}$ such that $L_X \omega = \lambda \omega$.

**Proof.** — We will use an argument similar to one in [9, Proposition 2.5]. Since $L_X \omega \wedge \omega = 0$ and $\omega$ has isolated singularities, the de Rham-Saito division theorem (cf. [17] or [8, Proposition 1.14]) ensures the existence of a holomorphic function $g$ on $\mathbb{C}^2$ such that $L_X \omega = g \omega$. The 1-form $\omega$ and the vector field $X$ being polynomials, $L_X \omega$ is also polynomial; therefore $g$ is rational and holomorphic on $\mathbb{C}^2$ hence polynomial. The vector field $X$ being affine we have $\deg L_X \omega \leq \deg \omega$; the equality $L_X \omega = g \omega$ implies that $g$ is constant.

If $\mathcal{F}$ is a foliation on $\mathbb{P}^2_\mathbb{C}$, we will denote by $\text{iso}(\mathcal{F})$ the Lie algebra of the algebraic group $\text{Iso}(\mathcal{F})$; $\text{iso}(\mathcal{F})$ is a Lie subalgebra of $\chi(\mathbb{P}^2_\mathbb{C})$ and it consists of symmetries of $\mathcal{F}$. We know from [9, Proposition 2.5] that if $\dim \text{iso}(\mathcal{F}) = 2$ then $\text{iso}(\mathcal{F})$ is affine, i.e. generated by two vector fields $X$ and $Y$ such that $[X, Y] = Y$. The following lemma classifies the affine Lie subalgebras of $\chi(\mathbb{P}^2_\mathbb{C})$ and it will be used to prove Theorem A.

**Lemma 2.2.** — Every affine Lie subalgebra of $\chi(\mathbb{P}^2_\mathbb{C})$ is linearly conjugated to one of the following models

(a) $\left\langle \gamma y^2 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}, \frac{\partial}{\partial y} \right\rangle$ with $\gamma \in \mathbb{C}^*;$. 
Let us write \( M : \tau \) up to replacing of \( s_\ell \) and \( a \) where

\[
\text{(b) } \langle y \frac{\partial}{\partial y}, \frac{\partial}{\partial y} \rangle;
\]
\[
\text{(c) } \langle \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \frac{\partial}{\partial y} \rangle;
\]
\[
\text{(d) } \langle x \frac{\partial}{\partial x} + (x+y) \frac{\partial}{\partial y}, \frac{\partial}{\partial y} \rangle;
\]
\[
\text{(e) } \langle x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y} + x \frac{\partial}{\partial y} \rangle.
\]

**Proof.** — Let \( g \) be an affine Lie subalgebra of \( \chi(\mathbb{P}_C^2) \). Then there exist \( X \) and \( Y \) in \( \chi(\mathbb{P}_C^2) \) such that \( g = \langle X, Y \rangle \) and \([X, Y] = Y \). Fixing homogeneous coordinates \([x : y : z] \) in \( \mathbb{P}_C^2 \) we have an isomorphism of Lie algebras

\[
\tau : s_3(\mathbb{C}) \to \chi(\mathbb{P}_C^2)
\]

defined, for \( A \in s_3(\mathbb{C}) \), by

\[
\tau(A) = (x \ y \ z) A \frac{\partial}{\partial z}.
\]

Notice that if \( A = \left( \begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array} \right) \in s_3(\mathbb{C}) \), then in the affine chart \( z = 1 \) the vector field \( \tau(A) \in \chi(\mathbb{P}_C^2) \) writes as

\[
\left( a_{31} + (a_{11} - a_{33})x + a_{21}y - a_{13}x^2 - a_{23}xy \right) \frac{\partial}{\partial x} + \left( a_{32} + a_{12}x + (a_{22} - a_{33})y - a_{13}xy - a_{23}y^2 \right) \frac{\partial}{\partial y}.
\]

Let \( M \) and \( N \) be the matrices of \( s_3(\mathbb{C}) \) associated to the vector fields \( X \) and \( Y \) respectively, i.e. \( M = \tau^{-1}(X) \) and \( N = \tau^{-1}(Y) \). Then the Lie bracket \([X, Y]\) corresponds to \([M, N] := MN - NM\) and therefore \([M, N] = N\).

Let us write \( M = \left( \begin{array}{ccc} -m_{22} - m_{33} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{array} \right) \). Taking into account the possible JORDAN forms of a matrix of \( s_3(\mathbb{C}) \), it suffices us to treat the following possibilities

\[
N = \left( \begin{array}{ccc} -a - b & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{array} \right), \quad N = \left( \begin{array}{ccc} -2c & 0 & 0 \\ 0 & c & 0 \\ 0 & 1 & c \end{array} \right), \quad N = \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right), \quad N = \left( \begin{array}{ccc} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right).
\]

where \( a, b \in \mathbb{C}, c \in \mathbb{C}^* \), with \((a, b) \neq (0, 0)\).

**1.** If \( N = \left( \begin{array}{ccc} -a - b & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{array} \right) \) then the equality \([M, N] = N\) implies that \( a = b = 0 \): contradiction.

**2.** If \( N = \left( \begin{array}{ccc} -2c & 0 & 0 \\ 0 & c & 0 \\ 0 & 1 & c \end{array} \right) \) then the \((1, 1)\) coefficient of the matrix \([M, N] - N\) is equal to \(2c\) and is therefore nonzero: contradiction.

**3.** Assume that \( N = \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right) \); the equality \([M, N] = N\) then leads to \( M = \left( \begin{array}{ccc} 1 - 2m_{33} & m_{12} & 0 \\ 0 & m_{33} - 1 & 0 \\ m_{31} & m_{32} & m_{33} \end{array} \right) \).

Up to replacing \( M \) by \( M - m_{32}N \) we can assume that \( m_{32} = 0 \). Now we will distinguish several eventualities:
3.1. When $(3m_{33} - 1)(3m_{33} - 2) \neq 0$ the matrix $P = \begin{pmatrix} 3m_{33} - 1 & m_{12} & 0 \\ 0 & 3m_{33} - 2 & 0 \\ -m_{31} & -m_{31}m_{12} & 3m_{33} - 2 \end{pmatrix}$ commutes with $N$ and $P^{-1}MP = \begin{pmatrix} 1 - 2m_{33} & 0 & 0 \\ 0 & m_{33} - 1 & 0 \\ 0 & 0 & m_{33} \end{pmatrix}$. Thus $g$ is linearly conjugated to

$$
\langle \tau(P^{-1}MP), \tau(N) \rangle = \langle (1 - 3m_{33})x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}, y \frac{\partial}{\partial y}\rangle = \langle \gamma x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, y \frac{\partial}{\partial y}\rangle,
$$

where $\gamma = 3m_{33} - 1 \in \mathbb{C}^*$. Therefore,

$$
\langle \tau(P^{-1}MP), \tau(N) \rangle = \langle (1 - 3m_{33})x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}, y \frac{\partial}{\partial y}\rangle = \langle \gamma x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, y \frac{\partial}{\partial y}\rangle,
$$

3.2. Assume that $m_{33} = \frac{1}{3}$. If $\delta \in \mathbb{C}^*$ then the matrix $P = \begin{pmatrix} \frac{1}{3} & -m_{12} & 0 \\ 0 & 1 & 0 \\ 0 & m_{12}m_{31} & 1 \end{pmatrix}$ commutes with $N$ and $P^{-1}MP = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & -\frac{2}{3} & 0 \\ \frac{m_{31}}{\delta} & 0 & \frac{1}{3} \end{pmatrix}$. As a result $g$ is linearly conjugated to

$$
\langle \tau(P^{-1}MP), \tau(N) \rangle = \langle \frac{m_{31}}{\delta} \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}, y \frac{\partial}{\partial y}\rangle = \langle \frac{m_{31}}{\delta} \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, y \frac{\partial}{\partial y}\rangle.
$$

The case where $m_{31} = 0$ leads to the model (b). If $m_{31} \neq 0$ then by taking $\delta = -m_{31}$ we get the model (c).

3.3. Assume that $m_{33} = \frac{2}{3}$. If $\delta \in \mathbb{C}^*$ then the matrix $P = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ -\delta m_{31} & -m_{12}m_{31} & 1 \end{pmatrix}$ commutes with $N$ and $P^{-1}MP = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & -\frac{2}{3} & 0 \\ \frac{m_{31}}{\delta} & 0 & \frac{1}{3} \end{pmatrix}$. As a consequence $g$ is linearly conjugated to

$$
\langle \tau(P^{-1}MP), \tau(N) \rangle = \langle \frac{m_{31}}{\delta} \frac{\partial}{\partial x} + \frac{m_{12}}{\delta} x + y \frac{\partial}{\partial y}, y \frac{\partial}{\partial y}\rangle = \langle \frac{m_{12}}{\delta} x + y \frac{\partial}{\partial y}, y \frac{\partial}{\partial y}\rangle.
$$

The case $m_{12} = 0$ leads to the model (a) with $\gamma = 1$. If $m_{12} \neq 0$ then by taking $\delta = -m_{12}$ we obtain the model (d).

4. Assume that $N = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$; then the equality $[M, N] = N$ implies that $M = \begin{pmatrix} -1 & 0 & 0 \\ m_{32} & 0 & 0 \\ m_{31} & m_{32} & 1 \end{pmatrix}$.

Up to replacing $M$ by $M - m_{32}N$ we can assume that $m_{32} = 0$. The matrix $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{m_{11}}{2} & 0 & 1 \end{pmatrix}$ commutes with $N$ and $P^{-1}MP = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Therefore $g$ is linearly conjugated to

$$
\langle \tau(P^{-1}MP), \tau(N) \rangle = \langle -2x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}, y \frac{\partial}{\partial y}\rangle = \langle 2x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, y \frac{\partial}{\partial y}\rangle.
$$

By permuting the coordinates $x$ and $y$ we obtain the model $(e)$. □

Proof of Theorem A. — Since dim $\text{iso}(F) = \text{dim} \text{Iso}(F) = 2$, [9, Proposition 2.5] implies that $\text{iso}(F)$ is affine. Therefore, up to linear conjugation, $\text{iso}(F)$ is one of the models (a)-(c) of Lemma 2.2.

Let $\omega$ be a 1-form defining $F$ in the affine chart $z = 1$

$$
\omega = A(x, y)dx + B(x, y)dy, \quad A, B \in \mathbb{C}[x, y], \quad \gcd(A, B) = 1.
$$
We will study the five possible models (a)-(e) of the Lie algebra $\mathfrak{iso}(\mathcal{F})$ and show that $\omega$ is linearly conjugated to one of the two 1-forms $\omega_1^1$ or $\omega_2^1$.

1. Assume that $\mathfrak{iso}(\mathcal{F})$ is of one of the types (a)-(d), i.e. that $\mathfrak{iso}(\mathcal{F}) = \langle X, Y \rangle$ where $X \in \{ \gamma x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, \varepsilon \frac{\partial}{\partial y} \}$, $x \in \{ 0, 1 \}$ and $\gamma \in \mathbb{C}^*$. By Lemma 2.1 there exist $\lambda, \mu \in \mathbb{C}$ such that $L_X \omega = \lambda \omega$ and $L_Y \omega = \mu \omega$. Since $L_X dx = dL_X x = 0$ and $L_Y dy = dL_Y y = 0$, we have $L_Y \omega = Y(A) dx + Y(B) dy = \frac{\partial A}{\partial y} dx + \frac{\partial B}{\partial y} dy$. Therefore $L_Y \omega = \mu \omega$ if and only if $\frac{\partial A}{\partial y} = \mu A$ and $\frac{\partial B}{\partial y} = \mu B$. Since $A, B \in \mathbb{C}[x,y]$ and $\mu \in \mathbb{C}$, it follows that $\mu = 0, A(x,y) = A(x)$ and $B(x,y) = B(x)$. Thus
\[ \omega = A(x) dx + B(x) dy, \quad A, B \in \mathbb{C}[x], \quad \gcd(A,B) = 1. \]

1.1. Let us consider the case where $X = \gamma x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ with $\gamma \in \mathbb{C}^*$. We have
\[ L_X \omega = (X(A)) dx + AdX(x) (X(B)) dy + BdX(x) = (xA' + \gamma A) dx + (xB' + B) dy, \]
so that $L_X \omega = \lambda \omega$ if and only if $\gamma x A' = (\lambda - \gamma) A$ and $\gamma x B' = (\lambda - 1) B$. By putting $\kappa = \frac{\lambda - \gamma}{\gamma}$ and $\nu = \frac{\lambda - 1}{\gamma}$, the last two equations can be rewritten as $xA' = \kappa A$ and $xB' = \nu B$ and can be immediately integrated to give $A(x) = \alpha x^\kappa$ and $B(x) = \beta x^\nu$, with $\alpha, \beta \in \mathbb{C}$. Since $A, B \in \mathbb{C}[x]$ and $\gcd(A,B) = 1$, we deduce that $\alpha, \beta \in \mathbb{C}^*$, $\kappa, \nu \in \mathbb{N}$ and $\kappa \nu = 0$. The equality $\deg f = d$ then implies that
- either $\kappa = 0$ and $\nu = d$, in which case $\omega = \alpha dx + \beta x^d dy$;
- or $\nu = 0$ and $\kappa = d$, in which case $\omega = \alpha x^d dx + \beta dy$.
If $\omega = \alpha dx + \beta x^d dy$, resp. $\omega = \alpha x^d dx + \beta dy$, by making the change of coordinates $(x, y) \mapsto \left( \frac{y}{x}, -\frac{\alpha}{\beta} x^d \right)$, we reduce ourselves to $\omega = \omega_1^d = y^d dx + x^d (x dy - y dx)$, resp. $\omega = \omega_2^d = x^d dx + y^d (x dy - y dx)$.

