Computation of transport coefficients around critical point based on novel diagrammatic method

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Abstract. We compute and discuss the temperature-dependence of the shear viscosity, focusing on that around the critical point, using the Nambu–Jona-Lasinio model in the leading order of $1/N_c$ expansion and the chiral limit. Our calculation is based on a systematic resummation method of the singular diagrams, developed recently by the present authors. In the leading order of the theory, $2 \to 2$ scattering between quark-quark, antiquark-antiquark and quark-antiquark contributes to the shear viscosity. The ratio of the shear viscosity to the entropy, $\eta/s$, decreases as temperature increases, only with a tiny cusp at $T = T_c$.

1. Introduction
Transport coefficients reflect dynamical properties of the medium slightly away from thermal equilibrium. This is also a parameter of hydrodynamics, which cannot be calculated by hydrodynamics itself. To determine the transport coefficient from a microscopic theory with accuracy is important to understand the property of the medium, and also necessary to determine the parameter of hydrodynamics such as viscosity. A diagrammatic method is a useful tool for computing the transport coefficient; there is, however, a technical difficulty to compute it due to the so called pinch singularity, which turns out to be circumvented by a resummation of an infinite number of diagrams. Very recently, the present authors [1] have developed a novel resummation technique for that use: Not only the leading order in the rearranged expansion, which is the linearized Boltzmann equation known in the literature [2], but also higher orders can be taken into account in this theory.

In this talk, we apply the method to a fermionic system with the chiral phase transition at finite temperature. For this purpose, we employ the Nambu–Jona-Lasinio model in the chiral limit and calculate the shear viscosity in the leading order of large-$N_c$ expansion, although our method can systematically take into account higher order corrections.

2. Diagrammatic Method for computing transport coefficients
2.1. Nambu–Jona-Lasinio model
The Lagrangian of NJL model has a form,
\[
\mathcal{L} = \overline{\psi} i \partial \psi + \frac{G_S}{2} \left[ (\overline{\psi} \psi)^2 + (\overline{\psi} i \gamma_5 \tau^a \psi)^2 \right],
\]
where $\psi$ is the quark field with color $N_c$ and flavor $N_f = 2$, $\partial = \gamma^\mu \partial_\mu$, and $\tau^a$'s are the Pauli matrices for the flavors. $G_S$ is the four point coupling constant. In the large-$N_c$ counting, $G_S$ is
of order $1/N_c$. In the leading order of $N_c$, the thermodynamic potential density is given by

$$\Omega(\sigma, T, \mu) = \frac{\sigma^2}{2G_S} - 4N_c \int \frac{d^3k}{(2\pi)^3} \left[ E_k + 2T \ln(1 + e^{-\beta E_k}) \right],$$

where $\sigma = G_S \langle \bar{\psi} \psi \rangle$ is the mean-field of the sigma channel, and $E_k = \sqrt{k^2 + m^2}$, with $m = -\sigma$. $\sigma$ can be determined by the gap equation $\partial \Omega / \partial \sigma = 0$. The pressure and entropy density are obtained from $\Omega$ as $p = -\Omega$ and $s = -\partial \Omega / \partial T$, respectively.

2.2. Diagrammatic method

Here, we briefly review a diagrammatic method for computing transport coefficients [1]. In the linear response theory [3], the shear viscosity is given by

$$\eta = \frac{1}{10} \lim_{\omega \to 0} \frac{1}{\omega} \Im \int d^4x \, e^{i\omega t} \Im \left\{ \left[ \pi_{ij}(x), \pi_{ij}(0) \right] \right\},$$

where $\pi_{ij}(x) = T_{ij}(x) - g_{ij} T^k(x)/3$ is the traceless stress tensor, and $T_{\mu\nu}(x) = \bar{\psi}(x)(\gamma_{\mu}\partial_{\nu} + \gamma_{\nu}\partial_{\mu})\psi(x)/2 - g_{\mu\nu}E$ is the energy-momentum tensor. We work in Minkowski space-time with a metric, $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. In our diagrammatic method [1], the shear viscosity can be shown to have the following form with the quark spectral function $\rho(k)$:

$$\eta = \frac{1}{10T} \int \frac{d^4k}{(2\pi)^4} \left( \rho(k) \left( 1 - n(k^0) \right) \Gamma_{ij}(k, s, s') \bar{\Gamma}^{ij}(k, s, s') \right),$$

