Black hole stability in Jordan and Einstein frames

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Abstract

We investigate the classical stability of Schwarzschild black hole in Jordan and Einstein frames which are related by the conformal transformations. For this purpose, we introduce two models of the Brans-Dicke theory and Brans-Dicke-Weyl gravity in Jordan frame and two corresponding models in the Einstein frame. The former model is suitable for studying the massless spin-2 graviton propagating around the Schwarzschild black hole, while the later is designed for the massive spin-2 graviton propagating around the black hole. It turns out that the black hole (in)stability is independent of the frame which shows that the two frames are equivalent to each other.

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1 Introduction

The Brans-Dicke theory, one of scalar-tensor theories (STT), has a non-minimally coupled scalar field $\phi$ to gravity in addition to metric $\mathbf{1}$. The original motivation of Brans and Dicke was the idea of Mach, in which they put Mach idea in general relativity to describe a varying gravitational constant. In connection with Einstein gravity, gravitational constant $G$ is related to an average value of a scalar field which is not constant. Since the Brans-Dicke theory was released, two versions of STT are possible: one version is on the Jordan and the other is on the Einstein frame which is related to the former by a conformal transformation and a redefinition of a scalar field. One may have a non-minimally coupled scalar in the Jordan frame, while one may have a minimally coupled scalar in the Einstein frame. However, the issues of lively debate which are not yet resolved completely include whether the two versions of STT are equivalent or not in the classical gravity and cosmology $\mathbf{2, 3, 4}$. Whereas many authors support the point of view that two frames are equivalent, others support the opposite viewpoint.

Here we wish to raise this issue on the stability of black holes $\mathbf{5, 6, 7}$. It was proposed that the stability of black holes does not depend on the frame because it is a classical solution which is considered as a ground state $\mathbf{8}$. Presumably, the ground state is stable against small perturbations. Usually, a non-minimally coupled scalar makes the linearized Einstein equation around the black hole complicated when one compares to a minimally coupled scalar in the Einstein frame $\mathbf{9}$. Because of this complication, some authors have made conformal transformations to find the corresponding theory in the Einstein frame where a minimally coupled scalar appears.

In this work, we show explicitly that the (in)stability of the Schwarzschild black hole is independent of choosing the frame by introducing the Brans-Dicke-Weyl (BDW) gravity and its conformal partner of the Einstein-scalar-Weyl (ESW) gravity. Especially, we focus on showing the instability of massive spin-2 graviton propagating around the black hole in the two gravity theories.
2 Brans-Dicke-Weyl gravity in two frames

Let us first consider the Brans-Dicke-Weyl (BDW) gravity whose action is given by

\[ S_{\text{BDW}} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ \phi R - \frac{\omega}{\phi} (\partial \phi)^2 - \frac{1}{2m^2} C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} \right], \quad (1) \]

where the Weyl-squared term is

\[ C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} = 2 \left( R_{\mu\nu} R_{\mu\nu} - \frac{1}{3} R^2 \right) + \left( R_{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R_{\mu\nu} + R^2 \right). \quad (2) \]

Here the quantities in the second parenthesis present the Gauss-Bonnet term, which could be neglected because it does not contribute to equation of motion. Also, we use the Planck units of \( c = \hbar = 1 \) and \( m \) is the mass of massive spin-2 graviton. We note that the Brans-Dicke action is conformally invariant only for \( \omega = -3/2 \) under full conformal transformations

\[ \hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad \hat{\phi} = \frac{\phi}{\Omega}, \quad (3) \]

while the Weyl-squared term of \( \sqrt{-g} C^2 \) is conformally invariant under conformal transformations. The \( \omega = -3/2 \) BDW gravity is related to the conformal massive gravity \[10\]. In this case, the author has found unstable s-mode of massive spin-2 graviton \[11\]. In fact, Brans-Dicke parameter \( \omega = -3/2 \) gives a border between a standard scalar field (\( \omega > -3/2 \)) and a ghost of negative kinetic energy (\( \omega < -3/2 \)).

