On the Cauchy problem for the Hunter-Saxton equation on the line

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Abstract

In this paper, we consider the Cauchy problem for the Hunter-Saxton (HS) equation on the line. Firstly, we establish the local well-posedness for the integral form of the (HS) equation by constructing some special spaces \(E_{p,r}^s\), which mix Lebesgue spaces and homogeneous Besov spaces. Then we present a global existence result and provide a sufficient condition for strong solutions to blow up in finite time for the equation. Finally, we give the ill-posedness and the unique continuation of the Hunter-Saxton equation.

Keywords: Hunter-Saxton equation, Local well-posedness, Global existence, Blow up, Ill-posedness, Unique continuation.

Mathematics Subject Classification: 35G25, 35A01, 35L03, 35L05, 35L60

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1 Introduction

In this paper, we consider the following Hunter-Saxton (HS) equation:

\[
(u_t + uu_x)_x = \frac{1}{2} u^2_x.
\]  

(1.1)

The equation (1.1) was derived by Hunter and Saxton as an asymptotic model of liquid crystals [16,20]. The HS equation is completely integrable [17,20] and has a bi-Hamiltonian structure [19]. Local well-posedness
and blow-up phenomena for the Cauchy problems of the (HS) equation on the circle were studied in \cite{16,24}. Global weak solutions of the (HS) equation was investigated in \cite{16,21,22}.

In particular, the (HS) equation is the limit form of the well-known Camassa-Holm (CH) equation:

\[(1 - \partial_x^2)u_t = 3uu_x - 2u_xu_xx - uu_xxx.\]

Local well-posedness and ill-posedness for the Cauchy problem of the (CH) equation were investigated in \cite{9,21,22}. Blow-up phenomena and global existence of strong solutions were discussed in \cite{6–9}. The existence and uniqueness of the global conservative solutions on the line were studied in \cite{10,11}. While the existence of global weak solutions and dissipative solutions were also investigated in \cite{2,3,23}. Moreover, the existence and uniqueness of the global conservative solutions on the circle were studied in \cite{13,14}.

However, the Cauchy problem of the Hunter-Saxton on the line has not been studied yet. In this paper we consider the Cauchy problem for the Hunter-Saxton (HS) on the line:

\[
\left\{
\begin{array}{ll}
    u_{xt} + (uu_x)_x = \frac{1}{2}u_x^2, \\
    u(0,x) = u_0(x), & x \in \mathbb{R}.
\end{array}
\right.
\tag{1.2}
\]

Taking \(\int_{-\infty}^{x} dz\) to the Hunter-Saxton equation (1.2) and letting \(g(t)\) is a bounded function, we get

\[
\left\{
\begin{array}{ll}
    u_t + uu_x = \int_{-\infty}^{x} \frac{1}{2}u_x^2(z)dz + g(t), \\
    u(0,x) = u_0(x), & x \in \mathbb{R}.
\end{array}
\right.
\tag{1.3}
\]

In fact, the mainly difficulty is that the term \(\int_{-\infty}^{x} \frac{1}{2}u_x^2(z)dz\) is not bounded in the inhomogeneous Besov spaces \(B^p_{p,r}\), even the Lebesgue spaces \(L^p\) (1 \(\leq p < \infty\)). Since \(u_x^2 \geq 0\) and \(\int_{-\infty}^{x} \frac{1}{2}u_x^2(z)dz\) is monotonic increasing, one can’t consider (1.3) in \(B^p_{p,r}\) on the line. To overcome this difficulty, we would like to study (1.3) in some new spaces \(E^p_{s,r}\) which mix Lebesgue spaces and homogeneous Besov spaces. In this way, one will get a bounded iterative sequence in \(E^p_{s,r}\), and finally obtain a solution for the Cauchy problem of (1.3).

The remaining part of the paper is organized as follows. In Section 2, we introduce some useful preliminaries. In Section 3, we prove the local well-posedness of (1.3) in some special space \(E^p_{s,r}\), the main approach is based on the Littlewood-Paley theory and transport equations theory. In Section 4, we obtain a global existence result and give a sufficient condition for strong solutions to blow up in finite time. In Section 5, we give the ill-posedness of (1.3) in the special Besov space \(L^\infty \cap \dot{B}^1_{p,r} \cap \dot{B}^{1+\frac{d}{2}}_{p,r}, r > 1\). In Section 6, we give the unique continuation of (1.3) when \(g(t) := C \int_{-\infty}^{x} u_x^2(z)dz\).

## 2 Preliminaries

In this section, we will recall some propositions on the Littlewood-Paley decomposition and Besov spaces.

**Proposition 2.1.** \cite{11} Let \(C\) be the annulus \(\{\xi \in \mathbb{R}^d : \frac{1}{2} \leq |\xi| \leq \frac{3}{2}\}\). There exist radial functions \(\chi\) and \(\varphi\), valued in the interval \([0,1]\), belonging respectively to \(D(B(0,\frac{1}{2}))\) and \(D(C)\), and such that

\[
\forall \xi \in \mathbb{R}^d, \quad \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j} \xi) = 1,
\]

\[
\forall \xi \in \mathbb{R}^d \setminus \{0\}, \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j} \xi) = 1,
\]

\[
|j - j'| \geq 2 \Rightarrow \text{Supp } \varphi(2^{-j}) \cap \text{Supp } \varphi(2^{-j'}) = \emptyset,
\]

\[
j \geq 1 \Rightarrow \text{Supp } \chi(\cdot) \cap \text{Supp } \varphi(2^{-j}) = \emptyset.
\]

The set \(\tilde{C} = B(0,\frac{2}{3}) + C\) is an annulus, and we have

\[
|j - j'| \geq 5 \Rightarrow 2^j \tilde{C} \cap 2^{j'} \tilde{C} = \emptyset.
\]
Further, we have
\[ \forall \xi \in \mathbb{R}^d, \quad \frac{1}{2} \leq \chi^2(\xi) + \sum_{j \geq 0} \varphi^2(2^{-j} \xi) \leq 1, \]
\[ \forall \xi \in \mathbb{R}^d \setminus \{0\}, \quad \frac{1}{2} \leq \sum_{j \in \mathbb{Z}} \varphi^2(2^{-j} \xi) \leq 1. \]

