The Dirichlet problem and prime ends

Denis Kovtonyuk, Igor’ Petkov and Vladimir Ryazanov

31 марта 2015 г.

Аннотация

It is developed the theory of the boundary behavior of homeomorphic solutions of the Beltrami equations $\overline{\partial}f = \mu \partial f$ of the Sobolev class $W^{1,1}_{\text{loc}}$ with respect to prime ends of domains. On this basis, under certain conditions on the complex coefficient $\mu$, it is proved the existence of regular solutions of its Dirichlet problem in arbitrary simply connected domains and pseudoregular as well as multivalent solutions in arbitrary finitely connected domains with continuous boundary data in terms of prime ends.

2010 Mathematics Subject Classification: Primary 30C62, 30D40, 37E30. Secondary 35A16, 35A23, 35J67, 35J70, 35J75.

Key words: Beltrami equations, Dirichlet problem, prime ends, boundary behavior, regular solutions, simply connected domains, pseudoregular and multivalent solutions, finitely connected domains.

1 Introduction

Theorems on the existence of homeomorphic solutions of the class $W^{1,1}_{\text{loc}}$ have been recently proved for many degenerate Beltrami equations, see, e.g., the monographs [4] and [12] and further references therein. The theory of boundary behavior of homeomorphic solutions with generalized derivatives and of the Dirichlet problem for the wide circle of degenerate Beltrami equations in Jordan domains was developed in the papers [7], [9], [10] and [14].

Recall some definitions. Let $D$ be a domain in the complex plane $\mathbb{C}$ and let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. (almost everywhere). Beltrami equation is an equation of the form

$$f_{\bar{z}} = \mu(z)f_z$$  \hspace{1cm} (1.1)
where \( f_z = \overline{\partial} f = (f_x + if_y)/2, \ f_z = \partial f = (f_x - if_y)/2, \ z = x + iy, \ f_x \) and \( f_y \) are partial derivatives of \( f \) in \( x \) and \( y \), correspondingly. The function is called its **complex coefficient** and

\[
K_\mu(z) = \frac{1 + \left| \mu(z) \right|}{1 - \left| \mu(z) \right|}
\]  

its **dilatation quotient**. Beltrami equation (1.1) is called **degenerate** if \( K_\mu \) is essentially unbounded, i.e., \( K_\mu \notin L^\infty(D) \).

Boundary values problems for the Beltrami equations are due to the famous dissertation of Riemann who considered the partial case of analytic functions when \( \mu(z) \equiv 0 \), and to the works of Hilbert (1904, 1924) and Poincare (1910) for the corresponding Cauchy–Riemann system, see, e.g., further references in the papers [10] and [14].

The classic **Dirichlet problem** for Beltrami equation (1.1) in a Jordan domain \( D \) is the search of continuous function \( f : D \to \mathbb{C} \), having partial derivatives of the first order a.e., satisfying (1.1) a.e. and also the boundary condition

\[
\lim_{z \to \zeta} \text{Re} \, f(z) = \varphi(\zeta) \quad \forall \ \zeta \in \partial D \quad (1.3)
\]

for a prescribed continuous function \( \varphi : \partial D \to \mathbb{R} \), see, e.g., [20].

To study the similar problem in domains with more complicated boundaries we need to apply the theory of prime ends by Caratheodory, see, e.g., his paper [1] or Chapter 9 in monograph [2].

The main difference in the case is that \( \varphi \) should be already a function of a boundary element (prime end \( P \)) but not of a boundary point. Moreover, (1.3) should be replaced by the condition

\[
\lim_{n \to \infty} \text{Re} \, f(z_n) = \varphi(P) \quad (1.4)
\]

for all sequences of points \( z_n \in D \) converging to prime ends \( P \) of the domain \( D \). Note that (1.4) is equivalent to the condition that

\[
\lim_{z \to P} \text{Re} \, f(z) = \varphi(P) \quad (1.5)
\]

along any ways in \( D \) going to the prime ends \( P \) of the domain \( D \).
Later on, $\overline{D}_P'$ denotes the completion of the domain $D$ by its prime ends and $E_D$ denotes the space of these prime ends, both with the topology of prime ends described in Section 9.4 of monograph [2]. In addition, continuity of mappings $f : \overline{D}_P \to \overline{D}_P'$ and boundary functions $\varphi : E_D \to \mathbb{R}$ should mean with respect to the given topology of prime ends.

Generalized homeomorphic solutions of the Beltrami equations are mappings with finite distortion whose boundary behavior with respect to prime ends in arbitrary finitely connected domains of $\mathbb{C}$ was studied in our last preprint [8] and we refer the reader to this text for historic comments, definitions and preliminary remarks.

2 Boundary behavior of solutions of the Beltrami equations

On the basis of results of Section 8 in [8], we obtain the corresponding results on the boundary behavior of solutions of the Beltrami equations.

**Theorem 2.1.** Let $D$ and $D'$ be bounded finitely connected domains in $\mathbb{C}$ and $f : D \to D'$ be a homeomorphic solution of the class $W^{1,1}_{\text{loc}}$ of (1.1) with $K_\mu \in L^1(D)$. Then $f^{-1}$ is extended to a continuous mapping of $\overline{D}_P'$ onto $\overline{D}_P$.

Furthermore, it is sufficient in Theorem 2.1 to assume that $K_\mu$ is integrable only in a neighborhood of $\partial D$ or even more weak conditions which are due to Lemma 5.1 in [8].

However, any degree of integrability of $K_\mu$ does not guarantee a continuous extendability of the direct mapping $f$ to the boundary, see, e.g., an example in the proof of Proposition 6.3 in [12]. Conditions for it have perfectly another nature. The principal relevant result is the following.

**Theorem 2.2.** Let $D$ and $D'$ be bounded finitely connected domains in $\mathbb{C}$ and $f : D \to D'$ be a homeomorphic solution of the class $W^{1,1}_{\text{loc}}$ of the Beltrami equation (1.1) with the condition

$$\int_0^{\varepsilon_0} \frac{dr}{||K_\mu||(z_0, r)} = \infty \quad \forall \ z_0 \in \partial D \quad (2.1)$$
where \(0 < \varepsilon_0 < d_0 = \sup_{z \in D} |z - z_0|\) and
\[
||K_\mu||(z_0, r) = \int_{S(z_0, r)} K_\mu \, ds . \tag{2.2}
\]

Then \(f\) can be extended to a homeomorphism of \(\overline{D}_P\) onto \(\overline{D'}_P\).

