ON THE FUZZY NATURE OF CONSTRUCTED ALGEBRAIC STRUCTURE

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1.0 INTRODUCTION
The concept of fuzzy sets was introduced by Zadeh(1965), by defining them in terms of mappings from a set into a unit interval on the real line. Fuzzy sets were introduced to provide means to describe situations mathematically which gives rise to ill-defined classes, i.e. collection of objects for which there is no precise criteria for membership, collections of this type have a vague boundaries (Fuzzy), there are objects for which it is impossible to determine whether or not they belong to the collection. The classical mathematical theories, by which certain types of certainty can be expressed, are the classical set theory and probability theory, in terms of set theory, uncertainty is expressed by any given set of possible alternatives in situations where only one of the alternatives may actually happen. Uncertainty expressed in terms of sets of alternatives results from the non-specificity inherent in each set. Probability theory expresses uncertainty in terms of a classical measure of subsets of a given set of alternatives. The set theory introduced by Zadeh, presents the notion of membership in a given subset as a matter of degree rather than of totally in or totally out. With a fuzzy set theory, one obtains a logic in which statements may be true or false to different degrees rather than the bivalent situations (on or off) of being true or false.

Permutation pattern have been used in the past decades to study mathematical structures. Audu(1986), Ibrahim(2005)studied the concept of permutation pattern using some elaborate scheme to determine the order of precedence and the position of each of the elements in a finite set of prime size have also been established in Ibrahim (2007), Garba (2018) and also an idea of embedment as an algebraic structure has yielded some interesting results by Ibrahim (2005), Garba (2018). They studied the structure and developed a scheme for the range of such cycles and use it to investigate further number theoretic and algebraic properties of \( G_p \) and furthermore a group theoretic properties was also investigated by Garba (2018) and the concept of Fuzzy nature of \( G_p \) has also been studied by Aremu (2017) and investigated the alpha-level cut of \( G_p \). Ibrahim (2007)studied the concept of permutation pattern using some elaborate scheme to determine the order of precedence and the position for each of the elements in a finite set of prime size, and establish a scheme the scheme for generating each element in the permutation. Garba (2009) studied the \( G_p \)structure using number theoretic properties of Catalan numbers, and also developed a scheme for range of such cycles defined to be \( |\Delta| \) where \( I \) is the last element in the cycle and \( f \) is the first element in the cycle, and established that for all cycles in \( G_p \) the range exist, they also use it to investigate further number theoretic properties of \( G_p \). Usman (2011)investigated the group theoretic properties of \( G_p \) using composition of functions, by investigating the properties of a group and established that the structure is an Abelian group, using additive group of integers modulo \( n \), where \( n \) is necessarily a prime.
The support of a fuzzy set (denoted Supp) is the set of all elements that have membership functions in the universe a Boolean state of obedience.

2.3 The \( \alpha - \text{Cut Level Set} \)

The \( \alpha - \text{level} \) of a fuzzy set \( \tilde{A} \) is a crisp set that contains the elements that have membership functions in \( \tilde{A} \) greater than or equal to \( \alpha \). The support of a fuzzy set (denoted Supp) is the crisp set of all \( x \in X \) for which \( \mu_i(x) > 0 \) (Zadeh 1965).

2.4 The Support of a Fuzzy Set

The support of a fuzzy set is the set of ordered pairs, \( \tilde{A} = \{(x, \mu_i(x)) : x \in X\} \), where \( \mu_i(x) : X \rightarrow [0,1] \) is called degree of membership of \( x \) in \( \tilde{A} \) (Zadeh 1965).

2.5 Cycle and Successor

Let \( \Omega \) be a non-empty, totally ordered and finite subset of \( \mathbb{N} \). Let \( G_p = \{w_1, w_2, ..., w_p-1\} \) be a structure such that each \( w_i \) is generated from the arbitrary set \( \Omega \) for any prime \( p \geq 5 \), using the scheme

\[
\pi(w_i) = (1(1 + i)_{mod p}(1 + 2i)_{mod p} \ldots (1 + (p - 1)i)_{mod p})
\]

Where \( mp \) is modulop

Then each \( w_i \) is called a cycle and the elements in each \( w_i \) are distinct and called successors (Ibrahim 2004).

