Sequent Calculi without Polarities for the Unity of Logic

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Abstract

The present work aims to establish the unity of logic in the same sense as Girard’s well-known work yet without having recourse to polarities. Our motivations are to reduce various logics into a single one, clarify the dichotomy between linearity and non-linearity (resp. intuitionism and classicality) in logic, and further prove that actually we do not need polarities for the unity of logic. Our starting conjecture is that there would be mathematically precise operations of unlinearization and classicalization on logic such that the unlinearization of classical linear logic (CLL) (resp. intuitionistic linear logic (ILL)) coincides with classical logic (CL) (resp. intuitionistic logic (IL)), and the classicalization of IL (resp. ILL) with CL (resp. CLL), where the two operations are compatible in the sense of the evident commutativity. Nevertheless, CLL, in contradiction to the name, is actually not the classicalization of ILL, and CL is not the unlinearization of CLL, both of which are obvious from the following game-semantic analysis: Linear negation of CLL brings the dichotomy between the positive and the negative polarities to logic and games, which is never true in (game semantics of) ILL, IL or CL because they accommodate only negative formulas; also, for the polarities, existing game semantics of CLL employs concurrency, which is rather exotic to the negative, sequential game semantics of ILL, IL or CL. That is, the game-semantic analysis tells us that classicality or linearity of logic, if any, has nothing to do with the positive polarity or concurrency (of the game semantics of) CLL, and hence, we perhaps need to replace CLL with its negative fragment for the conjecture. The main contribution of the present work is then to prove the conjecture in terms of sequent calculi except that CLL is replaced with its negative fragment, which we call classical linear logic negative (CLLN). Concretely, we first carve out a sequent calculus $\text{ILL}_{\mu}$ for intuitionistic linear logic extended (ILLE), a conservative extension of ILL, from the standard, two-sided sequent calculus for CLL by discarding linear negation and adding another cut rule. Next, we introduce a sequent calculus $\text{ILK}_{\mu}$ for a conservative extension of IL, called intuitionistic logic extended (ILE), and then define a translation of sequents $\Delta \vdash \Gamma$ in the standard sequent calculus $\text{LK}$ for CL into the ones $\Delta \vdash ?\Gamma$ in $\text{ILK}_{\mu}$ and another translation of sequents $\Theta \vdash \Xi$ in $\text{ILK}_{\mu}$ into the ones $!\Theta \vdash \Xi$ in $\text{ILK}_{\mu}$. Further, by reversing the order of applying these two translations, we arrive at a sequent calculus $\text{CLLK}_{\mu}$ for CLLN such that there are a translation of sequents $\Delta \vdash \Gamma$ in $\text{LK}$ into the ones $!\Delta \vdash \Gamma$ in $\text{CLLK}_{\mu}$ and another translation of sequents $\Theta \vdash \Xi$ in $\text{CLLK}_{\mu}$ into the ones $\Theta \vdash ?\Xi$ in $\text{CLLK}_{\mu}$. In this sense, CLLN is truly the classicalization of ILL(E) as well as the linearization of CL, where the maps $\mathcal{R} : \Delta \vdash \Gamma \mapsto !\Delta \vdash \Gamma$ and $\mathcal{F} : \Theta \vdash \Xi \mapsto \Theta \vdash ?\Xi$ are unlinearization and classicalization on logic, respectively, and the two maps commute.
up to permutations of rules in proof trees. The sequent calculi are, however, all undirected in the sense that there are non-canonical choices in designing cut-elimination procedures on them. For this point, we finally introduce the call-by-name and the call-by-value variants of the undirected unity of logic simply by modifying the cut rules of the calculi, both of which make the cut-elimination procedures canonical in their respective directions. These directed variants respectively form particular embeddings of the undirected one, but they both lose the commutativity between unlinearization and classicalization.

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1 Introduction

Classical logic (CL) [Sha18, TS00] was the only logic available until the advent of intuitionistic logic (IL) [Hey30, TVD88, TS00] and more recent linear logic (LL) [Gir87]. Even today, CL is still the (implicitly) official logic for most working mathematicians; IL and LL are proof-theoretically and category-theoretically more pleasing [LSS88, GTL89], and studied as mathematical structures in their own right. Consequently, one may say that they are the three most established logics.

However, fundamental questions remain unsolved: What are linearity and intuitionisity of logic in any mathematically precise sense? And perhaps more importantly, what is the relation between the three logics, or can we unify them into a single one? Although some answers to these questions have been given [LS00, Gir87, Gir93], neither has become the answer.
Motivated in this way, the present work is an attempt to establish the unity of logic in such a way that clarifies linearity and intuitionisity of logic. In addition, the work clarifies how the dichotomy between call-by-name and call-by-value computations arise in the unity of logic.

1.1 Linear Logic

LL is often said to be resource-conscious (or resource-sensitive) for it requires proofs to consume each premise exactly once in order to produce a conclusion. One of the striking achievements of LL is the following: Like CL it has an involutive negation, and more generally the De Morgan dualities \[ \neg \neg A \equiv A \] in the strict sense (i.e., not only up to logical equivalence), called linear negation \( \bot \), while like IL it has constructivity in the sense of non-trivial semantics \[ GTR89 \], where note that neither CL nor IL (in the form of the sequent calculi \( LK \) and \( LJ \) \[ Gen35, TS00 \]) achieves both of the dualities and the contructivity \[ GTR89, TS00 \].

Strictly speaking, LL has both classical and intuitionistic variants, CLL and ILL, respectively, and LL usually refers to CLL \[ Abr93, Mel09 \]. Let us call the standard (and two-sided) sequent calculi for CLL and ILL \[ Gir87, Abr93, TS00, Mel09 \] \( LLK \) and \( LLJ \), respectively.

1.2 Game Semantics

Game semantics \[ AM99a, Hy997 \] refers to a particular kind of mathematical (or denotational) semantics of logic and computation \[ Win93, Gun92, AC98 \], in which formulas (or types) and proofs (or programs) are interpreted as games and strategies, respectively.

A standard, in particular sequential and negative, game (n.b., in the present work, a game always refers to a sequential, negative one unless stated otherwise) is a certain kind of a rooted forest whose branches correspond to possible developments or (valid) positions of the ‘game in the usual sense’ (such as chess and poker). These branches are finite sequences of moves of the game; a play of the game proceeds as its participants, Player who represents a ‘mathematician’ (or a ‘computational agent’) and Opponent who represents a ‘rebutter’ (or a ‘computational environment’), alternately and separately (i.e., sequential) perform moves allowed by the rules of the game, in which it is always Opponent who performs the first move (i.e., negative).

On the other hand, a strategy on a game is what tells Player which move she should perform at each of her turns, i.e., ‘how she should play’, on the game.

1.3 Polarities and Concurrency in Logic and Games

A problem in the game semantics of CLL given by Andreas Blass \[ Bl92 \] was the starting point of game semantics in its modern, categorical form \[ AJ94 \]. Today, Guy McCusker’s variant \[ McC98 \] models ILL and IL in a unified manner, embodying Girard’s translation \( A \Rightarrow B \) \( \equiv \exists ! A \rightarrow B \) of IL into ILL \[ Gir87 \]. Even game semantics of computation with classical features has been proposed \[ Her97, Bl17 \] though game semantics of CL in general has not been established yet.

Now, recall that linear negation brings polarities into logic, which consist of the positive and the negative ones, and game semantics interprets positive (resp. negative) formulas as positive (resp. negative) games, in which Player (resp. Opponent) always initiates a play. Further, it interprets linear negation as the operation on games that switches their polarities.

A crucial point is, however, that games employed for the interpretation of ILL, IL and CL are all negative. Thus, it seems that the positive polarity in logic and games has nothing to do with ‘linearity’ or ‘classicality’ of logic. Moreover, because positive and negative games coexist in game semantics of CLL \[ AM99a, Mel05 \], it necessarily generalizes the negative, sequential games into concurrent ones, in which more than one participant may be active simultaneously, as opposed to
sequential games, in which only one participant performs a move at a time. However, *concurrency is not necessary at all for the negative, sequential game semantics of ILL, IL or CL* as mentioned above, i.e., it is not the game-semantic counterpart of ‘linearity’ or ‘classicality’ of logic.

Another approach is to model a fragment of CLL by sequential games [Lau02], but it employs both positive and negative games (yet in a restricted manner that keeps games sequential).

### 1.4 Unity of Logic without Polarities

Accordingly, we are concerned with a *negative fragment* of CLL, instead of CLL itself, which let us call *classical linear logic negative (CLLN)* without defining it precisely for now, and conjecture that there are universal operations of *unlinearization* and *classicalization* on logic that satisfy:

1. Unlinearization maps ILL to IL, and CLLN to CL;
2. Classicalization maps ILL to CLLN, and IL to CL;
3. Unlinearization and classicalization are *compatible* with each other in the sense that the following diagram commutes (perhaps up to some bijection):

\[
\begin{array}{ccc}
\text{ILL} & \xrightarrow{\text{unlinearization}} & \text{IL} \\
\text{CLLN} & \xrightarrow{\text{unlinearization}} & \text{CL} \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{classicalization} & \downarrow & \text{classicalization} \\
\end{array}
\]

If the conjecture is proven, then it gives another answer to the unity of logic in such a way that clarifies the dichotomy between linearity and non-linearity (resp. intuitionisity and classicality) of logic, achieving the goal of the present work.

Let us remark that such a unity of logic, as far as we are concerned, has not been established yet. For instance, Girard’s translation works as the unlinearization of ILL, but not that of CLL. Also, the well-known *negative translation* translates CL into IL [TS00], but it does not work for the linear logics. As the last example, note that the restriction of the number of elements on the RHS of sequents in sequent calculi works to obtain IL from CL, but not to go from CLL to ILL since for the linear logics we additionally need to exclude some logical constants and connectives.

Motivated in this way, we first carve out a conservative extension of ILL, called *intuitionistic linear logic extended (ILLE)*, from CLL, and a sequent calculus ILLKµ for ILE from LLK, by discarding linear negation and adding another cut rule. Next, we introduce a sequent calculus ILKµ for a conservative extension of IL, called *intuitionistic logic extended (ILE)*, and then define a translation of sequents Δ ⊢ Γ in LK into the ones Δ ⊢ ?Γ in ILKµ and another translation of sequents Θ ⊢ Ξ in ILKµ into the ones !Θ ⊢ Ξ in ILLKµ, where ? and ! are the why-not and the of-course exponentials coming from CLL, respectively. Note that the conservative extensions are to define classicalization internally by why-not ?. Further, by reversing the order of applying these two translations, we arrive at (the precise notion of) CLLN and its sequent calculus CLLKµ such that there are a translation of sequents Δ ⊢ Γ in LK into the ones !Δ ⊢ Γ in CLLKµ and another translation of sequents Θ ⊢ Ξ in CLLKµ into the ones Θ ⊢ ?Ξ in ILLKµ.

In this sense, CLLN is truly the classicalization of ILL as well as the linearization of CL, and the maps $\mathcal{T}_1 : \Delta \vdash \Gamma \mapsto !\Delta \vdash \Gamma$ and $\mathcal{T}_2 : \Theta \vdash \Xi \mapsto \Theta \vdash ?\Xi$ can be regarded reasonably as unlinearization and classicalization on logic, respectively, where the two maps indeed commute,
i.e., the following diagram commutes, up to permutations of rules occurring in proof trees:

\[
\begin{array}{ccc}
\text{LLJ} & \text{unlinearization } \mathcal{T}_1 & \text{LJ} \\
\downarrow \text{conservative extension} & & \downarrow \text{conservative extension} \\
\text{ILLK}_\mu & \text{unlinearization } \mathcal{T}_1 & \text{ILK}_\mu \\
\downarrow \text{classicalization } \mathcal{T}_1 & & \downarrow \text{classicalization } \mathcal{T}_1 \\
\text{CLLK}_\nu & \text{unlinearization } \mathcal{T}_1 & \text{LK}_\nu \\
\end{array}
\]

where the unlinearization between LLJ and LJ is just Girard’s translation. It also follows that unlinearization and classicalization are dual to each other.

1.5 Call-by-Name and Call-by-Value

Nevertheless, the sequent calculi ILLK\(_\mu\), ILK\(_\mu\), CLLK\(_\mu\) and LK in the above diagram are all undirected (signified by the subscript (\(\_\mu\))) in the sense that there are non-canonical choices in designing cut-elimination procedures on them. For this point, we introduce new sequent calculi ILLK\(_\eta\), ILK\(_\eta\) and LK\(_\eta\) for ILLE, ILE and CL, respectively, where the difference between ILLK\(_\mu\) and ILLK (resp. ILK\(_\mu\) and ILK\(_\eta\), LK and LK\(_\eta\)) is only in their cut rules, and another unity of logic in the sense of the following commutative diagram up to permutations of rules:

\[
\begin{array}{ccc}
\text{LLJ} & \text{unlinearization } \mathcal{T}_1 & \text{LJ} \\
\downarrow \text{conservative extension} & & \downarrow \text{conservative extension} \\
\text{ILLK} & \text{unlinearization } \mathcal{T}_1 & \text{ILK}_\eta \\
\downarrow \text{classicalization } \mathcal{T}_1 & & \downarrow \text{classicalization } \mathcal{T}_1 \\
\text{CLLK}_\mu & \text{unlinearization } \mathcal{T}_1 & \text{LK}_\eta \\
\end{array}
\]

where another variant of classicalization \(\mathcal{T}_?\) translates sequents \(\Delta \vdash \Gamma\) in LK\(_\eta\) into the sequents \(?\Delta \vdash ?\Gamma\) in ILK\(_\eta\). The point here is that there is only the canonical cut-elimination procedure on each of the newly introduced sequent calculi, where those on ILK\(_\eta\) and LK\(_\eta\) are call-by-name.

Dually, we introduce another series of sequent calculi CLLK\(_\nu\) and LK\(_\nu\) for CLLN and CL, respectively, where the difference between CLLK\(_\mu\) and CLLK\(_\nu\) (resp. LK and LK\(_\nu\)) is again only in their cut rules, and yet another unity of logic in the sense of the following commutative diagram up to permutations of rules:

\[
\begin{array}{ccc}
\text{LLJ} & \text{unlinearization } \mathcal{T}_1 & \text{LJ} \\
\downarrow \text{conservative extension} & & \downarrow \text{conservative extension} \\
\text{ILLK} & \text{unlinearization } \mathcal{T}_1 & \text{ILK}_\mu \\
\downarrow \text{classicalization } \mathcal{T}_1 & & \downarrow \text{classicalization } \mathcal{T}_1 \\
\text{CLLK}_\nu & \text{unlinearization } \mathcal{T}_1 & \text{LK}_\nu \\
\end{array}
\]

where another variant of unlinearization \(\mathcal{T}_?\) translates sequents \(\Delta \vdash \Gamma\) in LK\(_\nu\) into the sequents \(!\Delta \vdash !\Gamma\) in CLLK\(_\mu\). Furthermore, there is only the canonical, call-by-value choice in designing a cut-elimination procedure on each of CLLK\(_\nu\) and LK\(_\nu\).
In this manner, call-by-name and call-by-value computations arise naturally as particular embeddings of the undirected unity of logic, depending on the order of composing unlinearization and classicalization. Let us note, however, that these directed variants lose the commutativity and the duality between unlinearization and classicalization, i.e., their computational directions are inherently tied to the orders of composing the two operations. Let us also note that our call-by-name and call-by-value translations of LK into ILLK match the ones given in [DJS95, LR03].

1.6 Our Contribution and Related Work

Broadly, our contribution is to establish a novel unity of logic in terms of sequent calculi, for which we employ only negative formulas. In addition, the orthogonal, dual operations of unlinearization and classicalization are given, by which the dichotomy between linearity and non-linearity (resp. intuitionisity and classicality) in logic is formulated. Furthermore, we analyze how call-by-name and call-by-value computations arise through the unity of logic.

Our approach stands in sharp contrast to the polarized (in the two directions) and/or concurrent approaches [AM99b, Mel05, Lau02, Gir93, MT10, Mel12, LR03] for they stick to CLL or its fragments having both polarities, while we replace the logic with its negative fragment, viz., CLLN. For instance, Olivier Laurent and Laurent Regnier [LR03] achieve something similar to the present work: to establish a commutative diagram that unifies certain fragments of CL, IL and CLL; however, it employs polarized linear logic instead of CLLN, and the CPS-translation as classicalization between non-linear logics, which differs from our approach. Let us further point out that they employ another translation as classicalization between linear logics. In contrast, our classicalization is applied to both of the linear and the non-linear logics; the establishment of such a universal notion of classicalization is one of the advantages of the present work.

1.7 Structure of the Paper

The rest of the present paper is structured as follows. We first review the standard sequent calculi for CL, IL, ILL and CLL in Sect. 2 for convenience. Then, we introduce the new logics and the undirected sequent calculi for them, and establish the unity of logic in terms of these sequent calculi in Sect. 3. Further, we introduce the call-by-name (resp. call-by-value) variants of the sequent calculi, and establish another unity of logic in terms of them in Sect. 4 (resp. Sect. 5). Finally, we draw a conclusion and propose some future work in Sect. 6.

2 Review: Sequent Calculi for Existing Logics

Let us begin with recalling standard sequent calculi for the existing logics. We assume that the reader is familiar with the formal languages and the sequent calculi for classical logic (CL), intuitionistic logic (IL), classical linear logic (CLL) and intuitionistic linear logic (ILL) [Gen35, TS00, Gir87, Abr93], and just briefly recall them here. Throughout the paper, we focus on the propositional part of these logics [TS00, Sho67].

