Oscillations in Wave Map Systems and Homogenization of the Einstein Equations in Symmetry

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Abstract

In 1989, Burnett conjectured that, under appropriate assumptions, the limit of highly oscillatory solutions to the Einstein vacuum equations is a solution of the Einstein–massless Vlasov system. In a recent breakthrough, Huneau–Luk (Ann Sci l’ENS, 2024) gave a proof of the conjecture in $U(1)$-symmetry and elliptic gauge. They also require control on up to fourth order derivatives of the metric components. In this paper, we give a streamlined proof of a stronger result and, in the spirit of Burnett’s original conjecture, we remove the need for control on higher derivatives. Our methods also apply to general wave map equations.

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1. Introduction

In General Relativity, spacetime is represented by a 4-dimensional Lorentzian manifold which solves the Einstein equations with respect to some suitable matter fields; see (1.2). In describing complex gravitational systems, it is often useful to take a coarse-grained view and study effective models instead [16]. To date, the ΛCDM model in cosmology, consisting of an FLRW spacetime with empirically determined parameters, is the most successful effective model for our universe. However, it is not known how to derive these effective large scale models as limits of the Einstein equations at smaller scales, except for very simple toy problems, see e.g. [24]. In fact, it is not even clear what the correct notion of limit should be [9,23]!

In this paper we consider the simpler problem of determining the weak closure of the vacuum Einstein equations, i.e. the Einstein equations in the absence of matter. Fix a manifold \( M \), and consider a sequence \( (g_\varepsilon)_{\varepsilon} \) of vacuum Lorentzian metrics on \( M \):

\[
\text{Ric}(g_\varepsilon) = 0. \tag{1.1}
\]

If \( g_\varepsilon \) converges strongly to a Lorentzian metric \( g \) in \( C^0_{\text{loc}} \cap W^{1,2}_{\text{loc}} \) as \( \varepsilon \to 0 \), by the structure of the Ricci tensor, it is easy to see we can pass to the limit in (1.1); our effective model is then simply vacuum, i.e. \( \text{Ric}(g) = 0 \). On the other hand, if the convergence is only weak, then \((M, g)\) is no longer necessarily Ricci-flat, so the effective model is non-trivial. From the Einstein equations

\[
\text{Ric}(g) = 8\pi T, \tag{1.2}
\]

where \( T \) denotes a (trace-reversed) energy-momentum tensor, we are tempted to identify the Ricci tensor obtained in the limit as matter. However, in order for \( T \) to correspond to a true Einstein matter model, we must supplement (1.2) with a matter field equation coupled to the geometry of \((M, g)\) in order to get a closed system, see already Conjecture 1.1 below for an example.

As we have just seen, the effective model depends crucially on the convergence assumptions for the sequence \( g_\varepsilon \). In this paper we are concerned with the so-called high-frequency limit, in which small amplitude but high-frequency waves propagate on a fixed background. The high-frequency limit was studied in the physics literature [8,12,24,29,35,36,41,51,52] and we rely here on Burnett’s [10] and Green–Wald’s framework [23]. More precisely, we assume that there is a smooth Lorentzian metric \( g \) such that, for each compact set \( K \subset M \) with a fixed coordinate chart, there is a sequence \( \lambda_\varepsilon \searrow 0 \) such that

\[
\| \partial^k (g_\varepsilon - g) \|_{L^\infty(K)} \leq C(K)\lambda_\varepsilon^{1-k}, \quad \text{for } k = 0, 1, 2. \tag{1.3}
\]

Under these assumptions, Burnett [10] conjectured that the effective model is Einstein–massless Vlasov. We borrow a more precise formulation of Burnett’s conjecture from the recent work of Huneau–Luk [34].
Conjecture 1.1. (Burnett) Let \((g_\varepsilon)_{\varepsilon>0}\) and \(g\) be smooth Lorentzian metrics on \(\mathcal{M}\) satisfying (1.3). There is a finite, non-negative Radon measure \(\mu\) in \(T^*\mathcal{M}\) such that \((\mathcal{M}, g, \mu)\) is a solution to the Einstein–massless Vlasov system, that is

(a) the Einstein equation (1.2) holds, where \(T\) is defined by its action on a test vector field \(Y\) as

\[
\int_{\mathcal{M}} T(Y, Y) \, d\text{Vol}_g = \frac{1}{8\pi} \int_{T^*\mathcal{M}} \xi_a \xi_b Y^a Y^b \, d\mu;
\]

(b) \(\mu\) is solves the massless Vlasov equation with respect to \(g\):

(b1) \(\mu\) is supported on the zero mass shell \(\{(x, \xi) \in T^*\mathcal{M} : g^{ab}(x)\xi_a \xi_b = 0\}\);
(b2) the Vlasov equation holds distributionally: for any \(a \in C^\infty_c (T^*\mathcal{M}\backslash\{0\})\),

\[
\int_{T^*\mathcal{M}} \left( g^{ab} \xi_b \partial_x a - \frac{1}{2} \partial_x g^{ab} \xi_a \xi_b \partial_x a \right) \, d\mu = 0.
\]

We refer the reader to [4,47] for a general introduction to the Einstein–Vlasov model. According to Conjecture 1.1, lack of compactness in the Ricci tensor manifests itself as massless matter, which is propagated along the null directions of spacetime without collisions. In fact, Burnett went further and conjectured that, conversely, all Einstein–massless Vlasov systems can be realized as the weak limit of a sequence of vacuum spacetimes satisfying (1.3). We refer the reader to Huneau–Luk [32,33] for progress in that direction, as well as Touati [56] for preliminary work in a lower regularity setting.

Let us emphasize that although (1.3) are indeed weak convergence assumptions they forbid the occurrence of concentrations. For a setting where concentrations are allowed, and without symmetry assumptions, a complete characterization of the weak closure of the Einstein vacuum equations was recently obtained by Luk–Rodnianski [40]; see also [38,39] in \(T^2\)-symmetry.

The purpose of this paper is to prove Conjecture 1.1 under symmetry and gauge assumptions.

Main Theorem Conjecture 1.1 is true when all the metrics have \(U(1)\)-symmetry and can be put in elliptic gauge with respect to a fixed chart on \(\mathcal{M}\).

See Theorem 1 below for a more precise statement. We recall that a manifold \(\mathcal{M}\) has \(U(1)\)-symmetry if it has a one-dimensional spacelike group of isometries; the elliptic gauge conditions are more involved, see Sect. 1.1 and Appendix A.

A version of Theorem 1 where (1.3) was assumed up to \(k = 4\) was proved earlier by Huneau–Luk [34]. It is desirable to make assumptions only up to \(k = 2\) derivatives in (1.3), since the Einstein equations are second order. Besides this improvement, our proof is perhaps simpler, while remaining completely self-contained. On the way to Theorem 1 we also obtain results of independent interest for wave maps, see Sect. 1.2. Our proof consists of three steps.

1. The quasilinear terms: the \(U(1)\)-symmetry and gauge assumptions can be used to show that oscillations in the quasilinear terms in (1.1) do not contribute to the effective model in any way. Indeed, the massless Vlasov matter is produced by the oscillations in a semilinear wave map system.
2. **The semilinear terms:** to understand the oscillations in wave map systems, we rely on essentially classical bilinear and trilinear compensated compactness results due to Murat and Tartar [44, 54], of which we give simple proofs in Sect. 3. Through these results, and thanks to the *Lagrangian structure* of wave maps, we find that, surprisingly, the heart of the problem is to understand oscillation effects in a *linear* scalar wave equation with respect to an oscillating metric.

3. **The linear terms:** to study linear scalar wave equations with respect to oscillating metrics, our strategy is to take the metric oscillations as *sources* for a wave equation with respect to the limit metric. It turns out that the metric oscillations contribute to the propagation of lack of compactness via a commutator: this is shown by a careful integration by parts argument relying on the parity of the Vlasov equation. We estimate this commutator through a fine frequency analysis in Fourier space which exploits simultaneously the gauge choice, the cancellations encoded in the commutator and some of the rate assumptions (1.3).

The remainder of the introduction discusses each of these steps in detail.

1.1. **The Quasilinear Terms**

We begin by describing in further detail the setup of the Main Theorem. Fix a manifold $\mathcal{M}$ which can be trivialized along one direction, $\mathcal{M} = \mathcal{M} \times \mathbb{R}$. Here, $\mathcal{M}$ is also a fixed manifold of trivial topology, i.e. $\mathcal{M} = (0, T) \times \mathbb{R}^2$ for some $T > 0$. We take global coordinates $(t \equiv x^0, x^1, x^2)$ on $\mathcal{M}$, which we denote with greek indices, and coordinates $(x^0, x^1, x^2, x^3)$ on $\mathcal{M}$. Henceforth, all derivatives indicated by $\partial$, as well as all Sobolev norms, are considered with respect to this fixed chart.

Now take a sequence of Lorentzian metrics $(g_\varepsilon)_{\varepsilon > 0}$ on $\mathcal{M}$ of the form

$$g_\varepsilon \equiv e^{-2\psi_\varepsilon} g_\varepsilon + e^{2\psi_\varepsilon} \left( dx^3 + 2\mathcal{A}_{\alpha,\varepsilon} dx^\alpha \right)^2,$$

where $g_\varepsilon$ are Lorentzian metrics on $\mathcal{M}$ and $\psi_\varepsilon$ and $\mathcal{A}_\varepsilon \equiv \mathcal{A}_{\alpha,\varepsilon} dx^\alpha$ are, respectively, real-valued functions and 1-forms on $\mathcal{M}$. These conditions ensure that the vector field $\partial_3$ generates a one-dimensional spacelike group of isometries on each $(\mathcal{M}, g_\varepsilon)$, i.e. that these spacetimes are $U(1)$-symmetric.

In order to prove the Main Theorem we first note that, if $g_\varepsilon$ is bounded in $W^{1,\infty}_{\text{loc}}(\mathcal{M})$ and converges locally uniformly to some $g \equiv g_0$, then the weak limit $(\mathcal{M}, g)$ also has $U(1)$-symmetry. Using the $U(1)$-symmetric metric ansatz (1.4), we can show that if $(g_\varepsilon)_{\varepsilon > 0}$ are vacuum, then $[\text{Ric}(g_\varepsilon)]_{\alpha 3} = 0$ is a linear differential constraint, see [2] and [11, Chapter XVI.3] for details, which is therefore preserved in the limit:

$$[\text{Ric}(g_\varepsilon)]_{\alpha 3} = 0 \implies d\mathcal{A}_\varepsilon = e^{-4\psi_\varepsilon} \star_{g_\varepsilon} d\omega_\varepsilon, \quad \varepsilon \geq 0. \quad (1.5)$$

There $\omega_\varepsilon$ are functions on $\mathcal{M}$. We are now ready to state our main result precisely.

**Hypotheses 1.2.** Let $(g_\varepsilon)_{\varepsilon > 0} \equiv (g_\varepsilon, \psi_\varepsilon, \omega_\varepsilon)_{\varepsilon > 0}$ and $g \equiv (g \equiv g_0, \psi, \omega)$ satisfy.
(a) in the fixed chart we have introduced, the eigenvalues of $g_\varepsilon$ are uniformly bounded above and away from zero and $g$ is a smooth metric such that $g_\varepsilon \to g$ in $C^0_{\text{loc}}(\mathcal{M})$ as $\varepsilon \to 0$ and $g_\varepsilon$ is bounded in $W^{1,\infty}_{\text{loc}}(\mathcal{M})$; furthermore, for $\varepsilon \geq 0$, $g_\varepsilon$ are in an elliptic gauge, i.e.

\[ g_\varepsilon = -N^2_\varepsilon (dx^0)^2 + \tilde{g}_{ij,\varepsilon} (dx^i + \beta^i_\varepsilon dx^0)(dx^j + \beta^j_\varepsilon dx^0), \]

where $N_\varepsilon$ and $\beta_\varepsilon$ are, respectively, functions and vectors on $\mathcal{M}$ and $\tilde{g}_\varepsilon$ is a Riemannian metric on $\mathbb{R}^2$ which we can, and do, take to be conformally flat;

(a1) $g_\varepsilon$ has the form

\[ g_\varepsilon = -N^2_\varepsilon (dx^0)^2 + \tilde{g}_{ij,\varepsilon} (dx^i + \beta^i_\varepsilon dx^0)(dx^j + \beta^j_\varepsilon dx^0), \]

where $N_\varepsilon$ and $\beta_\varepsilon$ are, respectively, functions and vectors on $\mathcal{M}$ and $\tilde{g}_\varepsilon$ is a Riemannian metric on $\mathbb{R}^2$ which we can, and do, take to be conformally flat;

(b) $\psi_\varepsilon \to \psi$ in $C^0_{\text{loc}}(\mathcal{M})$, $\psi_\varepsilon \to \psi$ in $W^{1,4}_{\text{loc}}(\mathcal{M})$, and similarly replacing $\psi_\varepsilon$, $\psi$ with $\omega_\varepsilon$, $\omega$;

(c) $\|g_\varepsilon^{\alpha\beta} - g^{\alpha\beta}\|_{L^\infty(K)} (\|\partial^2(\psi_\varepsilon - \psi)\|_{L^4(K)} + \|\partial^2(\omega_\varepsilon - \omega)\|_{L^4(K)}) \lesssim_K 1$ for every compact $K \subset \mathcal{M}$.

We note that Hypotheses 1.2 are strictly weaker than the high-frequency limit conditions (1.3), when the latter are specialized to the $U(1)$-symmetric and elliptic gauge case. In particular, in Hypothesis 1.2(c) we make no assumptions on the comparative convergence (and divergence) rates of $(\psi_\varepsilon, \omega_\varepsilon)$ and $(\partial^2 \psi_\varepsilon, \partial^2 \omega_\varepsilon)$.

**Theorem 1.** Let $(g_\varepsilon)_{\varepsilon > 0}$ and $g$ satisfy Hypotheses 1.2 and assume that, for $\varepsilon > 0$, $g_\varepsilon$ solve (1.1). Then there is a non-negative Radon measure $\nu$ on $S^*\mathcal{M}$ such that $(\mathcal{M}, g, \nu)$ is a radially averaged measure-valued solution of the restricted Einstein–Vlasov equations in $U(1)$-symmetry. More precisely, we have

(a) **Limit equation:** For every vector field $Y \in C^\infty_0(\mathcal{M})$, the tensor $\text{Ric}(g)$ satisfies

\[ \int_\mathcal{M} [\text{Ric}(g)]_{\alpha\beta} Y^\alpha Y^\beta \, \text{dVol}_g = \int_{S^*\mathcal{M}} \xi^\alpha \xi^\beta Y^\alpha Y^\beta \, d\nu. \]  

(1.6)

(b) **Vlasov equation:** $(\mathcal{M}, g, \nu)$ is a radially averaged measure-valued solution of massless Vlasov:

(b1) Support property: $\nu$ is supported on the zero mass shell of $g$, i.e. for all $\varphi \in C^\infty_0(\mathcal{M})$

\[ \int_{S^*\mathcal{M}} \varphi(x) g^{\alpha\beta} \xi^\alpha \xi^\beta \, d\nu = 0. \]

(b2) Propagation property: for all $\tilde{a} \in C^\infty_0(S^*\mathcal{M})$, extended as a positively 1-homogeneous function to $T^*\mathcal{M}\setminus\{0\}$, the measure $\nu$ satisfies

\[ \int_{S^*\mathcal{M}} \left[ g^{\alpha\beta} \xi^\alpha \partial_{x^\beta} \tilde{a} - \frac{1}{2} \partial_{x^\mu} g^{\alpha\beta} \xi^\alpha \xi^\beta \partial_{x^\mu} \tilde{a} \right] \, d\nu = 0. \]

(1.7)

We note that, as long as $(\mathcal{M}, g)$ is globally hyperbolic, $(\mathcal{M}, g, \nu)$ naturally induces a non-radially averaged solution to the Einstein–massless Vlasov system, see [34, Section 2].
Remark 1.3. (Beyond the vacuum case) Our methods allow for an extension of Theorem 1 to a case where \( g_\varepsilon \) are not vacuum but are sourced by a tensor \( T_\varepsilon \). To be precise, we require that the \((\alpha, 3)\) components of \( T_\varepsilon \) must vanish and that there is a smooth tensor \( T \) such that \( T_\varepsilon \to T \) in \( C^0_{\text{loc}} \) and \( T_\varepsilon \to T \) in \( L^4_{\text{loc}} \). In that case, the analogue of (1.6) reads as

\[
[Ric(g)]_{\alpha 3} = 0, \quad [Ric(g)]_{33} = 8\pi T_{33},
\]

\[
\int_M [Ric(g)]_{\alpha \beta} Y^\alpha Y^\beta \, dVol_g = \int_M 8\pi \left[ T_{\alpha \beta} + T_{33} e^{-2\psi} g_{\alpha \beta} \right] Y^\alpha Y^\beta \, dVol_g
\]

\[
+ \int_{S^* M} \xi^\alpha \xi^\beta Y^\alpha Y^\beta \, dv,
\]

and the Vlasov equation in (1.7) has a source term related to the failure of compactness in \( T_\varepsilon \).

To understand the proof of Theorem 1, let us begin by computing the curvature of the limit spacetime \((M, g)\); we again use the notation \( g_0 \equiv g \). As we have seen above, this spacetime also has \( U(1) \)-symmetry, so our computations rely on the form of \( U(1) \)-metrics given in (1.4).

**Curvature in the \( U(1) \)-symmetry directions.** We have already seen that the vacuum condition passes to the limit in the \((\alpha, 3)\) direction, motivating us to introduce functions \( \omega_\varepsilon, \varepsilon \geq 0 \), on \( M \) as in (1.5). One can further show, see [2] and [11, Chapter XVI.3], that

\[
[Ric(g_\varepsilon)]_{\alpha 3} = 0 = \Box_{g_\varepsilon} \omega_\varepsilon - 4g_\varepsilon^{-1} (d\psi_\varepsilon, d\omega_\varepsilon), \quad \text{for all } \varepsilon \geq 0.
\]

(1.8)

In the \((3, 3)\) direction, (1.1) leads to a nonlinear wave equation, but the nonlinear terms are weakly continuous, see Lemma 1.8, and hence

\[
[Ric(g_\varepsilon)]_{33} = 0 = \Box_{g_\varepsilon} \psi_\varepsilon + \frac{1}{2} e^{-4\psi_\varepsilon} g_\varepsilon^{-1} (d\omega_\varepsilon, d\omega_\varepsilon), \quad \text{for all } \varepsilon \geq 0.
\]

(1.9)

Thus, the \((\alpha, 3)\) and \((3, 3)\) directions provide no contributions to any matter produced in the limit. Moreover, from (1.8) and (1.9) we obtain the wave map equation

\[
\begin{cases}
\Box_{g_\varepsilon} \psi_\varepsilon + \frac{1}{2} e^{-4\psi_\varepsilon} g_\varepsilon^{-1} (d\omega_\varepsilon, d\omega_\varepsilon) = 0, \\
\Box_{g_\varepsilon} \omega_\varepsilon - 4g_\varepsilon^{-1} (d\psi_\varepsilon, d\omega_\varepsilon) = 0,
\end{cases}
\]

(1.10)

from \((M, g_\varepsilon)\) to the Poincaré plane \((\mathbb{R}^2, g)\), where \( g = 2(d\psi)^2 + \frac{1}{2} e^{-4\psi} (d\omega)^2 \). We recall that (1.10), being a wave map system, is the Euler–Lagrange equation for a Lagrangian on the domain \((M, g_\varepsilon)\); in this case, the Lagrangian density is

\[
L_{\alpha \beta} [\psi_\varepsilon, \omega_\varepsilon] \equiv 2\partial_\alpha \psi_\varepsilon \partial_\beta \psi_\varepsilon + \frac{1}{2} e^{-4\psi_\varepsilon} \partial_\alpha \omega_\varepsilon \partial_\beta \omega_\varepsilon, \quad \varepsilon \geq 0.
\]

**Curvature in the non-symmetric directions.** Finally, we turn to the curvature in the \((\alpha, \beta)\) directions. From the vacuum condition (1.1) on \( g_\varepsilon \) and its \( U(1) \)-symmetry, we find that

\[
[Ric(g_\varepsilon)]_{\alpha \beta} = L_{\alpha \beta} [\psi_\varepsilon, \omega_\varepsilon] = 2\partial_\alpha \psi_\varepsilon \partial_\beta \psi_\varepsilon + \frac{1}{2} e^{-4\psi_\varepsilon} \partial_\alpha \omega_\varepsilon \partial_\beta \omega_\varepsilon, \quad \text{only for } \varepsilon > 0.
\]

(1.11)
Thus, for the $U(1)$-symmetric weak limit $g$, we easily compute

$$
[Ric(g)]_{\alpha\beta} = [Ric(g)]_{\alpha\beta} - \mathbb{L}_{\alpha\beta}[\psi, \omega]
= w^*- \lim_{\varepsilon \to 0} \mathbb{L}_{\alpha\beta}[\psi_\varepsilon, \omega_\varepsilon] - \mathbb{L}_{\alpha\beta}[\psi, \omega] + [Ric(g)]_{\alpha\beta} - w^*- \lim_{\varepsilon \to 0} [Ric(g_\varepsilon)]_{\alpha\beta}.
$$

(1.12)

From the symmetry assumptions alone, we find in (1.12) that there are two different types of contributions to the matter created in the limit in the $(\alpha, \beta)$ directions: those arising from the *semilinear* wave map equation (1.10) for $\varepsilon > 0$, and those arising from the *quasilinear* condition (1.11) which makes $g_\varepsilon$ in the wave map equation depend on the solution itself. However, it is easy to see that the latter contributions are *forbidden* under the gauge conditions we impose.

**Lemma 1.4.** If Hypotheses 1.2(a) hold, then $Ric(g_\varepsilon) \rightharpoonup^* Ric(g)$ in the sense of distributions.

For the convenience of the reader, we reprove this standard fact about elliptic gauge in Appendix A. Thus, the quasilinear terms do not contribute to the matter created in the limit, and (1.12) becomes

$$
[Ric(g)]_{\alpha\beta} = w^*- \lim_{\varepsilon \to 0} \mathbb{L}_{\alpha\beta}[\psi_\varepsilon, \omega_\varepsilon] - \mathbb{L}_{\alpha\beta}[\psi, \omega].
$$

(1.13)

We conclude that, in order to prove Theorem 1, it is enough to characterize the failure of compactness in the Lagrangian density associated to a *semilinear* wave map equation such as (1.10).

**Remark 1.5.** (Decoupling of the Einstein part) Equation (1.13) shows that, from the point of view of Theorem 1, the wave map and the Einstein parts of the system composed of (1.10) and (1.11) decouple *completely* thanks to the elliptic gauge conditions. Notice that this is in stark contrast with other types of analysis of the system composed of (1.10) and (1.11), such as understanding its well-posedness, see e.g. [32,56]: there, the quasilinearity is the main difficulty and it cannot be removed by any gauge condition.

