Time fractional thermoelastic problem of a thick cylinder with non homogeneous material properties

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Abstract
In this article, we assume a twodimensional thermoelastic problem of nonhomogeneous thick hollow cylinder within the context of fractional order derivative of order $0 < \alpha \leq 2$. Convective heat exchange boundary conditions are applied at the curved surface, whereas the lower surface and the upper surface of the cylinder are considered at zero temperature. Furthermore, cylinder is subjected to a sectional heating at the outer curved surface of cylinder. Let the material properties of the cylinder except Poisson’s ratio and density are considered to be expresses by a simple power law in axial direction. The solution of the thermoelastic problem is obtained in terms of trigonometric and Bessel’s functions. Both the thermal and mechanical behavior is analyzed by the influence of inhomogeneity. Numerical computations are carried out for a mixture of copper and tin metals for both homogeneous and nonhomogeneous cases. Results of numerical solutions are illustrated graphically for temperature distribution and thermal stresses for all the different values of the fractional-order parameter $\alpha$ with the help of Mathematica software.

Key Words: Time fractional, nonhomogeneous thick cylinder, temperature distribution, thermal stresses, convective boundary, sectional heating.

Introduction
Ootao [2] and Ootao-Tanigawa [1, 3] studied transient thermoelasticity for a multilayered hollow cylinder and a hollow circular disk (functionally graded) by using piecewise power law Nonhomogeneity. Sugano [4-6] determined the expression of transient thermal stresses in a doubly connected non-homogeneous region. Kassir [7] investigated the thermoelastic problems for the semi-infinite nonhomogeneous solids. Hata [8] studied the thermal behaviour for a nonhomogeneous plate under steady state temperature distribution. In [9-10] Ghorbanpour et al. investigated stress distribution in carbon reinforced nano tube cylinder and discussed its thermopiezoelectric behaviour with functionally graded material properties. Awaji and Sivakumar [11] numerically discussed One-dimensional transient temperature thermoelastic problem for a functionally graded circular hollow cylinder materials. Kim and Noda [12] calculated the two dimensional unsteady state thermoelasticity problems by using Green’s function approach for an infinite hollow circular cylinder. Kamdi and Lamba [13] did thermoelastic analysis of functionally graded hollow cylinder which was subjected to uniform temperature field. Al-Hajri and Kalla [14] developed a new modified integral transform to investigate a mixed boundary value problem which involves combination of Bessel’s function as a kernel. Lamba et al. [15-17] studied some structural thermoelastic problems for cylinder with
nonhomogeneous material properties. Hosseini and Akhlaghi [18] obtained analytically the transient thermoelastic solution of functionally graded thick hollow cylinders. Ehteram [19] determined the solution for temperature and thermal stresses due to circumferential loading using integral transform technique.

Aksoy et al. [20] determined thermoelastic stresses due to action of thermo mechanical loading in a laminated isotropic materials cylinder. Fu et al. [21] studied thermoelastic behaviour in a solid cylinder subjected to circumferential loading. Jabbari et al. [22] presented the thermoelastic buckling effect for a solid circular plate made of porous material. Some most important contribution of this nonhomogeneous theory of thermoelasticity has been discussed in [23-30].

There were two major imperfections in classical uncoupled thermoelasticity theory which leads to its modification. First was consideration of heat conduction equation without presence of elastic term, because of which theory fails to explain the concept of heat generation due to elastic change. Second, classical Fourier heat conduction equation in parabolic form are observed to be no longer perfect in many thermoelastic studies which based on theoretical and experimental approach of transport phenomenon in media with internal structure like porous, polymers, dielectrics, semiconductors, amorphous etc. The first imperfection was eliminated with the introduction of elastic term in heat conduction equation this modified was due to Biot [31] and this modified theory is known a classical coupled theory of thermoelasticity. The second imperfection was eliminated by several researchers who carried several modifications and development in different span of time, these developed theory is known as generalized thermoelasticity. The generalized thermoelastic theory includes various major contributions like Lord-Shulman [34]; Green-Lindsay [35] and Green- Naghdi [36]. Also microscopic level is quite essential for different physical situations but this ignores during processing by the classical Fourier law. This encourages for the formulation of nonclassical theories, which implies to replace the parabolic heat conduction equation and the Fourier law by more general equations. Further each heat conduction generalization turn out in constitutions of generalized theory of thermoelasticity. Caputo [37, 38]; Caputo and Mainardi [39, 40] analyzed Vibrations with a dissipative memory using fractional derivative and the discussed analysis resembles with the empirical evidences for an infinite viscoelastic materials. Povstenko [42] investigated the fundamental solution of the generalized nonhomogeneous telegraph equations with the assumption of space–time-fractional derivative. Also he formulated the theory of corresponding thermal stresses are obtained it for the axisymmetric case. Ezzat and El-Karamany [43] applied fraction thermoelasticity to solve perfectly conducting medium problem. Povstenko [44] successfully discussed fractional and advection diffusion equation for different kinds of boundary conditions. Raslan [45] solved axisymmetric problem of temperature distribution in a thick plate by the application of fractional-order theory of thermoelasticity.

Povstenko [46] determined central symmetric problem of thermal stress in a sphere with the fractional derivative of Caputo type by using Laplace transform and finite sin-Fourier integral transform. Povstenko and Kyrylych [47] investigated Cauchy source problems with one spatial variable by using Laplace and Fourier transform. He also discussed generalization of advection diffusion equation with space-time-fractional derivative. Povstenko et al. [48] studied of axial symmetry space-time-fractional diffusion equation with fractional order derivative of Caputo type and solution was obtained by using integral transform method. Povstenko and Kyrylych [49] investigated diffusion equation in a half-line domain of time-fractional Caputo derivative with Dirichlet boundary condition by using Laplace and sin-Fourier transform with
respect to time and the spatial coordinate respectively. Povstenko [50] discussed diffusive theory of stresses with time-fractional Liouville–Caputo derivative. Various significant thermoelastic problems based on fractional order derivative was studied by many researchers [55–61].

