Green matrix estimates of block Jacobi matrices I: 
Unbounded gap in the essential spectrum

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Abstract

This work deals with decay bounds for Green matrices and generalized eigenvectors of block Jacobi matrices when the real part of the spectral parameter lies in an infinite gap of the operator’s essential spectrum. We consider the cases of commutative and noncommutative matrix entries separately. An example of a block Jacobi operator with noncommutative entries and nonnegative essential spectrum is given to illustrate the results.

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1. Introduction

In this work, we consider block Jacobi operators acting in $\mathcal{H} = l_2(\mathbb{N}, \mathcal{K})$, the space of square-summable sequences whose elements lie in a Hilbert space $\mathcal{K}$ (see the precise definition in the next section). As in the case of scalar Jacobi operators, block Jacobi operators are associated with a second order difference equation, but instead of having scalar coefficients, this equation has operator coefficients (see (2.2)). These operators, $A_n$ and $B_n = B_n^*$ ($n \in \mathbb{N}$), are the entries of a block Jacobi matrix (see (2.3)). The class of block Jacobi operators under consideration, which is generically denoted by $J$, is such that the operators $A_n$ and $B_n$ are bounded and defined on the whole Hilbert space $\mathcal{K}$, that is, $A_n, B_n \in B(\mathcal{K})$ for all $n \in \mathbb{N}$. Additionally, we require that $J$ is self-adjoint and semi-bounded with its essential spectrum lying inside the interval $[b, +\infty)$. Under these assumptions, we provide estimates for the decay of the matrix entries of $(J - \lambda I)^{-1}$, i.e. the entries of the Green matrix of $J$, when $\text{Re}\lambda < b$. This in particular gives estimates of the so-called generalized eigenvectors. Moreover, when $\sigma(J) \cap (-\infty, b) \neq \emptyset$, we show that these estimates apply for the eigenvectors of $J$ corresponding to eigenvalues below $b$.

Similar questions for estimates of the generalized eigenvectors of scalar Jacobi matrices have been addressed in previous papers [11, 13] (references for results on this matter, preceding the ones of [11, 13], can be found in these papers). In [13], the case where the scalar Jacobi matrix $J$ satisfies the operator inequality $J \geq bI$ is studied. In the other work [11], the presence of a bounded discrete spectrum of $J$ is allowed. In [11, 13], there is a refinement of the Combes-Thomas method for obtaining estimates of the Green function which provides sharp coefficient estimates and establishes that the bounds depend inversely on the growth of the off-diagonal entries. In this work, on the basis of the methods developed in [11, 13], we establish estimates that include as a particular case the bounds found earlier in [11, 13]. The growth of the off-diagonal operators refers either to the growth of the operator norm (see Theorems 3.1 and 3.2) or to the absolute value of the operator (see Theorem 4.1). The latter case permits a wide range of interesting examples.

It should be mentioned that the semi-boundedness from below of $J$ is crucial as it already was in the scalar case. Indeed, in [17], an example of a scalar Jacobi matrix was produced with essential spectrum covering the interval $(0, \infty)$ and negative spectrum being discrete and accumulating to $-\infty$. For this example, the explicitly calculated asymptotics of generalized eigenvectors for $\lambda < 0$ does not satisfy the estimates of Theorem 3.2. Namely, the asymptotics found in [17] contains non-removable information on the main diagonal of the scalar Jacobi matrix while the estimates given by Theorem 3.2 do not depend on the main diagonal of the block Jacobi matrix. The fact that the asymptotic behavior of generalized eigenvectors of a semi-bounded Jacobi operator is independent of its main diagonal was known in the scalar case [11]. Theorems 3.1 and 3.2 extend that result to block Jacobi operators.

The growth estimates of generalized eigenvectors found in this work provide a
generalization of Combes-Thomas type estimates [2, 5] applicable to various discrete random models, including the Anderson model, with matrix potentials (an accessible and detailed survey on the matter in the scalar case is found in [15] and for Combes-Thomas estimates see Chapter 11 there). It is pertinent to mention here that there are various works dealing with random block-type operators, for instance [7, 9, 16], and some of them treating the problem of localization [7, 9]. In the context of our work, [9, Lem. 5.7] is of particular relevance since it gives a Combes-Thomas type estimate for random block operators.

Apart from random block-type operators, the estimates of generalized eigenvectors in the block Jacobi matrix case may have other interesting applications. One of them is related to the investigation of the spectral phase transition phenomena of the second kind (see [13] for a case of one-threshold transition) for discrete models with matrix entries. Block Jacobi matrices permit more freedom to construct models which exhibit multi-threshold spectral phase transitions.

Spectral properties of block Jacobi operators have also been studied in [19] which carries out averaging of the spectral measure over boundary conditions. More recently, [18] developed a framework which simplifies the general local Green function relations found in [8] and illustrates the power of the transfer matrix method.

In this work we deal exclusively with the case of unbounded gap in the essential spectrum. The case of a bounded gap in the essential spectrum of block Jacobi operators requires a different technique and will be considered in a forthcoming paper.

The following is a summary of the paper. Section 2 contains a short survey on the basics of the theory of block Jacobi matrices. Section 3 presents the main result of the paper (Theorem 3.1) which gives the estimates of the Green matrix entries decay. Theorem 3.2 also shows estimates of the eigenvector’s decay for the eigenvalue $\lambda < b$. Note that for such $\lambda$, the Green matrix, i.e. the resolvent of $J$, is not defined, but this problem can be easily overcome by a proper “small change” of the matrix $J$. Section 4 deals with the special case of commuting entries. In this case, more detailed estimates, counting the matrix character of the entries, are obtained (Theorem 4.1 and Corollary 4.1). This contrasts the results of Section 3, where just the norm of the entries are involved in the estimates. Section 5 presents an example of a block Jacobi matrix with a $2 \times 2$ matrix entries with nonnegative essential spectrum. The application of Theorem 3.2 and a heuristic analysis of solutions (the Levinson type asymptotics form) give a sort of arguments for proving the sharpness of Theorem 3.2. We hope that the true block matrix example of Section 5 is also of independent interest. It exhibits the matrix character role of the entries.

2. Block Jacobi matrices

Notation. The following notation is used throughout this work.

(I) By $\mathcal{H}$, we denote a separable infinite dimensional Hilbert space. This space
admits the decomposition
\[ \mathcal{H} = \bigoplus_{m=1}^{\infty} \mathcal{K}_m, \]
where, for all \( m \in \mathbb{N} \), \( \mathcal{K}_m = \mathcal{K} \) and \( \mathcal{K} \) is either an infinite or finite dimensional subspace of \( \mathcal{H} \). Therefore, \( \mathcal{K} \) is either unitarily equivalent to \( l_2(\mathbb{N}) \) or \( \mathbb{C}^d \) with a fixed \( d \in \mathbb{N} \). Here and throughout the text, \( l_2(\mathbb{N}) \) stands for the space of infinite, square-summable complex sequences. The Hilbert space \( \mathcal{H} \) is usually denoted by \( l_2(\mathbb{N}, \mathcal{K}) \).

(II) The symbol \( \| \cdot \| \) is used to denote the norm in \( \mathcal{H} \), while the norm in \( \mathcal{K} \) is denoted by \( \| \cdot \|_\mathcal{K} \). \( B(\mathcal{H}) \) and \( B(\mathcal{K}) \) denote the spaces of bounded linear operators defined on the whole space \( \mathcal{H} \) and \( \mathcal{K} \), respectively. The norms in \( B(\mathcal{H}) \) and \( B(\mathcal{K}) \) are denoted by \( \| \cdot \|_{B(\mathcal{H})} \) and \( \| \cdot \|_{B(\mathcal{K})} \), respectively.

(III) A vector \( u \) in \( \mathcal{H} \) can be written as a sequence
\[ u = \{ u_m \}_{m=1}^{\infty}, \quad u_m \in \mathcal{K}_m, \tag{2.1} \]
where \( \sum_{m=1}^{\infty} \| u_m \|_\mathcal{K}^2 < +\infty \). We also use the notation
\[ u = (u_1, u_2, u_3, \ldots)^T \]

(IV) Throughout this work, we use \( I \) to denote the identity operator in the spaces \( \mathcal{H} \) and \( \mathcal{K} \) since it will cause no confusion to use the same letter for these operators. The orthogonal projector in \( \mathcal{H} \) onto the subspace \( \mathcal{K}_m \) is denoted by \( P_m \) while the symbol \( \tilde{P}_M \) stands for the orthogonal projector onto \( \bigoplus_{m=1}^{M} \mathcal{K}_m \).

(V) Given a closed, densely defined operator \( A \) in a Hilbert space, we denote by \( |A| \) the operator \( (A^*A)^{1/2} \).

Let us turn to the definition of block Jacobi operators. For any sequence (2.1), consider the second order difference expressions
\[ (\Upsilon u)_k := A_{k-1}^* u_{k-1} + B_k u_k + A_k u_{k+1} \quad k \in \mathbb{N} \setminus \{1\}, \tag{2.2a} \]
\[ (\Upsilon u)_1 := B_1 u_1 + A_1 u_2, \tag{2.2b} \]
where \( B_k = B_k^*, A_k \in B(\mathcal{K}) \) for any \( k \in \mathbb{N} \).

