X-TORSION AND UNIVERSAL GROUPS

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Abstract. For a set $X \subseteq \mathbb{N}$, we define the $X$-torsion of a group $G$ to be all elements $g \in G$ with $g^n = e$ for some $n \in X$. With $X$ recursively enumerable, we give two independent proofs (group-theoretic, and model-theoretic) that there exists a universal finitely presented $X$-torsion-free group; one which contains all finitely presented $X$-torsion-free groups. We also show that, if $X$ is recursively enumerable, then the set of finite presentations of $X$-torsion-free groups is $\Pi^0_2$-complete in Kleene’s arithmetic hierarchy.

1. Introduction

Torsion is a well-studied object in group theory. We let $o(g)$ denote the order of a group element $g$. Recalling that $g \in G$ is torsion if $1 \leq o(g) < \omega$, we write $\text{Tor}(G) := \{g \in G \mid g \text{ is torsion}\}$. So, what if we were to ‘restrict’ the type of torsion we are looking at? Perhaps we are only concerned with 2-torsion, or torsion elements of prime order. So, for any set $X \subseteq \mathbb{N}$, we define the $X$-torsion of a group $G$, written $\text{Tor}^X(G)$, as

$$\text{Tor}^X(G) := \{g \in G \mid \exists n \in X \text{ with } g^n = e\}$$

It is clear that, for the case $X = \mathbb{N}$, we have $\text{Tor}^\mathbb{N}(G) = \text{Tor}(G)$ in the usual sense. For any set $X \subseteq \mathbb{N}$, we define the factor completion of $X$ to be $X^{fc} := \{n \in \mathbb{N} \mid \exists m \geq 1, nm \in X\}$, and we say $X$ is factor complete if $X^{fc} = X$. We say that a group $G$ is $X$-torsion-free if $\text{Tor}^X(G) = \{e\}$; equivalently, if $\text{Tord}(G) \cap X^{fc} = \emptyset$. We use the following notation for the set of orders of torsion elements in a group:

$$\text{Tord}(G) := \{n \in \mathbb{N} \mid \exists g \in \text{Tor}(G) \text{ with } o(g) = n \geq 2\}$$

One famous consequence of the Higman Embedding Theorem [6] is the fact that there is a universal finitely presented (f.p.) group; that is, a finitely presented group into which all finitely presented groups embed. Recently, Belegradek in [1] and Chiodo in [4] independently showed that there exists a universal f.p. torsion-free group; that is, an f.p. torsion-free group into which all f.p. torsion-free groups embed. In this paper we generalise this result further, in the context of $X$-torsion-freeness, as follows:

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Theorem 2.12. Let $X$ be a recursively enumerable set of integers. Then there is a universal finitely presented $X$-torsion-free group $G$. That is, $G$ is $X$-torsion-free, and for any finitely presented group $H$ we have that $H \hookrightarrow G$ if (and only if) $H$ is $X$-torsion-free.

There are two key steps in proving this theorem. The first is to construct a free product of all such groups, in an algorithmic way:

Theorem 2.9. Let $A$ be an r.e. set of integers. Then there is a countably generated recursive presentation $Q$ of a group $\overline{Q}$ which is $A$-torsion-free, and contains an embedded copy of every countably generated recursively presentable $A$-torsion-free group.

From this, we can construct a finitely presented example, using the following theorem of Higman:

Theorem 2.10. There is a uniform algorithm that, on input of a countably generated recursive presentation $P = (X| R)$, constructs a finite presentation $\overline{T(P)}$ such that $P \hookrightarrow \overline{T(P)}$ and $\text{Tord}(P) = \text{Tord}(\overline{T(P)})$, along with an explicit embedding $\phi : P \hookrightarrow \overline{T(P)}$.

We prove Theorem 2.9 in the following two independent ways, for reasons which we outline shortly. Firstly, in Section 2, we generalise the construction in [4, Theorem 3.10] of a universal f.p. torsion-free group, using arguments in group theory. Then, in Section 3, we generalise the construction in [1, Theorem A.1] of a universal f.p. torsion-free group, using arguments in model theory. The results in Section 3 provide the framework for proving that there might be other group properties for which there exist universal f.p. examples. Our main result in Section 3, which we apply directly to the theory of $X$-torsion-free groups to prove Theorem 2.9, is the following:

Theorem 3.8. If $T \supseteq T_{Grp}$ is an r.e. universal Horn $\mathcal{L}_{Grp}$-theory then there exists a recursively presented group $G \models T$ such that every recursively presented group $H \models T$ embeds into $G$.

