Convergence Results for the Double-Diffusion Perturbation Equations

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Abstract: We study the structural stability for the double-diffusion perturbation equations. Using the a priori bounds, the convergence results on the reaction boundary coefficients \(k_1, k_2\) and the Lewis coefficient \(L_e\) could be obtained with the aid of some Poincaré inequalities. The results showed that the structural stability is valid for the double-diffusion perturbation equations with reaction boundary conditions. Our results can be seen as a version of symmetry in inequality for studying the structural stability.

Keywords: structural stability; double-diffusion perturbation equations; Lewis coefficient; convergence result

1. Introduction

Many papers in the literature have studied the continuous dependence or convergence of solutions of different equations in porous media on construction coefficients. We give these studies a new name. We call these stabilities structural stability. This kind of stability is different from the traditional stability. We do not care about the stability with the initial data, but about their structural stability with the model itself. For an introduction to the nature of this structural stability, please see book [1]. It is important to establish the result of the structural stability in the problem of the continuum mechanics. In [2], the authors studied a variety of equations and obtained many results on structural stability. We think it is very important to study structural stability. In the process of establishing the model, the error always exists. We want to know whether a small error will cause a sharp change in the solution.

Straughan in paper [3] proposed a new type of double diffusion perturbation model in porous media. The Darcy approximation is used in the derivation of this type of equation. We usually call this type of equation Darcy equations. Details about such types of equations were introduced in [4,5].

There are many equations that describe fluids in porous media. In books [4,6,7], the authors studied many different types of equations. In [8–10], the saint-venant principle results were studied for the Brinkman, Darcy, and Forchheimer equations. The spatial decay results were obtained. In the literature, many results on the structural stability of equations in porous media have been obtained. Representative papers can be seen by [11–16]. It should be emphasized that some new results have also emerged recently, see [17–48]. These results all belong to the category of the study of structural stability.

In this article, we continue to consider the structural stability of such types of equations. We consider the following double-diffusion perturbation equations with velocity, pressure, temperature, and concentration perturbations:
We must adopt a new method to overcome the difficulty of not getting the maximum value.

where $u_i$, $\theta$, $\phi$, and $\pi$ are the velocity, temperature, concentration disturbance, and pressure, respectively. $\Delta$ is the Laplace operator. In Equation (1), $R$ is the Rayleigh coefficient and $C$ is the salinity Rayleigh coefficient, $\epsilon_1$ represents the porosity, and $L_i$ is the Lewis coefficient, $l = (0, 0, 1) = (l_1, l_2, l_3)$. The system of Equation (1) is established in the region $\Omega \times [0, \tau]$, where $\Omega$ is bounded in the strictly convex region in $R^3$, and $\tau$ is a given constant and satisfies $0 \leq \tau < \infty$. The boundary conditions are:

$$u_in_i = 0, \quad \frac{\partial \theta}{\partial n} = k_1\theta, \quad \frac{\partial \phi}{\partial n} = k_2\phi, \quad (x, t) \in \partial\Omega \times [0, \tau].$$  \quad (2)

The initial conditions are:

$$u_i(x, 0) = u_{i0}(x), \quad \phi(x, 0) = \phi_0(x), \quad \theta(x, 0) = \theta_0(x), \quad x \in \Omega.$$  \quad (3)

There are significant differences between the double-diffusion perturbation equations and the Brinkman, Forchheimer, Darcy equations. The main difficulty is that we cannot get the maximum value of the disturbance as the previous papers [11–16]. In the references, the maximum value of the disturbance is often used to obtain the required structural stability results. In this paper, we can get the maximum estimates of disturbance.

The structural stability results we need will not be obtained by using the previous methods. We will give some a priori bounds for the solutions.

**2. A Priori Bounds**

In the course of producing the results of convergence on the coefficient of (1), we find it is easy if we can derive some a priori bounds for the solutions. We will give some Lemmas that are useful in proving our main results.

**Lemma 1.** For the temperature $\theta$ and the concentration disturbance $\phi$, we have the following estimates:

$$\int_{\partial\Omega} \theta^2 \, dS \leq \left( \frac{m_1}{m_0} + \frac{\epsilon_0 m_2}{m_0^2} \right) \int_{\Omega} \theta^2 \, dS + \frac{1}{\epsilon_0} \int_{\Omega} \theta \phi_j \, dS,$$  \quad (4)
and
\[ \int_{\Omega} \phi^2 \, dS \leq \left( \frac{m_1}{m_0} + \epsilon_0 \frac{m_2}{m_0^2} \right) \int_{\Omega} \phi^2 \, dS + \frac{1}{\epsilon_0} \int_{\Omega} \phi \phi_{i,j} \, dS, \]

where \( m_0, m_1, m_2 \) are positive constants, and \( \epsilon_0 \) is an arbitrary positive constant.

**Proof.** We defined a function \( \zeta_i \) on \( \Omega \). The function \( \zeta_i \) satisfies the following conditions:
\[ \zeta_i n_i \geq m_0 > 0, \quad x \in \partial \Omega, \]
\[ |\zeta_i| \leq m_1, \quad x \in \Omega, \]
\[ |\zeta| \leq m_2, \quad x \in \Omega, \]
where \( n_i \) is the unit outward normal vector.

From the divergence theorem, we have:
\[ m_o \oint_{\partial \Omega} \theta^2 \, ds \leq \oint_{\partial \Omega} \zeta_i n_i \theta^2 \, ds \]
\[ = \int_{\Omega} (\zeta_i \theta)^2 \, dx \]
\[ = \int_{\Omega} \zeta_i \theta^2 \, dx + 2 \int_{\Omega} \zeta_i \theta \theta_{i,j} \, dx \]
\[ \leq m_1 \int_{\Omega} \theta^2 \, dx + 2m_2 \int_{\Omega} \theta \theta_{i,j} \, dx. \]

Using Schwarz’s inequality, we have:
\[ \oint_{\partial \Omega} \theta^2 \, ds \leq \left( \frac{m_1}{m_0} + \epsilon_0 \frac{m_2}{m_0^2} \right) \int_{\Omega} \theta^2 \, dx + \frac{1}{\epsilon_0} \int_{\Omega} \theta \theta_{i,j} \, dx. \]

Following the same procedures, we can also get:
\[ \int_{\partial \Omega} \phi^2 \, ds \leq \left( \frac{m_1}{m_0} + \epsilon_0 \frac{m_2}{m_0^2} \right) \int_{\Omega} \phi^2 \, dS + \frac{1}{\epsilon_0} \int_{\Omega} \phi \phi_{i,j} \, dS. \]

**Lemma 2.** For the velocity \( u_i \), temperature \( \theta \), and the concentration disturbance \( \phi \), we have the following estimates:
\[ \int_{\Omega} u_i u_i \, dx + \int_{\Omega} \theta^2 \, dx + \int_{\Omega} \phi^2 \, dx \leq n_1(t), \]
\[ \int_0^t \int_{\Omega} \theta_{i,j} \, dx \, dt \leq n_2(t), \]
\[ \int_0^t \int_{\Omega} \phi_{i,j} \, dx \, dt \leq n_3(t), \]
where \( n_1(t), n_2(t), \) and \( n_3(t) \) are non-negative monotonically increasing functions.