**Definition 1.** In \( \mathcal{H} \), define the operator \( J_0 \) whose domain is the set of sequences (2.1) having a finite number of non-zero elements and is given by \( J_0 f := \Upsilon f \). Since \( J_0 \) is symmetric (therefore closable), one can consider its closure which is denoted by \( J \).
We have defined the operator $J$ so that the block tridiagonal matrix

\[
\begin{pmatrix}
B_1 & A_1 & 0 & 0 & \cdots \\
A_1^* & B_2 & A_2 & 0 & \\
0 & A_2^* & B_3 & A_3 & \cdots \\
0 & 0 & A_3^* & B_4 & \cdots \\
\vdots & & & & \ddots \\
\end{pmatrix}
\]

(2.3)
can be regarded as the matrix representation of the operator $J$ (see [1, Sec. 47] where a definition of the matrix representation of an unbounded symmetric operator is given and also [6] and [20] for general questions on block Jacobi operators).

In this work, we impose conditions on the sequences $\{A_m\}_{m=1}^{\infty}$ and $\{B_m\}_{m=1}^{\infty}$ so that $J$ is self-adjoint. A sufficient condition for this to happen is the generalized Carleman criterion [3, Chap.7 Thm. 2.9], viz., if $\sum_{m=1}^{\infty} 1/\|A_m\|_K = +\infty$, then $J$ is self-adjoint.

Self-adjointness of $J$ implies that its domain, $\text{dom}(J)$, coincides with the maximal linear set in which the result of the “action” of the matrix (2.3) on a sequence (2.1) yields a sequence in $H$. Thus, it cannot lead to confusion if we use the same letter $J$ to denote both the operator and the matrix. Likewise,

\[
\text{diag}\{C_m\}_{m=1}^{\infty} := \begin{pmatrix}
C_1 & 0 & 0 & 0 & \cdots \\
0 & C_2 & 0 & 0 & \\
0 & 0 & C_3 & 0 & \cdots \\
0 & 0 & 0 & C_4 & \cdots \\
\vdots & & & & \ddots \\
\end{pmatrix},
\]

where $C_m \in B(K)$ for any $m \in \mathbb{N}$, is used for denoting the operator and the matrix (the operator being $\bigoplus_{m=1}^{\infty} C_m$ with $C_m \in B(K)$ for all $m \in \mathbb{N}$).

Consider the unilateral vector shift operator $S$ in $H$ given by

\[
S(u_1, u_2, u_3, \ldots)^T = (0, u_1, u_2, \ldots)^T
\]

and its adjoint $S^*$ for which

\[
S^*(u_1, u_2, u_3, \ldots)^T = (u_2, u_3, u_4, \ldots)^T
\]

It can be verified that the operator

\[
\text{diag}\{B_m\}_{m=1}^{\infty} + S \text{diag}\{A_m^*\}_{m=1}^{\infty} + \text{diag}\{A_m\}_{m=1}^{\infty} S^*
\]

(2.4)

coincides with the self-adjoint operator $J$.

**Definition 2.** Assume that the operator $J$ given in Definition 1 is self-adjoint. For
any $\lambda$ in the resolvent set of $J$, define

$$G_{jk}(\lambda) := P_j(J - \lambda I)^{-1} P_k.$$  

Note that by this definition $G_{jk}(\lambda)^* = G_{kj}(\lambda)$, and therefore

$$\|G_{jk}(\lambda)\|_{B(K)} = \|G_{kj}(\lambda)\|_{B(K)}.$$  

Due to the fact that for $\lambda \not\in \sigma(J)$, $(J - \lambda I)^{-1} \in B(H)$, one verifies that

$$(J - \lambda I)^{-1} u = \left( \sum_{k=1}^{\infty} G_{1k} u_k, \sum_{k=1}^{\infty} G_{2k} u_k, \sum_{k=1}^{\infty} G_{3k} u_k, \ldots \right)^T$$

Thus, $G_{jk}(\lambda)$ can be regarded as the entries of the matrix representation of the resolvent of $J$ at $\lambda$. We refer to $\{G_{jk}(\lambda)\}_{j,k=1}^{\infty}$ as the block Green matrix corresponding to $J$ at $\lambda$.

3. Estimates of generalized eigenvectors in an unbounded gap of the essential spectrum

In this section we find estimates for generalized eigenvectors of the operator $J$ given by Definition 1 when it is semi-bounded. There is no loss of generality in assuming the operator $J$ bounded from below. We consider that the real part of the spectral parameter is below the essential spectrum and obtain estimates for both cases: when this parameter is not an eigenvalue and when it is.

To simplify the writing of some formulae, we introduce the functions

$$\psi(x) := x^2 e^x, \quad 0 \leq x.$$  

(3.1)

and

$$\phi_\delta(x) := \begin{cases} 1/\sqrt{\delta} & \text{if } 0 \leq x < \delta \\ 1/\sqrt{x} & \text{if } \delta \leq x \end{cases}$$  

(3.2)

**Theorem 3.1.** Assume that the operator $J$ given in Definition 1 is self-adjoint and bounded from below. Take a real number $b$ such that $(-\infty, b) \cap \sigma_{\text{ess}}(J) = \emptyset$ and consider a complex number $\lambda$ with $\text{Re} \, \lambda < b$. Fix $\delta > 0$ and $\epsilon$ arbitrarily small in $(0, 1)$. If $\lambda \not\in \sigma(J)$, then

$$\|G_{jk}(\lambda)\|_{B(K)} \leq C \exp \left( -\gamma(\lambda) \sum_{m=\min(j,k)}^{\max(j,k)-1} \phi_\delta(\|A_m\|_{B(K)}) \right),$$

where $G_{jk}(\lambda)$ is given in Definition 2, $C$ does not depend on $j$ and $k$, $\phi_\delta$ is given in
\(3.2\) and
\[
\gamma(\lambda) := \sqrt{\delta \psi^{-1} \left( \frac{(b - \text{Re}\, \lambda)(1 - \epsilon)}{\delta} \right)} \tag{3.3}
\]
with \(\psi^{-1}\) being the inverse function of \(\psi\) given in (3.1).

**Proof.** Let \(E\) be the spectral measure of the self-adjoint operator \(J\), i.e., \(J = \int s dE_s\). Define
\[
K := (J - b)E(-\infty, b) .
\]
Due to the fact that there is no essential spectrum in \((-\infty, b)\), the operator \(K\) is compact and
\[
J - K \geq bI . \tag{3.4}
\]
Choose the number \(M\) so large that
\[
\|K(I - \tilde{P}_M)\|_{B(H)} \leq (b - \text{Re}\, \lambda)\frac{\epsilon}{2} \tag{3.5}
\]
for any fixed \(\lambda\) such that \(\text{Re}\, \lambda < b\). This can be done since the sequence \(\{\tilde{P}_M\}_{M=1}^{\infty}\) converges strongly to \(I\) as \(M \to \infty\). Clearly, our choice of \(M\) depends on \(\lambda, b, \epsilon, \) and \(J\).

For any fixed \(N \in \mathbb{N}\), let
\[
\Phi_m := \begin{cases} 
\exp \left( -\gamma \sum_{k=1}^{m-1} \phi_\delta(\|A_k\|_{B(K)}) \right) I, & m \leq N , \\
\exp \left( -\gamma \sum_{k=1}^{N-1} \phi_\delta(\|A_k\|_{B(K)}) \right) I, & m > N ,
\end{cases} \tag{3.6}
\]
where \(\phi_\delta\) is given in (3.2) and \(\gamma\) is to be determined later. Note that \(\Phi_m\) is a scalar matrix for all \(m \in \mathbb{N}\). Consider the following bounded operator in \(\mathcal{H}\)
\[
\Phi := \text{diag}(\Phi_m)_{m=1}^{\infty} .
\]
Besides depending on \(\delta\) and \(\gamma\), this operator depends on \(N\). When needed, we indicate this dependence explicitly, i.e., \(\Phi = \Phi(N)\). Note that, by freezing the sequence \(\{\Phi_m\}_{m=1}^{\infty}\) from \(\Phi_N\) onwards in (3.6), the operator \(\Phi(N)\) is a boundedly invertible contraction for any finite \(N\). At the end of this proof, we let \(N \to +\infty\).

