Natural $R$–Parity, $\mu$–term, and Fermion Mass Hierarchy
From Discrete Gauge Symmetries

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Abstract

In the minimal supersymmetric Standard Model with seesaw neutrino masses we show how $R$–parity can emerge naturally as a discrete gauge symmetry. The same discrete symmetry explains the smallness of the $\mu$–term (the Higgsino mass parameter) via the Giudice–Masiero mechanism. The discrete gauge anomalies are cancelled by a discrete version of the Green–Schwarz mechanism. The simplest symmetry group is found to be $Z_4$ with a charge assignment that is compatible with grand unification. Several other $Z_N$ gauge symmetries are found for $N = 10, 12, 18, 36$ etc, with some models employing discrete anomaly cancellation at higher Kac–Moody levels. Allowing for a flavor structure in $Z_N$, we show that the same gauge symmetry can also explain the observed hierarchy in the fermion masses and mixings.
1 Introduction

One of the challenging questions facing supersymmetric extensions of the Standard Model is an understanding of $R$–parity which is required for the stability of the proton. In the minimal supersymmetric Standard Model (MSSM), a discrete $Z_2$ symmetry is usually assumed. Under this symmetry the Standard Model (SM) particles are taken to be even while their superpartners are odd. The gauge symmetry of MSSM would allow baryon number and lepton number violating Yukawa couplings at the renormalizable level which would result in rapid proton decay. The $Z_2$ $R$–parity forbids such dangerous couplings.

The assumption of $R$–parity has profound implications for supersymmetric particle search at colliders as well as for cosmology. At colliders SUSY particles can only be produced in pairs, and the lightest SUSY particle (LSP), usually a neutralino, will be stable. This stable LSP is a leading candidate for cosmological cold dark matter.

Since $R$–parity is not part of the MSSM gauge symmetry, questions can be raised about its potential violation arising from quantum gravitational effects. These effects (associated with worm holes, black holes, etc) are believed to violate all global symmetries [1]. True gauge symmetries are however protected from such violations. When a gauge symmetry breaks spontaneously, often a discrete subgroup is left intact. Such discrete symmetries, called discrete gauge symmetries [2], are also immune to quantum gravitational effects. Not all discrete symmetries can however be gauge symmetries. For instance, since the original continuous gauge symmetry was free from anomalies, its unbroken discrete subgroup should be free from discrete gauge anomalies [3, 4]. This imposes a non–trivial constraint on the surviving discrete symmetry and/or on the low energy particle content [3, 4, 5, 6, 7, 8].

It will be of great interest to see if $R$–parity of MSSM can be realized as a discrete gauge symmetry, so that one can rest assured that it wont be subject to unknown quantum gravitational violations. This is the main question we wish to address in this paper.

A seemingly unrelated but equally profound problem facing the MSSM is an understanding of the origin of the $\mu$–term, the Higgsino mass parameter. The $\mu$ parameter is defined through the superpotential term $W \supset \mu H_u H_d$, where $H_u$ and $H_d$ are the two Higgs doublet superfields of MSSM. Since the $\mu$–term is SUSY–preserving and is a singlet of the SM gauge symmetry, its natural value would seem to be of order the Planck scale. But $\mu \sim 10^2$ GeV is required for consistent phenomenology. It will be desirable, and is often assumed, that the $\mu$ term is related to the supersymmetry breaking scale. An attractive scenario which achieves this is the Guidice–Masiero mechanism [9] wherein a bare $\mu$ term in the superpotential is forbidden through a higher dimensional Lagrangian term

$$\mathcal{L} = \int d^4 \theta \frac{H_u H_d Z^*}{M_{Pl}}$$

where $Z$ is a spurion field which parametrizes supersymmetry breaking via $\langle F_Z \rangle \neq 0$, with $\langle F_Z \rangle / M_{Pl} \sim M_{SUSY} \sim 10^2$ GeV. For this mechanism to work, there must exist a symmetry that forbids a bare $\mu$ term in the superpotential. Such a symmetry cannot be a continuous symmetry, consistent with the requirement of non–zero guaginos masses, and therefore must be discrete. It would be desirable to realize this as a discrete gauge symmetry.

The purpose of this paper is show that it is possible to realize $Z_N$ symmetries as discrete gauge symmetries which act as $R$–parity and which solve simultaneously the $\mu$–problem via
the Guidice–Masiero mechanism. We make use of a discrete version of the Green–Schwarz mechanism [10] for anomaly cancellation. Simple realizations of $R$–parity as a discrete gauge symmetry are possible which also solve the $\mu$–problem, without enlarging the particle content of MSSM. The simplest symmetry group we have found is a $Z_4$. Under this $Z_4$ all MSSM matter superfields and the gauginos carry equal charge of 1 while the Higgs superfields have zero charge. Such a simple charge assignment is compatible with grand unification. This charge assignment is anomaly–free by virtue of the discrete Green–Schwarz mechanism. Other $Z_N$ symmetries with $N = 10, 12, 18, 36$, etc are also identified, some realized at higher Kac–Moody level. Either lepton parity or baryon parity can be obtained as a discrete symmetry in this approach, with baryon parity requiring anomaly cancellation at higher Kac–Moody levels. By allowing for a family–dependent structure in $Z_N$, we show how it is possible in our framework to explain the observed fermion mass and mixing hierarchy in a simple way.

Attempts have been made in the past to derive $R$–parity as a discrete gauge symmetry within MSSM. Early analyses [5, 6] did not incorporate the seesaw mechanism for neutrino mass or the Guidice–Masiero mechanism for generating the $\mu$–term. A recent analysis which includes these features [7] has found $Z_9$ and $Z_{18}$ discrete gauge symmetries as possible candidates for $R$–parity by demanding these symmetries to be anomaly–free. It turns out that in these models [7] the Kahler potential violates $R$–parity at higher order, leading to cosmologically disfavored lifetime for the neutralino LSP. Furthermore, the discrete charge assignment in these models was not compatible with grand unification with the MSSM spectrum. The main difference in our approach is that we make use of the Green–Schwarz mechanism for discrete anomaly cancellation, which is less restrictive compared to the straightforward methods. The outcome differs in several ways, notably in the realization of simpler symmetries (eg. $Z_4$), exact $R$–parity without cosmological problems, and compatibility with grand unification. It should be mentioned that enlarging the particle content of MSSM has been proposed as a solution to the $R$–parity and $\mu$ problems [8]. In contrast to such approaches, in our framework, the low energy spectrum is identical to that of MSSM.

This paper is organized as follows. In Sec. 2 we review briefly the discrete version of Green–Schwarz anomaly cancellation mechanism. Sec. 3 contains our main results. In 3.1 we write down the constraints arising from the Lagrangian of MSSM and the discrete anomaly cancellation conditions. In Sec. 3.2 we identify possible discrete gauge symmetries at Kac–Moody level 1 which prevent $R$–parity violating terms. In Sec. 3.3 we embed these symmetries to a higher $Z_N$ to solve the $\mu$–problem. Sec. 3.4 is devoted to solutions based on higher Kac–Moody levels. In Sec. 4 we show how a simple discrete gauge symmetry can explain the fermion mass and mixing angle hierarchy. Finally we conclude in Sec. 5.

