ON THE ASYMPTOTIC EXPANSION OF MAPS WITH DISCONNECTED JULIA SET

JUAN RIVERA-LETIELIER

Abstract. We study the asymptotic expansion of smooth one-dimensional maps. We give an example of an interval map for which the optimal shrinking of components exponential rate is not attained for any neighborhood of a certain fixed point in the boundary of a periodic Fatou component. As a complement, we prove a general result asserting that when this happens the components do shrink exponentially, although the rate is not the optimal one. Finally, we give an example of a polynomial with real coefficients, such that all its critical points in the complex plane are real, and such that its asymptotic expansion as a complex map is strictly smaller than its asymptotic expansion as a real map.

1. Introduction

This note is concerned with non-uniform hyperbolicity notions for smooth one-dimensional maps. The Fatou and Julia sets of such a map are important in what follows. See [CG93, Mil06] for expositions in the complex case. For a non-degenerate smooth interval map we consider the theory of Fatou and Julia sets developed by Martens, de Melo, and van Strien in [MdMvS92], see also [dMvS93].

For a smooth one-dimensional map, several ways to measure the asymptotic expansion coincide, see [NS98, PRLS03, RL12]. This note is mainly focused on the asymptotic expansion of interval maps with disconnected Julia set. We show that to obtain the optimal shrinking of components exponential rate, in some cases it is necessary to exclude intervals that intersect a periodic Fatou component (Proposition A). This shows that a seemingly technical hypothesis of [RL12, Main Theorem] is in fact necessary. We also show a qualitative result that holds without this hypothesis (Theorem B).

We also study the relation between the asymptotic expansions of the real and the complex map defined by a polynomial with real coefficients. We give an example of a polynomial with real coefficients, such that all its critical points in $\mathbb{C}$ are real, and such that its asymptotic expansion as a map of $\mathbb{C}$ is different from its asymptotic expansion as a map of $\mathbb{R}$ (Proposition C).

We proceed to state our results more precisely.

1.1. Exponential rate of shrinking of components. Let $I$ be a compact interval of $\mathbb{R}$. A non-injective smooth map $f : I \to I$ is non-degenerate, if the set of points at which the derivative of $f$ vanishes is finite and if at each of these points a higher order derivative of $f$ is non-zero. In what follows, a connected component of the Fatou set of $f$ is called Fatou component.
For a non-degenerate smooth map $f : I \to I$, the number
\[ \chi_{\text{inf}}(f) := \left\{ \int |Df| d\mu : \mu \text{ probability measure on } J(f) \text{ invariant by } f \right\} \]
is a measure of the asymptotic expansion of $f$. The condition $\chi_{\text{inf}}(f) > 0$ can be seen as a strong form of non-uniform hyperbolicity in the sense of Pesin and has strong implications for the dynamics of $f$: It implies the existence of an exponentially mixing absolutely continuous invariant measure, see [RL12, Corollary A], and also [NS98, Theorem A] for maps with one critical point.

When $f$ is topologically exact on $J(f)$, several numbers that measure the asymptotic expansion of $f$ coincide with $\chi_{\text{inf}}(f)$. For example, for every sufficiently small interval $J$ contained in $I$ that is not a neighborhood of a periodic point in the boundary of a Fatou component, we have
\begin{equation}
\lim_{n \to +\infty} \frac{1}{n} \ln \max \{|W| : W \text{ connected component of } f^{-n}(J)\} = -\chi_{\text{inf}}(f),
\end{equation}
see [RL12, Main Theorem]. We note that the hypothesis that $J$ is not a neighborhood of a periodic point in the boundary of a Fatou component is only required in [RL12, Main Theorem] in the case where $\chi_{\text{inf}}(f) > 0$ and where $J(f)$ is not an interval. Our first result is that (1.1) can fail if this hypothesis is not satisfied.

**Proposition A.** There is a non-degenerate smooth map $f : I \to I$ that is topologically exact on its Julia set, and that satisfies $\chi_{\text{inf}}(f) > 0$ and the following property:

There is a fixed point $p$ of $f$ in the boundary of a Fatou component of $f$, such that for every $\delta > 0$ we have
\[ \limsup_{n \to +\infty} \frac{1}{n} \ln \max \{|W| : W \text{ connected component of } f^{-n}(B(p, \delta))\} > -\chi_{\text{inf}}(f). \]

We also show that a weak version of (1.1) does hold for every sufficiently small interval $J$ intersecting $J(f)$.

**Theorem B.** Let $f$ be a non-degenerate smooth interval map that is topologically exact on $J(f)$ and such that $\chi_{\text{inf}}(f) > 0$. Then there are $\delta > 0$ and $\lambda > 1$ such that for every point $x \in J(f)$, for every integer $n \geq 1$, and every pull-back $W$ of $B(x, \delta)$ by $f^n$, we have $|W| \leq \lambda^{-n}$.

By [RL12, Main Theorem] the constant $\lambda$ in Theorem B is less than or equal to $\exp(\chi_{\text{inf}}(f))$. On the other hand, Proposition A shows that in general $\lambda$ cannot be taken arbitrarily close to $\exp(\chi_{\text{inf}}(f))$ if $J$ is a neighborhood of a periodic point in the boundary of a Fatou component.

1.2. Real versus complex asymptotic expansion. For a complex rational map $f$ of degree at least 2, the limit (1.1), with $J$ replaced by any sufficiently small ball centered at a point in $J(f)$, is shown in [PRLS03 Main Theorem].

Note that a polynomial of degree at least 2 with real coefficients $f$ defines a map of $\mathbb{R}$ and a map of $\mathbb{C}$. In the following proposition we show that, even if all the critical points of $f$ in $\mathbb{C}$ are real, the asymptotic expansion of $f$ as a map of $\mathbb{C}$ might be strictly smaller than its asymptotic expansion as a map of $\mathbb{R}$.
Proposition C. There is a polynomial $P$ of degree 4 with real coefficients such that all its critical points in $\mathbb{C}$ are real and such that, if we denote by $\mathcal{M}^\mathbb{R}$ (resp. $\mathcal{M}^\mathbb{C}$) the space of all Borel probability measures on $\mathbb{R}$ (resp. $\mathbb{C}$) that are invariant by $P$, then

$$0 < \inf \left\{ \int |DP|d\mu : \mu \in \mathcal{M}^\mathbb{C} \right\} < \inf \left\{ \int |DP|d\mu : \mu \in \mathcal{M}^\mathbb{R} \right\}.$$ 

1.3. Organization. The proofs of Propositions A and C occupy §§2–6. In §2 we introduce a 2 parameter family of polynomials of degree 4 and reduce the proofs of Propositions A and C to the existence of a member of this family with properties analogous to those of the map $f$ in Proposition A (Main Lemma). In §3 we consider some general mapping properties of the maps introduced in §2, and in §4 we select parameters for which the corresponding maps have some special combinatorial properties. In §5 we choose the map used to prove the Main Lemma and make the main estimates used in the proof. The proof of the Main Lemma is given in §6.

The proof of Theorem B is in §7 and it is independent of the previous sections. Some general properties of non-degenerate smooth maps that are used in the proof are gathered in Appendix A.

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2. A 2 parameter family of polynomials

After introducing a 2 parameter family of polynomials, we state the Main Lemma and deduce Propositions A and C from it.

For real parameters $a$ and $\tau$, consider the real polynomial

$$f_{a,\tau}(x) = 1 - \tau + ax^2 - (a + 2 - \tau)x^4.$$ 

Note that $f_{a,\tau}(1) = f_{a,\tau}(-1) = -1$ and that for $a \neq 0$ the point $x = 0$ is a critical point of $f_{a,\tau}$.

The maximal invariant set $K_{a,\tau}$ of $f_{a,\tau}$ in $[-1,1]$ is important in what follows. When $a \geq 10$ and $\tau \leq 2$, all the critical points of $f_{a,\tau}$ in $\mathbb{C}$ are real, every point in $\mathbb{R} \setminus [-1,1]$ converges to $-\infty$ under forward iteration by $f_{a,\tau}$, and $x = 0$ is the only critical point of $f_{a,\tau}$ in $K_{a,\tau}$, see [3] So in this case $K_{a,\tau}$ is contained in $[-1,1]$.

The purpose of this section is to deduce Propositions A and C from the following lemma. For $a \geq 10$ and $\tau \leq 2$, put

$$\chi_{\text{per}}(f_{a,\tau}) := \inf \left\{ \frac{1}{n} \ln |Df_{a,\tau}^n(p)| : n \geq 1, p \text{ periodic point of } f_{a,\tau} \text{ of period } n \right\}.$$ 

Main Lemma. There are parameters $a \geq 10$ and $\tau \leq 2$ such that $f_{a,\tau}$ is topologically exact on $K_{a,\tau}$, and such that the following properties hold: $\chi_{\text{per}}(f_{a,\tau}) > 0$, the critical point $x = 0$ is in $K_{a,\tau}$, $f_{a,\tau}$ satisfies the Collet-Eckmann condition:

$$(2.1) \liminf_{n \to +\infty} \frac{1}{n} \ln |Df_{a,\tau}^n(f_{a,\tau}(0))| > 0.$$
and for every $\delta > 0$ we have:

$$\limsup_{m \to +\infty} \frac{1}{m} \ln \{|W| : W \text{ connected component of } f_{a,\tau}^{-m}((-1 - \delta, -1))\} > -\chi_{\text{per}}(f_{a,\tau}).$$

The proof of the Main Lemma occupies §3.

For the parameters $a$ and $\tau$ given by the Main Lemma there is no compact interval that is invariant by $f_{a,\tau}$. So we need to modify $f_{a,\tau}$ to deduce Proposition $\text{A}$ from the Main Lemma.

**Proof of Propositions $\text{A}$ and $\text{C}$ assuming the Main Lemma.** Let $a$ and $\tau$ be given by the Main Lemma. Note that we have $|Df_{a,\tau}| \geq 2a > 1$ on $\mathbb{R} \setminus [-1, 1]$, so for each $x < -1$ we have

$$f_{a,\tau}(-x) = f_{a,\tau}(x) < x.$$

Let $A > 3$ be sufficiently large so that $f_{a,\tau}([-2, 2])$ is contained in $(-A, A)$, put $I := [-A, A]$, and let $f_0 : I \to I$ be defined by $f_0(x) := |x|/2 - 3A/2$. Moreover, let $f : I \to I$ be a smooth function that coincides with $f_{a,\tau}$ on $[-2, 2]$, that coincides with $f_0$ on $[-A, -3] \cup [3, A]$, and such that for each $x$ in $(-A, -1)$ we have

$$Df(x) > 0 \text{ and } f(-x) = f(x) < x.$$

Note in particular that if $f$ is a non-degenerate smooth map, that $x = -A$ is a hyperbolic attracting fixed point of $f$, and that every point of $I \setminus [-1, 1]$ converges to $x = -A$ under forward iteration. Since $f$ coincides with $f_{a,\tau}$ on $[-1, 1]$ and since $f_{a,\tau}$ is topologically exact on $K_{a,\tau}$, it follows that the Julia set of $f$ is equal to $K_{a,\tau}$. So by $\text{RL12}$ Main Theorem' we have $\chi_{\text{inf}}(f) = \chi_{\text{per}}(f_{a,\tau})$ and by the Main Lemma $f$ satisfies the conclusions of Proposition $\text{A}$ with $p = -1$.

