Existence result for impulsive coupled systems on the half-line*

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Abstract

This work considers a second order impulsive coupled system of differential equations with generalized jump conditions in half-line, which can depend on the impulses of the unknown functions and their first derivatives.

The arguments apply the fixed point theory, Green’s functions technique, \(L^1\)-Carathéodory functions and sequences and Schauder’s fixed point theorem.

The method is based on Carathéodory concept of functions and sequences, together with the equiconvergence on infinity and on each impulsive moment, and it allows to consider coupled fully nonlinearities and very general impulsive functions.

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1 Introduction

Boundary-value problems in unbounded domains, can be applied to a large variety of contexts, (see for instance, [1, 3, 6, 11, 12, 13, 14, 19, 21]).

The theory of impulsive differential equations describes processes in which a sudden change of state occurs at certain moments. Some examples: in [2], Dishliev et al., make a very complete explanation of these equations, where we can also find applications about pharmacokinetic model, logistic model, Gompertz model (mathematical model for a time series), Lotka-Volterra model and population dynamics. In [4], Guo uses the fixed point theory to investigate the existence and uniqueness of solutions of two-point boundary value problems for second order non-linear impulsive integro-differential equations on infinite intervals in a Banach space. The same author, in [5], by a comparison result, obtains the existence of maximal and minimal solutions of the initial value problems for a class of second-order impulsive integro-differential equations in a Banach space. In [10], Lee and Liu study the existence of extremal solutions for a class of singular boundary value problems of second order impulsive differential equations. In [16], Minhós, and Carapinha study separated impulsive problems with a fully third order differential equation, including an increasing homeomorphism, and impulsive conditions given by generalized functions. In [20], Pang et al., consider a second-order impulsive differential equation with integral boundary conditions, where they proposed some sufficient conditions for the existence of solutions, by using the method of upper and lower solutions and Leray-Schauder degree theory. In [21], Lee and Lee combine the method of upper and lower solutions with fixed point index theorems on a cone to study the existence of positive solutions for a singular two point boundary value problem of second order impulsive equation with fixed moments.

In [22], Wang, Zhang and Liang, consider the initial value problem for second order impulsive integro-differential equations, which nonlinearity depend on the first derivative, in a Banach space $E$:

\[
\begin{align*}
  x''(t) &= f(t, x(t), x'(t), T x(t), S x(t)), \quad t \neq t_k, \; k = 1, 2, \ldots, m, \\
  \Delta x(t_k) &= I_k(x(t_k), x'(t_k)), \quad k = 1, 2, \ldots, m, \\
  \Delta x'(t_k) &= \overline{I}_k(x(t_k), x'(t_k)), \quad k = 1, 2, \ldots, m, \\
  x(0) &= x_0, \quad x'(0) = x'_0,
\end{align*}
\]
where \( f \in C[J \times E^4, E] \), \( J = [0, 1], 0 < t_0 < t_1 < \ldots < t_k < \ldots < t_m < 1 \).

\( I_k, \bar{T}_k \in C[E^2, E], k = 1, 2, \ldots, m, x_0, x_0^* \in E, \theta \) denotes de zero element of \( E, J' = J \setminus \{t_1, t_2, \ldots, t_m\} \) and \( J_0 = [0, t_1], J_k = (t_k, t_{k+1}], k = 1, 2, \ldots, m, \) \( t_{m+1} = 1 \),

\[
Tx(t) = \int_0^t k(t, s)x(s)ds, \quad Sx(t) = \int_0^1 h(t, s)x(s)ds, \quad \forall t \in J,
\]

where \( k \in C[D, \mathbb{R}_+] \), \( D = \{(t, s) : J \times J|s \geq t\} \), \( h \in C[J \times J, \mathbb{R}_+] \), \( \mathbb{R}_+ = [0, +\infty) \).

In [7], we found the study of second-order nonlinear differential equation

\[
(p(t)u'(t))' = f(t, u(t)), \quad t \in (0, \infty) \setminus \{t_1, t_2, \ldots, t_n\},
\]

where \( f : [0, +\infty) \times \mathbb{R} \to \mathbb{R} \) is continuous, \( p \in [0, +\infty) \cap C(0, +\infty) \) and \( p(t) \geq 0 \) for all \( t > 0 \), with the impulsive conditions

\[
\Delta u'(t_k) = I_k(u(t_k)), \quad k = 1, \ldots, n,
\]

where \( I_k : \mathbb{R} \to \mathbb{R}, k = 1, \ldots, n \), are Lipschitz continuous, \( n \geq 1 \), and the boundary conditions

\[
\alpha u(0) - \beta \lim_{t \to 0^+} p(t)u'(t) = 0, \quad \gamma \lim_{t \to \infty} u(t) + \delta \lim_{t \to \infty} p(t)u'(t) = 0.
\]

In order to ensure that the non-resonant scenario is considered, the condition

\[
\rho = \gamma \beta + \alpha \delta + \alpha \gamma \int_0^\infty \frac{d\tau}{p(\tau)} \neq 0
\]

is imposed.

In [8], the authors prove the existence of multiple positive solutions for a singular Gelfand type boundary value problem with the following second-order impulsive differential system:

\[
u''(t)\lambda h_1(t)f(u(t), v(t)) = 0, \quad t \in (0, 1), \quad t \neq t_k,
\]

\[
u''(t)\mu h_2(t)g(u(t), v(t)) = 0, \quad t \in (0, 1), \quad t \neq t_k,
\]

\[
\Delta u \big|_{t=t_1} = I_u(u(t_1)), \quad \Delta v \big|_{t=t_1} = I_v(v(t_1)),
\]

\[
\Delta u' \big|_{t=t_1} = N_u(u(t_1)), \quad \Delta v' \big|_{t=t_1} = N_v(v(t_1)),
\]

\[
u(0) = a \geq 0, \quad v(0) = b \geq 0, \quad u(1) = c \geq 0, \quad v(1) = d \geq 0,
\]

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where $\lambda, \mu$ are positive real parameters, $\Delta u|_{t=t_1} = u(t_1^+) - u(t_1^-)$, $\Delta u'|_{t=t_1} = u'(t_1^+) - u'(t_1^-)$, $f, g \in C(\mathbb{R}^2, (0, \infty))$, $I_u, I_v \in C(\mathbb{R}, \mathbb{R})$ satisfying $I_u(0) = 0 = I_v(0)$, $N_u, N_v \in C(\mathbb{R}, (-\infty, 0])$, and $h_1, h_2 \in C((0, 1), (0, \infty))$.

