Generating string field theory solutions with matter operators from $KBc$ algebra

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Abstract

The $KBc$ algebra is a subalgebra that has been used to construct classical solutions in Witten’s open string field theory, such as the tachyon vacuum solution. The main purpose of this paper is to give various operator sets that satisfy the $KBc$ algebra. In addition, since those sets can contain matter operators arbitrarily, we can reproduce the solution of Kiermaier, Okawa and Soler, and that of Erler and Maccaferri. Starting with a single D-brane solution on the tachyon vacuum, we replace the original $KBc$ in it with an appropriate set to generate each of the above solutions. Thus, it is expected that the $KBc$ algebra, combined with the single D-brane solution, leads to a more unified description of classical solutions.

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1 Introduction

A non-perturbative string theory should have an ability to describe the dynamics of the background, and should be formulated in a background independent way. String field theory (SFT) is a second-quantized string theory, where backgrounds are determined as classical solutions of the EOM. Since we can switch between backgrounds by shifting the string field, the background independence is also assured. Witten’s bosonic open string field theory [1] is known for that its EOM is relatively easy to analyze classically.

In Witten’s SFT, the background is defined through a first-quantized string theory which is described by a world-sheet CFT, and the string field $\Psi$ is a composite operator of the reference CFT. On the other hand, other backgrounds are realized by classical field configurations, solutions of the EOM, $Q_B \Psi + \Psi^2 = 0$, where $Q_B$ is the BRST operator of the reference CFT.

Since a bosonic string theory is unstable, any D-brane system will decay to the tachyon vacuum [2, 3], regardless of what CFT we choose at the beginning. Thus, this condensation phenomena must be characterized without any specific information of the reference CFT. In fact, the tachyon vacuum solution found in [4, 5] can be described by the so-called $KBc$ algebra [6], which is a universal subalgebra in the sense that any CFT possesses it. This universality of the $KBc$ algebra guarantees that any D-brane system falls into the tachyon vacuum.

However, in order to establish the background independence explicitly, any background should be expressed by a solution of the EOM on the reference D-brane system. Since a background is determined by a world-sheet CFT, the problem we would like to solve is whether
we can find a solution that corresponds to a given CFT. In particular, if we consider a one-parameter family of CFTs which includes the reference CFT, can we interpret it as a one-parameter family of SFT solutions? This problem has been addressed not only with the $KBc$ algebra, but also with matter operators of the reference CFT.

One type of such one-parameter modifications is the *marginal deformation*, which keeps the conformal invariance of the reference CFT. The marginal deformation means adding to the CFT a boundary term which is characterized by a matter primary operator having weight 1. There have been many attempts to find solutions which correspond to marginally deformed CFTs [7–16]. Among those attempts, the solution of Kiermaier, Okawa, and Soler (KOS) [16] is known as a successful analytical solution, where the *boundary condition changing operators* (bcc operators) were introduced. Since marginal deformations add boundary terms, their effects are regarded as changes of boundary conditions; the bcc operators play the role to change a boundary condition to another. The KOS solution is described by the bcc operators and the $KBc$ algebra. Though valid only when the matter operators are regular,\(^1\) it was generalized to non-regular cases by Erler and Maccaferri [17, 18]. However, some questions like how to explicitly find unknown bcc operators are still left, even though their approach has advanced our understanding of the background structure.

In this paper, we revisit the $KBc$ algebra to deepen our understanding further. In [19], we found operator sets which consist of the original $K, B$ and $c$ and satisfy the algebraic relations of the $KBc$ algebra. We call these operator sets representations of the $KBc$ algebra. Extending the previous method, we find in this paper a larger family of representations, which can arbitrarily contain matter operators. In addition, we can map the representations to the $KBc$ algebra on the tachyon vacuum. By using this, we explicitly show that each of the KOS and the Erler-Maccaferri solutions is reproduced by a specific representation of the $KBc$ algebra. Although our work is mostly devoted to finding what kind of representation gives each solution, we will also discuss a potential ability of our framework to connect the $KBc$ algebras in different CFTs. This is expected to lead to a deeper understanding of the whole background structure.

This paper is organized as follows. In section 2, we introduce various representations of the $KBc$ algebra with matter operators, both on a single D-brane solution and on the tachyon vacuum. Then in section 3, we apply the latter to construct the two solutions, the KOS and the Erler-Maccaferri solutions. In section 4, we summarize the paper and discuss how to find unknown physically meaningful solutions by using our framework. In appendices A and B, we present the technical details used in the text. We discuss representations of the extended algebra including matter operators in appendix C.

\section{Representations of the $KBc$ algebra}

In this section, we introduce representations of the $KBc$ algebra containing arbitrary number of matter operators. We first give representations on a single D-brane, and next on the tachyon vacuum. The latter representations will be used later in section 3, and hence play a key role in this paper. For other attempts to consider representations of the $KBc$ algebra, see [20–24].

\(^1\)Here “regular” means having finite self-OPE.


2.1 Representations on a single D-brane

Here we consider the world-sheet CFT which describes a single D-brane. As is well known, the $K\beta c$ algebra is a subalgebra consisting of three operators $K, B$ and $c$ in the CFT, which satisfy the following relations [6]:

$$ [K, B] = 0, \quad \{B, c\} = 1, \quad B^2 = 0, \quad c^2 = 0, \quad (2.1) $$

$$ Q_B K = 0, \quad Q_B B = K, \quad Q_B c = c K c. \quad (2.2) $$

Originally, $K, B$ and $c$ appearing in the above relations are defined in the sliver frame as

$$ K = \int_{-i\infty}^{i\infty} \frac{dz}{2\pi \imath} T(z), \quad B = \int_{-i\infty}^{i\infty} \frac{dz}{2\pi \imath} b(z), \quad c = c(0). \quad (2.3) $$

We call the triad $(K, B, c)$ in (2.3) the basic representation in this paper, and $(K, B, c)$ always denotes the basic representation. In the following, we will find other triads which satisfy (2.1) and (2.2) with the same BRST operator $Q_B$ on a single D-brane. We call them representation of the $K\beta c$ algebra (on a single D-brane) or $K\beta c$-representation for short.

Extending the idea of [19], we find that the following triad $(K(\xi), B(\xi), c(\xi))$ is also a representation of the $K\beta c$ algebra:

$$ K(\xi) = Q_{B\xi}^{1,1}, \quad B(\xi) = \xi^1, \quad c(\xi) = e^{-iQ_B\xi^2}ce^{iK\Xi^2}B_{\Xi^1}e^{-i\Xi^2K}ce^{iQ_B\xi^2}. \quad (2.4) $$

Let us explain the ingredients appearing in this expression. First, $\xi = (\xi^1, \xi^2)$ carries ghost number $-1$ and consists of $(K, B, c)$ and matter operators. Since matter operators commute with $B$ and $c$, $\xi$ can be expressed as

$$ \xi = (\xi^1, \xi^2) = B(\Xi^1, \Xi^2) = (\Xi^1, \Xi^2) B, \quad (2.5) $$

with $\Xi^{1,2}$ consisting of $K$ and matter operators. Note that $K\Xi^2 = \Xi^2 K$ does not hold in general, and that the triad (2.4) is real if $\Xi^{1,2}$ is real. Of course, all $K\beta c$-representations cannot be expressed as (2.4). A more general form of $K\beta c$-representations is given in appendix A.

