ASYMPTOTIC ESTIMATES FOR LARGEST VOLUME RATIO
OF A CONVEX BODY

DANIEL GALICER, MARIANO MERZBACHER, AND DAMIÁN PINASCO

Abstract. The largest volume ratio of given convex body $K \subset \mathbb{R}^n$ is defined as
$$\text{ivr}(K) := \sup_{L \subset \mathbb{R}^n} \text{vr}(K, L),$$
where the sup runs over all the convex bodies $L$. We prove the following sharp lower bound
$$c \sqrt{n} \leq \text{ivr}(K),$$
for every body $K$ (where $c > 0$ is an absolute constant). This result improves the former bound, of order $\sqrt{\frac{n}{\log \log(n)}}$.

We also study the exact asymptotic behavior of the largest volume ratio for some natural classes. In particular, if $K$ is the unit ball of an unitary invariant norm in $\mathbb{R}^{d \times d}$ (e.g., the unit ball of the $p$-Schatten class $S^p_d$ for any $1 \leq p \leq \infty$) or $K$ is unconditional, we show that $\text{ivr}(K)$ behaves as the square root of the dimension of the ambient space.

1. Introduction

For many applications in asymptotic geometric analysis, convex geometry or even optimization it is useful to approximate a given convex body by another one. For example, the classical Rogers-Shephard inequality [1, Theorem 1.5.2] states that, for a convex body $K \subset \mathbb{R}^n$, the volume of the difference body $K - K$ is “comparable” with the volume of $K$. Precisely, $|K - K|^\frac{1}{n} \leq 4|K|^\frac{1}{n}$ where $| \cdot |$ stands for the $n$-dimensional Lebesgue measure. Rogers and Shephard also showed, with the additional assumption that $K$ has barycenter at the origin, that the intersection body $K \cap (-K)$ has “large” volume. Namely, $|K \cap (-K)|^\frac{1}{n} \geq \frac{1}{2}|K|^\frac{1}{n}$. These inequalities imply that any given body is enclosed by (or contains) a symmetric body whose volume is “small” (“large”) enough. In many cases this allows us to take advantage of the symmetry of the difference body (or the intersection body) to conclude something about $K$.

Another interesting example of Milman and Pajor [23, Section 3] shows that

$$L_K \leq c \inf \left\{ \left( \frac{|W|}{|K|} \right)^\frac{1}{n} : W \text{ is unconditional and contains } K \right\},$$

2010 Mathematics Subject Classification. 52A23, 52A38, 52A40 (primary); 52A21, 52A20, 47B10 (secondary).

Key words and phrases. Volume Ratio, Unconditional Convex Bodies, Schatten Classes, Random Polytopes.

The second author was supported by a CONICET doctoral fellowship. This was partially supported by CONICET PIP 11220130100329, CONICET PIP 11220090100624, ANPCyT PICT 2015-2299.
where $L_K$ stands for the isotropic constant of $K \subset \mathbb{R}^n$ (see [6, Section 2.3.1]) and $c > 0$ is an absolute constant. Therefore, having a good approximation of $K$ by an unconditional convex body provides structural geometric information of $K$.

Perhaps the most notable application of these kind of approximations can be viewed when studying John/Löwner ellipsoid (maximum/minimum volume ellipsoid respectively). For example, if the Euclidean ball is the maximal volume ellipsoid inside $K$, we can decompose the identity as a linear combination of rank-one operators defined by contact points [1, Theorem 2.1.10]. This decomposition plays a key role in the study of distances between bodies, see [29] for a complete treatment on this. We also refer to [22, 13, 10, 20, 26] for many nice results/applications which involve these extremal ellipsoids. A natural quantity that relates a given body $K$ with its ellipsoid of minimal volume is given by the “standard” volume ratio

\[ \operatorname{vr}(K) = \inf \left\{ \left( \frac{|K|}{|\mathcal{E}|} \right)^{\frac{1}{n}} : \mathcal{E} \text{ is an ellipsoid contained in } K \right\} . \] (2)

Using the Brascamp-Lieb inequality, Ball showed that $\operatorname{vr}(K)$ is maximal when $K$ is a simplex. The extreme case, among all the centrally symmetric convex bodies, is given by the cube (see [1 Theorem 2.4.8]).

A natural generalization of this ratio is given by the following definition introduced by Giannopoulos and Hartzoulaki [9] and also studied by Gordon, Litvak, Meyer and Pajor [12]: given two convex bodies $K$ and $L$ in $\mathbb{R}^n$ the volume ratio of the pair $(K, L)$ is defined as

\[ \operatorname{vr}(K, L) := \inf \left\{ \left( \frac{|K|}{|T(L)|} \right)^{\frac{1}{n}} : T(L) \text{ is contained in } K \right\} , \] (3)

where the infimum (actually a minimum) is taken over all affine transformations $T$. In other words, $\operatorname{vr}(K, L)$ measures how well can $K$ be approximated by an affine image of $L$. Note that the classic value $\operatorname{vr}(K)$ is just $\operatorname{vr}(K, B_2^n)$ where $B_2^n$ is the Euclidean unit ball in $\mathbb{R}^n$.

Given a convex body $K$, it is natural to ask how “good” an approximation of this kind can be (in terms of the dimension of the ambient space). Namely, we want to known how large the value $\operatorname{vr}(K, L)$ is (for arbitrary convex bodies $L \subset \mathbb{R}^n$). Thus, it is important to compute the largest volume ratio of $K$, given by

\[ \operatorname{lvr}(K) := \sup_{L \subset \mathbb{R}^n} \operatorname{vr}(K, L), \]

where the sup runs over all the convex bodies $L$.

Khrabrov (based on the well-known construction due to Gluskin [11] provided to understand the diameter of Minkowski compactum), showed in [16, Theorem 5] the following:

For any convex body $K$ in $\mathbb{R}^n$ there is another body $L \subset \mathbb{R}^n$ such that

\[ c \sqrt{\frac{n}{\log \log(n)}} \leq \operatorname{vr}(K, L) \] (4)

where $c > 0$ is an absolute constant. The body $L$ is found using the probabilistic method (Khrabrov considered a random polytope whose vertices are sampled on the unit sphere and showed that, with high probability, it verifies Equation (3)).
On the other hand, it is very easy to see that \( \text{vr}(K, L) \leq n \) for every pair \((K, L)\). Using Chevet’s inequality together with clever positions of \( K \) and \( L \), Giannopoulos and Hartzoulaki \cite{9} were able to prove the following important and stronger result:

Let \( K \) and \( L \) be two convex bodies in \( \mathbb{R}^n \). Then

\[
\text{vr}(K, L) \leq c \sqrt{n} \log(n),
\]

where \( c > 0 \) is an absolute constant.

Combining the results of Khravrov and Giannopoulos-Hartzoulaki, Equations \((4)\) and \((5)\), we get:

For any convex body \( K \) in \( \mathbb{R}^n \), its largest volume ratio verifies

\[
\sqrt{\frac{n}{\log \log(n)}} \ll \text{lvr}(K) \ll \sqrt{n} \log(n).
\]

The well known result of John \cite[Theorem 2.1.3]{1} asserts that for any convex body \( L \subset \mathbb{R}^n \) we have \( \text{vr}(B_n^2, L) \ll \sqrt{n} \). It is not difficult to see that \( \sqrt{n} \ll \text{vr}(B_n^2, B_n^1) \), where \( B_n^1 \) stands for the unit ball of \( \ell_1^n \) thus,

\[
\text{lvr}(B_n^2) \sim \sqrt{n}.
\]

In \cite[Theorem 1.3]{8} the authors of this article showed that any convex body \( L \subset \mathbb{R}^n \) can be inscribed in a simplex \( S \) such that \( |S|_{\ell_1^n} \leq c \sqrt{n} |L|_{\ell_1^n}, \) where \( c > 0 \) is an absolute constant. In other words, if \( S \) is a simplex then \( \text{vr}(S, L) \ll \sqrt{n} \), for every convex body \( L \subset \mathbb{R}^n \). Since the regular simplex is the minimal volume simplex that contains the Euclidean unit ball (see \cite[Example 2.7]{8}), by computing volumes we have \( \sqrt{n} \ll \text{vr}(S, B_n^2) \). Therefore, for a simplex \( S \), we know the exact asymptotic behaviour of its largest volume ratio:

\[
\text{lvr}(S) \sim \sqrt{n}.
\]

Therefore the largest volume ratio of a convex body, in many cases, behaves as the square root of the dimension (of the ambient space).