To prove Proposition $\text{C}$ with $P = f_{a,\tau}$, we show that $f_{a,\tau}$ satisfies the Collet-Eckmann condition as a complex polynomial and that every invariant probability measure of $f_{a,\tau}$ on $\mathbb{C}$ is supported on the Julia set of $f_{a,\tau}$. This implies that $\chi^C_{\text{inf}}(f_{a,\tau}) > 0$ and that for every sufficiently small $\delta > 0$ the liminf in the Main Lemma is less than or equal to $-\chi^C_{\text{inf}}(f_{a,\tau})$, see $\text{PRLS03}$ Main Theorem'. So by the Main Lemma we have $\chi^C_{\text{inf}}(f_{a,\tau}) < \chi_{\text{per}}(f_{a,\tau})$ and then the inequality $\chi^C_{\text{inf}}(f_{a,\tau}) < \chi^C_{\text{inf}}(f_{a,\tau})$ follows from the fact that $\chi_{\text{per}}(f_{a,\tau})$ is equal to $\chi_{\text{inf}}(f) = \chi^C_{\text{inf}}(f_{a,\tau})$, see $\text{RL12}$ Main Theorem'. To prove that $f_{a,\tau}$ satisfies the Collet-Eckmann condition as a complex polynomial, note that the fact that all the critical points of $f_{a,\tau}$ in $\mathbb{C}$ are real and that $x = 0$ is the only critical point of $f_{a,\tau}$ in $K_{a,\tau}$, implies that each critical point of the complex polynomial $f_{a,\tau}^n$ different from $x = 0$ escapes to infinity in the Riemann sphere. So all the critical points of $f_{a,\tau}$ different from $x = 0$ are in the Fatou set of $f_{a,\tau}$. Together with (2.1), this implies that $f_{a,\tau}$ satisfies the Collet-Eckmann condition as a complex polynomial. On the other hand, it follows that the Fatou set of $f_{a,\tau}$ coincides with the attracting basin of $\infty$ of $f_{a,\tau}$, see $\text{GS98}$ Theorem 1]. So every invariant probability measure of $f_{a,\tau}$ on $\mathbb{C}$ is supported on the Julia set of $f_{a,\tau}$. This completes the proof of Proposition $\text{C}$. □

3. Mapping properties

The purpose of this section is to prove some mapping properties of the maps introduced in the previous section.
Note that for $a > 0$ and $\tau \leq 2$ we have $f_{a,\tau}(\mathbb{R} \setminus [-1,1]) = (-\infty, -1)$. We also have $|Df_{a,\tau}| \geq 2a$ on $(-\infty, -1]$; so, if in addition $a > 1/2$, then every point in $\mathbb{R} \setminus [-1,1]$ converges to $-\infty$ under forward iteration.

Note on the other hand that for every $a$ and $\tau$ the point $x = 0$ is a critical point of $f_{a,\tau}$ and for $\tau$ in $[0,2]$ the critical value $f_{a,\tau}(0) = 1 - \tau$ of $f_{a,\tau}$ is in $[-1,1]$. If in addition $a > 0$, then $f_{a,\tau}$ has 2 distinct critical points different from $x = 0$: a critical point $c_-$ in $(-1,0)$ and a critical point $c_+ \in (0,1)$. Since $f_{a,\tau}$ is a polynomial of degree 4, it follows that all of the critical points of $f_{a,\tau}$ are real and hence that $f_{a,\tau}$ has negative Schwarzian derivative on $\mathbb{R}$. By symmetry,

$$c_- = -c_+ \text{ and } v_{a,\tau} := f_{a,\tau}(c_+) = f_{a,\tau}(c_-).$$

On the other hand, for $a > 0$ and $\tau$ in $[0,2]$ we have

$$v_{a,\tau} = 1 - \tau + \frac{a^2}{4(a + 2 - \tau)} \geq \frac{a^2}{4(a + 2)} - 1. \tag{3.1}$$

So, for $a \geq 10$ we have $v_{a,\tau} > 1$ and therefore neither $c_-$ or $c_+$ is in $K_{a,\tau}$. Since $K_{a,\tau}$ contains $x = 1$ and $x = -1$, it follows that $K_{a,\tau}$ is disconnected. Note that $v_{a,\tau} > 1$ also implies that $f_{a,\tau}^{-1}([-1,1])$ consists of 3 connected components; denote by $I_{0,a,\tau}$ (resp. $V_{a,\tau}$, $I_{1,a,\tau}$) the connected component of this set containing $x = -1$ (resp. 0, 1). Note in particular that $K_{a,\tau}$ is contained in $I_{0,a,\tau} \cup V_{a,\tau} \cup I_{1,a,\tau}$.

**Lemma 3.1.** For every $\eta > 1$ there is $a_0 \geq 20$ such that for every $a \geq a_0$ and every $\tau$ in $[0,2]$ the following properties hold.

1. Suppose the forward orbit of $x = 0$ under $f_{a,\tau}$ is contained in $[-1,1]$. Then for every integer $m \geq 1$ and every interval $J$ such that $f_{a,\tau}^m$ maps $J$ diffeomorphically to $[-1,1]$, the map $f_{a,\tau}^m$ maps a neighborhood $\tilde{J}$ of $J$ diffeomorphically to $[-2,2]$ with distortion bounded by $\eta$.

2. The interval $I_{0,a,\tau}$ (resp. $V_{a,\tau}$, $I_{1,a,\tau}$) is contained in $[-1,1 + 2/a]$ (resp. $[-2\sqrt{\tau/a}, 2\sqrt{\tau/a}]$, $[1 - 2/a, 1]$).

3. If we put

$$\lambda_{a,\tau} := Df_{a,\tau}(-1) = 2(a + 4 - 2\tau),$$

then for every $x$ in $V_{a,\tau}$ such that $f_{a,\tau}(x)$ is in $I_{1,a,\tau}$, we have

$$\eta^{-1} \leq \frac{|x|}{(2\lambda_{a,\tau})|f_{a,\tau}^2(0) - f_{a,\tau}^2(x)|^{1/2}} \leq \eta$$

and

$$\eta^{-1} \leq \frac{|Df_{a,\tau}^2(x)|}{(2\lambda_{a,\tau})|f_{a,\tau}^2(0) - f_{a,\tau}^2(x)|^{1/2}} \leq \eta.$$

In the proof of this lemma we use the Koebe principle for maps with negative Schwarzian derivative, see for example [GMvS93, §IV, 1].

**Proof.** Recall that for $a \geq 10$ and $\tau$ in $[0,2]$ we have $v_{a,\tau} > 1$. This implies that $f_{a,\tau}(v_{a,\tau}) < -v_{a,\tau}$ and that the forward orbit of $v_{a,\tau}$ is disjoint from $(f_{a,\tau}(v_{a,\tau}), v_{a,\tau})$. So, if the orbit $x = 0$ is contained in $[-1,1]$ and if $J$ is an interval and $m \geq 1$ an integer such that $f_{a,\tau}^m$ maps $J$ diffeomorphically to $[-1,1]$, then $f_{a,\tau}^m$ maps a neighborhood of $J$ diffeomorphically to $(f_{a,\tau}(v_{a,\tau}), v_{a,\tau})$. In view of the Koebe principle, to prove part 1 it is enough to show that for every $A > 1$ there is $a_0 \geq 10$ such that for every $a \geq a_0$ and every $\tau$ in $[0,2]$ we have $v_{a,\tau} \geq A$ and $f_{a,\tau}(v_{a,\tau}) \leq -A$. 

To prove the first inequality, note that for $a \geq 8$ and $\tau$ in $[0, 2]$ we have by (3.1) that $v_{a, \tau} \geq a/5 - 1$. So for $a \geq \max\{10, 5(A+1)\}$ we have
\[ -f(v_{a, \tau}) > v_{a, \tau} \geq A. \]
This completes the proof of part 1.
To prove part 2, note that for $a \geq 10$ and $\tau$ in $[0, 2]$ there are 4 solutions to the equation $f_{a, \tau}(x) = 1$, counted with multiplicity. One the other hand, we have
\[ f_{a, \tau}\left(2\sqrt{\tau/a}\right) = 1 + 3\tau - 16\frac{(a+2-\tau)^2}{a^2} \geq 1 + \tau \left(3 - 32\frac{a+2}{a^2}\right). \]
Thus for $a \geq 20$ we have
\[ f_{a, \tau}\left(-2\sqrt{\tau/a}\right) = f_{a, \tau}\left(2\sqrt{\tau/a}\right) > 1. \]
Since $f_{a, \tau}(0) \leq 1$, it follows that $V_{a, \tau}$ is contained in $[-2\sqrt{\tau/a}, 2\sqrt{\tau/a}]$. On the other hand, for $a \geq 10$ and $\tau$ in $[0, 2]$ we have
\[ f_{a, \tau}\left(-1 + \frac{2}{a}\right) = f_{a, \tau}\left(1 - \frac{2}{a}\right) \geq f_{a, 2}\left(1 - \frac{2}{a}\right) = 4\frac{(a-2)^2(a-1)}{a^4} - 1 > 1. \]
This proves that $I_{0, a, \tau}$ is contained in $[-1, -1 + 2/a]$ and that $I_{1, a, \tau}$ is contained in $[1, 1 + 2/a]$.
To prove part 3, let $a_0 \geq 20$ be sufficiently large so part 2 holds. Then for every $a \geq a_0$ and every $x$ in $V_{a, \tau}$ we have
\[ \frac{|f_{a, \tau}(0) - f_{a, \tau}(x)|}{(\lambda_{a, \tau}/2)x^2} = \frac{a}{a+4-2\tau}\left(1 - \frac{a+2-\tau}{a}x^2\right) \in \left[\frac{a}{a+4}\left(1 - 8\frac{a+2}{a^2}\right), 1\right] \]
and
\[ \frac{Df_{a, \tau}(x)}{\lambda_{a, \tau}x} = \frac{a}{a+4-2\tau}\left(1 - 2\frac{a+2-\tau}{a}x^2\right) \in \left[\frac{a}{a+4}\left(1 - 16\frac{a+2}{a^2}\right), 1\right]. \]
This proves that there is a constant $a_1 \geq a_0$ such that if $a \geq a_1$ and $\tau$ is in $[0, 2]$, then for every $x$ in $V_{a, \tau}$ we have
\[ \eta^{-1/2} \leq \frac{|x|}{\sqrt{2/\lambda_{a, \tau}}|f_{a, \tau}(0) - f_{a, \tau}(x)|^{1/2}} \leq \eta^{1/2} \]
and
\[ \eta^{-1/2} \leq \frac{|Df_{a, \tau}(x)|}{\sqrt{2/\lambda_{a, \tau}}|f_{a, \tau}(0) - f_{a, \tau}(x)|^{1/2}} \leq \eta^{1/2}. \]
So, to prove part 3 it is enough to show that there is $a_2 \geq a_1$ such that for every $a \geq a_2$ and every $\tau$ in $[0, 2]$ we have for every $x'$ in $I_{1, a, \tau}$,
\[ (3.2) \quad \eta\lambda_{a, \tau} \leq |Df_{a, \tau}(x')| \leq \eta\lambda_{a, \tau}. \]
To prove this, just note that for $a \geq 20$, $\tau$ in $[0, 2]$, and $x$ in $I_{1, a, \tau}$, the number
\[ \frac{Df_{a, \tau}(x)}{-\lambda_{a, \tau}} = \frac{a}{a+4-2\tau}\left(2\frac{(a+2-\tau)}{a}x^2 - 1\right) \]
is in the interval
\[ \left[\frac{a}{a+4}\left(1 - \frac{2}{a}\right)\left(2\left(1 - \frac{2}{a}\right)^2 - 1\right), 1 + \frac{4}{a}\right]. \]
This implies (3.2) and completes the proof of the lemma. \qed
4. Combinatorial type

