Inflationary Dynamics with Runaway Potentials

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The evolution of two slow-rolling scalar fields with potentials of the form $V = V_0 \phi^{-\alpha} \exp(-\beta \phi^m)$ is studied. Considering different values of the parameters $\alpha$, $\beta$, and $m$, we derive several new inflationary solutions in which one field just evolves in the background and is not important for the inflationary dynamics. In addition, we find new solutions where both fields are dynamically important during inflation. Moreover, we discuss the evolution of perturbations in both scalar fields and the spacetime metric, concentrating on the production of entropy perturbations. We find that for a large region in parameter space and initial conditions tensor modes are negligible and that adiabatic and isocurvature perturbations are essentially uncorrelated.

I. INTRODUCTION

Since their advent just over twenty years ago, cosmological inflationary models have been developed to try to deal with several problems arising in the standard cosmology. The most important consequence of an inflationary stage in the very early universe is the development of perturbations in space–time and matter, which eventually evolve into the structures we observe in the universe today. The simplest model with an inflationary stage is where inflation is driven by a single scalar field, called the inflaton, with some potential $V(\phi)$. If the potential is flat enough, theory predicts that adiabatic perturbations are generated which obey the Gaussian statistics and have an almost scale-free spectrum (for recent reviews see e.g. [1] and [2]).

However, the most important drawback for inflationary cosmology is that there is no unique candidate for the inflaton field and its potential. Furthermore, according to our theories of particle physics, there are a large number of scalar fields which, potentially, could be important in the early universe. For example, hybrid inflation is a model, in which two scalar fields are important: whereas one field drives an inflationary epoch, the dynamics of the second field will end this period of inflation [3]. If inflation itself is driven by two or more scalar fields, the perturbations are no longer purely adiabatic or Gaussian [4]-[11]. Moreover, the transition to the normal radiation–dominated epoch depends on the decay of the scalar fields into radiation and matter, which could influence the evolution of perturbations as well. Cosmological observations by the Wilkinson Microwave Anisotropy Probe (WMAP), the Planck Surveyor, 2dF and SLOAN Digital Sky Survey will put strong constraints on the nature of the primordial perturbations and therefore put constraints on more complicated models of inflation and the process of reheating.

Although formalisms have been presented in order to study two–field inflation (see e.g. [8], [11] and [12]), no explicit model has been used in order to examine the background evolution of both fields and the consequences for the perturbations. In this paper, we investigate the evolution of two scalar fields, each with a potential of the form $V(\phi) = V_0 \phi^{-\alpha} \exp[-\beta \phi^m]$, and examine the effect of the parameters $\alpha$, $\beta$, $m$ on the evolution of the universe. This potential includes pure exponentials ($\alpha = 0$), pure power law ($\beta = 0$) and combinations of the two. For a single scalar field, this potential was investigated in detail in [13] and [14]. Exponential potentials appear in Kaluza–Klein theories as well as in supergravity and superstring models (for a review, see e.g. [15]). Models based on dynamical supersymmetry breaking involving fermion condensates motivate inverse power potentials [16]; supergravity corrections to these models in turn predict potentials of the form above [17]. Furthermore, models motivated by higher–dimensional theories and/or scalar–tensor theories easily incorporate these potentials. For example, the form of the potential was used in [18] to examine inflation with a background dilaton field.

The potentials just described were used both in inflationary cosmology as well as in models for dark energy, driving the observed acceleration of the universe at the present epoch (for a review, see [19]). Indeed, the potential above has interesting properties, allowing for scaling behavior and other attractor–like solutions (see e.g. [20] and [21]). However, our goal in this paper is to obtain an understanding of the dynamics of two scalar fields with this potential which drive an inflationary epoch in the early universe. This, in turn, allows us to study further the consequences for the perturbations, in the two scalar fields and the space–time metric, during the inflationary epoch. This can be seen as a first step to understanding the initial perturbations in the radiation dominated era, which eventually evolve into
the structures we observe today. We would like to emphasize, however, that in order to calculate the perturbations in the radiation dominated epoch, an understanding of the decay of both scalar fields is needed. There are some possibilities we would like to mention:

- Both scalar fields decay, one into radiation and baryonic matter, the other field only into dark matter. The second field could, in principle, also decay only partly into dark matter, whereas the “rest” provides an explanation for dark energy.

- Only one scalar field decays completely into matter and radiation, the other decays partly into some form of matter; the remains of the field play the role of dark energy.

- One of the fields plays the role of the curvaton field \[22\], i.e. it decays well after the inflationary epoch and is responsible for the curvature perturbation.

There are more possibilities, of course, but it is clear that the subsequent evolution of perturbations in the metric and matter fields strongly depends on which of these possibilities is realized. The situation is now far more complicated when compared to inflationary models based on one scalar field alone.

The paper is organized as follows: in Section 2 we present the equations of motion; section 3 discusses the case of two independent potentials for both fields, i.e. the total potential is given by \[ V = V_0 \phi^{\alpha} \exp[-\beta \phi^m] + V_0 \chi^{-\alpha} \exp[-\beta \chi^m]; \] in Section 4 we investigate a mixed potential of the form \[ V = V_0 \chi^{-\alpha} \exp[-\beta \phi^m]. \] In sections 3 and 4 we discuss all of the possible cases in detail. These two sections are necessarily technical. However, we have included a summary of our results at the end of each section. Readers who are not interested in the technical aspects may wish to refer to these summaries. The perturbations are discussed in Section 5; we present our conclusions in Section 6.

II. THE FIELD DYNAMICS

Our starting point shall be that of Einstein gravity with two scalar fields and a general potential \( V(\phi, \chi) \). The action for this (in Planck units) is described by

\[
S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} R - \frac{1}{2} \left( \partial_a \phi \partial^a \phi + \partial_a \chi \partial^a \chi \right) - V(\phi, \chi) \right].
\] (1)

This action leads to the usual Friedmann equation and two Klein–Gordon equations for both scalar fields:

\[
H^2 = \frac{1}{3} \left[ \frac{1}{2} \left( \dot{\phi}^2 + \dot{\chi}^2 \right) + V(\phi, \chi) \right],
\]
\[
\ddot{\phi} = -3H \dot{\phi} - V_\phi,
\]
\[
\ddot{\chi} = -3H \dot{\chi} - V_\chi,
\] (2)

where we use the notation \( V_\phi = \partial V / \partial \phi \) with its obvious extensions.

We shall find it convenient to make the slow–roll approximation which is appropriate when the slow–roll parameters

\[
\epsilon_\phi = \frac{1}{2} \left( \frac{V_\phi}{V} \right)^2, \quad \epsilon_\chi = \frac{1}{2} \left( \frac{V_\chi}{V} \right)^2,
\]
\[
\eta_{\phi \phi} = \frac{V_{\phi \phi}}{V}, \quad \eta_{\phi \chi} = \frac{V_{\phi \chi}}{V}, \quad \eta_{\chi \chi} = \frac{V_{\chi \chi}}{V}.
\] (3)

satisfy \( |\epsilon_i, \eta_{ij}| \ll 1, i, j = \phi, \chi \). These five conditions are sufficient for us to generate an inflationary epoch where \( \dot{a}(t) > 0 \). With this approximation the equations of motions (2) reduce to

\[
H^2 = \frac{1}{3} V(\phi, \chi),
\]
\[
3H \dot{\phi} = -V_\phi,
\]
\[
3H \dot{\chi} = -V_\chi.
\] (4)

For the most part, we shall see that slow–roll is an attractor in our model and that we shall be justified in making this approximation.
III. UNCOUPLED POTENTIALS

In this section we begin our investigation by considering two uncoupled potentials of the form

\[ V(\phi, \chi) = V_0, \phi^{-\alpha_1} \exp \left[ -\beta_1 \phi^{m_1} \right] + V_0, \chi^{-\alpha_2} \exp \left[ -\beta_2 \chi^{m_2} \right]. \]  

(7)

In this case the slow–roll parameters read

\[ \epsilon_\phi = \frac{1}{2} \left( \frac{\alpha_1}{\phi} + \beta_1 m_1 \phi^{m_1-1} \right)^2 \frac{V_0^2, \phi^{-2\alpha_1} \exp \left[ -2\beta_1 \phi^{m_1} \right]}{(V(\phi, \chi))^2}, \]

\[ \eta_{\phi \chi} = \left( \frac{\alpha_1}{\phi} + \beta_1 m_1 \phi^{m_1-1} \right)^2 + \frac{\alpha_1}{\phi^2} + \beta_1^2 m_1 (m_1 - 1) \phi^{m_1-2} \right) \frac{V_0, \phi^{-\alpha_1} \exp \left[ -\beta_1 \phi^{m_1} \right]}{V(\phi, \chi)}, \]

(8)

and similarly for \( \chi \), with \( \eta_{\phi \chi} = 0 \). It is clear that, with a judicious choice of initial conditions, these can always be made small. It is also clear that \( m_i = 1 \) will be a transitional value because, for \( m_i > 1 \), the slow–roll parameters will grow. We now begin to examine the evolution of the fields in this regime. We shall assume that \( \beta_1 > 0 \) but make no further assumptions about the other parameters in the potential terms.