The reason for providing both proofs of Theorem 2.9 is twofold. On the one hand, the group-theoretic proof in Section 2 is more direct, and gives a clear picture of why Theorem 2.9 holds. It is explicit, and algorithmic. On the other hand, the model-theoretic proof in Section 3 is more general, and might lead to showing that other group properties possess universal f.p. examples. Indeed, it was only by looking at the model-theoretic arguments in [1, Theorem A.1] that we realised we could consider $X$-torsion as an object, and that the results of [4] could be generalised to this; we would never have made the connection otherwise.
On this, it would be interesting further work to see what other group-theoretic properties $\rho$ possess universal f.p. examples. A potential proof technique would be as follows:

1. Show that the property $\rho$ satisfies the conditions of Theorem 3.7 (that is, closed under free products, identity, and subgroups).
2. Show that $\rho$ has an r.e. universal Horn $L_{Grp}$-theory, thus allowing us to apply Theorem 3.8.
3. Show that $\rho$ is possessed by finitely generated free groups, and preserved by free products and HNN extensions (which is what is needed to show that it is preserved under the Higman embedding of Theorem 2.10).

Of course, satisfying all of the above is quite difficult. One can easily find many group properties that satisfy (1), and some of these also satisfy (2) quite trivially. But there are very few properties of groups that satisfy (3), and therein lies the problem.

We finish our paper with a generalisation a result from [4] on the complexity of recognising torsion-freeness, and show the following:

**Theorem 2.13** Let $X \subseteq \mathbb{N}_{\geq 2}$ be a non-empty r.e. set of integers. Then the set of finite presentations of $X$-torsion-free groups is $\Pi^0_2$-complete.

This is remarkable; even if $X$ is a finite set, or stronger still, a single prime (say $X = \{2\}$), the set of finite presentations of groups with no $X$-torsion will still form a $\Pi^0_2$-complete set.

## 2. $X$-torsion and universality

This section is a generalisation of the definitions and results in [4, Section 3] on universal torsion-free quotients and universal finitely presented torsion-free groups.

### 2.1. Notation.

If $P$ is a group presentation, we denote by $\mathbf{P}$ the group presented by $P$, and $\mathbf{w}$ by the group element represented by the word $w$. A presentation $P = \langle X \mid R \rangle$ is said to be a recursive presentation if $X$ is a finite set and $R$ is a recursive enumeration of relations; $P$ is said to be a countably generated recursive presentation if instead $X$ is a recursive enumeration of generators. A group $G$ is said to be finitely (respectively, recursively) presentable if $G \cong \mathbf{P}$ for some finite (respectively, recursive) presentation $P$. If $P,Q$ are group presentations then we denote their free product presentation by $P \ast Q$: this is given by taking the disjoint union of their generators and relations. If $g_1, \ldots, g_n$ are elements of a group $G$, we write $\langle g_1, \ldots, g_n \rangle$ for the subgroup in $G$ generated by these elements and $\langle g_1, \ldots, g_n \rangle^G$ for the normal closure of these elements in $G$. Let $\omega$ denote the smallest infinite ordinal. Let $|X|$ denote the cardinality of a set $X$. If $X$ is a set, let $X^{-1}$ be a set of the same cardinality as and disjoint from $X$ along with a fixed bijection $*^{-1} : X \to X^{-1}$. Write $X^*$ for the set of finite words on $X \cup X^{-1}$. We will make use of $\Sigma_n$ sets and $\Pi_n$ sets; see [8] for an introduction to these.
2.2. $X$-torsion.

If $G, H$ are groups, and $X$ a set of integers with $H$ $X$-torsion-free, a surjective homomorphism $h : G \rightarrow H$ is universal if, for any $X$-torsion-free $K$ and any homomorphism $f : G \rightarrow K$, there is a homomorphism $\phi : H \rightarrow K$ such that $f = \phi \circ h : G \rightarrow K$, i.e., the following diagram commutes:

\[
\begin{array}{c}
\text{G} \\
\downarrow f \quad \downarrow \phi \\
\text{K}
\end{array}
\]

Note that if $\phi$ exists then it will be unique. Indeed, if $\phi'$ also satisfies $f = \phi' \circ h$, then $\phi = \phi' \circ h$, and hence $\phi = \phi'$ as $h$ is a surjection and thus is right-cancellative. Moreover, any such $H$ is unique, up to isomorphism. Such an $H$ is called the universal $X$-torsion-free quotient for $G$, denoted $G^{X}$-tf. Observe that if $G$ is itself $X$-torsion-free, then $G^{X}$-tf exists and $G^{X}$-tf $\subseteq G$, as the identity map $\text{id}_G : G \rightarrow G$ has the universal property above.