### 1.2. The Semilinear Terms

In the previous section we have shown that, in spite of the quasilinear nature of the Einstein equation (1.2), Theorem 1 *de facto* reduces to understanding the semilinear wave map equation (1.10). The study of oscillations in solutions to wave map equations in fact has much broader applications, as these are very widely studied systems of nonlinear hyperbolic PDEs, see e.g. the classical reference [50]. Accordingly, for $\varepsilon > 0$ consider a wave map from a Lorentzian manifold $(\mathcal{M}, g_\varepsilon)$ to a fixed Riemannian manifold $(\mathcal{N}, g)$:

$$
\Box_{g_\varepsilon} u^I_\varepsilon + \Gamma^I_{JK}(u_\varepsilon)g_\varepsilon^{-1}(du^J_\varepsilon, du^K_\varepsilon) = f^I_\varepsilon, \quad u^I_\varepsilon, f^I_\varepsilon : \mathcal{M} \to \mathbb{R}, \quad I, J, K \in \{1, \ldots, N\}.
$$

(1.14)
Here, $\Gamma^I_{JK}: \mathbb{R} \to \mathbb{R}$ are the Christoffel symbols of the Riemannian metric $g$ and depend continuously on $u^I$. For simplicity, we take $\mathcal{M} \subset \mathbb{R}^{1+n}$ and $\mathcal{N} \subset \mathbb{R}^N$ to be domains, and $\{x^0, x^1, \ldots , x^n\}$ to be coordinates on $\mathcal{M}$ represented with greek indices or, if $x^0$ is excluded, roman indices; however, in light of the assumptions ensuing, this restriction is without loss of generality. Indeed, we will assume the following

**Hypotheses 1.6.** Let $\mathcal{U}_\varepsilon \equiv (g_\varepsilon, (u^I_\varepsilon)_{I=1}^N, (f^I_\varepsilon)_{I=1}^N)$ and $\mathcal{U} \equiv (g, (u^I)_{I=1}^N, (f^I)_{I=1}^N)$ satisfy

(a) the eigenvalues of $g_\varepsilon$ are uniformly bounded above and away from zero and $g$ is a smooth metric such that $g_\varepsilon \to g$ in $C^0_{\text{loc}}$, $\partial_0 (g_\varepsilon)_{ij} \to \partial_0 g_{ij}$ strongly in $L^4_{\text{loc}}$, and $\delta^{ij} \partial^2_{ij} g_\varepsilon^{\alpha \beta} \to \delta^{ij} \partial^2_{ij} g^{\alpha \beta}$ is bounded in $L^2_{\text{loc}}$;

(b) $u^I_\varepsilon$ converges to $u^I$ uniformly in $C^0_{\text{loc}}$ and weakly in $W^{1,4}_{\text{loc}}$;

(c) $\|g_\varepsilon^{\alpha \beta} - g^{\alpha \beta}\|_{L^\infty(K)} \|\partial^2(u^I_\varepsilon - u^I)\|_{L^1(K)} \lesssim K$ for every compact $K \subset \mathcal{M}$;

(d) $f^I_\varepsilon \rightharpoonup f^I$ in $L^4_{\text{loc}}$.

**Remark 1.7.** For $n = 2$, Hypotheses 1.6(a) are implied by Hypotheses 1.2(a), see Appendix A.

The convergence of $\mathcal{U}_\varepsilon$ assumed in Hypotheses 1.6 is strong enough to easily ensure that $(g, u^1, \ldots, u^N)$ is itself a wave map. This is a substantially more difficult task under weaker hypotheses, see for instance [6,19,20,22] for several examples of oscillation and concentration effects in semilinear wave equations in lower regularity, albeit in settings where $g_\varepsilon = g$ is the Minkowski metric. On the other hand, Hypotheses 1.6 are weak enough that general quadratic quantities in the solutions, such as

$$L_{\alpha \beta}[u_\varepsilon] \equiv g_{IJ}(u_\varepsilon) \partial_\alpha u^I_\varepsilon \partial_\beta u^J_\varepsilon,$$

are not preserved in the limit as $\varepsilon \to 0$. We henceforth refer to $L_{\alpha \beta}$ as the Lagrangian density because this quantity features in the variational principle from which (1.14) is derived. With Theorem 1 and, specifically, (1.13) in view, our goal is precisely to characterize the failure of compactness in (1.15), i.e. to identify the compactness singularities and describe how they are propagated. For simplicity, we state our main result only for wave maps (1.14) without sources.

**Theorem 2.** Let $\Gamma^I_{JK}$ be continuous Christoffel symbols arising from a Riemannian metric

$$g = g_{IJ}(y) \, dy^I \otimes dy^J.$$ 

Let $\mathcal{U}_\varepsilon$ be a sequence of solutions to (1.14) with $f^I_\varepsilon \equiv 0$. There is a Radon measure $\nu$ on $S^*(\mathcal{M})$ such that

(a) **Limit equation.** $\mathcal{U}$ is a distributional solution of (1.14) and its Lagrangian energy density satisfies

$$\lim_{\varepsilon \to 0} \int_{\mathcal{M}} L_{\alpha \beta}[u_\varepsilon] Y^\alpha Y^\beta \, d\text{Vol}_{g_\varepsilon} = \int_{\mathcal{M}} L_{\alpha \beta}[u] Y^\alpha Y^\beta \, d\text{Vol}_{g} + \int_{S^*\mathcal{M}} \xi_\alpha \xi_\beta \, Y^\alpha Y^\beta \, dv, \quad \forall \, Y \in C^0_\infty(\mathcal{M}).$$
(b) **Vlasov equation.** The measure \( \nu \) is a (radially averaged) measure-valued solution of a massless Vlasov equation with respect to \( g \), in the sense that properties \((b_1)\) and \((b_2)\) of Theorem 1 hold.

Strictly speaking, in Theorem 2, as well as in Theorem 3 below, one may need to pass to a subsequence in \( g_\varepsilon \). In fact, throughout the paper we always work modulo subsequences. We also note that the case \( f_\varepsilon \neq 0 \) is very similar: \((a)\) still holds, and in \((b)\) the massless Vlasov equation becomes inhomogeneous with source related to the failure of compactness of \( (f_\varepsilon)_{\varepsilon>0} \).

The measure in Theorem 2 is essentially an \( H \)-measure induced by the sequence \( U_\varepsilon \), see Sect. 2. \( H \)-measures, often known as microlocal defect measures in the literature, were introduced independently by Gérard [21] and Tartar [53]. \( H \)-measures are ideal tools for proving Theorem 2: like other popular tools to study the failure of strong convergence, such as Young measures, they can be used to compute the difference between \( L_{\alpha\beta}[u] \) and \( \lim_{\varepsilon \to 0} L_{\alpha\beta}[u_\varepsilon] \), but crucially they also capture the way in which this difference propagates. We refer the reader to [48] for a comparison between Young measures and \( H \)-measures.

Any sequence \( (\partial_0 u_\varepsilon^I, \partial_1 u_\varepsilon^I, \ldots, \partial_n u_\varepsilon^I, f_\varepsilon^I)_{I=1}^N \) bounded in \( L^2_{\text{loc}} \) induces an \( H \)-measure

\[
\left( \left[ \tilde{\nu}^{IJ}, \tilde{\lambda}^{IJ} \right], \tilde{\mu}^{IJ} \right)_{I,J=1}^N,
\]

which is valued in \( N \times N \) block-matrices. The measures \( \tilde{\nu}^{IJ} \) takes values in \( (n+1) \times (n+1) \) matrices, while the measures \( \tilde{\mu}^{IJ} \) are scalar; they are essentially computed by respectively evaluating the limits

\[
\lim_{\varepsilon \to 0} \langle A(du_\varepsilon^I - du^I), du_\varepsilon^J - du^J \rangle \quad \text{and} \quad \lim_{\varepsilon \to 0} \langle B(f_\varepsilon^I - f^I), f_\varepsilon^J - f^J \rangle.
\]

Here and throughout \( \langle \cdot, \cdot \rangle \) denotes the Euclidean \( L^2 \)-inner product with respect to \( dx \), while \( A \) and \( B \) are zeroth order pseudo-differential operators. Finally, the measures \( \tilde{\lambda}^{IJ} \) capture the interaction between \( du_\varepsilon^I \) and \( f_\varepsilon^J \).

Our strategy to prove Theorem 2 is to rewrite (1.14) as

\[
\Box g_\varepsilon u_\varepsilon^I = Q_\varepsilon^I + f_\varepsilon^I, \quad Q_\varepsilon^I \equiv -\Gamma_{JK}^I (u_\varepsilon) g^{-1}(du_\varepsilon^K, du_\varepsilon^K),
\]

and to interpret the semilinearities in the wave map equations as source terms for a linear wave equation on an oscillating background. As will become clearer in the next subsection, source terms contribute to Theorem 2 only through the \( H \)-measure \( \tilde{\lambda}^{IJ} \). Hence, our goal is to compute

\[
\lim_{\varepsilon \to 0} \langle A \partial(u_\varepsilon^I - u^I), Q_\varepsilon^L - Q^L \rangle,
\]

where \( Q^L \) denotes the weak limit of \( Q_\varepsilon^L \) in \( L^2_{\text{loc}} \). Note that the uniform convergence of \( u_\varepsilon^I \) ensures that, in \( Q_\varepsilon^L \), only the null forms \( g^{-1}(du_\varepsilon^K, du_\varepsilon^K) \) are important. The null structure of the wave map nonlinearities translates into a \( \text{div-curl} \) structure both for bilinear and trilinear terms.
Lemma 1.8. (Murat and Tartar [44,54]) Under Hypotheses 1.6, we have

(a) \( Q^L \equiv w-\lim_{\varepsilon \to 0} Q^L_{\varepsilon} = -\Gamma^f_{JK}(u) g^{-1}(du^I, du^K); \)
(b) if \( u = 0 \) then \( w-\lim_{\varepsilon \to 0} \partial u_I \varepsilon g^{-1}(du^J, du^K) = 0 \), where \( \partial \) denotes an arbitrary partial derivative.

We reprove this classical result in Sect. 3.2 below using the geometric version of the div-curl lemma from [49] and the usual geometric framework of energy identities for covariant wave equations. We also alert the reader that (b) refers to as three-wave compensated compactness in [34].

When \( u = 0 \), Lemma 1.8 easily shows that (1.17) vanishes. However, this is not the case in general, as the trilinear quantity in (b) is weakly continuous only at zero. That such quantities even exist is only possible because \( \Box_g \), thought of as a first-order operator acting on \( \partial u \), does not have constant rank, c.f. [28] and Remark 3.7. The upshot is that in the general case \( u^I \neq 0 \) the nonlinearities create a coupling between the behavior of the measures \( \tilde{\nu}^{IJ} \) and \( \tilde{\nu}^{KL} \), so the lack of compactness in general quadratic quantities associated to wave maps does not admit a simple characterization.

For the particular quantity we are interested in, the Lagrangian density (1.15), something surprising occurs: the couplings between the different measures are added up so as to precisely cancel! Hence, through the classical Lemma 1.8, the nonlinear terms can be easily shown not to contribute to the failure of compactness of \( \mathbb{L}_{\alpha\beta}[u_{\varepsilon}] \) nor to its propagation. We conclude that, to establish Theorem 2, it is enough to characterize the failure of compactness in quadratic quantities associated to a linear scalar wave equation with oscillating coefficients.

1.3. The Linear Terms

We have reduced the proofs of Theorems 1 and 2 to understanding oscillations in a scalar linear wave equation with respect to oscillating background metrics. In other words, we take \( N = 1 \) in Hypotheses 1.6 and hence, for simplicity, we drop the superscripts.

Theorem 3. Let \( \mathcal{U}_\varepsilon = (g_{\varepsilon}, u_{\varepsilon}, f_{\varepsilon}) \) be a sequence satisfying Hypotheses 1.6 and such that \( \Box_{g_{\varepsilon}} u_{\varepsilon} = f_{\varepsilon} \).

(a) Limit equation. The triple \( \mathcal{U} = (g, u, f) \) is a solution of \( \Box_g u = f \).
(b) Vlasov equation. There are Radon measures \( \nu, \lambda \) such that \( \tilde{\nu}_{\alpha\beta} = \xi_{\alpha}\xi_{\beta}\nu, \tilde{\lambda}_{\gamma} = \xi_{\gamma}\lambda \). Moreover, \( \nu \) is a (radially averaged) measure-valued solution of an inhomogeneous massless Vlasov equation, in the sense that property (b1) of Theorem 1 holds, and for all \( \tilde{a} \in C_0^\infty(S^*M) \), extended as a positively 1-homogeneous function to \( T^*M\{0\} \), the measure \( \nu \) satisfies

\[
\int_{S^*M} \left[ g^{\alpha\beta} \xi_{\alpha} \partial_{x^\mu} \tilde{a} - \frac{1}{2} \partial_{x^\nu} g^{\alpha\beta} \xi_{\alpha} \xi_{\beta} \partial_{x^\mu} \tilde{a} \right] d\nu = -\int_{S^*M} \tilde{a} d(\Theta\lambda). \tag{1.18}
\]

Remark 1.9. (Initial value formulation) The transport equation (1.18) in Theorem 3(b) naturally inherits a suitable set of initial conditions in terms of initial
conditions for $\Box_{g_\varepsilon} u_\varepsilon = f_\varepsilon$, see [53, Section 3.4] as well as [18] for a detailed study when $g_\varepsilon = g$ is fixed. In other words, the failure of compactness seen in the evolution may be characterized in terms of failure of compactness of the initial data.

**Remark 1.10. (Regularity of $g$)** It is natural to ask whether $W^{1,\infty}_\text{loc}$-bounds on $g_\varepsilon$ in can be weakened to $W^{1,q}_\text{loc}$-bounds, for some $q < \infty$. This would affect the expected regularity of $g$, which would drop below $C^1$. Such a level of regularity seems problematic: indeed, the integrand in the left-hand side of (1.18) is the Poisson bracket between the symbol of $\Box_g$ and $\tilde{a}$, which in turn is the symbol of a commutator between the corresponding pseudo-differential operators that ought to be at least bounded, c.f. Remark 3.4.

Let us give an outline of the proof of Theorem 3. For a fixed Lorentzian metric, a full characterization of the H-measure associated to the linear wave equation is already essentially contained in Tartar’s original paper [53], see also [7,18]. For the sake of completeness, in Sect. 3, we extend these proofs to general covariant wave equations, relying on a standard geometric version of the energy identity, see e.g. [1].

The case of oscillating metrics $g_\varepsilon$, which takes up the entirety of Sect. 4 here, is much more involved, as predicted by Francfort–Murat [18]; it is, nonetheless, very natural from the point of view of Homogenization Theory [13]. An obvious additional difficulty of this case is that it is not clear what is the appropriate notion of convergence for the metrics. Though this is an interesting problem, we do not investigate it here: it turns out that Hypotheses 1.6 provide sets of convergence conditions under which the oscillations of $g_\varepsilon$ do not contribute to the propagation of non-compactness. With stronger conditions on the rates of convergence, as mentioned above, this remarkable fact is one of the key observations of Huneau–Luk [34], and it served as inspiration for our work.

Our strategy for dealing with the oscillations of $g_\varepsilon$ is to reduce to the case where $g$ is fixed, so we write

$$
\Box_{g_\varepsilon} u_\varepsilon = f_\varepsilon \quad \implies \quad \Box g u_\varepsilon = -H_\varepsilon + f_\varepsilon, \quad \text{where } H_\varepsilon \equiv (\Box g - \Box_{g_\varepsilon}) u_\varepsilon.
$$

Determining the contribution of the oscillations of $g_\varepsilon$ to the Vlasov equation amounts to calculating

$$
\lim_{\varepsilon \to 0} \langle H_\varepsilon, A e_0 (u_\varepsilon - u) \rangle, \quad \text{where } e_0 \equiv \partial_0 + \frac{g^{0i}}{g^{00}} \partial_i.
$$

Here $A \in \Psi^0$ is an arbitrary pseudo-differential operator corresponding to the test function $\tilde{a}$ in (1.18) and the upper indices denote components of the inverse metrics. A parity argument shows that we can assume that the symbol of $A$ is real and even; then, by a careful integration by parts argument, we obtain

$$
\lim_{\varepsilon \to 0} \langle H_\varepsilon, A e_0 (u_\varepsilon - u) \rangle
\begin{align*}
= & \frac{1}{2} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^{1+n}} \partial_\alpha (u_\varepsilon - u) [g_\varepsilon^{\alpha \beta} - g^{\alpha \beta}, A] \partial_\beta e_0 (u_\varepsilon - u) \, dx;
\end{align*}
$$

(1.19)
Fig. 1. The frequency space picture

see Lemma 4.6. By the Calderón commutator estimate, if \( g_\varepsilon \to g \) strongly in \( W^{1,\infty}_{\text{loc}} \), then (1.19) vanishes in the limit. However, even if all derivatives but one converge strongly, this simple proof fails, as the Calderón commutator estimate requires Lipschitz bounds. This is the case in Hypotheses 1.6: the assumptions imply that spatial derivatives of \( g_\varepsilon \) convergence strongly, with \( e_0 g_\varepsilon \) converging only weakly.

As is common in compensated compactness, see e.g. [30, Theorem 5.3.2], we examine the failure of compactness in \( e_0 g_\varepsilon \) in Fourier space, and we denote by \( \Lambda \) the region where the symbol of \( e_0 \) vanishes. This naturally induces a partition of Fourier space as follows, see Fig. 1.

**Low frequencies** \( (\mathcal{F}_{\text{low}}) \). In bounded regions of frequency space, \( W^{1,2}_{\text{loc}} \) and \( L^2_{\text{loc}} \) norms are comparable, hence \( e_0 g_\varepsilon \) is, in fact, compact in this range. Indeed, as a general principle, failure of compactness is a high-frequency phenomenon.

**High frequencies close to** \( \Lambda \) \( (\mathcal{F}_{\text{space}}) \). In this region, \( e_0 \) is not invertible, so the fact that \( u_\varepsilon \) appear in the commutator does not help. We instead compensate for the lack of compactness in \( e_0 g_\varepsilon \) by using the fact that the spatial laplacians of \( g_\varepsilon \) are bounded in \( L^2_{\text{loc}} \), see Hypotheses 1.6(a). We alert the reader that this, as well as the argument laid out in the next frequency regime, are referred to as elliptic-wave compensated compactness in [34].

**High frequencies away from** \( \Lambda \) \( (\mathcal{F}_{\text{time}}) \). This is the most difficult regime and, in some sense, the heart of the proof. To illustrate our strategy, let us consider the simple case where the limit metric \( g \) is the Minkowski metric and \( A \) is a multiplier, i.e. its symbol is merely a function \( m(\xi) \) for \( \xi \in S^* \mathcal{M} \) which is 0-homogeneous and even. Let us write \( w_\varepsilon \equiv u_\varepsilon - u \) and \( h_\varepsilon^{\alpha\beta} \equiv g_\varepsilon^{\alpha\beta} - g^{\alpha\beta} \). Then, from Plancherel
and the parity of \( m \), the sequence on the right hand side of (1.19) becomes

\[
\frac{1}{2} \int \partial_\alpha w_\varepsilon \{ h^{\alpha\beta}_\varepsilon \partial_\beta e_0 w_\varepsilon \} \, dx
= \frac{i}{4} \int \int \xi_\alpha \eta^\beta (\xi_0 + \eta_0) h^{\alpha\beta}_\varepsilon (\xi - \eta) \hat{w}_\varepsilon (-\xi) \hat{w}_\varepsilon (\eta) [m(\xi) - m(\eta)] \, d\xi \, d\eta. \tag{1.20}
\]

For simplicity, we take \((\alpha, \beta) = (i, j)\) and \( h^{ij}_\varepsilon = h^{ij}_\varepsilon \), as this is enough to illustrate the main point. We now manipulate the symbol \((\xi_0 + \eta_0)\xi_i \eta^i\) as follows:

\[
(\xi_0 - \eta_0)(\xi_0 + \eta_0)\xi_i \eta^i = \left[ \sigma_\square(\eta) - \sigma_\square(\xi) + \xi_k \eta^k \xi_i \eta^i \right] \xi_i \eta^i
\]

\[
= \sigma_\square(\eta)\xi_i (\eta^i - \xi^i) + \sigma_\square(\xi)\xi_i (\xi^i - \eta^i) + (\xi_k \eta^k + \eta_k \eta^k)(\xi_k - \eta_k)
\]

\[
\xi_i \eta^i \sigma_\square(\eta) - \eta_i \eta^i \sigma_\square(\xi) + \left( \eta^i \xi_0^2 + \xi^i \eta_0^2 \right) (\xi_i - \eta_i). \tag{1.21}
\]

Plugging this identity into (1.20), we find that terms which contain \( \partial^2 w_\varepsilon \) are always paired with \( h^{\alpha\beta}_\varepsilon \square u_\varepsilon \) or with \( \partial_i h^{\alpha\beta}_\varepsilon \partial w_\varepsilon \). The latter are obviously compact and the former are compact too, since by Hypothesis 1.6(c)

\[
\square_g u_\varepsilon \text{ is bounded in } L^4_{\text{loc}}, \tag{1.22}
\]

unlike general second order derivatives of \( u_\varepsilon \). Hence lack of compactness of \( e_0 g_\varepsilon \) is compensated by appealing to a differential condition on \( u_\varepsilon \). It is the last two manipulations in (1.21) that ensure we have no more than two derivatives on each \( w_\varepsilon \) and no more than one derivative on \( h^{\alpha\beta}_\varepsilon \). This extra step means that our Hypotheses 1.6 contain no assumptions on derivatives of order \( k > 2 \), c.f. [34] where assumptions on up to \( k = 4 \) are imposed.

**Remark 1.11.** (The role of rate assumptions) To apply compensated compactness methods it is crucial that we have differential information on the sequence with respect to fixed \( \varepsilon \)-independent differential operators, as in (1.22). Hypotheses 1.6(d) are not sufficient to deduce (1.22) and so, in the spirit of Conjecture 1.1, we require rate assumptions in Hypotheses 1.6(c).

The simple proof we have given here for the case where \( g \) is Minkowski is the template for the analysis of the quasilinear terms both in our work and in [34]. The proof in fact easily generalizes to any constant coefficient metric \( g \), as long as one still takes \( A \) to be a multiplier. However, when \( A \) is a true pseudo-differential operator with \( x \)-dependence and/or \( g \) is \( x \)-dependent, an application of Plancherel leads to convolutions, and the division by the symbol of \( e_0 \), which may itself be \( x \)-dependent, becomes tricky.
In order to make sense of the division by the symbol of $e_0$, [34] apply cutoffs to “freeze” the $x$-dependence of $g$ and $A$, making them locally constant in $x$: if the balls where the freezing is done shrink in an appropriate way as $\varepsilon \to 0$, the above argument works. Unfortunately, this procedure requires additional assumptions: in Hypotheses 1.6(c), we would also need information on the rate of uniform convergence of $u^\varepsilon$ compared not only to $g^\varepsilon$ but also to $\partial^2 u^\varepsilon$. In this paper, we avoid these additional assumptions by simply defining the inverse of $e_0$ as a pseudo-differential operator, which exists in the frequency regime we are considering. This simplifies the argument considerably, and yields a proof purely based on integration by parts. Concretely,

(a) We write $h^\alpha_\beta = e_0^{-1} h^{\alpha_\beta}_\varepsilon$. Integrating by parts brings the extra $e_0$ derivative onto $u^\varepsilon$; the trilinear form of (1.19) is then key.

