On the uniqueness of solutions to continuity equations  
V.I. Bogachev$^a$, G. Da Prato$^b$, M. Röckner$^c$, S.V. Shaposhnikov$^a$

$^a$ Department of Mechanics and Mathematics, Moscow State University, 119991 Moscow, Russia  
and National Research University Higher School of Economics, Moscow, Russia  
$^b$ Scuola Normale Superiore di Pisa, Piazza dei Cavalieri, 56100 Pisa, Italy  
$^c$ Fakultät für Mathematik, Universität Bielefeld, D-33501 Bielefeld, Germany

Abstract
We obtain sufficient conditions for the uniqueness of solutions to the Cauchy problem for the continuity equation in classes of measures that need not be absolutely continuous.

Keywords: Continuity equation, Cauchy problem, Uniqueness

INTRODUCTION

In this paper we study the uniqueness problem for the continuity equation
\[ \partial_t \mu_t + \text{div}(b\mu_t) = 0 \]
with respect to measures on $\mathbb{R}^d$. We consider solutions given by families of locally bounded Borel measures $(\mu_t)_{t \in [0,T]}$. A precise definition is given below.

There is a vast literature devoted to uniqueness and existence problems for the Cauchy problem for such equations. An important problem is to specify a class of measures $\mu_t$ in which, under reasonable assumptions about coefficients and initial data, there is a unique solution to the Cauchy problem. Certainly, if the coefficients are sufficiently regular, say, Lipschitzian or satisfy the Osgood type condition, then we can take the whole class of bounded measures $\mu_t$ (see, e.g., [4]). According to a well-known result of Ambrosio [3] (see also [29] for equations with a potential term) on representations of nonnegative bounded solutions by means of averaging with respect to measures concentrated on solutions to the corresponding ordinary equation $\dot{x} = b(x,t)$, any uniqueness condition for the ordinary equation guarantees uniqueness in the class of nonnegative bounded measures. However, in the class of signed measures there is no such representation.

In the case of non smooth coefficients a class convenient in many respects is the class of measures absolutely continuous with respect to Lebesgue measure, which is quite natural, in particular, taking into account existence results. A study of this class initiated by Cruzeiro [17], [18] and DiPerna and Lions [24] was continued by many researchers. A large number of papers are devoted to the so-called Lagrangian flows and their generalizations (see [2] and [3]). However, this class of absolutely continuous measures is rather narrow, in particular, it does not enable one to deal with singular initial data (and is essentially oriented towards vector fields having at least some minimal regularity such as the existence of their divergence or being BV). In addition, this class has no universal analogs in infinite dimensions. There are several papers (see [17], [32], [10], [34], [2], [20], and [27]) concerned with the infinite-dimensional case and using as reference measures certain special measures (all reducing to the absolute continuity with respect to Lebesgue measure when the infinite-dimensional state space is replaced by $\mathbb{R}^d$) such as Gaussian measures, convex measures, and differentiable measures, which becomes rather restrictive in infinite dimensions in spite of importance of such classes of measures in applications (e.g., when they are Gibbs measures). The recent paper [7] develops continuity equations in metric measure spaces, but again only considering solutions absolutely continuous with respect to the underlying fixed measure.

Thus, it is natural to look for other classes of measures, apart from absolutely continuous measures, in which the existence and uniqueness of solutions hold in the case of non-smooth coefficients. In this paper we consider the finite-dimensional case; it turns out that even in the one-dimensional case in the present framework new results can be obtained.

The main result in this direction obtained in our paper can be briefly formulated as follows: uniqueness holds in a certain class of measures with respect to which the given vector field $b$ can be suitably

$^{1}$corresponding author. E-mail addresses: vibogach@mail.ru (V. Bogachev), daprato@sns.it (G. Da Prato), roeckner@math.uni-bielefeld.de (M. Röckner), starticle@mail.ru (S. Shaposhnikov)
approximated by smooth vector fields (thus, the uniqueness class may depend on \( b \)). A precise formulation is given below (see Theorems 1.1 and 1.3), but we observe that this result is consistent with typical methods of constructing solutions when \( b \) is approximated by smooth fields \( b_k \) and the solution is obtained as a limit point of the sequence of solutions \( \mu^k_t \) for \( b_k \). As in many existing papers, our conditions admit discontinuous fields, but the hypotheses are mostly incomparable (with the already cited papers and, e.g., [22], [21], [12], [10], [24], [25], and [25]).

As an application we obtain some new results for the continuity equation with a merely continuous coefficient \( b \). In particular, we substantially improve the recent result from [15], where the uniqueness is proved in dimension one for absolutely continuous solutions under the assumptions that \( b \) is continuous and nonnegative, the trajectories of \( \dot{x} = b(x) \) do not blow up in finite time and the set of zeros \( Z = \{ x : b(x) = 0 \} \) consists of a finite union of points and closed intervals. For example, we prove the following assertion: if \( b \in C(\mathbb{R}) \) and \( 0 \leq \nu(x) \leq C + C|x| \) (which is a constructive condition to exclude a blowup), then the uniqueness holds in the class of all locally bounded (possibly signed) measures \( \mu \) on \( \mathbb{R}^d \times [0,T] \) given by families of locally bounded measures \( (\mu_t)_{t \in [0,T]} \) such that
\[
|\mu_t|(\partial Z) = 0 \quad \text{for almost all} \quad t \in [0,T],
\]
where \( \partial Z \) is the boundary of the set \( Z = \{ x : b(x) = 0 \} \). So there are no structure restrictions on the zero set of \( b \) and no assumption that \( \mu \) has a density. Moreover, we consider the multidimensional case. Finally, we prove an existence result which produces solutions with our uniqueness properties.

1. Main result

Let us consider the Cauchy problem
\[
\partial_t \mu + \text{div}(b\mu) = 0, \quad \mu|_{t=0} = \nu, \tag{1.1}
\]
where \( \nu \) is a locally bounded Borel measure on \( \mathbb{R}^d \), i.e., a real function on the class of all bounded Borel sets in \( \mathbb{R}^d \) that is countably additive on the class of Borel subsets of every compact set. Locally bounded Borel measures on \( \mathbb{R}^d \times [0,T] \) are defined similarly. In particular, for every compact set \( K \subset \mathbb{R}^d \times [0,T] \), the total variation of a locally bounded Borel measure \( \mu \) on \( K \) (denoted by \( |\mu|(K) \)) is finite. A Borel measure is bounded if it has a finite total variation.

We say that a locally bounded Borel measure \( \mu \) on \( \mathbb{R}^d \times [0,T] \) is given by a family of Borel locally bounded measures \( (\mu_t)_{t \in [0,T]} \) on \( \mathbb{R}^d \) if, for every bounded Borel set \( B \subset \mathbb{R}^d \), the mapping \( t \mapsto \mu_t(B) \) is measurable, \( |\mu_t|(K) \in L^1(0,T) \) for every compact \( K \subset \mathbb{R}^d \) and
\[
\int_0^T \int_{\mathbb{R}^d} u(x,t) \mu(dx,dt) = \int_0^T \int_{\mathbb{R}^d} u(x,t) \mu_t(dx) \text{ dt} \quad \forall \ u \in C^\infty_0(\mathbb{R}^d \times (0,T)).
\]
Clearly, the previous equality extends to bounded Borel measurable functions \( u \) that vanish for \( x \) outside a ball. In the considered framework there is no difference between measures on \( \mathbb{R}^d \times [0,T] \) and on \( \mathbb{R}^d \times (0,T) \), because we study only measures of the above form represented by families of measures on \( \mathbb{R}^d \) via Lebesgue measure. Not every measure on \( \mathbb{R}^d \times [0,T] \) has this property, of course.

