EXISTENCE AND NON-MONOTONICITY OF TRAVELING WAVE SOLUTIONS FOR GENERAL DIFFUSIVE PREDATOR-PREY MODELS

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Abstract. This paper is concerned with the existence and non-monotonicity of traveling wave solutions for general diffusive predator-prey models. By using Schauder’s fixed point theorem and the existence of contracting rectangles, we obtain the existence result. Then we investigate the asymptotic behavior of positive monotone traveling wave solutions by using the modified Ikehara’s Theorem. With the help of their asymptotic behavior, we provide a sufficient condition which guarantee that all positive traveling wave solutions of the system are non-monotone. Furthermore, to illustrate our main results, the existence and non-monotonicity of traveling wave solutions of Lotka-Volterra predator-prey model and modified Leslie-Gower predator-prey models with different kinds of functional responses are also discussed.

1. Introduction. The purpose of this work is to investigate the existence and non-monotonicity of traveling wave solutions for general diffusive predator-prey models governed by the following equations

\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= d_1 \Delta u_1 + u_1 f_1(u_1, u_2), \\
\frac{\partial u_2}{\partial t} &= d_2 \Delta u_2 + u_2 f_2(u_1, u_2),
\end{align*}
\] (1)

where \( d_1, d_2 > 0 \) are diffusion coefficients, and \( f_n: \mathbb{R}^2 \rightarrow \mathbb{R} \) are reaction functions for \( n = 1, 2 \). We can regard \( u_n = u_n(x, t) \) as the population density of species \( u_n \) at time \( t \) and position \( x \), then system (1) can be used to describe the interaction of species \( u_1 \) and \( u_2 \). In this article we assume the interaction between species \( u_1 \) and \( u_2 \) belongs to the predator-prey relationship, i.e. one species (predator) captures and feeds on the other species (prey). Such a interaction is governed by the functional responses \( f_1 \) and \( f_2 \).

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In past decades, traveling wave solutions for system (1) with different functional responses have been extensively studied by many researchers. For example, Dunbar [9, 10, 11] proved the existence of traveling wave solutions for system (1) with the specific functional responses:

\[
\begin{align*}
\circ \quad f_1(u_1, u_2) &= u_1(A - Bu_2), \quad f_2(u_1, u_2) = u_2(Eu_1 - C); \\
\circ \quad f_1(u_1, u_2) &= Au_1(1 - u_1/K) - Bu_1u_2, \quad f_2(u_1, u_2) = u_2(Eu_1 - C); \\
\circ \quad f_1(u_1, u_2) &= Au_1(1 - u_1/K) - B u_1u_2  + Eu_1, \quad f_2(u_1, u_2) = u_2(Du_1 + Eu_1 - C),
\end{align*}
\]

by using the shooting method, Wazewski Theorem, Liapunov function and LaSalle’s invariance principle. Different to Dunbar’s method, Gardner [14] used a modification of Conley index (called the connection index) to get the existence of traveling wave solutions of (1) for more abstract functional responses. However, many concrete models may not fit well with the frame established in Gardner’s abstract approaches. Later, the authors of the works [1, 12, 15, 18, 19, 20, 23, 26] improved the technique of Dunbar’s approach to investigate the traveling wave solutions of system (1) with different types of functional responses. For examples, Huang [20] used a geometric approach to derive the existence of traveling waves for some classes of non-monotone reaction-diffusion systems. Ai et al [1] also used the squeeze method and a Lyapunov function method to study the traveling waves for a generalized Holling-Tanner predator-prey model. Additionally, with the help of the upper-lower solutions and Schauder’s fixed point theorem, Ma [30] studied the existence of traveling wavefronts for reaction-diffusion systems with quasimonotonicity reactions. In 2010, Li et al [25] also proved the existence of such solutions by applying the cross iteration method. Improving the method of Li et al [25], Lin and Ruan [28] investigated the existence of traveling wave solutions for the following delayed reaction-diffusion systems

\[
\frac{\partial v_i}{\partial t} = d_i \Delta v_i(x, t) + f_i(v_t(x)),
\]

where \(x \in \mathbb{R}, t > 0, d_i > 0\) for \(i = 1, \cdots, n\), \(v_i(x) := v(x, t + s)\) for \(s \in [-\tau, 0]\) and \(\tau > 0\) is the time delay. Applying the Schauder’s fixed point theorem, the traveling wave problem of (3) can be reduced to the existence of a pair of upper and lower solutions. Using the technique of contracting rectangles (see Definitions 2.1 and 2.4), the asymptotic behavior of traveling wave solutions for delayed diffusive systems is then obtained. Furthermore, they applied their results to Lotka-Volterra reaction-diffusion system with distributed delay and discussed the existence of non-monotone traveling wave solutions of the system. Recently, Pan [31] also used contracting rectangles to study the convergence and traveling wave solutions for a predator-prey system with distributed delays. For the other related references, we refer to [6, 16, 17, 27, 29, 34, 35] and the references cited therein. Although there are several literature related to the existence of traveling wave solutions, it is worthy mentioning that there are few papers considering the non-monotonicity of traveling wave solutions for predator-prey models.

Since system (1) is not quasi-monotone, it becomes harder to study the traveling wave solutions and some new phenomena might occur, e.g., the existence of non-monotone traveling wave solutions. Motivated by above mentioned literature, we are concerned with the existence and non-monotonicity of traveling wave solutions for system (1) with more general functional responses. In this article, we assume the functional responses satisfy the following assumptions.
(H1) Let $g_1(\cdot)$ be a decreasing and concave function, and $g_2(\cdot)$ be an increasing and convex function such that
\[ g_1(0) > 0, \ g_2(0) > 0 \text{ and } f_1(x, g_1(x)) = f_2(x, g_2(x)) = 0, \text{ for any } x \in [0, \infty). \]

(H2) $g_1(\alpha) = 0$ for some $\alpha > 0$, and for any $(x_1, x_2) \in [0, 2\alpha] \times [0, g_1(0)]$, we have
\[ D_1f_1(x_1, x_2), D_2f_1(x_1, x_2), D_2f_2(x_1, x_2) < 0, \ D_1f_2(x_1, x_2) > 0, \]
where $D_j f_n(x_1, x_2)$ means the partial derivative of $f_n(\cdot, \cdot)$ with respect to $x_j$.

(H3) $g_2(2\alpha) < 2g_2(\alpha) < -\alpha g_1'(0)$ and $g_2(x) > x g_2'(x)$ for any $x \in [0, 2\alpha]$.

In the final section, we will illustrate some models which have functional responses satisfying the above assumptions, e.g. the Lotka-Volterra predator-prey model and diffusive predator-prey models with modified Leslie-Gower term and different kinds of functional response. Here we point out the significance of assumptions (H1)–(H3) to the proof of the main result. First, by (H1) and (H2) we know that $g_1(0) > 0$ and $g_2(0) > 0$ which implies
\[ f_1(0, 0) > f_1(0, g_1(0)) = 0 \text{ and } f_2(0, 0) > f_2(0, g_2(0)) = 0. \]

The conditions (5) will be used in proving the non-existence of traveling wave solutions (see Subsection 4.1). In addition, by (H1) and (H3) and the condition $g'(\alpha) = 0$, we have
\[ g_1(0) = g_1(0) - g_1(\alpha) > -\alpha g_1'(0) > 2g_2(\alpha) > g_2(2\alpha) > g_2(\alpha), \]
\[ g_1(\alpha) = 0 < g_2(0) < g_2(\alpha) < g_1(0). \]

Therefore, by Intermediate Value Theorem, there exists a number $k_1 \in (0, \alpha)$ such that $g_2(k_1) = g_1(k_1)$. Let us define $k_2 := g_2(k_1) = g_1(k_1)$, then it is clear that
\[ f_1(k_1, k_2) = f_1(k_1, g_1(k_1)) = 0 \text{ and } f_2(k_1, k_2) = f_2(k_1, g_2(k_1)) = 0. \]

Hence $(k_1, k_2)$ is a positive equilibrium of system (1). Moreover, by (6), it is clear that $(k_1, k_2)$ is the unique positive equilibrium (see Figure 1). By assumption (H2), it’s easy to verify that the equilibria $(0, 0)$ is unstable, and $(k_1, k_2)$ is stable for system of (8) with $d_1 = d_2 = 0$. Due to the importance of coexistence in population dynamics, we will look for the existence of monostable traveling wave solutions for system (1) which connect these two equilibria.
A solution \((u_1(x,t), u_2(x,t))\) of system (1) is called a traveling wave solution if there exist \(c > 0\) and smooth functions \(\phi_1(\cdot), \phi_2(\cdot)\) such that
\[
u_1(x,t) = \phi_1(x+ct) \quad \text{and} \quad \nu_2(x,t) = \phi_2(x+ct), \quad \text{for all } x, t \in \mathbb{R}. \tag{7}
\]
Note that \(\phi_1(\cdot), \phi_2(\cdot)\) are called the wave profiles, \(c\) is the wave speed and \(\xi := x + ct\) is the moving coordinate. Substituting the ansatzes of (7) into system (1), we can obtain the profile equations
\[
\left\{ \begin{array}{l}
c\phi_1''(\xi) = d_1\phi_1''(\xi) + \phi_1(\xi)f_1(\phi_1(\xi), \phi_2(\xi)), \\
c\phi_2''(\xi) = d_2\phi_2''(\xi) + \phi_2(\xi)f_2(\phi_1(\xi), \phi_2(\xi)).
\end{array} \right. \tag{8}
\]
In this paper, we look for the solutions of system (8) satisfying the conditions
\[
\lim_{\xi \to -\infty} (\phi_1(\xi), \phi_2(\xi)) = (0,0) \quad \text{and} \quad \lim_{\xi \to \infty} (\phi_1(\xi), \phi_2(\xi)) = (k_1, k_2). \tag{9}
\]
By using the Schauder’s fixed point theory via the existence of contracting rectangles (cf. [33, Theorem 5.2.5], Lin and Ruan [28] and Lemma 2.6), we can obtain the existence of traveling wave solutions of system (1) provided the wave speed is larger than the minimal speed \(c^*\). In addition, to study the monotonicity of traveling wave solutions, we first apply a modified version of Ikehara’s Theorem (see Lemma 3.1) to investigate the asymptotic behavior of positive monotone traveling wave solutions. With the help of their asymptotic behavior, we can derive the non-monotonicity of traveling wave solutions of system. Our main results are stated as follows.

**Theorem 1.1.** Assume (H1)–(H3) hold and \(c^* := \max\{2\sqrt{d_1 f_1(0,0)}, 2\sqrt{d_2 f_2(0,0)}\}\).

1. For any \(c > c^*\), system (8) has a positive solution satisfying (9).
2. For any \(0 < c < c^*\), system (8) has no positive solution satisfying (9).
3. Let \((\phi_1(\xi), \phi_2(\xi))\) be any positive solution of system (8) satisfying (9). If
\[
d_2 \geq d_1 \quad \text{and} \quad k_1 D_1 f_1(k_1, k_2) < k_2 D_2 f_2(k_1, k_2), \tag{10}
\]
then either \(\phi_1(\xi)\) or \(\phi_2(\xi)\) is non-monotone.

Moreover, we have the following existence result of traveling wave solution when the wave speed is equal to the minimal speed \(c = c^*\).

**Theorem 1.2.** Assume (H1)–(H3) hold, \(w f_2(0, w)\) is concave for \(w \in [0, g_2(0)]\),
\[
d_2 > d_1 \quad \text{and} \quad \frac{2d_2 - d_1}{d_2} < \frac{f_1(0,0)}{f_2(0,0)} < \frac{d_2}{d_1}. \tag{11}
\]
Then system (8) with \(c = c^* := 2\sqrt{d_2 f_2(0,0)}\) has a positive solution satisfying (9).

**Remark 1.** Following the similar arguments, we can also obtain the same results when the assumptions (H1)–(H3) are replaced by the following assumptions:

(A1) There are decreasing function \(g_1(\cdot)\) on \((0, \infty)\) and increasing function \(g_2(\cdot)\) on \([0, \infty)\) such that \(g_2(0) > 0\), \(f_1(x, g_1(x)) = f_2(x, g_2(x)) = 0\) on \((0, \infty)\) and \(g_1(0^+) = \infty\).

(A2) There exists an \(\alpha > 0\) such that \(g_1(\alpha) = 0\), \(g_1(\cdot)\) and \(g_2(\cdot)\) are convex on \((0, \alpha)\) and \([0, 2\alpha]\) respectively, and the condition (4) is true.