(b) Relying on parity arguments and the structure of the commutator, we can use the extra $e_0$ derivative to fashion $\Box_g u^\varepsilon$ out of the second derivatives on $u^\varepsilon$ which appear. Further integration by parts ensures that we have only up to two derivatives of $u^\varepsilon$ and one derivative of $g^\varepsilon$.

Combining the previous two points we show that (1.20) vanishes as $\varepsilon \to 0$, completing the proof.

2. Preliminaries on H-Measures and Compensated Compactness

2.1. Symbols and Pseudo-Differential Operators

In this section we gather some basic results about pseudo-differential operators. These can be found, for instance, in the books [31] and [25]. We take $\Omega \subset \mathbb{R}^N$ to be a fixed open set throughout.

**Definition 2.1.** For $m \in \mathbb{R}$, a function $a$ is called a symbol of order $m$, $a \in S^m \equiv S^m(\Omega, \mathbb{C}^d \times \mathbb{C}^d)$, if $a \in C^\infty(\Omega \times \mathbb{R}^N, \mathbb{C}^d \times \mathbb{C}^d)$ and, for each compact set $K \subset \Omega$,

$$|\partial_\alpha x \partial_\beta \xi a(x, \xi)| \lesssim_{\alpha, \beta, K} (1 + |\xi|)^{m-|\beta|}.$$

We write that $S^{-\infty} = \bigcap_{m \in \mathbb{R}} S^m$.

The following basic lemma gives meaning to asymptotic expansions of symbols:

**Lemma 2.2.** For $j \in \mathbb{N}_0$ let $a_j \in S^{m_j}$ and $m_j \searrow -\infty$. There is $a \in S^{m_0}$ such that, for every $k$, $a - \sum_{j < k} a_j \in S^{m_k}$. The symbol $a$ is unique modulo $S^{-\infty}$ and we write that $a \sim \sum_{j=0}^\infty a_j$ in $S^m$.

Each symbol $a \in S^m$ induces an operator $A$ acting on $v \in C^\infty_c(\mathbb{R}^N, \mathbb{C}^d)$ by

$$Av(x) \equiv \int_{\mathbb{R}^N} a(x, \xi) e^{2\pi i x \cdot \xi} \widehat{v}(\xi) \, d\xi,$$

where $\widehat{\cdot}$ denotes the Fourier transform. We say that $A$ is a pseudo-differential operator of order $m$. We write $\sigma(A) \equiv a$ and note that, for any pseudo-differential operator, the symbol $\sigma(A)$ is uniquely determined modulo $S^{-\infty}$.
Lemma 2.3. If \( a \in S^m \) then \( A \) extends a continuous operator \( A: H^s(\mathbb{R}^N, \mathbb{C}^d) \rightarrow H^{s-m}_{\text{loc}}(\Omega, \mathbb{C}^d) \). In particular, if \( m < 0 \) then \( A: L^2(\mathbb{R}^n, \mathbb{C}^d) \rightarrow L^2_{\text{loc}}(\Omega, \mathbb{C}^d) \) is compact.

We will work with a more restricted class of pseudo-differential operators, the so-called polyhomogeneous operators. To motivate the next definition, observe that if \( a \in C^\infty(\Omega \times \mathbb{R}^N) \) satisfies \( a(x, t\xi) = t^m a(x, \xi) \) for all \( t, |\xi| \geq 1 \), then \( a \in S^m \). Such functions are said to be positively \( m \)-homogeneous in \( \xi \) for \( |\xi| \geq 1 \).

Definition 2.4. A symbol \( a \in S^m \) is called polyhomogeneous if

\[
a \sim \sum_{j=0}^{\infty} a_{m-j} \quad \text{in } S^m,
\]

where \( a_{m-j} \in C^\infty(\Omega \times \mathbb{R}^N) \) is positively \((m-j)\)-homogeneous in \( \xi \) for \( |\xi| \geq 1 \). The term \( a_m \) is called the principal symbol and is denoted by \( \sigma^m(A) \).

The space of pseudo-differential operators with polyhomogeneous symbols in \( S^m(\Omega, \mathbb{C}^{d\times d}) \) is denoted by \( \Psi^m_d(\Omega) \); if their symbols are compactly supported in \( x \), we write \( \Psi^m_{d,c}(\Omega) \).

Lemma 2.5. Take \( P \in \Psi^l_d(\Omega) \) and \( Q \in \Psi^m_d(\Omega) \). Writing \( D \equiv \frac{1}{i} \partial \), we have the formulae

\[
\sigma(P^*) \sim \sum_{\alpha \in \mathbb{N}^n_0} \frac{1}{\alpha!} \partial_\xi^\alpha \partial_x^\alpha \sigma(P)^* \quad \text{in } S^m, \quad \sigma(PQ) \sim \sum_{\alpha \in \mathbb{N}^n_0} \frac{1}{\alpha!} \partial_\xi^\alpha \sigma(P) \partial_\xi^\alpha \sigma(Q) \quad \text{in } S^m.
\]

Thus, if \( [\sigma(P), \sigma(Q)] = 0 \), then \( [P, Q] \in \Psi^{l+m-1}_d(\Omega) \) with \( \sigma^{l+m-1}([P, Q]) = \frac{1}{i} \{\sigma^l(P), \sigma^m(Q)\} \).

Here, and in the sequel, \( [p, q] \equiv pq - qp \) and \( \{p, q\} \) denotes the Poisson bracket, that is,

\[
\{p, q\} \equiv \frac{\partial p}{\partial \xi_j} \frac{\partial q}{\partial x^j} - \frac{\partial p}{\partial x^j} \frac{\partial q}{\partial \xi_j}.
\]

Theorem 2.6. (Calderón Commutator) Let \( P \in \Psi^1_d(\mathbb{R}^N) \) and let \( a(x) \) be a Lipschitz function. Then, for any \( 1 < p < \infty \), \( [P, a]: L^p(\mathbb{R}^N) \rightarrow L^p(\mathbb{R}^N) \) is bounded and

\[
\| [P, a]f \|_{L^p} \leq C_p \| \nabla a \|_{L^\infty} \| f \|_{L^p}.
\]

Conversely, if \( [P, a]: L^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N) \) is bounded for \( P = \partial_{x_j}, j = 1, \ldots, N \), then \( a \) is Lipschitz.

We refer the reader to [42] for a proof of Theorem 2.6.
2.2. Existence and Properties of H-Measures

In this subsection we recall the definition of H-measures, as well as a few useful properties they possess. H-measures were introduced independently by Tartar [53,55] and Gérard [21], who called them microlocal defect measures. Here we adopt Tartar’s terminology and refer the reader to [55] for further details.

**Theorem 2.7. (Existence of H-measures)** Let \( v_\varepsilon \rightharpoonup v \) in \( L^2(\Omega, \mathbb{C}^d) \). Up to a subsequence, there are Radon measures \( \mu_{\alpha\beta} \), \( \alpha, \beta = 1, \ldots, d \), such that

\[
\mu_{\alpha\beta} = \mu_{\beta\alpha}, \quad \mu_{\alpha\beta} \xi^\alpha \bar{\xi}^\beta \geq 0 \quad \text{for all } \xi \in \mathbb{C}^d
\]

and, for any \( A \in \Psi^0_{d,c}(\Omega) \), we have

\[
\lim_{\varepsilon \to 0} \langle A(v_\varepsilon - v), v_\varepsilon - v \rangle = \lim_{\varepsilon \to 0} \int_{\Omega} A(v_\varepsilon - v) \cdot v_\varepsilon - v \, dx = \int_{S^*\Omega} \sigma^0(A)_{\alpha\beta} \, d\mu_{\alpha\beta} = \langle \mu, \sigma^0(A) \rangle. \tag{2.1}
\]

The matrix-valued measure \( \mu = (\mu_{\alpha\beta})_{\alpha,\beta} \) is called the H-measure associated with \( (v_\varepsilon) \).

In Theorem 2.7, as usual, \( S^*\Omega \equiv \Omega \times S^{N-1} \) denotes the cosphere bundle over \( \Omega \) and \( \cdot \) denotes the Euclidean inner product. Here, and in the rest of the paper, we will always write \( \langle f, g \rangle \equiv \int_{\Omega} f \bar{g} \, dx \) whenever this integral is meaningful.

**Remark 2.8.** The Stone–Weierstrass Theorem and a standard density argument show that it suffices to test (2.1) with symbols of the form \( \sigma^0(A)(x, \xi) = b(x)m(\xi) \), see also [17, Remark 2.7].

The following lemma, although simple, describes a very important property of H-measures.

**Lemma 2.9. (Localization property)** Let \( (v_\varepsilon) \) be a sequence such that \( v_\varepsilon \rightharpoonup v \) in \( L^2(\Omega, \mathbb{C}^d) \) and let \( \mu \) be its H-measure. Given \( P \in \Psi^m_{d,c}(\Omega) \), we have

\( (Pv_\varepsilon) \) is compact in \( H^{-m}_{loc} \iff \sigma^m(P)\mu = 0 \).

To conclude this subsection we define a way of generating, in a non-canonical fashion, an H-measure for a sequence that converges only locally in \( L^2 \).

**Definition 2.10.** By passing to a subsequence, \( v_\varepsilon \rightharpoonup v \) in \( L^2_{loc}(\Omega, \mathbb{C}^d) \) generates an H-measure \( \mu \),

\[
v_\varepsilon \rightharpoonup \mu,
\]

as follows: let \( (K_i)_{i=1}^\infty \) be a compact exhaustion of \( \Omega \) and let \( \chi_i \in C_c^\infty(K_{i+1}, [0, 1]) \) be such that \( \chi_i = 1 \) on \( K_i \). Consider a sequence of Radon measures \( (\mu_i) \) constructed as follows: \( \mu_1 \) is the H-measure generated by a subsequence \( (\chi_1 v_\varepsilon')_{\varepsilon'} \) of \( (\chi_1 v_\varepsilon)_\varepsilon \), \( \mu_2 \) is the H-measure generated by a subsequence of \( (\chi_2 v_\varepsilon')_{\varepsilon'} \), and so on. We define \( \mu \) through its action on \( \varphi \in C_c(S^*\Omega) \): let \( i \) be such that \( \text{supp } \varphi \subset S^*K_i \) and set \( \langle \mu, \varphi \rangle = \langle \mu_i, \varphi \rangle \). It is easy to see that \( \mu \) is well-defined.
2.3. Compensated Compactness

The next theorem, which is due to Robbin–Rogers–Temple [49] and generalizes an earlier result of Murat and Tartar [44], is the main compensated compactness result that we will use.

**Theorem 2.11. (Generalized div-curl lemma)** Let \( p_1, p_2 \in (1, \infty) \) be such that \( \frac{1}{p_1} + \frac{1}{p_2} = 1 \). For differential forms \( \omega_{i,e} \) over \( \Omega \) of degree \( k_i, i = 1, 2 \), such that \( k_1 + k_2 \leq N \),

\[
\omega_{i,e} \rightharpoonup \omega_i \text{ in } L^{p_i}_{\text{loc}}(\Omega) \\
d\omega_{i,e} \text{ is compact in } W^{-1,p_i}_{\text{loc}}(\Omega) \\
\implies \omega_{1,e} \wedge \omega_{2,e} \rightharpoonup \omega_1 \wedge \omega_2 \text{ in } \mathcal{D}'(\Omega).
\]

The case \( p = q = 2 \) can be proved easily using H-measures, but for the general case one needs to use the Hörmander–Mihlin multiplier theorem, which is applicable since the differential constraint in Theorem 2.11 has constant rank [45]. We refer the reader to [26,46] for characterizations of constant rank operators and to [27,28] for generalizations of Theorem 2.11 to this setting.

The \( L^p \)-theory of compensated compactness, even in the bilinear setting, is extremely useful to deal with higher-order nonlinearities, and in fact Theorem 2.11 extends straightforwardly to the general multilinear setting. However, it is worthwhile noting that the \( L^p \)-theory in the non-constant rank case is still poorly understood. The classical wave operator \( \Box \equiv -\partial_{tt} + \Delta_e \), if rewritten as a first-order system, is an important example of such an operator but, due to the particular structure of \( \Box \), Theorem 2.11 will be enough for our purposes.

3. The Linear Covariant Wave Equation

This section is concerned with a linear covariant wave equation

\[
\Box_g u = f, \quad u, f : \mathcal{M} \to \mathbb{R},
\]

where \( g \) is a smooth Lorentzian metric on an open domain \( \mathcal{M} \subset \mathbb{R}^{1+n} \). Recall that

\[
\Box_g u \equiv \frac{1}{\sqrt{|g|}} \partial_\alpha \left( \sqrt{|g|} g^{\alpha\beta} \partial_\beta u \right) = \nabla^\alpha \nabla_\alpha u,
\]

where \( g^{\alpha\beta} \equiv (g^{-1})^{\alpha\beta} \), \( |g| \equiv |\det g| \) and \( \nabla^\alpha \) is the covariant derivative with respect to \( g \). We will also write \( d\text{Vol}_g \equiv \sqrt{|g|} \, dx \) for the volume form induced by \( g \).

It will be convenient to work with a diagonalized form of the wave operator. To this end, define

\[
\beta^i \equiv -\frac{g^{0i}}{g^{00}}, \quad e_0 \equiv \partial_0 - \beta^i \partial_i, \quad g^{ij} \equiv g^{ij} - \frac{g^{0i} g^{0j}}{g^{00}},
\]

The symbol of the timelike vector field \( e_0 \) appears naturally in relation to the zero mass shell of \( g \); indeed,

\[
g^{\alpha\beta} \xi_\alpha \xi_\beta = g^{00}(\xi_0 - \beta^k \xi_k)^2 + g^{ij} \xi_i \xi_j.
\]
In order to use Stokes’ theorem, we define some useful geometric quantities associated with the covariant wave operator. Given functions $u_1, u_2: \mathcal{M} \to \mathbb{R}$ and a smooth vector field $X$ on $\mathcal{M}$, let us write

$$T_{\alpha\beta}[u_1, u_2] \equiv \partial_\alpha u_1 \partial_\beta u_2 - \frac{1}{2} g_{\alpha\beta} g^{\mu\nu} \partial_\mu u_1 \partial_\nu u_2,$$

$$J_X^\alpha[u_1, u_2] \equiv \frac{1}{2} \left[ X u_1 \partial_\alpha u_2 + X u_2 \partial_\alpha u_1 - X_\alpha g^{-1}(du_1, du_2) \right].$$

(3.5)

The energy-momentum tensor $T$ and the associated current $J_X^\alpha$ are related by the energy identity

$$\nabla^\alpha J_X^\alpha[u_1, u_2] = \frac{1}{2} (X u_1 \Box_g u_2 + X u_2 \Box_g u_1) + T_{\alpha\beta}[u_1, u_2] \nabla^\alpha X^\beta.$$  

(3.6)

When $u_1 = u_2 = u$ we recover the standard energy identity, see e.g. [1,15] for further details.

In this section we study the limiting behavior of sequences of solutions to (3.1). For the convenience of the reader, we state here a simplified form of Hypotheses 1.6.

**Hypotheses 3.1.** Let $u_\varepsilon, f_\varepsilon: \mathcal{M} \to \mathbb{R}$ be sequences such that $(u_\varepsilon, f_\varepsilon)$ satisfy, for each $\varepsilon > 0$, the linear wave equation (3.1). We consider the following regularity conditions:

(a) $g$ is smooth;
(b) $u_\varepsilon \rightharpoonup u$ in $W^{1,2}_{\text{loc}}(\mathcal{M})$;
(c) $f_\varepsilon \rightharpoonup f$ in $L^2_{\text{loc}}(\mathcal{M})$.

According to Definition 2.10 and Hypotheses 3.1, we may pass to a subsequence so that

$$(\partial_0 u_\varepsilon, \partial_1 u_\varepsilon, \ldots, \partial_n u_\varepsilon, f_\varepsilon)_\varepsilon \rightharpoonup \begin{bmatrix} \tilde{\nu} & \tilde{\lambda} \end{bmatrix}_{\tilde{\mu}}$$

(3.7)

where $\tilde{\nu}$ is a $\mathbb{C}^{(n+1)\times(n+1)}$-valued measure, generated by $(\partial_0 u_\varepsilon, \ldots, \partial_n u_\varepsilon)$, and $\tilde{\lambda}$ is $\mathbb{C}^{n+1}$-valued.

### 3.1. The H-Measure and Its Properties

We are now ready to state the main result of this section, which describes the structure, support and propagation properties of the H-measure defined in (3.7).

**Theorem 3.2.** Let $(u_\varepsilon, f_\varepsilon)$ satisfy Hypotheses 3.1 and define $\tilde{\nu}$ and $\tilde{\lambda}$ as in (3.7). Then

(a) **Limit equation.** $(u, f)$ satisfy (3.1) in the sense of distributions.
(b) **Energy density.** There are Radon measures $\nu$ and $\lambda$ on $S^*\mathcal{M}$ such that $\tilde{\nu}_{\alpha\beta} = \xi_\alpha \xi_\beta \nu$ and $\tilde{\lambda}_\gamma = \xi_\gamma \lambda$. Furthermore, $\nu$ and $\lambda$ satisfy the following conditions:
(b0) Parity: \( \nu \) is even and \( \lambda \) is odd, i.e. \( \langle \nu, \bar{a} \rangle = 0 \) for any \( \bar{a} \in C_c^\infty(S^*_M) \) which is odd in \( \xi \), and likewise for \( \lambda \).

(b1) Support property: for all \( \varphi \in C_c^\infty(M) \), \( \nu \) and \( \lambda \) satisfy
\[
\langle \nu, \varphi(x)g^{\alpha\beta}(x)\xi_\alpha\xi_\beta \rangle = 0, \quad \langle \lambda, \varphi(x)g^{\alpha\beta}(x)\xi_\alpha\xi_\beta \rangle = 0.
\]

(b2) Propagation property: for all \( \tilde{a}(x, \xi) \in C_c^\infty(S^*_M) \), though of as positively \( 1 \)-homogeneous functions in \( \xi \), the measure \( \nu \) satisfies
\[
\langle \nu, \{ g^{\alpha\beta}(x)\xi_\alpha\xi_\beta, \tilde{a} \} \rangle = -2\Re \langle \lambda, \tilde{a} \rangle.
\]

Theorem 3.2 follows by standard methods, and similar statements have appeared in [53, Theorem 3.12] and [5,17]. Comparing with these works, the main novelty here is that our proof holds for a general covariant wave operator where, unlike in these references, the coefficients of the operator are allowed to depend both on \( x \).

Before proceeding with the core of the proof, we show that we may assume that the convergence in Hypotheses 3.1 is global and not just local.

**Reduction to compact supports.** Let \( \chi \in C_c^\infty(M) \) satisfy \( \chi = 1 \) on a compact set \( K \). Then
\[
\square_g(\tilde{u}_\varepsilon) \equiv \square_g(\chi u_\varepsilon) = f_\varepsilon + 2g^{-1}(du_\varepsilon, d\chi) + u_\varepsilon \square_g(\chi) \equiv f_\varepsilon.
\]

Suppose that, for every such \( \chi \), the conclusion of Theorem 3.2 holds, with \( \tilde{\nu} \) and \( \tilde{\lambda} \) being now the H-measures generated according to (3.7), but with \( u_\varepsilon \) replaced with \( \tilde{u}_\varepsilon \) and \( f_\varepsilon \) replaced with \( \tilde{f}_\varepsilon \). Since \( (u_\varepsilon, f_\varepsilon) = (\tilde{u}_\varepsilon, \tilde{f}_\varepsilon) \) on \( K \), it is then clear, recalling Definition 2.10, that the original H-measure generated by \( (u_\varepsilon, f_\varepsilon) \) also satisfies the conclusion of Theorem 3.2.

Thus, from now onwards, we assume that the sequence \( (u_\varepsilon, f_\varepsilon) \varepsilon \) has uniformly bounded support.

**Proof of Theorem 3.2 (a,b0,b1).** Part (a) follows from the divergence structure of \( \square_g \), see Proposition 4.1 for a more general statement.

Noting that \( D_\alpha \partial_\beta u_\varepsilon = D_\beta \partial_\alpha u_\varepsilon \), Lemma 2.9 yields \( \xi_\varepsilon \tilde{\nu}_\varepsilon \gamma = \xi_\varepsilon \tilde{\nu}_\alpha \gamma \). It follows that \( \tilde{\nu}_\alpha\beta = \xi_\alpha \rho_\beta \) for some \( \mathbb{C}^d \)-valued Radon measure \( \rho \). Since \( \mu \) is Hermitian and non-negative, we must have \( \rho = \xi \nu \) for another non-negative Radon measure \( \nu \).

The support property of \( \nu \) in (b1) follows by applying again Lemma 2.9: since \( \square_g u_\varepsilon = f_\varepsilon \), by Hypotheses 3.1(c) we see that the sequence of vector fields \( (\sqrt{|g|}g^{\alpha\beta}v_\beta, \varepsilon) \alpha \) has a divergence which is compact in \( H^{-1}_{loc} \) and so \( g^{\alpha\beta}\xi_\alpha \xi_\beta v = 0 \). In turn, the support of \( \lambda \) is contained in the support of \( \nu \). Indeed, from (3.7) and the basic properties of H-measures, for any measurable set \( E \subset S^*M \),
\[
M \equiv \begin{bmatrix} \tilde{\nu}(E) & \tilde{\lambda}(E) \\ \tilde{\lambda}^*(E) & \mu(E) \end{bmatrix}
\]
is a positive semi-definite matrix and \( \mu(E) \geq 0 \), hence \( \mu(E) \geq 0 \implies \tilde{\lambda}(E) = 0 \).
To prove part (b0) we consider a real symbol $a(x, \xi) = b(x)m(\xi)$; the general case follows according to Remark 2.8. Suppose that $m$ is odd: then, using Plancherel’s identity,

$$\langle Ae_0u_\varepsilon, e_0u_\varepsilon \rangle = \int \int \hat{b}(\xi - \eta)m(\eta)\overline{\hat{e}_0u_\varepsilon}(\eta)\overline{\hat{e}_0u_\varepsilon}(\xi) d\xi d\eta = -\langle Ae_0u_\varepsilon, e_0u_\varepsilon \rangle,$$

where in the last line we made the change of variables $(\xi, \eta) \mapsto - (\xi, \eta)$, use the fact that $m$ is odd and that all functions are real. Hence

$$\langle \nu, (\xi_0 - \beta^i \xi_i)^2 a \rangle = \lim_{\varepsilon \to 0} \langle Ae_0u_\varepsilon, e_0u_\varepsilon \rangle = 0.$$

Note that, by (3.4), $\xi_0 - \beta^i \xi_i$ never vanishes on the zero mass shell \{ $g^{\alpha\beta} \xi_\alpha \xi_\beta = 0$ \} where, according to (b1), $\nu$ is supported. Hence we have shown that $\langle \nu, a \rangle = 0$ whenever $a$ is odd in $\xi$. An identical argument for $\lambda$, which is also supported in the zero mass shell, concludes the proof.

