Further properties of the Bergman spaces of slice regular functions

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Abstract

In this paper we continue the study of Bergman theory for the class of slice regular functions. In the slice regular setting there are two possibilities to introduce the Bergman spaces, that are called of the first and of the second kind. In this paper we mainly consider the Bergman theory of the second kind, by providing an explicit description of the Bergman kernel in the case of the unit ball and of the half space. In the case of the unit ball, we study the Bergman-Sce transform. We also show that the two Bergman theories can be compared only if suitable weights are taken into account. Finally, we use the Schwarz reflection principle to relate the Bergman kernel with its values on a complex half plane.

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1 Introduction

The literature on Bergman theory is wide and as classical reference books we mention, with no claim of completeness, the books [1], [2] and [21]. The theory has also been developed for hyperholomorphic functions such as the quaternionic regular functions in the sense of Fueter and the theory of monogenic functions, [3], [4], [15], [16], [17], [18], [22], [23], [24] and the literature therein. Recently we have started the study of Bergman theory in the slice hyperholomorphic setting, see [6], [7], [8], [9]. In this framework, it is possible to give two different notions of Bergman spaces, the so-called Bergman spaces of the first and of the
second kind. Bergman spaces of the first kind in the slice regular setting are mainly studied in [6]. They are defined as

$$A(\Omega) := \{ f \in \mathcal{SR}(\Omega) \mid \|f\|_{A(\Omega)}^2 := \int_{\Omega} |f|^2 d\mu < \infty \}$$

where $\Omega$ is an axially symmetric bounded open set in the space of quaternions $\mathbb{H}$ and $\mathcal{SR}(\Omega)$ denotes the set of slice regular functions on $\Omega$. The Riesz representation theorem allows to write the so-called slice regular Bergman kernel of the first kind $B(\cdot, \cdot)$ associated with $\Omega$, leading to the integral representation

$$f(q) = \int_{\Omega} B(q, \cdot) f d\mu, \quad \forall f \in A(\Omega).$$  \hspace{1cm} (1)

If we denote by $S^2$ the sphere of pure imaginary quaternions and by $\mathbb{C}(i)$ the complex plane with imaginary unit $i \in S^2$, then slice regular functions $f : \Omega \to \mathbb{H}$ are functions whose restrictions to $\Omega_i := \Omega \cap \mathbb{C}(i)$, for every $i \in S^2$, are holomorphic maps. For a study of this class of functions see the books [13], [19] and the references therein. When we suppose that the domain $\Omega$ intersects the real line and it is axially symmetric, i.e. symmetric with respect to the real axis, then the restriction of a slice regular function to two different planes, $\mathbb{C}(i)$ and $\mathbb{C}(j)$ with $i \neq j$, turn out to be strictly related by the Representation Formula

$$f(x + yi) = \frac{1}{2}(1 - ij)f(x + yj) + \frac{1}{2}(1 + ij)f(x - yj)$$

which asserts that if we know the value of the function $f$ on the complex plane $\mathbb{C}(j)$ then we can reconstruct $f$ in all the other planes $\mathbb{C}(i)$ for every $i \in S^2$. So the global behavior of these functions on these particular axially symmetric sets is in fact completely determined by their behavior on a complex plane $\mathbb{C}(i)$.

From this fact, it follows that there is a second way to define the Bergman space and the Bergman kernel: we can work with the restriction to a complex plane $\mathbb{C}(i)$ and then to extend it using the Representation Formula. This approach gives rise the so-called Bergman theory of the second kind.

In this paper we will mainly work with Bergman spaces of the second kind, see Section 3. In section 4, we will provide the explicit description of the Bergman kernel in the case of the unit ball and of the half space. The two Bergman theories of the first and of the second kind are compared in Section 5, where we show that they contain different elements, unless some suitable weights are taken into account. Then, in Section 6, we study the Bergman-Fueter transform which is an integral transform that associates to every slice regular function $f$ defined on $\Omega$ the Fueter regular function $\tilde{f}$ given by $\tilde{f} = \Delta f$, where $\Delta$ is the Laplace operator. Finally, in Section 7, we use the Schwarz reflection principle to relate the Bergman kernel of the second kind with its values on the half plane $\mathbb{C}^+(i) = \{ x + yi, y \geq 0 \}$ and also to write the inner product of two elements using an integral on an half plane.

2 Preliminary results

By $\mathbb{H}$ we denote the algebra of real quaternions. A quaternion $q$ is an element of the form $q = q_0 + iq_1 + jq_2 + kq_3$ where $i, j, k$ satisfy $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k, jk = -kj = i$,
A quaternion will also be denoted as $q = q_0 + q$ where $q_0$ and $q = iq_1 + jq_2 + kq_3$ are its real and imaginary part, respectively. The modulus of a quaternion $q$ is defined as $|q| = (q_0^2 + q_1^2 + q_2^2 + q_3^2)^{1/2}$. Let

$$S^2 := \{ q = iq_1 + jq_2 + kq_3 \mid |q|_{\mathbb{R}^3} = 1 \}.$$ 

It is important to note that an element $i \in S^2$ is such that $i^2 = -1$. Given $i \in S^2$, we denote by $C(i)$ the real linear space generated by 1 and i. It is immediate that $C(i) \cong \mathbb{C}$.

For any open set $\Omega \subset \mathbb{H}$ let

$$\Omega_i := \Omega \cap C(i), \quad \Omega_i^\perp = \{ x + yi \in \Omega_i \mid y \geq 0 \}.$$ 

We set $\mathbb{B}^4 := \{ q \in \mathbb{H} \mid |q| < 1 \}$, and $\mathbb{D}_i := \mathbb{B}^4 \cap C(i)$. Finally, for any function $f : \Omega \rightarrow \mathbb{H}$ we denote its restriction to $\Omega \cap C(i)$ by $f|_{\Omega_i}$.

Let us now recall the definition of slice regular functions.

**Definition 2.1** A real differentiable quaternion-valued function $f$ defined on an open set $\Omega \subset \mathbb{H}$ is called (left) slice regular on $\Omega$ if for any $i \in S^2$ the function $f|_{\Omega_i}$ is such that

$$\left( \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \right) f|_{\Omega_i}(x + yi) = 0 \text{ on } \Omega_i.$$ 

We denote by $\text{SR}(\Omega)$ the set of (left) slice regular functions on $\Omega$.

The function $f$ is called slice anti-regular on the right if for any $i \in S^2$ the function $f|_{\Omega_i}$ is such that

$$f|_{\Omega_i}(x + yi) \left( \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} \right) = 0 \text{ on } \Omega_i.$$ 

**Remark 2.2** With suitable modifications, we can define slice right regular functions and slice left anti-regular functions.

We now introduce a special subclass of domains on which slice regular functions have nice properties.

**Definition 2.3** Let $\Omega \subseteq \mathbb{H}$ be a domain. We say that $\Omega$ is a slice domain (s-domain for short) if $\Omega \cap \mathbb{R}$ is non empty and if $\Omega \cap C(i)$ is a domain in $C(i)$ for all $i \in S^2$.

**Definition 2.4** Let $\Omega \subseteq \mathbb{H}$. We say that $\Omega$ is axially symmetric if for every $x + iy \in \Omega$ we have that $x + jy \in \Omega$ for all $j \in S^2$.

Note that any axially symmetric open set $\Omega$ can be uniquely associated with an open set $O_\Omega \subseteq \mathbb{R}^2$, the so-called generating set, defined by

$$O_\Omega := \{ (x, y) \in \mathbb{R}^2 \mid x + yi \in \Omega, \ i \in S^2 \}.$$ 

In the sequel, we will denote by $\text{Hol}(\Omega_i)$ the set of $\mathbb{C}$-valued functions, holomorphic on $\Omega_i \subseteq \mathbb{C}_i$. Let $Z$ denote the complex conjugation, i.e. $Z(z) = \bar{z}, \ \forall z \in C(i)$. If $\Omega$ is an axially symmetric s-domain then for any $i \in S^2$ the domain $\Omega_i \subset C(i)$ satisfies that $Z(\Omega_i) = \Omega_i$. We also define

$$\text{Hol}_c(\Omega_i) := \{ f \in \text{Hol}(\Omega_i) \mid Z \circ f \circ Z = f \}.$$ 

Functions belonging to $\text{Hol}_c(\Omega)$ are called in the literature real or intrinsic.