Recently, Lamba and Khobragade [51-54] studied application of fractional order theory of thermoelasticity to hollow and solid cylindrical thermoelastic problems with different physical conditions on boundary and calculated deflection and corresponding thermal stresses.

From the literature survey it is observed that the maximum of research concern with temperature distribution with steady state in the case of nonhomogeneous cylinders having homogeneous material properties. But as a part of application within the context of fractional order theory of thermoelasticity with nonhomogeneous material properties are more practical and realistic in real life experiments. Presently due to complexity and tedious calculations in finding the analytical solution due to fractional order with nonhomogeneous structural materials is a big hurdle also heat production lead to various theoretical and technical problems during mechanical applications in which heat is generated and rapidly transferred from their surface in most of the solid cases. Hence by looking the practical applicability of the problem, the authors decided to conduct analysis of time fractional heat transfer on a thick cylinder with nonhomogeneous material properties and to examine its associated thermal stresses.

In this present study, we have assumed a two dimensional thermoelastic problem of nonhomogeneous thick hollow cylinder within the context of fractional order derivative theory. Further sectional heating is subjected at the outer curved surface of cylinder also the material properties of the cylinder except Poisson’s ratio and density are considered to be express by a simple power law in axial direction. Convective heat exchange boundary conditions are applied at the curved surface, whereas the lower surface and the upper surface of the cylinder are considered at zero temperature. The solution of the thermoelastic problem is obtained in terms of trigonometric and Bessel’s functions. Numerical computations are carried out for a mixture of copper and tin metals for both homogeneous and nonhomogeneous cases.

**Formulation of the problem**

We assume a two dimensional nonhomogeneous thick hollow cylinder of radius varying from $r = a$ to $r = b$ and thickness varying from $z = h_1$ to $z = h_2$. Mathematically, the space occupied by the cylinder is here by presented as $D = \{(x, y, z) \in \mathbb{R}^3 : h_1 \leq z \leq h_2 \text{ and } a_1 \leq (x^2 + y^2)^{1/2} \leq b_1\}$ where $r = (x^2 + y^2)^{1/2}$. The convective heat exchange boundary conditions are applied at the curved surface ($r = a$ and $r = b$), whereas the lower surface ($z = h_1$) and the upper surface ($z = h_2$) of the cylinder are considered at zero temperature. Furthermore cylinder is subjected to a sectional heating $Q(t)\delta (z-z_0)\delta (t)$ at the outer curved surface of cylinder.

The above said two dimensional nonhomogeneous thermoelastic problem of thick hollow cylinder is formulated as a mathematical model by assuming a nonlocal Caputo type time fractional heat conduction equation of order $\alpha$. Following Povstenko [33] the expression for Caputo type fractional derivative for a function $f(t)$ is given as

$$
\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{d^n f(\tau)}{d\tau^n} \, d\tau, \quad t > 0, \quad n-1 < \alpha < n,
$$

(1)
with the following Laplace transform rule, where Caputo derivative needs the initial values of the function \( f(t) \) and its corresponding integral derivatives of order \( k = 1, 2, 3, \ldots, n-1 \)

\[
L\left\{ \frac{d^\alpha f(t)}{dt^\alpha} \right\} = s^\alpha L\{f(s)\} - \sum_{k=0}^{n-1} f^{(k)}(0^+) s^{\alpha-k}, \quad n-1 < \alpha < n.
\]

in which \( s \) is the transform parameter and \( n \) is a positive integer.

### Temperature distribution

The governing transient heat conduction equation of a non-homogeneous thick hollow cylinder satisfies the following differential equation in the context of fractional order theory with convective heat exchange boundary conditions as

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( \overline{\alpha}(z) r \frac{\partial T}{\partial r} \right) + \frac{\partial}{\partial z} \left( \overline{\alpha}(z) \frac{\partial T}{\partial z} \right) = c(z) \rho \frac{\partial^\alpha T}{\partial t^\alpha}, \quad a \leq r < b, \quad h_1 \leq z \leq h_2
\]

With the boundary conditions,

\[
\left[ e_1 T - k_1 \frac{\partial T}{\partial r} \right]_{r=a} = 0, \quad h_1 \leq z \leq h_2, \quad t > 0
\]

\[
\left[ e_2 T + k_2 \frac{\partial T}{\partial r} \right]_{r=b} = Q_1 \delta(z-z_0)\delta(t), \quad h_1 \leq z \leq h_2, \quad t > 0
\]

\[
\left[ T \right]_{z=h_1} = 0, \quad a \leq r < b, \quad t > 0
\]

\[
\left[ T \right]_{z=h_2} = 0, \quad a \leq r < b, \quad t > 0
\]

with initial conditions are,

\[
T = Q_0 \delta(r-r_0)\delta(z-z_0), \quad \text{at} \quad t = 0, \quad 0 < \alpha \leq 2
\]

\[
\frac{\partial T}{\partial t} = 0, \quad \text{at} \quad t = 0, \quad 1 < \alpha \leq 2
\]

Where \( \overline{\alpha}(z) \) represents the thermal conductivity, \( \rho \) refers for the constant density and \( c(z) \) denotes the calorific capacity of the material for the inhomogeneous region. Here \( T(r,z,t) \) denotes the temperature function of the hollow cylinder at any time \( t \). Also \( \delta(\cdot) \) denotes the Dirac Delta function having \( a \leq r_0 \leq b \) and \( h_1 \leq z_0 \leq h_2 \). Here \( e_1, e_2 \) denote the external heat transfer coefficients & \( k_1, k_2 \) denote the thermal conductivity coefficients.

