CHARACTERIZING THE STRONG MAXIMUM PRINCIPLE

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ABSTRACT

In this paper we characterize the degenerate elliptic equations $F(D^2u) = 0$ whose subsolutions ($F(D^2u) \geq 0$) satisfy the strong maximum principle. We introduce an easily computed function $f$ on $(0, \infty)$ which is determined by $F$, and we show that the strong maximum principle holds depending on whether $\int_{0+} \frac{du}{f(y)}$ is infinite or finite. This complements our previous work characterizing when the (ordinary) maximum principle holds. Along the way we characterize radial subsolutions.

TABLE OF CONTENTS

1. Introduction.
2. Characterizing the Maximum Principle.
3. Characterizing the Strong Maximum Principle – Three Cases.
4. Characterizing the (SMP) for Cone Subequations.
5. The Radial Subequation Associated to $F$.
6. Increasing Radial Subharmonics for Borderline Subequations.
7. Proof of the (SMP) in the Borderline Case.
8. Radial (Harmonic) Counterexamples to the (SMP).
9. Subequations with the Same Increasing Radial Subharmonics.
10. Strong Comparison and Monotonicity.
11. Examples of Monotonicity Subequations which are not Cones.

Appendix A. A Theorem on Radial Subharmonics.

Appendix B. Uniform Ellipticity and the Borderline Condition.

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1. Introduction

We consider differential equations of the form $F(D^2u) = 0$ where $F$ is degenerate elliptic, and we focus attention on the set $F(X)$ of subsolutions ($F(D^2u) \geq 0$) on an open set $X \subset \mathbb{R}^n$. The main point of this paper is to answer the following.

**Question:** When do the subsolutions satisfy the strong maximum principle?

By the maximum principle and the strong maximum principle for $F$ we mean the following. Given a compact set $K \subset \mathbb{R}^n$ with connected interior, let $F(K)$ denote the space of upper semi-continuous functions on $K$ which are subsolutions on $\text{Int}K$. Consider the implications:

$$ u \in F(K) \quad \Rightarrow \quad \sup_K u \leq \sup_{\partial K} u \quad (MP) $$

$$ u \in F(K) \text{ has an interior maximum point} \quad \Rightarrow \quad u \text{ is constant} \quad (SMP) $$

We say that the (MP)/(SMP) holds for $F$ if it is true for all such $K$ and $u$. Of course, (SMP) $\Rightarrow$ (MP).

A characterization of when the (MP) holds for $F$ was given in [6, Remark 4.7]. This theorem is amplified in Section 2.

Given a non-zero vector $e \in \mathbb{R}^n$, let $P_e$ and $P_e^\perp$ denote orthogonal projection onto the line spanned by $e$ and the hyperplane perpendicular to $e$ respectively, so that $P_e + P_e^\perp = I$.

In two cases the answer to our (SMP) question is relatively simple and classical:

$$ F(0) < 0 \quad \Rightarrow \quad (SMP) \text{ holds,} $$

$$ F(-\mu P_e) \geq 0 \text{ for some } \mu > 0, e \neq 0 \quad \Rightarrow \quad (SMP) \text{ fails,} \quad (1.1) $$

so we concentrate on the remaining borderline cases.

In this paper a key role is played by the increasing radial subsolutions. They are determined by a “characteristic function” $f$, which is defined as follows. For simplicity we assume here the following form of invariance: For all $\lambda, \mu \in \mathbb{R},$

$$ F(\lambda P_e^\perp - \mu P_e) \geq 0 \text{ for one } e \neq 0 \quad \Rightarrow \quad F(\lambda P_e^\perp - \mu P_e) \geq 0 \text{ for all } e \neq 0. $$

This is true if $F$ is invariant under a group such as $O_n$ or $SU_{n/2}$ acting transitively on the $n-1$ sphere in $\mathbb{R}^n$. In this case the characteristic function $f$ associated to $F$ for $0 \leq \lambda < \infty$ is defined by

$$ f(\lambda) \equiv \sup \{ \mu : F(\lambda P_e^\perp - \mu P_e) \geq 0 \} \quad (1.2) $$

The borderline cases are exactly the cases where $f(0) = 0$ (see Lemma 3.4).

Now we can state our main result.

**THEOREM A.** Suppose $F$ is invariant and borderline. Then

$$ \text{The (SMP) holds for } F \quad \iff \quad \int_{0^+} \frac{dy}{f(y)} = \infty. $$
A version of this result for general (non-invariant) \( F \)'s is given below.

The characteristic function \( f \) determines the following one-dimensional variable coefficient operator

\[
(R^t_f \psi)(t) \equiv \min \left\{ \psi'(t), \psi''(t) + f \left( \frac{\psi'(t)}{t} \right) \right\}.
\] (1.3)

The next result is of general interest, and probably classical in the \( C^2 \)-case.

**Proposition B.** A radial function \( u(x) = \psi(|x|) \) with \( \psi \) increasing, is an \( F \)-subsolution if and only if \( \psi \) is an \( R^t_f \)-subsolution.

The “only if” part of this result requires a technical lemma for general upper semi-continuous functions, which is given in Appendix A.

These two results lead to the following.

**Problem:** Given an upper semi-continuous, increasing function \( f : [0, \infty) \to [0, \infty] \) with \( f(0) = 0 \), describe all the equations \( F \) which have \( f \) as their characteristic function, or equivalently (by Proposition B) have the same set of increasing radial subsolutions.

Such equations \( F \) always exist. Here are two examples. Let \( \lambda_1(A) \leq \cdots \leq \lambda_n(A) \) denote the ordered eigenvalues of a symmetric matrix \( A \) so that \( \lambda_{\min} = \lambda_1 \) and \( \lambda_{\max} = \lambda_n \). Define

\[
F_{f}^{\min/\max}(D^2u) \equiv \min \left\{ \lambda_{\max}(D^2u), \lambda_{\min}(D^2u) + f \left( \lambda_{\max}(D^2u) \right) \right\}, \quad \text{and}
\]
\[
F_{f}^{\min/2}(D^2u) \equiv \min \left\{ \lambda_{2}(D^2u), \lambda_{\min}(D^2u) + f \left( \lambda_{2}(D^2u) \right) \right\}.
\]

Both have associated characteristic function \( f \) (see Lemma 9.3). In fact, they are the largest and the smallest such examples.

**Proposition C.** If \( F \) is invariant and borderline with characteristic function \( f \), then its subsolutions satisfy

\[
F_{f}^{\min/2}(X) \subset F(X) \subset F_{f}^{\min/\max}(X).
\]

In Section 10 we consider strong comparison for \( F \) and establish a sufficient condition utilizing the “monotonicity subequation” \( M_F \) associated to \( F \).

**THEOREM D.** If the dual \( \widetilde{M}_F \) satisfies the strong maximum principle, then the strong comparison principle holds for \( F \).

We leave as an open question: When does the strong comparison principle for \( F \) imply the (SMP) for \( \widetilde{M}_F \)?

In Section 11 we construct many new examples of borderline equations for which strong comparison holds. Specifically, for each decreasing continuous function \( g : [0, \infty) \to \mathbb{R} \) with \( g(0) = 0 \) and \( g(x) < 0 \) for \( x > 0 \), we construct two equations \( M_g \) and \( \widetilde{M}_{g} \), with \( \widetilde{M}_{g} \) borderline, and compute the characteristic function \( f \) of \( \widetilde{M}_{g} \) in terms of \( g \).

**THEOREM E.** If \( g \) is subadditive and \( \int_{0^+} \frac{dy}{f(y)} = \infty \), where \( f \) is the characteristic function associated with \( \widetilde{M}_{g} \), then the strong comparison principle holds for \( M_g \) and \( \widetilde{M}_{g} \).
Many such functions exist. For further examples see (11.4) and [1], and see Example 11.7 for a specific example related to the Hopf function (11.10).

Our first main result, Theorem A above, extends to $F$’s which are not necessarily invariant as follows. We define the upper and lower characteristic functions $\bar{f}$ and $\underline{f}$ for $F$ by:

$$\bar{f}(\lambda) \equiv \sup \{\mu : F(\lambda P_{e} - \mu P_{e}) \geq 0 \text{ for some } e \neq 0\}$$

$$\underline{f}(\lambda) \equiv \sup \{\mu : F(\lambda P_{e} - \mu P_{e}) \geq 0 \text{ for all } e \neq 0\}$$

When $F(0) = 0$, we have $\underline{f}(0) = \bar{f}(0) = 0$.

**THEOREM A’.** Suppose that $F$ is borderline and has upper and lower characteristic functions $\bar{f}$ and $\underline{f}$.

(a) If $\int_{0^+} \frac{dy}{f(y)} = \infty$, then the (SMP) holds for $F$.

(b) If $\int_{0^+} \frac{dy}{f(y)} < \infty$, then the (SMP) fails for $F$.

Now that all of our results have been stated in the traditional manner using nonlinear operators $F$, we switch to our geometric point of view by replacing $F$ with the subset $F = \{F \geq 0\}$ in $\text{Sym}^2(\mathbb{R}^n)$, the space of $n \times n$ symmetric matrices. Let

$$\mathcal{P} \equiv \{A \in \text{Sym}^2(\mathbb{R}^n) : A \geq 0\}.$$

Instead of “operators” we consider subequations which by definition are closed subsets $F \subset \text{Sym}^2(\mathbb{R}^n)$ satisfying the following condition

$$F + \mathcal{P} \subset F,$$  \hspace{1cm} (P)

called positivity. Subsolutions are defined in the usual manner, except that one requires $D^2_x \varphi \in F$, rather than $F(D^2_x \varphi) \geq 0$, for test functions $\varphi$ at $x$. To emphasize the parallels with potential theory in several complex variables, we will use the terminology $F$-subharmonic rather than $F$-subsolution. The key topological property of $F$ is that:

$$F = \text{Int}F.$$  \hspace{1cm} (T)

This follows easily from (P) and the assumption that $F$ is closed.

Our notion of a supersolution $v$ is sometimes more restrictive than the classical notion $F(D^2v) \leq 0$. We require $-v$ to be subharmonic for the dual subequation $\tilde{F} = -(\sim \text{Int}F)$. This has an advantage over the standard notion of supersolution. For example, we were able to prove that comparison always holds for any subequation $F \subset \text{Sym}^2(\mathbb{R}^n)$ (Theorem 6.4 in [6]). The reader is referred to the “Pocket Dictionary” in Appendix A of [8] for a more complete translation of concepts.

We use “increasing” to mean non-decreasing throughout the paper.
2. Characterizing the Maximum Principle

In this section we review and amplify the (MP) results in [6].

Let \( \tilde{\mathcal{P}} \) denote the subset of \( A \in \text{Sym}^2(\mathbb{R}^n) \) with at least one non-negative eigenvalue, i.e., with \( \lambda_{\text{max}}(A) \geq 0 \). For the maximum principle we only need to consider subequations \( F \subset \tilde{\mathcal{P}} \), since if \( A \notin \tilde{\mathcal{P}} \), then \( A \) is negative definite and \( \langle Ax, x \rangle \) violates (MP). Note that \( \tilde{\mathcal{P}} \) is a subequation, that is, it is a closed set which satisfies (P). In fact, \( \tilde{\mathcal{P}} \) is universal for (MP) in the following sense.

**Theorem 2.1.** (MP) holds for a subequation \( F \iff F \subset \tilde{\mathcal{P}} \).

**Proof.** It remains to show that (MP) holds for \( \tilde{\mathcal{P}} \), which follows from Proposition 2.3.

**Definition 2.2.** A function \( u \) is subaffine on \( X \) if it is upper semi-continuous on \( X \) and for all compact sets \( K \subset X \) and affine functions \( a(x) \equiv \langle p, x \rangle + c \),

\[
  u \leq a \text{ on } \partial K \implies u \leq a \text{ on } K.
\]

Subaffine functions clearly satisfy (MP) (take \( a(x) = c = \text{constant} \) in Definition 2.2).

Furthermore, for any pure second-order subequation \( F \), the functions \( u \in F(X) \) satisfy (MP) if and only if they are subaffine, since the sum \( u + a \) of a function \( u \in F(X) \) and an affine function \( a \) is again in \( F(X) \).