We prove the following lower bound, that substantially improves \((4)\).

**Theorem 1.1.** For any convex body \( K \) in \( \mathbb{R}^n \) there is another body \( L \subset \mathbb{R}^n \) such that

\[
c \sqrt{n} \leq \text{vr}(K, L)
\]

where \( c > 0 \) is an absolute constant. In other words,

\[
\sqrt{n} \ll \text{lvr}(K).
\]

Moreover, we show that there are “many” (with high probability) random polytopes \( L \) which verify equation \((6)\). As we saw before in the previous examples, this lower bound cannot be improved in general.

To obtain Theorem \((1.1)\) we make some important changes in Khravrov’s proof (which require finer estimates) and use a deep result of Paouris regarding the mass distribution of an isotropic convex body together with Klartag’s solution to the isomorphic slicing problem (which asserts that, given any convex body, we can find another convex body, with absolutely bounded isotropic constant, that is geometrically close to the first one).
We also deal, for some natural classes of convex bodies, with the upper bounds. Our results are of probabilistic nature, so we will be interested in obtaining bounds with high probability.

First we treat the case of the Schatten trace classes which are the non-commutative version of the classical $\ell_p$ sequence spaces. They consist of all compact operators on a Hilbert space for which the sequence of their singular values belongs to $\ell_p$. Many different properties of them in the finite dimensional setting have been largely studied in the area of asymptotic geometric analysis. For example, König, Meyer and Pajor [18] established the boundedness of the isotropic constants of the unit balls of $S^d_p \subset \mathbb{R}^{d \times d}$ ($1 \leq p \leq \infty$), Guédon and Paouris [14] also studied concentration mass properties for the unit balls, Barthe and Cordero-Erasquin [4] analyzed variance estimates, Radke and Vritsiou [28] proved the thin-shell conjecture, and recently Kabluchko, Prochno and Thäle [15] exhibited the exact asymptotic behaviour of the volume and standard volume ratio; just to mention a few.

Therefore it is natural to try to understand what happens with the largest volume ratio of the unit ball. The following theorem provides an answer to this query.

**Theorem 1.2.** Let $1 \leq p \leq \infty$ and $S^d_p \subset \mathbb{R}^{d \times d}$ be the $p$-Schatten class. The largest volume ratio of its unit ball, $B_{S^d_p}$, behaves as

$$lvr(B_{S^d_p}) \sim d.$$  

Moreover, we show that this also holds for the unit ball of any unitary invariant norm in $\mathbb{R}^{d \times d}$ (which follows from Theorem 1.1 and Corollary 4.3 below).

Our approach is based on Giannopoulos-Hartzoulaki’s techniques. We show that if $L \subset \mathbb{R}^n$ is an arbitrary body then with “high probability” we can find transformations $T$ such that $T(L) \subset S^d_p$ and

$$\left( \frac{|S^d_p|}{|T(L)|} \right)^{\frac{1}{d}} \ll d.$$  

Recall that a body $K$ is called unconditional if for every choice of signs $(\varepsilon_k)_{k=1}^n \subset \{-1, +1\}^n$, the vector $(\varepsilon_1 x_1, \ldots, \varepsilon_n x_n)$ lies in $K$ if and only if $(x_1, \ldots, x_n)$ is in $K$. In other words, $\| (\varepsilon_1 x_1, \ldots, \varepsilon_n x_n) \|_{X_K} = \| (x_1, \ldots, x_n) \|_{X_K}$ for every vector $x \in \mathbb{R}^d$ and every sequence of signs $(\varepsilon_k)_{k=1}^n$. We also study the asymptotic behaviour of the largest volume ratio for unconditional bodies.

**Theorem 1.3.** Let $K \subset \mathbb{R}^n$ be an unconditional convex body. Then,

$$lvr(K) \sim \sqrt{n}.$$  

The fact that $lvr(K) \leq \sqrt{n}$ if $K$ is unconditional might be known for experts (although, as far as we know, is not explicitly stated elsewhere) and is a consequence of a mixture of the existence of Dvoretzky-Rogers’ parallelepiped and a result of Bobkov-Nazarov. As a result of a theorem Pivovarov, we present a random version of Dvoretzky-Rogers’ parallelepiped construction (which we believe is interesting in its own right) and show that, if $K \subset \mathbb{R}^n$ is unconditional and $L \subset \mathbb{R}^n$ is an arbitrary body, then with “high probability” we can find transformations $T$ such that $T(L) \subset K$ and

$$\left( \frac{|K|}{|T(L)|} \right)^{\frac{1}{d}} \ll \frac{\sqrt{n}}{L_L},$$
where $L_{L^e}$ stands for the isotropic constant of the polar body $L_{L^e}$.

2. Preliminaries

If $(a_n)_n$ and $(b_n)_n$ are two sequences of real numbers we write $a_n \ll b_n$ if there exists a constant $c > 0$ (independent of $n$) such that $a_n \leq cb_n$ for every $n$. We write $a_n \sim b_n$ if $a_n \ll b_n$ and $b_n \ll a_n$. We denote by $e_1, \ldots, e_n$ the canonical vector basis in $\mathbb{R}^n$ and by $S^{n-1}$, the unit sphere in $\mathbb{R}^n$. We denote by $\text{absconv}\{X_1, \ldots, X_m\}$ the absolute convex hull of the vectors $X_1, \ldots, X_m$. That is,

$$\text{absconv}\{X_1, \ldots, X_m\} := \left\{ \sum_{i=1}^{m} a_i X_i : \sum_{i=1}^{m} |a_i| \leq 1 \right\} \subset \mathbb{R}^n.$$

A convex body $K \subset \mathbb{R}^n$ is a compact convex set with non-empty interior. If $K$ is centrally symmetric (i.e., $K = -K$) we denote by $X_K$ the norm space $(\mathbb{R}^n, \| \cdot \|_{X_K})$ that has $K$ as its unit ball.

The polar set of $K$, denoted by $K^\circ$, is defined as

$$K^\circ = \{ x \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } y \in K \}.$$

The following result relates the volume of a body with the volume of its polar and is due to Blaschke-Santalo and Bourgain-Milman [1, Theorem 1.5.10 and Theorem 8.2.2]: If $K$ is centrally symmetric then

$$|K|^\frac{1}{n} |K^\circ|^\frac{1}{n} \sim \frac{1}{n}. \quad (10)$$

A probability measure $\mu$ on $\mathbb{R}^n$ is isotropic if its center of mass is the origin

$$\int_{\mathbb{R}^n} \langle x, \theta \rangle d\mu(x) = 0 \text{ for every } \theta \in S^{n-1},$$

and

$$\int_{\mathbb{R}^n} \langle x, \theta \rangle^2 d\mu(x) = 1 \text{ for every } \theta \in S^{n-1}. \quad (11)$$

A convex body $K \subset \mathbb{R}^n$ is said to be in isotropic position (or simply, is isotropic) if it has volume one and its uniform measure is, up to an appropriate re-scaling, isotropic. In that case, its isotropic constant $L_K$, is given by

$$L_K := \left( \int_{K} x_i^2 \, dx \right)^{1/2}.$$

Given a convex body $K$ in $\mathbb{R}^n$ with center of mass at the origin, there exists $A \in GL(n)$ such that $A(K)$ is isotropic [1 Proposition 10.1.3]. Moreover, this isotropic image is unique up to orthogonal transformations; consequently, the isotropic constant $L_K$ results an invariant of the linear class of $K$. In some sense, the isotropic constant $L_K$ measures the spread of a convex body $K$.

