Avoiding Haag’s Theorem with Parameterized Quantum Field Theory

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Under the normal assumptions of quantum field theory, Haag’s theorem states that any field unitarily equivalent to a free field must itself be a free field. Unfortunately, the derivation of the Dyson series perturbation expansion relies on the use of the interaction picture, in which the interacting field is unitarily equivalent to the free field but must still account for interactions. Thus, the traditional perturbative derivation of the scattering matrix in quantum field theory is mathematically ill defined. Nevertheless, perturbative quantum field theory is currently the only practical approach for addressing scattering for realistic interactions, and it has been spectacularly successful in making empirical predictions. This paper explains this success by showing that Haag’s Theorem can be avoided when quantum field theory is formulated using an invariant, fifth path parameter in addition to the usual four position parameters, such that the Dyson perturbation expansion for the scattering matrix can still be reproduced. As a result, the parameterized formalism provides a consistent foundation for the interpretation of quantum field theory as used in practice and, perhaps, for better dealing with other mathematical issues.

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I. INTRODUCTION

Haag’s Theorem states that, under the normal assumptions of quantum field theory (QFT), any field that is unitarily equivalent to a free field must itself be a free field. This is troublesome, because the usual Dyson perturbation expansion of the scattering matrix is based on the interaction picture, in which the interacting field is presumed to be related to the free field by a unitary transformation. And, as Streater and Wightman note, Haag’s theorem means that such a picture should not exist in the presence of actual interaction.

There has therefore been some consternation in the literature over the foundational implications of Haag’s Theorem (in addition to [1], see, for example, [4,5]). Nevertheless, currently popular textbooks (such as [5,8]) tend to simply ignore it. Indeed, the non-perturbative Lehmann-Symanzik-Zimmerman (LSZ) [9] and Haag-Ruelle [10,11] formalisms for scattering do not run afoul of Haag’s Theorem, so it is possible to formulate scattering theory for QFT rigorously.

Unfortunately, the LSZ and Haag-Ruelle formalisms are not useful for practical calculations of the S-matrix, for which perturbation theory is always used. So, the question remains as to why perturbation theory works so well for this, despite Haag’s Theorem (see [12] for a clear discussion of this point, and of Haag’s Theorem in general).

In this paper I will show that Haag’s Theorem does not apply to a parameterized formulation of QFT. Not surprisingly, this formulation starts from different assumptions than traditional QFT. A parameterized formalism is one in which field operators have a fifth, invariant parameter argument, in addition to the usual four position arguments of Minkowski space.

There is a long history of approaches using such a fifth parameter for relativistic quantum mechanics, going back to proposals of Fock [13] and, particularly, Stueckelberg [14,15], in the late thirties and early forties. The idea appeared subsequently in the work of a number of well-known authors, including Nambu [16], Feynman [17,18], Schwinger [19], DeWitt-Morette [20] and Cooke [21]. However, it was not until the seventies and eighties that the theory was more fully developed, particularly by Horwitz and Piron [22,23] and Fanchi and Collins [24,25], into what has come to be called relativistic dynamics.

A key feature of this approach is that time is treated comparably to the three space coordinates, rather than as an evolution parameter. The result is that relativistic quantum mechanics can be formulated in a way that is much more parallel to non-relativistic quantum theory. Further, the approach is particularly applicable to...
the study of quantum gravity and cosmology, in which the fundamental equations (such as the Wheeler-DeWitt equation) make no explicit distinction for the time coordinate (see, e.g., [28, 31]).

Extension of this approach to a second-quantized QFT has been somewhat more limited, focusing largely on application to quantum electrodynamics [22, 24]. The formalism I will use here is derived from related earlier work of mine on a foundational parameterized formalism for QFT and scattering [12, 43]. This formalism was developed previously in terms of spacetime paths, but, in the present work, it is presented entirely in field-theoretic mathematical language, without use of spacetime path integrals (though the concept of paths still remains helpful for intuitive motivation). The axiomatic approach used here is described in more detail in [46].

To show how Haag’s Theorem formally breaks down, I will step through an attempted proof of the equivalent of Haag’s Theorem for parameterized QFT, paralleling the proof of the theorem in traditional axiomatic QFT. In order to provide a context for this, Sec. II A gives a brief overview of the axioms of traditional QFT and Sec. II B describes free fields satisfying those axioms. Sec. II C outlines the formal proof of Haag’s Theorem in that theory.

Section III A then presents the parameterized QFT formalism axiomatically, and Sec. III B describes free fields satisfying those axioms. (The axiomatic formalism for parameterized QFT is only outlined here to the extent necessary for the purposes of this paper—for a fuller presentation see [46].) Section III C next shows the promised result, that Haag’s Theorem does not apply in the axiomatic parameterized formalism. This allows a straightforward interaction picture to be defined in Sec. III D in which interacting fields are related to free fields by a unitary transformation. Section IV gives some concluding thoughts.

In order to focus on the presentation of the key arguments in Sec. II C and Sec. III C, the proof of Haag’s Theorem, and its reconsideration, are just outlined semi-formally in these sections. However, the appendix includes formal statements of all theorems, along with their proofs (or references to standard proofs in the literature).

Throughout, I will use a spacetime metric signature of (- + + +) and take $\hbar = c = 1$.

II. AXIOMATIC QUANTUM FIELD THEORY

The first formal proof of Haag’s Theorem was based on axiomatic quantum field theory [1, 3]. This section summarizes the axiomatic basis for QFT necessary to prove the theorem. Section II C then outlines the proof, which will be subsequently reconsidered in the context of parameterized QFT.

A. Axioms

We begin with assumptions about the Hilbert space of states in traditional QFT, and then continue with axioms on the fields that operate on these states.

Axiom 0 (States). The states $|\psi\rangle$ are described by unit rays in a separable Hilbert space $\mathcal{H}$, such that the following are true.

1. (Transformation Law) Under a Poincaré transformation $\{\Delta x, \Lambda\}$ (where $\Delta x$ is a spacetime translation and $\Lambda$ is a Lorentz transformation), the states transform according to a continuous, unitary representation $\hat{U}(\Delta x, \Lambda)$ of the Poincaré group.

2. (Uniqueness of the Vacuum) There is a unique, invariant vacuum state $|0\rangle$ such that $\hat{U}(\Delta x, \Lambda)|0\rangle = |0\rangle$.

3. (Spectral Condition) Let

$$\hat{U}(\Delta x, 1) = e^{i\hat{P}_\mu \Delta x_\mu}.$$

Then $\hat{P}_\mu \hat{P}_\mu = -m^2$, where $m$ is interpreted as a mass, and the eigenvalues of $\hat{P}_\mu$ lie in or on the future light cone.

Note that the spectral condition given above captures the assumption that particles are on shell, meeting the mass-shell constraint, $p^2 = -m^2$, and that they advance into the future. In particular, time evolution is given by the (frame-dependent) Hamiltonian, $\hat{H} = \hat{P}^0$.