Large Field Values

A. \( m_1, m_2 > 0 \)

With \( m_1, m_2 > 0 \), the exponential part of the potential, for large enough \( \phi, \chi \) at least, will always dominate over the power law. Dividing equations (5-6) we generate the equation

\[ \frac{\dot{\phi}}{V_\phi} = \frac{\chi}{V_\chi}. \]

(9)

We are unable to integrate this directly but are able to do so to leading order in the fields by means of integration by parts. This then generates for large \( \phi, \chi \)

\[ \beta_1^2 m_1^2 \phi^{2m_1-2} V_0, \phi^{-m_1} \exp \left[ -\beta_1 \phi^{m_1} \right] = \beta_2^2 m_2^2 \chi^{2m_2-2} V_0, \chi^{-m_2} \exp \left[ -\beta_2 \chi^{m_2} \right]. \]

(10)

One then takes natural logarithms to find a simple expression between the fields. Since \( m_i > 0 \) the terms proportional to the powers of the fields will eventually dominate. This gives us

\[ \beta_1 \phi^{m_1} = \beta_2 \chi^{m_2}, \]

(11)

to leading order. Then using (11) and (11) we are able to show

\[ H(\phi) = \left( \frac{V_{0, \phi}}{3} \right)^{\frac{1}{2}} \phi^{-\alpha_1/2} \exp \left[ -\frac{\beta_1 \phi^{m_1}}{2} \right] \left( 1 + \frac{m_1^2}{m_2^2} \frac{\beta_1}{\beta_2} \frac{\phi^{2m_1}}{m_2^{2m_2-2}} \right)^{\frac{1}{2}}. \]

(12)

Substituting back into (4) we must solve the differential equation for \( \phi \)

\[ \dot{\phi} = \left( \frac{V_{0, \phi}}{3} \right)^{\frac{1}{2}} \frac{\alpha_1}{\phi} + \beta_1 m_1 \phi^{m_1-1} \left( 1 + \frac{m_1^2}{m_2^2} \frac{\beta_1}{\beta_2} \frac{\phi^{2m_1}}{m_2^{2m_2-2}} \right)^{\frac{1}{2}} \phi^{-\frac{\alpha_1}{2}} \exp \left[ -\frac{\beta_1 \phi^{m_1}}{2} \right], \]

(13)

\[ = \left( \frac{V_{0, \phi}}{3} \right)^{\frac{1}{2}} \beta_1 m_1 \phi^{\frac{\alpha_1}{2} + m_1 - 1} \left( 1 + \frac{m_1^2}{m_2^2} \frac{\beta_1}{\beta_2} \frac{\phi^{2m_1}}{m_2^{2m_2-2}} \right)^{\frac{1}{2}} \exp \left[ -\frac{\beta_1 \phi^{m_1}}{2} \right], \]

(14)

to leading order in \( \phi \). For large \( \phi \), we are able to integrate this expression under two different regimes, depending on whether \( m_1 < m_2 \) or \( m_1 = m_2 \). This corresponds to one field giving the dominant contribution to the Hubble parameter and both giving equal contributions. This is clear from the denominator in (14). We shall take both of these cases in turn.
In this case it is easy to see that the $\phi$ field is the dominant contributor. Solving for $\phi$ we integrate (14) by parts to show

$$t(\phi) = \left( \frac{3}{V_0} \phi \right)^{\frac{1}{3}} \frac{2}{m_1} \phi^{-2m_1 + 2} \exp \left[ \frac{\beta_1 \phi^m}{2} \right].$$

(15)

Immediately we are able to see that large $\phi$ implies large $t$ and so inflation will occur at late times with this set up. We are unable to invert this directly. However by taking $\ln$, we see that the exponential quickly dominates, and so to leading order

$$\phi(t) = \left[ \frac{2}{\beta_1} \ln \left( \frac{V_0}{3} \right) \frac{1}{2} \frac{\beta_1^2 m_1^2}{2} t \right]^{\frac{1}{m_1}}.$$ 

(16)

Writing (15) in the form

$$\frac{\beta_1 \phi^m}{2} = \ln \left( \frac{V_0}{3} \frac{1}{2} \frac{\beta_1^2 m_1^2}{2} \phi^{-2m_1 + 2} t \right),$$

(17)

we are able to substitute back into (16) to obtain a solution for $\phi(t)$ to next order in $t$. A little thought reveals why we want and need to do this. When we come to evaluate the Hubble parameter as a function of time we need to take the exponent of $\phi$. Every time we do this we must take the next to leading order expression. This is given by

$$\phi(t) = \left[ \frac{2}{\beta_1} \ln \left( \frac{V_0}{3} \right) \frac{1}{2} \frac{\beta_1^2 m_1^2}{2} t \right]^{\frac{1}{m_1}}.$$ 

(18)

Then we plug this into (12) to solve for $H(t)$ and then integrate to get the scalefactor which gives us,

$$H(t) = \frac{1}{\beta_1 m_1^2} \left( \frac{2}{\beta_1} \right)^{2m_1 - m_1} \left[ \frac{1}{t} [\ln(t)]^{2-m_1} \right].$$ 

(19)

$$a(t) \propto \exp \left[ \frac{1}{\beta_1 m_1 (2-m_1)} \left[ \frac{2}{\beta_1} \ln(t) \right]^{2-m_1} \right].$$ 

(20)

In the evolution of the fields we may drop most of the coefficients as they simply result in constant terms in the solution. This gives the final solution

$$\phi(t) = \left( \frac{2}{\beta_1} \ln t \right)^{\frac{1}{m_1}},$$

(21)

$$\chi(t) = \left( \frac{2}{\beta_2} \ln t \right)^{\frac{1}{m_2}}.$$ 

(22)

Now because $m_i < 1$, we see that the scale factor grows faster than power law. This regime is effectively equivalent to a single scalar field driving the dynamics of the universe as the second scalar $\chi$ sits in the background. Unsurprisingly, the solutions are in direct agreement with section III.A of [14].

2. $m_1 > m_2$

This is equivalent to the previous section by symmetry.
We now consider the case where both the fields give equal contributions to the evolution of our universe. Since, the second field is now significant, we expect to see something new. This case includes that of two straight exponential potentials which has already been studied in [23]. We are able to integrate (14) to leading order to show that

\[ t(\phi) = \left( \frac{3}{V_0} \right)^\frac{2}{3} \frac{1}{\beta_1^m - 2m^2} \phi^{\frac{1}{\beta_1} - 2m^2 + 2} \exp \left[\frac{\beta_1 \phi^m}{2}\right]. \]  

(23)

Once more the evolution of \( t(\phi) \) demonstrates that inflation occurs at late times for large \( \phi, \chi \). We then proceed in the same way and the solutions we get are as follows:

\[ \phi(t) = \left( \frac{2}{\beta_1} \ln t \right)^\frac{1}{m}, \]  

(24)

\[ \chi(t) = \left( \frac{2}{\beta_2} \ln t \right)^\frac{1}{m}, \]  

(25)

\[ H(t) = \left( 1 + \left( \frac{\beta_1}{\beta_2} \right)^\frac{1}{m} \right) \frac{1}{\beta_1 m^2} \left( \frac{2}{\beta_1} \right)^\frac{2m}{\beta_1 m^2} - \frac{1}{t} [\ln t]^\frac{2m}{\beta_1 m^2}, \]  

(26)

\[ a(t) \sim \exp \left[ \left( 1 + \left( \frac{\beta_1}{\beta_2} \right)^\frac{1}{m} \right) \frac{1}{\beta_1 m(2 - m)} \left[ \frac{2}{\beta_1} \ln t \right]^\frac{2m}{\beta_1 m^2} \right]. \]  

(27)

It should be noted that in the two exponential potential case this reduces to power law inflation

\[ a(t) \sim t^{\frac{2(\alpha_1 + \alpha_2 + \frac{1}{2})}{\alpha_1 \alpha_2}}, \]  

(28)

which is agreement with the dynamical system analysis of [23]. The “new” feature is that we generate more inflation than we would get with one scalar because of the extra factor, \( \exp \left[ 1 + \left( \frac{\beta_1}{\beta_2} \right)^\frac{1}{m} \right] \), in the solution for the scale factor. This concludes the analysis for the cases \( m_1, m_2 > 0 \).

\[ m_1 \leq 0, m_2 > 0 \]

It is possible when \( m_i < 0 \) and is also an even number, to generate a minimum of the potential at the origin. The nature of the slow-roll parameters dictates that this will certainly give a finite amount of inflation and that we may not get slow-roll here. We shall assume, initially at least, that the field begins in the part dominated by the power law part of the potential.

\[ \alpha_1 \neq 0, -4 \]

Following the same procedure as the previous section we are able to deduce that

\[ \dot{\phi} = \left( \frac{V_0}{3} \right)^\frac{1}{2} \frac{\dot{\phi}}{\alpha_1 \phi^{\frac{\alpha_1 + 2}{2}}} \exp \left[ -\frac{\beta_1 \phi^{m_1}}{2} \right]. \]  

(29)

Taking the \( \phi \) exponential as approximately constant, we are able to solve this to give,

\[ t(\phi) = \left( \frac{3}{V_0} \right)^\frac{2}{\alpha_1(\alpha_1 + 4)} \frac{1}{\alpha_1(\alpha_1 + 4)} \phi^{\frac{\alpha_1 + 4}{2}} \exp \left[ \frac{\beta_1 \phi^{m_1}}{2} \right]. \]  

(30)
Solving for the remaining fields and parameters we deduce

\[
\phi(t) = \left(\frac{V_{0}}{3}\right)^{\frac{1}{2}} \alpha_{1} \left(\frac{\alpha_{1} + 4}{2}\right)^{t} \frac{2}{\alpha_{1} + 4}, \quad (31)
\]

\[
\chi(t) = \left[\frac{2}{\beta_{2}} \alpha_{1} + 2 \ln t\right]^{\frac{1}{\alpha_{1} + 4}}, \quad (32)
\]

\[
H(t) = \left(\frac{V_{0}}{3}\right)^{\frac{1}{2}} \left(\frac{V_{0}}{3}\right)^{\frac{1}{2}} \alpha_{1} \left(\frac{\alpha_{1} + 4}{2}\right)^{t} \frac{2}{\alpha_{1} + 4}
\times \exp \left[\frac{-\beta_{1}}{2} \left(\frac{V_{0}}{3}\right)^{\frac{1}{2}} \alpha_{1} \left(\frac{\alpha_{1} + 4}{2}\right)^{t} \frac{2}{\alpha_{1} + 4}\right], \quad (33)
\]

\[
a(t) \propto \exp \left[\frac{4 + \alpha_{1}}{4} \left(\frac{V_{0}}{3}\right)^{\frac{1}{2}} \left(\frac{V_{0}}{3}\right)^{\frac{1}{2}} \alpha_{1} \left(\frac{\alpha_{1} + 4}{2}\right)^{t} \frac{2}{\alpha_{1} + 4} f_{1}(t)\right]. \quad (34)
\]

In this instance

\[
f_{1}(t) = \exp \left[\frac{-\beta_{1}}{2} \left(\frac{V_{0}}{3}\right)^{\frac{1}{2}} \alpha_{1} \left(\frac{\alpha_{1} + 4}{2}\right)^{t} \frac{2}{\alpha_{1} + 4}\right], \quad (35)
\]

and rapidly tends to unity as time increases. With this set of parameters, studying (36) one realizes that it is possible for inflation to occur in a different manner. It is helpful to write the scalefactor in the form

\[
a(t) \propto \exp \left[ct f_{1}(t)\right] \quad (36)
\]

\(\alpha_{1} > 0\): In this case, \(\alpha_{1}(\alpha_{1} + 4) > 0\) and so it follows, from equation (36) that \(t(\phi) \rightarrow \infty\) as \(\phi\) increases. Therefore, we must generate inflation at late times. In addition, \(c > 0, 0 < k < 1\) giving us a relatively slow expansion rate, \(-4 < \alpha_{1} < 0\): In this case \(\alpha_{1}(\alpha_{1} + 4) < 0\) and we get \(c > 0, 1 < k < \infty\). As \(\phi \rightarrow \infty\), \(t(\phi) \rightarrow -\infty\) and so inflation now occurs at early times. Potentially this can also give us a lot of inflation as the scale factor grows faster than \(e^{t}\).

