Global solutions of nonlinear wave equations with large energy

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Abstract

In this paper, we give a criterion on the Cauchy data for the semilinear wave equations satisfying the null condition in $\mathbb{R}^+ \times \mathbb{R}^3$ such that the energy of the data can be arbitrarily large while the solution is still globally in time in the future.

1 Introduction

In this paper, we study the Cauchy problem to the semilinear wave equations

$$\begin{cases}
\Box \phi = (-\partial_t^2 + \Delta) \phi = F(\phi, \partial \phi), \\
\phi(0, x) = \phi_0(x), \partial_t \phi(0, x) = \phi_1(x)
\end{cases} \quad (1)$$

in $\mathbb{R}^+ \times \mathbb{R}^3$. The nonlinearity $F$ is assumed to satisfy the null condition outside a large cylinder $\{(t, x)| |x| \leq R\}$, that is,

$$F(\phi, \partial \phi) = A^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + O(|\phi|^3 + |\partial \phi|^3 + |\phi|^N + |\partial \phi|^N), \quad |x| \geq R, \quad (2)$$

where $A^{\alpha\beta}$ are constants such that $A^{\alpha\beta} \xi_\alpha \xi_\beta = 0$ whenever $\xi_0^2 = \xi_1^2 + \xi_2^2 + \xi_3^2$ and $N$ is a given integer larger than 3. Inside the cylinder we assume $F$ is at least quadratic in terms of $\phi, \partial \phi$. The data $(\phi_0, \phi_1)$ are assumed to be smooth but can be arbitrarily large in the energy space.

The long time behavior of solutions of nonlinear wave equations has drawn considerable attention in the past decades. When the data are small, the classical result of Christodoulou [1] and Klainerman [9] shows that the solution of the nonlinear wave equation (1) is globally in time for all sufficiently small initial data. Generalizations and variants can be found in [6], [11], [14], [17], [18], [23], [24], [29] and references therein. One of the most remarkable related results should be the global nonlinear stability of Minkowski space first proved by Christodoulou-Klainerman [3] and later an alternative proof contributed by Lindblad-Rodnianski [13].

For the general large data case which is concerned in the current study, global existence results for the related wave map problems have been established with initial energy below that of any nontrivial harmonic maps, see e.g. [5], [25], [26], [28]. Failure of this energy condition may lead to finite time blow up, see e.g. [20], [19], [22], [27], [12]. For equation (1), the concrete example

$$\Box \phi = |\partial_t \phi|^2 - |\nabla \phi|^2$$

shows that the solution can blow up in finite time for general large data. For the details we refer to [7].

In a recent work [30] of Wang-Yu, they constructed an open set of Cauchy data for the semilinear wave equations satisfying the null condition such that the energy is arbitrarily large while the solution exists globally in the future. The construction is indirect. They in fact impose the radiation data at the past null infinity and then solve the equation to some time $t_0 < 0$ to obtain the Cauchy data. Their work relied on the short pulse method of Christodoulou in his monumental work [2] on the formation of trapped surface. Extensions and refinements of Christodoulou’s result are contributed by, e.g., Klainerman-Rodnianski [10], Luk-Rodnianski [15], [16], Klainerman-Luk-Rodnianski [21], Yu [35], [36], [31].
However, the nonlinear terms considered in Wang-Yu's work are quite restrictive. In fact, only quadratic null forms are allowed and cubic or higher order nonlinearities are excluded for consideration due to the short pulse method. In this paper, we use the new approach developed in [4], [33], [32], [34] to treat the nonlinear wave equations (1) with large data. We are able to give a criterion on the initial data such that the solution exists globally in the future while the energy can be arbitrarily large. In particular, our approach applies to equations with any higher order nonlinearities. Combined with the techniques developed in [34], our result here can even be extended to quasilinear wave equations. Moreover, we no longer require the data to have compact support as in [33], [32], [34]. This in particular implies that those results also hold for data satisfying conditions in this paper.

Before we state the main result, we define the necessary notations. We use the coordinate system \((t, x) = (x^0, x^1, x^2, x^3)\) of the Minkowski space. We denote \(\partial_0 = \partial_t, \partial_i = \partial_{x^i}, \partial = (\partial_t, \partial_1, \partial_2, \partial_3) = (\partial_t, \nabla)\). We may also use the standard polar coordinates \((t, r, \omega)\). Let \(\nabla\) denote the induced covariant derivative and \(\Delta\) the induced Laplacian on the spheres of constant \(r\). We also define the null coordinates \(u = \frac{t + r}{2}, v = \frac{t - r}{2}\) and denote the corresponding partial derivatives

\[
\partial_u = \partial_T - \partial_r, \quad \partial_v = \partial_t + \partial_r, \quad \nabla_i = (\partial_i, \nabla)
\]

for \(r > 0\). The vector fields that will be used as commutators are

\[
Z = \{\partial_t, \Omega_{ij} = x_i \partial_j - x_j \partial_i\}.
\]

Let \(\alpha\) be a positive constant. Without loss of generality, we may assume \(\alpha < \frac{1}{4}\). Denote

\[
E_0(R) = \sum_{k \leq 4} \int_{\{r \geq R\} \cap \mathbb{R}^3} r^{1+\alpha} |\nabla_v (rZ^k \phi)|^2 drd\omega \bigg|_{t=0} + \int_{\{r \leq R\} \cap \mathbb{R}^3} |\partial Z^k \phi|^2 dx \bigg|_{t=0},
\]

\[
E_1(R) = \sum_{k \leq 4} \int_{\{r \geq R\} \cap \mathbb{R}^3} |\partial_u (rZ^k \phi)|^2 + |rZ^k \phi|^2 drd\omega \bigg|_{t=0}.
\]

These quantities can be uniquely determined by the initial data \((\phi_0, \phi_1)\) together with the equation (1). We have the following main result:

**Theorem 1.** Consider the Cauchy problem for the semilinear wave equation (1) satisfying the null condition (2) with some integer \(N \geq 3\). For all \(\alpha \in (0, 1)\), there exists a constant \(R(\alpha)\), depending only on \(\alpha\), and a constant \(c_0\), depending only on the highest order \(N\) of the nonlinearity, such that if the initial data satisfy the estimate

\[
E_0(R) \leq R^{-2+\alpha}, \quad E_1(R) \leq R^{6\alpha}
\]

for some \(R \geq R(\alpha)\), then the solution \(\phi\) exists globally in the future and obeys the estimates:

\[
|\nabla_v \phi| \leq C_\delta (1 + r)^{-\frac{1}{2} + \delta}, \quad \delta > 0;
\]

\[
|\partial_u \phi| \leq C_\delta (1 + r)^{-1 + \delta} (1 + t - r + R)^{-\frac{1}{2} - \frac{\delta}{2}}, \quad \delta > 0, \quad t + R \geq r;
\]

\[
|\partial_v \phi| \leq C (1 + r)^{-1} R^{\frac{1}{2} + 4\alpha}, \quad t + R < r,
\]

where the constant \(C_\delta\) depends on \(\delta\), \(\alpha\) and the constant \(C\) depends only on \(\alpha\).

**Remark 1.** Similar result holds for equations in higher dimensions without assuming the null condition.

The Theorem implies that the energy of the initial data can be as large as \(R^{6\alpha}\). Since \(R\) can be any constant larger than a fixed constant \(R(\alpha)\), the energy can be arbitrarily large. Moreover, the amplitude of the solution, at least in a small region, can have size \(R^{2+\alpha}\). In Wang-Yu's work, the construction of the Cauchy data is indirect and only the size of the energy has a lower bound. The amplitude or the \(L^\infty\) estimates of the solution is unclear except the upper bound. From this point of view, the problem we consider here is a large data problem.
The existence of the initial data $(\phi_0, \phi_1)$ satisfying the conditions in the Theorem can be seen as follows: for any fixed $\alpha \in (0, 1)$ and any $R \geq R(\alpha)$, let $\phi_0$ be small in the ball with radius $R$ in $\mathbb{R}^3$. Here $R(\alpha)$ is a sufficiently large constant depending only on $\alpha$. Outside the ball, the energy of $\phi_0$ are allowed to be as large as $R^{2\alpha}$. Then for $\phi_1$, we require it to be small inside the ball with radius $R$. Outside the ball, it is close to $\partial_r \phi_0$. This will definitely give a large set of initial data $(\phi_0, \phi_1)$ satisfying the conditions in the Theorem.