Denote \( \mathcal{F} \) by the Fourier transform and \( \mathcal{F}^{-1} \) by its inverse. Let \( u \) be a tempered distribution in \( S'_h(\mathbb{R}^d) \). For all \( j \in \mathbb{Z} \), define
\[ \Delta_j u = \mathcal{F}^{-1}(\varphi(2^{-j} \cdot) \mathcal{F} u) \quad \text{and} \quad \dot{S}_j u = \sum_{j' < j} \Delta_j u. \]

Then the Littlewood-Paley decomposition is given as follows:
\[ u = \sum_{j \in \mathbb{Z}} \Delta_j u \quad \text{in} \quad S'_h(\mathbb{R}^d). \]

Let \( s \in \mathbb{R} \), \( 1 \leq p, r \leq \infty \). The homogeneous Besov space \( \dot{B}^s_{p,r}(\mathbb{R}^d) \) is defined by
\[ \dot{B}^s_{p,r} = \dot{B}^s_{p,r}(\mathbb{R}^d) = \{ u \in S'_h(\mathbb{R}^d) : \| u \|_{\dot{B}^s_{p,r}(\mathbb{R}^d)} = \left\| (2^s \| \Delta_j u \|_{L^r})_j \right\|_{l^p(\mathbb{Z})} < \infty \}. \]

Let \( u \) be a tempered distribution in \( S'(\mathbb{R}^d) \). For all \( j \in \mathbb{Z} \), define
\[ \Delta_j u = 0 \text{ if } j \leq -2, \quad \Delta_{-1} u = \mathcal{F}^{-1}(\chi_1 \mathcal{F} u), \quad \Delta_j u = \mathcal{F}^{-1}(\varphi(2^{-j} \cdot) \mathcal{F} u) \text{ if } j \geq 0, \quad S_j u = \sum_{j' < j} \Delta_j u. \]

Then the Littlewood-Paley decomposition is given as follows:
\[ u = \sum_{j \in \mathbb{Z}} \Delta_j u \quad \text{in} \quad S'(\mathbb{R}^d). \]

Let \( s \in \mathbb{R} \), \( 1 \leq p, r \leq \infty \). The nonhomogeneous Besov space \( B^s_{p,r}(\mathbb{R}^d) \) is defined by
\[ B^s_{p,r} = B^s_{p,r}(\mathbb{R}^d) = \{ u \in S'(\mathbb{R}^d) : \| u \|_{B^s_{p,r}(\mathbb{R}^d)} = \left\| (2^s \| \Delta_j u \|_{L^r})_j \right\|_{l^p(\mathbb{Z})} < \infty \}. \]

There are some properties about Besov spaces. For the homogeneous Besov space, we have

**Proposition 2.2.** \([1, 12]\) Let \( s \in \mathbb{R} \), \( 1 \leq p, p_1, p_2, r_1, r_2 \leq \infty \) with \( s < \frac{2}{p} \) or \( s = \frac{2}{p}, r = 1 \).

1. \( \dot{B}^s_{p,r} \) is a Banach space, and is continuously embedded in \( S'_h \).
2. If \( r < \infty \), then \( \lim_{j \to \infty} \| S_j u - u \|_{\dot{B}^s_{p,r}} = 0 \).
3. If \( p_1 \leq p_2 \) and \( r_1 \leq r_2 \), then \( \dot{B}^s_{p_1,r_1} \hookrightarrow \dot{B}^s_{p_2,r_2} \).
4. If \( p \in [1, 2] \), then \( \dot{B}^s_{p,p} \hookrightarrow L^p \). If \( p \in [2, \infty) \), then \( \dot{B}^s_{p,2} \hookrightarrow L^p \). If \( p = \infty \), then \( \dot{B}^s_{\infty,1} \hookrightarrow L^\infty \).
5. Fatou property: if \( (u_n)_{n \in \mathbb{N}} \) is a bounded sequence in \( \dot{B}^s_{p,r} \), then an element \( u \in B^s_{p,p} \) and a subsequence \( (u_{n_k})_{k \in \mathbb{N}} \) exist such that
\[ \lim_{k \to \infty} u_{n_k} = u \quad \text{in} \quad S'_h \quad \text{and} \quad \| u \|_{\dot{B}^s_{p,r}} \leq \liminf_{k \to \infty} \| u_{n_k} \|_{B^s_{p,r}}. \]
6. Let \( m \in \mathbb{R} \) and \( f \) be a \( S^m \)-multiplier, i.e. \( f \) is a smooth function and satisfies that \( \forall \alpha \in \mathbb{N}^d, \exists C = C(\alpha) \) such that \( |\partial^\alpha f(\xi)| \leq C|\xi|^{m-|\alpha|}, \forall \xi \in \mathbb{R}^d \). Then the operator \( f(D) = \mathcal{F}^{-1}(f\mathcal{F}) \) is continuous from \( B^s_{p,r} \) to \( B^{s-m}_{p,r} \).
7. If \( s > 0 \), then \( L^p \cap \dot{B}^s_{p,r} = B^s_{p,r} \).

For the nonhomogeneous Besov space, we have
Proposition 2.3. Let $s \in \mathbb{R}$, $1 \leq p, p_1, p_2, r, r_1, r_2 \leq \infty$.

1. $B^{s}_{p,r}$ is a Banach space, and is continuously embedded in $S'$.
2. If $r < \infty$, then $\lim_{j \to \infty} \|S_j u - u\|_{B^{s}_{p,r}} = 0$. If $p, r < \infty$, then $C^\infty_0$ is dense in $B^{s}_{p,r}$.
3. If $p_1 \leq p_2$ and $r_1 \leq r_2$, then $B^{s}_{p_1,r_1} \hookrightarrow B^{s-\frac{(r_2-1)}{r_2}}_{p_2,r_2}$. If $s_1 < s_2$, then the embedding $B^{s_2}_{p,r} \hookrightarrow B^{s_1}_{p,r_1}$ is locally compact.
4. If $p \in [1, 2]$, then $B^{s}_{p,p} \hookrightarrow L^p$. If $p \in [2, \infty)$, then $B^{s}_{p,2} \hookrightarrow L^p$. If $p = \infty$, then $B^{s}_{\infty,1} \hookrightarrow L^\infty$.
5. Fatou property: if $(u_n)_{n \in \mathbb{N}}$ is a bounded sequence in $B^{s}_{p,r}$, then an element $u \in B^{s}_{p,r}$ and a subsequence $(u_{n_k})_{k \in \mathbb{N}}$ exist such that
   \[ \lim_{k \to \infty} u_{n_k} = u \text{ in } S' \text{ and } \|u\|_{B^{s}_{p,r}} \leq \liminf_{k \to \infty} \|u_{n_k}\|_{B^{s}_{p,r}}. \]

