Infinite Minkowski sums of lattice polyhedra

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Abstract

Artinian integrally closed monomial ideals are characterized by their Newton polyhedra, which are lattice polyhedra inside the positive orthant having the positive orthant as their recession cone. Multiplication of such ideals correspond to Minkowski addition of their Newton polyhedra. In two dimensions, the isomorphic monoids of artinian, integrally closed monomial ideals under multiplication, or the class of lattice polyhedra described above, under Minkowski addition, are free abelian, as proved by Crispin-Quinonez.

Bayer and Stillman considered so-called monomial submodules of the Laurent polynomial ring. Inspired by this, we consider a family of such monomial submodules that can be (uniquely) expressed as an infinite product of monomial submodules isomorphic to integrally closed monomial ideals. Geometrically, their Newton polyhedras are expressed as an infinite Minkowski sum of simple lattice polyhedra. This gives another example of a topological ufd i.e. a topological abelian monoid in which every element can be uniquely written as a convergent (possibly infinite) product of irreducibles.

1 Introduction

Zariski [19, 18] showed that the set of \( m \)-primary, integrally closed ideals in a local, two-dimensional domain constitutes a free abelian monoid under multiplication. Crispin Quinonez [4, 12, 13] and Gately [7] studied integrally closed monomial ideals in two and three variables. The main result in [12] is an explicit description of the unique factorization of \( m \)-primary, integrally closed monomial ideals in two variables into simple ideals.

It is well known [5, 17] that the integral closure of a monomial ideal \( I \subset \mathbb{C}[x] \) is the ideal generated by the monomials whose exponent vectors lie in the convex hull (in \( \mathbb{R}^n \)) of \( \log(I) \subset \mathbb{N}^n \subset \mathbb{R}^n \), where \( \log(I) \) denotes the set of exponent vectors of monomials
in $I$. Hence, $m$-primary integrally closed monomial ideals can be identified with certain lattice polyhedra in $\mathbb{R}^n_+$ whose complement has finite volume. Since multiplication of ideals correspond to Minkowski addition of their polyhedra, the above result for two-dimensional monomial ideals can be stated as follows: the monoid (under Minkowski addition) of lattice polyhedra in $\mathbb{R}^2_+$ which

• are stable under Minkowski addition with $\mathbb{R}^2_+$,
• have complement to $\mathbb{R}^2_+$ which has finite volume,

is free abelian.

In the study of resolutions of monomial ideals, so-called monomial modules occur naturally [1]. A monomial module $M$ is a $\mathbb{C}[x]$-submodule of the ring $\mathbb{C}[x, x^{-1}]$ of Laurent polynomials which is generated by Laurent monomials $x^\alpha$, $\alpha \in \mathbb{Z}^n$.

We call such a monomial module integrally closed if the convex hull of $\log(M)$ contains no additional lattice points. Then a finitely generated, integrally closed monomial modules is isomorphic to a unique integrally closed ideal. However, there are plenty of non-finitely generated monomial modules, even if we impose the restriction that these modules should be what Bayer and Sturmfels [1] call co-artinian: i.e., containing no infinite descending (w.r.t. divisibility) sequence of Laurent monomials. In what way, may one ask, do integrally closed, co-artinian monomial modules factor? Since the corresponding question for monomial ideals is solved for $n = 2$ but open for larger $n$, it is reasonable to restrict to the case $n = 2$.

A monomial module $M$ is in some sense the limit of the monomial ideals obtained by intersecting its Newton polyhedron with translates of the positive quadrant, $I_a(M) = x^a y^a M \cap \mathbb{C}[x, y]$. If $M$ is the co-artinian monomial module with minimal generators $x^a y^{-a}$, $a \in \mathbb{Z}$, then the sequence of monomial ideal obtained in this way consists of powers of the maximal ideal $(x, y)$. Thus, the factorization of $M$ into simple integrally closed modules would, if it existed, be $M = (x, y)^\infty$. See Figure 1.

![Figure 1: An infinite power of the maximal ideal](image-url)
The problem here is that the Newton polyhedron \( \text{New}(M) \) has one unbounded face not parallel with the \( x \) or \( y \) axis. If we restrict to (generalized) polyhedra which have \( \mathbb{R}^2_+ \) as their recession cone, this particular problem does not arise. As we shall see, if we restrict ourselves even further, to a very particular type of integrally closed monomial submodules of \( \mathbb{C}[x, x^{-1}, y, y^{-1}] \), they can in fact be uniquely factored into a convergent, countable product of (translations of) simple integrally closed monomial ideals. This yields another example of a topological ufd, which is the author’s term for abelian topological monoid in which every element can be uniquely written as a convergent, possibly infinite, product of irreducibles \[16, 14, 15\].

