Pure U(1) Lattice Gauge Theory in Field Fourier Basis

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Abstract
In the basis of Fourier modes in field-space the elements of transfer-matrix for pure U(1) lattice gauge theory are explicitly obtained in terms of the plaquette-link matrix and summations on multiplicative Bessel functions. In this basis it is shown, 1) the transfer-matrix is block-diagonal, 2) all consisting vectors of a block are known based on an arbitrary block’s vector, 3) the ground-state belongs to the zero-mode’s block. The emergence of maximum in matrix elements as functions of gauge coupling is clarified. To illustrate the computational benefits in the Fourier basis, three matrix elements for $3 \times 3$ and $10 \times 10$ lattices in a range of gauge couplings are computed.

Keywords: Lattice gauge theories; Transfer-matrix method; Energy spectrum

Currently the numerical studies of gauge theories in the non-perturbative regime are mainly based on the lattice formulation of these theories [1-4]. The theoretical [5-9] as well as numerical [10-16] studies suggest that the compact U(1) gauge theory possesses two different phases, the so-called Coulomb and confined ones. Different studies suggest that the phase transition occurs at the critical coupling of order unity [5-16].

The main purpose of the present work is to formulate and study the pure U(1) lattice gauge theory in the field Fourier basis. As the partial outcome of the formulation, regarding the properties of the transfer-matrix in the Fourier basis, some mathematical statements are presented. Further, to show how the Fourier basis may be used for numerical purposes, some preliminary numerical results are presented. The matrix element of the transfer-matrix $\hat{V}$ between two adjacent times of $n_0$ and $n_0 + 1$ is given by

$$\langle n_0 + 1 | \hat{V} | n_0 \rangle \propto e^{S_E(n_0, n_0+1)}$$

in which $S_E(n_0, n_0 + 1)$ is the Euclidean action symmetrized in variables of two adjacent times. In the temporal gauge $A^0 \equiv 0$, it is convenient to replace the gauge variables at adjacent times $A^{(r,i)}_{n_0}$ and $A^{(r,i)}_{n_0+1}$ on spatial link $(r,i)$ by the new angle variables

$$\theta^{(r,i)} = a g A^{(r,i)}_{n_0}$$
$$\theta'^{(r,i)} = a g A^{(r,i)}_{n_0+1}$$

both taking values in $[-\pi, \pi]$ [1]. In above $a$ and $g$ are the lattice spacing parameter and the gauge coupling, respectively. The symmetrized Euclidean action in (1) for pure U(1) theory in temporal gauge on a lattice with $d$ spatial dimensions is explicitly given by [17,18]

$$S_E(n_0, n_0 + 1) = -\frac{1}{2 g^2} \sum_r \sum_{i \neq j}^d \left[ 2 - \cos \left( \theta^{(r,i)} + \theta'^{(r+i,j)} - \theta^{(r+j,i)} - \theta^{(r,j)} \right) \right]$$
$$\quad - \cos \left( \theta'^{(r,i)} + \theta'^{(r+i,j)} - \theta^{(r+j,i)} - \theta^{(r,j)} \right)$$
$$\quad - \frac{1}{g^2} \sum_r \sum_{i=1}^d \left[ 1 - \cos \left( \theta^{(r,i)} - \theta'^{(r,i)} \right) \right]$$

in which $\hat{i}$ is the unit-vector along the spatial direction $i$. For a spatial lattice with $N_P$ number

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of plaquettes and $N_L$ number of links, it is convenient to define the plaquette-link matrix $M$ of dimension $N_P \times N_L$, given explicitly by its elements as following

$$M^p_l = \begin{cases} +1, & \text{link } l = (r, i) \text{ belongs to oriented plaquette } p \\ -1, & \text{link } l = (r, -i) \text{ belongs to oriented plaquette } p \\ 0, & \text{otherwise}. \end{cases}$$ (4)