1.2. Let us examine the case where $X = \varepsilon \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ with $\varepsilon \in \{ 0, 1 \}$. Since $L_X dx = dL_X x = 0$ and $L_X dy = dL_X y = dy$, we have $L_X \omega = X(A) dx + X(B) dy + B dy = \varepsilon A' dx + (\varepsilon B' + B) dy$. Therefore $L_X \omega = \lambda \omega$ if and only if $\varepsilon A' = \lambda A$ and $\varepsilon B' = (\lambda - 1) B$. Since $A, B \in \mathbb{C}[x]$ and $\lambda \in \mathbb{C}$, it follows that $\lambda = 0$: contradiction with $\gcd(A,B) = 1$.

1.3. Let us study the eventuality: $X = x \frac{\partial}{\partial x} + (x + y) \frac{\partial}{\partial y}$. We have $dX(x) = dx$ and $dX(y) = dy + dx$, so that
\[ L_X \omega = X(A) dx + AdX(x) + (B dx + dy) = (xA' + A + B) dx + (xB' + B) dy. \]
The condition $L_X \omega = \lambda \omega$ is then equivalent to the system of differential equations $xA' + B = (\lambda - 1) A$ and $xB' = (\lambda - 1) B$, which can be easily integrated to yield $A(x) = (a - b ln x)^{\lambda - 1}$ and $B(x) = bx^{\lambda - 1}$, where $a, b \in \mathbb{C}$. Since $A \in \mathbb{C}[x]$, we deduce that $b = 0$ and therefore $B \equiv 0$: contradiction with $\gcd(A,B) = 1$.

2. Assume that $\mathfrak{iso}(\mathcal{F})$ is of type (e), i.e. $\mathfrak{iso}(\mathcal{F}) = \langle X, Y \rangle$ where $X = x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y}$, $Y = \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$. As before by writing explicitly that $L_X \omega = \lambda \omega$ and $L_Y \omega = \mu \omega$, with $\lambda, \mu \in \mathbb{C}$ (Lemma 2.1), we obtain the system of partial differential equations
\[ x \frac{\partial A}{\partial x} + 2y \frac{\partial A}{\partial y} = (\lambda - 1) A, \quad x \frac{\partial B}{\partial x} + 2y \frac{\partial B}{\partial y} = (\lambda - 2) B, \quad \frac{\partial A}{\partial x} + x \frac{\partial A}{\partial y} = \mu A, \quad \frac{\partial B}{\partial x} + x \frac{\partial B}{\partial y} = \mu B. \]
It follows in particular that
\[ (x^2 - 2y) \frac{\partial B}{\partial y} = (\mu x + 2 - \lambda) B \quad \text{and} \quad (x^2 - 2y) \frac{\partial A}{\partial y} = (\mu x + 1 - \lambda) A - x B. \]
Elementary integrations then lead to
\[ B(x,y) = b(x)(x^2 - 2y) \left( \frac{\mu x + 2 - \lambda}{x} \right) \quad \text{and} \quad A(x,y) = \left( a(x) \sqrt{x^2 - 2y} - xb(x) \right) (x^2 - 2y) \left( \frac{\mu x + 1 - \lambda}{x} \right), \]
3. Foliations of $F(d)$ degenerating onto $F_1^d$ and $F_2^d$

In this section we will study the problem of knowing whether the closure of the orbit of a foliation of $F(d)$ contains the foliations $F_1^d$ and $F_2^d$. The following definition will be useful.

**Definition 3.1** ([9]). — Let $F$ and $F'$ be two foliations of $F(d)$. We say that $F$ degenerates onto $F'$ if the closure $\overline{O(F)}$ (in $F(d)$) of the orbit $O(F)$ contains $O(F')$ and $O(F) \neq O(F')$.

**Remarks 3.2.** — Let $F$ and $F'$ be two foliations such that $F$ degenerates onto $F'$. Then:

(i) $\dim O(F') < \dim O(F)$;

(ii) $\deg I_{av}^F \leq \deg I_{av}^{F'}$, which is equivalent to $\deg I_1^F \geq \deg I_1^{F'}$. In particular, if $F$ is convex, $F'$ is also convex.

Corollary B is an immediate consequence of Theorem A and Remark 3.2 (i).

**Remark 3.3.** — Corollary B applies particularly to any foliation of $F(d)$ which is homogeneous, i.e. which is given, for a suitable choice of affine coordinates $(x,y)$, by a homogeneous 1-form $\omega = A(x,y)dx + B(x,y)dy$, where $A, B \in \mathbb{C}[x,y]$ and $\gcd(A, B) = 1$. Indeed, for such a foliation $\mathcal{H} \in F(d)$, we have (cf. [4])

$$\text{Iso}(\mathcal{H}) = \{ (\alpha x, \alpha y) \mid \alpha \in \mathbb{C}^* \};$$

in particular, $\dim O(\mathcal{H}) = 7$ and consequently

$$\overline{O(\mathcal{H})} \subset O(\mathcal{H}) \cup O(F_1^d) \cup O(F_2^d).$$

Assertion I. (resp. 2.) of the following proposition gives a necessary (resp. sufficient) condition for a foliation of $F(d)$ to degenerate onto the foliation $F_1^d$.

**Proposition 3.4.** — Let $F$ be an element of $F(d)$ such that $F_1^d \notin O(F)$. The following assertions hold:

1. If $F$ degenerates onto $F_1^d$, then $F$ possesses a non-degenerate singularity $m$ satisfying $BB(F, m) = 4$.
2. If $F$ possesses a quasi-radial singularity of maximal order $d - 1$, i.e. if $\text{QRad}(F, d - 1) \neq \emptyset$, then $F$ degenerates onto $F_1^d$.

**Proof.** — 1. Assume that $F$ degenerates onto $F_1^d$. Then there is an analytic family of foliations $(F_\varepsilon)$ defined by a family of 1-forms $(\omega_\varepsilon)$ such that $F_\varepsilon$ belongs to $O(F)$ for $\varepsilon \neq 0$ and $F_{\varepsilon=0} = F_1^d$. The non-degenerate singular point of $F_1^d$, denoted by $m_0$, is “stable” in the sense that there is an analytic family $(m_\varepsilon)$ of non-degenerate singular points of $F_\varepsilon$ such that $m_{\varepsilon=0} = m_0$. The $F_\varepsilon$’s being conjugated to $F$ for $\varepsilon \neq 0$, the foliation $F$ admits a non-degenerate singular point $m$ such that

$$\forall \varepsilon \in \mathbb{C}^*, \quad BB(F_\varepsilon, m_\varepsilon) = BB(F, m).$$

Since $\mu(F_\varepsilon, m_\varepsilon) = 1$ for any $\varepsilon \in \mathbb{C}$, the function $\varepsilon \mapsto BB(F_\varepsilon, m_\varepsilon)$ is continuous, hence constant on $\mathbb{C}$. As a consequence

$$BB(F, m) = BB(F_{\varepsilon=0}, m_{\varepsilon=0}) = BB(F_1^d, m_0) = 4.$$
2. Assume that $\mathcal{F}$ has a quasi-radial singularity $m$ of order $d - 1$. Then $\mu(\mathcal{F}, m) = 1, \mathbb{B}(\mathcal{F}, m) = 4$ and $\kappa(\mathcal{F}, m) = d$. This last equality ensures the existence of a line $\ell_m$ passing through $m$, not invariant by $\mathcal{F}$ and such that $\text{Tang}(\mathcal{F}, \ell_m, m) = d$. Let us choose an affine coordinate system $(x, y)$ such that $m = (0, 0)$ and $\ell_m = \{x = 0\}$. The foliation $\mathcal{F}$ is defined in these coordinates by a 1-form $\omega$ of type

$$\omega = C_d(x, y)(xy - ydx) + \sum_{i=1}^{d} (A_i(x, y)dx + B_i(x, y)dy),$$

where $A_i, B_i \in \mathbb{C}[x, y]_i, C_d \in \mathbb{C}[x, y]_d$. We have

$$\omega \wedge dx \bigg|_{x=0} = \sum_{i=1}^{d} B_i(0, 0)dy \wedge dx = \sum_{i=1}^{d} B_i(0, 1)y^i dy \wedge dx.$$ 

Then the equality Tang($\mathcal{F}, \ell_m, m) = d$ translates into $B_i(0, 1) = 0$ for $i \in \{1, 2, \ldots, d - 1\}$ and $B_d(0, 1) \neq 0$. This allows to write

$$B_1(x, y) = \alpha x, \quad B_d(x, y) = x\tilde{B}_{d-1}(x, y) + \gamma y^d,$$

where $\tilde{B}_{d-1} \in \mathbb{C}[x, y]_{i-1}, \tilde{B}_{d-1} \in \mathbb{C}[x, y]_{d-1}, \gamma \in \mathbb{C}^*, \alpha \in \mathbb{C}$. The equalities $\mu(\mathcal{F}, m) = 1$ and $\mathbb{B}(\mathcal{F}, m) = 4$ imply that $\alpha \neq 0$ and $A_1(x, y) = \delta x - \alpha y$ for some $\delta \in \mathbb{C}$. Thus $\omega$ is of type

$$\omega = \delta x dx + (x\tilde{B}_{d-1}(x, y) + \gamma y^d)dy + (C_d(x, y) + \alpha)(xy - ydx) + \sum_{i=2}^{d} A_i(x, y)dx + x \sum_{i=2}^{d-1} \tilde{B}_{i-1}(x, y)dy,$$

where $A_i \in \mathbb{C}[x, y]_i, \tilde{B}_{d-1} \in \mathbb{C}[x, y]_{d-1}, \delta, \gamma \in \mathbb{C}^*$.

By putting $\varphi = (e^{\delta}x, e^{\gamma}y)$ and $\theta = \alpha(xy - ydx) + \gamma y^d dy$, we obtain

$$\frac{1}{\epsilon^{d+1}} \varphi^* \omega = \theta + \epsilon \delta x dx + e^{\gamma}C_d(e^{\delta}x, y)(xy - ydx) + \sum_{i=2}^{d} \epsilon^{i-1} A_i(e^{\delta}x, y)dx + \sum_{i=2}^{d} \epsilon^{i-1} \tilde{B}_{i-1}(e^{\delta}x, y)dy$$

which tends to $\theta$ as $\epsilon$ tends to 0. By making the change of coordinates $(x, y) \mapsto \left(\frac{x}{\gamma}, \frac{y}{\gamma} \right)$, we reduce ourselves to $\theta = \omega_1^d = y^d dx + x^d (xy - ydx)$. As a result $\mathcal{F}$ degenerates onto $\mathcal{F}_1^d$.

**Example 3.5.** Let us consider the homogeneous foliation $\mathcal{H}_1^d$ defined in the affine chart $z = 1$ by the 1-form

$$\omega_1^d = y^d dx - x^d dy.$$ 

We know from [4, Proposition 4.1] that $\mathcal{H}_1^d$ is convex and admits the points $[1 : 0 : 0]$ and $[0 : 1 : 0]$ as radial singularities of maximal order $d - 1$. Therefore $\mathcal{H}_1^d$ degenerates onto $\mathcal{F}_1^d$ (Proposition 3.4) and it does not degenerate onto $\mathcal{F}_2^d$, because $\mathcal{F}_2^d$ is not convex. Thus, according to Remark 3.3, we have

$$\overline{\mathcal{O}(\mathcal{H}_1^d)} = \mathcal{O}(\mathcal{H}_1^d) \cup \mathcal{O}(\mathcal{F}_1^d).$$

**Example 3.6.** Let us consider the family $(\mathcal{G}^d(\gamma))_{\gamma \in \mathbb{C}}$ of foliations of degree $d$ on $\mathbb{P}^2_\mathbb{C}$ defined in the affine chart $z = 1$ by

$$\eta_1^d(\gamma) = (x - \gamma y)dy - ydx + x^d dx - y^d dy.$$ 

We remark that the point $m = [0 : 0 : 1]$ is a non-degenerate singularity of $\mathcal{G}^d(\gamma)$ with BAUM-BOTT index 4. Moreover, along the line $\ell = \{y = 0\}$ we have $\eta_1^d(\gamma) \wedge dy \bigg|_{y=0} = x^d dx \wedge dy$, so that Tang($\mathcal{G}^d(\gamma), \ell, m) = d$.

It follows that the singularity $m$ of $\mathcal{G}^d(\gamma)$ is quasi-radial of maximal order $d - 1$. As a consequence $\mathcal{G}^d(\gamma)$ degenerates onto $\mathcal{F}_1^d$ (Proposition 3.4).
The converse of assertion 2. of Proposition 3.4 is false as the following example shows.

