How many families survive for a long time*

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Abstract
A critical branching process \( \{Z_k, k = 0, 1, 2, \ldots\} \) in a random environment generated by a sequence of independent and identically distributed random reproduction laws is considered. Let \( Z_{p,n} \) be the number of particles at time \( p \leq n \) having a positive offspring number at time \( n \). A theorem is proved describing the limiting behavior, as \( n \to \infty \) of the distribution of a properly scaled process \( \log Z_{p,n} \) under the assumptions \( Z_n > 0 \) and \( p \ll n \).

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1 Introduction
We consider branching processes in random environment specified by sequences of independent identically distributed random laws. Denote by \( \Delta \) the space of proper probability measures on \( \mathbb{N}_0 = \{0, 1, 2, \ldots\} = \mathbb{N}_0 \cup \mathbb{N}_+ \). Let \( Q \) be a random variable taking values in \( \Delta \). An infinite sequence
\[
\Pi = (Q_1, Q_2, \ldots)
\]
of i.i.d. copies of \( Q \) is said to form a random environment. In the sequel we make no difference between the laws
\[
Q = (Q(\{0\}), Q(\{1\}), \ldots, Q(\{k\}), \ldots), \quad Q_n = (Q_n(\{0\}), Q_n(\{1\}), \ldots, Q_n(\{k\}), \ldots)
\]
and the generating functions
\[
f(s) = f(s; Q) := \sum_{k=0}^{\infty} Q(\{k\})s^k, \quad f_n(s) = f_n(s; Q) := \sum_{k=0}^{\infty} Q_n(\{k\})s^k, \quad n \in \mathbb{N}_+.
\]

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A sequence of \( \mathbb{N}_0 \)-valued random variables \( Z := (Z_0, Z_1, \ldots) \) is called a branching process in the random environment \( \Pi \), if \( Z_0 \) is independent of \( \Pi \) and

\[
\mathbb{E} \left[ s_n \bigg| Z_0, Z_1, \ldots, Z_{n-1}, Q_1, Q_2, \ldots, Q_n \right] = (f_n(s))^{Z_{n-1}}, \ n \in \mathbb{N}_+.
\]

Thus, \( Z_n \) is the \( n \)-th generation size of the population and \( Q_n \) is the distribution of the number of children of an individual at generation \( n-1 \). We assume that \( Z_0 = 1 \) for convenience and denote the corresponding probability measure on the underlying probability space by \( \mathbb{P} \). (If we refer to other probability spaces, then we use notation \( \mathbb{P}, \mathbb{E} \) and \( \mathbb{L} \) for the respective probability measures, expectations and laws.)

As it turns out the properties of \( Z \) are mainly determined by its associated random walk \( S := \{S_n, n \geq 0\} \). This random walk has initial state \( S_0 = 0 \) and increments \( X_n = S_n - S_{n-1}, \ n \geq 1 \), defined as

\[
X_n := \log f_n'(1),
\]

which are i.i.d. copies of the logarithmic mean offspring number

\[
X := \log f'(1).
\]

Following \([1]\) we call the process \( Z \) critical if and only if the random walk \( S \) is oscillating, that is,

\[
\limsup_{n \to \infty} S_n = \infty \quad \text{and} \quad \liminf_{n \to \infty} S_n = -\infty
\]

with probability 1.

It is shown in \([1]\) that the extinction moment of the critical branching process in a random environment is finite with probability 1. Moreover, it is established in \([1]\) that if

\[
\lim_{{n \to \infty}} \mathbb{P}(S_n > 0) =: \rho \in (0, 1), \quad (3)
\]

then, as \( n \to \infty \) (and some mild additional assumptions to be specified later on)

\[
\mathbb{P}(Z_n > 0) \sim \theta \mathbb{P}(\min(S_0, S_1, \ldots, S_n) \geq 0) = \theta n^{\rho-1} l(n), \quad (4)
\]

where \( l(n) \) is a slowly varying function and \( \theta \) is a known positive constant whose explicit expression is given by formula (23) below.

Let

\[
\mathcal{A} := \{0 < \alpha < 1; |\beta| < 1\} \cup \{1 < \alpha < 2; |\beta| < 1\} \cup \{\alpha = 1, \beta = 0\} \cup \{\alpha = 2, \beta = 0\}
\]

be a subset in \( \mathbb{R}^2 \). For \((\alpha, \beta) \in \mathcal{A} \) and a random variable \( X \) write \( X \in D(\alpha, \beta) \) if the distribution of \( X \) belongs to the domain of attraction of a stable law with characteristic function

\[
\mathcal{H}_{\alpha,\beta}(t) := \exp \left\{ -c|t|^\alpha \left( 1 + i\beta \frac{t}{|t|} \tan \frac{\pi \alpha}{2} \right) \right\}, \ c > 0, \quad (5)
\]
and, in addition, $\mathbf{E}X = 0$ if this moment exists.

Denote by $\{c_n, n \in \mathbb{N}_+\}$ a sequence of positive integers specified by the relation

$$c_n := \inf \{u \geq 0 : G(u) \leq n^{-1}\},$$

where

$$G(u) := \frac{1}{u^2} \int_{-u}^{u} x^2 \mathbf{P}(X \in dx).$$

It is known (see, for instance, [8, Ch. XVII, §5]) that, for every $X \in \mathcal{D}(\alpha, \beta)$ the function $G(u)$ is regularly varying with index $(-\alpha)$, and, therefore,

$$c_n = n^{1/\alpha} l_1(n), \quad n \in \mathbb{N}_+, \quad (6)$$

for some function $l_1(n)$, slowly varying at infinity. In addition, the scaled sequence $\{S_n/c_n, n \geq 1\}$ converges in distribution, as $n \to \infty$, to the stable law given by (5).

Observe that if $X \in \mathcal{D}(\alpha, \beta)$, then the quantity $\rho$ in (3) is calculated by the formula (see, for instance, [22])

$$\rho = \begin{cases} \frac{1}{2}, & \text{if } \alpha = 1 \text{ or } 2, \\ \frac{1}{2} + \frac{1}{\pi \alpha} \arctan \left(\beta \tan \frac{\pi}{2}\right), & \text{otherwise}. \end{cases} \quad (7)$$

In particular, $\rho \in (0, 1)$.

Denote

$$M_n := \max(S_1, ..., S_n), \quad L_{k,n} := \min_{k \leq j \leq n} S_j, \quad L_n := L_{0,n} = \min(S_0, S_1, ..., S_n)$$

and introduce two functions

$$V(x) : = 1 + \sum_{k=1}^{\infty} \mathbf{P}(-S_k \leq x, M_k < 0), \quad x \geq 0,$$

$$U(x) : = 1 + \sum_{k=1}^{\infty} \mathbf{P}(-S_k > x, L_k \geq 0), \quad x \leq 0,$$

and 0 elsewhere. In particular, $V(0) = U(0) = 1$.

The fundamental properties of $V, U$ are the identities

$$\mathbf{E}[V(x+X); x+X \geq 0] = V(x), \quad x \geq 0,$$

$$\mathbf{E}[U(x+X); x+X < 0] = U(x), \quad x \leq 0,$$

which hold for any oscillating random walk.

It follows from (8) that $V$ and $U$ give rise to probability measures $\mathbf{P}_x^+$ and $\mathbf{P}_x^-$ being important for subsequent arguments. The construction procedures of these probability measures are explained for $\mathbf{P}_x^+$ in [11] and for $\mathbf{P}_x^-$ in [2] and [15] in detail. We only recall that if $T_0, T_1, \ldots$ is a sequence of random variables and the random walk $S = (S_n, n \geq 0)$ with $S_0 = x$ are both adapted to some
filtration $\mathcal{F} = (\mathcal{F}_n, n \geq 0)$, then, for each fixed $n$ and a bounded and measurable function $g_n : \mathbb{R}^{n+1} \to \mathbb{R}$ the measures above are specified by the equalities
\[
E_x^+[g_n(T_0, \ldots, T_n)] = \frac{1}{V(x)} E_x[g_n(T_0, \ldots, T_n)V(S_n); L_n \geq 0],
\]
\[
E_x^-[g_n(T_0, \ldots, T_n)] = \frac{1}{U(x)} E_x[g_n(T_0, \ldots, T_n)U(S_n); M_n < 0].
\]

Observe that under the measure $P^+_x$ the sequence $S_0, S_1, \ldots$ is a Markov process with state space $\mathbb{R}^0 := [0, \infty)$ and transition probabilities
\[
P^+(x, dy) := \frac{1}{V(x)} P\{x + X \in dy\} V(y), \quad x \geq 0, y \geq 0.
\]

It is the random walk conditioned never to enter $\mathbb{R}^- := (-\infty, 0)$, while under the measure $P^-_x$ the process $S_0, S_1, \ldots$ becomes a Markov chain with state space $\mathbb{R}^-$ and transition kernel
\[
P^-(x, dy) := \frac{1}{U(x)} P\{x + X \in dy\} U(y), \quad x \leq 0, y < 0. \tag{9}
\]

Note that $P^-(x, [0, \infty)) = 0$, thus the Markov process never enters $[0, \infty)$ again. It may, however, start from the boundary $x = 0$.