For a centrally symmetric convex body $K \subset \mathbb{R}^n$, its $\ell$-norm is defined as

$$\ell(K) := \int_{\mathbb{R}^n} \| (x_1, \ldots, x_n) \|_{X_K} d\gamma(x),$$

where $d\gamma$ is the standard Gaussian probability in $\mathbb{R}^n$. For more information about this parameter see [29 Chapter 12]. Recall that $\ell(K) \sim \sqrt{n} w(K^\circ)$, where $w(\cdot)$ stands for the mean width (see [1 Chapter 1.5.5]).
Given a convex body $K$, $\text{Iso}(K)$ is the set of isometries of $K$, that is set of orthogonal transformations $O$ such that $O(K) = K$. We say that $K$ has enough symmetries if the only operator that commutes with every $T \in \text{Iso}(K)$ is the identity operator. A convex body with enough symmetries is almost in John position [2, Proposition 4.8]. More precisely, $\|id : \ell_2^d \rightarrow X_K\|^{-1}B_2^d$ is the maximal volume ellipsoid contained in $K$. That means that if $K$ has enough symmetries then

$$vr(K) = vr(K, B_2^d) \sim \|id : \ell_2^d \rightarrow X_K\|\frac{1}{\sqrt{n}|K|^{\frac{1}{n}}}.$$  

Natural examples of a convex bodies with enough symmetries are the unit balls of the Schatten classes. For every matrix $T \subset \mathbb{R}^{d \times d}$ consider $s(T) = (s_1(T), \ldots, s_d(T))$ the sequence of eigenvalues of $(TT^*)^{\frac{1}{2}}$ (the singular values of $T$). The $p$-Schatten norm of $T \in \mathbb{R}^{d \times d}$ is defined as

$$\sigma_p(T) = \|s(T)\|_{\ell_p^d};$$

that is, the $\ell_p$-norm of the singular values of $T$. The $p$-Schatten norm arises as a generalization of the classical Hilbert-Schmidt norm. The analysis of the Schatten norm has a long tradition in local Banach space theory and their properties are widely studied. We denote by $B_{S_p^d} \subset \mathbb{R}^{d \times d}$ the unit ball of $(\mathbb{R}^{d \times d}, \sigma_p)$. The norm $\sigma_p$ is one of the most important unitary invariant operator norms.

Recall that unitary invariant norm $N$ on $\mathbb{R}^{d \times d}$, that is a norm that satisfies $N(UTV) = N(T)$ for all $U, V \in O(d)$ (the group of distance-preserving linear transformations of a Euclidean space of dimension $d$). It is known that for any unitary invariant norm there is a $1$–symmetric norm $\tau$ such that for every $T \in \mathbb{R}^{d \times d}$

$$N(T) = \tau(s_1(T), \ldots, s_n(T)).$$

Assume that $\tau(e_i) = 1$ and set $u := \sum_{i=1}^d e_i$. By [6, Equation (4.3.6)]

$$\frac{1}{\tau(u)}S_{\infty}^d \subset B_N \subset \frac{d}{\tau(u)} S_1^d.  \tag{14}$$

Taking volumes we have that

$$\frac{1}{\tau(u)}|S_{\infty}^d|^{\frac{1}{d}} \leq \frac{d}{\tau(u)}|S_1^d|^{\frac{1}{d}}.$$

As $|S_{\infty}^d|^{\frac{1}{d}} \sim d|S_1^d|^{\frac{1}{d}}$ [9, Lemma 4.3.2], we conclude that

$$vr(B_N, S_{\infty}^d) \sim vr(S_1^d, B_N) \sim 1. \tag{15}$$

We now recall some basic properties of the volume ratio defined in Equation (3) which can easily be found in [16].

**Remark 2.1.** For every pair of centrally symmetric convex bodies $(K, L)$ in $\mathbb{R}^n$, the following holds:

1. $vr(K, L) = \left(\frac{|K|}{|L|}\right)^{\frac{1}{n}} \inf_{T \in SL(n, \mathbb{R})} \|T : X_L \rightarrow X_K\|,$

   where the infimum runs all over the linear transformations $T$ that lie on the special linear group of degree $n$ (matrices of determinant one).

2. $vr(K, L) \sim vr(L^o, K^o).$
(3) If \( T : X_L \to X_K \) is a linear operator we have that
\[
\frac{1}{\| T : X_L \to X_K \|} \cdot T(L) \subset K
\]
and so
\[
\vr(K, L) \leq \frac{\| T : X_L \to X_K \| |K|^{\frac{1}{n}}}{|\det T|^{\frac{1}{n}} |L|^{\frac{1}{n}}},
\]
(4) \( \vr(K, L) \leq \vr(K, Z) \cdot \vr(Z, L) \) for every convex body \( Z \) in \( \mathbb{R}^n \).
(5) \( \vr(K, L) = \vr(T(K), S(L)) \), for any affine transformations \( T \) and \( S \). In other words, the volume ratio between \( K \) and \( L \) depends exclusively on the affine classes of the bodies involved.

Notice that by Rogers-Shephard’s inequality, for every convex body \( L \subset \mathbb{R}^n \) we have \( \vr(L - L, L) \leq 4 \). Therefore, by the last property
\[
\vr(K, L) \leq \vr(K, L - L) \cdot 4.
\]
Thus, the largest volume ratio of the body \( K \) can be estimated by considering the sup over all symmetric bodies. Precisely,
\[
(16) \quad lvr(K) \leq 4 \sup_{L \subset \mathbb{R}^n} \vr(K, L),
\]
where the sup runs over all the centrally symmetric convex bodies \( L \). This will be useful since it allow us to deal only with bodies which are centrally symmetric.

3. Lower bound for the largest volume ratio

We now treat lower bounds for the largest volume ratio of a given convex body \( K \). Recall the statement of Remark 2.1 (1),
\[
\vr(K, L) = \left( \frac{|K|}{|L|} \right)^{\frac{1}{n}} \cdot \inf_{T \in SL(n, \mathbb{R})} \| T : X_L \to X_K \|.
\]
Therefore, to show “good” lower bounds for \( lvr(K) \) we need a body \( L \) such that its volume is “small” and the norm \( \| T : X_L \to X_K \| \) is large for every operator \( T \in SL(n, \mathbb{R}) \).

The key idea of [16] is to use the probabilistic method. Namely, Khrabrov considered the random body (based on Gluskin’s work [11])
\[
(17) \quad L^{(m)} := \text{absconv}\{X_1, \ldots, X_m, e_1, \ldots, e_n\},
\]
where \( \{X_i\}_{i=1}^m \) are independent vectors distributed according to the normalized Haar measure in \( S^{m-1} \). Note that as \( m \) grows, \( \inf_{T \in SL(n, \mathbb{R})} \| T : X_L^{(m)} \to X_K \| \) becomes larger but \( \frac{1}{|L^{(m)}|^{\frac{1}{n}}} \) decreases, so there is some sort of trade-off.

It should be noted that the volume of the random polytope \( L^{(m)} \) is bounded by (see [3])
\[
(18) \quad |L^{(m)}|^{\frac{1}{n}} \ll \frac{\sqrt{\log(\frac{2n}{m})}}{n}.
\]
In fact, this bound is the exact asymptotic growth of \( |L^{(m)}|^{\frac{1}{n}} \) with probability greater than or equal to \( 1 - \frac{1}{m} \) [3] Chapter 11.

