Pure tripartite entanglement types based on spectra of reduced density matrices

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Abstract. Given a specific ordered spectra of (shifted) one-qubit reduced density matrices, we discuss that the pure tripartite entanglement for a system consisting of three qubits is determined by only two local unitary invariant polynomials.

1. Introduction
Generally, for a composite quantum system in its pure state the classification of entanglement classes amounts to partitioning the associated projective space into orbits of the Local Unitary (LU) operations. However, according to a general principle in mathematics in order to simplify the problem one has to exploit the existing symmetries whenever possible. In our particular case of interest, the LU group is a Lie group $K$, which acts on the pure states space of a composite quantum system. This group action is a proper action, since the LU group $K$ is a compact Lie group. In addition, the multi-particle pure states space $P(H)$ is a Kähler manifold, for which the Kähler structure is induced from the Hermitian inner product on the Hilbert space $H$.

The symmetry is then the conservation of the spectra of each qubit’s reduced density matrix under the local unitary operation at each site. The appropriate mathematical machinery to exploit this symmetry is encoded in the Symplectic Geometry, in which there exists a map, so called the momentum map, whose components are preserved under the action of compact Lie group. In [1, 2], it is shown that the action of the LU group $K$ on the Kähler manifold $P(H)$ is symplectic, i.e. the symplectic structure $\omega$ (a closed non-degenerate 2-form) is preserved under the action of $K$. Also, the associated momentum map is given by sending each pure multipartite state to a collection of (shifted) one-body reduced density matrices.

More precisely, if a compact Lie group $G$ acts on a symplectic manifold $(N, \Omega)$ properly by preserving its symplectic structure, then there exists the momentum map $\mu : N \to g^*$, where $g^*$ is the dual space for the Lie algebra $g$ of the Lie group $G$. There exists an isomorphism $g \cong g^*$ by defining a $G$-invariant inner product on the Lie algebra. Then the quadruple $(N, \Omega, G, \mu)$ is called a Hamiltonian $G$-space, which can be reduced to a generally topological quotient space $N_\alpha = \mu^{-1}(\alpha)/K_\alpha$, where $\alpha \in g^*$, by exploiting the existing symmetry encoded in the components of the momentum map $\mu$. Here the subgroup $K_\alpha$ is the stabilizer of $\alpha$ by the coadjoint action of the Lie group $G$ on $g^*$. 

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2. Local Unitary Operations

For a composite quantum system consisting of \( N \) (distinguishable) particles the LU group equals to \( K = SU(n_1) \times SU(n_2) \times \cdots SU(n_N) \), where \( n_j \) denotes the \( \dim(\mathcal{H}_j) \) with \( \mathcal{H}_j = \mathbb{C}^{n_j} \) for the \( j \)th particle. The multipartite pure states space in then the projective Hilbert space \( \mathbb{P}(\mathcal{H}) \), as a Kähler manifold, where \( \mathcal{H} = \bigotimes_{j=1}^{N} \mathcal{H}_j \). The LU group \( K \) acts on the Hilbert space \( \mathcal{H} \) as follows

\[
k \cdot \phi = k_1 \phi_1 \otimes k_2 \phi_2 \otimes \cdots \otimes k_N \phi_N,
\]

where \( k = (k_1, k_2, \ldots, k_n) \in K \) and \( \phi = \phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_N \in \mathcal{H} \), with normalized \( \phi_j \in \mathcal{H}_j \). The action of the LU group \( K \) on the manifold \( \mathbb{P}(\mathcal{H}) \) is induced from its action (1) on \( \mathcal{H} \), which is given by conjugation, i.e.

\[
K \times \mathbb{P}(\mathcal{H}) \to \mathbb{P}(\mathcal{H}), (k, \rho) \mapsto k \cdot \rho = k \rho k^{-1},
\]

for all \( k \in K \) and \( \rho \in \mathbb{P}(\mathcal{H}) \). Recall that the projection \( \pi : \mathcal{H} \to \mathbb{P}(\mathcal{H}) \) is given by sending each state \( \phi \in \mathcal{H} \) to the corresponding trace-class, rank-one Hermitian operator \( \rho = \phi \phi^* \), where \( \phi^* \) is the complex conjugate of the multipartite pure state \( \phi \).

The orbits of the \( K \)-action on the manifold \( \mathbb{P}(\mathcal{H}) \) are given by the sets \( K \cdot \rho = \{ k \cdot \rho : \forall k \in K \} \).

