Critical Correlations of Wilson Lines in $SU(3)$ and the High Energy $\gamma^*p$ Cross Section

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Abstract

We discuss deep inelastic scattering at high energies as a critical phenomenon in $2 + 1$ space-time dimensions. In the limit of Bjorken $x \to 0$, QCD $SU(3)$ with quark fields becomes a critical theory with a diverging correlation length $\xi(x) \propto x^{-\frac{1}{2\lambda_2}}$ where the exponent $\lambda_2 = 2.52$ is obtained from the center group $Z(3)$ of $SU(3)$. We conjecture that the dipole wave function of the virtual photon for transverse sizes $1/Q < x_\perp < \xi$ obeys correlation scaling $\Psi \propto (x_\perp)^{-(1+n)}$ before exponentially decaying for distances larger than the correlation length. Using this behavior combined with different $x$-independent dipole proton cross sections we calculate the proton structure function and compare with the experimental data. We take the good agreement with the measured proton structure function $F_2(x,Q^2)$ as an indication that at high energies dimensional reduction to an effective three dimensional theory with a critical point occurs.

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I. INTRODUCTION

Deeply inelastic electron-proton scattering at very high energies has shown that the cross section of the virtual photon at high virtuality $Q^2$ increases faster with energy than at low virtuality. Small size objects experience a stronger energy dependent cross section than large size objects. Perturbative $QCD$ has been partially successful in explaining this physics. With suitable starting distributions next to leading order DGLAP - evolution can reproduce the experimental data. NNLO analysis \[\text{[1]}\] shows a slight improvement. But at small $x$ the DGLAP- summation of $log(Q^2)$ terms and neglect of $log(1/x)$ terms is theoretically unsatisfactory. Therefore there has been a considerable effort \[\text{[2–7]}\] to include the leading $log(1/x)$ contributions. It has been suggested \[\text{[3]}\] that the relevant collective variables at high energies are the gluon phase factors (Wilson lines) collected by the low $x-$ partons in the photon on near light like trajectories. We have followed this promising method in Ref. \[\text{[8]}\] relating high energy scattering to the behavior of Wilson line correlations in a 2 + 1 dimensional Hamiltonian near the light cone \[\text{[9,10]}\]. In this Ref. \[\text{[8]}\] the dynamics of Wilson lines in $QCD$ with two colors ($N_c = 2$) has been considered. In the present paper we address the realistic case of $SU(N_c)$ with three colors ($N_c = 3$). We propose that with increasing energy the effective dynamics reaches a critical point which is characterized by the symmetry properties of $SU(3)$. Universality tells us that $SU(3)$ Wilson lines have the same correlation functions as Potts spins in the center group $Z(3)$ in three dimensions. Since the correlation functions of Wilson lines influence the structure of the photon wave function at high energies or small $x$, the $Z(3)$ symmetry determines the strong increase of the $\gamma^* p$ cross section at high energies.

The discovery of asymptotic freedom \[\text{[11,12]}\] gives a fixed point of $SU(3)$ at $Q^2 \rightarrow \infty$, which has opened up the possibility of perturbative calculations in $QCD$. At infinitely high energies the longitudinal momentum transfers are minute, therefore $QCD$ at high energies reduces to a 2 + 1 dimensional theory, when the longitudinal space variables have been integrated out. Only transverse dynamics matters. The resulting effective Hamiltonian

\[\text{[90x672]}\]
has a fixed point at $1/x \to \infty$. Lattice simulations for $SU(2)$ have found an effective coupling at this fixed point which is not small. These lattice simulations still have to be extended to $SU(3)$. Therefore in the following paper we will use mostly symmetry arguments to characterize the physics near this critical point. This critical point influences the physics whenever a large number of low $x$ partons is involved. Beyond virtual photon-proton scattering other testing grounds for critical point dynamics will be RHIC and LHC.

As framework we use the formalism of a near light cone Hamiltonian. The light cone formulation of QCD has been a useful tool in perturbative calculations and extended beyond. Since the vacuum in the strictly light cone formulation is simple, the Hamiltonian must include all the complicated structures of QCD like condensates. Therefore we prefer a theoretical form developed in Refs. where the light cone is reached in a limiting procedure and quantization is always on spacelike surfaces. We start by choosing the following $\eta-$ coordinates which smoothly interpolate between the Lorentz and light front coordinates. The parameter $\eta$ specifies how near to the light cone the coordinate system is:

$$x^t = x^+ = \frac{1}{\sqrt{2}} \left\{ \left( 1 + \frac{\eta^2}{2} \right) x^0 + \left( 1 - \frac{\eta^2}{2} \right) x^3 \right\},$$

$$x^- = \frac{1}{\sqrt{2}} \left( x^0 - x^3 \right).$$  \hspace{1cm} (1)