We now describe in brief a construction of Lévy processes conditioned to stay nonnegative following basically the definitions given in [5] and [6].

Let $\Omega := D([0, \infty), \mathcal{R})$ be the space of real-valued càdlàg paths on the real half-line $[0, \infty)$ and let $\mathcal{B} := \{B_t, t \geq 0\}$ be the coordinate process defined by the equality $B_t(\omega) = \omega_t$ for $\omega \in \Omega$. In the sequel we consider also the spaces $\Omega_U := D([0, U], \mathcal{R}), U > 0$.

We endow the spaces $\Omega$ and $\Omega_U$ with Skorokhod topology and denote by $\mathcal{F} = \{\mathcal{F}_t, t \geq 0\}$ and by $\mathcal{F}^U = \{\mathcal{F}_t, t \in [0, U]\}$ (with some misuse of notation) the natural filtrations of the processes $\mathcal{B}$ and $\mathcal{B}^U = \{B_t, t \in [0, U]\}$.

Let $P_0$ be the law on $\Omega$ of an $\alpha$-stable Lévy process $\mathcal{B}$, $\alpha \in (0, 2]$ started at $x$ and let $P = P_0$. Denote by $\rho := P(B_1 \geq 0)$ the positivity parameter of the process $\mathcal{B}$ (in fact, this quantity is the same as in (7) for a random walk whose increments $X_t \in D(\alpha, \beta)$). We now introduce an analogue of the measure $P^+$ for Lévy processes. Namely, following [4] we specify for all $t > 0$ and events $\mathcal{A} \in \mathcal{F}_t$ the law $P^+_x$ on $\Omega$ of the Lévy process starting at point $x > 0$ and conditioned to stay nonnegative by the equality
\[
P^+_x(\mathcal{A}) := \frac{1}{x^{\alpha(1-\rho)}} E_x \left[ P_t^{\alpha(1-\rho)} I \{\mathcal{A}\} I \left\{ \inf_{0 \leq u \leq t} B_u \geq 0 \right\} \right],
\]
where $I \{\mathcal{C}\}$ is the indicator of the event $\mathcal{C}$.

Thus, $P^+_x$ is an $h$-transform of the Lévy process killed when it first enters the negative half-line. The corresponding positive invariant function is $H(x) = x^{\alpha(1-\rho)}$ in this case.
This definition has no sense for \( x = 0 \). However, it is shown in [5] that one can construct a law \( \mathbb{P}^+ = \mathbb{P}^+_0 \) and a càdlàg Markov process with the same semigroup as \((\mathcal{B}, \{\mathbb{P}^+_x, x > 0\})\) in such a way that \( \mathbb{P}^+ (B_0 = 0) = 1 \). Moreover,

\[
\mathbb{P}^+_x \implies \mathbb{P}^+, \text{ as } x \downarrow 0.
\]

Here and in what follows the symbol \( \implies \) stands for the weak convergence in the respective space of càdlàg functions endowed with the Skorokhod topology.

Let \( \mathbb{P}^{(m)} \) be the law on \( \Omega_1 \) of the meander of length 1 associated with \((\mathcal{B}, \mathbb{P})\), i.e.

\[
\mathbb{P}^{(m)} (\cdot) := \lim_{x \downarrow 0} \mathbb{P}_x \left( \inf_{0 \leq u \leq 1} B_u \geq 0 \right).
\]  

(10)

Thus, the law \( \mathbb{P}^{(m)} \) may be viewed as the law of the Lévy process \((\mathcal{B}, \mathbb{P})\) conditioned to stay nonnegative on the time-interval \((0, 1]\) while the law \( \mathbb{P}^+ \) specified earlier corresponds to the law of the Lévy process conditioned to stay nonnegative on the whole time interval \((0, \infty)\).

As shown in [5], \( \mathbb{P}^{(m)} \) and \( \mathbb{P}^+ \) are absolutely continuous with respect to each other: for every event \( A \in \mathcal{F}_1 \)

\[
\mathbb{P}^+ (A) = C_0 \mathbb{E}^{(m)} \left[ I \{ A \} B_1^\alpha (1 - \rho) \right],
\]

(11)

where (see, for instance, formulas (3.5)-(3.6) and (3.11) in [6])

\[
C_0 := \lim_{n \to \infty} V(c_n) \mathbb{P} (L_n \geq 0) = \left( \mathbb{E}^{(m)} \left[ B_1^\alpha (1 - \rho) \right] \right)^{-1} \in (0, \infty),
\]

(12)

and \( \mathbb{E}^{(m)} \) is the expectation with respect to \( \mathbb{P}^{(m)} \). In fact, one may extend the absolute continuity given in (11) to an arbitrary interval \([0, U]\) by considering the respective space \( \Omega_U \) instead of \( \Omega_1 \) and conditioning by the event \( \inf_{0 \leq u \leq U} B_u \geq 0 \) in (10).

We now come back to the branching processes in random environment and set

\[
\zeta(a) := \sum_{k=a}^\infty k^2 Q(\{k\}) (f'(1))^2, \quad a \in \mathbb{N}_0.
\]

In what follows we say that

1) **Condition A1 is valid** if \( X \in \mathcal{D} (\alpha, \beta) \);

2) **Condition A2 is valid** if

\[
\mathbb{E} \left( \log^+ \zeta(a) \right)^{\alpha + \varepsilon} < \infty
\]

for some \( \varepsilon > 0 \) and \( a \in \mathbb{N}_0 \);

3) **Condition A is valid** if Conditions A1- A2 hold true and, in addition, the parameter \( p = p(n) \) tends to infinity as \( n \to \infty \) in such a way that

\[
\lim_{n \to \infty} n^{-1} p = \lim_{n \to \infty} n^{-1} p(n) = 0.
\]

(13)
Branching processes in random environment meeting Condition A have been investigated in a recent paper [19]. The paper includes a Yaglom-type functional limit theorem describing the asymptotic properties of the process \( \{ Z_{[tp]}, 0 \leq t < \infty \} \) given \( Z_n > 0 \).

Here we investigate the structure of the process \( Z \) in more detail and prove a conditional limit theorem for the so-called reduced process \( \{ Z_{p,n}, 0 \leq p \leq n \} \), where \( Z_{p,n} \) is the number of particles in the process at time \( p \in [0, n] \), each of which has a nonempty offspring at time \( n \). Our main result looks as follows.

**Theorem 1** If Condition A is valid, then for any \( x \geq 0 \)

\[
\lim_{n \to \infty} \mathbb{P} \left( \frac{\log Z_{p,n}}{c_p} \geq x | Z_n > 0, Z_0 = 1 \right) = \mathbb{E}^+ \left[ \left( 1 - \frac{x}{B_1} \right)^{\alpha(1-\rho)} I\{B_1 \geq x\} \right] = \mathbb{P}^+ \left( \inf_{1 \leq v < \infty} B_v \geq x \right). \quad (14)
\]

Here and in what follows \( \mathbb{E}^+ \) is the expectation with respect to \( \mathbb{P}^+ \). This result complements papers [3] and [13] where it was established (under the assumptions \( \mathbb{E}[X] = 0, \sigma^2 = \mathbb{E}[X^2] < (0, \infty) \) and some additional technical conditions) that, as \( n \to \infty \)

\[
\mathcal{L} \left( \left\{ \frac{\log Z_{[nt]}, n}{\sigma \sqrt{n}}, 0 \leq t \leq 1 \right\} | Z_n > 0, Z_0 = 1 \right) \Rightarrow \mathcal{L}^{(m)} \left( \inf_{t \leq v \leq 1} B_v, 0 \leq t \leq 1 \right) = \mathcal{L} \left( \inf_{t \leq v \leq 1} B^+_v \right),
\]

where \( B^+ := \{ B^+_v, 0 \leq v \leq 1 \} \) is the standard Brownian meander.