The proof of part (b2) is more involved but follows essentially the outline of [53, Theorem 3.12]. The crucial technical ingredient is contained in the following lemma:

**Lemma 3.3.** Let $g$ be a smooth Lorentzian metric and take $A \in \Psi^0_{1,c}$. Then $[\Box_g, A] \in \Psi^{1/2}$ and

$$\sigma^1([\Box_g, A]) = \sigma^1(i P^\alpha D_\alpha) = \sigma^1(P^\alpha \partial_\alpha),$$

where $P^\alpha \in \Psi^0_{1,c}$ is such that

$$\sigma^0(P^\alpha) \equiv 2g^{\alpha\beta} \partial_\alpha a - \partial_{\nu\mu} g^{\alpha\beta} \xi_\beta \partial_{\xi^\nu} a.$$

Since $g$ is assumed to be smooth, Lemma 3.3 follows at once from the last part of Lemma 2.5. Nonetheless, the result still holds if $g \in C^1$, although this is much more difficult:

**Remark 3.4.** The Calderón Commutator (Theorem 2.6) shows that $[\Box_g, A] : H^1 \to L^2$ is bounded, even when $g$ is just $C^1$, but this assumption cannot be substantially weakened, c.f. [55, pages 336-337] and [14, 57].

**Proof of Theorem 3.2(b2).** Let us take $A \in \Psi^0_{1,c}$ to be a multiplier, so $a(x, \xi) \equiv m(\xi)$. We begin by applying $A$ and $\overline{A}$ to (3.1) to get, respectively,

$$\Box_g(Au_\varepsilon) = Af_\varepsilon + [\Box_g, A]u_\varepsilon, \quad \Box_g(Au_\varepsilon) = \overline{Af_\varepsilon} + [\Box_g, A]u_\varepsilon. \tag{3.8}$$

Given a smooth vector field $X$, we multiply the first equation by $X(Au_\varepsilon)$, the second equation by $X(Au_\varepsilon)$, and sum the two. Using the energy identity (3.6) we get

$$\nabla^\alpha J^{X}_\alpha[Au_\varepsilon, \overline{Au_\varepsilon}] - T_{\alpha\beta}[Au_\varepsilon, \overline{Au_\varepsilon}] \nabla^\alpha X^\beta$$

$$= \frac{1}{2} [X(Au_\varepsilon)Af_\varepsilon + X(Au_\varepsilon)\overline{Af_\varepsilon}] + \frac{1}{2} [X(Au_\varepsilon)[\Box_g, A]u_\varepsilon + X(Au_\varepsilon)[\Box_g, A]u_\varepsilon]. \tag{3.9}$$
Now let \( \varphi \in C^\infty_c(\mathbb{R}^{1+n}) \) and integrate (3.9) against \( \varphi \) with respect to \( \text{dVol}_g \). We deal with each of the corresponding terms separately.

**Step 1:** The left hand side of (3.9). For the first term, we integrate by parts and recall (3.5):

\[
\begin{align*}
\int \nabla^\alpha J^X_\alpha\left[ A u_\varepsilon, A u_\varepsilon \right] \varphi \, \text{dVol}_g &= -\frac{1}{2} \int \left( X(A u_\varepsilon) \partial_\alpha(A u_\varepsilon) + X(\overline{A u_\varepsilon}) \partial_\alpha(A u_\varepsilon) \right) \nabla^\alpha \varphi \, \text{dVol}_g \\
&\quad - \frac{1}{2} \int X_\alpha g^{\beta \gamma} \partial_\beta(A u_\varepsilon) \partial_\gamma(\overline{A u_\varepsilon}) \nabla^\alpha \varphi \, \text{dVol}_g.
\end{align*}
\]

Using the fact that \( A \) is a multiplier and that \( \sigma^0(A^*) = \sigma^0(A)* \), we have

\[
\begin{align*}
\lim_{\varepsilon \to 0} \int X(A u_\varepsilon) \partial_\alpha(A u_\varepsilon) \nabla^\alpha \varphi \, \text{dVol}_g &= \lim_{\varepsilon \to 0} \int A^* \left( A(\partial_\beta u_\varepsilon) X^\beta \nabla^\alpha \varphi \sqrt{|g|} \right) \partial_\alpha u_\varepsilon \, \text{d}x \\
&= \lim_{\varepsilon \to 0} \int A^* A(\partial_\beta u_\varepsilon) \partial_\alpha u_\varepsilon X^\beta \nabla^\alpha \varphi \sqrt{|g|} \, \text{d}x \\
&= \langle \tilde{v}_{\alpha \beta}, |m(\xi)|^2 X^\beta \nabla^\alpha \varphi \sqrt{|g|} \rangle = \langle v, g^{\alpha \gamma} \xi_\alpha \xi_\beta |m(\xi)|^2 X^\beta \nabla^\alpha \varphi \sqrt{|g|} \rangle,
\end{align*}
\]

where we also used the fact that \([A^*, X^\beta \nabla^\alpha \varphi \sqrt{|g|}] : L^2 \to L^2 \) is compact, c.f. Lemma 2.3. The second term on the right-hand side of (3.10) is treated identically and has the same limit. Finally, the last term on the right-hand side of (3.10) vanishes in the limit: indeed, arguing as before,

\[
\lim_{\varepsilon \to 0} \int X_\alpha g^{\beta \gamma} \partial_\beta(A u_\varepsilon) \partial_\gamma(\overline{A u_\varepsilon}) \nabla^\alpha \varphi \, \text{dVol}_g = \langle v, X_\alpha g^{\beta \gamma} \xi_\beta \xi_\gamma |m(\xi)|^2 \nabla^\alpha \varphi \sqrt{|g|} \rangle = 0,
\]

using the support condition on \( v \).

For the second term in (3.9), similar arguments yield

\[
\begin{align*}
\lim_{\varepsilon \to 0} \int T_{\alpha \beta}[A u_\varepsilon, A u_\varepsilon] \nabla^\alpha X^\beta \varphi \, \text{dVol}_g &= \langle v, (\xi_\alpha \xi_\beta \nabla^\alpha X^\beta - \frac{1}{2} g_{\alpha \beta} g^{\mu \nu} \xi_\mu \xi_\nu) |m(\xi)|^2 \varphi \sqrt{|g|} \rangle \\
&= \langle v, g^{\alpha \gamma} \xi_\alpha \xi_\beta \nabla^\gamma X^\beta |m(\xi)|^2 \varphi \sqrt{|g|} \rangle.
\end{align*}
\]

Setting \( \Phi^\beta(x, \xi) \equiv X^\beta |m(\xi)|^2 \varphi(x) \sqrt{|g|} \), \( \Phi \equiv \xi_\beta \Phi^\beta \), and using the fact that \( \nabla^\gamma g = 0 \), we have calculated the limit of the left-hand side of (3.9):

\[
\lim_{\varepsilon \to 0} \int \left( \nabla^\alpha J^X_\alpha[A u_\varepsilon, A u_\varepsilon] - T_{\alpha \beta}[A u_\varepsilon, A u_\varepsilon] \nabla^\alpha X^\beta \right) \varphi \, \text{dVol}_g = -\langle v, g^{\alpha \gamma} \xi_\alpha \xi_\beta \nabla^\gamma \Phi^\beta \rangle.
\]

**Step 2:** the right hand side of (3.9). For the first term we have

\[
\begin{align*}
\lim_{\varepsilon \to 0} \frac{1}{2} \int \left( X(\overline{A u_\varepsilon}) A f_\varepsilon + X(A u_\varepsilon) \overline{A f_\varepsilon} \right) \varphi \, \text{dVol}_g &= \lim_{\varepsilon \to 0} \frac{1}{2} \int \left( X u_\varepsilon A^* A f_\varepsilon + X(A^* A u_\varepsilon) f_\varepsilon \right) \varphi \, \text{dVol}_g \\
&= \frac{1}{2} \langle X^\beta (\lambda_\beta + \tilde{\lambda}_\beta^*), |m(\xi)|^2 \varphi \sqrt{|g|} \rangle = \langle \Re \lambda, \Phi \rangle.
\end{align*}
\]
According to Lemma 3.3, the last term yields
\[
\lim_{\varepsilon \to 0} \frac{1}{2} \int \left( X(Au_\varepsilon) [\Box_g, A]u_\varepsilon + X(Au_\varepsilon)[\Box_g, A]u_\varepsilon \right) \varphi \, d\text{Vol}_g \\
= \lim_{\varepsilon \to 0} -\langle v, \partial_{\xi^\mu} g^{\alpha \gamma} \xi_\alpha \xi_\gamma \xi_\beta X^\beta (m \partial_{\xi^\mu} \overline{m} + \overline{m} \partial_{\xi^\mu} m) \varphi \sqrt{|g|} \rangle \\
= -\frac{1}{2} \langle v, \partial_{\xi^\mu} g^{\alpha \gamma} \xi_\alpha \xi_\gamma \xi_\beta \partial_{\xi^\mu} \Phi^\beta \rangle.
\]

**Step 3:** putting everything together. Combining the last three computations we find that
\[
\langle v, -g^{\alpha \gamma} \xi_\alpha (\xi_\beta \nabla_{x^\gamma} \Phi^\beta) + \frac{1}{2} \partial_{x^\mu} g^{\alpha \gamma} \xi_\alpha \xi_\gamma (\xi_\beta \partial_{\xi^\mu} \Phi^\beta) \rangle = \langle \mathfrak{M} \lambda, \xi_0 \Phi \rangle.
\]
The left-hand side can be simplified further: note that, as $\nabla_\mu$ is the Levi-Civita connection,
\[
0 = \nabla_\mu g^{\alpha \gamma} = \partial_{x^\mu} g^{\alpha \gamma} + \delta_\mu^\beta \Gamma^\beta_{\mu \delta} g^{\delta \gamma} + \delta_\gamma^\beta \Gamma^\beta_{\mu \delta} g^{\alpha \delta}
\implies \frac{1}{2} \partial_{x^\mu} g^{\alpha \gamma} \xi_\alpha \xi_\gamma + g^{\alpha \gamma} \xi_\alpha \xi_\beta \Gamma^\beta_{\gamma \mu} = 0.
\]
Combining this identity with the two equations
\[
\partial_{\xi^\mu} \Phi = \xi_\beta \partial_{\xi^\mu} \Phi^\beta + \Phi^\mu, \quad \partial_{x^\gamma} \Phi = \xi_\beta \nabla_{x^\gamma} \Phi^\beta - \Gamma^\beta_{\gamma \mu} \Phi^\mu \xi_\beta
\]
we find that
\[
-g^{\alpha \gamma} \xi_\alpha (\xi_\beta \nabla_{x^\gamma} \Phi^\beta) + \frac{1}{2} \partial_{x^\mu} g^{\alpha \gamma} \xi_\alpha \xi_\gamma (\xi_\beta \partial_{\xi^\mu} \Phi^\beta) = -g^{\alpha \gamma} \xi_\alpha \partial_{x^\gamma} \Phi + \frac{1}{2} \partial_{x^\mu} g^{\alpha \gamma} \xi_\alpha \xi_\gamma \partial_{\xi^\mu} \Phi = -\frac{1}{2} \{ g^{\alpha \beta} \xi_\alpha \xi_\beta, \Phi \}.
\]

While the previous calculations hold for an arbitrary vector field $X$, we now take $X = e_0$, so that $X^\beta \xi_\beta = \xi_0 - \beta^i \xi_i$. As before we note that $\xi_0 - \beta^i \xi_i$ never vanishes on the zero mass shell, where $v$ is supported. Hence, we have shown that part (b2) of the theorem holds whenever $\tilde{a}$ is of the form $\tilde{a}(x, \xi) = \Phi(x, \xi) = b(x)q(\xi)$ with $q$ real and positively 1-homogeneous. The case of a general test function follows by considerations analogous to the ones in Remark 2.8.

### 3.2. Two Compensated Compactness Lemmas

This subsection contains two compensated compactness results for solutions of the wave system (3.1) which follow readily from the very classical Theorem 2.11. We begin with a *bilinear* result.
Lemma 3.5. Null forms are weakly continuous, i.e.

\[ u^I_{\varepsilon} \to u^I \text{ in } W^{1,2}_{\text{loc}} \]
\[ (\Box g u^I_{\varepsilon})_{\varepsilon} \text{ is compact in } W^{-1,2}_{\text{loc}} \]

\[ \implies g^{-1}(du^1_{\varepsilon}, du^2_{\varepsilon}) \to g^{-1}(du^1, du^2) \text{ in } \mathcal{D}' \]

Proof. It suffices to consider the case \( u^1_{\varepsilon} = u^2_{\varepsilon} \); indeed, one can use the polarization identity

\[ g^{-1}(du^1_{\varepsilon}, du^1_{\varepsilon}) + 2g^{-1}(du^1_{\varepsilon}, du^2_{\varepsilon}) + g^{-1}(du^2_{\varepsilon}, du^2_{\varepsilon}) = g^{-1}(du^1_{\varepsilon} + du^2_{\varepsilon}, du^1_{\varepsilon} + du^2_{\varepsilon}) \]

and pass to the limit on both sides to see that \( g^{-1}(du^1_{\varepsilon}, du^2_{\varepsilon}) \to g^{-1}(du^1, du^2) \) in the sense of distributions. We thus drop all superscripts from the sequences.

Let \( \star \) be the Hodge star with respect to the metric \( g^{-1} \). We have

\[ g^{-1}(du_{\varepsilon}, du_{\varepsilon}) \, d\text{Vol}_{g^{-1}} = du_{\varepsilon} \wedge (\star du_{\varepsilon}) \]

and, since \( u_{\varepsilon} \) is scalar, \( \Box g u_{\varepsilon} = \star d \star du_{\varepsilon} \). The conclusion follows from Theorem 2.11. \( \square \)

The next result is trilinear and was essentially known to Tartar: see [54, Lemma I.5], where it is proved when \( g \) is the Minkowski metric. The proof given below is the natural adaptation of Tartar’s proof, now in the language of geometric wave equations introduced at the beginning of the section. See also [34, Proposition 12.2] for an alternative proof.

Lemma 3.6. Let \( X \) be a smooth vector field. Then

\[ u^I_{\varepsilon} \to 0 \text{ in } W^{1,3}_{\text{loc}} \]
\[ (\Box g u^I_{\varepsilon})_{\varepsilon} \text{ is bounded in } L^3_{\text{loc}} \]

\[ \implies Xu^1_{\varepsilon} g^{-1}(du^2_{\varepsilon}, du^3_{\varepsilon}) \to 0 \text{ in } \mathcal{D}' \]

Proof. The assumptions imply that the sequence \( J^X_{\alpha}[u^1_{\varepsilon}, u^2_{\varepsilon}] \) is bounded in \( L^{3/2}_{\text{loc}} \) and, recalling (3.6), that \( \nabla^\alpha J^X_{\alpha}[u^1_{\varepsilon}, u^2_{\varepsilon}] \) is compact in \( W^{-1,3/2}_{\text{loc}} \). We note that

\[ 2J^X_{\alpha}[u^1, u^2] \partial_\beta u^3 g^{\alpha\beta} = Xu^1 g^{-1}(du^2, du^3) + Xu^2 g^{-1}(du^1, du^3) - Xu^3 g^{-1}(du^1, du^2), \]

where the left-hand side is a div-curl product. Using the polarization identity, as in Lemma 3.5, to prove the conclusion we can take \( u^2_{\varepsilon} = u^3_{\varepsilon} \) without loss of generality. Thus

\[ 2J^X_{\alpha}[u^1_{\varepsilon}, u^2_{\varepsilon}] \partial_\beta u^2_{\varepsilon} g^{\alpha\beta} = Xu^1_{\varepsilon} g^{-1}(du^2_{\varepsilon}, du^2_{\varepsilon}) \]

or, equivalently, writing again \( \star \) for the Hodge star with respect to \( g^{-1} \),

\[ Xu^1_{\varepsilon} g^{-1}(du^2_{\varepsilon}, du^2_{\varepsilon}) \, d\text{Vol}_{g^{-1}} = g^{-1}(2J^X_{\alpha}[u^1_{\varepsilon}, u^2_{\varepsilon}], du^2_{\varepsilon}) \, d\text{Vol}_{g^{-1}} = 2J^X[u^1_{\varepsilon}, u^2_{\varepsilon}] \wedge \star du^2_{\varepsilon}. \]

Since \( du^2_{\varepsilon} \to 0 \) in \( L^3_{\text{loc}} \), we can again use Theorem 2.11 to pass to the limit. \( \square \)
Remark 3.7. Taking \( u_1^\varepsilon = u_2^\varepsilon = u_3^\varepsilon \) in Lemma 3.6, we note that the trilinear quantity is weakly continuous solely at zero. That this happens is only possible because \( \Box_g \), thought of as a first-order operator acting on \( d\varepsilon \), does not have constant rank. Indeed, it is shown in [28] that, under constant rank constraints, nonlinearities which are weakly continuous at a point are necessarily weakly continuous everywhere. Furthermore, regardless of rank conditions, nonlinearities which are weakly continuous everywhere are polynomials with degree not exceeding the dimension of the domain, i.e. \( n + 1 \), see also [45]. In contrast, Lemma 3.6 is of course valid even when \( n = 1 \). See also [37,43] for other trilinear Compensated Compactness results without constant rank assumptions.

4. The Linear Covariant Wave Equation with Oscillating Coefficients

This section is devoted to the proof of Theorem 3. Our strategy is to reduce the analysis of the limiting behavior of sequences of solutions to
\[
\Box_g u^\varepsilon = f^\varepsilon, \quad u^\varepsilon, f^\varepsilon : (0, T) \times \mathbb{R}^n \to \mathbb{R}
\] (4.1)
to the case where \( (u^\varepsilon, f^\varepsilon) \) are solutions of a fixed wave equation, as in the previous section. Hence, we will frequently recast (4.1) in the form of (3.1), i.e.
\[
\Box_g u^\varepsilon = (\Box_g - \Box_g^\varepsilon) u^\varepsilon + f^\varepsilon,
\] (4.2)
Note that, by Hypotheses 1.6(c) and 1.6(d),
\[
\Box_g u^\varepsilon \text{ is uniformly bounded in } L^4_{\text{loc}}.
\] (4.3)

We begin by noting that part (a) of Theorem 3 poses no difficulty, as the covariant wave operator is an operator in divergence form. For later use, we state the result explicitly.

**Proposition 4.1.** (Limit equation) Let \( (g^\varepsilon, u^\varepsilon, f^\varepsilon) \varepsilon \) be a sequence satisfying Hypotheses 1.6 and solving (4.1). Then \( \Box_g u = f \) in the sense of distributions.

**Proof.** Note that \( \Box_g^\varepsilon u^\varepsilon \rightharpoonup \Box_g u \) in \( \mathcal{D}' \), and hence by Hypotheses 1.6(d) also weakly in \( L^2_{\text{loc}} \). Indeed, take a test function \( \varphi \); then, using the local uniform convergence of \( g^\varepsilon \) and integrating by parts, we find that
\[
\lim_{\varepsilon \to 0} \int \varphi \Box_g^\varepsilon u^\varepsilon \, d\text{Vol}_g = \lim_{\varepsilon \to 0} \int \varphi \Box_g^\varepsilon u^\varepsilon \, d\text{Vol}_g = - \lim_{\varepsilon \to 0} \int \partial_\alpha \varphi \partial_\beta u^\varepsilon g^{\alpha\beta} \, d\text{Vol}_g = - \int \partial_\alpha \varphi \partial_\beta u g^{\alpha\beta} \, d\text{Vol}_g = \int \varphi \Box_g u \, d\text{Vol}_g,
\]
since we have the product of weakly convergent terms with strongly convergent ones. Recall that \( d\text{Vol}_g \equiv \sqrt{|\det g|} \, dx \) and (3.2). As \( \Box_g^\varepsilon u^\varepsilon = f^\varepsilon \rightharpoonup f \) in \( L^2_{\text{loc}} \), by uniqueness of limits we see that \( \Box_g u = f \).
For part (b), our starting point is identity (4.2). We set
\[ h^\alpha_\beta \equiv g^\alpha_\beta - g^\alpha_\beta , \quad H^\varepsilon \equiv (\Box_g - \Box_g) u^\varepsilon ; \]
by (4.3) and Proposition 4.1, \( H^\varepsilon \) converges weakly in \( L^2_{\text{loc}} \) to zero. Besides the H-measures defined in (1.16), we will need the H-measure generated when \( d^\varepsilon u^\varepsilon \) is combined with the right-hand side in (4.2):
\[
(\partial_0 u^\varepsilon, \partial_1 u^\varepsilon, \ldots, \partial_n u^\varepsilon, H^\varepsilon + f^\varepsilon)_{\varepsilon} \xrightarrow{H} \left[ \tilde{\nu} \tilde{\sigma} \tilde{\sigma}^* \star \right].
\]

4.1. Elementary Reductions

Before proceeding with the core of the proof, we make a few basic observations. Firstly, both the structure of the H-measure and the localization part of Theorem 3(b) follow as in Sect. 3 since, by (4.3), \( \Box_g u^\varepsilon \) is bounded in \( L^2_{\text{loc}} \). Likewise, \( \tilde{\sigma}_\gamma = \xi_\gamma \sigma \) for some Radon measure \( \sigma \). Moreover, arguing once more as in Sect. 3, we can and will assume that the sequence \( u^\varepsilon \) is supported on a fixed bounded set \( \Omega_1 \). Hence we can and will also assume that \( g^\varepsilon = g \) for all \( \varepsilon \), outside a neighborhood of \( \Omega_2 \).

The final remark that we make here concerns the parity in \( \xi \) of equation (1.18): according to the parity of \( \nu \) and \( \lambda \), established in Theorem 3.2, we only need to test (1.18) against 1-homogeneous functions \( \tilde{a} \) which are odd in \( \xi \), which corresponds to testing against symbols \( a \) which are 0-homogeneous and even in \( \xi \). In particular, in the rest of the proof we will use implicitly the following straightforward lemma:

**Lemma 4.2.** For \( A \in \Psi^0 \) such that
\[
\sigma^0(A)(x, \xi) \text{ is real and even in } \xi, \quad (4.4)
\]
\( A\varphi \) and \( A^* \varphi \) are real whenever \( \varphi \in L^2 \) is real.

Due to Theorem 3.2 our task is to show that, as \( \varepsilon \to 0 \), \( H^\varepsilon \) does not contribute to the transport equation.

4.2. A Warm-Up: The Case of Strong Convergence of the Metrics

In this section we show that if we knew that \( g^\varepsilon \to g \) strongly in \( W^{1,\infty}_{\text{loc}} \) then Theorem 3 would follow easily. The first step is a reduction to estimating some commutators. The basic idea is to integrate by parts in order to try to distribute the derivatives in such a way that two derivatives do not land on the same term; this cannot be achieved completely, but the remaining terms have a commutator structure.