Leaving the construction of (2.4) in appendix A, here let us just confirm that (2.4) satisfies (2.1) and (2.2). Nontrivial relations are $\{B(\xi), c(\xi)\} = 1$ and $Q_B c(\xi) = c(\xi) K(\xi) c(\xi)$. To show the former, we first note that

$$ BQ_B \xi^{1,2} = B(K\Xi^{1,2} - BQ_B\Xi^{1,2}) = BK\Xi^{1,2} = K\Xi^{1,2} B, \quad (2.6) $$

Then we find that

$$ B(\xi)c(\xi) = \Xi^1 B e^{-iQ_B\xi^2}ce^{iK\Xi^2}B_{\Xi^1}e^{-i\Xi^2K}ce^{iQ_B\xi^2} = \Xi^1 e^{-iK\Xi^2}Bce^{iK\Xi^2}B_{\Xi^1}e^{-i\Xi^2K}ce^{iQ_B\xi^2} $$. 

2) When matter operators are not included, $\xi^1$ and $Q_B\xi^2$ in this paper correspond to $Be^{\xi^1}$ and $\xi^2$ in [19] (see (3.12) there), respectively.

3) Real means that the quantity is self double conjugate.
and similarly,
\[ c(\xi)B(\xi) = e^{-iQc\xi^2}cBe^{iQc\xi^2}. \]

Therefore \( \{B(\xi), c(\xi)\} = 1 \) holds due to \( \{B, c\} = 1 \).

Next, for showing \( Q_{Bc}(\xi) = c(\xi)K(\xi)c(\xi) \), we rewrite the r.h.s. as
\[ c(\xi)K(\xi)c(\xi) = -Q_B(c(\xi)B(\xi)c(\xi)) + (Q_Bc(\xi)B(\xi)c(\xi) + c(\xi)B(\xi)Q_Bc(\xi)). \]

where we have used \( \{B(\xi), c(\xi)\} = 1 \), which we have just proven, and in particular that \( c(\xi)B(\xi)c(\xi) = c(\xi) \). Since we can directly confirm from (2.4) that \( Q_{Bc}(\xi) \) is written as
\[ Q_{Bc}(\xi) = e^{-iQc\xi^2}c[\cdots]cBe^{iQc\xi^2}, \]
we see that \( (Q_{Bc}(\xi))c(\xi) = c(\xi)Q_{Bc}(\xi) = 0 \), and hence that \( Q_{Bc}(\xi) = c(\xi)K(\xi)c(\xi) \) holds.

Note that (2.4) does not in general respect relations which are not in (2.1) and (2.2) but are satisfied by the basic representation; for example, \( cKcKc = 0 \) which follows from \( [c, K] = \partial c \).

An example is \( \xi = B(1 + cK, 0) \) with \( \epsilon \) infinitesimal and real. For this \( \xi \), one can show \( c(\xi)K(\xi)c(\xi)K(\xi)c(\xi) = e\epsilon KcK^2c + O(\epsilon^2) \neq 0 \).

### 2.2 Representations on the tachyon vacuum

In the previous subsection, we introduced \( KBc \)-representations (2.4) on a D-brane. As we know that the tachyon vacuum is the most basic background, it seems useful to find various representations of the \( KBc \) algebra in its corresponding CFT with a trivial BRST cohomology. In this case, \( Q_B \) in the \( KBc \) algebra is replaced by \( Q_{tv} := Q_B + [\Psi_0, \cdot] \), where \( \Psi_0 \) is the tachyon vacuum solution; we call this algebra \( KBc_{tv} \).

We adopt the simple solution \([5]\) as the tachyon vacuum solution:
\[ \Psi_0 = UQ_BU^{-1} = \frac{1}{\sqrt{1 + K}}c(1 + K)Bc \frac{1}{\sqrt{1 + K}}, \]
\[ U = \left( \frac{1}{\sqrt{1 + K}} - \frac{1}{\sqrt{1 + K}}Bc \frac{1}{\sqrt{K}} \right), \]
\[ U^4 = U^{-1} = \left( \sqrt{K} + \frac{1}{\sqrt{K}}Bc \right) \frac{1}{\sqrt{1 + K}}. \]

The BRST operator \( Q_{tv} \) of the \( KBc_{tv} \) algebra is defined by this \( \Psi_0 \). Since we have
\[ (UXU^{-1})(UYU^{-1}) = UXYU^{-1}, \quad Q_{tv}(UXU^{-1}) = U(Q_BX)U^{-1}, \]
for any \( X \) and \( Y \), sandwiching (2.4) between \( U \) and \( U^{-1} \) directly gives \( KBc_{tv} \)-representations,
\[ K(\xi)_{tv} = U(Q_B\xi^1)U^{-1}, \quad B(\xi)_{tv} = U\xi^1U^{-1}. \]

\[^4|A, B| := AB - (-1)^{|A||B|}BA\] with \(|A| := +1 (-1)\) if \( A \) is Grassmann-even (-odd).
\[ c(\xi)_{tv} = U e^{-iQ_{tv} \xi^2} e^{iK_{tv} \Xi_{tv}} B_{tv} \Xi_{tv} e^{-i\Xi_{tv} K_{tv}} e^{iQ_{tv} \xi^2} U^{-1}. \]  
(2.15)

Namely, this triad satisfies
\[ [K(\xi)_{tv}, B(\xi)_{tv}] = 0, \quad \{B(\xi)_{tv}, c(\xi)_{tv}\} = 1, \quad B(\xi)_{tv}^2 = 0, \quad c(\xi)_{tv}^2 = 0, \]  
(2.16)

\[ Q_{tv} K(\xi)_{tv} = 0, \quad Q_{tv} B(\xi)_{tv} = K(\xi)_{tv}, \quad Q_{tv} c(\xi)_{tv} = c(\xi)_{tv} K(\xi)_{tv} c(\xi)_{tv}. \]  
(2.17)

In the following, we define \( X_{tv} \) for any \( X \) as 5)
\[ X_{tv} := UXU^{-1}. \]  
(2.18)

Using this notation and (2.14), we can rewrite (2.15) as
\[ K(\xi)_{tv} = Q_{tv} \xi^1_{tv}, \quad B(\xi)_{tv} = \xi^1_{tv}, \quad c(\xi)_{tv} = e^{-iQ_{tv} \xi^2_{tv}} c_{tv} e^{iK_{tv} \Xi_{tv}} B_{tv} \Xi_{tv} e^{-i\Xi_{tv} K_{tv}} c_{tv} e^{iQ_{tv} \xi^2_{tv}}. \]  
(2.19)

### 3 Construction of classical solutions

In this section, we explicitly show that \( K B c_{tv} \)-representations reproduce the KOS and the Erler-Maccaferri solutions. Hereafter, the string field \( \Psi \) represents the deviation from the tachyon vacuum, meaning that its EOM is given as
\[ Q_{tv} \Psi + \Psi^2 = 0. \]  
(3.1)

In [17], the single D-brane solution on the tachyon vacuum, \( \Psi = -\Psi_0 \), is deformed by using bcc operators [16] as \( -\Sigma \Psi_0 \bar{\Sigma} \), which was shown to give solutions corresponding to various backgrounds. Our strategy here is to deform \( -\Psi_0 \) by using \( K B c_{tv} \)-representations to express backgrounds.