**Example 3.7.** — Let \( \mathcal{F} \) be the foliation of degree \( d \geq 2 \) on \( \mathbb{P}^2_{\mathbb{C}} \) defined in the affine chart \( z = 1 \) by
\[
\omega = xdy - ydx + P(y)dy,
\]
where \( P \) is a polynomial of \( \mathbb{C}[y] \) of degree \( d \) admitting 0 as a root of multiplicity \( \leq d - 1 \), i.e. \( P \) is of the form
\[
P(y) = y^v(a_0 + a_1y + \cdots + a_{d-v}y^{d-v}), \quad \text{where} \quad v \in \{1,2,\ldots,d-1\}, \ a_i \in \mathbb{C}, \ a_0a_{d-v} \neq 0.
\]
The singular locus of \( \mathcal{F} \) consists of the two points \( m = [0:0:1] \) and \( m' = [1:0:0] \); moreover
\[
\mu(\mathcal{F},m) = 1, \quad \text{BB}(\mathcal{F},m) = 4, \quad \kappa(\mathcal{F},m) = v < d, \quad \mu(\mathcal{F},m') > 1.
\]
It follows that \( \mathcal{F} \) has no quasi-radial singularity of maximal order \( d - 1 \), i.e. \( \text{QRad}(\mathcal{F},d-1) = \emptyset \). However, \( \mathcal{F} \) degenerates onto \( \mathcal{F}_1^d \). Indeed, by putting \( \varphi = \left( \frac{d-v}{v}x, \frac{1}{v}y \right) \), we see that
\[
\lim_{v \to 0} \frac{e^{d+1}}{a_{d-v}} \varphi^* \omega = xdy - ydx + y^d dy.
\]

**Question 1.** — Let \( \mathcal{F} \) be a foliation of degree \( d \geq 2 \) on \( \mathbb{P}^2_{\mathbb{C}} \). Is it true that if \( \mathcal{F} \) degenerates onto \( \mathcal{F}_1^d \) then
\begin{itemize}
  \item either \( \mathcal{F} \) admits a quasi-radial singularity of maximal order \( d - 1 \),
  \item or \( \mathcal{F} \) is conjugated to Example 3.7, i.e. up to linear conjugation \( \mathcal{F} \) is given by a 1-form of type
\end{itemize}
\[
xdy - ydx + P(y)dy \quad \text{with} \quad P \in \mathbb{C}[y], \ \text{deg} P = d \ \text{and} \ P(0) = 0.
\]

**Proposition 3.8.** — Let \( d \) be an integer greater than or equal to 2. Let us denote by \( U_1(d) \) the subset of \( \mathbb{F}(d) \) defined by
\[
U_1(d) := \left\{ \mathcal{F} \in \mathbb{F}(d) \mid \forall s \in \text{Sing}(\mathcal{F}), \ \mu(\mathcal{F},s) = 1, \tau(\mathcal{F},s) = 1 \right\}.
\]

Then:
\begin{itemize}
  \item[(i)] \( U_1(d) \) is a non-empty ZARISKI open subset of \( \mathbb{F}(d) \); in particular, for any \( \gamma \in \mathbb{C}, \ \mathcal{G}^d(\gamma) \subset U_1(d) \) if and only if \( \gamma \left( d^{d+1} + \frac{(d+1)^{d+1}}{d^d} \right) \neq 0. \)
  \item[(ii)] Let \( \mathcal{F} \) be an element of \( U_1(d) \). For any singular point \( s \in \text{Sing}(\mathcal{F}) \), the set
\end{itemize}
\[
\Lambda(\mathcal{F},s) := \left\{ \ell_s \in \mathcal{L}_s(\mathcal{F}) \mid \text{Tang}(\mathcal{F},\ell_s,s) > 1 \right\}
\]
has at most 2 elements. In particular, the set \( \bigcup_{n=2}^d \text{QRad}(\mathcal{F},n-1) \) is finite.

To prove this proposition, we need the following lemma.

**Lemma 3.9.** — Let \( \mathcal{F} \) be a foliation of degree \( d \geq 2 \) on \( \mathbb{P}^2_{\mathbb{C}} \), \( s \) a singular point of \( \mathcal{F} \), \( \ell_s \) a line passing through \( s \) and not invariant by \( \mathcal{F} \) and \( X = A(x,y)\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y} \) a polynomial vector field defining \( \mathcal{F} \) in an affine chart \( (x,y) \) containing \( s \). Let us denote by \( (x_0,y_0) \) the coordinates of \( s \) and let \( a(x-x_0) + b(y-y_0) = 0 \) be an equation of the line \( \ell_s \). Then, for any \( n \in \{2,3,\ldots,d\} \), \( \text{Tang}(\mathcal{F},\ell_s,n) \geq n \) if and only if
\[
\left. \frac{d^j}{d^j A(x_0+bt,y_0-at) + bB(x_0+bt,y_0-at)} \right|_{t=0} = 0, \quad \forall j \in \{1,2,\ldots,n-1\}.
\]

In particular, the set \( \Lambda(\mathcal{F},s) := \left\{ \ell_s \in \mathcal{L}_s(\mathcal{F}) \mid \text{Tang}(\mathcal{F},\ell_s,s) > \tau(\mathcal{F},s) \right\} \) is finite and its cardinality is at most \( \tau(\mathcal{F},s) + 1 \).
Proof. — The 1-form \( \omega = A(x,y)dy - B(x,y)dx \) also defines the foliation \( \mathcal{F} \) because \( i_X \omega = 0 \). We have
\[
(\omega \wedge d(a(x-x_0) + b(y-y_0)))_{(x,y)=(x_0+bt,y_0-at)} = P(t)dy \wedge dx,
\]
where \( P(t) = aA(x_0 + bt, y_0 - at) + bB(x_0 + bt, y_0 - at) \). Thus \( \text{Tang}(\mathcal{F}, \ell_s, s) = \text{v}(P(t), 0) \). Notice that \( P(0) = 0 \) because the point \( s \) being singular for \( \mathcal{F} \), we have \( A(x_0, y_0) = B(x_0, y_0) = 0 \). Then \( \text{Tang}(\mathcal{F}, \ell_s, s) \geq n \) if and only if the root \( t = 0 \) of the polynomial \( P \) has multiplicity at least \( n \), that is if and only if \( P'(0) = P''(0) = \ldots = P^{(n-1)}(0) = 0 \), hence the announced equivalence holds.

By conjugating \( \omega \) by the translation \((x + x_0, y + y_0)\), we can assume that \( s = (0, 0) \). Let us denote \( \tau(\mathcal{F}, s) \) simply by \( \tau \). Then the vector field \( X \) decomposes in the form
\[
X = C_{\tau-2}(x,y)R + \sum_{i=\tau}^{d+1} X_i,
\]
where \( R = \frac{\partial}{\partial\tau} + \sum \frac{\partial}{\partial y_i} \), \( C_{\tau-2} \) is a polynomial of degree \( \leq \tau - 2 \), \( X_i = A_i(x,y)\frac{\partial}{\partial\tau} + B_i(x,y)\frac{\partial}{\partial y} \) is a homogeneous vector field of degree \( i \), with \( \text{det}(X_\tau, R) \neq 0 \). Thus, we have
\[
aA(bt, -at) + bB(bt, -at) = \left. \left( a\left( xC_{\tau-2}(x,y) + \sum_{i=\tau}^{d+1} A_i(x,y) \right) + b\left( yC_{\tau-2}(x,y) + \sum_{i=\tau}^{d+1} B_i(x,y) \right) \right) \right|_{(x,y) = (bt,-at)}
\]
\[
= \sum_{i=\tau}^{d+1} \left( aA_i(bt, -at) + bB_i(bt, -at) \right)
\]
\[
= \sum_{i=\tau}^{d+1} iQ_{i+1}(a,b),
\]
where \( Q_{i+1}(a,b) := aA_i(b, -a) + bB_i(b, -a) \) is a homogeneous polynomial of degree \( i + 1 \) in \((a, b)\). From this, we deduce that \( \text{Tang}(\mathcal{F}, \ell_s, s) > \tau \) if and only if \( Q_{i+1}(a,b) = 0 \). As a result
\[
\Lambda(\mathcal{F}, s) = \left\{ \ell_s = \{ax + by = 0\} \in \Sigma(\mathcal{F}) \mid Q_{i+1}(a,b) = 0 \right\}.
\]
Now, the polynomial \( Q_{i+1} \) is not identically zero because \( Q_{i+1}(a,b) = -\det(X_\tau, R)|_{(x,y) = (bt,-at)} \neq 0 \). It follows that \( \Lambda(\mathcal{F}, s) \) has cardinality at most \( \tau + 1 \).

Proof of Proposition 3.8. — We have
\[
U_1(d) = \left\{ \mathcal{F} \in F(d) \mid \forall s \in \text{Sing}(\mathcal{F}), \det(\text{Jac}X(s)) \neq 0, \det((J_1^t X_\tau, R_\tau) \neq 0 \right\},
\]
where \( X \) denotes a polynomial vector field defining \( \mathcal{F} \) in an affine chart containing \( s \) and \( R_\tau \) is the radial vector field centered at \( s \). It follows that \( U_1(d) \) is a Zariski open subset of \( F(d) \). To establish assertion (i), it remains to show that for any \( \gamma \in \mathbb{C} \), \( G^d(\gamma) \in U_1(d) \) if and only if \( \gamma \left( y^{d+1} + \frac{(d+1)d!}{d!} \right) \neq 0 \). In homogeneous coordinates, the foliation \( G^d(\gamma) \) is defined by the 1-form
\[
\Omega^d(\gamma) = z(x^d - yz^{d-1})dx - z(y^d + yz^{d-1} - xz^{d-1})dy + (y^{d+1} - x^{d+1} + y_0^2 z^{d-1})dz.
\]
The singular locus \( \text{Sing}(G^d(\gamma)) \) consists of the points
\[
s_0 = [0 : 0 : 1], \quad s_k = [x_k : x_k^d : 1], \quad s_l = [1 : x_l^d : 0], \quad k \in \{1, 2, \ldots, d^2 - 1\}, l \in \{0, 1, \ldots, d\}, \]
where \( \xi = \exp(\frac{2\pi i}{d+1}) \) and the \( x_k \)'s are the roots of the polynomial \( P(x) = x^{d^2 - 1} + \gamma x^{d-1} - 1 \).
In the affine chart $z = 1$, resp. $x = 1$, $G^d(\gamma)$ is given by the vector field
\[
Y = (y^d + \gamma y - x) \frac{\partial}{\partial x} + (x^d - y) \frac{\partial}{\partial y}, \quad \text{resp. } Z = (y^{d+1} + \gamma y^2 z - 1) \frac{\partial}{\partial y} + z(y^d + \gamma y z - z) \frac{\partial}{\partial z}.
\]
A direct computation shows that $\det(Jac Y(s_x)) = 1 \neq 0$, $\det(Jac Z(s_x)) = \gamma^2$ and
\[
\det(Jac Z(s_x')) = (d+1)x^2 - 1 \neq 0, \quad \det(Jac Z(s_x)) = 1 - d\gamma x^d + d^2 x^d = (d-1)(d\gamma x^d - d - 1), \quad \text{because } p(x) = 0.
\]
Indeed,
\[
det(Jac Z(s_x)) = d\gamma x^d - d - 1 \neq 0, \quad \text{because } x_k \neq 0.
\]
From these we deduce that $G^d(\gamma) \in U_1(d)$ if and only if $\gamma \neq 0$ and $d\gamma x^d - d - 1 \neq 0$, i.e. if and only if $\gamma \neq 0$ and $x_k^{d-1} \neq d+1$. Now, by putting $Q(t) = t^{d+1} + \gamma t - 1$, we have $P(x) = Q(x^{d-1})$ so that $t_0 \in C$ is a root of the polynomial $Q(t)$ if and only if there exists $k \in \{1, 2, \ldots, d-1\}$ such that $t_0 = x_k^{d-1}$. It follows that
\[
G^d(\gamma) \in U_1(d) \iff \gamma Q \left( \frac{d+1}{d\gamma} \right) \neq 0 \iff \gamma \left( \frac{d+1}{d\gamma} \right)^{d+1} \neq 0.
\]
Assertion (ii) is an immediate consequence of Lemma 3.9.

**Theorem 3.10.** — Let $d$ be an integer greater than or equal to 2. Let us denote by $\Sigma_1(d)$ the subset of $F(d)$ defined by
\[
\Sigma_1(d) := \left\{ \mathcal{F} \in F(d) \mid \text{QRad}(\mathcal{F}, d - 1) \neq \emptyset \right\}.
\]
Then
(a) $\emptyset \neq \Sigma_1(d) \subseteq B(\mathcal{F}_1^d)$;
(b) $\Sigma_1(d)$ is a constructible subset of $F(d)$ of dimension greater than or equal to $\dim F(d) - (d - 1)$.

**Proof.** — (a) $\Sigma_1(d)$ contains the foliations $\mathcal{H}_1^d$ and $G^d(\gamma), \gamma \in C$ (Examples 3.5 and 3.6) and is therefore non-empty. Assertion 2. of Proposition 3.4 ensures that $\Sigma_1(d) \subseteq B(\mathcal{F}_1^d)$; this inclusion is strict as Example 3.7 shows.

(b) Let us denote by $\mathbb{P}^2_C$ the dual projective plane of $\mathbb{P}^2_C$. Let $\pi : F(d) \times \mathbb{P}^2_C \times \mathbb{P}^2_C \to F(d)$ be the projection onto the first factor; we have $\Sigma_1(d) = \pi(W_1(d))$, where
\[
W_1(d) := \bigcup_{\mathcal{F} \in \Sigma_1(d)} \{ \mathcal{F} \} \times \text{QRad}(\mathcal{F}, d - 1)
\]
\[
= \left\{ (\mathcal{F}, s, \ell) \in F(d) \times \mathbb{P}^2_C \times \mathbb{P}^2_C \mid s \in \text{Sing}(\mathcal{F}), \ell \in \Sigma_s(\mathcal{F}), \mu(\mathcal{F}, s) = 1, \text{BB}(\mathcal{F}, s) = 4, \text{Tang}(\mathcal{F}, \ell, s) = d \right\}.
\]
According to Lemma 3.9, $W_1(d)$ can be rewritten as
\[
W_1(d) = \left\{ (\mathcal{F}, s, \ell) \in F(d) \times \mathbb{P}^2_C \times \mathbb{P}^2_C \left| \begin{array}{l}
\ell = (x_0, y_0) \in \ell = \{ a(x - x_0) + b(y - y_0) = 0 \}, \\
A(x_0, y_0) = 0, B(x_0, y_0) = 0, \det(JacX(s)) \neq 0, \frac{\text{tr}^2(JacX(s))}{\det(JacX(s))} = 4, \\
aA(x_0 + bt, y_0 + at) + bB(x_0 + bt, y_0 + at) \neq 0, \\
\frac{d}{dt} \left[ aA(x_0 + bt, y_0 + at) + bB(x_0 + bt, y_0 + at) \right] \bigg|_{t=0} = 0, j = 1, \ldots, d - 1
\end{array} \right. \right\},
\]
where $X = A(x, y) \frac{\partial}{\partial x} + B(x, y) \frac{\partial}{\partial y}$ is a polynomial vector field defining $\mathcal{F}$ in an affine chart $(x, y)$ containing $s$.