Here and later on, we set that \(K_\mu\) is equal to zero outside of the domain \(D\).

**Corollary 2.1.** In particular, the conclusion of Theorem 2.2 holds if
\[
k_{z_0}(r) = O\left(\log \frac{1}{r}\right) \quad \forall \, z_0 \in \partial D \tag{2.3}
\]
as \(r \to 0\) where \(k_{z_0}(r)\) is the average of \(K_\mu\) over the circle \(|z - z_0| = r\).

**Lemma 2.1.** Let \(D\) and \(D'\) be bounded finitely connected domains in \(\mathbb{C}\) and \(f : D \to D'\) be a homeomorphic solution of the class \(W^{1,1}_{\text{loc}}\) of the Beltrami equation \((1.1)\) with \(K_\mu \in L^1(D)\) and
\[
\int_{\varepsilon < |z - z_0| < \varepsilon_0} K_\mu(z) \cdot \psi_{z_0, \varepsilon}^2(|z - z_0|) \, dm(z) = o\left(I^2_{z_0}(\varepsilon)\right) \quad \forall \, z_0 \in \partial D \tag{2.4}
\]
as \(\varepsilon \to 0\) where \(0 < \varepsilon_0 < \sup_{z \in D} |z - z_0|\) and \(\psi_{z_0, \varepsilon}(t) : (0, \infty) \to [0, \infty], \varepsilon \in (0, \varepsilon_0),\) is a two-parametric family of measurable functions such that
\[
0 < I_{z_0}(\varepsilon) := \int_{\varepsilon}^{\varepsilon_0} \psi_{z_0, \varepsilon}(t) \, dt < \infty \quad \forall \, \varepsilon \in (0, \varepsilon_0) .
\]
Then \(f\) can be extended to a homeomorphism of \(\overline{D}_P\) onto \(\overline{D'}_P\).

**Theorem 2.3.** Let \(D\) and \(D'\) be bounded finitely connected domains in \(\mathbb{C}\) and \(f : D \to D'\) be a homeomorphic solution of the class \(W^{1,1}_{\text{loc}}\) of the Beltrami equation \((1.1)\) with \(K_\mu\) of finite mean oscillation at every point \(z_0 \in \partial D\). Then \(f\) can be extended to a homeomorphism of \(\overline{D}_P\) onto \(\overline{D'}_P\).

In fact, here it is sufficient for the function \(K_\mu(z)\) to have a dominant of finite mean oscillation in a neighborhood of every point \(z_0 \in \partial D\).

**Corollary 2.2.** In particular, the conclusion of Theorem 2.3 holds if
\[
\lim_{\varepsilon \to 0} \int_{B(z_0, \varepsilon)} K_\mu(z) \, dm(z) < \infty \quad \forall \, z_0 \in \partial D . \tag{2.5}
\]
Theorem 2.4. Let $D$ and $D'$ be bounded finitely connected domains in $\mathbb{C}$ and $f : D \to D'$ be a homeomorphic solution of the class $W^{1,1}_{\text{loc}}$ of the Beltrami equation (1.1) with the condition
\[
\int_{\varepsilon<|z-z_0|<\varepsilon_0} K_\mu(z) \frac{dm(z)}{|z-z_0|^2} = o \left( \left[ \log \frac{1}{\varepsilon} \right]^2 \right) \quad \forall \ z_0 \in \partial D . \tag{2.6}
\]
Then $f$ can be extended to a homeomorphism of $\overline{D}_P$ onto $\overline{D'}_P$.

Remark 2.1. Condition (2.6) can be replaced by the weaker condition
\[
\int_{\varepsilon<|z-z_0|<\varepsilon_0} \frac{K_\mu(z) \, dm(z)}{(|z-z_0| \log \frac{1}{|z-z_0|})^2} = o \left( \left[ \log \log \frac{1}{\varepsilon} \right]^2 \right) \quad \forall \ z_0 \in \partial D . \tag{2.7}
\]
In general, here we are able to give a number of other conditions of logarithmic type. In particular, condition (2.3), thanking to Theorem 2.2, can be replaced by the weaker condition
\[
k_{z_0}(r) = O \left( \log \log \frac{1}{r} \right) . \tag{2.8}
\]

Finally, we complete the series of criteria with the following integral condition.

Theorem 2.5. Let $D$ and $D'$ be bounded finitely connected domains in $\mathbb{C}$ and $f : D \to D'$ be a homeomorphic solution of the class $W^{1,1}_{\text{loc}}$ of the Beltrami equation (1.1) with the condition
\[
\int_{D} \Phi (K_\mu(z)) \, dm(z) < \infty \tag{2.9}
\]
for a nondecreasing convex function $\Phi : [0, \infty) \to [0, \infty)$ such that
\[
\int_{\delta_*}^{\infty} \frac{d\tau}{\tau \Phi^{-1}(\tau)} = \infty \tag{2.10}
\]
at some $\delta_* > \Phi(0)$. Then $f$ can be extended to a homeomorphism of $\overline{D}_P$ onto $\overline{D'}_P$.

Corollary 2.3. In particular, the conclusion of Theorem 2.5 holds if at some $\alpha > 0$
\[
\int_{D} e^{\alpha K_\mu(z)} \, dm(z) < \infty . \tag{2.11}
\]
Remark 2.2. Note that condition (2.10) is not only sufficient but also necessary for a continuous extension to the boundary of all direct mappings $f$ with integral restrictions of type (2.9), see, e.g., Theorem 5.1 and Remark 5.1 in [11]. Recall also that condition (2.10) is equivalent to each of conditions (7.14)–(7.18) in [8].

3 On the Dirichlet problem in simply connected domains

For $\varphi(P) \not\equiv \text{const}$, $P \in E_D$, a regular solution of Dirichlet problem (1.4) for Beltrami equation (1.1) is a continuous discrete open mapping $f : D \to \mathbb{C}$ of the Sobolev class $W_{\text{loc}}^{1,1}$ with its Jacobian

$$J_f(z) = |f_z|^2 - |f_{\overline{z}}|^2 \neq 0 \quad \text{a.e.}$$

satisfying (1.1) a.e. and condition (1.4) for all prime ends of the domain $D$. For $\varphi(P) \equiv c \in \mathbb{R}$, $P \in E_D$, a regular solution of the problem is any constant function $f(z) = c + ic'$, $c' \in \mathbb{R}$.

