Tachyon Condensation, Boundary State and Noncommutative Solitons

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(March 27, 2022)

Abstract

We discuss the tachyon condensation in a single unstable D-brane in the framework of boundary state formulation. The boundary state in the background of the tachyon condensation and the NS $B$-field is explicitly constructed. We show in both commutative theory and noncommutative theory that the unstable D-branes behaves like an extended object and eventually reduces to the lower dimensional D-branes as the system approaches the infrared fixed point. We clarify the relationship between the commutative field theoretical description of the tachyon condensation and the noncommutative one.

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The fate of the unstable D-branes due to the tachyon condensation has been one of centers of interest in string theory since the seminal papers by Sen [1], where he conjectured that the unstable D-branes may behave like solitons corresponding to lower dimensional D-branes. Since the tachyon condensation [2] is an off-shell phenomenon, we need to resort to the second quantized string theory in order to understand this noble phenomenon. There are two approaches available to this subject; the Witten’s open string field theory [3] with the level truncation [4,5] and the boundary string field theory (BSFT) [6,7]. As the tachyon condensation occurs the open string field acquires an expectation value. In general it is quite difficult to solve the string field equation which involves an infinite number of components. One may solve the string field equation by truncating the string field to the first few string levels. This has been a useful practical tool to discuss the tachyon condensation. Many aspects of the tachyon condensation physics [8] were explored by this method. However, this method provides numerical results in most cases and sometimes yields qualitative ones only.

The second approach, which is based on the background independent string field theory [6], has been developed recently [7] as an alternative tool to the string field theory with level truncation. It deals with the disk partition function of the open string theory, where the boundary of the disk is defined by the trace of the two ends of the open string. The configuration space for the BSFT is the space of two dimensional world-sheet theories on the disk with arbitrary boundary interactions. Endowing it with the antibracket structure in terms of the world-sheet path integral on the disk and the BRST operator, one can define \( S \), the action of the target space theory. The relationship between the BSFT action, \( S \) and the disk partition function \( Z \) is clarified in ref. [7] as

\[
S = \left( 1 + \beta^i \frac{\partial}{\partial g^i} \right) Z
\]

(1)

where \( g^i \) are the couplings of the boundary interactions and \( \beta^i \) are the corresponding world-sheet \( \beta \)-functions. Choosing the tachyon profile, \( T(X) \), as for the boundary interaction, one
can obtain the effective action for tachyon field \[ S \]. Evaluation of the effective action may be simplified considerably if we introduce a large \( NS \) B-field \[ 10–15 \]. Recent works also show that \( S \) may coincide with \( Z \) if we introduce supersymmetry to describe the \( D-\bar{D} \) system \[ 16–18 \]. The disk partition function \( Z \) for the \( D-\bar{D} \) system has been explicitly evaluated in recent papers \[ 16,19 \].

In this paper we discuss the tachyon condensation in the simplest setting, i.e., a single unstable D-brane in the bosonic string theory, adopting the boundary state formulation, which developed by Callan, Lovelace, Nappi and Yost \[ 20 \] sometime ago. The advantage of this approach is that one can explicitly construct the quantum states corresponding to the unstable D-brane systems. Hence, the couplings of the system to the various string states are readily obtained so it helps us to understand how the system evolves as the condensation occurs. This approach is closely related to the second one, the BSFT in that the normalization factor of the boundary state is simply the disk partition function \( Z \). Since the constructed boundary state is given as a quantum state of the closed string field, it may also help us to understand its relationship to the first approach based on the open string field theory, if we appropriately utilize the open-closed string duality. We extend the boundary state formulation to the case of the noncommutative open string theory in order to discuss the noncommutative tachyon in the same framework. The relationship of the commutative theory and the noncommutative theory on the tachyon condensation can be understood in this framework along the line of the equivalence between the commutative theory and the noncommutative theory of open string \[ 21,22 \].

II. BOUNDARY ACTION AND BOUNDARY STATE

We begin with the boundary state construction developed in ref. \[ 20 \] and establish the relationship between the boundary action and the boundary state in more general cases. The boundary state formulation is based on a rather simple observation: It utilizes the open-closed string duality. The disk diagram in the open string theory can be viewed equivalently...
as the disk diagram in the closed string theory.

![Disk Diagrams](image)

**Figure 1:** a. Disk diagram in open string theory, b. Disk diagram in closed string theory

In the open string theory it depicts an open string appearing from the vacuum, then subsequently disappearing into the vacuum while in the closed string theory it depicts a closed string propagating from the boundary of the disk, then disappearing. This open-closed string duality is also useful when we deal with the cylindrical diagram. The cylindrical diagram make an appearance both in the open string theory and in the closed string theory. However, it can be interpreted differently in two theories. In the open string theory it describes a one-loop amplitude while in the closed string it describes a tree level amplitude. Since the closed string description often turns out to be simpler than its open string counterpart, the boundary state formulation has been employed as a practical method to evaluate the open string diagrams by making use of this open-closed string duality. It is proved to be extremely useful especially when we discuss various interactions between the $D$-brane and the open string.
We define the boundary state $|X\rangle$ by the following eigenvalue equation in the closed string theory

$$\hat{X}^\mu |X\rangle = X^\mu |X\rangle. \quad (2)$$

Since $\hat{X}^\mu$ and $X^\mu$ are defined on $\partial M$, the boundary of the disk, we may expand them as

$$\hat{X}^\mu(\sigma) = \hat{x}_0^\mu + \sqrt{\frac{\alpha'}{2}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left\{ (a_n^\mu + \tilde{a}_n^\mu \dagger) e^{2i n \sigma} + (a_n^\mu + \tilde{a}_n^\mu \dagger) e^{-2i n \sigma} \right\} \quad (3a)$$

$$X^\mu(\sigma) = x_0^\mu + \sqrt{\frac{\alpha'}{2}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left( x_n^\mu e^{2i n \sigma} + \bar{x}_n^\mu e^{-2i n \sigma} \right) \quad (3b)$$

where $\sigma \in [0, \pi]$, $\mu = 0, \ldots, d-1$, and

$$a_n = \frac{i}{\sqrt{n}} \alpha_n^\mu, \quad a_n^\dagger = -\frac{i}{\sqrt{n}} \bar{\alpha}_n^\mu, \quad \tilde{a}_n = \frac{i}{\sqrt{n}} \tilde{\alpha}_n^\mu, \quad \tilde{a}_n^\dagger = -\frac{i}{\sqrt{n}} \bar{\tilde{\alpha}}_n^\mu.$$

They satisfy the following commutation relationship

$$[a_n^\mu, a_m^\nu \dagger] = (g^{-1})^{\mu\nu} \delta_{nm}, \quad [\tilde{a}_n^\mu, \tilde{a}_m^\nu \dagger] = (g^{-1})^{\mu\nu} \delta_{nm}. \quad (4)$$

Hence the eigenvalue equation can be rewritten in terms of the left movers and the right movers as

$$\hat{x}_0^\mu |X\rangle = x_0^\mu |X\rangle$$

$$\left(a_n^\mu + \tilde{a}_n^\mu \dagger\right) |X\rangle = x_n^\mu |X\rangle \quad (5)$$

$$\left(a_n^\mu + \tilde{a}_n^\mu \dagger\right) |X\rangle = \bar{x}_n^\mu |X\rangle.$$

This eigenvalue equations determine the boundary state $|X\rangle = |x, \bar{x}\rangle$ up to a normalization factor

$$|x, \bar{x}\rangle = N(x, \bar{x}) \prod_{n=1} \exp \left( -a_n^\mu \tilde{a}_n^\nu \dagger g_{\mu\nu} + a_n^{\mu\dagger} x_n^\nu g_{\mu\nu} + \bar{x}_n^\mu \tilde{a}_n^\nu \dagger g_{\mu\nu} \right) |0\rangle \quad (6)$$

where $a_n |0\rangle = \tilde{a}_n |0\rangle = 0$. Requiring the completeness relation

$$\int D[x, \bar{x}] |x, \bar{x}\rangle \langle x, \bar{x}| = I, \quad (7)$$
we may fix the normalization factor

$$\langle x, \bar{x} \rangle = \prod_{n=1}^{\infty} \exp \left\{ -\frac{1}{2} \bar{x}_n x_n - a_n^\dagger \bar{a}_n + a_n^\dagger x_n + \bar{x}_n \bar{a}_n^\dagger \right\} \langle 0 \rangle$$  \hspace{1cm} (8)$$

where the space-time indices are suppressed and the contraction with the metric $g_{\mu\nu}$ is implied.