6. Let $m \in \mathbb{R}$ and $f$ be a $S^m$-multiplier, (i.e. $f$ is a smooth function and satisfies that $\forall \alpha \in \mathbb{N}^d$, $\exists C = C(\alpha)$, such that $|\partial^\alpha f(\xi)| \leq C(1 + |\xi|)^{|\alpha|-m}$, $\forall \xi \in \mathbb{R}^d$). Then the operator $f(D) = F^{-1}(f\mathcal{F})$ is continuous from $B^{s}_{p,r}$ to $B^{s-m}_{p,r}$.

We then introduce some useful lemmas in nonhomogeneous Besov space (homogeneous Besov space is similar).

Lemma 2.4. Let $s_1 < s_2$, $\theta \in (0, 1)$, and $(p, r)$ is in $[1, \infty]^2$, then we have
   \[ \|u\|_{B^{s_2+(1-\theta)s_2}_{p,r}} \leq \|u\|_{B^{s_1}_{p,r}}^\theta \|u\|_{B^{s_2}_{p,r}}^{1-\theta}. \]

(2) If $s \in \mathbb{R}$, $1 \leq p \leq \infty$, $\varepsilon > 0$, a constant $C = C(\varepsilon)$ exists such that
   \[ \|u\|_{B^{s}_{p,1}} \leq C\|u\|_{B^{s}_{p,\infty}} \ln \left(1 + \frac{\|u\|_{B^{s}_{p,\infty}}}{\|u\|_{B^{s}_{p,1}}} \right). \]

Lemma 2.5. Let $s > 0$ and any $(p, r)$ in $[1, \infty]^2$, the space $L^\infty \cap B^{s}_{p,r}$ is an algebra, and a constant $C = C(s, d)$ exists such that
   \[ \|uv\|_{B^{s}_{p,r}} \leq C(\|u\|_{L^\infty} \|v\|_{B^{s}_{p,r}} + \|u\|_{B^{s}_{p,r}} \|v\|_{L^\infty}). \]

(2) If $1 \leq p, r \leq \infty$, $s_1 \leq s_2$, $s_2 > \frac{d}{p}(s_2 \geq \frac{d}{p}$ if $r = 1$) and $s_1 + s_2 > \max(0, \frac{2d}{p} - d)$, there exists $C = C(s_1, s_2, p, r, d)$ such that
   \[ \|uv\|_{B^{s_2}_{p,r}} \leq C\|u\|_{B^{s_1}_{p,r}} \|v\|_{B^{s_2}_{p,r}}. \]

(3) If $1 \leq p \leq 2$, there exists $C = C(p, d)$ such that
   \[ \|uv\|_{B^{\frac{d}{p}-d}_{p,\infty}} \leq C\|u\|_{B^{\frac{d}{p}-d}_{p,\infty}} \|v\|_{B^{\frac{d}{p}}_{p,1}}. \]

Now we state some useful results in the transport equation theory, which are crucial to the proofs of our main theorem later.

\[
\begin{align*}
  \left\{ \begin{array}{ll}
    f_t + v \cdot \nabla f = g, & x \in \mathbb{R}^d, \ t > 0, \\
    f(0, x) = f_0(x), &
  \end{array} \right. 
\end{align*}
\]

(2.1)

Lemma 2.6. Let $s \in \mathbb{R}$, $1 \leq p, r \leq \infty$. There exists a constant $C$ such that for all solutions $f \in L^\infty([0, T]; B^{s}_{p,r})$ of (2.1) in one dimension with initial data $f_0$ in $B^{s}_{p,r}$, and $g \in L^1([0, T]; B^{s}_{p,r})$, we have, for a.e. $t \in [0, T]$,

\[ \|f(t)\|_{B^{s}_{p,r}} \leq e^{CV(t)} \left(\|f_0\|_{B^{s}_{p,r}} + \int_0^t e^{-CV(t')} \|g(t')\|_{B^{s}_{p,r}} dt' \right) \]

with

\[ V(t) = \begin{cases} \|\nabla v\|_{B^{s+1}_{p,r}}, & \text{if } s > \max(-\frac{1}{2}, \frac{1}{p} - 1), \\ \|\nabla v\|_{B^{s}_{p,r}}, & \text{if } s > \frac{1}{p} \text{ or } (s = \frac{1}{p}, \ p < \infty, \ r = 1), \end{cases} \]

and when $s = \frac{1}{p} - 1$, $1 \leq p \leq 2$, $r = \infty$, and $V(t) = \|\nabla v\|_{B^{\frac{1}{p}}_{p,1}}$. 

4
Lemma 2.7. \[1\] Let \(1 \leq p \leq p_1 \leq \infty, 1 \leq r \leq \infty, s > -d \min(\frac{1}{p_1}, \frac{1}{p})\). Let \(f_0 \in B^{s}_{p,r}, g \in L^{1}([0,T]; B^{s}_{\tilde{p},r}),\) and let \(v\) be a time-dependent vector field such that \(v \in L^{p}([0,T]; B^{s}_{\tilde{p},r})\) for some \(\tilde{p} > 1\) and \(M > 0\), and

\[
\nabla v \in L^{1}([0,T]; B^{	ilde{p}}_{\tilde{p},\infty}), \quad \text{if } s < 1 + \frac{d}{p_1},
\nabla v \in L^{1}([0,T]; B^{	ilde{p}}_{p,r}), \quad \text{if } s > 1 + \frac{d}{p_1} \text{ or } (s = 1 + \frac{d}{p_1} \text{ and } r = 1).
\]

Then the equation (2.1) has a unique solution \(f\) in

- the space \(C([0,T]; B^{s}_{p,r}), \text{ if } r < \infty,\)
- the space \(\bigcap_{s' < s} C([0,T]; B^{s'}_{p,p}) \bigcap C_w([0,T]; B^{s}_{\tilde{p},r}), \text{ if } r = \infty.\)

