On globally static and stationary cosmologies with or without a cosmological constant and the dark energy problem

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Abstract
In the framework of spatially averaged inhomogeneous cosmologies in classical general relativity, effective Einstein equations govern the regional and the global dynamics of averaged scalar variables of cosmological models. A particular solution may be characterized by a cosmic equation of state. In this paper, it is pointed out that a globally static averaged dust model is conceivable without employing a compensating cosmological constant. Much in the spirit of Einstein’s original model we discuss consequences for the global, but also for the regional properties of this cosmology. We then consider the wider class of globally stationary cosmologies that are conceivable in the presented framework. All these models are based on exact solutions of the averaged Einstein equations and provide examples of cosmologies in an out-of-equilibrium state, which we characterize by an information-theoretical measure. It is shown that such cosmologies preserve high-magnitude kinematical fluctuations and so tend to maintain their global properties. The same is true for a \( \Lambda \)-driven cosmos in such a state despite exponential expansion. We outline relations to inflationary scenarios and put the dark energy problem into perspective. Here, it is argued, on the grounds of the discussed cosmologies, that a classical explanation of dark energy through backreaction effects is theoretically conceivable, if the matter-dominated universe emerged from a non-perturbative state in the vicinity of the stationary solution. We also discuss a number of caveats that furnish strong counter arguments in the framework of structure formation in a perturbed Friedmannian model.

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1. Introduction

The standard model of cosmology idealizes spatial sections in terms of constant curvature hypersurfaces, the matter and energy distributions being spatially constant. Friedmann’s and Lemaître’s solutions of Einstein’s equations are currently employed to describe the dynamics of the universe as a whole. Furthermore, the assumption is made that the spatially averaged inhomogeneous cosmos is described by a member of this family of solutions, a conjecture that can only be proved for Newtonian cosmologies [22, 31], but certainly corresponds to a restricted choice in general relativity.

Friedmann–Lemaître cosmologies describe a time-dependent, locally isotropic and, hence, on a simply connected domain homogeneous and isotropic expansion or contraction, respectively. Denoting the scale factor by $a(t)$, which can be defined through the volume of an arbitrary comoving domain within the space sections, $a = (V/V(t_i))^{1/3}$, normalized by the volume at some initial time $t_i$, we obtain from Einstein’s equations (restricted throughout this paper to a dust continuum) the well-known ‘acceleration law’

$$\frac{\dot{a}}{a} + \frac{4\pi G \varrho_H}{3} - \frac{\Lambda}{3} = 0,$$

with the homogeneous rest mass density $\varrho_H$ and the cosmological constant $\Lambda$. Its first integral yields a global expansion law, Friedmann’s differential equation,

$$\frac{\dot{a}^2}{a^2} - \frac{8\pi G \varrho_H}{3} + \frac{k}{a^2} - \frac{\Lambda}{3} = 0; \quad \varrho_H = \frac{\varrho_H(t_i)}{a^3} = \frac{M}{V(t_i)a^3},$$

with the total conserved rest mass $M$ enclosed within the arbitrary domain and an integration constant $k$ that is related to the spatially constant Ricci curvature $R_H$ of the space sections by $R_H = 6k/a^2$; $k := R_H(t_i)/6$.\footnote{In Friedmannian cosmology it is a common practice to normalize the curvature parameter to $k := 0, \pm 1$ by a suitable rescaling of the expansion factor $a(t)$ (then acquiring the dimension of time for units with $c = 1$). Although we formulated the equations for a dimensionless scale factor (to be consistent with the general equations to be discussed below), all numerical estimates in this section and especially in appendix A will be done with the implicit understanding that $k$ is normalized and $a$ has dimension of time.}

Throughout this paper we shall call a particular solution of Friedmann’s differential equation (2), a Friedmannian model or a Hubble flow. The only possible static Friedmannian model, $a(t) = a_E, a_E = \text{const}$, which has motivated Einstein [33] to introduce the (purely parametric) cosmological term $\Lambda$, follows from (2) with the assumption that $\dot{a} = 0$ in some finite time interval, and therefore also $\ddot{a} = 0$, and from equations (1) and (2):

$$4\pi G \varrho_E = \Lambda; \quad \varrho_E := \frac{\varrho_H(t_i)}{a_E^3} = \text{const};$$

$$\frac{k}{a_E^2} - \frac{8\pi G \varrho_E}{3} - \frac{\Lambda}{3} = 0.$$

Combining these two equations we obtain

$$\frac{k}{a_E^2} = \Lambda; \quad a_E = \frac{k}{\sqrt{4\pi G \varrho_E}},$$

and since $\varrho_E > 0$, $\Lambda$ has to be positive and hence also the curvature parameter $k$. Globally, we may introduce the total rest mass in the Einstein cosmos $M_E = \varrho_E V_E$ with its total volume $V_E := a_E^3 V$ and the global scalar curvature $R_E := 6k/a_E^2$. Since the curvature is spatially constant and positive, the volume can be calculated in terms of spherical space. In space units (adopting now the normalization of the curvature parameter to $k = +1$), the Riemannian...
volume of the Einstein cosmos is $V_E = 2\pi^2 c^3 a_E^3$, i.e., larger than the volume of a Euclidean sphere with the same radius $4\pi/3c^3 a_E^3$. However, note that a mass-preserving smoothing of the Einstein radius into a Euclidean geometry yields a corresponding Euclidean volume $4\pi/3c^3 a_E^3$ that is larger than the total volume of the spherical space of Einstein’s cosmos. Since $\rho_E$ is constant, we may rewrite the above equations as follows:

$$4\pi G M_E + \frac{R_E^2}{2} + 8\pi G M_E + \Lambda V_E = 0.$$  

(6)

$\Lambda V_E$ may be interpreted as the total dark energy in this model.

Einstein’s model requires a non-vanishing and positive cosmological constant to ‘balance’ the total rest mass content of the universe exactly (including the radiation density and pressure, which we consider here as being negligible in the matter-dominated era). Note that, by including pressure, there also exists a particular static model with $\Lambda = 0$ and equation of state $p_H = -1/3\rho_H$ ([50], p 383). The beauty of a closed spherical space, as emphasized by Eddington ([29], chapter II), is accompanied by its definite predictions, e.g., for known rest mass density we can determine its size as a strong boundary condition for any further studies.

To illustrate this we calculate in appendix A the Einstein radius by extrapolating the values of the cosmological parameters, as fitted to the Friedmannian model on the scale of our Hubble volume. Such estimates are rather naive as will become clear later, and we shall come back to this discussion within the more general setting of an inhomogeneous globally static cosmos.

Soon after the time when the Einstein static model was suggested, the observed redshifts of galaxies together with their interpretation as Doppler velocities indicated that the space defined by the galaxies in our environment is expanding, which led to the abandonment of Einstein’s model. This was actually a hasty decision, based on the restricted view that the global model by Einstein was asked to describe any patch of the universe, however small it was. These early discussions were based on the observational situation at the time which, following Hubble’s assessment [53], indicates that the seemingly uniform distribution of galaxies may already represent a ‘fair sample’ of the universe. As a consequence, these discussions were meant in a global sense: the observed—according to contemporary standards very small—patch of the universe was considered a representative for the whole. Actually, Eddington [30] already pointed out that the observed expansion might be a regional property of the universe rather than a global one; he said ‘that it is possible that the recession of the spirals is not the expansion theoretically predicted; it might be some local peculiarity masking a much smaller genuine expansion; but the temptation to identify the observed and the predicted expansions is very strong’.

Before the Einstein cosmos was disregarded as a reasonable description of the universe, there were many discussions following Einstein’s in 1917: Dingle [28] and Tolman [81, 82] have pointed out that the Einstein cosmos will soon develop into a highly irregular universe. This discussion is still referred to as the ‘instability of the Einstein cosmos’. Eddington [30] argued that the universe may evolve starting out from the Einstein cosmos, however, due to its instability, will start expanding or contracting, respectively. As an alternative, Lemaître [59, 60] advocated the singular ‘big bang solutions’, which expand until the matter density has dropped below the cosmological term which henceforth dominates, resulting in an accelerating phase thereafter (see also [69], section 3C for a review of these discussions). The nowadays favoured concordance model (cf appendix A), featuring a positive cosmological constant, describes such an evolution. We shall see that the ideas advanced at that time apply to the picture developed below, but their interpretation will be very different. In contrast to the historical flaw of realizing the instability of the Einstein cosmos, we nowadays view such instabilities as the origin of large-scale structure. All homogeneous world models are unstable, including the Einstein cosmos. As we shall discuss in detail below, the global instability
argument in the above form (i.e., within the class of homogeneous isotropic models) does not apply, when the cosmology acquires the status of describing the average dynamics on the largest scales.

Here, a disclaimer is in order: to advance a globally static, but regionally fluctuating cosmos as a viable model that could explain current observational results is premature. Instead, we revisit the ideas which led to the Einstein cosmos (the introduction of the cosmological constant) in light of a new framework and on the grounds of an ongoing discussion of the possibility that dark energy may be explained through ‘backreaction effects’ of structure formation. Globally expanding, stationary cosmologies are also conceivable in this framework. It was pointed out in [18] that the question whether the magnitude of backreaction effects is sufficient to explain acceleration of the observable universe and the question of whether a globally static or stationary cosmos bears physical justifications beyond a mere mathematical possibility are synonyms.

A thorough physical investigation of fluctuation-supported static and stationary cosmologies must be based on more general matter models (as a next step perfect fluids and scalar fields) that allow us to study dynamical scenarios of inflation and their exit details including the effect of radiation pressure. The corresponding effective equations are given in [16]. In this respect, the present investigation based on the effective equations for a dust continuum [15] provides a useful showcase for the presentation of the basic ideas. The present investigation of particular exact solutions offers more insight into the general formalism of averaged inhomogeneous cosmologies, and at the same time proposes new families of cosmologies that enjoy significantly more freedom than a rigid Friedmannian cosmology. Furthermore, by applying these ideas to inflationary scenarios, we can understand the relevance of matter and curvature fluctuations for the description of the early universe, in particular, the importance of the role played by a non-vanishing averaged scalar curvature.

2. Effective equations for inhomogeneous universe models

2.1. Averaged equations

For the sake of transparency we shall restrict all considerations in this paper to an irrotational dust continuum and recall a set of effective equations provided in [15]. The ideas presented can be carried over to studies of inhomogeneous cosmologies covering the early universe and radiation-dominated epochs with the help of the more general effective equations developed in [16].