QFT is then the theory of quantum fields that operate on these states. Define a field operator $\hat{\psi}(x)$ that, for each spacetime position $x$, acts on the states of the theory. Actually, in order to avoid improper states and delta functions, one should instead use the so-called smeared fields $\hat{\psi}[f]$ such that

$$\hat{\psi}[f] = \int d^4x f(x)\hat{\psi}(x),$$

for any $f$ drawn from an allowable set of test functions. However, for simplicity, the presentation here will freely use point fields instead of the smeared fields, though the mathematics can still be interpreted in terms of a more rigorous use of test functions and distributions [1, 46]. Also, I only consider scalar particles here, which is sufficient for addressing Haag’s Theorem.

The fields $\hat{\psi}(x)$ satisfy the following axioms, commonly known as the Wightman axioms [1] (presented here for fields that are not self-adjoint).

Axiom I (Domain and Continuity of Fields). The field $\hat{\psi}(x)$ and its adjoint $\hat{\psi}^\dag(x)$ are defined on a domain $D$ of states dense in $\mathcal{H}$ containing the vacuum state $|0\rangle$. The $\hat{U}(\Delta x, \Lambda)$, $\hat{\psi}(x)$ and $\hat{\psi}^\dag(x)$ all carry vectors in $D$ into vectors in $D$. 

Axiom II (Field Transformation Law). For any Poincaré transformation \( \{\Delta x, \Lambda\} \),
\[
\hat{U}(\Delta x, \Lambda)\hat{\psi}(x)\hat{U}^{-1}(\Delta x, \Lambda) = \hat{\psi}(\Lambda x + \Delta x).
\] (2)

Axiom III (Local Commutivity). For any spacetime positions \( x \) and \( x' \),
\[
[\hat{\psi}(x), \hat{\psi}(x')] = [\hat{\psi}^\dagger(x), \hat{\psi}^\dagger(x')] = 0.
\]
Further, if \( x \) and \( x' \) are space-like separated,
\[
[\hat{\psi}(x), \hat{\psi}^\dagger(x')] = 0.
\]

Axiom IV (Cyclicity of the Vacuum). The vacuum state \(| 0 \rangle \) is cyclic for the fields \( \hat{\psi}(x) \), that is, polynomials in the fields and their adjoints, when applied to the vacuum state, yield a set \( D_0 \) dense in \( \mathcal{H} \).

B. Free Fields

There is at least one class of field theories that satisfy the Wightman axioms, those in which the fields are free, that is, they describe particles that do not interact. The Hilbert space of a free-field theory is thus a Fock space that is the direct sum of subspaces of states with a fixed number of particles. The fields themselves are solutions of the Klein-Gordon equation
\[
\left( -\frac{\partial^2}{\partial x^2} + m_n^2 \right) \hat{\psi}(x) = 0,
\]
where \( \partial^2/\partial x^2 \) denotes the four-dimensional relativistic d’Alembertian, with the two-point vacuum expectation values
\[
\langle 0 | \hat{\psi}(x) \hat{\psi}^\dagger(x_0) | 0 \rangle = \Delta^+(x - x_0),
\] (3)
where
\[
\Delta^+(x - x_0) \equiv (2\pi)^{-3} \int d^3p \frac{e^{i[-\omega_p(x^0 - x_0^0) + p \cdot (x - x_0)]}}{2\omega_p}
\]
and \( \omega_p \equiv \sqrt{p^2 + m^2} \).

The usual approach for introducing interactions is to use the interaction picture, relating the interacting fields \( \hat{\psi}_I(x) \) to the free fields by a transformation
\[
\hat{\psi}_I(t, x) = \hat{G}(t)\hat{\psi}(t, x)\hat{G}(t)^{-1},
\]
where the time-dependence of \( \hat{G} \) is intended to reflect the presence of interactions. Such a \( \hat{\psi}_I(x) \) can be constructed that satisfies the Wightman axioms (though its Lorentz covariance is certainly not manifest, due to the time dependence of \( \hat{G}(t) \)). Unfortunately, Haag’s Theorem shows that, under the Wightman axioms, any field unitarily related to a free field will itself be a free field, so an interacting theory cannot be consistently constructed in this way.

C. Haag’s Theorem

This section outlines the proof of Haag’s Theorem, which actually follows from the results of two other theorems. All three theorems are summarized below. Appendix A contains the formal statement of these theorems, but the summary given here presents the key points necessary for understanding Haag’s theorem.

1. Let \( \hat{\psi}_1 \) and \( \hat{\psi}_2 \) be two field operators, defined in respective Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), satisfying the Wightman axioms. Suppose there exists a unitary operator \( \hat{G} \) such that, at a specific time \( t \), \( \hat{\psi}_2(t, x) = \hat{G}\hat{\psi}_1(t, x)\hat{G}^{-1} \). Then the equal-time vacuum expectation values of the fields at time \( t \) are the same.

2. Any free field satisfies Eq. (3). In addition, conversely, if the two-point expectation value for a field satisfies Eq. (3), then the field is a free field.

3. (Haag’s Theorem) Let \( \hat{\psi}_1 \) be a free field, so that it satisfies Eq. (3), and let \( \hat{\psi}_2 \) be a field unitarily related to \( \hat{\psi}_1 \) at time \( t \), as above. Then \( \hat{\psi}_2 \) will also satisfy Eq. (3) if \( x \) and \( x_0 \) are both at time \( t \). However, the Lorentz-covariance of \( \hat{\psi}_2 \) allows the satisfaction of Eq. (3) to be extended to any two spacelike positions and then, by analytic continuation, to any two positions. Therefore, \( \hat{\psi}_2 \) is also a free field.

Because of Haag’s Theorem, a QFT of interacting particles that satisfies the Wightman axioms cannot be unitarily equivalent to the known theory of free particles. So far, such an interacting theory (for the case of four-dimensional spacetime) has yet to be found. Nevertheless, as previously noted, empirically well-supported predictions of the standard model are based on calculations using perturbative QFT, which assumes that the interacting fields are, in fact, unitarily related to the free fields, in contradiction to Haag’s Theorem.

This naturally leads one to consider whether there might be an alternative set of axioms for QFT, for which Haag’s Theorem would not apply, but which could reproduce the results of traditional perturbative QFT. In particular, note that the proof of Theorem 3 essentially relies on a conflict between the presumption that the fields are Lorentz-covariant and the special identification of time in the assumptions of Theorem 1. This provides a motivation for considering parameterized QFT, which dispenses with any identification of time as an evolution parameter for fields.
equivalent to a free field, and outline how such a field can be constructed using an interaction picture in the parameterized theory.

III. PARAMETERIZED QUANTUM FIELD THEORY

This section presents the basic axioms of parameterized QFT, generally paralleling the axioms for traditional QFT as presented in Sec. [II]. This is not intended to be a complete presentation of the parameterized theory, but, instead, to just outline a basic axiomatic foundation parallel to the overview given for traditional QFT in Sec. [II]. For a full presentation of axiomatic, parameterized QFT, see [10].

