Semileptonic $B_c$ Decay and Heavy Quark Spin Symmetry

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Abstract

Semileptonic decay of the $B_c$ meson is studied in the heavy quark limit. The six possible form factors for $B_c \to B_s(B^0), B_s^*(B^{*0})$ semileptonic decay are determined by two invariant functions. Only one of these functions contributes at zero recoil, where it is calculable to lowest order in an operator product expansion in terms of the meson decay constant $f_B$ and the $B_c$ wavefunction. A similar result is found for $B_c \to D^0, D^{*0}$ and for $B_c \to \eta_c, J/\psi$ semileptonic decay for a restricted kinematic region. Semileptonic $B_c$ decay provides a means for determining the KM mixing angle $|V_{ub}|$. 
1. Introduction

The $B_c$ meson provides a unique probe of both strong and weak interactions. Unlike quarkonium systems which can decay strongly and electromagnetically, the $B_c$ can only decay weakly and thus is relatively long-lived. The $c$ and $b$ quark lifetimes are similar because of the small mixing angle $V_{cb}$, so $B_c$ decay proceeds through either quark at comparable rates. In this work, we study semileptonic weak decay of the $B_c$ meson by exploiting heavy quark spin symmetry. We will compute the weak decay amplitude for $B_c \rightarrow B_s$ in terms of the $B_s$ meson decay constant $f_{B_s}$. The two are related because the $B_c$ can be treated as a pointlike meson in the limit that the $b$ and $c$ quark masses are much larger than $\Lambda_{QCD}$. The weak decay of the $c$ quark produces a state which has a $\bar{b}$ and $s$ quark at the same point in space (to lowest order in an operator product expansion); the amplitude for this state to turn into a $B_s$ is $f_{B_s}$. A similar argument allows us to compute the amplitudes for $B_c \rightarrow D^0, D_s^0$ and $B_c \rightarrow \eta_c, J/\psi$ semileptonic decay.

The standard application of heavy quark symmetry is to hadrons containing a single heavy quark. We must take some care in defining the heavy quark effective theory when dealing with a system with two heavy quarks. It is well known that the static theory gives rise to severe infrared divergences for diagrams involving two heavy quarks with the same velocity [1]. These divergences are regulated by the kinetic energy term $h_Q(D^2/2m_Q)h_Q$; even though this term is higher order in $1/m_Q$, it may not be neglected in the $m_Q \rightarrow \infty$ limit. The kinetic energy term is different for $b$ and $c$ quarks, and breaks the heavy flavor symmetry. Physically, this is just the statement that the dynamics of heavy-heavy bound states is determined by balancing the kinetic and potential energies of the quarks; the $\Upsilon$ is not the same size as the $J/\psi$, and we cannot use the heavy flavor symmetry to relate these two states. The kinetic term in the effective Lagrangian breaks the heavy flavor symmetry, but it does not break the heavy quark spin symmetry. Thus we can still derive relations for hadrons with two heavy quarks using heavy quark spin symmetry.

2. Spin Symmetry

The invariance of the effective Lagrangian under individual spin rotations on the $b$ and $c$ quarks allows us to relate the form factors for vector and axial vector currents between the $B_c$ and various pseudoscalar and vector mesons in the same way as for heavy-light systems [2].
Let us first consider the semileptonic decays $B_c \rightarrow B_s(B_0^0) e^+ \nu$ or $B_c \rightarrow B_s^*(B^{*0}) e^+ \nu$ in which the the $B_c$ decays into a $B$ meson containing a light quark. These decays correspond to the semileptonic weak decay of the charm quark into a light $s$ or $d$ quark. Since the mass of the $b$ quark is much greater than that of the $c$ quark, the energy released in the decay of the $c$ quark is much smaller than $m_b$, and the $b$ quark is not deflected. Thus the velocity of the final meson is the same as the velocity of the initial meson. The initial momentum of the $B_c$ is $p^\mu = m_{B_c} v^\mu$, and the final momentum of the $B_a$ is $p'^\mu = m_{B_a} v^\mu + q^\mu$, where $q$ is a small residual momentum. The final $B_a$ is on shell, so $q \cdot v = O(1/m_B)$. The momentum transfer to the lepton system is

