GRADIENT BOUNDS AND MONOTONICITY OF THE ENERGY FOR SOME NONLINEAR SINGULAR DIFFUSION EQUATIONS

AGNID BANERJEE AND NICOLA GAROFALO

Abstract

We construct viscosity solutions to the nonlinear evolution equation (1.4) below which generalizes the motion of level sets by mean curvature (the latter corresponds to the case $p = 1$) using the regularization scheme as in [ES1] and [SZ]. The pointwise properties of such solutions, namely the comparison principles, convergence of solutions as $p \to 1$, large-time behavior and unweighted energy monotonicity are studied. We also prove a notable monotonicity formula for the weighted energy, thus generalizing Struwe’s famous monotonicity formula for the heat equation ($p = 2$).

1. Introduction

In $\mathbb{R}^n \times [0, \infty)$, or in the cylinder $\Omega \times [0, \infty)$, where $\Omega \subset \mathbb{R}^n$ is a bounded open set, we study the following equation

$$
\text{div}(\Phi(|Du|^2)Du) = \Phi(|Du|^2) u_t.
$$

Here, the function $\Phi$ is given by

$$
\Phi(s) = \frac{2}{p^2} s^p, \quad p \geq 1.
$$

When $p \to 1$, the equation (1.1) becomes

$$
u_t = |Du| \text{div}(\frac{Du}{|Du|}),
$$

which is the motion of level sets of $u$ by mean curvature. Most of our discussion will focus on the case $p > 1$ of equation (1.1), i.e., the equation

$$
|Du|^{p-2} u_t = \text{div}(|Du|^{p-2} Du).
$$

It is worth noting that the equation (1.3) can also be viewed as a generalization of the heat equation, which corresponds to the case $p = 2$. The heat equation is also embedded in the parabolic $p$-Laplacian,

$$
u_t = \text{div}(|Du|^{p-2} Du),
$$

which has also been well studied, see [D] and the references therein. Such equation, however, is quite different from (1.3) which, contrarily to the parabolic $p$-Laplacian, does not possess a divergence structure. The limiting case $p \to \infty$ of (1.3) is also extremely interesting in connection with the analysis of tug-of-war games with noise in which the number of rounds is bounded. The value functions for these games approximate a solution to the pde (1.3) above when the parameter that controls the size of the possible steps goes to zero. For this, see the interesting paper [MPR].

The equation (1.3) has been considered by several authors, see for instance [ES1]-[ES4], [SZ], [BG], [CGG], [CW], [GGIS], [ISZ]. In [ES1]-[ES4] the case of $\mathbb{R}^n \times [0, \infty)$ is treated, whereas in [SZ] the case of $\Omega \times [0, \infty)$ is studied, with $\Omega$ being a smooth domain with mean curvature.
bounded from below by a positive constant at each point on the boundary. The existence of viscosity solutions is proved using approximating evolution equations. In \[\text{CGG}\], equations of the form

\[u_t + F(Du, D^2u) = 0,\]

have been considered. Unlike what was done in \[\text{ES1}\] or \[\text{SZ}\], in the paper \[\text{CGG}\] the authors proved the existence of solutions using Perron’s method. Most of the discussion in \[\text{CGG}\], including solvability of Cauchy problem, pertains what the authors call geometric \(F\). For the \(F\) corresponding to the equation \[1.4\], being geometric in the sense of \[\text{CGG}\] is true for the case \(p = 1\), but not for \(p > 1\).

It should be noted that Proposition 2.2 in \[\text{BG}\] establishes that a solution for the generalized mean curvature flow in the sense of \[\text{ES1}\] is equivalent to being a solution in the sense of \[\text{CGG}\].

Finally, a Harnack type approach to the evolution of surfaces can be found in \[\text{CW}\].

The present paper is organized as follows. In Section 2 we introduce the relevant definitions of solution in the viscosity sense. In Section 3 we collect several comparison principles for viscosity solutions which generalize to the equation \[1.4\] those established in \[\text{ES1}\] and \[\text{CGG}\] when \(p = 1\). One notable aspect of the equation \[1.4\] is that it is invariant under the standard parabolic dilations \((x, t) \rightarrow (\lambda x, \lambda^2 t)\). Exploiting this invariance in Section 4 we have found a notable explicit solution \(G_p\) of \[1.4\], see Proposition 4.1. By means of a variant of this solution, we have been able to establish a comparison principle for solutions of \[1.4\] which resembles the classical result of Tychonoff for the heat equation, see Theorem 4.2.

In Section 5, following \[\text{ES1}\] and \[\text{SZ}\], we show the existence of solutions \(u\) to the Cauchy and Cauchy-Dirichlet problems as limits of solutions \(u^\varepsilon\) of the regularized problems \[5.3\]. We have preferred this regularization scheme since, on the one hand, it enables us to answer in the affirmative that, unlike what happens for the generalized mean curvature flow equation \[1.3\], in the case \(p > 1\) viscosity solutions of \[1.4\] do not have finite extinction time, or finite propagation speed. On the other hand, it facilitates in the subsequent sections our study of various pointwise properties, large time behavior, monotonicity results, etc. Continuing our discussion of the plan of the paper, in Section 6 we show that as \(p \rightarrow 1\), the corresponding solutions to \[1.4\] converge locally uniformly to the unique solution of generalized mean curvature flow \[1.3\]. In Section 7 we study the large-time behavior of the solutions for the Cauchy-Dirichlet problem. We first note that the \(p\)-energy is non-increasing as a function of time, thus generalizing a result in \[\text{SZ}\] for the case \(p = 1\). We then identify the double limit \(\lim_{\varepsilon \rightarrow 0} \lim_{t \rightarrow \infty} u^\varepsilon(x, t)\) as a solution of the \(p\)-Laplace equation subject to prescribed boundary conditions. In the case \(p = 1\), such limit solution corresponds to the function of least gradient as in \[\text{SZ}\]. Moreover, for \(1 < p \leq 2\), we show that

\[\lim_{\varepsilon \rightarrow 0} \lim_{t \rightarrow \infty} u^\varepsilon(x, t) = \lim_{t \rightarrow \infty} \lim_{\varepsilon \rightarrow 0} u^\varepsilon(x, t).\]

For the case \(p = 1\), it was shown in \[\text{SZ}\] that the limits in \(t\) and \(\varepsilon\) do not commute, in general.

In Section 8 we first establish the monotonicity of the energy of the unique bounded viscosity solutions to the Cauchy problem constructed in the existence theorems of Section 5, see Theorem 8.1 below. It is interesting to note that the proof of such result relies crucially on the decay of solutions which is obtained by comparison with the explicit solution \(G_p\) constructed in Proposition 4.1 in Section 4. This implies, in particular, energy estimates in terms of initial datum. In the second part of Section 8 we prove that, quite notably, viscosity solutions of the nonlinear singular equation \[1.4\] satisfy a monotonicity theorem similar to Struwe’s result monotonicity theorem for the heat equation in \[\text{S}\], see Theorem 8.2 below.

In closing, we mention two works in which the non-geometric case \((p > 1)\) of the equation \[1.1\] has been studied. The existence of solution to the Cauchy problem corresponding to \[1.1\] can also be obtained by an adaptation of Perron’s method, as it was done in the interesting work \[\text{OS}\], where a whole class of non-geometric equations was studied. Finally, the equation \[1.4\] with \(p > 1\) has also been studied in the interesting recent paper \[\text{MPR}\], where a solution
to the Cauchy-Dirichlet problem is obtained by using probabilistic methods as the limit of the value functions of tug-of-war games.

2. Preliminaries

We can formally rewrite (1.1) as a non-divergence form equation as follows
\begin{equation}
\Phi'(|Du|^2)\Delta u + 2\Phi''(|Du|^2)u_{ij}u_{ij} = \Phi'(|Du|^2)u_t,
\end{equation}
where we have let \( u_t = \frac{\partial u}{\partial t} \), \( u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j} \), etc. After substituting \( \Phi \) as in (1.2), we obtain
\begin{equation}
|Du|^{p-2}u_t = |Du|^{p-4}(\|Du\|^2\delta_{ij} + (p-2)u_{ij}u_{ij}).
\end{equation}
We now formally proceed to cancel off the powers of \( |Du| \) from both sides of (2.2), to find
\begin{equation}
u_t = \left(\delta_{ij} + (p-2)\frac{u_{ij}u_{ij}}{|Du|^2}\right)u_{ij}.
\end{equation}
Motivated by the above formal calculations, following Definitions 2.1-2.3 in [ES1], we now introduce the relevant notion of solutions to (1.1). Hereafter, whenever convenient we will write \( z = (x,t), z_0 = (x_0, t_0) \), etc., for points in \( \mathbb{R}^{n+1} \). Throughout this paper, \( \Omega \) indicates an open set in \( \mathbb{R}^n \) which of course be the whole of \( \mathbb{R}^n \), whereas \( T \) indicates an extended number satisfying \( 0 < T \leq \infty \).

**Definition 2.1.** A function \( u \in C(\Omega \times [0,T)) \cap L^\infty(\Omega \times [0,T)) \) is called a viscosity subsolution of (1.1), with \( \Phi \) as in (1.2), provided that if
\begin{equation}
u - \phi \text{ has a local maximum at } \ z_0 \in \Omega \times (0,T)
\end{equation}
for each \( \phi \in C^2(\Omega \times (0,T)) \), then
\begin{equation}
\left\{ \begin{array}{ll}
\phi_t \leq \delta_{ij} + (p-2)\frac{\phi_i\phi_j}{|Du|^2} & \text{at } z_0, \\
\text{if } D\phi(z_0) \neq 0,
\end{array} \right.
\end{equation}
and
\begin{equation}
\phi_t \leq (\delta_{ij} + (p-2)a_i a_j)\phi_{ij} \text{ at } z_0, \text{ for some } a \in \mathbb{R}^n \text{ with } |a| \leq 1,
\end{equation}
if \( D\phi(z_0) = 0 \).

**Definition 2.2.** A function \( u \in C(\Omega \times [0,T)) \cap L^\infty(\Omega \times [0,T)) \) is called a viscosity supersolution of (1.1), with \( \Phi \) as in (1.2), provided that if
\begin{equation}
u - \phi \text{ has a local minimum at } \ z_0
\end{equation}
for each \( \phi \in C^2(\Omega \times (0,T)) \), then
\begin{equation}
\left\{ \begin{array}{ll}
\phi_t \geq \delta_{ij} + (p-2)\frac{\phi_i\phi_j}{|Du|^2} & \text{at } z_0, \\
\text{if } D\phi(z_0) \neq 0,
\end{array} \right.
\end{equation}
and
\begin{equation}
\phi_t \geq (\delta_{ij} + (p-2)a_i a_j)\phi_{ij} \text{ at } z_0, \text{ for some } a \in \mathbb{R}^n \text{ with } |a| \leq 1,
\end{equation}
if \( D\phi(z_0) = 0 \).

**Definition 2.3.** A function \( u \in C(\Omega \times [0,T)) \cap L^\infty(\Omega \times [0,T)) \) is called a viscosity solution of (1.1) provided it is both a viscosity subsolution and supersolution.

**Remark 2.4.** When \( T < \infty \), a viscosity sub- or supersolution of (1.1) in \( \Omega \times [0,T] \) is to be understood as one in \( \Omega \times [0,T+\varepsilon) \) for some \( \varepsilon > 0 \).
Remark 2.5. In a standard fashion one can verify that, if $u$ is a viscosity solution of (1.1), then such is also $ku+c$, for any $k,c \in \mathbb{R}$. This simple, yet important property, will be repeatedly used in the present paper.

Following [ES1], it will be convenient to have the following equivalent definitions.

Definition 2.6 (Equivalent definition). A function $u \in C(\Omega \times [0,T) \cap L^\infty(\Omega \times [0,T))$ is called a viscosity subsolution of (1.1) if whenever $z_0 \in \Omega \times (0,T)$, and for some $q \in \mathbb{R}, \sigma \in \mathbb{R}^n$ and symmetric $(n+1) \times (n+1)$ matrix $R$, we have as $z \to z_0$,

$$u(z) \leq u(z_0) + q(t-t_0) + \left\langle \sigma, x-x_0 \right\rangle + \frac{1}{2} < R(z-z_0), z-z_0 > + o(|z-z_0|^2),$$

then

$$\begin{aligned}
q & \leq (\delta_{ij} + (p-2)a_i a_j) R_{ij}, \quad \text{if } \sigma \neq 0, \\
q & \leq (\delta_{ij} + (p-2)a_i a_j) R_{ij} \quad \text{for some } a \in \mathbb{R}^n, \text{ with } |a| \leq 1, \quad \text{if } \sigma = 0.
\end{aligned}$$

A viscosity supersolution is defined similarly. Finally, $u$ is a viscosity solution if it is at one time a viscosity sub- and supersolution.

Definitions 2.1, 2.2, 2.3 are each easily seen to be equivalent to the corresponding case in Definition 2.6. For this aspect we refer the reader to the seminal papers [I] and [J]. From Definition 2.0, one sees that a smooth enough viscosity solution is also a classical solution on the set where its spatial gradient does not vanish. Moreover, by adapting the argument which in Proposition 2.2 in [BG] is given in the case $p = 1$, we can conclude that, for an equation such as (1.1), the notion of solution in the sense of Definition 2.3 is equivalent to being a solution in the sense of Definition 2.1 on page 753 in [CGGI]. This is the content of the following proposition. But first, we recall the relevant definition from [CGGI], adapted to the equation (1.1).

Definition 2.7. A function $u \in C(\Omega \times [0,T)) \cap L^\infty(\Omega \times [0,T))$, with $\Phi$ as in (1.2), is called a viscosity subsolution of (1.1) in the sense of [CGGI], provided that if

$$(2.10) \quad u - \phi \quad \text{has a local maximum at } \quad z_0 \in \Omega \times (0,T)$$

for every $\phi \in C^2(\Omega \times (0,T))$, then either

$$(2.11) \quad \begin{aligned}
\phi_t & \leq \left( \delta_{ij} + (p-2) \frac{\partial \phi}{\partial x^i} \right) \phi_{ij} \quad \text{at } \quad z_0, \\
\text{if } D\phi(z_0) & \neq 0,
\end{aligned}$$

or

$$(2.12) \quad \begin{aligned}
\inf_{|a| = 1} \{ \phi_t - (\delta_{ij} + (p-2) a_i a_j) \phi_{ij} \} & \leq 0 \quad \text{at } \quad z_0, \\
\text{if } D\phi(z_0) & = 0.
\end{aligned}$$

Analogous definitions for supersolution, or solution, in the sense of [CGGI].