**Lemma 4.3.** Let \( A \in \Psi^0_{1, c} \) satisfy (4.4). If \( g^\varepsilon \to g \) in \( W^{1,\infty}_{\text{loc}} \) and Hypotheses (1.6) hold, then
\[
\lim_{\varepsilon \to 0} \langle H^\varepsilon, A \partial_\gamma (u^\varepsilon - u) \rangle = \lim_{\varepsilon \to 0} \int \partial_\gamma (u^\varepsilon - u) \left[ A, h^\alpha_\beta \right] \partial_\gamma^2 (u^\varepsilon - u) \, dx.
\]
Proof. Since derivatives of the metric coefficients converge strongly, \( H_\varepsilon = h_\varepsilon^{\alpha\beta} \partial_{\alpha\beta}^2 u_\varepsilon + o_{L^2}(1) \), where \( o_{L^2}(1) \) denotes a remainder which is compact in \( L^2 \). We begin by noting that

\[
\langle h_\varepsilon^{\alpha\beta} \partial_{\alpha\beta}^2 u_\varepsilon, A \partial_\gamma (u_\varepsilon - u) \rangle = \langle h_\varepsilon^{\alpha\beta} \partial_{\alpha\beta}^2 (u_\varepsilon - u), A \partial_\gamma (u_\varepsilon - u) \rangle + \langle h_\varepsilon^{\alpha\beta} \partial_{\alpha\beta}^2 u_\varepsilon, A \partial_\gamma (u_\varepsilon - u) \rangle + o(1) .
\]

Now, we evaluate the remaining term, setting \( w_\varepsilon \equiv u_\varepsilon - u \). First, we integrate by parts in \( \partial_\alpha \):

\[
\langle h_\varepsilon^{\alpha\beta} \partial_{\alpha\beta} w_\varepsilon, A \partial_\gamma w_\varepsilon \rangle = -\langle \partial_\alpha h_\varepsilon^{\alpha\beta} \partial_\beta w_\varepsilon, A \partial_\gamma w_\varepsilon \rangle + \langle h_\varepsilon^{\alpha\beta} \partial_\beta w_\varepsilon, [\partial_\alpha, A] \partial_\gamma w_\varepsilon \rangle
\]

\[
= -\langle h_\varepsilon^{\alpha\beta} \partial_\beta w_\varepsilon, A (\partial_{\alpha\gamma}^2 w_\varepsilon) \rangle + o(1) .
\]

Then, we integrate the remaining term by parts along \( \partial_\gamma \), and obtain

\[
-\langle h_\varepsilon^{\alpha\beta} \partial_\beta w_\varepsilon, A (\partial_{\alpha\gamma} w_\varepsilon) \rangle = -\langle h_\varepsilon^{\alpha\beta} \partial_\beta w_\varepsilon, [A, \partial_\gamma] \partial_\alpha w_\varepsilon \rangle + \langle \partial_\gamma h_\varepsilon^{\alpha\beta} \partial_\beta w_\varepsilon, A \partial_\alpha w_\varepsilon \rangle
\]

\[
+ \langle h_\varepsilon^{\alpha\beta} \partial_\beta \partial_\gamma w_\varepsilon, A \partial_\alpha w_\varepsilon \rangle
\]

\[
= \langle h_\varepsilon^{\alpha\beta} \partial_\beta \partial_\gamma w_\varepsilon, A \partial_\alpha w_\varepsilon \rangle + o(1) .
\]

Finally, integrating the remaining term along \( \partial_\beta \),

\[
\langle h_\varepsilon^{\alpha\beta} \partial_\beta \partial_\gamma w_\varepsilon, A \partial_\alpha w_\varepsilon \rangle = -\langle \partial_\beta h_\varepsilon^{\alpha\beta} \partial_\gamma w_\varepsilon, A \partial_\alpha w_\varepsilon \rangle + \langle h_\varepsilon^{\alpha\beta} \partial_\gamma w_\varepsilon, [\partial_\beta, A] \partial_\alpha w_\varepsilon \rangle
\]

\[
- \langle h_\varepsilon^{\alpha\beta} \partial_\gamma w_\varepsilon, A (\partial_{\alpha\gamma}^2 w_\varepsilon) \rangle
\]

\[
= \langle \partial_\gamma w_\varepsilon, [A, h_\varepsilon^{\alpha\beta}] \partial_{\alpha\beta}^2 w_\varepsilon \rangle
\]

\[
- \langle \partial_\gamma w_\varepsilon, A \left(h_\varepsilon^{\alpha\beta} \partial_{\alpha\beta}^2 w_\varepsilon\right) \rangle + o(1) .
\]

Combining the expressions above yields the identity

\[
\langle h_\varepsilon^{\alpha\beta} \partial_{\alpha\beta}^2 w_\varepsilon, A \partial_\gamma w_\varepsilon \rangle + \left\langle \partial_\gamma w_\varepsilon, A \left(h_\varepsilon^{\alpha\beta} \partial_{\alpha\beta}^2 w_\varepsilon\right) \right\rangle = \langle \partial_\gamma w_\varepsilon, [A, h_\varepsilon^{\alpha\beta}] \partial_{\alpha\beta}^2 w_\varepsilon \rangle + o(1) .
\]

Since \( A \) has real symbol and hence is self-adjoint, up to a compact operator, we conclude the proof.

Due to our strong-convergence assumptions, the Calderón commutator immediately yields:

**Proposition 4.4.** For all \( A \in \Psi_{1,c}^0 \) satisfying (4.4), if \( g_\varepsilon \rightarrow g \) in \( W_{\text{loc}}^{1,\infty} \) and Hypotheses 1.6(b) hold, then

\[
\lim_{\varepsilon \rightarrow 0} \langle H_\varepsilon, A \partial_\gamma (u_\varepsilon - u) \rangle = 0 .
\]

Proof. We need only observe that

\[
[A, h_\varepsilon^{\alpha\beta}] \partial_{\alpha\beta}^2 (u_\varepsilon - u) = [A \partial_\alpha, h_\varepsilon^{\alpha\beta}] \partial_\beta (u_\varepsilon - u) + o_{L^2}(1) .
\]

By Theorem 2.6 and the fact that \( \|\partial_\gamma h_\varepsilon^{\alpha\beta}\|_{L^\infty} \rightarrow 0 \), the \( L^2 \)-norm of the commutator on the right-hand side goes to zero. It now suffices to appeal to Lemma 4.3 and use Hölder’s inequality. □
4.3. The General Case

The remainder of this section deals with the more complicated case where we do not know that $g_\varepsilon \rightarrow g$ strongly in $W^{1,\infty}_{\text{loc}}$; instead, we only have the weaker Hypotheses 1.6 which do not yield strong convergence of $e_0(g_\varepsilon)_{00}$. Similarly to the previous subsection, our goal is to establish the following:

**Proposition 4.5.** Under Hypotheses 1.6, for all $A \in \Psi^0_{1,c}$ satisfying (4.4),

$$\lim_{\varepsilon \to 0} \langle H_\varepsilon, Ae_0(u_\varepsilon - u) \rangle = 0.$$

Before proceeding further, let us outline the proof of Proposition 4.5 as follows

- as in Sect. 4.2, we start by reducing Proposition 4.5 to estimating a commutator (Lemma 4.6);
- we then estimate this commutator by choosing an $\varepsilon$-dependent partition of Fourier space and estimating each regime independently (Lemmas 4.7 to 4.9).

In what follows, we use the frame introduced in (3.3), in which we have

$$\square_g = g^{00} e_0^2 + g^{ij} \partial_{ij}^2 + \frac{1}{2} \partial_0 g^{00} e_0 + \partial_t g^{i\beta} \partial_\beta + \frac{g^{i\beta} \partial_t \sqrt{|g|}}{\sqrt{|g|}} \partial_\beta$$

where $q = q(\tilde{g}^{ij})$ denotes the polynomial in $\tilde{g}^{ij}$ determined implicitly by $|g^{-1}| = -g^{00} q(\tilde{g}^{ij})$ (the existence of such a polynomial is readily verified by considering the LDU decomposition of the matrix-field $g$).

Under Hypotheses 1.6, general first derivatives of the metric coefficients do not converge strongly; however, spatial first derivatives of the metric coefficients do: since we assume that $g_\varepsilon = g$ outside a neighborhood of $\Omega$, by integration by parts and our hypotheses,

$$\| \partial_k h_\varepsilon^{\alpha\beta} \|_{L^2} \lesssim \| \partial_{kk} h_\varepsilon^{\alpha\beta} \|_{L^2} \| h_\varepsilon^{\alpha\beta} \|_{L^2} \lesssim \| h_\varepsilon^{\alpha\beta} \|_{L^2},$$

which converges to zero. We recall that $\partial_0(g_\varepsilon)_{ij}$, and hence, $\partial_0\tilde{g}_\varepsilon^{ij}$ (one may check that $\tilde{g}_\varepsilon^{ij}$ is the inverse of the Riemannian metric $(g_\varepsilon)_{ij}$) also converge strongly. It is now easy to see that, under our assumptions, the last four terms in (4.5) only involve strongly converging derivatives of the metric coefficients.

The proof of the next lemma follows the strategy used for Lemma 4.3, but it is much more involved.

**Lemma 4.6.** (Reduction to commutators) Under Hypotheses 1.6, let $A \in \Psi^0_{1,c}$ satisfy (4.4). Then,

$$2 \lim_{\varepsilon \to 0} \langle H_\varepsilon, Ae_0(u_\varepsilon - u) \rangle = \lim_{\varepsilon \to 0} \int \partial_\alpha (u_\varepsilon - u)[A, h_\varepsilon^{\alpha\beta}] \partial_\beta e_0(u_\varepsilon - u) \, dx.$$
Proof. Let us denote
\[
\begin{align*}
\tilde{h}^{00}_\varepsilon &\equiv g^{00}_\varepsilon - g^{00}, \\
\tilde{h}^{0i}_\varepsilon &\equiv -g^{00}_\varepsilon (\beta^i_\varepsilon - \beta^i), \\
\tilde{h}^{ij}_\varepsilon &\equiv g^{00}_\varepsilon (\beta^i_\varepsilon - \beta^i)(\beta^j_\varepsilon - \beta^j) + \tilde{g}^{ij}_\varepsilon - \tilde{g}^{ij}.
\end{align*}
\] (4.7)
From (4.5), we compute
\[
H_\varepsilon = (\Box g^\varepsilon - \Box \varepsilon) u_\varepsilon = \tilde{h}^{ij}_\varepsilon \partial_j^2 u_\varepsilon + \tilde{h}^{00}_\varepsilon e_0^2 u_\varepsilon + \frac{1}{2} e_0 h_\varepsilon^{00} e_0 u_\varepsilon + \tilde{h}^{0i}_\varepsilon e_0 \partial_i u_\varepsilon + e_0 (\tilde{h}^{0i}_\varepsilon \partial_i u_\varepsilon) + o_{L^2}(1) \\
= \tilde{h}^{ij}_\varepsilon \partial_j^2 (u_\varepsilon - u) + \left[ \tilde{h}^{00}_\varepsilon e_0 + \frac{1}{2} e_0 \tilde{h}^{00}_\varepsilon \right] e_0 (u_\varepsilon - u) \\
+ \tilde{h}^{0i}_\varepsilon e_0 \partial_i (u_\varepsilon - u) + e_0 (\tilde{h}^{0i}_\varepsilon \partial_i (u_\varepsilon - u)) \\
+ \frac{1}{2} e_0 \tilde{h}^{00}_\varepsilon e_0 u + e_0 \tilde{h}^{0i}_\varepsilon \partial_i u + o_{L^2}(1),
\] (4.8)
where \(o_{L^2}(1)\) denotes a remainder which is strongly converging in \(L^2\). The proof now proceeds in several steps. Step 5 deals with (4.9). In steps 1 through 4, we deal with (4.8) and we set \(w_\varepsilon \equiv u_\varepsilon - u\) to simplify the notation. We will also find it convenient to note the following identities for \(\varepsilon \to 0\): letting \(h_\varepsilon\) be a suitably regular function with \(h_\varepsilon \to 0\) in \(L^\infty\) and \(h_\varepsilon \partial^2_{\alpha\beta} u_\varepsilon\) uniformly bounded in \(L^2\), and using Lemma 2.3,
\[
\langle \partial_\gamma u_\varepsilon, [A, h_\varepsilon](\beta^k \partial_k e_0 u_\varepsilon) \rangle = \langle \partial_\gamma u_\varepsilon, [A, \beta^k h_\varepsilon] \partial_k e_0 u_\varepsilon \rangle \\
- \langle \partial_\gamma u_\varepsilon, h_\varepsilon, \beta^k, A \partial_k e_0 u_\varepsilon + A(\partial_k \beta^k e_0 u_\varepsilon) \rangle \\
= \langle \partial_\gamma u_\varepsilon, [A, \beta^k h_\varepsilon] \partial_k e_0 u_\varepsilon \rangle + o(1),\quad (4.10)
\]
\[
\langle \beta^k \partial_k u_\varepsilon, [A, h_\varepsilon] \partial_\gamma e_0 u_\varepsilon \rangle = \langle \partial_k u_\varepsilon, [A, \beta^k h_\varepsilon] \partial_\gamma e_0 u_\varepsilon \rangle + \langle \partial_k u_\varepsilon, [\beta^k, A](h_\varepsilon \partial_\gamma e_0 u_\varepsilon) \rangle \\
= \langle \partial_k u_\varepsilon, [A, \beta^k h_\varepsilon] \partial_\gamma e_0 u_\varepsilon \rangle + o(1).\quad (4.11)
\]
Step 1: first term in (4.8). By its special structure, both time and spatial derivatives of \(\tilde{h}^{ij}_\varepsilon\) converge strongly. Hence, by a straightforward adaptation of the proof of Lemma 4.3, we find that
\[
\lim_{\varepsilon \to 0} \left( \langle \tilde{h}^{ij}_\varepsilon \partial_j^2 w_\varepsilon, A e_0 w_\varepsilon \rangle \right) = \lim_{\varepsilon \to 0} \frac{1}{2} \int e_0 w_\varepsilon [A, \tilde{h}^{ij}_\varepsilon] \partial_j^2 w_\varepsilon \, dx.
\]
Integrating by parts in \(e_0\) and in \(\partial_j\) and using the compactness of derivatives of \(\tilde{h}^{ij}_\varepsilon\), we get
\[
\lim_{\varepsilon \to 0} \frac{1}{2} \int e_0 w_\varepsilon [A, \tilde{h}^{ij}_\varepsilon] \partial_j^2 w_\varepsilon \, dx = \lim_{\varepsilon \to 0} \frac{1}{2} \int \partial_i w_\varepsilon [A, \tilde{h}^{ij}_\varepsilon] \partial_j e_0 w_\varepsilon \, dx.
\]
Step 2: second term in (4.8). An integration by parts in \(e_0\) (which requires both an integration by parts in \(\partial_i\) and in a spatial direction) leads to
\[
\langle \tilde{h}^{00}_\varepsilon e_0^2 w_\varepsilon, A e_0 w_\varepsilon \rangle = \langle \partial_k \beta^k \tilde{h}^{00}_\varepsilon e_0 w_\varepsilon, A e_0 w_\varepsilon \rangle - \langle \tilde{h}^{00}_\varepsilon e_0 w_\varepsilon, [e_0, A] e_0 w_\varepsilon \rangle \\
- \langle e_0 \tilde{h}^{00}_\varepsilon e_0 w_\varepsilon, A e_0 w_\varepsilon \rangle - \langle \tilde{h}^{00}_\varepsilon e_0 w_\varepsilon, A e_0^2 w_\varepsilon \rangle.
\]
Thus, we have the identity
\[
\langle \tilde{h}_\varepsilon^{00} e_0^2 w_\varepsilon, A e_0 w_\varepsilon \rangle + \langle e_0 w_\varepsilon, A(\tilde{h}_\varepsilon^{00} e_0^2 w_\varepsilon) \rangle + \langle e_0 \tilde{h}_\varepsilon^{00} e_0 w_\varepsilon, A e_0 w_\varepsilon \rangle
= \langle e_0 w_\varepsilon, [A, \tilde{h}_\varepsilon^{00}] e_0^2 w_\varepsilon \rangle + o(1).
\]
Using the self-adjointness of \(A\) on the second term on the left hand side, we conclude that
\[
\lim_{\varepsilon \to 0} \langle 2\tilde{h}_\varepsilon^{00} e_0^2 w_\varepsilon + e_0 \tilde{h}_\varepsilon^{00} e_0 w_\varepsilon, A e_0 w_\varepsilon \rangle
= \lim_{\varepsilon \to 0} \int e_0 w_\varepsilon [A, \tilde{h}_\varepsilon^{00}] e_0^2 w_\varepsilon \, dx
= \lim_{\varepsilon \to 0} \int \left( \partial_0 w_\varepsilon [A, \tilde{h}_\varepsilon^{00}] \partial_0 e_0 w_\varepsilon + \partial_i w_\varepsilon [A, \tilde{h}_\varepsilon^{00} \beta^i \beta^j] \partial_j e_0 w_\varepsilon \right) \, dx
- \lim_{\varepsilon \to 0} \int \left( \partial_i w_\varepsilon [A, \beta^i \tilde{h}_\varepsilon^{00}] \partial_0 e_0 w_\varepsilon + \partial_0 w_\varepsilon [A, \tilde{h}_\varepsilon^{00} \beta^i] \partial_i e_0 w_\varepsilon \right) \, dx,
\]
where we have applied both (4.10) and (4.11) to conclude. 

**Step 3:** third term in (4.8). An integration by parts in \(\partial_i\) leads to
\[
\langle \tilde{h}_\varepsilon^{0i} e_0 \partial_i w_\varepsilon, A e_0 w_\varepsilon \rangle = \langle \tilde{h}_\varepsilon^{0i} [e_0, \partial_i] w_\varepsilon, A e_0 w_\varepsilon \rangle - \langle \partial_i \tilde{h}_\varepsilon^{0i} e_0 w_\varepsilon, A e_0 w_\varepsilon \rangle
- \langle \tilde{h}_\varepsilon^{0i} e_0 w_\varepsilon, [\partial_i, A] e_0 w_\varepsilon \rangle - \langle \tilde{h}_\varepsilon^{0i} e_0 w_\varepsilon, A [\partial_i, e_0] w_\varepsilon \rangle
+ \langle e_0 w_\varepsilon, [A, \tilde{h}_\varepsilon^{0i}] e_0 \partial_i w_\varepsilon \rangle - \langle e_0 w_\varepsilon, A(\tilde{h}_\varepsilon^{0i} e_0 \partial_i w_\varepsilon) \rangle.
\]
Thus, we have the identity
\[
\langle \tilde{h}_\varepsilon^{0i} e_0 \partial_i w_\varepsilon, A e_0 w_\varepsilon \rangle + \langle e_0 w_\varepsilon, A(\tilde{h}_\varepsilon^{0i} e_0 \partial_i w_\varepsilon) \rangle = \langle e_0 w_\varepsilon [A, \tilde{h}_\varepsilon^{0i}] e_0 \partial_i w_\varepsilon \rangle + o(1).
\]
Using the self-adjointness of \(A\), modulo a compact operator, (4.11), and interchanging \(e_0\) with \(\partial_i\), we conclude
\[
\lim_{\varepsilon \to 0} \left\{ \tilde{h}_\varepsilon^{0i} e_0 \partial_i w_\varepsilon, A e_0 w_\varepsilon \right\} = \lim_{\varepsilon \to 0} \frac{1}{2} \int e_0 w_\varepsilon [A, \tilde{h}_\varepsilon^{0i}] \partial_i e_0 w_\varepsilon \, dx
= \lim_{\varepsilon \to 0} \frac{1}{2} \int \partial_0 w_\varepsilon [A, \tilde{h}_\varepsilon^{0i}] \partial_i e_0 w_\varepsilon \, dx
- \lim_{\varepsilon \to 0} \frac{1}{2} \int \partial_i w_\varepsilon [A, \beta^i \tilde{h}_\varepsilon^{0i}] \partial_j e_0 w_\varepsilon \, dx.
\]

**Step 4:** fourth term in (4.8). To begin, recall that, for any \(h_\varepsilon \to 0\) in \(L^\infty\), by (4.5),
\[
h_\varepsilon e_0^2 w_\varepsilon + h_\varepsilon \frac{g_{jk}^0}{g_{00}} \partial_j \partial_k w_\varepsilon = h_\varepsilon \frac{\Box_g w_\varepsilon}{g_{00}} + o_{L^2}(1) = o_{L^2}(1),
\]
since \(\Box_g w_\varepsilon\) is bounded in \(L^2\) by (4.3). Consider the term \(e_0(\tilde{h}_\varepsilon^{0i} \partial_i w_\varepsilon)\); an integration by parts in \(e_0\) yields
\[
\langle e_0(\tilde{h}_\varepsilon^{0i} \partial_i w_\varepsilon), A e_0 w_\varepsilon \rangle = -\langle \partial_k \beta^k \tilde{h}_\varepsilon^{0i} \partial_i w_\varepsilon, A e_0 w_\varepsilon \rangle
- \langle \tilde{h}_\varepsilon^{0i} \partial_i w_\varepsilon, [e_0, A] e_0 w_\varepsilon \rangle - \langle \tilde{h}_\varepsilon^{0i} \partial_i w_\varepsilon, A e_0^2 w_\varepsilon \rangle
= -\langle \tilde{h}_\varepsilon^{0i} \partial_i w_\varepsilon, A e_0^2 w_\varepsilon \rangle + o(1).
\]
In the remaining term, we apply (4.12) to replace $e_0^2 w_\varepsilon$ with $\tilde{g}^{jk} w_\varepsilon$, and we commute $A$ with $\tilde{g}^{jk}/g^{00}$ using the Calderon commutator estimate:

$$-\langle \tilde{h}_\varepsilon^{0i} \partial_i w_\varepsilon, Ae_0^2 w_\varepsilon \rangle = \langle \tilde{h}_\varepsilon^{0i} \frac{\tilde{g}^{jk}}{g^{00}} \partial_i w_\varepsilon, A\partial_j^2 w_\varepsilon \rangle + o(1);$$

Integrating by parts in $j$, then $i$ and then $k$, as in Step 1, we obtain

$$\langle \tilde{h}_\varepsilon^{0i} \frac{\tilde{g}^{jk}}{g^{00}} \partial_i w_\varepsilon, A\partial_j^2 w_\varepsilon \rangle = -\langle \tilde{h}_\varepsilon^{0i} \frac{\tilde{g}^{jk}}{g^{00}} \partial_j^2 w_\varepsilon, A\partial_i w_\varepsilon \rangle + o(1)$$

$$= \langle \tilde{h}_\varepsilon^{0i} e_0^2 w_\varepsilon, A\partial_i w_\varepsilon \rangle + o(1),$$

where the last step follows from another application of (4.12). Combining the previous results, we finally arrive at the identity

$$-\langle \tilde{h}_\varepsilon^{0i} \partial_i w_\varepsilon, Ae_0^2 w_\varepsilon \rangle = \langle \partial_i w_\varepsilon, A(\tilde{h}_\varepsilon^{0i} \partial_i w_\varepsilon) \rangle - \langle \tilde{h}_\varepsilon^{0i} \partial_i w_\varepsilon, Ae_0^2 w_\varepsilon \rangle + o(1)$$

$$= \langle \partial_i w_\varepsilon, [A, \tilde{h}_\varepsilon^{0i}]e_0^2 w_\varepsilon \rangle + o(1).$$

Now we use the self-adjointness of $A$, modulo a compact operator, on all of the terms of the last expression, excluding the first term, to get that

$$-2\langle \tilde{h}_\varepsilon^{0i} \partial_i w_\varepsilon, Ae_0^2 w_\varepsilon \rangle = \langle [A, \tilde{h}_\varepsilon^{0i}]e_0^2 w_\varepsilon \rangle + o(1)$$

$$= \langle \partial_i w_\varepsilon, [A, \tilde{h}_\varepsilon^{0i}]e_0^2 w_\varepsilon \rangle + o(1).$$

Recalling (4.13), we arrive at

$$\lim_{\varepsilon \to 0} \left( e_0(\tilde{h}_\varepsilon^{0i} \partial_i w_\varepsilon), Ae_0 w_\varepsilon \right) = \lim_{\varepsilon \to 0} \frac{1}{2} \int \partial_i w_\varepsilon [A, \tilde{h}_\varepsilon^{0i}]e_0^2 w_\varepsilon \, dx$$

$$= \lim_{\varepsilon \to 0} \frac{1}{2} \int \partial_i w_\varepsilon [A, \tilde{h}_\varepsilon^{0i}] \partial_0 e_0 w_\varepsilon \, dx$$

$$- \lim_{\varepsilon \to 0} \frac{1}{2} \int \partial_i w_\varepsilon [A, \tilde{h}_\varepsilon^{0i} \beta^j] \partial_j e_0 w_\varepsilon \, dx,$$

where we use (4.10) in the last equality.

