One-loop matching of the type-II seesaw model onto the Standard Model effective field theory

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ABSTRACT: In this paper, we continue to construct the low-energy effective field theories (EFTs) of the canonical seesaw models, which are natural extensions of the Standard Model (SM) to accommodate tiny but nonzero neutrino masses. Different from three right-handed neutrino singlets in the type-I seesaw model, the Higgs triplet in the type-II seesaw model participates directly in the electroweak gauge interactions, rendering the EFT construction more challenging. By integrating out the heavy Higgs triplet in the functional-integral formalism, we carry out a complete one-loop matching of the type-II seesaw model onto the so-called Standard Model Effective Field Theory (SMEFT). It turns out that 41 dimension-six operators (barring flavor structures and Hermitian conjugates) in the Warsaw basis of the SMEFT can be obtained, covering all those 31 dimension-six operators in the case of type-I seesaw model. The Wilson coefficients for 41 dimension-six operators are computed up to $O(M^2)\Delta$ with $M\Delta$ being the mass scale of the Higgs triplet. Moreover, the branching ratios of rare radiative decays of charged leptons $l_\alpha^\to l_\beta^\to + \gamma$ are calculated in the EFT and compared with that in the full theory in order to demonstrate the practical application and the correctness of our EFT construction.

KEYWORDS: Effective Field Theories, Multi-Higgs Models, SMEFT

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1 Introduction

The experimental discovery of neutrino oscillations provides us with very compelling evidence that neutrinos are actually massive and lepton flavors are significantly mixed [1]. The origin of neutrino masses and flavor mixing definitely calls for new physics beyond the Standard Model (SM). On the other hand, the SM has so far been the most successful theory for strong, weak and electromagnetic interactions, passing essentially all the experimental tests except for neutrino oscillations [2]. Such a situation may hint at the widely-accepted idea that the SM just serves as an effective field theory (EFT) at the low-energy scale (i.e., the electroweak scale $\Lambda_{EW} \equiv 10^2$ GeV), where the higher-dimensional operators composed only of the SM fields respect the SM gauge symmetry and take the responsibility for all the observed deviations from the SM predictions.

As first pointed out by Steven Weinberg [3], the dimension-five operator in the Standard Model Effective Field Theory (SMEFT) is unique and leads to the generation of tiny Majorana neutrino masses after the spontaneous breakdown of the SM gauge symmetry. More explicitly, the effective Lagrangian of the SMEFT can be written as

$$\mathcal{L}_{\text{SMEFT}} = \mathcal{L}_{\text{SM}} + \sum_i \frac{C_i^{(5)} O_i^{(5)}}{\Lambda} + \sum_i \frac{C_i^{(6)} O_i^{(6)}}{\Lambda^2} + \cdots,$$

where $\Lambda$ is the cutoff energy scale for the SMEFT, $\mathcal{L}_{\text{SM}}$ stands for the SM Lagrangian, $O_i^{(5)}$ and $O_i^{(6)}$ are dimension-five (dim-5) and dimension-six (dim-6) operators with $C_i^{(5)}$ and $C_i^{(6)}$ being the associated Wilson coefficients, respectively. The dim-6 operators in the SMEFT have already been systematically studied in ref. [4], and recently revised in...
ref. [5], where the well-known Warsaw basis of 59 independent baryon-number-conserving
dim-6 operators (plus four baryon-number-violating operators) has been established. In
the precision era of particle physics, the SMEFT obviously offers an extraordinarily useful
and efficient way, which is independent of any specific ultraviolet (UV) models, to probe
new physics [6]. Recent years have seen tremendous progress in the developments of the
SMEFT itself [7–21] and its many interesting extensions [22–27]. However, as the mass
dimension of operators under consideration becomes higher, the number of independent
operators in the SMEFT will increase very rapidly [12], rendering a complete experimental
determination of all the relevant coefficients associated with the operators to be extremely
difficult or even impossible.

For this reason, we take another distinct attitude to the exploration of possible new
physics beyond the SM. If the renormalizable UV model is believed to exist, one can choose
one of the well-motivated UV models and match it onto the SMEFT by integrating out
the heavy degrees of freedom. In this way, only a fraction of the effective operators in the
SMEFT will be obtained and the corresponding Wilson coefficients are highly correlated.
Then the constructed EFT is confronted with the precision data from various experiments.
Apparently the disadvantage of such an approach is lacking of the model independence,
which is the primary motivation for the SMEFT. In light of the discovery of neutrino
oscillations, it is reasonable to argue that the canonical seesaw models for tiny Majorana
neutrino masses are strongly motivated, and the EFTs of these renormalizable UV models
will be indispensable for self-consistent phenomenological studies at the low-energy scale.

The basic strategy for the construction of EFTs from renormalizable UV theories has
been outlined in ref. [28]. By integrating out the heavy degrees of freedom, one can match
the UV model onto the SMEFT at the tree level, which has been accomplished in ref. [29]
for general field content and arbitrary types of interactions. At the one-loop level, a num-
ber of examples can be found in the literature [30–43] either by the diagrammatic method
or by the functional approach [32, 35–39, 44–54]. In the previous work [52], we have initi-
ated a program of performing a complete one-loop matching of the seesaw models onto the
SMEFT. For the type-I seesaw model [55–59], the one-loop matching has been achieved
and shown to be useful to investigate the radiative decays of charged leptons $l^-_\alpha \rightarrow l^-_\beta + \gamma$
in a self-consistent way [43, 60]. In the present paper, we continue to carry out the complete
one-loop matching for the effective operators up to dim-6 by integrating out the heavy Higgs
triplet in the type-II seesaw model [61–66] with the functional approach [67–69]. The main
motivation for such a study is two-fold. First, the Higgs triplet in the type-II seesaw model
transforms non-trivially under the SM gauge group, implying a more challenging construc-
tion of the EFT than that in the type-I seesaw model. Second, it is intriguing to see how
many and what kind of dim-6 operators in the Warsaw basis of the SMEFT can be obtained
in the type-II seesaw EFT (SEFT-II). The results should be compared to those in the type-
I seesaw EFT (SEFT-I), as derived in ref. [52]. The dim-6 operators that are not commonly
shared by the SEFT-I and the SEFT-II may indicate which kind of signatures in the preci-
sion measurements at the low-energy scale could serve as a discriminator for the type-I and
type-II seesaw models as the renormalizable UV theory for neutrino masses. Similar studies
can also be extended to the type-III seesaw model [70] and other neutrino mass models [71].
The remaining part of this paper is structured as follows. In section 2, we briefly recall the functional approach to the tree-level and one-loop matching of an UV model onto the SMEFT. In section 3, we introduce the type-II seesaw model in order to establish our notations and conventions, and further derive the operators up to dim-6 and the associated Wilson coefficients in the SEFT-II at the tree level. Section 4 is devoted to the one-loop matching of the type-II seesaw model onto the SMEFT by evaluating the supertraces via the publicly available package SuperTracer [51]. The dim-6 operators and the corresponding Wilson coefficients in the Warsaw basis are given. The results are also cross-checked by the diagrammatic approach. In section 5, as an illustrative example for the consistency between the UV model and the EFT, the branching ratios for the radiative decays of charged leptons \( l^-_\alpha \rightarrow l^-_\beta + \gamma \) are computed in the SEFT-II and compared to the results in the heavy-triplet limit of the full type-II seesaw model. Finally, we summarize our main conclusions in section 6.

2 The functional approach

The functional approach will be implemented to perform the tree-level and one-loop matching in the present work. In this section, following ref. [52], we briefly explain the general procedure to achieve this goal, and a more detailed account can be found in the literature, e.g., refs. [35, 46]. The basic idea to match a given UV theory onto the low-energy EFT is to identify the one-light-particle-irreducible (1LPI) effective action (i.e., \( \Gamma_{L,UV} \)) in the UV theory with the one-particle-irreducible (1PI) effective action (i.e., \( \Gamma_{EFT} \)) in the low-energy EFT at the matching scale, namely,

\[
\Gamma_{L,UV}[\phi_B] = \Gamma_{EFT}[\phi_B],
\]

where the effective actions on both sides are understood as the functionals of the light background fields \( \phi_B \). Some comments on the matching principle indicated in eq. (2.1) are helpful.

- The background field method will be always utilized in our discussions. See, e.g., ref. [72], for a pedagogical introduction to the background field method and earlier relevant works. In this framework, the generating functional of the Green functions in the UV theory reads

\[
Z_{UV}[J_\Phi, J_\phi] = \int D\Phi D\phi \exp \left\{ i \int d^d x \left( \mathcal{L}_{UV}[\Phi, \phi] + J_\Phi \Phi + J_\phi \phi \right) \right\},
\]

where \( \mathcal{L}_{UV}[\Phi, \phi] \) stands for the Lagrangian of the UV theory with a heavy field \( \Phi \) and a light field \( \phi \), while \( J_\Phi \) and \( J_\phi \) are the external sources for the heavy and light fields, respectively. Notice that we have denoted \( d \equiv 4 - 2\varepsilon \) (with \( \varepsilon \rightarrow 0 \)) as the spacetime dimension and will use the dimensional regularization of the divergent integrals throughout [73–75]. After splitting the relevant fields into the background and quantum fields, i.e., \( \Phi = \Phi_B + \Phi' \) and \( \phi = \phi_B + \phi' \), one needs to perform the
On the other hand, we can similarly derive the generating functional of Green functions in the corresponding low-energy EFT, for which the Lagrangian will be denoted thus denote the localized one by it to a given order of heavy field with the vanishing external source where the classical heavy field functional of connected Green functions with 1LPI effective action set namely, and take account of the extra degrees of freedom by including the complex-conjugate fields [cf. eq. (2.3)] and the minus sign for fermions. Then, the 1LPI effective action \( \Gamma_{\text{LUV}}[\phi_B] \) is defined as the Legendre transform of the generating functional of connected Green functions with \( J_\Phi = 0 \), namely, \[
\Gamma_{\text{LUV}}[\phi_B] \equiv \int d^d x L_{\text{UV}}[\Phi_c[\phi_B], \phi_B] + \frac{i}{2} \ln \det Q_{\text{UV}}[\Phi_c[\phi_B], \phi_B], \tag{2.6}
\]
where the classical heavy field \( \Phi_c[\phi_B] \equiv \Phi_B[J_\Phi = 0, J_\phi] \) is determined by the EOM
\[
\frac{\delta L_{\text{UV}}[\Phi, \phi]}{\delta \Phi} \bigg|_{\Phi = \Phi_c[\phi_B], \phi = \phi_B} = 0, \tag{2.7}
\]
with the vanishing external source \( J_\Phi = 0 \) for the heavy field. Since the classical heavy field \( \Phi_c[\phi_B] \) determined by eq. (2.7) is in general non-local, one can expand it to a given order of \( 1/M \) (where \( M \) represents the mass scale of heavy fields) and thus denote the localized one by \( \hat{\Phi}_c[\phi_B] \). The final result of the 1LPI effective action \( \Gamma_{\text{LUV}}[\phi_B] \) is given by eq. (2.6) with \( \Phi_c[\phi_B] \) everywhere replaced by \( \hat{\Phi}_c[\phi_B] \).

- On the other hand, we can similarly derive the generating functional of Green functions in the corresponding low-energy EFT, for which the Lagrangian will be denoted as \( L_{\text{EFT}}[\phi] \). At the one-loop level, the generating functional for the EFT reads
\[
Z_{\text{EFT}}[J_\phi] \propto \exp \left\{ i \int d^d x \left( L_{\text{EFT}}^{\text{tree}}[\phi_B] + L_{\text{EFT}}^{\text{1-loop}}[\phi_B] + J_\phi \phi_B \right) \right\} \times (\det Q_{\text{EFT}})^{-1/2}, \tag{2.8}
\]
where the EFT Lagrangian has been split into the tree-level one $\mathcal{L}_{\text{EFT}}^{\text{tree}}[\phi]$ and the one-loop-level one $\mathcal{L}_{\text{EFT}}^{\text{1-loop}}[\phi]$, and we have defined

$$Q_{\text{EFT}} \equiv -\frac{\delta^2 \mathcal{L}_{\text{EFT}}^{\text{tree}}}{\delta \phi^2} \bigg|_{\phi_B},$$

arising from the tree-level EFT Lagrangian. From the generating functional in eq. (2.8), one can get the 1PI effective action $\Gamma_{\text{EFT}}[\phi_B]$ up to the one-loop level

$$\Gamma_{\text{EFT}}[\phi_B] = \int d^d x \left( \mathcal{L}_{\text{EFT}}^{\text{tree}}[\phi_B] + \mathcal{L}_{\text{EFT}}^{\text{1-loop}}[\phi_B] \right) + \frac{i}{2} \ln \det Q_{\text{EFT}},$$

where the field multiplets [cf. eq. (4.3)] have been introduced, as in the UV theory, to account for the number of degrees of freedom for the light fields in the EFT.

- At the energy scale $\mu = M$, where $M$ can be identified as the masses of heavy fields in the UV theory, the matching condition in eq. (2.1) can be spelled out by using the 1LPI effective action $\Gamma_{\text{L,UV}}[\hat{\Phi}_c]$ in eq. (2.6) and the 1PI effective action $\Gamma_{\text{EFT}}[\phi_B]$ in eq. (2.10). Following the arguments given in refs. [50–52], we can complete the tree-level matching by implementing the EOM in eq. (2.7) for the classical heavy field $\Phi_c[\phi_B]$, i.e.,

$$\mathcal{L}_{\text{EFT}}^{\text{tree}}[\phi_B] = \mathcal{L}_{\text{UV}} \left[ \hat{\Phi}_c[\phi_B], \phi_B \right],$$

where $\Phi_c[\phi_B]$ has been replaced by its localized counterpart. Moreover, the one-loop effective action of the EFT is given by

$$\int d^d x \mathcal{L}_{\text{EFT}}^{\text{1-loop}}[\phi_B] = \Gamma_{\text{L,UV}}^{\text{1-loop}} \bigg|_{\text{hard}} = \frac{i}{2} \ln \det Q_{\text{UV}} \bigg|_{\text{hard}},$$

where the subscripts “hard” refer to the contributions from the hard-momentum region as the loop integrals are treated by the expansion-by-regions techniques [76–78]. In practice, the result in eq. (2.12) can be evaluated as [35]

$$\int d^d x \mathcal{L}_{\text{EFT}}^{\text{1-loop}}[\phi] = \frac{i}{2} \text{STr} \ln (-K) \bigg|_{\text{hard}} - \frac{i}{2} \sum_{n=1}^{\infty} \frac{1}{n} \text{STr} \left[ (K^{-1}X)^n \right] \bigg|_{\text{hard}},$$

where the supertrace “STr” is the generalization of the trace over both the internal degrees of freedom and the functional space (with the fermionic blocks assigned a minus sign). Notice that the inverse-propagator part $K$ and the interaction part $X$ stem from the explicit calculation of $\ln \det Q_{\text{UV}}$ on the rightmost side of eq. (2.12), where $K^{-1}X \sim M^{-1}$ has been taken into account [51]. As indicated in eq. (2.13), there are both log-type and power-type supertraces, corresponding to the first and second terms in eq. (2.13), respectively. The log-type supertrace is universal and receives contributions solely from heavy fields, leading to pure gauge-field operators. As we shall see later, the essential difference between the SEFT-I and the SEFT-II is that heavy fields are fermionic singlets in the former case while scalar triplets in the latter. Such a difference leads to extra dim-6 operators from the log-type supertraces in the SEFT-II.
Once the inverse-propagator part $K$ and the interaction part $X$ in the UV theory are obtained, the functional supertraces in eq. (2.13) can be evaluated by means of the covariant derivative expansion (CDE) method $^{[44, 49, 51]}$. Currently two Mathematica packages, i.e., SuperTracer $^{[51]}$ and STrEAM $^{[35]}$, making use of the CDE method to evaluate all functional supertraces, are publicly available. In the present work, we utilize the package SuperTracer (specifically, the first version of SuperTracer) in our calculations.

