New Short Proofs to Some Stability Theorems

Xizhi Liu *

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Abstract

We present new short proofs to both the exact and the stability results of two extremal problems. The first one is the extension of Turán’s theorem in hypergraphs, which was firstly studied by Mubayi [1]. The second one is about the cancellative hypergraphs, which was firstly studied by Bollobás [2] and later by Keevash and Mubayi [3]. Our proofs are concise and straightforward, but give a sharper version of stability theorems to both problems.

1 Introduction

Let $H$ be an $n$-vertex $r$-graph and let $F$ be a family of $r$-graphs. $H$ is $F$-free if it does not contain any $r$-graph in $F$ as a subgraph. The Turán number $ex(n,F)$ is the maximum number of edges in an $n$-vertex $F$-free $r$-graph. $F$ is called non-degenerate if the Turán density $\pi(F) := \lim_{n \to \infty} ex(n,F)/\binom{n}{r}$ is not 0.

Determining, even asymptotically, the value of $ex(n,F)$ for general non-degenerate $r$-graphs $F$ with $r \geq 3$ is known to be notoriously hard. On the other hand, many families $F$ have the property that there is a unique extremal family attain the value $ex(n,F)$, and any $F$-free hypergraph with close to $ex(n,F)$ edges is also structurally close to the extremal family. This property of $F$ is called stability. It is both an intersecting property of $F$ and also an extremely useful tool in determining the value of $ex(n,F)$. The Turán numbers for many families $F$ has been determined by using this method, and we refer the reader to a survey by Keevash [4] for results before 2011.

In the present paper, we mainly focus on the stability properties for two extremal problems. The first one is the extension of Turán’s theorem in hypergraphs, and it was firstly studied by Mubayi [1].

Let $V_1 \cup \ldots \cup V_\ell$ be a partition of $[n]$ with each part of size either $\lfloor n/\ell \rfloor$ or $\lceil n/\ell \rceil$. $T_r(n,\ell)$ is the family of all $r$-sets that intersect each $V_i$ in at most one vertex. Let $t_r(n,\ell)$ denote the number of edges in $T_r(n,\ell)$. $K^{(r)}_{\ell+1}$ is the family of all $r$-graphs $F$ with at most $\binom{\ell+1}{2}$ edges such that for some $(\ell+1)$-set $S$ every pair $x,y \in S$ is covered by an edge of $F$. Notice that $T_2(n,\ell)$ is just the ordinary Turán graph, and $K^{(2)}_{\ell+1}$ is just the ordinary complete graph on $\ell+1$ vertices, which is also denoted by $K_{\ell+1}$.

In [1] Mubayi proved both the exact and stability result for $K^{(r)}_{\ell+1}$-free $r$-graphs.

**Theorem 1.1** (Mubayi, [1]). Let $n,\ell, r \geq 2$. Then

$$ex(n,K^{(r)}_{\ell+1}) = t_r(n,\ell)$$

and $T_r(n,\ell)$ is the unique maximum $K^{(r)}_{\ell+1}$-free $r$-graph on $n$ vertices.

*Department of Mathematics, Statistics, and Computer Science, University of Illinois, Chicago, IL, 60607 USA. Email: xliu246@uic.edu
Theorem 1.2 (Stability; Mubayi, [1]). Fix \( l \geq r \geq 2 \). For every \( \delta > 0 \), there exists an \( \epsilon > 0 \) and an \( n_0 \) such that the following holds for all \( n \geq n_0 \). Let \( G \) be an \( n \)-vertex \( k^{(r)}_{\ell+1} \)-free \( r \)-graph with at least \((1-\epsilon)t_r(n, \ell)\) edges. Then the vertex set of \( G \) has a partition \( V_1 \cup \ldots \cup V_\ell \) such that all but at most \( \delta n^r \) edges have at most one vertex in each \( V_i \).

Note that in [1] Mubayi did not give an explicit relation between \( \epsilon \) and \( \delta \), but our proof will show that it suffices to choose \( \epsilon = (r-2)\delta \). Also, note that in [3] Contiero, Hoppen, et al. also proved a linear dependence between \( \delta \) and \( \epsilon \) by induction on \( \ell + r \), but our proof is different and much shorter.

