THREE FAMILIES OF \( q \)-LOMMEL POLYNOMIALS

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Abstract. Three \( q \)-versions of Lommel polynomials are studied. Included are explicit representations, recurrences, continued fractions, and connections to associated Askey–Wilson polynomials. Combinatorial results are emphasized, including a general theorem when \( R \) moments coincide with orthogonal polynomial moments. The combinatorial results use weighted Motzkin paths, Schröder paths, and parallelogram polyominoes.

1. Introduction

Lehmer [26] used the following Bessel function identity to study zeros of Bessel functions

\[
\frac{J_{\nu+1}(x)}{J_{\nu}(x)} = 2 \sum_{n=1}^{\infty} \sigma_{2n}(\nu)x^{2n-1}.
\]

In this identity \( \sigma_{2n}(\nu) \) is the \( 2n \)-th power sum of the inverses of the positive zeros \( j_{\nu,k} \) of \( J_{\nu}(x) \),

\[
\sigma_{2n}(\nu) = \sum_{k=1}^{\infty} j_{\nu,k}^{-2n}.
\]

Lehmer noted that \( \sigma_{2n}(\nu) \) is a rational function of \( \nu \), with a predictable denominator, and a numerator with nonnegative coefficients. Kishore [19] proved Lehmer’s positivity conjecture. Lalanne [24, Prop. 3.6], [25, Th. 4.7] proved \( q \)-versions of Kishore’s result using weighted binary trees and also weighted Dyck paths.

The Lommel polynomials are orthogonal with respect to the linear functional

\[
\mathcal{L}(P(x)) = \sum_{k=1}^{\infty} \left( P\left(j_{\nu,k}^{-1}\right) + P\left(-j_{\nu,k}^{-1}\right)\right) j_{\nu,k}^{-2}.
\]

Thus \( \sigma_{2n}(\nu) \) in (1.2) is effectively the \( (2n-2) \)-th moment for the Lommel polynomials, while (1.1) is the Lommel moment generating function.

The purpose of this paper is to study two sets of \( q \)-Lommel orthogonal polynomials, whose moment generating functions are quotients of \( q \)-Bessel functions. We also consider another set of polynomials, which is a type \( R \) polynomial, and whose moment generating function is again a quotient of \( q \)-Bessel functions. Koelink and Van Assche [23] and Koelink [22] analytically studied two of these \( q \)-Lommel polynomials. In this paper we concentrate on the combinatorial aspect of these three \( q \)-Lommel polynomials.

There are combinatorial results on the quotient of Bessel functions and the quotient of \( q \)-Bessel functions. Delest and Fédou [9] showed that a generating function for parallelogram polyominoes can be written as a ratio of Jackson’s third \( q \)-Bessel functions. Bousquet-Mélou and Viennot [4] generalized their result by adding one more parameter. A recounting of the history of the combinatorics of the \( q \)-analogue of the quotient of Bessel functions may be found.

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in [3, Sec. 1] (see also [25, Sec. 4]). It includes results by Klarner and Rivest [20, see (19)], Delest and Fedou [9, Fedou [12], Lalanne [21, 25], Brak and Guttmann [5], Bousquet-Mélou and Viennot [4], and Barcucci et al. [1, Cor. 3.5], [2, Th. 4.3, Th. 5.3].

In this paper we put these results in perspective by relating them to $q$-Lommel polynomials. The moment generating function has a continued fraction expansion. Using the general theory of orthogonal and type $R_I$ polynomials we give finite versions of the infinite continued fractions. We show that a generating function for bounded diagonal parallelogram polyominoes is given by a ratio of $q$-Lommel polynomials, which is a finite version of the result of Bousquet-Mélou and Viennot [4].

Even though the Lommel polynomials have a hypergeometric representation as a $\genfrac{[}{]}{0pt}{}{2}{3}$, they do not appear in the Askey scheme. In this paper we rectify this, by realizing two sets of $q$-Lommel polynomials as limiting cases of associated Askey–Wilson polynomials. One may ask for an associated Askey scheme which contains this limiting case.

The paper is organized in the following way. In Section 2 we define the three sets of $q$-Lommel polynomials using three-term recurrence relations. The classical connection between these polynomials and $q$-Bessel functions is given in Section 3. The associated Askey–Wilson polynomials are reviewed in Section 4, along with explicit limiting cases to the $q$-Lommel polynomials, see Theorems 4.7 and 4.8. In Section 5 we independently prove the continued fraction expansions for the moment generating functions, and give two surprising equalities of continued fractions in Corollary 5.6 and Theorem 5.12. Combinatorial interpretations of these continued fractions are given in Section 6, see Theorem 6.9 and Corollary 6.10. A general combinatorial result for the concurrence of type $R_I$ moments and orthogonal polynomials moments is given in Section 7, see Theorem 7.2. In Section 8 we propose some open problems.

We use the standard notations for both hypergeometric series and basic hypergeometric series [14].

2. $q$-Lommel Polynomials

In this section we give the defining recurrence relations for the Lommel, the classical $q$-Lommel, the even-odd $q$-Lommel, and the type $R_I$ $q$-Lommel polynomials.

Definition 2.1. The monic Lommel polynomials $h_n(x;c)$ are defined by

$$h_{n+1}(x;c) = xh_n(x;c) - \frac{1}{(c+n)(c+n-1)}h_{n-1}(x;c), \quad h_{-1}(x;c) = 0, \quad h_0(x;c) = 1.$$ 

We consider three versions of $q$-Lommel polynomials.

Definition 2.2. [15 §14.4] The classical $q$-Lommel polynomials are defined by

$$h_{n+1}(x;c,q) = xh_n(x;c,q) - \lambda_nh_{n-1}(x;c,q), \quad h_{-1}(x;c,q) = 0, \quad h_0(x;c,q) = 1,$$

where

$$\lambda_n = \frac{cq^n}{(1-cq^n-1)(1-cq^n)}.$$ 

Definition 2.3. The even-odd $q$-Lommel polynomials are defined by

$$p_{n+1}(x;c,q) = xp_n(x;c,q) - \lambda_np_{n-1}(x;c,q), \quad p_{-1}(x;c,q) = 0, \quad p_0(x;c,q) = 1,$$

where

$$\lambda_{2n} = \frac{cq^{2n+1}}{(1-cq^{2n})(1-cq^{2n+1})}, \quad \lambda_{2n+1} = \frac{cq^{2n}}{(1-cq^{2n})(1-cq^{2n+1})}.$$
Note that
\[
\lim_{q \to 1} (1 - q)^n h_n(x/(1 - q); q^c, q) = h_n(x; c), \quad \lim_{q \to 1} (1 - q)^n p_n(x/(1 - q); q^c, q) = h_n(x; c),
\]
so that each polynomial may be considered as a $q$-analogue of the classical Lommel polynomials.

**Definition 2.4.** The type $R_I$ $q$-Lommel polynomials are defined by
\[
r_{n+1}(x; c, q) = (x - b_n)r_n(x; c, q) - xa_nr_{n-1}(x; c, q), \quad r_{-1}(x; c, q) = 0, \quad r_0(x; c, q) = 1,
\]
where
\[
b_n = \frac{q^n}{1 - cq^n}, \quad a_n = \frac{cq^{2n-1}}{(1 - cq^{n-1})(1 - cq^n)}.
\]

Note that if
\[
\hat{r}_n(x; c) = \lim_{q \to 1} (1 - q)^{2n} r_n(x/(1 - q)^2; q^c, q),
\]
then
\[
(2.2) \quad \hat{r}_{n+1}(x; c) = x\hat{r}_n(x; c) - \frac{x}{(c + n - 1)(c + n)} \hat{r}_{n-1}(x; c).
\]

The polynomials $\hat{r}_n(x; c)$ in (2.2) are closely related to the monic Lommel polynomials. For example it is known that their moments are the same, see (7.3).

Koelink and Van Assche study the even-odd and the type $R_I$ $q$-Lommel polynomials in [22, Sec. 4], and Koelink continues this analytic study in [22].

Orthogonality relations for the classical $q$-Lommel are in [15, Theorem 14.4.3], while those for the even-odd $q$-Lommel and the type $R_I$ $q$-Lommel are in [23, Theorem 4.2] and [23, Theorem 3.4].

3. $q$-Bessel functions and $q$-Lommel polynomials

In this section we give the recurrence relation which connects $q$-Bessel functions to the classical $q$-Lommel polynomials and the type $R_I$ $q$-Lommel polynomials. This was the original motivation for Lommel polynomials.

**Definition 3.1.** The Bessel functions $J_\nu(x)$ are defined by
\[
J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu + 1)} \sum_{n \geq 0} \frac{(-z^2/4)^n}{n!(\nu + 1)n}.
\]

**Definition 3.2.** [6, p.188, (6.2)] The classical Lommel polynomials $R_{n,\nu}(z)$ are (non-monic) polynomials in $z^{-1}$ defined by $R_{0,\nu}(z) = 1$, $R_{1,\nu}(z) = 2\nu/z$, and
\[
(3.1) \quad R_{n+1,\nu}(z) = \frac{2(n + \nu)}{z} R_{n,\nu}(z) - R_{n-1,\nu}(z), \quad n \geq 1.
\]
Equivalently,
\[
h_n(x; c) = R_{n,0}(2/x)/(c)_n.
\]

The connection of Bessel functions to Lommel polynomials is the following proposition.

**Proposition 3.3.** [6, p.187] The Bessel functions and the classical Lommel polynomials are related by the recurrence
\[
(3.2) \quad J_{\nu+n}(z) = R_{n,\nu}(z)J_\nu(z) - R_{n-1,\nu+1}(z)J_{\nu-1}(z).
\]
We rescale these Laurent polynomials to obtain polynomials \( J_{\nu}^{(1)}(z; q) \) and second \( q \)-Bessel function \( J_{\nu}^{(2)}(z; q) \) are defined by

\[
\begin{align*}
J_{\nu}^{(1)}(z; q) &= \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} (z/2)^{\nu} \phi_1 (0; 0; q^{\nu+1}, q, -z^2/4), \\
J_{\nu}^{(2)}(z; q) &= \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} (z/2)^{\nu} \phi_1 (-; q^{\nu+1}; q, -q^{\nu+1}z^2/4).
\end{align*}
\]

In this paper we consider only the first and third \( q \)-Bessel function, as the second \( q \)-Bessel can be obtained from the first by changing \( q \) to \( q^{-1} \). Recall that we consider formal power series in \( z \), and have no restriction on \( q \).

