Note on the classification of the orientation reversing homeomorphisms of finite order of surfaces

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1 Introduction

The aim of this note is to establish the topological classification of finite period orientation reversing autohomeomorphisms of a closed oriented surface when the period is $2q$, with $q$ even. The classification of periodic orientation reversing autohomeomorphisms of a closed oriented surface has been made in [4] and [1] following different approaches. Weibiao Wang of the School of Mathematical Sciences of Peking University and Chao Wang of the School of Mathematical Sciences of East China Normal University in Shanghai, have pointed out certain errors in Theorems 4.4 and 4.5 of [1]. I thank both of them very much for the messages and communications on this subject. The case that must be corrected in [1] is the treated in this note: orientation reversing homeomorphisms of period multiple of 4. The approach to this problem in [1] is useful and we will correct the results here following the ideas in [1].

The results in [1] has been used in [3] and [2]. The last Section include the corrections to these articles following Section 3.

2 Preliminaries and notations

Let $S$ be an orientable closed surface and $\Phi$ be an orientation reversing autohomeomorphism $\Phi$ of finite order $2q$ of $S$, with $q$ even and greater than 2. The orbifold structure on the orbit space will be denoted by $S/\Phi$ and $|S/\Phi|$ is the quotient surface. The projection $\pi : S \rightarrow S/\Phi$ is an orbifold covering.

If $q$ is odd the surface $|S/\Phi|$ can have boundary and if $q$ is even $|S/\Phi|$ is a closed non-orientable surface. From now on $q$ will be consider even.

The orbifold covering $\pi : S \rightarrow S/\Phi$ has a finite set of singular values corresponding to the conic points of the orbifold structure of $S/\Phi$. Let $r$ be the number of conic points of $S/\Phi$. There are canonical presentations of the orbifold fundamental group $\pi_1 O(S/\Phi)$ as follows:

$$\langle d_1, \ldots, d_g, x_1, \ldots x_r : x_1 \ldots x_r d_1^2 \ldots d_g^2 = 1, x_i^{m_i} = 1, i = 1, \ldots, r \rangle$$
where $g$ is the topological genus of $|S/\Phi|$. The relation $x_1 \ldots x_r d_1^2 \ldots d_2^g = 1$ is called the long relation. The canonical generators $x_i$ of two canonical presentations are conjugate or inversed.

For the abelianization $H_1 O(S/\Phi)$ we define canonical generator system to be the generators obtained from a canonical presentation of $\pi_1 O(S/\Phi)$. The canonical generators of $H_1 O(S/\Phi)$ will be denoted by $X_1, \ldots, X_r, D_1, \ldots, D_g$. The capital letter denotes the homology class determined in $H_1 O(S/\Phi)$ by the generator of $\pi_1 O(S/\Phi)$ denoted by the corresponding small letter. We have the relation $2D_1 + \ldots + 2D_g + \sum_{i=1}^r X_i = 0$.

The covering $\pi : S \to S/\Phi$ is determined by the monodromy epimorphism $T : H_1 O(S/\Phi) \to \mathbb{Z}_{2g}$. This epimorphism determine and is determined by an epimorphism $\pi_1 O(S/\Phi) \to \mathbb{Z}_{2g}$. Note that the monodromy not only determines the covering $\pi$ but also the homeomorphism $\Phi$.

Two homeomorphisms $\Phi_1$ and $\Phi_2$ of a surface $S$ are said to be topologically equivalent if there is a homeomorphism $h : S \to S$ such that $\Phi_1 = h^{-1} \circ \Phi_2 \circ h$.

If $T_1$ is the monodromy of $\Phi_1$ and $T_2$ is the monodromy of $\Phi_2$ the homeomorphisms $\Phi_1$ and $\Phi_2$ are topologically equivalent if there is an automorphism $\alpha$ of $\pi_1 O(S/\Phi)$ inducing an automorphism $\alpha_*$ of $H_1 O(S/\Phi)$ such that $T_1 = T_2 \circ \alpha_*$.

Since each automorphism $\alpha_*$ sends canonical presentations to canonical presentations we have that the set $\{ \pm T(X_i), i = 1, \ldots, r \}$ is a topological invariant that we shall call the set of isotropies.

