On the propagation of gravitational waves in a $\Lambda$CDM universe

Jorge Alfaro$^1$, Domènec Espriu$^2$ and Luciano Gabbanelli$^2$

1 Pontificia Universidad Católica de Chile, Av. Vicuña Mackenna 4860, Santiago, Chile
2 Departament of Quantum Physics and Astrophysics and Institut de Ciències del Cosmos (ICCUB), Universitat de Barcelona, Martí i Franquès 1, 08028 Barcelona, Spain

E-mail: jalfaro@puc.cl, espriu@icc.ub.edu and gabbanelli@icc.ub.edu

Received 25 April 2018, revised 4 October 2018
Accepted for publication 5 December 2018
Published 27 December 2018

Abstract
We study here how the presence of non-zero matter density and a cosmological constant could affect the observation of gravitational waves in pulsar timing arrays. Conventionally, the effect of matter and cosmological constant is included by considering the redshift in frequency due to the expansion. However, there is an additional interesting effect due to the change of coordinates from the ones describing the geometry of the region where waves are produced to the ones used to measure the pulsar timing residuals. This change of coordinates is unavoidable as the strong gravitational field in a black hole merger distorts clocks and rules. Harmonic waves produced in such a merger become anharmonic when detected by a cosmological observer. The effect is tiny but appears to be nevertheless observable for the type of gravitational waves to which pulsar timing arrays are sensitive and for the favoured values of the cosmological parameters.

Keywords: gravitational waves, cosmological constant, dark energy, dark matter

(Some figures may appear in colour only in the online journal)

1. Introduction

In [1, 2] the influence of a non-vanishing cosmological constant on the detection of gravitational waves (GW) in pulsar timing arrays (PTA) was discussed$^3$. It was found that the value of the timing residuals [3] observed in PTA for a given range of angles subtended by the source

$^3$That is, beyond the frequency redshift due to the expansion of the universe.
and the pulsars were dependent on the value of \( \Lambda \). Given that the GW potentially observed in PTA would correspond to the final phase of the fusion of two very massive black holes (BH) lurking at the center of two colliding galaxies at relatively low redshifts, this result opens the possibility of ‘local’ measurements of the cosmological constant; namely at subcosmological distances.

To set up the framework for the discussion and avoid misconceptions it is important to get the correct physical picture from the very beginning. To simplify things, imagine a large mass \( M \) and a much smaller mass \( m \) orbiting around it. For the time being let us assume that there is no cosmological constant and no matter density. In the situation just described, the geometry is very approximately described by the well known Schwarzschild metric corresponding to a mass \( M \). This metric is expressed in terms of the radial coordinate \( r \) and a time coordinate \( t \).

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This time coordinate \( t \) is just the time ticked by a clock located at \( r \to \infty \), i.e. an observer at rest at infinity.

Next let us assume that a cosmological constant \( \Lambda \neq 0 \) is present. In this case the relevant metric is also well known [4]; the Schwarzschild–de Sitter (SdS) metric:

\[
\frac{1}{2} = \left( 1 - \frac{2GM}{r} - \frac{\Lambda}{3} r^2 \right) \left( 1 - \frac{2GM}{r} - \frac{\Lambda}{3} r^2 \right)^{-1} \left( \frac{d\theta^2 + \sin^2 \theta \, d\phi^2}{d\Omega^2} \right)
\]

with \( d\Omega^2 = d\theta^2 + \sin^2 \theta \, d\phi^2 \). This metric is unique once the requirements of spherical symmetry and time independence of the metric elements are imposed [5]. It exhibits two horizons: one at \( r = r_S = 2GM \) and another one at \( r_\Lambda = \sqrt{3/\Lambda} \). Here \( r \) has the same physical realization as in the case of the Schwarzschild metric mentioned above, but the metric is not anymore Minkowskian when \( r \to \infty \) so one must understand carefully what the meaning of the time coordinate \( t \) is. In this work we will consistently denote with lower case letters, i.e. \( t, r \), the coordinates relevant in the location where GW are emitted.

In the situation just described, if gravitational waves are produced it is obvious that they will be periodic in the time coordinate \( t \) (see also the discussion in section 3), and at a distance sufficiently large from the source but very small compared to cosmological distances they will be described by (a superposition of) approximately harmonic functions of the form

\[
h_{\mu\nu}^{(GW)}(w; t, r) = \frac{e_{\mu\nu}}{r} \cos \left[ w \left( t - r \right) \right],
\]

with \( e_{\mu\nu} \) being the polarization tensor. This is so because the metric is dominated by the strong gravitational field created by the mass \( M \), and in its vicinity the contribution of the \( \Lambda r^2 \) term (that produces very small corrections in the orbit parameters) is totally negligible. On the other hand, determining the precise form of the front wave (namely the actual superposition of harmonic functions) is a non-trivial task that in general requires a detailed analysis. Here we assume that the GW emission is strongly dominated by a single harmonic with a frequency that is governed by the period of rotation. There would be no problem to include in our considerations a dependence of the amplitude on \( t \), particularly relevant in the last stages of the spiraling down process, but we do not consider it here for simplicity.

If at long distances from the source the universe is approximately Minkowskian, as it would be the case if the metric is globally Schwarzschild, this wave front would be seen by a remote observer with exactly the same functional form. This is not the case when a cosmological constant or dust are present. The universe cannot be considered to be globally Minkowskian although at very short distances this is a good approximation, except in the case of strong
gravitational sources being present. In the vicinity of the very massive object of mass $M$, Einstein equations together with symmetry considerations dictate the form of the solution.

However, our spacetime is not Minkowskian far away from the GW source and we know that the distribution of matter and energy governs the way clocks tick. Exactly as clocks in the vicinity of a strong gravitational field created by a very massive object signal a time $t$ to a remote observer, at cosmological scales the universe being homogeneous and isotropic dictates at what rate comoving clocks, far from strong gravitational sources, tick. We know that the latter physical situation is described by a Friedmann–Lemaître–Robertson–Walker (FLRW) metric \[ \text{(3)} \]

\[
\text{d}s^2 = a(T)^2 \left( \text{d}T^2 - \text{d}R^2 + R^2 \text{d}\Omega^2 \right).
\]

$a(T)$ is the scale factor. Then a cosmological observer located very far away from the BH merger will not see the same functional dependence in the wave front simply because $T \neq t$ and $R \neq r$.

It is particularly interesting to consider the case where only a cosmological constant is present because then the coordinate transformation between the two coordinate systems is precisely known when one is placed far away from the source of the gravitational field, as is obviously the case if $r \gg 2MG$. The universe is globally de Sitter; SdS and FLRW coordinates correspond to slicing the same spacetime in two different ways. Both are solutions very far from the source. Clocks tick with time $t$ near the large mass $M$ and with time $T$ at the cosmological distances where we observe the phenomenon. In both cases the distribution of matter and energy dictates what time is.

The cosmological time and comoving space coordinates will in fact be non-trivial functions of the SdS coordinates, i.e. $T(t, r), R(t, r)$. These transformations are well known in the case where the universe is globally de Sitter, and will be reviewed in the next sections. Adding matter further modify these transformations. As a consequence of the different coordinate systems, some anharmonicities appear when (2) is written in terms of the observer coordinates $T, R$. Given that the cosmological constant is small, these anharmonicities are numerically small too, but the results of [1] suggest that nevertheless they could be measurable in a realistic PTA observation.