## 2 Discrete anomaly cancellation via Green–Schwarz mechanism

Let us first recall the essence of the Green–Schwartz (GS) anomaly cancellation mechanism for a $U(1)$ gauge symmetry. String theory when compactified to four dimensions generically contains an “anomalous $U(1)_A$” gauge symmetry. A subset of the gauge anomalies in the
axial vector $U(1)_A$ current can be cancelled via the GS mechanism in the following way \[1\]. In four dimensions, the Lagrangian for the gauge boson kinetic energy contains the terms
\[
\mathcal{L}_{\text{kinetic}} = \varphi(x) \sum_i k_i F_i^2 + i\eta(x) \sum_i k_i \bar{F}_i \tag{2}
\]
where $\varphi(x)$ denotes the string dilaton field and $\eta(x)$ is its axionic partner. The sum $i$ runs over the different gauge groups in the model, including $U(1)_A$. $k_i$ are the Kac–Moody levels for the different gauge groups, which must be positive integers for the non–Abelian groups, but may be non–integers for Abelian groups. The GS mechanism makes use of the transformation of the string axion field $\eta(x)$ under a $U(1)_A$ gauge variation $V_A^\mu \rightarrow V_A^\mu + \partial_\mu \theta(x)$,
\[
\eta(x) \rightarrow \eta(x) - \theta(x) \delta_{GS} \tag{3}
\]
where $\delta_{GS}$ is a constant. If the anomaly coefficients involving the $U(1)_A$ gauge boson and any other pair of gauge bosons are in the ratio
\[
\frac{A_1}{k_1} = \frac{A_2}{k_2} = \frac{A_3}{k_3}, ..., = \delta_{GS}, \tag{4}
\]
these anomalies will be cancelled by gauge variations of the $U(1)_A$ field arising from the second term of Eq. (2). $\delta_{GS}$ in Eq. (4) is also equal to the mixed gravitational anomaly, $\delta_{GS} = A_{\text{gravity}}/12$ \[11\]. All other crossed anomaly coefficients should vanish, since they cannot be removed by the shift in the string axion field.

Consider the case when the gauge symmetry in four dimensions just below the string scale is $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_A$. Let $A_3$ and $A_2$ denote the anomalies associated with $[SU(3)_C]^2 \times U(1)_A$ and $[SU(2)_L]^2 \times U(1)_A$ respectively. Then if $A_3/k_3 = A_2/k_2 = \delta_{GS}$ is satisfied, from Eq. (4), it follows that these mixed anomalies will be cancelled. The anomaly in $[U(1)_Y]^2 \times U(1)_A$ can also be cancelled in a similar way if $A_1/k_1 = \delta_{GS}$. However, in practice, this last condition is less useful, since $k_1$ is not constrained to be an integer as the overall normalization of the hypercharge is arbitrary. If the full high energy theory is specified, there can be constraints on $A_1$ as well. For example, if hypercharge is embedded into a simple group such as $SU(5)$ or $SO(10)$, $k_1 = 5/3$ is fixed since hypercharge is now quantized. $A_1/k_1 = \delta_{GS}$ will provide a useful constraint in this case. We shall remark on this possibility in our discussions. Note also that cross anomalies such as $[SU(3)] \times [U(1)_A]^2$ are automatically zero in the Standard Model, since the trace of $SU(N)$ generators is zero. Anomalies of the type $[U(1)_Y] \times [U(1)_A]^2$ also suffer from the same arbitrariness from the Abelian levels $k_1$ and $k_A$. Finally, $[U(1)_A]^3$ anomaly can be cancelled by the GS mechanism, or by contributions from fields that are complete singlets of the Standard Model gauge group.

The anomalous $U(1)_A$ symmetry is expected to be broken just below the string scale. This occurs when the Fayet–Iliopoulos term associated with the $U(1)_A$ symmetry is cancelled, so that supersymmetry remains unbroken near the string scale, by shifting the matter superfields that carry $U(1)_A$ charges \[12\]. Although the $U(1)_A$ symmetry is broken, a $Z_N$ subgroup of $U(1)_A$ can remain intact. Suppose that we choose a normalization wherein the $U(1)_A$ charges of all fields are integers. (This can be done so long as all the charges are relatively rational numbers.) Suppose that the scalar field which acquires a vacuum expectation
value (VEV) and breaks $U(1)_A$ symmetry has a charge $N$ under $U(1)_A$ in this normalization. A $Z_N$ subgroup is then left unbroken down to low energies. We shall identify $R$-parity of MSSM with this unbroken $Z_N$ symmetry.

The field that acquires a VEV and breaks $U(1)_A$ to $Z_N$ can supply large masses of order the string scale to a set of fermions which have Yukawa couplings involving this field. Such fields may include Majorana fermions and Dirac fermions. These heavy fields can carry SM gauge quantum numbers, but they must transform vectorially under the SM. In order that their mass terms be invariant under the unbroken $Z_N$, it must be that

$$2q_i = 0 \mod N \text{ (Majorana fermion)}$$
$$q_i + \bar{q}_i = 0 \mod N \text{ (Dirac fermion)}$$

where $q_i$ are the $U(1)_A$ charges of these heavy fermions. The index $i$ is a flavor index corresponding to different heavy fields. These heavy fermions, being chiral under the $U(1)_A$, contribute to gauge anomalies. Their contribution to the $[SU(3)_C]^2 \times U(1)_A$ gauge anomaly is given by $A_3 = \sum_i q_i \ell_i = (N/2) \sum_i p_i \ell_i$ (Majorana fermion) or $A_3 = \sum_i (q_i + \bar{q}_i) \ell_i = (N) \sum_i p_i \ell_i$ (Dirac fermion) where $\ell_i$ is the quadratic index of the relevant fermion under $SU(3)_C$ and the $p_i$ are integers. We shall adopt the usual normalization of $\ell = 1/2$ for fundamental of $SU(N)$. Then, for the case of heavy Dirac fermion, one has $A_3 = p(N/2)$ where $p$ is an integer, as the index of the lowest dimensional (fundamental) representations is 1/2 and those of all other representations are integer multiples of 1/2. The same conclusion follows for the case of Majorana fermions for a slightly different reason. All real representations of $SU(3)_C$ (such as an octet) have integer values of $\ell$, so that $\sum_i p_i \ell_i$ is an integer. Analogous conclusions follow for the $[SU(2)_L]^2 \times U(1)_A$ anomaly coefficient.

If the $Z_N$ symmetry that survives to low energies was part of $U(1)_A$, the $Z_N$ charges of the fermions in the low energy theory must satisfy a non-trivial condition: The anomaly coefficients $A_i$ for the full theory is given by $A_i$ from the low energy sector plus an integer multiple of $N/2$. These anomalies should obey Eq. (4), leading to the discrete version of the Green–Schwarz anomaly cancellation mechanism:

$$\frac{A_3 + \frac{p_1 N}{k_3}}{\frac{1}{2}} = \frac{A_2 + \frac{p_2 N}{k_2}}{\frac{1}{2}} = \delta_{GS}$$

with $p_1$, $p_2$ being integers. Since $\delta_{GS}$ is an unknown constant (from the effective low energy point of view), the discrete anomaly cancellation conditions of Eq. (6) are less stringent than those arising from conventional anomaly cancellations. If $\delta_{GS} = 0$ in Eq. (6), the anomaly is cancelled without assistance from the Green–Schwarz mechanism. We shall not explicitly use the condition that $\delta_{GS} \neq 0$, so our solutions will contain those obtained by demanding $\delta_{GS} = 0$ in Eq. (6), viz., $A_3 = -p_1(N/2)$, $A_2 = -p_2(N/2)$ with $p_1$, $p_2$ being integers.

In our analysis we shall not explicitly make use of the condition $A_1/k_1 = A_2/k_2$, since, as mentioned earlier, the overall normalization of hypercharge is arbitrary. However, once a solution to the various $Z_N$ charges is obtained, we can check for the allowed values $k_1$, and in particular, if $k_1 = 5/3$ is part of the allowed solutions. This will be an interesting case for two reasons. If hypercharge is embedded in a simple grand unification group such as $SU(5)$, one would expect $k_1 = 5/3$. Even without a GUT embedding $k_1 = 5/3$ is interesting.
We recall that unification of gauge couplings is a necessary phenomenon in string theory. Specifically, at tree level, the gauge couplings of the different gauge groups are related to the string coupling constant $g_{st}$ which is determined by the VEV of the dilaton field as \[ k_3g_3^2 = k_2g_2^2 = k_1g_1^2 = g_{st}^2 \] where $k_i$ are the levels of the corresponding Kac–Moody algebra. In particular, if $k_1 : k_2 : k_3 = 5/3 : 1 : 1$, we would have $\sin^2\theta_W = 3/8$ at the string scale, a scenario identical to that of conventional gauge coupling unification with simple group such as $SU(5)$. For these reasons, we shall pay special attention to the case $k_1 = 5/3$.