**Proposition 3.5.** \([15, (14.4.1)]\] The first \( q \)-Bessel functions satisfy

\[
q^{m+\nu} J_{\nu}^{(1)}(x; q) = R_{n, \nu}^{(1)}(x; q) J_{\nu}^{(1)}(x; q) - R_{n-1, \nu+1}^{(1)}(x; q) J_{\nu-1}^{(1)}(x; q).
\]

where \( R_{n, \nu}^{(1)}(x; q) = 1, R_{1, \nu}^{(1)}(x; q) = 2(1 - q^{\nu+1})/x \), and

\[
\frac{2}{x} (1 - q^{\nu+1}) R_{n, \nu}^{(1)}(x; q) = R_{n+1, \nu}^{(1)}(x; q) + q^{n+\nu} R_{n-1, \nu}^{(1)}(x; q), \quad n \geq 1.
\]

Again we need a rescaling to obtain the classical \( q \)-Lommel polynomials,

\[
h_n(x; c, q) = R_{n+1, \nu}^{(1)}(2/x; q)/(q^\nu; q)_n, \quad c = q^\nu.
\]

**Definition 3.6.** The Jackson’s third \( q \)-Bessel functions \( J_{\nu}^{(3)}(z; q) \) are defined by

\[
J_{\nu}^{(3)}(z; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} z^\nu \phi_1 (0; 0; q^{\nu+1}, q, z^2).
\]

Define the Laurent polynomials \( R_{m, \nu}^{(3)}(z; q) \) by

\[
R_{m+1, \nu}^{(3)}(z; q) = (z + z^{-1}(1 - q^{\nu+m})) R_{m, \nu}^{(3)}(z; q) - R_{m-1, \nu}^{(3)}(z; q).
\]

We rescale these Laurent polynomials to obtain polynomials

\[
r_{n}^{(3)}(x; c, q) := \frac{x^{n/2}}{(q^\nu; q^{-1})_n} R_{n, \nu}^{(3)}(x^{-1/2}; q^{-1}), \quad c = q^\nu.
\]

Then \( r_{n}^{(3)}(x; c, q) \) are the type \( R_I \) polynomials defined by \( r_{-1}^{(3)}(x; c, q) = 0, r_{0}^{(3)}(x; c, q) = 1 \), and

\[
r_{n+1}^{(3)}(x; c, q) = (x - \hat{b}_n) r_{n}^{(3)}(x; c, q) - x \hat{a}_n r_{n-1}^{(3)}(x; c, q), \quad n \geq 0,
\]

where

\[
\hat{b}_n = \frac{cq^n}{1 - cq^n}, \quad \hat{a}_n = \frac{c^2 q^{2n-1}}{(1 - cq^n)(1 - cq^n)}.
\]

Using the recurrences one can easily check that

\[
r_n(x; c, q) = \frac{r_n^{(3)}(cx; c, q)}{c^n},
\]

where \( r_n(x; c, q) \) are the type \( R_I \) \( q \)-Lommel polynomials \( r_n(x; c, q) \) in Definition 2.3.

Koelink and Swarttouw [21] (4.12)) showed that the third \( q \)-Bessel functions satisfy the following property analogous to (3.3) and (3.5).

**Proposition 3.7.** The third \( q \)-Bessel functions satisfy

\[
J_{\nu+m}^{(3)}(z; q) = R_{m, \nu}^{(3)}(z; q) J_{\nu}^{(3)}(z; q) - R_{m-1, \nu+1}^{(3)}(z; q) J_{\nu-1}^{(3)}(z; q).
\]
Koelink and Swarttouw [21 (4,24)] also showed that

\begin{equation}
\lim_{m \to \infty} z^m R^{(3)}_{m,\nu}(z; q) = \frac{(q; q)_\infty z^{1-\nu}}{(z^2; q)_\infty} J^{(3)}_{\nu-1}(z; q),
\end{equation}

which implies

\begin{equation}
\lim_{m \to \infty} \frac{R^{(3)}_{m,\nu+2}(z; q)}{R^{(3)}_{m+1,\nu+1}(z; q)} = \frac{J^{(3)}_\nu(z; q)}{J^{(3)}_{\nu+1}(z; q)}.
\end{equation}

By (3.7) and (3.11) we have

\begin{equation}
\frac{J^{(3)}_{\nu+1}(x^{1/2}; q^{-1})}{J^{(3)}_{\nu}(x^{1/2}; q^{-1})} = \lim_{n \to \infty} \frac{-q^{\nu+1} r_n^{(3)}(x^{-1}; q^{\nu+2}, q)}{x^{1/2}(1 - q^{\nu+1}) r_n^{(3)}(x^{-1}; q^{\nu+1}, q)}.
\end{equation}

The \(q\)-Bessel function relation for the even-odd \(q\)-Lommel polynomials which corresponds to Proposition 3.5 is given in [23, Proposition 4.1].

4. \(q\)-Lommel Polynomials and the Askey Scheme

The \(q\)-Lommel polynomials do not appear in the Askey scheme. In this section we realize both the classical \(q\)-Lommel and the even-odd \(q\)-Lommel polynomials as limiting cases of the associated Askey–Wilson polynomials, see Theorems 4.7 and 4.8. We then use results of Ismail and Masson [16] to give explicit formulas for each polynomial. Finally we prove that the moments for even-odd \(q\)-Lommel and the type \(R_t\) \(q\)-Lommel agree, see Theorem 4.14.

An explicit formula for the Lommel polynomial \(h_n(x; c)\) is

\[ h_n(x; c) = x^n F_3(-n/2, (1 - n)/2; c, 1 - c - n, -n; -4/x^2). \]

In this section we give explicit formulas for our three families of \(q\)-Lommel polynomials. The classical \(q\)-Lommel polynomials have a corresponding single sum formula [15, Theorem 14.4.1]:

\[ h_n(x; c, q) = \frac{1}{(c; q)_n} \sum_{j=0}^{[n/2]} (-1)^j (c, q; q)_n^{-j} (q, c, q; q)_{n-2j} x^{n-2j} q^{j(j-1)}. \]

Here are the main results for the even-odd \(q\)-Lommel polynomials.

**Theorem 4.1.** The even even-odd \(q\)-Lommel polynomials have the explicit formula

\[
p_{2n}(x; c, q) = (-1)^n \frac{q^{n^2}}{(c; q)_{2n}} \sum_{k=0}^{n} \frac{(q^{-n}, cq^n, c; q)_k}{(q; q)_k} q^{k+2k^2} x^{2k+1}\times \sum_{s=0}^{n-k} \frac{(c q^{-s+k-1}; q)_s}{(q; q)_s} \frac{1 - c q^{k-1+2s}}{1 - c q^{k-1}} \frac{(c q^{n+k}, q^{k-n}, c; q)_s}{(q^{-n}, cq^n, c; q)_s} c^s q^{-sk+s(s-1)}.
\]

**Theorem 4.2.** The odd even-odd \(q\)-Lommel polynomials have the explicit formula

\[
p_{2n+1}(x; c, q) = (-c)^n \frac{q^{n^2+(n+1)/2}}{(c q; 2n)} \sum_{k=0}^{n} \frac{(q^{-n}, cq^{n+1}, c; q)_k}{(q; q)_k} q^{-k^2} x^{2k+1}\times \sum_{s=0}^{n-k} \frac{(c q^{k}; q)_s}{(q; q)_s} \frac{1 - c q^{k+2s}}{1 - c q^{k}} \frac{(c q^{n+k+1}, q^{k-n}, q^{k+1}; q)_s}{(q^{-n}, cq^{n+1}, c; q)_s} c^s q^{-(3k+2)s-s(s-1)}.
\]
Proof. First we write the even even-odd polynomials as orthogonal polynomials in $x^2$ using the odd-even trick. Then we realize the new polynomials as limiting cases of associated Askey–Wilson polynomials, for which explicit formulas are known. The same method will work for the odd even-odd polynomials.

We begin with the associated Askey–Wilson polynomials. The monic Askey–Wilson polynomials satisfy

$$p_{n+1}(x) = (x - b_n)p_n(x) - \lambda_n p_{n-1}(x), \quad n \geq 1,$$

where

$$b_n = \frac{1}{2}(a + a^{-1} - A_n - C_n), \quad \lambda_n = \frac{1}{4}A_{n-1}C_n,$$

$$A_n = \frac{(1 - abq^n)(1 - acq^n)(1 - adq^n)(1 - abcdq^{n-1})}{a(1 - abcdq^{2n-1})(1 - abcdq^{2n})},$$

$$C_n = \frac{a(1 - q^n)(1 - bcaq^{n-1})(1 - bdq^{n-1})(1 - cdq^{n-1})}{(1 - abcdq^{2n-2})(1 - abcdq^{2n-1})}.$$

The associated Askey–Wilson polynomials replace $q^n$ by $aq^n$ in the three-term recurrence relation.