A Borel locally bounded measure \( \mu \) on \( \mathbb{R}^d \times [0,T] \) given by a family of locally bounded measures \( (\mu_t)_{t \in [0,T]} \) is a solution to the Cauchy problem (1.1) if \( b \in L^1(|\mu|, \mathbb{R}^d) \) for every ball \( U \subset \mathbb{R}^d \) and, for every function \( u \) in the class \( C^{1,1}(\mathbb{R}^d \times [0,T]) \) (consisting of functions that are continuous on \( \mathbb{R}^d \times [0,T] \) along with their first order derivatives in \( t \) and \( x \)) such that \( u(x,t) \equiv 0 \) if \( |x| > R \) for some \( R \), one has
\[
\int_{\mathbb{R}^d} u(x,t) \mu_t(dx) = \int_{\mathbb{R}^d} u(x,0) \nu(dx) + \int_0^t \int_{\mathbb{R}^d} \left[ \partial_t u + \langle b, \nabla u \rangle \right] \mu_s(dx) \text{ ds} \tag{1.2}
\]
for almost every \( t \in [0,T] \). Note that the integrals exist for almost all \( t \).

Let us observe that it is possible to pick a common full measure set \( S \subset [0,T] \) such that (1.2) will hold for each \( t \in S \) for all admissible functions \( u \). Indeed, such a set exists for a countable collection of functions \( u \), so it is sufficient to choose this countable collection in such a way that for any function \( u \) in the indicated class there is a sequence \( \{ u_n \} \) in this collection that is uniformly bounded along with the derivatives in \( t \) and \( x \) and \( u_n \to u, \partial_t u_n \to \partial_t u, \nabla u_n \to \nabla u \) pointwise.
Theorem 1.1. Let $\mu^1 = \mu^1(dx)dt$ and $\mu^2 = \mu^2(dx)dt$ be two solutions to the Cauchy problem (1.1). Assume that the measure $\mu = \mu_t(dx)dt$, where $\mu_t = \mu^1_t - \mu^2_t$, satisfies the following conditions: for some number $C_1 \geq 0$ and for every ball $U \subset \mathbb{R}^d$ one can find a number $C_2 \geq 0$, a sequence of vector fields $b_k \in C^\infty(\mathbb{R}^d \times \mathbb{R}^1, \mathbb{R}^d)$ and a sequence of positive functions $V_k \in C^1(\mathbb{R}^d)$ (all depending on the considered measure $\mu$, the constant $C_1$ and the ball $U$) such that

(i) $|b_k(x,t)| \leq C_1 + C_1|x|$

(ii) $\inf_k \inf_U V_k(x) > 0$ and

$$\langle b_k(x,t), \nabla V_k(x) \rangle \leq \left(C_2 - 2\max_{|\xi| = 1} \langle B_k(x,t)\xi,\xi \rangle \right) V_k(x)$$

for every $k$ and $(x,t) \in U \times [0,T]$, where $B_k = (\partial_x b_k^i)_{i,j \leq d}$.

(iii) $\lim_{k \to \infty} \| (b_k - b)\sqrt{V_k} \|_{L^1(\mu_t|dt \times [0,T])}$ = 0.

Then $\mu = 0$, i.e., $\mu^1 = \mu^2$.

Note that the equality $\mu^1 = \mu^2$ is equivalent to the equality $\mu^1_t = \mu^2_t$ for almost every $t \in [0,T]$; changing $\mu_t$ for $t$ in a measure zero set we do not change the solution. However, under broad assumptions about $b$, one can find a version of $t \mapsto \mu_t$ that is continuous in the sense of generalized functions (or even weakly continuous in the case of probability measures), and then such versions are uniquely defined. Moreover, in that case (1.2) holds pointwise and, as $t \to 0$, the measures $\mu_t$ converge to the initial measure $\nu$ in the respective sense (as distributions or weakly). This can be done by taking in (1.2) functions $u$ independent of $t$ (see [8] Lemma 2.1).

So in order to find a uniqueness class for the Cauchy problem (1.1) one should find approximations of $b$ satisfying conditions (i), (ii) and describe measures satisfying condition (iii).

Let us consider the standard example of an ordinary differential equation on the real line without uniqueness and see what kind of uniqueness for the associated continuity equation is offered by our theorem.

Example 1.2. Let $d = 1$ and $b(x) = \sqrt{|x|}$. Set $b_k(x) = (x^2 + k^{-2})^{1/4}$. It is obvious that $|b(x) - b_k(x)| \leq k^{-1/2}$.

Let us calculate $b_k'(x)$:

$$b_k'(x) = \frac{x}{2(x^2 + k^{-2})^{3/4}}.$$

Finally, we take

$$V_k(x) = \frac{1}{b_k(x)^2} = \frac{1}{(x^2 + k^{-2})^{1/2}}.$$

Clearly, (ii) is fulfilled. Moreover, we have

$$g_k(x) := |b(x) - b_k(x)|\sqrt{V_k(x)} = |b(x) - b_k(x)|b_k(x)^{-1}.$$  

Note that $g_k(x) = 1$ if $b(x) = 0$, i.e., $x = 0$. If $x \neq 0$, then

$$g_k(x) \leq \frac{k^{-1/2}}{|x|^{1/2}} \to 0 \text{ if } k \to \infty.$$  

Thus $|g_k| \leq 1$ and $g_k \to I_{\{0\}}$, where $I_{\{0\}}$ is the indicator of the set $\{b = 0\} = \{0\}$. Let $\mu = \mu_t dt$ be a locally bounded measure on $\mathbb{R}^1 \times [0,T]$. According to the Lebesgue dominated convergence theorem, one has

$$\lim_{k \to \infty} \int_0^T \int_U |b - b_k|\sqrt{V_k} d\mu_t dt = \int_0^T |\mu_t|(|\{x \in U : b(x) = 0\}) dt$$

for every interval $U$. Hence the uniqueness holds in the class of all locally bounded (possibly, signed) solutions $\mu = \mu_t dt$ such that $|\mu_t|(|\{0\}) = 0$ for almost all $t \in [0,T]$.

It is interesting that this result is sharp: there exist two solutions $\mu^1_t \equiv \delta_0$ and $\mu^2_t = \delta_{x/4}$ to the Cauchy problem with $b(x) = \sqrt{|x|}$ and $\nu = \delta_0$. Note that only $\mu^2_t$ belongs to our uniqueness class and that this solution is not absolutely continuous.

The proof of Theorem 1.1 is based on the maximum principle and Holmgren’s principle.
Lemma 1.3. Suppose that \( h \in C^\infty(\mathbb{R}^d \times \mathbb{R}^1, \mathbb{R}^d) \), \( |h(x,t)| \leq C_1 + C_1|x| \) for some number \( C_1 > 0 \) and for all \((x,t) \in \mathbb{R}^d \times [0,T] \). Assume also that for some positive function \( V \in C^1(\mathbb{R}^d) \), and number \( C_2 > 0 \) one has

\[
\langle h(x,t), \nabla V(x) \rangle \leq \left( C_2 - 2 \sup_{|\xi|=1} \langle \mathcal{H}(x,t)\xi, \xi \rangle \right) V(x),
\]

for all \((x,t) \in \mathbb{R}^d \times [-1,T] \), where \( \mathcal{H}(x,t) = (\partial_{x_i} h_j(x,t))_{1 \leq i,j \leq d} \).

Then, for any \( s \in (0,T) \), the Cauchy problem

\[
\partial_t f + \langle h, \nabla f \rangle = 0, \quad f|_{t=s} = \psi,
\]

where \( \psi \in C^\infty_0(\mathbb{R}^d) \), has a smooth solution \( f \) on \( \mathbb{R}^d \times (-1,s] \) such that

\[
|f(x,t)| \leq \max_y |\psi(y)|, \quad |\nabla f(x,t)|^2 \leq V(x)e^{C_2(s-t)} \sup_y \frac{|\nabla \psi(y)|^2}{V(y)}, \quad t \in [0,s].
\]

Moreover, \( f(x,t) = 0 \) if \( |x| > R \) for some number \( R = R(s, \psi, C_1) > 0 \).