**Proposition 2.3.**

\[
u \in \tilde{\mathcal{P}}(X) \iff u \text{ is subaffine on } X.
\]

**Proof.** Suppose \( u \) is not subaffine. Then there exists a compact set \( K \subset X \) and an affine function \( a \) so that (MP) fails for \( w \equiv u - a \) on \( K \), i.e., \( w \) has a strict interior maximum point on \( K \). This also holds for \( w + \epsilon \frac{|x|^2}{2} \) with \( \epsilon > 0 \) sufficiently small. Then \( \varphi = -\epsilon \frac{|x|^2}{2} \) is a test function for \( u \) at the maximum point \( \bar{x} \in \text{Int}K \). Since \( D_x^2 \varphi = -\epsilon I < 0 \), we conclude that \( w \notin \tilde{\mathcal{P}}(X) \) and so \( u \notin \tilde{\mathcal{P}}(X) \).

If \( u \notin \tilde{\mathcal{P}}(X) \), then there exists a test function \( \varphi \) for \( u \) at a point \( \bar{x} \in X \) with \( D_x^2 \varphi \notin \tilde{\mathcal{P}} \), i.e., \( A \equiv D_x^2 \varphi < 0 \). Set \( a(x) \equiv \langle D_x \varphi, x - \bar{x} \rangle + \varphi(\bar{x}) \). Then \( u(x) \leq a(x) + \frac{1}{4} \langle A(x - \bar{x}), x - \bar{x} \rangle \) near \( \bar{x} \), showing that \( u \) is not subaffine on a small ball \( K \) about \( \bar{x} \).

A consequence of this result is that any \( u \in \text{USC}(X) \) which is locally subaffine is subaffine. We will refer to \( \tilde{\mathcal{P}} \) as the subaffine subequation.

Note that in addition to Theorem 2.1 we have established the following two characterizations of the maximum principle. (The condition in (2.2) implies \( 0 \in \text{Int}F \) by positivity.)

**Corollary 2.4.**

\[
  (\text{MP}) \text{ holds for } F \iff 0 \notin \text{Int}F. \quad (2.1)
\]

\[
  (\text{MP}) \text{ fails for } F \iff -\epsilon \frac{|x|^2}{2} \text{ is } F \text{ subharmonic for some } \epsilon > 0. \quad (2.2)
\]
Remark 2.5. The function $-\epsilon \frac{|x|^2}{2}$ is a “universal” counterexample to the maximum principle in the sense that (2.2) is true. This complements Theorem 2.1 which says that $\tilde{P}$ is the “universal” subequation for the maximum principle.

Another obvious corollary is the following.

If two subequations $F$ and $G$ agree
in a neighborhood of the origin in $\text{Sym}^2(\mathbb{R}^n)$, then (MP) holds for $F$ $\iff$ (MP) holds for $G$.

A discussion of the subaffine subequation is not complete without mentioning duality.

Duality

For any subset $F$ of $\text{Sym}^2(\mathbb{R}^n)$, the Dirichlet dual $\tilde{F}$ is defined to be:

$$\tilde{F} = - (\sim \text{Int} F) = \sim (-\text{Int} F).$$

One can calculate that

$$F + A = \tilde{F} - A \quad \text{for each} \quad A \in \text{Sym}^2(\mathbb{R}^n).$$

This can be used to show that

$$F \text{ satisfies (P)} \Rightarrow \tilde{F} \text{ satisfies (P)}.$$

Other properties of the subequations and their dual subequations include:

$$F_1 \subset F_2 \Rightarrow \tilde{F}_2 \subset \tilde{F}_1, \quad F_1 \cap \tilde{F}_2 = \tilde{F}_1 \cup \tilde{F}_2, \quad \text{and} \quad \tilde{\tilde{F}} = F.$$  \hspace{1cm} (2.7)

$$\tilde{\tilde{F}} = F \quad \text{Int} \tilde{F} = -(\sim F) \quad \partial \tilde{F} = - \partial F.$$  \hspace{1cm} (2.8)

The first assertion in (2.8) follows from $\text{Int} F \subset \tilde{F} \subset F$ combined with condition (T) for $F$. The second assertion in (2.8) is a restatement of the first.

The Dirichlet dual of $\mathcal{P}$ can be computed as follows. Let $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the smallest and the largest eigenvalues of $A \in \text{Sym}^2(\mathbb{R}^n)$. By definition

$$\mathcal{P} = \{ A : \lambda_{\min}(A) \geq 0 \}. $$

Since $\lambda_{\min}(-A) = -\lambda_{\max}(A)$ it is easy to see that the dual of $\mathcal{P}$ is

$$\tilde{\mathcal{P}} = \{ A : \lambda_{\max}(A) \geq 0 \},$$

justifying the notation $\tilde{\mathcal{P}}$ for the subaffine subequation.
3. Characterizing the Strong Maximum Principle – Three Cases

Given a subequation \( F \), we consider the following three mutually exclusive cases.

**The Generic Case.** \( F \cap (-\mathcal{P}) = \emptyset \).

**The Borderline Case.** \( F \cap (-\mathcal{P}) = \{0\} \).

**The Counterexample Case.** \( (F - \{0\}) \cap (-\mathcal{P}) \neq \emptyset \).

The third case is the simplest to analyze.

If \( A \in F - \{0\} \) and \( A \leq 0 \),
then the function \( \langle Ax, x \rangle \) is a counterexample to the (SMP).

(3.1)

The first case is the generic case, for which the (SMP) always holds, i.e.

If \( F \cap (-\mathcal{P}) = \emptyset \), then the (SMP) for \( F \) holds.

(3.2)

Note that because of condition (P),

\[ F \cap (-\mathcal{P}) = \emptyset \iff 0 \notin F. \]

(3.3)

By Theorem 4.5 below, if \( 0 \notin F \), then the (SMP) holds for \( F \). This proves (3.2).

In the remaining borderline case the (SMP) may or may not hold, but the (MP) always does by Corollary 2.4. The rest of this section is devoted to discussing this case.

There are several equivalent ways of describing borderline subequations.

**Lemma 3.1.** A subequation \( F \) is **borderline** if and only if any of the following equivalent conditions holds for \( F \).

\[
\begin{align*}
(1) & \quad 0 \in \partial F \text{ and } F - \{0\} \subset \text{Int} \tilde{\mathcal{P}}. \\
(2) & \quad 0 \in \partial F \text{ and } -\mu P_e \notin F \quad \forall \mu > 0, e \neq 0.
\end{align*}
\]

\[
\begin{align*}
(1)' & \quad 0 \in \partial \tilde{F} \text{ and } \mathcal{P} - \{0\} \subset \text{Int} \tilde{F}. \\
(2)' & \quad 0 \in \partial \tilde{F} \text{ and } \mu P_e \in \text{Int} \tilde{F} \quad \forall \mu > 0, e \neq 0.
\end{align*}
\]

**Proof.** The equivalences (1) \( \iff \) (1)’ and (2) \( \iff \) (2)’ follow from (2.7) and (2.8).

Condition (1) implies Condition (2) because \( -\mu P_e \notin \text{Int} \tilde{\mathcal{P}} \) for \( \mu > 0 \). Condition (2)’ implies Condition (1)’ since, by (P), \( \text{Int} \tilde{F} + \mathcal{P} \subset \text{Int} \tilde{F} \), and \( \mathcal{P} - \{0\} \) is the convex hull the elements \( \mu P_e \) for \( \mu > 0 \) and \( e \in \mathbb{R}^n \).

The Characteristic Function of a Subequation

In order to analyze borderline subequations we associate two functions \( \underline{f} \leq \overline{f} \) with \( F \). We begin by considering a general subequation \( F \). The motivation and more details will be provided later in Section 5. First we associate the following two closed sets in \( \mathbb{R}^2 \) with \( F \), called the **upper and lower radial profiles of \( F \)**:

\[
\begin{align*}
\overline{\Lambda} & \equiv \{ (\lambda, \mu) : \lambda P_e \perp + \mu P_e \in F \text{ for some } e \neq 0 \}, \\
\underline{\Lambda} & \equiv \{ (\lambda, \mu) : \lambda P_e \perp + \mu P_e \in F \text{ for all } e \neq 0 \}. \tag{3.4}
\end{align*}
\]
Since $F$ is $\mathcal{P}$-monotone,

$$\overline{\Lambda} \text{ and } \underline{\Lambda} \text{ are } \mathbb{R}_+ \times \mathbb{R}_+ \text{ monotone.} \quad (3.5)$$

Closed subsets $\Lambda \subset \mathbb{R}^2$ which are $\mathbb{R}_+ \times \mathbb{R}_+$-monotone can be classified in several ways. The classification we need is in the following lemma.

**Lemma 3.2.** A set $\Lambda \subset \mathbb{R}^2$ is closed and $\mathbb{R}_+^2$-monotone $\iff$ there exists a lower semi-continuous, decreasing function $h : \mathbb{R} \to \mathbb{R} \cup \{\pm \infty\}$ such that $\Lambda = \{(\lambda, \mu) : \mu \geq h(\lambda)\}$.

**Proof.** Given $\Lambda$, for each $\lambda \in \mathbb{R}$, define $h(\lambda) = \inf\{\mu : (\lambda, \mu) \in \Lambda\}$, with $h(\lambda) = -\infty$ if this set is all of $\mathbb{R}$ and $h(\lambda) = \infty$ if this set is empty. The $\mathbb{R}_+^2$-monotonicity implies that $h$ is decreasing. Now $\Lambda$ is closed if and only if $h$ is lower semi-continuous. The remainder of the proof is left to the reader.

It is more convenient to replace $h$ by the function $f \equiv -h$ so that $f : \mathbb{R} \to \mathbb{R} \cup \{\pm \infty\}$ is upper semi-continuous, increasing and

$$\Lambda \equiv \{(\lambda, \mu) : \mu + f(\lambda) \geq 0\}. \quad (3.6)$$

Thus the radial profiles $\overline{\Lambda}$ and $\underline{\Lambda}$ of $F$ can be used interchangeably with the following associated functions $\overline{f}$ and $\underline{f}$ describing them.

**Definition 3.3.** Suppose that $F$ is a subequation. The upper (larger) and lower (smaller) characteristic functions $\overline{f}$ and $\underline{f}$ associated with $F$ are defined by:

$$\overline{f}(\lambda) \equiv \sup \{\mu : \lambda P_e^\perp - \mu P_e \in F \text{ for some } e \neq 0\}$$

$$\underline{f}(\lambda) \equiv \sup \{\mu : \lambda P_e^\perp - \mu P_e \in F \text{ for all } e \neq 0\}$$

Summarizing, we have

$$\lambda P_e^\perp + \mu P_e \in F \quad \text{for some } e \neq 0 \iff \mu + \overline{f}(\lambda) \geq 0. \quad (3.7)$$

$$\lambda P_e^\perp + \mu P_e \in F \quad \text{for all } e \neq 0 \iff \mu + \underline{f}(\lambda) \geq 0. \quad (3.8)$$

We will use the following fact to further analyze the borderline case.

**Lemma 3.4.**

$$F \text{ is borderline } \iff \underline{f}(0) = \overline{f}(0) = 0. \quad (3.9)$$

**Proof.** Use Definition 3.3 and condition (2) in Lemma 3.1.

The asymptotic structure of $F$ near 0 is reflected in the asymptotic behavior of $\underline{f}$ and $\overline{f}$ near 0. Now we can state the main result of this paper. Note that only the behavior of $\underline{f}(\lambda)$ and $\overline{f}(\lambda)$ for $\lambda$ positive (and small) affects the outcomes.
THEOREM 3.5. Suppose that $F$ is a borderline subequation with upper and lower characteristic functions $\overline{f}$ and $\underline{f}$.

(a) If $\int_{0^+} \frac{dy}{\overline{f}(y)} = \infty$, then the (SMP) holds for $F$.

(b) If $\int_{0^+} \frac{dy}{\underline{f}(y)} < \infty$, then the (SMP) fails for $F$.

The only case not covered is when $\int_{0^+} \frac{dy}{\underline{f}(y)} < \infty$ and $\int_{0^+} \frac{dy}{\overline{f}(y)} = \infty$.