For high energy photon proton scattering at small $x = Q^2/s$, with $s = W^2$ as c.m. energy squared we define two light like vectors using the photon vector $q, q^2 = -Q^2$ and the proton vector $p, p^2 = m^2 \approx 0$.

$$e_1 = q - \frac{q^2}{2pq} p$$

$$e_2 = p. \hspace{1cm} (2)$$

For finite energies the vector of the photon $q$ which points in the direction of $x^t$ can be calculated as a linear combination of the light like vector $e_1$ with a small amount of $e_2$ admixed

$$e_\eta = q + xp - \frac{\eta^2}{2} p$$

$$= e_1 - \frac{\eta^2}{2} e_2. \hspace{1cm} (3)$$
One sees that the mixing $\eta$ is related to the Bjorken variable $x$ and vanishes in the limit of infinite energies as $\frac{\eta^2}{2} = x$. Therefore it is natural to formulate high energy scattering in near light cone coordinates. In vacuum it is appropriate to use the Hamiltonian with periodic boundary conditions. In this work we only study the effective quark density function of the photon, i.e. a vacuum problem without theoretically modeling the proton target.

The outline of the paper is as follows: In Section II we give the near light cone Hamiltonian and its reduction to $(2 + 1)$ dimensions for small $x$. Section III is devoted to the approach of the critical point. Section IV gives the behavior of the correlations of Wilson lines near the critical point. In Section V this correlation length is used to model the effective quark density in the photon and calculate the structure function $F_2$ of the proton. A comparison to the HERA-data is given in the same section. Section VI contains the conclusions.

**II. NEAR LIGHT CONE $SU(3)$ QCD HAMILTONIAN**

For small $x$ the eikonal phases acquired by the quarks/antiquarks are the relevant collective variables. The light cone Hamiltonian on the finite light like $x^-$ interval of length $L$ has Wilson line or Polyakov operators similarly to QCD formulated on a finite interval in imaginary time at finite temperature

$$P(\vec{x}_\perp) = \frac{1}{N_c} \text{tr} \ P \exp \left( ig \int_{\vec{x}_\perp, x^-}^{L} dx^- A_-(\vec{x}_\perp, x^-) \right).$$

The dynamics of these Polyakov operators is determined by the near light cone Hamiltonian $H$ which has been derived in reference [9]. In the Weyl gauge $A_+ = 0$, the Gauss-law constraint can be resolved for $\Pi_-$ and one obtains a Hamiltonian which depends on the zero mode momentum $p_-$. The subsequent modified light front gauge $\partial_- A_+ = 0$ eliminates $x^-$ dependent fields $A_-(x^-, x_\perp)$, but allows fields $a_-(x_\perp)$ which are functions of the transverse coordinates only. The final QCD Hamiltonian near the light cone has the following form [4]:

4
\[ H = \int dx^- dx_\perp \mathcal{H}(x_\perp, x^-), \quad (5) \]

with

\[
\mathcal{H} = \text{tr} \left[ \partial_1 A_2 - \partial_2 A_1 - ig[A_1, A_2] \right]^2 \quad + \quad \frac{1}{\eta^2} \text{tr} \left[ \Pi_\perp - (\partial_- A_\perp - ig[a_-, A_\perp]) \right]^2 \quad + \quad \frac{1}{\eta^2} \text{tr} \left[ F_{\perp} - \nabla_\perp a_\perp \right]^2 \quad + \quad \frac{1}{2L^2} p_{\perp}^+(x_\perp)p_-(x_\perp) \quad + \quad \frac{1}{L^2} \int_0^L dy^- \int_0^L dy^- \sum_{p,q,n} \left[ \frac{G_{\perp pq}(x_\perp, y^-) G_{\perp pq}(x_\perp, y^-)}{2\pi n + g(a_q(x_\perp) - a_p(x_\perp))} \right]^2 e^{2\pi n(z^- - y^ -)/L} \quad - \quad \frac{i}{\eta^2} \psi_\perp^+(\partial_--iga_-)\psi_- - \frac{i}{\eta^2} \psi_\perp^+(\alpha_--iga_\perp)\psi + \frac{4\eta^2}{\eta} \psi_\perp^+ \beta_\psi, \quad (6) \]

The term \( e_\perp \) depends on an external source.