In Fig. 1 the above definition is presented graphically. An explicit example for definition (4) in $d=2$ case will be given later. In terms of this matrix, labeling links as $l = (r, i)$ and plaquettes as $p$, the action (3) comes to the following form:

$$S_E(n_0, n_0 + 1) = -\frac{1}{2g^2} \sum_p \left[ 2 - \cos (M^p_l \theta^k) - \cos (M^p_l \theta^k') \right]$$

$$- \frac{1}{g^2} \sum_l \left[ 1 - \cos (\theta^k - \theta^k') \right]$$ (5)

in which the summations over repeated indices are understood. Setting

$$\gamma = \frac{1}{g^2}$$ (6)

and by (5) the matrix-element (1) may be written as

$$\langle \theta' | \hat{V} | \theta \rangle = \mathcal{A} \prod_p \exp \left\{ -\frac{\gamma}{2} \left[ 2 - \cos (M^p_l \theta^k) - \cos (M^p_l \theta^k') \right] \right\}$$

$$\times \prod_l \exp \left\{ -\gamma \left[ 1 - \cos (\theta^k - \theta^k') \right] \right\}$$ (7)

in which $\mathcal{A}$ is inserted to fix the normalization. According to the recipe, by $\hat{V} = \exp(-a \hat{H})$, the Hamiltonian and the transfer-matrix eigenvalues are related as:

$$E_i = -\frac{1}{a} \ln v_i$$ (8)

As the main concern of this work, we formulate the theory in the field Fourier basis $|k_l\rangle$, which is related to the compact $\theta$-basis by:

$$\langle \theta' | k_l \rangle = \frac{\delta^{\theta'}_{\theta^k}}{\sqrt{2\pi}} \exp(i k_l \theta^k), \quad k_l = 0, \pm 1, \pm 2, \cdots$$ (9)

Using the identity for $I_n(x)$ as the modified Bessel function of the first kind

$$\exp(x \cos \phi) = \sum_n I_n(x) \exp(in \phi)$$ (10)
and the relation
\[ \int_{-\pi}^{\pi} d\theta \exp(i n \theta) = 2\pi \delta(n) \]  
(11)

one directly finds the matrix elements of \( \hat{V} \) in the field Fourier basis
\[ \langle k'| \hat{V} | k \rangle = A e^{-\gamma(N_p+N_L)} (2\pi)^N_L \sum_{\{n_p\} \{n'_p\}} \prod_p I_{n_p} \left( \frac{\gamma}{2} \right) \prod_l I_{n_l} \left( \frac{\gamma}{2} \right) \times \prod_l I_{m_l}(\gamma) \delta[(n_p + n'_p)M^p_l + k_l - k'_l], \]
(12)
in which \( m_l = k_l + \sum_p n_p M^p_l = k'_l - \sum_p n'_p M^p_l \). In above \( n_p, n'_p \) and \( m_l \) are integer-valued. The expression (12) is one of the basic results, based on which in the following some propositions (by \( Pn \)'s) and their proofs (by \( Pf \)'s) are presented:

**Pn 1**: For every matrix element we have the following properties:
1) non-negativity: \( \langle k'| \hat{V} | k \rangle \geq 0, \) 2) symmetricity: \( \langle k'| \hat{V} | k \rangle = \langle k| \hat{V} | k' \rangle, \) 3) reflectivity: \( \langle k'| \hat{V} | k \rangle = \langle -k'| \hat{V} | -k \rangle \).

**Pf 1**: All above properties are simply proven by the use of Bessel’s properties \( I_n(x) \geq 0 \) and \( I_\nu(x) = I_{-\nu}(x) \), and appropriate sign-changes of indices \( n_p, n'_p \) and \( m_l \).

It is obvious that not only all matrix elements are non-negative, but also each term is so in the sum (12). The vanishing of a matrix element means that the difference \( k' - k \) can not satisfy all the \( \delta \)'s in (12) for any set of integers \( \{n_p, n'_p\} \).

**Pn 2**: All diagonal elements are non-zero: \( \langle k| \hat{V} | k \rangle \neq 0 \).

**Pf 2**: It is easy to see that there are always surviving terms for \( k' = k \) in (12). On diagonal \( k = k' \), setting all \( n_p + n'_p = 0 \) is enough to satisfy all \( \delta \)'s in (12), leading to non-vanishing positive terms.

