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Modeling and simulation of the novel coronavirus in Caputo derivative

Muhammad Awais\textsuperscript{a}, Fehaid Salem Alshammari\textsuperscript{b}, Saif Ullah\textsuperscript{c}, Muhammad Altaf Khan\textsuperscript{d,e,*}, Saeed Islam\textsuperscript{a}

\textsuperscript{a} Department of Mathematics, Abdul Wali Khan University Mardan, Khyber Pakhtunkhwa, Pakistan
\textsuperscript{b} Department of Mathematics and Statistics, Faculty of Science, Imam Mohammed Ibn Saud Islamic University, Riyadh 13318, Saudi Arabia
\textsuperscript{c} Department of Mathematics University of Peshawar, Pakistan
\textsuperscript{d} Informetrics Research Group, Ton Duc Thang University, Ho Chi Minh City, Viet Nam
\textsuperscript{e} Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Viet Nam

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\textbf{ABSTRACT}

The Coronavirus disease or COVID-19 is an infectious disease caused by a newly discovered coronavirus. The COVID-19 pandemic is an inciting panic for human health and economy as there is no vaccine or effective treatment so far. Different mathematical modeling approaches have been suggested to analyze the transmission patterns of this novel infection. In this paper, we investigate the dynamics of COVID-19 using the classical Caputo fractional derivative. Initially, we formulate the mathematical model and then explore some of the basic and necessary analysis including the stability results of the model for the case when \( R_0 < 1 \). Despite the basic analysis, we consider the real cases of coronavirus in China from January 11, 2020 to April 9, 2020 and estimated the basic reproduction number as \( R_0 \approx 4.95 \). The present findings show that the reported data is accurately fit to the proposed model and consequently, we obtain more realistic and suitable parameters. Finally, the fractional model is solved numerically using a numerical approach and depicts many graphical results for the fractional order of Caputo operator. Furthermore, some key parameters and their impact on the disease dynamics are shown graphically.

\textbf{Introduction}

Coronavirus disease (COVID-19) is a highly contagious viral disease that affects millions of humans and caused reasonable death cases around the world. The infection was named by the World Health Organization (WHO) as COVID-19. The infection for the first time was identified in Wuhan China, which is the capital city of Hubei Province in Central China. Later, on 11th March 2020, WHO announced the COVID-19 disease as a pandemic spreading around the globe. According to the recent WHO statistics reported on 11 May 2020, there are 3976043 confirmed COVID-19 cases and it has taken the lives of more than 277708 people in the whole world [1]. About 215 countries, areas or territories are targeted by this novel infection so far and it is still posing a serious threat to the other countries on the globe. There are no specific vaccines or treatment for COVID-19 up till now and most of the patients have recovered without specific treatment or hospitalization. This infection affected people indiscriminately of all ages, nonetheless, people suffering from co-morbidities like diabetes, cardiovascular disease, cancer and chronic respiratory disease and with older age are at higher risk to develop serious illness leading to death.

This infection is extremely contagious, spreading from an infected person through droplets discharge from the nose while spitting, coughing, or sneezing. Although, the coronavirus affected people in different ways but the most common symptoms of this infection include dry cough, fever, aches and pains, headache and excessive fatigue, etc., while some of the serious symptoms are considered to the loss of speech or movement, chest pain or pressure, and the breath shortness. Normally, it takes five to six days from initial infection to show symptoms however, it can take up to fourteen days [1,2]. The people once infected with this virus should transmit the virus to other people by shaking hands, so, social distances, and restricted itself at home, and using masks and gloves during in open exposure when desired, should be followed to get safe.