We will make use of the set of the boundary states, \{\langle x, \bar{x} \rangle\} as a basis to construct various boundary states, which correspond to the disk diagrams with nontrivial backgrounds: For a given boundary action $S_{\partial M}$, the boundary state is defined as

$$|B\rangle = \int D[x, \bar{x}] e^{i S_{\partial M}[x, \bar{x}]} |x, \bar{x}\rangle.$$  \hspace{1cm} (9)$$

Here $S_{\partial M}[x, \bar{x}]$ is the boundary action evaluated with the boundary condition

$$X^\mu|_{\partial M} = x_0^\mu + \sqrt{\frac{\alpha'}{2}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left( x_n^\mu e^{2 in\sigma} + \bar{x}_n^\mu e^{-2 in\sigma} \right).$$  \hspace{1cm} (10)$$

The requirement of the completeness relation Eq.(7) and the definition of the boundary state, Eq.(9) are consistent with the closed string field theory. The boundary state in fact defines a quantum state of the closed string field. For a free string, $S_{\partial M} = 0$,

$$|B_{\text{Free}}\rangle = \int D[x, \bar{x}] |x, \bar{x}\rangle = \sqrt{\det g} \prod_{n=1}^{\infty} \exp \left( a_n^\dagger \bar{a}_n^\dagger g_{\mu\nu} \right) |0\rangle,$$  \hspace{1cm} (11)$$

which satisfies the boundary condition

$$a_n^\dagger |B_{\text{Free}}\rangle = \bar{a}_n^\dagger |B_{\text{Free}}\rangle, \quad \bar{a}_n^\dagger |B_{\text{Free}}\rangle = a_n^\dagger |B_{\text{Free}}\rangle.$$  \hspace{1cm} (12)$$

This boundary condition, of course, nothing but the Neumann boundary condition, $\partial_\tau X^\mu|_{\partial M} = 0$. The boundary state, satisfying the Dirichlet condition $\dot{X}^\mu |B\rangle = 0$, is simply obtained by taking $x^\mu = x_n^\mu = \bar{x}_n^\mu = 0$ in Eq.(8)

$$|B_{\text{Dirichlet}}\rangle = \sqrt{\det g} \prod_{n=1}^{\infty} \exp \left( -a_n^\dagger \bar{a}_n^\dagger g_{\mu\nu} \right) |0\rangle.$$  \hspace{1cm} (13)$$

It follows that the boundary state corresponding to a flat D-p-brane is given as

$$|Dp\rangle = \frac{T_p}{g_s} \sqrt{\det g} \prod_{n=1}^{\infty} \exp \left( a_n^\dagger \bar{a}_n^\dagger g_{ij} - a_n^\dagger \bar{a}_n^\dagger g_{ab} \right) |0\rangle.$$  \hspace{1cm} (14)$$
where \( i = 0, \ldots, p, a = p + 1, \ldots, d - 1 \). Here \( T_p \) and \( g_s \) are the tension of the \( Dp \)-brane and the string coupling constant respectively.

One of the well-known examples of the boundary state is the open string in a constant \( U(1) \) background. The \( U(1) \) background yields the following boundary action for an open string

\[
S_F = \int_{\partial M} d\hat{\tau} A_\mu \frac{\partial X^\mu}{\partial \hat{\tau}} = \frac{1}{2} \int_{\partial M} d\hat{\tau} F_{\mu\nu} \frac{\partial X^\mu}{\partial \hat{\tau}} \tag{15}
\]

where \( \hat{\tau} \) is a proper-time parameter along \( \partial M \)

\[
\hat{\tau} = \begin{cases} 
\tau - 1 : \hat{\tau} \in [-1, 0] \\
-\tau + 1 : \hat{\tau} \in [0, 1].
\end{cases} \tag{16}
\]

In the closed string world-sheet coordinates the boundary interaction reads as

\[
S_F = \frac{1}{2} \int_{\partial M} d\sigma F_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma} \tag{17}
\]

The boundary interaction Eq.(15) yields the boundary condition for the open string as

\[
\frac{1}{2\pi \alpha'} g_{\mu\sigma} \partial_\sigma X^\nu - F_{\mu\nu} \partial_\tau X^\nu = 0, \quad \text{on} \ \partial M. \tag{18}
\]

In the closed string world-sheet coordinates we get the boundary condition as

\[
\frac{1}{2\pi \alpha'} g_{\mu\sigma} \partial_\tau X^\nu - F_{\mu\nu} \partial_\sigma X^\nu = 0, \quad \text{on} \ \partial M. \tag{19}
\]

We may transcribe it into the boundary condition to be imposed on the boundary state

\[
a_n|B_F\rangle = (g + 2\pi \alpha' F)^{-1}(g - 2\pi \alpha' F)\tilde{a}_n^\dagger|B_F\rangle, \tag{20a}
\]

\[
\tilde{a}_n|B_F\rangle = (g - 2\pi \alpha' F)^{-1}(g + 2\pi \alpha' F)a_n^\dagger|B_F\rangle. \tag{20b}
\]

making use of the closed string mode expansion of \( \hat{X}^\mu(\tau, \sigma) \)

\[
\hat{X}^\mu(\tau, \sigma) = \hat{x}^\mu + 2\alpha' p^\mu \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \left( \alpha_n^\mu e^{-2i n(\tau - \sigma)} + \bar{\alpha}_n^\mu e^{-2i n(\tau + \sigma)} \right)
\]

\[
= \hat{x}^\mu + 2\alpha' p^\mu \tau + \sqrt{\frac{\alpha'}{2}} \sum_{n = 1} \frac{1}{\sqrt{n}} \left( a_n^\mu e^{-2i n(\tau - \sigma)} + \bar{a}_{n}^\mu e^{2i n(\tau - \sigma)} + \tilde{a}_n^\mu e^{-2i n(\tau + \sigma)} + \bar{\tilde{a}}_n^\mu e^{2i n(\tau + \sigma)} \right). \tag{21}
\]
With the given boundary condition Eq. (10), the boundary action is evaluated as

\[ S_F = (2\pi\alpha')^{\frac{i}{2}} \sum_{n=1}^{\bar{x}} F_{\mu\nu} x^\nu_n. \] (22)

Then a simple algebra Eq. (9) leads us to the boundary state \( |B_F \rangle \)

\[ |B_F \rangle = \mathcal{T}_s P \prod_{n=1} \frac{\det (g + 2\pi\alpha'F)}{\det (g - 2\pi\alpha'F)^{-1}} \exp \left\{ a_n^\dagger g (g + 2\pi\alpha'F)^{-1} (g - 2\pi\alpha'F) a_n^\dagger \right\} |0 \rangle \] (23)

which satisfies the desired boundary condition Eq. (20). It should be noted that the normalization factor of the boundary state is the well-known Dirac-Born-Infeld Lagrangian

\[ Z = \mathcal{T}_s P \prod_{n=1} \frac{\det (g + 2\pi\alpha'F)^{-1}}{\sqrt{\det (g + 2\pi\alpha'F)}}, \] (24)

where we make use \( \zeta(0) = \sum_{n=1} 1 = -\frac{1}{2} \). It can be also obtained by evaluating the Polyakov string path integral on a disk [24].