Given a foliation of spacetime into flow-orthogonal hypersurfaces (which is possible for irrotational dust) with the 3-metric $g_{ij}$ in the line element $ds^2 = -dt^2 + g_{ij}dX^i dX^j$, spatial averaging of any scalar field $\Psi$ is a covariant operation and is defined by the simple averager

$$\langle \Psi(t, X') \rangle_D := \frac{1}{V_D} \int_D J d^3 X \Psi(t, X'),$$  \hspace{1cm} (7)$$with $J := \sqrt{\det(g_{ij})}$; $g_{ij}$ is the metric of the spatial hypersurfaces and $X^i$ are coordinates that are constant along flow lines, which are here spacetime geodesics. Following [15] we define an effective scale factor by the volume of a simply connected domain $D$ in a $t$-hypersurface, normalized by the volume of the initial domain $D_i$,

$$a_D := \left( \frac{V_D}{V_{D_i}} \right)^{1/3}. \hspace{1cm} (8)$$
We recall the fact that, for a rest mass preserving domain $D$, volume averaging of a scalar function $\Psi$ does not commute with its time evolution:

$$\langle \theta, \Psi \rangle_D - \dot{\theta} \langle \Psi \rangle_D = (\Psi |_D) \langle \theta \rangle_D - \langle \Psi \theta \rangle_D,$$

where $\theta$ denotes the rate of expansion. Setting $\Psi \equiv \rho$ we obtain a regional continuity equation reflecting the conservation of the total rest mass $M_D$ within $D$:

$$\dot{\theta} M_D = 0 \iff \dot{\theta} \langle \rho \rangle_D + \langle \theta \rangle_D \langle \rho \rangle_D = 0. \quad (10)$$

Setting $\Psi \equiv \theta$ we can derive an effective equation for the spatially averaged expansion of the model

$$\langle \theta \rangle_D = \frac{\dot{\theta}}{\dot{a}/a} = \frac{3}{3 \dot{a}/a} = 3 H_D,$$

where we defined an effective Hubble functional on $D$ (an overdot denotes partial time derivative; inserting Raychaudhuri’s evolution equation, $\dot{\theta} = \Lambda - 4 \pi G \rho - \frac{1}{2} \dot{\theta}^2 - 2 \sigma^2$ (with the rate of shear $\sigma^2 = \frac{1}{2} \Theta_{ij} \Theta^{ij}$), into (9) and using the effective scale factor $a_D$ we obtain

$$\frac{\dot{a}_D}{a_D} + 4 \pi G \frac{M_D}{V a_D^3} - \Lambda = \mathcal{Q}_D. \quad (12)$$

The first integral of the above equation is directly given by averaging the Hamiltonian constraint:

$$\left( \frac{\dot{a}_D}{a_D} \right)^2 - \frac{8 \pi G M_D}{3 V a_D^3} + \frac{\langle R \rangle_D}{6} - \frac{\Lambda}{3} = - \frac{\mathcal{Q}_D}{6}. \quad (13)$$

where the total rest mass $M_D$, the averaged spatial Ricci scalar $\langle R \rangle_D$ and the kinematical backreaction term $\mathcal{Q}_D$ are domain dependent and, except for the mass, time-dependent functions. The backreaction source term is given by

$$\mathcal{Q}_D := 2 \langle tI \rangle_D - \frac{1}{2} \langle tI \rangle_D^2 = \frac{1}{2} \left( \langle \theta \rangle_D^2 - \langle \theta \rangle_D^2 \right) - 2 \langle \sigma^2 \rangle_D; \quad (14)$$

here, $I = \Theta^{ij}$ and $II = \frac{1}{2} \left( \langle \Theta^{ij} \rangle^2 - \langle \Theta^{ij} \rangle \langle \Theta^{ij} \rangle \right)$ denote the principal scalar invariants of the expansion tensor, defined as minus the extrinsic curvature tensor $K_{ij} := \Theta_{ij}$. In the second equality above it was split into kinematical invariants through the decomposition $\Theta_{ij} = \frac{1}{2} g_{ij} \theta + \sigma_{ij}$, with the rate of expansion $\theta = \Theta^{ij}$ and the shear tensor $\sigma_{ij}$. (Note that vorticity is absent in the present model; we adopt the summation convention.)

The time derivative of the averaged Hamiltonian constraint (13) agrees with the averaged Raychaudhuri equation (12) by virtue of the following integrability condition:

$$\dot{a} \mathcal{Q}_D + 6 \frac{\dot{a}_D}{a_D} \mathcal{Q}_D + \dot{a} \langle \mathcal{R} \rangle_D + 2 \frac{\dot{a}_D}{a_D} \langle \mathcal{R} \rangle_D = 0,$$

which we may write in the more compact form:

$$\frac{1}{a_D^2} \dot{a}_D \left( \mathcal{Q}_D a_D^2 \right) + \frac{1}{a_D} \dot{a}_D \left( \langle \mathcal{R} \rangle_D a_D^2 \right) = 0. \quad (15)$$

Formally integrating this condition yields

$$\frac{k_D}{a_D^2} = \frac{1}{3 a_D^2} \int_{t_i}^{t_f} dt' \mathcal{Q}_D \frac{d}{dt} a_D^2(t') = \frac{1}{6} \left( \langle \mathcal{R} \rangle_D + \mathcal{Q}_D \right), \quad (17)$$

i.e., besides the total material mass $M_D$ we have a further integral of motion given by the domain-dependent integration constant $k_D$ that we may also write as follows:

$$6k_D = Y_D + 2 \int_{t_i}^{t_f} dt' \mathcal{Q}_D \frac{d}{dt} a_D^2(t') = \mathcal{Q}_D a_D^2. \quad (18)$$
In the above equation we introduced the functional $Y_D$, which is a special case of the Yamabe functional; here $Y_D := \langle R \rangle_D V_D^{2/3}$ in three dimensions (see, e.g., the case $n = 3; \phi = \psi = 1$ in [4], p 150), which itself is an integral of motion for vanishing $Q_D$.\(^2\) Equation (16), having no Newtonian analogue, shows that the averaged intrinsic curvature and the averaged extrinsic curvature (encoded in the backreaction term) are dynamically coupled. Stating this genuinely relativistic property, we also note the surprising fact that inserting (17) into (13) results in an equation that is formally equivalent to its Newtonian counterpart [22]:

$$\frac{\ddot{a}_D + k_D}{a_D^2} = \frac{8\pi G}{3} \left( \frac{\dot{a}_D}{a_D^2} \right) - \frac{\Lambda}{3} = \frac{1}{3a_D^2} \int_0^t \frac{d}{dt'} Q_D^D \frac{d}{dt'} a_D^2(t').$$

(19)

The effective scale factor obeys the same equation as in Newtonian theory similar to the situation known for the homogeneous isotropic case. Note that these effective equations also cover anisotropic inhomogeneous cosmologies ([22], appendix B).

### 2.2. The cosmic quartet

For the purpose of comparing the model variables with observations it is comfortable to introduce dimensionless average characteristics as follows (in contrast to [15] we use the notation $\Omega^D_R$ for the curvature functional):

$$\Omega^D_m := \frac{8\pi GM_D}{3V_D a_D^3 H_D^2}; \quad \Omega^D_\Lambda := \frac{\Lambda}{3H_D^2}; \quad \Omega^D_R := -\frac{\langle R \rangle_D}{6H_D^2}; \quad \Omega^D_Q := -\frac{Q_D}{6H_D^2},$$

(20)

where we have employed the effective Hubble functional $H_D$ (11) that reduces to Hubble’s function in the homogeneous isotropic case. With these definitions the Hamiltonian constraint is written in the iconized form of a cosmic quartet:

$$\Omega^D_m + \Omega^D_\Lambda + \Omega^D_R + \Omega^D_Q = 1.$$  

(21)

These functionals, being scale dependent, are dynamically related in a complex way\(^3\), unlike the situation in a Friedmannian model that features a global cosmic triangle [7] where the cosmological parameters interact trivially, e.g., for an Einstein–de Sitter cosmology (vanishing curvature parameter and vanishing $\Lambda$) the curvature parameter remains zero throughout the entire evolution.

As in the Friedmannian cosmology we may also define other functionals, such as for example, an effective deceleration parameter,

$$q_{\text{eff}}^D := -\frac{\ddot{a}_D}{a_D H_D^2} = \frac{1}{2} \Omega_m^D - \Omega_\Lambda^D + 2\Omega_Q^D, \quad \text{(22)}$$

as well as corresponding effective parameters for the third derivative of $a_D$ like the recently introduced state finders [1] (see also [41] and references therein).

We also note the useful general evolution equation (a slightly different version is given in [15], appendix B)

$$\frac{1}{a_D^2} \ddot{a}_D \left( \Omega_Q^D a_D^6 \right) + \frac{1}{a_D^2} \dot{a}_D \left( \Omega_Q^D a_D^3 \right) - 3H_D \left( \Omega_Q^D + \Omega_R^D \right) \left( \Omega_m^D + \frac{2}{3} \Omega_R^D + 2\Omega_Q^D \right) = 0.$$  

(23)

\(^2\) The vanishing of $Q_D$ on every scale is necessary and sufficient for $a_D$ to be a (global) solution of Friedmann’s differential equation. Therefore, a non-vanishing $Q_D$ plays a key role and justifies the name ‘kinematical backreaction’.

\(^3\) The special case where $Q_D \propto a_D^3$, which is briefly discussed at the end of this section.
A clarifying remark concerning the definition of the 'curvature parameter' is in order. In the corresponding Newtonian problem the following curvature and kinematical backreaction functionals have been introduced [23]:

\[ \Omega_k^D := -\frac{k_D}{a_D^2 H_D^2}; \quad \Omega_{QN}^D := \frac{1}{3 a_D^2 H_D^2} \int_t^{t'} dt' Q_D \frac{d}{dt'} a_D^2 (t'). \] (24)

In view of equation (19) we may use the same functionals in place of those introduced above also in general relativity. However, the physical averaged curvature is not associated with \( \Omega_k^D \). The functionals (24) are related to the previously introduced ones by \( \Omega_k^D + \Omega_{QN}^D = \Omega_R^D + \Omega_Q^D \).

2.3. The cosmic equation of state

The above equations can formally be recast into standard zero-curvature Friedmann equations with new effective sources [16]4:

\[ \rho_D^{\text{eff}} = \langle \rho \rangle_D - \frac{1}{16 \pi G} \frac{\langle \mathcal{Q} \rangle_D}{a_D^4} - \frac{1}{16 \pi G} \langle \mathcal{R} \rangle_D; \quad p_D^{\text{eff}} = -\frac{1}{16 \pi G} Q_D + \frac{1}{48 \pi G} \langle \mathcal{R} \rangle_D. \] (25)

\[ 3 \frac{d^2}{dD} = \Lambda - 4 \pi G (\rho_D^{\text{eff}} + 3 p_D^{\text{eff}}); \quad 3 H_D^2 = \Lambda + 8 \pi G \rho_D^{\text{eff}}; \]

\[ \dot{\rho}_D^{\text{eff}} + 3 H_D (\rho_D^{\text{eff}} + p_D^{\text{eff}}) = 0. \] (26)

Equations (26) correspond to equations (12), (13) and (16), respectively. In these equations we have translated all that has been said before into a Friedmannian setting and a specific form of the fluctuating sources. Note that the kinematical backreaction term \( Q_D \) itself obeys a stiff equation of state mimicking a dilatonic field in the fluid analogy (for further implications see [16] and section 3.6).

Given an equation of state of the form \( p_D^{\text{eff}} = \beta (\rho_D^{\text{eff}}, a_D) \) that relates the effective sources (25) to a possible explicit dependence on the effective scale factor, the effective Friedmann equations (26) can be solved (one of equations (26) is redundant). Therefore, any question posed that is related to the evolution of scalar characteristics of inhomogeneous universe models may be ‘reduced’ to finding the cosmic state on a given spatial scale. (Note, however, that an equation of state must not exist in general.) Although formally similar to the situation in Friedmannian cosmology, here the equation of state is dynamical and depends on details of the evolution of inhomogeneities. In general, it describes non-equilibrium states.