The second one is about the cancellative hypergraphs, and it was firstly studied by Bollobás [2] and later by Keevash and Mubayi [3].

A hypergraph \( H \) is called cancellative if it does not contain three distinct sets \( A, B, C \) with \( A \triangle B \subset C \). Note that an ordinary graph \( G \) is cancellative iff it does not contain a triangle (i.e. \( K_3 \)), and Mantel’s theorem states that the maximum size of a cancellative graph is uniquely achieved by \( T_2(n, 2) \). Motivated by Mantel’s theorem, in the 1960’s, Katona raised the question of determining the maximum size of a cancellative 3-graph and conjectured that the maximum size of a cancellative 3-graph is achieved by \( T_3(n, 3) \). Katona’s conjecture was proved by Bollobás in [2].

Theorem 1.3 (Bollobás, [2]). A cancellative 3-graph on \( n \) vertices has at most \( t_3(n, 3) \) edges, with equality only for \( T_3(n, 3) \).

In [3] a new proof of Bollobás’ result was given by Keevash and Mubayi, and they also proved a stability theorem for cancellative 3-graphs.

Theorem 1.4 (Stability; Keevash and Mubayi, [3]). For any \( \delta > 0 \) there exists \( \epsilon > 0 \) and \( n_0 \) such that the following holds for all \( n \geq n_0 \). Any \( n \)-vertex cancellative 3-graph with at least \((1-\epsilon)t_3(n, 3)\) edges has a partition of vertex set as \( \{n\} = V_1 \cup V_2 \cup V_3 \) such that all but at most \( \delta n^3 \) edges of \( H \) has one vertex in each \( V_i \).

In their proof they also gave an explicit relation between \( \epsilon \) and \( \delta \), which is \( \epsilon < 27/2 \times 10^{-24} \delta^6 \). Our proof will show that it suffices to choose \( \epsilon = \delta/100 \).

The rest of this paper is organized as following. In Section 2 we introduce some definitions, useful theorems and lemmas. In Section 3 we prove Theorems 1.1 and 1.2. In Section 4 we prove Theorems 1.3 and 1.4. In Section 5 we present a short proof to the stability of a generalized Turán problem in graph theory. In the last section we present a brief discussion about the relation between \( \epsilon \) and \( \delta \).

2 Preliminaries

Let \( H \) be an \( r \)-graph on \( [n] \). The size of \( H \) is the number of edges in \( H \), which is denoted by \( |H| \). \( I \subset [n] \) is an independent set if every edge in \( H \) contains at most one vertex of \( I \). The shadow of \( H \), denoted by \( \partial H \), is defined as

\[
\partial H = \left\{ A \in {\binom{[n]}{r-1}} : \exists B \in H \text{ such that } A \subset B \right\}
\]

For every nonempty set \( S \subset [n] \), define the link \( L(S) \) of \( S \) in \( H \) to be

\[
L(S) = \{ A \in \partial H : A \cup \{s\} \in H, \ \forall s \in S \}
\]

For convenience, we use \( L(u) \) to represent \( L(\{u\}) \), and use \( L(u, v) \) to represent \( L(\{u, v\}) \). Note that in our proof \( L(u) \) also represents \( L(\{u\}) \).
Let $T \in \partial H$, the neighborhood of $T$ in $H$ is defined as

$$N(T) = \{v \in [n] : T \cup \{v\} \in H\}$$

and the degree of $T$ is $d(T) = |N(T)|$. It follows from an easy double counting that

$$\sum_{T \in \partial H} d(T) = 3|H|$$

The edge set of an ordinary graph $G$ can be viewed as a family of unordered pairs. To keep the calculations in our proof simply, we define an auxiliary family $\vec{G}$ of order pairs as $\vec{G} = \{(u, v) : \{u, v\} \in G\}$. Note that if $\{u, v\} \in G$, then $(u, v)$ and $(v, u)$ are both contained in $\vec{G}$ and hence we have $|\vec{G}| = 2|G|$. Let $N$ be a set, we use $N^2$ to denote the cartesian product $N \times N$, which is also the collection of all ordered pairs $(u, v)$ with $u, v \in N$. Here $u$ and $v$ might be the same.