3 The type-II seesaw model

Unlike the type-I seesaw model $^{[55–59]}$, where three right-handed neutrino singlets are introduced to the SM, the type-II seesaw model $^{[61–66]}$ extends the SM by an Higgs triplet with a hypercharge $Y = −1$. Now that the UV theory in question is the type-II seesaw model, the gauge-invariant Lagrangian for the UV theory is given by

$$\mathcal{L}_{\text{UV}} = \mathcal{L}_{\text{SM}} + \frac{1}{2} \text{Tr} \left[ (D_\mu \Delta) \, (D^\mu \Delta) \right] - V(H, \Delta) - \frac{1}{2} [\ell_L Y_\Delta \Delta \epsilon \ell_L + \text{h.c.}] ,$$  

(3.1)

where $\ell_L$ denotes the left-handed lepton doublet, $\epsilon = i\sigma^2$ is the two-dimensional antisymmetric tensor with $\epsilon_{12} = −\epsilon_{21} = 1$, where the indices of this tensor are referring to the weak isospin space, and $\ell^*_L = C \ell^*_L$ with $C = i\gamma^2\gamma^0$ being the charge-conjugation matrix is defined, and the Higgs triplet $\Phi \equiv (\Phi_1, \Phi_2, \Phi_3)$ has been cast in the matrix form $\Delta \equiv \sigma^I \Phi_I$ with $\sigma^I$ (for $I = 1, 2, 3$) being the Pauli matrices. In eq. (3.1), the SM Lagrangian $\mathcal{L}_{\text{SM}}$ reads

$$\mathcal{L}_{\text{SM}} = - \frac{1}{4} G^A_{\mu\nu} G^{A\mu\nu} - \frac{1}{4} W^I_{\mu\nu} W^{I\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + \left( D_\mu H \right)^\dagger (D^\mu H) - m^2 H^\dagger H - \lambda \left( H^\dagger H \right)^2 + \sum_{f} \bar{f} \not{\! D} f - \left( \bar{Q}_L \gamma^\mu \tilde{H} U_R + \bar{Q}_L \gamma^\mu \tilde{H} D_R + \bar{r}_L \gamma^\mu \tilde{H} E_R + \text{h.c.} \right) ,$$  

(3.2)

where $f = Q_L, U_R, D_R, \ell_L, E_R$ refer to the SM fermionic doublets and singlets, $H$ is the SM Higgs doublet with the hypercharge $Y = 1/2$, $\tilde{H}$ is defined as $\tilde{H} \equiv eH^*$, and the covariant derivative $D_\mu \equiv \partial_\mu - ig_1 Y B_\mu - ig_2 T^I W^I_\mu - ig_3 T^A G^A_{\mu\nu}$ has been defined as usual. For the fields in the fundamental representation of SU(2)$_L$, we have $T^I = \sigma^I/2$ (for $I = 1, 2, 3$). For the Higgs triplet $\Phi$ in the adjoint representation, we should take the representation matrices $(T^I)_{JK} = -ie^{IJK}$ (for $I, J, K = 1, 2, 3$), where $e^{IJK}$ is the totally antisymmetric Levi-Civita tensor. Note that $T^A = \lambda^A/2$ has been defined with $\lambda^A$ being the Gell-Mann matrices (for $A = 1, 2, \cdots , 8$), where the Latin letters $A, B, C$ refer to the adjoint representation of the SU(3)$_c$ group while $I, J, K$ to that of the SU(2)$_L$ group. In addition, the scalar potential $V(H, \Delta)$ without the quadratic and quartic terms of the Higgs doublet $H$ in eq. (3.1) can be written as

$$V(H, \Delta) = \frac{1}{2} M^2_{\Delta} \text{Tr} \left( \Delta^\dagger \Delta \right) - \left( \lambda_\Delta M_{\Delta} H^T \epsilon H + \text{h.c.} \right) + \frac{\lambda_1}{4} \left[ \text{Tr} \left( \Delta^\dagger \Delta \right) \right]^2 + \frac{\lambda_2}{4} \text{Det} \left( \Delta^\dagger \Delta \right) + \frac{\lambda_3}{2} \left( H^\dagger H \right) \text{Tr} \left( \Delta^\dagger \Delta \right) + \frac{\lambda_4}{2} \left( H^\dagger \sigma^I H \right) \text{Tr} \left( \Delta^\dagger \sigma^I \Delta \right) .$$  

(3.3)

It is worthwhile to mention that both the trilinear doublet-triplet term in $V(H, \Delta)$ and the Yukawa interaction term in eq. (3.1) are indispensable for the generation of tiny Majorana
operators induced by integrating out the heavy Higgs triplet field are all included in

two \Phi be vanishing when the classical triplet Higgs field

The terms in the square brackets satisfy the EOM of the Higgs triplet field, thus they will

mediately obtain the effective operators up to dim-6 in the SEFT-II at the tree level. But

order of \frac{\lambda_\Delta}{M_\Delta^2} are implied, and the relevant terms \hat{\lambda}_I and \hat{Y}_I are defined as

\hat{\lambda}_I \equiv -\lambda_\Delta M_\Delta H^\dagger \sigma^I H, \quad \hat{Y}_I \equiv \frac{1}{2} \sigma^I Y^\dagger_\Delta \ell_L.

(3.5)

Therefore, it is straightforward to derive the EOM of the Higgs triplet from eq. (3.4), i.e.,

\begin{align*}
D_\mu D^\mu - M_\Delta^2 \Phi \mp \hat{\lambda} + \hat{Y} &- \left(2\lambda_i + \frac{\lambda_2}{2}\right) (\Phi^\dagger_\Phi) + \frac{\lambda_4}{2} (\Phi^T_\Phi) (T^\dagger_\Phi)
\end{align*}

from which one can determine the classical triplet Higgs field \Phi_c. In order to find out the

operators in the SEFT-II up to dim-6, we need to localize \Phi_c by expanding it to a given

order of \frac{1}{M_\Delta^2} and maintain only the terms up to \mathcal{O}(M_\Delta^{-4}). Hence we finally arrive at

\hat{\Phi}_I \equiv \left[\frac{1}{M_\Delta^2} \delta_{IJ} \frac{1}{M_\Delta^2} \left(\frac{1}{M_\Delta^2} \hat{\lambda}_I + \hat{Y}_I\right) + \mathcal{O}(M_\Delta^{-6})\right].

(3.7)

Then inserting the classical triplet Higgs field in eq. (3.7) into eq. (3.4), we can immediately obtain the effective operators up to dim-6 in the SEFT-II at the tree level. But

before doing so, to make this procedure more transparent, we recast eq. (3.4) into

\begin{align*}
\mathcal{L}_{UV} \supset \Phi^\dagger \left[-D^2 \Phi - M_\Delta^2 \Phi + \hat{\lambda} + \hat{Y} &- \left(2\lambda_i + \frac{\lambda_2}{2}\right) (\Phi^\dagger_\Phi) + \frac{\lambda_4}{2} (\Phi^T_\Phi) (T^\dagger_\Phi)
\end{align*}

The terms in the square brackets satisfy the EOM of the Higgs triplet field, thus they will

be vanishing when the classical triplet Higgs field \hat{\Phi}_c in eq. (3.7) is inserted. Moreover, the
two \Phi^4 terms at the end of the second line in eq. (3.8) will only result in the operators

of mass dimension higher than six and can be safely neglected. Therefore, the tree-level

operators induced by integrating out the heavy Higgs triplet field are all included in

\begin{align*}
\mathcal{L}_{\text{SEFT-II}} \supset \left[\hat{\lambda}_I + \hat{Y}_I\right] \left[\frac{1}{M_\Delta^4} \delta_{IJ} \frac{1}{M_\Delta^4} \left(\frac{1}{M_\Delta^4} \right) \left(D^2_{\Phi} + \lambda_3 H^\dagger H \delta_{IJ} + \lambda_4 H^\dagger \sigma^K H T^\dagger_\Phi\right) \right] \left(\hat{\lambda}_J + \hat{Y}_J\right),
\end{align*}

(3.9)
where the classical triplet Higgs field given in eq. (3.7) has been applied. Some interesting observations can be made.

- Notice that the mass dimension of \( \hat{Y}_I \) in eq. (3.5) is three, contributed from two lepton doublets, whereas \( \hat{\lambda}_I \) contains only two Higgs doublets that are totally of mass dimension two. Thus one dimension-four operator comes out, i.e.,

\[
\frac{1}{M_\Delta} (\hat{\lambda}^I \hat{\lambda}) = \lambda_\Delta^2 \left( H^\dagger \sigma^I \tilde{H} \right)^\dagger \left( H^\dagger \sigma^I \tilde{H} \right) = 2 \lambda_\Delta^2 \left( H^\dagger H \right)^2 ,
\]

where the identity \( (\sigma^I)_{ab}(\sigma^I)_{cd} = 2 \delta_{ad} \delta_{bc} - \delta_{ab} \delta_{cd} \) has been utilized. The four-Higgs operator in eq. (3.10) leads to the tree-level threshold effect on the quartic Higgs coupling, namely,

\[
\lambda_{\text{eff}} = \lambda - 2 \lambda_\Delta^2 . \tag{3.11}
\]

The minus sign on the right-hand side of eq. (3.11) is crucially important, since the threshold shift in the quartic coupling of the Higgs doublet may help rescue the electroweak vacuum from instability, similar to the scenario of the SM with an extra scalar singlet considered in ref. [79]. In addition, the vacuum expectation value (vev) of the SM Higgs field will be modified to

\[
v = \sqrt{-m^2 / (\lambda - 2 \lambda_\Delta^2)} \approx 246 \text{ GeV}
\]

once the effects of the dimension-four operator in eq. (3.10) are considered.

- It is easy to see that the cross terms of \( \hat{\lambda} \) and \( \hat{Y} \) give rise to the unique dim-5 operator [3]

\[
\sum_i C_i^{(5)} O_i^{(5)} = \frac{1}{M_\Delta} (\hat{Y}^I \hat{\lambda}^I \hat{\lambda} + \hat{\lambda}^I \hat{Y}^I) = - \frac{(Y_\Delta)_{\alpha\beta} \lambda_\Delta}{M_\Delta} \left[ \bar{\ell}_a \tilde{H} \tilde{H}^T \ell^c_{\beta L} \right] + \text{h.c.} , \tag{3.12}
\]

Hence the dim-5 operator and its Wilson coefficient are given by

\[
O_{\alpha\beta}^{(5)} = \bar{\ell}_a \tilde{H} \tilde{H}^T \ell^c_{\beta L} , \quad C_{\alpha\beta}^{(5)} = - \lambda_\Delta (Y_\Delta)_{\alpha\beta} , \tag{3.13}
\]

with \( \alpha, \beta \) running over the lepton flavors \( e, \mu, \tau \), and the cutoff energy scale is identified as \( \Lambda = M_\Delta \). After the SM Higgs field acquires its vev, i.e., \( \langle H \rangle = v / \sqrt{2} \) and the SM gauge symmetry is spontaneously broken, the dim-5 operator gives rise to a Majorana neutrino mass term with the effective neutrino mass matrix

\[
M_\nu = \lambda_\Delta Y_\Delta v^2 / M_\Delta .
\]

Therefore, the smallness of neutrino masses can be attributed to the heaviness of the Higgs triplet (i.e., \( M_\Delta \gg v \)), manifesting the spirit of seesaw mechanisms.

- One can find that there are three dim-6 operators at the tree level. Among them one is the four-fermion operator stemming from the \( \hat{Y}^2 / M_\Delta^2 \) term, namely

\[
\frac{C_{4f}^{(6)} O_{4f}^{(6)}}{\Lambda^2} = \frac{1}{M_\Delta^2} \left( \hat{Y}^I \hat{Y}^I \right) = \frac{1}{4M_\Delta^2} \left( \bar{\ell}_a \sigma^I \epsilon_{\beta L}^c \sigma^I \epsilon_{\delta L}^c \right) \left( \bar{\ell}_a \sigma^I \epsilon_{\alpha L}^c \right) , \tag{3.14}
\]

from which we can identify the dim-6 operators with lepton flavors specified

\[
O_{4f,\alpha\beta\gamma\delta}^{(6)} = \left( \bar{\ell}_a \sigma^I \epsilon_{\beta L}^c \right) \left( \bar{\ell}_a \sigma^I \epsilon_{\delta L}^c \right) , \quad C_{4f,\alpha\beta\gamma\delta}^{(6)} = (Y_\Delta)^*_{\alpha\beta} (Y_\Delta)_{\gamma\delta} / 4 . \tag{3.15}
\]
Another two operators result from the $\lambda^2/M^4_\Delta$ terms. To be specific, we have

$$\frac{C^{(6)}_{D^2H^4}O^{(6)}_{D^2H^4}}{\Lambda^2} = \frac{1}{M^2_\Delta} \left(D_\mu \lambda^\dagger \left(D^\mu \lambda \right) = \frac{\lambda^2_3}{M^2_\Delta} \left[D_\mu \left(H^\dagger \sigma^I \bar{H} \right) \right] \left[D^\mu \left(H^\dagger \sigma^I \bar{H} \right) \right] \right), \quad (3.16)$$

and

$$\frac{C^{(6)}_{H^6}O^{(6)}_{H^6}}{\Lambda^2} = -\frac{\lambda^2_3}{M^2_\Delta} \left(H^\dagger H \right) \left(\lambda^\dagger \lambda \right) - \frac{\lambda^2_4}{M^2_\Delta} \left(H^\dagger \sigma^I H \right) \left(\lambda^\dagger T^I \lambda \right)$$

$$= -\frac{2(\lambda^2_3 - \lambda^2_4)}{M^2_\Delta} \left(H^\dagger H \right)^3, \quad (3.17)$$

from which one can easily extract those two dim-6 operators and the corresponding Wilson coefficients. More explicitly, from eq. (3.17), we get $O^{(6)}_{H^6} = \left(\lambda^\dagger H \right)^3$ and $C^{(6)}_{H^6} = -2(\lambda^2_3 - \lambda^2_4)/\lambda^2_\Delta$, and

$$O^{(6)}_{D^2H^4} = \left[D_\mu \left(H^\dagger \sigma^I \bar{H} \right) \right] \left[D^\mu \left(H^\dagger \sigma^I \bar{H} \right) \right] \right), \quad C^{(6)}_{D^2H^4} = \frac{\lambda^2_\Delta}{M^2_\Delta}, \quad (3.18)$$

from eq. (3.16).