Define
\[
F := S \text{diag}(\Phi_m^{-1} A_m^* \Phi_m - A_m^*) + \text{diag}(\Phi_m^{-1} A_m \Phi_{m+1} - A_m) S^* .
\]
By (3.6), \(F \in B(\mathcal{H})\) (actually, it is a “block-finite-rank” operator, viz., the sequences of the form (2.1) in the range of \(F\) have a finite number of nonzero elements). Using (2.4), one verifies that
\[
F = \Phi^{-1} J\Phi - J \tag{3.7}
\]
and
\[ \Phi^{-1}(J - \lambda I)\Phi = J + F - \lambda I. \] (3.8)

Also,
\[
2 \text{Re } F = F + F^* \\
= S \text{diag}\{\Phi_m^{-1}A_m\Phi_{m+1} - 2A_m + \Phi_m^*A_m(\Phi_{m+1}^*)^{-1}\}\nonumber\ast \\
+ \text{diag}\{\Phi_m^{-1}A_m\Phi_{m+1} - 2A_m + \Phi_m^*A_m(\Phi_{m+1}^*)^{-1}\} S^*.
\]

Since the matrix of the operator 2 Re F has only two block diagonals not necessarily zero and one diagonal is the adjoint of the other, one has
\[
\|\text{Re } F\|_{B(H)} \leq \sup_{m \in \mathbb{N}} \left\{ \left\| \Phi_m^{-1}A_m\Phi_{m+1} - 2A_m + \Phi_m^*A_m(\Phi_{m+1}^*)^{-1} \right\|_{B(K)} \right\}. \tag{3.9}
\]

Let us show that, by choosing \( \gamma \) appropriately, one can ensure that
\[
\|\text{Re } F\|_{B(H)} \leq (1 - \epsilon)(b - \text{Re } \lambda) \tag{3.10}
\]
under our assumption that \( \text{Re } \lambda < b \). First note that (3.9) implies
\[
\|\text{Re } F\|_{B(H)} \leq \sup_{m \leq N} \left\{ \left\| A_m \left( e^{-\gamma \phi_\delta(\|A_m\|_{B(K)})} - 2I + e^{\gamma \phi_\delta(\|A_m\|_{B(K)})} \right) \right\|_{B(K)} \right\} \nonumber\ast \\
\leq \sup_{m \in \mathbb{N}} \left\{ \left\| A_m \left( e^{-\gamma \phi_\delta(\|A_m\|_{B(K)})} - 2I + e^{\gamma \phi_\delta(\|A_m\|_{B(K)})} \right) \right\|_{B(K)} \right\}.
\]

On the basis of the inequality
\[
0 \leq e^x - 2 + e^{-x} \leq x^2 e^x \tag{3.11}
\]
valid for \( x \geq 0 \), one has
\[
\|\text{Re } F\|_{B(H)} \leq \sup_{m \in \mathbb{N}} \left\{ \|A_m\|_{B(K)} \gamma^2 \phi_\delta^2(\|A_m\|_{B(K)}) e^{\gamma \phi_\delta(\|A_m\|_{B(K)})} \right\}. \tag{3.12}
\]

Fix \( \delta > 0 \) and choose \( \gamma \) so small that the inequality
\[
\xi \psi(\gamma \phi_\delta(\xi)) \leq (1 - \epsilon)(b - \text{Re } \lambda) \tag{3.13}
\]
holds for all \( \xi \) in \([0, \delta]\). Taking into account the behaviour of the function \( \phi_\delta \) when its argument is not greater than \( \delta \) (see (3.2)), one verifies that the inequality holds whenever \( \gamma \) is given by (3.3). Actually the inequality also holds for \( \delta < \xi \) since the function \( \xi \psi(\gamma \phi_\delta(\xi)) \) is monotone decreasing in \( \xi \) for \( \xi < \delta \) as long as \( \gamma > 0 \). In view of (3.1) and (3.2), the inequalities (3.12) and (3.13) imply (3.10).
Using (3.4), one verifies that
\[ \text{Re}(J + F - K\tilde{P}_M - \lambda I) \geq (b - \text{Re} \lambda)I + K(I - \tilde{P}_M + \text{Re} F). \]

Thus, due to (3.5) and (3.10), the last inequality yields
\[ \text{Re}(J + F - K\tilde{P}_M - \lambda I) \geq (b - \text{Re} \lambda) \left( 1 - \frac{\epsilon}{2} - (1 - \epsilon) \right) I = (b - \text{Re} \lambda)\frac{\epsilon}{2}I. \]

Since \( J + F - K\tilde{P}_M \) is a self-adjoint operator perturbed by a bounded operator, it follows from the last inequality that
\[ J + F - K\tilde{P}_M - (\lambda + \frac{\epsilon}{2}(b - \text{Re} \lambda))I \]
is a maximal accretive operator. Thus, \( J + F - K\tilde{P}_M - \lambda I \) is invertible and the estimate (see [14])
\[ \| (J + F - K\tilde{P}_M - \lambda I)^{-1} \|_{B(H)} \leq \frac{2}{\epsilon(b - \text{Re} \lambda)} \quad (3.14) \]
is known to hold for \( \text{Re} \lambda < b \).

Below we use that the operator \( I + K\tilde{P}_M((J + F - K\tilde{P}_M - \lambda I)^{-1} \) is boundedly invertible. This fact is established as follows. Since \( K\tilde{P}_M((J + F - K\tilde{P}_M - \lambda I)^{-1} \) is compact, it suffices to show that \( \ker(I + K\tilde{P}_M((J + F - K\tilde{P}_M - \lambda I)^{-1} \) is trivial. Suppose on the contrary that
\[ 0 \neq v \in \ker(I + K\tilde{P}_M((J + F - K\tilde{P}_M - \lambda I)^{-1} \),
then
\[ (J + F - \lambda I)(J + F - K\tilde{P}_M - \lambda I)^{-1}v = 0, \]
which implies that \( \ker(J + F - \lambda I) \) is not empty since \( (J + F - K\tilde{P}_M - \lambda I)^{-1}v \neq 0. \)
Therefore, using (3.8) and taking into account that \( \Phi(N) \) and \( [\Phi(N)]^{-1} \) are bounded for any \( N < +\infty \), one concludes that \( J - \lambda I \) is not invertible which contradicts the fact that \( \lambda \) is in the resolvent set.

Due to the algebraic identity
\[ (J + F - \lambda I)^{-1} = (J + F - K\tilde{P}_M - \lambda I)^{-1} \left[ I + K\tilde{P}_M(J + F - K\tilde{P}_M - \lambda I)^{-1} \right]^{-1} \quad (3.15) \]
and (3.8), one has

$$K \tilde{P}_M \Phi^{-1}(J - \lambda I)^{-1} \Phi$$

$$= K \tilde{P}_M (J + F - K \tilde{P}_M - \lambda I)^{-1} [I + K \tilde{P}_M (J + F - K \tilde{P}_M - \lambda I)^{-1}]^{-1}$$

$$= (-I + K \tilde{P}_M (J + F - K \tilde{P}_M - \lambda I)^{-1}) [I + K \tilde{P}_M (J + F - K \tilde{P}_M - \lambda I)^{-1}]^{-1}$$

$$= I - [I + K \tilde{P}_M (J + F - K \tilde{P}_M - \lambda I)^{-1}]^{-1}.$$  

Thus,

$$\left\Vert [I + K \tilde{P}_M (J + F - K \tilde{P}_M - \lambda I)^{-1}]^{-1} \right\Vert_{B(H)} = \left\Vert I - K \tilde{P}_M \Phi^{-1}(J - \lambda I)^{-1} \Phi \right\Vert_{B(H)}$$

$$\leq 1 + \left\Vert K \right\Vert_{B(H)} \left\Vert \tilde{P}_M \Phi(N) \right\Vert_{B(H)}^{-1} \left\Vert (J - \lambda I)^{-1} \right\Vert_{B(H)} \left\Vert \Phi(N) \right\Vert_{B(H)}$$

$$\leq 1 + \left\Vert K \right\Vert_{B(H)} \left\Vert \tilde{P}_M \Phi(M) \right\Vert_{B(H)}^{-1} \left\Vert (J - \lambda I)^{-1} \right\Vert_{B(H)},$$

where in the last inequality, we have chosen $N > M$ and taken into account that $\Phi(N)$ is a contraction. This estimate, together with (3.8), (3.14), and (3.15), allows us to write

$$\left\Vert \Phi^{-1}(N)(J - \lambda I)^{-1} \Phi(N) \right\Vert_{B(H)}$$

$$\leq \frac{2}{\epsilon(b - \text{Re} \lambda)} \left( 1 + \left\Vert K \right\Vert_{B(H)} \left\Vert \tilde{P}_M \Phi(M) \right\Vert_{B(H)}^{-1} \left\Vert (J - \lambda I)^{-1} \right\Vert_{B(H)} \right)$$

$$=: C,$$  

where the $C$ does not depend on $N$. In view of Definition 2 and (3.6), the last inequality implies

$$\left\Vert \exp \left( \gamma \sum_{m=1}^{j-1} \phi_{\delta}(||A_m||_{B(K)}) \right) G_{jk}(\lambda) \exp \left( -\gamma \sum_{m=1}^{k-1} \phi_{\delta}(||A_m||_{B(K)}) \right) \right\Vert_{B(K)} \leq C$$  

(3.17)

for $j, m \leq N$. The estimate of the theorem is finally proven by combining both scalar exponential factors in (3.17) and letting $N \to \infty$. Formally, in this proof, $j \geq b$, but the other case is also covered by recurring to (2.5). \hfill \square