Inspired by these works, we follow arguments and techniques considered in [15] and [17], in particular, about impulsive problems on the half-line and second order coupled systems on the half-line, respectively. However, it is the first time where the existence of solutions is obtained for impulsive coupled systems, with generalized jump conditions in half-line and with full nonlinearities, that depend on the unknown functions and their first derivatives.

In particular, in the present paper, we consider the second order impulsive coupled system in half-line composed by the differential equations

$$\begin{cases}
  u''(t) = f(t, u(t), v(t), u'(t), v'(t)), & t \neq t_k, \\
  v''(t) = h(t, u(t), v(t), u'(t), v'(t)), & t \neq \tau_j,
\end{cases} \tag{1}$$

where $f, h : [0, +\infty[ \times \mathbb{R}^4 \to \mathbb{R}$ are $L^1$-Carathéodory functions, the boundary conditions

$$\begin{cases}
  u(0) = A_1, \quad v(0) = A_2, \\
  u'(+\infty) = B_1, \quad v'(+\infty) = B_2,
\end{cases} \tag{2}$$

for $A_1, A_2, B_1, B_2 \in \mathbb{R}$ and the generalized impulsive conditions

$$\begin{cases}
  \Delta u(t_k) = I_{0k}(t_k, u(t_k), u'(t_k)), \\
  \Delta u'(t_k) = I_{1k}(t_k, u(t_k), u'(t_k)), \\
  \Delta v(\tau_j) = J_{0j}(\tau_j, v(\tau_j), v'(\tau_j)) \\
  \Delta v'(\tau_j) = J_{1j}(\tau_j, v(\tau_j), v'(\tau_j)),
\end{cases} \tag{3}$$

where, $k, j \in \mathbb{N}$,

$$\Delta u^{(i)}(x_k) = u^{(i)}(x_k^+) - u^{(i)}(x_k^-), \quad \Delta v^{(i)}(\tau_j) = v^{(i)}(\tau_j^+) - v^{(i)}(\tau_j^-),$$

$I_{ik}, J_{ij} \in C([0, +\infty[ \times \mathbb{R}^2, \mathbb{R})$, $i = 0, 1$, with $t_k, \tau_j$ fixed points such that $0 < t_1 < \cdots < t_k < \cdots$, $0 < \tau_1 < \cdots < \tau_j < \cdots$ and

$$\lim_{k \to +\infty} t_k = +\infty, \quad \lim_{j \to +\infty} \tau_j = +\infty.$$

Some points play a key role, such as: Carathéodory functions and sequences, the equiconvergence at each impulsive moment and at infinity, Banach spaces with weighted norms, and Schauder’s fixed point theorem, to prove the existence of solutions.
The paper is organized in the following way: In section 2 we present some auxiliary results and definitions. Section 3 contains an existence result for the impulsive coupled systems with generalized jump conditions in half-line. In Section 4 the main existence theorem is applied to a real phenomena: a model of the motion of a spring pendulum.

2 Definitions and preliminary results

Define
\[ u(t_k^+) := \lim_{t \to t_k^+} u(t), \quad v(\tau_j^+) := \lim_{j \to \tau_j^+} \tau(j), \]
and consider the set
\[ PC_1([0, +\infty[) = \left\{ u : u \in C([0, +\infty[, \mathbb{R}) \text{ continuous for } t \neq t_k, u(t_k) = u(t_k^-), \quad u(t_k^+) \text{ exists for } k \in \mathbb{N} \right\}, \]
\[ PC_1^n([0, +\infty[) = \left\{ u : u^{(n)}(t) \in PC_1([0, +\infty[), n = 1, 2, \right\}, \]
\[ PC_2([0, +\infty[) = \left\{ v : v \in C([0, +\infty[, \mathbb{R}) \text{ continuous for } \tau \neq \tau_j, v(\tau_j) = v(\tau_j^-), \quad v(\tau_j^+) \text{ exists for } j \in \mathbb{N} \right\}, \]
and \[ PC_2^n([0, +\infty[) = \left\{ v : v^{(n)}(j) \in PC_2([0, +\infty[), n = 1, 2. \right\}. \]

Denote the space
\[ X_1 := \left\{ x : x \in PC_1([0, +\infty[) : \lim_{t \to +\infty} \frac{x(t)}{1 + t} \in \mathbb{R}, \lim_{t \to +\infty} x'(t) \in \mathbb{R} \right\}, \]
\[ X_2 := \left\{ y : y \in PC_2([0, +\infty[) : \lim_{t \to +\infty} \frac{y(t)}{1 + t} \in \mathbb{R}, \lim_{t \to +\infty} y'(t) \in \mathbb{R} \right\}, \]
and \[ X := X_1 \times X_2. \]