Remark 3.6. For most equations of interest, $\overline{f} = \underline{f}$, and in this case Theorem 3.5 gives a necessary and sufficient condition for $F$ to satisfy the (SMP). First note that $\overline{f} = \underline{f}$ if and only if $\Lambda = \overline{\Lambda}$, or equivalently, for all $\lambda, \mu$:

$$\text{If } \lambda P_{e^\perp} + \mu P_e \in F \text{ for some } e \neq 0, \text{ then } \lambda P_{e^\perp} + \mu P_e \in F \text{ for all } e \neq 0 \quad (3.10)$$

Note also that $P_{e^\perp}, P_e$ have the same span as $I, P_e$, and therefore, for any subequation $F$ which is invariant under the action of a group $G$ acting transitively on the unit sphere $S^{n-1} \subset \mathbb{R}^n$, the characteristic functions $\overline{f}$ and $\underline{f}$ are equal. Among possibilities for $G$ are $\text{SO}_n$ acting on $\mathbb{R}^n$, $\text{SU}_n$ acting on $\mathbb{R}^{2n} = \mathbb{C}^n$, $\text{Sp}_n$ acting on $\mathbb{R}^{4n} = \mathbb{H}^n$, $G_2$ acting on $\mathbb{R}^7$ and $\text{Spin}_7$ acting on $\mathbb{R}^8$.

Illustrative Examples 3.7. Suppose $\psi : \mathbb{R} \to \mathbb{R}$ is odd ($\psi(-t) = -\psi(t)$) and strictly increasing. Fix $1 \leq k \leq n$ and define $F = F_{\psi,k}$ to be the set of $A \in \text{Sym}^2(\mathbb{R}^n)$ such that

$$\sigma_\ell(\psi(A)) \geq 0 \quad \text{for } \ell = 1, \ldots, k,$$

where $\sigma_\ell$ denotes the $\ell$th elementary symmetric function. That is, $A \in F$ if and only if

$$\sigma_\ell(\psi(\lambda_1(A)), \ldots, \psi(\lambda_n(A))) \geq 0 \quad \text{for } \ell = 1, \ldots, k,$$

where $\lambda_1(A), \ldots, \lambda_n(A)$ are the eigenvalues of $A$. One checks that $F$ satisfies Condition (P) and is therefore a subequation. Direct calculation shows that the characteristic function $f = \overline{f} = \underline{f}$ is

$$f(\lambda) = \psi^{-1}\left\{\left(\frac{n}{k} - 1\right) \psi(\lambda)\right\}. \quad (3.11)$$

One concludes that if $\psi$ is smooth in a neighborhood of 0, then the (SMP) holds.

More interestingly, suppose $\psi(t) = \text{sign}(t)|t|^\alpha$ for $\alpha > 0$. Then $f(\lambda) = (\frac{n}{k} - 1)^\frac{1}{\alpha} \lambda$, and so the (SMP) holds.

A simple case is given by $k = 1$ and $\alpha = \frac{1}{3}$, where $F = \{A : \text{tr}(A^2) \geq 0\}$. 

9
4. Characterizing the (SMP) for Cone Subequations

A subset \( F \subset \text{Sym}^2(\mathbb{R}^n) \) is a cone if \( tF \subset F \) for all \( t > 0 \). For subequations which are cones, the (SMP) can be treated in a manner which parallels our treatment of the (MP) given in Section 2. The main theorem in this case is the following.

**THEOREM 4.1.** Suppose that \( F \) is a cone subequation. If \( F \) is borderline, then the (SMP) holds for \( F \). Otherwise, the (SMP) fails for \( F \).

A cone subequation \( F \) can never be generic for the (SMP) because \( 0 \in F \) (see (3.3)). Thus there are only two cases, the counterexample case and the borderline case. It remains to show that the (SMP) holds in the borderline case.

**Remark 4.2.** The discussion of the (MP) in Remark 2.5 has a parallel for the (SMP). Using condition (1) in Lemma 3.1 for \( F \) to be borderline, one sees that the set \( \text{Int} \tilde{P} \) is universal for the (SMP) for cones, in the sense that Theorem 4.1 can be restated as:

\[
\text{The (SMP) holds for } F \iff F - \{0\} \subset \text{Int} \tilde{P}. \tag{4.1}
\]

For cones the condition (2) in Lemma 3.1 for \( F \) to be borderline simplifies to:

\[
F \text{ is borderline} \iff -P_e \notin F \text{ for all } e \neq 0. \tag{4.2}
\]

Therefore, Theorem 4.1 can be restated as:

\[
\text{The (SMP) fails for } F \iff -\langle e, x \rangle^2 \text{ is a counterexample for some } e \neq 0, \tag{4.3}
\]

Thus the function \(-x_1^2\) is a universal counterexample to the (SMP) up to a linear coordinates change on \( \mathbb{R}^n \).

We find it convenient to use the following family of subequations to prove Theorem 4.1. Many other choices are available and some are discussed in Appendix B.

**Definition 4.3. (The \( \alpha \)-Max/Min Subequation).** For \( \alpha > 0 \), define \( \mathcal{P}^{\text{min}/\text{max}}_\alpha \) by

\[
\lambda_{\text{min}}(A) + \alpha \lambda_{\text{max}}(A) \geq 0. \tag{4.4}
\]

This defines a subequation because the ordered eigenvalues of \( A \in \text{Sym}^2(\mathbb{R}^n) \) are \( \mathcal{P} \)-monotone, and obviously each \( \mathcal{P}^{\text{min}/\text{max}}_\alpha \) is a cone. However, it is very important to note that \( \mathcal{P}^{\text{min}/\text{max}}_\alpha \) is not convex (unless \( n = 2 \)), and not uniformly elliptic (proof omitted). For our purposes the key fact is that:

The family \( \{ \mathcal{P}^{\text{min}/\text{max}}_\alpha : \alpha > 0 \text{ small} \} \) is a fundamental family of conical neighborhoods of \( \mathcal{P} \). \tag{4.5}

**Lemma 4.4.** If \( F \) is a borderline conical subequation, then \( F \subset \mathcal{P}^{\text{min}/\text{max}}_\alpha \) for some \( \alpha > 0 \) small enough.
**Proof.** Condition (1)' in Lemma 3.1 states that \( \text{Int} \tilde{F} \) is a conical neighborhood of \( P \). Hence (4.5) implies that for \( \alpha > 0 \) small, \( P_{\alpha}^{\min/\max} \subset \tilde{F} \). Now by (2.7) \( F \subset P_{\alpha}^{\min/\max} \).

Consequently, to prove Theorem 4.1 we only need to prove that

\[
\text{(SMP)} \text{ holds for } P_{\alpha}^{\min/\max}. \tag{4.6}
\]

Using the fact that \( \lambda_{\min}(-A) = -\lambda_{\max}(A) \) is easy to compute that

\[
\tilde{P}_{\alpha}^{\min/\max} = P_{\alpha}^{\min/\max}. \tag{4.7}
\]

We can prove a stronger result than (4.6) by localizing near the origin in \( \text{Sym}^2(\mathbb{R}^n) \).

**Theorem 4.5.** Suppose \( F \) is any subequation (not necessarily a cone). If there exists \( \alpha > 0 \) and a ball \( B(0, \delta) \) with

\[
F \cap B(0, \delta) \subset \tilde{P}_{\alpha}^{\min/\max},
\]

then the (SMP) holds for \( F \).

**Corollary 4.6.** Suppose \( F \) is a generic subequation. Then the (SMP) holds for \( F \).

**Proof.** By (3.3), \( F \) generic \( \Rightarrow 0 \notin F \), which implies that \( F \cap B(0, r) = \emptyset \) for \( r \) small.

**Proof of Theorem 4.5.**

The proof of Theorem 4.5 is classical, but it is included both for completeness and to point out that the standard Hopf function can be replaced by the simpler algebraic function

\[
\varphi(x) = \frac{\epsilon}{2}(R - |x|)^2, \quad r \leq |x| \leq R. \tag{4.8}
\]

Also the main part of the proof will be used later. This part is separated out as two inter-related lemmas, with no mention of the subequation \( F \) until the end of the second lemma.

**Lemma 4.7.** If the (SMP) fails for \( u \) on a compact set \( K \subset \mathbb{R}^n \), then for sufficiently small \( \bar{r} > 0 \) there must exist a ball \( B_{\bar{r}}(x_0) \) of radius \( \bar{r} \) with \( \overline{B_{\bar{r}}(x_0)} \subset \text{Int}K \) such that

\[
(a) \quad u(x) < \sup_K u \quad \text{for all } x \in B_{\bar{r}}(x_0), \quad \text{and}
\]

\[
(b) \quad u(\bar{x}) = \sup_K u \quad \text{for some } \bar{x} \in \partial B_{\bar{r}}(x_0). \tag{4.9}
\]

**Proof.** Set \( M = \sup_K u \). By hypothesis there exist points \( x_0 \in \text{Int}K \) which are not in the maximum set \( \{ u = M \} \). Pick such a point \( x_0 \) closer to \( \{ u = M \} \) than to \( \partial K \). Then (4.9) defines the radius \( \bar{r} \) and \( \overline{B_{\bar{r}}(x_0)} \subset \text{Int}K \).
We continue under the assumptions of this lemma with \( M \equiv \sup_K u = u(\bar{x}) \) and

\[
M(t) \equiv \sup_{\partial B_t} u
\]

where the ball \( B_t \) is centered at \( x_0 \). Choose any annulus

\[
A = A(r,R) \equiv \{ x : r \leq |x - x_0| \leq R \} \subset \text{Int}K,
\]  

(4.10)

containing \( \partial B_r \) in its interior, i.e. with \( r < \bar{r} < R \). Then

\[
u(\bar{x}) = M \text{ at } \bar{x} \in \text{Int}A, \text{ while on } \partial A : u|_{\partial B_r} < M \text{ and } u|_{\partial B_R} \leq M. \tag{4.11}
\]

**Lemma 4.8.** Suppose \( \varphi \) is a continuous function on the annulus \( A \) which is \( C^2 \) on the interior.

If \( \varphi|_{\text{Int}A} > 0 \) while \( \varphi|_{\partial B_r} < M - M(r) \) and \( \varphi|_{\partial B_R} \equiv 0 \),

(4.12)

then there must exist a point \( \bar{y} \in \text{Int}A(r,R) \) such that \(-\varphi\) is a test function for \( u \) at \( \bar{y} \). Hence,

\[
u \text{ is } F \text{ subharmonic on } \text{Int}K \implies -D_y^2 \varphi \in F. \tag{4.13}
\]

**Proof.** Combining (4.11) with (4.12) gives

\[
(u + \varphi)(\bar{x}) > M \text{ while on } \partial A : (u + \varphi)|_{\partial B_r} < M, \text{ and } (u + \varphi)|_{\partial B_R} \leq M.
\]

Hence, \( u + \varphi \) has an interior maximum at a point \( \bar{y} \) in the annulus \( A(r,R) \).

**Proof of Theorem 4.5.** Suppose that the (SMP) fails for \( F \). We employ Lemmas 4.7 and 4.8 with \( x_0 = 0 \) (for convenience) and \( \varphi \) defined by (4.8). By Corollary 5.2 below, the eigenvalues of \( A \equiv -D_y^2 \varphi \) are \( \lambda_{\text{max}}(A) = \varepsilon(R - |\bar{y}|)/|\bar{y}| \) (multiplicity \( n - 1 \)) and \( \lambda_{\text{min}}(A) = -\varepsilon \). Hence the ratio satisfies

\[
\frac{\lambda_{\text{max}}(A)}{-\lambda_{\text{min}}(A)} = \frac{R - |\bar{y}|}{|\bar{y}|} \leq \frac{R - r}{r}.
\]

Choose the annulus in Lemma 4.8 with \( \frac{R - r}{r} < \alpha \). Then \( \lambda_{\text{min}}(A) + \frac{1}{\alpha} \lambda_{\text{max}}(A) < 0 \), so that \(-D_y^2 \varphi \notin \mathcal{P}_{\alpha}^{\text{min/}
\text{max}}\). Consequently, under the hypothesis that (SMP) fails for \( u \), this produces a point \( \bar{y} \) with \(-\varphi\) a test function for \( u \), but \(-D_y^2 \varphi \notin \mathcal{P}_{\alpha}^{\text{min/}
\text{max}}\), which proves that \( u \) is not \( \mathcal{P}_{\alpha}^{\text{min/}
\text{max}}\)-subharmonic. Finally note that \(-D_y^2 \varphi\) can be made arbitrarily small, so the hypothesis \( F \cap B(0,\delta) \subset \mathcal{P}_{\alpha}^{\text{min/}
\text{max}}\) must be false.
5. The Radial Subequation Associated to $F$.