A. Axioms

We begin by stripping the assumptions on the Hilbert space of states down to the minimum necessary for a relativistic theory.

Axiom 0* (States). The states $|\psi\rangle$ are described by unit rays in a separable Hilbert space $\mathcal{H}$. Under a Poincaré transformation $\{\Delta x, \Lambda\}$ (where $\Delta x$ is a spacetime translation and $\Lambda$ is a Lorentz transformation), the states transform according to a continuous, unitary representation $\hat{U}(\Delta x, \Lambda)$ of the Poincaré group.

Note, in particular, that there is no equivalent here of the spectral condition. That is, we begin, for the parameterized theory, with states that are inherently off shell. Nevertheless, we can still define the Hermitian operator $\hat{P}$ such that

$$\hat{U}(\Delta x, 1) = e^{i\hat{P} \cdot \Delta x}$$

and interpret $\hat{P}$ as the relativistic energy-momentum operator, as usual. However, we will not require that the eigenvalues of $\hat{P}$ be on-shell.

Of course, $\hat{P}_\mu \hat{P}^\mu$ is still a Casimir operator of the Poincaré group and, so, acts on any irreducible representation as an invariant multiple of the identity. Thus, in the parameterized theory, we are not requiring that the $\hat{U}(\Delta x, \Lambda)$ be an irreducible representation for single-particle states. Position-representation states may essentially be superpositions of momentum states with squared-momentum $\hat{p}^2 = \mu$, for all $\mu$, positive and negative. This is sometimes considered explicitly as a superposition of different mass states [37, 47, 48] (see also the discussion at the end of Sec. IIA of [42]).

Moreover, we will no longer consider $\hat{P}_0$, the generator of time translation and the energy observable, to also be the Hamiltonian operator. Instead, we will allow any operator $\hat{H}$ to be considered a Hamiltonian operator if it has the following properties:

1. $\hat{H}$ is Hermitian.

2. $\hat{H}$ commutes with all spacetime transformations $\hat{U}(\Delta x, \Lambda)$ (and, hence, with $\hat{P}$).

3. $\hat{H}$ has an eigenstate $|0\rangle \in \mathcal{H}$ such that $\hat{H}|0\rangle = 0$, and this is the unique null (normalizable) eigenstate for $\hat{H}$.

Note that the concept of a vacuum state $|0\rangle$ has essentially been re-introduced here, but only relative to a choice of Hamiltonian operator. Each Hamiltonian operator must have a unique vacuum state, but different Hamiltonian operators defined on the same Hilbert space may have different vacuum states. On the other hand, the Hamiltonian operator is not required to be positive definite, since it is no longer considered to represent the energy observable, so this vacuum state is not a "ground state" in the traditional sense. However, when $\mathcal{H}$ is a Fock space with a particle interpretation, the vacuum state for an identified Hamiltonian will be the "no particle" state.

Given the choice of a Hamiltonian operator $\hat{H}$, we can consider the one-dimensional group generated by this operator. The frame-independent parameter $\lambda$ is the evolution parameter for this group. We can then define Schrödinger and Heisenberg pictures for evolution in $\lambda$, analogously to the definitions for time evolution.

That is, consider a state $|\psi\rangle \in \mathcal{H}$ and a Hamiltonian operator $\hat{H}$ defined on $\mathcal{H}$. The Schrödinger-picture evolution of the state is then given by

$$|\psi(\lambda)\rangle = e^{-i\hat{H}\lambda}|\psi\rangle.$$ 

Similarly, if $\hat{A}$ is an operator on $\mathcal{H}$, then its Heisenberg-picture evolution is given by

$$\hat{A}(\lambda) = e^{i\hat{H}\lambda} \hat{A} e^{-i\hat{H}\lambda}.$$ 

As mentioned in Sec. [II], $\lambda$ can be considered to parameterize the path of a particle in spacetime. Physically, then, a Hamiltonian operator $\hat{H}$ is the generator of evolution of a particle along its path. But note that there is no constraint that such a path have any fixed direction in time. In particular, the path is not constrained to be only a timelike trajectory, with $\lambda$ then being just the proper time for the particle. Such a restriction would be equivalent to re-introducing the mass-shell constraint, which would essentially return the theory to the traditional formulation.

Nevertheless, there is clearly an equivalence class of possible paths that a particle may take between fixed starting and ending points in spacetime, which need to be integrated over to obtain the complete amplitude for particle propagation from one point to another. And, after such integration, the path parameter should no longer appear in final, physical results. We will return to this point at the end of this section.

For now, though, we turn back to the field theoretic formalism. Fields in parameterized QFT are, in fact, defined in the same way as fields in traditional QFT, as
operators $\hat{\psi}(x)$, for each spacetime position $x$, that act on the states of the theory. The axioms are also similar to those of traditional QFT. However, they are altered to account for the lack of spectral constraints and the different conception of the Hamiltonian.

**Axiom I** (Domain and Continuity of Fields). The field $\hat{\psi}(x)$ and its adjoint $\hat{\psi}^\dagger(x)$ are defined on a domain $D$ of states dense in $\mathcal{H}$. The $\hat{U}(\Delta x, \Lambda)$, any Hamiltonian $\hat{H}$, $\hat{\psi}(x)$ and $\hat{\psi}^\dagger(x)$ all carry vectors in $D$ into vectors in $D$.

**Axiom II** (Field Transformation Law). For any Poincaré transformation $\{\Delta x, \Lambda\}$,

$$\hat{U}(\Delta x, \Lambda)\hat{\psi}(x)\hat{U}^{-1}(\Delta x, \Lambda) = \hat{\psi}(\Lambda x + \Delta x).$$

**Axiom III** (Commutation Relations). For any spacetime positions $x$ and $x'$,

$$[\hat{\psi}(x'), \hat{\psi}(x)] = [\hat{\psi}^\dagger(x'), \hat{\psi}^\dagger(x)] = 0$$

and

$$[\hat{\psi}(x'), \hat{\psi}^\dagger(x)] = \delta^4(x' - x).$$

**Axiom IV** (Cyclicity of the Vacuum). If $\hat{H}$ is a Hamiltonian operator, then its vacuum state $|0\rangle$ is in the domain $D$ of the field operator, and polynomials in the field $\hat{\psi}(x)$ and its adjoint $\hat{\psi}^\dagger(x)$, when applied to $|0\rangle$, yield a set $D_0$ of states dense in $\mathcal{H}$.