\(\alpha_{1} < 0\): Once more we get \(\alpha_{1}(\alpha_{1} + 4) > 0\), however we now have a negative power of \(\phi\) in (36) and so \(t(\phi) \rightarrow 0\) as \(\phi\) increases. This generates inflation at early times once more. In addition we have the constraints \(c, k > 0\).

Once more the setup is equivalent to that of a single scalar as the second scalar makes minimal contribution to the universal dynamics. We generate identical results to section III.B of [14].

Let us now return to the case where \(\chi\) dominates. This gives a Hubble parameter of

\[
H(\phi) = \left(\frac{V_{0} \alpha_{1}(\alpha_{1} + 2)}{3\beta_{2}^{2} m_{2}^{2}}\right)^{\frac{1}{2}} \left(\frac{\alpha_{1} + 2}{\beta_{2}}\right)^{-m_{2}} e^{-\frac{\alpha_{1}(\alpha_{1} + 2)}{2} \ln \phi - \frac{1}{2} - 1 \ln \phi} \exp \left[-\frac{\beta_{1} \phi^{m_{1}}}{2}\right], \quad (37)
\]

\[
\phi = \left(\frac{V_{0} \alpha_{1} \beta_{2}^{2} m_{2}^{2}}{3(\alpha_{1} + 2)}\right)^{\frac{1}{2}} \left(\frac{\alpha_{1} + 2}{\beta_{2}}\right)^{-m_{2}} e^{-\frac{\alpha_{1}}{2} \ln \phi} \exp \left[-\frac{\beta_{1} \phi^{m_{1}}}{2}\right]. \quad (38)
\]

Integrating up, again regarding the exponential as almost constant we get

\[
t(\phi) = \left(\frac{3(\alpha_{1} + 2)}{V_{0} \alpha_{1} \beta_{2}^{2}}\right)^{\frac{1}{2}} \left(\frac{\alpha_{1} + 2}{\beta_{2}}\right)^{-m_{2}} e^{-\frac{\alpha_{1}}{2} \ln \phi} \exp \left[-\frac{\beta_{1} \phi^{m_{1}}}{2}\right]. \quad (39)
\]

We are not able to invert this equation in order to obtain a meaningful description of \(\phi(t)\).

2. \(\alpha_{1} = -4\)

The evolution equation for \(\phi\) still holds – provided, of course, that we are still in a \(\phi\) dominated regime.

\[
\dot{\phi} = -4 \left(\frac{V_{0}}{3}\right)^{\frac{1}{2}} \phi \exp \left[-\frac{\beta_{1} \phi^{m_{1}}}{2}\right]. \quad (40)
\]
Then integrating we retrieve
\[ t(\phi) = -\frac{1}{4} \left( \frac{3}{V_0} \right)^{\frac{1}{2}} \ln \phi \exp \left[ \frac{\beta_1 \phi^{m_1}}{2} \right]. \] (41)

Notice that we generate a \( \ln \) rather than a power of \( \phi \) in the solution. This propagates through the calculation and will generate qualitatively different behavior in the Hubble parameter and scalefactor. However, we still see that \( t(\phi) \to -\infty \) for increasing \( \phi \) and so it is an early effect.

\[ \phi(t) = \exp \left[ -4 \left( \frac{V_0}{3} \right)^{\frac{1}{2}} t \right], \] (42)
\[ \chi(t) = \left[ \frac{8}{\beta_2} \left( \frac{V_0}{3} \right)^{\frac{1}{2}} \right]^{\frac{1}{2m_2}}, \] (43)
\[ H(t) = \left( \frac{V_0}{3} \right)^{\frac{1}{2}} \exp \left[ -8 \left( \frac{V_0}{3} \right)^{\frac{1}{2}} t - \frac{\beta_1}{2} \exp \left[ -4m_1 \left( \frac{V_0}{3} \right)^{\frac{1}{2}} t \right] \right], \] (44)
\[ a(t) \propto \exp \left[ -\frac{1}{8} \exp \left[ -8 \left( \frac{V_0}{3} \right)^{\frac{1}{2}} t \right] f_3(t) \right], \] (45)
where
\[ f_3(t) = \exp \left[ -\frac{\beta_1}{2} \exp \left[ -4m_1 \left( \frac{V_0}{3} \right)^{\frac{1}{2}} t \right] \right]. \] (46)

It is clear that \( f_3(t) \to 1 \) rapidly as \( t \) increases. Looking at equation (41), it is clear that \( t \) decreases for increasing \( \phi \) so the inflation we generate is an early time phenomenon.

3. \( \alpha_1 = 0 \)

In this special case the \( \phi \) field will quickly dominate once more, as the potential for \( \chi \) decays quickly. Thus we get behavior very similar to that for one field, with \( \chi \) just in the background as in [14]. Solving in the usual manner gives us

\[ \phi(t) = \left[ \left( \frac{V_0}{3} \right)^{\frac{1}{2}} \beta_1 m_1 (2 - m_1) \right]^{\frac{1}{2m_1}} t^{\frac{1}{2m_1}}, \] (47)
\[ \chi(t) = \left[ \frac{1}{\beta_2} \ln t \right]^{\frac{1}{m_2}}, \] (48)
\[ H(t) = \left( \frac{V_0}{3} \right)^{\frac{1}{2}} \exp \left[ -\frac{\beta_1}{2} \beta_1 m_1 (2 - m_1) \left( \frac{V_0}{3} \right)^{\frac{1}{2}} t^{\frac{m_1}{2m_1}} \right], \] (49)
\[ a(t) \propto \exp \left[ \left( \frac{V_0}{3} \right)^{\frac{1}{2}} t f_2(t) \right], \] (50)
where
\[ f_2(t) = \exp \left[ -\frac{\beta_1}{2} \beta_1 m_1 (2 - m_1) \left( \frac{V_0}{3} \right)^{\frac{1}{2}} t^{\frac{m_1}{2m_1}} \right]. \] (51)

Inflation here occurs at early times.
Proceeding in the usual manner we are able to show that

\[ H(\phi) = \left( \frac{V_{0x}}{3} \right)^{\frac{1}{3}} \phi^{\frac{2}{3}} \exp \left( -\frac{\beta_1 \phi^{m_1}}{2} \right) \left( 1 + \frac{\alpha_1 (\alpha_1 + 2) \chi^2}{\alpha_2 (\alpha_2 + 2) \phi^2} \right)^{\frac{1}{2}}, \]  

\[ = \left( \frac{V_{0x}}{3} \right)^{\frac{1}{3}} \phi^{\frac{2}{3}} \exp \left( -\frac{\beta_1 \phi^{m_1}}{2} \right) \times \left( 1 + \frac{\alpha_1 (\alpha_1 + 2)}{\alpha_2 (\alpha_2 + 2)} \left( \frac{V_{0x} \phi_2 (\alpha_2 + 2)}{V_{0x} \phi_1 (\alpha_1 + 2)} \right)^{\frac{\alpha_2 + 2}{2 \alpha_2 + 2}} \right)^{\frac{1}{\alpha_2 + 2}}. \]

It is then clear, for example, that \(-2 < \alpha_2 < \alpha_1 \) corresponds to \( \phi \) domination. We shall consider this case and that of equal contribution since \( \chi \) domination is identical in behavior to the former.

1. \( \alpha_2 > \alpha_1 > -2; \alpha_2 < -2, \alpha_1 > \alpha_2 \)

If is clear that the \( \phi \) field is dominant here and so the results from section \( \text{III B 1} \) hold for \( \phi(t), H(t) \) and \( a(t) \). In this case we do get different evolution for \( \chi(t) \) though, given by

\[ \chi(t) = \left( \frac{V_{0x} \phi_2 (\alpha_2 + 2)}{V_{0x} \phi_1 (\alpha_1 + 2)} \right)^{\frac{2 (\alpha_1 + 2) \chi^2}{\alpha_2 + 4 (\alpha_2 + 2) \phi^2}}. \]  

2. \( \alpha_1 = \alpha_2 = \alpha \)