Recall that a mapping $f : D \to \mathbb{C}$ is called discrete if the pre-image $f^{-1}(y)$ of every point $y \in \mathbb{C}$ consists of isolated points and open if the image of every open set $U \subseteq D$ is open in $\mathbb{C}$. Later on, $\mathbb{D}$ denotes the unit disk in $\mathbb{C}$.

Theorem 3.1. Let $D$ be a bounded simply connected domain in $\mathbb{C}$ and let $\mu : D \to \mathbb{D}$ be a measurable function with $K_\mu \in L_{\text{loc}}^1$ and, moreover,

$$\int_0^{\delta(z_0)} \frac{dr}{||K_\mu||(z_0, r)} = \infty \quad \forall \ z_0 \in \overline{D}$$

for some $0 < \delta(z_0) < d(z_0) = \sup_{z \in D} |z - z_0|$ and

$$||K_\mu||(z_0, r) := \int_{S(z_0, r)} K_\mu(z) \ ds .$$

Then the Beltrami equation (1.1) has a regular solution $f$ of the Dirichlet problem (1.4) for every continuous function $\varphi : E_D \to \mathbb{R}$.

Here and later on, we set that $K_\mu$ is equal to zero outside of the domain $D$. 
Corollary 3.1. In particular, the conclusion of Theorem 3.1 holds if
\[ k_{z_0}(\varepsilon) = O\left(\log \frac{1}{\varepsilon}\right) \quad \forall \ z_0 \in \mathbb{T} \] (3.3)
as \( \varepsilon \to 0 \) where \( k_{z_0}(\varepsilon) \) is the average of the function \( K_\mu \) over the circle \( S(z_0, \varepsilon) \).

Proof of Theorem 3.1. First of all note that \( E_D \) cannot consist of a single prime end. Indeed, all rays going from a point \( z_0 \in D \) to \( \infty \) intersect \( \partial D \) because the domain \( D \) is bounded, see, e.g., Proposition 2.3 in [13] or Proposition 13.3 in [12]. Thus, \( \partial D \) contains more than one point and by the Riemann theorem, see, e.g., II.2.1 in [3], \( D \) can be mapped onto the unit disk \( \mathbb{D} \) with a conformal mapping \( R \). However, then by the Caratheodory theorem there is one-to-one correspondence between elements of \( E_D \) and points of the unit circle \( \partial \mathbb{D} \), see, e.g., Theorem 9.6 in [2].

Let \( F \) be a regular homeomorphic solution of equation (1.1) in the class \( W^{1,1} \), which exists in view of condition (3.2), see, e.g., Theorem 5.4 in paper [16] or Theorem 11.10 in monograph [12].

Note that the domain \( D^* = F(D) \) is simply connected in \( \mathbb{C} \), see, e.g., Lemma 5.3 in [6] or Lemma 6.5 in [12]. Let us assume that \( \partial D^* \) in \( \mathbb{C} \) consists of the single point \( \{\infty\} \). Then \( \mathbb{C} \setminus D^* \) also consists of the single point \( \infty \), i.e., \( D^* = \mathbb{C} \), since if there is a point \( \zeta_0 \in \mathbb{C} \) in \( \mathbb{C} \setminus D^* \), then, joining it and any point \( \zeta_\ast \in D^* \) with a segment of a straight line, we find one more point of \( \partial D^* \) in \( \mathbb{C} \), see, e.g., again Proposition 2.3 in [13] or Proposition 13.3 in [12]. Now, let \( \mathbb{D}^* \) denote the exterior of the unit disk \( \mathbb{D} \) in \( \mathbb{C} \) and let \( \kappa(\zeta) = 1/\zeta, \kappa(0) = \infty, \kappa(\infty) = 0 \). Consider the mapping \( F_\ast = \kappa \circ F : \tilde{D} \to \mathbb{D}_0 \) where \( \tilde{D} = F^{-1}(\mathbb{D}^*) \) and \( \mathbb{D}_0 = \mathbb{D} \setminus \{0\} \) is the punctured unit disk. It is clear that \( F_\ast \) is also a regular homeomorphic solution of Beltrami equation (1.1) in the class \( W^{1,1}_{\text{loc}} \) in the bounded two-connected domain \( \tilde{D} \) because the mapping \( \kappa \) is conformal. By Theorem 2.2 there is a one-to-one correspondence between elements of \( E_D \) and 0. However, it was shown above that \( E_D \) cannot consists of a single prime end. This contradiction disproves the above assumption that \( \partial D^* \) consists of a single point in \( \mathbb{C} \).

Thus, by the Riemann theorem \( D^* \) can be mapped onto the unit disk \( \mathbb{D} \) with a conformal mapping \( R_\ast \). Note that the function \( g := R_\ast \circ F \) is again a regular homeomorphic solution in the Sobolev class \( W^{1,1}_{\text{loc}} \) of Beltrami equation (1.1) which
maps $D$ onto $\mathbb{D}$. By Theorem 2.2 the mapping $g$ admits an extension to a homeomorphism $g_* : \overline{D}_P \to \overline{\mathbb{D}}$.

We find a regular solution of the initial Dirichlet problem (1.4) in the form $f = h \circ g$ where $h$ is a holomorphic function in $\mathbb{D}$ with the boundary condition

$$\lim_{z \to \zeta} \text{Re } h(z) = \varphi(g_*^{-1}(\zeta)) \quad \forall \zeta \in \partial \mathbb{D}.$$ 

Note that we have from the right hand side a continuous function of the variable $\zeta$.

As known, the analytic function $h$ can be reconstructed in $\mathbb{D}$ through its real part on the boundary up to a pure imaginary additive constant with the Schwartz formula, see, e.g., § 8, Chapter III, Part 3 in [5],

$$h(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \varphi \circ g_*^{-1}(\zeta) \cdot \frac{\zeta + z}{\zeta - z} \cdot \frac{d\zeta}{\zeta}.$$ 

It is easy to see that the function $f = h \circ g$ is a desired regular solution of the Dirichlet problem (1.4) for Beltrami equation (1.1). \qed

Applying Lemma 2.2 in [15], see also Lemma 7.4 in [12], we obtain the following general lemma immediately from Theorem 3.1.