(A3) \(g_2(2\alpha) < 2g_2(\alpha) < -\alpha g_1(\alpha)/2\), \(g_1(\alpha/2) > g_2(\alpha/2)\) and \(g_2(x) > xg_2'(x)\) on \([0, 2\alpha]\).

In Subsection 5.2, we will show that the diffusive predator-prey models with modified Leslie-Gower term and Holling-type III functional response satisfies the assumptions (A1)–(A3).
Noting that although we follow the arguments similar to [28], there are some differences between our work and theirs. First, Lin and Ruan [28] proved the existence of strict contracting rectangle for a specific model. In this paper we can construct the strict contracting rectangle for more general models. Hence, our proof becomes more difficult and significantly different from that in [28]. Additionally, we provide some sufficient conditions which guarantee that any positive solution of system (8) satisfying (9) is non-monotone. To the best our knowledge, the proof arguments for this part are new. Moreover, our result can be applied to more ecology models, e.g. the Lotka-Volterra predator-prey model, the modified Leslie-Gower predator-prey system with different functional responses, and so on.

The rest of this paper is organized as follows. In section 2, we construct a pair of upper and lower solutions of system (8) and prove the existence of the strict contracting rectangle. In section 3, we apply the Ikehara’s theorem to study the asymptotic behavior of positive monotone solutions for system (8) satisfying (9). Then we prove the results of Theorem 1 in Section 4. In the final section, we apply our theorem to some predator-prey models.

2. Preliminaries. To prove Theorem 1.1 (1) and (2), we need to construct a pair of upper and lower solutions of system (8) and establish the strictly contracting rectangle. First we give the definition of the upper and lower solutions of system (8).

Definition 2.1. The functions \( \Phi(\xi) = (\phi_1(\xi), \phi_2(\xi)) \) and \( \Phi(\xi) = (\phi_1(\xi), \phi_2(\xi)) \) are called a pair of upper and lower solutions for system (8) if \( \Phi(\xi), \Phi'(\xi), \Phi''(\xi) \) are bounded and satisfy

\[
\begin{align*}
\phi_1'(\xi) & \leq d_1 \phi_1''(\xi) + \phi_1(\xi)f_1(\phi_1(\xi), \phi_2(\xi)), \\
\phi_2'(\xi) & \leq d_2 \phi_2''(\xi) + \phi_2(\xi)f_2(\phi_1(\xi), \phi_2(\xi)), \\
\phi_1'(\xi) & \geq d_1 \phi_1''(\xi) + \phi_1(\xi)f_1(\phi_1(\xi), \phi_2(\xi)), \\
\phi_2'(\xi) & \geq d_2 \phi_2''(\xi) + \phi_2(\xi)f_2(\phi_1(\xi), \phi_2(\xi)),
\end{align*}
\]

for \( \xi \in \mathbb{R} \) except at finite points \( \xi_1, \xi_2, \ldots, \xi_m \).

2.1. Construction of upper and lower solutions. To construct a pair of upper and lower solutions, we first consider the characteristic functions of system (8) given by

\[
\Delta_n(\lambda) = d_n\lambda^2 - c\lambda + f_n(0,0), \quad \text{for any } 1 \leq n \leq 2.
\]

It is clear that the function \( \Delta_n(\lambda) \) arises from the linearization of system (8) about the equilibrium \((0,0)\). One can see later that real roots of the characteristic functions play an important role in the construction of upper and lower solution. The following lemma follows from (16) obviously.

Lemma 2.2. Let \( \Delta_n(\lambda), n = 1, 2 \) be the functions defined in (16).

(1) For any \( 0 < c < 2\sqrt{d_n f_n(0,0)} \), the function \( \Delta_n(\lambda) \) has no real root.

(2) For any \( c > 2\sqrt{d_n f_n(0,0)} \), the function \( \Delta_n(\lambda) \) has two positive roots \( \lambda_n, \lambda_n^+ \), where

\[
\lambda_n := \left[ c - \sqrt{c^2 - 4d_n f_n(0,0)} \right] / 2d_n \quad \text{and} \quad \lambda_n^+ := \left[ c + \sqrt{c^2 - 4d_n f_n(0,0)} \right] / 2d_n
\]

Moreover, \( \Delta_n(\lambda) < 0 \) for any \( \lambda_n < \lambda < \lambda_n^+ \).
Based on Lemma 2.2, we can denote the threshold wave speed
\[ c^* := \max\{2\sqrt{d_1f_1(0,0)}, 2\sqrt{d_2f_2(0,0)}\}. \]

Next, let \( c > c^* \) and \( \eta \) be chosen such that
\[ 1 < \eta < \min_{1 \leq n,j \leq 2} \left\{ \ell_n^+ / \lambda_n, (\lambda_n + \lambda_j) / \lambda_n \right\}. \]  

(18)

Then, let \( q > 0 \) be fixed, we set the function \( \ell_n(\xi) \) and constants \( \bar{t}_n, t^*_n \) by
\[ \ell_n(\xi) := e^{\lambda_n \xi} - q e^{\eta \lambda_n \xi}, \quad \bar{t}_n := \frac{1}{\lambda_n(\eta - 1)} \ln \left( \frac{1}{q} \right), \quad t^*_n := \frac{1}{\lambda_n(\eta - 1)} \ln \left( \frac{1}{q^*} \right), \quad n = 1, 2. \]

It is easy to check that \( \ell_n(\xi) \) has a global maximum at \( \xi = \bar{t}_n \) with
\[ \ell_n(\bar{t}_n) = \left( 1 - \frac{1}{\eta} \right) \left( \frac{1}{q^*} \right)^{1/(\eta - 1)}, \]

(19)

(19)

\[ \ell_n(t^*_n) = 0, \quad \bar{t}_n < t^*_n \quad \text{and} \quad \lim_{q \to \infty} \bar{t}_n = \lim_{q \to \infty} t^*_n = -\infty, \quad \text{for} \quad n = 1, 2. \]

(20)

Furthermore, let \( t_4 \) be the number satisfying
\[ e^{\lambda_2 t_4} + q e^{\eta \lambda_2 t_4} = g_2(\alpha). \]

Thus we have
\[ \lim_{q \to \infty} t_4 = -\infty. \]  

(21)

By (18) and part (2) of Lemma 2.2, one can see that
\[ \Delta_1(\eta \lambda_1) < 0 \quad \text{and} \quad \Delta_2(\eta \lambda_2) < 0. \]  

(22)

According to (20), (21) and (22), we can choose \( q \) large enough such that
\[ \ell_n(\bar{t}_n) < \min\{\alpha, g_2(\alpha)\}, \quad \text{for any} \quad n = 1, 2, \]  

(23)

\[ q e^{\eta \lambda_1 \xi} \Delta_1(\eta \lambda_1) + M e^{\lambda_1 \xi} (e^{\lambda_1 \xi} + e^{\lambda_2 \xi} + q e^{\eta \lambda_1 \xi}) < 0, \quad \text{for any} \quad \xi < t^*_1, \]  

(24)

\[ q e^{\eta \lambda_2 \xi} \Delta_2(\eta \lambda_2) + M e^{\lambda_2 \xi} (q e^{\eta \lambda_1 \xi} + e^{\lambda_2 \xi} + M q e^{(\lambda_1 + \eta \lambda_2) \xi}) < 0, \quad \text{for any} \quad \xi < t^*_2, \]  

(25)

\[ q e^{\eta \lambda_2 \xi} \Delta_2(\eta \lambda_2) - M (e^{\lambda_2 \xi} + q e^{\eta \lambda_2 \xi}) e^{\lambda_1 \xi} < 0, \quad \text{for any} \quad \xi < t_4, \]  

(26)

where \( M > 0 \) is defined by
\[ M := \max \{ |D_j f_n(x_1, x_2)|, \quad 1 \leq j, n \leq 2 \quad \text{and} \quad (x_1, x_2) \in [0, \alpha] \times [0, g_2(\alpha)] \}. \]  

(27)

Additionally, by the facts \( f_2(0, 0) > 0 \) and \( f_1(0, g_2(\alpha)) > f_1(0, g_1(0)) = 0 \), there exists a small positive number \( \varepsilon \) satisfying the conditions
\[ \varepsilon < \min \{ \ell_1(\bar{t}_1), \ell_2(\bar{t}_2) \}, \quad f_1(\varepsilon, g_2(\alpha)) > 0 \quad \text{and} \quad f_2(0, \varepsilon) > 0. \]  

(28)

Based on the above parameters, we are ready to construct the upper and lower solutions of system (8) in the sequel.

Let’s define the functions
\[ \phi_n(\xi) := \begin{cases} e^{\lambda_1 \xi} - q e^{\eta \lambda_1 \xi}, & \text{if} \ \xi < t_1, \\ \varepsilon, & \text{if} \ \xi \geq t_1 \end{cases}, \quad \phi_n(\xi) := \begin{cases} e^{\lambda_1 \xi}, & \text{if} \ \xi < t_3, \\ \alpha, & \text{if} \ \xi \geq t_3. \end{cases} \]

(29)

\[ \phi_n(\xi) := \begin{cases} e^{\lambda_2 \xi} - q e^{\eta \lambda_2 \xi}, & \text{if} \ \xi < t_2, \\ \varepsilon, & \text{if} \ \xi \geq t_2 \end{cases}, \quad \phi_n(\xi) := \begin{cases} e^{\lambda_2 \xi} + q e^{\eta \lambda_2 \xi}, & \text{if} \ \xi < t_4, \\ g_2(\alpha), & \text{if} \ \xi \geq t_4. \end{cases} \]

(30)
where \( t_1 \in (t_1^*, t_1^*_0) \), \( t_2 \in (t_2^*, t_2^*_0) \) with \( \ell_1(t_1) = \ell_2(t_2) = \varepsilon \) and \( t_3, t_4 \in \mathbb{R} \) with
\[
e^{\lambda_1 t_3} = \alpha \quad \text{and} \quad e^{\lambda_2 t_4 + qe^{\eta \lambda_2 t_4}} = g_2(\alpha).
\]
The graphs of \( \overline{\phi}_n(\xi) \) and \( \underline{\phi}(\xi) \) with \( n = 1, 2 \) are given in Figure 2.

![Graphs](image)

**Figure 2:** Graphs of \( \overline{\phi}_n(\xi) \) and \( \underline{\phi} \) with \( n = 1, 2 \).

Note that \( \varepsilon < \min\{\ell_1(t_1^*), \ell_2(t_2^*)\} < \min\{\alpha, g_2(\alpha)\} \). It is easy to see that \( \overline{\phi}_n(\xi) \) is not differentiable at \( t_{n+2} \) and \( \underline{\phi}_n(\xi) \) is not differentiable at \( t_n \) for \( n = 1, 2 \). Denoting \( \overline{\Phi}(\xi) := (\overline{\phi}_1(\xi), \overline{\phi}_2(\xi)) \) and \( \Phi(\xi) := (\underline{\phi}_1(\xi), \underline{\phi}_2(\xi)) \), then we have the following results.

**Lemma 2.3.** Assume \( q \) is large and \( \varepsilon \) is small enough. Then \( \overline{\Phi}(\xi) \) and \( \Phi(\xi) \) is a pair of upper and lower solutions of system (8) such that
\[
(0, 0) \leq \Phi(\xi) \leq \overline{\Phi}(\xi) \leq (\alpha, g_2(\alpha)), \quad \text{for all} \ \xi \in \mathbb{R},
\]
\[
\overline{\Phi}(t^*_n) \leq \Phi(t^-_n) \quad \text{and} \quad \Phi(t^*_n) \geq \overline{\Phi}(t^-_n), \quad \text{for} \ n = 1, 2, 3, 4.
\]

**Proof.** According to (29) and (30), it is clear that \( \Phi(\xi) \leq \overline{\Phi}(\xi) \) for all \( \xi \in \mathbb{R} \) and the inequalities (31) holds. Therefore, we check the conditions (12), (13), (14) and (15) in the sequel.