An example may illustrate the cosmic equation of state: we look at a particular exact solution of the averaged Einstein equations that was given in [15], appendix B. From the integrability condition (16) we directly infer that the pair of solutions,

\[ Q_D = \frac{Q_D (t_i)}{a_D^4}; \quad \langle \mathcal{R} \rangle_D = \frac{\langle \mathcal{R} \rangle_D (t_i)}{a_D^2}, \] (27)

provides a special solution for which the averaged scalar curvature and the kinematical backreaction term decouple and evolve independently. The regional cosmic equation of state corresponding to this solution can be derived from (25)

\[ \rho_D^{\text{eff}} = \omega_D^{\text{eff}} = \frac{1 - \frac{1}{3} \gamma_1 a_D^2}{1 + \gamma_1 a_D^2 - \gamma_2 a_D^2}, \] (28)

4 Note that in this representation of the effective equations \( p_{\text{eff}} \) just denotes a formal 'pressure'. In the perfect fluid case with an inhomogeneous pressure function the foliation has to be differently chosen (in general the inhomogeneous lapse function is set equal to 1 here), and there is a further averaged pressure gradient term [16].
3.1. Globally static cosmos without a cosmological constant

Asymptotically, for an expanding domain \(|a| \) large, \( \gamma_2 \) tends to \(-\frac{1}{2} \), a property that is also shared by a non-flat Friedmannian domain for which we obtain \( u_{\text{Friedmann}}^{\text{eff}} = -\frac{1}{2}(1 + \gamma^2/a^2) \) with \( \gamma_3 := \Omega_m^D(t_i)/\Omega_Q^D(t_i) \). These Friedmannian models are subcases of the above inhomogeneous solution (\( \Omega_m^D(t_i) \neq 0; \Omega_Q^D(t_i) = 0 \)); \( u_{\text{Friedmann}}^{\text{eff}} \) follows from (28) by first multiplying with \( \Omega_m^D(t_i) \neq 0 \) (it is understood that the scale factor and all other quantities then no longer depend on \( D \)).

We end this section with a note on the definition of the effective sources in equation (25). There is some ambiguity in defining them. First, it may sometimes be useful to incorporate the global kinematical backreaction into the effective sources by defining \( \varrho_{\text{eff}, \Lambda}^D := \varrho_{\text{eff}}^D - \Lambda/8\pi G \) and \( p_{\text{eff}, \Lambda}^D := p_{\text{eff}}^D + \Lambda/8\pi G \). Second, we might add the ‘constant curvature term’ \( 3k_D/a^2_D \) to the left-hand side of the second equation in (26); if we wish to do so, then the effective sources can be represented solely through the kinematical backreaction term \( Q_D \) and its time integral. For this we have to exploit the ‘Newtonian form’, equation (19), and would have to define the effective sources as follows:

\[
\begin{align*}
\hat{\varrho}_{\text{eff}}^D := \langle \varrho \rangle_D + \frac{X_D}{16\pi G}, \\
\hat{p}_{\text{eff}}^D := -\frac{Q_D}{12\pi G} - \frac{X_D}{48\pi G}, \\
X_D := \frac{1}{a_D^2} \int_a^{t'} dt' Q_D \frac{d}{dt} a_D^2(t').
\end{align*}
\]

The integrated form of the integrability condition, equation (17), then allows \( X_D \) to be expressed again through the averaged scalar curvature, \( X_D = 6k_D - Q_D - \langle \mathcal{R} \rangle_D \), and we obtain the sources corresponding to (25), however, with a curvature source that captures the deviations from a constant curvature model:

\[
\begin{align*}
\hat{\varrho}_{\text{eff}}^D &= \langle \varrho \rangle_D - \frac{Q_D}{16\pi G} - \frac{\langle \mathcal{R} \rangle_D - 6k_D/a_D^2}{16\pi G}, \\
\hat{p}_{\text{eff}}^D &= -\frac{Q_D}{12\pi G} + \frac{\langle \mathcal{R} \rangle_D - 6k_D/a_D^2}{48\pi G}.
\end{align*}
\]

In the following section we present new families of exact solutions of the effective Einstein equations. While models can be obtained on any chosen simply connected spatial domain \( D \), most of the following solutions are global and the spatial domain \( D \) is extended to the whole Riemannian manifold \( \Sigma \), which we assume to be compact. (Note that regional solutions such as the example presented in equation (27) ‘appear’ to only depend on the matter distribution inside \( D \), but this is not the case, since the initial data are to be constructed non-locally from the whole distribution in \( \Sigma \); global solutions and their parameters therefore have a more robust status.)

3. Globally stationary effective universe models

3.1. Globally static cosmos without a cosmological constant

On the global scale we first require the effective scale factor \( a_{\Sigma} \) to be constant on some time interval; hence \( \dot{a}_{\Sigma} = \ddot{a}_{\Sigma} = 0 \) and equations (12) and (13) may be written in the form

\[
\begin{align*}
\langle \varrho \rangle_{\Sigma} &= 4\pi G \frac{M_{\Sigma}}{V_{a_{\Sigma}}} - \Lambda, \\
\langle \mathcal{R} \rangle_{\Sigma} &= 12\pi G \frac{M_{\Sigma}}{V_{a_{\Sigma}}} + 3\Lambda,
\end{align*}
\]

with the global kinematical backreaction \( Q_{\Sigma} \), the globally averaged 3-Ricci curvature \( \langle \mathcal{R} \rangle_{\Sigma} \) and the total rest mass \( M_{\Sigma} \) contained in \( \Sigma \).
Let us now consider the case of a vanishing cosmological constant, $\Lambda = 0$. The averaged scalar curvature is, for a non-empty universe, always positive, and the balance condition (31) replaces (3), while condition (32) replaces (4). Obviously, backreaction (31) and averaged scalar curvature (32) trivially satisfy the integrability condition (16). Thus, in view of (25), the globally static inhomogeneous cosmos without a cosmological constant is characterized by the cosmic equation of state:

$$\langle R \rangle = 3Q_S = \text{const} \Rightarrow p_S = \rho_S = 0.$$  \hspace{1cm} (33)

3.2. Interlude: kinematical backreaction as a cosmological constant

We first note that, apparently, in equation (12) $Q_D$ plays the role of a positive cosmological constant; however, in equation (13) $Q_S$ has the opposite sign. That the kinematical backreaction term may take the role of the cosmological constant has been suggested in [17] and discussed in connection with the backreaction problem in [23] as well as in the recent discussion on dark energy and backreaction [56, 75]. The above remark shows that caution is in order with a direct identification.

Of course, we may force the kinematical backreaction term to take exactly the role of the cosmological constant. Neglecting $\Lambda$ in the general equation (12), $Q_D$ may be regarded as a (in this equation possibly time-dependent) cosmological term. Then, from equation (13) a constraint equation on $Q_D$ follows, if we identify all sources that imply a deviation from Friedmann’s equation with the cosmological term; the correct requirement can be inferred from equation (19) and reads

$$\frac{2}{a_D^2} \int_t^t' dt' Q_D \frac{d}{dt'} a_D^2(t') \equiv Q_D,$$  \hspace{1cm} (34)

which implies $Q_D = Q_D(t_i) = \text{const}$ as the only possible solution. Physically, it appears contrived to freeze fluctuations to a constant value in an evolving model. However, it is noteworthy that $Q_D$ indeed is required to be constant, if we force it to behave like $\Lambda$. Equation (17) then implies for the averaged curvature

$$\langle R \rangle = \frac{6k_D}{a_D^2} - 3Q_D(t_i).$$  \hspace{1cm} (35)

The solution in the case $k_D = 0$ has been noticed in [56] too; however, freezing also the averaged curvature in an evolving model appears to be even more contrived. A different interpretation in the framework of an effective scalar field may imply a more meaningful interpretation of this solution (see section 3.6). We finally write the cosmic equation of state for this special (regional) solution

$$\frac{p_D^0}{\rho_D^0} = -\frac{Q_D(t_i) - k_D/a_D^2}{8\pi G \langle \varrho \rangle_D(t_i)/a_D^2 + Q_D(t_i) - 3k_D/a_D^2},$$  \hspace{1cm} (36)

which, for large $|a_D|$, approaches the equation of state $p_D^0 = -\rho_D^0$.

3.3. Local instability versus global stability

The problem of instability of the Einstein cosmos, as outlined in the introduction, must be thought of in two ways. (i) A static homogeneous cosmos is unstable within the class of Friedmann–Lemaître cosmologies, since for a small change in density the balance between the density and the cosmological constant in equation (3) is destroyed, leading to acceleration if $4\pi G \rho_H(t_i) > \Lambda$, and to deceleration otherwise. Suppose we take a slightly smaller
density, then equation (4) implies that $H(t) > 0$, and the model starts to expand; $\Lambda$ (being constant in time) cannot dynamically compensate for this expansion, which itself decreases the density further, and so gives rise to the instability. (ii) A homogeneous cosmos is unstable to inhomogeneous density perturbations. Such perturbations are amplified as a consequence of the attractive nature of the gravitational field tending to increase overdensities and to decrease underdensities. This is the content of local gravitational instability: inhomogeneities are amplified. The latter instability also applies to the other Friedmann–Lemaître cosmologies. Both types of scalar instabilities (and also vectorial and tensorial perturbations) have been recently clarified and detailed by Barrow et al [8] for the Einstein static universe model containing a perfect fluid, generalizing earlier works by Harrison [49] and Gibbons [46, 47]. Further insight was added by Losic and Unruh [63] who investigated the stability analysis to second order in scalar and metric fluctuations. A related analysis concerning the dynamical phase space as well as attractor or repellor properties of homogeneous solutions may be found in [32, 78].

Now, let us look at the same type of perturbations in the framework of the globally static, but inhomogeneous cosmos. Altering the density source would equally disturb the balance by virtue of equation (31), but it would not necessarily destroy it. The reason is that the kinematical backreaction term $Q_\Sigma$ indeed reacts back on this perturbation (justifying its name). For, unlike $\Lambda$, $Q_\Sigma$ can acquire a time dependence and is then dynamically coupled to the averaged scalar curvature $(\langle R \rangle)_\Sigma$ (in general through the integrability condition (16) and in particular, for a stationary cosmos, through equation (40)): as soon as the density is perturbed, taking, for example, a slightly lower average density as above, the (positive) curvature starts to evolve such that $\partial_t \left( \langle R \rangle_\Sigma V^2 / 3 \right) < 0$; in turn, $\partial_t \left( Q_\Sigma V^2 \right) > 0$, and, thus, fluctuations tend to decrease less rapidly. As we shall see in an exact solution below, the coupling to the averaged scalar curvature can be strong so that $Q_\Sigma$ decreases in proportion to the inverse volume similar to the density, not (as expected for fluctuations) in proportion to the square of the inverse volume. It does not matter if we consider homogeneous or inhomogeneous perturbations (as far as scalar perturbations are concerned), since the effective equations (12) and (13) govern both and are not narrowed to the class of homogeneous isotropic solutions as in the case of the standard Einstein cosmos. In other words, a perturbed effective cosmology stays within the same class of cosmologies governed by the effective equations, since the latter are general.

An interesting future task will be to analyse in detail—along the lines of a stability analysis of the homogeneous cosmos with symmetry [8, 63]—whether kinematical backreaction tends to stabilize perturbations of the dynamical balance between the globally averaged density source $4\pi G \langle \rho \rangle_\Sigma$ and the global kinematical backreaction term $Q_\Sigma$ due to its coupling to the averaged 3-Ricci curvature $(\langle R \rangle)_\Sigma$. In any case, we do not expect that a generic cosmology would dynamically approach this state for any initial setting. In particular, a perturbed Friedmannian state may not approach a globally static state, rather the question is whether perturbations of an already established balance would destabilize this state. This calls for an investigation of perturbation theory of the global out-of-equilibrium state, rather than of a Friedmannian state.