The study of the reduced processes has a rather long history. Reduced processes for ordinary Galton-Watson branching processes were introduced by Fleischmann and Prehn [9], who discussed the subcritical case. Critical and supercritical reduced Galton-Watson processes have been investigated by Zubkov [23] and Fleischmann and Siegmund-Schultze [10]. The first results for reduced branching processes in random environment were obtained by Borovkov and Vatutin [3] and Fleischmann and Vatutin [11]. They considered (under the annealed approach) the case when the support of the measure \( \mathbb{P} \) is concentrated only on the set of fractional-linear generating functions. Vatutin proved in [13] a limit theorem for the critical reduced processes under the annealed approach and general reproduction laws of particles. Papers [14], [15], [17] and [18] consider the properties of the critical reduced branching processes in random environment under the quenched approach (see also survey [20]).

### 2 Auxiliary results

To prove the main results of the paper we need to know the asymptotic behavior of the function \( V(x) \) as \( x \to \infty \). The following lemma gives the desired representation.
Lemma 2 (compare with Lemma 13 in [21] and Corollary 8 in [7]) If \( X \in \mathcal{D}(\alpha, \beta) \), then there exists a function \( l_0(x) \) slowly varying at infinity such that
\[
V(x) \sim x^{\alpha(1-\rho)} l_0(x)
\]
as \( x \to \infty \).

In the sequel we use the symbols \( K, K_1, K_2, \ldots \) to denote different constants. They are not necessarily the same in different formulas.

Our next result is a combination (with a slight reformulation) of Lemma 2.1 in [1] and Corollaries 3 and 8 in [7]:

Lemma 3 If \( X \in \mathcal{D}(\alpha, \beta) \), then (compare with (4)), as \( n \to \infty \)
\[
P(L_n \geq -r) \sim V(r) P(L_n \geq 0) \sim V(r) n^{\rho-1} l(n)
\]
uniformly in \( 0 \leq r \ll c_n \), and there exists a constants \( K_1 > 0 \) such that
\[
P(L_n \geq -r) \leq K_1 V(r) P(L_n \geq 0), \quad r \geq 0, \quad n \geq 1.
\]

For \( U > 0 \) and \( pU \leq n \) let
\[
Q_{U,n}^p := \left\{ \frac{S_{[pu]}}{c_p}, 0 \leq u \leq U \right\}, \quad Q_{\infty}^p := Q_{\infty}^n,
\]
\[
S_{U,n}^p := \left\{ \frac{S_{[pU]+(n-pU)t}}{c_n}, 0 \leq t \leq 1 \right\}, \quad S^n := S_0^n.
\]

Let \( \phi_1 : \Omega_1 \to \mathbb{R} \) be a bounded uniformly continuous functional and \( \{\varepsilon_n, n \in \mathbb{N}_+\} \) be a sequence of positive numbers vanishing as \( n \to \infty \).

Lemma 4 (see [19]) If Condition A1 is valid then
\[
E[\phi_1(S^n) | L_n \geq -x] \to E^{(m)}[\phi_1(B^1)]
\]
as \( n \to \infty \) uniformly in \( 0 \leq x \leq \varepsilon_n c_n \).

Lemma 5 (see [19]) If Conditions A1 and (13) are valid, then for any \( r \geq 0 \)
\[
L(Q_{p,n}^p | L_n \geq -r) \Rightarrow L^+(B)
\]
as \( n \to \infty \).

For \( x \geq 0 \) we set for brevity
\[
D(x) := C_0 E^{(m)}[(B_1 - x)^{\alpha(1-\rho)} I\{B_1 \geq x\}]
\]
\[
= E^+\left[ \left(1 - \frac{x}{B_1}\right)^{\alpha(1-\rho)} I\{B_1 \geq x\} \right].
\]
(17)
Lemma 6 For any $x > 0$

$$
\mathbb{P}^+ \left( \inf_{1 \leq u < \infty} B_u \geq x \right) = D(x).
$$

Proof. According to Theorem 5 in [4] for any pair $0 < x \leq y$

$$
\mathbb{P}^+_y \left( \inf_{0 \leq u < \infty} B_u \geq x \right) = \left( 1 - \frac{x}{y} \right)^{\alpha(1-\rho)} I \{ x \leq y \}.
$$
Hence,

$$
\mathbb{P}^+ \left( \inf_{1 \leq u < \infty} B_u \geq x \right) = \int_x^\infty \mathbb{P}^+ (B_1 \in dy) \mathbb{P}^+_y \left( \inf_{0 \leq u < \infty} B_u \geq x \right) = \int_x^\infty \mathbb{P}^+ (B_1 \in dy) \left( 1 - \frac{x}{y} \right)^{\alpha(1-\rho)} = D(x),
$$
as desired.

In the sequel we agree to write $a_n \ll b_n$ if $\lim_{n \to \infty} a_n/b_n = 0$. In particular, $\lim_{n \to \infty} a_n/b_n = 0$ means that the limit of the respective expression is calculated as $p, n \to \infty$ in such a way that $p_n - 1 \to 0$.

The following statement will be useful for the subsequent proofs.

Lemma 7 If Condition A1 is valid then, for any $x \geq 0$

$$
\lim_{n \to \infty} \mathbb{P} (L_{p,n} \geq xc_p \mid L_n \geq -r) = D(x).
$$

Proof. We select $N > x > 0$ and write

$$
0 \leq \mathbb{P} (L_{p,n} \geq xc_p; L_n \geq -r) - \mathbb{P} (S_p \leq Nc_p, L_{p,n} \geq xc_p; L_n \geq -r) \\
\leq \mathbb{P} (S_p > Nc_p; L_n \geq -r).
$$

(19)

It follows from Lemma 5 that for any $\varepsilon > 0$ one can find $N_0 = N_0 (\varepsilon)$ such that for all $N \geq N_0$

$$
\mathbb{P} (S_p > Nc_p; L_n \geq -r) \leq \varepsilon \mathbb{P} (L_n \geq -r).
$$

(20)

To proceed further we denote by $S^* := (S_0^*, S_1^*, \ldots, S_n^*, \ldots)$ an independent probabilistic copy of the random walk $S$ and let

$$
L_k^* := \min (S_0^*, S_1^*, \ldots, S_k^*).
$$

Then, for $N > x \geq 0$

$$
\mathbb{P} (S_p \leq Nc_p, L_{p,n} \geq xc_p; L_n \geq -r) \\
= \int_{xc_p}^{Nc_p} \mathbb{P} (S_p \in dy; L_p \geq -r) \mathbb{P} (L_{n-p}^* \geq xc_p - y) \\
= \int_x^N \mathbb{P} (S_p \in c_p dz; L_p \geq -r) \mathbb{P} (L_{n-p}^* \geq (z - c_p) c_p).
$$
Since $p \ll n$, we deduce by (15), (12) and properties of regularly varying functions that if $n \to \infty$ then for any $\varepsilon > 0$

$$
P (L^*_n - p \geq (x - z) c_p) \sim V((z - x) c_p) P (L_n \geq 0)$$

$$\sim (z - x)^{\alpha(1 - \rho)} V(c_p) P (L_n \geq 0)$$

$$= (z - x)^{\alpha(1 - \rho)} V(c_p) P (L_p \geq 0) \frac{P(L_n \geq 0)}{P(L_p \geq 0)}$$

$$\sim (z - x)^{\alpha(1 - \rho)} C_0 \frac{P(L_n \geq -r)}{P(L_p \geq -r)}$$

uniformly in $0 \leq x \leq z \varepsilon \leq N$. Hence we conclude that given condition (13) we have as $n \to \infty$

$$
P \left( S_{p} \leq N c_p; L_{p,n} \geq xc_p; L_{n} \geq -r \right)
\sim C_0 P (L_n \geq -r) \int_{x}^{N} (z - x)^{\alpha(1 - \rho)} P (S_p \in c_p dz | L_p \geq -r)$$

$$\sim C_0 P (L_n \geq -r) \int_{x}^{N} (z - x)^{\alpha(1 - \rho)} \nu(m) (B_1 \in dz)$$

$$= P (L_n \geq -r) \int_{x}^{N} \left(1 - \frac{x}{z}\right)^{\alpha(1 - \rho)} P^+ (B_1 \in dz).$$

Using (20) and (21) to evaluate (19) and letting $N$ to infinity we get (18).