**Remark 1.** It is possible to obtain a qualified estimate of the constant $C$. Indeed, it follows from (3.16) that

$$\left\Vert \exp \left( \gamma(\lambda) \sum_{k=\min(m,j)}^{\max(m,j)-1} \phi_{\delta}(||A_k||) \right) G_{mj}(\lambda) \right\Vert$$

$$\leq \frac{2}{\epsilon(b - \text{Re} \lambda)} \left( 1 + \frac{|b - \text{min} \sigma_p(J)|}{\text{dist}(\lambda, \sigma(J))} \left\Vert \tilde{P}_M \Phi(M) \right\Vert_{B(H)}^{-1} \right)$$  

(3.18)
for any $j, k \in \mathbb{N}$ and some $M \in \mathbb{N}$. Since

$$
\| P_M \Phi(M) \|_{B(\mathcal{H})}^{-1} = \| \exp \left( \gamma(\lambda) \sum_{k=1}^{M} \phi_\delta(\|A_k\|) \right) \| \leq e^{\gamma(\lambda) M \delta^{-1/2}},
$$

one obtains from (3.18) that

$$
\left\| \exp \left( \gamma(\lambda) \sum_{k=\min(m,j)}^{\max(m,j)-1} \phi_\delta(\|A_k\|) \right) G_{mj}(\lambda) \right\| \leq 2 \left( 1 + \frac{\|b - \min \sigma(J)\|}{\text{dist}(\lambda, \sigma(J))} e^{\gamma(\lambda) M \delta^{-1/2}} \right).
$$

Note that the choice of $M$ is given by (3.5) and is independent of $m$ and $j$. Admissible values of $M$ can be found by considering the canonical representation of the compact operator $K$ (see [4, Chap. 11, Sec. 1]). Let $\{\lambda_k\}_k$ be the sequence of eigenvalues of $J$ which are less than $b$ enumerated so that their multiplicity is taken into account. This sequence may be finite. Define

$$
s_k = \begin{cases} 
\lambda_k - b & \text{if } \lambda_k < b \\
0 & \text{otherwise}.
\end{cases}
$$

Thus, for $f \in \mathcal{H}$,

$$
\| K(I - P_M) f \| = \sum_{k=1}^{\infty} s_k^2 \left| \langle f, (I - P_M) \phi_k \rangle \right|^2,
$$

where $\{\phi_k\}_{k=1}^{\infty}$ is an orthonormal system of eigenvectors of $J$ corresponding to its spectrum in $(-\infty, d)$. Therefore, in view of (3.5), one arrives at the following condition for $M$.

$$
\sup_{\|f\|=1} \sqrt{\sum_k (\lambda_k - b)^2 \left| \langle f, (I - P_M) \phi_k \rangle \right|^2} \leq (b - \text{Re } \lambda) \frac{\epsilon}{2}.
$$

**Lemma 3.1.** Let $J = J^*$ be the operator given in Definition 1 and $L$ be a compact operator in $\mathcal{K}$ with trivial kernel such that $\|L\|_{B(\mathcal{K})} = 1$. If $A_m$ has trivial kernel for all $m \in \mathbb{N}$ and $\lambda$ is in the discrete spectrum of $J$, then, for any $\tau > 0$ sufficiently small, $\lambda$ is not in the spectrum of

$$
J(\tau) := J + \tau P_1 L^* L P_1
$$

(see Notation (IV)).

**Proof.** We prove the assertion by reductio ad absurdum. Note that, since $P_1 L^* L P_1$ is a compact operator in $\mathcal{H}$, Weyl perturbation theorem [14, Chp. 4, Thm. 5.35] tells us that $\lambda$ is not in the essential spectrum of $J(\tau)$. Suppose that for any neighborhood of
zero there is \( \tau > 0 \) such that \( \ker(J(\tau) - \lambda I) \) is not trivial. Pick a nonzero vector
\[
v_\tau \in \ker(J(\tau) - \lambda I) \tag{3.20}
\]
If \( E \) is the spectral measure of \( J \), one has
\[
E(\{\lambda\}) P_1 L^* L P_1 v_\tau = 0 \quad \text{for all } \tau > 0 . \tag{3.21}
\]
This is seen by applying the projector \( E(\{\lambda\}) \) to the equality
\[
(J - \lambda I) v_\tau = -\tau P_1 L^* L P_1 v_\tau \tag{3.22}
\]
which is obtained from (3.19) and (3.20). Now, (3.22) implies in turn that
\[
E(\{\lambda\}^\complement) (J - \lambda I) E(\{\lambda\}^\complement) E(\{\lambda\}^\complement) v_\tau = -\tau E(\{\lambda\}^\complement) P_1 L^* L P_1 v_\tau .
\]
where
\[
E(\{\lambda\}^\complement) \coloneqq E(\mathbb{R} \setminus \{\lambda\}) = I - E(\{\lambda\}) . \tag{3.23}
\]
Therefore
\[
E(\{\lambda\}^\complement) v_\tau = -\tau E(\{\lambda\}) \left( (J - \lambda I) \upharpoonright_{\text{ran}(E(\{\lambda\}^\complement))} \right)^{-1} E(\{\lambda\}^\complement) P_1 L^* L P_1 v_\tau .
\]
Taking into account (3.23), one obtains from the previous equality that
\[
\left\{ I + \tau \left( (J - \lambda I) \upharpoonright_{\text{ran}(E(\{\lambda\}^\complement))} \right)^{-1} E(\{\lambda\}^\complement) P_1 L^* L P_1 \right\} E(\{\lambda\}^\complement) v_\tau = -\tau \left( (J - \lambda I) \upharpoonright_{\text{ran}(E(\{\lambda\}^\complement))} \right)^{-1} E(\{\lambda\}^\complement) P_1 L^* L P_1 E(\{\lambda\}) v_\tau . \tag{3.24}
\]
Using the fact that
\[
\left\| \left( (J - \lambda I) \upharpoonright_{\text{ran}(E(\{\lambda\}^\complement))} \right)^{-1} \right\|_{B(\mathcal{H})} = \frac{1}{\text{dist}(\lambda, \sigma(J) \setminus \{\lambda\})}
\]
and choosing
\[
\tau < \frac{\text{dist}(\lambda, \sigma(J) \setminus \{\lambda\})}{2} , \tag{3.25}
\]
one verifies that the operator in braces in (3.24) is invertible and the following estimate holds
\[
\left\| E(\{\lambda\}^\complement) v_\tau \right\|_{B(\mathcal{H})} \leq \frac{2\tau}{\text{dist}(\lambda, \sigma(J) \setminus \{\lambda\})} \left\| L P_1 E(\{\lambda\}) v_\tau \right\|_{B(\mathcal{H})} . \tag{3.26}
\]
Now, it follows from (3.21) and (3.23) that
\[
E(\{\lambda\}) P_1 L^* L P_1 E(\{\lambda\}) v_\tau = -E(\{\lambda\}) P_1 L^* L P_1 E(\{\lambda\}^\complement) v_\tau ,
\]
whence
\[ \| E(\{\lambda\}) P_L^* L^\ast P_L E(\{\lambda\}) v_\tau \| \leq \| E(\{\lambda\}) v_\tau \| \]
\[ \leq \frac{2 \tau}{\text{dist}(\lambda, \sigma(J) \setminus \{\lambda\})} \| L P_L E(\{\lambda\}) v_\tau \| , \]
where, in the last inequality, we have used (3.26). If one defines
\[ T := L P_L^* E(\{\lambda\}) , \quad (3.27) \]
then the inequality above can be written as
\[ \| T^* T v_\tau \| \leq \frac{2 \tau}{\text{dist}(\lambda, \sigma(J) \setminus \{\lambda\})} \| T v_\tau \| \]
which in turn implies that
\[ \left\langle \left[ (T^* T)^2 - \frac{4 \tau^2}{\text{dist}(\lambda, \sigma(J) \setminus \{\lambda\})} T^* T \right] v_\tau , v_\tau \right\rangle \leq 0 . \quad (3.28) \]

We now show that \( T \) is not the zero operator. Indeed, if \( T = 0 \), then it follows from (3.27) that, for any \( u \in \mathcal{H} \),
\[ P_L^* E(\{\lambda\}) u = 0 \quad (3.29) \]
due to the fact that \( \ker(L) = \{0\} \). But, since the vector \( E(\{\lambda\}) u \) is in the kernel of \( J - \lambda I \), it should satisfy the equation (see (2.2))
\[ (\Omega E(\{\lambda\}) u)_1 = \lambda P_L^* E(\{\lambda\}) u \]
\[ = 0 , \]
where we have used (3.29) in the second equality. This last expression implies, via (2.2b), that
\[ B_1 P_L^* E(\{\lambda\}) u + A_1 P_2 E(\{\lambda\}) u = 0 \]
and from this, using again (3.29), one obtain \( A_1 P_2 E(\{\lambda\}) u = 0 \) which in turn implies that \( P_2 E(\{\lambda\}) u = 0 \) since \( \ker(A_1) \) is trivial. Having established that \( P_L^* E(\{\lambda\}) u = 0 \) and \( P_2 E(\{\lambda\}) u = 0 \), one finds recurrently from (2.2a), taking into account that \( \ker(A_m) = \{0\} \) for all \( m \in \mathbb{N} \), that
\[ P_3 E(\{\lambda\}) u = P_4 E(\{\lambda\}) u = \cdots = 0 . \]
Therefore \( E(\{\lambda\}) u = 0 \) for any \( u \in \mathcal{H} \) which is a contradiction since \( \lambda \) is in the discrete spectrum of \( J \). Thus we have shown that \( T \neq 0 \) and then \( T^* T \neq 0 \).

