SOLAR OBLIQUITY INDUCED BY PLANET NINE: SIMPLE CALCULATION

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ABSTRACT

Bailey et al. and Gomes et al. recently suggested that the $6^\circ$ misalignment between the Sun’s rotational equator and the orbital plane of the major planets may be produced by forcing from the hypothetical Planet Nine on an inclined orbit. Here, we present a simple yet accurate calculation of the effect, which provides a clear description of how the Sun’s spin orientation depends on the property of Planet Nine in this scenario.

Key words: planet–star interactions – planets and satellites: dynamical evolution and stability

1. INTRODUCTION

Batygin & Brown (2016) showed that a hypothetical planet (“Planet Nine”) in the outer solar system can explain several otherwise intriguing orbital properties of distant KBOs. Recently, Bailey et al. (2016) and Gomes et al. (2016) suggested that Planet Nine, which has an inclined orbit relative to the orbital plane of the major planets, may also be responsible for generating the $6^\circ$ solar obliquity (the misalignment angle between the Sun’s rotational equator and the solar system invariant plane). These studies were based on somewhat formal treatments and involved numerical integrations. In this note, we present a pedestrian yet accurate calculation of the solar obliquity generated by Planet Nine. This calculation yields a simple and transparent description of how the solar spin orientation depends on the property of Planet Nine.

2. EXPLICIT ANALYTIC CALCULATION

Batygin & Brown (2016) showed that to explain the spatial clustering of the orbits of distant KBOs, Planet Nine (labeled “p”) must have mass $m_p = (5-20)m_\oplus$, perihelion distance $q_p = a_p(1-e_p) \sim 250$ au, significant eccentricity $e_p \gtrsim 0.5$, and tens of degrees of orbital inclination with respect to the solar system invariant plane. The angular momentum of Planet Nine is $L_p = L_p \hat{l}_p$ (where $\hat{l}_p$ is the unit vector), with

$$L_p = 0.276 L_j \left( \frac{m_p}{10 m_\oplus} \right) \left( \frac{\hat{a}_p}{400 \text{ au}} \right)^{1/2} (1 - e_p^2)^{1/4},$$

where $L_j$ is the orbital angular momentum of Jupiter, and we have defined the “effective” semimajor axis $\hat{a}_p \equiv a_p \sqrt{1 - e_p^2}$.

Planet Nine exerts a torque on each of the “canonical” planets (labeled “j,” from Mercury to Neptune); this torque tends to induce a retrograde nodal precession of $\hat{l}_j$ (the orbital angular momentum unit vector of planet j) around $\hat{l}_p$ at the characteristic rate

$$\Omega_{lp} = \frac{3m_p}{4M_\odot} \left( \frac{a_j}{\hat{a}_p} \right)^3 n_j,$$

where $a_j, n_j$ are the semimajor axis and mean motion of planet j. Note that $\Omega_{lp}$ depends on $a_j$, so each planet has a different $\Omega_{lp}$. However, since the precession frequency due to mutual planet–planet interactions is much larger than the differential $\Omega_{lp}$, all of the canonical planets are strongly coupled, with their angular momentum axes aligned to each other, i.e., $\hat{l}_j = \hat{l}$ (see Lai & Pu 2016 for a precise calculation of the mutual inclinations induced by an inclined external perturber). The orbital angular momentum unit vector $\hat{l}$ of the canonical solar system planets then evolves according to the equation

$$\frac{d\hat{l}}{dt} = \Omega_L \cos \theta_p (\hat{l} \times \hat{l}_p) = \frac{J}{L_p} \Omega_L \cos \theta_p (\hat{l} \times \hat{j}),$$

where $\theta_p$ is the inclination of Planet Nine ($\cos \theta_p = \hat{l} \cdot \hat{l}_p$), $J$ is the total angular momentum

$$J = J \hat{j} = L + L_p = L \hat{l} + L_p \hat{l}_p,$$

with $L = \sum_j L_j = 1.624 L_j$ (note that the spin angular momentum of the Sun, $S_* \sim 0.01L_\odot$, is much smaller), and

$$\Omega_L = \frac{\sum_j L_j \Omega_{lp}}{L} = 2.74 \Omega_p,$$

Note that in Equation (3) we have neglected the torque from the Solar spin on $\hat{l}$—this is an excellent approximation, as $S_* \ll L$ (e.g., Boué & Fabrycky 2014; Lai 2014).