### Displacements and thermal stresses

By Following Hata [8], for a axisymmetric problem of temperature distribution throughout the thick hollow in coordinated \((r, \theta, z)\) cylinder the relation between Stress-strain can be expressed as

\[
\sigma_{rr} = 2\mu(z) \epsilon_{rr} + \lambda(z) \epsilon - \left( 3\lambda(z) + 2\mu(z) \right) \alpha_T(z) T
\]

\[
\sigma_{\theta \theta} = 2\mu(z) \epsilon_{\theta \theta} + \lambda(z) \epsilon - \left( 3\lambda(z) + 2\mu(z) \right) \alpha_T(z) T
\]
\[\sigma_{zz} = 2\mu(z)e_{zz} + \lambda(z)e - (3\lambda(z) + 2\mu(z))\alpha_\tau(z)T\quad (12)\]
\[\sigma_{rz} = 2\mu(z)e_{rz}\quad (13)\]

Where, \(e_{rr}, e_{\theta\theta}, e_{zz}\) denotes components of strain \((e = e_{rr} + e_{\theta\theta} + e_{zz})\), \(\lambda(z)\) and \(\mu(z)\) represents the Lame constants and \(\alpha_\tau(z)\) refers the coefficient of thermal expansion.

Condition for equilibrium in cylindrical coordinates are written as
\[
\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \quad (14)
\]
\[
\frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} = 0 \quad (15)
\]

Relation between Strain-displacement is
\[
e_{rr} = \frac{\partial u}{\partial r}, \quad e_{\theta\theta} = \frac{u}{r}, \quad e_{zz} = \frac{\partial w}{\partial z}, \quad e_{rz} = \frac{1}{2} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) \quad (16)
\]

Further using assumption made by Hata [8] in consideration of nonhomogeneous media that the shear modulus \(\mu(z)\) and coefficient of thermal expansion \(\alpha_\tau(z)\) are assumed vary in the axial direction given as following (provided that \(\rho \nu = 1 - 2\nu\) exist)
\[
\mu(z) = \mu_0 z^n, \quad \alpha_\tau(z) = \alpha_0 z^n
\]

In above equation \(\mu_0\) and \(\alpha_0\) are reference values of shear modulus and coefficient of thermal expansion also \(n \geq 0\).

On using equation (10)-(13) and (16) into (14)-(15), the equilibrium equation for displacement are obtained as following Lamba [16]
\[
\Delta^2 u - \frac{u}{r^2} + \frac{\nu}{1-2\nu} \frac{\partial e}{\partial r} + \frac{e_{rz}}{\mu(z)} \frac{\partial \mu(z)}{\partial r} + 2 \frac{\partial e_{rz}}{\partial z} - \frac{1 + \nu}{1 - 2\nu} \alpha_\tau(z) \frac{\partial T}{\partial r} = 0, \quad (17)
\]
\[
\Delta^2 w + \frac{\nu}{1-2\nu} \frac{\partial e}{\partial z} + \frac{e_{rr}}{\mu(z)} \frac{\partial \mu(z)}{\partial r} + \frac{e}{2\mu(z)} \frac{\partial \lambda(z)}{\partial z} - \frac{1 + \nu}{1 - 2\nu} \left[ \alpha_\tau(z) \frac{\partial T}{\partial z} + \theta \frac{\partial \alpha_\tau(z)}{\partial z} \right] - \frac{\alpha_\tau(z)}{2\mu(z)} \left[ 3 \frac{\partial \lambda(z)}{\partial z} + 2 \frac{\partial \mu(z)}{\partial z} \right] + \frac{1}{r} \frac{\partial u}{\partial z} = 0 \quad (18)
\]

Where \(\nabla^2\) is given by
\[
\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \quad (19)
\]

In the cylindrical coordinate system, the displacement functions are represented by the Goodier's thermoelastic displacement potential \(\phi\) and the Boussinesq harmonic functions \(\varphi\) and \(\psi\) as [15]
\[
u = \frac{\partial \phi}{\partial r} + \frac{\partial \psi}{\partial r} + z \frac{\partial \psi}{\partial z} \quad (20)
\]
\[
w = \frac{\partial \varphi}{\partial z} + \frac{\partial \psi}{\partial z} + z \frac{\partial \psi}{\partial z} - (3 - 4\nu) \psi \quad (21)
\]
In which Goodier’s thermoelastic displacement potential $\phi$ and the Boussinesq harmonic functions $\varphi$ and $\psi$ must satisfy the following equations

$$\nabla^2 \phi = K(z) \tau, \quad \nabla^2 \varphi = 0, \quad \text{and} \quad \nabla^2 \psi = 0$$  \hfill (22)

Where $K(z)$ denotes the restraint coefficient and is defined as $K(z) = \frac{(1 + \nu)}{(1 - \nu)} \alpha_r(z)$ and $\tau$ represents the temperature difference between system and surrounding medium written as $\tau = T - T_i$, here $T_i$ denotes the surrounding temperature.

For sake of brevity, we assume

$$- \int (\varphi + z \psi) dz = M$$  \hfill (23)

On using (23) in equation (20) and (21), we get Boussinesq harmonic functions $\varphi$ and $\psi$ are replaced by Michell’s function $M$ as defined below

$$u = \frac{\partial \varphi}{\partial \tau} - \frac{\partial M}{\partial r} \frac{\partial}{\partial \tau} \frac{\partial}{\partial z} \hfill (24)$$

$$w = \frac{\partial \varphi}{\partial \tau} + 2(1 - \nu) \nabla^2 M - \frac{\partial^2 M}{\partial z^2} \hfill (25)$$

Where Michell’s function must satisfy the condition given below

$$\nabla^2 \nabla^2 M = 0$$  \hfill (26)