A standard construction, showing that $G^{X}$-tf exists for every group $G$, is done via taking the quotient of $G$ by its $X$-torsion-free radical $\rho^X(G)$, where $\rho^X(G)$ is the intersection of all normal subgroups $N \triangleleft G$ with $G/N$ $X$-torsion-free (a generalisation of the torsion-free radical, $\rho(G)$, in [2]). It follows immediately that $G/\rho^X(G)$ has all the properties of an $X$-torsion-free universal quotient for $G$.

We present here an alternative construction for $G^{X}$-tf which, though isomorphic to $G/\rho^X(G)$, lends itself more easily to an effective procedure for finitely (or recursively) presented groups.

**Definition 2.1.** Given a group $G$, and a set of integers $X \subseteq \mathbb{N}$, we inductively define $\text{Tor}_n^X(G)$ as follows:

\[
\text{Tor}_0^X(G) := \{e\},
\]

\[
\text{Tor}_{n+1}^X(G) := \langle \{g \in G \mid g \text{ Tor}_n^X(G) \in \text{Tor}^X(G/\text{Tor}_n^X(G))\} \rangle_G,
\]

\[
\text{Tor}_\omega^X(G) := \bigcup_{n \in \mathbb{N}} \text{Tor}_n^X(G).
\]

Thus, $\text{Tor}_i^X(G)$ is the set of elements of $G$ which are annihilated upon taking $i$ successive quotients of $G$ by the normal closure of all $X$-torsion elements, and $\text{Tor}_\omega^X(G)$ is the union of all these.

**Lemma 2.2.** If $G$ is a group, then $G/\text{Tor}_\omega^X(G)$ is $X$-torsion-free.

*Proof.* Suppose $g \text{ Tor}_\omega^X(G) \in \text{Tor}^X(G/\text{Tor}_\omega^X(G))$. Then $g^n \text{ Tor}_\omega^X(G) = e$ in $G/\text{Tor}_\omega^X(G)$ for some $1 \leq n \in X$, so $g^n \in \text{Tor}_\omega^X(G)$. Thus there is some $i \in \mathbb{N}$ such that $g^n \in \text{Tor}_i^X(G)$, and hence $g \text{ Tor}_i^X(G) \in \text{Tor}^X(G/\text{Tor}_i^X(G))$. Thus $g \in \text{Tor}_{i+1}^X(G) \subseteq \text{Tor}_\omega^X(G)$, and so $g \text{ Tor}_\omega^X(G) = e$ in $G/\text{Tor}_\omega^X(G)$. \hfill \Box

**Proposition 2.3.** If $G$ is a group, then $\rho^X(G) = \text{Tor}_\omega^X(G)$.

*Proof.* Clearly $\rho^X(G) \subseteq \text{Tor}_\omega^X(G)$, by definition of $\rho^X(G)$ and the fact that $G/\text{Tor}_\omega^X(G)$ is torsion-free (Lemma 2.2). It remains to show that $\text{Tor}_\omega^X(G)$ $\subseteq$ $\rho^X(G)$. We proceed by contradiction, so assume $\text{Tor}_\omega^X(G) \nsubseteq \rho^X(G)$. Then
there is some \( N \triangleleft G \) with \( G/N \) \( X \)-torsion-free, along with some minimal \( i \) such that \( \text{Tor}_i^X(G) \notin N \) (clearly, \( i > 0 \), as \( \text{Tor}_0^X(G) = \{e\} \)). Then, by definition of \( \text{Tor}_i^X(G) \) and the fact that \( N \) is normal, there exists \( e \neq g \in \text{Tor}_i^X(G) \) such that \( g \text{Tor}_{i-1}^X(G) \in \text{Tor}_X(G/\text{Tor}_{i-1}^X(G)) \) and \( g \notin N \) (or else \( \text{Tor}_1^X(G) \subseteq N \)). But then \( g^n \in \text{Tor}_{i-1}(G) \) for some \( 1 < n \in X \). Since \( \text{Tor}_{i-1}^X(G) \subseteq N \) by minimality of \( i \), we have that \( gN \) is a (non-trivial) \( X \)-torsion element of \( G/N \), contradicting \( X \)-torsion-freeness of \( G/N \). Hence \( \text{Tor}_n^X(G) \in \rho^X(G) \).

\[ \square \]

**Corollary 2.4.** If \( G \) is a group, then \( G/\text{Tor}_n^X(G) \cong G^{X-\text{tf}} \).

What follows is a standard result, which we state without proof.