From the action (1), the Einstein equation is derived to be

\[ \left[ \phi G_{\mu\nu} - \frac{\omega}{\phi} \left( \partial_{\mu} \phi \partial_{\nu} \phi - \frac{1}{2} (\partial \phi)^2 g_{\mu\nu} \right) - \left( \nabla_{\mu} \nabla_{\nu} \phi - g_{\mu\nu} \nabla^2 \phi \right) \right] - \frac{W_{\mu\nu}}{m^2} = 0, \quad (4) \]

where the Einstein tensor is given by

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}, \quad (5) \]

and the Bach tensor \( W_{\mu\nu} \) takes the form

\[ W_{\mu\nu} = 2 \left( R_{\mu\nu\rho\sigma} R^{\rho\sigma} - \frac{1}{4} R^{\rho\sigma} R_{\rho\sigma} g_{\mu\nu} \right) - \frac{2}{3} R \left( R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu} \right) \]

\[ + \nabla^2 R - \frac{1}{6} \nabla^2 R g_{\mu\nu} - \frac{1}{3} \nabla_{\mu} \nabla_{\nu} R. \quad (6) \]

Its trace is zero (\( W^{\mu}_{\mu} = 0 \)). In the limit of \( m^2 \to \infty \), one recovers the Brans-Dicke theory.
On the other hand, the scalar equation is given by

$$\nabla^2 \phi - \frac{1}{2\phi} (\partial \phi)^2 + \frac{1}{2\omega} R \phi = 0. \quad (7)$$

Taking the trace of (4) leads to

$$R = \frac{\omega}{\phi^2} (\partial \phi)^2 + \frac{3}{\phi} \nabla^2 \phi. \quad (8)$$

Plugging (8) into (7), one finds a massless scalar equation for $\omega \neq -3/2$ as

$$\left(1 + \frac{3}{2\omega}\right) \nabla^2 \phi = 0 \rightarrow \nabla^2 \phi = 0. \quad (9)$$

Finally, we arrive at the trace equation

$$R = \frac{\omega}{\phi^2} (\partial \phi)^2 \quad (10)$$

and the Einstein equation

$$\left[\phi R_{\mu\nu} - \frac{\omega}{\phi} \partial_{\mu}\phi \partial_{\nu}\phi - \nabla_{\mu} \nabla_{\nu} \phi \right] - \frac{W_{\mu\nu}}{m^2} = 0. \quad (11)$$

Considering the background ansatz

$$\bar{R}_{\mu\nu} = 0, \quad \bar{R} = 0, \quad \bar{\phi} = \text{const}, \quad (12)$$

Eqs. (11) and (9) together with (10) provide the Schwarzschild black hole solution

$$ds^2_S = \bar{g}_{\mu\nu} dx^\mu dx^\nu = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \quad (13)$$

with the metric function

$$f(r) = 1 - \frac{r_0}{r}. \quad (14)$$

It is easy to show that the Schwarzschild black hole (13) is also the solution to the Brans-Dicke theory.

Now we transform the BDW action (1) into the corresponding action in the Einstein frame by choosing [2, 12, 13]

$$\hat{g}_{\mu\nu} = \phi g_{\mu\nu}, \quad \hat{C}^\mu_{\nu\rho\sigma} = C^\mu_{\nu\rho\sigma} \quad (15)$$

and the scalar field redefinition

$$\phi \rightarrow \hat{\phi} = \sqrt{2\omega + 3 \ln \phi}. \quad (16)$$
Then, the action of $\omega > -3/2$ BDW gravity in the Jordan frame is conformally equivalent to the (minimally coupled) Einstein-scalar-Weyl (ESW) gravity in the Einstein frame \[2\]

\[
\hat{S}_{BDW} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ \hat{R} - \frac{1}{2} \hat{g}^{\mu\nu} \partial_\mu \hat{\phi} \partial_\nu \hat{\phi} \right. \\
\left. - \frac{1}{2m^2} \hat{C}^{\mu\nu\rho\sigma} \hat{C}_{\mu\nu\rho\sigma} \right].
\]

(17)

Its Einstein equation takes the form

\[
\hat{G}_{\mu\nu} - \frac{1}{2} \left[ \partial_\mu \hat{\phi} \partial_\nu \hat{\phi} - \frac{1}{2} (\partial \hat{\phi})^2 \hat{g}_{\mu\nu} \right] - \frac{1}{m^2} \hat{W}_{\mu\nu} = 0
\]

(18)

and the scalar equation is given by

\[
\nabla^2 \hat{\phi} = 0.
\]

(19)

Tracing (18) leads to

\[
\hat{R} = \frac{1}{2} (\partial \hat{\phi})^2.
\]

(20)

For $\hat{\phi} = \bar{\hat{\phi}} =$const, one has $\bar{\hat{R}}_{\mu\nu} = 0$ and $\bar{\hat{R}} = 0$. This implies that the Schwarzschild metric \[13\] is a solution to the Eq. (18).

