Einstein-aether theory with a Maxwell field: General formalism

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We extend the Einstein-aether theory to include the Maxwell field in a nontrivial manner by taking into account its interaction with the time-like unit vector field characterizing the aether. We also include a generic matter term. We present a model with a Lagrangian that includes cross-terms linear and quadratic in the Maxwell tensor, linear and quadratic in the covariant derivative of the aether velocity four-vector, linear in its second covariant derivative and in the Riemann tensor. We decompose these terms with respect to the irreducible parts of the covariant derivative of the aether velocity, namely, the acceleration four-vector, the shear and vorticity tensors, and the expansion scalar. Furthermore, we discuss the influence of an aether non-uniform motion on the polarization and magnetization of the matter in such an aether environment, as well as on its dielectric and magnetic properties. The total self-consistent system of equations for the electromagnetic and the gravitational fields, and the dynamic equations for the unit vector aether field are obtained. Possible applications of this system are discussed. Based on the principles of effective field theories, we display in an appendix all the terms up to fourth order in derivative operators that can be considered in a Lagrangian that includes the metric, the electromagnetic and the aether fields.

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I. INTRODUCTION

The Einstein-aether theory is an alternative theory of gravity in which, in addition to the spacetime metric, there is a non-vanishing everywhere dynamic time-like unit vector field $U^i$ characterizing the velocity of a substratum, the aether (see, e.g., [1–10] for reviews and references).

The Einstein-aether theory has attracted some attention for at least five main motives. First, it is a pure field theory, i.e., a vector-tensor theory of gravity [11] (see also [12, 13]), admitting a rigorous formulation based on a Lagrange formalism. Second, it realizes the idea of a preferred frame of reference (see, e.g., [14–16]) associated with a world-line congruence for which the corresponding time-like four-vector $U^i$ is the tangent vector. Third, this time-like unit vector field $U^i$ can be interpreted as a velocity four-vector of some medium-like substratum (aether, vacuum, dark fluid, and so on), bringing into consideration well-verified ideas and well-elaborated methods from the relativistic theory of non-uniformly moving continuous media and their interactions with other fields, such as the electromagnetic field [17–20]. Fourth, the Einstein-aether theory is also a specific realization of the idea of dynamic self-interaction of complex systems moving with a spacetime dependent macroscopic velocity. Irregularities of the macroscopic motion are known to influence the internal structure of complex systems and evolution of their subsystems (see, e.g., [18]). When we deal with an accelerated expansion of the universe this dynamic self-interaction can produce the same cosmological effects as the ones prescribed to the dark energy, as it was shown in [21]. Fifth, the Einstein-aether theory, since it has a preferred unit vector field, is characterized by a violation of Lorentz invariance. Theories admitting Lorentz invariance violation are widely discussed in the literature. In this instance it is supposed that the quantum gravity scale provides a cutoff for the spacetime continuum, breaking thus, at some stage, Lorentz invariance [22, 26]. Various constraints for the scale of the breaking coming from astrophysical and cosmological observations have been obtained (see, e.g., [22, 26] for reviews, details and references).

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The Einstein-aether theory in the version elaborated by Jacobson and colleagues \[1-10\] is mainly motivated on the grounds that Lorentz symmetry is broken at some scale, and the decomposition of the Lagrangian used in their Einstein-aether theory and in its extensions, can be naturally interpreted in terms of a low energy effective theory \[27-31\]. The rationale and the heuristics for establishing such a low energy effective theory have a parallel to other effective field theories, notably in the effective quantum field theory generated by Goldstone bosons of a chiral theory with a spontaneously broken symmetry \[32,33\].

The Einstein-aether theory \[1-8\] contains four coupling parameters, which are to be estimated in gravitational tests. Indeed, the theory is within the sphere of analysis of three parameterized formalisms used to test gravity theories. These are the PPN (parameterized Post-Newtonian) \[11\] (see also \[7\]), the PPE (parameterized Post-Einsteinian) \[34\] and the PPF (parameterized Post-Friedmann) \[35\] formalisms. The classical PPN formalism is focused on tests in the solar system, whereas the PPE and PPF formalisms deal with strong gravitation. Constraints on the Einstein-aether theory have been already performed \[36,37\].

It is of course of interest to extend the Einstein-aether theory to include other fields. The next most ubiquitous field is the electromagnetic field. When Lorentz symmetry is broken, and somehow a preferred unit vector field pops up leading to an Einstein-aether theory, there are certainly other fields around which would interact with the metric and the aether. One of these fields could be the electromagnetic field as we know it now or some version of a 2-form field appropriate to the primordial universe. In this connection, an Einstein-scalar-aether theory, as an extension to the Einstein-aether theory, has been proposed in \[38\] to examine possible Lorentz invariance violations in an inflationary period. There is also a model, called the bumblebee model, which introduces a Lorentz violating vector field \(B_k\) subject to some potential, and explores the dynamics of its evolution that can be put interacted with the metric field (see, e.g., \[39-41\]). In this perspective, these theories, namely the Einstein-Maxwell-aether theory we propose here, the Einstein-scalar-aether proposed in \[38\], and the bumblebee theory \[39-41\], should be considered as effective field theories generated perhaps from a fundamental quantum gravity operating at the Planck scale, or from some other quantum field theory at a different scale, that has some of its symmetries spontaneously broken at some stage (see \[27-31\] and \[32,33\]). For instance, one might assume that the appropriate quantum fundamental theory, for which the Einstein-Maxwell-aether represents an effective field theory, is a quantum version of theories with effective metrics, i.e., theories that associate and unify through effective metrics, optical, color, and color-acoustic phenomena (see, e.g., \[12-19\] for details and references). In these settings, the classical optical metric is composed of a spacetime metric \(g_{ik}\) and unit vector field \(U_i\), and the metric has the form \(g_{ik} = A_{ik} + BU_iU_k\) for some appropriate scalar functions \(A\) and \(B\). Photon propagation in a medium that has a velocity four-vector \(U^i\) in the given spacetime metric \(g_{ik}\) is equivalent to photon propagation along a geodesic line in an effective spacetime with optical metric \(g_{ik}^\ast\). The corresponding version of quantized theory is not yet elaborated, but the effective metric approach is rather promising at a classical level.

In order to further justify extending the Einstein-aether theory to include the electromagnetic field at a classical level and take into account its interaction with the gravitational field and the dynamic unit vector field, one can consider several settings in which this interaction is important. The first setting is connected with cosmology. The accelerated expansion of the universe makes the aether motion irregular, and so the interaction of electromagnetic waves with a non-uniformly moving aether can change some fine details of the standard history of the relic photons. A refined structure of the relic photon distribution in the framework of the Einstein-Maxwell-aether theory could be tested using WMAP data for the cosmic microwave background radiation. The second situation in which the interaction is important appears in the context of objects with strong gravitation, such as black holes, wormholes and neutron stars. Much of the information we may have from these objects is from electromagnetic radiation coming from their vicinity. The interaction of this electromagnetic radiation with a deformed aether in a strong gravitational field will induce new dynamo-optical effects, which could be tested using observational data. A third situation that might be of relevance in this study is related to gravitational waves. The Einstein-Maxwell-aether theory should also break Lorentz invariance, since the dynamic unit vector field (velocity four-vector of the aether motion) remains one of the basic elements of the extended theory. The Einstein-Maxwell-aether theory, as a theory with a preferred frame of reference, is expected to predict new forms for gravitational wave propagation and consequent detection (here our expectations are connected with generalizations of the results obtained in the works \[36,37\]).

The Einstein-Maxwell-aether model should be experimentally verified. When one deals with an effective field theory coming from some Lorentz symmetry violation process, there are constraints coming from astrophysical and cosmological observations (see, e.g., the data published in \[24\]). Part of these data could be used to test the Einstein-Maxwell-aether theory also. For instance, an analysis of the gamma-ray burst observations (see, e.g., the results of the Fermi Large Area Telescope \[17\]) has shown that the method known as modified photon dispersion relation, gives an estimation for the quantum gravity energy scale \(E_{(QG)}\). The results are \(E_{(QG,1)} > 7.6 \times E_{(\text{Planck})}\) and \(E_{(QG,2)} > 1.3 \times 10^{11}\text{GeV}\) for the linear \(E_{(QG,1)}\) and quadratic \(E_{(QG,2)}\) leading order terms, in the decomposition of the dispersion function \(f(k) = (\omega^2-k^2c^2)\) with respect to the power law terms \(\left(\frac{k\omega}{E_{(QG,n)}}\right)^{2+n}\), \(n = 1,2\), related to corrections
induced by a Lorentz symmetry violation. The Einstein-Maxwell-aether theory predicts effects of electromagnetic polarization rotation which, in principle could be detected in X-ray and γ-ray data.

Our goal is thus to extend the Einstein-aether theory by including a Maxwell electromagnetic coupling to the gravitational field, to the aether time-like unit vector field, and to other matter fields, in short to study the Einstein-Maxwell-aether theory. For this purpose we insert into the Einstein-aether Lagrangian all possible cross-terms, which, on the one hand, are linear and quadratic in the Maxwell tensor and, on the other hand, linear, quadratic and of the second order in the covariant derivative of the aether velocity four-vector, as well as linear in the curvature tensor and its convolutions. In order to classify, in a phenomenological way, the coupling constants appearing in this Einstein-Maxwell-aether theory, we use the decomposition of this covariant derivative of the aether velocity four-vector with respect to its irreducible parts, namely, the acceleration four-vector, the shear and vorticity tensors, and the expansion scalar. This classification includes the set of independent coupling constants related, first, to the effects of induced polarization-magnetization of the matter in the moving aether, and second, to the phenomena associated with optical activity, birefringence, and so on. We should stress that these phenomenological coupling constants can be, in principle, estimated in electromagnetic and gravitational tests, thus extending the schemes of PPE and PPF formalisms. It is then possible to find solutions of our Einstein-Maxwell-aether theory. We give some hints how the symmetries of the theory can be used in some spacetime models, but we do not attempt to find exact or numerical solutions. The Einstein-scalar-aether theory proposed in [38] has interesting inflationary solutions, and displays the possibilities offered by the Einstein-Maxwell-aether theory we are proposing here. The discussion of observational effects of Lorentz invariance violation in our extended Einstein-aether theory, although of importance, is out of framework of this paper.

The paper is organized as follows. In Section II we review the basic elements of the Einstein-aether theory. In Section III based on our action functional for the extended Einstein-Maxwell-aether theory we derive the equations for the electromagnetic, aether time-like unit vector and gravitational fields. We also compare our theory with the bumblebee model and the work of Kostelecky and Mewes. In Section IV we decompose the polarization and magnetization four-vectors and permittivity tensors with respect to the acceleration four-vector, the shear and vorticity tensors, and the expansion scalar, and classify the corresponding coupling constants. We also discuss the Einstein-Maxwell-aether theory for spacetimes of three types, namely, homogeneous isotropic Friedmann cosmological models, static models with spherical symmetry, and plane-wave models. In Section V we draw some conclusions. The Appendix A displays all the terms up to fourth order in derivative operators that can be considered in a Lagrangian formalism of the phenomenological tensors introduced in the theory.

II. EINSTEIN-AETHER THEORY

Einstein’s theory is constructed from a Lagrangian with the metric $g_{ab}$ and its two or fewer derivatives. In order to construct an Einstein-aether theory with a dynamic unit time-like vector field $U^a$ associated to the four-velocity of a background aether one can think in adding to the Einstein theory terms involving $U^a$ and its two or fewer derivatives [1].

The Einstein-aether theory with a dynamic unit time-like vector field associated to the four-velocity of a background aether can be constructed using the following action functional [1]

$$S_{(EA)} = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa} [R + 2\Lambda + \lambda (g_{mn}U^mU^n - 1) + K^{abmn} \nabla_a U_m \nabla_b U_n] + L_{(m)} \right\}. \tag{1}$$

The determinant of the metric $g = \det(g_{ik})$, the Ricci scalar $R$, the cosmological constant $\Lambda$, and the matter Lagrangian $L_{(m)}$ are standard elements of the Einstein-Hilbert action. The new elements appearing in Eq. (1) are the terms involving the vector field $U^i$. The first such term $\lambda (g_{mn}U^mU^n - 1)$ ensures that the $U^i$ is normalized to one, and the second term $K^{abmn} \nabla_a U_m \nabla_b U_n$ is quadratic in the covariant derivative $\nabla_a U_m$ of the vector field $U^i$, with $K^{abmn}$ a tensor field constructed using the metric tensor $g^{ij}$ and the velocity four-vector $U^k$ only,

$$K^{abmn} = C_1 g^{ab} g^{mn} + C_2 g^{am} g^{bn} + C_3 g^{an} g^{bm} + C_4 U^a U^b g^{mn} \tag{2}$$

and where $C_1$, $C_2$, $C_3$ and $C_4$ are the Jacobson constants that can be found from experiments or from some fundamental theory.

The aether dynamic equations and the gravitational field equations are found by varying the action (1) with respect to the vector field $U^i$ and the gravitational field $g^{ij}$, respectively. This procedure is well documented. Nevertheless, we recall some of its details, to make clearer the development we propose, namely, to include the Maxwellian electromagnetic field, and to introduce the nomenclature and the standard definitions and relations. Let us find these equations.
First, the term $\lambda$ is a Lagrange multiplier. The variation of the action (1) with respect to $\lambda$ yields the equation

$$g_{m n} U^m U^n = 1,$$

which is the normalization condition of the time-like vector field $U^k$. Then, variation of the functional (1) with respect to $U^i$ yields that $U^i$ itself satisfies the equation

$$\nabla_m J_{m n}^{(A)} - I^n_{(A)} - \kappa I^n_{(m)} = \lambda U^n. \quad (4)$$

Here we are using the standard definition

$$J_{m n}^{(A)} = K^{l m s n} \nabla_l U_s,$$

and have introduced two four-vectors

$$I^n_{(A)} = \frac{1}{2} \nabla_l U_s \nabla_m U_j \frac{\delta K^{l s m j}}{\delta U_n} = C_i DU_m \nabla^n U^m,$$

and

$$I^n_{(m)} = \frac{\delta L_{(m)}}{\delta U_n}. \quad (7)$$

where $D$ appearing in Eq. (6) is defined as $D \equiv U^i \nabla_i$. In comparison with (1) a new contribution, $-\kappa I^n_{(m)} = -\kappa \frac{\delta L_{(m)}}{\delta U_n}$ has appeared in Eq. (4), since now we are assuming that the unit vector field is coupled to the matter. The Lagrange multiplier has the following form

$$\lambda = \lambda_{(A)} + \kappa \lambda_{(m)} \quad (8)$$

with

$$\lambda_{(A)} = U_n \left[ \nabla_m J_{m n}^{(A)} - I^n_{(A)} \right], \quad (9)$$

and

$$\lambda_{(m)} = -U_n \frac{\delta L_{(m)}}{\delta U_n}. \quad (10)$$

Using the projector $\Delta^j_n$ of tensors into the space orthogonal to $U^i$, $\Delta^j_n \equiv \delta^j_n - U^j_n U^n$, Eq. (4) can be rewritten as

$$\Delta^j_n \left[ \nabla_m J_{m n}^{(A)} - I^n_{(A)} - \kappa I^n_{(m)} \right] = 0. \quad (11)$$

The variation of the action (1) with respect to the metric $g^{i k}$ yields the gravitational field equations in the form

$$R_{i k} - \frac{1}{2} R g_{i k} - \Lambda g_{i k} = T_{i k}^{(U)} + \kappa T_{i k}^{(m)} + \kappa T_{i k}^{(int)}. \quad (12)$$

The term $T_{i k}^{(U)}$ describes the stress-energy tensor associated with the self-gravitation of the vector field $U^i$; it has the form:

$$T_{i k}^{(U)} = C_i \left( \nabla_m U_i \nabla^m U_k - \nabla_i U_m \nabla_k U^m \right) + C_i DU_i DU_k +$$

$$+ \frac{1}{2} g_{i k} J_{m n}^{(A)} \nabla_a U_m + \nabla_m \left[ U_{(i} J_{k)m}^{(A)} \right] - \nabla_m \left[ J_{m (A) U_k}^{(A)} \right] - \nabla_m \left[ J_{m U_k}^{(A)} \right] U^m + U_i U_k U_n \left[ \nabla_m J_{m n}^{(A)} - I^n_{(A)} \right], \quad (13)$$

where $p_i(q_k) = \frac{1}{2} (p_i q_k + p_k q_i)$ denotes symmetrization. The tensor $T_{i k}^{(U)}$ disappears when the motion of the aether is uniform, i.e., $\nabla_i U_k = 0$, and $\frac{\delta L_{(m)}}{\delta U_n} = 0$. The term

$$T_{i k}^{(m)} = -\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{i k}} \left[ \sqrt{-g} L_{(m)} \right]$$

(14)
describes as usual the stress-energy tensor of the matter. The standard algebraic decomposition of this tensor
\[
T_{ik}^{(m)} = W U_i U_k + I_i^{(H)} U_k + I_k^{(H)} U_i + \mathcal{P}_{ik} \tag{15}
\]
introduces the energy density \(W\), the heat-flux four-vector \(I_i^{(H)}\), and the pressure tensor \(\mathcal{P}_{ik}\), which are now determined in the preferred frame of reference associated with the aether velocity four-vector \(U^i\), i.e.,
\[
W = U^p T^{(m)}_{pq} U^q, \quad I_i^{(H)} = \Delta_i^k T^{(m)}_{pq} U^q, \quad \mathcal{P}_{ik} = \Delta_i^k T^{(m)}_{pq} \Delta^q_k = -P \Delta_{ik} + \Pi_{ik}. \tag{16}
\]
Here \(P\) is the Pascal (isotropic) pressure, and \(\Pi_{ik}\) is a non-equilibrium pressure. The last term in (12) is due to the interaction between the unit vector field and the matter and it is described by
\[
T_{ik}^{(int)} = \lambda_{(m)} U_i U_k = -U_i U_k U_n \frac{\delta L^{(m)}}{\delta U^n}. \tag{17}
\]
The compatibility conditions for the set of equations (12)
\[
\nabla^k \left[ T_{ik}^{(U)} + \kappa T_{ik}^{(m)} + \kappa T_{ik}^{(int)} \right] = 0, \tag{18}
\]
involve all three quantities, thus showing that the stress-energy tensor of the matter, \(T_{ik}^{(m)}\), is not itself a conserved quantity because of the coupling of the aether to the matter.