This establishes an equivalence relation in \( \mathbb{P}(\mathcal{H}) \), i.e. two states \( \rho_1 \) and \( \rho_2 \) are equivalent if and only if there exists an element \( k \in K \), such that \( \rho_2 = k \cdot \rho_1 \), i.e. \( \rho_1 \) and \( \rho_2 \) belong to the same orbit. According to the Hilbert’s theorem [7] for each orbit \( \mathcal{O}_{k,p} \), there exists an LU-invariant polynomial which is constant on it and the number of linearly independent LU-invariant polynomials is finite. Since the LU group is a compact Lie group the space of orbits, which is denoted by \( \mathbb{P}(\mathcal{H})/K \), is a Hausdorff space. Then the topological space of orbits \( \mathbb{P}(\mathcal{H})/K \) is mapped into a subset of the Euclidean space \( \mathbb{R}^m \), where \( m \) is the number of basis for the algebra of LU-invariant polynomials.

For the case of pure tripartite states, i.e. \( N = 3 \), the LU-invariant polynomials are related to the generalized Schmidt decomposition [5]. In fact, every pure three-qubit state \( \phi \in \mathcal{H} \) can be transformed to the following normal form

\[
|\phi_0\rangle = \lambda_0|000\rangle + \lambda_1 e^{i\phi}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle,
\]

by local unitary operations. The normal form (3), with \( \lambda_k \geq 0 \), for \( k = 0, \ldots, 4 \), and \( \sum_{k=0}^{4} \lambda_k^2 = 1 \) with \( \phi \in [0, \pi] \) is non-unique up to the permutation of qubits. In [5, 6], the LU-invariant polynomials are also constructed in terms of LU-invariant coefficients in the normal form (3), i.e.

\[
I_1 = |\lambda_1 \lambda_4 e^{i\phi} - \lambda_2 \lambda_3|^2, \quad I_2 = \lambda_0^2 \lambda_2^2, \quad I_3 = \lambda_0^2 \lambda_3^2, \quad I_4 = \lambda_0^2 \lambda_4^2, \\
I_5 = \lambda_0^2 (I_1 + \lambda_3^2 \lambda_4^2 - \lambda_1^2 \lambda_2^2),
\]

which satisfy some equalities and inequalities. These constraints determine image of the orbit space \( \mathbb{P}(\mathcal{H})/K \) as a semi-algebraic subset of the Euclidean space \( \mathbb{R}^m \), where \( m = 5 \), in terms of LU-invariant polynomials [8].
2.1. Momentum Map
Recall that the pure multipartite state space $P(H)$ is a Kähler manifold, for which the symplectic structure is given by the imaginary part of the Hermitian inner product on $H$.

\[ \omega_\rho(A, B) := \frac{i}{2} \text{Tr}(\rho [A, B]), \]  

(5)

for two Hermitian operators $A, B \in \mathfrak{su}^*(H)$. In Eq. (5) $\hat{A}, \hat{B} \in T_\rho(P(H))$ are the fundamental vector fields of the action of the Lie algebra $\mathfrak{su}(H)$ on the manifold given and $(\ldots)$ denotes the commutator between the operators. As mentioned above, the LU-group action on the complex projective manifold $P(H)$ is symplectic, and since the LU group $K$ is compact, there exists a momentum map $\mu : P(H) \rightarrow \mathfrak{k}^*$, which is defined as follows [1, 2]

\[ \mu(\rho) = \bigoplus_{l=1}^{N} \left( \rho(l) - \frac{1}{n_l} \mathbb{1}_{n_l} \right) \in \mathfrak{k}^*, \]  

(6)

where $\rho(l)$, for $l = 1, \ldots, N$, is the $l$th-qubit reduced density matrix.

3. Symmetry reduction of a Hamiltonian system
Let us first briefly review the symmetry reduction of a Hamiltonian $G$-manifold $(N, \Omega, G, \mu)$, when a compact Lie group $G$ acts symplectically on a symplectic manifold $(N, \Omega)$. Recall that the image of the manifold $N$ under the invariant momentum map $\psi : N \rightarrow \mathfrak{g}^*/G, \rho \mapsto \psi(\rho) \equiv \alpha_0$ is a convex polytope $\Delta$ which is a subset of the positive Weyl chamber $\mathfrak{g}^+$ [9]. The relation between $\mu(\rho) \equiv \alpha \in \mathfrak{g}^*$ and $\psi(\rho) \equiv \alpha_0 \in \mathfrak{g}^*/G$ is that $\alpha_0$ is the intersection of the coadjoint orbit $O_\alpha = \{ \text{Ad}_g \alpha : \forall g \in G \}$, where $\text{Ad}_g \alpha = g^{-1} \alpha g$, with $\mathfrak{g}^*/G$.

