Time crystals in primordial perturbations

Hao-Hao Li$^{1}$ and Yun-Song Piao$^{1,2}$

$^1$School of Physics, University of Chinese Academy of Sciences, Beijing 100049, China
$^2$Institute of Theoretical Physics, Chinese Academy of Sciences, P.O. Box 2735, Beijing 100190, China

(Dated: January 20, 2022)

Cosmological time crystal (TC) corresponds to a matter state where the periodic motion of field forms a limit cycle in its phase space. We explore what would happen if it existed in inflationary phase. It is found that the limit cycle responsible for TC will inevitably cause the periodic oscillation of the primordial perturbation spectrum. The oscillatory patterns of the spectrum depend on the TC parameters, and so encode the crystalline patterns of TC.

Introduction – Time crystals (TC) were put forward by Wilczek in 2012 [1][2], which described the time-periodic self-organized structures. In this new perspective, the spontaneous symmetry breaking occurs in the time-direction. Recently, the concept of TC has developed quickly in physical community, in particular condensed matter physics. The discrete TC has been discovered in the condensed matter experiments [3][4].

In Ref.[5], based on the $P(X,\phi)$ theory, Bains, Hertzberg and Wilczek proposed the cosmological TC, where $X = -(\nabla\phi)^2$, which corresponds that a limit cycle should exist in $(\phi, \dot{\phi})$ phase space, see FIG.1. However, such a cycle will inevitably cross the null energy condition (NEC) curve, which suggests the violation of NEC [5, 6]. It is well-known that the violation of NEC will cause the instability of perturbation on short scales. Recently, this instability has been cured by applying the higher-order derivative operator of $\phi$ [7] (equivalently $R^{(3)}\delta g^{00}$, as in [8–12], where $R^{(3)}$ is the Ricci scalar on the 3-dimensional spacelike surface). Is such a stable cosmological TC observable? It is still an open question.

According to [5], the cosmological TC may appear in inflationary phase. If so, the time-periodic patterns of TC might be imprinted in the power spectrum of primordial perturbation. In our work, we will explore this possibility, and show that the limit cycle in $(\phi, \dot{\phi})$ in phase space will result in the periodic oscillation of the primordial spectrum, which so encodes the crystalline patterns of TC, as in original concept [2]. Here, the TC model considered is a special subclass of G-inflation [13][14].

A stable cosmological TC – In Ref.[5], the cosmological TC was built by applying such a Langrangian,

$$\mathcal{L} = \frac{M_P^2}{2} R + \sum_{n,m\neq0} c_{n,m} \phi^{2n} (X/2)^m - V(\phi),$$

(1)

with the coefficient $c_{0,1} = c_{0,2} = 1$, $c_{1,1} = \beta > 0$, and the Higgs-like potential

$$V(\phi) = \Lambda + \frac{1}{4\alpha} (\alpha\phi^2 - 1)^2,$$

(2)

where $\beta > 0$ and $\alpha > 0$ are the parameters, both impact the period of cosmological TC [5, 6]. Here, although the Lagrangian (1) is that of k-inflation [15, 16], the inflation is actually potential-driven, and $\Lambda = \text{const.}$ is responsible for the inflation. We have

$$M_P^2 R^2 = \frac{\rho/3}{6} \dot{\phi}^2 + \frac{\dot{\phi}^4}{4} + \frac{V}{3}.$$  (3)

We solve the motive equation of $\phi$ numerically and plot it in FIG.1. We see that $\phi$ oscillate around the maximum point of its potential. As showed in Ref.[5], such solutions are the attractors, which form the limit cycles in $(\phi, \dot{\phi})$ phase space. The amplitude and period of oscillations keep constant, such a oscillatory matter state is called the cosmological TC.

FIG. 1. The evolution of background field $\phi$, its Higgs-like potential and the limit cycle. Here, the Planck mass $M_P \sim 10^{18}$. In $(\phi, \dot{\phi})$ phase space, the yellow one is the limit cycle, which is the attractor [5]. See the gray and brown curves for the trajectories of $\phi$ with different initial $(\phi_0, \dot{\phi}_0)$. Inside the green curve, the NEC is violated. Inside the red ellipse, the field is ghost-like.