3.4. Globally stationary effective cosmologies

Suppose that the universe indeed is hovering around a non-accelerating state on the largest scales. Still, the effective static cosmos discussed above may appear as a quite rigid model. We may think that a more natural condition would be stationarity. Indeed, at first sight the balance condition, if attained, does not necessarily imply that the model is static. We may look for a wider class of models that balances the fluctuations and the averaged sources by introducing
globally stationary effective cosmologies. The vanishing of the second time derivative of the scale factor would only imply \( \dot{a}/a \Sigma_1 = \text{const} = C \), i.e., \( a_\Sigma = a_\Sigma + C(t - t_i) \), where the integration constant \( a_\Sigma \) is generically non-zero, e.g., the model may emerge \cite{38, 39} from a globally static cosmos, \( a_\Sigma := 1 \), or from a "big bang", if \( a_\Sigma \) is set to zero. (In Friedmannian cosmology such an expansion law would correspond to a "curvature-dominated" model, since \( C^2 + k \approx 0 \) for negligible sources, hence resulting in a Hubble expansion that is determined by a constant negative curvature.)

On the global scale and for a stationary cosmos, equations (12) and (13) read

\[
\frac{Q}{a_\Sigma} = 4\pi G \frac{M}{V a_\Sigma^3} - \Lambda, \tag{37}
\]

\[
\frac{\langle R \rangle}{a_\Sigma} = 12\pi G \frac{M}{V a_\Sigma^3} + 3\Lambda - 6H^2, \quad H = \frac{C}{a_\Sigma}, \tag{38}
\]

By inserting (37) into (38), we can evaluate the constant \( C \) by looking at the resulting equation at initial time; for the normalization \( a/\Sigma_1(t_i) = 1 \) we get

\[
6C^2 = 6\Lambda + 3Q/\Sigma_1(t_i) - \langle R \rangle/\Sigma_1(t_i). \tag{39}
\]

We are now going to discuss the stationary cosmologies more explicitly by deriving an exact solution to the effective Einstein equations. (One can easily show from (2) that a Friedmannian cosmology does not allow for a stationary cosmos, since \( C^2 + k = \Lambda a^2 \) only allows for \( a = \text{const} \).)

First, note that the time derivatives of the above equations deliver a dynamical coupling relation between \( Q \Sigma \) and \( \langle R \rangle \Sigma \) as a special case of the integrability condition (16)

\[
-\partial_t Q/\Sigma_1 + \frac{1}{3} \partial_t \langle R \rangle/\Sigma_1 = \frac{4C^3}{a_\Sigma^4}. \tag{40}
\]

In view of the conservation of the total rest mass \( M/\Sigma_1 \), we infer directly from equation (37) that \( Q/\Sigma_1 \) evolves as

\[
Q/\Sigma_1 = -\Lambda + \frac{Q/\Sigma_1(t_i) + \Lambda}{a_\Sigma^2}. \tag{41}
\]

For the same reason we infer from equation (38) that \( \langle R \rangle/\Sigma_1 \) evolves as (inserting \( C^2 \), equation (39))

\[
\langle R \rangle/\Sigma_1 = 3\Lambda + \frac{\langle R \rangle/\Sigma_1(t_i) - 3Q/\Sigma_1(t_i) - 6\Lambda}{a_\Sigma^2} + \frac{3Q/\Sigma_1(t_i) + 3\Lambda}{a_\Sigma^2}. \tag{42}
\]

The solutions (41) and (42) satisfy the integrability condition (16); it provides the first example of a non-trivial solution to the effective Einstein equations, in which kinematical backreaction and averaged scalar curvature are dynamically coupled. (This solution, for \( \Lambda = 0 \), was first discussed in connection with the dark energy problem in \cite{18}.)

This solution stabilizes the (always positive-definite) combination \( 6\Lambda + 3Q/\Sigma_1 - \langle R \rangle/\Sigma_1 \), which evolves as \((6\Lambda + 3Q/\Sigma_1(t_i) - \langle R \rangle/\Sigma_1(t_i))/a_\Sigma^2 = 6C^2/a_\Sigma^2 = 6H^2 \), so that for large \( |a/\Sigma_1| \), it approaches zero. This relation governs the "crosstalk" between kinematical backreaction, averaged scalar curvature and \( \Lambda \). Although the stationary state approaches the condition needed for a static state, the time dependence of the individual terms changes the global picture drastically. For, if we prepare an initial state in the vicinity of the globally static cosmos, condition (37) is conserved in time, but the curvature evolves away from condition (38), which we shall explicitly discuss below for the subcase of a \( \Lambda \)-free stationary cosmos.

To work with the stationary models has an advantage: we can employ the dimensionless characteristics (20), which are valid on the global scale, and so ease the discussion of...
observational results. Directly from the stationarity conditions (37) and (38) we obtain for the dimensionless functionals
\[ \Omega_m^E := -4\Omega_m^\Sigma + 2\Omega_\Lambda^\Sigma \quad \text{and} \quad \Omega_R^E := 1 + 3\Omega_\Sigma^\Sigma - 3\Omega_\Lambda^\Sigma. \] (43)

A straightforward calculation, employing the solutions (41) and (42), provides their explicit solutions in terms of two parameters—chosen to be the mass density parameter at initial time and the parameter for the cosmological constant at initial time:
\[ \Omega_m^E := \frac{2(\Omega_\Sigma(t_i) + \Lambda)}{3C^2}; \quad \Omega_\Lambda^E := \frac{\Lambda}{3C^2}. \] (44)

Inserting the constant \( C \), equation (39), we get
\[ \Omega_m^E = \frac{\Omega_m^\Lambda}{a^3}; \quad \Omega_\Lambda^E = -\frac{1}{3}\frac{\Omega_m^\Lambda}{a^3} + \frac{1}{2}\Omega_\Lambda^\Sigma; \]
\[ \Omega_R^E = 1 - \frac{3}{2}\frac{\Omega_\Lambda^\Sigma}{a^2}; \quad \Omega_\Lambda^E = \Omega_\Lambda^\Sigma. \] (45)

The cosmic equation of state for a globally stationary cosmos can be obtained by inserting (37) and (38) into (25)
\[ p/\Sigma = -\rho/\Sigma; \]
\[ w_{\text{eff}} = -\frac{1}{3} \left[ 1 - \frac{3\Omega_\Lambda^\Sigma}{2a^2} \right]; \quad w_{\text{eff}}(t) = -\frac{1}{3} \left[ 1 + \frac{\Lambda}{3\Omega_\Sigma(t) - \langle R \rangle(t)} \right], \] (46)
which is time dependent for a non-vanishing cosmological constant and, although it approaches, for large \( |a/\Sigma| \), the cosmic state \( p_{\text{eff}} = -\rho_{\text{eff}} \), the asymptotic model is in a stationary and not in a de Sitter phase, since the cosmological constant is assumed to share the global balance condition. Below we shall assign another role to \( \Lambda \) that will allow for a de Sitter phase too. For this purpose let us first drop \( \Lambda \) as a possible source sharing the stationarity condition.

3.5. Globally stationary cosmos without a cosmological constant

Now, consider again the subcase of a vanishing cosmological constant. The cosmic equation of state for a globally stationary cosmos with \( \Lambda = 0 \) can be inferred from equation (46):
\[ p_{\text{eff}} = -\frac{1}{3}\rho_{\text{eff}}. \] (47)

It is interesting to compare this condition with the investigation of backreaction in inhomogeneous cosmon fields by Wetterich [87], in particular with the cosmon equation of state, which also points to a possibly interesting different interpretation of the backreaction term (cf section 3.6). As already mentioned in the introduction, in the case of a physical pressure source, the above equation of state allows for a static model without cosmological constant also in the homogeneous isotropic case ([50], p 383). Recently, the cosmic equation of state in the present formalism has been calculated in the framework of the Tolman–Bondi solution [68], also confirming a strong coupling between averaged scalar curvature and kinematical backreaction. For related discussions see [25, 67, 74].

The solution for this \( \Lambda \)-free cosmos, as a subcase of the general solutions (41) and (42), reads
\[ Q_\Sigma = \frac{Q_\Sigma(t_i)}{a_i^3}; \quad \langle R \rangle_\Sigma = \frac{\langle R \rangle_\Sigma(t_i) - 3Q_\Sigma(t_i)}{a_i^2} + \frac{3Q_\Sigma(t_i)}{a_i^2}. \] (48)
The total kinematical backreaction \( Q_\Sigma V_\Sigma = 4\pi GM_\Sigma \) is a conserved quantity in this case. This solution implies for the dimensionless functionals
\[ \Omega_m^E = \frac{\Omega_m^\Sigma}{a^2}; \quad \Omega_\Sigma^E = -\frac{\Omega_m^\Sigma}{4a^2}; \quad \Omega_R^E = 1 - \frac{3\Omega_m^\Sigma}{4a^2}; \quad \Omega_\Lambda^E = 0. \] (49)
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where this cosmos only depends on a single parameter, which we have chosen to be the rest mass density parameter at the initial time $t_i$, as in equation (44).

As already mentioned in the general case, the averaged curvature evolves strongly in this cosmology. In general, i.e., if the initial state is not the globally static state, it evolves away from the curvature of a static cosmos

$$\frac{\langle R \rangle}{12\pi G \langle \rho \rangle} = 1 - \frac{6C^2}{12\pi G \langle \rho \rangle a^2} = 1 - \left[ 1 - \frac{\langle R \rangle(t_i)}{12\pi G \langle \rho \rangle(t_i)} \right] a^\Sigma. \quad (50)$$

Stating this remark in terms of solution (42) for $\Lambda_1 = 0$,

$$\langle R \rangle = \frac{3Q \langle t_i \rangle}{a^3} = \frac{3Q \langle t_i \rangle - \langle R \rangle(t_i)}{a^3} \quad (51)$$

where the numerators in both terms are positive definite, we find that the second term will dominate after some time, and the averaged curvature can change sign in the course of evolution. In view of (49), this will happen when $\Omega_1^{\Sigma} = 1 - \frac{3Q}{4\omega} = 0$, i.e., when the density parameter has dropped to the ‘critical’ value $\Omega_1^{\Sigma} = 4/3$.

A comparison of this cosmology with a Friedmannian cosmology in terms of the evolution of the single parameter $\Omega_1^{\Sigma}$ is performed in appendix B.

3.6. Inflationary cosmogonies evolving into a globally static cosmos

The inhomogeneous globally static cosmos and also stationary cosmologies were built on the assumption that an exact balance between averaged material mass and its kinematical fluctuations (not its density fluctuations) is established. The cosmological constant, if present, takes the role of a further player in an established globally balanced state: if kinematical fluctuations vanish, then the Einstein static model arises, if we have complete balance due to kinematical fluctuations, then $\Lambda_1 = 0$. All intermediate states are conceivable.

Now, instead of preparing such a non-accelerating state, we could further widen our model assumptions by allowing for a non-zero global acceleration. To illustrate this we shall choose the simple assumption that this acceleration is entirely due to the cosmological constant, and also that the corresponding expansion rate is also only due to $\Lambda$. In this case, the balance condition is established only among the averaged material mass and the kinematical backreaction as in the globally static model. We shall see that this ‘driven balance’ (with the same cosmic equation of state as in the static model) is conserved and the fluctuating matter is subjected to a constant acceleration: conditions (31) and (32) are assumed to hold (for $\Lambda = 0$), but the respective terms evolve in time according to the solution

$$\langle R \rangle = \frac{\langle R \rangle(t_i)}{a^3} \quad ; \quad Q = \frac{Q(t_i)}{a^3} \quad ; \quad \langle \rho \rangle = \frac{\langle \rho \rangle(t_i)}{a^3}, \quad (52)$$

i.e., both averaged scalar curvature and kinematical backreaction obey conservation laws similar to the averaged density, and the ‘driven balance’ is maintained. With our assumptions, $a(t)$ is given by the exponential solution of the flat de Sitter model. (Note that $\langle R \rangle$ and $Q$ with $\langle R \rangle = 3Q$ solve the integrability condition (16) irrespective of the particular form of $H = \dot{a}/a\Sigma$.)