Due to the severity and life-threatening situation of the COVID-19 pandemic on health and society, different researchers, medical doctors, scientists and other health departments are adopting different...
approaches in order to explore the transmission patterns and possible control of this infection. Despite much scientific improvements, the WHO stresses that an urgent understating of the epidemiology and the evolution of the COVID-19 the outbreak is needed for the control strategies to stop the transmission. In this regards, one of the essential tools is the infectious mathematical modeling approach, which has adopted in recent days in order to explore the insights into the transmission, severity, and particularity of the disease, which could readily inform decision-making concerns in mitigating this infection [3,4]. In the last couple of months, the researchers around the world of different field published their research related to coronavirus. Regarding the mathematical models for the coronavirus, many researchers developed mathematical models to understand the transmission dynamics of coronavirus infection in different regions. In [5], Li et al., an epidemic COVID-19 model with the latency period has been developed and the model has been fitted to the incidence data reported in the mainland of China. In [6], the authors formulated SIR based model for the A COVID-19 infection and implemented the proposed model to explore and predict the transmission dynamics of this pandemic for the countries with a high number of cases such as France, China and Italy. In [7], Ribeiro et al. employed regression models to predict COVID-19 infected cases in Brazil. While in [8], Ndairou et al. developed a transmission model with super-spreaders infected class and implemented their model to the reported infected case of Wuhan. They have shown that the suggested mathematical model provides a better agreement to the reported COVID-19 cases and deaths that have accounted due to coronavirus infection. Recently, the role of non-pharmaceutical intervention on COVID-19 dynamics by considering the data of Pakistan has been studied in [9]. The authors used the real data of coronavirus of Pakistan and implemented a mathematical model with optimal control analysis.

The fractional epidemic model is another strong tool to explore the dynamics of infectious disease in a better way than the ordinary integer-order models. These models are based on fractional order differential operators, which generalizes the classical integer-order derivatives. Epidemic models with fractional derivatives give a greater degree of accuracy and provide a better fit to the real data in compression with classical integer-order models [10,11]. A diverse variety of fractional operators were introduced time to time in the literature with a different kernel. Some of the frequently used fractional orders (FO) operators are Caputo [12], Caputo-Fabrizio (CF) [13] and Atangana-Baleanu operator (AB) [14]. Although most of the COVID-19 models developed so far are based on classical integer-order derivative but only a few can be found with fractional operators. For example, in [15] Khan and Atangana formulated a fractional model of COVID-19 using Atangana-Bleaneu in Caputo sense (ABC) operator and provided a better fit to the reported case in Wuhan. Recently, in [16] Abdo et al. extended a classical COVID-19 model in the literature to fractional order using Mittag-Leffler kernel and carried out the Ulam stability analysis and simulation results. Atangana studied a new COVID-19 transmission model using different fractal-fractional operators [17]. The results are obtained both mathematically and statistically considering the real cases. Further, the lockdown effect in the model has been analyzed. The authors in [18] formulated a mathematical model for coronavirus from natural to human host. Some recent mathematical studied on corona and other infectious diseases are studied by authors see [19–22]. A novel results for the corona virus model has been discussed in [19]. The application of fractal-fractional differential equations to partial differential equations has been analyzed in [20]. The authors in [21] presented a theoretical results for fractional partial differential equations. Fractal fractional differential equations and its solutions in detailed with novel analysis is presented in [22].

Inspired by the above discussion, in the present study, we reformulate the model [15] using the fractional operator in the Caputo sense. In [15], they considered the reported cases for a short period of time from January 21, 2020 to January 28, 2020, in Wuhan city China. In this study, we consider the reported case in the mainland of China from January 21, 2020 to April 9, 2020 and explore a better theoretical and graphical analysis of the model. The data is taken from [23]. Using the reported data we will obtain the real parameter values for the model and will perform the simulation to predict the dynamics of coronavirus in the community. We will show that the model provides better fitting to the reported cases. Some important parameters that can help the disease under control will be explored in detail. The rest of the results are organized is as follows: The concepts and basic results regarding fractional and Caputo fractional derivative are recalled in Section 2. Formulation of the model, the integer case, the model, and their fitting to the coronavirus cases and parameters are discussed in Section 3. The COVID-19 model with Caputo fractional derivative, its basic mathematical properties as well as stability results are given in Section 4. The numerical iterative scheme and the simulation results of the fractional model are plotted and discussed in Section 5. Finally, a brief concluding remarks are given in Section 6.

Basics of the fractional calculus

In the following, we first recall some basic details of fractional calculus [12].