The existence of limit cycles requires that the NEC must be violated, because the sign of $H$ will change periodically. The NEC condition for any null vextors $k^\mu$ is
$T_{\mu\nu}k^\mu k^\nu = P_X (\nabla_\mu \phi k^\mu)^2 \geq 0$. In FIG.1(c), the boundary of NEC violation is an elliptic curve $P_X = 0$, which is $\phi^2 + \beta \phi^2 = 1$, where $P = \sum_{k,m \neq 0} c_{k,m} \phi^{2n} (X/2)^m$, see the Lagrangian (1). Another elliptic curve is $\rho_X = 2XP_{XX} + P_X = 0$, which is $3\phi^2 + \beta \phi^2 = 1$. Inside it, $\rho_X < 0$.

Because the limit cycle will not cross the ellipse with $\rho_X = 0$, but must cross the boundary of NEC violation, the bound on the oscillating amplitude of the field is [6]

$$\frac{1}{\sqrt{\beta}} < \phi_{\text{max}} < \sqrt{\frac{1 + \sqrt{1 + \alpha}}{\alpha}}.$$  

which shows how the width of the limit cycle depends on the parameters $\alpha$ and $\beta$.

It is well-known that if the NEC is violated, the Lagrangian (1) will be instable in perturbative level, since $c_s^2 < 0$ is inevitable, $c_s$ is the sound speed of perturbation. Here, we will cure it by considering Galileon field operator $\Box \phi$.

It is simple to work in unitary gauge $\phi = \phi(t)$. The EFT Lagrangian of cosmological perturbation corresponding to the Lagrangian (1) with the Galileon operator $\Box \phi$ is

$$\mathcal{L} = \frac{M_P^2}{2} \partial^2 \phi - \lambda(t) c(t) g^{00} + \frac{1}{2} M_s^2(t) (\delta g^{00})^2 - \frac{1}{2} m_3^2(t) \delta K \delta g^{00}$$

where

$$\delta g^{00} = 1 - \frac{X}{\phi^2(t)}$$

and the coefficients $c(t) = -M_P^2 \dot{H}$, $\lambda(t) = M_P^2 \left( \dot{H} + 3H^2 \right)$, $M_s^2(t) = \dot{\phi}^2 P_{XX}$, $m_3^2$ is free, $K$ is the extrinsic curvature on the 3-dimensional spacelike surface, $\Box \phi$ is related to $\delta K$ and $\phi^\sigma = \nabla^\sigma \phi$, see, e.g. Ref.[17].

The action of scalar perturbation $\zeta$ for (5) is

$$S^{(2)} = \int a^3 Q_s \left[ \zeta^2 - c_s^2 (\partial \zeta / \partial x)^2 \right] d^4 x,$$

where

$$Q_s = \frac{M_P^2}{(2M_P^2 H - m_3^2/2)^2} \left( 8M_P^2 \dot{\phi}^4 P_{XX} - 4M_P^2 \dot{H} + 3m_3^6 \right).$$

$$c_s^2 Q_s = d \left[ \frac{2M_P^2 a}{(2M_P^2 H - m_3^2)} \right] / (adt) - M_P^2.$$  

According to (10), without the operator $\delta K \delta g^{00}$, we have

$$c_s^2 Q_s = -M_P^2 H / H^2.$$  

When the NEC is violated, $\dot{H} > 0$. Thus we have else $c_s^2 < 0$, or $Q_s < 0$, the corresponding evolution is instable. Considering the operator $\delta K \delta g^{00}$, we have

$$c_s^2 Q_s = -M_P^2 H / (M_P^2 H - m_3^2/2)^2 - \frac{M_P^2}{(2M_P^2 H - m_3^2/2 - m_3^6/4)}.$$  

Now, if one properly adjusts the coefficient $m_3^6$ of $\delta K \delta g^{00}$, it is possible that both $Q_s > 0$ and $c_s^2 > 0$ are satisfied simultaneously. Thus the operator $\delta K \delta g^{00}$ may be applied to cure the instabilities of scalar perturbations in cosmological TC.

It has been found in Refs.[8–10], see also [11, 12, 18], that the operator $R^{(3)} \delta g^{00}$ is significant for building “healthy” nonsingular cosmologies (avoiding No-Go Theorem [19, 20]), see also [21, 22] for stable nonsingular cosmologies with the operators $N^2$, $N'K$ and $(\partial N)^2$ in DHOST theory [23], where $N$ is the lapse. In Ref.[7], $R^{(3)} \delta g^{00}$ has been adopted for stable cosmological TC. However, here we only consider the inflation scenario, which has been proved to be singular in the past (past-incomplete). Thus the operator $\delta K \delta g^{00}$ is enough to dispel the instabilities, as in G-inflation [13, 14].