Proposition 2.8. A bounded continuous function is a solution in the sense of Definition 2.5 if and only if it is a solution in the sense of Definition 2.7.

Proof. We only look at the case of subsolution since the other case is dealt similarly. The proof that Definition 2.7 $\implies$ Definition 2.1 is trivial, and we leave it to the reader. We thus focus on the implication Definition 2.1 $\implies$ Definition 2.7. Suppose that for every $\phi \in C^2(\Omega \times (0,T))$, $u - \phi$ has a local maximum at $z_0 \in \Omega \times (0,T)$. In the case when $D\phi(z_0) \neq 0$, the corresponding conditions in Definition 2.1 and Definition 2.7 are seen to be the same. So we look at the case when $D\phi(z_0) = 0$. Without loss of generality, by replacing $\phi$ with $\phi(x,t) + |x-x_0|^4 + (t-t_0)^4$, which does not affect the spatial and time derivatives at $z_0$, we can assume a strict local maximum at $z_0$ (say, in $C_{r_0}(z_0) = B_{r_0} x_0 \times [t_0 - r_0^2, t_0 + r_0^2]$). For $\varepsilon > 0$ we define

$$\xi_\varepsilon(x,t) = u(x,t) - \frac{|x-y|^4}{\varepsilon} - \phi(y,t).$$
Let \((x_\varepsilon, y_\varepsilon, t_\varepsilon)\) be the maximum of \(\xi_\varepsilon\) in the set \(\tilde{C}_{\varepsilon_0}(z_0) = \mathcal{B}_{\varepsilon_0}(x_0) \times \mathcal{B}_{\varepsilon_0}(y_0) \times [t_0 - r_0^2, t_0 + r_0^2]\).

We claim that, because of the strict maximum assumption of \(u - \phi\) at \((x_0, t_0)\), the sequence \((x_\varepsilon, y_\varepsilon, t_\varepsilon)\) must converge to \((x_0, x_0, t_0)\) as \(\varepsilon \to 0\), hence for all small enough \(\varepsilon\), the maximum of \(\xi_\varepsilon\) is attained at an interior point of \(C_{\varepsilon_0}(z_0)\). To see this, suppose on the contrary \((x_\varepsilon, y_\varepsilon, t_\varepsilon)\) stays away from \((x_0, x_0, t_0)\) and

\[
\xi_\varepsilon(x_\varepsilon, y_\varepsilon, t_\varepsilon) = \xi_\varepsilon(x_0, x_0, t_0) = u(x_0, t_0) - \phi(x_0, t_0).
\]

Now since \(u, \phi\) are bounded, because of the term \(-\frac{|x-y|^4}{\varepsilon}\) in \(\xi_\varepsilon\) and the extremum condition at \((x_\varepsilon, y_\varepsilon, t_\varepsilon)\), we conclude that, after possibly passing to a subsequence, we have \((x_\varepsilon, y_\varepsilon, t_\varepsilon)\to (x_1, x_1, t_1)\neq (x_0, x_0, t_0)\), because of our assumption. So, from (2.13), we obtain

\[
u(x_1, t_1) - \phi(x_1, t_1) \geq u(x_0, t_0) - \phi(x_0, t_0),
\]

which contradicts the strict maximum condition at \((x_0, t_0)\).

Now, if we consider the function \((y, t) \to \xi_\varepsilon(x_\varepsilon, y, t_\varepsilon)\), we easily see that at the point \((y_\varepsilon, t_\varepsilon)\) we have

\[
\phi(y_\varepsilon, t_\varepsilon) = 4\frac{|x-y|^2(x-y)}{\varepsilon},
\]

\[
\phi^2(y_\varepsilon, t_\varepsilon) \geq -4\frac{|x-y|^2}{\varepsilon} - 8\frac{(x-y)^2}{\varepsilon}.
\]

Two cases occur:

1) \(\phi(y_\varepsilon, t_\varepsilon) = 0\) for all \(\varepsilon\) small enough.

2) \(\phi(y_\varepsilon, t_\varepsilon) \neq 0\) for a subsequence \(\varepsilon \to 0\).

In case 1), from the first equation in (2.13) we have \(y_\varepsilon = x_\varepsilon\). We thus fix \(y = y_\varepsilon\), and arguing in the \(x\) variable with the test function \(\frac{|x-y|^4}{\varepsilon} + \phi(y, t)\), we obtain from Definition (2.11) \(\phi(y_\varepsilon, t_\varepsilon) \leq 0\) (extrema condition at \((x_\varepsilon = y_\varepsilon, t_\varepsilon)\)) and so in the limit \(\phi(x_0, t_0) \leq 0\) as \((y_\varepsilon, t_\varepsilon) \to (x_0, t_0)\). Also from (2.15), we obtain \(D^2\phi(y_\varepsilon, t_\varepsilon) \geq 0\) as \(y_\varepsilon = x_\varepsilon\), and so likewise \(D^2\phi(x_0, t_0) \geq 0\).

Now from this information it is easily seen that for all \(p > 1\) we obtain at \((x_0, t_0)\)

\[
\Delta \phi \geq (2 - p)\phi_{11} = (2 - p)\phi_{ij} \alpha_i \alpha_j \quad \text{with} \quad a = e_1.
\]

To see that (2.10) holds for \(1 \leq p \leq 2\) we have \((2 - p)\phi_{11} \leq \phi_{11} \leq \Delta \phi\). For \(p > 2\) the right-hand side in (2.10) is nonpositive because of nonnegativity of \(D^2 \phi\). So for all \(p > 1\) and \(a = e_1\), we obtain at \((x_0, t_0)\), \(\phi_t + (\Delta \phi + (p - 2)\phi_{ij} \alpha_i \alpha_j) \leq 0\), which implies Definition (2.7).

In case 2), then

\[
(x, t) \to v_\varepsilon(x, t) = u(x, t) - |x - y_\varepsilon|^4/\varepsilon - \phi(x - (x - y_\varepsilon), t)
\]

has local maximum at \((x_\varepsilon, t_\varepsilon)\). To see this, given any point \((x + a, t + b)\) in the neighborhood of \((x_\varepsilon, t_\varepsilon)\) which lies in \(C_{\varepsilon_0}(z_0)\), we have

\[
v_\varepsilon(x + a, t + b) = u(x + a, t + b) - \frac{|(x + a) - (y + a)|^4}{\varepsilon} - \phi(y + a, t + b)
= \xi_\varepsilon(x + a, y + a, t + b).
\]

(adding and subtracting \(a\) in the term \(\frac{|x-y|^4}{\varepsilon}\)) So

\[
v_\varepsilon(x + a, t + b) \leq \xi_\varepsilon(x + a, t + b) = u(x_\varepsilon, t_\varepsilon) - \frac{|x_\varepsilon - y|^4}{\varepsilon} - \phi(y_\varepsilon, t_\varepsilon)
= v_\varepsilon(x_\varepsilon, t_\varepsilon),
\]
which justifies the local maximum of \( u_\varepsilon \) at \((x_\varepsilon, t_\varepsilon)\). (since maximum of \( \xi_\varepsilon \) is attained at \((x_\varepsilon, y_\varepsilon, t_\varepsilon)\) in \( \bar{C}_r(0,z_0) \)) Thus, by applying Definition 2.1 to the test function \(|x_\varepsilon - y_\varepsilon|^4/\varepsilon + \phi(x - (x_\varepsilon - y_\varepsilon), t)\), by treating it as a function of the variable \((x, t)\), we obtain
\[
\phi_t - (\delta_{ij} + (p - 2)\frac{\phi_i\phi_j}{|Du|^2})\phi_{ij} \leq 0,
\]
at \((x_\varepsilon - (x_\varepsilon - y_\varepsilon), t_\varepsilon) = (y_\varepsilon, t_\varepsilon)\). Let \( a_\varepsilon^i = \phi_i/|Du| \) at \((y_\varepsilon, t_\varepsilon)\). After passing to a subsequence, we may assume that \( a_\varepsilon \to a \) with \(|a| = 1\). Therefore, by letting \( \varepsilon \to 0 \), we obtain
\[
\phi_t - (\Delta \phi + (p - 2)\phi_{ij}a_ia_j) \leq 0
\]
at \((x_0, t_0)\), which again implies Definition 2.1.

The fact that equation (1.1) is truly an evolution equation associated with the \( p \)-Laplacian is seen from the following lemma.

**Lemma 2.9.** For \( p > 1 \) a time-independent continuous function \( u(x) \) is a \( p \)-harmonic function if and only \( U(x, t) = u(x) \) is a viscosity solution of (1.1).

**Proof.** Suppose \( u \) be a \( p \)-harmonic function for some \( p > 1 \), i.e., a \( W^{1,p}_\text{loc} \) weak solution of the \( p \)-Laplacian
\[
\text{div}(|Du|^{p-2}Du) = 0.
\]
We want to show that \( U(x, t) = u(x) \) is a viscosity solution of (1.1). Suppose that \( U - \phi \) have a strict local maximum at \((x_0, t_0)\). Now, in a compact neighborhood \( \omega \) of \((x_0, t_0)\) in space-time, there are smooth functions \( u_\varepsilon \) such that \( u_\varepsilon \to u \) uniformly, along with their first derivatives, in that neighborhood. Furthermore, in \( \omega \) the functions \( u_\varepsilon \) solve the equation
\[
(\delta_{ij} + (p - 2)\frac{u_\varepsilon^i u_\varepsilon^j}{|Du_\varepsilon|^2 + \varepsilon^2})u_\varepsilon^{ij} = 0,
\]
see for instance [L]. Let \( z_\varepsilon = (x_\varepsilon, t_\varepsilon) \in K \) be a point at which \( u_\varepsilon - \phi \) has its absolute maximum. We claim that \( z_\varepsilon \to z_0 \) and hence for small enough \( \varepsilon \), \( z_\varepsilon \) is an interior point of \( K \). This would imply
\[
Du_\varepsilon(z_\varepsilon) = Du_\varepsilon(x_\varepsilon) = D\phi(z_\varepsilon), \quad \phi_t(z_\varepsilon) = u_\varepsilon^t(z_\varepsilon) = 0, \quad D^2u_\varepsilon(z_\varepsilon) = D^2u_\varepsilon(x_\varepsilon) \leq D^2\phi(z_\varepsilon).
\]
This implies at \( z_\varepsilon \)
\[
(\delta_{ij} + (p - 2)\frac{\phi_i\phi_j}{|D\phi|^2 + \varepsilon^2})\phi_{ij} \geq 0 = \phi_t.
\]
Suppose on the contrary , there exists a subsequence of \( \{z_\varepsilon\}_{\varepsilon > 0} \), which we continue to denote by \( z_\varepsilon \), which converges by compactness to a point \( z_1 \neq z_0 \in K \). In fact, since \( u_\varepsilon - \phi \) attains its absolute maximum at \( z_\varepsilon \in K \), we have
\[
u_\varepsilon(z_\varepsilon) - \phi(z_\varepsilon) \geq u_\varepsilon(z_0) - \phi(z_0).
\]
Because of uniform convergence, passing to the limit in the latter inequality, we would have
\[
u(z_1) - \phi(z_1) \geq u(z_0) - \phi(z_0).
\]
This would contradict the assumption that \( u - \phi \) has a strict maximum at \( z_0 \). In conclusion, \( z_\varepsilon \to z_0 \) as \( \varepsilon \to 0 \), and consequently, if \( D\phi(z_0) \neq 0 \), we obtain at \( z_0 \), after passing to the limit in the inequality (2.17),
\[
\phi_t \leq (\delta_{ij} + (p - 2)\frac{\phi_i\phi_j}{|D\phi|^2})\phi_{ij}.
\]
If instead \( D\phi(z_0) = 0 \), then consider the vectors \( a_\varepsilon = \frac{D\phi(z_0)}{\sqrt{(\|D\phi(z_0)\|^2 + \varepsilon^2)^{1/2}}} \). Since \( |a_\varepsilon| \leq 1 \) by compactness there exists a subsequence, which we continue to indicate \( a_\varepsilon \), such that \( a_\varepsilon \to a \), with \( |a| \leq 1 \). Letting \( \varepsilon \to 0 \) in (2.17) we obtain at \( z_0 \)
\[
\phi_t \leq (\delta_{ij} + (p-2)a_ia_j)\phi_{x_j}.
\]
This shows that \( u \) is a subsolution. A similar argument proves that \( u \) is a supersolution.

Conversely, let \( u \) be a viscosity solution of (1.1). The fact that \( u \) is a \( p \)-harmonic function is a consequence of the equivalence of definition of viscosity and weak solution for the \( p \)-Laplacian established in [JLM].

\[ \square \]

3. Comparison principles

In this section we collect some comparison principles for viscosity solutions of (1.1), both on a bounded open set \( \Omega \subset \mathbb{R}^n \), and in the whole space. Assume that \( u : \mathbb{R}^n \times (0, \infty) \) is a continuous and bounded function. Let us denote by \( u^\varepsilon \) and \( u_\varepsilon \) respectively the sup and inf convolution of \( u \), see the definitions (3.1), (3.2) and Lemma 3.1 in [ES1]. It is easily seen that the conclusions of Lemma 3.1 in [ES1] continue to hold in our setting. By this we mean that, in addition to the analytic properties (i)-(vi) claimed in that lemma, if \( u \) is a viscosity (sub-) supersolution of (1.1) in \( \mathbb{R}^n \times (0, \infty) \), then \( (u^\varepsilon)_{\varepsilon} \) is a (sub-) supersolution of (1.1) in \( \mathbb{R}^n \times [\sigma(\varepsilon), \infty) \). Using this fact, and the above stated equivalent Definition 2.6 of solution, the proof of the following Theorem 3.3 follows. It is a slight modification of the argument in [ES1], and it has been described in [SZ] which remains unchanged for other \( p \)'s. This in particular allow us to assert uniqueness of solutions to Cauchy-Dirichlet problem. However, the result will also be used now and then at other places.

**Theorem 3.1.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain. Assume \( u \) and \( v \) are two solutions of (1.1) in the cylinder \( \Omega_\infty = \Omega \times [0, \infty) \), with possibly different initial and boundary data. Then, the maximum of \( |u - v| \) is achieved on the parabolic boundary \( \partial \Omega_\infty = \partial \Omega \times (0, \infty) \cup \Omega \times \{0\} \).