In this section we show there parameters $a$ and $\tau$ for which the corresponding map $f_{a,\tau}$ has some special combinatorial properties (Lemma 4.1). We also prove some general facts about the dynamics of these maps (Lemma 1.2).

Let $M := (M_n)_{n=0}^{+\infty}$ be a sequence of integers such that $M_0 = 2$ and such that for every $n \geq 0$ we have $M_{n+1} \geq 2M_n + 1$. Given $a \geq 20$ and $\tau$ in $[0, 2]$, we say that the combinatorics of $f_{a,\tau}$ is of type $M$ if there is an increasing sequence of points $(x_n)^{+\infty}_{n=0}$ in $(-1, 0)$, such that the following properties hold for every $n \geq 0$.

A. The interval $V_n := [x_n, -x_n]$ satisfies $f_{a,\tau}(V_n) \subset I_{1,a,\tau}$ and there is an interval $J_n$ containing $f_{a,\tau}^2(V_n)$, such that $f_{a,\tau}^{M_n-2}$ maps $J_n$ diffeomorphically to $[-1, 1]$, preserving the orientation. Moreover

$$f_{a,\tau}^{M_n}(x_n) = -1, f_{a,\tau}^{M_n}(x_{n+1}) = x_n,$$

and $f_{a,\tau}^{M_n}(0) \in V_n \cap (-1, 0)$.

B. For every $j$ in $\{0, \ldots, M_{n+1} - 2M_n - 1\}$ the point $f_{a,\tau}^{2M_n+j}(0)$ is in $I_{0,a,\tau}$.

Note that if the combinatorics of $f_{a,\tau}$ is of type $M$, then $V_0 = V_{a,\tau}$ and for every $n \geq 0$ the interval $V_{n+1}$ is the pull-back of $V_n$ by $f_{a,\tau}^{M_n}$ containing $x = 0$.

**Lemma 4.1.** Let $M := (M_n)_{n=0}^{+\infty}$ be a sequence of integers such that $M_0 = 2$ and such that for every $n \geq 0$ we have $M_{n+1} \geq 2M_n + 1$. Then for every $a \geq 20$ there is $\tau$ in $[0, 2]$ such that the combinatorics of $f_{a,\tau}$ is of type $M$.

**Proof.** We define recursively a nested sequence of closed intervals $(T_n)_{n=0}^{+\infty}$ and for every $n \geq 0$ and $\tau$ in $T_n$ a point $x_{n,\tau}$, such that the following properties hold for each $n \geq 0$:

- The point $x_{n,\tau}$ depends continuously with $\tau$ and $f_{a,\tau}^{M_n}(x_{n,\tau}) = -1$.
- The point $f_{a,\tau}^{M_n}(0)$ is equal to $-1$ if $\tau$ is the left end point of $T_n$ and it is equal to $1$ if $\tau$ is the right end point of $T_n$.
- The set $f_{a,\tau}^{2M_n}([-x_{n,\tau}, -x_{n,\tau}])$ is contained in $[-1, 1]$ and the pull-back of $[-1, 1]$ by $f_{a,\tau}^{M_n-2}$ containing $f_{a,\tau}^{2M_n}(x_{n,\tau})$ is diffeomorphic and $f_{a,\tau}^{M_n-2}$ preserves the orientation on this set.

For each $\tau$ in $[0, 2]$ let $x_{0,\tau}$ be the left end point of $V_{a,\tau}$. Clearly $x_{0,\tau}$ depends continuously with $\tau$ on $[0, 2]$ and $f_{a,\tau}^2(x_{0,\tau}) = -1$. To define $T_0$, note that by the intermediate value theorem there is $\tau$ in $[0, 2]$ such that $f_{a,\tau}(0)$ is equal to the left end point of $I_{1,a,\tau}$. Let $\tau_0$ be the least number with this property and put $T_0 := [0, \tau_0]$. Then for every $\tau$ in $T_0$ the set $f_{a,\tau}(V_{a,\tau})$ is contained in $I_{1,a,\tau}$ and therefore $f_{a,\tau}^2(V_{a,\tau})$ is contained in $[-1, 1]$. Moreover, note that $f_{a,\tau}^2(0) = -1$ and $f_{a,\tau}^2(0) = 1$, so all of the properties above are satisfied when $n = 0$.

Let $n \geq 0$ be an integer and suppose by induction that a closed interval $T_n$ and for every $\tau$ in $T_n$ a point $x_{n,\tau}$ are already defined. By the intermediate value theorem we can find parameters $\tau^-_{n+1}$ and $\tau^+_{n+1}$ in $T_n$ such that,

$$\tau^-_{n+1} < \tau^+_{n+1}, f_{a,\tau^-_{n+1}}^{M_n}(0) = x_{n,\tau}, f_{a,\tau^+_{n+1}}^{M_n}(0) = 0,$$

and such that for every $\tau$ in $T'_{n+1} := [\tau^-_{n+1}, \tau^+_{n+1}]$ we have

$$x_{n,\tau} \leq f_{a,\tau}^{M_n}(0) \leq 0.$$
For each \( \tau \) in \( T_{n+1}' \), let \( x_{n+1, \tau} \) be the unique point in \([-1, 0]\) that is mapped to \( x_{n, \tau} \) by \( f_{a, \tau}^{M_n} \). Clearly, \( x_{n+1, \tau} \) depends continuously with \( \tau \) on \( T_{n+1}' \). Moreover,
\[
 f_{a, \tau}^{M_n+1}(x_{n+1, \tau}) = f_{a, \tau}^{M_n+1-M_n}(x_{n, \tau}) = f_{a, \tau}^{M_n+1-2M_n}(-1) = -1.
\]
So for each \( \tau \) in \( T_{n+1}' \) the first property above is satisfied with \( n \) replaced by \( n+1 \). By the induction hypothesis for every \( \tau \) in \( T_{n+1}' \) we have
\[
 f_{a, \tau}^{2M_n}([x_{n+1, \tau}, -x_{n+1, \tau}]) \subset f_{a, \tau}^{M_n}([x_{n, \tau}, -x_{n, \tau}]) \subset [-1, 1].
\]
Moreover, the pull-back of \( I_{0, a, \tau} \) by \( f_{a, \tau}^{2M_n-2} \) containing \( f_{a, \tau}^0(x_{n+1, \tau}) \) is diffeomorphic and \( f_{a, \tau}^{2M_n-2} \) preserves the orientation on this set. Since \( M_{n+1} \geq 2M_n + 1 \), it follows that there is a parameter \( \tau \) in \( T_{n+1}' \) such that \( f_{a, \tau}^{M_{n+1}}(0) = 1 \). Let \( \tau_{n+1} \) be the least number with this property and put \( T_{n+1} := [\tau_{n+1}, \tau_{n+1}] \). Since we also have,
\[
 f_{a, \tau_{n+1}}^{M_{n+1}}(0) = f_{a, \tau_{n+1}}^{M_{n+1}-M_n}(x_{n, \tau}) = f_{a, \tau_{n+1}}^{M_{n+1}-2M_n}(-1) = -1,
\]
the second point is satisfied with \( n \) replaced by \( n+1 \). To prove that for each \( \tau \) in \( T_{n+1} \) the third point is satisfied with \( n \) replaced by \( n+1 \), note that by the definition of \( \tau_{n+1} \) for each \( j \) in \( \{0, 1, \ldots, M_{n+1} - 2M_n - 1\} \) we have
\[
 f_{a, \tau}^{2M_n+j}([x_{n+1, \tau}, -x_{n+1, \tau}]) \subset I_{0, a, \tau}.
\]
Moreover,
\[
 f_{a, \tau}^{M_{n+1}}([x_{n+1, \tau}, -x_{n+1, \tau}]) \subset [-1, 1].
\]
It follows that the pull-back of \([-1, 1]\) by \( f_{a, \tau}^{M_{n+1}-2M_n} \) containing \( x = -1 \) is diffeomorphic and contained in \( I_{0, a, \tau} \) and that \( f_{a, \tau}^{M_{n+1}-2M_n} \) preserves the orientation on this set. By the considerations above this implies that the pull-back of \([-1, 1]\) by \( f_{a, \tau}^{M_{n+1}-2} \) containing \( f_{a, \tau}^2(0) \) is diffeomorphic and that \( f_{a, \tau}^{M_{n+1}-2} \) preserves the orientation on this set. This proves that the third point with \( n \) replaced by \( n+1 \).

To prove the lemma, let \( \tau \) be a parameter in the intersection \( \bigcap_{n=0}^{\infty} T_n \) and for each integer \( n \geq 0 \) put \( x_n := x_{n, \tau} \). Property A follows directly from the induction hypothesis and the definitions above. Property B is given by (4.1). The proof of the lemma is thus complete. \( \square \)

**Lemma 4.2.** Let \( \underline{M} := (M_n)_{n=0}^{\infty} \) be a sequence of integers such that \( M_0 = 2 \) and such that for every \( n \geq 0 \) we have \( M_{n+1} \geq 2M_n + 1 \). Suppose moreover that
\[
 M_{n+1} - 2M_n \to +\infty \text{ as } n \to +\infty
\]
and let \( a \geq 20 \) and \( \tau \) in \([0, 2]\) be such that the combinatorics of \( f_{a, \tau} \) is of type \( \underline{M} \).