**Definition 4.3.** The associated Askey–Wilson polynomials $p_n^{(\alpha)}(x)$ are defined as a solution to

$$p_{n+1}^{(\alpha)}(x) = (x - b_n^{(\alpha)})p_n^{(\alpha)}(x) - \lambda_n^{(\alpha)}p_{n-1}^{(\alpha)}(x), \quad n \geq 1,$$

$$b_n^{(\alpha)} = \frac{1}{2}(a + a^{-1} - A_n^{(\alpha)} - C_n^{(\alpha)}), \quad \lambda_n^{(\alpha)} = \frac{1}{4}A_{n-1}^{(\alpha)}C_n^{(\alpha)},$$

$$A_n^{(\alpha, q)} = \frac{(1 - abaq^n)(1 - acaq^n)(1 - adaq^n)(1 - abdaq^{n-1})}{a(1 - abdaq^{2n-1})(1 - abdaq^{2n})},$$

$$C_n^{(\alpha, q)} = \frac{a(1 - q^n)(1 - bcaaq^{n-1})(1 - bdqq^{n-1})(1 - cdaq^{n-1})}{(1 - abdaq^{2n-2})(1 - abdaq^{2n-1})}.$$

There are two linearly independent solutions to (4.2), depending on the initial conditions. Ismail and Rahman [15, (4.15), (8.9)] gave these two independent solutions as double sums, the inner sum a very well poised

**Theorem 4.4.** Two linearly independent solutions $\psi_n^{(\alpha, \epsilon)}(x, q), \epsilon = 1, 2$ to (4.2) are given by

$$\psi_n^{(\alpha, \epsilon)}(x; q) = K_n \sum_{k=0}^{n} \frac{(q^{-n}, abcaq^{n-1}, abcaq^{n-1}, az/az)_{k}^{q^{-n}}}{(q, abca, ada, abca/q, q; q)_{k}^{q^{-n}}} x^{k}$$

$$\times 10W_9(abdaq^{k-2}; a, bca/q, bda/q, cda/q, S, abdaq^{n+k-1}, q^{k-n}; q; T)$$

where

$$K_n = (2a)^{-n}(aab, aca, ada, abca/q; q; q)_{n}$$

and the two choices for $\epsilon$ correspond to

$$(S, T) = (q^{k+1}, a^{2}), \quad \text{for } \epsilon = 1, \quad (S, T) = (q^{k}, qa^{2}), \quad \text{for } \epsilon = 2.$$

We next explain how Theorem 4.4 follows from Theorem 4.3. First we rewrite the recurrence relation [6] in terms of polynomials in $x^2$. 

Proposition 4.5. If \( p_{2n}(x; c, q) = t_n(x^2) \), then
\[
t_{n+1}(x) = (x - B_n)t_n(x) - \Lambda_n t_{n-1}(x), \quad t_{-1} = 0, \quad t_0(x) = 1.
\]
where
\[
B_0 = \frac{1}{(1 - c)(1 - cq)}, \quad B_n = \lambda_{2n} + \lambda_{2n+1}, \quad n \geq 1,
\]
\[
\Lambda_n = \lambda_{2n-1}\lambda_{2n}, \quad n \geq 1.
\]

Proposition 4.6. If \( p_{2n+1}(x; c, q) = x s_n(x^2) \), then
\[
s_{n+1}(x) = (x - B_n)s_n(x) - \Lambda_n s_{n-1}(x), \quad s_{-1} = 0, \quad s_0(x) = 1.
\]
where
\[
B_n = \lambda_{2n+2} + \lambda_{2n+1}, \quad n \geq 1,
\]
\[
\Lambda_n = \lambda_{2n+1}\lambda_{2n}, \quad n \geq 1.
\]

We shall obtain the recurrence relations in Propositions 4.5 and 4.6 by an appropriate limiting case of Theorem 4.4. Our goal is to obtain \((A_n, C_n) = (\lambda_{2n+1}, \lambda_{2n})\) for \( t_n(x) \) and \((A_n, C_n) = (\lambda_{2n+2}, \lambda_{2n+1})\) for \( s_n(x) \). Then we match the initial conditions to find the correct linear combination of the two solutions.

First choosing \( a = e^{-1}q^{-1} \alpha, b = c = d = 1/\alpha \), we obtain
\[
A_n(\alpha, 1/q) = \frac{\alpha(1 - cq^{n+1}/\alpha)^3(1 - \alpha cq^n)}{cq(1 - cq^{2n})(1 - cq^{2n+1})},
\]
\[
C_n(\alpha, 1/q) = \frac{cq(1 - q^n/\alpha)(1 - \alpha q^{n-1})^3}{\alpha(1 - q^{2n-1})(1 - cq^{2n})}.
\]

By rescaling \( x \) by \( B\alpha^2 x/2 \), i.e., \( \tilde{p}_n(x) = 2^n\alpha^{-2n} B^{-n} p_n^{(\alpha)}(B\alpha^2 x/2) \), we have
\[
\tilde{p}_{n+1}(x) = (x - \tilde{b}_n(\alpha))\tilde{p}_n(x) - \tilde{\lambda}_n(\alpha)\tilde{p}_{n-1}(x),
\]
\[
\tilde{b}_n(\alpha) = \frac{1}{B\alpha^2} \left( \frac{cq}{\alpha} + \frac{\alpha}{cq} - A_n(\alpha, 1/q) - C_n(\alpha, 1/q) \right),
\]
\[
\tilde{\lambda}_n(\alpha) = \frac{1}{B^2\alpha^2} A_n(\alpha, 1/q) C_n(\alpha, 1/q).
\]

If \( \alpha \to \infty \), the first two terms in \( \tilde{b}_n(\alpha) \) vanish. Choosing \( B = 1/q \), we obtain the desired values for Proposition 4.5
\[
\lim_{\alpha \to \infty} -\frac{1}{B\alpha^2} A_n(\alpha, 1/q) = \frac{q^n}{(1 - cq^{2n})(1 - cq^{2n+1})} = \lambda_{2n+1},
\]
\[
\lim_{\alpha \to \infty} -\frac{1}{B\alpha^2} C_n(\alpha, 1/q) = \frac{cq^{3n-1}}{(1 - cq^{2n-1})(1 - cq^{2n})} = \lambda_{2n}.
\]

The first degree limiting polynomial matches the second Ismail-Rahman solution in Theorem 4.4 with \((a, b, c, d) = (\alpha/cq, 1/\alpha, 1/\alpha, 1/\alpha)\),
\[
x - \frac{1}{(1 - c)(1 - cq)}
\]
so that
\[
\lim_{\alpha \to \infty} \tilde{p}_n(x) = \lim_{\alpha \to \infty} \phi_n^{(\alpha,2)}(x; 1/q),
\]
which is the stated explicit formula in Theorem 4.1. \( \square \)
For the odd even-odd polynomials in Proposition 4.6 we choose \((a, b, c, d) = (cq^2\alpha, 1/\alpha, 1/\alpha, 1/\alpha)\),

\[
A_n(\alpha, q) = \frac{(1 - cq^{n+2})(1 - cq^{n+1}/\alpha)}{\alpha cq^2(1 - cq^{2n+1})(1 - cq^{2n+2})},
\]

\[
C_n(\alpha, q) = \frac{\alpha cq^2(1 - cq^n)(1 - q^{n-1}/\alpha)^3}{(1 - cq^{2n})(1 - cq^{2n+1})}.
\]

As before choosing \(\hat{p}_n(x) = 2^n\alpha^{2n}B^{-n}\tilde{p}_n(\alpha)(B\alpha^2x/2)\) and \(B = -cq\) we find

\[
\lim_{\alpha \to \infty} \frac{1}{B\alpha^2} A_n(\alpha, q) = \frac{cq^{3n+2}}{(1 - cq^{2n+1})(1 - cq^{2n+2})} = \lambda_{2n+2},
\]

\[
\lim_{\alpha \to \infty} \frac{1}{B\alpha^2} C_n(\alpha, q) = \frac{q^n}{(1 - cq^{2n})(1 - cq^{2n+1})} = \lambda_{2n+1}.
\]

The first degree limiting polynomial matches the first Ismail–Rahman solution in Theorem 4.4 with \((a, b, c, d) = (cq^2\alpha, 1/\alpha, 1/\alpha, 1/\alpha)\),

\[
x = \frac{1 + cq}{(1 - c)(1 - cq^2)}
\]

so that

\[
\lim_{\alpha \to \infty} \hat{p}_n(x) = \lim_{\alpha \to \infty} \psi_n^{(1)}(x; q),
\]

which is the stated explicit formula in Theorem 4.2.

We summarize these limits for the even-odd \(q\)-Lommel polynomials.

**Theorem 4.7.** The even-odd \(q\)-Lommel polynomials are the following limits of associated Askey–Wilson polynomials

\[
p_{2n}(x; c, q) = \lim_{\alpha \to \infty} \frac{(2q)^n}{\alpha^{2n}} \psi_n^{(2)}\left(\alpha^2x^2/2q; 1/\alpha, 1/\alpha, 1/\alpha\right), \quad (a, b, c, d) = (cq, 1/\alpha, 1/\alpha, 1/\alpha),
\]

\[
p_{2n+1}(x; c, q) = x \lim_{\alpha \to \infty} \frac{(-2/cq^2)^n}{\alpha^{2n}} \psi_n^{(1)}\left(-cq^2\alpha^2x^2/2q; \alpha, 1/\alpha, 1/\alpha, 1/\alpha\right).
\]

For the classical \(q\)-Lommel polynomials \(h_n(x; c, q)\), for the even polynomials choose

\[
(a, b, c, d) = (1, q/\alpha^2, c, 1/\alpha),
\]

\[
\hat{\tilde{p}}_n(x) = 2^n\alpha^{-2n}(-1)^n\tilde{p}_n(\alpha)(-\alpha^2x/2),
\]

\[
t_{2n}(x) = 2^n\alpha^{-2n}(-1)^n \lim_{\alpha \to \infty} \psi_n^{(2)}\left(-\alpha^2x^2/2; q\right)
\]

and for the odd polynomials choose

\[
(a, b, c, d) = (1, q^2/\alpha^2, c, 1/\alpha),
\]

\[
\hat{\tilde{p}}_n(x) = 2^n\alpha^{-2n}(-q)^n\tilde{p}_n(\alpha)(-\alpha^2x/2q),
\]

\[
t_{2n}(x) = 2^n\alpha^{-2n}(-q)^n \lim_{\alpha \to \infty} \psi_n^{(1)}\left(-\alpha^2x^2/2q; q\right).
\]

**Theorem 4.8.** The classical \(q\)-Lommel polynomials are the following limits of associated Askey–Wilson polynomials

\[
h_{2n}(x; c, q) = \lim_{\alpha \to \infty} \frac{(-2)^n}{\alpha^{2n}} \psi_n^{(2)}\left(-\alpha^2x^2/2; q\right), \quad (a, b, c, d) = (1, q/\alpha^2, c, 1/\alpha),
\]

\[
h_{2n+1}(x; c, q) = x \lim_{\alpha \to \infty} \frac{(-2q)^n}{\alpha^{2n}} \psi_n^{(1)}\left(-\alpha^2x^2/2q; q\right), \quad (a, b, c, d) = (1, q^2/\alpha^2, c, 1/\alpha).
\]
Theorem 4.9 is [15, Theorem 14.4.1].