#### 3.1 Generating other solutions from a single D-brane solution

We first rewrite \( -\Psi_0 \) in terms of \( (K_{tv}, B_{tv}, c_{tv}) \), which is obtained by applying (2.18) to the basic representation \( (K, B, c) \). Noting that
\[ U_{tv} = UUU^{-1} = U, \]  
(3.2)

we can rewrite \( -\Psi_0 \) as
\[ -\Psi_0 = -UQ_{B} U^{-1} = U^{-1} U(Q_{B} U) U^{-1} = U^{-1} Q_{tv}(UUU^{-1}) = U_{tv}^{-1} Q_{tv} U_{tv}, \]  
(3.3)

where we have also used (2.14) at the third equality. By using (2.14) again, we see from (2.12) and (2.13) that
\[ U = U_{tv} = \left( \frac{1}{\sqrt{1 + K_{tv}}} - \frac{1}{\sqrt{1 + K_{tv}}} B_{tv} c_{tv} \right) \frac{1}{\sqrt{K_{tv}}}, \]  
(3.4)

\[ ^{5)} \text{In this notation, (2.14) is written as } X_{tv} Y_{tv} = (XY)_{tv} \text{ and } Q_{tv} X_{tv} = (Q_B X)_{tv}. \]
\[ U^{-1} = U_{v}^{-1} = \left( \sqrt{K_{v}} + \frac{1}{\sqrt{K_{v}}}B_{v}c_{v} \right) \frac{1}{\sqrt{1 + K_{v}}}, \]  
\[ (3.5) \]

and hence we can regard \(-\Psi_{0}\) as a string field expressed completely by \((K_{v}, B_{v}, c_{v})\):

\[-\Psi_{0} = U_{v}^{-1}Q_{v}U_{v} =: \Psi_{1}(K_{v}, B_{v}, c_{v}). \tag{3.6}\]

Since we only need \(KBc_{v}\) algebra to show that (3.6) is a solution to the EOM (3.1), string field \(\Psi_{1}(K(\xi)_{v}, B(\xi)_{v}, c(\xi)_{v})\) is also a solution. Explicitly, we have

\[ \Psi_{1}(K(\xi)_{v}, B(\xi)_{v}, c(\xi)_{v}) = [U_{v}^{-1}Q_{v}U_{v}]_{\xi} = -\left[ \frac{1}{\sqrt{K_{v}}}c_{v}B_{v}\frac{K^2_{v}}{1 + K_{v}} - c_{v}\frac{1}{\sqrt{K_{v}}} \right], \tag{3.7}\]

where \([\cdots]_{\xi}\) is the symbol to mean that \((K_{v}, B_{v}, c_{v})\) inside the bracket is replaced by \((K(\xi)_{v}, B(\xi)_{v}, c(\xi)_{v})\). If we choose \(\xi\) real, the triad \((K(\xi)_{v}, B(\xi)_{v}, c(\xi)_{v})\) is real, and hence the solution (3.7) is. In the rest of this section, we will show that (3.7) reproduces the KOS and the Erler-Maccaferri solutions by choosing suitable real \(\xi\)’s.

### 3.2 Reproducing KOS solution for marginal deformation

The KOS solution [16] describes a marginal deformation of the CFT characterized by a regular matter primary operator \(V\) with its weight 1, where “regular” means that \(cV\) has a vanishing self-OPE. They introduced the bcc operators \(\sigma\) and \(\bar{\sigma}\), which are matter primary operators of weight 0 and are related to \(V\) through

\[ V = -\sigma \partial \bar{\sigma}, \quad e^{-\alpha(K + V)} = \sigma e^{-\alpha K} \bar{\sigma}. \tag{3.8}\]

The following is also an important relation:

\[ \sigma \bar{\sigma} = \bar{\sigma} \sigma = 1. \tag{3.9}\]

The solution is given as

\[ \Psi_{\text{KOS}} = -\frac{1}{\sqrt{1 + K}} c(1 + K)\sigma B(1 + K)\bar{\sigma} \frac{1}{\sqrt{1 + K}}, \tag{3.10}\]

which satisfies (3.1).

Our problem is to find \(\xi\) such that

\[ [\Psi_{1}(K_{v}, B_{v}, c_{v})]_{\xi} = \Psi_{\text{KOS}}. \tag{3.11}\]

Multiplied by \(U^{-1}\) and \(U\), this condition reads

\[ \frac{1}{\sqrt{K(\xi)}} c(\xi) \frac{K(\xi)^2}{1 + K(\xi)} B(\xi)c(\xi) \frac{1}{\sqrt{K(\xi)}} = \frac{1}{\sqrt{K}} cK\sigma \frac{B}{1 + K} \bar{\sigma}Kc \frac{1}{\sqrt{K}} (= U^{-1}\Psi_{\text{KOS}}U), \tag{3.12}\]

where we have used (3.7) and that \(U^{-1}[X_{v}]_{\xi}U\) is equal to \(X\) with \((K, B, c)\) in it replaced by \((K(\xi), B(\xi), c(\xi))\).
By considering $B(3.12)B$, we obtain a necessary condition for $\Xi^1$:

$$B \frac{1}{\sqrt{K\Xi^1}} \frac{1}{1 + \Xi^1K} \frac{1}{\sqrt{\Xi^1K}} = B\sqrt{K}\sigma \frac{1}{1 + K}\bar{\sigma}\sqrt{K}. \quad (3.13)$$

In deriving the l.h.s., we have used (2.6) and $B = B(\xi)/\Xi^1$. Since $B$ commutes with all the other operators appearing in (3.13), we can remove $B$ from it. Using $\sqrt{K\Xi^1} = K\sqrt{\Xi^1KK}^{-1}$, (3.13) with $B$ removed is rewritten as

$$K \frac{1}{1 + \Xi^1K} = \sqrt{K}\sigma \frac{1}{1 + K}\bar{\sigma}\sqrt{K}, \quad (3.14)$$

which determines $\Xi^1$:

$$\Xi^1 = 1 + \frac{1}{\sqrt{K}} \left[ \left( \frac{\sigma}{1 + K} \right)^{-1} - 1 - K \right] \frac{1}{\sqrt{K}} = 1 + \frac{1}{\sqrt{K}} V \frac{1}{\sqrt{K}}. \quad (3.15)$$

Here we have used (3.9) and $\sigma K\bar{\sigma} = K + V$, the latter of which follows from the $\alpha$-derivation of the second equation in (3.8). This $\Xi^1$ is real since we are taking a real $V$ (hence $\sigma^\dagger = \bar{\sigma}$).