**Lemma 3.1.** Let $D$ be a bounded simply connected domain in $\mathbb{C}$ and $\mu : D \to \mathbb{D}$ be a measurable function with $K_\mu \in L^1(D)$. Suppose that, for every $z_0 \in \overline{D}$, there exist $\varepsilon_0 < d(z_0) := \sup_{z \in D} |z - z_0|$ and a one-parametric family of measurable functions $\psi_{z_0, \varepsilon} : (0, \infty) \to (0, \infty), \varepsilon \in (0, \varepsilon_0)$ such that

$$0 < I_{z_0}(\varepsilon) := \int_{\varepsilon}^{\varepsilon_0} \psi_{z_0, \varepsilon}(t) \, dt < \infty \quad \forall \varepsilon \in (0, \varepsilon_0) \quad (3.4)$$

and as $\varepsilon \to 0$

$$\int_{D(z_0, \varepsilon, \varepsilon_0)} K_\mu(z) \cdot \psi_{z_0, \varepsilon}^2(|z - z_0|) \, dm(z) = o(I_{z_0}^2(\varepsilon)) \quad (3.5)$$

where $D(z_0, \varepsilon, \varepsilon_0) = \{ z \in D : \varepsilon < |z - z_0| < \varepsilon_0 \}$. Then the Beltrami equation (1.1) has a regular solution $f$ of the Dirichlet problem (1.4) for every continuous function $\varphi : E_D \to \mathbb{R}$. 

Remark 3.1. In fact, it is sufficient here to request instead of the condition $K_\mu \in L^1(D)$ only a local integrability of $K_\mu$ in the domain $D$ and the condition $\|K_\mu\|(z_0, r) \neq \infty$ for a.e. $r \in (0, \varepsilon_0)$ at all $z_0 \in \partial D$.

By Lemma 3.1 with the choice $\psi_{z_0, \varepsilon}(t) \equiv 1/(t \log \frac{1}{\varepsilon})$ we obtain the following result, see also Lemma 6.1 in [8].

Theorem 3.2. Let $D$ be a bounded simply connected domain in $\mathbb{C}$ and let $\mu : D \to \mathbb{D}$ be a measurable function such that

$$K_\mu(z) \leq Q(z) \in \text{FMO}(\overline{D}).$$  \hspace{1cm} (3.6)

Then the Beltrami equation (1.1) has a regular solution $f$ of the Dirichlet problem (1.4) for every continuous function $\varphi : E_D \to \mathbb{R}$.

Corollary 3.2. In particular, the conclusion of Theorem 3.2 holds if $K_\mu(z) \leq Q(z) \in \text{BMO}(\overline{D})$.

By Corollary 6.1 in [8] we obtain from Theorem 3.2 the following statement.

Corollary 3.3. The conclusion of Theorem 3.2 also holds if

$$\limsup_{\varepsilon \to 0} \int_{B(z_0, \varepsilon)} K_\mu(z) \, dm(z) < \infty \quad \forall \, z_0 \in \overline{D}.$$  

The next statement follows from Lemma 3.1 under the choice $\psi(t) = 1/t$, see also Remark 3.1.

Theorem 3.3. Let $D$ be a bounded simply connected domain in $\mathbb{C}$ and let $\mu : D \to \mathbb{D}$ be a measurable function such that

$$\int_{\varepsilon < |z-z_0| < \varepsilon_0} K_\mu(z) \frac{dm(z)}{|z-z_0|^2} = o \left( \left[ \log \frac{1}{\varepsilon} \right]^2 \right) \quad \forall \, z_0 \in \overline{D}. \hspace{1cm} (3.7)$$

Then the Beltrami equation (1.1) has a regular solution $f$ of the Dirichlet problem (1.4) for every continuous function $\varphi : E_D \to \mathbb{R}$.

Remark 3.2. Similarly, choosing in Lemma 3.1 $\psi(t) = 1/(t \log 1/t)$ instead of $\psi(t) = 1/t$, we obtain that condition (3.7) can be replaced by the condition

$$\int_{\varepsilon < |z-z_0| < \varepsilon_0} \frac{K_\mu(z) \, dm(z)}{(|z-z_0| \log \frac{1}{|z-z_0|})^2} = o \left( \left[ \log \log \frac{1}{\varepsilon} \right]^2 \right) \quad \forall \, z_0 \in \overline{D}. \hspace{1cm} (3.8)$$
Here we are able to give a number of other conditions of a logarithmic type. In particular, condition (3.3), thanking to Theorem 3.1, can be replaced by the weaker condition
\[ k_{z_0}(r) = O\left( \log \frac{1}{r} \log \log \frac{1}{r} \right). \] (3.9)

Finally, by Theorem 3.1, applying also Theorem 3.1 in [17], we come to the following result.

**Theorem 3.4.** Let \( D \) be a bounded simply connected domain in \( \mathbb{C} \) and let \( \mu : D \to \mathbb{D} \) be a measurable function such that
\[
\int_D \Phi(K_\mu(z)) \, dm(z) < \infty
\] (3.10)
where \( \Phi : [0, \infty) \to [0, \infty) \) is a nondecreasing convex function such that
\[
\int_\delta^\infty \frac{d\tau}{\tau \Phi^{-1}(\tau)} = \infty
\] (3.11)
for some \( \delta > \Phi(0) \). Then the Beltrami equation (1.1) has a regular solution \( f \) of the Dirichlet problem (1.4) for every continuous function \( \varphi : E_D \to \mathbb{R} \).

