Solving the Stock Option Forecast problem by a numerical method for the Black-Scholes Equation with Machine Learning Classification Model

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Abstract

We proposed classification models that utilize the result from the Quasi-Reversibility Method[2], which solves the Black-Scholes equation to forecast the option prices one day in advance. Combining the minimizer from QRM with our machine learning classifications, we can classify the option as an increase or decrease in value. Based on the different classifications of the options, we can apply various trading strategies which we aim to figure out ways to improve the results from QRM’s extrapolations. To further test the viability of our model, we collected 23548 options data from the real-world market for our model, and we will then feed in the data along with the minimizer from QRM to form decision trees and random forests, which we will later test for accuracy, precision, and recall.

1 Introduction and Overview

The stock market, also known as an equity market, contains buyer and seller that trades the ownership of a business. In order to make a profit in such a market, we need to make intuitive guesses about the market trend; in other words, we want to buy low and sell it high. However, since the market is volatile and we are solving ill-posed problems because the data are very noisy and the model is very sensitive to the noise in the data, it is very hard to correctly predict the exact price the stock will be even after just a day. Thus, we can make a more generalized prediction, such as estimating a market gain or loss. On the other hand, options trading is the contract that gives you the right to sell or buy the stock in the future. If we can predict if the option is going to trade higher, we can make a profit out of the market. Hence, we came up with QRM(Quasi Reversibility Method) to capture the market’s movement by solving an ill-posed problem(Black-Scholes model) to predict the option price one day in advance.[2] In our research, we will utilize a minimizer which is the prediction of future option data from the QRM along with other features from the stock market to forecast the market movement. We proposed using ensemble methods such as gradient boosting trees and random forests to classify the future data.
We use the classification model because we believe that by combining it with QRM, we can capture some critical features from the market that determine the movement of an option's prices with the assumption that the market will not be too volatile in a short period which allows us to split the stock into different categories. We will first design weak learners to classify the stock into different classes. Then, we will improve the result by minimizing the loss function.

2 Quasi-reversibility model and Black–Scholes model

2.1 Black-Scholes equation

The Black-Scholes equation is a backward parabolic partial differential equation that is used to solve questions in mathematical finance. The proposing model is to use some initial boundary conditions for stock that we are interested in and simulate a more accurate options trading strategy. Since it is an ill-posed problem, we will introduce variables \( \tau \) to solve the equation backward, which at any given time \( t \), we have

\[
\tau = T - t.
\] 

(2.1)

Give the Black-Scholes equation to be:

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0
\]

(2.2)

Assume that the function \( u(s, \tau) \) satisfy the Black-Scholes equation with a volatility coefficient \( \sigma \). We can rewrite the equation as:

\[
\frac{\partial u(x, \tau)}{\partial \tau} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 u(x, \tau)}{\partial x^2} = 0
\]

(2.3)

with the initial condition

\[
u(x, 0) = f(x)
\]

(2.4)

Given the payoff function \( f(x) \), which \( x \) is the stock price, and at maturity time, we have \( \tau = 0 \). With the assumption that gives risk-free interest rate to be zero, the function \( u(x, \tau) \) is the price of an option at time \( t \). [9]

2.2 Quasi-Reversibility method

We introduce the method to solve Black-Scholes equations with some approximations. We want to solve the below equation

\[
P u = u_t + \frac{1}{2} \sigma^2(t) x^2 u_{xx} = 0
\]

(2.5)

with the Dirichlet boundary conditions

\[
u(x_b, t) = o_b(t), u(x_a, t) = o_a(t)
\]

(2.6)
and initial conditions
\[
u(x, 0) = f(x)
\] (2.7)
where
\[
f_x = \frac{x_a}{x_b} - 1
\] (2.8)
\[
f_o = \frac{o_a}{o_b} - 1
\] (2.9)
where \( t \in [0, 2\tau] \), \( x \in (x_b(0), x_a(0)) \)
P represents the Black-Scholes equation’s partial differential operator. \( x \) represents the stock price. \( \sigma \) represent the volatility of the option from the data. \( o_b, o_a \) represent the bid and ask price of the option and \( x_b, x_a \) represent the bid and ask price of the stock. Also, we have the assumption that the bid price is always less than the ask price.