Next, let us determine $\Xi^2$. For this purpose, we multiply (3.12) by $B$ only from the left side to get a necessary condition for $\Xi^2$:

$$\sqrt{K}\sigma \frac{1}{1 + K} \bar{\sigma}\sqrt{K} \left[ \sqrt{\Xi^1K}e^{-i\Xi^2K}Bc e^{iQ_B\xi^2} \frac{1}{\sqrt{Q_B\xi^1}} - \sqrt{K}Bc \frac{1}{\sqrt{K}} \right] = 0, \quad (3.16)$$

where we have used (3.14). As proven in appendix B, (3.12) actually follows from the condition that the quantity inside the bracket in (3.16) vanishes. Thus our task is to determine $\Xi^2$ from

$$\sqrt{\Xi^1K}e^{-i\Xi^2K}Bc e^{iQ_B\xi^2} \frac{1}{\sqrt{Q_B\xi^1}} = \sqrt{K}Bc \frac{1}{\sqrt{K}}. \quad (3.17)$$

For this purpose, note first the following expression for the quantity on the l.h.s. of (3.17) obtained by using the singular homotopy operator $B/K$ and the formula (2.6):

$$e^{iQ_B\xi^2} \frac{1}{\sqrt{Q_B\xi^1}} = Q_B \left( \frac{B}{K} e^{iQ_B\xi^2} \frac{1}{\sqrt{Q_B\xi^1}} \right) = Q_B \left( \frac{B}{K} e^{iK\Xi^2} \frac{1}{\sqrt{K\Xi^1}} \right) = Q_B \left( \frac{B}{\sqrt{K}} F^{-1} \frac{1}{K} \right), \quad (3.18)$$

where we have defined $F$ by

$$F := \sqrt{\Xi^1K}e^{-i\Xi^2K} \frac{1}{\sqrt{K}}. \quad (3.19)$$

Using (3.18), the l.h.s. of (3.17) is rewritten as

$$\sqrt{\Xi^1K}e^{-i\Xi^2K}Bc e^{iQ_B\xi^2} \frac{1}{\sqrt{Q_B\xi^1}} = F\sqrt{K}Bc Q_B \left( \frac{B}{\sqrt{K}} F^{-1} \frac{1}{K} \right)$$
\[ F\sqrt{K}c\sqrt{K}F^{-1} \frac{1}{K} + B(Q_B F)F^{-1} \frac{1}{K}, \]  

(3.20)

where we have used \(Q_B F^{-1} = -F^{-1}(Q_B F)F^{-1}\) at the second equality. Therefore, the condition (3.17) for \(\Xi^2\) (and hence for \(F\)) is now reduced to

\[ Q_B F = [\sqrt{K}c\sqrt{K}, F]. \]  

(3.21)

Besides (3.21), we have to take into account that \(F\) is defined as a particular quantity (3.19). Using \((\Xi^1)^\dagger = \Xi^{1,2}\) (we will check \((\Xi^2)^\dagger = \Xi^2\) below for the resultant \(\Xi^2\)), we obtain

\[ FF^\dagger = \sqrt{\Xi^1}K e^{-i\Xi^2 K} \frac{1}{K} e^{iK\Xi^2} \sqrt{\Xi^1} = \sqrt{\Xi^1}K e^{-i\Xi^2 K} e^{i\Xi^2 K} \sqrt{\Xi^1}K \frac{1}{K} = \Xi^1. \]  

(3.22)

Let us take the simplest one as \(F\) satisfying (3.22):

\[ F = \sqrt{\Xi^1}. \]  

(3.23)

This \(F\) actually satisfies (3.21) as we shall show. From (3.15), \(\Xi^1\) is Taylor-expanded as

\[ \sqrt{\Xi^1} = \sum_{n=0}^{\infty} a_n \left( \frac{1}{\sqrt{K}} V \frac{1}{\sqrt{K}} \right)^n = \frac{1}{\sqrt{K}} \sum_{n=0}^{\infty} a_n \left( V \frac{1}{K} \right)^n \sqrt{K}, \]  

(3.24)

with numerical coefficients \(a_n\). From the fact that \(V\) is a matter primary operator of weight 1 and \([V, c] = 0\), we have

\[ Q_B V = [K, cV], \]  

(3.25)

from which we also have

\[ Q_B \left( V \frac{1}{K} \right) = [Kc, V \frac{1}{K}]. \]  

(3.26)

Since both \(Q_B\) and the commutator follow the Leibniz rule, we have

\[ Q_B \left( V \frac{1}{K} \right)^n = [Kc, \left( V \frac{1}{K} \right)^n], \]  

(3.27)

for all \(n \in \mathbb{N}\). This implies that \(F = \sqrt{\Xi^1}\) given by the series (3.24) satisfies (3.21). From (3.19), \(\Xi^2\) corresponding to \(F\) of (3.23) is given in terms of \(\Xi^1\) by

\[ e^{-i\Xi^2 K} = \frac{1}{\sqrt{\Xi^1}K} \sqrt{\Xi^1}K, \quad \Xi^2 = i \ln \left( \frac{1}{\sqrt{\Xi^1}K} \sqrt{\Xi^1}K \right) \frac{1}{K}. \]  

(3.28)

Finally, let us confirm \((\Xi^2)^\dagger = \Xi^2\) for \(\Xi^2\) of (3.28), which we used above. Note that

\[ e^{i(\Xi^2)^\dagger K} = \frac{1}{K} e^{iK(\Xi^2)^\dagger} K = \frac{1}{K} \left( \sqrt{K} \sqrt{\Xi^1} \frac{1}{\sqrt{K} \Xi^1} \right) K = \frac{1}{\sqrt{K} \sqrt{\Xi^1} K} \frac{1}{\sqrt{\Xi^1} K}, \]  

(3.29)
where we have used the double conjugation of the first equation of (3.28) at the second equality. Multiplying this and (3.28) together, we obtain
\[ e^{-i\Xi^2 K} e^{i(\Xi^2)^+ K} = 1. \] (3.30)

This implies the reality of \( \Xi^2 \).

In the above, we adopted (3.23) as \( F \) satisfying (3.22). We saw that this \( F \) satisfies the condition (3.21) and that the corresponding \( \Xi^2 \) is real. However, we can take a more general \( F \) of the following form:
\[ F = \sqrt{\Xi^1} e^{if(\Xi^1)}, \] (3.31)
where \( f(\Xi^1) \) is an arbitrary real function of \( \Xi^1 \) alone. This \( F \) satisfies (3.22), and, as seen by looking back upon the above proofs of (3.21) and the reality of \( \Xi^2 \) for \( F = \sqrt{\Xi^1} \), we can easily see that the two properties hold also for \( F \) of (3.31). Note, in particular, that the series (3.24) satisfies (3.21) for any coefficients \( a_n \). The same \( \Psi_1(K_{tv}, B_{tv}, c_{tv}) \) is reproduced for any \( f(\Xi^1) \).

Finally, \( \xi_{tv}^{1,2} \) for generating the KOS solution from the single D-brane solution (3.6) on the tachyon vacuum are obtained from \( \Xi_{tv}^{1,2}(K, V) \) we have determined above as \( \xi_{tv}^{1,2} = U\xi_{tv}^{1,2}U^{-1} = B_{tv}\xi_{tv}^{1,2} \) with \( \Xi_{tv}^{1,2} = \Xi_{tv}^{1,2}(K_{tv}, V_{tv}) \). Expressed in terms of the original \( (K, B, c, V) \), \( \xi_{tv}^{1,2} \) is given by
\[ \xi_{tv}^{1,2} = UB\Xi_{tv}^{1,2}U^{-1} = B\sqrt{\frac{K}{1+K}}\Xi_{tv}^{1,2}\sqrt{\frac{K}{1+K}}. \] (3.32)

Note that, although \( \Xi^1 \) (3.15) is singular at \( K = 0 \), this singularity is resolved in \( \xi_{tv}^{1,2} \). This is the case also for \( \xi_{tv}^{2} \) obtained from \( \Xi^2 \) of (3.28). The singularities at \( K = 0 \) appearing in various places in our construction are artifacts of solving (3.12) for \( \xi_{tv}^{1,2} \) instead of the original (3.11) for \( \xi_{tv}^{1,2} \).