Suppose $\psi$ is of class $C^2$ on an interval contained in the positive real numbers. Consider $\psi(|x|)$ as a function on the corresponding annular region in $\mathbb{R}^n$.

Lemma 5.1.

$$D_x^2 \psi = \frac{\psi'(|x|)}{|x|} P_{x\perp} + \psi''(|x|) P_x.$$ 

**Proof.** First note that $D(|x|) = \frac{x}{|x|}$, and therefore $D^2(|x|) = D(\frac{x}{|x|}) = \frac{1}{|x|} I - \frac{x}{|x|^2} \circ \frac{x}{|x|} = \frac{1}{|x|} (I - P_x) = \frac{1}{|x|} P_{x\perp}$. Hence,

$$D_x \psi = \psi'(|x|) \frac{x}{|x|} \quad \text{and}$$

$$D_x^2 \psi = \psi'(|x|) D \left( \frac{x}{|x|} \right) + \psi''(|x|) \frac{x}{|x|} \circ \frac{x}{|x|} = \psi'(|x|) \frac{x}{|x|} P_{x\perp} + \psi''(|x|) P_x. \quad \blacksquare$$

Corollary 5.2. The second derivative $D_x^2 \psi$ has eigenvalues $\frac{\psi'(|x|)}{|x|}$ with multiplicity $n - 1$ and $\psi''(|x|)$ with multiplicity 1.

For simplicity we shall now assume that $F$ has characteristic functions $f = \overline{f}$ which we denote by $\overline{f}$. Recall from (3.7), or (3.8), that

$$\lambda P_{e\perp} + \mu P_e \in F \quad \forall e \neq 0 \quad \iff \quad \mu + f(\lambda) \geq 0. \quad (5.1)$$

With motivation from Lemma 5.1 this leads to a subequation $R_F$ on $(0, \infty)$. Let $p = \psi'(t)$ and $a = \psi''(t)$ denote jet coordinates.

**Definition 5.3.** The radial subequation $R_F$ associated to $F$ is defined by

$$R_F : a + f \left( \frac{P}{t} \right) \geq 0 \quad 0 < t < \infty \quad (5.2)$$

where $f$ is the characteristic function associated with the subequation $F$.

It follows immediately from these definitions and Lemma 5.1 that if $\psi(t)$ is a $C^2$-function defined on a subinterval of $(0, \infty)$, with $u(x) \equiv \psi(|x|)$ defined on the corresponding annular region in $\mathbb{R}^n$, then

$$u(x) \equiv \psi(|x|) \text{ is } F \text{ subharmonic } \iff \psi(t) \text{ is } R_F \text{ subharmonic } \quad (5.3)$$

This is extended to upper semi-continuous functions in Appendix A (Theorem A.1). The proof of the implication $\Rightarrow$ is elementary, whereas the proof of $\Leftarrow$ requires some details. However, note that $u(x) = \psi(|x|)$ is upper semi-continuous $\iff \psi(t) \text{ is upper semi-continuous}$.
Remark 5.4. The radial subequation $R$ associated to $F$ satisfies the topological conditions (T) in [7]. Namely,

\[(i) \quad R = \text{Int}R, \quad (ii) \quad R_t = \text{Int}R_t, \quad (iii) \quad \text{Int}R_t = (\text{Int}R)_t.\]

where $R_t$ is the fibre of $R$ above $t$. Note that $\text{Int}R$ is not defined by $a + f(\frac{\rho}{t}) > 0$ but by $a + f_-(\frac{\rho}{t}) > 0$ where $f_-(y) \equiv \lim_{z \to y^-} f(z)$ is lower semi-continuous. The proof is left to the reader.

6. Increasing Radial Subharmonics for Borderline Subequations

As in the last section, we assume for simplicity that $f = \overline{f} = \underline{f}$. Because of the next result we focus on radial subharmonics which are increasing.

Lemma 6.1. Suppose that $F$ is borderline and $u(x) = \psi(|x|)$ is a radial $F$-subharmonic function. There is only one way that $u(x)$ can violate the (SMP). Namely, for some $r$, $\psi(t)$ must satisfy:

\[\psi(t) < M \text{ for } t < r \quad \text{and} \quad \psi(t) \equiv M \text{ for } t \geq r. \quad (6.1)\]

Moreover,

\[\text{\psi must be increasing on } (\bar{a}, r), \text{ for some } \bar{a} < r. \quad (6.2)\]

Proof. By the borderline hypothesis the (MP) holds for $F$. Since $u$ satisfies the (MP), so does $\psi(t)$. If $\psi$ has an interior maximum point at $t_0$ on an interval $[a, b]$, then either $\psi$ is equal to the maximum value $M$ on $[a, t_0]$ or on $[t_0, b]$ since otherwise $\psi$ violates the (MP) on an interval about $t_0$. If $\psi$ equals this maximum value on $[a, t_0]$, we can extend $u(x)$ to the ball of radius $b$ (to be constant on the ball of radius $a$) as an $F$-subharmonic function which violates the (MP). This proves (6.1).

Pick $\bar{a}$ to be a minimum point for $\psi$ on $[a, b]$. Then $\psi$ must be increasing on $[\bar{a}, b]$. Otherwise, there exist $\alpha, \beta$ with $\bar{a} < \alpha < \beta < r$ and $\psi(\alpha) > \psi(\beta)$. In this case $\psi(\alpha) > \psi(\bar{a})$ also (since $\psi(\beta) \geq \psi(\bar{a})$), and this violates the maximum principle on $[\bar{a}, \beta]$. □

Definition 6.2. Suppose that $F$ is borderline. The increasing radial subharmonic equation $R^+_F$ on $(0, \infty)$ is defined by

\[R^+_F : a + f\left(\frac{p}{t}\right) \geq 0 \text{ and } p \geq 0. \quad (6.3)\]

For $C^2$-functions $\psi(t)$ it is obvious that:

\[\psi(t) \text{ is } R^+_F \text{ subharmonic } \iff \psi(|x|) \text{ is } F \cap \{x \cdot p \geq 0\} \text{ subharmonic} \quad (6.4)\]

where $F \cap \{x \cdot p \geq 0\}$ is a variable coefficient subequation on $\mathbb{R}^n$ depending on both the first and second derivatives. The equivalence (6.4) is extended using Theorem A.1.
THEOREM 6.3. (Increasing Radial Subharmonics). Suppose that \( F \) is borderline. The function \( u(x) = \psi(|x|) \) is \( F \)-subharmonic and radially increasing on an annular region in \( \mathbb{R}^n \) if and only if \( \psi(t) \) is \( R_F^\uparrow \)-subharmonic on the corresponding subinterval of \((0, \infty)\).

Proof. Theorem A.1 states that: \( u \) is \( F \)-subharmonic \( \iff \psi \) is \( R_F \)-subharmonic. By definition \( u(x) \) is radially increasing if \( u \) satisfies the first-order variable coefficient subequation \( \{ p \cdot x \geq 0 \} \). It remains to show that

\[
\begin{align*}
u & \text{ satisfies the subequation } \{ p \cdot x \geq 0 \} \iff \psi \text{ satisfies } \psi'(t) \geq 0. \tag{6.5}\end{align*}
\]

Suppose \( \psi(|x|) \) is \( \{ x \cdot p \geq 0 \} \)-subharmonic and that \( \varphi(t) \) is a test function for \( \psi(t) \) at a point \( t_0 \). Then \( \varphi(|x|) \) is a test function for \( u(x) \) at \( x_0 \) if \( |x_0| = t_0 \). Now

\[
D_{x_0} \varphi = \varphi'(|x_0|) \frac{x_0}{|x_0|} \quad \text{and hence} \quad x_0 \cdot D_{x_0} \varphi = |x_0| \varphi'(|x_0|). \tag{6.6}
\]

Thus \( \varphi'(t_0) \geq 0 \) proving that \( \psi(t) \) is increasing. Conversely, if \( \psi(t) \) is increasing and \( \varphi(x) \) is a test function for \( u(x) \) at \( x_0 \), then \( \varphi(t) \equiv \varphi\left(\frac{tx_0}{|x_0|}\right) \) is a test function for \( \psi(t) \) at \( t_0 = |x_0| \). Hence, \( \varphi'(t_0) \geq 0 \). However, \( \varphi'(t_0) = (D_{x_0} \varphi) \cdot x_0 \).

Remark 6.4. (Decreasing Radial Subharmonics). For borderline \( F \) we define the decreasing radial subharmonic equation \( R_F^{-\downarrow} \) on \((0, \infty)\) by

\[
R_F^{-\downarrow} : a - f\left(\frac{p}{t}\right) \geq 0 \quad \text{and} \quad p \leq 0. \tag{6.7}
\]

We leave it to the reader to show the following. For \( \psi \) upper semi-continuous,

\[
\psi(t) \text{ is } R_F^{-\downarrow} \text{ subharmonic } \iff \psi(|x|) \text{ is } F \cap \{ x \cdot p \leq 0 \} \text{ subharmonic} \tag{6.8}
\]

7. Proof of the (SMP) in the Borderline Case

In this section we prove Part (a) of Theorem 3.5.

We can assume that the subequation \( F \) is \( O_n \)-invariant because of the following construction. Set

\[
F^\# \equiv \bigcup_{g \in O_n} g(F). \tag{7.1}
\]

First note that \( F^\# \) is also a subequation. Now from Definition 3.3 of the characteristic function \( \overline{f} \) of \( F \) and the fact that \( P_{e^\perp}, P_e \) have the same span as \( I, P_e \), it is easy to see that the characteristic function for \( F^\# \) is \( \overline{f} \). Moreover, \( F^\# \) is an \( O_n \)-invariant subequation which contains \( F \) so that it suffices to prove Theorem 3.5(a) for \( F^\# \).

From now on we assume that \( F \) is an \( O_n \)-invariant borderline subequation, and we let \( f \) denote the restriction of \( \overline{f} = f \) to \([0, \infty)\). Hence \( f(0) = 0 \) and \( f \) is increasing. Furthermore, let both \( R_f^\uparrow \) and \( R_F^\uparrow \) denote the subequation defined by (6.4).
Part (a) of Theorem 4.3 follows from two implications:

\[ \int_{0^+} \frac{dy}{f(y)} = \infty \Rightarrow \text{(SMP) for } R_F^\uparrow, \quad \text{and} \quad (7.2) \]

\[ \text{(SMP) for } R_F^\uparrow \Rightarrow \text{(SMP) for } F \]  \hspace{1cm} (7.3)

We consider the second implication first.

**Proposition 7.1.** If the (SMP) holds for the associated increasing radial subequation

\[ R_F^\uparrow : \quad a + f \left( \frac{p}{t} \right) \geq 0 \quad \text{and} \quad p \geq 0 \quad \text{on } (0, \infty), \]

then the (SMP) holds for \( F \) on \( \mathbb{R}^n \).

**Proof.** Suppose \( u \) is a counterexample to the (SMP) for \( F \) on a compact set \( K \). We will show this leads to a counterexample to the (SMP) for \( R_F^\uparrow \).

Recall the construction of the annulus \( A(r, R) \), the number \( \bar{r} \), and the function

\[ M(t) \equiv \sup_{\partial B_t} u \quad \text{for } r < t < R. \]

from the discussion preceding Lemma 4.8.

Since \( F \) is borderline, \( 0 \in \partial F \), and hence by Corollary 2.4 the (MP) holds for \( u \) on \( B_t \) since \( u \) is \( F \)-subharmonic on \( \text{Int} K \). Therefore \( M(t) \) must be increasing for \( r < t < R \). Hence

\[ M(t) < M \quad \text{for } r < t < \bar{r} \quad \text{and} \quad M(t) = M \quad \text{for } \bar{r} \leq t < R. \]

(7.5)

That is, the (SMP) for \( M(t) \) on \( r \leq t \leq R \) fails. It remains to show that \( M(t) \) is \( R_F^\uparrow \)-subharmonic.

**Lemma 7.2.** For any upper semi-continuous function \( u \), the function \( M(t) \equiv \sup_{\partial B_t} u \) is upper semi-continuous.