This is the only case where we do not get domination by one field over the other. Again the results will be very similar to that of section \( \text{III B 1} \) except that we will generate more inflation because the Hubble parameter will be larger than before. Then we find that:

\[ t(\phi) = \left( \frac{3}{V_{0x}} \right)^{\frac{4}{3}} 2 \left( 1 + \left( \frac{V_{0x}}{V_{0x}} \right)^{\frac{\alpha_2 + 2}{\alpha + 4}} \right)^{\frac{1}{2}} \phi^{\frac{2}{\alpha + 4}} \exp \left( -\frac{\beta_1 \phi^{m_1}}{2} \right), \]  

\[ \phi(t) = \left( \frac{V_{0x}}{3} \right)^{\frac{1}{3}} \left( 1 + \left( \frac{V_{0x}}{V_{0x}} \right)^{\frac{\alpha_2 + 2}{\alpha + 4}} \right)^{-\frac{1}{\alpha + 4}} \times \frac{V_{0x}}{V_{0x}} \phi_1 (\alpha_1 + 2) \left( \frac{V_{0x}}{V_{0x}} \phi_2 (\alpha_2 + 2) \right)^{-\frac{2}{\alpha + 4}} \right)^{\frac{\alpha + 2}{\alpha_2 + 2}} \]  

\[ \times \exp \left[ -\frac{\beta_1}{2} \left( \frac{V_{0x}}{3} \right)^{\frac{1}{3}} \left( 1 + \left( \frac{V_{0x}}{V_{0x}} \right)^{\frac{\alpha_2 + 2}{\alpha + 4}} \right)^{-\frac{1}{\alpha + 4}} \phi_1 (\alpha_1 + 2) \phi_2 (\alpha_2 + 2) \right] \]  

\[ a(t) \propto \exp \left[ \frac{4 + \alpha}{4} \left( \frac{V_{0x}}{3} \right)^{\frac{1}{3}} 2 \left( 1 + \left( \frac{V_{0x}}{V_{0x}} \right)^{\frac{\alpha_2 + 2}{\alpha + 4}} \right)^{-\frac{1}{\alpha + 4}} \left( \frac{V_{0x}}{3} \right)^{\frac{1}{3}} \phi_1 (\alpha_1 + 2) \phi_2 (\alpha_2 + 2) \right] t^{\frac{2}{\alpha + 4}} f_4 (t). \]
where
\[ f_4(t) = \exp \left[ -\frac{\beta_1}{2} \left( \frac{V_{0\phi}}{3} \right)^{\frac{1}{2}} \frac{\alpha}{2} (\alpha + 4) \left( 1 + \left( \frac{V_{0\phi}}{V_{0\chi}} \right)^{\frac{3}{2}} \right)^{-\frac{1}{2}} t \right]^{\frac{2m_1}{\alpha + 4}} \right]. \tag{60} \]

\( f_4(t) \to 1 \) rapidly as \( t \) increases. This gives us inflation at late times once more.

**Small Field Values**

A moment's thought reveals that in order to generate slow-roll at small values of either \( \chi \) or \( \phi \) we must have \( \alpha_i = 0, m \geq 1 \). This can be seen easily by examining the slow-roll parameters \( \xi \). Our analysis will then follow a similar pattern to before. Again, we shall see that one field will become dominant if \( m_1 \neq m_2 \) and we proceed as before.

**D. \( m_1 < m_2, m_1 \neq 2 \)**

Once more the \( \phi \) will provide the dominant contribution and so
\[ H(\phi) = \left( \frac{V_{0\phi}}{3} \right)^{\frac{1}{2}} \exp \left[ -\frac{\beta_1 \phi^{m_1}}{2} \right]. \tag{61} \]

Then we find that
\[ \dot{\phi} = \left( \frac{V_{0\phi}}{3} \right)^{\frac{1}{2}} \beta m_1 \phi^{m_1 - 1} \exp \left[ -\frac{\beta_1 \phi^{m_1}}{2} \right]. \tag{62} \]

This is identical to the case we considered previously in section III.B.3 and so all the results there still hold. The one difference is that the evolution of \( \chi \) changes slightly.
\[ \chi(t) = \left[ \left( \frac{V_{0\phi}}{3} \right)^{\frac{1}{2}} \frac{V_{0\phi}}{V_{0\chi}} \beta_2 m_2 (2 - m_2) t \right]^{\frac{1}{m_2 - 2}}. \tag{63} \]

One should note that as \( \phi \to 0, t \) decreases and so we are looking at an early time effect. This is equivalent to section IV.A of [14]. The cases where either one or both of \( m_i = 2 \) are easily covered and does not result in any analytical difficulties. The behavior is not markedly different from the above.

**E. \( 2 = m_1 < m_2 \)**

The \( \phi \) is still dominant and so we can integrate (62) to achieve
\[ t(\phi) = \left( \frac{3}{V_{0\phi}} \right)^{\frac{1}{2}} \frac{1}{2\beta_1} (\ln \phi) \exp \left[ -\frac{\beta_1 \phi^{2}}{2} \right]. \tag{64} \]

Solving the equation for small \( \phi \) we find that
\[ \phi(t) = \exp \left[ 2 \left( \frac{V_{0\phi}}{3} \right)^{\frac{1}{2}} \beta t \right], \tag{65} \]
\[ \chi(t) = \left[ 4 \left( \frac{V_{0\phi}}{3} \right)^{\frac{1}{2}} \frac{V_{0\phi}}{V_{0\chi}} \beta_2 m_2 (2 - m_2) t \right]^{-\frac{1}{m_2 - 2}}. \tag{66} \]
Note that although the behavior $\chi(t) \sim (-\kappa_1 t)^{m_2}$, $\kappa_1 > 0$ appears problematic, this would be resolved by including the integration constants. This would then give us

$$\chi(t) = \left[ K - 4 \left( \frac{V_{0\phi}}{3} \right) \frac{\beta_1}{V_0} \frac{\beta_1}{\beta_2 m_2 (2 - m_2)} t \right]^{\frac{1}{m_2 - 2}}, \quad K > 0. \tag{67}$$

Now since $\phi(t)$ in a monotonic increasing function, the slow–roll conditions will be violated within finite time and so our solution will no longer be valid. This should occur before $t = \frac{K}{\frac{3}{V_{0\phi}} \frac{\beta_1}{\beta_2 m_2 (2 - m_2)}}$ since there are no physical grounds upon which to rule out this solution.

**F. $m_1 = m_2 = 2$**

In this instance we find that

$$H(\phi) = \frac{1}{\sqrt{3}} \left( V_{0\phi} + V_{0\chi} \right)^{\frac{1}{2}} \exp \left[ \frac{\beta_1 \phi^2}{2} \right]. \tag{68}$$

It is not too difficult to show that

$$t(\phi) = \frac{\sqrt{3}}{2 \beta_1 V_{0\phi}} \left( V_{0\phi} + V_{0\chi} \right)^{\frac{1}{2}} \left( \ln \phi \right) \exp \left[ \frac{\beta_1 \phi^2}{2} \right]. \tag{69}$$

It is clear that as $\phi \to 0$ we have $t \to -\infty$ and so inflation will occur at early times. Then, proceeding in the familiar manner, we can show

$$\phi(t) = \exp \left[ \frac{V_{0\phi}}{\sqrt{3}} \frac{2\beta_1}{V_{0\phi} + V_{0\chi}} t \right], \tag{70}$$

$$\chi(t) = \exp \left[ \frac{V_{0\chi}}{\sqrt{3}} \frac{2\beta_2}{V_{0\phi} + V_{0\chi}} t \right], \tag{71}$$

$$H(t) = \frac{1}{\sqrt{3}} \left( V_{0\phi} + V_{0\chi} \right)^{\frac{1}{2}} \exp \left[ \frac{\beta_1}{2} \exp \left[ \frac{V_{0\phi}}{\sqrt{3}} \frac{2\beta_1}{V_{0\phi} + V_{0\chi}} t \right] \right], \tag{72}$$

$$a(t) \propto \exp \left[ H(t)t \right]. \tag{73}$$

where we have treated all the exponentials as being approximately equal to unity.

**Large and small field values**

If we take one of the fields large and one small we will always have the large field dominant and so this relates to the cases considered already. For example, small $\chi$ means that $m_1 \geq 1$ and so $m_1 < m_2$. Therefore the $\phi$ field will be dominant. This applies to all possible cases.

**G. Summary for Uncoupled Potential**

We have seen that for two fields with uncoupled potentials often one field will be dominant, and we return to an effective single field theory – for the background at least– as studied in [14]. There are, however, several cases where the second field is important and we have demonstrated them. We have seen that it is possible to generate inflation at both late and early times and for large and small values of both of the fields. As a result, these background solutions do not differ qualitatively from the single field set-up [14]. The results are summarized in the table below. If $m_1 < m_2$ then the $\phi$ field is dominant so we only outline the results for $m_1$. If $m_1 = m_2 = m$ the same qualitative behavior exists except that we generate more inflation than we would normally with one field.
TABLE I: Summary of Inflationary Behavior for uncoupled potentials

| Early Times | Late Times | All Times |
|-------------|------------|-----------|
| $m_1 \leq 0, \alpha_1 \leq 0$ | $0 \leq m_1 \leq 1$ | $m_1 = 1, \alpha_1 = 0$ |
| or | or | or |
| $m_1 \geq 1, \alpha_1 = 0$ | $m_1 \leq 0, \alpha_1 \geq 0$ | $m_1 = \alpha_1 = 0$ |

IV. A COUPLED POTENTIAL

We now turn our attention to a potential of the form

$$V(\phi, \chi) = V_0 \phi^{-\alpha} e^{-\beta \chi^m}$$

and repeat the processes of the previous section. The motivation for such a potential arises, for example, in higher–dimensional theories of the form

$$S = \int d^N x \sqrt{-g_N} \left[ \frac{1}{2\kappa_N} R - \frac{1}{2} (\partial \phi)^2 - V(\phi) \right].$$

Writing the $N$–dimensional metric in the form

$$ds_N^2 = g^{(4)}_{\mu\nu} dx^\mu dx^\nu + b^2 ds_D^2,$$

and compactifying the theory on a torus which has zero curvature, one ends up with a potential of the form (74), where $\chi \propto \ln b$ describing the size of the extra dimensions (see e.g. [24]).

We still have the three coupled equations (4-6) to solve which are equivalent to

$$H = \left(\frac{V_0}{3}\right)^\frac{1}{2} \phi^{-\alpha} e^{-\beta \chi^m},$$

$$3H \dot{\phi} = \alpha V_0 \phi^{-(\alpha+1)} e^{-\beta \chi^m},$$

$$3H \dot{\chi} = \beta m \chi^{m-1} V_0 \phi^{-\alpha} e^{-\beta \chi^m},$$

with solution

$$\phi^2 = \begin{cases} \frac{2\alpha}{\beta \ln \chi}, & m \neq 2, \\ \frac{\beta m^2}{2} \chi^{2-2m}, & m = 2. \end{cases}$$

There are certain cases where it is not clear that equation (80) means anything– for example $m > 2, \alpha > 0$– since this would generate an imaginary value for at least one of the fields. At this stage one should remember that we have omitted all integration constants up to this point. As there are no physical grounds on which to rule out such cases, when one puts back these constants (dependent on the initial conditions), any such problems may be overcome.

