THE MAIN CONJECTURE FOR IMAGINARY QUADRATIC FIELDS FOR THE SPLIT PRIME \( p = 2 \)

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ABSTRACT. Let \( \mathbb{K} \) be an imaginary quadratic field such that 2 splits into two primes \( p \) and \( \mathfrak{p} \). Let \( \mathbb{K}_\infty \) be the unique \( \mathbb{Z}_2 \)-extension of \( \mathbb{K} \) unramified outside \( p \). Let \( \mathfrak{f} \) be an ideal coprime to \( p \) and \( \mathbb{L} \) be an arbitrary extension of \( \mathbb{K} \) contained in the ray class field \( \mathbb{K}(\mathfrak{p}^2\mathfrak{f}) \). Let \( \mathbb{L}_\infty = \mathbb{K}_\infty\mathbb{L} \) and let \( \mathbb{M} \) be the maximal \( p \)-abelian, \( p \)-ramified extension of \( \mathbb{L}_\infty \). We set \( X = \text{Gal}(\mathbb{M}/\mathbb{L}_\infty) \). In this paper we prove the Iwasawa main conjecture for the module \( X \).

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1. INTRODUCTION

Let \( \mathbb{K} \) be an imaginary quadratic field in which 2 splits into two distinct primes \( p \) and \( \mathfrak{p} \). By class field theory there exists exactly one \( \mathbb{Z}_2 \)-extension \( \mathbb{K}_\infty/\mathbb{K} \) which is unramified outside \( p \). Let \( \mathbb{L} = \mathbb{K}(\mathfrak{p}^2) \), \( L_\infty = \mathbb{K}_\infty\mathbb{L} \) and \( \mathbb{L}_n \) subextensions such that \( [\mathbb{L}_n : \mathbb{L}] = 2^n \). We will denote the Euler system of elliptic units in \( \mathbb{L}_n \) by \( C_n \).

Let \( \mathfrak{f} \) be coprime to \( p \) and \( \mathbb{K} \subset \mathbb{L'} \subset \mathbb{L} \) be an abelian extension such that \( \mathbb{L} \) is the smallest ray class field of the type \( \mathbb{K}(\mathfrak{p}^2) \) containg \( \mathbb{L'} \). Analogous to \( \mathbb{L}_\infty \) we let \( L'_\infty = L_\infty\mathbb{L'} \) and \( L'_n \) be the intermediate fields. Let \( U_n \) be the local units congruent to 1 in \( L'_n \) and \( U_\infty = \lim_{\infty\leftarrow n} U_n \). We define the elliptic units in \( L'_n \) by \( C_n(L'_n) = N_{L_n/L'_n}(C_n) \). Let \( E_n \) be the units of \( L'_n \) congruent to 1 modulo \( p \) and define \( \overline{E} = \lim_{\infty\leftarrow n} E_n \). We define further \( \overline{C} = \lim_{\infty\leftarrow n} C_n \), where the overline denotes in both cases the \( p \)-adic closure of the groups \( E_n \) and \( C_n \), respectively (i.e. we embedd the groups \( C \) and \( E \) in the local units and consider their topological closure). We denote by \( A_n \) the \( 2 \)-part of the class group of \( L'_n \) and define \( A_\infty = \lim_{\infty\leftarrow n} A_n \). Let further \( \Omega \) be the maximal \( 2 \)-abelian \( p \)-ramified extension of \( L'_\infty \). We will use the notation \( X := \text{Gal}(\Omega/L'_\infty) \).

There is a natural decomposition \( \text{Gal}(L'_\infty/\mathbb{K}) \cong H \times \Gamma' \), where \( H = \text{Gal}(L'_\infty/\mathbb{K}_\infty) \) and \( \Gamma' \cong \text{Gal}(\mathbb{K}_\infty/\mathbb{K}) \). We will fix once and for all such a decomposition. Let \( \chi \) be a character of \( H \) and \( M \) an arbitrarly \( \Lambda = \mathbb{Z}_2[\Gamma' \times H] \)-module. Let \( \mathbb{Z}_2(\chi) \) be the extension of \( \mathbb{Z}_2 \) generated by the values of \( \chi \) and define \( M_\chi = M \otimes_{\mathbb{Z}_2[H]} \mathbb{Z}_2(\chi) \). So \( M_\chi \) is largest quotient on which \( H \) acts via \( \chi \). The modules \( M_\chi \) are \( \Lambda \)-modules. The main aim of this paper is to understand their structure in more detail, i.e. to prove the following main conjecture.

**Theorem 1.1.** \( \text{Char}(A_\infty \chi) = \text{Char}(\overline{E}/\overline{C})\chi \) and \( \text{Char}(X\chi) = \text{Char}(U_\infty/\overline{C})\chi \).

To do so we will use the following useful reduction step: Let \( \mathfrak{f}' \) be a principal ideal coprime to \( p \) in \( \mathbb{K} \) such that \( \omega_p = 1 \), where \( \omega_p \) denotes the number of roots of unity of \( \mathbb{K} \) congruent to 1 mod \( \mathfrak{f}' \).
Lemma 1.2. If Theorem 1.1 holds for $\mathcal{K} (f^p^n) := \cup_{n \in \mathbb{N}} \mathcal{K}(f^p^n)$, then it holds for every $L_\infty$.

Theorem 1.1 was addressed before by Rubin in [Ru-1] and [Ru-2] for $p \geq 3$ and $[L' : K]$ coprime to $p$. Bley proved the conjecture in [Bl] for $p \geq 3$ and general ray class fields $L'$ under the assumption that the class number of $K$ is coprime to $p$.

The most recent work on this problem is due to Kezuka [Ke-2] for the prime $p = 2$ and $K = \mathbb{Q}(\sqrt{-q})$ where $q$ is a prime congruent to 7 modulo 8. She proves the main conjecture in the case $L' = \mathbb{H}$ the Hilbert class field of $K$. Note that in Kezuka’s case the definition of $K$ ensures that $K$ has odd class number. In this article we drop the assumption that the class number has to be odd and allow $[L : K]$ to be even.

Our proof will follow closely the methods developed by Rubin and generalized by Bley and Kezuka. We will first construct a suitable measure on the group $\text{Gal}(L_\infty/K)$ and use it to define a $p$-adic $L$-function. This part of the paper is a summary of section 2 of [Cr-M]. Using properties of the Euler system of elliptic units developed by Rubin and Tchebotarev’s Theorem we will prove that $\text{Char}(A_{\infty, \chi})$ divides $\text{Char}(\mathcal{E}/C_\chi)$. In section 4 we will finish the proof by showing that they are generated by polynomials of the same degree and hence are equal.

An analogue of the relation between the galois groups $\Gamma'$ and $\text{Gal}(L_\infty/K)$ explained in section 3.2 holds for $p \geq 3$ as well. Thus, all results of section 3.2 can be proved for general $p$ and $L$ as well. In fact most of them are in [Bl]. Thus, the proof given here can also be used to prove the main conjecture for general ray class fields $L$ and any prime $p$ without the assumption that the class number of $K$ has to be coprime to $p$. It is not stated here for the general case as it is given in [Bl] up to the slight modification in section 3.2 and to avoid technical case distinctions for example in section 3.1, where the statements for $p \geq 3$ and $p = 2$ are actually different.

2. $p$-adic Measures

Before we start defining the $p$-adic measures we will need later we prove Lemma 1.2.

Proof of Lemma 1.2. Let $M \in \{A_{\infty}, U_{\infty}/C, E/C, X\}$. Let $\chi$ be a character of $\text{Gal}(L_\infty'/K_\infty)$. By restriction $\chi$ is also a character of $\text{Gal}(\mathcal{K}(f^p\infty)/K_\infty)$. In particular, it is trivial on $\text{Gal}(\mathcal{K}(f^p\infty)/L_\infty')$. As $f$ is coprime to $p$ and none of the characteristic ideals is divisible by 2 (this follows from Theorem 1.1 for $K(f^p\infty)$ and the fact that $\text{Char}(X)$ and $\text{Char}(A_{\infty})$ are not divisible by 2 as shown in Theorem 3.21 and Corollary 3.22) the $\text{Gal}(\mathcal{K}(f^p\infty)/L_\infty')$-invariant parts of $M(\mathcal{K}(f^p\infty))$ are pseudoisomorphic to the norm $N_{\mathcal{K}(f^p\infty)/L_\infty'} M(\mathcal{K}(f^p\infty))$ which is pseudoisomorphic to $M(L_\infty')$. Thus, we obtain $\text{Char}(M(L_\infty')_\chi) = \text{Char}(M(L_\infty)_\chi)$. □

For the rest of the paper we will only consider the case $L = \mathcal{K}(fp^2)$ for $f$ being coprime to $p$, principal and such that $\omega f = 1$. Define $F_n = \mathcal{K}(fp^n)$ and note that $L_n = F_{n+2}$. We will use the notation $F_0 = F = \mathcal{K}(f)$. To define our elliptic units we will use the following exposition from [Cr-M].

Lemma 2.1. [Cr-M] Lemma 2] There exists an elliptic curve $E/F$ which satisfies the following properties.

a) $E$ has CM by the ring of integers $O_K$ of $K$;
b) $\mathbb{F}(E_{\text{tors}})$ is an abelian extension of $\mathbb{K}$;  
c) $E$ has good reduction at primes in $\mathbb{F}$ lying above $p$.

Let $\phi$ be a Grossencharacter of $\mathbb{K}$ of infinity type $(1,0)$ and conductor $\mathfrak{f}$. Let $E/\mathbb{F}$ be the elliptic curve defined in Lemma 2.1. Then we can assume that the Grossencharacter $\psi$ associated to $E$ satisfies

$$\psi_{E/\mathbb{F}} = \phi \circ N_{\mathbb{F}/\mathbb{K}}.$$ 

In the sequel we have to describe the points on our elliptic curve explicitly. Therefore, we fix once and for all a minimal Weierstrass model of $E$.

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$ 

Using the fact that $E$ has good reduction at all primes above $p$ we can assume that the discriminant $\Delta(E)$ is coprime to $p$. Choosing a suitable embedding $\mathbb{F} \hookrightarrow \mathbb{C}$ we can further assume that there is a complex number $\Omega_{\infty}$ such that the period lattice $\mathcal{L}$ of $E$ satisfies $\mathcal{L} = \Omega_{\infty} \mathcal{O}(\mathbb{K})$ (see [Co-Go] for details).

Let $\sigma \in \text{Gal}(\mathbb{F}/\mathbb{K})$ be an arbitrary element. Then $\sigma$ acts on the coefficients of the model (1) and defines another curve $E_{\sigma}$ over $\mathbb{F}$. The curve $E_{\sigma}$ satisfies point a)-c) of Lemma 2.1. From point b) we obtain that the two curves $E$ and $E_{\sigma}$ have the same Grossencharacter. In particular, all the $E_{\sigma}$ are isogenous to each other.

Let $a$ be an ideal in $\mathbb{K}$ coprime to $fp$ and $a \in a$ an arbitrary element. By point a) of Lemma 2.1 we see that the multiplication by $a$ is a well defined endomorphism of $E$ and we can consider its kernel $E_{a}$. We define further $E_{\sigma a} = \bigcap_{a \in a} E_{a}$. Let $\sigma_{a}$ be the Artin symbol of $a$ in $\text{Gal}(\mathbb{F}/\mathbb{K})$. Then the main theorem of complex conjugation allows us to define an isogeny $\eta_{\sigma}(a) : E_{\sigma} \to E_{\sigma a}$ over $\mathbb{F}$, of degree $N(a)$. This isogeny has the property that for every $g$ coprime to $a$ and any $u \in E_{g}$ we have

$$\sigma_{a}(u) = \eta_{g}(a)(u).$$

Moreover, the kernel of this isogeny is precisely the subgroup $E_{\sigma a}$ (see [Co-Go] proof of Lemma 4). Whenever $\sigma$ is trivial we will drop the subscript $\sigma$ and write $\eta(a)$ instead of $\eta_{id}(a)$.

Let $\omega$ be the Neron-differential associated to the model (1) of $E$ then

$$\omega = \frac{dx}{2y + a_1x + a_3}.$$ 

Due to [Co-Go] p. 341, there exists a unique $\Lambda(a) \in \mathbb{F}^{\times}$ such that

$$\omega_{\sigma a} \circ \eta(a) = \Lambda(a)\omega.$$ 

This defines a map $\Lambda : \{\text{ideals coprime to } \mathfrak{f}\} \to \mathbb{F}^{\times}$ satisfying the cocyle condition. Therefore it can be extended to a cocyle of all fractional ideals coprime to $\mathfrak{f}$. This cocyle $\Lambda$ plays also an important role in determining the period lattice for the curve $E_{\sigma a}$. Let $\mathcal{L}_{\sigma a}$ be the period lattice of $E_{\sigma a}$. Then

$$\Lambda(a)\Gamma_{\infty}a^{-1} = \mathcal{L}_{\sigma a}.$$ 

(see [Co-Go] p. 342 for details).