Given the above conditions, the method will take 5 steps to solve for the Black-Scholes equations.

- First, we will need to solve for the variable that is dimensionless, namely the \( x \)'s.
- Secondly, we want to perform some interpolation and extrapolation to forecast the option price.
- Thirdly, we want to apply the initial boundary condition to the PDE.
- Fourthly, we would like to apply regularization to try to solve the method numerically.
- Lastly, solve the minimizing problem with finite difference and regularization. [9]

### 2.3 Analysis
Assume we have \( Y_{2\tau} = (x_b(0), x_a(0)) \times (0, 2\tau) \) the function
\[
o_b(t), o_a(t) \in H^2[0, 2\tau], \sigma(t) \in C^1[0, 2\tau]
\] (2.10)
which we can find the solution \( u \in H^2(Y_{2\tau}) \) which satisfy the following conditions
\[
Au = 0 \in Y_{2\tau}
\] (2.11)
\[
u(0, t) = o_b(t), u(1, t) = o_a(t)
\] (2.12)
\[
u(x, 0) = f(x), x \in (0, 1)
\] (2.13)
We are going to perform a more general convergence analysis base on a new Carleman estimate, where we assume the time interval is within a short period. Let’s formulate a rather general case with the given property.
Given an non-negative constant \( L \) and let \( Z_L \) be \( (x, t) \in (0, 1) \times (0, L) \). Then, let’s have
two non-negative number $\gamma_1, \gamma_2$ which $\gamma_2 > \gamma_1$. Suppose there are function $\Gamma(x, t) \in C^1(\bar{Z}_L)$ that satisfy
\[
\|b\|_{C^1(\bar{Z}_L)} \leq \gamma_2, \Gamma(x, t) \geq \gamma_1 \in Z_L.
\] (2.14)

Then, the rather general version of the original problem becomes
\[
Mk = k_t + \Gamma(x, t)k_x x = 0 \in Z_L
\] (2.15)

With the boundary condition
\[
k(0, t) = o_b(t), k(1, t) = o_a(t)
\] (2.16)

and the initial condition
\[
k(x, 0) = k(x) = o_b(0)(1 - x) + o_a(0)x, x \in (0, 1)
\] (2.17)

We are trying to find a convergence solution to the general solutions. \[\[ \]

3 Classification algorithms

3.1 Decision Tree:

A decision tree is a classification algorithm based on a tree model. Such a tree consists of a root node with no incoming edge, a decision node with precisely one incoming edge and one outgoing edge, and leaf nodes with only an incoming edge with no outgoing edge. The incoming edge is the result of the previous decision, and the outgoing edge is the result of current decisions. Also, it is possible to have a branch that consists of multiple nodes. Note that all the data will classify the data into a final class. The idea of a decision tree is elementary. By having some features of your data, we will split the data based on what features are in the node, where each edge coming out from the node represent the possible answer from the decision we made. \[\[\]

The decision tree does not have to be a binary tree where you can have more than two decisions made out of one node. In our case, we will only use a binary tree because we only wanted to capture the option movement as increase or decrease. We will feed the tree with minimizer and market features and split the data into training, validation, and test sets. Then, we will create trees with different combinations of features of our data to generate outcomes for trading strategy. Below is a graph showing the basic binary decision, which we call a classification. \[\[\]
Figure 1: Binary Decision Tree
3.2 Data Split:

Data split is the procedure to determine the decisions on each node. Generally, we will have the feature which splits the data roughly in half as our root node which makes the tree balanced while making the leaf node have bias toward one of the classifying classes so we can have a final decision.

3.2.1 Entropy:

Entropy is the measure of the disorder of the feature relative to our data. In another word, it determines how well we can classify the data into a certain class. The formula is given below:

$$H(S) = \sum_{i=1}^{c} -p_i \log_2 p_i,$$

(3.1)

where $H(S)$ is the Entropy, or disorder, of the feature, $p_i$ is the probability of feature $i$, and $c$ is the total feature that we have. Entropy can range from $[0, 1]$, with the extremes, 0 representing the lowest Entropy, signaling no disorder, which is bad for training because everything got classified into just one class. On the other hand, an entropy of 1 signals a good learning set because it splits up the data evenly. Furthermore, we can use Entropy to determine the information gain of each node which is how much information we can learn from a feature.