**Definition 1.** Let \( x(t) \in C^r(0, \infty), \mathbb{R} \) be a function then the fractional derivative in Caputo sense having order \( r \) for \( p \leq n \) where \( n \in \mathbb{N} \) is given as:

\[
^C D_t^p x(t) = \frac{1}{\Gamma(n-p)} \int_0^t \frac{X^p(t)}{(t-s)^{n-p}} ds, \quad t > 0.
\]

Clearly, \(^C D_t^p x(t)\) approaches \( x(t)\) as \( p \to 1\).

**Definition 2.** A fixed point say, \( x^* \) satisfying \( f(t, x^*) = 0 \), then it is called the equilibrium point of the model formulated via Caputo derivative given by:

\[
^C D_t^p x(t) = f(t, x(t)), \quad p \in [0, 1],
\]

As proved in [24], the point \( x^* \) will be locally asymptotically stable if all eigenvalues \( \Pi \) of the Jacobian matrix calculated around the equilibrium point satisfy the following condition:

\[
\arg \left( \frac{\Pi}{\pi} \right) > \frac{\alpha \pi}{2}, \quad \alpha \in (0, 1].
\]

For the global stability results of the fractional system in Caputo derivative considering the Lyapunov method, we need to have the following important results [25,26].

**Theorem 0.1.** Suppose \( x^* \) be an equilibrium point of the system (1) and \( \Psi \in \mathbb{R}^n \) be a domain containing \( x^* \), and let \( L : [0, \infty) \times \Psi \to \mathbb{R} \) be a continuously differentiable function satisfying

\[
M_1(x) \leq L(t, x(t)) \leq M_2(x),
\]

and

\[
^C D_t^p L(t, x(t)) \leq -M_3(x),
\]

\[
0 < p < 1 \quad \text{and} \quad x \in \Psi. \quad \text{Where} \ M_1(x), M_2(x) \quad \text{and} \ M_3(x) \quad \text{denote} \quad \text{the continuous positive definite functions on} \quad \Psi. \quad \text{Then}, \quad x^* \quad \text{will be uniformly} \quad \text{asymptotically stable equilibrium point of the (1).}
\]

**Mathematical model**

In this section, we briefly explain the model formulation using classical integer-order derivative. To develop the model total human population denoted by \( N(t) \) is further classified into five mutually-exclusive epidemiological classes, which are the susceptible \( S(t) \),
exposed $E(t)$, infected (symptomatic) $I(t)$, asymptotically infected showing no clinical symptoms $A(t)$, and the recovered people $R(t)$. The class $M(t)$ denotes the COVID-19 in reservoir or the seafood place or market. The recruitment rate is shown by the $\Lambda$ while the natural death rate is represented by $\mu$. The symptomatic and asymptomatic infected people could export the virus into $M$ at the rate $\eta$ and $\nu$, respectively. The virus in $M$ will subsequently leave the $M$ class at a rate of $\nu$, where $1/\nu$ accounts the lifetime of the COVID-19 virus. The susceptible individuals acquired infection after effective contacts with the people in $I$ and $A$ compartments at the rate $\frac{\eta I}{N}$, where the parameter $\eta$ is the disease transmission coefficient and $\psi$ is the transmissibility multiple of $A$ to $I$ and $0<\psi \leq 1$. Further, we assume that susceptible people could be infected after the interaction with $M$, given by $\eta_S M$ where, $\eta_S$ is the transmission rate from $M$ to $S$. The exposed people develop the COVID-19 infection after the completion of incubation period and then move to either class $I$ or $A$ at the transmission rate $\omega$ and $\rho$ respectively. The parameter $\theta$ is the proportion of asymptomatic infection. The infectious period of symptomatic $I$ and asymptomatic $A$ individuals are defined as $1/\tau$ and $1/\tau_A$ respectively after which they enter to recovered class $R$. The nonlinear differential equations that describes the dynamics of the COVID-19 disease are given by [15]:

\[
\begin{align*}
\frac{dS}{dt} &= \Lambda - \frac{\eta(I + \psi A)}{N} S - \eta_S M S - \mu S, \\
\frac{dE}{dt} &= \frac{\eta(S + \psi A)}{N} S + \eta_S M S - \left( \theta \rho + \left( 1 - \theta \right) \omega + \mu \right) E, \\
\frac{dI}{dt} &= \left( 1 - \theta \right) \omega E - \left( \tau + \mu \right) I, \\
\frac{dA}{dt} &= \theta \rho E - \left( \tau_A + \mu \right) A, \\
\frac{dR}{dt} &= \tau_A + \tau I - \mu R, \\
\frac{dM}{dt} &= \psi I + \omega A - \nu M.
\end{align*}
\]  