Recall that we choose an embedding $\overline{\mathbb{Q}}$ into $\mathbb{C}$ such that $\mathcal{L} = \Omega_{\infty} \mathcal{O}(\mathbb{K})$. Choose a place $v$ above $p$ induced by this embedding and let $\mathcal{I}_{p}$ be the ring of integers of the maximal unramified extension of $\mathbb{F}_{v}$. For every $\sigma \in \text{Gal}(\mathbb{F}/\mathbb{K})$ we denote by $\mathcal{E}_{\sigma v}$ the formal group given by the kernel of the reduction modulo $v$ of $E_{\sigma}/\mathbb{F}$. Note
that the formal parameter of this group is \( t_\sigma = -x_\sigma/y_\sigma \). If \( \sigma \) is trivial we omit the sub- and superskript \( \sigma \). With these notations we obtain:

**Lemma 2.2.** [Cr-M, Lemma 3] There exists an isomorphism \( \beta^v \) between the formal multiplicative group \( \widehat{G}_m \) and the formal group \( \widehat{E^v} \), which can be written as a power series \( t = \beta^v(w) \in \mathcal{I}_p[[w]] \).

We fix once and for all such an isomorphism and denote the coefficient of \( w \) in this power series by \( \Omega \). The isogenies \( \eta(a) \) induce homomorphisms \( \widehat{\eta(a)} : \widehat{E}^v \rightarrow \widehat{E}^{\sigma_a.v} \). As long as \( a \) is coprime to \( f \) they are even isomorphisms. Define \( \beta_a = \eta(a) \circ \beta^v \). If we denote the first coefficient of \( \beta_a^v \) by \( \Omega_{a,v} \), we obtain from [Co-Go] Lemma 6] that \( \Omega_{a,v} = \Lambda(a) \Omega_v \).

These notations and definitions allow us to define our basic rational function: Let \( a \) be an integral element in \( \mathbb{K} \) that is coprime to \( 6 \) and not a unit. Let \( P^a \) be a generic point on \( E^a \) and denote its \( x \)- and \( y \)-coordinates by \( x(P^a) \) and \( y(P^a) \). Then we define

\[
\xi_{a,\sigma}(P^a) = c_\sigma(a) \prod_{\beta \in V_{a,\sigma}} (x(P^a) - x(S)),
\]

where \( V_{a,\sigma} \) is a set of representatives of the non-zero \( \sigma \)-division points on \( E^a \) modulo \( \{\pm 1\} \) and \( c_\sigma(a) \) is a canonical 12th root in \( \mathbb{F} \) of \( \Delta((a^{-1}L_\sigma)/\Delta(L_\sigma))^{\frac{1}{12}}(\alpha) \) (here \( \Delta \) stands for the Ramanujan’s \( \Delta \)-function). -see also [Co, Appendix, Proposition 1] and [Co, Appendix, Theorem 8]. Recall that we assumed that \( f = (f) \) is principal. By the definition of \( L \) we see that \( \rho = \Omega_\infty/f \) defines a primitive \( f \) division point on \( E \).

Using this we define the rational function

\[
\xi_{a,\sigma}(P^a) = \xi_{a,\sigma}(P^a + \Lambda(a)\rho),
\]

where \( a \) is an integer ideal coprime to \( f \) such that \( \left(\frac{a}{\mathbb{K}}\right) = \sigma \). We fix a set of ideals \( \mathfrak{c}_0 \) such that the artinsymbols of \( a \in \mathfrak{c}_0 \) run through every element of \( \text{Gal}(\mathbb{F}/\mathbb{K}) \) exactly once and define

\[
Y_{a,\sigma}(P^a) = \frac{\xi_{a,\sigma}(P^a)^{\rho}}{\xi_{a,\sigma_p}(\eta_{\sigma}(p))(P^a)^{\rho}}.
\]

We obtain the following result ([Cr-M, Lemma 4] and [Cr-M, equation 15]):

**Lemma 2.3.** For an integral ideal \( a \) of \( \mathcal{O}_K \) coprime to \( f \), let \( \sigma_a \) denote the Artin symbol of \( a \) in \( \text{Gal}(\mathbb{F}/\mathbb{K}) \). Then the series \( Y_{a,\sigma_a}(t_{\sigma_a}) \) lies in \( 1 + \mathfrak{m}_v[[t_{\sigma_a}] \) and the series \( h_{a,\sigma}(t_{\sigma_a}) := \frac{1}{2} \log(Y_{a,\sigma_a}(t_{\sigma_a})) \) has coefficients in \( \mathcal{O}(\mathbb{F}_v) \). Further, the function \( Y_{a,\sigma}(P^a) \) satisfies the relation \( \prod_{R \in \mathbb{E}_F} Y_{a,\sigma}(P^a \oplus R) = 0 \).

There is a one to one correspondence between \( \mathcal{I}_p \)-valued measures on \( \mathbb{Z}_p^{\times} \) and the ring of power series \( \mathcal{I}_p[[T]] \) given by Mahler’s Isomorphism

\[
F_v(w) = \int_{\mathbb{Z}_p^{\times}} (1 + w)^x dv(x).
\]

For every \( a \in \mathfrak{c}_0 \) we can consider the power series \( B_{a,\sigma}(w) = h_{a,\sigma}(\beta_a^v(w)) \). Using Lemma 2.3 we obtain that these series correspond to measures on \( \mathbb{Z}_p^{\times} \). We will denote these measures by \( \nu_{a,\sigma} \). Using [Si, Lemma 1.1] together with the second claim of Lemma 2.3 we see that the measures \( \nu_{a,\sigma} \) coincide with their restriction to
\[ \mathbb{Z}_2^\times. \text{ As } \text{Gal}(\mathbb{L}_\infty/\mathbb{F}) \cong \mathbb{Z}_2^\times \text{ we can see the measures } \nu_{\alpha,a} \text{ as measures on } \text{Gal}(\mathbb{L}_\infty/\mathbb{F}). \]

If we extend them by zero outside \( \text{Gal}(\mathbb{L}_\infty/\mathbb{F}) \) we can even see them as measures on \( \text{Gal}(\mathbb{L}_\infty/\mathbb{K}) \). For \( a \in \mathcal{C}_0 \) let \( \nu_{\alpha,a} \circ \sigma_a \) be the pushforward measure of \( \nu_{\alpha,a} \) on \( \sigma_a^{-1}\text{Gal}(\mathbb{L}_\infty/\mathbb{F}) \). Then we can define

\[
\nu_\alpha := \sum_{a \in \mathcal{C}_0} \nu_{\alpha,a} \circ \sigma_a, \]

which is by the choice of \( \mathcal{C}_0 \) a measure on \( \text{Gal}(\mathbb{L}_\infty/\mathbb{K}) \). The measures \( \nu_\alpha \) have nice interpolation properties with respect to \( L \)-functions when it comes to integration of characters of the form \( \varepsilon = \chi \phi^k \) for a character of finite order \( \chi \). Let \( \chi \) be a character of conductor \( gp^n \) with \( g \mid f \) and consider the set

\[
S = \left\{ \gamma \in \text{Gal}(\mathbb{K}(fp^n/\mathbb{F})/\mathbb{K}) : \left| \gamma|_{\mathbb{K}(p^n)} \right| = \left( \frac{\mathbb{K}(fp^n)/\mathbb{K}}{p^n} \right) \right\}. \]

With this definition we can define the Gauss-like sum

\[
G(\varepsilon) = \frac{\phi^k(p^n)}{p^n} \sum_{\gamma \in S} \chi(\gamma)\zeta_{p^n}^{-\gamma}. \]

These notations allow us to state:

**Theorem 2.4.** [Cr-M, Theorem 4] Let \( \mathcal{D}_p = \mathcal{I}_p(\zeta_m) \), where \( \zeta_m \) denotes an \( m \)-th root of unity and \( m = |H| \). Then there exists a unique measure \( \nu \) on \( \text{Gal}(\mathbb{F}_\infty/\mathbb{K}) \) taking values in \( \mathcal{D}_p \) such that for any \( \varepsilon = \phi^k \chi \), with \( k \geq 1 \) and \( \chi \) a character of conductor dividing \( fp^n \) for some \( n \geq 0 \), one has

\[
\Omega_{\nu}^{-k} \int_{\text{Gal}(\mathbb{L}_\infty/\mathbb{K})} \varepsilon \, d\nu = \Omega_{\nu}^{-k} (-1)^k(k-1)!f^k u_\chi G(\varepsilon) \left( 1 - \frac{\varepsilon(p)}{p} \right) L_1(\pi,k), \]

with a unit \( u_\chi \) only depending on \( \chi \). Further, \( \nu_\alpha = (N\alpha - \sigma_a)\nu \).

If \( \chi \) is a character of \( H \) it can be extended linearly to the ring of \( \mathcal{D}_p \) valued measures on \( \text{Gal}(\mathbb{L}_\infty/\mathbb{K}) \). It follows that the \( \chi(\nu_\alpha) \) generate the trivial ideal.

**Proof.** Only the first claim is stated in Theorem 4 of [Cr-M]. But the other claims are stated as intermediate steps in the proof. \( \square \)

To prove the main conjecture we will not only need measures on \( \text{Gal}(\mathbb{L}_\infty/\mathbb{K}) \) but also on \( \text{Gal}(\mathbb{K}(gp^n)/\mathbb{K}) \) for \( g \mid f \). Therefore we define the pseudomesure

\[
(4) \quad \nu(g) := \nu(f)|_{\text{Gal}(\mathbb{K}(gp^n)/\mathbb{K})} \prod_{\sigma \mid gp^n} \left( 1 - \left| \sigma|_{\mathbb{K}(p^n)} \right|^{-1} \right)^{-1}, \]

where \( \nu(f)|_{\text{Gal}(\mathbb{K}(gp^n)/\mathbb{K})} \) is the measure on \( \text{Gal}(\mathbb{K}(gp^n)/\mathbb{K}) \) induced from \( \nu(f) \). Note that these pseudomesures are in fact measures as soon as \( g \neq (1) \), while \( (1 - \sigma)(\nu(1)) \) is actually a measure for every \( \sigma \) in \( \text{Gal}(\mathbb{K}(p^n)/\mathbb{K}) \). It is an easy verification that in the case \( \omega_a = 1 \) the measure \( \nu(g) \) is actually the measure one would obtain by starting with an elliptic curve \( E/\mathbb{K}(g) \) and do all the constructions we did so far directly for \( \text{Gal}(\mathbb{K}(gp^n)/\mathbb{K}) \) (compare with [dS, comments after II 4.12]).

Having all these definitions in place allows us to define our \( p \)-adic \( L \)-function.
Definition 2.5. Fix an isomorphism
\[ \kappa : \Gamma' \to 1 + 4\mathbb{Z}_2, \]
and let \( \chi \) be a character of \( H \). We denote by \( \mathfrak{g}_\chi \) the prime to \( p \)-part of its conductor and define the \( p \)-adic \( L \)-function of the character \( \chi \) as
\[
L_p(s, \chi) = \int_{\text{Gal}(\mathbb{K}(\mathfrak{g}_\chi \infty)/\mathbb{K})} \chi^{-1} \kappa^s d\nu(\mathfrak{g}_\chi)
\]
if \( \chi \neq 1; \)
\[
L_p(s, \chi) = \int_{\text{Gal}(\mathbb{K}(\mathfrak{p}_\infty)/\mathbb{K})} \chi^{-1} s \cdot (1 - \gamma) d((1 - \gamma)\nu(1))
\]
if \( \chi = 1. \)

3. Elliptic units and Euler Systems

It is well known that for every \( m \) torsion point \( P_{\sigma}^m \) on \( E_{\sigma}^m \) the elements \( \xi_{\alpha,\sigma}(P_{\sigma}^m) \) are contained in \( \mathbb{K}(m) \) [dS, Proposition II 2.4]. The following proposition will be very useful in the course of our proof.