Let us stress, that this interaction term describing a possible coupling between the matter and the unit vector field has to be postulated here, since we intend to generalize the Einstein-aether theory by introduction a coupling between the electromagnetic field and the unit vector field. The interaction term guarantees consistency of the whole theory.

III. EINSTEIN-MAXWELL-AETHER THEORY

A. The inclusion of the Maxwell field and of terms up to fourth order in the derivatives in the Einstein-aether theory and the ansatz

We want to include an electromagnetic gauge vector field \(A_i\) to extend the Einstein-aether theory into an Einstein-Maxwell-aether theory. The corresponding gauge invariant Maxwell tensor \(F_{ik}\) is
\[
F_{ik} = \nabla_i A_k - \nabla_k A_i. \tag{19}
\]

According to the principles of effective field theories (see, e.g., [27–31]) one can establish some interrelations between the terms in the action functional and differential operators of the first, second, and higher orders. In the Appendix A we give the complete set of terms that could be included in the action if one selects terms with derivatives up to the fourth order. The theory that we propose does not require all these terms, since we impose further requirements.

Indeed, to set our ansatz we impose three requirements that our theory should satisfy:
(a) The electrodynamics of the theory must be linear in the Maxwell tensor \(F_{ik}\) and of second order in the partial derivatives of the electromagnetic potential four-vector \(A_i\).
(b) The dynamical equations for the unit vector field \(U^i\) are considered to be a set of quasilinear equations of second order in their partial derivatives. A note is in order: According to the standard terminology in mathematical physics quasilinear means that the equations can be nonlinear in the four-vector \(U^i\) itself, nonlinear in the first covariant derivative \(\nabla_i U_k\), but the second partial derivatives \(\partial_i \partial_k U^s\) enters the equations linearly with tensorial coefficients that can depend on \(U^i\) and \(F_{mn}\), but can not contain \(\nabla_i U_k\).
(c) The equations for the gravitational field are considered to be equations of second order in the partial derivatives of the metric (similarly to the standard Einstein’s and Einstein-aether theories).

This ansatz (composed of requirements (a), (b), and (c)) can be reformulated as the assumption that the discarded terms have coefficients, phenomenologically introduced, that are small enough in comparison with the non-discarded coupling constants.

In some sense, we have followed the rationale used for the Einstein-aether theory, namely, that for regions where quantum gravity is not anymore dominant and Lorentz symmetry is already broken by those very quantum effects an Einstein-aether theory can naturally appear [1]. Thus the energy scale for this model is below the Planck scale. It is clear, that the quantum theory behind the Einstein-aether theory, as an effective low energy theory, is still beyond grasp. In the same way that one expects that general relativity is a low energy phenomenon from some quantum gravity, or other fundamental theory, one also expects that the Einstein-aether theory is a low energy phenomenon of
such a theory. Which of the theories is the correct one, experiments can in principle tell. Now, as Solomon and Barrow have recently proposed, such an Einstein-aether theory might be operating at the inflationary energy scale. Thus an Einstein-scalar-aether in this setting is in action from after the Planck scale to the inflationary scale and after. Since a generic 2-form field, like the Maxwell field, can appear at very high energy scales an Einstein-Maxwell-aether could be in operation between these energies and the Planck scale. Or it could be in playing after inflation decays and the matter fields, such as the Maxwell field, make their appearance.

**B. Action functional**

Keeping in mind the restrictions discussed in the Sec. III A, we start with the following ansatz for the action functional

\[ S_{\text{(total)}} = S_{\text{(EA)}} + S_{\text{(EMA1)}} + S_{\text{(EMA2)}} + S_{\text{(EMA3)}} + S_{\text{(NM1)}} + S_{\text{(NM2)}}. \]  

(20)

This Einstein-Maxwell-aether action contains five new terms relative to the Einstein-aether action of Eq. (1). These new terms are \( S_{\text{(EMA1)}} \), \( S_{\text{(EMA2)}} \), \( S_{\text{(EMA3)}} \), \( S_{\text{(NM1)}} \) and \( S_{\text{(NM2)}} \).

The first additional term

\[ S_{\text{(EMA1)}} = \frac{1}{2} \int d^4 x \sqrt{-g} \left[ A^{mnpq} \nabla_m U_n + B^{mnlpq} (\nabla_l U_s)(\nabla_m U_n) \right] F_{pq} \]  

(21)

is linear in the Maxwell tensor and does not contain the Riemann tensor and its convolutions. It contains the covariant derivative of the aether velocity four-vector.

The second additional term

\[ S_{\text{(EMA2)}} = \frac{1}{4} \int d^4 x \sqrt{-g} \left\{ \frac{1}{\mu} \left[ F_{ik} F^{ik} + 2(\varepsilon \mu - 1) F_{im} U^m U_n + X^{mnkpa} + Y^{mnkpa} (\nabla_l U_s)(\nabla_m U_n) \right] F_{ik} F_{pq} \nabla_m U_n \right\} \]  

(22)

is quadratic in \( F_{mn} \) and does not contain the Riemann tensor. It contains the covariant derivative of the aether velocity four-vector. The scalar quantities \( \varepsilon \) and \( \mu \) are the dielectric and magnetic permittivities, respectively, of the matter immersed in the aether. They are equal to one if one deals with pure aether.

The third term

\[ S_{\text{(EMA3)}} = \frac{1}{4} \int d^4 x \sqrt{-g} F_{pq} [2B^{mplq} \nabla_m U_s + Y^{mplq} F_{ik} \nabla_m (\nabla l U_s)] \]  

(23)

contains a second covariant derivative of the unit vector field. The tensor quantities \( A^{mnpq} \), \( B^{mplq} \), \( B^{mplq} \), \( X^{mnkpa} \), \( Y^{mnkpa} \), \( Y^{mnkpa} \), and \( Y^{mnkpa} \) describe electrodynamic properties of the matter in the moving aether. They are constructed using the metric \( g_{ik} \), the covariant constant Kronecker tensors (\( \delta_{ik} \), \( \delta_{ik}^{ab} \) and higher order Kronecker tensors), the Levi-Civita tensor \( \epsilon^{ikab} \), and the unit vector field \( U^k \).

The first nonminimal term is

\[ S_{\text{(NM1)}} = \frac{1}{4} \int d^4 x \sqrt{-g} R^{ikmn} F_{ik} F_{mn}, \]

(24)

and does not contain the unit vector field \( U^k \). Here,

\[ R^{ikmn} = q_1 R g^{ikmn} + q_2 R g^{ikmn} + q_3 R^{ikmn} \]

(25)

is the nonminimal susceptibility tensor with

\[ g^{ikmn} \equiv \frac{1}{2} (g^{im} g^{kn} - g^{in} g^{km}), \]

(26)

and

\[ R^{ikmn} \equiv \frac{1}{2} (R^{i} g^{mn} - R^{mn} g^{ik} + R^{km} g^{im} - R^{km} g^{in}). \]

(27)

The constants \( q_1 \), \( q_2 \) and \( q_3 \) are nonminimal parameters describing a linear coupling of the Maxwell tensor \( F_{mn} \) with the curvature (see, e.g., [49] for details).
The second nonminimal term can be represented as

\[ S_{(\text{NM2})} = \frac{1}{4} \int d^4x \sqrt{-g} \left\{ S_{ikmnsiq} F_{is} R_{ikmn} F_{pq} + 2Q F_{pq} R^p_k U^q U^r \right\}, \tag{28} \]

and includes the terms listed in Eqs. (A14) and (A27).

After displaying all the terms of importance, we can now display the total action functional. It is given by

\[ S_{(\text{total})} = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa} \left[ R + 2\Lambda + \lambda (g_{mn}U^m U^n - 1) + K_{\alpha \beta \gamma \delta} (\nabla_\alpha U_\beta) (\nabla_\gamma U_\delta) \right] + L_{(m)} + \right. \]

\[ + \frac{1}{2} \left[ A^{mnpq} + B^{mnspq} (\nabla_i U_s) \right] F_{pq} \nabla_m U_n + \frac{1}{2} Q R^p_k U^q U^r F_{pq} + \]

\[ + \frac{1}{4} R^{ikmn} F_{ik} F_{mn} + \frac{1}{4} S_{ikmnsiq} F_{is} R_{ikmn} F_{pq} + \frac{1}{4} \left[ 2B^{mlspq} + Y^{mlskpq} F_{ik} \right] F_{pq} \nabla_m (\nabla_l U_s) + \]

\[ + \frac{1}{4} \left[ F_{ik} F^{ik} + 2(\varepsilon\mu - 1)F_{im} U^m F_{in} U^n \right] + \frac{1}{4} \left[ X^{mnikpq} + Y^{mlskpq} (\nabla_i U_s) \right] (\nabla_m U_n) F_{ik} F_{pq} \right\}, \tag{29} \]

The coefficients involved into the decompositions of these quantities can be interpreted as coupling constants (see for the interpretation of the nonminimal coupling constants \( q_1, q_2, q_3 \), and Section IV for the interpretation of other coupling constants).

Given the action, Eq. (29), we can now obtain the electrodynamic equations, the aether dynamic equations and the gravitational field equations, by variation of the corresponding appropriate quantities.

### C. Electrodynamic equations

The electrodynamic equations can be obtained by variation of the action (29) with respect to the electromagnetic potential four-vector \( A_i \), which enters the Maxwell tensor \( F_{ik} \) via \( F_{ik} = \nabla_i A_k - \nabla_k A_i \). The result of the variational procedure can be written in the following standard form

\[ \nabla_k H^{ik} = -4\pi I^i, \tag{30} \]

where \( H^{ik} \) is the excitation tensor linear in the Maxwell tensor and given by

\[ H^{ik} = \mathcal{H}^{ik} + C^{ikmn} F_{mn}, \tag{31} \]

where \( \mathcal{H}^{ik} \), \( C^{ikmn} \), and \( I^i \) have their own physical meanings. The skew-symmetric tensor \( \mathcal{H}^{ik} \) is given by

\[ \mathcal{H}^{ik} = A^{mnik} \nabla_m U_n + B^{mlsk} (\nabla_i U_s) (\nabla_m U_n) + B^{mlsk} \nabla_m (\nabla_l U_s) U^l U_n + Q U_m R^m [U^k], \tag{32} \]

and describes the spontaneous polarization-magnetization of the matter influenced by the moving aether. The tensor \( C^{ikmn} \) is a linear response tensor, and in turn, can be decomposed into four terms, namely,

\[ C^{ikmn} = C^{ikmn}_{(0)} + C^{ikmn}_{(D)} + C^{ikmn}_{(R)} + C^{ikmn}_{(DD)}. \tag{33} \]

The first term is given by

\[ C^{ikmn}_{(0)} = \frac{1}{2\mu} \left[ (g_{im} g_{kn} - g_{in} g_{km}) + (\varepsilon\mu - 1) (g_{im} U^k U^n - g_{in} U^k U^m + g_{kn} U^i U^m - g_{km} U^i U^n) \right]. \tag{34} \]

It contains the four-vector \( U^i \) but does not include the covariant derivative \( \nabla_m U_n \). The second term is given by

\[ C^{ikmn}_{(D)} = X^{liskmn} \nabla_l U_s + Y^{abliskmn} (\nabla_a U_b) (\nabla_l U_s). \tag{35} \]

It contains terms linear and quadratic in the covariant derivative of the aether velocity four-vector \( U^i \). The third term is given by

\[ C^{ikmn}_{(R)} = R^{ikmn} + S^{pqabikmn} R_{pqab}. \tag{36} \]
It is a generalized nonminimal susceptibility tensor. The fourth term is given by

\[ C^{ikmn}_{(DD)} = \mathcal{Y}^{\alpha i} \nabla_{\alpha} \nabla_{\beta} U_{\beta} . \]  

(37)

It relates to the linear response induced by a second covariant derivative of the unit vector field \( U^i \). The electric current four-vector \( I^i \), appearing in the right-hand side of Eq. (30), is defined as follows

\[ I^i = \frac{1}{4\pi} \frac{\partial L_m}{\partial A_i} . \]  

(38)

As usual, we have to add the Maxwell equation

\[ \nabla_k F^{*ik} = 0 , \]  

(39)

where the asterisk indicates dualization, i.e.,

\[ F^{*ik} = \frac{1}{2} \epsilon^{ikmn} F_{mn} . \]  

(40)

Here \( \epsilon^{ikmn} = \epsilon^{ikmn} \sqrt{-g} \) is the Levi-Civita tensor with \( \epsilon^{ikmn} \) being the completely skew-symmetric symbol (\( \epsilon^{0123} = 1 \)).

Electrodynamics of continuous media can be formulated in terms of four-vectors representing physical fields. These four-vector fields are the electric field \( E^i \), the magnetic field \( H^i \), the electric excitation \( D^i \), and the magnetic excitation \( B^i \) [19]: They are defined in terms of \( F^{ik} \) and \( H^{ik} \) as

\[ E^i = F^{ik} U_k , \quad B^i = F^{*ik} U_k , \quad D^i = H^{ik} U_k , \quad H^i = H^{*ik} U_k . \]  

(41)

Other quantities useful to interpret the phenomenological coupling constants that appear in this formalism are the polarization vector, \( P^i \), and the magnetization vector, \( M^i \). These two four-vectors are defined as

\[ P^i = D^i - E^i , \quad M^i = H^i - B^i , \]  

(42)

respectively. The electrodynamic equations in terms of these quantities contain covariant derivatives of the velocity four-vector \( U^i \). The corresponding equations are written in the Appendix B.

To be complete, inverting Eq. (41), we find that the tensors \( F^{ik} \) and \( H^{ik} \) can be written in terms of \( E^i, B^i, D^i, \) and \( H^i \) as

\[ F^{ik} = \delta^{ik}_{mn} E^m U^n - \epsilon^{ikmn} B^m U^n , \quad H^{ik} = \delta^{ik}_{mn} D^m U^n - \epsilon^{ikmn} H^m U^n , \]  

(43)

where \( \delta^{ik}_{mn} \) and \( \epsilon^{ikmn} \) are the generalized Kronecker delta and the Levi-Civita tensor, respectively.