We will use the new approach developed in [4], [33], [32], [34] to prove the main Theorem. A key ingredient of this new approach is the $p$-weighted energy inequality originally introduced by Dafermos-Rodnianski [4]. This inequality can be obtained by using the vector field $r^p \partial_r$ as multipliers in a neighborhood of the null infinity. It in particular implies that the $p$-weighted energy $E_0(R)$, see the definition before the main Theorem, keeps small if initially it is. This allows us to relax the size of the transversal derivative the solution, which is $E_1(R)$ in the theorem.

We will first construct the solution of the nonlinear wave equation outside the light cone, that is the region $r \geq R + t$, and show that the energy flux through the outgoing null hypersurface $r = t + R$ is small. And then we prove the solutions exists globally inside the light cone, for which we are not able to apply the results, e.g., in [34] directly. In the previous results, the smallness needed in order to close the bootstrap argument for nonlinear problem is guaranteed by assuming the data to be sufficiently small. Hence it is not necessary to keep track of the dependence of the radius $R$ of the constants in the argument. However, in this paper, the smallness comes from the radius $R$ and thus we need an argument with all the dependence on $R$.

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2 Preliminaries and energy identities

We briefly recall the energy identity for wave equations, for details we refer to [34]. Let $m$ be the Minkowski metric. We make a convention that the Greek indices run from 0 to 3 while the Latin indices run from 1 to 3. We raise and lower indices of any tensor relative to the metric $g_{\mu\nu}$ of the general relativity.

Recall the energy-momentum tensor

$$T_{\mu\nu}[\phi] = \partial_{\mu}\phi\partial_{\nu}\phi - \frac{1}{2}m_{\mu\nu}\partial_{\gamma}\phi\partial_{\gamma}\phi.$$ 

Given a vector field $X$, we define the currents

$$J^X_{\mu}[\phi] = T_{\mu\nu}[\phi]X^\nu, \quad K^X[\phi] = T^\mu_{\nu}[\phi]\pi^X_{\mu\nu},$$

where $\pi^X_{\mu\nu} = \frac{1}{2}\mathcal{L}_X g_{\mu\nu}$ is the deformation tensor of the vector field $X$. For any function $\chi$, we define the vector field $J^X[\phi]$ as

$$J^X[\phi] = J^X_{\mu}[\phi]\partial^\mu = \left( J^X_{\mu}[\phi] - \frac{1}{2}\partial_{\mu}\chi \cdot \phi^2 + \frac{1}{2}\chi\partial_{\mu}\phi^2 \right) \partial^\mu.$$ 

For any bounded region $D$ in $\mathbb{R}^{3+1}$, using Stokes' formula, we have the energy identity

$$\int_D \Box_{\phi}(\chi\phi + X(\phi)) + K^X[\phi] + \chi\partial_{\gamma}\phi\partial_{\gamma}\phi - \frac{1}{2}\Box_{\phi}\phi^2 d\text{vol} = \int_{\partial D} i_{J^X[\phi]} d\text{vol},$$

where $\partial D$ denotes the boundary of the domain $D$ and $i_Y d\text{vol}$ denotes the contraction of the volume form $d\text{vol}$ with the vector field $Y$ which gives the surface measure of the boundary.
3 The solution on the region \( \{ r \geq t + R \} \)

In this section, we construct the solution of the nonlinear wave equation (1) on the region \( \{ r \geq R + t \} \). First we define some notations. For \( R \leq r_1 \leq r_2 \), we use \( S_{r_1, r_2} \) to denote the following outgoing null hypersurface emanating from the sphere with radius \( r_1 \)

\[
S_{r_1, r_2} := \{ u = -\frac{r_1}{2}, \quad r_1 \leq r \leq r_2 \}.
\]

Similarly define \( \bar{C}_{r_1, r_2} \) to be the following incoming null hypersurface emanating from the sphere with radius \( r_2 \)

\[
\bar{C}_{r_1, r_2} := \{ v = \frac{r_2}{2}, \quad r_1 \leq r \leq r_2 \}.
\]

On the initial hypersurface \( \mathbb{R}^3 \), the annulus with radii \( r_1, r_2 \) is

\[
B_{r_1, r_2} := \{ t = 0, \quad r_1 \leq r \leq r_2 \}.
\]

We use \( S_r \) to be short for \( S_{r, \infty} \). Similarly we have \( \bar{C}_r \) and \( B_r \).

We use \( D_{r_1, r_2} \) to denote the region bounded by \( S_{r_1, r_2}, B_{r_1, r_2}, \bar{C}_{r_1, r_2} \). Let \( E[\phi](\Sigma) \) to be the energy flux for \( \phi \) through the hypersurface \( \Sigma \) in the Minkowski space. In particular,

\[
E[\phi](S_{r_1, r_2}) = \int_{S_{r_1, r_2}} |\overline{\partial_\nu}\phi|^2 r^2 d\omega, \quad E[\phi](\bar{C}_{r_1, r_2}) = \int_{\bar{C}_{r_1, r_2}} |\overline{\partial_u}\phi|^2 r^2 d\omega,
\]

where \( \overline{\partial_u} = (\partial_u, \nabla) \). On the initial hypersurface

\[
E[\phi](B_{r_1, r_2}) = \int_{B_{r_1, r_2}} |\partial \phi|^2 dx.
\]

3.1 Energy estimates

In the energy identity (5), take the region \( D \) to be \( D_{r_1, r_2} \), the vector field \( X = \partial_t \) and the function \( \chi = 0 \). We obtain the classical energy estimate

\[
2 \int_D (\overline{\Delta} \phi \cdot \partial_t \phi) d\text{vol} + E[\phi](S_{r_1, r_2}) + E[\phi](\bar{C}_{r_1, r_2}) = E[\phi](B_{r_1, r_2}). \tag{6}
\]

We also need an integrated energy estimate adapted to the region \( D_{r_1, r_2} \). For some small positive constant \( \epsilon \), depending only on \( \alpha \), we construct the vector field \( X \) and choose the functions \( f, \chi \) as follows

\[
X = f(r) \partial_r, \quad f = 2\epsilon^{-1} - \frac{2\epsilon^{-1}}{(1 + r)^{\epsilon}}, \quad \chi = \frac{1}{r^{\epsilon}}.
\]

We then can derive from the energy identity (5) that

\[
I^r[\phi]_{r_1}^{r_2} \leq C_\epsilon (E[\phi](B_{r_1}) + E[\phi](S_{r_1, r_2}) + E[\phi](\bar{C}_{r_1, r_2}) + D^r[\overline{\Delta} \phi]_{r_1}^{r_2}), \tag{7}
\]

where we denote

\[
I^r[\phi]_{r_1}^{r_2} := \int_{D_{r_1, r_2}} |\overline{\Delta} \phi|^2 \frac{1}{(1 + r)^{\epsilon}} dxdt, \quad D^r[F]_{r_1}^{r_2} := \int_{D_{r_1, r_2}} (1 + r)^{1+\epsilon} |F|^2 dxdt.
\]

Here \( \overline{\Delta} \phi = (\partial_\phi, \frac{\phi}{1+r}) \). The constant \( C_\epsilon \) depends only on \( \epsilon \) and is independent of \( r_1, r_2 \). For the derivation of the above estimate (7), it is almost the same as Proposition 1 of [34] or Proposition 2 of [33]. The only point we have to point out here is that we use the fact that the solution \( \phi \) goes to zero as \( r \to \infty \) on the initial hypersurface. We thus can use a Hardy’s inequality to control the integral of \( \frac{|\phi|^2}{(1+r)^{\epsilon}} \). This is also the reason that we have \( E[\phi](B_{r_1}) \), which is \( E[\phi](B_{r_1, \infty}) \) according to our notations, instead of \( E[\phi](B_{r_1, r_2}) \) on the right hand side of the above estimate (7).

Combine the above two estimates (6), (7). We derive the following integrated energy estimates
Proposition 1. We have
\[ E[\phi](S_{r_1,r_2}) + E[\phi](\bar{C}_{r_1,r_2}) + I'[\phi]_{r_1}^2 \leq C_\epsilon(E[\phi](B_{r_1}) + D'[\Box\phi]_{r_1}^2) \tag{8} \]
for some constant $C_\epsilon$ depending only on $\epsilon$.