In general, the resulting reduced space, which is denoted by $N_{O_{\alpha_0}} = \psi^{-1}(\alpha_0)/K$, is topologically a stratified space such that each piece (stratum) is a symplectic manifold and it contains all the points in the fiber $\psi^{-1}(\alpha_0)$, for $\alpha_0 \in \Delta$, whose stabilizers are conjugate with a subgroup of the Lie group $G$. For more details the interested readers can refer to [10, 11]. The crucial in our analysis is the properness of the Lie group and equivariance property of the momentum map. The following theorem will be used in continue to obtain the image of the reduced space of Hamiltonian $G$-manifold in terms of the Hilbert map.

**Theorem 3.1** [10] Let $(N, \Omega, G, \mu)$ be a Hamiltonian $G$-manifold. The space of orbits $M/G$ is embedded into a Euclidean space $\mathbb{R}^m$ by the Hilbert map $\sigma : N/G \rightarrow \mathbb{R}^m$, where $m$ is the number of bases of the $G$-invariant polynomials. The embedding of reduced space $N_{O_{\alpha_0}} = \psi^{-1}(O_{\alpha_0})/G$, for $\alpha_0 \in \mathfrak{g}^*/G$, is obtained by restriction of the map $\sigma$ to $N_{O_{\alpha_0}}$ as a subspace of the Euclidean space $\mathbb{R}^m$.

4. Three-qubit case
Recall that the three-qubit polytope $\Delta$ is determined by the following polygonal inequalities [12, 13]

\[ \gamma_k \leq \sum_{j \neq k} \gamma_j + \frac{1}{2}, \]  

(7)

where $-\frac{1}{2} \leq \gamma_l \leq 0$ for $l = 1, 2, 3$ are the minimal eigenvalues of matrices $\rho^{(k)} - \frac{1}{2} \mathbb{1}_2$. The momentum value of the normal form $\mu(\rho_0)$, where with $\rho_0 = \phi_0 \phi_0^*$, amounts to finding these matrices by considering the Eq. (6). Furthermore, the value of invariant momentum map for the normal form (3), i.e. $\psi(\rho_0)$, can be obtained by diagonalization and re-ordering of the
eigenvalues of matrices $\rho^{(k)} - \frac{1}{2} I_2$. The resulting minimal eigenvalues are obtained in terms of the LU-invariant polynomials (4) as follows

$$2\gamma_k = -\sqrt{1 - 4c_k}, \quad c_k = \sum_{j=1, j\neq k}^4 I_j.$$  \hspace{1cm} (8)

By restriction of the Hilbert map image of the space of orbits $\mathbb{P}(\mathcal{H})/K$ to the reduced space $\psi^{-1}(\alpha_0)/K$, for $\alpha_0 \in \Delta$, its Hilbert map image under the LU-invariant polynomials is determined. Let us choose $\alpha_0$ in the relative interior of the polytope $\Delta$ (7), with equal $\gamma_k$s (for simplicity) such that $-\frac{1}{6} < \gamma < 0$ (see [3] for further details). The restriction of LU-invariant polynomials (4) to the constraints obtained from Eqs. (8) lead to the following inequality

$$I_5^2 \leq \frac{1}{2} \left(1 - 4 \left(I_4 + \gamma^2\right)^2\right),$$  \hspace{1cm} (9)

for a given $\alpha_0 = \psi(\rho_0) \in \text{rel. int.}(\Delta) \cap (-\frac{1}{6} < \gamma < 0)$. In other words, the LU-invariant polynomials $(I_4, I_5)$ such that they satisfy the inequality (9) are considered as the Hilbert map image of the reduced space of the Hamiltonian $K$-manifold $(\mathbb{P}(\mathcal{H}), \omega, K, \psi)$ for pure tripartite states.

5. Summary
In this paper we reviewed that by fixing the spectra of one-qubit (shifted) reduced density matrices, the entanglement types of pure tripartite states are determined in terms of two LU-invariant polynomials.

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