Lemma 2.8. \[18\] Let \(1 \leq p \leq \infty, 1 \leq r < \infty, s > \frac{d}{p} \text{ (or } s = \frac{d}{p}, p < \infty, r = 1).\) Denote \(\tilde{N} = \mathbb{N} \cup \{\infty\}.\)

Let \((v^n)_{n \in \tilde{N}} \in C([0,T]; B^{s+1}_{p,r}).\) Assume that \((f^n)_{n \in \tilde{N}} \in C([0,T]; B^{s}_{p,r})\) is the solution to

\[
\begin{aligned}
\begin{cases}
f^n + v^n \cdot \nabla f^n = g, & x \in \mathbb{R}^d, \ t > 0, \\
f^n(0,x) = f_0(x)
\end{cases}
\end{aligned}
\tag{2.2}
\]

with initial data \(f_0 \in B^{s}_{p,r}, g \in L^{1}([0,T]; B^{s}_{p,r})\) and that for some \(\alpha \in L^{1}([0,T]), \sup_{n \in \tilde{N}} \|v^n(t)\|_{B^{s+1}_{p,r}} \leq \alpha(t)\). If \(v^n \to v^\infty \text{ in } L^{1}([0,T]; B^{s}_{p,r}),\) then \(f^n \to f^\infty \text{ in } C([0,T]; B^{s}_{p,r}).\)

### 3 Local well-posedness

In this section, we establish local well-posedness of (1.3) in Besov spaces. In fact, though \(u \in B^{s}_{p,r} (1 \leq p < \infty, s > 0),\) and \(u\) is not a constant, the right hand side of (1.3)

\[
\int_{-\infty}^{x} \frac{1}{2} u^2(z) dz + g(t) \notin B^{s}_{p,r}.
\]

For example, if \(u \in H^2,\) let \(g(t) = 0, f(t,x) := \int_{-\infty}^{x} \frac{1}{2} u^2(z) dz,\) we have

\[
f_x(t,x) = \frac{1}{2} u^2(z) \geq 0.
\]

This implies \(f(t,x) \notin L^{p}(\mathbb{R}) (1 \leq p < \infty),\) let along \(f(t,x) \in B^{s}_{p,r}.\) To overcome this difficulty, we firstly introduce the following new function spaces.

**Definition 3.1.** Let \(s \geq 2\) and \(1 \leq p, r \leq \infty.\) Set

\[
E^s_{p,r} \triangleq \begin{cases} L^\infty \cap \dot{B}^{s-1}_{p,r} \cap \dot{B}^{s-2}_{p,r} \cap \dot{W}^{1,q}, & \text{when } s > 2, \\ E^s_{p,r}, & \text{when } s = 2, \end{cases}
\tag{3.1}
\]

\[
\tilde{E}^s_{p,r} \triangleq \begin{cases} L^\infty \cap \dot{B}^{s-1}_{p,r}, & \text{when } s > 2, \\ \tilde{E}^s_{p,r}, & \text{when } s = 2, \end{cases}
\tag{3.2}
\]

where

\[
E^2_{p,r} \triangleq \begin{cases} L^\infty \cap \dot{B}^1_{p,r} \cap \dot{B}^2_{p,r} \cap \dot{W}^{1,q}, & r = p, \text{ when } 1 \leq p \leq 2, \\ L^\infty \cap \dot{B}^1_{p,r} \cap \dot{B}^{2}_{p,r} \cap \dot{W}^{1,q}, & r = 2, \text{ when } 2 \leq p \leq \infty, \end{cases}
\tag{3.3}
\]

\[
\tilde{E}^2_{p,r} \triangleq \begin{cases} L^\infty \cap \dot{B}^1_{p,r}, & r = p, \text{ when } 1 \leq p \leq 2, \\ L^\infty \cap \dot{B}^1_{p,r}, & r = 2, \text{ when } 2 \leq p \leq \infty, \end{cases}
\tag{3.4}
\]

and \(\frac{1}{p} + \frac{1}{q} = 1.\)

For example, if \(s = p = r = 2,\) then \(E^2_{2,2} = L^\infty \cap \dot{H}^1 \cap \dot{H}^{2},\) which implies \(u \in B^{\frac{3}{2}}_{\infty,2} \text{ and } u_x \in H^1\) by proposition \(222 (\dot{H}^2 \to B^{\frac{3}{2}}_{\infty,2}, L^\infty \cap B^{\frac{3}{2}}_{\infty,2} = B^{\frac{3}{2}}_{\infty,2}).\) If \(s = 2, p = r = 1, E^2_{1,1} = L^\infty \cap \dot{B}^1_{1,1} \cap \dot{B}^{1}_{1,1},\) which implies \(u \in B^{1}_{\infty,1} \text{ and } u_x \in B^{1}_{1,1}.\) Actually, when \(1 \leq p \leq 2,\) we have \(\dot{B}^{s-1}_{p,p} \cap \dot{B}^{s-2}_{p,p} \subset \dot{W}^{1,q},\) so the space \(\dot{W}^{1,q}\) is unnecessary. Moreover, we can easily prove that both \(E^2_{p,r}\) and \(\tilde{E}^s_{p,r}\) are Banach spaces.

Here is our main result.
Theorem 3.2. Let \( u_0 \in L^p_{p,r} \). Then there exists a time \( T > 0 \) such that (3.1) has a unique solution \( u \) in \( C([0,T]; E^p_{p,r}) \cap C^1([0,T]; E^p_{p,r}) \). Moreover the solution depends continuously on the initial data.

Proof. We use six steps to prove Theorem 3.2. Without loss of generality, we consider the case \( g(t) = 0 \) in (1.3) for \( g(t) \) is just a bound term independent of \( x \). And for simplicity, we first consider the case \( E^2_{2,2} = L^\infty \cap H^1 \cap H^2 = B^2_{\infty,2} \cap H^1 \cap H^2 \).

Step one. Constructing approximate solution.