It follows that $W_1(d)$ is a quasi-projective subvariety of $F(d) \times \mathbb{P}^2_C \times \mathbb{P}^2_C$. Thus, by Chevalley's Theorem [11, Exercise II.3.19], the set $\Sigma_1(d) = \pi(W_1(d))$ is constructible.
According to the above discussion and Proposition 3.8 (i), the intersection \( U_1(d) \cap \Sigma_1(d) \) contains the foliations \( \mathcal{G}^d(\gamma) \), with \( \gamma \left( \frac{q+d^2+1}{d^2} \right) \neq 0 \), and is therefore non-empty (\( U_1(d) \) being the set of \( \mathcal{F} \in \mathcal{F}(d) \) such that for any \( s \in \text{Sing} \mathcal{F} \), \( \mu(\mathcal{F}, s) = 1 \) and \( \tau(\mathcal{F}, s) = 1 \)). Then there exists an irreducible component \( \Sigma_1(d) \) of \( \Sigma_1(d) \) such that \( U_1(d) \cap \Sigma_1(d) \neq \emptyset \). Let \( W_1(d) = \bigcup_{i=1}^{k} W_1^i(d) \) be the decomposition of \( W_1(d) \) into its irreducible components. Let us denote by \( \pi_0 : \Sigma_1(d) \to \mathcal{F}(d) \) the restriction of \( \pi \) to \( W_1(d) \). Then, there is \( n \in \{1, \ldots, k\} \) such that \( \pi_0(W_1^i(d)) = \Sigma_1(d) \). Indeed, since \( \Sigma_1(d) = \pi_0(W_1(d)) \), we have \( \Sigma_1(d) \subset \Sigma_1(d) = \bigcup_{i=1}^{k} \pi_0(W_1^i(d)) \). The irreducibility of \( \Sigma_1(d) \) therefore ensures the existence of \( n \in \{1, \ldots, k\} \) such that \( \Sigma_1(d) \subset \Sigma_1(d) \subset \Sigma_1(d) \). Since \( \Sigma_1(d) \) is an irreducible component of \( \Sigma_1(d) \) and since \( \pi_0(W_1^n(d)) \) is irreducible by continuity of \( \pi_0 \), we deduce that \( \pi_0(W_1^n(d)) = \Sigma_1(d) \).

Thus, since \( U_1(d) \) is a ZARISKI open subset of \( \mathcal{F}(d) \) (Proposition 3.8 (i)), the morphism \( \pi_0 \) induces by restriction a dominant morphism of quasi-projective varieties \( \pi_0^n : W_1^n(d) \cap \pi_0^{-1}(U_1(d)) \to \Sigma_1(d) \cap U_1(d) \). Notice that all the fibers of \( \pi_0 \) over the elements of \( U_1(d) \cap \Sigma_1(d) \) are finite and non-empty. Indeed, if \( \mathcal{F} \in U_1(d) \cap \Sigma_1(d) \) then, by Proposition 3.8 (ii), the set \( \text{QRad}(\mathcal{F}, d-1) \) is finite and non-empty; therefore so is \( \pi_0^{-1}(\mathcal{F}) = \{ \mathcal{F} \} \times \text{QRad}(\mathcal{F}, d-1) \). Since \( \pi_0(W_1^n(d) \cap \pi_0^{-1}(U_1(d))) \subset U_1(d) \cap \Sigma_1(d) \), it follows that all the non-empty fibers of \( \pi_0^n \) are finite and therefore zero-dimensional. The fiber dimension theorem (cf. [15, Theorem 3, page 49]) then implies that \( \dim(W_1^n(d) \cap \pi_0^{-1}(U_1(d))) = \dim(\Sigma_1(d) \cap U_1(d)) \); since \( W_1^n(d) \cap \pi_0^{-1}(U_1(d)) \) and \( \Sigma_1(d) \cap U_1(d) \) are non-empty open subsets of the irreducible varieties \( W_1^n(d) \) and \( \Sigma_1(d) \) respectively, we have

\[
\dim \Sigma_1(d) = \dim(\Sigma_1(d) \cap U_1(d)) = \dim(W_1^n(d) \cap \pi_0^{-1}(U_1(d))) = \dim W_1^n(d).
\]

Now, from (3.1) we deduce that each irreducible component \( W_1^i(d) \) of \( W_1(d) \) has dimension

\[
\dim W_1^i(d) \geq \dim(\mathcal{F}(d) \times \mathbb{P}^2_v \times \mathbb{P}^2_v) - 4 - (d - 1) = \dim(\mathcal{F}(d) - (d - 1),
\]

hence

\[
\dim \Sigma_1(d) = \dim(\Sigma_1(d)) \geq \dim(\Sigma_1(d) \cap U_1(d)) = \dim(W_1^n(d) \cap \pi_0^{-1}(U_1(d))) = \dim W_1^n(d) \geq \dim(\mathcal{F}(d) - (d - 1),
\]

Assertion \textbf{I.} (resp. \textbf{2.}) of the following proposition gives a necessary (resp. sufficient) condition for a foliation of \( \mathcal{F}(d) \) to degenerate onto the foliation \( \mathcal{F}^d_2 \).

\textbf{Proposition 3.11.} — Let \( \mathcal{F} \) be an element of \( \mathcal{F}(d) \) such that \( \mathcal{F}^d_2 \not\in \mathcal{O}(\mathcal{F}) \). The following assertions hold:

\textbf{1.} If \( \mathcal{F} \) degenerates onto \( \mathcal{F}^d_2 \), then \( \deg \mathcal{I}_{\mathcal{F}}^d \geq d - 1 \).

\textbf{2.} If \( \mathcal{F} \) admits an inflection point of maximal order \( d - 1 \), i.e. if \( \text{Flex}(\mathcal{F}, d - 1) \neq \emptyset \), then \( \mathcal{F} \) degenerates onto \( \mathcal{F}^d_2 \).

\textbf{Proof.} — \textbf{1.} If \( \mathcal{F} \) degenerates onto \( \mathcal{F}^d_2 \), then \( \deg \mathcal{I}_{\mathcal{F}}^d \geq \deg \mathcal{I}_{\mathcal{F}}^d \). An immediate computation shows that \( \mathcal{I}_{\mathcal{F}}^d = y^{d-1} \) so that \( \deg \mathcal{I}_{\mathcal{F}}^d = d - 1 \), hence the announced inequality holds.

\textbf{2.} Assume that \( \mathcal{F} \) possesses such a point. We choose an affine coordinate system \((x, y)\) such that \( p = (0, 0) \) is an inflection point of order \( d - 1 \) of \( \mathcal{F} \) and \( x = 0 \) is the tangent line to the leaf of \( \mathcal{F} \) passing through \( p \).
Let $\omega$ be a 1-form defining $\mathcal{F}$ in these coordinates. Since $T^*_p\mathcal{F} = \{x = 0\}$, $\omega$ is of type

$$\omega = C_d(x,y)(xdy - ydx) + \alpha dx + \sum_{i=1}^d (A_i(x,y)dx + B_i(x,y)dy), \quad \text{where} \ A_i, B_i \in \mathbb{C}[x,y]_i, \ C_d \in \mathbb{C}[x,y]_d, \ \alpha \in \mathbb{C}^*.$$  

We have

$$\omega \land dx \bigg|_{x=0} = \sum_{i=1}^d B_i(0,y)dy \land dx = \sum_{i=1}^d B_i(0,1)y^i dy \land dx.$$  

Therefore the hypothesis that $(0,0)$ is an inflection point of order $d - 1$ of $\mathcal{F}$ translates into $B_i(0,1) = 0$ for $i \in \{1,2,\ldots,d-1\}$ and $B_d(0,1) \neq 0$. Then we can write

$$B_i(x,y) = x\tilde{B}_{i-d}(x,y) + \beta y^d, \quad \quad B_i(x,y) = x\tilde{B}_{i-d}(x,y) \text{ for } i \in \{1,2,\ldots,d-1\},$$

where $\tilde{B}_{i-d} \in \mathbb{C}[x,y]_{i-1}$, $\tilde{B}_{d-1} \in \mathbb{C}[x,y]_{d-1}$, $\beta \in \mathbb{C}^*$. Thus $\omega$ is of type

$$\omega = \alpha dx + (\tilde{B}_{d-1}(x,y) + \beta y^d)dy + C_d(x,y)(xdy - ydx) + \sum_{i=1}^d A_i(x,y)dx + x \sum_{i=1}^{d-1} \tilde{B}_{i-1}(x,y)dy,$$

where $A_i \in \mathbb{C}[x,y]_i$, $\tilde{B}_{i-1} \in \mathbb{C}[x,y]_{i-1}$, $\tilde{B}_{d-1} \in \mathbb{C}[x,y]_{d-1}$, $\alpha, \beta \in \mathbb{C}^*$.

Let us consider the family of automorphisms $\phi = \phi_e = (\varepsilon^{d+1}x, \varepsilon y)$. We have

$$\frac{1}{e^{d+1}} \phi^* \omega = \alpha dx + (\varepsilon^d x \tilde{B}_{d-1}(\varepsilon^d x, y) + \beta y^d)dy + e^{d+1}C_d(\varepsilon^d x, y)(xdy - ydx) + \sum_{i=1}^d \varepsilon^i A_i(\varepsilon^d x, y)dx + x \sum_{i=1}^{d-1} \varepsilon^i \tilde{B}_{i-1}(\varepsilon^d x, y)dy.$$

which tends to $\alpha dx + \beta y^d dy$ as $\varepsilon$ tends to 0. Clearly $\alpha dx + \beta y^d dy$ defines a foliation conjugated to $\mathcal{F}^d_2$; as a result $\mathcal{F}$ degenerates onto $\mathcal{F}^d_2$.

**Example 3.12.** — Let us consider the homogeneous foliation $\mathcal{H}^d_2$ defined in the affine chart $z = 1$ by the 1-form

$$\omega_2^d = x^d dx - y^d dy.$$

We know from [4, Proposition 4.1] that $\mathcal{H}^d_2$ has no non-degenerate singularity with BAUM-BOTT index 4 and that

$$\text{Flex}(\mathcal{H}^d_2, d-1) = \{xy = 0\} \setminus \{(0 : 0 : 1)\} \neq 0.$$  

Thus $\mathcal{H}^d_2$ degenerates onto $\mathcal{F}^d_2$ (Proposition 3.11) and it does not degenerate onto $\mathcal{F}^d_1$ (Proposition 3.4). Consequently, according to Remark 3.3, we have

$$\overline{O(\mathcal{H}^d_2)} = O(\mathcal{H}^d_2) \cup O(\mathcal{F}^d_2).$$

**Example 3.13 (JOUANOLOU’s foliation).** — Let us consider the foliation $\mathcal{F}_j^d$ of degree $d \geq 2$ on $\mathbb{P}^2_C$ defined, in the affine chart $z = 1$, by

$$\omega_j^d = (x^d y - 1)dx + (y^d - x^{d+1})dy.$$  

This example is due to JOUANOLOU and is historically the first explicit example of foliation without invariant algebraic curve ([12]). The point $p = (0,0)$ is an inflection point of maximal order $d - 1$ of $\mathcal{F}_j^d$ because $T^*_p\mathcal{F}_j^d = \{x = 0\}$ and $\omega_j^d \land dx \bigg|_{x=0} = y^d dy \land dx$. As a result $\mathcal{F}_j^d$ degenerates onto $\mathcal{F}^d_2$ (Proposition 3.11).

However, we know from [13, Section 3] that every singularity $s$ of $\mathcal{F}_j^d$ is non-degenerate with BAUM-BOTT index

$$\text{BB}(\mathcal{F}_j^d, s) = \frac{(d+2)^2}{d^2+d+1} \neq 4,$$  

while
so that $\mathcal{F}^d$ does not degenerate onto $\mathcal{F}^1$ (Proposition 3.4).

The converse of assertion 2. of Proposition 3.11 is false as the following example shows.

**Example 3.14.** — Let $\mathcal{F}$ be the foliation of degree $d \geq 2$ on $\mathbb{P}^2_{\mathbb{C}}$ defined in the affine chart $z = 1$ by

$$\omega = dx + P(y)dy,$$

where $P \in \mathbb{C}[y]$, $\deg P = d$.

It is easy to check that $\text{Sing}(\mathcal{F}) = \{(1:0:0)\}$ and $1^p_{\mathcal{F}} = P'(y)$. If the derivative $P'$ has a single root, i.e. if $P$ is of the form $P(y) = a(y - \alpha)^d + b$, where $\alpha, a, b \in \mathbb{C}, \alpha \neq 0$, then $\mathcal{F}$ is conjugated to $\mathcal{F}^d_2$; indeed, we have

$$\frac{1}{a}\varphi^*\omega = dx + y^d dy,$$

where $\varphi = (ax - by, y + \alpha)$.

We assume that the derivative $P'$ has at least two distinct roots; this implies that $d \geq 3$. A straightforward computation shows that $\mathcal{F}$ has no inflection point of maximal order $d - 1$, i.e. $\text{Flex}(\mathcal{F}, d - 1) = \emptyset$. However, $\mathcal{F}$ degenerates onto $\mathcal{F}^d_2$. Indeed, by writing $P(y) = a_0 + a_1 y + \cdots + a_d y^d$, $a_i \in \mathbb{C}, a_d \neq 0$, and by putting $\psi = \left( \frac{a_d}{\varepsilon^{d+1}}, \frac{1}{\varepsilon} \right)$, we obtain that

$$\lim_{\varepsilon \to 0} \frac{\varepsilon^{d+1}}{a_d} \psi^*\omega = dx + y^d dy.$$

**Question 2.** — Let $\mathcal{F}$ be a foliation of degree $d \geq 3$ on $\mathbb{P}^2_{\mathbb{C}}$. Is it true that if $\mathcal{F}$ degenerates onto $\mathcal{F}^d_2$ then

- either $\mathcal{F}$ possesses an inflection point of maximal order $d - 1$,
- or $\mathcal{F}$ is conjugated to Example 3.14, i.e. up to linear conjugation $\mathcal{F}$ is given by a 1-form of type $dx + P(y)dy$ with $P \in \mathbb{C}[y]$, $\deg P = d$?