Proposition 3.1. Let \( m \) be an ideal coprime to \( \alpha f \) and \( P \in E_{\sigma}^m \) a primitive \( m \)-division point. Let \( r \) be a prime and \( m = rm' \) with \( \mathbb{K}(m') \neq \mathbb{K}(1) \). Then
\[
N_{\mathbb{K}(m)/\mathbb{K}(m')} \xi_{\alpha,\sigma}(P) = \begin{cases} 
\xi_{\alpha,\sigma}(\eta_{\sigma}(r)P) & \text{if } r \mid m' \\
\xi_{\alpha,\sigma}(\eta_{\sigma}(r)P)1 - \text{Frob}_{r}^{-1} & \text{if } r \nmid m'
\end{cases}
\]

Proof. This proof follows [Ke, Proposition 4.3.2]. The unit group \( \mathcal{O}^\times = \mathcal{O}(\mathbb{K})^\times \) has exactly two elements. Hence, the map \( \mathcal{O}^\times \to (\mathcal{O}/m)^\times \) is injective. It follows that the kernel of the projection
\[
\phi: (\mathcal{O}/m)^\times /\mathcal{O}^\times \to (\mathcal{O}/m')^\times /\mathcal{O}^\times
\]
is isomorphic to the kernel of
\[
\phi': (\mathcal{O}/m)^\times \to (\mathcal{O}/m')^\times.
\]
Hence,
\[
[\mathbb{K}(m) : \mathbb{K}(m')] = \begin{cases} 
Nr - 1 & \text{if } r \mid m' \\
Nr & \text{if } r \nmid m'
\end{cases}
\]
The conjugates of \( P \) under \( \text{Gal}(\mathbb{K}(m)/\mathbb{K}(m')) \) are the set
\[
\{P + Q \mid Q \in E_{\sigma}^m \text{ such that } P + Q \notin E_{m'}^\sigma\}
\]
if \( r \mid m' \) and
\[
\{P + Q \mid Q \in E_{\sigma}^m\}
\]
if \( r \nmid m' \). In the first case there is exactly one \( r \)-torsion point \( Q_0 \) such that \( P + Q_0 \) is contained in \( E_{m'}^\sigma \). We obtain
\[
\xi_{\alpha,\sigma}(P + Q_0) N_{\mathbb{K}(m)/\mathbb{K}(m')} \xi_{\alpha,\sigma}(P) = \prod_{Q \in E_{\sigma}^m} \xi_{\alpha,\sigma}(P + Q) = \xi_{\alpha,\sigma}(\eta_{\sigma}(r)P).
\]
By the definition of \( \eta \) we obtain further that
\[
\xi_{\alpha,\sigma}(P + Q_0)^{\text{Frob}_r} = \xi_{\alpha,\sigma}(\eta_{\sigma}(r)(P + Q_0)) = \xi_{\alpha,\sigma}(\eta_{\sigma}(r)P),
\]
which implies the claim in this case.
In the case $r \mid m$ we obtain the claim directly from

$$N_{K(m)/K(m')} \xi_{\alpha,\sigma}(P) = \prod_{Q \in E'} \xi_{\alpha,\sigma}(P + Q) = \xi_{\alpha,\sigma}(\eta_\sigma(r)P).$$

$\square$

Before we can define our Euler system we still need one further concept. Let $S_{n,l}$ be the set of square free ideals of $\mathcal{O}$ that are only divisible by prime ideals $q$ satisfying the following two conditions

i) $q$ is totally split in $\mathbb{L}_n = K(fp^{n+2})$

ii) $Nq \equiv 1 \mod 2^{l+1}$

**Lemma 3.2.** Let $\mathbb{H}_n = K(p^{n+2})$. Given a prime $q$ in $S_{n,l}$ there exists a cyclic extension $\mathbb{H}_n(q)/\mathbb{H}_n$ of degree $2^l$ inside $\mathbb{H}_nK(q)$. Furthermore, $\mathbb{H}_n(q)/\mathbb{H}_n$ is totally ramified at the primes above $q$ and unramified outside $q$. Let $\mathbb{V}$ be any subfield $\mathbb{H}_n \subset \mathbb{V} \subset \mathbb{L}_n$ and $\mathbb{V}(q) = \mathbb{H}_n(q)\mathbb{V}$ then $\text{Gal}(\mathbb{V}(q)/\mathbb{V}) \cong \text{Gal}(\mathbb{H}_n(q)/\mathbb{H}_n)$ and the ramification behavior is the same.

Note that from now on $K(q)$ denotes a ray class field of conductor $q$, while we denote for any $\mathbb{V} \neq K$ the field constructed in Lemma [3.2] by $V(q)$.

**Proof.** As $q$ is unramified in $\mathbb{H}_n/K$ it follows that $K(q) \cap \mathbb{H}_n = K(1) \cap \mathbb{H}_n = K(1)$. Hence, $\text{Gal}(\mathbb{H}_nK(q)/\mathbb{H}_n) = \text{Gal}(K(q)/K(1)) \cong (\mathcal{O}/q)/\mathcal{O}^\times$. As $|\mathcal{O}^\times| = 2$ and $Nq \equiv 1 \mod 2^{l+1}$ we can extract a cyclic extension of degree $2^l$ over $\mathbb{H}_n$. By definition $q$ is totally ramified in $\mathbb{H}_n(q)/\mathbb{H}_n$ and the extension is unramified outside $q$. The rest of the claim is an immediate consequence of the fact that $q$ is unramified in $\mathbb{L}_n$. $\square$

If $r = \prod q_i$ with $q_i$ distinct primes in $S_{n,l}$ then we define $V(r)$ as the compositum of the $V(q_i)$.

Having this in place we can define Euler systems.

**Definition 3.3.** An Euler system is a set of global elements

$$\{\alpha^\sigma(n, r) \mid n \geq 0, r \in S_{n,l}, \sigma \in \text{Gal}(K(p^{2})/K)\}$$

satisfying

i) $\alpha^\sigma(n, r) \in \mathbb{L}_n(r)^\times$ is a global unit in $\mathbb{L}_n(r)$ for $r \neq (1)$.

ii) If $q$ is a prime such that $qr \in S_{n,l}$ then

$$N_{\mathbb{L}_n(qr)/\mathbb{L}_n(r)}(\alpha^\sigma(n, rq)) = \alpha^\sigma(n, r)^{\text{Frob}_{r}^{-1}}$$

iii) $\alpha^\sigma(n, rq) \equiv \alpha^\sigma(n, r)(Nq_{\ell} - 1)/2^{l} \mod \lambda$ for every prime $\lambda$ above $q$.

Note that if we fix $\sigma$ and $n$ and let only $r$ vary we obtain an Euler system in the sense of [Ru-2] for the field $\mathbb{L}_n$. So in Rubin’s language our Euler-System is a system of Euler systems indexed by the pairs $(\sigma, n)$.

We now give a precise definition of the elliptic units.

**Definition 3.4.** Let $g | f$ be a non-trivial ideal. We define the elliptic units $C_{g,n}$ in $\mathbb{L}_n$ as the group of units (they are units by [IS Chapter II Proposition 2.4 iii]) generated by all the $\xi_{\alpha,\sigma}Q_g(P_{n+2}^\sigma)$, where $Q_g$ is a primitive $g$ division point and $P_{n+2}^\sigma$ is a $p^n$-torsion point on $E^\sigma$. If $g = (1)$ we define $C_{1,n}$ as the group generated by all the units of the form $\prod_{\ell=1}^s \xi_{\alpha,\sigma}(P_{n+2}^\sigma)^{m_\ell}$ with $\sum_{\ell=1}^s m_\ell(N\alpha_\ell - 1) = 0$ (they are units by [IS Chapter II Exercise 2.4]). We define further the groups $C_g = \lim_{\infty \leftarrow n} C_{g,n}$.
and the group \( C(\mathfrak{g}) = \prod_{h \in \mathfrak{g}} C_h \). We will also use the notation \( C_n \) and \( C \) instead of \( C_n(\mathfrak{g}) \) and \( C(\mathfrak{g}) \) if the conductor is clear from the context.

This allows us to prove the following Lemma.

**Lemma 3.5.** For every \( u \in C_g \) there exists an Euler system such that \( \alpha^\sigma(n, 1) = u \).

**Proof.** As the properties defining an Euler-system are multiplicative it suffices to consider the case of \( u \) being one of the generators, i.e. \( u = \xi_{\alpha, \sigma}(P^\sigma_{n+2} + Q_\emptyset) \). Assume first that \( g \neq (1) \) and let \( \mathcal{V}_n = \mathbb{K}(\mathfrak{g}p^{n+2}) \). Define

\[
\alpha^\sigma(n, r) = N_{\mathbb{K}(\mathfrak{g}p^{n+2})/\mathcal{V}_n(r)}(P^\sigma_{n+2} + Q_\emptyset + \sum_{l|r} Q_l),
\]

where \( l \) are primes in \( S_{l,n} \). Then \( \alpha^\sigma(n, 1) = u \). It remains to show that \( \alpha \) generates an Euler system. Using that \( \sigma_q = 1 \) and that \( \text{Gal}(\mathbb{L}_n(rq)/\mathbb{L}_n(r)) = \text{Gal}(\mathcal{V}_n(rq)/\mathcal{V}_n(r)) \) we obtain:

\[
N_{\mathbb{L}_n(rq)/\mathbb{L}_n(r)}(\alpha^\sigma(n, rq)) = N_{\mathbb{V}_n(q)/\mathcal{V}_n(r)}N_{\mathbb{K}(\mathfrak{g}p^{n+2}rq)/\mathcal{V}_n(q)}(P^\sigma_{n+2} + Q_\emptyset + \sum_{l|q} Q_l) \]

\[
= N_{\mathbb{K}(\mathfrak{g}p^{n+2}rq)/\mathcal{V}_n(r)}(P^\sigma_{n+2} + Q_\emptyset + \sum_{l|q} Q_l)\]

\[
= N_{\mathbb{K}(\mathfrak{g}p^{n+2}rq)/\mathcal{V}_n(r)}(P^\sigma_{n+2} + Q_\emptyset + \sum_{l|q} Q_l)\]

\[
= N_{\mathbb{K}(\mathfrak{g}p^{n+2}rq)/\mathcal{V}_n(r)}(P^\sigma_{n+2} + Q_\emptyset + \sum_{l|q} Q_l)\]

\[
= (\alpha^\sigma(n, r))^\text{Frob}_{q-1}
\]

It remains to check property iii): The group \( \text{Gal}(\mathbb{K}(\mathfrak{g}p^{n+2}rq)/\mathcal{V}_n(rq)) \) acts only on the \( q \)-torsion points. By definition we obtain that

\[
|\text{Gal}(\mathbb{K}(\mathfrak{g}p^{n+2}rq)/\mathcal{V}_n(rq))\mathbb{K}(\mathfrak{g}p^{n+2}rq))| = (Nq - 1)/2^l
\]

due to the fact that \( \mathbb{K}(\mathfrak{g}p^{n+2}) \neq \mathbb{K} \) is non-trivial. Using the fact that \( q \)-torsion points reduce to zero modulo \( \lambda \) and that \( \text{Gal}(\mathbb{K}(\mathfrak{g}p^{n+2}rq)/\mathcal{V}_n(rq)) \) restricts surjectively to \( \text{Gal}(\mathbb{K}(\mathfrak{g}p^{n+2}rq)/\mathcal{V}_n(r)) \) the claim is an easy consequence of the definitions.

If \( g = (1) \) we choose \( \alpha^\sigma(n, r) = \prod_{i=1}^s \xi_{\alpha_i, \sigma}(P^\sigma_{n+2} + \sum_{l|r} Q_l)^{m_i} \) and proceed as above. \( \square \)

For every prime \( q \in S_{l,n} \) we fix a generator \( \tau_q \) of \( G_q = \text{Gal}(\mathbb{L}_n(q)/\mathbb{L}_n) \) and define the following group ring elements

\[
N_q = \sum_{i=0}^{2^l-1} \tau_q^i \quad D_q = \sum_{i=0}^{2^l-1} i \tau_q^i.
\]

For \( r = \prod_{k=1}^s q_k \) we define \( D_r = \sum_{k=1}^s D_{q_k} \in \mathbb{Z}[\text{Gal}(\mathbb{L}_n(r)/\mathbb{L}_n)] \).

With these definitions we have the following lemma.

**Lemma 3.6.** [Ru-2 Proposition 2.2] For every Euler system \( \alpha^\sigma(n, r) \) there exists a canonical map

\[
\kappa: S_{l,n} \to \mathbb{L}_n^\times / (\mathbb{L}_n^\times)^{2^l}
\]

such that \( \kappa(r) = (\alpha^\sigma(n, r))^{D_r} \mod (\mathbb{L}_n(r))^{2^l} \).
For every prime ideal \( q \in S_{n,l} \) of \( K \) we define the free group of ideals in \( L_n \)
\[ I_q = \bigoplus_{[y]_q} \mathbb{Z}\Omega = \mathbb{Z}[\text{Gal}(L_n/K)]\Omega. \]

For every \( y \in L_n^\times \) denote by \([y]_q\) the coset of the principal ideal \((y)\) in \( I_q/2^l I_q\). Let \( \tilde{\Omega} \) be a prime above \( \Omega \) in \( L_n(q) \) and note that for every \( x \in L_n(q)^\times \) the element \( x^{1-\tau q} \) lies in \((O(L_n(q))/\tilde{\Omega})^\times\). As \( O(L_n(q))/\tilde{\Omega} \cong O(L_n)/\Omega \) there is a well defined image \( x^{1-\tau q} \in (O(L_n)/\Omega)^\times \). Thus, we can define a map
\[ L_n(q)^\times \to (O(L_n)/\Omega)^\times/((O(L_n)/\Omega)^\times)^{2^l} \quad x \mapsto (x^{1-\tau q})^1/d, \]
where \( d = (Nq - 1)/2^l \). This map is surjective and the kernel of this map consists precisely of the elements whose \( \tilde{\Omega} \) valuation is divisible by \( 2^l \). Let now \( w \in (O(L_n)/\Omega)^\times/((O(L_n)/\Omega)^\times)^{2^l} \) and let \( x \) be a preimage. Define
\[ l_{\Omega}(w) = \text{ord}_{\tilde{\Omega}}(x) \mod 2^l \in \mathbb{Z}/2^l\mathbb{Z}. \]

Note that \( l_{\tilde{\Omega}} \) is a well defined isomorphism. Thus, we can define
\[ \varphi_q : (O(L_n)/q)^\times/((O(L_n)/q)^\times)^{2^l} \to I_q/I_q^{2^l} \quad w \mapsto \sum_{\Omega | q} l_{\Omega}(w)\Omega. \]

With these notations we have the following proposition.