Let \( M > 0 \) be the constant defined by (27). For any \( \xi < t_1 \), by direct computations and (24), it is clear that
\[
- c\phi''_1(\xi) + d_1 \phi''_2(\xi) + \phi_{\alpha}(\xi) f_1(\phi_{\alpha}(\xi), \overline{\phi}_2(\xi))
\]
\[
= - c\phi''_1(\xi) + d_1 \phi''_2(\xi) + f_1(0, 0) \phi_{\alpha}(\xi) + \phi_{\alpha}(\xi) \left[ f_1(\phi_{\alpha}(\xi), \overline{\phi}_2(\xi)) - f_1(0, 0) \right]
\]
\[
= - c\phi''_1(\xi) + d_2 \phi''_2(\xi) + f_1(0, 0) \phi_{\alpha}(\xi) + \phi_{\alpha}(\xi) \left[ D_1 f_1(X_1) \phi_{\alpha}(\xi) + D_2 f_1(X_1) \overline{\phi}_2(\xi) \right]
\]
\[
\geq - c\phi''_1(\xi) + d_2 \phi''_2(\xi) + f_1(0, 0) \phi_{\alpha}(\xi) + e^{\lambda_1 \xi} \left[ D_1 f_1(X_1) e^{\lambda_1 \xi} + D_2 f_1(X_1) e^{\lambda_2 \xi} + qe^{\eta \lambda_2 \xi} \right]
\]
\[
\geq e^{\lambda_1 \xi} \Delta_1(\lambda_1) - qe^{\eta \lambda_1 \xi} \Delta_1(\eta \lambda_1) - Me^{\lambda_1 \xi} (e^{\lambda_1 \xi} + e^{\lambda_2 \xi} + qe^{\eta \lambda_1 \xi}) \geq 0,
\]
where \( X_1 \) lies between \( (0, 0) \) and \( (\phi_{\alpha}(\xi), \overline{\phi}_2(\xi)) \). For any \( \xi > t_1 \), it follows from (27) that
\[
- c\phi''_1(\xi) + d_1 \phi''_2(\xi) + \phi_{\alpha}(\xi) f_1(\phi_{\alpha}(\xi), \overline{\phi}_2(\xi)) = \varepsilon f_1(\varepsilon, \overline{\phi}_2(\xi)) \geq \varepsilon f_1(\varepsilon, g_2(\alpha)) > 0.
\]

Given any \( \xi < t_2 \), it is easy to see that
\[
- c\phi''_1(\xi) + d_2 \phi''_2(\xi) + \phi_{\alpha}(\xi) f_2(\phi_{\alpha}(\xi), \overline{\phi}_2(\xi))
\]
\[
= - c\phi''_1(\xi) + d_2 \phi''_2(\xi) + f_2(0, 0) \phi_{\alpha}(\xi) + \phi_{\alpha}(\xi) \left[ D_1 f_2(X_2) \phi_{\alpha}(\xi) + D_2 f_2(X_2) \overline{\phi}_2(\xi) \right]
\]
\[
\geq e^{\lambda_2 \xi} \Delta_2(\lambda_2) - qe^{\eta \lambda_2 \xi} \Delta_2(\eta \lambda_2) - Me^{\lambda_2 \xi} (qe^{\eta \lambda_2 \xi} + e^{\lambda_2 \xi}) - Me^{\lambda_2 \xi} (qe^{\eta \lambda_2 \xi} + e^{\lambda_2 \xi}) \geq 0.
\]
because of (25), where $X_2$ lies between $(0,0)$ and $(\phi_1(\xi), \phi_2(\xi))$. Given any $\xi > t_2$, by (27), one can see that

$$-c\phi_1'(\xi) + d_1\phi_2''(\xi) + \phi_2(\xi)f_2(\phi_1(\xi), \phi_2(\xi)) = \varepsilon f_2(\phi_1(\xi), \varepsilon) \geq \varepsilon f_2(0, \varepsilon) > 0.$$  

For any $\xi < t_3$, from the definition of $\lambda_1$, we know that

$$-c\phi_1'(\xi) + d_1\phi_2''(\xi) = \phi_2(\xi)f_1(\phi_1(\xi), \phi_2(\xi))$$

$$= -c\phi_2'(\xi) + d_1\phi_1''(\xi) + f_1(0,0)\phi_1(\xi) + \phi_1(\xi)(D_1f_1(X_3)\phi_1(\xi) + D_2f_1(X_3)\phi_2(\xi))$$

$$\leq \Delta_1(\lambda_1) = 0,$$

where $X_3$ lies between $(0,0)$ and $(\phi_1(\xi), \phi_2(\xi))$. For any $\xi > t_3$, it is clear that

$$-c\phi_1'(\xi) + d_1\phi_2''(\xi) + \phi_1(\xi)f_1(\phi_1(\xi), \phi_2(\xi)) = \alpha f_1(\alpha, \phi_2(\xi)) \leq \alpha f_1(\alpha, 0) = 0. \quad (33)$$

Given any $\xi < t_4$, by (26), it is also clear that

$$-c\phi_2'(\xi) + d_2\phi_1''(\xi) = \phi_1(\xi)f_1(\phi_1(\xi), \phi_2(\xi))$$

$$= -c\phi_2'(\xi) + d_2\phi_1''(\xi) + f_2(0,0)\phi_2(\xi) + \phi_2(\xi)[D_1f_2(X_4)\phi_1(\xi) + D_2f_2(X_4)\phi_2(\xi)]$$

$$\leq e^{\lambda_2\xi}\Delta_2(\lambda_2) - qe^{|\lambda_2|}\Delta_2(\eta\lambda_2) + M(e^{\lambda_2\xi} + qe^{|\lambda_2|})e^{\lambda_2\xi} \leq 0,$$

where $X_4$ lies between $(0,0)$ and $(\phi_1(\xi), \phi_2(\xi))$. Given any $\xi > t_4$, we have

$$-c\phi_1'(\xi) + d_2\phi_2''(\xi) + \phi_2(\xi)f_2(\phi_1(\xi), \phi_2(\xi)) = g_2(\alpha)f_2(\phi_1(\xi), g_2(\alpha))$$

$$\leq g_2(\alpha)f_2(\alpha, g_2(\alpha)) = 0.$$

The proof is complete. \qed

By Lemma 2.3, we can prove the existence of solutions for system (8) (in Section 4) by using crossing iteration method and Schauder’s fixed point theorem. However, from the formulas (29) and (30), we only know that $\Phi(\infty) = \Phi(\infty) = (0,0)$, and which implies the solutions of (8) satisfying the left condition of (9). To guarantee the solutions satisfying the right condition of (9), we need to establish the strictly contracting rectangles in the following subsection.

2.2. Existence of strictly contracting rectangles. First, we give the definition of the strictly contracting rectangle (cf. [33, Section 5.2]).

**Definition 2.4.** For any $s \in [0, 1]$, let $a(s) = (a_1(s), a_2(s))$, $b(s) = (b_1(s), b_2(s)) \in \mathbb{R}^2$. Then the set $[a(s), b(s)]$ is called a strict contracting rectangle if the following conditions hold:

$$a(0) = 0, \quad a(1) = b(1) = (k_1, k_2), \quad b(0) = (\alpha, g_2(\alpha)), \quad (34)$$

$$a_n(s) > 0 \text{ and } b_n(s) < 0, \text{ for any } s \in (0, 1), \text{ } n = 1, 2, \quad (35)$$

$$f_1(a_1(s), y) > 0, \text{ for any } s \in (0, 1) \text{ and } a_2(s) \leq y \leq b_2(s), \quad (36)$$

$$f_1(b_1(s), y) < 0, \text{ for any } s \in (0, 1) \text{ and } a_2(s) \leq y \leq b_2(s), \quad (37)$$

$$f_2(x, a_2(s)) > 0, \text{ for any } s \in (0, 1) \text{ and } a_1(s) \leq x \leq b_1(s), \quad (38)$$

$$f_2(x, b_2(s)) < 0, \text{ for any } s \in (0, 1) \text{ and } a_1(s) \leq x \leq b_1(s). \quad (39)$$

Let’s remark that the notation $a \leq b$ means the standard partial ordering in $\mathbb{R}^2$ and the notation $[a, b]$ stands for the rectangle $\{x \in \mathbb{R}^2 \mid a \leq x \leq b\}$. 
In the sequel, to prove the existence of the strictly contracting rectangle, we define the regions (see Figures 3–5).

\[ \Omega_1 := \{(x, y) \mid 0 \leq x < k_1, 0 \leq y < g_1(x)\}, \quad \Omega_2 := \{(x, y) \mid k_1 < x \leq 2\alpha, g_1(x) < y\}, \]
\[ \Omega_3 := \{(x, y) \mid 0 \leq x \leq 2\alpha, g_2(s) < y\}, \quad \Omega_4 := \{(x, y) \mid 0 \leq x \leq 2\alpha, 0 \leq y < g_2(s)\}. \]

Then we have the following result.

**Lemma 2.5.** Let’s define \( a_n(s) = sk_n \) for \( n = 1, 2 \),

\[ b_1(s) = sk_1 + 2(1 - s)\alpha \quad \text{and} \quad b_2(s) = sk_2 + 2(1 - s)g_2(\alpha). \]

For \( s \in (0, 1) \), we have

1. \( (a_1(s), b_2(s)) \in \Omega_1 \);
2. \( (b_1(s), a_2(s)) \in \Omega_2 \);
3. \( (b_1(s), b_2(s)) \in \Omega_3 \);
4. \( (a_1(s), a_2(s)) \in \Omega_4 \).

**Proof.** (1) Let \( L_1 \) be the line segment connecting \((0, 2g_2(\alpha))\) and \((k_1, k_2)\), i.e., the graph of the set \(\{(a_1(s), b_2(s)) \mid 0 \leq s \leq 1\}\). From (6) we know that \( g_1(0) > 2g_2(\alpha) = b_2(0) \). This implies that \((0, b_2(0))\) lies below the point \((0, g_1(0))\). Additionally, we also have \((k_1, k_2)\) lies on the graph of the function \(g_1\). Since \(g_1\) is concave, \(L_1\) lies below the graph of the function \(g_1\) on \([0, k_1]\) except the point \((k_1, k_2)\). Moreover, we also know that \(0 \leq a_1(s) < k_1\) and \(0 \leq b_2(s)\) for all \(s \in (0, 1)\). Therefore, we can conclude that \(L_1\backslash\{(k_1, k_2)\} \subseteq \Omega_1\) and it follows that \((a_1(s), b_2(s)) \in \Omega_1\) for any \(s \in (0, 1)\). See the left graph of Figure 3.

(2) Let \( L_2 \) be the line segment connecting \((2\alpha, 0)\) and \((k_1, k_2)\), i.e., the graph of the set \(\{(b_1(s), a_2(s)) \mid 0 \leq s \leq 1\}\). Note that the slope of the tangent \(L_{2T}\) to the graph of function \(g_1\) at \((k_1, k_2)\) is \(g_1'(k_1)\). Since \(g_1'\) is decreasing and \(g_2\) is increasing, by (H3) and the fact \(k_1 < \alpha\), it is easy to see that

\[ -g_1'(k_1) > -g_1'(0) > g_2(\alpha)/\alpha > g_2(k_1)/(2\alpha - k_1) = k_2/(2\alpha - k_1). \]

Then the slope of line \(L_{2T}\) is less than the slope of line \(L_2\). Since \(g_1\) is concave, \(L_2\) lies above the graph of function \(g_1\) on \([k_1, 2\alpha]\) except the point \((k_1, k_2)\). Thus, it is easy to check that \((b_1(s), a_2(s)) \in \Omega_2\) for any \(s \in (0, 1)\). See the right graph of Figure 3.

(3) Let \( L_3 \) be the line segment connecting \((2\alpha, 2g_2(\alpha))\) and \((k_1, k_2)\), i.e., the graph of the set \(\{(b_1(s), b_2(s)) \mid 0 \leq s \leq 1\}\). By (H3), we know that \(g_2(2\alpha) < 2g_2(\alpha)\). Since \(g_2\) is convex, \(L_3\) lies above the graph of function \(g_2\) on \([k_1, 2\alpha]\) except the point \((k_1, k_2)\). Then one can easily see that desired result is true. See the left graph of Figure 4.

(4) Let \( L_4 \) be the line segment connecting \((0, 0)\) and \((k_1, k_2)\), i.e., the graph of the set \(\{(a_1(s), a_2(s)) \mid 0 \leq s \leq 1\}\). Note also that the slope of the tangent \(L_{4T}\) to the graph of function \(g_2\) at \((k_1, k_2)\) is \(g_2'(k_1)\). It follows from (H3) that

\[ k_2/k_1 = g_2(k_1)/k_1 > g_2'(k_1). \]

This implies that the slope of line \(L_{4T}\) is less than the slope of the line \(L_4\). Since \(g_2\) is convex, \(L_4\) lies below the graph of function \(g_2\) on \([0, k_1]\) except the point \((k_1, k_2)\). Therefore, it is easy to see that desired result is true. See the right graph of Figure 4. The proof is complete. □
Lemma 2.6. For any $n = 1, 2$ and $s \in [0, 1]$, let us define

\[ a_n(s) = sk_n, \quad b_1(s) = sk_1 + 2(1-s)\alpha, \quad b_2(s) = sk_2 + 2(1-s)g_2(\alpha), \quad (40) \]

\[ a(s) = (a_1(s), a_2(s)) \quad \text{and} \quad b(s) = (b_1(s), b_2(s)). \]

Then $[a(s), b(s)] \text{ with } s \in [0, 1]$ is a strictly contracting rectangle (see Figure 5).