2.5.1 \( n \text{th successor} \)

Then \( n \text{th successor} \) of a cycle \( w_i \) is given by \( a_n = (1 + (n - 1)i)_{mod p} \) where \( 1 \leq n \leq p \), and \( 1 \leq i \leq p - 1 \). The number of distinct successors in a cycle is called the length of the cycle (Ibrahim 2004).

2.5.2 Range of Cycle

The range of a cycle \( w_i \) is defined as \( \Delta^i(w) := [\Delta^i(w)] \), where \( \Delta^i(w) \) is the difference between the first and last successor in a cycle \( w \) (Garba, 2009).

2.5.3 Definition of \( G_p' \)

Let \( G_p = \{w_1, w_2, ..., w_{p-1}\} \) be as defined above then \( G_p' := G_p \cup \{w_p\} \), where \( w_p := \{pp \ldots p\} \). This is \( G_p' = \{w_1, w_2, ..., w_{p-1}, w_p\} \).

Using the above setting, if \( p=5 \), then we have the following set of permutations \( w_1 = (12345), w_2 = (13524), w_3 = (14253), w_4 = (15432) \) and this shows that \( S = \{(12345), (13524), (14253), (15432)\} \).

Note that 0 and 5 are equivalent in modulo 5, thus instead of using 0 in modulo \( p \) we will be using \( p \) (Garba, 2009).

3.0 RESULT AND DISCUSSION

In this section, the discussion of the result is carried out by figures, tables and proofs.

3.1 Fuzzy Nature of \( G_p \)

Let \( G_p := G_p \cup \{w_p\} \) and \( G_p \subseteq G_p' \), then \( G_p \) is a fuzzy set defined by

\[
\tilde{G}_p = \{(\mu_{i_p}(w_i)) : w_i \in G_p\}
\]

Where \( \mu_{i_p}(w_i) = \left( i, \frac{\pi(w_i)}{p+2} \right) \), \( i < p \)

\[
\pi(w_i) = |\Delta^i_w(w)|
\]

\( \sigma \) is the last successor and \( f \) is the first successor.

Illustration: consider \( G_5' \) where \( p=5 \), \( G_5' = \{w_1, w_2, w_3, w_4, w_5\} \) let \( G_5 \subseteq G_5' \)

Then \( G_5 = \{w_1, w_2, w_3, w_4, w_5\} \), Defined \( \mu_{G_5}(w_1) = \left( i, \frac{\pi(w_i)}{5} \right) \), \( i < 5 \)

\[
\pi(w_i) = |\Delta^i_w(w)|
\]

\( \mu_{G_5}(w_1) = (1, 0.6) \)

\( \mu_{G_5}(w_2) = (2, 0.4) \)

\( \mu_{G_5}(w_3) = (3, 0.3) \)

\( \mu_{G_5}(w_4) = (4, 0.1) \)

\( \tilde{G}_5 = \{(\mu_{G_5}(w_i)) : w_i \in G_5'\} \)

\( \tilde{G}_5 = \{(1,0.6), (2,0.4), (3,0.3), (4,0.1)\} \), then \( \tilde{G}_5 \) is a fuzzy set.
3.2 Proposition: The $\alpha$–cut level of any $G_p$ is $G_p | w_{p-1}$

Proof

An $\alpha$–cut level is a set that contains values from the membership functions greater than or equal to $\alpha$. $\alpha$ is an arbitrary value with the range of fuzzy $[0,1]$. let $G_p$ be a fuzzy set and $G_p \subseteq G_p$, then the $\alpha$–cut level is a set $G_{p\alpha} = \{ w_i : \mu_{G_p}(w_i) \geq \alpha \}$ for $\alpha = \frac{1}{p}$, where $p \geq 5$.