The corresponding initial conditions are

\[
S(t_0) = S_0 \geq 0, \quad E(t_0) = E_0 \geq 0, \quad I(t_0) = I_0 \geq 0, \quad A(t_0) = A_0 \geq 0, \quad R(t_0) = R_0 \geq 0, \quad M(t_0) = M_0 \geq 0. 
\]  

For simplicity, let us denote

\[
\lambda(t) = \frac{\eta(I + \psi A)}{N}, \quad k_1 = (\theta \rho + (1 - \theta) \omega + \mu), \quad k_2 = \left( \tau + \mu \right), \quad k_3 = (\tau_A + \mu).
\]

Then, the above model can be written as

\[
\begin{align*}
\frac{dS}{dt} &= \Lambda - \lambda \left( t \right) S - \mu S, \\
\frac{dE}{dt} &= \lambda \left( t \right) S - k_1 E, \\
\frac{dI}{dt} &= \left( 1 - \theta \right) \omega E - k_2 I, \\
\frac{dA}{dt} &= \theta \rho E - k_3 A, \\
\frac{dR}{dt} &= \tau_A + \tau I - \mu R, \\
\frac{dM}{dt} &= \psi I + \omega A - \nu M.
\end{align*}
\]  

Next, we estimate the parameters of the model and provide the method with brief explanation.

Data fitting and estimation of the parameters

We investigating the parameters estimations and curve fitting for the reported data of coronavirus cases from January 21, 2020 to April 9, 2020, in the mainland of China to the model (5). We utilized the approach first by selecting some of available parameters from the literature and then the rest of the parameters are fitted to the coronavirus cases. We consider the time unit per day. In the following, we provide a brief explanations of the parameters estimation:

(i) Natural mortality rate $\mu$: The life span an average in China for the year 2019 is 76.79 years [27], so, the estimated value of natural death rate is $\mu = 1/76.79$ per year.

(ii) Recruitment rate $\Lambda$: Population of China for the year 2019 is about $N(0) = 1.43$ billion [27], so, we can obtain the recruitment rate parameter by computing $\Lambda/\mu = N(0)$, and also under the assumption that in the disease absence it is the limiting population, so we have $\Lambda = 46381$ per day.

Using the above parameters values and the rest of the parameters are simulated and fitted to the actual cases that is depicted in Fig. 1 using the method of least-square curve fitting briefly discussed and utilized in [28,29]. The best fitted model parameters are calculated by minimizing the error between the reported infected data and the infected cases obtained from the model (5). The objective function implemented in the estimation procedure is as follows:
The Caputo fractional model

We extend the classical integer model by replacing the derivative with the Caputo fractional-order derivative in order to provide a better understanding of the COVID-19 infection. The parameter \( p \in (0, 1) \) in the model is used for the order of Caputo derivative. Hence, the model is given through the following system of nonlinear fractional differential equations:

\[
\begin{align*}
\mathcal{C}D^p_t S(t) &= \Lambda - \lambda(t)S - \mu S, \\
\mathcal{C}D^p_t E(t) &= \lambda(t)S - (\beta + (1 - \theta)\omega + \mu)E, \\
\mathcal{C}D^p_t I(t) &= (1 - \theta)\omega E - (\tau_a + \mu)I, \\
\mathcal{C}D^p_t A(t) &= \beta E - (\tau_a + \mu)A, \\
\mathcal{C}D^p_t R(t) &= \tau_a A + \tau_l - \mu R, \\
\mathcal{C}D^p_t M(t) &= qI + mA - kM,
\end{align*}
\]

with the initial conditions given in (6).

Existence and positivity of the Caputo model solution

To proceed the model analysis, first we explore the basic necessary mathematical properties. For the non-negativity of the model solution, let us consider the following set

\[ \mathbb{R}_+^m = \left\{ (S, E, I, A, R, M) \in \mathbb{R}_+^m : S \geq 0, E \geq 0, I \geq 0, A \geq 0, R \geq 0, M \geq 0 \right\}. \]

We follow the approach used in [30] in order to prove the desired results and provide the following theorem.