**Proof.** Multiplying both sides of the Equation (1) by \( 2u_i \), and integrating over \( \Omega \), we can get:
\[ \frac{d}{dt} \int_{\Omega} u_i u_i \, dx = 2C \int_{\Omega} \phi i u_i \, dx - 2R \int_{\Omega} \theta i u_i \, dx + 2 \int_{\Omega} \pi_{i} u_i \, dx \]
\[ \leq 2 \int_{\Omega} u_i u_i \, dx + C^2 \int_{\Omega} \phi^2 \, dx + R^2 \int_{\Omega} \theta^2 \, dx. \]
Multiplying both sides of the Equation (1) by 2θ, and integrating over Ω, we can get:

\[
\frac{d}{dt} \int_\Omega \theta^2 dx = 2 \int_\Omega u_3 \theta dx + 2 \int_\Omega \theta \Delta \theta dx - 2 \int_\Omega u_\theta \theta dx.
\] (16)

Using (4) and taking \( \varepsilon_0 = 2k_1 \), we can get:

\[
\int_\partial \Omega \theta^2 dS \leq \left[ \frac{m_1}{m_0} + \frac{2k_1m_2^2}{m_0^2} \right] \int_\Omega \theta^2 dx + \frac{1}{2k_1} \int_\Omega \theta \theta dx.
\] (17)

For the second term on the right side of Equation (16), we have:

\[
2 \int_\Omega \theta \Delta \theta dx = 2 \int_\Omega \theta \frac{\partial \theta}{\partial n} dS - 2 \int_\Omega \theta, \theta dx
= 2k_1 \int_\partial \Omega \theta^2 dS - 2 \int_\Omega \theta, \theta dx
\leq \left( \frac{2k_1m_1}{m_0} + \frac{4k_1^2m_2^2}{m_0^2} \right) \int_\Omega \theta^2 dx - \int_\Omega \theta, \theta dx.
\] (18)

Combining (16) and (18), and using the Hölder’s inequality, we can get:

\[
\frac{d}{dt} \int_\Omega \theta^2 dx + \int_\Omega \theta, \theta dx \leq \int_\Omega u_\theta u dx + \left( \frac{2k_1m_1}{m_0} + \frac{4k_1^2m_2^2}{m_0^2} + 1 \right) \int_\Omega \theta^2 dx.
\] (19)

Multiplying both sides of Equation (1) by 2\varphi, and integrating over Ω, we can obtain:

\[
\varepsilon_1 \frac{d}{dt} \int_\Omega \varphi^2 dx + \int_\Omega \varphi, \varphi dx \leq \int_\Omega u_\varphi u dx + \left( \frac{2k_2m_1}{m_0} + \frac{4k_2^2m_2^2}{m_0^2} + 1 \right) \int_\Omega \varphi^2 dx.
\] (20)

We define a new function:

\[
F_1(t) = \int_\Omega u_\theta u dx + \int_\Omega \theta^2 dx + \int_\Omega \varphi^2 dx.
\]

Combining (15), (19), and (20), we obtain:

\[
F_1(t) \leq m_4 + m_3 \int_0^t F_1(\eta) d\eta.
\] (21)

where

\[
m_3 = \max \left\{ 3 + \frac{1}{\varepsilon_1}, R^2 + \frac{2k_1m_1}{m_0} + \frac{4k_1^2m_2^2}{m_0^2} + 1, C^2 + \frac{2k_2m_1}{m_0\varepsilon_1} + \frac{4k_2^2m_2^2}{m_0^2\varepsilon_1} + 1/\varepsilon_1 \right\},
\]

and

\[
m_4 = \int_\Omega u_\theta u_\theta dx + \int_\Omega \theta_\theta \theta_\theta dx + \int_\Omega \varphi_\theta \varphi_\theta dx.
\]

Using Gronwall’s inequality, we can get:

\[
F_1(t) \leq m_3 m_4 e^{m_3 t} \int_0^t e^{-m_3 \eta} d\eta = n_1(t).
\] (22)

Inserting (22) into (19), we can get:

\[
\int_0^t \int_\Omega \theta, \theta dx d\eta \leq \left( \frac{2k_1m_1}{m_0} + \frac{4k_1^2m_2^2}{m_0^2} + 1 \right) \int_0^t n_1(\eta) d\eta = n_2(t).
\]
Inserting (22) into (20), we can get:

\[ \int_0^t \int_\Omega \varphi_i \varphi_j dx \, d\eta \leq \left( \frac{2k_2m_1}{m_0} + \frac{4k_2^2m_0^2}{m_0^2} + 1 \right) \int_0^t n_1(\eta) \, d\eta = n_3(t). \]

\[ \square \]

**Lemma 3.** For velocity \( u_i \), we have the following estimates:

\[ \left[ \int_\Omega (u_i u_i)^2 \, dx \right]^\frac{1}{2} \leq m_1(t), \quad (23) \]

where \( m_1(t) \) is a positive function to be defined later.

**Proof.** We have the following identity:

\[ \int_\Omega u_i u_j u_i u_j \, dx = \int_\Omega u_i (u_i - u_j) u_j \, dx + \int_\Omega u_i u_j u_i u_j \, dx. \quad (24) \]

Since \( \partial \Omega \) is bounded, we know from the result of [44]:

\[ | \int_\Omega u_i u_j u_i u_j \, dx | \leq k_0 | \int_\partial \Omega u_i u_j \, ds |, \]

where \( k_0 \) is the Gaussian curvature depending on \( \partial \Omega \).