\[
e^\perp_{c_0} = g \nabla_\perp \int dy^- dy_- \frac{1}{\sqrt{\perp}} \left( f^{c_0 ab}_\perp A^a_\perp(y_\perp, y^-) \Pi^b_\perp(y_\perp, y^-) + \rho^{c_0}(y_\perp, y^-) \lambda^{c_0}/2 \right). \quad (7) \]

In the \( x \to 0 \) limit, those pieces \( \mathcal{H}^\perp \) of the Hamiltonian \( \mathcal{H} \) dominate which are most singular at \( \eta = 0 \) and do not couple to the three dimensional gauge fields \( A_\perp \) and \( \psi_\perp \). This reduced Hamiltonian has collective variables \( a_{c_0}^\perp \) with the color indices \( c_0 = 3, 8 \)

\[
a_{c_0}^\perp = \frac{1}{L} \int_0^L dx^- A_{c_0}^\perp(\vec{x}_\perp, x^-), \quad (8) \]

which determine the Wilson lines and live in a 2 + 1 dimensional space. The Wilson line operators \( P(x_\perp) \) can always be parameterized in terms of the diagonal color matrices \( a^3_\perp \lambda_3/2 \) and \( a^8_\perp \lambda_8/2 \) by suitable gauge transformations. We consider the dynamics of the fields \( a_- \) in vacuum, i.e. without the source term \( e_\perp \). A possible external source, e.g., a Gaussian distributed random color charge [15], can be taken into account via this term which shifts the zero mode fields \( \nabla_\perp a_{c_0}^\perp \) to fluctuate around the classical fields \( e^\perp_{c_0} \). We recall that the \( \eta^- \) coordinates correspond to the physics in a fast moving frame and factorize the reduced energy from the Lorentz boost factor \( \propto 1/\eta \) and a transverse lattice cut off \( a \)

\[
h_{\text{red}} = 2\eta a \int dx^- dx_\perp \mathcal{H}^\perp \quad = \quad \int dx^- dx_\perp \sum_{c_0=3,8} \left( \frac{2a}{\eta} \text{tr} \left( \frac{1}{L} e_{c_0}^\perp - \nabla_\perp a_{c_0}^\perp \right)^2 + \frac{2a}{L^2} p_{c_0}^+ p_{c_0}^+ - \frac{4a}{\eta} \psi_\perp^+ ga_{c_0} \lambda_{c_0}^\perp \psi_- \right). \quad (9) \]
Then we redefine modified zero mode fields and their conjugate momenta on a transverse lattice at positions $b_\perp$:

$$\varphi_{c_0}(\vec{b}_\perp) = \frac{1}{2} g L a_{c_0}(b_\perp),$$

$$\frac{\delta}{\delta \varphi_{c_0}(b_\perp)} = a^2 \frac{\delta}{\delta \varphi_{c_0}(x_\perp)}. \quad (10)$$

Their dynamics on the lattice is determined by the lattice Hamiltonian $h_{\text{lat}}$ (cf. also Ref. [10] for the case of $SU(2)$),

$$h_{\text{lat}} = \sum_{\vec{b},c_0} \left[ -g_{\text{eff}}^2 \frac{1}{J} \frac{\delta}{\delta \varphi_{c_0}(\vec{b})} J \frac{\delta}{\delta \varphi_{c_0}(\vec{b})} + \frac{1}{g_{\text{eff}}^2} \sum_{\vec{\epsilon}} \left( (\varphi_{c_0}(\vec{b}) - \varphi_{c_0}(\vec{b} + \vec{\epsilon}))^2 \right) - \frac{4a}{\eta L} <\psi_+^\dagger \psi_- >^{c_0} \varphi(\vec{b})^{c_0} \right]. \quad (11)$$

The summation goes over color indices $c_0 = 3, 8$ and all 2-dimensional lattice sites. The geometry of the $SU(3)$ manifold enters via the Jacobian $[16,17]$:

$$J = \sin^2(\varphi^3) \sin^2\left(\frac{1}{2}(\varphi^3 - \sqrt{3}\varphi^8)\right) \sin^2\left(\frac{1}{2}(\varphi^3 + \sqrt{3}\varphi^8)\right). \quad (12)$$

Because of dimensional reduction the effective coupling constant depends on the length $L$ of the interval in $x_-$ direction and the QCD-coupling $g^2$ multiplied by the parameter $\eta$ characterizing the nearness to the lightcone

$$g_{\text{eff}}^2 = \frac{g^2 L \eta}{4a}. \quad (13)$$

The phase factors $\varphi^3, \varphi^8$ are defined in the fundamental domain (cf. Fig. 1)

$$0 \leq \varphi^3 \leq \pi$$

$$-\varphi^3/\sqrt{3} \leq \varphi^8 \leq \varphi^3/\sqrt{3}.$$ 

The Jacobian vanishes on the boundaries of the fundamental domain. The coupling to the quarks is introduced by the external field

$$<\psi_+^\dagger \psi_- >^{c_0} = \int dx^- \psi_+^{1c_0} \psi_-^{c_0}. \quad (14)$$

