POST-POST-NEWTONIAN LIGHT PROPAGATION WITHOUT INTEGRATING THE GEODESIC EQUATIONS

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ABSTRACT. A new derivation of the propagation direction of light is given for a 3-parameter family of static, spherically symmetric space-times within the post-post-Newtonian framework. The emitter and the observer are both located at a finite distance. The case of a ray emitted at infinity is also treated.

1. INTRODUCTION

The aim of this work is to present a new calculation of the propagation direction of light rays in a 3-parameter family of static, spherically symmetric space-times within the post-post-Newtonian framework. Rather than deriving the results from an integration of the geodesic equations, we obtain the desired expressions by a straightforward differentiation of the time delay function (see, e.g., Teyssandier & Le Poncin-Lafitte 2008 and Refs. therein). This study is motivated by the fact that any in-depth discussion of the highest accuracy tests of gravitational theories requires to evaluate the corrections of order higher than one in powers of the Schwarzschild radius (see, e.g., Ashby & Bertotti 2010 for the Cassini experiment). Even for the Gaia mission, a discrepancy between the analytical post-Newtonian solution and a computational estimate has recently necessitated a thorough analysis of the post-post-Newtonian propagation of light (see Klioner & Zschocke 2010 and Refs. therein).

2. LIGHT DIRECTION IN SPHERICALLY SYMMETRIC SPACE-TIMES

The gravitational field is assumed to be generated by an isolated spherically symmetric body of mass \( M \). Setting \( m = GM/c^2 \), the metric is supposed to be of the form

\[
ds^2 = \left(1 - \frac{2m}{r} + 2\beta \frac{m^2}{r^2} + \cdots \right) (dx^0)^2 - \left(1 + 2\gamma \frac{m}{r} + \frac{3}{2} \frac{m^2}{r^2} + \cdots \right) \delta_{ij} dx^i dx^j,
\]

where \( r = \sqrt{\delta_{ij} x^i x^j} \), \( \beta \) and \( \gamma \) are the usual post-Newtonian parameters, and \( \epsilon \) is a post-post-Newtonian parameter (\( \beta = \gamma = \epsilon = 1 \) in general relativity). We put \( x^0 = ct \) and \( x = (x^i) \), with \( i = 1, 2, 3 \).

Consider a photon emitted at a point \( x_A \) at an instant \( t_A \) and received at a point \( x_B \) at an instant \( t_B \). The propagation direction of this photon at any point \( x \) of its path is characterized by the triple

\[
\hat{l}_i = \left(\frac{l_i}{l_0}\right) = \left(\frac{l_1}{l_0}, \frac{l_2}{l_0}, \frac{l_3}{l_0}\right),
\]

where \( l_0 \) and \( l_i \) are the covariant components of the vector tangent to the ray, i.e. the quantities defined by \( l_\alpha = g_{\alpha\beta} dx^\beta/d\lambda \), \( g_{\alpha\beta} \) denoting the metric components and \( \lambda \) an arbitrary parameter along the ray.

Denote by \( \hat{l}_A \) and \( \hat{l}_B \) the expressions of \( \hat{l} \) at points \( x_A \) and \( x_B \), respectively. In any stationary space-time, these triples can be derived from the relations (see Le Poncin-Lafitte et al. 2004)

\[
\left(\frac{l_i}{l_0}\right)_A = c \frac{\partial T(x_A, x_B)}{\partial x^i_A}, \quad \left(\frac{l_i}{l_0}\right)_B = -c \frac{\partial T(x_A, x_B)}{\partial x^i_B},
\]

where \( T(x_A, x_B) \) is the expression giving the travel time of a photon as a function of \( x_A \) and \( x_B \):

\[
t_B - t_A = T(x_A, x_B).
\]
For the metric (1), \( \mathcal{T}(x_A, x_B) \) is given by (see, e.g., Teyssandier & Le Poncin-Lafitte 2008):