Now substituting using equation (24) and (25) in (10)-(13) and (17) to (18), the following thermoelastic fields are obtained as

$$\begin{align*}
\Delta^2 \phi - \frac{\partial}{\partial z} \left( \nabla^2 M \right) &- \frac{1}{r^3} \left( \phi - \frac{\partial M}{\partial z} \right) - \frac{(1 + \nu)}{(1 - 2\nu)} \alpha_r(z) T = 0, \\
\Delta^2 \phi + 2(1 - \nu) \left( \nabla^2 M + \frac{\partial}{\partial r} \left( \nabla^2 M \right) + \frac{\partial}{\partial z} \left( \nabla^2 M \right) \right) + 2(1 - \nu) \left[ \frac{2}{r^3} \frac{\partial M}{\partial r} - \frac{2}{r^2} \frac{\partial^2 M}{\partial r^2} \right] \\
- \frac{(1 + \nu)}{(1 - 2\nu)} \alpha_r(z) T - \left( 3\lambda + 2\mu \right) \frac{\alpha_r(z)}{2\mu(z)} T + \frac{1}{r} \left[ \frac{\partial \phi}{\partial r} - \frac{\partial^2 M}{\partial r \partial z} \right] &= 0.
\end{align*}$$  \hfill (27) (28)

The corresponding radial, tangential, axial and shear stresses are expressed by the use of the Goodier’s thermoelastic displacement potential $\phi$ and Michell’s function $M$ as

$$\begin{align*}
\sigma_{rr} &= 2\mu(z) \left[ \frac{\partial \phi}{\partial r} - \frac{\partial^2 M}{\partial r \partial z} \right] + \lambda(z) \left[ \nabla^2 \phi + (1 - 2\nu) \frac{\partial}{\partial z} \left( \nabla^2 M \right) \right] - \left( 3\lambda(z) + 2\mu(z) \right) \alpha_r(z) T \\
\sigma_{\theta \theta} &= 2\mu(z) \left[ \frac{\partial \phi}{\partial r} - \frac{\partial^2 M}{\partial r \partial z} \right] + \lambda(z) \left[ \nabla^2 \phi + (1 - 2\nu) \frac{\partial}{\partial z} \left( \nabla^2 M \right) \right] - \left( 3\lambda(z) + 2\mu(z) \right) \alpha_r(z) T \\
\sigma_{zz} &= 2\mu(z) \left[ \frac{\partial \phi}{\partial z} + 2(1 - \nu) \nabla^2 M - \frac{\partial^2 M}{\partial z^2} \right] + \lambda(z) \left[ \nabla^2 \phi + (1 - 2\nu) \frac{\partial}{\partial z} \left( \nabla^2 M \right) \right] \\
- \left( 3\lambda(z) + 2\mu(z) \right) \alpha_r(z) T
\end{align*}$$  \hfill (29) (30)
\[
\sigma_{rz} = \mu(z) \left[ \frac{\partial}{\partial z} \left( \frac{\partial \phi}{\partial r} - \frac{\partial^2 M}{\partial r \partial z} \right) + \frac{\partial}{\partial r} \left( \frac{\partial \phi}{\partial z} + (1 - 2\nu) \nabla^2 M - \frac{\partial^2 M}{\partial z^2} \right) \right]
\]  
(31)

For the traction free surface boundary conditions for stress functions are as
\[
\begin{align*}
\sigma_{rr} &= 0 \quad \text{at} \quad r = a \\
\sigma_{rr} &= 0 \quad \text{at} \quad r = b
\end{align*}
\]  
(33)

Above equations (3) to (33) represents the mathematical modeling of the time fractional thermoelastic problem of nonhomogeneous hollow cylinder.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{geometry.png}
\caption{Geometry of nonhomogeneous hollow cylinder}
\end{figure}

Solution of the problem
Solution of the heat conduction problem
From equation (3) with the below considering (for sake of brevity as)
\[
\bar{\lambda}(z) = \bar{\lambda}_0 z^\beta, \quad c(z) = c_0 z^\beta, \quad \rho = \rho_0
\]  
(34)

We get
\[
\left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right) + \left( \frac{\partial^2 T}{\partial z^2} + \frac{\beta}{z} \frac{\partial T}{\partial z} \right) = \frac{1}{\kappa} \frac{\partial^\alpha T}{\partial t^\alpha}, \quad a \leq r \leq b, \quad h_1 \leq z \leq h_2
\]  
(35)

Where
\[
\kappa = \left( \frac{\bar{\lambda}_0}{c_0 \rho_0} \right)
\]
with the corresponding boundary and initial conditions as defined in equations (4) to (9).
Here the reference value of thermal conductivity is denoted as $\bar{\kappa}$, calorific capacity as $c_0$ and density by $\rho_0$ respectively.

Now, let using the variable transformation as $T = z^{((1-\beta)/2)} \theta(r,z,t)$ in order to remove $\beta$ from the numerator of equation (35).

Hence, equation (35) becomes

$$\left( \frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \frac{\partial \theta}{\partial r} \right) + \left( \frac{\partial^2 \theta}{\partial z^2} + \frac{1}{z} \frac{\partial \theta}{\partial z} + \frac{\gamma^2}{z^2} \theta \right) = \frac{1}{\kappa} \frac{\partial^\alpha \theta}{\partial t^\alpha}$$

(36)

Where $\gamma^2 = (\beta - 1)/2$

with corresponding boundary and initial conditions (following equations (4) to (9)) given as

$$\left( e_1 \theta - k_1 \frac{\partial \theta}{\partial r} \right)_{r=a} = 0, \quad h_1 \leq z \leq h_2, \quad t > 0$$

(37)

$$\left( e_2 \theta + k_2 \frac{\partial \theta}{\partial r} \right)_{r=b} = Q_1 z^{((\beta-1)/2)} \delta(z-z_0) \delta(t), \quad h_1 \leq z \leq h_2, \quad t > 0$$

(38)

$$\theta \bigg|_{z=h_1} = 0, \quad a \leq r \leq b, \quad t > 0$$

(39)

$$\theta \bigg|_{z=h_2} = 0, \quad a \leq r \leq b, \quad t > 0$$

(40)

$$\theta = Q_0 z^{((\beta-1)/2)} \delta(r-r_0) \delta(z-z_0), \quad \text{at } t = 0, \quad 0 < \alpha \leq 2$$

(41)

$$\frac{\partial \theta}{\partial t} = 0, \quad \text{at } t = 0, \quad 1 < \alpha \leq 2$$

