WALKS ON GRAPHS AND LATTICES – EFFECTIVE BOUNDS AND APPLICATIONS

IGOR RIVIN

Abstract. We continue the investigations started in [7, 8]. We consider the following situation: $G$ is a finite directed graph, where to each vertex of $G$ is assigned an element of a finite group $\Gamma$. We consider all walks of length $N$ on $G$, starting from $v_i$ and ending at $v_j$. To each such walk $w$ we assign the element of $\Gamma$ equal to the product of the elements along the walk. The set of all walks of length $N$ from $v_i$ to $v_j$ thus induces a probability distribution $F_{N,i,j}$ on $\Gamma$. In [7] we give necessary and sufficient conditions for the limit as $N$ goes to infinity of $F_{N,i,j}$ to exist and to be the uniform density on $\Gamma$ (a detailed argument is presented in [8]). The convergence speed is then exponential in $N$.

In this paper we consider $(G, \Gamma)$, where $\Gamma$ is a group possessing Kazhdan’s property $T$ (or, less restrictively, property $\tau$ with respect to representations with finite image), and a family of homomorphisms $\psi_k : \Gamma \to \Gamma_k$ with finite image. Each $F_{N,i,j}$ induces a distribution $F_{N,i,j}^k$ on $\Gamma_k$ (by push-forward under $\psi_k$). Our main result is that, under mild technical assumptions, the exponential rate of convergence of $F_{N,i,j}^k$ to the uniform distribution on $\Gamma_k$ does not depend on $k$.

As an application, we prove effective versions of the results of [8] on the probability that a random (in a suitable sense) element of $\text{SL}(n, \mathbb{Z})$ or $\text{Sp}(n, \mathbb{Z})$ has irreducible characteristic polynomial, generic Galois group, etc.

Introduction

The following set-up was first brought up in [7], and then fleshed out and applied in a somewhat unexpected direction in [8]:

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Firstly, let $G$ be a finite “ergodic” undirected graph, which means that the adjacency matrix of $G$ has a unique Perron-Frobenius eigenvalue with a strictly positive eigenvector.

Secondly, let $\Gamma$ be a finite group, and assign to each vertex $v$ of $G$ an element $\gamma(v) \in \Gamma$.

Finally, consider the set of walks $W_{N,i,j}$ on $G$ of length $N$ starting at $v_i$ and ending at $v_j$. Each walk $w \in W_{N,i,j}$ defines an element $\gamma(w) \in \Gamma$: the element $\gamma(w)$ is simply the product (in order) of elements $\gamma(v)$ along $w$. The set $W_{N,i,j}$ thus induces a probability distribution $F_{N,i,j}$ on $\Gamma$, where the probability $p_{N,i,j}(\nu)$ assigned to $\nu \in \Gamma$ is defined as:

$$p_{N,i,j}(\nu) = \frac{|\{w \in W_{N,i,j} \mid \gamma(w) = \nu\}|}{|W_{N,i,j}|}.$$

A priori, it is not clear that $F_{N,i,j}$ ever has full support, but, rather surprisingly, the following holds:

**Theorem A** ([7, 8]). If the set $\{\gamma(v) \mid v \in V(G)\}$ generates $\Gamma$ and there is no one-dimensional complex representation $\rho$ of $\Gamma$ which maps all of $\gamma(v)$ to the same complex number, then the distributions $F_{N,i,j}$ converge to the uniform distribution on $\Gamma$. The speed of convergence is exponential in $N$.

The proof of Theorem A is recalled below. The application to irreducibility of random matrices in [8] requires the use of Theorem A for finite quotients of $\text{SL}(n, \mathbb{Z})$ and $\text{Sp}(2n, \mathbb{Z})$. To get effective bounds, we need to have uniform bounds on the exponential speed of convergence in Theorem A, and this is the main subject of the current paper. The setup is as before, but $\Gamma$ is no longer (necessarily) finite, but it is assumed to have property $\tau$ for representations with finite image (see [5] for discussion of Property $\tau$). Any finite homomorphism $\psi$ of $\Gamma$ with finite image $\psi G$ induces a family of distributions $F_{N,i,j}^\psi$ on $\psi G$. We then have the following:

**Theorem B.** Let $G, \Gamma$ be as above. With the assumptions as in Theorem A, and the additional assumption that the set $\{\gamma(v)^{-1}\gamma(w) \mid v, w \in V(G)\}$ generates $\Gamma$ the exponential convergence rate of $F_{N,i,j}^\psi$ to the uniform distribution on $\psi(\Gamma)$ can be bounded independently of $\psi$.