The aforementioned operators in the SEFT-II have already been partially or completely derived in the previous works [80–82].\(^1\) Now we should convert those operators and the corresponding Wilson coefficients into those in the Warsaw basis by applying the Fierz transformation, integration by parts and the EOM of the Higgs doublet at the tree level. As a result, the tree-level matching leads to

$$L^{\text{tree}}_{\text{SEFT-II}} = L_{\text{SM}} + 2\lambda^2_\Delta \left[ 1 + \frac{2m^2}{M^2_\Delta} \right] \left(\lambda^\dagger \lambda \right)^2 + \left( \frac{C^{(5)}_{\alpha\beta}O^{(5)}_{\alpha\beta}}{M^2_\Delta} + \text{h.c.} \right) + \left( \frac{C^{\text{tree}}_{\alpha\beta\gamma\delta}}{M^2_\Delta} \right) O^{\alpha\beta\gamma\delta}_{\text{tree}} \right)$$

$$+ \left( \frac{C^{\text{tree}}_{\alpha\beta\gamma\delta}}{M^2_\Delta} \right) O^{\alpha\beta\gamma\delta}_{\text{tree}} \right) + \left( \frac{C^{\text{tree}}_{\alpha\beta\gamma\delta}}{M^2_\Delta} \right) O^{\alpha\beta\gamma\delta}_{\text{tree}} \right) + \left( \frac{C^{\text{tree}}_{\alpha\beta\gamma\delta}}{M^2_\Delta} \right) O^{\alpha\beta\gamma\delta}_{\text{tree}} \right)$$

$$+ \left( \frac{C^{\text{tree}}_{\alpha\beta\gamma\delta}}{M^2_\Delta} \right) O^{\alpha\beta\gamma\delta}_{\text{tree}} \right) + \left( \frac{C^{\text{tree}}_{\alpha\beta\gamma\delta}}{M^2_\Delta} \right) O^{\alpha\beta\gamma\delta}_{\text{tree}} \right) + \left( \frac{C^{\text{tree}}_{\alpha\beta\gamma\delta}}{M^2_\Delta} \right) O^{\alpha\beta\gamma\delta}_{\text{tree}} \right)$$

where the dim-6 operators $O_H$, $O_{\square}$, $O_{\bar{H}D}$, $O_{\bar{e}H}$, $O_{\bar{u}H}$, $O_{\bar{d}H}$ and $O_{\ell\ell}$ are in the Warsaw basis, and their specific forms and the associated Wilson coefficients are respectively given by

$$O_H = \left(\lambda^\dagger H \right)^3, \quad O_{\square} = \left(\lambda^\dagger H \right) \square \left(\lambda^\dagger H \right), \quad O_{\bar{H}D} = \left(\lambda^\dagger D_\mu H \right) \left(\lambda^\dagger D^\mu H \right),$$

$$O_{\bar{e}H} = \left(\lambda^\dagger H \right) \left(\bar{\epsilon}_{\alpha\bar{L}} E_{\beta\bar{R}} H \right), \quad O_{\bar{u}H} = \left(\lambda^\dagger H \right) \left(\bar{Q}_{\alpha\bar{L}} U_{\beta\bar{R}} \bar{H} \right), \quad O_{\bar{d}H} = \left(\lambda^\dagger H \right) \left(\bar{Q}_{\alpha\bar{L}} D_{\beta\bar{R}} \bar{H} \right)$$

$$O_{\ell\ell} = \left(\epsilon_{\alpha\mu} \epsilon^\mu_{\beta\bar{L}} \right) \left(\bar{\epsilon}_{\alpha\mu} \epsilon^\mu_{\beta\bar{L}} \right). \quad (3.20)$$

\(^1\)It is worthwhile to mention that the dim-6 operators $O^{(6)}_{H^6}$ and $O^{(6)}_{D^2H^4}$ from the tree-level matching have not been discussed in refs. [80, 81] but in ref. [82]. We have compared our results in eqs. (3.10)–(3.18) with those obtained in ref. [82] and found an excellent agreement.
and
\[
C_{H}^{\text{tree}} = 2(4\lambda - \lambda_3 + \lambda_4)\lambda_3^2 - 16\lambda_4^4, \quad C_{H\Box}^{\text{tree}} = 2\lambda_3^2, \quad C_{HD}^{\text{tree}} = 4\lambda_3^2, \quad (C_{eH}^{\text{tree}})_{\alpha\beta} = 2\lambda_3^2 (Y_l)_{\alpha\beta}, \quad (C_{uH}^{\text{tree}})_{\alpha\beta} = 2\lambda_3^2 (Y_u)_{\alpha\beta}, \quad (C_{dH}^{\text{tree}})_{\alpha\beta} = 2\lambda_3^2 (Y_d)_{\alpha\beta}
\]
\[(C_{ll}^{\text{tree}})_{\alpha\beta\gamma\delta} = \frac{1}{4} (Y_\Delta)_{\alpha\gamma} (Y_\Delta^\dagger)_{\beta\delta}.
\]

Thus far we have accomplished the tree-level matching of the type-II seesaw model onto the SMEFT. In addition to the unique dim-5 Weinberg operator, there are seven dim-6 operators in the Warsaw basis. The decoupling of the Higgs triplet also leads to the threshold shift in the quartic Higgs coupling 

\[
\lambda \to \lambda - 2\lambda_3^2 \left(1 + \frac{2m^2}{M_\Delta^2}\right) \quad \text{below the decoupling scale (i.e., } \mu < M_\Delta). \quad \text{As a direct consequence, the vev of the SM Higgs field will be modified.}
\]

4 One-loop matching

As has been explained in section 2, one should calculate the matrices $K$ and $X$ in the UV theory in order to accomplish the one-loop matching. In the type-II seesaw model, the inverse-propagator part can be written as

\[
K_\Phi = P^2 - M_\Delta^2,
\]

where $P_\mu \equiv iD_\mu$ with $D_\mu$ being the covariant derivative. Note that the gauge boson fields in the covariant derivative should be replaced by their background fields, and the inverse-propagator $K_\Phi$ is universal for the heavy Higgs triplet $\Phi_I$.

Furthermore, we define the field multiplets $\varphi_i$ that are relevant for one-loop matching in the type-II seesaw model as below

\[
\varphi_i \in \{\varphi_\Phi, \varphi_\ell, \varphi_E, \varphi_Q, \varphi_U, \varphi_D, \varphi_H, \varphi_W, \varphi_B\},
\]

where

\[
\varphi_\Phi = \begin{pmatrix} \Phi \\ \Phi^* \end{pmatrix}, \quad \varphi_F = \begin{pmatrix} F \\ F^c \end{pmatrix}, \quad \varphi_H = \begin{pmatrix} H \\ H^* \end{pmatrix}, \quad \varphi_V = V,
\]

with $F = \ell, E, Q, U, D$ and $V = W, B$. To extract the interaction matrix $X$, one needs to first figure out the fluctuation operator in eq. (2.5) and then follow the procedure from eq. (2.12) to eq. (2.13), see, e.g., ref. [35], for more details. In the following, we list all the relevant $X$ terms arising from the type-II seesaw model

- $X_{HH}$

\[
X_{HH} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}
\]

with

\[
X_{11} = m^2 + 2\lambda(|H|^2 + H^H) + \lambda_3(\Phi^T \Phi) + \lambda_4 \sigma^I (\Phi^T T^I \Phi)
\]
\[
X_{12} = 2\lambda HH^T + \lambda_3 M_\Delta \Phi_I \sigma^I 
\]
\[
X_{21} = 2\lambda H^* H^T - \lambda_3 M_\Delta (\epsilon \sigma^I) \Phi_I 
\]
\[
X_{22} = m^2 + 2\lambda(|H|^2 + H^* H^T) + \lambda_3(\Phi^T \Phi) + \lambda_4 \sigma^T (\Phi^T T^I \Phi)
\]
\[
X_{\ell \Phi} = \begin{pmatrix}
\frac{Y_\Delta}{2} (\sigma^I \epsilon \ell_L) & 0 \\
0 & \frac{Y^*}{2} (\epsilon \sigma^I \ell_L)
\end{pmatrix}
\] (4.6)

\[
X_{\Phi\ell} = \begin{pmatrix}
\frac{Y^\dagger}{2} (\ell_L^T C \epsilon \sigma^I) & 0 \\
0 & \frac{Y^T}{2} (\epsilon \sigma^I \ell_L)
\end{pmatrix}
\] (4.7)

\[
X_{W\Phi} = i g_2 \left( \left( D_\mu \Phi \right)^\dagger T^I - \left( D_\mu \Phi \right)^T T^{I*} \right) + g_2 \left( -\Phi^I T^I \Phi^T T^{I*} \right) i D_\mu
\] (4.8)

\[
X_{\Phi W} = 2i g_1 \left( \left( D_\mu \Phi \right)^\dagger \left( D_\mu \Phi \right)^T \Phi \right) + g_1 \left( \Phi^I - \Phi^T \Phi^T \right) i D_\mu
\] (4.9)

\[
X_{B\Phi} = 2i g_1 \left( \left( D_\mu \Phi \right)^\dagger \left( D_\mu \Phi \right)^T \Phi \right) + g_1 \left( \Phi^I - \Phi^T \Phi^T \right) i D_\mu
\] (4.10)

\[
X_{H\Phi} = \begin{pmatrix}
X_{11} & X_{12} \\
X_{21} & X_{22}
\end{pmatrix}
\] (4.12)

with

\[
X_{11} = \lambda_3 H \Phi^I + \lambda_4 \left( \sigma^J H \right) \left( \Phi^T T^J \right)
\]

\[
X_{12} = 2 \lambda_\Delta M_\Delta \left( \sigma^I \epsilon H^* \right) + \lambda_3 H \Phi^T + \lambda_4 \left( \sigma^J H \right) \left( \Phi^T T^{J*} \right)
\]

\[
X_{21} = -2 \lambda_\Delta M_\Delta \left( \epsilon \sigma^I H \right) + \lambda_3 H^* \Phi^I + \lambda_4 \left( \sigma^J H^* \right) \left( \Phi^T T^J \right)
\]

\[
X_{22} = \lambda_3 H^* \Phi^T + \lambda_4 \left( \sigma^J H^* \right) \left( \Phi^T T^{J*} \right)
\] (4.13)

\[
X_{\Phi H} = \begin{pmatrix}
X_{11} & X_{12} \\
X_{21} & X_{22}
\end{pmatrix}
\] (4.14)
with
\[ X_{11} = \lambda_3 H^\dagger \Phi + \lambda_4 \left( H^\dagger \sigma^I \right) \left( T^I \Phi \right) \]
\[ X_{12} = 2\lambda_5 M_\Delta \left( H^\dagger \sigma^I \epsilon \right) + \lambda_3 H^T \Phi + \lambda_4 \left( H^T \sigma^I \Phi^* \right) \left( T^J \Phi^* \right) \]
\[ X_{21} = -2\lambda_5 M_\Delta \left( H^T \epsilon \sigma^I \right) + \lambda_3 H^\dagger \Phi^* + \lambda_4 \left( H^\dagger \sigma^J \right) \left( T^J \Phi^* \right) \]
\[ X_{22} = \lambda_3 H^T \Phi^* + \lambda_4 \left( H^T \sigma^J \Phi^* \right) \left( T^J \Phi^* \right) \]

• \( X_{\Phi \Phi} \)
\[ X_{\Phi \Phi}^{JK} = \left( \begin{array}{c} X_{11} \\ X_{21} \end{array} \right) \]

with
\[ X_{11} = \left( 2\lambda_1 + \frac{\lambda_2}{2} \right) \left( |\Phi|^2 + \Phi \Phi^\dagger \right) - \frac{\lambda_2}{2} \left[ T^I \left( \Phi \Phi^\dagger T^I \right) + \left( T^I \Phi \right) \left( \Phi^\dagger T^I \right) \right] + \lambda_3 \left( H^\dagger H \right) + \lambda_4 \left( H^\dagger \sigma^J H \right) T^I \]
\[ X_{12} = \left( 2\lambda_1 + \frac{\lambda_2}{2} \right) \Phi \Phi^\dagger - \frac{\lambda_2}{2} \left( T^I \Phi \right) \left( \Phi^\dagger T^I \right) \]
\[ X_{21} = \left( 2\lambda_1 + \frac{\lambda_2}{2} \right) \Phi^* \Phi^\dagger - \frac{\lambda_2}{2} \left( T^I \Phi^* \right) \left( \Phi^\dagger T^I \right) \]
\[ X_{22} = \left( 2\lambda_1 + \frac{\lambda_2}{2} \right) \left( \Phi \Phi^\dagger + \Phi^* \Phi^\dagger \right) - \frac{\lambda_2}{2} \left[ T^I \left( \Phi \Phi^\dagger T^I \right) + \left( T^J \Phi^* \right) \left( \Phi^\dagger T^I \right) \right] + \lambda_3 \left( H^\dagger H \right) + \lambda_4 \left( H^\dagger \sigma^J H \right) T^I \]

• \( X_{HB} \)
\[ X_{HB} = ig_1 \left( \frac{-D_\nu H}{(D_\nu H)^\dagger} \right) + \frac{g_1}{2} \left( \frac{-H^*}{H^{\dagger*}} \right) iD_\nu \]

• \( X_{BH} \)
\[ X_{BH} = \frac{ig_1}{2} \left( (D_\nu H)^\dagger - (D_\nu H)T \right) + \frac{g_1}{2} \left( -H^\dagger H^T \right) iD_\nu \]

• \( X_{HW} \)
\[ X_{HW}^i = ig_2 \left( -\sigma^I D_\nu H \right) + \frac{g_2}{2} \left( \frac{-\sigma^I H^*}{\sigma^I H^{\dagger*}} \right) iD_\nu \]

• \( X_{WH} \)
\[ X_{WH}^i = \frac{ig_2}{2} \left( (D_\nu H)^\dagger \sigma^I - (D_\nu H)^T \sigma^I \right) + \frac{g_2}{2} \left( \frac{-H^\dagger \sigma^I H^T \sigma^I \Phi^*}{H^\dagger \sigma^I H^{\dagger*} \Phi^*} \right) iD_\nu \]

In the \( X \) terms, the heavy Higgs triplet field \( \Phi \) has to be replaced by the solution to its EOM, namely, the right-hand side of eq. (3.7). The corresponding \( X \) terms for the SM interactions can be found in the appendix B of ref. [35], whereas the \( X \) terms for the interactions between the Higgs boson and gauge bosons should take the forms given in eqs. (4.18)–(4.21) due to different conventions used in the package SuperTracer [51] and in
ref. [35]. In addition, different conventions of the quartic Higgs coupling in the literature should be noted.