Slice regular functions satisfy the following property (see [5][11][12]).
Theorem 2.5 (Representation Formula) Let $f$ be a slice regular function on an axially symmetric s-domain $\Omega \subseteq \mathbb{H}$. Choose any $j \in S^2$. Then the following equality holds for all $q = x + iy \in \Omega$:

$$f(x + yj) = \frac{1}{2}(1 - ij)f(x + yj) + \frac{1}{2}(1 + ij)f(x - yj). \quad (2)$$

Remark 2.6 By the Splitting Lemma, the restriction of a slice regular function to any complex plane $\mathbb{C}(i)$ can be written as $f|_{\Omega_i(z)} = F(z) + G(z)j$ where $j$ is an element in $S^2$ orthogonal to $i$ and $F, G : \Omega_i \to \mathbb{C}(i)$ are two holomorphic functions. Conversely, the Representation formula allows to extend a function of the form $f(z) = F(z) + G(z)j$, defined on $\Omega_i$ to a slice regular function defined on $\Omega$. Thus we have the so-called extension operator

$$P_i : Hol(\Omega_i) + Hol(\Omega_i)j \rightarrow SR(\Omega),$$

defined for any $f \in Hol(\Omega_i) + Hol(\Omega_i)j$ by

$$P_i[f](q) = P_i[f](x + I_q y) = \frac{1}{2}[(1 + I_q i)f(x - iy) + (1 - I_q i)f(x + iy)], \quad (3)$$

where $q = x + I_q y$.

2.1 The slice regular Bergman theory of the first kind

Let $\Omega \subset \mathbb{H}$ be a bounded axially symmetric s-domain. We will consider the space $L^2(\Omega, \mathbb{H})$ formed by functions $f : \Omega \to \mathbb{H}$ such that

$$\int_{\Omega} |f|^2 d\mu < \infty,$$

where $d\mu$ denotes the Lebesgue volume element in $\mathbb{R}^4$. The functional $\langle \cdot, \cdot \rangle_{L^2(\Omega, \mathbb{H})} : L^2(\Omega, \mathbb{H}) \times L^2(\Omega, \mathbb{H}) \to \mathbb{H}$, defined by

$$\langle f, g \rangle_{L^2(\Omega, \mathbb{H})} = \int_{\Omega} \bar{f}g d\mu,$$

is an $\mathbb{H}$-valued inner product on $L^2(\Omega, \mathbb{H})$. The space $L^2(\Omega, \mathbb{H})$ is then equipped with the norm

$$\|f\|_{L^2(\Omega, \mathbb{H})} := \left( \int_{\Omega} |f|^2 d\mu \right)^{\frac{1}{2}}. \quad (4)$$

Definition 2.7 The set $A(\Omega) := SR(\Omega) \cap L^2(\Omega, \mathbb{H})$, equipped with the norm and the inner product inherited from $L^2(\Omega, \mathbb{H})$, is called the slice regular Bergman space of the first kind associated with $\Omega$.

The following results have been proved in [6], see Proposition 3.4 and Theorem 3.5, but we repeat them since they will be useful in the sequel.

Proposition 2.8 Let $\Omega$ be a bounded axially symmetric s-domain. For any compact set $K \subset \Omega \setminus \mathbb{R}$, there exists $\lambda_K > 0$ such that

$$\sup\{|f(q)| \mid q \in K\} \leq \lambda_K \|f\|_{A(\Omega)}, \quad \forall f \in A(\Omega).$$

Theorem 2.9 The set $A(\Omega)$ is a complete space.

Theorem 2.9 implies that $A(\Omega)$ is a quaternionic right linear Hilbert space. For further properties on the Bergman theory of the first kind we refer the reader to [6].
3 The slice regular Bergman theory of the second kind

Let $\Omega \subseteq \mathbb{H}$ be a bounded axially symmetric s-domain. We introduce here a family of Bergman-type spaces. For every $i \in S^2$ we set 

$$A(\Omega_i) := \left\{ f \in S\mathcal{R}(\Omega) \mid \|f\|_{A(\Omega_i)}^2 := \int_{\Omega_i} |f|_{\Omega_i}^2 d\sigma < \infty \right\},$$

where $d\sigma$ denotes the Lebesgue measure in the plane $\mathbb{C}_i$ and $d\sigma_i(x + iy) = dx dy$. On $A(\Omega_i)$ we define the norm

$$\|f\|_{A(\Omega_i)} := \left( \int_{\Omega_i} |f|_{\Omega_i}^2 d\sigma \right)^{\frac{1}{2}}, \quad \forall f \in A(\Omega_i), \tag{5}$$

and the inner product

$$\langle f, g \rangle_{A(\Omega_i)} = \int_{\Omega_i} f \overline{g} d\sigma. \tag{6}$$

**Proposition 3.1** Let $i, j$ be any unit vectors, and let $\Omega$ be a bounded axially symmetric s-domain and $f \in S\mathcal{R}(\Omega)$. Then $f \in A(\Omega_i)$ if and only if $f \in A(\Omega_j)$.

**Proof.** Using formula (2), we have that

$$|f(x + yi)|^2 \leq |f(x + yj)|^2 + 2|f(x + yj)||f(x - yj)| + |f(x - yj)|^2.$$

As

$$|f(x + yj)||f(x - yj)| \leq \frac{1}{2}(|f(x + yj)|^2 + |f(x - yj)|^2),$$

then

$$|f(x + yi)|^2 \leq 2 \left[ |f(x + yj)|^2 + |f(x - yj)|^2 \right], \tag{7}$$

and integrating both sides of (7) on the generating set $\Omega_i$, one obtains that

$$\int_{\Omega_i} |f|_{\Omega_i}^2 d\sigma_i \leq 2 \int_{\Omega_i} |f|_{\Omega_j}^2 d\sigma_j.$$

Now replacing $i$ by $j$ in (7), we get

$$\int_{\Omega_j} |f|_{\Omega_j}^2 d\sigma_j \leq 2 \int_{\Omega_i} |f|_{\Omega_i}^2 d\sigma_i,$$

and the assertion follows. \hfill $\blacksquare$

**Corollary 3.2** Given any $i, j \in S^2$, the Bergman spaces $A(\Omega_i)$ and $A(\Omega_j)$ contain the same elements and have equivalent norms.

We also recall the following result, see [6]:

**Theorem 3.3 (Completeness of $A(\Omega_i)$)** Let $\Omega$ be a bounded axially symmetric s-domain. The spaces $(A(\Omega_i), \| \cdot \|_{A(\Omega_i)})$ are complete for every $i \in S^2$. 

Remark 3.4 Let $\Omega$ be a given bounded axially symmetric s-domain in $\mathbb{H}$. Theorem 3.3 implies that the Bergman spaces $A(\Omega_i)$, for $i \in S^2$, are quaternionic right linear Hilbert spaces.

Definition 3.5 We say that a function $f \in S\mathcal{R}(\Omega)$ belongs to the Bergman space of the second kind $A(\Omega)$ on the bounded, axially symmetric s-domain $\Omega$ if $f \in A(\Omega_i)$ for some $i \in S^2$.

The norm $\| \cdot \|_{A(\Omega)}$ in $A(\Omega)$ is defined to be one of the equivalent norms $\| \cdot \|_{A(\Omega_i)}$ for some fixed $i \in S^2$. To make the norm independent of the choice of $i$ one may also define the norm

$$\|f\|_{A(\Omega)} = \sup_{i \in S^2} \|f\|_{A(\Omega_i)}.$$ 

In any case, the previous discussion immediately shows that on a given bounded axially symmetric s-domain $\Omega$ in $\mathbb{H}$, the linear space $A(\Omega)$ endowed with one of the previous norms is complete.

Note also that $A(\Omega)$ can be seen as an inner product linear space, if endowed with the inner product defined in $A(\Omega_i)$, for some fixed $i \in S^2$. This inner product yields to one of the equivalent norms $\| \cdot \|_{A(\Omega_i)}$.