The spin axis $\hat{s}_*$ (unit vector) of the Sun evolves due to the torques from all planets,

$$\frac{d\hat{s}_*}{dt} = \Omega_{sps} \cos \theta_{sl} (\hat{s}_* \times \hat{l}),$$

where $\theta_{sl}$ is the angle between $\hat{s}_*$ and $\hat{l}$ and the characteristic spin precession frequency is given by

$$\Omega_{sps} = \sum_j \Omega_{sj} = \sum_j \frac{3k_{pj}}{2k_*} \left( \frac{m_j}{M_*} \right) \left( \frac{R_*}{a_j} \right)^3 \Omega_*.$$

Here, $\Omega_* = 2\pi/P_*$ is the angular frequency of the Sun, and $k_{sj}, k_{pj}$ are defined through the Sun’s moment of inertia and quadrupole moment: $I_3 = k_* M_\odot R_*^2$ and $I_3 - I_1 = k_{pj} \hat{s}_* \hat{s}_* M_\odot R_*^2$, with $\hat{s}_* = \Omega_* (GM_\odot R_*^3)^{-1/2}$. Normalizing to the values $k_* \approx 0.06$ and $k_{pj} \approx 0.01$ (corresponding to $J_2 = k_{pj} \hat{s}_*^2 \approx 2.2 \times 10^{-7}$;
The rotation rate of the Sun decreases over time due to magnetic braking. We ignore these complications and treat $\lambda_s / P_s$ as a free parameter.

Mecheri et al. 2004), we find

$$\Omega_{\ast, \text{ps}} = 2.88 \Omega_{\ast, J} = \frac{2 \pi}{55.8 \text{ Gyr}} \lambda_s \left( \frac{P_s}{10 \text{ day}} \right)^{-1},$$

where $\lambda_s \equiv 6 k_{g_1} / k_s \approx 1$. Note that in Equation (6), we have neglected the torque from $m_p$ on $\hat{s}_s$—this is obviously an excellent approximation.

Equations (3) and (6) completely determine the evolution of the spin axis of the Sun.

To solve $\hat{s}_s(t)$ analytically, we note that Equation (3) implies that $\hat{l}$ precesses around the constant unit vector $\hat{j}$ at the rate $-J / L_p \Omega_{\ast, \text{ps}} \cos \theta_p$. We transform Equation (6) into the frame corotating with $\hat{l}$, giving

$$\left( \frac{\text{d} \hat{s}_s}{\text{d} t} \right)_{\text{rot}} = \frac{J}{L_p} \Omega_{\ast, \text{ps}} \cos \theta_p \hat{j} - \Omega_{\ast, \text{ps}} \cos \theta_{\text{id}} \hat{l} \times \hat{s}_s.$$  

In this rotating frame, $\hat{j}$ and $\hat{l}$ are constant in time, and for $\theta_{\text{id}} \ll 1$ and constant $P_s$, Equation (9) describes a uniform rotation of $\hat{s}_s$.

Figure 1. Parameters of Planet Nine required to produce the solar obliquity $\theta = 6^\circ$ and the relative longitude of ascending node $\Delta \Omega$ (between Planet Nine and the solar equator). The effective semimajor axis of Planet Nine $a_p \equiv a_p (1 - e_p^2)^{1/2}$ (where $a_p$ is the semimajor axis) is shown as a function of $e_p$, the inclination of Planet Nine relative to the orbital plane of the canonical solar system planets. The mass of Planet Nine is set to $m_p = 10 m_\oplus$, and the solar rotation parameter is set to $P_s / \lambda_s = 20$ days. The three solid lines depict Equation (17) with three values of $\Delta \Omega$ (covering the allowed range, $12^\circ$—$52^\circ$). The dashed and dotted lines depict Equation (18) for different $\Delta \Omega$ and $e_p$ (the eccentricity of Planet Nine). The intersect of a solid line and a corresponding dashed/dotted line of the same color marks the values of $a_p$ and $\theta$ (with the corresponding $m_p$ and $e_p$) required to generate the observed $\theta = 6^\circ$ and $\Delta \Omega$.

Figure 2. Similar to Figure 1, except that the different curves correspond to different values of the solar rotation parameter $P_s / \lambda_s = 24$ days (black; the current solar rotation period), 15 days (blue), and $\infty$ (red; implying that the spin axis of the Sun is constant in time, the limit that was considered by Gomes et al. 2016), all for $\Delta \Omega = 45^\circ$. Note that the solid line (depicting Equation (17)) does not depend on $P_s / \lambda_s$.