For convenience we introduce the notation

$$
\tau_n := \min \{ j : S_j = L_n \}, \quad A_{\text{a.u.s.}} := \{ Z_n > 0 \text{ for all } n \geq 0 \}
$$

and recall that by Corollary 1.2 in [1], (4) and (12)

$$
P (Z_n > 0) \sim \theta P (L_n \geq 0) \sim \theta(n)n^{\rho - 1} \sim \frac{\theta c_0}{V(c_n)}$$

as $n \to \infty$, where

$$
\theta := \sum_{k=0}^{\infty} \mathbb{E} [P^+_{Z_k} (A_{\text{a.u.s.}}); \tau_k = k].
$$

Let

$$
\hat{L}_{k,n} := \min_{0 \leq j \leq n-k} (S_{k+j} - S_k)
$$

and let $\hat{F}_k$ be the $\sigma-$algebra generated by the tuple $\{Z_0, Z_1, ..., Z_k; Q_1, Q_2, ..., Q_k\}$ (see (1)).

For further references we formulate two statements borrowed from [1].

**Lemma 8** (see Lemma 2.5 in [1]) Assume Condition A1. Let $Y_1, Y_2, ...$ be a uniformly bounded sequence of random variables adapted to the filtration $\mathcal{F} = \{ \hat{F}_k, k \in \mathbb{N} \}$, which converges $P^+$-a.s. to some random variable $Y_\infty$. Then, as $n \to \infty$

$$
\mathbb{E} [Y_n | L_n \geq 0] \to \mathbb{E}^+ [Y_\infty].
$$
Lemma 9 (see Lemma 4.1 in [1]) Assume Condition A1 and let \( l \in \mathbb{N}_0 \). Suppose that \( \zeta_1, \zeta_2, \ldots \) is a uniformly bounded sequence of real-valued random variables, which, for every \( k \geq 0 \) meets the equality

\[
E \left[ \zeta_n; Z_{k+l} > 0, \hat{L}_{k,n} \geq 0 \right] = P \left( L_n \geq 0 \right) (\zeta_{k,\infty} + o(1)) \quad P\text{-a.s.}
\]

as \( n \to \infty \) with random variables \( \zeta_1, \infty = \zeta_1, \infty (l), \zeta_2, \infty = \zeta_2, \infty (l), \ldots \). Then

\[
E \left[ \zeta_n; Z_{\tau_n+l} > 0 \right] = P \left( L_n \geq 0 \right) \left( \sum_{k=0}^{\infty} E \left[ \zeta_{k,\infty}; \tau_k = k \right] + o(1) \right)
\]

as \( n \to \infty \), where the right-hand side series is absolutely convergent.

For \( q \leq p \leq n \) and \( u > 0 \) denote

\[
m(u; n) = m(u; n, p, q) := \min \{ q + \lfloor u(p - q) \rfloor, n \}
\]

and set

\[
X^{q,p}_u := \left\{ X_u^{q,p} = e^{-S_{m(u;n)}} Z_{m(u;n)}, 0 \leq u < \infty \right\}.
\]

The next statement is an evident corollary of Theorem 1.3 in [1].

Corollary 10 Assume Condition A. Let \( (q_1, p_1), (q_2, p_2), \ldots \) be a sequence of pairs of positive integers such that \( q_n \ll p_n \ll n \) and \( q_n \to \infty \) as \( n \to \infty \). Then

\[
\mathcal{L} \left( X^{q_n,p_n}_u \mid Z_n > 0, Z_0 = 1 \right) \Rightarrow \mathcal{L} \left( W_u, 0 \leq u < \infty \right),
\]

where

\[
P \left( W_u = W, 0 \leq u < \infty \right) = 1
\]

for some random variable \( W \) such that

\[
P \left( 0 < W < \infty \right) = 1.
\]

We conclude this section by recalling asymptotic properties of the distribution of the number of particles in a critical branching process in random environment at moment \( p \ll n \) given \( Z_n > 0 \).

Theorem 11 (see [12]) If Condition A is valid, then, as \( n \to \infty \)

\[
\lim_{n \geq p \to \infty} P \left( \frac{\log Z_p}{c_p} \leq z \mid Z_n > 0, Z_0 = 1 \right) = P^+ (B_1 \leq z)
\]

for any \( z > 0 \).
3 Reduced processes

The proof of Theorem 1 will be divided into several steps which we formulate as lemmas.

For \( f_n(s), n = 1, 2, \ldots \), specified by (2) set

\[
 f_{p,n}(s) := f_{p+1}(f_{p+2}(\ldots (f_n(s)) \ldots)), \quad 0 \leq p \leq n - 1, \quad f_{n,n}(s) \equiv 1.
\]

We label \( Z_p \) particles of the \( p \)th generation by positive numbers \( 1, 2, \ldots, Z_p \) in an arbitrary but fixed way and denote by \( Z_n^{(i)}(p) \), \( i = 1, 2, \ldots, Z_p \), \( 0 \leq p \leq n \), the offspring size at moment \( n \) of the population generated by the \( i \)th particle of the \( p \)th generation.

For fixed positive \( x \) and \( N \) introduce the events

\[
 A_{p,n}(x) := \{ \ln(e + Z_{p,n}) \geq xc_p \},
\]

\[
 B_{p,n}(N) := \left\{ \sum_{i=1}^{Z_p} \left( I \left\{ Z_n^{(i)}(p) > 0 \right\} - (1 - f_{p,n}(0)) \right) > \sqrt{NZ_p(1 - f_{p,n}(0))} \right\},
\]

and

\[
 C_{p,n} := \{ Z_p(1 - f_{p,n}(0)) < e^{\sqrt{c_p}} \}
\]

and use the notation \( \bar{C}_{p,n} \) for the event complementary to \( C_{p,n} \).

Finally, we use for brevity the notation \( P_{j}(\bullet) := P(\bullet | Z_0 = j) \) with the natural agreement that \( P(\bullet) := P(\bullet | Z_0 = 1) \). In particular, \( P(Z_n > 0) = P(Z_n > 0 | Z_0 = 1) \).

**Lemma 12** If the conditions of Theorem 1 are valid then, for any \( j \in \mathbb{N}_+ \)

\[
 \lim_{N \to \infty} \lim_{n \to p \to \infty} \frac{P_j(A_{p,n}(x)B_{p,n}(N); L_n \geq 0)}{P(Z_n > 0)} = 0 \quad (24)
\]

and

\[
 \lim_{n \to p \to \infty} \frac{P_j(A_{p,n}(x)C_{p,n}; L_n \geq 0)}{P(Z_n > 0)} = 0. \quad (25)
\]

**Proof.** First we establish the validity of (24). To this aim we temporary introduce the notation \( P^{(F)}(\bullet), E^{(F)}(\bullet) \) and \( D^{(F)}(\bullet) \) for the probability, expectation and variance calculated for the fixed \( \sigma \)-algebra \( F = F_{p,n} \) generated by the random variables \( \{Z_0, Z_1, \ldots, Z_p\} \) and random probability generating functions \( f_1(s), f_2(s), \ldots, f_n(s) \).