(42)

To find the expression for $\theta$ from the differential equation (36) we follow the the extended integral transform defined by Al-Hajri and Kalla [14] of order $i$ over the variable $z$ as

$$T[f(z), a, b; \gamma_i] = \bar{f}(\gamma_i) = \int_z^{h_2} f(z) S(\gamma_i, z) \, dz$$

(43)

where $S(\gamma, z)$ represent the kernel of the integral transform given as

$$S(\gamma, z) = Z_i \cos(\gamma_i \log z) - W_i \sin(\gamma_i \log z), \quad z > 0 \quad \text{and} \quad \gamma_i (i = 1, 2, 3, \ldots)$$

(44)

$$Z_i = \sin(\gamma_i \log h_1) + \sin(\gamma_i \log h_2), \quad W_i = \cos(\gamma_i \log h_1) + \cos(\gamma_i \log h_2),$$

and $\gamma_i$ denotes the real and positive roots of the transcendental equation

$$\sin(\gamma \log h_1) \cos(\gamma \log h_2) - \sin(\gamma \log h_2) \cos(\gamma \log h_1) = 0$$

(45)

The inversion transformation is given as

$$f(z) = \sum_{i=1}^{\infty} \frac{\bar{f}(\gamma_i)}{S(\gamma_i)} S(\gamma_i, z),$$

(46)

Where
\[ h_i \int z S(\gamma_i z) S(\gamma_j z) dz = \begin{cases} S(\gamma_i), & i = j, \\ 0, & i \neq j. \end{cases} \quad (47) \]

Hence, equations (36) to (42) become
\[
\left( \frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \frac{\partial \theta}{\partial r} \right) - \gamma_i^2 \theta = \frac{1}{\kappa} \frac{\partial^\alpha \theta}{\partial t^\alpha} \quad (48) \\
\left( e_1 \frac{\partial \theta}{\partial r} - k_1 \frac{\partial \theta}{\partial r} \right)_{r=a} = 0, \quad h_1 \leq z \leq h_2, \quad t > 0 \quad (49) \\
\left( e_2 \frac{\partial \theta}{\partial r} + k_2 \frac{\partial \theta}{\partial r} \right)_{r=b} = Q_0 g_0 \delta(t), \quad h_1 \leq z \leq h_2, \quad t > 0 \quad (50)
\]

\[ \bar{\theta} = Q_0 g_0 \delta(r-r_0), \quad \text{at} \quad t = 0, \quad 0 < \alpha \leq 2 \quad (51) \]
\[ \frac{\partial \bar{\theta}}{\partial t} = 0, \quad \text{at} \quad t = 0, \quad 1 < \alpha \leq 2 \quad (52) \]

Where
\[ g_0 = \int_{h_1}^{h_2} z^{(\mu+1)/2} \delta(z-z_0) S(\gamma_i z) dz \]

Now on using the integral transform defined by Al-Hajri and Kalla [14], to the equations (48) and to the transformed boundary conditions (49) to (52), we get
\[ \frac{\partial^\alpha \bar{\theta}}{\partial t^\alpha} + A_1 \bar{\theta} = A_2 \delta(t), \quad (53) \]
\[ \bar{\theta} = Q_0 g_0 r_0 M(q_n r_0), \quad \text{at} \quad t = 0, \quad 0 < \alpha \leq 2 \quad (54) \]
\[ \frac{\partial \bar{\theta}}{\partial t} = 0, \quad \text{at} \quad t = 0, \quad 1 < \alpha \leq 2 \quad (55) \]

Where
\[ A_1 = \kappa(q_n^2 + \gamma_i^2), \quad A_2 = \frac{\kappa h_i}{k_2} M(q_n h_i) Q_0 g_0 \]

Here, \( M(q_n r) \) denotes the kernel of the transform as
\[ M(q_n r) = [B(q_n a, e_1, k_1) + B(q_n b, e_2, k_2)] J_0(q_n r) - [A(q_n a, e_1, k_1) + A(q_n b, e_2, k_2)] J_0(q_n r) \]

Where
\[ A(q_n r, e_n, k_n) = e_n J_0(q_n r) + k_n q_n J_0(q_n r), \quad n = 1, 2; \quad r = a, b \]
\[ B(q_n r, e_n, k_n) = e_n Y_0(q_n r) + k_n q_n Y_0(q_n r), \quad n = 1, 2; \quad r = a, b \]
Here \( J_0 \) is Bessel’s function of first kind and \( Y_0 \) is of second kind and \( q_n \) are the positive roots of the transcendental equation:

\[
[B(q_n a, e_1, k_1) \times A(q_n b, e_1, k_1) - A(q_n a, e_1, k_1) \times B(q_n b, e_2, k_2)] = 0
\]

On using Laplace transform and its inversion to equation (53) by using the initial condition (54) and (55), we get

\[
\bar{\theta}(n, t) = Q_0 g_0 r_0 M(q_n r_0) E_{\alpha,1}(-A_1 t^\alpha) + A_2 t^{\alpha-1} E_{\alpha,\alpha}(-A_1 t^\alpha)
\]

(56)

On taking the inverse transform of the equation (56), we get

\[
\theta(r, t) = \sum_{n=1}^{\infty} Q_0 g_0 r_0 M(q_n r_0) E_{\alpha,1}(-A_1 t^\alpha) + A_2 t^{\alpha-1} E_{\alpha,\alpha}(-A_1 t^\alpha) \times M(q_n r)
\]

(57)

Now, using the inverse transform defined in equation (46) to the equation (57), we get

\[
\theta(r, z, t) = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \frac{\bar{\theta}(r, t)}{S(y_i)} \times S(y_i z), \ z > 0
\]

(58)

