The Wiener–Hopf technique (Wiener and Hopf, 1931; Hopf, 1934) was first proposed in the 1930s as a method for solving an integral equation of the form

$$g(\tau) = \int_0^\infty h(t)c(\tau - t)dt, \quad \text{for} \quad \tau \in [0, \infty),$$

in terms of $h(\cdot)$, where $c(\cdot)$ is a known difference kernel and $g(\cdot)$ is a specified function. The above integral equation and the Wiener–Hopf technique have been widely used in many applications in applied mathematics and engineering (see Lawrie and Abrahams, 2007 for a review). In the 1940s, Wiener (1949) reformulated the problem within discrete time, which is commonly referred to as the Wiener (causal) filter. The discretization elegantly encapsulates several problems in time series analysis. For example, the best fitting finite order autoregressive parameters fall under this framework. The autoregressive parameters can be expressed as a solution of a system of finite interval Wiener–Hopf equations (commonly referred to as the FIR Wiener filter), for which Levinson (1947) and Durbin (1960) proposed a $O(n^2)$ method for solving these equations. More broadly, the best linear predictor of a causal stationary time series naturally gives rise to the Wiener filter, for example, the prediction of hidden states in a Kalman filter model. The purpose of this article is to revisit the discrete-time Wiener–Hopf equations (it is precisely defined in (1.2)) and derive an alternative solution using the tools of linear prediction. Below we briefly review some classical results on the Wiener filter.
Suppose that \( \{X_t : t \in \mathbb{Z}\} \) is a real-valued, centered, weakly stationary time series defined on the probability space \((\Omega, \mathcal{F}, P)\) and \(c(\tau) = \text{cov}(X_0, X_{\tau})\) is the autocovariance function of \(\{X_t\}\). Let \(H_\infty\) and \(H_t\) \((t \in \mathbb{Z})\) denote closed sub-spaces of the real Hilbert space \(L_2(\Omega, \mathcal{F}, P)\) spanned by \(\{X_j : j \in \mathbb{Z}\}\) and \(\{X_j : j \leq t\}\) respectively. We denote the orthogonal projection onto the closed subspace \(V \in L_2(\Omega, \mathcal{F}, P)\) as \(P_V\). For \(Y \in L_2(\Omega, \mathcal{F}, P)\), the orthogonal projection of \(Y\) onto \(H_0\) is

\[
P_{H_0}(Y) = \sum_{j=0}^{\infty} h_j X_{-j},
\]

where \(P_{H_0}(Y) = \arg \min_{U \in H_0} \mathbb{E}|Y_0 - U|^2\). To evaluate \(\{h_j : j \geq 0\}\), we rewrite (1.1) as a system of normal equations. By using that \(P_{H_0}(Y)\) is an orthogonal projection onto \(H_0\), it is easily shown that (1.1) leads to the system of normal equations

\[
c_{YX}(\ell') = \sum_{j=0}^{\infty} h_j (\ell' - j) \quad \text{for} \quad \ell' \geq 0,
\]

where \(c_{YX}(\ell') = \text{cov}(Y, X_{\ell'})\). The above set of equations is typically referred to as the discrete-time Wiener–Hopf equations (or semi-infinite Toeplitz equations). There are two well-known methods for solving this equation in the frequency domain; the Wiener–Hopf technique (sometimes called the gapped function, see Wiener, 1949) and the prewhitening method proposed by Bode and Shannon (1950) and Zadeh and Ragazzini (1950). Both solutions solve for \(H(\omega) = \sum_{j=0}^{\infty} h_j e^{i\omega j}\) (see Kailath, 1974; Kailath, 1980; and Orfanidis, 2018, Sections 11.3-11.8). The Wiener–Hopf technique is based on the spectral factorization and a comparison of Fourier coefficients corresponding to the negative and non-negative indices in Fourier series expansion. The prewhitening method, as the name suggests, is more in the spirit of time series where the time series \(\{X_t\}\) is “whitened” using an autoregressive filter.

To state the solution, we assume the spectral density \(f(\omega) = \sum_{r \in \mathbb{Z}} c(\tau) e^{i\omega \tau}\) satisfies the condition \(0 < \inf_\omega f(\omega) \leq \sup_\omega f(\omega) < \infty\). Then, \(\{X_t\}\) admits an infinite order Wold-type MA and AR representation (Pourahmadi, 2001; Sections 5-6 and Krampe et al., 2018, page 706)

\[
X_t = \epsilon_t + \sum_{j=1}^{\infty} \psi_j \epsilon_{t-j}, \quad X_t - \sum_{j=1}^{\infty} \phi_j X_{t-j} = \epsilon_t, \quad t \in \mathbb{Z},
\]

where \(\sum_{j=1}^{\infty} \psi_j < \infty, \sum_{j=1}^{\infty} \phi_j^2 < \infty\), and \(\{\epsilon_t\}\) is a uniquely determined white noise process with \(\mathbb{E}\epsilon_t^2 = \sigma^2 > 0\) and \(\epsilon_t\) is orthogonal to \(H_{t-1}\) for \(t \in \mathbb{Z}\). Insights into how the Wold representation in (1.3) connects linear prediction through the infinite order MA and AR coefficients is given in Cheng and Pourahmadi (1993) and Meyer and Kreiss (2015). Note that (1.3) holds under the weaker condition that \(\inf_\omega f(\omega) > 0\) (see, e.g., Wiener and Masani, 1958).

From (1.3), we immediately obtain the spectral factorization \(f(\omega) = \sigma^2 |\phi(\omega)|^{-2}\), where \(\phi(\omega) = 1 - \sum_{j=1}^{\infty} \phi_j e^{i\omega j}\). Given \(A(\omega) = \sum_{j=-\infty}^{\infty} a_j e^{i\omega j}\), we use the notation \(|A(\omega)|_+ = \sum_{j=1}^{\infty} |a_j| e^{i\omega j}\) and \(|A(\omega)|_- = \sum_{j=-\infty}^{0} |a_j| e^{i\omega j}\). Both the Wiener–Hopf technique and prewhitening method yield the solution

\[
H(\omega) = \sigma^{-2} \phi(\omega)|\psi(\omega)|^{1/2} f_{YX}(\omega)|_+, \quad |\omega| < \pi,
\]

where \(f_{YX}(\omega) = \sum_{\ell \in \mathbb{Z}} c_{YX}(\ell') e^{i\omega \ell'}\) and \(\phi(\omega)^*\) is a complex conjugate of \(\phi(\omega)\).

We mention that the special case of \(m\)-step ahead forecasts falls under this framework. By setting \(Y = X_{m}\), the coefficients \(\{h_j : j \geq 0\}\) are the \(m\)-step ahead prediction coefficients and the solution for \(H(\omega)\) is

\[
H_m(\omega) = \phi(\omega)|\psi(\omega)| e^{-i\omega m}|_+, \quad m > 0,
\]

where \(\psi(\omega) = (\phi(\omega))^{-1} = 1 + \sum_{j=1}^{\infty} \psi_j e^{i\omega j}\) and the MA coefficients \(\{\psi_j\}\) is from (1.3).
The normal equations in (1.2) belong to the general class of Wiener–Hopf equations of the form
\[ g_{\ell} = \sum_{j=0}^{\infty} h_{\ell-j} \quad \text{for} \quad \ell \geq 0, \]  
where \( \{c(r) : r \in \mathbb{Z}\} \) is a symmetric, positive definite sequence. The Wiener–Hopf technique yields the solution
\[ H(\omega) = \sigma^{-2}\phi(\omega)[\phi(\omega)^*G_+^+(\omega)], \]  
where \( G_+^+(\omega) = \sum_{\ell=0}^{\infty} g_{\ell} e^{i\omega \ell} \) (the derivation is well known, but for completeness we give a short proof in Section 2.3). An alternative method for solving for \( \{h_j : j \geq 0\} \) is within the time domain. This is done by representing (1.5) as the semi-infinite Toeplitz system
\[ g_+ = T(f)h_+ , \]  
where \( g_+ = (g_0, g_1, g_2, \ldots) \) and \( h_+ = (h_0, h_1, \ldots) \). Let \( \{\phi_j : j \geq 0\} \) denote the infinite order AR coefficients corresponding to \( T \) (setting \( \phi_0 = -1 \)) define as in (1.3) and \( \phi(\cdot) \) be its Fourier transform. By letting \( \phi_j = 0 \) for \( j < 0 \), we define the lower triangular Toeplitz matrix \( T(\phi) = (\phi_j; \mathbb{T}, \tau \geq 0) \). Provided that \( 0 < \inf_{\omega} |f(\omega)| \leq \sup_{\omega} |f(\omega)| < \infty \), it is well known that \( T(\phi) \) is invertible on \( \ell_2^+ = \{(v_0, v_1, \ldots) : \sum_{j=0}^{\infty} |v_j|^2 < \infty \} \), and the inverse is \( T(f)^{-1} = \sigma^{-2}T(\phi)T(\phi)^* \) (see, e.g., Theorem III of Widom, 1960). Thus, the time domain solution to (1.5) is
\[ h_+ = T(f)^{-1}g_+ = \sigma^{-2}T(\phi)T(\phi)^*g_+. \]

In this article, we study the Wiener–Hopf equations from a time series perspective, combining the prediction theory developed in the time domain with the deconvolution method in the frequency domain. Observe that (1.5) is a system of semi-infinite convolution equations (since the equations only hold for non-negative index \( \ell \)), thus the standard deconvolution approach is not possible. In Subba Rao and Yang (2021), we used the tools of linear prediction to rewrite the Gaussian likelihood of a stationary time series within the frequency domain. We transfer some of these ideas to solving the Wiener–Hopf equations. In Section 2.2, we show that we can circumvent the constraint \( \ell \geq 0 \), by using linear prediction to yield the normal equations in (1.2) for all \( \ell \in \mathbb{Z} \). In Section 2.3, we show that there exists a stationary time series \( \{X_t\} \) and the random variable \( Y \in \mathcal{H}_0 \) where \( Y \) and \( \{X_t\} \) induce the general Wiener–Hopf equations of the form (1.5). This allows us to use the aforementioned technique to reformulate the Wiener–Hopf equations as a bi-infinite Toeplitz system, and thus obtain a solution to \( H(\omega) \) as a deconvolution. The same technique is used to obtain an expression for entries of the inverse Toeplitz matrix \( T(f)^{-1} \).