First we note that, for \( 1 \leq i \leq Z_p \)

\[
 E^{(F)} \left[ I \left\{ Z_n^{(i)}(p) > 0 \right\} \right] = 1 - f_{p,n}(0),
\]

\[
 D^{(F)} \left[ I \left\{ Z_n^{(i)}(p) > 0 \right\} \right] = (1 - f_{p,n}(0))f_{p,n}(0). \quad (26)
\]
Besides,
\[ Z_{p,n} = \sum_{i=1}^{Z_p} I \left\{ Z_n^{(i)}(p) > 0 \right\}. \]

Using these relations and applying Chebyshev’s inequality to evaluate the probability under the expectation sign we obtain for sufficiently large \( n \)
\[
\Pr_j(A_{p,n}(x)B_{p,n}(N); L_n \geq 0) \leq \Pr_j(Z_p > 0, B_{p,n}(N); L_n \geq 0)
\]
\[
= E \left[ P_j^{(F)}(B_{p,n}); Z_p > 0, L_n \geq 0 \right]
\]
\[
\leq E \left[ I \{ Z_p > 0, L_n \geq 0 \} \frac{1}{NZ_p(1 - f_{p,n}(0))} D^{(F)} \left( \sum_{i=1}^{Z_p} I \left\{ Z_n^{(i)}(p) > 0 \right\} \right) \right]
\]
\[
\leq N^{-1} \Pr(L_n \geq 0).
\]

This estimate along with (22) implies (24).

To establish (25) observe that
\[
\Pr_j(A_{p,n}(x)C_{p,n}; L_n \geq 0) = E \left[ P_j^{(F)}(A_{p,n}(x)C_{p,n}); L_n \geq 0 \right]
\]
\[
\leq \frac{1}{x^2c_p^2} E_j \left[ I \{ C_{p,n}; L_n \geq 0 \} E^{(F)}(ln^2(e + Z_{p,n})) \right]. \quad (27)
\]

Since
\[
(ln^2(e + x))'' = \frac{2}{(e + x)^2}(1 - \ln(e + x)),
\]
the function \( ln^2(e + x) \) is concave on the set \( x > 0 \). This fact allows us to apply Jensen’s inequality to the internal expectation in the right-hand side of (27) and to obtain the estimate
\[
E^{(F)}(ln^2(e + Z_{p,n})) \leq ln^2(e + E^{(F)}[Z_{p,n}]) = ln^2(e + Z_p(1 - f_{p,n}(0))).
\]

By the first equality in (26) we find that, for all sufficiently large \( n \),
\[
\Pr_j(A_{p,n}(x)C_{p,n}; L_n \geq 0) \leq \frac{1}{x^2c_p^2} E_j \left[ I \{ C_{p,n}; L_n \geq 0 \} ln^2(e + Z_p(1 - f_{p,n}(0))) \right]
\]
\[
\leq \frac{1}{x^2c_p^2} \ln^2(e) \Pr(L_n \geq 0) \leq \frac{K_1}{x^2c_p} \Pr(L_n \geq 0).
\]

These estimates imply (26).

The lemma is proved.

**Lemma 13** If Condition A is valid, then, for any \( j = 1, 2, \ldots \) and \( x \geq 0 \)
\[
\lim_{n \to \infty} \Pr_j(L_{p,n} \geq xc_p; L_n \geq 0, Z_n > 0) = D(x).
\]
Proof. Clearly,

\[ P_j (L_{p,n} \geq xc_p; L_n \geq 0, Z_n > 0) = E \left[ I (p,n;x) \left( 1 - f_{0,n}^j (0) \right) \right], \quad (28) \]

where

\[ I (p,n;x) := I \{ L_{p,n} \geq xc_p; L_n \geq 0 \}. \]

We now select \( \gamma > 1 \) and \( l < p \) and write the right-hand side of (28) as follows:

\[ E \left[ I (p,n;x) \left( 1 - f_{0,n}^j (0) \right) \right] = G_1 (m,p,n; x, \gamma) + G_2 (p,n;x, \gamma) + G_3 (m,p,n; x, \gamma), \]

where

\[ G_1 (l,p,n; x, \gamma) := E \left[ I (p,n;x) \left( f_{0,n}^j (0) - f_{0,l}^j (0) \right) I \{ L_n \geq 0 \} \right], \]
\[ G_2 (p,n;x, \gamma) := E \left[ I (p,n;x) \left( 1 - f_{0,n}^j (0) \right) I \{ L_n \geq 0 \} - I \{ L_n \geq 0 \} \right], \]
\[ G_3 (l,p,n; x, \gamma) := E \left[ I (p,n;x) \left( 1 - f_{0,l}^j (0) \right) I \{ L_n \geq 0 \} \right]. \]

By (16) we have

\[ G_1 (l,p,n; x, \gamma) \leq E \left[ (f_{0,n}^j (0) - f_{0,l}^j (0)) I \{ L_n \geq 0 \} \right] \]
\[ = E \left[ (f_{0,n}^j (0) - f_{0,l}^j (0)) I \{ L_n \geq 0 \} P \left( L_n (\gamma - 1) \geq -S_n | S_n \right) \right] \]
\[ \leq K_1 P \left( L_n (\gamma - 1) \geq 0 \right) E \left[ (f_{0,n}^j (0) - f_{0,l}^j (0)) I \{ L_n \geq 0 \} V (S_n) \right] \]
\[ = K_1 P \left( L_n (\gamma - 1) \geq 0 \right) E^+ \left[ f_{0,n}^j (0) - f_{0,l}^j (0) \right] \]
\[ \leq K_2 (\gamma - 1)^{p-1} P \left( L_n \geq 0 \right) E^+ \left[ f_{0,n}^j (0) - f_{0,l}^j (0) \right]. \quad (29) \]

where we have used (15) to justify the last inequality. Since \( f_{0,l} (0) \to f_{0,\infty} (0) \in (0, 1) \) \( P^+ \) a.s. as \( t \to \infty \), letting in (29) first \( n \to \infty \) and then \( l \to \infty \), we get in account of (22)

\[ \lim_{n \to \infty} G_1 (l,p,n; x, \gamma) = 0. \]

Further,

\[ G_2 (p,n;x, \gamma) \leq P \left( L_n \geq 0 \right) - P \left( L_n \gamma \geq 0 \right) \leq K_1 \left( 1 - \gamma^{-(1-\rho)} \right) P \left( L_n \geq 0 \right) \]

implying

\[ \lim_{n \to \infty} \lim_{\gamma \to 1} P \left( L_n \geq 0 \right) = 0. \]

By (16) we conclude that, given \( p \ll n \)

\[ G_3 (l,p,n; x, \gamma) = E \left[ (1 - f_{0,l}^j (0)) I \{ L_p \geq 0 \} P \left( L_p^* \geq xc_p - S_p | S_p \right) \right] \]
\[ \sim P \left( L_n \gamma \geq 0 \right) E \left[ (1 - f_{0,l}^j (0)) I \{ L_p \geq 0 \} V (S_p - xc_p) \right]. \]
as \( n \to \infty \). Using Lemma 2 and properties of regularly varying functions we get, as \( p \to \infty \)

\[
E \left[ (1 - f_{0,l}^j(0)) I \{ L_p \geq 0 \} V(S_p - xc_p) \right]
\]

\[
= E \left[ (1 - f_{0,l}^j(0)) I \{ L_p \geq 0 \} \frac{V(S_p - xc_p)}{V(c_p)} \times V(c_p) I \left\{ \frac{S_p}{c_p} \geq x \right\} \right]
\]

\[
\sim E \left[ (1 - f_{0,l}^j(0)) I \{ L_p \geq 0 \} \left( \frac{S_p}{c_p} - x \right)^{\alpha(1 - \rho)} V(c_p) I \left\{ \frac{S_p}{c_p} \geq x \right\} \right].
\]

Further,

\[
E \left[ (1 - f_{0,l}^j(0)) I \{ L_p \geq 0 \} \left( \frac{S_p}{c_p} - x \right)^{\alpha(1 - \rho)} V(c_p) I \left\{ \frac{S_p}{c_p} \geq x \right\} \right]
\]

\[
= E \left[ (1 - f_{0,l}^j(0)) I \{ L_l \geq 0 \} V(c_p) P \left( L_{p-l}^* \geq -S_l | S_l \right) \times \right.
\]

\[
\times E \left[ \left( \frac{S_{p-l}^* + S_l}{c_p} - x \right)^{\alpha(1 - \rho)} I \left\{ \frac{S_{p-l}^* + S_l}{c_p} \geq x \right\} | L_{p-l}^* \geq -S_l \right].
\]