With the above information, it is now ready to evaluate the supertraces in eq. (2.13) and then to derive the one-loop-level operators by using the package SuperTracer. Then, the generated operators from SuperTracer can be further converted into the independent operators in the Warsaw basis of the SMEFT [5]. However, in order to cross-check our results via diagrammatic approach, we first convert these operators into those in Green’s basis [33, 42, 83] by utilizing the algebraic, Fierz identities and integration by parts. In the following subsections, we shall discuss the one-loop threshold corrections to the renormalizable terms existing in the SM Lagrangian, and clarify how to implement the EOMs to remove the redundant dim-6 operators in a consistent way. The concept of field redefinitions and its impact on the Wilson coefficients of dim-6 operators in the Warsaw basis will be emphasized [84].

4.1 Threshold corrections

Generally, the one-loop matching can result in threshold corrections to the renormalizable terms already existing in the SM [85], i.e.,

$$\delta L = \delta Z_G G^A_{\mu\nu} G^{A\mu\nu} + \delta Z_W W^I_{\mu\nu} W^{I\mu\nu} + \delta Z_B B_{\mu\nu} B^{\mu\nu}$$

$$+ \sum_f f D f + \left( \bar{Q} \delta Y_u \tilde{H} U_R + \bar{Q}_l \delta Y_d H D_R + \bar{E}_L \delta Y_e H E_R + \text{h.c.} \right)$$

$$+ \delta Z_H \left( D_{\mu} H \right)^\dagger \left( D^\mu H \right) + \delta m^2 H^\dagger H + \delta \lambda \left( H^\dagger H \right)^2,$$

(4.22)

where $f = Q_L, U_R, D_R, \ell_L, E_R$. For a given UV model, not all the above terms are induced, which is of course dependent on the interactions of the heavy fields. In the type-II seesaw model, the one-loop matching leads to

$$(4\pi)^2 \delta Z_B^G = -\frac{g_2^2 L_\Delta}{4},$$

$$(4\pi)^2 \delta Z_W^G = -\frac{g_2^2 L_\Delta}{6},$$

$$(4\pi)^2 \delta Z_f^G = 6\lambda_3^2 + 6m^2 \frac{\lambda_3^2}{M_\Delta^2} (5 + 2 L_\Delta),$$

$$(4\pi)^2 \left( \delta Z_f^G \right)_{\alpha\beta} = \frac{3}{4} \left( 1 + 2 L_\Delta \right) \left( Y_{\Delta} Y_{\Delta} \right)_{\alpha\beta},$$

$$(4\pi)^2 \left( \delta \ell^G \right) = 3 \left[ M_\Delta^2 \left( 4 \lambda_3^2 + \lambda_3 \right) + 4m^2 \frac{\lambda_3^2}{M_\Delta^2} \right] \left( 1 + L_\Delta \right),$$

$$(4\pi)^2 \delta \lambda^G = \frac{1}{2} \left( 3 \lambda_3^2 + 2\lambda_2^2 \right) L_\Delta + 2 \lambda_3^2 \left[ (20\lambda + 8\lambda_1 + \lambda_2)(1 + L_\Delta) - 2\lambda_3 (5 + 2 L_\Delta) \right.\left. \left. + 4\lambda_4 (3 + 2 L_\Delta) - 20 \lambda_3 (2 + L_\Delta) \right] + \frac{4m^2 \lambda_3^2}{M_\Delta^2} \left[ 20\lambda (1 + L_\Delta) - \lambda_3 (8 + 5 L_\Delta) \right]$$

$$+ 2\lambda_4 (4 + 3 L_\Delta) - 20 \lambda_3 (3 + 2 L_\Delta),$$

(4.23)
with

\[ L_\Delta \equiv \ln \left( \frac{\mu^2}{M_\Delta^2} \right) + \frac{1}{\varepsilon} - \gamma_E + \ln(4\pi), \tag{4.24} \]

up to \( O \left( M_\Delta^{-2} \right) \) in the Green’s basis, where the other terms in eq. (4.22) do not appear, the superscript “G” signifies the results in the Green’s basis, and \( \gamma_E \) in eq. (4.24) is the Euler constant. It is worth pointing out that the divergences in \( L_\Delta \) come from the hard part of loop integrals and usually consist of both the UV and infrared (IR) divergences in the dimensional regularization. The UV divergences can be absorbed by the renormalization constants in the UV model but with heavy fields replaced with their classical EOMs, such as eq. (3.7) in the type-II seesaw model, while the IR divergences can be regarded as part of the counterterms of the EFT to cancel corresponding UV divergences of the EFT [36]. Here, one can simply remove the \( 1/\varepsilon - \gamma_E + \ln(4\pi) \) terms in \( L_\Delta \) to get the renormalized couplings or Wilson coefficients in the \( \overline{\text{MS}} \) scheme [30, 36].

In addition, there also exist one-loop corrections to the Wilson coefficient of the dimension-five operator \( (\delta C^{(5)})^{G}_{\alpha\beta} O^{(5)}_{\alpha\beta} \) in eq. (3.13), namely,

\[ (4\pi)^2 \left( \delta C^{(5)} \right)^G_{\alpha\beta} = \lambda_\Delta (1 + L_\Delta) \left[ (2\lambda_3 - 4\lambda_4 - 8\lambda_1 - \lambda_2)(Y_\Delta)_{\alpha\beta} + (Y_l Y_l^\dagger Y_\Delta)_{\alpha\beta} + (Y_\Delta Y_\Delta^* Y_l Y_l^\dagger)_{\alpha\beta} \right]. \tag{4.25} \]

This result will be important for us to find the threshold correction to the Wilson coefficient of the Weinberg operator in the Warsaw basis.

With the help of the EOMs of relevant fields, we can obtain the operators together with the Wilson coefficients in the Warsaw basis from those in the Green’s basis. This is normally done in the diagrammatic calculations for one-loop matching, as in refs. [42, 43]. However, as has been stressed in ref. [84], the implementation of the EOMs of fields to get rid of the operator redundancy, even including higher-order corrections, may not completely reproduce the correct effective Lagrangian. Without repeating the general arguments in ref. [84], we just explain how to consistently use the EOMs and demonstrate that it is necessary to supplement the results with additional terms from field redefinitions.

- First, it is important to clarify which EOMs of fields should be implemented. Apart from the existing renormalizable terms in the tree-level effective Lagrangian in the Green’s basis, we have to take into account the threshold corrections in eq. (4.22) from one-loop matching. The overall Lagrangian in the Green’s basis with both tree-level and one-loop corrections should be used to derive the lowest-order EOMs in the SEFT-II, namely,

\[ i\dot{\Phi} E_R = H^\dagger Y_l^\dagger \ell_L, \]
\[ i\dot{\Phi} \ell_L = \left( 1 - \delta Z^G \right) Y_l HE_R, \]
\[ D'^\nu B_{\mu\nu} = \frac{g_1}{2} \left[ (1 + \delta Z^G_H + 4\delta Z^G_B) H^\dagger i\partial_\mu H + 2 \left( 1 + 4\delta Z^G_B \right) \sum_f Y(f) \bar{\gamma}_\mu f \right], \]
Second, the application of the lowest-order EOMs in eq. (4.26) can indeed remove the Wilson redundancies in the Green’s basis, but cannot always give rise to the correct Wilson coefficients of dim-6 operators in the Warsaw basis, as indicated by the general discussions in ref. [84]. Following the arguments in ref. [84], we now briefly explain the main idea. The action $S[\phi]$ of the effective field theory can be organized as a power series in $\zeta \equiv 1/\Lambda$, i.e.,

$$S[\phi] = \sum_{n=-2}^{2} \zeta^n S_n[\phi].$$

(4.27)

In eq. (4.27), the summation on the right-hand side begins with $n = -2$ instead of $n = 0$, since the threshold correction to the quadratic coupling of the Higgs doublet (i.e., $m^2$) may be of order $O(\Lambda^2)$, as shown in eq. (4.23) in the type-II seesaw model under consideration. Furthermore, the summation ends with $n = 2$, as only the operators up to dim-6 will be taken into account. Therefore, to eliminate redundant dim-6 operators, one can perform the following perturbative field redefinitions

$$\phi^a \rightarrow \phi^a - \zeta^2 f_2^a(\phi),$$

(4.28)

at the order of $O(\zeta^2)$, where $f_2^a(\phi)$ denotes the functional corresponding to the redefinition of $\phi^a$. Then the action $S[\phi]$ changes into $S'[\phi]$, namely

$$S'[\phi] = S[\phi] - \zeta^2 f_2^a(\phi) \frac{\delta}{\delta \phi^a} \sum_{n=-2}^{0} \zeta^n S_n[\phi] + \frac{1}{2} \zeta^2 f_2^a(\phi) f_2^b(\phi) \frac{\delta^2 S_{-2}[\phi]}{\delta \phi^a \delta \phi^b},$$

(4.29)

where only the terms up to the order of $O(\zeta^2)$ are retained. As one can see from eq. (4.29), the field redefinitions in eq. (4.28) can also modify $S_k$ (for $k = 0, 1$) due to the existence of $S_{-2}[\phi]$ and $S_{-1}[\phi]$, which are absent in the scenarios considered in ref. [84] but will be present in general. If we choose $f_2^a(\phi)$ by requiring $f_2^a(\phi) \delta \mathcal{K}/\delta \phi^a$ to be a redundant term in $S_2[\phi]$, where $\mathcal{K}$ can be any term in $\sum_{n=-2}^{0} \zeta^n S_n$, but usually taken to be the kinetic term. In this way, the redundant operator in $S_2[\phi]$ can be eliminated by the second term in eq. (4.29), which is equivalent to substitute the EOMs in the action $S[\phi]$. However, the last term of order $O(\zeta^2)$ in eq. (4.29) cannot
be reproduced by the utilization of the EOMs. Therefore, the application of the EOMs and the field redefinitions are not equivalent in the presence of \( S_{-2}[\phi] \) in the action.

In the SEFT-II, we find that \( S_{-2}[H, H^\dagger] = \int d^4 x [3/(4\pi)^2] \cdot (4\lambda_3^2 + \lambda_3)(1 + L_\Delta) H^\dagger H \) with \( \zeta = 1/M_\Delta \) can be obtained with the help of eq. (4.23), which is induced at the one-loop level. Thus we only need to take account of the field redefinitions at the tree level to evaluate the last term in eq. (4.29). First of all, after the tree-level dim-6 operator \( O_{DH}^{G} \) in eq. (3.18) is decomposed into those in the Green’s basis, we shall find that the operator

\[
O_{HD}^{G} = (H^\dagger H)(D_\mu H)^\dagger(D^\mu H)
\]

with the tree-level Wilson coefficient \( C_{\text{tree}}^{G}(H^\dagger H) H \) appearing in the Green’s basis, but it is absent in the Warsaw basis. Then, to remove this redundant operator \( O_{HD}^{G} \) [more exactly the last two operators in eq. (4.30), as the first one appears in the Warsaw basis], one can redefine the Higgs doublet as \( H \rightarrow H - [1/(2M_\Delta^2)]C_{\text{tree}}^{G}(H^\dagger H) H \), and likewise for \( H^\dagger \), namely, \( f_2^2(\phi) = (1/2) \cdot C_{\text{tree}}^{G}(H^\dagger H) \), \( f_2^2(\phi) = (1/2) \cdot C_{\text{tree}}^{G}(H^\dagger H) \), and \( \mathcal{K} = (D^\mu H)^\dagger(D_\mu H) \) with \( \phi^1 \) and \( \phi^2 \) referring to \( H \) and \( H^\dagger \), respectively. Therefore, in addition to the results obtained from the direct use of the EOMs in eq. (4.26), there will be an extra contribution

\[
\frac{1}{2} \zeta^2 f_2^2(\phi) f_2(\phi) \delta^2 S_{-2}[\phi] = \frac{1}{M_\Delta^2} \cdot \frac{1}{4} (C_{\text{tree}}^{G}(H^\dagger H))^2 \cdot \frac{3}{(4\pi)^2} (4\lambda_3^2 + \lambda_3)(1 + L_\Delta) \cdot (H^\dagger H)^6
\]

where \( \alpha, \beta = 1, 2 \) and the repeated indices on the left-hand side should be summed. The one-loop contribution in eq. (4.31) will be missed by simply applying the EOMs and must be added to the Wilson coefficient of the dim-6 operator \( O_H = (H^\dagger H)^6 \) in the Warsaw basis.