We now give the following definition (see [9]) which will be used in Section 7:

Definition 3.6 Let $\Omega$ be an axially symmetric s-domain. We define

$$A_c(\Omega) := P_i(Hol_c(\Omega_i)) \cap L_2(\Omega_i, \mathbb{C}(i)),$$

where $L_2(\Omega_i, \mathbb{C}(i))$ denotes the space of square integrable functions defined on $\Omega_i$ and with values in $\mathbb{C}(i)$.

As explained in [6], given any $q \in \Omega_i$ the evaluation functional $\phi_q : A(\Omega_i) \to \mathbb{H}$, given by

$$\phi_q[f] := f(q), \quad \forall f \in A(\Omega_i),$$

is a bounded quaternionic right linear functional on $A(\Omega_i)$ for every $i \in S^2$. Thus the Riesz representation theorem for quaternionic right linear Hilbert space shows the existence of the unique function $K_q(i) \in A(\Omega_i)$ such that

$$\phi_q[f_{\Omega_i}] = \langle K_q(i), f_{\Omega_i} \rangle_{A(\Omega_i)}. \quad (8)$$

We set $\mathcal{K}_{\Omega_i}(q, \cdot) := K_q(i)$ and we have:

Definition 3.7 (Slice regular Bergman kernel associated with $\Omega_i$) The function

$$\mathcal{K}_{\Omega_i}(\cdot, \cdot) : \Omega_i \times \Omega_i \to \mathbb{H}$$

will be called the slice Bergman kernel associated with $\Omega_i$.

This kernel satisfies the expected properties, see Remark 5 in [6]. We can now give the following:
Definition 3.8 Slice Bergman kernel of the second kind associated with $\Omega$. Let $K_{\Omega}(\cdot, \cdot) : \Omega \times \Omega \to \mathbb{H}$ be slice Bergman kernel associated with $\Omega_i$, we will call slice Bergman kernel of the second kind associated with $\Omega$ the function
\[
K_{\Omega} : \Omega \times \Omega \to \mathbb{H},
\]
\[
K_{\Omega}(x + y, r) := \frac{1}{2}(1 - ij)K_{\Omega_i}(x + y, r) + \frac{1}{2}(1 + ij)K_{\Omega_j}(x + y, r).
\]
Also this kernel satisfies the expected properties, see [5], Proposition 7. In particular, $K_{\Omega}(\cdot, \cdot)$ is a reproducing kernel on $\Omega$: let $i \in S^2$ then
\[
f(q) = \int_{\Omega_i} K_{\Omega}(q, \cdot) f \, d\sigma_i,
\]
and the integral does not depend on $i \in S^2$. This fact follows from the observation that, see (8):
\[
f(q) = \langle K(q, \cdot), f(\cdot) \rangle_{A(\Omega_i)} = \langle K_q i, f(\cdot) \rangle_{A(\Omega_i)}.
\]

4 The Bergman kernel $K_{\Omega}$ on the unit ball and on the half space

As it has been pointed out in [21], even in the classical complex case, the Bergman kernel cannot be computed explicitly on an arbitrary domain, at least not in general. However, if one considers the unit disk with center at the origin in the complex plane $\mathbb{C}$, then it is well known that the Bergman kernel is given by
\[
K(z, \zeta) = \frac{1}{\pi} \frac{1}{(1 - z\bar{\zeta})^2}.
\]
We will use this formula to obtain an explicit expression for the Bergman kernel of the second kind on the unit ball $\mathbb{B}^4$.

Theorem 4.1 The slice regular Bergman kernel of the second kind for the unit ball $\mathbb{B}^4$ is
\[
K_{\mathbb{B}^4}(q, r) = \frac{1}{\pi} \left( 1 - 2\bar{q}r + q^2 r^2 \right) \left( 1 - 2\text{Re}(q)\bar{r} + |q|^2 |\bar{r}|^2 \right)^{-2}.
\]

Proof. Let us consider any $r \in \mathbb{H}$ and let us choose $z$ such that $z, r$ belong to the same complex plane $\mathbb{C}(j)$. In this case the Bergman kernel for the unit disc in the complex plane $\mathbb{C}(j)$ is the function $K(z, r)$ defined in [21]. The Bergman kernel we are looking for is the slice regular extension of $K(z, r)$, which is unique, to the whole unit ball $\mathbb{B}^4$. The extension is given by the formula (9). Thus we have
\[
K_{\mathbb{B}^4}(q, r) = \frac{1}{\pi} \left\{ \frac{1}{2}(1 - ij) \frac{1}{(1 - (x + jy)\bar{r})^2} + \frac{1}{2}(1 + ij) \frac{1}{(1 - (x - jy)\bar{r})^2} \right\},
\]
where $q = x + iy$. After some standard computations we obtain
\[
K_{\mathbb{B}^4}(q, r) = \frac{1}{\pi} \left( 1 - 2\bar{q}r + q^2 r^2 \right) \left( 1 - 2\text{Re}(q)\bar{r} + |q|^2 |\bar{r}|^2 \right)^{-2}.
\]
Proposition 4.2 The kernel $K_{\mathbb{B}^4}(q,r)$ is slice regular in $q$ and slice right anti-regular in the variable $r$.

Proof. By construction, the Bergman kernel is slice regular in the variable $q$. The fact that $K_{\mathbb{B}^4}(q,r)$ is slice right anti-regular in the variable $r$ can be verified directly: the function $p(q,r) := (1 - 2q\bar{r} + q^2r^2)$ is a polynomial in the variable $\bar{r}$ with coefficients on the left thus it is slice right anti-regular in $r$; the function $h(q,r) := (1 - 2\text{Re}(q)\bar{r} + |q|^2r^2)^{-2}$ is a rational function in $r$ with real coefficients thus it is slice right anti-regular in $r$. Then $K(q,r) = p(q,r)h(q,r)$ is the product of two slice right anti-regular functions in $r$, where $h$, as a function in the variable $r$, has real coefficients. The statement follows.

In the paper [12], the authors show that the slice hyperholomorphic Cauchy kernel admits two different analytic expressions (see also [5] for the specific case of regular functions). An interesting feature of the slice regular Bergman kernel for the unit ball is that it also admits two analytic expressions. In fact, we have the following result.

Proposition 4.3 The kernel $K_{\mathbb{B}^4}(q,r)$ can be written in the form

$$K_{\mathbb{B}^4}(q,r) = \frac{1}{\pi}(1 - 2q\text{Re}[r] + q^2|r|^2)^{-2}(1 - 2qr + q^2r^2).$$

Proof. First of all, we note that the function $\psi(q,r) := \frac{1}{\pi}(1 - 2q\text{Re}[r] + q^2|r|^2)^{-2}(1 - 2qr + q^2r^2)$ is slice regular in the variable $q$ since it is the product of a rational function with real coefficients and a polynomial in $q$ with coefficients on the right. Then we observe that for any fixed $r = u + vi$ if we consider $q$ belonging to the plane $\mathbb{C}(i)$ we have:

$$K_{\mathbb{B}^4}(q,r)|_{\mathbb{C}(i)} = \psi(q,r)|_{\mathbb{C}(i)} = \frac{1}{(1 - qr)^2}.$$

By the Identity Principle, the two functions $K_{\mathbb{B}^4}$ and $\psi$ coincide (and $\psi$ turns out to be slice right anti-regular in $r$).

Let us now consider the half space $\mathbb{H}^+ = \{q \in \mathbb{H} | \text{Re}(q) > 0\}$. To construct the Bergman kernel associated with $\mathbb{H}^+$, we work in the complex plane and note that the transformation $\alpha(z) = \frac{1}{2}(z + \xi)$ maps the complex half plane of numbers with positive real part $\mathbb{C}_+$ to the unit disc $\mathbb{D}$. By a well known formula, see [25], p. 37, we have that the Bergman kernel of $\mathbb{C}_+$ is the function $K_{\mathbb{C}^+} = \frac{1}{\pi(z + \xi)^2}$. We will extend this function in order to obtain the Bergman kernel of $\mathbb{H}^+$.