An interesting example of a discrete gauge symmetry in the MSSM (or the SM) with seesaw neutrino masses is the $Z_6$ subgroup of $B - L$. The introduction of the right–handed neutrino for generating small neutrino masses makes $B - L$ a true gauge symmetry. When the $\nu^c$ fields acquire super-large Majorana masses, $U(1)_{B-L}$ breaks down to a discrete $Z_6$ subgroup. The $Z_6$ charges of the MSSM fields arising from $B - L$ are displayed in Table 1. Here we have used the standard notation for the fermion fields ($Q$ and $L$ being the left–handed quark and lepton doublets, $u^c$, $d^c$ being the quark singlets, and $e^c$, $\nu^c$ being the (conjugates of) the right–handed electron and the right–handed neutrino singlets). To obtain the unbroken $Z_6$ charge, we first multiply the $B - L$ charge by 3 so that they become integers, then observe that the $\nu^c\nu^c$ Majorana mass term carries 6 units of this integer $B - L$ charge. Thus this $Z_6$ subgroup is left unbroken.

It is worth mentioning that the $Z_6$ symmetry has a $Z_2$ and a $Z_3$ subgroups as well. In the analysis that follows in the next section we will be making connections with the $Z_6$ subgroup of $B - L$ and its $Z_2$ and $Z_3$ subgroups.

Anomalous $U(1)$ symmetry has found applications in addressing the fermion mass and mixing hierarchy problem \cite{14}, doublet–triplet splitting problem in GUT \cite{15}, the strong CP problem \cite{16}, the $\mu$ problem of SUSY \cite{17} and for SUSY breaking \cite{18}.

### 3 Discrete gauge symmetries in the MSSM

In this section we turn to the identification of discrete gauge symmetries in the MSSM that can serve as $R$–parity and simultaneously explain the origin of the $\mu$ term. We stay with the minimal particle content of MSSM, with the inclusion of the right–handed neutrinos needed for generating neutrino masses via the seesaw mechanism \cite{19}. Anomalies associated with the discrete gauge symmetry will be cancelled by the Green–Schwarz mechanism as discussed in Sec. 2.
3.1 Constraints from the Lagrangian and discrete anomalies

We have displayed in Table 2 the particle content of MSSM along with their charges under an anomalous \( U(1) \) gauge symmetry. The Grassmannian variable \( \theta \) also carries a charge (equal to \( \alpha \)), which allows for the \( U(1) \) to be identified as an \( R \) symmetry. \( Z \) is a spurion superfield that acquires a non–zero \( F \) component and breaks supersymmetry with \( \langle F_Z \rangle / M_{Pl} \sim M_{SUSY} \sim 10^2 \) GeV. We shall assume family–independent \( U(1) \) symmetry in this section. Any unbroken discrete symmetry must be family–independent to be consistent with MSSM phenomenology, that is the reason for focusing on such symmetries. In Sec. 4 we shall extend this analysis to flavor–dependent symmetries, even in that case, we will demand that a flavor–independent \( Z_M \) symmetry is left intact.

| Field | \( Q \) | \( u^c \) | \( d^c \) | \( L \) | \( e^c \) | \( \nu^c \) | \( H_u \) | \( H_d \) | \( \theta \) | \( Z \) |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| \( SU(3)_C \) | 3 | \( \bar{3} \) | \( \bar{3} \) | 1 | 1 | 1 | 1 | 1 | 1 |
| \( SU(2)_L \) | 2 | 1 | 1 | 2 | 1 | 1 | 2 | 2 | 1 |
| \( U(1)_Y \) | \( 1/6 \) | \( -2/3 \) | 1/3 | \( -1/2 \) | 1 | 0 | 1/2 | \( -1/2 \) | 0 |
| \( U(1)_A \) | \( q \) | \( u \) | \( d \) | \( l \) | \( e \) | \( n \) | \( h \) | \( \bar{h} \) | \( \alpha \) |

Table 2: The matter superfields of MSSM along with their anomalous \( U(1) \) charges. \( \theta \) is the Grassmann variable. \( Z \) is the spurion field which is responsible for supersymmetry breaking.

The superpotential of the model, including small neutrino masses via the seesaw mechanism is

\[
W = Q u^c H_u + Q d^c H_d + L e^c H_d + L \nu^c H_u + M_R \nu^c \nu^c ;
\] (8)

where \( M_R \) is the heavy right–handed neutrino Majorana mass. We have suppressed Yukawa couplings and generation indices, which must be understood.

In order to avoid a supersymmetric \( \mu \) term in the Lagrangian, \( \mathcal{L} \supset \mu \int d^2 \theta H_u H_d \), so that the magnitude of \( \mu \) may be related to the SUSY breaking scale, we impose the condition

\[
h + \bar{h} \neq 2\alpha .
\] (9)

A \( \mu \)–parameter of the right order is induced through the Guidice–Masiero mechanism \[ \square \] via the Lagrangian term, \( \mathcal{L} \supset \int d^4 \theta H_u H_d \frac{Z}{M_{Pl}} \). Invariance of this term under the \( U(1) \) symmetry requires

\[
h + \bar{h} - z = 0 .
\] (10)

The gaugino masses arise through the Lagrangian term

\[
\int d^2 \theta W_\alpha W_\alpha \frac{Z}{M_{Pl}}
\] (11)

once \( \langle F_Z \rangle \neq 0 \) is induced. Combining the invariance of this term with that of the gauge kinetic term \( \int d^2 \theta W_\alpha W_\alpha \), we see that the spurion field \( Z \) must have zero charge under the
It is clear that the simultaneous presence of the gaugino mass term and the \( \mu \) term reduces the \( U(1) \) symmetry to a discrete subgroup \( Z_N \). Therefore, one has to start with a discrete symmetry \( Z_N \) with the spurion superfield \( Z \) having a charge \( 0 \mod N \). Under \( Z_N \), the conditions Eqs. (9)-(11) become

\[
\begin{align*}
  z &= 0 \mod N \\
  h + \bar{h} &= 0 \mod N \\
  h + \bar{h} &\neq 2\alpha \mod N
\end{align*}
\]  

(12)

which also implies that \( 2\alpha \neq 0 \mod N \).

Based on the invariance of the Yukawa couplings of Eq. (8) under \( Z_n \) and the conditions listed in Eq. (12), we obtain the following set of constraint equations:

\[
\begin{align*}
  z &= m_1 N \\
  h + \bar{h} &= m_2 N \\
  q + u + h &= 2\alpha + m_3 N \\
  q + d + \bar{h} &= 2\alpha + m_4 N \\
  l + e + \bar{h} &= 2\alpha + m_5 N \\
  2n &= 2\alpha + m_6 N \\
  l + n + h &= 2\alpha + m_7 N
\end{align*}
\]  

(13)

where \( m_i \ (i = 1 - 7) \) are all integers.

The discrete \( Z_N \) anomaly coefficients for the \( SU(3)_C \) and the \( SU(2)_L \) gauge groups are

\[
\begin{align*}
  A_3 &= -3\alpha + 3q + \frac{3}{2}u + \frac{3}{2}d, \\
  A_2 &= -5\alpha + \frac{9}{2}q + \frac{3}{2}l + \frac{1}{2}(h + \bar{h}).
\end{align*}
\]  

(14)

Here we note that the fermionic charge of the \( u^c \) field, for example, relevant for the anomaly coefficient, is \((u - \alpha)\) since \( \theta \) carries charge \( \alpha \). \( A_3 \) and \( A_2 \) include contributions from the gauginos as well. We shall cancel these anomalies by applying the Green–Schwarz mechanism as given in Eq. (6).