In fact, \( X_1, X_2 \) and \( X \) are Banach spaces with the norms
\[ \|u\|_{X_1} = \max \left\{ \|u\|_0, \|u'\|_1 \right\}, \quad \|v\|_{X_2} = \max \left\{ \|v\|_0, \|v'\|_1 \right\}, \]
and
\[ \|(u, v)\|_X = \max \left\{ \|u\|_{X_1}, \|v\|_{X_2} \right\}, \]
respectively, where
\[ \|\Upsilon\|_0 := \sup_{t \in [0, +\infty[} \frac{|\Upsilon(t)|}{1 + t} \quad \text{and} \quad \|\Upsilon\|_1 := \sup_{t \in [0, +\infty[} |\Upsilon(t)|. \]
Definition 1 A function \( g : [0, +\infty[ \times \mathbb{R}^4 \to \mathbb{R} \) is \( L^1 \)-Carathéodory if

i) for each \( (x, y, z, w) \in \mathbb{R}^4 \), \( t \mapsto g(t, x, y, z, w) \) is measurable on \( [0, +\infty[ \);

ii) for almost every \( t \in [0, +\infty[ \), \((x, y, z, w) \mapsto g(t, x, y, z, w) \) is continuous on \( \mathbb{R}^4 \);

iii) for each \( \rho > 0 \), there exists a positive function \( \phi_\rho \in L^1([0, +\infty[) \) such that, for \((x, y, z, w) \in \mathbb{R}^4 \) with

\[
\sup_{t \in [0, +\infty[} \left\{ \left| \frac{x}{1 + t} \right|, \left| \frac{y}{1 + t} \right|, \left| \frac{z}{1 + t} \right|, \left| w \right| \right\} < \rho,
\]

one has

\[
|g(t, x, y, z, w)| \leq \phi_\rho(t), \text{ a.e. } t \in [0, +\infty[.
\]

Definition 2 A sequence \((c_n)_{n \in \mathbb{N}} : [0, +\infty[ \times \mathbb{R}^2 \to \mathbb{R} \) is a Carathéodory sequence if it verifies

i) for each \((a, b) \in \mathbb{R}^2 \), \((a, b) \mapsto c_n(t, a, b) \) is continuous for all \( n \in \mathbb{N} \);

ii) for each \( \rho > 0 \), there are nonnegative constants \( \chi_{n, \rho} \geq 0 \) with \( \sum_{n=1}^{+\infty} \chi_{n, \rho} < +\infty \) such that for \( |a| < \rho(1 + t) \), \( t \in [0, +\infty[ \) and \( |b| < \rho \) we have

\[
|c_n(t, a, b)| \leq \chi_{n, \rho}, \text{ for every } n \in \mathbb{N}, t \in [0, +\infty[.
\]

Lemma 3 Let \( f, h : [0, +\infty[ \times \mathbb{R}^4 \to \mathbb{R} \) be \( L^1 \)-Carathéodory functions. Then the system \((7)\) with conditions \((2), (3)\), has a solution \((u(t), v(t))\) expressed by

\[
\begin{align*}
  u(t) &= A_1 + B_1 t + \sum_{0 < t_k < t < +\infty} [I_{0k}(t_k, u(t_k), u'(t_k)) + I_{1k}(t_k, u(t_k), u'(t_k))(t - t_k)] \\
  &\quad - t \sum_{k=1}^{+\infty} I_{1k}(t_k, u(t_k), u'(t_k)) + \int_{0}^{+\infty} G(t, s)f(s, u(s), v(s), u'(s), v'(s))ds,
\end{align*}
\]

\[
\begin{align*}
  v(t) &= A_2 + B_2 t + \sum_{0 < \tau_j < t < +\infty} [J_{0j}(\tau_j, v(\tau_j), v'(\tau_j)) + J_{1j}(\tau_j, v(\tau_j), v'(\tau_j))(t - \tau_j)] \\
  &\quad - t \sum_{j=1}^{+\infty} J_{1j}(\tau_j, v(\tau_j), v'(\tau_j)) + \int_{0}^{+\infty} G(t, s)h(s, u(s), v(s), u'(s), v'(s))ds,
\end{align*}
\]

where

\[
  G(t, s) = \begin{cases} 
    -t, & 0 \leq t \leq s \leq +\infty \\
    -s, & 0 \leq s \leq t \leq +\infty.
  \end{cases}
\]
The proof follows standard techniques and it is omitted.

**Definition 4** The operator $T : X \rightarrow X$ is said to be compact if $T(D)$ is relatively compact, for $D \subseteq X$.

The existence tool will be given by Schauder’s fixed point theorem:

**Theorem 5** (23) Let $Y$ be a nonempty, closed, bounded and convex subset of a Banach space $X$, and suppose that $P : Y \rightarrow Y$ is a compact operator. Then $P$ has at least one fixed point in $Y$.

### 3 Existence result

In this section we prove the existence of solution for the problem (1)-(3).

**Theorem 6** Let $f, h : [0, +\infty[ \times \mathbb{R}^4 \rightarrow \mathbb{R}$ be $L^1$– Carathéodory functions verifying (5) with $\Phi_\rho(t), \Psi_\rho(t)$, respectively. If $I_{0k}, I_{1k}, J_{0j}, J_{1j} : [0, +\infty[ \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are Carathéodory sequences with nonnegative constants $\varphi_{k,\rho} \geq 0, \psi_{k,\rho} \geq 0, \phi_{j,\rho} \geq 0, \vartheta_{j,\rho} \geq 0$ and

$$
\sum_{k=1}^{+\infty} \varphi_{k,\rho} < +\infty, \quad \sum_{k=1}^{+\infty} \psi_{k,\rho} < +\infty, \quad \sum_{j=1}^{+\infty} \phi_{j,\rho} < +\infty, \quad \sum_{j=1}^{+\infty} \vartheta_{j,\rho} < +\infty, \tag{7}
$$

such that

$$
|I_{0k}(t_k, x, y)| \leq \varphi_{k,\rho}, \quad |I_{1k}(t_k, x, y)| \leq \psi_{k,\rho},
$$

$$
|J_{0k}(t_k, x, y)| \leq \phi_{j,\rho}, \quad |J_{1k}(t_k, x, y)| \leq \vartheta_{j,\rho}, \tag{8}
$$

for $|x| < \rho(1 + t), |y| < \rho, t \in [0, +\infty[$ then there is at least a pair $(u, v) \in \left( PC_1^2([0, +\infty[) \times PC_2^2([0, +\infty[) \right) \cap X$, solution of (1)-(3).