Theorem 4.4 The slice regular Bergman kernel of the second kind for the half space $\mathbb{H}^+$ is

$$K_{\mathbb{H}^+}(q,r) = \frac{1}{\pi}(q^2 + 2q\bar{r} + \bar{r}^2)(q^2 + 2\text{Re}[q]\bar{r} + \bar{r}^2)^{-2}$$

$$= \frac{1}{\pi}(q^2 + 2\text{Re}[r]q + |r|^2)^{-2}(q^2 + 2qr + r^2).$$

(12)
Proof. The proof is based on computations similar to those done to obtain formulas (10) and (11), thus we will not repeat them.

Remark 4.5 Another possibility to prove the result, but with more complicated computations, is to use (10) and Proposition 4.2 in [8].

5 Comparison between the slice regular Bergman spaces of first and second kind

The properties presented so far are satisfied in a domain $\Omega \subset \mathbb{H}$, however the concept of slice regular function deals with the fact that the restriction of a function to open sets $\Omega_i \subset \mathbb{C}(i)$ is a holomorphic map. Therefore the behavior of a slice regular function is related with that one of its restrictions to the sets $\Omega_i$. Purpose of this section is to compare the sets $A(\Omega)$ and $A(\Omega_i)$. We will show that they contain different elements unless a suitable weight function is taken into account.

Definition 5.1 Let $\Omega$ be a bounded axially symmetric s-domain.

1. For any $i \in \mathbb{S}^2$ define the weight function $\rho(z) = \text{Im}(z)^2$, $\forall z \in \Omega_i$ and the weighted Bergman space on $\Omega_i$ with weight $\rho$:

$$A_{\rho}(\Omega_i) := \{f \in \mathcal{SR}(\Omega) \mid \int_{\Omega_i} |f|_{\Omega_i}^2 \rho d\sigma_i < \infty\},$$

endowed with the inner product

$$\langle f, g \rangle_{A_{\rho}(\Omega_i)} := \int_{\Omega_i} \bar{f}g \rho d\sigma_i,$$

and the corresponding norm

$$\|f\|_{A_{\rho}(\Omega_i)} := \left(\int_{\Omega_i} |f|_{\Omega_i}^2 \rho d\sigma_i \right)^{1/2}.$$

2. Define the weight function $\delta(q) = \frac{1}{|q|^2}$, and the weighted Bergman space on $\Omega$ with weight $\delta$:

$$A_{\delta}(\Omega) := \{f \in \mathcal{SR}(\Omega) \mid \int_{\Omega} |f|^2 \delta d\mu < \infty\},$$

endowed with the inner product

$$\langle f, g \rangle_{A_{\delta}(\Omega)} := \int_{\Omega} \bar{f}g \delta d\mu,$$

and the corresponding norm

$$\|f\|_{A_{\delta}(\Omega)} := \left(\int_{\Omega} |f|^2 \delta d\mu \right)^{1/2}.$$
Proposition 5.2 Let $f \in A(\Omega)$ and $i$ be a fixed element in $\mathbb{S}^2$. Then

$$\frac{1}{4} \int_\Omega |f(x + iy)|^2 d\mu \leq \|f\|^2_{\mathcal{A}(\Omega)} \leq 4 \int_\Omega |f(x + iy)|^2 d\mu.$$  \hfill (13)

Proof. To prove the result, we use the fact that for any $i, j \in \mathbb{S}^2$ one has that $|f(x + iy)|^2 \leq 2||f(x + jy)||^2 + |f(x - jy)|^2$, see (7). Let us take the integral on $\Omega$ of both sides of this inequality and let us consider the right hand side. A change of variable in the second integral below and the axial symmetry of $\Omega$ give:

$$\int_\Omega |f(x + jy)|^2 d\mu + \int_\Omega |f(x - jy)|^2 d\mu = 2 \int_\Omega |f(x + jy)|^2 d\mu. \hfill (14)$$

Let $q = x + jy$ vary in $\Omega$, and keep $i \in \mathbb{S}^2$ fixed. Then (14) and (7) give

$$\frac{1}{4} \int_\Omega |f(x + iy)|^2 d\mu \leq \int_\Omega |f(x + jy)|^2 d\mu = \|f\|^2_{\mathcal{A}(\Omega)}.$$

Exchanging the role of $i$ and $j$ and repeating the reasoning, we obtain

$$\|f\|^2_{\mathcal{A}(\Omega)} = \int_\Omega |f(x + jy)|^2 d\mu \leq 4 \int_\Omega |f(x + iy)|^2 d\mu,$$

and the statement follows. \hfill $\blacksquare$

Proposition 5.3 Let $i$ be a fixed element in $\mathbb{S}^2$.

1. If $f \in A(\Omega)$ then there exist $K_{\Omega} > 0$ such that

$$\int_{\Omega_i} |f_{|\Omega_i}|^2 \delta d\sigma_i \leq K_{\Omega}\|f\|^2_{\mathcal{A}(\Omega)};$$

if $f \in A_\rho(\Omega_i)$ then there exists $M_{\Omega} > 0$ such that

$$\|f\|^2_{\mathcal{A}(\Omega)} \leq M_{\Omega} \int_{\Omega_i} |f_{|\Omega_i}|^2 \rho d\sigma_i.$$

2. If $f \in A(\Omega_i)$ then there exist $K'_{\Omega} > 0$ such that

$$\int_{\Omega} |f|^2 \delta d\mu \leq K'_{\Omega}\|f\|^2_{\mathcal{A}(\Omega_i)};$$

if $f \in A_\delta(\Omega)$ then there exists $M'_{\Omega} > 0$ such that

$$\|f\|^2_{\mathcal{A}(\Omega_i)} \leq M'_{\Omega} \int_{\Omega} |f|^2 \delta d\mu.$$

Proof.
1. Assume that \( f \in \mathcal{A}(\Omega) \). Consider \( i \in \mathbb{S}^2 \) fixed and the integral

\[
\int_{\Omega} |f(q_0 + g|i)|^2 d\mu.
\]

This integral exists, finite, by virtue of our assumption and by (13). Consider the change of coordinates \( q = q_0 + i|q| = r \cos \theta + i r \sin \theta \), where \( i = (\sin \delta \cos \varphi, \sin \delta \sin \varphi, \cos \delta) \), and the values of \( r, \theta, \delta, \varphi \) are such that \( q \in \Omega \). Note that the Jacobian of the change of coordinate is \( r^3 \sin^2 \theta \sin \delta \). We have that

\[
\int_{\Omega} |f(q_0 + g|i)|^2 d\mu = \lambda_\Omega \int_r \int_{\theta} |f|_{i|\Omega}|^2 (r \sin \theta)^2 r dr d\theta = \lambda_\Omega \int_{\Omega} |f|_{i|\Omega}|^2 r d\sigma_1. \tag{15}
\]

The first part of the statement is a direct consequence of (15) and Proposition 5.2. To get the second part, assume that \( f \in \mathcal{A}_{\rho}(\Omega) \). Using backward the equalities in (15) together with Proposition 5.2 the statement follows.

2. Let \( f \in \mathcal{A}(\Omega_i) \) with \( i \in \mathbb{S}^2 \) fixed. Let us compute

\[
\int_{\Omega} |f(q_0 + g|i)|^2 \left( \frac{1}{y} \right)^2 d\mu.
\]

Using the change of coordinate described in point 1 we have:

\[
\int_{\Omega} |f(q_0 + g|i)|^2 \left( \frac{1}{y} \right)^2 d\mu = \lambda_\Omega \int_r \int_{\theta} |f|_{i|\Omega}|^2 \left( \frac{1}{r \sin \theta} \right)^2 (r \sin \theta)^2 r dr d\theta = \lambda_\Omega \int_{\Omega} |f|_{i|\Omega}|^2 r d\sigma_1,
\]

where \( \lambda_\Omega \) is a constant depending of \( \Omega \). Reasoning as in the proof of Proposition 5.2 we deduce

\[
\int_{\Omega} |f|^2 \delta d\mu \leq 4 \int_{\Omega} |f(x + iy)|^2 \left( \frac{1}{y} \right)^2 d\mu,
\]

from which the first part of the statement follows.