Our proof of theorems 1.1 and 1.2 is based on two results. The first one is the stability of triangle-free graphs. For completeness we include its proof here.

**Theorem 2.1** (Füredi, [6]). Let $t \geq 0$ and let $G$ be an $n$-vertex $K_{\ell+1}$-free graph with $t_2(n, \ell) - t$ edges. Then $G$ contains an $\ell$-partite subgraph $G'$ with at least $t_2(n, \ell) - 2t$ edges.

The second one describes an relation between the number of copies of $K_{r_1}$ and $K_{r_2}$ in a $K_{\ell+1}$-free graph, where $r_1$ and $r_2$ are two positive integers less that $\ell + 1$.

**Theorem 2.2** (Fisher and Ryan, [7]). Let $G$ be an $n$-vertex $K_{\ell+1}$-free graph. For every $i \in [\ell]$, let $k_i$ denote the number of copies of $K_i$ in $G$. Then

$$\left(\frac{k_\ell}{\binom{\ell}{\ell}}\right)^{\frac{1}{\ell}} \leq \left(\frac{k_{\ell-1}}{\binom{\ell-1}{\ell-1}}\right)^{\frac{1}{\ell-1}} \leq \left(\frac{k_2}{\binom{2}{2}}\right)^{\frac{1}{2}} \leq \left(\frac{k_1}{\binom{1}{1}}\right)^{\frac{1}{1}}$$

(1)

To prove theorems 1.3 and 1.4 we first present two simply properties of cancellative 3-graphs.

**Lemma 2.3.** Let $H$ be a cancellative 3-graph, and $v$ is a vertex in $H$. Then the link graph $L(v)$ is triangle-free.

**Proof.** Suppose $\{x, y, z\}$ is a triangle in $L(v)$. Then $\{v, x, y\}, \{v, x, z\}, \{v, y, z\}$ are all contained in $H$, but

$$\{v, x, y\} \triangle \{v, x, z\} = \{y, z\} \subset \{v, y, z\}$$

which is a contradiction. Therefore, $L(v)$ is triangle-free. \qed

**Lemma 2.4.** Let $H$ be a cancellative 3-graph, and $T \in \partial H$. Then $N(T)$ is an independent set.

**Proof.** Let $u, v \in N(T)$ and let $A_1 = \{u\} \cup T$ and $A_2 = \{v\} \cup T$. Note that $A_1$ and $A_2$ are contained in $H$. Since $A_1 \triangle A_2 = \{u, v\}$ and by assumption there is no edge in $H$ containing $\{u, v\}$. Therefore, $N(T)$ is an independent set. \qed

In the proof of theorem 1.4 we need the following lemma, which is essentially the stability of triangle-free graphs. For completeness we include its proof here.

Let $G$ be an ordinary graph and let $v$ be a vertex in $G$. We use $N_G(v)$ to denote the neighborhood of $v$ in $G$, and use $d_G(v)$ to denote the degree of $v$ in $G$. 

3
Lemma 2.5. Let $G$ be a triangle-free graph on $[n]$ with at least $(1 - \epsilon)(n/2)^2$ edges. Then $G$ contains two vertices $v_1$ and $v_2$ such that $N_G(v_1)$ and $N_G(v_2)$ are disjoint and $|N_G(v_1)| + |N_G(v_2)| \geq (1 - \epsilon)n$.

Proof. Since $G$ is triangle-free. So $N_G(u)$ and $N_G(v)$ are disjoint for all edge $uv$ in $G$. Therefore, it suffices to find an edge $uv$ in $G$ such that $d_G(u) + d_G(v) \geq (1 - \epsilon)n$. Combining an easy counting argument with the Jensen Inequality we obtain

$$
\sum_{uv \in E(G)} (d_G(u) + d_G(v)) = \sum_{v \in V(G)} d_G^2(v) \geq \frac{\left(\sum_{v \in V(G)} d_G(v)\right)^2}{n} = 4e^2(G) \frac{n}{n} = 4e^2(G) / n \geq (1 - \epsilon)n.
$$

It follows from an averaging argument that there exists an edge $uv$ with $d_G(u) + d_G(v) \geq 4e(G) / n \geq (1 - \epsilon)n$.