Notation. Following the standard convention in proof theory [TS00], capital letters $A, B, C$, etc. range over formulas, and capital Greek letters $\Delta, \Sigma, \Theta$, etc. over finite sequences of formulas.

2.1 Sequent Calculi for Classical and Intuitionistic Logics

Let us begin with recalling the standard sequent calculi $LK$ and $LJ$ for CL and IL, respectively [Gen35, TS00]. As minor points, we rather define negation in terms of implication and bottom, and include top and the right-rule on bottom for our unified approach. Also, we notationally
distinguish between classical and intuitionistic conjunctions, classical and intuitionistic implications, and classical and intuitionistic negations, whose convenience would be clear shortly.

**Definition 2.1** (Formulas of CL [TS00]). Formulas $A, B$ of classical logic (CL) are defined by the following grammar:

$$A, B \overset{df}{=} X \mid \top \mid \bot \mid A \land B \mid A \lor B \mid A \Rightarrow B$$

where $X$ ranges over propositional variables [Sho67, TS00], and we define $A^\neg \overset{df}{=} A \Rightarrow \bot$. We call $\top$ top, $\bot$ bottom, $\land$ (classical) conjunction, $\lor$ (non-linear) disjunction, $\Rightarrow$ (classical) implication, and $\neg$ (classical) negation.

**Remark.** The adjectives (in the parentheses) on conjunction $\land$, disjunction $\lor$, implication $\Rightarrow$ and negation $\neg$ would make sense by the unity of logic given in Sect. 3, but for now it is perhaps better to simply ignore them.

**Definition 2.2** (LK for CL [Gen35, TS00]). The sequent calculus $LK$ for CL consists of the axioms and the rules in Fig. 1.

**Figure 1:** Sequent calculus $LK$ for CL

**Definition 2.3** (Formulas of IL [TS00]). Formulas $A, B$ of intuitionistic logic (IL) are defined by the following grammar:

$$A, B \overset{df}{=} X \mid \top \mid \bot \mid A \& B \mid A \lor B \mid A \Rightarrow B$$

where $X$ ranges over propositional variables, and we define $A^\neg \overset{df}{=} A \Rightarrow \bot$. We call $\top$ top, $\bot$ bottom, $\&$ (intuitionistic) conjunction, $\lor$ (non-linear) disjunction, $\Rightarrow$ (intuitionistic) implication, and $\neg$ (intuitionistic) negation.
Remark. As in the case of CL, we may ignore the adjectives on conjunction $\&$, disjunction $\lor$, implication $\Rightarrow$ and negation $(\bot)^\sim$ until Sect. 3. In addition, we shall see later that intuitionistic conjunction coincides with additive conjunction of linear logic as the notation $\&$ suggests.

**Definition 2.4** (LJ for IL [Gen35, TS00]). The sequent calculus $\text{LJ}$ for IL consists of the axioms and the rules of $\text{LK}$ that contain only intuitionistic sequents, i.e., ones such that the number of elements on the RHS is $\leq 1$, in Fig. 2 where we replace $\land$ (resp. $\Rightarrow$) with $\&$ (resp. $\sim$).

| Rule        | Premises                                                                 | Conclusion                                      |
|-------------|---------------------------------------------------------------------------|-------------------------------------------------|
| $(\land L)$ | $\Delta, A_1, B \vdash B$                                                | $\Delta, \land A_1 \land A_2 \vdash B$         |
| $(\land R)$ | $\Delta \vdash B$                                                       | $\Delta \vdash A_1 \land A_2 \vdash B$         |
| $(\lor L)$  | $\Delta, A_1 \vdash B$                                                  | $\Delta, A_2 \vdash B$                         |
| $(\lor R)$  | $\Delta \vdash A_1$                                                     | $\Delta \vdash A_2$                           |
| $(\Rightarrow L)$ | $\Delta \vdash A$                                   | $\Delta, A \Rightarrow B \vdash C$          |
| $(\Rightarrow R)$ | $\Delta, A \vdash B$                               | $\Delta \vdash A \Rightarrow B$               |
| $(\bot R)$  | $\Delta \vdash A$                                                      | $\Delta \vdash \bot, A$                        |

Figure 2: Sequent calculus LJ for IL

A traditional view regards CL as IL augmented with a classical axiom such as the **law of excluded middle** (LEM) $\vdash A \lor A$ and the **double negation elimination** (DNE) axiom $\vdash (A \lor A)^\sim \Rightarrow A$ [TS00]. In Hilbert systems or natural deductions [TS00], one needs to add a classical axiom explicitly in order to obtain CL from IL, as opposed to the case of sequent calculi (which we have seen in Def. 2.2 and 2.4). In fact, the following is a valid proof of LEM in $\text{LK}$:

| Rule        | Premises                                                                 | Conclusion                                |
|-------------|---------------------------------------------------------------------------|-------------------------------------------|
| $(\bot L)$  | $\Delta \vdash A$                                                      | $\Delta \vdash A \lor A$                 |
| $(\Rightarrow L)$ | $\Delta \vdash A$                                    | $\Delta, A \Rightarrow A \lor A \vdash A$|
| $(\Rightarrow R)$ | $\Delta, A \vdash A$                                      | $\Delta \vdash A \Rightarrow A \lor A$   |

Note that some of the sequents in the proof have two elements on the RHS; thus, the proof is not valid in LJ (even if we replace classical implication $\Rightarrow$ and negation $(\bot)^\sim$ with the intuitionistic ones $\Rightarrow$ and $(\bot)^\sim$, respectively). In this way, sequent calculi systematically **unify** CL and IL.

Nevertheless, this unity of CL and IL is **not** applied to linear logics; in fact, the standard sequent calculus for ILL (Def. 2.8) is not obtained from the standard one for CLL (Def. 2.7) simply by the restriction of the number of elements on the RHS of sequents (because we also...
have to discard some logical constants and connectives of CL L). In other words, we have not yet established the ‘classicality’ vs. ‘intuitionisity’ distinction of logic in a unified manner.

Finally, let us recall the following fundamental theorem in proof theory, which was originally established by Gerhard Gentzen:

**Theorem 2.5** (Cut-elimination on LK and LJ [Gen35] [TS00]). There is an effective algorithm that transforms a given proof of a sequent in the sequent calculus LK (resp. LJ) into a proof of the same sequent in the calculus that does not use the rule Cut.

*Proof.* See the original article [Gen35] or the standard textbook [TS00].

Such an algorithm is generally called a cut-elimination procedure, whose existence is regarded as a fundamental property of sequent calculi in general.

### 2.2 Sequent Calculi for Classical and Intuitionistic Linear Logics

Let us call the standard, two-sided sequent calculi for CLL and ILL [Gir87] [TS00] LLK and LLJ, respectively. For completeness, let us recall them as well:

**Definition 2.6** (Formulas of linear logics [Gir87] [TS00] [Mel09]). Formulas $A, B$ of classical linear logic (CLL) are defined by the following grammar:

$$A, B \overset{df}{=} X | \top | \bot | 1 | 0 | A \otimes B | A \& B | A \oplus B | A^\bot | !A | ?A$$

where $X$ ranges over propositional variables. We call $\top$ top, $\bot$ bottom, 1 one, 0 zero, $\otimes$ tensor, $\&$ par, $\&$ with, $\oplus$ plus, $(\_)^\bot$ linear negation, ! of-course, and ? why-not.

Formulas $A, B$ of intuitionistic linear logic (ILL) are defined by the following grammar:

$$A, B \overset{df}{=} X | \top | 1 | A \otimes B | A & B | A \oplus B | A \Rightarrow B | !A$$

where again $X$ ranges over propositional variables, and we call $\Rightarrow$ linear implication.

**Convention.** We write $\top$ and 1 for the units of tensor $\otimes$ and with $\&$, respectively, i.e., we swap the traditional notations for the units [Gir87] (n.b., but we keep the names unchanged), for it is notationally more systematic. Accordingly, the conventional divisions [Gir87] becomes:

- **Multiplicatives** refer to top $\top$, bottom $\bot$, tensor $\otimes$, par $\&$ and linear implication $\Rightarrow$;
- **Additives** refer to one 1, zero 0, with $\&$ and plus $\oplus$;
- **Exponentials** refer to of-course ! and why-not ?;
- **Conjunctions** refer to tensor $\otimes$ and with $\&$;
- **Disjunctions** refer to par $\&$ and plus $\oplus$.

**Remark.** It slightly varies among authors which logical constants and connectives of CLL to include in the construction of formulas of ILL. Our choice in Def. 2.6 is for Cor. 3.4 to hold.

**Notation.** Given $n \in \mathbb{N}$, we define $\pi \overset{df}{=} \{1, 2, \ldots, n\}$. In formulas, every unary operation precedes any binary operation; every binary operation except (linear) implication is left associative, while (linear) implication is right associative. Given a finite sequence $s$, we write $|s| \in \mathbb{N}$ for its length, i.e., the number of elements of $s$. We often use parentheses to clear ambiguity in formulas.
However, it does not work for the classical case; in fact, any of the existing translations of CL [See87]. Hence, one may wonder if Girard’s translation is to be called ‘unlinearization’ of logic.

Recall that there is a well-known translation of sequents $\Delta \vdash B$, called Girard’s translation [Gir87], for CLL consists of the axioms $\neg \neg A \vdash A$ in LLK for CL.

**Definition 2.7 (LLK for CLL [Gir87]).** The sequent calculus $\text{LLK}$ for CLL consists of the axioms and the rules in Fig. 3.

**Definition 2.8 (LLJ for ILL [Gir87, Abr93, Mel09]).** The sequent calculus $\text{LLJ}$ for ILL consists of those of LLK on top $\top$, one 1, tensor $\otimes$, with $\&$, plus $\oplus$ and of-course $!$ that contain only intuitionistic sequents, as well as the following two rules on linear implication $\neg -$:

$\neg L \quad \Delta, A \vdash \Gamma$  $\nRightarrow \Delta, \neg A \vdash \Gamma$

That is, they are the ones in Fig. 4.

Recall that there is a well-known translation of sequents $\Delta \vdash B$ in LJ into those $!\Delta \vdash B$ in LLJ, called Girard’s translation [Gir87], which has been abstracted by category theory as well [Seo87]. Hence, one may wonder if Girard’s translation is to be called ‘unlinearization’ of logic. However, it does not work for the classical case; in fact, any of the existing translations of CL into (a variant of) CLL is different from Girard’s translation [Gir87, Gir05, LR08].

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1 As mentioned in the introduction, the work [LR08] does not give such a distinction either.
As a summary of the present section, let us restate that operations on logic to be called unlinearization and classicalization that give the unity of logic in the sense that the diagram

\[ \text{ILL} \xrightarrow{\text{unlinearization}} \text{IL} \]

\[ \text{CLL} \xrightarrow{\text{unlinearization}} \text{CL} \]

commutes have not been established yet. For instance, Girard’s translation works as the unlinearization ILL \( \rightarrow \) IL, but not as the one CLL \( \rightarrow \) CL; also, the well-known negative translation [TS00] works as the classicalization IL \( \rightarrow \) CL, but not as the one ILL \( \rightarrow \) CLL. In addition, we have also seen that the restriction of the number of elements on the RHS of sequents works for obtaining IL from CL, but not ILL from CLL. The lack of such universal operations of unlinearization and classicalization on logic is a main problem we address in the present work, where we replace CLL with its negative fragment. Let us call the negative fragment of CLL classical linear logic negative (CLLN), which we define in Sect. 3.3.

3 Undirected Unity of Logic without Polarities

Having reviewed the existing logics and the standard sequent calculi for them, main contributions of the present work start from now on.

In this section, we introduce a conservative extension of ILL, called intuitionistic linear logic extended (ILLE), to which IL, CLLN and CL shall be all reduced, and a sequent calculus ILLK\(\mu\) for ILLE. We then define a translation \(\mathcal{T}_\gamma\) of LK into ILLK\(\mu\). Further, we decompose the translation
$$\mathcal{F}_\mathcal{L}$$ into two translations, \textit{unlinearization} \(\mathcal{F}\) and \textit{classicalization} \(\mathcal{F}_\mathcal{C}\), such that the diagram

\[
\begin{array}{c}
\text{LLJ} \quad \text{conservative extension}
\end{array}
\begin{array}{c}
\downarrow
\end{array}
\begin{array}{c}
\text{LJ}
\end{array}
\begin{array}{c}
\text{conservative extension}
\end{array}
\begin{array}{c}
\text{ILLK}_\mu \quad \text{unlinearization}
\end{array}
\begin{array}{c}
\downarrow
\end{array}
\begin{array}{c}
\text{ILK}_\mu
\end{array}
\begin{array}{c}
\text{classicalization}
\end{array}
\begin{array}{c}
\text{CLLK}_\mu
\end{array}
\begin{array}{c}
\downarrow
\end{array}
\begin{array}{c}
\text{LK}
\end{array}
\]

commutes up to permutations of rules occurring in proof trees, where \(\text{ILK}_\mu\) is a sequent calculus for a conservative extension of IL, called \textit{intuitionistic logic extended (ILE)}, and \(\text{CLLK}_\mu\) is a sequent calculus for CLLN. Let us remark that the conservative extensions ILLE and ILE accommodate only negative formulas, and the extensions are to define the classicalization \(\mathcal{F}_\mathcal{C}\). In this manner, we give the unity of logic without polarities in terms of sequent calculi in such a way that establishes the universal operations of unlinearization and classicalization on logic in a mutually compatible manner, achieving the main aim of the present work.

On the other hand, these sequent calculi are all \textit{undirected} in the sense that we may define both \textit{call-by-name} and \textit{call-by-value} cut-elimination procedures on them. The subscript \(\omega_{\mu}\) on the sequent calculus signifies this undirected nature of them.

### 3.1 Conservative Extension of Intuitionistic Linear Logic

We begin with defining the formal language of ILLE, on which our idea is as follows. As explained in the introduction, linear negation brings polarities, which consist of the positive and the negative ones, into logic \([\text{Gir93}]\) and games \([\text{Lau02}]\), but positive formulas never occur in ILL, IL or CL, and positive games never do in their game semantics \([\text{McC98, Her97, Blo17}]\). Hence, we regard linear negation as a main problem in CLL that prohibits us from obtaining the unity of logic in the sense of the commutative diagram, and we simply replace it with its negative fragment:

**Definition 3.1 (Formulas of IILLE).** The language of \textit{intuitionistic linear logic extended (ILLE)} is obtained from that of CLL by replacing linear negation \(\bot\) with \textit{unpolarized (u-)} linear negation \(\neg\), i.e., formulas \(A, B\) of IILLE are defined by the following grammar:

\[
A, B \overset{\text{df}}{=} X \mid \top \mid \bot \mid 1 \mid 0 \mid A \otimes B \mid A \forall B \mid A \& B \mid A \oplus B \mid \neg A \mid !A \mid ?A
\]

where \(X\) ranges over propositional variables, and we define \(A \rightarrow B \overset{\text{df}}{=} \neg A \forall B\). The naming of the logical constants and connectives other than u-linear negation follows that of CLL (Def. 2.6).