Furthermore, since our notion of sub- and supersolution is the same as that of [CGG], see Proposition 2.8 above, we also have the usual parabolic comparison principle (see [CGG], Theorem 4.1 or [GGIS]).

**Theorem 3.2.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain, and let \( u \) and \( v \) respectively be sub- and supersolution of (1.1) in \( \Omega \times [0, T] \). If \( u \leq v \) on the parabolic boundary, then \( u \leq v \) on \( \Omega \times [0, T] \).

Using again Proposition 2.8 we obtain a comparison theorem when \( \Omega = \mathbb{R}^n \), see Theorem 2.1 in [GGIS]. This implies uniqueness in the Cauchy problem (the reader should keep in mind that, for the notions adopted in this paper, all solutions are a priori assumed to be bounded).

**Theorem 3.3.** Let \( u \) and \( v \) respectively be sub- and supersolution of (1.1) in \( \mathbb{R}^n \times [0, \infty) \) such that \( u(x, 0) \leq v(x, 0) \). Let \( u(\cdot, 0) \) and \( v(\cdot, 0) \) be uniformly continuous. Then, \( u \leq v \) in \( \mathbb{R}^n \times [0, \infty) \).

We note explicitly that the assumption that the initial datum be uniformly continuous in Theorems 3.3 and 3.4 is needed to guarantee that the condition (A2) in Theorem 2.1 of [GGIS] be satisfied. After a minor modification in the proof of Theorem 3.3 we can assert the following.

**Theorem 3.4.** Let \( u \) and \( v \) be two solutions of (1.1) in \( \mathbb{R}^n \times [0, \infty) \). Let \( u(\cdot, 0) \) and \( v(\cdot, 0) \) be uniformly continuous. Then,
\[
\sup_{\mathbb{R}^n \times [0, \infty)} |u - v| = \sup_{\mathbb{R}^n} |u(\cdot, 0) - v(\cdot, 0)|.
\]
Similarly to Proposition 2.4 in [GGIS], we can find sufficiently small $T > 0$ such that
\[
\sup_{\mathbb{R}^n \times [0,T]} (u - v) > \sup_{\mathbb{R}^n} [u(\cdot, 0) - v(\cdot, 0)]^+ > 0.
\]
Otherwise, if $\sup_{\mathbb{R}^n} [u(\cdot, 0) - v(\cdot, 0)]^+ = 0$, and Theorem 3.3 would imply $\sup_{\mathbb{R}^n \times [0,T]} (u - v) \leq 0$.

Following [GGIS], we set $\omega(x, y, t) = u(x, t) - v(y, t)$, $B(x, y, t) = \delta(|x|^2 + |y|^2) + \frac{\gamma - t}{T-t}$, ($B$ plays the role of barrier), and $\xi(x, y, t) = \frac{|x-y|^4}{\varepsilon} + B(x, y, t)$. We then consider
\[
\Phi(x, y, t) = \omega(x, y, t) - \xi(x, y, t).
\]

By the contradiction assumption we have
\[
\alpha = \lim_{\theta \to 0} \sup_{\mathbb{R}^n} \{\omega(x, y, t) \mid |x - y| < \theta, 0 \leq t \leq T\} \geq \sup_{0 \leq t \leq T} \omega(x, x, t)
= \beta > \sup_{\mathbb{R}^n} [u(\cdot, 0) - v(\cdot, 0)]^+ = \beta_1.
\]

Similarly to Proposition 2.4 in [GGIS], we can find sufficiently small $\delta_0, \gamma_0$ such that
\[
\sup_{\mathbb{R}^n} \Phi(x, y, t) > \beta_1,
\]
for all $\delta < \delta_0, \gamma < \gamma_0$, and because of the barrier function $B$, we obtain as in Proposition 2.5 in [GGIS] that the sup is attained at some $(x_1, y_1, t_1)$ with $t_1 < T$. Now, by making use of the condition that the initial datum is uniformly continuous, which guarantees the validity of (A2) in Theorem 2.1 in [GGIS], as in Proposition 2.6 in [GGIS] we can find $\varepsilon_0$ small enough such that for all $\varepsilon < \varepsilon_0$, $\sup \Phi$ is attained at $(x_1, y_1, t_1)$ with $t_1 > 0$ (i.e., at an interior point). The rest of the proof is now identical to that in [GGIS].

In closing, we mention the following additional properties of solutions which can be established as in [ES1]:

1. Assume $u_k$ is a viscosity solution of (1.1) for $k = 1, 2, \ldots$, and that $u_k \to u$ locally uniformly on $\mathbb{R}^n \times [0, \infty)$. Then, $u$ is a viscosity solution. An analogous assertion holds for subsolutions and supersolutions.

2. (Convexity preserving property, see Theorem 3.1 in [GGIS]) If $u$ is a (bounded) solution to the Cauchy problem and if the initial datum $g$ is concave/convex and globally Lipschitz, then $u(\cdot, t)$ is concave/convex for all $t$. This property continues to be true if instead of boundedness, linear growth in $x$ is allowed.

3. If $p = 1$ and $\Phi$ is any smooth function, then $\Phi(u)$ is a viscosity solution if such is $u$, see [ES1]. This property is no longer true in general when $p > 1$. For instance, it is violated by the standard heat equation ($p = 2$).

4. **AN EXPLICIT SOLUTION FOR THE CASE $p > 1$**

In this section we exploit the scaling properties of (1.1) to construct an interesting explicit global viscosity solution of (1.1), which we later use to establish a Tichonoff type maximum principle for the equation (1.1), see Theorem 4.2 below. In fact, such explicit solution will also be used in a crucial way in the proof of the monotonicity results in Theorem 5.1 and Theorem 8.2 below. Here is the relevant result.

**Proposition 4.1.** For any $p > 1$ the function
\[
G_p(x, t) = t^{-\frac{n+p-2}{2(p-1)}} \exp \left( -\frac{|x|^2}{4(p-1)t} \right),
\]

where $G_p(x, t)$ is a global viscosity solution of (1.1) for some $\varepsilon > 0$ such that $G_p(x, t) < \Phi(\cdot, t)$. This function is defined for $p > 1$ and satisfies the equation (1.1) in the viscosity sense. Further, $G_p(x, t)$ is a viscosity solution of (1.1) for $p > 1$.

The proof of this proposition involves the construction of a suitable barrier function and the verification of the maximum principle for the given function $G_p(x, t)$. The details of the proof are omitted for brevity, but it involves the use of the scaling properties of (1.1) and the construction of a barrier function that is adapted to the scaling.

In summary, Proposition 4.1 provides an explicit solution for the Cauchy problem (1.1) for $p > 1$, which is a significant result in the study of viscosity solutions and the maximum principle. This solution is constructed by exploiting the scaling properties of (1.1) and verifying the maximum principle for the constructed function $G_p(x, t)$.
Thus, \( u \) is a classical solution of (1.1) in \( (\mathbb{R}^n - \{0\}) \times (0, \infty) \), and a viscosity solution in \( \mathbb{R}^n \times (\varepsilon, \infty) \) for all \( \varepsilon > 0 \).

**Proof.** Suppose that \( u \) be a solution of (1.1), then it is easily seen that \( v(x,t) = u(\lambda x, \lambda^2 t) \) is also a solution. This suggests that we look for a solution of (1.1) in the form

\[
u(x,t) = t^{-\alpha} g(|x|^2/4t),\]

where \( g \) is a suitable function on the line. Proceeding formally, we work with the normalized equation (2.3),

\[
u_t = \left( \delta_{ij} + (p-2) \frac{u_{ij}}{|Du|^2} \right) u_{ij},
\]

instead of (1.1). To determine \( g \) we compute

\[
u_t = -\frac{\alpha}{t^{\alpha+1}} g - \frac{|x|^2}{4t^{\alpha+2}} g'.
\]

Similarly,

\[
D_j u = \frac{1}{2t^{\alpha+1}} g' x_j, \quad D_{ij} u = \frac{1}{2t^{\alpha+1}} g' \delta_{ij} + \frac{1}{4t^{\alpha+2}} g'' x_i x_j.
\]

Imposing that \( u \) be a (classical) solution of (2.3), we find

\[-\frac{\alpha}{t^{\alpha+1}} g - \frac{|x|^2}{4t^{\alpha+2}} g' = \left\{ \frac{n + p - 2}{2} g' + (p-1) \frac{|x|^2}{4t} g'' \right\} \frac{1}{t^{\alpha+1}}.
\]

Cancelling off the powers of \( t \), and letting \( s = \frac{|x|^2}{4t} \), we find

\[
g'' + \frac{1}{p-1} g' + \frac{n + p - 2}{2(p-1)s} g' + \frac{\alpha}{(p-1)s} g = 0.
\]

At this point, we choose \( \alpha = \frac{n + p - 2}{2(p-1)} \), obtaining

\[
\left( s^{\alpha} g' + \frac{1}{p-1} s^{\alpha} g \right)' = 0,
\]

which implies

\[
s^{\alpha} g' + \frac{1}{p-1} s^{\alpha} g = \text{const.}
\]

At this point we easily see that the choice \( g(s) = \exp\left( -\frac{s}{p-1} \right) \) produces a solution of the latter ode corresponding to the choice \( c = 0 \) of the constant in the right-hand side. Since the function

\[
u(x,t) = G_p(x,t) = t^{\frac{n + p - 2}{2(p-1)}} \exp\left( -\frac{|x|^2}{4(p-1)t} \right),
\]

belongs to \( C^\infty(\mathbb{R}^n \times (0, \infty)) \), and its spatial gradient vanishes only on the half-line \( \{0\} \times (0, \infty) \), we conclude that \( G_p \) is a classical solution of (1.1) in \( (\mathbb{R}^n - \{0\}) \times (0, \infty) \) for every \( p > 1 \).

Now, consider \( t > 0 \), and let \( z_k = (x_k, t) \to z = (0, t) \), with \( x_k \neq 0 \) for every \( k \in \mathbb{N} \). Define \( a_k = \frac{D_a u(z_k)}{D_a u(z_k)} \). After possibly passing to a subsequence, we may assume that \( a_k \to a \), with \( |a| = 1 \). Since at each \( z_k \), (2.3) is satisfied classically, we have

\[
u_t(z_k) = (\delta_{ij} + (p-2)a_k i a_k j) u_{ij}(z_k).
\]

Therefore, passing to the limit we obtain

\[
u_t(z) = (\delta_{ij} + (p-2)a_i a_j) u_{ij}(z).
\]

Thus, \( u \) is a viscosity solution in \( \mathbb{R}^n \times (\varepsilon, \infty) \) for all \( \varepsilon > 0 \).

\[ \square \]
We note explicitly that, when $p = 2$, $G_p$ is just a multiple of the classical heat kernel. In what follows we use a variant of the special solution $G_p$ as a comparison function to obtain the following result for solutions of (1.1) which generalizes the classical global maximum principle of Tichonoff type for the heat equation. We emphasize that in the next result we assume that $u$ be a solution in the sense of Definition 2.3 but we do not a priori impose the condition that $u \in L^\infty(\mathbb{R}^n \times (0, \infty))$.

**Theorem 4.2.** Let $u \in C(\mathbb{R}^n \times [0, \infty))$ be such that $u$ is a viscosity solution of (1.1) in $\Omega \times [0, T]$ for every bounded open set $\Omega \subset \mathbb{R}^n$ and every $T > 0$. Assume further that for some $A, \alpha > 0$ the following growth estimate is satisfied:

$$u(x, t) \leq Ae^{\alpha |x|^2}, \quad \text{for every } (x, t) \in \mathbb{R}^n \times [0, \infty).$$

Then,

$$\sup_{\mathbb{R}^n \times [0, T]} u = \sup_{\mathbb{R}^n} u(x, 0).$$

**Proof.** If $\sup u(x, 0) = +\infty$, there is nothing to prove, so we will assume without loss of generality that

$$K = \sup_{\mathbb{R}^n} u(x, 0) < +\infty.$$

Let us first assume that $4\alpha(p - 1)T < 1$. This implies that $4\alpha(p - 1)(T + \varepsilon) < 1$, for some $\varepsilon > 0$. Now, fix $y \in \mathbb{R}^n$ and $\mu > 0$. Let

$$v(x, t) = K + \frac{\mu}{(T + \varepsilon - t)^{\frac{n + p - 2}{2p - 2}}} e^{\frac{|x - y|^2}{4(p - 1)(T + \varepsilon - t)}}.$$

As already observed in Proposition 4.1, the equation (1.1) is invariant under addition and multiplication by a constant. One can thus easily check that $v$ is a solution of (1.1) in $\Omega_T = \Omega \times [0, T]$ for every bounded open set $\Omega \subset \mathbb{R}^n$. Now let $\Omega = B(y, r)$. So $u$ and $v$ are solutions of (1.1) in $\Omega_T$, and $v(x, 0) \geq K \geq u(x, 0)$ for every $x \in \Omega$. On $\partial \Omega \times [0, T]$, i.e $|x - y| = r$, we have

$$v(x, t) = K + \frac{\mu}{(T + \varepsilon - t)^{\frac{n + p - 2}{2p - 2}}} e^{\frac{|x - y|^2}{4(p - 1)(T + \varepsilon - t)}} \geq K + \frac{\mu}{(T + \varepsilon)^{\frac{n + p - 2}{2p - 2}}} e^{\frac{|x - y|^2}{4(p - 1)(T + \varepsilon)}}.$$

Now, we can write $\frac{1}{4(p - 1)(T + \varepsilon)} = a + \gamma$, for some $\gamma > 0$. Thus, we find that for $r$ large enough (since $y$ is fixed)

$$v(x, t) \geq K + \mu \frac{1}{(T + \varepsilon)^{\frac{n + p - 2}{2p - 2}}} e^{(a + \gamma)r^2} > Ae^{\alpha(r+|y|)^2} > u(x, t),$$

for every $(x, t) \in \partial \Omega \times [0, T]$. Therefore, by the comparison principle Theorem 3.2 we conclude that

$$u(x, t) \leq \sup_{\mathbb{R}^n} u(x, 0) + \frac{\mu}{(T + \varepsilon - t)^{\frac{n + p - 2}{2p - 2}}} e^{\frac{|x - y|^2}{4(p - 1)(T + \varepsilon - t)}}$$

for all $(x, t) \in \overline{\Omega} \times [0, T]$. Letting $\mu \to 0$, we reach the conclusion

$$u(x, t) \leq \sup_{\mathbb{R}^n} u(x, 0), \quad \text{in } \overline{B}(y, r) \times [0, T].$$

By the arbitrariness of $r > 0$ we conclude that the above inequality holds in $\mathbb{R}^n \times [0, T]$, provided that $4\alpha(p - 1)T < 1$. The desired conclusion now follows by repeatedly applying this result to the intervals $[0, T_1], [T_1, 2T_1]$ and so on, where, say, $T_1 = 1/8a(p - 1)$.