Taking \( \theta = u_i, \epsilon_0 = 2k_0 \) in (4), we have:

\[ | \int_\Omega u_i u_j u_i u_j \, dx | \leq k_0 \left( \frac{m_1}{m_0} + \frac{2k_0 m_0^2}{m_0^2} \right) \int_\Omega u_i u_j u_i u_j \, dx \]

\[ \leq k_0 \left( \frac{m_1}{m_0} + \frac{2k_0 m_0^2}{m_0^2} \right) n_1(t) + \frac{1}{2} \int_\Omega u_i u_j u_i u_j \, dx. \quad (25) \]

Combining (24) and (25), we get:

\[ \int_\Omega u_i u_j u_i u_j \, dx \leq 2 \int_\Omega u_i (u_i - u_j) u_j \, dx + 2k_0 \left( \frac{3}{m} + \frac{2k_0 d^2}{2m^2} \right) n_1(t). \quad (26) \]

Using Equation (1), we obtain:

\[ \frac{d}{dt} \int_\Omega u_i (u_i - u_j) \, dx = 2 \int_\Omega (u_i - u_j) u_i u_j \, dx \]

\[ \leq 4 \int_\Omega u_i (u_i - u_j) \, dx + C^2 \int_\Omega \varphi_i \varphi_j \, dx + R^2 \int_\Omega \varphi_i \varphi_j \, dx. \quad (27) \]

We define \( E(t) = \int_\Omega u_i (u_i - u_j) \, dx \). From (26), we obtain:

\[ E(t) \leq 4e^{\delta t} \int_0^t m(y) e^{-\delta \gamma} \, dy + R^2 n_2(t) + C^2 n_3(t) = m_2(t), \quad (28) \]

with \( m(t) = \int_\Omega u_i (x, 0) [(u_i(x, 0) - u_j(x, 0)] \, dx \).

Inserting (28) into (26), we have:

\[ \int_\Omega u_i u_j u_i u_j \, dx \leq 2m_2(t) + 2k_0 \left( \frac{m_1}{m_0} + \frac{2k_0 m_0^2}{m_0^2} \right) n_1(t). \quad (29) \]
Using the result of (B.17) in [26], we have:

\[
\left( \int_{\Omega} |u|^4 dx \right)^{\frac{1}{2}} \leq M \left( \frac{5}{4} \int_{\Omega} |u|^2 dx + \frac{3}{4} \int_{\Omega} |\nabla u|^2 dx \right)
\leq M \left( \frac{5}{4} n_1(t) + \frac{6}{4} m_2(t) \right)
+ \frac{6k_0}{4} \left( \frac{m_1}{m_0} + \frac{2k_0 m_2}{m_0^2} \right) n_1(t) = m_1(t),
\]

where \( M \) is a positive constant. \( \Box \)

**Lemma 4.** For the temperature \( \theta \), concentration disturbance \( \varphi \), we have the following estimates:

\[
\left( \int_{\Omega} \theta^4 dx \right)^{\frac{1}{2}} \leq n_4(t), \tag{31}
\]

\[
\left( \int_{\Omega} \varphi^4 dx \right)^{\frac{1}{2}} \leq n_5(t), \tag{32}
\]

with \( n_4(t) \) and \( n_5(t) \) are all monotonically increasing functions greater than zero.

**Proof.** Multiplying both sides of the \((1)\) by \( \theta^3 \) and integrating over \( \Omega \), we have:

\[
\frac{1}{4} \frac{d}{dt} \int_{\Omega} \theta^4 dx + \int_{\Omega} u \theta \theta^3 dx = \int_{\Omega} u_3 \theta^3 dx + \int_{\Omega} \theta^3 \theta dx
= \int_{\Omega} u_3 \theta^3 dx - 3 \int_{\Omega} \theta^2 \theta \theta dx + k_1 \int_{\partial \Omega} \theta^4 dS
\leq \frac{1}{4} \int_{\Omega} u_4^2 dx + \frac{3}{4} \int_{\Omega} \theta^4 dx - 3 \int_{\Omega} \theta^2 \theta \theta dx + k_1 \int_{\partial \Omega} \theta^4 dx.
\]

Replacing \( \theta \) by \( \theta^2 \) and choosing \( c_0 = 2k_1 \) in (4), we get:

\[
k_1 \int_{\partial \Omega} \theta^4 dx \leq \frac{k_1 m_1}{m_0} \int_{\Omega} \theta^4 dx + \frac{2k_1^2 m_2^2}{m_0^2} \int_{\Omega} \theta^4 dx + \frac{1}{2} \int_{\Omega} \theta^2 dx \cdot (\theta^2 dx).
\]

Inserting (34) into (33), we obtain:

\[
\frac{d}{dt} \int_{\Omega} \theta^4 dx \leq \int_{\Omega} u_4^2 dx + 3 \int_{\Omega} \theta^4 dx + \frac{4m_1 k_1}{m_0} \int_{\Omega} \theta^4 dx + \frac{8k_1^2 m_2^2}{m_0^2} \int_{\Omega} \theta^4 dx.
\]

Inserting (23) into (35), we obtain:

\[
\frac{d}{dt} \int_{\Omega} \theta^4 dx \leq \int_{\Omega} u_4^2 dx + 3 \int_{\Omega} \theta^4 dx + \left( 3 + \frac{4m_1 k_1}{m_0} + \frac{8k_1^2 m_2^2}{m_0^2} \right) \int_{\Omega} \theta^4 dx.
\]

An integration of (36) leads to

\[
\int_{\Omega} \theta^4 dx \leq e^{\left( 3 + \frac{4m_1 k_1}{m_0} + \frac{8k_1^2 m_2^2}{m_0^2} \right) t} \left[ \int_{\Omega} \theta^0_1 dx + \int_{0}^{t} \left( m_1(\eta) \right)^2 d\eta \right].
\]

We obtain:

\[
\left( \int_{\Omega} \theta^4 dx \right)^{\frac{1}{2}} \leq n_4(t). \tag{38}
\]

Following the same procedures, we can also get:

\[
\left( \int_{\Omega} \varphi^4 dx \right)^{\frac{1}{2}} \leq n_5(t), \tag{39}
\]
with
\[ n_4(t) = e^{\left(3 + \frac{4a_1^2}{m_0^2} + \frac{8a_2^2}{m_0^4}\right)t} \left[ \int_{\Omega} \theta_0^2 dx + \int_0^t (m_1(\eta))^2 d\eta \right], \]
and
\[ n_5(t) = e^{\left(3 + \frac{4a_1^2}{m_0^2} + \frac{8a_2^2}{m_0^4}\right)t} \left[ \int_{\Omega} \psi^4 dx + \int_0^t (m_1(\eta))^2 d\eta \right]. \]