**Lemma 2.5.** Let \( P = (X|R) \) be a countably generated recursive presentation. Then the set of words \( \{w \in X^* \mid \overline{w} \in \overline{P}\} \) is r.e.

**Lemma 2.6.** Let \( P = (X|R) \) be a countably generated recursive presentation, and \( A \) an r.e. set of integers. Then the set of words \( \{w \in X^* \mid \overline{w} \in \text{Tor}^A(\overline{P})\} \) is r.e.

**Proof.** Take any recursive enumeration \( \{w_1, w_2, \ldots\} \) of \( X^* \). Using Lemma 2.5 start checking if \( \overline{w_i} = e \in \overline{P} \) for each \( w_i \in X^* \) and each \( n \in A \) (by proceeding along finite diagonals). For each \( w_i \) we come across which is \( A \)-torsion, add it to our enumeration. This procedure will enumerate all words in \( \text{Tor}^A(\overline{P}) \), and only words in \( \text{Tor}^A(\overline{P}) \). Thus the set of words in \( X^* \) representing elements in \( \text{Tor}^A(\overline{P}) \) is r.e. \( \square \)

We use this to show the following:

**Lemma 2.7.** Given a countably generated recursive presentation \( P = (X|R) \), and r.e. set of integers \( A \), the set \( T^A_i := \{w \in X^* \mid \overline{w} \in \text{Tor}^A(\overline{P})\} \) is r.e., uniformly over all \( i \) and all such presentations \( P \). Moreover, the union \( T^A_0 := \bigcup T^A_i \) is r.e., and is precisely the set \( \{w \in X^* \mid \overline{w} \in \text{Tor}^A(\overline{P})\} \).

**Proof.** We proceed by induction. Clearly \( \text{Tor}_1^A(\overline{P}) \) is r.e., as it is the normal closure of \( \text{Tor}^A(\overline{P}) \), which is r.e. by Lemma 2.6. So assume that \( \text{Tor}_i^A(\overline{P}) \) is r.e. for all \( i \leq n \). Then \( \text{Tor}_{n+1}^A(\overline{P}) \) is the normal closure of \( \text{Tor}^A(\overline{P}) / \text{Tor}_n^A(\overline{P}) \), which again is r.e. by the induction hypothesis and Lemma 2.6. The rest of the lemma then follows immediately. \( \square \)

**Proposition 2.8.** There is a uniform algorithm that, on input of a countably generated recursive presentation \( P = (X|R) \) of a group \( \overline{P} \), and an r.e. set of integers \( A \), outputs a countably generated recursive presentation \( P^{A-\text{tf}} = (X|R') \) (on the same generating set \( X \), and with \( R \subseteq R' \) as sets) such that \( P^{A-\text{tf}} \cong \overline{P}^{A-\text{tf}} \), with associated surjection given by extending \( \text{id}_X : X \rightarrow X \).

**Proof.** By Corollary 2.4, \( \overline{P}^{A-\text{tf}} \) is the group \( \overline{P}/\text{Tor}_n^A(\overline{P}) \). Then, with the notation of Lemma 2.7, it can be seen that \( P^{A-\text{tf}} := (X|R \cup T_n) \) is a countably generated recursive presentation for \( \overline{P}^{A-\text{tf}} \), uniformly constructed from \( P \). \( \square \)
2.3. Universality and complexity of $X$-torsion.

With the above machinery, we can now prove the main technical result of this section; Theorem 2.9. We will re-prove this result again in Section 3 using tools from model theory.

**Theorem 2.9.** Let $A$ be an r.e. set of integers. Then there is a countably generated recursive presentation $Q$ of a group $\overline{Q}$ which is $A$-torsion-free, and contains an embedded copy of every countably generated recursively presentable $A$-torsion-free group.