On the other hand, consider \( f \in \mathcal{A}_{\delta}(\Omega) \), then

\[
\int_{\Omega} |f(x + iy)|^2 \left( \frac{1}{y} \right)^2 d\mu \leq 4 \int_{\Omega} |f|^2 \delta d\mu.
\]

Using the previous inequalities one has the result.

\[ \blacksquare \]

**Corollary 5.4** The sets of functions \( \mathcal{A}_{\rho}(\Omega_i) \) and \( \mathcal{A}(\Omega) \) contain the same elements and have equivalent norms. Similarly, for the sets \( \mathcal{A}(\Omega_i) \) and \( \mathcal{A}_{\delta}(\Omega) \).

**Remark 5.5** Consider the unit ball \( \mathbb{B}^4 := \{ q \in \mathbb{H} \mid |q| < 1 \} \), the unit disc \( \mathbb{D}_i = \mathbb{B} \cap \mathbb{C}(i) \) and the following slice regular function on \( \mathbb{B}^4 \):

\[
f(q) = \frac{1}{1 - q}, \quad \forall q \in \mathbb{B}^4.
\]
For any $i \in \mathbb{S}^2$ the integral
\[
\int_{D_i} \frac{1}{|1 - z|^2} d\sigma_i
\]
is not finite. However, both the integrals
\[
\int_{D_i} \frac{1}{|1 - z|^2} (y^2) d\sigma_i,
\]
\[
\int_{\mathbb{B}} \frac{1}{|1 - q|^2} d\mu
\]
are finite. Therefore $f \in \mathcal{A}(\mathbb{B}) \setminus \mathcal{A}(D_i)$ and $f \in \mathcal{A}_\rho(D_i)$.

6 The Bergman-Fueter transform on the unit ball

In this section we introduce an integral transform which associates to every slice regular function $f$ defined on the unit ball $\mathbb{B}^4$ of $\mathbb{H}$ a Fueter regular function $\hat{f}$ defined on the same set. We will call the mapping $f \rightarrow \hat{f}$ slice Bergman-Fueter integral transform (for short BF-integral transform). Its definition is based on the Fueter mapping theorem and on the slice regular Bergman kernel of the second kind. It is inspired by the paper [14], where an integral transform that generates Fueter regular functions from slice regular functions is defined. Precisely, in [14] we consider the function
\[
\mathcal{F}(s, q) := \Delta S^{-1}(s, q) = -4(s - \bar{q})(s^2 - 2\text{Re}[q]s + |q|^2)^{-2},
\]
where $\Delta$ is the Laplace operator and $S^{-1}(s, q)$ is the slice regular Cauchy kernel. Let $W \subset \mathbb{H}$ be an axially symmetric open set and let $f$ be a slice regular function on $W$. Let $\Omega$ be a bounded axially symmetric open set such that $\Omega \subset W$. Suppose that the boundary of $\Omega \cap \mathbb{C}(i)$ consists of a finite number of continuously differentiable Jordan curves for any $i \in \mathbb{S}^2$. Then, if $q \in \Omega$, the Cauchy-Fueter regular function $\hat{f}(q)$ given by
\[
\hat{f}(q) := \Delta f(q)
\]
has the integral representation
\[
\hat{f}(q) = \frac{1}{2\pi} \int_{\partial(\Omega \cap \mathbb{C}(i))} \mathcal{F}(s, q) ds_1 f(s), \quad ds_1 = -dsi,
\]
and the integral does not depend neither on $\Omega$ and nor on the imaginary unit $i \in \mathbb{S}^2$.

In the case under study, we restrict our attention to the case of the unit ball $\mathbb{B}^4$, the case in which we know the explicit expression of the Bergman kernel, but such definition can be given for more general domains. We make the following preliminary observations.

- It is well known that slice regular functions on axially symmetric $s$-domains are of class $C^\infty$.
- The Fueter mapping theorem, for $f \in \mathcal{A}(\mathbb{B}^4)$ gives:
\[
\hat{f}(q) = \int_{\partial(\mathbb{B}^4 \cap \mathbb{C}(i))} \Delta \mathcal{K}_{\mathbb{B}^4}(q, r)f(r)d\sigma(r).
\]
**Definition 6.1** The integral

\[ \hat{f}(q) = \int_{\partial(\mathbb{B}^4 \cap C(q))} \Delta K_{B\mathbb{B}^4}(q, r)f(r)d\sigma(r). \]  

(17)

is called the Bergman-Fueter integral transform of \( f \) and its kernel

\[ K_{BF}(q, r) := \Delta K_{B\mathbb{B}^4}(q, r) \]

is called Bergman-Fueter kernel; here \( K_{B\mathbb{B}^4}(q, r) \) is the slice Bergman kernel of the second kind for the unit ball \( \mathbb{B}^4 \).

We will study in another paper the Sobolev regularity of the Bergman-Fueter integral transform.

**Theorem 6.2** The following formula holds:

\[ \Delta K_{B\mathbb{B}^4}(q, r) = -\frac{4}{\pi}|(1 - 2Re|q\bar{r}| + |q|^2r^2)^{-2} + 2(1 - 2\bar{q}\bar{r} + \bar{q}^2r^2)(1 - 2Re|q\bar{r}| + |q|^2r^2)^{-3}|r^2. \]

Proof. To compute the derivatives of the Bergman kernel written in the form (11), we set for simplicity

\[ \tilde{K}(q, r) := \pi K_{B\mathbb{B}^4}(q, r) = (1 - 2\bar{q}\bar{r} + \bar{q}^2r^2)(1 - 2Re|q\bar{r}| + |q|^2r^2)^{-2}. \]

So we have:

\[
\partial_{x_0} \tilde{K}(q, r) = (-2\bar{r} + 2\bar{q}r^2)(1 - 2Re|q\bar{r}| + |q|^2r^2)^{-2}
- 2(1 - 2\bar{q}\bar{r} + \bar{q}^2r^2)(-2\bar{r} + 2x_0r^2)(1 - 2Re|q\bar{r}| + |q|^2r^2)^{-3}
\]

and

\[
\partial_{x_0}^2 \tilde{K}(q, r) = 2\bar{r}^2(1 - 2Re|q\bar{r}| + |q|^2r^2)^{-2}
- 4(-2\bar{r} + 2\bar{q}r^2)(-2\bar{r} + 2x_0r^2)(1 - 2Re|q\bar{r}| + |q|^2r^2)^{-3}
- 4(1 - 2\bar{q}\bar{r} + \bar{q}^2r^2)r^2(1 - 2Re|q\bar{r}| + |q|^2r^2)^{-3}
+ 6(1 - 2\bar{q}\bar{r} + \bar{q}^2r^2)(-2\bar{r} + 2x_0r^2)^2(1 - 2Re|q\bar{r}| + |q|^2r^2)^{-4}.
\]

We now calculate the first and the second derivatives with respect to \( x_1 \). We have:

\[
\partial_{x_1} \tilde{K}(q, r) = (2it - i\bar{q}r^2 - \bar{q}ir^2)(1 - 2Re|q\bar{r}| + |q|^2r^2)^{-2}
- 2(1 - 2\bar{q}\bar{r} + \bar{q}^2r^2)2x_1r^2(1 - 2Re|q\bar{r}| + |q|^2r^2)^{-3}.
\]

The second derivative with respect to \( x_1 \) is

\[
\partial_{x_1}^2 \tilde{K}(q, r) = -2\bar{r}^2(1 - 2Re|q\bar{r}| + |q|^2r^2)^{-2}
- 8(2it - i\bar{q}r^2 - \bar{q}ir^2)x_1r^2(1 - 2Re|q\bar{r}| + |q|^2r^2)^{-3}
- 4(1 - 2\bar{q}\bar{r} + \bar{q}^2r^2)r^2(1 - 2Re|q\bar{r}| + |q|^2r^2)^{-3}
+ 24(1 - 2\bar{q}\bar{r} + \bar{q}^2r^2)x_1^2r^4(1 - 2Re|q\bar{r}| + |q|^2r^2)^{-4}.
\]
The derivatives with respect to $x_2$ and $x_3$ can be computed in a similar way. With some calculations we obtain:

\[
\Delta \tilde{K}(q,r) = -4r^2(1 - 2\text{Re}[q\bar{r}] + |q|^2r^2)^{-2} - 16\left(1 - 2q \bar{r} + q^2 \bar{r}^2\right)r^2\left(1 - 2\text{Re}[q\bar{r}] + |q|^2r^2\right)^{-3} - 16(1 - 2q \bar{r} + q^2 \bar{r}^2)r^2 \left(1 - 2\text{Re}[q\bar{r}] + |q|^2r^2\right)^{-3} + 24(1 - 2q \bar{r} + q^2 \bar{r}^2)\left(1 - 2\text{Re}[q\bar{r}] + |q|^2r^2\right)r^2\left(1 - 2\text{Re}[q\bar{r}] + |q|^2r^2\right)^{-4}.
\]