Then the forward orbit of \( x = 0 \) is contained in \([-1, 1]\) and \( f_{a, \tau} \) is topologically exact on \( K_{a, \tau} \).

**Proof.** To prove the first assertion note that for every \( n \geq 0 \) the point \( f_{a, \tau}^{M_n}(0) \) is in \([-1, 1]\). Since \( M_n \to +\infty \) as \( n \to +\infty \), it follows that for every integer \( m \geq 0 \) the point \( f_{a, \tau}^m(0) \) is in \([-1, 1]\).

To prove that \( f_{a, \tau} \) is topologically exact on \( K_{a, \tau} \), we first make some preparatory remarks. Property B and (4.2) imply that \( x = 0 \) cannot be asymptotic to a periodic cycle. Using that \( f_{a, \tau} \) has negative Schwarzian derivative, Singer’s theorem implies that all the periodic points of \( f_{a, \tau} \) are hyperbolic repelling, see for example [MvS93, Theorem II.6.1]. Since \( f_{a, \tau} \) has no wandering intervals intersecting \( K_{a, \tau} \), see for example [MvS93, Theorem A in §IV], it follows that the interior of \( K_{a, \tau} \) is empty. In particular, \( x_n \to 0 \) as \( n \to +\infty \).
We proceed to the proof that $f_{a,\tau}$ is topologically exact on $K_{a,\tau}$. To do this, let $U$ be an open interval intersecting $K_{a,\tau}$. If for some integer $m \geq 1$ the interval $f_{a,\tau}^m(U)$ contains $x = 0$, then there is an integer $n \geq 0$ such that $f_{a,\tau}^m(U)$ contains either $[x_n, 0]$ or $[0, -x_n]$. In both cases $I_{0,a,\tau} \subset f_{a,\tau}^{M_n+m}(U)$ and therefore $[-1, 1] \subset f_{a,\tau}^{M_n+m+1}(U)$. It remains to consider the case where for each integer $m \geq 0$ the interval $f_{a,\tau}^m(U)$ does not contain $x = 0$. Since $K_{a,\tau}$ has empty interior, it follows that there is an integer $m \geq 0$ such that $f_{a,\tau}^m(U)$ is not contained in $I_{0,a,\tau} \cup V_{a,\tau} \cup I_{1,a,\tau}$. If $m$ is the least integer with this property, then $f_{a,\tau}^m$ maps $U$ diffeomorphically to $f_{a,\tau}^m(U)$. Thus $f_{a,\tau}^n(U)$ contains one of the points $\partial I_{0,a,\tau} \cup \partial V_{a,\tau} \cup \partial I_{1,a,\tau}$ in its interior and therefore $f_{a,\tau}^{m+2}(U)$ contains a neighborhood of the hyperbolic repelling fixed point $x = -1$. Again using that $K_{a,\tau}$ has empty interior we conclude that there is an integer $N \geq 0$ such that $f_{a,\tau}^{m+N+2}(U)$ contains $I_{0,a,\tau}$ and therefore $f_{a,\tau}^{m+N+3}(U) \supset [-1,1]$. This completes the proof that $f_{a,\tau}$ is topologically exact on $K_{a,\tau}$ and of the lemma. \hfill $\square$

5. Main estimates

We start this section chosing the map used to prove the Main Lemma, through its combinatorial type. The rest of this section is devoted to prove some estimates concerning this map (Lemmas 5.1 and 5.2), that are used in the proof of the Main Lemma in the next section.

Fix $\eta$ in $(1, 2)$, let $a_0 \geq 20$ be given by Lemma 3.1 for this choice of $\eta$, and fix a satisfying

$$a > \max\{a_0, 128, \eta^{20}\}.$$  \hfill (5.1)

Define a sequence of integers $M := (M_n)_{n=0}^{\infty}$ inductively by $M_0 = 2$ and by the property that for every $n \geq 0$ we have

$$\eta^{M_{n+1}} \geq 4\eta^{5M_n/2}(2a + 8)^{M_n/2}. $$  \hfill (5.2)

Note that for each $n \geq 0$ we have

$$M_{n+1} \geq 5M_n/2 $$  \hfill (5.3)

and $M_{n+1} - 2M_n \geq M_n/2 \geq 2^n$. In particular,$$M_{n+1} - 2M_n \to +\infty \text{ as } n \to +\infty.$$

Let $\tau$ in $[0, 2]$ be such that the combinatorics of $f_{a,\tau}$ is of type $M$ (Lemma 4.1). In what follows we put

$$f := f_{a,\tau}, K := K_{a,\tau}, I_0 := I_{0,a,\tau}, \text{etc.}$$

Note that, by (5.1) and by the definition of $\lambda$, we have

$$\lambda > \max\{256, \eta^{20}\}. $$  \hfill (5.4)

Moreover, for every integer $n \geq 0$ we have by (5.2),

$$\eta^{M_{n+1}} \geq 4\eta^{5M_n/2}\lambda^{M_n/2}. $$  \hfill (5.5)

**Lemma 5.1.** For every integer $n \geq 0$ the following properties hold.

1. On $J_n$ we have

$$\eta^{-(M_n-2)}\lambda^{M_n-2} \leq |Df^{M_n-2}| \leq \eta^{M_n-2}\lambda^{M_n-2}.$$
2. We have
\[ \eta^{M_n/2} \lambda^{-M_n/2} \leq |x_n| \leq \sqrt{2} \eta^{M_n/2} \lambda^{-M_n/2} \]
and
\[ \eta^{-(3M_n/2-2)} \lambda^{M_n/2} \leq |Df^{M_n}(x_n)| \leq \sqrt{2} \eta^{3M_n/2-2} \lambda^{M_n/2}. \]

Proof. Since \( M_0 = 2 \), part 1 holds trivially when \( n = 0 \). Suppose by induction that part 1 holds for some integer \( n \geq 0 \). Property A implies \( f^{M_n}(V_n) = [-1, f^{M_n}(0)]. \) Since \( f^{M_n}(0) \) is \( V_n \cap (-1, 0) \) (property A) and since this last set is contained in \([-1/4, 0]\) (5.1) and part 2 of Lemma 5.1, the induction hypothesis implies that
\[ \frac{1}{2} \eta^{-(M_n-2)} \lambda^{-(M_n-2)} \leq |f^2(V_n)| \leq \eta^{M_n-2} \lambda^{-(M_n-2)}. \]

Thus, by part 3 of Lemma 5.1 we obtain (5.6) and
\[ \eta^{M_n/2} \lambda^{-M_n/2+2} \leq |Df^2(x_n)| \leq \sqrt{2} \eta^{M_n/2} \lambda^{-M_n/2+2}. \]

Using the induction hypothesis again we obtain (5.7).

To prove part 1 with \( n \) replaced by \( n + 1 \), note first that by the induction hypothesis we have
\[ \eta^{-(M_n-2)} \lambda^{M_n-2} \leq |Df^{M_n-2}(f^2(x_{n+1}))| \leq \eta^{M_n-2} \lambda^{M_n-2}. \]

Since \( f^{M_n}(x_{n+1}) = x_n \), by (5.7) we have
\[ \eta^{-(5M_n/2-4)} \lambda^{3M_n/2-2} \leq |Df^{2M_n-2}(f^2(x_{n+1}))| \leq \sqrt{2} \eta^{5M_n/2-4} \lambda^{3M_n/2-2}. \]

Using that \( f^{2M_n}(x_{n+1}) = -1 \), we obtain
\[ \eta^{-(5M_n/2-4)} \lambda^{M_{n+1}-M_n/2-2} \leq |Df^{M_{n+1}-2}(f^2(x_{n+1}))| \]
\[ \leq \sqrt{2} \eta^{5M_n/2-4} \lambda^{M_{n+1}-M_n/2-2}. \]

Using part 1 of Lemma 5.1 and (5.5) we obtain part 1 of the lemma with \( n \) replaced by \( n + 1 \). This completes the proof of the induction step and of the lemma. \( \square \)

Define the sequence of intervals \( \{U_n\}_{n=0}^{\infty} \) inductively as follows. Put \( U_0 = [-1, 1] \setminus (I_0 \cup I_1) \) and note that \( U_0 \) contains \( V_0 \). Let \( n \geq 0 \) be an integer such that \( U_n \) is already defined and contains \( V_n \). Then, let \( U_{n+1} \) be the pull-back of \( U_n \) by \( f^{M_n} \) containing \( x = 0 \). By definition \( U_{n+1} \) contains \( V_{n+1} \).

Using that the closure of \( U_0 \) is contained in \((-1, 1)\), an induction argument shows that for every \( n \geq 0 \) the interval \( V_n \) contains the closure of \( U_{n+1} \) in its interior. So, if for each \( n \geq 0 \) we denote by \( y_n \) the left end point of \( U_n \), then for every \( n \geq 0 \) we have
\[ f^{M_n}(y_{n+1}) = y_n \text{ and } y_n < x_n < y_{n+1} < 0. \]

**Lemma 5.2.** For every integer \( n \geq 0 \) we have
\[ |y_{n+1}| \geq |x_{n+1} - y_{n+1}| \geq \eta^{-M_n} \lambda^{-M_n/2}. \]
Moreover, on \( V_n \setminus U_{n+1} \) we have
\[ |Df^{M_n}| \geq \eta^{-2M_n} \lambda^{M_n/2}. \]
Proof. Note that, since $I_0$ is contained in $[-1,-7/8]$ and $V$ in $[-1/4,1/4]$ (6.1) and part 2 of Lemma 3.1, we have $|x_0 - y_0| \geq 5/8$. On the other hand, for every integer $n \geq 0$ the point $f^2(y_{n+1})$ is contained in $f^2(V_n) \subset J_n$, so by part 1 of Lemma 5.1 we have

$$|f^2(0) - f^2(y_{n+1})| \geq |f^2(x_{n+1}) - f^2(y_{n+1})| \geq |x_n - y_n| \eta^{-(M_n-2)} \lambda^{-1}(M_n-2).$$

Combined with part 3 of Lemma 5.1 this implies

$$|y_{n+1}| \geq \sqrt{2} |x_n - y_n|^{1/2} \eta^{-M_n/2} \lambda^{-M_n/2}.$$  

When $n = 0$ we obtain $|y_1| \geq \sqrt{5/4} \eta^{-1} \lambda^{-1}$. Together with (5.4), with (5.5) with $n = 0$, and with (5.6) with $n = 1$, we have

$$|x_1| \leq \sqrt{2} \eta^{2M_1/2} \lambda^{-M_1/2} \leq \sqrt{2} \eta^{-2M_1} \lambda^{M_1/2} \leq |y_1| - \eta^{-2} \lambda^{-1}.$$  

This implies (5.8) when $n = 0$. To prove that (5.8) holds for $n \geq 1$, we proceed by induction. Let $n \geq 1$ be an integer such that (5.8) holds with $n$ replaced by $n - 1$. Using (6.9), (6.10), and the induction hypothesis, we obtain

$$|y_{n+1}| \geq \sqrt{2} \eta^{M_n/2} \lambda^{-M_n/2} \geq \sqrt{2} \eta^{-M_n} \lambda^{M_n/2}.$$  

Together with (5.3), (5.4), (5.5), and (5.6) with $n$ replaced by $n + 1$, we have

$$|x_{n+1}| \leq \sqrt{2} \eta^{M_n+2} \lambda^{-M_n+2} \leq \sqrt{2} \eta^{-M_n+1} \leq |y_{n+1}| - \eta^{-M_n} \lambda^{-M_n/2}.$$  

This completes the proof of the induction step and of the fact that (5.8) holds for every $n \geq 0$.