In practice, evaluating \( H(\omega) \) in (1.4) requires the spectral factorization of the underlying spectral density. One strategy is to assume that the spectral density is rational, which allows one to obtain a computationally tractable solution for \( H(\omega) \). Of course, this leads to an approximation error in \( H(\omega) \) when the underlying spectral density is not a rational function. In Section 3, we show that Baxter’s inequality (Baxter, 1962; Baxter, 1963) can be utilized to obtain a bound between \( H(\omega) \) and its approximation based on a rational approximation of the general spectral density. The proof of the results in Sections 2 and 3 can be found in the Appendix.

2. A PREDICTION APPROACH

2.1. Notation and Assumptions

Here, we collect together the notation introduced in Section 1 and some additional notation necessary for the article.

Let \( L_2([0, 2\pi]) \) be the space of all square-integrable complex functions on \([0, 2\pi]\) and \( \ell_2^+ \) the space of all bi-infinite complex column vectors \( v = (\ldots, v_{-1}, v_0, v_1, \ldots) \) where \( \sum_{j \in \mathbb{Z}} |v_j|^2 < \infty \). Similarly, let \( \ell_2^+ = \{v_+ = \ldots, v_{-1}, v_0, v_1, \ldots) \).
(\(v_0, v_1, \ldots\)) \(\rightarrow \sum_{j=0}^{\infty} |v_j|^2 < \infty\) denote the space of all semi-infinite square summable vector sequences. To connect the time and frequency domain through an isomorphism, we define the Fourier transform \(F : \ell_2 \rightarrow L_2([0, 2\pi])\)

\[
F(v)(\omega) = \sum_{j \in \mathbb{Z}} v_j e^{ij\omega}.
\]

For \(f(\omega) = \sum_{r \in \mathbb{Z}} c(r)e^{ir\omega} \in L_2([0, 2\pi])\), define the semi- and bi-infinite Toeplitz operators \(T(f)\) and \(T_{\pm}(f)\) on \(\ell_2^+\) and \(\ell_2\) with the matrix form \(T(f) = (c(t - r); t, r \geq 0)\) and \(T_{\pm}(f) = (c(t - r); t, r \in \mathbb{Z})\) respectively. This article will make frequent use of the convolution theorem: If \(h \in \ell_2\), then \(F(T_{\pm}(f)h)(\omega) = f(\omega)F(h)(\omega)\).

**Assumption 2.1.** Let \(\{c(r) : r \in \mathbb{Z}\}\) be a symmetric positive definite sequence on \(\ell_2\) and \(f(\omega) = \sum_{r \in \mathbb{Z}} c(r)e^{ir\omega}\) be its Fourier transform. Then,

(i) \(0 < \inf_{\omega} f(\omega) \leq \sup_{\omega} f(\omega) < \infty\).

(ii) For some \(K > 1\) we have \(\sum_{r \in \mathbb{Z}} |r^K c(r)| < \infty\).

Under Assumption 2.1(i), we have the unique factorization

\[
f(\omega) = \sigma^2 |\psi(\omega)|^2 = \sigma^2 |\phi(\omega)|^{-2},
\]

where \(\sigma^2 > 0\), \(\psi(\omega) = 1 + \sum_{j=1}^{\infty} \psi_j e^{ij\omega}\), and \(\phi(\omega) = (\psi(\omega))^{-1} = 1 - \sum_{j=1}^{\infty} \phi_j e^{ij\omega}\). The characteristic polynomials \(\Psi(z) = 1 + \sum_{j=1}^{\infty} \psi_j z^j\) and \(\Phi(z) = 1 - \sum_{j=1}^{\infty} \phi_j z^j\) do not have zeros in \(|z| \leq 1\) thus the AR(\(\infty\)) parameters are causal or equivalently are said to have minimum phase (see Szegö, 1921; Inoue, 2000, pages 68-69).

We mention that Assumption 2.1(i) is used in all the results in this article, whereas Assumption 2.1(ii) is only required in the approximation theorem in Section 3. Under Assumption 2.1(ii), both \(\sum_{j=1}^{\infty} |\psi_j|\) and \(\sum_{j=1}^{\infty} |\phi_j|\) are finite (see Cheng and Pourahmadi, 1993; Meyer and Kreiss, 2015).

### 2.2. Solving Wiener–Hopf equations using linear prediction

We now give an alternative formulation for the solution of (1.2) and (1.5), which utilizes properties of linear prediction to solve it using a standard deconvolution method. To integrate our derivation within the Wiener causal filter framework, we start with the classical Wiener filter. For \(Y \in L_2(\Omega, F, P)\), consider the projection of \(Y\) onto \(H_0\)

\[
P_{H_0}(Y) = \sum_{j=0}^{\infty} h_j X_{-j}.
\]

We observe that by construction, (2.2) gives rise to the half-convolution equations

\[
\text{cov}(Y, X_{-\ell}) = \sum_{j=0}^{\infty} h_j c(\ell - j) \quad \ell \geq 0.
\]

Since (2.3) only holds for non-negative \(\ell\), this prevents one using deconvolution to solve for \(H(\omega)\). Instead, we define a “proxy” set of variables for \(\{X_{-\ell} : \ell < 0\}\) such that (2.3) is valid for \(\ell < 0\). By using the property of orthogonal projections, we have

\[
\text{cov}(Y, P_{H_0}(X_{-\ell})) = \text{cov}(P_{H_0}(Y), X_{-\ell}) \quad \ell < 0.
\]
This gives
\[
\text{cov}(Y, P_{H_0}(X_{-\ell})) = \sum_{j=0}^{\infty} h_j \text{cov}(X_{-j}, X_{-\ell}) = \sum_{j=0}^{\infty} h_j c(\ell - j) \quad \ell < 0.
\] (2.4)

Equations (2.3) and (2.4) allow us to represent the solution of \(H(\omega)\) as a deconvolution. We define the semi- and bi-infinite sequences \(c_\pm = (\text{cov}(Y, P_{H_0}(X_{-\ell})); \ell' < 0)', c_\pm = (\text{cov}(Y, X_{-\ell}); \ell' \geq 0)', \) and \(c_{\pm}' = (c_\ell', c_\ell')'\). Taking the Fourier transform of \(c_\pm\) and using the convolution theorem gives \(F(c_\pm)(\omega) = H(\omega)f(\omega)\). Thus
\[
H(\omega) = \frac{F(c_\pm)(\omega)}{f(\omega)} = \sum_{\ell=0}^{\infty} \text{cov}(Y, X_{-\ell}) e^{i\ell\omega} + \sum_{\ell=1}^{\infty} \text{cov}(Y, P_{H_0}(X_{\ell})) e^{-i\ell\omega}.
\] (2.5)

This forms the key to the following theorem.

**Theorem 2.1.** Suppose \(\{X_t\}\) is a centered weakly stationary time series defined on the probability space \((\Omega, \mathcal{F}, P)\) whose spectral density satisfies Assumption 2.1(i). Let \(\phi(\cdot)\) and \(\psi(\cdot)\) be defined as in (2.1) and \(\phi(\ell)(\omega) = \sum_{j=0}^{\infty} \phi_{\ell+j}(\omega)\) for \(\ell \geq 0\). For \(Y \in L_2(\Omega, \mathcal{F}, P)\), let \(P_{H_0}(Y) = \sum_{j=0}^{\infty} h_j X_{-j}\). Then, \((h_j; j \geq 0)' \in L_2^\ast\), \((c_{XY}(\ell)) = \text{cov}(Y, X_{-\ell}); \ell \geq 0)' \in L_2^\ast\) and
\[
H(\omega) = \sum_{\ell=0}^{\infty} c_{XY}(\ell) \left( e^{i\ell\omega} + \psi(\omega)\phi(\ell)(\omega)\right) f(\omega).
\] (2.6)

The above solution can alternatively be expressed as
\[
H(\omega) = \sigma^{-2}\psi(\omega) \sum_{\ell=0}^{\infty} c_{XY}(\ell) \left( e^{i\ell\omega} - \sum_{j=1}^{\ell} \phi_j e^{i(j-\ell)\omega} \right),
\] (2.7)

where for \(\ell = 0, \sum_{j=1}^{\ell} = 0\).

**Proof.** See Appendix A. ■

We thank an anonymous referee for pointing out that the representation in (2.6) is equivalent to (2.7). The benefit of the latter representation is that both \(\psi(\omega)\) and \(e^{i\ell\omega} - \sum_{j=1}^{\ell} \phi_j e^{i(j-\ell)\omega}\) for \(\ell \geq 0\) are in terms of the power series of \(e^{i\omega}\), thus it is transparent that \(H(\omega)\) is causal.