3.2.2 Information Gain:

Information Gain is the measure of change in Entropy due to splitting the data on a feature. The formula is as follows:

$$\text{Information Gain}(S, X) = H(S) - \sum_{i=1}^{x_i} \frac{x_i}{S} H(x_i),$$

(3.2)

where $H(S)$ is the measure of the disorder before the split, $x_i$ is the number of observations on the $ith$ split, and $H(x_i)$ is the disorder for the target feature on split $x_i$. By subtracting the combined Entropy of all the child nodes from the Entropy of the parent node, we can observe how much ”information” we have gained or lost. The greater the information gain, the more Entropy, or disorder, decreases. Information Gain is utilized to tell whether or not a split was more or less beneficial based on how much ”noise” was removed due to the split.

3.2.3 Gini Impurity:

Gini Impurity is the measure of the probability that an instance of a random variable is incorrectly classified if that new instance were randomly classified according to the distribution of class labels from the data set. The formula is presented below:

$$Gini = 1 - \sum_{i=1}^{n} (p_i)^2,$$

(3.3)
n where n is the number of classes and \( p_i \) is the probability of an item with label \( i \). Thus, to calculate the Gini Impurity, one summates the probability, \( p_i \), times the probability of a mistake in classifying that item, \( \sum_{k \neq i} p_k \). This leads Gini Impurity to vary between values of 0 and 1, where a 0 represents complete purity of classification, a 0.5 represents an equal distribution of elements over some classes, and a 1 represents the random distribution of elements over multiple classes.

3.3 Hyper Parameter Selection:

Hyper-parameters define a model much like how the rules of a sport define the sport. One can differentiate between models through the difference in Hyper-Parameters as Hyper-Parameters address questions regarding the model design. These questions might include the maximum depth of a tree, the maximum amount of nodes, and the learning rate of gradient descent. These Hyper-Parameters are not features of the model, and instead make up the structure of the model. These metrics determine how the algorithm behaves when it is ran, and so tuning for these Hyper-parameters proves essential to minimize noise and provide higher accuracy. The selection process can vary, as you can either change the Hyper-parameters yourself and hope to see better results, or one can use a method such as Grid Search to tune for the most efficient combinations of the Hyper-parameters. The Grid Search is essentially building a model for each and every possible assortment of Hyper-parameters, grading each model, and selecting the one with the most promising results. Another selection method would be to generate random Hyper-Parameter values from a data distribution for each parameter. This process is called Random Search, due to the randomness created when not using a definite set of values, but rather random sampling the values from a distribution.

3.4 Ensemble methods:

The ensemble method uses groups of based learners or models to form a final decision by taking the average among different learners or models. It is more or less like voting, where the decision is based on the majority opinion. Since individual models might create high bias or variance, using the model combination will likely offer a lower bias and variance by taking the random samples of the same model or proposing a different model to train the data. It is the key to the bagging and boosting method, where bagging trains the weaker learner in parallel, whereas boosting trains them sequentially. We would like to use such methods because we would want to eliminate the possible high bias.

3.5 Gradient Descent:

Gradient descent is a first-order iterative optimization method that tries to navigate the local minima of some algorithm. It is a boosting method that is trying to reduce the inaccuracy of our tree where it turns a weak model into a stronger model by fixing the weakness. The idea is to look at the gradient of the current location and then take an opposite step to the gradient, which results in the steepest step towards the minima. The method is analogous to going down the hill with the steepest path. Also, we have something called the learning rate, which is the step size we are taking each time we are trying to go down the hill. A larger learning rate indicates a bigger step that might go over the
minima and get to uphills. While taking a small step will grant you more accuracy but sufferers from inefficiency and saddle points which tricks the machine into that we have reached a minima. In the method, we will use something called a loss function to determine our movement on the hill. It is essential the difference between the actual output of our tree compares to our prediction, which we call the error. We will try to minimize the loss function until it is about zero, which is hopefully the global minimum. There are also challenges in using the algorithm. It is a greedy method that only considers the current location, and it can lead to a local minima, which is not necessarily the best solution.