**Proof.** Define the operators $T_1 : X \rightarrow X_1, T_2 : X \rightarrow X_2$, and $T : X \rightarrow X$ by

$$
T(u, v) = (T_1(u, v), T_2(u, v)), \tag{9}
$$

with

$$(T_1(u, v))(t) = A_1 + B_1t + \sum_{0 < t_k < t < +\infty} \left[ I_{0k}(t_k, u(t_k), u'(t_k)) + I_{1k}(t_k, u(t_k), u'(t_k))(t - t_k) \right] - t \sum_{k=1}^{+\infty} I_{1k}(t_k, u(t_k), u'(t_k)) + \int_{0}^{+\infty} G(t, s)f(s, u(s), v(s), u'(s), v'(s))ds,
$$

$$(T_2(u, v))(t) = A_2 + B_2t + \sum_{0 < t_k < t < +\infty} \left[ J_{0k}(t_k, u(t_k), u'(t_k)) + J_{1k}(t_k, u(t_k), u'(t_k))(t - t_k) \right] - t \sum_{k=1}^{+\infty} J_{1k}(t_k, u(t_k), u'(t_k)) + \int_{0}^{+\infty} G(t, s)f(s, u(s), v(s), u'(s), v'(s))ds,
$$
\[ (T_2 (u, v)) (t) = A_2 + B_2 t \]
\[ + \sum_{0 < \tau_j < t < +\infty} \left[ J_{0j} (\tau_j, v(\tau_j), v'(\tau_j)) + J_{1j} (\tau_j, v(\tau_j), v'(\tau_j)) (t - \tau_j) \right] \]
\[ - t \sum_{j=1}^{+\infty} J_{1j} (\tau_j, v(\tau_j), v'(\tau_j)) + \int_{0}^{+\infty} G(t, s) h(s, u(s), v(s), u'(s), v'(s)) \, ds, \]

where \( G(t, s) \) is defined in [6].

Let \((u, v) \in X\).

The proof will follow several steps which, for clearness, are detailed for operator \( T_1 (u, v) \). The technique for operator \( T_2 (u, v) \) is similar.

**Step 1:** \( T \) is well defined and continuous on \( X \).

By the Lebesgue dominated convergence theorem, (7) and (8),

\[
\lim_{t \to +\infty} \frac{T_1 (u, v) (t)}{1 + t} = \lim_{t \to +\infty} \left( \frac{A_1 + B_1 t}{1 + t} \right) \\
+ \frac{1}{1 + t} \sum_{0 < t_k < t < +\infty} \left[ I_{0k} (t_k, u(t_k), u'(t_k)) + I_{1k} (t_k, u(t_k), u'(t_k)) (t - t_k) \right] \\
- \frac{t}{1 + t} \sum_{k=1}^{+\infty} I_{1k} (t_k, u(t_k), u'(t_k)) \\
+ \int_{0}^{+\infty} \lim_{t \to +\infty} \frac{G(t, s)}{1 + t} f(s, u(s), v(s), u'(s), v'(s)) \, ds \\
\leq B_1 + \sum_{0 < t_k < t < +\infty} I_{1k} (t_k, u(t_k), u'(t_k)) - \sum_{k=1}^{+\infty} I_{1k} (t_k, u(t_k), u'(t_k)) \\
+ \int_{t}^{+\infty} |f(s, u(s), v(s), u'(s), v'(s))| \, ds \\
\leq B_1 + 2 \sum_{k=1}^{+\infty} \psi_{k, \rho} + \int_{0}^{+\infty} \Phi_{\rho}(s) \, ds < +\infty.
\]
for \( \rho > 0 \) given by (4) and
\[
\lim_{t \to +\infty} (T_1(u,v))'(t) = B_1 + \sum_{0 < t_k < t < +\infty} I_1(t_k, u(t_k), u'(t_k))
- \sum_{k=1}^{+\infty} I_1(t_k, u(t_k), u'(t_k))
- \lim_{t \to +\infty} \int_t^{+\infty} f(s, u(s), v(s), u'(s), v'(s)) ds
\leq B_1 + 2 \sum_{k=1}^{+\infty} \psi_k, \rho + \int_0^{+\infty} \Phi(s) ds < +\infty.
\]

So, \( T_1 X \subset X_1 \). Analogously, \( T_2 X \subset X_2 \). Therefore, \( T \) is well defined in \( X \) and, as \( f \) and \( h \) are \( L^1 \)-Carathéodory functions, by Definition 2, \( T \) is continuous.

To prove that \( TD \) is relatively compact, for \( D \subseteq X \), it is enough to show that:

i) \( TD \) is uniformly bounded, for \( D \) a bounded set in \( X \);

ii) \( TD \) is equicontinuous on each interval \( [x_k, x_{k+1}] \times [\tau_j, \tau_{j+1}] \), for \( k, j = 1, 2, ... \);

iii) \( TD \) is equiconvergent at each impulsive point and at infinity.

Step 2: \( TD \) is uniformly bounded, for \( D \) a bounded set in \( X \).