Next, let us introduce our sequent calculus \(\text{ILLK}_\mu\) for IILLE, which is simply the corresponding negative fragment of LLK (Def. 2.7) augmented with a new cut rule:

**Definition 3.2 (ILLK\(_\mu\)).** The sequent calculus \(\text{ILLK}\) for IILLE consists of the axioms and the rules in Fig. 5 where \(e(A_1, A_2, \ldots, A_k) \overset{\text{df}}{=} eA_1, eA_2, \ldots, eA_k\) for all \(e \in \{!, ?\}\), and the sequent calculus \(\text{ILLK}_\mu\) for IILLE consists of the axioms and the rules of ILLK as well as the following:

\[
(Cut'_{\mu}) \quad \frac{\Delta \vdash B, ?T \quad \Delta', !B \vdash ?T'}{\Delta, !\Delta' \vdash ?T', !T'}
\]
(XL) $\Delta, A, A', \Delta' \vdash \Gamma$
$\Delta, A', A, \Delta' \vdash \Gamma$

(IC) $\Delta, !A, !A \vdash \Gamma$
$\Delta, !A \vdash \Gamma$

(?!A) $\Delta, !A \vdash \Gamma$

(?C) $\Delta, ?B, \Gamma$
$\Delta \vdash ?B, \Gamma$

(ID) $\Delta, A \vdash \Gamma$
$\Delta, !A \vdash \Gamma$

(?D) $\Delta \vdash B, \Gamma$
$\Delta \vdash ?B, \Gamma$

(L) $\Delta \vdash B, \Gamma$
$\Delta \vdash \Delta', B \vdash \Gamma'$

(Cut) $\Delta, \Delta' \vdash \Gamma'$

(TL) $\Delta \vdash \Gamma$
$\Delta, T \vdash \Gamma$

(TR) $\vdash T$

(LR) $\vdash !\Lambda$

(\&L) $\Delta, A_1, A_2 \vdash \Gamma$
$\Delta, A_1 \& A_2 \vdash \Gamma$

(i \in \mathbb{I})

(\&R) $\Delta, A_1 \vdash \Gamma$
$\Delta, A_2 \vdash \Gamma$

(\&R) $\Delta, B_1, \Gamma$
$\Delta, B_2, \Gamma$

(\&R) $\Delta, B_1, \Gamma$
$\Delta, B_2, \Gamma$

(\&R) $\Delta, B_1 \& B_2, \Gamma$

(\&R) $\Delta, B_1 \& B_2, \Gamma$

(\&R) $\Delta, B_1 \& B_2, \Gamma$

(\&R) $\Delta, B_1 \& B_2, \Gamma$

(\&R) $\Delta, B_1 \& B_2, \Gamma$

(L) $\Delta, A \vdash \Gamma$
$\Delta, \neg B \vdash \Gamma$

(\neg) $\Delta, A \vdash \Gamma$
$\Delta, \neg A \vdash \Gamma$

(\neg) $\Delta, A \vdash \Gamma$
$\Delta, \neg A \vdash \Gamma$

(\neg) $\Delta, A \vdash \Gamma$
$\Delta, \neg A \vdash \Gamma$

Figure 5: Sequent calculus ILLK for ILLE

The new cut rule $\text{Cut}_\mu^n$ is for the translation $\mathcal{T}_T$ of LK into ILLK$_\mu$ given in the next section, where $\text{Cut}_\mu^n$ interprets Cut of LK, for which the following proof in ILLK$_\mu$ plays a key role:

\begin{align*}
(\text{Id}) & \quad \frac{A \vdash A}{A \vdash A} \\
(\text{!R}) & \quad \frac{A \vdash A}{!A \vdash !A} \\
(\text{?R}) & \quad \frac{A \vdash A}{?A \vdash ?A} \\
(\text{Cut}_\mu^n) & \quad \frac{?A \vdash ?A}{\Delta, ?A \vdash \Gamma}
\end{align*}

Let us call this derived axiom the weakly distributive axiom and write $\text{Dist}$ for it. Moreover, for convenience, let us write $?!$ and $?R$, respectively, for the following derived rules in ILLK$_\mu$:

\begin{align*}
(\text{?D}) & \quad \frac{?! \vdash ?A}{?! \vdash ?A} \\
(\text{Cut}) & \quad \frac{?! \vdash ?A}{\Delta, ?A \vdash \Gamma}
\end{align*}

\begin{align*}
(\text{Dist}) & \quad \frac{?! \vdash ?A}{?! \vdash \Delta, ?A \vdash \Gamma} \\
(\text{Cut}) & \quad \frac{?! \vdash \Delta, ?A \vdash \Gamma}{\Delta, ?A \vdash \Gamma}
\end{align*}

Let us announce beforehand that there are two mutually symmetric, non-canonical choices in designing a cut-elimination procedure on the new cut rule $\text{Cut}_\mu^n$, which correspond to call-by-name and call-by-value computations, while there is no such non-canonical choice for the
standard cut rule Cut of ILLK. In the next section, the difference between Cut and Cut\textsubscript{I} also explains why cut-elimination on LK is problematic, but that on LLK is not \cite{LLS96}.

At this point, one may wonder why we call ILLL \textit{intuitionistic} even though we allow more than one formulas occurring on the RHS of sequents in ILLK and ILLK\textsubscript{µ}. We shall give a general answer to this question later, but here let us point out that LEM w.r.t. plus \(\otimes\), viz., \(\vdash \neg A \otimes A\) for an arbitrary formula \(A\), is not provable in ILLK or ILLK\textsubscript{µ}, but it is in the classicalization of ILLK or ILLK\textsubscript{µ} introduced later. In addition, let us also remark that we regard LEM w.r.t. par \(\Box\) as intuitionistically valid; in fact, ILLK and ILLK\textsubscript{µ} permit the following proof:

\[
\begin{array}{c}
(\text{Id}) \quad \frac{}{A \vdash A} \\
(\neg \text{R}) \quad \frac{}{\vdash \neg A, A} \\
(\forall \text{R}) \quad \frac{}{\vdash \neg A \forall A}
\end{array}
\]

Let us proceed to prove the following \textit{cut-elimination} theorem on ILLK\textsubscript{µ}:

\textbf{Theorem 3.3 (Cut-elimination on ILLK\textsubscript{µ}).} Given a proof of a sequent in ILLK\textsubscript{µ}, there is a proof of the same sequent in ILLK\textsubscript{µ} that does not use the rule Cut or Cut\textsubscript{I}.

\textbf{Proof.} By the cut-elimination procedures given in Appx. A (on Cut) and B (on Cut\textsubscript{I}). \(\blacksquare\)

An important corollary of the theorem is, as announced previously, that ILLK\textsubscript{µ} is indeed a conservative extension of LLJ:

\textbf{Corollary 3.4 (ILLK\textsubscript{µ} as a conservative extension of LLJ).} ILLK\textsubscript{µ} is a conservative extension of LLJ, i.e., every sequent provable in LLJ is also provable in ILLK\textsubscript{µ}, and if a sequent \(\Delta \vdash \Gamma\) is provable ILLK\textsubscript{µ}, where at most only \(\top, \otimes, \neg, \wedge, \vee, \&\) and \(\otimes\) occur in \(\Delta, \Gamma\), then \(\Gamma = \epsilon\) or \(\Gamma = B\) for some formula \(B\), and the sequent \(\Delta \vdash \Gamma\) is also provable in LLJ.

\textbf{Proof.} Let us first show that \(\Gamma\) must be the empty sequence \(\epsilon\) or a singleton sequence \(B\). Note that ILLK\textsubscript{µ} enjoys the subformula property \cite{TS00} by Thm. 3.3. Note also that, among the rules of ILLK\textsubscript{µ} on the logical constants and connectives of ILL, only \(\forall\) and \(\neg\) increase the number of formulas occurring on the RHS of a sequent. However, \(\exists\) generates a formula containing \(\forall\), which does not occur in ILL, and thus we may ignore it by the subformula property of ILLK\textsubscript{µ}.

On the other hand, for \(\neg\), again by the subformula property, only \(\forall\)R can delete \(\neg\) thanks to the equation \(\neg X \forall Y = X \rightarrow Y\). Hence, \(\neg\) cannot increase the number of formulas occurring on the RHS of a sequent either. Consequently, the number of elements on the RHS of any sequent provable in ILLK\textsubscript{µ}, in which only the logical constants and connectives of ILL occur, is less than or equal to 1; thus, in particular, we may assume \(\Gamma = \epsilon\) or \(\Gamma = B\).

Let us proceed to show that the sequent \(\Delta \vdash \Gamma\) is also provable in LLJ. Dually to the above argument, among the rules of ILLK\textsubscript{µ} on the logical constants and connectives of ILL, only \(\forall\)R and \(\neg\)L decrease the number of formulas occurring on the RHS of a sequent. Nevertheless, \(\forall\)R generates a formula containing \(\forall\), and hence we may ignore it again by the subformula property.

On the other hand, \(\neg\) is trickier, but we may replace it with \(\neg\)L of LLJ as follows. The point is that, again by the subformula property, only \(\forall\)L can delete \(\neg\), and the last one of such applications of \(\forall\)L in a proof tree (constructed in ILLK\textsubscript{µ}) that satisfies the assumption of the corollary must be of the following form:

\[
(\forall \text{L}) \quad \frac{\Theta, \neg X \vdash \Xi, Y \vdash \Gamma}{\Theta, \Xi, X \rightarrow Y \vdash \Gamma}
\]

where \(\Theta, \Xi, X \rightarrow Y = \Delta\). Then, for such a proof in ILLK\textsubscript{µ}, we may clearly delay each application of \(\neg\)L until right before the application of \(\forall\)L that deletes \(\neg\), obtaining another proof of the same
sequent. Therefore, without loss of generality, it suffices to consider consecutive applications of \( \neg L \) and \( \forall L \) in the following form:

\[
\Theta \vdash X \\
\Theta, \neg X \vdash \Xi, Y \vdash \Gamma
\]

which can be replaced with \( \neg \circ L \) of LLJ. In this way, we inductively replace each of the consecutive applications of \( \neg L \) and \( \forall L \) occurring in the proof with \( \neg \circ L \). Hence, it is easy to see that each rule occurring in the resulting proof in \( \text{ILLS}_\mu \) is available in LLJ, from which the corollary follows. \( \square \)

**Remark.** The corollary would not hold if we included 1 or 0 as a logical constant of ILL.

The corollary supports our claim that ILLE is intuitionistic even though the sequent calculus \( \text{ILLS}_\mu \) for ILLE allows more than one formula to occur on the RHS of sequents.

At the end of the present section, let us point out that there is no non-canonical choice in designing the cut-elimination of Cut given in Appx. [A] (up to a reasonable invariant of proofs), but there is one in the cut-elimination of Cut\(_{\mu}^!\) in Appx. [B] due to the following critical pair [GTL89]: A proof in \( \text{ILLS}_\mu \) of the form

\[
\begin{align*}
(\text{Cut}\_!^\mu) & \quad p \\
\left[\begin{array}{c}
?W^! \\
\Pi^! \vdash !A, ?T^! \\
\Pi^!, !A \vdash ?T^!
\end{array}\right)
\end{align*}
\]

can be transformed into

\[
\begin{align*}
(\text{Cut}\_!^\mu) & \quad p' \\
\left[\begin{array}{c}
?W^* \\
\Pi^* \vdash !A, ?T^!
\end{array}\right)
\end{align*}
\]

but also into

\[
\begin{align*}
(\text{Cut}\_!^\mu) & \quad p'' \\
\left[\begin{array}{c}
?W^* \\
\Pi^* \vdash !A, ?T^!
\end{array}\right)
\end{align*}
\]

The two transformations are in fact symmetric, and thus neither is canonical.

Also, recall that the former corresponds to call-by-value computation, while the latter to call-by-name computation [CH00, Wad03].

### 3.2 Conservative Extension of Intuitionistic Logic

In this section, we establish a translation \( \mathcal{T}_{!} \) of LK into \( \text{ILLS}_\mu \) and decompose it into two translations, unlinearization \( \mathcal{T} \) and classicalization \( \mathcal{T}_{!} \), i.e., \( \mathcal{T}_{!} = \mathcal{T} \circ \mathcal{T}_{!} \). For such a decomposition, we need some intermediate logic between ILLE and CL. As mentioned in the introduction, we conjecture that such an intermediate logic would be a conservative extension of IL, which let us call intuitionistic logic extended (ILE) even before defining it precisely for convenience.

Our idea on ILE and its sequent calculus, which we write \( \text{ILLS}_\mu \), is as follows. First, by taking Girard’s translation as the unlinearization \( \mathcal{T} \) on ILLE, we regard sequents \( !\Delta \vdash B \) in \( \text{ILLS}_\mu \) as sequents \( \Delta \vdash B \) in \( \text{ILLS}_\mu \). In addition, we employ why-not ? in such a way that:
1. $\text{ILK}_\mu$ extends $\text{LJ}$ in a conservative manner;

2. We can define classicalization $\mathcal{F}_\mu$ on $\text{ILK}_\mu$ that translates $\text{LK}$ into $\text{ILK}_\mu$.

Then, let us generalize sequents in $\text{ILK}_\mu$ to sequents in $\text{ILLK}_\mu$ of the form $!\Delta \vdash (B), ?\Gamma$, which we take as sequents $\Delta \vdash (B), ?\Gamma$ in $\text{ILK}_\mu$, where $(B)$ denotes a singleton sequence of $B$ or the empty sequence $\epsilon$, and require that the principal formula of each right rule of $\text{ILK}_\mu$ must be the distinguished element $B$ (so that $\text{ILK}_\mu$ extends $\text{LJ}$ in a conservative manner, while we may define the classicalization $\mathcal{F}_\mu$). Of course, we may simply write $\Gamma$ in place of $(B), ?\Gamma$ for most of the left rules of $\text{ILK}_\mu$ since they do not break down the required form of the RHS of sequents in $\text{ILK}_\mu$.

As a consequence, some logical constants and connectives of $\text{ILLE}$ become redundant for $\text{ILE}$. For instance, we may substitute zero $0$ with bottom $\bot$ because there is the following proof of the sequent $!\Delta, \bot \vdash ?\Gamma$ in $\text{ILLK}_\mu$ for any $\Delta$ and $\Gamma$:

\[
\frac{(\bot L)}{(\bot L)} \hspace{1cm} \frac{(?W^*)}{!\Delta, \bot \vdash ?\Gamma} \hspace{1cm} \frac{!(W^*)}{\bot, !\Delta \vdash ?\Gamma} \hspace{1cm} \frac{!(XL^*)}{!\Delta, \bot \vdash ?\Gamma}
\]

which simulates the following axiom:

\[
(0L) \quad !\Delta, 0 \vdash ?\Gamma
\]

Similarly, we may substitute tensor $\otimes$ with with $\&$. In fact, the following rule in $\text{ILLK}_\mu$:

\[
(\otimes L) \quad \frac{!\Delta, !A_1, !A_2 \vdash \Gamma}{!\Delta, !A_1 \otimes !A_2 \vdash \Gamma}
\]

can be simulated by the following proof in $\text{ILLK}_\mu$:

\[
\begin{align*}
&\text{(Id)} & A_2 \vdash A_2 & \\
&\text{($&L$)} & A_1 & \vdash A_1 & \\
&\text{(Id)} & A_1 & \vdash A_1 & \\
&\text{($&R$)} & (A_1 \& A_2) & \vdash A_2 & \\
&\text{(Cut)} & !((A_1 \& A_2) \& A_1) & \vdash !A_1 & \\
&\text{($XL^*$)} & !((A_1 \& A_2), !A_1 \& !A_2) & \vdash \Gamma & \\
&\text{($!C$)} & !A_1 \& !A_2 & \vdash \Gamma & \\
&\text{($!R$)} & !A_1 \vdash B_1, ?T_1 & \\
&\text{($!L$)} & !A_2 \vdash B_2, ?T_2 & \\
\end{align*}
\]

where note that there is an isomorphism $(A_1 \& A_2) \cong !A_1 \otimes !A_2$ in $\text{ILLK}_\mu$. Also, the following rule in $\text{ILLK}_\mu$:

\[
(\otimes R) \quad \frac{!A_1 \vdash B_1, ?T_1 \vdash B_2, ?T_2}{!A_1 \& !A_2 \vdash B_1 \otimes B_2, ?T_1, ?T_2}
\]

can be simulated by the following proof in $\text{ILLK}_\mu$:

\[
\begin{align*}
&\text{($!W^*$)} & !\Delta_1 \vdash B_1, ?T_1 & \\
&\text{($?W^*$)} & !\Delta_1 \& !\Delta_2 \vdash B_1, ?T_1 & \\
&\text{($XR^*$)} & !\Delta_1 \& !\Delta_2 \vdash B_1, ?T_1, ?T_2 & \\
&\text{($!R$)} & !\Delta_2 \vdash B_2, ?T_2 & \\
&\text{($!L$)} & !\Delta_1 \& !\Delta_2 \vdash B_1 \& B_2, ?T_1, ?T_2 & \\
\end{align*}
\]

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Dually, we may substitute par \( \exists \) with plus \( \oplus \), but we also have to replace plus \( \oplus \) with (non-linear) disjunction \( \lor \) given by \( A \lor B \overset{\text{df}}{=} !A \oplus !B \) [Gir87] because we cannot peel off of-course \( \top \) occurring on the LHS of sequents in \( \text{ILK}_\mu \).

Finally, \( \text{ILK}_\mu \) dispenses with of-course \( \top \) as well because it is placed on each element on the LHS of a sequent by default, which we take as unlinearity of logic. Thus, linear implication \( \to \) is replaced with (intuitionistic) implication \( \Rightarrow \) given by \( A \Rightarrow B \overset{\text{df}}{=} !A \Rightarrow !B \) [Gir87].

Accordingly, we define the formal language of ILE as follows:

**Definition 3.5 (Formulas of ILE).** Formulas \( A, B \) of **intuitionistic logic extended (ILE)** are defined by the following grammar:

\[
A, B \overset{\text{df}}{=} X \mid \top \mid \bot \mid 1 \mid A \& B \mid A \lor B \mid A \Rightarrow B \mid ?A
\]

where \( X \) ranges over propositional variables, and we define \( A \overset{\text{df}}{=} \bot \Rightarrow \bot \). We call \( \top \) **top**, \( \bot \) **bottom**, 1 one, \& (**intuitionistic**) conjunction, \( \lor \) (non-linear) disjunction, \( \Rightarrow \) (**intuitionistic**) implication, \( ? \) **why-not**, and \( (\_) \) **(intuitionistic) negation**.

