Verification of the GGS conjecture for $\mathfrak{sl}(n), n \leq 12$.

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Abstract

In the 1980’s, Belavin and Drinfeld classified non-unitary solutions of the classical Yang-Baxter equation (CYBE) for simple Lie algebras [1]. They proved that all such solutions fall into finitely many continuous families and introduced combinatorial objects to label these families, Belavin-Drinfeld triples. In 1993, Gerstenhaber, Giaquinto, and Schack attempted to quantize such solutions for Lie algebras $\mathfrak{sl}(n)$. As a result, they formulated a conjecture stating that certain explicitly given elements $R \in \text{Mat}_n(\mathbb{C}) \otimes \text{Mat}_n(\mathbb{C})$ satisfy the quantum Yang-Baxter equation (QYBE) and the Hecke condition [2]. Specifically, the conjecture assigns a family of such elements $R$ to any Belavin-Drinfeld triple of type $A_{n-1}$. Until recently, this conjecture has only been known to hold for $n \leq 4$. In 1998 Giaquinto and Hodges checked the conjecture for $n = 5$ by direct computation using Mathematica [3]. Here we report a computation which allowed us to check that the conjecture holds for $n \leq 12$. The program is included which prints an element $R$ for any triple and checks that $R$ satisfies the QYBE and Hecke conditions.

1 Belavin-Drinfeld triples

Let $(e_i), 1 \leq i \leq n$, be a basis for $\mathbb{C}^n$. Set $\Gamma = \{e_i - e_{i+1}: 1 \leq i \leq n-1\}$. We will use the notation $\alpha_i \equiv e_i - e_{i+1}$. Let $(,)$ denote the inner product on $\mathbb{C}^n$ having $(e_i)$ as an orthonormal basis.

**Definition 1.1** A Belavin-Drinfeld triple of type $A_{n-1}$ is a triple $(\tau, \Gamma_1, \Gamma_2)$ where $\Gamma_1, \Gamma_2 \subset \Gamma$ and $\tau: \Gamma_1 \to \Gamma_2$ is a bijection, satisfying two conditions:

(a) $\forall \alpha, \beta \in \Gamma_1, (\tau \alpha, \tau \beta) = (\alpha, \beta)$.
(b) $\tau$ is nilpotent: $\forall \alpha \in \Gamma_1, \exists k \in \mathbb{N}$ such that $\tau^k \alpha \notin \Gamma_1$.

We employ three isomorphisms of Belavin-Drinfeld triples:

a) Any triple $(\tau, \Gamma_1, \Gamma_2)$ is isomorphic to the triple $(\tau', \Gamma_1', \Gamma_2')$ obtained as follows: $\Gamma_1' = \{\alpha_m: \alpha_{n-m} \in \Gamma_1\}, \tau'(\alpha_m) = \alpha_k$ where $\tau(\alpha_{n-m}) = \alpha_{n-k}$.

b) Any triple $(\tau, \Gamma_1, \Gamma_2)$ is isomorphic to the triple $(\tau^{-1}, \Gamma_2, \Gamma_1)$.

c) The product of isomorphisms (a), (b).

Modulo these isomorphisms, we found all Belavin-Drinfeld triples for $n \leq 13$ by computer. The number of such triples is given below:

| $n$ | # of triples | $n$ | # of triples | $n$ | # of triples |
|-----|--------------|-----|--------------|-----|--------------|
| 2   | 1            | 3   | 2            | 4   | 4            |
| 3   | 2            | 6   | 41           | 10  | 10434        |
| 4   | 4            | 7   | 161          | 11  | 45069        |
| 5   | 13           | 8   | 611          | 12  | 201300       |
| 6   | 9            | 9   | 2490         | 13  | 919479       |
2 The GGS conjecture

Let \( \mathfrak{g} = \mathfrak{sl}(n) \) be the Lie algebra of \( n \times n \) matrices of trace zero. Set \( \mathfrak{h} \subset \mathfrak{g} \) to be the subset of diagonal matrices. Elements of \( \mathbb{C}^n \) define linear functions on \( \mathfrak{h} \) by \( (\sum_i \lambda_i e_i)(\sum_i a_i e_{ii}) = \sum_i \lambda_i a_i \). Set \( \sigma = \sum_{1 \leq i,j \leq n} e_{ij} \otimes e_{ji} \), and let \( P \) be the orthogonal projection of \( \sigma \) to \( \mathfrak{g} \otimes \mathfrak{g} \) with respect to the form \( \langle X,Y \rangle = Tr(XY) \) on \( \text{Mat}_n(\mathbb{C}) \). Then, set \( P^0 \) to be the projection of \( P \) to \( \mathfrak{h} \otimes \mathfrak{h} \). Thus \( P^0 = \sum_i \frac{1}{n} e_{ii} \otimes e_{ii} - \sum_{i \neq j} \frac{1}{n} e_{ij} \otimes e_{ji} \).