Let \( D \subset X \) be a bounded subset. Thus, there is \( \rho_1 > 0 \) such that, for \((u,v) \in D\),
\[
\|(u,v)\|_X = \max \{\|u\|_{X_1}, \|v\|_{X_2}\}
= \max \{\|u\|_0, \|u'\|_1, \|v\|_0, \|v'\|_1\} < \rho_1.
\] (10)

Define
\[
K_i := \sup_{t \in [0, +\infty] \left( \frac{|A_i| + |B_i|}{1 + t} \right)}, i = 1, 2, \quad Q(s) := \sup_{t \in [0, +\infty]} \frac{|G(t,s)|}{1 + t},
\] (11)
with \( 0 \leq Q(s) \leq 1, \forall s \in [0, +\infty] \).
As, \( f \) is a \( L^1 \)-Carathéodory function, then

\[
\| T_1(u, v) \|_0 = \sup_{t \in [0, +\infty)} \frac{|T_1(u, v)(t)|}{1 + t} 
\]

\[
\leq \sup_{t \in [0, +\infty]} \left( \frac{|A_1| + |B_1|}{1 + t} + \sum_{0 < t_k < t < +\infty} |I_{0k}(t_k, u(t_k), u'(t_k)) + I_{1k}(t_k, u(t_k), u'(t_k)) (t - t_k)| 
\right) 
\]

\[
+ \frac{t}{1 + t} \sum_{k=1}^{+\infty} |I_{1k}(t_k, u(t_k), u'(t_k))| 
\]

\[
+ \int_{0}^{+\infty} \sup_{t \in [0, +\infty]} \frac{|G(t, s)|}{1 + t} |f(s, u(s), v(s), u'(s), v'(s))| ds 
\]

\[
\leq K_1 + \sup_{t \in [0, +\infty]} \left( \frac{1}{1 + t} \sum_{0 < t_k < t < +\infty} [\varphi_{k, \rho} + \psi_{k, \rho}(t - t_k)] 
\right) 
\]

\[
+ \sup_{t \in [0, +\infty]} \left( \frac{t}{1 + t} \sum_{k=1}^{+\infty} \psi_{k, \rho} \right) + \int_{0}^{+\infty} Q(s) \Phi_{\rho}(s) ds 
\]

\[
\leq K_1 + \sum_{k=1}^{+\infty} \varphi_{k, \rho} + 2 \sum_{k=1}^{+\infty} \sum_{k=1}^{+\infty} \psi_{k, \rho} + \int_{0}^{+\infty} Q(s) \Phi_{\rho}(s) ds < +\infty, \forall (u, v) \in D, 
\]

and

\[
\| (T_1(u, v))' \|_1 = \sup_{t \in [0, +\infty]} |(T_1(u, v))'(t)| 
\]

\[
\leq |B_1| + \sup_{t \in [0, +\infty]} \sum_{0 < t_k < t < +\infty} |I_{1k}(t_k, u(t_k), u'(t_k))| 
\]

\[
+ \sum_{k=1}^{+\infty} |I_{1k}(t_k, u(t_k), u'(t_k))| 
\]

\[
+ \sup_{t \in [0, +\infty]} \int_{t}^{+\infty} |f(s, u(s), v(s), u'(s), v'(s))| ds 
\]

\[
\leq |B_1| + 2 \sum_{k=1}^{+\infty} \psi_{k, \rho} + \int_{0}^{+\infty} \Phi_{\rho}(s) ds < +\infty. 
\]

Therefore, \( T_1D \) is bounded and by similar arguments, \( T_2D \) is also bounded.
Furthermore, \( \|T(u, v)\|_X < +\infty \), that is \( TD \) is uniformly bounded on \( X \).

**Step 3:** \( TD \) is equicontinuous on each interval \( [x_k, x_{k+1}] \times [\tau_j, \tau_{j+1}] \), that is, \( T_1D \) is equicontinuous on each interval \( [x_k, x_{k+1}] \), for \( k \in \mathbb{N}, 0 < t_1 < \ldots < t_k < \ldots \), and \( T_2D \) is equicontinuous on each interval \( [\tau_j, \tau_{j+1}] \), for \( j \in \mathbb{N} \) and \( 0 < \tau_1 < \ldots < \tau_j < \ldots \).

Consider \( I \subseteq [x_k, x_{k+1}] \) and \( \iota_1, \iota_2 \in I \) such that \( \iota_1 \leq \iota_2 \). For \((u, v) \in D\), we have

\[
\lim_{\iota_1 \rightarrow \iota_2} \left| \frac{T_1(u, v)(\iota_1)}{1 + \iota_1} - \frac{T_1(u, v)(\iota_2)}{1 + \iota_2} \right| \leq \lim_{\iota_1 \rightarrow \iota_2} \left| \frac{A_1 + B_1 \iota_1}{1 + \iota_1} - \frac{A_1 + B_1 \iota_2}{1 + \iota_2} \right|
\]

\[
\quad + \left| \frac{1}{1 + \iota_1} \sum_{0 < t_k < \iota_1} [I_{0k}(t_k, u(t_k), u'(t_k)) + I_{1k}(t_k, u(t_k), u'(t_k)) (\iota_1 - t_k)] \right|
\]

\[
\quad \quad - \left| \frac{1}{1 + \iota_2} \sum_{0 < t_k < \iota_2} [I_{0k}(t_k, u(t_k), u'(t_k)) + I_{1k}(t_k, u(t_k), u'(t_k)) (\iota_2 - t_k)] \right|
\]

\[
\quad - \left| \frac{\iota_1}{1 + \iota_1} \sum_{k=1}^{+\infty} I_{1k}(t_k, u(t_k), u'(t_k)) + \frac{\iota_2}{1 + \iota_2} \sum_{k=1}^{+\infty} I_{1k}(t_k, u(t_k), u'(t_k)) \right|
\]

\[
\quad + \int_{0}^{+\infty} \lim_{\iota_1 \rightarrow \iota_2} \left| \frac{G(\iota_1, s)}{1 + \iota_1} - \frac{G(\iota_2, s)}{1 + \iota_2} \right| \left| f(s, u(s), v(s), u'(s), v'(s)) \right| ds = 0,
\]