Following the transformation \( T = z^{[(1-\beta)/2]} \theta(r, z, t) \), we get

\[
T(r, z, t) = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \left[ \frac{Q_0 g_0 r_0 M(q_n r_0) E_{\alpha,1}(-A_1 t^\alpha) + A_2 t^{\alpha-1} E_{\alpha,\alpha}(-A_1 t^\alpha)}{M(q_n) \times S(y_i)} \right] \times \left[ [B(q_n h_1, e_1, k_1) + B(q_n h_2, e_2, k_2)] J_0(q_n r) - [A(q_n h_1, e_1, k_1) + A(q_n h_2, e_2, k_2)] Y_0(q_n r) \right] \times z^{[(1-\beta)/2]} S(y_i z)
\]

(59)

**Thermoelastic equations**

The expression of Goodier’s thermoelastic displacement potential governed by equation (22) is obtained by referring to the heat conduction equation (35) and its solution given by equation (59) as

\[
\phi = \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \left[ K(z) \left( \frac{Q_0 g_0 r_0 M(q_n r_0) E_{\alpha,1}(-A_1 t^\alpha) + A_2 t^{\alpha-1} E_{\alpha,\alpha}(-A_1 t^\alpha)}{M(q_n) \times S(y_i)} \right) \times \left[ [B(q_n h_1, e_1, k_1) + B(q_n h_2, e_2, k_2)] J_0(q_n r) - [A(q_n h_1, e_1, k_1) + A(q_n h_2, e_2, k_2)] Y_0(q_n r) \right] \times g_i^2(z) \right]
\]

\[
- \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \left[ K(z) T_i(r, z) \left( \frac{M(q_n) \times S(y_i)}{Q_0 g_0 r_0 M(q_n r_0) E_{\alpha,1}(-A_1 t^\alpha) + A_2 t^{\alpha-1} E_{\alpha,\alpha}(-A_1 t^\alpha)} \right) \times \left[ [B(q_n h_1, e_1, k_1) + B(q_n h_2, e_2, k_2)] J_0(q_n r) - [A(q_n h_1, e_1, k_1) + A(q_n h_2, e_2, k_2)] Y_0(q_n r) \right] \times \frac{1}{g_i^2(z)} \right]
\]
The expression of components of stresses can be obtained by using the displacement components obtained as

\[ u = \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \left( \frac{\partial\phi}{\partial r} - q_n z \frac{(\beta+\beta_2)}{2} \left( \frac{1+\beta}{2} \cos(\log z) + \frac{3-\beta}{2} \sin(\log z) \right) \omega a t^{n-\alpha} E_{1,n+1-\alpha+1} \left( \omega t \right) \right) \times \left[ C_m J_1 (q_n r) + D_m Y_1 (q_n r) \right] \]

\[ w = \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \left( \frac{\partial\phi}{\partial z} + (2\nu - 2) q_n z \exp(\omega t) \left[ C_m J_0 (q_n r) + D_m Y_0 (q_n r) \right] \right) \]

\[ + \frac{1-2\nu}{4} \left( \frac{\beta+\beta_2}{2} \left[ (\beta+5)(\beta-1)\cos(\log z) - (\beta+1)(\beta-5)\sin(\log z) \right] \omega a t^{n-\alpha} E_{1,n+1-\alpha+1} \left( \omega t \right) \right) \times \left[ C_m J_0 (q_n r) + D_m Y_0 (q_n r) \right] \]

The expression of components of stresses can be obtained by using the displacement components given by equations (62) and (63) in equation (29) to (32). Also the values of constants \( C_m \) and \( D_m \) can be determined by using the traction free boundary conditions given by equation (33). We have not mentioned the large mathematical equations of stresses and constants, However numerical computations are carried out by using Mathematica software.

**Numerical calculations**

Mixtures of Copper and Tin metals assumed for numerical computations in the ratio 70:30 respectively, with non-dimensional variables are as \([8]\) given below:

\[ T^* = \frac{T}{\Theta_R}, \quad r^* = \frac{r}{a}, \quad z^* = \frac{z - h_t}{a}, \quad \kappa^* = \frac{\kappa t}{a^2}, \quad h^* = \frac{h_t}{a}, \quad u^* = \frac{u}{\alpha_0 \Theta_R a}, \quad w^* = \frac{w}{\alpha_0 \Theta_R a}, \]

\[ \sigma_{rr}^* = \frac{\sigma_{rr}}{\alpha_0 E \Theta_R}, \quad \sigma_{\theta\theta}^* = \frac{\sigma_{\theta\theta}}{\alpha_0 E \Theta_R}, \quad \sigma_{zz}^* = \frac{\sigma_{zz}}{\alpha_0 E \Theta_R}, \quad \sigma_{rz}^* = \frac{\sigma_{rz}}{\alpha_0 E \Theta_R}, \]

The constants used during the numerical calculation are given as:
Inner radius of a cylinder \( a = 1 cm \), Outer radius of a cylinder \( b = 2 cm \), \( t_0 = 2 \text{ sec} \), Thickness of cylinder \( h_1 = 2 cm \), Thickness of cylinder \( h_2 = 5 cm \), \( r_0 = 1.5 cm \) and Reference temperature \( T_R = 32^\circ C \)

For the forgoing analysis mathematical simplicities are done by setting the radiation coefficients constants as \( k_1 = 0.86, k_2 = 0.86 \), and the convective heat transfer coefficients \( e_1 = 1, e_2 = 1 \)

The other associated values are taken as:

Thermal diffusivity \( \kappa = 1.11 cm^2/s^\alpha \), Coefficient of linear thermal expansion \( \alpha_0 = 17 \times 10^{-6}/^\circ C \), Young’s modulus \( E = 4.41 \times 10^7 N/cm^2 \), the relation between the parameter \( p \), the Poisson’s ratio \( \nu \) and Shear modulus \( \mu_0 \) is \( p = \frac{1-2\nu}{\nu} \) and \( \mu_0 = \frac{E}{2(1+\nu)} \)