**Example 1.** For a concrete example, consider the monomial module \( N \) with minimal generators

\[
\{1\} \cup \{ x^{1+2+\cdots+r}y^{-r} \mid r \in \mathbb{N}_+ \},
\]

and let \( M \) be its integral closure. Then

\[
N \simeq \prod_{i=1}^{\infty} E_i,
\]

where \( E_i \) is the integral closure of the monomial ideal \((x^i, y)\). For the corresponding Newton polyhedra, this becomes

\[
\text{New}(N) = \sum_{i=1}^{\infty} \text{New}(E_i),
\]

and infinite Minkowski sum of polyhedra. This is illustrated in Figure 2.

![Figure 2: The Newton polyhedra of a non-finitely generated integrally closed monomial module, expressed as an infinite Minkowski sum of simple polyhedra.](image)

**2 Monomial modules and monoid submodules**

The additive monoid \( \mathbb{N}^2 \) is a submonoid of \( \mathbb{Z}^2 \), its difference group (sometimes called Grothendieck group). This gives an inclusion of semigroup algebras \( \mathbb{C}[\mathbb{N}^2] \subset \mathbb{C}[\mathbb{Z}^2] \), which gives the Laurent polynomial ring \( \mathbb{C}[\mathbb{Z}^2] \) a natural \( \mathbb{C}[\mathbb{N}^2] \) module structure.

Note that \( \mathbb{C}[\mathbb{N}^2] = \mathbb{C}[x, y] \) and that \( \mathbb{C}[\mathbb{Z}^2] = \mathbb{C}[x, x^{-1}, y, y^{-1}] \). We will write \( T = \{ x^ay^b \mid (a, b) \in \mathbb{Z}^2 \} \) and \( S = \{ x^ay^b \mid (a, b) \in \mathbb{N}^2 \} \) for the corresponding, isomorphic but multiplicatively written, monoids, and use \( \log : T \rightarrow \mathbb{Z}^2 \) and \( \exp : \mathbb{Z}^2 \rightarrow T \) for the natural
isomorphisms. We will also view $T$ as a subset (or indeed a $C$-basis) of $C[Z^2]$ and $S$ as a subset (or indeed a $C$-basis) of $C[N^2]$, and alternatively write these semigroup rings as $C[T] = C[Z^2]$ and $C[S] = C[N^2]$.

A monomial ideal $I \subset C[S]$ is an ideal generated by its monomials $I \cap S$; thus $I$ is the $C$-vector space span of $I \cap S$. The corresponding monoid ideal $(I \cap S) \subset S$ is a filter with respect to the divisibility order on $S$; similarly $\log(I \cap S) \subset \mathbb{N}^2$ is a filter with respect to the cartesian order $\leq$ on $\mathbb{N}^2$. An equivalent definition of a monoid ideal $J \subset S$ is to demand that $SJ \subset J$.

The integral closure $\bar{I}$ of $I$ is the set of all elements $u \in C[S]$ such that
\[
u^m + a_1 \nu^{m-1} + \cdots + a_{m-1} \nu + a_m = 0, \quad a_i \in I.
\]
This is a monomial ideal so the $C$-span of $\bar{I} \cap S$; the monoid ideal $\bar{J} = \bar{I} \cap S$ is the integral closure of $J = I \cap S$, where the integral closure of a monoid ideal is
\[
\bar{J} = \{ \alpha \in \mathbb{N}^2 | \exists r > 0 : r\alpha \in rJ \};
\]
here $r$ should be an integer.

It holds that
\[
\log(\bar{J}) = \text{conv}(\log(J)) \cap \mathbb{N}^2
\]
where the convex hull is taken inside $\mathbb{R}^2$. The ideal $I$, (or the monoid ideal $J$), is closed whenever it is equal to its closure.

The so-called Newton polyhedra of a monomial ideal $I \subset C[S]$ is
\[
\text{New}(I) = \text{conv}(\log(I \cap S)) \subset \mathbb{R}^2;
\]
it is in fact contained in the positive quadrant $\mathbb{R}^2_+$. Note that $I$ and $\bar{I}$ have the same Newton polyhedra. If $I$ is artinian (equivalently, $m$-primary) then $S \setminus (I \cap S)$ is finite, and $\text{vol}(\mathbb{R}^2_+ \setminus \text{New}(I)) < \infty$, and the recession cone of $\text{New}(I)$ is $\mathbb{R}^2_{++}$. For the latter condition to hold, it is enough that $I$ contains some monomial which is not a power of $x$ and some monomial which is not a power of $y$.