3 Proofs of Theorems 1.1 and 1.2

Let $H$ be a $K_{\ell+1}^{(r)}$-free $r$-graph on $[n]$. Define an auxiliary graph

$$
G = \left\{ A \in \binom{[n]}{2} : \exists B \in H \text{ such that } A \in B \right\}.
$$

Let us state two easy facts about the relation between $H$ and $G$ without proof.

Lemma 3.1. (a). $H$ is $K_{\ell+1}^{(r)}$-free iff $G$ is $K_{\ell+1}$-free.

(b). The number of edges in $H$ is at most the number of copies of $K_r$ in $G$.

Proof of theorem 1.1: Combining lemma 3.1 with equation (1), we obtain that $|H| \leq \binom{\ell}{2} \binom{n}{2} r^{\ell}$. This proves theorem 1.1 for the case $\ell$ divides $n$, and we omit the proof of the other case.

Proof of theorem 1.2: Choose $\epsilon = (r - 2)!\delta$, and let $n$ be sufficiently large. By assumption we have $|H| \geq (1 - \epsilon)t_r(n, \ell) \geq (1 - 2\epsilon)\binom{\ell}{2} \binom{n}{2} r^{\ell}$. Combining lemma 3.1 with equation (1) we know that the number of edges $e$ in $G$ satisfies

$$
e \geq (1 - 2\epsilon)^{2/r} \binom{\ell}{2} \binom{n}{2} r^2 \geq (1 - 2\epsilon) \binom{\ell}{2} \binom{n}{2} r^2 \geq (1 - 2\epsilon)t_2(n, \ell)
$$

Therefore, by theorem 2.1, $G$ has a vertex set partition $V_1 \cup \ldots \cup V_t$ such that all but at most $2\epsilon t_2(n, \ell)$ edges of $G$ have at most one vertex in each $V_i$. It follows that all but at most $2\epsilon t_2(n, \ell) \binom{n - 2}{r - 2} \leq \epsilon n^r/(r - 2)! \leq \delta n^r$ edges of $H$ have at most one vertex in each $V_i$. This completes the proof of theorem 1.2.

4 Proofs of Theorems 1.3 and 1.4

The most important step in this section is building an relation between $H$ and $\partial H$, which is equation (2).

Proof of theorem 1.3: Let us count the number of ordered pairs $(u, v)$ in $[n]^2 \setminus \overline{\partial H}$. By lemma 2.4, if $\{u, v\}$ is contained in $N(e)$ for some $e \in \partial H$, then $\{u, v\}$ can not be contained in $\partial H$. Since every set $S \subset [n]$ is contained in exactly $|L(S)|$ sets in $\{N(T) : T \in \partial H\}$. Therefore, we have

$$
\sum_{T \in \partial H} \sum_{(u,v) \in N^2(T)} \frac{1}{|L(u,v)|} \leq n^2 - 2|\partial H|
$$

(2)
Combining lemma 2.3 with Mantel’s theorem we obtain that $|L(u, v)| \leq (n - d(T))^2/4$ for every $(u, v) \in [n]^2$. It follows from (2) that

$$\sum_{T \in \partial H} 4 \left( \frac{d(T)}{n - d(T)} \right)^2 \leq n^2 - 2|\partial H|$$

Since $(x/(n - x))^2$ is convex for $x \in [0, n]$, it follows from Jensen’s inequality that

$$4 \left( \frac{3|H|/|\partial H|}{n - 3|H|/|\partial H|} \right)^2 |\partial H| \leq n^2 - 2|\partial H|$$