The plan of the rest of the paper is as follows:

The starting point for the proof of the theorems above is Fourier Transform on finite groups, which is discussed in Section 1. In particular, we will be using Theorem 1.2 and Corollary 1.3 to reduce the question of whether a probability distribution is close to uniform to the proving that the Fourier Transform is small at every non-trivial
representation. The reader might well wonder how moving the problem to Fourier transform space helps us – the answer is that it turns out that we can reduce the estimation of the “fourier coefficients” to questions in linear algebra, through the construction in Section 2.

In Sections 4, 5, 6 we prove the additional estimates we need to prove Theorem B. Finally, in Section 7 we use Theorem B to show that the probability that a matrix in SL(n, Z) or in Sp(2n, Z) given by a word of length N in a symmetric generating set has reducible characteristic polynomial decreases exponentially with N.

1. Fourier Transform on finite groups

For a thorough introduction to the topic of this section the reader is referred to [9, 11]. Let Γ be a finite group, and let f : Γ → C be a function on Γ. Furthermore, let ̂Γ be the unitary dual of Γ: the set of all irreducible complex unitary representations of Γ. To f we can associate its Fourier Transform ̂f. This is a function which associates to each d-dimensional unitary representation ρ a d × d matrix ̂f(ρ) as follows:

\[ ̂f(ρ) = \sum_{γ ∈ Γ} f(γ)ρ(γ). \]

There is an inverse transformation, as well. Given a function g on ̂Γ which associates to each d-dimensional representation ρ a d × d matrix g(ρ), we can write:

\[ g^{♯}(γ) = \frac{1}{|Γ|} \sum_{ρ ∈ ̂Γ} d_ρ \text{tr}(g(ρ)ρ(γ^{-1})), \]

where d_ρ is the dimension of ρ. We mean “inverse” in the most direct way possible:

\[ ̂f^{♯} = f. \]

The following result is classical (see, eg, [11]):

**Theorem 1.1.**

\[ \sum_{ρ ∈ ̂Γ} d_ρ^2 = |Γ|, \]

and, together with the Fourier inversion formula, implies

**Theorem 1.2.** Let g be a function on ̂Γ, such that for every nontrivial ρ ∈ ̂Γ,

\[ \|g(ρ)\|_{op} < ε, \]
where \( \| \bullet \|_{op} \) denotes the operator norm (see Section 3). Then, for any \( \gamma_1, \gamma_2 \in \Gamma \),

\[
|g^\#(\gamma_1) - g^\#(\gamma_2)| < 2\epsilon.
\]

**Proof.** First, note that for the trivial representation \( \rho_0 \), the quantity

\[
d_{\rho_0}g(\rho_0)\rho_0(\gamma) = g(\rho_0),
\]

so does not depend on \( \gamma \). By the Fourier inversion formula, then,

\[
|g^\#(\gamma_1) - g^\#(\gamma_2)| = \left| \frac{1}{|\Gamma|} \sum_{\rho \in \Gamma} d_\rho \text{tr}(g(\rho)(\rho(\gamma_1) - \rho(\gamma_2))) \right| \leq \sum_{i=1}^{2} \left| \frac{1}{|\Gamma|} \sum_{\rho \in \Gamma, \rho \neq \rho_0} d_\rho \text{tr}(g(\rho)) \right| \leq \sum_{i=1}^{2} \frac{2}{|\Gamma|} \sum_{\rho \in \Gamma} d_\rho^2 \| g(\rho) \|_{op} < 2\epsilon.
\]

\( \square \)

**Corollary 1.3.** Under the assumption of Theorem 1.2 and assuming in addition that \( g \) is real valued, if

\[
\sum_{\gamma \in \Gamma} g(\gamma) = 1,
\]

then

\[
g(\gamma) - 1/|\Gamma| < 2\epsilon \quad \forall \gamma \in \Gamma.
\]

Furthermore, if \( \Omega \in \Gamma \),

(1)

\[
\left| \sum_{\gamma \in \Omega} g(\gamma) - \frac{\Omega}{|\Gamma|} \right| < 2\epsilon|\Omega|.
\]