\[
\mathcal{T}(x_A, x_B) = \frac{|x_B - x_A|}{c} + \frac{(\gamma + 1)m}{c} \ln \left( \frac{r_A + r_B + |x_B - x_A|}{r_A - r_B - |x_B - x_A|} \right) + m^2 \frac{|x_B - x_A|}{c} \left[ \arccos(\frac{n_A \cdot n_B}{|x_A \times x_B|}) - \frac{(\gamma + 1)^2}{r_A r_B + x_A \cdot x_B} \right] + \cdots,
\]

where

\[
L = \frac{x_A}{r_A} - \frac{x_B}{r_B} = \kappa = \frac{8 - 4\beta + 8\gamma + 3\epsilon}{4}.
\]

Substituting for \( \mathcal{T}(x_A, x_B) \) from Eq. (5) into Eqs. (3) yields \( \hat{l}_A \) and \( \hat{l}_B \) as linear combinations of \( n_A \) and \( n_B \). However, it is more convenient to introduce the unit vector \( N_{AB} \) defined by

\[
N_{AB} = \left( \frac{x_B - x_A}{|x_B - x_A|} \right).
\]

and the unit vector \( P_{AB} \) orthogonal to \( N_{AB} \) defined as \( OH/|OH| \), \( H \) being the orthogonal projection of the center \( O \) of the mass \( M \) on the straight line passing through \( x_A \) and \( x_B \), that is

\[
P_{AB} = N_{AB} \times \left( \frac{n_A \times n_B}{n_A \times n_B} \right).
\]

Using Eqs. (5)-(8), we deduce the following proposition from Eqs. (3).

**Proposition 1.** The triples \( \hat{l}_A \) and \( \hat{l}_B \) are given by

\[
\hat{l}_A = -N_{AB} - \frac{m}{r_A} \left\{ (\gamma + 1) + \frac{m}{r_A} \left[ \kappa - \frac{(\gamma + 1)^2}{1 + n_A \cdot n_B} \right] \right\} N_{AB}
\]

\[
- \frac{m}{r_A} \left\{ (\gamma + 1) \frac{n_A \times n_B}{1 + n_A \cdot n_B} + \frac{1}{r_A \cdot n_A \cdot n_B} \left( \kappa \frac{\arccos(n_A \cdot n_B)}{|n_A \times n_B|} \left( 1 - \frac{r_A}{r_B} n_A \cdot n_B \right) \right) \right\} P_{AB}
\]

and

\[
\hat{l}_B = -N_{AB} - \frac{m}{r_B} \left\{ (\gamma + 1) + \frac{m}{r_B} \left[ \kappa - \frac{(\gamma + 1)^2}{1 + n_A \cdot n_B} \right] \right\} N_{AB}
\]

\[
+ \frac{m}{r_B} \left\{ (\gamma + 1) \frac{n_A \times n_B}{1 + n_A \cdot n_B} + \frac{1}{r_B \cdot n_A \cdot n_B} \left( \kappa \frac{\arccos(n_A \cdot n_B)}{|n_A \times n_B|} \left( 1 - \frac{r_B}{r_A} n_A \cdot n_B \right) \right) \right\} P_{AB},
\]

respectively.

In any static, spherically symmetric space-time the geodesic equations imply that the vector \( L \) defined as \( L = -x \times \hat{l} \) is a constant of the motion. The null geodesics considered here are assumed to be unbound. Consequently the magnitude of \( L \) is such that \( |L| = \lim_{|x| \to \infty} |x \times dx/dt| \) since \( \hat{l} \to -(dx/dt) \) when \( |x| \to \infty \). So the quantity \( b \) defined by

\[
b = | -x \times \hat{l} |\]