**Lemma 0.1.** [30] Consider, \( \mathcal{A}(t) \in \mathcal{C}[r_1, r_2] \) and \( \mathcal{C}C^p_t \mathcal{A}(t) \in \mathcal{C}[r_1, r_2] \), then

\[
\mathcal{A}(t) = \mathcal{A}(r_1) + \frac{1}{\Gamma(p)} \mathcal{C}C^p_t \mathcal{A}(\xi) (t - r_1)^{p-1},
\]

with \( r_1 < \xi < t \), \( t \in (r_1, r_2] \).

**Corollary 1.** [30] Suppose that \( \mathcal{A}(t) \in \mathcal{C}[r_1, r_2] \) and \( \mathcal{C}C^p_t \mathcal{A}(t) \in \mathcal{C}[r_1, r_2] \), where \( p \in (0, 1) \), then the following conclusions are drawn if

(i) \( \mathcal{C}C^p_t \mathcal{A}(t) \geq 0, \forall t \in (r_1, r_2) \), then \( \mathcal{A}(t) \) is non-decreasing.

(ii) \( \mathcal{C}C^p_t \mathcal{A}(t) \leq 0, \forall t \in (r_1, r_2) \), then \( \mathcal{A}(t) \) is non-increasing.

**Theorem 0.2.** For the fractional model (9) there exists a solution, say, \( y(t) \) which is unique and will remain in \( \mathbb{R}_+^m \) and non-negative.

**Proof.** Utilizing the approach presented in [31], it is easy to show the existence of the Caputo model (9). Further, the uniqueness of the solution can be obtained by making use of the Remark 3.2 in [31] for all values of \( t > 0 \). For the positivity of the solution, it is necessary to show that on each hyperplane bounding the positive orthant, the vector field points to \( \mathbb{R}_+^m \). From the system (9), we deduced that

\[
\begin{align*}
\mathcal{C}D^p_t S(t) |_{t=0} &= \Lambda \geq 0, \\
\mathcal{C}D^p_t E(t) |_{t=0} &= \lambda(t)S |_{t=0} \geq 0, \\
\mathcal{C}D^p_t I(t) |_{t=0} &= (1 - \theta)\omega E |_{t=0} \geq 0, \\
\mathcal{C}D^p_t A(t) |_{t=0} &= \beta E |_{t=0} \geq 0, \\
\mathcal{C}D^p_t R(t) |_{t=0} &= \tau_a A |_{t=0} \geq 0, \\
\mathcal{C}D^p_t M(t) |_{t=0} &= qI |_{t=0} \geq 0.
\end{align*}
\]

Hence, based on the above corollary, we concluded that the solution will remain in \( \mathbb{R}_+^m \), for all \( t \geq 0 \).

Finally, to confirm the boundedness of the fractional model, we proceed by summing the first five equations of the model (9), which give

\[
\mathcal{C}D^p_t N(t) = \Lambda - \mu N(t). \tag{10}
\]

Applying the Laplace transform of (10), we obtained

\[
L[\mathcal{C}D^p_t N(t) + \mu N(t)] = L[\Lambda],
\]

\[
\mathcal{F}(s)N(s) + \frac{\mu}{s}N(0) + \mu N(s) = \frac{\Lambda}{s}.
\]

\[
N(s) = \frac{\Lambda}{s(\mathcal{F} + \mu)} - N(0) \frac{e^{-st}}{\mathcal{F} + \mu}.
\]

Applying Laplace inverse, we obtained

\[
N(t) = N(0)E_{\mu}(-\mu t^\mu) + \Lambda E_{\mu+1}(-\mu t^\mu).
\]

The Mittag–Leffler function describe by infinite power series i.e;

\[
E_{\mu,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k + \beta)}
\]

and laplace transform of Mittag–leffler function is Thus keeping the fact in mind that the Mittag–Leffler function has asymptotic behavior [12], we obtain

\[
\lim_{t \to \infty} N(t) = \frac{\Lambda}{\mu}.
\]