Before we begin, we shall find it instructive to examine some plots of the potential in question. These are shown in figure 1 where we have plotted $V(\phi, \chi) = \phi^{-\alpha} e^{\pm \beta \chi^m}$ respectively.

Large field Values

A. $1 \geq m > 0$

This guarantees us slow–roll for large values of the fields. We generate the following equation for $\chi$

$$\dot{\chi} = \left(\frac{V_0}{3}\right)^\frac{1}{2} \beta m \left(\frac{\beta m^2 (2 - m)}{2\alpha}\right)^\frac{1}{2} \chi^{\alpha(-m-2)} \chi^{-2m-1} e^{-\beta \chi^m}.$$

We are then able to solve this to produce the following expression,

$$t(\chi) = \left(\frac{3}{V_0}\right)^\frac{1}{2} \frac{2}{\beta^2 m^2} \left(\frac{\beta m^2 (2 - m)}{2\alpha}\right)^\frac{1}{2} \chi^{\alpha(2m-1)} \chi^{2-2m} e^{-\beta \chi^m}.$$
FIG. 1: Plot of the Coupled potential.
Proceeding in a similar way we are able to solve the system producing, to leading order in \( t \), the results below.

\[
\phi(t) = \left( \frac{2\alpha}{\beta m(2 - m)} \right)^{\frac{1}{2}} \left[ \frac{2}{\beta \ln t} \right]^{\frac{2}{2m}} \left[ \frac{2m}{2m - 2} \right], \tag{83}
\]

\[
\chi(t) = \left[ \frac{2}{\beta \ln t} \right]^{\frac{1}{m}}, \tag{84}
\]

\[
H(t) = \frac{2}{\beta^2 m^2 t} \left[ \frac{2}{\beta \ln t} \right]^{\frac{2 - 2m}{2m}}, \tag{85}
\]

\[
a(t) \propto \exp \left[ \frac{1}{\beta m(2 - m)} \left[ \frac{2}{\beta \ln t} \right]^{\frac{2 - 2m}{m}} \right]. \tag{86}
\]

One should note that this gives very similar behavior to section (III A 1). This should not be entirely unexpected because the exponential part of the potential will always dominate for \( m > 0 \).

**B. \( m < 0 \)**

Due to the nature of the slow–roll parameters, we must once more consider large values for both of the fields. Now solving for \( \phi \), treating the exponential approximately constant, we find

\[
t(\phi, \chi) = \left( \frac{3}{V_0} \right)^{\frac{1}{2}} \frac{2}{\alpha(\alpha + 4)} \phi^{\frac{\alpha + 4}{2}} \exp \left[ \frac{\beta \chi^m}{2} \right]. \tag{87}
\]

Substituting for either field using (80) we able to invert this finding

\[
\chi(t) = \left( \frac{\beta m(2 - m)}{2\alpha} \right)^{\frac{m}{2}} \left[ \frac{V_0}{V_0} \right]^{\frac{1}{2}} \left( \frac{\alpha}{2} \right) (\alpha + 4) t \tag{88}
\]

\[
\phi(t) = \left( \frac{V_0}{3} \right)^{\frac{1}{2}} \left( \frac{\alpha}{2} \right) (\alpha + 4) t \tag{89}
\]

One should also note that the onset of inflation occurs in analogy with section (III B 1). It is also worth pointing out that this gives the solution one would expect when \( m \to 0 \). That is, we generate the same results as III B 1 except that the \( \chi \) field freezes out. This should not surprise us since the potential becomes equivalent to that of an inverse power. We then find the form for the Hubble parameter

\[
H(t) = \left( \frac{V_0}{3} \right)^{\frac{1}{2}} \left[ \frac{V_0}{V_0} \right]^{\frac{1}{2}} \left( \frac{\alpha}{2} \right) (\alpha + 4) t^{\frac{4 - m}{(2 - m)(\alpha + 4)}} \tag{90}
\]

\[
\times \exp \left[ -\frac{\beta}{2} \left( \frac{\beta m(2 - m)}{2\alpha} \right)^{\frac{2m}{2m}} \left[ \frac{V_0}{V_0} \right]^{\frac{1}{2}} \left( \frac{\alpha}{2} \right) (\alpha + 4) t \right]^{\frac{4m}{(2 - m)(\alpha + 4)}}, \tag{90}
\]

where, of course, the exponential part tends rapidly to 1. Solving for the scale factor we deduce to leading order

\[
a(t) \propto \exp \left[ \frac{4 + \alpha}{4} t \left( \frac{V_0}{3} \right)^{\frac{1}{2}} \left( \frac{\alpha}{2} \right) (\alpha + 4) t^{\frac{4m}{(2 - m)(\alpha + 4)}} \right], \tag{91}
\]

where

\[
g_1(t) = \exp \left[ -\frac{\beta}{2} \left( \frac{\beta m(2 - m)}{2\alpha} \right)^{\frac{2m}{2m}} \left[ \frac{V_0}{V_0} \right]^{\frac{1}{2}} \left( \frac{\alpha}{2} \right) (\alpha + 4) t \right]^{\frac{4m}{(2 - m)(\alpha + 4)}}. \tag{92}
\]
It is obvious that the special case of $\alpha = 0$ gives us just a single scalar field that evolves. This has already been studied. We must however consider what happens when $\alpha = -4$. We generate the following equation

$$ t(\phi, \chi) = -\frac{1}{4} \left( \frac{3}{V_0} \right)^{\frac{1}{2}} \left[ \ln \phi \right] \exp \left[ -\frac{\beta \chi m}{2} \right]. $$

(93)

Then solving for the fields we find

$$ \phi(t) = \exp \left[ -4 \left( \frac{V_0}{3} \right)^{\frac{1}{2}} t \right], $$

(94)

$$ \chi(t) = \exp \left[ -\frac{8}{2-m} \left( \frac{V_0}{3} \right)^{\frac{1}{2}} t \right], $$

(95)

$$ H(t) = \left( \frac{V_0}{3} \right)^{\frac{1}{2}} \exp \left[ -8 \left( \frac{V_0}{3} \right)^{\frac{1}{2}} t - \frac{\beta}{2} \exp \left[ -\frac{8m}{2-m} \left( \frac{V_0}{3} \right)^{\frac{1}{2}} t \right] \right], $$

(96)

$$ a(t) \propto \exp \left[ -\frac{1}{8} \exp \left[ -8 \left( \frac{V_0}{3} \right)^{\frac{1}{2}} t \right] g_2(t) \right] $$

(97)

where

$$ g_2(t) = \exp \left[ -\frac{\beta}{2} \exp \left[ -\frac{8m}{2-m} \left( \frac{V_0}{3} \right)^{\frac{1}{2}} t \right] \right], $$

(98)

which rapidly tends to unity.

C. 1 < $m$ < 2

If one considers the slow–roll parameters we may expect only to generate inflation for small values of $\chi$. However, we must still take large $\phi$ for slow–roll to be a valid approximation. In this instance, we are able to approximate the exponential by unity. Provided $m \neq 2$, we generate the same set of results as for $m < 0$ with large $\chi$, as in section IV.B.

In the special case where $m = 2$, we get slightly different behavior. We summarize the results below:

$$ \phi(t) = \left[ \left( \frac{V_0}{3} \right)^{\frac{1}{2}} \frac{\alpha}{2} (\alpha+4) t \right]^{\frac{1}{\alpha+4}}, $$

(99)

$$ \chi(t) = \exp \left[ \frac{\beta}{\alpha} \left[ \left( \frac{V_0}{3} \right)^{\frac{1}{2}} \frac{\alpha}{2} (\alpha+4) t \right]^{\frac{1}{\alpha+4}} \right], $$

(100)

$$ H(t) = \left( \frac{V_0}{3} \right)^{\frac{1}{2}} \left[ \left( \frac{V_0}{3} \right)^{\frac{1}{2}} \frac{\alpha}{2} (\alpha+4) t \right]^{\frac{-\alpha}{\alpha+4}} \exp \left[ -\frac{\beta}{2} \exp \left[ m \frac{\beta}{\alpha} \left[ \left( \frac{V_0}{3} \right)^{\frac{1}{2}} \frac{\alpha}{2} (\alpha+4) t \right]^{\frac{-\alpha}{\alpha+4}} \right] \right], $$

(101)

$$ a(t) \propto \exp \left[ \frac{4 + \alpha}{4} \left( \frac{V_0}{3} \right)^{\frac{1}{2}} \left[ \left( \frac{V_0}{3} \right)^{\frac{1}{2}} \frac{\alpha}{2} (\alpha+4) \right]^{\frac{-\alpha}{\alpha+4}} t^{\frac{-\alpha}{\alpha+4}} g_3(t) \right], $$

(102)

where

$$ g_3(t) = \exp \left[ -\frac{\beta}{2} \exp \left[ m \frac{\beta}{\alpha} \left[ \left( \frac{V_0}{3} \right)^{\frac{1}{2}} \frac{\alpha}{2} (\alpha+4) t \right]^{\frac{-\alpha}{\alpha+4}} \right] \right]. $$

(103)
D. Summary for the Coupled Potential

The summary of the inflationary behavior is effectively equivalent to Table 1 with the obvious changes to the parameters. We are able to generate inflation at late, early or all times by tuning the parameters in the potential. If \( m > 0 \), the exponential part is dominant and the solutions produced are similar to that of a single scalar, \( \chi \) with the second field, \( \phi \), in the background. When \( m \leq 0 \), the exponential part is subdominant and the results resemble those for a single scalar, \( \phi \), with a \( \phi^{-\alpha} \) potential. The quantitative modifications are small and quickly decrease with time.

In all cases one of the fields provides the dominant contribution to \( H(t) \) and \( a(t) \). The effects of the second field are present but decay quickly so that the field is only in the background.