We firstly set \( u^0 \equiv 0 \), and define a sequence \( (u^n)_{n \in \mathbb{N}} \) of smooth functions by solving the following linear transport equations:

\[
\begin{cases}
  u^n_{t} + u^n u^n z = \int_{-\infty}^{x} \frac{1}{2} (u^n_z)^2 (z) dz, \\
  u^n |_{t=0} = u_0.
\end{cases}
\]  (3.5)

Define

\[
G^n = \int_{-\infty}^{x} \frac{1}{2} (u^n_z)^2 (z) dz.
\]

We assume that \( u^n \in C([0,T]; E^2_{2,2}) \cap C([0,T]; E^2_{2,2}) \) for all \( T > 0 \), then we have

\[
\|G^n\|_{L^\infty} \leq C\|u^n\|_{L^2} \leq C\|u^n\|^2_{E^2_{2,2}},
\]

\[
\|G^n_{xx}\|_{L^2} \leq C\|u^n_x\|_{L^\infty}\|u^n_x\|_{L^2} \leq C\|u^n\|^2_{E^2_{2,2}},
\]

\[
\|G^n_{xx}\|^2_{L^2} \leq C\|u^n_x\|_{L^\infty}\|u^n_{xx}\|_{L^2} \leq C\|u^n\|^4_{E^2_{2,2}}.
\]

This implies that

\[
\|G^n\|^2_{E^2_{2,2}} \leq C\|u^n\|^2_{E^2_{2,2}}.
\]  (3.6)

So \( G^n \in L^\infty([0,T]; E^2_{2,2}) \). Similar to the proof of Lemma 2.7 we deduce that (3.5) has a global solution \( u^{n+1} \) which belongs to \( C([0,T]; E^2_{2,2}) \cap C([0,T]; E^2_{2,2}) \) for all \( T > 0 \).

Step two. Uniform bounds.

Using Lemma 2.6 we have

\[
\|u^{n+1}(t)\|_{L^\infty} \leq C \int_0^t \|u^n\|_{E^2_{2,2}} dt' \left( \|u_0\|_{E^2_{2,2}} + \int_0^t \|G^n\|_{E^2_{2,2}} dt' \right),
\]  (3.7)

\[
\|u^{n+1}(t)\|_{H^1} \leq C \int_0^t \|u^n\|_{E^2_{2,2}} dt' \left( \|u_0\|_{E^2_{2,2}} + \int_0^t \|G^n\|_{E^2_{2,2}} dt' \right),
\]  (3.8)

Combining (3.7), (3.8) and (3.6), we get

\[
\|u^{n+1}(t)\|_{E^2_{2,2}} \leq C \int_0^t \|u^n\|_{E^2_{2,2}} dt' \left( \|u_0\|_{E^2_{2,2}} + \int_0^t \|G^n\|_{E^2_{2,2}} dt' \right).
\]  (3.9)

Then, we fix a \( T > 0 \) such that \( 2C^2T\|u_0\|_{E^2_{2,2}} < 1 \) and suppose that

\[
\forall t \in [0,T], \|u^n(t)\|_{E^2_{2,2}} \leq \frac{C\|m_0\|_{E^2_{2,2}}}{(1 - 2C^2t\|u_0\|_{E^2_{2,2}})}.
\]  (3.10)

Plugging (3.10) into (3.9) and using a simple calculation yield

\[
\|u^{n+1}(t)\|_{E^2_{2,2}} \leq C\|u_0\|_{E^2_{2,2}} \left( 1 - 2C^2t\|u_0\|_{E^2_{2,2}} \right)^{-\frac{1}{2}} \left( 1 + C^2\|u_0\|_{E^2_{2,2}} \int_0^t \left( 1 - 2C^2t\|u_0\|_{E^2_{2,2}} \right)^{-\frac{1}{2}} dt' \right).
\]  (3.11)
Using interpolation inequalities implies that
\[
\partial_t u^n + u^{n+1} \partial_x (u^{n+1} - u^n) = -(u^{n+m} - u^n)u_x^{n+1} + \int_{\infty}^{x} \frac{1}{2} (u_x^{n+m} - u_x^n)(u_x^{n+m} + u_x^n)dz.
\]

By virtue to Lemma 2.6 and (3.11), we have
\[
\|u^{n+m+1} - u^{n+1}\|_{L^\infty} \leq C_{u^n} \int_0^t \|u^{n+m} - u^n\|_{H^1 \cap L^\infty} ds.
\]

Combining (3.14) and (3.15) yields that
\[
\|u^{n+m+1} - u^{n+1}\|_{H^1 \cap L^\infty} \leq C_{u^n} \int_0^t \|u^{n+m} - u^n\|_{H^1 \cap L^\infty} ds.
\]

This implies \(\{u^n\}_{n \in \mathbb{N}}\) is a Cauchy sequence in \(C([0, T]; L^\infty \cap \dot{H}^1)\). Since \(\{u^n\}_{n \in \mathbb{N}}\) is bounded in \(L^\infty ([0, T]; E^2_{2,2})\). Using interpolation inequalities implies that
\[
u^n \to u \quad \text{in} \quad B^{\frac{3}{2} - \varepsilon}_{\infty, 2}, \quad \varepsilon > 0
\]
\[
u_x^n \to u_x \quad \text{in} \quad H^{1-\varepsilon}, \quad \varepsilon > 0
\]

We then prove that \(u \in E^2_{2,2}\) and satisfies (1.3). Applying the Fatou property we deduce that
\[
\left\{ \begin{array}{l}
u^n \text{ is bounded in } B^\frac{3}{2}_{\infty, 2} \implies \nu_n \to u \in B^\frac{3}{2}_{\infty, 2}, \quad \nu_{nx} \to u_x \in B^\frac{1}{2}_{\infty, 2} \\
u_x^n \text{ is bounded in } H^1 \implies \nu_{nx} \to u_x \in H^1 \subset B^\frac{7}{2}_{\infty, 2}
\end{array} \right.
\]

Because \(H^1 \subset B^\frac{7}{2}_{\infty, 2}\), we get \(\bar{u}_x = u_x\). In fact,
\[
< \bar{u}_x - u_x, \phi > = < u_x - u^n_x, \phi > + < u_x - u^n_x, \phi > \to 0, \quad \forall \phi \in C^\infty_0.
\]

Then we may pass to the limit in (3.16) easily and conclude that \(u\) is indeed a solution of (1.3) in the sense of distributions.
Finally, as \( u \) belongs to \( L^\infty([0,T]; E^2_{2,2}) \), the right-hand side of (3.3) also belongs to \( L^\infty([0,T]; L^\infty \cap \dot{H}^1) \), which implies \( u_t \) is in \( C([0,T]; L^\infty \cap \dot{H}^1) \). Similar to the proof of Lemma 2.7 in [1], we can easily deduce that \( u \) belongs to \( C([0,T]; E^2_{2,2}) \cap C^1([0,T]; \dot{E}^2_{2,2}) \).