### D. Aether dynamic equations

The dynamic equations for the aether are obtained by the variation of the action functional [29] with respect to the velocity four-vector \( U^i \). The variation procedure yields four equations

\[ \nabla_m \left[ J^{mn}_{(A)} + \kappa J^{mn}_{(M)} \right] = I^n_{(A)} + \kappa I^n_{(m)} + \kappa I^n_{(M)} + \lambda \ U^m . \]  

(44)

The Lagrangian multiplier is now

\[ \lambda = \lambda_{(A)} + \kappa \lambda_{(m)} + \kappa \lambda_{(M)} , \]  

(45)

where \( \lambda_{(A)} \) and \( \lambda_{(m)} \) are defined in Eqs. (9)-(10), respectively, and

\[ \lambda_{(M)} = U_n \left[ \nabla_m J^{mn}_{(M)} - I^n_{(M)} \right] . \]  

(46)

Eliminating the Lagrange multiplier \( \lambda \) in Eq. (44) one obtains the following compact equation,

\[ \Delta_n \left\{ \nabla_m \left[ J^{mn}_{(A)} + \kappa J^{mn}_{(M)} \right] - \left[ I^n_{(A)} + \kappa I^n_{(m)} + \kappa I^n_{(M)} \right] \right\} = 0 . \]  

(47)
The quantities $J'_{mn}$, $I'_{in}$, $T'_{mn}$, in Eqs. (44-47), are defined in Eqs. (5-17), respectively. The other two quantities that appear in Eqs. (44-47) are defined as,

$$J'_{(M)} = \frac{1}{2} F_{pq} \left( A^{mnpq} + 2B^{mnlspq} \nabla_l U_s \right) -$$

$$- \frac{1}{4} \nabla_l \left[ \left( 2B^{(lm)npq} + \gamma^{(lm)nijkp} F_{ik} \right) F_{pq} \right] + \frac{1}{4} F_{ik} F_{pq} \left( X^{mnikpq} + 2\gamma^{mnlsikpq} \nabla_i U_s \right),$$  \hspace{1cm} (48)

$$I'_{(M)} = \left( \varepsilon - \frac{1}{\mu} \right) F^{kn} F_{km} U^m + \frac{1}{2} F_{pq} \nabla_l U_s \left( \frac{\delta A_{spq}}{\delta U_n} + \nabla_m U_j \frac{\delta B_{jslpq}}{\delta U_n} \right) +$$

$$+ \frac{1}{4} F_{ik} F_{pq} \nabla_l U_s \left( \frac{\delta X^{lsikpq}}{\delta U_n} + \nabla_m U_j \frac{\delta Y^{mjsikpq}}{\delta U_n} \right) + \frac{1}{4} R_{iklmjn} F_{ls} F_{pq} \frac{\delta}{\delta U_n} S^{ikmlspq} +$$

$$+ \frac{1}{4} F_{pq} \left[ 2\frac{\delta}{\delta U_n} B^{mnlspq} + F_{ik} \frac{\delta}{\delta U_n} \gamma^{mnlspq} \right] \nabla_m \nabla_l U_s + \frac{1}{2} QU^l \left( R^{nm} F_{ml} - F^{nm} R_{ml} \right).$$  \hspace{1cm} (49)

E. The gravitational field equations

1. The general equations

The variation of the action functional $\delta [S]$ with respect to the metric $g^{ik}$ yields

$$R_{ik} - \frac{1}{2} R g_{ik} - \Lambda g_{ik} = T^{(U)}_{ik} + \kappa \left[ T^{(m)}_{ik} + T^{(int)}_{ik} + T^{(EM0)}_{ik} + T^{(EMA1)}_{ik} + T^{(EMA2)}_{ik} + T^{(EMA3)}_{ik} + T^{(NM1)}_{ik} + T^{(NM2)}_{ik} \right].$$  \hspace{1cm} (50)

The terms $T^{(U)}_{ik}$, $T^{(m)}_{ik}$, and $T^{(int)}_{ik}$ are given by the formulas (13), (14)-(16), and (17), respectively. Let us discuss in detail the new elements of this decomposition.

2. Stress-energy tensor of the electromagnetic field in a uniformly moving aether

The part of the stress-energy tensor of the electromagnetic field indicated as $T^{(EM0)}_{ik}$ is given by

$$T^{(EM0)}_{ik} = \frac{1}{\mu} \left\{ \frac{1}{4} g_{ik} F_{mn} F^{mn} - F_{im} F^{'m}_{k} \right\} + \left( \varepsilon \mu - 1 \right) U^p U^q \left[ \left( \frac{1}{2} g_{ik} - U_i U_k \right) F^{mp} F_{mq} - F_{ip} F_{kp} \right].$$  \hspace{1cm} (51)

In vacuum, $\varepsilon = \mu = 1$, $T^{(EM0)}_{ik}$ gives the usual Maxwell term. Clearly, the tensor (51) is symmetric and traceless, i.e.,

$$T^{(EM0)}_{ik} = T^{(EM0)}_{ki}, \quad T^{(EM0)}_{ik} g^{ik} = 0.$$  \hspace{1cm} (52)

Other interesting quantities are connected with the energy density scalar $W^{(EM0)}_{ik}$ and with the energy flux four-vector $I^{(EM0)}_{ik}$ associated with this tensor. They are related to $T^{(EM0)}_{pq}$ and defined as

$$W^{(EM0)}_{ik} \equiv U^p T^{(EM0)}_{pq} U^q = - \frac{1}{2} \left( \varepsilon E^m E_m + \frac{1}{\mu} B^m B_m \right),$$  \hspace{1cm} (53)

$$I^{(EM0)}_{ik} \equiv \Delta^{ip} T^{(EM0)}_{pq} U^q = - \epsilon^{imns} U_s E_m \mathcal{H}_n.$$  \hspace{1cm} (54)

Clearly, $W^{(EM0)}_{ik}$ coincides with the standard definition of the energy density scalar in a spatially isotropic medium \cite{19}, and $I^{(EM0)}_{ik}$ coincides with the Poynting vector. All these properties allow us to identify this tensor $T^{(EM0)}_{iq}$ with the stress-energy tensor of the electromagnetic field in the Abraham version \cite{17} (see also, \cite{19}). Thus it is interesting to note that, on the one hand, the tensor $T^{(EM0)}_{ik}$ is an effective stress-energy tensor, since it is obtained by variation with respect to the metric \cite{48}, on the other hand, it coincides with the Abraham tensor, which appears from an analysis of the balance equations in the electrodynamics of a moving continuous medium \cite{17}. 

3. Stress-energy tensor associated with a spontaneous polarization-magnetization of matter or vacuum induced by a non-uniformly moving aether

The quantity \( T_{ik}^{(EMA1)} \) appearing in Eq. [50] is given by

\[
T_{ik}^{(EMA1)} = \frac{1}{2} g_{ik} F_{pq} \left( A_{ik}^{pq} + 2 B_{ik}^{lsmpq} \nabla_l U_s \right) \nabla_m U_n - \frac{1}{2} \nabla_m \left\{ F_{pq} U^m \left[ A_{(ik)}^{pq} + 2 B_{(ik)}^{lsmpq} \nabla_l U_s \right] \right\} - \\
\frac{1}{2} \nabla_m \left\{ F_{pq} \left[ U_i A_k^{mpq} + 2 U_i B_k^{mpq} \nabla_l U_s \right] \right\} + \frac{1}{2} \nabla_m \left\{ F_{pq} \left[ U_i A_k^{mpq} + 2 U_i B_k^{mpq} \nabla_l U_s \right] \right\} + \\
\frac{1}{2} U_i U_k U_n \left\{ \nabla_m \left[ F_{pq} \left( A_{ik}^{mpq} + 2 B_{ik}^{mpq} \nabla_l U_s \right) \right] - F_{pq} \nabla_l U_s \left( \frac{\delta}{\delta U_n} A_{ik}^{mpq} + \nabla_m U_j \frac{\delta}{\delta U_n} B_{ik}^{mpq} \right) \right\} - \\
- F_{pq} \nabla_l U_s \left( \frac{\delta}{\delta U_n} A_{ik}^{mpq} + \nabla_m U_j \frac{\delta}{\delta U_n} B_{ik}^{mpq} \right), \tag{55} \]

This stress-energy tensor is linear in the Maxwell tensor \( F_{pq} \) and therefore is generated by a spontaneous polarization-magnetization induced in the system due to a non-uniformly moving aether.

4. Stress-energy tensor of the electromagnetic field, quadratic in the Maxwell tensor and quadratic in the vector field covariant derivative

The quantity \( T_{ik}^{(EMA2)} \) appearing in Eq. [50] is given by

\[
T_{ik}^{(EMA2)} = \frac{1}{4} g_{ik} F_{ab} F_{pq} \left( X_{i(k)}^{mabpq} + Y_{i(k)}^{mabpq} \nabla_l U_s \right) \nabla_m U_n - \\
\frac{1}{4} \nabla_m \left\{ F_{ab} F_{pq} U^m \left[ X_{(ik)}^{mabpq} + 2 Y_{(ik)}^{mabpq} \nabla_l U_s \right] \right\} - \frac{1}{4} \nabla_m \left\{ F_{ab} F_{pq} \left[ U_i X_k^{mabpq} + 2 U_i Y_k^{mabpq} \nabla_l U_s \right] \right\} + \\
\frac{1}{4} \nabla_m \left\{ F_{ab} F_{pq} \left[ U_i X_k^{mabpq} + 2 U_i Y_k^{mabpq} \nabla_l U_s \right] \right\} + \frac{1}{4} U_i U_k U_n \left\{ \nabla_m \left[ F_{ab} F_{pq} \left( X_{i(k)}^{mabpq} + 2 Y_{i(k)}^{mabpq} \nabla_l U_s \right) \right] - \\
- F_{ab} F_{pq} \nabla_l U_s \left( \frac{\delta}{\delta U_n} X_{i(k)}^{mabpq} + \nabla_m U_j \frac{\delta}{\delta U_n} Y_{i(k)}^{mabpq} \right) \right\} \right\} \frac{1}{2} F_{ab} F_{pq} \nabla_m U_n \left( \frac{\delta}{\delta g^{ik}} X_{i(k)}^{mabpq} + \nabla_l U_s \frac{\delta}{\delta g^{ik}} Y_{i(k)}^{mabpq} \right), \tag{56} \]

This \( T_{ik}^{(EMA2)} \) is quadratic in the Maxwell tensor and quadratic in the vector field covariant derivative. Note that, in general, \( T_{ik}^{(EMA1)} \) and \( T_{ik}^{(EMA2)} \) are not traceless and are not conserved quantities.

5. Stress-energy tensors of the electromagnetic field in the aether environment, nonminimally coupled to gravity

In Eq. [50], there are three nonminimal terms included in the total stress-energy tensor, namely \( T_{ik}^{(NM1)} \), \( T_{ik}^{(NM2)} \), and \( T_{ik}^{(EMA3)} \). We assume these terms have a negligible contribution in the gravitational field and do not discuss them further.

F. Remarks

1. Analogy with the bumblebee model

The set of master equations just derived points to an analogy to the bumblebee model \([39][41]\). The bumblebee model introduces a Lorentz violating vector field \( B_k \) in such a way that a scalar potential \( V(B_k B_k) \) is inserted into
Comparing the bumblebee model with the Einstein-Maxwell-aether theory we propose here we mention three important facts. First, the Einstein-Maxwell-aether model deals with two independent vector fields, namely, the unit four-vector $U^i$ and the potential four-vector $A_k$; on the other hand, the bumblebee model contains a unique vector field $B_k$. Second, the Einstein-Maxwell-aether theory describes an aether field plus a Maxwell field, while with the bumblebee model is suited to study or the aether, or the Maxwell field. Third, in our Lagrangian, we have considered terms in the square of the covariant derivatives of the vector field $U_k$, terms for the pure electromagnetic field, and then the cross-terms, which contain both $\nabla_i U_k$ and $F_{ik}$. These cross-terms do not appear in the original bumblebee model. The effects that come from the coupling of a non-uniformly moving aether with the electromagnetic field are specially interesting in the Einstein-Maxwell-aether theory.

2. Comparison with the work of Kostelecky and Mewes

In a quite general theory, from which the standard model extension is incorporated, Kostelecky et. al. [22] display the Lorentz violating terms containing tensor coefficients. It is of importance to give a comparison of the terms we use in our action Eq. (20) and subsequent equations with the terms given by Kostelecky and Mewes [22]. In particular, let us compare briefly the structure of these Lorentz violating coefficients of [22], which are formally similar to the ones introduced in our work (see, e.g., our Eqs. (22), (23)), and emphasize the novelty of our approach.

First, is it possible to extract all our tensor coefficients from the terms $(k_{DF}^{kl})$, and $(k_{EF}^{kl})\kappa_{\alpha\mu}$ appearing in Eqs. (9) and (10) of the paper [22] with derivative operators of zero order? The answer is negative, since our tensors in Eqs. (22) and (23) contain covariant derivatives of the aether velocity unit four-vector $\nabla_i U_k$, namely, terms of second order $(\nabla_i U_k)(\nabla_m U_n)$ and $\nabla_i \nabla_k U_m$. That is why in our case we deal, in fact, with tensors of coefficients possessing five and six indices in those terms. In contrast, in [22] the authors use coefficients with three and four indices. The physical interpretations of these coefficients differ indeed, and the strategies to their experimental verification also do not coincide.

Second, there is a similarity in that the Lagrangians in both works are quadratic in the electromagnetic potential $A_i$. In the paper [22] one can find terms of the type $A_i \partial_{\alpha_1}...\partial_{\alpha_2} A_{i_2}$ (see, e.g., Eq. (1) in [22]). There are also gauge-invariant terms, in which $A_m^i$ is replaced by $F_{\mu\nu}$ (see, e.g., Eqs. (8), (9), and (10) in [22]). In the Appendix A we give the terms cubic and of the fourth order in $F_{mn}$, but when we set up our ansatz we discard these terms cubic and of the fourth order in $F_{mn}$, and thus consider the terms linear and quadratic in $F_{mn}$.

Third, in our work we excluded from the action functional all the terms that contain derivatives of the Maxwell tensor $\nabla_k F_{mn}$. These terms disappear either through an integration by parts, or by using our ansatz. In the work [22] the derivatives of the Maxwell tensor enter the basic decomposition as an essential part. So in this particular item, our classification can be considered as a subclassification of the terms that are indicated as zero-order in the derivatives of $F_{ik}$ in the work [22].

IV. CLASSIFICATION OF THE COUPLING CONSTANTS

A. Motivation

The Einstein-Maxwell-aether theory proposed here contains several coupling constants. Since the theory has been constructed in a phenomenologically manner, these couplings have to be estimated and determined experimentally. In addition, underlying symmetries of the models under study can be of help in finding some of the coupling constants.

How many constants should we consider as key parameters for the theory? In order to clarify this question, let us start with the known discussion in the Einstein-aether theory about the number of independent constants appearing in the formulation of the tensor $K^{abmn}$. This tensor possesses the following symmetry of indices: $K^{abmn} = K^{bnma}$. Also, since $U^m \nabla_m U_m = 0$, a number of components of $K^{abmn}$ can be connected by the relations $K^{abmn} U_m = 0 = K^{abmn} U_n$. Thus, there are, in principle, $\frac{1}{2} \times 12 \times 13 = 78$ independent components for this tensor. However, usually one deals with only four coupling constants $C_1$, $C_2$, $C_3$, and $C_4$, related to $K^{abmn}$. We now recall how this problem is solved in the Einstein-aether theory and then use the same idea in the extended Einstein-Maxwell-aether theory.
B. Decomposition of $\nabla _i U_k$ and interpretation of Jacobson’s coupling constants

The tensor $\Psi _{ik} \equiv \nabla _i U_k$ can be decomposed, as usual, into a sum of its irreducible parts, namely, the acceleration four-vector $DU^i$, the shear tensor $\sigma _{ik}$, the vorticity tensor $\omega _{ik}$, and the expansion scalar $\Theta$. The decomposition is given by

$$\Psi _{ik} = \nabla _i U_k = U_i DU_k + \sigma _{ik} + \omega _{ik} + \frac{1}{3} \Delta _{ik} \Theta,$$  \hspace{1cm} (57)

where the basic quantities are defined as

$$U_i DU_k \equiv U_i U^m \nabla _m U_k, \quad \sigma _{ik} \equiv \frac{1}{2} \Delta _i ^m \Delta _k ^n (\nabla _m U_n + \nabla _n U_m) - \frac{1}{3} \Delta _{ik} \Theta,$$

$$\omega _{ik} \equiv \frac{1}{2} \Delta _i ^m \Delta _k ^n (\nabla _m U_n - \nabla _n U_m), \quad \Theta \equiv \nabla _m U^m, \quad D \equiv U^i \nabla _i.$$  \hspace{1cm} (58)

Now, when we construct the scalar $K^{abmn}(\nabla _a U_m)(\nabla _b U_n)$ using the unit vector field $U^m$ itself and the geometric quantities $g^{ik}$, $\delta _k ^i$, $\delta _{ik} ^m$, $\epsilon ^{ikmn}$, we find that there are 4 and only 4 non-vanishing independent second-order scalars expressed in terms of $DU_i$, $\sigma _{mn}$, $\omega _{mn}$, $\Theta$. These are $DU_i DU_k$, $\sigma _{ik} \sigma ^{ik}$, $\omega _{ik} \omega ^{ik}$ and $\Theta ^2$ \[10\]. Clearly, non-vanishing cross-terms cannot be constructed. Thus, in these terms the scalar $K^{abmn}(\nabla _a U_m)(\nabla _b U_n)$ which appears in the action functional given by Eq. \[29\] should be represented as

$$K^{abmn}(\nabla _a U_m)(\nabla _b U_n) = (C_1 + C_4) DU_k DU^k + (C_1 + C_3) \sigma _{ik} \sigma ^{ik} + (C_1 - C_3) \omega _{ik} \omega ^{ik} + \frac{1}{3} (C_1 + 3C_2 + C_4) \Theta ^2.$$  \hspace{1cm} (59)

This shows explicitly that there are only four independent Jacobson’s coupling constants. Note that the squared acceleration, shear, vorticity and expansion terms enter the scalar \[59\] with equal weight (i.e., with equal coefficients), when $C_2 = C_3 = C_4 = 0$ and $C_1$ is free. In this particular symmetric situation we obtain the case analyzed in \[21\], namely,

$$K^{abmn}(\nabla _a U_m)(\nabla _b U_n) = C \Psi _{ik} \Psi ^{ik} \equiv C \Psi ^2 = C \left[ DU_k DU^k + \sigma _{ik} \sigma ^{ik} + \omega _{ik} \omega ^{ik} + \frac{1}{3} \Theta ^2 \right],$$  \hspace{1cm} (60)

where $C \equiv C_1$. Thus, we have shown that the maximal number of components of the tensor $K^{abmn}$ is 78, Jacobson’s theory admits 4 independent components, and this number can be reduced to one in the case with high symmetry mentioned above.