To prove (5.9), let $n \geq 0$ be an integer and note that by (5.8) and by part 3 of Lemma 3.1 for each $x$ in $[x_n,y_{n+1}]$ we have

$$|Df^2(x)| = |Df^2(x)| \geq \eta^{-2} \lambda^2 |x| \geq \eta^{-2} \lambda^2 |y_{n+1}| \geq \eta^{-M_n-2} \lambda^{-M_n/2 + 2}.$$  

Together with part 1 of Lemma 5.1 this implies (5.9). The proof of the lemma is thus complete.

6. PROOF OF THE MAIN LEMMA

After showing some general properties of the dynamics of the map chosen in the previous section (Lemma 6.1), in this section we give the proof of the Main Lemma.

Throughout this section we use the notation introduced in the previous section. Note that part 1 of Lemma 3.1 implies that on $I_0 \cup I_1$ we have

$$\eta^{-1} \lambda \leq |Df| \leq \eta \lambda.$$  

Lemma 6.1. The map $f$ is topologically exact on the maximal invariant set $K$ of $f$ in $[-1,1]$ and $f$ satisfies the Collet-Eckmann condition.

Proof. That $f$ is topologically exact on $K$ is given by Lemma 4.2. To prove that $f_{0,\tau}$ satisfies the Collet-Eckmann condition, in part 1 we show that for every integer $n \geq 0$ we have

$$|Df^{M_n}(f(0))| \geq \eta^{-2M_n-2} \lambda^{-M_n/2}.$$  

In part 2 we complete the proof of the lemma using this fact.

1. By part 1 of Lemma 5.1 and (6.1), we have

$$|Df^{M_n-1}(f(0))| = |Df^{M_n-2}(f^2(0))| \cdot |Df(f(0))| \geq (\eta^{-1} \lambda)^{M_n-1}.$$
To estimate $D f(Mn(0))$, note that the property that $f^{2Mn}(0)$ is in $I_0$, implies $x_n \leq f^{Mn}(0) < x_{n+1}$. Since $(y_{n+1}, x_{n+1}) \subset U_{n+1} \setminus V_{n+1}$ is disjoint from $K$, it follows that $x_n \leq f^{Mn}(0) \leq y_{n+1}$. By (6.1), by part 3 of Lemma 3.1 by (5.8), and by (6.9), we have

$$|D f(Mn(0))| = |D f^2(Mn(0))| \cdot |D f(Mn+1(0))|^{-1}$$

$$\geq \eta^{-1} \lambda^{-1} |D f^2(Mn(0))|$$

$$\geq \eta^{-3} \lambda |y_{n+1}| \geq \eta^{-Mn-3} \lambda^{-Mn/2+1}.$$ 

Combined with (6.3), this implies (6.2).

2. To prove that $f$ satisfies the Collet-Eckmann condition we show by induction that for every integer $m \geq 1$ we have

$$|D f^m(f(0))| \geq (\eta^{-3} \lambda^{1/2})^m.$$ 

When $m = 1$ this inequality follows from (6.1) and from the fact that $f(0)$ is in $I_1$. Let $m \geq 2$ be an integer and suppose by induction that (6.4) holds with $m$ replaced by each element of $\{1, \ldots, m-1\}$. Let $n \geq 0$ be the largest integer such that $M_n \leq m$. When $m = M_n$, the inequality (6.4) follows from (6.2). If $m = M_n+1$, then $f^{M_n}(0)$ is in $I_1$ and (6.4) follows from (6.1) together with the induction hypothesis. Suppose that $n \geq 1$ and that $m$ is in $\{M_n+2, \ldots, 2M_n-1\}$. Note that the pull-back $J_n$ of $J_0$ by $f$ containing $f(0)$ is contained in $I_1$ and it is mapped diffeomorphically to $[-1, 1]$ by $f^{M_n-1}$. Since $f(0)$ and $f^{M_n+1}(0)$ are both in $f(V_n)$ and hence in $J_n$, we have by part 1 of Lemma 3.1 and by the induction hypothesis

$$|D f^{M_n}(f^{M_n+1}(0))| \geq \eta^{-1} |D f^{M_n}(f(0))| \geq \eta^{-1}(\eta^{-3} \lambda^{1/2})^{m-M_n}.$$ 

Combined with (6.2), this implies (6.4). Finally, if $m$ is in $\{2M_n, \ldots, M_n+1-1\}$, then $f^m(0)$ is in $I_1$ and (6.4) follows from (6.1) and the induction hypothesis. This completes the proof of the induction step and of the fact that $f$ satisfies the Collet-Eckmann condition.

**Proof of the Main Lemma.** In part 1 below we show

$$\chi_{\text{per}}(f) \geq \frac{1}{2} \ln \lambda - 2 \ln \eta > 0.$$ 

Parts 2 and 3 are devoted to show the last inequality of the Main Lemma; the remaining statements are given by Lemma 6.1. In part 2 we reduce the proof of the desired inequality to a lower estimate on the size of a certain sequence of pull-backs. The lower estimate is given in part 3.

1. To prove (6.5), let

$$m : [-1, 1] \setminus \{0\} \rightarrow \mathbb{N}$$

be the function that is constant equal to 1 on $[-1, 1] \setminus V_0$ and that for every integer $n \geq 0$ is constant equal to $M_n$ on $V_n \setminus V_{n+1}$. We show that for every $x$ in $K \setminus \{0\}$, we have

$$|D f^m(x)(x)| \geq (\eta^{-2} \lambda^{1/2})^{m(x)}.$$ 

If $x$ is in $K \setminus V_0$, then actually $x$ is in $K \setminus U_0 \subset I_0 \cup I_1$ and in this case (6.6) is given by (6.1). On the other hand, if for some integer $n \geq 0$ the point $x$ is in $(V_n \setminus V_{n+1}) \cap K$, then $x$ is in $V_n \setminus U_{n+1}$ and (6.6) is given by (6.9). Thus (6.6) is proved for each $x$ in $K \setminus \{0\}$. To prove (6.5), let $p$ be a hyperbolic repelling
periodic point of \( f \). Then \( x = 0 \) is not in the forward orbit of \( p \). Thus we can define inductively a sequence of integers \((m_\ell)_{\ell=0}^{+\infty}\), by \( m_0 := m(p) \) and for \( \ell \geq 1 \), by
\[
m_\ell := m(f^{m_{\ell-1}}(p)) + m_{\ell-1}.
\]
Then by (6.6) we have
\[
\chi_p(f) = \lim_{m \to +\infty} \frac{1}{m} \ln |Df^m(p)| = \lim_{\ell \to +\infty} \frac{1}{m_\ell} \ln |Df^{m_\ell}(p)| \geq \frac{1}{2} \ln \lambda - 2 \ln \eta.
\]
This proves (6.5).

2. Let \( \delta > 0 \) be given, let \( N_0 \geq 0 \) be an integer such that \( \lambda^{-N_0} \leq \delta \), and put
\[
J := (-1 - \lambda^{-N_0}, -1].
\]
Note that for every integer \( n \geq 0 \) we have \( f^{M_{n+1}+2M_n}(x_{n+2}) = -1 \). For each integer \( n \geq 1 \) satisfying \( M_{n+1} - 2M_n \geq N_0 \), we show in part 3 below that the pull-back \( W_n \) of \( J \) by \( f^{2M_{n+1}+2-2N_0} \) containing \( x_{n+2} \) satisfies,
\[
|W_n| \geq (4\eta \lambda)^{-1}(\eta \lambda)^{-3M_{n+1}/4}.
\]
In view of (5.4) and (6.3), this implies
\[
\liminf_{m \to +\infty} \frac{1}{m} \ln \{|W| : W \text{ connected component of } f^{-m}((-1 - \delta, -1])\} \leq \frac{3}{8} \ln(\eta \lambda) < \frac{1}{2} \ln \lambda - 2 \ln \eta \leq \chi_{\text{per}}(f).
\]
So to complete the proof of the proposition it is enough to show that for every integer \( n \geq 0 \) such that \( M_{n+1} - 2M_n \geq N_0 \), we have (6.7).