**Proposition 3.7.** [Ru-2 Proposition 2.4] Let \( \alpha^\sigma(n,v) \) be an Euler system and \( \kappa \) be the map defined in Lemma 3.6. Let \( v \) be a prime in \( S_{n,l} \) and \( q \) be a prime in \( K \). Then

i) If \( q \nmid v \) then \([\kappa(v)]_q = 0.\)

ii) Assume that \( q | v \) and \( v/q \neq (1) \). Then \([\kappa(v)]_q = \varphi_q(\kappa(v/q))\)

iii) Assume that \( v = q \) and that the \( \Omega \)-valuation of \( (\alpha^\sigma(n, 1)) \) is divisible by \( 2^l \) for all \( \Omega \) above \( q \) in \( L_n \). Then \([\kappa(v)]_q = \varphi_q(\kappa(v/q))\).

Note that Rubin does not distinguish between the cases ii) and iii). But as Bley [Bl Proposition 3.3] points out, the extra assumption in iii) is necessary.

Let \( y \) be any element in the kernel of \([\cdot]\). Then \( y = B^2 C \), where \( B \) is an ideal only divisible by primes above \( q \) and \( C \) is coprime to \( q \). Let \( (\beta) = BC \) for some ideal \( \mathfrak{D} \) coprime to \( q \). Then \( y = \beta^2 u \) and \( u \) is coprime to \( q \). In particular, \( u \) is a unit at all ideals above \( q \). Thus, \( \varphi_q(u) \) is well defined and we can extend \( \varphi_q \) on \( \ker([\cdot]_q) \).

### 3.1. An Application of Tchebotarev’s Theorem

This section follows ideas of Bley in [Bl] and of Greither in [Gr]. As some steps of the proofs are slightly different for the case \( p = 2 \) we will carry them out in detail. The main goal of this section is to prove the following Theorem.

**Theorem 3.8.** Let \( M = L_n \) for some \( n \) and write \( G = \text{Gal}(M/K) \). Assume that \( \overline{\mu}^\times \) is the precise power of \( \overline{\mu} \) dividing the conductor of the extension \( M/K \). Let \( M = 2^l \) for some \( l \) and let \( W \subset \mathbb{M}^\times/((\mathbb{M}^\times)^M \) be a finite \( \mathbb{Z}[G] \)-module. Assume that there is a \( \mathbb{Z}[G] \)- homomorphism \( \psi : W \to \mathbb{Z}/MZ[G] \). Let \( C \in A(M) \) be an arbitrary ideal class. Then there are infinitely many primes \( \Omega \) in \( M \) satisfying:

i) \([\Omega] = 2^{3k+4}C.\)

ii) If \( q = \Omega \cap \mathbb{K} \) then \( Nq \equiv 1 \mod 2M \) and \( q \) is totally split in \( M \).

iii) For all \( w \in W \) one has \([w]_q = 0 \) and there exists a unit \( u \) in \( \mathbb{Z}/MZ \) such that \( \varphi_q(w) = 2^{3k+4}u\psi(w)\Omega. \)
The proof of Theorem 3.8 relies on several lemmas which we will prove in the following. We fix the following Notation: Let $\mathbb{H}$ be the Hilbert class field of $\mathbb{M}$ and define $\mathbb{M}' = \mathbb{M}(\zeta_{2M})$ and $\mathbb{M}'' = \mathbb{M}'(W^{1/M})$.

**Lemma 3.9.** $|\mathbb{H} \cap \mathbb{M}' : \mathbb{M}| \leq 2^{c-1}$ if $c \geq 1$. The extension $\mathbb{H} \cap \mathbb{M}'/\mathbb{M}$ is trivial if $c = 0$.

**Proof.** As $2$ is totally split in $K/Q$ the ideal $\mathfrak{p}$ is totally ramified in $K(\zeta_{2M})/K$ and the ramification index is $M$. If $c = 0$ then $\mathbb{M}/K$ is unramified at $\mathfrak{p}$ and $\mathbb{M}'/\mathbb{M}$ is totally ramified at all primes above $\mathfrak{p}$. Hence, $\mathbb{M}' \cap \mathbb{H} = \mathbb{M}$ and the claim follows in this case. Assume now that $c \geq 1$, then the ramification index of $\mathfrak{p}$ in $\mathbb{M}/K$ is at most $|(O(\mathbb{K})/\mathfrak{p}^c)^\times|$ . Hence, the ramification index of every divisor of $\mathfrak{p}$ in $\mathbb{M}'/\mathbb{M}$ is at least $M/2^{c-1}$. In particular, $|\mathbb{M}' : \mathbb{M} \cap \mathbb{H}| \geq M/2^{c-1}$. Using that $|\mathbb{M}' : \mathbb{M}| \leq M$ it follows that $|\mathbb{H} \cap \mathbb{M}' : \mathbb{M}| \leq 2^{c-1}$. □

**Lemma 3.10.** If $c = 0$ then the group $\text{Gal}(\mathbb{M}'' \cap \mathbb{H}/\mathbb{M})$ is annihilated by $4$. If $c > 1$ then $\text{Gal}(\mathbb{M}'' \cap \mathbb{H}/\mathbb{M})$ is annihilated by $2^{c+2}$. In both cases it is annihilated by $2^{2c+2}$.

**Proof.** By definition we have $|K(\zeta_{2M}) : \mathbb{M} \cap K(\zeta_{2M})| \geq \min(M, M/2^{c-1})$. Consider first the case $c \geq 1$. As $\text{Gal}(K(\zeta_{2M})/K) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/(M/2)\mathbb{Z}$ we can choose an element $j$ in $\text{Gal}(K(\zeta_{2M})/\mathbb{M} \cap K(\zeta_{2M}))$ of order $M/2^{c}$. Choose $r \in \mathbb{Z}$ such that $j(\mathbb{M}) = \mathbb{M}^r$. It follows that $rM/2^c \equiv 1 \mod 2M$ and $r \not\equiv 1 \mod 2M$ for every $0 < b < M/2^c$. The element $j$ has a lift to $\text{Gal}(\mathbb{M}'/\mathbb{M})$ of the same order. Let $\sigma$ be in $\text{Gal}(\mathbb{M}''/\mathbb{M}')$ and $\alpha$ in $\mathbb{M}''$ such that $\alpha^M = w$ generates a maximal cyclic subgroup of $W$. By Kummer Theory there exists an even integer $t_w$ such that $\sigma(\alpha) = \mathbb{M}_2^w \alpha$. If we fix for every generator $w \in W$ one such $\alpha_w$ we can find a lift of $j$ in $\text{Gal}(\mathbb{M}''/\mathbb{M})$ such that $j(\alpha_w) = \alpha_w$ for our chosen generators $w$. By abuse of notation we will denote the lift by $j$ as well. We obtain

$$j\sigma j(\alpha_w) = j\sigma(\alpha_w) = j(\mathbb{M}_2^w \alpha_w) = \mathbb{M}_2^w \alpha_w.$$ 

Thus,

$$j\sigma j = \sigma^r.$$ 

The extension $(\mathbb{M}'' \cap \mathbb{H} \mathbb{M}')/\mathbb{M}$ is clearly abelian. Hence $\text{Gal}(\mathbb{M}'/\mathbb{M})$ acts trivially on the group $H = \text{Gal}(\mathbb{M}'' \cap \mathbb{H} \mathbb{M}'/\mathbb{M}')$. Together with (5) this implies that $H$ is annihilated by $r - 1$. On the other hand it is a Kummer extension of exponent at most $M$. Therefore, $H$ is annihilated by $2^d = \gcd(M, r - 1)$. Then $r \equiv 1 \mod 2^d$. Assume now that $v \equiv 0 \mod 2^v$ for some $v \geq d$. Then $v^{e+1-d} \equiv 1 \mod 2^{v+1}$. This shows that $2^{v+1-d} \equiv 1 \mod 2t+1$. Recall that $M = 2^t$ and that $r \not\equiv 1 \mod 2M$ for all $0 < b < M/2^c$. It follows that $M/2^c | 2M/2^d$ and $c \geq d - 1$. Therefore $2^{c+1}$ annihilates $H$. There is a natural surjective projection

$$H \rightarrow \text{Gal}(\mathbb{M}'' \cap \mathbb{H} \mathbb{M}'/\mathbb{M}'' \cap \mathbb{H}).$$

Using Lemma 3.9 this gives the claim in the case $c \neq 0$.

In the case $c = 0$ we choose $j$ of order $M/2$. Then $r^{M/2} \equiv 1 \mod 2M$ and $r^b \not\equiv 1 \mod 2M$ for all $0 < b < M/2$. Using the same arguments as for the case $c = 1$ we obtain that the extension $\mathbb{M}'' \cap \mathbb{H} \mathbb{M}'/\mathbb{M}$ is annihilated by $4$. This implies the claim in the case $c = 0$.

Using the Kummer pairing we see that there is a homomorphism

$$F : \text{Gal}(\mathbb{M}''/\mathbb{M}') \rightarrow \text{Hom}(W, \zeta_M)$$
Theorem. To do so we note that

Using that Lemma 3.11.

Proof. Let \( W' \) be the image of \( W \) in \( M'/\langle M' \rangle \). By Kummer duality we have \( \text{Hom}(W', \langle \zeta_M \rangle) \cong \text{Gal}(M''/M') \). Let \( f : (M'/\langle M' \rangle)_M^M \to (M''/\langle M' \rangle)_M^M \) be the natural map. Using the exact sequence

\[
0 \to \ker(f) \to W \to W' \to 0
\]

we obtain a second exact sequence

\[
\text{Hom}(W', \langle \zeta_M \rangle) \to \text{Hom}(W, \langle \zeta_M \rangle) \to \text{Hom}(\ker(f), \langle \zeta_M \rangle).
\]

Hence, to prove the lemma it suffices to prove that the kernel of \( f \) is annihilated by \( 2^{c+2} \). Let \( u \in \ker(f) \) and choose an element \( v \in M' \) such that \( u = v^M \). We define \( \delta_v : \text{Gal}(M'/M) \to \langle \zeta_M \rangle \) by \( \delta_v(g) = g(v)/v \). As

\[
\delta_v(gh) = gh(v)/g(v) \cdot g(v)/v = \delta_v(g) \cdot g \delta_v(h).
\]

It follows that \( \delta_v \) is a cocycle. Note that \( v \) is unique up to \( M \)-th roots of unity. If we choose \( v' = v \zeta_M^c \), we obtain \( \delta_v'(g) = g(v)/v \cdot g \zeta_M^c / \zeta_M^c \). Hence, \( \delta_v \) is uniquely defined up to coboundaries and \( \delta_v \) has a well defined image in \( H^1(\text{Gal}(M'/M), \langle \zeta_M \rangle) \). Thus, we have an injective map \( \ker(f) \hookrightarrow H^1(\text{Gal}(M'/M), \langle \zeta_M \rangle) \). Therefore it suffices to bound \( H^1(\text{Gal}(M'/M), \langle \zeta_M \rangle) \). If the group \( \text{Gal}(M'/M) \) is cyclic we see that \( \langle \zeta_M \rangle \) has a trivial Herbrandt quotient. So it suffices to consider

\[
|H^0(\text{Gal}(M'/M), \langle \zeta_M \rangle)| \leq |\langle \zeta_M \rangle \cap M| \leq 2^{c+1}.
\]

If \( \text{Gal}(M''/M) \) is not cyclic then it is isomorphic to \( \Delta \times C_r \) where \( C_r \) is cyclic and \( \Delta \cong \mathbb{Z}/2\mathbb{Z} \). Using the exact sequence

\[
H^1(\Delta, \langle \zeta_M \rangle) \to H^1(\text{Gal}(M'/M), \langle \zeta_M \rangle) \to H^1(C_r, \langle \zeta_M \rangle)
\]

and the fact that the last term is annihilated by \( 2^{c+1} \) while the first one is annihilated by \( 2^{c+2} \) we obtain that the middle term is annihilated by \( 2^{c+2} \) proving the lemma. \( \square \)