The relationship between the boundary action and the boundary state observed in the case of the \( U(1) \) background can be established in more general cases. In order to see this explicitly let us introduce a boundary action of more general form as

\[ S = S_M + S_{\partial M}, \] (25)

\[ = -\frac{1}{4\pi\alpha'} \int_M d\tau d\sigma \sqrt{-h} h^{\alpha\beta} g_{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu + \frac{i}{2} \int_{\partial M} d\sigma X^\mu M_{\mu\nu} X^\nu. \]

Here \( M_{\mu\nu} = M_{\mu\nu} \left( \frac{1}{i} \frac{\partial}{\partial \sigma} \right) \) is a differential/integral operator in \( \sigma \). From this action we get a bulk equation on \( M \) as usual

\[ (\partial_\tau^2 - \partial_\sigma^2) X^\mu = 0, \] (26)

and a boundary condition to be imposed on \( \partial M \)

\[ \frac{1}{2\pi\alpha'} g_{\mu\nu} \partial_\tau X^\nu - i M_{\mu\nu} \left( \frac{1}{i} \frac{\partial}{\partial \sigma} \right) X^\nu = 0. \] (27)

Making use of the mode expansion of \( \hat{X}^\mu \), Eq. (21), we may transcribe the boundary condition into operator equations acting on the boundary state as
\[ \hat{p}^\mu |B\rangle = i\pi (g^{-1}M)^{\mu\nu}\hat{x}^\nu |B\rangle, \]
\[ a^\nu_n |B\rangle = \left( g + \frac{\pi\alpha'}{n}M(2n) \right)^{-1} \left( g - \frac{\pi\alpha'}{n}M(2n) \right) \hat{a}^\nu_n |B\rangle, \]
\[ \hat{a}^\nu_n |B\rangle = \left( g + \frac{\pi\alpha'}{n}M(-2n) \right)^{-1} \left( g - \frac{\pi\alpha'}{n}M(-2n) \right) a^\nu_n |B\rangle. \] (28)

From the action we see that
\[ M(2n)^T = M(-2n), \] (29)
which ensures consistency of our construction.

Given the boundary conditions Eq.(28) one can determine the boundary state, but only up to a normalization factor. Since the normalization factor is also often important, we should find a way to fix it. To this end one may calculate couplings of the system to the closed string degrees of freedom. Interpreting them as linear variations of the effective action, which is supposed to be obtained as the disk partition function, one may fix the normalization factor. This procedure has been applied to the open string in the constant \( U(1) \) background in ref. [25]. However, it would be involved in more general cases. Here, in order to fix the normalization factor we simply take Eq.(9) as the definition of the boundary state. With the given boundary condition Eq.(10) we find that the boundary action \( S_{\partial M}[x, \bar{x}] \) is obtained as
\[ S_{\partial M} = i\pi \frac{x^\mu M_{\mu\nu}(0)x^\nu}{2} + i\pi\frac{\alpha'}{2} \sum_{n=1}^\infty \frac{1}{n} \bar{x}_n M(2n)_{\mu\nu}x_n. \] (30)

Then a Gaussian integral of Eq.(30) brings us to an explicit expression of the boundary state
\[ |B\rangle = Z_{\text{Disk}} \exp \left\{ a^\dagger_n g \left( g + \frac{\pi\alpha'}{n}M(2n) \right)^{-1} \left( g - \frac{\pi\alpha'}{n}M(2n) \right) \hat{a}^\dagger_n \right\} |0\rangle, \] (31)
\[ Z_{\text{Disk}} = \frac{T_p}{g_s} \frac{1}{\sqrt{\det M(0)}} \prod_{n=1}^\infty \det \left( g + \frac{\pi\alpha'}{n}M(2n) \right)^{-1}. \]

Here, \( Z_{\text{Disk}} \) is the disk partition function with the boundary action \( S_{\partial M} \). It is easy to see that the boundary state \( |B\rangle \) readily satisfies the desired boundary condition, Eq.(28). Eq.(31) exhibits clearly the relationships between the normalization factor, the disk partition function and the boundary condition in the framework of boundary state formulation.
III. TACHYON CONDENSATION AND BOUNDARY STATE

Being equipped with the boundary state formulation given in the previous section, we construct the boundary state in the background of the tachyon condensation and discuss some important physical properties of the unstable D-brane systems. The tachyon condensation introduces the following boundary interaction in the closed string world-sheet coordinates

$$S_T = i \int_{\partial M} d\sigma T(X).$$

(32)

In order to construct the boundary state explicitly for the string in the tachyon background we consider a simple tachyon profile, \( T(X) = u_{\mu\nu}X^\mu X^\nu \). In terms of the normal modes \( S_T \) is written as

$$S_T = i\pi x^\mu u_{\mu\nu}x^\nu + i\pi \alpha' \sum_{n=1}^1 \frac{1}{n} \bar{x}^\mu_n u_{\mu\nu} x^\nu_n$$

(33)

With this boundary action we construct the boundary state, following Eq.(9),

$$|B_T\rangle = \int D[x, \bar{x}] e^{iS_T} |x, \bar{x}\rangle$$

$$= \int D[x, \bar{x}] e^{-\pi x_n x_n} \exp \left\{ -\frac{1}{2} \bar{x}^\mu_n \left( g + \frac{2\pi \alpha'}{n} u \right) x_n - \bar{a}_n^\dagger g a_n^\dagger + \bar{a}_n^\dagger g x_n + \bar{x}_n g \tilde{a}_n^\dagger \right\} |0\rangle$$

$$= Z_{\text{Disk}} \exp \left\{ \bar{a}_n^\dagger g \left( g + \frac{2\pi \alpha'}{n} u \right)^{-1} \left( g - \frac{2\pi \alpha'}{n} u \right) \tilde{a}_n^\dagger \right\} |0\rangle,$$

(34)

$$Z_{\text{Disk}} = \frac{T_p}{g_s} \frac{1}{\sqrt{\det(u)}} \prod_{n=1}^1 \det \left( g + \frac{2\pi \alpha'}{n} u \right)^{-1}$$

where we suppress the space-time indices. It is easy to confirm that this boundary state satisfies the appropriate boundary condition

$$\left( \frac{1}{2\pi \alpha'} g_{\mu\nu} \partial_\tau \bar{X}^\nu - 2iu_{\mu\nu} \bar{X}^\nu \right) |B_T\rangle = 0.$$
\[ S_{\partial M} = S_T + S_F, \]  

(36)

which corresponds to the case where

\[ M(0) = 2u, \quad M(2n) = 2nF + u, \quad n \geq 1. \]  

(37)

Accordingly the boundary state is constructed to be

\[
|B_{F+T}\rangle = \int D[x, \bar{x}] e^{iS_F + iS_T} |x, \bar{x}\rangle
\]

\[
= Z_{\text{Disk}} \exp \left\{ a_n^\dagger g \left( g + 2\pi \alpha' F + 2\pi \alpha' \frac{u}{n} \right)^{-1} \left( g - 2\pi \alpha' F - 2\pi \alpha' \frac{u}{n} \right) \bar{a}_n^\dagger \right\} |0\rangle, \]