Conventionally, the expansion of the universe is taken into account by replacing $w$ by its redshifted counterpart in (2). If $z$ is small

\[
w' \approx w (1 - z),
\]

with $z$ being the redshift factor. We will see that this effect on the frequency is indeed reproduced, but there is more than this.

Let us justify why in PTA observations the coordinates that an astronomer uses to study the phenomenon are (up to some modifications described in section 6) $T$ and $R$ to a high degree of accuracy. Galaxies are following the cosmic expansion with respect the source of the GW and therefore the relevant change of coordinates for an observer sitting at, say, the center of mass of a galaxy would be the one relating $(t, r)$ with $(T, R)$. Needless to say, gravitational fields inside the galaxy locally modify this relation. These modifications are truly negligible, except for those that will be considered in section 6. Therefore in this work we will consistently denote the coordinates used by an Earth-bound observer by $(T, R)$. Naturally, the Earth and pulsars are gravitationally bound to the Galaxy and the distance of a given pulsar to an observer on the Earth is a constant distance $L$ in coordinate $R$. On the other hand, the GW source is not gravitationally bound, so it feels the expansion of the universe. In short, the effect that will be discussed and analyzed in this work has nothing to do with an hypothetical
expansion of the distance between the pulsar and the observer (which is not present) but rather with the non-trivial transformation relating coordinates \((t, r)\) and \((T, R)\).

The results obtained in [1, 2] were not yet directly applicable to a realistic measurements of PTA timing residuals because at the very least non-relativistic dust needs to be included to get a realistic description of the cosmology. This is in fact the main purpose of the present article.

We have organized our presentation as follows. In the next section we briefly review for notation purposes some basic ingredients of the \(\Lambda\)CDM cosmology. After this, we study the linearization of Einstein’s equations and argue why this is the proper conceptual framework. Next we discuss issues related to the choice of the coordinate system in de Sitter spacetime. This discussion actually contains the essential ingredient of the present study. Later we consider the case where there is no cosmological constant but only non-relativistic matter and, finally, extend our considerations to the combined \(\Lambda\)CDM case, and derive the coordinate transformation between the system of FLRW coordinates and the spherically symmetric coordinates, relevant to describe the coalescence of two BH. We conclude with some preliminary analysis that indicates the relative importance of dark energy and dark matter in the effects described.

In this work we focus on the more fundamental aspects concerning the definition of and relation amongst the different coordinate systems. We understand that this merits a very detailed discussion, as many of the results obtained do not seem to be available in the literature. A preliminary observational study is postponed to a subsequent publication.

2. Conventions and notations in \(\Lambda\)CDM cosmology

The Einstein equations, including the cosmological constant and the energy-momentum tensor reads

\[
R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} - \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}. \tag{5}
\]

The Ricci tensor is constructed by the contraction of the first and third indices of the Riemann curvature tensor \(R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}\). The Ricci scalar is obtained as usual \(R = g^{\mu\nu} R_{\mu\nu}\). The signature convention for the metric is \((+ - - -)\) and \(\kappa = 8\pi G\). We have included the cosmological constant term \(g_{\mu\nu} \Lambda\).

The energy momentum tensor for a perfect fluid in thermodynamic equilibrium takes the form

\[
T^{\mu\nu} = (\rho + P) U^\mu U^\nu - P g^{\mu\nu}, \tag{6}
\]

with the chosen convention. In FLRW comoving coordinates \(\{T, R, \theta, \phi\}\), it is clear that \(U^T = U_R = 1 \) whereas \(U_\theta = U_\phi = 0\). The normalization condition \(g_{\mu\nu} U^\mu U^\nu = 1\) is fulfilled. In order to determine the time evolution of the scale factor, Einstein field equation (5) are required together with some equation of state that relates the pressure and the density for every single component present, \(P_i = \omega_i \rho_i\).

The cosmological constant can be moved, if desired, to the right hand side of Einstein equations and included in the form of an energy-momentum tensor

\[
T^{(\Lambda)}_{\mu\nu} = \rho_{\Lambda} g_{\mu\nu}. \tag{7}
\]

Here the cosmological constant density \(\rho_{\Lambda}\) is a constant. In this interpretation, the cosmological constant arises from an intrinsic energy density of the vacuum, \(\rho_{\Lambda} \equiv \rho_{\text{vac}}\); that is, the new component in the stress-energy tensor can be interpreted as an ideal fluid source of energy
density which has negative pressure (opposing the pressure of matter if there were) [7]. Here \( \omega_{\Lambda} = -1 \), hence
\[
\rho_{\Lambda} = -p_{\Lambda} = \frac{\Lambda}{\kappa}.
\] (8)

By Gauss’s outflux theorem, the cosmological constant is equivalent to a repulsive gravitational field of magnitude \( \Lambda r/3 \) away from any chosen center. The current preferred value is \( \Lambda = 1.11 \times 10^{-52} \text{ m}^{-2} \) [8].

Using the spatially flat generic metric (3) we get from the 1st Friedmann equation
\[
\frac{\ddot{a}(T)}{a(T)} \equiv H = \sqrt{\frac{\Lambda}{3}} \quad \Rightarrow \quad a(T) = a_0 e^{\sqrt{\frac{\Lambda}{3}} \Delta T}.
\] (9)

The dot stands for derivatives with respect to the cosmological time \( T \). The cosmological constant sets the expansion rate, given by the Hubble parameter \( H \). \( \Delta T = T - T_0 \) is the time interval and \( T_0 \) is taken such that \( a(T_0) = a_0 \).

Dust is a special case of perfect fluid as well. The gravitational field is produced entirely by matter characterized by a positive mass density, given by the scalar function \( \rho_{\text{dust}} \), but with the absence of pressure. The stress-energy tensor is
\[
T_{\mu\nu} = \rho_{\text{dust}} U_{\mu} U_{\nu}.
\] (10)

Several independent techniques like cluster mass-to-light ratios [9], baryon densities in clusters [10, 11], weak lensing of clusters [12, 13], and the existence of massive clusters at high redshift [14] have been used to obtain a handle on \( \Omega_{\text{matter}} = 0.31 \). \( \Omega_i = \rho_i / \rho_{cr} \) with \( \rho_{cr} \) being the critical density for which the spatial geometry is Euclidean. This density is known to be dominated by dark matter, \( \Omega_{\text{DM}} = 0.26 \).

As said above, in FLRW comoving coordinates the four-velocity is given by \( U^\mu = (1, 0, 0, 0) \); so the stress-energy tensor reduces to \( T_{\mu\nu} = \text{diag}(\rho, 0, 0, 0) \). The Einstein equations derived for this metric structure together with the equation of state corresponding to \( \omega_{\text{dust}} = 0 \), give the solution [15]
\[
a(T) = a_0 \left( \frac{T}{T_0} \right)^{2/3}, \quad \rho_{\text{dust}}(T) = \frac{4}{3 \kappa T^2}.
\] (11)

Therefore, we are able to write the scale factor and the metric with a density dependence exclusively
\[
d s^2 = dT^2 - a_0^2 \left[ \frac{\rho_{00}}{\rho_{\text{dust}}(T)} \right]^{2/3} \left( dR^2 + R^2 d\Omega^2 \right).
\] (12)

Here, \( \rho_{00} = 4/(3 \kappa T_0^2) \) is the density measured at a particular cosmological time \( T_0 \).