3 Black hole (in)stability in the Einstein frame

We briefly describe the stability analysis of the Schwarzschild black hole found from the ESW gravity in the Einstein frame. For this purpose, we introduce the perturbations around the black hole

\[
\hat{g}_{\mu\nu} = \bar{g}_{\mu\nu} + \hat{h}_{\mu\nu}, \quad \hat{\phi} = \bar{\hat{\phi}} + \hat{\phi}.
\]

(21)

Then, Eq. (20) yields the non-propagation of the linearized Ricci scalar as

\[
\delta \hat{R} = 0.
\]

(22)

Taking into account $\delta \hat{R} = 0$, the linearized Einstein equation \[18\] is given by

\[
\delta \bar{\hat{R}}_{\mu\nu} = \frac{1}{m^2} \left[ \bar{\nabla}^2 \delta \hat{R}_{\mu\nu} + 2 \bar{\hat{R}}_{\rho\mu\nu} \delta \hat{R}^{\rho\sigma} \right].
\]

(23)

and the linearized scalar equation for \[19\] is

\[
\nabla^2 \hat{\phi} = 0.
\]

(24)
which is surely a massless scalar equation propagating on the Schwarzschild black hole. It turned out that the scalar mode does not have any unstable modes \[9, 14, 15\]. Explicitly, introducing the tortoise coordinate \(r^* = r + r_0 \ln[r/r_0 - 1]\) and the scalar perturbation

\[
\hat{\phi}(t, r, \Theta, \Phi) = e^{-ikt} \frac{\hat{\psi}(r)}{r} Y_{lm}(\Theta, \Phi), \tag{25}
\]

the linearized equation (24) reduces to the Schrödinger-type equation as

\[
\frac{d^2 \hat{\psi}}{dr^*^2} + (k^2 - V_{\hat{\psi}}) \hat{\psi} = 0 \tag{26}
\]

with the potential

\[
V_{\hat{\psi}} = \left(1 - \frac{r_0}{r}\right) \left[l(l + 1) \frac{r^2}{r^2} + \frac{r_0}{r^3}\right]. \tag{27}
\]

The potential \(V_{\hat{\psi}}\) is always positive exterior the event horizon \(r = r_0\) for \(l \geq 0\), implying that the black hole is stable against the scalar perturbation. It is well known that the Schwarzschild black hole is stable \[5, 6, 7\] against the odd- and even-perturbations with the same potentials in Einstein gravity because its linearized Einstein equation (23) is given by

\[
\delta \hat{R}_{\mu\nu}(\hat{h}) = 0 \tag{28}
\]

in the limit of \(m^2 \to \infty\).

We mention that counting the number of DOF, it might be helpful to explain intuitively why the Schwarzschild black hole is physically stable in the Einstein gravity of \(m^2 \to \infty\) \[5, 6, 7\], whereas the Schwarzschild black hole can be unstable in the ESW gravity. We wish to point out that \(h_{\mu\nu}\) is used to describe a massless spin-2 graviton, while \(\delta \hat{R}_{\mu\nu}\) can be taken to describe massive spin-2 graviton to avoid ghost states. The number of DOF for the massless spin-2 graviton \(h_{\mu\nu}\) is 2 in the Einstein gravity (\(m^2 \to \infty\) ESW gravity), since one requires \(-3\) further for a residual diffeomorphism after a gauge-fixing. We know that these 2 DOF correspond to the transverse modes. On the other hand, from Eqs. (23) and (22) together with the linearized Bianchi identity (\(\nabla^\mu \delta \hat{R}_{\mu\nu} = 0\)), the number of DOF for massive spin-2 graviton \(\delta \hat{R}_{\mu\nu}\) in the ESW gravity \[16\] \[17\] is \(10 - 5 = 5\), which includes the longitudinal (would be unstable) modes \[18\] \[19\] as well as transverse modes.