Now we have all ingredients to prove Theorem 3.8.

of Theorem 3.8 Consider the map \( \iota : (\mathbb{Z}/M\mathbb{Z})[G] \to \langle \zeta_M \rangle \) defined by \( \sum a_\sigma \sigma \to \zeta_M^a \). Then \( \iota \psi \in \text{Hom}(W, \langle \zeta_M \rangle) \). Using Lemma 3.11 we see that \( 2^{c+2} \iota(\psi) \) has a preimage \( \gamma \) in \( \text{Gal}(M''/M') \). Let \( \gamma_1 = 2^{c+2} \left( \frac{C}{M/M} \right) \) and choose \( \delta \in \text{Gal}(M''/M') \) such that \( \delta \mid_{M'} = 2^{c+2} \gamma_1 \) and \( \delta \mid_{M''} = 2^{c+2} \gamma_2 \). Note that this is possible as \( \text{Gal}(M'' \cap H/M) \) is annihilated by \( 2^{3c+2} \) due to Lemma 3.10 Using Tchebotarev’s Theorem we can find infinitely many primes \( \mathcal{Q} \in M \) of degree 1 such that

\[
\left( \frac{\mathcal{Q}}{M''/M} \right) = \text{conjugacy class of } \delta.
\]

As \( \delta \mid_{M''} = 2^{3c+2} \gamma_2 \mid_{M''} = \text{id} \) we see that \( \mathcal{Q} \) is totally split in \( M'/M \). Let \( q = \mathcal{Q} \cap K \). Then \( q \) is totally split in \( M'/K \) and \( Nq \equiv 1 \mod 2M \). Further \( \delta \mid_H = 2^{3c+2} \gamma_1 \mid_H = 2^{3c+4} \left( \frac{C}{H/M} \right) \). It follows that \( \mathcal{Q} \) is a preimage of \( 2^{3c+4} \psi(w) \). It remains to prove point iii) of the Theorem. To do so we note that

\[
\text{ord}_\mathcal{Q}(2^{3c+4} \psi(w) \mathcal{Q}) \equiv 0 \mod M \iff 2^{3c+4} \psi(w) = 1.
\]

Using that \( \gamma \) is the preimage of \( 2^{c+2} \psi \) we see that

\[
2^{3c+4} \psi(w) = 1 \iff (2^{2c+2} \gamma)w^{1/M}/w'^{1/M} = 1.
\]
As $\mathcal{Q}$ has Artin-symbol $2^{2c+2}\gamma$ in $\mathcal{M}''/\mathcal{M}$ we see that
\[ \text{ord}_\mathcal{Q}(2^{2c+4}\psi(w)\mathcal{Q}) \equiv 0 \pmod{M} \iff w \text{ is an } M\text{-th power modulo } \mathcal{Q}. \]
w is an $M$-th power in $\mathcal{M}''$ and $\mathcal{Q}$ is not ramified in $\mathcal{M}''/\mathcal{M}$. Therefore, $[(w)]_q = 0$. By definition $\varphi_q(w) = 0 \iff w$ is an $M$-th power modulo $\mathcal{Q}$. It follows that
\[ \text{ord}_\mathcal{Q}(2^{2c+4}\psi(w)\mathcal{Q}) = u'\text{ord}_\mathcal{Q}(\varphi_q(w)) \text{ for some unit } u'. \]
From this the claim follows as in [Ru-3, page 403].

3.2. $\chi$-components on the class group and on $E/C$. Recall that we fixed a decomposition $\text{Gal}(\mathbb{L}_\infty/\mathbb{K}) \cong \Gamma' \times H$ with $H = \text{Gal}(\mathbb{L}_\infty/\mathbb{K}_\infty)$. Let $\gamma'$ be a topological generator of $\Gamma'$. To simplify notation we will use the notation $\gamma_n'$ for the element $\gamma'^n$. Let $\Gamma = \text{Gal}(\mathbb{L}_\infty/\mathbb{L})$. There exists a power of 2 such that $\Gamma^{2^m}$ is contained in $\text{Gal}(\mathbb{L}_\infty/\mathbb{K}(\mathfrak{p}^2))$. In particular $\mathbb{L}_\infty/\mathbb{L}_{\Gamma^{2^m}}$ is totally ramified at all primes above $\mathfrak{p}$ and $\Gamma^{2^m+\mu} = \Gamma^{2m'+\mu}$ for some $m' \leq m$ independent of $n$. Recall that $A_n$ denotes the class group of $\mathbb{L}_n$, i.e. $\gamma'^{2^m+\mu}$ acts trivial on $A_n$ and $L_n$ for $n \geq m'$. We fixed the notations $\Lambda = \lim_{\substack{\longrightarrow \\mathbb{Z}_p[[\text{Gal}(\mathbb{L}_n/\mathbb{K})]]}}$ and $A_\infty = \lim_{\substack{\longleftarrow \\mathbb{Z}_p}} A_n$. Let $\chi$ be a character of $H$. Then $A_{\infty,\chi}$ and $(\overline{E}/\overline{C})_\chi$ are $\Lambda_\chi$-modules. Let $\Lambda_{\chi,n}$ be the quotient of $\Lambda_{\chi}$ by $1 - \gamma_n'$. In particular, there is a pseudo-isomorphism
\[ A_{\infty,\chi} \sim \bigoplus_{i=1}^k \Lambda_{\chi}/g_i, \]
with finite kernel and cokernel.

Lemma 3.12. The kernel of the multiplication of $(1 - \gamma_n')$ on $A_\infty$ is finite for every $n$.

Proof. This follows directly from the fact that all finite subextension of $\mathbb{L}_\infty/\mathbb{K}$ are abelian over $\mathbb{K}$ and that the Leopoldt conjecture holds for any abelian extensions of imaginary quadratic fields. In particular, Leopoldt’s conjecture holds for every field $\mathbb{L}_n$ (see [Gr] for more details). \hfill \Box

Lemma 3.13. Let $\chi$ be a character of $H$ and $n \geq m'$. Then there is a $\Lambda_{\chi,n}$ homomorphism
\[ A_{\chi,n} \to \bigoplus_{i=1}^k \Lambda_{\chi,m+n-m'}/(\overline{g}_i), \]
with uniformly bounded cokernel. Here, $\overline{g}_i$ is the restriction of $g_i$ to level $n$.

Proof. This proof is very similar to [Gr, Lemma 3.11]: By [Wash, page 281] the module $A_n$ is isomorphic to $A_\infty/\nu_{m+n-m',m'}Y$ for some submodule $Y$. Consider the map
\[ \phi_n : A_\infty/(1 - \gamma_{m+n-m'})A_\infty \to A_n. \]
By definition the kernel is isomorphic to $\nu_{m+n-m',m'}Y/(1 - \gamma_{m+n-m'})A_\infty$ which is bounded by the size of $Y/(1 - \gamma_{m'})A_\infty \leq A_\infty/(1 - \gamma_{n'})A_\infty$. By Lemma 3.12 this quotient is finite and the kernel of $\phi_n$ is uniformly bounded. Thus, the kernel of the natural projections
\[ A_{\infty,\chi}/(1 - \gamma_{m+n-m'})A_{\infty,\chi} \to A_{\chi,n} \]
has a uniformly bounded kernel and we can deduce the claim from (6). \hfill \Box
Let $\Gamma'_{n_2,n_1} = \Gamma^{q_1}/\Gamma^{q_2}$ for $n_2 > n_1$. Recall that $\Gamma^{q_2}$ fixes the field $\mathbb{L}_{m+n-m}$ for $n > m$. Hence $\text{Gal}(\mathbb{L}_{m+n-m}/\mathbb{L}_{n_1-m+m'}) = \Gamma'_{n_2,n_1}$.

**Lemma 3.14.** There is a constant $k$ such that

$$|(1 - \gamma_m')H^1(\Gamma'_{n_2,n_1}, E_{m'+n_2-m})| \leq 2^k \text{ and } |(1 - \gamma_m')H^0(\Gamma'_{n_2,n_1}, E_{m'+n_2-m})| \leq 2^k$$

for any pair $(n_1, n_2)$ with $n_2 > n_1 \geq m$.

**Proof.** The proof follows the ideas of [Ri-1, Lemma 1.2]. But it is restated here as we use weaker assumptions. Let $E'_{m'+n_2-m}$ be the $p$-units in $\mathbb{L}_{m'+n_2-m}$ and $R_{m'+n_2-m}$ be the $\mathbb{Z}_p$-free group defined by the exact sequence

$$0 \to E_{m'+n_2-m} \to E'_{m'+n_2-m} \to R_{m'+n_2-m} \to 0$$

As $\mathbb{L}_\infty/\mathbb{L}_{m'}$ is totally ramified we see that $\Gamma^m$ acts trivially on $R_{m'+n_2-m}$. We know from [PW, page 267] that $|H^1(\Gamma_{n_2,n_1}, E_{m'+n_2-m})|$ is uniformly bounded$ootnote{In Iwasawa's notation $E'$ are the p-units, but the proof of Theorem 12 works for the p-units as well.}$. Further, we have the exact sequence

$$H^0(\Gamma'_{n_2,n_1}, R_{m'+n_2-m}) \to H^1(\Gamma'_{n_2,n_1}, E_{m'+n_2-m}) \to H^1(\Gamma_{n_2,n_1}, E_{m'+n_2-m})$$

The first term is annihilated by $1 - \gamma_m'$ and the last term is uniformly bounded. It follows that $(1 - \gamma_m')H^1(\Gamma'_{n_2,n_1}, E_{m'+n_2-m})$ is uniformly bounded.

It is an immediate consequence from [Ja, V Theorem 2.5] that $q(E'_{m'+n_2-m}) = 2^{(n_2-n_1)(1-s)}$, where $s$ is the number of primes above 2. Thus,

$$2^{(n_2-n_1)(s-1)}|H^1(\Gamma'_{n_2,n_1}, E_{m'+n_2-m})| = |H^0(\Gamma'_{n_2,n_1}, E_{m'+n_2-m})|.$$ 

Consider the map $H^0(\Gamma'_{n_2,n_1}, E_{m'+n_2-m}) \to N_{n_1,m}E'_{m'+n_1-m}/N_{n_1,m}E_{m'+n_2-m}$ induced by $N_{n_1,m} = (\gamma_m' - 1)/(\gamma_m' - 1)$. Using that $N_{n_1,m}(1 - \gamma_m') = (1 - \gamma_m')$ and that $\Gamma^{q_1}$ is precisely the group fixing $\mathbb{L}_{m'+n_1-m}$ we see that the subgroup

$$((1 - \gamma_m')E'_{m'+n_1-m} + N_{n_2,n_1}E_{m'+n_2-m})/N_{n_2,n_1}E_{m'+n_2-m}$$

is certainly contained in the kernel. Note that $N_{n_1,m}E'_{m'+n_1-m}/N_{n_1,m}E_{m'+n_2-m}$ is the kernel of the natural map $H^0(\Gamma'_{n_2,n_1}, E_{m'+n_2-m}) \to H^0(\Gamma'_{n_2,n_1}, E_{m'+n_2-m})$. Thus, we obtain:

$$|(1 - \gamma_m')H^0(\Gamma'_{n_2,n_1}, E_{m'+n_2-m})| \leq \frac{|H^0(\Gamma'_{n_2,n_1}, E_{m'+n_2-m})||H^0(\Gamma_{n_2,m}, E'_{m'+n_1-m})|}{|H^0(\Gamma'_{n_2,n_1}, E_{m'+n_2-m})|} \leq \frac{2^{(n_2-n_1)(s-1)+k}}{2^{(n_2-n_1)(s-1)}} = 2^k,$$

where $2^k$ is the uniform bound on $H^1(\Gamma'_{n_2,n_1}, E_{m'+n_2-m})$. It is easy to verify that the natural map $H^0(\Gamma'_{n_2,n_1}, E_{m'+n_2-m}) \to H^0(\Gamma'_{n_2,n_1}, E_{m'+n_2-m})$ is an injection and the claim follows. \hfill $\square$

**Lemma 3.15.** Let $n \geq m'$ and consider the projection

$$\pi_n: \overline{E}_\infty/(1 - \gamma_m'E_{m-n}) \to \overline{E}_n.$$ 

There exists an integer $k$ such that $2^k(1 - \gamma_m')$ annihilates the kernel and the cokernel of $\pi_n$ for all $n \geq m$.\footnote{In Iwasawa's notation $E'$ are the p-units, but the proof of Theorem 12 works for the p-units as well.}
Proof. We have an exact sequence
\[
\lim_{\infty \to n'} H^1(\Gamma_{m+n'-m',m+n-m}, E_{n'}) \to \\
\to E_{\infty}/(1 - \gamma'_{m+n-m}) E_{\infty} \to E_n \to \lim_{\infty \to n} H^0(\Gamma_{m+n'-m',m+n-m}, E_n)
\]
The first and the last term are annihilated by $2^k(1 - \gamma'_{m})$ due to Lemma\ref{lem:Bley} and the claim follows.