**Proof.** Without loss of generality, suppose that \( g(\gamma) > 1/|\Gamma| \). Then there is a \( \gamma_2 \), such that \( g(\gamma_2) < 1/|\Gamma| \). Thus,

\[
g(\gamma) - 1/|\Gamma| < g(\gamma) - g(\gamma_2) < 2\epsilon.
\]

The estimate (1) follows immediately by summing over \( \Omega \). \( \square \)
2. Fourier estimates via linear algebra

In order to prove Theorem A, we would like to use Theorem 1.2, and to show the equidistribution result, we would need to show that for every nontrivial irreducible representation \( \rho \),

\[
\lim_{N \to \infty} \frac{1}{|W_{N,i,j}|} \sum_{w \in W_{N,i,j}} \rho(\gamma_w) = 0.
\]

To demonstrate Eq. (2), suppose that \( \rho \) is \( k \)-dimensional, so acts on a \( k \)-dimensional Hilbert space \( H_\rho = H \). Let \( Z = L^2(G) \) – the space of complex-valued functions from \( V(G) \) to \( \mathbb{C} \), let \( e_1, \ldots, e_n \) be the standard basis of \( Z \), and let \( P_i \) be the orthogonal projection on the \( i \)-th coordinate space. We introduce the matrix

\[
U_\rho = \sum_{i=1}^{n} P_i \otimes \rho(t_i) = \begin{pmatrix}
\rho(t_1) & 0 & \cdots & 0 \\
0 & \rho(t_2) & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \rho(t_n)
\end{pmatrix},
\]

and also the matrix \( A_\rho = A(G) \otimes I_H \), where \( I_H \) is the identity operator on \( H \). Both \( U_\rho \) and \( A_\rho \) act on \( Z \otimes H \). The following is immediate:

**Lemma 2.1.** Consider the matrix \((U_\rho A_\rho)^l\), and think of it as an \( n \times n \) matrix of \( k \times k \) blocks. Then the \( ij \)-th block equals the sum over all paths \( w \) of length \( l \) beginning at \( v_i \) and ending of \( v_j \) of \( \rho(\gamma_w) \).

Now, let \( T_{ji} \) be the operator on \( Z \) which maps \( e_k \) to \( \delta_{kj}e_i \).

**Lemma 2.2.**
\[
\text{tr} \left( \left( (T_{ji} P_j) \otimes I_H \right) \left( U_\rho \otimes A_\rho \right)^N (P_i \otimes I_H) \right) = \text{tr} \sum_{w \in W_{N,i,j}} \rho(\gamma_w)
\]

**Proof.** The argument of trace on the left hand side simply extracts the \( ij \)-th \( k \times k \) block from \( (U_\rho \otimes A_\rho)^N \).

By submulticativity of operator norm, we see that

\[
\| (T_{ji} P_j) \otimes I_H (U_\rho \otimes A_\rho)^N P_i \otimes I_H \|_{\text{op}} \leq \| (U_\rho \otimes A_\rho)^N \|_{\text{op}},
\]

and so proving Theorem A reduces (thanks to Theorem 1.2) to showing

**Theorem 2.3.**
\[
\lim_{N \to \infty} \frac{\| (U_\rho \otimes A_\rho)^N \|_{\text{op}}}{|W_{N,i,j}|} = 0,
\]

for any non-trivial \( \rho \).
Notation 2.4. We will denote the spectral radius of an operator $A$ by $\mathcal{R}(A)$.

Since $|W_{N,t}| \propto R^N(A(G))$, and by Gelfand’s Theorem (Theorem 3.3),
\[ \lim_{N \to \infty} \|B^N\|^{1/N} = \mathcal{R}(B), \]
for any matrix $B$ and any matrix norm $\| \cdot \|$, Theorem 2.3 is equivalent to the statement that the spectral radius of $U_\rho \otimes A_\rho$ is smaller than that of $A(G)$.

Theorem 2.3 is proved in Section 2.1.

2.1. Proof of Theorem 2.3

Lemma 2.5. Let $A$ be a bounded hermitian operator $A : H \to H$, and $U : H \to H$ a unitary operator on the same Hilbert space $H$. Then the spectral radius of $UA$ is smaller than the spectral radius of $A$, and the inequality is strict unless an eigenvector of $A$ with maximal eigenvalue is also an eigenvector of $U$.