Now let $z = \frac{3|H|/|\partial H|}{n - 3|H|/|\partial H|}$. Then (4) implies

$$|\partial H| \leq \frac{n^2}{2(2z^2 + 1)}$$

Substitute $|H| = \frac{z}{3(z + 1)}|\partial H|$ into the equation above we obtain

$$|H| \leq \frac{z}{6(z + 1)(2z^2 + 1)} n^3$$

Since the maximum of $\frac{z}{6(z + 1)(2z^2 + 1)}$ is $1/27$. Therefore, we have $|H| \leq \left( \frac{n}{3} \right)^3$. This proves theorem 1.3 for the case 3 divides $n$, and we omit the proof of the other case. \(\blacksquare\)

Choose $\epsilon = \delta/100$. Let $H$ be a cancellative 3-graph on $[n]$ with at least $(1 - \epsilon)t_3(n, 3) > (1 - 2\epsilon)(n/3)^3$ edges. Before we prove theorem 1.4, let us present a lemma follows from equation (2).

**Lemma 4.1.** There exists $T \in \partial H$ such that

$$\sum_{(u,v) \in N^2(T)} |L(u, v)| \geq (1 - 100\epsilon)d^2(T) \left( \frac{n - d(T)}{2} \right)^2$$

**Proof.** Suppose that (3) is false for all $T \in \partial H$. Since $1/x$ is convex for $x > 0$, it follows from Jensen’s inequality that

$$\sum_{(u,v) \in N^2(T)} \frac{1}{|L(u, v)|} \geq \frac{d^2(T)}{\sum_{(u,v) \in N^2(T)} |L(u, v)|/d^2(T)} \geq \frac{4d^2(T)}{(1 - 100\epsilon)(n - d(T))^2}$$

Substitute (4) into (2) we obtain

$$\sum_{T \in \partial H} \frac{4d^2(T)}{(1 - 100\epsilon)(n - d(T))^2} \leq n^2 - 2|\partial H|$$

Similar argument as in the proof of theorem 1.3 yields

$$|H| \leq \frac{z}{6(z + 1) \left( \frac{2z^2 + 1}{1 - 100\epsilon} \right)} n^3$$

By assumption we have $\frac{3|H|}{|\partial H|} \geq \frac{3(1 - \epsilon)(n/3)^3}{n^2/2} \geq 1/9$. Therefore, we may assume that $z > 1/8$. It follows that $\frac{2z^2 + 1}{1 - 100\epsilon} > \frac{2z^2 + 1}{1 - 2\epsilon}$. So we obtain

$$\frac{z}{6(z + 1) \left( \frac{2z^2 + 1}{1 - 100\epsilon} \right)} < (1 - 2\epsilon) \frac{z}{6(z + 1) (2z^2 + 1)} \leq \frac{1}{27} (1 - 2\epsilon)$$

This implies that $|H| < (1 - 2\epsilon) \left( \frac{n}{3} \right)^3 < (1 - \epsilon)t_3(n, 3)$, which is a contradiction. \(\blacksquare\)
Proof of theorem 1.4: Choose $T \in \partial H$ such that (3) holds for $T$. Let $V'_1 = N(T)$. By Pigeonhole principle, there exists a pair $(u, v) \in N^2(T)$ such that $|L(u, v)| \geq (1 - 100\epsilon)((n - d(T))/2)^2$. Let $L$ denote the graph $L(u, v)$ and let $U$ denote the vertex set $[n] \setminus N(T)$. Combining lemma 2.4 with lemma 2.5 we know that there exist two vertices $x$ and $y$ in $U$ such that $N_L(x)$ and $N_L(y)$ are disjoint and $N_L(x) + N_L(y) \geq (1 - 100\epsilon)(n - d(T))$. Let $V_2 = N_L(x)$ and $V_3 = N_L(y)$. Note that $N_L(x) = N(u|x)$ and $N_L(y) = N(u|y)$ and hence $V_2$ and $V_3$ are independent sets in $H$.

Now we have independent sets $V'_1, V_2$ and $V_3$, and $|V'_1| + |V_2| + |V_3| \geq d(T) + (1 - 100\epsilon)(n - d(T)) > n - 100\epsilon n$. Let $V_1 = [n] \setminus (V_2 \cup V_3)$. The number of edges in $H$ that has at least two vertices in some $V_i$ is at most \( \left( \binom{100\epsilon n}{3} \right) + \left( \binom{100\epsilon n}{2} \right) < 100\epsilon n^3 = \delta n^3 \). This completes the proof of theorem 1.4.