**Proof.** Take an enumeration $P_1, P_2, \ldots$ of all countably generated recursive presentations of groups, and construct the countably generated recursive presentation $Q := P_1^{A-tf} * P_2^{A-tf} * \ldots$; this is the countably infinite free product of the universal $A$-torsion-free quotient of all countably generated recursively presentable groups (with some repetition). As each $P_i^{A-tf}$ is uniformly constructible from $P_i$ (by Proposition 2.8), we have that our construction of $Q$ is indeed effective, and hence $Q$ is a countably generated recursive presentation. Also, Proposition 2.8 shows that $\overline{Q}$ is an $A$-torsion-free group, as we have successfully annihilated all the $A$-torsion in the free product factors, and the free product of $A$-torsion-free groups is again $A$-torsion-free. Moreover, $\overline{Q}$ contains an embedded copy of every $A$-torsion-free countably generated recursively presentable group, as the universal $A$-torsion-free quotient of an $A$-torsion-free group is itself. □

As detailed in [3, Lemma 6.9 and Theorem 6.10], the following is implicit in Rotman’s proof [9, Theorem 12.18] of the Higman Embedding Theorem.

**Theorem 2.10.** There is a uniform algorithm that, on input of a countably generated recursive presentation $P = \langle X \mid R \rangle$, constructs a finite presentation $T(P)$ such that $P \hookrightarrow T(P)$ and $\text{Tord}(\overline{P}) = \text{Tord}(T(P))$, along with an explicit embedding $\phi : P \hookrightarrow T(P)$.

We can now prove our main result:

**Theorem 2.11.** Let $A$ be an r.e. set of integers. Then there is a finitely presentable group $G$ which is $A$-torsion-free, and contains an embedded copy of every countably generated recursively presentable $A$-torsion-free group.

**Proof.** We construct $Q$ as in Theorem 2.9 and then use Theorem 2.10 to embed $\overline{Q}$ into a finitely presentable group $T(Q)$. By construction, $\text{Tord}(\overline{Q}) = \text{Tord}(T(Q))$, so $\overline{T(Q)}$ is $A$-torsion-free. Finally, $\overline{T(Q)}$ has an embedded copy of every countably generated recursively presentable $A$-torsion-free group, since $\overline{Q}$ did. Taking $G$ to be $\overline{T(Q)}$ completes the proof. □

As all f.p. groups are recursively presentable, we have the following corollary:

**Theorem 2.12.** Let $A$ be an r.e. set of integers. Then there is a universal finitely presented $A$-torsion-free group $G$. That is, $G$ is $A$-torsion-free, and for any finitely presented group $H$ we have that $H \hookrightarrow G$ if (and only if) $H$ is $A$-torsion-free.

The following is an unexpected and very strong generalisation of [4, Theorem 4.2], classifying the computational complexity of recognising f.p. $A$-torsion-free groups.
Theorem 2.13. Let $A \subseteq \mathbb{N}_{\geq 2}$ be a non-empty set of integers. Then the set of finite presentations of $A$-torsion-free groups is $\Pi^0_2$-hard. Moreover, if $A$ is r.e., then this set of presentations is $\Pi^0_2$-complete.

Proof. This follows the proofs of [3, Lemma 6.11] and [4, Theorem 4.2]. First, there must be some element $1 < a \in A$. Now, given $n \in \mathbb{N}$, form the recursive presentation $P_n := \langle x_1, x_2, \ldots \mid x_i^a = e \forall i \in \mathbb{N}, x_j = e \forall j \in W_n \rangle$. Then form the finite presentation $Q_n$ using Higman’s Embedding Theorem (Theorem 2.10) so that $\text{Tord}(Q_n) = \text{Tord}(P_n)$. Now note that $Q_n$ is $A$-torsion-free $\iff \forall n \in A \forall i \in \mathbb{N} \forall j \in W_n$; the latter being a $\Pi^0_2$-complete set ([4, Lemma 4.1]). So the set of finite presentations of $A$-torsion-free groups is $\Pi^0_2$-hard, for any non-empty $A \subseteq \mathbb{N}_{\geq 2}$.

Moreover, when $A$ is also r.e., this set has the following $\Pi^0_2$ description:

$$G \text{ is } A\text{-torsion-free } \iff (\forall w \in G)(\forall n \in A)(w^n \neq_G e \text{ or } w =_G e)$$

and is thus $\Pi^0_2$-complete.

2.4. $X$-torsion-length.

We finish this section by generalising the notion of torsion length, which was first introduced in [5, Definition 2.5].

Definition 2.14. Given $X \subseteq \mathbb{N}$, we define the $X$-Torsion Length of $G$, $\text{TorLen}^X(G)$, by the smallest ordinal $n$ such that $\text{Tor}^X_n(G) = \text{Tor}^X_\omega(G)$.

