Explosive phenomena in modified gravity

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Abstract

Observational manifestations of some models of modified gravity, which have been suggested to explain the accelerated cosmological expansion, are analyzed for gravitating systems with time dependent mass density. It is shown that if the mass density rises with time, the system evolves to the singular state with infinite curvature scalar. The corresponding characteristic time is typically much shorter than the cosmological time.
Contemporary astronomical data strongly indicate that at the present epoch the universe expands with acceleration. A possible way to explain this accelerated expansion is to assume that there is a new component in the cosmological energy density, the so-called dark energy. The latter can be either a small vacuum energy, which is identical to the cosmological constant, or the energy density associated with an unknown, presumably scalar field, which slowly varies in the course of the cosmological evolution.

A competing possibility to create cosmological acceleration is to modify gravity itself introducing additional terms into the usual action of General Relativity [1]; for recent reviews see [2, 3]. To this purpose the models with the following action were considered:

\[ S = \frac{m^2_{Pl}}{16\pi} \int d^4x \sqrt{-g} f(R) + S_m, \]  

(1)

where \( m_{Pl} = 1.22 \cdot 10^{19} \) GeV is the Planck mass, \( R \) is the scalar curvature, and \( S_m \) is the matter action. In the usual Einstein gravity function \( f(R) \) has the form \( f(R) = R \), in the modified gravity \( f(R) \) acquires an additional term:

\[ f(R) = R + F(R), \]  

(2)

which changes gravity at large distances and is responsible for cosmological acceleration. In the pioneering papers [1] function \( F(R) \) at small \( R \) behaves as:

\[ F(R) = -\mu^4 R, \]  

(3)

where \( \mu \) is a small parameter with dimension of mass. However, as it was shown in ref. [1], such a choice of \( F(R) \) leads to a strong exponential instability near massive objects and so the usual gravitational fields would be drastically distorted. An attempt to cure this ill-behavior by adding to the action \( gR^2 \)-term [5] was only partially successful. It could terminate the instability with reasonably small coefficient \( g \) for sufficiently dense objects with \( \rho > 1 \frac{g}{cm^3} \), while for the objects with smaller mass density the coefficient \( g \) would be too large and incompatible with the existing bound on the \( R^2 \)-gravity. We will discuss this problem in a more detailed paper which is under preparation.

A choice of \( F(R) \), which leads to an accelerated cosmological expansion and is devoid of the above mentioned instability and of some other problems was suggested in several papers [6–9]. In the present work we examine a very interesting model of modified gravity with \( F(R) \) function suggested in ref. [6] :

\[ F(R) = \lambda R_0 \left[ \left( 1 + \frac{R^2}{R_0^2} \right)^{-n} - 1 \right]. \]  

(4)

Here constant \( \lambda \) is chosen to be positive to produce an accelerated cosmological expansion, \( n \) is a positive integer, and \( R_0 \) is a constant with dimension of the curvature scalar. The latter is assumed to be of the order of the present day average curvature of the universe, i.e. \( R_0 \sim 1/t_U^2 \), where \( t_U \approx 4 \cdot 10^{17} \) sec is the universe age.

The corresponding equations of motion have the form

\[ (1 + F') R_{\mu\nu} - \frac{1}{2} (R + F) g_{\mu\nu} + (g_{\mu\nu} D_\alpha D^\alpha - D_\mu D_\nu) F' = \frac{8\pi\tau^{(m)}_{\mu\nu}}{m^2_{Pl}}, \]  

(5)
where $F' = dF/dR$, $D_\mu$ is the covariant derivative, and $T^{(m)}_{\mu\nu}$ is the energy-momentum tensor of matter.

By taking trace over $\mu$ and $\nu$ in eq. (5) we obtain the equation of motion which contains only the curvature scalar $R$ and the trace of the energy-momentum tensor of matter:

$$3D^2F' - R + RF' - 2F = T,$$

(6)

where $T = 8\pi T_\mu^\mu/m_{P_l}^2$. Note that our sign convention is different from that of paper [6] and is the same as in ref. [4].

