Linear constrained combinatorial optimization on well-described sets

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Abstract. A Horizontal method (LCCO-HM) for linear constrained combinatorial optimization (LCCO) on well-described combinatorial sets (CWDs) is offered. LCCO-HM utilizes an essential feature of the sets to solve linear unconstrained programs in polynomial time. The method belongs to the Branch and Bound group. It reduces a search domain to solutions of auxiliary unconstrained combinatorial problems and performs pruning branches based on properties of a linear function on CWDs. Applicability of LCCO-HM is justified for sets of multipermutations and partial multipermutations embedded in Euclidean space. LCCO-HM is illustrated by an example of solving a permutation-based linear optimization problem.

1. Introduction
Problems of combinatorial optimization (COPs) are widely spread in theory and practice [1][2][3][4][5][6][7]. Many of them are modeled as linear combinatorial optimization problems (linear COPs, LCOPs) [1][3][4]. For instance, the widely known np-complete 0-1 Knapsack Problem is formulated as a constrained linear Boolean optimization problem, where the objective function expresses the knapsack utility, and a constraint is imposed on its weight or volume. Moreover, if this constraint was missing, a solution of the corresponding unconstrained linear program on the Boolean set \( B_n = \{0^n, 1^n\} \) would be easily found. Indeed, it would be a vector of units reflecting the fact that all items worth to be put in the knapsack since they are all useful by assumption. In addition to Boolean optimization problems, the same feature is possessed by permutation-based problems [8][9], since the minimum of a linear function on a numerical permutation set can be found in explicit form by properties of a linear function on the set [6].

In literature, these two and some other such sets are known as well-described (WDSs) [10]. Namely, according to [10], a WDS is a set, where the optimum of a linear function can be found efficiently, i.e., in polynomial time on the problem dimension. Thus, for example, permutation matrix set \( \Pi_n \) [11] belongs to these set’s class because a linear problem on \( \Pi_n \) can be solved in time \( O(n^3) \) [12], where \( n \) is the dimension of permutation matrices.

In this paper, we formulate the general problem of constrained linear optimization on a combinatorial WDS (CWDS) subject some additional requirements and propose a method called a Horizontal method for solving LCOPs on CWDSs (LCCO-HM). It is an exact method, which is based on a decomposition of the search space on lower-dimensional WDSs, solving unconstrained LCOPs on these sets, and analyzing auxiliary two-dimensional grid graphs [13][14].

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So, consider the following problem (further referred to as a CWDS-LCOP):

\[ z = cx \rightarrow \text{extr}, \]  
\[ A'x \leq b', \]  
\[ A''x = b'', \]  
\[ x \in E \subset \mathbb{R}^n, \]

where \( c \neq 0, \ A' \in \mathbb{R}^{m' \times n}, \ A'' \in \mathbb{R}^{m'' \times n}, \ b' \in \mathbb{R}^{m'}, \ b'' \in \mathbb{R}^{m''}, \ m', m'' \in \mathbb{Z}_+, \ m' + m'' \geq 1. \)

\[ E \text{ is a combinatorial WDS,} \]

i.e., \( E \) is a finite WDS.

It is necessary to find a complete solution to it, which is a tuple

\[ z^\text{extr} = \text{extr} cx, \quad X^\text{extr} = \text{Arg}\min_{x \in E'} cx, \]

where \( E' \) is a feasible domain of CWDS-LCOP given by (2)-(4).

This work continues a study begun in [14][15][16][17] on the application of graph approaches to solving combinatorial optimization problems, as well as on exploring and exploiting properties of basic sets of Euclidean combinatorial configurations (\( C_b \)-sets) [18][19][21][22] that are images in Euclidean space of classical combinatorial sets such as permutations, partial permutations, combinations, partitions, compositions, etc. It significantly extends results of [14][15][16][17] in many directions, because deals with much wider problem statement (1)-(5) with few additional requirements (see Section 2).

This paper is organized as follows. Section 2 introduces the necessary terminology used throughout this paper and contains required theoretical information about structural and extreme properties of WDSs \( E_{nk}(G), \ E_{nk}^m(G) \). Section 3 is dedicated to detailed description of the Horizontal method (LCCO-HM) of linear constrained optimization on the generic CWDs. Section 4 justifies applicability of LCCO-HM for linear optimization on the classes of sets \( E_{nk}(G), \ E_{nk}^m(G) \). An illustrative example of optimization on \( E_{nk}(G) \) by LCCO-HM is given in Section 5. Finally, Section 6 presents our conclusions.

2. Prerequisites

There are two standard combinatorial optimization approaches to solve CWDS-LCOP exactly – cutting-plane methods (CPMs) and branch and bound techniques (B&B) [2][3][4][23][24]. They deal with polyhedral relaxations of the original problem and similar problems induced by the original CWDS-LCOP and its polyhedral relaxation. Here, an issue is that combinatorial polytopes are often described by an exponential number of constraints. It results in the necessity of developing specific methods such as presented in [5][22][25]. Meanwhile, a possibility to efficiently solve any CWDS-LCOP with \( m', m'' \) depending polynomially on \( n \) is justified by the existence of a polynomial separating oracle [26] for a point of \( \mathbb{R} \) and the polytope \( \text{conv} E' \), wherefrom it follows a polynomial solvability of their polyhedral relaxations by ellipsoid methods [24][27].

Let us consider two classes of sets [19]:

\[ E_{nk}(G) = \{ x = (x_1, ..., x_n) \in \mathbb{R}^n : \{x_1, ..., x_n\} = G \} \]

is a basic \( C_b \)-set of multipermutations [28] induced by a multiset \( G = \{g_i\}_{i \in J_n} \subset \mathbb{R}^1 \) with \( k \) different elements;

\[ E_{nk}^m(G) = \{ x = (x_1, ..., x_n) \in \mathbb{R}^n : \{x_1, ..., x_n\} \subset G \} \]
is a basic $C_b$-set of partial multipermutations ($n$-multipermutations) induced by a multiset $G = \{g_i\}_{i \in J_q} \subset \mathbb{R}^1$ with $k$ different elements and $\eta > k$.

This means that there exist $e_1, ..., e_k \in \mathbb{R}^1$ such that

$$S(G) = \{e_1, ..., e_k\},$$

where $e_1 < ... < e_k$,

is a ground set [29] of $G$.