In [16], for \( m = n \log(n) \), it is shown that, with high probability, the norm \( \| T : X_L^{(m)} \to X_K \| \) is “large” for every \( T \in SL(n, \mathbb{R}) \). To achieve all this he proved the following interesting inequality:
If $K \subset \mathbb{R}^n$ is in Löwner position then for every $m \in \mathbb{N}$ and every $\beta > 0$,

\begin{equation}
\mathbb{P}\left\{ \text{There exists } T \in SL(n, \mathbb{R}) : \|T : X_{L(m)} \to X_K\| \leq \beta \left( \frac{|B^n_2|}{|K|} \right)^{1/n} \right\} \\
\leq \left( C \sqrt{n} \right)^n \left( \frac{|B^n_2|}{|K|} \right)^n \beta^{nm-n^2}.
\end{equation}

(19)

In order to prove our main contribution, Theorem 1.1, we present the following refinement of the previous estimate.

**Proposition 3.1.** Let $K \subset \mathbb{R}^n$ be centrally symmetric convex body and $L^{(m)}$ the random polytope defined in (17), then for every $\beta > 0$ we have

\begin{equation}
\mathbb{P}\left\{ \text{There exists } T \in SL(n, \mathbb{R}) : \|T : X_{L^{(m)}} \to X_K\| \leq \beta \left( \frac{|B^n_2|}{|K|} \right)^{1/n} \right\} \\
\leq C^n \left( \ell(K)|K|^{1/n} + \|id : \ell^n_2 \to X_K\| \sqrt{|n|} |K|^{1/n} \right)^n (2\beta)^{nm}.
\end{equation}

To prove Proposition 3.1 we need a couple of lemmas. The first one is a technical tool which bounds the number of points in an $\epsilon$-net for an adequate set. This should be compared with [16, Lemma 5]: note that the set and the metric differ. This subtle but important modification is the key ingredient we need.

**Lemma 3.2.** Let $K \subset \mathbb{R}^n$ be a convex body, $\gamma > 0$ and

$$M^K_\gamma := \{ T \in SL(n, \mathbb{R}) \text{ and } \|T : \ell^n_1 \to X_K\| \leq \gamma \}.$$ 

There is a $\gamma$-net, $N^K_\gamma$ for $M^K_\gamma$ in the metric $\mathcal{L}(\ell^n_2, X_K)$ such that

$$\#N^K_\gamma \leq C^n \left( \ell(K)|K|^{1/n} + \|id : \ell^n_2 \to X_K\| \sqrt{|n|} |K|^{1/n} \right)^n.$$ 

Proof. Let $U$ be the unit ball of $\mathcal{L}(\ell^n_2, X_K)$. By the standard identification we consider $M^K_\gamma$ and $U$ as subsets of $\mathbb{R}^{n \times n}$. Let $N^K_\gamma$ be a maximal collection of elements of $M^K_\gamma$-separated. These elements form an $\gamma$-net and, for every $\xi \in N^K_\gamma$, the balls $\xi + \frac{\gamma}{2} U$ are disjoints. Since

$$\|T : \ell^n_1 \to X_K\| = \|T : \ell^n_2 \to X_K\|,$$

we have that $\gamma U \subset \{ T : \|T : \ell^n_1 \to X_K\| \leq \gamma \}$ and then

$$\bigcup_{\xi \in N^K_\gamma} \xi + \frac{\gamma}{2} U \subset \frac{3}{2} \{ T : \|T : \ell^n_1 \to X_K\| \leq \gamma \}.$$

Computing the volume on both sides, we get the following bound for $\#N^K_\gamma$,

$$\#N^K_\gamma \left( \frac{3}{2} \right)^n |U| \leq \left( \frac{3}{2} \right)^n \|\{ T : \|T : \ell^n_1 \to X_K\| \leq \gamma \}||

\begin{equation}
\#N^K_\gamma \leq \left( \frac{3}{\gamma} \right)^n |\{ T : \|T : \ell^n_1 \to X_K\| \leq \gamma \}|.
\end{equation}

(20)
Now notice that
\[
\{ T \in \mathcal{L}(\ell_1^n, X_K) : \| T : \ell_1^n \to X_K \| \leq \gamma \}
\subset \{ X \in \mathbb{R}^{n \times n} : X_i \in \gamma \cdot K \text{ for all } i \}
\subset (\gamma K)^n 
\tag{21}
\]
and hence
\[
| \{ T : \| T : \ell_1^n \to X_K \| \leq \gamma \} | \leq (\gamma^n)^n | K |^n. 
\tag{22}
\]
In order to bound Equation (20) we need a lower bound for $| U |$. By passing to spherical coordinates it can be checked that
\[
\frac{| U |}{| B_2^n |} = \int_{S^{n-1}} \| T \|_{\mathcal{L}(\ell_2^n, X_K)}^{-2} d\sigma(T), 
\tag{23}
\]
where $\sigma$ is the normalized Haar measure on $S^{n-1}$. Now we apply Hölder’s inequality to get
\[
1 \leq \left( \int_{S^{n-1}} \| T \|_{\mathcal{L}(\ell_2^n, X_K)}^{-2} d\sigma(T) \right)^{1/2} \left( \int_{S^{n-1}} \| T \|_{\mathcal{L}(\ell_2^n, X_K)}^2 d\sigma(T) \right)^{1/2} 
\leq \left( \int_{S^{n-1}} \| T \|_{\mathcal{L}(\ell_2^n, X_K)}^2 d\sigma(T) \right)^{1/2} \left( \int_{S^{n-1}} \| T \|_{\mathcal{L}(\ell_2^n, X_K)}^{-n} d\sigma(T) \right)^{1/n^2}.
\]
Therefore,
\[
\frac{| U |}{| B_2^n |} \geq \left( \int_{S^{n-1}} \| T \|_{\mathcal{L}(\ell_2^n, X_K)}^{-2} d\sigma(T) \right)^{-n/2}.
\]
By comparing spherical and Gaussian means and applying Gaussian Chevet’s inequality \[29, Equations (12.7),(43.1)], we have that
\[
\left( \int_{S^{n-1}} \| T \|_{\mathcal{L}(\ell_2^n, X_K)}^{-2} d\sigma(T) \right)^{1/2} \ll \frac{n}{\sqrt{n}} \left( \ell(K) + \| \text{id} : \ell_2^n \to X_K \| \sqrt{n} \right), 
\]
which implies
\[
\| \ell(K) + \| \text{id} : \ell_2^n \to X_K \| \sqrt{n} \cdot C^{-n} \geq | U |. 
\tag{24}
\]
Using (22) and (24) in Equation (20) we get the desired bound.

We also need the following result.

**Lemma 3.3** \[29, Lemma 38.3\]. Let $K \subset \mathbb{R}^n$ be a convex body, $L^{(m)}$ the random polytope in (17), $T \in SL(n, \mathbb{R})$ and $\alpha > 0$. Then
\[
P \{ \| T : X_{L^{(m)}} \to X_K \| \leq \alpha \} \leq \alpha^{mn} \left( \frac{| K |}{| B_2^n |} \right)^m. 
\tag{25}
\]

We present the proof of Proposition 3.1.
Proof of Proposition 3.1. Let \( \{X_i\}_{i=1}^m \subset S^n \) and \( L^{(m)} \) be the polytope in (17) such that there exists \( T \in SL(n, \mathbb{R}) \) with \( \|T : X_{L^{(m)}} \to X_K\| \leq \gamma \). As \( \ell_1^n \subset L^{(m)} \), \( T \) lies in the set \( \mathcal{M}^K \) defined in Lemma 3.2. Consider a \( \gamma \)-net, \( \mathcal{N}_\gamma^K \) for \( \mathcal{M}^K \) for the metric \( \ell(\ell_2^n, X_K) \) such that

\[
#\mathcal{N}_\gamma^K \leq Cn^2 \left( \ell(K) |K|^\frac{1}{n} + \|id : \ell_2^n \to X_K\| \sqrt{n} |K|^{\frac{1}{n}} \right)^n.
\]