Hence, the biologically feasible region is constructed as:

\[
\Omega = \left\{ (S, E, I, A, R, M) \in \mathbb{R}_+^m : S, E, I, A, R, M \geq 0 : N \leq \frac{\Lambda}{\mu} \right\}.
\]

Next, we investigate the equilibria and the associated basic reproduction number \( R_0 \).
Model equilibria and basic reproduction number

The equilibrium points of the Caputo fractional model \( (9) \) can be evaluated by setting
\[
\begin{align*}
\frac{d^\alpha S(t)}{dt^\alpha} &= 0, \\
\frac{d^\alpha E(t)}{dt^\alpha} &= 0, \\
\frac{d^\alpha I(t)}{dt^\alpha} &= 0, \\
\frac{d^\alpha A(t)}{dt^\alpha} &= 0, \\
\frac{d^\alpha R(t)}{dt^\alpha} &= 0, \\
\frac{d^\alpha M(t)}{dt^\alpha} &= 0.
\end{align*}
\] (12)

The disease free equilibrium (DFE) denoted by \( \mathcal{D}_0 \) is obtained by solving the system (12) at disease free state
\[
\mathcal{D}_0 = (S^0, 0, 0, 0, 0, 0) = (\Lambda/\mu, 0, 0, 0, 0).
\]

Further, using next generation approach we evaluate the following expression of the basic reproduction number:
\[
\mathcal{R}_0 = \frac{\theta (\psi \mu + \Lambda \psi \eta)}{\psi \mu k_1} + \frac{(1 - \theta)\omega (\psi \mu + \Lambda \psi \eta)}{\psi \mu k_2} x_1.
\]

Existence of endemic equilibrium

The endemic equilibrium (EE) of the Caputo model (9) is represented by \( \mathcal{D}_e \) and is given as follows:
\[
\begin{align*}
S^* &= \frac{\Lambda}{\lambda^* + \mu}, \\
E^* &= \frac{\lambda^* S^*}{k_1}, \\
I^* &= \frac{(1 - \theta)\omega}{k_2} E^*, \\
A^* &= \frac{\theta \rho}{k_2} E^*, \\
R^* &= \frac{\tau_a A^* + \tau I^*}{\mu} , \\
M^* &= \frac{\alpha A^* + \phi I^*}{\nu},
\end{align*}
\]
and
\[
\lambda^* = \frac{\eta (I^* + \omega A^*)}{S^* + E^* + I^* + A^* + R^*} + \eta M^*,
\]

satisfying the following equation
\[
P(\lambda^*) = m_1(\lambda^*)^2 + m_2 \lambda^*.
\] (13)

The coefficients in (13) are as follows:
\[
\begin{align*}
m_1 &= \psi k_1 k_3, \\
m_2 &= \psi k_2 k_3 (1 - \mathcal{R}_0).
\end{align*}
\] (14)

Clearly, \( m_1 > 0 \) and \( m_2 > 0 \) if \( \mathcal{R}_0 < 1 \), and \( \lambda^* = -\min_{m_1} \). Thus, no endemic equilibrium will exist if \( \mathcal{R}_0 < 1 \).

Stability of the DFE

In order to confirm the local stability of DFE it is enough to show that the eigenvalues of the matrix of dynamics given (16) lie outside the closed angular sector as proved in [24].

The following theorem deals with the desired result.

**Theorem 0.3.** DFE \( \mathcal{D}_0 \) of the model is locally asymptotically stable (LAS) if for all eigenvalues \( \lambda_i \) of the Jacobian matrix \( J_{\mathcal{D}_0} \) of the model (9) satisfy the following condition [24]:
\[
\left|\arg \left( \lambda_i \right) \right| > \frac{\psi \theta}{2}, \quad i = 1, 2, 3, 4, 5, 6.
\] (15)

**Proof.** The Jacobian of linearization matrix of model is given as:
\[
J_{\mathcal{D}_0} = \begin{pmatrix}
-\mu & 0 & -\eta & -\psi & 0 & \frac{\eta \nu}{\mu} \\
0 & -k_1 & \eta & \psi & 0 & \frac{\eta \nu}{\psi} \\
0 & (1 - \theta)\omega & -k_2 & 0 & 0 & 0 \\
0 & \theta \rho & 0 & -k_2 & 0 & 0 \\
0 & 0 & \tau & \tau_a & -\mu & 0 \\
0 & 0 & \tau & 0 & -\nu & 0
\end{pmatrix}.
\] (16)