The idea of inflation is related to matter creation. In this line a global stationarity condition singles out a state which assumes that, as soon as matter is created, also its kinematical fluctuations are large and are trying to establish the balance condition on the global scale. To address this question properly, the above model is too simple. A more ambitious model would first attempt to understand kinematical backreaction and averaged curvature by means
of effective inhomogeneous scalar fields. We shall now discuss some aspects of such an attempt.

In the absence of matter and extrinsic curvature fluctuations, equations (12), (13) and (16) imply (we always consider the global scale here) that

\[ 3 \frac{\ddot{a}_\Sigma}{a_\Sigma} = \Lambda; \quad 3 \left( \frac{\dot{a}_\Sigma}{a_\Sigma} \right)^2 + \frac{\langle R \rangle_{\Sigma}(t_i)}{a_\Sigma^2} = \Lambda, \]

leaving the exponentially expanding de Sitter cosmos as the only solution, when the initial scalar curvature is put to zero: \( H/\Lambda = \sqrt{1/3}. \) As a next step, let us consider a cosmos without matter, but with intrinsic and extrinsic curvature fluctuations; the effective equations (13) and (16) then read

\[ 3 \frac{\ddot{a}_\Sigma}{a_\Sigma} = \Lambda + Q/\Sigma; \quad 3 \left( \frac{\dot{a}_\Sigma}{a_\Sigma} \right)^2 + \frac{\langle R \rangle_{\Sigma} + Q/\Sigma}{2} = \Lambda. \]

As in (26) the above equations can be recast into standard zero-curvature Friedmann equations with new effective sources, now implying an interpretation in terms of an effective scalar field,

\[ \Theta := \frac{\dot{\varphi}}{\varphi} = -\frac{1}{16\pi G} \frac{\dot{Q}_\Sigma}{\varphi} - \frac{1}{16\pi G} \frac{\langle R \rangle_{\Sigma}}{\varphi}; \quad P := \frac{p}{\varphi} = -\frac{1}{16\pi G} \frac{\dot{Q}_\Sigma}{\varphi} + \frac{1}{48\pi G} \frac{\langle R \rangle_{\Sigma}}{\varphi}. \]

In this matter-free case the above equations suggest an interpretation of the geometrical degrees of freedom in the extrinsic curvature fluctuations as the kinetic energy part of a scalar field. \( Q/\Sigma \) obeys a stiff equation of state; also, depending on the sign of \( Q/\Sigma, \) we may have ‘phantom energy’ [24] (see also, e.g., [71]). An important remark here is that the simplified case of vanishing averaged scalar curvature implies through the integrability condition (16) that \( Q/\Sigma = Q/\Sigma(t_i) a_\Sigma^-6, \) and the effective sources decay in proportion to the square of the inverse volume. As the previous considerations about globally stationary cosmologies have shown, the average curvature plays a significant role for the maintenance of a large \( Q/\Sigma \) and therefore the presence of curvature is crucial and should not be neglected in this picture.

If we now take matter sources into account, the cosmic equation of state changes and it is here where detailed models of matter creation would have to be analysed to understand whether and if, physically, such a creation process would entail, e.g., strong global expansion fluctuations, favouring an inhomogeneous state at the time when \( \Lambda \) no longer rules the expansion. If \( Q/\Sigma \) is interpreted as a kinetic energy term of a scalar field (we may call it a morphon), then its very presence would also shape the extrinsic curvature of created matter fluctuations; its role in the equations above shows that a possible balance condition in the process of conversion of kinetic energy into matter would be \( Q/\Sigma = 4\pi G \langle \varphi \rangle_{\Sigma}, \) as required for a globally static or stationary cosmos.

3.7. Global far-from-equilibrium states

Expressing a fluctuation-dominated cosmos in a thermodynamic language, the globally stationary state is in an out-of-equilibrium state compared with the Friedmannian ‘equilibrium state’ (in the sense defined below). In [52] an entropy measure has been advanced that we can employ to characterize both states. Looking at the non-commutativity relation (9) for \( \Psi = \varphi \) on the global scale, Hosoya et al [52] found that the source of non-commutativity is given by the production of relative information entropy, defined as to measure the deviations from the average mass density due to the development of inhomogeneities

\[ \langle \partial_\varphi \varphi \rangle_{\Sigma} - \partial_\varphi \langle \varphi \rangle_{\Sigma} = \frac{\dot{a}_\Sigma S_\varphi(\varphi) \langle \varphi \rangle_{\Sigma}}{V_{\Sigma}}, \]
with the, for positive-definite density, positive-definite Kullback–Leibler functional

\[
S[\varrho \| \langle \varrho \rangle_\Sigma] := \int_\Sigma J d^3x \varrho \ln \frac{\varrho}{\langle \varrho \rangle_\Sigma}.
\]  

(57)

This measure vanishes for Friedmannian cosmologies (‘zero structure’) and so defines the notion ‘equilibrium state’ introduced above. It attains some positive time-dependent value otherwise. The source in (56) shows that relative entropy production and volume evolution are competing: commutation can be reached, if the volume expansion is faster than the production of information contained within the same volume.

In [52] the following conjecture was advanced:

The relative information entropy of a dust matter model \( S[\varrho \| \langle \varrho \rangle_\Sigma] \) is, for sufficiently large times, globally (i.e., averaged over the whole compact manifold \( \Sigma \)) an increasing function of time.

This conjecture can be proven for linearized scalar perturbations on a Friedmannian background (the growing-mode solution of the linear theory of gravitational instability implies \( \partial_t S > 0 \) and \( S \) is, in general, time convex, i.e., \( \partial^2_t S > 0 \)). However, in a generic non-perturbative situation it may not hold. Below we shall approach this question for the case of a globally stationary cosmos.

Let us first consider the general situation. We calculate the second time derivative of (57) and first obtain [52]

\[
\frac{\ddot{S}}{V_\Sigma} = -\langle \delta \varrho \delta \dot{\theta} \rangle_\Sigma + \langle \varrho \rangle_\Sigma (\Delta \theta)^2,
\]

(58)

where, for any scalar field \( \Psi \), \( \delta \Psi := \Psi - \langle \Psi \rangle_\Sigma \) denotes the deviation from the global average value and \( \Delta \Psi := \sqrt{\langle \delta \Psi \rangle^2_\Sigma} \) the global amplitude of its fluctuations. Raychaudhuri’s equation specifies the deviations \( \delta \dot{\theta} \) for the time evolution of the expansion rate

\[
\delta \dot{\theta} = -4\pi G \delta \varrho - \frac{2}{3} \delta (\theta^2) - 2 \delta (\sigma^2), \quad \delta \Lambda = 0,
\]

(59)

which, together with the commutation rule (9), yields

\[
\frac{\ddot{S}}{V_\Sigma} = 4\pi G (\Delta \varrho)^2 + \langle \varrho \rangle_\Sigma (\Delta \theta)^2 + \frac{1}{3} \langle \delta \varrho \delta \theta^2 \rangle_\Sigma + 2 \langle \delta \varrho \delta \sigma^2 \rangle_\Sigma.
\]

(60)

For our purpose the above equation may be recast by explicitly writing out the last second terms and expressing the expansion fluctuation amplitude in terms of \( Q_\Sigma = \frac{2}{3} (\Delta \theta)^2 - \frac{2}{3} (\sigma^2)^2 \rangle_\Sigma \) (cf equation (14)). We obtain the general equation

\[
\frac{\ddot{S}}{V_\Sigma} = B_1 + \langle \varrho \rangle_\Sigma \left[ Q_\Sigma - \frac{3}{2} \langle \theta \rangle_\Sigma \right],
\]

(61)

with \( B_1 := 4\pi G (\Delta \varrho)^2 + \frac{1}{3} \langle \varrho \theta^2 \rangle_\Sigma + 2 \langle \varrho \sigma^2 \rangle_\Sigma \geq 0 \).

From this equation we infer that, for a globally static universe model with \( Q_\Sigma = 4\pi G \langle \varrho \rangle_\Sigma \) and \( \langle \theta \rangle_\Sigma = 0 \), the Kullback–Leibler functional is always time convex.

For the case of a stationary universe model (with or without \( \Lambda \)) we can obtain sufficient conditions for time convexity as follows. Directly from (61) we conclude that \( Q_\Sigma - \frac{3}{2} \langle \theta \rangle_\Sigma^2 = Q_\Sigma - 3H^2 \) should then be positive and, employing the dimensionless characteristics (20) on the global scale, that the condition

\[
\Omega_\Sigma^2 \leq -\frac{1}{2}
\]

(62)

is sufficient. Inserting the stationarity condition \( Q_\Sigma = 4\pi G \langle \varrho \rangle_\Sigma - \Lambda \) into (61) we instead need the condition \( 4\pi G \langle \varrho \rangle_\Sigma - \Lambda - \frac{3}{2} \langle \theta \rangle_\Sigma^2 \geq 0 \) which, expressed through the average characteristics (20) on the global scale, reads

\[
\frac{1}{2} \Omega_m^2 - \Omega_\Lambda \geq 1.
\]

(63)
For a $\Lambda$-free stationary cosmos we conclude that a sufficient condition for time convexity of the Kullback–Leibler functional is $\Omega^\Sigma_m \geq 2$, which implies a sufficiently positive global average curvature through equation (43)

$$\Omega^\Sigma_R = 1 - \frac{1}{4} \Omega^\Sigma_m \Rightarrow \Omega^\Sigma_R \leq -\frac{1}{2}. \quad (64)$$

Given the above results, we have three possibilities for the evolution of the entropy functional (57) in a globally static cosmology (and also in a globally stationary cosmology, if the conditions above are met), depending on the sign of its first time derivative, the relative entropy production rate (equation (56); note that for the special case of a globally static cosmos $\partial_t \langle \varrho \rangle_\Sigma = 0$).

First, if $\dot{S}$ is positive at initial time $t_i$, then the functional (57) is always growing; second, $\dot{S}$ becomes positive at some time $t > t_i$, in which case the functional (57) is always growing thereafter; third, $\dot{S}$ never becomes positive, i.e., the functional (57) approaches some constant $S_0 \geq 0$ from above. This latter case would violate the entropy conjecture advanced in [52].

In the first two cases, looking back to equation (56), the static model leaves the volume unchanged, while information is created. This enhances the source of non-commutativity, which may be interpreted as a signature of a non-Friedmannian state.

The globally stationary cosmos does not necessarily imply a time convex evolution of the relative information entropy. In this context it is interesting to think of a floating equilibrium realized in open biological systems (since gravity is long range, a gravitational system is also not isolated). In this analogy a stationary far-from-equilibrium may be characterized, according to Prigogine [72], by a minimum of the entropy production rate, i.e., for our inhomogeneity measure (57) we would require the condition $\ddot{S} = 0$. Whether such a saturation is possible within a dust cosmology is doubtful in view of a lack of physics like pressure gradients. However, the above analogy prevails.