For $P, Q$ two polyhedra in $\mathbb{R}^2$, their Minkowski sum is
\[
P + Q = \{ a + b | a \in P, b \in Q \}.
\]
It holds that
\[
\text{New}(I_1 I_2) = \text{New}(I_1) + \text{New}(I_2)
\]
whenever $I_1, I_2$ are integrally closed monomial ideals.

When $J \subset S$ is a monoid ideal, the set $\text{mg}(J)$ of its minimal elements with respect to divisibility is finite, and
\[
J = \bigcup_{m \in \text{mg}(J)} mS.
\]
Similarly, we define the minimal generators of a monomial ideal $I$ as the monomials corresponding to the minimal generators of $J = I \cap S$; they do indeed generate $I$ minimally as an ideal of $C[S]$. When $I$ is integrally closed, its minimal generators corresponds to the vertices of the Newton polyhedron.
Bayer and Stillman \cite{1} called a $\mathbb{C}[S]$-submodule of $\mathbb{C}[T]$ a \textit{monomial module} and used such objects in their study of homological properties of monomial ideals. If $M \subset \mathbb{C}[T]$ is such a monomial module, then it is the $\mathbb{C}$-span of $L = M \cap T$. The subset $L \subset T$ is not a monoid ideal of $T$, nor a submodule; it is however a filter with respect to divisibility, and satisfies $SL \subset L$ (but \textbf{not} $TL \subset L$). We say that $L$ is an $S$-submodule of the free abelian group $T$. We define the integral closure of $L$ as

$$\bar{L} = \{ \alpha \in \mathbb{Z}^2 \mid \exists r > 0 : r\alpha \in rJ \}$$  \hspace{1cm} (4)

It holds, similar to the monoid ideal case, that

$$\log(\bar{L}) = \text{conv}(\log(L)) \cap \mathbb{Z}^2.$$  \hspace{1cm} (5)

We will in this article refer to $\bar{M}$, the $\mathbb{C}$-span of $\bar{L}$, as the integral closure of $M$, and call $\text{conv}(\log(L)) \subset \mathbb{R}^2$ the Newton polyhedron of $M$, even though it strictly speaking need not be a polyhedron, since it can be the intersection of infinitely many halfplanes.

Following Bayer and Stillman, we call a monomial module $M$ \textit{co-artinian} if $L = M \cap T \subset T$ contains no infinite descending (with respect to divisibility) chain; equivalently, if $L$ contains no infinite principal order ideal. A co-artinian monomial module can still extend infinitely far “to the left”, as Figure 3 shows.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{A co-artinian monomial module}
\end{figure}

3 \textbf{Factorization of integrally closed monomial ideals}

The following are known for monomial ideals in two variables and three variables: \cite{13, 17, 12, 4, 7}:

- the integral closure of a monomial ideal is a monomial ideal,
- the product of integrally closed monomial ideals is an integrally closed monomial ideal,
• multiplication of integrally closed monomial ideals correspond to Minkowski addition of their Newton polyhedra,

• in two variables, the monoid of integrally closed artinian monomial ideals under ideal multiplication is a free abelian monoid of infinite rank, with basis given by the simple ideals $E_{r,s} = (x^r, y^s)$, $\gcd(r, s) = 1$,

• in three variables, factorization is no longer unique, nor is the length of factorizations of a given ideal always the same.

In Figure 4 we illustrate the factorization

$$E_{4,3}E_{5,2} = (x^4, y^3)(x^2, y^2) = (x^9, x^4y^2, y^5).$$

![Figure 4: Factorization of $I = (x^9, x^4y^2, y^5)$](image)

The corresponding monoid ideals, Newton polyhedron, and the boundary of the Newton polyhedron, is shown in Figure 5.

![Figure 5: Associated monoid ideal and Newton polyhedra of $I = (x^9, x^4y^2, y^5)$](image)

### 4 Infinite factorization

We say that two monomial submodules $M_1, M_2$ are equivalent if $M_2 = mM_1$ for some $m \in T$. If $M_1, M_2$ are integrally closed, this is happens iff there is an integral vector translating one Newton polyhedron into the other. A finitely generated monomial module $M$ is thus equivalent with a monomial ideal. Furthermore, if $M$ is integrally closed and $\text{New}(M)$ has recession cone $\mathbb{R}_+^2$, we can translate it so that “aligns” with $\mathbb{R}_+^2$, and so $M$ is equivalent to a artinian integrally closed monomial ideal.