Rather than go and re-work all the theory developed in [5], we simply state here the main results [5, Theorem 3.3 and Theorem 3.10], generalised to $X$-torsion. Going through the work in [5], it is straightforward to see that all results there generalise to $X$-torsion, and thus we refrain from doing so here.

Theorem 2.15. Given any $\emptyset \neq X \subseteq \mathbb{N}_{\geq 2}$, there is a family of finite presentations $\{P_n\}_{n \in \mathbb{N}}$ of groups satisfying $\text{TorLen}^X(P_n) = n$ and $P_n/\text{Tor}^Y_1(P_n) \cong P_{n-1}$.

Theorem 2.16. Given any $\emptyset \neq X \subseteq \mathbb{N}_{\geq 2}$, there exists a 2-generator recursive presentation $Q$ for which $\text{TorLen}^X(Q) = \omega$. If $X$ is r.e., then we can algorithmically construct such a finite presentation $Q$ from $X$.

3. Presentations

The purpose of this section is to re-prove Theorem 2.9 using model-theoretic arguments. We follow the idea in [1, Theorem A.1].

3.1. Notation.

Throughout this section $\mathcal{L}_{\text{Grp}}$ will be used to denote the language of group theory, that is the language of first-order logic supplemented with a binary function symbol $\cdot$ whose intended interpretation is the group operation, a constant symbol $e$ whose intended interpretation is the identity element, and a unary function symbol $\Box^{-1}$ whose intended interpretation is the function that sends
elements to their inverses. We use the standard abbreviation of writing $x^n$ instead of $\underbrace{x \cdots x}_n$. If $X$ is a set of constant symbols that are not in $\mathcal{L}$ endowed with a well-ordering of its elements then we write $\mathcal{L}_X$ for the language obtained by adding the constant symbols in $X$ to $\mathcal{L}$. If $\mathcal{L}$ is a language, $X$ is a set of new constant symbols that are not in $\mathcal{L}$ endowed with an implicit well-ordering, $\mathcal{M}$ is an $\mathcal{L}$-structure, and $A \in \mathcal{M}$ with a canonical bijection witnessing that $|A| = |X|$, then we write $\langle \mathcal{M}, A \rangle$ for the $\mathcal{L}_X$-structure obtained by interpreting the constant symbols in $X$ with the elements of $A$. If $\mathcal{M}$ is an $\mathcal{L}$-structure and $A \in \mathcal{M}$, then we say that $\mathcal{M}$ is generated by $A$ if all of $\mathcal{M}$ is obtained by closing $A$ and the constants of $\mathcal{M}$ under applications of the interpretation of functions from $\mathcal{L}$ in $\mathcal{M}$.

3.2. Model theory preliminaries.

We begin by recalling some definitions and results from Chapter 9 of [7].

**Definition 3.1.** Let $\mathcal{L}$ be a language. We say that an $\mathcal{L}_{\infty \infty}$-formula $\phi$ is basic Horn if $\phi$ is in the form

$$\bigwedge \Phi \Rightarrow \psi,$$

where $\Phi$ is a, possibly infinite, set of atomic $\mathcal{L}$-formulae, and $\psi$ is either an atomic $\mathcal{L}$-formula or 1. We say that an $\mathcal{L}_{\infty \infty}$-formula $\phi$ is universal Horn, and write $\forall_1$ Horn, if $\phi$ is in the form $\forall \vec{x} \theta(\vec{x})$ where $\theta$ is basic Horn. We say that an $\mathcal{L}$-theory $T$ is universal Horn if $T$ has an axiomatisation that only consists of $\forall_1$ Horn sentences.

Let $T_{\text{Grp}}$ be the obvious $\mathcal{L}_{\text{Grp}}$-theory that axiomatises the class of groups. It is clear that that $T_{\text{Grp}}$ can be written as a finite set of finitary $\forall_1$ Horn sentences.

**Definition 3.2.** Let $X \subseteq \mathbb{N}$. We write $T_{X-\text{tf}}$ for the $\mathcal{L}_{\text{Grp}}$-theory with axioms:

$$T_{\text{Grp}} \cup \{ \forall x (x^n = e \Rightarrow x = e) \mid n \in X \}.$$

It is clear that for all $X \subseteq \mathbb{N}$, $T_{X-\text{tf}}$ is a finitary universal Horn theory that axiomatises the class of $X$-torsion-free groups.