Now, since \( T^* T \) is a compact nonzero operator, one can take \( \tau \) such that, in addition
to (3.25), satisfies
\[
\frac{4\tau^2}{\text{dist}(\lambda, \sigma(J) \setminus \{\lambda\})^2} < \min \sigma(T^*T) \setminus \{0\}.
\]

By our choice of \(\tau\), the quadratic form in (3.28) cannot be negative. Indeed, if \(\sigma(T^*T) \setminus \{0\} = \{\mu_k\}_{k=1}^p (p \in \mathbb{N} \text{ since rank}(T^*T) \text{ is finite due to the fact that } \lambda \in \sigma_{\text{discr}}(J))\), then
\[
\mu_k^2 - \frac{4\tau^2}{\text{dist}(\lambda, \sigma(J) \setminus \{\lambda\})^2}\mu_k > 0
\]
for all \(k = 1, \ldots, p\) in view of the fact that \(\min \sigma(T^*T) \setminus \{0\} > 0\). This conclusion and (3.28) imply that
\[
v_\tau \in \ker \left[ (T^*T)^2 - \frac{4\tau^2}{\text{dist}(\lambda, \sigma(J) \setminus \{\lambda\})^2}T^*T \right].
\]

Therefore \(v_\tau \in \ker(T^*T) = \ker(T)\) which yields, recalling (3.27) and the fact that \(\ker(L) = \{0\}\), that \(P_1v_\tau = 0\).

We now show that \(P_1v_\tau = 0\) leads to \(v_\tau = 0\) which is a contradiction to our assumption. Since \(v_\tau\) is an eigenvector of \(J(\tau)\) at \(\lambda\), it follows from (2.2b) that
\[
(B_1 + \tau L^*L)P_1v_\tau + A_1P_2v_\tau = \lambda P_1v_\tau
\]
and therefore \(P_2v_\tau = 0\) since the kernel of \(A_1\) is trivial. As before, having established that \(P_1v_\tau, P_2v_\tau = 0\), it follows by iteration of (2.2a) that
\[
P_3v_\tau = P_4v_\tau = \cdots = 0
\]
since \(\ker(A_m) = \{0\}\) for all \(m \in \mathbb{N}\). \(\square\)

**Theorem 3.2.** Assume that the operator \(J\) given in Definition 1 is self-adjoint and bounded from below and \(\ker(A_m) = \{0\}\) for all \(m \in \mathbb{N}\). Consider real numbers \(b\) and \(\lambda\) such that \(\langle -\infty, b \rangle \cap \sigma_{\text{ess}}(J) = \emptyset\) and \(\lambda \in \langle -\infty, b \rangle \cap \sigma_p(J)\). Fix \(\delta > 0\) and \(\epsilon \in (0, 1)\). If \(u\) is an eigenvector corresponding to \(\lambda\), normalized so that \(\|u\| = 1\), then
\[
\|u_m\|_K \leq C \exp \left( -\gamma(\lambda) \sum_{k=1}^{m-1} \phi_\delta (\|A_k\|_{B(K)}) \right).
\]

where \(C\) does not depend on \(m\) and the functions \(\phi_\delta\) and \(\gamma\) are given in (3.2) and (3.3), respectively.

**Proof.** By Lemma 3.1, one can choose \(\tau > 0\) such that \(\lambda \notin \sigma(J(\tau))\). According to perturbation theory, \(J(\tau)\) is bounded from below and \(\sigma_{\text{ess}}(J) = \sigma_{\text{ess}}(J(\tau))\) (see [14, Chp. 4, Thm. 5.35 and Chp. 5, Thm. 4.11]). Thus, the estimates of Theorem 3.1 can be applied to \(J(\tau)\).
If \( u \) is a nonzero vector in \( \ker(J - \lambda I) \), then
\[
(J(\tau) - \lambda I)u = \tau P_1 L^* L P_1 u.
\]
Therefore,
\[
u = \tau(J(\tau) - \lambda I)^{-1} P_1 L^* L P_1 u \tag{3.30}
\]
which in turn implies
\[
\|u_m\|_K = \|P_m u\|_H
= \|P_m (J(\tau) - \lambda I)^{-1} P_1 L^* L P_1 u\|_H
\leq \tau \|P_m (J(\tau) - \lambda I)^{-1} P_1\|_{B(H)} \|P_1 u\|_H
\leq C \exp \left(-\gamma(\lambda) \sum_{k=1}^{m-1} \phi(\|A_k\|_{B(K)})\right) \|P_1 u\|_H,
\]
where in the first inequality we use that \( \|L\|_{B(K)} = 1 \) and in the second we resort to Theorem 3.1.

\[\square\]

**Remark 2.** The semi-boundedness of \( J \) is essential. Indeed, there are examples of (scalar) Jacobi operators [17], where accumulation of \( \sigma_{\text{discr}}(J) \) at infinity is allowed, whose generalized eigenvalues has an asymptotic behavior depending on the main diagonal entries of the Jacobi operator. The estimates given in the previous theorem show that, in the semi-bounded case, the diagonal block entries do not play any role in the component-wise estimates of the generalized eigenvectors.

Under additional conditions on the asymptotic behaviour of the sequence \( \{\|A_k\|\}_{k=1}^\infty \), it is possible to simplify the expression for the estimates in Theorems 3.1 and 3.2. This is done in the following assertion.

**Corollary 3.1.** Assume that the operator \( J \) given in Definition 1 is self-adjoint and bounded from below and that \( \|A_k\| \xrightarrow{k \to \infty} \infty \). Let the real number \( b \) be such that \( (-\infty, b) \cap \sigma_{\text{ess}}(J) = \emptyset \). Fix \( \epsilon \) arbitrarily small in \( (0, 1) \).

(a) If \( \lambda \notin \sigma(J) \) and \( \Re \lambda < b \), then
\[
\|G_{m,j}(\lambda)\|_{B(K)} \leq C_a \exp \left(-(1 - \epsilon) \sqrt{b - \Re \lambda} \sum_{k=\min(m,j)}^{\max(m,j)-1} 1/\sqrt{\|A_k\|_{B(K)}}\right).
\]

(b) Under the assumption that \( \ker(A_k) = \{0\} \) for all \( k \in \mathbb{N} \), if \( \lambda \in \sigma_p(J) \cap (-\infty, b) \) and \( u \) is the corresponding eigenvector, normalized so that \( \|u\| = 1 \), then
\[
\|u_m\|_K \leq C_b \exp \left(-(1 - \epsilon) \sqrt{b - \lambda} \sum_{k=1}^{m-1} 1/\sqrt{\|A_k\|_{B(K)}}\right).
\]
The constant $C_a$ does not depend on $m$ and $j$, and $C_b$ does not depend on $m$.

Proof. By choosing $\delta$ appropriately (essentially sufficiently large), one obtains

$$\frac{(b - \Re \lambda)(1 - \epsilon)}{\delta} < \epsilon_1 \ll 1.$$  

Given $b$ and $\epsilon$, the choice of $\delta$ depends on $\lambda$.

The presence of the factor $(1 - \epsilon)\sqrt{b - \Re \lambda}$ instead of $\gamma(\lambda)$ follows from the fact that, if $0 < t < \epsilon_1$, then

$$\psi^{-1}(t) \geq \sqrt{t}(1 - \epsilon_2),$$

where $\epsilon_2$ is arbitrarily small whenever $\epsilon_1$ is sufficiently small. Indeed, if (3.31) holds, then the choice of $\gamma$ in (3.3) can be replaced by

$$\gamma = \sqrt{\delta \sqrt{\frac{(b - \Re \lambda)(1 - \epsilon)}{\delta}}} = \sqrt{(b - \Re \lambda)(1 - \epsilon)},$$

where $\bar{\epsilon}$ is arbitrarily small.