Proof. By (40), it is clear that (34) and (35) hold. According to Definition 2.4, we only need to verify the conditions (36)~(39). Since

\[ f_1(x_1, x_2) \begin{cases} > 0, & \text{if } (x_1, x_2) \in \Omega_1, \\ < 0, & \text{if } (x_1, x_2) \in \Omega_2, \end{cases} \quad \text{and} \quad f_2(x, y) \begin{cases} > 0, & \text{if } (x_1, x_2) \in \Omega_3, \\ < 0, & \text{if } (x_1, x_2) \in \Omega_4. \end{cases} \]

(41)

By (4), (41) and Lemma 2.5 (1), it is clear that

\[ f_1(a_1(s), y) > f_1(a_1(s), b_1(s)) > 0, \text{ for any } 0 < s < 1 \text{ and } a_2(s) \leq y \leq b_2(s). \]

Therefore the condition (36) holds. Similarly, we claim that (37), (38) and (39) hold by (4), (41) and Lemma 2.5 again. This completes the proof. \qed
3. **Asymptotic behavior of traveling wave solutions.** In this section, we will investigate the asymptotic behavior of positive monotonic traveling wave solutions of system (1), provided that such solutions exist. With the help of their asymptotic behavior, we can prove the statement of part (3) of Theorem 1.1 (see Section 5.)

To this end, we first introduce the following modified Ikehara’s Theorem.

**Lemma 3.1 (Cf. [13, 36])**. Let \( u(\xi) \) be a positive and decreasing function, and

\[
L(\lambda) := \int_0^\infty u(\xi)e^{-\lambda \xi} \, d\xi.
\]

Assume that \( L(\lambda) \) has the representation \( L(\lambda) = H(\lambda)/(\lambda - \Lambda_\ast)^{k+1} \), where \( k > -1 \) and \( H(\lambda) \) is analytic on the strip \( \Lambda_\ast \leq \text{Re} \lambda < 0 \). Then we have

\[
\lim_{\xi \to \infty} \frac{u(\xi)\xi^k e^{-\lambda \xi}}{\Gamma(-\Lambda_\ast + 1)} = \frac{H(\Lambda_\ast)}{\Gamma(-\Lambda_\ast + 1)}.
\]

Assume that system (8) admits a positive monotone solution \((\phi_1(\xi), \phi_2(\xi)) \) satisfying (9). Then we have the following properties.

**Lemma 3.2.** Let \((\phi_1(\xi), \phi_2(\xi)) \) be a positive monotone solution of system (8) satisfying (9). Then \( k_n > \phi_n(\xi) \) and \( \phi'_n(\xi) > 0 \) for all \( \xi \in \mathbb{R} \), \( n = 1, 2 \).

**Proof.** Let \( H_2(\Phi)(\xi) := \beta_2 \phi_2(\xi) + \phi_2(\xi)f_2(\phi_1(\xi), \phi_2(\xi)) \), where \( \beta_2 > 0 \) is chosen such that

\[
\frac{d}{dx_2} \left[ \beta_2 x_2 + x_2f_2(x_1, x_2) \right] > 0, \text{ for any } (x_1, x_2) \in [0, 2a] \times [0, g_1(0)]. \tag{42}
\]

By the same argument in [4, Lemma 2], we have that

\[
\phi_2(\xi) = \frac{1}{d_2(\rho_{22} - \rho_{12})} \left( \int_{-\infty}^{\xi} e^{\rho_{12}(\xi - s)}H_2(\Phi)(s)ds + \int_{\xi}^{\infty} e^{\rho_{22}(\xi - s)}H_2(\Phi)(s)ds \right), \tag{43}
\]

for all \( \xi \in \mathbb{R} \), where

\[
\rho_{12} = \frac{c - \sqrt{c^2 + 4\beta_2 d_2}}{2d_2} \quad \text{and} \quad \rho_{22} = \frac{c + \sqrt{c^2 + 4\beta_2 d_2}}{2d_2}. \tag{44}
\]

By (42), (H2) and the fact

\((\phi_1(-\infty), \phi_2(-\infty)) = (0, 0) \) and \((0, 0) \leq (\phi_1(\xi), \phi_2(\xi)) \leq (k_1, k_2)\).
one can see that

$$
H_2(\Phi)(\xi) \begin{cases} 
\leq \beta_2 k_2, & \text{for any } \xi \in \mathbb{R}, \\
< \beta_2 k_2, & \text{for any } \xi < -1.
\end{cases}
$$

(45)

According to (43) and (45), we have

$$
\phi_2(\xi) < \frac{1}{d_2(\rho_{22} - \rho_{12})} \left( \int_{-\xi}^{\xi} e^{\rho_{22}(s-\xi)} \beta_2 k_2 ds + \int_{\xi}^{\infty} e^{\rho_{22}(s-\xi)} \beta_2 k_2 ds \right) = k_2,
$$

for all $\xi \in \mathbb{R}$. On the other hand, if $\phi_1(\xi_0) = k_1$ for some $\xi_0$, then it is clear that

$$
\phi_1(\xi) = k_1 \text{ and } \phi'_1(\xi) = \phi''_1(\xi) = 0, \text{ for any } \xi \geq \xi_0,
$$

(46)

because $\phi_1(\xi)$ is monotone, smooth and satisfies (9). By (46), (H2) and the fact $\phi_2(\xi_0) < k_2$, one can easily see that

$$
0 = c_0 \phi'_1(\xi_0) = d_1 \phi''_1(\xi_0) + \phi_1(\xi_0) f_1(\phi_1(\xi_0), \phi_2(\xi_0)) > k_1 f_1(k_1, k_2) = 0.
$$

This leads a contradiction. Hence we can conclude that $\phi_1(\xi) < k_1$ for all $\xi \in \mathbb{R}$.

Next, we claim that $\phi'_1(\xi), \phi''_1(\xi) > 0$ for all $\xi \in \mathbb{R}$. Note first that $\phi'_1(\xi), \phi''_1(\xi) \geq 0$, for any $\xi \in \mathbb{R}$. If $\phi'_1(\xi_0) = 0$ for some $\xi_0$, then $\phi'_1(\xi_0)$ is a local minimum of $\phi'_1(\cdot)$ and hence $\phi''_1(\xi_0) = 0$. Due to $\phi''_1(\xi_0) = 0$, $0 < \phi_n(\xi_0) < k_n$ and (H2), it is clear that

$$
0 = c_0 \phi'_1(\xi_0) = d_1 \phi''_1(\xi_0) + \phi_1(\xi_0) f_1(\phi_1(\xi_0), \phi_2(\xi_0)) > \phi_1(\xi_0) f_1(k_1, k_2) = 0.
$$

This leads a contradiction. Similarly, if $\phi'_1(\xi_0) = 0$ for some $\xi_0$, we also have $\phi''_1(\xi_0)$ is a local minimum, $\phi''_1(\xi_0) = 0$ and $\phi''_1(\xi_0) \geq 0$. Differentiating the second equation of system (8), we can obtain

$$
0 = c_0 \phi''_1(\xi_0) = d_1 \phi''''_1(\xi_0) + \phi_2(\xi_0) f_2(\phi_1(\xi_0), \phi_2(\xi_0))
$$

$$
+ \phi_2(\xi_0) \left[ D_1 f_2(\phi_1(\xi_0), \phi_2(\xi_0)) \phi'_1(\xi_0) + D_2 f_2(\phi_1(\xi_0), \phi_2(\xi_0)) \phi''_1(\xi_0) \right]
$$

$$
\geq \phi_2(\xi_0) D_1 f_2(\phi_1(\xi_0), \phi_2(\xi_0)) \phi''_1(\xi_0) > 0
$$

by (H2) and the fact $\phi'_1(\xi_0) > 0$. This also gives a contradiction. Hence, we can conclude that $\phi'_1$ and $\phi''_1$ are both positive. The proof is complete.

According to Lemma 3.2, we consider the functions

$$
\psi_1(\xi) := k_1 - \phi_1(\xi) \quad \text{and} \quad \psi_2(\xi) := k_2 - \phi_2(\xi),
$$

(47)

where $(\phi_1(\xi), \phi_2(\xi))$ is a positive monotone solution of system (8) satisfying (9). Then $\psi_1(\xi)$ and $\psi_2(\xi)$ are positive and decreasing functions which satisfy

$$
\begin{cases}
0 = -c \psi'_1(\xi) + d_1 \psi''_1(\xi) + h_1(\psi_1(\xi), \psi_2(\xi)), \\
0 = -c \psi'_2(\xi) + d_2 \psi''_2(\xi) + h_2(\psi_1(\xi), \psi_2(\xi)) \quad \text{(48)}
\end{cases}
$$

where $h_1(\cdot, \cdot) : \mathbb{R}^2 \to \mathbb{R}$ are functions defined by

$$
h_n(x_1, x_2) := \begin{cases} 
(x_1 - k_1) f_1(k_1, k_2, x_2), & \text{if } n = 1, \\
(x_2 - k_2) f_2(k_1 - x_1, k_2, x_2), & \text{if } n = 2.
\end{cases}
$$

(49)

It is worthy mentioning that the value $\Lambda_*$ in Lemma 3.1 is related to the root of the characteristic function $\Delta_3(\lambda)$ which arises from the linearization of system (48) at $(0, 0)$ and is given by

$$
\Delta_3(\lambda) = p_1(\lambda)p_2(\lambda) - D_2 h_1(0, 0) D_1 h_2(0, 0),
$$

(50)

where

$$
p_n(\lambda) = d_n \lambda^2 - c \lambda + D_n h_n(0, 0), \quad \text{for } n = 1, 2.
$$

(51)
Therefore, we investigate the properties of roots for $\Delta_3(\lambda)$ in the following lemma.

**Lemma 3.3.** Assume that (10) holds. Then $\Delta_3(\lambda)$ has the following properties.

1. If $\Delta_3(\lambda)$ has a negative root, then $\Delta_3(\lambda)$ also has a positive root.
2. One of the following statements holds.
   1. $\Delta_3(\lambda)$ has exactly two negative roots $\mu_1, \mu_2$; and they are simple and satisfy
      \[ p_1(\mu_n) < 0 < p_2(\mu_n) \text{ and } \Delta_3(\lambda) \neq 0 \text{ for any } \lambda \neq \mu_n \text{ with } \text{Re}\lambda = \mu_n, \]  
      \[ \text{Moreover, by (49) and direct computations, we know that} \]
      \[ p_1(\mu) < 0 < p_2(\mu) \text{ and } \Delta_3(\lambda) \neq 0 \text{ for any } \lambda \neq \mu \text{ with } \text{Re}\lambda = \mu. \]
   2. $\Delta_3(\lambda)$ has exactly two negative roots $\mu$; and it is a double root and satisfies
      \[ p_1(\mu) < 0 < p_2(\mu) \text{ and } \Delta_3(\lambda) \neq 0 \text{ for any } \lambda \neq \mu \text{ with } \text{Re}\lambda = \mu. \]
   3. $\Delta_3(\lambda)$ has no negative root.