**Remark 2.1.** (Relationship to concurrent filters). There is a close relationship between Theorem 2.1 and solutions to concurrent filters (that are frequently used by the U.S. Census Bureau). Notable applications are the multi-step ahead forecasts used in the derivation of the X-11 and X-11-ARIMA seasonal filters (see Dagum, 1975; Dagum, 1982; and Ladiray and Quenneville, 2012 for a review). In relation to the Wiener filter, this is the technique of using multi-step ahead forecasts to obtain a solution to concurrent filter \(P_{H_0}(Y) = \sum_{j=0}^{\infty} h_j X_{-j}\) from the two-sided filter \(P_{H_0}(Y) = \sum_{j=-\infty}^{\infty} a_j X_{-j}\) where \(\sum_{j=-\infty}^{\infty} a_j e^{i\omega j} = \sum_{r \in \mathbb{Z}} c_{XY}(r) e^{i\omega r}/f(\omega)\); See Bell and Martin (2004) and Wildi and McElroy (2016), Section 2 (Proposition 1). We summarize the technique below. By standard projection arguments, we have
\[
P_{H_0}(Y) = P_{H_0}P_{H_0^\perp}(Y) = \sum_{j=0}^{\infty} a_j X_{-j} + \sum_{r=1}^{\infty} a_r P_{H_0}(X_{-j}) = \sum_{j=0}^{\infty} a_j + \sum_{r=1}^{\infty} a_r \phi_j(\ell) X_{-j},
\] (2.8)

where \(P_{H_0}(X_{-j}) = \sum_{r=0}^{\infty} \phi_j(\ell) X_{-j}\) (an expression for these coefficients in terms of AR and MA coefficients is given in Appendix, (A2)). Therefore, by comparing the above to (2.2) we have \(h_j = a_j + \sum_{r=1}^{\infty} a_r \phi_j(\ell)\) for \(j \geq 0\). Note that both (2.7) and (2.8) yield different solutions to the same Wiener–Hopf equations.
Remark 2.2. (Relationship to prediction). It is clear that \( \sum_{\ell=1}^{\infty} X_{\ell} e^{i\ell \omega} \) is not a well-defined random variable. However, it is interesting to note that under Assumption 2.1(ii) (for \( K = 1 \)) \( \sum_{\ell=1}^{\infty} P_{H_0}(X_{\ell}) e^{i\ell \omega} \) is a well-defined random variable in \( H_0 \) and

\[
\sum_{\ell=1}^{\infty} P_{H_0}(X_{\ell}) e^{i\ell \omega} = \psi(\omega) \sum_{j=0}^{\infty} \phi_j(\omega) X_{-j}. \tag{2.9}
\]

In other words, despite \( \sum_{\ell=1}^{\infty} X_{\ell} e^{i\ell \omega} \) not being well-defined, informally, its projection onto \( H_0 \) does exist.

2.3. General Wiener–Hopf equations

We now generalize the prediction approach in the previous section to general Wiener–Hopf linear equations that satisfy

\[
g_\ell = \sum_{j=0}^{\infty} h_j c(\ell - j), \quad \ell \geq 0, \tag{2.10}
\]

where \( \{g_\ell : \ell \geq 0\} \) and \( \{c(r) : r \in \mathbb{Z}\} \) (which is assumed to be a symmetric, positive definite sequence) are known. We will obtain a solution similar to (2.6) but for the normal equations in (2.10). We first describe the classical Wiener–Hopf method to solve (2.10). Since \( \{c(r)\} \) is known for all \( r \in \mathbb{Z} \), we extend (2.10) to the negative index \( \ell < 0 \), and define \( \{g_\ell : \ell < 0\} \) as

\[
g_\ell = \sum_{j=0}^{\infty} h_j c(\ell - j) \quad \text{for} \quad \ell < 0. \tag{2.11}
\]

Note that \( \{g_\ell : \ell < 0\} \) is not given, but it is completely determined by \( \{g_\ell : \ell \geq 0\} \) and \( \{c(r)\} \) (this can be seen from (2.15), below). The Wiener–Hopf technique evaluates the Fourier transform of the above and isolates the non-negative indices in the Fourier series expansion to yield the solution for \( H(\omega) \). Specifically, evaluating the Fourier transform of (2.10) and (2.11) gives

\[
f(\omega)H(\omega) = G_-(\omega) + G_+(\omega) \tag{2.12}
\]

where \( G_-(\omega) = \sum_{\ell=-\infty}^{-1} g_\ell e^{i\ell \omega} \) and \( G_+(\omega) = \sum_{\ell=0}^{\infty} h_j c \). Replacing \( f(\omega) \) with \( \sigma^2|\psi(\omega)|^2 \) and dividing the above with \( \sigma^2 \psi(\omega)^* \) yields

\[
H(\omega)\psi(\omega) = \frac{G_-(\omega)}{\sigma^2 \psi(\omega)^*} + \frac{G_+(\omega)}{\sigma^2 \psi(\omega)^*} = \sigma^{-2} \phi(\omega)^* G_-(\omega) + \sigma^{-2} \phi(\omega)^* G_+(\omega). \tag{2.13}
\]

Isolating the non-negative indices in (2.13) gives the solution

\[
H(\omega) = \sigma^{-2} \phi(\omega)[\phi(\omega)^* G_+(\omega)]_. \tag{2.14}
\]

this proves the result stated in (1.6). Similarly, by isolating the negative indices, we obtain the expression \( G_-(\omega) = \sum_{\ell=-\infty}^{-1} g_\ell e^{i\ell \omega} \) in terms of \( f(\omega) \) and \( G_+(\omega) \)

\[
G_-(\omega) = -\psi(\omega)^*[\phi(\omega)^* G_+(\omega)]_. \tag{2.15}
\]
Thus (2.14) and (2.15) provide explicit solutions to $H(\omega)$ and $G_-(\omega)$ respectively. However, from a time series perspective, it is difficult to interpret these formulae. We now obtain an alternative expression for these solutions based on the linear prediction of random variables.

We consider the matrix representation, $T(f)h_+ = g_+$ in (1.7). We solve $T(f)h_+ = g_+$ by embedding the semi-infinite Toeplitz matrix $T(f)$ on $\ell^+_2$ into the bi-infinite Toeplitz system on $\ell^+_2$. To relate $T(f)$ and $T_\pm(f)$, we partition the bi-infinite Toeplitz matrix $T_\pm(f)$ into four sub-matrices $C_{00} = (c(t - \tau); t, \tau < 0)$, $C_{01} = (c(t - \tau); t < 0, \tau \geq 0)$, $C_{10} = (c(t - \tau); t \geq 0, \tau < 0)$, and $C_{11} = (c(t - \tau); t, \tau \geq 0)$. We observe that $C_{11} = T(f)$. Further, we let $h_\pm = (0', h'_+)' = (\ldots, 0, 0, h_0, h_1, h_2, \ldots)'$ and $g_\pm = (g', g'_+)' = (\ldots, g_{-2}, g_{-1}, g_0, g_1, g_2, \ldots)'$ where $g_\pm = C_{00}C_{11}^{-1}g_\pm$. Then, we obtain the following bi-infinite Toeplitz system on $\ell^+_2$:

$$T_\pm(f)h_\pm = \begin{pmatrix} C_{00} & 0 \\ C_{10} & C_{11} \end{pmatrix} \begin{pmatrix} h_0 \\ h_+ \end{pmatrix} = \begin{pmatrix} C_{00}h_+ \\ C_{10}h_+ \\ C_{11}h_+ \end{pmatrix} = \begin{pmatrix} C_{01}C_{11}^{-1}g_\pm \\ g_+ \end{pmatrix} = \begin{pmatrix} g_- \\ g_+ \end{pmatrix} = g_\pm. \quad (2.16)$$

We note that the non-negative indices in the sequence $g_\pm$ are \{g_\ell : \ell \geq 0\}, but for the negative indices, where $\ell < 0$, it is $g_\ell = [C_{01}C_{11}^{-1}g_\pm]_\ell$, which is identical to $g_\ell$ defined in (2.11). The Fourier transform on both sides in (2.16) gives $f(\omega)H(\omega) = F(g_\pm(\omega))$, which is identical to (2.12). We now reformulate the above equation through the lens of prediction. To do this, we construct a stationary process $\{X_\ell\}$ and a random variable $Y$ on the same probability space, which yields (2.10) as their normal equations.