Finally, let us consider the combined situation where dust and a cosmological constant are both present, and let us derive the evolution of the corresponding cosmological scale factor in FLRW coordinates. These components are considered as non-interacting fluids; each with such an equation of state. Then follows that the 2nd Friedmann equation holds separately for each such fluid \( i \)
\[
\dot{\rho}_i = -3 H (\rho_i + P_i),
\] (13)

from where we get
\[
\rho_i \propto a^{-3(1+\omega_i)}.
\] (14)
The total density is the addition of the density for each component, using $\omega_\Lambda = -1$ and $\omega_{\text{dust}} = 0$.

$$\rho_{\text{Eff}}(T) = \rho_{\text{dust}}(T) + \rho_\Lambda = \rho_{\text{dust}} \left( \frac{a_0}{a(T)} \right)^3 + \rho_\Lambda;$$

where $\rho_{\text{dust}}$ is the ‘initial’ density of ‘dust’ and $\rho_\Lambda$ is the (constant) density of ‘dark energy’ when $a(T_0) = a_0$.

To obtain an expression for the scale factor, we have to solve the 1st Friedmann equation coming from the temporal Einstein field equation combined with (15) for the density

$$\frac{\dot{a}}{a} = \sqrt{\frac{\kappa \rho_{\text{Eff}}}{3}}.$$

The scale factor is

$$a(T) = a_0 \left[ \sqrt{1 + \frac{\rho_{\text{dust}}}{\rho_\Lambda}} \sinh \left( \sqrt{\frac{3}{2}} \kappa \rho_\Lambda \frac{\Delta T}{2} \right) + \cosh \left( \sqrt{\frac{3}{2}} \kappa \rho_\Lambda \frac{\Delta T}{2} \right) \right]^{2/3}.$$

In the limit when $\rho_{\text{dust}} \to 0$ we recover the one component scale factor (9); or if $\rho_\Lambda \to 0$, we get (11) as it should be.

3. Why linearized gravity

The linearized theory of gravity is nothing else than perturbation theory around Minkowski spacetime. This is, starting with a small deviation $|h_{\mu\nu}| \ll 1$ from a flat metric tensor,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \quad (18)$$

The usual procedure for obtaining the linearized Einstein tensor can be found in any textbook on general relativity (see e.g. [17]). However, a gauge choice is mandatory to obtain a solution to these equations in order to avoid redundancy under coordinate transformations $x^\mu \to x^\mu + \xi^\mu(x)$. Although the following discussion is valid in any gauge, it is simplest in the familiar Lorenz gauge

$$\partial_\nu h_{\mu\nu} - \frac{1}{2} \partial_\mu h = 0, \quad (19)$$

or defining the trace-reversed perturbation variable

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu} \Rightarrow \partial_\nu \tilde{h}^{\mu\nu} = 0. \quad (20)$$

In this gauge the linearized Einstein equations, including the cosmological constant and the energy momentum tensor read

$$\Box \tilde{h}_{\mu\nu} = -2 \Lambda \eta_{\mu\nu} - 2 \kappa T_{\mu\nu}. \quad (21)$$

As discussed in [3], up to first order in $\Lambda$ the perturbation $h_{\mu\nu}$ in equation (18) can be decomposed into a gravitational wave perturbation $h_{\mu\nu}^{(GW)}$ and a background modification due to the cosmological constant $h_{\mu\nu}^{(\Lambda)}$, being both ‘small’ in magnitude. It is straightforward to propose a new contribution due to dust $h_{\mu\nu}^{(\text{dust})}$. This component would contribute as an independent new background that will not interact with the other one, the one corresponding to $\Lambda$, and that will
be also ‘small’ enough so that the linearized approximation to be meaningful. Therefore, the total metric would be written as

\[ g_{\mu\nu} = \eta_{\mu\nu} + h^{(GW)}_{\mu\nu} + h^{(A)}_{\mu\nu} + h^{(dust)}_{\mu\nu}. \]  

(22)

The first two contributions have been already discussed in [18]. The two independent background would satisfy each a simpler equation of motion

\[ \Box \tilde{h}^{(A)}_{\mu\nu} = -2 \Lambda \eta_{\mu\nu} \]  

(23)

for the cosmological constant and

\[ \Box \tilde{h}^{(dust)}_{\mu\nu} = -2 \kappa T_{\mu\nu} \]  

(24)

for a dark matter background uniformly distributed along the spacetime. Finally the perturbation due to the gravitational wave itself satisfies the usual homogeneous wave equation

\[ \Box \tilde{h}^{(GW)}_{\mu\nu} = 0. \]  

(25)

In the previous discussion we have not included the modification due to the gravitational field created by the very massive object of mass \( M \) as this is negligible at large distances, but the perturbation is also additive as is easily seen by expanding Schwarzschild in powers of \( G \). Thus the discussion is easiest in linearized gravity. The contribution of the different terms simply adds to construct the total metric. And because the deviations w.r.t. Minkowski spacetime are small away from the GW source, linearization is a priori well justified.

If we stress the importance of the linearization process it is because it is not always possible to accomplish in all coordinate systems. For instance, it is impossible in cosmological coordinates. This will be discussed in more detail in section 4. As a consequence there are no harmonic GW waves in FLRW coordinates. In SdS coordinates, however, linearization is possible [1, 3], and linearization remains valid after including dust, which is one of the main results of this paper.

The solution of equation (23) as explained in [3] should correspond to the linearization (i.e. expanding in \( \Lambda \) at first order) of equation (26); i.e. the metric (28). In fact this statement is not totally correct because in (21) the Lorenz gauge condition is assumed, while the SdS metric does not fulfill this gauge condition, as it can be easily checked. In fact, as explained in [3] a trivial coordinate transformation turns the SdS metric into one that complies with the Lorenz gauge. This coordinate transformation is (a) time-independent and (b) is of order \( \Lambda \). The discussion on GW and the separation in background and waves is however simplest in the Lorenz gauge and it is the one just presented. In fact, the relevant effects to be discussed later are all of order \( \sqrt{\Lambda} \) and order \( \Lambda \) corrections are safe to be neglected.

4. The choice of coordinate systems

De Sitter geometry models a spatially flat universe and neglects matter, so the dynamic of the universe is dominated by the cosmological constant \( \Lambda \). Obviously there exist a number of useful coordinate choices for this spacetime. These consist in picking a convenient time choice and thus defining a family of spacelike surfaces. Some slices make explicit a cosmologically expanding space with flat curved spatial part, whereas some other do not.