**Proof.** For the derivation of the integrated energy estimate (7), we refer to [33] or [34]. Then from the energy identity (6), we can estimate
\[ E[\phi](S_{r_1,r_2}) + E[\phi](\bar{C}_{r_1,r_2}) \leq E[\phi](B_{r_1,r_2}) + \frac{1}{2}C_\epsilon^{-1}I'[\phi]_{r_1}^2 + 2C_\epsilon D'[\Box\phi]_{r_1}^2, \]
where $C_\epsilon$ is the constant in the integrated energy estimate (7). Then the integrated energy estimate (7) can be improved to be
\[ I'[\phi]_{r_1}^2 \leq 4C_\epsilon(E[\phi](B_{r_1}) + D'[\Box\phi]_{r_1}^2). \]
This together with the previous estimate proves the proposition. Here according to our notation $C_\epsilon$ is a constant depending only on the small constant $\epsilon$. \qed

Next we consider the $p$-weighted energy inequality. In the energy identity (5), we take
\[ X = f \partial_v, \quad \chi = r^{p-1}, \quad f = r^p, \quad 0 \leq p \leq 2. \]
We can compute
\[
\int_{B_{r_1,r_2}} i j^X[\phi] d\text{vol} = \frac{1}{2} \int_{B_{r_1,r_2}} f(\partial_v \psi)^2 + |\nabla \psi|^2 - \partial_v(f \psi^2) + \frac{1}{2} r \psi^2 drd\omega,
\]
\[
\int_{S_{r_1,r_2}} i j^X[\phi] d\text{vol} = \int_{S_{r_1,r_2}} f(\partial_v \psi)^2 - \frac{1}{2} \partial_v(f \psi^2) dvd\omega,
\]
\[
\int_{\bar{C}_{r_1,r_2}} i j^X[\phi] d\text{vol} = -\int_{\bar{C}_{r_1,r_2}} f|\nabla \psi|^2 + f' r \psi^2 + \frac{1}{2} \partial_v(f \psi^2) dud\omega,
\]
\[
\int \int_{D_{r_1,r_2}} K^X[\phi] + \chi \partial^\gamma \phi \partial_\gamma \phi - \frac{1}{2} \Box \chi \phi^2 d\text{vol}
\]
\[
= \int \int_{D_{r_1,r_2}} \frac{1}{2} f'(\partial_v \psi)^2 + (\chi - \frac{1}{2} f')|\nabla \psi|^2 - \frac{1}{2} \partial_v(f' r \psi^2) drdt.
\]
Here $\psi = r \phi$. We can do integration by parts on $D_{r_1,r_2}$ to estimate the integral of $\partial_v(f' r \psi^2)$. Alternatively, we can modify the current vector field $j^X[\phi]$ defined in line (4) to be
\[ j^X[\phi] = j^X[\phi] + \frac{1}{2} f' r \phi^2 \partial_v. \]
Notice that
\[-\int_{B_{r_1,r_2}} \partial_v(f \psi^2) drd\omega - \int_{\bar{C}_{r_1,r_2}} \partial_u(f \psi^2) dud\omega + \int_{S_{r_1,r_2}} \partial_v(f \psi^2) dvd\omega = 0. \]
Then from the energy identity (5) and the above calculations, we obtain
\[
\int \int_{D_{r_1,r_2}} r^{p-1}(p|\partial_v \psi|^2 + (2-p)|\nabla \psi|^2) drdtd\omega + \int \int_{\bar{C}_{r_1,r_2}} r^p|\nabla \psi|^2 dud\omega
\]
\[
+ \int \int_{S_{r_1,r_2}} r^p|\partial_v \psi|^2 dvd\omega = \int \int_{B_{r_1,r_2}} r^p|\partial_v \psi|^2 drd\omega - 2 \int \int_{D_{r_1,r_2}} r^{p-1} \Box \phi \partial_v \psi dxdt. \tag{9}
\]
3.2 Bootstrap argument

We assume initially
\[ E_0(R) \leq R^{-\beta}, \quad E_1(R) \leq R^4. \]
for some positive constant \( \beta \), which will be determined later. The definition of \( E_0(R) \) can be found in the introduction before the statement of the main Theorem. We impose the following bootstrap assumption on the nonlinearity \( F(\partial \phi) \) in the equation (1)
\[
\sum_{k \leq 4} \iint_{D_R} |Z^k F|^2 r^{2+\alpha} dx dt \leq 2 R^{-\beta}.
\]
(10)

Then the \( p \)-weighted energy inequality (9) obtained in the end of the previous subsection implies that
\[
\int_{D_{r_1 \cdot r_2}} r^{p-1}(p|\partial_{\omega} Z^k \psi|^2 + (2-p)|\nabla Z^k \psi|^2) dr dt d\omega + \int_{S_{r_1 \cdot r_2}} r^p |\partial_{\omega} Z^k \psi|^2 dv d\omega + \int_{C_{r_1 \cdot r_2}} r^p |\nabla Z^k \psi|^2 dv d\omega \\
\leq r_1^{p-1-\alpha} \int_{B_{r_1 \cdot r_2}} r^{1+\alpha} |\nabla \phi|^2 dr d\omega + \int_{D_{r_1 \cdot r_2}} \frac{P}{2} r^{p-1} |\partial_{\omega} Z^k \psi|^2 dr dt d\omega + \int_{D_{r_1 \cdot r_2}} \frac{2}{p} r^{2p+1} |Z^k \psi|^2 dt \lesssim r_1^{p-1-\alpha} E_0(R) + \frac{P}{2} \int_{D_{r_1 \cdot r_2}} r^{p-1} |\partial_{\omega} Z^k \psi|^2 dr dt d\omega + r_1^{p-1-\alpha} \int_{D_{r_1 \cdot r_2}} \frac{2}{p} r^{2+\alpha} |Z^k \psi|^2 dt.
\]

The second term in the last line can be absorbed. Then let \( r_2 \) goes to infinity, we can obtain the following \( p \)-weighted energy estimate
\[
\int_{D_{r_1}} r^{p-1} |\nabla_{\omega} Z^k \psi|^2 dr d\omega + \int_{S_{r_1}} r^p |\partial_{\omega} Z^k \psi|^2 dv d\omega + \int_{C_{r_1 \cdot r_2}} r^p |\nabla Z^k \psi|^2 dv d\omega \lesssim R^{-\beta} r_1^{p-1-\alpha}
\]
(11)
for all \( k \leq 4, \ r_2 \geq r_1 \geq R, \ 0 < p \leq 1 + \alpha \). Here and in the following we make a convention that \( A \lesssim B \) means \( A \leq CB \) for some constant \( C \) depending only on \( \alpha \) and is independent of \( R, r_1 \).

Note that the assumption (3) in particular implies that
\[
\int_\omega r^2 |Z^k \phi(0, r, \omega)|^2 d\omega \lesssim R^4, \quad k \leq 4, \quad r \geq R.
\]
(12)

Using the \( p \)-weighted energy inequality (11) when \( p = 1 + \alpha \), on \( S_{r_1} \), we can estimate
\[
\int_\omega |r Z^k \phi|^2(t, r, \omega) d\omega \lesssim \int_\omega |r Z^k \phi(0, r_1, \omega)|^2 d\omega + \int_{S_{r_1}} r^{1+\alpha} |\partial_{\omega}(r Z^k \phi)|^2 dv d\omega \cdot \alpha^{-1} r_1^{-\alpha} \\
\lesssim \int_\omega |r Z^k \phi(0, r_1, \omega)|^2 d\omega + R^{-\beta} r_1^{-\alpha}, \quad k \leq 4.
\]
(13)

In particular, we have
\[
\int_\omega |r Z^k \phi|^2(t, r, \omega) d\omega \lesssim R^4, \quad k \leq 4.
\]

We also need an inequality to estimating \( \partial_u \psi, \psi = r \phi \). From the energy inequality (8), we can show that
\[
\int_{C_{r_1 \cdot r_2}} |\partial_u Z^k \psi|^2 dv d\omega \lesssim E[\phi](C_{r_1 \cdot r_2}) + \int_{\omega} |r Z^k \phi|^2(u_{r_2}, v_{r_2}, \omega) d\omega \lesssim R^4, \quad k \leq 4, \quad r_2 \geq R,
\]
where \( u_r = \frac{r-R}{2}, \ v_r = \frac{r+R}{2}, \ \overline{\partial_u} = (\partial_u, \overline{\psi}). \)

We now improve the bootstrap assumption (10). The quadratic part of the nonlinearity \( F \) is a null form \( Q(\phi, \phi) \). Note that
\[
Z^k Q(\phi, \phi) = \sum_{k_1 + k_2 \leq k} Q(Z^{k_1} \phi, Z^{k_2} \phi).
\]
Here $Q$ denotes a general null form. They may stand for different null forms with different constants $A^{\mu\nu}$. For details we refer to e.g. [8]. We denote

$$\phi_1 = Z^{k_1}\phi, \quad \phi_2 = Z^{k_2}\phi, \quad \psi_1 = r\phi_1, \quad \psi_2 = r\phi_2.$$  