These axioms are presented in terms of the essentially Schrödinger-picture operators $\hat{\psi}(x)$. However, for a Hamiltonian $\hat{H}$, we can now also define the (parameterized) Heisenberg-picture form of a field operator:

$$\hat{\psi}(x; \lambda) = e^{i\hat{H}\lambda} \hat{\psi}(x) e^{-i\hat{H}\lambda}.$$

Taking the limit of infinitesimal $\Delta \lambda$ then gives the dynamic evolution of the field as

$$i \frac{\partial}{\partial \lambda} \hat{\psi}(x; \lambda) = [\hat{\psi}(x; \lambda), \hat{H}].$$

**B. Free Fields**

As for traditional QFT, we can define a parameterized field theory of free particles that satisfies the above axioms. And, again, the Hilbert space of such a theory is a Fock space of states of a fixed number of non-interacting particles. Instead of the Klein-Gordon equation, however, for a parameterized free field, we require that $\hat{H}$ be chosen such that

$$[\hat{\psi}(x; \lambda), \hat{H}] = \left(-\frac{\partial^2}{\partial x^2} + m^2\right) \hat{\psi}(x; \lambda),$$

where $m$ is the particle mass. Together with Eq. (11), this gives the Stückelberg-Schrödinger field equation [44, 42]

$$i \frac{\partial}{\partial \lambda} \hat{\psi}(x; \lambda) = \left(-\frac{\partial^2}{\partial x^2} + m^2\right) \hat{\psi}(x; \lambda).$$

We can also define the momentum field $\hat{\psi}(p; \lambda)$ as the four-dimensional Fourier transform of $\hat{\psi}(x; \lambda)$. Like $\hat{\psi}(x; \lambda)$, $\hat{\psi}(p; \lambda)$ evolves in $\lambda$ using $\hat{H}$. Taking the Fourier transform of Eq. (6) gives the field equation

$$i \frac{\partial}{\partial \lambda} \hat{\psi}(p; \lambda) = (p^2 + m^2) \hat{\psi}(p; \lambda).$$

Recall that the four-momentum here is not necessarily on-shell. Since $p^2$ is relative to the Minkowski metric, $p^2 + m^2$ can take on any value, positive, negative or zero.

Equation (7) may be integrated to get

$$\hat{\psi}(p; \lambda) = \Delta(p; \lambda - \lambda_0) \hat{\psi}(p; \lambda_0),$$

where

$$\Delta(p; \lambda - \lambda_0) = e^{-i(p^2 + m^2)(\lambda - \lambda_0)}.$$

Taking the inverse Fourier transform of the momentum field in Eq. (8) then gives

$$\hat{\psi}(x; \lambda) = \int \frac{d^4x_0}{(2\pi)^4} \Delta(x - x_0; \lambda - \lambda_0) \hat{\psi}(x_0; \lambda_0),$$

where

$$\Delta(x - x_0; \lambda - \lambda_0) \equiv (2\pi)^{-4} \int d^4p \ e^{ip(x - x_0)} \Delta(p; \lambda - \lambda_0).$$

The two-point vacuum expectation value of the parameterized free field is given by

$$\langle 0 | \hat{\psi}(x; \lambda) \hat{\psi}^\dagger(x_0; \lambda_0) | 0 \rangle = \Delta(x - x_0; \lambda - \lambda_0).$$

This represents the propagation of a particle from four-position $x_0$ to four-position $x$ over any path $q$ with $q(\lambda_0) = x_0$ and $q(\lambda) = x$.

As mentioned earlier, paths in general are invariant under continuous transformation of the path parameterization. This is a well-known gauge invariance of spacetime path formalisms. After integrating out the gauge volume of all possible functional re-parameterizations, the only gauge variance that is left is the so-called intrinsic length of the path, $|\lambda - \lambda_0|$. (For details see [28, 42].)

To represent propagation over all possible paths, it thus remains necessary to integrate over all intrinsic lengths. This may be implemented by integrating Eq. (11) over all $\lambda > \lambda_0$, which gives

$$\int_{\lambda_0}^{\lambda} d\lambda \ \langle 0 | \hat{\psi}(x; \lambda) \hat{\psi}^\dagger(x_0; \lambda_0) | 0 \rangle = \Delta(x - x_0),$$

where

$$\Delta(x - x_0) \equiv \int_{\lambda_0}^{\lambda} d\lambda \Delta(x - x_0; \lambda - \lambda_0)$$

$$= \int_0^\infty d\lambda \Delta(x - x_0; \lambda)$$

$$= -i(2\pi)^{-4} \int d^4p \ e^{ip(x - x_0)} \frac{1}{p^2 + m^2 - i\epsilon}.$$
is just the Feynman propagator \[ \Delta \]. Note that the left-hand side of Eq. (12) still involves the arbitrary parameter value \( \lambda_0 \), even though the resulting propagator on the right-hand side does not depend on it. This simply reflects the remaining global freedom to shift together the starting parameter values of all paths.

As shown again by Eq. (12), particles are considered to be fundamentally off shell in the parameterized formalism. Nevertheless, it is always possible to divide the Feynman propagator into future-directed and past-directed parts (see, e.g., [7] or [8]):

\[
\Delta(x-x_0) = \theta(x^0-x^0_0)\Delta^+(x-x_0) + \theta(x^0_0-x^0)\Delta^+(x-x_0)^*,
\]

where \( \Delta^+(x-x_0) \) is the on-shell propagator from Eq. (5).

Thus, in the special case of a particle that is unambiguously future-directed (i.e., \( x^0 > x^0_0 \)), \( \Delta(x-x_0) = \Delta^+(x-x_0) \), and the particle can be considered to be on shell. This is essentially just the reverse of the usual derivation in traditional QFT, in which “physical” particles are considered to be fundamentally on shell, but in which future and past-directed on-shell propagators are combined to form the Feynman propagator for off-shell “virtual” particles.

C. Haag’s Theorem Reconsidered

This section reconsiders Haag’s Theorem in the context of parameterized QFT. The conclusion is that Haag’s Theorem does not, in fact, hold under the axioms given for parameterized QFT in Sec. IIIA. To see this, consider carrying out a proof of Haag’s Theorem for parameterized fields similar to the one outlined in Sec. IIC for traditional fields. As in Sec. IIC, this argument is summarized below, with the formalization given in App. A2.

1*. Let \( \hat{\psi}_1 \) and \( \hat{\psi}_2 \) be two off-shell field operators, defined in respective Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), satisfying the axioms of parameterized QFT. Suppose there exists a unitary operator \( \hat{G} \) such that \( \psi_2(x) = \hat{G}\hat{\psi}_1(x)\hat{G}^{-1} \). Then the equal-\( \lambda \) vacuum expectation values of the fields are the same.

2*. Any free field satisfies Eq. (11). In addition, conversely, if the two-point expectation value for a field satisfies Eq. (11), then the field is a free field.

3*. (Haag’s Theorem?) Let \( \hat{\psi}_1 \) be a free field, so that it satisfies Eq. (11), and let \( \hat{\psi}_2 \) be a field unitarily related to \( \hat{\psi}_1 \), as above. Then \( \hat{\psi}_2 \) will also satisfy Eq. (11) for \( \lambda = \lambda_0 \). But now, if we try to generalize this to unequal parameter values \( \lambda \) and \( \lambda_0 \), there is no equivalent of the Lorentz transformation to use in order to bring the parameter values back to equality. All that is available is parameter translation, which would maintain the difference \( \lambda - \lambda_0 \). Thus, Theorem [14] does not apply and the proof of Haag’s theorem does not go through.