Proof. The spectral radius of $UA$ does not exceed the operator norm of $UA$, which is equal to the spectral radius of $A$. Suppose that the two are equal, so that there is a $v$, such that $\|UAv\| = \mathcal{R}(A)v$, and $v$ is an eigenvector of $UA$. Since $U$ is unitary, $v$ must be an eigenvector of $A$, and since it is also an eigenvector of $UA$, it must also be an eigenvector of $U$.

In the case of interest to us, $\rho$ is a $k$-dimensional irreducible representation of $\Gamma$, $U = \text{Diag}(\rho(t_1), \ldots, \rho(t_n))$, while $A = A(G) \otimes I_k$. We assume that $A(G)$ is an irreducible matrix, so that there is a unique eigenvalue of modulus $\mathcal{R}(A(G))$, that eigenvalue $\lambda_{\text{max}}$ (the Perron-Frobenius eigenvalue) is positive, and it has a strictly positive eigenvector $v_{\text{max}}$. We know that the spectral radius of $A$ equals the spectral radius of $A(G)$, and the eigenspace of $\lambda_{\text{max}}$ is the set of vectors of the form $v_{\text{max}} \otimes w$, where $w$ is an arbitrary vector in $\mathbb{C}^k$. If $v_{\text{max}} = (x_1, \ldots, x_n)$, we can write $v_{\text{max}} \otimes w = (x_1w, \ldots, x_nw)$, and so $U(v_{\text{max}} \otimes w) = (x_1\rho(t_1)w, \ldots, x_n\rho(t_n)w)$. Since all of the $x_i$ are nonzero, in order for the inequality in Lemma 2.5 to be nonstrict, we must have some $w$ for which $\rho(t_i)w = cw$ (where the constant $c$ does not depend on $i$). Since the elements $t_i$ generate $\Gamma$, the existence of such a $w$ contradicts the irreducibility of $\rho$, unless $\rho$ is one dimensional. This proves Theorem 2.3.
3. Some remarks on matrix norms

In this note we use a number of matrix norms, and it is useful to summarize what they are, and some basic relationships and inequalities satisfied by them. For an extensive discussion the reader is referred to the classic [3]. All matrices are assumed square, and $n \times n$.

A basic tool in the inequalities below is the singular value decomposition of a matrix $A$.

**Definition 3.1.** The singular values of $A$ are the non-negative square roots of the eigenvalues of $AA^*$, where $A^*$ is the conjugate transpose of $A$.

Since $AA^*$ is a positive semi-definite Hermitian matrix for any $A$, the singular values $\sigma_1 \overset{\text{def}}{=} \sigma_{\max} \geq \sigma_2 \geq \ldots$ are non-negative real numbers. For a Hermitian $A$, the singular values are simply the absolute values of the eigenvalues of $A$.

The first matrix norm is the Frobenius norm, denoted by $\| \cdot \|$. This is defined as

$$\| A \| = \sqrt{\text{tr} AA^*} = \sqrt{\sum_i \sigma_i^2}.$$ 

This is also the sum of the square moduli of the elements of $A$.

The next matrix norm is the operator norm, $\| \|_{\text{op}}$, defined as

$$\| A \|_{\text{op}} = \max_{\| v \| = 1} \| Av \| = \sigma_{\max}.$$ 

Both the norms $\| \cdot \|$ and $\| \|_{\text{op}}$ are submultiplicative (submultiplicativity is part of the definition of matrix norm: saying that the norm $\| \cdot \|$ is submultiplicative means that $\| AB \| \leq \| A \| \| B \|$.)

From the singular value interpretation of the two matrix norms and the Cauchy-Schwartz inequality we see immediately that

$$\| A \| / \sqrt{n} \leq \| A \|_{\text{op}} \leq \| A \|.$$ 

We will also need the following simple inequalities:

**Lemma 3.2.** Let $U$ be a unitary matrix:

$$\| \text{tr} AU \| \leq \| A \| \sqrt{n} \leq n \| A \|_{\text{op}}.$$ 

**Proof.** Since $U$ is unitary, $\| U \| = \| U^t \| = \sqrt{n}$. So, by the Cauchy-Schwartz inequality, $\text{tr} AU \leq \| A \| \| U \| = \sqrt{n} \| U \|$. The second inequality follows from the inequality (3). \[Q.E.D.\]
The final (and deepest result) we will have the opportunity to use is:

**Theorem 3.3** (Gelfand). For any operator $M$, the spectral radius $\mathcal{R}(M)$ and any matrix norm $\|\|\|$, 

$$\mathcal{R}(M) = \lim_{k \to \infty} \|M^k\|^{1/k},$$

### 4. Some remarks on Kazhdan’s property T

A group $G$ is said to have Kazhdan’s Property $T$ if there exists an $\epsilon > 0$ and a compact subset $K \subseteq G$ such that for every nontrivial irreducible representation $(H, \rho)$ of $G$ and every vector $v \in H$ of norm one, $\|\rho(k)v - v\| > \epsilon$ for some $k \in K$. This definition is the one given in A. Lubotzky’s book [6]. For finitely generated discrete groups $K$ can be taken to be any set of generators (though the $\epsilon$ will depend on the generating set, it is obvious that knowing Kazhdan’s constant for some generating set will give bounds for any other generating set. It is known that lattices in semi-simple Lie groups have property $T$ and Kazhdan’s constants have been explicitly computed by Y. Shalom (see [10]). Related results have also been obtained by A. Zuk [13].

We will need the following

**Lemma 4.1.** Let $G$ have Kazhdan’s property $T$ and let $t_1, \ldots, t_n$ be a generating set of $G$, such that the set of all products $t_{j}^{-1} t_i$ is also a generating set. Then, there exists an $\epsilon > 0$ such that for any irreducible representation $(H, \rho)$ and any pair $v, w \in H$ there exists $i \leq n$ such that $\|\rho(t_i)v - w\| > \epsilon$.

**Proof.** Suppose not. By the triangle inequality, $\|\rho(t_i)v - \rho(t_j)v\| < 2\epsilon$, for all pairs $i, j$. Since $\rho$ is unitary, we see that $\|\rho(t_i^{-1} t_j)v - v\| < 2\epsilon$. It follows that the we can choose the $\epsilon$ whose existence is postulated in the Lemma to be half the Kazhdan constant of $G$ with respect to the generating set consisting of all products $t_{j}^{-1} t_i$. $\square$

To show that the condition in the statement of Lemma 4.1 is often met, first note:

**Lemma 4.2.** Let $S = \{t_1, \ldots, t_n\}$ be a symmetric generating set for $G$. Then, the subgroup $H$ generated by all products of the form $t_{j}^{-1} t_i$ has index at most two in $G$ (hence is always normal).

**Proof.** Since $S$ is symmetric, $H$ has every element which can be written as a word of even length in the elements of $S$. If $H \neq G$, then the index

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\[\text{known as Kazhdan’s constant}\]
of $H$ clearly equal to two (the other coset being the set of “odd” elements of $G$.

\[ \square \]

**Corollary 4.3.** If $G$ is one of
\[
\text{SL}(n,\mathbb{Z}), \text{SL}(n,\mathbb{Z}/p\mathbb{Z}), \text{Sp}(n,\mathbb{Z}), \text{Sp}(n,\mathbb{Z}/p\mathbb{Z})
\]
for $n \geq 2$, and $S$ is a symmetric generating set, then $S^{-1}S$ generates $G$.

5. **Linear algebra estimates**

**Lemma 5.1.** Let $U, A$ be as in Lemma 2.5. Assume that the spectral radius of $A$ equals 1 (for simplicity of notation), that the second biggest (in absolute value) eigenvalue of $A$ has absolute value $\lambda < 1$. Let $A_{\text{max}}$ be the eigenspace of $A$ corresponding to the eigenvalue 1, and let $P_{\text{max}}$ be the orthogonal projection on $A_{\text{max}}$. Assume now that for any $v \in A_{\text{max}}$,
\[
\|P_{\text{max}}Uv\| \leq d\|v\|,
\]
for some $0 \leq d < 1$. Then, there is a function $f(\lambda, d) < 1$, such that the spectral radius of $UA$ is smaller than $f(\lambda, d)$.