This $\phi_1$ is only a notation and should not be confused with the initial data $\phi$. Note that

$$|r^2 Z^k Q(\phi,\phi)| \lesssim \sum_{k_1+k_2 \leq k} |\partial \psi_1| |\partial \psi_2| + |\nabla \psi_1||\nabla \psi_2| + |\partial \psi_1||\partial \psi_2|.$$  

For the proof of the above inequality, see [33]. Then by using Sobolev embedding, for $k \leq 4$, we have the estimate

$$\int_\Omega |r^2 Z^k Q(\phi,\phi)|^2 d\omega \lesssim \sum_{k_1 \leq 4 \text{, } k_2 \leq 4} \int_\Omega |\partial \psi_1|^2 d\omega \cdot \int_\Omega |\partial \psi_2|^2 d\omega + \sum_{k_1 \leq 4 \text{, } k_2 \leq 2} \int_\Omega |\partial \psi_1|^2 d\omega \cdot \int_\Omega |\partial \psi_2|^2 d\omega$$

$$+ \sum_{k_1 \leq 4 \text{, } k_2 \leq 2} \int_\Omega |\partial \psi_1|^2 d\omega \cdot \int_\Omega |\partial \psi_2|^2 d\omega + \int_\Omega |\nabla Z^4 \psi||\partial \psi|^2 d\omega. \quad (15)$$  

We use estimate (12) to bound $||\psi_2||_{L^2(\Omega)}$, $k_2 \leq 4$. For $||\partial \psi_2||_{L^2(\Omega)}$, $k_2 \leq 3$, we can estimate

$$\sup_{v=\frac{1}{2}} \int_\Omega |r^\alpha |\partial \psi_2||_{L^2(\Omega)}^2 \lesssim \int_\Omega |\partial \psi_2|^2 (0, r, \omega) d\omega + \int_\Omega \partial_v (r^\alpha |\partial \psi_2|^2) v d\omega$$

$$\lesssim \int_\Omega |\partial \psi_2|^2 (0, r, \omega) d\omega + \int_\Omega r^{\alpha-1} |\partial \psi_2|^2 + r^\alpha |\partial \psi_2||\partial \psi_2| d\omega$$

$$\lesssim \int_\Omega |\partial \psi_2|^2 (0, r, \omega) d\omega + \int_\Omega r^{\alpha-1} |\partial \psi_2|^2 d\omega$$

$$+ \int_\Omega r^{\alpha+1} (|\nabla \Omega \phi_2|^2 + |rZ^2F|^2) d\omega,$$

where we have used the equation for $\psi_2$ in null coordinates $(u, v, \omega)$. But the coordinate $(0, r, \omega)$ appeared in the previous estimate is with respect to the polar coordinate $(t, r, \omega)$. Integrate the above estimate with respect to $r$ from $R$ to infinity. We obtain

$$\int_R^\infty \sup_{v=\frac{1}{2}} \int_\Omega |r^\alpha |\partial \psi_2||_{L^2(\Omega)}^2 d\omega \lesssim \int_B |\partial \psi_2|^2 d\omega + \int_D r^{\alpha-1} |\partial \psi_2|^2 d\omega$$

$$+ \int_D r^{\alpha-1} |\nabla \Omega \psi_2|^2 + r^{\alpha+3} |Z^2 F|^2 d\omega dr \quad (16)$$  

Here we have used the assumption on the initial data that $E_0(R) \leq R^{-\beta}$. The bound for $F$ follows from the bootstrap assumption (10). The estimates for $|\partial \psi_2|^2$, $|\nabla \Omega \psi_2|^2$ are due to the $p$-weighted energy inequality (11) and the fact that $k_2 \leq 3$.

Similarly, for $||\partial \psi_1||_{L^2(\Omega)}$, $k_1 \leq 3$, we have

$$\int_R^\infty \sup_{v=\frac{1}{2}} \int_H |r^{-1} |\partial \psi_1||_{L^2(\Omega)}^2 d\omega \lesssim R^{-1} \int_B |\partial \psi_1|^2 d\omega + \int_D r^{-2} |\partial \psi_1|^2 d\omega$$

$$+ \int_D |\nabla \Omega \phi_1|^2 + |rZ^2 F|^2 d\omega dr \lesssim R^{-1+\epsilon} + R^{-1+\epsilon} I^\epsilon |\phi_1|_R^\infty + R^{2-\alpha-\beta} \lesssim R^{-1+\epsilon}.$$  

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Here the estimate for the integrated energy estimate $J^*|\phi_1|^\infty_R$ follows from (8) in which the bounds for $D^*[Z^kF]^\infty_R$ are guaranteed by the bootstrap assumption (10).

On the right hand side of the null form estimate (15), we are left to estimate the special term $|\nabla \psi_1| |\partial_t \psi_1|$, $\psi_1 = Z^\alpha \psi$. We can show that

$$
\int_{D_R} |\nabla \psi_1|^2 |\partial_t \psi_1|^2 r^\alpha drdtd\omega \lesssim \int R \int_{-\frac{R}{2}} \int_{\frac{R}{2}} (v + \frac{r}{2})^\alpha \int_{\omega} |\nabla \psi_1|^2 \cdot \int_{\omega} |\psi_2|^2 d\omega drd\nu
$$

$$
\lesssim \int R \int_{-\frac{R}{2}} \int_{\frac{R}{2}} (v + \frac{r}{2})^\alpha \int_{\omega} |\nabla \psi_1|^2 \cdot (\int_{\omega} |\psi_2|^2 (r, \frac{r}{2}, \omega) d\omega + R^{-\beta} (v + \frac{r}{2})^{-\alpha}) drd\nu
$$

$$
\lesssim R^{-2\beta - \alpha} + \int R \int_{-\frac{R}{2}} \int_{\frac{R}{2}} (v + \frac{r}{2} - u)^\alpha |\nabla \psi_1|^2 d\omega \cdot \int_{\omega} |\psi_2|^2 (r, \frac{r}{2}, \omega) d\omega dr.
$$

Here we have used estimate (13) and $\kappa_2 \leq 3$. We note that when $u$ is fixed, the $p$-weighted energy inequality (11) implies that

$$
\int_{-\frac{R}{2}} \int_{\omega} (\frac{r}{2} - u)^{1+\alpha} |\nabla \psi_1|^2 d\omega du \leq \int_{\Omega_{R,\omega}} r^{1+\alpha} |\nabla \psi_1|^2 d\omega \lesssim R^{-\beta}.
$$

Thus we can show that

$$
\int_{D_R} |\nabla \psi_1|^2 |\partial_t \psi_1|^2 r^\alpha drdtd\omega \lesssim R^{-2\beta - \alpha} + \int R R^{-1} \int_{\omega} |\psi_2|^2 (0, r, \omega) d\omega dr \lesssim R^{-2\beta - \alpha} + R^{-1-\beta + \epsilon_1}.
$$

Therefore from the null form estimate (15), we can derive that

$$
\int_{D_R} r^\alpha |r^2 Z^k Q(\phi, \phi)|^2 drdtd\omega
$$

$$
\lesssim R^{c_1} \int_{D_R} r^{-2+\alpha} |\partial_t \psi|^2 dt d\omega d\nu + \int_{D_R} \sup_{u} r^\alpha \|\partial_u \psi_2\|^2_{L^2(S^2)} \int_{\Omega_{R,\omega}} |\nabla \psi_1|^2 d\omega dr
$$

$$
+ \int R \sup_{u} r^{1+\alpha} |\partial_u \psi_2|^2 d\omega dr = R^{-2\beta - \alpha} + R^{-1+\beta + \epsilon_1}
$$

$$
\lesssim R^{2\epsilon_1 - \beta + c} + R^{-1-\beta + \epsilon_1} + R^{1+\epsilon_1} + R^{23-\beta} + R^{-1-\beta + \epsilon_1}.
$$

For cubic or higher order nonlinearities, we first conclude from estimate (16) that

$$
\int_{0}^{r_1 - R} \int_{\omega} r_1^\alpha (|\partial_t Z^k \psi|^2 + |\partial_t \partial_t Z^k \psi|^2) d\omega d\nu \lesssim R^{-1-\beta}, \quad k \leq 2.
$$

In particular, we have

$$
\int_{\omega} |\partial_t Z^k \psi|^2 d\omega \lesssim R^{-1-\beta R_1^{-\alpha}}, \quad k \leq 2.
$$