The characteristic equation in term of \( \lambda \) for \( J_{\mathcal{D}_0} \) is given below:
\[
(\lambda + \mu)^2 (\lambda^4 + a_1 \lambda^3 + a_2 \lambda^2 + a_3 \lambda + a_4) = 0,
\] (17)

where,
\[
a_1 = k_1 + k_2 + \nu + \lambda, \\
a_2 = (k_1 k_3 - \eta \theta \psi) + (k_2 - \eta (1 - \theta) \omega) + (k_1 + k_2 + \nu) k_3, \\
a_3 = \psi k_1 [k_3 (1 - \mathcal{R}_1) + k_3 (1 - \mathcal{R}_2)] + k_2 [k_3 (1 - \mathcal{R}_3)] + \eta \theta \psi k_3 + k_3 (k_2 - \eta \omega), \\
a_4 = \psi k_1 [k_1 (1 - \mathcal{R}_0)].
\]

From (17) the argument of the first two eigenvalues \( -\mu \), satisfy the condition given in (15) for all \( p \in (0, 1] \). Further, it is clear that if \( \mathcal{R}_0 < 1 \), then all \( a_i > 0 \), and it is easy to show that \( a_1 a_2 a_3 > a_2^2 + a_2^2 a_4 \). Thus, following [24,35], the arguments of all eigenvalues satisfy the necessary condition given in (15). This completes the proof.

Global asymptotic stability of the DFE

In this part, we prove the global asymptotical stability (GAS) of the fractional model (9). For this purpose, the following result is presented.

**Theorem 0.4.** For \( p \in (0, 1] \) the DFE \( \mathcal{D}_0 \) of the proposed model (9) is GAS on \( \Omega \) if \( \mathcal{R}_0 < 1 \).

**Proof.** To present the proof, let us define the following suitable Lyapunov function:
\[
\mathcal{F}(t) = \mathcal{A}_1 E(t) + \mathcal{A}_2 I(t) + \mathcal{A}_3 A(t) + \mathcal{A}_4 M(t),
\]

where \( \mathcal{A}_j \), for \( j = 1, \ldots, 4 \), which are positive constants to be consider later. Taking the time Caputo fractional derivative of \( \mathcal{F}(t) \) we obtain
\[
\frac{d^\alpha \mathcal{F}(t)}{dt^\alpha} = \mathcal{A}_1 D^\alpha E + \mathcal{A}_2 D^\alpha I + \mathcal{A}_3 D^\alpha A + \mathcal{A}_4 D^\alpha M.
\]

Using system (9), we get
\[
\frac{d^\alpha \mathcal{F}(t)}{dt^\alpha} = \mathcal{A}_1 \left( \frac{\eta (I + \psi A)S}{N} + \eta M_S - k_1 E \right) + \mathcal{A}_2 (\psi E - \lambda k_1) \mathcal{A}_1 + \mathcal{A}_3 \left( \theta \psi E - \lambda k_2 \right) \mathcal{A}_1 + \mathcal{A}_4 \left( \psi \mu E - \lambda M \right) \mathcal{A}_1
\]
\[
\leq \mathcal{A}_1 \left( \eta (I + \psi A) + \eta M_S - k_1 E \right) + \mathcal{A}_2 (\psi E - \lambda k_1) \mathcal{A}_1 + \mathcal{A}_3 \left( \psi \mu E - \lambda M \right) \mathcal{A}_1
\]
\[
= \mathcal{A}_1 \left( \theta \psi E - \lambda k_3 \right) A + \mathcal{A}_2 \left( \psi E - \lambda k_4 \right) A + \mathcal{A}_3 \left( \theta \psi E - \lambda k_5 \right) A + \mathcal{A}_4 \left( \psi \mu E - \lambda M \right) A
\]
\[
+ \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 \mathcal{A}_4 \left( k_3 E \right) = \mathcal{A}_1 \mathcal{A}_2 \mathcal{A}_3 \mathcal{A}_4 \left( k_3 E \right)
\]
\[
\leq \mathcal{A}_1 S^0 \mathcal{A}_2 S^0 \mathcal{A}_3 S^0 \mathcal{A}_4 S^0.
\]