Proof. The solution \( f \) is given by the equality \( f(x_0,t_0) = \psi(x(s)) \), where \( x(\cdot) \) is a solution to the ordinary equation \( \dot{x}(t) = h(x(t),t), \quad x(t_0) = x_0 \). Note that

\[
\frac{d}{dt}|x(t)|^2 = 2\langle h(x(t),t), x(t) \rangle \geq -C_1 - 3C_1|x(t)|^2.
\]

Hence

\[
|x(t)|^2 \geq |x_0|^2 e^{3C_1(t_0-t)} + C_1(t_0-t)e^{-3C_1t} \geq |x_0|^2 e^{-3C_1s} - C_1s.
\]

Assume that \( \psi(x) = 0 \) if \( |x| > r \). Then there exists a number \( R = R(s, r, C_1) \) such that

\[
|x_0|^2 e^{-3C_1s} - C_1s \geq r \quad \text{if} \, \quad |x_0| \geq R.
\]

Therefore, \( |x(s)| \geq r \) and \( f(x_0,t_0) = 0 \).

Let us prove the announced gradient estimate. Set \( u = 2^{-1} \sum_{k=1}^d |\partial_{x_k} f|^2 \). Differentiating the equation \( \partial_t f + \langle h, \nabla f \rangle = 0 \) with respect to \( x_k \) and multiplying by \( \partial_{x_k} f \) we find that

\[
\partial_t u + \langle h, \nabla u \rangle + \langle \mathcal{H} f, \nabla f \rangle = 0.
\]

Since \( \langle \mathcal{H} f, \nabla f \rangle \leq 2u \sup_{|\xi|=1} \langle \mathcal{H}(x,t)\xi, \xi \rangle \), we have

\[
\partial_t u + \langle h, \nabla u \rangle + 2u \sup_{|\xi|=1} \langle \mathcal{H} \xi, \xi \rangle \geq 0.
\]

Set \( u = wV \). Then

\[
\partial_t w + \langle h, \nabla w \rangle + qw \geq 0,
\]

where

\[
q = 2 \sup_{|\xi|=1} \langle \mathcal{H} \xi, \xi \rangle + \langle h, \nabla V \rangle V^{-1} \leq C_2.
\]

Note that \( w(x,s) \leq \max_y |\nabla \psi(y)|^2/V(y) \). Then the maximum principle (see [33, Theorem 3.1.1]) yields that \( w(x,t) \leq e^{C_2(s-t)} \max_y |\nabla \psi(y)|^2/V(y) \), which completes the proof. \( \square \)

Proof of Theorem 1.1. Let us fix \( \psi \in C^\infty_0(\mathbb{R}^d) \) and \( s \) in the full measure set \( S \subset (0,T) \) mentioned before the theorem (such that (1.2) holds for all admissible \( u \) and each \( t \in S \)). Let \( R = R(s, \psi, C_1) \) be a number from Lemma 1.3. According to the hypotheses of the theorem, for the ball \( \{x: |x| < 2R\} \) there exist sequences \( b_k \) and \( V_k \) satisfying all conditions (i)–(iii).

According to Lemma 1.3 applied with \( h = b_k \) there exists a smooth solution \( f \) to the Cauchy problem

\[
\partial_t f + \langle b_k, \nabla f \rangle = 0, \quad f|_{t=s} = \psi
\]

satisfying the estimate

\[
|f(x,t)| \leq \max_y |\psi(y)|, \quad |\nabla f(x,t)|^2 \leq V_k(x)e^{C_2(s-t)} \max_y \frac{|\nabla \psi(y)|^2}{2V_k(y)}, \quad (x,t) \in \mathbb{R}^d \times [0,s].
\]

Moreover, \( f(x,t) = 0 \) if \( |x| > R \). Certainly, \( f \) depends on several parameters \( (k, s, \psi, \text{etc.}) \), which is suppressed in our notation.
Let $U$ be a ball containing the support of $\psi$. By our assumptions, $C(U) = \inf_k \inf_U V_k(x) > 0$, hence it follows that
\[ |\nabla f(x,t)| \leq (2C(U))^{-1} e^{C_2(s-t)/2} \sqrt{V_k(x)} \max |\nabla \psi| . \]
Substituting the function $u = f$ in (1.2) for the solution $\mu_t = \mu_1 - \mu_2$, we arrive at the following equality:
\[ \int_{\mathbb{R}^d} \psi d\mu_s = \int_0^s \int_{\mathbb{R}^d} (b - b_k, \nabla f) d\mu_t dt. \]
Hence
\[ \int_{\mathbb{R}^d} \psi d\mu_s \leq \tilde{C} \int_0^s \int_{|x| < 2R} |b - b_k| \sqrt{V_k(x)} d\mu_t |dt| dt, \]
where $\tilde{C} = (2C(U))^{-1} e^{C_2T/2} \max |\nabla \psi|$ does not depend on $k$. Letting $k \to \infty$, we conclude that
\[ \int_{\mathbb{R}^d} \psi d\mu_s \leq 0. \]
Recall that $\psi$ was an arbitrary function in $C^0_0(\mathbb{R}^d)$. Then $\mu_s = \mu_1 - \mu_2 = 0$ for almost all $s$. \qed

We also give a result without restrictions on the growth of $b_k$, but with some additional conditions on the solution.

**Theorem 1.4.** Let $\mu^1 = \mu^1(dx) dt$ and $\mu^2 = \mu^2(dx) dt$ be two solutions to the Cauchy problem (1.1). Assume that the measure $\mu = \mu_t(dx) dt$, where $\mu_t = \mu^1_t - \mu^2_t$, satisfies the following condition:

(i) $\lim_{N \to \infty} N^{-1} \int_0^T \int_{N<|x|<2N} |b(x,t)| |\mu_t|(dx) dt = 0$,

(ii) for every ball $U \subset \mathbb{R}^d$, one can find numbers $C > 0$ and $\delta > 0$, a sequence of vector fields $b_k \in C^\infty(U \times \mathbb{R}^1, \mathbb{R}^d)$ and a sequence of positive functions $V_k \in C^1(U)$ (all depending on the considered measure $\mu$ and the ball $U$) such that

\[ \inf_k \inf_U V_k(x) > 0, \]

\[ \langle b_k(x,t), \nabla V_k(x) \rangle \leq \left( C - \delta |b_k(x,t)|^2 - 2 \max_{|\xi| = 1} \langle B_k(x,t) \xi, \xi \rangle \right) V_k(x) \]

for every $k$ and all $(x,t) \in U \times [0,T]$, where $B_k = (\partial_x b_k^j)_{i,j \leq d}$,

(iii) $\lim_{k \to \infty} \| (b_k - b)/\sqrt{V_k} \|_{L^1(|\mu_t| dt, U \times [0,T])} = 0$.

Then $\mu = 0$, i.e., $\mu^1 = \mu^2$.

**Remark 1.5.** Note that if the sequence of functions $|b_k|$ on $U \times [0,T]$ is uniformly bounded, by making $\delta$ smaller and $C$ larger we can restate the second estimate in (i) as

\[ \langle b_k(x,t), \nabla V_k(x) \rangle \leq \left( C - 2 \max_{|\xi| = 1} \langle B_k(x,t) \xi, \xi \rangle \right) V_k(x). \]

It is worth noting that $V_k$ looks like a so-called Lyapunov function, but there is an important difference: its sublevel sets need not be compact.

As above, the proof of Theorem 1.4 is based on the maximum principle and Holmgren’s principle. Let $\zeta \in C^\infty_0(\mathbb{R}^d)$ be such that $0 \leq \zeta \leq 1$ and there exists a number $C(\zeta) > 0$ such that $|\nabla \zeta(x)|^2 \zeta^{-1}(x) \leq C(\zeta)$ for every $x$. Let $U$ be a ball containing the support of $\zeta$.

We need an analog of Lemma 1.3.

**Lemma 1.6.** Suppose that $h \in C^\infty(\mathbb{R}^d \times \mathbb{R}^1, \mathbb{R}^d)$ and that for some positive function $V \in C^1(\mathbb{R}^d)$ and numbers $C > 0$, $\delta > 0$ one has

\[ \langle h(x,t), \nabla V(x) \rangle \leq \left( C - 2 \sup_{|\xi| = 1} \langle H(x,t) \xi, \xi \rangle - \delta |h(x,t)|^2 \right) V(x), \]

for all $(x,t) \in U \times [0,T]$, where $H(x,t) = (\partial_x h^j(x,t))_{1 \leq i, j \leq d}$.