**Proposition 3.15.** — Let $d$ be an integer greater than or equal to 2. Let us denote by $U_2(d)$ the set of foliations $\mathcal{F} \in \mathcal{F}(d)$ whose inflection divisor $1_{\mathcal{F}}$ is transverse (i.e. $1_{\mathcal{F}} = 1^p_{\mathcal{F}}$) and reduced. Then

(i) $U_2(d)$ contains the JOUANOLOU’s foliation $\mathcal{F}^d_2$ and it is a (non-empty) ZARISKI open subset of $\mathcal{F}(d)$;

(ii) for any $d \geq 3$, every foliation $\mathcal{F} \in U_2(d)$ has a finite number (possibly zero) of transverse inflection points of order greater than or equal to 2; in other words, the set $\bigcup_{k=3}^d \text{Flex}(\mathcal{F}, k - 1)$ is finite.

To establish this proposition, let us first prove the following lemma.

**Lemma 3.16.** — Let $\mathcal{F}$ be a foliation of degree $d \geq 2$ on $\mathbb{P}^2_{\mathbb{C}}$, $p$ a regular point of $\mathcal{F}$ and $X$ a polynomial vector field defining $\mathcal{F}$ in an affine chart $(x, y)$ containing $p$. Then, for any $k \in \{2, 3, \ldots, d\}$, $\text{Tang}(\mathcal{F}, T^p_\mathcal{F}, p, k) \succeq k$

if and only if the matrix

$$\begin{pmatrix}
X_x(X^1, x^2, \ldots, x^k) & \cdots & X_x(x^k) \\
X_y(X^1, x^2, \ldots, x^k) & \cdots & X_y(x^k)
\end{pmatrix}
$$

has rank 1.

**Remark 3.17.** — If $X = \sum_{i=1}^n X_i(z_1, \ldots, z_n) \frac{\partial}{\partial z_i}$ is a holomorphic vector field on $\mathbb{C}^n$ and if $t \mapsto \alpha(t)$ is an integral curve of $X$, then we have the following formula which can be easily proved by induction on $j$:

$$(3.2) \frac{d^j}{dt^j} \alpha(t) = (X^j(z_1), \ldots, X^j(z_n)) \circ \alpha(t).$$

**Proof.** — Let $t \mapsto \alpha(t)$ be the integral curve of $X$ passing through $p$ at $t = 0$. The point $p$ being regular for $\mathcal{F}$, we have $T_p \mathcal{F} \ni \alpha'(0) = X(p) \neq 0$. Up to linear conjugation, we can assume that $p = (0, 0)$ and $T^p_\mathcal{F} = \{y = 0\}$. We can then write $\alpha(t) = \left( \sum_{i \geq 1} x_i t^i, \sum_{i \geq 1} y_i t^i \right)$ with $y_1 = 0$ and $x_1 \neq 0$. 


Thus, \( \text{Tang}(\mathcal{F}, T_p^\mathcal{F}, p) = \nu(g(t), 0) \), where \( g(t) = \sum_{i \geq 2} y_i t^i \). As a result, \( \text{Tang}(\mathcal{F}, T_p^\mathcal{F}, p) \geq k \) if and only if \( y_2 = y_3 = \cdots = y_k = 0 \), or equivalently if and only if the matrix \( \begin{pmatrix} x_1 & x_2 & \cdots & x_k \\ 0 & y_2 & \cdots & y_k \end{pmatrix} \) has rank 1.

Now, by using formula (3.2), we see that

\[
\begin{pmatrix} x_1 & x_2 & \cdots & x_k \\ 0 & y_2 & \cdots & y_k \end{pmatrix} = \begin{pmatrix} X(x) & X^2(x) & \cdots & X^k(x) \\ X(y) & X^2(y) & \cdots & X^k(y) \end{pmatrix} \bigg|_{(x,y)=(0,0)},
\]

hence the lemma follows.

\[\square\]

**Proof of Proposition 3.15.** — (i) For \( \mathcal{F} \in \mathcal{F}(d) \), to say that \( I_\mathcal{F} \) is transverse and reduced means that \( \mathcal{F} \) has no invariant line and that \( I_\mathcal{F} \) has no multiple component, which shows that \( U_2(d) \) is a ZARISKI open subset of \( \mathcal{F}(d) \).

As we have already mentioned in Example 3.13, the JOUANOLOU's foliation \( \mathcal{F}^d \) has no invariant algebraic curve [12]; in particular, it has no invariant line and consequently \( I_{\mathcal{F}^d} = I_{\mathcal{J}}^d \). To establish the first announced assertion, it remains to prove that \( I_{\mathcal{F}^d} \) is reduced. In homogeneous coordinates, the foliation \( \mathcal{F}^d \) is defined by the vector field \( y^d \frac{\partial}{\partial x} + z^d \frac{\partial}{\partial y} + x^d \frac{\partial}{\partial z} \); an immediate computation, using formula (1.1), shows that \( I_{\mathcal{F}^d} \) has equation \( F(x,y,z) = 0 \), where

\[
F(x,y,z) = x^{2d+1}z^{d-1} + y^{2d+1}x^{d-1} + z^{2d+1}y^{d-1} - 3x^{d}y^{d}z^{d}.
\]

We must show that \( F \) has no multiple factor in \( \mathbb{C}[x,y,z] \). Since \( F \in \mathbb{Z}[x,y,z] \), it suffices to show that \( F \) has no multiple factor in \( \mathbb{F}_2[x,y,z] \). Indeed, if \( F \) had a multiple factor in \( \mathbb{C}[x,y,z] \), then one of the resultants \( \text{Res}_y(F, \frac{\partial F}{\partial x}) \in \mathbb{Z}[y,z] \) or \( \text{Res}_z(F, \frac{\partial F}{\partial y}) \in \mathbb{Z}[x,z] \) or \( \text{Res}_x(F, \frac{\partial F}{\partial z}) \in \mathbb{Z}[x,y] \) would be identically zero and therefore so would be its reduction modulo 2; so that \( F \) would also have a multiple factor in \( \mathbb{F}_2[x,y,z] \). We have to show that \( \gcd(F, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}) = 1 \) in \( \mathbb{F}_2[x,y,z] \), or equivalently that

\[
\gcd(F, \frac{\partial F}{\partial x}) = 1 \text{ in } \mathbb{F}_2(y,z)[x], \quad \gcd(F, \frac{\partial F}{\partial y}) = 1 \text{ in } \mathbb{F}_2(x,z)[y], \quad \gcd(F, \frac{\partial F}{\partial z}) = 1 \text{ in } \mathbb{F}_2(x,y)[z].
\]

The coordinates \( x, y, z \) playing a symmetric role, it suffices again to show that \( \gcd(F, \frac{\partial F}{\partial x}) = 1 \) in \( \mathbb{F}_2(y,z)[x] \). In \( \mathbb{F}_2[x,y,z] \) we have

\[
F = x^{2d+1}z^{d-1} + y^{2d+1}x^{d-1} + z^{2d+1}y^{d-1} + x^d y^d z^d \quad \text{and} \quad \frac{\partial F}{\partial x} = x^{d-2} \left( x^{d+2}z^{d-1} + dxy^d z^d + (d+1)y^{2d+1} \right) .
\]

Then \( x = 0 \) is not a root of \( F \in \mathbb{F}_2(y,z)[x] \) and consequently

\[
\mathbb{F}_2(y,z)[x] \ni \gcd(F, \frac{\partial F}{\partial x}) = \gcd(F, \varphi), \quad \text{where} \quad \varphi = x^{d+2} + dxy^d + (d+1)^{\frac{2d+1}{d}}.
\]

Moreover, a straightforward computation shows that

\[
x^3 F = \left( x^{d+2}z^{d-1} - (d-1)x^d y^d d^2 + 1 \right) \varphi + y^{d-1}z^{d+1} \left( x^d + \frac{y^{d+1}}{d^2} \right) \left( x^2 + (d^2 - d - 1)^{\frac{d^2+1}{d^2}} z^d + d(d+1)^{\frac{2d+2}{d}} \right),
\]
so that
\[ \mathbb{F}_2(y, z)[x] \ni \gcd(F, \varphi) = \gcd \left( x + \frac{y^{d+1}}{z^d}, x - \frac{y^{d+1}}{z^d} \right), \] because \( d^2 - d \equiv d(d+1) \equiv 0 \mod 2 \)
\[ = \gcd \left( x - \frac{y^{d+1}}{z^d}, x^{d+2} + dxyz^d + (d+1)\frac{y^{2d+1}}{z^{d-1}} \right) \]
\[ = \gcd \left( x - \frac{y^{d+1}}{z^d}, x^{d+2} - \frac{y^{2d+1}}{z^{d-1}} \right) \]
\[ = 1, \]
because \( \left( \frac{y^{d+1}}{z^d} \right)^{d+2} \neq \frac{y^{2d+1}}{z^{d-1}} \) in the field \( \mathbb{F}_2(y, z) \). As a result \( \mathbb{F}_2(y, z)[x] \ni \gcd(F, \frac{\partial F}{\partial \varphi}) = 1. \)

(ii) Let \( \mathcal{F} \) be a foliation of degree \( d \geq 3 \) on \( \mathbb{P}^2_\mathbb{C} \) with reduced and transverse inflection divisor \( I_\mathcal{F} \), i.e. \( \mathcal{F} \in U_2(d) \). We want to show that the set \( \Gamma(\mathcal{F}) := \bigcup_{k=3}^{d} \text{Flex}(\mathcal{F}, k-1) \) is finite. By definition of \( \Gamma(\mathcal{F}) \) we have
\[ \text{(3.3)} \]
\[ \Gamma(\mathcal{F}) \subset \left\{ p \in \mathbb{P}^2_\mathbb{C} \mid p \notin \text{Sing}(\mathcal{F}), \quad \text{Tang}(\mathcal{F}, T_p \mathcal{F}, p) \geq 3 \right\}. \]

Let \( X \) be a vector field defining \( \mathcal{F} \) in an affine chart \( \mathbb{C}^2 = \{(x, y)\} \subset \mathbb{P}^2_\mathbb{C} \). Lemma 3.16 and inclusion (3.3) imply that \( \Gamma(\mathcal{F}) \cap \mathbb{C}^2 \) is contained in the set of points \( p \in \mathbb{C}^2 \) such that
\[ \begin{pmatrix} X(x) \\ X(y) \end{pmatrix} (p) \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad I_X(p) := \begin{vmatrix} X(x) & X^2(x) \\ X(y) & X^2(y) \end{vmatrix} (p) = 0, \quad X(I_X)(p) = \begin{vmatrix} X(x) & X^3(x) \\ X(y) & X^3(y) \end{vmatrix} (p) = 0. \]

Now, the affine chart \( \mathbb{C}^2 = \{(x, y)\} \subset \mathbb{P}^2_\mathbb{C} \) being arbitrary, \( \Gamma(\mathcal{F}) \) is finite if and only if \( \Gamma(\mathcal{F}) \cap \mathbb{C}^2 \) is finite. It suffices therefore to show that the algebraic curves \( I_\mathcal{F} \cap \mathbb{C}^2 = \{I_X(x, y) = 0\} \) and \( \mathcal{C} := \{X(I_X)(x, y) = 0\} \)
intersect at a finite number of points, i.e. that they have no common component. Let us argue by contradiction and assume that there exist \( K, L, L' \subset \mathbb{C}[x, y] \), with \( \deg K > 0 \), such that \( I_X = KL \) and \( X(I_X) = K'L' \). Then \( KL' = X(KL) = X(K)L + KX(L) \) and therefore \( X(K)L = K(L' - X(L)) \). Moreover, the hypothesis that \( I_\mathcal{F} \) is reduced implies that \( \gcd(K, L) = 1 \). It follows that there is \( L'' \subset \mathbb{C}[x, y] \) such that \( X(K) = KL'' \), which means that the algebraic curve \( \mathcal{C}' := \{K(x, y) = 0\} \), contained in \( I_\mathcal{F} \), is invariant by \( \mathcal{F} \), contradicting the hypothesis that \( I_\mathcal{F} \) is transverse.

**Theorem 3.18.** — Let \( d \) be an integer greater than or equal to 2. Let us denote by \( \Sigma_2(d) \) the subset of \( \mathcal{F}(d) \) defined by
\[ \Sigma_2(d) := \left\{ \mathcal{F} \in \mathcal{F}(d) \mid \text{Flex}(\mathcal{F}, d-1) \neq \emptyset \right\}. \]

Then
(a) \( \mathcal{B}(\mathcal{F}_2^d) = \mathcal{F}(2) \setminus \mathcal{F}_2(2) = \Sigma_2(2) \) and, for any \( d \geq 3 \), we have \( \emptyset \neq \Sigma_2(d) \subsetneq \mathcal{B}(\mathcal{F}_2^d) \);
(b) \( \Sigma_2(d) \) is a constructible subset of \( \mathcal{F}(d) \);
(c) for any \( d \geq 3 \), we have \( \dim \Sigma_2(d) \geq \dim \mathcal{F}(d) - (d - 3) \).
In particular, the set \( \Sigma_2(3) \), and therefore \( \mathcal{B}(\mathcal{F}_2^3) \), contains a non-empty Zariski open subset of \( \mathcal{F}(3) \).

**Proof.** — (a) As we have already said in Introduction, the first equality \( \mathcal{B}(\mathcal{F}_2^d) = \mathcal{F}(2) \setminus \mathcal{F}_2(2) \) follows from [9, Theorem 3]. The second equality \( \mathcal{F}(2) \setminus \mathcal{F}_2(2) = \Sigma_2(2) \) is a consequence of the following obvious remark: if \( \mathcal{F} \in \mathcal{F}(2) \setminus \mathcal{F}_2(2) \) then every transverse inflection point of \( \mathcal{F} \) is of order 1.
The set $\Sigma_2(d)$ contains the foliations $\mathcal{J}_2^d$ and $\mathcal{J}_2^d$ (Examples 3.12 and 3.13) and is therefore non-empty. According to assertion 2. of Proposition 3.11, we have $\Sigma_2(d) \subset B(\mathcal{J}_2^d)$; this inclusion is strict for any $d \geq 3$ as Example 3.14 shows.