It was shown in perturbation theory [18] that the product of “bad” light cone components $\psi_-$ has a zero mode contribution. In structure functions such a singular piece is generated
at Bjorken $x = 0$. In QCD on the light cone $[19,20]$ the negative energy states obey a constraint equation which allows a constant $x^-$ independent solution. Therefore we assume that the fermion negative energy states develop an expectation value $\langle \psi^\dagger \psi^- \rangle \geq 0$. Further work is still needed to show how such a nonvanishing zero mode density $\langle \psi^\dagger \psi^- \rangle$ is realized near the light cone.

In the continuum limit of the transverse lattice theory the lattice size $a$ goes to zero and/or the extension $L$ of the lattice to infinity. This limit combined with the light cone limit $\eta \to 0$ leads to an indefinite behavior of the effective coupling constant (cf. Eq. (13)). The critical behavior of the zero-mode theory resolves this ambiguity. In Ref. $[10]$ we have done a Finite Size Scaling (FSS) analysis for $SU(2)$ QCD obtaining a second order transition as a function of the coupling $g^2_{\text{eff}}$ between a phase with massive excitations at strong coupling and a phase with massless excitations at weak coupling. In the strong coupling domain of $g^2_{\text{eff}}$ the energy of the rotators $\varphi^c_0$ is dominated by the electric energy $\propto g^2_{\text{eff}} \frac{1}{J} \frac{\delta}{\delta \varphi^c_0(b)} \frac{\delta}{\delta \varphi^c_0(b)}$ which corresponds to the Laplacian in the group manifold. Each site has an energy spectrum with a gap $\varepsilon_n = n(n+2)\varepsilon_0$ in $SU(2)$ or $\varepsilon_n = n(n+4)\varepsilon_0$ in $SU(3)$ $[17]$. With decreasing $g^2_{\text{eff}}$ the magnetic coupling $\propto \frac{1}{g^2_{\text{eff}}} (\varphi^c_0(\vec{b}) - \varphi^c_0(\vec{b} + \vec{\varepsilon}))^2$ becomes stronger. A larger nearest neighbor coupling leads to a coherently aligned ground state which has massless excitations. Consequently the mass gap vanishes at a sufficiently small $g^2_{\text{eff}}$. The resulting critical $SU(2)$ theory is in the same universality class as the $Z(2)$ theory or the Ising model in 3-dimensions, which has been checked in the lattice simulations $[10]$ with the available numerical accuracy. Barring any unusual effects arising from the coupling to the three dimensional gluon fields, we think that the full theory may shift the exact value of the critical coupling, but does not change the critical exponents which are determined by the symmetry of the problem. The Abelian collective fields in $SU(2)$ have reflection symmetry around $\pi/2$ which corresponds to the up-down symmetry of the Ising spins.
III. HOW TO REACH THE CRITICAL POINT IN $SU(3)$ AT SMALL $X$

The reduced $SU(3)$ Hamiltonian has rather different symmetry properties than the $SU(2)$ Hamiltonian. A closer look at the fundamental domain in $SU(3)$ (cf. Fig. 1) shows three regions separated by dashed lines. An element from the center group $Z(3)$ can be mapped from one region to another by large gauge transformations. In $SU(2)$ these large gauge transformations are reflections around $\pi/2$ on the $\varphi^3$ axis. In previous work \[10\] we found that the universality class of the zero mode $SU(2)$ theory is the same as $Z(2)$ theory, therefore we think that the universality class of the reduced Hamiltonian in $SU(3)$ is the three state Potts model $Z(3)$. In each subregion of the fundamental domain the zero mode variables $\varphi^3, \varphi^8$ are represented by one spin orientation. The relevant center group $Z(3)$ has a weak first order transition whose critical line ends in a second order point in the presence of an external field. We conjecture that this external field is provided by the fermion zero mode density near the light cone.