![Figure 2: Binary Decision Tree](image)

### 3.5.1 Gradient Classification:

Here, we would like to apply gradient descent to our decision tree to boost our accuracy. First, we would like to introduce the mean squared error, which is used to evaluate the quality of our splits. Since we are solving an optimization problem, in particular, trying to have a classification to determine the label of different classes, we would use the log-loss function and minimize it for better accuracy. Needs to compute the log-loss derivative here.

The idea is we are going to have some input data and a differentiable loss function. In the classification case, we are going to use a log-loss function, which has the form:

\[
\text{total loss} = -\frac{1}{N} \sum_{i=1}^{N} y_i \cdot \log(p(y_i)) + (1 - y_i) \cdot (1 - p(y_i))
\]  

(3.4)
The reason is that the logarithmic loss is a metric that tests the prediction probability corresponding to the true label. With the loss function, we can now try to do the iterations.

Step 1

\[ F_0(x) = \arg \min \sum_{i=1}^{n} L(y_i, \gamma) \]  

1. \( y_i \) - The output (observed value)

2. \( L \) - The loss function

3. \( \gamma \) - value for log(odds)

Now we want to add up the loss function for each observed value. We want to find the gamma value that minimizes the sum. We will simply take the derivative of each loss function.

Step 2: for \( m = 1 \) to \( M \)

\[ r_{im} = -[\frac{\partial L(y_i, F(x_i))}{\partial F(x_i)}]_{F(x)=F(m-1)(x)} \forall i = 1, 2, ..., n \]  

We want to calculate the pseudo residuals because it is the intermediate error term between the actual value and the intermediate predicted value. We have

\[ Observed - Predicted = PseudoResidual \]  

Then, we want to fit a regression tree that is closed under scaling \( h_m(x) \) to pseudo residuals. After that, we want to compute multiplier \( \gamma_m \) by solving the one-dimensional optimization problem

\[ \gamma_m = \arg \min \sum_{i=1}^{n} L(y_i, F_{m-1}(x_i) + \gamma h_m(x_i)) \]  

Finally, we will update our model to be

\[ F_m(x) = F_{m-1}(x) + \gamma_m h_m(x) \]  

We will simply output the \( F_m(x) \) to be our new decision tree model. 

By differentiating the log-loss function at very iterations, we are able to improve our decision tree accuracy by a large margin. Also, the use of cross-validation eliminates some of the over-fitting when we get deeper trees that exhibit higher train accuracy.
3.6 Random Forest

We are using a Bagging method which selects a subset of the data and uses it to create many different training sets. Then, we will train these sets independently and average the majority of the prediction that has been made to acquire a more accurate estimation while eliminating over-fitting during the process. In this project, we are intended to use a random forest algorithm, which utilizes the bagging method to create many random uncorrelated decision trees, which is a forest. When generating such decision trees, we purposely generate trees with low correlation, known as "the random subspace method." While the decision tree utilized all the provided features, random forest only considered a fraction of the features. The data split is done by either Gini, entropy, or log-loss criterion. We will decide how we want to split the tree based on hyper-parameter turnings. On the other hand, to extract an important feature, we will calculate how important a certain feature is by calculating when the probability of reaching the node decreases and how much-weighted node impurity will decrease.

\[ \text{imp}(i) = w_i N_i - w_{\text{left}(i)} \cdot N_{\text{left}(i)} - w_{\text{right}(i)} \cdot N_{\text{right}(i)} \]  

- \( w_i \) = weighted samples that reaches node \( i \)
- \( N_i \) = impurity of node \( i \)
- left = left split on node \( i \)
- right = right split on node \( i \)