**Step 5:** the two terms in (4.9). We integrate by parts in $e_0$ to get that

$$\langle e_0 \tilde{h}_\varepsilon^{00} e_0 u + 2e_0 \tilde{h}_\varepsilon^{0k} \partial_k u, Ae_0 (u_\varepsilon - u) \rangle$$

$$= -\langle \partial_j \beta^j (\tilde{h}_\varepsilon^{00} e_0 u + 2\tilde{h}_\varepsilon^{0k} \partial_k u), Ae_0 (u_\varepsilon - u) \rangle$$

$$- \langle \tilde{h}_\varepsilon^{00} e_0^2 u + 2\tilde{h}_\varepsilon^{0k} e_0 \partial_k u, Ae_0 (u_\varepsilon - u) \rangle$$

$$- \langle \tilde{h}_\varepsilon^{00} e_0 u + 2\tilde{h}_\varepsilon^{0k} \partial_k u, [e_0, A]e_0 (u_\varepsilon - u) \rangle$$

$$- \langle \tilde{h}_\varepsilon^{00} e_0 u + 2\tilde{h}_\varepsilon^{0k} \partial_k u, Ae_0^2 (u_\varepsilon - u) \rangle$$

$$= -\langle \tilde{h}_\varepsilon^{00} e_0 u + 2\tilde{h}_\varepsilon^{0k} \partial_k u, Ae_0^2 (u_\varepsilon - u) \rangle + o(1),$$
where the last line follows by the uniform convergence of $\tilde{h}_e^{\alpha \beta}$. For the remaining term, we may apply the same reasoning as in the previous step: from (4.12), we have

\[
- \langle \tilde{h}_e^{00} e_0 u + 2\tilde{h}_e^{0k} \partial_k u, A e_0^2 (u_e - u) \rangle
= \langle \tilde{h}_e^{00} e_0 u + 2\tilde{h}_e^{0k} \partial_k u, A \frac{g^{ij}}{g^{00}} \partial_j (u_e - u) \rangle + o(1)
= -\langle \partial_t \tilde{h}_e^{00} e_0 u + 2\partial_t \tilde{h}_e^{0k} \partial_k u, A \frac{g^{ij}}{g^{00}} \partial_j (u_e - u) \rangle + o(1) = o(1),
\]

with the second line following from an integration by parts. Thus, (4.9) does not contribute to the limit.

**Step 6:** Conclusion. Combining the previous steps yields

\[
\langle 2H_e, Ae_0 w_e \rangle = \int \partial_0 w_e [A, \tilde{h}_e^{00}] \partial_0 e_0 w_e \, dx
+ \int \left( \partial_0 w_e [A, \tilde{h}_e^{0i} - \tilde{h}_e^{00} \beta^i] \partial_i + \partial_i w_e [A, \tilde{h}_e^{i0} - \tilde{h}_e^{00} \beta^i] \partial_0 \right) e_0 w_e \, dx
+ \int \partial_i w_e [A, \tilde{h}_e^{ij} - \tilde{h}_e^{0i} \beta^j - \tilde{h}_e^{0j} \beta^i + \tilde{h}_e^{00} \beta^i \beta^j] \partial_j e_0 w_e \, dx + o(1),
\]

and using the definitions in (4.7) the conclusion follows. \(\square\)

By passing to subsequences if need be, by Hypothesis 1.6(c) we may find a sequence $\omega_e \downarrow 0$ such that

\[
\sup_{\alpha, \beta} \| \partial^2_{\alpha \beta} u_e \|_{L^4(\Omega)} \lesssim \omega_e^{-1}, \quad \sup_{\alpha, \beta} \| h^{\alpha \beta}_e \|_{L^\infty(\Omega)} \lesssim \omega_e.
\]

In order to prove Proposition 4.5, we move to Fourier space. Let $\zeta : \mathbb{R}^+_0 \rightarrow [0, 1]$ be a smooth function such that $\zeta(x) = 1$ for $x \leq 1$ and $\zeta = 0$ for $x \geq 2$.

We consider an $\varepsilon$-dependent partition of frequency space into low frequencies, spatially-dominated high frequencies and time-dominated high frequencies, c.f. Fig. 1. This partition is associated to the smooth functions $0 \leq \Theta_{\text{low}, \varepsilon}, \Theta_{\text{spa}, \varepsilon}, \Theta_{\text{time}, \varepsilon} \leq 1$ defined by

\[
\Theta_{\text{low}, \varepsilon}(\rho) \equiv \zeta \left( \omega_e^{3\delta_1} |\rho_{\text{tot}}| \right) \implies \sup \Theta_{\text{low}, \varepsilon} \subseteq \{ |\rho_{\text{tot}}| \leq 2\omega_e^{-3\delta_1} \},
\]
\[
\Theta_{\text{spa}, \varepsilon}(\rho) \equiv (1 - \Theta_{\text{low}, \varepsilon})(\rho) \left[ 1 - \zeta \left( \frac{|\rho_{\text{spa}}|}{|\rho_{\text{tot}}|^{3\delta_2}} \right) \right] \implies \sup \Theta_{\text{spa}, \varepsilon} \subseteq \{ |\rho_{\text{spa}}| \geq |\rho_{\text{tot}}|^{3\delta_2} \geq \omega_e^{-3\delta_1 \delta_2} \},
\]
\[
\Theta_{\text{time}, \varepsilon}(\rho) \equiv (1 - \Theta_{\text{low}, \varepsilon})(\rho) \zeta \left( \frac{|\rho_{\text{spa}}|}{|\rho_{\text{tot}}|^{3\delta_2}} \right) \implies \sup \Theta_{\text{time}, \varepsilon} \subseteq \left\{ \rho_0^2 \geq |\rho_{\text{tot}}|^2 - 4|\rho_{\text{tot}}|^{2\delta_2} \text{ and } |\rho_{\text{tot}}| \geq \omega_e^{-3\delta_1} \right\},
\]
where \( |\rho_{\text{tot}}|^2 = \sum_\alpha \rho_\alpha^2 \) and \( |\rho_{\text{spa}}|^2 = \sum_i \rho_i^2 \). Here, \( \delta_1, \delta_2 > 0 \) are parameters to be fixed. Clearly

\[
\Theta_{\text{low}, \varepsilon} + \Theta_{\text{spa}, \varepsilon} + \Theta_{\text{time}, \varepsilon} = 1.
\]

For an \( L^2 \) function \( h_\varepsilon \), we define its projections on a range of frequencies according to

\[
P_{\text{range}, \varepsilon}[h_\varepsilon] \equiv \mathcal{F}^{-1} \left( \Theta_{\text{range}, \varepsilon} \mathcal{F}(h_\varepsilon) \right), \quad \text{range} \in \{\text{low}, \text{spa}, \text{time}\},
\]

where \( \mathcal{F} \) denotes the spacetime Fourier transform and \( \mathcal{F}^{-1} \) denotes its inverse. These projections are linear and commute with derivatives.

We focus on the low frequencies first. Note that, if the frequency parameter is capped, then terms with derivatives, which in frequency space correspond to multiplication by the frequency variable, are comparable to zeroth order terms. Thus, the strategy of Sect. 4.2 still works under the current convergence assumptions on \( g_\varepsilon \) as long as one restricts to low frequencies.

**Lemma 4.7.** Under Hypotheses 1.6, as long as \( \delta_1 < 1 \),

\[
\lim_{\varepsilon \to 0} \int \partial_\alpha (u_\varepsilon - u) [A, P_{\text{low}, \varepsilon}[h_\varepsilon^{\alpha \beta}]] \partial_\beta e_0 (u_\varepsilon - u) \, dx = 0.
\]

**Proof.** Without loss of generality, set \( u \equiv 0 \). Consider the identity

\[
\partial_\alpha u_\varepsilon [A, P_{\text{low}, \varepsilon}[h_\varepsilon^{\alpha \beta}]] \partial_\beta e_0 u_\varepsilon = \partial_\alpha u_\varepsilon [A \partial_\beta, P_{\text{low}, \varepsilon}[h_\varepsilon^{\alpha \beta}]] e_0 u_\varepsilon - \partial_\alpha u_\varepsilon A(\partial_\beta P_{\text{low}, \varepsilon}[h_\varepsilon^{\alpha \beta}] e_0 u_\varepsilon).
\]

We estimate the second term directly and, for the first term, apply the Theorem 2.6: for small \( \varepsilon \),

\[
\left| \int \partial_\alpha u_\varepsilon [A, P_{\text{low}, \varepsilon}[h_\varepsilon^{\alpha \beta}]] \partial_\beta e_0 u_\varepsilon \, dx \right| \lesssim \sup_{\alpha, \beta} \|P_{\text{low}, \varepsilon}[h_\varepsilon^{\alpha \beta}]\|_{W^{1,\infty}} \|\partial u_\varepsilon\|_{L^2}^2 \\
\lesssim \omega_\varepsilon^{-\delta_1} \sup_{\alpha, \beta} \|h_\varepsilon^{\alpha \beta}\|_{L^\infty} \lesssim \omega_\varepsilon^{1-\delta_1},
\]

where we use Bernstein’s inequality \( \hat{f} \subset B_R(0) \implies \|Df\|_\infty \lesssim R \|f\|_\infty \) in the second inequality.

For high frequencies this method fails, as we do not have sufficient control over \( h_\varepsilon^{00} \) and \( h_\varepsilon^{0i} \). If the spatial frequencies dominate, however, we can compensate for this issue by appealing to control on higher order spatial derivatives of \( h_\varepsilon^{\alpha \beta} \). This is independent of the commutator structure.

**Lemma 4.8.** Under Hypotheses 1.6, as long as \( \delta_1 \delta_2 > \frac{1}{2} \),

\[
\lim_{\varepsilon \to 0} \int \partial_\alpha (u_\varepsilon - u) [A, P_{\text{spa}, \varepsilon}[h_\varepsilon^{\alpha \beta}]] \partial_\beta e_0 (u_\varepsilon - u) \, dx = 0.
\]
Proof. Without loss of generality, set $u \equiv 0$. We have

$$
\left| \int \partial_\alpha u_\varepsilon [A, \mathcal{P}_{\text{spa}, \varepsilon} [h_{e}^{\alpha \beta}]] \partial_\beta e_0 u_\varepsilon \, dx \right| \leq \left| \int \partial_\alpha u_\varepsilon A (\mathcal{P}_{\text{spa}, \varepsilon} [h_{e}^{\alpha \beta}]) \partial_\beta e_0 u_\varepsilon \, dx \right| + \left| \int \mathcal{P}_{\text{spa}, \varepsilon} [h_{e}^{\alpha \beta}] \partial_\alpha u_\varepsilon A \partial_\beta e_0 u_\varepsilon \, dx \right|
$$

$$
\lesssim \sup_{\alpha, \beta, \gamma} \| \partial_\beta e_0 u_\varepsilon \| _{L^4} \| \partial_\alpha u_\varepsilon \| _{L^4} \| \mathcal{P}_{\text{spa}, \varepsilon} [h_{e}^{\alpha \beta}] \| _{L^2} \lesssim \sup_{\alpha, \beta} \omega_\varepsilon^{-1} \| \mathcal{P}_{\text{spa}, \varepsilon} [h_{e}^{\alpha \beta}] \| _{L^2} .
$$

Using the assumptions directly would imply that the term above is bounded, but not necessarily converging to zero. However, by Plancherel’s theorem,

$$
\| \mathcal{P}_{\text{spa}, \varepsilon} [h_{e}^{\alpha \beta}] \| _{L^2}^2 = \int \left| \Theta_{\text{spa}, \varepsilon} \hat{h}_{e}^{\alpha \beta} \right|^2 \, d\xi = \int \frac{|\Theta_{\text{spa}, \varepsilon} (\xi)|^2}{|\xi_{\text{spa}}|^4} \left| \delta^{ij} \xi_i \xi_j \hat{h}_{e}^{\alpha \beta} (\xi) \right|^2 \, d\xi
$$

$$
\lesssim \omega_\varepsilon^{2\delta_1 \delta_2} \left\| \delta^{ij} \partial_{ij} h_{e}^{\alpha \beta} \right\| _{L^2}^2 ,
$$

since $|\xi_{\text{spa}}|^2 \gtrsim |\xi|^2 \delta_2 \gtrsim \omega_\varepsilon^{-2\delta_1 \delta_2}$ in the support of $\Theta_{\text{spa}, \varepsilon} (\xi)$. By the boundedness of the spatial laplacian of the metric coefficients, we obtain our result. \qed

Finally, we are left with the regime of high frequencies where it is the time frequency which dominates. Here, the lack of control over $\partial h_{e}^{\alpha \beta}$ is compensated by control over $\Box_{\varepsilon} u_\varepsilon$, see (4.3). Crucial to the argument is the commutator structure yielded by Lemma 4.6 and the invertibility of $e_0$ in this frequency regime.

**Lemma 4.9.** Under Hypotheses 1.6 and assuming that $A$ satisfies (4.4), if $\delta_1 > \frac{1}{2}$ and $\delta_2 < 1$,

$$
\int \partial_\alpha (u_\varepsilon - u) [A, \mathcal{P}_{\text{time}, \varepsilon} [h_{e}^{\alpha \beta}]] \partial_\beta e_0 (u_\varepsilon - u) \, dx = 0 .
$$

Proof. Without loss of generality, set $u \equiv 0$. Throughout the proof, we let $h_{e}^{\alpha \beta} = \mathcal{P}_{\text{time}} [h_{e}^{\alpha \beta}]$ and we assume that $\delta_1 > 0$ and $\delta_2 < 1$. We also note that, for sufficiently small $\varepsilon_0$,

$$
|\beta^j(x) \xi_j / \xi_0| \ll 1 , \quad |\xi_0 + \beta^j(x) \xi_j| \gtrsim \omega_\varepsilon^{-\delta_1} \gg 1 \quad \text{when} \; \xi \in \text{supp} \; \Theta_{\text{time}, \varepsilon} ,
$$

whenever $\varepsilon \leq \varepsilon_0$. Hence we may find an operator $Q \in \Psi^{-1}_{1,c}$ and $R \in \Psi_{1,c}^0$ such that

$$
\sigma (Q) (x, \xi) = q (x, \xi) \equiv \left[ g^{00}(x)(\xi_0 + \beta^j(x) \xi_j) \right]^{-1} = \frac{1 + \sigma (R) (x, \xi)}{g^{00}(x) \xi_0} ,
$$

$$
\sigma (R) (x, \xi) \equiv \sum_{\ell=1}^{\infty} \left( -\beta^j(x) \frac{\xi_j}{\xi_0} \right)^\ell ,
$$

whenever $(x, \xi) \in \Omega \times \text{supp} \; \Theta_{\text{time}, \varepsilon_0}$.
It is now easy to see that we have the estimates
\[ \| Q h_\varepsilon^{\alpha\beta} \|_{L^2(\Omega)} \lesssim \omega_\varepsilon^{1+\delta_1}, \quad (4.14) \]
\[ \| \partial_0 (Q h_\varepsilon^{\alpha\beta}) \|_{L^2(\Omega)} \lesssim \omega_\varepsilon^{\delta_1}, \quad \sup_j \| \partial_j (Q h_\varepsilon^{\alpha\beta}) \|_{L^2(\Omega)} \lesssim \omega_\varepsilon^{1+\delta_1}. \quad (4.15) \]
Indeed, for (4.14), we compute
\[ \| Q h_\varepsilon^{\alpha\beta} \|_{L^2(\Omega)} \lesssim \| (g^{00})^{-1} \|_{L^\infty(\Omega)} \| \varepsilon_j \|_{L^\infty(\text{supp } \theta_\varepsilon, \text{time})} \| h_\varepsilon^{\alpha\beta} \|_{L^2} \times \sum_{\ell=0}^\infty \left( \| \beta^i \|_{L^\infty(\Omega)} \| \varepsilon_j \varepsilon_\ell \|_{L^\infty(\text{supp } \theta_\varepsilon, \text{time})} \right)^\ell \lesssim \| \varepsilon_j \|_{L^\infty(\text{supp } \theta_\varepsilon, \text{time})} \| h_\varepsilon^{\alpha\beta} \|_{L^2} \lesssim \omega_\varepsilon^{1+\delta_1}. \]
The estimates in (4.15) follow similarly. Note that we only require $L^2$ norms in $\Omega$ in what follows as we will always be testing against $u_\varepsilon$ and its derivatives, which have compact support in $\Omega$.

Using $Q$, we may rewrite our commutator as
\[ \langle \partial_\alpha u_\varepsilon, [A, h_\varepsilon^{\alpha\beta}] \partial_\beta e_0 u_\varepsilon \rangle = \langle \partial_\alpha u_\varepsilon, A \left( g^{00} e_0 Q h_\varepsilon^{\alpha\beta} \partial_\beta e_0 u_\varepsilon \right) \rangle - \langle \partial_\alpha u_\varepsilon, g^{00} e_0 Q h_\varepsilon^{\alpha\beta}, A \partial_\beta e_0 u_\varepsilon \rangle. \quad (4.16) \]

**Step 1**: integration by parts in $e_0$. In this step, we show that
\[ \lim_{\varepsilon \to 0} \langle \partial_\alpha u_\varepsilon, [A, h_\varepsilon^{\alpha\beta}] \partial_\beta e_0 u_\varepsilon \rangle = \lim_{\varepsilon \to 0} \langle \partial^2 u_\varepsilon, [A, Q h_\varepsilon^{\alpha\beta}] (g^{00} e_0^2 u_\varepsilon) \rangle + \lim_{\varepsilon \to 0} \langle \partial_\alpha u_\varepsilon, [A, \partial_\beta (Q h_\varepsilon^{\alpha\beta})] (g^{00} e_0^2 u_\varepsilon) \rangle. \quad (4.17) \]
To begin, we seek to move the $e_0$ derivative on $Q h_\varepsilon^{\alpha\beta}$ in (4.16) onto $u_\varepsilon$ through integration by parts in $e_0$. Note that, whenever a derivative hits a coefficient of the limit metric $g$, that term is $o(1)$: using (4.14),
\[ \langle \partial g Q h_\varepsilon^{\alpha\beta} \partial^2 u_\varepsilon, A \partial u_\varepsilon \rangle \lesssim \| \partial^2 u_\varepsilon \|_{L^4} \| \partial u_\varepsilon \|_{L^4} \| Q h_\varepsilon^{\alpha\beta} \|_{L^2} \lesssim \omega_\varepsilon^{\delta_1} \to 0, \quad (4.18) \]
where $\partial$ denotes an arbitrary partial derivative and $g$ an arbitrary metric coefficient. We note that the order of the terms in the left-hand side is unimportant. We will use (4.18) and its variants implicitly in the sequel.

The first term of (4.16) becomes
\[ \langle \partial_\alpha u_\varepsilon, A \left( g^{00} e_0 Q h_\varepsilon^{\alpha\beta} \partial_\beta e_0 u_\varepsilon \right) \rangle = \langle \partial_\alpha u_\varepsilon, [A, g^{00} e_0] (Q h_\varepsilon^{\alpha\beta} \partial_\beta e_0 u_\varepsilon) \rangle + \left\{ \partial_\alpha u_\varepsilon g^{00}, e_0 A (Q h_\varepsilon^{\alpha\beta} \partial_\beta e_0 u_\varepsilon) \right\} - \langle \partial_\alpha u_\varepsilon, A (Q h_\varepsilon^{\alpha\beta} g^{00} e_0 \partial_\beta e_0 u_\varepsilon) \rangle - \langle \partial_\alpha u_\varepsilon, A (Q h_\varepsilon^{\alpha\beta} \partial_\beta (g^{00} e_0^2 u_\varepsilon)) \rangle = -\langle \partial_\alpha u_\varepsilon, A (Q h_\varepsilon^{\alpha\beta} \partial_\beta (g^{00} e_0^2 u_\varepsilon)) \rangle - (g^{00} e_0 \partial_\alpha u_\varepsilon, A (Q h_\varepsilon^{\alpha\beta} \partial_\beta e_0 u_\varepsilon)) + o(1), \]
and the second term yields

$$\langle \partial_\alpha u_\epsilon, g^{00} e_0 Q h_\epsilon^{\alpha \beta}, A \partial_\beta e_0 u_\epsilon \rangle$$

$$= \langle \partial_\alpha u_\epsilon, Q h_\epsilon^{\alpha \beta}, [g^{00} e_0, A \partial_\beta] e_0 u_\epsilon \rangle$$

$$+ \langle \partial_\alpha u_\epsilon Q h_\epsilon^{\alpha \beta}, A \partial_\beta (g^{00} e_0^2 u_\epsilon) \rangle + \langle g^{00} e_0 \partial_\alpha u_\epsilon Q h_\epsilon^{\alpha \beta}, A \partial_\beta e_0 u_\epsilon \rangle$$

$$+ \langle (e_0 g^{00} + \partial_k \beta^k g^{00}) \partial_\alpha u_\epsilon Q h_\epsilon^{\alpha \beta}, A \partial_\beta e_0 u_\epsilon \rangle$$

$$= \langle \partial_\alpha u_\epsilon Q h_\epsilon^{\alpha \beta}, A \partial_\beta (g^{00} e_0^2 u_\epsilon) \rangle + \langle g^{00} e_0 \partial_\alpha u_\epsilon Q h_\epsilon^{\alpha \beta}, A \partial_\beta e_0 u_\epsilon \rangle + o(1).$$

Combining the previous computations gives

$$\langle \partial_\alpha u_\epsilon, [A, h_\epsilon^{\alpha \beta}] \partial_\beta e_0 u_\epsilon \rangle = \langle \partial_\alpha u_\epsilon, [Q h_\epsilon^{\alpha \beta}, A] \partial_\beta (g^{00} e_0^2 u_\epsilon) \rangle$$

$$+ \langle g^{00} e_0 \partial_\alpha u_\epsilon, [Q h_\epsilon^{\alpha \beta}, A] \partial_\beta e_0 u_\epsilon \rangle + o(1)$$

$$= \langle \partial_\alpha u_\epsilon, [Q h_\epsilon^{\alpha \beta}, A] \partial_\beta (g^{00} e_0^2 u_\epsilon) \rangle + o(1).$$

(4.19)

because, by the symmetry of $h_\epsilon^{\alpha \beta}$, the self-adjointness of $A$ (up to a compact operator), and (4.18), we have

$$\langle g^{00} e_0 \partial_\alpha u_\epsilon, [Q h_\epsilon^{\alpha \beta}, A] \partial_\beta e_0 u_\epsilon \rangle = \langle g^{00} e_0 \partial_\beta u_\epsilon, [Q h_\epsilon^{\alpha \beta}, A] \partial_\alpha e_0 u_\epsilon \rangle$$

$$= \langle A(e_0 \partial_\beta u_\epsilon g^{00} Q h_\epsilon^{\alpha \beta}), \partial_\alpha e_0 u_\epsilon \rangle - \langle A(e_0 \partial_\beta u_\epsilon g^{00} Q h_\epsilon^{\alpha \beta}), \partial_\alpha e_0 u_\epsilon \rangle + o(1)$$

$$= \langle A(\partial_\beta e_0 u_\epsilon Q h_\epsilon^{\alpha \beta}), g^{00} e_0 \partial_\alpha u_\epsilon \rangle - \langle A(\partial_\beta e_0 u_\epsilon), Q h_\epsilon^{\alpha \beta} g^{00} e_0 \partial_\alpha u_\epsilon \rangle + o(1)$$

$$= -\langle g^{00} e_0 \partial_\alpha u_\epsilon, [Q h_\epsilon^{\alpha \beta}, A] \partial_\beta e_0 u_\epsilon \rangle + o(1).$$

(4.20)

To obtain (4.17), we need only integrate (4.19) by parts in $\partial_\beta$.

