How to solve the Maxwell equations?

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Abstract. This paper contains the general solution for potential $A_\mu$ of electromagnetic field with given sources. It consists of two terms having clear physical sense and includes gauge additive.

1. Introduction. The Maxwell equations and their solution

The division of Maxwell equations into two pairs [1] — Bianchi identity

$$\partial_\mu F^\mu_\nu = 0$$

(1)

and second pair — Ampere law

$$\partial_\mu F^\mu_\nu = J^\nu.$$ 

(2)

clearly indicates not only relativistic kinematics but also their intimate dynamics.

The first pair (1) does not contain sources, these are the equations of the free field. It is satisfied by introducing 4-vector potential $A_\mu$ instead of strength $F^\mu_\nu$:

$$F^\mu_\nu = \partial_\mu A^\nu - \partial_\nu A^\mu,$$

(3)

after which the second pair [eq. (2)] takes on the following form

$$\left(\delta^\nu_\mu \partial^2 - \partial_\mu \partial_\nu\right)A_\mu = J^\nu.$$ 

(4)

We must solve only this one equation.

According to well known property of linear equations their solution is a sum of two terms — particular solution of the inhomogeneous equation and general solution of homogeneous one.

In our case the first is the known Liénard-Wiechert potential

$$\int \frac{J_\mu(y)}{(x-y)^2} \, d^4 y,$$

(5)

written in momentum representation it is equal to

$$\int J_\mu(k) e^{i k x} \frac{d^4 k}{k^2 - i0}.$$ 

(6)

The general solution of the homogeneous equation

$$C \int \exp(i k_\mu x_\mu) e_\mu(k) \delta(k^2) \, d^4 k,$$

(7)

contains the transverse polarization vector $e_\mu(k)$

$$e_\mu(k) k_\mu = 0,$$

(8)

$C$ is an arbitrary constant.

Thus, the general solution of the equation (4) is following
\[ A_\mu(x) = \int d^4k \exp(ik_\alpha x_\alpha) \left[ \frac{j_\mu(k)}{k^2-i\delta} + C e_\mu(k)\delta(k^2) \right] + \partial_\mu \chi(x). \] (9)

The third term \( \partial_\mu \chi(x) \) in this formula reflects the well-known gauge uncertainty of potential.

2. On problems of electrodynamics

Representation of the potential in the form (9) sets up two types of the problems for electrodynamics.

1. The influence of sources – current and charge – on the field (the first term on the right side of formula (9)). This problem, in turn, splits into two tasks:
   - for a given source, find the field (there is no reverse field effect on the sources);
   - for a given field, find the motion of charged bodies (the equation of motion (10) is used).

   The most difficult case is when the field and its sources change each other. In this case, it is necessary to use perturbation theory.

2. Evolution of the field itself (the second term). This is essentially diffraction – the propagation of waves in a medium with obstacles. The exact solutions are known – diffraction on a wedge (or on the edge of a plane) and on the sphere. A generalization to the obstacles defined by the metric (and symmetry) is possible.

   The fundamental question of the origin of the waves in this case is not important, since their sources are very remote.

3. Connection with mechanics

The equation of motion of a charged particle in a given electromagnetic field

\[ \frac{dp_\mu}{d\tau} = F_{\mu\nu} j_\nu \] (10)

contains momentum \( p_\mu = \frac{mdx_\mu}{d\tau} = (my; m\gamma \vec{v}) \) and current density \( j_\mu = \frac{e}{m} p_\mu \) \( (\gamma = \frac{dt}{d\tau} = \frac{1}{\sqrt{1-v^2}}). \)

The H.Lorentz force \( F_{\mu\alpha} j_\alpha \) increases energy and rotates the momentum, conserving its invariant length, mass: \( p_\mu^2 = m^2 \). It can be said that the field \( F_{\mu\alpha} \) realizes the local orthogonal transformation of the 4-momentum.

4. Two examples (an illustration of general formula)

Let us illustrate these general equations by two examples:

A) The choice of gauge

Usual choice of static potential as \( A_\mu = (A_0, \vec{A}) \) with zero vector part is not the best one – the gauge \( A_\mu = (0, \vec{A}) \) is much better. Indeed, \( d\vec{p}/dt = -e\vec{E} = ed\vec{A}/dt \) immediately gives \( \vec{p} - e\vec{A} = \vec{h} = \text{const} \), equation of hodograph (Hodograph is a geodetic line on the 4-sphere (V. Fock, 1935). W. Hamilton in 1847 has obtained an explicit expression \( e\vec{A} = \vec{L} \times \vec{n}/\rho \) (\( \rho \) is ellipse parameter)). This simple expression contains very much information: \( \text{Kepler’s ellipse} (\vec{p} \times \vec{h} \text{ reproduces 1-st Kepler law } r = \rho(1 + e\cos\theta)^{-1}) \) (1609), \( \text{Rutherford’s scattering} (1911), \text{Bohr’s orbits} (1913) \) and \( \text{fine structure of hydrogen (relativistic correction)} \) by Sommerfeld (1916).

   Moreover, the Hamilton integral \( \vec{h} \) may be generalized up to General Relativity. In Quantum Mechanics it serves as generator of gauge transformations.

B) Mie scattering

This is a kind of diffraction on the dielectric droplet: the ingoing wave is splitting into two waves – scattered and diffracted. All three waves have similar multipole expansion (It is the expansion of
exponential \( \exp(ikx) \) from 2-nd term of Eq. (9), \( \sum_{lm} a_l Z_l(kr)Y_{lm}(\theta, \varphi) \) but different \( a_l \)'s (partial amplitudes).

These are defined from the border conditions – continuity of each additives and their derivatives on the border \( r = R \) of two substances. An equality of logarithmic derivatives gives (Here \( \alpha = kr, \beta = KR \); \( Z \)'s are cylinder functions: \( Z_d = j_l(\beta), Z_{sc} = h_l(\alpha) \)).

\[
a_{sc} = -a_{in} \frac{\text{Re}M}{M}, \quad a_d = -a_{in} \frac{\text{Re}N}{N}
\]

where \( M = (\beta \partial_{\beta} - \alpha \partial_{\alpha})Z_d(\beta)Z_{sc}(\alpha) \) and \( N = (\alpha \partial_{\beta} - \beta \partial_{\alpha})Z_d(\beta)Z_{sc}(\alpha) \).

The structure \( \frac{\text{Re}M}{M} \) is dictated by unitarity of scattering process, \( M \) and \( N \) are equal to corresponding scattering phase \( \exp(-i\delta_l) \). Both are used in the final expressions of fields given by G.Mie.

Another question: why the wave speed changes when crossing the boundary (\( k \) turns into \( K \)). The answer is the Ewald–Oseen extinction theorem: the primary wave is cancelled just at the boundary as a result of the medium’s response (polarization and magnetization). A complete calculation of the corresponding susceptibilities is possible only within the framework of the quantum theory of matter.

There are, of course, many other problems binding both terms of Eq. (9), but we will not to touch them.

5. Conclusion
The expression (9) for potential may be considered as integral counterpart to differential equation (4). This point of view was defended by L. V. Lorenz [2], who strongly underlined the role of retardation of action (see also [3]). But limiting yourself with only first term of (9) is not a correct operation – how would you explain an existing of electromagnetic waves without of second term of (9).

We also remark the possibility of free choice of gauge contained in (9). This freedom greatly facilitates the quantization of the field thanks to absence of gauge condition.

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References
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