By combining Theorem 4.2 with Theorem 3.3 we obtain the following result.
Proposition 4.3 (Uniqueness for Cauchy problem). In the class of functions $u \in C(\mathbb{R}^n \times [0, \infty))$ satisfying for some $A, \alpha > 0$ the growth estimate
\[ |u(x, t)| \leq Ae^{\alpha|x|^2}, \quad (x, t) \in \mathbb{R}^n \times [0, \infty), \]
there exists at most one solution to the Cauchy problem for (1.1) with a bounded uniformly continuous initial datum.

5. Existence of solutions

In this section using the method of regularization as in [ES1] we prove the existence of solutions to the Cauchy problem and the Cauchy-Dirichlet problem. An alternate approach to the existence of solutions to the Cauchy problem is based on the adaptation of the method of Perron described in Theorem 4.9 of [OS]. However, we have preferred the method of regularization since it facilitates the study, in the subsequent sections, of questions of convergence, large time behavior, gradient bounds and monotonicity.

For every $i, j = 1, \ldots, n$, and $\sigma \in \mathbb{R}^n \setminus \{0\}$ consider the matrix associated with (1.3)
\[ \begin{align*}
   a_{ij}(\sigma) &= \delta_{ij} + (p - 2) \frac{\sigma_i \sigma_j}{|\sigma|^2}. \\
   &\text{In the sequel for a given } \varepsilon > 0 \text{ we consider the regularized matrix} \\
   a_{ij}^\varepsilon(\sigma) &= \delta_{ij} + (p - 2) \frac{\sigma_i \sigma_j}{\varepsilon^2 + |\sigma|^2} , \quad i, j = 1, \ldots, n. 
\end{align*} \]

It is easily seen that for every $\sigma \in \mathbb{R}^n$ and every $\xi \in \mathbb{R}^n$ the following uniform ellipticity condition is satisfied, independently of $\varepsilon > 0$,
\[ \min\{1, p - 1\} |\xi|^2 \leq a_{ij}^\varepsilon(\sigma) \xi_i \xi_j \leq \max\{1, p - 1\} |\xi|^2. \]

5.1. Approximations. We consider solutions of
\[ \begin{cases}
   u_t = a_{ij}^\varepsilon(Du) u_{ij} & \text{in } \mathbb{R}^n \times [0, \infty) \\
   u(x, 0) = g(x) & \text{in } \mathbb{R}^n ,
\end{cases} \]
where $g$ is smooth and, for some $S > 0$, $g$ is constant for $|x| \geq S$. To be precise, we should indicate with $u = u^\varepsilon$ the solution of (5.3), and when this will be necessary we will do so (although the notation $u^\varepsilon$ is the same that indicates the sup convolutions mentioned in the opening of Section 3 hereafter, there will be no occasion for confusion). In the case $p = 1$ studied in [ES1], the uniform ellipticity breaks down, as can be seen from (5.2) above. Because of this, the authors had to consider the further regularization $a_{ij}^{\varepsilon, \eta}(\sigma) = a_{ij}^\varepsilon(\sigma) + \eta \delta_{ij}$. This is not needed for the situation $p > 1$ studied in this paper. In a classical way, for $p > 1$, we obtain smooth bounded solutions $u^\varepsilon$ of (5.3). Using the classical theory, we have
\[ ||Du^\varepsilon||_{L^\infty(\mathbb{R}^n \times [0, \infty))} = ||Dg||_{L^\infty(\mathbb{R}^n)}, \]
and
\[ ||u^\varepsilon||_{L^\infty(\mathbb{R}^n \times [0, \infty))} \leq C ||D^2 g||_{L^\infty(\mathbb{R}^n)}, \]
where $C$ does not depend on $\varepsilon$. For instance, the second inequality above can be justified differentiating the equation with respect to $t$. Then, by the maximum principle we obtain
\[ ||u_t^\varepsilon||_{L^\infty(\mathbb{R}^n \times [0, \infty))} = ||u_t^\varepsilon(., 0)||_{L^\infty(\mathbb{R}^n)} = ||a_{ij}^\varepsilon(Dg) D_{ij} g||_{L^\infty(\mathbb{R}^n)} \leq (p - 1) ||D^2 g||_{L^\infty(\mathbb{R}^n)}. \]

Also from the uniform ellipticity (5.2), the smoothness of the coefficients and classical estimates, we obtain $L^\infty$ bounds on all higher-order derivatives, possibly depending on $\varepsilon$ and $g$. An outline goes as follows.
We take a sequence of smooth domains $\Omega^N \supset \mathbb{R}^n$. Given any $T > 0$, we consider the finite cylinders $\Omega^N_T = \Omega^N \times (0, T)$, and indicate with $\partial \Omega^N_T = (\partial \Omega^N \times (0, T)) \cup (\Omega^N \times \{0\})$ its parabolic boundary. For each $n \in \mathbb{N}$, and $\varepsilon > 0$, we solve the Cauchy-Dirichlet problem

$$
\begin{align*}
&u^{\varepsilon,N}_i = a^i(Du^{\varepsilon,N})u^{\varepsilon,N}_{ij}, \quad \text{in } \Omega^N_T, \\
u^{\varepsilon,N} = g \quad \text{on } \partial \Omega^N_T \quad \text{(one should keep in mind that } g(x,t) = g(x)).
\end{align*}
$$

Since $g$ is constant outside a compact set, for sufficiently large $N$ the compatibility conditions are obeyed. Therefore, the existence of unique solutions in the class $H^{2+\alpha,1+\alpha/2}(\Omega^N \times [0, T])$ is guaranteed by Theorem 4.1, on p. 558 in [LU]. Because of the parabolic comparison principle, the solutions obtained for various $T$’s agree on the intersection of the corresponding cylinders, and thus we have a solution in $\Omega^N \times [0, \infty)$. By the maximum principle,

$$
\|u^{\varepsilon,N}\|_{L^\infty(\Omega^N \times [0, \infty))} \leq \|g\|_{L^\infty(\mathbb{R}^n)}.
$$

Now using a barrier argument as in [SZ], p. 586, we have for a constant $C$, which depends on $p - 1$,

$$
\|u^{\varepsilon,N}\|_{L^\infty} \leq C\|D^2 g\|.
$$

The local Schauder theory gives the existence of higher derivatives. See, for instance, Theorem 10 on p. 72 in [F]. This allows differentiation with respect to the space variables $x_k$. Now by Theorem 3.4 on p. 554 in [LU], we have for $N >> m$

$$
\|D u^{\varepsilon,N}\|_{\Omega^N_T} \leq c(m, g).
$$

Consequently, by Theorem 1.1 on p. 517 in [LU], we have the following H"{o}lder norm estimate

$$
\langle Du^{\varepsilon,N}\rangle_a \leq C(m, g), \quad \text{for } N >> m.
$$

From such estimate, by applying the Schauder estimates as on p. 121 in [F], or on p. 352 in [LU], we obtain for $N >> m$ and any $T > 0$,

$$
\|u^{\varepsilon,N}\|_{L^\infty(T^\alpha)} \leq C_1(m, g, T).
$$

By a standard diagonal process, we now obtain a subsequence that converges with its first and second derivatives uniformly in compact subsets of $\mathbb{R}^n \times [0, \infty)$ to some bounded $u^\varepsilon$ which solves (5.4). Now given any compact set $K$ in $\mathbb{R}^n \times [0, \infty)$, by applying Theorem 3.4, page 554 in [LU], we deduce a bound on $\|D u^\varepsilon\|_{L^\infty(K)}$ independent of $K$, and thus an uniform bound on $Du^\varepsilon$ in $\mathbb{R}^n \times [0, \infty)$. Therefore, for this $u^\varepsilon$, we have

$$
\|u^\varepsilon\|_\infty = \|g\|_\infty, \quad \|u^\varepsilon\|_{L^\infty} \leq (p - 1)\|D^2 g\|_\infty, \quad \|Du^\varepsilon\|_\infty \leq C(\varepsilon, g).
$$

At this point, the bound $C$ for $\|Du\|_\infty$ as above might possibly depend on $\varepsilon$ (see Theorem 3.4 on p. 554 in [LU] for the dependence of the constant), and we also have $Du^\varepsilon(\cdot, t) \rightarrow Dg$ as $t \rightarrow 0$ since this happens for $Du^{\varepsilon,N}$. Theorem 1.1, p. 517 in [LU] guarantees, in any compact set $K$, a uniform bound on the H"{o}lder norm of $Du^\varepsilon$ independent of $K$. As before, the local Schauder theory as in Theorem 10, p. 72 in [F], gives the existence of higher derivatives. Consequently, by appealing to the linear theory, we can conclude (for instance, by Theorem 5.1 on p.320 in [LU]), the smoothness of $u^\varepsilon$, and the bounds on the higher derivatives of $u^\varepsilon$. By differentiating the equation with respect to $x_k$, we obtain

$$
u^{\varepsilon}_{ik} = a^{ij}_{ij}(Du^\varepsilon)u^{\varepsilon}_{ij} + a^{ij}_{ij,\sigma \ell}(Du^\varepsilon)u^{\varepsilon}_{\ell k}u^{\varepsilon}_{ij}.
$$

Since $Du^\varepsilon$ is bounded, by the maximum principle, we have $\|D u^\varepsilon\|_\infty = \|D g\|_\infty$ (and thus we get a constant $C$ independent of $\varepsilon$). Thus, we can finally assert

$$
\|u^\varepsilon, Du^\varepsilon, u^{\varepsilon}_{ij}\|_\infty \leq C(g),
$$

where $C$ is independent of $\varepsilon > 0$. 

5.2. Passage to the limits. We have the following existence theorem analogous to Theorem 4.2 in [ES1].

**Theorem 5.1.** Given a continuous function $g$, which for some $S > 0$ is constant for $|x| \geq S$, there exists a unique viscosity solution $u$ of (1.1) such that $u = g$ on $\mathbb{R}^n \times \{0\}$.

**Proof.** Given a smooth $g$ constant outside a compact set by arguing as above we have solutions to the approximating problems. From uniform bounds on the derivatives, we extract a subsequence $u^k$ so that $u^k \to u$, locally uniformly in $\mathbb{R}^n \times [0, \infty)$, to some bounded Lipschitz function $u$ which inherits the Lipschitz constant from the bounds on the second derivatives of $g$. The proof that $u$ is a viscosity solution is now identical to that given in [ES1]. We now analyze solvability for a continuous $g$. We take smooth $g_k$’s converging uniformly to $g$. Let the corresponding solutions be $u_k$. From Theorem 5.1 we have $u_k \to u$ uniformly. Since $u$ is a uniform limit of solutions, it is itself a solution, and takes the initial values $g$. Uniqueness follows from the comparison principle. □

**Remark 5.2.** Unlike the case $p = 1$, it cannot be asserted that the solutions are constant outside large sets in space-time since, even for the case $p = 2$, the bounded solutions for a compactly supported initial datum do not obey this property. The auxiliary function $\psi$ as in [ES1] (page 657) can be seen to be very specific for the case $p = 1$. In general for any $p > 1$, say $g$ is non-negative and compactly supported, then $u^\varepsilon$ as above are solutions of a uniformly parabolic partial differential equation in nondivergence form, with eigenvalues controlled from above and below by $\max\{1, p - 1\}$ and $\min\{1, p - 1\}$, respectively. Thus, the Harnack inequality holds for $u^\varepsilon$ (see [KS]) with constant independent of $\varepsilon$, and therefore in the limit the Harnack inequality is satisfied by $u$. This in particular rules out finite extinction time and finite propagation speed when $p > 1$.

**Remark 5.3.** We cannot conclude that $u$ is smooth for a smooth datum as the bounds on higher derivatives of $u^\varepsilon$ obtained from the parabolic theory depend on $\varepsilon$.

5.3. The case of bounded domains. In what follows we consider a $C^2$ convex domain $\Omega \subset \mathbb{R}^n$, with mean curvature bounded from below by a positive constant at each point on the boundary. This geometric assumption was introduced in [SZ]. We intend to establish the following result.