The \(s\)-mode analysis is suitable for investigating the massive graviton propagation in the ESW gravity, but not for studying the massless graviton propagation in the Einstein gravity. In general, the \(s\)-mode analysis of the massive graviton with 5 DOF shows the Gregory-Laflamme instability \[20\] which never appears in the massless spin-2 analysis \[21\] \[22\]. The
even-parity metric perturbation is used to define a $s(l = 0)$-mode analysis in the ESW gravity and whose form is given by $\delta \hat{R}_{tt}$, $\delta \hat{R}_{tr}$, $\delta \hat{R}_{rr}$ and $\delta \hat{R}_{\Theta \Theta}$ as [16]

$$
\delta \hat{R}_{\mu \nu} = e^{\Omega t} \begin{pmatrix}
\delta \hat{R}_{tt}(r) & \delta \hat{R}_{tr}(r) & 0 & 0 \\
\delta \hat{R}_{tr}(r) & \delta \hat{R}_{rr}(r) & 0 & 0 \\
0 & 0 & \delta \hat{R}_{\Theta \Theta}(r) & 0 \\
0 & 0 & 0 & \sin^2 \Theta \delta \hat{R}_{\Theta \Theta}(r)
\end{pmatrix}. \tag{29}
$$

Even though one starts with 4 DOF, they are related to each other when one uses the transverse-traceless condition of $\bar{\nabla}^\mu \delta \hat{R}_{\mu \nu} = 0$ and $\delta \hat{R} = 0$. Hence, we obtain one decoupled equation for $\delta \hat{R}_{tr}$ from the massive graviton equation. Since Eq.(23) is the same linearized equation for four-dimensional metric perturbation around five-dimensional black string as [21, 22]

$$
\bar{\nabla}^2 h_{\mu \nu} + 2 \bar{R}_{\mu \sigma \nu \rho} h^{\sigma \rho} = m^2 h_{\mu \nu}, \quad \bar{\nabla}^\mu h_{\mu \nu} = 0, \quad h = 0, \tag{30}
$$

we use the GL instability analysis to reveal unstable modes [20]. Actually, Eq. (23) is considered as a boosted-up version of the massive graviton equation (30) [23]. In addition, the massive spin-2 polarizations could be described by the linearized Ricci tensor well [18, 19, 24, 25]. We stress to note that taking the linearized Ricci tensor is the only prescription to avoid ghosts because the linearized equation (23) becomes a forth-order differential equation when it is expressed in terms of the metric perturbation $h_{\mu \nu}$.

Eliminating all but $\delta \hat{R}_{tr}$, Eq.(23) reduces to a second-order radial equation for $\delta \hat{R}_{tr}$

$$
A \delta \hat{R}_{tr}'' + B \delta \hat{R}_{tr}' + C \delta \hat{R}_{tr} = 0, \tag{31}
$$

where $A$, $B$ and $C$ are given by

$$
A = - m^2 f - \Omega^2 + \frac{f'}{2} - \frac{f f''}{2} - \frac{f f'}{r}, \tag{32}
$$

$$
B = - 2m^2 f' - \frac{3f f''}{2} - \frac{3\Omega^2 f'}{f} + \frac{3f^3}{4f} + \frac{2m^2 f'}{2} + \frac{2\Omega^2}{2} + \frac{3f'^2}{2} + \frac{f f''}{r} - \frac{2f f'}{r^2}, \tag{33}
$$

$$
C = m^4 + \Omega^4 f + \frac{2m^2 \Omega^2}{f} - \frac{5\Omega^2 f'^2}{4f^2} + \frac{m^2 f'^2}{4f} + \frac{f^4}{4f^2} - \frac{m^2 f''}{2} - \frac{\Omega^2 f''}{2f} + \frac{f'^2 f''}{4f} - \frac{f''^2}{2} - \frac{2m^2 f'}{r} - \frac{\Omega^2 f'}{rf} + \frac{f^3 f'}{r f} + \frac{2\Omega^2}{r^2} + \frac{2m^2 f}{r^2} - \frac{5f'^2}{2r^2} + \frac{f f''}{r^2} + \frac{2f f'}{r^3}. \tag{34}
$$
with the metric function $f = 1 - r_0/r$ [14]. It is worth noting that the $s$-mode perturbation is described by single DOF but not 5 DOF. The boundary conditions are that $\delta R_{tr}$ should be regular on the future horizon and vanishing at infinity.

Now we are in a position to solve (31) numerically and find unstable modes. See Fig. 1 that is generated from the numerical analysis. From the observation of Fig. 1 with $O(1) \simeq 0.86$, we find unstable modes [21] for the small Schwarzschild black hole

$$0 < m < \frac{O(1)}{r_0}$$

(35)

with mass $m$. As a consequence, this shows that the region of instability becomes progressively smaller, as the horizon size $r_0$ increases.