To match the Hamiltonian lattice with scattering we consider the same physical picture as in Ref. \[8\]. The lattice constant $a$ is chosen to coincide with the photon resolution $\approx 1/Q$. The longitudinal extension $L$ must be larger than the color coherence length of the $q\bar{q}$ state in the photon-proton c.m. system. The photon and proton move on almost light like trajectories and the coherence length grows with $x$ as:

$$\Delta x^- = \frac{1}{Q\sqrt{x}}.$$}

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\end{equation}

We demand therefore that $\frac{L}{a} > \frac{1}{Q\sqrt{x}}$. In the limit of small $x$ the parameter $\eta$ decreases as $\eta = \sqrt{2x}$. Assuming that a stable fixed point in the effective running coupling $g_{\text{eff}}^2 = g^2 \frac{L\ln}{4a}$ exists, we get

$$\frac{L}{a} \approx \frac{c}{g^2 \sqrt{x}}.$$}

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Since $1/a \approx Q$, we can choose a positive constant $N_0$ such that the interval $L$ exceeds the color coherence length of the $q\bar{q}$ state in the photon, namely
\[ \frac{L}{a} = \frac{1}{g^2} \left( N_0 + \frac{c}{aQ\sqrt{x}} \right). \]  

In \( Z(3) \) theory the critical behavior depends on two parameters, the effective field \( \hat{h} \) and the coupling \( \hat{\tau} \): Using equation (17) one finds that the external field \( h = \frac{4a\langle \psi^\dagger \psi^- \rangle}{nL} \) converges in the limit \( x \to 0 \) to \( h^* \):

\[ h^* = \frac{4g^2}{\sqrt{2c}} \langle \psi^\dagger \psi^- \rangle. \]  

The limiting external field \( h^* \) is independent of the short distance cutoff \( a \) under the assumption that the light cone density of the quarks \( \langle \psi^\dagger \psi^- \rangle \) behaves in the same way with \( a \) as the perturbative quark density in the photon namely \( \propto \int dr_\perp K_2(r_\perp^2) \). It is specific to high energy scattering that the external field and the coupling are approaching their critical values in a similar way for \( x \to 0 \)

\[ \hat{h} = |h - h^*| \approx h_0 \sqrt{x}, \]  

\[ \hat{\tau} = \frac{g^2 L \eta}{4a} - g_{\text{eff}}^2 \approx t_0 \sqrt{x}. \]  

It would be important to find another physical situation where the experimental conditions regulate both quantities independently.

**IV. BEHAVIOR OF THE CORRELATION LENGTH NEAR THE CRITICAL POINT**

The earliest work on critical behavior of pure \( SU(3) \) gauge theory [21] has been done in the context of the finite temperature phase transition. Recent studies [22,23] have established common features of \( SU(3) \) QCD and the three state Potts model \( Z(3) \) with a first order phase transition line which ends in a critical point where the transition is second order. In Refs. [22,23] the universality class has been identified as the 3-dimensional Ising model in three dimensions which has the critical exponents :

\[ \lambda_1 = \nu^{-1} = 1.56, \]  

\[ \lambda_2 = d - \beta \lambda_1 = 2.52. \]
In order to assess the divergence of the correlation length, we rescale the lattice constant by a factor \( b \) and the external field and coupling with powers of \( b \) according to their respective anomalous dimensions

\[
\xi_a = \frac{b \xi}{f(b^{\lambda_1 \hat{\tau}}, b^{\lambda_2 \hat{h}})}.
\]

(21)

When the external field \( \hat{h} \to 0 \) and \( \hat{\tau} \hat{h}^{-\lambda_1/\lambda_2} \) is sufficiently close to zero, the critical behavior of the correlation length is calculable by choosing \( b^{\lambda_2 \hat{h}} = 1 \), then one obtains

\[
\frac{\xi}{a} = \hat{h}^{-1/\lambda_2} f(b^{\lambda_1 \hat{\tau}}, 1)
= \hat{h}^{-1/\lambda_2} f_h(\hat{\tau} \hat{h}^{-\lambda_1/\lambda_2}).
\]

(22)

The function \( f_h(r) = f(r, 1) \). Note the power \( 1/\lambda_2 = 0.4 \) governing the low \( x \) behavior of the correlation length is considerably smaller than the power \( 1/\lambda_1 = 0.63 \) which determines the power for a small ratio \( \hat{h} \hat{\tau}^{-\lambda_2/\lambda_1} \). In the latter case one gets with the function \( f_{\tau}(s) = f(1, s) \) the behavior

\[
\frac{\xi}{a} = \hat{\tau}^{-1/\lambda_1} f(1, \hat{h} \hat{\tau}^{-\lambda_2/\lambda_1})
= \hat{\tau}^{-1/\lambda_1} f_{\tau}(\hat{h} \hat{\tau}^{-\lambda_1/\lambda_1}).
\]

(23)

In the previous work on \( SU(2) \) we considered a second order transition in the absence of an external field. Since the universality class was also of the Ising type a critical behavior of the correlation length \( \xi \propto \tau^{-1/\lambda_1} \) resulted. Such a power gave a qualitative description of the scarce data on the longitudinal structure function \[8\], but definitely cannot describe the \( F_2 \)-data as shown in Ref. [24].