(38)

\[
Z_{\text{Disk}} = \frac{T_p}{g_s} \frac{1}{\sqrt{\det(u)}} \prod_{n=1}^{d/2} \det \left( g + 2\pi \alpha' F + 2\pi \alpha' \frac{u}{n} \right)^{-1}. 
\]

The boundary state \(|B_{F+T}\rangle\) satisfies the following boundary condition

\[
\left( g_{\mu\nu} \partial_\tau \hat{X}^\nu - 2\pi \alpha' F_{\mu\nu} \partial_\sigma \hat{X}^\nu - 4\pi \alpha' i u_{\mu\nu} \hat{X}^\nu \right) |B_{F+T}\rangle = 0. \]  

(39)

When \( F \) is skew-diagonal with \( F_{2\mu-1,2\mu} = f_\mu, \) \( u \) is diagonal, \( u_{\mu\nu} = u_\mu \delta_{\mu\nu}, \) \( g_{\mu\nu} = \delta_{\mu\nu}, \) and \( 2\pi \alpha' = 1, \) the normalization factor reduces to \([26]\)

\[
Z_{\text{Disk}} = \frac{T_p}{g_s} \frac{1}{\sqrt{\det(u)}} \prod_{n=1}^{d/2} \left\{ \left( 1 + \frac{u_{2\mu-1}}{n} \right) \left( 1 + \frac{u_{2\mu}}{n} \right) + f_\mu^2 \right\}^{-1}. \]  

(40)

If we are concerned the unstable \( Dp \)-brane in \( d \) dimensions,

\[
F_{ab} = F_{ai} = F_{ia} = 0, \quad u_{ab} = u_{ai} = u_{ia} = 0,
\]

where \( i, j = 0, \ldots, p \) and \( a, b = p + 1, \ldots, d - 1, \) thus,

\[
|B_{F+T}\rangle = Z_{\text{Disk}} \prod_{n=1}^{d/2} \exp \left\{ a_n^\dagger g \left( g + 2\pi \alpha' F + 2\pi \alpha' \frac{u}{n} \right)^{-1} \left( g - 2\pi \alpha' F - 2\pi \alpha' \frac{u}{n} \right) \bar{a}_n^\dagger \right\} \exp \left( -a_n^\dagger \bar{a}_n^\dagger g_{ab} \right) |0\rangle, \]

(41)

\[
Z_{\text{Disk}} = \frac{T_p}{g_s} \frac{1}{\sqrt{\det(u)}} \prod_{n=1}^{d/2} \det \left( g + 2\pi \alpha' F + 2\pi \alpha' \frac{u}{n} \right)^{-1}. \]

In Eq.(41) \( g, \) \( u \) and \( F \) are \((p + 1) \times (p + 1)\) matrices.

Here one can make a simple observation on the effect of the tachyon condensation on the boundary state wavefunction. In order to see it we may leave the integration over \( x, \)
Then we find

$$|B; x\rangle \sim e^{-\pi x u x}.$$  \hspace{1cm} (43)

It implies that the spatial dimension of the system is order of $1/\sqrt{\det u}$ if we use a closed string as a probe. The unstable D-brane may behave like a soliton in the lower dimensions and it becomes sharply localized as the system approaches the infrared fixed point, $u \to \infty$. One may be concerned about its behaviour at the infrared fixed point since the boundary state may become singular as the system is sharply localized. In order to examine the behaviour of a $Dp$-brane near the infrared fixed point, let us suppose that tachyon condensation takes place only in one direction, i.e., $u_{ij} = u_{ip} = u_{pi} = 0$, $i, j = 0, \ldots, p-1$. Then as $u_{pp} = u \to \infty$,

$$Z_{\text{Disk}} = \lim_{u \to \infty} \frac{T_p}{g_s} \int d^{p+1}x e^{-\pi x u x^2} \prod_{n=1}^{p-1} \det [g + 2\pi \alpha' F + 2\pi \alpha' u/n]_{(p+1)\times(p+1)}^{-1}$$

$$= \frac{T_p}{g_s} \int d^{p}x \frac{1}{\sqrt{u}} \prod_{n=1}^{p-1} \det [g + 2\pi \alpha' F]_{p\times p}^{-1} \frac{1}{2\pi \alpha'} \left( \frac{u}{n} \right)^{-1}$$

$$= 2\pi \sqrt{\alpha'} \frac{T_p}{g_s} \int d^{p}x \sqrt{\det [g + 2\pi \alpha' F]_{p\times p}}$$

(44)

where $[A]_{m \times m}$ denotes a $m \times m$ matrix and the zeta function regularization is used [19].

$$\prod_{n=1}^{p-1} \frac{1}{n + \epsilon} = \exp \left\{ \frac{d}{ds} \left( \zeta(s, \epsilon) - \epsilon^{-s} \right) \right\}_{s=0} = \frac{\epsilon \Gamma(\epsilon)}{\sqrt{2\pi}}.$$ \hspace{1cm} (45)

This is precisely the disk partition function for a ($p-1$) dimensional D-brane in the $U(1)$ background. Note also it gives us the correct relationship between the tension of a $Dp$-brane and that of a $D(p-1)$-brane

$$T_{p-1} = 2\pi \sqrt{\alpha'} T_p$$ \hspace{1cm} (46)

Accordingly as $u \to \infty$,

$$|B_{F+T}\rangle = \frac{T_{p-1}}{g_s} \int d^p x \sqrt{\det [g + 2\pi \alpha' F]}_{p\times p}$$

$$\prod_{n=1}^{p-1} \exp \left\{ a^i_n \left[ g(g + 2\pi \alpha' F)^{-1} (g - 2\pi \alpha' F) \right]_{ij} a^j_n - a^i_n a^a_n g_{ab} \right\} |0\rangle$$

(47)
where $i, j = 1, \ldots, p - 1$ and $a, b = p, \ldots, d - 1$. The first term in the exponent of Eq.(47) describes the boundary state corresponding to an open string in $(p - 1 + 1)$ dimensions with a constant $U(1)$ background and the second term implies that the boundary conditions along the directions $(x^p, \ldots, x^{d-1})$ are Dirichlet. Thus, the unstable D-brane turns into the low dimensional D-brane at the infrared fixed point. As the system reaches the infrared fixed point, it becomes sharply localized. And as we may expect, its wavefunction gets a divergent contribution from the zero mode, but it is cancelled by the those from the higher modes. The cancellation occurs only when we include contributions of all higher modes. This phenomenon is also observed in the $D$-$\overline{D}$ system [16,19,27].