Non–zero gauginos masses arise through the VEV \( \langle F_Z \rangle \neq 0 \) (see Eq. (11)). Let us denote the \( Z_N \) charge of \( F_Z \) to be \( M \). \( \langle F_Z \rangle \neq 0 \) breaks the original \( Z_N \) symmetry down to a subgroup \( Z_M \):

\[
Z_N \rightarrow Z_M.
\]  

(15)

(It must be that \( M > 1 \) for an unbroken discrete symmetry to survive after SUSY breaking.) Since \( M = z - 2\alpha = m_1 N - 2\alpha \) where \( m_1 \) is an integer, we have

\[
\alpha = \frac{n_1}{2} M
\]  

(16)

with \( n_1 \) being an integer. Let \( N = n_0 M \) where \( n_0 \) is an integer. Since invariance of the Yukawa couplings under the \( Z_N \) symmetry requires invariance under the subgroup \( Z_M \), we

\[\text{SUSY breaking scalar masses are invariant under the } U(1) \text{ symmetry and do not provide any constraint.}\]
can solve the constraints of Eqs. (13)-(14) along with Eq. (6) to determine the various charges by first confining to the invariance under the smaller group $Z_M$. Under this $Z_M$, a superpotential term $\mu H_u H_d$ will be allowed. Once a solution is found, we can embed the $Z_M$ symmetry into a higher $Z_N$ symmetry that would forbid the $\mu$ term. Making some change of variables, viz., $n_2 = n_0 m_2$, $n_4 = n_0 m_6$, $n_5 = -n_0 p_1$, $n_6 = -n_0 p_2$, and applying the anomaly cancellation condition of Eq. (6), we obtain the charges of the various fields from Eqs. (13)-(14) as

$$
\begin{align*}
    z &= M n_0 \\
    h &= 3q + M \left( \frac{n_2 - n_6}{3} - \frac{n_4}{2} - \frac{7n_1}{6} \right) + \frac{M}{b} (n_2 - n_3 + \frac{n_5}{3} - n_1) \\
    \bar{h} &= -3q + M \left( \frac{2n_2 + n_6}{3} + \frac{n_4}{2} + \frac{n_1}{6} \right) + \frac{M}{b} (-n_2 + n_3 - \frac{n_5}{3} + n_1) \\
    u &= -4q + M \left( -\frac{n_2 - n_6}{3} + \frac{n_4}{2} + \frac{n_1}{6} \right) + \frac{M}{b} (-n_2 + n_3 - \frac{n_5}{3} + n_1) \\
    d &= 2q + M \left( \frac{n_2 - n_6}{3} - \frac{n_4}{2} - \frac{7n_1}{6} \right) + \frac{M}{b} (n_2 - n_3 + \frac{n_5}{3} - n_1) \\
    l &= -3q + M \left( -\frac{n_2 - n_6}{3} + \frac{2n_1}{3} \right) + \frac{M}{b} (-n_2 + n_3 - \frac{n_5}{3} + n_1) \\
    e &= 6q + M \left( -\frac{n_2 - n_6}{3} - \frac{n_4}{2} - \frac{5n_1}{6} \right) + \frac{2M}{b} (n_2 - n_3 + \frac{n_5}{3} - n_1) \\
    n &= M \left( \frac{n_4 + n_1}{2} \right) \\
    \alpha &= M \frac{n_1}{2} .
\end{align*}
$$

Here we have defined $b \equiv k_3/k_2$. The $n_i$ in Eq. (17) are all integers. A specific choice of the integers $n_i$ will fix the charge assignment explicitly. We note that the terms proportional to $q$ in Eq. (17) are proportional to the SM hypercharge $Y$. One can remove these terms and set $q = 0$ in Eq. (17) without loss of generality by making a shift of the $Z_M$ charges proportional to $Y$. The quark doublet $Q$ will then have zero charge under the unbroken $Z_M$. It should be kept in mind that to each solution we find, one can add $Z_M$ charges proportional to $Y$ to obtain equivalent solutions.

Based on Eq. (17), one can compute the anomaly coefficients under $Z_M$. They are

$$
\begin{align*}
    A_3 &= \frac{3}{2} M (n_1 - n_2) \\
    A_2 &= \frac{1}{2b} M (3n_1 - 3n_2 + bn_6) .
\end{align*}
$$

Note that from the last of Eq. (17), we have $2\alpha = 0 \mod M$. So the superpotential is invariant under $Z_M$. Also, under $Z_M$, one has $h + \bar{h} = 0 \mod M$, so a $\mu$ term in the superpotential is allowed by this symmetry. (Such a term will be forbidden when the $Z_M$ symmetry is embedded into a higher $Z_N$ symmetry, which we shall do in subsection 3.3.) In
order to avoid $R$–parity breaking couplings, the total charges of the corresponding superpotential terms should be non–integer multiples of $M$, which puts extra constraints on the $Z_M$ charges. There are four types of $R$–parity violating terms. Their $Z_M$ charges are given by

\[
\begin{align*}
   u^c d^c d^c & : M \left( \frac{5n_1}{6} - \frac{2n_2 + n_6}{3} - \frac{n_4}{2} \right) + \frac{M}{b}(n_2 - n_3 + \frac{n_5}{3} - n_1) \\
   LLe^c & : M \left( \frac{n_1 - n_4}{2} \right) \\
   LH_u & : M \left( \frac{n_1 - n_4}{2} \right) \\
   QLd^c & : M \left( \frac{n_1 - n_4}{2} \right).
\end{align*}
\]  

(19)

It is easy to show that the largest $Z_M$ symmetry is $Z_{6k_3}$ from Eq. (17). We shall now find solutions to these sets of equations for various values of the parameter $b = k_3/k_2$.

### 3.2 Green–Schwarz anomaly cancellation at Kac–Moody level 1

The simplest possibility for the parameters $k_2$ and $k_3$ to take is $k_3 = k_2 = 1$, so that $b = 1$ in Eqs. (17)-(18). This is the case of Kac–Moody algebra realized at level 1. Since the constraint equations depend only on the ratio $b = k_3/k_2$, the case of higher levels will coincide with that of level 1 as long as the levels are the same for both $SU(3)_C$ and $SU(2)_L$. From a theoretical point of view this case is the most attractive, since it allows for both $SU(3)_C$ and $SU(2)_L$ to emerge from the same gauge group as in a GUT. The charge assignment and possible discrete symmetries for this case $k_2 = k_3$ are shown explicitly in Table 3.5.

The procedure we have followed to obtain Table 3 is as follows. First we set $b = 1$. Then we choose a set of integers $n_i$ in the range $n_i \subset (0 - 5)$. Any $n_i$ larger than or equal to 6 (or any negative $n_i$) can be absorbed into the $mod M$ piece. The highest $Z_M$ symmetry is then found to be $Z_6$. For every choice of the integer set $n_i$ we demand that the $R$–parity breaking couplings of Eq. (19) be forbidden. (This requires $n_1 - n_4$ to be an odd integer and that the charge of $d^c$ should be non–zero under $Z_M$.) Then we solve for the charges of the various fields, setting $q = 0$ as mentioned above. If the Green–Schwarz anomaly cancellation condition is satisfied we accept the solution. Upto overall conjugation and shifts proportional to hypercharge, the complete set of solutions is as given in Table 3. We have also listed the anomaly coefficients ($A_2, A_3$) in Table 3.

Several remarks are in order about the results shown in Table 3.

(i) Models I and II differ only in the value of $\alpha$. In the effective low energy Lagrangian, what matters is $2\alpha$, which is the same for both models. Although the two models look

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5 Among the solutions, we remove those which either are conjugate of the listed solution or can be realized as linear combination of the known solution and hypercharge. For example, in model III and IV, we have chosen $q = 0 \ mod 6$. The charge $q$ need not be actually zero. Since there exists an unbroken $U(1)_Y$ hypercharge which is anomaly–free, one can always take a linear combination of model III (or IV) with $U(1)_Y$ to find equivalent solutions with $q \neq 0 \ mod M$. This comment also applies to the models listed in all the other tables.
identical from low energy point of view, their embedding into a high scale theory will not be the same. This is the reason for listing them separately. We shall see that when $Z_M$ is embedded into a higher symmetry $Z_N$ so as to forbid a large $\mu$ term, Models I and II will look different. Similar remarks apply to Models III and IV.

(ii) The $Z_2$ symmetries in Table 3 (I and II) are actually subgroups of the $Z_6$ symmetry (III and IV). Their embedding into $Z_N$ will however lead to different solutions. Note also that the $Z_3$ subgroup of $Z_6$ does not show up as a solution, since that would allow for lepton number violation.