For any Belavin-Drinfeld triple, consider the following equations:

\[
\begin{align*}
    r_{12}^0 + r_{21}^0 &= P^0, \\
    \forall \alpha \in \Gamma_1, (\tau \alpha \otimes 1) r^0 + (1 \otimes \alpha) r^0 &= 0.
\end{align*}
\]

Belavin and Drinfeld showed that nonunitary solutions of the CYBE correspond to solutions of these equations. Define \( r^0 = r^0 - P^0/2 \).

The GGS conjecture gives an explicit form of a matrix \( R \in \text{Mat}_n(\mathbb{C}) \otimes \text{Mat}_n(\mathbb{C}) \) for any given triple and any given \( r^0 \in \mathfrak{h} \otimes \mathfrak{h} \) satisfying (2.1), (2.2) as follows:

Set \( \tilde{\Gamma}_1 = \{ v \in \text{Span}(\Gamma_1) : v = e_i - e_j, 0 \leq i < j \leq n, i \neq j \} \), and define \( \tilde{\Gamma}_2 \) similarly. Then, extend \( \tau \) to a map \( \tilde{\Gamma}_1 \rightarrow \tilde{\Gamma}_2 \) so that \( \tau \) is additive, i.e. \( \tau(a + b) = \tau(a) + \tau(b) \) provided \( a, b, (a + b) \in \tilde{\Gamma}_1 \). Further, define \( \alpha < \beta \) if \( \alpha \in \tilde{\Gamma}_1 \) and \( \tau^k(\alpha) = \beta \), for some \( k \geq 1 \). It is clear from the conditions on \( \tau \) that this means, given \( \alpha = \alpha_i + \ldots + \alpha_{i+p} \), that \( \beta = \alpha_j + \ldots + \alpha_{j+p}, \ 0 \leq p \leq n-2, 1 \leq i, j \leq n, i \neq j \). Assume \( \beta = \tau^k(\alpha), k \geq 1 \). If, in this case, \( \tau^k(\alpha) = \alpha_{j+p} \), that is, \( \tau^k \) sends the left endpoint of \( \alpha \) to the right endpoint of \( \beta \), then define \( \text{sign}(\alpha, \beta) = (-1)^p \). Otherwise, set \( \text{sign}(\alpha, \beta) = 1 \).

We will use the notation \( x \wedge y = \frac{1}{2}(x \otimes y - y \otimes x) \). Furthermore, for all matrices \( x \in \text{Mat}_n(\mathbb{C}) \otimes \text{Mat}_n(\mathbb{C}) \) we will use the notation \( x = \sum_{i,j,k,l} x_{ij}^{kl} e_{ij} \otimes e_{kl} \). Let \( q \) be indeterminate and set \( \hat{q} \equiv q - q^{-1} \). Finally, for any \( \alpha = e_i - e_j \), set \( e_\alpha = e_{ij} \), and say \( \alpha > 0 \) if \( i < j \), otherwise \( \alpha < 0 \). Now, we can define the matrix \( R \) as follows:

\[
\begin{align*}
    a &= 2 \sum_{\alpha, \beta > 0, \alpha < \beta} \text{sign}(\alpha, \beta) e_{-\alpha} \wedge e_\beta, \\
    c &= \sum_{\alpha > 0} e_{-\alpha} \wedge e_\alpha, \\
    \epsilon &= \alpha c + \alpha c + \alpha c, \\
    \tilde{a} &= \sum_{i,j,k,l} a_{ik}^j q^{-k} q^{-i} q^{-l}, \\
    R_s &= q \sum_{i} e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ii} \otimes e_{jj} + \hat{q} \sum_{i > j} e_{ij} \otimes e_{ji}, \\
    R &= q^{r^0} (R_s + \hat{q} \tilde{a}) q^{r^0}.
\end{align*}
\]

**Conjecture 2.1 (GGS)** The matrix \( R \) satisfies the quantum Yang-Baxter equation, \( R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} \), and \( PR \) satisfies the Hecke relation, \( (PR - q)(PR + q^{-1}) = 0 \).

3 Checking GGS by computer

We checked the GGS conjecture through a program written in C, which takes as input any list of Belavin-Drinfeld triples. For each triple, it finds a valid \( r^0 \), constructs the matrix \( R \),
and checks the QYBE and Hecke conditions. Following is a more detailed description of the procedure.