as \( \iota_1 \rightarrow \iota_2 \), and

\[
\quad \lim_{\iota_1 \rightarrow \iota_2} \left| (T_1(u, v)(\iota_1))' - (T_1(u, v)(\iota_2))' \right|
\]

\[
\quad \leq \lim_{\iota_1 \rightarrow \iota_2} \left| \sum_{0 < t_k < \iota_1} I_{1k}(t_k, u(t_k), u'(t_k)) - \sum_{0 < t_k < \iota_2} I_{1k}(t_k, u(t_k), u'(t_k)) \right|
\]

\[
\quad \quad - \int_{\iota_1}^{+\infty} f(s, u(s), v(s), u'(s), v'(s)) ds + \int_{\iota_1}^{+\infty} f(s, u(s), v(s), u'(s), v'(s)) ds \right|
\]

\[
\quad \leq \lim_{\iota_1 \rightarrow \iota_2} \left| I_{1k}(t_k, u(t_k), u'(t_k)) \right| + \int_{\iota_1}^{\iota_2} |f(s, u(s), v(s), u'(s), v'(s))| ds
\]

\[
\quad \leq \lim_{\iota_1 \rightarrow \iota_2} \left| \psi_{k, \rho} \right| + \int_{\iota_1}^{\iota_2} \Phi_{\rho}(s) ds = 0.
\]

Therefore, \( T_1D \) is equicontinuous on \( X_1 \). Similarly, we can show that \( T_2D \) is equicontinuous on \( X_2 \), too. Thus, \( TD \) is equicontinuous on \( X \).
Step 4: TD is equiconvergent at each impulsive point and at infinity, that is \(T_1D\), is equiconvergent at \(t = t_i^+\), \(i = 1, 2, \ldots\), and at infinity, and \(T_2D\), is equiconvergent at \(\tau = \tau_l^+\), \(l = 1, 2, \ldots\), and at infinity.

First, let us prove the equiconvergence at \(t = t_i^+\), for \(i = 1, 2, \ldots\). The proof for the equiconvergence at \(\tau = \tau_l^+\), for \(l = 1, 2, \ldots\), is analogous. Thus, it follows

\[
\frac{T_1(u, v)(t)}{1 + t} - \lim_{t \to t_i^+} \frac{T_1(u, v)(t)}{1 + t} \leq A_1 + B_1t - \frac{A_1 + B_1t_i}{1 + t_i}
\]

\[
+ \frac{1}{1 + t} \sum_{0 < t_k < t < +\infty} \left[ I_{0k}(t_k, u(t_k), u'(t_k)) + I_{1k}(t_k, u(t_k), u'(t_k))(t - t_k) \right]
\]

\[- \frac{1}{1 + t_i} \sum_{0 < t_k < t_i^+} \left[ I_{0k}(t_k, u(t_k), u'(t_k)) + I_{1k}(t_k, u(t_k), u'(t_k))(t_i - t_k) \right]
\]

\[- \frac{t}{1 + t} \sum_{k=1}^{+\infty} I_{1k}(t_k, u(t_k), u'(t_k)) + \frac{t_i}{1 + t_i} \sum_{k=1}^{+\infty} I_{1k}(t_k, u(t_k), u'(t_k)) \]

\[+ \int_0^{+\infty} \left| \frac{G(t, s)}{1 + t} - \frac{G(t, s)}{1 + t_i} \right| \Phi_p(s)ds \to 0.\]
uniformly on \((u, v) \in D\), as \(t \to t_i^+\), for \(i = 1, 2, \ldots\) and

\[
\left( T_1 (u, v) (t) \right)' - \lim_{t \to t_i^+} \left( T_1 (u, v) (t) \right)'
\]

\[
= \sum_{0 < t_k < t < +\infty} I_{1k}(t_k, u(t_k), u'(t_k)) - \sum_{0 < t_k < t_i^+} I_{1k}(t_k, u(t_k), u'(t_k))
\]

\[
- \int_t^{+\infty} f(s, u(s), v(s), u'(s), v'(s))ds + \int_{t_i}^{+\infty} f(s, u(s), v(s), u'(s), v'(s))ds
\]

\[
\leq \sum_{0 < t_k < t < +\infty} I_{1k}(t_k, u(t_k), u'(t_k)) - \sum_{0 < t_k < t_i^+} I_{1k}(t_k, u(t_k), u'(t_k))
\]

\[
+ \left| - \int_t^{+\infty} f(s, u(s), v(s), u'(s), v'(s))ds + \int_{t_i}^{+\infty} f(s, u(s), v(s), u'(s), v'(s))ds \right|
\]

\[
\leq \sum_{0 < t_k < t < +\infty} I_{1k}(t_k, u(t_k), u'(t_k)) - \sum_{0 < t_k < t_i^+} I_{1k}(t_k, u(t_k), u'(t_k))
\]

\[
+ \int_{t_i}^{t} \phi_\rho(s)ds \to 0,
\]

uniformly on \((u, v) \in D\), as \(t \to t_i^+\), for \(i = 1, 2, \ldots\)

Therefore, \(T_1 D\) is equiconvergent at each point \(t = t_i^+\), for \(i = 1, 2, \ldots\).

Analogously, it can be proved that \(T_2 D\) is equiconvergent at each point \(\tau = \tau_i^+\), for \(l = 1, 2, \ldots\).