**Remark 3.3.** Recall that condition (3.11) is equivalent to each of conditions (7.14)–(7.18) in [8]. Moreover, condition (3.11) is not only sufficient but also necessary to have a regular solution of the Dirichlet problem (1.4) for every Beltrami equation (1.1) with integral restriction (3.10) for every continuous function \( \varphi : E_D \to \mathbb{R} \). Indeed, by the Stoilow theorem on representation of discrete open mappings, see, e.g., [19], every regular solution \( f \) of the Dirichlet problem (1.4) for Beltrami equation (1.1) with \( K_\mu \in L^1_{\text{loc}} \) can be represented in the form of composition \( f = h \circ F \) where \( h \) is a holomorphic function and \( F \) is a regular homeomorphic solution of (1.1) in the class \( W^{1,1}_{\text{loc}} \). Thus, by Theorem 5.1 in [18] on the nonexistence of regular homeomorphic solutions of (1.1) in the class \( W^{1,1}_{\text{loc}} \), if (3.11) fails, then there is a measurable function \( \mu : D \to \mathbb{D} \) satisfying integral condition (3.10) for which Beltrami equation (1.1) has no regular solution of the Dirichlet problem (1.4) for any nonconstant continuous function \( \varphi : E_D \to \mathbb{R} \).
Corollary 3.4. In particular, the conclusion of Theorem 3.4 holds if at some \( \alpha > 0 \)
\[
\int_D e^{\alpha K_\mu(z)} \, dm(z) < \infty .
\] (3.12)

4 On pseudoregular solutions in multiply connected domains

As it was first noted by Bojarski, see, e.g., § 6 of Chapter 4 in [20], the Dirichlet problem for the Beltrami equations, generally speaking, has no regular solution in the class of continuous (single–valued) in \( \mathbb{C} \) functions with generalized derivatives in the case of multiply connected domains \( D \). Hence the natural question arose: whether solutions exist in wider classes of functions for this case? It is turned out to be solutions for this problem can be found in the class of functions admitting a certain number (related with connectedness of \( D \)) of poles at prescribed points. Later on, this number will take into account the multiplicity of these poles from the Stoilow representation.

Namely, a pseudoregular solution of such a problem is a continuous (in \( \overline{\mathbb{C}} \)) discrete open mapping \( f : D \to \overline{\mathbb{C}} \) of the Sobolev class \( W^{1,1}_{\text{loc}} \) (outside of poles) with its Jacobian \( J_f(z) = |f_z|^2 - |f_{\bar{z}}|^2 \neq 0 \) a.e. satisfying (1.1) a.e. and the boundary condition (1.4).

Arguing similarly to the case of simply connected domains and applying Theorem V.6.2 in [3] on conformal mappings of finitely connected domains onto circular domains and also Theorems 4.13 and 4.14 in [20], we obtain the following result.

**Theorem 4.1.** Let \( D \) be a bounded \( m \)-connected domain in \( \mathbb{C} \) with nondegenerate boundary components and \( \mu : D \to \mathbb{D} \) be a measurable function with \( K_\mu \in L^1_{\text{loc}} \) and
\[
\int_0^{\delta(z_0)} \frac{dr}{||K_\mu||(z_0, r)} = \infty \quad \forall \ z_0 \in \overline{D}
\] (4.1)
for some \( 0 < \delta(z_0) < d(z_0) = \sup_{z \in D} |z - z_0| \) and
\[
||K_\mu||(z_0, r) := \int_{S(z_0, r)} K_\mu(z) \, ds .
\]
Then the Beltrami equation (1.1) has a pseudoregular solution \( f \) of the Dirichlet problem (1.4) with \( k \geq m - 1 \) poles at prescribed points in \( D \) for every continuous function \( \varphi : E_D \to \mathbb{R} \).

Here, as before, we set \( K_\mu \) to be extended by zero outside of the domain \( D \).

**Corollary 4.1.** In particular, the conclusion of Theorem 4.1 holds if

\[
k_{z_0}(\varepsilon) = O\left( \log \frac{1}{\varepsilon} \right) \quad \forall \ z_0 \in \overline{D}
\]

as \( \varepsilon \to 0 \) where \( k_{z_0}(\varepsilon) \) is the average of the function \( K_\mu \) over the circle \( S(z_0, \varepsilon) \).

**Proof of Theorem 4.1.** Let \( F \) be a regular solution of equation (1.1) in the class \( W^{1, 1}_{\text{loc}} \) that exists by condition (4.1), see, e.g., Theorem 5.4 in the paper [16] or Theorem 11.10 in the monograph [12].

Note that the domain \( D^* = F(D) \) is \( m \)-connected in \( \overline{\mathbb{C}} \) and there is a natural one-to-one correspondence between components \( \gamma_j \) of \( \gamma = \partial D \) and components \( \Gamma_j \) of \( \Gamma = \partial D^* \), \( \Gamma_j = C(\gamma_j, F) \) and \( \gamma_j = C(\Gamma_j, F^{-1}) \), \( j = 1, \ldots, m \), see, e.g., Lemma 5.3 in [6] or Lemma 6.5 in [12]. Moreover, by Remark 1.1 in [8] every subspace \( E_j \) of \( E_D \) associated with \( \gamma_j \) consists of more than one prime end, even it is homeomorphic to the unit circle.

Next, no one of \( \Gamma_j \), \( j = 1, \ldots, m \), is degenerated to a single point. Indeed, let us assume that \( \Gamma_{j_0} = \{ \zeta_0 \} \) first for some \( \zeta_0 \in \mathbb{C} \). Let \( r_0 \in (0, d_0) \) where \( d_0 = \inf_{\zeta \in \Gamma \setminus \Gamma_{j_0}} |\zeta - \zeta_0| \). Then the punctured disk \( D_0 = \{ \zeta \in \mathbb{C} : 0 < |\zeta - \zeta_0| < r_0 \} \) is in the domain \( D^* \) and its boundary does not intersect \( \Gamma \setminus \Gamma_{j_0} \). Set \( \tilde{D} = F^{-1}(D_0) \). Then by the construction \( \tilde{D} \subset D \) is a 2-connected domain, \( \overline{\tilde{D}} \cap \gamma \setminus \gamma_{j_0} = \emptyset \), \( C(\gamma_{j_0}, \tilde{F}) = \{ \zeta_0 \} \) and \( C(\zeta_0, \tilde{F}^{-1}) = \gamma_{j_0} \) where \( \tilde{F} \) is a restriction of the mapping \( F \) to \( \tilde{D} \). However, this contradicts Theorem 2.2 because, as it was noted above, \( E_{j_0} \) contains more than one prime end.