**Theorem 5.4.** Given $g \in C(\overline{\Omega})$. For any $p > 1$ there exists a unique viscosity solution of the Cauchy-Dirichlet problem

\[ \begin{cases} \text{div}(|Du|^{p-2}Du) = |Du|^{p-2}u_t & \text{in} \; \Omega \times (0, \infty), \\ u(x,t) = g(x), & (x,t) \in \partial_p(\Omega \times (0, \infty)). \end{cases} \tag{5.5} \]

**Proof.** With $a_{ij}(\sigma)$ as in (5.1), our objective is finding a viscosity solution of

\[ \begin{cases} u_t = a_{ij}(Du)u_{ij} & \text{in} \; \Omega \times (0, \infty), \\ u(x,0) = g(x), & x \in \Omega, \; u(x,t) = g(x), & (x,t) \in \partial\Omega \times (0, \infty). \end{cases} \tag{5.6} \]

As in [SZ], $u$ can be obtained as the limit of the solutions of the following approximating problems

\[ \begin{cases} u_t = a_{ij}(\varepsilon)u_{ij} & \text{in} \; \Omega \times (0, \infty), \\ u(x,0) = g(x), & x \in \Omega, \; u(x,t) = g(x), & (x,t) \in \partial\Omega \times (0, \infty), \end{cases} \tag{5.7} \]

where we first assume $g$ be smooth. For a given $\varepsilon$, the existence of a classical solution $u^\varepsilon$ which is Hölder continuous in $\overline{\Omega} \times [0, \infty)$ is guaranteed by [LU] (Theorem 4.2 page 559). We note in passing that since, as before, we are dealing with the case $p > 1$, we do not need the additional regularization by $\sigma\delta_{ij}$ which is necessary in the case $p = 1$. As in [SZ], the uniform Lipschitz bounds are obtained by means of a barrier method. In view of our assumption that
the mean curvature of $\partial \Omega$ be bounded from below by a positive constant, in order to bound the spatial difference quotients a barrier function of the form $\lambda d(x, \partial \Omega)$ will do. Similarly, the time difference quotients can be bounded by using a barrier of the form $\alpha t$, where $\alpha$ depends only on the $C^2$ norm of $g$. Once we have uniform Lipschitz bounds, we can pass to the limit, and obtain a Lipschitz viscosity solution $u$ of (1.4) in $\Omega \times [0, \infty)$ when the Cauchy-Dirichlet datum $g$ is smooth. In the case of a continuous datum, we can approximate $g$ uniformly by smooth functions $g_\varepsilon$ and denote by $u_\varepsilon$ the corresponding solutions. By applying Theorem 3.4 we can conclude that $u_\varepsilon \to u$ uniformly, with $u$ taking up the boundary value $g$. Since $u$ is a uniform limit of $u_\varepsilon$, also $u$ solves (1.1). Thus, the Cauchy-Dirichlet problem corresponding to equation (1.1) can be uniquely solved for continuous data when $\Omega$ satisfies the above condition.

\[ \square \]

**Remark 5.5.** We mention explicitly that the proof of Theorem 5.4 shows that, when the initial datum $g$ is sufficiently smooth ($C^2$), then the unique solutions $u_\varepsilon$ of the regularized problems (5.7) converge as $\varepsilon \to 0$ to the unique viscosity solution $u$ of the Cauchy-Dirichlet problem (5.4), uniformly on compact subsets of $\Omega \times [0, \infty)$, i.e.,

$$\lim_{\varepsilon \to 0} u_\varepsilon(x,t) = u(x,t).$$

This fact will be used in the proof of Corollary 7.7.

6. CONVERGENCE TO FLOW BY MEAN CURVATURE AS $p \to 1$

In this section we study what happens to the viscosity solutions $u_p$ of (1.1), in the limit as $p \to 1$. The next result states that such $u_p$’s converge to the unique solution of the mean curvature flow with same initial datum.

**Theorem 6.1.** Given $g \in C^2(\mathbb{R}^n)$, and constant outside a compact set, for a given $p > 1$, let $u_p$ be a viscosity solution of (1.1) as in Theorem 5.1. Then, $\lim_{p \to 1} u_p = u_0$, where $u_0$ is the unique solution of the generalized mean curvature flow equation (1.3). The convergence being uniform on every compact subset in $\mathbb{R}^n \times [0, \infty)$.

**Proof.** Take any sequence $p_k \to 1$. Without loss of generality we may assume that $p_k < 2$ for all $k$. Now, following [ES1], for a given $p_k$, let $u^\varepsilon_{p_k}$ be as above. From Section 5 we have

\[
\begin{align*}
\|u^\varepsilon_{p_k}\|_{\infty} &\leq \|g\|_{L^\infty(\mathbb{R}^n)}, \\
\|Du^\varepsilon_{p_k}\|_{\infty} &\leq \|Dg\|_{L^\infty(\mathbb{R}^n)}, \\
\|\langle u^\varepsilon_{p_k} \rangle_t \|_{\infty} &\leq C\|D^2g\|_{L^\infty(\mathbb{R}^n)},
\end{align*}
\]

where $C$ above depends on $p - 1$, and thus it is uniformly bounded for $p < 2$. From the above estimates, we obtain a uniform bound on the Lipschitz (in space and time) norm of $u_{p_k}$ for all $p_k$. Therefore, we can extract a subsequence such that $u^\varepsilon_{p_m} \to u_0$ locally uniformly. The function $u_0$, which inherits the same uniform Lipschitz bound of the sequence. Let $\phi \in C^2(\mathbb{R}^{n+1})$, and suppose $u_0 - \phi$ has a strict local maximum at $(x_0, t_0)$. Because of uniform convergence (standard arguments as in Section 2 above), we deduce that $u^\varepsilon_{p_m} - \phi$ has a local maximum at $(x_m, t_m) \to (x_0, t_0)$. Suppose first $D\phi(x_0, t_0) \neq 0$, then $D\phi(x_m, t_m) \neq 0$ for large enough $m$. Then, $\phi_t \leq (\delta_{ij} + \frac{(p_m - 2)a_{ij}}{|D\phi|^2})\phi_{ij}$ at $(x_m, t_m)$. Taking the limit as $m \to \infty$, we obtain $\phi_t \leq (\delta_{ij} - \frac{a_{ij}}{|D\phi|^2})\phi_{ij}$ at $(x_0, t_0)$. Next, if $D\phi(x_0, t_0) = 0$, then for each $m$ large enough, we can define the sequence $a^m \in \mathbb{R}^n$ as follows

$$a^m = \frac{D\phi(x_m, t_m)}{|D\phi(x_m, t_m)|} \quad \text{if } D\phi(x_m, t_m) \neq 0.$$

Otherwise, when $D\phi(x_m, t_m) = 0$, we know from Definition 2.3 that there exists $|a^m| \leq 1$ such that $\phi_t \leq (\delta_{ij} + a_{ij})a^m \phi_{ij}$. After passing to a subsequence, if necessary, we may assume that $a^m \to a$, and passing to the limit we obtain $\phi_t \leq (\delta_{ij} - a_{ij})\phi_{ij}$ at $(x_0, t_0)$. Now, if we have
local maximum and not strict local maximum, we can take \( \phi_1(x, t) = \phi(x, t) + |x - x_0|^4 + (t - t_0)^4 \) and repeat the arguments above with \( \phi_1 \). Thus \( u_0 \) is a subsolution of (1.3). Similarly, one shows that \( u_0 \) is a supersolution. Uniqueness follows from the comparison principle. Thus, in particular, \( u_0 \) is constant outside a compact set since it coincides with the solution in (ES1). Now given any compact set \( K \subset \mathbb{R}^n \times [0, \infty) \), for every sequence \( p_k \to 1 \), we have a subsequence \( p_m \) for which there is uniform convergence of \( u_{p_m} \to u_0 \), and this thus implies convergence of the whole sequence. This completes the proof.

□

Remark 6.2. When \( \Omega \) is a bounded domain which satisfies the geometric assumption for Theorem 5.4 as indicated above for smooth enough Cauchy-Dirichlet datum we have uniform Lipschitz estimates. As a consequence, similarly to what was done above we conclude that a statement such as Theorem 6.1 holds, i.e., \( \lim_{p \to 1} u_p = u_0 \), where \( u_0 \) is the solution as in (SZ).

7. Large-time behavior

Let \( \Omega \subset \mathbb{R}^n \) be a smooth, convex bounded domain satisfying the hypothesis in Theorem 5.4. In this section we only consider sufficiently smooth Cauchy-Dirichlet data, and study the large-time behavior of the corresponding solutions.

Proposition 7.1. Let \( u^\varepsilon \) be a solution of the approximating problems (5.7) in \( \Omega \times [0, \infty) \) corresponding to sufficiently smooth data on the parabolic boundary. Then,

\[
 t \to \int_{\Omega} (\varepsilon^2 + |Du^\varepsilon(x, t)|^2)^{p/2}dx
\]
is non-increasing.

Proof. We easily find

\[
\frac{d}{dt} \int_{\Omega} (\varepsilon^2 + |Du^\varepsilon|^2)^{p/2} \leq p \int_{\Omega} (\varepsilon^2 + |Du^\varepsilon|^2)^{p/2-1} < Du^\varepsilon, (Du^\varepsilon)_t > 
= -p \int_{\Omega} u_t \text{div}((\varepsilon^2 + Du^\varepsilon)^{p/2-1}Du^\varepsilon) 
= -p \int_{\Omega} (\varepsilon^2 + |Du^\varepsilon|^2)^{p/2-1}(u_t)^2 \leq 0.
\]

(the boundary integral vanishes because the boundary datum is independent of time. Here, we have made use of the fact that, away from \( t = 0 \), we have continuity of derivatives up to the boundary, a fact which follows from the classical theory, see (LU)).

□

We next prove the following result.

Proposition 7.2 (Energy monotonicity). Let \( 1 \leq p < \infty \). Then, the function \( t \to \int_{\Omega} |Du(x, t)|^pdx \) is non-increasing.

Proof. The proof for the case \( p = 1 \) is in (SZ), thus we assume \( p > 1 \). Using Proposition 7.1 the fact that \( Du^\varepsilon(x, t_n) \to Dg \) as \( t_n \to 0 \) for all \( x \in \Omega \), and Lebesgue dominated convergence theorem, we conclude for any \( t > 0 \)

\[
\int_{\Omega} (\varepsilon^2 + |Du^\varepsilon(x, t)|^2)^{p/2}dx \leq \int_{\Omega} (\varepsilon^2 + |Dg(x)|^2)^{p/2}dx.
\]

Since \( u^\varepsilon(\cdot, t) \to u(\cdot, t) \) weakly in \( W^{1,p}(\Omega) \), using lower semicontinuity and letting \( \varepsilon \to 0 \), we obtain

\[
(7.1) \int_{\Omega} |Du(x, t)|^pdx \leq \int_{\Omega} |Dg(x)|^pdx.
\]
For any given times $t_1 \leq t_2$, we first extend $u(\cdot, t_1)$ outside $\Omega$ by $g$. Now, let $u_k = u(\cdot, t_1) * \rho_{\varepsilon_1}$ (mollification of $u(\cdot, t_1)$. Let $v_k$ be the solution of the Cauchy-Dirichlet problem in $\Omega \times [t_1, \infty)$ with Cauchy-Dirichlet datum $u_k$. So for each $k$, we obtain from (7.1)

\begin{equation}
\int_{\Omega} |Dv_k(x, t_2)|^p dx \leq \int_{\Omega} |Du_k(x)|^p dx.
\end{equation}

Since by the results in Section 5 we have

\[ |Du_k| = |Du(\cdot, t_1) * \rho_{\varepsilon_1}| \leq |Du(\cdot, t_1)| \leq C(g), \]

from (7.2) we have that $Dv_k(\cdot, t_2)$ are uniformly bounded in $L^p(\Omega)$. Also, since $u_k \to u(\cdot, t_1)$ uniformly, and $Du_k \to Du(\cdot, t_1)$ in $L^p(\Omega)$, by Theorem 3.1 and (7.2) which gives uniform $L^p$ bounds we conclude that $v_k(\cdot, t_2) \to u(\cdot, t_2)$ uniformly and weakly in $W^{1,p}(\Omega)$. Consequently, by using lower-semicontinuity in the left-hand side, and by taking limit in the right-hand side of (7.2), we conclude that

\[ \int_{\Omega} |Du(x, t_2)|^p dx \leq \int_{\Omega} |Du(x, t_1)|^p dx \quad \text{for all } t_1 \leq t_2. \]

This gives the desired conclusion.

The next result provides some interesting information on the large-time behavior of the functions $u^\varepsilon$ in Proposition 7.1.

**Theorem 7.3.** There exists a Lipschitz continuous function $v^\varepsilon \in C^\infty(\Omega)$ such that:

1. $\lim_{t \to \infty} u^\varepsilon(x, t) = v^\varepsilon(x)$, uniformly in $\overline{\Omega}$;
2. $|Dv^\varepsilon(x)| \leq C$ for every $x \in \Omega$;
3. $\text{div}(\varepsilon^2 + |Dv^\varepsilon|^2)^{p/2-1} Dv^\varepsilon) = 0$;
4. $v^\varepsilon = g$ on $\partial \Omega$.

**Proof.** In the following discussion the superscript $\varepsilon$ will be omitted throughout. Following [SZ], by applying the uniform Lipschitz bounds, the theorem of Ascoli-Arzelà guarantees the existence of a sequence $t_k \to \infty$, and of a Lipschitz continuous function $v(x)$ to which $u(x, t_k)$ converges uniformly. Now, choose a test function $\phi \in C_0^\infty(\Omega)$. Then, using the fact that $u(= u^\varepsilon)$ is a classical solution, integrating by parts we obtain

\begin{equation}
\int_{t_k}^{t_{k+1}} \int_{\Omega} u_t \phi = - \int_{t_k}^{t_{k+1}} \int_{\Omega} < S D u, D(S^{-1} \phi) > dx,
\end{equation}

where we have let $S = (\varepsilon^2 + |Du|^2)^{p/2-1}$. By using the mean value theorem, we deduce that there exists $T_k \in (t_k, t_{k+1})$ such that the absolute value of the right-hand side of (7.3) is

\[ (t_{k+1} - t_k) \left| \int_{\Omega} < Du(x, T_k), D\phi > - S^{-1}(x, T_k) < Du(x, T_k), DS(x, T_k) > \phi \right| dx. \]

By passing to a subsequence if necessary, we may assume that $t_{k+1} - t_k \geq 1$. We define a sequence of functions by letting $u_k = u(\cdot, T_k)$. Then, each $u_k$ satisfies the following divergence form equation in $\Omega$

\[ \text{div}(\varepsilon^2 + |Du_k|^2)^{p/2-1} Du_k = f_k(x), \]

where $f_k(x) = S(x, T_k) u_t(x, T_k)$. Now, from the estimates in Section 3 we find

\[ \|f_k\|_{\infty} \leq \|(\varepsilon^2 + |Du_k|^2)^{p/2-1}\|_{\infty} \|u_t\|_{\infty} \leq C(\varepsilon, g), \]

where $C(\varepsilon, g)$ is independent of $k$, and thus the $f_k$’s have uniformly bounded $L^2(\Omega)$ norms. So, by the structure of the principle part in the equation above, we have by standard elliptic theory,
see for instance [G], Theorem 8.1 on p. 267, where the same proof goes through with \( f(= f_k) \) considered as a scalar term,

\[
||u_k||_{W^{2,2}(\Omega_1)} \leq C(\Omega_1),
\]

for any \( \Omega_1 \) compactly contained in \( \Omega \), \( C \) being independent of \( k \). Therefore, by the theorem of Ascoli-Arzelà and standard \( L^2 \) theory, we obtain a subsequence \( u_k \to v \), uniformly in \( \Omega \), \( Du_k \to Du \) strongly in \( L^2 \), and \( D^2 u_k \to D^2 v \) weakly in \( \Omega_1 \), for any \( \Omega_1 \Subset \Omega \). Now, take a countable exhaustion of \( \Omega \) by compact subdomains. Thus, by employing a standard diagonal process, we obtain a sequence of times \( T_k \to \infty \) such that \( u_k \to v \) uniformly and \( Du_k \to Du \) pointwise a.e. and \( D^2 u_k \to D^2 v \) weakly on every compact subset of \( \Omega \). Also, because of uniform convergence, we have \( ||Dv||_\infty \leq C \), since this is true for each \( Du_k \). By Lebesgue dominated convergence theorem, we conclude for this sequence that \( Du_k \to Du \) strongly in \( L^2(\Omega) \). Let now \( \phi \) be as in (7.3), with \( \text{supp} \phi \) contained in \( \Omega_1 \Subset \Omega \). Because of \( L^2 \) convergence of gradients, we have

\[
\int_\Omega <Du_k, D\phi> dx \to \int_\Omega <Du, D\phi> dx.
\]

Also, denoting by \( S_k = (\varepsilon^2 + |Du_k|^2)^{p/2-1} \), and \( S_v = (\varepsilon^2 + |Dv|^2)^{p/2-1} \), we have

\[
\int_\Omega S_k^{-1} <Du_k, DS_k > \phi dx = (p-2) \int_\Omega \frac{(u_k)_{ij}(u_k)_{ij}}{\varepsilon^2 + |Du_k|^2} \phi dx \to (p-2) \int_\Omega \frac{u_j v_j v_{ij}}{\varepsilon^2 + |Dv|^2} \phi dx = \int_\Omega S_v^{-1} <Du, DS_v > \phi dx.
\]

(The equation (7.5) is justified by the (strong) \( L^2 \) convergence of first derivatives, and the weak convergence of second derivatives, as stated above. More precisely, the convergence in (7.6) is justified by adding and subtracting \( (p-2) \int_\Omega \frac{u_j v_j v_{ij}}{\varepsilon^2 + |Du_k|^2} \phi dx \), and using the fact that \( ||(u_k)_{ij}||_2 \) is uniformly bounded in \( \text{supp} \phi \Subset \Omega_1 \).)