Let us show that (3.31) holds. From (3.1), if $t = y e^{\sqrt{y}}$, one has

$$\psi^{-1}(t) = \sqrt{y} e^{-\frac{1}{2} \sqrt{y}} = \sqrt{y} \exp \left( -\frac{1}{2} \sqrt{y} e^{-\frac{1}{2} \sqrt{y}} \right) \geq \sqrt{y} e^{-\frac{1}{2} \sqrt{y}}.$$

Thus, if $t < \epsilon_1$, then

$$\psi^{-1}(t) \geq \sqrt{y} e^{-\frac{1}{2} \sqrt{y}} = \sqrt{t}(1 - \epsilon_2).$$

Finally, observe that

$$\sum_{k=1}^{m} \phi_{\delta} (\|A_k\|) = \sum_{\|A_k\| \leq \delta, k \leq m} \frac{1}{\sqrt{\delta}} + \sum_{\|A_k\| > \delta, k \leq m} \frac{1}{\sqrt{\|A_k\|}}$$

$$= \sum_{k=1}^{m} \frac{1}{\sqrt{\|A_k\|}} + \sum_{\|A_k\| \leq \delta, k \leq m} \frac{1}{\sqrt{\delta}} - \sum_{\|A_k\| \leq \delta} \frac{1}{\sqrt{\|A_k\|}}$$

where the second and third terms can be absorbed into the constant which does not depend on $m \gg 1$ since $\|A_k\| \to \infty$.

Remark 3. The choice of the function $\gamma$ in Theorems 3.1 and 3.2 is rather optimal. If one assumes that $b$ is the border of the essential spectrum of $J$, then the behavior of $\gamma(\lambda)$, when $\lambda$ approaches $b$, is the most interesting in applications. Formula (3.1) leads to the following asymptotic behavior

$$\gamma(\lambda) \simeq \sqrt{(b - \Re \lambda)(1 - \epsilon)}$$

as $\lambda \to b$. Note that $\sqrt{1 - \epsilon}$ may be chosen arbitrarily close to 1, but it has been
shown in the case of scalar Jacobi matrices [11] that the coefficient \((1 - \epsilon)\) cannot be replaced by any number greater than 1, although arbitrarily close to 1. This proves the sharpness of the estimates of Theorems 3.1 and 3.2. Namely, the factor \((1 - \epsilon)\) in the definition of \(\gamma\) (see (3.3)) cannot be replaced by \((1 + \epsilon)\) for any arbitrary small \(\epsilon > 0\).

4. Estimates of generalized eigenvectors in the case of commuting entries

If in Theorems 3.1 and 3.2 one additionally requires that the matrices \(A_m, B_m, A^*_m\) commute for \(m \in \mathbb{N}\), then a refinement of the previous results occurs so that it is possible to estimate the growth of the generalized eigenvectors along any spatial “direction” in \(\mathcal{K}\). The refined results are given in the next assertion, where Notation (V) is used. We draw the reader’s attention to the fact that in this section we use functions of operators given by the functional calculus of self-adjoint operators.

**Theorem 4.1.** Assume that the operator \(J\) given in Definition 1 is self-adjoint and bounded from below and the system of operators

\[
\{A_m, B_m, A^*_m\}_{m \in \mathbb{N}}
\]

commutes pairwise. Take a real number \(b\) such that \((-\infty, b) \cap \sigma_{ess}(J) = \emptyset\) and consider a complex number \(\lambda\) with \(\text{Re}\lambda < b\). Fix \(\delta > 0\) and \(\epsilon\) arbitrarily small in \((0, 1)\).

(i) If \(\lambda \notin \sigma(J)\) and \(\text{Re}\lambda < b\), then

\[
\left\| \exp \left( \gamma(\lambda) \sum_{k=\min(m,j)}^{\max(m,j)-1} \phi_\delta(|A_k|) \right) G_{mj}(\lambda) \right\| \leq C. \tag{4.1}
\]

(ii) Under the assumption that \(\ker(A_k) = \{0\}\) for all \(k \in \mathbb{N}\), if \(\lambda \in \sigma_p(J) \cap (-\infty, b)\) and \(u\) is the corresponding eigenvector, normalized so that \(\|u\|_{\mathcal{H}} = 1\), then

\[
\left\| \exp \left( \gamma(\lambda) \sum_{k=1}^{m-1} \phi_\delta(|A_k|) \right) u_m \right\|_{\mathcal{K}} \leq \tilde{C}. \tag{4.2}
\]

In both (i) and (ii), \(\phi_\delta\) and \(\gamma\) are given by (3.2) and (3.3), respectively. The constant \(C\) does not depend on \(m\) and \(j\), and \(\tilde{C}\) does not depend on \(m\).

**Proof.** First, we prove (i). We consider again the operators \(J_b\) and \(K\) defined in the proof of Theorem 3.1, but modify the definition of the operators \(\Phi\). For any fixed
\(N \in \mathbb{N}\), define the bounded operators on \(\mathcal{K}\)

\[
\Phi_m := \begin{cases} 
\exp\left(-\gamma \sum_{k=1}^{m-1} \phi_\delta(|A_k|)\right), & m \leq N, \\
\exp\left(-\gamma \sum_{k=1}^{N-1} \phi_\delta(|A_k|)\right), & m > N, 
\end{cases}
\]

and the bounded operator on \(\mathcal{H}\) by

\[
\Phi := \text{diag}\{\Phi_m\}_{m=1}^\infty
\]

Similar to what we had in the proof of Theorem 3.1, \(\Phi\) depends on \(N\) and \(\Phi(N)\) is a boundedly invertible contraction for any finite \(N\). Note that this time the block operator \(\Phi_m\) is not a scalar operator.

Consider the operator \(F \in B(\mathcal{H})\) such that (3.7) is satisfied with our new \(\Phi\). Repeating the argumentation in the proof of Theorem 3.1, one arrives at (3.9). Using (4.3) and the fact that the system \(\{A_m, B_m, A_m^*\}_{m \in \mathbb{N}}\) commutes, one obtains from (3.9) that

\[
\|\Re F\|_{B(\mathcal{H})} \leq \sup_{m \in \mathbb{N}} \left\{ \|A_m|e^{-\gamma \phi_\delta(|A_m|)} - 2I + e^{\gamma \phi_\delta(|A_m|)}\|_{B(\mathcal{K})} \right\}. \tag{4.4}
\]

Due to the inequality

\[
e^X - 2I + e^{-X} \leq X^2 e^X
\]

valid for any positive operator \(X\) and obtained from (3.11) by the spectral theorem, one derives from (4.4) the estimate

\[
\|\Re F\|_{B(\mathcal{H})} \leq \sup_{m \in \mathbb{N}} \left\{ \|A_m|\gamma^2 \phi_\delta^2(|A_m|)e^{\gamma \phi_\delta(|A_m|)}\| \right\}.
\]

But

\[
\sup_{m \in \mathbb{N}} \left\{ \|A_m|\gamma^2 \phi_\delta^2(|A_m|)e^{\gamma \phi_\delta(|A_m|)}\| \right\} \leq \gamma^2 e^{\gamma/\sqrt{\delta}}
\]

since, again by the spectral theorem and the definition of \(\phi_\delta\) given in (3.2), one has

\[
|A_m|\phi_\delta^2(|A_m|) \leq I.
\]

Following the reasoning of the proof of Theorem 3.1, one verifies that (3.10) holds as long as \(\gamma\) is given by (3.3). The rest of the proof repeats the one of Theorem 3.1 up to (3.16) from which, in view of (4.3), one obtains

\[
\left\| \exp \left( \gamma \sum_{m=1}^{j-1} \phi_\delta(|A_m|) \right) G_{jk}(\lambda) \exp \left( -\gamma \sum_{m=1}^{k-1} \phi_\delta(|A_m|) \right) \right\|_{B(\mathcal{K})} \leq C.
\]

The assertion (i) follows from this inequality by combining the operators on both sides of \(G_{jk}\) and letting \(N \to \infty\). In this proof, \(j \geq k\), but the other case is also covered by recurring to (2.5).
To prove (ii), one resorts to Lemma 3.1 and choose \( \tau > 0 \) so that \( \lambda \not\in \sigma(J(\tau)) \). As in the proof of Theorem 3.2, it follows from (3.19) that if \( u \) is in \( \ker(J - \lambda I) \), then (3.30) holds. Hence

\[
\|u_m\|_K \leq \tau \left\| P_m(J(\tau) - \lambda I)^{-1}P_1 \right\|_{B(H)} \|P_1u\|_H.
\]

For finishing the proof, it only remains to note that \( J(\tau) \) satisfies the hypothesis of (i).

**Remark 4.** Qualified estimates of the constants \( C \) and \( \tilde{C} \) given in (4.1) and (4.2), respectively, can be obtained by following the reasoning of Remark 1 with the scalar \( \phi_{\delta}(\|A_k\|) \) substituted by the operator \( \phi_{\delta}(|A_k|) \) for all \( k \in \mathbb{N} \). Note that the operator

\[
\exp \left( \gamma(\lambda) \sum_{k=\min(m,j)}^{\max(m,j)-1} \phi_{\delta}(|A_k|) \right)
\]

in (4.1) and (4.2) governs the growth of the generalized eigenvectors.