Since the boundary state in the background of the tachyon condensation is explicitly constructed, the couplings to the closed string states are readily obtained. The massless closed string states are given in the bosonic string theory as

$$e_{\mu\nu}a_1^{\mu} a_1^{\nu}|k\rangle$$

(48)

where $\mu, \nu = 0, \ldots, d - 1$. Here $e_{\mu\nu}$ is chosen as

$$e_{\mu\nu} = h_{\mu\nu}, \quad h_{\mu\nu} = h_{\nu\mu}, \quad k^\mu h_{\mu\nu} = \eta^\mu h_{\mu\nu} = 0$$

(49)

for the graviton,

$$e_{\mu\nu} = \frac{\phi}{2\sqrt{2}}(\eta_{\mu\nu} - k_\mu l_\nu - k_\nu l_\mu), \quad l^2 = 0, \quad k \cdot l = 1$$

(50)

for the dilaton, and

$$e_{\mu\nu} = \frac{1}{\sqrt{2}}\Lambda_{\mu\nu}, \quad \Lambda_{\mu\nu} = -\Lambda_{\nu\mu}, \quad k^\mu \Lambda_{\mu\nu} = 0$$

(51)

for the Kalb-Ramond field. (It may be more appropriate to discuss the couplings to the closed string states in the super-string theory. But the general features of the couplings in the tachyon background discussed here remain unchanged.) The coupling of the boundary state to the massless closed string state is given as
\[ \langle k | e_{ij} a_i^1 \tilde{a}_1^j | B \rangle = \langle k | e_{ij} a_i^1 \tilde{a}_1^j | B \rangle + \langle k | e_{ab} a_i^a \tilde{a}_1^b | B \rangle \]

\[ = e_{ij} \left( g + 2 \pi \alpha' F + 2 \pi \alpha' \frac{u}{n} \right)^{-1} \left( g - 2 \pi \alpha' F - 2 \pi \alpha' \frac{u}{n} \right) g^{-1} \]

where \( i = 0, 1, \ldots, p \) and \( a = p + 1, \ldots, d - 1 \). Here the boundary state \( | B \rangle \) is given by Eq.(41). (An improved form of the coupling of the boundary state to the massless closed string state has been discussed recently in ref. [28].) Let us suppose that \( u_{pp} = u \to \infty \), then

\[ | B \rangle \to | B' \rangle = Z_{Disk} \prod_{n=1}^{p} \exp \left\{ a_{n}^{i'} \left[ g \left( g + 2 \pi \alpha' F + 2 \pi \alpha' \frac{u}{n} \right)^{-1} \right] \left( g - 2 \pi \alpha' F - 2 \pi \alpha' \frac{u}{n} \right) a_{n}^{i' j'} - a_{n}^{i'} \tilde{a}_{n}^{j'} \right\} | 0 \rangle \]

where \( i', j' = 0, 1, \ldots, p - 1, \) and \( a', b' = p, \ldots, d - 1 \). This is exactly the coupling of the boundary state in lower dimensions to a massless closed string state. At a glance one can realize that the couplings of the massive closed string states also turn into the couplings to the lower dimensional D-brane as the system reaches the infrared fixed point. Thus, if we use a closed string as a probe, the unstable D-brane looks identical with the lower dimensional D-brane at the infrared fixed point.

**IV. TACHYON CONDENSATION AND NONCOMMUTATIVE SOLITONS**

In recent papers [10–15] it has been pointed out that the tachyon condensation may be greatly simplified if one introduces a large NS \( B \)-field on the world-sheet. Some properties of the unstable systems, such as the tachyon potential and the D-brane tension, can be
calculated exactly. Here in this section we will discuss the noncommutative tachyon in the same framework of the boundary state formulation. We may recall that in the canonical quantization [22] one can establish the equivalence between the noncommutative open string theory with the commutative one in the presence of the NS $B$-field background. Along this line we can establish the relationship between the noncommutative tachyon theory and the commutative one.

The bosonic part of the classical action for an open string ending on a $Dp$-brane with a NS $B$-field is given by

$$ S_M + S_B = -\frac{1}{4\pi\alpha'} \int_M d^2\xi \left[ g_{\mu\nu} \sqrt{\eta} \frac{\partial X^\mu}{\partial \xi^\alpha} \frac{\partial X^\nu}{\partial \xi^\beta} - 2\pi\alpha' B_{ij} e^{\alpha\beta} \frac{\partial X^i}{\partial \xi^\alpha} \frac{\partial X^j}{\partial \xi^\beta} \right] $$

where $\mu, \nu = 0, 1, \ldots, d-1$ and $i, j = 0, 1, \ldots, p$. Let us consider the commutative description first. With a constant $B$ the second term yields the boundary action

$$ S_B = \frac{1}{2} \int_{\partial M} d\tau B_{ij} X^i \partial \tau X^j. $$

As we transform the open string world-sheet coordinates to the closed string world-sheet coordinates, the boundary action turns into

$$ S_B = \frac{1}{2} \int_{\partial M} d\sigma B_{ij} X^i \partial \sigma X^j, $$

which is of the same form as the constant $U(1)$ background discussed in the previous section. Hence, we get the corresponding boundary state in the background of tachyon condensation by replacing $F$ with $B$ in Eq.(41)

$$ |B_{B+T} \rangle = Z_{Disk} \prod_{n=1} \exp \left\{ a^{i\dagger}_n (g_M B_{B+T})_{ij} \bar{a}^j_n - a^{a\dagger}_n a^b_n g_{ab} \right\} |0\rangle, $$

$$ (M_{B+T}) = \left( g + 2\pi\alpha' B + 2\pi\alpha' \frac{u}{n} \right)^{-1} \left( g - 2\pi\alpha' B - 2\pi\alpha' \frac{u}{n} \right), $$

$$ Z_{Disk} = T_p \frac{1}{g_s} \prod_{n=1} \det \left( g + 2\pi\alpha' B + 2\pi\alpha' \frac{u}{n} \right)^{-1} \prod_{n=1} \det \left( g + 2\pi\alpha' B + 2\pi\alpha' \frac{u}{n} \right)^{-1} $$

where $i, j = 0, \ldots, p$ and $a, b = p+1, \ldots, d-1$. Since we are interested to compare the commutative description with the noncommutative one, we consider the case where $Dp$-brane $\rightarrow D(p-2)$-brane. In this case we take
\[ u_{pp} = u_{p-1,p-1} = u \to \infty, \quad u_{ij} = B_{ij} = 0, \quad (59) \]

for \( i, j = 0, \ldots, p - 2 \) and the boundary state describes the \( D(p - 2) \)-brane

\[ |B_{B+T} \rangle = Z_{\text{Disk}} \prod_{n=1}^{p-2} \exp \left\{ a_n^i g_{ij} \tilde{a}_n^{ij} - a_n^{a\dagger} \tilde{a}_n^{b\dagger} g_{ab} \right\} |0\rangle \quad (60a) \]

\[ Z_{\text{Disk}} = \frac{T_{p-2}}{g_s} \sqrt{\det g} \quad (60b) \]

where \( i, j = 0, \ldots, p - 2 \) and \( a, b = p - 1, \ldots, d - 1 \). Note that the role of the NS \( B \)-field is rather trivial in the commutative description of the tachyon condensation.

Let us turn to the noncommutative description. Since the NS \( B \)-field term is quadratic in string variables, \( X \), one may define the world-sheet Green function with respect to \( S_M + S_B \) instead of \( S_M \). Then it leads us to the noncommutative open string theory, of which world-sheet Hamiltonian and the string variables in the longitudinal directions are given as [22]

\[ H = (2\pi \alpha') \frac{1}{2} \mu_i (G^{-1})^{ij} p_j + (2\pi \alpha') \sum_{n=1}^{n} \left\{ \frac{1}{2} K_{n i} (G^{-1})^{ij} K_{nj} + \frac{1}{(2\pi \alpha')^2} \frac{n^2}{2} Y_{ni} G_{ij} Y_{nj} \right\}, \quad (61a) \]

\[ X^i(\sigma) = x^i + i \theta^{ij} p_j \left( \sigma - \frac{\pi}{2} \right) + \sqrt{2} \sum_{n=1}^{n} \left( Y^i_n \cos n \sigma + \frac{i}{n} \theta^{ij} K_{jn} \sin n \sigma \right), \quad (61b) \]

where \((Y^i_n, K_{jn})\) are the canonical pairs. Here \( \theta \) and \( G \) are the noncommutativity parameter and the open string metric respectively

\[ \theta^{ij} = -(2\pi \alpha')^2 \left( \frac{1}{g + 2\pi \alpha' B} \frac{1}{g - 2\pi \alpha' B} \right)^{ij} \quad (62a) \]

\[ (G^{-1})^{ij} = \left( \frac{1}{g + 2\pi \alpha' B} \frac{1}{g - 2\pi \alpha' B} \right)^{ij} \quad (62b) \]