On the other hand, the last missing piece to define \( \text{ILK}_\mu \) is a cut rule. Then, we simply adopt \( \text{Cut}_\mu^? \), interpreted appropriately, as a cut rule of \( \text{ILK}_\mu \), arriving at:

**Definition 3.6 (\( \text{ILK}_\mu \)).** The sequent calculus \( \text{ILK}_\mu \) for ILE consists of the axioms and the rules in Fig. 6, where \( ?(A_1, A_2, \ldots, A_n) \overset{\text{df}}{=} ?A_1, ?A_2, \ldots, ?A_n \).

\[
\begin{align*}
&(\text{XL}) \quad \Delta, A, A', \Delta' \vdash \Gamma \quad \Delta', A, \Delta \vdash \Gamma \quad (\text{XR}) \quad \Delta \vdash \Gamma, B, B', \Gamma' \quad \Delta, B, B', B, \Gamma' \\
&(\text{CL}) \quad \Delta, A \vdash \Gamma, \Delta, A \vdash \Gamma \quad (\text{C}) \quad \Delta \vdash ?B, B, \Gamma \quad \Delta \vdash ?B, B, \Gamma \quad (\text{WL}) \quad \Delta \vdash \Gamma, A \vdash \Gamma \quad (\text{W}) \quad \Delta \vdash \Gamma, \Delta \vdash ?B, \Gamma
\end{align*}
\]

\[
\begin{align*}
&(\text{Id}) \quad \Delta \vdash A \quad (\text{Cut}_\mu^?) \quad \Delta \vdash ?B, ?T \quad \Delta', B \vdash ?T' \quad (1\text{R}) \quad \Delta \vdash 1, ?T
\end{align*}
\]

\[
\begin{align*}
&(\text{TL}) \quad \Delta \vdash \Gamma, \top \vdash \Gamma \quad (\top R) \quad \vdash \top \quad (\bot L) \quad \bot \vdash \bot \quad (\bot R) \quad \Delta \vdash ?T, \Delta \vdash \bot, ?T
\end{align*}
\]

\[
\begin{align*}
&(\&L) \quad \Delta, A_1 \vdash \Gamma, \Delta, A_1 \& A_2 \vdash \Gamma \quad (i \in \mathcal{T}) \quad (\&R) \quad \Delta \vdash B_1, ?T, A \vdash B_2, ?T
\end{align*}
\]

\[
\begin{align*}
&(\lor L) \quad \Delta, A_1 \vdash \Gamma, \Delta, A_2 \vdash \Gamma \quad (\lor R) \quad \Delta \vdash B_1 \lor B_2, ?T
\end{align*}
\]

\[
\begin{align*}
&(\Rightarrow L) \quad \Delta, B \vdash \Gamma, \Theta \vdash A, ?T \quad (\Rightarrow R) \quad \Delta, A \vdash B, ?T
\end{align*}
\]

As expected, the cut-elimination theorem holds also for \( \text{ILK}_\mu \):

**Theorem 3.7 (Cut-elimination on \( \text{ILK}_\mu \)).** Given a proof of a sequent in \( \text{ILK}_\mu \), there is a proof of the same sequent in \( \text{ILK}_\mu \) that does not use the rule \( \text{Cut}_\mu^? \).
Let us leave it to the reader to translate \( XL, XR, WL, WR, CR, \ldots \), in the sense that if the sequent \( \Delta \vdash \Gamma \) is provable in \( \text{ILK}_\mu \), where at most only \( \top, \bot, \lor, \land \) and \( \Rightarrow \) occur in \( \Delta, \Gamma \), then \( \Gamma = \epsilon \) or \( \Gamma = B \) for some formula \( B \), and the sequent \( \Delta \vdash B \) is also provable in \( \text{LJ} \).

**Proof.** Immediate from the subformula property of \( \text{ILK}_\mu \), which holds thanks to Thm. 3.7.

It is straightforward to see that there is a non-canonical choice in designing a cut-elimination procedure on the rule \( \text{Cut}_\mu^\text{L} \), similarly to the case of the rule \( \text{Cut}_\mu^\text{R} \), because weakening is possible on both of the LHS and the RHS of sequents. For this reason, we call \( \text{ILK}_\mu \) **undirected**.

As announced above, the point of \( \text{ILK}_\mu \) is that we may decompose the translation \( \text{LK} \overset{?=\delta}{\rightarrow} \text{ILK}_\mu \) into \( \text{LK} \overset{?=\delta}{\rightarrow} \text{ILK}_\mu \overset{?=\delta}{\rightarrow} \text{ILK}_\mu \), where we regard the new translations \( \mathcal{T} \) and \( \mathcal{R} \) as [classicalization and unlinearization](#) of logic, respectively, as the following two theorems prove:

**Theorem 3.9 (Translation \( \mathcal{T} \) of \( \text{LK} \) into \( \text{ILK}_\mu \)).** There is a translation \( \mathcal{T} \) of formulas and proofs that assigns, to each proof \( p \) of a sequent \( \Delta \vdash \Gamma \) in \( \text{LK} \), a proof \( \mathcal{T}(p) \) of a sequent \( \mathcal{T}^* \Delta \vdash ? \mathcal{T}^* \Gamma \) in \( \text{ILK}_\mu \), where \( \mathcal{T}(\top) \overset{df}{=} \top, \mathcal{T}(\bot) \overset{df}{=} \bot, \mathcal{T}(A \land B) \overset{df}{=} ? \mathcal{T}(A) \& ? \mathcal{T}(B), \mathcal{T}(A \lor B) \overset{df}{=} ? \mathcal{T}(A) \lor ? \mathcal{T}(B) \) and \( \mathcal{T}(A \Rightarrow B) \overset{df}{=} ? \mathcal{T}(A) \Rightarrow ? \mathcal{T}(B) \). Moreover, it is conservative in the sense that if the sequent \( \mathcal{T}^* \Delta \vdash ? \mathcal{T}^* \Gamma \) contains only formulas of \( \text{CL} \), and it is provable in \( \text{ILK}_\mu \), then the sequent \( \Delta \vdash \Gamma \) is provable in \( \text{LK} \).

**Proof.** Let us first translate each of the axioms and the rules of \( \text{LK} \) into a proof tree in \( \text{ILK}_\mu \). Let us leave it to the reader to translate XL, XR, WL, WR, CR, Id, TL, TR, LL and LR because they are just straightforward.

Cut of \( \text{LK} \) is translated in \( \text{ILK}_\mu \) as:

\[
\begin{array}{c}
\text{(Cut)}^\mu \\
\Delta \vdash ?, \top & \Delta', B \vdash ?, \top' & \Delta, \Delta' \vdash ?, \top, ?, \top'
\end{array}
\]

\( \land \) of \( \text{LK} \) is translated in \( \text{ILK}_\mu \) as:

\[
\begin{array}{c}
\text{(?L)} \\
\Delta, A_1 \vdash ?, \top & \Delta, A_1 \vdash ?, \top & \Delta, ?, \top
\end{array}
\]

\( \land \) as:

\[
\begin{array}{c}
\text{(?R)} \\
\Delta \vdash ?, B_1, ?, \top & \Delta \vdash ?, B_2, ?, \top & \Delta \vdash ?, B_1 \& B_2, ?, \top
\end{array}
\]

Dually, \( \lor \) of \( \text{LK} \) is translated in \( \text{ILK}_\mu \) as:

\[
\begin{array}{c}
\text{(vL)} \\
\Delta, A_1 \vdash ?, \top & \Delta, A_2 \vdash ?, \top & \Delta, A_1 \lor A_2 \vdash ?, \top
\end{array}
\]

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Let us first translate each of the axioms and the rules of Proof.

Let us leave it to the reader to translate XL, XR, WL, ?W, CL, ?C, ?D, ?L

Next, ⇒L of LK is translated in ILKµ as:

(Id) \[ \frac{A \vdash B, A}{B} \]

(?L') \[ \frac{A \Rightarrow B, A \vdash B}{A \Rightarrow B} \]

(?D) \[ \frac{A \Rightarrow B, ?(A \Rightarrow B)}{A \Rightarrow ?B} \]

(\nabla R') \[ \frac{A \Rightarrow ?B, ?T}{A \Rightarrow ?B, ?T} \]

We have defined the translation \( \mathcal{T} \) of LK into ILKµ.

Finally, given a proof of a sequent \( \mathcal{T}^*_\mu(\Delta) \vdash ?\mathcal{T}^*_\mu(\Gamma) \) in ILKµ, we obtain another proof of the same sequent in ILKµ such that the rule ?D (resp. ?L') is applied as early as possible (resp. as late as possible) in the proof, which, by the subformula property of ILKµ, clearly has a corresponding proof of the sequent \( \Delta \vdash \Gamma \) in LK, showing that the translation \( \mathcal{T} \) is conservative. \( \square \)

Remark. The translation \( \mathcal{T} \) is, as far as we are concerned, a novel one. In particular, it stands in sharp contrast with well-known transformations of CL into IL such as the negative translation and the intuitionistic restriction \( \text{TS00} \).

Theorem 3.10 (Translation \( \mathcal{T} \) of ILKµ into ILKµ). There is a translation \( \mathcal{T} \) of formulas and proofs that assigns, to each proof p of a sequent \( \Delta \vdash \Gamma \) in ILKµ, a proof \( \mathcal{T}(p) \) of a sequent \( !\mathcal{T}^*\mu(\Delta) \vdash ?\mathcal{T}^*\mu(\Gamma) \) in ILKµ, where \( \mathcal{T}(\top) \equiv \top, \mathcal{T}(\bot) \equiv \bot, \mathcal{T}(\text{Id}) \equiv 1, \mathcal{T}(A \land B) \equiv \mathcal{T}(A) \land \mathcal{T}(B), \mathcal{T}(A \lor B) \equiv \mathcal{T}(A) \lor \mathcal{T}(B), \mathcal{T}(A \Rightarrow B) \equiv \mathcal{T}(A) \Rightarrow \mathcal{T}(B) \) and \( \mathcal{T}(\bot) \equiv \bot \). Moreover, it is conservative in the sense that if the sequent \( !\mathcal{T}^*\mu(\Delta) \vdash ?\mathcal{T}^*\mu(\Gamma) \) contains only formulas of ILE, and it is provable in ILKµ, then the sequent \( \Delta \vdash \Gamma \) is provable in LKµ.

Proof. Let us first translate each of the axioms and the rules of ILKµ into a proof tree in ILKµ. Let us leave it to the reader to translate XL, XR, WL, ?W, CL, ?C, ?D, ?L', Id, ⊤L, ⊤R, ⊥L, ⊥R and 1R because they are just straightforward.

Cutµ of ILKµ is translated in ILKµµ as:

(\text{Cut}^\mu) \[ \frac{\Delta \vdash ?B, ?T}{\Delta \vdash ?\Delta', ?B \vdash ?T'} \]
\&L of ILK\(\mu\) is translated in ILLK\(\mu\) as:

\[
\begin{align*}
(\text{Id}) \quad & A_i \vdash A_i \\
(\&L) \quad & A_1 \& A_2 \vdash A_i \\
(\text{ID}) \quad & !((A_1 \& A_2) \vdash A_i) \\
(\text{IR}) \quad & !(A_1 \& A_2) \vdash !A_i \\
(\text{Cut}) \quad & !(A_1 \& A_2), !\Delta \vdash \Gamma \\
(\text{XL}^*) \quad & !((A_1 \& A_2), !\Delta \vdash \Gamma) \\
\end{align*}
\]

and \&R\(^7\) as:

\[
\begin{align*}
(\&R) \quad & !\Delta \vdash B_1, !\Gamma \quad !\Delta \vdash B_2, !\Gamma \\
& !\Delta \vdash B_1 \& B_2, !\Gamma
\end{align*}
\]

Dually, \forall L of LK is translated in ILLK\(\mu\) as:

\[
\begin{align*}
(\oplus L) \quad & !\Delta, !A_1 \vdash \Gamma \quad !\Delta, !A_2 \vdash \Gamma \\
(\text{ID}) \quad & !\Delta, !A_1 \oplus !A_2 \vdash !\Gamma \\
(\text{IR}) \quad & ![A_1 \oplus A_2] !\Delta \vdash \Gamma
\end{align*}
\]

and \forall R\(^7\) as:

\[
\begin{align*}
(\text{IR}) \quad & ![A_1 \oplus A_2] !\Delta \vdash \Gamma \\
& ![A_1 \oplus A_2] !\Delta \vdash B_1 \& B_2, !\Gamma
\end{align*}
\]

Next, \Rightarrow L\(^7\) of ILK\(\mu\) is translated in ILLK\(\mu\) as:

\[
\begin{align*}
(\neg L) \quad & ![A \rightarrow \neg A] A \rightarrow \neg A \\
(\text{Id}) \quad & ![A \rightarrow \neg A] \neg A \rightarrow B \\
(\neg R) \quad & ![A \rightarrow \neg A] !A \rightarrow \neg A \\
(\text{IR}) \quad & ![A \rightarrow \neg A] !A \rightarrow \neg A \\
(\text{IR}) \quad & ![A \rightarrow \neg A] !A \rightarrow \neg A \\
(\text{Cut}) \quad & ![A \rightarrow \neg A] !A \rightarrow \neg A
\end{align*}
\]

and \Rightarrow R\(^7\) as:

\[
\begin{align*}
(\neg R) \quad & ![A \rightarrow \neg A] !A \rightarrow B, !\Gamma \\
(\neg R) \quad & ![A \rightarrow \neg A] !A \rightarrow B, !\Gamma
\end{align*}
\]

Finally, we show that the translation \(\hat{\mathcal{R}}\) is conservative in the symmetric way to that of the proof of Thm. \(\ref{thm:conservativeness}\) completing the proof.

As promised before, by composing the translations \(\mathcal{R}_\mu : LK \rightarrow ILK\(\mu\) and \(\hat{\mathcal{R}} : ILK\(\mu\) \rightarrow ILLK\(\mu\), we finally obtain a translation \(\mathcal{R}_\mu \circ \hat{\mathcal{R}} : LK \rightarrow ILLK\(\mu\):
Corollary 3.11 (Translation $R_\gamma$ of LK into ILLK$_\mu$). The composition $R_\gamma \overset{df}{=} R \circ R_\gamma$ assigns, to each proof $p$ of a sequent $\Delta \vdash \Gamma$ in LK, a proof $R_\gamma(p)$ of a sequent $!R_\gamma(\Delta) \vdash ?R_\gamma(\Gamma)$ in ILLK$_\mu$, where $R_\gamma(\top) \overset{df}{=} \top$, $R_\gamma(\bot) \overset{df}{=} \bot$, $R_\gamma(A \land B) \overset{df}{=} R_\gamma(A) \& R_\gamma(B)$, $R_\gamma(A \lor B) \overset{df}{=} R_\gamma(A) \lor R_\gamma(B)$. Moreover, it is conservative in the sense that if the sequent $!R_\gamma(\Delta) \vdash ?R_\gamma(\Gamma)$ contains only formulas of CL, and it is provable in ILLK$_\mu$, then the sequent $\Delta \vdash \Gamma$ is provable in LK.

Proof. By Thm. 3.9 and 3.10.

The translation $R_\gamma$ of CL into the negative fragment of CLL, viz., ILLE, given in Cor. 3.11 is, as far as we are concerned, a novel one. In contrast to the existing translations of CL into CLL which we regard as $!\Delta$, and dispense with why-not ?. In addition, we simply adopt Cut and the classicalization $\vdash$ as a cut rule of ILLK$_\mu$.

3.3 Classical Linear Logic Negative

Let us decompose again the translation $R_\gamma$ into the unlinearization $R$ and the classicalization $\gamma$ yet in the reverse order and up to permutations of rules occurring in proof trees.

For this decomposition, the intermediate logic between ILE and CL, and its sequent calculus, which we call classical linear logic negative (CLLN) and ILLK$_\mu$, respectively, are obtained from ILE and ILLK$_\mu$ in the dual manner to how we have obtained ILE and ILLK$_\mu$ from them. (The naming of CLLN comes from its place in the decomposition, i.e., it should be classical yet linear and negative.) Spelling it out, we define sequents in ILLK$_\mu$ to be sequents $!\Delta, (A) \vdash ?\Gamma$ in ILLK$_\mu$, which we regard as $!\Delta, (A) \vdash ?\Gamma$ in CLLK$_\mu$, and require that the principal formula of each left rule of CLLK$_\mu$ must be the distinguished one $A$. Also, we enforce the implicit placement of why-not ? on each element on the RHS of sequents and take it as the definition of classicality of logic.

Dually to the case of ILE, we substitute one 1 (resp. par $\land$, tensor $\otimes$) with top $\top$ (resp. plus $\oplus$, with $\&$); further, we replace with $\&$ (resp. linear implication $\Rightarrow$) with (classical) conjunction $\land$ (resp. (classical linear) implication $\Rightarrow$) given by $A \land B \overset{df}{=} ?A?B$ (resp. $A \Rightarrow B \overset{df}{=} A \Rightarrow ?B$), and dispense with why-not ?. In addition, we simply adopt Cut$^\mu$ as a cut rule of CLLK$_\mu$.

Consequently, the formal language of CLLN and the sequent calculus CLLK$_\mu$ are as follows:

Definition 3.12 (Formulas of CLLN). Formulas $A, B$ of classical linear logic negative (CLLN) are defined by the following grammar:

$$A, B \overset{df}{=} X | \top | \bot | 0 | A \land B | A \lor B | A \Rightarrow B | !A$$

where $X$ ranges over propositional variables, and we define $A_\bot \overset{df}{=} A \Rightarrow \bot$. We call $\top$ top, $\bot$ bottom, 0 zero, $\land$ (classical) conjunction, $\lor$ (linear) disjunction, $\Rightarrow$ (classical linear) implication, $!$ of-course, and $\bot$ (classical linear) negation.