**Proof.** We will use Gelfand’s Theorem 3.3. For our result, we will use the operator norm, and Lemma 5.1 will follow immediately from Theorem 5.2 with $f(\lambda, d) = \sqrt{g(\lambda, d)}$, where $g$ is the function in the statement of Theorem 5.2.

\[ \square \]

**Theorem 5.2.** For $U, A$ as in the statement of Lemma 5.1 and $v$ an arbitrary vector. Then
\[
\|(UA)^2v\| \leq g(\lambda, d)\|v\|,
\]
for some function $g(\lambda, d) < 1$, and so
\[
\|(UA)^k\|_{\text{op}} \leq g^{\lceil k/2 \rceil}(\lambda, d),
\]
where $\|M\|_{\text{op}}$ denotes the operator norm of $M$.

**Proof.** Since $U$ is unitary, $\|(UA)^2v\| = \|AUAv\|$, for any $v$. Now write $v = x \oplus y$, with $x \in A_{\text{max}}$, and $y \in A_{\text{max}}^\perp$.

Our first observation is that
\[
\|Av\|^2 \leq \|x\|^2 + \lambda^2\|y\|^2 = \lambda^2\|v\|^2 + (1 - \lambda^2)\|x\|^2.
\]
It follows that
\[
\|AUAv\| \leq d\|x\| + \lambda\|y\|,
\]
where $d$ is the constant from Lemma 5.1.
and so by (6),

\[ \lambda^2\|Av\|^2 \leq \|AUAv\|^2 \leq \lambda^2\|Av\|^2 + (1 - \lambda^2)(d\|x\| + \lambda\|y\|)^2 \leq \]

\[ \lambda^2(\|x\|^2 + \lambda^2\|y\|^2) + (1 - \lambda^2)(d\|x\| + \lambda\|y\|)^2 = \]

\[ (1 - (1 - d^2)(1 - \lambda^2))\|x\|^2 + \lambda^2\|y\|^2 + 2(1 - \lambda^2)d\lambda yx. \]

Let us now write \( \|y\| = \alpha\|x\| \).

\( \lambda > 0 \).

Eq. (9) gives us

\[ \frac{|AUAv\|^2}{\|v\|^2} = \]

\[ \frac{1 - (1 - \lambda^2)(1 - d^2) + \lambda^2\alpha^2 + 2(1 - \lambda^2)d\lambda \alpha}{1 + \alpha^2} \leq \]

\[ 1 - (1 - \lambda^2)(1 - d^2) + \lambda^2\alpha^2 + 2(1 - \lambda^2)d\lambda \alpha = h(\lambda, d, \alpha). \]

Note that \( h(\lambda, d, 0) = 1 - (1 - \lambda^2)(1 - d^2) < 1 \), and \( h(\lambda, d, \alpha) \) is a monotonically increasing function of \( \alpha \) when \( \alpha \geq 0 \), and \( 0 \leq \lambda, d < 1 \). This means that we can find \( 0 < \alpha_0 \) such that \( h(\lambda, d, \alpha_0) = 1 - (1 - \lambda^2)(1 - d^2)/2 \), namely

\[ \alpha_0 = \frac{1 - \lambda^2}{\lambda} \left( \sqrt{d^2 + \frac{1 - d^2}{2(1 - \lambda^2)}} \right), \]

Putting together all the inequalities, we see that if \( \|y\|/\|x\| \leq \alpha_0 \), then

\[ \|UAUAv\| \leq \sqrt{1 - (1 - \lambda^2)(1 - d^2)/2\|v\|}, \]

while if \( \|y\|/\|x\| > \alpha_0 \), then

\[ \|UAUAv\| \leq \frac{1 + \alpha_0 \lambda}{1 + \alpha_0} \|v\|, \]

so setting

\[ g(\lambda, d) = \min \left( \sqrt{\frac{1 + \alpha_0 \lambda}{1 + \alpha_0}}, \sqrt{\frac{1 - (1 - \lambda^2)(1 - d^2)/2}{1 + \alpha_0}} \right), \]

the Lemma is proved.

\( \lambda = 0 \). In this case, the computation is much simpler:

\[ \frac{|AUAv\|^2}{\|v\|^2} = \frac{d^2}{1 + \alpha^2} \leq d^2, \]
and so the Lemma is proved here too. □

6. Applications of Theorem 5.2 to speed of convergence in Theorem A

Let us apply Theorem 5.2 to the setting of Theorems A and B. We will be using the argument and the notation of Sections 2.1 and 5. Let \( S = \{t_1, \ldots, t_n\} \), let \( \Gamma \) be the group generated by \( S \), and let \( \Gamma_1 \) be the group generated by \( S^{-1}S \).