around a fixed axis (see Lai 2014). We set up a Cartesian coordinate system where $\hat{l} = \hat{x}$ and $\hat{l}_p = -(\sin \theta_p) \hat{y} + (\cos \theta_p) \hat{z}$ (so that the polar and azimuthal angles of $\hat{l}_p$ are $\theta_p$ and $\phi_p = 270^\circ$). In this coordinate system, Equation (9) reduces to

$$\frac{\text{d} \hat{s}_{x\perp}}{\text{d} t} = -\Omega_y \hat{s}_{z\perp} + \Omega_{\perp} \hat{s}_{x\parallel},$$

$$\frac{\text{d} \hat{s}_{x\parallel}}{\text{d} t} = -\Omega_z \hat{s}_{x\parallel},$$

where

$$\Omega_y = \Omega_{\ast, \text{ps}} \cos \theta_p,$$

$$\Omega_{\perp} = \Omega_{\ast, \text{ps}} - \Omega_{\ast, \text{ps}} \cos \theta_{\text{id}} \left( \frac{L}{L_p} + \cos \theta_p \right).$$

For $\hat{s}_{x\perp} \approx 1$ (consistent with $\theta_{\text{id}} \ll 1$), Equations (10)–(11) can be solved (assuming that $\hat{s}_{x\perp}$ is aligned with $\hat{l}$ at $t = 0$):

$$\hat{s}_{x\parallel} = \frac{-\Omega_y}{\Omega_z} \sin \Omega_z t, \quad \hat{s}_{x\parallel} = \frac{\Omega_y}{\Omega_z} (1 - \cos \Omega_z t).$$

Thus, the polar and azimuthal angles of $\hat{s}_s$ are given by

$$\theta_{x\parallel} \approx (\hat{s}_{x\perp}^2 + \hat{s}_{x\parallel}^2)^{1/2} = \sqrt{\frac{2 \Omega_y}{\Omega_z} \sin \Omega_z t},$$

$$\phi_{x\parallel} \approx (\hat{s}_{x\perp}^2 + \hat{s}_{x\parallel}^2)^{1/2} = \sqrt{\frac{2 \Omega_y}{\Omega_z} \sin \Omega_z t}.$$
3. DEPENDENCE AND CONSTRAINT ON PLANET NINE PARAMETERS

For a given $P_\star/\lambda_\star$, the values of $\theta_d$, $\phi_\ast$, and $m_\star$ at $t = 4.5$ Gyr depend on the parameters of Planet Nine ($\tilde{a}_p$, $m_p$, $\theta_p$, $\epsilon_p$) through the combination of two frequencies, $\Omega_\star$ and $\Omega_*$. Batygin & Brown (2016) suggested that the longitude of the ascending node of Planet Nine (relative to that of the solar equator), $\Delta \Omega \equiv \phi_\ast - \phi_\ast$, is about $45^\circ$ and ranges from $12^\circ$ to $52^\circ$. To produce this $\Delta \Omega$ and $\theta_d = 6^\circ$ over time 4.5 Gyr, the parameters of Planet Nine must satisfy the following conditions, derived from Equations (15)–(16):

$$\tilde{a}_p \simeq 462 \left( \frac{m_p}{10 \, m_\oplus} \frac{\sin 2\theta_p}{\theta_d f} \right)^{1/3} \text{au},$$

(17)

$$\frac{L}{L_p} \simeq \left[ 15g + 4.84\lambda_\star \left( \frac{P_\star}{10 \text{ days}} \right)^{-1} \sin \theta_p \right] \frac{\sin \theta_p}{\theta_d f} - \cos \theta_p,$$

(18)

where $L = 1.624L_\odot$ and $L_p$ is given by Equation (1), and we have defined

$$\theta_d \equiv \theta_d/6^\circ, \quad g \equiv \left( \frac{\pi/2 - \Delta \Omega}{\pi/4} \right), \quad f \equiv \left( \frac{\pi/2 - \Delta \Omega}{\cos \Delta \Omega} \right).$$

(19)

Figures 1–3 illustrate the parameter space of Planet Nine required to produce $\theta_d = 6^\circ$ and $12^\circ < \Delta \Omega < 52^\circ$. Figure 1 shows the effective semimajor axis $\tilde{a}_p$ as a function of $\theta_p$ for several values of planet mass and eccentricity, assuming an “averaged” $P_\star/\lambda_\star$ or 20 days. Figure 2 illustrates how the result depends on the solar rotation parameter $P_\star/\lambda_\star$. Figure 3 shows $\tilde{a}_p$ as a function of $m_p$ for several values of $\theta_p$ and $\epsilon_p$. In general, a larger $m_p$ requires a smaller $\theta_p$, with a modest change in $\tilde{a}_p$. There exists a minimum value of $\theta_p$, as indicated by Equation (18). In all cases, $\tilde{a}_p$ lies in the range between 340 and 480 au in order to produce the desired solar spin orientation.

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