So, \(TD\) is equiconvergent at each impulsive point.
To prove the equiconvergence at infinity, for the operator $T_1$, we have

$$\left| \frac{T_1(u, v)(t)}{1 + t} - \lim_{t \to +\infty} \frac{T_1(u, v)(t)}{1 + t} \right| \leq \left| A_1 + B_1 t \frac{1}{1 + t} - B_1 \right|$$

$$+ \left| \frac{1}{1 + t} \sum_{0 < t_k < t < +\infty} \left[ I_{0k}(t_k, u(t_k), u'(t_k)) + I_{1k}(t_k, u(t_k), u'(t_k)) \right] \right|$$

$$- \lim_{t \to +\infty} \frac{1}{1 + t} \sum_{0 < t_k < t < +\infty} \left[ I_{0k}(t_k, u(t_k), u'(t_k)) + I_{1k}(t_k, u(t_k), u'(t_k)) \right]$$

$$+ \left| - \frac{t}{1 + t} \sum_{k=1}^{+\infty} I_{1k}(t_k, u(t_k), u'(t_k)) + \lim_{t \to +\infty} \frac{t}{1 + t} \sum_{k=1}^{+\infty} I_{1k}(t_k, u(t_k), u'(t_k)) \right|$$

$$+ \left| \int_{0}^{+\infty} \left| G(t, s) \frac{1}{1 + t} - \lim_{t \to +\infty} \frac{G(t, s)}{1 + t} \right| f(s, u(s), v(s), u'(s), v'(s))ds \right|$$

$$\leq \left| A_1 + B_1 t \frac{1}{1 + t} - B_1 \right|$$

$$+ \left| \frac{1}{1 + t} \sum_{0 < t_k < t < +\infty} \left[ I_{0k}(t_k, u(t_k), u'(t_k)) + I_{1k}(t_k, u(t_k), u'(t_k)) \right] \right|$$

$$- \sum_{k=1}^{+\infty} I_{1k}(t_k, u(t_k), u'(t_k))$$

$$+ \left| - \frac{t}{1 + t} \sum_{k=1}^{+\infty} I_{1k}(t_k, u(t_k), u'(t_k)) + \sum_{k=1}^{+\infty} I_{1k}(t_k, u(t_k), u'(t_k)) \right|$$

$$+ \left| \int_{0}^{+\infty} \left| G(t, s) \frac{1}{1 + t} - \lim_{t \to +\infty} \frac{G(t, s)}{1 + t} \right| \Phi(s)ds \to 0, \right.$$
Analogously,

\[
\left| (T_1(u, v)(t))' - \lim_{t \to +\infty} (T_1(u, v)(t))' \right| \\
= \left| \sum_{0 < t_k < t < +\infty} I_{1k}(t_k, u(t_k), u'(t_k)) - \sum_{k=1}^{+\infty} I_{1k}(t_k, u(t_k), u'(t_k)) \right| \\
- \int_t^{+\infty} f(s, u(s), v(s), u'(s), v'(s))ds \\
- \lim_{t \to +\infty} \sum_{0 < t_k < t < +\infty} I_{1k}(t_k, u(t_k), u'(t_k)) + \lim_{t \to +\infty} \sum_{k=1}^{+\infty} I_{1k}(t_k, u(t_k), u'(t_k)) \\
+ \lim_{t \to +\infty} \int_t^{+\infty} f(s, u(s), v(s), u'(s), v'(s))ds \\
\leq \left| \sum_{k=0}^{+\infty} I_{1k}(t_k, u(t_k), u'(t_k)) - \sum_{k=1}^{+\infty} I_{1k}(t_k, u(t_k), u'(t_k)) \right| \\
+ \int_t^{+\infty} |f(s, u(s), v(s), u'(s), v'(s))|ds \\
\leq \left| \sum_{0 < t_k < t < +\infty} I_{1k}(t_k, u(t_k), u'(t_k)) - \sum_{k=1}^{+\infty} I_{1k}(t_k, u(t_k), u'(t_k)) \right| \\
+ \int_t^{+\infty} \Phi_\rho(s)ds \to 0,
\]

uniformly on \((u, v) \in D\), as \(t \to +\infty\).

So, \(T_1D\) is equiconvergent at \(+\infty\). Following the same arguments, \(T_2D\) is equiconvergent at \(+\infty\), too. Therefore, \(TD\) is equiconvergent at \(+\infty\).

Therefore, \(TD\) is relatively compact and, by Definition \(\square\) \(T\) is compact.

In order to apply Theorem \(\square\) we need the next step:

**Step 5:** \(T\Omega \subset \Omega\) for some \(\Omega \subset X\) a closed and bounded set.

Consider

\[
\Omega := \{(u, v) \in E : \|(u, v)\|_X \leq \rho_2\},
\]
with \( \rho_2 > 0 \) such that

\[
\rho_2 := \max \left\{ \rho_1, \ K_1 + \sum_{k=1}^{\infty} \varphi_{k,\rho} + 2 \sum_{k=1}^{\infty} \psi_{k,\rho} + \int_0^{+\infty} Q(s)\Phi_\rho(s)ds, \right. \]

\[
\left. \ K_2 + \sum_{k=1}^{\infty} \phi_{j,\rho} + 2 \sum_{k=1}^{\infty} \varphi_{j,\rho} + \int_0^{+\infty} Q(s)\Phi_\rho(s)ds, \right. \]

\[
\left. |B_1| + 2 \sum_{k=1}^{\infty} \psi_{k,\rho} + \int_0^{+\infty} \Phi_\rho(s)ds, \right. \]

\[
\left. |B_2| + 2 \sum_{k=1}^{\infty} \phi_{j,\rho} + \int_0^{+\infty} \Phi_\rho(s)ds, \right. \]

(12)

with \( \rho_1 \) given by (10). According to Step 2 and \( K_1, K_2 \) and \( Q(s) \) given by (11), we have

\[
\| T(u,v) \|_X = \| (T_1(u,v), T_2(u,v)) \|_{X_1}
\]

\[
= \max \left\{ \| T_1(u,v) \|_X, \| T_2(u,v) \|_{X_2} \right\}
\]

\[
= \max \left\{ \| T_1(u,v) \|_0, \| (T_1(u,v))' \|_1, \| T_2(u,v) \|_0, \| (T_2(u,v))' \|_1 \right\}
\]

\[
\leq \rho_2.
\]

So, \( T \Omega \subset \Omega \), and by Theorem 5, the operator \( T(u,v) = (T_1(u,v), T_2(u,v)) \), has a fixed point \( (u,v) \).