(iii) The $Z_6$ symmetric solutions of Table 3 are actually identical to the $Z_6$ subgroup of $B-L$ shown in Table 1 which can prevent $R$-parity violation in MSSM [20]. This can be recognized by taking linear combinations of Models III-IV and hypercharge. Suppose we normalize hypercharge so that all MSSM fields have integer values denoted by $\hat{Y}$ (with $Q$ field having $\hat{Y} = 1$). Take now the combination $3(IV) + \hat{Y}$ (mod 6). This redefined charge is identical to the $Z_6$ subgroup of $B-L$ of Table 1. The $Z_2$ models are identified as the $Z_2$ subgroups of $B-L$. We conclude from our systematic analysis that even with GS anomaly cancellation, the only allowed discrete symmetries at the $Z_M$ level (which admits a superpotential $\mu$ term) are the subgroups of $B-L$. This will however be not the case when $Z_M$ is embedded into $Z_N$. Note also that the anomaly coefficients $A_2$ and $A_3$ in Table 3 are all equal to $M/2$, so GS mechanism is not playing a role in anomaly cancellations. This remark will also be different in the $Z_N$ embedding.

### 3.3 Embedding $Z_M$ into $Z_N$ and solving the $\mu$–problem

We recall that the original $Z_N$ symmetry broke down to a subgroup $Z_M$ once the spurion field $Z$ acquired a VEV along its $F$-component. At the level of $Z_M$, a superpotential $\mu$–term is allowed. Now we turn to the task of identifying the original $Z_N$ symmetry needed for explaining the $\mu$ term. We look for the simplest higher symmetry into which the $Z_M$ solutions of Table 3 can be embedded. Each of the model in Table 3 has a different embedding into $Z_N$.

Consider the embedding of Model I in Table 3 into $Z_N$. The smallest $Z_N$ group that contains a $Z_2$ subgroup is $Z_4$. This embedding is shown in Table 4. There are two possible charge assignments indicated as Models 1a and 1b. These models are obtained as follows. First we choose the value of $\alpha$ to be either 1 or 3 under $Z_4$ (since it must reduce to 1 under

| Model | $Z_M$ | $q$ | $u$ | $d$ | $l$ | $e$ | $n$ | $h$ | $\bar{h}$ | $\alpha$ | $(A_2, A_3)$ |
|-------|------|----|----|----|----|----|----|----|-------|-------|-----------|
| I     | $Z_2$| 2  | 1  | 1  | 2  | 1  | 1  | 1  | 1     | 1     | (2, 2)     |
| II    | $Z_2$| 2  | 1  | 1  | 2  | 1  | 1  | 1  | 1     | 0     | (1, 1)     |
| III   | $Z_6$| 6  | 5  | 1  | 2  | 5  | 3  | 1  | 5     | 3     | (6, 6)     |
| IV    | $Z_6$| 6  | 5  | 1  | 2  | 5  | 3  | 1  | 5     | 0     | (3, 3)     |

Table 3: MSSM charge assignment when $k_2 = k_3$. $\alpha$ denotes the charge of the gaugino and the Grassmanian variable $\theta$. When $\alpha \neq 0$, the $Z_M$ acts as an $R$–symmetry. Also shown are the anomaly coefficients $(A_2, A_3)$. 


These two values correspond to the two models in Table 4. Then we demand that a bare $\mu$ term in the superpotential is prevented by the $Z_4$ symmetry. That determines the charges ($h$, $\bar{h}$) to be either $(1, 3)$ or $(3, 1)$. It turns out that the charges in the latter case are the conjugates of the former, and so we discard it. Then we set the charge $n$ to be either 1 or 3, consistent with it being 1 under $Z_2$ subgroup. This fixes the charges of all fields. For each case the anomaly coefficients $A_2$ and $A_3$ are computed. If the anomalies are cancelled by the GS mechanism, we accept the solution. Only two solutions are found to survive, as displayed in Table 4.

$$
\begin{array}{cccccccc}
  & q & u & d & l & e & n & h & \bar{h} & \alpha & (A_2, A_3) \\
 Ia & 4 & 1 & 3 & 4 & 3 & 1 & 1 & 3 & 1 & (1, 3) \\
 Ib & 4 & 1 & 3 & 4 & 3 & 1 & 1 & 3 & 3 & (3, 1) \\
\end{array}
$$

Table 4: Embedding of the $Z_2$ symmetry of Model I of Table 3 into $Z_4$ symmetry.

Note that the discrete $Z_4$ anomalies are cancelled by the GS mechanism. Individually $A_2$ and $A_3$ are not multiples of $N/2 = 2$, but the two coefficients differ only by $N/2 = 2$. We conclude that this simple solution would not have been possible without GS anomaly cancellation.

The models of Table 4 can be recast in a very simple form by forming the linear combination \{Ia + $Y$ (mod 4)\}, or \{Ib + $Y$ (mod 4)\} with $Y$ being the integer values of SM hypercharge. We display in Table 5 Model Ia recast in this form. The charge assignment is very simple, all matter fields of MSSM carry charge 1 under $Z_4$, while the Higgs superfields carry charge 0. The gauginos also have charge 1. The contribution to the $Z_4$ anomaly from the matter fields are the same for $A_2$ and $A_3$ (the number of color triplets is the same as the number of $SU(2)_L$ doublets in MSSM). While the gluino contributes an amount equal to 3 to $A_3$, the sum of the $SU(2)_L$ gaugino (= 2) and the Higgsinos (= −1) add to $A_3 = +1$. We see that $A_2$ and $A_3$ differ by $N/2 = 2$, signaling anomaly cancellation via GS mechanism.

$$
\begin{array}{cccccccc}
  & q & u & d & l & e & n & h & \bar{h} & \alpha & (A_2, A_3) \\
 Ia & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & (3, 1) \\
\end{array}
$$

Table 5: Model 1a recast with a shift proportional to hypercharge.

The charge assignment shown in Table 5 is clearly compatible with grand unification. The Kac–Moody level associated with hypercharge will be $k_1 = 5/3$ with a GUT embedding. Gauge coupling unification is then predicted, since $\sin^2 \theta_W = 3/8$ near the string scale. This is true even if there were no covering GUT symmetry.

Now we turn to Model II of Table 3. Embedding this $Z_2$ into a $Z_4$ is not viable, since a large $\mu$ term cannot be prevented in that case. The next simplest possibility is $Z_6$, which also does not work as the $Z_6$ anomalies do not cancel. The simplest embedding is found to be into a $Z_{10}$ with the charge assignment as shown in Table 6.
In the models of Table 6, one might consider a $Z_5$ subgroup of $Z_{10}$. This subgroup is sufficient to forbid the $\mu$–term in the superpotential $W$, as well as to prevent dangerous $R$–parity violating couplings in $W$. With invariance only under $Z_5$, the term $u^c d^c d^c$ will have zero charge. A Lagrangian term arising from the Kahler potential $\mathcal{L} \supset \int d^4\theta(u^c d^c d^c Z^*/M_{Pl}^2)$ will then be allowed. Once $F_Z$ acquires a non–zero VEV, this term will lead to a superpotential term $\mathcal{L} \supset \int d^2\theta(M_{SUSY}/M_{Pl})u^c d^c d^c$. Such a term violates $R$–parity, although very weakly. Signals of such a weak violation will be unobservable in collider experiments. However, this scenario will not fit well with cosmological constraints. The LSP will decay through this induced $R$–parity violating Yukawa coupling $\lambda''$, which has a strength of order $M_{SUSY}/M_{Pl} \sim 10^{-15}$. We can estimate the lifetime of the LSP to be $\tau \sim [(\lambda'')^2 M_{LSP}/(8\pi)]^{-1} \sim 10^4$ sec. Such a lifetime falls into the cosmologically disfavored range and would be in violation of nucleosynthesis constraints. (This situation is analogous to the gravitino problem of supergravity, but is slightly worse, since the LSP mass is expected to be order 100 GeV, rather than a TeV for the gravitino, making the LSP lifetime somewhat longer than that of the gravitino.) We consider the $Z_5$ solution to be unacceptable for this reason. Since $Z_5$ symmetry does not contain a $Z_2$ subgroup, exact $R$–parity could not be defined after SUSY breaking, unlike in the case of the $Z_{10}$ model.