Cosmology with gravitational action (4), as well as some other cosmological scenarios with modified gravity were critically analyzed in recent paper [2]. It was shown that, taken literally, the models suffer from several serious problems. Though the instability of ref. [4] was eliminated, still there remain some other types of singular behavior. In particular, there exists the past singularity, when $R \to \infty$ at some finite time in the past. It was argued that the problem can be solved by an addition to the action of $R^2$-term with sufficiently small coefficient allowed by the present observational data.

The singularity similar to that considered in the present work was first noticed in ref. [10] in the case of cosmological evolution back to the past. In a sense the future singularity considered here is the time reversal of the past singularity on the quoted paper. So mathematically both singularities are quite similar, despite of some difference due to effects of the universe contraction (when one goes backward in time). The Hubble anti-friction favors the approach to the singularity in the contracting universe. However, despite mathematical similarities there is an important difference between the two systems. According the ref. [10], the singularity may be avoided with a certain range of initial conditions. In our case singularity emerges for any initial conditions.

In ref. [11,12] it was argued that infinite $R$ singularity could arise in the future, unless the initial conditions for $R$ are not fine-tuned. This is similar to the cosmological situation of ref. [10]. The systems considered in these works are different from that discussed here and the singularity of the quoted papers appears only for certain initial conditions, while in our case, as we have already mentioned above, the singularity arises for an arbitrary initial state. All the singularities can be eliminated by an addition of $R^2$-term to the action and we study the effects of this term below.

In what follows we will consider a different physical situation than those discussed in the above mentioned references. Namely we study behavior of astronomical objects with mass density which rises with time and show that curvature, $R$, reaches infinitely large value during the time interval which is very short in comparison with the cosmological time scale. This singularity cannot be eliminated by fine-tuning of the initial conditions. An addition of $R^2$-term could prevent from the singular behavior but at expense of quite large values of $n$ which may be at odds with the standard cosmological evolution.

We study objects with mass density which is much larger than the cosmological one, $\rho_m \gg \rho_c$. The cosmological energy density at the present time is $\rho_c \approx 10^{-29}$ g/cm$^3$, while matter density of, say, a dust cloud in a galaxy could be about $\rho_m \sim 10^{-24}$ g/cm$^3$. Since the magnitude of the curvature scalar is proportional to the mass density of a nonrelativistic
system, we find $R \gg R_0$. In this limit:

$$F(R) \approx -\lambda R_0 \left[ 1 - \left( \frac{R_0}{R} \right)^{2n} \right]. \quad (7)$$

Let us start from the initial state in which modified gravity around or inside some massive objects is not much different from the usual Einstein (Newtonian) gravity and correspondingly $R \approx -T$, as can be seen from the normal Einstein equations.

We analyze temporary evolution of solutions of eq. (6) for the gravitational field of some massive object with time varying density. We assume that the gravitational field of this object is weak, as is usually the case. Correspondingly the background metric is approximately flat and the covariant derivatives can be replaced by the usual flat space ones. Hence:

$$D^2F' = (\partial^2_t - \Delta) F' = F''(\partial^2_t - \Delta) R + F'''[(\dot{R})^2 - (\nabla R)^2], \quad (8)$$

where $\Delta$ is the usual Laplacian, and $\nabla$ is the gradient.

Substituting expression (7) for $F(R)$ at large $R$ into eq. (6), we obtain:

$$(\partial^2_t - \Delta) R - (2n+2) \frac{\dot{R}^2 - (\nabla R)^2}{R} + \frac{R^2}{3n(2n+1)} \left( \frac{R_0^{2n}}{R^{2n}} - (n+1) \right) \frac{R^{2n+2}}{6 n (2n+1) \lambda R_0^{2n+1}} (R + T) = 0. \quad (9)$$

However, due to the presence of the nonlinear terms containing derivatives, this equation is difficult to analyze and we instead study the equation for $F'(R)$ and express $R$ through $F'$ using:

$$F' = -2n\lambda \left( \frac{R_0}{R} \right)^{2n+1}. \quad (10)$$

Notice that infinite $R$ corresponds to $F' = 0$ and if $F'$ reaches zero, it would mean that $R$ becomes infinitely large.