We find that the open string action is written in this representation as

\[ S_M + S_B = -\frac{1}{4\pi \alpha'} \int_M d^2 \xi \sqrt{-h} h^{\alpha \beta} \left( G_{ij} \frac{\partial Y^i}{\partial \xi^\alpha} \frac{\partial Y^j}{\partial \xi^\beta} + g_{ab} \frac{\partial Y^a}{\partial \xi^\alpha} \frac{\partial Y^b}{\partial \xi^\beta} \right) \quad (63) \]

where \( Y^\mu \) is the commutative open string variable

\[ Y^\mu(\sigma) = x^\mu + \sqrt{2} \sum_{n=1}^{n} Y^\mu_n \cos n \sigma. \quad (64) \]
The boundary interaction for tachyon condensation for the open string may be written as

\[ S_T = i \int_{\sigma=0} d\tau T(X) + i \int_{\sigma=\pi} d\tau T(X). \]  

(65)

The end points of open string are given in the noncommutative theory as

\[ X^i(0) = x^i + \frac{\pi}{2} \theta^{ij} p_j + \sqrt{2} \sum_{n=1} Y_n^i, \quad X^i(\pi) = x^i - \frac{\pi}{2} \theta^{ij} p_j + \sqrt{2} \sum_{n=1} Y_n^i (-1)^n. \]  

(66)

They differ from those in the commutative theory by the zero mode of the momentum and do not commute with each other

\[ [X^i(0), X^j(0)] = -i\pi \theta^{ij}, \quad [X^i(0), X^j(\pi)] = 0, \quad [X^i(\pi), X^j(\pi)] = i\pi \theta^{ij}. \]  

(67)

On the boundary \( \partial M \) if we adopt the proper-time \( \hat{\tau} \) Eq.(16) instead of \( \tau \) as the worldsheet time coordinate, the string variables are written on \( \partial M \) as

\[ X^i|_{\partial M} = x^i - \frac{\pi}{2} \theta^{ij} p_j + Y|_{\partial M}. \]  

(68)

Thus, we may write the boundary interaction of the tachyon condensation as

\[ S_T = i \int_{\partial M} d\hat{\tau} T(\zeta + Y), \quad [\zeta^i, \zeta^j] = i\pi \theta^{ij}. \]  

(69)

It can be read in the closed string world-sheet coordinates as

\[ S_T = i \int_{\partial M} d\sigma T(\zeta + Y), \]  

(70)

where \( Y^i \) is the usual closed string variable which can be expanded on \( \partial M \) as

\[ Y^i(\sigma) = y^i_0 + \sqrt{\alpha'} \sum_{n=1} \frac{1}{\sqrt{n}} \left( y_n^i e^{2i\sigma} + \bar{y}_n^i e^{-2i\sigma} \right). \]  

(71)

When we transcribe the open string representation into the closed string representation we keep noncommutative zero mode part unchanged, i.e., we treat \( \zeta \) as noncommutative operators. For the sake of simplicity we only consider hereafter the case where \( Dp \)-brane \( \rightarrow D(p - 2) \)-brane. Extension to the more general cases is straightforward. So we take
\[ B_{ij} = u_{ij} = 0 \text{ for } i, j = 0, p - 2 \text{ and } B_{ij} \neq 0, u_{ij} \neq 0 \text{ for } i, j = p - 1, p. \] It is convenient to introduce ‘creation’ and ‘annihilation’ operators as

\[ b = \frac{1}{\sqrt{2\pi\theta}}(\zeta^{p-2} + i\zeta^{p-1}), \quad b^\dagger = \frac{1}{\sqrt{2\pi\theta}}(\zeta^{p-2} - i\zeta^{p-1}), \quad [b, b^\dagger] = 1. \quad (72) \]

We can introduce the creation and annihilation operators similarly also for higher \( D_p \)-branes as we cast \((\theta)\) into the standard skew-diagonal form, \( \theta^{2i-1,2i} = \theta^i \)

\[
\begin{pmatrix}
0 & \theta_1 \\
-\theta_1 & 0 \\
& \ddots \\
0 & \theta_{\frac{p}{2}} \\
-\theta_{\frac{p}{2}} & 0
\end{pmatrix}.
\]

The excitations in the zero mode can be easily described as we introduce a complete set of the number eigenstates \( \{|n\rangle_{NC}\} \)

\[ |n\rangle_{NC} = \frac{(b^\dagger)_n}{\sqrt{n!}}|0\rangle_{NC}, \quad b|0\rangle_{NC} = 0. \]

Thus, the quantum state on \( \partial M \), can be specified by \( |n\rangle_{NC} \otimes |Y\rangle \), where

\[ \hat{Y}^i|Y\rangle = Y^i|Y\rangle. \]

We may expand \( \hat{Y}^i \) on \( \partial M \) as

\[ \hat{Y}^i(\sigma) = \dot{y}^i + \sqrt{\frac{\alpha'}{2}} \sum_{n=1}^{\alpha'} \frac{1}{\sqrt{n}} \left\{ (a_n^i + \tilde{a}_n^i) e^{2in\sigma} + (a_n^i + \tilde{a}_n^i) e^{-2in\sigma} \right\}, \quad (73) \]

where \( i, j = p - 1, p, \)

\[ [a_n^i, a_m^j] = (G^{-1})^{ij}\delta_{nm}, \quad [\tilde{a}_n^i, \tilde{a}_m^j] = (G^{-1})^{ij}\delta_{nm}. \quad (74) \]

Since the action for the higher modes are identical to that in the absence of the NS \( B \)-field except for the space-time metric \( g \) being replaced by the open string metric \( G \), we find that the boundary action in the closed string representation is given in the noncommutative theory as
\[ S_T = i\pi \zeta^i u_{ij} \zeta^j + i\pi \alpha' \sum_{n=1}^{\infty} \frac{1}{n} \bar{y}_n^i u_{ij} y_n^j. \] (75)

From the analysis of the noncommutative open string theory, we may define the boundary state in the noncommutative theory as

\[ |B_T\rangle_{NC} = \frac{T_p}{G_s} \int D[y, \bar{y}] \sqrt{\det(2\pi \theta)} \text{tr} \left( e^{iS_T[y, \bar{y}]} \right) |y, \bar{y}\rangle \] (76)

where \( G_s \) is the string coupling constant in the noncommutative theory. The integration measure \( D[y, \bar{y}] \) does not take the zero modes into account and the integration over the zero modes is taken care of by \( \sqrt{\det(2\pi \theta)} \text{tr}. \) By a simple algebra we find

\[ |B_T\rangle_{NC} = Z \exp \left\{ a_n^{a\dagger} g_{ij} \bar{a}_n^{j\dagger} + a_n^{b\dagger} (G\mathcal{M})_{kl} \bar{a}_n^{l\dagger} - a_n^{a\dagger} g_{ab} \bar{a}_n^{b\dagger} \right\} |0\rangle, \] (77a)

\[ \mathcal{M} = \left( G + \frac{2\pi \alpha'}{n} \right)^{-1} \left( G - \frac{2\pi \alpha'}{n} \right) \] (77b)