**Case 1:** Homogeneous Case: \( p = 0, \nu = 0.5 \) and \( \mu_0 = 1.47 \times 10^7 N/cm^2 \)

**Case 2:** Nonhomogeneous Case: \( p = 1.5, \nu = 0.286 \) and

**Numerical Analysis of Solution**

The obtained mathematical results of temperature distribution, radial stress distribution, tangential stress distribution, axial stress and shear stress distribution for fractional-order parameter \( \alpha = 0.5, \alpha = 1, \alpha = 1.5, \alpha = 2 \) (depicting weak, normal and strong conductivity) are computed numerically by MATHEMATICA software for the finite hollow cylinder.
Fig. 2 shows the distribution of dimensionless temperature \( T^* \) along the radial direction for different values of the fractional-order parameter \( \alpha \) in the thick hollow cylinder. The above distribution is plotted for different values of dimensionless thickness \( z^* = 0.5, z^* = 1.5 \) and \( z^* = 2.5 \) for both the homogeneous and non-homogeneous case. Due to the effect of surrounding temperature, a finite value of temperature occurs at the inner radius \( r^* = 1 \) and outer curved radii \( r^* = 2 \). Maximum distribution of temperature exists at the central region \( r^* = 1.5 \) which is because of the thermal deformation due to thermal energy accumulation at the centre of cylinder caused by applied sectional heating at the curved surface. Further, temperature gradually increases towards the middle region and slightly decreases towards the inner and outer radii. Also, absolute temperature value is high at the middle portion whereas less at the lower and upper surfaces. From the graphical view, it can be concluded that magnitude of temperature is found low for homogeneous case as compared to non-homogeneous case. It is also observed that temperature distribution is found more for large value of dimensionless thickness parameter \( z^* \) in both the homogeneous and non-homogeneous case also dominating behaviour of variation is noted in homogeneous case. Furthermore, it is analyzed that the speed of thermal signals propagation is varying directly proportional to the values of the different values of fractional-order parameter \( \alpha \).

| Homogeneous case | Inhomogeneous case |
|------------------|--------------------|

Fig. 2: Temperature Distribution
Fig. 3: Displacement Distribution

Fig. 3 shows the dimensionless displacement distribution $w^*$ in dimensionless radial direction $r^*$, the above distribution is plotted for different values of dimensionless thickness $z^* = 0.5$, $z^* = 1.5$ and $z^* = 2.5$ for both the homogeneous and non homogeneous case. For all the values of fractional order parameter $\alpha = 0.5$, $\alpha = 1$, $\alpha = 1.5$, $\alpha = 2$ displacement goes on increasing with increase in radius and attains peak value at the mid after mid it becomes sinusoidal in nature for homogeneous case but decreases finitely in nonhomogeneous case and a nonuniform pattern is observed. Also with increase in value of dimensionless thickness the displacement distribution acquires dominating characteristic.

| Homogeneous case | Inhomogeneous case |
Fig. 4: Radial Stress Distribution

Fig. 4 shows the dimensionless radial stress distribution \( \left( \sigma_{rr}^* \right) \) along the dimensionless radial direction \( r^* \) for different values of the fractional-order parameter \( \alpha \) in the thick hollow cylinder. The above distribution is plotted for different values of dimensionless thickness \( z^* = 0.5 \), \( z^* = 1.5 \) and \( z^* = 2.5 \) for both the homogeneous and non homogeneous case. It is seen that for all the values of fractional order parameter \( \alpha = 0.5, \alpha = 1, \alpha = 1.5, \alpha = 2 \) radial stress behaves sinusoidal in nature with increase in radius and shows nonuniform pattern. Peak value of the radial stress is observed at the centre for both the cases. Initially small scale stress distribution occurs but it goes on fluctuating behaviour noted towards the outer radii. Also for
both the cases the radial stress distribution is zero at both the radial ends \( (r^* = 1 \ & \ r^* = 2) \), which satisfies and agrees with the traction free boundary conditions defined in Eqn. (33). It is also observed that stress distribution is found more for large value of dimensionless thickness parameter \( z^* \) in both the homogeneous and non homogeneous case also dominating behaviour of variation is noted in nonhomogeneous case.

---

**Homogeneous case**

| \( z^* = 0.5 \) | \( z^* = 1.5 \) | \( z^* = 2.5 \) |
|-------------------|-------------------|-------------------|
| ![Graph](image)   | ![Graph](image)   | ![Graph](image)   |

**Inhomogeneous case**

| \( z^* = 0.5 \) | \( z^* = 1.5 \) | \( z^* = 2.5 \) |
|-------------------|-------------------|-------------------|
| ![Graph](image)   | ![Graph](image)   | ![Graph](image)   |
Fig. 5: Tangential Stress Distribution

Fig. 5 shows the dimensionless tangential stress distribution \( \sigma_{\theta^*} \) along the dimensionless radial direction \( r^* \) for different values of the fractional-order parameter \( \alpha \) in the thick hollow cylinder. It is seen that for all the values of fractional order parameter \( \alpha = 0.5, \alpha = 1, \alpha = 1.5, \alpha = 2 \) tangential stress gradually decreases in the range \( 1 \leq r^* \leq 1.5 \) and the increases towards the outer radii. Peak value of the tangential stress is observed near to the centre of the cylinder for both the cases. It is seen that tangential stress is tensile in nature the range \( 2.11 \leq r^* \leq 28.1 \) and \( 28.1 \leq r^* \leq 2 \), while behaves compressive in the remaining range. Further it is noted that, the magnitude of stress found high at the curved boundary surface and is slowly decreasing towards the inner radii. For both the homogeneous and non homogeneous cases the magnitude of tangential stress is high at both the lower and upper surfaces, while found low for in the middle portion. But a significant effect of fractional parameter is observed for different dimensionless value of thickness.