We first note that since $\{c(r) : r \in \mathbb{Z}\}$ is a symmetric, positive definite sequence, there exists a stationary time series $\{X_\ell\}$ with $\{c(r) : r \in \mathbb{Z}\}$ as its autocovariance function (see Brockwell and Davis, 2006, Theorem 1.5.1). Using this, define the random variable

$$Y = \sum_{j=0}^{\infty} h_jX_{-j}. \quad (2.17)$$

Provided that $h_+ \in \ell^+_2$, then $\mathbb{E}[Y^2] < \infty$ and thus $Y \in H_0$ (we show in Theorem 2.2 that this is true if $g_+ \in \ell^+_2$). By (2.10), we observe that $\text{cov}(Y, X_{-\ell}) = \sum_{j=0}^{\infty} h_j(c_{-\ell} - j) = g_\ell$ for all $\ell \geq 0$. We now show that for $\ell < 0$,

$$\text{cov}(Y, X_{-\ell}) = [C_{01}C_{11}^{-1}g_\pm]_\ell = g_\ell. \quad (2.18)$$

First, since $Y \in H_0$, then $\text{cov}(Y, X_{-\ell}) = \text{cov}(P_{H_0}(Y), X_{-\ell}) = \text{cov}(Y, P_{H_0}(X_{-\ell}))$. Further, for $\ell < 0$, the $\ell$th row (where we start the enumeration of the rows from the bottom) of $C_{01}C_{11}^{-1}$ contains the coefficients of the best linear predictor of $X_{-\ell}$ given $H_0$

$$P_{H_0}(X_{-\ell}) = \sum_{j=0}^{\infty} [C_{01}C_{11}^{-1}]_{\ell j}X_{-j} \quad \ell \leq 0. \quad (2.18)$$

A detailed calculation of (2.18) is given in the Appendix. Using the above, we evaluate $\text{cov}(Y, P_{H_0}(X_{-\ell}))$ for $\ell < 0$:

$$\text{cov}(Y, P_{H_0}(X_{-\ell})) = \text{cov}\left( Y, \sum_{j=0}^{\infty} [C_{01}C_{11}^{-1}]_{\ell j}X_{-j} \right)$$

$$= \sum_{j=0}^{\infty} [C_{01}C_{11}^{-1}]_{\ell j}\text{cov}(Y, X_{-j}) \quad \text{(from (2.17), } g_j = \text{cov}(Y, X_{-j}))$$

$$= \sum_{j=0}^{\infty} [C_{01}C_{11}^{-1}]_{\ell j}g_j = [C_{01}C_{11}^{-1}g_\pm]_\ell = g_\ell.$$
Thus the entries of $g_{\pm} = (g', g)^T$ are indeed the covariances: $g_- = (\text{cov}(Y, P_{H'}(X_e)); \ell < 0)'$ and $g_+ = (\text{cov}(Y, X_e)); \ell \geq 0)'$. This allows us to use Theorem 2.1 to solve general Wiener–Hopf equations. Further, it gives an intuition to (2.11) and (2.16).

**Theorem 2.2.** Suppose that $\{c(r) : r \in \mathbb{Z}\}$ is a symmetric, positive definite sequence and its Fourier transform $f(\omega) = \sum_{r \in \mathbb{Z}} c(r)e^{i\omega r}$ satisfies Assumption 2.1(i). We define the (semi) infinite system of equations

$$
g(\ell) = \sum_{j=0}^{\infty} h_j c(\ell - j), \quad \ell \geq 0,
$$

where $(g(\ell), \ell \geq 0)' \in \ell^\infty$. Then, $(h_j, j \leq 0)' \in \ell^\infty$ and

$$
H(\omega) = \sum_{\ell=0}^{\infty} e^{i\omega \ell} \left( e^{i\omega \ell} + \psi(\omega)^* \phi(\omega)^* \right). \tag{2.19}
$$

Moreover, as in Theorem 2.1, $H(\omega)$ can be rewritten as

$$
H(\omega) = \sigma^{-2} \phi(\omega) \sum_{\ell=0}^{\infty} g(\ell) \left( e^{i\omega \ell} - \sum_{s=1}^{\ell} \phi_s e^{i(s-\omega)\ell} \right). \tag{2.20}
$$

**Proof.** See Appendix A. 

It is interesting to observe that the solution for $H(\omega)$ given in (2.14) was obtained by comparing the Fourier coefficients, whereas the solution in Theorem 2.2 was obtained using linear prediction. The two solutions are algebraically different. We now show that they are the same by direct verification.

**Lemma 2.1.** Suppose the same set of assumptions and notation as in Theorem 2.2 hold. Then

$$
[\phi(\omega)^* G(\omega)]_+ = \sum_{\ell=0}^{\infty} g(\ell) \left( e^{i\omega \ell} - \sum_{s=1}^{\ell} \phi_s e^{i(s-\omega)\ell} \right), \tag{2.21}
$$

where $G(\omega) = \sum_{\ell=0}^{\infty} g(\ell) e^{i\omega \ell}$.

Theorem 2.2 can be used to obtain an exact elementwise expression for $T(f)^{-1}$. As mentioned in Section 1, the time domain solution for the inverse Toeplitz matrix is $T(f)^{-1} = \sigma^{-2} T(\phi) T(\phi)^*$. We show below that an alternative expression for the entries of $T(f)^{-1} = (d_{kj}; k, j \geq 0)$ can be deduced using the deconvolution method described in Theorem 2.2.

**Corollary 2.1.** Suppose the same set of assumptions and notation as in Theorem 2.2 hold. Let $d_{kj} = (d_{kj}, j \geq 0)$ denote the $k$th row of $T(f)^{-1}$. Then, $d_{kj}' \in \ell^\infty$ for all $k \geq 0$ and the Fourier transform $D_k(\omega) = \sum_{j=0}^{\infty} d_{kj} e^{ij\omega}$ is

$$
D_k(\omega) = \frac{e^{ik\omega} + \psi(\omega)^* \phi(\omega)^*}{f(\omega)} = \sigma^{-2} \phi(\omega) \left( e^{ik\omega} - \sum_{s=1}^{k} \phi_s e^{i(k-s)\omega} \right), \quad k \geq 0.
$$

Therefore,

$$
d_{kj} = \frac{\sigma^{-2}}{2\pi} \int_0^{2\pi} \phi(\omega) \left( e^{ik\omega} - \sum_{s=1}^{k} \phi_s e^{i(k-s)\omega} \right) e^{-ij\omega} d\omega, \quad j, k \geq 0. \tag{2.22}
$$

**Proof.** See Appendix A.
Remark 2.3. (Connection to the inverse of finite order Toeplitz matrix). Consider the \( n \times n \) Toeplitz matrix \( T_n(f) = (c(s-t); 0 \leq s, t \leq n-1) \) and \( d^{(n)}_{kj} = (T_n(f)^{-1})_{kj} \). There are several different expressions for \( d^{(n)}_{kj} \) including the Cholesky decomposition given in Akaike (1969), Pourahmadi (2001), and Jentsch and Meyer (2021) or expressions based on a dual process representation; Subba Rao and Yang (2021) and Inoue (2021). The arguments in this article can also be used to obtain an alternative expression for the inverse of a finite dimensional Toeplitz matrix. Using similar arguments to those used to prove Corollary 2.1, we obtain

\[
\phi^{(n)}(\ell) = \frac{1}{2\pi} \int_0^{2\pi} \phi(\omega) e^{i\ell \omega} d\omega = \sum_{\ell \in \mathbb{Z}} \phi^{(n)}(\ell) \gamma(\ell) \quad 0 \leq j, k \leq n - 1,
\]

where \( \gamma(k) = (2\pi)^{-1} \int_0^{2\pi} \phi(\omega) e^{-ik\omega} d\omega \) (usually called the inverse autocovariance function) and \( \phi^{(n)}(\ell) \) are the multi-step ahead finite prediction coefficients; \( P_{H_{[-(n-1),0]}}(X_r) = \sum_{s=0}^{n-1} \phi^{(n)}(\ell) X_{s-r} \), where \( H_{[-(n-1),0]} = \text{sp}\{X_r : -(n-1) \leq r \leq 0\} \).

It is interesting to compare and contrast (2.23) with the entries of the finite dimension Cholesky decomposition \( T_n(f)^{-1} = L_n(\hat{\phi}) \hat{L}_n(\phi)^T \) (see the aforementioned references). Equation (2.23) is in terms of products of coefficients of finite predictors for \( X_r \) “outside” the interval \( -(n-1), \ldots, 0 \), while the Cholesky decomposition is based on the coefficients of the best linear predictor of \( X_r \) “inside” the interval \( -(n-1), \ldots, 0 \).

Remark 2.4. (Multivariate extension). The case that the (autocovariance) sequence \( \{C(r) : r \in \mathbb{Z}\} \) is made up of \( d \times d \)-dimensional, has not been considered in this article. However, if \( \Sigma(\omega) = \sum_{r \in \mathbb{Z}} C(r) e^{i\omega r} \) is a positive definite matrix with Vector MA(\( \infty \)) and Vector AR(\( \infty \)) representations (see, Wiener and Masani, 1958) then it may be possible to extend the above results to the multivariate setting.

3. FINITE ORDER AUTOREGRESSIVE APPROXIMATIONS

In many applications, it is often assumed the spectral density is rational (Cadow, 1982; Ahlén and Sternad, 1991; and Ge and Kerrigan, 2016). Obtaining the spectral factorization (such as that given in (2.1)) of a rational spectral density is straightforward, and is one of the reasons that rational spectral densities are widely used. However, a rational spectral density is usually only an approximation of the underlying spectral density. We obtain a bound for the approximation when the rational spectral density corresponds to a finite order autoregressive process. The expressions in (2.19) and (2.20) easily lend themselves to obtaining a rational approximation. Further, one can use Baxter’s inequality to obtain a bound for the approximation.