If \( \lambda_1 \) is the Perron-Frobenius eigenvalue of \( G \), and \( \lambda_2 \) is the second largest (in absolute value) eigenvalue, we set \( \lambda = |\lambda_2|/|\lambda_1| \). Let \( X = (x_1, \ldots, x_n) \) be the (unit) Perron-Frobenius eigenvector of \( A(G) \). We know that \( A_1 \) is the space of all vectors of the form \( Y = X \otimes v = (x_1v, \ldots, x_nv) \), where \( v \in \mathbb{R}^k \). Such a vector is a unit vector precisely if \( \|v\| = 1 \).

Assume now that the group \( \Gamma_1 \) has the analogue of Kazhdan’s property \( T \), but with respect to the set of restrictions of irreducible representations of \( \Gamma \) – these are not necessarily irreducible when restricted to \( \Gamma_1 \) – with the constant \( \epsilon_1 \) corresponding to the generating set \( S^{-1}S \). We know (by Lemma 4.1) that there is an \( i \leq n \), such that \( \|\rho(t_i)v - w\| \geq \epsilon_1/2 \), and so, by the Law of Cosines,

\[
\langle \rho(t_i)v, w \rangle \leq 1 - \epsilon_1^2/8,
\]

and so, by Eq. (13),

\[
\langle UV, W \rangle \leq 1 - x_i^2 \epsilon_1^2/8
\]

Lemma 5.1 now gives us:

**Lemma 6.1.** The operator norm of \((UA)^k \) is at most \( g^{(k/2)}(\lambda, 1 - x_i^2 \epsilon_1^2/8) \), where \( g \) is the function computed in Theorem 5.2.

This completes the proof of Theorem B.

7. Applications to irreducibility

In this section, Theorem B is used to show that the probability that a random walk of length \( N \) on a graph \( G \) decorated with elements of \( \text{SL}(n, \mathbb{Z}) \) or \( \text{Sp}(2n, \mathbb{Z}) \) represents a matrix with reducible characteristic polynomial goes to 0 exponentially fast with the length \( N \) of the walks considered.
The results above show that for a fixed graph \( G \) and the series of groups \( \Gamma_p \), where \( \Gamma_l = \text{SL}(n, l) \) or \( \Gamma_l = \text{Sp}(2n, l) \) there exist a constant \( c > 1 \), such that the probability \( p_\gamma \) that one of the random walks of length \( N \) over \( G \) (decorated with elements of \( \Gamma_p \)) hits a subset \( \Omega \subseteq \Gamma_l \) satisfies

\[
|p_\Omega - |\Omega|/|\Gamma_l| | \leq 2c^{-N}|\Omega|,
\]

where \( c > 1 \) does not depend on \( l \).

7.1. \( \text{SL}(n) \). We know (see [8]) that the set \( R_l \in \text{SL}(n, l) \) has cardinality bounded by

\[
|R_p| \leq c_2 |\text{SL}(n, p)|,
\]

for \( p \) prime. Now, for given \( N \gg 1 \), there is a prime \( p_N \) satisfying

\[
(1 - \epsilon)c^{N/(n^2 - 1)} \leq p_N \leq (1 + \epsilon)c^{N/(n^2 - 1)}.
\]

By estimates (14) and (15), it follows that a random walk on \( G \) of length \( N \) represents a reducible element in \( \text{SL}(n, p_N) \) with probability \( P_N \) bounded above by:

\[
P_N \leq \frac{c_2}{p_N^N}(1 + (1 + \epsilon)c_2) = O(c^{N/(n^2 - 1)}).
\]

Since an element in \( \text{SL}(n, \mathbb{Z}) \) is reducible only if it is reducible in \( \text{SL}(n, l) \) (for every \( l \)), (16) gives an upper bound on the probability that an element represented by a random walk of length \( N \) is reducible over the integers.

7.2. \( \text{Sp}(2n) \). Here, the method in the last section does not work (since we only have \( O(1) \) bounds for individual primes).

Therefore, define

\[
q_k = \prod_{i=1}^{k} p_k
\]

(so \( q_k \) is the product of the first \( k \) primes). The prime number theorem tells us that \( q_k \sim k^k \).