3. Let \( n \geq 1 \) be an integer such that \( M_{n+1} - 2M_n \geq N_0 \). It follows from property A that the pull-back of \( I_0 \) by \( f^{M_n} \) containing \( x_n \) is diffeomorphic. This implies that the pull-back of \( I_0 \) by \( f^{2M_n-2} \) containing \( f^2(x_{n+1}) \) is diffeomorphic and hence that for every integer \( M \geq 2M_n - 1 \) the pull-back of \( I \) by \( f^M \) containing \( f^2(x_{n+1}) \) is diffeomorphic. So by part 1 of Lemma 3.1 the pull-back \( J' \) of \( J \) by \( f^{M_{n+1}-N_0} \) containing \( f^2(x_{n+1}) \) satisfies
\[
\eta^{-1}\lambda^{-N_0}|Df^{M_{n+1}-N_0}(f^2(x_{n+1}))|^{-1} \leq |J'|
\leq \eta^M \lambda^{-N_0}|Df^{M_{n+1}-N_0}(f^2(x_{n+1}))|^{-1}.
\]
So, by part 1 of Lemma 5.1 and 5.7, we have
\[
\sqrt{1/2}\eta^{-(5M_n/2-3)}\lambda^{-M_{n+1}+M_n/2} \leq |J'| \leq \eta^{5M_n/2-3}\lambda^{-M_{n+1}+M_n/2}.
\]
On the other hand, by (5.6) with \( n \) replaced by \( n + 1 \) and by part 3 of Lemma 3.1 applied to \( x = x_{n+1} \), we have
\[
|f^2(0) - f^2(x_{n+1})| \leq \eta^{M_{n+1}+2M_n+2}.
\]
Together with (5.5) and (6.8) this implies that, if we denote by \( z' \) the left end point of \( J' \), then
\[
|f^2(0) - z'| \leq 2\eta^{M_{n+1}+2}\lambda^{-M_{n+1}+2}.
\]
So by part 3 of Lemma 3.1 we have
\[
|Df^2| \leq 2\eta^{M_{n+1}/2+2}\lambda^{-M_{n+1}/2+2}.
\]
on the pull-back $J'$ of $J$ by $f^2$ containing $x_{n+1}$. Note that $J''$ is the pull-back of $J$ by $f^{M_{n+1}+2-N_0}$ containing $x_{n+1} = f^{M_{n+1}}(x_{n+2})$. Combining (6.8) and (6.10), we obtain

$$\sqrt{1/2\eta} - (5M_n/2 - 3)\lambda^{-M_{n+1}+M_n/2} \leq |J'| \leq |J''| 2^{M_{n+1}+2+2} \lambda^{-M_{n+1}/2+2}$$

Together with (5.3) this implies

$$|J''| \geq 2^{-2}\eta^{-M_{n+1}/2} \lambda^{-M_{n+1}/2-2}.$$  

By part 1 of Lemma 5.1 with $n$ replaced by $n+1$, this implies that the pull-back $J'''$ of $J''$ by $f^{M_{n+1}+1} - 2$ containing $f^2(x_{n+2})$ satisfies

$$(6.11) |J'''| \geq 2^{-2}\eta^{-3M_{n+1}/2} \lambda^{-3M_{n+1}/2}.$$  

Combined with (5.4), with (5.5) with $n$ replaced by $n+1$, and with (6.9) with $n+1$ replaced by $n+2$, we obtain

$$|f^2(0) - f^2(x_{n+2})| \leq \eta^{M_{n+2}+2} \lambda^{-M_{n+2}+2} \leq \eta^{-3M_{n+2}} \leq |J''|.$$  

Noting that $W_n$ is the pull-back of $J'''$ by $f^2$ containing $x_{n+2}$, if we denote by $w_n$ the left end point of $W_n$, then we have

$$f^2(w_n) = z'''$$

and

$$|f^2(0) - f^2(w_n)| \leq 2|J'''|.$$  

So by part 3 of Lemma 5.1 we have

$$(6.12) |Df^2| \leq 2\eta\lambda |J'''|^{1/2}$$

on $W_n$ and therefore

$$(6.12) |J'''| \leq 2\eta\lambda |J'''|^{1/2} \cdot |W_n|.$$  

Together with (6.11) this implies

$$|W_n| \geq (2\eta\lambda)^{-1} |J'''|^{1/2} \geq (4\eta\lambda)^{-1} |J'''|^{-3M_{n+1}/4} \lambda^{-3M_{n+1}/4}.$$  

This proves (6.7) and completes the proof of the proposition. 

7. Proof of Theorem B

The purpose of this section is to prove Theorem B. First we use [RL12, Main Theorem] to reduce the proof of Theorem B to show the exponential shrinking of components of a small neighborhood of a periodic point in the boundary of a Fatou component (Proposition 7.1). For future reference we state this last result for a more general class of maps. The proof of this result occupies most of this section.

A non-injective interval map $f : I \to I$ is of class $C^3$ with non-flat critical points if:

- The map $f$ is of class $C^3$ outside $\text{Crit}(f)$.
- For each critical point $c$ of $f$ there exists a number $\ell_c > 1$ and diffeomorphisms $\phi$ and $\psi$ of $\mathbb{R}$ of class $C^3$, such that $\phi(c) = \psi(f(c)) = 0$ and such that on a neighborhood of $c$ on $I$ we have,

$$|\psi \circ f| = \pm |\phi|^{\ell_c}.$$  

The number $\ell_c$ is the order of $f$ at $c$. 


Denote by $\mathcal{A}$ the collection of interval maps of class $C^1$ with non-flat critical points, whose Julia set is completely invariant. Note that each smooth non-degenerate map whose Julia set is completely invariant is contained in $\mathcal{A}$. In Appendix A we gather some general properties of maps in $\mathcal{A}$ that are used below.

By [MdMvS92, §1], each interval map in $\mathcal{A}$ has at most a finite number of periodic Fatou components. Combined with Lemma [A,2] and part 1 of the Main Theorem’ of [RL12], Theorem [13] is a direct consequence of the following proposition.

**Proposition 7.1.** Let $f$ be a map in $\mathcal{A}$ that is topologically exact on $J(f)$ and such that $\chi_{\inf}(f) > 0$. Then for every periodic point $p$ in the boundary of a Fatou component of $f$, there are $\chi > 0$ and $\delta > 0$ such that

$$\liminf_{n \to \infty} \frac{1}{n} \ln \max \{|W| : W \text{ connected component of } f^{-n}(B(p, \delta))\} < -\chi.$$

**Proof.** Fix $\chi_0$ in $(0, \chi_{\inf}(f))$, put $\ell := \prod_{c \in \text{Crit}(f)} \ell_c$, and denote by $\pi$ the forward orbit of $p$. Let $\delta_0 > 0$ and $\varepsilon$ by given by Lemma [A,1] with $K = 2$. Reducing $\delta_0$ if necessary we assume $f$ satisfies the conclusions of Lemma [A,3] with $\delta_2 = \delta_0$. Moreover, we assume that for every critical point $c$ of $f$, every interval $\hat{J}$ contained in $I$, and every pull-back $\hat{W}$ of $J$ by $f$ that is contained in $B(c, \delta_0)$, the following property holds for every $\kappa$ in $(0, 1/2)$: If $W$ is a pull-back of $\kappa \hat{J}$ by $f$ contained in $\hat{W}$, then

$$(3\kappa)^{-1/\ell} W \subset \hat{W}.$$ 

We also assume there is $M_0 > 1$ such that for every critical point $c$ of $f$ and every $x$ in $B(c, \delta_0)$ we have

$$|Df(x)| \leq M_0|x - c|^\ell - 1.$$ 

Taking $M_0$ larger if necessary we assume

$$M_0 > \sup_f |Df|.$$ 

In view of Lemma [A,2] and part 1 of the Main Theorem’, there is $\delta_\ast$ in $(0, \delta_0)$ such that for every $y$ in $f^{-1}(O) \setminus O$, every integer $m \geq 1$, and every pull-back $W$ of $B(y, \delta_\ast)$ by $f^m$, we have

$$|W| \leq \min \{\exp(-(m + 1)\chi_0), \delta_0\}.$$ 

Moreover, if we denote by $U$ the Fatou component of $f$ containing $p$ in its boundary and denote by $K$ the interval $B(p, \delta_\ast) \setminus U$, then a similar property holds with $B(y, \delta_\ast)$ replaced by $K$: For every integer $m \geq 1$ and every pull-back $W$ of $K$ by $f^m$, we have

$$|W| \leq \min \{\exp(-(m + 1)\chi_0), \delta_0\}.$$ 

Note that the hypothesis $\chi_{\inf}(f) > 0$ implies that the periodic point $p$ is hyperbolic repelling, so $\chi_p(f) > 0$. It follows that there is a constant $\delta_1 > 0$ and an integer $\mu_1 \geq 1$ such that for every $\delta$ in $(0, \delta_1)$, every integer $m \geq 1$, and every pull-back $W$ of $B(p, \delta)$ by $f^m$ containing a point $p'$ of $O$, we have

$$B(p', \delta \exp(-(m + \mu_1)\chi_p(f))) \subset W \subset B(p', \delta \exp(-(m - \mu_1)\chi_p(f))).$$ 

Let $\gamma_1$ in $(0, 1)$ be sufficiently close to 1 so that

$$\chi_1 := (1 - \gamma_1)\chi_0 < \chi_p(f)/\ell$$ 

and $M_0^{1-\gamma_1} < \exp(\gamma_1\chi_0(\ell_\varepsilon - 1)/2)$.

Let $n_1 \geq 1$ be a sufficiently large integer so that

$$n_1 \gamma_1 \geq 2(\pi + 1),$$
\( (7.4) \) \quad 12^{\# \text{Crit}(f)+1} \exp\left(-n_{\dagger}\gamma_{\dagger}\chi_{p}(f)/(2\ell)\right) \\
\quad \quad \leq \min \{\frac{\varepsilon}{2}, \exp\left(-\left(\pi + 2\mu_{1}\right)\chi_{\dagger}\right)\exp\left(-n_{\dagger}\gamma_{\dagger}\chi_{\dagger}/2\right)\}, \\
\( (7.5) \) \quad \left(M_{0}^{1-\gamma_{\dagger}} \exp\left(-\gamma_{\dagger}(\ell_{c}-1)\chi_{0}/2\right)\right)^{n_{\dagger}} \\
\quad \quad < \frac{1}{4} \left(2^{\ell_{c}-1} \exp\left(\pi + 2\mu_{1}(\ell_{c}-1)\chi_{0}\right)\right)^{-1}.

Reducing \( \delta_{1} \) if necessary, we assume that \( 2\delta_{1} \exp(\mu_{1}\chi_{p}(f)) < 1 \) and that the following properties hold:

1. For every \( p' \) in \( \mathcal{O} \), every pull-back of \( B(p', \delta_{1} \exp(\mu_{1}\chi_{p}(f))) \) by \( f \) that is disjoint from \( \mathcal{O} \) contains a point \( y \) of \( f^{-1}(\mathcal{O}) \) and is contained in \( B(y, \delta_{*}) \).

2. For every \( n \) in \( \{1, \ldots, n_{\dagger}\} \) and every pull-back \( W \) of \( B(p, \delta_{1}) \) by \( f^{n} \), we have

\[ |W| \leq \exp(-n\chi_{1}). \]

To prove the proposition we show that for every integer \( n \geq 1 \) and every pull-back \( W \) of \( B(p, \delta_{1}) \) by \( f^{n} \), we have

\[ |W| \leq \exp(-n\chi_{1}). \]

We proceed by induction in \( n \). By our choice of \( \delta_{1} \) the desired assertion holds for each \( n \) in \( \{1, \ldots, n_{\dagger}\} \). Let \( n \geq n_{\dagger}+1 \) be an integer for which this property holds with \( n \) replaced by \( n-1 \) and let \( W \) be a pull-back of \( B(x, \delta_{1}) \) by \( f^{n} \). If \( W \) intersects \( \mathcal{O} \), then by (7.2) with \( m \) replaced by \( n \) we have by our choice of \( \delta_{1} \),

\[ |W| \leq 2\delta_{1} \exp(-\left(n-m\right)\chi_{p}(f)) < \exp(-n\chi_{1}). \]

Since \( \chi_{p}(f) \geq \chi_{\inf}(f) > \chi_{0} > \chi_{1} \), this proves the induction hypothesis in this case.