where $s$ denotes spin, flavor and color indices, $n(k^0) = 1/(\exp(k^0/T) + 1)$ is the Fermi-Dirac distribution function, $\bar{\Gamma}_{ij}(k, s, s')$ the vertex function including thermal and quantum corrections and $\Sigma_R(k)$ the retarded self-energy taken the spin, flavor and color average. We remark that $\bar{\Gamma}_{ij}(k, s, s')$ does not includes collision effects unlike $\Gamma(k, s, s')$, as will be elucidated below; see Eq. (6). In the leading order, $\bar{\Gamma}_{ij}(k, s, s')$ becomes

$$\bar{\Gamma}^{ij}(k, s, s') = \frac{1}{2} \bar{u}(k, s) \left( k^i \gamma^j + \gamma^i k^j - 2\delta^{ij} \frac{k \cdot \gamma}{3} \right) u(k, s') = 2\delta_{s,s'} B_0^{ij}(k),$$

with $B_0^{ij}(k) = k^i k^j - \delta^{ij} k^2/3$ and $u(k, s')$ being a wave function of a quark. In the quasiparticle approximation, $\rho(k) = (2\pi)\delta(k^2 - m^2)$. Equation (4) shows that $\eta$ diverges if $\Im \Sigma_R(k) = 0$. This is the pinch singularity mentioned above. In the $1/N_c$ expansion, $\Im \Sigma_R(k)$ is of order $1/N_c$, implying that $\eta/s$ is of order $N_c$. $\Gamma^{ij}(k, s, s') \equiv 2\delta_{s,s'} B^{ij}(k)$ is the resummed vertex satisfying the following self-consistent equation:

$$B^{ij}(k) = B_0^{ij}(k) - \frac{1}{4N_c} \sum_{s, s'} \Im T_{AARR}(k, s, s; k', s', s') \frac{\rho(k')}{\Im \Sigma_R(k')} B^{ij}(k'),$$
where $T_{AARR}$ is a four point function that does not include diagrams with the pinch singularity. The diagrams contributing to the leading order in $1/N_c$ is shown in Fig. 1, where the indices $A$ and $R$ stand for advanced and retarded, respectively. In the leading order approximation, $T_{AARR}$ is of order $1/N_c$, which cancels with the contribution by the inverse of $\text{Im } \Sigma_R(k)$ in RHS of Eq. (6), so the kernel in that equation is found to be of order 1. This means that the ladder diagram shown in Fig. 1 must be summed over. For later use, we mention that the self-energy and the four-point function are related through Ward-Takahashi identity [1, 5],

$$-2\text{Im } \Sigma_R(k) + \int \frac{d^4k'}{(2\pi)^4} \rho(k') \frac{1}{4N_c} \sum_{s,s'} \text{Im } T_{AARR}(k, s, s'; k', s', s') = 0.$$  (7)

2.3. Reduction of the self-consistent equation to a Boltzmann equation

It is useful to introduce the following collision operator to clarify the physical process in Eq. (6);

$$\mathcal{L} \phi^{ij}(k) \equiv -\frac{1}{k^0} \text{Im } \Sigma_R(k) \phi^{ij}(k) + \frac{1}{2k^0} \int \frac{d^4k'}{(2\pi)^4} \frac{1}{4N_c} \sum_{s,s'} \text{Im } T_{AARR}(k, k'; s, s) \rho(k') \phi^{ij}(k')$$

$$= \frac{1}{16N_c k^0} \int \frac{d^4k'}{(2\pi)^4} n(k^0) \rho(k') \int \frac{d^4q}{(2\pi)^4} (1-n(q^0)) \rho(q) \int \frac{d^4q'}{(2\pi)^4} n(q'^0) \rho(q') \times |M|^2(2\pi)^4 \delta^{(4)}(k + q - k' - q') (\phi^{ij}(k)-\phi^{ij}(k') + \phi^{ij}(q) - \phi^{ij}(q')) ,$$

where $\phi^{ij}(k) \equiv B^{ij}(k)/(-2\text{Im } \Sigma_R(k))$. Using the collision operator, we can rewrite Eq. (8) as

$$\frac{1}{2k^0} B^{ij}(k) = \mathcal{L} \phi^{ij}(k).$$  (9)

It can be shown [1] that Eq. (9) is equivalent to a linearized Boltzmann equation. Equation (8) contains 1 → 3 and 3 → 1 processes in addition to 2 → 2 process; however, in the leading order, only 2 → 2 process contributes to the shear viscosity, because 1 → 3 and 3 → 1 cannot satisfy the on-shell condition of the quarks. 2 → 2 processes contain quark-quark, antiquark-antiquark, and quark-antiquark scattering. Scatterings of the $\pi$ and $\sigma$ modes are higher order in the large-$N_c$ expansion. The $\mathcal{M}$ of quark-quark scattering part is written as

$$|\mathcal{M}_{qqqq}|^2 = \sum_{\text{spin,flavor,color}} |\mathcal{M}_{qqqq}|^2 = |\mathcal{M}_\sigma|^2 + |\mathcal{M}_\pi|^2 + 2 \text{Re } (\mathcal{M}_\sigma^* \mathcal{M}_\pi) .$$  (10)