Now, let assume that \( \Gamma_{j_0} = \{ \infty \} \). Then the component of \( \overline{\mathbb{C}} \setminus D^* \) associated with \( \Gamma_{j_0} \), see Lemma 5.1 in [6] or Lemma 6.3 in [12], is also consists of the single point \( \infty \) because if the interior of this component is not empty, then choosing there an arbitrary point \( \zeta_0 \) and joining it with a point \( \zeta_* \in D^* \) by a segment of a straight line we would find one more point in \( \Gamma_{j_0} \), see, e.g., Proposition 2.3 in [13] or Proposition 13.3 in [12].
Thus, applying if it is necessary an additional stretching (conformal mapping), with no loss of generality we may assume that $D^*$ contains the exteriority $\mathbb{D}_*$ of the unit disk $\mathbb{D}$ in $\mathbb{C}$. Set $\kappa(\zeta) = 1/\zeta$, $\kappa(0) = \infty$, $\kappa(\infty) = 0$. Consider the mapping $F_* = \kappa \circ F : \widetilde{D} \to \mathbb{D}_0$ where $\widetilde{D} = F^{-1}(\mathbb{D}_*)$ and $\mathbb{D}_0 = \mathbb{D}\setminus\{0\}$ is the punctured unit disk. It is clear that $F_*$ is also a homeomorphic solution of Beltrami equation (1.1) of the class $W^{1,1}_\text{loc}$ in 2–connected domain $\widetilde{D}$ because the mapping $\kappa$ is conformal. Consequently, by Theorem 2.2 elements of $E_{j_0}$ should be in a one-to-one correspondence with 0. However, it was already noted, $E_{j_0}$ cannot consists of a single prime end. The obtained contradiction disproves the assumption that $\Gamma_{j_0} = \{\infty\}$.

Thus, by Theorem V.6.2 in [3], see also Remark 1.1 in [8], $D^*$ can be mapped with a conformal mapping $R_*$ onto a bounded circular domain $\mathbb{D}^*$ whose boundary consists of mutually disjoint circles. Note that the function $g := R_* \circ F$ is again a regular homeomorphic solution in the Sobolev class $W^{1,1}_\text{loc}$ for Beltrami equation (1.1) that maps $D$ onto $\mathbb{D}^*$. By Theorem 2.2 the mapping $g$ admits an extension to a homeomorphism $g_* : \overline{D}_P \to \overline{\mathbb{D}}^*$.

Let us find a solution of the initial Dirichlet problem (1.4) in the form $f = h \circ g$ where $h$ is a meromorphic function in $\mathbb{D}^*$ with the boundary condition

$$\lim_{z \to \zeta} \text{Re } h(z) = \varphi(g_*^{-1}(\zeta)) \quad \forall \ \zeta \in \partial \mathbb{D}^* \quad (4.3)$$

and $k \geq m - 1$ poles corresponding under the mapping $g$ to those at prescribed points in $D$. Note that the function from the right hand side in (1.3) is continuous in the variable $\zeta$. Thus, such a function $h$ exists by Theorems 4.13 and 4.14 in [20]. It is clear that the function $f$ associated with $h$ is by the construction a desired pseudoregular solution of the Dirichlet problem (1.4) for Beltrami equation (1.1). \square

Applying Lemma 2.2 in [15], see also Lemma 7.4 in [12], we obtain immediately from Theorem 4.1 the next lemma.

**Lemma 4.1.** Let $D$ be a bounded $m$–connected domain in $\mathbb{C}$ with nondegenerate boundary components and $\mu : D \to \mathbb{D}$ be a measurable function with $K_\mu \in L^1(D)$. Suppose that for every $z_0 \in \overline{D}$ there exist $\varepsilon_0 < d(z_0) := \sup_{z \in \overline{D}} |z - z_0|$ and one-parameter family of measurable functions $\psi_{z_0, \varepsilon} : (0, \infty) \to (0, \infty)$, $\varepsilon \in (0, \varepsilon_0)$ such
that
\[ 0 < I_{z_0}(\varepsilon) := \int_{\varepsilon}^{\varepsilon_0} \psi_{z_0,\varepsilon}(t) \, dt < \infty \quad \forall \, \varepsilon \in (0, \varepsilon_0) \]  
(4.4)

and
\[ \int_{\varepsilon <|z-z_0|<\varepsilon_0} K_\mu(z) \cdot \psi^2_{z_0,\varepsilon}(|z-z_0|) \, dm(z) = o(I^2_{z_0}(\varepsilon)) \quad \text{as } \varepsilon \to 0 . \]  
(4.5)

Then the Beltrami equation \((1.1)\) has a pseudoregular solution \(f\) of the Dirichlet problem \((1.4)\) with \(k \geq m - 1\) poles at prescribed points in \(D\) for every continuous function \(\varphi : E_D \to \mathbb{R}\).

**Remark 4.1.** In fact, here it is sufficient to assume instead of the condition \(K_\mu \in L^1(D)\) the local integrability of \(K_\mu\) in the domain \(D\) and the condition \(||K_\mu||(z_0, r) \neq \infty\) for a.e. \(r \in (0, \varepsilon_0)\) and all \(z_0 \in \partial D\).

By Lemma 4.1 with the choice \(\psi_{z_0,\varepsilon}(t) \equiv 1/t \log \frac{1}{t}\) we obtain the following result, see also Lemma 6.1 in [8].

**Theorem 4.2.** Let \(D\) be a bounded \(m\)-connected domain in \(\mathbb{C}\) with nondegenerate boundary components and \(\mu : D \to \mathbb{D}\) be a measurable function such that
\[ K_\mu(z) \leq Q(z) \in \text{FMO}(\overline{D}) . \]  
(4.6)

Then the Beltrami equation \((1.1)\) has a pseudoregular solution \(f\) of the Dirichlet problem \((1.4)\) with \(k \geq m - 1\) poles at prescribed points in \(D\) for every continuous function \(\varphi : E_D \to \mathbb{R}\).

**Corollary 4.2.** In particular, the conclusion of Theorem 4.2 holds if \(K_\mu(z) \leq Q(z) \in \text{BMO}(\overline{D})\).