There is as well a unique choice of coordinates in which the metric does not depend on time at all; as is described in equation (1) if the BH contribution were omitted
8

\[ ds^2 = \left( 1 - \frac{\Lambda}{3} r^2 \right) dr^2 - \left( 1 - \frac{\Lambda}{3} r^2 \right)^{-1} dr^2 - r^2 d\Omega^2. \]  

(26)

The mere existence of such a choice is enough to tell us that there is no fundamental sense in which this is an expanding cosmological spacetime. The notion of expansion is a concept that is linked to a coordinate choice. Yet, the choice of coordinates is not totally arbitrary. Observers’ clocks tick at a given rate, and they can measure the local space geometry at a fixed time.

Cosmological observers detect that the universe as we see it is spatially flat and that gravitationally unbound objects separate from each other at a rate that is governed by the cosmological scale factor \( a(T) \) present in equation (3). Then the physical coordinate system for discussing cosmology is given by the FLRW metric with coordinates \( (T, R) \), as this coordinate system encodes the observed homogeneity and isotropy of the matter sources summarized in the cosmological principle [16]. Of course, space redefinitions of the metric are innocuous. It is of no consequence to choose to describe the world around us using Cartesian or polar coordinates. As long as coordinate transformations do not involve time in any essential way, any coordinate choice will be equally good to describe a universe conforming to the cosmological principle.

Time and space will however look very different to an observer in the vicinity of a BH or, for that matter, in the vicinity of any strong gravitational source that is spherically symmetric, centered at one point that we conventionally denote by \( r = 0 \). The spacetime there is isotropic, but not homogeneous. If the BH is static, we know that the solution is unique and it is given by the Schwarzschild metric. If in addition, there is a cosmological constant, the corresponding metric will be the one given in equation (1).

The SdS metric approximates a Schwarzschild space for ‘small’ \( r \); and for ‘large’ \( r \) the space approximates a de Sitter space. We are focusing right now on effects on gravitational wave propagation, hence we are in a ‘large’ \( r \) regime. Far away from a super massive BH with a typical Schwarzschild radius of \( \sim 10^{11} \) m, the term \( r_s/r \) can be safely neglected from our considerations and only the \( \Lambda \) dependence needs to be taken into account; i.e. under this regime the metric is given by (26). In this case the transformation relating the SdS coordinates to the FLRW ones are well known (see e.g. [3])

\[
\begin{align*}
    t(T, R) &= T - \sqrt{\frac{3}{\Lambda}} \log \sqrt{1 - \frac{\Lambda}{3} a^2(T) R^2}; \\
    r(T, R) &= a(T) R = a_0 e^{\sqrt{\frac{3}{\Lambda^2}}} \Delta T R; \\
\end{align*}
\]

(27)

the \( \theta \) and \( \phi \) coordinates do not transform. Note that even though for \( r \gg r_s \) the influence of the BH is negligible, its presence sets up a global coordinate system, and \( r \) and \( t \) have a well defined physical meaning.

The SdS metric in the ‘far away from the source’ limit can be linearized expanding (26) in powers of \( \Lambda \). For \( r \to \infty \), but \( \Lambda r^2 \ll 1 \) (i.e. well before the cosmological horizon)

\[
    ds^2 = \left( 1 - \frac{\Lambda}{3} r^2 \right) dr^2 - \left( 1 + \frac{\Lambda}{3} r^2 \right)^{-1} dr^2 - r^2 d\Omega^2.
\]

(28)

This metric satisfies a linearized version of Einstein field equation (5). On the contrary, the FLRW metric does not fulfill any linearized version of Einstein equations\(^4\); in fact, no metric that depends only on time can be solution of the linearized Einstein equations with a cosmological constant, whatever the gauge choice selected.

\(^4\)This is easy to understand by realizing that in the FLRW metric \( \Lambda \) appears in a non-analytic form.
If linearization were possible in FLRW coordinates, GW could be harmonic in those coordinates; i.e. they could obey a (flat) wave equation. But this as we have seen is impossible. As a consequence, GW are necessarily anharmonic in FLRW coordinates. Therefore the proper procedure is to find a coordinate system where linearization is possible and the wave perturbation and the background can be treated additively. Luckily this is accomplished in SdS coordinates because they are analytic in $\Lambda$ and have a smooth Minkowski limit. Linearization and, consequently, harmonic waves remain valid when dust is included. The change from SdS to FLRW is necessarily not a ‘soft’ one—it must have a singularity somewhere and it does.

If one attempts to solve directly the wave equation with a non-zero cosmological constant in FLRW using perturbation theory in $\Lambda$, one generates secular terms that grow with $(T, R)$ and invalidate perturbation theory.

The coordinate transformations relating SdS and FLRW can be expanded for small values of $\Lambda T^2$ or $\Lambda R^2$ (this will always be the relevant situation in the ensuing discussions)

$$
\begin{align*}
t(T, R) & \approx T + a_0 \left( \frac{\Lambda T^2}{2} \sqrt{\frac{2}{3}} + \Delta T R^2 \frac{\Lambda}{3} \right) + \ldots \\
r(T, R) & \approx a_0 \left[ 1 + \Delta T \sqrt{\frac{2}{3}} + \Delta T^2 \frac{\Lambda}{6} \right] R + \ldots.
\end{align*}
$$

(29)

The change is of course still non–analytical in $\Lambda$. Applying this approximate transformation to the SdS metric, one ends up with an approximate version of the FLRW metric

$$
\text{d}s^2 \approx \text{d}T^2 - a_0^2 \left( 1 + 2 \sqrt{\frac{\Lambda}{3}} \Delta T + 2 \frac{\Lambda}{3} \Delta T^2 \right) (\text{d}R^2 + R^2 \text{d}\Omega^2)
$$

(30)

which is the expansion of (3) for small values of the cosmological constant. In practice, the leading $\sqrt{\Lambda}$ term will be the relevant one for our purposes. The extension of these considerations to the case where a cosmological constant and non-relativistic matter coexist comes next.

5. Including non-relativistic matter

First we would like to repeat the discussion previously done in section 4 in the case where there is no cosmological constant, just dust. Namely we ask ourselves what would be the equivalent of equation (27) in such a case.

With the aim of deriving a full picture on how dust influences the observation of gravitational waves, we need to find the coordinate system where dust is described in ‘static’ spherically symmetric coordinates with an origin coincident with the object emitting gravitational waves.