**Proof.** Assume the balls \( B_t \) are centered at the origin. Given \( \delta > 0 \),

\[ N_\delta \equiv \{ x : u(x) < M(t) + \delta \} \]

is an open set containing \( \partial B_t = \{ x : |x| = t \} \). Hence the annulus \( \{ x : t - \epsilon \leq |x| \leq t + \epsilon \} \) is contained in \( N_\delta \) for \( \epsilon > 0 \) small. Thus \( M(r) < M(t) + \delta \) if \( t - \epsilon \leq r \leq t + \epsilon \). This proves that \( M(t) \) is upper semi-continuous.

Since \( M(t) \) satisfies the subequation \( \{ p \geq 0 \} \) it remains to show that \( M(t) \) satisfies the subequation \( R_F \). By Theorem A.1 it suffices to show that \( M(|x|) \) is \( F \)-subharmonic on \( r < |x| < R \). The next result completes the proof of Proposition 7.1.

**Lemma 7.3.** If \( u \) is \( F \)-subharmonic on an annulus, then \( M(|x|) \) is also \( F \)-subharmonic on the same annulus where \( M(t) \equiv \sup_{|x|=t} u \).
Proof. By Lemma 7.2 $M(t)$ is upper semi-continuous, and hence $M(|x|)$ is upper semi-
continuous. Let $u_g(x) \equiv u(gx)$ with $g \in O_n$. Each $u_g$ is $F$-subharmonic since $F$ is
$O_n$-invariant. Thus
\[ M(|x|) = \sup_{g \in O_n} u_g(x) \quad (7.6) \]
is $F$-subharmonic by the standard “families locally bounded above” property for $F$. ■

A One-Variable Result

The point of this subsection is to prove the one-variable result (7.2) which completes
the proof of Theorem 4.3 part (a). We assume throughout that $f : [0, \infty) \to [0, \infty]$ is an
upper semi-continuous, increasing function with $f(0) = 0$, and we define the subequation
$R_f^1$ on $(0, \infty)$ by (6.4).

Proposition 7.4.
\[ \int_{0+} dy f(y) = \infty \quad \Rightarrow \quad \text{The (SMP) holds for } R_f^1. \]

To prove this we first consider the following one-variable constant coefficient subequa-
tion $E$ defined by
\[ E : \quad a + f(p) \geq 0 \quad \text{and} \quad p \geq 0. \quad (7.7) \]

Proposition 7.5.
\[ \int_{0+} dy f(y) = \infty \quad \Rightarrow \quad \text{The (SMP) holds for } E. \]

Proof that 7.5 ⇒ 7.4. Suppose that the (SMP) fails for $R_f^1$ on $[r_1, r_2] \subset (0, \infty)$. Choose
$r$ with $0 < r < r_1$. Consider the constant coefficient subequation $E_r$ defined by
\[ E_r : \quad a + f \left( \frac{p}{r} \right) \geq 0 \quad \text{and} \quad p \geq 0. \quad (7.8) \]

If $t > r$, then $a + f \left( \frac{p}{r} \right) \geq 0$ implies that $a + f \left( \frac{p}{r} \right) \geq 0$ since $f$ is increasing. That is, each
fibre $(R_f^1)_t \subset E_r$ if $t > r$, so that on a neighborhood of $[r_1, r_2]$, if $\psi$ is $R_f^1$-subharmonic, then
$\psi$ is $E_r$-subharmonic. Therefore the (SMP) fails for $E_r$. The function $f \left( \frac{p}{r} \right)$ satisfies the
same conditions as the function $f$. Hence, by Proposition 7.5, $\int_{0+} dy f(y) = \frac{1}{r} \int_{0+} dy f(y) < \infty$.

Proof of Proposition 7.5. Suppose that $\psi$ is a counterexample to the (SMP) for $E$.
Since $\psi$ is upper semi-continuous and increasing, there exists a point $r_0$ such that
\[ \psi(t) < M \quad \text{for} \quad t < r_0, \quad \text{and} \quad \psi(t) \equiv M \quad \text{for} \quad r_0 \leq t. \quad (7.9) \]
By sup-convolution we may assume that \( \psi \) is quasi-convex and still satisfies \( E \) with a new \( r_0 \) slightly smaller than the old one. Since \( f \) is increasing we have the following.

**Lemma 7.6.** The derivative \( \psi' \) can be assumed to be absolutely continuous.

**Proof.** Since \( \psi(t) + \frac{1}{2} \lambda t^2 \) is convex for some \( \lambda > 0 \), the second distributional derivative \( \psi'' = \mu - \lambda \) where \( \mu \geq 0 \) is a non-negative measure. Consider the Lebesgue decomposition \( \mu = \alpha + \nu \) of \( \mu \) into its \( L^1_{\text{loc}} \)-part \( \alpha \) and its singular part \( \nu \). Since \( \nu \) is supported on \( t \leq r_0 \), there exists a unique convex function \( \beta \) with \( \beta'' = \nu \) and \( \beta \equiv 0 \) on \( r_0 \leq t \). It follows easily that \( \beta(t) \geq 0 \) and \( \beta \) is decreasing. Therefore \( \psi(t) \equiv \psi(t) - \beta(t) \leq \psi(t) \) and \( \dot{\psi}(t) \) is increasing. Hence \( \psi \) also satisfies (7.9). Now \( \psi'' = \alpha - \lambda \), and therefore \( \psi' \) is absolutely continuous. Since \( \nu \) is singular, \( \beta''(t) = 0 \) a.e., and since \( \beta \) is decreasing, \( \dot{\psi}'(t) = \psi'(t) - \beta'(t) \geq \psi'(t) \) a.e.. Therefore, since \( f \) is increasing and \( \dot{\psi} \) is \( E \)-subharmonic,

\[
\dot{\psi}''(t) + f(\dot{\psi}'(t)) \geq 0 \quad \text{a.e.} \quad (7.10)
\]

This almost-everywhere inequality is all that will be used to complete the proof of Proposition 7.5. However, in general, if a quasi-convex function satisfies a subequation \( F \) a.e., then it must be \( F \)-subharmonic (see Corollary 7.5 in [6] for pure second-order case and (8.3) below for the general case).

Now let \( \varphi(t) \equiv \psi'(t) \). This function \( \varphi \) is absolutely continuous since \( \varphi'(t) \equiv \alpha(t) - \lambda \). The properties that \( \psi \) is increasing and \( \psi(t) \equiv M \) for \( t \geq r_0 \) translate into the properties:

\[
\varphi(t) \geq 0 \quad \text{and} \quad \varphi(t) = 0 \quad \text{if} \quad t \geq r_0. \quad (7.11)
\]

The inequality (7.10) states that

\[
\varphi'(t) + f(\varphi(t)) \geq 0 \quad \text{a.e.} \quad (7.12)
\]

Note that at a point \( t \) where \( \varphi \) is differentiable, if \( \varphi(t) = 0 \), then this implies that \( \varphi'(t) \geq 0 \). Thus (7.12) can be rewritten as

\[
\frac{-\varphi'(t)}{f(\varphi(t))} \leq 1 \quad \text{a.e.} \quad (7.13)
\]

where the LHS equals \(-\infty\) at points where \( \varphi(t) = 0 \). Therefore, for any measurable set \( B \) we have

\[
- \int_B \frac{\varphi'(t)}{f(\varphi(t))} \leq |B|. \quad (7.14)
\]

On the set \( B^- \) where \( \varphi \) is differentiable and \( \varphi'(t) < 0 \), the inequality (7.14) has content. Otherwise the integrand \( \frac{-\varphi'(t)}{f(\varphi(t))} \leq 0 \).

Choose \( s_1 \) and \( s_0 \) so that \( r_1 < s_1 < s_0 < r_0 \) and \( 0 < \varphi(s_0) < \varphi(s_1) \). We will show that

\[
\int_{\varphi(s_0)}^{\varphi(s_1)} \frac{dy}{f(y)} \leq r_0 - r_1 \quad \text{for all such} \quad s_0 > s_1. \quad (7.15)
\]
Because of (7.11) the point \( s_0 \) with \( \varphi(s_0) > 0 \) can be chosen arbitrarily close to \( r_0 \). Then taking the limit as \( s_0 \) increases to \( r_0 \) proves that

\[
\int_0^{\varphi(s_1)} \frac{dy}{f(y)} \leq r_0 - r_1 < \infty.
\]

It remains to prove (7.15). Let \( N(\varphi|_A, y) \) denote the cardinality of \( \{ t \in A : \varphi(t) = y \} \). Set \( A = [s_1, s_0] \), and let \( V_A(\varphi) \) denote the total variation of \( \varphi \) on \( A \). Since \( \varphi \) is absolutely continuous, we have, by Theorem 2.10.13 (p. 177) in [5], that

\[
V_A(\varphi) \text{ is finite}, \quad \text{and } V_A(\varphi) = \int N(\varphi|_A, y) \, dy \quad (7.16)
\]

Now set

\[
f_\epsilon(y) \equiv \max\{f(y), \epsilon\} \quad \text{where } \epsilon > 0.
\]

Then

\[
\int \frac{1}{f_\epsilon(y)} N(\varphi|_A, y) \, dy \leq \frac{1}{\epsilon} V_A(\varphi) < \infty.
\]

Hence, the second half of Theorem 3.2.6 (p.245) in [5] applies to yield

\[
\int_{\varphi(s_1)}^{\varphi(s_0)} \frac{1}{f_\epsilon(y)} \, dy = -\int_{s_1}^{s_0} \frac{\varphi'(t)}{f_\epsilon(\varphi(t))} \, dt. \quad (7.17)
\]

Since \( \frac{1}{f_\epsilon(y)} \leq \frac{1}{f(y)} \) on the set \( B^- \) where \( \varphi \) is differentiable and \( \varphi'(t) < 0 \), we have

\[
\int_{B^-} \frac{-\varphi'(t) \, dt}{f_\epsilon(\varphi(t))} \leq \int_{B^-} \frac{-\varphi'(t) \, dt}{f(\varphi(t))} \leq |B^-| \leq r_0 - r_1 \quad (7.18)
\]

by (7.14). Combining (7.17) and (7.18) proves that

\[
\int_{\varphi(s_1)}^{\varphi(s_0)} \frac{dy}{f_\epsilon(y)} \leq r_0 - r_1,
\]

since \( \int_{B^-} \frac{-\varphi'(t) \, dt}{f_\epsilon(\varphi(t))} \leq 0. \) By the Monotone Convergence Theorem this proves (7.15). \( \blacksquare \)

**Remark 7.7.** In the proof of Proposition 7.5, the fact that \( f \) is increasing was only used in Lemma 7.6. Therefore, if a subequation \( E \) is defined by an upper semi-continuous function \( f : [0, \infty) \to [0, \infty] \) with \( f(0) = 0 \) and \( f(y) > 0 \) for \( y > 0 \), then we have that:

\[
\int_0^y \frac{du}{f(u)} = \infty \Rightarrow \text{the (SMP) holds for all } E\text{-subharmonic functions } \psi \text{ for which } \psi' \text{ is absolutely continuous}.
\]
8. Radial (Harmonic) Counterexamples to the (SMP)

In this section we give the proof of Part (b) of Theorem 3.5 by constructing a radial counterexample to the (SMP) for $F$. Let $f$ denote the restriction of $f$ to $[0, \infty)$, where $f$ is the (smaller) characteristic function (Definition 3.3) of the given borderline subequation $F$. Then

$$f : [0, \infty) \rightarrow [0, \infty] \text{ is upper semicontinuous, increasing and } f(0) = 0. \quad (8.1)$$

More precisely we prove the following.

**THEOREM 8.1.** Suppose that $F$ is a borderline subequation with $f$ as described above. If $\int_0^\infty \frac{dy}{f(y)} < \infty$, then there exists a radially increasing $F$-subharmonic function $u(x) = \psi(|x|)$ on $|x| > 1$ where $\psi$ is of class $C^{1,1}$ on $(1, \infty)$ and satisfies

$$\psi(t) < m \text{ for } 1 < t < t_0 \text{ and } \psi(t) = m \text{ for } t \geq t_0. \quad (8.2)$$

By Theorem 6.3, it suffices to construct an increasing $C^{1,1}$-function which is $R^\uparrow_f$-subharmonic and satisfies (8.2).