**Corollary 4.1.** Let \( J \) be the operator given in Definition 1 such that it is self-adjoint, bounded from below, and the operators \( A_m \) and \( B_m \) satisfy the conditions of Theorem 4.1 for all \( m \in \mathbb{N} \). Assume, additionally that \( \|A_m^{-1}\| \xrightarrow{m \to \infty} 0 \). Fix an \( \epsilon \in (0, 1) \).

a) If \( \lambda \not\in \sigma(J) \), then

\[
\left\| \exp \left( (1 - \epsilon)\sqrt{b - \Re \lambda} \sum_{k=\min(m,j)}^{\max(m,j)-1} \frac{1}{\sqrt{|A_k|}} \right) G_{mj}(\lambda) \right\| \leq C_a.
\]

b) If \( \lambda \in \sigma_p(J) \) and \( u \) is the corresponding eigenvector, normalized so that \( \|u\|_H = 1 \), then

\[
\left\| \exp \left( (1 - \epsilon)\sqrt{b - \Re \lambda} \sum_{k=1}^{m-1} \frac{1}{\sqrt{|A_k|}} \right) u_m \right\|_K \leq C_b.
\]

The constant \( C_a \) does not depend on \( m \) and \( j \), and \( C_b \) does not depend on \( m \).

**Proof.** We prove the claim in (b). The assertion (a) is proven analogously. We repeat part of the argumentation of the proof of Corollary 3.1. After having shown that \( \gamma(\lambda) \)
can be substituted by $(1 - \epsilon)\sqrt{b - \text{Re} \lambda}$, one arrives at
\[
\left\| \exp\left((1 - \epsilon)\sqrt{b - \text{Re} \lambda} \sum_{k=1}^{m-1} \phi_\delta(|A_k|)\right) u_m \right\|_\mathcal{K}
\]
\[
\leq \left\| \exp\left((1 - \epsilon)\sqrt{b - \text{Re} \lambda} \left( \sum_{k=1}^{m-1} (|A_k|)^{-\frac{1}{2}} + \sum_{|A_k|\leq \delta I} \left( \frac{1}{\sqrt{\delta}} I - (|A_k|)^{-\frac{1}{2}} \right) \right) \right) u_m \right\|_\mathcal{K}
\]
\[
\leq \tilde{C} \left\| \exp\left((1 - \epsilon)\sqrt{b - \text{Re} \lambda} \sum_{k=1}^{m-1} (|A_k|)^{-\frac{1}{2}} \right) u_m \right\|_\mathcal{K},
\]
where in passing to the last inequality, we have used the pairwise commutativity of the elements of the sequence $\{A_k\}_{k=1}^\infty$. Note also that, under the assumption that $m$ is sufficiently large, the operator
\[
\sum_{|A_k|\leq \delta I} \left( \frac{1}{\sqrt{\delta}} I - (|A_k|)^{-\frac{1}{2}} \right)
\]
is uniformly (with respect to $m$) bounded.

**Remark 5.** The form of the estimates given in Theorem 4.1 are optimal. Taking into consideration that $\phi_\delta(x) = 1/\sqrt{x}$ when $x > \delta$ and $\gamma(\lambda) \simeq \sqrt{(b - \text{Re} \lambda)(1 - \epsilon)}$ as $\lambda$ approaches $b$, one proves the estimate sharpness as in the scalar case $d = 1$ repeating the reasoning given in Remark 3. However the sharpness of Theorem 4.1 has a deeper character even in the “trivial” case where our block Jacobi matrix $J$ is the orthogonal sum of $d$ different copies of scalar Jacobi matrices (all entries are diagonal matrices). Indeed, in this case, Theorem 4.1 provides us with a sharp estimate for each scalar copy separately. Note that in the commuting case the reduction of $J$ to the orthogonal sum of $d$ copies of scalar Jacobi matrices generally cannot be performed effectively. A possible exception is the very special case when the matrix entries are scalar proportional to some fixed commuting matrices.

**5. An example with noncommuting entries**

This example corresponds to the case $d = 2$. For $\alpha \in (0, 1)$, define
\[
A_n = A_n^* := \begin{pmatrix} 0 & r_n \\ r_n & 0 \end{pmatrix}, \quad r_n = n^\alpha,
\]
and
\[
B_n := \begin{pmatrix} s_n & 0 \\ 0 & t_n \end{pmatrix}, \quad s_n = s n^\alpha, \quad t_n = t n^\alpha, \quad s, t > 0.
\]
By the moment, we consider arbitrary values for $s$ and $t$. Later, we will impose extra conditions on them.

Note that the block Jacobi operator $J$ whose matrix representation is (2.3) with the entries given above cannot be reduced to a orthogonal sum of scalar Jacobi matrices because $A_n$, $B_n$ do not commute if $s \neq t$. Let

$$B_n := \begin{pmatrix} 0 & I \\ -A_n^{-1}A_{n-1} & A_n^{-1}(\lambda I_2 - B_n) \end{pmatrix}, \quad \lambda \in \mathbb{R},$$

be the transfer matrix of $J$ associated with sequences $A_n$, $B_n$. Here $I$ is the unit matrix in $\mathcal{K} = \mathbb{C}^2$ (recall the notation given in Section 2). For the spectral analysis of $J$ we need to find asymptotic formulae for the eigenvalues of $B_n$. First recall that a necessary and sufficient condition for the invertibility of the $4 \times 4$ matrix $B_n - \mu I$ is the invertibility of the $2 \times 2$ matrix

$$X_n := A_n^{-1}(\lambda I - B_n) - \mu I - [-A_n^{-1}A_{n-1}(-\mu^{-1}I)].$$

This is clear if one uses the Schur-Frobenius complement. Multiplying $X_n$ by $\mu$, one has

$$\mu X_n = \begin{pmatrix} -\mu^2 - 1 + O(\frac{1}{n}) & \mu(\frac{\lambda}{n^\alpha} - t) \\ \mu(\frac{\lambda}{n^\alpha} - s) & -\mu^2 - 1 + O(\frac{1}{n}) \end{pmatrix}$$

for $n$ sufficiently large. Moreover, the diagonal elements of $\mu X_n$ are equal and $\det(\mu X_n)$ vanishes if and only if

$$(-\mu^2 - 1 + O(n^{-1}))^2 - \mu^2 \left( \frac{\lambda}{n^\alpha} - t \right) \left( \frac{\lambda}{n^\alpha} - s \right) = 0.$$

The last equation is equivalent to

$$\mu^2 + 1 + O(n^{-1}) = \pm \mu \sqrt{ \left( \frac{\lambda}{n^\alpha} - t \right) \left( \frac{\lambda}{n^\alpha} - s \right) },$$

which yields the following four eigenvalues of $B_n$ corresponding to the four possible choices of signs $+$, $-$ below.

$$\mu_n = \pm \frac{1}{2} \sqrt{ \left( \frac{\lambda}{n^\alpha} - t \right) \left( \frac{\lambda}{n^\alpha} - s \right) } \pm \frac{1}{4} \left( \frac{\lambda}{n^\alpha} - t \right) \left( \frac{\lambda}{n^\alpha} - s \right) - 1 + O\left( \frac{1}{n} \right).$$

Note that the $O(n^{-1})$ terms are all real. If one chooses $\alpha \in (\frac{1}{2}, 1)$, then, in the special
case \( st = 4 \), the last formula can be written as

\[
\mu_n = \pm \frac{1}{2} \sqrt{\frac{\lambda}{n^\alpha}} \pm \sqrt{-\lambda(s + t) + \frac{1}{4n^\alpha}} + O\left(\frac{1}{n}\right)
\]

\[
= \pm \frac{1}{2} \sqrt{\frac{\lambda}{n^\alpha}} \pm \frac{i}{2} n^{-\alpha/2} \sqrt{\lambda} \sqrt{s + t} \left(1 + O\left(n^{\alpha-1}\right)\right)
\]

\[
= \pm \left[ 1 \pm \frac{i\sqrt{\lambda}}{2} n^{-\alpha/2} \sqrt{s + t} - \frac{(s + t)\lambda}{4n^\alpha} + O\left(n^{\alpha/2-1}\right) \right]
\]

(5.1)

This formula will be used to give an estimate of the growth of generalized eigenvectors of \( J \). Note that if \( st \neq 4 \), then \( \mu_n \to \pm \frac{1}{2} \sqrt{st} \pm \frac{1}{2} \sqrt{st - 4} \) as \( n \to \infty \). Since this value does not coincide with \( \pm 1 \), it provides uniformly with respect to \( \lambda \) elliptic (if \( st < 4 \)) or hyperbolic (if \( st > 4 \)) behavior of solutions of the formal spectral equation. Therefore the only case where the value \( \lambda \) is essential (producing the unbounded gap) is the situation where \( st = 4 \).