Let us introduce the new notation $w = -F'$. Equation (9) for $w$ takes the simple form describing an unharmonic oscillator:

$$(\partial^2_x - \Delta) w + U'(w) = 0. \quad (11)$$

Potential $U(w)$ is equal to:

$$U(w) = \frac{1}{3} (T - 2\lambda R_0) w + \frac{R_0}{3} \left[ \frac{q^\nu}{2 n \nu} w^{2n\nu} + \left( q^\nu + \frac{2\lambda}{q^\nu} \right) \frac{w^{1+2n\nu}}{1+2n\nu} \right], \quad (12)$$

where $\nu = 1/(2n+1)$, $q = 2n\lambda$, and in eq. (11) $U'(w) = dU/dw$. It is useful to remember that $T \gg R_0$. Their ratio is about $T/R_0 \sim \rho_m/\rho_c \gg 1$ and hence $w \ll 1$. Thus the first term in square brackets in eq. (12) dominates. Potential $U$ would depend upon time, if the mass density of the object under scrutiny changes with time, $T = T(t)$. 

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If only the dominant terms are retained in equations (11), (12) and if the space derivatives are neglected, equation (11) simplifies to:

\[ \ddot{w} + \frac{T}{3} - \frac{q'(R_0)}{3w^\nu} = 0. \] (13)

It is convenient to introduce dimensionless quantities:

\[ t = \gamma \tau, \ w = \beta z \] (14)

where \( \beta \) and \( \gamma \) are so chosen that the equation for \( z \) becomes very simple:

\[ z'' + z^{-\nu} + (1 + \kappa \tau) = 0. \] (15)

Here prime means differentiation with respect to \( \tau \) and the trace of the energy-momentum tensor of matter is parametrized as:

\[ T(t) = T_0(1 + \kappa \tau). \] (16)

Constants \( \gamma \) and \( \beta \) are equal to

\[ \gamma^2 = \frac{3q}{(-R_0)} \left( \frac{-R_0}{T_0} \right)^{2(n+1)}, \] (17)

\[ \beta = \gamma^2 T_0/3 = q \left( \frac{-R_0}{T_0} \right)^{2n+1}. \] (18)

Thus \( \beta \) is a small dimensionless number, and \( \gamma \) has dimension of time. It is essential that \( \gamma \), which determines characteristic time scale, may be much shorter than the universe age, \( t_U \), due to the small factor \( (R_0/T_0)^{n+1} \). Assuming that \( 3q \sim 1 \) and \( R_0 \sim 1/t_U^2 \), we find for \( n = 2 \) and \( \rho_m = 10^{-24} \text{g/cm}^3 \): \( \gamma \approx 400 \text{ sec} \). It would be much smaller for larger \( n \) or \( \rho_m \).

For example if \( n = 3 \) and the same \( \rho_m \) we find \( \gamma = 0.004 \text{ sec} \).

In the case of constant \( T (\kappa = 0) \) or very slowly varying \( T (\kappa \ll 1) \) the solution of eq. (15) is evident. If the initial values \( z(0) \) and \( z'(0) \) are sufficiently small, \( z(\tau) \) oscillates near the minimum of the potential, which is situated at

\[ z_{\text{min}} = (1 + \kappa \tau)^{-1/\nu}. \] (19)

If by some reason the magnitude of \( z(0) \) takes a sufficiently large value, \( z > (1 - \nu)^{1/\nu} \), such that potential

\[ U(z) = z - z^{1-\nu}/(1 - \nu). \] (20)

becomes positive, evidently at some stage \( z(\tau) \) would overjump potential \( U(z) \) which is equal to zero at \( z = 0 \) ("the waves are cresting over"). In other words, \( z(\tau) \) would reach zero, which corresponds to infinite \( R \), and so the singularity can be reached in finite time. Analogous situation can be realized if the initial velocity, \( z'(0) \), is sufficiently large.

The singularity can be also reached in finite time even if \( z \) was initially situated at the minimum of the potential and the initial velocity was zero. It would take place if
$\kappa$ is positive, i.e. the energy density rises with time. The motion of $z_{\text{min}}$ to zero and simultaneous diminishing of the depth of the potential well make it easier for $z(\tau)$ to reach zero. On the other hand, it is not evident that for decreasing energy density $z(\tau)$ initially resting at the minimum of $U(z)$ could reach zero, most probably it would not.