5 Further Applications

In this section we present some applications of equation (1) in the generalized Turán problems.

Let $T$ and $H$ be two ordinary graphs. Let $ex(n, T, H)$ denote the maximum possible number of copies of $T$ in an ordinary $H$-free graph on $n$ vertices. The function $ex(n, T, H)$ is called the generalized Turán number.

Fix $\ell \geq r \geq 3$. In [3] Erdős proved that $ex(n, K_r, K_{\ell+1}) \leq t_{r}(n, \ell)$. Actually a similar argument as in the proofs of theorems 1.1 and 1.2 also gives an exact and stability result to $ex(n, K_r, K_{\ell+1})$. Here we state the stability result without proof.

**Theorem 5.1.** Fix $\ell \geq r \geq 3$, and $\delta > 0$. Then there exists an $\epsilon > 0$ and an $n_0$ such that the following holds for all $n \geq n_0$. If $G$ is an $n$-vertex $K_{\ell+1}$-free graph containing at least $(1 - \epsilon)\binom{\ell}{r}t_{r}(n, \ell)$ copies of $K_r$, then $G$ has a vertex set partition $V_1 \cup \ldots \cup V_{\ell}$ such that all but at most $\delta n^2$ edges have at most one vertex in each $V_i$.

Note that our proof implies that it suffices to choose $\epsilon = \delta$.

In [9] Alon and Shkbalman studied the function $ex(n, T, H)$ for other combinations of $T$ and $H$. In particular they proved that $ex(n, K_r, H) = (1 + o(1))t_{r}(n, \ell)$ holds for every graph $H$ with chromatic number $\chi(H) = \ell + 1$. Later their result was improved by Ma and Qiu [10], who proved that $ex(n, K_r, H) = t_{r}(n, \ell) + \text{biex}(n, H) \cdot \Theta(n^{-2})$, where $\text{biex}(n, H)$ is the Turán number of the decomposition family of $H$. Moreover they proved a stability result for $ex(n, K_r, H)$.

**Theorem 5.2** (Ma and Qiu, [10]). Fix $\ell \geq r \geq 3$, and $\delta > 0$. For every graph $H$ with chromatic number $\ell + 1$, there exists an $\epsilon > 0$ and an $n_0$ such that the following holds for all $n \geq n_0$. If $G$ is an $n$-vertex $H$-free graph containing at least $(1 - \epsilon)\binom{\ell}{r}t_{r}(n, \ell)$ copies of $K_r$, then $G$ has a vertex set partition $V_1 \cup \ldots \cup V_{\ell}$ such that all but at most $\delta n^2$ edges have at most one vertex in each $V_i$.

Here we present a short proof to theorem 5.2 using theorem 5.1 and the Removal Lemma, and our proof implies that it is suffices to choose $\epsilon = \delta/3$.

**Theorem 5.3** (Removal Lemma, e.g. see [6], [11]). Let $H$ be a graph with chromatic number $\ell + 1$. For every $\delta > 0$ there exists an $n_0$ such that the following holds for all $n \geq n_0$. Every $n$-vertex $H$-free graph $G$ can be made $K_{\ell+1}$-free by removing at most $\delta n^2$ edges.
Proof of Theorem 5.2: Let $n$ be sufficiently large. Choose $\epsilon = \delta/3$. Let $G$ be an $n$-vertex $H$-free graph containing at least $(1 - \epsilon)\binom{n}{\ell}t_r(n, \ell)$ copies of $K_r$. By the Removal Lemma, $G$ contains a $K_{r+1}$-free subgraph $G'$ with at least $\epsilon(G) - \epsilon n^2/\ell^2$ edges. Since every edge $e$ in $G$ is contained in at most $\binom{n}{2}$ copies of $K_r$ in $G$. Therefore, the number of copies of $K_r$ in $G'$ is at least $(1 - 2\epsilon)\binom{n}{\ell}t_r(n, \ell)$. By theorem 5.1, $G'$ has a vertex partition $V_1 \cup \ldots \cup V_j$ such that all but at most $2\epsilon n^2$ edges in $G'$ have at most one vertex in each $V_i$. Therefore, all but at most $3\epsilon n^2$ edges in $G$ have at most one vertex in each $V_i$. □