By Corollary 6.1 in [8] we have by Theorem 4.2 the next:

**Corollary 4.3.** The conclusion of Theorem 4.2 holds if
\[ \limsup_{\varepsilon \to 0} \int_{B(z_0, \varepsilon)} K_\mu(z) \, dm(z) < \infty \quad \forall \, z_0 \in \overline{D}. \]

The following statement follows from Lemma 4.1 through the choice \(\psi(t) = 1/t\), see also Remark 4.1.
Theorem 4.3. Let $D$ be a bounded $m$–connected domain in $\mathbb{C}$ with nondegenerate boundary components and $\mu : D \to \mathbb{D}$ be a measurable function such that

$$\int_{\varepsilon < |z-z_0| < \varepsilon_0} K_\mu(z) \frac{dm(z)}{|z-z_0|^2} = o \left( \left[ \log \frac{1}{\varepsilon} \right]^2 \right) \quad \forall \ z_0 \in \overline{D}. \quad (4.7)$$

Then the Beltrami equation \((1.1)\) has a pseudoregular solution $f$ of the Dirichlet problem \((1.4)\) with $k \geq m - 1$ poles at prescribed points in $D$ for every continuous function $\varphi : E_D \to \mathbb{R}$.

Remark 4.2. Similarly, choosing in Lemma 4.1 $\psi(t) = 1/(t \log 1/t)$ instead of $\psi(t) = 1/t$ we obtain that the condition \((4.7)\) can be replaced by the condition

$$\int_{\varepsilon < |z-z_0| < \varepsilon_0} \frac{K_\mu(z) dm(z)}{\left[ |z-z_0| \log \frac{1}{|z-z_0|} \right]^2} = o \left( \left[ \log \log \frac{1}{\varepsilon} \right]^2 \right) \quad \forall \ z_0 \in \overline{D}. \quad (4.8)$$

Here we are able to give a number of other conditions of the logarithmic type. In particular, condition \((4.2)\), thanking to Theorem 4.1, can be replaced by the weaker condition

$$k_{z_0}(r) = O \left( \log \frac{1}{r} \log \log \frac{1}{r} \right). \quad (4.9)$$

Finally, by Theorem 4.1 applying also Theorem 3.1 in the paper [17], we come to the following result.

Theorem 4.4. Let $D$ be a bounded $m$–connected domain in $\mathbb{C}$ with nondegenerate boundary components, $k \geq m - 1$ and $\mu : D \to \mathbb{D}$ be a measurable function such that

$$\int_D \Phi(K_\mu(z)) \ dm(z) < \infty \quad (4.10)$$

where $\Phi : [0, \infty) \to [0, \infty)$ is nondecreasing convex function with the condition

$$\int_{\delta}^{\infty} \frac{d\tau}{\tau \Phi^{-1}(\tau)} = \infty \quad (4.11)$$

for some $\delta > \Phi(0)$. Then the Beltrami equation \((1.1)\) has a pseudoregular solution $f$ of the Dirichlet problem \((1.4)\) with $k$ poles at prescribed points in $D$ for every continuous function $\varphi : E_D \to \mathbb{R}$.
Recall that condition (4.11) is equivalent to every of conditions (7.14)–(7.18) in [8].

Corollary 4.4. In particular, the conclusion of Theorem 4.4 holds if for some \( \alpha > 0 \)

\[
\int_D e^{\alpha K_\mu(z)} \, dm(z) < \infty .
\] (4.12)

5 On multivalent solutions in finitely connected domains

In multiply connected domains \( D \subset \mathbb{C} \), in addition to pseudoregular solutions, Dirichlet problem (1.4) for Beltrami equations (1.1) admits multivalent solutions in the spirit of the theory of multivalent analytic functions.

We say that a discrete open mapping \( f : B(z_0, \varepsilon_0) \to \mathbb{C} \), where \( B(z_0, \varepsilon_0) \subset D \), is a local regular solution of equation (1.1) if \( f \in W^{1,1}_{\text{loc}} \), \( J_f \neq 0 \) and \( f \) satisfies (1.1) a.e. Two local regular solutions \( f_0 : B(z_0, \varepsilon_0) \to \mathbb{C} \) and \( f_* : B(z_*, \varepsilon_*) \to \mathbb{C} \) of equation (1.1) is called an extension of each to other if there is a chain of such solutions \( f_i : B(z_i, \varepsilon_i) \to \mathbb{C} \), \( i = \overline{1,m} \), that \( f_1 = f_0, f_m = f_* \) and \( f_i(z) \equiv f_{i+1}(z) \) for \( z \in E_i := B(z_i, \varepsilon_i) \cap B(z_{i+1}, \varepsilon_{i+1}) \neq \emptyset \), \( i = \overline{1,m-1} \). A collection of local regular solutions \( f_j : B(z_j, \varepsilon_j) \to \mathbb{C} \), \( j \in J \), is said to be a multivalent solution of equation (1.1) in \( D \), if the disks \( B(z_j, \varepsilon_j) \) cover the whole domain \( D \) and \( f_j \) are mutually extended each to other through this collection and the collection is maximal by inclusion. A multivalent solution of (1.1) is called multivalent solution of Dirichlet problem (1.4) if \( u(z) = Ref(z) = Ref_j(z) \), \( z \in B(z_j, \varepsilon_j) \), \( j \in J \), is a single–valued function in \( D \) and satisfies condition (1.4).

The proof of the existence of multivalent solutions of Dirichlet problem (1.4) for Beltrami equation (1.1) in finitely connected domains is reduced to the Dirichlet problem for harmonic functions in circular domains, see, e.g., § 3 of Chapter VI in [3].

Theorem 5.1. Let \( D \) be a bounded \( m- \) connected domain in \( \mathbb{C} \) with nondegenerate boundary components and \( \mu : D \to \mathbb{D} \) be a measurable function which satisfies
hypotheses of Theorems 4.1–4.4 or Corollaries 4.1–4.4. Then Beltrami equation (1.1) has a multivalent solution of Dirichlet problem (1.4) for every continuous function \( \varphi : \mathcal{E}_D \to \mathbb{R} \).

**Proof.** It is sufficient to prove the statement of the theorem under the hypotheses of Theorem 4.1 because the hypotheses of the rest theorems and corollaries imply the hypotheses of Theorem 4.1 as it was shown above.

Next, similarly to the first part of Theorem 4.1, we first prove that there is a regular homeomorphic solution \( g \) of Beltrami equation (1.1) mapping the domain \( D \) onto a circular domain \( D^* \) whose boundary consists of mutually disjoint circles. By Theorem 2.2 the mapping \( g \) admits an extension to a homeomorphism \( g^* : \overline{D}_P \to \overline{D}^* \).