Definition 3.13 (CLLK$_\mu$). The sequent calculus CLLK$_\mu$ for CLLN consists of the axioms and the rules in Fig. 7 where $!(A_1, A_2, \ldots, A_k) \overset{df}{=} !A_1, !A_2, \ldots, !A_k$. 
There is the following proof of LEM w.r.t. disjunction $\oplus$ in $\text{CLLK}_\mu$ for any formula $A$:

\[
\begin{align*}
\text{(Id)} & \quad \frac{}{A \vdash A} \\
\text{(\rightharpoonup R)} & \quad \frac{A \vdash \bot, A}{\vdash A, \rightharpoonup A} \\
\text{(\oplus R)} & \quad \frac{A \vdash A, A \oplus A}{\vdash A, A, \rightharpoonup A} \\
\text{(XR)} & \quad \frac{A, A \oplus A \vdash A}{\vdash A, A, A, A} \\
\text{(\oplus L')} & \quad \frac{!A, !B \vdash \Gamma, \Theta \vdash A, \Xi}{\vdash A, \Theta, (A \rightharpoonup B) \vdash \Gamma, \Xi}
\end{align*}
\]

which supports our claim that CLLN is classical.

Dually to the case of $\text{ILK}_\mu$, the cut-elimination theorem holds also for $\text{CLLK}_\mu$:

**Theorem 3.14** (Cut-elimination on $\text{CLLK}_\mu$). Given a proof of a sequent in $\text{CLLK}_\mu$, there is a proof of the same sequent in $\text{CLLK}_\mu$ that does not use the rule $\text{Cut}'_\mu$.

**Proof.** Left to the reader. \hfill $\square$

Similarly to the case of $\text{ILK}_\mu$, there is a non-canonical choice in designing a cut-elimination procedure on $\text{Cut}'_\mu$, where we leave the details to the reader. Hence, we call $\text{CLLK}_\mu$ undirected.

Let us then decompose the translation $\mathcal{R}_\gamma$ as follows:

**Theorem 3.15** (Translation $\mathcal{R}$ of LK into $\text{CLLK}_\mu$). There is a translation $\mathcal{R}$ of formulas and proofs that assigns, to each proof $p$ of a sequent $\Delta \vdash \Gamma$ in LK, a proof $\mathcal{R}(p)$ of a sequent $\mathcal{R}^*(\Delta) \vdash \mathcal{R}^*(\Gamma)$ in $\text{CLLK}_\mu$, where $\mathcal{R}(\top) \overset{\text{df}}{=} \top$, $\mathcal{R}(\bot) \overset{\text{df}}{=} \bot$, $\mathcal{R}(A \land B) \overset{\text{df}}{=} \mathcal{R}(A) \land \mathcal{R}(B)$, $\mathcal{R}(A \lor B) \overset{\text{df}}{=} \mathcal{R}(A) \lor \mathcal{R}(B)$ and $\mathcal{R}(A \Rightarrow B) \overset{\text{df}}{=} \mathcal{R}(A) \Rightarrow \mathcal{R}(B)$. Moreover, it is conservative.
in the sense that if the sequent $\mathcal{R}^*(\Delta) \vdash \mathcal{R}^*(\Gamma)$ contains only formulas of CL, and it is provable in $\text{CLLK}_\mu$, then the sequent $\Delta \vdash \Gamma$ is provable in $\text{LK}$.

**Proof.** Let us first translate each of the axioms and the rules of $\text{LK}$ into a proof tree in $\text{CLLK}_\mu$. Let us leave it to the reader to translate $\text{WL}$, $\text{WR}$, $\text{CL}$, $\text{CR}$, $\text{XL}$, $\text{XR}$, $\text{Id}$, $\top_L$, $\top_R$, $\bot_L$ and $\bot_R$ because they are just straightforward.

Cut of $\text{LK}$ is translated in $\text{CLLK}_\mu$ as:

$$(\text{Cut}_\mu) \quad \frac{\Delta, B, \Gamma \vdash \Delta', !B \vdash \Gamma'}{\Delta, \Delta' \vdash \Gamma, \Gamma'}$$

$\land$R of $\text{LK}$ is translated as that of $\text{CLLK}_\mu$, and $\land$L of $\text{LK}$ as:

$$(\text{Id}) \quad A_i \vdash A_i$$

$$(\land') \quad A_1 \land A_2 \vdash A_i$$

$$(\text{Cut}_\mu') \quad \frac{\vdash (A_1 \land A_2) \vdash A_i \quad \Delta, !A_i \vdash \Gamma}{\Delta, (A_1 \land A_2) \vdash \Gamma}$$

$$(\text{XL'}) \quad \frac{\vdash (A_1 \land A_2), !\Delta \vdash \Gamma}{\Delta, (A_1 \land A_2) \vdash \Gamma}$$

Next, $\lor$L of $\text{LK}$ is translated in $\text{CLLK}_\mu$ as:

$$(\oplus') \quad \frac{\Delta, !A_1 \vdash \Gamma \quad \Delta, !A_2 \vdash \Gamma}{\Delta, !A_1 \oplus !A_2 \vdash \Gamma}$$

$$(\lor') \quad \frac{\Delta, !\Delta \vdash \Gamma}{\Delta, !\Delta \vdash B_1 \oplus !B_2, \Gamma}$$

and $\lor$R as:

$$(\text{Id}) \quad B_i \vdash B_i$$

$$(\text{Id}) \quad !A \vdash !A$$

$$(\text{Id}) \quad \vdash (A_1 \oplus A_2) \vdash \Gamma$$

$$(\text{R'}) \quad \frac{\Delta \vdash B_i, \Gamma}{\Delta \vdash !B_i \oplus !B_2, \Gamma}$$

Next, $\Rightarrow$R of $\text{LK}$ is translated in $\text{CLLK}_\mu$ as:

$$(\Rightarrow') \quad \frac{\Delta, !A \vdash B, \Gamma}{\Delta \vdash !A \Rightarrow B, \Gamma}$$

and $\Rightarrow$L as:

$$(\text{Id}) \quad \vdash B$$

$$(\text{Id}) \quad !A \vdash !A$$

$$(\text{R'}) \quad \frac{\vdash (A \Rightarrow B) \vdash B \quad !A \vdash (A \Rightarrow B) \vdash B}{\vdash (A \Rightarrow B), !A \vdash B}$$

$$(\text{XL'}) \quad \frac{\vdash (A \Rightarrow B), !A \vdash B}{\vdash (A \Rightarrow B) \vdash !A \Rightarrow !B}$$

$$(\Rightarrow') \quad \frac{\Delta \vdash B, \Gamma}{\Delta \vdash \Theta, !A \Rightarrow B \vdash \Gamma, \Xi}$$

$$(\text{Id}) \quad \vdash \Theta, !A \Rightarrow B \vdash \Gamma, \Xi$$

which completes the translation $\mathcal{T}$.

Finally, we show that the translation $\mathcal{T}$ is conservative essentially in the same way as the proof of Thm. 3.10, completing the proof. \qed
Theorem 3.16 (Translation $\mathcal{F}_i$ of CLLK$_\mu$ into ILLK$_\mu$). There is a translation $\mathcal{F}_i$ of formulas and proofs that assigns, to each proof $p$ of a sequent $\Delta \vdash \Gamma$ in CLLK$_\mu$, a proof $\mathcal{F}_i(p)$ of a sequent $\mathcal{F}_i^*(\Delta) \vdash \mathcal{F}_i^*(\Gamma)$ in ILLK$_\mu$, where $\mathcal{F}_i(\top) \equiv \top$, $\mathcal{F}_i(\bot) \equiv \bot$, $\mathcal{F}_i(0) \equiv 0$, $\mathcal{F}_i(A \land B) \equiv \mathcal{F}_i(A) \land \mathcal{F}_i(B)$, $\mathcal{F}_i(A \lor B) \equiv \mathcal{F}_i(A) \lor \mathcal{F}_i(B)$, $\mathcal{F}_i(A \Rightarrow B) \equiv \mathcal{F}_i(A) \Rightarrow \mathcal{F}_i(B)$, $\mathcal{F}_i(A) \Leftarrow \mathcal{F}_i(B)$ and $\mathcal{F}_i(\Delta)$ as:

Prove that assigns, to each proof $\mathcal{F}_i(p)$ of a sequent $\mathcal{F}_i^*(\Delta) \vdash \mathcal{F}_i^*(\Gamma)$ contains only formulas of CLLN, and it is provable in ILLK$_\mu$, then the sequent $\Delta \vdash \Gamma$ is provable in ILLK$_\mu$.

Proof. Let us first translate each of the axioms and the rules of CLLK$_\mu$ into a proof tree in ILLK$_\mu$.

It is trivial to translate XL, XR, !W, WR, IC, CR, !D, !R, Id, $\top L$, $\top R$, $\bot L$, $\bot R$ and 0L; hence, let us leave them to the reader.

\textbf{Cut}$_L^\prime$ of CLLK$_\mu$ is translated in ILLK$_\mu$ as:

\begin{align*}
\text{(Cut)}: \quad & \Delta \vdash \Theta \vdash \exists \Xi \\
& \Delta, \Theta \vdash \exists \Xi
\end{align*}

\textbf{L} of CLLK$_\mu$ is translated in ILLK$_\mu$ as:

\begin{align*}
\text{(L)}: \quad & \Delta, A_i \vdash \exists \top \\
& \Delta, \Theta \vdash \exists \Xi
\end{align*}

\textbf{R} as:

\begin{align*}
\text{(R)}: \quad & \Delta \vdash \exists \top, \exists \Xi \\
& \Delta, \Theta \vdash \exists \Xi
\end{align*}

Next, $\exists L$ of CLLK$_\mu$ is translated in ILLK$_\mu$ as:

\begin{align*}
\text{(Id)}: \quad & B \vdash B \\
\text{(6)}: \quad & A \vdash A
\end{align*}

\textbf{L} as:

\begin{align*}
\text{(L)}: \quad & \Delta, A \vdash \exists \top \\
& \Delta, \Theta \vdash \exists \Xi
\end{align*}

\textbf{R} as:

\begin{align*}
\text{(R)}: \quad & \Delta \vdash \exists \top, \exists \Xi \\
& \Delta, \Theta \vdash \exists \Xi
\end{align*}

Next, $\forall L$ of CLLK$_\mu$ is translated in ILLK$_\mu$ as:

\begin{align*}
\text{(Id)}: \quad & B \vdash B \\
\text{(6)}: \quad & A \vdash A
\end{align*}
where the of-course ⋄ on A ⊸ B in the rule ⊸-L ′ is vital for this interpretation, and ⊸-R ′ as:

\[
\begin{align*}
(\text{L}^-) & \quad !\Delta, A \vdash ?B, ?T \\
(\text{R}^-) & \quad !\Delta \vdash \neg A, ?B, ?T \\
\end{align*}
\]

which completes the translation \( \mathcal{F} \).

Finally, we show that the translation \( \mathcal{F} \) is conservative essentially in the same way as the proof of Thm. 3.9, completing the proof.

**Corollary 3.17** (Translation \( \mathcal{F}_T \) of \( \text{LK} \) into \( \text{ILLK}_\mu \)). The composition \( \mathcal{F}_T \models \mathcal{F} \circ \mathcal{F} \) assigns, to each proof \( p \) of a sequent \( \Delta \vdash \Gamma \) in \( \text{LK} \), a proof \( \mathcal{F}_T(p) \) of a sequent \( !\mathcal{F}_T(\Delta) \vdash ?\mathcal{F}_T(\Gamma) \) in \( \text{ILLK}_\mu \), where \( \mathcal{F}_T(\top) \defeq \top \), \( \mathcal{F}_T(\bot) \defeq \bot \), \( \mathcal{F}_T(A \land B) \defeq \mathcal{F}_T(A) \& \mathcal{F}_T(B) \), \( \mathcal{F}_T(A \lor B) \defeq !\mathcal{F}_T(A) \lor !\mathcal{F}_T(B) \), \( \mathcal{F}_T(A \Rightarrow B) \defeq !\mathcal{F}_T(A) \Rightarrow \mathcal{F}_T(B) \).

Proof. By Thms. 3.15 and 3.16, where the last statement is by induction on \( p \).

### 4 Call-by-Name Unity of Logic without Polarities

Having accomplished the main aim of the present work, the rest of the paper is dedicated to another goal, namely, to give the call-by-name and the call-by-value variants of the unity of logic.

First, let us obtain a call-by-name sequent calculus for ILE from \( \text{ILLK}_\mu \) simply by modifying the cut rule slightly:

**Definition 4.1** (\( \text{ILLK}_\eta \)). The sequent calculus \( \text{ILLK}_\eta \) for ILE consists of the axioms and the rules in Fig. 8 where \( !(A_1, A_2, \ldots, A_n) \defeq A_1, A_2, \ldots, A_n \).

**Theorem 4.2** (Cut-elimination on \( \text{ILLK}_\eta \)). Given a proof of a sequent in \( \text{ILLK}_\eta \), there is a proof of the same sequent in \( \text{ILLK}_\eta \) that does not use the rule \( \text{Cut}_\eta \).

Proof. Left to the reader.

By Thm. 3.7 and 4.2 we know that the choice between \( \text{ILLK}_\mu \) and \( \text{ILLK}_\eta \) does not change the notion of ILE (at the level of provability, not proofs themselves).

We may then give another variant of classicalization \( \mathcal{F}_\tau \), which translates \( \text{LK} \) into \( \text{ILLK}_\eta \):

**Theorem 4.3** (Translation \( \mathcal{F}_\tau \) of \( \text{LK} \) into \( \text{ILLK}_\eta \)). There is a translation \( \mathcal{F}_\tau \) of formulas and proofs that assigns, to each proof \( p \) of a sequent \( \Delta \vdash \Gamma \) in \( \text{LK} \), a proof \( \mathcal{F}_\tau(p) \) of a sequent \( \mathcal{F}_\tau(\Delta) \vdash \mathcal{F}_\tau(\Gamma) \) in \( \text{ILLK}_\eta \), where \( \mathcal{F}_\tau(\top) \defeq \top \), \( \mathcal{F}_\tau(\bot) \defeq \bot \), \( \mathcal{F}_\tau(A \land B) \defeq \mathcal{F}_\tau(A) \& \mathcal{F}_\tau(B) \), \( \mathcal{F}_\tau(A \lor B) \defeq \mathcal{F}_\tau(A) \lor \mathcal{F}_\tau(B) \), \( \mathcal{F}_\tau(A \Rightarrow B) \defeq !\mathcal{F}_\tau(A) \Rightarrow \mathcal{F}_\tau(B) \).

Proof. Let us first translate each of the axioms and the rules of \( \text{LK} \) into a proof tree in \( \text{ILLK}_\eta \). Let us leave it to the reader to translate \( \text{XL} \), \( \text{XR} \), \( \text{WL} \), \( \text{WR} \), \( \text{CL} \), \( \text{CR} \), \( \text{Id} \), \( \top \text{L} \), \( \top \text{R} \), \( \bot \text{L} \) and \( \bot \text{R} \) because they are just straightforward.

Cut of \( \text{LK} \) is translated in \( \text{ILLK}_\eta \) as:

\[
(\text{Cut}_\eta^\tau) \quad \frac{\mathcal{F}_\tau(\Delta) \vdash \mathcal{F}_\tau(\Gamma)}{\mathcal{F}_\tau(\Delta, \mathcal{F}_\tau(\Gamma))}
\]
Figure 8: Sequent calculus $\mathbf{ILK}_\eta$ for ILE

\[\text{\&L of LK is translated in } \mathbf{ILK}_\eta \text{ as:}\]
\[
\frac{\Delta, A_1 \vdash \Gamma}{\Delta, \text{?}, A_1 \vdash \text{T}}\quad \frac{\Delta, A_1 \& A_2 \vdash \Gamma}{\Delta, \text{?}, (A_1 \& A_2) \vdash \text{T}}
\]

and \text{\&R as:}
\[
\frac{\Delta \vdash \text{T}}{\Delta \vdash \text{T}}\quad \frac{\Delta \vdash \text{T}}{\Delta \vdash \text{T}}\quad \frac{\Delta \vdash \text{T}}{\Delta \vdash \text{T}}\quad \frac{\Delta \vdash \text{T}}{\Delta \vdash \text{T}}\quad \frac{\Delta \vdash \text{T}}{\Delta \vdash \text{T}}\quad \frac{\Delta \vdash \text{T}}{\Delta \vdash \text{T}}
\]

Dually, $\forall L$ of LK is translated in $\mathbf{ILK}_\eta$ as:
\[
\frac{\Delta, A \vdash \Gamma}{\Delta, \exists A, \exists \Xi \vdash B \vdash \Gamma, \exists \Xi} \quad \frac{\Delta, A \vdash \Gamma}{\Delta, \exists A, \exists \Xi \vdash B \vdash \Gamma, \exists \Xi}
\]

and $\forall R$ as:
\[
\frac{\Delta \vdash \text{T}}{\Delta \vdash \text{T}}\quad \frac{\Delta \vdash \text{T}}{\Delta \vdash \text{T}}\quad \frac{\Delta \vdash \text{T}}{\Delta \vdash \text{T}}\quad \frac{\Delta \vdash \text{T}}{\Delta \vdash \text{T}}\quad \frac{\Delta \vdash \text{T}}{\Delta \vdash \text{T}}\quad \frac{\Delta \vdash \text{T}}{\Delta \vdash \text{T}}
\]
Next, \( \exists L \) of LK is translated in \( \text{ILK}_\eta \) as:

\[
\begin{align*}
(\exists L') & \quad \frac{\Delta \vdash A, ?, \Gamma}{\Delta, ?, B \vdash \Gamma} \\
(\exists L^+) & \quad \frac{\Delta, ?, A \Rightarrow B \vdash \Gamma}{\Delta, ?, B \vdash \Gamma} \\
(\exists \text{L}^+) & \quad \frac{\Delta, ?, A \Rightarrow B \vdash \Gamma}{\Delta, ?, B \vdash \Gamma} \\
(\exists \text{C}^+) & \quad \frac{\Delta, ?, (? A \Rightarrow B) \vdash \Gamma}{\Delta, ?, B \vdash \Gamma}
\end{align*}
\]

and \( \exists R \) as:

\[
\begin{align*}
(\exists R) & \quad \frac{\Delta, ?, A \vdash ?, B, ?, \Gamma}{\Delta \vdash ?, A \Rightarrow ?, B, ?, \Gamma} \\
(\exists \text{D}) & \quad \frac{\Delta \vdash ?, A \Rightarrow ?, B, ?, \Gamma}{\Delta \vdash ?, (? A \Rightarrow B), ?, \Gamma}
\end{align*}
\]

We have defined the translation \( \mathcal{T}_T \) of LK into \( \text{ILK}_\eta \).