In order to explicate the proof we will use the “almost-everywhere theorem” for quasi-convex functions, which holds for the most general possible subequations $F$. This theorem states that for a quasi-convex function $u$

If $u$ has its $2^-$-jet in $F$ a.e., then $u$ is $F$ subharmonic. \quad (8.3)

This general result will be established in [11]. We will also make use of the fact that

$$u \text{ is of class } C^{1,1} \iff u \text{ and } -u \text{ are quasiconvex.} \quad (8.4)$$

**Proof of Theorem 8.1.** We start by solving the constant coefficient subequation $E$ on $\mathbb{R}$ defined by

$$E : \quad a + f(p) \geq 0 \quad \text{and} \quad p \geq 0, \quad (8.5)$$

which is simpler than $R^\uparrow_f$.

**Lemma 8.2.** If $\int_0^\infty \frac{dy}{f(y)} < \infty$, then there exists an $E$-subharmonic function $\varphi(s)$ of class $C^{1,1}$ on $(0, \infty)$ with

$$\varphi(s) < m \text{ strictly increasing on } (0, s_0) \quad \text{and} \quad \varphi(s) \equiv m \text{ on } [s_0, \infty)$$

**Proof.** Set $s(y) = \int_0^y \frac{dy}{f(y)}$ for $y \geq 0$. For $0 \leq y_1 < y_2 \leq y_0$ we have

$$\frac{y_2 - y_1}{f(y_2)} \leq \int_{y_1}^{y_2} \frac{dt}{f(t)} = s_2 - s_1. \quad (8.6)$$
Therefore, this function $s(y)$ is strictly increasing until $f$ equals $+\infty$ (and is constant afterwards). In particular, it is a homeomorphism from $[0, y_0]$ to $[0, s_0]$ for some $y_0 > 0$ with $s_0 = s(y_0) < \infty$. Let $y(s)$ denote the inverse, which is also strictly increasing with $y(0) = 0$. The inequality (8.6) implies that $y(s)$ is Lipschitz on $[0, s_0]$ with Lipschitz constant $f(y_0)$, since $f(y_2) \leq f(y_0)$ if $y_2 \leq y_0$.

Taking $y_1 = 0$, $y_2 = y(s)$ yields $y(s) \leq sf(y(s))$ which implies that $y$ is differentiable from the right at $s = 0$ with $y'(0) = 0$. Moreover, since $y(s)$ is Lipschitz, it is differentiable a.e. and

$$y'(s) = f(y(s)) \quad \text{a.e.} \tag{8.7}$$

Fix $m$ and consider the function $\varphi(s)$ defined on $(0, \infty)$ by $\varphi(s_0) = m$ and

$$\varphi'(s) \equiv \begin{cases} y(s_0 - s) & \text{if } 0 < s \leq s_0 \\ 0 & \text{if } s \geq s_0. \end{cases}$$

Since $\varphi'(s)$ is continuous and strictly decreasing to zero on $(0, s_0]$, $\varphi(s)$ must be strictly increasing to $m$ on $(0, s_0]$ and identically equal to $m$ afterwards.

Since $\varphi$ is twice differentiable at $s = s_0$, with $\varphi'(s_0) = \varphi''(s_0) = 0$, the function $\varphi$ is class $C^{1,1}$ on all of $(0, \infty)$. Moreover, (8.7) implies that

$$\varphi''(s) + f(\varphi'(s)) = 0 \quad \text{a.e. on } (0, \infty). \tag{8.8}$$

By (8.4) and (8.3) this implies that $\varphi$ is $E$-subharmonic on $(0, \infty)$.

We will use Lemma 8.2 applied to the subequation $E'$ defined by

$$E' : a + p + f(p) \geq 0 \quad \text{and} \quad p \geq 0, \tag{8.9}$$

rather than $E$. Now consider the radial subequation $R^+_f$ on $(0, \infty)$ defined by

$$R^+_f : a + f \left( \frac{p}{t} \right) \geq 0, \quad \text{and} \quad p \geq 0 \tag{8.10}$$

which depends on the variable $t \in (0, \infty)$.

**Proposition 8.3.** Suppose $\varphi(s)$ is the $E'$-subharmonic function given by Lemma 8.2 applied to $E'$ rather than $E$. Then the function $\psi(t)$ defined on $(1, \infty)$ by

$$\psi'(t) = t\varphi' \log(t) \quad \text{and} \quad \psi(t_0) = m, \tag{8.11}$$

where $t_0 = e^{s_0}$, is a $C^{1,1}$ subharmonic for $R^+_f$. Moreover,

$$\psi(t) \text{ is strictly increasing with } \psi(t) < m \text{ on } 1 < t < t_0 \quad \text{and} \quad \psi(t) \equiv m \text{ on } t_0 \leq t. \tag{8.12}$$
Proof. That $\varphi'$ is Lipschitz implies that $\psi'$ is Lipschitz. Therefore $\psi$ is class $C^{1,1}$. At a point of differentiability we have $\psi''(t) = \varphi'(\log t) + \varphi''(\log t)$, and hence $\psi''(t) + f\left(\frac{\psi'(t)}{t}\right) = \varphi''(\log t) + \varphi'(\log t) + f(\varphi'(\log t)) = 0$. Therefore $\psi(t)$ satisfies (8.10) a.e. (Since $\varphi'(s)$ is continuous and $>0$ on $(0,s_0)$, $\psi'(t)$ is also continuous and $>0$ on $(1,t_0)$. Thus $\psi$ is strictly increasing on $(1,t_0)$.) Thus by (8.4) and (8.3), $\psi$ is $R^*_f$-subharmonic. The properties (8.12) are straightforward.

Remark 8.4. (F-Harmonicity). The $F$-subharmonic function $u(x) = \psi(|x|)$ constructed in this section is, in fact, $F$-harmonic if $F$ is invariant as in Remark 3.6. We leave it to the reader to show that $-\psi$ is $\tilde{R}^*_f$-subharmonic and hence $-u$ is $\tilde{F}$-subharmonic. One can show that

$$a + f(p) = 0, \quad p \geq 0 \quad \Rightarrow \quad (p,a) \in \partial E,$$

but the converse is not true if $f$ has a jump.

9. Subequations with the Same Increasing Radial Subharmonics.

In this section we describe all subequations with the same set of increasing radial subharmonics. Recall that a class of such subharmonics is completely determined by an upper semi-continuous increasing function $f : [0, \infty) \rightarrow [0, \infty]$, (9.1)

with $f(0) = 0$, via the one-variable subequation

$$R^*_f : \quad \psi'(t) \geq 0 \quad \text{and} \quad \psi''(t) + f\left(\frac{\psi'(t)}{t}\right) \geq 0,$$

(9.2)
on $(0, \infty)$ (see Theorem 6.3). The problem is to determine all subequations with this characteristic function $f$.

We start with two examples. Given $A \in \text{Sym}^2(\mathbb{R}^n)$, let $\lambda_1(A) \leq \cdots \leq \lambda_n(A)$ denote the ordered eigenvalues of $A$. In particular, the minimum and maximum eigenvalues are $\lambda_{\min}(A) = \lambda_1(A)$ and $\lambda_{\max}(A) = \lambda_n(A)$ respectively. Recall the monotonicity $\lambda_k(A + P) \geq \lambda_k(A)$ for $P \in P$.

Example 9.1. (The Min/Max Subequation).

$$F^{\min/\max}_f \equiv \{A : \lambda_{\max}(A) \geq 0 \quad \text{and} \quad \lambda_{\min}(A) + f(\lambda_{\max}(A)) \geq 0\}$$

Example 9.2. (The Min/2 Subequation).

$$F^{\min/2}_f \equiv \{A : \lambda_2(A) \geq 0 \quad \text{and} \quad \lambda_{\min}(A) + f(\lambda_2(A)) \geq 0\}$$

Proposition 9.3. The sets $F^{\min/\max}_f$ and $F^{\min/2}_f$ are subequations which are borderline and $O_n$-invariant. Moreover, for both subequations, the characteristic function restricted to $[0, \infty)$ equals $f$. 

22
**Proof.** Since \( f \) is upper semi-continuous, both sets are closed. Since \( f \) is increasing, positivity follows from the \( \mathcal{P} \)-monotonicity of the ordered eigenvalues. To prove these subequations are borderline, suppose \( A \) lies in the larger subequation \( F_f^{\min/\max} \) and \( A \in -\mathcal{P} \), i.e., \( \lambda_{\max}(A) \leq 0 \). Then \( \lambda_{\max}(A) = 0 \) and since \( f(0) = 0, \lambda_{\min}(A) = 0 \). Hence, \( A = 0 \). Invariance follows because the ordered eigenvalues themselves are \( O_n \)-invariant.

To complete the proof we compute the full radial profiles (not just the increasing part).

The subequation \( F_f^{\min/\max} \) has radial profile

\[
\{(\lambda, \mu) : \lambda \geq 0 \text{ and } \mu + f(\lambda) \geq 0 \} \cup \{(\lambda, \mu) : \mu \geq 0 \text{ and } \lambda + f(\mu) \geq 0 \}.
\]

For \( n \geq 3 \), the subequation \( F_f^{\min/2} \) has radial profile

\[
\{(\lambda, \mu) : \lambda \geq 0 \text{ and } \mu + f(\lambda) \geq 0 \}.
\]

We see this as follows. Note that the radial profile of \( F_f^{\min/\max} \) is symmetric about the diagonal. Recall that if \( A \equiv \lambda P_{e_1} + \mu P_e \) belongs to any borderline subequation, then either \( \lambda \geq 0 \) or \( \mu \geq 0 \).

For (9.3), suppose \( A \equiv \lambda P_{e_1} + \mu P_e \in F_f^{\min/\max} \). If \( \lambda \geq \mu \), then \( \lambda = \lambda_{\max} \geq 0 \) and \( \mu = \lambda_{\min} \) satisfies \( \lambda_{\min} + f(\lambda_{\max}) \geq 0 \). If \( \mu \geq \lambda \), then \( \mu = \lambda_{\max} \geq 0 \) and \( \lambda = \lambda_{\min} \) satisfies \( \lambda_{\min} + f(\lambda_{\max}) \geq 0 \).

For (9.4), suppose \( A \equiv \lambda P_{e_1} + \mu P_e \in F_f^{\min/2} \). Since \( n \geq 3 \), \( \lambda = \lambda_2 \geq 0 \) and either \( \mu = \lambda_1 \) or \( \mu > \lambda \). In either case \( \lambda \geq 0 \) and \( \mu + f(\lambda) \geq 0 \).

**Corollary 9.4.** Both \( F_f^{\min/\max} \) and \( F_f^{\min/2} \) have their increasing radial subharmonics \( u(x) = \psi(|x|) \) determined by the subequation \( R^1_f \) defined in (9.2).

The subequations \( F_f^{\min/\max} \) and \( F_f^{\min/2} \) are the largest and smallest possible under the following “invariance” hypothesis on \( F \) (cf. Remark 3.6):

\[
\lambda P_{e_1} + \mu P_e \in F \text{ for some } e \neq 0 \quad \Rightarrow \quad \lambda P_{e_1} + \mu P_e \in F \text{ for all } e \neq 0.
\]

**THEOREM 9.5.** Suppose \( F \) is invariant as in (9.5). Then the radial increasing subharmonics \( u(x) = \psi(|x|) \) for \( F \) are determined by \( R^1_f \) as in (9.2) if and only if

\[
F_f^{\min/2} \subset F \subset F_f^{\min/\max}.
\]

**Proof.** Each \( A \in \text{Sym}^2(\mathbb{R}^n) \) can be written as a sum \( A = \lambda_1 P_{e_1} + \cdots + \lambda_n P_{e_n} \) using the ordered eigenvalues of \( A \). Set \( B_0 \equiv \lambda_1 P_{e_1} + \lambda_2 P_{e_2} \) and \( B_1 \equiv \lambda_1 P_{e_1} + \lambda_n P_{e_n} \), and note that \( B_0 \leq A \leq B_1 \).

If \( A \in F_f^{\min/2} \), then \( \lambda_2 \geq 0 \) and \( \lambda_1 + f(\lambda_2) \geq 0 \). Thus \( B_0 \equiv \lambda_1 P_{e_1} + \lambda_2 P_{e_2} \in F_f^{\min/2} \).