Theorem [14] here can be considered a kind of inverse to the well-known Wightman reconstruction theorem (see [1], Theorem 3-7). A consequence of the reconstruction theorem is that any fields over spacetime that have the same vacuum expectation values will be unitarily equivalent. Theorem [14] shows that fields that are so unitarily related will (not surprisingly) have equal vacuum expectation values. (For parameterized fields, we do not require the expectation values to meet a spectral condition or for the vacuum state to be absolutely unique, but the proof of the reconstruction theorem otherwise goes through, for a QFT that also does not meet such conditions [16].)

However, the key point here is that the assumptions of Theorem [14] and the resulting expectation values are all for parameterized fields at equal values of \( \lambda = \lambda_0 \). If a similar condition of equal expectation values also held for different values of \( \lambda \) for the two fields, then the unitary operator \( \hat{G} \) relating the fields would have to be independent of \( \lambda \). In this case, it is clear that, if one field satisfies the free field equation Eq. (6), then the other will, too.

But the fact that Theorem [14] only requires equality of expectation values for equal values of \( \lambda \) means that the fields can be related by a different value of \( \hat{G}(\lambda) \) for each value of \( \lambda \). Theorem [3] Haag’s Theorem, shows that the requirement of Lorentz-covariance of the fields prevents a similar construction in traditional QFT in which \( \hat{G} \) depends on time (as required by the traditional interaction picture). In parameterized QFT, however, the translation group of \( \lambda \) does not mix with the group of Lorentz transformations, so dependence of \( \hat{G} \) on \( \lambda \) is not a problem.

D. Interacting Fields

In the absence of Haag’s Theorem, we are free to use an interaction picture representation for developing an interacting parameterized QFT, which is what will be done in this section. Begin by considering a free field \( \hat{\psi}(x) \) defined on a Hilbert space \( \mathcal{H} \) with a free Hamiltonian operator \( \hat{H} \) and vacuum state \( |0\rangle \). Now consider a unitary transformation \( \hat{G} \) of \( \mathcal{H} \) onto itself. This then induces a transformed field \( \hat{\psi}'(x) = \hat{G}\hat{\psi}(x)\hat{G}^{-1} \) with a Hamiltonian \( \hat{H}' = \hat{G}\hat{H}\hat{G}^{-1} \) and a vacuum state \( |0'\rangle = \hat{G}|0\rangle \).

Now, in this case,

\[
\hat{\psi}'(x; \lambda) = e^{iH'\lambda}\hat{\psi}(x)e^{-iH'\lambda} = e^{iH'\lambda}\hat{G}e^{-iH\lambda}\hat{\psi}(x; \lambda)e^{iH\lambda}\hat{G}e^{-iH'\lambda} = \hat{G}\hat{\psi}(x; \lambda)\hat{G}^{-1},
\]

since \( \hat{H}'\hat{G} = \hat{G}\hat{H} \). So, the Heisenberg-picture forms of the fields are also unitarily related by \( \hat{G} \), which means that \( \hat{\psi}'(x; \lambda) \) is also a free field under the Hamiltonian \( \hat{H}' \).
However, since $\hat{\psi}'$ is defined on the same Hilbert space $\mathcal{H}$ as $\hat{\psi}$, it can be evolved equally well using $\hat{H}$ rather than $\hat{H}'$. Doing so results in the interaction-picture form of $\hat{\psi}'$: 

$$\hat{\psi}'(x; \lambda) = e^{iH\lambda} \hat{\psi}'(x)e^{-iH\lambda} = \hat{G}(\lambda)\hat{\psi}(x; \lambda)\hat{G}^{-1}(\lambda),$$

where 

$$\hat{G}(\lambda) = e^{iH\lambda} Ge^{-iH\lambda}.$$ 

So the interaction-picture field $\hat{\psi}'(x; \lambda)$ is still unitarily related to the Heisenberg-picture free field $\hat{\psi}(x; \lambda)$, but now by a transformation $\hat{G}(\lambda)$ that is not constant in $\lambda$ (presuming that $\hat{G}$ and $\hat{H}$ do not commute).

The result is that the field equation for $\hat{\psi}'(x; \lambda)$ is no longer that of a free field. Consider that 

$$\frac{\partial}{\partial \lambda} \hat{\psi}'(x; \lambda) = \frac{\partial \hat{G}(\lambda)}{\partial \lambda} \hat{\psi}(x; \lambda)\hat{G}^{-1}(\lambda)$$

$$+ \hat{G}(\lambda) \frac{\partial}{\partial \lambda} \hat{\psi}(x; \lambda)\hat{G}^{-1}(\lambda)$$

$$- \hat{G}(\lambda) \hat{\psi}(x; \lambda) \frac{\partial \hat{G}(\lambda)}{\partial \lambda} \hat{G}^{-1}(\lambda)$$

$$= \frac{\partial \hat{G}(\lambda)}{\partial \lambda} \hat{G}^{-1}(\lambda) \hat{\psi}'(x; \lambda) + i\hat{G}(\lambda)\hat{H}\hat{\psi}(x; \lambda)\hat{G}^{-1}(\lambda)$$

$$- \hat{\psi}'(x; \lambda) \frac{\partial \hat{G}(\lambda)}{\partial \lambda} \hat{G}^{-1}(\lambda)$$

$$= i\hat{H}'(\lambda) \hat{\psi}(x; \lambda)$$

$$+ \left[i\frac{\partial \hat{G}(\lambda)}{\partial \lambda} \hat{G}^{-1}(\lambda), \hat{\psi}(x; \lambda)\right],$$

where 

$$\hat{H}'(\lambda) = e^{iH\lambda} \hat{H} e^{-iH\lambda} = \hat{G}(\lambda)\hat{H}\hat{G}^{-1}(\lambda).$$ 

But, since $\hat{\psi}'(x; \lambda)$ evolves according to $\hat{H}$, 

$$\frac{\partial}{\partial \lambda} \hat{\psi}'(x; \lambda) = i[\hat{H}, \hat{\psi}'(x; \lambda)],$$

so we can take 

$$\hat{H}'(\lambda) = \hat{H} + \Delta \hat{H}(\lambda),$$

were 

$$\Delta \hat{H}(\lambda) = i\frac{\partial \hat{G}(\lambda)}{\partial \lambda} \hat{G}^{-1}(\lambda).$$ 

Now, as a free field, $\hat{\psi}(x; \lambda)$ satisfies the commutation condition with $\hat{H}$ given in Eq. (13). However, since $\hat{G}(\lambda)$ does not depend on $x$, this also implies that 

$$[\hat{\psi}'(x; \lambda), \hat{H}'(\lambda)] = \left(-\frac{\partial^2}{\partial x^2} + m^2\right)\hat{\psi}'(x; \lambda).$$

Substituting this into Eq. (13) then gives 

$$i\frac{\partial}{\partial \lambda} \hat{\psi}'(x; \lambda) = \left(-\frac{\partial^2}{\partial x^2} + m^2\right)\hat{\psi}'(x; \lambda)$$

$$+ [\Delta \hat{H}(\lambda), \hat{\psi}'(x; \lambda)].$$ 

Thus, the field equation for $\hat{\psi}'(x; \lambda)$ has the form of that for an interacting field, with the interaction Hamiltonian $\Delta \hat{H}(\lambda)$.