Due to its correctness, the random forest can be an extremely time-consuming process because it trains many trees simultaneously. On the other hand, it is also hard to interpret the result of the decision tree due to the gigantic forest that was being created. That is why we use feature importance to help us interpret the result. Finally, We would like to apply random forest to our data because we believe that the market follows certain patterns. If we can make the right decision while the data exhibit certain feature, we are able to extract from it and improve the prediction we made from QRM. Furthermore, since our only goal is to classify future options as either increase or decrease, having majority votes with different trees can reduce over-fitting and bias within the short period of option price predictions.
4 Result and Data

4.1 Classification matrix

We define the option price of a certain day to be the average of the bid and the asking price. All the result below was calculated based on the following matrix

- Predicted label of 1 represents that we predicted the option price would increase the next day.
- Predicted label of -1 represents that we predicted the option price would decrease the next day.
- True label of 1 represents that the real option price increase the next day.
- True label of -1 represents that the real option price decrease the next day.

4.2 Decision Tree

| Method      | Accuracy | Precision | Recall  |
|-------------|----------|-----------|---------|
| Decision Tree | 53.90%   | 58.24%    | 62.66%  |

Figure 3: Confusion Matrix
4.3 Decision Tree with Gradient Boosting

| Method            | Accuracy | Precision | Recall  |
|-------------------|----------|-----------|---------|
| Gradient Boosting | 65.74%   | 74.26%    | 56.50%  |

Figure 4: Decision Tree
4.4 Random Forest

| Method       | Accuracy | Precision | Recall  |
|--------------|----------|-----------|---------|
| Random Forest| 69.02%   | 70.96%    | 75.68%  |

Since the decision tree is too deep for the purpose of demonstration. We will only show a partial tree which represents the first few layers of the tree.
4.5 **Stock market Data:**

The data that has been used for this project are from bloomberg.com[3].
4.6 Result

| Method                | Accuracy | Precision | Recall  |
|-----------------------|----------|-----------|---------|
| Decision Tree         | 53.90%   | 58.24%    | 62.66%  |
| Gradient Boosting     | 65.74%   | 74.26%    | 56.50%  |
| Random Forest         | 69.02%   | 70.96%    | 75.68%  |

5 Summary and Conclusions

Although we cannot correctly predict the exact option price due to the market volatility, we can make a good prediction of whether the option price will increase or decrease based on the market features and the minimizer we acquired from QRM. Based on our assumptions, we will consider the percentage of precision as our profitable buy for the option. Notice that we had 58.24% recall when only the Decision Tree was being applied. Improving our result with gradient boosting, we are able to get 74.26%. Last but not least, we can achieve 70.96% precision with gradient decision forest. The data suggest that we are able to make a profit out of the market by applying the above methods.

However, since the current metric only examines if a stock option will increase or decrease in value one day in advance, it does not consider the fees and the duration of the option. Furthermore, the data gathered is within a short time frame, which does not reflect the market’s volatility. In the future, we will implement methods that predict the...
percentage of increase or decrease relative to the current option price, so there is a high margin and chance for people to make money in options trading. We can improve the model accuracy by applying different machine learning models than tree classifications while supplying a finer minimizer from the QRM.
6 Appendix

Derivation of the log_loss function

First, we have some input for our classification:

\[ Data(x_i, y_i)_{i=1}^n \]

Also, we have some loss function

\[ L(y_i, F(x)) \]

1. \(x_i\) - The input data that we feed into the model

2. \(y_i\) - The output we are trying to predict

Likelihood Function: The likelihood function describes the joint probability of the observed data as a function of parameters of our input.

\[ L(\theta) = L(\theta; y) = f_y(y; \theta) \]

Likelihood function is going to give us the product of all such probability

\[ L(\theta) = \prod_{i=1}^{n} f_i(y_i; \theta) \]

Here we are using the log likelihood function which is

\[ log_{\text{likelihood}} = [y_i \cdot log(p) + (1 - y_i) \cdot log(1 - p)] \]

1. we require \(y_i\) to be binary values which in this case 0 or 1.

2. \(p\) represent the predicted probability.