4 Black hole (in)stability in the Jordan frame

We now turn to performing the stability analysis of the Schwarzschild black hole [13] in the Jordan frame. To this end, we introduce the metric and scalar perturbations around the black hole

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \quad \phi = \bar{\phi}(1 + \varphi).$$

(36)
Then, the linearized Einstein equation (11) takes the form

\[
m^2 \phi \left[ \delta R_{\mu\nu} - \bar{\nabla}_\mu \bar{\nabla}_\nu \phi \right] = \left[ \bar{\nabla}^2 \delta G_{\mu\nu} + 2 \bar{R}_{\rho\mu\sigma\nu} \delta G^{\rho\sigma} \right] + \frac{1}{3} \left[ g_{\mu\nu} \bar{\nabla}^2 - \bar{\nabla}_\mu \bar{\nabla}_\nu \right] \delta R,
\]

where the linearized Einstein tensor, Ricci tensor, and Ricci scalar are given by

\[
\delta G_{\mu\nu} = \delta R_{\mu\nu} - \frac{1}{2} \delta R \bar{g}_{\mu\nu},
\]

\[
\delta R_{\mu\nu} = \frac{1}{2} \left( \bar{\nabla}^\rho \bar{\nabla}_\mu h_{\nu\rho} + \bar{\nabla}^\rho \bar{\nabla}_\nu h_{\mu\rho} - \bar{\nabla}^2 h_{\mu\nu} - \bar{\nabla}_\mu \bar{\nabla}_\nu h \right),
\]

\[
\delta R = \bar{g}^{\mu\nu} \delta R_{\mu\nu} = \bar{\nabla}^\mu \bar{\nabla}_\nu h_{\mu\nu} - \bar{\nabla}^2 h
\]

with \( h = h^\rho \rho \). From (10), we obtain the non-propagation of the linearized Ricci scalar

\[
\delta R = 0.
\]

Substituting (31) into (37), one finds the linearized Ricci tensor equation

\[
\bar{\nabla}^2 \phi \left[ \delta R_{\mu\nu} - \bar{\nabla}_\mu \bar{\nabla}_\nu \phi \right] = \frac{1}{m^2} \left[ \bar{\nabla}^2 \delta R_{\mu\nu} + 2 \bar{R}_{\rho\mu\sigma\nu} \delta R^{\rho\sigma} \right] = - \frac{1}{m^2} \Delta_L \delta R_{\mu\nu},
\]

where the Lichnerowicz operator \( \Delta_L \) acting on the transverse-traceless tensor \( \delta R_{\mu\nu} \) is introduced to have the simplicity. From (39), we derive the linearized scalar equation

\[
\bar{\nabla}^2 \phi = 0,
\]

which implies, as shown in the previous section, that the black hole is stable against the scalar perturbation. Here we note that \( \delta R_{\mu\nu} \) is taken to describe massive spin-2 graviton to avoid ghost states, while \( h_{\mu\nu} \) is used to describe a massless spin-2 graviton.

Now, we mention briefly the stability of the black hole in the limit of \( m^2 \to \infty \), reducing to the stability of the black hole in the Brans-Dicke theory. In addition to the massless scalar equation (43), the linearized equation (42) is taken to be \( \frac{1}{2} \Delta_L \delta R_{\mu\nu} = 0 \),

\[
\delta R_{\mu\nu}(h) - \bar{\nabla}_\mu \bar{\nabla}_\nu \phi = 0,
\]

where \( \delta R_{\mu\nu}(h) \) is given by (39). Its trace equation is satisfied automatically when one uses (31) and (33). The metric perturbation \( h_{\mu\nu} \) is classified depending on the transformation properties under parity, namely odd (axial) and even (polar). Using the Regge-Wheeler
gauge [5], and Zerilli gauge [6], one obtains two distinct perturbations: odd and even perturbations. For odd parity, one has with two off-diagonal components $h_0$ and $h_1$

$$h^o_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & h_0(r) \\ 0 & 0 & 0 & h_1(r) \\ 0 & 0 & 0 & 0 \\ h_0(r) & h_1(r) & 0 & 0 \end{pmatrix} e^{-ikt} \sin \Theta \frac{dp_l}{d\Theta},$$