### 3.3 Reproducing Erler-Maccaferri solution

The Erler-Maccaferri solution [17] is a generalization of the KOS solution to the case where \( V \) is not necessarily regular. In this case, the boundary condition changing operators are modified to be accompanied by some matter operators, resulting in that \( \sigma\bar{\sigma} = 1 \) and the associativity\footnote{For example, \((\sigma\bar{\sigma})\sigma \neq \sigma(\bar{\sigma}\sigma)\) in general.} no longer hold, while \( \sigma\bar{\sigma} = 1 \) is kept. Since (3.8) does not hold anymore, we have to express all equations in terms of \( \sigma \) and \( \bar{\sigma} \), not using \( V \). Despite those differences, the solution is still of the same form as (3.10):
\[ \Psi_{EM} = -\frac{1}{\sqrt{1+K}}c(1+K)\sigma B\frac{\bar{\sigma}(1+K)c}{\sqrt{1+K}}. \] (3.33)

Our problem is to find \( \xi \) satisfying \([\Psi_1(K_{tv}, B_{tv}, c_{tv})]_\xi = \Psi_{EM}\).
Even in the present case, we can reach (3.14) with no obstacle, and hence we obtain

$$
\Xi^1 = 1 + \frac{1}{\sqrt{K}} \left[ \left( \sigma - \frac{1}{1+K} \bar{\sigma} \right)^{-1} - 1 - K \right] \frac{1}{\sqrt{K}}.
$$

(3.34)

Following the previous process, we get (3.17) without any change. Therefore, we naively expect that $\Xi^2$ is again given by (3.28). The only difference from the KOS case is the process of showing that $F$ of (3.23) satisfies (3.21), because we explicitly used $V$ there.

Let us show that (3.21) is satisfied by $F = \sqrt{\Xi^1}$ here as well. We define

$$
S := \sigma \frac{1}{1+K} \bar{\sigma}, \quad W := S^{-1} - (1 + K).
$$

(3.35)

Since $\Xi^1$ is written as

$$
\Xi^1 = 1 + \frac{1}{\sqrt{K}} W \frac{1}{\sqrt{K}},
$$

(3.36)

we have only to show

$$
Q_B W = [K, cW], \quad [c, W] = 0,
$$

(3.37)

which correspond to (3.25) and $[c, V] = 0$. The latter relation can be shown as follows:

$$
S[c, W]S = Sc - Sc(1 + K)S - cS + S(1 + K)cS
\begin{align*}
&= \sigma \left[ \frac{1}{1+K}, c \right] \bar{\sigma} + \sigma \frac{1}{1+K} \bar{\sigma} \left[ 1 + K, c \right] \sigma \frac{1}{1+K} \bar{\sigma} \\
&= 0.
\end{align*}
$$

(3.38)

In the last equality, we have used that $[1 + K, c] = -\partial c$ is a ghost operator and commutes with $\sigma$. Next let us consider the former relation in (3.37). We have

$$
Q_B W = -S^{-1}(Q_B S)S^{-1} = -S^{-1} \left( c[1 + K, \sigma] \frac{1}{1+K} \bar{\sigma} + \sigma \frac{1}{1+K} [1 + K, \bar{\sigma}] c \right) S^{-1},
$$

(3.39)

where we have used $Q_B \sigma = c[K, \sigma] = c[1 + K, \sigma]$ and $[c, [1 + K, \sigma]] = 0$ which follow from that $\sigma$ is a matter primary operator of weight 0 ($\bar{\sigma}$ as well). As $\sigma$ and $\bar{\sigma}$ commute with $c$, we obtain the desired relation:

$$
Q_B W = -S^{-1} c(1 + K) + (1 + K)cS^{-1} = [K, cW] + [c, W](1 + K) = [K, cW].
$$

(3.40)

As indicated in [17], (3.33) includes a solution which describes the change of the dimension of the D-brane or of the number of D-branes (see also [25] for the latter). In the latter case, $\sigma$ and $\bar{\sigma}$ have multi-components, but our construction here is still valid because $\xi$ can contain arbitrary number of matter operators. Since (3.33) is expected to describe any backgrounds, the discussion here implies that any open string backgrounds can also be described by $KB_{tv}$-representations and (3.7), namely, $[\Psi_1(K_{tv}, B_{tv}, c_{tv})]_\xi$.

---

8) According to [17], $\bar{\sigma} \sigma = 1$ but in general $\sigma \bar{\sigma} = g \neq 1$ with $g$ being a constant. Using this fact, we have $S^{-1} = (\sigma(1 + K)\bar{\sigma})/g$ for $S$ defined in (3.35).
4 Discussions

In this paper, we showed that there are various representations of $KBc$ algebra which contains matter operators, both on a single D-brane and the tachyon vacuum. Each of the KOS and the Erler-Maccaferri solutions was reproduc ed by modifying the single D-brane solution on the tachyon vacuum to the one in a proper representation. However, though our framework has an ability to determine the representation that reproduces a given solution, there is no guiding principle in our framework to find new physically meaningful solutions. Here we will discuss two possible directions to address this subject.

The first approach is to find the basic representation of other CFTs. In SFT, there exists a solution corresponding to any world-sheet CFT. In addition, given a CFT, there is always the basic representation, and hence $\Psi_0$ (2.11) written by it gives the tachyon vacuum solution of the D-brane system described by the CFT. Since $\Psi_0$ consists of the basic representation, we can construct any backgrounds if we find the basic representation of each CFT in the language of the reference CFT. Our method has provided how to find various $KBc$ triads, thus has opened a new way to realize this idea.

However, what we have found in this paper are representations on a single D-brane and on the tachyon vacuum, hence our method at glance does not seem to connect $KBc$ algebras of different CFTs. But, there is a clue. Recalling how we have obtained $KBc_{tv}$-representations from $KBc$-representations, we can use a unitary transformation to go to representations in other CFTs. Let us consider a solution of the form $[\Psi_1]_\xi = U(\xi)_{tv}^{-1}U(\xi)_{tv}$, where $U(\xi)_{tv}$ denotes the unitary operator obtained from $U_{tv}$ (3.4) by replacing $(K_{tv}, \cdots)$ with $(K(\xi)_{tv}, \cdots)$. Following the same logic as in subsection 2.2, a new triad

$$U(\xi)_{tv}^{-1}(K(\xi)_{tv}, B(\xi)_{tv}, c(\xi)_{tv})U(\xi)_{tv}, \quad (4.1)$$

forms a representation of $KBc$ algebra with its BRST operator given as $Q_\xi := Q_{tv} + [[\Psi_1]_\xi, \cdot]$. Starting with this triad, we can again generate a class of other representations, among which the basic representation of $KBc$ algebra in the CFT corresponding to $[\Psi_1]_\xi$ might exist. Thus, our method has a potential connection among different CFTs.

The other direction is to investigate the manifold-like structure explained in our previous work [19]. Witten’s open SFT has a similar structure to Chern-Simons theory, and several correspondences are known. For example, the star product corresponds to the wedge product, and the BRST operator to the exterior derivative. Though restricted to the $KBc$ algebra, the interior product, Lie derivative and Wilson line in SFT were introduced in [19]. The Lie derivative there can generate $KBc$-representations which do not include matter operators, so we defined the $KBc$ manifold whose points are those representations.