Since $G_{p\alpha} = \{ \mu_{G_p}(w_i) : \mu_{G_p}(w_i) \geq \alpha, i < p \}$

Without loss of generality, $\mu_{G_p}(w_{p-1}) = \left( i, \frac{\pi(w_i)}{p+2} \right)$, $i < p$

Where $G_p = \{ w_1, w_2, \ldots w_{p-1} \}$

$\mu_{G_p} = \{ w_1, w_2, \ldots w_{p-2} \} \geq \frac{1}{p}$ but $\mu_{G_p} < \frac{1}{p}$

$=> G_p | w_{p-1}$ is the domain of the alpha-cut level of the set $G_p$.

Illustration consider when $p=5$,

$G_5 = \{ w_1, w_2, w_3, w_4 \}$

$\mu_{G_5}(w_1) = (1, 0.6)$

$\mu_{G_5}(w_2) = (2, 0.4)$

$\mu_{G_5}(w_3) = (3, 0.3)$

$\mu_{G_5}(w_4) = (4, 0.1)$

if $\alpha = \frac{1}{p}$, then $\alpha = \frac{1}{5} = 0.2$,

$=> w_{p-1} < \alpha$

From the illustration above it implies that, the $\alpha$–cut level is the domain $G_p | w_{p-1}$. The table below gives a complete description of the alpha-cut-level of the constructed algebraic structure, the alpha level of each $G_p$ exist, and is unique.

| s/n | $w_i$ | $\mu_{G_p}(w_i)$ |
|-----|-------|------------------|
| 1   | $w_1$ | 0.6              |
| 2   | $w_2$ | 0.4              |
| 3   | $w_3$ | 0.3              |
| 4   | $w_4$ | 0.1              |
| 5   | $\alpha$–cut level | 0.2              |

Table 3.1: Membership Functions of $w_i$ and $\alpha$–Cut Level

Figure 3.1: Alpha Cut Level Set

Figure 3.1 illustrate the $\alpha$–cut level of the $G_p$ the vertical axis represents the membership functions while the horizontal axis represents the permutations $w_i$.

For $G_5$, the $\alpha$–cut level is $G_5 | w_4$.

For $G_7$, the $\alpha$–cut level is $G_7 | w_6$.

For $G_9$, the $\alpha$–cut level is $G_9 | w_{10}$.

For $G_{10}$, the $\alpha$–cut level is $G_{10} | w_{12}$.

This generalize the proof.
3.3 Proposition: The Support of the fuzzy set $\hat{G}_{p}$ of any $G_{p}$ is the entire domain.

Proof

The support of a fuzzy set (denoted by Supp) are those members of the set in which their membership degree is $> 0$,

$$\text{Supp}(\hat{G}_{p}) = \{ \mu_{G_{p}}(w_i) : \mu_{G_{p}}(w_i) > 0 \}$$

And $\pi(\omega_i)$ is never zero, then, the result follows.

Table 3.2: Membership Functions of $w_i$

| s/n | $w_i$ | $\mu_{G_{p}}(w_i)$ |
|-----|------|------------------|
| 1   | $w_1$ | 0.6              |
| 2   | $w_2$ | 0.4              |
| 3   | $w_3$ | 0.3              |
| 4   | $w_4$ | 0.1              |

Figure 3.2: Support of a Fuzzy Set $G_{p}$

Figure 3.2 gives the description of the Supp($G_{p}$), and it can be seen that all the values are $> 0$, the vertical axis represents the membership functions while the horizontal axis represents the permutations $\omega$ and is true for any $G_{p}$, then the support of any fuzzy set in $G_{p}$ is the entire domain.

3.5 Conclusion

The construction of an algebraic structures and investigating their algebraic properties cannot be over emphasized as it has a lot of applications in different field of mathematics, in this paper we investigated some fuzzy nature of an algebraic structure $G_{p}$ that was constructed earlier, where we discovered that if $\hat{G}_{p}$ is a fuzzy set, then the $\alpha$ - cut level set of any $G_{p}$ is a set $G_{p}|w_{p-1}$ and the support of $\hat{G}_{p}$ is the entire domain, In the above constructed algebraic structure the first element of the permutation is always fixed.

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