**Step five. Uniqueness and continuous dependence.**

We will prove the uniqueness of solutions to (1.3) next. This proof is based on the way we have in Step 3. Suppose that \((m_1, m_2)\) are two solutions of (1.3), set \( w = m_1 - m_2 \), we obtain

\[
\partial_t w + m_1 \partial_x w = -wm_2 + \int_{-\infty}^{\infty} \frac{1}{2} w_x (m_1 + m_2) dz.
\]

By virtue of Lemma 2.6 we have

\[
\|w(t)\|_{L^\infty} \leq \|w(0)\|_{L^\infty} + C_{u_0} \int_0^t \|w\|_{H^1 \cap L^\infty} ds. \tag{3.18}
\]

and

\[
\|w(t)\|_{H^1} \leq \|w(0)\|_{H^1} + C_{u_0} \int_0^t \|w\|_{H^1 \cap L^\infty} ds. \tag{3.19}
\]

Combining (3.18), (3.19) and Gronwall’s inequality yield that

\[
\|w(t)\|_{H^1 \cap L^\infty} \leq \|w(0)\|_{H^1 \cap L^\infty} + C_{u_0} \int_0^t \|w\|_{H^1 \cap L^\infty} ds \leq C_{u_0} \|w(0)\|_{H^1 \cap L^\infty}. \tag{3.20}
\]

Therefore, the uniqueness is obvious in view of (3.20). Moreover, an interpolation argument ensures that the continuity with respect to the initial data holds for the norm \( C([0,T]; B^{2^{-\epsilon}}_{\infty,2} \cap \dot{H}^2) \) for \( \epsilon > 0 \) sufficient small. In fact, similar to the proof of [18], we can raise the continuity with respect to the initial data until \( C([0,T]; E^2_{2,2}) \).

**Step six. Other cases.**

Since we have prove the local well-posedness for (1.3) in \( C([0,T]; E^2_{2,2}) \cap C^1([0,T]; \dot{E}^2_{2,2}) \), other cases are similar. In fact, we should only take some modifications in step three (step six is similar).

Consider the Cauchy sequence \( w^{n+1} := u^{n+m+1} - u^{n+1} \):

\[
\frac{d}{dt} w^{n+1} + u^{n+1} w_x^{n+1} = -w^n u_x^{n+m+1} + \int_{-\infty}^{\infty} w_x^{n+1} (u_x^{n+m} + u_x^{n}) dz. \tag{3.21}
\]

Similar to Step 2, it’s easy to deduce that \( \{u_x^n\} \) is bounded in \( L^q \) if \( u_{x0} \in L^q \). Then by virtue of Lemma 2.6 and the Holder inequality, we have

\[
\|w^{n+1}(t)\|_{L^\infty} \leq C \int_0^t \|w^{n+m+1}\|_{L^\infty} \|w^{n+1}\|_{L^\infty} + \|w_x^{n+m} + u_x^n\|_{L^p} \|w^n\|_{W^{1,p}} ds
\]

\[
\leq C_{u_0} \int_0^t \|w^n\|_{L^\infty \cap W^{1,p}} ds \tag{3.22}
\]

and

\[
\|w_x^{n+1}(t)\|_{W^{1,p}} \leq C \int_0^t (\|w_x^{n+m+1}\|_{L^\infty} + \|u_x^n\|_{L^\infty}) (\|w^n\|_{W^{1,p}} + \|w_x^{n+1}\|_{W^{1,p}} + (\|u_x^{n+m+1}\|_{L^p}) \|w^n\|_{L^\infty} ds
\]

\[
\leq C_{u_0} \int_0^t \|w^n\|_{L^\infty \cap W^{1,p}} ds. \tag{3.23}
\]
where the last inequality from the Gronwall's inequality. Then Combining \(3.24\) and \(4.5\), we get

\[
\|w^{n+1}(t)\|_{L^\infty \cap \dot{W}^{1,p}} \leq C u_0 \int_0^t \|w^n\|_{L^\infty \cap \dot{W}^{1,p}} \, ds
\]

(3.24)

After some calculations we still get \(\{\nu^n\}_{n=1}^\infty\) is a Cauchy sequence in \(L^\infty \cap \dot{W}^{1,p}\).

This complete the proof.

Remark 3.3. For \(1 \leq p \leq 2\), we have \(\dot{B}^{1,1}_{p,p} \cap \dot{B}^{2}_{p,p} \hookrightarrow \dot{W}^{1,q}\). But for \(p > 2\), we have to consider it for an extra space \(\dot{W}^{1,q}\) for the initial data.

4 Blow-up and global existence

First we prove a conservation inequality for \(1.3\).

Lemma 4.1. Let \(u_0 \in E_{2,2}^s\), \(s > \frac{5}{2}\) and \(T^*\) be the maximal existence time of the corresponding solution \(u\) to \(1.3\), then we have

\[
\|u(t)\|_{\dot{H}_1} \leq \|u_0\|_{\dot{H}_1}.
\]

Proof. Differentiating the equation \(1.3\) and multiplying with \(2u_x\), we have

\[
\frac{d}{dt} u_x^2 + (u u_x^2)_x = 0.
\]

(4.1)

This implies

\[
\frac{d}{dt} \|u(t)\|_{\dot{H}_1} = 0.
\]

Next we state a blow-up criterion for \(1.3\).