We will use the notation $\tau(\alpha) = 0$ if $\alpha \notin \tilde{\Gamma}_1$. Given a triple, the first step is to find an appropriate $\tilde{r}^0$. We rewrite the equations (2.1), (2.2) as follows:

$$\tilde{r}_{12}^0 + \tilde{r}_{21}^0 = 0, \quad \forall \alpha \in \Gamma_1, ((\alpha - \tau \alpha) \otimes 1)\tilde{r}^0 = \frac{1}{2}((\alpha + \tau \alpha) \otimes 1)P^0.$$

(3.1)

(3.2)

As before, we view elements of $\mathbb{C}^n$ as linear functions on $\mathfrak{h}$. Then, it is easy to check that $(\alpha_i)_{1 \leq i \leq n}$ and $(\alpha_i - \tau \alpha_i)_{1 \leq i \leq n-1}$ are bases of $\mathfrak{h}^*$. Let $(g_i)$ and $(f_i)$ be dual to the bases $(\alpha_i)$ and $(\alpha_i - \tau \alpha_i)$, respectively. Then, if we view $\tilde{r}^0$ as an element of $\text{Mat}_{n-1}(\mathbb{C})$ in the basis $(f_i)$, it is clear that $\tilde{r}^0 = (b_{ij})$ where $b_{ij} = \frac{1}{2}(\alpha_i + \tau \alpha_i, \alpha_j - \tau \alpha_j), i \in \Gamma_1$, where the inner product is the same we defined earlier on $\mathbb{C}^n$, and $b_{ij} = -b_{ji}, i \notin \Gamma_1, j \in \Gamma_1$. Then, the free components of $\tilde{r}^0$ are those $b_{ij}$ with $i, j \notin \Gamma_1, i < j$, which determine those $b_{ij}, i, j \notin \Gamma_1, i > j$ since $\tilde{r}^0$ is skew-symmetric. Thus, the dimension of the space of all valid $\tilde{r}^0$ is $\binom{n-1}{2}$.

The computer program merely chooses $b_{ij} = 0$ whenever $i, j \notin \Gamma_1$. It is known that it is sufficient to consider one element from the family of possible $\tilde{r}^0$ in verifying the GGS conjecture. Namely, this follows from

**Proposition 3.1** If $R$ of the form (3.3) satisfies the QYBE and $PR$ satisfies the Hecke relation for a given $\tilde{r}^0$ satisfying (3.1), (3.2), then for any other solution $\tilde{r}^0, \tilde{r}'$ of (3.1), (3.2), $q^rRq^r'$ also satisfies the QYBE and $Pq^rRq^r'$ satisfies the Hecke relation.

**Proof.** It is clear that $Pq^rRq^r' = q^{r_21}PRq^{r'}$. Since $r_{21}' = -r'$ by (3.1), the Hecke relation may be rewritten as $q^{r'}(PR - q)(PR + q^{-1})q^{r'} = 0$, which is true iff $PR$ satisfies the Hecke relation.

To see that $q^{r'}Rq^{r'}$ satisfies the QYBE, we take the following steps. By (3.2),

$$((\alpha - \tau \alpha) \otimes 1)r' = 0.$$  

(3.3)

Suppose that $r' = \sum_i a_i \otimes b_i$ where the $b_i$ are linearly independent. By (1.3), we know that $\alpha(a_i) = \beta(a_i)$ whenever $\alpha < \beta$. Then we consider the commutator $[a_i \otimes 1 + 1 \otimes a_i, R] = [a_i \otimes 1 + 1 \otimes a_i, q^{R_0}R^0 + \hat{q}^2 \hat{q}^0 q^0]$. First note that $[a_i, e_\alpha] = \alpha(a_i)e_\alpha$ for any $a_i \in \mathfrak{h}$. Then, it is clear $[a_i \otimes 1 + 1 \otimes a_i, q^{R_0}R^0] = [a_i \otimes 1 + 1 \otimes a_i, \sum_{i > j} d_{ij} e_{ij} \otimes e_{ji}] = \sum_{i > j} d_{ij} (\alpha(a_i) - \alpha(a_j))e_{ij} \otimes e_{ji} = 0$ for the appropriate coefficients $d_{ij}$. Now, we see that

$$[a_i \otimes 1 + 1 \otimes a_i, q^{R_0} \hat{q}^0 q^0] = [a_i \otimes 1 + 1 \otimes a_i, \sum_{\alpha, \beta > 0, \alpha < \beta} (f_{\alpha, \beta} e_{-\alpha} \otimes e_{\beta} + g_{\alpha, \beta} e_{\beta} \otimes e_{-\alpha})]$$

$$= \sum_{\alpha, \beta > 0, \alpha < \beta} (\beta(a_i) - \alpha(a_i))(f_{\alpha, \beta} e_{-\alpha} \otimes e_{\beta} + g_{\alpha, \beta} e_{\beta} \otimes e_{-\alpha}) = 0.$$