C. Coupling constants related to a spontaneous polarization-magnetization of the matter or vacuum induced by an aether non-uniform motion

1. Preliminary analysis

We now analyze the spontaneous polarization-magnetization tensor $\mathcal{H}^{ik}$. This tensor splits naturally into four terms (see Eq. \[52\]). The first, second, and third terms, have as multipliers the tensors $A^{kmn}$, $B^{kmnpq}$, $B^{(ml)sk}$, respectively, which depend on metric $g_{ik}$, the covariant constant Kronecker tensors $(\delta _k ^i$, $\delta _{ik} ^m$ and higher order Kronecker tensors), the Levi-Civita tensor $\epsilon ^{ikab}$, and the unit vector field $U^k$. The fourth term has $Q$ as a multiplier and due to its simplicity does not require special consideration. To analyze the first two terms, i.e., those containing $A^{kmn}$ and $B^{kmnpq}$, we use a decomposition of the polarization $\mathcal{P}^m$ and magnetization $\mathcal{M}^m$ four-vectors with respect to the irreducible parts of the covariant derivative of the unit vector field $U^k$. This approach is useful as it gives a direct method of interpretation of the corresponding coupling constants. In the analysis of the third term $B^{(ml)sk}$ in Eq. \[52\], we follow another route, as this method of using the decomposition of $\mathcal{P}^m$ and $\mathcal{M}^m$ is not effective since $B^{(ml)sk}$ contains a second covariant derivative of the velocity four-vector. Instead, we use the standard decomposition of $B^{(ml)sk}$ with respect to the metric $g_{ik}$, the covariant constant Kronecker tensors $(\delta _k ^i$, $\delta _{ik} ^m$ and higher order Kronecker tensors), the Levi-Civita tensor $\epsilon ^{ikab}$, and the unit vector field $U^k$.

Generically, the skew-symmetric tensor $\mathcal{H}^{ik}$ appearing in Eq. \[52\] can be represented as (see, e.g., \[19\])

$$\mathcal{H}^{ik} = \tilde{g}^{ik} U^m \mathcal{P}^m - \epsilon ^{ikmn} U_n \mathcal{M}^m,$$  \hspace{1cm} (61)
where $\mathcal{P}^m$ is the spontaneous polarization four-vector and $\mathcal{M}^m$ is the spontaneous magnetization pseudo four-vector. One can invert the relation (61) and find

$$\mathcal{P}^i \equiv \mathcal{H}^{ik}U_k, \quad \mathcal{M}^i \equiv \frac{1}{2} \epsilon^{ikmn}\mathcal{H}_{mn}U_k.$$  

The four-vectors $\mathcal{P}^m$ and $\mathcal{M}_m$ are orthogonal to the velocity four-vector, i.e.,

$$\mathcal{P}^i U_i = 0, \quad \mathcal{M}_i U^i = 0,$$

and this fact simplifies the decomposition of these quantities with respect to irreducible parts of the covariant derivative of the velocity four-vector (57). Our scheme of analysis and interpretation of the coupling constants is the following: we decompose the four-vectors $\mathcal{P}^i$ and $\mathcal{M}^i$ with respect to $DU^i$, $\sigma^{ik}$, $\omega^{jk}$, $\Theta \Delta^{ik}$ using unknown coupling constants, and then reconstruct $\mathcal{H}^{ik}$ using (61).

Note that the term spontaneous, in spontaneous polarization and spontaneous magnetization, is being used following the terminology of classical electrodynamics. One usually distinguishes between non-spontaneous polarization or non-spontaneous magnetization induced by an electromagnetic field on one hand, and the spontaneous polarization or spontaneous magnetization of non-electromagnetic origin, produced, e.g., by medium deformation or heating, on the other hand. In this sense, the term spontaneous is appropriate for spontaneous polarization or spontaneous magnetization caused by the interaction of the medium with a non-uniformly moving aether.

2. Reconstruction of the tensors $A^{mnik}$ and $B^{mnlsik}$

(a) Polarization-magnetization linear in the covariant derivative of the unit vector field, $\mathcal{H}^{ik}_{(1)} = A^{mnik} \nabla_m U_n$

Let us start with the analysis of the first term in Eq. (62),

$$\mathcal{H}^{ik}_{(1)} = A^{mnik} \nabla_m U_n,$$

that contributes to the tensor of spontaneous polarization-magnetization. Studying the symmetry of the tensor coefficients $A^{ikmn}$, appearing in Eq. (62), and in analogy with $K^{abmn}$, we find that $A^{ikmn} = -A^{iknm}$, since the Maxwell tensor is skew-symmetric. Again, we can also put $A^{ikmn} U_m = 0$, since $U_m \nabla_m U_n = 0$. Formally speaking, there are, in general, $6 \times 12 = 72$ independent components of the tensor $A^{ikmn}$. Nevertheless, we intend to show that using the unit vector field $U^m$ itself and the geometric quantities $g^{ik}$, $\delta^i_k$, $\delta^j_{ik}$, $\epsilon^{ikmn}$, we can reconstruct this tensor based on two and only two independent coupling constants.

Searching for first order, i.e., linear, terms in the decomposition of $\mathcal{P}^i$ we can find only one natural four-vector, $DU^j$, and for the decomposition of $\mathcal{M}^i$ we can find only one natural pseudo four-vector, $\omega^{*i} = \omega^{*ik} U_k$. Thus, the corresponding first order decompositions are

$$\mathcal{P}^i_{(1)} = \pi_1 DU^i, \quad \mathcal{M}^i_{(1)} = \mu_1 \epsilon^{ikpq} U_k \omega_{pq},$$

with $\pi_1$ and $\mu_1$ being independent coupling constants.

(b) Polarization-magnetization quadratic in the covariant derivative of the unit vector field, $\mathcal{H}^{ik}_{(2)} = B^{mnlsik}(\nabla_i U_s)(\nabla_m U_n)$

Let us now analyze the second term in Eq. (62),

$$\mathcal{H}^{ik}_{(2)} = B^{mnlsik}(\nabla_i U_s)(\nabla_m U_n).$$

The tensor $B^{ikmnpq}$ possesses the following symmetries

$$B^{mnlsik} = -B^{mnlski} = B^{ikmn}, \quad B^{mnlsik} U_n = 0, \quad B^{mnlsik} U_s = 0.$$  

Thus, in general, $B^{ikmnpq}$ can be characterized by $\frac{1}{2} \times 12 \times 13 \times 6 = 468$ independent components. Nevertheless, below we show that the reconstruction of the tensor $B^{ikmnpq}$ requires the introduction of only five coupling constants.

To deal then with second-order terms, represented by $\mathcal{H}^{ik}_{(2)} = B^{mnlsik}(\nabla_i U_s)(\nabla_m U_n)$, we find that there are the following quadratic terms: $\Theta^2$, $DU^j DU^j$, $\sigma_{ik} \sigma_{mn}$, $\omega_{ik} \omega_{mn}$. With these quantities we cannot construct neither a vector, nor a pseudo vector orthogonal to $U^i$. There are also quadratic cross-terms, namely, $DU^i \Theta$, $DU^j \sigma_{mn}$, $DU^j \omega_{mn}$, $\Theta \omega_{ij}$, $\sigma_{ij} \omega_{mn}$ and $\Theta \sigma_{ij}$. The first term, $DU^i \Theta$, is a four-vector, the next two terms, i.e., $DU^i \sigma_{mn}$, and
$DU^i \omega_{mn}$, can be contracted to form vectors that enter into the decomposition of $\mathcal{P}^i$. The next two terms, i.e., $\Theta \omega_{ij}$ and $\sigma_{ij} \omega_{mn}$, can be contracted with the Levi-Civita tensor and with the velocity four-vector to form pseudo vectors that enter into the decomposition of $\mathcal{M}^i$. The last term, i.e., $\Theta \sigma_{ij}$, cannot form a vector or pseudo vector in any way, as the trace of the symmetric tensor $\sigma_{ij}$ is zero.

Thus, in summary, there are only one linear and three quadratic terms in the decomposition of the polarization four-vector, $\mathcal{P}^i$. Therefore $\mathcal{P}^i$ can be written as

$$\mathcal{P}^i = DU_k \left[g^{ik} \left(\pi_1 + \pi_2 \Theta\right) + \pi_3 \sigma^{ik} + \pi_4 \omega^{ik}\right],$$

(68)

with $\pi_1$, $\pi_2$, $\pi_3$, and $\pi_4$ being four independent coupling constants. An interesting aspect of this representation is that the polarization four-vector $\mathcal{P}^i$ is proportional to the acceleration four-vector $DU_k$, and so $\mathcal{P}^i$ vanishes when $DU_k$ is equal to zero, $DU_k=0$.

Similarly, we can construct $\mathcal{M}^i$ with one linear term and three quadratic terms. $\mathcal{M}^i$ can then be written as

$$\mathcal{M}^i = \epsilon^{ikpq} U_k \omega_{nm} \left[g^{p\delta n} (\mu_1 + \mu_2 \Theta) + \mu_3 (\sigma^m \delta^n - \sigma^n \delta^m)\right],$$

(69)

where $\mu_1$, $\mu_2$, and $\mu_3$, are 3 new independent coupling constants. There is no magnetization when there is no vorticity, i.e., $\omega_{nm}=0$. One can also indeed add another term to the right hand side of Eq. (69), namely, $\mu_4 \Delta^i \epsilon^{ikpq} \omega_{pq} DU_k$, in which case $\mathcal{M}^i$ gets another coupling constant $\mu_4$. However, when we reconstruct the tensor $\mathcal{H}^{ik}$ in (61), the term with $\mu_4$ disappears because of the properties of products of two Levi-Civita symbols together with the contractions of $U^i$ with $\omega_{pq}$ and $DU_k$. Thus $\mu_4$ is a hidden coupling constant which does not enter into the dynamics, so it can be put to zero without loss of generality, $\mu_4=0$.

(c) Reconstruction of the tensors $A^{mnik}$ and $B^{mnlsik}$

Now we put Eqs. (68) and (69) into Eq. (61) and compare the result with Eq. (32). It is then possible to reconstruct the tensor $A^{mnik}$, namely,

$$A^{mnik} = \pi_1 g^{ikmn} U^m U^l - \mu_1 \Delta^{ikmn},$$

(70)

where we have introduced the following auxiliary tensors

$$g^{mnpq} \equiv g^{mp} g^{nq} - g^{mq} g^{np},$$

(71)

$$\Delta^{mnpq} \equiv \Delta^{mp} \Delta^{nq} - \Delta^{mq} \Delta^{np},$$

(72)

Similarly, we can reconstruct the tensor $B^{mnlsik}$. It is given by

$$B^{mnlsik} = \frac{1}{2} \pi_2 U^l \left[\Delta^{mn} U^l g^{iknp} + \Delta^{ls} U^m g^{iknp}\right] +$$

$$+ \frac{1}{2} \pi_3 \delta^{[l[i} U^k]} \left[U^l \left[\Delta^{m(p \Delta s)m} - \frac{1}{3} \Delta^{ps} \Delta^{mn}\right] + U^m \left[\Delta^{l(p \Delta s)n} - \frac{1}{3} \Delta^{pn} \Delta^{ls}\right]\right] +$$

$$+ \frac{1}{4} \pi_4 \delta^{[l[i} U^k]} \left[U^l \Delta^{psmn} + U^m \Delta^{pnl}\right] - \frac{1}{2} \mu_2 \left(\Delta^{mn} \Delta^{ikls} + \Delta^{ls} \Delta^{ikmn}\right) -$$

$$- \frac{1}{2} \mu_3 \left[\frac{2}{3} (\Delta^{mn} \Delta^{ikls} + \Delta^{ls} \Delta^{ikmn}) + \Delta^{n[i} \Delta^{k]} mls + \Delta^{m[i} \Delta^{k]} nls + \Delta^{e[i} \Delta^{k]} lmn + \Delta^{l[i} \Delta^{k]} smn\right].$$

(73)

The tensor $A^{mnik}$ contains two coupling constants, and the tensor $B^{mnlsik}$ contains five coupling constants.

3. Polarization-magnetization associated with the second covariant derivative of the unit vector field

Now we analyze the third term in Eq. (32), namely,

$$\mathcal{H}^{ik} = B^{(ml)sik} \nabla_{(m} \nabla_{l)} U_s.$$  

(74)
The tensor \( B^{(ml)sk} \) is symmetric with respect to the indices \((ml)\). Generally it possesses \( \frac{3}{2} \times 4 \times 5 \times 4 \times 6 = 240 \) independent components, and thus can be described using 240 independent coupling constants. However, in fact five coupling constants are enough to represent this tensor using the metric \( g_{ik} \), the Kronecker tensors \((\delta^i_k, \delta^{ik} \text{ and higher order Kronecker tensors})\), the Levi-Civita tensor \( \epsilon^{ikl} \), and the unit vector field \( U^k \).

The tensor \( B^{(ml)sk} \) cannot be decomposed as was done for the terms \( H^{ik}_{(1)} \) and \( H^{ik}_{(2)} \). We use here another approach. Keeping in mind the symmetry of the tensor \( B^{(ml)sk} \) and that there are only two natural symmetric tensors, i.e., \( g^{ik} \) and \( U^i U^k \), and only two natural skew-symmetric pure tensors, i.e., \( \delta^{ik} \) and \( \delta^{ik} \), this tensor has to be of the form

\[
B^{(ml)sk} = \delta^{ik}_{pq} U^q \left[ g^{ps} \left( \rho_0 g^{ml} + \rho_2 U^m U^l \right) + \rho_3 \delta^{(m} g^{lk)} + \rho_4 U^s g^{(ml)k} \right] + \rho_5 \Delta^{iks}(mU^l),
\]

where five new coupling constants \( \rho_1, \rho_2, \rho_3, \rho_4, \) and \( \rho_5 \) have appeared.

4. Nonminimal polarization-magnetization

The nonminimal part of the polarization-magnetization tensor

\[
H^{ik}_{(NM1)} = QU_m R^{mi} [U^k]
\]

is associated with the vanishing magnetization four-vector \( M^i_{(NM)} = 0 \) and the polarization four-vector of the form

\[
P^i_{(NM)} = \frac{1}{2} QU_m R^{mk} \Delta_k^i.
\]

Only one coupling constant, \( Q \), describes the polarization of the medium/vacuum, induced by the interaction with curvature in the presence of unit vector field.

D. Coupling constants related to the permittivity tensors of the matter or vacuum in a non-uniformly moving aether

1. Susceptibilities linear and quadratic in the covariant derivatives of the unit vector field

In order to represent the tensors \( X^{lsikmn} \) and \( Y^{ablsikmn} \) in \( C^{ikmn}_{(D)} \) (see (35)), we use a similar scheme, as for the coefficients \( A^{mnik} \) and \( B^{mnlsik} \). We start with the linear response tensor \( C^{ikmn} \) given in Eq. (33). It admits the standard decomposition

\[
C^{ikmn} = \frac{1}{2} \left[ \varepsilon^{im} U^k U^n - \varepsilon^{in} U^k U^m + \varepsilon^{km} U^i U^m - \varepsilon^{km} U^i U^m \right] - \frac{1}{2} \eta^{ikl} (\mu^{-1})_{ls} \eta^{mns} +
\]

\[
+ \frac{1}{2} \left[ \eta^{ikl} \left( U^m \nu^l_m - U^m \nu^l_m \right) + \eta^{lmn} (U^i \nu^k_m - U^k \nu^i_m) \right],
\]

where \( \varepsilon^{im} \) is the dielectric permittivity tensor, \( (\mu^{-1})_{pq} \) is the magnetic impermeability tensor, \( \nu^m_p \) is the tensor of magneto-electric coefficients, i.e.,

\[
\varepsilon^{im} = 2 C^{ikmn} U_k U_n, \quad (\mu^{-1})_{pq} = -\frac{1}{2} \eta_{pqk} C^{ikmn} \eta_{mnq}, \quad \nu^m_p = \eta_{pqk} C^{ikmn} U_n = U_k C^{mkln} \eta_{lnp}.
\]

As usual, the tensors \( \eta_{mn} \) and \( \eta^{ik} \) are skew-symmetric tensors orthogonal to \( U^i \),

\[
\eta_{mn} = \epsilon_{mnls} U^s, \quad \eta^{ik} = \epsilon^{ikls} U^s,
\]

and obey the following identities

\[
- \eta^{ik} \eta_{lnp} = \delta^{ik} \eta_{lns} U^s = \Delta_m^i \Delta_n^k - \Delta_n^i \Delta_m^k, \quad -\frac{1}{2} \eta^{ik} \eta_{klm} = \delta^{ik} \eta_{lns} U^s = \Delta_m^i.
\]
We now decompose explicitly the permittivity tensors $\varepsilon^{im}$, $(\mu^{-1})_{pq}$ and $\nu^{pm}$ using the irreducible parts of the covariant derivative of the velocity four-vector (namely, $DU^i$, $\sigma_{ik}$, $\omega_{pq}$, and $\Theta$). The properties

$$\varepsilon_{ik}U^k = 0, \quad (\mu^{-1})_{ik}U_k = 0, \quad \nu^{ik}U_k = 0 = \nu^{ik}U_i$$

simplify the decomposition of $\varepsilon^{im}$, $(\mu^{-1})_{pq}$ and $\nu^{pm}$, and the results are the following. The dielectric permittivity tensor is decomposed as

$$\varepsilon_{ik} = \Delta_{ik}(\varepsilon + \alpha_1\Theta) + \Delta_{ik}(\alpha_2DU_mDU^m + \alpha_3\Theta^2 + \alpha_4\sigma_{mn}\sigma^{mn} + \kappa_5\omega_{mn}\omega^{mn}) +$$

$$+\kappa_6\sigma_{ik} + \kappa_7\Theta\sigma_{ik} + \kappa_8DU^iDU^k + \kappa_9\sigma^{ip}\sigma_k^{p} + \kappa_{10}\omega^{ip}\omega_k^{p} + \kappa_{11}\sigma_p^{(i}k^{p)}$$,

where $\varepsilon$ and $\alpha_1, \ldots, \alpha_{11}$ form twelve independent coupling constants. The magnetic impermeability tensor is decomposed as

$$\nu^{ik} = \Delta_{ik}(\frac{1}{\mu} + \gamma_1\Theta) + \Delta_{ik}[\gamma_2DU_mD_U^m + \gamma_3\Theta^2 + \gamma_4\sigma_{mn}\sigma^{mn} + \gamma_5\omega_{mn}\omega^{mn}] +$$

$$+\gamma_6\sigma_{ik} + \gamma_7\Theta\sigma_{ik} + \gamma_8DU^iDU^k + \gamma_9\sigma^{ip}\sigma_k^{p} + \gamma_{10}\omega^{ip}\omega_k^{p} + \gamma_{11}\sigma_p^{(i}k^{p)}$$,

where $\mu$ and $\gamma_1, \ldots, \gamma_{11}$ form also twelve independent coupling constants. The magneto-electric cross-effect pseudo tensor is decomposed as

$$\nu^{pm} = (\nu_1 + \nu_3\Theta)\Delta^{p}_q\Delta^{m}_q\omega^{*q}n + (\nu_2 + \nu_4\Theta)\eta^{pml}DU_l +$$

$$+\nu_5\Delta^{p}(\omega^{*q}\sigma^{m})q + \nu_6\Delta^{p}(\omega^{*q}\sigma^{m})q + \nu_7\omega^{(p}\Delta^{m)}q\omega^{*s}q +\nu_8\omega^{(p}\Delta^{m)}q\omega^{*s}q +$$

$$+\nu_9DU^{(p}\omega^{*m)}qU_q + \nu_{10}DU^{(p}\omega^{*m)}qU_q + \nu_{11}\eta^{(p}\eta^{m)}qU_q + \nu_{12}\eta^{(p}\eta^{m)}qU_q$$,

where $\nu_1, \ldots, \nu_{12}$ form another twelve independent coupling constants.