Finally, let us show that the translation \( \mathcal{T}_T \) is conservative. Note that, given a proof of a sequent in \( \text{ILK}_\eta \), we may obtain another proof of the same sequent in \( \text{ILK}_\mu \) by putting forward each application of the rules \( ? \text{D} \) and \( ? \text{L} \) in the proof as much as possible. Then, it is easy to see that the resulting proof in \( \text{ILK}_\eta \) has a corresponding proof in \( \text{LK} \), completing the proof. \( \square \)

Further, an advantage of \( \text{ILK}_\eta \) over \( \text{ILK}_\mu \) is in the point that it is possible to translate the former into \( \text{ILK}_\mu \), not \( \text{ILK}_\eta \).

**Theorem 4.4** (Translation \( \mathcal{T} \) of \( \text{ILK}_\eta \) into \( \text{ILK} \)). There is a translation \( \mathcal{T} \) of formulas and proofs that assigns, to each proof \( p \) of a sequent \( \Delta \vdash \Gamma \) in \( \text{ILK}_\eta \), a proof \( \mathcal{T}(p) \) of a sequent \( \mathcal{T}^*(\Delta) \vdash \mathcal{T}^*(\Gamma) \) in \( \text{ILK} \), where \( \mathcal{T}(\top) \equiv \top \), \( \mathcal{T}(\bot) \equiv \bot \), \( \mathcal{T}(1) \equiv 1 \), \( \mathcal{T}(A \& B) \equiv \mathcal{T}(A) \& \mathcal{T}(B) \), \( \mathcal{T}(A \lor B) \equiv \mathcal{T}(A) \lor \mathcal{T}(B) \), \( \mathcal{T}(A \Rightarrow B) \equiv \mathcal{T}(A) \Rightarrow \mathcal{T}(B) \), and \( \mathcal{T}(\forall A) \equiv \forall A \). Moreover, it is conservative in the sense that if the sequent \( \mathcal{T}^*(\Delta) \vdash \mathcal{T}^*(\Gamma) \) contains only formulas of \( \text{CL} \), and it is provable in \( \text{ILK} \), then the sequent \( \Delta \vdash \Gamma \) is provable in \( \text{ILK}_\eta \).

**Proof.** It suffices to translate \( \text{Cut}_T^2 \) because the other translations are the same as the translation \( \mathcal{T} \) of \( \text{ILK}_\mu \) into \( \text{ILK}_\mu \) (Thm. 3.10). Then, \( \text{Cut}_T^2 \) of \( \text{ILK}_\eta \) is translated in \( \text{ILK} \) as:

\[
\begin{align*}
(\text{IR}) & \quad \frac{\Delta \vdash B, ?, \Gamma}{\Delta, ?, B \vdash \Gamma} \\
(\text{Cut}) & \quad \frac{\Delta \vdash B, ?, \Gamma}{\Delta, ?, B \vdash \Gamma}
\end{align*}
\]

which completes the proof. \( \square \)

**Corollary 4.5** (Translation \( \mathcal{T}_T \) of LK into \( \text{ILK} \)). The composition \( \mathcal{T}_T \equiv \mathcal{T} \circ \mathcal{T}_T \) assigns, to each proof \( p \) of a sequent \( \Delta \vdash \Gamma \) in LK, a proof \( \mathcal{T}_T(p) \) of a sequent \( \mathcal{T}_T(\Delta) \vdash \mathcal{T}_T(\Gamma) \) in \( \text{ILK} \), where \( \mathcal{T}_T(\top) \equiv \top \), \( \mathcal{T}_T(\bot) \equiv \bot \), \( \mathcal{T}_T(A \& B) \equiv \mathcal{T}_T(A) \& \mathcal{T}_T(B) \), \( \mathcal{T}_T(A \Rightarrow B) \equiv \mathcal{T}_T(A) \Rightarrow \mathcal{T}_T(B) \), \( \mathcal{T}_T(A \lor B) \equiv \mathcal{T}_T(A) \lor \mathcal{T}_T(B) \), \( \mathcal{T}_T(\forall A) \equiv \forall A \). Moreover, it is conservative in the sense that if the sequent \( \mathcal{T}_T(\Delta) \vdash \mathcal{T}_T(\Gamma) \) contains only formulas of \( \text{CL} \), and it is provable in \( \text{ILK} \), then the sequent \( \Delta \vdash \Gamma \) is provable in LK.

**Proof.** By Thm. 3.6 and 4.3 \( \square \)

Note that the translation \( \mathcal{T}_T : \text{LK} \rightarrow \text{ILK}_\eta \) does not employ the undirected cut rule \( \text{Cut}_T^2 \). Seeing it more closely, Cut of LK is translated by \( \mathcal{T}_T \) as the following proof tree in \( \text{ILK} \):

\[
\begin{align*}
(\text{IR}) & \quad \frac{\Delta \vdash B, ?, \Gamma}{\Delta, ?, B \vdash \Gamma} \\
(\text{Cut}) & \quad \frac{\Delta \vdash ?, B, ?, \Gamma}{\Delta, ?, B \vdash \Gamma}
\end{align*}
\]
for which the cut-elimination procedure on Cut given in Appx. A computes in the call-by-name fashion. Hence, we have established another, call-by-name variant of the unity of logic.

Note that the translation $\mathcal{T}$ coincides with the $T$-translation given by Danos et al. [DJS95]. Note also that there is the evident embedding of proofs of a given sequent $\gamma \vdash \tau$ in ILLK into proofs of the sequent $\Delta \vdash \tau$ in ILLK. Thus, the call-by-name variant of the unity of logic presented in this section forms a particular part of the undirected one given in Sect. 3.

On the other hand, the call-by-name variant loses the commutativity between unlinearization and classicalization of the undirected one. In fact, the reverse composition $\mathcal{T}^{-1} \circ \mathcal{T}$ interprets sequents $\Delta \vdash \tau$ in LK as sequents $\gamma \vdash \tau$ in ILLK, but then it is clearly impossible to interpret Cut of LK between such interpreted sequents in ILLK.

Writing LK$\eta$ as the sequent calculus for CL, which is the image $\mathcal{T}^{-1}(\text{ILLK}) \hookrightarrow \text{LK}$, the present section is summarized by the following commutative (up to permutations of rules) diagram:

5 Call-by-Value Unity of Logic without Polarities

Let us then dualize the call-by-name unity of logic in the previous section in order to obtain the call-by-value variant.

Definition 5.1 (CLLK$\nu$). The sequent calculus $\text{CLLK}_\nu$ for CLN consists of the axioms and the rules in Fig. 9, where $\gamma(A_1, A_2, \ldots, A_k) \overset{\text{df.}}{=} !A_1, !A_2, \ldots, !A_k$.

Theorem 5.2 (Cut-elimination on $\text{CLLK}_\nu$). Given a proof of a sequent in $\text{CLLK}_\nu$, there is a proof of the same sequent in $\text{CLLK}_\mu$ that does not use the rule Cut$\nu$.

Proof. Left to the reader. □

Again, by Thms. 3.1 and 5.2, it does not matter $\text{CLLK}_\mu$ or $\text{CLLK}_\nu$ to employ as a sequent calculus for CLN.

Because the following results are just symmetric to the corresponding ones in the previous section, let us present them without proofs:

Theorem 5.3 (Translation $\mathcal{R}$ of LK into $\text{CLLK}_\nu$). There is a translation $\mathcal{R}$ of formulas and proofs that assigns, to each proof $p$ of a sequent $\Delta \vdash \Gamma$ in LK, a proof $\mathcal{R}(p)$ of a sequent $\mathcal{R}^*(\Delta) \vdash \mathcal{R}^*(\Gamma)$ in $\text{CLLK}_\nu$, where $\mathcal{R}(\top) \overset{\text{df.}}{=} \top$, $\mathcal{R}(\bot) \overset{\text{df.}}{=} \bot$, $\mathcal{R}(A \land B) \overset{\text{df.}}{=} \mathcal{R}(A) \land \mathcal{R}(B)$, $\mathcal{R}(A \lor B) \overset{\text{df.}}{=} \mathcal{R}(A) \lor \mathcal{R}(B)$ and $\mathcal{R}(A \Rightarrow B) \overset{\text{df.}}{=} \mathcal{R}(A) \Rightarrow \mathcal{R}(B)$. Moreover, it is conservative in the sense that if the sequent $\mathcal{R}^*(\Delta) \vdash \mathcal{R}^*(\Gamma)$ contains only formulas of CL, and it is provable in $\text{CLLK}_\nu$, then the sequent $\Delta \vdash \Gamma$ is provable in LK.
Theorem 5.4 (Translation $\mathcal{F}_c$ of CLLK$_\nu$ into ILLK). There is a translation $\mathcal{F}_c$ of formulas and proofs that assigns, to each proof $p$ of a sequent $\Delta \vdash \Gamma$ in CLLK$_\nu$, a proof $\mathcal{F}_c(p)$ of a sequent $\mathcal{F}_c(\Delta) \vdash \mathcal{F}_c(\Gamma)$ in ILLK, where $\mathcal{F}_c(\top) \equiv \top$, $\mathcal{F}_c(\bot) \equiv \bot$, $\mathcal{F}_c(0) \equiv 0$, $\mathcal{F}_c(\Delta \land \Delta') \equiv \mathcal{F}_c(\Delta) \land \mathcal{F}_c(\Delta')$, $\mathcal{F}_c(\Delta \lor \Delta') \equiv \mathcal{F}_c(\Delta) \lor \mathcal{F}_c(\Delta')$, $\mathcal{F}_c(\Delta \rightarrow \Delta') \equiv \mathcal{F}_c(\Delta) \rightarrow \mathcal{F}_c(\Delta')$. Moreover, it is conservative in the sense that if the sequent $\mathcal{F}_c(\Delta) \vdash \mathcal{F}_c(\Gamma)$ contains only formulas of CLLN, and it is provable in ILLK, then the sequent $\Delta \vdash \Gamma$ is provable in CLLK$_\nu$.  

Corollary 5.5 (Translation $\mathcal{F}_m$ of LK into ILLK). The composition $\mathcal{F}_m \equiv \mathcal{F}_c \circ \mathcal{F}_t$ assigns, to each proof $p$ of a sequent $\Delta \vdash \Gamma$ in LK, a proof $\mathcal{F}_m(p)$ of a sequent $\mathcal{F}_m(\Delta) \vdash \mathcal{F}_m(\Gamma)$ in ILLK, where $\mathcal{F}_m(\top) \equiv \top$, $\mathcal{F}_m(\bot) \equiv \bot$, $\mathcal{F}_m(\Delta \land \Delta') \equiv \mathcal{F}_m(\Delta) \land \mathcal{F}_m(\Delta')$, $\mathcal{F}_m(\Delta \lor \Delta') \equiv \mathcal{F}_m(\Delta) \lor \mathcal{F}_m(\Delta')$, $\mathcal{F}_m(\Delta \rightarrow \Delta') \equiv \mathcal{F}_m(\Delta) \rightarrow \mathcal{F}_m(\Delta')$. Moreover, it is conservative in the sense that if the sequent $\mathcal{F}_m(\Delta) \vdash \mathcal{F}_m(\Gamma)$ contains only formulas of CLL, and it is provable in ILLK, then the sequent $\Delta \vdash \Gamma$ is provable in LK.

The translation $\mathcal{F}_m$ interprets Cut of LK as the following proof tree in CLLK$_\nu$:

$$
\text{(Cut)} \quad \frac{\Delta \vdash \mathcal{F}_m(\beta), \beta \Gamma}{\Delta, \mathcal{F}_m(\beta) \Gamma}
$$

for which the cut-elimination procedure on Cut computes in the call-by-value fashion.

Note that the translation $\mathcal{F}_m$ coincides with the Q-translation introduced in [DJS95]. Also, similarly to the case of the call-by-name variant, the call-by-value unity of logic forms a particular embedding of the undirected one, and the commutativity between unlinearization and classicalization of the undirected one is lost in the call-by-value variant.
Finally, writing $\mathcal{LK}_\nu$ for the sequent calculus $\mathcal{T}_\nu(\mathcal{ILLK}) \leftrightarrow \mathcal{LK}$ for CL, the present section is summarized by the following commutative (up to permutations of rules) diagram:

\[
\begin{array}{c}
\text{LLJ} \quad \text{unlinearization } \mathcal{T} \quad \text{conservative extension} \\
\downarrow \quad \downarrow \quad \downarrow \\
\text{ILLK} \quad \text{unlinearization } \mathcal{T} \quad \text{conservative extension} \\
\downarrow \quad \downarrow \\
\text{CLLK}_\nu \quad \text{unlinearization } \mathcal{T} \quad \text{classicalization } \mathcal{T} \quad \text{LK}_\nu \\
\downarrow \quad \downarrow \\
\text{LK}
\end{array}
\]

6 Conclusion and Future Work

We have established a novel unity of logic in terms of sequent calculi, which does not have recourse to (both) polarities unlike existing methods. Another distinguished feature of the present work is the universal operations of unlinearization and classicalization on logic such that they are compatible and dual to each other. Further, we have analyzed that the sequent calculi for the aforementioned unity of logic are all undirected, and then carved out the call-by-name and the call-by-value fragments of the undirected unity, where the compatibility and the duality between unlinearization and classicalization break down, and the two computational paradigms inherently correspond to the order of composing the two operations.

As immediate future work, we plan to establish term calculi on the sequent calculi introduced by the present work and translations between them that correspond to the unity of logic without polarities. We would also like to establish categorial semantics of the unity of logic. Finally, it should be possible to give sequential, negative game semantics of the unity of logic; indeed, such a game-semantic analysis is the starting point of the present work.

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A Cut-Elimination on Cut

In this appendix, we define a cut-elimination procedure on Cut of the sequent calculus $\mathcal{ILLK}$ (Def. 3.2), following the method for $\mathcal{LLJ}$ (Def. 2.8) given in Gavin Bierman’s PhD thesis [Bie94].

The cut-elimination procedure removes applications of Cut in a proof that are topmost (w.r.t. the height of each application of Cut) among the ones with the highest rank in the proof, where the rank of an application of Cut is roughly the complexity of the cut formula. We may basically follow the proof given in [Bie94], where the height and the rank are made precise, to show that our cut-elimination procedure eliminates all the applications of Cut in a given proof in $\mathcal{ILLK}$.

Therefore, in the following, let us just describe how the cut-elimination procedure transforms each application of Cut. Strictly speaking, however, the procedure actually deals with a multiple, consecutive applications of Cut at a time, regarding them as the following single rule:

Definition A.1 (Multiple Cut). Let us define left multiple Cut to be the following rule:

\[
(\text{Cut}^n) \quad \frac{\Delta \vdash A, \Gamma \quad \Delta', A^n \vdash \Gamma'}{\Delta^n, \Delta' \vdash \Gamma^n, \Gamma'} \quad (n \in \mathbb{N}^+)
\]
and symmetrically, right multiple Cut to be the following rule:

\[(\text{CutR}^n) \quad \frac{\Delta' \vdash A^n, \Gamma'}{\Delta', \Delta \vdash \Gamma', \Gamma} \quad (n \in \mathbb{N}^+)\]

The left and the right multiple Cut rules are derivable in ILLK by consecutive applications of Cut from left and right, respectively, in the evident manner. The cut-elimination procedure has to take multiple Cut, not Cut, as a single rule to handle applications of Cut whose cut formula is the principal formula of a contraction rule; see [TS00, Gał93] for the details.

Let us now list all the cases of an application of Cut in ILLK, where the cases are divided into four patterns, and describe how the cut-elimination procedure transforms it. The first pattern is an application of Cut such that the principal formulas of the rules applied at the end of the left and the right hypotheses of the application of Cut are both the cut formula.