Then, for any $s \in (0,T)$, the Cauchy problem

\[ \partial_t f + \zeta \langle h, \nabla f \rangle = 0, \quad f|_{t=s} = \psi, \]
where $\psi \in C_0^\infty(\mathbb{R}^d)$, has a smooth solution $f$ such that
\[ |f(x, t)| \leq \max_y |\psi(y)|, \quad |\nabla f(x, t)|^2 \leq V(x)e^{M(s-t)} \max_y \frac{|\nabla \psi(y)|^2}{V(y)}, \quad t \in [0, s], \]
where $M = C + \delta^{-1}C(\zeta)$.

Proof. The reasoning is similar to that of Lemma [3.3] but we explain the necessary changes. The existence of a smooth bounded solution with bounded derivatives is well known. Set $u = 2^{-1} \sum_{k=1}^d |\partial_{x_k} f|^2$. Differentiating the equation $\partial_t f + \zeta(h, \nabla f) = 0$ with respect to $x_k$ and multiplying by $\partial_{x_k} f$ we obtain
\[ \partial_t u + \zeta(h, \nabla u) + \zeta(\mathcal{H} \nabla f, \nabla f) + \langle \nabla \zeta, \nabla f \rangle = 0. \]

Note that
\[ \langle \mathcal{H} \nabla f, \nabla f \rangle \leq 2u \sup_{|\xi|=1} \langle \mathcal{H}(x, t) \xi, \xi \rangle, \quad \langle \nabla \zeta, \nabla f \rangle \leq 2|\nabla \zeta||h||u|. \]

Since
\[ 2|\nabla \zeta||h| \leq \delta^{-1} \frac{|\nabla \zeta|^2}{\zeta M} + \delta |h|^2 \leq \delta^{-1}C(\zeta) + \delta |h|^2, \]
we have
\[ \partial_t u + \zeta(h, \nabla u) + u \left( 2 \zeta \sup_{|\xi|=1} \langle \mathcal{H} \xi, \xi \rangle + \delta \zeta |h|^2 + \delta^{-1}C(\zeta) \right) \geq 0. \]

Set $u = wV$. Then
\[ \partial_t w + \zeta(h, \nabla w) + qw \geq 0, \]
where
\[ q = \zeta \left( 2 \sup_{|\xi|=1} \langle \mathcal{H} \xi, \xi \rangle + |h|^2 + \langle h, \nabla V \rangle V^{-1} \right) + \delta^{-1}C(\zeta). \]

By our assumptions $q \leq C + \delta^{-1}C(\zeta) = M$. Note that $w(x, s) \leq \max_x |\nabla \psi(x)|^2 / V(x)$. Then again the maximum principle (see [3.3] Theorem 3.1.1) yields that $w(x, t) \leq e^{M(s-t)} \max_y |\nabla \psi(y)|^2 / V(y)$, which completes the proof.

Proof of Theorem [7.4]. Let us fix a function $\varphi \in C_0^\infty(\mathbb{R}^d)$ such that $0 \leq \varphi \leq 1$, $\varphi(x) = 1$ if $|x| < 1$ and $\varphi(x) = 0$ if $|x| > 2$. Assume also that $|\nabla \varphi(x)|^2 \varphi(x)^{-1} \leq C(\varphi)$ for every $x$. Set $\varphi_{4N}(x) = \varphi(x/N)$. For every ball $U_{4N} = \{ x : |x| < 4N \}$, there exist numbers $C > 0$, $\delta > 0$ and sequences $\{ b_k \}$, $\{ V_k \}$ such that all conditions (ii) and (iii) are fulfilled on $U_{4N}$. Let us fix $\psi \in C_0^\infty(\mathbb{R}^d)$ and $s \in (0, T)$ in the same full measure set $S$ as in the proof of Theorem [1.1]. According to Lemma [1.6] there exists a smooth solution $f$ to the Cauchy problem
\[ \partial_t f + \varphi_{2N}(b_k, \nabla f) = 0, \quad f|_{t=s} = \psi \]
satisfying the estimate
\[ |f(x, t)| \leq \max_y |\psi(y)|, \quad |\nabla f(x, t)|^2 \leq V_k(x)e^{M(s-t)} \max_y \frac{|\nabla \psi(y)|^2}{2V_k(y)}, \]
where $M = C + \delta^{-1}C(\varphi_{2N})$. Certainly, $f$ depends on several parameters $(k, s, N, \varphi, \psi, \varphi, \psi)$, which is suppressed in our notation.

Let $U$ be a ball containing the support of $\psi$. By our assumptions, $C(U) = \inf_{k} \inf_{U} V_k(x) > 0$, hence it follows that
\[ |\nabla f(x, t)| \leq (2C(U))^{-1} e^{M(s-t)/2} \sqrt{V_k(x)} \max \nabla \psi. \]
Substituting the function $u = f\varphi_N$ in [1.2] for the solution $\mu_t = \mu_{t_1} - \mu_{t_2}$ and noting that $\varphi_{2N}(x) = 1$ if $\varphi_N(x) \neq 0$, we arrive at the following equality:
\[ \int_{\mathbb{R}^d} \psi \, d\mu_s = \int_0^s \int_{\mathbb{R}^d} \left[ \varphi_N(b - b_k, \nabla f) + \langle b, \nabla \varphi_N \rangle f \right] \, d\mu_t \, dt. \]
Then we obtain
\[ \int_{\mathbb{R}^d} \psi \, d\mu_s \leq \int_0^s \int_{|x| < 2N} \left[ C \varphi_N(b - b_k) \sqrt{V_k(x)} + |b||\nabla \varphi_N| \right] \, d|\mu_t| \, dt. \]
where $\bar{C} = (2C(U))^{-1}e^{MT/2} \max |\nabla \psi|$ does not depend on $k$. Letting $k \to \infty$, we conclude that

$$\int_{\mathbb{R}^d} \psi \, d\mu_s \leq \int_0^s \int_{|x|<2N} |b| |\nabla \varphi_N| \, d|\mu_t| \, dt.$$  

Since $|\nabla \varphi_N| \leq N^{-1} \max |\nabla \varphi|$ and $\nabla \varphi_N$ vanishes outside of the set $\{N < |x| < 2N\}$, we have

$$\int_{\mathbb{R}^d} \psi \, d\mu_s \leq N^{-1} \max |\nabla \varphi| \int_0^s \int_{N<|x|<2N} |b| \, d|\mu_t| \, dt.$$  

Letting $N \to \infty$, we obtain that

$$\int_{\mathbb{R}^d} \psi \, d\mu_s \leq 0,$$

which means that $\mu_s = \mu_1^k - \mu_2^k = 0$ for almost all $s$. \qed

Thus, the main difficulty is to construct the required approximations.

2. Corollaries and examples

In this section we obtain some corollaries of the main result and consider some examples. Our first corollary generalizes a result of [15] by omitting restrictions on the structure of the boundary of the zero set of the field.

**Corollary 2.1.** Let $d = 1$, $b \in C(\mathbb{R}^1)$ and $0 \leq b(x) \leq C + C|x|$ for every $x \in \mathbb{R}^1$ and some number $C > 0$. Then the corresponding Cauchy problem (1.1) has at most one solution in the class of all locally bounded (possibly, signed) measures $\mu$ on $\mathbb{R}^d \times [0,T]$ given by families of locally bounded measures $(\mu_t)_{t \in [0,T]}$ such that

$$|\mu_t|(\partial Z) = 0 \quad \text{for almost all} \quad t \in [0,T],$$

where $\partial Z$ is the boundary of the set $Z = \{x: b(x) = 0\}$. In particular, the latter holds for absolutely continuous $\mu_t$ provided that $b^{-1}(0)$ has Lebesgue measure zero.

**Proof.** Set $Z^0 = \{x: b(x) = 0\} \setminus \partial Z$ and suppose that $Z^0$ is not empty. Obviously, the measure $\nu_0 = \nu|_{Z^0}$ is a stationary solution to the equation $\partial_t \mu_t + \text{div}(b \mu_t) = 0$ and, for every solution $\mu_t$, we have $\mu_t = \nu_0$ on $Z^0$, because the term with $\text{div}(b \mu)$ vanishes in the domain $Z^0$. Replacing $\mu_t$ by $\mu_t - \nu_0$, we can trivially prove the uniqueness of solutions $\mu_s$ such that $\mu_s(Z^0) = 0$.