3. Convergence Result for the Reaction Boundary Coefficients \(k_1\) and \(k_2\)

Let \((u_i, \theta, \varphi, \pi)\) be the solution of (1)–(3) with \(k_1 = \hat{k}_1, k_2 = \hat{k}_2(u_i^*, \theta^*, \varphi^*, \pi^*)\) be the solution of (1)–(3) with \(k_1 = 0, k_2 = 0\). We define \(\omega_i = u_i - u_i^*, \hat{\theta} = \theta - \theta^*, \hat{\varphi} = \varphi - \varphi^*, \hat{\pi} = \pi - \pi^*\), then \((\omega_i, \hat{\theta}, \hat{\varphi}, \hat{\pi})\) satisfies the following equations:

\[
\begin{align*}
\frac{\partial \omega_i}{\partial t} &= C \phi_i l_i - R \hat{\theta}_i l_i + \hat{\pi}_i, \\
\frac{\partial \omega_i}{\partial x_i} &= 0, \\
\frac{\partial \hat{\theta}}{\partial t} + \omega_i \hat{\theta}_i + u_i^* \hat{\theta}_i &= \omega_i^3 + \Delta \hat{\theta}, \\
\varepsilon \frac{\partial \hat{\varphi}}{\partial t} + L_\varepsilon (\omega_j \varphi_j + u_j^* \varphi_j) &= \omega_i^3 + \Delta \hat{\varphi}.
\end{align*}
\]

The boundary conditions are:

\[ \omega_i n_i = 0, \frac{\partial \hat{\theta}}{\partial n} = \hat{k}_1 \hat{\theta}, \frac{\partial \hat{\varphi}}{\partial n} = \hat{k}_2 \varphi, (x, t) \in \partial \Omega \times [0, \tau]. \] (41)

The initial conditions are:

\[ \omega_i(x, 0) = 0, \hat{\theta}(x, 0) = 0, \hat{\varphi}(x, 0) = 0, x \in \Omega. \] (42)

In deducing our main result, we will use the following Lemma.

**Lemma 5.** For the difference of the velocity \(\omega_i\), we can get the following estimates:

\[ \int_{\Omega} \omega_{i,j} \omega_{i,j} dx \leq 2 \int_{\Omega} \omega_{i,j} (\omega_{i,j} - \omega_{j,i}) dx + 2k_0 \left[ \frac{m_1 \eta_0}{m_0} + \frac{2k_0 m_2^2}{m_0^2} \right] \int_{\Omega} \omega_{i,j} dx, \] (43)

with \(k_0\) as a positive constant.

**Proof.** We know the fact:

\[ \int_{\Omega} \omega_{i,j} \omega_{i,j} dx = \int_{\Omega} \omega_{i,j} (\omega_{i,j} - \omega_{j,i}) dx + \int_{\Omega} \omega_{i,j} \omega_{j,i} dx. \] (44)

Since the boundary of \(\Omega\) is bounded, we have:

\[ | \int_{\Omega} \omega_{i,j} \omega_{j,i} dx | \leq k_0 \int_{\partial \Omega} \omega_i \omega_j dS, \] (45)

with \(k_0\) as a positive constant depending on the Gaussian curvature of \(\partial \Omega\) (see [44]).

Using the result (4) with \(\varepsilon_0 = 2k_0\), we can obtain:

\[ \int_{\partial \Omega} \omega_i \omega_j dS \leq \left[ \frac{m_1}{m_0} + \frac{2k_0 m_2^2}{m_0^2} \right] \int_{\Omega} \omega_i \omega_j dx + \frac{1}{2k_0} \int_{\Omega} \omega_{i,j} \omega_{j,i} dx. \] (46)
Inserting (45) and (46) into (44), we can get:

\[
\int_{\Omega} \omega_{i,j} \omega_{i,j} \, dx \leq 2 \int_{\Omega} \omega_{i,j} (\omega_{i,j} - \omega_{j,i}) \, dx + 2k_0 \left[ \frac{m_1}{m_0} + \frac{2k_0 m^2}{m_0} \right] \int_{\Omega} \omega_{i,j} \, dx.
\]

In this part, we will get the following Theorem. □

**Theorem 1.** Let \((u_i, \theta, \varphi, \pi)\) be the classical solution of the initial value problem (1)–(3) with \(k_1 = k_1, k_2 = k_2,\) and \((u_i^*, \theta^*, \varphi^*, \pi^*)\) be the classical solution of the initial boundary value problem (1)–(3) with \(k_1 = 0, k_2 = 0.\) \((\omega_i, \hat{\theta}, \hat{\varphi}, \hat{\pi})\) is the difference of these two solutions. When \(k_1\) and \(k_2\) tend to zero, the solution \((u_i, \theta, \varphi, \pi)\) converges to the solution \((u_i^*, \theta^*, \varphi^*, \pi^*)\).

The difference of the solution \((\omega_i, \hat{\theta}, \hat{\varphi}, \hat{\pi})\) satisfies:

\[
\int_{\Omega} \omega_i \omega_i \, dx + \int_{\Omega} \hat{\theta}^2 \, dx + \epsilon_1 \int_{\Omega} \hat{\varphi}^2 \, dx + \int_{\Omega} \omega_{i,j} (\omega_{i,j} - \omega_{j,i}) \, dx \\
\leq k_1^2 m_8 \varepsilon_n \, n_6(t) + k_2^2 m_8 \varepsilon n_7(t),
\]

where \(m_8\) is a positive constant and \(n_6(t)\) and \(n_7(t)\) are positive functions.

**Proof.** Multiplying both sides of Equation (40) by \(2\omega_i,\) and integrating over \(\Omega,\) we can get:

\[
\frac{d}{dt} \int_{\Omega} \omega_i \omega_i \, dx = 2C \int_{\Omega} \hat{\varphi}_i \omega_i \, dx - 2R \int_{\Omega} \hat{\theta}_i \omega_i \, dx + 2 \int_{\Omega} \hat{\pi}_i \omega_i \, dx \\
\leq 2 \int_{\Omega} \omega_i \omega_i \, dx + C^2 \int_{\Omega} \hat{\varphi}^2 \, dx + R^2 \int_{\Omega} \hat{\theta}^2 \, dx.
\]

From Equation (40), we know:

\[
\frac{d}{dt} \int_{\Omega} \omega_{i,j} (\omega_{i,j} - \omega_{j,i}) \, dx \\
= 2 \int_{\Omega} (\omega_{i,j} - \omega_{j,i}) \omega_{i,j} \, dx \\
= 2C \int_{\Omega} (\omega_{i,j} - \omega_{j,i}) \hat{\varphi}_j \, dx + 2 \int_{\Omega} (\omega_{i,j} - \omega_{j,i}) \hat{\pi}_j \, dx - 2R \int_{\Omega} (\omega_{i,j} - \omega_{j,i}) \hat{\theta}_j \, dx.
\]