$$k^\mu = p^\mu - p'^\mu = (m_{B_c} - m_{B_a}) v^\mu - q^\mu. \quad (2.1)$$

Heavy quark spin symmetry implies that the pseudoscalar $B_c$ meson is degenerate with the vector $B_c^*$ meson. The consequences of spin symmetry for hadronic matrix elements may be derived using the commutation relations of Isgur and Wise [2], or more compactly using the well-known trace formalism [3]. The lowest-lying $b\bar{c}$ bound states are represented by a $4 \times 4$ matrix

$$H^{(b\bar{c})} = \frac{1 + \not{v}}{2} \left[ B_c^{* \mu} \gamma_\mu - B_c \gamma_5 \right] \frac{1 - \not{v}}{2}, \quad (2.2)$$

where $B_c$ and $B_c^*$ annihilate pseudoscalar and vector meson $b\bar{c}$ bound states of velocity $v$, respectively. A subscript $v$ on the heavy meson fields has been suppressed. Under spin symmetries on the heavy quark and antiquark, the heavy meson field transforms as

$$H^{(b\bar{c})} \rightarrow S_c \ H^{(b\bar{c})} \ S_b^\dagger. \quad (2.3)$$

An analogous definition for $H^{(c\bar{c})}$ describes the $(\eta_c, J/\psi)$ spin multiplet. The spin multiplet for the $B_a$ and $B_a^*$ is given by

$$H^{(b\bar{c})} = \frac{1 + \not{v}}{2} \left[ B_a^{* \mu} \gamma_\mu - B_a \gamma_5 \right] \frac{1 - \not{v}}{2}, \quad (2.4)$$

where the subscript $a = 1, 2, 3$ (or $u, d, s$) is an $SU(3)_V$ flavor index. The field $H^{(b\bar{c})}_a$ is a doublet under heavy quark spin symmetry and a 3 under flavor $SU(3)_V$ symmetry [4],

$$H^{(b\bar{c})}_a \rightarrow \left( U H^{(b\bar{c})}_a \right) S_b^\dagger. \quad (2.5)$$

Note that the pseudoscalar and vector meson fields $B$ and $B^*$ have dimension $3/2$ because they contain factors of $\sqrt{m_B}$ and $\sqrt{m_{B^*}}$ relative to the standard normalization for scalar and vector fields.
The amplitudes for semileptonic $B_c$ decay to $B_a$ and $B_a^*$ are determined by the matrix elements of the weak hadronic current $\bar{q}_a \gamma_\mu (1 - \gamma_5) c$ between the meson states. The most general form for the matrix element of the current which respects the heavy quark spin symmetry is

$$\langle B_a^*, v, q | \gamma_\mu | B_c, v \rangle = \sqrt{m_{B_c} m_{B_a}} \text{Tr} \left( \overline{\Phi}_a(\bar{c}) \Gamma (v, a_0 q) \Gamma (c, \bar{b}) \right),$$

where

$$\Omega(v, a_0 q) = \Omega_1(a_0 q) + a_0 \Omega_2(a_0 q) \not{\phi},$$

is the most general Dirac matrix that can be written in terms of the vectors $q$ and $v$ (recall that $q \cdot v = 0$). Terms with factors of $\not{\phi}$ can be omitted because of the identities

$$\not{\phi} H(c, \bar{b}) = H(c, \bar{b}), \quad H(c, \bar{b}) \not{\phi} = -H(c, \bar{b}), \quad \not{\phi} H_a = H_a, \quad H_a \not{\phi} = -H_a.$$  

Note that the factor of $\Gamma$ multiplying $H(c, \bar{b})$ in Eq. (2.6) is required by the heavy quark spin symmetry on the $c$ quark. The radius of the $B_c$ meson, $a_0$, is the typical scale for the variation of the form factors (as will be shown in the next section). For a Coulomb bound state, $a_0^{-1} \sim \alpha_s (a_0^{-1}) m_c$; the linear confining term in the potential makes the state somewhat smaller than this estimate. In our case, $a_0(B_c) \simeq a_0(J/\psi) \simeq (500 \text{ MeV})^{-1}$. [3]