V. COSMOLOGICAL PERTURBATIONS

We now turn our attention to the evolution for cosmological perturbations in both of the scalar fields and the spacetime metric during a period of inflation. Again, we assume that the fields are slow–rolling, implying that the parameters (3) are small. Furthermore, we shall assume that both fields contribute approximately equally to the expansion rate. Only then can we expect new features in the perturbation spectra resulting from the second field.

A convenient formalism to study both adiabatic and isocurvature perturbations in inflation was presented in [8], which we will use here. Instead of working with the fields \( \phi \) and \( \chi \) it is useful to perform a rotation as follows:

\[
\delta \sigma = (\cos \theta) \delta \phi + (\sin \theta) \delta \chi \\
\delta s = -(\sin \theta) \delta \phi + (\cos \theta) \delta \chi,
\]

with

\[
\cos \theta = \frac{\dot{\phi}}{\sqrt{\dot{\phi}^2 + \dot{\chi}^2}} \\
\sin \theta = \frac{\dot{\chi}}{\sqrt{\dot{\phi}^2 + \dot{\chi}^2}}
\]

\( \sigma \) is called the adiabatic field and \( s \) is called the entropy field. The motivation for their names becomes clear when one considers fluctuations of them.

The line–element for arbitrary scalar perturbations of the Robertson–Walker metric for a spatially flat universe reads, (using the notation of [8])

\[
ds^2 = -(1 + 2A)dt^2 + 2a^2B_i dx^i dt + a^2 (1 - 2\psi) \delta_{ij} + 2E_{ij} dx^i dx^j.
\]

The gauge–invariant curvature perturbation, defined as

\[
\mathcal{R} = \psi + \frac{H \delta \rho}{\rho},
\]

is, on very large scales, constant for purely adiabatic perturbations (see e.g. [22] and references therein). However, entropy perturbations are a source for the curvature perturbation [108]. The entropy perturbation between two species \( A \) and \( B \), defined as

\[
\mathcal{S} = \frac{\delta n_A}{n_A} - \frac{\delta n_B}{n_B},
\]

where \( n_i \) are the number densities of the particle species \( i \) can evolve in time, even on superhorizon scales. Therefore, it was argued [12] that on very large scales in general we have the following equations describing the evolution of \( \mathcal{R} \) and \( \mathcal{S} \):

\[
\dot{\mathcal{R}} = \gamma H \mathcal{S}, \\
\dot{\mathcal{S}} = \delta H \mathcal{S}.
\]

For the case of the two slow–rolling scalar fields and in the spatial flat gauge (\( \psi = 0 \)), \( \mathcal{R} \) and \( \mathcal{S} \) are given by

\[
\mathcal{R} \approx \frac{H (\dot{\phi} \delta \phi + \dot{\chi} \delta \chi)}{\dot{\phi}^2 + \dot{\chi}^2} = \frac{H \delta \sigma}{\sigma}.
\]
and
\[
S = \frac{H(\dot{\phi}\delta \chi - \dot{\chi}\delta \phi)}{\dot{\phi}^2 + \dot{\chi}^2} = \frac{H\delta s}{\sigma}.
\]
(113)

For $S$ we use the same normalization as in [12]. Fluctuations in the field $\sigma$ are adiabatic perturbations, whereas fluctuations in $s$ are entropic perturbations. On very large scales ($k \ll aH$) and in flat gauge, the evolution of fluctuations are described as
\[
(\delta \sigma)^{\cdot^2} + 3H(\delta \sigma)^{\cdot} + \left(V_{\sigma\sigma} - \dot{\theta}^2\right)\delta \sigma = -2V_\sigma A + \dot{\sigma}A + 2(\dot{\delta} s)^{\cdot} - 2\frac{V_\sigma}{\dot{\sigma}}\dot{\delta} s
\]
(114)
and
\[
(\delta s)^{\cdot^2} + 3H(\delta s)^{\cdot} + \left(V_{ss} - \dot{\theta}^2\right)\delta s = -2\frac{\dot{\theta}}{\dot{\sigma}}[\dot{\sigma}(\delta \sigma)^{\cdot} - \dot{\sigma}A] - \ddot{\delta} \sigma
\]
(115)

The metric perturbation $A$ can be obtained from Einstein’s equation and is given, in the flat gauge, by
\[
HA = 4\pi G \left(\dot{\phi}\delta \phi + \dot{\chi}\delta \chi\right).
\]
(116)

An important point is that $\dot{\theta}$ must be non–vanishing in order for the adiabatic field to be sourced by the entropy field. If $\theta$ is constant, the entropy field does not contribute to perturbations in the gravitational potential.

In [12] the formalism presented was applied to slow–roll inflation with two scalar fields and it was shown that $\gamma$ and $\delta$ in (110) and (111) are given in terms of the slow–roll parameters
\[
\gamma = -2\eta_{ss},
\]
\[
\delta = -2\epsilon + \eta_{\sigma\sigma} - \eta_{ss},
\]
(117) (118)
and are, therefore, specified by the potential $V(\phi, \chi)$. In the last two equations, the slow–roll parameters are constructed from the usual slow–roll parameters for $\phi$ and $\chi$ (3) and are given by
\[
\epsilon = \frac{1}{2} \left(\frac{V_{\phi}}{V}\right)^2 \approx \epsilon_{\phi} + \epsilon_{\chi}.
\]
(119)

and
\[
\eta_{\sigma\sigma} = \eta_{\phi\phi} \cos^2 \theta + 2\eta_{\phi\chi} \cos \theta \sin \theta + \eta_{\chi\chi} \sin^2 \theta,
\]
\[
\eta_{ss} = \eta_{\phi\phi} \sin^2 \theta - 2\eta_{\phi\chi} \cos \theta \sin \theta + \eta_{\chi\chi} \cos^2 \theta,
\]
\[
\eta_{\sigma s} = (\eta_{\chi\chi} - \eta_{\phi\phi}) \sin \theta \cos \theta + \eta_{\phi\chi} (\cos^2 \theta - \sin^2 \theta).
\]
(120)

The time evolution of $R$ and $S$ between horizon crossing and some later time is given by
\[
\begin{pmatrix}
R \\
S
\end{pmatrix} = \begin{pmatrix}
1 & T_{RS} \\
0 & T_{SS}
\end{pmatrix}
\begin{pmatrix}
R \\
S
\end{pmatrix}_*,
\]
(121)
where the asterisk marks the time of horizon crossing. The transfer functions $T_{RS}$ and $T_{SS}$ are given by
\[
T_{SS}(t, t_*) = \exp \left(\int_{t_*}^t \delta(t')H(t')dt'\right)
\]
\[
T_{RS}(t, t_*) = \int_{t_*}^t \gamma(t')T_{SS}(t, t')H(t')dt'.
\]
(122)

Thus, in the case of two slow–rolling scalar fields, the transfer functions are completely specified by the potential through the slow–roll parameters.

We are now in a position to calculate some of these transfer functions for our general potentials. We shall consider the most simple cases only in order to show how the background solutions derived in Section 3 and Section 4 can be used in order to study perturbations.
A. Analytic Calculations

We begin with analytic calculations for two simple cases. The first case considers two exponential potentials (a special case of assisted inflation). The second case is a simple model of a coupled potential.

1. Two Exponentials

Let us now take
\[ V(\phi, \chi) = V_0 e^{-\beta_1 \phi} + V_0 e^{-\beta_2 \chi}. \]  

(123)

Although this setup was studied earlier [27], we would like to check the formalism for the perturbations. We return to the first approximation (10) to make the calculation. Then in this instance we find
\[ V_0 e^{-\beta_1 \phi} = \frac{\beta_2}{\beta_1} V_0 e^{-\beta_2 \chi}, \]  

(124)
\[ \dot{\phi} = \frac{\beta_2}{\beta_1} \dot{\chi}. \]  

(125)

These two equations enable us to calculate all the parameters. We find that
\[ \cos \theta = \frac{\beta_2}{\sqrt{\beta_1^2 + \beta_2^2}}, \sin \theta = \frac{\beta_1}{\sqrt{\beta_1^2 + \beta_2^2}}. \]  

(126)

This shows that we get a constant angle in the phase plane. A quick glance at equations (114-115) reveals that the adiabatic and entropy perturbations decouple in this case. We should expect our analysis to reflect this. It is also straightforward to show that
\[ 2\epsilon = \eta_{\sigma \sigma} = \eta_{ss} = \frac{\beta_1^2 \beta_2^2}{\beta_1^2 + \beta_2^2}, \]  

(127)
\[ \eta_{\sigma s} = 0. \]  

(128)

The functions \( \gamma(t), \delta(t) \) then turn out to be constant given by
\[ \gamma = 0, \]  

(129)
\[ \delta = -\frac{\beta_1^2 \beta_2}{\beta_1^2 + \beta_2^2} < 0. \]  

(130)

Then we simply compute the transfer functions using (122) to see that
\[ T_{SS}(t, t_*) = \left( \frac{a(t)}{a(t_*)} \right)^\delta, \]  

(131)
\[ T_{RS}(t, t_*) = 0. \]  

(132)

Now since \( \delta < 0 \), it is immediately obvious that \( T_{SS}(t, t_*) \to 0 \) as time passes, provided \( t > t_* \). This means that \( R \) remains constant and \( S \to 0 \) from (121). Therefore, in this case we only expect adiabatic perturbations and no entropy perturbations if inflation lasts long enough. This is in agreement with the results in [27]. One should add that the transfer functions are calculated from the time of horizon crossing. However, when we calculate the transfer parameters, \( \gamma(t), \delta(t) \), we assume that \( k/aH \ll 1 \). This is only true a few e–folds after horizon crossing. This means that our calculated transfer functions will be accurate a few e–folds after horizon crossing and so, in effect, they have the wrong initial conditions. Whilst this does not affect their predictions, this means they differ from what we observe numerically. In this instance there is some evolution of \( R \) before it freezes in and some evolution of \( S \) before it follows the decay predicted. This can be seen in Figure [2] in the first few e–folds after horizon exit.
2. A coupled example