### Homogeneous case

| \( \alpha \) | \( r^* \) |
|---|---|
| 0.5 | 0.5 |
| 1 | 0.5 |
| 1.5 | 0.5 |
| 2 | 0.5 |

### Inhomogeneous case

| \( \alpha \) | \( r^* \) |
|---|---|
| 0.5 | 0.5 |
| 1 | 0.5 |
| 1.5 | 0.5 |
| 2 | 0.5 |
Fig. 6: Axial Stress Distribution

Fig. 6 shows the variation of radial stress distribution \( \sigma_{zz}^* \) along the dimensionless radial direction \( r^* \) for different values of the fractional-order parameter \( \alpha \) in the thick hollow cylinder. The above distribution is plotted for different values of dimensionless thickness \( z^* = 0.5 \), \( z^* = 1.5 \) and \( z^* = 2.5 \) for both the homogeneous and non homogeneous case. It is observed that for all the values of fractional order parameter \( \alpha = 0.5, \alpha = 1, \alpha = 1.5, \alpha = 2 \) tensile behaviour of the axial stress are seen after the mid towards outer curved surface, whereas compressive behaviour is noted in the remaining first half range from inner radii. Also it converges to zero at the centre of cylinder. Further due to thermal energy accumulation at inner and outer radii, the magnitude of crest and trough is found high near \( r^* = 1 \) & \( r^* = 2 \). But low magnitude is noted near the middle region. The magnitude of axial stress is decreasing from the upper towards the lower surface in the range \( 1.21 \leq r^* \leq 1.49 \), while increasing in the region \( 1.51 \leq r^* \leq 1.79 \).
Fig. 7 shows the variation of radial stress distribution \( \sigma_{r\zeta} \) along the dimensionless radial direction \( r^* \) for different values of the fractional-order parameter \( \alpha \) in the thick hollow cylinder. It is observed that for all the values of fractional order parameter \( \alpha \), the shear stress goes on decreasing with increase in radius till middle in the range \( 1 \leq r^* \leq 1.49 \) attains extremum at \( r^* = 1.5 \) then after stress goes on increasing in the region \( 1.5 \leq r^* \leq 1.7 \) of cylinder and after it sinusoidal behaviour is observed towards outer curved region. It is noted that magnitude of shear stress increasing from the upper towards the lower surface in the range \( 1.2 \leq r^* \leq 1.7 \). Hence graphical view indicates that the overall magnitude is high in homogeneous case as compared to the non homogeneous case.
Fig. 8: Variation of dimensionless temperature for different inhomogeneity parameter $p$

Fig. 8 shows the dimensionless temperature distribution $T^*$ along the dimensionless radial direction $r^*$ for different values of the fractional-order parameter $\alpha$ in the thick hollow cylinder by fixing the different inhomogeneity parameter $p$. It is seen that the magnitude of temperature is decreases with increasing in the inhomogeneity parameter. It is also analyzed that the speed of thermal signals propagation is varying inversely proportional to the values of the different values of fractional-order parameter $\alpha$. 

$p = 3$

$p = 1.5$

$p = 2.5$
Fig. 9: Variation of dimensionless displacement for different inhomogeneity parameter $p$

Fig. 9 shows the dimensionless displacement distribution $w^*$ along the dimensionless radial direction $r^*$ for different values of the fractional-order parameter $\alpha$ by fixing the different inhomogeneity parameter $p$. It is seen that the magnitude of displacement is decreases with increasing in the inhomogeneity parameter. Also from the graphical view it is analyzed that the different values of fractional-order parameter $\alpha$ significantly affect the variation with a fixed inhomogeneity parameter.
Fig. 10: Variation of dimensionless radial stress for different inhomogeneity parameter $p$

$\mathcal{G}_p$,

$p = 3$

$p = 1.5$

$p = 2.5$

Fig. 11: Variation of dimensionless tangential stress for different inhomogeneity parameter $p$
Fig. 12: Variation of dimensionless axial stress for different inhomogeneity parameter \( p \)
Figures 10, 11, 12 and 13 show the variation of dimensionless radial stress, tangential stress, shear stress and axial stress respectively in radial direction for different values of $p$. For all the values of fractional order parameter $\alpha = 0.5, \alpha = 1, \alpha = 1.5, \alpha = 2$ a common character in seen in all the stresses is that their magnitude decreases with increasing in the inhomogeneity parameter. The radial stress, tangential stress and axial stress are compressive in nature near central region and tensile on curved surfaces while shear stress found compressive through the range.

**Conclusion**

In the study, we investigated temperature, displacement and thermal stresses in the time fractional thermoelastic problem of a thick hollow cylinder subjected to sectional heating on the curved surface by using the integral transform method. The material properties are assumed to vary by simple power law along axial direction. We solved two-dimensional transient conductivity equation for a thick hollow cylinder with inhomogeneous material properties. The thermoelastic solutions of the problem are obtained in the form of Bessel’s and trigonometric functions. Numerical calculations were carried out by considering a mixture of copper and tin metals respectively and the obtained numerical solutions of transient state temperature field and thermal stresses are examined and illustrated graphically. Also analysis of inhomogeneity grading is observed for various value of $p$. It is seen that from the numerical analysis that plot of temperature, displacement and stresses for different value of dimensionless thickness $z^*$, a different variation is observed for all the values of fractional order parameter $\alpha = 0.5, \alpha = 1, \alpha = 1.5, \alpha = 2$ in both the homogeneous and nonhomogeneous cases. Also graphical plot shows that with increase in the inhomogeneity parameter, the magnitude of temperature, displacement and stresses are affected a large. Hence it is concluded that above both the factor can plays an important and useful roles in designing of new materials which is applicable to real life situations. Further for homogeneous cylinder magnitude of temperature, displacement and stresses found high as compared to nonhomogeneous cylinder. Fractional order parameter range $0 < \alpha < 1$ corresponds to weak conductivity and $1 < \alpha < 2$ corresponds to strong conductivity while $\alpha = 1$ corresponds to normal conductivity. Hence we say that nonhomogeneous hollow cylinder with sectional heating on curved surface within the context of
fractional order theory approach predicts lagging response to physical stimulus, as observed in nature. Hence, we conclude that above study useful for the design of new materials.

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