We now use the expressions in (2.19) to obtain an approximation of \( H(\omega) \) in terms of the best fitting AR(\( p \)) coefficients. In particular, using that \( \psi(\omega)^* = [\phi(\omega)^*]^{-1} \), we replace the infinite order AR coefficients in

\[
H(\omega) = \frac{\sum_{r=0}^{\infty} g_r (e^{i\omega r} + [\phi(\omega)^*]^{-1}\phi_r(\omega)^*)}{f(\omega)}
\]

with the best fitting AR(\( p \)) coefficients. More precisely, suppose that \( \phi_{p,1}, \ldots, \phi_{p,p} \) are the best fitting AR(\( p \)) coefficients in the sense that it minimizes the mean squared prediction error

\[
(\phi_{p,1}, \ldots, \phi_{p,p})^* = \arg\min_{\phi_{p,1}, \ldots, \phi_{p,p}} \mathbb{E} \left\| X_0 - \sum_{j=1}^{p} a_j X_{-j} \right\|^2 = \arg\min_{\phi_{p,1}, \ldots, \phi_{p,p}} \frac{1}{2\pi} \int_0^{2\pi} \left\| 1 - \sum_{j=1}^{p} a_j e^{i\omega j} \right\|^2 f(\omega) d\omega,
\]

where \( a = (a_1, \ldots, a_p) \). The corresponding best fitting AR(\( p \)) spectral density is \( f_p(\omega) = \sigma_p^2 |\phi(\omega)|^{-2} \) where \( \sigma_p^2 = \mathbb{E} |X_0 - \sum_{j=1}^{p} \phi_{p,j} X_{-j}|^2 \) and \( |\phi(\omega)| = 1 - \sum_{j=1}^{p} |\phi_{p,j}| e^{i\omega j} \). We note that the zeros of the characteristic polynomial \( 1 - \sum_{j=1}^{p} \phi_{p,j} z^j \) lie outside the unit circle (see Brockwell and Davis, 2006, Problem 8.3). Then, we define the
approximation of $H(\omega)$ as

$$H_p(\omega) = \frac{\sum_{\ell=0}^{\infty} g_\ell \left(e^{i\ell\omega} + [\phi^{(p)}(\omega)]^{-1} \phi^{(p)}(\omega)^* \right)}{f_p(\omega)},$$

(3.2)

where $\phi^{(p)}(\omega) = \sum_{\ell=1}^{p-\ell} \phi_{p,\ell} e^{i\ell\omega}$ for $0 \leq \ell < p$ and $0$ for $\ell \geq p$. We observe that the Fourier coefficients of $H_p(\omega)$ are the solution of $T(f_p)h_p = g$, where $h_p = (h_{p,0}, h_{p,1}, \ldots)'$ with $h_{p,j} = (2\pi)^{-1} \int_0^{2\pi} H_p(\omega) e^{-ij\omega} d\omega$. Thus $T(f_p)$ and $T(f_p)^{-1}$ are approximations of $T(f)$ and $T(f)^{-1}$ respectively. By using Lemma 2.1 and (2.14) we can show that

$$H_p(\omega) = \sigma_p^{-2} [\phi^{(p)}(\omega) [\phi^{(p)}(\omega)^* G_+(\omega)]_+]_+.$$

(3.3)

From a practical perspective, the best fitting AR($p$) coefficients can be estimated from the data. The AR coefficients $\{\phi_{p,j} : 1 \leq j \leq p\}$ in (3.3) can be replaced by its estimate and the result used as an estimator of $H(\omega)$.

Below we obtain a bound for $H(\omega) - H_p(\omega)$.

**Theorem 3.1.** (Approximation theorem). Suppose that $\{c(r) : r \in \mathbb{Z}\}$ is a symmetric, positive definite sequence that satisfies Assumption 2.1(ii) and its Fourier transform $f(\omega) = \sum_{r \in \mathbb{Z}} c(r) e^{i\omega r}$ satisfies Assumption 2.1(i). We define the (semi) infinite system of equations

$$g_\ell = \sum_{j=0}^{\infty} h_j c(\ell - j), \quad \ell \geq 0,$$

where $(g_\ell; \ell \geq 0)' \in \ell_2^+$. Let $H(\omega)$ and $H_p(\omega)$ be defined as in (2.6) and (3.2). Then, there exist a constant $0 < C < \infty$, such that

$$\left|H(\omega) - H_p(\omega)\right| \leq C \left[p^{-K+1} \sup_{s} |g_s| + p^{-K} |G_+(\omega)|\right],$$

where $G_+(\omega) = \sum_{\ell=0}^{\infty} g_\ell e^{i\ell\omega}$.

**Proof.** See Appendix A.

**Remark 3.1.** (Alternative approximation methods). There are other ways to obtain the spectral factorization for non-rational spectral density. For example, using the Fourier coefficients of the log spectral density $\log(\phi)(\omega)$ (usually called the cepstral coefficients), Pourahmadi (1984) proposed a recursive algorithm for obtaining the AR and MA coefficients (see, also, Bauer, 1955; Tunnicliffe-Wilson, 1972; and McElroy and Politis, 2019, Chapter 7.7, Fact 7.7.6). As this is an infinite recursion based on an infinite number of cepstral coefficients, typically the number of non-zero cepstral coefficients is truncated to a finite number in order to terminate the recursion. The truncation will lead to an approximation error, which we do not investigate here.

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**DATA AVAILABILITY STATEMENT**

Data sharing is not applicable to this article as no new data were created or analyzed in this study.
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APPENDIX : PROOFS

The purpose of this appendix is to give the technical details behind the results stated in the main section.

Proof of Theorem 2.1. To prove that \( h^+ = (h_{j:}^+ \geq 0)^{\ell}$ \( \in \ell^2_+ \), we note that since \( \mathbb{E}[Y^2] < \infty \), then \( P_{\mathcal{H}_0}(Y) = \sum_{j=0}^{\infty} h_j X_{-j} \) is a well-defined random variable in \( \mathcal{H}_0 \) with

\[
\text{var}[P_{\mathcal{H}_0}(Y)] = \langle h^+, T(f)h^+ \rangle < \infty.
\]

Furthermore, we note that

\[
\langle h^+, T(f)h^+ \rangle = \frac{1}{2\pi} \int_{0}^{2\pi} \left| \sum_{j=0}^{\infty} h_j e^{ij\omega} \right|^2 f(\omega) d\omega \geq \inf_{\omega} f(\omega) \cdot \frac{1}{2\pi} \int_{0}^{2\pi} \left| \sum_{j=0}^{\infty} h_j e^{ij\omega} \right|^2 d\omega.
\]

Since \( \inf_{\omega} f(\omega) > 0 \), we have \( \sum_{j=0}^{\infty} h_j e^{ij\omega} \in L_2([0, 2\pi]) \) and thus \( h^+ \in \ell^2_+ \).

To prove that \( c^+ = (c_{XY}(\ell); \ell \geq 0)^{\ell} \in \ell^2_+ \), we recall that (2.2) leads to the matrix equation \( c^+ = T(f)c^+ \). Let \( \|A\|_{sp} = \sup_{\|e\|_0 \leq 1} \|Ae\|_2 \) be the spectral norm. Then, since \( \sup_{\omega} f(\omega) < \infty \), \( \|T(f)\|_{sp} \leq \sup_{\omega} f(\omega) < \infty \), we have that

\[
\|c^+\|_2 = \|T(f)c^+\|_2 \leq \|T(f)\|_{sp} \|c^+\|_2 < \infty.
\]

Thus, \( c^+ \in \ell^2_+ \).

From (2.5), we have \( H(\omega) = F(c^+)\omega / f(\omega) \). Our goal is to express \( F(c^+)\omega \) in terms of the infinite order AR and MA coefficients of \( \{X_{\ell}\} \). To do this we observe

\[
F(c^+)\omega = \sum_{\ell=0}^{\infty} c_{XY}(\ell)e^{i\ell\omega} + \sum_{\ell=1}^{\infty} \text{cov} \left( Y, P_{\mathcal{H}_0}(X_{\ell}) \right) e^{-i\ell\omega}.
\]  

(A1)

The second term on the right-hand side of (A1) looks quite unwieldy. However, we show below that it can be expressed in terms of the infinite order AR coefficients associated with \( f \). It is well known that the \( \ell \)-step ahead forecast \( P_{\mathcal{H}_0}(X_{\ell}) \) \( (\ell > 0) \) has the representation \( P_{\mathcal{H}_0}(X_{\ell}) = \sum_{j=0}^{\infty} \phi_j(\ell)X_{-j} \) with \( \ell \)-step ahead prediction coefficients

\[
\phi_j(\ell) = \sum_{s=1}^{\ell} \phi_{j+s} \psi_{\ell-s},
\]  

(A2)

where \( \{\phi_j : j \geq 1\} \) and \( \{\psi_j : j \geq 0\} \) are the infinite order AR and MA coefficients defined in (2.1) (setting \( \psi_0 = 1 \)) respectively. We now obtain an expression for \( \text{cov}(Y, P_{\mathcal{H}_0}(X_{\ell})) \). Using (A2),

\[
\text{cov}(Y, P_{\mathcal{H}_0}(X_{\ell})) = \text{cov} \left( Y, \sum_{j=0}^{\ell} \sum_{s=1}^{\ell} \phi_{j+s} \psi_{\ell-s} X_{-j} \right) = \sum_{j=1}^{\ell} \sum_{s=1}^{\ell} c_{XY}(j)\phi_{j+s} \psi_{\ell-s}.
\]  