By standard techniques, and Lemma 3, it can be shown that this fixed point is a solution of problem (1)-(3).

4 Motion of a spring pendulum

Consider the motion of the spring pendulum of a mass attached to one end of a spring and the other end attached to the ceiling. By [18], we represent this motion by the system,

\[
\begin{align*}
\quad l''(t) &= \frac{1}{l^2} \left( l(t)\theta'(t) - g \cos(\theta(t)) - \frac{k}{m} (l(t) - l_0) \right), \quad t \in [0, +\infty[ \\
\theta''(t) &= \frac{1}{l^2} \left( -g(l(t) \sin(\theta(t)) - 2l(t)l'(t)\theta'(t)) \right), \quad \text{where:} \\
\end{align*}
\]

(13)

- \( l(t), l_0 \) are the length at time \( t \) and the natural length of the spring, respectively;
- \( \theta(t) \) is the angle between the pendulum and the vertical;
- $m, k, g$ are the mass, the spring constant and gravitational force, respectively;

together with the boundary conditions
\[
\begin{align*}
\ell(0) &= 0, \quad \theta(0) = 0, \\
\ell'(+\infty) &= B_1, \quad \theta'(+\infty) = B_2,
\end{align*}
\]
with $B_1, B_2 \in [0, \pi]$, and the generalized impulsive conditions
\[
\begin{align*}
\Delta \theta_1(t_k) &= \frac{1}{k^3}(\alpha_1 \theta_1(t_k) + \alpha_2 \theta_1'(t_k)), \\
\Delta \theta_1'(t_k) &= \frac{1}{k^3}(\alpha_3 \theta_1(t_k) + \alpha_4 \theta_1'(t_k)), \quad \beta \geq 3, \\
\Delta \theta_2(\tau_j) &= \frac{1}{j^3}(\alpha_5 \theta_2(\tau_j) + \alpha_6 \theta_2'(\tau_j)), \quad \gamma \geq 3 \\
\Delta \theta_2'(\tau_j) &= \frac{1}{j^3}(\alpha_7 \theta_2(\tau_j) + \alpha_8 \theta_2'(\tau_j)),
\end{align*}
\]
with $\alpha_i \in \mathbb{R}, \ i = 1, 2, \ldots, 8$ and for $k, j \in \mathbb{N}, \ 0 < t_1 < \cdots < t_k < \cdots, \ 0 < \tau_1 < \cdots < \tau_j < \cdots$.

Figure 1: Motion of the spring pendulum.
The system (13)-(15) is a particular case of the problem (1)-(3), with
\[
\begin{align*}
  f(t,x,y,z,w) &= xw - g \cos(y) - \frac{k}{m}(x - l_0), \\
  h(t,x,y,z,w) &= \frac{-gx \sin(y) - 2xz}{x^2}, \\
  I_{0k}(t_k,x,z) &= \frac{1}{k^3}(\alpha_1 x + \alpha_2 z), \quad I_{1k}(t_k,x,z) = \frac{1}{k^\beta}(\alpha_3 x + \alpha_4 z), \quad \beta \geq 3, \\
  J_{0j}(\tau_j,y,w) &= \frac{1}{j^3}(\alpha_5 y + \alpha_6 w), \quad \gamma \geq 3, \quad J_{1j}(\tau_j,y,w) = \frac{1}{j^3}(\alpha_7 y + \alpha_8 w),
\end{align*}
\]
with \( t_k = k, \tau_j = j, k,j \in \mathbb{N}, \alpha_i \in \mathbb{R}, i = 1,2,...,8. \)

In fact, \( f \) and \( h \) are \( L^1 \)-Carathéodory functions, with
\[
\begin{align*}
  f(t,x,y,z,w) &\leq \frac{1}{t^3} \left( \rho^2(1 + t) + g + \frac{k}{m}(\rho(1 + t) + l_0) \right) := \phi_\rho(t), \\
  h(t,x,y,z,w) &\leq \frac{1}{t^3} \left( \frac{g \rho(1 + t) + 2\rho^3(1 + t)}{l^2(t)} \right) \\
  &\leq \frac{1}{t^3} \left( \frac{g \rho(1 + t) + 2\rho^3(1 + t)}{l^2(t)} \right) \\
  &\leq \frac{1}{t^3} \left( \frac{1}{(\min_{t \in [0,\infty]} l(t))^2} \right) \left( g \rho(1 + t) + 2\rho^3(1 + t) \right) \\
  &:= \varphi_\rho(t),
\end{align*}
\]
and \( \phi_\rho(t), \varphi_\rho(t) \) verifying Definition [1].

Besides \( I_{0k}, I_{1k}, J_{0j}, J_{1j} \), are Carathéodory sequences and verify (8), as
\[
\begin{align*}
  I_{0k}(t_k,x,z) &\leq \frac{\rho[\alpha_1(1 + k) + \alpha_2]}{k^3}, \quad I_{1k}(t_k,x,z) \leq \frac{\rho[\alpha_3(1 + k) + \alpha_4]}{k^\beta}, \quad \beta \geq 3, \\
  J_{0j}(\tau_j,y,w) &\leq \frac{\rho[\alpha_5(1 + k) + \alpha_6]}{j^\gamma}, \quad \gamma \geq 3, \quad J_{1j}(\tau_j,y,w) \leq \frac{\rho[\alpha_7(1 + k) + \alpha_8]}{j^3}.
\end{align*}
\]

So, by Theorem 6, there is at least a pair \((l, \theta) \in \left( PC^2_1([0,\infty]) \times PC^2_2([0,\infty]) \right) \cap X, \) solution of problem (13), (14) and (15).

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The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors contributions
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