Since \( F_f^{\min/2} \) and \( F \) have the same radial profile in the half-plane \( \{\lambda \geq 0\} \) by (9.4), we conclude that \( B_0 \in F \). However, \( B_0 \leq A \) proving that \( A \in F \).
For the other inclusion, pick \( A \in F \). Since \( F \subset \tilde{\mathcal{P}} \) we have \( \lambda_{\text{max}} \geq 0 \). Now \( A \leq B_1 \) implies \( B_1 \in F \). By the invariance hypothesis and (9.3), \( F \) and \( F_{f_{\text{min/max}}} \) have the same same radial profile in the half-plane \( \{ \lambda \geq 0 \} \). Therefore, \( B_1 \in F_{f_{\text{min/max}}} \), i.e., \( \lambda_n \geq 0 \) and \( \lambda_1 + f(\lambda_n) \geq 0 \). This implies by definition that \( A \in F_{f_{\text{min/max}}} \).

**Remark 9.6.** Dropping the invariance assumption (9.5), the proof of Theorem 9.5 shows that for any borderline subequation \( F \) with characteristic functions \( \underline{f} \) and \( \overline{f} \)

\[
F_{\underline{f}}^{\text{min/2}} \subset F \subset F_{\overline{f}}^{\text{min/\text{max}}}.
\]

The subequation \( F^\# \) defined by (7.1) as the \( O_n \)-orbit of \( F \), has characteristic function \( \overline{f} \). It satisfies

\[
F \subset F^\# \subset F_{\overline{f}}^{\text{min/\text{max}}}.
\]

and is the smallest \( O_n \)-invariant subequation containing \( F \).

10. Strong Comparison and Monotonicity.

By the **strong comparison principle** for a subequation \( F \) we mean the following.

If \( u \in F(K) \) and \( v \in \tilde{F}(K) \), then the (SMP) holds for \( u + v \) on \( K \). 

\((SCP)\)

**Monotonicity**

Associated to \( F \) is its **monotonicity subequation**

\[
M_F \equiv \{ A \in \text{Sym}^2(\mathbb{R}^n) : F + A \subset A \}. \tag{10.1}
\]

We leave it as an exercise to show that:

1. \( M_F \) is a subequation,
2. \( 0 \in \partial M_F \),
3. \( M_F \) is its own monotonicity subequation,
4. If \( M_F \) is a cone, then \( M_F \) is a convex cone subequation.

**Definition 10.1.** A subequation \( M \) such that \( 0 \in M \) and \( M \) is additive, i.e., \( M + M \subset M \), will be called a **monotonicity subequation**.

Each monotonicity subequation \( M \) arises as the monotonicity subequation \( M_F \) for some subequation \( F \).

**Proof.** In fact \( M \) is its own monotonicity subequation, because if \( M + A \subset M \), then \( 0 \in M \Rightarrow A \in M \).
Strong Comparison

Using the fact that $\tilde{F} + A = \tilde{F} - A$ one can show that

$$F + M \subset \tilde{F}, \quad \tilde{F} + M \subset \tilde{F}, \quad F + \tilde{F} \subset \tilde{M}$$  \hfill (10.4)

are equivalent for any two subequations $F, M$.

**THEOREM 10.2. (Strong Comparison).** If the (SMP) holds for $\tilde{M}$ and $F + M \subset F$, then $F$ satisfies the (SCP).

**Proof.** As noted in (10.4), $F + \tilde{F} \subset \tilde{M}$. It follows from this algebraic fact that if $u \in F(K)$ and $v \in \tilde{F}(K)$, then $u + v \in \tilde{M}(K)$. See [6] Section 7 for the proof when $M = \mathcal{P}$. This same proof works in general. Therefore since the (SMP) holds for $\tilde{M}$, the (SCP) holds for $F$. \hfill □

**Corollary 10.3.** If $M$ is a monotonicity subequation and its dual $\tilde{M}$ satisfies the (SMP), then the (SCP) holds for $M$ (and equivalently for $\tilde{M}$).

**Proof.** Take $F = M$ in Theorem 10.2. \hfill □

Examples are provided in the next section.

11. Examples of Monotonicity Subequations which are not Cones

The examples will be constructed as follows.

**Definition 11.1.** Suppose $g : [0, \infty) \to \mathbb{R}$ is a continuous decreasing function with $g(0) = 0$ and $g(x) < 0$ for $x > 0$. Set

$$M(g) \equiv \{ A : \text{tr}A \geq 0 \text{ and } \lambda_{\min}(A) \geq g(\text{tr}A) \}$$  \hfill (11.1)

**Proposition 11.2.** $M(g)$ is a subequation which is orthogonally invariant with

$$M(g) \cap \{ \text{tr}A = 0 \} = \{ 0 \} \quad \text{and} \quad \mathcal{P} - \{ 0 \} \subset \text{Int}M(g).$$  \hfill (11.2)

**Proof.** Since $g$ is continuous, $M(g)$ is a closed set. Recall that

$$\lambda_{\min}(A + B) \geq \lambda_{\min}(A) + \lambda_{\min}(B).$$  \hfill (11.3)

This combined with the fact that $g$ is decreasing easily implies that positivity (P) holds for $M(g)$. Obviously $M(g)$ is $O_n$-invariant.

If $A \in M(g)$ and $\text{tr}A = 0$, then since $g(0) = 0$, the minimum eigenvalue $\lambda_{\min}(A) \geq g(0) = 0$. But then $\text{tr}A = 0$ implies $A = 0$. 25
If $P \geq 0$ and $P \neq 0$, then $\text{tr}P > 0$. Since $x > 0$ implies $g(x) < 0$, we have $g(\text{tr}P) < 0$. Thus $\lambda_{\min}(P) > 0 > g(\text{tr}P)$ which implies that $P \in \text{Int}M(g)$, since $g$ is continuous. □

**Corollary 11.3.** The dual subequation $\widetilde{M}(g)$ is borderline.

**Proof.** The first part of (11.2) implies that $0 \in \partial M(g) = -\partial \widetilde{M}(g)$. Combined with the second part of (11.2), this is condition (1)$'$ in Lemma 3.1 for the subequation $F = \widetilde{M}(g)$, which proves that $M(g)$ is borderline. □

**Proposition 11.4.** The subequation $M(g)$ is additive, i.e., $M(g) + M(g) \subset M(g)$, if and only if $g$ is subadditive, i.e., $g(x+y) \leq g(x) + g(y)$.

**Proof.** Use (11.3) and $\text{tr}(A + B) = \text{tr}A + \text{tr}B$. □

If $g(x) \equiv -\delta x$ ($\delta > 0$), then $\widetilde{M}(g) \equiv P(\delta)$ is the convex cone subequation discussed in Appendix B. However, there are plenty of other subadditive decreasing functions $g$.

Suppose $g$ is concave on $[0,a]$ with $g(0) = 0$. Then, as noted in the introduction to [1], the extension of $g(x)$ from $[0,a]$ to $[0,\infty)$ defined by

$$g(x) \equiv ng(a) + g(x - na), \quad na \leq x \leq (n+1)a \quad (11.4)$$

is subadditive on $[0,\infty)$ and has the property that $g \geq h$ for any other subadditive function $h$ on $[0,\infty)$ which agrees with $g$ on $[0,a]$. The elementary proof is omitted. Summarizing, we have the following.

**Theorem 11.5.** Suppose that $g : [0,\infty) \to \mathbb{R}$ is the extension of a decreasing concave function on $[0,a]$ defined by (11.4) with $g(0) = 0$. Then $M(g)$ is a monotonicity subequation (orthogonally invariant), and its dual $\widetilde{M}(g)$ is borderline.

**Lemma 11.6.** The dual subequation $\widetilde{M}(g)$ is defined by

$$\widetilde{M}(g) : \text{tr}A \geq 0 \text{ or } \lambda_{\max}(A) \geq -g(-\text{tr}A) \quad \text{(with } \text{tr}A \leq 0)$$

**Proof.** Note that

$$A \in \widetilde{M}(g) \iff -A \notin \text{Int}M(g) \iff \lambda_{\min}(-A) \leq g(-\text{tr}A) \text{ or } \text{tr}(-A) \leq 0$$

$$\iff \lambda_{\max}(A) \geq -g(-\text{tr}A) \text{ or } \text{tr}(A) \geq 0,$$

since $\lambda_{\max}(A) = -\lambda_{\min}(-A)$. □

**Proposition 11.7.** The characteristic function $f$ for $\widetilde{M}(g)$ is $f(\lambda) = g^{-1}(-\lambda) + (n-1)\lambda$ for $\lambda \geq 0$.

**Proof.** The increasing radial profile of $\widetilde{M}(g)$ is by definition

$$\Lambda \equiv \{ (\lambda,\mu) : \lambda P_{e\perp} + \mu P_{e} \in \widetilde{M}(g) \text{ and } \lambda \geq 0 \}.$$
Note that $\text{tr}A = (n-1)\lambda + \mu$ if $A \equiv \lambda P_e^+ + \mu P_e$. If $\lambda \geq 0$ and $A \in \widetilde{M}(g)$ with $\text{tr}A \leq 0$, then
\[
\lambda \equiv \lambda_{\max} \geq 0, \quad \mu \leq 0, \quad \text{and hence} \quad \lambda \geq -g(-(n-1)\lambda - \mu)
\]
Set $x \equiv -(n-1)\lambda - \mu \geq 0$ and $y \equiv -\lambda \leq 0$. Then $y \leq g(x)$ is equivalent to $x \leq g^{-1}(y)$ since $g$ is decreasing and $g(0) = 0$. Thus $-(n-1)\lambda - \mu \leq g^{-1}(-\lambda)$, or $\mu + g^{-1}(-\lambda) + (n-1)\lambda \geq 0$. Since $f$ is defined by $\mu + f(\lambda) \geq 0$ for such pairs $(\lambda, \mu)$, this completes the proof. \hfill \Box

**Example 11.8.** Define $g : [0, a] \to [-b, 0]$ via its inverse by
\[
g^{-1}(-\lambda) \equiv \lambda(\alpha - 2\log \lambda) \quad 0 \leq \lambda \leq a.
\] (11.5)
Here $\alpha$ is a constant chosen first, and then $a$ is chosen small enough so that $h(\lambda) \equiv g^{-1}(-\lambda)$ is strictly increasing on $[0, a]$, and finally we set $-b = g(a)$. Note that $h'(\lambda) = \alpha - 2 - 2\log \lambda$. Also, $h''(\lambda) = -\frac{2}{\lambda} < 0$. Therefore $g$ is concave and strictly decreasing on $[0, a]$ with $g(0) = 0$. Applying Theorem 11.4 we see that
\[
M(g) \text{ is a monotonicity subequation whose dual } \widetilde{M}(g) \text{ is borderline.} \quad (11.6)
\]
By Proposition 11.7

The dual $\widetilde{M}(g)$ has characteristic function $f(\lambda) = \lambda(\alpha + n - 1 - 2\log \lambda)$ on $[0, a]$. \quad (11.7)

The indefinite integral of $1/f(y)$ is $-\frac{1}{2}\log(\alpha + n - 1 - 2\log y)$ and hence
\[
\int_{0^+} \frac{dy}{f(y)} = \infty. \quad (11.8)
\]
Therefore

The (SMP) holds for $\widetilde{M}(g)$, and the (SCP) holds for $M(g)$. \quad (11.9)

It is easy to see that $\widetilde{M}(g)$ is not contained in a uniformly elliptic subequation since $f(\lambda)/\lambda = \alpha + n - 1 - 2\log \lambda \to \infty$ as $\lambda \to \infty$.

**Example 11.9. (The Hopf Function).** Consider
\[
\psi(t) = e^{-\beta R^2/2} - e^{-\beta t^2/2} \quad (11.10)
\]

This function is increasing on $[0, \infty)$ and satisfies the radial subequation
\[
\psi'' + f \left( \frac{\psi'}{t} \right) = 0, \quad \text{where } f(\lambda) \equiv \lambda \left( \log \frac{\beta^2}{\lambda^2} - 1 \right).
\]
Hence, (by Section 6) $\psi(|x|)$ is an $F$-harmonic for any subequation with characteristic function $f$.