Now we turn to the proof of the nonnegativity of \( J \), modulo compact operators, i.e., the existence of a compact operator \( K \) such that \( J + K \geq 0 \). Consider the quadratic form of \( J \), viz.,

\[
(Ju, u) = \sum_{n=1}^{\infty} (A_{n-1}u_{n-1} + B_n u_n + A_n u_{n+1}, u_n)_K.
\]

Since the vectors \( u = \{u_n\}_{n=1}^{\infty} \) with finitely many nonzero elements form a core for \( J \), for calculating the quadratic form of \( J \) it suffices to calculate it in such vectors. Write \( u_n = d_nv_n \) where \( d_n = n^{-\alpha/2} \). Using the identities

\[
d_n d_{n-1} (n - 1)^\alpha = 1 + O(n^{-1}), \quad d_n d_{n+1} n^\alpha = 1 + O(n^{-1})
\]

one obtains

\[
(Ju, u) = \sum_{n=1}^{\infty} \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) v_{n-1} + \left( \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix} \right) v_n + \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) v_{n+1}, v_n \right)_K
\]

\[
+ \sum_{n=1}^{\infty} \left( O(n^{\alpha-1}) u_{n-1} + O(n^{\alpha-1}) u_{n+1}, u_n \right)_K.
\]

The last series in the equality above corresponds to the Jacobi operator in \( \mathcal{H} = l_2(\mathbb{N}, \mathbb{C}^2) \) with the subdiagonals decaying as \( O(n^{\alpha-1}) \) with \( \alpha < 1 \). Therefore it defines a compact operator in \( \mathcal{H} \). Hence the problem of positivity has been reduced to the question of positivity of the block Jacobi matrix \( J_c \) defined by constant entries

\[
A_n := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B_n := \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix}.
\]
We prove the following general result for $s, t > 0$.

**Lemma 5.1.** If $s, t > 0$, then

$$J_c \geq \frac{st - 4}{\frac{t+s}{2} + \left[\left(-\frac{s}{2}\right)^2 + 4\right]^{1/2}}.$$  

**Proof.** For $v_n = (f_n, g_n)^T$, we have

$$(J_c v, v) = \sum_{n=1}^{\infty} \left( \left( \frac{g_{n-1} + sf_n + g_{n+1}}{f_{n-1} + tg_n + f_{n+1}} \right, \left( f_n \right) \right)_{C^2}$$

$$\geq \sum_{n=1}^{\infty} (s|f_n|^2 + t|g_n|^2 - 2|f_n||g_{n-1}| - 2|f_{n-1}||g_n|)$$

$$\geq \sum_{n=1}^{\infty} (s|f_n|^2 + t|g_n|^2 - \frac{1}{\epsilon}|f_n|^2 - \epsilon|g_{n-1}|^2 - \eta|g_n|^2 - \frac{1}{\eta}|f_{n-1}|^2)$$

for any $\epsilon, \eta > 0$. Let us optimize the choice of $\epsilon$ and $\eta$. Note that the last sum can be written as

$$\sum_{n=1}^{\infty} \left[ \left( s - \frac{1}{\epsilon} - \frac{1}{\eta} \right) |f_n|^2 + (t - \epsilon - \eta) |g_n|^2 \right].$$

Choose $\epsilon, \eta$ so that

$$s - \frac{\epsilon + \eta}{\epsilon \eta} = t - (\epsilon + \eta),$$

and define the new variable $k := \frac{\eta}{\epsilon}$. Then, for a fixed $k$, the identity (5.2) is equivalent to

$$\epsilon^2 + (1 + k)^{-1}(s - t)\epsilon - \frac{1}{k} = 0.$$  

The positive solution of the last equation is given by

$$\epsilon = \left[2(1 + k)\right]^{-1}(t - s) + \left[\left\{\frac{1}{2}(1 + k)^{-1}(t - s)\right\}^2 + k^{-1}\right]^{1/2}. $$

Since

$$\epsilon \eta = (1 + k)\epsilon = \frac{t - s}{2} + \left[\frac{(t - s)^2}{4} + \frac{(k + 1)^2}{k}\right]^{1/2},$$

one checks that the minimum of $(1 + k)\epsilon$ taken for $k > 0$ is equal to

$$\frac{t - s}{2} + \left[\frac{(t - s)^2}{4} + 4\right]^{1/2}.$$
and it is attained for \( k = 1 \). Note that (5.2) is also satisfied when

\[
\eta = \epsilon = \frac{1}{2} \left( \frac{t - s}{2} + \left[ \frac{(t-s)^2}{2} + 4 \right]^{1/2} \right).
\]

(5.4)

Finally, one has

\[
J_c \geq t - (\epsilon + \eta) = t - 2\epsilon = \frac{t + s}{2} - \left[ \frac{(t-s)^2}{2} + 4 \right]^{1/2}
\]

\[
= \left\{ \left[ \frac{t + s}{2} \right] - \left[ \frac{(t-s)^2}{2} + 4 \right] \right\} \left\{ \frac{t + s}{2} + \left[ \frac{(t-s)^2}{2} + 4 \right]^{1/2} \right\}^{-1}
\]

\[
= (ts - 4) \left\{ \frac{t + s}{2} + \left[ \frac{(t-s)^2}{2} + 4 \right]^{1/2} \right\}^{-1}
\]

Now, we turn to the asymptotics of the decreasing generalized eigenvectors of the semi-bounded block Jacobi matrix \( J \). Denote by \( \mathcal{B}_\infty := \lim_{n \to \infty} \mathcal{B}_n \). The following arguments are heuristic. Let \( \{\mu_q(n)\}_{q=1}^4 \) be the eigenvalues of \( \mathcal{B}_n \), then, on the basis of a formal Levinson type formula for \( \lambda < 0 \), one has, for \( n_0 \) sufficiently large,

\[
u_n \approx \left( \prod_{k=n_0}^n \mu_q(k) \right) e_q,
\]

(5.5)

where \( e_q \) are the eigenvectors of \( \mathcal{B}_\infty \) (the proof of similar asymptotic formulae are found in [17]).

Using (5.1) and (5.5), one obtains the following estimate of the decreasing generalized eigenvectors

\[
\|u_n\|_2 \leq \text{const.} \prod_{k=n_0}^n \left[ 1 - \frac{(1 - \epsilon)\sqrt{-\lambda(t + s)}}{2k^{\alpha/2}} \right]
\]

for arbitrary small \( \epsilon > 0 \) and \( n \gg 1 \). The last product can be estimated from above by

\[
C_{\epsilon_0} \exp \left[ -(1 - \epsilon_0)\frac{\sqrt{-\lambda(t + s)}}{2 \left( 1 - \frac{\alpha}{2} \right)} n^{1-\alpha/2} \right]
\]

for some constant \( C_{\epsilon_0} \) and arbitrary \( \epsilon_0 > \epsilon \). Since \( st = 4 \), one has \( \sqrt{\frac{st}{2}} \geq 1 \). Thus, one can write

\[
\|u_n\|_2 \leq \exp \left[ -(1 - \epsilon_0)\frac{\sqrt{-\lambda}}{1 - \frac{\alpha}{2}} n^{1-\alpha/2} \right].
\]

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This estimate and the one obtained rigorously in Theorem 3.2 satisfy
\[
\exp \left( -\gamma(\lambda) \sum_{k=1}^{m-1} \phi_{\delta}(\|A_k\|_{B(K)}) \right) \asymp \exp \left[ -(1 - \epsilon_0) \sqrt{-\lambda \sum_{k=1}^{n-1} \frac{1}{k^{\alpha/2}}} \right]
\asymp \exp \left[ -(1 - \epsilon_0) \frac{\sqrt{-\lambda}}{1 - \frac{4}{\epsilon} \eta^{1-\alpha/2}} \right].
\]

This formal reasoning shows sharpness of Theorem 3.2, provided one chooses \(s\) and \(t\) arbitrary close to 2 and therefore \(\frac{1}{2} \sqrt{s + t}\) is arbitrary close to 1.

**Section’s concluding remarks**

i Weyl Theorem and the results of this section prove that \(\sigma_{\text{ess}}(J) \subset \mathbb{R}_+\).

ii Lemma 5.1 and the decomposition \(J = J_c + K\) show that \(J\) is bounded from below and \(\sigma(J) \cap \mathbb{R}_-\) is discrete and can accumulate only at zero.

iii One can prove that \(\sigma_{\text{ess}}(J) = \mathbb{R}_+\) by using the formal Levinson type asymptotics of solutions as an Ansatz for approximation of Weyl sequences corresponding to each \(\lambda > 0\). This idea is described in detail in [13].

iv Concerning the assumption \(st = 4\), one can check that for \(st > 4\), \(\sigma_{\text{ess}}(J) = \emptyset\) and if \(st < 4\), then \(\sigma_{\text{ess}}(J) = \mathbb{R}\). This explains the role of the condition \(st = 4\). In the case \(s = t\), the matrix \(J\) can be written as an orthogonal sum of two (unitarily equivalent) scalar Jacobi matrices by diagonalizing the matrix \(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\). In that case, the above results for \(st > 4\) and \(st < 4\) follow directly from [10]. Moreover the result of (iii) for \(s = t = 2\) immediately follows from [12]. Finally note that \(J\), our class of block Jacobi matrices depending on parameters \(s, t\), exhibits a spectral phase transition phenomenon of first kind (see [10]) with the threshold corresponding to the condition \(st = 4\).

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