(b) Let $\pi : \mathbf{F}(d) \times \mathbb{P}^2_{\mathbb{C}} \to \mathbf{F}(d)$ be the projection onto the first factor; notice that $\Sigma_2(d) = \pi(W_2(d))$, where

$$W_2(d) := \bigcup_{\mathcal{F} \in \Sigma_2(d)} \{ (\mathcal{F}, p) \in \mathbf{F}(d) \times \mathbb{P}^2_{\mathbb{C}} \mid p \not\in \text{Sing}(\mathcal{F}), \text{Tang}(\mathcal{F}, T_0^d \mathcal{F}, p) = d \}.$$  

By Lemma 3.16, $W_2(d)$ can be rewritten as

$$W_2(d) = \left\{ (\mathcal{F}, p) \in \mathbf{F}(d) \times \mathbb{P}^2_{\mathbb{C}} \mid \left( \begin{array}{c} X(x) \\ X(y) \end{array} \right) \right\},$$

where $X$ denotes a polynomial vector field defining $\mathcal{F}$ in an affine chart $(x, y)$ containing $p$. It follows that $W_2(d)$ is a quasi-projective subvariety of $\mathbf{F}(d) \times \mathbb{P}^2_{\mathbb{C}}$. Therefore, by Chevalley’s theorem [11, Exercise II.3.19], the set $\Sigma_2(d) = \pi(W_2(d))$ is constructible.

(c) From the above discussion and Proposition 3.15 (i), we have $\mathcal{J}_0^d \subset U_2(d) \cap \Sigma_2(d) \neq \emptyset$ ($U_2(d)$ being the set of foliations of $\mathbf{F}(d)$ with reduced and transverse inflection divisor). Therefore there exists an irreducible component $\Sigma^*_0(d)$ of $\Sigma_2(d)$ such that $U_2(d) \cap \Sigma^*_0(d) \neq \emptyset$. We denote by $\pi_0 : W_2(d) \to \mathbf{F}(d)$ the restriction of $\pi$ to $W_2(d)$. Let $W_2(d) = \bigcup_{i=1}^n W^i_2(d)$ be the decomposition of $W_2(d)$ into its irreducible components. Then, by arguing as in the proof of Theorem 3.10, we see that there is $k \in \{1, \ldots, n\}$ such that $\pi_0(W^k_2(d)) = \Sigma^*_0(d)$. Since $U_2(d)$ is a Zariski open subset of $\mathbf{F}(d)$ (Proposition 3.15 (i)), the morphism $\pi_0$ therefore induces by restriction a dominant morphism of quasi-projective varieties $\pi_0 : W^k_2(d) \cap \Sigma^*_0(d) \to \Sigma^*_0(d) \cap U_2(d)$. Notice that, for any $\mathcal{F} \in U_2(d) \cap \Sigma_2(d)$, the fiber $\pi_0^{-1}(\mathcal{F})$ is finite and non-empty, because $\pi_0^{-1}(\mathcal{F}) = \{ (\mathcal{F}, p) \times \text{Flex}(\mathcal{F}, d-1) \}$ and $\text{Flex}(\mathcal{F}, d-1)$ is finite and non-empty by assertion (ii) of Proposition 3.15. Since $\pi_0(W^k_2(d) \cap \Sigma^*_0(U_2(d))) \subset U_2(d) \cap \Sigma_2(d)$, we deduce that all the non-empty fibers of $\pi_0$ are finite and therefore zero-dimensional. The fiber dimension theorem (cf. [15, Theorem 3, page 49]) then ensures that $\dim(W^k_2(d) \cap \pi_0^{-1}(U_2(d))) = \dim(\Sigma^*_0(d) \cap U_2(d))$; since $W^k_2(d) \cap \pi_0^{-1}(U_2(d))$ and $\Sigma^*_0(d) \cap U_2(d)$ are non-empty open subsets of the irreducible varieties $W^k_2(d)$ and $\Sigma^*_0(d)$ respectively, we have

$$\dim(\Sigma^*_0/d) = \dim(\Sigma^*_0/d \cap U_2(d)) = \dim(W^k_2(d) \cap \pi_0^{-1}(U_2(d))) = \dim(W^k_2(d)).$$

Now, it follows from (3.4) that each irreducible component $W^j_2(d)$ of $W_2(d)$ has dimension

$$\dim W^j_2(d) = \dim(\mathbf{F}(d) \times \mathbb{P}^2_{\mathbb{C}}) - (d - 1) = \dim(\mathbf{F}(d)) - (d - 3),$$

hence

$$\dim \Sigma_2(d) = \dim \Sigma^*_0/d \geq \dim(\Sigma^*_0/d \cap U_2(d)) \geq \dim(W^k_2(d) \cap \pi_0^{-1}(U_2(d))) \geq \dim(W^k_2(d)) \geq \dim(\mathbf{F}(d)) - (d - 3).$$

The subset $\Sigma_2(d) \subset \mathbf{F}(d)$ being constructible, it contains a dense open subset of its closure $\Sigma_2(d)$. In degree $d = 3$ we have $\dim \Sigma_2(3) \geq \dim \mathbf{F}(3)$ and therefore $\dim \Sigma_2(3) = \dim \mathbf{F}(3)$, so that $\Sigma_2(3) = \mathbf{F}(3)$ because $\mathbf{F}(3)$ is irreducible. It follows that $\Sigma_2(3)$ contains a dense open subset of $\mathbf{F}(3)$. This ends the proof of the theorem. □
Remark 3.19. — The set \( \mathbf{F}(d) \) contains elements which degenerate onto both \( \mathcal{F}_1^d \) and \( \mathcal{F}_2^d \), e.g. the family of foliations \( \mathcal{G}^d(\gamma) \), \( \gamma \in \mathbb{C} \). Indeed, on the one hand, we have seen (Example 3.6) that \( \mathcal{G}^d(\gamma) \) degenerates onto \( \mathcal{F}_1^d \). On the other hand, by putting \( \varphi = (\frac{1}{x}, \frac{1}{y}) \) we obtain that \( \lim_{t \to 0} \epsilon^{d+1}\varphi^*\eta^d(\gamma) = \mathbf{O}_2^d \), which shows that \( \mathcal{G}^d(\gamma) \) degenerates onto the homogeneous foliation \( \mathcal{H}_2^d \) (Example 3.12) and therefore, by transitivity, onto \( \mathcal{F}_2^d \).

Example 3.20. — Let us consider the homogeneous foliation \( \mathcal{H}_{1,2}^d \) defined in the affine chart \( z = 1 \) by the 1-form

\[
\mathbf{0}_{1,2}^d = (x^d + y^d)dx + x^d dy.
\]

This foliation degenerates onto both \( \mathcal{F}_1^d \) and \( \mathcal{F}_2^d \). Indeed, on the one hand, \( \mathcal{H}_{1,2}^d \) is given in the affine chart \( y = 1 \) by

\[
\mathbf{0}_{1,2}^d = xdz - zdz + x^d dz + x^d (xdz - zdx);
\]

we see that the point \([0 : 1 : 0]\) is a radial singularity of maximal order \( d - 1 \) of \( \mathcal{H}_{1,2}^d \). Thus, by Proposition 3.4, \( \mathcal{H}_{1,2}^d \) degenerates onto \( \mathcal{F}_1^d \). On the other hand, a straightforward computation shows that

\[
\text{Flex}(\mathcal{H}_{1,2}^d, d - 1) = \{y = 0\} \setminus \{[0 : 0 : 1]\} \neq \emptyset;
\]

consequently, \( \mathcal{H}_{1,2}^d \) also degenerates onto \( \mathcal{F}_2^d \) (Proposition 3.11).

Since \( O(\mathcal{H}_{1,2}^d) \subset O(\mathcal{H}_{1,2}^d) \cup O(\mathcal{F}_1^d) \cup O(\mathcal{F}_2^d) \) (Remark 3.3), we deduce that in fact

\[
O(\mathcal{H}_{1,2}^d) = O(\mathcal{H}_{1,2}^d) \cup O(\mathcal{F}_1^d) \cup O(\mathcal{F}_2^d).
\]

Theorem 3.21. — Let \( d \) be an integer greater than or equal to 2. Then

(a) \( \emptyset \neq \Sigma_1(d) \cap \Sigma_2(d) \subset B(\mathcal{F}_1^d) \cap B(\mathcal{F}_2^d) \supset B(\mathcal{H}_{1,2}^d) \);

(b) \( B(\mathcal{H}_{1,2}^d) \) contains a quasi-projective subvariety of \( \mathbf{F}(d) \) of dimension equal to \( \dim \mathbf{F}(d) - 3d \).

Proof. — (a) The intersection \( \Sigma_1(d) \cap \Sigma_2(d) \) contains the homogeneous foliation \( \mathcal{H}_{1,2}^d \) (Example 3.20) and is therefore non-empty. The inclusion \( \Sigma_1(d) \cap \Sigma_2(d) \subset B(\mathcal{F}_1^d) \cap B(\mathcal{F}_2^d) \) follows from Theorems 3.10 and 3.18.

Let us show the inclusion \( B(\mathcal{H}_{1,2}^d) \subset B(\mathcal{F}_1^d) \cap B(\mathcal{F}_2^d) \). Let \( \mathcal{F} \in B(\mathcal{H}_{1,2}^d) \), i.e. \( \mathcal{F} \in \mathbf{F}(d) \) such that \( \mathcal{H}_{1,2}^d \in \overline{O(\mathcal{F})} \).

Since \( \mathcal{H}_{1,2}^d \) degenerates onto \( \mathcal{F}_i^d \), \( i = 1, 2 \), it follows that \( \mathcal{F}_i^d \in \overline{O(\mathcal{H}_{1,2}^d)} \subset \overline{O(\mathcal{F})} \), hence \( \mathcal{F} \in B(\mathcal{F}_1^d) \cap B(\mathcal{F}_2^d) \).

(b) Let us denote by \( \Sigma(\mathcal{H}_{1,2}^d) \) the subset of \( \mathbf{F}(d) \) defined as follows: an element \( \mathcal{F} \) of \( \mathbf{F}(d) \) belongs to \( \Sigma(\mathcal{H}_{1,2}^d) \) if and only if

(1) \( \mathcal{F} \) admits an invariant line \( \ell \);

(2) there is a system of homogeneous coordinates \( [x : y : z] \in \mathbb{P}_2^d \) in which \( \ell = \{z = 0\} \) and \( \mathcal{F} \) is defined in the affine chart \( z = 1 \) by a 1-form \( \omega \) of type

\[
\omega = \sum_{i=0}^{d-1} \omega_i + \lambda \mathbf{0}_{1,2}^d = \sum_{i=0}^{d-1} \omega_i + \lambda \left( (x^d + y^d)dx + x^d dy \right),
\]

where \( \lambda \in \mathbb{C}^* \) and the \( \omega_i \)'s are homogeneous 1-forms of degree \( i \).

Notice that \( \Sigma(\mathcal{H}_{1,2}^d) \subset B(\mathcal{H}_{1,2}^d) \). Indeed, by putting \( \varphi = (\frac{1}{x}, \frac{1}{y}) \) and by writing \( \omega_i = P_i(x,y)dx + Q_i(x,y)dy \), where \( P_i, Q_i \in \mathbb{C}[x,y] \), we obtain

\[
\epsilon^{d+1}\varphi^* \omega = \sum_{i=0}^{d-1} (\epsilon^{d-i}P_i(x,y)dx + \epsilon^{d-i}Q_i(x,y)dy) + \lambda \mathbf{0}_{1,2}^d.
\]
which tends to $\lambda \mathcal{O}_{1,\mathbf{2}}$ as $\varepsilon$ tends to 0. It follows that $\mathcal{H}_{1,\mathbf{2}}^d \in \overline{\mathcal{O}(\mathcal{F})}$ for any $\mathcal{F} \in \Sigma(\mathcal{H}_{1,\mathbf{2}}^d)$, hence the inclusion $\Sigma(\mathcal{H}_{1,\mathbf{2}}^d) \subset \mathbf{B}(\mathcal{H}_{1,\mathbf{2}}^d)$ holds.

Moreover, every foliation $\mathcal{F} \in \mathcal{F}(d)$ is given in the affine chart $z = 1$ by a 1-form of type

$$\sum_{i=0}^{d} (A_i(x,y)dx + B_i(x,y)dy) + C_d(x,y)(xdy - ydx),$$

where $A_i, B_i \in \mathbb{C}[x,y], C_d \in \mathbb{C}[x,y]_d$ with $\text{gcd}(yC_d - \sum_{i=0}^{d} A_i, xC_d + \sum_{i=0}^{d} B_i) = 1$. Condition (2) is then equivalent to taking $C_d \equiv 0$, $A_d(x,y) = \lambda(x^d + y^d), B_d(x,y) = \lambda x^d$. Since the set of foliations of $\mathcal{F}(d)$ admitting an invariant line is a ZARISKI closed subset of $\mathcal{F}(d)$, we deduce that $\Sigma(\mathcal{H}_{1,\mathbf{2}}^d)$ are quasi-projective subvarieties of $\mathcal{F}(d)$.