As known, in the circular domain \( \mathbb{D}^* \), there is a solution of the Dirichlet problem

\[
\lim_{z \to \zeta} u(z) = \varphi(g^{-1}_*(\zeta)) \quad \forall \zeta \in \partial \mathbb{D}^* \tag{5.1}
\]

for harmonic functions \( u \), see, e.g., § 3 of Chapter VI in [3]. Let \( B_0 = B(z_0, r_0) \) is a disk in the domain \( D \). Then \( B_0 = g(B_0) \) is a simply connected subdomain of the circular domain \( \mathbb{D}^* \) where there is a conjugate function \( v \) determined up to an additive constant such that \( h = u + iv \) is a single–valued analytic function. The function \( h \) can be extended to, generally speaking multivalent, analytic function \( H \) along any path in \( \mathbb{D}^* \) because \( u \) is given in the whole domain \( \mathbb{D}^* \).

Thus, \( f = H \circ g \) is a desired multivalent solution of the Dirichlet problem (1.4) for Beltrami equation (1.1). \( \square \)

**Remark 5.1.** Note that it can be proved an analog of the known theorem on monodromy for analytic functions stating that any multivalent solution of Beltrami equation (1.1) in a simply connected domain \( D \) is its regular single–valued solution.

Note also, here the hypothesis that the boundary components of the domain is not degenerate to single points is essential as it is shown by the simplest case \( \mu(z) \equiv 0 \) of analytic functions in the punctured unit disk because the isolated singularities of harmonic functions are removable and by the maximum principle harmonic functions in the unit disk are uniquely determined by its continuous boundary values.
Список литературы

[1] CARATHEODORY C. Über die Begrenzung der einfachzusammenhängender Gebiete // Math. Ann. – 1913. – 73. – P. 323 – 370.

[2] COLLINGWOOD E. F., LOHWATOR A.J. The Theory of Cluster Sets. – Cambridge Tracts in Math. and Math. Physics 56. – Cambridge: Cambridge Univ. Press, 1966.

[3] GOLUZIN G. M. Geometric Theory of Functions of a Complex Variable. – Transl. of Math. Monographs 26. – Providence: AMS, 1969.

[4] GUTLYANSKII V., RYAZANOV V., SREBRO U., YAKUBOV E. The Beltrami Equations: A Geometric Approach. – Developments in Math. 26. – New York etc.: Springer, 2012.

[5] HURWITZ A., COURANT R. The Function theory. – Moscow: Nauka, 1968 [in Russian].

[6] IGNA'T'EV A., RYAZANOV V. Finite mean oscillation in the mapping theory // Ukr. Mat. Vis. – 2005. – 2, no. 3. – P. 395–417, 443 [in Russian]; transl. in Ukr. Math. Bull. – 2005. – 2, no. 3. – P. 403–424.

[7] KOVTONYUK D., PETKOV I., RYAZANOV V. On the boundary behaviour of solutions to the Beltrami equations // Complex Variables and Elliptic Equations. – 2013. – 58, no. 5. – P. 647 – 663.

[8] KOVTONYUK D., PETKOV I., RYAZANOV V. On boundary behavior of mappings with finite distortion in the plane // ArXiv: 1502.01603v2 [math.CV] 8 Feb 2015, 30 p.

[9] KOVTONYUK D.A., PETKOV I.V., RYAZANOV V.I., SALIMOV R.R. Boundary behavior and Dirichlet problem for Beltrami equations // Algebra and Analysis. - 2013. - 25, no. 4. - P. 101-124 [in Russian]; transl in St. Petersburg Math. J. - 2014. - 25. - P. 587- 603.

[10] KOVTONYUK D., PETKOV I., RYAZANOV V., SALIMOV R. On the Dirichlet problem for the Beltrami equation // J. Anal. Math. - 2014. - 122, no. 4. - P. 113- 141.

[11] KOVTONYUK D., RYAZANOV V. On the boundary behavior of generalized quasi-isometries // J. Anal. Math. – 2011. – 115. – P. 103–119.

[12] MARTIO O., RYAZANOV V., SREBRO U., YAKUBOV E. Moduli in Modern Mapping Theory. – Springer Monographs in Mathematics. – New York etc.: Springer, 2009.

[13] RYAZANOV V.I., SALIMOV R.R. Weakly flat spaces and bondaries in the mapping theory // Ukr. Mat. Vis. – 2007. – 4, № 2. – P. 199 – 234 [in Russian]; transl. in Ukrainian Math. Bull. – 2007. – 4, no. 2. – P. 199 – 233.

[14] RYAZANOV V., SALIMOV R., SREBRO U., YAKUBOV E. On Boundary Value Problems for the Beltrami Equations // Contemporary Math. - 2013. - 591. - P. 211-242.
[15] Ryazanov V., Sevost'yanov E. Equicontinuous classes of ring $Q$-homeomorphisms // Sibirsk. Math. Zh. - 2007. - 48, № 6. - P. 1361–1376 [in Russian]; transl. in Siberian Math. J. - 2007. - 48, no. 6. - P. 1093–1105.

[16] Ryazanov V., Srebro U., Yakubov E. On ring solutions of Beltrami equation // J. Anal. Math. – 2005. – 96. – P. 117–150.

[17] Ryazanov V., Srebro U., Yakubov E. On integral conditions in the mapping theory // Ukrainian Math. Bull. – 2010. – 7, no. 1. – P. 73-87.

[18] Ryazanov V., Srebro U., Yakubov E. Integral conditions in the theory of the Beltrami equations // Complex Var. Elliptic Equ. – 2012. – 57, no. 12. – P. 1247–1270.

[19] Stoilow S.: Lecons sur les Principes Topologue de le Theorie des Fonctions Analytique. Gauthier-Villars (1938). Riemann, Gauthier-Villars, Paris (1956) [in French].

[20] Vekua I.N. Generalized analytic functions. – London: Pergamon Press, 1962.

Denis Kovtonyuk, Igor’ Petkov and Vladimir Ryazanov,
Institute of Applied Mathematics and Mechanics,
National Academy of Sciences of Ukraine,
74 Roze Luxemburg Str., Donetsk, 83114, Ukraine,
denis_kovtonyuk@bk.ru, igorpetkov@list.ru,
vl_ryazanov@mail.ru