For a $C_b$-set of permutations, the number of irredundant constraints in H-representation of the generalized permutohedron $P_{nk}(G) = \text{conv} E_{nk}(G)$ reaches the value of $N = 2^n - 1$ for the $C_b$-set $E_n(G) = E_{nn}(G)$ of permutations [6]. For the generalized partial permutohedron $P_{nk}^p(G) = \text{conv} E_{nk}^p(G)$, $N$ reaches the value of $N = 2^{n+1} - 2$ on the partial permutohedron $P_{nk}^p(G) = \text{conv} E_{nk}^p(G)$ [5], where $E_{nk}^p(G) = E_{nn}(G)$. The same bounds are valid for any sets of these classes induced by $G$ with elements $e_1, e_2$ of multiplicity 1 [30][31]. Also, CPMs require a comprehensive study of structural and geometric properties of $C_b$-sets in order to form deep cuts and overcome an issue of slow convergence of CPMs [26]. For example, for sets coinciding with a vertex set of their convex hull (vertex-located sets, VLSs [32][33]), a combinatorial cutting plane method to solve LCOPs was proposed in [34]. It utilizes an absence of points of VLS $E$ in an interior of the faces of an arbitrary dimension of combinatorial polytope $P = \text{conv} E$. Effectiveness of the method was confirmed by the results of computational experiments [35]. Later, this method was generalized in many directions, e.g., onto a non-vertex located $C_b$-set of partial permutations [36], sets inscribed into a hypersphere or other strictly convex smooth surface [37], for nonlinear optimization problems [39] etc.

Regarding known so far B&B techniques for solving linear and nonlinear COPs on CWDS, they are based on partitions of these sets into sets of the same combinatorial type and lower dimension lying in parallel hyperplanes to coordinate ones [40][41][42][43][44].

### 3. A Horizontal Method For Solving LCOPs On CWDSs

Let us consider the following CWDS-LCOP, where, in addition to (1)-(5), the following requirements are satisfied: $E$ allows the following partition:

$$E = \bigcup_{i \in I} E(i),$$

$$E(i) \cap E(j) = \emptyset, \forall i \neq j, \ i, j \in I = J_{|I|},$$

$$E(i) \subset E$$ is a WDS, $i \in I$, (8)

where $J_n = \{1, ..., n\}$,

$$z_{\min}(i) \geq z_{\min}(i + 1), \ i \in J_{|I|-1},$$

$$z_{\max}(i) \geq z_{\max}(i + 1), \ i \in J_{|I|-1}. \ \text{(10)}$$

Here,

$$z_{\min}(i) = \min_{x \in E(i)} cx, \ z_{\max}(i) = \max_{x \in E(i)} cx. \ \text{(12)}$$

We also assume that the value $|I|$ depends polynomially on $n$, i.e.,

$$\exists \nu \in \mathbb{Z}_+, \exists \phi : \mathbb{R}^n \to \mathbb{R}^1 - \text{a polynomial of degree } \nu, \ |I| = \phi(n^\nu). \ \text{(13)}$$

Using a notation for a feasible region:

$$E' = \{x \in E: (2), (3) \text{ hold}\},$$

(14)
problem (1)-(11), (13) (further referred to as a CWDS-LCOP1) can be rewritten in the form of (1), (5)-(11), (13),
\[ x \in E', \]  
where \( E' \) is given by (14).

Without loss of generality, let us assume that \( \text{extr} = \min \), i.e., the following minimization problem is considered:
\[ z = cx \rightarrow \min \]  
subject to constraints (5)-(11), (13), (15).

We aim to find a tuple \( \langle X^*, z^* \rangle \):
\[ z^* = \min_{x \in E'} cx, \quad X^* = \text{Arg min}_{x \in E'} cx, \]  
implying that our goal is finding a complete solution of the problem.

Let us introduce notations \( x_{\min}, X_{\min}, z_{\min} \) (\( x_{\max}, X_{\max}, z_{\max} \)) for a certain and a complete solution of the unconstrained optimization problem (1), (4) on minimum and maximum, respectively. Then, due to (8)-(12), the following conditions holds:
\[ \langle X_{\min}, z_{\min} \rangle = \langle X_{\min}(I), z_{\min}(|I|) \rangle, \]
\[ \langle X_{\max}, z_{\max} \rangle = \langle X_{\max}(1), z_{\max}(1) \rangle. \]

In addition to (10), (11), a condition is satisfied
\[ z_{\min}(i) \leq z_{\max}(i), \quad i \in I, \]  
which can be represented by a grid graph \( G_r \) of dimension \(|I| \times 2\) depicted in Fig. 1. It is seen that \( \langle x_{\max}, z_{\max} \rangle = \langle x_{\max}(1), z_{\max}(1) \rangle \) is its source, while \( \langle X_{\min}, z_{\min} \rangle = \langle X_{\min}(1), z_{\min}(1) \rangle \) is its sink. \( G_r \) is digraph, and a direction of its arcs indicates a direction of non-increasing the objective function. Note that conditions (9), (13) ensure that all components of \( G_r \) can be found in polynomial time.

\[ \langle x_{\max}(1), z_{\max}(1) \rangle \quad \langle x_{\min}(1), z_{\min}(1) \rangle \]
\[ \langle x_{\max}(2), z_{\max}(2) \rangle \quad \langle x_{\min}(2), z_{\min}(2) \rangle \]
\[ \langle x_{\min}(|I|-1), z_{\min}(|I|-1) \rangle \quad \langle x_{\max}(|I|-1), z_{\max}(|I|-1) \rangle \]
\[ \langle x_{\min}(|I|), z_{\min}(|I|) \rangle \quad \langle x_{\max}(|I|), z_{\max}(|I|) \rangle \]

**Figure 1.** Directed grid graph \( G_r \).

An idea of our method to solve CWDS-LCOP1 is to make branchings based on the decomposition (8) and analyze only solutions of the corresponding unconstrained CWDS-LCOP1 at each vertex of a search tree. It results in that if the search is limited by a polynomial number
of nodes of a search tree, then a solution to the problem is also formed in polynomial time on the dimension of the problem.