Since $\omega$ and $\mu$ define the same foliation if $\mu \neq 0$, and the choice of a line $\ell \subset \mathbb{P}^2_\mathbb{C}$ is equivalent to the choice of a point in $\mathbb{P}^2_\mathbb{C}$, conditions (1) and (2) imply that

$$\dim \Sigma(\mathcal{H}_{1,\mathbf{2}}^d) = 2 + 2 \sum_{i=0}^{d-1} (i + 1) = d^2 + d + 2 = \dim \mathcal{F}(d) - 3d.$$

4. A family of foliations of $\mathcal{F}(d)$ with orbits of dimension less than or equal to 7

In this section we will establish some properties of the family $(\mathcal{F}_0^d(\lambda))_{\lambda \in \mathbb{C}^*}$ of foliations of degree $d$ on $\mathbb{P}^2_\mathbb{C}$ defined in the affine chart $z = 1$ by

$$\omega_0^d(\lambda) = xdy - \lambda ydx + y^d dy.$$

In homogeneous coordinates, $\mathcal{F}_0^d(\lambda)$ is given by

$$\Omega_0^d(\lambda) = -\lambda yz^d dx + z\left(xz^{d-1} + y^d\right) dy + y\left((\lambda - 1)xz^{d-1} - y^d\right) dz.$$

Thus, the singular locus of $\mathcal{F}_0^d(\lambda)$ consists of the two points $s_1 = [0:0:1]$ and $s_2 = [1:0:0]$. The singularity $s_1$ is non-degenerate with BAUM-BOTT index $BB(\mathcal{F}_0^d(\lambda), s_1) = 2 + \lambda + \frac{1}{\lambda}$ and the singular point $s_2$ has maximal algebraic multiplicity $d$. We see that for $\lambda = 1$ the 1-form $\Omega_0^d(1)$ writes in the affine chart $x = 1$ as

$$z^d dy + y^d (zdy - ydz);$$

we deduce that $\mathcal{F}_0^d(1)$ is conjugated to the foliation $\mathcal{F}_1^d$ and is therefore convex.

In the sequel we assume that $\lambda \in \mathbb{C} \setminus \{0, 1\}$. A direct computation, using formula (1.1), leads to

$$I_{\mathcal{F}_0^d(\lambda)}^{\text{inv}} = yz^{2d-1} \quad \text{and} \quad I_{\mathcal{F}_0^d(\lambda)}^{\text{fr}} = (\lambda - 1)x - ((d-1)\lambda + 1)y^d;$$

it follows that, for any $\lambda \in \mathbb{C} \setminus \{0, 1\}$, $\mathcal{F}_0^d(\lambda)$ is not convex.

A straightforward computation shows that the algebraic curve $(1 - \lambda d)x + y^d = 0$ is invariant by $\mathcal{F}_0^d(\lambda)$.

What is more, the rational 1-form $\eta_0^d(\lambda) = \frac{\omega_0^d(\lambda)}{y((1 - \lambda d)x + y^d)}$ is closed. For $\lambda = \frac{1}{d}$ we note that $\eta_{0}^{d}(\frac{1}{d}) = \frac{\omega_0^d(\lambda)}{y^{d+1}}$ has as first integral $\frac{x}{dy} - \ln y$; this allows to see that $\text{Iso}(\mathcal{F}_0^d(\frac{1}{d}))$ is the group $\{(\alpha^d x, \alpha y) \mid \alpha \in \mathbb{C}^*\}$.

When $\lambda \in \mathbb{C} \setminus \{0, 1, \frac{1}{d}\}$ a straightforward computation shows that $\eta_{0}^{d}(\lambda)$ integrates into

$$\lambda \ln((1 - \lambda d)x + y^d) - \ln y,$$
which allows to verify that the isotropy group is here again
\[ \text{Iso}(\mathcal{F}_0^d(\lambda)) = \{ (\alpha^d x, \alpha y) \mid \alpha \in \mathbb{C}^* \}. \]

It follows in particular that, for any \( \lambda \in \mathbb{C} \setminus \{0, 1\} \), \( O(\mathcal{F}_0^d(\lambda)) \) has dimension 7.

Notice that two foliations \( \mathcal{F}_0^d(\lambda) \) and \( \mathcal{F}_0^d(\lambda') \) are conjugated if and only if \( \lambda = \lambda' \).

**Proposition 4.1.** — Let \( \lambda \) be a nonzero complex number. Let \( \mathcal{F} \) be an element of \( \mathcal{F}(d) \) such that \( \mathcal{F}_0^d(\lambda) \not\in O(\mathcal{F}) \).

1. If \( \mathcal{F} \) degenerates onto \( \mathcal{F}_0^d(\lambda) \), then \( \mathcal{F} \) admits a non-degenerate singular point \( m \) satisfying \( \text{BB}(\mathcal{F}, m) = 2 + \lambda + \frac{1}{\lambda} \).

2. If \( \mathcal{F} \) possesses a non-degenerate singular point \( m \) such that
\[ \text{BB}(\mathcal{F}, m) = 2 + \lambda + \frac{1}{\lambda} \quad \text{and} \quad \kappa(\mathcal{F}, m) = d, \]
then \( \mathcal{F} \) degenerates onto \( \mathcal{F}_0^d(\lambda) \).

**Proof.** — It suffices to argue as in the proof of Proposition 3.4, replacing the foliation \( \mathcal{F}_1^d \) by \( \mathcal{F}_0^d(\lambda) \) and the equality \( \text{BB}(\mathcal{F}, m) = 4 \) by \( \text{BB}(\mathcal{F}, m) = 2 + \lambda + \frac{1}{\lambda} \).

**Proposition 4.2.** — The orbit \( O(\mathcal{F}_0^d(\lambda)) \) is closed in \( \mathcal{F}(d) \) in the following two cases:

(i) \( d \geq 3 \) and \( \lambda = -\frac{1}{d-1} \);

(ii) \( d \in \{3, 4, 5\} \) and \( \lambda \in \mathbb{C}^* \).

The proof of this proposition uses the following lemma.

**Lemma 4.3.** — Let \( \lambda \) be a nonzero complex number. Then, the orbit \( O(\mathcal{F}_0^d(\lambda)) \) is closed in \( \mathcal{F}(d) \) if and only if \( \mathcal{F}_0^d(\lambda) \) does not degenerate onto \( \mathcal{F}_2^d \).

**Proof.** — The direct implication is obvious. Let us prove the converse. From the above discussion, \( \mathcal{F}_0^d(\lambda) \) is conjugated to the convex foliation \( \mathcal{F}_1^d \); therefore its orbit \( O(\mathcal{F}_0^d(\lambda)) \) is closed in \( \mathcal{F}(d) \). For any \( \lambda \in \mathbb{C} \setminus \{0, 1\} \), the unique non-degenerate singular point \( s_1 = [0:0:1] \) of \( \mathcal{F}_0^d(\lambda) \) has BAUM-BOTT index \( \text{BB}(\mathcal{F}_0^d(\lambda), s_1) = 2 + \lambda + \frac{1}{\lambda} \neq 4 \); this implies, according to assertion I. of Proposition 3.4, that \( \mathcal{F}_0^d(\lambda) \) does not degenerate onto \( \mathcal{F}_1^d \). Moreover, for any \( \lambda \in \mathbb{C} \setminus \{0, 1\} \), \( O(\mathcal{F}_0^d(\lambda)) \) has dimension 7. The converse implication then follows immediately from Corollary B.

**Proof of Proposition 4.2.** — (i) Let us put \( \lambda_0 = -\frac{1}{d-1} \); according to (4.1) we have \( \Gamma_{\mathcal{F}_0^d(\lambda_0)} = (\lambda_0 - 1)x \), hence \( \deg_{\mathcal{F}_0^d(\lambda_0)} = 1 < d - 1 \) for any \( d \geq 3 \). According to the first assertion of Proposition 3.11, it follows that, for any \( d \geq 3 \), the foliation \( \mathcal{F}_0^d(\lambda_0) \) does not degenerate onto \( \mathcal{F}_2^d \), so that its orbit \( O(\mathcal{F}_0^d(\lambda_0)) \) is closed in \( \mathcal{F}(d) \) (Lemma 4.3).

(ii) Let \( [x:y:z] \) be homogeneous coordinates in \( \mathbb{P}_2^\mathbb{C} \). For \( n \in \mathbb{N} \), let us denote by \( \Lambda_n^d \) the \( \mathbb{C} \)-vector space of \( d \)-forms in the variables \( x, y, z \), whose coefficients are homogeneous polynomials of degree \( n \). Let us put \( \alpha = ydz - zdy, \beta = zdx - xdz \) and \( \gamma = xdy - ydx \). We have the identification
\[ \mathcal{F}(d) = \{ [\Omega] \in \mathcal{P}(\Lambda_{d+1}^1) \mid \Omega = pdx + qdy + rdz, \ p, q, r \in \mathbb{C}[x, y, z]_{d+1}, \ xp + yq + zr = 0, \gcd(p, q, r) = 1 \} \]
\[ = \{ [\Omega] \in \mathcal{P}(\Lambda_{d+1}^1) \mid \Omega = \Lambda \alpha + B \beta + C \gamma, \ A, B, \in \mathbb{C}[x, y, z]_d, \ C \in \mathbb{C}[x, y]_d, \ \gcd(\Lambda A - xB, \ zB - yC, \ xC - zA) = 1 \}. \]
By writing

\[
A = \xi_1 x^d + \xi_3 x^{d-1} y + \ldots + \xi_{2d+1} x^2 + (\xi_{2d+3} x^{d-2} + \ldots + \xi_{2d+1} x^2) z + (\xi_{2d+4} x^{d-3} y + \ldots + \xi_{2d+1} x^2) z^2 + \ldots + \xi_{2d+1} x^2 z^d,
\]

\[
B = \xi_2 x^d + \xi_4 x^{d-1} y + \ldots + \xi_{2d+2} x^2 + (\xi_{2d+4} x^{d-2} + \ldots + \xi_{2d+2} x^2) z + (\xi_{2d+5} x^{d-3} y + \ldots + \xi_{2d+2} x^2) z^2 + \ldots + \xi_{2d+2} x^2 z^d,
\]

\[
C = \xi_3 x^d + \xi_5 x^{d-1} y + \ldots + \xi_{2d+3} x^2 + (\xi_{2d+5} x^{d-2} + \ldots + \xi_{2d+3} x^2) z + (\xi_{2d+6} x^{d-3} y + \ldots + \xi_{2d+3} x^2) z^2 + \ldots + \xi_{2d+3} x^2 z^d
\]

we can identify the class \([\Omega]\) of \(\Omega = A\alpha + B\beta + C\gamma\) to the element \([\xi_1 : \xi_2 : \ldots : \xi_{2d+4d+3}]\) \(\in \mathbb{P}^{d+4d+2}\). Thus, we can identify \(F(d)\) with the Zariski open set:

\[
\left\{ [\xi_1 : \xi_2 : \ldots : \xi_{2d+4d+3}] \in \mathbb{P}^{d+4d+2} / G : \begin{array}{l}
A = \xi_1 x^d + \xi_3 x^{d-1} y + \ldots + \xi_{2d+1} x^2 + (\xi_{2d+3} x^{d-2} + \ldots + \xi_{2d+1} x^2) z + (\xi_{2d+4} x^{d-3} y + \ldots + \xi_{2d+1} x^2) z^2 + \ldots + \xi_{2d+1} x^2 z^d \\
B = \xi_2 x^d + \xi_4 x^{d-1} y + \ldots + \xi_{2d+2} x^2 + (\xi_{2d+4} x^{d-2} + \ldots + \xi_{2d+2} x^2) z + (\xi_{2d+5} x^{d-3} y + \ldots + \xi_{2d+2} x^2) z^2 + \ldots + \xi_{2d+2} x^2 z^d \\
C = \xi_3 x^d + \xi_5 x^{d-1} y + \ldots + \xi_{2d+3} x^2 + (\xi_{2d+5} x^{d-2} + \ldots + \xi_{2d+3} x^2) z + (\xi_{2d+6} x^{d-3} y + \ldots + \xi_{2d+3} x^2) z^2 + \ldots + \xi_{2d+3} x^2 z^d
\end{array} \right\}.
\]

Then, via this identification, we have

\[
G^d = [\Omega^d] = [x^d \beta + y^d \gamma] = [0 : 1 : 0 : 0 : \ldots : 0 : 1]
\]

and

\[
G^d_0(\lambda) = [\omega^d_\lambda] = [(y^d + x^{d-1}) \alpha + \lambda y^{d-1} \beta] = [0 : 0 : \ldots : 0 : 1 : 0 : 0 : \ldots : 0 : 1 : 0 : \lambda : 0 : 0 : \ldots : 0 : 0].
\]

In addition, the orbit of a foliation \(F = [\Omega] \in F(d)\) is

\[
O(F) = \left\{ [\varphi^* \Omega] \mid \varphi = [a_1 x + a_2 y + a_3 z : a_4 x + a_5 y + a_6 z : a_7 x + a_8 y + a_9 z] \in \operatorname{Aut}(\mathbb{P}^2) \right\}.
\]