**Theorem 4.9.** The classical $q$-Lommel polynomials are

$$h_n(x; c, q) = \sum_{k=0}^{n/2} {n-k\choose k} \frac{(-c)^k q^{2k-k}}{q^{n-k^2}} x^{n-2k}.$$  

**Proof.** We consider the even case, the proof for the odd case is similar. The inner sum becomes an evaluable very well poised $_6W_5$

$$\begin{align*}
_6W_5 \left( cq^{k-1}; q^k, cq^{n+k}, q^{n-k} \left| q, q^{-2k} \right. \right) = \frac{(-c q^{n-k}; q^{-1})_k (q^{n+1}; q)_k}{(c q; q^k, q^{k+1}; q)_k} q^{-k(n-k)}.
\end{align*}$$

By considering the coefficient of $x^{2n-2k}$, we arrive at Theorem 4.9 with $n$ replaced by $2n$. \hfill \Box

For the type $R_I$ $q$-Lommel polynomials there is a simple generating function which gives an explicit expression.

**Proposition 4.10.** The type $R_I$ $q$-Lommel polynomials have the generating function

$$\begin{align*}
\sum_{n=0}^{\infty} (c^{-1}; q^{-1})_n r_n(x; c, q)t^n = \sum_{k=0}^{\infty} \frac{(-x t/c) q^{-k(2)}}{(t/c, tx; q^{-1})_{k+1}}.
\end{align*}$$

**Proof.** If $G(x, t)$ is the generating function on the left side, then Definition 2.4 implies

$$\begin{align*}
G(x, t) - 1 = (x + 1/c) t G(x, t) - x t/c G(x, t q^{-1}) - x t^2/c G(x, t),
\end{align*}$$

whose iterate is the result. \hfill \Box

**Theorem 4.11.** The type $R_I$ $q$-Lommel polynomials have the explicit formula

$$r_n(x; c, q) = \frac{1}{(c^{-1}; q^{-1})_n} \sum_{k=0}^{n-k} \sum_{a=0}^{n} \frac{(-x/c)^k q^{-k(2)}}{q^{-a} \choose k} c^{-a} \left[ n - a \right] q^{-k(n-k-a)}.$$  

**Proof.** Apply the $q^{-1}$-binomial theorem to Proposition 4.10 to find the resulting coefficient of $t^n$. \hfill \Box

**Proposition 4.12.** We have the connection coefficient relation

$$r_n(x^2; c, q) = \sum_{k=0}^{n} {n \choose k} \frac{c^k q^{n^2-(n-k)^2}}{q^{n-k^2}} p_{2n-2k}(x; c, q).$$

**Proof.** Induction on $n$ using the three term relations. \hfill \Box

**Proposition 4.13.** If $L_p$ is the linear functional for the even-odd polynomials $p_n(x; c, q)$, then

$$L_{p}(r_n(x^2; c, q)) = \frac{c^n q^{n^2}}{(c, cq; q)_n}.$$  

**Proof.** Apply $L_p$ to both sides of Proposition 4.12. By orthogonality, $L_p(p_j(x)) = 0$ for $j > 0$, so only the $k = n$ term survives. \hfill \Box

**Theorem 4.14.** The moments of the type $R_I$ $q$-Lommel polynomials are equal to the even moments of the even-odd $q$-Lommel polynomials,

$$L_r(x^m) = L_p(x^{2m}), \quad m \geq 0.$$
Proof. The type $R_t$ moments $L_r(x^m)$ are recursively determined by [17] Corollary 3.15

$$L_r(r_n(x; c, q)) = a_1a_2 \cdots a_n = \frac{e^n q^n}{(c, cq; q)_n}, \quad n \geq 0.$$ 

By Proposition [4,12] the moments $L_p(x^{2m})$ satisfy the same recurrence. □

For completeness, we give the inverse relation to Proposition 4.12.

Proposition 4.15. We have the connection coefficient relation

$$p_{2n}(x; c, q) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-c)^k q^{2nk-(k+1)}}{(cq^{n-1}, cq^{2n-1}; q-1)_k} r_{n-k}(x^2; c, q).$$

Proposition 4.16. The even-odd $q$-Lommel polynomials have the explicit expressions

$$p_{2n}(x; c, q) = \frac{1}{(c; q)_{2n}} \sum_{k=0}^{n} (-1)^k x^{2n-2k} (cq^k; q)_{2n-2k} \sum_{j=0}^{k} \binom{n-j}{k-j} q^{j} c^j q^{jn+\binom{j}{2}},$$
$$p_{2n+1}(x; c, q) = \frac{1}{(c; q)_{2n+1}} \sum_{k=0}^{n} (-1)^k x^{2n-2k+1} (cq^k; q)_{2n-2k+1} \sum_{j=0}^{k} \binom{n-j}{k-j} q^{j} c^j q^{jn+\binom{j}{2}}.$$ 

Proof. This may be verified from Definition [23] by considering the coefficients of $x^{2n-2k-1}$. □

5. Moments and Continued fractions

In this section we review the known facts which connect continued fractions to moment generating functions. We independently prove the continued fractions for the moments of each of the three $q$-Lommel polynomials.

Definition 5.1. Take a sequence of orthogonal polynomials $p_n(x)$ which satisfy $p_{-1}(x) = 0$, $p_0(x) = 1$, and

$$p_{n+1}(x) = (x - b_n)p_n(x) - \lambda_n p_{n-1}(x), \quad n \geq 0,$$

and whose linear functional for orthogonality is $L_p$. Define

$$\mu_n(\{b_k\}_{k \geq 0}, \{\lambda_k\}_{k \geq 0}) = L_p(x^n).$$

The moment generating function for $L_p$ is

$$\sum_{n=0}^{\infty} L_p(x^n) t^n = \sum_{n=0}^{\infty} \mu_n(\{b_k\}_{k \geq 0}, \{\lambda_k\}_{k \geq 0}) t^n.$$ 

A Jacobi continued fraction also exists for $M_p(t)$, converging as formal power series in $t$,

$$\sum_{n=0}^{\infty} L_p(x^n) t^n = \frac{1}{1 - b_0 t - \frac{\lambda_1 t^2}{1 - \lambda_2 t^2}}.$$  

(5.1)

Definition 5.2. [17] For general type $R_t$ orthogonal polynomials

$$r_{n+1}(x) = (x - b_n)r_n(x) - (a_n x + \lambda_n) r_{n-1}(x), \quad n \geq 0,$$

with linear functional $L_r$, define

$$\mu_n(\{b_k\}_{k \geq 0}, \{a_k\}_{k \geq 0}, \{\lambda_k\}_{k \geq 0}) = L_r(x^n).$$
The corresponding continued fraction for the type $R_I$ moment generating function is [17, Corollary 3.7]

\[
\sum_{n=0}^{\infty} L_r(x^n) t^n = \frac{1}{1 - b_0 t - \frac{a_1 t + \lambda_1 t^2}{1 - b_1 t - \frac{a_2 t + \lambda_2 t^2}{1 - \cdots}}}.
\]

Note that both continued fractions in (5.1) and (5.2) are explicitly given in terms of the three term recurrence coefficients. We shall evaluate the continued fractions as quotients of basic hypergeometric series, namely $q$-Bessel functions, using contiguous relations.

For the Lommel polynomials $h_n(x; c)$, it is known that the moment generating function is a quotient of Bessel functions, with

\[
\sum_{n=0}^{\infty} L_h(x^n) t^n = \frac{0F_1(c + 1; -t^2)}{0F_1(c; -t^2)} = \frac{1}{1 - \frac{\lambda_1 t^2}{1 - \frac{\lambda_2 t^2}{1 - \cdots}}}.
\]

The moment generating function for the classical $q$-Lommel polynomials is a quotient of $q$-Bessel functions. In this section we shall see that a corresponding result holds for our other two $q$-Lommel polynomials, and in fact they are equal.

**Theorem 5.3.** [15, Theorem 14.4.3] The moment generating function for the classical $q$-Lommel polynomials $h_n(x; c, q)$ is a quotient of Jackson’s first $q$-Bessel functions

\[
\sum_{n=0}^{\infty} L_h(x^n) t^n = \frac{2\phi_1 (0, 0; cq; q^{-t^2})}{2\phi_1 (0, 0; c; q^{-t^2})} = \frac{1}{1 - \frac{\lambda_1 t^2}{1 - \frac{\lambda_2 t^2}{1 - \cdots}}},
\]

with $\lambda_n = cq^{n-1}/(1 - cq^{n-1})(1 - cq^n)$.

**Theorem 5.4.** The moment generating function for the even-odd $q$-Lommel polynomials $p_n(x; c, q)$ is a quotient of Jackson’s third $q$-Bessel functions

\[
\sum_{n=0}^{\infty} L_p(x^n) t^n = \frac{1\phi_1 (0; cq; q; q^{-t^2})}{1\phi_1 (0; c; q; q^{-t^2})} = \frac{1}{1 - \frac{\lambda_1 t^2}{1 - \frac{\lambda_2 t^2}{1 - \cdots}}},
\]

with

\[
\lambda_{2n} = \frac{cq^{2n-1}}{(1 - cq^{2n-2})(1 - cq^{2n})}, \quad \lambda_{2n+1} = \frac{q^n}{(1 - cq^{2n})(1 - cq^{2n+2})}.
\]

**Theorem 5.5.** The moment generating function for the type $R_I$ $q$-Lommel polynomials $r_n(x; c, q)$ is a quotient of Jackson’s third $q$-Bessel functions

\[
\sum_{n=0}^{\infty} L_r(x^n) z^n = \frac{1\phi_1 (0; cq; q; qz)}{1\phi_1 (0; c; q; qz)} = \frac{1}{1 - b_0 z - \frac{a_1 z}{1 - b_1 z - \frac{a_2 z}{1 - b_2 z - \cdots}}}.
\]
Theorem 4.14 implies that the two continued fractions in Theorems 5.4 and 5.5 with \( z = t^2 \) are equal.