In the mathematical framework of the \( SU(3) \) critical theory exposed here, it is clear that the growth of the correlation function with \( \hat{h}^{-1/\lambda_2} \) is realized, since for small \( x \) the argument \( r \) of the scaling function \( f_h \) is small: \( r = \hat{\tau} \hat{h}^{-\lambda_1/\lambda_2} \propto x^{-0.18} \). The argument of \( f_{\tau} \), however, \( \hat{h} \hat{\tau}^{-\lambda_2/\lambda_1} \propto x^{-0.3} \) is large for small \( x \). Therefore we can well approximate the argument of \( f_h \) by \( r = 0 \) for high energy scattering. Recently the scaling function \( f_h \) has been calculated in lattice simulations [25] and found to become constant at \( r = 0 \). Substituting \( \hat{h} \propto x^{1/2} \) in Eq. (22) one finds that the correlation length \( \xi \) increases with \( x \to 0 \) as
Near the critical point the Wilson lines experience long range correlations which means that dipoles in the photon wave function are correlated over large distances. For finite correlation length there exists an intermediate range where $1/Q < x_\perp < \xi$ for which the correlation function of Wilson lines is power behaved:

$$< P(x_\perp)P(0) > \approx \frac{1}{x_\perp^{1+n}} \quad (25)$$

where $n = 0.04$ in Ising like systems. This region is responsible for the well known effect of critical opalescence in the gas liquid transition. For larger distances $x_\perp > \xi$ the correlation function decreases exponentially

$$< P(x_\perp)P(0) > \approx e^{-x_\perp/\xi}. \quad (26)$$

V. PROTON STRUCTURE FUNCTION $F_2$ AND CRITICAL BEHAVIOR

The photon is ideal to investigate high energy cross sections of hadronic objects with a variable size. In the course of $x$-evolution the photon wave function develops many dipoles which in general diffuse into distance scales beyond the original size $1/Q$. This increase in parton density and/or size of the photon wave function is generally believed to be the origin of the increasing high energy cross section. In this work, we do not follow the development of the photon dipole state in detail, we only give a qualitative description of the effective photon size as a function of $x$ using the results of the $2 + 1$ dimensional critical $QCD \ SU(3)$ theory as a guiding principle. We parameterize the longitudinal and transverse photon probability densities as:

$$\rho_\gamma^T = \frac{6\alpha}{4\pi^2} \sum_f \epsilon_f^2 \epsilon^2 [z^2 + (1 - z)^2] F_T(\epsilon x_\perp),$$

$$\rho_\gamma^L = \frac{6\alpha}{4\pi^2} \sum_f \epsilon_f^2 4Q^2 z^2 (1 - z)^2 F_L(\epsilon x_\perp), \quad (27)$$

$$\epsilon = \sqrt{Q^2 z(1 - z)}. \quad (28)$$
The perturbative scale for the photon quark density is given by $1/\varepsilon$. We modify the photon wave function depending on the relation of the correlation length $\xi$ of the Wilson loops to the perturbative scale $1/\varepsilon$. We set

$$\xi = \frac{1}{\varepsilon} \left( \frac{x}{x_0} \right)^{-\frac{1}{2\lambda_2}}.$$  

(29)

For the reference Bjorken parameter $x_0 = 10^{-2}$ the correlation length is fixed at the perturbative scale. The critical exponent $\frac{1}{2\lambda_2} = 0.2$ determines the Wilson line correlations for $x < x_0$. If the transverse size of the dipole is smaller than the perturbative length scale $x_\perp < 1/\varepsilon$ we use the perturbative quark densities $F_T(\varepsilon x_\perp) = K_1(\varepsilon x_\perp)^2$ and $F_L(\varepsilon x_\perp) = K_0(\varepsilon x_\perp)^2$. This is the uninteresting case. For $1/\varepsilon < x_\perp < \xi$ we modify the perturbative photon density using the correlation functions of the critical theory, Eqs. (25,26),

$$F_{T/L}(\varepsilon x_\perp) = K_1/0(\frac{1}{\varepsilon x_\perp})^{2+2n} \quad \text{for } x_\perp < \frac{1}{\varepsilon},$$

$$= K_0(\varepsilon x_\perp)^2 \quad \text{for } \frac{1}{\varepsilon} < x_\perp < \xi,$$

$$= K_1/0(x/\xi)^{2+2n} \quad \text{for } x_\perp > \xi.$$  

(30)