In Tables 7 and 8 we show the simplest embedding of Models III and IV into $Z_{12}$ and $Z_{18}$ respectively. The procedure we have adopted is identical to that for Models I and II.

| $Z_{10}$ | $q$ | $u$ | $d$ | $l$ | $e$ | $n$ | $h$ | $\bar{h}$ | $\alpha$ | $(A_2, A_3)$ |
|---------|-----|-----|-----|-----|-----|-----|-----|-------|-------|---------|
| IIa     | 10  | 1   | 7   | 4   | 3   | 7   | 3   | 7     | 2     | (6, 6)   |
| IIb     | 10  | 7   | 9   | 8   | 1   | 9   | 1   | 9     | 4     | (2, 2)   |

Table 6: $Z_{10}$ embedding of Model II.

| $Z_{12}$ | $q$ | $u$ | $d$ | $l$ | $e$ | $n$ | $h$ | $\bar{h}$ | $\alpha$ | $(A_2, A_3)$ |
|---------|-----|-----|-----|-----|-----|-----|-----|-------|-------|---------|
| IIIa    | 12  | 5   | 7   | 8   | 11  | 9   | 1   | 11    | 3     | (9, 9)   |
| IIIb    | 12  | 11  | 1   | 8   | 5   | 3   | 7   | 5     | 3     | (9, 9)   |
| IIIc    | 12  | 5   | 7   | 8   | 11  | 9   | 1   | 11    | 9     | (3, 3)   |
| IIId    | 12  | 11  | 1   | 8   | 5   | 3   | 7   | 5     | 9     | (3, 3)   |

Table 7: $Z_{12}$ embedding of Model III.

As in the case of the $Z_{10}$ model of Table 6, we may consider taking a $Z_9$ subgroup of $Z_{18}$ in Table 8. However, since $Z_9$ does not contain $Z_2$ or $Z_6$ as subgroups, after SUSY breaking, small $R$–parity violating Yukawa couplings of the type $W \supset LLe^c$ will be generated from the Kahler potential with coupling constants of order $10^{-15}$. Such couplings would violate constraints from big bang nucleosynthesis since the lifetime of the LSP will be of order $10^4$ sec. We shall not consider the $Z_9$ subgroup any further.
3.4 Discrete anomaly cancellation at higher Kac–Moody level

Thus far we have assumed the parameter $b \equiv k_3/k_2 = 1$. This is the case when the Kac–Moody levels for $SU(3)_C$ and $SU(2)_L$ are the same, the simplest possibility being $k_3 = k_2 = 1$. It is also possible that $k_2$ and $k_3$ are not the same. It is not clear to us how easy it is to construct string models with different values for $k_3$ and $k_2$. Although it might appear less attractive theoretically, it is nevertheless a logical possibility. In this section we analyze discrete anomaly cancellation for values of $k_3/k_2 \neq 1$.

From a technical point of view it appears to be difficult to construct models with levels higher than 3 in string theory [21]. Motivated by this observation, we shall confine our discussions to $k_2$ and $k_3$ being less than or equal to 3. This allows for the cases when $b \equiv k_3/k_2 = 1/2, 1/3, 1/2, 2/3$ and 3/2. The case of $b = 1$ has already been analyzed in the previous section, so we turn to the other cases.

From the solution Eq. (17) which applies to $Z_M$ invariance (that allows a bare $\mu$ term, but forbids all $R$–parity violations), a few simplifications can be found. The case where $b = 1/2$ and $b = 1/3$ are identical to the case of $b = 1$. This is because the $b$–dependent terms only contribute to the various charges proportional to $n_5$ in Eq. (17). But this $n_5$ contribution can be absorbed into the $n_2$ term in all equations. No new solutions will then be generated under $Z_M$. Similarly, it is easy to see that the cases $b = 2/1$ and $b = 2/3$ are equivalent under $Z_M$. And the case where $b = 3/1$ becomes identical to the case of $b = 3/2$. Among these equivalent cases under $Z_M$, we shall only consider one possibility. Although it is possible that when the resulting models are embedded into a higher symmetry $Z_N$, new models at higher levels may emerge, we shall not pursue it here.

We shall then focus on the case where $b \equiv k_3/k_2 = 2$, and $b = 3$ for anomaly cancellation at higher Kac–Moody level. Following the same procedure as in the previous section, we obtain the corresponding discrete symmetry and charge assignment. The solutions for the case of $k_3/k_2 = 2$, which is the same for $k_3/k_2 = 2/3$, are shown in Table 9. The discrete $Z_M$ symmetry is $Z_6$ in this case. Note that the discrete GS mechanism cancels the gauge anomalies of $Z_6$. For example, $A_2 = 9/2$, $A_3 = 6$ is anomaly free since with $k_2 = 1$, $k_3 = 2$, the cancellation condition is that $2A_2$ and $A_3$ differ by an integer multiple of $N/2 = 3$ (see Eq. (6)).

The two $Z_6$ models of Table 9 have been embedded into the simplest possible $Z_N$ model in Tables 10 and 11. The $Z_N$ symmetries are found to be $Z_{12}$ and $Z_{18}$. The discrete gauge anomalies are cancelled by GS mechanism, as before. Take Model Vd for example, which has $A_2 = 3/2, A_3 = 9$ under $Z_{12}$. $2A_2$ and $A_3$ differ by 6, which is an integer multiple of $N/2 = 6$.

The next case is when $b = k_3/k_2 = 3$, which gives at the $Z_M$ level the same models

| $Z_{18}$ | q | u | d | l | e | n | h | $\hat{h}$ | $\alpha$ | $(A_2, A_3)$ |
|---------|---|---|---|---|---|---|---|---|---|-----------|
| IVa     | 18 | 11 | 13 | 14 | 17 | 15 | 1 | 17 | 6 | (9, 9) |
| IVb     | 18 | 5  | 1  | 8  | 11 | 15 | 7 | 11 | 6 | (18, 9) |

Table 8: $Z_{18}$ embedding of model IV.
| Model | $Z_M$ | $q$ | $u$ | $d$ | $l$ | $e$ | $n$ | $h$ | $\bar{h}$ | $\alpha$ | $(A_2, A_3)$ |
|-------|-------|-----|-----|-----|-----|-----|-----|-----|---------|---------|-------------|
| V     | $Z_6$ | 6   | 2   | 4   | 5   | 5   | 3   | 4   | 2       | 3       | (9/2, 6)    |
| VI    | $Z_6$ | 6   | 2   | 4   | 5   | 5   | 3   | 4   | 2       | 0       | (3/2, 3)    |

Table 9: Discrete symmetries and the corresponding charge assignment when $k_3/k_2 = 2$ or $2/3$.

| $Z_{12}$ | $q$ | $u$ | $d$ | $l$ | $e$ | $n$ | $h$ | $\bar{h}$ | $\alpha$ | $(A_2, A_3)$ |
|-----------|-----|-----|-----|-----|-----|-----|-----|---------|---------|-------------|
| Va        | 12  | 8   | 4   | 5   | 11  | 3   | 10  | 2       | 3       | (9/2, 9)    |
| Vb        | 12  | 2   | 10  | 5   | 5   | 9   | 4   | 8       | 3       | (9/2, 9)    |
| Vc        | 12  | 2   | 10  | 11  | 11  | 3   | 4   | 8       | 3       | (3/2, 9)    |
| Vd        | 12  | 8   | 4   | 11  | 5   | 9   | 10  | 2       | 3       | (3/2, 9)    |

Table 10: Embedding of $Z_6$ of Model V into $Z_{12}$.

| $Z_{18}$ | $q$ | $u$ | $d$ | $l$ | $e$ | $n$ | $h$ | $\bar{h}$ | $\alpha$ | $(A_2, A_3)$ |
|----------|-----|-----|-----|-----|-----|-----|-----|---------|---------|-------------|
| VIa      | 18  | 2   | 10  | 11  | 5   | 15  | 4   | 14      | 6       | (9/2, 18)   |
| VIb      | 18  | 14  | 16  | 5   | 17  | 15  | 10  | 8       | 6       | (27/2, 9)   |

Table 11: $Z_{18}$–embedding of the $Z_6$ model VI.
as \( b = 3/2 \). In Table 12 we list the allowed \( Z_M \) models, with \( M = 18 \). Table 13 has the embedding of Model VII into \( Z_{18} \) that prevents a large \( \mu \) term, Table 14 has the embedding of Model VIII, the simplest possibility for which being \( Z_{90} \). In all cases the discrete anomalies are cancelled by the GS mechanism.