**Corollary 5.6.** We have the equality of continued fractions

\[
\frac{1}{1 - b_0 z - \frac{a_1 z}{1 - b_1 z - \frac{a_2 z}{1 - b_2 z - \cdots}}} = \frac{1}{1 - \frac{\lambda_1 z}{1 - \frac{\lambda_2 z}{1 - \cdots}}},
\]

where

\[
a_n = \frac{cq^{2n-1}}{(1 - cq^{n-1})(1 - cq^n)}, \quad b_n = \frac{q^n}{1 - cq^n},
\]

\[
\lambda_{2n} = \frac{cq^{3n-1}}{(1 - cq^{2n-1})(1 - cq^{2n})}, \quad \lambda_{2n+1} = \frac{q^n}{(1 - cq^{2n})(1 - cq^{2n+1})}.
\]

Theorems 5.3, 5.4, and 5.5 may all be proven using contiguous relations for hypergeometric and basic hypergeometric series.

To prove Theorems 5.3 and 5.4 we use Heine’s contiguous relation [10, 17.6.19] which is

\[
\frac{\phi_2 (aq, b; cq; q, z)}{\phi_2 (a, b; c; q, z)} = \frac{(1 - b)(a - c)z}{(1 - c)(1 - cq)} \frac{\phi_2 (aq, bq; cq^2; q, z)}{\phi_2 (bq, aq; cq; q, z)}.
\]

Equivalently,

\[
(5.3) \quad \frac{\phi_2 (aq, b; cq; q, z)}{\phi_2 (a, b; c; q, z)} = \frac{1}{1 - \frac{(1 - b)(a - c)z}{(1 - c)(1 - cq)} \frac{\phi_2 (aq, bq; cq^2; q, z)}{\phi_2 (bq, aq; cq; q, z)}}.
\]

Applying (5.3) iteratively, we obtain Heine’s continued fraction, which is a \( q \)-analogue of Gauss’s continued fraction.

**Lemma 5.7 (Heine’s fraction).** We have

\[
\frac{\phi_2 (aq, b; cq; q, z)}{\phi_2 (a, b; c; q, z)} = \frac{1}{1 - \frac{\beta_1 z}{1 - \frac{\beta_2 z}{1 - \cdots}}},
\]

where

\[
\beta_{2n+1} = \frac{(1 - bq^n)(a - cq^n)q^n}{(1 - cq^{2n})(1 - cq^{2n+1})}, \quad \beta_{2n} = \frac{(1 - aq^n)(b - cq^n)q^{n-1}}{(1 - cq^{2n-1})(1 - cq^{2n})}.
\]

Theorem 5.3 is the special case \( a = b = 0 \) and \( z = -t^2 \) of Lemma 5.7. Theorem 5.4 is also the limiting case \( z = -t^2/a, \ b = 0, \ a \to \infty \) of Lemma 5.7.

For Theorem 5.5 we need the \( q \)-Nörlund fraction [8, (19.2.7)]. However, to simplify the expressions we need some notation for continued fractions.
Definition 5.8. For sequences $a_i$ and $b_i$, let
\[
\begin{align*}
\sum_{i=0}^{m} \left( \frac{a_i}{b_i} \right) &= \frac{a_0}{b_0 + \frac{a_1}{b_1 + \cdots + \frac{a_m}{b_m}}} = \frac{\infty}{\sum_{i=0}^{m} \left( \frac{a_i}{b_i} \right)}, \\
\end{align*}
\]

The following lemma will be used later.

Lemma 5.9. For any sequences $\{a_i : 0 \leq i \leq m\}$, $\{b_i : 0 \leq i \leq m\}$, and $\{c_i : -1 \leq i \leq m\}$, we have
\[
\begin{align*}
\sum_{i=0}^{m} \left( \frac{a_i}{b_i} \right) &= 1 - c - (a + b - ab - abq)z + \frac{1}{1 - c} \sum_{m=1}^{\infty} \frac{e_m(z)}{e_m + d_m z}, \\
\end{align*}
\]

Proof. By multiplying $c_i$ to the numerator and denominator of the $i$th fraction, we obtain
\[
\begin{align*}
\frac{a_0}{b_0 + \frac{a_1}{b_1 + \cdots + \frac{a_m}{b_m}}} &= \frac{a_0 c_0}{b_0 c_0 + \frac{a_1 c_0 c_1}{b_1 c_0 + \cdots + \frac{a_m c_{m-1} c_m}{b_m c_m}}},
\end{align*}
\]
which is equivalent to the equation in the lemma. □

Lemma 5.10 ($q$-Nörlund fraction). We have
\[
\begin{align*}
\frac{2\phi_1(a, b; c, q, z)}{2\phi_1(aq, bq; cq; q, z)} &= 1 - c - (a + b - ab - abq)z + \frac{1}{1 - c} \sum_{m=1}^{\infty} \frac{c_m(z)}{c_m + d_m z}, \\
\end{align*}
\]

where
\[
\begin{align*}
c_m(z) &= (1 - aq^m)(1 - bq^m)(cz - abq^m z^2)q^{m-1}, \\
e_m &= 1 - cq^m, \\
d_m &= -(a + b - abq^m - abq^{m+1})q^m.
\end{align*}
\]

The $q$-Nörlund fraction can be restated in the form of a continued fraction for type $R_I$ orthogonal polynomials.

Proposition 5.11 ($q$-Nörlund fraction restated). We have
\[
\begin{align*}
\frac{2\phi_1(aq, bq; cq, q, z)}{2\phi_1(a, b; c, q, z)} &= \frac{1}{1 - b_0 z - \frac{a_1 z^2}{1 - b_1 z - \frac{a_2 z^2}{1 - b_2 z - \cdots}}}, \\
\end{align*}
\]

where
\[
\begin{align*}
b_m &= \frac{(a + b - abq^m - abq^{m+1})q^m}{1 - cq^m}, \\
a_m &= \frac{(1 - aq^m)(1 - bq^m)q^{m-1}}{(1 - cq^m)(1 - cq^m)}, \\
\lambda_m &= \frac{(1 - aq^m)(1 - bq^m)abq^{2m-1}}{(1 - cq^m)(1 - cq^m)},
\end{align*}
\]
Proof. By taking the inverse on each side of the equation in Lemma 5.10 we obtain
\[
2 \Phi_1 \left( aq, bq; cq; q, z \right) \frac{2 \Phi_1 (a, b; c; q, z)}{2 \Phi_1 (a, b; c; q, z)} = \frac{1 - c}{c_0(z)} \sum_{m=0}^\infty \left( \frac{c_m(z)}{e_m + d_m z} \right).
\]
Applying Lemma 5.9 with \( c_i = 1/(1 - c q') \) and \( m \to \infty \) yields
\[
2 \Phi_1 \left( aq, bq; cq; q, z \right) = \frac{(1 - c q^{-1})(1 - c)}{c_0(z)} \sum_{m=0}^\infty \left( \frac{c_m(z)/(1 - c q^{-m})(1 - c q^m)}{e_m/(1 - c q^m) + d_m z/(1 - c q^m)} \right),
\]
which is the same as the desired identity. \( \square \)

Proof of Theorem 5.5. Replace \( z \) by \( z/b \), put \( a = 0 \), and let \( b \to \infty \) in Proposition 5.11. The result is Theorem 5.5. \( \square \)

Note that when \( b = 0 \) both Lemma 5.7 and Proposition 5.11 give a continued fraction expression for
\[
2 \Phi_1 \left( aq, 0; cq; q, z \right) = \frac{2 \Phi_1 (a, 0; c; q, z)}{2 \Phi_1 (a, 0; c; q, z)}.
\]
Therefore we obtain the following theorem.

**Theorem 5.12.** We have the equality of continued fractions
\[
\frac{1}{1 - b_0 z - \frac{a_1 z}{1 - b_1 z - \frac{a_2 z}{1 - b_2 z - \ldots}}} = \frac{1}{1 - \frac{\lambda_1 z}{1 - \frac{\lambda_2 z}{1 - \ldots}}},
\]
where
\[
a_n = \frac{(aq^n - 1) cq^{n-1}}{(1 - c q^{n-1})(1 - c q^n)}, \quad b_n = \frac{aq^n}{1 - c q^n},
\]
\[
\lambda_2n = \frac{-c q^{2n-1}(1 - a q^n)}{(1 - c q^{2n-1})(1 - c q^{2n})}, \quad \lambda_{2n+1} = \frac{(a - c q^n) q^n}{(1 - c q^{2n})(1 - c q^{2n+1})}.
\]

When Theorem 5.12 is interpreted as an equality for moment generating functions, we find the following generalization of Theorem 4.14 which holds for \( q \)-Lommel polynomials.

**Corollary 5.13.** Let \( \lambda_n, a_n \) and \( b_n \) be given by Theorem 5.12. The \( 2n \)th moment of the orthogonal polynomials defined by \( p_{n+1}(x) = x p_n(x) - \lambda_n p_{n-1}(x) \) is equal to the \( n \)th moment of the type \( R_I \) polynomials defined by \( r_{n+1}(x) = (x - b_n) r_n(x) - a_n x r_{n-1}(x) \).

6. **Combinatorics of moments of type \( R_I \) \( q \)-Lommel polynomials**

The moment generating function for type \( R_I \) polynomials is given by the continued fraction in (5.2). For type \( R_I \) \( q \)-Lommel polynomials we give in this section a general combinatorial interpretation for this infinite continued fraction in terms of parallelogram polyominoes. We also interpret the finite continued fraction and give an explicit rational expression using \( q \)-Lommel polynomials. To be specific we give a combinatorial interpretation for the ratio
\[
r_{n}^{(3)}(x^{-1}; q^{+1}, q) / n_{n+1}^{(3)}(x^{-1}; q^{+1}, q)
\]
of (rescaled) type \( R_I \) \( q \)-Lommel polynomials, Theorem 6.9 This is a finite version of the result of Bousquet-Mélou and Viennot [4]. The \( n \to \infty \) limit of Theorem 6.9 yields a quotient of \( q \)-Bessel functions,
\[
J_{\nu}^{(3)}(x^{1/2}; q^{-1}) / J_{\nu}^{(3)}(x^{1/2}; q^{-1})
\]
which is the moment generating function for the type $R_1$ $q$-Lommel polynomials. This material appears in our unpublished manuscript [18, Section 5].