Since we want to have a better prediction which means that we would want to maximize the log likelihood function. If we want to use this as our loss function, we will simply invert the function which

\[ log_{\text{loss}} = -log_{\text{likelihood}} \]

If we minimize the loss function, we essentially will have a better fitting. Since the likelihood function is a function of probability, we want to make it into a function of corresponding odds is the logit function which has the form

\[ logit(p) = \log\left(\frac{p}{1-p}\right) \]

which \(\frac{p}{1-p}\) represents the corresponding odds of the given probability. Let’s do some basic arithmetic by plugging in variables

\[ log_{\text{loss}} = -[y_i \cdot log(p) + (1 - y_i) \cdot log(1 - p)] \] (6.1)

\[ = -[y_i \cdot log(p)] - log(1 - p) + y_i \cdot log(1 - p) \] (6.2)

\[ = -y_i \cdot (log(p) - log(1 - p)) - log(1 - p) \] (6.3)

\[ = -y_i \cdot log\left(\frac{p}{1-p}\right) - log(1 - p) \] (6.4)

\[ (6.5) \]
First, let’s have $\log\left(\frac{p}{1-p}\right) = y$ Then,

$$y = \log\left(\frac{p}{1-p}\right)$$  \hspace{2cm} (6.6)

$$e^y = \frac{p}{1-p}$$  \hspace{2cm} (6.7)

$$e^y = -1 + \frac{1}{1-p}$$  \hspace{2cm} (6.8)

$$e^y + 1 = \frac{1}{1-p}$$  \hspace{2cm} (6.9)

$$1 - p = \frac{1}{e^y + 1}$$  \hspace{2cm} (6.10)

$$-p = \frac{1 - e^y - 1}{e^y + 1}$$  \hspace{2cm} (6.11)

$$p = \frac{e^y}{1 + e^y}$$  \hspace{2cm} (6.12)

$$p = \frac{e^{\log\left(\frac{p}{1-p}\right)}}{1 + e^{\log\left(\frac{p}{1-p}\right)}}$$  \hspace{2cm} (6.13)

Then, we have

$$\log(1 - p) = \log(1 - \frac{e^{\log\left(\frac{p}{1-p}\right)}}{1 + e^{\log\left(\frac{p}{1-p}\right)}}$$  \hspace{2cm} (6.14)

$$= \log\left(\frac{1 + e^{\log\left(\frac{p}{1-p}\right)} - 1}{1 + e^{\log\left(\frac{p}{1-p}\right)}}\right)$$  \hspace{2cm} (6.15)

$$= \log\left(\frac{1}{1 + e^{\log\left(\frac{p}{1-p}\right)}}\right)$$  \hspace{2cm} (6.16)

$$= \log(1) - \log(1 + e^{\log\left(\frac{p}{1-p}\right)})$$  \hspace{2cm} (6.17)

$$= -\log(1 + e^{\log\left(\frac{p}{1-p}\right)})$$  \hspace{2cm} (6.18)

Let’s put the equation back into log-loss function.

$$\text{log-loss} = -y_i \cdot \log\left(\frac{p}{1-p}\right) - \log(1 - p)$$  \hspace{2cm} (6.19)

$$= -y_i \cdot \log\left(\frac{p}{1-p}\right) + \log(1 + e^{\log\left(\frac{p}{1-p}\right)})$$  \hspace{2cm} (6.20)

Let’s differential the loss function which

$$\frac{d}{d\log\left(\frac{p}{1-p}\right)}(-y_i \cdot \log\left(\frac{p}{1-p}\right) + \log(1 + e^{\log\left(\frac{p}{1-p}\right)})) = -y_i + \frac{e^{\log\left(\frac{p}{1-p}\right)}}{1 + e^{\log\left(\frac{p}{1-p}\right)}}$$  \hspace{2cm} (6.21)

Which we can rewrite it as, $\text{Predicted} - \text{Observed}$ where

$$\text{Observed} = y_i$$  \hspace{2cm} (6.22)

$$\text{predicted} = \frac{e^{\log\left(\frac{p}{1-p}\right)}}{1 + e^{\log\left(\frac{p}{1-p}\right)}}$$  \hspace{2cm} (6.23)
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