Since we have shown that

$$
\int_{\omega} |Z^k \psi|^2 d\omega \lesssim R^{c_1}, \quad k \leq 4,
$$

we then have

$$
\int_{\omega} |\partial Z^k \psi|^2 d\omega \lesssim R^{c_1}, \quad k \leq 2.
$$

Thus for cubic or higher order nonlinearities, we can bound

$$
\int_{D_R} |Z^k (F - Q)|^2 r^{2+\alpha} dx d\nu \lesssim \sum_{k \leq 4} \int_{D_R} |\partial Z^k \phi|^2 r^{-4+2+\alpha} R^{2(N-2)\epsilon_1} dxd\nu \lesssim R^{(2N-3)\epsilon_1 + \alpha + \epsilon_1}.
$$
Here we recall that \( N \) is the order of the highest order nonlinearity. To summarize, we have shown that
\[
\int \int_{\mathcal{D}_R} |Z^k F|^2 r^{2+\alpha} dxdt \lesssim R^{2N-3}\epsilon_1 + \epsilon + \epsilon = R^{-1+\epsilon} + R^{-2\beta-\alpha}.
\]
If we take
\[
\beta = 1 - 2\alpha, \quad \epsilon = \frac{\alpha}{20}, \quad \epsilon_1 = \frac{\alpha}{2N},
\]
we then have
\[
\int \int_{\mathcal{D}_R} |Z^k F|^2 r^{2+\alpha} dxdt \lesssim R^{-\frac{1}{2}}.
\]
According to our notations, the implicit constant in the above estimate depends only on \( \alpha \). Hence let the constant \( R \) be sufficiently large, depending only on \( \alpha \), we then can improve the bootstrap assumption (10). Once we have improved the bootstrap assumption (10), the proof for the existence of a unique solution of the equation (1) on the region \( \{r \geq R + t\} \) is standard, see the end of [33].

**Remark 2.** In particular, the small constant \( \epsilon_0 \) in the main Theorem can be \( \epsilon_0 = \frac{1}{2N} \).

## 4 The solution on \( \{r \leq R + t\} \)

We have constructed the solution of the equation (1) outside the light cone \( \{r \geq R + t\} \). In this section, we will prove that the solution also exists globally in the future inside the light cone which is the region \( \{r \leq R + t\} \). We use the foliation
\[
S_r := \{u = u_r = \frac{\tau - R}{2}, \quad \tau + \frac{R}{2} = v_r \leq v\}, \quad \Sigma_r := \{t = \tau, \quad r \leq R\} \cup S_r.
\]
The energy flux through \( \Sigma_r \) for the scalar field \( \phi \) is \( E[\phi](\tau) \). For \( \tau_2 \geq \tau_1 \), we define
\[
I_r[\phi]_{\tau_1}^{\tau_2} := \int_{\tau_1}^{\tau_2} \int_{\Sigma_r} \frac{|\phi|^2}{(1 + r)^{1+\epsilon}} dxdt, \quad D_r[F]_{\tau_1}^{\tau_2} := \int_{\tau_1}^{\tau_2} \int_{\Sigma_r} (1 + r)^{1+\epsilon} |F|^2 dxdt.
\]
We have the integrated energy estimate and the energy estimate
\[
E[\phi](\tau_2) + I_r[\phi]_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} \int_{S_r} \frac{|\nabla \phi|^2}{1 + r} dxdt \lesssim E[\phi](\tau_1) + D_r[F]_{\tau_1}^{\tau_2}, \quad (18)
\]
see Proposition 1 of [34] or Proposition 2 of [33]. As before, the implicit constant here depends only on \( \epsilon \).

### 4.1 The \( p \)-weighted energy inequality

As we have discussed in the introduction, the smallness needed to close the bootstrap argument for nonlinear problem in this paper comes from the radius \( R \) while in the previous work e.g. [33] the smallness comes from the data. In particular, the previous argument can not be applied directly to the settings in this paper. Instead we need an argument with all the dependence of the constants on the radius \( R \). To be more precise, we first consider one of the key ingredients the \( p \)-weighted energy inequality. We recall the \( p \)-weighted energy identity originally introduced by Dafermos-Rodnianski in [4]
\[
\int_{S_{\tau_2}^{\tau_1}} r^p (\partial_\psi \phi)^2 dv d\omega + \int_{\tau_1}^{\tau_2} \int_{S_{\tau}^\omega} 2r^{p+1} F \cdot \partial_\psi \psi dv dr d\omega
\]
\[
+ \int_{\tau_1}^{\tau_2} \int_{S_{\tau_2}^\omega} r^{p-1} (p(\partial_\psi \phi)^2 + (2 - p)|\nabla \psi|^2) dv dr d\omega + \int_{C(\tau_1, \tau_2, \psi)} r^p |\nabla \psi|^2 dv d\omega
\]
\[
= \int_{S_{\tau_1}^{\tau_2}} r^p (\partial_\psi \phi)^2 dv d\omega + \int_{\tau_1}^{\tau_2} r^p \left(|\nabla \psi|^2 - (\partial_\psi \phi)^2\right) d\omega dr|_{r=R}.
\]
where \( \psi = r\phi \), \( F = \square \phi \). Note that the boundary term on \( \{ r = R \} \) is proportional to \( R^p \). Hence we can simply take \( p = 0 \) to estimate it. First for any \( \tau \), we have

\[
\int_{S^r_{\tau}} (\partial_v \psi)^2 dv d\omega \leq 5E[\phi](\tau).
\]

For the proof of this inequality, see e.g. Corollary 1 in [33]. For the inhomogeneous term \( F \partial_v \psi \) when \( p = 0 \), we can estimate it as follows:

\[
| \int_{\tau_1}^{\tau_2} \int_{S^r_{\tau}} r F \cdot \partial_v \psi dv d\omega | \lesssim D^v[F]_{\tau_1} + E[\phi](\tau_1).
\]

Therefore for general \( p \), we have the estimate for the boundary term

\[
\left| \int_{\tau_1}^{\tau_2} r^p (|\nabla \psi|^2 - (\partial_v \psi)^2) dv d\omega |_{r = R} \right| \lesssim R^p(D^v[F]_{\tau_1} + E[\phi](\tau_1)).
\]

Since the boundary term on the incoming null hypersurface \( \bar{C}(\tau_1, \tau_2, v) \) has a good sign, to obtain a useful estimate from the \( p \)-weighted energy identity, it suffices to estimate the integral of the inhomogeneous term \( r^{p+1}F \partial_v \psi \) in the above \( p \)-weighted energy identity. On \( S_r \), we control it as follows

\[
2r^{p+1}|F \partial_v \psi| \leq r^p|\partial_v \psi|^2 r_+^{-1-\epsilon} + r^{p+2}|F|^2 r_+^{1+\epsilon}, \quad r_+ = 1 + \tau.
\]

The integral of the first term \( r^p|\partial_v \psi|^2 r_+^{-1-\epsilon} \) will be bounded by using Gronwall’s inequality. Thus we derive

\[
\int_{S^r_{\tau_2}} r^p(\partial_v \psi)^2 dv d\omega + \int_{\tau_1}^{\tau_2} \int_{S_r} r^{p-1}(p|\partial_v \psi|^2 + (2-p)|\nabla \psi|^2) dv d\omega d\tau
\]

\[
\lesssim R^p(E[\phi](\tau_1) + D^v[F]_{\tau_1}) + \int_{S^r_{\tau_1}} r^p|\partial_v \psi|^2 dv d\omega + \int_{\tau_1}^{\tau_2} r_+ D^v_+D^o_+ [F]_{\tau_1} d\tau + (\tau_1)^{1+\epsilon} D^o_+ [F]_{\tau_1},
\]

where

\[
D^s_+[F]_{\tau_1} := \int_{\tau_1}^{\tau_2} \int_{S_r} (1+r)^{1+\alpha}|F|^2 dx dr.
\]

### 4.2 The data

To study the equation on the region \( \{ r \leq t + R \} \), we need the initial data on the outgoing null hypersurface \( S_0 \), that is \( \{ v \geq \frac{R}{t}, u = -\frac{R}{t} \} \). The data on the ball with radius \( R \) can be arbitrarily small according to our assumptions. It suffices to understand the solution on the outgoing null hypersurface \( S_0 \) (or using the notation in Section 3 \( S_{R, \infty} \)). Recall that we already constructed the solution on the region \( \{ r \geq t + R \} \) in the previous section. From the \( p \)-weighted energy inequality (11) we have

\[
\int_{S_0} r^{1+\alpha}|\partial_v Z^k \psi|^2 dv d\omega \lesssim R^{-\beta}, \quad \beta = 1 - 2\alpha, \quad k \leq 4.
\]