(45)

while for even parity, the metric tensor takes the form with four components $H_0$, $H_1$, $H_2$, and $K$ as

$$h^e_{\mu\nu} = \begin{pmatrix} H_0(r)f & H_1(r) & 0 & 0 \\ H_1(r) & H_2(r)f^{-1} & 0 & 0 \\ 0 & 0 & r^2K(r) & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \Theta K(r) \end{pmatrix} e^{-ikt} p_l,$$

(46)

where $p_l$ is Legendre polynomial with angular momentum $l$ and $f$ is the metric function given by (14). Also the scalar perturbation is

$$\varphi(t, r, \Theta, \Phi) = e^{-ikt} \frac{\psi(r)}{r} Y_{lm}(\Theta, \Phi).$$

(47)

For the odd-parity perturbation, its linearized equation takes a simple form as

$$\delta R_{\mu\nu}(h) = 0,$$

(48)

which shows that the odd-perturbation is stable, since this is the same equation as the Eq. (28). For the even-perturbation (46), however, we have to use the linearized equation (44) because the scalar field $\psi(r)$ contributes to making an even mode $\hat{M}$ together with $H_0$, $H_1$, $H_2$, and $K$. For example, one has a relation of $H_2 = H_0/f^2 - 2\psi/rf$. In this case, we have the Zerilli’s equation [6]

$$\frac{d^2\hat{M}}{dr^2} + \left[ k^2 - V_Z \right] \hat{M} = 0,$$

(49)

where $\hat{M}$ and the Zerilli potential are given by [9 14 15]

$$\hat{M} = \frac{1}{pq - h} \left[ p(K + \frac{\psi}{r}) - \frac{H_1}{k} \right],$$

(50)

$$V_Z(r) = \left( 1 - \frac{r_0}{r} \right) \left[ \frac{2\lambda^2(\lambda + 1)r^3 + 3\lambda^2 r_0^2 + 9\lambda r_0^2/2 + 9r_0^2/4}{r^3(\lambda r + 3r_0/2)^2} \right]$$

(51)
with
\[ \lambda = \frac{1}{2}(l - 1)(l + 2). \]  
(52)
The Zerilli potential \( V_Z \) is always positive for whole range of \(-\infty < r^* < \infty \) and \( l \geq 2 \). Also, it is a barrier-type localized around \( r^* = 0 \) which implies that the even-perturbation is stable, even though the scalar is coupled to the even-parity perturbations.

The above all statements show clearly that the Schwarzschild black hole is stable against metric and scalar perturbations (3 = 2 + 1 DOF) in the Brans-Dicke theory.

Now let us go back to the linearized massive equation (42) in the BDW gravity. It might be difficult to solve (42) directly because it is a coupled second-order equation for \( \delta R_{\mu\nu} \) and \( \varphi \) (6 = 5 + 1 DOF). Surely, this is a nontrivial task. Curiously, however, the equation (42) could be rewritten in terms of \( \delta \tilde{R}_{\mu\nu} = \delta R_{\mu\nu} - \bar{\nabla}_\mu \bar{\nabla}_\nu \varphi \) as
\[ \bar{\nabla}^2 \delta \tilde{R}_{\mu\nu} + 2 \bar{R}_{\rho\mu\sigma\nu} \delta \tilde{R}^{\rho\sigma} = \tilde{m}^2 \delta \tilde{R}_{\mu\nu}, \]  
(53)
where \( \tilde{m}^2 = m^2 \bar{\phi} \) and we used
\[ \Delta_L \delta \tilde{R}_{\mu\nu} = \Delta_L \delta R_{\mu\nu}. \]  
(54)
Explicitly, we have the following relation
\[ \Delta_L (\bar{\nabla}_\mu \bar{\nabla}_\nu \varphi) = \frac{1}{2} \Delta_L \left( \bar{\nabla}_\mu \bar{\nabla}_\nu + \bar{\nabla}_\nu \bar{\nabla}_\mu \right) \varphi \]
\[ = -\frac{1}{2} \left( \bar{\nabla}_\mu \bar{\nabla}_\nu + \bar{\nabla}_\nu \bar{\nabla}_\mu \right) \bar{\nabla}^2 \varphi = 0, \]  
(55)
where in the second line, we used the scalar equation (43). See Appendix for a detailed proof of the relation (55).