**Step 2:** introducing $\Box_g$. In this step, we show:

$$\langle \partial_\alpha u_\epsilon, [A, h_\epsilon^{\alpha \beta}] \partial_\beta e_0 u_\epsilon \rangle = \langle \partial_\alpha^2 u_\epsilon, [A, Q h_\epsilon^{\alpha \beta} \Box_g u_\epsilon] \rangle + \langle \partial_\alpha u_\epsilon, [A, \partial_\beta (Q h_\epsilon^{\alpha \beta}) \Box_g u_\epsilon] \rangle$$

$$+ \langle \partial_\alpha u_\epsilon, [\partial_\gamma (Q h_\epsilon^{\alpha \beta}), A] \left( \tilde{g}^{ij} \partial_\gamma (g^{ij} \partial_\beta u_\epsilon) \right) \rangle + o(1).$$

(4.21)

From (4.5), it is clear that terms $g^{00} e_0^2$ in (4.17) may be replaced by $\Box_g - \tilde{g}^{ij} \partial_{ij}$, as the remaining terms in (4.5), which involve derivatives of $g$, do not contribute, c.f. (4.18). Thus, we have

$$\langle \partial_\alpha u_\epsilon, [A, h_\epsilon^{\alpha \beta}] \partial_\beta e_0 u_\epsilon \rangle$$

$$= -\langle \partial_\alpha^2 u_\epsilon, [A, Q h_\epsilon^{\alpha \beta} \tilde{g}^{ij} \partial_{ij} u_\epsilon] \rangle$$

$$- \langle \partial_\alpha u_\epsilon, [A, \partial_\beta (Q h_\epsilon^{\alpha \beta}) \tilde{g}^{ij} \partial_{ij} u_\epsilon] \rangle + \langle \partial_\alpha^2 u_\epsilon, [A, Q h_\epsilon^{\alpha \beta} \Box_g u_\epsilon] \rangle$$

$$+ \langle \partial_\alpha u_\epsilon, [A, \partial_\beta (Q h_\epsilon^{\alpha \beta}) \Box_g u_\epsilon] \rangle + o(1)$$

$$= \langle \partial_\alpha u_\epsilon, [A, Q h_\epsilon^{\alpha \beta}] \partial_\beta (\tilde{g}^{ij} \partial_{ij} u_\epsilon) \rangle + \langle \partial_\alpha^2 u_\epsilon, [A, Q h_\epsilon^{\alpha \beta}] \Box_g u_\epsilon \rangle$$

$$+ \langle \partial_\alpha u_\epsilon, [A, \partial_\beta (Q h_\epsilon^{\alpha \beta}) \Box_g u_\epsilon] \rangle + o(1).$$
integrating by parts in $\partial_\beta$ to arrive at the final equality. Now, we integrate the first term in $\partial_i$,

$$
\langle \partial_{\alpha} u_\varepsilon, [A Q_{h_\varepsilon}^{\alpha\beta}, A] \left( \tilde{g}^{ij} \partial^2_{j\beta} u_\varepsilon \right) \rangle = \left\{ \partial_{\alpha} u_\varepsilon, [Q_{h_\varepsilon}^{\alpha\beta}, A] \left( \tilde{g}^{ij} \partial^2_{j\beta} u_\varepsilon \right) \right\} + o(1),
$$

where we use (4.18) as needed. To obtain our claim, it only remains to show that the first term in the above formula vanishes in the limit. To see this, we argue as before, invoking the symmetry of $h_\varepsilon^{\alpha\beta}$ and $\tilde{g}^{ij}$ in their indices and self-adjointness of $A$ (up to a compact operator):

$$
\langle \partial_{\alpha} u_\varepsilon, [Q_{h_\varepsilon}^{\alpha\beta}, A] \left( \tilde{g}^{ij} \partial^2_{j\beta} u_\varepsilon \right) \rangle = \langle \partial_{\beta}^2 u_\varepsilon, [Q_{h_\varepsilon}^{\alpha\beta}, A] \left( \tilde{g}^{ij} \partial^2_{i\alpha} u_\varepsilon \right) \rangle - \langle \partial_{\beta}^2 u_\varepsilon, [Q_{h_\varepsilon}^{\alpha\beta}, A] \left( \tilde{g}^{ij} \partial^2_{j\beta} u_\varepsilon \right) \rangle + o(1).
$$

(4.22)

Step 3: Conclusion. From (4.21), and estimates (4.3), (4.14), (4.15), we obtain

$$
\lim_{\varepsilon \to 0} \int \partial_{\alpha} u_\varepsilon \left[ A, P_{\text{time}} \varepsilon [h_\varepsilon^{\alpha\beta}] \right] \partial_\beta e_0 u_\varepsilon \, dx \
\lesssim \| Q_{h_\varepsilon} \|_{L^2(\Omega)} \| \partial^2 u_\varepsilon \|_{L^4} \| \Box u_\varepsilon \|_{L^4} + \| \partial (Q_{h_\varepsilon}) \|_{L^2(\Omega)} \| \partial u_\varepsilon \|_{L^4} \| \Box u_\varepsilon \|_{L^4} \
+ \| \partial_j (Q_{h_\varepsilon}) \|_{L^2(\Omega)} \| \partial^2 u_\varepsilon \|_{L^4} \| \partial u_\varepsilon \|_{L^4} \
\lesssim \omega_\varepsilon^{\delta_1} + \omega_\varepsilon^{\delta_1 - \frac{1}{2}} \to 0,
$$
as long as $\delta_1 > \frac{1}{2}$. \hfill \Box

Proof of Proposition 4.5. It suffices to pick $\delta_1 \in (\frac{1}{2}, 1)$ and $\delta_2 \in (\frac{1}{2}\delta_1, 1)$; for instance, $\delta_1 = \frac{5}{6}$ and $\delta_2 = \frac{4}{5}$. Now combine Lemma 4.6 with Lemmas 4.7, 4.8 and 4.9.

4.4. Conclusion of the Proof

Combining the results from the previous subsections we finish the proof of Theorem 3.

Proof of (1.18) in Theorem 3(b). By Proposition 4.5, whenever $A$ satisfies (4.4),

$$
0 = \lim_{\varepsilon \to 0} \langle A e_0 (u_\varepsilon - u), H_\varepsilon \rangle = \langle (\xi_0 - \beta^i \xi_i) \sigma, a \rangle - \langle (\xi_0 - \beta^i \xi_i) \lambda, a \rangle.
$$

Since $\nu$ is supported on the zero mass shell $\{g^{\alpha\beta} \xi_\alpha \xi_\beta = 0\}$, and since $\xi_0 - \beta^i \xi_i$ never vanishes on that set, see (3.4), it follows that $\langle \lambda, \tilde{a} \rangle = \langle \sigma, \tilde{a} \rangle$ for any $\tilde{a} \in C^\infty_c(S^* M)$.
which is odd and 1-homogeneous in $\xi$. Thus, according to Theorem 3.2(b2), for any such $\tilde{a}$,

$$
\int_{S^*M} \left[ g^{a\beta} \xi_\alpha \partial_{x^\beta} \tilde{a} - \frac{1}{2} \partial_{x^\alpha} g^{a\beta} \xi_\alpha \xi_\beta \partial_{x^\alpha} \tilde{a} \right] d\nu = - \int_{S^*M} \tilde{a} d(\nu\sigma) = - \int_{S^*M} \tilde{a} d(\nu\lambda).
$$

(4.23)

However, $\nu$ is even and $\lambda$ is odd, c.f. Theorem 3.2(b0); thus, whenever $\tilde{a}$ is even in $\xi$, $\langle \nu\lambda, \tilde{a} \rangle = 0$, and likewise the right-hand side of (4.23) vanishes as well in that case. Hence we see that (4.23) actually holds for any $\tilde{a} \in C^\infty_c(S^*M)$, as wished.

5. Nonlinear Wave Map Systems with Oscillating Coefficients

5.1. Proof of Theorem 2

We recall that Theorem 2 is concerned with sequences of solutions to

\begin{align*}
\Box g u_I^\varepsilon &= - \Gamma^f_{JK} (u_\varepsilon) g^{-1}_\varepsilon (du_J^\varepsilon, du_K^\varepsilon) + f_I^\varepsilon, \\
u, f : \mathbb{R}^{1+n} &\rightarrow \mathbb{R}, \quad I, J, K \in \{1, \ldots, N\}.
\end{align*}

(5.1)

We will reduce the study of the wave map system (5.1) to the case of wave maps into a flat target, as studied in Sect. 3 and 4. By repeating the arguments detailed in Sect. 3 we see that, by replacing $u_\varepsilon$ with $\chi u_\varepsilon$ for an arbitrary smooth cut-off function $\chi$, there is no loss of generality in assuming that the sequence $(u_\varepsilon)$ has uniformly bounded support.

Before proceeding with the proof, let us introduce the notation

$$
H_I^\varepsilon \equiv (\Box g - \Box g) u_I^\varepsilon, \quad Q_I^\varepsilon \equiv - \Gamma^f_{JK} (u_\varepsilon) g^{-1}_\varepsilon (du_J^\varepsilon, du_K^\varepsilon), \quad Q^I \equiv w- \lim_{\varepsilon \rightarrow 0} Q_I^\varepsilon.
$$

Hence we may rewrite (5.1) as

$$
\Box g u_I^\varepsilon = F_I^\varepsilon, \quad \text{where } F_I^\varepsilon \equiv - H_I^\varepsilon + Q_I^\varepsilon + f_I^\varepsilon.
$$

In addition to the H-measures defined in (1.16), we will need the H-measure

\begin{align*}
\left( (\partial_0 u_I^\varepsilon, \partial_1 u_I^\varepsilon, \ldots, \partial_n u_I^\varepsilon, F_I^\varepsilon)_{I=1}^N \right)_\varepsilon \overset{H}{\Rightarrow} \left( \left[ \tilde{\nu}^IJ^* \tilde{\sigma}^{IJ}^* \right]_{I,J=1}^N \right).
\end{align*}

We deal with the terms $H_I^\varepsilon$ and $Q_I^\varepsilon$ separately. For the former, it suffices to apply, with minor modifications, the arguments in Sect. 4.

**Lemma 5.1.** Under Hypotheses 1.6 and assuming that $u_\varepsilon \rightarrow u$ in $C^0_{loc}$, for any $A \in \Psi^0_{1,c}$ satisfying (4.4),

$$
\lim_{\varepsilon \rightarrow 0} \langle A e_0 (u_I^\varepsilon - u_I^*), g_{IL}(u_\varepsilon) H_{IL}^\varepsilon \rangle = 0.
$$
Proof. The proof consists of a small modification of the arguments used to prove Propositions 4.4 and 4.5. Here we only point out the modifications needed in the proof of Proposition 4.5, as the former is much simpler. Note that, by the local uniform convergence of \( u_\varepsilon \), it is enough to show that

\[
\lim_{\varepsilon \to 0} \langle A e_0(u_\varepsilon^I - u^I), g_{IL}(u) H_{\varepsilon}^L \rangle = 0.
\]

For simplicity of notation we suppress the dependence of \( g_{IL} \) on \( u \).

Similarly to Lemmas 4.3 and 4.6, we have

\[
2 \lim_{\varepsilon \to 0} \langle A e_0(u_\varepsilon^I - u^I), g_{IL} H_{\varepsilon}^L \rangle = \lim_{\varepsilon \to 0} \int g_{IL} \partial_\alpha (u_\varepsilon^I - u^I) [A, h_{\varepsilon}^{\alpha\beta}] \partial_\beta e_0(u_\varepsilon^L - u^L) \, dx.
\]

(5.2)

Indeed, the proofs of these lemmas consists of integrating by parts using the self-adjointness of \( A \) to produce commutators. With \( g_{IL} \) now in the bracket, the integration by parts generates terms with derivatives of \( g_{IL} \), which however are compact, as they have one fewer derivative on \( u_\varepsilon^I \). Using the self-adjointness also yields the same conclusion: e.g. in Step 2 of Lemma 4.6, again writing \( w_\varepsilon^I \equiv u_\varepsilon^I - u^I \), we find the commutator

\[
\langle g_{IL} e_0 w_\varepsilon^I, [A, \tilde{h}_{\varepsilon}^{00}] e_0^2 w_\varepsilon^I \rangle
\]

\[
= \langle g_{IL} \tilde{h}_{\varepsilon}^{00} e_0^2 w_\varepsilon^L, A e_0 w_\varepsilon^I \rangle + \langle e_0 w_\varepsilon^L, A (g_{IL} \tilde{h}_{\varepsilon}^{00} e_0^2 w_\varepsilon^I) \rangle + o(1)
\]

\[
= \langle g_{IL} \tilde{h}_{\varepsilon}^{00} e_0^2 w_\varepsilon^L, A e_0 w_\varepsilon^I \rangle + \langle g_{IL} \tilde{h}_{\varepsilon}^{00} e_0^2 w_\varepsilon^I, A e_0 w_\varepsilon^L \rangle + o(1)
\]

\[
= 2 \langle g_{IL} \tilde{h}_{\varepsilon}^{00} e_0^2 w_\varepsilon^L, A e_0 w_\varepsilon^I \rangle + o(1),
\]

due to the symmetry of \( g_{IL} \) in \( I, L \). Arguing similarly in the other steps, (5.2) is established.

The proofs of Proposition 4.4 and Lemmas 4.7 and 4.8 only require cosmetic modifications. In the proof of Lemma 4.9, the fact that \( I = L \) is used in a non-trivial way in the arguments involving the symmetry in \( \alpha, \beta \) of \( h_{\varepsilon}^{\alpha\beta} \) in (4.20) and (4.22). However, since we now sum over all \( I, L \) and \( g_{IL} = g_{LI} \), these arguments still apply: for instance, the analogue of (4.22) is now

\[
\langle g_{IL} \partial_\alpha^2 u_\varepsilon^I, [Q h_{\varepsilon}^{\alpha\beta}, A] \left( \tilde{g}^{ij} \partial_{ij}^2 u_\varepsilon^L \right) \rangle
\]

\[
= \langle g_{IL} \partial_\beta^2 u_\varepsilon^L, [Q h_{\varepsilon}^{\alpha\beta}, A] (g_{IL} \partial_\beta^2 u_\varepsilon^L) \rangle
\]

\[
= \langle g_{IL} \partial_\alpha^2 u_\varepsilon^I, [Q h_{\varepsilon}^{\alpha\beta}, A] \left( \tilde{g}^{ij} \partial_{ij}^2 u_\varepsilon^L \right) \rangle + o(1),
\]

where we exchanged \( \alpha \) with \( \beta \), \( I \) with \( L \) and \( i \) with \( j \) in the first equality. Here, we have also commuted through \( g_{IL} \) to place it on the right hand side; this follows similarly as for the commutation of \( \tilde{g}^{ij} \), since \( g_{IL} \) is independent of \( \varepsilon \).

In light of Theorem 3, the main remaining point in the proof of Theorem 2 is to characterize the contribution of \( Q^I \) to the transport equation. This is done in the next lemma.
Lemma 5.2. Assuming that Hypotheses 1.6 hold and that \( u^I_\varepsilon \to u^I \) in \( C^0_{\text{loc}} \), then for any \( A \in \Psi_{1,c}^0 \),

\[
\lim_{\varepsilon \to 0} (A(\partial_\gamma u^I_\varepsilon - \partial_\gamma u^I), Q^L_\varepsilon - Q^L) = (g^{\alpha\beta}[\Gamma^L_{JK}(u) + \Gamma^L_{KL}(u)] \partial_\beta u^K \tilde{\nu}^{IJ}_{\gamma\alpha}, \sigma^0(A)).
\]

Additionally, if \( \Gamma^L_{JK} \) are Christoffel symbols with respect to a Riemannian metric \( g = g_{IJ}(u) du^I \otimes du^J \), then

\[
\lim_{\varepsilon \to 0} (A(\partial_\gamma u^I_\varepsilon - \partial_\gamma u^I), g_{IL}(Q^L_\varepsilon - Q^L)) = -i(g^{\alpha\beta}(\partial_\gamma g_{IJ} \tilde{\nu}^{IJ}_{\gamma\alpha}, \sigma^0(A)) + 2i(g^{\alpha\beta}(\partial_\gamma g_{IK} \partial_\beta u^K \tilde{\nu}^{IJ}_{\gamma\alpha}, \sigma^0(A)).
\]

Proof. We have that \( w^- \lim Q^L_\varepsilon = w^- \lim -\Gamma^L_{JK}(u)g^{-1}(du^J, du^K) \), by continuity of \( \Gamma^L_{JK} \) and uniform convergence of \( g_\varepsilon \) and \( u_\varepsilon \). Hence, by Lemma 3.5,

\[
Q^L_\varepsilon \to Q^L = -\Gamma^L_{JK}(u)g^{-1}(du^J, du^K) \text{ in } L^2.
\]

The first part of the lemma is now a direct consequence of the trilinear compensated compactness of Lemma 3.6. Indeed, for any \( A \in \Psi_{1,c}^0 \), we have

\[
\lim_{\varepsilon \to 0} (A(\partial_\gamma u^I_\varepsilon - \partial_\gamma u^I), Q^L_\varepsilon - Q^L)
\]

\[
= \lim_{\varepsilon \to 0} \left( \Gamma^L_{JK}(u) A(\partial_\gamma u^I_\varepsilon - \partial_\gamma u^I), g^{-1}(du^J, du^K) - g^{-1}(du^J, du^K) \right)
\]

\[
= \lim_{\varepsilon \to 0} -\left( \Gamma^L_{JK}(u) A(\partial_\gamma u^I_\varepsilon - \partial_\gamma u^I), g^{-1}(du^J - u_\varepsilon^J, du^K - u_\varepsilon^K) \right)
\]

\[
+ \lim_{\varepsilon \to 0} \left( \Gamma^L_{JK}(u) A(\partial_\gamma u^I_\varepsilon - \partial_\gamma u^I), g^{-1}(du^J - u_\varepsilon^J, du^K) \right)
\]

\[
+ g^{-1}(du^J, du^K - u_\varepsilon^K)
\]

again by continuity of \( \Gamma^L_{JK} \) and uniform convergence of \( u_\varepsilon \) and \( g_\varepsilon \). By Lemma 3.6, the first limit on the right-hand side vanishes, hence we arrive at

\[
\lim_{\varepsilon \to 0} (A(\partial_\gamma u^I_\varepsilon - \partial_\gamma u^I), Q^L_\varepsilon - Q^L) = \left( \Gamma^L_{JK}(u) \tilde{\nu}^{IJ}_{\gamma\alpha}, g^{\alpha\beta}(\partial_\gamma u^K \sigma^0(A)) \right)
\]

\[
+ \left( \Gamma^L_{JK}(u) \tilde{\nu}^{IK}_{\gamma\alpha}, g^{\alpha\beta}(\partial_\gamma u^K \sigma^0(A)) \right).
\]

For the second part, we begin by recalling the formula for the Christoffel symbols,

\[
g_{IL}\Gamma^L_{JK} = \frac{1}{2} (g_{IKL} + g_{IKJ} - g_{JKI}),
\]

where \( g_{JKI} \equiv \partial_\gamma u^K \). and likewise for the other terms. Then

\[
2g_{IL}\Gamma^L_{JK}\partial_\beta u^K \tilde{\nu}^{IJ} = g_{IJ,K} \partial_\beta u^K \tilde{\nu}^{IJ} + (g_{IK,J} - g_{JKI}) \partial_\beta u^K \tilde{\nu}^{IJ}
\]

\[
= g_{JK}\partial_\beta u^K \tilde{\nu}^{IJ} + g_{IK,J} \partial_\beta u^K \tilde{\nu}^{IJ} - g_{IK,J} \partial_\beta u^K \tilde{\nu}^{IJ}
\]

\[
= \partial_\gamma g_{IJ} \tilde{\nu}^{IJ} + 2i g_{IK,J} \partial_\beta u^K \tilde{\nu}^{IJ}
\]

where in the last line we used the fact that \( \tilde{\nu}^{IJ} = (\tilde{\nu}^{IJ})^* \). To conclude, it now suffices to use the first part of the lemma, recalling that \( \Gamma^L_{JK} = \Gamma^L_{KJ} \). \( \square \)
Proof of Theorem 2.} We first note that \((g, (u^I)^{N}_{I=1}, (f^I)^{N}_{I=1})\) is a distributional solution of (5.1); this follows at once from Proposition 4.1 and (5.4).