Having decomposed explicitly the permittivity tensors $\varepsilon^{im}$, $(\mu^{-1})_{pq}$ and $\nu^{pm}$ using the irreducible parts of the covariant derivative of the velocity four-vector we can now reconstruct the tensors $X^{lsikmn}$ and $Y^{alsikmn}$ in $C^{ikmn}_{(D)}$ given in Eq. 35, keeping in mind their symmetry,

$$X^{lsikmn} = -X^{lsikmn} = X^{lsikmn} = X^{lsmnik}$$,

$$Y^{alsikmn} = -Y^{alsikmn} = Y^{alsikmn} = Y^{alsikmn}$$.

The reconstructed tensors $X^{lsikmn}$ and $Y^{alsikmn}$ are presented in Appendix B.

The given representation of the permittivity tensors allows us to interpret and classify the coupling constants appearing in this decomposition. Two constants, $\varepsilon$ and $\mu$, have a standard interpretation in terms of an aether uniform motion. Other coupling constants can be classified with respect to electrodynamic effects which can exist when the aether is in a state of non-uniform motion. For instance, the magneto-electric coefficients, described by the non-symmetric tensor $\nu^{pm}$, represent the effect of optical activity, and it can be splitted into a sum of symmetric and skew-symmetric parts. Thus, the term with the coupling constant $\nu_1$ is related to the polarization rotation phenomenon linear in the vorticity tensor $\omega^{pm}$ and is purely skew-symmetric contribution to the linear term. Similarly, the parameters $\nu_7$ and $\nu_8$ relate to quadratic effects, symmetric and skew-symmetric, respectively. The coupling constant $\nu_2$ is connected with the optical activity caused by an acceleration of the aether with the effect being linear in $DU^i$. The parameters $\nu_3, \ldots, \nu_6$ and $\nu_9, \ldots, \nu_{12}$ are connected to the corresponding cross-effects.

2. Susceptibilities containing second covariant derivatives of the unit vector field

The term $C^{ikmn}_{(DD)} = Y^{jlsikmn}\nabla_j\nabla_lU_s$ in Eq. 37 relates to a linear response induced by a second covariant derivative of the unit vector field $U^i$. The tensor $Y^{jlsikmn} = Y^{j[ls[i][kmn]}$, in general, possesses $\frac{1}{2} \times 4 \times 5 \times 4 \times \frac{1}{2} \times 6 \times 7 = 840$
components. In our setting it can be characterized by twelve coupling constants only, and similarly to the tensor $B^{(ml)s[ik]}$ in Eq. (73), it can be represented as follows,

$$\gamma^{jikmn} = g^{ikmn} \left[ U^s \left( \rho_0 \gamma^{ijl} + \rho_7 \gamma^{jil} \right) + \rho_8 U^s \left( g^{ik[j]m}[g^{m][j]} + g^{ik[j]}[m][j] \right) + \rho_{10} \left[ U^l \left( g^{ik[s][j]m} + g^{ik[s]m[j]} \right) + U^j \left( g^{ik[s][j]m} + g^{ik[s]m[j]} \right) \right] \right] + \rho_{11} U^s \left( g^{ik[j][m]n} U^l + g^{ikl[m][n]} U^j + g^{ikjn}[U^k] U^l + g^{imjn}[U^k] U^j \right) + \rho_{12} U^l U^j U^s \left( g^{ik[s][j]m} + g^{ik[s]m[j]} \right) + \rho_{13} g^{ijl} U^s + \rho_{14} g^{s[j]il} + \rho_{15} U^s U^j U^l \left[ U^l \gamma^{ik[j]}[m][n] \right] + \rho_{16} U^s \left( g^{ij[k]}[l][m][n] + g^{l}[U^k] g^{s[m][n]} + U^j \gamma^{ij[k]}[m][n] \right) + \rho_{17} \left( U^l \gamma^{ij[k]}[m][n] + U^j \gamma^{ij[k]}[m][n] \right) \right), \quad (88)$$

3. Nonminimal susceptibilities

This term is given in Eq. (37) as $C^{ikmn} = R^{ikmn} + S^{pqabikmn} R_{pqab}$, and is a term that contributes to the total linear response tensor $R^{ikmn}$. According to Eq. (25) the nonminimal susceptibility tensor $R^{ikmn}$ contains three independent coupling constants, namely, $q_1$, $q_2$ and $q_3$. The tensor $S^{pqabikmn}$ is skew-symmetric with respect to indices $pq$, $ab$, $ik$, $mn$, is symmetric with respect to transpositions $pq \rightarrow ab$, $ik \rightarrow mn$, and thus can be characterized, in general, by \( \frac{1}{2} \times 6 \times 7 \times \frac{1}{2} \times 6 \times 7 = 441 \) components. When we only use the metric $g_{ik}$, the Kronecker tensors ($\delta_k^i$, $\delta_{ik}$ and higher order Kronecker tensors), the Levi-Civita tensor $\epsilon^{ikab}$, and the unit vector field $U^k$, in its reconstruction, this tensor has 11 independent terms, so 11 coupling constants. It has thus the form, see Appendix B for details,

$$S^{pqabikmn} = S^{pqabikmn}_{(1)} + S^{pqabikmn}_{(2)} + S^{pqabikmn}_{(3)}, \quad (89)$$

where

$$S^{pqabikmn}_{(1)} = \left( q_1 g^{pqab} + q_5 \Delta^{pqab} \right) \left( \Delta^{ikmn} - g^{ikmn} \right) + q_6 g^{ikmn} \left( U^l \gamma^{j[k][m][n]}[j][k] + U^j \gamma^{i[k][m][n]}[i][k] \right), \quad (90)$$

$$S^{pqabikmn}_{(2)} = q_7 \left( U^l \gamma^{p[q][m][n][i]}[i] + U^l \gamma^{p[q][m][n][i]}[i] \right) + q_8 \left( U^l \gamma^{p[q][m][n][i]}[i] + U^l \gamma^{p[q][m][n][i]}[i] \right) + q_9 \left( U^l \gamma^{p[q][m][n][i]}[i] + U^l \gamma^{p[q][m][n][i]}[i] \right) + q_{10} \left( g^{n[m][n]} U^l \gamma^{p[q][m][n][i]}[i] + g^{n[m][n]} U^l \gamma^{p[q][m][n][i]}[i] \right), \quad (91)$$

and the last term $S^{pqabikmn}_{(3)}$ can be obtained from $S^{pqabikmn}_{(2)}$ with introduction of new coupling constants $q_{11}, ..., q_{14}$ instead of $q_7, ..., q_{10}$, respectively, and by the substitution

$$U^l \gamma^{p[q][m][n][i]}[i] \rightarrow -g^{ikpq} U_q \rightarrow \epsilon^{ikpq} U_q. \quad (92)$$

Clearly, the tensor $S^{pqabikmn}$ possesses the required symmetries, and the unit vector field enters this quantity in even combinations of second and fourth orders.
E. Summary of the decompositions

Let us sum up the independent parameters of the Einstein-Maxwell-aether theory we are interested in. We started with the four parameters $C_1, C_2, C_3, C_4$ introduced in the pure Einstein-aether theory. Then we added thirteen parameters $\rho_1, \rho_2, \rho_3, \rho_4, \mu, \omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8$, appearing in the decomposition of the spontaneous polarization-magnetization tensor. Also there are thirty-six coupling constants $\varepsilon, \mu, \alpha_1, \ldots, \alpha_{11}, \gamma_1, \ldots, \gamma_{11}$, and fourteen nonminimal coupling constant parameters $q_1, \ldots, q_{14}$. Finally, we have introduced twelve parameters $\rho_6, \ldots, \rho_{17}$. In total the theory has a set of 79 independent parameters.

F. Three spacetime models with high symmetry: Remarks on the structure of the unit vector field $U^i$ based on the analysis of the compatibility conditions

1. Motivation

Keeping in mind applications of this Einstein-Maxwell-aether theory, we would like to call the attention to three interesting consequences coming from the analysis of the structure of the unit vector field. Indeed, three spacetime models with high symmetry are prone to be solutions, possibly analytical solutions, of the Einstein-Maxwell-aether theory presented here. These spacetimes models are the spatially homogeneous cosmological models, static spherically symmetric structures, and plane-wave spacetimes. We do not intend here to analyze the total system of reduced master equations, but would like to mention the consequences, which follow from the compatibility conditions related to our ansatz on the structure of the unit vector field.

2. Three spacetime models

(a) Spatially homogeneous cosmological models

Let us consider first, the Friedmann-Lemaître-Robertson-Walker (FLRW) cosmological models with line element

$$ds^2 = dt^2 - a^2(t) \left( dx^2 + dy^2 + dz^2 \right),$$

(93)

where $a(t)$ is the Friedmann scale factor as a function of the cosmological time $t$, and $x, y, z$ are spatial homogeneous coordinates. Within these models we can assume that the aether velocity four-vector is of the form $U^i = \delta^i_t$, and thus the tensor $\nabla_m U_n$ has the following irreducible terms

$$U_m DU_n = 0, \quad \sigma_{ik} = 0, \quad \omega_{pq} = 0, \quad \Theta = 3 \frac{\dot{a}}{a} = 3H(t),$$

(94)

where $H(t) \equiv \frac{\dot{a}}{a}$ is the Hubble function. In such a case we find, that $P_i = 0$, $M_i = 0$ and

$$K^{abmn}(\nabla_a U_m)(\nabla_b U_n) = \frac{1}{3} \Theta^2 (C_1 + 3C_2 + C_3).$$

(95)

The spacetime symmetries require that the global electromagnetic field obeys $F_{ik} = 0$, and the corresponding electrodynamical equations are satisfied identically, since $I^i = 0$, $P^i = 0$ and $M^i = 0$. We obtain the standard FLRW cosmological model, if we prove that the equations for the aether velocity are satisfied identically, when $U^i = \delta^i_t$. Indeed, $J^{mn}_{[i]} = 0$ and $I^n_{[i]} = 0$, since $F_{pq} = 0$. If we suppose that $\frac{\delta L_{[i][m]}}{\delta U^m} = 0$, then, $I^n_{[A]} = 0$, since $DU^n = 0$. The term $J^{mn}_{(A)}$ yields

$$J^{mn}_{(A)} = \frac{1}{3} \Theta \left[ (C_1 + 3C_2 + C_3)g^{mn} - (C_1 + C_3)U^m U^n \right],$$

(96)

and the reduced equation

$$\Delta^n \nabla_m J^{mn}_{(A)} = 0,$$

(97)

is satisfied identically. Thus we have checked that in the spatially homogeneous FLRW cosmological models without a global electromagnetic field, the aether coupling parameters remain hidden, the unit vector field being of the form $U^i = \delta^i_t$.

A non-uniform aether motion may provide the appearance of unlighted cosmological epochs similar to the ones described in [54]. In these unlighted epochs, the square of the effective refraction index is negative, and the corresponding electromagnetic waves can not propagate.
(b) Static spherically symmetric models
We now assume a static spherically symmetric metric spacetime with line element
\[
\text{d}s^2 = B(r)\text{d}t^2 - A(r)\text{d}r^2 - r^2(\text{d}\theta^2 + \sin^2\theta \text{d}\phi^2),
\]
where \(t\) is the global time, \((r, \theta, \phi)\) are the spherical symmetric spatial coordinates, and \(B(r), A(r)\) are the metric functions. Let us assume that the aether velocity four-vector is of the form
\[
U^i = \delta^i_1 \frac{1}{\sqrt{B(r)}}.
\]
This assumption, that the aether is aligned with the timelike Killing vector is not the most general, and can be put under scrutiny on physical grounds, as in general the aether has radial and time components as it falls into a central body, see \([5, 8, 9]\) for a more general class of spherical symmetric solutions. Nevertheless, we maintain here
\[
\text{Eq. (99)}
\]
and leave for another work the study of more general examples of exact spherically symmetric solutions to the Einstein-Maxwell-aether theory.

The irreducible parts of the covariant derivative are then
\[
U_mD_{U_n} = -\frac{B'}{2B}\delta^r_n U_m, \quad \sigma_{ik} = 0, \quad \omega_{pq} = 0, \quad \Theta = 0,
\]
where a \(\prime\) means a derivative with respect to \(r\). The reduced quantity \(K^{abmn}(\nabla_a U_m)(\nabla_b U_n)\) is given by
\[
K^{abmn}(\nabla_a U_m)(\nabla_b U_n) = (C_1 + C_4)DU_mDU^n.
\]
In this case there is no magnetization, \(M'=0\). The polarization \(P^i\) four-vector is non-vanishing, its linear part being of the form \(P^i = r_1 DU^i\), and thus contains the radial component \(P^r\) only. The compatibility conditions for the electromagnetic equations require then that a static radial electric field should appear in the system, \(E_{\text{radial}} = \sqrt{AB} F^{r\theta} \neq 0\), which in turn is supported by the polarization induced by the aether non-uniform state. Concerning the gravitational field equations, one sees that they can be reduced to a pair of equations for \(A(r)\) and \(B(r)\), but here we do not intend to specify this set of equations.

The compatibility of the model as a whole depends on the question of whether the equation for the aether velocity four-vector \(U_i = \sqrt{B}\) is satisfied identically. In fact, in this case one obtains
\[
\mathcal{J}^{\alpha\beta}_{(A)} = [(C_1 + C_4)\delta^\alpha_\beta \delta^\gamma_\nu + C_3 \delta^\alpha_\nu \delta^\gamma_\beta] \frac{B'}{2AB\sqrt{B}}, \quad \Delta^m_r \nabla_m \mathcal{J}^{\alpha\beta}_{(A)} = 0, \quad I^n_\alpha = \kappa \frac{\delta L^{(m)}}{\delta U_n} = 0.
\]
Only the equation for \(n=r\)
\[
\nabla_m \mathcal{J}^{mn}_{(M)} = I^n_{(M)},
\]
needs to be analyzed. Eq. (103) can be reduced to an identity when \(\mathcal{J}^{rr}_{(M)} = 0\) and \(I^n_{(M)} = 0\). This is possible, e.g., for a special choice of the coupling parameters. We will return to this problem in the future.