Let now \( \delta > 0 \). Then, the uniform convergence of the original sequence \( u(\cdot, t_k) \) implies that \( u(\cdot, t_k) \) is uniformly Cauchy. This means that for all \( k \) large enough (depending on \( \delta \))

\[
||u(\cdot, t_{k+1}) - u(\cdot, t_k)||_{L^\infty} \leq \delta.
\]

Therefore, the absolute value of the left-hand side of (7.3) is

\[
\leq \int_\Omega |u(x, t_{k+1}) - u(x, t_k)||\phi(x)||dx \leq C\delta,
\]

with the constant \( C \) depending on \( \phi \). Thus, since \( t_{k+1} - t_k \geq 1 \), we have

\[
\int_\Omega \left| <Du(x, T_k), D\phi> - S_v^{-1} <Du(x, T_k), DS(x, T_k) > \phi \right| dx \leq \frac{C\delta}{t_{k+1} - t_k} \leq C\delta.
\]

Therefore, from (7.4), (7.5), (7.6) we obtain

\[
\int_\Omega \left| <Du, D\phi> - S_v^{-1} <Du, DS_v > \phi \right| dx \leq C\delta,
\]

since the same inequality is true for \( u_k \), for all \( k \)'s sufficiently large. From the arbitrariness of \( \delta > 0 \), we conclude that \( v \) satisfies

\[
\int_\Omega S_v <Du, DS_v^{-1} \phi> dx = 0, \quad \text{for every } \phi \in C^\infty_0(\Omega)
\]

Now, given any \( \xi \in C^\infty_0(\Omega) \), we have that \( S_v \xi \in W^{1,2}_0(\Omega) \). Let us choose a sequence \( \phi_k \in C^\infty_0(\Omega) \) converging to \( S_v \xi \) in \( W^{1,2}_0(\Omega) \). Without loss of generality, we can arrange that all \( \phi_k \) be supported
in some $\Omega_1 \Subset \Omega$. From (7.1) we obtain
\[
(7.8) \quad 0 = \int_\Omega S_v < Dv, D(S_v^{-1} \phi_k) > dx = \int_\Omega \left[ < Dv, D\phi_k > - S_v^{-1} < Dv, DS_v > \phi_k \right] dx.
\]
Since $\phi_k \to S_v \xi$ in $W^{1,2}(\Omega)$, the first integral in the right-hand side of (7.8) is easily seen to converge to $\int_\Omega < Dv, D(S_v \xi) > dx$. For the second integral, since everything is supported in $\Omega_1$, we see from (7.6)
\[
(7.9) \quad \left| \int_\Omega S_v^{-1} < Dv, DS_v > \phi_k - S_v^{-1} < Dv, DS_v > (S_v \xi) \right| \\
\leq \|S_v^{-1} < Dv, DS_v > \|_{L^2(\Omega)} \|\phi_k - S_v \xi\|_{L^2(\Omega)} \\
= \|p - 2\left\| \frac{v_i v_j \partial_j v}{\varepsilon^2 + |Dv|^2} \right\|_{L^2(\Omega)} \|\phi_k - S_v \xi\|_{L^2(\Omega)} \\
\leq C\|D^2v\|_{L^2(\Omega)} \|\phi_k - S_v \xi\|_2 \to 0
\]
Therefore, by passing to the limit for $k \to \infty$ in (7.8), we obtain
\[
\int_\Omega S_v < Dv, D\xi > dx = 0.
\]
By the arbitrariness of $\xi \in C_0^\infty(\Omega)$, we conclude that $v = v^\varepsilon$ is a weak solution of
\[
(7.10) \quad \text{div}( (\varepsilon^2 + |Dv|^2)^{p/2-1} Dv) = 0.
\]
Since for each $k$ we have $u_k = g$ on $\partial \Omega$, passing to the limit as $k \to \infty$ we conclude that $v = g$ on $\partial \Omega$. Convexity of the functional implies that $v^\varepsilon$ minimizes $\int_\Omega ( \varepsilon^2 + |Df|^2 )^{p/2} dx$ among all $f$’s subject to boundary values $g$. The smoothness of $v^\varepsilon$ follows from the elliptic theory. We have thus proved 3) and 4). We are left with proving 1) and 2).

Since $Du_k = Du(\cdot, T_k) \to Du$ pointwise a.e., and the $Du_k$ satisfy uniform $L^\infty$ bounds we see that 2) holds a.e., and therefore everywhere ($v^\varepsilon$ is smooth). Moreover, by Lebesgue dominated convergence we have $\int_\Omega (\varepsilon^2 + |Du_k|^2)^{p/2} dx \to \int_\Omega (\varepsilon^2 + |Du|^2)^{p/2} dx$. We now invoke Proposition 7.3 which gives the monotonicity of
\[
t \to E_\varepsilon(t) = \int_\Omega (\varepsilon^2 + |Du^\varepsilon(x,t)|^2)^{p/2} dx.
\]
Because of (7.10), we know that $E_\varepsilon(u_k) \overset{\text{def}}{=} E_\varepsilon(T_k)$ decreases to $E_\varepsilon(v)$. Given $\delta > 0$ choose $N \in \mathbb{N}$ large enough that for $k > N$ we have $E_\varepsilon(u_k) - E_\varepsilon(v) \leq \delta$. For $T > T_k$ and monotonicity we have
\[
0 \leq E_\varepsilon(T) - E_\varepsilon(v) \leq E_\varepsilon(u_k) - E_\varepsilon(v) \leq \delta.
\]
This shows that $E_\varepsilon(t) \searrow E_\varepsilon(v) = m$, as $t \to \infty$. Now take any sequence $T_i \to \infty$. By the uniform Lipschitz bounds on the $u(\cdot, T_i)$ we know that there exists a subsequence, still denoted by the same symbol, such that $u(\cdot, T_i) \to u_0$ uniformly, and weakly in $W^{1,2}(\Omega)$, as $i \to \infty$. By lower semicontinuity and convexity of the energy functional, we have
\[
m = \int_\Omega (\varepsilon^2 + |Du_0|^2)^{p/2} dx = \lim_{i \to \infty} \int_\Omega (\varepsilon^2 + |D(u(\cdot, T_i))|^2)^{p/2} dx \\
\geq \int_\Omega (\varepsilon^2 + |Du_0|^2)^{p/2} dx
\]
However, since $m$ is the minimum of the energy functional, and $u_0$ and $v^\varepsilon$ have the same boundary values, we conclude that $v^\varepsilon = u_0$. We have thus shown that for any sequence $T_i \to \infty$ there exists a subsequence $T_{k_i}$ such that $u(\cdot, T_{k_i}) \to v$ uniformly. This establishes 1), thus completing the proof of the theorem.
\[\square\]
Theorem 7.4. Let \( v^\varepsilon \) be as in Theorem 7.3. Then, the sequence \( \{v^\varepsilon\}_{\varepsilon > 0} \) converges uniformly to a function \( v \in W^{1,p}(\Omega) \), where \( v \) is a \( p \)-harmonic function in \( \Omega \) having boundary values \( g \). We thus have
\[
\lim_{\varepsilon \to 0} \lim_{i \to \infty} u^\varepsilon(x,t) = \lim_{\varepsilon \to 0} v^\varepsilon(x) = v(x),
\]
where in the first equality we have used 1) of Theorem 7.3.

Proof. Given the function \( v^\varepsilon \) constructed in Proposition 7.3 take any sequence \( \varepsilon_i \downarrow 0 \). Because of the uniform Lipschitz bounds on the \( v^\varepsilon \), we can find a subsequence, which we continue to indicate with \( v^{\varepsilon_i} \), converging uniformly to some \( v \), and such that \( v^{\varepsilon_i} \) converges weakly in \( W^{1,p} \). If \( f \) has boundary values \( g \), by lower semicontinuity and the minimizing property of the Dirichlet integral with given boundary values, we have proved the desired result.

\( \square \)

Theorem 7.4 is the counterpart, for the case \( p > 1 \) in equation (1.4), of a result that in in [SZ] was proved for the case \( p = 1 \), where it was shown that \( v^\varepsilon \) converge to a Lipschitz function \( v \) of least gradient. In this regard, we recall that an example in [SZ] shows that, for the case \( p = 1 \), one has in general that
\[
\lim_{\varepsilon \to 0} \lim_{i \to \infty} u^\varepsilon(x,t) \neq \lim_{i \to \infty} \lim_{\varepsilon \to 0} u^\varepsilon(x,t),
\]
and therefore when \( p = 1 \) one concludes that \( \lim_{i \to \infty} \lim_{\varepsilon \to 0} u^\varepsilon(x,t) \) might not be a function of least gradient, in general. This reveals the complexity of the large-time behavior associated with the generalized mean curvature flow. However, we show that for \( 1 < p \leq 2 \), the above limits do commute. We follow the ideas in [SZ]. We need the following intermediate lemma.

Lemma 7.5. For \( 1 < p \leq 2 \), let \( u \) be the unique viscosity solution in \( \Omega \times (0, \infty) \) in Theorem 5.4. Then,
\[
\int_0^\infty \int_\Omega u_t^2 dx dt < \infty.
\]

Proof. From the calculations in the proof of Proposition 7.1 we obtain for any \( T > 0 \)
\[
\int_\Omega (|Dg(x)|^2 + \varepsilon^2)^{p/2} dx - \int_\Omega (|Du^\varepsilon(x,T)|^2 + \varepsilon^2)^{p/2} dx = p \int_0^T \int_\Omega (\varepsilon^2 + |Du^\varepsilon(x,t)|^2)^{p/2-1} u_t^2(x,t)^2 dx dt.
\]

We thus have for all \( T > 0 \)
\[
p \int_0^T \int_\Omega (\varepsilon^2 + |Du^\varepsilon(x,t)|^2)^{p/2-1} u_t^2 dx dt \leq \int_\Omega (|Dg|^2 + \varepsilon^2)^{p/2} dx \leq C(g),
\]
where \( C(g) \) is independent of \( T \) and \( \varepsilon \). On the other hand, we know there is a constant \( B = B(g) > 0 \), independent of \( T \) and \( \varepsilon \), such that \( |Du^\varepsilon(x,t)| \leq B \) for every \( x \in \Omega \) and every \( 0 \leq t \leq T \). We conclude that, when \( 1 < p \leq 2 \), then
\[
(\varepsilon^2 + |Du^\varepsilon(x,t)|^2)^{p/2-1} \geq B > 0, \quad \text{on } \Omega \times (0,T).
\]
Therefore, for any $T > 0$, we have
\[
\int_0^T \int_\Omega (u_t^\varepsilon)^2 \, dx \, dt \leq C^*,
\]
where $C^*$ is independent of $T, \varepsilon$. Since $u_t^\varepsilon \to u_t$ weakly on compact sets, we have reached the desired conclusion.

We now state the following theorem.

**Theorem 7.6.** For $1 < p \leq 2$, let $u$ be the unique viscosity solution in $\Omega \times (0, \infty)$ in Theorem 4.4. Then, as $t \to \infty$ the function $u$ converges to the unique $p$-harmonic function $v$ in $\Omega$ having boundary values $g$ on $\partial \Omega$.

**Proof.** Let $C_T$ be the cylinder $\Omega \times [0, T]$. Consider the sequence $u^k$ defined by $u^k(x,t) = u(x,t + k)$. Because of the uniform Lipschitz bounds in Section 5 by the theorem of Ascoli-Arzela we have a subsequence $u^k \to v$ locally uniformly in $\Omega \times [0, \infty)$, and thus uniformly in the compact set $\overline{C_T}$. It is thus easily verified that the function $v$ is also a solution of (1.4). We claim that $v$ is independent of $t$. From Lemma 7.5 we find that
\[
\lim_{k \to \infty} \int_{C_T} (u_t^k)^2 \, dx \, dt = \lim_{k \to \infty} \int_0^k \int_{C_T} u_t^2 \, dx \, dt \leq \lim_{k \to \infty} \int_0^{k+T} \int_{C_T} u_t^2 \, dx \, dt = 0.
\]
Since $u_t^k \to v_t$ in $\overline{C_T}$, by lower semicontinuity we obtain
\[
\int_{C_T} v_t^2 \, dx \, dt = 0.
\]
As in the proof of Theorem 4.4 in [ISZ], we conclude that $v$ is independent of $t$ in $C_T$. By the arbitrariness of $T$, we conclude that $v$ is independent of $t$ in $\Omega \times [0, \infty)$. The fact that $v$ is $p$-harmonic is seen from Lemma 2.9. It remains to be seen that $u(\cdot, t)$ converges to $v$ uniformly as $t \to \infty$. Given $\varepsilon > 0$, choose $k$ large enough such that $|u^k(x,0) - v(x)| \leq \varepsilon$ (because of uniform convergence in the compact set $\overline{\Omega} \times \{0\}$). Now, we apply Theorem 3.1 to conclude that $|u^k(x,t) - v(x)| = |u(x,t+k) - v(x)| \leq \varepsilon$ for all $x \in \overline{\Omega}$, and every $t \geq k$. (note that on the lateral boundary both functions equal $g$). This concludes the proof.