The above parameterizations extends the photon density into the scaling region using the scaling index $n = 0.04$. For distances beyond the scaling region the density decays exponentially. The connections are made in such a way that the density is continuous at the respective boundaries of each region. The photon density combined with an energy independent dipole-proton cross section determines the structure function $F_2$ and the photon-proton cross section:

$$F_2(x, Q^2) = \frac{Q^2}{4\pi^2\alpha^2}(\sigma_{\gamma p}^{T,\text{tot}} + \sigma_{\gamma p}^{L,\text{tot}}),$$

(31)

$$\sigma_{\gamma p}^{T/L,\text{tot}} = \int d^2x_\perp \int_0^1 dz \rho_{\gamma}^{T/L}(x_\perp, z)\sigma_{\text{dip}}(x_\perp)$$  

(32)

First, we use the Golec-Biernat-Wüsthoff [26] dipole-proton cross section at fixed $x_0 = 10^{-2}$ with $\sigma_0 = 23 mb$ to calculate the proton structure function,

$$\sigma_{\text{GBW}}(x_\perp, R_0) = \sigma_0(1 - e^{-\frac{x_\perp^2}{4R_0^2}}),$$

(33)

$$R_0 = \frac{1}{1 GeV} \left( \frac{x_0}{3 \times 10^{-4}} \right)^{0.145}.$$  

(34)
The numerical values entering the above formulas are taken over from the original reference [26]. Note, the value $R_0 = 0.33 \text{fm}$ at $x_0 = 10^{-2}$ is independent of $x$. In Fig. 2 we present the proton structure function $F_2$ measured at HERA [27,28] as a function of $x$ for various $Q^2$. The solid theoretical curve describes the data rather accurately. One must be cautious not to overestimate the agreement, since due to the fixed beam energy the values of $Q^2$ and $x$ are correlated in electron-proton scattering. For intermediate values $5 GeV^2 < Q^2 < 15 GeV^2$ the theory slightly overestimates the data.

If one approximates the GBW-dipole proton cross section by a simple quadratic function at small distances $r < 2R_0$ and a constant function for $r > 2R_0$ [29],

$$\sigma(r) \approx \sigma_0 \left( \frac{r^2}{4R_0^2} \Theta(2R_0 - r) + \Theta(r - 2R_0) \right),$$  \hspace{1cm} (35)

one can estimate the photon proton cross section rather simply. Small errors are introduced if one further neglects the exponentially suppressed part of the photon density in the integral over large transverse distances and sets the anomalous dimension $n \rightarrow 0$. Then one can integrate the dominant transverse cross section in $F_2$ up to the correlation length $\xi$ analytically.

$$\sigma_T = \frac{3\alpha}{\pi} \sum e_j^2 Q^2 \sigma_0 \int_0^1 dz (z^2 + (1-z)^2) \int_0^\infty r dr \frac{z(1-z)}{r^2 z^2} \Theta(1 - \frac{r^2}{\xi^2}) \frac{\sigma(r)}{\sigma_0}$$ \hspace{1cm} (36)

Redefining the integration variable as $r' = r(x/x_0)^{\frac{1}{2\lambda_2}}$ one obtains the $\gamma^* - p$ cross section as if the original GBW cross section $\sigma(r, R(x))$ with running $R(x)$ would have been used.

$$\sigma_T = \frac{3\alpha}{\pi} \sum e_j^2 Q^2 \sigma_0 \int_0^1 dz (z^2 + (1-z)^2) \int_0^\infty r' dr' \frac{z(1-z)}{r'^2 z^2} \Theta(1 - \frac{z^2 r'^2}{\xi^2}) \frac{\sigma_{GBW}(r', R(x))}{\sigma_0}$$ \hspace{1cm} (37)

where

$$R(x) = R_0 \left( \frac{x}{x_0} \right)^{\frac{1}{2\lambda_2}},$$ \hspace{1cm} (38)

$$\frac{1}{2\lambda_2} = 0.2.$$ \hspace{1cm} (39)

Therefore the agreement with the data can be understood analytically. The critical theory gives the phenomenologically obtained power dependence of $R$ with $x$. The favored $x$-dependence of GBW is in the range 0.145 and 0.20. Without a model for the proton source,
it is not possible to obtain the absolute length $R_0$. The structure function \[30\] scales as a function of $Q^2 R_0^2 (x/x_0)^{1/\lambda_2}$ as can be easily derived for the simplified dipole cross section given in Eq. \[35\].