Using the divergence theorem and Hölder’s inequality, we can get:

\[
\frac{d}{dt} \int_{\Omega} \omega_{i,j} (\omega_{i,j} - \omega_{j,i}) \, dx \\
\leq (2C^2 + 2R^2) \int_{\Omega} (\omega_{i,j} - \omega_{j,i}) (\omega_{i,j} - \omega_{j,i}) \, dx + \frac{1}{2} \int_{\Omega} \hat{\varphi}_j \hat{\varphi}_j \, dx + \frac{1}{2} \int_{\Omega} \hat{\pi}_j \hat{\pi}_j \, dx \\
= (4C^2 + 4R^2) \int_{\Omega} \omega_{i,j} (\omega_{i,j} - \omega_{j,i}) \, dx + \frac{1}{2} \int_{\Omega} \hat{\varphi}_j \hat{\varphi}_j \, dx + \frac{1}{2} \int_{\Omega} \hat{\pi}_j \hat{\pi}_j \, dx.
\]

Multiplying both sides of Equation (40) by \(2\hat{\theta}_i,\) and integrating over \(\Omega,\) we can get:

\[
\frac{d}{dt} \int_{\Omega} \hat{\theta}^2 \, dx = 2 \int_{\Omega} \omega_i \hat{\theta} \, dx + 2 \int_{\Omega} \hat{\theta} \Delta \hat{\theta} \, dx - 2 \int_{\Omega} \omega_i \hat{\theta} \, dx - 2 \int_{\Omega} \hat{u}_i \hat{\theta} \, dx \\
= 2 \int_{\Omega} \omega_i \hat{\theta} \, dx + 2 \int_{\Omega} \hat{\theta} \Delta \hat{\theta} \, dx + 2 \int_{\Omega} \omega_i \hat{\theta} \, dx.
\]

The first term on the right side of Equation (51) can be bounded by:

\[
2 \int_{\Omega} \omega_i \hat{\theta} \, dx \leq \int_{\Omega} \omega_i \omega_i \, dx + \int_{\Omega} \hat{\theta}^2 \, dx.
\]
Using the result (4), and taking \( \varepsilon_0 = 1 \), we can get:

\[
\int_{\partial \Omega} \phi^2 dS \leq \left( \frac{m_1}{m_0} + \frac{m_2}{m_0^2} \right) \int_{\Omega} \phi^2 dx + \int_{\Omega} \phi \phi_j dx.
\]  

(53)

We now take the second term on the right side of Equation (51):

\[
2 \int_{\Omega} \theta \dot{\theta} dx = 2 \int_{\partial \Omega} \theta \frac{\partial \theta}{\partial n} dS - 2 \int_{\Omega} \theta \dot{\theta} dx
\]

\[
= 2k_1 \int_{\partial \Omega} \theta dS - 2 \int_{\Omega} \theta \dot{\theta} dx
\]

\[
\leq k_1 \int_{\partial \Omega} \theta^2 dS + \int_{\Omega} \theta^2 dx - 2 \int_{\Omega} \theta \dot{\theta} dx
\]

(54)

Combining (51), (52), and (54), we can get:

\[
\frac{d}{dt} \int_{\Omega} \theta^2 dx + \frac{1}{2} \int_{\partial \Omega} \theta \dot{\theta} dx
\]

\[
\leq k_1 \int_{\partial \Omega} \theta^2 dS + \int_{\Omega} \omega_i \omega_i dx + m_s \int_{\Omega} \theta^2 dx + 2 \int_{\Omega} \omega_i \theta^2 dx
\]

\[
\leq k_1 \int_{\partial \Omega} \theta^2 dS + \int_{\Omega} \omega_i \omega_i dx + m_s \int_{\Omega} \theta^2 dx + 2 \left( \int_{\Omega} (\omega_i \omega_i)^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega} \theta^2 dx \right)^{\frac{1}{2}},
\]

where \( m_s = \left( \frac{m_1}{m_0} + \frac{m_2}{m_0^2} \right) + 1 \).

Using the result of (B.17) in [26] and (31), we can get:

\[
\frac{d}{dt} \int_{\Omega} \theta^2 dx + \frac{1}{2} \int_{\partial \Omega} \theta \dot{\theta} dx
\]

\[
\leq k_1 \int_{\partial \Omega} \theta^2 dS + \int_{\Omega} \omega_i \omega_i dx + m_s \int_{\Omega} \theta^2 dx + 2 \left( \int_{\Omega} (\omega_i \omega_i)^2 dx \right)^{\frac{1}{2}} n_4(t)
\]

\[
\leq k_1 \int_{\partial \Omega} \theta^2 dS + \int_{\Omega} \omega_i \omega_i dx + m_s \int_{\Omega} \theta^2 dx + 2M \left( \frac{5}{4} \int_{\Omega} \omega_i \omega_i dx + \frac{3}{4} \int_{\Omega} \omega_i \omega_i dx \right) n_4(t)
\]

\[
\leq k_1 \int_{\partial \Omega} \theta^2 dS + m_s \int_{\Omega} \omega_i \omega_i dx + m_s \int_{\Omega} \theta^2 dx + 3Mn_4(\tau) \int_{\Omega} \omega_i (\omega_i - \omega_i) dx,
\]

where \( m_s = M n_4(\tau) \left[ \frac{3k_0 (\frac{m_1}{m_0} + \frac{2k_0 m_2}{m_0^2}) + \frac{3}{2} \right] + 1 \).