Explicit evaluation of Eq. (2.6) gives

$$\langle B_a, v, q | V_\mu | B_c, v \rangle = \sqrt{2m_{B_c} 2m_{B_a}} [\Omega_1 v_\mu + a_0 \Omega_2 q_\mu],$$

$$\langle B_a^*, v, q | V_\mu | B_c, v \rangle = -i \sqrt{2m_{B_c} 2m_{B_a}^*} a_0 \Omega_2 \epsilon_{\mu\nu\alpha\beta} \epsilon^{*\nu} q^\alpha v^\beta,$$

$$\langle B_a^*, v, q | A_\mu | B_c, v \rangle = \sqrt{2m_{B_c} 2m_{B_a}^*} \left[ \Omega_1 \epsilon^{*\mu} + a_0 \Omega_2 \epsilon^{*} \cdot q v_\mu \right],$$

where $V_\mu$ and $A_\mu$ refer to the vector and axial vector currents $\overline{q}_a \gamma_\mu c$ and $\overline{q}_a \gamma_\mu \gamma_5 c$, respectively, and $\epsilon_\mu$ is the polarization vector of the $B_a^*$. The form factor $\Omega_2$ is irrelevant for semileptonic $B_c \to B_s(B^0)$ decay because the contribution of $\Omega_2$ to the decay amplitude will be proportional to the lepton mass. In addition, $\Omega_2$ does not contribute to decay amplitudes at zero recoil, $q = 0$. Note that the dimensionless functions $\Omega_i(a_0 q)$ are independent of the light quark flavor index in the $SU(3)_V$ limit. Thus, the ratio of KM mixing angles $|V_{cs}/V_{cd}|$ can be extracted from comparison of $B_c$ semileptonic decay to $B_s, B_s^*$ and $B^0, B^{*0}$. Leading $SU(3)_V$-violating light quark flavor dependence of the form factors may be estimated in chiral perturbation theory [3].

A similar analysis applies to the decays $B_c \to D^0$ and $B_c \to D^{*0}$ in which the $\bar{b}$ quark decays to a $\tau$. In this case, however, the light antiquark will typically recoil with
momentum comparable to or larger than the $c$ quark mass. In order for the final meson to be bound, there must be a correspondingly large momentum transfer to the spectator $c$ quark, and the effective theory breaks down. Higher dimensional operators which we have neglected will be of order $q/m_c$ and will dominate for large momentum transfer to the light degrees of freedom. Our results are thus valid only for $q \ll m_c$, i.e. near the zero recoil point. With this caveat, the analysis proceeds exactly as before. The amplitudes can be written in terms of two invariant functions $\Sigma_1(a_0q)$ and $\Sigma_2(a_0q)$,

\begin{align}
\langle D^0, v, q | V_\mu | B_c, v \rangle &= \sqrt{2m_{B_c} 2m_D} \left[ \Sigma_1 v_\mu + a_0 \Sigma_2 q_\mu \right], \\
\langle D^{*0}, v, q | V_\mu | B_c, v \rangle &= -i \sqrt{2m_{B_c} 2m_D} a_0 \Sigma_2 \epsilon_{\mu\nu\alpha\beta} \epsilon^{*\nu} q^\alpha v^\beta, \\
\langle D^{*0}, v, q | A_\mu | B_c, v \rangle &= \sqrt{2m_{B_c} 2m_D} \left[ \Sigma_1 \epsilon^{*}_\mu + a_0 \Sigma_2 \epsilon^{*} \cdot q v_\mu \right].
\end{align}

Measurement of this decay provides a means of determining the KM angle $|V_{ub}|$.

Finally, we analyze the semileptonic decays of the $B_c$ to the charmonium mesons $\eta_c$ and $\psi$. Once again, the momentum transfer to the produced $\bar{c}$ quark may be large and our results are only valid near the zero recoil point. In this case, however, there is an additional spin symmetry of the produced antiquark, which forbids a form factor proportional to $q$. Thus, the matrix elements for the semileptonic decay of $B_c$ to $\eta_c$ and $\psi$ near zero recoil are determined by a single function $\Delta(a_0q)$:

\begin{align}
\langle \eta_c, v, q | V_\mu | B_c, v \rangle &= \sqrt{2m_{B_c} 2m_{\eta_c}} \Delta v_\mu, \\
\langle \psi, v, q | A_\mu | B_c, v \rangle &= \sqrt{2m_{B_c} 2m_{\psi}} \Delta \epsilon^{*}_\mu,
\end{align}

where $V_\mu$ and $A_\mu$ refer to the vector and axial vector currents $\bar{b}\gamma_\mu c$ and $\bar{b}\gamma_\mu\gamma_5 c$, respectively, and $\epsilon_\mu$ is the polarization vector of the $\psi$.