Note also that the $\lambda$-dependent transformation $\hat{G}(\lambda)$ essentially induces a different effective vacuum state 

$$|0'; \lambda\rangle = \hat{G}(\lambda)|0\rangle = e^{iH\lambda}|0\rangle$$

for each value of $\lambda$, such that $\hat{H}'(\lambda)|0'; \lambda\rangle = 0$. However, consider the construction of multi-particle basis states from the interaction-picture field adjoints, all at a fixed value of $\lambda = \lambda_0$:

$$|x_1, \ldots, x_N; \lambda_0\rangle_I = \hat{\psi}'(x_1; \lambda_0)\cdots \hat{\psi}'(x_N; \lambda_0)|0'; \lambda_0\rangle_I$$

$$= \hat{G}(\lambda_0)\hat{\psi}^\dagger(x_1; \lambda_0)\cdots \hat{\psi}^\dagger(x_N; \lambda_0)|0\rangle$$

$$= \hat{G}(\lambda_0)|x_1, \ldots, x_N; \lambda_0\rangle.$$ 

Further, since $|x_1, \ldots, x_N; \lambda_0\rangle \in \mathcal{H}$, it can be expanded in the free-particle basis states for any $\lambda$, such that 

$$|x_1, \ldots, x_N; \lambda_0\rangle_I = \sum_{N'=0}^{\infty} \left(\prod_{i=1}^N \int d^4x_i\right) |x'_1, \lambda_1; \ldots; x'_{N'}, \lambda_{N'}\rangle$$

$$\langle x'_1, \lambda_1; \ldots; x'_{N'}, \lambda_{N'}; |\hat{G}(\lambda_0)|x_1, \ldots, x_N; \lambda_0\rangle,$$

where 

$$|x'_1, \lambda_1; \ldots; x'_{N'}, \lambda_{N'}\rangle = \hat{\psi}^\dagger(x_1; \lambda_1)\cdots \hat{\psi}^\dagger(x_N; \lambda_{N'})|0\rangle.$$ 

The matrix elements 

$$G_{\lambda_0; \lambda_1, \ldots, \lambda_{N'}}(x'_1, \ldots, x'_{N'}; x_1, \ldots, x_N)$$

then represent the probability amplitude for $N$ particles at spacetime positions $x_1, \ldots, x_N$ at the starting path parameter $\lambda_0$ to result, under interaction, in $N'$ particles at spacetime positions $x'_1, \ldots, x'_{N'}$ at the ending path parameter values $\lambda_1, \ldots, \lambda_{N'}$.

Now suppose that the interaction represented by $\hat{G}(\lambda)$ is restricted to a limited region of spacetime, and that the $x_i$ and $x'_i$ are all outside this region. Then $G_{\lambda_0; \lambda_1, \ldots, \lambda_{N'}}(x'_1, \ldots, x'_{N'}; x_1, \ldots, x_N)$ is the probability amplitude for $N$ particles to propagate into the given region, interact and result in $N'$ particles propagating out of the region. Note that each of the outgoing particles is allowed a different ending parameter value. This is important, because, as discussed at the end of Sec. [III] it
is necessary to integrate over the ending parameter values to get the correct full propagation factors for each particle:

$$G_{\lambda_0}(x'_1, \ldots, x'_{N'}; x_1, \ldots, x_N) = \left( \prod_{i=1}^{N'} \int_{\lambda_0}^\infty d\lambda_i \right) G_{\lambda_0, \lambda_1, \ldots, \lambda_N'}(x'_1, \ldots, x'_{N'}; x_1, \ldots, x_N).$$

(14)

This, then, is just the probability amplitude for scattering from $x_1, \ldots, x_N$ to $x'_1, \ldots, x'_{N'}$. Now consider the limiting case in which the $x_1, \ldots, x_N$ are in the infinite past and the $x'_1, \ldots, x'_{N'}$ in the infinite future. In this case, the initial propagation of a particle from one of the $x_1, \ldots, x_N$ will always be future-directed, as will the final propagation of a particle to one of the $x'_1, \ldots, x'_{N'}$. Therefore, per the comment at the end of Sec. III B, these propagations will be on shell, as required for the incoming and outgoing external legs of the Feynman diagram for an interaction.

Finally, take

$$\hat{G}(\lambda) = e^{-i\hat{V}(\lambda)},$$

where $\hat{V}(\lambda)$ is an (appropriately integrated) product of field operators and their adjoints, representing an individual interaction vertex. Expanding $\hat{G}(\lambda)$ in a Taylor series then gives a sum of Feynman diagrams, with vertices generated by $\hat{V}(\lambda)$ and Feynman propagators on internal edges. Indeed, as shown in detail in [46], the expression in Eq. (14) can be expanded to exactly duplicate, term for term, the Dyson series for the scattering operator, as derived using perturbation theory in traditional QFT.

But now the derivation from parameterized QFT does not suffer from the mathematical inconsistency resulting from Haag’s theorem.

IV. CONCLUSION

The original argument by Haag for what has become known as Haag’s Theorem was based on the requirement that the vacuum state $|0\rangle$ be the unique state invariant relative to Euclidean transformations [2]. For the free theory, $|0\rangle$ is a null eigenstate of the free Hamiltonian $\hat{H}$. Haag observed that the effective vacuum state $|0'\rangle$ of the interacting field should also be invariant under Euclidean transformations, which, under the assumption of the uniqueness of $|0\rangle$ implies that $|0'\rangle$ equals $|0\rangle$ up to a phase. And, since $|0'\rangle$ is a null eigenstate of the interacting Hamiltonian $\hat{H}'$, it follows that $|0\rangle$ is, too. However, interaction terms in $\hat{H}'$ generally include the interaction of the field with itself, such that $\hat{H}'$ does not annihilate $|0\rangle$ (it “polarizes the vacuum”). This is then a contradiction.

A similar argument could be made in the case of a parameterized theory, if the assumption is made that the vacuum state $|0\rangle$ is the unique state that is invariant under Poincaré transformations. Then, since $\hat{G}(\lambda)$ must transform as a Lorentz-invariant scalar not dependent on position, the interacting vacuum $|0'; \lambda\rangle = \hat{G}(\lambda)|0\rangle$ would also be Poincaré invariant, and, thus, equal to $|0\rangle$ up to a phase. That means that $|0'; \lambda\rangle$ would actually be independent of $\lambda$, which implies that $\hat{G}(\lambda)$ would commute with the free Hamiltonian $\hat{H}$ and $\hat{V}_f'$ would be a free field.