is the Euclidean distance between the asymptote to the ray and the line parallel to this asymptote passing through the center \( O \) as measured by an inertial observer at rest at infinity. Hence \( b \) may be considered as the *impact parameter* of the ray (see, e.g., Chandrasekhar 1983). Besides its geometric meaning, \( b \) presents the interest to be *intrinsic*, since it corresponds to a quantity which could be really measured.
Substituting for \( \tilde{L}_a \) from Eq. (10) into Eq. (11), introducing the zeroth-order distance of closest approach \( r_c \) defined as

\[
r_c = \frac{r_a r_b}{|x_B - x_A|} |n_A \times n_B|,
\]

and then using \((r_a + r_b)|n_A \times n_B|/|x_B - x_A| = |N_{AB} \times n_A| + |N_{AB} \times n_B|\), we get

\[
b = r_c \left[ 1 + \frac{(\gamma + 1)m |N_{AB} \times n_A| + |N_{AB} \times n_B|}{1 + n_A \cdot n_B} \right].
\]

Using this expansion of \( b \), we obtain the proposition which follows.

**Proposition 2.** In terms of the impact parameter \( b \), the triples \( \tilde{L}_a \) and \( \tilde{L}_b \) may be written as

\[
\tilde{L}_a = -N_{AB} - \frac{m|N_{AB} \times n_A|}{b} \left\{ \gamma + 1 + \frac{m}{b} \left[ \kappa |N_{AB} \times n_A| + (\gamma + 1)2|N_{AB} \times n_B| \right] \right\} N_{AB}
\]

\[
- \frac{m|N_{AB} \times n_A|}{b} \left\{ (\gamma + 1) \frac{n_A \times n_B}{1 + n_A \cdot n_B}
+ \frac{\kappa m}{b} \frac{\arccos(n_A \cdot n_A)}{n_A \times n_B} N_{AB} \cdot n_B - N_{AB} \cdot n_A \right\} P_{AB},
\]

\[
\tilde{L}_b = -N_{AB} - \frac{m|N_{AB} \times n_A|}{b} \left\{ \gamma + 1 + \frac{m}{b} \left[ \kappa |N_{AB} \times n_A| + (\gamma + 1)2|N_{AB} \times n_B| \right] \right\} N_{AB}
\]

\[
+ \frac{m|N_{AB} \times n_A|}{b} \left\{ (\gamma + 1) \frac{n_A \times n_B}{1 + n_A \cdot n_B}
- \frac{\kappa m}{b} \frac{\arccos(n_A \cdot n_A)}{n_A \times n_B} N_{AB} \cdot n_A - N_{AB} \cdot n_B \right\} P_{AB}.
\]

### 3. Deflection of a Light Ray Emitted at Infinity

Assume now that the ray arriving at \( x_B \) is emitted at infinity in a direction defined by a unit vector \( N_c \). Substituting \( N_c \) for \( N_{AB} \) and \( -N_c \) for \( n_a \) in Eq. (15) yields the expression of \( \tilde{L}_b \), where \( b \) is furnished by the limit of Eqs. (12) and (13) when \( r_a \rightarrow \infty \) and \( n_a \rightarrow -N_c \). We can set a proposition as follows.

**Proposition 3.** For a light ray emitted at infinity in a direction \( N_c \) and arriving at \( x_B \), \( \tilde{L}_b \) is given by

\[
\tilde{L}_b = -N_c - \frac{m|N_c \times n_B|}{b} \left[ \gamma + 1 + \frac{\kappa m|N_c \times n_B|}{b} \right] N_c
\]

\[
+ \frac{m}{b} \left\{ (\gamma + 1)(1 + N_c \cdot n_B) + \frac{\kappa m}{b} \pi - \arccos(N_c \cdot n_B)
+ |N_c \times n_B| N_c \cdot n_B \right\} P_a(N_c),
\]

where \( P_a(N_c) \) is the unit vector orthogonal to \( N_c \) defined as

\[
P_a(N_c) = -N_c \times \frac{N_c \times n_B}{|N_c \times n_B|}
\]

and \( b \) is the impact parameter of the ray, namely

\[
b = r_c \left[ 1 + \frac{(\gamma + 1)m |N_c \times n_B|}{r_c |N_c \times n_B|} + \ldots \right],
\]

with \( r_c = r_B |N_c \times n_B| \).
The deflection of the ray at point \( x_B \) may be characterized by the angle \( \Delta \chi_B \) made by the vector \( N_c \) and a vector tangent to the ray at \( x_B \). We have

\[
\Delta \chi_B = \left| \frac{N_c \times \hat{L}_B}{|\hat{L}_B|} \right| + O(1/c^6).
\]  