Introduce a notation CWDS-LCOP1([☆]) for a problem of the form of (16),

\[
A'(☆) x \leq b'(☆), \\
A''(☆) x = b''(☆), \\
x \in E(☆),
\]

where [☆] ⊆ G, \(A'(☆) \in \mathbb{R}^{m'(☆) \times n}\), \(A''(☆) \in \mathbb{R}^{m''(☆) \times n}\), \(b'(☆) \in \mathbb{R}^{m'(☆)}\), \(b''(☆) \in \mathbb{R}^{m''(☆)}\), \(m'(☆) \leq m'\), \(m''(☆) \leq m''\),

\[
E(☆) \text{ is a WDS},
\]

\[
E(☆) = \bigcup_{i=1}^{\vert I(☆) \vert} E(☆, i), \quad E(☆, i) \bigcap E(☆, j) = \emptyset, \; \forall i \neq j, \; i, j \in I(☆),
\]

\[
E(☆, i) \text{ is a WDS}, \; i \in I(☆),
\]

\[
z_{\min}(☆, i) \leq z_{\min}(☆, i+1), \; i \in J_{I(☆)} - 1,
\]

\[
z_{\max}(☆, i) \leq z_{\max}(☆, i+1), \; i \in J_{I(☆)} - 1,
\]

\[
|I(☆)| \text{ polynomially depends on } n(☆),
\]

where

\[
I(☆) \subseteq I, \; n(☆) = n - |I(☆)|,
\]

\[
z_{\min}(☆) = \min_{x \in E(☆)} cx, \quad z_{\max}(☆) = \max_{x \in E(☆)} cx.
\]

Let also

\[
X_{\min}^\ast(☆) = \arg \min_{x \in E(☆)} cx, \quad X_{\max}^\ast(☆) = \arg \max_{x \in E(☆)} cx.
\]

Then CWDS-LCOP1 (5)-(11),(13)-(16) is a special case of CWDS-LCOP1([☆]) with [☆] = ∅. In the decision tree, CWDS-LCOP1([☆]) with [☆] ≠ ∅ will serve as a node induced a branch \(B(☆, z_{\min}(☆))\), while \(B(0, z_{\min})\) will correspond to CWDS-LCOP1 being a root of this tree.

We associate CWDS-LCOP1([☆]) with the unconstrained linear combinatorial optimization problem (further referred to as CWDS-ULCOP1([☆])) obtained from CWDS-LCOP1([☆]) by eliminating the additional constraints (18),(19). For instance, CWDS-ULCOP1=CWDS-ULCOP(0) is a problem (4)-(11),(13),(16).

3.1. CWDS-LCOP1([☆])-properties

**Property P1** If

\[
m'(☆) + m''(☆) = 0,
\]

then CWDS-LCOP1([☆]) = CWDS-ULCOP1([☆]). On this branch, the problem solutions can only be among the elements of \(X_{\min}^\ast([☆])\). Therefore, after its examining, this branch can be cut off.

**Property P2** Let \((x^\ast, z^\ast)\) be a current solution, then

(i) if

\[
z^\ast < z_{\min}^\ast(☆),
\]

then the branch CWDS-LCOP1([☆]) can be pruned;
value of since a minimization problem is solved, when choosing a node for branching, we will rely on the
then we try to update the current solution:
Combining Remark 1. and, in some cases, the usage of property P1 if the condition (30) holds.
(ii) if
then, within this branch, CWDS-LCOP1-solutions can only be among elements of \(X_{\text{min}}([\bullet])\).
So, after examining them, this branch can be cut off;
(iii) CWDS-LCOP1([\bullet]) is incompatible if at least one of the below conditions is met:
where
\[ z^{\text{extr}, j}([\bullet]) = \text{extr}_{x \in E([\bullet])} a'_j([\bullet]) x, \ j \in J_{m'([\bullet])}, \]
\[ A'([\bullet]) = (a'_j([\bullet]))_{j \in J_{m'([\bullet])}}, \ A''([\bullet]) = (a''_j([\bullet]))_{j \in J_{m'([\bullet])}}. \]
Property P3 For \(j \in J_{m'([\bullet])}\), constraint \(a'_j([\bullet]) x \leq b'_j([\bullet])\) is redundant if
\[ z^{\text{max}, j}([\bullet]) \leq b'_j([\bullet]). \]
The condition (35) allows reducing the number of additional constraints in CWDS-LCOP1([\bullet]) and, in some cases, the usage of property P1 if the condition (30) holds.
Remark 1. Combining (34) with (35) we obtain a necessary condition
\[ z^{\text{min}, j}([\bullet]) \leq b'_j([\bullet]) < z^{\text{max}, j}([\bullet]) \]
that CWDS-LCOP1([\bullet]) is compatible and the constraint \(a'_j([\bullet]) x \leq b_j([\bullet])\) cannot be eliminated \((j \in J_{m'([\bullet])})\).
As an alternative to the condition (36), the following one can be used
\[ a'_j([\bullet]) x^{\text{min}}([\bullet]) \leq b'_j([\bullet]) < a'_j([\bullet]) x^{\text{max}}([\bullet]), \ j \in J_{m'([\bullet])}. \]
Clearly, \(z_{\text{min}}([\bullet])\) is the lower bound on \(f(x)\) for the branch CWDS-LCOP1([\bullet]). That is why, since a minimization problem is solved, when choosing a node for branching, we will rely on the value of \(z_{\text{min}}([\bullet])\) choosing, first of all, a node with the least value attained.
If at some step of the recursive process we obtain \(y \in E'([\bullet])\), where
\[ E'([\bullet]) = \{x \in E([\bullet]) : (18), (19) \text{ hold}\}, \]
then we try to update the current solution:
\[ \text{if } cy < z^{**}, \text{ then } X^{**} = \{y\}, \]
\[ \text{if } cy = z^{**}, \text{ then } X^{**} = X^{**} \cup \{y\}. \]
Let us denote \(\text{sol}^{\text{extr}}([\bullet], j) = \langle X^{\text{extr}}([\bullet]), z^{\text{extr}}([\bullet]) \rangle\).
3.2. CWDS-LCOP1(_) Examination

A branch CWDS-LCOP1([•]) will be examined in two stages:

(i) Based on P3, we form the system of constraints (18), (19), then verify properties P1, P2 and try to prune this branch.

(ii) If the branch is not pruned, we build graph $G_r([•])$ shown in Fig. 2.

(iii) We examine the $G_r([•])$-nodes attempting to find a feasible solution $x^{**}$ or improve the current solution $x^{**}$ by elements of $X^{\text{min}}([•])$, $X^{\text{max}}([•])$ and observing the nodes in the following order:

(a) $\text{sol}^{\text{min}}([•], |I([•])|)$, $\text{sol}^{\text{min}}([•], |I([•])| - 1)$, ..., $\text{sol}^{\text{min}}([•], j)$, where

\[ j : j = 1 \text{ or } z^{\text{min}}([•], j) \leq z^{**} \text{ and } z^{\text{min}}([•], j - 1) > z^{**}; \]

(40)

(b) $\text{sol}^{\text{max}}([•], |I([•])|)$, $\text{sol}^{\text{max}}([•], |I([•])| - 1)$, ..., $\text{sol}^{\text{max}}([•], j')$, where

\[ j' : j' = 1 \text{ or } z^{\text{max}}([•], j') \leq z^{**} \text{ and } z^{\text{max}}([•], j' - 1) > z^{**}; \]

(41)

(c) replace the branch CWDS-LCOP1([•]) in the queue by CWDS-LCOP1([•], $i$), $j \in J_{|I([•])|} \backslash J_{j-1}$, where $i$ is given by (40).