In fact for satisfying \( \delta \)'s in (12) with \( k = k' \), it is sufficient to set \( n_p + n'_p = n^0_p \) with the condition \( \sum_p n^0_p M^p_l = 0 \), presented in vector notation as
\[ n^0 \cdot M = 0 \]  
(13)

Later we will give the general form of the non-zero elements based on the \( n^0 \) vectors.

**Pn 3**: Transitivity: If \( \langle k| \hat{V} | k' \rangle \neq 0 \) and \( \langle k'| \hat{V} | k'' \rangle \neq 0 \), then \( \langle k| \hat{V} | k'' \rangle \neq 0 \).

**Pf 3**: This simply follows by two successive uses of the \( \delta \)'s in (12).

By **Pn 2 & 3**, having a non-zero matrix element is an equivalence relation, by which the set of all vectors is partitioned into equivalence classes. Later by explicit examples we will see that there are more than one class (in fact, an infinite number of classes) even for a finite extent lattice. As the consequence, the transfer-matrix \( \hat{V} \) appears in the block-diagonal form based on the classes, with all elements of each block being non-zero. The remarkable fact is that, given by a vector one can simply construct all of its co-blocks. This is simply done by taking \( n_p + n'_p = q_p + n^0_p \) in all \( \delta \)'s of (12), by which using \( \sum_p n^0_p M^p_l = 0 \), another vector in the class is obtained as \( k^q_l = k_l + \sum_p q_p M^p_l \), presented in the vector notation by
\[ k^q = k + q \cdot M \]  
(14)
The vector $\mathbf{q}$ is arbitrary, and if it is kind of $n^0$ of (13), the two vectors are the same ($k^q = k$). For two vectors related as above then the non-zero matrix element simply comes to the form

$$\langle k^q | \hat{V} | k \rangle = A e^{-\gamma(N_p+N_L)} (2\pi)^{NL} \sum_{\{n^0_p\}} \delta \left( \sum_{p} n^0_p M^p_l \right) \sum_{\{n_p\}} \prod_{p} I_{n_p} \left( \frac{\gamma}{2} \right) I_{q_p+n^0_p-n_p} \left( \frac{\gamma}{2} \right) \prod_{l} I_{k_l+\sum_p n^0_p M^p_l} (\gamma)$$ (15)

The important fact is that the allowed $n^0_p$'s are not dependent on $k$, but only on the matrix $M$. As seen later in an explicit example, for periodic spatial lattices the sub-space by vectors $n^0$ satisfying (13) is one dimensional with the general form

$$n^0 = n^0 (1,1,\ldots,1) = n^0 \ s$$ (16)

By the above for the periodic lattice, the matrix-element (15) comes to the form

$$\langle k^q | \hat{V} | k \rangle = A e^{-\gamma(N_p+N_L)} (2\pi)^{NL} \sum_{n^0} \sum_{\{n_p\}} \prod_{p} I_{n_p} \left( \frac{\gamma}{2} \right) I_{q_p+n^0_p-n_p} \left( \frac{\gamma}{2} \right) \prod_{l} I_{k_l+\sum_p n^0_p M^p_l} (\gamma)$$ (17)

The above expression, with no restriction on summations, is quite adequate for numerical purposes and will be used later.

Pn 4: Each block of $\hat{V}$ is infinite dimensional.

Pf 4: This simply follows by the infinite possible choices for the integer sets $\{q_p\}$.

For definiteness, throughout this work we consider the normalization $A$ to be constant (i.e. independent of $g$); for other choices and their consequences see (19). The limit $\gamma \ll 1$ (large coupling limit $g \gg 1$) of the matrix elements is obtained easily by the expansion of exponentials in (7), by which in the lowest orders one finds:

$$\langle k^q | \hat{V} | k \rangle = A e^{-\gamma(N_p+N_L)} (2\pi)^{NL} \left\{ \prod_{l} \delta(k_l) \delta(k'_l) \right. + \frac{\gamma}{4} \sum_{p} \left[ \prod_{l} \delta(k_l + M^p_l) \delta(k'_l) + \prod_{l} \delta(k_l - M^p_l) \delta(k'_l) \right. \\
+ \left. \prod_{l} \delta(k_l) \delta(k'_l - M^p_l) + \prod_{l} \delta(k_l) \delta(k'_l + M^p_l) \right] \\
+ \left. \gamma \left[ \prod_{l} \delta(k_l + 1) \delta(k'_l - 1) + \prod_{l} \delta(k_l - 1) \delta(k'_l + 1) \right] + O(\gamma^2) \right\}$$ (18)

By above we have the next important proposition:

Pn 5: Provided that the ground-state is unique, it belongs to $k = 0$'s block.