3.8. Discussion: the global picture

An investigation of models in the framework of the effective equations while still including a cosmological constant shows that the inflationary picture of cosmogonical scenarios in the early universe does not have to be abandoned. The cosmology can still undergo several phase transitions and inflationary phases. As in the early days after Einstein’s suggestion of the static cosmos, it is a subject of controversy related to the initial conditions of the universe model: either the singular ‘big bang’ initial state, or the static initial state eventually evolving into an expanding universe model is preferred (see the discussion in [8]). In the context of inhomogeneous inflationary scenarios, we are entitled to single out an averaged model that is globally stationary, i.e., we assume that inflation may end and, during a phase of damped oscillations, exit into a globally stationary phase, instead of a homogeneous Friedmannian phase. If such an inhomogeneous ‘initial condition’ at the end of inflation is prepared, then e.g., a globally static inhomogeneous cosmos certainly is in a stronger position than its homogeneous predecessor with regard to its more general properties, especially if its stability could be proven. Similar ideas have actually been already discussed within chaotic inflation, e.g., [61, 62], being mirrored here within a classical cosmos.

Justifying such an inhomogeneous initial state physically is another issue. If we follow Einstein’s thoughts leading to the static cosmos, the relevant arguments were not empirical, but rather philosophical. These thoughts are far from conceiving a ‘fitting model’ to observational facts. The cosmological principle that elevates the Friedmannian models to a matter of principles is not always the underlying reason to employ homogeneous isotropic cosmologies. As the stationary but inhomogeneous cosmology shows, there is no local isotropy and, hence,
the *cosmological principle* must be replaced by another principle of ‘global prejudice’ (in the constructive way of the word) such as for example ‘the universe as a whole cannot create momentum out of itself’ (a ‘non-Münchhausen’ principle⁵). The formulation of a sound principle is itself a considerable task; the tentative balance condition that was helpful to construct exact examples, should be replaced by a cosmic virial theorem averaged on the global scale, which in turn calls for a generalization of the dust matter model. Also, global topological constraints should be important. The way to deal with a cosmological constant, e.g., the question whether we should include it in a virialized state [66], and the more fundamental approach of an effective scalar field dynamics in the early universe is another issue of concern in the formulation of a global principle. For example, a global net acceleration can be a result of initial conditions, and in fact this is the case in generic models of Friedmannian cosmology.

While the globally static cosmos features properties similar to its homogeneous predecessor, e.g., the global density and the global curvature are constant in time, the globally stationary cosmos evolves very differently. We have shown that, although the relation among kinematical backreaction and averaged curvature tends to the corresponding relation in the globally static cosmos, the time dependence of the individual terms may affect the global properties strongly. The stationary state tends to the static state only in the sense that, e.g., in the case of an expanding cosmos, the rate of expansion slows down, but the steady increase of the scale factor allows for a global change of the sign of curvature. As equation (51) shows, an initially positive averaged scalar curvature would decrease, and eventually would become negative. This may not necessarily be regarded as a signature of a global topology change, as a corresponding sign change in a Friedmannian model would suggest; the averaged scalar curvature is only a weak descriptor for the topology in the general case, and information on the sectional curvatures is required for definite conclusions on this issue (see [4], chapter 9; [3]). However, if a global change of topology arises, we could imagine that the cosmos started with a finite volume of a closed spaceform, then the averaged curvature of the inhomogeneous space may become negative and a globally hyperbolic curvature arises that would suggest a finite-volume compact spaceform with (now no longer simply connected) sections of negative sectional curvatures and, generically, a ‘horned topology’ (for topology-related issues see [5, 6, 58, 86]).

3.9. Discussion: the regional picture

The implications that are furnished by the effective static model featuring *regional* expansion and contraction are qualitatively very different from a cosmogony with *global* expansion. First, the former is more ‘violent’ than the latter, since fluctuations grow exponentially rather than at a moderate power-law pace. Second, an expanding region must be counterbalanced by a contracting region. The second property implies that, depending on estimates of the total size of the cosmos, we should be close to seeing a large ‘blue excess’ in galaxy number counts, e.g., at the borders of the currently drawn Sloan survey. The microwave sky would be redshifted in expanding domains alternating with a blue-shifted sky in contracting domains, while we would have to claim an underdensity in our regional Hubble volume [2, 83, 85, 88] counterbalanced by overdense regions beyond the horizon; we would be surrounded (in space and time direction) by high-density walls. This spatially and temporally alternating scenario implies a large paradigmatic change compared with the standard model of cosmology.

Exponential fluctuations imply the likely situation that, e.g., the global curvature cannot be seen on the scale of our Hubble volume. Not unlikely, since we are seeing a large

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⁵ Baron v. Münchhausen was trying to pull himself out of the pond without external help.
Hubble expansion, the regional curvature is (relatively) more negative, i.e., in a special case seemingly zero or negative. As already noted, a globally static cosmos would necessarily call for a replacement and, regionally, for an obvious refinement of the cosmological principle including the question, whether and how close our observers have to be at the centre of such a regional ‘Hubble bubble’ [2, 62, 84]. The scale of this ‘reduced curvature region’ likely exceeds scales that have been discussed in connection with peculiar-velocity catalogues that are statistically affected by boundary conditions [48, 90]. A strong constraint is the isotropic microwave sky, a question related to assigning the whole dipole to our proper motion [27].

The globally stationary cosmology evolves very differently and, hence, also its regional properties would be different. Since already the global curvature can approach 0, the expanding stationary cosmos may show similar regional fluctuations compared with a Friedmannian model, however, in a much narrower range of times. The differences to a Friedmannian model can be clearly seen in the detailed comparison given in appendix B.

We shall not embark here further into speculations about implications of a globally static or stationary cosmos. It is clear that such a drastic change of view stimulates thoughts, but detailed calculations needed for proper statements do not exist yet. Rather, we shall put a currently held discussion about the dark energy problem into perspective, which turns out to lie at the heart of the balance condition required for globally stationary cosmologies.

4. The dark energy problem

Let us first recall the contemporary view on the global universe model. The standard model of cosmology idealizes the matter distribution in the universe to be homogeneous and isotropic, neglecting structure. Observations point to an averaged matter content of at most 30% (including dark matter) and almost flat space sections [79]. The missing gap to fill the global cosmic triangle (A.1) is modelled by a (spatially constant) cosmological term in the simplest case [70] (the sum of the parameters has to be equal to 1). This is the concordance model (roughly 0.3 + 0.0 + 0.7 = 1), which can be fitted to a large set of orthogonal observational data. (There are, however, other voices [12, 13].) Modelling this dark energy gap with a cosmological constant results in an accelerated phase. The ‘coincidence’ that dark energy starts to dominate exactly when structure also enters the nonlinear regime suggests that there could be a physical relation between the effect of structure on the average expansion (known as backreaction effect) and the dark energy gap found in the standard model, thus providing a natural solution to this coincidence problem. (It is clear here that sub-horizon fluctuations are made responsible.)

From equation (12) the condition for an accelerating patch \( D \) of the universe directly follows

\[
Q_D > 4 \pi G \langle \rho \rangle_D. \tag{65}
\]

This regional condition is weaker than the requirement of global acceleration, since it accounts for the regional nature of our observations. It has been recently claimed [56] that condition (65) could be satisfied within our regional Hubble volume, hence providing a smart explanation of the dark energy problem [9, 87]. In [18] the following argument has been advanced. The considerations in this paper and the ‘classical’ explanation of the dark energy problem in terms of backreaction effects are intimately related through condition (65). If this condition would hold, then the cosmologies presented in this work would also attain the status of physically

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6 The only direct support for acceleration comes from high-redshift supernovae observations [77]; however, the ‘fitting models’ for the interpolation between low- and high-redshift observations and the measured distances do not take inhomogeneities into account, see [37] for a discussion and references.
viable models on the global scale, since the physical basis of these models is exactly the
possibility that kinematical expansion fluctuations and averaged sources are of the same order.
A crucial support for condition (65) to survive in time could also be given in [18] employing
a particular example of the averaged Einstein equations, listed as globally stationary cosmos
without $\Lambda$ in this paper. This example of an exact solution indeed points to a strong coupling
between kinematical backreaction and averaged scalar curvature: the rate of decay of $Q^\Sigma$ is
in proportion to $\langle \varrho \rangle_\Sigma$ and can therefore be of the same order as $4\pi G \langle \varrho \rangle_\Sigma$ today.

4.1. Caveats

There is, however, a large body of opponents including myself who do not think that condition
(65) can be attained within the standard model of cosmology. However, this opposition
rests on the picture that fluctuations grow through gravitational instability of a homogeneous
universe model. If we allow for the very different picture of inhomogeneous initial conditions,
then the above outlined explanation can be a natural consequence. In the standard picture of
cosmological structure formation from cold dark matter initial conditions there are essentially
three caveats that have to be overcome, which we are going to put into perspective now in
some more detail.

4.1.1. Caveat 1: explicit calculations of kinematical backreaction. The above summarized
suggestion of solving the dark energy problem has a long history; there have been many
tries to calculate backreaction effects following the advent of the averaging problem
initiated by George Ellis [34] (see [10, 11, 14, 40, 42–45, 56, 73, 76, 87] and references therein).
The new input into the discussion concerns the quantitative importance of backreaction
summarized by the ‘bold claim’ (65). Earlier efforts to support this claim point to a negative
answer. Let us therefore look into the details of explicit calculations of the effect.

Cosmologists are mostly thinking in Newtonian terms when modelling structures (e.g.,
by $N$-body simulations which are constructed such that their spatial average evolves as in the
standard model). Computation of backreaction in the Newtonian framework is possible in great
detail [23], but there is a drawback (often giving rise to misunderstanding in the literature):
the relevance of kinematical backreaction for the global evolution cannot be estimated
within Newtonian cosmology, simply because one can in general prove that an averaged
Newtonian cosmology has zero global kinematical backreaction $Q^D$. Nevertheless, as
non-perturbative calculations in [23] show, backreaction has indirect impact on the evolution
of the standard cosmological parameters on (relatively large) regional scales. However,
the dimensionless backreaction parameter (20) $\Omega^D_\Omega := -\frac{\Omega_\Omega}{6 H_0^2}$ on large expanding regions
is quantitatively negligible compared with the effective density parameter $\Omega^D_m := \frac{\kappa n m H_0}{3 \sqrt{\Omega_m^D H_0^2}}$,
i.e., condition (65) rewritten in terms of dimensionless characteristics,

$$\Omega^D_\Omega > \frac{\Omega^D_m}{4},$$

(implying for the effective deceleration parameter (22) $q^D_{eff} < 0$) is not fulfilled on large scales
in these calculations. In fact, $\Omega^D_\Omega$ is not only small, but positive, i.e., with regard to (14),
shear fluctuations dominate over expansion fluctuations. In [23] it was achieved to fulfil this
condition on a patch of 100 Mpc (for \( h = 0.5 \)) by saying that, on these smaller scales, we may have different expectation values for velocity fluctuations (the kinematical components entering \( Q_D \) in Newtonian theory) and for the density fluctuations (they are theoretically independent and, on 100 Mpc, it may well happen that the expectation value for \( Q_D \) is \( 1 - \sigma \) off the expectation value for the density fluctuations, i.e., we could prepare an ‘untypical’ sample). Although this situation is likely to happen (the probability distribution of \( Q_D \) is non-Gaussian), we cannot apply this argument to very large samples.