3. The Scale of Variation of Form Factors

The invariant tensors in Eq. (2.6) are multiplied by dimensionless form factors $\Omega_i(a_0q)$. In this section, we explain in greater detail why the scale of variation of the form factors is set by $a_0$, the radius of the $B_c$ bound state. Before discussing the case of interest, it is useful to first consider the scale of variation of form factors for two different circumstances — the matrix elements for a heavy quark current between two heavy-light mesons and the matrix elements for a light quark current between two heavy-light mesons. Let us first analyze the well-known example of the matrix elements of a heavy quark current between meson
states containing a single heavy quark. For concreteness, consider the matrix elements of
the current $\bar{b}\Gamma b$ between $B$ and $B^*$ mesons. The matrix elements can be evaluated using
the trace formalism,

$$\langle B, v' | \bar{b}\Gamma b | B, v \rangle = m_B \text{ Tr} \left( \overline{H_v^{(b)}} \Omega(v, v') H_{v'}^{(b)} \Gamma \right), \quad (3.1)$$

where $v$ and $v'$ are the velocities of the initial and final meson fields, respectively. The Dirac
matrix coupling the heavy quark spin indices in the trace must be $\Gamma$ by the spin symmetry.
The matrix $\Omega$ coupling the light quark indices is not constrained, and is the most general
possible Dirac matrix that can be constructed out of $v$ and $v'$. From Eq. (2.8), it follows
that there is only one possible invariant for $\Omega$, the Isgur-Wise function $\xi(v \cdot v')$. This
dimensionless nonperturbative function is the form factor for the light degrees of freedom
in the heavy-light mesons. The light degrees of freedom of a $B$ meson with definite velocity
$v$ carry a momentum which is typically of order $\Lambda_{\text{QCD}} v$. Thus, the momentum transfer
to the light degrees of freedom in the above transition is of order $\Lambda_{\text{QCD}} (v - v')$. Since
hadronic form factors for the light degrees of freedom vary on the momentum scale $\Lambda_{\text{QCD}}$,
the variation of the Isgur-Wise function is controlled by $\Lambda_{\text{QCD}} (v - v')/\Lambda_{\text{QCD}} = v - v'$. Thus the Isgur-Wise function $\xi(v \cdot v')$ varies on the scale over which $v \cdot v'$ changes by order
one.$^1$

The scale of variation of form factors is different for the matrix elements of a light quark
current between the same states. In the following, we will assume that the momentum
transfer of the transition is small compared with the mass of the heavy quark, so that the
velocity of the $B$ meson is not changed. The matrix elements of the light quark current
$\bar{d}\gamma_\mu d$ are given by

$$\langle B, v, q | \bar{d}\gamma_\mu d | B, v, 0 \rangle = m_B \text{ Tr} \left( \overline{H_v^{(b)}} \Omega_\Gamma(v, q) H_{v'}^{(b)} \right), \quad (3.2)$$

where the states are described by both a velocity $v$ and a residual momentum $q$. The
Dirac matrix coupling the heavy quark indices is the identity matrix because of the heavy
quark spin symmetry. The Dirac matrix $\Omega_\Gamma$ coupling the light quark indices is the most

$^1$ The $B^* \to B\gamma$ electromagnetic transition amplitude due to the $b$ quark current $\bar{b}\gamma^\mu b$ can be
computed from Eq. (3.1) with the substitution $\Gamma = \gamma^\mu$. The amplitude vanishes; thus the $b$ quark
magnetic moment transition is a $1/m_b$ effect.
general Dirac matrix which transforms as $\Gamma$ under Lorentz transformations. For $\Gamma = \gamma^\mu$, we find\[\Omega_\Gamma = \Omega_1(q^2/\Lambda_{QCD}^2)\, v^\mu + \Omega_2(q^2/\Lambda_{QCD}^2)\, \sigma^{\mu\nu} q_\nu / \Lambda_{QCD}, \quad (3.3)\]

where we have used current conservation in writing Eq. (3.3). In contrast to the first example, the momentum transfer to the light degrees of freedom is $q$, so that the scale of variation of the form factors is now $q/\Lambda_{QCD}$. The form factors $\Omega_i$ in Eq. (3.3) have a variation on the scale $p \cdot p' \sim \Lambda_{QCD}^2$ (or $v \cdot v' \sim \Lambda_{QCD}^2/m_B^2$) instead of the scale $v \cdot v' \sim 1$ for the Isgur-Wise function $\xi(v \cdot v')$.