(A3)
For the second identity above, we use Fubini’s theorem; noting that coefficients are absolutely summable since
\[
\ell \psi_{\|\cdot\|_\infty} \sum_{j=0}^{\infty} |c_{YX}(j)\phi_{j+\ell}| \leq \left( \sum_{j=0}^{\infty} |\psi_{\|\cdot\|_\infty}| \right) \left( \sum_{j=0}^{\infty} c(j)^2 \right)^{1/2} \left( \sum_{j=0}^{\infty} \phi(j)^2 \right)^{1/2} < \infty.
\]
Using (A3) we have
\[
\sum_{\ell=1}^{\infty} \text{cov}(Y, P_{H_0}(X_{\ell})) e^{-i\ell \omega} = \sum_{\ell=1}^{\infty} \left( \sum_{j=0}^{\infty} \psi_{\|\cdot\|_\infty} c_{YX}(j)\phi_{j+\ell} \right) e^{-i\ell \omega}.
\]
The Fourier coefficients of the right-hand side of above has a convolution form, thus, we use the convolution theorem and rewrite
\[
\sum_{\ell=1}^{\infty} \text{cov}(Y, P_{H_0}(X_{\ell})) e^{-i\ell \omega} = \sum_{\ell=1}^{\infty} \left( \sum_{j=0}^{\infty} \psi_{\|\cdot\|_\infty} c_{YX}(j)\phi_{j+\ell} \right) e^{-i\ell \omega} = \left( \sum_{j=0}^{\infty} \psi_{\|\cdot\|_\infty} c_{YX}(j)\phi_{j+\ell} e^{-i\ell \omega} \right) = \psi(\omega)^* \sum_{j=0}^{\infty} c_{YX}(j)\phi_{j}(\omega)^*.
\]
where \(\phi_{j}(\omega) = \sum_{\ell=0}^{\infty} \phi_{j+\ell} e^{i\ell \omega}\) for \(j \geq 0\). Substituting the above into (A1) gives
\[
F(c_{\pm})(\omega) = \sum_{\ell=0}^{\infty} c_{YX}(\ell) e^{i\ell \omega} + \psi(\omega)^* \sum_{j=0}^{\infty} \phi_j(\omega)^* c_{YX}(j) = \sum_{\ell=0}^{\infty} c_{YX}(\ell) \left( e^{i\ell \omega} + \psi(\omega)^* \phi_j(\omega)^* \right). \tag{A4}
\]
Since \(\psi(\omega)^*\) is bounded, it is easily seen that \(F(c_{\pm})(\omega) \in L_{\infty}(0, 2\pi)\). Finally, substituting the above into \(H(\omega) = F(c_{\pm})(\omega)/f(\omega)\) proves (2.6).

To prove the alternative expression in (2.7), we rearrange the expression \(e^{i\ell \omega} + \psi(\omega)^* \phi_j(\omega)^*\) which appears in \(F(c_{\pm})(\omega)\). Using the definition \(\phi(\omega)^* = 1 - \sum_{j=1}^{\infty} \phi_j e^{-ij\omega}\), we have
\[
e^{i\ell \omega} + \psi(\omega)^* \phi_j(\omega)^* = e^{i\ell \omega}(1 + \psi(\omega)^* \sum_{j=1}^{\infty} \phi_j e^{-ij\omega})
= e^{i\ell \omega} \left( 1 + \psi(\omega)^* \left[ -\phi(\omega)^* + 1 - \sum_{j=1}^{\ell} \phi_j e^{-ij\omega} \right] \right)
= e^{i\ell \omega} \psi(\omega)^* \left( 1 - \sum_{j=1}^{\ell} \phi_j e^{-ij\omega} \right). \tag{A5}
\]
Therefore, substituting (A5) into (A4) and using that \( H(\omega) = F(c_{\pm})(\omega)/f(\omega) \) gives

\[
H(\omega) = \frac{\sum_{\ell=0}^{\infty} c_{\ell}X(\ell)e^{i\omega\ell}\psi(\omega)^* \left( 1 - \sum_{j=0}^{\ell} \phi_j e^{-j\omega} \right)}{f(\omega)}
\]

\[
= \frac{\sum_{\ell=0}^{\infty} c_{\ell}X(\ell) \left( e^{i\omega\ell} - \sum_{j=0}^{\ell} \phi_j e^{i(\ell-j)\omega} \right)}{\sigma^2 \psi(\omega)} \quad \text{(using } f(\omega) = \sigma^2 \psi(\omega)\psi(\omega)^*)
\]

\[
= \sigma^2 \phi(\omega) \sum_{\ell=0}^{\infty} c_{\ell}X(\ell) \left( e^{i\omega\ell} - \sum_{j=0}^{\ell} \phi_j e^{i(\ell-j)\omega} \right).
\]

This shows (2.7) and thus proves the Theorem.

**Proof of equation (2.9) in Remark 2.2.** For fixed \( j \geq 0 \), the coefficient of \( X_{-j} \) in \( \sum_{\ell=0}^{\infty} P_{H_0}(X_{\ell})e^{i\omega\ell} \) is \( \sum_{\ell=1}^{\infty} \phi_j(\ell)e^{i\omega\ell} \). Using (A2), we get

\[
\sum_{\ell=1}^{\infty} \phi_j(\ell)e^{i\omega\ell} = \sum_{\ell=1}^{\infty} \sum_{j=1}^{\infty} \phi_{j+\ell}\psi_{\ell-j}e^{i\omega\ell}
\]

\[
= \sum_{j=1}^{\infty} \phi_{j+\ell}e^{i\omega\ell} \sum_{\ell=1}^{\infty} \psi_{\ell-j}e^{i(\ell-j)\omega}
\]

\[
= \psi(\omega) \sum_{j=1}^{\infty} \phi_{j+\ell}e^{i\omega\ell} = \psi(\omega)\phi_j(\omega).
\]

The second identity above is also due to the fact that \( \sum_{\ell=1}^{\infty} \sum_{j=1}^{\infty} |\phi_{j+\ell}\psi_{\ell-j}| < \infty \) under Assumption 2.1(ii) for \( K = 1 \). We now show that under Assumption 2.1(ii) for \( K = 1 \), \( \psi(\omega)\sum_{j=0}^{\infty} X_{-j}\phi_j(\omega) \) converges in \( H_0 \). To show this, we define the partial sum

\[
S_n = \psi(\omega) \sum_{j=0}^{n} X_{-j}\phi_j(\omega) \in H_0.
\]

Then, for any \( n < m \)

\[
\mathbb{E}|S_m - S_n|^2 = \text{var}\left( \sum_{j=0}^{m} \psi(\omega)\phi_j(\omega)X_{-j} \right) = |\psi(\omega)|^2 (\phi_n^m(\omega))^T T_{m-n}(f)(\phi_n^m(\omega)),
\]

where \( (\phi_n^m(\omega)) = (\phi_n(\omega), \ldots, \phi_m(\omega))^T \) and \( T_{m-n}(f) = (c(t - \tau); 0 \leq t, \tau \leq m - n) \). Therefore,

\[
\mathbb{E}|S_m - S_n|^2 \leq |\psi(\omega)|^2 \|T_{m-n}(f)\|_{\text{spec}}\|\phi_n^m(\omega)\|_2^2
\]

\[
\leq |\psi(\omega)|^2 \|(\sup(\omega))\|_2\|\phi_n^m(\omega)\|_2^2.
\]

If Assumption 2.1(ii) is satisfied for \( K = 1 \), then it is easy to show \( \sum_{\ell=1}^{\infty} |\phi_j(\omega)|^2 < \infty \). Therefore, by Cauchy’s criterion, \( \|\phi_n^m(\omega)\|_2 \to 0 \) as \( n, m \to \infty \), which implies \( \mathbb{E}|S_m - S_n|^2 \to 0 \) as \( n, m \to \infty \). Again applying Cauchy’s criterion (on the Hilbert space \( H_0 \)), we conclude that \( \psi(\omega)\sum_{j=0}^{\infty} X_{-j}\phi_j(\omega) \) converges in \( H_0 \). This shows \( \sum_{\ell=1}^{\infty} P_{H_0}(X_{\ell})e^{i\omega\ell} \) is well-defined in \( H_0 \) and satisfies (2.9).
Proof of equation (2.18). Representing

\[ P_{H_0}(X_{-\ell}) = \sum_{j=0}^{\infty} A_{\ell,j}X_j \quad \ell < 0, \]  

(A6)

we will show that matrix \( A = (A_{\ell,j}; \ell < 0, j \geq 0) = C_{01}C_{11}^{-1} \). We first evaluate the covariance \( \text{cov}(X_{-\ell}, P_{H_0}(X_{-\ell})) \) (for \( t \geq 0 \)) using (A6). The left hand side of (A6) is

\[ \text{cov}(X_{-\ell}, P_{H_0}(X_{-\ell})) = \text{cov}(P_{H_0}(X_{-\ell}), X_{-\ell}) = \ell < 0, t \geq 0. \]

Whereas the right-hand side of (A6) is

\[ \text{cov}(X_{-\ell}, \sum_{j=0}^{\infty} A_{\ell,j}X_j) = \sum_{j=0}^{\infty} A_{\ell,j}c(j-t) = \sum_{j=0}^{\infty} A_{\ell,j}[C_{11}]_{\ell,j} = [AC_{11}]_{\ell,j} \quad \ell < 0, t \geq 0. \]

Comparing coefficients gives \( C_{01} = AC_{11} \), that is, \( A = C_{01}C_{11}^{-1} \).