It is not difficult to conclude by Lemma 4 and (17) that, for any fixed \( l \)

\[
\lim_{p \to \infty} E \left[ \left( \frac{S_{p-l}^* + S_l}{c_p} - x \right)^{\alpha(1 - \rho)} I \left\{ \frac{S_{p-l}^* + S_l}{c_p} \geq x \right\} | L_{p-l}^* \geq -S_l \right] = C_0^{-1} D(x).
\]

Besides, for \( 0 \leq x \ll c_p \)

\[
V(c_p) P \left( L_{p-l}^* \geq -x \right) \sim V(c_p) P \left( L_{p-l}^* \geq 0 \right) V(x) \sim C_0 V(x)
\]

as \( p \to \infty \), leading after evident transformations to

\[
E \left[ (1 - f_{0,l}^j(0)) I \{ L_p \geq 0 \} V(S_p - xc_p) \right]
\]

\[
\sim D(x) E \left[ (1 - f_{0,l}^j(0)) I \{ L_l \geq 0 \} V(S_l) \right] = E^+ \left[ 1 - f_{0,l}^j(0) \right] D(x).
\]

Hence we obtain

\[
\lim_{\gamma \downarrow 1} \lim_{l \to \infty} \lim_{n \to \infty} \frac{G_3(l, p, n; x, \gamma)}{P(L_n \geq 0)} = E^+ \left[ 1 - f_{0,\infty}^j(0) \right] D(x).
\]

To complete the proof of the lemma it remains to note that

\[
P \left( L_n \geq 0, Z_n > 0, Z_0 = j \right) = E \left[ \left( 1 - f_{0,n}^j(0) \right) | L_n \geq 0 \right] P \left( L_n \geq 0 \right)
\]

and that

\[
\lim_{n \to \infty} E \left[ \left( 1 - f_{0,n}^j(0) \right) | L_n \geq 0 \right] = E^+ \left[ 1 - f_{0,\infty}^j(0) \right]
\]

according to Lemma 8.

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The lemma is proved.

Set
\[ \eta_l := \frac{f''_l(1)}{(f'_l(1))^2}, \quad l \in \mathbb{N}_+, \]
and let
\[ J^+(p, r) := \sum_{l=p}^{r-1} \eta_l e^{S_p - S_l} + e^{S_p - S_r}, \quad J^-(p, r) := \sum_{l=p}^{r-1} \eta_l e^{S_r - S_l}, \]
\[ \hat{J}^-(0, r) := \sum_{l=0}^{r-1} \eta_{l+1} e^{S_l+1}. \]

It is known (see, for instance, Lemma 2.7 in [1]) that if Conditions $A_1$ and $A_2$ are valid then
\[ J^+(0, \infty) < \infty \quad \text{P}^+ - \text{a.s.} \]
and, for any $y > 0$
\[ \lim_{n \to \infty} \mathbf{P} \left( J^+(0, n) < y | L_n \geq 0 \right) = \mathbf{P}^+ \left( J^+(0, \infty) < y \right). \] (31)

In addition (compare with Lemma 2.7 in [1] or see Lemma 6 in [16]), if Conditions $A_1$ and $A_2$ are valid then, for $\mathbf{P}^-$ defined in (9)
\[ \hat{J}^- (0, \infty) = \sum_{l=0}^{\infty} \eta_{l+1} e^{S_l+1} < \infty \quad \mathbf{P}^- - \text{a.s.}, \]
and for any $y \geq 0$
\[ \lim_{n \to \infty} \mathbf{P} \left( \hat{J}^-(0, n) > y | M_n \leq 0 \right) = \mathbf{P}^- \left( \hat{J}^-(0, \infty) > y \right). \] (32)

Set
\[ \tau_{p,n} := \min \left\{ p \leq j \leq n : S_j - S_p = \hat{L}_{p,n} \right\}. \]

**Lemma 14** If Condition $A$ is valid, then, for any $j = 1, 2, ...$
\[ \lim_{y \to \infty} \lim_{n \to \infty} \mathbf{P}_j \left( J^+ (\tau_{p,n}, n) > y | L_n \geq 0, Z_n > 0 \right) = 0, \] (33)
\[ \lim_{y \to \infty} \lim_{n \to \infty} \mathbf{P}_j \left( J^- (p, \tau_{p,n}) > y | L_n \geq 0, Z_n > 0 \right) = 0. \] (34)

**Proof.** We write
\[
\mathbf{P}_j \left( J^+ (\tau_{p,n}, n) > y | L_n \geq 0, Z_n > 0 \right) \\
= \mathbf{E} \left[ I \left\{ J^+ (\tau_{p,n}, n) > y \right\} \left( 1 - f_{0,n}(0) \right) ; L_n \geq 0 \right] \\
\leq \mathbf{E} \left[ I \left\{ J^+ (\tau_{p,n}, n) > y \right\} \left( 1 - f_{0,p}(0) \right) ; L_n \geq 0 \right] \\
= \int_0^\infty \mathbf{E} \left[ \left( 1 - f_{0,p}(0) \right) ; S_p \in c_p dz, L_p \geq 0 \right] \mathbf{P} \left( J^+ (\tau_{n-p,n-p}) > y ; L_{n-p} \geq -c_p z \right).
\]
Note that

\[
P \left( J^+ (\tau_{n-p}, n - p) > y; L_{n-p} \geq -c_p z \right)
= \sum_{k=0}^{n-p} P \left( J^+ (k, n - p) > y; \tau_{n-p} = k, L_{n-p} \geq -c_p z \right)
= \sum_{k=0}^{n-p} P (M_k < 0; S_k \geq -c_p z) P \left( J^+ (0, n - p - k) > y; L_{n-p-k} \geq 0 \right).
\]

In view of (31) for any \( \varepsilon > 0 \) there exists \( y_0 \) and \( N = N(y_0, \varepsilon) \) such that, for all \( y \geq y_0 \) and \( n - p - k \geq N \)

\[
P \left( J^+ (0, n - p - k) > y, L_{n-p-k} \geq 0 \right) \leq P \left( J^+ (0, n - p - k) > y_0, L_{n-p-k} \geq 0 \right) \leq P^* \left( J^+ (0, \infty) > y_0 \right) P (L_{n-p-k} \geq 0) \leq \varepsilon P (L_{n-p-k} \geq 0).
\]

On the other hand, for each fixed \( N \) one can find a sufficiently large \( y_1 \geq y_0 \) such that

\[
P \left( J^+ (0, j) > y, L_j \geq 0 \right) \leq \varepsilon P (L_j \geq 0)
\]

for all \( y \geq y_1 \). These estimates and (30) imply

\[
P \left( J^+ (\tau_{n-p}, n - p) > y; L_{n-p} \geq -c_p z \right)
\leq \varepsilon \sum_{k=0}^{n-p} P (M_k < 0; S_k \geq -c_p z) P (L_{n-p-k} \geq 0) = \varepsilon P (L_{n-p} \geq -c_p z).
\]

Thus,

\[
\int_0^\infty E \left[ (1 - f_{0,p}^j(0); S_p \in c_p dz, L_p \geq 0) \right] P \left( J^+ (\tau_{n-p}, n - p) > y; L_{n-p} \geq -c_p z \right)
\leq \varepsilon \int_0^\infty E \left[ (1 - f_{0,p}^j(0); S_p \in c_p dz, L_p \geq 0) \right] P (L_{n-p} \geq -c_p z)
= \varepsilon E \left[ (1 - f_{0,p}^j(0); L_n \geq 0) \right] \leq \varepsilon K P_j (L_n \geq 0, Z_n > 0).
\]

This proves (31), since \( \varepsilon > 0 \) may be chosen arbitrary small.