Applying the EOMs in eq. (4.26) to the dim-6 operators in the Green’s basis, one obtains one-loop corrections to the renormalizable terms in the Warsaw basis:

\[
\begin{align*}
\delta Z_H &= \delta Z_H^G, \\
\delta Z_W &= \delta Z_W^G, \\
\delta Z_H &= \delta Z_H^G, \\
\delta Z_L &= \delta Z_L^G, \\
\delta m^2 &= \left( \delta m^2 \right)^G + C_{DH}^G \frac{m^4}{M_\Delta^2}, \\
\delta \lambda &= \delta \lambda^G + \frac{g_2^2}{2} C_{2W}^G + 2g_2 C_{W DH}^G + 4(\lambda - 2\lambda_3^2) C_{DH}^G + C_{DH}^G \frac{m^2}{M_\Delta^2}, \\
&\quad - \left( \delta m^2 \right)^G + m^2 \delta Z_H^G \frac{C_{DH}^{G, \text{tree}}}{M_\Delta^2}.
\end{align*}
\]
$$\begin{align*}
(\delta Y_l)_{\alpha\beta} &= (Y_l)_{\alpha\beta} C^{G \Delta}_{DH} \frac{m^2}{M^2}, \\
(\delta Y_u)_{\alpha\beta} &= (Y_u)_{\alpha\beta} C^{G \Delta}_{DH} \frac{m^2}{M^2}, \\
(\delta Y_d)_{\alpha\beta} &= (Y_d)_{\alpha\beta} C^{G \Delta}_{DH} \frac{m^2}{M^2},
\end{align*}
$$

(4.32)

where $C^{G \Delta}_{DH}$, $C^{G \Delta}_{2W}$, $C^{G \Delta}_{W DH}$ and $C^{G \Delta}_{H D}$ denote the one-loop Wilson coefficients of the dim-6 operators $O^{G \Delta}_{DH}$, $O^{G \Delta}_{2W}$, $O^{G \Delta}_{W DH}$ and $O^{G \Delta}_{H D}$ in the Green’s basis, respectively. One can see that $\delta Y_l$, $\delta Y_u$ and $\delta Y_d$ absent in the Green’s basis now appear in the Warsaw basis. They are induced by the operator $O^{G \Delta}_{DH}$ in the Green’s basis after applying the EOM of $H$. Moreover, $O^{G \Delta}_{DH}$ also gives an additional one-loop contribution to the Wilson coefficient of the dim-5 operator

$$\left(\delta C^{(5)}\right)_{\alpha\beta} = \left(\delta C^{(5)}\right)^G_{\alpha\beta} - 2C^{G \Delta}_{DH} C^{G \Delta}_{DH} \frac{m^2}{M^2},$$

(4.33)

but this contribution is of the order of $\mathcal{O}\left(M^{-3}\right)$, since the dim-5 operator term itself is suppressed by $M^{-1}$ in the Lagrangian, and will be omitted. Then the kinetic terms of gauge bosons $W^I_{\mu}$, $B_{\mu}$, and the SU(2)$_L$ doublets $H$ and $\ell_L$ need to be normalized. For the gauge bosons, one needs to redefine the gauge boson fields and gauge couplings $g_1$ and $g_2$ simultaneously, namely

$$\begin{align*}
B_{\mu} &\to (1 + 2 \delta Z_B) B_{\mu}, \\
g_1 &\to g_1^{\text{eff}} = (1 - 2 \delta Z_B) g_1, \\
W^I_{\mu} &\to (1 + 2 \delta Z_W) W^I_{\mu}, \\
g_2 &\to g_2^{\text{eff}} = (1 - 2 \delta Z_W) g_2,
\end{align*}
$$

(4.34)

which give additional one-loop contributions to the effective couplings in the EFT via the corresponding tree-level terms. From eqs. (4.32)–(4.36), we obtain the effective couplings
in the EFT

\[ m_{\text{eff}}^2 = m^2(1 - \delta Z_H) - \delta m^2 \]

\[ = m^2 - \frac{1}{(4\pi)^2} \left[ 3M_H^2 (4\lambda_1 + \lambda_2) (1 + L) + 6m^2 \lambda_1^2 (3 + 2L) + \frac{4m^4}{M_H^2} \lambda_1^2 (11 + 6L) \right], \]

\[ \lambda_{\text{eff}} = \left[ \lambda - 2\lambda_1^2 \left( 1 + \frac{2m^2}{M_H^2} \right) \right] (1 - 2\delta Z_H) - \delta \lambda \]

\[ = \lambda - 2\lambda_1^2 \left( 1 + \frac{2m^2}{M_H^2} \right) + \frac{1}{(4\pi)^2} \left[ -\frac{1}{2} \left( 3\lambda_1^2 + 2\lambda_2^2 \right) L_\Delta + \left( g_1^2 - 20\lambda_1^2 \right) \frac{m^2}{30M_H^2} \right. \]

\[ + \left. \left[ 3g_1^2 (5 + 6L_\Delta) + g_2^2 (61 + 86L_\Delta) - 24\lambda (59 + 34L_\Delta) - 48(8\lambda_1 + \lambda_2)(1 + L_\Delta) \right. \]

\[ + 12\lambda_3 (29 + 18L_\Delta) - 8\lambda_4 (59 + 42L_\Delta) \right] \frac{m^2 \lambda_1^2}{6M_H^2} - 2\lambda_3^2 (2\lambda (13 + 10L_\Delta) \]

\[ + (8\lambda_1 + \lambda_2)(1 + L_\Delta) - 2\lambda_3 (8 + 5L_\Delta) + 4\lambda_4 (3 + 2L_\Delta)] + \frac{8\lambda_4^4}{3} \left[ 3(19 + 11L_\Delta) \right. \]

\[ + 20(13 + 6L_\Delta) \frac{m^2}{M_H^2} \] \]

\[ \left( Y_{\text{eff}}^{(i)} \right)_{\alpha\beta} = (Y_i)_{\alpha\beta} (1 - \delta Z_H/2) - \frac{1}{2} \delta Z_i Y_i - (\delta Y_i)_{\alpha\beta} \]

\[ = (Y_i)_{\alpha\beta} - \frac{1}{(4\pi)^2} \left[ \left( 3 + \frac{m^2}{M_H^2} (17 + 6L_\Delta) \right) \lambda_1^2 (Y_i)_{\alpha\beta} + \frac{3}{8} (Y_\Delta Y_\Delta Y_i^{(\alpha)} Y_i^{(\beta)}) (1 + 2L_\Delta) \right], \]

\[ \left( Y_{\text{eff}}^{(2)} \right)_{\alpha\beta} = (Y_\alpha)_{\alpha\beta} (1 - \delta Z_H/2) - (\delta Y_\alpha)_{\alpha\beta} \]

\[ = (Y_\alpha)_{\alpha\beta} - \frac{1}{(4\pi)^2} \left[ 3 + \frac{m^2}{M_H^2} (17 + 6L_\Delta) \right] \lambda_2^2 (Y_\alpha)_{\alpha\beta}, \]

\[ \left( Y_{\text{eff}}^{(3)} \right)_{\alpha\beta} = (Y_\alpha)_{\alpha\beta} (1 - \delta Z_H/2) - (\delta Y_\alpha)_{\alpha\beta} \]

\[ = (Y_\alpha)_{\alpha\beta} - \frac{1}{(4\pi)^2} \left[ 3 + \frac{m^2}{M_H^2} (17 + 6L_\Delta) \right] \lambda_3^2 (Y_\alpha)_{\alpha\beta}, \]

(4.37)

up to $O(M_H^{-2})$. Similarly, the Wilson coefficient of the dim-5 operator is given by

\[ \left( C_{\text{eff}}^{(5)} \right)_{\alpha\beta} = \left[ C^{(5)} (1 - \delta Z_H) - \frac{1}{2} \delta Z_t C^{(5)} + \frac{1}{2} C^{(5)} \delta Z_t^T + \delta C^{(5)} \right]_{\alpha\beta} \]

\[ = -\lambda_\Delta (Y_\Delta)_{\alpha\beta} + \frac{\lambda_\Delta}{(4\pi)^2} \left[ 6\lambda_2^2 (Y_\Delta)_{\alpha\beta} + \frac{3}{4} (1 + 2L_\Delta) (Y_\Delta Y_\Delta Y_i^{(\alpha)} Y_i^{(\beta)}) + (1 + L_\Delta) \right. \]

\[ \times \left. \left[ (2\lambda_3 - 4\lambda_4 - 8\lambda_1 - \lambda_2)(Y_\Delta)_{\alpha\beta} + (Y_i Y_\Delta Y_i^{(\alpha)} Y_\Delta^{(\beta)}) + (Y_\Delta Y_i^{(\alpha)} Y_i^{(\beta)} Y_i^{(\gamma)}) \right) \right]. \] (4.38)

Those in eqs. (4.35), (4.37) and (4.38) are the complete one-loop matching results, which can be used together with the two-loop renormalization-group equations (RGEs) of relevant Wilson coefficients in the SEFT-II.
4.2 Dimension-six operators

In the type-II seesaw model, the dim-6 operators $O_H$, $O_{H\Box}$, $O_{HD}$, $O_{eH}$, $O_{uH}$, $O_{dH}$ and $O_{\ell\ell}$ in the Warsaw basis have already appeared after integrating out the heavy triplet scalar at the tree level, as shown in section 3, and the tree-level contributions to their Wilson coefficients are shown in eq. (3.21). Those tree-level Wilson coefficients in eq. (3.21) result in extra one-loop contributions to the total Wilson coefficients of $O_H$, $O_{H\Box}$, $O_{HD}$, $O_{eH}$, $O_{uH}$, $O_{dH}$ and $O_{\ell\ell}$ via the normalizations of the kinetic terms of $H$ and $\ell_L$ given in eq. (4.36), the one-loop parts of EOMs in eq. (4.26), and also the term in eq. (4.31), i.e.,

$$
\delta C_H = -3C_{H}^{\text{tree}} \delta Z_H - 2C_{HD}^{\text{tree}} \left[ \delta \lambda^G + \left( \lambda - 2\lambda_3^2 \right) \delta Z_H^G \right] + \frac{12}{(4\pi)^2} \lambda_\Delta^4 \left( 4\lambda_3^2 + \lambda_3 \right)(1+L_\Delta)
$$

$$
= -\frac{4\lambda_\Delta^2}{(4\pi)^2} \left\{ \left( 3\lambda_3^2 + 2\lambda_3^2 \right) L_\Delta + [16\lambda(8+5L_\Delta) + 4(8\lambda_3 + \lambda_2)(1+L_\Delta) - \lambda_3(52+19L_\Delta)] + \lambda_4(57+32L_\Delta) \right\},
$$

$$
\delta C_{H\Box} = -2C_{H\Box}^{\text{tree}} \delta Z_H = -\frac{24}{(4\pi)^2} \lambda_\Delta^4,
$$

$$
\delta C_{HD} = -2 \left( C_{HD}^{\text{tree}} \right)^{\alpha\beta} \delta Z_H = -\frac{48}{(4\pi)^2} \lambda_\Delta^4.
$$

$$
\delta C_{eH}^{\alpha\beta} = -\frac{3}{2} \left( C_{eH}^{\text{tree}} \right)^{\alpha\beta} \delta Z_H - \frac{1}{2} \left( C_{eH}^{\text{tree}} \right)^{\alpha\beta} - \frac{1}{2} C_{HD}^{\text{tree}} (Y_{\alpha\beta} \delta Z_H^G)
$$

$$
= -\frac{30}{(4\pi)^2} \lambda_\Delta^4 (Y_{\alpha\beta}) - \frac{3}{4(4\pi)^2} (1+2L_\Delta) \lambda_\Delta^2 \left( Y_{\alpha\beta} Y_{ij} \right)_{ij},
$$

$$
\delta C_{uH}^{\alpha\beta} = -\frac{3}{2} \left( C_{uH}^{\text{tree}} \right)^{\alpha\beta} \delta Z_H - \frac{1}{2} C_{HD}^{\text{tree}} (Y_u)_{\alpha\beta} \delta Z_H^G
$$

$$
= -\frac{30}{(4\pi)^2} \lambda_\Delta^4 (Y_u)_{\alpha\beta},
$$

$$
\delta C_{dH}^{\alpha\beta} = -\frac{3}{2} \left( C_{dH}^{\text{tree}} \right)^{\alpha\beta} \delta Z_H - \frac{1}{2} C_{HD}^{\text{tree}} (Y_d)_{\alpha\beta} \delta Z_H^G
$$

$$
= -\frac{30}{(4\pi)^2} \lambda_\Delta^4 (Y_d)_{\alpha\beta},
$$

$$
\delta C_{\ell\ell}^{\alpha\beta\gamma\delta} = \frac{1}{8} \left( \delta Z_H \right)_{\alpha\gamma} \left( Y_{ij} \right)_{\beta\delta} + \left( \delta Z_H \right)_{\alpha\gamma} \left( Y_{\beta\delta} \right)_{ij} + \left( \delta Z_H \right)_{\alpha\gamma} \left( Y_{\beta\delta} \right)_{ij}
$$

$$
+ \left( \lambda_\Delta \right)_{\alpha\gamma} \left( Y_{ij} \delta Z_H \right)_{\beta\delta} = -\frac{3}{16(4\pi)^2} (1+2L_\Delta) \left[ \left( Y_{\alpha\beta} Y_H \right)_{\gamma\delta} \left( Y_{ij} \right)_{\beta\delta} + \left( Y_{\alpha\beta} \right)_{\gamma\delta} \left( Y_{ij} Y_H \right)_{\beta\delta} \right],
$$

which will be added into the total one-loop-level Wilson coefficients of the corresponding operators.

All dim-6 operators in the Warsaw basis induced by integrating out the heavy triplet scalar at the one-loop level in the type-II seesaw model are listed in table 1 and the associated one-loop-level Wilson coefficients up to $O\left( M_\Delta^{-2} \right)$ are explicitly given in the remaining part of this subsection, where an overall loop factor $1/(4\pi)^2$ is implied in all Wilson coefficients, and the contributions shown in eq. (4.39) have been added into the corresponding Wilson coefficients of $O_H$, $O_{H\Box}$, $O_{HD}$, $O_{eH}$, $O_{uH}$, $O_{dH}$ and $O_{\ell\ell}$.

- $X^3$

$$
C_W = \frac{g_2^3}{90}.
$$
Table 1. Summary of the dimension-six operators in the Warsaw basis in the SMEFT [5], where all the 41 operators induced at the tree and one-loop level by the heavy Higgs triplet in the type-II seesaw model are shown in the light gray region and those absent in the type-I seesaw model are further highlighted in blue in the dark gray region. The remaining 31 dimension-six operators are exactly those present in the low-energy EFT of the type-I seesaw model [52].

\[ X^2H^2 \]

\[ C_{HB} = \frac{g_1^2}{4} \lambda_3 - g_1^2 \lambda_3^2, \tag{4.41} \]

\[ C_{HWB} = -\frac{g_1 g_2}{3} \lambda_4 - \frac{5g_1 g_2}{3} \lambda_4^2, \tag{4.42} \]

\[ C_{HW} = \frac{g_2^2}{6} \lambda_3 - \frac{g_2^2}{3} \lambda_3^2. \tag{4.43} \]
• $H^4D^2$

\[
C_{H^\Box} = \frac{g_1^4}{80} - \frac{g_2^4}{40} - \frac{\lambda_2^2}{4} + \frac{\lambda_4^2}{6} + \left[ \frac{g_2^2}{3} (1 - 3L_\Delta) - \frac{g_3^2}{2} (9 + 14L_\Delta) + 8\lambda (3 + 2L_\Delta) \right. \\
+ 4(8\lambda_1 + \lambda_2)(1 + L_\Delta) - \lambda_3 (7 + 8L_\Delta) + \frac{2}{3} \lambda_4 (25 + 24L_\Delta) \left. \right] \lambda_2^2 \\
- \frac{4}{3} (67 + 24L_\Delta) \lambda_4^4 , \\
(4.44)
\]

\[
C_{HD_\Box} = - \frac{g_1^4}{20} - \frac{2\lambda_3^2}{3} + \left[ \frac{g_2^2}{6} (23 - 6L_\Delta) + \frac{g_3^2}{2} (5 + 6L_\Delta) + 4(3 + 2L_\Delta) \lambda_\lambda \right. \\
+ 8(8\lambda_1 + \lambda_2 - 2\lambda_3)(1 + L_\Delta) - \frac{8}{3} \lambda_4 (11 + 12L_\Delta) \left. \right] \lambda_3^2 - \frac{16}{3} (17 + 3L_\Delta) \lambda_4^4 . \\
(4.45)
\]