\[ Z = \frac{T_p}{G_s} \sqrt{\det(2\pi \theta)} \text{tr} \left( e^{-2\pi \zeta^i u_{ij} \zeta^j} \right) \prod_{n=1}^{\infty} \det \left( G + \frac{2\pi \alpha'}{n} \right)^{-1} \] (77c)

where \( i, j = 0, \ldots, p-2, k, l = p-1, p \) and \( a, b = p+1, \ldots, d-1. \) Let us take the large \( B \)-field limit where

\[ 2\pi \alpha' B \to \infty \] (78)

with \( g \) kept fixed, or equivalently the decoupling limit, where

\[ g \sim \epsilon, \quad \alpha' \sim \sqrt{\epsilon}, \quad \epsilon \to 0, \] (79)

while \( G, \theta \) are kept fixed. Note that in this limit

\[ \theta \to \frac{1}{B}, \quad G^{-1} \to -\frac{1}{(2\pi \alpha')^2} \frac{1}{B^2} \frac{1}{g} \frac{1}{B} \] (80)

and the effect of the tachyon condensation on the higher modes of \( Y^i \) are suppressed. Thus, in the large \( B \)-field limit, we have

\[ Z = \frac{T_p}{G_s} \sqrt{\det(2\pi \theta)} \text{tr} \left( e^{-2\pi \zeta^i u_{ij} \zeta^j} \right) \sqrt{\det G}. \] (81)

Now let us suppose that the system reaches the infrared fixed point where \( u \to \infty, \)
\[
\text{tr} \left( e^{-2\pi \zeta^i u_{ij}\zeta^j} \right) = \sum_{n=0} e^{-2\pi^2 \theta (u_{p-1,p-1} + u_{pp})n} \rightarrow 1. \tag{82}
\]

It implies that the unstable D-brane behaves like a noncommutative soliton \cite{29} and the most symmetric one \( |0\rangle_{NC} \langle 0 |_{NC} \) is singled out at the infrared fixed point. Thus, at the infrared fixed point, we find

\[
Z = \frac{T_p}{G_s} \sqrt{\det(2\pi \theta)} \sqrt{\det G}. \tag{83}
\]

We may recall the relationship between the string coupling in the commutative theory and that in the noncommutative one \cite{21,22}

\[
\frac{G_s}{g_s} = \left( \frac{\det G}{\det g} \right)^{\frac{1}{4}} = \frac{\sqrt{\det G}}{\sqrt{\det(g + 2\pi \alpha' B)}}, \tag{84}
\]

which implies in the large \( B \)-field limit

\[
\frac{G_s}{g_s} = \frac{\sqrt{\det G}}{\sqrt{\det(2\pi \alpha' B)}}. \tag{85}
\]

Then it follows from Eq.(80) that

\[
Z = (2\pi)^2 \alpha' T_p \frac{T_p}{g_s}. \tag{86}
\]

Since it can be identified with

\[
Z = \frac{T_{p-2}}{g_s},
\]

we obtain the relationship between the tension of \( Dp \)-brane and that of \( D(p-2) \)-brane

\[
T_{p-2} = (2\pi)^2 \alpha' T_p. \tag{87}
\]

In the noncommutative theory the unstable D-brane is described by the noncommutative soliton, which corresponds to

\[
\phi_0(r) = 2e^{-r^2/\sqrt{\det \theta}}. \tag{88}
\]

In the large \( B \)-field limit, the soliton becomes sharply localized but its contribution to the partition function is finite in contrast to the commutative case. So the cancellation observed
in the commutative theory is no longer needed. In fact the contributions from the higher modes to the partition function are rather trivial in the noncommutative theory. It should be noted that if we reverse the limit procedure, i.e., take the limit, \( u \to \infty \) first then the large B-field limit later, we cannot get the correct result.

Now let us call our attention to the quantum state of the system \( |B_T\rangle \), which has never been discussed explicitly in the literature. If we take the large B-field limit first, the boundary state reduces to

\[
|B_T\rangle_{NC} = Z \prod_{n=1}^{\infty} \exp \left\{ a^i_n g_{ij} \tilde{a}^j_n + a^k_l G_{kl} \tilde{a}^l_n - a^{a\dagger}_n g_{ab} \tilde{a}^b_n \right\} |0\rangle \tag{89}
\]

where \( i, j = 0, \ldots, p-2 \), \( k, l = p-1, p \) and \( a, b = p+1, \ldots, d-1 \). To our surprise, the resultant boundary state satisfies the Neumann boundary condition along the directions, \( p-1, p \). It seems that the boundary state still describes the \( Dp \)-brane instead of \( D(p-2) \)-brane. It is certainly against our expectation. We may be tempted to take the limit of the infrared fixed point, \( u \to \infty \) prior to the large B-field limit to get the boundary state satisfying the Dirichlet boundary condition as in the case of the commutative theory. But then as we pointed out, we cannot obtain the correct partition function. The resolution of this problem can be found if we take notice that the open string metric \( G \) becomes singular in the large B-field limit and the left movers and the right movers \( (a, a\dagger, \tilde{a}, \tilde{a}\dagger) \) in Eq.(89) respect the open string metric \( G \) as in Eq.(74). In order to compare the boundary state \( |B_T\rangle_{NC} \) in the noncommutative theory with that in the commutative theory \( |B_{B+T}\rangle \), we should rewrite the left and right movers in Eq.(89) in terms of the left and right movers in the commutative theory, which respect the closed string metric \( g \).

If we define the background metric \( E \) as

\[
E = E_S + E_A = g + 2\pi\alpha' B, \tag{90}
\]

equivalently,

\[
g = \frac{1}{2}(E + E^T), \quad B = \frac{1}{2\pi\alpha'} \frac{1}{2}(E - E^T), \tag{91}
\]
the string action with the NS B-field may be written in the closed string world-sheet coordinates as

\[ S_M + S_B = -\frac{1}{4\pi\alpha'} \int_M d^2\xi \sqrt{\text{h}^{\alpha\beta} \text{E}_{ij} \frac{\partial X^i}{\partial \xi^\alpha} \frac{\partial X^j}{\partial \xi^\beta}}. \] (92)

Here we are concerned with the action only for the string variables in the directions of \( p - 1, p \). If we apply the open-closed string duality to Eq.(63), we obtain the string action in the noncommutative theory as

\[ S_M + S_B = -\frac{1}{4\pi\alpha'} \int_M d^2\xi \sqrt{\text{h}^{\alpha\beta} \text{G}_{ij} \frac{\partial Y^i}{\partial \xi^\alpha} \frac{\partial Y^j}{\partial \xi^\beta}}. \] (93)

Comparing the string action in the commutative theory Eq.(92) with that in the noncommutative theory Eq.(93), we find that two actions are related by the well-known T-dual transformation \[30]\n
\[ E' = (aE + b)(cE + d)^{-1} \] (94)

where \( a, b, c \) and \( d \) satisfy the following \( O(2, 2, R) \) condition

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}^T
\begin{pmatrix}
0 & I \\
I & 0
\end{pmatrix}
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
= 
\begin{pmatrix}
0 & I \\
I & 0
\end{pmatrix}.
\] (95)

Under the T-dual transformation the left and right movers transform as

\[
a_n(E) \rightarrow (d - cE^T)^{-1}a_n(E'), \quad a^\dagger_n(E) \rightarrow a^\dagger_n(E')(d^T - Ec^T)^{-1},
\] (96a)

\[
\tilde{a}_n(E) \rightarrow (d + cE)^{-1}\tilde{a}_n(E'), \quad \tilde{a}^\dagger_n(E) \rightarrow \tilde{a}^\dagger_n(E')(d^T + E^Tc^T)^{-1}
\] (96b)