The next simplest case will prove to be that with a coupled potential with a straight exponential. That is

\[ V(\phi, \chi) = V_0 \phi^\alpha e^{-\beta \chi}, \]

(133)

We shall find it most convenient to work in terms of the field \( \phi \) before substituting in its solution (83) to compute our integrals. We find that our \( \theta \) now evolves according to

\[ \cos \theta = \frac{\alpha}{\sqrt{\alpha^2 + \beta^2 \phi^2}}, \quad \sin \theta = \frac{\beta \phi}{\sqrt{\alpha^2 + \beta^2 \phi^2}}. \]

(134)

We also have the slow–roll parameters

\[ \epsilon = \frac{1}{2} \left[ \frac{\alpha^2}{\phi^2} + \beta^2 \right], \]

\[ \eta_{\phi\phi} = \frac{\alpha(\alpha + 1)}{\phi^2}, \quad \eta_{\phi \chi} = \frac{\alpha \beta}{\phi}, \quad \eta_{\chi \chi} = \beta^2. \]

(135)

(136)

It is then straightforward to show that

\[ \eta_{\sigma\sigma} = \frac{\alpha^2}{\alpha^2 + \beta^2 \phi^2} \left[ \frac{\alpha(\alpha + 1)}{\phi^2} + 2 \beta^2 + \frac{\beta^4}{\alpha^2 \phi^2} \right], \]

\[ \eta_{ss} = \frac{\alpha^2}{\alpha^2 + \beta^2 \phi^2} \frac{\beta^2}{\alpha}, \quad \eta_{\sigma s} = -\frac{\alpha^2}{\alpha^2 + \beta^2 \phi^2} \frac{\beta}{\phi}. \]

(137)

(138)

We then find that the transfer parameters are given as

\[ \gamma(t) = \frac{2 \beta \alpha}{\phi(\alpha^2 + \beta^2 \phi^2)} = \beta^2 \alpha^\frac{1}{2} \frac{1}{(\ln t)^\frac{1}{2}(\alpha + 4 \ln t)}, \]

\[ \delta(t) = \frac{\alpha^2}{\alpha^2 + \beta^2 \phi^2} \left[ \frac{\alpha}{\phi^2} - \frac{\beta^2}{\alpha} \right] = \frac{\beta^2(\alpha - 4 \ln t)}{4(\ln t)(\alpha + 4 \ln t)}. \]

(139)

(140)

We have also verified these relations numerically. Now it remains to compute the transfer functions which we are able to do exactly. From (85) we see that

\[ H(t) = \frac{2}{\beta^2 t}. \]

(141)

This then gives

\[ T_{SS}(t, t_*) = \exp \left( I(t, t_*) \right), \]

(142)

where

\[ I(t, t_*) = \int_{t_*}^{t} \frac{(\alpha - 4 \ln t')}{2(\ln t')(\alpha + 4 \ln t')} \frac{1}{t'} \, dt' \]

\[ = \frac{1}{2} \ln \left[ \frac{\ln t}{\ln t_*} \right] - \ln \left[ \frac{\alpha + 4 \ln t}{\alpha + 4 \ln t_*} \right]. \]

(143)

(144)

Plugging this back in we find that

\[ T_{SS}(t, t_*) = \left[ \frac{\ln t}{\ln t_*} \right]^\frac{1}{2} \left[ \frac{\alpha + 4 \ln t_*}{\alpha + 4 \ln t} \right]. \]

(145)

Then using this result

\[ T_{RS}(t, t_*) = \beta^2 \alpha^\frac{1}{2} \frac{\alpha + 4 \ln t_*}{(\ln t_*)^\frac{1}{2}} \int_{t_*}^{t} \frac{2}{\beta^2 t' (\alpha + 4 \ln t')^2} \, dt' \]

\[ = 2 \alpha^\frac{1}{2} \frac{\alpha + 4 \ln t_*}{(\ln t_*)^\frac{1}{2}} \left[ \frac{1}{\frac{1}{2} \alpha + 4 \ln t'} \right]_{t_*}^{t}, \]

\[ = \alpha^\frac{1}{2} \left( \frac{1}{(\ln t_*)^\frac{1}{2}} - \frac{\alpha + 4 \ln t_*}{(\ln t_*)^\frac{1}{2}} \frac{1}{\alpha + 4 \ln t} \right). \]

(146)

(147)

(148)
Thus at late times we see
\[ T_{SS} \to 0, \quad T_{RS} \to \frac{\alpha^2}{2} \frac{1}{(\ln t_*)^2}. \] (149)

Again, this implies that at late times \( R \) remains constant, although its value on a certain length scale depends on the time of horizon crossing \( t_* \).

### B. Numerical Calculations

In principle, with the background solutions given in Sections 3 and 4, it is possible to calculate any transfer function that we wish to consider. However, in practice, this requires making a large number of assumptions in order to do the integrals analytically. Therefore, to get precise answers, we will now resort to a numerical investigation of the perturbations. We begin with the case of two exponential potentials, where we can check our numerical results with analytical solutions. Then we study two inverse power potentials, then the case of a mixed potential and finally the coupled potential. The initial conditions for background are chosen such that both fields contribute approximately equally to the expansion rate. With this prescription a change in the initial field values is equivalent to changing the amount of inflation with the same parameters.

We choose to run our simulations in terms of the variables \( \delta \sigma \) and \( \delta s \) – this gives a more accurate solution. The equations for their evolution in the spatially flat gauge are given by

\[
\begin{align*}
\ddot{\delta \sigma} + 3H\dot{\delta \sigma} &+ \left[ \frac{k^2}{a^2} + V_{\sigma \sigma} - \frac{\dot{\theta}^2}{\dot{\theta}} - \frac{1}{\alpha^2} \frac{d}{dt} \left( \frac{\alpha^3 \dot{\sigma}^2}{H} \right) \right] \delta \sigma \\
&= 2(\dot{\delta s})^2 - 2 \left( \frac{V_{\sigma}}{\dot{\sigma}} + \frac{\dot{H}}{H} \right) \dot{\delta s} \\
\ddot{\delta s} + 3H\dot{\delta s} &+ \left[ \frac{k^2}{a^2} + V_{ss} + 3\dot{\theta}^2 \right] \delta s = 2 \frac{V_s}{H} \frac{d}{dt} \left( \frac{H\dot{\sigma}}{\dot{\sigma}} \right).
\end{align*}
\] (150, 151)

The general recipe is as follows: we evolve the background through 100 e–folds of inflation to make sure that the fields are in the slow–roll regime. We then switch on the perturbations so that each mode starts inside the horizon with \( k = 1000aH \) at different times. We then follow the perturbations for another 60 e–folds after horizon exit, where inflation is assumed to end.

To set the initial conditions one treats the modes, \( \delta \phi \) and \( \delta \chi \), as independent stochastic variables deep inside the horizon. With this prescription, it can be shown that \[ a\delta \phi = \frac{1}{\sqrt{2k}} e^{-ik\tau}, \] (152)
deep inside the horizon, and similarly for \( \delta \chi \). In the last expression, \( d\tau = dt/a \) is the conformal time. To calculate the spectra numerically, we use the method described in [10]. One makes two runs: the first run begins in the Bunch–Davis vacuum for \( \delta \phi, \delta \chi = \dot{\delta \chi} = 0 \), and the second in the Bunch–Davis vacuum for \( \delta \chi, \delta \phi = \dot{\delta \phi} = 0 \). The power spectra are then given by

\[
\begin{align*}
\mathcal{P}_R &= \frac{k^3}{2\pi^2} \left( |R_1|^2 + |R_2|^2 \right), \\
\mathcal{P}_S &= \frac{k^3}{2\pi^2} \left( |S_1|^2 + |S_2|^2 \right), \\
\mathcal{P}_C &= \frac{k^3}{2\pi^2} |R_1 S_1 + R_2 S_2|, \\
\end{align*}
\] (153, 154, 155)

where the subscripts 1, 2 are the results for the two different runs. Furthermore, for the tensor modes one finds

\[ P_T = 8 \left( \frac{H_*}{2\pi} \right)^2, \quad n_T = -\left( \frac{\dot{\sigma}}{H} \right)^2. \] (156)

We also define

\[ r_C = \frac{P_C}{\sqrt{\mathcal{P}_R \mathcal{P}_S}}, \quad r_T = \frac{P_T}{16\mathcal{P}_R}. \] (157)
Note that our normalization of \( r_T \) is 16 times smaller than that used in Ref. [26]. As a test for our code, we are able to check it against the results for assisted inflation with two exponential potentials [27]. The analytic results give

\[
\begin{align*}
    n_R - 1 &= -\frac{2\beta_1^2 \beta_2^3}{2(\beta_1^2 + \beta_2^2) - \beta_1^2 \beta_2^3}, \\
    n_S &= 3 \left( 1 - \frac{\sqrt{(\beta_1^2 + \beta_2^2 - 3\beta_1^2 \beta_2^2) (\beta_1^2 + \beta_2^2 - \frac{1}{3} \beta_1^2 \beta_2^3)}}{\beta_1 + \beta_2^3} \right).
\end{align*}
\]

Finally, in the case of two–field slow–roll inflation there are two consistency relations:

\[
\begin{align*}
    r_T &= \frac{n_T}{2} (1 - r_T^2), \\
    (n_C - n_S) r_T &= \frac{n_T}{4} (2n_C - n_R - n_S).
\end{align*}
\]

In the models with runaway potentials considered here we generally find that the spectral index of the correlation spectrum is much larger than the indices for the adiabatic and isocurvature perturbations. Furthermore, the perturbations are essentially uncorrelated so that the two consistency relations are effectively equivalent. We find that they are both satisfied to within a few percent either side.