**Definition 3.3.** Let $\mathcal{L}$ be a language and let $\mathbb{K}$ be a class of $\mathcal{L}$-structures. A $\mathbb{K}$-presentation is a tuple $\langle X, \Phi \rangle$ such that $X$ is a set of new constant symbols endowed with an implicit well-ordering, called generators, that are not in $\mathcal{L}$, and $\Phi$ is a set of atomic $\mathcal{L}_X$-sentences. We will write $\langle X, \Phi \rangle$ instead of $\langle X, \Phi \rangle_{\mathbb{K}}$ when $\mathbb{K}$ is clear from the context. If both $\Phi$ and $X$ are finite then we say that $\langle X, \Phi \rangle_{\mathbb{K}}$ is a finitely presented $\mathbb{K}$-presentation. If both $\Phi$ and $X$ are r.e. then we say that $\langle X, \Phi \rangle_{\mathbb{K}}$ is a recursively presented $\mathbb{K}$-presentation.

**Definition 3.4.** Let $\mathcal{L}$ be a language and let $\mathbb{K}$ be a class of $\mathcal{L}$-structures. Let $\langle X, \Phi \rangle_{\mathbb{K}}$ be a $\mathbb{K}$-presentation. We say that an $\mathcal{L}_X$-structure $\langle \mathcal{M}, A \rangle$ is a model of $\langle X, \Phi \rangle_{\mathbb{K}}$ if

$$\mathcal{M} \in \mathbb{K} \text{ and } \langle \mathcal{M}, A \rangle \models \bigwedge \Phi.$$

**Definition 3.5.** Let $\mathcal{L}$ be a language and let $\mathbb{K}$ be a class of $\mathcal{L}$-structures. Let $\langle X, \Phi \rangle_{\mathbb{K}}$ be a $\mathbb{K}$-presentation. We say that $\langle X, \Phi \rangle_{\mathbb{K}}$ presents an $\mathcal{L}_X$-structure $\langle \mathcal{M}, A \rangle$ if
(i) \(\langle M, A \rangle\) is a model of \(\langle X, \Phi \rangle_K\),
(ii) \(M\) is generated by \(A\),
(iii) for every model \(\langle N, B \rangle\) of \(\langle X, \Phi \rangle_K\), there exists a homomorphism \(f : M \rightarrow N\) such that \(f(c^M) = c^N\) for all \(c \in X\).

We say that \(K\) admits presentations if every \(K\)-presentation presents a structure \(\langle M, A \rangle\) with \(M \in K\).

Note that if \(\langle X, \Phi \rangle\) presents \(\langle M, A \rangle\) and \(\langle N, B \rangle\) is a model of \(\langle X, \Phi \rangle\), then, since \(M\) is generated by \(A\), the homomorphism whose existence is guaranteed by Definition 3.5(iii) is unique.

The following is Lemma 9.2.1 of [7]:

**Lemma 3.6.** Let \(\Phi\) be a set of atomic \(\mathcal{L}_X\)-sentences where \(X\) is a set of constants not in \(\mathcal{L}\). Let \(\langle X, \Phi \rangle\) be a \(K\)-presentation and let \(\langle M, A \rangle\) be an \(\mathcal{L}_X\)-structure with \(M \in K\). The following are equivalent:

(i) \(\langle X, \Phi \rangle\) presents \(\langle M, A \rangle\),
(ii) \(A\) generates \(M\); and for every atomic formula \(\psi(\bar{x})\) of \(\mathcal{L}\) and for every \(\bar{a} \in A\), \(M \models \psi(\bar{a})\) if and only if every structure in \(K\) is a model of \(\forall \bar{x} (\bigwedge \Phi \Rightarrow \psi)\).

And this is Lemma 9.2.2 of [7]:

**Lemma 3.7.** Let \(\mathcal{L}\) be a language and let \(K\) be a class of \(\mathcal{L}\)-structures which is closed under isomorphic copies. The following are equivalent:

(i) \(K\) is closed under products, 1 and substructures,
(ii) \(K\) admits presentations,
(iii) \(K\) is axiomatised by a universal Horn theory in the language \(\mathcal{L}_{\infty\infty}\).

### 3.3. A proof of Theorem 2.9 using model theory.

The following is an adaptation of the proof of Theorem A.1 in [1].

**Theorem 3.8.** If \(T \supseteq T_{\text{Grp}}\) is an r.e. universal Horn \(\mathcal{L}_{\text{Grp}}\)-theory then there exists a recursively presented group \(G \models T\) such that every recursively presented group \(H \models T\) embeds into \(G\).