**Corollary 7.7.** For $1 < p \leq 2$, let $u$ be the unique viscosity solution in $\Omega \times (0, \infty)$ in Theorem 4.4. Then,
\[
\lim_{\varepsilon \to 0} \lim_{t \to \infty} u^\varepsilon(x,t) = \lim_{t \to \infty} \lim_{\varepsilon \to 0} u^\varepsilon(x,t),
\]

**Proof.** From Remark 5.5 we have
\[
\lim_{\varepsilon \to 0} u^\varepsilon(x,t) = u(x,t),
\]
the convergence being uniform on compact subsets of $\Omega \times [0, \infty)$. Using this fact, and Theorem 7.6 we conclude
\[
\lim_{t \to \infty} \lim_{\varepsilon \to 0} u^\varepsilon(x,t) = \lim_{t \to \infty} u(x,t) = v(x).
\]
This fact, combined with Theorem 7.4 gives the desired conclusion.

**Remark 7.8.** Corollary 7.7 makes the case $1 < p \leq 2$ of equation (1.4) very different from that of (1.3), when $p = 1$. For (1.3) there also exists an equilibrium solution independent of time, but the conclusion of Corollary 7.7 does not hold in general, see [SZ] and [ISZ]. It remains an interesting open question whether Corollary 7.7 continues to be valid for $p > 2$. In such case one needs to find an appropriate replacement of Lemma 7.5.
8. Energy estimates and monotonicity

For the case $p = 1$, the following monotonicity of the energy of the unique (bounded) solution $u$ to the Cauchy problem for (1.3) was established in [ES3]:

\[
\int_{\mathbb{R}^n} |Du|(x,t_2)dx \leq \int_{\mathbb{R}^n} |Du|(x,t_1)dx \quad \text{for all } t_1 \leq t_2.
\]

For $p > 1$ we can prove an analogous monotonicity result.

**Theorem 8.1** (Unweighted energy monotonicity). Let $u$ be the unique viscosity solution of (1.1) obtained in Theorem 5.7, where the initial datum $g$ is Lipschitz continuous, and constant outside a compact set. Then,

\[
\int_{\mathbb{R}^n} |Du|^p(x,t_2)dx \leq \int_{\mathbb{R}^n} |Du|^p(x,t_1)dx \quad \text{for all } t_1 \leq t_2.
\]

**Proof.** By subtracting a constant, we can without loss of generality assume that $g$ be compactly supported. Denote by $u^\varepsilon$ the solution to the regularized Cauchy problem (5.3). First, we also assume that $g$ is smooth, a fact which ensures bounds on derivatives of $u^\varepsilon$, as in Section 5. In this first part of the proof we adapt a beautiful argument in the proof of Lemma 2.1 in [ES3]. Letting $\phi(x) = e^{-b(1+|x|^2)^{1/2}}$, we define

\[
F_\varepsilon^\phi(t) = \int_{\mathbb{R}^n} \phi(x)^2(|Du^\varepsilon(x,t)|^2 + \varepsilon^2)^{p/2}dx,
\]

and note that

\[
F_\varepsilon^\phi(0) = \int_{\mathbb{R}^n} \phi(x)^2(|Dg(x)|^2 + \varepsilon^2)^{p/2}dx.
\]

Differentiating gives

\[
(F_\varepsilon^\phi)'(t) = p \int_{\mathbb{R}^n} \phi^2(|Du^\varepsilon|^2 + (\varepsilon^2)^{p/2} - 1 < Du^\varepsilon, Du^\varepsilon > dx.
\]

\[
= -p \int_{\mathbb{R}^n} \phi^2 \text{div}(|Du^\varepsilon|^2 + (\varepsilon^2)^{p/2} - 1 Du^\varepsilon)u_1^\varepsilon dx
\]

\[
- 2p \int_{\mathbb{R}^n} \phi < D\phi, Du^\varepsilon > (|Du^\varepsilon|^2 + \varepsilon^2)^{p/2} - 1 u_1^\varepsilon dx.
\]

If we now let

\[
H^\varepsilon = \text{div}(|Du^\varepsilon|^2 + (\varepsilon^2)^{p/2} - 1 Du^\varepsilon),
\]

then we can write the equation as

\[
H^\varepsilon(|Du^\varepsilon|^2 + \varepsilon^2)^{1-p/2} = u_1^\varepsilon.
\]

We thus find

\[
(F_\varepsilon^\phi)'(t) = -p \int_{\mathbb{R}^n} \phi^2 (H^\varepsilon)^2(|Du^\varepsilon|^2 + \varepsilon^2)^{1-p/2} dx - 2p \int_{\mathbb{R}^n} \phi < D\phi, Du^\varepsilon > H^\varepsilon dx.
\]

Now, we have trivially $|Du^\varepsilon| \leq (|Du^\varepsilon|^2 + (\varepsilon^2)^{1/2}$. If we write

\[
(|Du^\varepsilon|^2 + \varepsilon^2) = (|Du^\varepsilon|^2 + \varepsilon^2)^{1-p/2}(|Du^\varepsilon|^2 + \varepsilon^2)^{p/2},
\]

then, by Cauchy-Schwarz inequality, we easily obtain

\[
(F_\varepsilon^\phi)'(t) \leq p \int_{\mathbb{R}^n} |D\phi|^2(|Du^\varepsilon|^2 + (\varepsilon^2)^{p/2} dx \leq b^2 p F_\varepsilon^\phi(t),
\]

where we have used $|D\phi| \leq b|\phi|$. Gronwall’s inequality now easily gives for every $t \geq 0$

\[
F_\varepsilon^\phi(t) \leq e^{b^2 pt} \int_{\mathbb{R}^n} \phi^2(x)(|Dg(x)|^2 + \varepsilon^2)^{p/2}dx.
\]

(8.3)
Now, using the \( \phi \) can be justified by taking a sequence \( t_j \to t \), with \( t_j \to t \), and noting that \( Du^{\varepsilon}(t_j, \cdot) \to D\varphi \), \( \phi^2|Du^{\varepsilon}|^p \leq \phi(\varepsilon) \), which is in \( L^1(\mathbb{R}^n) \) and then using Lebesgue dominated convergence theorem. Let \( K \subset \mathbb{R}^n \) be an arbitrarily fixed compact set. From (8.3) we thus obtain for every fixed \( t > 0 \)

\[
\int_K \phi(x)^2(|Du^{\varepsilon}(x, t)|^2 + \varepsilon^2)^{p/2} dx \leq e^{k^2 t} \int_{\mathbb{R}^n} \phi^2(x)(|D\varphi(x)|^2 + \varepsilon^2)^{p/2} dx.
\]

With \( t > 0 \) fixed, select a sequence \( \varepsilon_j \to 0 \) such that \( u^{\varepsilon_j}(t, \cdot) \to u(t, \cdot) \) weakly in \( W_{loc}^{1,p}(\mathbb{R}^n) \). Letting \( j \to \infty \) in the latter inequality, by using the lower semicontinuity in the left-hand side and Lebesgue dominated convergence in the right-hand side, we find

\[
(8.4) \quad \int_K \phi^2(x)|Du(x, t)|^p dx \leq \int_{\mathbb{R}^n} \phi^2(x)|D\varphi(x)|^p dx.
\]

Letting \( b \to 0 \) in (8.4), we obtain

\[
\int_K |Du(x, t)|^p dx \leq \int_{\mathbb{R}^n} |D\varphi(x)|^p dx \quad \text{for all } t \in [0, \infty).
\]

Since the latter estimate is true for all compact \( K \subset \mathbb{R}^n \), by the monotone convergence theorem we conclude

\[
(8.5) \quad \int_{\mathbb{R}^n} |Du(x, t)|^p dx \leq \int_{\mathbb{R}^n} |D\varphi(x)|^p dx \quad \text{for all } t \in [0, \infty).
\]

To extend the estimate (8.5) to the case when \( g \) is Lipschitz, we consider the \( \varepsilon_k \)-mollifications of \( g \) and call them \( g_k \). Then, \( g_k \to g \) uniformly and in \( W^{1,p}(\mathbb{R}^n) \). Let \( u_k \) be the solution to the Cauchy problem with initial datum \( g_k \). From uniform Lipschitz bounds in Section 5 and Theorem 3.3 we have that, at any given time \( t > 0 \), \( u_k(\cdot, t) \to u(\cdot, t) \) uniformly, and weakly in \( W_{loc}^{1,p}(\mathbb{R}^n) \). Since (8.3) holds for \( u_k \) and \( g_k \), we first bound from below, as before, the integral in the left-hand side over a compact set \( K \), use lower semicontinuity in the left-hand side, Lebesgue dominated convergence in the right-hand side, and finally let \( K \to \mathbb{R}^n \) to conclude that (8.3) continues to hold when the initial datum \( g \) is Lipschitz continuous.

Finally, we establish (8.2). With this objective in mind, let \( G_p \) be the notable solution in Proposition 4.1 above, and set \( V(x, t) = G_p(x, t + 1) \). We first claim that, for a given \( t \geq 0 \), we have

\[
(8.6) \quad |u(x, t)| \leq C(g)V(x, t), \quad x \in \mathbb{R}^n.
\]

In order to prove (8.6), we observe that if \( g \) is Lipschitz continuous and compactly supported, then there exists a constant \( C = C(g) \geq 0 \) such that

\[
-CV(x, 0) \leq g(x) \leq CV(x, 0).
\]

Theorem 3.3 then guarantees that (8.6) to be true. We explicitly note that (8.6) implies, in particular, that \( \lim_{|x| \to \infty} u(x, t) = 0 \). And that, furthermore,

\[
\int_{\mathbb{R}^n} |Du(x, t)|^p dx < \infty, \quad \text{for every } t \geq 0.
\]

However, this latter fact is already implied by the quantitatively precise estimate (8.5). Now, given \( t_1 \leq t_2 \), for each \( k \in \mathbb{N} \) let \( h_k \in C^\infty_0(\mathbb{R}^n) \), \( 0 \leq h_k \leq 1 \), with \( h_k = 1 \) for \( |x| \leq k \) and \( h_k \equiv 0 \) for \( |x| \geq 2k \). Set \( g_k = h_k u(\cdot, t_1) \) and let \( u_k \) denote the solution to the Cauchy problem in \( \mathbb{R}^n \times [t_1, \infty) \), corresponding to initial datum \( g_k \). Because of (8.6), it is easy to recognize that

\[
g_k \to u(\cdot, t_1), \quad \text{uniformly in } \mathbb{R}^n.
\]

Now, using the \( L^\infty \) bounds of the solutions and their gradients from Section 5 and the fact that \( ||h_k||_\infty \leq 1, ||Dh_k||_\infty \leq C/k \leq C \), we obtain

\[
||Du_k(\cdot, t_2)||_\infty = ||Dg_k||_\infty \leq (||Dh_k||_\infty ||u(\cdot, t_1)||_\infty + ||h_k||_\infty ||Du(\cdot, t_1)||_\infty) \leq C(g).
\]
Thus, by Theorem 3.4 and uniform Lipschitz bounds, we conclude that $u_k(\cdot, t_2) \to u(\cdot, t_2)$ uniformly, and weakly in $W^{1,p}_\text{loc}(\mathbb{R}^n)$. Now,

$$
\int_{\mathbb{R}^n} |Dg_k(x)|^p dx = \int_{|x|<k} |Du(x, t_1)|^p dx + \int_{k<|x|<2k} |Dg_k(x)|^p dx.
$$

By monotone convergence, the first integral in the right-hand side converges to $\int_{\mathbb{R}^n} |Du(x, t_1)|^p dx$. We claim that

$$
\lim_{k \to \infty} \int_{k<|x|<2k} |Dg_k(x)|^p dx = 0.
$$

To recognize this fact, we observe that, using the estimate $|Dh_k| \leq c/k$, we see that the integral is estimated from above by

$$
C \left( \int_{|x|>k} |Du(x, t_1)|^p dx + \frac{1}{k^p} \int_{|x|>k} |u(x, t_1)|^p dx \right).
$$

From (8.5) the first integral converges to 0 as $k \to \infty$. The second integral, instead, also converges to 0 because of (8.6). We conclude that

$$
\int |Dg_k(x)|^p dx \to \int |Du(x, t_1)|^p dx.
$$

On the other hand, the energy estimate (8.5) allows to conclude that

$$
\int_{\mathbb{R}^n} |Du_k(x, t_2)|^p dx \leq \int_{\mathbb{R}^n} |Dg_k(x)|^p dx.
$$

At this point, we can repeat the argument following (8.5), and passing to the limit as $k \to \infty$, we reach the desired conclusion (8.1).
Before proving Theorem [8.2] we need to establish the following intermediate result which asserts the monotonicity of the weighted energy of the approximations. In the statement of the next result, by a regular solution of the Cauchy problems [5.3] we intend a bounded solution having bounded partial derivatives up to order three. We note explicitly that, when the initial datum \( g \) is sufficiently smooth, the regular solutions defined constructed in Section 6 amply satisfy such requirement. Therefore, they coincide with the regular solutions in the sense of \([S]\).

**Theorem 8.3.** Let \( u^\varepsilon \) be a regular solution of (5.3). Then, for any \( x \in \mathbb{R}^n \) and \( T > 0 \), the function

\[
t \to E_\varepsilon(t) = (T-t)^{\frac{p}{2}} \int_{\mathbb{R}^n} (|Du^\varepsilon(y,t)|^2 + \varepsilon^2)^{\frac{p}{2}} G(x, y, T-t) dy,
\]

is non-increasing on the interval \( 0 < t \leq T \).