Lemma 4.2. Let \(u_0 \in E_{2,2}^s = L^\infty \cap \dot{H}^1 \cap \dot{H}^s\) with \(s > \frac{5}{2}\) being as in Theorem 3.2 and let \(T^*\) be the maximal existence time of the corresponding solution \(u\) to \(1.3\). Then \(u\) blows up in finite time \(T^*\) if and only if

\[
\int_0^{T^*} \|u_x(t')\|_{L^\infty} \, dt' = \infty.
\]

Proof. Taking the \(L^\infty\) norm to \(1.3\) both sides, we have

\[
\|u(t)\|_{L^\infty} \leq C(\|u_0\|_{L^\infty} + \int_0^t \|u_x\|_{L^\infty} \|u\|_{L^\infty} + \|u_0x\|_{L^2}^2) \, ds,
\]

(4.2)

Then differentiating the equation \(1.3\),

\[
\frac{d}{dt} u_x + uu_{xx} = -\frac{1}{2} u_x^2.
\]

(4.3)

By virtue to Lemma 2.6 and \(u_x \in H^{s-1}, s > \frac{5}{2}\), we get

\[
\|u_x(t)\|_{H^{s-1}} \leq C(\|u_0\|_{H^{s-1}} + \int_0^t \|u_x\|_{L^\infty} \|u_x\|_{H^{s-1}} \, ds,
\]

(4.4)

Combining \(1.2\), \(1.3\) and the Gronwall inequality, we have

\[
\|u(t)\|_{E_{2,2}^s} \leq Ce^{\int_0^t \|u_x\|_{L^\infty} \, ds}[\|u_0\|_{E_{2,2}^s} + \|u_0x\|_{L^2}^2].
\]

(4.5)

If \(T^*\) is finite, and \(\int_0^{T^*} \|u_x\|_{L^\infty} \, dt' < \infty\), then \(u \in L^\infty([0, T^*]; E_{2,2}^s)\), which contradicts the assumption that \(T^*\) is the maximal existence time.

On the other hand, by Theorem 5.2 and the fact that \(E_{2,2}^s \hookrightarrow \dot{W}^{1,\infty}\), if \(\int_0^{T^*} \|u_x\|_{L^\infty} \, dt' = \infty\), then \(u\) must blow up in finite time.
Remark 4.3. By the algebra interpolation, we can easily get a weaker blow-up criterion for (1.3):
\[
\lim_{t \to T} ||u_x||_{L^0_{0,\infty}} = \infty.
\]
Let us consider the ordinary differential equation:
\[
\begin{cases}
q_t(t, x) = u(t, q(t, x)), & t \in [0, T), \\
q(0, x) = x, & x \in \mathbb{R}.
\end{cases}
\]
(4.6)
If \( u \in E^s_t \) with \( s \geq 2 \) being as in Theorem 3.2 then \( u \in C([0, T); C^{0,1}) \). By the classical results in the theory of ordinary differential equations, we can easily infer that (4.6) have a unique solution \( q \in C^1([0, T) \times \mathbb{R}; \mathbb{R}) \) such that the map \( q(t, \cdot) \) is an increasing diffeomorphism of \( \mathbb{R} \) with
\[
q_x(t, x) = \exp \left( \int_0^t u(t', q(t', x)) \, dt' \right) > 0, \quad \forall (t, x) \in [0, T) \times \mathbb{R}.
\]
Now the following theorem shows that under particular condition for the initial data, the corresponding solution \( u \) of (1.3) exists globally in time.

Theorem 4.4. Let \( u_0 \in E^s_{2,2}, s > \frac{7}{2} \). Assume \( u_{0,xx}(x) \geq 0 \) when \( x \leq x_0 \), \( u_{0,xx}(x) \leq 0 \) when \( x \geq x_0 \) for some \( x_0 \in \mathbb{R} \), and \( u_{0,xx}(x_0) > 0 \). Then the corresponding solution \( u \) of (1.3) exists globally in time.

Proof. Arguing by density, now we assume \( s > \frac{7}{2} \). Differentiating (4.10), we have
\[
\frac{d}{dt} u_{xx} + uu_{xxx} = -2u_x u_{xx}
\]
(4.7)
Then we have
\[
\frac{d}{dt} u_{xx}(t, q(t, x)) = -2u_x u_{xx}.
\]
Hence,
\[
u_{xx}(t, q(t, x)) = u_{0,xx} \exp \left( \int_0^t -2u_x(t', q(t', x)) \, dt' \right),
\]
which implies that \( u_{xx} \) doesn’t change sign, so we can deduce that
\[
u_{xx}(t, x) \geq 0, \quad \text{when} \quad x \leq q(t, x_0) \quad \text{and} \quad u_{xx}(t, x) \leq 0, \quad \text{when} \quad x \geq q(t, x_0).
\]
(4.8)
Moreover, we have
\[
\frac{d}{dt} u_x(t, q(t, x)) = -\frac{1}{2} u_x^2 \leq 0.
\]
(4.9)
Using the fact that the flow \( q(t, x) \) is a diermorphsim and \( u_{0,xx}(x_0) > 0 \), and by (4.8) we deduce that \( q(t, x_0) \) is the maximum value point. (4.9) tells us \( u_x(t) \) will decrease monotonically at the point \( q(t, x_0) \) along the flow. Moreover, since \( u_x(t) \) belongs to \( H^{s-1} \), it will decrease but will not be less than zero. Otherwise, it will contradict with the decay of infinity in \( H^s \).

As a result, we can deduce that the maximum point is at the initial point \( x_0 \) such that
\[
\|u_x\|_{L^\infty} \leq \|u_{0,x}\|_{L^\infty},
\]
which means the solution is global by Lemma 4.2.

Remark 4.5. The initial datas satisfying the conditions of Theorem 4.4 are exist such as \( u_0(x) = \int_{-\infty}^x e^{-z^2} \, dz \). It’s easy to deduce that \( u_0(x) \in L^\infty \cap H^1 \cap H^2, u_{0,xx}(0) > 0 \), and \( u_{0,xx}(x) \geq 0 \) when \( x \leq 0 \) and \( u_{0,xx}(x) \leq 0 \) when \( x \geq 0 \).

We then shows that the corresponding solution of (1.3) will blow up by giving some particular condition for the initial data.

Theorem 4.6. Let \( u_0 \in E^s_{2,2}, s \geq 2 \). Assume that there exists a point \( x_0 \) such that \( u_{0,xx}(x_0) < 0 \), Then the corresponding solution \( u \) of (1.3) blows up in finite time.
Proof. Arguing by density, now we assume \( s > \frac{5}{2} \). From (4.10), we have

\[
\frac{d}{dt} u_x(t, q(t, x)) = -\frac{1}{2} u_x^2.
\]

Solving the above equality, we finally get

\[
u_x(t, q(t, x)) = \frac{2}{t + \frac{2}{u_0}}
\]

As \( u_{0x}(x_0) < 0 \), we can easily deduce that the maximal time \( T < -\frac{2}{u_0(x_0)} \).