| Model | \( Z_M \) | \( q \) | \( u \) | \( d \) | \( l \) | \( e \) | \( n \) | \( h \) | \( \bar{h} \) | \( \alpha \) | \( (A_2, A_3) \) |
|-------|-----------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| VII   | \( Z_{18} \) | 18    | 1     | 17    | 10    | 7     | 9     | 17    | 1     | 9     | (6,18) |
| VIII  | \( Z_{18} \) | 18    | 1     | 17    | 10    | 7     | 9     | 17    | 1     | 18    | (15, 9) |

Table 12: Discrete symmetries and the charge assignments for \( k_3/k_2 = 3 \) or 3/2.

| \( Z_{36} \) | \( q \) | \( u \) | \( d \) | \( l \) | \( e \) | \( n \) | \( h \) | \( \bar{h} \) | \( \alpha \) | \( (A_2, A_3) \) |
|------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| VIIa       | 36    | 1     | 35    | 28    | 25    | 9     | 17    | 19    | 9     | (33, 27) |
| VIIb       | 36    | 19    | 17    | 10    | 7     | 9     | 35    | 1     | 9     | (6, 27)  |

Table 13: \( Z_{36} \)-embedding of the \( Z_{18} \) model VII.

| \( Z_{90} \) | \( q \) | \( u \) | \( d \) | \( l \) | \( e \) | \( n \) | \( h \) | \( \bar{h} \) | \( \alpha \) | \( (A_2, A_3) \) |
|------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| VIIIa      | 90    | 55    | 71    | 46    | 25    | 9     | 53    | 37    | 54    | (89, 27) |
| VIIIb      | 90    | 55    | 53    | 28    | 25    | 27    | 89    | 1     | 72    | (42, 36) |

Table 14: Embedding of \( Z_{18} \) of Model VIII into \( Z_{90} \).

It should be mentioned that at the level of \( Z_M \), it is easy to realize an \( R \)-parity that allows for lepton number violation, but conserves baryon number. Rapid proton decay will be prevented in this case. Lepton number violating processes and neutrino masses do provide some constraints, but these are much less stringent.

Consider the \( Z_6 \) models of Table 3 (Models III and IV). Suppose we impose invariance only under the \( Z_3 \) subgroup of \( Z_6 \) with the same charge assignment as in Table 3. The lepton number violating couplings \( LH_u \), \( LLe^c \), and \( Qld^c \) all have charge 3 under the \( Z_6 \) symmetry, so with only \( Z_3 \) invariance imposed, these couplings will be allowed in the superpotential. Since the original \( Z_6 \) symmetry is free from discrete gauge anomalies, the subgroup \( Z_3 \) is also free from such anomalies. One cannot however embed this \( Z_3 \) symmetry to any higher \( Z_N \) in order to explain the \( \mu \)-parameter. Consider the \( LH_u \) term in the superpotential. \( Z_N \) invariance of this term would imply \( l + h = 2\alpha \mod N \). The last two relations of Eq. (13) would imply \( 2\alpha = 0 \mod N \), implying that a bare \( \mu \) term in the superpotential will be allowed. An alternative explanation for the \( \mu \)-term will have to be found in the case of lepton number violating \( R \)-parity.
It is also possible, although somewhat non–trivial, to have baryon number violating
$R$–parity without dangerous lepton number violation. (Neutrino masses violate lepton number
by two units, but that does not result in rapid proton decay.) At the level of $Z_N$ we
can show that anomalies associated with such an $R$–parity will have to be cancelled at
higher Kac–Moody levels. If the coupling $u^c d^c d^c$ is allowed in the superpotential, we find
that the $Z_N$ discrete symmetry has anomalies given by $A_3 = 3\alpha$, $A_2 = 5/2\alpha – (3/4)pN$
where $p$ is an integer. Imposing the anomaly cancellation condition, Eq. (6), we find $\alpha = \frac{(m_1k_2 + m_2 – (k_3/2)p)N}{(6k_3 – 5k_3)}$ with $m_1, m_2, p$ being integers. When $k_2 = k_3 = 1$, this
relation shows that $2\alpha = 0$ mod $N$, meaning that a bare $\mu$–term will be allowed in
the superpotential. If we choose $k_2$ and $k_3$ differently, this problem will not arise. Consider
for example, $k_3 = 2, k_2 = 1$. A consistent $Z_N$ charge assignment corresponding to a $Z_4$
symmetry is shown in Table 15 for this case. This model allows for the coupling $u^c d^c d^c$, while
preventing other $R$–parity violating couplings. The $Z_4$ anomalies are cancelled by GS
mechanism, which in this case reads as $2A_2 – A_3 = m – 2n$, with $m, n$ being integers.

| $Z_4$ | $q$ | $u$ | $d$ | $l$ | $e$ | $n$ | $h$ | $\bar{h}$ | $\alpha$ | $(A_2, A_3)$ |
|-------|----|----|----|----|----|----|----|--------|-------|-------------|
| A     | 4  | 2  | 2  | 1  | 1  | 1  | 2  | 2      | 1     | (1/2, 3)    |
| B     | 0  | 2  | 2  | 3  | 3  | 3  | 0  | 0      | 3     | (5/2, 5)    |

Table 15: Examples of a $Z_4$ symmetry that allows for baryon number violation without dangerous lepton
number violations. The $Z_4$ anomalies are cancelled by GS mechanism at levels $k_2 = 1, k_3 = 2$.

4 Discrete flavor symmetries and the the fermion mass hierarchy

As indicated earlier, there must be an unbroken $Z_M$ symmetry which is flavor–independent
that survives to low energy scales, to be identified as an $R$–parity. We can however introduce
flavor dependence in the original symmetry, provided that a subgroup of the flavor group
remains unbroken and can be identified as one of the $Z_M$ symmetries of Table 3. In this
section we embark on this question. Our aim will be to seek an understanding of the observed
hierarchy in the fermion masses and mixings without introducing such hierarchy by hand.

Anomalous $U(1)_A$ symmetry is widely used for the explanation of fermion mass and
mixing hierarchy \[23\]. The general superpotential in this has the following expression:

$$W = Q_i u^c_i H_u \left( \frac{S}{M_{Pl}} \right)^{(h_1)_{ij}} + Q_i d^c_j H_d \left( \frac{S}{M_{Pl}} \right)^{(h_2)_{ij}} + L_i e^c_j H_d \left( \frac{S}{M_{Pl}} \right)^{(h_3)_{ij}} + L_i \nu^c_j H_u \left( \frac{S}{M_{Pl}} \right)^{(h_4)_{ij}} + \nu^c \nu^c S \left( \frac{S}{M_{Pl}} \right)^{(h_5)_{ij}}$$

where $S$ is an MSSM singlet field with a non–trivial anomalous $U(1)_A$ charge. $S$ acquires
a VEV near the string scale and disappears from the low energy spectrum. Here $(h_\alpha)_{ij}$ is
a set of integers for $\alpha = 1, 2, 3, 4, 5$ and $i, j = 1, 2, 3$ are the generation indices. We assume that all the Yukawa couplings are of order one. After $S$ field develops a VEV, near but somewhat below the string scale, a small parameter $\epsilon = \langle S \rangle / M_{Pl} \sim 1/5$ is generated. This factor appears in various powers with the Yukawa couplings, explaining the observed mass and mixing hierarchy \cite{22}. It is possible to suppress all the MSSM Yukawa couplings to the desired level by choosing appropriate set of $U(1)$ charges \cite{14}.