The first step is to find a set of coordinates, spherically symmetric, where the angular element that in FLRW reads $a(T)^2 R^2 \text{d}\Omega^2$ should become simply $r^2 \text{d}\Omega^2$. The coordinate transformation will be given by

$$
\begin{align*}
T(t, r) & = \text{unknown function} \\
R(t, r) & = \frac{r}{a(T)}.
\end{align*}
$$

(31)

Next we take into account the transformation properties of a rank 2 tensor

$$
g'_{\mu'\nu'} = \frac{\partial X^\mu}{\partial x^\mu'} \frac{\partial X^\nu}{\partial x^\nu'} g_{\mu\nu}.
$$

(32)
Spherical symmetry is a requirement for both metrics, then the transformation for the two angular variables is characterized by the identity. This means that in the angular diagonal components of the metric the Jacobian corresponds to the identity and the transformation for these components do not bring any new information. Analogous is the case for the angular and off-diagonal elements of the metric. The transformation is trivial and in both pictures the components of the metric are null. However, there is one off-diagonal element that is not null per se; the $g_{rt} = g_{tr}$ element. If a diagonal metric is desired we must impose the vanishing of this element. This translates into the following differential equation for the cosmological time $T$

$$\partial_r T = - \frac{6r}{{9T^2 - 4r^2}} \quad \iff \quad \partial_r \rho_{\text{dust}} = \frac{\kappa \rho_{\text{dust}}^2 r}{1 - \frac{2\rho_{\text{dust}}}{3} r^2}. \quad (33)$$

Of course we already know the structure for the bijection between the cosmological time and the content of the universe, i.e. $|T| \propto 1/\sqrt{\rho_{\text{dust}}}$ coming from equation (12). This allows to write the diagonal metric condition (33) in terms of the non-relativistic matter density only. We have two remaining components of the metric to be transformed: the $TT$ and $RR$–components. If we use only the requirement for the radial transformation (31) and impose a diagonal structure by (33), the metric reads

$$ds^2 = \left(\frac{\partial_t \rho_{\text{dust}}}{3 \kappa \rho_{\text{dust}}}\right)^2 \left[1 - \frac{\kappa \rho_{\text{dust}}}{3} r^2\right] dr^2 - \frac{1}{1 - \frac{2\rho_{\text{dust}}}{3} r^2} dr^2 - r^2 d\Omega^2. \quad (34)$$

Note that $\rho_{\text{dust}}$ is a (scalar) known function of $T$ and therefore it is expected to depend on the new temporal variable $t$; but this dependence is unknown a priori (because $t$ is unknown so far). In any case we see that the new metric necessarily contains a $t$ dependence via $\rho_{\text{dust}}(t)$.

Before continuing, let us reduce the number of dimensionful parameters of the theory. The limit of weak coupling ($\kappa \rightarrow 0$) and vanishing matter density ($\rho_{\text{dust}} \rightarrow 0$) are essentially the same because both parameters appear together in all the equations—except in the energy momentum conservation which is trivially satisfied. Hence we define: $\tilde{\rho}_{\text{dust}} = \kappa \rho_{\text{dust}}$. Then the solution of (33) is

$$\frac{6 + r^2 \tilde{\rho}_{\text{dust}}}{\tilde{\rho}_{\text{dust}}^{1/3}} = C(t); \quad (35)$$

where $C(t)$ is a constant with respect to $r$, meaning that it can only depends on $t$. Dimensional analysis do the rest of the work. In natural units, $[\tilde{\rho}_{\text{dust}}] = L^{-2}$, then the units for the ‘constant’ should be $[C(t)] = [\tilde{\rho}_{\text{dust}}]^{-1/3} = L^{2/3}$. In this picture, $\kappa$ does not belong to the theory, hence there is only one dimensionful parameter to give units to $C(t)$; namely $t$. As $[t] = L$, then $C(t) = A t^{2/3}$ with $A$ a positive dimensionless parameter to be determined.

It is expected for later times the dust density to be homogeneously diluted. This requirement automatically fixes the remaining free parameter of the theory, $A$. This is translated into the metric (34) to be Minkowskian–like when $t \rightarrow \infty$. This is automatically fulfilled except for the factor in front of the brackets in the $g_{rt}$ element. If a Minkowskian limit is expected, the following enforcement of the prefactor is needed $(\partial_t \rho_{\text{dust}})^2/(3\kappa \rho_{\text{dust}}^3) \rightarrow 1$. We have a relation for obtaining the temporal variation of the dust density, namely (35) with the corresponding $C(t)$ function. Therefore, reintroducing $\kappa$ and deriving both sides of the equality, one arrives to

$$\partial_t (\kappa \rho_{\text{dust}}) = \left(\frac{\kappa \rho_{\text{dust}}}{3} \right)^{1/3} \frac{A}{3 r^{1/3}}. \quad (36)$$
Squaring the last result, dividing by $3 \left( \kappa \rho \right)^{1/3} t^{2/3}$, and finally using equation (35) again for replacing $1/r^{2/3}$, the prefactor for the $g_{tt}$ component in the limit of $\rho_{\text{dust}} \to 0$ becomes

$$\frac{A^2}{3^3 \left( \kappa \rho \right)^{1/3} t^{2/3}} \to \frac{A^3}{3^3 \sqrt{6}}. \quad (37)$$

After all, imposing the flat asymptotic limit to this factor, the constant is fixed to $A = 3\sqrt{6}$. Once $A$ is known, $\partial_t \rho_{\text{dust}}$ can be expressed as a function of $\rho_{\text{dust}}$ and $r$ uniquely. Following the same steps as before; replacing $1/r^{2/3}$ from the prefactor by $A/C(t)$ from equation (35), we get simply

$$\left( \frac{\partial_t \rho_{\text{dust}}}{\sqrt{6}} \right)^3 = \frac{1}{\left( 1 + \frac{\kappa \rho_{\text{dust}}}{6} r^2 \right) \left( 1 - \frac{\kappa \rho_{\text{dust}}}{3} r^2 \right)^2}$$

where the asymptotic limit is shown explicitly. Then the sought for metric turns out to be

$$ds^2 = \left( 1 + \frac{\kappa \rho_{\text{dust}}}{6} r^2 \right) ^{1/2} dr^2 - \left( 1 + \frac{\kappa \rho_{\text{dust}}}{3} r^2 \right) ^{1/2} d\Omega^2. \quad (38)$$

This metric has a good Minkowskian limit for $\rho_{\text{dust}} \to 0$. It has similar characteristics to the ones of the SdS metric and it is clearly the relevant metric to describe the Keplerian problem we alluded to in the introduction, in the presence of dust but without cosmological constant, when properly extended with the BH gravitational potential (see the following).

Further discussions concerning dust in these coordinates are included in the appendix. With the full structure for $C(t)$, we present a detailed discussion on the physical solutions to equation (35). In appendix A we show that the mere existence of a solution for (35) entails the presence of a horizon in such coordinates. In appendix B, we include the full computation of the non-diagonal stress-energy tensor also in these coordinates.

The corresponding change of variables to and from FLRW is given by

$$\begin{align*}
  t &= \frac{\left[ 6 + \left( \kappa \rho_{\text{dust}} \right)^{1/3} \left( \kappa \rho_{\text{dust}} \right)^{1/3} R \right]^{1/2}}{9 \sqrt{2} \kappa \rho_{\text{dust}}} \\
  r &= \sqrt{\frac{\rho_{\text{dust}}}{\rho_{\text{dust}}}} R
\end{align*} \quad (39)$$

with $\rho_{\text{dust}}$ given by equation (12). The metric (38) can be linearized for $\rho_{\text{dust}} \to 0$ to

$$ds^2 = \left( 1 + \frac{\kappa \rho_{\text{dust}}}{6} r^2 \right)^{1/2} dr^2 - \left( 1 + \frac{\kappa \rho_{\text{dust}}}{3} r^2 \right)^{1/2} d\Omega^2. \quad (40)$$

Within this linearized approximation it is of course trivial to include the BH gravitational field. This would give rise to

$$ds^2 = \left( 1 - \frac{\rho s}{r} + \frac{\kappa \rho_{\text{dust}}}{6} r^2 \right) dr^2 - \left( 1 + \frac{\rho s}{r} + \frac{\kappa \rho_{\text{dust}}}{3} r^2 \right) d\Omega^2. \quad (41)$$

This expression is valid in the region $r \gg \rho s$ and $\kappa \rho_{\text{dust}} r^2 \ll 1$. Note that it is not time-independent, unlike its SdS counterpart, but the time dependence enters only via the matter density.