Pf 5: According to (18), at the extreme $\gamma \rightarrow 0$ all the elements of $\hat{V}$ are approaching zero, except the diagonal element $V_{00} = \langle 0 | \hat{V} | 0 \rangle$. By the relation (9) between energy eigenvalues and $\hat{V}$-eigenvalues, all energies are going to infinity in limit $\gamma \rightarrow 0$ except the one in $V_{00}$'s block, appearing as the lowest energy. Since lowering the coupling (increasing $\gamma$) does not
cause the mixing among the blocks, by the uniqueness assumption, the ground-state belongs to the $k = 0$’s block at any coupling.

The other interesting limit is at $\gamma \to \infty$ ($g \to 0$), which is expected to recover the ordinary formulation of the gauge theory on the continuum. This limit can be reached by using the asymptotic behavior of Bessel functions for large arguments, for which using the saddle-point approximation we have

$$I_n(x) \simeq \frac{e^x}{\sqrt{2\pi x}} e^{-n^2/2x+1/8x} \left(1 + O(1/x^2)\right), \quad x \to \infty$$

By using the above the summations in matrix element (17) can be treated as Gaussian integrals in the limit $\gamma \to \infty$, leading to the asymptotic behavior:

$$\langle k | \hat{V} | k \rangle \simeq A \frac{(2\pi)^{\frac{1}{2}(N_L+N_P)}}{\pi^{N_P-\frac{1}{2}}} \frac{1}{\sqrt{\det C}} \frac{1}{\sqrt{s^TDs}} e^{-B(k,q)/\gamma} \gamma^{\frac{1}{2}(N_P+N_L)} e^{b/\gamma}$$

in which $b = (4N_P + N_L)/8$, and $B(k, q)$, in terms of symmetric matrices $C$, $D$ and $F$, is

$$B(k, q) = q^TDq + \frac{1}{2} k^TFk + 2q^TC^{-1}Mk - \frac{s^T(Dq + C^{-1}Mk)^2}{s^TDs}$$

with $s$ given in (16), and

$$C = 4 \mathbb{1}_{N_P} + MM^T, \quad D = \mathbb{1}_{N_P} - 2C^{-1}, \quad F = \mathbb{1}_{N_L} - M^TC^{-1}M$$

in which $\mathbb{1}_N$ is the identity matrix of dimension $N$. It is obvious by the above that the matrix elements tend to zero in the limit $\gamma \to \infty$. On the other hand, by (18), we already know that only $V_{00}$ may survive in the limit $\gamma \to 0$. An immediate conclusion is:

**Pn 6:** Except perhaps $V_{00}$, all non-zero matrix elements are to develop maximum.

**Pf 6:** As by **Pn 1** all non-zero matrix elements are positive, for the mentioned elements the increasing behavior at small $\gamma$ and the decreasing one at large $\gamma$ are to be connected through at least one maximum.

The numerical results presented later demonstrate clearly the appearance of precisely one maximum. The existence of the maximum in matrix elements of $\hat{V}$ is particularly important in connection to the nature of the phase transition by the model, as by (8) a maximum in matrix elements may lead to a minimum in the energy spectrum. The appearance of minimum in the spectrum, specially in the ground-state, is considered as the signature of a first order phase transition.