Let \( S \in \mathcal{N}_\gamma^K \) such that \( \|S - T\|_{\mathcal{L}(\ell_2^n, X_K)} \leq \gamma \), then

\[
\|S : X_{L^{(m)}} \to X_K\| \leq \|T : X_{L^{(m)}} \to X_K\| + \|S - T : \ell_2^n \to X_K\|
\leq \gamma + \|S - T : \ell_2^n \to X_K\|,
\]

where we have used the fact that \( \|S - T : X_{L^{(m)}} \to X_K\| \leq \|S - T : \ell_2^n \to X_K\| \) since by construction \( L^{(m)} \subset B_2^n \). Hence,

\[
\mathcal{B}_\gamma := \{\text{There exists } T \in SL(n, \mathbb{R}) : \|T : X_{L^{(m)}} \to X_K\| \leq \gamma\}
\subset \bigcup_{S \in \mathcal{N}_\gamma^K} \{\|S : X_{L^{(m)}} \to X_K\| \leq 2\gamma\}.
\]

Take \( \gamma_0 := \beta \left( \frac{|B_2^n|}{|K|} \right)^\frac{1}{n} \), by the union bound, Equation (26) and Lemma 3.3

\[
\mathbb{P}(\mathcal{B}_{\gamma_0}) \leq Cn^2 \left( \ell(K) |K|^\frac{1}{n} + \|id : \ell_2^n \to X_K\| \sqrt{n} |K|^{\frac{1}{n}} \right)^n (2\beta)^{nm},
\]

which concludes the proof. \( \square \)

As a consequence of Proposition 3.1, we obtain the following result.

**Proposition 3.4.** Let \( K \subset \mathbb{R}^n \) be centrally symmetric convex body such that

\[
\ell(K) |K|^\frac{1}{n} + \|id : \ell_2^n \to X_K\| \sqrt{n} |K|^{\frac{1}{n}} \sim 1.
\]

Given \( \delta > 1 \), with probability greater than or equal to \( 1 - e^{-n^2} \) the random polytope \( L^{(\delta n)} \) in (17) verifies

\[
\sqrt{n} \ll \mathrm{vr}(K, L^{(\delta n)}).
\]

In particular, \( \sqrt{n} \ll lvr(K) \).

**Proof.** By Proposition 3.1 we know that there is an absolute constant \( C > 0 \) such that, for every \( \beta > 0 \),

\[
\mathbb{P} \left\{ \text{There exists } T \in SL(n, \mathbb{R}) : \|T : X_{L^{(m)}} \to X_K\| \leq \beta \left( \frac{|B_2^n|}{|K|} \right)^{1/n} \right\}
\leq Cn^2 (2\beta)^{nm}.
\]

If \( m = \lceil \delta n \rceil \) and \( \beta \leq \frac{1}{2} (Ce)^{-\frac{1}{n}} \), then with probability at least \( 1 - e^{-n^2} \) the random polytope verifies

\[
\|T : X_{L^{(\delta n)}} \to X_K\| \geq \beta \left( \frac{|B_2^n|}{|K|} \right)^{1/n} \sim \frac{1}{\sqrt{n} |K|^{1/n}}
\]

for every \( T \in SL(n, \mathbb{R}) \).
Hence, by Equations (18) and (28) and Remark 2.1 (1) we have
\[ \sqrt{n} \ll \vr(K, L^{(\ell n)}) , \]
which concludes the proof. \( \square \)

In order to prove Theorem 3.6 we will show that any given convex body can be approximated by another one which fulfils the hypothesis of the previous proposition. To achieve this we will make use of two deep and important results in the theory for isotropic convex bodies: Paouris’ result on the concentration of mass and Klartag’s perturbation with uniformly bounded isotropic constant (also known as Klartag’s solution to the isomorphic slicing problem).

**Theorem 3.5** ([24], Theorem 1.1). There is an absolute constant \( c > 0 \) such that if \( K \subset \mathbb{R}^n \) is an isotropic convex body, then
\[ \mathbb{P}\{ x \in K : \|x\|_2 \geq cL_K \sqrt{n}t \} \leq e^{-\sqrt{n}t} \]
for every \( t \geq 1 \).

**Theorem 3.6** ([17], Theorem 1.1). Let \( K \subset \mathbb{R}^n \) be a convex body and let \( \varepsilon > 0 \). Then there is a convex body \( T \subset \mathbb{R}^n \) such that
\begin{enumerate}
  \item \( d(K, T) < 1 + \varepsilon \),
  \item \( L_T < \frac{1}{\varepsilon} \).
\end{enumerate}
Here \( c > 0 \) is an absolute constant and
\[ d(K, T) = \inf\{ab : a, b > 0, \exists x, y \in \mathbb{R}^n, \frac{1}{a}(K + x) \subset T + y \subset b(K + x)\} . \]

**Remark 3.7.** Given a convex body \( K \subset \mathbb{R}^n \) there is a convex body \( T \subset \mathbb{R}^n \) such that \( \vr(T, K) \sim \vr(K, T) \sim 1 \) and \( L_T \leq c \), where \( c > 0 \) is an absolute constant.

Indeed, given \( K \), by Theorem 3.4 (using \( \varepsilon = 1 \)) there is \( T \subset \mathbb{R}^n \) with \( L_T \leq c \) and \( d(K, T) \leq 2 \). Notice that if for certain \( x, y \in \mathbb{R}^n \) and \( a, b > 0 \) we have that \( \frac{1}{a}(K + x) \subset T + y \subset b(K + x) \). Then,
\[ \vr(T, K) \leq \frac{|T|^\frac{1}{n}}{\frac{1}{a}|K|^\frac{1}{n}} \leq ab \frac{|K|^\frac{1}{n}}{|K|^\frac{1}{n}} \leq ab. \]

Hence \( \vr(T, K) \leq d(T, K) \), and by symmetry, the same holds for \( \vr(K, T) \).

**Proposition 3.8.** For every convex body \( K \subset \mathbb{R}^n \) there is a convex body \( W \) with \( \vr(W, K) \sim 1 \) such that
\[ (29) \quad \ell(W)|W|^{\frac{1}{n}} + \|\text{id} : \ell^n_2 \rightarrow X_W\| \sqrt{n}|W|^{\frac{1}{n}} \sim 1. \]

**Proof of Proposition 3.8.** By Remark 3.7 and the Roger-Shephard inequality (replacing the body if necessary) we can assume that \( K^* \) is a centrally symmetric isotropic convex body and \( L_{K^*} \) is uniformly bounded.

Consider \( W \) such that \( W^* = K^* \cap c\sqrt{n}B^n_2 \), with \( c > 0 \) the absolute constant in Theorem 3.5. This theorem also implies that \( |W^*|^{\frac{1}{n}} \geq (1 - \exp(-\sqrt{n}))^{\frac{1}{n}} \geq \frac{1}{2} \) and hence \( \vr(W, K) \sim \vr(K^*, W^*) \sim 1 \).

Since \( W^* \subset c\sqrt{n}B^n_2 \) we have that \( w(W^*) \ll \sqrt{n} \) and then \( \ell(W) \sim \sqrt{n}w(W^*) \sim n \). By the same inclusion we conclude that \( \|\text{id} : \ell^n_2 \rightarrow X_W\| = \|\text{id} : X_{W^*} \rightarrow \)
Finally, as $|W^n|^\frac{1}{d} \sim 1$, we have that $|W|^\frac{1}{d} \sim \frac{1}{n}$ (applying the Blaschke-Santaló/Bourgain-Milman inequality, Equation (10)). Therefore

$$\ell(W)|W|^\frac{1}{d} + \|id : \ell_2^n \to X_W\| |\sqrt{n}|W|^\frac{1}{d} \sim 1,$$

which concludes the proof. \hfill \Box

The following theorem contains, as a consequence, Theorem 1.1.