Proof of Theorem 2.2. We first prove that \( h_+ = (h_0, h_1, \ldots) \in \ell^+_2 \). Under Assumption 2.1(i), \( T(f) \) is invertible on \( \ell^+_2 \) (Widom (1960), Theorem III). Using that \( \|T(f)^{-1}\|_\infty \leq \left[ \text{inf}_{\omega \in \mathbb{R}} f(\omega) \right]^{-1} \) we have

\[ ||h_+||_2 \leq ||T(f)^{-1}||_p ||g_+||_2 \leq \left[ \text{inf}_{\omega \in \mathbb{R}} f(\omega) \right]^{-1} ||g_+||_2, \]

where \( g_+ = (g_0, g_1, \ldots) \). Since \( g_+ \in \ell^+_2 \), from the above inequality, we get \( h_+ \in \ell^+_2 \) and its Fourier transform \( H(\omega) \) is well-defined. Thus, using the construction described as in (2.17), there exists a stationary time series \( \{X_t\} \) and random variable \( Y \in \mathbb{R}(X_t ; t \leq 0) \), whose normal equations satisfy

\[ \text{cov}(Y, X_{-\ell}) = g_\ell = \sum_{j=0}^{\infty} h_j \ell(j - \ell) \quad \ell \geq 0. \]

This allows us to use Theorem 2.1 to prove the result.

Proof of Lemma 2.1. Using that \( \phi(\omega)^* = 1 - \sum_{j=0}^{\infty} \phi e^{-j\omega} \), the right-hand side of (2.21) is

\[ \sum_{\ell=0}^{\infty} g_\ell e^{i\ell\omega} \left( e^{i\omega} - \sum_{j=1}^{\ell} \phi_j e^{i(j-1)\omega} \right) = \sum_{\ell=0}^{\infty} g_\ell e^{i\ell\omega} \left( 1 - \sum_{s=1}^{\ell} \phi_s e^{-is\omega} \right) 
\]

\[ = \sum_{\ell=0}^{\infty} g_\ell e^{i\ell\omega} \left( \phi(\omega)^* + \sum_{s=\ell+1}^{\infty} \phi_s e^{-is\omega} \right) 
\]

\[ = G_+(\omega)\phi(\omega)^* + \sum_{\ell=1}^{\infty} g_\ell \sum_{s=\ell+1}^{\infty} \phi_s e^{i(\ell-s)\omega}. \]

It is straightforward that the second term on the right-hand side above is anti-causal. Therefore, take the causal part of above gives

\[ \left[ \sum_{\ell=0}^{\infty} g_\ell e^{i\ell\omega} \left( e^{i\omega} - \sum_{s=1}^{\ell} \phi_s e^{i(j-1)\omega} \right) \right]_+ = [G_+(\omega)\phi(\omega)^*]_+. \]

This proves the lemma.
Proof of Corollary 2.1. Let $\delta_{\ell,k}$ denotes the indicator variable where $\delta_{\ell,k} = 1$ if $\ell = k$ and zero otherwise. Since $T(f)^{-1} = (d_{j,k}; j, k \geq 0)$ is the inverse of $T(f) = (c(j-k); j, k \geq 0)$, $\{d_{j,k}\}$ and $\{c(r)\}$ satisfy the normal equations

$$
\delta_{\ell,k} = \sum_{j=0}^{\infty} d_{j,k} c(\ell - j) \quad \ell, k \geq 0.
$$

(A7)

Thus for each $k \geq 0$, we obtain a system of Wiener–Hopf equations. To derive $d_{j,k}$ we apply Theorem 2.2 to (A7). For each (fixed) $k \geq 0$ we obtain

$$
D_k(\omega) = \frac{1}{f(\omega)} \sum_{r=0}^{\infty} \delta_{\ell,k} \left( e^{i\ell\omega} + \psi(\omega)^* \phi_f(\omega)^* \right) = \frac{\psi(\omega)^* + \phi_f(\omega)^*}{f(\omega)},
$$

(A8)

where $D_k(\omega) = \sum_{j=0}^{\infty} d_{j,k} e^{ij\omega}$. Using the identity (A5) we can replace the above with

$$
D_k(\omega) = \frac{\psi(\omega)^* (e^{ik\omega} + \sum_{j=1}^{k} \phi_{f} e^{i(k-j)\omega})}{f(\omega)} = \sigma^{-2} \phi(\omega) \left( e^{ik\omega} + \sum_{j=1}^{k} \phi_{f} e^{i(k-j)\omega} \right).
$$

(A9)

Taking an inverse Fourier transform in (A8) and (A9) yields the entries

$$
d_{k,j} = \frac{1}{2\pi} \int_{0}^{2\pi} \left( \frac{e^{ik\omega} + \psi(\omega)^* \phi_f(\omega)^*}{f(\omega)} \right) e^{-ij\omega} d\omega
$$

$$
= \sigma^{-2} \frac{1}{2\pi} \int_{0}^{2\pi} \phi(\omega) \left( e^{ik\omega} + \sum_{j=1}^{k} \phi_{f} e^{i(k-j)\omega} \right) e^{-ij\omega} d\omega \quad j, k \leq 0.
$$

Thus proving the Corollary.

As an aside it is interesting to construct the random variable $Y_{-k}$ which yields the Wiener–Hopf equation (A7). Let $\{X_t\}$ be a mean zero weakly stationary process with $\{c(r)\}$ as its autocovariance. We define a sequence of random variables $\{\varepsilon_{-k}; k \geq 0\}$ where for $k \geq 0$

$$\varepsilon_{-k} = X_{-k} - P_{(\varepsilon_{-k})} (X_{-k})$$

and $P_{(\varepsilon_{-k})}$ denotes the orthogonal projection onto the closed subspace $\mathbb{sp}(X_t; t \leq 0 \text{ and } r \neq -k)$. We standardize $\varepsilon_{-k}$, where $Y_{-k} = \varepsilon_{-k} / \sqrt{\text{var}(\varepsilon_{-k})}$, noting that $\text{var}(\varepsilon_{-k}) = \text{cov}(\varepsilon_{-k}, X_{-k})$. Thus by definition $\text{cov}(Y_{-k}, X_{-f}) = \delta_{F,k}$ and $Y_{k} = \sum_{j=0}^{\infty} d_{k,j} X_{-j}$.

Proof of Theorem 3.1. We note that under Assumption 2.1(ii), $\sum_{j=1}^{\infty} |\hat{\phi}_j| < \infty$.

To prove the result, we use Baxter’s inequality, that is for the best fitting AR($p$) coefficients (see equation (3.1)), we have

$$
\sum_{j=1}^{p} |\phi_{p,j} - \phi_j| \leq C_p \sum_{j=p+1}^{\infty} |\phi_j|,
$$

(A10)

where $C_p$ is a constant that solely depends on $f(\omega) = \sigma^{-2} |\phi(\omega)|^{-2}$.

Returning to the proof, the difference $H(\omega) - H_p(\omega)$ can be decomposed as

$$
H(\omega) - H_p(\omega) = \sum_{\ell=0}^{\infty} \phi_{\ell} e^{i\ell\omega} \left( 1 - \frac{1}{f(\omega)} \right) + \sum_{\ell=0}^{\infty} g_{\ell} \left( \frac{[\phi(\omega)^*]^{-1} \phi_f(\omega)^*}{f(\omega)} - \frac{[\phi(\omega)^*]^{-1} \phi_f(\omega)^*}{f(\omega)} \right).
$$
\[
= \sum_{\ell=0}^{\infty} g_{\ell} e^{i\ell \omega} \left( \frac{1}{f(\omega)} - \frac{1}{f_p(\omega)} \right) + \sum_{\ell=0}^{\infty} g_{\ell} \left[ \sigma^{-2} \phi(\omega) \phi_{\ell}(\omega)^* - \sigma_p^{-2} \phi^{(p)}(\omega) \phi_{\ell}^{(p)}(\omega)^* \right]
\]
\[
= A(\omega) + B(\omega) + C(\omega),
\]
where

\[
A(\omega) = \left( \frac{1}{f(\omega)} - \frac{1}{f_p(\omega)} \right) \sum_{\ell=0}^{\infty} g_{\ell} e^{i\ell \omega},
\]
\[
B(\omega) = \left[ \sigma^{-2} \phi(\omega) - \sigma_p^{-2} \phi^{(p)}(\omega) \right] \sum_{\ell=0}^{\infty} g_{\ell} \phi_{\ell}(\omega)^*,
\]
\[
C(\omega) = \sigma_p^{-2} \phi^{(p)}(\omega) \sum_{\ell=0}^{\infty} g_{\ell} \left[ \phi_{\ell}(\omega)^* - \phi_{\ell}^{(p)}(\omega)^* \right].
\]

To bound each of the terms above, we derive some auxiliary bounds that we use later to bound \(A(\omega)\) to \(C(\omega)\). We use \(C\) to denote a generic constant that may change from line to line (and depends on \(C_f\)).