To proceed let us have an explicit representation of the plaquette-link matrix $M$. In the following we consider the lattice with two spatial dimensions $d = 2$. For the 2-dim cubic periodic lattice with $N_s$ sites in each direction, it is convenient to define the $N_s \times N_s$ translation-matrix $T$ by its elements as follows:

$$T_{ab} = \delta_{ab} - \delta_{a+1,b} - \delta_{a,N_s} \delta_{b1}, \quad a, b = 1, \cdots, N_s$$

For $N_s = 3$ the explicit form of $T$ is

$$T = \begin{pmatrix} + & - & 0 \\ 0 & + & - \\ - & 0 & + \end{pmatrix}$$
For $N_s$ sites in each direction of a 2-dim periodic cubic lattice there are $N_P = N_s^2$ plaquettes and $N_L = 2N_s^2$ links. Then, by the numbering of plaquettes and links as shown in Fig. 2, it is easy to check that the matrix $M$ can be constructed by gluing the two $N_s^2 \times N_s^2$ matrices next to each other, as follows:

$$M = \left( \begin{array}{c|c} 1_{N_s} \otimes T & -T \otimes 1_{N_s} \end{array} \right)$$

(25)

By construction, the matrix $M$ is the $N_s^2 \times 2N_s^2$ dimensional, as it should. For $N_s = 3$ the matrix comes to the form:

$$M = \begin{pmatrix} + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ - & 0 & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & - & 0 & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & + & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & - & 0 & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -T \otimes 1_3 \\ 1_3 \otimes T \end{pmatrix}$$

(26)

As announced earlier, by the above explicit form it is obvious that the subspace by $n^0$ of (16) is one-dimensional. As two vectors in an equivalence class consider the followings:

$$|0\rangle \rightarrow k = (0, 0, \cdots, 0)$$

(27)

$$|1\rangle \rightarrow k^{q_1} = q_1 \cdot M$$

(28)

in which $q_1 = (1, 0, \cdots, 0)$ with $N_P = N_s^2$ elements. By (18) and (19), the non-vanishing elements $V_{00}, V_{01} = V_{10}$, and $V_{11}$ belong to the ground-state’s block. To see how a vanishing element may occur, consider:

$$|1'\rangle \rightarrow k_{1'} = (1, 0, 0, \cdots, 0)$$

(29)

$$|1''\rangle \rightarrow k_{1''} = (0, 1, 0, \cdots, 0)$$

(30)

$$|2\rangle \rightarrow k_2 = (2, 0, 0, \cdots, 0)$$

(31)
By an explicit representation like (26), it is seen that any pair of above vectors can not satisfy (14), leading to vanishing elements $V_{11''} = V_{1''2} = V_{1'2} = 0$. In other words, by the given representation for $M$ and by any pair of (29)-(31) one can see there is no solution for the $\{n_p + n'_p\}$’s inside the $\delta$’s of (12). The same is true between each of (29)-(31) and one of (27)-(32). Hence, the five vectors (27)-(32) belong to four different blocks.

Using the explicit form of $M$, the expression (17) with summations on $N_s^2 + 1$ integers in 2-dim case is quite adequate for numerical purposes. In the following we present some preliminary results to see how the formalism can lead to numerical results practically. First of all is the issue of cut-off in summations. For small $\gamma$ limit we have

$$I_s(\gamma) \simeq \frac{1}{s!} \left( \frac{\gamma}{2} \right)^{|s|}, \quad \gamma \ll |s|$$

by which for small arguments the Bessel’s of low degrees are quite dominant. The subtle point is about large arguments, for which an initial guess of cut-off is $s_\ast \simeq \sqrt{2} \gamma$, at which by (19) we have $I_{s_\ast}(\gamma) \propto 1/\sqrt{\gamma}$. However, in practice the lower cut-off is sufficient, as in the summations there are multiple of Bessel functions rather than a single one, making the convergence to the desired significant digits faster. As examples the numerical evaluations of elements $V_{00}$, $V_{01}$, and $V_{11}$ by (27) and (28) are presented in Fig. 3, by the choice $A = 1$ and for $3 \times 3$ and $10 \times 10$ lattices. The results are generated on a desktop PC in a reasonable time. As expected by $P_n \delta$, except $V_{00}$ the two other elements develop maximum. As mentioned earlier, the appearance of maximum in the elements may lead to minimum in the energy spectrum, in connection to a first order phase transition. Further studies based on more numerical results, specially those related to the energy spectrum, will be presented in a separate paper.

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