We now prove that for all locally bounded measures $\mu = \mu_t \, dt$ such that $|\mu_t|(\partial Z) = 0$ for almost all $t \in [0,T]$ there exist appropriate $b_k$ and $V_k$ such that all conditions (i)–(iii) in Theorem 1.1 are fulfilled.

Let us fix an interval $[-N, N]$. In order to apply Theorem 1.1 one has to find suitable approximations for $b$ on $[-N, N]$. Let $\omega$ be the modulus of continuity of $b$ on $[-1 - N, N + 1]$. If $\omega(\delta) = 0$ for some $\delta > 0$, then $b = \text{const}$ and one can take $b_k = b$ and $V_k = 1$. Let $\omega(\delta) > 0$ if $\delta > 0$. Let $k \geq 1$, $g_1/(x) = kg(kx)$, where $g \in C^0_0(\mathbb{R}^1)$, $g \geq 0$ and $\|g\|_{L^1} = 1$. We use the following approximations:

$$b_k = b * g_1/k + \omega(k^{-1}), \quad V_k = b_k^{-2}.$$  

Let us verify conditions (ii) and (iii) (since (i) is obviously true for such $b_k$ with $C_1 = 2C$). Since $b_k(x) \leq \max[-1-N,N+1](x) = C_N$ on $[-N, N]$, we have $\inf_k \inf_{[-N,N]} V_k \geq (4C_N)^{-2} > 0$. Moreover, $b_k V_k' = -2b_k' V_k$. Thus, condition (ii) in Theorem 1.1 is fulfilled.

Set $g_k(x) := |b - b_k|/\sqrt{V_k} = |b - b_k| b_k^{-1}$. Note that if $b(x) = 0$, then $g_k(x) = 1$. If $b(x) \neq 0$, then $g_k(x) \to 0$. Finally,

$$g_k = \frac{|b - b_k|}{b_k} \leq \frac{2\omega(k^{-1})}{b * g_1/k + \omega(k^{-1})} \leq 2.$$

This yields the equality

$$\lim_{k \to \infty} \int_0^T \int_{-N}^N |b - b_k|\sqrt{V_k} \, d|\mu_t| \, dt = \int_0^T \int_{-N}^N \chi_{\{b = 0\}}(x) |\mu_t|(dx) \, dt = 0.$$  

So condition (iii) is fulfilled. \qed
Remark 2.2. The case of nonpositive \( b \) can be reduced to the considered situation: it suffices to replace \( x \) by \(-x\) and \( b(x) \) by \(-b(-x)\). By the way, there is no direct generalization of the last corollary to the case of a signed drift \( b \). Indeed, it fails even for \( b(x) = x^{1/3} \). But we can assert the uniqueness for solutions concentrated on the set where \( b \geq 0 \) (or \( b \leq 0 \)) and satisfying condition (2.1). This is a trivial observation because such a solution satisfies the equation with \( b^+ = b \wedge 0 \) and we can apply Corollary 2.1. There is a more interesting case where one can extract a negative part of \( b \) with some better regularity.

Corollary 2.3. Let \( d = 1 \). Assume that \( b \in C(\mathbb{R}^1) \), \( |b(x)| \leq C + C|x| \) and \( b = g + f \), where \( f \geq 0 \), \( g \) is a Lipschitzian function with Lipschitz constant \( \Lambda \), and

\[
g(x) < 0 \implies f(x) = 0.
\]

Then the corresponding Cauchy problem (1.1) has at most one solution in the class of all locally bounded measures \( \mu \) given by families of locally bounded measures \( \{\mu_t\}_{t \in [0,T]} \) such that

\[
|\mu_t|(\partial Z_f) = 0 \quad \text{for almost all} \quad t \in [0,T],
\]

where \( \partial Z_f \) is the boundary of the set \( Z_f = \{x : f(x) = 0\} \).

Proof. As in the previous corollary, we prove that all conditions (i)–(iii) in Theorem 1.1 are fulfilled for all locally bounded measures \( \mu = \mu_t \, dt \) such that \( |\mu_t|(\partial Z_f) = 0 \) for almost all \( t \in [0,T] \).

There exists a 1-Lipschitzian function \( \tilde{h} \) such that \( h \geq \tilde{h}, \, h(x) = 0 \) if \( f(x) > 0 \), and the set of zeros of \( h + f \) coincides with \( \partial Z_f \). Let \( \bar{g} = g - h \) and \( \tilde{f} = f + h \). As above, we fix an interval \([-N, N]\) and find suitable approximations of \( b \) on \([-N, N]\). Let \( \omega \) be the modulus of continuity of \( f \) on \([-1 - N, N + 1]\). Let \( k \geq 1 \), \( \varrho_1/k(x) = k \varrho(kx) \), where \( \varrho \in C_0^\infty(\mathbb{R}^1) \), \( \varrho \geq 0 \), \( \varrho(x) = 0 \) if \( |x| < 1 \), and \( \|\varrho\|_{L^1} = 1 \). Set

\[
\varepsilon_k = 8\Lambda k^{-1/2},
\]

\[
\tilde{f}_k = (f - 2\omega(3/k))^+ + h, \quad b_k = (\bar{g} + \tilde{f}_k) * \varrho_1/k + \varepsilon_k, \quad V_k = (|\bar{g}| * \varrho_1/k + \tilde{f}_k * \varrho_1/k + \varepsilon_k)^{-2}.
\]

Let us verify condition (ii). We shall prove that

\[
b_k V_k' \leq -2(-C + b_k') V_k \tag{2.3}
\]

for some \( C > 0 \). Since \( \varrho \) and \( |\varrho| \) are Lipschitzian functions, we can replace \( b_k' \) in the latter inequality by the \( |\bar{g}| * \varrho_1'/k + \tilde{f}_k' * \varrho_1'/k \). Note that

\[
b_k V_k' = -2 V_k \frac{(\bar{g} * \varrho_1/k + \tilde{f}_k * \varrho_1/k + \varepsilon_k)(|\bar{g}| * \varrho_1'/k + \tilde{f}_k' * \varrho_1'/k)}{|\bar{g}| * \varrho_1/k + \tilde{f}_k * \varrho_1/k + \varepsilon_k}.
\]

In order to prove (2.3) it is enough to show that

\[
\frac{(\bar{g} - \bar{g}) * \varrho_1/k)(|\bar{g}| * \varrho_1'/k + \tilde{f}_k' * \varrho_1'/k)}{|\bar{g}| * \varrho_1/k + \tilde{f}_k * \varrho_1/k + \varepsilon_k} \geq -\gamma
\]

for some \( \gamma > 0 \). If \( (\bar{g} - \bar{g}) * \varrho_1/k(x) \neq 0 \), then there is a point \( z \) in the interval \((x - k^{-1}, x + k^{-1})\) such that \( \bar{g}(z) = g(z) - h(z) < 0 \). If \( g(z) < 0 \) or \( h(z) > 0 \), then \( f(z) = 0 \) and \((f - 2\omega(3/k))^+(x) = 0\) on the interval \((z - 3k^{-1}, z + 3k^{-1})\). In particular, \( \tilde{f}_k = h \) on \((x - k^{-1}, x + k^{-1})\) and

\[
|\tilde{f}_k * \varrho_1'/k(x)| = |h * \varrho_1'/k(x)| \leq 1.
\]

We use here that \( h \) is a 1-Lipschitzian function. Moreover, for every \( y \in (x - k^{-1}, x + k^{-1}) \) we have

\[
|\bar{g}(y)| \geq |\bar{g}(z)| - 2\Lambda k^{-1} \geq 2^{-1}(|\bar{g}| - \bar{g})(y) - 4\Lambda k^{-1}.
\]