Here note that we have fixed \( \beta \) in line (17). For the energy flux, we can assume

\[
\sum_{k \leq 4} \int_{S_0} |\partial_v Z^k \psi|^2 r^2 dv d\omega \lesssim R^{-\beta - \alpha - 1}.
\]

This is consistent with the previous inequality as \( |\partial_v Z^k \psi|^2 \) is the main part of \( |\partial_v Z^k \phi|^2 r^2 \). A rigorous way to see this is to use the \( p \)-weighted energy inequality (11). We have

\[
\int_{R} \int_{S_{r, \infty}} r^\alpha |\partial_v Z^k \psi|^2 dv d\omega dr \lesssim R^{-\beta}.
\]
Since the data inside the ball with radius $R$ is small, we also can show (simply replacing $R$ in Section 3 with $\frac{1}{2}R$) that

$$\int_{\frac{1}{2}R}^{R} \int_{S_{r,\infty}} r^{\alpha} |\partial_{r} Z^{k}\psi|^{2} dr \lesssim R^{-\beta}.$$  

In particular, we can choose a slice such that

$$E[Z^{k}\phi](S_{r,\infty}) \leq \int_{S_{r,\infty}} |\partial_{r} Z^{k}\psi|^{2} dv d\omega + r_{0} \int_{\omega} |\phi|^{2}(0, r_{0}, \omega) dv \lesssim R^{-\beta-\alpha-1} = R^{-2+\alpha}$$

for some $r_{0} \in (\frac{1}{2}R, R)$. Here we note that the initial data on the ball with radius $R$ can be arbitrarily small.

In particular the data on $S_{0}$ satisfy the above two estimates (20) and (21). Here recall that the implicit constant depends only on $\alpha$.

### 4.3 Bootstrap argument

We now use the above boundary conditions to establish the decay of the energy flux. We impose the following bootstrap assumptions on the nonlinearity $F$ for all $k \leq 4$

$$D^{\alpha}[Z^{k}F]_{r_{1}}^{2} \leq 2 \min\{R^{-\beta}(\tau_{1})^{-1-\alpha}, R^{-2+\alpha}, R^{-1-\beta-\alpha}(\tau_{1})^{-\alpha}\}, \quad D_{r_{1}}^{\alpha}[Z^{k}F]_{r_{1}}^{2} \leq 2 \tau_{1}^{-1-\alpha} R^{-\beta}. \quad (22)$$

We show the decay of $E[Z^{k}\phi](\tau)$. Let $p = 1 + \alpha$ in the $p$-weighted energy inequality (19). We have

$$\int_{S_{r_{2}}} r^{1+\alpha} |\partial_{v} \psi|^{2} dv d\omega + \int_{\tau_{1}}^{\tau_{2}} \int_{S_{\tau}} r^{\alpha} |\partial_{v} \psi|^{2} dv d\omega d\tau \lesssim R^{1+\alpha-2+\alpha} + R^{-\beta} = R^{-\beta}.$$  

Hence we can choose a dyadic sequence $\{\tau_{n}\}$ such that

$$\int_{S_{\tau_{n}}} r^{\alpha} |\partial_{v} \psi|^{2} dv d\omega \lesssim (\tau_{n})^{-1 \beta}.$$  

Interpolation leads to

$$\int_{S_{\tau_{n}}} r^{\beta} |\partial_{v} \psi|^{2} dv d\omega \lesssim R^{-\beta}(\tau_{n})^{-\alpha}.$$  

Then take $p = 1$ in the $p$-weighted energy inequality (19). We derive

$$\int_{\tau_{n}}^{\tau'} E[\phi](\tau) d\tau \lesssim R^{-\beta}(\tau_{n})^{\alpha} + RE[\phi](\tau_{n}) + R^{1+\epsilon}(E[\phi](\tau_{n}) + (\tau_{n})^{\alpha} R^{-1-\beta-\epsilon})$$

$$\lesssim R^{-\beta}(\tau_{n})^{\alpha} + R^{1+\epsilon} E[\phi](\tau_{n}), \quad \tau' \geq \tau_{n}.$$  

In the energy estimate (18), set $\tau_{1} = 0$. We have

$$E[\phi](\tau) \lesssim R^{-2+\alpha}.$$  

For $\tau' \geq \tau$, we have

$$E[\phi](\tau') \lesssim E[\phi](\tau) + R^{-\beta}\tau_{n}^{-1-\alpha}.$$  

We thus can conclude that

$$(\tau' - \tau_{n}) E[\phi](\tau') \lesssim R^{-\beta}(\tau_{n})^{\alpha} + R^{1+\epsilon} E[\phi](\tau_{n}).$$

In particular, we have

$$E[\phi](\tau) \lesssim \tau_{n}^{-1}(R^{-\beta} + R^{1+\epsilon} R^{-2+\alpha}) \lesssim \tau_{n}^{-1} R^{-\beta}.$$  

This then implies that

$$E[\phi](\tau_{n+1}) \lesssim R^{-\beta}(\tau_{n})^{1-\alpha} + R^{1+\epsilon-\beta}(\tau_{n})^{2}.$$  

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As $\tau_n$ is dyadic, we then infer that
\[
E[\phi](\tau) \lesssim R^{-\beta} r_+^{-1-\alpha} + R^{1+\epsilon-\beta} r_+^{-2}.
\]
Summarizing, we have the following energy decay estimate

**Proposition 2.** For any $k \leq 4$, we have
\[
I^\epsilon[Z^k\phi]_{r_1}^2 + D^\epsilon[Z^k F]_{r_1}^2 + E[Z^k\phi](\tau) \lesssim A(\tau),
\]
where
\[
A(\tau) := \min\{R^{-\beta} r_+^{-1-\alpha} + R^{1+\epsilon-\beta} r_+^{-2}, \quad R^{-2\alpha}, \quad R^{-\beta} r_+^{-1}\}.
\]
In particular, we have
\[
E[Z^k\phi](\tau) \lesssim \min\{R^{-\gamma} r_+^{-1-\alpha}, R^{-2\alpha}\}, \quad \gamma = \beta - (1+\epsilon)\alpha.
\]

**Proof.** The estimate for the energy flux $E[\phi](\tau)$ follows from the above argument. The estimate for the integrated energy $I^\epsilon[Z^k\phi]_{r_1}^2$ follows from (18) and the bound for the inhomogeneous term $F$ is a restatement of the bootstrap assumption (22).

The following lemma will be used to show the $C^1$ estimate of the solution.

**Lemma 1.**
\[
\int_{\tau_1}^{\tau_2} \int_{\Sigma_{\tau} \cap \{r \geq 1\}} r^{1-\epsilon}|\partial_u \partial_\nu Z^k \phi|^2 \, dx \, d\tau \lesssim A(\tau_1), \quad \forall k \leq 3.
\]

**Proof.** Using the equation for $Z^k \phi$ (commutation of the equation (1) with $Z^k$) we have
\[
\int_{\tau_1}^{\tau_2} \int_{\Sigma_{\tau} \cap \{r \geq 1\}} r^{1-\epsilon}|\partial_u \partial_\nu Z^k \phi|^2 \, dx \, d\tau \lesssim \int_{\tau_1}^{\tau_2} \int_{\Sigma_{\tau} \cap \{r \geq 1\}} r^{1-\epsilon}(r^{-1}|\partial Z^k \phi| + |\Delta Z^k \phi| + |Z^k F|)^2 \, dx \, d\tau
\lesssim I^\epsilon[Z^k \phi]_{r_1}^2 + I^\epsilon[\Omega Z^k \phi]_{r_1}^2 + D^\epsilon[Z^k F]_{r_1}^2 \lesssim A(\tau_1)
\]
for all $k \leq 3$.