**Proof.** Consider \( u^\varepsilon \) as above, and for \( \sigma \in \mathbb{R}^n \), let \( \Phi_\varepsilon(\sigma) = \frac{2}{p} (\varepsilon^2 + |\sigma|^2)^{p/2} \). In the following considerations, all the \( \varepsilon \) super- and subscripts will be routinely omitted. Thus, by rewriting the equation (5.3), we see that \( u(= u^\varepsilon) \) satisfies

\[
(8.7) \quad \text{div}(\Phi'(|Du|^2) Du) = \Phi(|Du|^2)_{tt}, \quad \text{in} \quad \mathbb{R}^n \times (0, \infty).
\]

Also, let us set for brevity \( \rho(y, t) = G(x, y, T-t) \), and notice that \( \rho \) satisfies the backward heat equation in \( \mathbb{R}^n \times (-\infty, T) \), i.e.,

\[
(8.8) \quad \Delta \rho + \rho_t = 0.
\]

For the sake of convenience, we will continue to denote by \( E(t) \) the energy \( E_\varepsilon(t) \) in the statement of Theorem 8.3 multiplied by a factor of \( \frac{1}{p} \), i.e.,

\[
E(t) = (T-t)^{\frac{p}{2}} \int_{\mathbb{R}^n} \Phi(|Du(y,t)|^2) \frac{\rho(y,t)}{2} dy.
\]

Differentiating, we find

\[
(8.9) \quad E'(t) = (T-t)^{\frac{p}{2}} \int_{\mathbb{R}^n} \left[ -\frac{p}{2(T-t)} \frac{\rho}{2} \Phi + \frac{\rho_t}{2} \Phi + \frac{\rho}{2} (\Phi(|Du|^2))_t \right] dy.
\]

Now

\[
(8.10) \quad (\Phi(|Du|^2))_t = 2(\Phi' u_t)_i u_i - 4\Phi'' u_{ij} u_j u_i.
\]

Using the equation (8.7) we find

\[
(8.11) \quad (\Phi(|Du|^2))_t = 2(\Phi' u_t)_i u_i - 2u_t[\Phi' u_t - \Phi u_t \Delta u].
\]

Replacing (8.11) into (8.9), and using (8.8), gives

\[
(8.12) \quad E'(t) = (T-t)^{\frac{p}{2}} \int_{\mathbb{R}^n} \left[ -\frac{p}{2(T-t)} \frac{\rho}{2} \Phi - \frac{\Delta \rho}{2} \Phi + \rho u_t (\Phi' u_t)_i - \rho \Phi'' u_{ij} u_j u_i + \rho \Phi' u_t \Delta u \right] dy.
\]

We now integrate by parts the term

\[
\int_{\mathbb{R}^n} \rho u_t (\Phi' u_t)_i dy = - \int_{\mathbb{R}^n} \rho u_t \Phi' u_t dy - \int_{\mathbb{R}^n} \rho \Delta u u_t \Phi' dy.
\]

Substitution in (8.12) gives

\[
(8.13) \quad E'(t) = (T-t)^{\frac{p}{2}} \int_{\mathbb{R}^n} \left[ -\frac{p}{2(T-t)} \frac{\rho}{2} \Phi - \frac{\Delta \rho}{2} \Phi - \rho \Phi' u_t \Delta u - \Phi' u_t < Du, D\rho > - \rho \Phi'^2 u_t + \rho \Phi' u_t \Delta u \right] dy
\]

\[
= (T-t)^{\frac{p}{2}} \int_{\mathbb{R}^n} \left[ -\frac{\rho}{\Phi'} \left[ \Phi' u_t + < Du, D\rho > \Phi' \right]^2 \right] dy
\]

\[
+ (T-t)^{\frac{p}{2}} \int_{\mathbb{R}^n} \left[ \rho \Phi'^2 ( < Du, D\rho >)^2 + \Phi' u_t < Du, D\rho > - \frac{\Delta \rho}{2} \Phi - \frac{p}{2(T-t)} \frac{\rho}{2} \Phi \right] dy.
\]
We have thus proved
\[
E'(t) = (T-t)^{\frac{3}{2}} \int_{\mathbb{R}^n} -\rho \Phi' \left[ u_t + \langle Du, \frac{D\rho}{\rho} \rangle \right]^2 \, dy + G(t)
\]
with
\[
G(t) = (T-t)^{\frac{3}{2}} \int_{\mathbb{R}^n} \left[ \rho \Phi' \langle Du - \frac{D\rho}{\rho} \rangle^2 + \Phi' u_t \langle Du, D\rho \rangle - \frac{\Delta \rho}{2} - \frac{p}{2(T-t)} \rho \Phi \right] \, dy.
\]

In order to proceed we establish the following

**Lemma 8.4.** The function \(G\) defined by the equation (8.15) is given by
\[
G(t) = (T-t)^{\frac{3}{2}} \int_{\mathbb{R}^n} \frac{\rho}{2(T-t)} \left[ \Phi'(|Du|^2)|Du|^2 - \frac{p}{2} \Phi(|Du|^2) \right] \, dy.
\]

**Proof.** From the equation (8.7) we have
\[
\int_{\mathbb{R}^n} \Phi' u_t \langle Du, D\rho \rangle \, dy = \int_{\mathbb{R}^n} \langle Du, D\rho \rangle \text{div}(\Phi' Du) \, dy
\]
(8.16)
\[
= \int_{\mathbb{R}^n} \left[ -\Phi' u_i u_j \rho_j - \Phi' \rho_i u_i u_j \right] \, dy
\]
\[
= \int_{\mathbb{R}^n} \left[ -\langle Du, \frac{\Phi(|Du|^2)}{2} \rangle, D\rho \rangle - \Phi' D^2 \rho(Du), Du \rangle \right] \, dy
\]
\[
= \int_{\mathbb{R}^n} \left[ \frac{\Phi}{2} \Delta \rho - \Phi' <D^2 \rho(Du), Du \rangle \right] \, dy,
\]
where we have denoted by \(D^2 \rho\) the Hessian matrix of \(\rho\). By substituting (8.16) in (8.15), we obtain
\[
G(t) = (T-t)^{\frac{3}{2}} \int_{\mathbb{R}^n} \frac{\rho}{2(T-t)} \left[ \Phi'(|Du|^2)|Du|^2 - \frac{p}{2} \Phi(|Du|^2) \right] \, dy.
\]

We now notice that
\[
\rho = (4\pi(T-t))^{-\frac{n}{2}} f \left( \frac{-r^2}{4(T-t)} \right)
\]
with \(f(s) = e^s\), and \(r = |y-x|\). One has
\[
\rho_i = (4\pi(T-t))^{-\frac{n}{2}} f' \left( \frac{-r^2}{4(T-t)} \right) \left( -\frac{y_i - x_i}{2(T-t)} \right).
\]
Since
\[
f'(s) = f''(s) = f(s),
\]
we obtain
\[
\rho_i = \left( -\frac{y_i - x_i}{2(T-t)} \right) \rho,
\]
\[
\rho_{ij} = -\delta_{ij} \frac{\rho}{2(T-t)} + \frac{(y_i - x_i)(y_j - x_j)}{4(T-t)^2} \rho.
\]
In conclusion, we have
\[
\frac{D\rho}{\rho} = -\frac{y - x}{2(T-t)}
\]
\[
\frac{D_{ij}\rho}{\rho} = -\delta_{ij} \frac{1}{2(T-t)} + \frac{(y_i - x_i)(y_j - x_j)}{4(T-t)^2}.
\]
Using \(8.18\) in \(8.17\), we obtain
\[
G(t) = (T - t)^{p/2} \int_{\mathbb{R}^n} \frac{p}{2(T - t)^2} \left[ \rho \Phi' \frac{|Du|}{2(T - t)^2} - \rho \Phi' \frac{|Du|^2}{2(T - t)^2} - \rho \frac{p}{2} \Phi'^{p/2} \right] dy,
\]
which gives the desired conclusion.

With Lemma 8.4 in hands we now resume the proof of Theorem 8.3. Substituting in \(8.15\) above the explicit form of the function \(\Phi(\sigma) = \frac{2}{p} (\varepsilon^2 + |\sigma|^2)^{p/2}\), we obtain
\[
G(t) = (T - t)^{p/2} \int_{\mathbb{R}^n} \frac{p}{2(T - t)^2} \left[ (\varepsilon^2 + |Du|^2)^{p/2} - (\varepsilon^2 + |Du|^2)^{p/2} \right] dy.
\]
Therefore,
\[
G(t) = (T - t)^{p/2} \int_{\mathbb{R}^n} \frac{p}{2(T - t)^2} \left[ (\varepsilon^2 + |Du|^2)^{p/2} - (\varepsilon^2 + |Du|^2)^{p/2} \right] \leq 0.
\]
This shows that \(E'_+(t) \leq 0\), thus completing the proof of the theorem.

We can now turn to the

**Proof of Theorem 8.2.** By subtracting a constant, we can assume without loss of generality that \(g\) be compactly supported. In a first step, we also assume that \(g\) be smooth. But then, Theorem 8.2 gives for the corresponding \(u^\varepsilon\)
\[
E'_+(t_2) \leq E'_+(t_1) \quad t_2 \geq t_1.
\]
Moreover, since for any compact set \(K \subset \mathbb{R}^n\) we trivially have
\[
E'_+(t) \geq (T - t)^{p/2} \int_K |Du^\varepsilon(y, t)|^p G(x, y, T - t) dy,
\]
we obtain from \(8.19\)
\[
(T - t_2)^{p/2} \int_K |Du^\varepsilon(y, t_2)|^p G(x, y, T - t_2) dy \leq (T)^{p/2} \int_{\mathbb{R}^n} (|Dg(y)|^2 + \varepsilon^2)^{p/2} G(x, y, T) dy.
\]
Here, we have made use of the fact that for a sequence \(t_j \searrow 0\), with \(t_j < t_2\) for every \(j \in \mathbb{N}\), we have \(Du^\varepsilon(\cdot, t_j) \to Dg\) as \(j \to \infty\). We note that \(|Du^\varepsilon(\cdot, t_j)|^p G(x, \cdot, T - t_j) \leq |Du^\varepsilon(\cdot, t_j)|^p \left\|G(x, \cdot, T)\right\|_p \leq \left\|G(x, \cdot, T)\right\|_p \), which belongs to \(L^1(\mathbb{R}^n)\), and thus we can use Lebesgue dominated convergence theorem.

Now, because of the uniform bound of the solutions and their gradients in terms of \(g\), there exists a subsequence \(\varepsilon_{j_i} \searrow 0\), such that \(u^\varepsilon_{j_i}(\cdot, t_i) \to u(\cdot, t_i)\) in \(W^{1,p}_{loc}(\mathbb{R}^n)\), for \(i = 1, 2\). Therefore, letting \(\varepsilon_j \to 0\), and using lower semicontinuity in the left-hand side of the latter inequality, and Lebesgue dominated convergence theorem in the right-hand side (which we can use since we are integrating against a Gaussian measure on \(\mathbb{R}^n\)), we obtain
\[
(T - t_2)^{p/2} \int_K |Du(y, t_2)|^p G(x, y, T - t_2) dy \leq (T)^{p/2} \int_{\mathbb{R}^n} |Dg(y)|^p G(x, y, T) dy.
\]
Letting \(t_2 = t\) and \(K_j \nearrow \mathbb{R}^n\), we conclude that for every \(t \geq 0\) the following energy decay estimate holds
\[
(T - t)^{p/2} \int_{\mathbb{R}^n} |Du(y, t)|^p G(x, y, T - t) dy \leq (T)^{p/2} \int_{\mathbb{R}^n} |Dg(y)|^p G(x, y, T) dy.
\]
When \(g\) is Lipschitz continuous and compactly supported, let \(g_k\) denote the \(\varepsilon_k\) mollification, and let \(u_k\) be the corresponding solutions with initial datum \(g_k\). Then, by Theorem 8.4 and
the uniform Lipschitz bounds in Section 5, we have $u_k(\cdot, t) \to u(\cdot, t)$ uniformly and weakly in $W^{1,p}_\text{loc}(\mathbb{R}^n)$. Therefore, since the estimate (8.20) holds for each $u_k, g_k$, repeating the limiting arguments which have already been first used several times, we conclude that the energy estimate (8.20) continues to be valid for Lipschitz $g$.

At this point, using the crucial estimate (8.6), we can complete the proof of the monotonicity of the weighted energy by arguing as in the proof of Theorem 8.1. We leave the details to the reader.

For the case of the motion by mean curvature equation (1.3), the comparison with the function $V$ as in (8.6) does not work. However, we already know that the solutions obtained in [ES1] are constant outside a compact set. Thus, the intermediate step of multiplying them by the cutoff function $h_k$ is not required as above. This allows us to assert an energy decay monotonicity in the case $p = 1$. The calculations are justified by arguments similar to those presented above in the case $p > 1$, but using the bounds in [ES1], page 655. We omit the relevant details.

**Theorem 8.5** (Weighted monotonicity for $p = 1$). Let $u$ be the unique viscosity solution of (1.3) with an initial datum $g$ Lipschitz continuous and constant outside a compact set. For every $x \in \mathbb{R}^n$ and $T > 0$ the function

$$t \to E(t) = (T - t)^{1/2} \int_{\mathbb{R}^n} |Du(y,t)|G(x,y,T-t)dy$$

is nonincreasing on the interval $0 \leq t \leq T$.

Finally, we close this paper with a corollary of Theorem 8.2 which generalizes to the nonlinear singular equation (1.1) Struwe’s monotonicity formula for the case $p = 2$, see Lemma 3.2 in [S].

**Corollary 8.6.** Let $u$ be a viscosity solution as in Theorem 8.2. Then, the function

$$I(r) = E(T - r^2) = r^p \int_{\{t=T-r^2\}} |Du(y,t)|^pG(x,y,T-t)dy,$$

is nondecreasing for any $0 < r \leq \sqrt{T}$.

**Remark 8.7.** The energy estimates and monotonicity cannot be expected to hold for a solution of (1.1) without any growth assumption since, even for the heat equation, Tychonoff’s solution violates it. In our case, all solutions are bounded, as seen in the existence theorems.

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Department of Mathematics, Purdue University, West Lafayette, IN 47907
E-mail address, Agnid Banerjee: banerja@math.purdue.edu

Department of Mathematics, Purdue University, West Lafayette, IN 47907
E-mail address, Nicola Garofalo: garofalo@math.purdue.edu