We now consider the scale of variation for the form factors found in Sect. 2. The $B_c \to B_s$ transition amplitude is an example of a matrix element of an operator containing both a light quark and a heavy quark. The matrix elements of the current $\sigma \Gamma c$ are given by

\[
\langle B_s, v, q | \bar{s} \Gamma c | B_c, v, 0 \rangle = -\sqrt{m_{B_c} m_{B_s}} \, \text{Tr} \left( H^{(c)}_{s \, v} \left[ \Omega_1 + a_0 \Omega_2 \bar{q} \right] \Gamma H^{(s)}_{v \, \bar{q}} \right), \quad (3.4)
\]

where the matrix $\Gamma$ multiplies $H^{(c)}_{s \, v}$ on the $c$ quark index because of the $c$ quark spin symmetry, and $\Omega_1 + a_0 \Omega_2 \bar{q}$ is the most general possible scalar matrix, and multiplies the light quark index. In the limit $m_b \gg m_c \gg \Lambda_{QCD}$, the addition of momentum $q$ does not change the velocity of the meson. The scale of variation of the form factors is controlled by the size of the $B_c$ bound state. The matrix element Eq. (3.4) measures the overlap of the $c$ quark distribution in the $B_c$, the $s$ quark distribution in the $B_s$, and $e^{iq \cdot x}$. Equivalently, it measures the overlap of the $s$ quark distribution in the $B_s$ with the $c$ quark distribution in the $B_c$ shifted by momentum $q$. The width of the momentum distribution of the $s$ quark is $\Lambda_{QCD}$, and the width of the momentum distribution of the $c$ quark is of order the inverse radius of the $B_c$ bound state, $a_0^{-1} \gg \Lambda_{QCD}$. Thus a shift in the $c$ quark momentum distribution by an amount $q \ll a_0^{-1}$ does not affect the overlap amplitude. Consequently, the scale of variation of the form factors is $a_0^{-1}$, not $\Lambda_{QCD}$.

\[\text{2 The } \Omega_1 \text{ form factor is the electric coupling, and the } \Omega_2 \text{ form factor is the magnetic coupling. The } \Omega_2 \text{ form factor gives a } B^* \to B\gamma \text{ transition amplitude that is not suppressed by powers of } 1/m_B, \text{ and corresponds to a light quark magnetic moment transition in a quark model.}\]
4. The Zero Recoil Limit

The weak currents $\pi \Gamma c$, $\bar{d} \Gamma c$, $\bar{b} \Gamma u$ and $\bar{b} \Gamma c$ do not generate symmetries of the effective theory, so their matrix elements at zero recoil cannot be normalized by symmetry considerations. However, in the limit $1/a_0 \gg \Lambda_{QCD}$ in which the $B_c$ is pointlike on the hadronic scale $\Lambda_{QCD}$, it is possible to calculate these matrix elements at zero recoil in terms of heavy-heavy bound state wavefunctions and the meson decay constants $f_B$ and $f_D$.

We begin by considering semileptonic $B_c \to B_s$ decay. In the $m_b \to \infty$ limit, the kinematics of $B_c \to B_s$ decay is analogous to that of neutron $\beta$-decay in that the entire energy of the decay is transferred to the lepton system. In the rest frame of the $B_c$, $k^0 = E_\ell + E_\nu = m(B_c) - m(B_s)$, where $k$ is defined in Eq. (2.1), or equivalently $q^0 = 0$. (The recoil energy of the $B_s$ is of order $\vec{q}^2/m_b$.) Thus the hadronic form factors only depend on $\vec{q}$, and the zero recoil point is $\vec{q} = 0$.

The calculation of the form factor at zero recoil proceeds as follows. The initial $B_c$ state is written as $3$

$$|B_c, v\rangle_{HQ} = \int d^3x \, \Psi(x) \left[ c_v^{(+)}(x)(1 + \frac{v}{c})\frac{i\gamma_5}{2} \frac{(1 - \frac{v}{c})}{2} b_v^{(-)}(0) \right] |0\rangle,$$

where $b_v^{(-)}(0)$ creates a $\bar{b}$ quark with velocity $v$ at the origin, $c_v^{(+)}(x)$ creates a $c$ quark with velocity $v$ at the point $x$, and $\Psi(x)$ is the wavefunction of the $B_c$. The superscripts $(+)$ and $(-)$ refer to the $\hat{\nu}$ eigenvalue,

$$\hat{\nu} c_v^{(+)} = +c_v^{(+)} , \quad \hat{\nu} c_v^{(-)} = -c_v^{(-)} , \quad (4.2)$$

and similarly for $b_v^{(\pm)}$. To compute the $\Omega_1$ form factor in Eq. (2.9) for $B_c \to B_s$ semileptonic decay, we consider the matrix element of the vector current between the $B_c$ and $B_s$ at finite three momentum transfer $\vec{q}$.