**Lemma 3.16.** Let $U_\infty$ be defined as in the introduction. $U_\infty \otimes_{Z_p} Q_p \cong \Lambda \otimes_{Z_p} Q_p$ and $U_{\infty,\chi} \otimes_{Z_p} Q_p \cong \Lambda_\chi \otimes_{Z_p} Q_p$.

**Proof.** The first claim follows as in \cite{Bley} Lemma 3.5 Claim 2]. Bley gives two references for this proof. Note that the second one is only stated for $p > 2$ but the proof works for $p = 2$ as well.

The second claim can be proved as follows:

\[
U_\infty \otimes_{Z_p} Z_p(\chi) \otimes_{Z_p} Q_p \cong \Lambda \otimes_{Z_p} Z_p(\chi) \otimes_{Z_p} Q_p.
\]
Let $I_\chi \subset Z(\chi)[H]$ be the module generated by $\sigma - \chi(\sigma)$ for $\sigma \in H$. It is an easy verification that

\[
U_\infty \otimes_{Z_p} Z_p(\chi) \otimes_{Z_p} Q_p / I_\chi(U_\infty \otimes_{Z_p} Z_p(\chi) \otimes_{Z_p} Q_p)
\cong \Lambda \otimes_{Z_p} Z_p(\chi) \otimes_{Z_p} Q_p / I_\chi(\Lambda \otimes_{Z_p} Z_p(\chi) \otimes_{Z_p} Q_p).
\]

It is proved in \cite{Bley} Lemma 2.1 that $M_\chi \cong (M \otimes_{Z_p} Z_p(\chi)) / I_\chi(M \otimes_{Z_p} Z_p(\chi))$. Further, for any module $M$ we see that

\[
M \otimes_{Z_p} Z_p(\chi) \otimes_{Z_p} Q_p / I_\chi(M \otimes_{Z_p} Z_p(\chi) \otimes_{Z_p} Q_p)
= (M \otimes_{Z_p} Z_p(\chi) / I(\chi)(M \otimes_{Z_p} Z_p(\chi))) \otimes Q_p = M_\chi \otimes_{Z_p} Q_p.
\]
Using this for $U_\infty$ and $\Lambda$ the second claim follows.

**Lemma 3.17.** Let $h_\chi$ be the characteristic ideal of $(\overline{E}/\overline{C})_\chi$. Let $n \geq m$. Then there exist constants $n_0$, $c_1$ and $c_2$ independent of $n$, a divisor $h'_\chi$ of $h_\chi$ and a $\text{Gal}(\mathbb{L}_{m'+n-m}/\mathbb{K})$-homomorphism

\[
\partial_{m'+n-m} : \overline{E}_{m'+n-m,\chi} \to \Lambda_\chi
\]
such that

i) $h'_\chi$ is prime to $1 - \gamma'_{m}$ for all $v$,

ii) $(\gamma_{n_0} - 1)^{-1}2^{c_2} h'_\chi \Lambda_{n,\chi} \subset \partial_{m'+n-m}(\text{im}(\overline{C}_{m'+n-m,\chi}))$, where $\text{im}(\overline{C}_{m'+n-m,\chi})$ denotes the image of $\overline{C}_{m'+n-m,\chi}$ in $\overline{E}_{m'+n-m,\chi}$.

**Proof.** From the second claim of Lemma\ref{lem:Bley} and the fact that $\Lambda_\chi \otimes_{Z_p} Q_p$ is a principal ideal domain we obtain that the submodule $E_{\infty,\chi}$ is free cyclic over the ring $\Lambda_\chi \otimes_{Z_p} Q_p$. Consider the map

\[
\pi_n : \overline{E}/(1 - \gamma'_{n}) \to \overline{E}_{m'+n-m}.
\]
The rest of the proof is exactly the same as \cite{Bley} Lemma 3.5]; we just have to substitute $E_n$ by $E_{m'+n-m}$ and $(1 - \gamma')$ by $(1 - \gamma'_{m})$ in all computations due to the new definition of $\pi_n$ and the fact that Lemma\ref{lem:Bley} is weaker than the corresponding claim in Bley’s case. Note that Bley states the lemma only for non trivial characters but this assumption is not necessary. \qed
Lemma 3.18. Let $\mathcal{M} = \mathbb{L}_n$ for some $n$ and let $\Delta$ be a subgroup of $Gal(\mathcal{M}/K)$. Let $\chi$ be a character of $\Delta$, $M = 2^i$ and $\mathfrak{A} = \prod_{i=1}^{n} \mathfrak{q}_i \in S_{n,1}$. Let $\Omega$ be a divisor of $\mathfrak{q}_i$ in $\mathcal{M}$. Let $C = [\Omega]$ the ideal class of $\Omega$. Let $B \subset \mathcal{M}$ be the subgroup generated by ideals dividing $\mathfrak{A}/\mathfrak{q}_i$. Let $x \in \mathcal{M}/(\mathcal{M}/A)^M$ be such that $[x]_e = 0$ for all $(x, \mathfrak{A}) = 1$. Let $W \subset \mathcal{M}/(\mathcal{M}/A)^M$ be such that $\eta \in \mathcal{M}$-submodule generated by $x$. Assume that there are elements $\eta, g \in \mathcal{M}[G]$ such that

\[ (8) \quad \psi : W \to (\mathbb{Z}/M\mathbb{Z})[G] \]

such that $(g\psi(x))_\chi = (\eta g[x]_\chi)$ in $I_q/MI_q$.

Proof. This is [Bl] Lemma 3.8. The proof is the same as [Gr] Lemma 3.12. \(\square\)

To prove the central theorem of this section we need the following lemma.

Lemma 3.19. [Gr] Lemma 3.13 Let $\Delta$ be any finite group and $N$ a $\mathcal{M}[\Delta]$-module. Let $\chi$ be a character of $\Delta$ and $n : N \to N_\chi$ the natural projection. Then there exists a $\mathcal{M}[\Delta]$-homomorphism

\[ \varepsilon : N_\chi \to N \]

such that $n \circ \varepsilon = |\Delta|$.

Proof. Let $\mathfrak{q}$ be an element in $S_{n,1}$ and $\mathfrak{A}$ in $I_q$. Then there is an element $v_\Omega(\mathfrak{A})$ in $\mathbb{Z}/2^l\mathbb{Z}[Gal([\mathfrak{L}])\mathbb{K}]$ such that $\mathfrak{A} = v_\Omega(\mathfrak{A})[\mathfrak{L}]$. The following theorem allows us to relate the characteristic ideal of $A_\chi$ to the one of $(\mathbb{E}/\mathbb{C})_\chi$. The proof follows the ideas of [Bl].

Theorem 3.20. Let $\mathcal{M} = \mathbb{L}_n$ and $G = Gal(\mathcal{M}/K)$ for $n$ large enough. Let $\chi$ be a character of $H \subset Gal(\mathcal{M}/K)$. For $1 \leq i \leq k$ let $C_i \in A_\chi(M)$ be such that $t(C_i) = (0, 0, \cdots, 2c_3, 0, \cdots, 0)$ in $\mathbb{Z}/2^l\mathbb{Z}[\mathcal{M}]$ where $t$ is the map defined in Lemma 3.13 and $2c_3$ annihilates the cokernel. Let $C_{k+1}$ be arbitrary. Let $d = 3c + 4$ where $c$ is defined in Theorem 3.38. Then there are prime ideals $\Omega_i$ in $\mathcal{M}$ such that

\[ (7) \quad (v_{\Omega_i}(\kappa(q_1))_\chi = u_i[H](\gamma_n - 1)2^{d+c_2}h_\chi^i \mod 2^n \]

\[ (8) \quad (g_{i-1}v_{\Omega_i}(\kappa(q_1 \cdots q_{i-1}))_\chi = u_i[H](\gamma_n - 1)2^{d+c_2}h_\chi^{i-1} \mod 2^n \] for $2 \leq i \leq k + 1$.

Proof. By Lemma 3.17 there exists an element $\xi'$ in $im(C_{n,\chi})$ with the property $\vartheta_n(\xi') = (1 - \gamma_n)2^{c_2}h_\chi^i$. By approximating $\xi$ with a global elliptic unit we can find $\xi \in C_n$ such that $\vartheta_n(\xi) = (1 - \gamma_n)2^{c_2}h_\chi^i \mod MA_{\chi + n - m}$. We can apply Lemma 3.25 to find an Euler system $\alpha'(n, t)$ such that $\alpha'(n, (1)) = \xi$. Let $i = 1$ and $C$ be a preimage of $C_i$ under the map $A_n \to A_{\chi - n}$. Choose $M = 2^n$ and $W = \mathcal{O}(\mathcal{M})/\mathcal{O}(\mathcal{M}/C)^M$. Consider

\[ \psi : W \to \mathbb{Z}/M\mathbb{Z}[G] \quad x \mapsto (\varepsilon \circ \vartheta_n)(x^n), \]
where \( v \) is such that \( x^v \in E_n \) for all \( x \) and \( \varepsilon_\chi \) is defined as in Lemma 3.19. Then Theorem 3.8 implies that we can find an ideal \( \Omega_1 \) satisfying i) and ii). We know further from Theorem 3.8 that \( \varphi_q(w) = 2^d u \psi(w) \Omega_1 \). As \( \alpha^\sigma(n, 1) \) is a unit we can apply Proposition 3.7 and obtain

\[
v_{\Omega_1}(\kappa(q_1))\Omega_1 = [\kappa(q_1)]_{q_1} = \varphi_{q_1}(\kappa(1))
= \varphi_{q_1}(\xi) = 2^d u \psi(\xi) \Omega_1 = 2^d u \varepsilon_\chi((\gamma^\prime_{n_0} - 1)^{c_1} 2^{c_2} h'_\chi) \mod 2^n.
\]

Projecting to the \( \chi \)-component and using the definition of \( \varepsilon_\chi \) we get (17).

We will now define the ideals \( \Omega_i \) inductively. Assume that we have already found the \( \Omega_1, \ldots, \Omega_{i-1} \) and let \( a_{i-1} = \prod_{j=1}^{i-1} q_i \). Using point iii) recursively we see

\[
\prod_{j \leq i-2} g_j(v_{\Omega_{i-1}}(\kappa(a_{i-1}))) = [H^{i-1} u 2^{(i-2)(2d+c_0)+d+c_2}(\gamma^\prime_{n_0} - 1)^{c_1} \Sigma_{j=1}^{i-2} c_j h'_\chi].
\]

Let \( D_i = |H^{i-1} u 2^{(i-2)(2d+c_0)+d+c_2} \) be by enlarging \( c_1 \) we can assume that \( c_1 + \Sigma_{j=1}^{i-2} c_j \) is bounded by \( c_i^{i-1} \) and set \( t_i = c_i^{i-1} \). It follows that \( v_{\Omega_{i-1}}(\kappa(a_{i-1})) \) is bounded by \( D_i h'_\chi(\gamma^\prime_{n_0} - 1)^{t_i} \).

Define \( N = (\gamma^\prime_{n_0} - 1)^{t_i}(I_{a_{i-1}/(\text{TM}_{a_{i-1}} + \mathbb{Z}_p[G][\kappa(a_{i-1})]_{q_{i-1}})}) \). As \( h'_\chi \) is coprime to every \( \gamma^\prime_n - 1 \) we see that \( A_{\chi,m+n-m'/h'_\chi} \) is finite and further

\[
|N| \leq |A_{\chi,m+n-m'/h'_\chi}|\cdot |A_{\chi,m+n-m'/h'_\chi}|.
\]

Choose now \( 2^d = M > \max(|A_{\chi}(M)|, |A_{\chi,m+n-m'/h'_\chi}|) \) and we want to apply Lemma 3.18 with \( E = 2^{c_3+d}, \eta = (\gamma^\prime_{n_0} - 1)^{t_i}, g = g_{i-1}, \mathfrak{A} = a_{i-1}, \) and \( x = \kappa(a_{i-1}) \). To do so we have to check the assumptions. It follows directly from Proposition 3.7 i) that \( [x]_r = 0 \) for all \( r \) coprime to \( a_{i-1} \). We now have to check the conditions i)-iii) from Lemma 3.18.

i) By definition \( C = [\mathfrak{Q}_{i-1}] = 2^d C_{i-1} \). The annihilator of \( t(C) \) is \( g_{i-1}/(2^{c_3+d}, g_{i-1}) \) and we obtain that \( E \cdot \text{ann}_{\mathbb{Z}_p[G]}(C_\chi) \subset g_{i-1} \mathbb{Z}_p[G]_\chi \).

ii) It is immediate from Lemma 3.13 that \( \mathbb{Z}_p[G]_\chi/\mathbb{Z}_p[G]_\chi \) is finite.

iii) \( M > |A_{\chi}| \cdot |N| = |A_{\chi}| \cdot |\eta(I_{a_{i-1}/(\text{TM}_{a_{i-1}})})| \).