Here, it is convenient to assume that $m_3^6$ is proportional to $H$, $m_3^6 = xM_P^2 H (x \neq 0)$. According to (9) and (10), we have

$$c_s^2 = \frac{(2x - x^2)}{3x^2} + O(\epsilon, \dot{\phi}^4 P_{XX} / H^2),$$

noting that $\dot{\phi}^4 P_{XX}$ is far smaller than $H^2$ during inflation. We keep $c_s^2 \approx 1$, so get $x = 1/2$ (which brings $Q_s \approx M_P^2/3 > 0$). We plot $c_s^2$ in FIG.2. The blue line is $c_s^2$ without the operator $\delta K \delta g^{00}$. In contrast, the magenta line is that with $\delta K \delta g^{00}$. Due to the existence of limit cycle, both $H$ and $\dot{H}$ are oscillating. Thus although $c_s^2 \approx 1$, it is affected by $\epsilon$, $\dot{\phi}^4 P_{XX} / H^2$ and is also oscillating. In certain sense, the oscillations of both $H$ and $c_s^2$ actually encode the crystal shape of cosmological TC.

Oscillatory patterns of power spectrum – The power spectrum of primordial perturbation is given by

$$P(\kappa) = \frac{k^3}{2\pi^2} \kappa^2. \kappa = aH.$$  

In the momentum space, the motive equation of $\zeta(k)$ is

$$\ddot{\zeta}_k + \left( 3H + \frac{Q_s}{Q_s} \right) \dot{\zeta}_k + c_s^2 \frac{k^2}{a^2} \zeta_k = 0.$$  

(15)
During the inflation, \(|\epsilon| = |\dot{H}/H^2| \ll 1\), but \(|\frac{Q_s}{\dot{\phi}}| \ll 1\) is not always valid. Thus we will solve Eq.(15) numerically.

Defining the Mukhanov-Sasaki variable \(u_k = y\dot{\zeta}_k\), we rewrite Eq.(15) as

\[
\ddot{u}_k + \left(k^2 - \frac{y''}{y}\right)u_k = 0,
\]

where \(y = a\sqrt{2Q_s}c_b\), e.g.,[24], \(dn = c_s dt/a\) and \(d' = d/d\eta\).

This suggests that the initial state for \(u_k\) may be the Bunch-Davies state \(u_k = e^{-ik\eta}\). In physical time, it equals to

\[
\zeta_k = e^{-ikf}\frac{y\omega}{y\sqrt{2k}},
\]

\[
\dot{\zeta}_k = \left(-\frac{ikc_s}{a} - \frac{\dot{y}}{y}\right)\zeta_k.
\]

We plot the perturbation spectrum \(P_\zeta(k)\) in FIG.3 for different limit cycles. As expected, \(P_\zeta(k)\) is oscillating periodically, which is actually an inevitable result of the limit cycle existing in \((\phi, \dot{\phi})\) phase space.

It is interesting to search for the possible oscillating patterns of power spectrum \(P_\zeta(k)\) for different parameters \(\alpha\) and \(\beta\), which control the shapes of limit cycles. As found in Ref.[5], the larger \(\beta\) is, the larger allowed range of \(\alpha\) is. But a bigger \(\beta\) will bring a shorter minor axis of ghost elliptic curve, see Eq.(4). We plot some power spectrums in FIG.4 with \(\beta = 30\) and different values of \(\alpha\). We show just four kinds of oscillating patterns. When \(\alpha\) increases, the “ripple” in the bottom of spectrum will move up, and after crossing the peak, it will descend. When \(\alpha\) is fixed and \(\beta\) is changed, the case is reverse. Thus the shape of limit cycle (equivalently the crystalline patterns of cosmological TC) will be encoded in the different oscillating patterns of power spectrum.

Discussion – Recently, as a special matter state, the cosmological TC has been implemented stably. How to look for such a matter state in cosmological observations is a significant issue.

In our work, we explored what would happen if the cosmological TC existed in the inflationary phase. We apply the operator \(m_3(t)\delta K\delta g^{90} \sim \Box \phi\) to remove the instability relevant with \(c_s^2 < 0\), and calculated the spectrum of corresponding primordial perturbation. It is found that the limit cycle responsible for TC in \((\phi, \dot{\phi})\) phase space will inevitably cause the oscillation of spectrum. We point out that the oscillatory patterns of the primordial spectrum depends on the TC parameters, and so
encode the crystalline patterns of TC. Thus if the cosmological TC existed in inflationary phase, the imprint of TC might be preserved in the CMB. It will be interesting to search for the corresponding signals in Planck data.