In equation (39) the coordinate transformation is expressed in terms of $\rho_{\text{dust}}$ but the dependence of $\rho_{\text{dust}}$ on $T$ is known; see equation (12). We can therefore expand in powers of $\Delta T = T - T_0$, where $T_0$ is the time where the density of dust is $\rho_{d0}$. Linearization of the coordinate transformation leads to
\[
\begin{aligned}
  &\begin{cases}
    t = T + \frac{R^2}{2} \sqrt{\frac{\Lambda + \kappa \rho_{\text{dust}}}{3}} + \Delta T R^2 \frac{\kappa \rho_{\text{dust}}}{12} + \ldots \\
    r = (1 + \Delta T \sqrt{\frac{\Lambda + \kappa \rho_{\text{dust}}}{3}} - \Delta T^2 \frac{\kappa \rho_{\text{dust}}}{12}) R + \ldots
  \end{cases}
  \\
\end{aligned}
\]

where the scale factor \( a_0 \) is taken equal to 1 from now on. Note that the initial condition \( T = T_0 \) for the cosmological time does not correspond to a unique value for \( t \) as the relation does depend on \( R \).

Now we have all the ingredients to combine both components, dust and cosmological constant. In linearized gravity this is exceedingly simple as previously discussed. The metric will be

\[
ds^2 = \left(1 + \frac{\kappa \rho_{\text{dust}}}{6} r^2 - \frac{\Lambda}{3} r^2 \right) \, dt^2 - \left(1 + \frac{\kappa \rho_{\text{dust}}}{3} r^2 + \frac{\Lambda}{3} r^2 \right) \, dr^2 - r^2 \, d\Omega,
\]

where the BH contribution \( \sim r_S/r \) has been omitted because is not relevant at large distances.

The additive picture permits to recover the individual linearized theory for each component (taking the corresponding limit of the other going to zero). One would expect the same additive behaviour in the perturbations for the change of variables, but if one analyze how the full scale factor (17) expands in Taylor series,

\[
a(T) \simeq 1 + \sqrt{\frac{\Lambda + \kappa \rho_{\text{dust}}}{3}} \Delta T + \ldots
\]

it is clear that the additivity takes place inside the square root. Indeed it is easy to see that the following coordinate transformation does the job of moving from FLRW coordinates to the ones in equation (43)

\[
\begin{aligned}
  &\begin{cases}
    t \sim T + \frac{R^2}{2} \sqrt{\frac{\Lambda + \kappa \rho_{\text{dust}}}{3}} + \Delta T R^2 \left( \frac{\Lambda}{3} + \frac{\kappa \rho_{\text{dust}}}{12} \right) + \ldots \\
    r \sim a(T) R \sim \left(1 + \Delta T \sqrt{\frac{\Lambda + \kappa \rho_{\text{dust}}}{3}} + \ldots \right) R.
  \end{cases}
\end{aligned}
\]

Of course in both limits, \( \rho_{\text{dust}} \to 0 \) and \( \Lambda \to 0 \), we recover each 1-component universe coordinate transformation.

### 6. Relevance for pulsar timing arrays

After the transformation to FLRW coordinates and keeping the changes of order \( \sqrt{\Lambda} \) only, equation (2), gets modified to

\[
\begin{aligned}
  &h^{(\text{FLRW})}_{\mu \nu} \equiv h^{(\text{GW})}_{\mu \nu} = \rho'_{\mu \nu} \cos \left[ \frac{\Lambda}{3} T \right] + h^{(\text{GW})}_{\nu \mu} \cos \left[ \frac{\Lambda}{3} T \right].
\end{aligned}
\]

\( \rho'_{\mu \nu} \) is the transformed polarization tensor that will not be very relevant for the following discussion, although its precise form is essential for precise comparison with observations. This last expression can also be written as

\[
\begin{aligned}
  &h^{(\text{GW})}_{\mu \nu} = \rho'_{\mu \nu} \cos \left[ \frac{\Lambda}{3} T - k_{\text{eff}} R \right].
\end{aligned}
\]
with \( w_{\text{eff}} = w\left(1 - \sqrt{\Lambda/3}R\right) \) and \( k_{\text{eff}} = w\left(1 - \sqrt{\Lambda/3}R/2\right) \). Notice that \( w_{\text{eff}} \) agrees with the usual frequency redshift, so this well known result is well reproduced. However, the wave number is different. This discrepancy lies at the root of a possible measurement of \( \Lambda \) in PTA.

As emphasized before, the corrections of order \( \Lambda \) and beyond are subleading when the actual presumed value for the cosmological constant is considered and distances to the GW source of the magnitude considered here are involved; only \( \sqrt{\Lambda} \) terms really matter. In any case, the expressions could be improved to order \( \Lambda \) accuracy easily.

In the Lorenz gauge the only spatial components of the metric that are different from zero are the \( X, Y \) entries of \( e^{\prime}_{\mu\nu} \). Although some temporal components are also non-zero in these coordinates, they are several orders of magnitude smaller than the spatial ones and therefore will not be relevant for the present study.

The phase velocity of GW propagation in such coordinates is \( v_p \sim 1 - \sqrt{\Lambda/3}T + \mathcal{O}(\Lambda) \). On the other hand, with respect to the ruler distance traveled (computed with \( g_{ij} \)) the velocity is still 1.

An analogous transformation to FLRW has been found when the two components are considered. As abundantly mentioned before we will only need the \( \sqrt{(\Lambda + \kappa\rho_d)/3} \) correction because higher powers of this square root can be neglected. The result is rather compact and we immediately know the form that a GW adopts in FLRW coordinates

\[
H_{\mu\nu}^{(GW)} = \frac{e'_{\mu\nu}}{R} \left[ 1 + \sqrt{\frac{\Lambda + \kappa\rho_d}{3}} T \right] \cos \left[ w(T - R) + \sqrt{\frac{\Lambda + \kappa\rho_d}{3}} \left( \frac{R^2}{2} - TR \right) \right].
\]

(48)

This implies

\[
w_{\text{eff}} = w\left(1 - R\sqrt{\frac{\Lambda + \kappa\rho_d}{3}}\right), \quad k_{\text{eff}} = w\left(1 - \frac{R}{2}\sqrt{\frac{\Lambda + \kappa\rho_d}{3}}\right).
\]

(49)

The first result agrees with the expectations of the cosmological redshift, but the second one is a novel effect. From the analysis in [1], we already know that the fact that \( \sqrt{(\Lambda + \kappa\rho_d)/3} \) is non-zero may produce observational effects in PTA.