(c) Spacetimes with plane-wave symmetry
As an illustration for this class of spacetimes we can consider the metric
\[
ds^2 = 2\text{d}udv - L^2 \left(e^{2\beta} \text{d}x^2 + e^{-2\beta} \text{d}x^3\right),
\]
where \(u\) and \(v\) are the retarded and advanced times, respectively, given in terms of the time \(t\) and spatial coordinate \(x^1\) by \(u = \sqrt{2}(t-x^1), \quad v = \sqrt{2}(t+x^1), \quad \) and \(x^2, x^3\) are the other spatial coordinates. \(L(u)\) and \(\beta(u)\) are functions of the retarded time \(u\) only. When the aether velocity four-vector is assumed to be of the form
\[
U^i = \frac{1}{\sqrt{2}} (\delta^i_u + \delta^i_v) = \delta^i_t,
\]
i.e., the aether is at rest in the spacetime reference frame, the covariant derivative of the velocity four-vector reduces to the following equation
\[
\nabla U^k = \frac{1}{\sqrt{2}} \left[\delta^2_i \delta^k_2 \left(\frac{L'}{L} + \beta'\right) + \delta^3_i \delta^k_3 \left(\frac{L'}{L} - \beta'\right)\right],
\]
where a prime here denotes a derivative with respect to the retarded time \( u \). Thus we obtain

\[
DU^k = 0, \quad \omega_{pq} = 0, \quad \Theta = \frac{\sqrt{2} L'(u)}{L},
\]

(107)
i.e., the acceleration four-vector and the vorticity tensor are equal to zero for this unit vector field \( U^i \). The corresponding shear tensor is non-vanishing and can be written as a sum of two traceless tensors, i.e.,

\[
\sigma^k_i = \frac{\Theta}{2} \left( \frac{1}{3} \Delta^k_i - \delta^k_i \delta^1_1 \right) + \frac{\beta'}{\sqrt{2}} \left( \delta^2_2 \delta^k_3 - \delta^3_3 \delta^k_2 \right).
\]

(108)

The gravitational field equations for this case are known to be compatible when the total stress-energy tensor is of the null-type, i.e., it can be presented in the form \( W_{kij} \) with \( k \) a null four-vector, \( k^i k_i = 0 \). The analysis of the equations of the aether motion shows that, when \( U^i = \delta^i_1 \), they can be satisfied with some restrictions for the coupling parameters, but we refrain from discussing details here.

V. CONCLUSIONS

A. On the interpretation of the coupling constants

1. Motivation

We have followed the rationale used for the Einstein-aether theory, that for regions where quantum gravity is not anymore dominant and Lorentz symmetry is already broken by those quantum effects an Einstein-aether theory can naturally appear [1]. One expects then that the Einstein-aether theory is a low energy phenomenon of some fundamental quantum theory. An Einstein-aether theory can be in action at the inflationary period, giving rise to an Einstein-scalar-aether theory [38]. A generic 2-form field, like the Maxwell field, can appear at very high energy scales giving rise to some form of an Einstein-Maxwell-aether as discussed by us (see also [22]). Or it could be in operation after inflation decays and the matter fields, such as the Maxwell field, make their appearance.

2. Coupling constants associated with a spontaneous polarization-magnetization induced by a non-uniform aether motion

In the Einstein-Maxwell-aether theory we have proposed, the new cross-terms containing both the Maxwell tensor and the covariant derivatives of the aether velocity four-vector, allows us not only to give a formal interpretation of the new coupling constants, but to propose ways of how one can try to estimate them in the frameworks of the PPE and PPF formalisms. Such a work requires detailed analysis and is beyond the scope here. Nevertheless, we would like to expand our ideas in three examples.

According to Eqs. (68) and (69), eight constants describe the effects of spontaneous polarization and magnetization of the matter or vacuum, which can appear due to an aether non-uniform motion. The coupling constant \( \pi_1 \) introduces the polarization produced by a pure acceleration of the aether. This parameter can pop up in a static spherically symmetric system, since there the radial component of the acceleration four-vector is non-vanishing, \( DU_r \neq 0 \). However, in static spherically symmetric systems the parameters \( \pi_2, \pi_3, \pi_4 \) are hidden, since there are no shear, vorticity and expansion in such spacetimes. The parameter \( \pi_2 \) can appear when the vector field has acceleration and expansion, \( \Theta \neq 0 \). Similarly, a combination of acceleration and shear brings into the open the parameter \( \pi_3 \). The combination of acceleration and vorticity reveals the parameter \( \pi_4 \). Similar interpretation can be done with the parameters \( \mu_1, \mu_2, \mu_3 \) (see Eq. (69)). However, instead of the acceleration we have to use here the vorticity tensor \( \omega_{ik} \). The degeneracy with respect to the parameters \( \pi_2, \pi_3, \pi_4, \mu_1, \mu_2, \mu_3 \) altogether can be removed, if the spacetime contains a rotating object, like a neutron star and thus is not spherically symmetric, or contains gravitational waves propagating non-co-axially with respect to the aether motion.

3. Coupling constants associated with optical activity produced by an aether non-uniform motion

Optical activity is associated with the rotation of the polarization of the electromagnetic waves propagating in a medium. The presence of optical activity amounts to the non-vanishing of the magneto-electric coefficients tensor \( \nu^{pm} \) (see, e.g., [18]). According to Eq. (65) the couplings \( \nu_1, ..., \nu_{12} \) describe the optical activity of the matter or vacuum
when the aether motion is non-uniform. More precisely, the optical activity appears when the aether is accelerated \((DU^i \neq 0)\) or its velocity is characterized by a non-vanishing vorticity tensor \((\omega_{ik} \neq 0)\). Linear effects in the vorticity tensor and in the acceleration four-vector appear when \(\nu_1\) and \(\nu_2\) are non-vanishing, respectively. Nonlinear effects appear when \(\omega_{ik} \neq 0\) or \(DU_k \neq 0\) and at least one of the two quantities \(\sigma\) and \(\sigma^{ik}\) is not equal to zero. The removal of a degeneracy with respect to \(\nu_2\) is possible for spherically symmetric objects. Other coupling constants can appear in systems with rotating bodies or in systems with gravitational waves with arbitrary direction of propagation. When gravitational waves are present, the effects of optical activity are similar to the ones described in [51].

4. Coupling constants associated with dynamo-optical effects and birefringence

Dynamo-optical effects are connected with the variation of the dielectric and magnetic permittivity tensors in non-uniformly moving media (see, e.g., [18]). When these permittivity tensors become anisotropic, birefringence can take place, i.e., the phase velocity of electromagnetic waves is a function of the wave polarization. According to Eqs. (83) and (84) linear dynamo-optical effects induced by an aether motion are connected to the presence of a shear tensor and an expansion scalar, bringing into play the couplings \(\alpha_1\), \(\alpha_6\), \(\gamma_1\), and \(\gamma_6\). The other coupling constants \(\alpha_2\), ..., \(\alpha_{11}\) and \(\gamma_2, ..., \gamma_{11}\) describe quadratic and nonlinear cross-effects. One of the most interesting application of these effects is the analysis of the phase and group velocities of the electromagnetic waves propagating in the medium or vacuum interacting with an aether non-uniform motion. Similar effects caused by an interaction with curvature have been considered in [50–56].

B. How can we reduce the number of coupling parameters introduced phenomenologically?

The Einstein-Maxwell-aether theory under consideration includes 79 independent coupling constants. It seems to be useful to reduce the number of these parameters using some underlying symmetry, similarly to what has been done in nonminimal gravito-electric theories (see, e.g., [49, 50]). For instance, one can put \(C_2=C_3=C_4=0\) and keep only one constant \(C_1 \equiv C\), if we admit that the squared contributions for the acceleration, shear, vorticity and expansion are equivalent, see Eq. (60). One can also put \(\pi_2=\pi_3=\pi_4\) and \(\mu_2=\mu_3\) in order to guarantee that the nonlinear terms enter in an equal manner into the spontaneous polarization-magnetization tensor. This procedure can be used for the permittivity tensors also. In this case the total number of independent coupling constants can be reduced to 14, say.

C. Outlook

The applications of this formalism to cosmology and astrophysics are the next steps in the study of the Einstein-Maxwell-aether theory proposed here. It is of interest to discuss Bianchi type I solutions in this theory, as well as static and spherically symmetric solutions. Of course, it will be important to make an analysis of the Einstein-Maxwell-aether theory in the frameworks of PPN, PPF and PPE formalisms.

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Appendix A: Inclusion of all the terms up to the fourth order in an Einstein-Maxwell-aether theory and the choice for the ansatz

1. Extension of the Einstein-aether theory to include all the terms up to fourth order in the derivatives

In the Secs. III A and III B we have given a motivation and the requirements to choose the ansatz of the action functional as in Eq. (20) and subsequent equations. Here we give the details for such a choice. We follow in part the structure of the action functional for the pure Einstein-aether theory, see Eq. (1), as discussed from several points of view (see, e.g., [7] for a review), and we return to this question in order to justify further generalizations that include
the electromagnetic gauge vector field $A_i$ and the corresponding gauge invariant Maxwell tensor $F_{ik}$. According to the principles of effective field theories (see, e.g., [27–31]) one can establish some interrelations between the terms in the action functional and differential operators of the first, second, and higher orders.

The tensor $F_{ik}$ is defined as

$$F_{ik} = \nabla_i A_k - \nabla_k A_i.$$  \hfill (A1)

The Maxwell tensor $F_{ik}$ seems to contain a covariant derivative. However, due to the symmetry of the Christoffel symbols $\Gamma^i_{km} = \Gamma^i_{mk}$ it can be rewritten using partial derivatives $F_{ik} = \partial_i A_k - \partial_k A_i$ only. In other words, $F_{ik}$ contains neither metric coefficients, nor Christoffel symbols, and thus this quantity does not change upon variation of the action functional with respect to metric. For this reason we consider, that the electromagnetic field has derivatives independent of the derivatives involving the metric. In a sense this means that the electromagnetic field introduces a scale parameter $l_{(em)}$ which is an independent scale. For instance, in a cosmological setting, when we deal with, e.g., the cosmic microwave background radiation, the electromagnetic derivatives, and so $l_{(em)}$, are of the order of the wavelength of the radiation.

Now, in a theory of gravitation the covariant derivative, $\nabla_i$, is the basic differential operator. The commutator $\nabla_i \nabla_k - \nabla_k \nabla_i$ of some vector field $U^m$ is known to produce the Riemann tensor $R_{nik}^m$ according to the relationship

$$(\nabla_i \nabla_k - \nabla_k \nabla_i) U^m = U^{rn} R_{nik}^m.$$  \hfill (A2)

This means that, when we consider the Riemann tensor, the Ricci tensor $R_{pq} = R_{mpq}$ and the Ricci scalar $R = \sum_p R_{pp}$, we deal, in fact, with quantities of second order with respect to the covariant derivative $\nabla_k$. Equivalently, these tensors are quantities up to second order in the partial derivative of the metric. This, in turn, means that the metric introduces a gravitational scale parameter $l_{(g)}$ which is another independent scale. Such a scale can be a cosmological distance, a radius of a star, or any other relevant parameter. In addition, concerning the terms of the type $\nabla_a U_m$, we treat it as a quantity of the first order in a metric derivative, as this covariant derivative contains a partial derivative of the metric and we suppose that the gravity field alone makes the aether non-uniform. Thus a derivative of the aether velocity also picks the gravitational scale parameter $l_{(g)}$. Note that the covariant derivative of the Maxwell tensor, $\nabla_m F_{ik}$ contains both types of derivative, namely, a second and first order electromagnetic derivative and a first order metric derivative.

The electromagnetic derivatives and the metric derivatives are, in general, of different character. For instance, in a cosmological setting, when we deal with, e.g., the cosmic microwave background radiation, the electromagnetic derivative is related to the wavelength of the electromagnetic wave and is of the order of 1 micrometer, while at the same time the metric derivative could be of cosmological scale.

Based on these consideration below we use the following classification for the scalar terms that can enter into the action functional: a scalar term is of the type $(M, N)$ if it contains $M$th order metric derivatives (i.e., it is of the $M$th order with respect to $l_{(g)}^{-1}$), and if it contains $N$th order electromagnetic derivatives (i.e., it is of the $N$th order with respect to $l_{(em)}^{-1}$). This classification scheme is directly related to the order $d$ scheme elaborated in [22] where $d = M + N$. Our two parameter version of the classification of the Lagrangian terms does not contradict this $d$ scheme and is, in fact, its concretization. Within a given Lagrangian and action with their corresponding coupling constants and terms, our classification scheme is useful to pick up the important terms in a given concrete physical setting.

We want to display all the terms up to four orders in the derivatives. Thus, let us discuss the structure of all the terms for which $M + N \leq 4$. This means there is one type of zero-order terms: $(0,0)$; two types of the first order terms: $(1,0)$ and $(0,1)$; three types of the second order terms: $(2,0)$, $(1,1)$ and $(0,2)$; four types of the third order terms: $(3,0)$, $(2,1)$, $(1,2)$ and $(0,3)$; five types of the fourth order terms: $(4,0)$, $(3,1)$, $(2,2)$, $(1,3)$ and $(0,4)$.

- $(0,0)$. There is one term of this type involving the aether velocity $U^i$. It is,

$$U^i U_i,$$  \hfill (A3)

and is included in the action functional of the standard Einstein-aether theory. There are other two scalars, namely, $A_m A^m$ and $U_m A^m$, but, since they are not gauge invariant, we omit them.

- $(1,0)$. The terms of this type are of the form

$$\alpha^{ik} \nabla_i U_k.$$  \hfill (A4)

For the tensorial coefficients $\alpha^{ik}$, constructed using the metric $g_{ik}$, the Kronecker tensors $(\delta^i_k, \delta^{ik}$ and higher order Kronecker tensors), the Levi-Civita tensor $\epsilon^{ikab}$, and the unit vector field $U^k$, there is only one appropriate scalar of the type $(1,0)$, namely, $\alpha \Theta$, where $\Theta = \nabla_k U^k$ is the expansion scalar, and $\alpha$ is a coupling constant introduced phenomenologically.
• (0,1). There are no gauge-invariant scalars of the type (0,1) that would contain the Maxwell tensor in combination with the metric $g_{ik}$, the Kronecker tensors ($\delta^i_k$, $\delta^k_i$ and higher order Kronecker tensors), the Levi-Civita tensor $\epsilon^{ikab}$, and the unit vector field $U^k$.

• (2,0). The type (2,0) is exhausted by the terms

$$ R_, $$

and

$$ K^{abmn} \nabla_a U_m \nabla_b U_n, $$

which enter in the action functional of the Einstein-aether theory.

There are other terms, but these can be absorbed or discarded. Indeed, terms with second-order covariant derivatives $K^{ijkl} \nabla_i \nabla_k U_l$, in which $K^{ijkl}$ contains the metric $g_{ik}$, the Kronecker tensors ($\delta^i_k$, $\delta^k_i$ and higher order Kronecker tensors), the Levi-Civita tensor $\epsilon^{ikab}$, and the unit vector field $U^k$, can be rewritten as follows $K^{ijkl} \nabla_i \nabla_k U_l = \nabla_i \left[ K^{ijkl} \nabla_k U_l \right] - \left( \nabla_k U_l \right) \nabla_i \left( K^{ijkl} \right)$. Since the metric $g_{ik}$, the Kronecker tensors ($\delta^i_k$, $\delta^k_i$ and higher order Kronecker tensors), and the Levi-Civita tensor $\epsilon^{ikab}$, are covariantly constant tensors, i.e., $\nabla_i(g_{ik}) = 0$, $\nabla_i\delta^k_i = 0$, $\nabla_i\delta^i_k = 0$, $\nabla_k^{ijkl} = 0$, we obtain from the above mentioned term, $K^{ijkl} \nabla_i \nabla_k U_l = \nabla_i \left[ K^{ijkl} \nabla_k U_l \right] - \left( \nabla_k U_l \right) \nabla_i \left( K^{ijkl} \right)$.

The first term in the right-hand side of this relationship is a perfect four-divergence, which can be omitted, and the second term can be included into $K^{abmn} \nabla_a U_m \nabla_b U_n$ by redefinition of the tensor $K^{abmn}$.

As for the nonminimal term $R_{ik} U^i U^k$ it can also be absorbed and discarded. Using (A2), $R_{ik} U^i U^k$ can be rewritten as $R_{ik} U^i U^k = \nabla_i \left[ U^k \nabla_k U^i - U^i \nabla_k U^k \right] + \left( \nabla_i U^i \right) \left( \nabla_k U^k \right) - \left( \nabla_m U^m \right) \left( \nabla_k U^k \right)$. The first term in this relationship is a perfect four-divergence, and the other terms can be included in the construction of the Jacobson’s type term $K^{abmn}(\nabla_a U_m)(\nabla_b U_n)$. Here and below we use the parentheses in the expressions of the form $(\nabla_a U_m)T$ just to indicate that the covariant derivative operator acts on $U_m$ only.

• (1,1). The gauge-invariant terms of the type (1,1) can be listed using the representation

$$ A^{mnpq} F_{pq} \nabla_m U_n, $$

where the tensor coefficients $A^{mnpq}$ are constructed using the metric $g_{ik}$, the covariant constant Kronecker tensors ($\delta^i_k$, $\delta^k_i$ and higher order Kronecker tensors), the Levi-Civita tensor $\epsilon^{ikab}$, and the unit vector field $U^k$. There are also terms of the type $A^{mn} \delta_m \nabla_m F_{pq}$. However, these can be reduced to the terms given in Eq. (A7) using the relationships $A^{mpq} \nabla_m F_{pq} = \nabla_m \left[ A^{mpq} F_{pq} \right] - F_{pq} \left( \nabla_m U^j \right) \frac{\partial A^{mpq}}{\partial U^j}$, with the corresponding redefinition of the quantity $A^{mnpq}$.