![Directed grid graph $G_r([•])$.](image)

Figure 2. Directed grid graph $G_r([•])$.

The presented method will be called a Horizontal method for solving LCOPs on CWDSs (or abbreviated LCCO-HM).

4. LCCO-HM For Optimization On $E_{nk}(G)$, $E_{nk}^n(G)$

In this section, it is justified the applicability of LCCO-HM to optimization on the following $C_k$-sets:

\[ E \in \{E_{nk}(G), E_{nk}^n(G)\}. \]

Let

\[ \{i_j\}_{j \in J_n} : \{i_j\}_{j \in J_n} = J_n, c_{i_j} \leq c_{i_{j+1}}, j \in J_{n-1}, \]

(42)

\[ s \in J_n^0 : c_{i_s} \leq 0, c_{i_{s+1}} > 0, \]

where $i_0 = 0$, $i_{n+1} = n + 1$, $c_0 = M$, $c_{n+1} = -M$, $M > 0$ is a large constant.
Theorem 1. \cite{43} For,

\[ E = E_{nk}(G), \]  

\[ x_{ij}^{\text{max}} = g_j, \ j \in J_n; \ z^{\text{max}} = \sum_{j=1}^{n} c_i g_j; \]  

\[ x_{ij}^{\text{min}} = g_{n-j+1}, \ j \in J_n; \ z^{\text{min}} = \sum_{j=1}^{n} c_i g_{n-j+1}. \]

Theorem 2. \cite{5} For

\[ E = E_{nk}^n(G), \]  

\[ x_{ij}^{\text{max}} = g_j, \ j \in J_s; x_{i,n-j+1}^{\text{max}} = g_{n-j+1}, \ j \in J_{n-s}; \]  

\[ z^{\text{max}} = \sum_{j=1}^{s} c_i g_j + \sum_{j=1}^{n-s} c_{n-j+1} g_{n-j+1}; \]  

\[ x_{ij}^{\text{min}} = g_{n-j+1}, \ j \in J_s; x_{i,n-j+1}^{\text{min}} = g_j, \ j \in J_{n-s}; \]  

\[ z^{\text{min}} = \sum_{j=1}^{s} c_i g_{n-j+1} + \sum_{j=1}^{n-s} c_{n-j+1} g_j. \]

Corollary 1. Sets (6), (7) are WDSs.

Corollary 2. If coefficients of the objective function (1) are ordered such that \( c_1 \leq c_2 \leq \ldots \leq c_n \),

then:

(i) for the set (43),

\[ x_j^{\text{max}} = g_j, \ j \in J_n; \ z^{\text{max}} = \sum_{j=1}^{n} c_j g_j; \]  

\[ x_j^{\text{min}} = g_{n-j+1}, \ j \in J_n; \ z^{\text{min}} = \sum_{j=1}^{n} c_j g_{n-j+1}; \]

(ii) For the set (45),

\[ x_j^{\text{max}} = g_j, \ j \in J_s; x_j^{\text{max}} = g_{n-j+1}, \ j \in J_{n-s}; \]  

\[ z^{\text{max}} = \sum_{j=1}^{s} c_{n-j+1} g_j + \sum_{j=1}^{n-s} c_j g_{n-j+1}; \]  

\[ x_j^{\text{min}} = g_{n-j+1}, \ j \in J_s; x_j^{\text{min}} = g_j, \ j \in J_{n-s}; \]  

\[ z^{\text{min}} = \sum_{j=1}^{s} c_j g_{n-j+1} + \sum_{j=1}^{n-s} c_{n-j+1} g_j. \]
Theorem 3. (i) For set (43), $X^{extr}$ is formed from a point $x^{extr}$ found by formula (44) by permuting its coordinates within groups of identical coefficients of the objective function; (ii) for set (45), $X^{extr}$ is formed from a point $x^{extr}$ found by formula (46) first by permuting its coordinates within groups of identical coefficients of the objective function other than zero, and then within the group zero coefficients of the function by forming partial permutations of the corresponding dimension from unused so far elements of $G$ in a formation of a current point $x \in X^{extr}$.

Corollary 3. (i) If $c_1 < c_2 < \ldots < c_n$, then, for the set (43),

$$X^{extr} = x^{extr}.$$  \hspace{1cm} (51)

If, in addition, $c_s < 0 < c_{s+1}$, then, for the set (45), (51) holds.

(ii) If (50) does not hold and $(n_1, \ldots, n_l)$ are multiplicities of a ground set of $C = \{e_1, \ldots, e_n\}$, then, for set (43), $X^{extr}$ is a Cartesian product of $l$ basic $C_b$-sets of multipermutations and $|X^{extr}| \leq n_1! \cdot \ldots \cdot n_l!$. The same holds for (45), if, in addition, (52) is satisfied.

(iii) If conditions (50) and (52) are not satisfied, then $X^{extr}$ is a Cartesian product of $l - 1$ basic $C_b$-sets of multipermutations and a basic $C_s$-set of $n_j$-multipermutations, $n_j$ is the multiplicity of zero, and $|X^{extr}| \leq n_1! \cdot \ldots \cdot n_j^{-1}! \cdot \frac{(n-n_j)!}{(n-2n_j)} \cdot n_j+1! \cdot \ldots \cdot n_l!$.

Corollary 3 says that, for sets (43), (45), $X^{extr}$ can be found in polynomial time on $n$ in terms of $C_b$-sets of multipermutations and partial multipermutations. If, in addition, the multiplicities $n_1, \ldots, n_l, \eta - n_j$ do not depend in $n$, $X^{extr}$ can be enumerated in polynomial time.

Theorem 4. If (47) holds, then sets (42) satisfy constraints (8)-(11) when choosing

$$E(i) = \{x \in E : x_n = e_{k-i+1}, \ i \in J_k\}. \hspace{1cm} (53)$$

Proof. (i) For sets (6), (7), condition (5) is satisfied by Cor. 1.