By Borel’s estimate and the strong approximation property for \( \text{Sp}(2n) \) (see [9]) we know that probability that an element of \( \text{Sp}(2n, q_k) \) is reducible is bounded above by \( c_3^k \), for some \( c_3 <, \) and so by (14) we

\[\text{If we wished to keep this discussion completely elementary, Chebyshev’s elementary bound tells us that } q_k = O(k^{ak}) \text{ for some } a > 1, \text{ which is sufficient for what we are about to do.}\]
know that the probability $P_N$ that a walk on $G$ of length $N$ gives us a reducible element modulo $q^k$ is bounded above by

$$P_N \leq c_3^{-k}(1 + 2c^{-Nk^{(2n^2+n)}}).$$

If we pick

$$k \approx \frac{N}{2n^2 + n} \log c \log \frac{N}{2n^2 + n}$$

(so that the second term in parentheses is $O(1)$), we see that

$$P_N = O(\exp (\log c_3 \log c(N/(2n^2 + n) - \varepsilon))),$$

for any $\varepsilon > 0$, and as before, the same bound obtains for the probability that a random walk of length $N$ on $G$ gives a reducible element in $Sp(2n, \mathbb{Z})$.

7.3. Remarks. The first observation is that the argument in Section 7.2 applies, mutatis mutandis to the problem of counting elements in $Sp(n, \mathbb{Z})$ whose Galois group is not the full symmetric group.

Secondly, presumably sharper bounds can be given using more sophisticated sieve machinery (see, eg, [1]). As evidence for this, if the argument above is used to estimate the probability that a polynomial of degree $d$ with coefficient height bound $H$, reducible, our argument gives $O(H^{\log(d-1) - \log d})$, Gallagher’s large sieve argument [2] gives $O(H^{-1/2})$, while the truth is $O(1/H)$. Since the arguments above are completely elementary (even the use of the Prime Number Theorem can be avoided), and we get the result we want (that the probability decays exponentially) it seems wise to leave sieve methods to the experts. In fact, related results have been obtained by Emmanuel Kowalski, using his deep generalization of the large sieve [4] (also monograph, in preparation).

References

[1] Alina Carmen Cojocaru and M. Ram Murty. An introduction to sieve methods and their applications, volume 66 of London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 2006.

[2] P. X. Gallagher. The large sieve and probabilistic Galois theory. In Analytic number theory (Proc. Sympos. Pure Math., Vol. XXIV, St. Louis Univ., St. Louis, Mo., 1972), pages 91–101. Amer. Math. Soc., Providence, R.I., 1973.

[3] Roger A. Horn and Charles R. Johnson. Matrix analysis. Cambridge University Press, Cambridge, 1990. Corrected reprint of the 1985 original.

[4] Emmanuel Kowalski. The principle of the large sieve. arxiv.org, math.NT/0610021, 2006.
[5] Alex Lubotzky. What is... property (τ)? Notices Amer. Math. Soc., 52(6):626–627, 2005.
[6] Alexander Lubotzky. *Discrete groups, expanding graphs and invariant measures*, volume 125 of Progress in Mathematics. Birkhäuser Verlag, Basel, 1994. With an appendix by Jonathan D. Rogawski.
[7] Igor Rivin. Growth on groups (and other stories). Technical Report math.CO/9911076, arxiv.org, 1999.
[8] Igor Rivin. Walks on groups, counting reducible matrices, polynomials, and surface and free group automorphisms. 2006. arxiv preprint.
[9] Jean-Pierre Serre. *Représentations linéaires des groupes finis*. Hermann, Paris, revised edition, 1978.
[10] Yehuda Shalom. Explicit Kazhdan constants for representations of semisimple and arithmetic groups. Ann. Inst. Fourier (Grenoble), 50(3):833–863, 2000.
[11] Barry Simon. *Representations of finite and compact groups*, volume 10 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 1996.
[12] John von Neumann. Some matrix inequalities and metrization of matrix space. Tomsk University Review, 1:286–300. In collected works, Pergamon, Oxford, 1962, Volume IV, 205-218.
[13] A. Žuk. Property (T) and Kazhdan constants for discrete groups. Geom. Funct. Anal., 13(3):643–670, 2003.

Department of Mathematics, Temple University, Philadelphia
E-mail address: rivin@math.temple.edu

Current address: Mathematics Department, Stanford University, Stanford, California
E-mail address: rivin@math.stanford.edu