As explained in appendix A, representations introduced in this paper are obtained by extending the previous construction in [19]. In appendix A, we redefine the interior product and Lie derivative so that the resultant $KBc$-representations contain matter operators. However, for incorporating matter operators, we have to ignore some of their properties which were previously respected. For example, the interior product was nilpotent before, but now its second operation is not defined, because the operation of the interior product on matter operators is not defined (see appendix A).9)

---

9) This fact does not affect our results in this paper. We discuss the operation of the interior product and
The $KBc$ manifold seems mathematically parallel to the ordinary manifold to some extent, hence it could be possible for it to have a more profound structure. In order to recover the lost properties, we have to impose further conditions on $\xi$, meaning that representations we obtain are limited. If the $KBc$ manifold is really a physical object, some proper restrictions may lead us to find a correct family of $KBc$-representations, which is expected to have a control over classical solutions in SFT.

Acknowledgement

D.T. thanks Koji Hashimoto for valuable comments on his talk at “Strings and Fields 2021” in YITP, Kyoto. The work of D.T. was supported by Grant-in-Aid for JSPS Fellows No. 22J20722. J.Y. thanks Hiroshi Kunitomo for daily discussions on SFT.

A Generating $KBc$-representations

We extend the method in [19] to the case where matter operators are included in $KBc$-representations. We first find an operation lowering the ghost number by 1, which is an analog of interior product in differential geometry. Second, we define an analog of Lie derivative by using the interior product, which provides a differential equation to generate (2.4).

The interior product

We first construct as general an operation as possible that is linear and lowers the ghost number by 1, which we write $I_X$. The subscript $X$ is a (yet undetermined) quantity which characterizes the operation. Let us start with the following ansatz:

1. The operation of $I_X$ on $(K, B, c)$ is written by $K, B, c$ and matter operators.
2. Acted by $I_X$, the both hand sides of each relation in (2.1) return the same result.
3. Against any product of the original $(K, B, c)$, $I_X$ acts according to the (anti-)Leibniz rule just as $Q_B$ does.

One might think it is a problem that we have not defined how $I_X$ acts on matter operators. But, we will never need such operations. Here, we do not consider the “$KBc$ manifold” as we did in [19]. Instead, we only focus on finding $KBc$-representations including matter operators. If one would like to construct the manifold, $I_X$ will be further restricted than what we will obtain here. The operation of $I_X$ on matter operators is discussed in appendix C.

The generic form of $I_X(K, B, c)$ is given as follows by ansatz 1:

$$
I_X K = i B f, \quad I_X B = 0, \quad I_X c = g + \sum_n h_L^{(n)} B h_R^{(n)}. \tag{A.1}
$$

the Lie derivative on matter operators in appendix C.
Here \( f, g \) and \( h^{(n)}_{L,R} \) are general operators consisting of \( K \) and matter operators. We determine \( f, g \) and \( h^{(n)}_{L,R} \) from ansatz 2, by using ansatz 3 for \( I_X \) acting on products. Among the relations in (2.1), \([K,B]=0\) and \(B^2=0\) do not impose any condition on those undetermined quantities; ansatz 2 is trivial for the two relations, because there is no quantity carrying ghost number less than \(-1\).

Next, let us consider ansatz 2 for \( c^2=0 \). The linearity of \( I_X \) gives \( I_X 0 = 0 \), and hence \( I_X c^2 \) must vanish. As a necessary condition, \( B I_X c^2 = 0 \) must also hold, which reads

\[
g B c - B c g - \sum h^{(n)}_{L,R} B c h^{(n)}_{R} = 0. \tag{A.2}
\]

From this, \( I_X c \) is now written as

\[
I_X c = g + [g, B c] = \{c, g B\}. \tag{A.3}
\]

By using the Jacobi identity for (anti-)commutators, one can straightforwardly show \( I_X c^2 = 0 \) from (A.3), meaning that (A.2) is sufficient. Under (A.3), let us show that \( I_X \{B, c\} = I_X 1 \) holds for any \( f \) and \( g \). For the r.h.s., we have \( I_X 1 = 0 \) because of the Leibniz rule. From (A.3), the l.h.s. also vanishes as follows:

\[
I_X \{B, c\} = \{(c, g B), B\} = g B c B - B c g B = 0. \tag{A.4}
\]

Note that both \( f \) and \( g \) are always multiplied by \( B \) in \( I_X (K, B, c) \). Thus, defining \( X^1 := B f = f B \) and \( X^2 := B g = g B \), the final result is

\[
I_X K = i X^1, \quad I_X B = 0, \quad I_X c = \{c, X^2\}, \tag{A.5}
\]

where \( X^{1,2} \) consists of \( B, K \) and matter operators, and its ghost number is \(-1\).\(^{10})

### Lie derivative

The definition of the Lie derivative here is

\[
\mathcal{L}_X := -i \{Q_B, I_X\}. \tag{A.6}
\]

This is an analog of the ordinary Lie derivative given by the anti-commutator between the interior product and the exterior derivative. Note that \( I_X \) and \( \mathcal{L}_X \) are defined only for the original \((K, B, c)\) and not for matter operators. Since the operation of \( I_X \) is not closed in the \( KBc \) algebra while \( Q_B \) is, one may be afraid that our definition of \( I_X \) and \( \mathcal{L}_X \) is incomplete. However, this is not a problem for our purpose of constructing \( KBc \)-representations as we shall see below. Here is the result of the action of \( \mathcal{L}_X \) on \((K, B, c)\):

\[
\mathcal{L}_X K = Q_B X^1, \quad \mathcal{L}_X B = X^1, \quad \mathcal{L}_X c = -c (X^1 + i [X^2, K]) c + i [c, Q_B X^2]. \tag{A.7}
\]

\(^{10})\) \(X^1\) and \(X^2\) in this paper correspond to \(BX^1\) and \(B(X^2/K)\) in \([19]\) (see (2.6) there), respectively. Recalling the correspondence between the form number in differential geometry and the ghost number in open SFT, which was indicated in \([19]\), it is natural that \(X^{1,2}\) which we expect is an analog of the tangent vector has ghost number \(-1\).
The key property of the Lie derivative is that it keeps the $KBc$ algebra in the following sense. Let us consider a new triad,

$$(\tilde{K}, \tilde{B}, \tilde{c}) := (K, B, c) + \epsilon L_X(K, B, c),$$  
(A.8)

with $\epsilon$ infinitesimal. Note that

$$[Q_B, L_X] = 0$$  
(A.9)

holds due to $Q_B^2 = 0$. Using this, we see that (2.2) holds for $(\tilde{K}, \tilde{B}, \tilde{c})$ to $O(\epsilon)$; for example,

$$Q_B \tilde{B} = (1 + \epsilon L_X)Q_B B = (1 + \epsilon L_X)K = \tilde{K}.$$  
(A.10)

In addition, one can easily confirm that $L_X$ follows the Leibniz rule, so (2.1) is satisfied by the new triad to $O(\epsilon)$; for example,

$$\{\tilde{B}, \tilde{c}\} = \{B, c\} + \epsilon L_X\{B, c\} = 1 + \epsilon L_X 1 = 1.$$  
(A.11)

**Generating $KBc$-representations**

What we have learned above is that our Lie derivative can generate new $KBc$-representations which are infinitesimally changed from the basic representation. Thus we expect that we obtain finitely changed $KBc$-representations if we successively apply the Lie derivative.