Thus, \( |\bar{g}| * \varrho_1/k(x) \geq 2^{-1}(|\bar{g}| - \bar{g}) * \varrho_1/k(x) - 4\Lambda k^{-1} \) and

\[
|\bar{g}| * \varrho_1/k(x) + \tilde{f}_k * \varrho_1/k(x) + \varepsilon_k \geq 2^{-1}(|\bar{g}| - \bar{g}) * \varrho_1/k(x),
\]

because \( \varepsilon_k - 4\Lambda k^{-1} > 0 \). Therefore, we have

\[
\frac{|(\bar{g} - \bar{g}) * \varrho_1/k)(|\bar{g}| * \varrho_1'/k + \tilde{f}_k' * \varrho_1'/k)}{|\bar{g}| * \varrho_1/k + \tilde{f}_k * \varrho_1/k + \varepsilon_k} \leq 2(\Lambda + 1).
\]
Thus, condition (ii) is fulfilled. Let us verify condition (iii). We have
\[ |b - b_k| \sqrt{V_k} = \left| \frac{\tilde{g} \ast \varrho_{1/k} - \tilde{g} + \tilde{f}_k \ast \varrho_{1/k} - \tilde{f} + \varepsilon_k}{|\tilde{g}| \ast \varrho_{1/k} + \tilde{f}_k \ast \varrho_{1/k} + \varepsilon_k} \right|. \]
If \( \tilde{f}(x) = 0 \), then \( \tilde{f}_k \ast \varrho_{1/k}(x) = 0 \) and
\[ -\varepsilon_k^{-1} \Lambda k^{-1} + 1 \leq |b(x) - b_k(x)| \sqrt{V_k(x)} \leq \varepsilon_k^{-1} \Lambda k^{-1} + 1. \]
Since \( \varepsilon_k^{-1} \Lambda k^{-1} \to 0 \), we have \( |b(x) - b_k(x)| \sqrt{V_k(x)} \to 1 \). If \( \tilde{f}(x) \neq 0 \), then \( |b - b_k| \sqrt{V_k} \to 0 \). Thus, we obtain
\[ \lim_{k \to \infty} \int_0^T \int_{-N}^N |b - b_k| \sqrt{V_k} d|\mu_t| dt = \int_0^T \int_{-N}^N \chi_{\tilde{f}(x) = 0}(x) |\mu_t|(dx) dt = 0 \]
and condition (iii) is fulfilled as well. \( \square \)

**Remark 2.4.** (i) In the same way one can prove the above result under the assumptions that \( b = g + f \), where
\[ (g(x + y) - g(x))y \leq C|y|^2 \]
and \( f \) is an increasing nonnegative function that is constant on the set \( \{x: g(x) < 0\} \).

(ii) The same assertion is true in the case, where \( b = g - f \), \( g \) is a Lipschitzian function and \( f \) satisfies all conditions of the last corollary.

(iii) According to Corollaries 2.1 and 2.3, the uniqueness holds in the class of solutions with the following property: “you must go if you can”.

Let us now consider the case \( d \geq 2 \). First we mention a known example of uniqueness (see, e.g., [4]), in which it is easy to check our hypotheses.

**Example 2.5.** Assume that \( b \in C(\mathbb{R}^d \times \mathbb{R}^1, \mathbb{R}^d) \) and there exist numbers \( C_1, C_2 \) and \( C_3 \) such that
\[ |b(x, t)| \leq C_1 + C_2|x|, \quad (b(x + \xi, t) - b(x, t), \xi) \leq C_3|\xi|^2 \]
for all \( (x, t) \in \mathbb{R}^d \times [0, T] \) and every \( \xi \in \mathbb{R}^d \). Then there exists at most one solution to \( (1.1) \) in the class of all locally bounded measures \( \mu \) given by families of locally bounded measures \( (\mu_t)_{t \in [0, T]} \).

**Proof.** All hypotheses of Theorem 1.1 are fulfilled with the functions \( V_k = 1 \) and \( b_k = b \ast \varrho_{1/k} \), where \( \varrho_{1/k} = k^d g(kx) \) and \( g \in C_0^\infty(\mathbb{R}^{d+1}) \), \( g \geq 0 \), \( \|g\|_{L^1} = 1 \). \( \square \)

Let us consider a more specific situation, where
\[ b(x) = -\beta(|x|^2)x \]
is a radially symmetric vector field.

**Example 2.6.** Assume that \( \beta \geq 0 \) is bounded and continuous on \([0, +\infty)\) and that for every interval \([0, N]\) there exists a number \( \Lambda_N > 0 \) such that \( s \mapsto \beta(s) - \Lambda_N s \) is a decreasing function on \([0, N]\). Then the Cauchy problem \( (1.1) \) with \( b(x) = -\beta(|x|^2)x \) has at most one solution in the class of all locally bounded measures \( \mu \) given by families of locally bounded measures \( (\mu_t)_{t \in [0, T]} \) such that
\[ |\mu_t|(\partial Z \setminus \{0\}) = 0 \] for almost all \( t \in [0, T] \),
where \( \partial Z \) is the boundary of the set \( Z = \{x: \beta(|x|^2) = 0\} \).

**Proof.** Let us fix a natural number \( N \). Let \( \omega \) be the modulus of continuity of \( \beta \) on \([0, N + 1]\). If \( \omega(\delta) = 0 \) for some \( \delta > 0 \), then \( \beta = \text{const} \) on \([0, N]\) and one can take \( b_k = b \) and \( V_k = 1 \). Let us consider the case where \( \omega(\delta) > 0 \) if \( \delta > 0 \). Applying the reasoning from the proof of Corollary 2.1 we arrive at the case where \( \mu_t = 0 \) on the set \( \{x: \beta(|x|^2) = 0\} \setminus \partial Z \). Let
\[ b_k(x) = -(\beta_k(|x|^2) + \omega(k^{-1}))x, \quad V_k(x) = (\beta_k(|x|^2) + \omega(k^{-1}))^{-2}, \]
where \( \beta_k(z) = \beta \ast \varrho_{1/k} \). Then
\[ \langle b_k(x), \nabla V_k(x) \rangle = 4|x|^2\beta'_k(|x|^2)V_k(x) \]
and

\[ \sum_{i,j} \partial_x b_k^j(x) h_i h_j = -2 \beta_k'(|x|^2) \sum_{i,j} x_i h_i x_j h_j - (\beta_k(|x|^2) + \omega(k^{-1}))|h|^2 \]

\[ \leq -2(\beta_k'(|x|^2) \wedge 0)|x|^2|h|^2. \]

Note also that \( \beta_k(z) \leq \Lambda N + 1 \) if \( z \in [0, N] \). Hence \( 4|x|^2 \beta_k'(|x|^2) \leq 4N \Lambda N + 1 \) and

\[ \langle b_k(x), \nabla V_k(x) \rangle \leq \left( 4N \Lambda N + 1 + 4(\beta_k(|x|^2) \wedge 0)|x|^2 \right)V_k(x) \]

if \( |x| \leq \sqrt{N} \). Thus, condition (ii) in the definition of \( \mathcal{M}_k \) required in Theorem 1 holds. Let us verify condition (iii). We have

\[ g_k(x) = |b_k(x) - b(x)| \sqrt{V_k(x)} = \frac{\beta_k(|x|^2) - \beta(|x|^2) + \omega(k^{-1})|x|}{\beta_k(|x|^2) + \omega(k^{-1})}. \]

Note that if \( x \neq 0 \) and \( \beta(|x|^2) = 0 \), then \( g_k(x) = 1 \). If \( \beta(|x|^2) \neq 0 \), then \( g_k(x) \to 0 \). Since \( g_k(x) \leq 2|x| \), we obtain the equality

\[ \lim_{k \to \infty} \int_0^T \int_{|x| \leq \sqrt{N}} g_k(x)|\mu_t|(dx) dt = \int_0^T \int_{|x| \leq \sqrt{N}} \chi(\beta(|x|^2) = 0)|\mu_t|(dx) dt = 0. \]

Thus, condition (iii) is fulfilled as well. \( \square \)

**Remark 2.7.** In the same way one can consider a more general situation:

\[ b(x) = -\beta(W(x))\nabla W(x), \]

where \( W \) is a smooth bounded function on \( \mathbb{R}^d \) with bounded derivatives and \( \beta \) is a nonnegative continuous function on \( \mathbb{R}^1 \) and such that \( s \mapsto \beta(s) - \Lambda s \) is a decreasing function for some number \( \Lambda > 0 \). Then the uniqueness holds in the class of all bounded measures \( \mu \) given by families of bounded measures \( (\mu_t)_{t \in [0, T]} \) such that

\[ |\mu_t|(\partial Z \setminus \mathcal{W}) = 0 \quad \text{for almost all} \quad t \in [0, T], \]

where \( \partial Z \) is the boundary of the set \( Z = \{ x : \beta(W(x)) = 0 \} \) and \( \mathcal{W} = \{ x : |\nabla W(x)| = 0 \} \).