Obtaining the linearized equation (53) is our main result for carrying out the stability analysis of the Schwarzschild black hole in the Jordan frame. It is important to note that the equation (53) actually describes the massive spin-2 field (5 DOF) propagating around the Schwarzschild black hole, because \( \delta \tilde{R}_{\mu\nu} \) satisfies the transverse and traceless gauge condition:
\[ \bar{\nabla}^\mu \delta \tilde{R}_{\mu\nu} = \bar{\nabla}_\nu \delta \tilde{R} = 0, \]  
(56)
where the Eq. (43) was used. Note also that the linearized equation (53) is exactly the same as the one (23) obtained in the Einstein frame, when replacing
\[ \delta \tilde{R}_{\mu\nu} \rightarrow \delta \hat{R}_{\mu\nu}, \quad \bar{R}_{\rho\mu\sigma\nu} \rightarrow \bar{\hat{R}}_{\rho\mu\sigma\nu}, \quad \tilde{m}^2 \rightarrow m^2. \]  
(57)
Therefore, the corresponding result can be seen that the unstable modes for the Schwarzschild black hole in the Jordan frame are given by the region [see Fig.1]:

$$0 < \tilde{m} < \frac{\mathcal{O}(1)}{r_0}.$$  \hspace{1cm} (58)

This states clearly that the instability of black hole in the Jordan and Einstein frames are equivalent.

5 Discussions

It was well known that the Schwarzschild black hole is stable against metric and scalar perturbations in the Brans-Dicke theory and Einstein gravity. We note that the metric perturbation $h_{\mu\nu}$ was used to describe a massless spin-2 graviton, whereas the linearized Ricci tensors of $\delta\hat{R}_{\mu\nu}$ and $\delta\tilde{R}_{\mu\nu}$ were taken to describe massive spin-2 graviton to avoid ghost states. In this work, we have found unstable $s$-mode from the massive spin-2 graviton described by the linearized Ricci tensors in the ESW gravity and BDW gravities. This implies that the (in)stability of black holes does not depend on the frame.

Let us question what it means that the instability of Schwarzschild black hole is given by the $s$-mode of massive spin-2 graviton. The Schwarzschild black hole stands out among all possible solutions of Einstein gravity as the only static regular solution to the vacuum Einstein equation in asymptotically flat spacetimes. The Schwarzschild solution also solves many other equations of STT, $f(R)$ gravity, and Chern-Simons gravity including the BDW and ESW gravity theories. These properties are consistent with various no-hair proofs which states that the Schwarzschild black hole could not support regular scalar, nor other fields. The stability of the black hole implies that the black hole is really existed as a truly solution in the Einstein and Brans-Dicke gravity. Hence, the presence of unstable $s$-mode around the black hole in the BDW and ESW gravity theories indicates that these massive gravity theories could not accommodate the static black holes. Naively, the black holes decay to something and, the final state may be a spherically symmetric black hole [22, 26]. Alternatively, it implies that there is no propagating massive graviton around the stable Schwarzschild black hole because the massive graviton is unstable. Therefore, it may happen that the massive graviton decays to other fields around the small stable black holes.
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Appendix: Proof of the relation (55)

The Lichnerowicz operator acting on a symmetric second rank tensor $M_{ab}$ in the Schwarzschild background is given by [27]

$$\Box_L M_{ab} = 2\tilde{R}^c_{\ abd}M^d_c - \tilde{\nabla}^2 M_{ab}, \quad (59)$$

where $M_{ab} \equiv \tilde{\nabla}(aV_b) \equiv \tilde{\nabla}(a\tilde{\nabla}b)c$.

We note that $\tilde{\nabla}^2 M_{ab}$ can be arranged into the form:

$$\tilde{\nabla}^2 M_{ab} = \frac{1}{2} \tilde{\nabla}^2 \tilde{\nabla}_a V_b + (a \leftrightarrow b)$$

$$= \frac{1}{2} \left[ \tilde{\nabla}_c \tilde{\nabla}^c, \tilde{\nabla}_a \right] V_b + \frac{1}{2} \tilde{\nabla}_a \tilde{\nabla}^2 V_b + (a \leftrightarrow b)$$

$$= \frac{1}{2} \left( \tilde{\nabla}_c \tilde{\nabla}^c, \tilde{\nabla}_a \right] V_b + \frac{1}{2} \left[ \tilde{\nabla}_c, \tilde{\nabla}_a \right] \tilde{\nabla}^c V_b + \frac{1}{2} \tilde{\nabla}_a \tilde{\nabla}^2 V_b + (a \leftrightarrow b)$$