Therefore, from (4.11) we know \( u_x(t) \to -\infty \) as \( t \to T \). By Lemma 4.2, the solution \( u \) will blow up in finite time.

\section{ill-posedness}

In this section, we are going to prove the norm inflation in \( A \triangleq L^\infty \cap \dot{B}^\frac{1}{2}_{p,r} \cap \dot{B}^{1+\frac{1}{r}}_{p,r} \) with \( 1 \leq p \leq \infty, r > 1 \).

\begin{theorem}
 Let \( 1 < r \leq \infty, 1 \leq p \leq \infty \). For any \( \epsilon > 0 \), there exists \( u_0 \in H^\infty \) such that

\begin{enumerate}[(1)]
\item \( \| u_0 \|_A \leq \epsilon \).
\item There exists a unique solution \( u \in E_{p,r}^\infty \) with maximal \( T < \epsilon \).
\item \( \lim_{t \to T} \| u_x \|_{B^{1+\frac{1}{r}}_{p,r}} \geq \lim_{t \to T} \| u_x \|_{B^{1+\frac{1}{r}}_{p,r}} = \infty \).
\end{enumerate}
\end{theorem}

\begin{proof}
 We first define

\[ h(x) = \sum_{k \geq 1} \frac{h_k(x)}{\gamma^{2k}} \]

where \( h_k(\xi) = i2^{-k}\xi \varphi(2^{-k}\xi) \), and \( \varphi(\xi) \) is even non-negative, non-zero \( C_0^\infty \) function in \([22]\). Similar to the proof of \([22]\), we can easily prove that

\[ \| h \|_A \leq \| h \|_{B^{1+\frac{1}{r}}_{p,r}} \leq C, \quad h'(0) = -\infty \]

For any \( \epsilon > 0 \) small enough, let \( u_0 := \frac{\epsilon S \gamma^k}{\| h \|_{B^{1+\frac{1}{r}}_{p,r}}} \), where \( N \) is sufficient large such that \( u_0(0) < -2\epsilon^{-1} \).

Then \( \| u_0 \|_A \leq \epsilon \) and \( u_0 \in E_{p,r}^\infty \) for fixed \( N \).

Finally, by Theorem 5.2 and Theorem 4.2 we get a unique local solution \( u \in E_{p,r}^\infty \) and \( u(t, x) \) blow up in finite time \( T \), we deduce that

\[ \lim_{t \to T} \| u_x \|_{B^{1+\frac{1}{r}}_{p,r}} \geq \lim_{t \to T} \| u_x \|_{B^{1+\frac{1}{r}}_{p,r}} = \infty. \]

Then we find the example:

\[ \| u_0 \|_A \leq \epsilon, \quad \| u(t) \|_A \to \infty \]

This implies the ill-posedness of (1.3).
\end{proof}

\begin{remark}
 For \( s = 2, p = r = 1 \), we get the local well-posedness of (1.3) in \( E_{2,1}^2 \cap \dot{B}_{1,1}^{1+\frac{1}{2}} \), the above theorem implies the solution will be ill-posed in \( E_{2,1}^2 \cap \dot{B}_{1,1}^{1+\frac{1}{2}} \), the above theorem implies the solution will be ill-posed in \( E_{2,r}^{2 \frac{3}{2}} \).

For \( s = 2, p = r = 2 \), we also get the local well-posedness of (1.3) in \( E_{2,2}^2 \cap \dot{B}_{1,1}^{1+\frac{1}{2}} \), the above theorem implies the solution will be ill-posed in \( L^s \cap \dot{H}^\frac{3}{2} \cap \dot{H}^\frac{1}{2} \). However, the above theorem only implies the solution will be ill-posed in \( L^s \cap \dot{H}^1 \cap \dot{H}^\frac{1}{2} \), we don’t whether the solution will be well-posed or ill-posed in \( L^\infty \cap \dot{H}^s \cap \dot{H}^\frac{3}{2} \), \( \frac{3}{2} \leq s < 2 \).
\end{remark}
6 Unique continuation

In this final section, we consider the unique continuation of (1.3) with \( g(t) = C \int_{-\infty}^{\infty} u_x^2(z)dz, \quad C \in \mathbb{R} \). We first recall the equation:

\[
\begin{align*}
    u_x + uu_x &= \int_{-\infty}^{x} \frac{1}{2} u_x^2(z)dz + C \int_{-\infty}^{\infty} u_x^2(z)dz, \\
    u(0, x) &= u_0(x), \quad x \in \mathbb{R},
\end{align*}
\]

(6.1)

**Theorem 6.1.** Let \( u(t, x) \) be a real strong solution of (1.3). If there exists an open set \( \Omega = (a, b) \times [t_1, t_2] \), \((a, b \in \mathbb{R}, t_1, t_2 \geq 0)\) such that 

\[ u(t, x) = 0, \quad (x, t) \in \Omega, \]

and \( u(t, x) \) meets one of the following cases:

1) For \( C > 0 \),
2) For \( C = 0, b = \infty \),
3) For \(-\frac{1}{2} < C < 0, b = \infty \) or \( a = -\infty \),
4) For \( C = -\frac{1}{2}, a = -\infty \),
5) For \( C < -\frac{1}{2} \),

then \( u \equiv 0, \quad (x, t) \in \mathbb{R}^+ \times \mathbb{R} \).

**Proof.** 1) For \( C > 0 \), (1.3) can be wrote as

\[ u_x + uu_x = \int_{-\infty}^{x} \frac{1}{2} u_x^2(z)dz + C \int_{-\infty}^{\infty} u_x^2(z)dz. \]

From the hypothesis it follows that

\[ u_x + uu_x = 0, \quad (x, t) \in \Omega. \]

This implies that

\[ \int_{-\infty}^{x} \frac{1}{2} u_x^2(z)dz + C \int_{-\infty}^{\infty} \frac{1}{2} u_x^2(z)dz = 0, \quad (x, t) \in \Omega. \]

As \( \Omega = (a, b) \times [t_1, t_2] \), we can easily get that \( u_x = 0, \quad (x, t) \in \mathbb{R} \times [t_1, t_2] \). The continuity of the strong solution implies \( u = 0, \quad (x, t) \in \mathbb{R} \times [t_1, t_2] \). Then we get \( u \equiv 0, \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+ \) by the uniqueness.

Other cases are similar to 1), we omit it here.

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