To exhibit the phenomena of infinite factorization, we will restrict to co-artinian integrally closed monomial modules $M$, with monoid submodule $L = M \cap T$, and Newton polyhedra $P$, such that
1. $L$ contains $\mathbb{N}$ but no lattice point strictly smaller than $\text{vek}0$, i.e., nothing in the third and fourth quadrant, (except 0).

2. $P = \text{New}(M)$ contains no infinite ray $\{t(a, b) | t \geq 0\}$ with $b < 0 < a$.

3. There is no infinite facet of $P$ except the positive $y$-axis and (possibly) an infinite ray parallel with the $x$-axis.

4. The finite facets are line segments joining vertices $(c, d)$ and $(c + k, d - 1)$ with $d < 0 < c$, i.e., they have outward normals $(-1, -k)$.

An example of an acceptable monomial module is the right-most part of Figure 2.

In what follows we will be using some facts and notions about topological spaces, topological monoids, and topological groups; references may be found in [2, 3, 10].

**Definition 2.** We denote this collection of monomial modules by $\mathcal{A}$, and the denote by $\mathcal{A}_n$ the subcollection consisting of those $M \in \mathcal{A}$ where the outward normals of $\text{New}(M)$ belong to $\{(-1, -k) | 1 \leq k \leq n\}$.

**Lemma 3.** If $M \in \mathcal{A}_n$ then $P = \text{New}(P)$ has an infinite face with outward normal $(0, -1)$, and is contained in the strip $\{(a, b) | b \geq -c\}$ for some non-negative integer $c$.

**Proof.** Obvious. □

**Theorem 4.** For any positive integer $n$, the set $\mathcal{A}_n$ is (under multiplication of monomial modules) a free abelian module of rank $n$.

**Proof.** Take $M \in \mathcal{A}_n$. Then there is a unique non-negative integer $c$ such that $y^c M$ is equivalent to an artinian, integrally closed ideal $I \subset \mathbb{C}[S]$. It follows from Crispin-Quinonez characterization of such ideals [12, 13, 4] that $I = \prod_j E_{(r_j, s_j)}$. This factorization is unique, and the factors can be read off from the outward normals; thus $r_j \in \{1, 2, \ldots, n\}$ and $s_j = 1$. We get that

$$I = \prod_{j=1}^{n} E_{(j,1)}^{a_j}, \quad a_j \in \mathbb{N}. \tag{6}$$

In other words, $I$ belongs to the submonoid generated by $E_{(1,1)}, E_{(2,1)}, \ldots, E_{(n,1)}$; this is a submonoid of a free abelian monoid generated by $n$ different irreducible elements, hence is isomorphic to a free abelian module with these generators as a basis.

To each $E_{(j,1)}$ with $1 \leq j \leq n$ we associate the monomial module $F_j = y^{-1} E_{(j,1)}$, which is $I$ translated so that the minimal generator $y$ of $I$ is placed on the origin. It is clear that $F_j \in \mathcal{A}_j \subset \mathcal{A}_n$. Hence our original $M \in \mathcal{A}_n$ satisfies

$$M = \prod_{j=1}^{n} F_j^{a_j}, \quad a_j \in \mathbb{N}. \tag{7}$$

□
Example 5. The monomial module $M$ minimally generated by $\{1, xy^{-1}, x^3y^{-2}, x^6y^{-3}\}$ factors as

$$M = F_1 F_2 F_3,$$

as can be seen in Figure 2.

We now turn to the question about factorization of monomial modules in $A$.

Definition 6. If $M, N \in A$ we put $d(M, N) = 2^{-s}$ where

$$s = \min\{(a, b) \in \log((M \Delta N) \cap T)\} a^2 + b^2,$$

i.e. $s$ is the smallest squared distance to the origin for a lattice point in the symmetric difference of $M$ and $N$.

Proposition 7. The function $d$ is a metric on $A$. A sequence $(N_j)_{j=1}^{\infty}$ of monomial modules in $A$ converges to $N \in A$ with respect to $d$ if and only if

$$\forall \alpha \in T : \exists v(\alpha) : \alpha \in P \iff \forall \ell > v(\alpha) : \alpha \in P_\ell \quad (8)$$

Furthermore, the topology induced by $d$ on $A$ turns it into a topological monoid, and each $A_n \subset A$, when equipped with the subspace topology, is discrete.

Proof. Obvious. $\square$

It is clear that the above proposition holds when “sequence” is replaced by “net”. Thus, we get:

Proposition 8. Let $A \subset A$, $N \in A$. Then

$$\prod_{M \in A} M = N$$

if and only if, for every $\alpha \in T$ there is some finite subset $B \subset A$ such that, for any finite $B \subset C \subset A$, $\alpha \in N$ if and only if $\alpha \in \prod_{K \in C} K$.