The cases of the first pattern are the following:

- **(⊤R, ⊤L)-Cut.** A proof of the form

  \[(\text{TR}) \quad \vdash \top \quad \quad (\text{TL}) \quad \top^n, \Delta \vdash \Gamma \quad \frac{\Delta \vdash \Gamma}{\top^{n+1}, \Delta \vdash \Gamma} \quad \text{(CutL}^{n+1})\]

  is transformed into

  \[(\text{TR}) \quad \vdash \top \quad \quad (\text{TL}) \quad \top^n, \Delta \vdash \Gamma \quad \frac{\Delta \vdash \Gamma}{\top^{n+1}, \Delta \vdash \Gamma} \quad \text{(CutL}^n)\]

- **(⊥R, ⊥L)-Cut.** A proof of the form

  \[(\text{⊥R}) \quad \Delta \vdash \bot^n, \Gamma \quad (\text{⊥L}) \quad \bot^n, \Delta \vdash \Gamma \quad \frac{\bot \vdash \bot^{n+1}, \Gamma}{\Delta \vdash \Gamma} \quad (\text{CutR}^{n+1})\]

  is transformed into

  \[(\text{⊥R}) \quad \Delta \vdash \bot^n, \Gamma \quad (\text{⊥L}) \quad \bot^n, \Delta \vdash \Gamma \quad \frac{\bot \vdash \bot^{n+1}, \Gamma}{\Delta \vdash \Gamma} \quad (\text{CutR}^n)\]

- **(¬R, ¬L)-Cut.** A proof of the form

  \[(\text{¬R}) \quad \Delta_1, A \vdash \Gamma_1 \quad (\text{¬L}) \quad \Delta_2, \neg A^n \vdash A, \Gamma_2 \quad \frac{\Delta_1 \vdash \neg A, \Gamma_1}{\Delta_1^{n+1}, \Delta_2 \vdash \Gamma_1, \Gamma_2} \quad \frac{\Delta_2 \vdash \neg A^{n+1} \vdash \Gamma_2}{\Delta_1^{n+1}, \Delta_2 \vdash \Gamma_1^{n+1}, \Gamma_2} \quad \text{(CutL}^{n+1})\]

  is transformed into

  \[(\text{¬R}) \quad \Delta_1, A \vdash \Gamma_1 \quad (\text{¬L}) \quad \Delta_2, \neg A^n \vdash A, \Gamma_2 \quad \frac{\Delta_1 \vdash \neg A, \Gamma_1}{\Delta_1^{n+1}, \Delta_2 \vdash \Gamma_1, \Gamma_2} \quad \frac{\Delta_2 \vdash \neg A^{n+1} \vdash \Gamma_2}{\Delta_1^{n+1}, \Delta_2 \vdash \Gamma_1^{n+1}, \Gamma_2} \quad \text{(CutL}^n)\]

  \[(\text{¬R}) \quad \Delta_1, A \vdash \Gamma_1 \quad (\text{¬L}) \quad \Delta_2, \neg A^n \vdash A, \Gamma_2 \quad \frac{\Delta_1 \vdash \neg A, \Gamma_1}{\Delta_1^{n+1}, \Delta_2 \vdash \Gamma_1, \Gamma_2} \quad \frac{\Delta_2 \vdash \neg A^{n+1} \vdash \Gamma_2}{\Delta_1^{n+1}, \Delta_2 \vdash \Gamma_1^{n+1}, \Gamma_2} \quad \text{(CutL}^n)\]

The case of the right multiple Cut is just symmetric, and thus we omit it.
• ($\odot$R, $\odot$L)-Cut. A proof of the form

\[
\frac{\Delta_1 \vdash A_1, \Gamma_1 \quad \Delta_2 \vdash A_2, \Gamma_2}{\Delta_1, \Delta_2 \vdash A_1 \odot A_2, \Gamma_1, \Gamma_2}
\]

is transformed into

\[
\frac{p_1}{\Delta_1 \vdash A_1, \Gamma_1} \quad \frac{p_2}{\Delta_2 \vdash A_2, \Gamma_2} \quad \frac{p_3}{\Delta_3, (A_1 \otimes A_2)^n, A_1, A_2 \vdash \Gamma_3}
\]

\[
\Delta_1^{n+1}, \Delta_2^{n+1}, \Delta_3 \vdash \Gamma_1^{n+1}, \Gamma_2^{n+1}, \Gamma_3
\]

The case of the right multiple Cut is just symmetric, and thus we omit it.

• ($\exists$R, $\exists$L)-Cut. A proof of the form

\[
\frac{\Delta \vdash A_1, A_2, (A_1 \exists A_2)^n, \Gamma}{\Delta, \Delta_1^{n+1}, \Delta_2^{n+1} \vdash \Gamma, \Gamma_1^{n+1}, \Gamma_2^{n+1}}
\]

is transformed into

\[
\frac{p}{\Delta \vdash A_1, A_2, (A_1 \exists A_2)^n, \Gamma} \quad \frac{p_1}{\Delta, \Delta_1^{n+1}, \Delta_2^{n+1} \vdash \Gamma, \Gamma_1^{n+1}, \Gamma_2^{n+1}}
\]

The case of the left multiple Cut is just symmetric, and thus we omit it.

• ($\&$R, $\&$L)-Cut. A proof of the form

\[
\frac{\Delta \vdash A_1, \Gamma \quad \Delta \vdash A_2, \Gamma}{\Delta \vdash A_1 \& A_2, \Gamma}
\]

is transformed into

\[
\frac{p_1}{\Delta \vdash A_1, \Gamma} \quad \frac{p_2}{\Delta \vdash A_2, \Gamma} \quad \frac{p' \quad i \in \mathbb{Z}}{\Delta', (A_1 \& A_2)^n, A_i \vdash \Gamma'}
\]

\[
\Delta^{n+1}, \Delta' \vdash \Gamma^{n+1}, \Gamma'
\]

The case of the right multiple Cut is just symmetric, and thus we omit it.
• \((⊕R, ⊕L)\)-Cut. A proof of the form

\[
\Delta', \Delta'_{n+1} \vdash A_i, (A_1 ⊕ A_2)^n, \Gamma'
\]

is transformed into

\[
\Delta', \Delta'_{n+1} \vdash A_i, (A_1 ⊕ A_2)^n, \Gamma'
\]

The case of the left multiple cut is just symmetric, and thus we omit it.

• \((!R, !D)\)-Cut. A proof of the form

\[
\Delta', \Delta'_{n+1} \vdash A, \Gamma, \Delta', \Delta'_{n+1} \vdash A, \Gamma
\]

is transformed into

\[
\Delta, \Delta_{n+1} \vdash A, \Gamma, \Delta, \Delta_{n+1} \vdash A, \Gamma
\]

Remark. The case of the right multiple cut, i.e., a proof of the form

\[
\Delta', \Delta'_{n+1} \vdash A, \Gamma, \Delta', \Delta'_{n+1} \vdash A, \Gamma
\]

for \(n > 0\) cannot occur since otherwise the application of the rule \(!R\) would be invalid.

• \((!R, !W)\)-Cut. A proof of the form

\[
\Delta', \Delta'_{n+1} \vdash A, \Gamma, \Delta', \Delta'_{n+1} \vdash A, \Gamma
\]

is transformed into

\[
\Delta', \Delta'_{n+1} \vdash A, \Gamma, \Delta', \Delta'_{n+1} \vdash A, \Gamma
\]

\[
\Delta', \Delta'_{n+1} \vdash A, \Gamma, \Delta', \Delta'_{n+1} \vdash A, \Gamma
\]
Remark. The case of the right multiple Cut cannot occur since otherwise the application of the rule !R would be invalid.

- (?R, !C)-Cut. A proof of the form

\[
\frac{\Delta \vdash A, \Gamma}{\Delta' \vdash A, \Gamma} \quad \frac{\Delta', A^n, A \vdash \Gamma'}{\Delta'' \vdash A^n + 1 \vdash \Gamma'}
\]

is transformed into

\[
\frac{\Delta \vdash A, \Gamma}{\Delta' \vdash A, \Gamma} \quad \frac{\Delta', A^n, A \vdash \Gamma'}{\Delta'' \vdash A^n + 1 \vdash \Gamma'}
\]

Remark. The case of the right multiple Cut cannot occur since otherwise the application of the rule !R would be invalid.

- (?D, ?L)-Cut. A proof of the form

\[
\frac{\Delta \vdash \Delta^n, \Gamma}{\Delta' \vdash \Delta^n + 1, \Gamma} \quad \frac{\Delta', \Delta, \Delta' \vdash \Gamma''}{\Delta'' \vdash \Delta^n, \Gamma''}
\]

is transformed into

\[
\frac{\Delta \vdash \Delta^n, \Gamma}{\Delta' \vdash \Delta^n + 1, \Gamma} \quad \frac{\Delta', \Delta, \Delta' \vdash \Gamma''}{\Delta'' \vdash \Delta^n, \Gamma''}
\]

Remark. The case of the left multiple Cut cannot occur since otherwise the application of the rule ?L would be invalid.

- (?W, ?L)-Cut. A proof of the form

\[
\frac{\Delta \vdash \Delta^n, \Gamma}{\Delta' \vdash \Delta^n, \Gamma} \quad \frac{\Delta', \Delta, \Delta' \vdash \Gamma''}{\Delta'' \vdash \Delta^n, \Gamma''}
\]

is transformed into

\[
\frac{\Delta \vdash \Delta^n, \Gamma}{\Delta' \vdash \Delta^n, \Gamma} \quad \frac{\Delta', \Delta, \Delta' \vdash \Gamma''}{\Delta'' \vdash \Delta^n, \Gamma''}
\]

Remark. The case of the left multiple Cut cannot occur since otherwise the application of the rule ?L would be invalid.
Remark. The case of the left multiple Cut cannot occur since otherwise the application of the rule \(?L\) would be invalid.

- \((?C, ?L)\)-Cut. A proof of the form

\[
\begin{align*}
\Delta' \vdash A, A, A^n, \Gamma' & \\
\Delta' \vdash A^{n+1}, \Gamma' & \\
\Delta', \Delta^{n+1} \vdash \Gamma', \top^{n+1}
\end{align*}
\]

is transformed into

\[
\begin{align*}
\Delta' \vdash A, A, A^n, \Gamma' & \\
\Delta, \Delta \vdash \top & \\
\Delta', \Delta^{n+1} \vdash \Gamma', \top^{n+1}
\end{align*}
\]

Remark. The case of the left multiple Cut cannot occur since otherwise the application of the rule \(?L\) would be invalid.

The second pattern is an application of Cut such that the principal formula of the rule applied at the end of the right hypothesis of the application of Cut is not the cut formula. The cases of the second pattern are the following:

- Right-Minor 1R-Cut. A proof of the form

\[
\begin{align*}
\Delta' \vdash A, A, A^n, \Gamma' & \\
\Delta, \Delta \vdash \top & \\
\Delta', \Delta^{n+1} \vdash \Gamma', \top^{n+1}
\end{align*}
\]

is transformed into

\[
\begin{align*}
\Delta' \vdash A, A, A^n, \Gamma' & \\
\Delta \vdash \top & \\
\Delta', \delta^{n+1} \vdash \Gamma', \top^{n+1}
\end{align*}
\]

The case of the right multiple Cut is analogous, and thus we omit it.

- Right-Minor 0L-Cut. A proof of the form

\[
\begin{align*}
\Delta' \vdash A, A, A^n, \Gamma' & \\
\Delta, \Delta \vdash \top & \\
\Delta', \delta^{n+1} \vdash \Gamma', \top^{n+1}
\end{align*}
\]

is transformed into

\[
\begin{align*}
\Delta' \vdash A, A, A^n, \Gamma' & \\
\Delta \vdash \top & \\
\Delta', \delta^{n+1} \vdash \Gamma', \top^{n+1}
\end{align*}
\]

The case of the right multiple Cut is analogous, and thus we omit it.

- Right-Minor \(\top\)L-Cut. A proof of the form

\[
\begin{align*}
\Delta' \vdash A, A, A^n, \Gamma' & \\
\Delta, \Delta \vdash \top & \\
\Delta', \delta^{n+1} \vdash \Gamma', \top^{n+1}
\end{align*}
\]

is transformed into

\[
\begin{align*}
\Delta' \vdash A, A, A^n, \Gamma' & \\
\Delta \vdash \top & \\
\Delta', \delta^{n+1} \vdash \Gamma', \top^{n+1}
\end{align*}
\]
is transformed into

\[
\frac{p_1}{(\text{Cut} L^n)} \frac{p_2}{(\top L)} \frac{\Delta_1 \vdash A, \Gamma_1, \Delta_2, A^n \vdash \Gamma_2}{\Delta_1^n, \Delta_2 \vdash \bot, \Gamma_1^n, \Gamma_2}
\]

The case of the right multiple Cut is analogous, and thus we omit it.

• Right-Minor $\top R$-Cut. This case is impossible because there is no occurrence of a formula on the LHS of the conclusion of the rule $\top R$.

• Right-Minor $\bot L$-Cut. This case is impossible because there is only one occurrence of bottom $\bot$ on the LHS of the conclusion of the rule $\bot L$.

• Right-Minor $\bot R$-Cut. A proof of the form

\[
\frac{p_1}{(\text{Cut} L^n)} \frac{p_2}{(\bot R)} \frac{\Delta_1 \vdash A, \Gamma_1, \Delta_2, A^n \vdash \bot, \Gamma_2}{\Delta_1^n, \Delta_2 \vdash \bot, \Gamma_1^n, \Gamma_2}
\]

is transformed into

\[
\frac{p_1}{(\text{Cut} L^n)} \frac{p_2}{(\bot R)} \frac{\Delta_1 \vdash A, \Gamma_1, \Delta_2, A^n \vdash \Gamma_2}{\Delta_1^n, \Delta_2 \vdash \bot, \Gamma_1^n, \Gamma_2}
\]

The case of the right multiple Cut is analogous, and thus we omit it.

• Right-Minor $\otimes L$-Cut. A proof of the form

\[
\frac{p_1}{(\text{Cut} L^n)} \frac{p_2}{(\otimes L)} \frac{\Delta_1 \vdash A, \Gamma_1, \Delta_2, A^n \vdash B, C \vdash \Gamma_2}{\Delta_1^n, \Delta_2 \vdash B \otimes C \vdash \Gamma_1^n, \Gamma_2}
\]

is transformed into

\[
\frac{p_1}{(\text{Cut} L^n)} \frac{p_2}{(\otimes L)} \frac{\Delta_1 \vdash A, \Gamma_1, \Delta_2, A^n \vdash B, C \vdash \Gamma_2}{\Delta_1^n, \Delta_2 \vdash B \otimes C \vdash \Gamma_1^n, \Gamma_2}
\]

The case of the right multiple Cut is analogous, and thus we omit it.

• Right-Minor $\otimes R$-Cut. A proof of the form

\[
\frac{p_1}{(\text{Cut} L^{n_1+n_2})} \frac{p_2}{(\otimes R)} \frac{\Delta \vdash A, \Gamma}{\Delta_1, \Delta_2 \vdash \Gamma_1, \Gamma_2} \frac{\Delta_1 \vdash A^{n_1} \vdash B_1, \Gamma_1, \Delta_2, A^{n_2} \vdash B_2, \Gamma_2}{\Delta_{n_1+n_2} \vdash B_1 \otimes B_2, \Gamma_1, \Gamma_2}
\]

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is transformed into

\[
\frac{p}{\textsc{cutl}^{n_1 + n_2}} \quad \frac{p_1}{\frac{\Delta \vdash A, \Gamma \quad \Delta_1, A_{n_1 + 1} \vdash B_1, \Gamma_1}{\Delta_{n_1 + n_2}, \Delta_1, B_1 \otimes B_2, \Gamma_1, \Gamma_2}} \quad \frac{p_2}{\frac{\Delta_2, A_{n_2} \vdash B_2, \Gamma_2}{\Delta_{n_1 + n_2}, \Delta_2, A_{n_1} \otimes B_1, \Gamma_1, \Gamma_2}}
\]

The case of the right multiple Cut is analogous, and thus we omit it.

- **Right-Minor \(\forall L\)-Cut.** A proof of the form

\[
\frac{p}{\textsc{cutl}^{n_1 + n_2}} \quad \frac{p_1}{\frac{\Delta \vdash A, \Gamma \quad \Delta_1, A_{n_1 + 1} \vdash B_1, \Gamma_1}{\Delta_{n_1 + n_2}, \Delta_1, B_1 \otimes B_2, \Gamma_1, \Gamma_2}} \quad \frac{p_2}{\frac{\Delta_2, A_{n_2} \vdash B_2, \Gamma_2}{\Delta_{n_1 + n_2}, \Delta_2, A_{n_1} \otimes B_1, \Gamma_1, \Gamma_2}}
\]

is transformed into

\[
\frac{p}{\textsc{cutl}^{n_1}} \quad \frac{p_1}{\frac{\Delta \vdash A, \Gamma \quad \Delta_1, A_{n_1 + 1} \vdash B_1, \Gamma_1}{\Delta_{n_1}, \Delta_1, B_1 \otimes B_2, \Gamma_1, \Gamma_2}} \quad \frac{p_2}{\frac{\Delta_2, A_{n_2} \vdash B_2, \Gamma_2}{\Delta_{n_1}, \Delta_2, A_{n_1} \otimes B_1, \Gamma_1, \Gamma_2}}
\]

The case of the right multiple Cut is analogous, and thus we omit it.

- **Right-Minor \(\exists R\)-Cut.** A proof of the form

\[
\frac{p}{\textsc{cutl}^{n}} \quad \frac{p'}{\frac{\Delta', A^n \vdash B_1, B_2, \Gamma'}{\Delta', A^n \vdash B_1 \otimes B_2, \Gamma'}}
\]

is transformed into

\[
\frac{p}{\textsc{cutl}^{n}} \quad \frac{p'}{\frac{\Delta', A^n \vdash B_1, B_2, \Gamma'}{\Delta', A^n \vdash B_1 \otimes B_2, \Gamma'}}
\]

The case of the right multiple Cut is analogous, and thus we omit it.