We shall use the following automorphisms of $H_1 O(S/\Phi)$ induced by automorphisms of $\pi_1 O(S/\Phi)$:

- $H_1(i,j)(D_i) = 2D_j + D_i$; $H_1(i,j)(D_j) = -D_j$,
- $H_2(i,j)(D_i) = D_j + X_j$; $H_2(i,j)(X_j) = -X_j$,
- $H_3(i,j)(X_i) = X_j$; $H_3(i,j)(X_j) = X_i$, if order($X_i$) = order($X_j$).
- $H_4(i,j)(D_i) = D_j$; $H_4(i,j)(D_j) = D_i$.

The remaining generators of $H_1 O(S/\Phi)$ that do not appear in the above lines are unchanged by the automorphisms.

### 3 Classification of orientation reversing autohomeomorphisms of period a multiple of 4

In order to establish the classification we need to define two topological invariants $h_1$ and $h_2$, the definition is very close to the given in $\Pi$ but we need to make some essential changes in order to establish the classification.

We have an orientable closed surface $S$ and an orientation reversing autohomeomorphism $\Phi$ of finite order $2g$ of $S$.

The invariant $h_1$ is defined for orientation reversing homeomorphisms without isotropies of type $T(X_i) = [q]$.

Let $D_1, \ldots, D_g, X_1, \ldots, X_r$ be a canonical generator system of $H_1 O(S/\Phi)$. Using the automorphism $H_2$ of section 2 we may modify the generators such that the isotropies satisfy: $T(X_i) \in \{[2], \ldots, [q - 2]\}$. Using the relation $2D_1 + \ldots + 2D_g + \sum_{i=1}^r X_i = 0$, the element $T(D_1 \ldots D_g)$ is determined by the $T(X_i)$ up
multiplication by $[q]$. We define the invariant $h_1(\Phi) = 0$ if $T(D_1...D_g) \in \{[0], ..., [q-1]\}$ and $h_1(\Phi) = 1$ if $T(D_1...D_g) \in \{[q], ..., [2q-1]\}$.

The geometrical interpretation of this invariant is given by the monodromy of a homology class $c$ of $H_1O(S/\Phi)$ that is represented by a closed curve $\gamma$ such that cutting $|S/\Phi|$ through $\gamma$ we obtain an orientable surface and the position of this curve with respect to the conical points of $S/\Phi$ is given by the condition $T(X_i) \in \{[2], ..., [2q-1]\}$.

The invariant $h_2$ is necessary in the case where $|S/\Phi|$ has genus two. Let $l$ be the smallest integer such that the element $[2l]$ is a generator of the subgroup of $\mathbb{Z}_{2q}$ generated by the isotropies. Let $|S/\Phi|^{2l}$ be the orbit surface of action of $\Phi^{2l}$. The homeomorphism $\Phi$ defines an orientation reversing finite order and fixed point free homeomorphism $\Phi_{\text{free}}$ on $|S/\Phi|^{2l}$. The invariant $h_2$ is given by the topological type of $\Phi_{\text{free}}$. If we note by $T_{\text{free}}$ the monodromy of $\Phi_{\text{free}}$ then the invariant $h_2$ is given by $\{\pm T_{\text{free}}(D_1) + \varepsilon T_{\text{free}}(D_1 + D_2) : \varepsilon = 1, 0\}$.

With this invariants now it is possibly to establish a classification theorem that corrects the Theorem 0.2 in the introduction of [1]:

**Theorem 1** Let $\Phi_1$ and $\Phi_2$ be two orientation reversing autohomeomorphisms of finite order $2q$ of a surface $S$. Assume $q$ is even.

The homeomorphisms $\Phi_1$ and $\Phi_2$ are topologically equivalent if and only if

(i) $\Phi_1$ and $\Phi_2$ have the same set of isotropies

(ii) if there is no any isotropy of order 2, $h_1(\Phi_1) = h_1(\Phi_2)$.

(iii) if $|S/\Phi|$ has genus two and $\Phi_{\text{free}}$ and $\Phi_{\text{free}}$ have order greater than one (equivalently the set of isotropies is not a generator system of $\langle [2] \rangle \leq \mathbb{Z}_{2q}$), $h_2(\Phi_1) = h_2(\Phi_2)$.

**Proof.**

The set of isotropies, $h_1$ and $h_2$ are topological invariants by the way of their definitions, then we must prove that these invariants determine the topological type. The way of proving that is checking that these invariants determine completely the monodromy $T$ of a given orientation reversing autohomeomorphism $\Phi$.

We consider $S/\Phi$ of genus $g$ and with $r$ conical points. Let

$$D_1, ..., D_g, X_1, ..., X_r$$

be a canonical system of generators of $H_1O(S/\Phi)$.