• $H^6$

\[
C_{H} = - \frac{g_1^4}{15} - \frac{\lambda_3^2}{2} + \frac{4\lambda_4^2}{3} - \lambda_3 \lambda_4^2 + \left[ \frac{2g_2^4}{15} - g_3^2 \lambda (5 + 6L_\Delta) - \frac{1}{3} g_3^2 \lambda (61 + 86L_\Delta) \right. \\
+ 120\lambda_2^2 (3 + 2L_\Delta) + 16\lambda (8\lambda_1 + \lambda_2)(1 + L_\Delta) - 60\lambda_3 (3 + 2L_\Delta) \\
+ \frac{8}{3} \lambda_4 (83 + 60L_\Delta) - 2\lambda_3 (8\lambda_1 + \lambda_2 + 16\lambda_4)(2 + L_\Delta) + 4\lambda_1 (17 + 7L_\Delta) \\
+ 2\lambda_2 \lambda_4 (2 + 3L_\Delta) + 26\lambda_3 + \frac{4}{3} \lambda_4^2 (25 + 12L_\Delta) \left. \right] \lambda_2^2 + \frac{2}{3} \lambda_3^4 \left[ 3g_1^2 (5 + 6L_\Delta) \\
+ g_3^2 (61 + 86L_\Delta) - 32\lambda (107 + 60L_\Delta) - 24\lambda_1 (29 + 24L_\Delta) - 72\lambda_2 (1 + L_\Delta) \\
+ 6\lambda_3 (171 + 82L_\Delta) - 2\lambda_4 (605 + 342L_\Delta) \right] + \frac{128}{3} \lambda_5^4 (71 + 30L_\Delta) \\
+ 12\lambda_3^4 (4\lambda_2^2 + \lambda_3^2)(1 + L_\Delta) . \\
(4.46)
\]

• $\psi^2HX$

\[
C_{eB}^{\alpha\beta} = \frac{g_1}{4} \left( Y_\lambda \tilde{Y}_\lambda^i Y_i^j \right)_{\alpha\beta} , \tag{4.47}
\]

\[
C_{eW}^{\alpha\beta} = - \frac{g_2}{8} \left( \tilde{Y}_\lambda Y_\lambda^i Y_i^j \right)_{\alpha\beta} . \tag{4.48}
\]

• $\psi^2DH^2$

\[
C_{Hq}^{(1)\alpha\beta} = - \frac{g_1^4}{120} \delta_{\alpha\beta} + \frac{g_2^2}{36} (19 + 6L_\Delta) \lambda_2^2 \delta_{\alpha\beta} - \frac{3}{4} (5 + 2L_\Delta) \lambda_2^2 \left( Y_d^i Y_d^j - Y_u^i Y_u^j \right)_{\alpha\beta} , \tag{4.49}
\]

\[
C_{Hq}^{(3)\alpha\beta} = - \frac{g_1^4}{60} \delta_{\alpha\beta} + \frac{g_2^2}{12} (7 + 2L_\Delta) \lambda_2^2 \delta_{\alpha\beta} - \frac{1}{4} (5 + 2L_\Delta) \lambda_2^2 \left( Y_d^i Y_d^j + Y_u^i Y_u^j \right)_{\alpha\beta} , \tag{4.50}
\]

\[
C_{Hd}^{\alpha\beta} = - \frac{g_1^4}{30} \delta_{\alpha\beta} + \frac{g_2^2}{9} (19 + 6L_\Delta) \lambda_2^2 \delta_{\alpha\beta} - \frac{3}{2} (5 + 2L_\Delta) \lambda_2^2 \left( Y_u^i Y_u^j \right)_{\alpha\beta} , \tag{4.51}
\]

\[
C_{Hd}^{\alpha\beta} = - \frac{g_1^4}{60} \delta_{\alpha\beta} - \frac{g_2^2}{18} (19 + 6L_\Delta) \lambda_2^2 \delta_{\alpha\beta} + \frac{3}{2} (5 + 2L_\Delta) \lambda_2^2 \left( Y_d^i Y_d^j \right)_{\alpha\beta} . \tag{4.52}
\]

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\[ C^{(1)\alpha\beta}_{H\ell} = \frac{g_1^4}{40} \delta_{\alpha\beta} - \frac{g_1^2}{12} (19 + 6L_\Delta) \lambda_3^2 \delta_{\alpha\beta} - \frac{g_1^2}{12} (5 + 3L_\Delta) \left( Y_{\Delta} Y_i^\dagger \right)_{\alpha\beta} \]
\[ - \frac{3}{4} \left[ (5 + 2L_\Delta) \lambda_3^2 \left( Y_{\Delta} Y_i^\dagger + 2Y_{\Delta} Y_i^\dagger \right)_{\alpha\beta} + (1 + L_\Delta) \left( Y_{\Delta} Y_i^\dagger Y_i^T Y_{\Delta} \right)_{\alpha\beta} \right], \quad (4.53) \]
\[ C^{(2)\alpha\beta}_{H\ell} = - \frac{g_2^4}{60} \delta_{\alpha\beta} + \frac{g_2^2}{12} (7 + 2L_\Delta) \lambda_3^2 \delta_{\alpha\beta} + \frac{g_2^2}{12} (2 + L_\Delta) \left( Y_{\Delta} Y_i^\dagger \right)_{\alpha\beta} \]
\[ - \frac{1}{4} \left[ (5 + 2L_\Delta) \lambda_3^2 \left( Y_{\Delta} Y_i^\dagger - 4Y_{\Delta} Y_i^\dagger \right)_{\alpha\beta} + (1 + L_\Delta) \left( Y_{\Delta} Y_i^\dagger Y_i^T Y_{\Delta} \right)_{\alpha\beta} \right], \quad (4.54) \]
\[ C^{(3)\alpha\beta}_{H\ell} = \frac{g_1^4}{40} \delta_{\alpha\beta} - \frac{g_1^2}{6} (19 + 6L_\Delta) \lambda_3^2 \delta_{\alpha\beta} + \frac{3}{2} (5 + 2L_\Delta) \lambda_3^2 \left( Y_i^\dagger Y_i \right)_{\alpha\beta} \]
\[ + \frac{1}{4} \left( Y_i^\dagger Y_{\Delta} Y_i^\dagger Y_{\Delta} \right)_{\alpha\beta}. \quad (4.55) \]

- $\psi^2 H^3$

\[ C^{(\alpha\beta)}_{uH} = \left\{ \begin{array}{c}
\frac{g_2^4}{60} + \frac{\lambda_3^2}{3} + \left[ \frac{1}{4} g_1^2 (5 + 6L_\Delta) - \frac{1}{12} g_2^2 (61 + 86L_\Delta) + 16 \lambda (3 + L_\Delta) \\
+ 4 (8 \lambda_1 + \lambda_2) (1 + L_\Delta) - \lambda_3 (13 + 8L_\Delta) + \frac{2}{3} \lambda_4 (35 + 24L_\Delta) \right] \lambda_3^2 \\
- \frac{2}{3} \lambda_3^2 (205 + 48L_\Delta) \right\} (Y_u)_{\alpha\beta} + \frac{1}{2} \lambda_3^2 \left[ (7 + 6L_\Delta) \left( Y_d Y_d^\dagger Y_u \right)_{\alpha\beta} \\
- 5 (1 + 2L_\Delta) \left( Y_u Y_d^\dagger Y_u \right)_{\alpha\beta} \right], \quad (4.56) \]
\[ C^{(\alpha\beta)}_{dH} = \left\{ \begin{array}{c}
\frac{g_2^4}{60} + \frac{\lambda_3^2}{3} + \left[ \frac{1}{4} g_1^2 (5 + 6L_\Delta) - \frac{1}{12} g_2^2 (61 + 86L_\Delta) + 16 \lambda (3 + L_\Delta) \\
+ 4 (8 \lambda_1 + \lambda_2) (1 + L_\Delta) - \lambda_3 (13 + 8L_\Delta) + \frac{2}{3} \lambda_4 (35 + 24L_\Delta) \right] \lambda_3^2 \\
- \frac{2}{3} \lambda_3^2 (205 + 48L_\Delta) \right\} (Y_d)_{\alpha\beta} + \frac{1}{2} \lambda_3^2 \left[ (7 + 6L_\Delta) \left( Y_u Y_d^\dagger Y_d \right)_{\alpha\beta} \\
- 5 (1 + 2L_\Delta) \left( Y_d Y_d^\dagger Y_d \right)_{\alpha\beta} \right], \quad (4.57) \]
\[ C^{(\alpha\beta)}_{cH} = \left\{ \begin{array}{c}
\frac{g_2^4}{60} + \frac{\lambda_3^2}{3} + \left[ \frac{1}{4} g_1^2 (5 + 6L_\Delta) - \frac{1}{12} g_2^2 (61 + 86L_\Delta) + 16 \lambda (3 + L_\Delta) \\
+ 4 (8 \lambda_1 + \lambda_2) (1 + L_\Delta) - \lambda_3 (13 + 8L_\Delta) + \frac{2}{3} \lambda_4 (35 + 24L_\Delta) \right] \lambda_3^2 \\
- \frac{2}{3} \lambda_3^2 (205 + 48L_\Delta) \right\} (Y_i)_{\alpha\beta} + \frac{1}{2} \lambda_3^2 \left[ (7 + 6L_\Delta) \left( Y_i Y_i^\dagger Y_i \right)_{\alpha\beta} \\
- \frac{1}{4} \left[ 3 \lambda_3 + 2 \lambda_4 - \lambda_3 (7 + 6L_\Delta) \right] \left( Y_{\Delta} Y_i^\dagger Y_i \right)_{\alpha\beta} + \frac{1}{2} \left( Y_{\Delta} Y_i^\dagger Y_i^T Y_{\Delta} Y_i \right)_{\alpha\beta} \\
+ \frac{1}{4} \left( Y_i Y_i^\dagger Y_{\Delta} Y_i^\dagger Y_i \right)_{\alpha\beta} \right], \quad (4.58) \]
• Four-quark

\[ C^{(1)\alpha\beta\gamma\delta}_{qq} = -\frac{g_1^4}{120} \delta_{\alpha\beta}\delta_{\gamma\delta}, \]  
(4.59)

\[ C^{(3)\alpha\beta\gamma\delta}_{qq} = -\frac{g_2^2}{120} \delta_{\alpha\beta}\delta_{\gamma\delta}, \]  
(4.60)

\[ C^{\alpha\beta\gamma\delta}_{uu} = -\frac{g_1^4}{45} \delta_{\alpha\beta}\delta_{\gamma\delta}, \]  
(4.61)

\[ C^{\alpha\beta\gamma\delta}_{dd} = -\frac{g_1^4}{180} \delta_{\alpha\beta}\delta_{\gamma\delta}. \]  
(4.62)

\[ C^{(1)\alpha\beta\gamma\delta}_{ud} = \frac{g_1^4}{45} \delta_{\alpha\beta}\delta_{\gamma\delta}, \]  
(4.63)

\[ C^{(1)\alpha\beta\gamma\delta}_{qu} = -\frac{g_1^4}{90} \delta_{\alpha\beta}\delta_{\gamma\delta} - \frac{1}{3} \lambda_3^2(Y_u)_{\alpha\delta}(Y_u^\dagger)_{\gamma\beta}, \]  
(4.64)

\[ C^{(8)\alpha\beta\gamma\delta}_{qu} = -2(Y_u)_{\alpha\delta}(Y_u^\dagger)_{\gamma\beta} \lambda_3^2, \]  
(4.65)

\[ C^{(1)\alpha\beta\gamma\delta}_{qd} = \frac{g_1^4}{180} \delta_{\alpha\beta}\delta_{\gamma\delta} - \frac{1}{3} \lambda_3^2(Y_d)_{\alpha\delta}(Y_d^\dagger)_{\gamma\beta}, \]  
(4.66)

\[ C^{(8)\alpha\beta\gamma\delta}_{qd} = -2(Y_d)_{\alpha\delta}(Y_d^\dagger)_{\gamma\beta} \lambda_3^2, \]  
(4.67)

\[ C^{(1)\alpha\beta\gamma\delta}_{quqd} = 2(Y_u)_{\alpha\beta}(Y_d)_{\gamma\delta} \lambda_3^2. \]  
(4.68)

• Four-lepton

\[ C^{\alpha\beta\gamma\delta}_{\ell\ell} = -\frac{g_1^4}{120} (2\delta_{\alpha\delta}\delta_{\gamma\beta} - \delta_{\alpha\beta}\delta_{\gamma\delta}) - \frac{g_1^4}{80} \delta_{\alpha\beta}\delta_{\gamma\delta} + \frac{g_2^2}{12}(5 + 3L_\Delta)(Y_\Delta Y_\Delta^\dagger)_{\alpha\beta}\delta_{\gamma\delta} \]  
\[ + \frac{g_2^2}{12}(2 + L_\Delta) \left[ 2(Y_\Delta Y_\Delta^\dagger)_{\alpha\beta}\delta_{\gamma\delta} + 2(Y_\Delta Y_\Delta^\dagger)_{\alpha\delta}\delta_{\gamma\beta} - \frac{1}{8} (Y_\Delta Y_\Delta^\dagger)_{\alpha\beta}(Y_\Delta Y_\Delta^\dagger)_{\gamma\delta} \right] \]  
\[ - \frac{1}{2} (Y_\Delta Y_\Delta^\dagger)_{\alpha\delta}(Y_\Delta Y_\Delta^\dagger)_{\gamma\beta} + \frac{1}{4} (8\alpha_1 + \lambda_2)(1 + L_\Delta)(Y_\Delta)_{\alpha\gamma}(Y_\Delta^\dagger)_{\beta\delta} \]  
\[ - \frac{3}{16} (1 + 2L_\Delta) \left[ \left( Y_\Delta Y_\Delta^\dagger Y_\Delta^\dagger \right)_{\alpha\gamma}(Y_\Delta^\dagger)_{\beta\delta} + (Y_\Delta)_{\alpha\gamma}(Y_\Delta Y_\Delta^\dagger Y_\Delta^\dagger)_{\beta\delta} \right], \]  
(4.69)

\[ C^{\alpha\beta\gamma\delta}_{\ell e} = -\frac{g_1^4}{20} \delta_{\alpha\beta}\delta_{\gamma\delta} - \lambda_3(Y_e)_{\alpha\delta}(Y_e^\dagger)_{\gamma\beta} + \frac{g_2^2}{6}(5 + 3L_\Delta)(Y_\Delta Y_\Delta^\dagger)_{\alpha\beta}\delta_{\gamma\delta} \]  
\[ - \frac{3}{8} (3 + 2L_\Delta)(Y_\Delta Y_\Delta^\dagger)_{\gamma\alpha}(Y_\Delta Y_\Delta^\dagger)_{\beta\delta} \]  
(4.70)

\[ C^{\alpha\beta\gamma\delta}_{ee} = -\frac{g_1^4}{20} \delta_{\alpha\beta}\delta_{\gamma\delta}. \]  
(4.71)