Choosing \( E' = G \), we find the T-dual transformation \[31\], which connects the left movers and the right movers in the commutative theory Eq.(92) and those in the noncommutative theory Eq.(93)

\[
T = 
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} = 
\begin{pmatrix}
I & 0 \\
-(2\pi\alpha')^{-1}\theta & I
\end{pmatrix}.
\] (97)
If we denote the left and right movers in the commutative theory, Eq.(92) by \( a, a^\dagger, \tilde{a}, \tilde{a}^\dagger \), and those in the noncommutative theory, Eq.(93) by \( a', a'^\dagger, \tilde{a}', \tilde{a}'^\dagger \), the relationship between them are given by

\[
\begin{align*}
  a'_n &= \left( I + \frac{\theta E^T}{2\pi\alpha'} \right) a_n, \\
  a'^\dagger_n &= a^\dagger_n \left( I - \frac{E\theta}{2\pi\alpha'} \right),
\end{align*}
\] (98a)

\[
\begin{align*}
  \tilde{a}'_n &= \left( I - \frac{\theta E}{2\pi\alpha'} \right) \tilde{a}_n, \\
  \tilde{a}'^\dagger_n &= \tilde{a}^\dagger_n \left( I + \frac{E^T\theta}{2\pi\alpha'} \right).
\end{align*}
\] (98b)

Note that

\[
I - \frac{E\theta}{2\pi\alpha'} = EG^{-1}, \quad I + \frac{E^T\theta}{2\pi\alpha'} = E^T G^{-1}.
\] (99)

Using Eq.(99) and Eq.(80) in the large B-field limit, we obtain

\[
\begin{align*}
\exp \left( a'^\dagger_n G \tilde{a}'^\dagger_n \right) |0\rangle = \exp \left( a^\dagger_n EG^{-1} \tilde{a}^\dagger_n \right) |0\rangle = \exp \left( -a^\dagger_n g \tilde{a}^\dagger_n \right) |0\rangle.
\end{align*}
\] (100)

Therefore, the boundary state turns out to satisfy the the correct Dirichlet boundary condition along the directions of \( p - 1, p \)

\[
|B_T\rangle_{NC} = Z \prod_{n=1} \exp \left\{ a^{i\dagger}_n g_{ij} \tilde{a}^j_n - a^{a\dagger}_n g_{ab} \tilde{a}^b_n \right\} |0\rangle
\] (101)

where \( i, j = 0, \ldots, p - 2 \) and \( a, b = p - 1, \ldots, d - 1 \). In the noncommutative theory the boundary state also describes the \( D(p - 2) \)-brane as desired. One can reach the same conclusion also by taking the Seiberg-Witten limit Eq.(79). One may attempt to get the same result in the commutative theory, by taking the large B-field limit first. As one may expect,

\[
|B_{F + T} \rangle \rightarrow Z_{Disk} \prod_{n=1} \exp \left\{ a^{i\dagger}_n g_{ij} \tilde{a}^j_n - a^{a\dagger}_n g_{ab} \tilde{a}^b_n \right\} |0\rangle
\] (102)

where \( i, j = 0, \ldots, p - 2 \) and \( a, b = p - 1, \ldots, d - 1 \). However, in this limit \( Z_{Disk} \) does not yield the correct tension of the D-brane. Hence, this procedure only works for the noncommutative theory. In order to get a consistent description of tachyon condensation in the noncommutative theory, we should take the large B-field limit prior to the infrared fixed point limit. It is also consistent with the work of Gopakumar, Minwalla and Strominger [29].
V. DISCUSSIONS AND CONCLUSIONS

In this paper we discussed the tachyon condensation, which is one of the noble phenomena in string theory, in the simplest setting, i.e., in a single D-brane system in the bosonic string theory, using the boundary state formulation. We do not need to deal with the non-Abelian supersymmetric formulation to understand the tachyon condensation in this simplest system unlike in the $D$-$\bar{D}$-brane system. For the purpose of applying the boundary state formulation to the system with boundary actions of general form, we improved the boundary state formulation of ref. \[20\]. We show that in general the normalization factor of the boundary state corresponds to the disk partition function for the given boundary interaction and obtain the general form of the boundary state. As the tachyon condensation develops the boundary state wavefunction becomes sharply localized in the directions where the tachyon condensation occurs and eventually reduces to that of a lower dimensional D-brane. Both tension and boundary state wavefunction of the lower dimensional D-brane are correctly derived from the boundary state formulation of the unstable D-brane at the infrared fixed point. Since one may obtain the non-BPS $D2p$-brane ($D(2p + 1)$-brane) of type IIB (IIA) string theory, starting from a $D2p$-$\bar{D}2p$-brane ($D(2p + 1)$-$\bar{D}(2p + 1)$-brane) pair in type IIA (IIB) string theory \[1\], it is interesting to discuss these descent relations among BPS and the non-BPS $D$-brane in the boundary state formulation, extending the present work. The relationship between the present work and those on the $D$-$\bar{D}$ system \[16,18,27\] may be clarified in this context.

As we point out that the boundary state formulation, discussed in this paper, has some advantages over other approaches, in that the it provides not only the disk partition function but also the quantum state of the system. Since the boundary state depicts the system in terms of the quantum state of the closed string and the disk partition function corresponds to the normalization of the closed string wavefunction, it may be possible to embed it in a large framework, the closed string field theory. The open-closed string duality then may lead us to the open string field theory of the tachyon condensation. It may improve our understanding
of the open string field theory of tachyon condensation, which has been discussed only within
the limits of the level truncation [5].

One of the main results of the present work is that the noncommutative theory of the
tachyon condensation is derived from the boundary state formulation and its relationship
to the commutative theory is established along the line of the equivalence between the
noncommutative open string theory and the noncommutative one in canonical quantization.
We show that in the noncommutative theory the unstable D-brane precisely reduces to the
lower dimensional D-brane in the large B-field. The boundary state formulation explains
why some of the results obtained in the large $B$-field limit in the noncommutative theory
are exact. The zero mode contribution is the most important and the contributions of other
higher modes are suppressed. However, there is a subtle point, yet important that at its
appearance the resultant boundary state of the unstable D-brane satisfies the Neumann
boundary condition in stead of the Dirichlet boundary condition along the directions where
the tachyon condensation develops. But taking notice that the left and the right movers
along the directions of the tachyon condensation respect the open string metric $G$, which
is singular in the large B-field limit, we find a resolution in the framework of the boundary
state formulation. Using the T-dual relationship between the left and right movers in the
commutative theory and those in the noncommutative theory, we find that the boundary
state correctly reduces to that of the lower dimensional D-brane.

We may take a step forward in the noncommutative theory by introducing the $U(1)$
background in addition to the NS B-field background. The extension along this direction
is important in connection with the confinement in $D$-$\bar{D}$ system [33] and the matrix model
description of the tachyon condensation [34]. It may be also fruitful to extend the present
work along other directions, such as the quantum corrections at the one loop level to the
tachyon condensation [35,36] and the tachyon condensation on noncommutative tori [37],
which have been discussed recently in the literature.
ACKNOWLEDGEMENT

This work was supported by grant No. 2000-2-11100-002-5 from the Basic Research Program of the Korea Science & Engineering Foundation. Part of this work was done during the author’s visit to APCTP (Korea), KIAS (Korea) and PIMS (Canada). The author thanks G. Semenoff for his contribution at the early stage of this work and for hospitality during the author’s visit to PIMS. He also thanks Piljin Yi, S. J. Rey and S. Hyun for useful discussions.
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