(ii) For $E = E_{nk}(G)$,

$$E(i) = \{x \in E_{nk}(G) : x_n = e_{k-i+1}\} \simeq E_{n-1,k(i)}(G(i)), \ i \in J_k,$$  \hspace{1cm} (54)

where

$$G(i) = G \setminus \{e_{k-i+1}\}, \ k(i) = |S(G(i))|, \ i \in J_k.$$  \hspace{1cm} (55)

(iii) For $E = E_{nk}^n(G)$,

$$E(i) = \{x \in E_{nk}^n(G) : x_n = e_{k-i_1}\}, i \in J_k \simeq E_{n-1,k(i)}^n(G(i)),$$  \hspace{1cm} (56)

where $G(i), k(i)$ satisfy constraint (56).

From (54), it is seen that CWDS-LCOP1(j) will also be a linear constrained optimization problem on a WDS. Thus, the condition (9) is satisfied for

$$I = J_k.$$  \hspace{1cm} (57)

(iv) Condition (8) also holds, since formulas (53), (56) yield an expansion of the corresponding set $E$ into parallel hyperplanes. From that, it follows that the condition (13) is satisfied.
(v) The constraints (10), (11) are also satisfied. Indeed, by (47) and (55), from Col. 3, it follows that $x_n^\text{min}(k-i+1) = e_i$, while first $n-1$ coordinates of $x_n^\text{max}$ are elements of $G \setminus \{e_i\}$ ordered non-decreasingly. Let us find $j \in J_n$ such that $g_j = e_i, g_j+1 = e_i+1$. Then, by (48),
\[
x_n^\text{max}(k-i+1) = (g_1, \ldots, g_{j-1}, g_{j+1}, \ldots, g_n, g_j) = (g_1, \ldots, g_{j-1}, e_{i+1}, \ldots, g_n, e_i),
\]
where from, $z_n^\text{max}(k-i) = (g_1, \ldots, g_{j-1}, g_j, \ldots, g_n, g_{j+1}) = (g_1, \ldots, g_{j-1}, e_i, \ldots, g_n, e_{i+1}).$

From (58), (59) it is seen that $x_n^\text{max}(i), x_n^\text{max}(i+1)$ are adjacent permutations. For them, $z_n^\text{max}(k-i) - z_n^\text{max}(k-i+1) = c_j(e_i - e_{i+1}) + c_n(e_{i+1} - e_i) = (c_n - c_j)(e_{i+1} - e_i) \geq 0$, where from, $z_n^\text{max}(k-i+1) \leq z_n^\text{max}(k-i)$ for $i \in J_{k-1}$. Thus, condition (11) also holds.

Similarly,
\[
x_n^\text{min}(k-i+1) = (g_n, \ldots, g_{n-j+2}, g_{n-j'}, \ldots, g_1, g_{n-j'+1}) = (g_n, \ldots, g_{n-j'+1}, e_{i+1}, \ldots, g_1, e_i),
\]
\[
x_n^\text{min}(k-i) = (g_n, \ldots, g_{n-j+2}, g_{n-j'+1}, \ldots, g_1, g_{n-j'}) = (g_n, \ldots, g_{n-j'+1}, e_i, \ldots, g_1, e_{i+1}),
\]
where $j' \in J_k$ such that $g_{n-j'+1} = e_i, g_{n-j'} = e_{i+1}$.

From that, $z_n^\text{min}(k-i) - z_n^\text{min}(k-i+1) = c_j(e_i - e_{i+1}) + c_n(e_{i+1} - e_i) = (c_n - c_j)(e_{i+1} - e_i) \geq 0$, where from, $z_n^\text{min}(k-i+1) \leq z_n^\text{min}(k-i)$ for $i \in J_{k-1}$. Hence, condition (10) holds.

In the same way, the fairness of (10), (11) is justified for the set (45).

(vi) (57) implies that (13) holds with $\nu = 0$.

\[\square\]

**Remark 2.** Applying the branching schemes (56), (54) iteratively means a sequential fixation of the last $|\{\star\}|$ coordinates in CWDS-LCOP1(\{\star\})-solutions.

For instance, for sets (6), (7) under consideration, CWDS-LCOP1(\{\star\}) induces a set of problems CWDS-LCOP1(i, j), $i \in J_k, j(i) \in J_k$, such that
\[
E(i, j(i)) = \{x \in E : x_n = e_{k-i+1}, x_{n-1} = e_{k(i) - j(i)+1}\}, i \in J_k, j(i) \in J_k.
\]

In this case, much more convenient to represent the branches (60) as follows:
\[
E(\Omega) = E(\omega_1, \omega_2) = \{x \in E : x_{n-1} = \omega_1, x_n = \omega_2\},
\]
where
\[
\text{for (6), } \Omega \in E^2_{nk}(G);
\]
\[
\text{for (7), } \Omega \in E^2_{nqk}(G).
\]

Now, taking $[\star] = \Omega$ in (18)-(28), we obtain a representation of CWDS-LCOP(\Omega) for sets (6), (7). In particular,
\[
E(\Omega) = \bigcup_{i \in I(\Omega)} E(e_i, \Omega) = \{x \in E : x_{n-j} = e_i, x_{n-j+1} = \omega_{\Omega-j+1}, j \in J_{[\Omega]} \} \text{ is a WDS,}
\]
\[
i \in I(\Omega) = \{i \in J_k : \{e_i, \Omega\} \subseteq G\},
\]
\[
z_n^\text{min}(e_i, \Omega) \leq z_n^\text{min}(e_i, \Omega), i < j, i, j \in I(\Omega),
\]
\[
z_n^\text{max}(e_i, \Omega) \leq z_n^\text{max}(e_i, \Omega), i < j, i, j \in I(\Omega).
\]
Remark 3. This scheme can be modified, for instance, by sequential choosing the coordinates in decreasing order of absolute values of the objective function coefficients.

In this case, in addition to the ordered set $[\star]$ of $G$-elements, one needs also consider a set of $\Lambda$ of indices of fixed coordinates similar.

5. LCCO-HM Illustration

Solve CWDS-LCOP1 with the following input $n = 8$, $m' = 2$,
\[
c = (-6 -5 -2 -1 3 5 8 10)\; ;
\]
\[
(A'|b') = \begin{pmatrix}
-4 & -1 & 1 & 10 & 2 & 4 & 7 & 8 & | & 95 \\
3 & 5 & -4 & -5 & -7 & -1 & -2 & 4 & | & -12 \\
\end{pmatrix} ;
\]

$E = E_{nk}(G)$, where $G = \{1, 2, 2, 3, 4, 4, 6\}$.

Here, $G = \{1, 2^2, 3, 4^3, 6\}$, hence, $S(G) = \{1, 2, 3, 4, 6\}$, $k = 5$, thus, the problem is solved on $E = E_{nk}(G) = E_{85}(G)$.