Consider the situation shown in figure 1 describing the relative situation of a GW source (possibly a very massive black hole binary), the Earth and a nearby pulsar [19]. The electromagnetic field phase \( \phi_0 \) is associated to the pulsed emission at the pulsar. The same magnitude measured from Earth will be

\[
\phi(T) = \phi_0 \left[ T - \frac{L}{c} - \tau_0(T) - \tau_{\text{GW}}(T) \right].
\]

(50)

The corrections due to the spatial motion of the Earth within the Solar system and the solar system with respect to the pulsar, together with the corrections when the electromagnetic wave propagates through the interstellar medium are taken into account in \( \tau_0 \). \( \tau_{\text{GW}} \) involves the corrections owing exclusively to the gravitational wave strength \( h_{ij}^{(\text{FLRW})} \) computed in comoving coordinates.

In the following we will focus in the phase corrections due to the passage of a GW; this shift is approximately given by

\[
\tau_{\text{GW}} = -\frac{1}{2} \dot{\mathcal{H}} \hat{n}^i \mathcal{H}_i(T)
\]

(51)
\( \hat{n} \) is the unit vector pointing from the pulsar to the Earth \((-\sin \alpha \cos \beta, -\sin \alpha \sin \beta, \cos \alpha)\) and \( H_{ij} \) is the integral of the transverse-traceless metric perturbation along the null geodesic from the pulsar to the Earth parametrized by \( \vec{R}(x) = \vec{P} + L(1 + x)\hat{n} \). \( \vec{P} \) the pulsar location and \( L \) the distance to the pulsar. Making the dependences explicit, this magnitude is given by

\[
H_{ij}(T_E, L, \alpha, \beta, Z_E, w, \varepsilon, \Lambda) = \frac{L}{c} \int_{-1}^{0} dx \, h^{(\text{FLRW})}_{ij}[T_E + \frac{L}{c} x; \vec{P} + L(1 + x)\hat{n}].
\]

(52)

\( Z_E \) is the distance from Earth to the GW source, \( T_E \) the time of arrival of the wave to the local system and \( \varepsilon \sim |\varepsilon'|, i, j = X, Y \). The speed of light has been restored in this formula. We have assumed that from the pulsar to the Earth the electromagnetic signal follows the trajectory given by the line of sight and is the distortion created by the intervening GW what modifies the timing residual. The real question is whether the observationally preferred exceedingly small value of the cosmological constant affects the timing residuals from a pulsar at all. This question was answered in the affirmative in [1].

We take average values for the parameters: \( \varepsilon = 1.2 \times 10^9 \) m for a GW generated by some source placed at \( Z_E = 3 \times 10^{24} \) m gives a strength \( |h| \sim \varepsilon/R \sim 10^{-15} \) which is within the expected accuracy of PTA [20]. For one pulsar location \( L = 10^{19} \) m, we plot a snapshot of the resulting timing residuals as a function of the angle \( \alpha \) for the time of arrival of the wave to the local system, \( T_E \). In figure 2 we compare the different cases discussed throughout this article. Depending on which GW strength is used to compute the timing residual, different enhancements of the signal are produced.

The figure speaks by itself and it strongly suggests that the angular dependence of the timing residual is somehow influenced by the value of the cosmological constant and/or the presence of dust, despite of their small values. In fact the curves are extremely sensitive to small changes in the later parameters. Another feature that catches the eye immediately is an
enhancement of the signal for a specific small angle $\alpha$, corresponding generally to a source of low galactic latitude or a pulsar nearly aligned with the source (but not quite as otherwise $e_i^j \hat{n}_i \hat{n}_j = 0$, although the total timing residual is non-vanishing due to the $O(\Lambda)$ time components for waves).

This enhancement is relatively easy to understand after a careful analysis of the integral in equation (52), as was explained in [1]. We will not enter into details regarding the analysis of the possible modifications that the presence of dust can bring about and we will approach it in future work. For us it is only necessary to know that the effect is visible, and therefore the need to consider carefully the different coordinate systems where GW are produced and measured.

The effect discussed in the previous sections is largely degenerate with respect to the details of the source of GW. Not much depends on the polarization or frequency of the wave front (see e.g. [1]). This is so because the enhancement is entirely a consequence of the characteristics of the FLRW metric. The enhancement for a particular value of the angle subtended by the source of the gravitational wave and the pulsar cannot be mimicked by changing frequency.

Figure 2. Raw timing residual as a function of the angle $\alpha$ subtended by the source and the measured pulsar as seen from the observer. The figures represent a snapshot of the timing residual at a given time $T_E$: (a) in flat space time, (b) in a universe with only dust, (c) with only cosmological constant, and (d) in a combined $\Lambda$CDM universe. Clearly, at least in a superficial analysis, it seems plausible to be able to disentangle both contributions. The figures depict the signal at the time $T_E = Z_E/c$. Time evolution or small changes in the parameters will change the phase of the oscillatory signal but not the enhancement clearly visible at relatively low values of the angle $\alpha$. The figures are symmetrical for $\pi \leq \alpha \leq 2\pi$. (a) Minkowski universe. (b) Dust universe. (c) Schwarzschild–de Sitter universe. (d) $\Lambda$CDM universe.
or amplitude of a gravitational wave, or polarization. All these changes would enhance or suppress the signal for all angles, not a restricted range of values. The phenomenon described is thus indeed a telltale signal.

The observable defined in (51) is not the preferred one in PTA collaborations. Typically, PTA use two-pulsars correlator and use the assumption that the gravitational wave signal in the location in the two pulsars are uncorrelated (see Hellings and Downs [21] and Anholm et al [22]). This is of course questionable, but the direct observable presented here would normally be too small to be observed directly without the remarkable enhancement reported in this work.

Unfortunately, the averaged pulsar-pulsar correlator when supplemented with the uncorrelation hypothesis likely turns into a local observable. Local observables are not affected by modifications in the wave number. One needs to be able to measure the GW in two widely separated points, for the wave number to be relevant. Otherwise only the frequency of the GW is relevant and as we have seen this is not at all varied with respect to the usual treatment. It is also for this reason that this effect is totally invisible in an interferometric experiment such as LIGO. For the effect to be observable one needs a non-local observable.

All these issues will be discussed more extensively in a forthcoming investigation.

7. Conclusions and outlook

In this paper we have reviewed how a non-zero cosmological constant modifies the propagation of a gravitational wave, with an additional contribution to the one that is usually taken into account—the redshift in frequency. The effect is entirely due to the change of coordinate systems between the reference frame where the wave originates (e.g. the merger of two gigantic black holes) and the reference frame where waves are measured in PTA, namely cosmological FLRW coordinates.

We have then proceed to extending these results to the case where non-relativistic dust is present, and later to the combined and more realistic situation where non-relativistic matter and vacuum energy are both present. The results are derived in a linearized approximation where only the first deviations are considered. This approximation is however enough for the case of study, as subsequent corrections are seen to be extremely small when the measured value of the matter density and cosmological constant are used, combined with the distances involved in the problem.

It came out as a surprise that these deviations were at all measurable, at least in principle, in PTA observations. So far no clear positive measurement of GW has come out from the existing collaborations. Perhaps the sort of signal predicted here could help. In any case the effect is actually reinforced by the presence of non-relativistic matter and the possibility of local measurements of $\Lambda$ in this way remains.