We finally get:

\[
\Delta \tilde{K}(q,r) = -4r^2(1 - 2\text{Re}[q\bar{r}] + |q|^2r^2)^{-2} - 2\left(1 - 2q \bar{r} + q^2 \bar{r}^2\right)\left(1 - 2\text{Re}[q\bar{r}] + |q|^2r^2\right)r^2\left(1 - 2\text{Re}[q\bar{r}] + |q|^2r^2\right)^{-3} - 3\left(1 - 2q \bar{r} + q^2 \bar{r}^2\right)r^2\left(1 - 2\text{Re}[q\bar{r}] + |q|^2r^2\right)^{-3} - 3\left(1 - 2\text{Re}[q\bar{r}] + |q|^2r^2\right)r^2\left(1 - 2\text{Re}[q\bar{r}] + |q|^2r^2\right)^{-3} - 4.
\]

We rewrite the kernel $\Delta \tilde{K}(q,r)$ in a shorter way defining the function

\[Q(q,r) := (1 - 2\text{Re}[q\bar{r}]) + |q|^2r^2 - 1.\]

**Corollary 6.3** Let $Q(q,r)$ be as above. Then the kernel $\Delta \tilde{K}(q,r)$ is given by

\[\Delta \tilde{K}(q,r) = -\frac{4}{\pi} [Q(q,r) + 2K_{BF}(q,r)]Q(q,r)r^2.\]

**Theorem 6.4** The Bergman-Fueter kernel $K_{BF}(q,r)$ is Fueter regular in $q$ and slice anti-regular in $r$.

Proof. The fact that $K_{BF}(q,r)$ is Fueter regular in $q$ is a direct consequence of the Fueter mapping theorem since by definition it is $K_{BF}(q,r) := \Delta K_{BS}(q,r)$ and $K(q,r)$ is slice regular in the variable $q$. We have to verify that it is slice anti-regular in $r$. In fact observe that the function $1 - 2\text{Re}[q\bar{r}] + |q|^2r^2$ is slice anti-regular in $r$ and it has real coefficients so also $Q(q,r) := (1 - 2\text{Re}[q\bar{r}]) + |q|^2r^2$ is slice anti-regular in $r$ and for the same reason also $Q^2(q,r)$. Since $K_{BS}(q,r)$ is slice anti-regular in $r$ then the product $K(q,r)Q(q,r)$ is slice anti-regular in $r$ because $Q(q,r)$ is a rational function of a polynomial with real coefficients. The statement follows from Corollary 6.3.

**7 Schwarz reflection principle**

In the theory of one complex variable the Schwarz reflection principle is well known. It shows how to extend functions defined on domains in the upper half plane and with real boundary values on the real axis to domains which are symmetric with respect to the real axis. Here we use this property to obtain results for slice regular functions and the Bergman theory.
We begin by recalling some facts in classical complex analysis. Consider a domain $\Omega \subset \mathbb{C}$ such that $\Omega \cap \mathbb{R} \neq \emptyset$ and for any $z \in \Omega$ one has that $\bar{z} \in \Omega$, or equivalently, $\overline{\Omega} = \Omega$. Set

$$C^+ := \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \}, \quad \text{and} \quad C^- := \{ z \in \mathbb{C} \mid \text{Im}(z) < 0 \},$$

and $\Omega^+ := \Omega \cap C^+$, $\Omega^- := \Omega \cap C^-$. Recall that the spaces $\operatorname{Hol}(\Omega)$ and $\operatorname{Hol}_c(\Omega)$ have been defined in Section 2.

**Definition 7.1** Let $\Omega \subset \mathbb{C}$ be such that $\overline{\Omega} = \Omega$. We define:

1. $\widehat{\operatorname{Hol}}(\Omega^+) = \{ f \in \operatorname{Hol}(\Omega^+) \mid \text{for any } x \in \mathbb{R} \text{ there exists } \lim_{\Omega^+ \ni w \to x} f(w) \in \mathbb{R} \};$

2. the function
   $$E[f](z) = \begin{cases} 
   f(z) & \text{if } z \in \Omega^+ \\
   \lim_{\Omega^+ \ni w \to z} f(w) & \text{if } z \in \Omega \cap \mathbb{R} \\
   \overline{f(\bar{z})} & \text{if } z \in \Omega^- 
   \end{cases}$$
   for any $f \in \widehat{\operatorname{Hol}}(\Omega^+)$;

3. $E( \widehat{\operatorname{Hol}}(\Omega^+) ) = \{ E[f] \mid f \in \widehat{\operatorname{Hol}}(\Omega^+) \}$.

**Remark 7.2** With the notations in Definition 7.1 the Schwarz reflection principle asserts that $\operatorname{Hol}_c(\Omega) = E( \widehat{\operatorname{Hol}}(\Omega^+) )$.

Denoting by $i$ the imaginary unit of the complex plane $\mathbb{C}$, we have that $\operatorname{Hol}(\Omega) = \operatorname{Hol}_c(\Omega) + \operatorname{Hol}_c(\Omega)i$, and so we immediately obtain the formula

$$\operatorname{Hol}(\Omega) = E( \widehat{\operatorname{Hol}}(\Omega^+) ) + E( \overline{\widehat{\operatorname{Hol}}(\Omega^+)} )i.$$

The previous formula says that any holomorphic function $f \in \operatorname{Hol}(\Omega)$ can be written in terms of two elements of $f_1, f_2 \in \widehat{\operatorname{Hol}}(\Omega^+)$ as $f = E(f_1) + E(f_2)i$, where

$$f_1(z) = \frac{1}{2} \left( f \mid_{\Omega^+} (z) + \overline{f \mid_{\Omega^+} (\bar{z})} \right), \quad \text{and} \quad f_2(z) = -\frac{i}{2} \left( f \mid_{\Omega^+} (z) - \overline{f \mid_{\Omega^+} (\bar{z})} \right), \quad \forall z \in \Omega^+.$$

Let us now consider the function spaces

$$\tilde{A}(\Omega^+) = \widehat{\operatorname{Hol}}(\Omega^+) \cap L_2(\Omega^+),$$
$$A(\Omega) = \operatorname{Hol}(\Omega) \cap L_2(\Omega),$$
$$A_c(\Omega) = \operatorname{Hol}_c(\Omega) \cap L_2(\Omega).$$

Obviously, $A_c(\Omega) \subset A(\Omega)$. The facts $\operatorname{Hol}(\Omega) = \operatorname{Hol}_c(\Omega) + \operatorname{Hol}_c(\Omega)i$ and $\operatorname{Hol}_c(\Omega) = E( \widehat{\operatorname{Hol}}(\Omega^+) )$ imply that

$$A(\Omega) = E( \tilde{A}(\Omega^+) ) + E( \tilde{A}(\Omega^+) )i.$$

The following results are presented with their proofs for the sake of completeness and for lack of reference.
Proposition 7.3 Let \( \Omega \subseteq \mathbb{C} \) be an open set such that \( \bar{\Omega} = \Omega \). Then:

1. If \( f, g \in A_c(\Omega) \) then
   \[
   <f, g >_{A(\Omega)} = 2 \operatorname{Re} \left( \int_{\Omega^+} \bar{f}g \sigma \right).
   \]

2. Let \( f, g \in A(\Omega) \) and let \( f_1, f_2, g_1, g_2 \in A_c(\Omega) \) be such that \( f = f_1 + f_2i \) and \( g = g_1 + g_2i \) then
   \[
   <f, g >_{A(\Omega)} = 2 \left[ \operatorname{Re} \left( \int_{\Omega^+} \bar{f}_1 g_1 + \bar{f}_2 g_2 \sigma \right) + \operatorname{Re} \left( \int_{\Omega^+} (\bar{f}_1 g_2 - \bar{f}_2 g_1) \sigma \right) \right].
   \]