To give another example (out of many that are not quoted here), the answer attempted in the work by Russ et al [76] was based on a second-order perturbative calculation in general relativity. However, initial and boundary conditions were assumed such that, already from their assumptions, the global backreaction term, according to our definition, has to vanish (see the discussion in [15]). Although it is obvious from the above remarks about Newtonian calculations that the global value can only be determined in the framework of general relativity, the quoted relativistic calculation was ‘designed’ after a Newtonian model. This also reveals inherent difficulties for treating the problem with sufficient generality, if standard assumptions like periodic boundary conditions are adopted.

Given these remarks we could say that caveat 1 is due to Newtonian or quasi-Newtonian work, or (since calculations have been mostly done perturbatively) is generally due to the fact that backreaction effects need a non-perturbative and background-free calculation (see [57] for suggestions of non-perturbative variables in general relativity). Until more general calculations have been done, we may say that the above quoted results cannot be extrapolated. However, basically a non-perturbative calculation in general relativity has already been done; we have pointed out that equation (19) for the scale factor is identical to the corresponding Newtonian equation. The non-perturbative Newtonian model investigated in [23] was based on integrating this exact equation for a specific fluctuation source term, the (perturbative) ‘Zel’dovich approximation’. Employing instead the relativistic version of this approximation as given by Kasai [55] does not lead to differences concerning the results either. Therefore, such a relativistic calculation for \( a_D \) yields the same result in terms of \( Q_D \). The difference is that the latter is linked to the scalar curvature (and this may give rise to different interpretations as will be examined in a forthcoming work). But, the quantitative conclusions given in [23] can, to some extent, be extrapolated. This remark also points to the necessity of considering initial conditions other than those assumed in [23] in order to meet condition (65).

4.1.2. Caveat 2: observational constraints. The effective cosmological parameters defined in (20) can be considered to provide a fair representation of the values which an observer would also measure in a sufficiently shallow survey region \( D \) (the light-cone effect is not taken into account). We may therefore discuss estimates of those parameters in comparison with observed values. Note, however, that the interpretation of observations is mostly done by employing a standard Friedmannian cosmology as a ‘fitting model’ and therefore geometrical inhomogeneities (that are hidden in the definition of the spatial averages in the Riemannian volume element) are ignored; below we shall argue why we should not ignore them.

On the grounds that the parameters (20) can be ‘observed’, we examine condition (66), which itself would ‘only’ require a contribution of \(|\Omega_Q^D| > 0.075\) for the value \( \Omega_m^D = 0.3 \) to the cosmic quartet. Assuming that this condition holds on some large domain \( \mathcal{D} \), which we may take to be as large as our observable universe, then Hamilton’s constraint in the form (21) also implies

\[
\Omega_A^D + \Omega_R^D = 1 - \Omega_m^D - \Omega_Q^D > 1 - \frac{3}{4} \Omega_m^D.
\]
showing that we need a substantial negative curvature (positive $\Omega^D_k$) on the domain $D$, if we put the cosmological constant to zero. (To reconcile this condition with a non-vanishing cosmological constant would need an even larger value of $\Omega^D_k$ than that suggested by the concordance model.) The fact that a large value of kinematical backreaction entails a substantial average Ricci curvature has also been stressed by Räsänen [75].

Condition (67) seems to contradict the widely agreed expectation that the curvature should be very small, the concordance model assumes an exactly zero scalar curvature. However, the Friedmannian curvature parameter is only directly related to the averaged scalar curvature in the homogeneous case. If we would—mistakenly—estimate the averaged scalar curvature through $\Omega^D_k = -k_D \left( a_D^2 H_D^2 \right)$, then we would have to estimate the (regional) time integral of $Q_D$ in the relation $\Omega^D_k + \Omega^D_m = 1 - \Omega^D_m - \Omega^D_N$ (cf equation (24)), and thus our estimate would depend on the dynamical evolution of the kinematical backreaction parameter and not just on its value. (In the (global) example presented in appendix B we would have $\Omega^D_\Sigma = 1 - \Omega^D_m = \text{const}$, while the true curvature functional $\Omega^D_k$ is strongly time dependent.)

If we ask whether the kinematical backreaction $Q_D$ is observable, the answer within our setup above is yes. On the observable domain $D$, $Q_D$ is built from invariants of the peculiar-velocity gradient in a Newtonian model. Ignoring again geometrical fluctuations (and with them the fact that in a relativistic setting $Q_D$ cannot be represented through invariants of a gradient, which is derived from a vector field) good high-resolution maps of peculiar velocities could in principle determine the value of backreaction. Existing catalogues are, however, too small and they would only return the cosmic variance around the assumed Friedmannian background in a likely untypical patch of the universe.

On the other hand, we know several observational facts that could place constraints on the value of kinematical variables [35]. ‘Global’ bounds on $Q_D$, where $D$ is of the order of the CMB (cosmic microwave background) scale, can be inferred from works of Maartens et al [64, 65]. Observational upper bounds were given in terms of covariant and gauge invariant quantities, e.g., for an upper bound on the shear fluctuations $\langle \sigma^2 \rangle_D$ we use the limit $\sigma^2/\theta^2 < 16 \alpha^2 \times 10^{-10}$, where both $\sigma$ and $\theta$ have to be interpreted as averaged values in our setting, i.e., $\sigma_* := \langle \sigma \rangle_D$ and $\theta_* := 3 H_D$. The parameter $\alpha$ is determined by observations and depends on scale. For large-scale CMB observations we set $\alpha \approx 1$. In the case when shear fluctuations would dominate, we may assume $\langle \theta^2 \rangle_D = \langle \theta^2 \rangle_D$. In that case, $Q_D$ reduces to $-2 \langle \sigma^2 \rangle_D$ (is therefore negative); the corresponding dimensionless parameter obeys the bound

$$0 < -\frac{Q_D}{6 H_D^2} = \Omega^D_\sigma < 2.4 \alpha \times 10^{-9} \quad \text{for} \quad \langle \theta^2 \rangle_D = \langle \theta^2 \rangle_D.$$  

This bound is comfortably satisfied in the calculations quoted above, but it is not what we need for explaining dark energy; the opposite situation must be examined, namely, we assume shear fluctuations to be negligible and search bounds for a positive $Q_D \approx \frac{1}{2} \langle \theta^2 \rangle_D - \langle \theta \rangle_D^2$.

On the Hubble length $H_H = 2c/H \approx 6$ Gpc h$^{-1}$ Maartens and his colleagues [65] obtained an upper bound on the density fluctuations that we may convert into expansion fluctuations using, for simplicity, the proportionality relation stemming from the standard linear theory of gravitational instability on an Einstein–de Sitter reference background with density $\varrho_H$ and Hubble expansion parameter $H$ (suppressing the decaying mode)

$$\frac{\langle \rho - \langle \rho \rangle_D \rangle_D}{\varrho_H^2} = \frac{\langle \theta - \langle \theta \rangle_D \rangle_D^2}{V^H_D H^2},$$

where the initial volume $V_H = 4/3\pi (L_H/2)^3$ accounts for the dimension and the scale. With these assumptions and the upper bound on the density fluctuations quoted in [65], the
The backreaction parameter is bound from below:

$$0 > -\frac{\Omega_D}{6H_0^2} = \Omega_Q^D > -\left(\frac{14}{5\Omega_m^D} + \frac{1}{10}\right)^2 \alpha^2 \times 10^{-8} \quad \text{for} \quad \langle \sigma^2 \rangle_D = 0, \quad (70)$$

i.e., for $\Omega_m^D = 0.3$ on the Hubble scale we would have $\Omega_Q^D > -8.9\alpha^2 \times 10^{-7}$.

However, a precise relation of the bounds on gauge-invariant variables given by Maartens and colleagues with the relevant variables discussed here requires more efforts and is the subject of a forthcoming work. Notwithstanding, the above order-of-magnitude estimates demonstrate that observational constraints will also provide obstacles to advocate condition (65).

The loophole here is that the strong coupling of density fluctuations to expansion fluctuations, as given by a direct proportionality for small perturbations of a Friedmannian model, equation (69), was strongly used to place constraints on expansion fluctuations. In an out-of-equilibrium state these two types of fluctuations (being independent variables) are decoupled and an observational constraint must be based directly on an upper bound on expansion fluctuations.

4.1.3. Caveat 3: geometrical fluctuations, curvature backreaction and the choice of foliation.

It is a reasonable question to ask whether a cosmological model is accelerating or not given the framework of the kinematically averaged Einstein equations, provided we consider the chosen foliation of spacetime appropriate. Caveat 3 consists of the fact that those averages are performed on an inhomogeneous hypersurface. This is all right as long as model and observations are to be compared, since observations themselves are carried out in an inhomogeneous geometry. The problem we speak about here comes into the fore by ‘fitting’ a Friedmannian cosmology to observational data, and this is the standard procedure. The question whether a cosmological constant is needed to explain the observations is due to this fitting process. Consequently, there is a need to compare averages of the dynamical model to averages on a homogeneous geometry, which is a difficult task because the averaging of tensors is not straightforward [34, 37, 40]. There is obviously an interpretation problem in the standard model (having Euclidean geometry in the case of the concordance model). Averages on inhomogeneous spaces do not have the same values as averages on homogeneous spaces. Riemannian volumes may substantially differ from Euclidean volumes and are not small perturbations per se (a Riemannian ball has $\pi^2/6$ less volume than an Euclidean ball with the same geodesic radius [21]). It is possible to define a ‘geometrical renormalization’ flowing averages on inhomogeneous spaces into Friedmannian averages [19]. The consequences for the regional cosmological parameters are summarized in [20] implying further ‘backreaction effects’ such as Ricci curvature backreaction that lead to a ‘dressing’ of cosmological parameters by smoothed-out geometrical inhomogeneities. These effects may provide a further key for explaining dark matter and dark energy.

The answer to the question whether these effects are significant needs detailed realistic models for ‘volume roughening’ of spatial slices in the universe, and robust estimators for intrinsic curvature fluctuations. Since this problem has been largely overlooked when considering averages in cosmology, there are not many results. A naive Swiss cheese model by just gluing Riemannian balls in place of Euclidean balls in the slice yields an effect of 67%, reducing the necessary dark energy roughly from 70% to 50% [21]. A mismatch of similar magnitude has been reported by Hellaby [51] using volume matching as suggested by Ellis and Stoeger [40] (compare [80]). However, the quantitative significance of these effects on cosmological scales has yet to be explored.
Related to the averaging of both extrinsic and intrinsic inhomogeneities Zalaletdinov [89] proposed a macroscopic description of gravitation based on a covariant spacetime averaging procedure. The geometry of the macroscopic spacetime follows from averaging Cartan’s structure equations, leading to a definition of correlation tensors. Macroscopic field equations (averaged Einstein equations) can be derived in this framework. Within this approach Coley et al [26] have recently shown that for a spatially homogeneous and isotropic macroscopic spacetime, the correlation tensor is of the form of a spatial curvature term. In this context it was also speculated that the extra correlation terms might help to stabilize a globally static universe model.

As for any averaging procedure the choice of spatial slices on which one considers average quantities is crucial. In [19] this problem has been identified and, in the framework of smoothing the geometry with the help of the Ricci flow, one has strong means to approach the slicing problem, e.g., by aiming at ‘minimizing’ artificial gauge effects [19]. Recently, Ishibashi and Wald [54] gave particular examples that show why caution is in order concerning a straightforward observational interpretation of averaged kinematical quantities on given spatial slices. Here, a clear advantage of volume-averaging scalar quantities is that the resulting averages of 4-covariant variables and their governing equations can be transformed to a different choice of slicing, which is a feature of the covariant fluid gauge [36] underlying the averaged equations in the present paper; examples that illustrate this have been given in [16].