**Proof.** (1) Assume that $\Delta_3(\lambda)$ has a negative root $\lambda^-$, by (H2) and (49), we have
\[ p_1(\lambda^-)p_2(\lambda^-) = D_2h_1(0, 0)D_1h_2(0, 0) < 0. \]  
Since $\Delta_3(\lambda)$ is a polynomial of degree four, it suffices to show that $\Delta_3(\lambda_0) < 0$ for some $\lambda_0 > 0$. Since $\lambda^- < 0$, by (10), we have
\[ 0 < \frac{c}{2d_2} < \frac{c}{d_2} - \lambda^- < \frac{c}{d_1} - \lambda^- . \]  
Moreover, by (49) and direct computations, we know that
\[ D_nh_n(0, 0) = k_nD_nf_n(k_1, k_2), \text{ for } n = 1, 2. \]  
Then (10) and (56) imply that
\[ p_1(\lambda) = d_1\lambda^2 - c\lambda + D_1h_1(0, 0) \leq d_2\lambda^2 - c\lambda + D_2h_2(0, 0) = p_2(\lambda), \]  
for any $\lambda \in \mathbb{R}$. By (54), (55) and (57), it is clear that
\[ p_1(c/d_1 - \lambda^-) = p_1(\lambda^-) < 0 < p_2(\lambda^-) = p_2(c/d_2 - \lambda^-) < p_2(c/d_1 - \lambda^-). \]
Therefore, we know that
\[ \Delta_3(c/d_1 - \lambda^-) = p_1(c/d_1 - \lambda^-)p_2(c/d_1 - \lambda^-) - D_2h_1(0, 0)D_1h_2(0, 0) < p_1(\lambda^-)p_2(\lambda^-) - D_2h_1(0, 0)D_1h_2(0, 0) = \Delta_3(\lambda^-) = 0. \]
It is clear that $\lambda_0 := c/d_1 - \lambda^- > 0$. Hence the assertion of this part follows.

(2) First, we denote
\[ \omega_1^+ := \frac{c \pm \sqrt{c^2 - 4d_1D_1h_1(0, 0)}}{2d_1} \text{ and } \omega_2^+ := \frac{c \pm \sqrt{c^2 - 4d_2D_2h_2(0, 0)}}{2d_2}. \]  
It is clear that $\omega_n^+$ and $\omega_n^-$ are two distinct roots of polynomial $p_n(\lambda)$, where $n = 1, 2$. Therefore, it follows from (H2) that
\[ \Delta_3(\omega_1^+) = \Delta_3(\omega_2^+) = -D_2h_1(0, 0)D_1h_2(0, 0) > 0. \]  
Additionally, the fact (10) implies that
\[ D_1h_1(0, 0) = k_1D_1f_1(k_1, k_2) < k_2D_2f_2(k_1, k_2) = D_2h_2(0, 0). \]  
By (10) and (60), we have that $P_1(\lambda) \leq P_2(\lambda)$ on $\mathbb{R}$ and it follows that
\[ \omega_1^- < \omega_2^- < 0 < \omega_2^+ < \omega_1^+. \]
By (59) and Rolle’s theorem, there are three numbers $\nu_1, \nu_2$ and $\nu_3$ satisfy
\[ \Delta_3(\nu_1) = \Delta_3(\nu_2) = \Delta_3(\nu_3) = 0 \text{ and } \omega_1^- < \nu_1 < \omega_2^- < \nu_2 < \omega_2^+ < \nu_3 < \omega_1^+. \]
Since \( \Delta'_3(\lambda) \) is a polynomial of degree three with positive leading number, we have
\[
\Delta'_3(\lambda) \begin{cases} 
> 0, & \text{if } \lambda \in (\nu_1, \nu_2) \cup (\nu_3, \infty), \\
< 0, & \text{if } \lambda \in (-\infty, \nu_1) \cup (\nu_2, \nu_3).
\end{cases}
\tag{62}
\]
That is, \( \Delta_3(\lambda) \) is increasing on \((\nu_1, \nu_2) \cup (\nu_3, \infty)\) and decreasing on \((-\infty, \nu_1) \cup (\nu_2, \nu_3)\). Moreover, one can easily check that \( \Delta_3(\nu_1) \) is the minimum of \( \Delta_3(\lambda) \) on \((\nu_1, \nu_2) \cup (\nu_3, \infty)\).

If \( \Delta_3(\nu_1) \) is negative, by Intermediate Value Theorem, \( \Delta_3(\lambda) \) has exactly two negative roots \( \mu_1 \) and \( \mu_2 \) satisfying
\[
\omega^-_1 < \mu_1 < \nu_1 < \mu_2 < \omega^-_2.
\tag{63}
\]
By (62), it is clear that \( \mu_1 \) and \( \mu_2 \) are both simple roots. Additionally, since \( \omega^+_n \) are roots of polynomial \( P_n(\lambda) \), by (63), it is clear that
\[
p_1(\mu_n) < 0 < p_2(\mu_n), \text{ for } n = 1, 2.
\]
Moreover, by Lemma 3.3, we know that \( \Delta_3(\lambda) \) has at least one positive root. Since \( \Delta_3(\lambda) \) is a real polynomial of degree four, it follows that \( \Delta_3(\lambda) \) has no complex root, i.e.,
\[
\Delta_3(\lambda) \neq 0 \text{ for any } \lambda \neq \mu_n \text{ with } \Re \lambda = \mu_n.
\]
On the other hand, if the condition \( \Delta_3(\nu_1) = 0 \) holds, by similar arguments, we know that \( \Delta_3(\lambda) \) has exactly one negative root \( \mu \) which is a double root and satisfies (63). If the condition \( \Delta_3(\nu_1) > 0 \) holds, it is clear that \( \Delta_3(\lambda) \) has no negative root because \( \Delta_3(\nu_1) \) is the minimum of \( \Delta_3(\lambda) \) on \((\nu_1, \nu_2) \cup (\nu_3, \infty)\). The proof is complete. □

**Lemma 3.4.** Assume system (8) has a positive monotone solution \((\phi_1(\xi), \phi_2(\xi))\) satisfying (9). Then the characteristic function \( \Delta_3(\lambda) \) has a negative root \( \Lambda \) satisfying
\[
\int_0^\infty e^{-\lambda \xi} \psi_n(\xi) \, d\xi \begin{cases} 
< \infty, & \text{if } \lambda \in \mathbb{C} \text{ with } \Re \lambda \in (\Lambda, 0), \\
= \infty, & \text{if } \lambda \in \mathbb{C} \text{ with } \Re \lambda \in (-\infty, \Lambda),
\end{cases}
\tag{64}
\]
for any \( 1 \leq n \leq 2 \), where \( \psi_1(\xi) = k_1 - \phi_1(\xi) \) and \( \psi_1(\xi) = k_1 - \phi_1(\xi) \).

**Proof.** Given any \( 1 \leq n \leq 2 \), let us define
\[
\overline{\Lambda}_n := \inf \left\{ \lambda < 0 \mid \int_0^\infty e^{-\lambda \xi} \psi_n(\xi) \, d\xi < \infty \right\},
\tag{65}
\]
\[
\Lambda_n := \inf \left\{ \lambda < 0 \mid \psi_n(\xi) = O(e^{\lambda \xi}) \text{ and } \psi_n'(\xi) = O(e^{\lambda \xi}) \text{ for } \xi \gg 1 \right\}.
\]
Since \((\psi_1(\xi), \psi_2(\xi))\) satisfies (48), one can verify that
\[
(\psi_1(\infty), \psi_1'(\infty), \psi_2(\infty), \psi_2'(\infty)) = (0, 0, 0, 0).
\]
Then it follows from the stable manifold theorem that
\[
\left\{ \lambda < 0 \mid \psi_n(\xi) = O(e^{\lambda \xi}) \text{ and } \psi_n'(\xi) = O(e^{\lambda \xi}) \text{ for } \xi \gg 1 \right\} \neq \emptyset,
\]
for any \( 1 \leq n \leq 2 \). This implies that \( \Lambda_n \) and \( \overline{\Lambda}_n \) are well defined because of
\[
\left\{ \lambda < 0 \mid \psi_n(\xi) = O(e^{\lambda \xi}) \text{ and } \psi_n'(\xi) = O(e^{\lambda \xi}) \text{ for } \xi \gg 1 \right\}
\subseteq \left\{ \lambda < 0 \mid \int_0^\infty e^{-\lambda \xi} \psi_n(\xi) \, d\xi < \infty \right\}, \text{ where } 1 \leq n \leq 2.
Now we multiply both sides of the first and second equations in system (48) by $e^{-\lambda \xi}$ and integrate them over $(-\infty, s)$. By integration by parts, one can see

$$e^{-\lambda s}[(d_1 \lambda - c)\psi_1(s) + d_1\psi_1'(s)]$$

$$+ \int_{-\infty}^{s} e^{-\lambda \xi}[p_1(\lambda)\psi_1(\xi) + D_2 h_1(0,0)\psi_2(\xi)]d\xi = J_1(\lambda), \tag{66}$$

$$e^{-\lambda s}[(d_2 \lambda - c)\psi_2(s) + d_2\psi_2'(s)]$$

$$+ \int_{-\infty}^{s} e^{-\lambda \xi}[p_2(\lambda)\psi_2(\xi) + D_1 h_2(0,0)\psi_1(\xi)]d\xi = J_2(\lambda), \tag{67}$$

where $p_n(\lambda)$ is defined by (51) and

$$J_n(\lambda) = \int_{-\infty}^{s} e^{-\lambda \xi}[D_1 h_n(0,0)\psi_1(\xi) + D_2 h_n(0,0)\psi_2(\xi) - h_n(\psi_1(\xi), \psi_2(\xi))]d\xi.$$  

Let $\lambda \in \mathbb{C}$ with max$\{\Lambda_1, \Lambda_2\} < \text{Re}\lambda < 0$ be fixed and $s$ tend to positive infinity. Then one can see

$$\begin{pmatrix} p_1(\lambda) & D_2 h_1(0,0) \\ D_1 h_2(0,0) & p_2(\lambda) \end{pmatrix} \begin{pmatrix} Q_1(\lambda) \\ Q_2(\lambda) \end{pmatrix} = \begin{pmatrix} J_1(\lambda) \\ J_2(\lambda) \end{pmatrix}, \tag{68}$$

where

$$Q_n(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda \xi} \psi_n(\xi)d\xi. \tag{69}$$

In the following, we want to use (66), (67) and (68) to claim $\Lambda_1 = \Lambda_2 = \overline{\Lambda}_1 = \overline{\Lambda}_2$. To this end, assume the desired result is not true. Here we just focus on the condition $\Lambda_1 < \Lambda_2$ because the other conditions can be proved by similar arguments. Additionally, due to $\overline{\Lambda}_2 \leq \Lambda_2$, we divide this proof into two parts.

Case 1: $\overline{\Lambda}_2 = \Lambda_2$. By (68), it is easy to check that

$$D_2 h_1(0,0) \int_{0}^{s} e^{-\lambda \xi} \psi_2(\xi) d\xi = J_1(\lambda) - p_1(\lambda)Q_1(\lambda) - D_2 h_1(0,0) \int_{-\infty}^{0} e^{-\lambda \xi} \psi_2(\xi) d\xi, \tag{70}$$

for any $\Lambda_2 = \max\{\Lambda_1, \Lambda_2\} < \text{Re}\lambda < 0$. By the definition of $\overline{\Lambda}_2$ and the property of Laplace transforms [23, page 58], the left side in (70) is singular at $\Lambda_2$. However, the right side in (70) is nonsingular at $\Lambda_2$ because of the definition of $J_1(\lambda)$ and the fact $\Lambda_1 < \Lambda_2$.

Case 2: $\overline{\Lambda}_2 < \Lambda_2$. By the second equation in (68), we have

$$\int_{-\infty}^{s} \left[p_2(\lambda)e^{-\lambda \xi} \psi_2(\xi) + D_1 h_2(0,0)e^{-\lambda \xi} \psi_1(\xi)\right]d\xi = J_2(\lambda), \tag{71}$$

where $\Lambda_2 = \max\{\Lambda_1, \Lambda_2\} < \text{Re}\lambda < 0$. It is observed that the left and right sides in (71) are both analytic on the strip max$\{\overline{\Lambda}_1, \overline{\Lambda}_2\} < \text{Re}\lambda < 0$. Thus, by [5, Theorem 3.7], we know that (71) is true on the strip max$\{\overline{\Lambda}_1, \overline{\Lambda}_2\} < \text{Re}\lambda < 0$. By this fact and (67), we have

$$\lim_{s \to -\infty} e^{-\lambda s} \psi_2(s) = \lim_{s \to -\infty} d_2 e^{-\lambda s} \psi_2'(s) = 0,$$

for any max$\{\overline{\Lambda}_1, \overline{\Lambda}_2\} < \lambda < \Lambda_2 = \max\{\Lambda_1, \Lambda_2\}$, because of the facts

$$(d_1 \lambda - c)e^{-\lambda s} \psi_1(s) < 0, \quad d_1 e^{-\lambda s} \psi_1'(s) < 0.$$

This is contrary to the definition of the value $\Lambda_2$. Therefore, we can conclude that $\Lambda_1 = \Lambda_2$ and the claim holds.
Based on the previous claim, let’s define
\[ \Lambda := \Lambda_1 = \Lambda_2 = \overline{\Lambda}_1 = \overline{\Lambda}_2. \] (72)

Now we show that \( \Lambda > -\infty \). From (68), it is easy to check that
\[
\begin{pmatrix}
Q_1(\lambda) \\
Q_2(\lambda)
\end{pmatrix}
= \frac{1}{\Delta_3(\lambda)}
\begin{pmatrix}
p_2(\lambda) & -D_2 h_2(0, 0) \\
-D_1 h_2(0, 0) & p_1(\lambda)
\end{pmatrix}
\begin{pmatrix}
J_1(\lambda) \\
J_2(\lambda)
\end{pmatrix}
\] (73)
for any \( \Lambda < \Re\lambda < 0 \) with \( \Delta_3(\lambda) \neq 0 \). Add the first equation in (73) to second equation in (73) and then we get
\[
\Delta_3(\lambda) [Q_1(\lambda) + Q_2(\lambda)] + [D_1 h_2(0, 0) - p_2(\lambda)] J_1(\lambda)
+ [D_1 h_2(0, 0) - p_1(\lambda)] J_2(\lambda) = 0,
\] (74)
for any \( \Lambda < \Re\lambda < 0 \). If the condition \( \Lambda = -\infty \) holds, one can choose \( \lambda < 0 \) with \( |\lambda| \) being large such that the sum of the integrand in each term of equation (74) is positive. This leads a contradiction. Thus, we have that \( \Lambda > -\infty \). Additionally, it follows from (72) and the definition of \( \Lambda_n \) that (64) is true.