An acceptable flavor texture which gives the correct pattern of fermion masses and mixings is:

$$
U_{ij} = \begin{pmatrix}
    e^6 & e^5 & e^3 \\
    e^5 & e^4 & e^2 \\
    e^3 & e^2 & 1
\end{pmatrix} H_u, \quad D_{ij} = \begin{pmatrix}
    e^4 & e^3 & e^3 \\
    e^3 & e^2 & e^2 \\
    e & 1 & 1
\end{pmatrix} e^p H_d, \\
L_{ij} = \begin{pmatrix}
    e^4 & e^3 & e \\
    e^3 & e^2 & e \\
    e^3 & e^2 & 1
\end{pmatrix} e^p H_d, \quad \nu^D_{ij} = \begin{pmatrix}
    e^2 & e & e \\
    e & 1 & 1 \\
    e & 1 & 1
\end{pmatrix} e^{a_1} H_d, \quad (21)
$$

where $U_{ij}, D_{ij}, L_{ij}$ and $\nu^D_{ij}$ correspond to up–quark, down quark, charged lepton and Dirac neutrino Yukawa matrices resulting from the appropriate powers of the $S$ field in Eq. (20).

The integer $p$ can be either 0, 1 or 2, corresponding to large, medium and small $\tan \beta$ respectively.

Once the charged lepton sector and Dirac neutrino sector are constructed, we can uniquely define the form of the heavy Majorana neutrino mass matrix. In the present example it is

$$
\nu^M_{ij} = \begin{pmatrix}
    e^2 & e & e \\
    e & 1 & 1 \\
    e & 1 & 1
\end{pmatrix} e^{a_2}. \quad (22)
$$

As mentioned before, the MSSM superpotential does not possess any unbroken $U(1)$ symmetry, apart from the gauge symmetry. Therefore we seek solutions of a $Z_N$ discrete symmetry that would generate the Yukawa matrices of Eq. (21).

$Z_N$ invariance of the Yukawa couplings in Eq. (20) imposes constraints in the up–quark sector given by

$$
q_i + u_j + h + ps = 2\alpha \text{ mod } N, \quad (23)
$$

where $p$ is the power of $\epsilon$ appearing in the appropriate element of the $U_{ij}$ matrix, which is equal to the power in the field $S$. $s$ denotes the $U(1)$ charge of $S$ field. Similar conditions apply for the charges of the other SM fermions as well. By construction, the flavor structure of the matrices obey the “determinant rule”, viz., that in any 2×2 sub–block, the determinant is a homogeneous function of $\epsilon$. This means that out of the 18 conditions for the up–quark and the down–quark sector, only 8 will be independent.

We wish to have an unbroken $Z_M$ symmetry that is a subgroup of $Z_N$ which is flavor–independent. Since the flavon field $S$ has a $Z_N$ charge of $s$, once it acquires a VEV of order the string scale, the $Z_N$ will be broken down to $Z_s$. We shall attempt to embed the $Z_M$ of Table 3 into $Z_s$.

To be specific, let us work out an example with the $Z_2$ model of Table 3 and embed this $Z_2$ into a higher $Z_N$ symmetry that allows for the desired flavor structure. The $Z_N$ symmetry
must be $Z_{14+2n}$ for this embedding to be consistent, with $n$ being any integer. The smallest such symmetry is then $Z_{14}$. The flavon field $S$ must carry zero charge under $Z_2$ and should transform non-trivially under the $Z_N$. The simplest possibility is $s = 2$. Now, the Yukawa textures of Eq. (21) makes use of $S^6$ terms in the superpotential, which should be different from $S^0$. This requirement makes the smallest $Z_N$ symmetry to be $Z_{14}$. If this symmetry were $Z_{12}$, for example, $S^6$ will be neutral under $Z_{12}$, making the $(11)$ and the $(33)$ entries of the up–quark Yukawa matrix of the same order. We can generalize this statement to any low scale discrete symmetry. The corresponding flavor–dependent symmetry must be $Z_{M(k+1)+n}$, where $Z_M$ is the low scale surviving discrete symmetry, $k$ corresponds the highest power of the $S$ field in the general superpotential, Eq. (20), and $n$ is a positive integer. This choice will guarantee the existence of $Z_M$ discrete $R$–parity at low energy scales.

Three examples of $Z_{14}$ symmetric models are presented in Table 16. We have chosen the charge of $S$ to be 2 and fixed the charge of $\theta$ to be 7 in these examples. Discrete anomaly cancellation is enforced via GS mechanism at Kac–Moody level 1. We have also imposed the conditions that the $Z_{14}$ symmetry forbid all $R$–parity violating couplings.

The $Z_N$ symmetry group would depend on the highest power of $\epsilon$ appearing in the fermion Yukawa matrices. If we want to have symmetry smaller than $Z_{14}$, we should reduce the power of $S$ field in Eq. (21). One way is to re-parameterize the value of $\epsilon$. For example, if $\epsilon$ is taken to be of order 1/10, rather than 1/5 as was assumed in Eq. (21), it might suffice to use cubic powers of $S$ at most. A $Z_8$ discrete symmetry would then suffice to forbid the $R$–parity breaking terms. We consider the expansion given in Eq. (21) to be more realistic.

In Table 17 we present three models based on $Z_{28}$ symmetry that forbid all $R$–parity violating couplings, explain the fermion mass and mixing hierarchy via the texture of Eq. (21) and also solve the $\mu$–problem via the Guidice–Masiero mechanism. As before, the discrete gauge anomalies are cancelled by the GS mechanism. We find it remarkable that a single discrete symmetry can do all these jobs. It may be mentioned that $Z_{28}$ is not a large symmetry unlikely to be realized in string theory. For example, if the particle spectrum contains fields carrying charges of $(1,1/4,1/7)$ under the anomalous $U(1)$, and if a scalar field with charge 1 acquires a VEV, the unbroken $Z_N$ symmetry will be $Z_{28}$.

|   | $Q_i$ | $u^c_i$ | $d^c_i$ | $L_i$ | $e^c_i$ | $\nu^c_i$ | $H_u$ | $H_d$ | $\theta$ | $S$ | $A_2$ | $A_3$ |
|---|------|--------|--------|------|--------|----------|------|------|--------|----|------|------|
| A | 0,2,6 | 1,3,7  | 3,5,5  | 4,6,6| 13,1,5 | 5,7,7    | 1    | 13   | 7      | 2  | 6    | 13   |
| B | 4,6,10| 13,1,5 | 11,13,13| 6,8,8| 9,11,1 | 5,7,7    | 13   | 1    | 7      | 2  | 13   | 13   |
| C | 6,8,12| 5,7,11 | 1,3,3  | 0,2,2| 7,9,13 | 5,7,7    | 9    | 5    | 7      | 2  | 13   | 6    |

Table 16: Examples of flavor–dependent $Z_{14}$ symmetry which forbids all $R$–parity breaking terms. $i = 1,2,3$ is the flavor index and charges in the brackets are in order of 1-3. We are considering $p = 2$ and $q = 0$ in Eq. (21) which corresponds to medium values of $\tan \beta \sim 10$. We have taken $a_2 = 0$ in Eq. (22) for simplicity.
5 Conclusion

In this paper we have investigated the possibility of realizing $R$–parity of MSSM as a discrete gauge symmetry. Simultaneously we have demanded that this discrete symmetry should provide a natural explanation for the $\mu$–term, the Higgsino mass parameter in the MSSM superpotential, via the Guidice–Masiero mechanism. We have adopted a discrete version of the Green–Schwarz anomaly cancellation mechanism in our search for discrete gauge symmetries, which is is less constraining than the conventional methods.

We have found simple examples of $Z_N$ symmetries that act as $R$–parity and simultaneously solve the $\mu$–problem, without extending the particle content of the MSSM. The simplest example is a $Z_4$ symmetry with a simple charge assignment that is compatible with grand unification. The Green–Schwarz mechanism plays a crucial role in cancelling the $Z_4$ anomalies. Other examples of $Z_N$ symmetries are provided with $N = 10, 12, 18, 36$ etc. In some cases the discrete anomalies are cancelled by the GS mechanism at higher Kac–Moody levels. We have found that it is easy to realize lepton number violating $R$–parity as a discrete symmetry, but implementing the Guidice–Masiero mechanism for the $\mu$ term is difficult in this case. Baryon number violating $R$–parity can be realized, along with a natural explanation of the $\mu$ term, but the discrete gauge anomalies are cancelled in this case at higher Kac–Moody level.

It has been shown that a simple $Z_N$ symmetry can also explain the observed hierarchical structure of quark and lepton masses and mixings, while preserving the desired $R$–parity and the solution to the $\mu$–problem. Examples of a $Z_{28}$ symmetry doing all these have been presented.

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