Thus, we obtain a homomorphism

\[
\psi_i : W_\chi \rightarrow \mathbb{Z}/M \mathbb{Z}[G]
\]

with \( g_{i-1} \psi_i(\kappa(a_{i-1}))_\chi = (2^{c_3+d}(\gamma^\prime_{n_0} - 1)^{t_i} v_{\Omega_{i-1}}(\kappa(a_{i-1})))_\chi \). Let \( \Pi_\chi \) be the projection \( W \rightarrow W_\chi \) and define \( \psi = \varepsilon_\chi \circ \psi_i \circ \Pi_\chi \). Let \( M \) be as in the previous paragraph and \( C \) a preimage of \( C_i \). Then Theorem 3.8 gives us a prime ideal \( \Omega_i \) satisfying i) and ii) (recall that \( 2^n \mid M \)). Further, \( \varphi_q(\kappa(a_{i-1})) = 2^d u \psi(\kappa(a_{i-1})) \Omega_i \). Then we obtain

\[
v_{\Omega_i}(\kappa(q_1 \ldots q_i)) \Omega_i = [\kappa(q_1 \ldots q_i)]_{q_i} = \varphi_{q_i}(\kappa(q_1 \ldots q_i))
= 2^d u \psi(\kappa(a_{i-1})) \Omega_i.
\]

Projecting to the \( \chi \)-component and using the definition of \( \psi \) we obtain

\[
(g_{i-1} u \varphi_q(\kappa(q_1 q_2 \ldots q_i)))_\chi = u_i[H](\gamma^\prime_{n_0} - 1)^{c_i^{i-1} 2^{2d+c_2}(v_{\Omega_{i-1}}(\kappa(q_1 \ldots q_i)))_\chi}
\]

which finishes the proof. \( \Box \)

To derive a relation between \( h_\chi \) and \( \prod_{i=1}^s g_i \) we need the following result which is proved in [Cr-M], Theorem 1] and [O-V] Theorem 1.1. In the case of \( \mathbb{L} \) being the Hilbert class field and \( K = \mathbb{Q}(\sqrt{-q}) \) with \( q \) a prime congruent to 7 modulo 8 this is [C-K-L] Theorem 1.1.
Theorem 3.21. Let $\mathbb{M}/\mathbb{K}$ be an arbitrary abelian extension and $\Omega/\mathbb{K}_\infty \mathbb{M}$ be the maximal $p$-abelian $p$-ramified extension of $\mathbb{M}\mathbb{K}_\infty$ then $\text{Gal}(\Omega/\mathbb{K}_\infty \mathbb{M})$ is finitely generated as $\mathbb{Z}_p$-module.

Corollary 3.22. Let $\mathbb{M}$ be as above and consider $\mathbb{H}/\mathbb{K}_\infty \mathbb{M}$ (the maximal $p$-abelian unramified extension of $\mathbb{K}_\infty \mathbb{M}$). Then $A_\infty = \text{Gal}(\mathbb{H}/\mathbb{K}_\infty \mathbb{M})$ is finitely generated as a $\mathbb{Z}_p$-module.

Proof. $\text{Gal}(\mathbb{H}/\mathbb{K}_\infty \mathbb{M})$ is a quotient of $\text{Gal}(\Omega/\mathbb{K}_\infty \mathbb{M})$ and therefore finitely generated. □

Theorem 3.23. $\text{Char}(A_{\infty,\chi}) | \text{Char}((\overline{E}/\mathcal{C})_\chi)$.

Proof. The main argument of this proof is analogous to [Wash, page 371]. From (7) and (8) we obtain that $\prod_{i=1}^{k} g_{i} \nu_{q_{k+1}}(\kappa(q_{1} \ldots q_{k+1})) = \eta h'_{\chi} \mod 2^{n}$, where $\eta = \tilde{u} | H|^{k+1} 2^{k(2d+c_{0})+d+c_{2}(n_{0} - 1)} \gamma + \sum_{j=1}^{c} c_{j}^{2}$ for some unit $\tilde{u}$. It follows that $\prod_{i=1}^{k} g_{i}$ divides $\eta h'_{\chi}$ in $\Lambda_{\chi,m+n-m'/2^{n}}$. For every $n$ we can find an element $z_{n}$ such that $\prod_{i=1}^{k} g_{i} z_{n} = \eta h'_{\chi}$ in $\Lambda_{\chi,n}/2^{n} \Lambda_{\chi,m+n-m'}$. The $z_{n}$’s have a convergent subsequence and we obtain that $\prod_{i=1}^{k} g_{i} | \eta h'_{\chi}$ in $\Lambda_{\chi}$. By Lemma 3.12 and Corollary 3.22 $\text{Char}(A_{\infty,\chi})$ is coprime to $\eta$ and the claim follows. □

Corollary 3.24. $\text{Char}(A_{\infty}) | \text{Char}(\overline{E}/\mathcal{C})$

Proof. As Theorem 3.23 holds for all characters and $\text{Char}(A_{\infty})$ is coprime to 2 this is immediate. □

4. Characteristic ideals and the main conjecture

Consider the exact sequence

$$0 \to \overline{E}/\mathcal{C} \to U_{\infty}/\mathcal{C} \to X \to A_{\infty} \to 0,$$

where $X = \text{Gal}(\Omega/\mathbb{L}_\infty)$. Then

$$\text{Char}(A_{\infty}) \text{Char}(U_{\infty}/\mathcal{C}) = \text{Char}(X) \text{Char}(\overline{E}/\mathcal{C}).$$

From Corollary 3.24 we deduce

$$\text{Char}(X) | \text{Char}(U_{\infty}/\mathcal{C}).$$

In the following we will establish a relation between $p$-adic $L$-functions and elliptic units to show that $\text{Char}(X)$ is in fact equal to $\text{Char}(U_{\infty}/\mathcal{C})$.

Let $u \in U_{\infty}$ and let $g_{u}(w)$ be the Coleman power series of $u$ (see [dS I Theorem 2.2]). Let $\tilde{g}_{u}(W) = \log g_{u}(W) - \frac{1}{p} \sum_{w \in \mathbb{F}_{p}} \log g_{u}(W + w)$. There exists a measure $\nu_{u}$ on $\mathbb{Z}_{p}^{\times}$ having $\tilde{g}_{u} \circ \beta^{v}$ as characteristic series [dS I 3.4]. Recall that $D_{p} = I_{p}(\zeta_{m})$ and let $\Lambda(D_{p}, \Gamma' \times H)$ be the algebra of $D_{p}$-valued measures on $\Gamma' \times H$. Define

$$\iota(f): U_{\infty} \to \Lambda(D_{p}, \Gamma' \times H), \quad u \mapsto \sum_{\sigma \in \text{Gal}(\mathbb{F}/\mathbb{K})} \nu_{u} \circ \sigma.$$

Note that this construction of measures coincides with the one from section 2 for elliptic units.

Lemma 4.1. $\iota(f)$ is a pseudoisomorphism.
Proposition 4.2. Let \( \chi \) be a character of \( H \) of conductor \( g \) or \( gp \). The module \( \chi \circ \iota(f)(C(f)) \) is pseudoisomorphic to \( \chi(\nu(g))\Lambda(D_p, \Gamma' \times H) \), if \( \chi \) is non-trivial. If \( \chi \) is trivial \( \chi \circ \iota(f)(C(f)) \) is pseudoisomorphic to \( (\gamma' - 1)\chi(\nu(1))\Lambda(D_p, \Gamma' \times H) \).

Proof. Analogous to \([1S, III Lemma 1.10]\). As \( \chi \) has conductor \( g \) or \( gp \) it follows that \( \chi \circ \iota(f)(C(f)) = \chi \circ \pi_{g,a} \circ \iota(f)(C(f)) = \chi \circ \iota(g)N_{g,a}(C(f)) \). Assume first that \( g \neq 1 \). It is immediate that \( \sum_{\sigma \subseteq \text{Gal}(K(\mathbb{p}^\infty))} \chi(\sigma) = 0 \). Hence,

\[
(11) \quad \chi \circ \iota(g)(C(g)) = \chi \circ \iota(g)(\overline{C(g)}).
\]

If \( \omega_g = 1 \) we can construct the measure \( \nu(g) \) as in section 2 and obtain that \( \iota(g)(C(g)) = \iota(g)(\overline{C(g)}) \) is the ideal generated by \( \mathcal{J} \nu(g) \) where \( \mathcal{J} \) is the ideal generated by all the \( \nu_\alpha \). If \( \omega_g \neq 1 \) there exists an integer \( k \) such that \( \omega_g^k = 1 \) and then we can define the measure \( \nu(g^k) \). But by (11) we have \( \nu(g) = \nu(g^k) \) and \( N_{g^k,g} \) is surjective on the elliptic units. So in both cases the image under \( \iota(g) \) is precisely \( \mathcal{J} \nu(g) \).

If the norm \( N_{g,a} : \overline{C(f)} \to \overline{C(g)} \) is not surjective it follows that the cokernel of the module \( \chi \circ \iota(g) \circ N_{g,a}(C(f)) \) in \( \chi \circ \iota(g)(\overline{C(g)}) \) is annihilated by \( \text{Gal}(\mathbb{p}^\infty : \mathbb{K}(\mathbb{p}^\infty)) \) and the product \( \prod_{i \mid f,g} (1 - \chi(\sigma_i)\sigma_i^{-1}) \). These elements are certainly coprime and we see that \( \chi \circ \iota(f)(\overline{C(f)} \sim \chi \circ \iota(g)(\overline{C(g)}) \) due to (11), where \( A \sim B \) means that \( A \) and \( B \) are pseudoisomorphic. But the \( \chi(\nu_\alpha) \) are coprime due to Theorem 2.4 and the claim follows for \( g \neq 1 \).

Assume now that \( g = 1 \). Let \( \tau \in \text{Gal}(\mathbb{K}(\mathbb{p}^\infty) / \mathbb{K}) \) then the elements \( \xi_{\alpha,\sigma}(P_n^\alpha)^{-1} \) are norms of elliptic units in \( \mathbb{K}(\mathbb{p}^\infty) \), where \( \mathbb{p} \) is a prime having Artin symbol \( \tau^{-1} \) in \( \text{Gal}(\mathbb{p}^\infty / \mathbb{K}) \). It follows that the element \( \xi_{\alpha,\sigma}(P_n^\alpha)^{-1} \) corresponds to the measure \( \nu_\alpha(\tau - 1)\nu(1) \) under \( \iota(1) \). The group \( \overline{C(1)} \) is generated by products \( \prod_{i=1}^g \xi_{\alpha,\sigma}(P_n^\alpha)^{m_i} \) with \( \sum m_i(\tau - 1) = 0 \). Let \( \nu_\alpha \) be the measure corresponding to such a product. Then we obtain \( \tau - 1) = \sum m_i(\tau - 1)\nu(1) \). As \( \tau - 1) \nu(1) \) is not contained in the augmentation of \( \Lambda(D_p, \text{Gal}(\mathbb{K}(\mathbb{p}^\infty) / \mathbb{K}) \) we obtain that the ideal generated by \( \sum m_i(\tau - 1) \) is the augmentation ideal and that \( \iota((1))(\overline{C(1)}) = A\nu((1)) \), where \( A \) denotes the augmentation of \( \Lambda(D_p, \text{Gal}(\mathbb{K}(\mathbb{p}^\infty) / \mathbb{K}) \). Analogously to the case \( g \neq 1 \) we can conclude that \( \chi(\nu((1))(\overline{C(1)}) \) has finite index in \( \chi(\iota((1))((1)) \). Hence, it suffices to consider the image \( \chi(\nu((1))(\overline{C(1)}) \). If \( \chi \) is a non-trivial character, then \( \chi(A) \) contains \( \chi(\tau) - 1 \) as well as \( \gamma' - 1 \). Thus \( \chi(\nu((1))(\overline{C(1)}) \sim \chi(\nu(1)) \). If \( \chi \) is the trivial character then \( \chi(\nu((1))(\overline{C(1)}) \sim (\gamma' - 1) \chi(\nu(1)) \).

Corollary 4.3. Let \( F(w, \chi) \) be the Iwasawa function associated to \( L(\chi) \) defined in Definition 2.4. Then \( \text{Char}((U_\infty / C)\chi) = F(w, \chi) \).