Finally, we claim that \( \Lambda \) is a negative root of \( \Delta_3(\lambda) \). By (73) again, it is clear that
\[
\int_{0}^{\infty} e^{-\lambda \psi_1(\xi)} d\xi = \frac{-D_2 h_2(0, 0) J_2(\lambda) + p_2(\lambda) J_1(\lambda)}{\Delta_3(\lambda)} - \int_{-\infty}^{0} e^{-\lambda \psi_1(\xi)} d\xi.
\] (75)
for any \( \Lambda < \Re\lambda < 0 \) and \( \Delta_3(\lambda) \neq 0 \). From the definition of \( \Lambda \) and the property of Laplace transforms [23, page 58], one can see the left side of equation (75) is non-singular at \( \Lambda \). But, if the condition \( \Delta_3(\lambda) \neq 0 \) holds, it is easy to check that the right side of equation (75) is non-singular at \( \Lambda \), which gives a contradiction. The proof is complete. 

Combining the results of Lemmas 3.2 and 3.3, we can use Lemma 3.1 to derive the asymptotic behavior of functions \( \psi_1(\xi) \) and \( \psi_2(\xi) \), where \( \psi_n(\xi) \) is defined by (47). The result is as follows.

**Lemma 3.5.** Let \( (\psi_1(\xi), \psi_2(\xi)) \) be defined by (47) and \( \Lambda \) be defined by Lemma 3.4.

1. If the condition (2-1) in Lemma 3.3 holds, then we have
\[
\lim_{\xi \to \infty} \frac{\psi_1(\xi)}{e^{\Lambda\xi}} = m_1 > 0 \quad \text{and} \quad \lim_{\xi \to \infty} \frac{\psi_2(\xi)}{e^{\Lambda\xi}} = m_2 > 0.
\] (76)

2. If the condition (2-2) in Lemma 3.3 holds, then we have
\[
\lim_{\xi \to \infty} \frac{\psi_1(\xi)}{\xi e^{\Lambda\xi}} = m_1 > 0 \quad \text{and} \quad \lim_{\xi \to \infty} \frac{\psi_2(\xi)}{\xi e^{\Lambda\xi}} = m_2 > 0.
\]

**Proof.** We only prove the statement of (1), since the statement of (2) can be proved by the same way. Let us define
\[
H(\lambda) := (\lambda - \Lambda) \int_{0}^{\infty} e^{-\lambda \psi_1(\xi)} d\xi,
\]
for any \( \Lambda < \Re\lambda < 0 \); and define
\[
H(\lambda) := \frac{-D_2 h_2(0, 0) J_2(\lambda) + p_2(\lambda) J_1(\lambda)}{\Delta_3(\lambda)/\lambda - \Lambda} - (\lambda - \Lambda) \int_{-\infty}^{0} e^{-\lambda \psi_1(\xi)} d\xi,
\]
for any \( Re\lambda = \Lambda \), where \( p_n(\lambda) \) and \( I_n(\lambda) \) are defined by (51) and (69), respectively. It follows from Lemma 3.3 (2-1) and Lemma 3.4 that \( \Lambda \) is a simple root of \( \Delta_3(\lambda) \) and

\[
\Delta_3(\lambda) \neq 0, \text{ for any } \lambda \in \mathbb{C} \text{ with } \lambda \neq \Lambda \text{ and } Re\lambda = \Lambda. \tag{77}
\]

Then it is easy to check that \( H(\lambda) \) is well defined and analytic on the strip \( \Lambda \leq Re\lambda < 0 \) because of the facts (75) and (77). Moreover, by Lemma 3.2, one can also see \( \psi_1(\xi) > 0 \) and \( \psi_2(\xi) < 0 \) on \( \mathbb{R} \). Thus, Lemma 3.1 implies that

\[
\lim_{\xi \to \infty} \psi_1(\xi)/e^{\lambda \xi} = H(\Lambda)/\Gamma(-\Lambda + 1).
\]

In the sequel, we claim that \( H(\Lambda) \neq 0 \). Suppose the claim is false, i.e. \( H(\Lambda) = 0 \). By the definition of \( H(\lambda) \), it is clear that

\[
-D_2h_1(0,0)J_2(\lambda) + p_2(\lambda)J_1(\lambda) = 0. \tag{78}
\]

Since \( \Lambda \) is a simple root of \( \Delta_3(\lambda) \), by (75) and (78), we know that the left hand term in equation (75) is non-singular at \( \lambda = \Lambda \) which is contrary to the property of Laplace transformation (see [23, page 58]). Thus, we have that \( H(\Lambda) \neq 0 \).

Additionally, we have \( H(\Lambda)/\Gamma(-\Lambda + 1) > 0 \) because of the facts \( \psi_1(\xi) > 0 \) and \( H(\Lambda) \neq 0 \). Now let us set \( m_1 := H(\Lambda)/\Gamma(-\Lambda + 1) \). Moreover, by similar arguments, one can also prove that there exists a positive number \( m_2 \) such that

\[
\lim_{\xi \to \infty} \psi_2(\xi)/e^{\lambda \xi} = m_2. \tag{79}
\]

The proof is complete.

4. **Proofs of Theorem 1.1 and Theorem 1.2.** According to the results of previous sections, we are ready to proof the result of parts (1) and (2) of Theorem 1.1 by using the Schauder’s fixed point combining with the strictly contracting rectangles. In addition, we show the non-monotonicity of traveling wave solutions by the result of Lemma 3.5.

To this end, we first choose positive large numbers \( \beta_1 \) and \( \beta_2 \) such that

\[
\frac{d}{dx_n}[\beta_n x_n + x_n f_n(x_1, x_2)] > 0, \text{ for any } (x_1, x_2) \in [0, 2\alpha] \times [0, g_1(0)]. \tag{79}
\]

Then we define the space \( S \) and operators \( H_n \) and \( G = (G_1, G_2) \) by

\[
S := \{ \Phi | \Phi(\xi) \text{ is a bounded and uniformly continuous function from } \mathbb{R} \text{ to } \mathbb{R}^2 \},
\]

\[
H_n(\Phi)(\xi) := \beta_n \phi_n(\xi) + \phi_n(\xi) f_n(\phi_1(\xi), \phi_2(\xi)),
\]

\[
G_n(\Phi)(\xi) := \frac{1}{d_n(\rho_{2n} - \rho_{1n})} \left( \int_{-\infty}^{\xi} e^{\rho_{1n}(\xi-s)} H_n(\Phi)(s) ds + \int_{\xi}^{\infty} e^{\rho_{2n}(\xi-s)} H_n(\Phi)(s) ds \right),
\]

for any \( \Phi(\xi) = (\phi_1(\xi), \phi_2(\xi)) \in S, n = 1, 2, \) where

\[
\rho_{1n} = c - \sqrt{c^2 + 4\beta_n d_n} \quad \text{and} \quad \rho_{2n} = c + \sqrt{c^2 + 4\beta_n d_n}. \tag{80}
\]

By direct computations, we see that

\[
d_n G_n''(\Phi)(\xi) - cG_n''(\Phi)(\xi) - \beta_n G_n(\Phi)(\xi) + H_n(\Phi)(\xi) = 0, \text{ for } n = 1, 2.
\]

It’s obvious that any fixed points of operator \( G \) are solutions of system (8). Hence our goal is to prove the existence of fixed points of \( G \).
4.1. Proofs of parts (1) and (2) of Theorem 1.1. (1) Assume that $0 < c < c^*$. By (5), we have that $f_1(0,0) > 0$ and $f_2(0,0) > 0$. Then one can use the arguments in [29, Theorem 3.6] to prove that system (8) has no positive solution satisfying (9) for any $0 < c < c^*$. Since the proof is completely the same as [29, Theorem 3.6], we skip the details here.

(2) Assume that $c > c^*$. Let $\Phi_1(\xi)$ and $\Phi_2(\xi)$ be upper and lower solutions of system (8) obtained in Lemma 2.3. We set the subspace of $S$

$$\Gamma := \{ \Phi \mid \Phi(\xi) \in S \text{ and } \Phi_1(\xi) \leq \Phi(\xi) \leq \Phi_2(\xi) \}.$$ In addition, we equip the space $S$ with the decay norm $\| \cdot \|_\rho$ defined by

$$\| \Phi \|_\rho := \sup \{ |\Phi(\xi)|e^{-\rho|\xi|} \mid \xi \in \mathbb{R} \},$$ (81)

where $\rho = \min\{-\rho_{11}, -\rho_{12}, -\rho_{21}, -\rho_{22}\}$ and $\rho_{jn}$ is defined by (80). It’s easy to verify that $S$ is a Banach space endowed with the norm $\| \cdot \|_\rho$. Similar to the proof of [18, 22], one can see that operator $G : \Gamma \rightarrow \Gamma$ is continuous and compact; and $\Gamma$ is nonempty, convex, closed and bounded with the decay norm $\| \cdot \|_\rho$. Therefore, by the Schauder’s fixed point theorem, the operator $G$ admits a fixed point $\Phi^*(\xi) = (\phi_1^*(\xi), \phi_2^*(\xi))$ in $\Gamma$. This implies that system (8) has a solution $\Phi^*(\xi)$.

Next, we show that $\Phi^*(\xi)$ satisfies the condition (9). Since $\Phi_1(\xi) \leq \Phi^*(\xi) \leq \Phi_2(\xi)$, by the construction of $\Phi_1(\xi)$ and $\Phi_2(\xi)$, we have that $\Phi^*(-\infty) = (0,0)$ and $\liminf_{\xi \rightarrow \infty} \phi_n^*(\xi) > 0$, for any $n = 1, 2$.

In addition, by Lemma 2.6, we know that there exists a strict contracting rectangle. Note that $\Phi''(\xi)$ and $\Phi^{*''}(\xi)$ are uniform bounded. Then, following the same argument as that in [28, Theorem 3.2], we can conclude that $\Phi^*(\infty) = (k_1, k_2)$. The proof is complete.

4.2. Proof of part (3) of Theorem 1.1. Suppose that system (8) has a positive monotone solution $(\phi_1(\xi), \phi_2(\xi))$ satisfying (9). Then we define

$$\psi_1(\xi) := k_1 - \phi_1(\xi) \quad \text{and} \quad \psi_2(\xi) := k_2 - \phi_2(\xi).$$

Recall that $(\psi_1(\xi), \psi_2(\xi))$ satisfies system (48). Since the condition (10) holds, by Lemma 3.3 (2), we divide our proof into three cases.