We have solved equation (15) numerically and found that indeed the singularity, $R \to \infty$, is reached in finite time for rising $T(\tau)$ under quite general conditions. The solution for $n = 2$, $\kappa = 0.01$, and $\rho_m/\rho_c = 10^5$ is presented in Fig. 1, where the ratio $z(\tau)/z_{\text{min}}(\tau)$ (left) and functions $z(\tau)$ and $z_{\text{min}}(\tau)$ separately (right) are depicted. The initial conditions are taken as $z(0) = 1$ and $z'(0) = 0$.

In Figs. 2 and 3 the same quantities are presented for $n = 3$ and $n = 4$ respectively. It is clearly seen that $z(\tau)$ reaches zero after a finite number of oscillations around $z_{\text{min}}(\tau)$. When $z_{\text{min}}(\tau)$ shifts to smaller values, function $z(\tau)$ initially remains behind but when the displacement from the equilibrium point becomes large enough, $z(\tau)$ started to run after it with an increasing speed, then overtakes the position of the minimum, and oscillates back. After a few oscillations the retarded position of $z(\tau)$ happens to be above the point $z_0(\tau)$, where the potential is zero. It is essential that the position of this point moves to smaller values with rising $T(\tau)$. Because of that it is easier to overjump the potential at $z = 0$. It is intriguing that the magnitude of the ratio $z/z_{\text{min}}$ is approximately equal to 3 in the last maximum before the singular point $z = 0$ is reached. We have not found an explanation for that.

In terms of physical time, $t$, the evolution of the energy density can be presented as $T(t) = T_0(1 + t/t_{\text{ch}})$, where $t_{\text{ch}}$ is the characteristic time of the variation. Coefficient $\kappa$ in equation (16) is expressed through $t_{\text{ch}}$ as:

$$\kappa = \gamma/t_{\text{ch}}.$$  \hspace{1cm} (21)

Thus the presented in Fig. 1 case of $\kappa = 0.01$, $n = 2$, and $\rho_m/\rho_c = 10^5$ corresponds to $t_{\text{ch}} = 4 \cdot 10^4$ sec, while $n = 3$ (Fig. 2) corresponds to $t_{\text{ch}} = 0.4$ sec. The characteristic time of the density variation can be estimated as $t_{\text{ch}} \sim d/v$, where $d$ is the size of the system and $v$ is the velocity of the constituent particles in the process of the collapse of the cloud or in the collision of the clouds. In the first case the velocity would be quite low and the characteristic time is expected to be close to the Newton free-fall time but in the case of
Figure 2: Ratio $z(\tau)/z_{\text{min}}(\tau)$ (left) and functions $z(\tau)$ and $z_{\text{min}}(\tau)$ (right) for $n = 3$, $\kappa = 0.01$, $\rho_m/\rho_c = 10^5$.

Figure 3: Ratio $z(\tau)/z_{\text{min}}(\tau)$ (left) and functions $z(\tau)$ and $z_{\text{min}}(\tau)$ (right) for $n = 4$, $\kappa = 0.01$, $\rho_m/\rho_c = 10^5$. 
colliding clouds the velocities are typically galactic ones, about 300 km/sec. The velocity may be even larger at the collision of the supernova ejecta with galactic or intergalactic clouds.

It is more informative to act another way around, namely to estimate $\kappa$ knowing size, $d$, of the object with changing mass density or sizes of the colliding objects:

$$\kappa = \frac{\gamma v}{d}.$$  \hspace{1cm} (22)

For $n > 2$ and astronomically large clouds one should expect $\kappa \ll 1$. For very small $\kappa$ our numerical calculations with quickly oscillating functions are not reliable but it seems natural to expect that the system would reach singularity according to the analysis presented above.

As it is seen from the numerical calculations, the singularity is reached when $t \sim t_{ch}$. This is much shorter than the cosmological time for clouds of denser matter in galaxies or a collapsing cloud forming a star or another denser body.