Proof. Let \( g \) be such that the conductor of \( \chi \) is \( gp \) or \( g \). By Lemma 11 we see that \( \text{Char}(U_\infty / C) \) equals \( \text{Char}(\Lambda(D_p, \text{Gal}(\mathbb{K}(\mathbb{p}^\infty) / \mathbb{K})) / \iota(f)(C(f)) \). But the latter equals \( \chi(\nu(g)) \) if \( \chi \) is non-trivial and \( (1 - \gamma' \chi(\nu(1)) \) if \( \chi \) is trivial. But these are precisely
the measures used to define $L_p(s, \chi)$. As $\int_G \kappa^* \chi d(1 - \gamma)^e \nu(g) = \int_G \kappa^* d(1 - \gamma)^e \nu(g)$, where $e = 1$ if $\chi$ is trivial and $e = 0$ in all other cases, the claim follows. \hfill \square

4.1. Matching the invariants. In the following we will show how the $\lambda$- and $\mu$-invariants of $F(w, \chi)$ match with the ones of $X$. This section follows closely Section 4 of [Cr-M]. Recall that $L_n = \mathbb{K}(p^{n+2})$. To start with we need the following result from [Cr-M]. 

Let $t$ be such that $\mathbb{K}_t = F \cap \mathbb{K}_\infty$.

**Corollary 4.4.** If $G \in \mathbb{Z}_p[[\Gamma']]$ is a characteristic power series for $Gal(\mathbb{M}(\mathbb{L}_\infty)/\mathbb{L}_\infty)$, then for all sufficiently large $n$ one has

$$\mu(G) 2^{t+n-1} + \lambda(G) = 1 + \text{ord}_2 \left[ \frac{h(\mathbb{L}_n) R_p(\mathbb{L}_n)}{\omega(\mathbb{L}_n) \sqrt{\Delta_p(\mathbb{L}_n/\mathbb{K})}} \frac{h(\mathbb{L}_{n-1}) R_p(\mathbb{L}_{n-1})}{\omega(\mathbb{L}_{n-1}) \sqrt{\Delta_p(\mathbb{L}_{n-1}/\mathbb{K})}} \right].$$

Note that $D_p[[\Gamma']] \cong D_p[[w]]$. Consider any character $\rho$ of $\Gamma'$ of finite order. We say level $(\rho) = m$ if $\rho\left((\Gamma')^{2m}\right) = 1$, but $\rho\left((\Gamma')^{2m-1}\right) \neq 1$.

To determine the invariants of the Iwasawa function $F(w, \chi)$ we need the following two results ([dS Chapter III, Lemma 2.9] and [dS Chapter III, Proposition 2.10]).

**Lemma 4.5.** For any power series $F \in D_p[[w]]$ and all sufficiently large $n$, one has

$$\mu(F) 2^{n+t-1} + \lambda(F) = \text{ord}_2 \left\{ \prod_{(\rho) = t+n} \rho(F) \right\},$$

where $\rho(F)$ means that the action of $\rho$ is extended to $D_p[[\Gamma']]$ by linearity and $\text{ord}_p$ is the valuation on $\mathbb{C}_p$ normalized by taking $\text{ord}_2(2) = 1$.

**Proposition 4.6.** For any ramified character $\varepsilon$ of $Gal(\mathbb{F}_\infty/\mathbb{K})$, we let $g$ be the conductor of $\varepsilon$ and $g$ the least positive integer in $g \cap \mathbb{Z}$. We define $G(\varepsilon)$ as in Theorem 2.7 and we define $S_p(\varepsilon)$ by

$$S_2(\varepsilon) = -\frac{1}{12g\omega_\varepsilon} \sum_{\sigma \in Gal(\mathbb{K}(\varepsilon)/\mathbb{K})} \varepsilon^{-1}(\sigma) \log \varphi_{g}(\sigma).$$

Let $A_n$ be the collection of all $\varepsilon$ for which $n$ is the exact power of $p$ dividing their Artin conductor. Then for all sufficiently large $n$ one has

$$\text{ord}_2 \left( \prod_{\varepsilon \in A_{n+1}} G(\varepsilon)S_p(\varepsilon) \right) = \text{ord}_2 \left[ \frac{h(\mathbb{L}_n) R_p(\mathbb{L}_n)}{\omega(\mathbb{L}_n) \sqrt{\Delta_p(\mathbb{L}_n/\mathbb{K})}} \frac{h(\mathbb{L}_{n-1}) R_p(\mathbb{L}_{n-1})}{\omega(\mathbb{L}_{n-1}) \sqrt{\Delta_p(\mathbb{L}_{n-1}/\mathbb{K})}} \right].$$

Using [Cr-M Theorem 5], for a character $\rho$ of $\Gamma'$ of sufficiently large finite order, one has

$$\rho(F(w, \chi^{-1})) \sim \begin{cases} G(\rho \chi) S_2(\rho \chi) & \text{if } \chi \neq 1; \\ (\rho(\gamma_0) - 1) G(\rho \chi) S_2(\rho \chi) & \text{if } \chi = 1, \end{cases}$$

where $u \sim v$ denotes the fact that $u/v$ is a $p$-adic unit. Let

$$F = \prod_{\chi \in \mathbb{H}} F(w, \chi).$$
It follows that for all sufficiently large $n$ one has
\begin{equation}
\prod_{\text{level}(\rho) = t + n} \rho(F) \sim 2 \prod_{\varepsilon \neq 0 \atop \text{level}(\rho) = t + n} G(\varepsilon)S_p(\varepsilon),
\end{equation}

since in the product on the right hand side we range over all $\chi$ (including $\chi = 1$) and
\[
\prod_{\text{level}(\rho) = t + n} (\rho(\gamma_0) - 1) = 2.
\]

As $\mathbb{L}_0/F$ is ramified at $p$ of degree 2 and $p$ is unramified in $F/K$ we see that a character in $A_{n+1}$ is of level $t + n$. Combining Corollary 4.4, Lemma 4.5 and (12) we obtain that
\[
\mu(F) 2^{n+t-1} + \lambda(F) = \mu(G)2^{t+n-1} + \lambda(G) \quad \forall n \gg 0
\]

This implies together with Theorem 3.21

**Theorem 4.7.** $\mu(G) = \mu(F) = 0$ and $\lambda(G) = \lambda(F)$.

4.2. **Proving the main conjecture.** In this section we use all the results proved before to prove the main conjecture.

**Lemma 4.8.** $\text{Char}(X) = \text{Char}(U_{\infty}/\overline{C})$ and $\text{Char}(A_{\infty, \chi}) = (\overline{E}/\overline{C})_{\chi}$.

**Proof.** The first claim follows directly from (10) and Theorem 4.7. From (9) we also obtain that $\text{Char}(A_{\infty}) = \text{Char}(\overline{E}/\overline{C})$. Further Theorem 3.23 establishes that $\text{Char}(A_{\infty, \chi})$ divides $\text{Char}(\overline{E}/\overline{C})_{\chi}$. Both together imply the second claim. \qed

This has also the following consequence:

**Theorem 4.9.** $\text{Char}(X_{\chi}) = \text{Char}((U_{\infty}/\overline{C})_{\chi})$ for any $\chi$.

**Proof.** For any $\Lambda$-module we denote by $M^\chi$ the largest submodule in $M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(\chi)$ on which $H$ acts via $\chi$. By [15] page 5 there exists a homomorphism between $M^\chi$ and $M^X$ such that the kernel and the cokernel are annihilated by $|H|$. As none of the characteristic ideals involved is divisible by 2 we can consider the characteristic ideals of $M^X$ instead of $M^\chi$ for any $M$ in $\{A_{\infty}, U_{\infty}/\overline{C}, X, \overline{E}/\overline{C}\}$. The sequence
\[
0 \to (\overline{E}/\overline{C})^X \to (U_{\infty}/\overline{C})^X \to X^X
\]
is exact. Let $e_\chi$ be the idempotent induced by the character $\chi$. Then $e_\chi |H|$ is an element in $\mathbb{Z}_p(\chi)[H]$. In particular, $e_\chi |H|M \subset M^X$. It follows that the cokernel of the natural homomorphism $\phi_\chi : X^X \to A_{\infty, \chi}$ is annihilated by $|H|$. As $A_{\infty}$ has bounded rank it follows that the Coker $(\phi_\chi)$ is finite. The module $\ker(\phi_\chi)$ equals $X^X \cap \im(U_{\infty}/\overline{C})$. Again the exponent of $X^X \cap \im((U_{\infty}/\overline{C}))/\im((U_{\infty}/\overline{C})^X)$ is bounded by $|H|$. Hence, $\text{Char}(A_{\infty, \chi})\text{Char}(\im(U_{\infty}/\overline{C})^X) = \text{Char}(X^X)$. Using the exactness of the sequence above we obtain
\[
\text{Char}(A_{\infty, \chi})\text{Char}((U_{\infty}/\overline{C})^X) = \text{Char}(\overline{E}/\overline{C})^X\text{Char}(X^X).
\]

The claim follows now from Lemma 4.8. \qed

The second claim of Lemma 4.8 and Theorem 4.9 prove Theorem 1.1 for $L_{\infty}$.

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REFERENCES

[Bl] Bley, W. (2006) Equivariant Tamagawa Conjecture for Abelian Extensions of a Quadratic Imaginary Field: Documenta Mathematica 11, pp 73-118

[C-K-L] Choi, J., Kezuka, Y., Li, Y. (2018). Analogues of Iwasawas $\mu = 0$ conjecture and weak Leopoldt theorem for certain non-cyclotomic $\mathbb{Z}_2$-extensions: Asian Journal of Mathematics, Vol. 23, No. 3, pp 383-400

[Co] Coates, J. (1991). Elliptic curves with complex multiplication and Iwasawa theory: Bull. London Math. Soc. 23, pp. 321-350.

[Co-Go] Coates, J., Goldstein, C. (1983). Some remarks on the main conjecture for elliptic curves with complex multiplication: American J. of Mathematics 105, pp. 337-366.

[Col79] Coleman, R. (1979). Division values in local fields: Invent. Math., 53, pp. 91-116.

[Cr-M] Crişan, V., Müller, K. The Vanishing of the $\mu$-Invariant for Split Prime $\mathbb{Z}_p$-extensions over Imaginary Quadratic Fields To appear in The Asian Journal of Mathematics.

[DS] de Shalit, E. (1987). The Iwasawa theory of elliptic curves with complex multiplication: Progress. Math. Vol.3.

[Gr] Greither, C. (1992). Class groups of abelian fields, and the main conjecture Annales de L’Institut Fourier 42, no 3, pp 449-499.

[Ja] Janusz, G. J. (1973), Algebraic Number Fields Pure and Applied Mathematics Volume 55, Academic Press.

[Iw] Iwasawa, Kenkichi (1973) On $\mathbb{Z}_l$ Extensions of Algebraic Number Fields Annals of Mathematics Second Series, Vol. 98, no.2, pp. 246-326.

[Ke] Kezuka, Y. (2016) On the Main Conjecture of Iwasawa Theory for certain elliptic curves with complex multiplication, PhD-Thesis, University of Cambridge, Cambridge.

[Ke-2] Kezuka, Y. (2019) On the Main Conjecture of Iwasawa Theory for Certain Non-Cyclotomic $\mathbb{Z}_p$-Extension. J. London Math. Soc., Vol. 100, pp. 107-136.

[Lu] Lubin, J. (1964). One Parameter Formal Lie Groups over $p$-adic Integer Rings: Annals of Mathematics, Second Series, Vol. 80, No. 3, pp. 464-484.

[O-V] Oukhaba, H., Viguié, S. (2016). On the $\mu$-invariant of Katz $p$-adic $L$-functions attached to imaginary quadratic fields: Forum Math. 28, no. 3, pp. 507-525.

[Ru-1] Rubin, K. (1988) On the main conjecture of Iwasawa theory for imaginary quadratic fields: Invent. Math. 93, pp 701-7013

[Ru-2] Rubin, K. (1991). The "main conjectures" of Iwasawa Theory for imaginary quadratic fields: Invent. Math., 103, pp. 25-68.

[Ru-3] Rubin, K. (1990). The Main Conjecture, Appendix to the second edition of S. Lang: Cyclotomic Fields I and II Graduate Texts in Mathematics 121, Springer.

[Si] Silverman, J.H. (1986). The Arithmetic of Elliptic Curves, Graduate Texts in Mathematics 106: Springer.

[Si] Sinnott, W. (1984). On the $\mu$-invariant of the $\Gamma$-transform of a rational function: Invent. Math., 75, pp. 273-282.

[Ts] Tsuji, T. (1999) Semi-Local Units modulo Cyclotomic Units J. Nubmer Theory, Volume 78, Issue 1, pp 1-26.

[Wash] Washington, L.C. (1997) Introduction to cyclotomic fields, 2nd Edition, Graduate Texts in Mathematics 83: Springer.

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