$$\mathcal{M}^{\nu}(\vec{q}) = \int d^3z \, e^{i\vec{q} \cdot \vec{z}}_{HQ}\langle B_s, v|V^{\mu}|B_c, v\rangle_{HQ}$$

$$= \int d^3z \, e^{i\vec{q} \cdot \vec{z}}_{HQ}\langle B_s, v|\bar{c}(z)\gamma^{\mu}c_v(z)|B_c, v\rangle_{HQ} , \quad (4.3)$$

where $\vec{q}$ is the three-momentum transfer to the leptonic system in the decay. Inserting Eq. (4.1) into Eq. (4.3), and using heavy field contractions,

$$\langle 0| c_v(x)\bar{c}_v^{(+)}(y) |0\rangle = \frac{1 + \hat{\nu}}{2} \delta(x - y) , \quad (4.4)$$

$3$ States with a subscript HQ are normalized to $v^0$ rather than to $2E$. 

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yields
\[ \mathcal{M}^\mu(q) = i \int d^3 x \, e^{i \vec{q} \cdot \vec{x}} \Psi(x) \, \text{HQ} \langle B_s, v | \overline{\psi}(x) \gamma^\mu \gamma_5 b_v(0) | 0 \rangle . \] (4.5)

Performing an operator product expansion
\[ \overline{\psi}(x) \gamma^\mu \gamma_5 b_v(0) = \overline{\psi}(0) \gamma^\mu \gamma_5 b_v(0) + x^k \partial_k \overline{\psi}(0) \gamma^\mu \gamma_5 b_v(0) + \ldots , \] (4.6)

and using the definition
\[ \langle 0 | \overline{\psi} \gamma^\mu \gamma_5 b | B_s, v \rangle_{\text{HQ}} \equiv i f_{B_s} m_{B_s} v^\mu / \sqrt{2 m_{B_s}} , \] (4.7)
of the $B_s$ meson decay constant gives
\[ \Omega_1(a_0 \vec{q}) = \frac{1}{\sqrt{2}} f_{B_s} \sqrt{m_{B_s}} \int d^3 x \, e^{i \vec{q} \cdot \vec{x}} \Psi(x) , \] (4.8)

where we have retained only the first term in the operator product expansion, Eq. (4.6).

The above computation also applies to the case where the final meson is $B_d$ instead of $B_s$, with the replacement $f_{B_s} \sqrt{m_{B_s}} \rightarrow f_{B_d} \sqrt{m_{B_d}}$. In the $SU(3)_V$ limit, $f_{B_d} \sqrt{m_{B_d}} = f_{B_s} \sqrt{m_{B_s}}$. The leading $SU(3)_V$-violating correction to this result can be found in ref. [9].

In the $m_b \gg m_c \gg \Lambda_{\text{QCD}}$ limit, the wavefunction $\Psi(x)$ for the $B_c$ is a Coulomb wavefunction, so that the form factor $\Omega_1$ can be computed explicitly,
\[ \Omega_1(a_0 \vec{q}) = \frac{1}{\sqrt{2}} f_{B_s} \sqrt{m_{B_s}} \frac{8 \pi^{1/2} a_{0}^{3/2}}{(1 + a_{0}^{2} \vec{q}^{2})^{2}} . \] (4.9)

There will be corrections to the Coulomb form of the $B_c$ wavefunction, because confinement effects are significant at the $c$ quark mass. Confinement effects in the meson wavefunction at the $c$ quark scale have been studied in detail in the $\psi$ system. A quark model with a modified Coulomb potential provides a very good description of the spectrum and radiative decays for the $\psi$. A similar computation for the $B_c$ should provide a good description of the $B_c$ wavefunction for use in Eq. (4.8).