**Theorem 3.9.** Let $K \subset \mathbb{R}^n$ be centrally symmetric convex body. Given $\delta > 1$ with probability greater than or equal to $1 - e^{-n^2}$ the random polytope $L(\lceil \delta n \rceil)$ verifies

$$\sqrt{n} \ll v_r(K, L(\lceil \delta n \rceil)).$$

In particular, $\sqrt{n} \ll lvr(K)$.

**Proof.** By Proposition 3.8 there is $W$ with $v_r(W, K) \sim 1$ such that

$$\ell(W)|W|^\frac{1}{d} + \|id : \ell_2^n \to X_W\| |\sqrt{n}|W|^\frac{1}{d} \sim 1. \tag{30}$$

Applying Proposition 3.2 given $\delta > 1$, with probability greater than or equal to $1 - e^{-n^2}$ the random polytope $L(\lceil \delta n \rceil)$ in (17) verifies

$$\sqrt{n} \ll v_r(W, L(\lceil \delta n \rceil)).$$

Then,

$$\sqrt{n} \ll v_r(W, L(\lceil \delta n \rceil)) \leq v_r(W, K)v_r(K, L(\lceil \delta n \rceil)) \sim v_r(K, L(\lceil \delta n \rceil)),$$

as wanted. \hfill \Box

4. Upper bounds

We now provide upper estimates for $lvr(K)$ for different classes of convex bodies. Together with this inequalities we derive sharp asymptotic estimates for the largest volume ratio.

4.1. *Schatten trace classes.* To bound $v_r(K, L)$, Giannopoulos and Hartzoulaki [10] managed to find randomly a unitary operator $T : X_K \to X_L$ with small norm. To do this, they used Chevet’s inequality for an adequate position of $L$.

To our purposes we will use the following high probability version of the Gaussian Chevet’s inequality (tail inequality).

**Proposition 4.1.** Let $A = (g_{ij})_{1 \leq i, j \leq n} \in \mathbb{R}^{n \times n}$ be a random matrix with independent gaussian entries $g_{ij} \sim \mathcal{N}(0, 1)$ and $K, L \subset \mathbb{R}^n$ two convex bodies. Then, for all $u \geq 0$, with probability greater than $1 - e^{-n^2}$ we have

$$\|A : X_L \to X_K\| \ll \ell(K)\|id : \ell_2^n \to X_L\| + \ell(L)\|id : \ell_2^n \to X_K\| + u\|id : \ell_2^n \to X_L\| \cdot \|id : \ell_2^n \to X_K\| \tag{31}$$

Although the previous proposition is probably known for specialist we were not able to find an explicit reference of it (the closest statement we found is [30, Exercise 8.7.3]). We include a sketch of its proof for completeness.
Sketch of the proof of Proposition 4.2. We define in $L \times K^\circ$ the distance
\[ d((x, y^*), (\tilde{x}, \tilde{y}^*)) := \|x - \tilde{x}\|_2 \|id : \ell_2^n \to X_K\| + \|y^* - \tilde{y}^*\|_2 \|id : \ell_2^n \to X_{L^*}\|, \]
where $\| \cdot \|_2$ stands for the Euclidean norm.

Consider the random process in $L \times K^\circ$ given by
\[ X_{(x,y^*)} := \langle Ax, y^* \rangle. \]
It is not hard to see that this process is subgaussian for $d$ (see the proof of [30, Theorem 8.7.1]); i.e.,
\[ \|X_{(x,y^*)} - X_{(\tilde{x},\tilde{y}^*)}\|_{\psi_2} \leq Cd((x, y^*), (\tilde{x}, \tilde{y}^*)). \]
Note that if we consider the Gaussian process
\[ Y_{(x,y^*)} := \langle g, x \rangle \|id : \ell_2^n \to X_K\| + \langle h, y^* \rangle \|id : \ell_2^n \to X_{L^*}\|, \]
where $g = (g_1, \ldots, g_n)$, $h = (h_1, \ldots, h_n)$ and $(g_i)_{i=1}^n, (h_j)_{j=1}^n$ are independent standard Gaussian variables; we have
\[ \|Y_{(x,y^*)} - Y_{(\tilde{x},\tilde{y}^*)}\|_2 = d((x, y^*), (\tilde{x}, \tilde{y}^*)). \]
Combining the generic chaining (tail bound) [30, Theorem 8.5.5] and Talagrand’s majorizing measure theorem [30, Theorem 8.6.1] we get
\[ \|A : X_L \to X_K\| = \sup_{(x,y^*) \in L \times K^\circ} X_{(x,y^*)} \ll \left( \mathbb{E} \left[ \sup_{(x,y^*) \in L \times K^\circ} Y_{(x,y^*)} \right] + u \text{ diam}(K \times L^\circ) \right)^{1/2}, \]
with probability at least $1 - e^{-u^2}$.

The result follows by the fact that
\[ \mathbb{E} \left[ \sup_{(x,y^*) \in L \times K^\circ} Y_{(x,y^*)} \right] = \ell(K) \|id : \ell_2^n \to X_{L^*}\| + \ell(L^\circ) \|id : \ell_2^n \to X_K\|. \]
and $\text{diam}(L \times K^\circ) \sim \|id : \ell_2^n \to X_{L^*}\| \cdot \|id : \ell_2^n \to X_K\|$. \hfill \Box

We will also need the following inequality which can be found in [27, Proposition 1]:

Let $A = (g_{ij})_{1 \leq i,j \leq n} \in \mathbb{R}^{n \times n}$ as in Proposition 4.2 then with probability at least $1 - e^{-n}$ we have
\[ \det(A) \gg \sqrt{n}. \]

Given a convex body $W \subset \mathbb{R}^n$ we need to introduce a position $\tilde{W}$ highly related with the well-known $\ell$-position. Its existence can be tracked in the proof of the main theorem of the paper of Giannopoulos and Hartzoulaki [9]. It verifies
\begin{itemize}
  \item $\ell(\tilde{W}) \ll \sqrt{n \log(n)}$,
  \item $\ell(\tilde{W}^\circ) \ll \sqrt{n}$,
  \item $\|id : \ell_2^n \to X_{\tilde{W}^*}\| \ll \sqrt{n \log(n)}$.
\end{itemize}
In particular, $\frac{1}{|W|} \leq \ell(\tilde{W}) \ll \sqrt{n \log(n)}$.

When a convex body in $\mathbb{R}^n$ satisfies the previous estimates we say it is in $GH$-position.
Theorem 4.2. Let $B_N$ be the unit ball of any unitary invariant norm $N$ in $\mathbb{R}^{d \times d}$ and $L \subset \mathbb{R}^{d^2}$ a convex body in $GH$-position, and let $A = (g_{ij})_{1 \leq i,j \leq d^2} \in \mathbb{R}^{d \times d^2}$ be a random matrix with independent Gaussian entries $g_{ij} \sim \mathcal{N}(0,1)$. Then with probability greater than $1 - 2e^{-d}$, the body $\hat{L} := \frac{A}{\|A\| \tau(u)} \subset B_N$ and also

$$\frac{|B_N|^{\frac{1}{d}}}{|L|^{\frac{1}{d}}} \ll d.$$ 

As a consequence of Theorem 4.1, the previous result and Remark 2.1 (5) we obtain the following corollary.