$$= \frac{1}{2} \left( \tilde{\nabla}_c \tilde{\nabla}^c, \tilde{\nabla}_a \right] V_b + \frac{1}{2} \tilde{\nabla}_c \tilde{\nabla}^c V_b + \frac{1}{2} \tilde{\nabla}_a \tilde{\nabla}^2 V_b + (a \leftrightarrow b)$$

$$= \frac{1}{2} \left( \tilde{\nabla}_c \tilde{\nabla}^c, \tilde{\nabla}_a \right] V_b + \frac{1}{2} \tilde{\nabla}_c \tilde{\nabla}^c V_b + \frac{1}{2} \tilde{\nabla}_a \tilde{\nabla}^2 V_b + (a \leftrightarrow b)$$

$$= \frac{1}{2} \tilde{\nabla}_c \tilde{\nabla}^c \tilde{R}_{ab\lambda\mu} V^\lambda + \tilde{R}_{ab\lambda\mu} \tilde{\nabla}_c V^\lambda + \frac{1}{2} \tilde{\nabla}_a \tilde{\nabla}^2 V_b + (a \leftrightarrow b)$$

$$= \tilde{\nabla}_c \tilde{R}_{(ab)\lambda\mu} V^\lambda + 2\tilde{R}_{ab\lambda\mu} M^\lambda_c + \tilde{\nabla}(a\tilde{\nabla}^2 V_b). \quad (60)$$

Substituting (60) into Eq. (59) leads to

$$\Box_L M_{ab} = - \tilde{\nabla}_c \tilde{R}_{(ab)\lambda\mu} V^\lambda - \tilde{\nabla}(a\tilde{\nabla}^2 V_b). \quad (61)$$

From the Bianchi identity, the first term of the r.h.s. in Eq. (61) vanishes:

$$\tilde{\nabla}_{[\mu} \tilde{R}_{ab]cd} = 0$$

$$\Rightarrow \tilde{\nabla}_{\mu} \tilde{R}_{abcd} + \tilde{\nabla}_a \tilde{R}_{b\mu cd} + \tilde{\nabla}_b \tilde{R}_{a\mu cd} = 0 \quad (\times \tilde{g}^{ac})$$

$$\Rightarrow \tilde{\nabla}_{\mu} \tilde{R}_{bd} + \tilde{\nabla}^c \tilde{R}_{b\mu cd} - \tilde{\nabla}_b \tilde{R}_{\mu d} = 0 \quad (62)$$

$$\Rightarrow \tilde{\nabla}_c \tilde{R}^c_{db\mu} = 0. \quad (63)$$
The second term of the r.h.s. in Eq. (61) can be re-written as follows:

\[ \bar{\nabla}^2 V_b = \bar{\nabla}^2 \bar{\nabla}_b \varphi \]

\[ = [\bar{\nabla}_c \bar{\nabla}^c, \bar{\nabla}_b] \varphi + \bar{\nabla}_b \bar{\nabla}^2 \varphi \]

\[ = \bar{\nabla}_c [\bar{\nabla}^c, \bar{\nabla}_b] \varphi + [\bar{\nabla}_c, \bar{\nabla}_b] \bar{\nabla}^c \varphi + \bar{\nabla}_b \bar{\nabla}^2 \varphi \]

\[ = \bar{R}_{ab} \bar{\nabla}^b \varphi + \bar{\nabla}_b \bar{\nabla}^2 \varphi \]  \hspace{1cm} (64)

\[ = \bar{\nabla}_b \bar{\nabla}^2 \varphi \]

\[ \Rightarrow \bar{\nabla}_a \bar{\nabla}^2 V_b = \bar{\nabla}_a \bar{\nabla}^2 \bar{\nabla}_b \varphi = \bar{\nabla}_a \bar{\nabla}_b \bar{\nabla}^2 \varphi. \]  \hspace{1cm} (65)

Note that in Eqs. (62) and (64), we used the Ricci flat condition of \( \bar{R}_{ab} = 0 \), which shows the Schwarzschild background. Plugging the Eqs. (63) and (65) into Eq. (61), we finally get

\[ \triangle L M_{ab} = \triangle L \bar{\nabla}_a \bar{\nabla}_b \varphi = - \bar{\nabla}_a \bar{\nabla}_b \bar{\nabla}^2 \varphi. \]

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