**Case 1.** Assume that (2-1) in Lemma 3.3 holds.

Let $m_1$, $m_2$ and $\Lambda$ be the numbers defined in (76) and Lemma 3.4, respectively, $p_n(\Lambda)$ is the polynomial defined by (51) and $h_n(\cdot, \cdot)$ is function given by (49). We first claim that

$$0 = p_1(\Lambda)m_1 + D_2 h_1(0,0)m_2,$$ (82)

To prove the claim, we divide the first equation in (48) by $e^{\lambda \xi}$. Then one can see

$$0 = -c \psi_1'(\xi) e^{\Lambda \xi} + \frac{d_1 \psi_1''(\xi)}{e^{\lambda \xi}} + \frac{h_1(\psi_1(\xi), \psi_2(\xi))}{e^{\lambda \xi}}.$$ (83)

From (83), we see the claim holds provided that

$$\lim_{\xi \rightarrow \infty} \frac{\psi_1'(\xi)}{e^{\lambda \xi}} = m_1 \Lambda, \quad \lim_{\xi \rightarrow \infty} \frac{\psi_1''(\xi)}{e^{\lambda \xi}} = m_1 \Lambda^2$$

and

$$\lim_{\xi \rightarrow \infty} \frac{h_1(\psi_1(\xi), \psi_2(\xi))}{e^{\lambda \xi}} = \sum_{n=1}^{2} D_n h_1(0,0)m_n.$$
By Mean Value Theorem, we know that
\[
\lim_{\xi \to \infty} \frac{h_1(\psi_1(\xi), \psi_2(\xi))}{e^{\Lambda \xi}} = \lim_{\xi \to \infty} \frac{D_1 h_1(x(\xi))\psi_1(\xi) + D_2 h_1(x(\xi))\psi_2(\xi)}{e^{\Lambda \xi}} = D_1 h_1(0, 0)m_1 + D_2 h_1(0, 0)m_2,
\]
where \(x(\xi)\) lies on the line segment connecting \((0, 0)\) and \((\psi_1(\xi), \psi_2(\xi))\).

Next, we claim that \(\lim_{\xi \to \infty} \psi_1'(\xi)/e^{\Lambda \xi} = \Lambda\). To this end, we integrate the first equation of system (48) over \((\xi, \infty)\) and obtain
\[
0 = c\psi_1(\xi) - d_1 \psi_1'(\xi) + \int_{\xi}^{\infty} h_1(\psi_1(s), \psi_2(s))ds,
\]
since \(\psi_1(\infty) = \psi_1'(\infty) = 0\). By (91) and the fact \(\psi_1(\infty) = \psi_1'(\infty) = 0\) again, we have
\[
\lim_{\xi \to \infty} \int_{\xi}^{\infty} h_1(\psi_1(s), \psi_2(s))ds = 0.
\]
Then (84) and L’Hospital’s rule imply that
\[
\lim_{\xi \to \infty} \int_{\xi}^{\infty} h_1(\psi_1(s), \psi_2(s))ds/e^{\Lambda \xi} = \lim_{\xi \to \infty} \frac{-h_1(\psi_1(\xi), \psi_2(\xi))}{\Lambda e^{\Lambda \xi}} = \frac{-D_1 h_1(0, 0)m_1 - D_2 h_1(0, 0)m_2}{\Lambda}.
\]
By (76), (91) and (86), we can conclude that \(\lim_{\xi \to \infty} \psi_1'(\xi)/e^{\Lambda \xi}\) exists. By this fact, (83) and (84), one also can see \(\lim_{\xi \to \infty} \psi_1''(\xi)/e^{\Lambda \xi}\) exists. Thus, it follows from L’Hospital’s Rule and the fact \(\psi_1(\xi) = \psi_1'(\xi) = \psi_1''(\xi) = 0\) that
\[
\lim_{\xi \to \infty} \frac{\psi_1(\xi)}{e^{\Lambda \xi}} = \lim_{\xi \to \infty} \frac{\psi_1'(\xi)}{\Lambda e^{\Lambda \xi}} = \lim_{\xi \to \infty} \frac{\psi_1''(\xi)}{\Lambda^2 e^{\Lambda \xi}}.
\]
Then (76) and (87) imply that
\[
\lim_{\xi \to \infty} \frac{\psi_1'('\xi)}{e^{\Lambda \xi}} = m_1 \Lambda \text{ and } \lim_{\xi \to \infty} \frac{\psi_1''(\xi)}{e^{\Lambda \xi}} = m_1 \Lambda^2.
\]
by (83), (84) and (88), it is easy to see that (82) is true.

However, Lemma 3.4 and Lemma 3.3 (2-1), it is clear that \(p_1(\Lambda) < 0\). Moreover, by direct computations and assumption (H2), we know that
\[
D_2 h_1(0, 0) = k_1 D_2 f_1(k_1, k_2) < 0.
\]
Then we have \(0 = p_1(\Lambda)m_1 + D_2 h_1(0, 0)m_2 < 0\) which leads a contradiction.

**Case 2.** Assume that (2-2) in Lemma 3.3 holds.

The proof for this part is similar to previous condition and hence the proof is skipped.

**Case 3.** Assume that (2-3) in Lemma 3.3 holds.

Under this condition, we know that \(\Delta_3(\lambda)\) has no negative root. However, it follows from Lemma 3.4 that \(\Lambda\) is a negative root of \(\Delta_3(\lambda)\). This leads a contradiction.

With the help of above results, we know that system (8) has no positive monotone solution satisfying (9). In other words, if \((\phi_1(\xi), \phi_2(\xi))\) is such a solution, then either \(\phi_1(\xi)\) or \(\phi_2(\xi)\) is non-monotone. The proof of Theorem 1.1 is complete.

Next, motivated by the works of [16, 24, 31, 32], we prove the results of Theorem 1.2 as follows.
4.3. Proof of Theorem 1.2. Under the assumption of Theorem 1.2, we know that 
\(d_1 f_1(0,0) < d_2 f_2(0,0)\) and \(\lambda_2 < \lambda_1\) (by (11)). Let’s set \(f(w) := w f_2(0, w)\), then \(f(w)\) is concave for \(w \in [0, g_2(0)]\). Now we consider the following equation

\[ w_t(x,t) = d_2 \Delta w(x,t) + f(w(x,t)), \quad \text{for } x, t \in \mathbb{R}. \]  

(89)

According to the assumptions of \(f_2(\cdot, \cdot)\), one can verify that

\[ f(0) = f(g_2(0)) = 0, \quad f'(0) = f_2(0, 0) > 0, \quad f'(g_2(0)) = g_2(0) D_2 f_2(0, g_2(0)) < 0. \]

Hence (89) is a Fisher’s equation. From the known literature (see e.g., [37]), equation (89) admits a traveling wave \(w(x,t) = \varphi(x + e^t) = \varphi(\xi)\) satisfying the equation

\[ c^* \varphi'(\xi) = d_2 \varphi''(\xi) + \varphi(\xi) f_2(0, \varphi(\xi)) \]

and the following properties:

\[ 0 < \varphi(\xi) < g_2(0), \quad \lim_{\xi \to -\infty} \varphi(\xi) = 0, \quad \lim_{\xi \to \infty} \varphi(\xi) = g_2(0) \quad \text{and} \quad \lim_{\xi \to -\infty} \frac{\varphi(\xi)}{\xi e^{\lambda_2 \xi}} = -K, \]

(90)

for some \(K > 0\).

For convenience, we denote the constants \(M\) and \(M^-\) by

\[
M := \max \{ |D_2 f_2(x_1, x_2)| \mid 1 \leq i, j \leq 2 \text{ and } (x_1, x_2) \in [0, \alpha] \times [0, g_2(\alpha)] \},
\]

\[
M^- := \max \{ D_2 f_2(x_1, x_2) \mid (x_1, x_2) \in [0, \alpha] \times [0, g_2(\alpha)] \}.
\]

By (90), we may choose \(\xi_0 < 0\) such that

\[ 0 < \varphi(\xi) < (K + 1)|\xi|e^{\lambda_2 \xi}, \quad \forall \xi < \xi_0. \]

Then we define the functions

\[ \phi_2(\xi) := \varphi(\xi), \forall \xi \in \mathbb{R} \quad \text{and} \quad \overline{\phi}_2(\xi) := \begin{cases} -L\xi e^{\lambda_2 \xi}, & \text{if } \xi < t_4, \\ g_2(\alpha), & \text{if } \xi \geq t_4, \end{cases} \]

where \(t_4 < \min\{\xi_0, -1/\lambda_2\}\) and satisfies

\[ L := \frac{g_2(\alpha)}{-t_4 e^{\lambda_2 t_4}} > K + 1, \quad M < -M L e^{(\lambda_2 - \lambda_1) t_4} \text{ and } -\xi e^{\lambda_2 \xi} < e^{\lambda_2 \xi/2}, \quad \forall \xi < t_4. \]  

(91)

In addition, we also denote the functions \(\phi_3(\xi)\) and \(\overline{\phi}_3(\xi)\) as the form of (29) with \(q > 0\), small \(\varepsilon > 0\) and \(\eta\) satisfying the following condition:

\[ 1 < \eta < \min \{ 2, \lambda_1^+ / \lambda_1, (\lambda_1 + \lambda_2/2)/\lambda_1 \}. \]

Let \(q\) be large enough such that \(t_1\) in (29) is less than \(t_4\). Then, apply the similar arguments given in Subsection 2.1 (see also [16, 31]), one can show that \(\Phi(\xi) = (\phi_3(\xi), \phi_2(\xi))\) and \(\overline{\Phi}(\xi) = (\phi_3(\xi), \overline{\phi}_2(\xi))\) constitute a pair of upper and lower solutions of system (8) satisfying the conditions of (31). Hence, following the same proof arguments for part (1) of Theorem 1.1, we can derive the assertion of Theorem 1.2. The proof is complete.

5. Some applications. In this section, we will apply our main theorem to some diffusive predator-prey models.
5.1. The Lotka-Volterra Predator-Prey model. The Lotka-Volterra predator-prey model is described as the following equations

$$\begin{cases}
\frac{\partial u_1}{\partial t} = d_1 \Delta u_1 + r_1 u_1 (1 - a_{11} u_1 - a_{12} u_2), \\
\frac{\partial u_2}{\partial t} = d_2 \Delta u_2 + r_2 u_2 (1 + a_{21} u_1 - a_{22} u_2),
\end{cases} \quad (92)$$

where $u_n := u_n(x,t) : \mathbb{R}^2 \to \mathbb{R}$ and all parameters are positive. The traveling wave solutions for this model with delays has been considered in Lin et al [25].

Now let us set

$$\alpha = \frac{1}{a_{11}}, \quad g_1(x) = \frac{1}{a_{12}} - \frac{a_{11}}{a_{12}} x \quad \text{and} \quad g_2(x) = \frac{1}{a_{22}} + \frac{a_{21}}{a_{22}} x.$$ 

Then it is easy to check that (H1)~(H3) are true under the assumption

$$2a_{21}/a_{11} + 2 < a_{22}/a_{12}. \quad (93)$$

By direct computations, one can easily see that

$$k_1 D_1 f_1(k_1, k_2) < k_2 D_2 f_2(k_1, k_2) \iff r_1 a_{11} (a_{11} + a_{21}) > r_2 a_{22} (a_{22} - a_{12}),$$

where $f_1(x_1, x_2) = r_1 (1 - a_{11} x_1 - a_{12} x_2)$, $f_2(x_1, x_2) = r_2 (1 + a_{21} x_1 - a_{22} x_2)$ and

$$(k_1, k_2) := \left( \frac{a_{11} + a_{21}}{a_{11} a_{22} + a_{12} a_{21}}, \frac{a_{22} - a_{12}}{a_{11} a_{22} + a_{12} a_{21}} \right).$$

Hence, as a consequence of Theorem 1.1, we have the following result.

**Theorem 5.1.** Assume the condition (93) holds. Then The statements of (1) and (2) of Theorem 1.1 hold for system (92). In addition, if

$$d_2 \geq d_1 \quad \text{and} \quad r_1 a_{11} (a_{11} + a_{21}) > r_2 a_{22} (a_{22} - a_{12}),$$

we also obtain the assertion (3) of Theorem 1.1.