We shall need several definitions related to parallelogram polyominoes and Motzkin paths.

**Definition 6.1.** An NE-path is a lattice path from $(0,0)$ to $(a,b)$ for some positive integers $a,b$ consisting of north steps $(0,1)$ and east steps $(1,0)$. A parallelogram polyomino is a set of unit squares enclosed by two NE-paths with the same ending points that do not intersect except the starting and ending points. Denote by $P$ the set of parallelogram polyominoes.

For a parallelogram polyomino $\alpha \in P$ let $U(\alpha)$ be the upper boundary path and $D(\alpha)$ the lower boundary path, see Figure 1. A diagonal of $\alpha$ is the set of squares in $\alpha$ whose centers are on the line $x + y = i$ for some integer $i$. The size of a diagonal is the number of squares in it. See Figure 2.

**Definition 6.2.** We denote by $P^{\leq k}$ the set of parallelogram polyominoes in which every diagonal has size at most $k$.

Consider $\alpha \in P$ and a diagonal $\tau$ of $\alpha$. Let $u$ (resp. $d$) be the northwest (resp. southeast) corner of the topmost (resp. bottom-most) square of $\tau$. We say that $d$ is an NN-diagonal (resp. NE-diagonal, EN-diagonal, and EE-diagonal) if the step in $U(\alpha)$ starting at $u$ is a north (resp. north, east, and east) step and the step in $D(\alpha)$ starting at $d$ is a north (resp. east, north, and east) step. See Figure 3.

For sequences $\{a_n\}_{n \geq 0}$, $\{b_n\}_{n \geq 0}$, $\{c_n\}_{n \geq 0}$, and $\{d_n\}_{n \geq 0}$, define the weight $\text{wt}(\alpha; a,b,c,d)$ of $\alpha \in P$ to be the product of $a_n$ (resp. $b_n$, $c_n$, and $d_n$) for each NN-diagonal (resp. EE-diagonal, NE-diagonal, and EN-diagonal) of size $n + 1$.

Now we review Flajolet’s theory [13] on continued fraction expressions for Motzkin path generating functions.
Definition 6.3. A Motzkin path is a lattice path from \((0,0)\) to \((n,0)\) consisting of up steps \((1,1)\), down steps \((1,-1)\), and horizontal steps \((1,0)\) that never goes below the \(x\)-axis. A \(2\)-Motzkin path is a Motzkin path in which every horizontal step is colored red or blue. The height of a \(2\)-Motzkin path is the largest integer \(y\) for which \((x,y)\) is a point in the path.

Denote by \(\text{Motz}_2\) the set of all \(2\)-Motzkin paths and by \(\text{Motz}_2^\leq m\) the set of all \(2\)-Motzkin paths with height at most \(m\).

For sequences \(\{a_n\}_{n \geq 0}, \{b_n\}_{n \geq 0}, \{c_n\}_{n \geq 0}\), and \(\{d_n\}_{n \geq 0}\), define the weight \(\text{wt}(p; a, b, c, d)\) of a \(2\)-Motzkin path \(p\) to be the product of \(a_n\) (resp. \(b_n\), \(c_n\), and \(d_n\)) for each red horizontal step (resp. blue horizontal step, up step, and down step) starting at height \(n\), see Figure 4.

Flajolet’s theory [13] proves the following lemma for a finite continued fraction.

Lemma 6.4. Given sequences \(\{a_n\}_{n \geq 0}, \{b_n\}_{n \geq 0}, \{c_n\}_{n \geq 0}\), and \(\{d_n\}_{n \geq 0}\), we have

\[
\sum_{p \in \text{Motz}_2^\leq m} \text{wt}(p; a, b, c, d) = \frac{1}{1 - a_0 - b_0 - \frac{c_0 d_1}{1 - a_1 - b_1 - \cdots - \frac{c_{m-1} d_m}{1 - a_m - b_m}}}
\]

There is a well known bijection between \(2\)-Motzkin paths and parallelogram polyominoes.

Definition 6.5 (The map \(\phi : \text{Motz}_2^\leq m \to \mathcal{P}^{\leq m+1}\)). Let \(p \in \text{Motz}_2^\leq m\). Then \(\phi(p) = \alpha\) is the parallelogram polyomino whose upper and lower boundary paths \(U, D\) are constructed by the following algorithm.

1. The first step of \(U\) (resp. \(D\)) is a north (resp. east) step.
2. For \(i = 1, 2, \ldots, n\), where \(n\) is the number of steps in \(p\), the \((i+1)\)th steps of \(U\) and \(D\) are defined as follows.
   a. If the \(i\)th step of \(p\) is an up step, then the \((i+1)\)th step of \(U\) (resp. \(D\)) is a north (resp. east) step.
(b) If the \(i\)th step of \(p\) is a down step, then the \((i + 1)\)st step of \(U\) (resp. \(D\)) is a east (resp. north) step.

(c) If the \(i\)th step of \(p\) is a red horizontal step, then the \((i + 1)\)st steps of \(U\) and \(D\) are both north steps.

(d) If the \(i\)th step of \(p\) is a blue horizontal step, then the \((i + 1)\)st steps of \(U\) and \(D\) are both east steps.

(3) Finally, the last step of \(U\) (resp. \(D\)) is an east (resp. north) step.

For example, if \(p\) is the 2-Motzkin path in Figure 1 then \(\phi(p)\) is the parallelogram polyomino \(\alpha\) in Figure 1.

It is easy see from the construction that \(\phi: \text{Mot}_z \rightarrow \mathcal{P}\) is a bijection such that if \(\phi(p) = \alpha\), then \(\text{wt}(\alpha; a, b, c, d) = d_0 \text{wt}(p; a, b, c, d)\).

Therefore we obtain the following proposition from Lemma 6.4, which changes the weighted 2-Motzkin paths into weighted parallelogram polyominoes.

**Proposition 6.6.** Given sequences \(\{a_n\}_{n \geq 0}, \{b_n\}_{n \geq 0}, \{c_n\}_{n \geq 0},\) and \(\{d_n\}_{n \geq 0}\), we have

\[
\sum_{\alpha \in \mathcal{P}} \text{wt}(\alpha; a, b, c, d) = \frac{d_0}{1 - a_0 - b_0 - \frac{c_0d_1}{1 - a_1 - b_1 - \cdots - \frac{c_{m-1}d_m}{1 - a_m - b_m}}}.
\]

As a special case in Proposition 6.6 if \(\{a_n\}_{n \geq 0}, \{b_n\}_{n \geq 0}, \{c_n\}_{n \geq 0},\) and \(\{d_n\}_{n \geq 0}\) are the sequences given by \(a_n = q^{n+1}Y, \ b_n = q^{n+1}X, \ c_n = q^{n+1}XY,\) and \(d_n = q^{n+1}\), then one can easily check that

\[
XY \cdot \text{wt}(\alpha; a, b, c, d) = X^{\text{col}(\alpha)}Y^{\text{row}(\alpha)}q^{\text{area}(\alpha)}.
\]

Thus we obtain the following corollary.

**Corollary 6.7.** We have

\[
\sum_{\alpha \in \mathcal{P}} X^{\text{col}(\alpha)}Y^{\text{row}(\alpha)}q^{\text{area}(\alpha)} = \frac{qXY}{1 - q(X + Y) - \frac{q^3XY}{1 - q^2(X + Y) - \cdots - \frac{q^{2m+1}XY}{1 - q^{m+1}(X + Y)}}}.
\]

For the rest of this section we will find a finite version of the following result due to Bousquet-Mélou and Viennot [4].

**Theorem 6.8** ([9] for \(\nu = 0\) and [4] for general \(\nu\)). The tri-variate generating function for parallelogram polyominoes is

\[
\sum_{\alpha \in \mathcal{P}} (q^\nu x)^{\text{col}(\alpha)}(q^\nu y)^{\text{row}(\alpha)} q^{\text{area}(\alpha)} = -q^\nu x^{1/2}J_{\nu + 1}^{(3)}(x^{1/2}; q^{-1})/J_{\nu}^{(3)}(x^{1/2}; q^{-1}).
\]

In fact Delest and Fédou [9] (for \(\nu = 0\)), and Bousquet-Mélou and Viennot [4] state their results in the following equivalent form:

\[
\sum_{\alpha \in \mathcal{P}} x^{\text{col}(\alpha)}y^{\text{row}(\alpha)}q^{\text{area}(\alpha)} = \frac{qxy}{1 - qy} \frac{1\phi_1 (0; q^2y; q, q^2x)}{1\phi_1 (0; qy; q, qx)}.
\]

Bousquet-Mélou and Viennot [4] also showed that
Let \( X_1 \ldots \) JANG SOO KIM AND DENNIS STANTON

Let \( \text{ominoes is} \) Theorem 6.9.

Then \( P \) use Theorem 4.11 to write the finite continued fraction as an explicit rational function.

The tri-variate generating function for bounded diagonal parallelogram poly- Corollary 6.10.

We also note that there are similar results in [1].

Cigler and Krattenthaler [7] found a different finite version of Theorem 6.8.

Now we are ready to prove a finite version of Theorem 6.8.

By (3.12), taking the limit for a sequence \( b = \{b_n\}_{n \geq 0}, a = \{a_n\}_{n \geq 0}, \) and \( \lambda = \{\lambda_n\}_{n \geq 0}, \) and for a nonnegative integer \( k, \)

\[
\begin{align*}
  \frac{x^m P_m(x^{-1}; \delta b, \delta a, \delta \lambda)}{x^{m+1} P_{m+1}(x^{-1}; b, a, \lambda)} = \frac{1}{-a_0 x - \lambda x^2} \frac{m}{m+1} K_0 (b_0 x, \lambda x) \\
\end{align*}
\]

Now we are ready to prove a finite version of Theorem 6.8.