(19)

Substituting for \( \hat{L}_B \) from Eq. (16) into Eq. (19), and then introducing the angle \( \phi_B \) between \( N_c \) and \( n_B \) defined by

\[
N_c \cdot n_B = \cos \phi_B, \quad 0 \leq \phi_B \leq \pi,
\]

(20)

we get

\[
\Delta \chi_B = \frac{(\gamma + 1)GM}{c^2 b} (1 + \cos \phi_B) + \frac{G^2 M^2}{c^4 b^2} \left[ \kappa \left( \pi - \phi_B + \frac{1}{2} \sin 2 \phi_B \right) - (\gamma + 1)^2 (1 + \cos \phi_B) \sin \phi_B \right].
\]

(21)

where the impact parameter given by Eq. (18) may be rewritten as

\[
b = r_c \left[ 1 + \frac{(\gamma + 1)GM}{c^2 r_c} \frac{\sin \phi_B}{1 - \cos \phi_B} + \cdots \right], \quad r_c = r_B \sin \phi_B.
\]

(22)

It may be seen from the formulas given in Teyssandier & Le Poncin-Lafitte 2006 that \( \phi_B + \Delta \chi_B \) is the angular distance between the center \( O \) and the source at infinity as measured at \( x_B \) by a static observer, i.e. an observer at rest with respect to the coordinates \( x' \). It will be shown in a subsequent paper that this property implies that \( \Delta \chi_B \) can be regarded as an intrinsic quantity.

The \( 1/c^2 \) term in Eq. (21) is currently used in VLBI astrometry. If \( b \) is replaced by its coordinate expression (22), it may be seen that \( \Delta \chi_B \) is given by an expression as follows

\[
\Delta \chi_B = \frac{(\gamma + 1)GM}{c^2 r_c} (1 + \cos \phi_B) + \frac{G^2 M^2}{c^4 r_c^2} \left[ \kappa \left( \pi - \phi_B + \frac{1}{2} \sin 2 \phi_B \right) - (\gamma + 1)^2 (1 + \cos \phi_B) \sin \phi_B \right.
\]

\[
- (\gamma + 1)^2 \frac{(1 + \cos \phi_B)^2}{\sin \phi_B} \right].
\]

(23)

For a ray grazing a mass \( M \) of radius \( r_0 \), the underbraced term in the r.h.s. of Eq. (23) generates a post-post-Newtonian contribution \( (\Delta \chi_B)^{(2)} \) grazing \( \approx -4(\gamma + 1)^2 (GM/c^2 r_0)^2 (r_B/r_0) \) which can be great if \( r_B \gg r_0 \). For Jupiter, \( (\Delta \chi_B)^{(2)} \) grazing \( = 16.1 \mu \) as if the observer is located at a distance from Jupiter \( r_B = 6 \) AU: this value is appreciably greater than the level of accuracy expected for Gaia. However, this ‘enhanced’ term is due to the use of the coordinate-dependent quantity \( r_c \) instead of the intrinsic impact parameter \( b \). This result confirms the conclusion recently drawn in Klioner & Zschocke 2010.

4. CONCLUSION

Deriving the second-order terms in the propagation direction of light from the time transfer function rather than from the null geodesic equations is a very elegant and powerful procedure. The application of this method to a ray emitted at infinity and received by a static observer located at a finite distance from the central mass is easy and yields an intrinsic characterization of the gravitational bending of light.

5. REFERENCES

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