In the parameterized formalism presented here, however, the free vacuum $|0\rangle$ is not required to be the unique Poincaré-invariant state. It only needs to be the unique (normalizable) null eigenstate of the free Hamiltonian. Unlike $|0\rangle$, the parameterized interacting vacuum $|0'; \lambda\rangle$ depends non-trivially on $\lambda$. It is the null eigenstate of the effective Hamiltonian $\hat{H}'(\lambda)$ (which is itself dependent on $\lambda$), not $\hat{H}$, and can exist without compromising the uniqueness of $|0\rangle$.

Put another way, the vacuum $|0\rangle$ is symmetric relative to translations in $\lambda$. This symmetry is preserved in the interacting theory in the sense that the physics is not affected by a translation in $\lambda$, and, in this sense, all $|0'; \lambda\rangle$ are equivalent. However, fixing on a specific $\lambda = \lambda_0$ breaks the underlying symmetry for the interacting theory, choosing a specific $|0'; \lambda_0\rangle$, distinct from $|0\rangle$, as the effective vacuum state for constructing interacting particle states. In this sense, the additional degree of freedom in the parameterized formalism provides the possibility for the interacting vacuum to be different from the non-interacting vacuum.

Thus, in parameterized QFT, it is possible for an interacting field to be unitarily related to the corresponding free field. And, as shown in [46], it is possible to choose this transformation so that the traditional Dyson perturbation expansion for scattering amplitudes can be reproduced term by term in the new formalism. This explains why such an expansion works, despite Haag’s Theorem—the result in traditional QFT was essentially correct, only the derivation was lacking.

Further, as argued in [49], different formulations of QFT may lead to different interpretations, even while being empirically equivalent. Clearly, one would like to base any interpretation on a formulation that is rigorously defined mathematically. But this is problematic for traditional canonical quantum field theory, since models of realistic interactions using the canonical formulation run afoul of Haag’s Theorem.

The parameterized formulation presented here resolves this problem. Further, by allowing the Fock representation of a free field to be extended to the corresponding interacting field, this approach allows the intuitive particle interpretation of the free theory to be carried over to the interacting theory. Indeed, it can also provide for a fuller interpretation in terms of spacetime paths, decoherence and consistent histories over spacetime [43, 45].

Of course, this does not resolve all the mathematical issues with traditional QFT, such as those involved in renormalization. And the approach still needs to be extended to cover gauge field theories and non-Abelian in-
teractions. It is also worth noting that the parameterized interacting vacuum states have superficial similarity to the theta vacua that result from instantons in Yang-Mills theory (see, for example, §25), which also have a $U(1)$ group symmetry. Whether there is a deeper connection is a subject for further investigation.

But, in any case, addressing the problem of Haag’s Theorem is a step toward building a firmer foundation, both mathematically and interpretationally, for QFT in general.

Appendix A: Formalization

This appendix presents the formal statements of Haag’s Theorem and related theorems, as considered in Sec. [II.C] and reconsidered in Sec. [III.C]

1. Haag’s Theorem

As noted in Sec. [II.C] Haag’s Theorem follows from two other theorems, which I present here using notation consistent with that of this paper, but without proof. For details on the proofs, see [1].

Theorem 1. Let $\hat{\psi}_1(x)$ and $\hat{\psi}_2(x)$ be two field operators defined in respective Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Suppose there are continuous, unitary representations $U_i(\Delta x, R)$ of the inhomogeneous Euclidean group of translations $\Delta x$ and three-dimensional rotations $R$, defined on each $\mathcal{H}_i$, for $i = 1, 2$, such that, for a specific time $t$,

$$U_i(\Delta x, R)\hat{\psi}_i(t, x)U_i^{-1}(\Delta x, R) = \hat{\psi}_i(t, Rx + \Delta x).$$

Suppose the representations possess unique invariant states $|0\rangle_i$ such that $U_i(\Delta x, R)|0\rangle_i = |0\rangle_i$.

Suppose, finally, that there exists a unitary operator $\hat{G}$ such that, at time $t$,

$$\hat{\psi}_2(t, x) = \hat{G}\hat{\psi}_1(t, x)\hat{G}^{-1}.$$  

Then

$$U_2(\Delta x, R) = \hat{G}U_1(\Delta x, R)\hat{G}^{-1}$$

and

$$c|0\rangle_2 = \hat{G}|0\rangle_1,$$  

where $c$ is a complex number of modulus one.

This theorem immediately implies the following corollary.

Corollary 1. In any two theories satisfying the hypotheses of Theorem [1], the equal-time vacuum expectation values are the same:

$$1 \langle 0 | \hat{\psi}_1(t, x_1') \cdots \hat{\psi}_1(t, x_N') \hat{\psi}_1^\dagger(t, x_1) \cdots \hat{\psi}_1^\dagger(t, x_N) | 0 \rangle_1 = 2 \langle 0 | \hat{\psi}_2(t, x_1') \cdots \hat{\psi}_2(t, x_N') \hat{\psi}_2^\dagger(t, x_1) \cdots \hat{\psi}_2^\dagger(t, x_N) | 0 \rangle_2.$$  

The second theorem is a general result originally from [51].

Theorem 2. If $\hat{\psi}(x)$ is a field for which the vacuum is cyclic, and if

$$\langle 0 | \hat{\psi}(x)\hat{\psi}^\dagger(x_0) | 0 \rangle = \Delta^+(x - x_0),$$

then $\hat{\psi}(x)$ is a free field.

Using these theorems, we can prove Haag’s Theorem.

Theorem 3 (Haag’s Theorem). Suppose that $\hat{\psi}_1(x)$ is a free field and $\hat{\psi}_2(x)$ is a local, Lorentz-covariant field. Suppose further that the fields $\hat{\psi}_1(x)$, $\hat{\psi}_2(x)$, $\hat{\psi}_3(x)$ and $\hat{\psi}_4(x)$ satisfy the hypotheses of Theorem [1]. Then $\hat{\psi}_2(x)$ is also a free field.

Proof. Since $\hat{\psi}_1(x)$ is a free field, its vacuum expectation value is given by Eq. [51]. The corollary to Theorem [1] then implies that, at a specific time $t_0$,

$$2 \langle 0 | \hat{\psi}_2(t_0, x')\hat{\psi}_3^\dagger(t_0, x_0) | 0 \rangle = \Delta^+(0, x - x_0).$$ (A1)

Any two position vectors $(t, x)$ and $(t_0, x_0)$ can be brought into the equal time plane $t = t_0$ by a Lorentz transformation, if their separation is spacelike. Along with the given covariance of $\hat{\psi}_2(x)$, this means that Eq. (A1) can be extended to any two spacelike positions and then, by analytic continuation, to any two positions:

$$2 \langle 0 | \hat{\psi}_2(x)\hat{\psi}_3^\dagger(x_0) | 0 \rangle = \Delta^+(x - x_0).$$

Haag’s Theorem is then an immediate consequence of this and Theorem [2].