The squared amplitudes for each channel are given by

$$|\mathcal{M}_\sigma|^2 = 16N_c^2 \left[(t - 4m^2)^2 |D_\sigma(k - k')|^2 + (u - 4m^2)^2 |D_\sigma(k - q')|^2 - \frac{1}{2N_c} (4m^2 s - tu) \text{Re } D_\sigma(k - k') D_\sigma^*(k - q') \right],$$

$$|\mathcal{M}_\pi|^2 = 48N_c^2 \left[u^2 |D_\pi(k - k')|^2 + u^2 |D_\pi(k - q')|^2 - \frac{1}{2N_c} tu \text{Re } D_\pi(k - k') D_\pi^*(k - q') \right],$$

$$\text{Re } (\mathcal{M}_\sigma^* \mathcal{M}_\pi^*) = 12N_c \left[u (4m^2 - t) \text{Re } D_\sigma(k - k') D_\pi^*(k - q') + t(4m^2 - u) \text{Re } D_\sigma(k - q') D_\pi^*(k - k') \right],$$  (11)

where $t = (k - k')^2$ and $u = (k - q')^2$ are Mandelstam variables and $D_\sigma$ and $D_\pi$ denote propagators of the $\sigma$ and $\pi$ modes, respectively. We remark that the interference terms are necessary to satisfy the unitarity of $S$ matrix, although they are higher order in the $1/N_c$ expansion. The antiquark-antiquark scattering and the quark-antiquark scattering are obtained by using crossing symmetry: $|\mathcal{M}_{qqqq}|^2(k, q, k', q') = |\mathcal{M}_{qqqq}|^2(-k', -q', -k, -q)$, and $|\mathcal{M}_{qqqq}|^2(k, q, k', q') = |\mathcal{M}_{qqqq}|^2(k, -q', k', -q)$.
Figure 2. The preliminary results of the chiral condensate (left) and the ratio of the shear viscosity to the entropy (right) as a function of temperature.

3. Numerical results
Here we show the numerical result for the chiral condensate and the ratio of the shear viscosity to the entropy, $\eta/s$, as a function of temperature in Fig. 2. We fixed the parameters as cutoff $\Lambda = 650\text{MeV}$, $N_c = 3$, $N_f = 2$, $G_S\Lambda^2 = 4.233$. The ratio $\eta/s$ decreases as temperature increases, and shows apparently no singular behavior even at $T = T_c$, except for a tiny cusp. This cusp is attributed to the vanishing of the on-shell coupling between the quarks and the mesons at $T = T_c$. One might anticipate a much more singular behavior of the viscosity owing to the softening of the $\sigma$ mode at $T = T_c$ in the chiral limit. The $\sigma$-exchange interaction in the $t$-channel reads $|M_\sigma|^2 \sim (t - 4m^2)^2/(t - m_\sigma^2)^2$ around $T = T_c$. Since $m_\sigma = 0$ at $T = T_c$, the denominator vanishes as $t \to 0$ as expected, but so does the numerator in the chiral limit, and hence $|M_\sigma|^2$ is not divergent at $T = T_c$ and $t \to 0$. This situation may change with finite (current) quark mass and chemical potential, in which case there may exist a critical point. In this case, the coupling of quarks to the critical mode that fluctuation diverges might make the shear viscosity singular. This is beyond the scope of the present talk.

4. Summary and outlook
We have calculated the temperature dependence of the shear viscosity in the leading order of $1/N_c$ expansion in the NJL model, using a systematic resummation method. The large fluctuation of the order parameter near $T_c$ affects the shear viscosity through the $\sigma$ meson exchange in the $t$-channel of quark-quark scattering. However, the shear viscosity shows apparently no singular behavior because the dynamical quark mass vanishes at the same time. In reality, beyond the leading order, the fluctuation of the order parameter also couples to hydrodynamics modes, which become important near $T_c$. It is interesting to develop a diagrammatic method taking into account such effects of hydrodynamic mode. We leave this problem in our future work.

Acknowledgments
This work was supported by the Grant-in-Aid for Scientific Research (No. 20540265) and for the Global COE Program “The Next Generation of Physics, Spun from Universality and Emergence” by the Ministry of Education, Culture, Sports, Science and Technology (MEXT) of Japan.

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