**Case 1.** Genus $g$ of $|S/\Phi|$ different from two.

We shall use the following Lemma (see Lemma 3.1(2) of [1]):

**Lemma 2** If $|S/\Phi|$ has genus $g > 2$ then there is an automorphism $h$ of $S/\Phi$ such that: $T(h(D_1)) = ... = T(h(D_{g-1})) = [1], h(X_i) = X_i, i = 1, ..., r$.

Using the Lemma we can assume $T(D_i) = [1], i \neq g$.

Subcase 1. There is at least one isotropy of order 2.
By automorphisms $H_3$ we can assume that $T(X_1) = \ldots = T(X_s) = [q], s \geq 1$ and using automorphisms $H_2(i,j)$ we can have $T(X_i) \in \{[2], \ldots, [q-2]\}$ $i > s$.

By the long relation $2D_1 + \ldots + 2D_g + \sum_{i=1}^r X_i = 0$ and $H_2(i,1)$, we can have $T(D_g) \in \{[1], \ldots, [q-1]\}$ and this fact determines $T(D_g)$ and then completely $T$.

Subcase 2. There is no $T(X_j) = [q]$.

By automorphisms $H_2(i,j)$ we can assume $T(X_i) \in \{[2], \ldots, [q-2]\}$. Then using the long relation $2D_1 + \ldots + 2D_g + \sum_{i=1}^r X_i = 0$ and invariant $h_1$ we determine completely $T(D_g)$ and the topological type of $\Phi$.

**Case 2.** $|S/\Phi|$ has genus two.

By automorphisms $H_2$ and $H_4$ we can assume $T(X_i) = [q]$ and $T(X_i) \in \{[2], \ldots, [q-2]\}, i = 1, \ldots, s, j = s + 1, \ldots, r$, and using the isotropies of order 2 or the invariant $h_1$, as in the previous case, we can determine $T(D_1 + D_2)$. Let $l$ be the smallest integer such that the element $[zl]$ is a generator of the subgroup of $Z_{2q}$ generated by the isotropy invariants. By the automorphisms $H_2(1,j)$ and $H_2(2,j)$ we can obtain $T(D_1) \in \{[1], \ldots, [l]\}$ and using if necessary $H_1(1,2)$ and $h_2$ the value of $T(D_1)$ is determined. \(\square\)

**Note.** In [1] Theorems 4.4 and 4.5 are wrong as established. The hypothesis in Theorem 4.4: “$(\Phi_j/4)$, $j = 1, 2$, have fixed points” must be replaced by “there is some $T_j(X_i) = [q]$”. In Theorems 4.5 and 4.6 the hypothesis “$(\Phi_j/4)$, $j = 1, 2$, are fixed point free” must be replaced by “there is no any $T_j(X_i) = [q]$”. And the invariants $h_1$ and $h_2$ must be as defined above. Another incorrectness that should be pointed out is that in the case (that is not consider here) where $S/\Phi$ is orientable the invariant given by the set of isotropies is the couple of sets $\{T(X_i) : i = 1, \ldots, r\}, \{-T(X_i) : i = 1, \ldots, r\}$.

### 4 Consequences and corrections in [3] and [2]

The classification in [1] has been used in [3] and in [2]. We correct here the statements following Section 3.

In the Proposition 2 in [3] the hypothesis $f^q$ have fixed points must be replaced by there are branched points with isotropy groups of order 2.

Finally in the Proposition 1.2 of [2] the Condition 2.1 must be $q = 2$ and $f^2$ has fixed points. Note that in Condition 2.2 of such Proposition 1.2 of [2], when $f^2$ is fixed point free the invariant $h_1$ in [1] and the $h_1$ in this paper are equal. Finally for the case $q > 2$ and $f^2$ with fixed points it is necessary to make a new Condition 2.3 in the Proposition 1.2: using automorphism of the quotient orbifold we can consider $T(X_1) = \ldots = T(X_r) = [m] \in \{[2], \ldots, [q-2]\}, r \geq 1$, where $\langle [m] \rangle = \langle [2] \rangle$. If $\lfloor \frac{m}{r} \rfloor \in \{0, \ldots, \lfloor [q-1] \rfloor \}$ we have $h_1(\Phi) = 1$ and $h_1(\Phi) = 0$ in case $\lfloor \frac{m}{r} \rfloor \in \{[q], \ldots, [2q-1]\}$.
References

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