However, since we have not defined the action of the Lie derivative on $X$ containing matter operators, such a naive successive operation of Lie derivative is forbidden. In other words, we have to consider an appropriate Lie derivative on (A.8) and on the series of subsequent triads. To construct such a proper new derivative, let us consider the following new interior product:

$$\tilde{I}_Y \tilde{K} = iY^1, \quad \tilde{I}_Y \tilde{B} = 0, \quad \tilde{I}_Y \tilde{c} = \{\tilde{c}, Y^2\},$$  
(A.12)

with $Y^{1,2}$ carrying ghost number $-1$. We again demand that $\tilde{I}_Y$ follows the (anti-)Leibniz rule, and define a new Lie derivative,

$$\tilde{L}_Y := -i\{Q_B, \tilde{I}_Y\}.$$  
(A.13)

From this definition and the (anti-)Leibniz rule of $\tilde{I}_Y$, we find that $\tilde{L}_Y$ commutes with $Q_B$ and that $\tilde{L}_Y$ follows the Leibniz rule, respectively. This means that

$$(1 + \epsilon \tilde{L}_Y)(\tilde{K}, \tilde{B}, \tilde{c})$$  
(A.14)

is also a $KBc$-representation. The action of $\tilde{L}_Y$ on (A.8) is given by

$$\tilde{L}_Y \tilde{K} = Q_B Y^1, \quad \tilde{L}_Y \tilde{B} = Y^1, \quad \tilde{L}_Y \tilde{c} = -\tilde{c}(Y^1 + i[Y^2, \tilde{K}])\tilde{c} + i[\tilde{c}, Q_B Y^2],$$  
(A.15)

which is of the same form as (A.12).

Thus, we conclude that the triad $(K_s, B_s, c_s)$ with parameter $s$ which is determined through the following differential equations is a $KBc$-representation:

$$\dot{K}_s = Q_B \xi^1(s), \quad \dot{B}_s = \dot{\xi}^1(s),$$

15
\[ \dot{c}_s = -c_s(\dot{\xi}^1(s) + i[\dot{\xi}^2(s), K_s])c_s + i[c_s, Q_B\dot{\xi}^2(s)]. \]  

(A.16)

Here each of \( \xi^{1,2}(s) \) is a function of \( s \) which consists of \( K, B \) and matter operators, and the dot means \( s \)-derivative. The problem left for us is to solve (A.16). We put \( (\xi^1(0), \xi^2(0)) = (B, 0) \) and adopt as the initial condition \( (K_0, B_0, c_0) = (K, B, c) \).

The first two equations in (A.16) can be solved easily to give

\[ K_s = Q_B\xi^1(s), \quad B_s = \xi^1(s). \]  

(A.17)

Then using

\[ [\dot{\xi}^2(s), K_s] = [\dot{\xi}^2(s), Q_B\xi^1(s)] = [Q_B\dot{\xi}^2(s), \xi^1(s)] = [Q_B\dot{\xi}^2(s), B_s], \]  

(A.18)

we can rewrite the last equation in (A.16) as

\[ \dot{c}_s = -c_s(\dot{B}_s + i[Q_B\dot{\xi}^2(s), B_s])c_s + i[c_s, Q_B\dot{\xi}^2(s)]. \]  

(A.19)

Since our construction of \( (K_s, B_s, c_s) \) assures that it must be a \( KBc \)-representation, we can use the \( KBc \) algebra among them in advance. In concrete, using \( \{B_s, c_s\} = 1 \), we can simplify the differential equation (A.19) to

\[ \dot{c}_s = -c_s\dot{B}_sc_s + i[c_s(Q_B\dot{\xi}^2(s))c_s, B_s]. \]  

(A.20)

To solve this, we multiply it by \( B_s \) from the left side to obtain

\[ \frac{d}{ds}(B_sc_s) = -i[Q_B\dot{\xi}^2(s), B_sc_s]. \]  

(A.21)

Here we have used \( B_s\dot{\xi}^{1,2}(s) = 0 \). This is solved as

\[ B_sc_s = E(s)^{-1}BcE(s), \quad E(s) := S \exp \left[ i \int_0^s ds' Q_B\dot{\xi}^2(s') \right], \]  

(A.22)

where \( S \) denotes \( s \)-ordering which puts an operator with larger \( s \) to the right. Using \( \{B_s, c_s\} = 1 \) again, we also have

\[ c_sB_s = E(s)^{-1}cBE(s). \]  

(A.23)

By expressing \( \xi^1(s) \) as \( \xi^1(s) = B\Xi^1(s) \) with \( \Xi^1(s) \) consisting of \( K \) and matter operators (this is always possible), (A.22) reads

\[ Bc_s = \frac{1}{\Xi^1(s)}E(s)^{-1}BcE(s). \]  

(A.24)

Multiplying this by (A.23) \( \times c \) from the left side and using \( B_scB = B_s \) and \( c_sc_sc_s = c_s \), we obtain

\[ c_s = E(s)^{-1}cBE(s)c_s \frac{1}{\Xi^1(s)}E(s)^{-1}BcE(s). \]  

(A.25)
Since the original differential equation for \( c_s \) is a first-order ODE, we conclude that (A.25) is the unique solution we have been looking for. Note that \( c_s \) of (A.25) is not determined only by \( \xi(s) \) but it in general depends on the whole of \( \xi^2(s') \) with \( 0 \leq s' \leq s \).

If we choose as \( \xi(s) \) a special one with

\[
\xi^1(1) = \xi^1, \quad \xi^2(s) = s\xi^2
\]  

(A.26)

for given \( \xi^{1,2} \) (\( \xi^1(s) \) for \( 0 < s < 1 \) can be arbitrary), the triad \( (K_1, B_1, c_1) \) turns out to be that of (2.4), the representation introduced in the text (we need (2.6) to obtain \( c(\xi) \) in (2.4) from \( c_1 \)). One can also confirm directly that (A.17) and (A.25) form a \( KBc \)-representation, by following the same procedure shown in the text (subsection 2.1).

**B Proof of (3.17) ⇒ (3.12)**

In section 3, (3.17) was derived as a necessary condition for (3.12) by multiplying it by \( B \) from the left side. Here, we show that (3.17) is also sufficient for (3.12).