There would not be an inflationary stage forever. TC will dissolve due to interactions with a ‘waterfall’ field \( \sigma \). During inflation phase, field \( \sigma \) becomes tachyonic and ends inflation. Another field coupled with \( \phi \) may offer a large no-gaussianity which can be observable in the feature experiment. It is a new view to observe cosmological TC. In addition, adding external matter can have very strong implications for stability of models with non-standard kinetic terms where the perturbations of \( \phi \) near the ghost region \( c_\phi^2 < 0 \) [26]. However, in our model, the \( m_\phi^2(t)\delta K\delta \eta^{00} \) term will stay the perturbations far away from the ghost region, so it is still a stable model.

It is well-known that the inflation scenario itself can not avoid the initial singularity. The cosmological TC has been also investigated in certain theories of modified gravity [27–29], in particular Ref.[7] with the operator \( R^2\delta \eta^{00} \). This might provide a link of the cosmological TC to the nonsingular cosmologies, which is worthy of deeply exploring.

FIG. 4. The possible oscillatory patterns of \( P_\zeta(k) \), which actually encode different crystalline patterns of cosmological TC with \( \Lambda = 10 \).

**ACKNOWLEDGMENTS**

We thank Gen Ye and Yong Cai for helpful discussions. This work is supported by NSFC, Nos.11575188, 11690021.

**APPENDIX**

In this paper, we set units \( c = \hbar = 1 \) and the sign \((-+++)\).

It is possible to work out the analytic solution of \( P_\zeta \), if we assume that, like the slow-roll parameter \( \epsilon = -\frac{H}{\dot{H}} \), other parameters

\[
\epsilon_s = \frac{\dot{c}_s}{H c_s}, \quad \kappa = \frac{\dot{c}_g}{H c_g}, \quad \delta = \frac{\dot{\kappa}}{H \kappa}
\]

are also slow-roll, where \( \epsilon_g = Q_c c_s \).

Defining the parameter \( \epsilon_r = \epsilon + \epsilon_s \), we have

\[
\frac{d}{d\eta} \left( \frac{c_s}{a H} \right) = \epsilon_r - 1,
\]

which gives \( aH \frac{c_s}{c} \simeq - \frac{1}{\eta} (1 + \epsilon_r) \). In slow-roll approximation, \( y''/y \) in Eq.(16) is

\[
y''/y \approx \frac{\nu^2 - 1/4}{\eta^2},
\]

where \( \nu = \frac{3}{2} + \epsilon_r + \frac{3}{2} \kappa \). We choose Bunch-Davies vacuum \( u_k = \frac{\pi^{1/2}}{k^{1/2}} \) as the initial condition of \( u_k \) when \( aH \ll c_s k \).

On super-Hubble scales \((-k\eta \ll 1)\), the exact solution of Eq.(16) is [30, 31]

\[
u = e^{i(\nu - \frac{1}{2}) \pi} \sqrt{2}^{\nu - 1} \Gamma(\nu) (k\eta)^{\frac{1}{2} - \nu}.
\]

Thus

\[
P_\zeta(k) = \frac{k^3}{2\pi^2} \left| \frac{u_k}{y} \right|^2 \left. \right|_{c_s k = aH} \simeq \frac{2^{2\nu-3}\Gamma(\nu)H^2}{2\pi^5 c_s c^2} (1 + \epsilon_r)^{2\nu/1-
u}.
\]

However, generally, \( \epsilon_s \) and \( \epsilon_g \) are not slow-roll, so the slow-roll approximation is not precise. We have checked numerically that \( P_\zeta(k) \) in (23) is not exactly similar to the numerical result.