Corollary 4.3. Let $B_N$ be the unit ball of any unitary invariant norm $N$ in $\mathbb{R}^{d \times d}$. Then,

$$\mu(L^2) \sim d.$$ 

Proof of Theorem 4.2. Note that by [6, Lemma 4.3.2] we know that $|B_{S_N^d}|^{\frac{1}{d}} \sim d^{-\frac{d}{2}}$ and also by [2, Exercise 7.24] $\ell(B_{S_N^d}) \sim d^{\frac{d}{2}}$, hence $\ell(K)|K|^{\frac{1}{d}} \sim 1$. On the other hand, since $S_N^d$ has enough symmetries; by Equation (12) and by [2, Exercise 7.24] (see also [15]) we know that

$$\mu(B_{S_N^d}) = \|id : \ell_2^d \to S_N^d\| \cdot d \cdot |B_{S_N^d}|^{\frac{1}{d}} \sim \sqrt{d}.$$ 

Using the previous information, the fact that $L$ is in $GH$-position, together with Proposition 1.1 we have

$$\|A : X_L \to S_N^d\| \ll \ell(S_N^d) \|id : \ell_2^d \to X_L\| + \ell(L^2) \|id : \ell_2^d \to S_N^d\|$$

$$+ u \|id : \ell_2^d \to X_L\| + \|id : \ell_2^d \to S_N^d\|$$

$$\ll \left( \frac{d^2}{\log(d)} + u \frac{d}{\log(d)} \right),$$

with probability greater than $1 - e^{-u^2}$. Choosing $u = \sqrt{d}$ and using Equation (32) we can ensure that, with probability greater than $1 - 2e^{-d}$, the random matrix $A$ fulfills simultaneously

$$\|A : X_L \to S_N^d\| \ll \frac{d^2}{\log(d)} \text{ and } \det(A)^{\frac{1}{d}} \gg d.$$ 

Since $A(L) \subset \|A\|S_N^d$ then, by Equation (13),

$$\hat{L} := \frac{1}{\tau(u)} A(L) \subset \frac{1}{\tau(u)} S_N^d \subset B_N.$$

Using the fact that $\frac{1}{\tau(u)} S_N^d \sim |B_N|^{\frac{1}{d}},$

$$\frac{|B_N|^{\frac{1}{d}}}{|L|^{\frac{1}{d}}} \sim \frac{|\frac{1}{\tau(u)} S_N^d|^{\frac{1}{d}}}{|\hat{L}|^{\frac{1}{d}}}$$

$$= \frac{\|A\|}{|\det A|^{\frac{1}{d}} |L|^{\frac{1}{d}}}$$

As $|S_N^d|^{\frac{1}{d}} \sim d^{-\frac{d}{2}}$ and $\frac{1}{|L|^{\frac{1}{d}}} \ll d \log d$ we obtain the desired bound. \qed
4.2. Largest volume ratio for unconditional bodies and random Dvoretzky-Rogers’ parallelepiped. Let $K$ be an unconditional convex body in $\mathbb{R}^n$ and $L$ be a centrally symmetric convex body; the following statement shows a way to find positions of $L$, say $\tilde{L}$, with extremely high probability such that the ratio $\left(\frac{|K|}{|\tilde{L}|}\right)^{\frac{1}{n}}$ is bounded by $\sqrt{n}$.

**Theorem 4.4.** Let $L \subset \mathbb{R}^n$ be a centrally symmetric convex body such that $L^o$ is in isotropic position and consider the random matrix $T := \sum_{j=1}^{n} X_j \otimes e_j$, where $X_1, \ldots, X_n$ are independently chosen accordingly to the uniform measure in the isotropic body $L^o$. With probability greater than or equal to $1 - e^{-n}$, for every unconditional isotropic body $K \subset \mathbb{R}^n$, the position $\tilde{L} := \frac{1}{2\sqrt{\pi e}} \cdot T(L)$ lies inside $K$ and

$$\left(\frac{|K|}{|\tilde{L}|}\right)^{\frac{1}{n}} \ll \sqrt{n}.$$  \hspace{1cm} (34)

Note that as a direct consequence of Theorem 1.1, the previous theorem and Equation (16) we have

$$\text{ivr}(K) \sim \sqrt{n},$$  \hspace{1cm} (35)

for every unconditional body $K \subset \mathbb{R}^n$ (an unconditional body is isotropic and unconditional up to a diagonal operator), which shows the upper estimates in Theorem 1.3.

Recall the following result of Bobkov and Nazarov [5, Proposition 2.4 and Proposition 2.5] (see also [21] or [6, Proposition 4.2.4]), which asserts that the normalized $\ell_1$-ball ($\ell_\infty$-ball) in $\mathbb{R}^n$ is the largest set (smallest set) within the class of all unconditional isotropic bodies (up to some universal constants).

**Proposition 4.5.** [5, Proposition 2.4 and Proposition 2.5] Let $K \subset \mathbb{R}^n$ be an unconditional isotropic convex body. Then,

$$\frac{1}{2\sqrt{\pi e}} \cdot B_1^n \subset K \subset \sqrt{\frac{6}{\pi e}} \cdot B_\infty^n,$$  \hspace{1cm} (36)

where $B_\infty^n$ and $B_1^n$ stand for the unit balls of $\ell_\infty^n$ and $\ell_1^n$ respectively.

It should be noted that (35) can be obtained by a direct use of a classical result of Dvoretzky and Rogers. Indeed, given a centrally symmetric convex body $L \subset \mathbb{R}^n$, by [7, Theorem 5A] (see also [24]) there is a centrally symmetric parallelepiped $P \supset L$ such that

$$\left(\frac{|P|}{|L|}\right)^{1/n} \leq c \sqrt{n},$$  \hspace{1cm} (37)

for some absolute constant $c > 0$. Thus, by Remark 2.1 (5), $\text{vr}(B_\infty^n, L) \ll \sqrt{n}$. If $K$ is an unconditional body, by Proposition 4.5 we have $\text{vr}(K, B_\infty^n) \sim 1$. By Remark 2.1 (4) we obtain

$$\text{vr}(K, L) \leq \text{vr}(K, B_\infty^n) \cdot \text{vr}(B_\infty^n, L) \ll \sqrt{n}.$$  \hspace{1cm} (38)

Observe that, in general, understanding how the parallelepiped $P$ in Equation (37) looks like seems difficult (its construction depends on certain contact points when $L$ is in John’s position, which are not easy to find explicitly), thus
Theorem 4.4 seems much stronger since it provides a random algorithm that works with high probability.

We therefore state the following novel probabilistic construction of the Dvoretzky-Rogers' parallelepiped, which can be derived from a result of Pivovarov. Note that Theorem 4.4 is a direct consequence of the next theorem together with the first inclusion of Proposition 4.5.

**Theorem 4.6.** Let $L \subset \mathbb{R}^n$ be a centrally symmetric convex body such that $L^\circ$ is in isotropic position and consider the random matrix $T := \sum_{j=1}^n X_j \otimes e_j$, where $X_1, \ldots, X_n$ are independently chosen according to the uniform measure in the isotropic body $L^\circ$. With probability greater than or equal to $1 - e^{-n}$, the parallelepiped $P = T^{-1}(B_\infty^n)$ contains $L$ and

\[
\left( \frac{|P|}{|L|} \right)^{\frac{1}{n}} \ll \frac{\sqrt{n}}{L_{L^\circ}}.
\]

**Proof.** By [27, Proposition 1] we know that

\[
\mathbb{P}\left\{ \left| \det \left( \sum_{j=1}^n X_j \otimes e_j \right) \right|^{1/n} \gg \sqrt{n} L_{L^\circ} \right\} > 1 - e^{-n}.
\]

On the other hand since $|\langle X_i, y \rangle| \leq 1$ for all $y \in L$ and $1 \leq i \leq n$ we have that $||T : X_L \to \ell_\infty^n|| \leq 1$, where $T := \sum_{j=1}^n X_j \otimes e_j$.