Multiplying both sides of Equation (40) by \( 2 \phi \), and integrating over \( \Omega \) we can get:

\[
\epsilon_1 \frac{d}{dt} \int_{\Omega} \phi^2 dx = 2 \int_{\Omega} \omega_3 \phi dx + 2 \int_{\Omega} \phi \Delta \phi dx - 2L_\phi \int_{\Omega} \omega_i \phi_j \phi dx - 2L_\phi \int_{\Omega} u_i \, \psi \, \psi_i \, dx
\]

\[
= 2 \int_{\Omega} \omega_3 \phi dx + 2 \int_{\Omega} \phi \Delta \phi dx + 2L_\phi \int_{\Omega} \omega_i \phi_j \phi dx.
\]

(57)

The first term on the right side of Equation (57) can be bounded by:

\[
2 \int_{\Omega} \omega_3 \phi dx \leq \int_{\Omega} \omega_i \omega_i dx + \int_{\Omega} \phi^2 dx.
\]

(58)

Using (4), and taking \( \varepsilon_0 = 1 \), we can get:

\[
\int_{\partial \Omega} \phi^2 dS \leq \left( \frac{m_1}{m_0} + \frac{m_2}{m_0^2} \right) \int_{\Omega} \phi^2 dx + \int_{\Omega} \phi \phi_j dx.
\]

(59)
We now take the second term on the right side of Equation (57). We have:

\[
2 \int_{\Omega} \phi \Delta \phi \, dx = 2 \int_{\Omega} \phi \frac{\partial \phi}{\partial n} \, dS - 2 \int_{\Omega} \phi, \phi, \phi \, dx
\]
\[
= 2k_2 \int_{\Omega} \phi \, dS - 2 \int_{\Omega} \phi, \phi, \phi \, dx
\]
\[
\leq \hat{k}^2 \int_{\Omega} \phi^2 \, dS + \int_{\Omega} \omega_i \omega_i \, dx + m_5 \int_{\Omega} \phi^2 \, dx + 2L_1^2 \int_{\Omega} \omega_i \phi \, dx
\]
\[
\leq \hat{k}^2 \int_{\Omega} \phi^2 \, dS + \int_{\Omega} \omega_i \omega_i \, dx + m_5 \int_{\Omega} \phi^2 \, dx + 2L_1^2 \left( \int_{\Omega} (\omega_i \phi)^2 \, dx \right)^{\frac{1}{2}} \left( \int \phi^i \, dx \right)^{\frac{1}{2}}.
\]  

Combining (57)–(60), we can obtain:

\[
\epsilon_1 \frac{d}{dt} \int_{\Omega} \phi^2 \, dx + \frac{1}{2} \int_{\Omega} \phi, \phi, \phi \, dx
\]
\[
\leq \hat{k}^2 \int_{\Omega} \phi^2 \, dS + \int_{\Omega} \omega_i \omega_i \, dx + m_5 \int_{\Omega} \phi^2 \, dx + 2L_1^2 \int_{\Omega} \omega_i \phi \, dx
\]
\[
\leq \hat{k}^2 \int_{\Omega} \phi^2 \, dS + \int_{\Omega} \omega_i \omega_i \, dx + m_5 \int_{\Omega} \phi^2 \, dx + 2L_1^2 \left( \int_{\Omega} (\omega_i \phi)^2 \, dx \right)^{\frac{1}{2}} \left( \int \phi^i \, dx \right)^{\frac{1}{2}}.
\]

We can also get:

\[
\epsilon_1 \frac{d}{dt} \int_{\Omega} \phi^2 \, dx + \frac{1}{2} \int_{\Omega} \phi, \phi, \phi \, dx
\]
\[
\leq \hat{k}^2 \int_{\Omega} \phi^2 \, dS + \int_{\Omega} \omega_i \omega_i \, dx + m_5 \int_{\Omega} \phi^2 \, dx + 2L_1^2 M \left( \frac{5}{4} \int_{\Omega} \omega_i \omega_i \, dx + \frac{3}{4} \int_{\Omega} \omega_i, \omega_i, \omega_i \, dx \right) n_5(t).
\]

where \( m_5 = ML_1^2 n_5(\tau) \left[ 3k_0 \left( \frac{m_1}{m_0} + \frac{2k_0 n_5^2}{m_0^2} \right) + \frac{5}{2} \right] + 1. \)

Combining (48), (50), (56), (61), and (62), we can get:

\[
\frac{d}{dt} \left[ \int_{\Omega} \omega_i \omega_i \, dx + \int_{\Omega} \phi^2 \, dx + \epsilon_1 \int_{\Omega} \phi^2 \, dx + \int_{\Omega} \omega_i, \omega_i, \omega_i \, dx \right]
\]
\[
\leq (m_6 + m_7 + 2) \int_{\Omega} \omega_i \omega_i \, dx + (R^2 + m_5) \int_{\Omega} \phi^2 \, dx + (C^2 + m_5) \int_{\Omega} \phi^2 \, dx
\]
\[
+ \hat{k}^2 \int_{\Omega} \phi^2 \, dS + \hat{k}^2 \int_{\Omega} \phi^2 \, dS + \int_{\Omega} \phi^2 \, dx \left( 3MN_4(\tau) + 3ML_1^2 n_5(\tau) + 4C^2 + 4R^2 \right) \int_{\Omega} \omega_i, \omega_i, \omega_i \, dx.
\]

Let

\[
F_2(t) = \int_{\Omega} \omega_i \omega_i \, dx + \int_{\Omega} \phi^2 \, dx + \epsilon_1 \int_{\Omega} \phi^2 \, dx + \int_{\Omega} \omega_i, \omega_i, \omega_i \, dx,
\]
\[
m_8 = \max \{m_6 + m_7 + 2, R^2 + m_5, \frac{C^2 + m_5}{\epsilon_1}, 3MN_4(\tau) + 3ML_1^2 n_5(\tau) + 4C^2 + 4R^2 \}.
\]

From (63), it can be seen that:

\[
\frac{d}{dt} F_2(t) \leq \hat{k}^2 \int_{\Omega} \phi^2 \, dS + \hat{k}^2 \int_{\Omega} \phi^2 \, dS + m_8 F_2(t).
\]

Integrating (64), and using (4), (5), (13), and (14), we can get:

\[
F_2(t) \leq \hat{k}^2 m_8 e^{m_8 t} n_6(t) + \hat{k}^2 m_8 e^{m_8 t} n_7(t),
\]

with \( n_6(t) = \int_0^t \left( \frac{m_1}{m_0} + \frac{m_2}{m_0} \right) n_1(\eta) \, d\eta + n_2(t) \) and \( n_7(t) = \int_0^t \left( \frac{m_1}{m_0} + \frac{m_2}{m_0} \right) n_1(\eta) \, d\eta + n_3(t). \)
Inequality (65) shows that when $\hat{k}_1, \hat{k}_2$ simultaneously tend to zero, the energy $F_2(t)$ also tends to zero as the indicated norm. \qed