Indeed, we can take

\[ b_k(x) = -((\beta_k(W(x)) + \omega(k^{-1}))\nabla W(x) \quad \text{and} \quad V_k(x) = (\beta_k(W(x)) + \omega(k^{-1}))^{-2}, \]

where \( \beta_k(z) = \beta \ast \varphi_{1/k} \), and repeat the reasoning from the proof of the previous example.

Finally, let us discuss the existence of solutions possessing our uniqueness properties.

We observe that not every approximation of \( b \) yields a solution from the considered uniqueness class. Indeed, let \( b(0) = 0 \) and take the approximations by smooth functions \( b_k \) such that \( b_k(0) = 0 \). Then for \( \nu = \delta_0 \) we obtain the unique solution \( \mu_t^k = \delta_0 \) for every \( k \). So, the limit measure of such \( \mu_t^k \) is again \( \delta_0 \).

**Proposition 2.8.** Assume that \( b : \mathbb{R}^d \times [0, T] \to \mathbb{R}^d \) is continuous. Suppose that there exist a sequence of vector fields \( b_k \in C^\infty(\mathbb{R}^d \times \mathbb{R}^1, \mathbb{R}^d) \) and a sequence of positive functions \( V_k \in C^1(\mathbb{R}^d) \) such that

(i) there exist numbers \( C_1, C_2 \) and \( C_3 \) such that \( V_k \leq C_1 V_m + C_2 \) for every \( k \leq m \) and

\[ \langle b_k(x, t), \nabla V_k(x) \rangle \leq C_3 V_k(x) \quad \forall (x, t) \in \mathbb{R}^d \times [0, T], \]

(ii) the function \( (1 + |x|)^{-1}|b_k(x, t)| \) is bounded for every \( k \) and \( b_k \to b \) uniformly on \( U \times [0, T] \) for every ball \( U \).

Then, for every initial nonnegative finite measure \( \nu \) such that \( \sup_k \| V_k \|_{L^1(\nu)} < \infty \), there exists a family of nonnegative finite measures \( (\mu_t)_{t \in [0, T]} \) solving the Cauchy problem \( (1.1) \) with the following property: for every ball \( U \) one has

\[ \lim_{k \to \infty} \int_0^T \int_U |b - b_k| \sqrt{V_k} d\mu_t dt = 0. \quad (2.4) \]
Hence, one can find a subsequence by compactness arguments and the diagonal procedure (see [30] for details in a similar situation), one that Example 2.9.

Let us illustrate the previous proposition in the situation of Corollary 2.1. Assume where the first factor tends to zero and the second one is uniformly bounded.

Proof. Since \(b_k\) is a smooth vector field of linear growth, there exists a nonnegative finite solution \((\mu^k_t)_{t \in [0,T]}\) to the Cauchy problem with \(b_k\). Moreover, \(\mu^k_t(\mathbb{R}^d) \leq \nu(\mathbb{R}^d)\). Using the standard compactness arguments and the diagonal procedure (see [30] for details in a similar situation), one can find a subsequence \(\{k_l\}\) such that on every compact set in \(\mathbb{R}^d\) the sequence \(\{\mu^k_{t_l}\}\) converges weakly to some solution \(\mu_t\) for every \(t \in [0,T]\).

Let \(\psi_N(x) = \psi(x/N)\). We have

\[
\int_{\mathbb{R}^d} V_{k_l} \psi_N d\mu^k_{t_l} = \int_{\mathbb{R}^d} V_{k_l} \psi_N d\nu + \int_0^t \int_{\mathbb{R}^d} \left[ (b_{k_l}, \nabla \psi_N) V_{k_l} + \psi_N (b_{k_l}, \nabla V_{k_l}) \right] d\mu^k_{s_l} \, ds.
\]

Using conditions (i) and (ii) and letting \(N \to \infty\), we arrive at the equality

\[
\int_{\mathbb{R}^d} V_{k_l} d\mu^k_{t_l} = \int_{\mathbb{R}^d} V_{k_l} d\nu + C_3 \int_0^t \int_{\mathbb{R}^d} V_{k_l} d\mu^k_{s_l} \, ds.
\]

Applying Gronwall’s inequality we obtain that

\[
\int_{\mathbb{R}^d} V_{k_l} d\mu^k_{t_l} \leq e^{C_3 t} \int_{\mathbb{R}^d} V_{k_l} d\nu.
\]

Since \(V_k \leq C_1 V_m + C_2\) for every \(k, m\) such that \(k \leq m\), we obtain the estimate

\[
\int_{\mathbb{R}^d} V_k d\mu \leq \lim_{l \to \infty} \int_{\mathbb{R}^d} V_{k_l} d\mu^k_{t_l} \leq C_2 \nu(\mathbb{R}^d) + C_1 \lim_{l \to \infty} \int_{\mathbb{R}^d} V_{k_l} d\mu^k_{t_l}.
\]

Hence

\[
\int_{\mathbb{R}^d} V_k d\mu \leq C_2 \nu(\mathbb{R}^d) + C_1 e^{C_3 t} \sup_k \int_{\mathbb{R}^d} V_k d\nu.
\]

Finally, we have

\[
\int_0^T \int_U |b - b_k| \sqrt{V_k} d\mu \, dt \leq \left( \int_0^T \int_U |b - b_k|^2 d\mu \, dt \right)^{1/2} \left( \int_0^T \int_U V_k d\mu \, dt \right)^{1/2},
\]

where the first factor tends to zero and the second one is uniformly bounded. \(\square\)

Example 2.9. Let us illustrate the previous proposition in the situation of Corollary 2.1. Assume that \(b\) on \(\mathbb{R}^1\) satisfies the following additional condition:

\[(b(x + y) - b(x))y \geq -C|y|^2\]

for all \(x, y\) and some \(C > 0\). Then all conditions of Proposition 2.8 are fulfilled with the approximations \(b_k = b \ast \varrho_{1/k} + 3\omega(1/k)\) and \(V_k = b_k - b_k^2\). Indeed, one has \(b_k' \geq -C\) and \(b_k' V_k' = -2b_k^2 V_k \leq 2C V_k\).

Let \(k > m\). Note that the sequence of numbers \(\omega(k^{-1})\) is decreasing and

\[b \ast \varrho_{1/k}(x) + 3\omega(1/k) \geq b(x) + 2\omega(1/k) \geq b(x) + 2\omega(1/m) \geq b \ast \varrho_{1/m}(x) + w(1/m)\]

Hence we have \(V_k \leq 9V_m\). Assume that

\[
\int_{\mathbb{R}^1 \setminus Z_0} \frac{1}{b^2} \, d\nu < \infty.
\]

As above we replace \(\nu\) by \(\nu - \nu_0\), where \(\nu_0 = \nu|_{Z_0}\) is a stationary solution. Then

\[
\sup_k \int_{\mathbb{R}^1} V_k d\nu \leq \int_{\mathbb{R}^1 \setminus Z_0} \frac{1}{(b + 2\omega(1/k))^2} \, d\nu = \int_{\mathbb{R}^1 \setminus Z_0} \frac{1}{b^2} \, d\nu < \infty.
\]

By Proposition 2.8 there exists a solution \((\mu_t)_{t \in [0,T]}\) such that equality (2.4) holds. Moreover,

\[
\int_{\mathbb{R}^1 \setminus Z_0} \frac{1}{b^2} \, d\mu_t < \infty
\]

and \(\mu_t(\partial Z) = 0\) for almost all \(t \in [0,T]\).
Remark 2.10. There is another way of constructing a solution to the Cauchy problem (1.1). Given \( \varepsilon > 0 \), let us consider the Cauchy problem

\[
\partial_t \mu_{\varepsilon} - \varepsilon \Delta \mu_{\varepsilon} + \text{div}(b \mu_{\varepsilon}) = 0 , \quad \mu_{\varepsilon}|_{t=0} = \nu .
\]

Under suitable conditions on \( b \), there exists a limit point \( \mu \) of the solutions \( \mu_{\varepsilon} \). To describe the set of all limit points \( \mu \) is a well-known problem. Assume that \( d = 1, b \geq 0, b \) has a single zero \( b(0) = 0 \) and \( 1/b \) is Lebesgue integrable near the origin. Let \( \nu = \delta_0 \). It turns out (see [28]) that the measure \( \mu \) is given by a family of measures \( \mu_t = \delta_{x_t} \), where \( x_t \) is the upper extreme solution of the Cauchy problem \( \dot{x} = b(x), \ x(0) = 0 \). Note that the upper extreme solution does not stay at \( x = 0 \) and \( \mu_t(\{0\}) = 0 \) if \( t > 0 \). Thus, we obtain a solution from our uniqueness class.