In the analysis of eq. (11) the spatial derivatives have been neglected. At first sight, the account of these terms could inhibit formation of singularity, as e.g. happens in the process of structure formation due to gravitational (Jeans) instability. However the situation is opposite here and the inhomogeneities stimulate singularity formation. Indeed, the effect of inhomogeneities can be described by an appearance of the term $w/d^2$ in eq. (11) with positive coefficient. Such a term is equivalent to an addition of an extra attractive force pushing $w$ or $z$ to zero, i.e. to $R \to \infty$.

There are several possible cases when the conditions leading to singularity can be realized: collapse of gas cloud leading finally to star formation, collision of two gas clouds in a galaxy, stellar ejecta colliding with interstellar or intergalactic matter, and many others. From the calculational point of view such processes can be either adiabatic, when the mass density changes slowly (this is the case analyzed above) or fast, when the mass density changes instantly in an explosive way. Seemingly the latter would result in a faster approach to singularity. This case will be analyzed elsewhere.

If $R$ becomes large, the approximation of flat space-time would be invalid and the derivatives in the equations of motion should be changed into covariant ones. We have not analyzed if in this case the approach to singularity is terminated. However, even if it is terminated, it takes place at high curvatures when gravity becomes strongly different from the Newtonian one.

Another possible way to avoid singularity is to introduce $R^2$-terms into the gravitational action:

$$\delta F(R) = -R^2/6m^2,$$  \hspace{1cm} (23)

where $m$ is a constant parameter with dimension of mass.

With such an extra term in the gravitational action it becomes impossible to express analytically $\dot{R}$ through $F'(R)$. So one needs to work with the equation of motion for $\dot{R}$. In the homogeneous case and in the limit of large ratio $R/R_0$ equation (11) is modified as

$$\left[ 1 - \frac{R^{2n+2}}{6\lambda n(2n+1)R_0^{2n+1}m^2} \right] \ddot{R} = (2n+2) \frac{\dot{R}^2}{R} - \frac{R^{2n+2}(R+T)}{6\lambda n(2n+1)R_0^{2n+1}} = 0.$$  \hspace{1cm} (24)
To analyze eq. (24) let us introduce, as is done above, dimensionless curvature and time:

\[
y = \frac{-R}{T_0}, \quad \tau_1 = t \left[ -\frac{T_0^{2n+2}}{6\lambda n(2n+1)R_0^{2n+1}} \right]^{1/2}.
\]  

(25)

Correspondingly eq. (24) is transformed into:

\[
(1 + gy^{2n+2})y'' - 2(n + 1) \left( \frac{y'}{y} \right)^2 + y^{2n+2} [y - (1 + \kappa_1 \tau_1)] = 0,
\]  

(26)

where prime means derivative with respect to \( \tau_1 \) and

\[
g = -\frac{T_0^{2n+2}}{6\lambda n(2n+1)m^2 R_0^{2n+1}} > 0.
\]  

(27)

For very large \( m \), or small \( g \), when the second term in the coefficient of the second derivatives in eqs. (24) and (26) can be neglected, the numerical solution demonstrates that \( R \) would reach infinity in finite time in accordance with the results presented above. Nonzero \( g \) would terminate the unbounded rise of \( R \). To avoid too large deviation of \( R \) from the usual gravity coefficient \( g \) should be larger than or of the order of unity. Notice that the factor \((1 + gy^{2n+2}) \) is always non-zero because \( g > 0 \).

Keeping in mind the bound on \( m > 10^{-2.5} \text{ eV} \), which follows from the laboratory tests of gravity [13], we find \( n \geq 6 \), demanding that the gravity of objects with \( \rho \sim 10^{-24} \text{ g/cm}^3 \) is not noticeably distorted. In ref. [2] a stronger bound is presented, \( m \gg 10^5 \text{ GeV} \). If this is the case, then \( n \geq 9 \). A natural value is \( m \sim m_{Pl} \) and correspondingly \( n \geq 12 \). For smaller values of \( T_0 \) the bounds on \( n \) are noticeably stronger.