Assume \( W \) does not intersect \( \mathcal{O} \) and denote by \( m \) the largest element of \( \{0, \ldots, n-1\} \) such that the pull-back \( W_{0} \) of \( B(p, \delta_{1}) \) by \( f^{m} \) containing \( f^{n-m}(W) \) intersects \( \mathcal{O} \). Let \( p_{0} \) be the point of \( W_{0} \) in \( \mathcal{O} \). By maximality of \( m \), the pull-back \( W' \) of \( B(p, \delta_{1}) \) by \( f^{m+1} \) containing \( f^{n-(m+1)}(W) \) is disjoint from \( \mathcal{O} \). By our choice of \( \delta_{1} \) it follows that \( W' \) contains a point \( y \) of \( f^{-1}(\mathcal{O}) \setminus \mathcal{O} \) and is contained in \( B(y, \delta_{*}) \). So by our choice of \( \delta_{*} \) we have,

\[ |W'| \leq \exp(-\left(n-m\right)\chi_{0}). \]

In the case where \( m \leq \gamma_{1}n \), we obtain

\[ |W'| \leq \exp(-n(1-\gamma_{1})\chi_{0}) = \exp(-n\chi_{1}), \]

so the induction hypothesis is verified in this case.

Suppose \( m \geq \gamma_{1}n \). Note that by (7.2) we have

\[ W_{0} \subset B(p_{0}, \delta_{1} \exp(-(m-\mu_{1})\chi_{p}(f))). \]

By (7.2) and our choice of \( n_{\dagger} \) there is an integer \( \tilde{m} \) satisfying

\( (7.6) \) \quad \frac{m}{2} - 2\mu_{1} - \pi \leq \tilde{m} \leq \frac{m}{2} + \pi, \)

such that \( \tilde{m} - m \) is divisible by \( \pi \), and such that the pull-back \( \hat{W}_{0} \) of \( B(p, \delta_{1}) \) by \( f^{\tilde{m}} \) containing \( p_{0} \) contains

\[ B(p_{0}, \delta_{1} \exp(-(m-\mu_{1})\chi_{p}(f))). \]

For each \( j \) in \( \{1, \ldots, n-m\} \) denote by \( W_{j} \) (resp. \( \hat{W}_{j} \)) the pull-back of \( W_{0} \) (resp. \( \hat{W}_{0} \)) by \( f^{j} \) containing \( f^{n-(m+j)}(W) \) and let \( \kappa_{j} > 0 \) be the smallest number such that

\[ \kappa_{j}^{-1}W_{j} \subset \hat{W}_{j}. \]
Note that \( \kappa_0 \leq \exp(-m\chi_\ell(f)/2) \). By (7.3) and (7.6) we have \( \hat{m} + n - m \leq n - 1 \).
So by the induction hypothesis and (7.6) we have
\[
|\hat{W}_{n-m}| \leq \exp(-(\hat{m} + n - m)\chi_1) \leq \exp((\pi + 2\mu_\ell)\chi_1)\exp(-(n-m/2)\chi_1).
\]
So, in view of (7.4), to complete the proof of the induction step it is enough to prove
\[
\kappa_{n-m} \leq 12^\#\text{Crit}(f)^{1/\ell}\kappa_0^{1/\ell}.
\]
To prove this inequality, we show that for each critical point \( c \) of \( f \) there are at most 1 element \( j \) of \( \{1, \ldots, n - m\} \) such that \( \hat{W}_j \) contains \( c \). Part 1 of Lemma A.3 implies that for each \( j \) in \( \{1, \ldots, n - m\} \) the set \( \hat{W}_j \) intersects \( J(f) \), so \( f^{\hat{m}+j}(\hat{W}_j) \) intersects \( K \). Moreover, the pull-back \( \hat{K}_j \) of \( K \) by \( f^{\hat{m}+j} \) contained in \( \hat{W}_j \) is an interval. By our choice of \( \delta_\ell \), for every \( j \geq 1 \) we have
\[
|\hat{K}_j| \leq \exp(-(\hat{m} + j)\chi_0).
\]
Suppose by contradiction there is a critical point \( c \) of \( f \) and elements \( j \) and \( j' \) of \( \{1, \ldots, n - m\} \) such that \( \hat{W}_{j'} \) and \( \hat{W}_j \) contain \( c \) and such that \( j' \geq j + 1 \). By part 1 of Lemma A.3 this implies that \( c \) is in \( \hat{K}_j \) and in \( \hat{K}_{j'} \). By (7.9) this implies that \( f^{j'-j}(c) \) is in \( B(c, \exp(-\hat{m}\chi_0)) \). By our choices of \( \delta_\ell \) and \( M_0 \) and by (7.6), the derivative of \( f^{j'-j} \) on \( B(c, 2\exp(-\hat{m}\chi_0)) \) is bounded from above by
\[
M_0^{j'-j}(2\exp(-\hat{m}\chi_0))^\ell\chi_0^{-1} \leq 2^\ell\chi_0^{-1}\exp((\pi + 2\mu_\ell)(\ell\chi_0 - 1)\chi_0)M_0^{1-\gamma_\ell}\exp(-\gamma_\ell(\ell\chi_0 - 1)\chi_0/2).
\]
By (7.9) it follows that the derivative of \( f^{j'-j} \) on \( B(c, 2\exp(-\hat{m}\chi_0)) \) is bounded from above by a number that strictly less than 1/4. Since \( f^{j'-j}(c) \) is in \( B(c, 2\exp(-\hat{m}\chi_0)) \), this implies that \( f^{j'-j} \) contains a hyperbolic attracting fixed point and that \( c \) converges to this point under forward iteration by \( f^{j'-j} \). This contradicts the fact that \( c \) is in \( J(f) \).

To prove (7.8), let \( k \) be the number of those \( j \) in \( \{1, \ldots, n - m\} \) such that \( \hat{W}_j \) contains a critical point of \( f \). If \( k = 0 \), then \( f^{n-m} \) maps \( \hat{W}_{n-m} \) diffeomorphically to \( \hat{W}_0 \). Noting that by (7.4) we have \( \kappa_0 \leq \epsilon/2 \), by Lemma A.1 this implies \( \kappa_{n-m} \leq 4\kappa_0 \) and thus (7.8).

Suppose \( k \geq 1 \). Put \( j_0 := 0 \) and let \( j_1 < \cdots < j_k \) be all the elements \( j \) of \( \{1, \ldots, n - m\} \) such that \( \hat{W}_j \) contains a critical point of \( f \). By part 1 of Lemma A.3 for each \( s \) in \( \{1, \ldots, k\} \) the set \( \hat{W}_{j_s} \) contains a unique critical point of \( f \); denote it by \( c_s \). Moreover, by our choice of \( \delta_\ell \) the set \( \hat{W}_{j_s} \) is contained in \( B(c_s, \delta_0) \). We prove by induction that for every \( s \) in \( \{0, \ldots, k\} \) we have
\[
\kappa_{j_s} \leq 12^s\kappa_0^{1/(1-\ell_s)}.
\]
This is trivially true when \( s = 0 \). Let \( s \) in \( \{0, \ldots, k - 1\} \) be such that this property holds for \( s \). First note that by our choice of \( \delta_0 \) we have
\[
\kappa_{j_{s+1}} \leq (3\kappa_{j_{s+1}-1})^{1/\ell_{s+1}} \leq 3\kappa_{j_{s+1}-1}^{1/\ell_{s+1}}.
\]
If \( j_{s+1} = j_s + 1 \), then the induction step follows from (7.10). If \( j_{s+1} \geq j_s + 2 \), then \( f^{j_{s+1} - j_s} \) maps \( W_{j_{s+1}} \) diffeomorphically to \( W_{j_s} \). Noting that (7.4) and (7.10) imply \( \kappa_{j_k} \leq \varepsilon/2 \), by Lemma A.1 we have \( \kappa_{j_{s+1} - 1} \leq 4\kappa_{j_s} \). Together with (7.10) and (7.11), this completes the proof of the induction step.

By (7.10) with \( s = k \) we have \( \kappa_{j_k} \leq 12\#\text{Crit}(f)_{0}/\ell \). This proves (7.8) if \( j_k = n - m \). In the case where \( j_k \leq n - m - 1 \), using that \( f^{n-m-j_k} \) maps \( W_{n-m} \) diffeomorphically to \( W_{j_k} \) and that we have \( \kappa_{j_k} \leq \varepsilon/2 \) by (7.4), by Lemma A.1 we have

\[
\kappa_{n-m} \leq 4\kappa_{j_k} \leq 12\#\text{Crit}(f)_{0}/\ell.
\]

This proves (7.8) and completes the proof of the induction step and of the proposition. \( \square \)

**Appendix A. General properties of smooth interval maps**

In this section we gather a few general facts of maps in \( \mathcal{A} \), that are used in the proof of Proposition 7.1.

The following version of the Koebe principle follows from [VSV04, Theorem C (2)(ii)]. A periodic point \( p \) of period \( n \) of a map \( f \) in \( \mathcal{A} \) is hyperbolic repelling if \( |Df^n(p)| > 1 \).

**Lemma A.1** (Koebe principle). Let \( f : I \to I \) be an interval map in \( \mathcal{A} \) all whose periodic points in \( J(f) \) are hyperbolic repelling. Then there is \( \delta_0 > 0 \) such that for every \( K > 1 \) there is \( \varepsilon \in (0, 1) \) such that the following property holds. Let \( J \) be an interval contained in \( I \) that intersects \( J(f) \) and satisfies \( |J| \leq \delta_0 \). Moreover, let \( n \geq 1 \) be an integer and \( W \) a diffeomorphic pull-back of \( J \) by \( f^n \). Then for every \( x \) and \( x' \) in the unique pull-back of \( \varepsilon J \) by \( f^n \) contained in \( W \) we have

\[
K^{-1} \leq |Df^n(x)|/|Df^n(x')| \leq K.
\]

**Lemma A.2.** Let \( f : I \to I \) be a multimodal map in \( \mathcal{A} \) having all of its periodic points in \( J(f) \) hyperbolic repelling and that is essentially topologically exact on \( J(f) \). Then, for every \( \kappa > 0 \) there is \( \delta_1 > 0 \) such that for every \( x \) in \( J(f) \), every integer \( n \geq 1 \), and every pull-back \( W \) of \( B(x, \delta_1) \) by \( f^n \), we have \( |W| \leq \kappa \).

This lemma is a direct consequence of part 2 of the following lemma.