First, note that, using \((\Xi^{1,2})^\dagger = \Xi^{1,2} \) the double conjugation of (3.17) reads

\[
\frac{1}{\sqrt{Q_B\xi}} e^{-iQ_B\xi^2} cBe^{iK\Xi^2} \sqrt{K\Xi^1} = \frac{1}{\sqrt{K}} eB\sqrt{K}.
\]  

(B.1)

Plugging (2.4) into the l.h.s. of (3.12) and using (3.17) and (B.1), we obtain

\[
\frac{1}{\sqrt{K(\xi)}} c(\xi) K(\xi)^2 B(\xi)c(\xi) \frac{1}{\sqrt{K(\xi)}}

= \frac{1}{\sqrt{Q_B\xi}} e^{-iQ_B\xi^2} cBe^{iK\Xi^2} \sqrt{K\Xi^1} \sqrt{K} \sigma \frac{1}{1 + K} \bar{\sigma} \sqrt{K} \left( \sqrt{\Xi^1} K e^{-i\Xi^2} B e^{iQ_B\xi^2} \frac{1}{\sqrt{Q_B\xi}} \right)

= \frac{1}{\sqrt{Q_B\xi}} e^{-iQ_B\xi^2} cBe^{iK\Xi^2} \sqrt{K\Xi^1} \sqrt{K} \sigma \frac{1}{1 + K} \bar{\sigma} Kc \frac{1}{\sqrt{K}}

= \left( \frac{1}{\sqrt{Q_B\xi}} e^{-iQ_B\xi^2} cBe^{iK\Xi^2} \sqrt{K\Xi^1} \right) \sqrt{K} \sigma \frac{1}{1 + K} \bar{\sigma} Kc \frac{1}{\sqrt{K}}

= \frac{1}{\sqrt{K}} cB \sqrt{K} \sqrt{K} \sigma \frac{1}{1 + K} \bar{\sigma} Kc \frac{1}{\sqrt{K}} = \frac{1}{\sqrt{K}} cK \sigma \frac{1}{\sqrt{K}} \sqrt{K} \sigma \frac{1 + K}{\sqrt{K}}

\]  

(B.2)

which is nothing but the r.h.s of (3.12). Note that we have used (3.17) and (B.1) at the second and the fifth equalities, respectively, for quantities inside the parentheses.

**C Interior product and Lie derivative for matter operators**

We extend the action of the interior product and the Lie derivative to matter primary operators. For simplicity, we consider only one matter operator \( V \) with weight \( h \). This \( V \) satisfies
the following relations:

\[
[B, V] = 0, \quad [c, V] = 0, \quad (C.1)
\]

\[
Q_B V = c[K, V] + h[K, c] V. \quad (C.2)
\]

We call (2.1), (2.2), (C.1) and (C.2) together \( KBcV \) algebra.

First, we extend the interior product \( \mathcal{I}_X \) introduced in appendix A to define \( \mathcal{I}_X V \). In addition to the three ansatz in appendix A, we impose the following two:

4. \( \mathcal{I}_X V \) is written by \( K, B, c \) and the matter operator \( V \).

5. Acted by \( \mathcal{I}_X \), the both hand sides of each relation in (C.1) return the same result.

The generic form of \( \mathcal{I}_X V \) is given by ansatz 4 as

\[
\mathcal{I}_X V = BJ, \quad (C.3)
\]

where \( J \) is a general operator written by \( K \) and \( V \). We determine \( J \) through ansatz 5. Since \( \mathcal{I}_X [B, V] = 0 \) automatically holds, we just have to consider \( \mathcal{I}_X [c, V] = 0 \). From (A.5) and (C.5), we have

\[
\mathcal{I}_X [c, V] = [\{c, X^2\}, V] = 0.
\]

Therefore, choosing \( BJ = [X^2, V] \), \( \mathcal{I}_X V \) is determined as

\[
\mathcal{I}_X V = [X^2, V]. \quad (C.5)
\]

Note that, although \( \mathcal{I}_X \) respects (2.1) and (C.1), it does not necessarily keep other relations. Especially, the relation \( [V, [K, c]] = 0 \) which follows from \( [K, c] = -\partial c \) is not kept by \( \mathcal{I}_X \).

We can also define the Lie derivative for \( V \) as we did for \( (K, B, c) \) in appendix A:

\[
\mathcal{L}_X V = -i\{Q_B, \mathcal{I}_X\} V = -i[Q_BX^2, V] + \{X^1 - i[K, X^2], cV\} + (h - 1)\{X^1 - i[K, X^2], c\} V. \quad (C.6)
\]

Then, let us solve the differential equation,

\[
\dot{V}_s = -i[Q_B \xi^2(s), V_s] + \{\xi^1(s) - i[K_s, \xi^2(s)], c_s V_s\} + (h - 1)\{\dot{\xi}^1(s) - i[K_s, \xi^2(s)], c_s\} V_s, \quad (C.7)
\]

together with (A.16), by adopting \( V_0 = V \) as the initial condition. The resulting \( (K_s, B_s, c_s, V_s) \) constitute a \( KBcV \)-representation.

Solving the differential equation (C.7) for an arbitrary \( h \) seems a difficult problem. However, we can solve it in the special case of \( h = 1 \), where \( V \) is a marginal operator. Therefore, we restrict the following arguments to \( h = 1 \). First, we consider

\[
\frac{d}{ds}(c_s V_s) = \mathcal{L}_{\xi(s)}^{(s)}(c_s V_s) = -i\{Q_B, \mathcal{I}_{\xi(s)}^{(s)}\}(c_s V_s), \quad (C.8)
\]

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The solution to (C.9) is given by
\[ X \text{real} E \]
where
\[ V \]  
This implies that
\[ Q \]
If we choose \( \xi \) such that \( \xi = 1 \), then \( iQ_B \xi^2 \) is defined in (A.22). Using this result, \( \xi \) is given by
\[ \xi^2 = \xi^2 \]  
Starting with a solution containing matter operators, we can construct new solutions by using \( \xi \) and this property ensures that the triad \( (K, B, c, V) \) is defined in (A.22). Using this result, \( \xi \) is given by
\[ \xi \]
For a real \( \xi \), we have \( \xi = 1 \) and \( \xi = 1 \). This reality problem of \( \xi \) together with \( (K(\xi), B(\xi), c(\xi)) \) of (2.4) constitutes to a \( KBcV \)-representation. However, the ordering problem and the resulting ambiguity of \( \xi \) is a different quantity from \( \xi \). For a real \( \xi \), we have from (C.6)
\[ \xi \]
This implies that \( \xi \) determined by (C.7) is no longer real unless \( h = 1 \).

\[ (\mathcal{L}_X(K, B, c))^\dagger = \mathcal{L}_X(K, B, c), \]  
and this property ensures that the triad \( (K(\xi), B(\xi), c(\xi)) \) is real as long as \( \xi \) is real. On the other hand, for a real \( V \), we have from (C.6)
\[ \mathcal{L}_X V = \mathcal{L}_X V + (h - 1)\mathcal{I}_X [V, [K, c]]. \]  
This implies that \( \xi \) determined by (C.7) is no longer real unless \( h = 1 \).

Though this reality problem of \( \xi \) is resolved for \( V \) with weight 1, there still remains another problem existent even in the case \( h = 1 \). This is a problem that the expression of \( \mathcal{L}_X V \) is not unique. This expression depends on the order of \( [K, c] \) and \( V \) in (C.2). If we rewrite (C.2) as
\[ Q_B V = c[K, V] + hV[K, c], \]  
the corresponding \( \mathcal{L}_X V = -i\{Q_B, \mathcal{L}_X \}V \) is a different quantity from (C.6), but is given by
\[ \mathcal{L}_X V = -i[Q_B X^2, V] - c[X^1 - i[K, X^2], V] + hV\{X^1 - i[K, X^2], c\}. \]  
Of course, if we choose one expression of \( \mathcal{L}_X V \), we obtain a family of \( KBcV \)-representations. However, the ordering problem and the resulting ambiguity of \( \mathcal{L}_X V \) may cause inconveniences, if one applies the algebraic tools we have developed here to other problems in SFT.
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