This implies that $r' \in \Lambda^2K$ where $K$ is the space of symmetries of $R$, that is, $K = \{x \in \text{Mat}_n(\mathbb{C}) : [1 \otimes x + x \otimes 1, R] = 0\}$. Furthermore, it is well-known and easy to check that if $x \in \Lambda^2K$ and $R$ satisfies the QYBE, then $e^x Re^x$ also satisfies the QYBE. Thus, in our case, we have proved that $q^rRq^r'$ satisfies the QYBE. The proposition is proved. □
Now, given the chosen \( r^0 \) in the basis \( (f_i) \), the computer program changes bases to \( (g_i) \). This is accomplished via the transformation \([r^0]_{(g_i)} = \left( [(1 - \tau)^{-1}]_{(\alpha_i)} \right)^T [r^0]_{(f_i)} \left( (1 - \tau)^{-1} \right)_{(\alpha_i)}\), where \((1 - \tau)\) is considered to be a linear transformation on \( h^* \), with \((1 - \tau)\alpha_i = \alpha_i - \tau \alpha_i \). Denote this new matrix by \( (b'_{ij}) \).

Then, the computer program obtains the matrix \([\tilde{r}^0]_{(e_{ii})} \in Mat_n(\mathbb{C}) \) from this matrix by two quick transformations. First it finds the intermediate matrix \((b''_{ij}) = [\tilde{r}^0]_{(e_{ii}), (g_i)} \in Mat_{n \times (n - 1)}(\mathbb{C}) \) by \( b''_{i1} = \frac{1}{n}((n - 1)b'_{i1} + (n - 2)b'_{i2} + \ldots + b'_{i(n - 1)}) \), and the other terms follow easily. The same technique on the other side finally gives \([\tilde{r}^0]_{(e_{ii})}\).

Once \( \tilde{r}^0 \) is obtained, the computer constructs the matrix \( R \in Mat_n(\mathbb{C}) \otimes Mat_n(\mathbb{C}) \) in the basis \( e_{ij} \otimes e_{kl}, 1 \leq i, j, k, l \leq n \). First it computes \( a, c, \) and \( \epsilon \) by (2.3). Then, formulas (2.4), (2.5) are implemented for each entry separately. Elements \( x \in Mat_n(\mathbb{C}) \otimes Mat_n(\mathbb{C}) \) are implemented as 3-dimensional arrays \( (x_{ik}') \), since all matrices presented in the GGS conjecture take the form \( \sum_{i,j,k} x'_{ik} e_{ij} \otimes e_{k,i+k-j} \). Polynomials in \( q \) are implemented as structures containing two arrays of integers, one for positive and one for negative powers of \( q \). The sizes of the arrays are determined in the input of the program.

The computer checks the QYBE and Hecke conditions in the following manner. For the QYBE condition, the corresponding entries of \( R_{12}R_{13}R_{23} \) and \( R_{23}R_{13}R_{12} \) are computed and compared individually; both take the form \( \sum_{i,j,k,l,m} d_{ikm}^{jl} e_{ij} \otimes e_{kl} \otimes e_{m,i+k+m-j-l} \). The same method is applied to the Hecke condition with matrices \( \sum_{i,j,k} d_{ik}^{j} e_{ij} \otimes e_{k,i+k-j} \). Explicitly, if \( R = \sum_{i,j,k} r_{ik}^{j} e_{ij} \otimes e_{k,i+k-j} \), the QYBE and Hecke conditions become, respectively:

\[
\sum_p r_{ik}^{j+i-p} r_{k+i-p,m}^{j} = \sum_p r_{ik}^{p} r_{m+k+i-p}^{j+l-p} r_{j+l-p,m}^{j}, \quad \forall i, j, k, l, m. \tag{3.4}
\]

\[
\sum_l r_{k,l}^{j} r_{k+l}^{j} = \delta_{ij} + \hat{q} r_{ki}^{j}, \quad \forall i, j, k. \tag{3.5}
\]

Then, the computer prints the matrices \( \tilde{r}^0 \) and \( R \) and reports whether or not the conditions passed.

After generating all Belavin-Drinfeld triples for \( n \leq 13 \) as described in the previous section, all tests were performed on each triple where \( n \leq 12 \) with this procedure, all of which passed. Thus, by application of the previous proposition, we have the following result:

**Proposition 3.2** The GGS conjecture is true for Lie algebras \( \mathfrak{sl}(n) \) with \( n \leq 10 \).

The computer program is included with this paper, with instructions on usage included with the program itself.

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