• Semileptonic

\[ C^{(1)\alpha\beta\gamma\delta}_{\ell q} = \frac{g_1^4}{120} \delta_{\alpha\beta}\delta_{\gamma\delta} - \frac{g_1^2}{36}(5 + 3L_\Delta)(Y_\Delta Y_\Delta^\dagger)_{\alpha\beta}\delta_{\gamma\delta}, \]  
(4.72)

\[ C^{(3)\alpha\beta\gamma\delta}_{\ell q} = -\frac{g_2^2}{60} \delta_{\alpha\beta}\delta_{\gamma\delta} + \frac{g_2^2}{12}(2 + L_\Delta)(Y_\Delta Y_\Delta^\dagger)_{\alpha\beta}\delta_{\gamma\delta}, \]  
(4.73)

\[ C^{\alpha\beta\gamma\delta}_{e\ell} = \frac{g_1^4}{15} \delta_{\alpha\beta}\delta_{\gamma\delta}. \]  
(4.74)
\[ C_{\alpha \beta \gamma \delta}^{ed} = -\frac{g_1^4}{30} \delta_{\alpha \beta} \delta_{\gamma \delta}, \]  
(4.75) 
\[ C_{\alpha \beta \gamma \delta}^{qe} = \frac{g_1^4}{60} \delta_{\alpha \beta} \delta_{\gamma \delta}, \]  
(4.76) 
\[ C_{\alpha \beta \gamma \delta}^{elu} = \frac{g_1^4}{30} \delta_{\alpha \beta} \delta_{\gamma \delta} - \frac{g_1^2}{9} (5 + 3L_\Delta) (Y_\Delta Y_\Delta^\dagger)_{\alpha \beta} \delta_{\gamma \delta}, \]  
(4.77) 
\[ C_{\alpha \beta \gamma \delta}^{ld} = -\frac{g_1^4}{60} \delta_{\alpha \beta} \delta_{\gamma \delta} + \frac{g_1^2}{18} (5 + 3L_\Delta) (Y_\Delta Y_\Delta^\dagger)_{\alpha \beta} \delta_{\gamma \delta}, \]  
(4.78) 
\[ C_{\alpha \beta \gamma \delta}^{le dq} = 2 (Y_l)_{\alpha \beta} (Y_u^\dagger)_{\gamma \delta} \lambda_2^2, \]  
(4.79) 
\[ C_{\alpha \beta \gamma \delta}^{le qu} = -2 (Y_l)_{\alpha \beta} (Y_u)_{\gamma \delta} \lambda_2^2. \]  
(4.80) 

Now, with all above results, we can write out the complete Lagrangian of the SEFT-II up to the one-loop level, that is

\[ \mathcal{L}_{\text{SEFT-II}} = \mathcal{L}_{\text{SM}} \left( m_2 \rightarrow m_{\text{eff}}, \lambda \rightarrow \lambda_{\text{eff}}, Y_l \rightarrow Y_{\text{eff}}^l, Y_u \rightarrow Y_{\text{eff}}^u, Y_d \rightarrow Y_{\text{eff}}^d, g_1 \rightarrow g_{\text{eff}}^1, g_2 \rightarrow g_{\text{eff}}^2 \right) \]

\[ + \frac{1}{M_\Delta} \left[ \left( C_{\text{tree}}^{H} \right)_{\alpha \beta} O_{\alpha \beta} + \left( C_{\text{tree}}^{H \Box} \right)_{\alpha \beta} O_{\alpha \beta} + \left( C_{\text{tree}}^{H D} \right)_{\alpha \beta} O_{\alpha \beta} + \left( C_{\text{tree}}^{U} \right)_{\alpha \beta} O_{\alpha \beta} + \left( C_{\text{tree}}^{D} \right)_{\alpha \beta} O_{\alpha \beta} + \text{h.c.} \right] \]  
\[ + \sum_i \frac{C_i}{M_\Delta} O_i, \]  
(4.81)

where the couplings in the SM have been replaced by the effective coupling given in eqs. (4.35) and (4.37), \( C_{\text{eff}}^{(5)} \) is given in eq. (4.38) and includes both the tree-level and one-loop contributions, \( C_{\text{tree}}^{H}, C_{\text{tree}}^{H \Box}, C_{\text{tree}}^{H D}, C_{\text{tree}}^{U}, C_{\text{tree}}^{D} \) are the tree-level contributions to the Wilson coefficients of the associated dim-6 operators and are listed in eq. (3.21), \( O_i \) denote the dim-6 operators listed in table 1 including Hermitian conjugations of the non-Hermitian operators, and \( C_i \) refer to the one-loop contributions to the corresponding Wilson coefficients. Unlike what we have done for the unique dimension-five operator, the tree- and one-loop-level contributions to the Wilson coefficients of \( O_H, O_{H \Box}, O_{H D}, O_{eH}, O_{uH}, O_{dH} \) and \( O_{\ell \ell} \) are not summed up.

### 4.3 Diagrammatic approach

The one-loop matching of an UV theory onto the SMEFT can also be achieved by calculating the 1LPI Feynman diagrams and extracting the hard-part contributions of the amplitudes by virtue of expansion by regions. As an independent cross-check of the functional approach, the tree-level and one-loop matching of a general UV model onto arbitrary effective theories by diagrammatic approach has recently been automated in the publicly available package Matchmaker [86]. It is worthwhile to mention that the one-loop matching of the type-I seesaw model onto the SMEFT has been done in ref. [87] through the diagrammatic calculations, which should be compared with the functional approach in ref. [52].
Since we will calculate the radiative decays of charged lepton $l^+_\alpha \rightarrow l^-_\beta + \gamma$ in the SEFT-II in the next section, it is instructive to make use of the Feynman diagrammatic approach to match the relevant operators, i.e., $O_{eB}$ and $O_{eW}$ in the Warsaw basis, following the procedure in ref. [43]. In the type-II seesaw model, only the operators, i.e.,

$$O_{\ell D}^{\alpha\beta} = \frac{i}{2} \alpha L \left( D^2 \phi + \bar{\psi} D \phi \right) \ell_{\beta L},$$

$$O_{B\ell}^{\alpha\beta} = \frac{i}{2} \alpha L \gamma^\mu \phi^\dagger \ell_{\beta L} B_{\mu \nu},$$

$$O_{\bar{W}\ell}^{\alpha\beta} = \frac{i}{2} \alpha L \gamma^\mu \phi^\dagger \ell_{\beta L} \bar{W}_{\mu \nu},$$

in the Green’s basis contribute to $O_{eB}$ and $O_{eW}$, whereas others in the Green’s basis contributing to $O_{eB}$ and $O_{eW}$ are absent due to the special interactions of the triplet scalar. Thus we need only to derive the Wilson coefficients of those operators listed in eq. (4.82) via the Feynman diagrammatic approach. For this purpose, we calculate the amplitudes corresponding to the Feynman diagrams (a) and (b) in figure 1. There are another two operators in the Green’s basis contributing to these amplitudes, namely,

$$O_{\bar{W}B}^{\alpha\beta} = \frac{i}{2} \alpha L \gamma^\mu \phi^\dagger \ell_{\beta L} \bar{W}_{\mu \nu} \ell^\dagger_{\alpha L},$$

(4.83)

When the one-loop matching is carried out by calculating the Feynman diagrams in figure 1, those two operators in eq. (4.83) should also be taken into consideration, though they do not contribute to $O_{eB}$ and $O_{eW}$ in the Warsaw basis.

The contributions from the operators given in eqs. (4.82) and (4.83) to the amplitudes $(\ell\ell B)$ and $(\ell\ell W)$ in the EFT are respectively found to be

$$iM_{B,EFT} = \frac{i\sigma_{\ell B}}{M} \sigma(p_2) P_R \left[ \gamma^\mu q_2 \left( C_{B\ell}^\dagger \right)_{\beta\alpha} + \frac{1}{2} g_1 \gamma^\mu \phi_2 (C_{LD})_{\beta\alpha} + \frac{1}{2} g_1 \gamma^\mu \phi_2 (C_{LD})_{\beta\alpha} + \frac{1}{2} g_1 \gamma^\mu \phi_2 (C_{LD})_{\beta\alpha} + \frac{1}{2} g_1 \gamma^\mu \phi_2 (C_{LD})_{\beta\alpha} \right] u(p_1) \epsilon_\mu(q),$$

(4.84)

and

$$iM_{W,EFT} = \frac{i\sigma_{\ell W}}{M} \sigma(p_2) P_R \left[ \gamma^\mu q_2 \left( C_{W\ell}^\dagger \right)_{\beta\alpha} + \frac{1}{2} g_2 C_{W\ell} + \frac{1}{2} g_2 C_{W\ell} \right] u(p_1) \epsilon_\mu(q).$$

(4.85)

\[\text{The threshold correction to the kinetic term of the lepton doublet, i.e., } \delta Z_{L\ell}^\gamma \ell_\ell \ell_\ell \phi \ell_\ell \text{ in the Green’s basis, also contributes to these amplitudes from figure 1. However, since the contributions from this dim-4 term can be easily distinguished from those induced by the dim-6 operators given in eqs. (4.82) and (4.83). Hence we do not consider this threshold correction here.}\]
Figure 1. Feynman diagrams for the amplitude $\langle \ell \ell S \rangle$ in the UV model, which should be matched by operators $O_{\ell D}, O_{St}, O_{St}'$ and $O_{St}'$ (for $S = B, W$) in the Green’s basis in the EFT.

Now we proceed to calculate the amplitudes $\langle \ell \ell B \rangle$ and $\langle \ell \ell W \rangle$ in the UV model. It is straightforward to find the corresponding amplitudes for diagrams (a) and (b) in figure 1 for $B_\mu$ as below

$$iM_{\mu}^{B,UV} = \pi(p_2) \frac{k - p_2}{(2\pi)^4 (k^2 - M_\Delta^2)(k - p_2)^2 (k - p_2 - q)^2} u(p_1) \epsilon_\mu(q)$$

$$\times \left( -\frac{3}{2} \right) g_1 \delta_{ab} \left( Y_\Delta Y_\Delta^\dagger \right)_{\beta \alpha},$$

$$iM_{\mu}^{B,UV} = \pi(p_2) \int \frac{k - p_2}{(2\pi)^4 (k^2 - M_\Delta^2)(k - p_2 - q)^2 (k - q)^2 - M_\Delta^2} u(p_1) \epsilon_\mu(q)$$

$$\times (-3) g_1 \delta_{ab} \left( Y_\Delta Y_\Delta^\dagger \right)_{\beta \alpha}. \quad (4.86)$$

The hard-momentum parts of these two amplitudes can be obtained by expanding the integrands in the limit of $p \ll k, M_\Delta$ with $p$ being any external momentum (i.e., $p = p_1, p_2, q$ in the present case). Since the combination of the external fields $\ell \ell B$ is already of mass-dimension four, the relevant terms for dim-6 operators should be proportional to the second power of the external momentum. The latter is equivalent to the square of the spacetime derivative. With the help of eq. (4.86), we can get the contributions from the hard-momentum region

$$iM_{\mu}^{B,UV} \bigg|_{\text{hard}} = iM_{\mu}^{B,UV} \bigg|_{\text{hard}} + iM_{\mu}^{B,UV} \bigg|_{\text{hard}}$$

$$= \frac{-i g_1 \delta_{ab}}{12(4\pi)^2 M_\Delta^2} \pi(p_2) \frac{\gamma_\mu}{(2\pi)^2} \left[ -9 \phi \vec{p}_2 + 3 \phi \vec{p}_2 + q^2 (7 + 6L_\Delta) + 12 q \cdot p_2 \right]$$

$$+ 6 (\phi - \vec{q}) p_2^\mu + 2 \left[ 6 \phi (2 + 3L_\Delta) q_\mu \right] u(p_1) \epsilon_\mu(q), \quad (4.87)$$

where only the terms proportional to $p^2$ (or equivalently $M_\Delta^{-2}$) have been retained. Similarly, we can obtain the contributions from the hard-momentum region to the amplitude...
\langle \ell\ell W \rangle \) in the UV model, that is
\begin{align}
 i\mathcal{M}_{\text{tot}}^{W,\text{UV}}\bigg|_{\text{hard}} &= \frac{\frac{g_3 g_\alpha}{12(4\pi)^2} M_\Delta^2}{\mathcal{F}_\alpha(p_2)} \mathcal{P}_R \left\{ \gamma^\mu \left[ -3q_\mu p_2 + 3p_\mu^2 + q^2 (4 + 2\Delta) + 6q \cdot p_2 \right]
 + 6q_\mu p_2^\mu + \left[ 6q_\mu - (1 + 2\Delta)q \right]q_\mu \right\} u(p_1) c_\mu(q) .
\end{align}

Equating eq. (4.84) with eq. (4.87), and eq. (4.85) with eq. (4.88) as well, one can arrive at
\begin{align}
 C_{\ell D} &= \frac{1}{2(4\pi)^2} \left( Y_{\Delta}^l Y_{\Delta}^{\dagger} \right) , & C_{\ell B} &= C_{W\ell} = 0 , \\
 C'_{\ell B} &= \frac{3g_1}{4(4\pi)^2} \left( Y_{\Delta} Y_{\Delta}^{\dagger} \right) , & C'_{W\ell} &= \frac{g_2}{4(4\pi)^2} \left( Y_{\Delta} Y_{\Delta}^{\dagger} \right) , \\
 C_{B\ell} &= \frac{g_1}{24(4\pi)^2} \left( 11 + 12\Delta \right) \left( Y_{\Delta} Y_{\Delta}^{\dagger} \right) , & C_{W\ell} &= \frac{g_2}{24(4\pi)^2} \left( 5 + 4\Delta \right) \left( Y_{\Delta} Y_{\Delta}^{\dagger} \right) .
\end{align}

Given the results in eq. (4.89) and the connection between the operators in the Green’s basis and those in the Warsaw basis, we can obtain the Wilson coefficients of \( O_{\ell B} \) and \( O_{\ell W} \), namely,
\begin{align}
 C_{\ell B} &= -\frac{g_1}{8} C_{\ell D} Y_l - \frac{i}{4} C'_{\ell B} Y_l + \frac{1}{4} C'_{B\ell} Y_l = + \frac{g_1}{4(4\pi)^2} \left( Y_{\Delta} Y_{\Delta}^{\dagger} \right) , \\
 C_{W\ell} &= -\frac{g_2}{8} C_{\ell D} Y_l - \frac{i}{4} C'_{W\ell} Y_l + \frac{1}{4} C'_{W\ell} Y_l = - \frac{g_2}{8(4\pi)^2} \left( Y_{\Delta} Y_{\Delta}^{\dagger} \right) ,
\end{align}

which are exactly same as those obtained by the functional approach [cf. eqs. (4.47) and (4.48)].