4. Convergence Result for the Lewis Coefficient $L_e$

Let $(u_i, \theta, \varphi, \pi)$ be the solution of (1)–(3) when $Le = \hat{L}e$, and let $(u_i^*, \theta^*, \varphi^*, \pi^*)$ be the solution of (1)–(3) when $Le = 0$. We assume $\omega_i = u_i - u_i^*, \hat{\theta} = \theta - \theta^*, \hat{\varphi} = \varphi - \varphi^*, \hat{\pi} = \pi - \pi^*$, then $(\omega_i, \hat{\theta}, \hat{\varphi}, \hat{\pi})$ satisfies the following equations:

\[
\frac{d\omega_i}{dt} = C\phi_i - R\hat{\theta}_i + \hat{\pi}_j, \\
\frac{d\omega_i}{dx_j} = 0, \\
\frac{d\hat{\theta}}{dt} + \omega_i\hat{\theta}_j + u_i^*\hat{\theta}_j = \omega_3 + \Delta\hat{\theta}, \\
\epsilon_1\frac{d\hat{\varphi}}{dt} + L_eu_i\phi_i, \hat{\pi} = \omega_3 + \Delta\hat{\varphi}.
\] (66)

The boundary conditions are:

\[
\omega_i n_i = 0, \quad \frac{\partial\hat{\theta}}{\partial n} = k_1\hat{\theta}, \quad \frac{\partial\hat{\varphi}}{\partial n} = k_2\hat{\varphi}, \quad (x, t) \in \partial\Omega \times [0, \tau].
\] (67)

The initial conditions are:

\[
\omega_i(x, 0) = 0, \frac{\partial\varphi}{\partial \Omega} = 0, \hat{\theta}(x, 0) = 0, x \in \Omega.
\] (68)

**Theorem 2.** Let $(u_i, \theta, \varphi, \pi)$ be the solution of (1)–(3) when $Le = \hat{L}e$, $(u_i^*, \theta^*, \varphi^*, \pi^*)$ be the solution of (1)–(3) when $Le = 0$. We assume $\omega_i = u_i - u_i^*, \hat{\theta} = \theta - \theta^*, \hat{\varphi} = \varphi - \varphi^*, \hat{\pi} = \pi - \pi^*$, then $(\omega_i, \hat{\theta}, \hat{\varphi}, \hat{\pi})$ satisfies the following estimates:

\[
\int_\Omega \omega_i \omega_i dx + \int_\Omega \hat{\theta}^2 dx + \epsilon_1 \int_\Omega \hat{\varphi}^2 dx + \int_\Omega \omega_i \omega_i dx \leq 2L_e^2 m_{11} e^{m_{11} t} \int_0^t m_1(\eta)n_5(\eta)e^{-m_{11} \eta} d\eta,
\] (69)

where $m_{11}$ is a constant greater than zero.

**Proof.** Multiplying both sides of Equation (66) by $2\omega_i$, and integrating over $\Omega \times [0, t]$, we can obtain:

\[
\frac{d}{dt} \int_\Omega \omega_i \omega_i dx = 2C \int_\Omega \phi_i \omega_i dx - 2R \int_\Omega \hat{\theta}_i \omega_i dx + 2 \int_\Omega \hat{\pi}_j \omega_i dx \\
\leq 2 \int_\Omega \omega_i \omega_i dx + C^2 \int_\Omega \hat{\theta}^2 dx + R^2 \int_\Omega \hat{\pi}^2 dx.
\] (70)

\[\]

Multiplying both sides of Equation (66) by $2\hat{\theta}$, and integrating over $\Omega$, we can get,

\[
\frac{d}{dt} \int_\Omega \hat{\theta}_i dx = 2 \int_\Omega \omega_i \hat{\theta}_i dx + 2 \int_\Omega \hat{\theta} \Delta \hat{\theta}_i dx - 2 \int_\Omega \omega_i \hat{\theta}_i dx - 2 \int_\Omega \omega_i \hat{\theta}_i dx \\
= 2 \int_\Omega \omega_i \hat{\theta}_i dx + 2 \int_\Omega \hat{\theta} \Delta \hat{\theta}_i dx + 2 \int_\Omega \omega_i \hat{\theta}_i dx.
\] (71)

The first term on the right side of Equation (71) can be obtained from Hölder’s inequality:

\[
2 \int_\Omega \omega_i \hat{\theta}_i dx \leq \int_\Omega \omega_i \omega_i dx + \int_\Omega \hat{\theta}^2 dx.
\] (72)
Using a method similar to (4), and taking \( \varepsilon_0 = 2k_1 \), we can get:

\[
\int_{\partial \Omega} \dot{\theta}^2 dS \leq \left( \frac{m_1}{m_0} + 2k_1 \frac{m_2^2}{m_0^2} \right) \int_{\Omega} \dot{\theta}^2 dx + \int_{\Omega} \dot{\theta} \dot{\theta}, dx. \tag{73}
\]

We now deal with the second term on the right side of Equation (71). We have:

\[
2 \int_{\Omega} \dot{\theta} \Delta \dot{\theta} dx = 2 \int_{\Omega} \dot{\theta} \frac{\partial \dot{\theta}}{\partial n} dS - 2 \int_{\Omega} \dot{\theta}_j \dot{\theta}_j dx
\]

\[
= 2k_1 \int_{\partial \Omega} \dot{\theta}^2 dS - 2 \int_{\Omega} \dot{\theta}_j \dot{\theta}_j dx
\]

\[
\leq k_1^2 \int_{\partial \Omega} \dot{\theta}^2 dS - 2 \int_{\Omega} \dot{\theta}_j \dot{\theta}_j dx \tag{74}
\]

Combining (71)–(74), we can get:

\[
\frac{d}{dt} \int_{\Omega} \dot{\theta}^2 dx + \frac{1}{2} \int_{\Omega} \dot{\theta}_j \dot{\theta}_j dx
\]

\[
\leq \int_{\Omega} \omega_j \omega_j dx + \lambda_0 \int_{\Omega} \dot{\theta}^2 dx + 2 \left( \int_{\Omega} (\omega_j \omega_i)^2 dx \right)^{\frac{1}{2}} \int_{\Omega} \theta^4 dx \tag{75}
\]

where \( \lambda_0 = 2k_1 \left( \frac{m_1}{m_0} + 2k_1 \frac{m_2^2}{m_0^2} \right) + 1. \)

Using the result of (B.17) in [26] again, we can also get:

\[
\frac{d}{dt} \int_{\Omega} \dot{\theta}^2 dx + \frac{1}{2} \int_{\Omega} \dot{\theta}_j \dot{\theta}_j dx
\]

\[
\leq \int_{\Omega} \omega_j \omega_j dx + \lambda_0 \int_{\Omega} \dot{\theta}^2 dx + 2 \left( \int_{\Omega} (\omega_j \omega_i)^2 dx \right)^{\frac{1}{2}} \int_{\Omega} \theta^4 dx \tag{76}
\]