**Theorem 6.9.** The tri-variate generating function for bounded diagonal parallelogram poly-ominoes is

\[
\sum_{\alpha \in P \leq m+1} (q^\alpha x)^{\text{col}(\alpha)} (q^\alpha y\text{row}(\alpha)) q^{\text{area}(\alpha)} = \frac{q^x y}{1 - q(x + y)} - \frac{q^3 x y}{1 - q^2(x + y)} - \cdots
\]

Proof. Let \( b = \{b_i\}_{i \geq 0}, a = \{a_i\}_{i \geq 0}, \) and \( \lambda = \{\lambda_i\}_{i \geq 0}, \) where

\[
\begin{align*}
  b_i = \frac{q^{\nu(i+1)}}{1 - q^{\nu(i+1)}}, \quad a_i = \frac{q^{2\nu(i+1)}}{(1 - q^{i+1})(1 - q^{\nu(i+1)})}, \quad \lambda_i = 0.
\end{align*}
\]

Then \( P_m(x; b, a, \lambda) = r_m(3)(x; q^{\nu+1}, q) \) and \( P_m(x; \delta b, \delta a, \delta \lambda) = r_m(3)(x; q^{\nu+2}, q). \) By (6.2),

\[
\frac{r_m(3)(x; q^{\nu+2}, q)}{r_m(3)(x; q^{\nu+1}, q)} = \frac{x^m P_m(x^{-1}; \delta b, \delta a, \delta \lambda)}{x^{m+1} P_{m+1}(x^{-1}; b, a, \lambda)} = \frac{1}{-a_0 x - \lambda x^2} \frac{m}{m+1} K_0 (b_0 x, \lambda x).
\]

By Lemma 6.9 with \( c_i = 1 - q^{\nu+1}, \)

\[
\begin{align*}
  \frac{1}{-a_0 x - \lambda x^2} \frac{m}{m+1} K_0 (b_0 x, \lambda x) &= \frac{1}{1 - q^{2\nu+1} x} \frac{m}{m+1} K_0 (b_0 x, \lambda x) = \frac{1}{1 - q^{2\nu+1} x} \frac{m}{m+1} K_0 (b_0 x, \lambda x).
\end{align*}
\]

Letting \( X = q^{\nu} x \) and \( Y = q^\nu, \) and combining the above equations, we obtain

\[
\begin{align*}
  \frac{q^{2\nu+1}}{1 - q^{2\nu+1}} \frac{r_m(3)(x; q^{\nu+2}, q)}{r_m(3)(x; q^{\nu+1}, q)} &= -\frac{m}{m+1} \left( \frac{-q^{2\nu+1} X Y}{1 - q^{2\nu+1}(X + Y)} \right).
\end{align*}
\]

Corollary 6.7 then completes the proof. \( \square \)

By (6.12), taking the limit \( m \to \infty \) in Theorem 6.9 we obtain Theorem 6.8. We may also use Theorem 4.11 to write the finite continued fraction as an explicit rational function.

**Corollary 6.10.** The tri-variate generating function for bounded diagonal parallelogram poly-ominoes is

\[
\sum_{\alpha \in P \leq m+1} (x^{\text{col}(\alpha)} y^{\text{row}(\alpha)}) q^{\text{area}(\alpha)} = \frac{x \sum_{k=0}^{n-k} \sum_{i=0}^{n-k} (-1)^{i} x^{a q^{k+n-a}} q^{(i-k)(k+a)} [a]_{q^{-1}} [k]_{q^{-1}}}{\sum_{k=0}^{n+1} \sum_{a=0}^{n+1-k} (-1)^{i} x^{a q^{k+n-a}} q^{(i-k)(k+a)} [a]_{q^{-1}} [k]_{q^{-1}}}.
\]

Cigler and Krattenthaler [7] found a different finite version of Theorem 6.8.
There is another concurrence of moments, which follows from [17, Corollary 3.7]

odd-even tricks
the first and second
in Section 5. There is a concurrence of moments (see Propositions 4.5 and 4.6), which we call

For the even-odd

1

Remark 6.12. The second odd-even trick (7.2) with $\lambda_{2k-1} = q^k y$ and $\lambda_{2k} = q^k$ gives

\[
1 + \frac{qy}{1 - q(x + y) - \frac{q^3 xy}{1 - q^2(x + y) - \frac{q^5 xy}{\cdots}}} = \frac{1}{1 - \frac{qy}{1 - \frac{q^2 y}{1 - \frac{q^4 y}{\cdots}}}}.
\]

Remark 6.13. There are also finite versions of Theorem 6.9 for the classical $q$-Lommel polynomials and the even-odd $q$-Lommel polynomials. The rational function is again a quotient of orthogonal polynomials while the weights on $P_{\leq n+1}$ depend upon the diagonals.

Here are the infinite continued fractions for these two cases. For the classical $q$-Lommel polynomials, Theorem 6.3 becomes

\[
\frac{\phi_1(0,0; q^2 y; q; -qx)}{\phi_1(0,0; qy; q; -qx)} = \frac{1 - qy}{1 - qy - \frac{q^2 xy}{1 - q^2 y - \frac{q^4 xy}{\cdots}}}.
\]

For the even-odd $q$-Lommel polynomials, Theorem 5.4 becomes

\[
\frac{\phi_1(0; q^2 y; q^2; -qx)}{\phi_1(0; qy; q; qx)} = \frac{1 - qy}{1 - qy - \frac{A_1}{1 - q^2 y - \frac{A_2}{1 - q^3 y - \frac{A_3}{\cdots}}}}.
\]

where $A_{2k-1} = xy^k$ and $A_{2k} = xyq^{3k/2+1}$.

7. Concurrence of moments

Recall the notation for the moments $\mu_n (\{k\}_{k \geq 0}, \{\lambda_k\}_{k \geq 0})$ and $\mu_n (\{k\}_{k \geq 0}, \{a_k\}_{k \geq 0}, \{\lambda_k\}_{k \geq 0})$ in Section 5. There is a concurrence of moments (see Propositions 4.3 and 4.6), which we call the first and second odd-even tricks

\[
\mu_{2n} (\{0\}, \{\lambda_k\}) = \mu_n (\{\lambda_{2k} + \lambda_{2k+1}\}, \{\lambda_{2k}\}, \{\lambda_{2k+1}\})
\]

\[
\mu_{2n+2} (\{0\}, \{\lambda_k\}) = \lambda_1 \mu_n (\{\lambda_{2k+2} + \lambda_{2k+1}\}, \{\lambda_{2k}\}, \{\lambda_{2k+1}\})
\]

The classical orthogonal polynomial moments are a special case of type $R_I$ moments

$\mu_n (\{k\}, \{0\}, \{\lambda_k\}) = \mu_n (\{k\}, \{\lambda_k\})$.

There is another concurrence of moments, which follows from [17, Corollary 3.7]

\[
\mu_{2n} (\{0\}, \{a_k\}) = \mu_n (\{0\}, \{a_k\}, \{0\}).
\]
It is known [17] that a type $R_I$ moment $\mu_n(\{b_k\}, \{a_k\}, \{\lambda_k\})$ is a nonnegative polynomial in the recurrence coefficients. Besides [7.3] Theorem 4.14 is another example of classical orthogonal polynomial moments being equal to type $R_I$ moments

$$(7.4) \quad \mu_{2n}(\{0\}, \{\Lambda_k\}) = \mu_n(\{b_k\}, \{a_k\}, \{0\}).$$

The main result in this section is Theorem 7.2 which expresses the $\Lambda_k$ as a function of the sequences $a_k$ and $b_k$, thereby providing the concurrence (7.4).

To prove Theorem 7.2 we need to recall a classical result and notation. The Hankel determinant [6, Theorem 4.2] will be used:

$$\det(\mu_{i+j}(\{b_k\}_{k \geq 0}, \{\lambda_k\}_{k \geq 0}))_{i,j=0}^n = \lambda_1^n \lambda_2^{n-1} \cdots \lambda_n^1.$$ 

Recall that for a sequence $a = \{a_k\}_{k \geq 0}$ we write $\delta a = \{a_{k+1}\}_{k \geq 0}$. We also define $\delta^{-1}a = \{a_{k-1}\}_{k \geq 0}$, where $a_{-1}$ is irrelevant for our purpose.

**Definition 7.1.** A Schröder path is a lattice path from $(r, 0)$ to $(s, 0)$, for some integers $r, s$, consisting of northeast steps $(1, 1)$, east steps $(1, 0)$, and south steps $(0, -1)$ that never goes below the $x$-axis. Given sequences $b = \{b_k\}_{k \geq 0}$ and $a = \{a_k\}_{k \geq 0}$, the weight $\text{wt}(P)$ of a Schröder path $P$ is the product of $b_i$ for each east step starting at height $i$ and $a_i$ for each south step starting at height $i$.

Our main theorem of this section is the next theorem.