• (0,2). The representatives of the type (0,2) are given by $F_{mn} F^{mn}$ and $F_{mn} U^m F^{ml} U_l$. Generically, such terms can be described as

$$ C^{ikmn}_{(2)} F_{ik} F_{mn}, $$

where $C^{ikmn}_{(2)}$ is called the linear response tensor. The subscript (2) indicates here that this term is quadratic in the Maxwell tensor $F_{ik}$.

• (3,0). The type (3,0) includes terms of three subtypes:

$$ Z^{ikmnls}_{(1)} \left( \nabla_i U_k \right) \left( \nabla_m U_n \right) \left( \nabla_l U_s \right), $$

$$ Z^{imlns}_{(2)} \left( \nabla_i U_m \right) \left( \nabla_m U_n \right) \left( \nabla_l U_s \right), $$

$$ Z^{iklnms}_{(3)} R_{ikmn} \nabla_l U_s. $$

There are also terms of the type $Z^{imlns}_{(4)} \nabla_i \nabla_m \nabla_l U_s$. However, these can be transformed into a combination of the terms given in Eqs. (A9) and (A10) by the procedure described for the (2,0) type terms. In addition, terms of the type $Z^{ikmnls}_{(5)} \nabla_l R_{ikmn}$ can be expressed as the terms in Eq. (A11) using integration by parts, namely

$$ Z^{ikmnls}_{(5)} \nabla_l R_{ikmn} = \nabla_l \left[ Z^{ikmnls}_{(5)} R_{ikmn} \right] - R_{ikmn} \left( \nabla_l U_j \right) \frac{\partial Z^{ikmnls}_{(5)}}{\partial U_j}. $$
• (2,1). The list of independent terms of the type (2,1) is:

\[ B^{mnlspq} F_{pq}(\nabla_m U_n)(\nabla_l U_s), \]  
\[ B^{mlspq} F_{pq} \nabla_m \nabla_l U_s, \]  
\[ Q^{ikmnpq} R_{ikmn} F_{pq} = Q R^{ik} U_k F_{lm} U^m. \]

There are also terms of the type \( Q^{iklpq} (\nabla_q F_{pq})(\nabla_i U_k) \), but due to the relationships \( Q^{iklpq} (\nabla_i U_k) = \nabla_i \left[ Q^{iklpq} F_{pq}(\nabla_q U_k) \right] - F_{pq}(\nabla(U))^j_i (\nabla_i U_k) \partial Q^{iklpq} / \partial U^j \) - \( Q^{iklpq} F_{pq}(\nabla_j \nabla_i U_k) \) these terms can be reduced to the terms given in Eqs. \( \text{A12} \) and \( \text{A13} \). Similarly, terms in the second derivative of the Maxwell tensor, i.e., \( Q^{lpq} (\nabla_i \nabla_j F_{pq}) \) can be transformed into terms of the type \( Q^{iklpq} (\nabla_i F_{pq})(\nabla_j U_k) \) and then be reduced again to the terms given in Eqs. \( \text{A12} \) and \( \text{A13} \).

• (1,2). The terms of the type (1,2) can be specified as:

\[ X^{mnikpq}(\nabla_m U_n) F_{ik} F_{pq}. \]  

There are also terms of the type \( X^{mnikpq} F_{ik}(\nabla_1 F_{pq}) \) which, again, can be transformed into the terms given in Eq. \( \text{A15} \) by integration by parts.

• (0,3). The terms of the type (0,3) can be written as

\[ C^{ikmns} R_{ik} F_{lm} F_{qs}, \]  

where \( c^{ikmns} \) is a second-order response tensor.

• (4,0). We divide the type (4,0) into three subtypes. The first one contains various quadratic combinations of the Ricci scalar, Ricci and Riemann tensors, and the unit four-vector \( U^j \), and can be written in an abbreviated form as

\[ Z^{ikmns} R_{ikmn} R_{lspq}. \]

The second subtype consists of combinations of the Ricci scalar, Ricci and Riemann tensors multiplied by covariant derivatives of the unit four-vector, and can be written as two terms, namely,

\[ Z^{ikmns} R_{ikmn}(\nabla_i U_k)(\nabla_p U_q), \quad Z^{ikmns} R_{ikmn}(\nabla_i \nabla_p U_q). \]

Again, all other terms, which contain \( \nabla_i \nabla_q R_{ikmn} \) and \( (\nabla_q U^j)(\nabla_i \nabla_k U_n) \) can be transformed into combinations of the already listed terms in Eq. \( \text{A18} \).

The third subtype does not include the Riemann tensor, contains combinations of the covariant derivatives of the unit four-vector, and can be written as three terms, namely,

\[ Z^{ikmnpq}(\nabla_i U_k)(\nabla_m U_n)(\nabla_l U_s)(\nabla_p U_q), \quad Z^{imnspq}(\nabla_i \nabla_m U_n)(\nabla_l U_s)(\nabla_p U_q), \quad Z^{imnlpq}(\nabla_i \nabla_m U_n)(\nabla_l \nabla_p U_q). \]

Similarly, the terms that contain \( \nabla_i \nabla_q \nabla_a U_j \) and \( \nabla_p \nabla_q \nabla_a U_j \) can be transformed into combinations of the already listed terms in Eq. \( \text{A19} \).

Let us stress, that in fact we can consider in Eq. \( \text{A19} \) only the symmetrized terms \( \nabla_{(i} \nabla_{m)} U_n \), since, according to \( \text{A2} \), its skew-symmetric part \( \nabla_{[i} \nabla_{m]} U_n \) can be expressed using the Riemann tensor, and the corresponding scalar, \( Z^{imnlpq}(\nabla_{[i} \nabla_{m]} U_n)(\nabla_l U_s)(\nabla_p U_q) = \frac{1}{2} Z^{imnlpq} U^j R_{(mn} U_{ij} U_s)(\nabla_p U_q), \) can be reduced to the terms given in Eq. \( \text{A18} \). We are using the standard symbols for symmetrization \( T_{(ik)} = \frac{1}{2} \left[ T_{(ik)} + T_{(ki)} \right] \), and skew-symmetrization \( T_{[ik]} = \frac{1}{2} \left[ T_{(ik)} - T_{(ki)} \right] \).
• (3,1). The terms of the type (3,1) can be written by a simple extension of the nomenclature used for the terms of the type (3,0), i.e.,

\[ Z_{(1)}^{ikmnlpq} F_{pq} (\nabla_i U_k)(\nabla_m U_n)(\nabla_l U_s) , \]  

(A20)

\[ Z_{(2)}^{ikmnlpq} F_{pq} (\nabla_i \nabla_m U_n)(\nabla_l U_s) , \]  

(A21)

\[ Z_{(3)}^{ikmnpq} F_{pq} \nabla_i \nabla_k \nabla_m U_n , \]  

(A22)

\[ Z_{(4)}^{ikmnlpq} F_{pq} R_{ikmn} \nabla_l U_s , \]  

(A23)

\[ Z_{(5)}^{ikmnlpq} F_{pq} \nabla_i R_{ikmn} . \]  

(A24)

The terms in the covariant derivative of the Maxwell tensor, \( \nabla_j F_{pq} \), can be reduced to the listed terms in Eqs. (A20)–(A24) by an integration by parts, not being necessary to repeat the procedure here.

• (2,2). The type (2,2) is relevant in our considerations. There are four subtypes.

The first subtype contains the covariant derivatives of the unit vector field, but does not include the nonminimal terms constructed using the Ricci scalar, and the Ricci and Riemann tensors. It has two terms, namely,

\[ \gamma^{mnlspq} F_{ik} F_{pq} (\nabla_m U_n)(\nabla_l U_s) , \quad \gamma^{mnlspq} F_{ik} F_{pq} (\nabla_m \nabla_l U_s) . \]  

(A25)

The second subtype contains only nonminimal terms (see also [44–50]), it does not contain the unit vector field \( U^l \). The independent terms are three, namely,

\[ RF_{ik} F^{ik} , \quad R^{ik} F_{lm} F_{km} , \quad R^{ikmn} F_{ik} F_{mn} . \]  

(A26)

There are other nonminimal terms, i.e., \( R^{ikmn} F_{ik} F_{mn}^{*} , \quad * F_{ik} F_{mn}^{*} , \quad * F_{ik} F_{mn} F_{mn}^{*} , \quad * F_{ik} F_{mn} F_{mn} , \quad R^{ikmn} F_{ik} F_{mn} , \quad * R^{ikmn} F_{ik} F_{mn} \), and so on, where an asterisk means we are taking the dual of the respective tensor with the Levi-Civita tensor.

However, these terms can be reduced to a combination of the terms given in Eq. (A26).

The third subtype includes independent combinations of the following scalars, \( RF_{im} U_m F_{im} U_n , \quad R^{ip} U^q U^q F_{im} U_m F_{ip} F_{mn} \), \( R^{ikmn} U_k U_m F_{ip} F_{lp} F_{mn} \), \( R^{ikmn} U_k U_m F_{ip} F_{lp} F_{mn} \), and their analogs containing the pairs of dual quantities \( * R^{ikmn} \) with \( F_{pq}^{*} \) and \( * R^{ikmn} \) with \( F_{pq}^{*} \). Generically, all these terms can be written as

\[ S_{ikmnlpq} R_{ikmn} F_{ls} F_{pq} . \]  

(A27)

They are extensions of the nonminimal terms.

The fourth subtype includes the irreducible terms which are quadratic in the covariant derivatives of the Maxwell tensor, namely,

\[ G^{ikpqmn} (\nabla_p F_{ik})(\nabla_q F_{mn}) . \]  

(A28)

There are other terms that could be included. However, the terms that contain first covariant derivatives of the Maxwell tensor \( G^{ikpqmn} F_{ik}(\nabla_p F_{mn})(\nabla_q U_j) \) can be reduced to a combination of the terms given in Eq. (A25).

In addition, the scalars in the second covariant derivative of the Maxwell tensor \( G^{ikpqmn} F_{ik}(\nabla_p \nabla_q F_{mn}) \) can also be transformed into terms of the type (2,2) already listed above.

• (1,3). Similarly to the case (0,3) one obtains that all the terms of this subtype can be written as

\[ C_{(5)}^{ikmnlpq} F_{ik} F_{mn} F_{ls} \nabla_p U_q . \]  

(A29)

The terms containing \( \nabla_i F_{pq} \) again can be reduced to the scalars of the type given in Eq. (A29).

• (0,4). Similarly to the cases (0,2) and (0,3) one obtains the terms

\[ C_{(4)}^{ikmnlpq} F_{ik} F_{mn} F_{ls} F_{pq} . \]  

(A30)

where the tensor \( C_{(4)}^{ikmnlpq} \) describes a nonlinear electromagnetic response of the third order.
2. Remarks

Some remarks related to our classification \( M + N \leq 4 \) scheme are in order.

(i) Following the study of the dynamical evolution of a scalar field \( \phi \) in the primordial universe [28], in the framework of effective field theory, one has that the derivative of a scalar field \( \nabla_k \phi = \partial_k \phi \) is considered as a metric derivative, i.e., it is a quantity of the order \( l_{(g)}^{-1} \), since the variations of the scalar field are produced by the dynamics of the gravitational field. Thus, terms of the type \( R_{ik} \nabla^i \phi \nabla^j \phi \) and \((g^{ik} \nabla_i \phi)(\nabla^j \phi)(\nabla_k \phi)\) that appear in the Lagrangian presented in [28] are metric derivatives of the fourth order. There is a correspondence to our case. First, \( \nabla_m U_n \) is a metric derivative of first order, i.e., of the order \( l_{(g)}^{-1} \). Second, terms of the type \( R^{ik} \nabla_i U_m \nabla_k U^m \) and \( (U^p \nabla_p \nabla_j U^q) (\nabla_k U_l) (\nabla_k U^l) \) are then metric derivatives of the fourth order, i.e., of the type \((4,0)\) in our classification.

(ii) Following [58], the electromagnetic derivatives and the metric derivatives are independent, i.e., the parameters \( l_{(g)}^{(a)} \) and \( l_{(em)}^{(b)} \) to be independent. This means, for instance, that the terms of the type \((2,2)\) (see Eq. (A22)) can be of the same order of magnitude as the terms of the type \((1,1)\), when the wavelength of an electromagnetic wave \( \lambda_{(em)} \approx l_{(em)} \) is of the order of \( l_{(g)}^{(a)} \frac{\mu}{c} \), where \( Y \) and \( A \) are the typical values of the components of the tensors \( Z_{mnlsikpq} \) and \( A_{mnpq} \), respectively. Similarly, there are cases when the terms of the type \((2,2)\) can be considered as leading order terms in comparison with, e.g., terms of the type \((4,0)\). There are also special cases, when \( l_{(em)} \) and \( l_{(g)} \) are of the same order, and we should consider terms of the type \((2,2)\) to be of the same order of magnitude, as terms, e.g., of the type \((4,0)\) and \((3,1)\). That is why we listed all the terms of the type \((M,N)\), for which \( M + N \leq 4 \).

(iii) We note that a general formulation of the Einstein-Maxwell-aether theory does not allow the explicit introduction of the parameter \( l_{(g)} \) and the definition of the corresponding dimensionless coupling constants. However, this becomes possible, when one deals with applications of the theory to cosmology and astrophysics. For instance, in [56] studying nonminimal traversable electric wormholes we have introduced three parameters with the dimension of length, namely the gravitational \( r_M^{(a)} \) and the electric \( r_Q^{(a)} \) radii, related to the mass \( M \) and charge \( Q \) of the object, respectively, as well as, the nonminimal radius \( r_Q^{(a)} \) connected with the nonminimal coupling parameter \( q_1 \). Dimensionless parameters \( r_M^{(a)} / r_Q^{(a)} \) and \( r_Q^{(a)} / r_M^{(a)} \) became then the guiding parameters in the analysis of the wormhole solution. We expect that in applications of the Einstein-Maxwell-aether theory the introduction of coupling parameters will appear naturally.

(iv) In addition, from the point of view of dimensional units, the coefficients \( A_{mnpq} \), \( B_{mnpq} \), \( X_{mnpq} \), \( Y_{mnpq} \) (see Eq. (29)) differ from each others by powers in units of length. In a generic formulation there is no interest in introducing multipliers to provide the same dimensionality for these tensorial objects. On the other hand, a units redefinition of the coupling parameters will perhaps be of interest when one deals with applications of the theory.

3. The ansatz

a. Requirements for the ansatz

We impose now three requirements that our theory should satisfy.

(a) The electrodynamics of the theory must be linear in the Maxwell tensor \( F_{ik} \) and of second order in the partial derivatives of the electromagnetic potential four-vector \( A_i \). These requirements imply that the terms of the type \((0,3)\), \((1,3)\), \((0,4)\), given in Eqs. (A10), (A20), and (A30), respectively, and the term given in Eq. (A25) quadratic in the derivative of the Maxwell tensor of the type \((2,2)\) are not present in the theory.

(b) The dynamical equations for the unit vector field \( U^i \) are considered to be a set of quasilinear equations of second order in their partial derivatives. This requirement implies that the terms of the type \((3,0)\) given in Eqs. (A9) and (A10), of the type \((4,0)\) given in Eq. (A19), and of the type \((3,1)\) given in Eqs. (A20), (A21), and (A22), are not present in the theory. A note is in order: According to the standard terminology in mathematical physics quasilinear means that the equations can be nonlinear in the four-vector \( U^i \) itself, nonlinear in the first covariant derivative \( \nabla_i U_k \), but the second partial derivatives \( \partial_i \partial_k U_s \) enters the equations linearly with tensorial coefficients that can depend on \( U^i \) and \( F_{mn} \), but can not contain \( \nabla_i U_k \).

(c) The equations for the gravitational field are considered to be equations of second order in the partial derivatives of the metric (similarly to the standard Einstein’s and Einstein-aether theories). This requirement implies that the terms of the type \((3,0)\) given in Eq. (A11), of the type \((4,0)\) given in Eqs. (A17) and (A18), and of the type \((3,1)\) given in Eqs. (A23) and (A24), are not present in the theory. All the other terms are included into the action functional of the theory we propose, see next section.

This set of requirements (a), (b), and (c) can be reformulated as the assumption that the discarded terms have coefficients, phenomenologically introduced, that are small enough in comparison with the non-discarded coupling constants.
b. The ansatz

With these requirements, the ansatz for the Lagrangian and the action can then be given as in Eq. (20) and subsequent equations.