3. If \( f \in A_c(\Omega) \), then
   \[
   \|f\|_{A(\Omega)} = \sqrt{2} \left( \int_{\Omega^+} |f|^2 \sigma \right)^{\frac{1}{2}},
   \]
   and if \( f \in A(\Omega) \) is written as \( f = f_1 + f_2i \) with \( f_1, f_2 \in A_c(\Omega) \), then
   \[
   \|f\|_{A(\Omega)} = \sqrt{2} \left( \int_{\Omega^+} (|f_1|^2 + |f_2|^2) \sigma \right)^{\frac{1}{2}}.
   \]

Proof. To prove point 1 we compute

\[
<f, g >_{A(\Omega)} = \int_{\Omega} \bar{f}g \sigma = \int_{\Omega^+} \bar{f}g \sigma + \int_{\Omega^-} \bar{f}g \sigma
= \int_{\Omega^+} \bar{f}(\zeta)g(\zeta) \sigma + \int_{\Omega^-} f(\zeta)g(\zeta) \sigma = \int_{\Omega^+} \bar{f}(\zeta)g(\zeta) \sigma + \int_{\Omega^+} f(\zeta)\overline{g(\zeta)} \sigma = 2\operatorname{Re} \left( \int_{\Omega^+} \bar{f}g \sigma \right).
\]

Points 2. and 3. are consequence of 1.

In the next result \( B(\cdot, \cdot) \) denotes the Bergman kernel associated with \( \Omega \). By Remark 7.2 there exist \( B_{\Omega,1}, B_{\Omega,2} : \Omega \times \Omega^+ \to \mathbb{C} \) be such that

\[
B_{\Omega}(\cdot, z) = E(B_{\Omega,1}(\cdot, z)) + E(B_{\Omega,2}(\cdot, z))i.
\]

Proposition 7.4 Let \( \Omega \subseteq \mathbb{C} \) be an open set such that \( \bar{\Omega} = \Omega \), let \( z \in \Omega \setminus \Omega_0 \), and let \( B_{\Omega,1}, B_{\Omega,2} : \Omega \times \Omega^+ \to \mathbb{C} \) be such that

\[
B_{\Omega}(\cdot, z) = E(B_{\Omega,1}(\cdot, z)) + E(B_{\Omega,2}(\cdot, z))i.
\]

1. For any \( f \in A(\Omega) \), let \( f_1, f_2 \in A_c(\Omega) \) be such that \( f = f_1 + f_2i \). Then

\[
f(z) = 2 \left[ \operatorname{Re} \left( \int_{\Omega^+} (B_{\Omega,1}(z, \zeta)f_1(\zeta) + B_{\Omega,2}(z, \zeta)f_2(\zeta)) \sigma \right) + \right. \]
\[
+ \operatorname{Re} \left( \int_{\Omega^+} \left( B_{\Omega,1}(z, \zeta)f_2(\zeta) - B_{\Omega,2}(z, \zeta)f_1(\zeta) \right) \sigma \right) i \right] \]
2. If $u,v$ are the real component functions of $f$, that is $f = u + v\mathbf{i}$, then

$$u(z) = 2\text{Re} \left( \int_{\Omega^+} (\mathbb{B}_{\Omega, 1}(z, \zeta)f_1(\zeta) + \mathbb{B}_{\Omega, 2}(z, \zeta)f_2(\zeta)) d\sigma \right)$$

$$v(z) = 2\text{Re} \left( \int_{\Omega^+} (\mathbb{B}_{\Omega, 1}(z, \zeta)f_2(\zeta) - \mathbb{B}_{\Omega, 2}(z, \zeta)f_1(\zeta)) d\sigma \right).$$

3. If $f \in \mathcal{A}_c(\Omega)$ then

$$f(z) = 2 \left[ \text{Re} \left( \int_{\Omega^+} \mathbb{B}_{\Omega, 1}(z, \zeta)f(\zeta) d\sigma \right) - i \int_{\Omega^+} \mathbb{B}_{\Omega, 2}(z, \zeta)f(\zeta) d\sigma \right].$$

Proof.

1. It is a consequence of Proposition 7.3.

2. Point 2 follows from 1. by direct computations.

3.

$$f(z) = \int_{\Omega^+} \left[ \left(\mathbb{B}_{\Omega, 1}(z, \zeta) - \mathbb{B}_{\Omega, 2}(z, \zeta)\mathbf{i}\right) f(\zeta) + \frac{\mathbb{B}_{\Omega, 1}(z, \zeta) + \mathbf{i}\mathbb{B}_{\Omega, 2}(z, \zeta)}{2\mathbb{B}_{\Omega, 2}(z, \zeta)} \bar{f}(\zeta) \right] d\sigma$$

$$= \int_{\Omega^+} \left[ \left(\mathbb{B}_{\Omega, 1}(z, \zeta) + \mathbb{B}_{\Omega, 2}(z, \zeta)\mathbf{i}\right) f(\zeta) - 2\mathbb{B}_{\Omega, 2}(z, \zeta)\mathbf{i}f(\zeta) + \frac{\mathbb{B}_{\Omega}(z, \zeta)f(\zeta)}{2\mathbb{B}_{\Omega}(z, \zeta)f(\zeta)} \right] d\sigma$$

$$= \int_{\Omega^+} \left[ \mathbb{B}_{\Omega}(z, \zeta)f(\zeta) - 2\mathbb{B}_{\Omega, 2}(z, \zeta)\mathbf{i}f(\zeta) + \frac{\mathbb{B}_{\Omega}(z, \zeta)f(\zeta)}{2\mathbb{B}_{\Omega}(z, \zeta)f(\zeta)} \right] d\sigma$$

We now come to the case of slice regular functions. In the sequel, we will denote by $W$ the operator acting on function defined from $\Lambda$ to $\mathbb{C}$ as follows:

$$W[f](z) = \mathbb{f}(\overline{z}), \quad \forall z \in \Lambda,$$

for any complex valued function $f$ with domain $\Lambda$.

Let now $\Omega \subset \mathbb{H}$ be an axially symmetric s-domain. For any $\mathbf{i} \in \mathbb{S}^2$ denote $\Omega^+_\mathbf{i} := \{ x + \mathbf{i}y \in \Omega \mid y > 0 \}$, $\Omega^-_\mathbf{i} := \{ x + \mathbf{i}y \in \Omega \mid y < 0 \}$, and $\Omega_0 := \Omega \cap \mathbb{R}$.

**Definition 7.5** Let $\Omega$ be an axially symmetric s-domain.

1. We define $\overline{\text{Hol}}(\Omega^+_\mathbf{i}) = \{ f \in \text{Hol}(\Omega^+_\mathbf{i}) \mid \text{for any } x \in \Omega_0 \text{ there exists } \lim_{\Omega^+_\mathbf{i} \ni z \to x} f(z) \in \mathbb{R}\}$.

2. For any $f \in \overline{\text{Hol}}(\Omega^+_\mathbf{i})$ we define the function on $\Omega_1 \subset \mathbb{C}_\mathbf{i}$ by

$$E_\mathbf{i}[f](z) = \begin{cases} 
  f(z) & \text{if } z \in \Omega^+_\mathbf{i} \\
  \lim_{\Omega^+_\mathbf{i} \ni \mathbf{w} \to z} f(w) \in \mathbb{R} & \text{if } z \in \Omega_0 \\
  \mathbb{f}(\overline{z}) & \text{if } z \in \Omega^-_\mathbf{i}
\end{cases}$$
3. Finally, let
\[ E_i(\overline{\text{Hol}}(\Omega_i^+)) = \{ E_i[f] \mid f \in \overline{\text{Hol}}(\Omega_i^+) \}. \]
According to the notation of the previous definition, the Schwarz reflection principle implies that \( \text{Hol}_c(\Omega_i) = E_i(\overline{\text{Hol}}(\Omega_i^+)) \), and as \( \text{Hol}(\Omega_i) = \text{Hol}_c(\Omega_i) + \text{Hol}_c(\Omega_i)i \), then
\[ \text{Hol}(\Omega_i) = E_i(\overline{\text{Hol}}(\Omega_i^+)) + E_i(\overline{\text{Hol}}(\Omega_i^+) )i. \]

Note that \( \text{Hol}_c(\Omega_i) \) and \( \overline{\text{Hol}}(\Omega_i^+) \) are linear spaces over \( \mathbb{R} \).