Let us choose the constants values as
Fig. 3. Dynamics of fractional COVID-19 model when $p = 1, 0.95, 0.90, 0.85, 0.80$. 
\[ A_1 = \nu \mu, \quad A_2 = \nu \eta \mu + \Lambda \rho \eta \omega, \quad A_3 = \nu \nu \eta \mu + \Lambda \nu \omega \eta \omega, \quad A_4 = \Lambda \eta \omega, \]

and then after some simplification, we have,

\[ \text{C}D_p t F(t) \leq \nu \mu k_1 \{ R_0 - 1 \} E. \]

It is clear that when \( R_0 < 1 \) then \( \text{C}D_p t \mathcal{S}(t) \) is negative, because all the model parameters are non-negative. Thus, it follows from the results given in Theorem 0.1, that the DFE \( \mathcal{S}_0 \) is GAS in \( \Omega \).

**Numerical solution of fractional model**

This section investigates the numerical solution of the fractional model (9) and to present the simulation results for various values of model parameters and \( p \in (0, 1) \). In order to do this, we utilize the Euler’s type approach presented discussed in the recent literature and references therein [32,33]. To obtain the iterative scheme, let us express the fractional model (9) in the following simple form:

\[
\begin{aligned}
&\quad \text{C}D_p t \mathcal{S}(t) = \mathcal{S}(t, g(t)), \\
&\quad g(0) = g_0, \quad 0 < \mathcal{S} < \infty,
\end{aligned}
\]

where \( g = (S, E, I, A, R, M) \in \mathbb{R}^6 \), \( \mathcal{S}(t, g(t)) \) is used for a continuous real valued vector function, which additionally satisfies the Lipschitz condition and \( g_0 \) stands for initial state vector. Taking integral on both sides of (18) we get

\[ g(t) = g_0 + \frac{1}{\Gamma(p)} \int_0^t (t - \lambda)^{p-1} \mathcal{S}(\lambda, g(\lambda)) d\lambda. \]  

(19)

In order to formulate an iterative scheme, we consider a uniform grid on \([0, \mathcal{S}]\) with \( h = \frac{\mathcal{S}}{m} \) is the step size and \( m \in \mathbb{N} \). Thus, the Eq. (19) gets the structure as follows after make use of the Euler method [34]

\[
\begin{aligned}
&\quad g_{n+1} = g_n + \frac{h^p}{(p+1)} \sum_{j=0}^{n} \left( (n-j+1)^p - (n-j)^p \right) \mathcal{S}(t_j, g(t_j)), \\
&\quad n = 0, 1, 2, \ldots, m.
\end{aligned}
\]

(20)

Thus, utilizing the above scheme (20), we deduced the following iterative formulae for the corresponding classes of the model (9)

---

**Fig. 4.** Influence of \( \eta_w \) (disease transmission coefficient) on symptomatic infected individuals \( I(t) \) where (a) \( p = 1 \), (b) \( p = 0.90 \), (c) \( p = 0.85 \), (d) \( p = 0.80 \).
The parameter \( M \) (disease transmission coefficient) on asymptomatic infected \((A(t))\) people for various cases of fractional order \( p \). It is observed that the population in both infected classes is decreasing significantly with the decrease in \( \eta_w \) as shown in subplots (a-d) of Figs. 4 and 5. Further, the same behavior is observed for all values of fractional order \( p \) but the decrease in infected individuals is more significant for the smaller values of \( p \).

Conclusion

The dynamics of novel coronavirus (COVID-19) pandemic is presented using a mathematical model. The model was first analyzed in [15] with Atangana-Baleanu fractional operator. In this study, we reformulated the model using the Caputo fractional operator and provides a better theoretical and graphical analysis of the model. Additionally, in the present paper, we considered the reported COVID-19 confirmed cases from the early stage until the end of pandemic in the mainland China. The findings of the present study show a better agreement of the model prediction to the real reported infected cases and ultimately, a more suitable model parameters are estimated. Initially, the model is briefly explored with classical integer-order
The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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