5. Concluding remarks

In this work we have contrasted Friedmannian cosmology to particular choices of inhomogeneous cosmologies. Two very different points of view concerning the comparison of an idealized Friedmannian cosmos and an averaged inhomogeneous cosmos are conceivable.

First, an averaged inhomogeneous model could be a small perturbation of a Friedmannian model in the sense that the kinematical backreaction on the global scale, $Q_\Sigma$, is small and, hence, the kinematics of both models measured by the evolution of the effective scale factor $a_\Sigma$ are comparable. Still, as shown in detail within Newtonian cosmology in [23], this must not imply that the parameters of a Friedmannian model evolve as in the inhomogeneous model; in contrast, even for negligible backreaction, there are significant differences. We may place such a universe model into a near-equilibrium state as measured, e.g., by the information-theoretical measure (57) that vanishes for a Friedmannian cosmology.

Second, and this point of view was exploited in the present work, we may conceive a universe model in a far-from-equilibrium state, characterized by strong averaged expansion fluctuations on the global scale. Such a model is no longer a perturbation of a Friedmannian model. We have shown that these highly inhomogeneous models may then evolve in the vicinity of the balance condition $Q_\Sigma = 4\pi G \langle \rho \rangle_\Sigma$ (disregarding the cosmological constant).

Which of these two points of view is realized depends on the choice of the initial state of the universe at the exit epoch of an eventual inflationary phase in the early universe. We have argued why a possible solution of the dark energy problem through kinematical backreaction effects is possible from the second point of view, but unlikely from the first point of view. Given such a global state we have shown that strong kinematical fluctuations are conserved, and the state is maintained by a strong coupling of backreaction to the averaged Ricci curvature. This has been exemplified by an exact solution of the averaged Einstein equations for a globally stationary cosmos. We may say that large kinematical fluctuations are maintained at the cost of averaged scalar curvature in this solution.
We have considered three families of cosmologies that belong to the out-of-equilibrium state of vanishing global acceleration. The first family covers globally stationary cosmologies without a cosmological constant; as a subcase, a globally static cosmos is possible that revives the original Einstein cosmos concerning its global properties, however, having very different properties on regional scales. The second family covers models that obey the stationarity condition of vanishing global acceleration, but include a cosmological constant sharing the balance condition. In this case we have also given the exact solution. Finally, the third family rests on the stationarity conditions of the first family, but the global acceleration is non-zero due to the cosmological constant alone. Fluctuations and average characteristics in such \( \Lambda \)-driven cosmologies have very similar properties to the previous cosmologies, however, with the presence of a global exponential expansion.

Although, formally, these new families of cosmologies obey simple equations and are in this respect very close to Friedmannian cosmologies, their dynamical properties must be regarded as significantly different. Investigating details of the new models must be considered as an endeavour of high magnitude, especially if one is interested in the regional properties. One important reason for this difficulty lies in the scale dependence of the average characteristics. For example, the Einstein static model has globally and locally a constant curvature and a constant density, while the globally static, but inhomogeneous cosmos features strong matter and curvature fluctuations on regional scales. The latter model enjoys the freedom that the averages can be scale dependent, e.g., the Einstein static model has (like all Friedmannian cosmologies) a scale-independent density, while the globally static cosmos allows for a decay of the average density by going to larger scales. It is possible that fluctuations, encoded in \( Q_D \), on a regional domain \( D \) are large relative to the averaged density \( 4\pi G \langle \rho \rangle_D \) on that domain, but \( Q_\Sigma \) could be very small on the global scale \( |\Sigma| \) in absolute terms, and still of the order of \( 4\pi G \langle \rho \rangle_\Sigma \).

These new models need detailed investigation before any conclusive result relevant to observations can be obtained. In the light of the present investigation three lines of research appear to be fruitful strategies. First, investigating global stability properties of the presented cosmologies, i.e. studying perturbations on non-Friedmannian ‘backgrounds’ including studies of gravitational instability on regional domains. Second, investigating the relativistic generalization of [23] and comparing standard with globally inhomogeneous initial conditions and, third, investigating inhomogeneous inflationary scenarios and the properties of fluctuations at the exit epoch, i.e., studying the role of kinematical backreaction in the early universe.

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Appendix A. The size of a globally static cosmos

In order to get an idea about typical numbers characterizing a globally static cosmos, we are going to discuss some numerical estimates. We already imply that there exist regional fluctuations of this global state; we take the classical Einstein cosmos as the global model and infer our estimates from regionally observed properties.
Dividing Friedmann’s differential equation (2) by \( H^2 \) (here, the static case is excluded), we obtain Hamilton’s constraint for the homogeneous isotropic case, written in the iconized form of a cosmic triangle [7]:

\[
\Omega_m + \Omega_\Lambda + \Omega_k = 1; \quad \Omega_m := \frac{8\pi G \rho_0}{3H^2}, \quad \Omega_\Lambda := \frac{\Lambda}{3H^2}, \quad \Omega_k := \frac{-k}{a^2H^2}. \tag{A.1}
\]

Confining ourselves now to our regional Hubble volume, we write \( \Omega_m (\text{Hubble scale}) =: \alpha_m \) and express \( H \) through \( h := H/100 \text{ km s}^{-1} \text{ Mpc}^{-1} \). From (5) with \( k = +1 \) and (6) we first obtain

\[
\alpha_E = \frac{1}{\sqrt{4\pi G \rho_0}} = \frac{1}{\pi}2GM_E = \frac{1}{\pi}a_{\text{Schwarzschild}}. \tag{A.2}
\]

Extrapolating the value of the matter density \( \rho_h = \frac{1}{8\pi G} \alpha_m H^2 = \rho_E \), we find for the Einstein radius in space units

\[
r_E = c a_E = \frac{c}{\sqrt{\alpha_m h}} \sqrt{\frac{2}{3}} \times 10^{-8} \text{ s Gpc m}^{-1}, \tag{A.3}
\]

which for \( \alpha_m \approx 0.3 \) and \( h \approx 0.6 \) yields \( r_E \approx 7.5 \text{ Gpc} \). For comparison, if \( \alpha_m = 1 \) and \( h \approx 0.46 \) [13] we would get \( r_E \approx 5.3 \text{ Gpc} \), for \( \alpha_m \approx 0.3 \) and \( h \approx 0.46 \), \( r_E \approx 9.7 \text{ Gpc} \).

Defining a Hubble volume roughly by \( V_H = \frac{4\pi}{3} (c/H)^3 \) (where we took the horizon radius of a Friedmannian model, \( r_H := c/H \approx 3 \text{ Gpc h}^{-1} \), as the radius of an Euclidean sphere randomly placed within the Einstein cosmos), we get for the first set of values above \( V_H \approx 174.5 \text{ Gpc}^3 \), and with \( V_E = 2\pi^2 r_E^3 \approx 8319 \text{ Gpc}^3 \), we conclude that a fluctuating static cosmos with the radius of the homogeneous Einstein cosmos would roughly contain 50 Hubble volumes. Thus, the total volume \( V_E \) is comfortably large to allow for significant regional fluctuations on the Hubble scale.

The regional curvature that we would observe on the Hubble scale will not be the global curvature of the Einstein cosmos. However, if we assume for simplicity that the curvature is non-fluctuating, but that we have a fluctuating expansion being globally zero and regionally given by \( \dot{h} \) (as will be possible in the general model discussed in the text), we could estimate the global Friedmannian curvature parameter from regional observations, setting \( \Omega_k \) (Hubble scale) := \( \alpha_k = \frac{-k}{a^2H^2} \); we obtain with the first set of values above \( \alpha_k = -\frac{c^2}{H^2} \times 10^{-16} \text{ s}^2 \text{ Gpc}^2 \text{ m}^2 \approx -0.44 \).

Thus, if we would ‘fit’ a Friedmannian model within the regional Hubble volume, we would obtain from (A.1) (0.3; -0.44; 1.14) for values of the regional cosmic triangle \( (\alpha_m; \alpha_k; \alpha_\Lambda) \), i.e., straight application of this (quickly cooked) model would be in trouble compared with the values of the so-called concordance model as a ‘best-fit’ Friedmannian cosmology in comparison with observations (0.3; ±0; 0.7).

However, these estimates are naive in a variety of ways. First, the averaged curvature is not given by the Friedmannian curvature parameter, if kinematical fluctuations are present. Second, a fluctuating cosmos implies fluctuations in scalar curvature too, i.e., it is easily conceivable that the magnitude of the regional curvature could be smaller than the above estimate. Moreover, as we have learned, if expansion fluctuations are assumed large, then we can as well dismiss the cosmological constant altogether. Third, a further feature of globally static inhomogeneous cosmological models is that they would allow for a significantly larger size compared to the values quoted above, since the average density may likely be scale dependent and may decrease towards the global scale.
Appendix B. Evolution of $\Omega_1$-parameters in the globally stationary cosmos compared with their evolution in Friedmannian cosmologies for $\Lambda = 0$

At first glance, the reader may think that a globally stationary cosmos, introduced in section 3, looks very similar to a Friedmannian cosmos, since the scale factor has a power-law form, $a_S \propto t$, and the global expansion $H_S \propto t^{-1}$, but in fact both cosmologies are drastically different in nature. This can be nicely illustrated with the evolution of the density parameter, which is the only free parameter in both cosmologies, if $\Lambda = 0$.

For this comparison we employ equation (19) on the global scale, and evaluate this equation with the solution of the stationary cosmology $Q_S = Q_S(t_i)a_S^{-3}$

$$H_S^2(1 - \Omega_m^S)a_S^2 + k_S = \frac{2}{3} \int_0^t dt' Q_S a_S^2 a_{\Sigma} = -\frac{2}{3} Q_S(t_i) \left( \frac{1}{a_S} - 1 \right),$$  \hspace{0.5cm} \text{(B.1)}

with $k_S = (Q_S(t_i) + \langle R \rangle_S(t_i))/6 = H_S^2(\Omega_m^S - 1)$, and the density parameter $\Omega_m^S$ in the general model (20). From the definition of the latter we have a second equation

$$\Omega_m^S = \Omega_m \frac{H_S^2}{H_S^2 a_S^3}, \hspace{0.5cm} \text{(B.2)}$$

Inserting (B.2) into (B.1), a simple calculation provides

$$\Omega_m^S = \frac{\Omega_m^S}{\Omega_m(1 - \kappa) + [1 - \Omega_m^S(1 - \kappa)] a_S}, \hspace{0.5cm} \text{(B.3)}$$

with $\kappa = 1$ for the stationary inhomogeneous model with non-zero kinematical backreaction, $Q_S(t_i) = \frac{1}{2} \Omega_m^S H_S^2$; and $\kappa = 0$ for the Friedmannian models. This minor formal difference amounts to a substantially different evolution. The reader may insert numbers to compare their evolution in Friedmannian cosmologies for $\kappa = 1$.

These conditions can easily be derived from the fact that $3Q_S + \langle R \rangle_S = 6H_S^2 > 0$. The infinities are reached at the ‘big crunch’ of a contracting stationary model with $C < 0$, in which case $a_S \rightarrow 0$ in a finite time.

$$+\infty > \Omega_m^S > +\frac{4}{3}; \hspace{0.5cm} -\infty < \Omega_{\delta}^S < +\frac{2}{3}; \hspace{0.5cm} -\infty < \Omega_{\delta}^2 < -\frac{1}{2}. \hspace{0.5cm} \text{(B.6)}$$
It should be emphasized that these are the global values of the cosmological parameters; on regional scales very different values may be measured, whereas in the standard model of cosmology global and regional values have to be equal.

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