Multiplying both sides of Equation (66) by \( 2\dot{\phi} \), and integrating over \( \Omega \times [0, t] \), we can get:

\[
\varepsilon_1 \frac{d}{dt} \int_{\Omega} \dot{\phi}^2 dx = 2 \int_{\Omega} \omega_j \dot{\phi}_j dx + 2 \left( \int_{\Omega} \phi \Delta \phi dx - 2L_\varepsilon \int_{\Omega} u_i \phi_j \phi_j dx \right)
\]

\[
= 2 \int_{\Omega} \omega_j \dot{\phi}_j dx + 2 \left( \int_{\Omega} \dot{\phi} \Delta \phi dx + 2L_\varepsilon \int_{\Omega} u_i \phi_j \phi_j dx \right). \tag{77}
\]

The first term on the right side of Equation (77) can be bounded by:

\[
2 \int_{\Omega} \omega_j \dot{\phi}_j dx \leq \int_{\Omega} \omega_j \omega_j dx + \int_{\Omega} \dot{\phi}^2 dx. \tag{78}
\]

Using the result (4), and taking \( \varepsilon_0 = 2k_2 \), we can get:

\[
\int_{\partial \Omega} \dot{\phi}^2 dS \leq \left( \frac{m_1}{m_0} + 2k_2 \frac{m_2^2}{m_0^2} \right) \int_{\Omega} \dot{\phi}^2 dx + \int_{\Omega} \dot{\phi}_j \dot{\phi}_j dx. \tag{79}
\]
The second term on the right side of (77) can be bounded as follows:

\[
2 \int_{\Omega} \phi \Delta \phi \, dx = 2 \int_{\partial \Omega} \frac{\partial \phi}{\partial n} \, dS - 2 \int_{\Omega} \phi_{,ij} \phi_{,ij} \, dx
\]

\[
= 2k_2 \int_{\partial \Omega} \phi^2 \, dS - 2 \int_{\Omega} \phi_{,ij} \phi_{,ij} \, dx
\]

\[
\leq 2k_2 \left( \frac{m_1}{m_0} + 2k_2 \frac{m_2^2}{m_0^2} \right) \int_{\Omega} \phi^2 \, dx - \int_{\Omega} \phi_{,ij} \phi_{,ij} \, dx. \tag{80}
\]

Combining (77), (78), and (80), we can obtain:

\[
\epsilon_1 \frac{d}{dt} \int_{\Omega} \phi^2 \, dx + \frac{1}{2} \int_{\Omega} \phi_{,ij} \phi_{,ij} \, dx
\]

\[
\leq \int_{\Omega} \omega_i \omega_i \, dx + m_{10} \int_{\Omega} \phi^2 \, dx + 2L_c^2 \int_{\Omega} u_i u_i \phi^2 \, dx \tag{81}
\]

\[
\leq \int_{\Omega} \omega_i \omega_i \, dx + m_{10} \int_{\Omega} \phi^2 \, dx + 2L_c^2 \left( \int_{\Omega} (u_i u_i)^2 \, dx \right)^{\frac{3}{4}} \left( \int \phi^4 \, dx \right)^{\frac{1}{4}},
\]

where \( m_{10} = 2k_2 \left( \frac{m_1}{m_0} + 2k_2 \frac{m_2^2}{m_0^2} \right) + 1 \).

Using the results (23) and (32), we can obtain:

\[
\epsilon_1 \frac{d}{dt} \int_{\Omega} \phi^2 \, dx + \frac{1}{2} \int_{\Omega} \phi_{,ij} \phi_{,ij} \, dx
\]

\[
\leq \int_{\Omega} \omega_i \omega_i \, dx + m_{10} \int_{\Omega} \phi^2 \, dx + 2L_c^2 m_1(t)n_5(t). \tag{82}
\]

A combination of (70), (76), (82), and (50) gives:

\[
\frac{d}{dt} \left[ \int_{\Omega} \omega_i \omega_i \, dx + \int_{\Omega} \phi^2 \, dx + \epsilon_1 \int_{\Omega} \phi_{,i} \phi_{,j} \, dx + \int_{\Omega} \omega_{i,j} (\omega_{i,j} - \omega_{j,i}) \, dx \right]
\]

\[
\leq (m_6 + 3) \int_{\Omega} \omega_i \omega_i + (R^2 + m_9) \int_{\Omega} \phi^2 \, dx + (C^2 + m_{10}) \int_{\Omega} \phi^2 \, dx
\]

\[
+ (3Mn_4(\tau) + 4C^2 + 4R^2) \int_{\Omega} \omega_{i,j} (\omega_{i,j} - \omega_{j,i}) \, dx + 2L_c^2 m_1(t)n_5(t). \tag{83}
\]

Let,

\[
F_3(t) = \int_{\Omega} \omega_i \omega_i \, dx + \int_{\Omega} \phi^2 \, dx + \epsilon_1 \int_{\Omega} \phi_{,i} \phi_{,j} \, dx + \int_{\Omega} \omega_{i,j} (\omega_{i,j} - \omega_{j,i}) \, dx,
\]

\[m_{11} = \max \left\{ m_6 + 3, \frac{R^2 + m_9}{\epsilon_1}, \frac{C^2 + m_{10}}{\epsilon_1}, 3Mn_4(\tau) + 4C^2 + 4R^2 \right\}. \]

From (83), it can be seen that:

\[
\frac{d}{dt} F_3(t) \leq 2L_c^2 m_1(t)n_5(t) + m_{11} F_2(t). \tag{84}
\]

by an integration of (84) leads to:

\[
F_3(t) \leq 2L_c^2 m_{11} e^{m_{11} t} \int_0^t m_1(\eta)n_5(\eta)e^{-m_{11} \eta} \, d\eta. \tag{85}
\]

Inequality (85) shows that when \( L_c \) tends to zero, the energy \( F_3(t) \) also tends to zero.

5. Conclusions

In this paper, we studied the convergence results for the double-diffusion perturbation equations in a bounded domain. The convergence result of solutions were gained for
the reaction boundary coefficients $k_1, k_2$ and the Lewis coefficient $L_e$. Using the method in this paper, similar results for other coefficients could also be gained. Our method is useful for studying the structural stability of bounded regions. However, for unbounded regions, because the regions become more complex, and the inequalities that can be used in bounded regions cannot be used in unbounded regions, essential difficulties will arise. Methods of dealing with stress terms will be the biggest difficulty in unbounded areas. It is an open problem now that we could solve by constructing special functions in relevant future research. In this paper, we only give a theoretical proof and a numerical simulation will be given in another paper.

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