The previous calculations clearly exemplify the diagrammatic approach to one-loop matching. However, owing to the complexity of interactions in the full type-II seesaw model, numerous Feynman diagrams have to be calculated for a complete one-loop matching, so the time-consuming manual calculations seem to be impractical. In our work, we actually implement the diagrammatic approach in a semi-automatic way. First, we use FeynRules [88] to generate the FeynArts [89] model file. The interface FeynHelper [90] is utilized to realize the connection among FeynArts, FeynCalc [91] and Package-X [92]. More explicitly, FeynArts draws all the Feynman diagrams and generates the corresponding amplitudes, FeynCalc calculates the loop integrals involved in the amplitudes, and Package-X automatically converts the amplitudes into Passarino-Veltman functions [93], whose analytical expressions are provided. All these tools have been implemented in Mathematica, and thus it is convenient for us to compare the results from diagrammatic calculations with those from functional approach by SuperTracer. Through the Feynman diagrammatic approach, we have cross-checked all the operators up to dim-6 and the Wilson coefficients from SuperTracer, and found a complete agreement.

5 Radiative decays of charged leptons

As a demonstrative example, we shall calculate the branching ratios of radiative decays of charged leptons \( l_{\alpha}^{-} \rightarrow l_{\beta}^{-} + \gamma \) in the SEFT-II and compare the results with those calculated
in the full type-II seesaw model. In the EFT, the relevant Lagrangian after the spontaneous gauge symmetry breaking is given by

\[ \mathcal{L}_{\text{SEFT-II}} \supset -i a_{\alpha} (M_{\alpha} \alpha_{\beta} \mu R) - \frac{1}{2} g_{a} M_{\nu} \nu_{\alpha} + \frac{g_{2}}{\sqrt{2}} a_{\alpha} \alpha_{\beta} \mu W_{\mu}^{\alpha} + \frac{v}{\sqrt{2} M_{\Delta}} \left( \cos \theta_{w} c_{\alpha} \beta - \sin \theta_{w} c_{\alpha} \beta \right) l_{\alpha} \sigma_{\mu} \mu l_{\beta} F_{\mu \nu} + \text{h.c.}, \]  

(5.1)

where only the terms related to radiative decays of charged leptons are kept. In eq. (5.1), we have defined the charged-lepton mass matrix \( M_{l} = v Y_{l} / \sqrt{2} \) and the effective neutrino mass matrix \( M_{\nu} = v^{2} \lambda_{\Delta} Y_{\Delta} / M_{\Delta} \) induced by the dim-5 Weinberg operator. The electromagnetic dipole operator in the second line of eq. (5.1) results from the two dim-6 operators \( O_{\mu B} \) and \( O_{\mu W} \), for which the Wilson coefficients are respectively \( C_{\mu B} = g_{1} (Y_{\Delta} Y_{\Delta}^{*} Y_{l}) / (64 \pi^{2}) \) and \( C_{\mu W} = -g_{2} (Y_{\Delta} Y_{\Delta}^{*} Y_{l}) / (128 \pi^{2}) \). In addition, \( \theta_{w} = \arctan(g_{1} / g_{2}) \) is the weak mixing angle, and \( F_{\mu \nu} = \partial_{\mu} \nu_{\nu} - \partial_{\nu} \nu_{\mu} \) is the gauge field strength with \( A_{\mu} \) being the photon field. Note that the terms in the first line of eq. (5.1) appear at the tree level while those in the second line at the one-loop level.

Working in the flavor basis where the charged-lepton mass matrix is diagonal, namely, \( M_{l} = \text{Diag} \{ m_{e}, m_{\mu}, m_{\tau} \} \), we can make a field transformation \( \nu_{L} \to U \nu_{L} \) with \( U \) being the unitary matrix to diagonalize the effective neutrino mass matrix via \( U \dagger M_{\nu} U^{*} = \tilde{M}_{\nu} = \text{Diag} \{ m_{1}, m_{2}, m_{3} \} \). Then the Lagrangian in eq. (5.1) turns out to be

\[ \mathcal{L}_{\text{SEFT-II}} \supset -i \eta_{L} M_{l} l_{R} - \frac{1}{2} g_{a} M_{\nu} \nu_{\alpha} + \frac{g_{2}}{\sqrt{2}} \eta_{L} \alpha_{\mu} \mu W_{\mu}^{\alpha} + \frac{3 e}{8 (4 \pi)^{2} M_{\Delta}} \eta_{L} \sigma_{\mu} \mu Y_{\Delta} Y_{\Delta}^{*} M_{l} l_{R} F_{\mu \nu} + \text{h.c.}, \]  

(5.2)

in which \( e = g_{1} \cos \theta_{w} = g_{2} \sin \theta_{w} \) has been used, and the unitary matrix \( U \) is the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrix [94–96]. From the Lagrangian given in eq. (5.2), one can observe that there are two kinds of contributions to the branching ratios of \( l_{\alpha} \to l_{\beta} + \gamma \) in the EFT. One is from massive neutrinos via one-loop diagrams induced by the charged-current interactions of leptons, i.e., the diagrams (a)-(c) in figure 2. The other one comes directly from the dim-6 operators \( O_{\mu B} \) and \( O_{\mu W} \) [i.e., the electromagnetic dipole operator in the second line of eq. (5.2)], corresponding to the diagram (d) in figure 2. In fact, the former contribution is entirely included in the full theory, whereas the latter corresponds to the contributions from the singly- and doubly-charged scalars \( \Delta^{-} \) and \( \Delta^{-} \) in the full theory. The triplet-mediated diagrams have been shown in figure 3.

The radiative decays of charged leptons induced by massive neutrinos [i.e., the processes corresponding to diagrams (a)-(c) in figure 2] have been investigated a long time ago [55, 97–102]. The amplitudes of these rare decays are highly suppressed due to the smallness of neutrino masses, and to the unitarity of the PMNS matrix (i.e., the Glashow-Iliopoulos-Maiani mechanism [103] in the leptonic sector) as well. As a consequence, in the type-II seesaw model, the contributions given by the diagrams (a)-(c) in figure 2 are usually omitted and only those from \( \Delta^{-} \) and \( \Delta^{-} \) shown in figure 3 need to be taken into account.
Figure 2. Feynman diagrams for radiative decays of charged leptons $l_\alpha^- \to l_\beta^- + \gamma$ at one-loop level in the EFT. The unitary gauge has been adopted. (a)-(c) are mediated by massive neutrinos via the charged current interactions of leptons, (d) is generated by the dimension-six operators at one-loop level.

Figure 3. Feynman diagrams mediated by the singly- and doubly-charged scalars for radiative decays of charged leptons $l_\alpha^- \to l_\beta^- + \gamma$ at the one-loop level in the full type-II seesaw model. The unitary gauge has been adopted.

Similarly, we shall also ignore the contributions from massive neutrinos to the radiative decays of charged leptons, and focus only on those from the electromagnetic dipole operator in the second line of eq. (5.2). It is easy to obtain the amplitude corresponding to the diagram (d) in figure 2:

$$i\mathcal{M} = \frac{3e}{4(4\pi)^2 M_\Delta^4} \left( Y_\Delta Y_\Delta^\dagger \right)_{\beta\alpha} \bar{\nu}(p_2)\sigma^{\mu\nu}q_\nu \left( m_\alpha P_R + m_\beta P_L \right) u(p_1)\epsilon_\mu^\dagger(q).$$ \hspace{1cm} (5.3)$$

Then, the decay rate is given by

$$\Gamma(l_\alpha^- \to l_\beta^- + \gamma) = \frac{1}{2m_\alpha} \cdot \frac{1}{8\pi} \left( 1 - \frac{m_\beta^2}{m_\alpha^2} \right) \cdot \frac{1}{2} \sum |\mathcal{M}|^2$$

$$= \frac{9\alpha_{em} m_\alpha^5}{64(4\pi)^4 M_\Delta^4} \left( 1 + \frac{m_\beta^2}{m_\alpha^2} \right) \left( 1 - \frac{m_\beta^2}{m_\alpha^2} \right)^3 \left| \left( Y_\Delta Y_\Delta^\dagger \right)_{\beta\alpha} \right|^2$$

$$\approx \frac{9\alpha_{em} m_\alpha^5}{64(4\pi)^4 M_\Delta^4} \left| \left( Y_\Delta Y_\Delta^\dagger \right)_{\beta\alpha} \right|^2,$$ \hspace{1cm} (5.4)$$

If the masses of triplet scalars are very large, the contributions from massive neutrinos via diagrams (a)-(c) in figure 2 may be comparable to those from $\Delta^-$ and $\Delta^{--}$ via the diagrams in figure 3. In this case, all the contributions should be considered though all of them are very small.
where $\alpha_{\text{em}} \equiv e^2/(4\pi)$ is the electromagnetic fine-structure constant, and the tiny ratio $m_\beta^2/m_\alpha^2 \ll 1$ in the last step has been neglected. For convenience, one can define a dimensionless ratio between the rate of radiative decays and that of purely leptonic decays $l^-_\alpha \to l^-_\beta + \bar{\nu}_\beta + \nu_\alpha$, that is

$$
\xi(l^-_\alpha \to l^-_\beta + \gamma) \equiv \frac{\Gamma(l^-_\alpha \to l^-_\beta + \gamma)}{\Gamma(l^-_\alpha \to l^-_\beta + \bar{\nu}_\beta + \nu_\alpha)} \approx \frac{27\alpha_{\text{em}}}{256\pi G_F M_\Delta^4} \left| \left( Y_\Delta Y^\dagger_\Delta \right)_{e\mu} \right|^2, \quad (5.5)
$$

where $G_F$ is the Fermi constant. For $\mu \to e\gamma$, the dimensionless ratio is approximately the branching ratio, i.e., $\text{BR}(\mu \to e\gamma) \approx \xi(\mu \to e\gamma)$, as the purely leptonic decay $\mu^- \to e^- + \bar{\nu}_\mu + \nu_\mu$ dominates over all other channels. One can make use of eq. (5.5) to constrain the Yukawa coupling matrix $Y_\Delta$ if the experimental constraints on the radiative decays of charged leptons are taken into account.

The radiative decay of muon $\mu \to e\gamma$ in the type-II seesaw model has also been extensively calculated [104–110]. Once the diagrams in figure 3 are evaluated, the branch ratio of $\mu \to e\gamma$ contributed from $\Delta^-$ and $\Delta^{--}$ will be [104–110]$^4$

$$
\text{BR}'(\mu \to e\gamma) \approx \frac{\alpha_{\text{em}}}{192\pi G_F^2} \left( \frac{1}{M_{\Delta^-}^2} + \frac{8}{M_{\Delta^{--}}^2} \right) \frac{1}{4} \left| \left( Y_\Delta Y^\dagger_\Delta \right)_{e\mu} \right|^2 \approx \frac{27\alpha_{\text{em}}}{256\pi G_F M_\Delta^4} \left| \left( Y_\Delta Y^\dagger_\Delta \right)_{e\mu} \right|^2. \quad (5.6)
$$

In the second line of eq. (5.6), the mass spectrum $M_{\Delta^-}^2 = M_\Delta^2$ and $M_{\Delta^{--}}^2 = M_\Delta^2 (1 + 2\lambda_\Delta^2 v^2/M_\Delta^2) \approx M_\Delta^2$ has been considered.

As is expected, the results in eqs. (5.5) and (5.6) are completely consistent with each other for the radiative decay of muon $\mu \to e\gamma$. In a similar way, one can check other decay channels.

6 Summary

In the present paper, we have accomplished a complete one-loop matching of the type-II seesaw model onto the SMEFT. The primary motivation for the construction of the low-energy EFT for the type-II seesaw model is two-fold. First, neutrino oscillations have provided us with very strong evidence that neutrinos are massive particles. The origin of neutrino masses definitely calls for a renormalizable UV theory beyond the SM. As one of the simplest and most natural models for tiny Majorana neutrino masses, the type-II seesaw model has to be scrutinized by experimental tests. Second, as the particle physics has entered the precision era, more accurate data require more precise calculations. The latter could be either the loop-level matching condition between the UV theory and the EFT or the higher-order calculations in the EFT. The one-loop matching of the type-II seesaw model onto the SMEFT sets up a self-consistent framework to investigate low-energy

$^4$Note that different conventions of the Yukawa coupling matrix exist in the literature, namely, $Y_\Delta$ in our work and $h$ in some references. The connection between these two conventions is given by $h = Y_\Delta^\dagger/\sqrt{2}$. 

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phenomenology of the EFT and to analyze experimental results of possible deviations from the SM predictions.

The functional approach has been implemented in this work to derive all the effective operators up to dim-6 in the Warsaw basis and the corresponding Wilson coefficients. In addition, the one-loop threshold corrections to the mass parameter and the couplings in the SM and also to the coefficient of the dim-5 operator are given up to $\mathcal{O}(M^{-2})$, which are the very matching conditions for the two-loop RGEs of these physical parameters. At the one-loop level, we have found 41 dim-6 operators in the SEFT-II, covering all the 31 dim-6 operators in the SEFT-I. Ten dim-6 operators present in the SEFT-II but not in the SEFT-I arise solely from the gauge interactions of the Higgs triplet in the type-II seesaw model. It is worthwhile to mention that the dim-6 operators and their Wilson coefficients derived by the functional approach have been cross-checked by the diagrammatic approach. The calculations by using both functional and diagrammatic approaches are necessary to ensure the correctness of the final results.

Finally, we point out that there are several important issues left for further studies. First, the EFT has been constructed by one-loop matching at the decoupling scale characterized by the masses of heavy degrees of freedom. The two-loop RGEs of the SM parameters and the Wilson coefficients in the EFT must be derived and used to run all the physical parameters from the high-energy scale to the low-energy one. Second, now that the EFTs for both type-I and type-II seesaw models are available, it is interesting to explore the distinct experimental signatures induced by the dim-6 operators that are only present in the SEFT-II. Third, once the EFTs at the low-energy scale are obtained, a global-fit analysis of all relevant experimental data in the SEFT-II framework will be indispensable to probe the fundamental parameters in the UV theory. We hope to come back to these issues in the near future.

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Note added. After completing one-loop matching of the type-II seesaw model onto the SMEFT, we are kindly informed that a similar work has also been done independently in ref. [111]. When our paper is being finalized, the package MatchmakerEFT for one-loop matching via diagrammatic approach is released [86]. In this updated version, we have cross-checked our results by using MatchmakerEFT and found an excellent agreement except for the additional contribution in eq. (4.31) to the Wilson coefficient for the operator $O_H$ [cf. eq. (4.46)]. Such a discrepancy clearly originates from the simple implementation of EOMs in changing from the Green’s basis to the Warsaw basis in ref. [86], as we have explained in section 4.1.
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