Appendix B: Reconstruction of the electrodynamic tensors $X'^{sikmn}$ and $\gamma'^{alsikmn}$ in terms of electrodynamic constants and spacetime tensors

We recall that in a medium moving with velocity $U^i$ the currentless equations of electrodynamics can be rewritten as the four Maxwell equations and two constitutive equations, as stated in Sect. III C. The four Maxwell equations are the Gauss law

$$\Delta^m \nabla_m D^k = \omega^k \mathcal{H}^k,$$  \hfill (B1)

the law of the magnetic flux conservation

$$\Delta^m \nabla_m B^k = -\omega^k E^k,$$  \hfill (B2)

the Ampère law

$$\Delta^k \mathcal{D}_k - \eta^{ikm} \nabla_k \mathcal{H}_m = -2\Delta^k \mathcal{H}_m \omega^{skm} + \left( \sigma^{ik} - \omega^{ik} - \frac{2}{3} \Theta \Delta^{ik} \right) \mathcal{D}^k,$$  \hfill (B3)

and the Faraday law,

$$\Delta^k \mathcal{D}_k + \eta^{ikm} \nabla_k E_m = 2\Delta^k E_m \omega^{skm} + \left( \sigma^{ik} - \omega^{ik} - \frac{2}{3} \Theta \Delta^{ik} \right) B_k.$$  \hfill (B4)

The constitutive equations are

$$\mathcal{D}^i = \mathcal{P}^i - \frac{\varepsilon_i}{4} E^k - \nu^k B_k, \quad \mathcal{H}^i = \mathcal{M}^i - \left( \mu^{-1} \right)^{ik} B_k + \nu^k E_k.$$  \hfill (B5)

We used here the standard definition $\omega^i \equiv -\epsilon^{ikm} U_n \nabla_k U_m$ for the local angular rotation velocity of the medium.

The reconstruction started in Sect. IV D of the quantities $X'^{mnabpq}$ and $\gamma'^{mnlsabpq}$ yields, respectively,

$$X'^{sikmn} = \frac{1}{2} \left( \alpha_1 - \frac{1}{3} \alpha_9 \right) \Delta^s \left( g^{ikmn} - \Delta^{ikmn} \right) + \frac{1}{4} \alpha_6 U_p U_q \left[ g^{iklp} g^{mnsq} + g^{mnlp} g^{iksq} \right] +$$

$$+ \frac{1}{2} \left( \gamma_1 - \frac{1}{3} \gamma_6 \right) \Delta^s \Delta^{ikmn} - \frac{1}{2} \gamma_6 \eta^{ikl} \eta^{jmn} - \nu_2 U^q \left\{ \Delta^{iks} B^m U_n + \Delta^m ns U^l \right\},$$  \hfill (B6)

$$\gamma'^{alsikmn} = \frac{1}{2} \left( g^{ikmn} - \Delta^{ikmn} \right) \left[ \alpha_2 U^l U^q \Delta^{ls} + \left( 3 - \frac{1}{3} \alpha_4 + \frac{1}{9} \alpha_9 \right) \Delta^{ab} \Delta^{ls} + \alpha_4 \Delta^{a(l \Delta s)b} + \frac{1}{2} \alpha_5 \Delta^{als} \right] -$$

$$- \frac{1}{4} \alpha_7 \left\{ U_p U_q \left[ \Delta^{ab} g^{ikp(l g^s)qmn} + \Delta^{ls} g^{ikp(a g^b)qmn} \right] + \frac{2}{3} \Delta^{ab} \Delta^{ls} \left( g^{ikmn} - \Delta^{ikmn} \right) \right\} -$$

$$- \frac{1}{2} \alpha_9 U^a U^q \left[ g^{ikp(b g^s)qmn} + \right.$$

$$+ \frac{1}{2} \alpha_9 U_p U_q \left\{ \frac{1}{3} \left[ \Delta^{ab} g^{ikp(l g^s)qmn} + \Delta^{ls} g^{ikp(a g^b)qmn} \right] - \frac{1}{2} \left[ g^{ikp(a \Delta b)(l g^s)qmn} + g^{ikp(l \Delta s)(a g^b)qmn} \right] \right\} +$$

$$+ \frac{1}{8} \alpha_{10} U_p \left[ g^{ikp U^m \Delta n a ls} - g^{ikap U^m \Delta n b ls} + g^{mnbp U^i \Delta k a ls} - g^{mnap U^i \Delta k b ls} + \right.$$
\[+ g^{\alpha \beta \gamma \delta} U^{[a} \frac{\partial}{\partial \alpha_{[b}} U^{[c] \partial_{c]}}} - \frac{1}{8} \alpha_{11} \eta^{a} \left\{ g^{\alpha \beta \gamma \delta} U^{[a} \frac{\partial}{\partial \alpha_{[b}} U^{[c] \partial_{c]}}} + g^{\alpha \beta \gamma \delta} U^{[a} \frac{\partial}{\partial \alpha_{[b}} U^{[c] \partial_{c]}}} + g^{\alpha \beta \gamma \delta} U^{[a} \frac{\partial}{\partial \alpha_{[b}} U^{[c] \partial_{c]}}} + g^{\alpha \beta \gamma \delta} U^{[a} \frac{\partial}{\partial \alpha_{[b}} U^{[c] \partial_{c]}}} \right\} +
\]
\[+ g^{\alpha \beta \gamma \delta} U^{[a} \frac{\partial}{\partial \alpha_{[b}} U^{[c] \partial_{c]}}} + g^{\alpha \beta \gamma \delta} U^{[a} \frac{\partial}{\partial \alpha_{[b}} U^{[c] \partial_{c]}}} + g^{\alpha \beta \gamma \delta} U^{[a} \frac{\partial}{\partial \alpha_{[b}} U^{[c] \partial_{c]}}} + g^{\alpha \beta \gamma \delta} U^{[a} \frac{\partial}{\partial \alpha_{[b}} U^{[c] \partial_{c]}}} \right\} +
\]
\[+ \frac{1}{2} \Delta^{i j k m n} \left[ \gamma_{2} U^{a} U^{b} \Delta^{i j k m n} + \left( \frac{\gamma_{3} - \frac{1}{3} \gamma_{4}}{2} \right) \Delta^{a i} \Delta^{i j k m n} + \frac{1}{2} \Delta^{a i} \Delta^{i j k m n} \right] -
\]
\[- \frac{1}{4} \gamma_{8} U^{a} U^{b} \left\{ \eta^{a b} \eta^{c d} \eta^{e f} \eta^{g h} \eta^{i j k m n} + \frac{1}{2} \gamma_{9} \left\{ \frac{1}{3} \left[ \Delta^{a i} \eta^{i j k m n} + \Delta^{i j k m n} \eta^{a i} \right] -
\]
\[- \frac{1}{4} \gamma_{11} \left\{ \eta^{a b} \eta^{c d} \eta^{e f} \eta^{g h} \gamma^{i j k m n} \right\} + \frac{1}{2} \gamma_{4} \left[ \Delta^{a i} \eta^{i j k m n} + \Delta^{i j k m n} \eta^{a i} \right] -
\]
\[- \frac{1}{2} \nu_{4} \left[ \Delta^{a i} U^{[a} \frac{\partial}{\partial \alpha_{[b}} U^{[c] \partial_{c]}}} + \Delta^{i j k m n} U^{a} \left[ \Delta^{a i} \eta^{i j k m n} + \Delta^{i j k m n} \eta^{a i} \right] \right] +
\]
\[+ \frac{1}{8} \left( \nu_{5} - \nu_{6} \right) \Delta^{a i} \left[ U^{[a} \frac{\partial}{\partial \alpha_{[b}} U^{[c] \partial_{c]}}} \right] + \Delta^{i j k m n} \left[ U^{[a} \frac{\partial}{\partial \alpha_{[b}} U^{[c] \partial_{c]}}} \right] +
\]
\[+ \frac{1}{32} \left( \nu_{7} + \nu_{8} \right) \Delta^{a i} \left[ U^{[a} \frac{\partial}{\partial \alpha_{[b}} U^{[c] \partial_{c]}}} \right] + \Delta^{i j k m n} \left[ U^{[a} \frac{\partial}{\partial \alpha_{[b}} U^{[c] \partial_{c]}}} \right] +
\]
\[+ \frac{1}{8} \nu_{9} \left\{ U^{a} \left[ \eta^{a b} \eta^{c d} \eta^{e f} \eta^{g h} \eta^{i j k m n} \right] + \eta^{a b} \eta^{c d} \eta^{e f} \eta^{g h} \eta^{i j k m n} \right\} + \Delta^{a i} \left[ U^{[a} \frac{\partial}{\partial \alpha_{[b}} U^{[c] \partial_{c]}}} \right] +
\]
\[+ \nu_{10} \left\{ U^{a} \left[ \eta^{a b} \eta^{c d} \eta^{e f} \eta^{g h} \eta^{i j k m n} \right] + \eta^{a b} \eta^{c d} \eta^{e f} \eta^{g h} \eta^{i j k m n} \right\} + \Delta^{a i} \left[ U^{[a} \frac{\partial}{\partial \alpha_{[b}} U^{[c] \partial_{c]}}} \right] +
\]
\begin{align}
&+ \frac{1}{8} \nu_{11} \left\{ U^a \left[ \eta^{iks} U^{[m} \eta^n]^{[ib} + \eta^{ikl} U^{[m} \eta^n]^{sb} + \eta^{mn} U^{[i} \eta_k]^{[lb} + \eta^{mn} U^{[i} \eta_k]^{sb} - \\
&\quad - \Delta^{ikb} U^{[m} \Delta^{n]s} - \Delta^{iks} U^{[m} \Delta^{n]l} - \Delta^{mln} U^{[i} \Delta^{k]s} - \Delta^{mn} U^{[i} \Delta^{k]l} \right] + \\
&\quad + U^l \left[ \eta^{ikb} U^{[m} \eta^{n]as} + \eta^{ik} U^{[m} \eta^{n]bs} + \eta^{mn} U^{[i} \eta^{k]as} + \eta^{mnt} U^{[i} \eta^{k]bs} - \\
&\quad - \Delta^{ik} U^{[m} \Delta^{n]b} - \Delta^{iks} U^{[m} \Delta^{n]a} - \Delta^{mnas} U^{[i} \Delta^{k]b} - \Delta^{mnb} U^{[i} \Delta^{k]a} \right] \right\} + \\
&+ \frac{1}{4} \nu_{12} \left\{ - \frac{2}{3} U^a \Delta^{ls} \left( U^{[m} \Delta^{n]}^{l} \eta^s + U^{[i} \Delta^{k]}^{j} b m n \right) - \frac{2}{3} U^l \Delta^{ab} \left( U^{[m} \Delta^{n]}^{a} \eta^b + U^{[i} \Delta^{k]}^{j} \eta^b m n \right) + \\
&\quad + U^a \left[ U^{[m} \Delta^{n]}(l) \Delta^{s} \eta^{s} \eta^{ik} + U^{[i} \Delta^{k]}(l) \Delta^{s} \eta^{j} b m n - U^{[m} \eta^{n]}(l) \eta^{j} \eta^{ik} - U^{[i} \eta^{k]}(l) \eta^{j} \eta^{b} m n \right] + \\
&\quad + U^l \left[ U^{[m} \Delta^{n]}(a) \Delta^{b} \eta^{s} \eta^{ik} + U^{[i} \Delta^{k]}(a) \Delta^{b} \eta^{s} \eta^{j} m n - U^{[m} \eta^{n]}(a) \eta^{b} \eta^{ik} - U^{[i} \eta^{k]}(a) \eta^{b} \eta^{j} m n \right] \right\}. \quad (B7)
\end{align}

References

[1] T. Jacobson and D. Mattingly, Phys. Rev. D 64 (2001) 024028.
[2] C. Heinicke, P. Baekler and F. W. Hehl, Phys. Rev. D 72 (2005) 025012.
[3] B. Z. Foster, Phys. Rev. D 73 (2006) 024005.
[4] C. Eling and T. Jacobson, Class. Quant. Grav. 23 (2006) 5625.
[5] C. Eling and T. Jacobson, Class. Quant. Grav. 23 (2006) 5643.
[6] C. Eling, T. Jacobson and M. C. Miller, Phys. Rev. D 76 (2007) 042003.
[7] T. Jacobson, PoS-QG-Ph (2007) 020.
[8] E. Barausse, T. Jacobson and T. P. Sotiriou, Phys. Rev. D 83 (2011) 124043.
[9] P. Berglund, J. Bhattacharyya and David Mattingly, Phys. Rev. Lett. 110 (2013) 071301.
[10] T. Jacobson, [arXiv:1310.5115] [gr-qc].
[11] C. M. Will, Theory and experiment in gravitational physics, Cambridge University Press, Cambridge, 1993.
[12] K. Bamba, S. Nojiri and S. D. Odintsov, Phys. Rev. D, 77 (2008) 123532.
[13] J. B. Jimenez and A. L. Maroto, Phys. Rev. D 80 (2009) 063512.
[14] C. M. Will and K. Nordtvedt, Astrophys. J., 177 (1972) 757.
[15] K. Nordtvedt and C. M. Will, Astrophys. J., 177 (1972) 775.
[16] R. W. Hellings and K. Nordtvedt, Phys. Rev. D 7 (1973) 3593.
[17] C. Moller, The Theory of Relativity, Clarendon Press, Oxford, 1952.
[18] L. D. Landau, E. M. Lifshitz, and L. P. Pitaevskii, Electrodynamics of Continua, Media, Butterworth-Heinemann, Oxford, 1960, (second edition, Elsevier Butterworth Heinemann, Oxford, 1984).
[19] A. C. Eringen and G. A. Maugin, Electrodynamics of Continua, Volumes I and II, Springer-Verlag, New York, 1990.
[20] F. W. Hehl and Yu. N. Obukhov, Foundations of classical electrodynamics: Charge, flux, and metric, Birkhäuser, Boston, 2003.
[21] A. B. Balakin and H. Dehnen, Phys. Lett. B 681 (2009) 113.
[22] A. Kostecke and M. Mewes, Phys. Rev. D 80 (2009) 015020.
[23] S. Liberati and L. Maccione, Ann. Rev. Nucl. Part. Sci. 59 (2009) 245.
[24] A. Kostecke and N. Russell, Rev. Mod. Phys. 83 (2011) 11.
[25] S. Nojiri and S. D. Odintsov, Phys. Rept. 505 (2011) 59.
[26] S. Liberati, Class. Quant. Grav. 30 (2013) 133001.
[27] C. P. Burgess, Ann. Rev. Nucl. Part. Sci. 57 (2007) 329.
[28] S. Weinberg, Phys. Rev. D 77 (2008) 123541.
[29] B. Withers, Class. Quant. Grav. 26 (2009) 225009.
30

[30] S. Weinberg, Effective Field Theory, Past and Future, arXiv:0908.1964 [hep-th].
[31] S. Liberati, Lect. Notes Phys. 870 (2013) 297.
[32] H. Leutwyler, Annals Phys. 235 (1994) 165.
[33] T. Becher and H. Leutwyler, Eur. Phys. J. C 9 (1999) 643.
[34] N. Yunes and F. Pretorius, Phys. Rev. D 80 (2009) 122003.
[35] T. Baker, P. G. Ferreira and C. Skordis, Phys. Rev. D 87 (2013) 024015.
[36] T. Jacobson and D. Mattingly, Phys. Rev. D 70 (2004) 024003.
[37] K. Yagi, D. Blas, E. Barausse and N. Yunes, Phys. Rev. D xx (2014) xxxxxx; arXiv:1311.7144 [gr-qc].
[38] A. R. Solomon and J. D. Barrow, arXiv:1309.4778 [astro-ph.CO].
[39] R. Bluhm, N.L. Gagne, R. Potting and A. Vrublevskis, Phys. Rev. D 77 (2008) 125007.
[40] M.D. Seifert, Phys. Rev. D 81 (2010) 065010.
[41] A. Kostelecky and J. Tasson, Phys. Rev. D 83 (2011) 016013.
[42] W. Gordon, Ann. Phys. 72 (1923) 421.
[43] V. Perlick, Ray Optics, Fermat’s Principle, and Applications to General Relativity, Springer-Verlag, Berlin (2000).
[44] A.B. Balakin and W. Zimdahl, Gen. Rel. Grav. 37 (2005) 1731.
[45] A.B. Balakin, H. Dehnen and A.E. Zayats, Phys. Rev. D 76 (2007) 124011.
[46] A.B. Balakin, H. Dehnen and A.E. Zayats, Annals Phys. 323 (2008) 2183.
[47] V. Vasileiou et al., arXiv:1305.3463 [astro-ph.HE].
[48] A. B. Balakin, Gravit. Cosmol. 13 (2007) 163.
[49] A. B. Balakin and J. P. S. Lemos, Class. Quant. Grav. 22 (2005) 1867.
[50] A. B. Balakin, V. V. Bochkarev and J. P. S. Lemos, Phys. Rev. D 85 (2012) 064015.
[51] A. B. Balakin and J. P. S. Lemos, Class. Quantum Grav. 19 (2002) 4897.
[52] A. B. Balakin, Class. Quant. Grav. 14 (1997) 2881.
[53] A. B. Balakin and J. P. S. Lemos, Class. Quant. Grav. 18 (2001) 941.
[54] A. B. Balakin, R. Kerner and J. P. S. Lemos, Class. Quant. Grav. 18 (2001) 2217.
[55] T. Yu. Alpin and A. B. Balakin, Gravit. Cosmol. 12 (2006) 307.
[56] A. B. Balakin and W.-T. Ni, Class. Quant. Grav. 27 (2010) 055003.
[57] F. W. Hehl and Yu. N. Obukhov, Lect. Notes Phys. 562 (2001) 479.
[58] I. T. Drummond and S. J. Hathrell, Phys. Rev. D 22 (1980) 343.
[59] A. B. Balakin, J. P. S. Lemos and A. E. Zayats, Phys. Rev. D 81 (2010) 084015.