Remark 4. On each step, as soon as a new point $x \in E(\Omega)$ is found, we will check its feasibility, i.e., satisfying additional constraints (18), (19) becoming
\[
A'
(\Omega) x \leq b'
(\Omega),
\]
\[
A''
(\Omega) x = b''
(\Omega).
\]

For that, residuals
\[
\Delta'
(x, \Omega) = A'
(\Omega) x - b'
(\Omega),
\]
\[
\Delta''
(x, \Omega) = A''
(\Omega) x - b''
(\Omega)
\]
will be evaluated. If
\[
\Delta'
(x, \Omega) \geq 0, \Delta''
(x, \Omega) = 0 \Rightarrow x \in E',
\]
otherwise,
\[
x \notin E'.
\]

In this example, equality constraints are missing, and $\langle A'(\Omega), b'(\Omega) \rangle = \langle A', b' \rangle$, that is why (68), (69) become:
\[
\Delta'
(x) = A' x - b',
\]
\[
if \Delta'
(x) \geq 0 \Rightarrow x \in E',
\]
otherwise (70) holds.

Solution. The coefficients of the objective function satisfy condition (50), that is why, in accordance with Cor. 3,
\[
\forall \Omega \subseteq J_n, X_{\text{max}}(\Omega) = x_{\text{max}}(\Omega), X_{\text{min}}(\Omega) = x_{\text{min}}(\Omega).
\]

1. Set $\Omega = [\cdot] = \emptyset$. By (48),
\[
\text{sol}_{\text{max}}(\Omega) = \text{sol}_{\text{max}} = (x_{\text{max}}, z_{\text{max}}) = ((1, 2, 2, 3, 4, 4, 6), 101) ;
\]
\[
\text{sol}_{\text{min}}(\Omega) = \text{sol}_{\text{min}} = (x_{\text{min}}, z_{\text{min}}) = ((6, 4, 4, 4, 3, 2, 2, 1), -23).}
Queue is \( Q = \{ B(\emptyset, -23) \} \).
Check (69) for \( x^\text{min}, x^\text{max} \). (71) becomes:

\[
\Delta'(x^{\text{extr}}) = A' x^{\text{extr}} - b',
\]

\[
\Delta'(x^{\text{min}}) = (43, -9) \not\geq 0 \text{ hence, } x^{\text{min}} \notin E'; \Delta'(x^{\text{max}}) = (-31, -14) \not\geq 0 \text{ hence, } x^{\text{max}} \notin E'.
\]
Verify condition (36) becoming

\[
z^{\text{min}, i} \leq b'_i < z^{\text{max}, i}, \; j = 1, 2. \tag{72}
\]

Points \( x^{\text{min}, 1} = (6, 4, 4, 1, 4, 3, 2, 2), \; x^{\text{max}, 1} = (1, 2, 2, 6, 3, 4, 4, 4), \; x^{\text{min}, 2} = (4, 6, 2, 2, 1, 4, 3, 4) \), \( x^{\text{max}, 2} = (2, 1, 4, 4, 6, 3, 4, 2) \) and values \( \{(z^{\text{min}, i}, z^{\text{max}, i})\}_{j=1,2} = \{(36, 138), (-23, 70)\} \) are found by (44). (72) holds since \( b'_1 = 95 \in [36, 138], b'_2 = 12 \in [-23, 70]. \)

Form thr grid graph \( G_r \) (see Fig. 3): based on the decomposition (54) having the form of:

\[
E(i) = \{ x \in E_{85}(G) : x_n = e_{6-i} \} \simeq E_{7,k(i)}(G(i)), \; i \in J_5,
\]
where, for instance, \( G(i) = G' \{ \emptyset \} = \{ 2, 2, 3, 4, 4, 4, 6 \}, k(1) = 4, E(1) \simeq E_{7,4}(G(1)) \). It has the form of graph shown in Fig. 3, its dimension is \( k \times 2 = 5 \times 2 \). A node

\[
\langle x^{\text{max}}(1), z^{\text{max}}(1) \rangle = \langle x^{\text{max}}(1), z^{\text{max}}(1) \rangle = \langle (1, 2, 2, 3, 4, 4, 4, 6), 101 \rangle
\]
is a source of the directed graph, while a node \( \langle x^{\text{min}}(5), z^{\text{min}}(5) \rangle = \langle x^{\text{min}}(5), z^{\text{min}}(5) \rangle = \langle (6, 4, 4, 4, 3, 2, 2, 1), -23 \rangle \) is its sink. Direction of arcs is from source toward sink. Tab. 1 contains the residuals (68) for nodes of \( G_r \). We explore nodes \( \text{sol}^{\text{min}, 5}, \; \text{sol}^{\text{min}, 4}, \ldots, \; \text{sol}^{\text{min}, 1} \) in order to find \( x^{**} \in E \). From Tab. 1, \( \text{sol}^{\text{min}, 5}, \; \text{sol}^{\text{min}, 4} \notin E', \) but \( \text{sol}^{\text{min}, 2} \in E' \) (highlighted by green in Fig. 3 and by bold in Tab. 1). Thus, the current solution is \( \text{sol} \langle x^{**}, z^{**} \rangle = \langle (6, 4, 4, 4, 2, 2, 1, 3), -14 \rangle \). This means that, by (39)-(40), branches corresponding to \( j = 1, 2, 3 \) or \( \Omega \in \{6, 4, 3\} \) are discarded (white nodes), remaining nodes need exploration (highlighted by grey). Rest nodes of \( G_r \) are not able to improve or complement the current solution. \( Q = \{ B(\emptyset, -23) \} \) becomes \( Q = \{ B(1, -23), B(2, -21) \} \).

| Table 1. \( G_r \)-nodes: residuals |
|---|---|---|---|---|
| i \( \Omega \) | \( x = x^{\text{max}}(\Omega) \) | \( x = x^{\text{min}}(\Omega) \) | \( \Delta_i(x) \) | \( \Delta_i(x) \) |
| 1 | 6 | -31 | -14 | 14 | 19 |
| 2 | 4 | -29 | -26 | 38 | 17 |
| 3 | 3 | -31 | -35 | 36 | 8 |
| 4 | 2 | -24 | -43 | 42 | 3 |
| 5 | 1 | -12 | -44 | 43 | 9 |

2. Branch \( B(1, -23) \). Information on \( G_r(1) \) and residuals corresponding \( \Omega = \{ 1 \} \) combine in Tab. 2. Here, inputs in the shadowed cells are only evaluated and analyzed. Order of examination is \( \text{sol}^{\text{min}, 2}(1), \; \text{sol}^{\text{min}, 3}(1), \; \text{sol}^{\text{min}, 3}(1), \) after that \( B(\{4, 1\}, -9) \) is pruned, and \( \Omega = \{ 6, 1 \} \) is not examined. Then \( \text{sol}^{\text{max}}(2, 1) \) is found. Since \( z^{\text{max}}(2, 1) = 29 > z^{**} \), remaining nodes of \( G_r(1) \) are not examined. Moreover, residuals are evaluated only for \( \text{sol}^{\text{min}, 2}(2), \; \text{sol}^{\text{min}, 3}(1), \) where a new record can be found, which is also indicated in Tab. 2. No feasible new feasible points were found; therefore, cells shadowed by green are missing. \( Q = Q \{ B(1, -23) \} + \{ B(\{2, 1\}, -23), B(\{3, 1\}, -18) \} \).
Figure 3. Directed grid graph Example: $G_r$.