To prove (34) we write

\[
P_j \left( J^- (p, \tau_{p,n}) > y; L_n \geq 0, Z_n > 0 \right)
\leq E \left[ (1 - f_{0,p}^j(0); J^- (p, \tau_{p,n}) > y; L_n \geq 0) \right]
= \int_0^N \left[ (1 - f_{0,p}^j(0); S_p \in c_p dz, L_p \geq 0) \right] P \left( J^- (0, \tau_{p,n}) > y; L_{n-p} \geq -c_p z \right)
+ E \left[ (1 - f_{0,p}^j(0); S_p > N c_p, J^- (p, \tau_{p,n}) > y; L_n \geq 0) \right].
\]
By Lemma 5 with \( r = 0 \) for any \( \varepsilon > 0 \) one can find \( N_0 \) such that the inequality
\[
E \left[ \left( 1 - f_{0,p}^0(0) \right) ; S_p > Nc_p, J^- (p, \tau_{p,n}) > y; L_n \geq 0 \right] \leq P \left( S_p > Nc_p, L_n \geq 0 \right) \leq \varepsilon P \left( L_n \geq 0 \right)
\]
is valid for all \( N \geq N_0 \). Further,
\[
P \left( J^- (0, \tau_{0,n-p}) > y; L_{n-p} \geq c_p z \right)
= \sum_{k=0}^{n-p-1} P \left( J^- (0, k) > y; L_{n-p} \geq c_p z; \tau_{n-p} = k \right)
= \sum_{k=0}^{n-p-1} P \left( J^- (0, k) > y; L_k \geq c_p z; \tau_k = k \right) P \left( L_{n-p-k} \geq 0 \right)
= \sum_{k=0}^{n-p-1} P \left( j^- (0, k) > y; M_k \leq 0, S_k \geq -c_p z \right) P \left( L_{n-p-k} \geq 0 \right),
\]
where at the last step we have used the duality principle for random walks. In view of (36) for any \( \varepsilon > 0 \) there exists \( y_0 = y_0(\varepsilon) \) such that
\[
P \left( j^- (0, k) > y; M_k \leq 0, S_k \geq -c_p z \right) \leq P \left( j^- (0, k) > y; M_k \leq 0 \right) \leq \varepsilon P \left( M_k \leq 0 \right)
\]
for all \( y \geq y_0 \). On the other hand, one can show (compare with a similar statement in [6] for random walk conditioned to stay positive) that there exists a proper distribution \( G(\cdot) \) with \( G(z) \in (0,1) \) for all \( z > 0 \) such that for any \( R > 0 \)
\[
P \left( S_{pR} \geq -c_p z \mid M_{pR} \leq 0 \right) \to 1 - G(zR^{1/\alpha})
\]
as \( p \to \infty \) uniformly in \( z \in [0, N] \). This leads to the following chain of estimates being valid for \( y \geq y_0 \), a large but fixed \( T > 0 \) and \( k \leq pT \)
\[
P \left( j^- (0, k) > y; M_k \leq 0, S_k \geq -c_p z \right) \leq P \left( j^- (0, k) > y; M_k \leq 0 \right) \leq \varepsilon P \left( M_k \leq 0 \right) \leq \varepsilon K_1 P \left( M_k \leq 0, S_k \geq -c_p z \right).
\]
This, in account of (13) allows us to proceed with one more chain of estimates
\[
\sum_{k=0}^{pT} P \left( J^- (0, k) > y; L_k \geq -c_p z; \tau_k = k \right) P \left( L_{n-p-k} \geq 0 \right)
\]
\[
\leq \varepsilon K_1 \sum_{k=0}^{pT} P \left( M_k \leq 0, S_k \geq -c_p z \right) P \left( L_{n-p-k} \geq 0 \right)
\]
\[
\leq \varepsilon K_1 P \left( L_{n-p} \geq -c_p z \right) \leq \varepsilon K_2 V(c_p) P \left( L_{n-p} \geq 0 \right)
\]
\[
\leq \varepsilon K_3 z^{\alpha(1-\rho)} V(c_p) P \left( L_n \geq 0 \right)
\]
(36)
being valid for all \( z \in [0, N] \).

To consider the case \( k \geq Tp \) we use Theorem 4 of [21] according to which

\[
P ( M_k \leq 0, S_k \geq -c_p z ) \leq K_1 \frac{V(c_p z) z c_p}{k c_k}
\]

if \( z c_p \leq \varepsilon c_k \). Using this bound we get

\[
\sum_{k=pT}^{n-1} \frac{1}{k c_k} P ( L_{n-p-k} \geq 0 ) \leq K_1 V(c_p z) z c_p \sum_{k=pT}^{n-1} \frac{1}{k c_k} P ( L_{n-p-k} \geq 0 ).
\]

By (36) and properties of regularly varying functions we conclude that

\[
\sum_{k=pT}^{n-1} \frac{1}{k c_k} P ( L_{n-p-k} \geq 0 ) = \sum_{k=pT}^{n/2+p} \frac{1}{k c_k} P ( L_{n-p-k} \geq 0 ) + \sum_{k=n/2+p+1}^{n-1} \frac{1}{k c_k} P ( L_{n-p-k} \geq 0 )
\]

\[
\leq K_1 P ( L_n \geq 0 ) \sum_{k=pT}^{n/2+p} \frac{1}{k c_k} + \frac{K_2}{n c_n} \sum_{j=0}^{n/2} P ( L_j \geq 0 )
\]

\[
\leq \frac{K_3}{c_p T} P ( L_n \geq 0 ) + \frac{K_4}{n c_n} n P ( L_n \geq 0 )
\]

\[
\leq \frac{K_5}{c_p T} P ( L_n \geq 0 ).
\]

As a result we get

\[
\sum_{k=pT}^{n-1} P ( J^- (0, k) > y; L_k \geq -c_p z; \tau_k = k ) P ( L_{n-p-k} \geq 0 )
\]

\[
\leq K_1 V(c_p z) z c_p \sum_{k=pT}^{n-1} \frac{1}{k c_k} P ( L_{n-p-k} \geq 0 ) \leq K_6 z^{\alpha(1-\rho)+1} \frac{c_p}{c_p T} V(c_p) P ( L_n \geq 0 )
\]

\[
\leq K_7 z^{\alpha(1-\rho)+1} \frac{1}{T^{1/\alpha}} V(c_p) P ( L_n \geq 0 ). \tag{37}
\]

Combining (36) and (37) we see that

\[
P ( J^- (0, \tau_{0,n-p}) > y; L_{n-p} \geq -c_p z ) \leq K_8 z^{\alpha(1-\rho)} \left( \frac{z}{T^{1/\alpha}} + \varepsilon \right) V(c_p) P ( L_n \geq 0 ).
\]
Thus, assuming that $N \leq \epsilon T^{1/\alpha}$ we get in account of (12)

$$\int_0^N \mathbf{E} \left[ \left(1 - f_{0,p}(0)\right) ; S_p \in c_p dz, L_p \geq 0 \right] \mathbf{P} \left( J^+ (0, \tau_{n-p}) > y; L_{n-p} \geq -c_p z \right)$$

$$\leq \varepsilon K_1 \mathbf{P} \left( L_n \geq 0 \right) V(c_p) \times$$

$$\times \int_0^N \mathbf{E} \left[ \left(1 - f_{0,p}(0)\right) ; S_p \in c_p dz, L_p \geq 0 \right] z^{\alpha(1-\rho)} \left( \frac{z}{T^{1/\alpha}} + \varepsilon \right)$$

$$\leq \varepsilon K_1 \mathbf{P} \left( L_n \geq 0 \right) V(c_p) \int_0^N z^{\alpha(1-\rho)} \mathbf{P} \left( S_p \in c_p dz, L_p \geq 0 \right)$$

$$\leq \varepsilon K_1 \mathbf{P} \left( L_n \geq 0 \right) V(c_p) \mathbf{P} \left( L_p \geq 0 \right) \int_0^N z^{\alpha(1-\rho)} \mathbf{P} \left( S_p \in c_p dz \left| L_p \geq 0 \right. \right)$$

This estimate combined with (35) proves (34).