Note also that there are many results on existence of solutions to continuity equations based on Lyapunov functions conditions (see, e.g., [3, Corollary 3.4]), but our purpose here is to ensure existence of solutions in our uniqueness classes. In a separate paper the infinite-dimensional case will be considered; some related results are obtained in our forthcoming paper [9].

Acknowledgment. Our work was supported by the DFG through SFB 701 at Bielefeld University and the RFBR projects 14-01-00237 and 14-01-90406.

References

[1] G. Alberti, S. Bianchini, G. Crippa, A uniqueness result for the continuity equation in two dimensions, J. Eur. Math. Soc. 16 (2) (2014) 201–234.
[2] L. Ambrosio, Transport equation and Cauchy problem for BV vector fields, Invent. Math. 158 (2004) 227–260.
[3] L. Ambrosio, Transport equation and Cauchy problem for non-smooth vector fields, In “Calculus of Variations and Nonlinear Partial Differential Equations”, Lecture Notes in Math., V. 1927, pp. 1–41. Springer, Berlin, 2008.
[4] L. Ambrosio, P. Bernard, Uniqueness of signed measures solving the continuity equation for Osgood vector fields, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 19 (3) (2008) 237–245.
[5] L. Ambrosio, G. Crippa, A. Figalli, L.V. Spinolo, Some new well-posedness results for continuity and transport equations, and applications to the chromatography system, SIAM J. Math. Anal. 41 (5) (2009) 1890–1920.
[6] L. Ambrosio, G. Crippa, Existence, uniqueness, stability and differentiability properties of the flow associated to weakly differentiable vector fields, In: Transport Equations and Multi-D Hyperbolic Conservation Laws, Lecture Notes Unione Mat. Ital. 5, , pp. 3–57. Springer, Berlin, 2008.
[7] L. Ambrosio, D. Trevisan, Well-posedness of Lagrangian flows and continuity equations in metric measure spaces, Analysis & PDE 7 (5) (2014) 1179–1234.
[8] V.I. Bogachev, G. Da Prato, M. Röckner, On parabolic equations for measures, Comm. Partial Differential Equations 33 (1-3) (2008) 397–418.
[9] V.I. Bogachev, G. Da Prato, M. Röckner, S.V. Shaposhnikov, An analytic approach to infinite-dimensional continuity and Fokker–Planck–Kolmogorov equations, Annali Scuola Normale Super. Pisa 15 (2015).
[10] V.I. Bogachev, E. Mayer-Wolf, Absolutely continuous flows generated by Sobolev class vector fields in finite and infinite dimensions, J. Funct. Anal. 167 (1) (1999) 1–68.
[11] V.I. Bogachev, M. Röckner, S.V. Shaposhnikov, Nonlinear evolution and transport equations for measures, Dokl. Akad. Nauk 429 (1) (2009) 7–11 (in Russian); English transl.: Dokl. Math. 80 (3) (2009) 785–789.
[12] F. Bouchut, G. Crippa, Lagrangian flows for vector fields with gradient given by a singular integral, J. Hyperbolic Differential Equations 10 (2) (2013) 235–282.
[13] F. Bouchut, F. James, One-dimensional transport equations with discontinuous coefficients, Nonlinear Anal. 32 (7) (1998) 891–933.
[14] F. Cipriano, A.B. Cruzeiro, Flows associated with irregular \( R^d \)-vector fields, J. Differential Equations 219 (1) (2005) 183–201.
[15] G. Crippa, Lagrangian flows and the one-dimensional Peano phenomenon for ODEs, J. Differential Equations 250 (7) (2011) 3135–3149.
[16] G. Crippa, C. De Lellis, Estimates and regularity results for the DiPerna–Lions flow, J. Reine Angewandte Math. 616 (2008) 15–46.
[17] A.B. Cruzeiro, Équations différentielles ordinaires: non explosion et mesures quasi-invariantes, J. Funct. Anal. 54 (2) (1983) 193–205.
[18] A.B. Cruzeiro, Équations différentielles sur l’espace de Wiener et formules de Cameron–Martin non-linéaires, J. Funct. Anal. 54 (2) (1983) 206–227.
[19] A.B. Cruzeiro, Unicité de solutions d’équations différentielles sur l’espace de Wiener, J. Funct. Anal. 58 (3) (1984) 335–347.
[20] G. Da Prato, F. Flandoli, M. Röckner, Uniqueness for continuity equations in Hilbert spaces with weakly differentiable drift, Stoch. Partial Differ. Equ. Anal. Comput. 2 (2) (2014) 121–145.
[21] C. De Lellis, Notes on hyperbolic systems of conservation laws and transport equations, In: Handbook of differential equations: evolutionary equations. Vol. III, pp. 277–382, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2007.
[22] C. De Lellis, ODEs with Sobolev coefficients: the Eulerian and the Lagrangian approach, Discrete Contin. Dyn. Syst. Ser. S 1 (3) (2008) 405–426.
[23] B. Desjardins, A few remarks on ordinary differential equations, Comm. Partial Differential Equations 21 (11-12) (1996) 1667–1703.
[24] R.J. DiPerna, P.-L. Lions, Ordinary differential equations, transport theory and Sobolev spaces, Invent. Math. 98 (1989) 511–547.
[25] M. Hauray, C. Le Bris, A new proof of the uniqueness of the flow for ordinary differential equations with BV vector fields, Ann. Mat. Pura Appl. (4) 190 (1) (2011) 91–103.
[26] P.-E. Jabin, Differential equations with singular fields, J. Math. Pures Appl. (9) 94 (6) (2010) 597–621.
[27] A.V. Kolesnikov, M. Röckner, On continuity equations in infinite dimensions with non-Gaussian reference measure, J. Funct. Anal. 266 (7) (2014) 4490–4537.
[28] I.G. Krykun, S.Ya. Makhno, The Peano phenomenon for Itô equations, Ukr. Mat. Visn. 10 (1) (2013) 87–109 (in Russian); English transl.: J. Math. Sci. (New York) 192 (4) (2013) 441–458.
[29] S. Maniglia, Probabilistic representation and uniqueness results for measure-valued solutions of transport equations, J. Math. Pures Appl. (9) 87 (6) (2007) 601–626.
[30] O.A. Manita, Shaposhnikov S.V., On the well-posedness of the Cauchy problem for Fokker–Planck–Kolmogorov equations with potential terms on arbitrary domains. 2013, arXiv:1307.3662.
[31] P.B. Mucha, Transport equation: extension of classical results for div $b \in BMO$, J. Differential Equations 249 (8) (2010) 1871–1883.
[32] G. Peters, Anticipating flows on the Wiener space generated by vector fields of low regularity, J. Funct. Anal. 142 (1) (1996) 129–192.
[33] D.W. Stroock, S.R.S. Varadhan, Multidimensional diffusion processes, Berlin – New York, Springer-Verlag, 1979.
[34] A.-S. Üstünel, M. Zakai, Transformation of measure on Wiener space, Springer-Verlag, Berlin, 2000.