**Theorem 7.2.** Suppose that sequences $b = \{b_k\}_{k \geq 0}$, $a = \{a_k\}_{k \geq 0}$, and $\Lambda = \{\Lambda_k\}_{k \geq 0}$ satisfy

$$\mu_{2n}(\{0\}, \{\Lambda_k\}) = \mu_n(\{b_k\}, \{a_k\}, \{0\}).$$

Then

$$\Lambda_1 \Lambda_2 \cdots \Lambda_{2n} = \frac{f_n(a, b)}{f_{n-1}(a, b)},$$

where

$$f_n(a, b) = \sum_p \text{wt}(p),$$

and the sum is over all $n$-tuples $p = (P_0, P_1, \ldots, P_n)$ of non-intersecting Schröder paths, $P_k : (-k, 0) \to (k, 0), 0 \leq k \leq n$. Moreover,

$$\Lambda_1 \Lambda_2 \cdots \Lambda_{2n-1} = a_0^{-1} \frac{f_n(\delta^{-1}a, \delta^{-1}b)}{f_{n-1}(\delta^{-1}a, \delta^{-1}b)},$$

and if $a_k = b_k = 1$ then

$$f_n(\{1\}, \{1\}) = 2^{\binom{n+1}{2}}.$$ 

**Proof.** Let

$$\rho_n := \mu_{2n}(\{0\}, \{\Lambda_k\}) = \mu_n(\{b_k\}, \{a_k\}, \{0\}),$$

$$\Delta_n := \det(\rho_{i+j})_{0 \leq i, j \leq n}.$$ 

Using the odd-even trick $B_n = \Lambda_{2n+1} + \Lambda_{2n}$ and $\Theta_n = \Lambda_{2n-1} \Lambda_{2n}$, we have

$$\rho_n = \mu_{2n}(\{0\}, \{\Lambda_k\}) = \mu_n(\{B_k\}, \{\Theta_k\}).$$

Therefore

$$\Delta_n = \det(\mu_{i+j}(\{B_k\}, \{\Theta_k\}))_{0 \leq i, j \leq n} = \Theta_1^n \Theta_2^{n-1} \cdots \Theta_n^1 = \Lambda_1^n \Lambda_2^{n-1} \cdots \Lambda_2^{n-1} \Lambda_1^1,$$

which shows $\Lambda_1 \Lambda_2 \cdots \Lambda_{2n} = \Delta_n / \Delta_{n-1}$.

Kim and Stanton [17, Corollary 3.7] showed that $\mu_n(\{b_k\}, \{a_k\}, \{0\})$ is the sum of weights of all Schröder paths from $(0, 0)$ to $(n, 0)$. Since $\Delta_n = \det(\mu_{i+j}(\{b_k\}, \{a_k\}, \{0\}))_{0 \leq i, j \leq n}$, the
Thus we may apply the Lindström–Gessel–Viennot lemma of tail swapping to reduce this sum to non-intersecting paths, \( \sigma' \). As in the even case, \( \Delta \) used the idea relating \( \Delta \) to \( \Delta' \) and we obtain the identity for \( \Lambda_1 \Lambda_2 \cdots \Lambda_2n \).

Now using the second odd-even trick \( B'_n = \Lambda_2n+2 + \Lambda_2n+1 \) and \( \Lambda'_n = \Lambda_2n+1\Lambda_2n \), we have

\[
\rho_{n+1} = \mu_{2n+2} (\{0\}, \{\Lambda_k\}) = \Lambda_1\mu_n (\{B'_k\}, \{\Lambda'_k\}).
\]

Then

\[
\Delta'_n := \det (\rho_{i+j+1})_{0 \leq i, j \leq n-1} = \Lambda_n^n \det (\mu_{i+j} (\{B'_k\}, \{\Lambda'_k\}))_{0 \leq i, j \leq n-1}
\]

\[
= \Lambda_n^n \Lambda_n^{n-1} \Lambda_3 \cdots \Lambda_{2n-2} \Lambda_{2n-1},
\]

so

\[
\Lambda_1 \Lambda_2 \cdots \Lambda_{2n-1} = \Delta'_n / \Delta_n^{n-1}.
\]

As in the even case, \( \Delta'_n = \det (\mu_{i+j+1} (\{b_k\}, \{a_k\}, \{0\}))_{0 \leq i, j \leq n-1} \) is the generating function for \( n \)-tuples non-intersecting Schröder paths \( p' = (P'_1, \ldots, P'_n) \), \( P'_k : (-k+1, 0) \to (k, 0) \). For \( 1 \leq k \leq n \), let \( P'_k \) be the path from \((-k-1, 0) \to (k, -1) \) obtained from \( P'_k \) by adding a northeast step at the beginning and a south step at the end, and let \( P_0 \) be the empty path from \((0, -1) \to (0, 0) \). This gives a bijection from \( n \)-tuples non-intersecting Schröder paths \( p' = (P'_1, \ldots, P'_n) \), \( P'_k : (-k+1, 0) \to (k, 0) \) to \( (n+1) \)-tuples non-intersecting Schröder paths \( p = (P_0, P_1, \ldots, P_n) \), \( P_k : (-k-1, 0) \to (k, -1) \). Note that the starting point of \( P_k \) has height \(-1\), which shifts the indices of \( a_k \) and \( b_k \) down by one. This shows that

\[
\Delta'_n = a_0^n \det (\mu_{i+j} (\{b_k-1\}, \{0\}, \{a_k-1\}))_{0 \leq i, j \leq n} = a_0^n f_n (\delta^{-1} a, \delta^{-1} b),
\]

and we obtain the identity for \( \Lambda_1 \Lambda_2 \cdots \Lambda_{2n-1} \).

Finally the fact that \( \Delta_n = 2 \binom{n+1}{2} \) and \( \Delta'_n = 2 \binom{n+1}{2} \) if \( a_k = b_k = 1 \) for all \( k \) follows from [17, Theorem 6.15, \( A = B = 1, C = 0 \)].

The first few values of \( \Lambda_1 \cdots \Lambda_k \) in Theorem 7.2 are

\[
\Lambda_1 = a_0^{-1} f_1 (\delta^{-1} a, \delta^{-1} b) = \frac{a_1 + b_0}{1},
\]

\[
\Lambda_1 \Lambda_2 = f_1 (a, b) = a_2 + b_1,
\]

\[
\Lambda_1 \Lambda_2 \Lambda_3 = a_0^{-1} f_2 (\delta^{-1} a, \delta^{-1} b)
\]

\[
= \frac{a_1 a_2 a_3 + a_2^2 b_0 + a_2 a_3 b_1 + 2 a_2 b_0 b_1 + b_0 b_1^2 + a_1 a_2 b_2 + a_2 b_0 b_2}{a_1 + b_0},
\]

\[
\Lambda_1 \Lambda_2 \Lambda_3 \Lambda_4 = f_2 (a, b)
\]

\[
= \frac{a_1 a_2 a_3 a_4 + a_2^2 b_1 + a_2 a_3 b_1 + 2 a_2 b_1 b_2 + b_1 b_2^2 + a_2 a_3 b_3 + a_3 b_1 b_3}{a_2 + b_1}.
\]

**Remark 7.3.** En and Fu [11] used the idea relating \( \Delta_n \) and \( \Delta'_{n-1} \) in the proof of Theorem 7.2 to give a simple proof of the Aztec diamond theorem, which is equivalent to the result \( \Delta_n = 2 \binom{n+1}{2} \) when \( a_k = b_k = 1 \).
8. Open problems

Recall that Kishore's theorem is a statement about the power series coefficients of the ratio \( J_{\nu+1}(x)/J_\nu(x) \) of two Bessel functions.

**Theorem 8.1** (Kishore, [19]). We have

\[
\frac{J_{\nu+1}(z)}{J_\nu(z)} = \sum_{n=1}^{\infty} \frac{N_{n,\nu}(z)}{D_{n,\nu}} \left( \frac{z}{2} \right)^{2n-1},
\]

where

\[
D_{n,\nu} = \prod_{k=1}^{n} (k + \nu)^{\lfloor n/k \rfloor},
\]

and \( N_{n,\nu} \) is a polynomial in \( \nu \) with nonnegative integer coefficients.

We conjecture the following finite version of Kishore's theorem on a ratio of Lommel polynomials \( R_{m,\nu}(x) \) defined in Section 3.

**Conjecture 8.2.** Let

\[
\frac{R_{m,\nu+2}(x)}{R_{m+1,\nu+1}(x)} = \sum_{n=0}^{\infty} \frac{N_{n,\nu}^{(m)}(x)}{D_{n,\nu}^{(m)}} \left( \frac{x}{2} \right)^{2n+1},
\]

where

\[
D_{n,\nu}^{(m)} = \prod_{k=0}^{m} (\nu + k + 1)^{f(m,n,k)},
\]

\[
f(m,n,k) = \begin{cases} 
\max \left( \frac{n + 1}{k + 1}, \frac{n + m - 2k + 1}{m - k + 1} \right), & \text{if } k \neq m/2, \\
1, & \text{if } k = m/2.
\end{cases}
\]

Then \( N_{n,\nu}^{(m)} \) is a polynomial in \( \nu \) with nonnegative integer coefficients.

In Section 3 we saw that the ratio

\[
\frac{J_{\nu+1}^{(3)}(z; q^{-1})}{J_\nu^{(3)}(z; q^{-1})} = \frac{-q^{\nu+1}z}{1 - q^{\nu+1}} \frac{\phi_1(0; q^{\nu+2}; q, q^{\nu+2}z^2)}{\phi_1(0; q^{\nu+1}; q, q^{\nu+1}z^2)}
\]

has two generalizations, the \( q \)-Nörlund continued fraction and Heine's continued fraction. These two generalizations seem to have a similar property as follows.

**Conjecture 8.3.** Let

\[
\sum_{n \geq 0} \gamma_n(a, b, c) z^n = \frac{2\phi_1(aq; cq; q, z)}{2\phi_1(a, b; c, q, z)}.
\]

Then

\[
\frac{\gamma_n(a, b, c)}{1 - c} = \frac{P_n(a, b, c)}{\prod_{k=0}^{n} (1 - cq^k)^{\lfloor n/k \rfloor}},
\]

for some polynomial \( P_n(a, b, c) \) in \( a, b, c, q \) with integer coefficients.

**Conjecture 8.4.** Let

\[
\sum_{n \geq 0} \gamma'_n(a, b, c) z^n = \frac{2\phi_1(a; bq; cq; q, z)}{2\phi_1(a, b; c, q, z)}.
\]

Then

\[
\frac{\gamma'_n(a, b, c)}{1 - c} = \frac{P'_n(a, b, c)}{\prod_{k=0}^{n} (1 - cq^k)^{\lfloor n/k \rfloor}},
\]
for some polynomial $P_n^q(a, b, c)$ in $a, b, c, q$ with integer coefficients.

**Problem 8.5.** Find a combinatorial proof of Theorem 5.12 which contains the Bousquet-Méloû–Viennot result.

**Problem 8.6.** Find an Askey scheme whose top element is the associated Askey–Wilson polynomial which contains the $q$-Lommel polynomials.

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