EULER’S DIFFERENCE TABLE AND DECOMPOSITION OF TENSOR POWERS OF ADJOINT REPRESENTATION OF $A_n$ LIE ALGEBRA

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ABSTRACT. By using of Euler’s difference table, we obtain simple explicit formula for the decomposition of $k$-th tensor power of adjoint representation of $A_n$ Lie algebra at $2k \leq n+1$.

1. INTRODUCTION

Decomposition of tensor products of representations of simple complex Lie algebras is complicate problem. For the Lie algebras of small rank $A_2, C_2, G_2$ and $A_3$ some results were obtained in recent papers [FGP1] - [FGP4] (see also references there). For the general case see, for example, papers [Ha], [BD] and references there.

In present note we consider the problem of decomposition of $k$-th tensor power of adjoint representation of $A_n$ Lie algebra at stability domain $2k \leq n + 1$, where the results dont depend on $n$ [Ma]. Using Euler’s difference table [Eu1]-[Eu2], (see also [Ri]) we obtain simple explicit formulas (19), (20) for such decomposition, which on best author knowlidge are new one.

2. EULER’S DIFFERENCE TABLE

In 1753 Euler with relation to the card game "Jeu de Recontre" investigated in details the permutations of $k$ numbers $[1,2,...,k]$ without fixed points [Eu1], [Eu2] (see also [Ri]). Such permutation has the name derangement and we denote the number of derangements as $d_k$.

In order to find derangement numbers $d_k$ Euler constructed first the difference table from numbers $e^j_k, j \leq k$, that determined by basic recurrence relation

\[ e^j_k = e^{j+1}_k - e^{j-1}_k, \quad e^0_j = j! , \]

and proved that

\[ e^0_k = d_k . \]

Note that $e^j_k$ is divided to $j!$ and it is useful to introduce higher derangement numbers

\[ d^j_k = \frac{e^j_k}{j!} . \]

For derangement numbers we have two basic recurrence formulae [9], [11]

\[ d_k = (k - 1)(d_{k-1} + d_{k-2}), \quad d_0 = 1, d_1 = 0, \]
\[ d_k = k d_{k-1} + (-1)^k, \quad d_0 = 1, \]

\[ \]

\[ ^{1}\text{For the basic notations of Lie algebras see for example [Ov].} \]
and the generating function

\[
e^{-x} \frac{1}{1 - x} = \sum_{k=0}^{\infty} \frac{d_k}{k!} x^k.
\]

The first ten derangement numbers are

\[
\begin{array}{ccccccccccc}
  k & | & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
  d_k & | & 1 & 0 & 1 & 2 & 9 & 44 & 265 & 1854 & 133496 & 1334961 \\
\end{array}
\]

From (1) follows recurrence relation for numbers \(d_n^k\)

\[
d_n^k = \frac{1}{k} (d_{n-1}^{k-1} + d_{n-1}^{k-1}).
\]

Iterating it we obtain

\[
d_n^k = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} d_{n-j}.
\]

Let us give also the generating function

\[
\sum_{n=0}^{\infty} \frac{d_{n+k}}{n!} x^n = \frac{e^{-x}}{(1 - x)^{k+1}}
\]

and the tables of numbers \(e_n^k\) and \(d_n^k\)

\[
\begin{array}{cccccccccccc}
  k & | & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
  d_k & | & 1 & 0 & 1 & 2 & 9 & 44 & 265 & 1854 & 133496 & 1334961 \\
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\end{array}
\]
3. Decomposition of tensor powers

It is convenient to consider adjoint representation \( ad \) of Lie algebra \( A_n \) as tensor

\[
X_i^j, \quad i, j = 1, \ldots, n + 1,
\]

which satisfies the condition

\[
\sum_{i=1}^{n+1} X_i^i = 0.
\]

The \( k \)th tensor power of \( ad \)

\[
X_k = X_{i_1}^{j_1} X_{i_2}^{j_2} \cdots X_{i_k}^{j_k}
\]

has the decomposition

\[
X_k = c_k^0 Y_0 + c_k^1 Y_1 + \cdots + c_k^k Y_k.
\]

The quantities \( Y_p, p = 0, 1, \ldots, k \) decompose into irreducible representations of Lie algebra \( A_n \)

\[
Y_p = R_p + \ldots, \quad R_p = X[p, 0, \ldots, 0, p].
\]

The coefficients of terms denoted as ... are the Littlewood - Richardson coefficients, see [Ma].

The quantity \( Y_p \) obtained by contractions in (15) \( k - p \) upper indeces with \( k - p \) lower indeces. So we choose ordered subset of \( p \) quantities \( X_i^j \) from \( k \) such quantities. The number of such subsets is \( \frac{k!}{p!(k-p)!} \). Then we choose from \( p \) quantities \( X_i^j \) \( l \) quantities of type \( X_i^i \). This gives factor \( \binom{p}{l} \). The contraction on rest \( k - l \) indeces gives the quantity \( d_{k-l} \). As result we have the formula

\[
c_k^p = \binom{k}{p} \frac{1}{p!} \sum_{l=0}^{p} \binom{p}{l} d_{k-l}.
\]

Taking into account (9) we obtain the main formula of present note

\[
c_k^j = \binom{k}{j} d_k^j,
\]

where \( \binom{k}{j} \) are binomial coefficients, \( d_k^j \) are higher derangement numbers (3).

Note that coefficients \( c_k^j \) satisfied the recurrence relation

\[
c_{j+1}^k = \frac{1}{(j+1)^2} \left( (k-j) c_j^k + k c_j^{k-1} \right), \quad c_0^k = d_k,
\]

which is consequence of recurrence relation (8).
4. Conclusion

In conclusion we give the decomposition of first ten powers of adjoint representation of Lie algebra $A_n$ at $2k \leq n + 1$.

\begin{align*}
X_1 &= Y_1, \\
X_2 &= Y_0 + 2Y_1 + Y_2, \\
X_3 &= 2Y_0 + 9Y_1 + 6Y_2 + Y_3, \\
X_4 &= 9Y_0 + 44Y_1 + 42Y_2 + 12Y_3 + Y_4, \\
X_5 &= 44Y_0 + 265Y_1 + 320Y_2 + 130Y_3 + 20Y_4 + Y_5, \\
X_6 &= 265Y_0 + 1854Y_1 + 2715Y_2 + 1420Y_3 + 315Y_4 + 30Y_5 + Y_6, \\
X_7 &= 1854Y_0 + 14833Y_1 + 25494Y_2 + 16275Y_3 + 4690Y_4 + 651Y_5 + 42Y_6 + Y_7, \\
X_8 &= 14833Y_0 + 133496Y_1 + 263284Y_2 + 198184Y_3 + 70070Y_4 + 12712Y_5 + 1204Y_6 + 56Y_7 + 56Y_8, \\
X_9 &= 133496Y_0 + 1334961Y_1 + 2970288Y_2 + 2573508Y_3 + 1076544Y_4 + 240534Y_5 + 29904Y_6 + 2052Y_7 + 72Y_8 + Y_9, \\
X_{10} &= 1334961Y_0 + 14684570Y_1 + 36377685Y_2 + 35636040Y_3 + 17199210Y_4 + 4558428Y_5 + 699930Y_6 + 63240Y_7 + 3285Y_8 + 90Y_9 + Y_{10}.
\end{align*}

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