Next, we improve the bootstrap assumption (22). We mainly consider the quadratic nonlinearity $Q(\phi, \phi)$, which satisfies the null condition. We first estimate $D^\alpha[F]_{r_1}^2$. On $S_r$, we can estimate
\[
\int_\omega |r Z^k \phi|^2(\tau, r, \omega) \, d\omega \leq \int_\omega |r Z^k \phi(\tau, R, \omega)|^2 \, d\omega + \int_{S_r} r^{1+\alpha}|\partial_\nu (r Z^k \phi)|^2 \, dv \cdot d\omega \cdot \alpha^{-1} R^{-\alpha}
\lesssim \int_\omega |r Z^k \phi(\tau, R, \omega)|^2 \, d\omega + R^{-\beta} R^{-\alpha} \lesssim R^{-1+\alpha}.
\]
Let $\tilde{C}_{\tau_1, \tau_2, v_1}$ be the incoming null hypersurface between $\Sigma_{\tau_1}$ and $\Sigma_{\tau_2}$, defined as follows:
\[
\tilde{C}_{\tau_1, \tau_2, v_1} := \{v = v_1, \quad u_{\tau_1} \leq u \leq u_{\tau_2}\}.
\]
The energy estimate on the region $\{v \geq v_1, \quad u_{\tau_1} \leq u \leq u_{\tau_2}\}$ then implies that
\[
\int_{C_{\tau_1, \tau_2, v_1}} |\partial_u Z^k \phi|^2 \, dv \, d\omega \lesssim A(\tau_1), \quad k \leq 4.
\]
For the detailed proof of this estimate, we refer to e.g. Lemma 8 in [33] or Lemma 11 in [32]. Then from estimate (15), we can show that
\[
D^\alpha[Z^k Q]_{r_1}^2 \lesssim R^{-1+\alpha} \int_{\tau_1}^{\tau_2} \int_{S_r} |\partial_\psi_1|^2 r^{-3+\alpha} \, dv \, d\omega \cdot \sup_{k_1 \leq 2} R^{-\beta} \int_{\tau_1}^{\tau_2} \sup_{r \geq 2} r^{-2} \int_\omega |\partial_u \psi_1|^2 \, dv \, d\tau
\]
\[
+ A(\tau_1) \sum_{k_2 \geq 2} \int_{\tau_1}^{\tau_2} \sup_{r \geq 1} r^{\alpha-1} \int_\omega |\partial_u \psi_2|^2 \, dv \, d\omega + \int_{\tau_1}^{\tau_2} \int_{S_r} |\nabla Z^4 \psi|_r^2 r^\alpha E[Z^3 \phi](\tau) \, dv \, d\tau.
\]
Here we still use the notation that $\phi_1 = Z^{k_1}\phi$, $\phi_2 = Z^{k_2}\phi$, $\psi_1 = r\phi_1$. Now on $S_r$, $\tau_1 \leq \tau \leq \tau_2$, we can estimate

$$r^{-\alpha}\int_\omega (\partial_u \psi_1)^2 d\omega \lesssim r^{-\alpha}\int_\omega (\partial_u \psi_1)^2 d\omega \bigg|_{v=v_{\tau_2}} + \int_{S_r} r^{-1-\alpha}|\partial_u \psi_1|^2 d\omega \tau d\omega$$
$$+ \int_{S_r} r^{-1-\alpha}(\partial_u \psi_1)^2 d\omega \tau d\omega + \int_{S_r} r^{-\alpha}(\partial_u \partial_u \psi_1)^2 d\omega \tau d\omega$$
$$\lesssim \int_\omega (\partial_u \psi_1)^2 d\omega \bigg|_{v=v_{\tau_2}} + \int_{S_r} r^{-1-\epsilon}(|\partial u_1|^2 + |\partial \Omega \psi_1|^2) d\omega \tau d\omega \times \int_{S_r} r^{3-\alpha}|Z^{k_1}F|^2 d\omega \tau d\omega.$$

Similarly, on $C_{\tau_1, \tau_2, v}$, we have

$$r^\alpha\int_\omega (\partial_u \psi_2)^2 d\omega \lesssim r^\alpha\int_\omega (\partial_u \psi_2)^2 d\omega \bigg|_{u=u_{\tau_1}} + \int_{C_{\tau_1, \tau_2, v}} r^\alpha(\partial_u \psi_2)^2 d\omega \tau d\omega$$
$$+ \int_{C_{\tau_1, \tau_2, v}} r^\alpha(\partial_u \partial_u \psi_2)^2 d\omega \tau d\omega + \int_{C_{\tau_1, \tau_2, v}} r^{\alpha-1}(\partial_u \psi_2)^2 d\omega \tau d\omega$$
$$\lesssim r^\alpha\int_\omega (\partial_u \psi_2)^2 d\omega \bigg|_{u=u_{\tau_1}} + \int_{C_{\tau_1, \tau_2, v}} r^\alpha(\partial_u \psi_2)^2 + r^\alpha(\Delta \psi_2)^2 + r^{\alpha+2}|Z^{k_2}F|^2 d\omega \tau d\omega.$$

Therefore we can show that

$$D^\alpha\!_+[Z^kQ]^{\tau_2}_{\tau_1} \lesssim R^{-3+2\alpha+\epsilon}A(\tau_1) + R^{-\beta-2+\alpha}A(\tau_1) + A(\tau_1)R^{-1-\beta} + A(\tau_1)R^{-\beta} \lesssim A(\tau_1)R^{-\beta}.$$

The estimate for cubic or higher order nonlinearities is better and we can conclude that

$$D^\alpha\!_+[Z^kF]^{\tau_2}_{\tau_1} \lesssim A(\tau_1)R^{-\beta}, \quad k \leq 4.$$ (23)

Next we estimate the integral inside the cylinder with radius $R$. We have

$$\int_{\tau_1}^{\tau_2} \int_{r \leq R} (1 + r)^{1+\epsilon}|Z^{k}F|^2 dx d\tau \lesssim \int_{\tau_1}^{\tau_2} \int_{r \leq R} (1 + r)^{1+\epsilon}|\partial \phi_1|^2|\partial \phi_2|^2 d\tau d\tau$$
$$\lesssim \int_{\tau_1}^{\tau_2} \int_{r \leq 1} |\partial \phi_1|^2|\partial \phi_2|^2 d\tau d\tau + \int_{\tau_1}^{\tau_2} \int_{r \leq R} r^{1+\epsilon}|\partial \phi_1|^2|\partial \phi_2|^2 d\tau d\tau.$$

Here we omitted the summation sigh for simplicity and the right hand side should be interpreted as the sum for all $k_1 + k_2 \leq k \leq 4$. The integral on the cylinder with radius 1 can be estimated by using elliptic estimates, which relies on the commutator $\partial_t$. To estimate the second part, we claim that

$$\int_\omega r|\partial(Z^{k}\phi)|^2 d\omega \lesssim A(\tau), \quad k \leq 2, \quad 1 \leq r \leq R.$$ (24)

In fact from Lemma 1, we have

$$\int_{1 \leq r \leq R} r^{1-\epsilon}|\partial_u \partial_u Z^{k}\phi|^2 dx \lesssim A(\tau), \quad k \leq 2,$$
$$\int_{1 \leq r \leq R} |\partial_u \partial_t Z^{k}\phi|^2 dx \lesssim E[\partial_t Z^{k}\phi](\tau) \lesssim A(\tau), \quad k \leq 3.$$

This implies that

$$\int_{1 \leq r \leq R} |\partial_u \partial_t Z^{k}\phi|^2 dx \lesssim A(\tau), \quad k \leq 2.$$