Using the localization lemma 2.9, just as in the proof of Theorem 3.2, we find that

\[
\tilde{v}_{\alpha\beta}^{IJ} = \xi_\alpha \xi_\beta v^{IJ}, \quad \tilde{\lambda}_0^{IJ} = \tilde{\xi}_0^{IJ}, \quad \tilde{\sigma}_0^{IJ} = \tilde{\xi}_0^{IJ},
\]

for some Radon measures \(v^{IJ}, \lambda^{IJ}, \sigma^{IJ}\) and for each \(I\) and \(J\). Likewise, the measures \(v^{IJ}, \lambda^{IJ}, \sigma^{IJ}\) are supported on the zero mass shell of \(g\), and hence the measures \(v \equiv g_{IJ}v^{IJ}, \lambda \equiv g_{IJ}\lambda^{IJ}\) and \(\sigma \equiv g_{IJ}\sigma^{IJ}\) are also supported on the same set. Furthermore, \(v^{IJ}\) and hence also \(v\) are even, whereas \(\sigma\) and \(\lambda\) are odd. By the uniform convergence of both \(u_\varepsilon\) and \(g_\varepsilon\), and using the polarization identity,

\[
\lim_{\varepsilon \to 0} \int \mathbb{L}_{\alpha\beta}[u_\varepsilon]Y^\alpha Y^\beta \, dVol_{g_\varepsilon} - \int \mathbb{L}_{\alpha\beta}[u]Y^\alpha Y^\beta \, dVol_g = \lim_{\varepsilon \to 0} \int g_{IJ}(u) \left[ \partial_\alpha u_\varepsilon^I \partial_\beta u_\varepsilon^J - \partial_\alpha u_\varepsilon^J \partial_\beta u_\varepsilon^I \right] Y^\alpha Y^\beta \, dVol_g = \lim_{\varepsilon \to 0} \int g_{IJ}(u) \partial_\alpha(u_\varepsilon^I - u^I) \partial_\beta(u_\varepsilon^J - u^J) Y^\alpha Y^\beta \, dVol_g
\]

for any test vector field \(Y\). This proves part (a).

It remains to prove the propagation property of \(v\). We first note that

\[
\int_{S^*M} \left[ g^{\alpha\beta} \xi_\alpha \partial_{\xi^\beta} \tilde{a} - \frac{1}{2} \partial_{\xi^\gamma} g^{\alpha\beta} \xi_\alpha \xi_\beta (\partial_{\xi^\gamma} \tilde{a}) \right] \, d\nu^{IJ} = - \int_{S^*M} \tilde{a} \, d(\Omega \sigma^{IJ}). \tag{5.7}
\]

This is proved by repeating verbatim the arguments in the proof of Theorem 3.2(b2): the only difference is that we multiply the equation for \(\Box_g(Au_\varepsilon^I)\) with \(X(Au_\varepsilon^2)\) and the one for \(\Box_g(Au_\varepsilon^2)\) with \(X(Au_\varepsilon^1)\), c.f. (3.8). It follows that \(v^{IJ}\) satisfies the equation

\[
\int_{S^*M} \left[ g^{\alpha\beta} \xi_\alpha \partial_{\xi^\beta} \tilde{a} g_{IJ} - \frac{1}{2} \partial_{\xi^\gamma} g^{\alpha\beta} \xi_\alpha \xi_\beta (\partial_{\xi^\gamma} \tilde{a}) g_{IJ} \right] \, d\nu^{IJ} = - \int_{S^*M} \tilde{a} \, g_{IJ} \, d(\Omega \sigma^{IJ}), \tag{5.8}
\]

which is obtained from (5.7) by replacing \(a\) with \(a_{IJ}\). Setting \(\sigma \equiv g_{IJ}\sigma^{IJ}\), we have

\[
\int_{S^*M} \left[ g^{\alpha\beta} \xi_\alpha \partial_{\xi^\beta} \tilde{a} - \frac{1}{2} \partial_{\xi^\gamma} g^{\alpha\beta} \xi_\alpha \xi_\beta (\partial_{\xi^\gamma} \tilde{a}) \right] \, dv = - \langle g^{\alpha\beta} \xi_\alpha \partial_{\xi^\beta} g_{IJ} \, v^{IJ}, \tilde{a} \rangle - \langle \Omega \sigma, \tilde{a} \rangle.
\]

Here, repeating the arguments in Sect. 4.1, by the parity of the measures involved, it is clear that we need only consider \(\tilde{a}(x, \xi)\) to be odd in \(\xi\), or equivalently, to let \(A\) in Lemmas 5.1 and 5.2 satisfy (4.4). Then, Lemma 5.1 shows that no contribution to \(\Omega \sigma\) is made by the metric oscillations, \(H_\varepsilon\). The contributions from \(Q_\varepsilon\)
are non-trivial, as shown in the last part of Lemma 5.2. Since the second term in the right-hand side of (5.3) is imaginary, it follows from (5.6) that

\[ \langle \Re \sigma, a \rangle = -\langle g^{\alpha \beta} \xi_\alpha \partial_\chi g^{IJ} \nu^{IJ}, a \rangle + \langle \Re \lambda, a \rangle. \]

Combining the previous two computations yields

\[ \int_{S^* M} \left[ g^{\alpha \beta} \xi_\alpha \partial_\chi \tilde{a} - \frac{1}{2} \partial_\chi g^{\alpha \beta} \xi_\alpha \partial_\chi \tilde{a} \right] d\nu = -\langle \Re \lambda, \tilde{a} \rangle, \quad (5.9) \]

and we may take \( \lambda = 0 \) to recover the statement of Theorem 2.

### 5.2. Proof of Theorem 1

Theorem 1 follows as a simple application of Theorem 2.

**Proof of Theorem 1.** Setting \( N = 2 \), and labeling \( u^1 = \psi, u^2 = \omega, (1.10) \) is a wave map system from \( \mathcal{M} \times \mathbb{R} \) into the Poincaré plane, equipped with metric \( g \) and with Christoffel symbols \( \Gamma^I_{JK} \) as follows:

\[ g = 2 d\psi \otimes d\psi + \frac{1}{2} e^{-4\psi} d\omega \otimes d\omega, \]

\[ \Gamma^1_{JK} = \begin{cases} \frac{1}{2} e^{-4\psi}, & J = K = 2, \\ 0, & \text{otherwise}, \end{cases} \quad \Gamma^2_{JK} = \begin{cases} -2, & J \neq K, \\ 0, & \text{otherwise}. \end{cases} \]

Following the notation of Theorem 2, we set \( \mathbb{L}_{\alpha \beta} [\psi, \omega] = 2 \partial_\alpha \psi \partial_\beta \psi + \frac{1}{4} e^{-4\psi} \partial_\alpha \omega \partial_\beta \omega. \)

Let \( \tilde{\nu}^{II} \) and \( \tilde{\lambda}^{II} \) be as in the proof of Theorem 2. Further introduce the Radon measure

\[ \tilde{\mu} \equiv 2 \left[ \tilde{\nu}^{11} (\tilde{\lambda}^{11}) + \frac{1}{2} e^{-4\psi} \left[ \tilde{\nu}^{22} (\tilde{\lambda}^{22}) + \frac{1}{2} e^{-4\psi} \tilde{\nu}^{11} \tilde{\lambda}^{11} \right] \right]. \]

From the proof of Theorem 2, we have \( \tilde{\nu}_{\alpha \beta} = \tilde{\xi}_\alpha \tilde{\xi}_\beta \nu \) and \( \tilde{\lambda}_\gamma = \tilde{\xi}_\gamma \lambda \), where

\[ \nu \equiv 2\nu^1 + \frac{1}{2} e^{-4\psi} \nu^2, \quad \lambda \equiv 2\lambda^1 + \frac{1}{2} e^{-4\psi} \lambda^2, \]

where \( \nu \) satisfies the localization property stated as Theorem 1(b1) and the propagation property (5.9), which for \( \lambda = 0 \) yields Theorem 1(b2).

Let us now turn to the proof of Theorem 1(a). In Sect. 1.1, we have already justified that \( \text{Ric}(g)_{\alpha \lambda} = \text{Ric}(g)_{33} = 0 \). For the \((\alpha, \beta)\) direction, the result follows from (1.13) and Theorem 2(a).

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Declarations

Conflict of interest The authors have no financial or proprietary interests in any material discussed in this article.

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A. On Elliptic Gauge Conditions

In this appendix we collect some standard facts concerning elliptic gauge. In fact, the results that we require hold in the more general setting of three-dimensional spacetimes allowing a constant mean curvature spacelike foliation, see Definition A.1 below.

Let $M = (0, T) \times \mathbb{R}^n$ be a smooth manifold, covered by global coordinates $(t \equiv x^0, x^1, \ldots, x^n)$. As before, we take greek indices to range in $\{0, 1, \ldots, n\}$ and roman indices to range in $\{1, \ldots, n\}$, and assume the Einstein summation convention. Let $M$ be equipped with a Lorentzian metric $g = g_{\alpha\beta} dx^\alpha dx^\beta$, with inverse $g^{-1} = g^{\alpha\beta} \partial x^\alpha \partial x^\beta$. It will be convenient to consider the Cauchy frame defined in (3.3):

$$
\beta^i \equiv -g^{0i} g_{00}, \quad e_0 \equiv \partial_0 - \beta^i \partial_i, \quad \tilde{g}^{ij} \equiv g^{ij} - g^{0i} g^{0j} g_{00}.
$$

Define $N$ via $g^{00} = -N^{-2}$ and $\tilde{g}_{ij}$ as the inverse of the Riemannian metric $\tilde{g}^{ij}$. Then, $g$ may be written as

$$
g = -N^2 (dx^0)^2 + \tilde{g}_{ij} (dx^i + \beta^i dx^0) (dx^j + \beta^j dx^0).
$$

Note that $g_{ij} = \tilde{g}_{ij}$. The second fundamental form associated to constant $x^0$ hypersurfaces is

$$
K_{ij} \equiv -\frac{1}{2N} \mathcal{L}_{e_0} \tilde{g}_{ij} = -\frac{1}{2N} \left[ e_0 \tilde{g}_{ij} - \partial_j \beta^k \tilde{g}_{ki} - \partial_i \beta^k \tilde{g}_{kj} \right] = H_{ij} + \frac{1}{n} \tilde{g}_{ij} \tau,
$$

where $\tau$ and $H_{ij}$ are, respectively, the trace and the traceless part of $K_{ij}$. We easily compute that
\[ \tau \equiv \tilde{g}^{ij} K_{ij} = -\frac{1}{2N} \left[ \tilde{g}^{ij} e_0 \tilde{g}_{ij} - 2\partial k \beta^k \right], \quad \text{(A.2)} \]

\[ H_{ij} \equiv K_{ij} - \frac{1}{n} \tilde{g}_{ij} \tau = -\frac{1}{2N} \left[ e_0 \tilde{g}_{ij} - \frac{1}{n} \tilde{g}_{ij} \tilde{g}^{kl} e_0 \tilde{g}_{kl} \right] + \frac{1}{2N} \left[ \partial_j \beta^k \tilde{g}_{ki} + \partial_i \beta^k \tilde{g}_{kj} - \frac{2}{n} \partial_k \beta^k \tilde{g}_{ij} \right]. \quad \text{(A.3)} \]

Let \( R_{\alpha \beta} \) denote the Ricci tensor components of \( g \). We denote by \( \tilde{\mathcal{D}}^i \) and \( \tilde{R}_{ij} \) the covariant derivative and Ricci tensor components, respectively, of the Riemannian metric \( \tilde{g}_{ij} \). We have, see \([11, \text{Chapter VI.3}]\), that

\[ \tilde{\mathcal{D}}_j (H_{ij}) = -N^{-1} R_{0j} + (1 - 1/n) \partial_j \tau, \quad \text{(A.4)} \]

\[ \Delta \tilde{g} N = N^{-1} R_{00} + N |H|^2 - e_0 \tau + N \frac{\tau^2}{n}, \quad \text{(A.5)} \]

\[ \tilde{g}^{ij} \tilde{R}_{ij} = 2N^{-2} \left( R_{00} - \frac{1}{2} g_{00} \sigma^\alpha \beta R_{\alpha \beta} \right) + |H|^2 + \tau^2 (1 - 1/n), \quad \text{(A.6)} \]

\[ \tilde{\mathcal{D}}^i \partial_j \beta_i = -2 R_{0i} + 2 \partial^k N H_{ik} - \tilde{D}_k \partial_i \beta^k + \frac{2}{n} \tilde{D}_i \partial_k \beta^k + \left( N^{-1} \partial^i N + \tilde{\mathcal{D}}^i \right) \left[ \frac{1}{2N} (e_0 \tilde{g}_{ij} - \frac{1}{n} \tilde{g}_{ij} \tilde{g}^{kl} e_0 \tilde{g}_{kl}) \right] + 2N (1 - 1/n) \partial_i \tau, \quad \text{(A.7)} \]

with indices raised and lowered through \( \tilde{g}^{ij} \) and \( \tilde{g}_{ij} \), respectively. Furthermore,

\[ R_{ij} = \tilde{R}_{ij} - \frac{1}{N} \left( \tilde{D}_i \partial_j N + e_0 H_{ij} - \partial_j \beta^k H_{ik} - \partial_i \beta^k H_{jk} + e_0 \tau \frac{1}{n} \tilde{g}_{ij} \right) - 2H_{li} H^l_j - 2\tau \left( H_{ij} - \frac{\tau}{n} \tilde{g}_{ij} \right). \quad \text{(A.8)} \]

**Definition A.1.** We say that a metric \( g \) on \( \mathcal{M} \) of the form (A.1)

(a) is **spatially conformally flat** if the metric induced on constant \( x^0 \) hypersurfaces, \( \tilde{g}_{ij} \), satisfies \( \tilde{g}_{ij} = e^{2\gamma} \delta_{ij} \) for some conformal factor \( \gamma \) defined on \( \mathcal{M} \).

(b) has a **constant mean curvature** spacelike foliation if \( \tau \) is constant on each \( \{x^0 = \text{constant}\} \) slice, and we let the constant be either \( \tau = x^0 \) or \( \tau = 0 \). In the latter case, we say that the foliation is **maximal**.

**Remark A.2.** (Elliptic gauge) For \( n = 2 \), Riemannian manifolds are locally conformally flat. Hence, the condition that \( g \) (globally) takes the form (A.1) and that \( \tilde{g}^{ij} \) is (globally) conformally flat is sometimes referred to as an **elliptic gauge** condition for Ricci-flat \( g \) in \( 1 + 2 \) dimensions, see e.g. [34]. That \( g \) has a constant mean curvature spacelike foliation is also sometimes referred to as the **CMC gauge condition**, though, strictly speaking, it requires a geometric condition of constant mean curvature on the initial data, see [3] and [32, Footnote 3].

The following lemma justifies Hypotheses 1.6(a) as well as Lemma 1.4:

**Lemma A.3.** Let \( (g_\epsilon) \epsilon \) be a sequence of metrics on \( \mathcal{M} \) of the form (A.1) which are spatially conformally flat with a fixed constant mean curvature time foliation. Suppose, further, that these satisfy
• $g_\varepsilon \to g$ in $C_{\text{loc}}^0$, for some Lorentzian metric $g$ which is also spatially conformally flat with the same constant mean curvature time foliation, and that $g_\varepsilon$ is uniformly bounded in $W_{\text{loc}}^{1,2}$ entrywise;
• the Ricci tensor of $g_\varepsilon$, $\text{Ric}(g_\varepsilon)$, is uniformly bounded in $L_{\text{loc}}^2$ entrywise.

Thus, for any $c \in \mathbb{N}$, assumption, (A.5) reduces to $\partial_k H_{ij}$ showing that $\text{spacetime}$. Then, if $n = 2$, we also have

\begin{itemize}
  \item $\delta^{ij} \partial_i^2 g_{\varepsilon}^\alpha\beta$ are bounded in $L_{\text{loc}}^2$ and $e_0 \tilde{g}_{ij}^{\varepsilon} \to e_0 \tilde{g}_{ij}$ strongly in $L_{\text{loc}}^4$;
  \item $\text{Ric}(g_\varepsilon) \to \text{Ric}(g)$ in the sense of distributions.
\end{itemize}

**Proof.** We consider equations (A.4)–(A.8) in the spatially conformally flat case, where $\tilde{g}_{ij} = e^{2\gamma} \delta_{ij}$ for some function $\gamma$. In what follows, raised indices indicate contractions with the Euclidean metric.

**Step 1:** showing $\partial_k H_{\varepsilon}^{ij}$ are uniformly bounded in $L_{\text{loc}}^2$ and $H_{\varepsilon}^{ij}$ are bounded in $L_{\text{loc}}^4$. Since $H_{ij}$ is trace-free, (A.4) takes the form

$$\delta^{ij} H_{ij} = -N^{-1} R_{0j} + (1 - 1/n) \partial_j \tau = -N^{-1} R_{0j} ,$$

where the right hand side is clearly bounded in $L_{\text{loc}}^2$ by the assumptions. Thus, the spatial divergence of $H$ is in $L_{\text{loc}}^2$. The spatial divergence is an elliptic operator when acting on two-by-two traceless symmetric matrices, hence we conclude that $\partial_k H_{ij} \in L_{\text{loc}}^2$ for any spatial derivative $k$. For $c \in (0, T)$,

$$\| H_{ij} \|^2_{L_{\text{loc}}^4(\{x^0 = c\})} \leq \| H_{ij} \|^2_{L_{\text{loc}}^2(\{x^0 = c\})} + \| \partial_k H_{ij} \|^2_{L_{\text{loc}}^2(\{x^0 = c\})} \lesssim 1,$$

independently of $c$, hence after integration in $x^0$ we have that $H_{ij}$ are bounded in $L_{\text{loc}}^4$ in spacetime.

**Step 2:** showing that $\partial_i H_{\varepsilon}^{ij}$ are uniformly bounded in $L_{\text{loc}}^2$ and $\partial_i g_{\varepsilon}^\alpha\beta$ are compact in $L_{\text{loc}}^4$.

Let us begin with the case $\alpha = \beta = 0$, where we recall $g^{00} = -N^{-2}$. By the conformal assumption, (A.5) reduces to

$$\Delta N = (n - 2) \delta^{ij} N \partial_i \gamma + N^{-1} R_{00} + N [H]^2 - e_0 \tau + N \frac{\gamma^2}{n} .$$

(A.10)

From the previous step we find that, if $n = 2$, the right hand side is bounded in $L_{\text{loc}}^2$. Thus, for any $c \in (0, T)$,

$$\sup_{i,j} \| \partial_i^2 \partial_j N \|_{L_{\text{loc}}^2(\{x^0 = c\})} \lesssim \| \delta^{ij} \partial_i^2 \partial_j N \|_{L_{\text{loc}}^2(\{x^0 = c\})} + \| N \|_{L_{\text{loc}}^2(\{x^0 = c\})}$$

$$\implies \sup_{i,j} \| \partial_i^2 \partial_j N \|_{L_{\text{loc}}^2} \lesssim \| \delta^{ij} \partial_i^2 \partial_j N \|_{L_{\text{loc}}^2} + \| N \|_{L_{\text{loc}}^2} \lesssim 1 ,$$

(A.11)

after integration in $x^0$. Since $W_{\text{loc}}^{2,2}(\{x^0 = c\})$ embeds compactly into $W_{\text{loc}}^{1,4}(\{x^0 = c\})$, independently of $c$, integration over $x^0$ yields the claim for $g_{\varepsilon}^{00}$.

Similarly, we have the equation

$$\Delta \gamma = -\frac{1}{2} (n - 2) \delta^{ij} \partial_i \gamma \partial_j \gamma$$

$$- \left[ 2 N^{-2} \left( R_{00} - \frac{1}{2} g^{00} g_{\varepsilon}^{\alpha\beta} R_{\alpha\beta} \right) + |H|^2 + \frac{n - 1}{n} \gamma^2 \right] \frac{e^{2\gamma}}{2(n - 1)} .$$

(A.12)
which is the simplification of (A.6) under the spatial conformal flatness assumption. Thus, by repeating the arguments used for $g^0$, and using Step 1, one can show that $\partial^2_{ij} \gamma$ are uniformly bounded in $L^2_{loc}$ and hence that $\partial_t \gamma$ is compact in $L^1_{loc}$. This concludes the step in the case $(\alpha, \beta) = (i, j)$.

Finally, let us deal with the case $(\alpha, \beta) = (0, i)$. We consider (A.7) or, more precisely, the equation

$$
\Delta \beta_i = -\frac{n-2}{n} \partial^2_{ij} \beta^j - 2R_{0i} + 2\partial^k N H_{ik} + 2 \left( \partial_i \gamma \partial_j \beta^j - \partial_j \gamma \partial^j \beta_i \right) + 2N(1 - 1/n)\partial_t \tau ,
$$

which is obtained from the former when $\tilde{g}_{ij} = e^{2\gamma} \delta_{ij}$. If $n = 2$, the first term on the right hand side vanishes. By the previous steps, for $c \in (0, T)$, the right hand side is bounded in $L^{4/3}_{loc}((x^0 = c))$ independently of $c$, and hence $\beta$ is bounded in $W^{2,4/3}_{loc}((x^0 = c)) \subset W^{1,4}_{loc}((x^0 = c))$ independently of $c$ as well. With this improvement in the regularity of $\beta$, we can now see that the right hand side of (A.13) is in fact bounded in $L^2_{loc}((x^0 = c))$ independently of $c$, and so $\beta$ is bounded in $W^{2,2}_{loc}((x^0 = c))$ independently of $c$. As before we deduce that, entrywise, $\partial^2_{ij} \beta$ are uniformly bounded in $L^2_{loc}$ and $\partial_t \beta$ is compact in $L^4_{loc}$.

**Step 3:** showing $e_0 \tilde{g}^{ij}_{\varepsilon}$ converges strongly in $L^4_{loc}$. From (A.2), noting that $\tilde{g}^{ij}_{\varepsilon} e_0 \tilde{g}_{ij} = 2 n e_0 \gamma$ under the spatial conformal flatness assumption, we deduce by the previous step that $e_0 \gamma$ is compact in $L^4_{loc}$. Thus, so is $e_0 \tilde{g}^{ij}_{\varepsilon} = -2 e_0 \gamma \delta_{ij}$, entrywise.

**Step 4:** showing that $H^{ij}_{\varepsilon}$ is uniformly bounded in $W^{1,2}_{loc}$, hence compact in $L^4_{loc}$. From Step 1, it remains only to show that $e_0 H_{ij}$ is bounded in $L^2_{loc}$. For this, we appeal to (A.8), which under our assumptions takes the form

$$
R_{ij} = -\delta_{ij} \nabla \gamma - (n - 2)[\partial^2_{ij} \gamma + \delta_{ij} \partial_k \gamma \partial^k \gamma - \partial_i \gamma \partial_j \gamma] \\
- \frac{1}{N} \left( \partial^2_{ij} N^2 - 2\partial_i \gamma \partial_j N + \partial_{ij} \partial^k \gamma \partial_k N \right) \\
- \frac{1}{N} \left( e_0 H_{ij} - \partial_j \beta^k H_{ik} - \partial_i \beta^k H_{jk} + \frac{e_0 \tau}{n} \tilde{g}_{ij} \right) - 2 H_{ij} H_j^l \\
- 2\tau \left( H_{ij} - \frac{\tau}{n} \tilde{g}_{ij} \right),
$$

(A.14)

By the previous steps all terms other than $e_0 H_{ij}$ are bounded in $L^2_{loc}$, hence we can conclude.

**Step 5:** the convergence of $\text{Ric}(g_{\varepsilon})$. Consider now the equations (A.10), (A.13) and (A.14) as equations for $R_{00}$, $R_{0j}$ and $R_{ij}$, respectively. By the previous steps, we see that all the nonlinearities are compact in $L^1_{loc}$. Hence, we deduce that $\text{Ric}(g_{\varepsilon})$ converges to $\text{Ric}(g)$ in the sense of distributions.

\[ \square \]

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