### 1. Two Exponential Potentials

Let us discuss first the case of two exponential potentials. We have seen from the analytical treatment that in this case the entropic perturbations decay. We now give the fields a power in the exponent, \( V(\phi, \chi) = V_0 \exp \left[ -\beta_1 \phi^{m_1} \right] + V_0 \exp \left[ -\beta_2 \chi^{m_2} \right] \). Using equation (160), one can show that

\[
\tan \theta = \frac{\dot{\chi}}{\dot{\phi}} = \frac{\beta_2 m_2 \chi^{m_2 - 1}}{\beta_1 m_1 \phi^{m_1 - 1}}.
\]

Then, provided the two fields have different potentials and initial conditions, one would expect the entropy perturbations to be sourced because \( \theta \neq 0 \). To gain further understanding, if one makes the field redefinition, \( \psi = \chi^{m_2} \) we may write the Lagrangian as

\[
\mathcal{L}_\psi = \frac{1}{m_2^2} \psi^{\frac{2 - 2m_2}{m_2}} (\partial \psi)^2 + V_0 \exp \left[ -\beta_2 \phi \right].
\]

We see that we generate a non–canonical kinetic term. It has been shown, [28], that such a term sources perturbations in \( \delta s \) and hence in entropy. Because these source terms are proportional to derivatives of this non–canonical factor, these decay for \( m_i \leq 1 \). Furthermore, since the potential is then reduced to that of straight exponential, we would still expect the entropy perturbations to decay, the difference being its duration.

Let us now consider some examples and begin with two exponentials. We set the normalization such that \( \mathcal{R} \sim 10^{-5} \) at the end of inflation.

In our first example, we set \( \beta_1 = 0.05, \beta_2 = 0.10, m_1 = m_2 = 1, V_0 = 10^{-10} \). Figure 2 shows the evolution of \( \mathcal{R} \) and \( S \) for a given mode from the time of horizon crossing. We can see that the entropy perturbation begins to decay almost immediately as one would expect from our previous calculation of the transfer functions. Further, one can see that there is significant evolution in the first few e–folds after horizon crossing, \( N_e \approx 8 \). Following this \( \mathcal{R} \) remains constant as expected. As a second example we consider a case with a power in one of the exponentials. The parameters are as before setting \( m_1 = 0.3 \). Above we have shown that this should source the entropy perturbation which is realized in Figure 2. Let us now observe how the spectral indices behave for all of parameter space. We set \( \beta_1 = 0.05, \beta_2 = 0.10 \) and vary the powers \( m_1, m_2 \). We maintain the initial conditions \( \phi_0 = 1, \chi_0 = 1 \). The results are summarized in Figure 3. We find that \( n_C \) always runs significantly so we do not include it here.

We observe in Figure 3 that the spectrum for the tensor modes is flat– to the extent that it is indistinguishable from a flat spectrum with present observations– and the amount of tensor production is very small. Again this is below the level of current experiments. In general, the spectral indices are functions of the slow–roll parameters which decrease in time with run–away potentials. Thus the more inflation we allow the smaller the indices will be and the closer to scale invariance we will be. In fact, in all cases we can get arbitrarily close to a Harrison–Zel’doovich spectrum by allowing an arbitrarily large amount of inflation. Furthermore, one may expect the plots to be symmetric but
we remind the reader that $\beta_1 \neq \beta_2$ so the two fields always have different potentials. Our results do not completely agree with the analytic results given above. The reason is that we have not yet reached the attractor solution for two exponential potentials. However, we have verified that once we are on the attractor, the analytic and numerical results agree sufficiently.

2. Two Inverse Power Potentials

To gain further insight here, it is helpful to use the field re-definition, $e^{-\psi} = \chi^{-\alpha}$. Then,

$$L_\psi = \frac{1}{2\alpha^2} e^{2\psi}(\partial\psi)^2 + V_0 e^{-\psi}. \quad (164)$$

Therefore, this case corresponds to two scalar fields with non–canonical kinetic term and exponential potential. Because the non–canonical kinetic terms are a source of entropy perturbations, we expect that the entropy perturbations to be more strongly sourced when compared with the previous example of two straight exponentials. The data are shown in Figure 4. Compared with the case of two exponential potentials, the contribution of the entropy to the total perturbation is larger, but we see that the entropy modes are always smaller in magnitude than the curvature perturbation. Furthermore, adiabatic and isocurvature perturbations are effectively uncorrelated. The cleft in the plot arises from the background fields having the same evolution as they each have the same $\alpha$. It is the clear that from our definition of $P_C$ that this quantity should be almost zero in this cleft. Once more the spectral index of the tenors is very small.

3. Mixed Case

We now consider the case of one field having an inverse power potential and the other an exponential potential. Again, we can transform the inverse power law potential into a exponential potential. It is then clear from the discussion of the last two subsections that the results for the spectra must be similar, as one can observe in Figure 5.

4. Coupled Potential

Once more the features observed here are very similar to the cases studied so far. We set $\beta = 0.05$ and vary the other parameters. The data are shown in Figures 6.

FIG. 2: Evolution of $R$ and $S$ for a wavelength that leaves the horizon, $N_* = 8$, 60 e–folds before the end of inflation. The left figure shows the case $\beta_1 = 0.05$, $\beta_2 = 0.10$, $m_1 = m_2 = 1$, $V_0 = 10^{-10}$ with initial conditions $\phi_0 = 1$, $\chi_0 = 1$. On the right hand side we now set $m_1 = 0.3$ leaving the other parameters unchanged. The plots show $|P_R|^\frac{1}{2}$ and $|P_S|^\frac{1}{2}$. 
FIG. 3: Properties of the spectra for two exponentials. We set $\beta_1 = 0.05, \beta_2 = 0.10$ and vary the powers $m_1, m_2$ and we maintain the initial conditions $\phi_0 = 1, \chi_0 = 1$. All quantities are defined in the text. We remind the reader that the scale of $P_S$ is in general arbitrary but here it is equal to $P_R$ at horizon crossing.
FIG. 4: Properties of the spectra for two inverse power potentials. We vary the powers $\alpha_1, \alpha_2$ and we maintain the initial conditions $\phi_0 = 1, \chi_0 = 1$. All quantities are defined in the text. We remind the reader that the scale of $P_S$ is in general arbitrary but here it is equal to $P_R$ at horizon crossing.
FIG. 5: Properties of the spectra for the mixed potential. We set $\beta_1 = 0.05$ and vary the powers $m_1, \alpha_2$. We maintain the initial conditions $\phi_0 = 1, \chi_0 = 1$. All quantities are defined in the text. We remind the reader that the scale of $P_S$ is in general arbitrary but here it is equal to $P_R$ at horizon crossing.
FIG. 6: Properties of the spectra for the coupled potential. We set $\beta = 0.05$, vary the powers $m, \alpha$ and we maintain the initial conditions $\phi_0 = 1, \chi_0 = 1$. All quantities are defined in the text. We remind the reader that the scale of $P_S$ is in general arbitrary but here it is equal to $P_R$ at horizon crossing.
5. Compatibility with WMAP

In general, all our models generate almost scale–invariant spectra with minimal tensor production. This also allows both red and blue spectra as seen in Figure 7. Furthermore, we are able to get arbitrarily close to scale–invariance by allowing more inflation. In addition, we find that the perturbations produced are essentially uncorrelated.

Let us comment on the compatibility with current experiments. The WMAP results have provided strong constraints on the initial power spectrum and hence on inflationary models. If one examines Figure 7, one sees that it possible to have

$$0.9 \leq n_R \leq 1.3 \text{ at } 2\sigma. \quad (165)$$

We include the equivalent plot for our models.

![Graph showing allowed region for curvature index from the WMAP data alone. The plot scales the tensor/scalar ratio to agree with the WMAP team’s plot. One can clearly see the degeneracy in the predictions of $r_T$ and $n_R$ among the possible models. For example as $m \to 0$ in the coupled case we approach the results of the inverse power potential with the same dominant $\alpha$. This is what we would have expected from the background solutions of sections 3 and 4.](image)

One open issue we have not discussed is the end inflation. This may be caused by another scalar field. This process is, in general, non–adiabatic and would generate further entropy production which would add to that from the two fields. Hence, the spectra deep in the radiation dominated epoch depend on the details of reheating. Without a detailed study of this mechanism it is impossible to draw firm conclusions from the entropic values in all cases.

VI. CONCLUSIONS

In this paper we have studied in considerable depth inflationary solutions generated by two scalar fields and potentials of the form $V = V_0 \phi^{-\alpha} \exp(-\beta \phi^m)$. This potential is very general and encompasses potentials motivated by supergravity and string theory and more general theories motivated by higher–dimensional gravity.

We have seen that it is possible to generate a large range of different inflating universes by varying the parameters in the theory. We find that slow–roll is a valid approximation by means of numerical calculations and we are able to verify that the analytic solutions presented in this paper hold. For two uncoupled fields (Section 3), we found that, often, only one of the fields is dominant. However, we also found solutions where both fields are important for the inflationary dynamics. For two coupled fields with the above potential (Section 4), we have found that only one of the fields dominates inflation, whereas the other field is only a background field. This case might be interesting if the background field acted as a curvaton field, for example.

We have used some of the background solutions we found to describe the evolution of the perturbations in two specific cases and, potentially, one would be able to find the transfer functions for all the solutions discussed here. We have demonstrated this with two examples. In general, however, one has to resort to numerical methods in order
to evaluate the transfer functions. The solutions presented here thus allow one to make predictions about the relative size of $R$ and $S$ for a large class of inflationary scenarios.

We made use of numerical calculations to investigate consequences for cosmological perturbations. We discussed the case where both fields contribute approximately equally to the expansion rate during a period of slow–roll inflation. We found that our models produce scale–invariant spectra with a small scalar–tensor ratio. By varying the parameters of the potential, it is possible to produce both red and blue spectra for the curvature perturbation. This is still consistent with observational constraints. Although the exact details depend on the initial condition of the fields, we do not expect our conclusions will be changed qualitatively. In the case of double inflation, one can generate uncorrelated [5, 6] as well as correlated perturbations, [29]. In contrast, in multi–field inflationary models with run–away potentials, the perturbations are always uncorrelated.

In order to make predictions for the anisotropies in the cosmic microwave background radiation, one has to follow the perturbations into the radiation dominated era, which involves a detailed calculation of the decay of both fields. Ending inflation in the class of models discussed here will involve presumably a third field, which becomes important just at the end of the inflationary stage.

In future work we will use some of these insights gained in this work in order study inflationary models in models with extra dimensions [30].

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