First, we bound the difference \(|\phi(\omega) - \phi^{(p)}(\omega)|\). Using (A10), we have

\[
|\phi(\omega) - \phi^{(p)}(\omega)| = \left| \sum_{j=1}^{p} (\phi_j - \phi_{p,j}) e^{i\omega} + \sum_{j=p+1}^{\infty} \phi_j e^{i\omega} \right|
\]
\[
\leq \sum_{j=1}^{p} |\phi_j - \phi_{p,j}| + \sum_{j=p+1}^{\infty} |\phi_j| \leq (C_f + 1) \sum_{j=p+1}^{\infty} |\phi_j|.
\]

(A11)

Furthermore, using that \(|\phi(\omega)| \leq 1 + \sum_{j=1}^{\infty} |\phi_j| < \infty\), we have

\[
|\phi^{(p)}(\omega)| \leq |\phi^{(p)}(\omega) - \phi(\omega)| + |\phi(\omega)| \leq (C_f + 1) \sum_{j=p+1}^{\infty} |\phi_j| + 1 + \sum_{j=1}^{\infty} |\phi_j| < \infty.
\]

Using above two bounds, we obtain a bound for \(|\phi(\omega)|^2 - |\phi^{(p)}(\omega)|^2\). By the triangular inequalities \(|A^2 - B^2| \leq |A^2 - 2|A|| + |B|^2\) and \(|A + B| \leq |A| + |B|\) together with (A11) gives

\[
||\phi(\omega)||^2 - |\phi^{(p)}(\omega)|^2 \leq |\phi(\omega))^2 - \phi^{(p)}(\omega))^2| \leq |\phi(\omega) - \phi^{(p)}(\omega)||\phi(\omega)| + |\phi^{(p)}(\omega)|)
\]
\[
\leq C \sum_{j=p+1}^{\infty} |\phi_j|.
\]

(A12)

Next, we bound \(|\sigma^{-2} - \sigma_p^{-2}|\). We recall that \(\sigma^2 = (2\pi)^{-1} \int |\phi(\omega)|^2 f(\omega) d\omega\) and \(\sigma_p^2 = \mathbb{E}[X_i - \sum_{j=1}^{p} \phi_{p,j} X_{i-j}]^2 = (2\pi)^{-1} \int |\phi_p(\omega)|^2 f(\omega) d\omega\). Using these expression we have

\[
|\sigma^{-2} - \sigma_p^{-2}| \leq \frac{1}{2\pi} \int_0^{2\pi} ||\phi(\omega)||^2 - |\phi^{(p)}(\omega)|^2| f(\omega) d\omega.
\]

Combining the above with (A12) and Assumption 2.1(i) gives

\[
|\sigma^{-2} - \sigma_p^{-2}| \leq \frac{1}{2\pi} \int_0^{2\pi} ||\phi(\omega)||^2 - |\phi^{(p)}(\omega)|^2| f(\omega) d\omega
\]
where

\[ \frac{1}{2\pi} \int_{0}^{2\pi} |\phi(\omega)|^2 - |\phi(\omega)|^2 |d\omega \leq C \sum_{j=p+1}^{\infty} |\phi_j|. \]

As an immediate consequence of above, we have \( \sigma_p^2 > \sigma^2 - |\sigma^2 - \sigma_p^2| \geq \sigma^2/2 > 0 \) for large \( p \). Since \( \sigma_p^2 > 0 \) for all \( p \), we have

\[ \inf_p \sigma_p^2 > 0. \quad (A13) \]

Therefore, we obtain the bound

\[ |\sigma^{-2} - \sigma_p^{-2}| \leq \sigma^{-2} \sigma_p^{-2} |\sigma^2 - \sigma_p^2| \leq C \sum_{j=p+1}^{\infty} |\phi_j|. \quad (A14) \]

Lastly, by Assumption 2.1(ii),

\[ \sum_{j=p+1}^{\infty} |\phi_j|^q \leq p^{-K+q} \sum_{j=p+1}^{\infty} |\phi_j|^q = O(p^{-K+q}) \quad \text{for} \quad 0 \leq \alpha \leq K. \quad (A15) \]

Now we are ready to bound each term in \( H(\omega) - H_p(\omega) \). First, we bound \( A(\omega) \). Note that

\[ \left| (f(\omega))^{-1} - (f_p(\omega))^{-1} \right| = |\sigma^{-2} |\phi(\omega)|^2 - \sigma_p^{-2} |\phi(\omega)|^2| \]

\[ \leq \sigma^{-2} |\phi(\omega)|^2 - |\phi(\omega)|^2 + |\sigma^2 - \sigma_p^{-2} ||\phi(\omega)|^2|. \]

Using (A12) and (A14), we get

\[ \left| (f(\omega))^{-1} - (f_p(\omega))^{-1} \right| \leq \sigma^{-2} |\phi(\omega)|^2 - |\phi(\omega)|^2 + |\sigma^2 - \sigma_p^{-2} ||\phi(\omega)|^2| \leq C \sum_{j=p+1}^{\infty} |\phi_j|. \]

Therefore, substituting (A15) for \( \alpha = 0 \) into \( A(\cdot) \) gives

\[ |A(\omega)| \leq \left| (f(\omega))^{-1} - (f_p(\omega))^{-1} \right| \sum_{\ell=0}^{\infty} g_\ell e^{i\ell \omega} \leq C \left( \sum_{j=p+1}^{\infty} |\phi_j| \right) \left( \sum_{\ell=0}^{\infty} g_\ell e^{i\ell \omega} \right) = O \left( p^{-K} |G_s(\omega)| \right), \]

where \( G_s(\omega) = \sum_{\ell=0}^{\infty} g_\ell e^{i\ell \omega} \).

To bound \( B(\omega) \) we note that from (A11) and (A14)

\[ |\sigma^{-2} \phi(\omega) - \sigma_p^{-2} \phi(\omega)| \leq \sigma^{-2} |\phi(\omega) - \phi(\omega)| + |\sigma^2 - \sigma_p^{-2} \phi(\omega)| \leq C \sum_{j=p+1}^{\infty} |\phi_j|. \]

Therefore, we have

\[ |B(\omega)| \leq |\sigma^{-2} \phi(\omega) - \sigma_p^{-2} \phi_p(\omega)| \sum_{\ell=0}^{\infty} |g_\ell \phi(\omega)| \leq C \left( \sum_{j=p+1}^{\infty} |\phi_j| \right) \left( \sum_{\ell=0}^{\infty} |g_\ell \phi(\omega)| \right). \]
The second summand on the right-hand side of the above is bounded with
\[
\sum_{\ell' = 0}^{\infty} |g_{\ell'} e(\omega)| \leq \sum_{\ell' = 0}^{\infty} |g_{\ell'}| \sum_{s = 1}^{\infty} |\phi_{\ell'+s}| = \sum_{s = 1}^{\infty} \sum_{\ell' = 0}^{\infty} |g_{\ell'}| |\phi_s| \leq \sup_s |g_s| \cdot \sum_{s = 1}^{\infty} u|\phi_s|.
\]

Thus by using the above two bounds and (A15) (for \(\alpha = 0\)), we have
\[
|B(\omega)| \leq C \left( \sum_{j = p+1}^{\infty} |\phi_j| \right) \cdot \sup_s |g_s| \cdot \sum_{s = 1}^{\infty} u|\phi_s| = O \left( \sup_s |g_s| \cdot p^{-K} \right).
\]

Finally, we obtain a bound for \(C(\omega)\). Since \(\phi^{(p)}(\omega) = 0\) for \(\ell' \geq p\) we split \(C(\omega) = C_1(\omega) + C_2(\omega)\) where
\[
C_1(\omega) = \sigma_p^{-2} \phi^{(p)}(\omega) \sum_{\ell' = 0}^{p-1} g_{\ell'} [\phi_{\ell'}(\omega)^* - \phi_{\ell'}^{(p)}(\omega)^*]
\]
\[
C_2(\omega) = \sigma_p^{-2} \phi^{(p)}(\omega) \sum_{\ell' = p}^{\infty} g_{\ell'} \phi_{\ell'}(\omega)^*.
\]

To bound \(C_1(\omega)\), we note by (A10) and (A15) (for \(\alpha = 0\))
\[
|\phi_{\ell'}(\omega)^* - \phi_{\ell'}^{(p)}(\omega)^*| = \sum_{s = 1}^{\ell' - p} |\phi_{\ell'+s}| + \sum_{s = p - \ell' + 1}^{\infty} |\phi_{\ell'+s}|
\]
\[
\leq C \sum_{s = p+1}^{\infty} |\phi_s| = O(p^{-K}) \quad 0 \leq \ell' < p.
\]

where the \(O(p^{-K})\) bound above is uniform over \(0 \leq \ell' < p\). Therefore, combining the above with (A13) gives
\[
|C_1(\omega)| \leq \sigma_p^{-2} |\phi^{(p)}(\omega)| \sum_{\ell' = 0}^{p-1} g_{\ell'} |\phi_{\ell'}(\omega)^* - \phi_{\ell'}^{(p)}(\omega)^*|
\]
\[
\leq C \sup_s |g_s| \cdot \sum_{\ell' = 0}^{p-1} |\phi_{\ell'}(\omega)^* - \phi_{\ell'}^{(p)}(\omega)^*| = O \left( \sup_s |g_s| \cdot p^{-K+1} \right). \tag{A16}
\]

To bound \(C_2(\omega)\), we use (A15) for \(\alpha = 1\),
\[
\sum_{\ell' = p}^{\infty} |g_{\ell'}| \phi_{\ell'}(\omega)| \leq \sup_s |g_s| \sum_{\ell' = p}^{\infty} |\phi_{\ell'+s}| \leq \sup_s |g_s| \sum_{s = p+1}^{\infty} u|\phi_s| = O \left( \sup_s |g_s| \cdot p^{-K+1} \right).
\]

Therefore, we have
\[
|C_2(\omega)| \leq \sigma_p^{-2} |\phi^{(p)}(\omega)| \sum_{\ell' = p}^{\infty} g_{\ell'} |\phi_{\ell'}(\omega)| = O \left( \sup_s |g_s| \cdot p^{-K+1} \right). \tag{A17}
\]
Combining (A16) and (A17) gives

\[ |C(\omega)| \leq |C_1(\omega)| + |C_2(\omega)| = O\left( \sup_s |g_s| \cdot p^{-K+1} \right). \]

Altogether, this yields the bound

\[ |H(\omega) - H_p(\omega)| \leq C \left[ p^{-K+1} \cdot \sup_s |g_s| + p^{-K} \cdot |G_\omega(\omega)| \right]. \]

This proves the result. \(\blacksquare\)