Table 2. $Gr(1)$

| i | $\Omega$ | $sol^{\text{max}}(\Omega)$ | $x = x^{\text{max}}(\Omega)$ | $sol^{\text{min}}(\Omega)$ | $x = x^{\text{min}}(\Omega)$ |
|---|---|---|---|---|---|
| 1 | 6,1 | (2,2,3,4,4,6,1) | 58 | -12 | 44 | (4,4,3,2,6,1) | 19 | 19 | -11 |
| 2 | 4,1 | (2,2,3,4,6,1) | 52 | -6 | -42 | (6,4,4,3,2,4,1) | 9 | 41 | -1 |
| 3 | 3,1 | (2,2,4,4,4,6,3) | 42 | 0 | -44 | (6,4,4,4,2,3,1) | -18 | -38 | -4 |
| 4 | 2,1 | (2,3,4,4,6,2,1) | 29 | 8 | -37 | (6,4,4,3,2,2,1) | -23 | 43 | -9 |

Table 3. $Gr(\{2,\})$

| i | $\Omega$ | $sol^{\text{max}}(\Omega)$ | $x = x^{\text{max}}(\Omega)$ | $sol^{\text{min}}(\Omega)$ | $x = x^{\text{min}}(\Omega)$ |
|---|---|---|---|---|---|
| 1 | 6,2,1 | (2,3,4,4,6,2,1) | 29 | 8 | -37 | (4,4,3,2,6,2,1) | 7 | 31 | -7 |
| 2 | 4,2,1 | (2,3,4,4,6,2,1) | 25 | 12 | -49 | (6,4,4,3,2,4,1) | -15 | 47 | 1 |
| 3 | 3,2,1 | (2,4,4,4,6,3,2) | 15 | 17 | -43 | (6,4,4,4,2,3,2) | -21 | 41 | -3 |
| 4 | 2,2,1 | (3,4,4,4,6,2,2,1) | -4 | 9 | -35 | (6,4,4,3,2,2,1) | -23 | 43 | -9 |

3. Branch $B(\{2,\},-23)$ is analyzed in Tab. 3. Similar to $B(2,-23)$, only four out of 4 out of 8 nodes of $Gr(\{2,\})$ are analyzed, residuals are evaluated for two of them, no records are found. $Q = Q\{B(\{2,\},-23)} + B(\{2,1\},-23, B(\{3,2,1\},-21})$.

4. Analysis of $B(\{2,2,1\},-23)$ is shown in Tab. 4. $Q = Q\{B(\{2,2,1\},-23)} + B(\{3,2,2,1\},-23, B(\{4,2,2,1\},-19})$.

5. Analysis of $B(\Omega,-23)$ for $\Omega = \{3,2,2,1\}$.

$E(\Omega) \sim E' = E_{43}(G(\Omega))$ induced by $G(\Omega) = G(\Omega) = G(\Omega) = \{4,6\}$;

$E'(\Omega) = \{x^4\}_{x \in J_4} = \{x^4, (4,6,4^2,4^2,6,4), (6,4^3)\} \quad \text{(73)}$

is combinatorialy equivalent to a $C_{23}$-set of multipermutations of cardinality 4, half of which
Table 4. $Gr\{(2, 2, 1)\}$

| i | $\Omega$ | $s o l_{\text{max}}(\Omega)$ | $x = x_{\text{max}}^{(\Omega)}$ | $s o l_{\text{min}}(\Omega)$ | $x = x_{\text{min}}^{(\Omega)}$ |
|---|---|---|---|---|---|
| 1 | 6,2,2,1 | $(2,3,4,6,2,2,1)$ | 15 | 20 | -47 | 2 | 50 | -22 |
| 2 | 4,2,2,1 | $(3,4,4,6,2,2,1)$ | -4 | 9 | -35 | 6,4,4,3,4,2,2 | 1 | -19 | 51 | -11 |
| 3 | 3,2,2,1 | $(4,4,4,3,2,2,1)$ | -13 | 15 | -25 | 6,4,4,4,3,2,2 | 2 | -23 | 43 | -9 |

$(x^1, x^4)$ has already been analyzed. For two rest ones $\{c^T(x_j^j, \Omega), \Delta_1^j(x_j^j(\Omega)), \Delta_2^j(x_j^j(\Omega))\}_{j=2,3} = \{(-21,37,-5), (-15,33,-23)\}$. No improvements of the objective function are found;

$$Q = Q\backslash \{B(\{3,2,2,1\}, -23)\}.$$  

6. Now, branch $B(2)$ with the least lower bound $z_{\text{min}}(\Omega) = -21$ is considered. Results of its analysis is presented in Tab. 5. The same as in Tab. 2-Tab. 4, the number of analyzed nodes is four, half of which are explored partially;

$$Q = Q\backslash \{B(\{2\}, -21)\} + \{B(\{1,2\}, -21), B(\{2,2\}, -18)\}.$$  

7. Explore the branch $B(\{2,1\}, -21)$ (see Tab. 6). It is seen that only three nodes of $Gr(\{2,1\})$ need to be analyzed. A new record of the objective function is found. Namely, $z_{\text{min}}((\Omega,2)) = -21 > z^{**}$, $(\Delta_{j}^j(\Omega,2)))_{j=2,3} = (40,3) \geq 0$, hence, by (38),

$$(x^{**}, z^{**}) = (x_{\text{min}}(\Omega,2), z_{\text{min}}(\Omega,2)) = (6,4,4,4,2,3,1,2), -19).$$

Now, in the queue, $B(\{2,1\}, -21)$ is replaced by $B(\{2,2,1\}, -21)$. Also, branches with low bound less than $-19$ are removed. Thus,

$$Q = Q\backslash \{B(\{1,2\}, -21), B(\{2,2\}, -18), B(\{3,1\}, -18), B(\{2,2\}, -18)\} + \{B(\{2,1,2\}, -21)\}.$$  