6 Concluding Remarks

Note that we showed that a linear dependence between $\delta$ and $\epsilon$ is sufficient for Theorems 1.2, 1.4, 5.1 and 5.2, and in [6] Füredi showed that a linear dependence is also sufficient for Theorem 2.1. So one might wondering if the linear dependence between $\delta$ and $\epsilon$ is tight (up to a constant) for the stability theorems above. In other words, if there exists an absolute constant $C > 0$ such that for every $\epsilon > 0$ there exists a construction with $\delta \geq C\epsilon$.

We did not try to answer the question above in full generality, but our example below of $K_3$-free graphs shows that the answer seems to be negative.

Fix $\epsilon > 0$. Let $G = (V, E)$ be an $n$-vertex $K_3$-free graph with $(1/4 - \epsilon)n^2$ edges. Let $V_1 \cup V_2$ be a partition of $V$ such that the number of edges in the bipartite graph $G[V_1, V_2]$ is maximum. Define the set of bad edges $B$ and the set of missing edges $M$ as following.

$$B = \{uv \in E(G) : u, v \in V_i \text{ for some } i \in \{1, 2\}\}$$

and

$$M = \{uv \notin E(G) : u \in V_1 \text{ and } v \in V_2\}$$

Therefore, in order to make $G$ bipartite one has to remove all edges in $B$.

Assume that $|B| = \delta n^2$. Let $B_1 = B \cap \binom{V_1}{2}$ be the set of bad edges contained in $V_1$. Without lose of generality we may assume that $|B_1| \geq \delta n^2/2$.

For every vertex $v \in V_1$, let $N_1(v)$ be the neighborhood of $v$ in $V_1$, and let $d_1(v) = |N_1(v)|$. Let $N_2(v)$ be the neighborhood of $v$ in $V_2$, and let $d_2(v) = |N_2(v)|$. By the maximality of the partition $V_1 \cup V_2$, we know that $d_2(v) \geq d_1(v)$ since otherwise one can move $v$ from $V_1$ to $V_2$ to get a larger bipartite subgraph of $G$. Also we know that there is no edge between $N_1(v)$ and $N_2(v)$ since $G$ is $K_3$-free.

Now let $\Delta = \max\{d_1(v) : v \in V_1\}$.

Case 1: $\Delta \geq \delta^{1/3}n$. Then choose $v \in V_1$ of maximum degree $\Delta$. Since there is no edge between $N_1(v)$ and $N_2(v)$. Therefore, $|M| \geq (\Delta n)^2 \geq \delta^{2/3}n^2$. On the other hand, we have $|M| \leq cn^2 + \delta n^2$. So

$$\delta^{2/3} \leq \epsilon + \delta$$

which implies that $\lim_{\epsilon \to 0} \delta/\epsilon = 0$.

Case 2: $\Delta < \delta^{1/3}n$. Using a greedy strategy one can choose a matching $\mathcal{M}$ with at least $(\delta n^2/2)/ (2\delta^{1/3}n) = \delta^{2/3}n/4$ edges from $B_1$. Let $u_1v_1, \ldots, u_mv_m$ be the edges in $\mathcal{M}$. Since $G$ is $K_3$-free. Therefore, we have $d_2(u_i) + d_2(v_i) \leq |V_2|$ and hence

$$|M| \geq \sum_{i=1}^{m} (2|V_2| - d_2(u_i) - d_2(v_i)) \geq m|V_2| \geq \frac{\delta^{2/3}}{4}n \times \frac{n}{3} = \frac{\delta^{2/3}}{12}n^2$$

Similarly we obtain that $\lim_{\epsilon \to 0} \delta/\epsilon = 0$.

Our example above shows that for $K_3$-free graphs there is no absolute constant $C > 0$ such that $\delta/\epsilon \geq C$ holds for all $\epsilon > 0$.
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