[1] F. Wilczek, Phys. Rev. Lett. 109, 160401(2012) [arXiv:1202.2539 [quant-ph]].
[2] A. Shapere and F. Wilczek, Phys. Rev. Lett. 109, 160402(2012) [arXiv:1202.2537 [cond-mat.other]].
[3] N. Y. Yao, A. C. Potter, I. D. Potirniche and A. Vishwanath, Phys. Rev. Lett. 118, 030401(2017) [arXiv:1608.02589 [cond-mat.dis-nn]].
[4] J. Zhang, P. W. Hess, A. Kyprianidis, P. Becker, A. Lee, J. Smith, G. Pagano, I. D. Potirniche, A. C. Potter and A. Vishwanath, Nature. 543, 217-220(2016) [arXiv:1609.08684v1 [quant-ph]].
[5] J. S. Bains, M. P. Hertzberg and F. Wilczek, JCAP 1705, no.05, 011(2017) [arXiv:1512.02304 [hep-th]].
[6] D. A. Easson and A. Vikman, [arXiv:1607.00996 [gr-qc]].
[7] D. A. Easson and T. Manton, Phys. Rev. D 99, no.4, 043507(2019) [arXiv:1802.03693 [hep-th]].
[8] Y. Cai, Y. Wan, H. G. Li, T. Qiu and Y. S. Piao, JHEP 1701, 090(2017) [arXiv:1610.03400 [gr-qc]].
[9] Y. Cai, H. G. Li, T. Qiu and Y. S. Piao, Eur. Phys. J. C 77, no.6, 369(2017) [arXiv:1701.04330 [gr-qc]].
[10] P. Creminelli, D. Pirtskhalava, L. Santoni and E. Trincherini, JCAP 1611, no.11, 047(2016) [arXiv:1610.04207 [hep-th]].
[11] Y. Cai and Y. S. Piao, JHEP 1709, 027(2017) [arXiv:1705.03401 [gr-qc]].
[12] R. Kolevatov, S. Mironov, N. Sukhov and V. Volkova, JCAP 1708, no. 08, 038 (2017) [arXiv:1705.06626[hep-th]].
[13] T. Kobayashi, M. Yamaguchi and J. Yokoyama, Phys. Rev. Lett. 105, 231302 (2010) [arXiv:1008.0603 [hep-th]].
[14] C. Burrage, C. de Rham, D. Seery and A. J. Tolley, JCAP 1101, 014 (2011) [arXiv:1009.2497 [hep-th]].
[15] C. Armendariz-Picon, T. Damour and V. F. Mukhanov, Phys. Lett. B 458, 209 (1999) [hep-th/9904075].
[16] J. Garriga and V. F. Mukhanov, Phys. Lett. B 458, 219(1999) [hep-th/9904176].
[17] C. Cheung, P. Creminelli, A. L. Fitzpatrick, J. Kaplan and L. Senatore, JHEP 0803, 014 (2008) [arXiv:0709.0293 [hep-th]].
[18] S. Mironov, V. Rubakov and V. Volkova, JCAP 1810, no. 10, 050 (2018) [arXiv:1807.08361 [hep-th]].
[19] M. Libanov, S. Mironov and V. Rubakov, JCAP 1608, no. 08, 037 (2016) [arXiv:1605.05992 [hep-th]].
[20] T. Kobayashi, Phys. Rev. D 94, no. 4, 043511 (2016) [arXiv:1606.05831 [hep-th]].
[21] G. Ye and Y. S. Piao, Commun. Theor. Phys. 71, no. 4, 427 (2019) [arXiv:1901.02202 [gr-qc]].
[22] G. Ye and Y. S. Piao, Phys. Rev. D 99, no. 8, 084019 (2019) [arXiv:1901.08283 [gr-qc]].
[23] D. Langlois, Int. J. Mod. Phys. D 28, no. 05, 1942006 (2019) [arXiv:1811.06271 [gr-qc]].
[24] J. Khoury and F. Piazza, JCAP 0907, 026(2009) [arXiv:0811.3633 [hep-th]].
[25] A. D. Linde, Phys. Rev. D 49, 748 (1994) [astro-ph/9307002].
[26] D. A. Easson, I. Sawicki and A. Vikman, JCAP 1307, 014 (2013) [arXiv:1304.3903 [hep-th]].
[27] P. Das, S. Pan, S. Ghosh and P. Pal, Phys. Rev. D 98 (2018) no.2, 024004 [arXiv:1801.07970 [hep-th]].
[28] X. H. Feng, H. Huang, S. L. Li, H. L and H. Wei, arXiv:1807.01720 [hep-th].
[29] P. Das, S. Pan and S. Ghosh, Phys. Lett. B 791, 66(2019) [arXiv:1810.06606 [hep-th]].
[30] E. D. Stewart and D. H. Lyth, Phys. Lett. B 302, 171(1993) [gr-qc/9302019].
[31] A. Riotto, ICTP Lect. Notes Ser. 14, 317(2003) [hep-ph/0210162].