Thus, $T(L) \subset B_\infty^n$, or equivalently $L \subset T^{-1}(B_\infty^n) := P$ and the ratio

\[
\left( \frac{|P|}{|L|} \right)^{\frac{1}{n}} = \frac{|B_\infty^n|^{\frac{1}{n}}}{|\det T|^{\frac{1}{n}} |L|^{\frac{1}{n}}}.
\]

Therefore, by Equations (40) and (39) and taking into account that $|L|^{\frac{1}{n}} \sim \frac{1}{n}$ (which comes by applying the Blaschke-Santalo/Bourgain-Milman inequality, Equation (10)), since $|L^2| = 1$ we have, with probability greater than or equal to $1 - e^{-n}$,

\[
\left( \frac{|P|}{|L|} \right)^{\frac{1}{n}} \ll \frac{\sqrt{n}}{L_{L^\circ}},
\]

which concludes the proof. \qed

We finish the article with a consequence of Theorem 4.4.

**Corollary 4.7.** For every centrally symmetric convex body $L \subset \mathbb{R}^n$ we have

\[
\text{vr}(B_\infty^n, L) \cdot L_{L^\circ} \ll \sqrt{n}.
\]

This seems to be an improvement of the well-known inequality [4, Proposition 3.5.13]

\[
L_L \cdot L_{L^\circ} \ll \sqrt{n}.
\]

Indeed, by Equation (1) we known that

\[
L_L \ll \text{vr}(B_\infty^n, L),
\]

but in general $\text{vr}(B_\infty^n, L)$ can be larger than $L_L$: according to Theorem 3.9 and [6, Theorem 4.4.1] there is a polytope $L^{(2n)}$ which verifies

\[
\text{vr}(B_\infty^n, L^{(2n)}) \gg \sqrt{n}; \text{ and } L_{L^{(2n)}} \ll \log(n).
\]
In Corollary 4.7, at least at first instance, one should be tempted to change \( \text{vr}(B_{\infty}^n, L) \) by \( \sup_{K \subset \mathbb{R}^n \text{unc}} \text{vr}(K, L) \), where the infimum run all over unconditional convex bodies; but using Proposition 4.5, it can be seen that

\[
\text{vr}(B_{\infty}^n, L) \sim \sup_{K \subset \mathbb{R}^n \text{unc}} \text{vr}(K, L).
\]

References

[1] Shiri Artstein-Avidan, Apostolos Giannopoulos, and Vitali D Milman. Asymptotic Geometric Analysis, Part I, volume 202. American Mathematical Soc., 2015.

[2] Guillaume Aubrun and Stanislaw J Szarek. Alice and Bob meet Banach. Mathematical Surveys and Monographs, 223, 2017.

[3] I Bárány and Z Füredi. Approximation of the sphere by polytopes having few vertices. Proceedings of the American Mathematical Society, 102(3):651–659, 1988.

[4] Franck Barthe and Dario Cordero-Erausquin. Invariances in variance estimates. Proceedings of the London Mathematical Society, 106(1):33–64, 2013.

[5] Sergey G Bobkov and Fedor L Nazarov. On convex bodies and log-concave probability measures with unconditional basis. In Geometric aspects of functional analysis, pages 53–69. Springer, 2003.

[6] Silouanos Brazitikos, Apostolos Giannopoulos, Petros Valettas, and Beatrice-Helen Vritsiou. Geometry of isotropic convex bodies, volume 196. American Mathematical Society Providence, 2014.

[7] Aryeh Dvoretzky and Claude A Rogers. Absolute and unconditional convergence in normed linear spaces. Proceedings of the National Academy of Sciences, 36(3):192–197, 1950.

[8] Daniel Galicer, Mariano Merzbacher, and Damian Pinasco. The minimal volume of simplices containing a convex body. The Journal of Geometric Analysis, pages 1–16, 2017.

[9] Apostolos Giannopoulos and Marianna Hartzoulaki. On the volume ratio of two convex bodies. Bulletin of the London Mathematical Society, 34(06):703–707, 2002.

[10] Apostolos Giannopoulos, Irini Perissinaki, and Antonis Tsolomitis. John’s theorem for an arbitrary pair of convex bodies. Geometriae Dedicata, 84(1-3):63–79, 2001.

[11] Efim D Gluskin. Diameter of the Minkowski compactum is approximately equal to \( n \). Functional Analysis and Its Applications, 15(1):57–58, 1981.

[12] Y Gordon, AE Litvak, M Meyer, and A Pajor. John’s decomposition in the general case and applications. Journal of Differential Geometry, 68(1):99–119, 2004.

[13]Peter Gruber. Convex and discrete geometry, volume 336. Springer Science & Business Media, 2007.

[14] O Guédon and G Paouris. Concentration of mass on the Schatten classes. In Annales de l’Institut Henri Poincare (B) Probability and Statistics, volume 43, pages 87–99. No longer published by Elsevier, 2007.

[15] Zakhar Kabluchko, Joscha Prochno, and Christoph Thaele. Exact asymptotic volume and volume ratio of Schatten unit balls. arXiv preprint arXiv:1804.03407, 2018.

[16] Alexander Igorevich Khrabrov. Generalized volume ratios and the Banach–Mazur distance. Mathematical Notes, 70(5-6):838–846, 2001.

[17] Bo’az Klartag. On convex perturbations with a bounded isotropic constant. Geometric & Functional Analysis GAFA, 16(6):1274–1290, 2006.

[18] Hermann König, Mathieu Meyer, and Alain Pajor. The isotropy constants of the Schatten classes are bounded. Mathematische Annalen, 312(4):773–783, 1998.

[19] Marek Lassak. On the Banach–Mazur distance between convex bodies. J. Geom, 41:11–12, 1992.

[20] Marek Lassak. Approximation of convex bodies by centrally symmetric bodies. Geometriae Dedicata, 72(1):63–68, 1998.

[21] G Ya Lozanovskii. On some Banach lattices. Siberian Mathematical Journal, 10(3):419–431, 1969.

[22] Jiří Matoušek. Lectures on discrete geometry, volume 108. Springer New York, 2002.

[23] Vitali D Milman and Alain Pajor. Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed \( n \)-dimensional space. In Geometric aspects of functional analysis, pages 64–104. Springer, 1989.
[24] Grigoris Paouris. Concentration of mass on convex bodies. *Geometric & Functional Analysis (GAFA)*, 16(5):1021–1049, 2006.

[25] A Pelczynski and SJ Szarek. On parallelepipeds of minimal volume containing a convex symmetric body in \(\mathbb{R}^n\). In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 109, pages 125–148. Cambridge University Press, 1991.

[26] Aleksander Pelczynski. Structural theory of Banach spaces and its interplay with analysis and probability. In *Proceedings of the ICM*, pages 237–269, 1983.

[27] Peter Pivovarov. On determinants and the volume of random polytopes in isotropic convex bodies. *Geometriae Dedicata*, 149(1):45–58, 2010.

[28] Jordan Radke and Beatrice-Helen Vritsiou. On the thin-shell conjecture for the Schatten classes. *arXiv preprint arXiv:1602.06924*, 2016.

[29] Nicole Tomczak-Jaegermann. *Banach-Mazur distances and finite-dimensional operator ideals*, volume 38. Longman Sc & Tech, 1989.

[30] Roman Vershynin. *High-dimensional probability: An introduction with applications in data science*, volume 47. Cambridge University Press, 2018.

Departamento de Matemática - IMAS-CONICET, Facultad de Cs. Exactas y Naturales Pab. I, Universidad de Buenos Aires (1428) Buenos Aires, Argentina
E-mail address: dgalicer@dm.uba.ar

Departamento de Matemática - IMAS-CONICET, Facultad de Cs. Exactas y Naturales Pab. I, Universidad de Buenos Aires (1428) Buenos Aires, Argentina
E-mail address: mmerzbacher@dm.uba.ar

Departamento de Matemáticas y Estadísticas, Universidad T. Di Tella, Av. Figueroa Alcorta 7350 (1428), Buenos Aires, Argentina and CONICET
E-mail address: dpinasco@utdt.edu