The $c$ and $b$ quark fields were treated as free fields in the computation of Eq. (4.8). Radiative gluon corrections are in principle important. Gluon exchange between the $\overline{b}$ and $c$ quarks of the $B_c$ and between the $\overline{b}$ and $s$ quarks of the $B_s$ has already been included exactly in the definition of the states. The only gluon contributions that are not included are radiative gluon corrections where gluons are exchanged between the $B_c$ and $B_s$. This gluon exchange leads to a violation of factorization in the computation of the decay amplitude. The $B_c$ does not couple to gluons in the limit that its radius becomes
zero. For a finite radius $a_0$, the leading gluon coupling of the $B_c$ is to a two gluon operator with coefficient proportional to $a_0^3$. This produces a small (and incalculable) correction to the decay form factors.

The decay $B_c \rightarrow D$ proceeds through the quark decay $\bar{b} \rightarrow \pi$. The $D$ meson is light compared with the energy released in the decay, and so can have a large recoil momentum. The approximation methods used in this paper cannot be applied in this case. However, there is a region of phase space near zero recoil where $q \lesssim m_c$, where the heavy $c$ quark expansion is still valid. The computation of the $\Sigma_1$ form factor in this region is almost identical to the computation described above for $\Omega_1$, with the result

$$\Sigma_1(a_0 \vec{q}) = \frac{1}{\sqrt{2}} f_D \sqrt{m_D} \int d^3x \ e^{i \vec{q} \cdot \vec{x}} \Psi(x).$$

(Eq. 4.10)

Eqs. (4.8) and (4.10) both depend on the wavefunction of the $B_c$ meson. One can obtain a more reliable extraction of KM mixing angles by considering the ratio $\Sigma_1(0)/\Omega_1(0)$, which should be insensitive to the detailed form of the $B_c$ wavefunction, and thus provides a way of measuring $|V_{ub}|/|V_{cs}|$ and $|V_{ub}|/|V_{cd}|$.

The decay $B_c \rightarrow \eta_c$ proceeds through the quark decay $\bar{b} \rightarrow \tau$. As for $B \rightarrow D$, the heavy quark expansion is only valid in the region $q \lesssim m_c$ near the zero recoil point. The form factor $\Delta$ is calculable in terms of the wavefunctions of the $bc$ and $c\tau$ bound states. A straight-forward derivation yields

$$\Delta(a_0 \vec{q}) = 2 \int d^3x e^{-i \vec{q} \cdot \vec{x}/2} \Psi_{\eta_c}^*(x) \Psi_{B_c}(x),$$

(Eq. 4.11)

where the convolution of the two wavefunctions depends on the radii $a_0$ and $a_\eta$ of the $B_c$ and $\eta_c$. The details of this computation are nearly identical to those found in ref. [10] for the semileptonic decay of baryons containing two heavy quarks. In the limit that both states are described by a Coulomb wavefunction,

$$\Delta(a_0 \vec{q}) = 16 \frac{a_0^{3/2} a_\eta^{3/2}}{(a_0 + a_\eta)^3} \left[ 1 + \frac{q^2 a_0^2 a_\eta^2}{4(a_0 + a_\eta)^2} \right]^{-2}.$$  

(Eq. 4.12)

5. Corrections

Corrections to the results of the previous sections can be divided into two categories. The first set of corrections are corrections to heavy quark spin symmetry, and affect the
relations derived in Sect. 2. The second set of corrections relate to the validity of factor-
ization and the operator product expansion used in Sect. 4, as well as to the details of the $B_c$ wavefunction.

Let us first consider corrections to heavy quark spin symmetry. In this paper, we have
worked in the limit $m_b \gg m_c \gg \Lambda_{\text{QCD}}$. There are corrections to the heavy quark theory
due to the finite mass of the $b$ quark, which are of order $\Lambda_{\text{QCD}}/m_b$ and $m_c/m_b$. These
corrections are small and will not be discussed further. In addition, there are violations
of the $c$ quark spin symmetry in the $B_c$. These arise from interactions of the $c$ quark spin
with the $b$ quark spin, with the orbital angular momentum of the $c$ quark, and with light
degrees of freedom. The $c$-$b$ spin-spin interaction is a $1/m_b$ effect, and is small. There is
no $c$ quark spin-orbit interaction for the $B_c$ because the $c$ quark is in an $s$-wave. At lowest
order, the $B_c$ is made up of a $\bar{b}$ quark and $c$ quark in a bound state. There are, however,
corrections to this form in which the $B_c$ wavefunction also contains additional light degrees
of freedom. In a bag model, this would correspond to exciting gluonic excitations in the
bag. The spin coupling of the $c$ quark to these light degrees of freedom violates the $c$
quark spin symmetry. The interaction energy is of order $\Lambda_{\text{QCD}}^2/m_c$. Since the energy cost
of exciting a light degree of freedom is of order $\Lambda_{\text{QCD}}$, the net spin symmetry violation in
the matrix element is of order $\Lambda_{\text{QCD}}/m_c$.