As follows from eq. (26), the frequency of small oscillations of \( y \) around \( y_0 = 1 + \kappa_1 \tau_1 \) in dimensionless time \( \tau_1 \) is

\[
\omega^2 = \frac{1}{g} \left( \frac{gy_0^{2n+2}}{1 + gy_0^{2n+2}} \right) \leq \frac{1}{g}.
\]  

(28)

It means that in physical time the frequency would be

\[
\omega \sim \frac{1}{U} \left( \frac{T_0}{R_0} \right)^{n+1} \frac{y_0^{n+1}}{\sqrt{1 + gy_0^{2n+2}}} \leq m.
\]  

(29)

In particular, for \( n = 5 \) and for a galactic gas cloud with \( T_0/R_0 = 10^5 \), the oscillation frequency would be \( 10^{12} \text{ Hz} \approx 10^{-3} \text{ eV} \). Higher density objects e.g. those with \( \rho = 1 \text{ g/cm}^3 \) would oscillate with much higher frequency, saturating bound (29), i.e. \( \omega \sim m \). All kind of particles with masses smaller than \( m \) might be created by such oscillating field.

On the other hand, as we have seen above, for denser objects the variation of \( T \) in terms of \( \tau \) or \( \tau_1 \) is very slow because of very small \( \kappa \). As a result the amplitude of the oscillations around the equilibrium point would be also small and possibly such oscillations are of no danger from the observational point of view. Still it is possible that there might be intermediate cases when the oscillations would lead to observable phenomena.
Thus we have shown that the impact of the considered above versions of modified gravity on the systems with time dependent mass density in the contemporary universe could be catastrophic, leading to the singularity \( R \to \infty \) during finite time in the future. This time is typically much shorter than the cosmological one. The problem can be fixed by the \( R^2 \)-term if the power \( n \) is sufficiently large, \( n \geq 6 \) (or maybe \( n \geq 9 \)). So either the versions of the theory with large \( n \) or theories with another form of \( F(R) \) should be considered. A more exciting possibility is that the explosive phenomena predicted by modified gravity are observed in the sky.

**References**

[1] S. Capozziello, S. Carloni, A. Troisi, RecentRes. Dev. Astron. Astrophys. 1, 625 (2003); arXiv:astro-ph/0303041.
S.M. Carroll, V. Duvvuri, M. Trodden, M.S. Turner, Phys.Rev. D 70, 043528 (2004); arXiv:astro-ph/0306438.

[2] S.A. Appleby, R.A. Battye, A.A. Starobinsky, JCAP 1006, 005 (2010); arXiv:0909.1737v2.

[3] S. Nojiri, S. Odintsov, arXiv:1011.0544v2.

[4] A.D. Dolgov, M.Kawasaki, Phys. Lett. B 573, 1 (2003).

[5] S. Nojiri, S. Odintsov Phys. Rev. D 68, 123512 (2003).

[6] A.A. Starobinsky, JETP Lett. 86, 157 (2007).

[7] W.Hu, I. Sawicki, Phys. Rev. D 76, 064004 (2007).

[8] A.Appleby, R. Battye, Phys. Lett. B 654, 7 (2007).

[9] S. Nojiri, S. Odintsov, Phys. Lett.B 657, 238 (2007);
S. Nojiri, S. Odintsov, Phys. Rev.D 77, 026007 (2008);
G. Cognola, E. Elizalde, S. Nojiri, S.D. Odintsov, L. Sebastiani, S. Zerbini, Phys. Rev. D 77, 046009 (2008);
G. Cognola, E. Elizalde, S.D. Odintsov, P. Tretyakov, S. Zerbini, Phys. Rev. D 79, 044001 (2009).

[10] S.A. Appleby, R.A. Battye, JCAP 0805, 019 (2008).

[11] A. Dev, D. Jain, S. Jhingan, S. Nojiri, M. Sami, I. Thongkool, Phys. Rev. D 78 083515 (2008);
I. Thongkool, M. Sami, R. Gannouji, S. Jhingan, Phys. Rev. D 80 043523 (2009);
I. Thongkool, M. Sami, S. Rai Choudhury, Phys. Rev. D 80 127501 (2009).

[12] A.V. Frolov, Phys. Rev. Lett. 101, 061103 (2008).

[13] D.J. Kapner, T.S. Cook, E.G. Adelberger, J.H. Gundlach, B.R. Heckel, C.D. Hoyle, H.E. Swanson, Phys. Rev. Lett. 98 021101 (2007).