**Lemma A.3.** Let \( f : I \to I \) be a multimodal map in \( \mathcal{A} \) having all of its periodic points in \( J(f) \) hyperbolic repelling. Then there is \( \delta_2 > 0 \) such that for every \( x \) in \( J(f) \) the following properties hold.

1. For every integer \( n \geq 1 \), every pull-back \( W \) of \( B(x, \delta_2) \) by \( f^n \) intersects \( J(f) \), contains at most 1 critical point of \( f \), and is disjoint from \((\text{Crit}(f) \cup \partial I) \setminus J(f)\).
2. If in addition \( f \) is essentially topologically exact on \( J(f) \), then

\[
\lim_{n \to +\infty} \max \{|W| : W \text{ connected component of } f^{-n}(B(x, \delta_2))\} = 0.
\]

In the proof of Lemma A.3 below we use the fact that every Fatou component is mapped to a periodic Fatou component under forward iteration, see [MieS92, Theorem A']. We also use the fact that each interval map in \( \mathcal{A} \) has at most a finite number of periodic Fatou components, see [MieS92, §1].

**Proof of Lemma A.3** The assertion in part 1 that \( W \) contains at most 1 critical point of \( f \) is a direct consequence of part 2. In part 1 below we prove the rest of
the assertions in part 1. In part 2 below we complete the proof of the proposition by proving part 2. Put
\[ \mathcal{S} := \{ \text{Crit}(f) \cup \partial I \} \setminus J(f). \]

1. Let \( V \) be a periodic Fatou component of \( f \) and let \( p \geq 1 \) be its period. Then each boundary point \( y \) of \( V \) in \( J(f) \) is such that \( f^p(y) \) is in \( \partial V \). This implies that \( f^p(y) \) is a periodic point of \( f \) in \( J(f) \) and our hypotheses imply that \( f^p(y) \) is hyperbolic repelling. It follows that there is a compact interval \( K_V \) contained in \( V \), such that for every integer \( n \geq 0 \) the set \( f^n(\mathcal{S}) \) is disjoint from \( V \setminus K_V \). Since \( f \) has at most a finite number of periodic Fatou components and since every Fatou component of \( f \) is mapped to a periodic Fatou component of \( f \) under forward iteration, it follows that there is \( \delta_* > 0 \) such that for every integer \( n \geq 1 \) the distance between \( f^n(\mathcal{S}) \) and \( J(f) \) is at least \( \delta_* \).

To prove part 1 with \( \delta_2 = \delta_* \), let \( x \) be a point in \( J(f) \), let \( n \geq 1 \) be an integer, and let \( W \) be a pull-back of \( B(x, \delta_*) \) by \( f^n \). By definition of \( \delta_* \), the set \( W \) is disjoint from \( \mathcal{S} \). Put \( W_0 := B(x, \delta_*) \) and for every \( j \in \{ 1, \ldots, n \} \) let \( W_j \) be the pull-back of \( W_0 \) by \( f^j \) that contains \( f^{j-1}(W) \). We show by induction in \( j \) that \( W_j \) intersects \( J(f) \). By definition \( W_0 \) intersects \( J(f) \). Let \( j \) be an integer in \( \{ 0, \ldots, n - 1 \} \) such that \( W_j \) intersects \( J(f) \) and suppose by contradiction that \( W_{j+1} \) does not intersect \( J(f) \). Since \( W_{j+1} \) is disjoint from \( \mathcal{S} \), it follows that \( f \) is injective on \( W_{j+1} \) and that \( f(W_{j+1}) = W_j \). Since by hypothesis \( J(f) \) is completely invariant, this implies that \( W_j \) is disjoint from \( J(f) \) as well. We obtain a contradiction that completes the proof of the induction hypothesis and of the fact that for each \( j \) in \( \{ 0, \ldots, n \} \) the set \( W_j \) intersects \( J(f) \). Since \( W_n = W \), this proves that \( W \) intersects \( J(f) \).

2. Let \( \delta_* \) be as in part 1. Our hypothesis that all the periodic points of \( f \) in \( J(f) \) are hyperbolic repelling implies that there are \( \delta_1 \) in \( (0, \delta_*) \) and \( \gamma \) in \( (0, 1) \) such that for every periodic point \( y \) in the boundary of a Fatou component the following property holds: For every integer \( n \geq 1 \) the pull-back \( W \) of \( B(f^n(y), \delta_1) \) by \( f^n \) that contains \( y \) satisfies \( |W| < \gamma^n \).

Since \( f \) is essentially topologically exact on \( J(f) \), there is an interval \( I_0 \) contained in \( J(f) \) that contains all critical points of \( f \) and such that the following properties hold: \( f(I_0) \subset I_0 \), \( f|_{I_0} \) is topologically exact on \( J(f|_{I_0}) \), and \( \bigcup_{n=0}^{+\infty} f^{-n}(I_0) \) contains an interval whose closure contains \( J(f) \). Notice in particular that \( J(f|_{I_0}) \) is not reduced to a point. Reducing \( \delta_1 \) if necessary we assume
\[ \delta_1 < \text{diam}(J(f|_{I_0})), \]
\[ \delta_1 < \min\{|V| : V \text{ periodic Fatou component of } f\}. \]

To prove part 2 with \( \delta_2 = \delta_1/2 \), we proceed by contradiction. If the desired assertion does not hold, then there is \( \kappa > 0 \), a sequence of positive integers \( (n_j)_{j=1}^{+\infty} \) such that \( n_j \to +\infty \) as \( j \to +\infty \), a sequence of points \( (x_j)_{j=1}^{+\infty} \) in \( J(f) \), and a sequence of intervals \( (J_j)_{j=1}^{+\infty} \) such that for every \( j \) we have
\[ |J_j| \geq \kappa, x_j \in f^{n_j}(J_j), \quad \text{and } f^{n_j}(J_j) \subset B(x_j, \delta_1/2). \]

Note that by part 1 for each \( j \geq 1 \) the interval \( J_j \) intersects \( J(f) \); denote by \( K_j \) the convex hull of \( J_j \cap J(f) \). In part 2.1 we prove that for every sufficiently large \( j \) we have \( |K_j| \geq \kappa/3 \) and in part 2.2 we use this fact to complete the proof of part 2.
2.1. If $J(f) = I$, then for each $j$ we have $K_j = J_j$ and therefore $|K_j| \geq \kappa$, so there is nothing to prove in this case. Assume $J(f) \neq I$ and let $P \geq 1$ be the largest period of a periodic Fatou component of $f$. Given a Fatou component $V$ of $f$, let $n(V)$ be the least integer $n \geq 0$ such that the Fatou component of $f$ containing $f^n(V)$ is periodic. Clearly, $|V| \to 0$ as $n(V) \to +\infty$. Let $N_0 \geq 1$ be such that for every Fatou component $V$ satisfying $n(V) \geq N_0$ we have $|V| \leq \kappa/3$. Let $\delta_1 > 0$ be sufficiently small so that for every $x$ in $J(f)$, every $n$ in $\{0, \ldots, N_0 + P\}$, and every pull-back $W$ of $B(x, \delta_1)$ by $f^n$, we have $|W| \leq \kappa/3$. Finally, let $N_1 \geq 1$ be such that $\gamma^N \leq \delta_1$.

We prove that for every $j$ such that $n_j \geq P + N_0 + N_1$ we have $|K_j| \geq \kappa/3$. It is enough to show that for every connected component $U$ of $J_j \setminus K_j$ we have $|U| \leq \kappa/3$. Let $V$ be the Fatou component of $f$ containing $U$. If $n(V) \geq N_0$, then $|U| \leq \kappa/3$. It remains to consider the case where $n(V) \leq N_0 - 1$. Then $f^{n(V)}(J_j)$ intersects a periodic Fatou component, so it must contain one of its boundary points. By the definition of $P$ this implies that $f^{n(V)+P}(J_j)$ contains a periodic point $y$ in the boundary of a Fatou component. Noting that by our choice of $j$ we have $n_j - P - n(V) \geq N_1$, by (A.3) we conclude that

$$f^{n_j-P-n(V)}(y) \in f^{n_j}(J_j) \subset B(x_j, \delta_1/2) \subset B(f^{n_j-P-n(V)}(y), \delta_1).$$

Using the definition of $\delta_1$, $\gamma$, and $N_1$, we obtain

$$|f^{P+n(V)}(J_j)| \leq \gamma^{n_j-P-n(V)} \leq \gamma^{N_1} \leq \delta_1.$$

Since $n(V) \leq N_0 - 1$, by definition of $\delta_1$ we have $|U| \leq |J_j| \leq \kappa/3$. This completes the proof that for every sufficiently large $j$ we have $|K_j| \geq \kappa/3$.

2.2. Taking subsequences if necessary we assume $(K_j)_{j=1}^{+\infty}$ converges to an interval $K$. We have $|K| \geq \kappa/3$ and $\partial K \subset J(f)$.

Suppose the interior of $K$ intersects $J(f)$. Then there is a compact interval $K'$ contained in the interior of $K$ and an integer $n \geq 1$ such that $f^n(K') \supset J(f|_{I_0})$. This implies that for every sufficiently large $j$ we have $f^{n_j}(K_j) \supset J(f|_{I_0})$ and therefore by (A.3) we have

$$|f^{n_j}(I_j)| \geq |f^{n_j}(K_j)| \geq \text{diam}(J(f|_{I_0})) > \delta_1.$$ 

We get a contradiction with (A.3).

It remains to consider the case where the interior of $K$ is contained in $F(f)$. Then the interior $V$ of $K$ is a Fatou component of $f$. Since for each $j$ we have $\partial K_j \subset J(f)$, for large $j$ the interval $K_j$ contains $V$. Taking $j$ larger if necessary we assume $n_j \geq n(V)$. If $V$ contains no turning point of $f^{n_j}$, then $f^{n_j}(V)$ is a periodic Fatou component of $f$ and therefore by (A.2) we have

$$|f^{n_j}(J_j)| \geq |f^{n_j}(K_j)| \geq |f^{n_j}(V)| > \delta_1.$$ 

We thus obtain a contradiction with (A.3). It follows that $V$ contains a turning point of $f^{n_j}$. Using that $K_j$ intersects $J(f)$, we conclude by the definition of $\delta_4$ that

$$|f^{n_j}(J_j)| \geq |f^{n_j}(K_j)| \geq \delta_4 > \delta_1.$$ 

We obtain a contradiction that completes the proof of part 2 and of the proposition. \qed
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JUAN RIVERA-LETELIER, FACULTAD DE MATEMÁTICAS, PONTIFICIA UNIVERSIDAD CATÓLICA DE CHILE, AVENIDA VICUÑA MACKENNA 4860, SANTIAGO, CHILE
E-mail address: riveraletelier@mat.puc.cl