**Remark 3.9.** A group property \(\rho\) having an r.e. universal Horn theory does not imply that the finite presentations of groups with \(\rho\) are r.e. Indeed, for \(X\) a non-empty r.e. set, \(T_{X-\text{uf}}\) is an r.e. universal Horn theory (Definition 3.2), but by Theorem 2.13 the set of finite presentations of such groups is \(\Pi^0_2\)-complete.

**Proof.** Let \(T \supseteq T_{\text{Grp}}\) be a r.e. universal Horn \(\mathcal{L}_{\text{Grp}}\)-theory. Let \(K\) be the class of \(\mathcal{L}_{\text{Grp}}\)-structures that satisfy \(T\). Let \(K'\) be the class of \(\mathcal{L}_{\text{Grp}}\)-structures that satisfy \(T_{\text{Grp}}\). By Lemma 3.7 both \(K\) and \(K'\) admit presentations. If \(\tau\) is a presentation, then we will write \(G^\tau\) for the element of \(K\) presented by \(\tau\), and \(G_\tau\) for the element of \(K'\) presented by \(\tau\). Let \(\langle \pi_n \mid n \in \mathbb{N} \rangle\) be effective enumeration of recursive presentations. Let \(\pi\) be the disjoint union of all the \(\pi_n\)’s; \(\pi := \pi_1 \ast \pi_2 \ast \cdots\). Therefore, \(\pi\) is a recursive presentation. We claim that \(G^\pi\) is the desired universal group satisfying \(T\). It is immediate that \(G^\pi \models T\).
If \(\tau\) is a recursive presentation such that \(G_\tau \models T\) then \(G_\tau = G^\tau\), and so \(G_\tau\) embeds into \(G^\pi\). This shows that \(G^\pi\) is universal. It remains to show that \(G^\pi\)
is recursively presented in $K'$. Let $\pi = \langle Y, \Phi \rangle$ and let $Y^{G_{\pi}}$ be the interpretations of the constant symbols $Y$ in $G^{\pi}$. Let $\Phi'$ be the set of atomic $L_{Grp,Y}$-sentences (the language obtained by adding new constants for the generators in $Y$) that hold in $(G^{\pi}, Y^{G_{\pi}})$. Lemma 3.6 implies that $\langle Y, \Phi' \rangle_{K'}$ presents $(G^{\pi}, Y^{G_{\pi}})$. We need to show is that $\Phi'$ is r.e. Let $S = T \cup \Phi$. We claim that for all atomic $L_{Grp,Y}$-sentences $\sigma$,

(1) $S \vdash \sigma$ if and only if $\sigma \in \Phi'$.

Since $G^{\pi} \models S$, it follows that for all atomic $L_{Grp,Y}$-sentences $\sigma$, if $S \vdash \sigma$ then $\sigma \in \Phi'$. We need to show the converse. Let $\sigma \in \Phi'$, and suppose that $S \not\vdash \sigma$. Note that, since $\sigma$ is atomic, it is of the form $w_1 = w_2$ where $w_1$ and $w_2$ are words in the generators $Y$. Let $H$ be a group such that $Y^{H}$ is the interpretation of the constant symbols $Y$ in $H$, $\langle H, Y^{H} \rangle \models S$, and $H = (w_1 \neq w_2)$. Therefore $\langle H, Y^{H} \rangle$ is a model of $\langle Y, \Phi \rangle_{K'}$. It follows from (iii) of Definition 3.5 that the map $g : Y^{G_{\pi}} \rightarrow Y^{H}$ defined by the implicit ordering on $Y$ must lift to a homomorphism $f : G^{\pi} \rightarrow H$. But this map is not well-defined since $w_1 = w_2$ in $G^{\pi}$ and $f(w_1) \neq f(w_2)$ in $H$, which is a contradiction. Therefore (1) holds. Since $S$ is r.e., (1) yields a recursive enumeration of the elements of $\Phi'$.

If $X \subseteq \mathbb{N}$ is r.e. then Definition 3.2 shows that $T_{X}$ is an r.e. universal Horn theory that extends $T_{Grp}$. This immediately re-proves Theorem 2.9. It is then an application of Theorem 2.10 again, to re-prove Theorem 2.12. This time, we have done most of the work using model-theoretic techniques and in a way which could be applied to other group properties (not just $X$-torsion-freeness), as discussed in the introduction.

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