If $\log \beta^2 - 1 = \alpha + n - 1$ defines $\alpha$, then this $f$ is the same as the $f$ in (11.7) For $\beta$ large, $\psi'(t) \in [0, a]$ for all $t$, and hence in this case
\[
\psi(|x|) \equiv e^{-\beta R^2/2} - e^{-\beta|x|^2/2} \text{ is } \widetilde{M}(g) \text{ harmonic on } \mathbb{R}^n \quad (11.11)
\]
with $g$ defined by (11.5). This function $\psi(|x|)$ is also a harmonic for the subequations $F_{f_{\min/2}} \subset \widetilde{M}(g) \subset F_{f_{\min/\max}}$ described in Theorem 9.5.
Appendix A. Radial Subharmonics

Since our characterization of radial subharmonics is useful for many purposes, it is separated out in this appendix. Recall the characteristic lower function \( f \) associated with a subequation \( F \) and the radial subequation \( R_f \) defined by

\[
\psi'' + f \left( \frac{\psi'}{t} \right) \geq 0 \quad \text{on } 0 < t < \infty.
\]

In the following we drop the bar, letting \( f \) denote \( F \).

**THEOREM A.1. (Radial Subharmonics).** The function \( u(x) \equiv \psi(|x|) \) is \( F \)-subharmonic on an annular region in \( \mathbb{R}^n \) if and only if \( \psi(t) \) is \( R_f \)-subharmonic on the corresponding sub-interval of \( (0, \infty) \).

**Proof.** \((\Rightarrow):\) Suppose \( u(x) \equiv \psi(|x|) \) is \( F \)-subharmonic. If \( \varphi(t) \) is a test function for \( \psi(t) \) at \( t_0 \), then \( \varphi(|x|) \) is a test function for \( \psi(|x|) \) at any point on the \( t_0 \)-sphere in \( \mathbb{R}^n \). Therefore \( D^2_{x_0} \varphi \in F \). Applying the formula (Lemma 5.1) for \( D^2_{x_0} \varphi \) in terms of \( \varphi'(t_0) \) and \( \varphi''(t_0) \), the equivalence (5.1), and the definition of \( (R_F)_t \), we have \( J^2_{t_0} \varphi(t) \in R_F \). This proves that \( \psi(t) \) is \( R_F \)-subharmonic.

\((\Leftarrow):\) Suppose that \( \psi(t) \) is \( R_F \)-subharmonic. We must show that \( u(x) \equiv \psi(|x|) \) is \( F \)-subharmonic. That is, given a test function \( \varphi(x) \) for \( u(x) \) at a point \( x_0 \), we must show that \( D^2_{x_0} \varphi \in F \).

Suppose that there exists a smooth function \( \overline{\psi}(t) \), defined near \( t_0 = |x_0| \), such that \( \overline{\varphi}(x) \equiv \overline{\psi}(|x|) \) satisfies

\[
u(x) \leq \overline{\varphi}(x) \leq \varphi(x) \quad (A.1)
\]

near \( x_0 \). Then \( \overline{\psi}(t) \) is a test function for \( \psi(t) \) at \( t_0 \). Hence, the 2-jet of \( \overline{\psi} \) at \( t_0 \) belongs to \( R_F \). By Lemma 5.1 and the discussion above, this implies that \( D^2_{x_0} \overline{\varphi} \in F \). The inequality \( \overline{\varphi}(x) \leq \varphi(x) \) (with equality at \( x_0 \)) implies that \( D^2_{x_0} \varphi = D^2_{x_0} \overline{\varphi} + P \) for some \( P \geq 0 \), which proves that \( D^2_{x_0} \varphi \in F \) as desired.

To complete this argument by finding \( \overline{\psi}(t) \) there is some flexibility given by Lemma 2.4 in [7] so that not all test functions \( \varphi(x) \) need be considered. First we may choose new coordinates \( z = (t, y) \) near \( x_0 \) so that \( t \equiv |x| \). (Thus \( t = \text{constant} \) defines the sphere of radius \( t \) near \( x_0 \).) Furthermore, we may assume that \( \varphi(z) \) is a polynomial of degree \( \leq 2 \) in \( z = (t, y) \) and that it is a strict local test function, i.e., \( u(z) < \varphi(z) \) for \( z \neq z_0 \). Now Lemma A.2 below ensures the existence of \( \overline{\varphi}(x) = \overline{\psi}(|x|) \) satisfying (A.1).

Let \( z = (t, y) \) denote standard coordinates on \( \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^\ell \). Fix a point \( z_0 = (t_0, y_0) \) and let \( u(t) \) be an upper semi-continuous function (of \( t \) alone) and \( \varphi(z) \) a \( C^2 \)-function, both defined in a neighborhood of \( z_0 \).

**Lemma A.2.** Suppose \( u(t) < \varphi(z) \) for \( z \neq z_0 \) with equality at \( z_0 \). If \( \varphi(z) \) is a polynomial of degree \( \leq 2 \), then there exists a polynomial \( \overline{\varphi}(t) \) of degree \( \leq 2 \) with

\[
u(t) \leq \overline{\varphi}(t) \leq \varphi(z) \quad \text{near } z_0. \quad (A.2)
\]
Proof. We may assume \(z_0 = 0\) and \(u(0) = \varphi(0) = 0\). Then
\[
\varphi(z) = \langle p, t \rangle + \langle q, y \rangle + \langle At, t \rangle + 2\langle Bt, y \rangle + \langle Cy, y \rangle.
\]
We assume \(u(t) < \varphi(t, y)\) for \(|t| \leq \epsilon\) and \(|y| \leq \delta\) with \((t, y) \neq (0, 0)\).

Setting \(t = 0\), we have \(0 = u(0) < \langle q, y \rangle + \langle Cy, y \rangle\) for \(y \neq 0\) sufficiently small. Therefore, \(q = 0\) and \(C > 0\) (positive definite). Now define
\[
\overline{\varphi}(t) \equiv \langle p, t \rangle + \langle (A - B^tC^{-1}B)t, t \rangle.
\] (A.3)
The inequalities in (A.2) follow from the fact that for \(t\) sufficiently small,
\[
\overline{\varphi}(t) = \inf_{|y| \leq \delta} \varphi(z) = \langle p, t \rangle + \langle At, t \rangle + \inf_{|y| \leq \delta} \{2\langle Bt, y \rangle + \langle Cy, y \rangle\}.
\] (A.4)
To prove (A.4) fix \(t\) and consider the function \(2\langle Bt, y \rangle + \langle Cy, y \rangle\). Since \(C > 0\), it has a unique minimum point at the critical point \(y = -C^{-1}Bt\). The minimum value is \(-\langle B^tC^{-1}Bt, t \rangle\). If \(t\) is sufficiently small, the critical point \(y\) satisfies \(|y| < \delta\), which proves (A.4).

Appendix B. Uniform Ellipticity and Borderline

For cone subequations being borderline is closely related to uniform ellipticity. We add another set of conditions to the conditions (1), (1)', and (2), (2)' in Lemma 3.1.

Proposition B.1. A cone subequation \(F\) is borderline if one of the following equivalent conditions holds for \(F\).

\[(3)\quad F \subset G\] for some uniformly elliptic cone subequation \(G\).

\[(3)\quad \tilde{G} \subset \tilde{F}\] for some uniformly elliptic cone subequation \(\tilde{G}\).

Before proving this proposition we start with a general discussion of uniform ellipticity.

A family of convex cone subequations \(\{M_\delta\}\) is said to be a fundamental neighborhood system for \(\mathcal{P}\) if given any conical neighborhood \(G\) of \(\mathcal{P}\) (this means that \(\mathcal{P} - \{0\} \subset \text{Int}G\) and \(G\) is a cone), there exists \(\delta\) with \(M_\delta \subset G\). Given such a family \(\{M_\delta\}\), a subequation \(F\) is uniformly elliptic if there exists an \(M_\delta\) with
\[
F + M_\delta \subset F.
\]
This definition is easily seen to be independent of the choice of the neighborhood system \(\{M_\delta\}\) for \(\mathcal{P}\). (The monotonicity condition (B.1) can always be rephrased classically as two inequalities – see, for example, (4.5.1)' in [9]).

The standard choice made in the literature consists of the Pucci cones
\[
\mathcal{P}_{\lambda,\Lambda} \equiv \{A : \lambda \text{tr} A^+ + \Lambda \text{tr} A^- \geq 0\}
\]
with $0 < \lambda < \Lambda$, where $A = A^+ + A^-$ is the decomposition of $A$ into positive and negative parts. Another good choice is the $\delta$-uniformly elliptic regularization $\mathcal{P}(\delta)$ of $\mathcal{P}$

$$\mathcal{P}(\delta) \equiv \{ A : A + \delta(\text{tr} A)I \geq 0 \} \quad (\delta > 0).$$

Both $\mathcal{P}_{\lambda,\Lambda}$ and $\mathcal{P}(\delta)$ are convex cone subequations as required. See Section 4.5 of [8] for more details regarding $\mathcal{P}_{\lambda,\Lambda}$ and $\mathcal{P}(\delta)$ (The Riesz characteristics are computed in Example 6.2.5.)

**Lemma B.2.** The conditions (3), (3)$'$ and

$$4) \quad F \subset \mathcal{P}(\delta) \quad \text{for some} \quad \delta > 0.$$

are equivalent.

**Proof.** We note that $G$ is uniformly elliptic if and only if $\tilde{G}$ is. This follows from the fact that $G + M_\delta \subset G \iff \tilde{G} + M_\delta \subset \tilde{G}$ (cf. [8, Lemma 4.1.2]). Consequently, (3) $\iff$ (3)$'$.

Now assume (3). Since $\tilde{G}$ is uniformly elliptic and $0 \in \tilde{G}$, there exists $\delta > 0$ with $0 + \mathcal{P}(\delta) = \mathcal{P}(\delta) \subset \tilde{G}$. Thus $F \subset G \subset \mathcal{P}(\delta)$, which proves (4). Since $\mathcal{P}(\delta)$ is uniformly elliptic, (4) $\Rightarrow$ (3).

**Proof of Proposition B.1.** Because of Lemma 3.1 it suffices to prove that (4) $\Rightarrow$ (2)$'$ and (1)$'$ $\Rightarrow$ (4). If condition (4) (that $F \subset \mathcal{P}(\delta)$) holds, then $\mathcal{P}(\delta) \subset \tilde{F}$, and since $P_e \in \text{Int}\mathcal{P}(\delta)$, this implies that $P_e \in \text{Int}\tilde{F}$, that is, condition (2)$'$ holds. Finally, (1)$'$ says that $\text{Int}\tilde{F}$ is a conical neighborhood of $\mathcal{P}$, and hence there exists $\delta > 0$ with $\mathcal{P}(\delta) \subset \tilde{F}$. Thus (1)$'$ $\Rightarrow$ (4).

**Remark B.3.** For simplicity suppose that $f = \underline{f} = \overline{f}$ is the characteristic function for a cone subequation $F$. Then $f(t\lambda) = tf(\lambda)$ for $t > 0$, and hence the characteristic function reduces to two numerical invariants

$$\alpha \equiv f(1) \quad \text{and} \quad \alpha^* \equiv -f(-1), \quad 0 \leq \alpha, \alpha^* \leq \infty \quad (B.1)$$

where we have

$$f(\lambda) = \alpha \lambda \quad \text{for} \quad \lambda > 0 \quad \text{and} \quad f(\lambda) = \alpha^* \lambda \quad \text{for} \quad \lambda < 0. \quad (B.2)$$

The radial profile $\Lambda$ is defined by

$$\mu + \alpha \lambda \geq 0 \quad \text{if} \quad \lambda \geq 0 \quad \text{and} \quad \mu + \alpha^* \lambda \geq 0 \quad \text{if} \quad \lambda \leq 0. \quad (B.3)$$

Note that $\alpha = \infty \iff P_e \perp -\mu P_e \in F$ for all $\mu \iff -P_e \in F \iff F$ is not borderline. That is,

$$F \quad \text{satisfies the (SMP)} \iff \alpha \equiv \alpha_F < \infty. \quad (B.4)$$

The invariant $p_F \equiv \alpha_F + 1$ is called the **Riesz characteristic of $F$** because of its connection with Riesz kernels. See [9], [10] for applications, examples and a fuller discussion, where it is proved, in particular, that $\alpha \alpha^* \geq 1$. 

30
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