Definition 9. For $M \in A$, a positive integer, let $M(n)$ denote the monomial module generated by monomials $x^a y^b$ with $b \geq -n$. Let $\rho_n(M) = M(c(n))$ for the smallest $c(n)$ such that $\text{New}(M(c(n)))$ contains no boundary segment with outward normal $(-k, -1)$ with $k > n$. Then $\rho_n(M) \in A_n$ and hence

$$\rho_n(M) = \prod_{j=1}^{n} w_M(j, n)$$

for some non-negative integers $w_M(j, n)$. Since $w_M(j, n) = w_M(j, s)$ whenever $j \leq n, s$, we call this common value $w_M(j)$.
Theorem 10. The function
\[ \rho_n : \mathcal{A} \to \mathcal{A}_n \]  
(9)
is a continuous, surjective monoid homomorphism. For \( s > t \), the maps
\[ \rho_{s,t} : \mathcal{A}_s \to \mathcal{A}_t \]
\[ \rho_{s,t}(M) = M[t] \]
(10)
are surjective monoid homomorphisms between discrete, free abelian monoids of finite rank. Furthermore, for \( s > t > r \) it holds that \( \rho_{t,r} \circ \rho_{s,t} = \rho_{s,r} \). It also holds that \( \rho_{s,t} \circ \rho_s = \rho_t \). Consequently, we have an inverse system of discrete, free abelian monoids of finite rank, indexed over the positive integers. For the inverse limit of this system, we have that
\[ \lim_{\substack{n \to \infty \n}} \mathcal{A}_n \simeq \mathcal{A} \]
(11)
as topological monoids.

The author studied inverse limits of discrete, free abelian monoids (not necessarily of finite rank). The resulting topological monoids, called topological ufd’s, have the property that any element can be uniquely factored as a convergent (possibly infinite) product of irreducible elements. Examples of such topological ufd’s, and similar topological monoids having infinite factorizations, are

- Halter-Koch’s [8, 9] “formal polynomials” are \( \lim_{\bullet} k[x_1, \ldots, x_n] \) where the truncations are the ring homomorphisms setting the last variable to zero
- Entire functions with prescribed sets of zeroes, under Weierstrass factorization
- Entire functions with prescribed sets of poles, as per the Mittag-Leffler theorem

The purpose of this article has been to exhibit yet another example of a topological ufd. Hence:

Theorem 11. Any \( M \in \mathcal{A} \) can be uniquely written as a convergent (possibly infinite) product
\[ M = \prod_{j=1}^{\infty} F_j^{w_M(j)} \]
(12)
Similarly, \( \text{New}(M) \) is in a unique way expressable as a convergent Minkowski sum of the generalized lattice polyhedra \( N(F_j) = \mathbb{R}_+^2 + \ell_j \), where \( \ell_j \) is the line segment from the origin to the lattice point \((j, -1)\).

In Figure 6 one such \( N(F_j) \) is shown.

Example 12. The first partial sums of the generalized polyhedron \( \sum_{j=1}^{\infty} N(F_j) \) are shown in figure 7.
Figure 6: \((x^2, y^2) = y^2 N(F^2_1)\) as the Minkowski sum of line segment and \(\mathbb{R}^2_+\).

Figure 7: Infinite Minkowski sum of generalized lattice polyhedra.

5 Further questions

1. This manuscript considers a very restricted set of monomial modules. At the very least, one should be able to easily treat products of (translates of) \(E(k, 1)\) and \(E(1, k)\), by anchoring the vertex where the slope of the boundary of the Newton polyhedra increases past 1 to the origin; see figure 8.

2. For a general co-artinian \(M\), one could guess that the set of slopes are a subset of \(\mathbb{Q}\) which is the image of an order-preserving map \(\mathbb{Z} \rightarrow \mathbb{Q}\). Such an \(M\) should be the product of the corresponding simple monomiod modules, suitably translated.

3. It is probably not possible to assign meaning to something like \(\prod_{q \in \mathbb{Q}} F_q\).

4. Infinite exponents, such as in \(M = (x, y)\infty\), shown in Figure 1, pose a problem that may or may not be surmountable.

5. In higher dimensions factorization is no longer unique [7].

6. Lewis [11] recently studied factorization of integrally closed monomial ideals via Newton Polyhedra and the so-called integral polytope group, also studied by Funke [6]. An important aspect of those works is factoring out with the equivalence relation stemming from translations. We have attached the monomial modules we considered to the origin in a particular way, in effect choosing representatives; perhaps it is better not to?
Figure 8: Infinite Minkowski sum of generalized lattice polyhedra, properly translated

6 Bibliography

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