**Lemma 15** If Condition $A$ is valid, then, for any $j = 1, 2, \ldots$ and $x \geq 0$

$$\lim_{n \to p \to \infty} \mathbf{P}_j \left( A_{p,n}(x) | L_n \geq 0, Z_n > 0 \right) = D(x).$$

**Proof.** It follows from Lemma 12 and the inequalities

$$\mathbf{P}_j \left( A_{p,n}(x) \bar{B}_{p,n}(N) \bar{C}_{p,n}; L_n \geq 0 \right) \leq \mathbf{P}_j \left( A_{p,n}(x); L_n \geq 0 \right)$$

$$\leq \mathbf{P}_j \left( A_{p,n}(x) \bar{B}_{p,n}(N) \bar{C}_{p,n}; L_n \geq 0, Z_n > 0 \right)$$

$$+ \mathbf{P}_j \left( A_{p,n}(x) C_{p,n}; L_n \geq 0 \right) + \mathbf{P}_j \left( A_{p,n}(x) B_{p,n}(N); L_n \geq 0 \right)$$

that, in fact, we need to show that

$$\lim_{N \to \infty} \lim_{n \to p \to \infty} \frac{\mathbf{P}_j \left( A_{p,n}(x) \bar{B}_{p,n}(N) \bar{C}_{p,n}; L_n \geq 0, Z_n > 0 \right)}{\mathbf{P}_j \left( Z_n > 0, L_n \geq 0 \right)} = D(x).$$

Using the equality

$$Z_{p,n} = Z_p \left(1 - f_{p,n}(0)\right) + \sum_{l=1}^{Z_p} \left( I \left\{ Z_{n,l}^p > 0 \right\} - (1 - f_{p,n}(0)) \right),$$

the estimate

$$\left| \sum_{l=1}^{Z_p} \left( I \left\{ Z_{n,l}^p > 0 \right\} - (1 - f_{p,n}(0)) \right) \right| \leq \sqrt{N} Z_p (1 - f_{p,n}(0)),$$

being valid on the set $\bar{B}_{p,n}(N)$, and recalling (24), (25) and the fact that

$$Z_p (1 - f_{p,n}(0)) \geq e^{\sqrt{p}}$$
on the set $\tilde{C}_{p,n} \cap \{Z_p > 0\}$, we conclude that for any $\delta > 0$ there exists a number $n_0 = n_0(\delta)$ such that for $n \geq n_0$
\[ P_j(A_{p,n}(x)B_{p,n}(N)\bar{C}_{p,n} \mid L_n \geq 0, Z_n > 0) \]
\[ \leq P_j(\ln(e + Z_p(1 - f_{p,n}(0))(1 + \delta)) \geq xc_p; \bar{C}_{p,n} \mid L_n \geq 0, Z_n > 0) + \alpha_1(p,n) \]
\[ \leq P_j(\ln Z_p + \ln(1 - f_{p,n}(0)) + \ln(1 + 2\delta) \geq xc_p; L_n \geq 0, Z_n > 0) + \alpha_1(p,n), \]
where
\[ \lim_{n \to \infty} |\alpha_1(p,n)| = 0. \]
In view of
\[ 1 - f_{p,n}(0) \leq f'_j(1) (1 - f_{p+1,n}(0)) \leq \min_{p \leq j \leq n} f'_j(1) = e^{L_{p,n}}, \]
we get
\[ P_j(\ln Z_p + \ln(1 - f_{p,n}(0)) + \ln(1 + 2\delta) \geq xc_p; L_n \geq 0, Z_n > 0) \]
\[ \leq P_j \left( \frac{1}{c_p} \ln \frac{Z_p}{e^{c_p x}} + \frac{1}{c_p} L_{p,n} + \ln(1 + 2\delta) \geq x; L_n \geq 0, Z_n > 0 \right). \]
Using the equivalences
\[ P_j(L_n \geq 0, Z_n > 0) \sim P_j(Z_n > 0 \mid L_n \geq 0)P(L_n \geq 0) \]
\[ \sim \frac{1}{\theta} e^x \left( 1 - f'_{0,\infty}(0) \right) P(Z_n > 0) \]
valid as $n \to \infty$, and following the proof of Theorem 1.1 in [1], we obtain
\[ P_j(A_{p,n}(x)B_{p,n}(N)\bar{C}_{p,n} \mid L_n \geq 0, Z_n > 0) \]
\[ \leq P_j \left( \frac{1}{c_p} \ln \frac{Z_p}{e^{c_p x}} + \frac{1}{c_p} L_{p,n} + \ln(1 + 3\delta) \geq x; L_n \geq 0, Z_n > 0 \right). \]
This inequality, Corollary [10] and Lemma [13] yield
\[ \lim_{N \to \infty} \limsup_{n \to \infty} P_j(A_{p,n}(x)B_{p,n}(N)\bar{C}_{p,n} \mid L_n \geq 0, Z_n > 0) \]
\[ \leq \lim_{\delta \to 0} \lim_{n \to \infty} P_j \left( \frac{1}{c_p} L_{p,n} + \ln(1 + 3\delta) \geq x \mid L_n \geq 0, Z_n > 0 \right) = D(x). \]
To get a similar estimate from below observe that according to relations (2.2) and (2.3) of [12]
\[ 1 - f_{p,n}(0) \geq \left( \sum_{i=p}^{n-1} \eta_i e^{-(S_i - S_p)} + e^{-(S_n - S_p)} \right)^{-1} \]
\[ = e^{L_{p,n}} \left( \sum_{i=p}^{n-1} \eta_i e^{-(S_i - L_{p,n})} + e^{-(S_n - L_{p,n})} \right)^{-1} \]
\[ = e^{L_{p,n}} \left( J^- (p, \tau_{p,n}) + J^+ (\tau_{p,n}, n) \right)^{-1}. \]
Thus,
\[ \log (1 - f_{p,n}(0)) \geq \hat{L}_{p,n} - \log \left( J^{-}(p, \tau_{p,n}) + J^{+}(\tau_{p,n}, n) \right). \]

According to Lemma 14
\[ \lim_{y \to \infty} \lim_{n \gg p \to \infty} P_j \left( \log \left( J^{-}(p, \tau_{p,n}) + J^{+}(\tau_{p,n}, n) \right) > y | L_n \geq 0, Z_n > 0 \right) = 0. \]

Hence for any \( \delta > 0 \) we get for all sufficiently large \( p \) and \( n \)
\[
P_j \left( \log Z_p + \log (1 - f_{p,n}(0)) + \log (1 - \delta) \geq x; L_n \geq 0, Z_n > 0 \right) \geq P_j \left( \frac{1}{c_p} \log Z_p + \frac{1}{c_p} L_{p,n} + \log (1 - 2\delta) \geq x; L_n \geq 0, Z_n > 0 \right).
\]

leading by Lemma 13 to the following estimate from below:
\[
\lim_{n \gg p \to \infty} \inf \ P_j (A_{p,n}(x) B_{p,n}(N) C_{p,n} | L_n \geq 0, Z_n > 0) \geq P_j \left( \frac{1}{c_p} L_{p,n} + \log (1 - 3\delta) \geq x | L_n \geq 0, Z_n > 0 \right) = D(x).
\]

Lemma 15 is proved.

Proof of Theorem 1. First we note that to check the validity of (14) it is sufficient to investigate the asymptotic behavior of the probability of the event \( A_{p,n}(x) \). We use Lemma 9 to this aim. For \( z, p, n \in \mathbb{N}_0 \) with \( p \leq n \) set
\[
\psi(z, p, n) := P_z (A_{p,n}(x), L_n \geq 0).
\]

Clearly, \( \psi(0, p, n) = 0 \). We know by Lemmas 15 and 8 that if \( n \gg p = p(n) \to \infty \) then
\[
\psi(z, p, n) \sim P_z (L_n \geq 0, Z_n > 0) D(x) = P (L_n \geq 0) \theta P (Z_n > 0).
\]

In addition, for \( k \leq p \leq n \)
\[
E \left[ I \{ A_{p,n}(x) \}, L_{k,n} \geq 0 | \bar{F}_k \right] = \psi(Z_k, p - k, n - k).
\]

Relations (38) and (39) show that we may apply Lemma 9 to
\[
\zeta_n := I \{ A_{p,n}(x) \}, \ z, n := P_{z}^{+} (A_{u,s}) D(x)
\]
and \( l = 0 \) to conclude that
\[
P (A_{p,n}(x)) = P (A_{p,n}(x); Z_n > 0) \approx D(x) \theta P (L_n \geq 0) \sim D(x) P (Z_n > 0).
\]

This completes the proof of (14).
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