Let \([x_1 : x_2 : \ldots : x_{2d+4d+3}]\) be a system of homogeneous coordinates in \(\mathbb{P}^{d+4d+2}\). For \(d = 3\), let us consider the following homogeneous polynomial in \(x_1, x_2, \ldots, x_{24}\) of degree 5:

\[
P_5 = -90x_1 (x_1 (294x_1 - 269x_2) + 10x_2 (29x_1 + 4x_6) + 86x_2^3) x_{22} x_{24} - 1125x_1^2 (21x_1 - 23x_2) x_{23} x_{24} + 45x_2 (23x_1 (29x_1 + 13x_4) - x_6 (552x_1 - 271x_4) + 125x_2 x_{23}) x_{22} x_{24} + 28125x_1 x_2 x_{24} + 21x_1 (21x_1 - 23x_2) x_{23} x_{24} + 25 ((108x_9 - 2x_{12}) (3x_1 - 4x_4) + 9x_{10} (112x_3 - 93x_4) + 675x_2 x_{11}) x_{23} x_{24} - 600x_1 x_{23} x_{24} - 5 (2x_9 (207x_9 - 116x_4) - x_{12} (153x_3 - 314x_4) + 5x_{10} (356x_3 - 395x_4) + 1350x_{11}) x_{23} x_{24} x_{22} + 1875 (x_{11} (2x_3 - x_4 + x_5 + e_2 - 3x_2 - 2x_1 - 4x_2) x_{22} x_{24} - 35x_1 (2x_3 - x_4 + x_5 - 3x_2 - 2x_1 + 8x_2^2) x_{23} x_{24} + 50 (3x_{10} (39x_3 - 38x_4) - 3x_2 (9x_9 - 32x_1) x_{22} x_{24} - 35x_1 (2x_3 - 37x_4) - 3x_2 (3x_2 - 2x_1 + 8x_2^2) x_{23} x_{24} + 15 (3x_{11} (21x_1 + 22x_4) - 8x_3 (14x_9 - 43x_1) + 6x_6 (13x_9 - 56x_1) - 350x_{10} x_{11}) x_{23} x_{24} + R x_{23} x_{24} - 5 (20x_{11} (24x_1 - 7x_4) + 4x_9 (97x_3 - 43x_6) + x_{12} (94x_3 - 211x_4) - 600x_5 x_{10}) x_{23} x_{24} + S x_{23} x_{24} - 75 (2x_{10} (78x_1 - 29x_4) - 15x_2 (29x_9 - 19x_{12}) x_{23} x_{24} + 125x_{23} x_{24} x_{23} x_{24} + T x_{23} x_{24} + U x_{23} x_{24} + V x_{23} x_{24},
\]
where

\[ R = 5568x_1x_3 (3x_1 - 4x_4) - 18x_1x_5 (1612x_1 - 1941x_4) + 6x_3^2 (1952x_3 - 4389x_6) + 3x_5^2 (7057x_3 - 2136x_6) - 11250x_2x_5^2 + 2700\pi (3x_1 - 4x_3)^2 + 54\pi (3x_1 - 4x_4) (106x_3 - 89x_6), \]

\[ S = 27000x_2x_5 (3x_1 - 4x_4) - 24\pi^2 (658x_1 - 249x_4) + 1512x_4x_6 (11x_1 - 4x_4) + 252\pi^2 \left( 83x_3 - 36x_6 \right) - 90x_2x_5 (329x_5 - 318x_6) - 2x_4x_5 (1707x_3 - 6047x_4) + 3x_1x_6 (8712x_3 - 3599x_6) - x_4x_6 (11658x_3 - 6041x_6) + 90x_2x_6 (226x_5 - 267x_6), \]

\[ T = 20x_1x_3 (294x_1 - 253x_3) - 40x_1x_5 (159x_1 - 152x_4) + 1900x_2x_3 (x_3 - x_4) + 20\pi^2 (68x_3 - 95x_6) - 25x_2x_6 (40x_3 - 33x_6) + 60x_1x_2 (361x_5 - 252x_6) - 10x_2x_4 (983x_5 - 756x_6) + 67500x_2^2x_7, \]

\[ U = 90x_1x_3 (98x_1 - 117x_4) - 30x_1x_6 (171x_1 - 284x_4) - 150x_2x_6 (68x_3 - 35x_6) - 30x_2x_4 (167x_3 + 396x_6) + 7050x_2x_5^2 + 20\pi^2 (73x_3 - 157x_6) + 270\pi x_2 (41x_5 + 33x_6), \]

\[ V = 5x_2x_3 (1604x_3 - 611x_6) - 30x_1^2 (294x_3 - 563x_6) - 30\pi^2 (355x_1 - 86x_6) - 30x_1x_2 (463x_3 - 242x_6) - 75x_2^2 (109x_5 - 198x_6). \]

A computation carried out with Maple shows that evaluating \( P_3 \) at an arbitrary element \( [\xi_1 : \xi_2 : \cdots : \xi_{24}] \) of \( O(F_0^3(\lambda)) \), we find \( P_3([\xi_1 : \xi_2 : \cdots : \xi_{24}]) = 0 \), i.e. \( O(F_0^3(\lambda)) \) is contained in the zero locus of \( P_3 \)

\[ \text{Zeros}(P_3) := \{ [x_1 : x_2 : \cdots : x_{24}] \in \mathbb{P}^{23} | P_3([x_1 : x_2 : \cdots : x_{24}]) = 0 \}, \]

which is a Zariski closed subset of \( \mathbb{P}^{23} \). Therefore we have \( O(F_0^3(\lambda)) \subset \text{Zeros}(P_3) \) for any \( \lambda \in \mathbb{C}^* \). Moreover, we have

\[ P_3 (0, 1, 0, 0, \cdots, 0, 0, 1) = -50625 \neq 0, \]

hence \( F_2^3 \not\subset \text{Zeros}(P_3) \). It follows that, for any \( \lambda \in \mathbb{C}^* \), we have \( F_2^3 \not\subset O(F_0^3(\lambda)) \), so that \( F_0^3(\lambda) \) does not degenerate onto \( F_2^3 \). Consequently, according to Lemma 4.3, the orbit \( O(F_0^3(\lambda)) \) is closed in \( \mathbb{C}^3 \).

To show that the orbit \( O(F_0^3(\lambda)) \), resp. \( O(F_0^5(\lambda)) \), is closed in \( \mathbb{F}(4) \), resp. \( \mathbb{F}(5) \), it suffices to argue as in degree \( d = 3 \), replacing the polynomial \( P_3 \) by the following polynomial \( P_4 \), resp. \( P_5 \):

\[ P_4 = (3x_1 (129x_3 - 212x_6) + 3x_3 (178x_5 + 15x_8) + 12x_4 (223x_5 - 3x_6) + 5184x_2x_7 - 20\pi^2) x_{11} + 1728x_5x_4^2, \]

\[ -432 (2x_1 - x_3) x_1 x_3 + 48 (2x_1 - 31x_4) x_3 x_1 x_3 - 18 (24x_1 - 19x_4) x_3^2 - 162x_2 (4x_1 - 15x_4) x_3 x_4 - 18 (2x_1 (27x_3 - 20x_6) - x_3 (15x_5 - x_6) + x_2 (170x_5 - 69x_6)) x_3 x_4 + 4212x_2 x_3 x_4 x_5 - 486x_1 x_5 x_3 x_4 + 36 (3 (x_1 - x_4) (12x_2 - x_4) + 2x_2 (3x_3 - 2x_4)) x_3 x_4 - 10368x_2 x_3 x_4, \]

resp.

\[ P_5 = (50x_1 (4906x_3 - 4793x_6) - 27040x_1 (5x_1 - 6x_4) - 5x_5 (10596x_3 - 13469x_6) + 20x_8 (1019x_3 - 208x_6) + 569100x_2x_5) x_3 x_4 + 14227x_5 x_4 x_3^2 - 11690x_1 x_3 x_4 x_5 - 98140x_3 x_4 x_3 x_4 + 140x_2 (1800x_1, - 691x_4) x_4^2 + 35 (156x_3 x_1 - 1645x_1 x_4) x_4 x_3 + 8620x_2 (2x_1 - x_4) - 50x_5 (141x_1 - 11x_4) + 10x_3 (513x_5 - 158x_6) + 70x_2 (2779x_7 - 2704x_10) + 9875x_4 x_5 x_4 - 35 (x_1 - x_4) (295x_5 + 68x_4) - x_5 (3776x_3 - 4427x_6) x_4^2 + 70 (323x_1 x_3 - 253x_1 x_4) x_3 x_4 + 7 (686x_3 x_3 - 293x_1 x_4) x_4 x_5 - 2975x_1 x_3 x_4 - 15946x_1 x_4 x_3 - 142275x_3 x_4 x_5 + 14x_5 (15x_1 + 112x_4) - 14x_6 (10x_1 + 112x_4) - 595x_2 (221x_5 - 250x_8) x_4 x_5 + 49210x_1 x_4 x_4 x_6. \]

\[ \square \]
For $d \geq 6$, we propose:

**Conjecture 1.** — Let $d$ be an integer greater than or equal to 6 and $\lambda$ a nonzero complex number. A homogeneous coordinate system $[x_1 : x_2 : \cdots : x_d : x_{d+4}]$ being fixed in $\mathbb{P}^d_{\mathbb{C}}$, there exists a homogeneous polynomial $Q_d \in \mathbb{C}[x_1,x_2,\cdots,x_{d+4}]$ of degree 3, not depending on $\lambda$, which vanishes on the orbit $O(\mathcal{F}_0^d(\lambda))$ and does not vanish at the point $\mathcal{F}_2^d = [0 : 1 : 0 : \cdots : 0 : 1]$.

Computations made with Maple by the first author show the validity of this conjecture for $d$ small ($d \leq 30$) by taking the polynomial $Q_d$ in the following form:

$$Q_d = x_d + 3x^3 + \left( \sum_{i=1}^{d-4} a_{i}x_{2d+2i+1}x_d^2 + 4x_d^{d+2} - i + \sum_{i=0}^{d-4} b_{i}x_{2d+2i+4}x_d^{d+2} - i \right) + (x_1 x_2 \cdots x_{d+1})M \begin{pmatrix} x_d^2 + 4x^3 + 3 \\ x_d^2 + 4x + 2 \\ \vdots \\ x_d^2 + 3x + 1 \end{pmatrix},$$

where $M = \begin{pmatrix} L_1 \\ L_2 \\ \vdots \\ L_{d+1} \end{pmatrix}$ is a square matrix of order $d+1$ whose lines are of the form:

$L_1 = [0 \ 0 \ a_{1,3x} + b_{1,4x} + a_{1,5x} + b_{1,6x} + \cdots + a_{1,d+1,2x} + b_{1,d+1,2x}]$

$L_2 = [b_{2,1,2x} + b_{2,2,4x} + a_{2,3,5x} + b_{2,4,6x} + a_{2,5,7x} + b_{2,6,8x} + a_{2,7,9x} + b_{2,8,10x} + \cdots + a_{2,d+1,2x} + b_{2,d+1,2x}]$

$\vdots$

$L_{d-1} = \begin{bmatrix} 0 \ 0 \ \cdots \ a_{d-1,2x} + b_{d-1,2x} + a_{d-1,3x} + b_{d-1,4x} + a_{2d-2,2x} + b_{2d-2,2x} + a_{2d-1,2x} + b_{2d-1,2x} \end{bmatrix}$

$L_d = \begin{bmatrix} 0 \ 0 \ \cdots \ b_{2d-1,2x} + a_{2d-1,2x} + b_{2d-1,3x} + a_{2d-1,4x} + \cdots + a_{2d,d+1,2x} + b_{2d,d+1,2x} \end{bmatrix},$

where $a_{i,j}, b_{i,j}, \delta_{i,j}, a_{i,j}, b_{i,j} \in \mathbb{C}$ with $b_{2,1} \neq 0$.

It is clear that Conjecture 1 and Lemma 4.3 imply the following conjecture.

**Conjecture 2.** — For any integer $d \geq 6$ and any $\lambda \in \mathbb{C}^*$, the orbit $O(\mathcal{F}_0^d(\lambda))$ is closed in $\mathbb{F}(d)$.

**References**

[1] C. R. Alcántara and R. Ronzón-Lavie, *Classification of foliations on $\mathbb{C}P^2$ of degree 3 with degenerate singularities*, J. Singul. 14:52–73, 2016.

[2] P. Baum and R. Bott, *Singularities of holomorphic foliations*, J. Differential Geometry, 7:279–342, 1972.

[3] S. Bedrouni, “Feuilletages de degré trois du plan projectif complexe ayant une transformée de Legendre plate”, PhD thesis, University of Sciences and Technology Houari Boumediene, 2017. Available on [https://arxiv.org/abs/1712.03895](https://arxiv.org/abs/1712.03895).

[4] S. Bedrouni and D. Marín, *Tissus plats et feuilletages homogènes sur le plan projectif complexe*, Bull. Soc. Math. France, 146(3):479–516, 2018.
[5] S. Bedrouni and D. Marín, *Classification of foliations of degree three on $\mathbb{P}^2_{\mathbb{C}}$ with a flat Legendre transform*, Ann. Inst. Fourier (Grenoble), 71(4):1757–1790, 2021.

[6] S. Bedrouni and D. Marín, *Une nouvelle démonstration de la classification des feuilletages convexes de degré deux sur $\mathbb{P}^2_{\mathbb{C}}$*, Bull. Soc. Math. France, 148(4):613–622, 2020.

[7] M. Brunella, “Birational geometry of foliations”, volume 1 of IMPA Monographs, Springer, Cham, 2015.

[8] F. Cano, D. Cerveau, and J. Déserti, “Théorie élémentaire des feuilletages holomorphes singuliers”, Echelles. Belin, 2013.

[9] D. Cerveau, J. Déserti, D. Garba Belko, and R. Meziani, *Géométrie classique de certains feuilletages de degré deux*, Bull. Braz. Math. Soc. (N.S.), 41(2):161–198, 2010.

[10] C. Favre and J. V. Pereira, *Webs invariant by rational maps on surfaces*, Rend. Circ. Mat. Palermo (2), 64(3):403–431, 2015.

[11] R. Hartshorne, “Algebraic geometry”, Graduate Texts in Mathematics, No. 52, Springer-Verlag, New York-Heidelberg, 1977.

[12] J. P. Jouanolou, “Équations de Pfaff algébriques”, volume 708 of Lecture Notes in Mathematics, Springer, Berlin, 1979.

[13] A. Lins Neto and J. V. Pereira, *The generic rank of the Baum-Bott map for foliations of the projective plane*, Compos. Math., 142(6):1549–1586, 2006.

[14] D. Marín and J. V. Pereira, *Rigid flat webs on the projective plane*, Asian J. Math. 17(1):163–191, 2013.

[15] D. Mumford, “The red book of varieties and schemes”, volume 1358 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1988.

[16] J. V. Pereira, *Vector fields, invariant varieties and linear systems*, Ann. Inst. Fourier (Grenoble), 51(5):1385–1405, 2001.

[17] K. Saito, *On a generalization of de-Rham lemma*, Ann. Inst. Fourier (Grenoble), 26(2):vii, 165–170, 1976.