2. Haag’s Theorem Reconsidered

First, note that Theorems [1] and [2] may be directly adapted for parameterized fields. The axioms of the parameterized theory, though, do not require a unique vacuum state, so we must simply assume the relationship between vacuum states of the two fields in Theorem [1] below. This simplifies the derivation of the relevant conclusion of equality of expectation values.

Theorem 1*. Let $\hat{\psi}_1(x)$ and $\hat{\psi}_2(x)$ be two (off-shell, Schrödinger-picture) field operators, defined in respective Hilbert spaces, and $\hat{H}_1$ and $\hat{H}_2$ be Hamiltonian operators defined on those Hilbert spaces, with vacuum states $|0\rangle_1$ and $|0\rangle_2$ respectively.
and $|0\rangle_2$. Suppose, that there exists a unitary operator $\hat{G}$, such that

$$\hat{\psi}_2(x) = \hat{G}\hat{\psi}_1(x)\hat{G}^{-1}$$

and

$$|0\rangle_2 = \hat{G}|0\rangle_1.$$ 

Then the equal-$\lambda$ vacuum expectation values of the fields are the same:

$$1\langle 0|\hat{\psi}_1(x'_1; \lambda)\cdot\cdot\cdot\hat{\psi}_1(x'_N; \lambda)\hat{\psi}_1(x_1; \lambda)\cdot\cdot\cdot\hat{\psi}_1(x_N; \lambda)|0\rangle_1 = 2\langle 0|\hat{\psi}_2(x'_1; \lambda)\cdot\cdot\cdot\hat{\psi}_2(x'_N; \lambda)\hat{\psi}_2(x_1; \lambda)\cdot\cdot\cdot\hat{\psi}_2(x_N; \lambda)|0\rangle_2.$$ 

Proof. With the given assumptions, the expectation values are clearly equal for the Schrödinger-picture fields. The parameterized Heisenberg-picture fields are related to the Schrödinger picture fields by the unitary transformations $\exp(-i\hat{H}_\lambda)$, and the vacuum states $|0\rangle_i$ are invariant under the corresponding such transformations. Therefore, the vacuum expectation values of the parameterized fields are equal to the corresponding vacuum expectation values for the respective unparameterized fields and, hence, to each other, for all $\lambda$.

Theorem 2*. If $\hat{\psi}(x; \lambda)$ is a field with a Hamiltonian $\hat{H}$ having vacuum state $|0\rangle$, and if

$$\langle 0|\hat{\psi}(x; \lambda)\hat{\psi}(x_0; \lambda)|0\rangle = \Delta(x - x_0; \lambda - \lambda_0)$$

where $\Delta(x - x_0; \lambda - \lambda_0)$ is given by Eq. [9], then $\hat{\psi}(x; \lambda)$ is a free field.

Proof. Let $\langle x; \lambda \rangle = \langle 0|\hat{\psi}(x; \lambda)$. Then,

$$\langle x; \lambda \rangle = \int d^4x_0 \langle x; \lambda|x_0; \lambda_0\rangle\langle x_0; \lambda_0\rangle$$

$$= \int d^4x_0 \Delta(x - x_0; \lambda - \lambda_0)\langle x_0; \lambda_0\rangle.$$ 

Therefore,

$$i\frac{\partial}{\partial \lambda}\langle x; \lambda \rangle = \left(-\frac{\partial^2}{\partial x^2} + m^2\right)\langle x; \lambda \rangle.$$ 

However,

$$i\frac{\partial}{\partial \lambda}\langle x; \lambda \rangle = i\frac{\partial}{\partial \lambda}\langle 0|\hat{\psi}(x; \lambda) = \langle 0|\hat{\psi}(x; \lambda), \hat{H}\rangle$$

$$= \langle 0|\hat{\psi}(x; \lambda)\hat{H} = \langle x; \lambda|\hat{H},$$

since $\langle 0|\hat{H} = 0$. Thus, the Hamiltonian $\hat{H}$ acts as

$$\hat{H} = -\frac{\partial^2}{\partial x^2} + m^2,$$ 

which is the position representation of the required free-field Hamiltonian (see Sec. III). Further, the equal-$\lambda$ expectation value for $\hat{\psi}$ is

$$\langle 0|\hat{\psi}(x; \lambda)\hat{\psi}(x_0; \lambda)|0\rangle = \Delta(x - x_0; 0) = \delta^4(x - x_0).$$

This, together with Axiom III, implies that

$$\langle 0|\hat{\psi}(x; \lambda)\hat{\psi}(x_0; \lambda)|0\rangle = 0,$$

for all $x$ and $x_0$. In particular, in the limit of $x_0 \to x$,

$$\langle 0|\hat{\psi}(x; \lambda)\hat{\psi}(x; \lambda)|0\rangle = |\hat{\psi}(x; \lambda)|^2 = 0,$$

so $\hat{\psi}(x; \lambda)|0\rangle = 0$, for all $x$ and $\lambda$. This and the commutation relations of Axiom III indicate that $\hat{\psi}(x)$ and its adjoint can be used to build the states of a free-particle Fock space. 

Now, even given Theorems 1* and 2* the following proposition corresponding to Theorem 3 turns out not to be true.

Proposition 3* (Haag’s Theorem?). Suppose that $\hat{\psi}_1(x; \lambda)$ is a free field and $\hat{\psi}_2(x; \lambda)$ is a local, Lorentz-covariant field. Suppose further that the fields $\hat{\psi}_1(x; \lambda)$ and $\hat{\psi}_2(x; \lambda)$ satisfy the hypotheses of Theorem 1*. Then $\hat{\psi}_2(x; \lambda)$ is also a free field.

Given the assumptions of this proposition and Theorem 1, we can easily deduce the equivalent of Eq. (A1):

$$2\langle 0|\hat{\psi}_2(x; \lambda)\hat{\psi}_1(x_0; \lambda)|0\rangle_2 = \Delta(x - x_0; 0) = \delta^4(x - x_0).$$

However, just as Eq. (A1) was at the single time $t_0$, the equivalent equation for the parameterized theory is at the single parameter value $\lambda_0$. But now, if we try to generalize this to unequal parameter values $\lambda$ and $\lambda_0$, there is no equivalent of the Lorentz transformation to use in order to bring the parameter values back to equality. All that is available is parameter translation, which would maintain the difference $\lambda - \lambda_0$.

Therefore, it is possible for $\hat{\psi}_1(x; \lambda)$ and $\hat{\psi}_2(x; \lambda)$ to have the same equal-$\lambda$ two-point vacuum expectation value, but for their unequal-$\lambda$ expectation values to differ. Thus, Theorem 2* does not apply and the proof of Haag’s theorem does not go through.