In particular, we can show that

$$r \int_\omega |\partial_u Z^{k}\phi|^2 d\omega \lesssim A(\tau), \quad 1 \leq r \leq R, \quad k \leq 2.$$
This leads to the above claim (24). Hence, we can show that
\[
\int_{\tau_1}^{\tau_2} \int_{1 \leq r \leq R} r^{1 + \epsilon} |\partial \phi_1|^2 |\partial \phi_2|^2 \, dx \, d\tau \lesssim R^\epsilon \int_{\tau_1}^{\tau_2} A(\tau)^2 \, d\tau \lesssim A(\tau_1) R^{-\beta}.
\]
Inside the cylinder with radius 1, by using elliptic theory, we can show that
\[
|\partial Z^k \phi|^2 \lesssim A(\tau), \quad k \leq 2.
\]
For the details, we refer to e.g. the end of the second last section of [34]. Therefore, we can estimate
\[
\int_{\tau_1}^{\tau_2} \int_{r \leq 1} |\partial \phi_1|^2 |\partial \phi_2|^2 \, dx \, d\tau \lesssim \int_{\tau_1}^{\tau_2} A(\tau)^2 \, d\tau \lesssim A(\tau_1) R^{-\beta + \epsilon}.
\]
Combined with the estimate (23), we then have shown that
\[
D^k[Z^k F]_{\tau_1}^{\tau_2} \lesssim D^k_+[Z^k F]_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} \int_{r \leq R} (1 + r)^{1 + \epsilon} |Z^k F|^2 \, dx \, d\tau \lesssim A(\tau_1) R^{-\beta}, \quad \forall k \leq 4.
\]
Simply considering the total decay in \( R \) and \( (\tau_1)_+ \), we see from the definition of \( A(\tau) \) in Proposition 2 that
\[
A(\tau) \leq 2 \tau_1^{1-\alpha} R^{+\alpha-\beta}.
\]
Therefore we have the estimate for \( D^k_+[Z^k F]_{\tau_1}^{\tau_2} \)
\[
D^k_+[Z^k F]_{\tau_1}^{\tau_2} \lesssim (\tau_1)_+^{1-\alpha} R^{-\beta} R^{+\alpha-\beta}, \quad \forall k \leq 4.
\]
For \( D^k[Z^k F]_{\tau_1}^{\tau_2} \), when \( (\tau_1)_+ \leq R \), we have
\[
A(\tau_1) \leq R^{-2+\alpha} \leq R^\epsilon \min\{R^{-\beta}(\tau_1)_+^{1-\alpha}, R^{-2+\alpha}, R^{-1-\beta-\epsilon}(\tau_1)_+^{\alpha}\}.
\]
Here recall that \( \beta = 1 - 2\alpha, \epsilon = \frac{\alpha}{20} \). When \( (\tau_1)_+ \geq R \), we can show that
\[
A(\tau_1) \leq R^{-\beta}(\tau_1)_+^{1-\alpha} + R^{1+\epsilon-\beta}(\tau_1)_+^{2} \leq 2R^{1+\epsilon-\beta}(\tau_1)_+^{2} \leq 2R^{2\alpha} \min\{R^{-\beta}(\tau_1)_+^{1-\alpha}, R^{-2+\alpha}, R^{-1-\beta-\epsilon}(\tau_1)_+^{\alpha}\}
\]
In any case, we have
\[
A(\tau_1) \leq 2R^{2\alpha} \min\{R^{-\beta}(\tau_1)_+^{1-\alpha}, R^{-2+\alpha}, R^{-1-\beta-\epsilon}(\tau_1)_+^{\alpha}\}
\]
Therefore we have
\[
D^k[Z^k F]_{\tau_1}^{\tau_2} \lesssim A(\tau_1) R^{-\beta} \lesssim R^{3\alpha-\beta} \min\{R^{-\beta}(\tau_1)_+^{1-\alpha}, R^{-2+\alpha}, R^{-1-\beta-\epsilon}(\tau_1)_+^{\alpha}\}
\]
for all \( k \leq 4 \). Recall that \( \epsilon = \frac{\alpha}{20}, \beta = 1 - 2\alpha \) and \( \alpha \leq \frac{1}{4} \). We conclude that for sufficiently large \( R \), depending only on \( \alpha \), we can improve the bootstrap assumption (22). Then the construction of the solution on the region \( \{r \leq t + R\} \) will be the same as that in e.g. [33] (the last section). Hence we can conclude our main Theorem.

References

[1] D. Christodoulou. Global solutions of nonlinear hyperbolic equations for small initial data. Comm. Pure Appl. Math., 39(2).

[2] D. Christodoulou. The formation of black holes in general relativity. EMS Monographs in Mathematics. European Mathematical Society (EMS), Zürich, 2009.
[3] D. Christodoulou and S. Klainerman. The global nonlinear stability of the Minkowski space, volume 41 of Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 1993.

[4] M. Dafermos and I. Rodnianski. A new physical-space approach to decay for the wave equation with applications to black hole spacetimes. In XVIth International Congress on Mathematical Physics, pages 421–432. World Sci. Publ., Hackensack, NJ, 2010.

[5] Joachim J. Krieger and W. Schlag. Concentration compactness for critical wave maps. EMS Monographs in Mathematics. European Mathematical Society (EMS), Zürich, 2012.

[6] M. Keel, H. Smith, and C. Sogge. Global existence for a quasilinear wave equation outside of star-shaped domains. J. Funct. Anal., 189(1):155–226, 2002.

[7] S. Klainerman. Global existence for nonlinear wave equations. Comm. Pure Appl. Math., 33(1):43–101, 1980.

[8] S. Klainerman. Long time behaviour of solutions to nonlinear wave equations. In Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983), pages 1209–1215, Warsaw, 1984. PWN.

[9] S. Klainerman. The null condition and global existence to nonlinear wave equations. In Nonlinear systems of partial differential equations in applied mathematics, Part 1 (1984), volume 23 of Lectures in Appl. Math., pages 293–326. Amer. Math. Soc., Providence, RI, 1986.

[10] S. Klainerman and I. Rodnianski. On the formation of trapped surfaces. Acta Math., 208(2):211–333, 2012.

[11] S. Klainerman and T. Sideris. On almost global existence for nonrelativistic wave equations in 3d. Comm. Pure Appl. Math., 49(3).

[12] J. Krieger, W. Schlag, and D. Tataru. Renormalization and blow up for charge one equivariant critical wave maps. Invent. Math., 171(3):543–615, 2008.

[13] H. Lindblad and I. Rodnianski. The global stability of Minkowski space-time in harmonic gauge. Ann. of Math. (2), 171(3):1401–1477, 2010.

[14] J. Luk. The Null Condition and Global Existence for Nonlinear Wave Equations on Slowly Rotating Kerr Spacetimes. 2010. arXiv: 1009.4109.

[15] J. Luk and I. Rodnianski. Local Propagation of Impulsive Gravitational Waves. 2012. arXiv:1209.1130.

[16] J. Luk and I. Rodnianski. Nonlinear interaction of impulsive gravitational waves for the vacuum Einstein equations. 2013. arXiv:1301.1072.

[17] J. Metcalfe, M. Nakamura, and C. D. Sogge. Global existence of solutions to multiple speed systems of quasilinear wave equations in exterior domains. Forum Math., 17(1):133–168, 2005.

[18] J. Metcalfe and C. Sogge. Global existence of null-form wave equations in exterior domains. Math. Z., 256(3):521–549, 2007.

[19] P. Raphaël and I. Rodnianski. Stable blow up dynamics for the critical co-rotational wave maps and equivariant Yang-Mills problems. Publ. Math. Inst. Hautes Études Sci., pages 1–122, 2012.

[20] I. Rodnianski and J. Sterbenz. On the formation of singularities in the critical O(3) σ-model. Ann. of Math. (2), 172(1):187–242, 2010.

[21] J. Luk S. Klainerman and I. Rodnianski. A fully anisotropic mechanism for formation of trapped surfaces in vacuum. 2013. arXiv:1302.5951.
[22] J. Shatah. Weak solutions and development of singularities of the SU(2) $\sigma$-model. *Comm. Pure Appl. Math.*, 41(4):459–469, 1988.

[23] T. Sideris and S.-Y. Tu. Global existence for systems of nonlinear wave equations in 3D with multiple speeds. *SIAM J. Math. Anal.*, 33(2):477–488 (electronic), 2001.

[24] C. Sogge. Global existence for nonlinear wave equations with multiple speeds. In *Harmonic analysis at Mount Holyoke (South Hadley, MA, 2001)*, volume 320 of *Contemp. Math.*, pages 353–366. Amer. Math. Soc., Providence, RI, 2003.

[25] J. Sterbenz and D. Tataru. Energy dispersed large data wave maps in 2 + 1 dimensions. *Comm. Math. Phys.*, 298(1):139–230, 2010.

[26] J. Sterbenz and D. Tataru. Regularity of wave-maps in dimension 2 + 1. *Comm. Math. Phys.*, 298(1):231–264, 2010.

[27] M. Struwe. Equivariant wave maps in two space dimensions. *Comm. Pure Appl. Math.*, 56(7):815–823, 2003. Dedicated to the memory of Jürgen K. Moser.

[28] T. Tao. Global regularity of wave maps VII. Control of delocalised or dispersed solutions. 2009. arXiv:0908.0776.

[29] C. Wang and X. Yu. Global existence of null-form wave equations on small asymptotically Euclidean manifolds. 2012. arXiv:1207.5218.

[30] J. Wang and P. Yu. A Large Data Regime for non-linear Wave Equations. 2012. arXiv:1210.2056.

[31] J. Wang and P. Yu. Long time solutions for wave maps with large data. *J. Hyperbolic Differ. Equ.*, 10(2):371–414, 2013.

[32] S. Yang. Global stability of large solutions to nonlinear wave equations. 2012. arXiv: 1205.4216.

[33] S. Yang. Global solutions of nonlinear wave equations in time dependent inhomogeneous media. *Arch. Ration. Mech. Anal.*, 209(2):683–728, 2013.

[34] S. Yang. On the quasilinear wave equations in time dependent inhomogeneous media. preprint.

[35] P. Yu. Dynamical formation of black holes due to the condensation of matter field. 2011. arXiv:1105.5898.

[36] P. Yu. Energy estimates and gravitational collapse. *Comm. Math. Phys.*, 317(2):273–316, 2013.

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