**Remark 7.6** Let \( i, j \in \mathbb{S}^2 \) be such that \( j \perp i \). Then
\[ \mathcal{S}\mathcal{R}(\Omega) = P_1 \circ E_i[\overline{\text{Hol}}(\Omega_i^+) ] + P_1 \circ E_i[\overline{\text{Hol}}(\Omega_i^+) ]i + P_1 \circ E_i[\overline{\text{Hol}}(\Omega_i^+) ]j + P_1 \circ E_i[\overline{\text{Hol}}(\Omega_i^+) ]ij. \]

Thus any slice regular function can be written in terms of four elements, each of them belonging \( \overline{\text{Hol}}(\Omega_i^+) \). By changing \( \Omega_i^+ \) with \( \Omega_i^- \) we obtain a decomposition of \( \mathcal{S}\mathcal{R}(\Omega) \) in terms of \( \overline{\text{Hol}}(\Omega_i^-) \).

**Proposition 7.7** Let \( f \in \overline{\text{Hol}}(\Omega_i^+) \), then its extension \( P_i[f] \) to the whole \( \Omega \) is given by
\[ \text{Ref}(q_0 + i|q|) + I_q \text{Im}(q_0 + i|q|), \]
for all \( q \in \Omega \).

Proof. Formula (3) immediately gives
\[ P_i[f](q) = P_i[E_i[f]](q_0 + I_q|q|) = \frac{1}{2} \left[ (1 + I_qi)f(q_0 + |q|i) + (1 - I_qi)f(q_0 + |q|i) \right] \]
\[ = \text{Ref}(q_0 + i|q|) + I_q \text{Im}(q_0 + i|q|). \]

**Proposition 7.8** Let \( f \in \mathcal{S}\mathcal{R}(\Omega) \), then there exist conjugated harmonic functions \( u_n, v_n \in C^2(\Omega_i^+, \mathbb{R}) \), \( n = 0, 1, 2, 3 \), with \( \lim_{\Omega^+ \ni w \to z} v_n(w) = 0 \), for each \( z \in \Omega_0 \), such that
\[ f(q) = f(q_0 + I_q|q|) = \sum_{n=0}^{3} u_n(q_0 + i|q|)e_n + I_q \sum_{n=0}^{3} v_n(q_0 + i|q|)e_n, \]
for all \( q \in \Omega \), where \( e_0 = 1, e_1 = i, e_2 = j, e_3 = ij \).

Proof. It is a direct consequence of Remark 7.6 and Proposition 7.7.

From the previous proposition one has that:

1. If \( q = q_0 \in \Omega_0 \) then \( f(q_0) = \sum_{n=0}^{3} u_n(q_0)e_n \).

2. \( f(q) = \sum_{n=0}^{3} u_n(q_0 + i|q|)e_n - I_q \sum_{n=0}^{3} v_n(q_0 + i|q|)e_n. \)
Definition 7.9 Let $i \in S^2$. We set:

1. $\tilde{A}(\Omega_i^+) = \overline{\operatorname{Hol}(\Omega_i^+) \cap \mathcal{L}_2(\Omega_i^+)}$

2. $\mathcal{A}(\Omega_i) = SR(\Omega) \cap \mathcal{L}_2(\Omega_i)$.

The following result shows that the Bergman type space $\mathcal{A}(\Omega_i)$ can be written in terms of the Bergman type spaces on half of a slice $\Omega_i^+$. Moreover, the inner product of two elements can be computed on half of a slice.

Proposition 7.10 The following facts hold:

1. 
\[
\mathcal{A}(\Omega_i) = P_1 \circ E(\tilde{A}(\Omega_i^+)) + P_1 \circ E(\tilde{A}(\Omega_i^+)) i + P_1 \circ E(\tilde{A}(\Omega_i^+)) j + P_1 \circ E(\tilde{A}(\Omega_i^+)) ij. \tag{18}
\]

2. Let $f, g \in \mathcal{A}(\Omega_i)$ and let $f_1 f_2, g_1, g_2 \in \operatorname{Hol}(\Omega_i) \cap \mathcal{L}_2(\Omega_i, \mathbb{C}(i))$ be such that $f = P_1[f_1] + P_1[f_2] i$ and $g = P_1[g_1] + P_1[g_2] j$. Then
\[
< f, g >_{\mathcal{A}(\Omega_i)} = \left( < f_1 |_{\Omega_i^+}, g_1 |_{\Omega_i^+} >_{\mathcal{L}(\Omega_i^+)} + < W[f_1] |_{\Omega_i^+}, W[g_1] |_{\Omega_i^+} >_{\mathcal{L}(\Omega_i^+)} \right) + \left( < f_2 |_{\Omega_i^+}, g_2 |_{\Omega_i^+} >_{\mathcal{L}(\Omega_i^+)} + < W[f_2] |_{\Omega_i^+}, W[g_2] |_{\Omega_i^+} >_{\mathcal{L}(\Omega_i^+)} \right) j + \left( < f_1 |_{\Omega_i^+}, g_2 |_{\Omega_i^+} >_{\mathcal{L}(\Omega_i^+)} - < W[f_1] |_{\Omega_i^+}, W[g_2] |_{\Omega_i^+} >_{\mathcal{L}(\Omega_i^+)} \right)
\]

3. Let $f \in \mathcal{A}(\Omega_i)$. Then
\[
\| f \|_{\mathcal{A}(\Omega_i)}^2 = 2 \left[ \int_{\Omega_i^+} | f |_{\Omega_i^+}^2 d\sigma \right]
\]

Proof.

1. The decomposition (18) is a consequence of Remark 7.6

2. Let us compute the inner product $< f, g >_{\mathcal{A}(\Omega_i)}$. We have:
\[
< f, g >_{\mathcal{A}(\Omega_i)} = \int_{\Omega_i^+} (f_1 + f_2 j)(g_1 + g_2 j) d\sigma = \int_{\Omega_i^+} \overline{f_1 g_1} d\sigma - j \int_{\Omega_i^+} \overline{f_2 g_2} d\sigma j + \int_{\Omega_i^+} \overline{f_1 g_2} d\sigma j - j \int_{\Omega_i^+} \overline{f_2 g_1} d\sigma j = < f_1 g_1 >_{\mathcal{L}(\Omega_i)} - j < f_2 g_2 >_{\mathcal{L}(\Omega_i)} j + < f_1 g_2 >_{\mathcal{L}(\Omega_i)} j - j < f_2 g_1 >_{\mathcal{L}(\Omega_i)} = \left( < f_1, g_1 >_{\mathcal{L}(\Omega_i)} + < f_2, g_2 >_{\mathcal{L}(\Omega_i)} \right) j + \left( < f_1, g_2 >_{\mathcal{L}(\Omega_i)} - < f_2, g_1 >_{\mathcal{L}(\Omega_i)} \right) j
\[
= \left( <f_1|_{\Omega_i^+}, g_1|_{\Omega_i^+}>_{L(\Omega_i^+, C)} + <W[f_1]|_{\Omega_i^+}, W[g_1]|_{\Omega_i^+}>_{L(\Omega_i^+, C)} \right) + \\
\left( <f_2|_{\Omega_i^+}, g_2|_{\Omega_i^+}>_{L(\Omega_i^+, C)} + <W[f_2]|_{\Omega_i^+}, W[g_2]|_{\Omega_i^+}>_{L(\Omega_i^+, C)} \right)
\]

\[
\|f\|_{A(\Omega_i)}^2 = \|f_1\|_{A(\Omega_i)}^2 + \|f_2\|_{A(\Omega_i)}^2
\]

\[
= 2 \left[ \int_{\Omega_i^+} |f_1|_{\Omega_i}^2 d\sigma + \int_{\Omega_i^+} |f_2|_{\Omega_i}^2 d\sigma \right] = 2 \left[ \int_{\Omega_i^+} |f|_{\Omega_i}^2 d\sigma \right]
\]

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