8. The branch $B(\{2,2,1\}, -21)$ analysis is presented in Tab. 7, namely, two nodes are analyzed fully and one – partially,

$$Q = Q\backslash \{B(\{2,1,2\}, -21)\} + \{B(\{3,2,1\}, -21)\}.$$  

9. Perform exploring the branch $B(\{\Omega,\}) = B(\{3,2,1,2\}, \cdot)$ in the same way, as $B(\{3,2,2,1\}, -23)$. For $E(\Omega), G(\Omega)$, conditions (73) are satisfied. Points $x^1, x^4$ has already been analyzed in Tab. 6. Rest two are $x^2(\Omega) = (4,6,4,4,3,2,1,2), x^3(\Omega) = (4,6,4,4,3,2,1,2)$ with $\{c^T x^j(\Omega), \Delta_1^j(x_j^j(\Omega)), \Delta_2^j(x_j^j(\Omega))\}_{j=2,3} = \{(-19,36,1), (-13,32,17)\}$ $x^2(\Omega) \in E'$ and $c^T x^2(\Omega) = -19 = z^{**}$. That is why, by (39), the current $X^{**}$ is complemented resulting in $X^{**} = x^{**} \cup x^2(\Omega), Q = Q\backslash \{B(\{3,2,1,2\}, -21)\}$.  

10. To $B(\{3,2,1\}, \cdot)$, the lower bound $-21$ corresponds. Analysis of this branch is presented in Tab. 8 and conducted similar to steps 7, 8 with only difference that $Q = Q\backslash \{B(\{3,2,2,1\}, -21)\} + \{B(\{2,3,2,1\}, -21)\}$ on the first stage and $Q = Q\backslash \{B(\{2,3,2,1\}, -21)\}$ on the second. No improvement of the objective function or updating $X^{**}$ occur.

11. The last branch for examination is $B(\{4,2,2,1\})$ with the lower bound $-19$ equal to the current upper bound. $z_{\text{min}}(\{4,2,2,1\}) \notin E'$. In the rest points of $E(\{4,2,2,1\}$, the objective function takes higher values and in accordance with $P2.b$, thus, the branch can be discarded.

$$Q = Q\backslash \{B(\{4,2,2,1\}, -19)\} = \emptyset.$$  

The queue is empty, the search is completed.  

**Final solution:**

$$(X^{*}, z^{*}) = (X^{**}, z^{**}) = \{(6,4,4,4,2,3,1,2), (4,6,4,4,3,2,1,2)\}, -19).$$

**Conclusion.** $|E| = \frac{n \cdot 2^n}{2}$ = 3360. Taking into account that the bottom-right nodes in Tab. 2-Tab. 8 were found in Tab. 1, the total number of analyzed nodes is 28.
Table 5. \(Gr(2)\)

| i  | \(\Omega\) | \(sol_{\text{max}}(\Omega)\) | \(x = x_{\text{max}}(\Omega)\) | \(sol_{\text{min}}(\Omega)\) | \(x = x_{\text{min}}(\Omega)\) |
|----|------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| 1  | 6.2        | (1.2,3,4,4,4,6,2)            | 74                           | -24                           | -43                          |
| 2  | 4.2        | (1.2,3,4,4,6,4,2)            | 68                           | -18                           | -41                          |
| 3  | 3.2        | (1.2,4,4,4,6,3,2)            | 58                           | -12                           | -43                          |
| 4  | 2.2        | (1.3,4,4,4,6,2,2)            | 45                           | -4                            | -36                          |
| 5  | 1.2        | (2,3,4,4,4,6,1,2)            | 31                           | 7                             | -31                          |

Table 6. \(Gr(\{1,2\})\)

| i  | \(\Omega\) | \(sol_{\text{max}}(\Omega)\) | \(x = x_{\text{max}}(\Omega)\) | \(sol_{\text{min}}(\Omega)\) | \(x = x_{\text{min}}(\Omega)\) |
|----|------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| 1  | 6.1,2      | (1.2,3,4,4,6,1,2)            | 44                           | 3                             | -35                          |
| 2  | 4.1,2      | (2,3,4,4,4,6,1,2)            | 27                           | 11                            | -43                          |
| 3  | 3.1,2      | (2.4,4,4,4,6,3,1,2)          | 17                           | 16                            | -37                          |
| 4  | 2.1,2      | (3,4,4,4,6,2,1,2)            | 6                            | 24                            | -33                          |

Table 7. \(Gr(\{2,1,2\})\)

| i  | \(\Omega\) | \(sol_{\text{max}}(\Omega)\) | \(x = x_{\text{max}}(\Omega)\) | \(sol_{\text{min}}(\Omega)\) | \(x = x_{\text{min}}(\Omega)\) |
|----|------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| 1  | 6.2,1,2    | (3,4,4,4,6,2,1,2)            | 6                            | 24                            | -33                          |
| 2  | 4.2,1,2    | (3,4,4,6,4,2,1,2)            | -2                           | 8                             | -29                          |
| 3  | 3,2,1,2    | (4,4,4,6,3,2,1,2)            | -11                          | 14                            | -19                          |

Table 8. \(Gr(\{3,2,1\})\)

| i  | \(\Omega\) | \(sol_{\text{max}}(\Omega)\) | \(x = x_{\text{max}}(\Omega)\) | \(sol_{\text{min}}(\Omega)\) | \(x = x_{\text{min}}(\Omega)\) |
|----|------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| 1  | 6,3,2,1    | (2,4,4,4,6,3,2,1)            | 15                           | 17                            | -43                          |
| 2  | 4,3,2,1    | (2,4,4,6,4,3,2,1)            | 7                            | 1                             | -39                          |
| 3  | 2,3,2,1    | (4,4,4,6,2,3,2,1)            | -11                          | 13                            | -19                          |

6. Conclusion

In this paper, an important issue of single outing polynomially solvable combinatorial problems and developing new optimization methods utilizing the problems’ specifics is attacked. The class of sets is well-described sets allowing to solve unconstrained linear programs easily. A Horizontal Method for linear constrained combinatorial optimization on such sets (LCCO-HM) is developed and applied to linear optimization on the generic sets of multipermutations and partial multipermutations in Euclidean space. An illustrative example of LCCO-HM is given for a set of multipermutations, which shows that for the exact solution of the problem a small number of nodes of auxiliary graphs need to be analyzed compared to the cardinality of the multipermutation set.
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