Acknowledgments

We acknowledge the financial support of projects FPA2013-46570-C2-1-P, FPA2016-76005-C2-1-P and MDM-2014-0369. The work of JA is partially supported by grants Fondecyt 1150390 and CONICYT-PIA-ACT14177 (Government of Chile). LG acknowledges the FPI grant BES-2014-067939, from MINECO (Spain).
Appendix A

Let us make a short comment on the equation that relates the density of dust and the variables $t$ and $r$ in (35). Once the temporal function $C(t) = A t^{3/2}$ is fixed, the following change of variables

$$u^3 = r^2 \kappa \rho, \quad x = A \sqrt{\frac{t^2}{r^2}}$$  \hspace{1cm} (A.1)

makes the relation (35) to correspond with the roots of an extended function

$$f(u) = u^3 - x u + 6.$$  \hspace{1cm} (A.2)

The procedure goes as follows: at any particular time $t$, the density profile is determined at each point $r$. The density takes a value such that the function $f$ vanishes, $f(u = \sqrt{r^2 \kappa \rho}) = 0$. Or in other words, at a certain point of the spacetime, the relation between the magnitudes $u$ and $x$ is given by the roots of the function $f(u)$.

The new function accomplishes $f(-\infty) = -\infty$ and $f(0) = 6$, hence at least one of the roots of $f(u)$ has a real and negative value, $u_1 < 0$. This root of (A.2) has no physical interpretation because as it has been defined in (A.1), $u$ must be positive. Therefore, we know that a positive solution must exist; as $f(\infty) = \infty$ there must be at least one double root (or two more roots).

To find it (them), let us analyze the function $f(u)$. The critical points are obtained by the roots of the first derivative of $f(u)$

$$f'(u_c) = 3 u_c^2 - x = 0 \quad \implies \quad u_c = \pm \sqrt{\frac{x}{3}},$$  \hspace{1cm} (A.3)

The second derivative classifies whether the critical points are a maximum, a minimum or a saddle point

$$f''(u_c) = 6 u_c.$$  \hspace{1cm} (A.4)

Therefore, being also $x > 0$, $u_{c_1} = \sqrt{x/3}$ is a local minimum and $u_{c_2} = -\sqrt{x/3}$ is a local maximum. We are interested in $u > 0$. The next step is to locate the image of the positive critical point $u_{c_1}$ (the minimum), because this value will determine the existence of the wanted remaining root(s)

$$f(u_{c_1} = \sqrt{x/3}) = u_{c_1}^3 - 3 u_{c_1}^3 + 6 \leqslant 0.$$  \hspace{1cm} (A.5)

We are searching for positive real roots of (A.2), hence the value of the minimum must be zero or negative (one double root or two simple roots). It is interesting to note that the latter bound can be written as $3^{5/3} \leqslant x$. Bringing the value of $x$ back, we will see that the previous constraint reflects the presence of a horizon. The mere existence of a positive solution for $f(u) = 0$ with $u \propto \rho$ entails the presence of a horizon in such coordinates. With the value of the constant $A = 3 \sqrt{6}$, the cosmological horizon equation is written as

$$\frac{r}{t} \leqslant \sqrt{\frac{2}{3}}.$$  \hspace{1cm} (A.6)
Appendix B

In this appendix we will give the components of the stress-energy tensor explicitly for a dust universe in the new set of coordinates. Dust is a pressureless perfect fluid with an energy-momentum tensor given by \( \text{(10)} \); this definition is coordinate independent. Once the coordinates are fixed one can find the explicit stress-energy tensor by means of Einstein equation \( \text{(5)} \). The Einstein tensor is straightforward once the metric is chosen; from \( \text{(34)} \) for instance the computations are particularly simple—it would be equivalent to use the metric \( \text{(38)} \), at this point \( \partial_t \rho_{\text{dust}} \) is already know.

The product of the two nonnull off-diagonal components of the Einstein tensor lead us to

\[
G^t_r G^r_t = -\frac{(\kappa \rho_{\text{dust}})^3 r^2}{3 \left( 1 - \frac{\kappa \rho_{\text{dust}}}{3} r^2 \right)^2}. \tag{B.1}
\]

Notice that \( G^t_r \neq G^r_t \). The r.h.s. of Einstein equation \( \text{(5)} \) equals the previous relation to

\[
\kappa^2 T^r_r T^t_t = (\kappa \rho)^2 U^t U^r U_t U_r. \tag{B.2}
\]

We aim at solving the four-velocity and we have another equation to do so: the normalization condition over the four-velocity that yields

\[
g_{\mu\nu} U^\mu U^\nu = U_t U^t + U_r U^r = 1. \tag{B.3}
\]

The system of two equations: \( \text{(B.1)} \) – \( \text{(B.2)} \) and \( \text{(B.3)} \) has the following solution

\[
U^t U^t = \frac{1}{1 - \frac{\kappa \rho_{\text{dust}}}{3} r^2} \tag{B.4};
\]

\[
U^r U^r = -\frac{\kappa \rho_{\text{dust}}}{3} \frac{r^2}{1 - \frac{\kappa \rho_{\text{dust}}}{3} r^2}. \tag{B.5}
\]

We have chosen the solution noticing that \( U^t U^t > 0 \) and \( U_r U^r < 0 \). The next step is to isolate each component of the velocity; using the prescription \( (U_\mu)^2 = U_\mu U^\nu g_{\mu\nu}, \) for \( \mu = t, r \) we obtain

\[
U_t = \frac{|\partial_t \rho_{\text{dust}}|}{(3 \kappa \rho_{\text{dust}})^{1/2}}; \tag{B.6}
\]

\[
U_r = -\sqrt{\frac{\kappa \rho_{\text{dust}}}{3}} \frac{r}{1 - \frac{\kappa \rho_{\text{dust}}}{3} r^2}. \tag{B.7}
\]

A relevant comment is that if the temporal Einstein equation

\[
G^t_t = \kappa \frac{\partial_t \rho_{\text{dust}} + 3 \rho}{3} = \kappa \rho \frac{U_t}{U^t}, \tag{B.8}
\]

is combined with the corresponding velocity in equation \( \text{(B.4)} \), and then one solves for the radial derivative one gets \( \text{(33)} \).

Finally
\[
T_t' = \frac{3 \rho_{dust}}{3 - \kappa \rho_{dust}} \frac{1}{r^2} \\
T_r' = \frac{9 \kappa \rho_{dust}^2}{|\partial_t \rho_{dust}| (3 - \kappa \rho_{dust}^2)} \frac{1}{r} \\
T_r' = -\frac{1}{3} \frac{\partial_t \rho_{dust}}{\rho_{dust}} \frac{1}{r^2} \\
T_r' = -\kappa \rho_{dust}^2 \frac{1}{3 - \kappa \rho_{dust}^2} \frac{1}{r^2}.
\]

\hspace{1cm} (B.9)

ORCID iDs

Jorge Alfaro https://orcid.org/0000-0002-2091-1325
Domènec Espriu https://orcid.org/0000-0002-2725-5279
Luciano Gabbanelli https://orcid.org/0000-0002-4864-3967

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