In order to test the sensitivity of our model to the assumed form of the dipole cross section we also tried the rather different dipole cross section of Forshaw et al. \[31\] with $m_q = 0$ which parameterizes the weak energy dependence of the cross section for large dipoles separately from the strong energy dependence of the cross section for small dipoles.

$$\sigma(s,r) = \sigma_{soft}(s,r) + \sigma_{hard}(s,r),$$  \hspace{1cm} (40)

$$\sigma_{soft}(s,r) = g_1(r)s^{0.06},$$  \hspace{1cm} (41)

$$\sigma_{hard}(s,r) = g_2(r)s^{0.38}.$$  \hspace{1cm} (42)

In the expression for $s = Q^2/x$ we fix $x_0 = 10^{-2}$. The $Q^2$ dependence in the cross section remains. These cross sections for different values of $Q^2$ look rather different from the GBW cross section and its simplified form (cf. Fig. 3). Using the same modified photon wave function inside the scaling region we find the theoretical (dashed) curves in Fig. 2. In spite of the rather different shape of the dipole proton cross section the agreement of the theory with the data is again rather good. In our opinion this represents a strong indication that the theory does not rely on a specific choice of the dipole proton cross section. Any reasonable choice of dipole-proton cross section at fixed $x_0$ should do.

**VI. CONCLUSIONS**

The presented calculation of the proton structure function suggests the discovery of a new critical point in QCD. Besides the fixed point of asymptotically vanishing coupling at infinite momentum transfer $Q^2 \to \infty$ we propose that QCD has another critical point at infinite energies $1/x \to \infty$. It is characterized by the exponents of the center group $Z(3)$ of $SU(3)$ and determines the correlation functions of Wilson lines near the light cone. A correlation length which increases as $\xi \propto (\frac{x}{x_0})^{-\frac{1}{2\lambda_2}}$ with the critical exponent $\lambda_2 = 2.52$ given
by the 3-state Potts model $Z(3)$ in three dimensions gives the correct effective photon wave function to explain the growing photon proton cross sections observed in HERA [27,28].

This result is independent of the specific form of the dipole proton cross section which is used as an input at the scale $x_0 = 10^{-2}$. For the Golec-Biernat-Wuesthoff [26] and the Forshaw et al. [31] cross sections good agreement can be achieved using the effective photon dipole density which has been modified according to the critical scaling Greens function. In this paper we have made a further step extending the picture of a dilute parton gas into a liquid like phase at small $x$. At small $x$ a critically diverging correlation length in transverse space leads to increasing cross sections in qualitative agreement with the observed growth of structure functions for high $Q^2$ deep inelastic scattering. In various studies [32,33] of high energy scattering the conformal invariance of the resulting effective theory has been pointed out. In 2+1 dimensions near the critical point the effective theory is also conformal invariant, whether the Greens functions in transverse space show this symmetry is not clear.

Clearly on the theoretical side it is necessary to take into account the target at $x^- = 0$ in the calculation. As a next step in a lattice calculation, we plan to include an external source representing the proton target and calculate the Wilson line correlation functions with twisted boundary conditions. Averaging over a stochastic source distribution is also possible [15]. After such a calculation one does not have to rely anymore on parameterized dipole proton cross sections which make the extraction of the critical dynamics difficult. The agreement so far between the reduced $QCD$ Hamiltonian near the light cone with experiment gives a strong motivation to pursue the investigation of the full Hamiltonian in $SU(3)$.

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FIG. 1. The fundamental domain of gauge field variables $\varphi_3 = \frac{gLa^3}{2}$ and $\varphi_8 = \frac{gLa^8}{2}$. 
FIG. 2. Proton structure function $F_2$ as a function of $x$ for various $Q^2$. The experimental data are from H1 and ZEUS at HERA [27,28]. The curves are the theoretical results of the critical theory with the functions $F_{T/L}(\epsilon x_\perp)$ of Eq. (30).
FIG. 3. Different dipole-proton cross sections as a function of the dipole size $r$. The full drawn curve is the Golec-Biernat-Wuesthoff cross section at $x_0 = 10^{-2}$, the dashed curves are the Forshaw et al. cross sections at the same $x_0 = 10^{-2}$ but different $Q^2 = 10\text{GeV}^2$, $50\text{GeV}^2$ and $100 \text{GeV}^2$. The curve marked (GBW-simple) gives the approximation in Eq. (35) to the GBW cross section.