5.2. The modified Leslie-Gower Predator-Prey system with different functional responses. The diffusive predator-prey models with modified Leslie-Gower term and functional response $F(u_1)$ can be described by the following equations

$$\begin{cases}
\frac{\partial u_1}{\partial t} = d_1 \Delta u_1 + r_1 \left( u_1 - b u_1^2 - F(u_1, u_2) u_2 \right), \\
\frac{\partial u_2}{\partial t} = d_2 \Delta u_2 + r_2 u_2 \left( 1 - \frac{a_2 u_2}{\gamma + u_1} \right),
\end{cases} \quad (94)$$

where $u_n := u_n(x,t) : \mathbb{R}^2 \to \mathbb{R}$, all parameters are positive and $\gamma$ is called the Leslie-Gower term. The non-diffusive system (94) with Holling-type II functional response $F(u_1, u_2) = a_1 u_1 / (\gamma_1 + u_1)$ was first proposed by Aziz-Alaoui and Okiye [2]. From their work, we know that the parameter $r_1$ means the growth rate of prey $u_1$; $b$ measures the strength of competition among individuals of species $u_1$; $a_1$ is the maximum value which per capita reduction rate of $u_1$ can attain; $\gamma_1$ (respectively, $\gamma$) measures the extent to which environment provides protection to prey $u_1$ (respectively, to predator $u_2$); $r_2$ means the growth rate of $u_2$ and $a_2$ has a similar meaning to $a_1$. Furthermore, the authors considered the boundedness of solutions, existence of an attracting set and global stability of the coexisting interior equilibrium. Additionally, Zhou [38] investigated the existence, multiplicity and stability of positive steady-state solutions of system (94) with Holling-type II functional response on a bounded domain with smooth boundary. Further results about the study of steady-state solutions can be found in the literature [7, 8, 38] and
references cited therein. However, to the best of our knowledge, there is no results about the existence of traveling wave solutions for system (94). In the sequel, we apply our main results to obtain the existence and non-monotonicity of traveling wave solutions of system (94) with different functional responses.

5.2.1. Holling-Type II functional response. Let’s consider system (94) with Holling-type II functional response:

$$F(u_1, u_2) := \frac{a_1 u_1}{\gamma_1 + u_1}.$$ 

According to the notations used in Section 1, we set

$$\alpha = 1/b, \quad g_1(x) = (-bx^2 + (1 - b\gamma_1)x + \gamma_1)/a_1, \quad g_2(x) = (x + \gamma)/a_2,$n

$$f_1(u_1, u_2) = r_1(1 - bu_1 - \frac{a_1 u_2}{\gamma_1 + u_1}) \quad \text{and} \quad f_2(u_1, u_2) = r_2(1 - \frac{a_2 u_2}{\gamma + u_1}).$$

Then $g_1(0) = \gamma_1/a_1, g_2(0) = \gamma/a_2,

$$g'_1(x) = [-2bx + (1 - b\gamma_1)]/a_1, \quad g''_1(x) = -2b/a_1,$n

$$D_1 f_1(x_1, x_2) = r_1[-b + \frac{a_1 x_2}{(\gamma_1 + x_1)^2}], \quad D_2 f_1(x_1, x_2) = -\frac{r_1 a_1}{\gamma_1 + x_1},$$n

$$D_2 f_2(x_1, x_2) = -\frac{a_2 r_2}{x_1 + \gamma} \quad \text{and} \quad D_1 f_2(x_1, x_2) = \frac{a_2 r_2 x_2}{(x_1 + \gamma)^2}.$$

In addition, $g_2(x) > x g'_2(x)$ for all $x \in [0, 2a]$, and $g_2(2a) < 2g_2(\alpha) < -\alpha g'_1(0)$ when

$$\frac{2 + b\gamma}{2a b} < \frac{2(1 + b\gamma)}{a_2 b} < \frac{b\gamma_1 - 1}{a_1 b}.$$ 

Then one can easily verify that (H1)~(H3) hold under the assumption

$$b > \frac{2a_1 + a_2}{a_2 \gamma_1 - 2a_1 \gamma} > 0. \quad (95)$$

Moreover, by direct computations, the conditions of (10) are equivalent to

$$d_2 \geq d_1 \quad \text{and} \quad r_1 k_1 [b - \frac{a_1 k_2}{(\gamma_1 + k_1)^2}] \geq r_2. \quad (96)$$

Since $k_1$ and $k_2$ are independent of $r_1$ and $r_2$, we can find parameters satisfying (96). Hence, as a consequence of Theorem 1.1, we have the following result.

**Theorem 5.2.** Assume the condition (95) holds. Then the statements of (1) and (2) of Theorem 1.1 hold for system (92). In addition, if the condition (96) holds, we also obtain the assertion (3) of Theorem 1.1.

5.2.2. Holling-Type III functional response. Now consider system (94) with Holling-type III functional response:

$$F(u_1, u_2) := \frac{a_1 u_1^2}{\gamma_1 + \gamma_2 u_1^2}.$$ 

Similarly, we set

$$\alpha = 1/b, \quad g_1(x) = \frac{(1 - bx)(\gamma_1 + \gamma_2 x^2)}{a_1 x}, \quad g_2(x) = \frac{x + \gamma}{a_2},$$n

$$f_1(x_1, x_2) = r_1(1 - bx_1 - \frac{a_1 x_1 x_2}{\gamma_1 + \gamma_2 x_1}) \quad \text{and} \quad f_2(x_1, x_2) = r_2(1 - \frac{a_2 x_2}{\gamma + x_1}).$$
It is clear that \( g_1(0^+) = \infty, g_2(0) = \gamma/a_2 > 0, \)
\[
g_1'(x) = \frac{\gamma_2 x^2(1 - 2bx) - \gamma_1}{a_1 x^2}, \quad g_1''(x) = \frac{-2b\gamma x^4 + 2\gamma_1 x}{a_1 x^4},
\]
\[
D_1f_1(x_1, x_2) = \gamma_1 \left[ -b - \frac{a_1 x_2 (\gamma_1 - \gamma_2 x_2^2)}{(\gamma_1 + \gamma_2 x_1^2)^2} \right], \quad D_2f_1(x_1, x_2) = -\frac{a_1 x_1}{\gamma_1 + \gamma_2 x_1^2},
\]
\[
D_2f_2(x_1, x_2) = -\frac{a_2 x_2}{x_1 + \gamma} \quad \text{and} \quad D_1f_2(x_1, x_2) = \frac{a_2 x_2^2}{(x_1 + \gamma)^2}.
\]

It is clear that (A1) and (A2) hold when \( b^2 \gamma_1 > \gamma_2 \). In addition, we have \( g_2(x) > xg_2'(x) \) on \( x \in [0, 2a] \), \( g_2(2a) < 2g_2(a) < -ag_1'(a)/2 \) when
\[
a_2 \gamma_1 b^2 - 4a_1 \gamma b + a_2 \gamma_2 - 4a_1 > 0, \quad (97)
\]
and \( g_1(\alpha/2) > g_2(\alpha/2) \) when
\[
4a_2 \gamma_1 b^2 - 4a_1 \gamma b + a_2 \gamma_2 - 2a_1 > 0. \quad (98)
\]

Thus (A3) holds when the parameters satisfy (97) and (98). Moreover, the conditions in (10) for system (94) are equivalent to
\[
d_2 \geq d_1 \quad \text{and} \quad k_1 r_1 \left[ -b - \frac{a_1 k_2 (\gamma_1 - \gamma_2 k_2^2)}{(\gamma_1 + \gamma_2 k_1^2)^2} \right] < -k_2 \frac{a_2 r_2}{k_1 + \gamma} = -r_2. \quad (99)
\]

Since \( k_1 \) and \( k_2 \) are independent of \( r_1 \) and \( r_2 \), we can find parameters satisfying the above inequality. Therefore, we have the following results.

**Theorem 5.3.** Assume that \( b^2 \gamma_1 > 4\gamma_2 \), (97) and (98) hold. Then the statements of (1) and (2) of Theorem 1.1 hold for system (94). In addition, if (99) hold, then we also obtain the assertion (3) of Theorem 1.1.

**Remark 2.** (1) It is easy to verify that the conditions of Theorem 5.3 hold when
\[
b > \max\{\sqrt{\gamma_2/\gamma_1}, \frac{4a_1 \gamma}{a_2 \gamma_1}\} \quad \text{and} \quad a_2 \gamma_2 > 4a_1.
\]

In fact, (97) and (98) hold for all \( b \) when \( 4a_2^2 \gamma_2^2 < a_2 \gamma_1(a_2 \gamma_2 - 4a_1) \).

(2) No matter the functional response is Holling-type II or III, system (94) always admit traveling wave solutions when the strength of competition among individuals of species \( u_1 \) (i.e. \( b \)) is large enough.

5.2.3. **Beddington-DeAngelis functional response.** Let’s consider system (94) with Beddington-DeAngelis functional response:
\[
F(u_1, u_2) = \frac{a_1 u_1}{\gamma_1 + \gamma_2 u_1 + \gamma_3 u_2}.
\]

Note that the parameters \( a_1, \gamma_1, \gamma_2, \gamma_3 \) represent the consumption rate, the saturation constant, the saturation constant for an alternative prey, and the predator interference, respectively. This Beddington-DeAngelis functional response was offered by Beddington [3] for describing parasite host interactions. Some qualitative behavior of solutions for non-diffusive system (94) with Beddington-DeAngelis functional response was studied by Khelfa and Hamri [21]. Similarly, we set
\[
\alpha = 1/b, \quad g_1(x) = \frac{(\gamma_1 + \gamma_2 x)(1 - bx)}{a_1 - \gamma_3 (1 - bx)}, \quad g_2(x) = \frac{1}{a_2} (x + \gamma),
\]
\[
f_1(x_1, x_2) = r_1(1 - b_1 x_1 - \frac{a_1 x_2}{\gamma_1 + \gamma_2 x_1 + \gamma_3 x_2}) \quad \text{and} \quad f_2(x_1, x_2) = r_2(1 - \frac{a_2 x_2}{\gamma + x_1}).
\]
Then $g_1(0) = \gamma_1/(a_1 - \gamma_3)$, $g_2(0) = \gamma/a_2$, 
\[ g'_1(x) = -\gamma_2 a_3 b^2 x^2 - 2\gamma_2 (a_1 - \gamma_3) b x + \gamma_2 (a_1 - \gamma_3) - \gamma_1 a_1 b, \]
\[ g''_1(x) = \frac{2a_1 a_3 b^2 - 2b_2 a_1\gamma_2 (a_1 - \gamma_3)}{(a_1 - \gamma_3(1 - bx))^2}, \]
\[ D_1 f_1(x_1, x_2) = -b r_1 + \frac{a_1 r_1 \gamma_2 x_2}{(\gamma_1 + \gamma_2 x_1 + \gamma_3 x_2)^2}, \]
\[ D_2 f_2(x_1, x_2) = -\frac{a_2 r_2}{x_1 + \gamma} \quad \text{and} \quad D_1 f_2(x_1, x_2) = \frac{a_2 r_2 x_2}{(x_1 + \gamma)^2}. \]

Hence (H1) and (H2) hold provided that $a_1 > \gamma_3$ and $\gamma_2 \in (\gamma_2, \bar{\gamma}_2)$ where 
\[ \bar{\gamma}_2 := \frac{\gamma_1 \gamma_3 b}{a_1 - \gamma_3} \quad \text{and} \quad \gamma_2 := \frac{b \gamma_1 (a_1 - \gamma_3)}{a_1}. \]

In addition, we see that $g_2(x) > x g_2'(x)$ for all $x \in [0, 2a]$ and $g_2(2a) < 2g_2(0) < -\alpha g_1'(0)$ when 
\[ \frac{2 + \beta}{a_2 b} < \frac{2(1 + \beta)}{a_2 b} < \frac{\gamma_1 a_1 b - \gamma_2 (a_1 - \gamma_3)}{b(a_1 - \gamma_3)^2}. \]  
(100)

It’s easy to verify that (100) holds if 
\[ \gamma_2 < \bar{\gamma}_2 := \frac{a_1 a_2 \gamma_1 b - 2(1 + \beta)(a_1 - \gamma_3)}{a_2 (a_1 - \gamma_3)}. \]  
(101)

Therefore, (H3) holds when the condition (101) holds. It’s easy to see that 
$(\gamma_2, \min\{\bar{\gamma}_2, \gamma_2\})$ is a non-empty interval when $\gamma_3$ is small enough. Moreover, the conditions in (10) for (94) are equivalent to 
\[ d_2 \geq d_1 \quad \text{and} \quad k_1 r_1 \left( -b + \frac{a_1 \gamma_2 k_2}{(\gamma_1 + \gamma_2 k_1 + \gamma_3 k_2)^2} \right) < -k_2 \frac{a_2 r_2}{k_1 + \gamma} = -r_2. \]  
(102)

Since $k_1$ and $k_2$ are independent of $r_1$ and $r_2$, we can find parameters satisfying the above inequality. Therefore, we have the following results.

**Theorem 5.4.** Assume $a_1 > \gamma_3$ and $\gamma_2 \in (\gamma_2, \min\{\bar{\gamma}_2, \gamma_2\})$. Then the statements of (1) and (2) of Theorem 1.1 hold for system (94). In addition, if the condition (102) hold, we also obtain the assertion (3) of Theorem 1.1 for system (94).

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