The results of Sect. 4 depend not only on taking $m_c \gg \Lambda_{\text{QCD}}$, but also on the size
of the $B_c$ being much smaller than $\Lambda_{\text{QCD}}^{-1}$. The higher derivative terms in the operator
product expansion, Eq. (4.6) produce corrections of order $a_0 \Lambda_{\text{QCD}}$, because each factor of
$\partial$ on the light quark operator produces a factor of $\Lambda_{\text{QCD}}$ in the matrix element, and each
factor of $x$ produces a factor of the size of the bound state $a_0$. There are also violations
of factorization in the operator matrix element Eq. (4.3). As discussed in the previous
section, gluon interactions with the $B_c$ are of order $a_0^3$. There is no suppression factor
for the interaction of gluons with the $B_s$, since the $B_s$ has a size of order $\Lambda_{\text{QCD}}$. Thus
the gluon interactions produce corrections of order $(\Lambda_{\text{QCD}} a_0)^3$. For the $B_c \to \eta_c, \psi$ decay,
there is an additional suppression factor of $a_0^3$ for the interaction of gluons with the $\eta_c, \psi$,
so that the net correction is of order $\Lambda_{\text{QCD}}^6 a_0^3 a_0^3 a_0^3$. There are non-perturbative corrections
to the Coulomb wavefunction of the $B_c$, which are of order $\Lambda_{\text{QCD}}/m_c$. As discussed in the
previous section, most of these effects can be included by modeling the $B_c$ by a realistic
potential which is adjusted to correctly reproduce the $B_c$ excitation spectrum. There are
also radiative corrections which produce corrections of order $\alpha_s(m_b)$ and $\alpha_s(m_c)$. Finally,
there are $1/m_c$ recoil corrections for $B_c \to D$ and $B_c \to \eta_c, \psi$ which are of order $\vec{q}/m_c$.11
6. Conclusions

The semileptonic decay $B_c \rightarrow D\ell\nu$ provides a way of extracting the weak mixing angle $|V_{ub}|$. Theoretical uncertainties can be minimized by extracting the ratio $|V_{ub}/V_{cs}|$ using the ratio of the $B_c \rightarrow D$ and $B_c \rightarrow B_s$ form factors near zero recoil. The number of $B_c$’s produced in hadron collisions is much smaller than the number of $B$’s. Nevertheless, the $B_c$ meson still provides an alternative measurement of $|V_{ub}|$ to the value which will be obtained by comparing semileptonic $B \rightarrow \rho\ell\nu$ and $D \rightarrow \rho\ell\nu$ decays. Both of these extractions will have corrections due to the finite mass of the $c$ quark whose numerical importance will have to be determined experimentally.

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References

[1] B. A. Thacker and G. P. Lepage, Phys. Rev. D43 (1991) 196.
[2] N. Isgur and M.B. Wise, Phys. Lett. 232B (1989) 113;
   Phys. Lett. 237B (1990) 527.
[3] A. F. Falk, H. Georgi, B. Grinstein and M. B. Wise, Nucl. Phys. B343 (1990) 1;
   J. D. Bjorken, talk given at Les Rencontres de la Valle d’Aoste La Thuile, Aosta Valley, Italy, March 1990, SLAC preprint SLAC-PUB-5278 (1990).
[4] M.B. Wise, CALT-68-1765 (1992);
   G. Burdman and J. Donoghue, UMHEP-365 (1992).
[5] C. Quigg and J. L. Rosner, Phys. Rev. D23 (1981) 2625.
[6] E. Jenkins and M.J. Savage, UCSD/PTH 92-07.
[7] H. Georgi, Phys. Lett. 240B (1990) 447.
[8] M. J. Dugan, M. Golden and B. Grinstein, Harvard preprint HUTP-91/A045 (1991).
[9] B. Grinstein, E. Jenkins, A.V. Manohar, M.J. Savage and M.B. Wise, UCSD/PTH 92-05 (1992).
[10] M.J. White and M.J. Savage, Phys. Lett. 271B (1991) 410.
[11] E. Eichten and C. Quigg, private communication.
[12] M. Adler, Caltech Ph.D. Thesis 1989, (unpublished).