- **Right-Minor \&L-Cut.** A proof of the form

\[
\frac{p}{\textsc{cutl}^{n}} \quad \frac{p'}{\frac{\Delta', A^n \vdash B_1 + B_2, \Gamma'}{\Delta', A^n \vdash B_1 \otimes B_2, \Gamma'}}
\]

(i ∈ \(\exists\))
is transformed into

\[
\frac{p}{\Delta \vdash A, \Gamma} \quad \frac{\Delta', A^n, B_1 \vdash \Gamma'}{\Delta', A^n \vdash B_1, B_2 \vdash \Gamma'}
\]

&L

\[
\frac{\Delta^n, \Delta', B_1 \vdash \Gamma^n, \Gamma'}{\Delta^n, \Delta' \vdash \Gamma^n, B_1 \& B_2 \vdash \Gamma'}
\]

The case of the right multiple Cut is analogous, and thus we omit it.

- **Right-Minor &R-Cut.** A proof of the form

\[
\frac{p \quad p_1' \quad p_2'}{\Delta \vdash A, \Gamma} \quad \frac{\Delta', A^n \vdash B_1, \Gamma'}{\Delta', A^n \vdash B_1, B_2, \Gamma'}
\]

is transformed into

\[
\frac{\Delta \vdash A, \Gamma}{\Delta, A^n \vdash B_1, \Gamma'} \quad \frac{\Delta', A^n \vdash B_2, \Gamma'}{\Delta', A^n \vdash B_1 \& B_2, \Gamma'}
\]

&L

\[
\frac{\Delta^n, \Delta', B_1 \vdash \Gamma^n, \Gamma'}{\Delta^n, \Delta' \vdash \Gamma^n, B_1 \& B_2 \vdash \Gamma'}
\]

&L

The case of the right multiple Cut is analogous, and thus we omit it.

- **Right-Minor ⊕L-Cut.** A proof of the form

\[
\frac{p \quad p_1' \quad p_2'}{\Delta \vdash A, \Gamma} \quad \frac{\Delta', A^n, B_1 \vdash \Gamma'}{\Delta', A^n \vdash B_1 \& B_2, \Gamma'}
\]

is transformed into

\[
\frac{\Delta \vdash A, \Gamma}{\Delta, A^n, B_1 \vdash \Gamma'} \quad \frac{\Delta', A^n \vdash B_2, \Gamma'}{\Delta', A^n \vdash B_1 \& B_2, \Gamma'}
\]

&L

\[
\frac{\Delta^n, \Delta', B_1 \vdash \Gamma^n, \Gamma'}{\Delta^n, \Delta' \vdash \Gamma^n, B_1 \& B_2 \vdash \Gamma'}
\]

&L

The case of the right multiple Cut is analogous, and thus we omit it.

- **Right-Minor ⊕R-Cut.** A proof of the form

\[
\frac{p \quad p_1'}{\Delta \vdash A, \Gamma} \quad \frac{\Delta', A^n \vdash B_1, \Gamma'}{\Delta', A^n \vdash B_1 \& B_2, \Gamma'}
\]

is transformed into

\[
\frac{\Delta \vdash A, \Gamma}{\Delta, A^n \vdash B_1, \Gamma'} \quad \frac{\Delta', A^n \vdash B_2, \Gamma'}{\Delta', A^n \vdash B_1 \& B_2, \Gamma'}
\]

&L

\[
\frac{\Delta^n, \Delta', B_1 \vdash \Gamma^n, \Gamma'}{\Delta^n, \Delta' \vdash \Gamma^n, B_1 \& B_2 \vdash \Gamma'}
\]

&L

\[
\frac{\Delta^n, \Delta' \vdash \Gamma^n, B_1 \& B_2 \vdash \Gamma'}{\Delta^n, \Delta' \vdash \Gamma^n, B_1 \& B_2 \vdash \Gamma'}
\]

\((i \in \mathbb{Z})\)

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The case of the right multiple Cut is analogous, and thus we omit it.

- Right-Minor ¬L-Cut. A proof of the form

\[
\frac{p}{\Delta^p, A^n \vdash B, \Gamma'}
\]

is transformed into

\[
\frac{p'}{\Delta^p, A^n, \neg B \vdash \Gamma^n, \Gamma'}
\]

The case of the right multiple Cut is analogous, and thus we omit it.

- Right-Minor ¬R-Cut. A proof of the form

\[
\frac{p}{\Delta^p, A^n, B \vdash \Gamma'}
\]

is transformed into

\[
\frac{p'}{\Delta^p, A^n, \neg B \vdash \Gamma^n, \Gamma'}
\]

The case of the right multiple Cut is analogous, and thus we omit it.

- Right-Minor ¬D-Cut. A proof of the form

\[
\frac{p}{\Delta^p, A^n, B \vdash \Gamma'}
\]

is transformed into

\[
\frac{p'}{\Delta^p, A^n, \neg B \vdash \Gamma^n, \Gamma'}
\]

The case of the right multiple Cut is analogous, and thus we omit it.

- Right-Minor ¬W-Cut. A proof of the form

\[
\frac{p}{\Delta^p, A^n, B \vdash \Gamma'}
\]

is transformed into

\[
\frac{p'}{\Delta^p, A^n, \neg B \vdash \Gamma^n, \Gamma'}
\]

The case of the right multiple Cut is analogous, and thus we omit it.
is transformed into

\[
\frac{p}{\Delta, A, \Gamma} \quad \frac{\Delta', A^n \vdash \Gamma'}{\Delta', \Gamma' \vdash B, !B \vdash \Gamma'}
\]

\[
\text{(CutL)}
\]

The case of the right multiple Cut is analogous, and thus we omit it.

- **Right-Minor !C-Cut.** A proof of the form

\[
\frac{p}{\Delta, A, \Gamma} \quad \frac{\Delta', A^n, !B, !B \vdash \Gamma'}{\Delta', \Gamma' \vdash B, \Gamma'}
\]

\[
\text{(CutL)}
\]

is transformed into

\[
\frac{p}{\Delta, A, \Gamma} \quad \frac{\Delta', A^n, !B, !B \vdash \Gamma'}{\Delta', \Gamma' \vdash B, \Gamma'}
\]

\[
\text{(CutL)}
\]

The case of the right multiple Cut is analogous, and thus we omit it.

- **Right-Minor !R-Cut.** A proof of the form

\[
\frac{p}{\Delta, A, \Gamma} \quad \frac{\Delta', A^n, !B, !B \vdash \Gamma'}{\Delta', \Gamma' \vdash B, \Gamma'}
\]

\[
\text{(CutL)}
\]

is transformed into

\[
\frac{p}{\Delta, A, \Gamma} \quad \frac{\Delta', A^n, !B, !B \vdash \Gamma'}{\Delta', \Gamma' \vdash B, \Gamma'}
\]

\[
\text{(CutL)}
\]

where it is easy to see by induction on the proof \(p\) that \(\Delta = \Delta''\) and \(\Gamma = \Gamma''\) for some finite sequences \(\Delta''\) and \(\Gamma''\) of formulas (so that the last application of the rule !R is valid).

**Remark.** The case of the right multiple Cut cannot occur since otherwise the application of the rule !R would be invalid.

- **Right-Minor ?D-Cut.** A proof of the form

\[
\frac{p}{\Delta, A, \Gamma} \quad \frac{\Delta', A^n \vdash \Gamma'}{\Delta', \Gamma' \vdash B, \Gamma'}
\]

\[
\text{(CutL)}
\]

is transformed into

\[
\frac{p}{\Delta, A, \Gamma} \quad \frac{\Delta', A^n \vdash \Gamma'}{\Delta', \Gamma' \vdash B, \Gamma'}
\]

\[
\text{(CutL)}
\]

The case of the right multiple Cut is analogous, and thus we omit it.
• Right-Minor ?W-Cut. A proof of the form

\[
\begin{array}{c}
\Delta \vdash A, \Gamma \\
\Delta^n, \Delta' \vdash \Gamma^n, ?B, \Gamma'
\end{array}
\]

is transformed into

\[
\begin{array}{c}
\Delta \vdash A, \Gamma \\
\Delta', A^n \vdash \Gamma'
\end{array}
\]

\[
\begin{array}{c}
\Delta^n, \Delta' \vdash \Gamma^n, ?B, \Gamma'
\end{array}
\]

The case of the right multiple Cut is analogous, and thus we omit it.

• Right-Minor ?C-Cut. A proof of the form

\[
\begin{array}{c}
\Delta \vdash A, \Gamma \\
\Delta^n, \Delta' \vdash \Gamma^n, ?B, \Gamma'
\end{array}
\]

is transformed into

\[
\begin{array}{c}
\Delta \vdash A, \Gamma \\
\Delta', A^n \vdash ?B, ?B, \Gamma'
\end{array}
\]

\[
\begin{array}{c}
\Delta^n, \Delta' \vdash \Gamma^n, ?B, \Gamma'
\end{array}
\]

The case of the right multiple Cut is analogous, and thus we omit it.

• Right-Minor ?L-Cut. A proof of the form

\[
\begin{array}{c}
\Delta \vdash \neg A, \Gamma \\
\Delta^n, \Delta' \vdash \Gamma^n, \neg B, \Gamma'
\end{array}
\]

is transformed into

\[
\begin{array}{c}
\Delta \vdash \neg A, \Gamma \\
\Delta', \neg A^n \vdash \Gamma'
\end{array}
\]

\[
\begin{array}{c}
\Delta^n, \Delta' \vdash \Gamma^n, \neg B, \Gamma'
\end{array}
\]

where again \(\Delta = \neg \Delta''\) and \(\Gamma = \neg \Gamma''\) for some finite sequences \(\Delta''\) and \(\Gamma''\) of formulas (so that the last application of the rule ?L is valid).

\textit{Remark.} The case of the right multiple Cut cannot occur since otherwise the application of the rule ?L would be invalid.
The third pattern is an application of Cut such that the principal formula of the rule applied at the end of the left hypothesis of the application of Cut is not the cut formula. Because it is just dual to the second pattern, let us omit the cases of the third pattern.

Finally, the fourth pattern is an application of Cut such that at least one of the hypotheses is a singleton proof of the axiom Id.

The cases of the fourth pattern are the following:

- **Left Id-Cut.** A proof of the form

  \[
  \begin{array}{c}
  \text{(Id)} \quad A \vdash A \\
  \text{(CutL}^n \text{)} \\
  \hline
  \Delta, A^n \vdash \Gamma \\
  \end{array}
  \]

  is transformed into

  \[
  \begin{array}{c}
  p \\
  \text{(XL}^* \text{)} \\
  \hline
  \Delta, A^n \vdash \Gamma \\
  \end{array}
  \]

- **Right Id-Cut.** A proof of the form

  \[
  \begin{array}{c}
  p \\
  \text{(CutR}^n \text{)} \\
  \hline
  \Delta \vdash A^n, \Gamma \quad \text{(Id)} \quad A \vdash A \\
  \end{array}
  \]

  is transformed into

  \[
  \begin{array}{c}
  p \\
  \text{(XR}^* \text{)} \\
  \hline
  \Delta \vdash A^n, \Gamma \\
  \end{array}
  \]

We have considered all the cases and thus completed the description of our cut-elimination procedure on Cut in the sequent calculus \( \text{ILLK} \).

### B Cut-Elimination on \( \text{Cut}^{??}_\mu \)

Similarly, let us give our cut-elimination procedure on \( \text{Cut}^{??}_\mu \) in the sequent calculus \( \text{ILLK}_\mu \). Again, we just describe how the cut-elimination procedure transforms each application of \( \text{Cut}^{??}_\mu \), where we omit the cases of the second, the third and the fourth patterns because the former two are handled essentially in the same way as the computation of the cut-elimination procedure on Cut given in the previous section, and the last one cannot occur by the form of \( \text{Cut}^{??}_\mu \).

Similarly to the cut-elimination on Cut given in the previous section, the cut-elimination procedure on \( \text{Cut}^{??}_\mu \) also regards the following derivable rules as single ones:

**Definition B.1** (Multiple \( \text{Cut}^{??}_\mu \)). Let us define **left multiple \( \text{Cut}^{??}_\mu \)** to be the following rule:

\[
\begin{array}{c}
\text{(CutL}^n \text{)} \\
\hline
\Delta \vdash ?A, ?T \\
\Delta, A^n \vdash ?T' \\
\end{array} \quad (n \in \mathbb{N}^+) \]

and symmetrically, **right multiple \( \text{Cut}^{??}_\mu \)** to be the following rule:

\[
\begin{array}{c}
\text{(CutR}^n \text{)} \\
\hline
\Delta \vdash ?A, ?T' \\
\Delta, A^n \vdash ?T \\
\end{array} \quad (n \in \mathbb{N}^+) \]

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- $(?D, !D)$-$\text{Cut}_{\mu}^{\eta}$. A proof of the form

\[
\frac{p}{?\Delta \vdash A, ?T} \quad \frac{p'}{?\Delta, !A^n, A \vdash ?T'}
\]

is transformed into

\[
\frac{?\Delta \vdash A, ?T}{?\Delta, !A^n, A \vdash ?T'}
\]

The case of the right multiple cut is just symmetric, and thus we omit it.

- $(?D, !W)$-$\text{Cut}_{\mu}^{\eta}$. A proof of the form

\[
\frac{p}{?\Delta \vdash A, ?T} \quad \frac{p'}{?\Delta, !A^n, A \vdash ?T'}
\]

is transformed into

\[
\frac{?\Delta \vdash A, ?T}{?\Delta, !A^n, A \vdash ?T'}
\]

Moreover, a proof of the form

\[
\frac{p'}{?\Delta'n \vdash A^n, ?T'} \quad \frac{p}{?\Delta, !A^n, A \vdash ?T'}
\]

is transformed into

\[
\frac{?\Delta, !A^n, A \vdash ?T'}{?\Delta, !A^n, A \vdash ?T'}
\]

- $(?D, !C)$-$\text{Cut}_{\mu}^{\eta}$. A proof of the form

\[
\frac{p}{?\Delta \vdash A, ?T} \quad \frac{p'}{?\Delta, !A^n, A \vdash ?T'}
\]

is transformed into

\[
\frac{?\Delta \vdash A, ?T}{?\Delta, !A^n, A \vdash ?T'}
\]
is transformed into
\[
\begin{align*}
\text{(Cut)}^{p}_{\mu} & \frac{p}{\Delta \vdash ?, \Delta} \\
\text{(Cut)}^{p}_{\mu, L^{n+2}} & \frac{\Delta \vdash ?, \Delta, \Delta, \Delta \vdash ?^{n+2}, T'}{\Delta^{n+1}, \Delta' \vdash ?^{n+2}, T'}
\end{align*}
\]

Moreover, a proof of the form
\[
\begin{align*}
\text{(Cut)}^{p'}_{\mu, R^{n+1}} & \frac{p'}{\Delta' \vdash A, ?^{n+1}, T'} \\
\text{(IC)} & \frac{\Delta, !A \vdash ?}{\Delta, !A \vdash T}
\end{align*}
\]

is transformed into
\[
\begin{align*}
\text{(Cut)}^{p}_{\mu, R^{n}} & \frac{p'}{\Delta' \vdash A, ?^{n}, T'} \\
\text{(IC)} & \frac{\Delta, !A \vdash ?}{\Delta, !A \vdash T}
\end{align*}
\]

- **(?W, !D)-Cut\(^{p}_{\mu}**. A proof of the form
\[
\begin{align*}
\text{(Cut)}^{p}_{\mu, L^{n+1}} & \frac{p}{\Delta \vdash ?, \Delta} \\
\text{(Cut)}^{p}_{\mu, L^{n}} & \frac{\Delta \vdash ?, \Delta, \Delta, \Delta \vdash ?^{n+1}, T'}{\Delta^{n+1}, \Delta' \vdash ?^{n+1}, T'}
\end{align*}
\]

is transformed into
\[
\begin{align*}
\text{(Cut)}^{p}_{\mu, L^{n}} & \frac{p}{\Delta \vdash ?, \Delta} \\
\text{(Cut)}^{p}_{\mu, L^{n}} & \frac{\Delta \vdash ?, \Delta, \Delta, \Delta \vdash ?^{n}, T'}{\Delta^{n+1}, \Delta' \vdash ?^{n}, T'}
\end{align*}
\]

Moreover, a proof of the form
\[
\begin{align*}
\text{(Cut)}^{p'}_{\mu, R^{n+1}} & \frac{p'}{\Delta' \vdash A, ?^{n+1}, T'} \\
\text{(ID)} & \frac{\Delta, !A \vdash ?}{\Delta, !A \vdash T}
\end{align*}
\]

is transformed into
\[
\begin{align*}
\text{(Cut)}^{p'}_{\mu, R^{n}} & \frac{p'}{\Delta' \vdash A, ?^{n}, T'} \\
\text{(ID)} & \frac{\Delta, !A \vdash ?}{\Delta, !A \vdash T}
\end{align*}
\]

• **(!W*)-Cut\(^{p}_{\mu}**. A proof of the form
\[
\begin{align*}
\text{(Cut)}^{p}_{\mu, R^{n}} & \frac{p}{\Delta \vdash ?, \Delta} \\
\text{(ID)} & \frac{\Delta, !A \vdash ?}{\Delta, !A \vdash T}
\end{align*}
\]

is transformed into
\[
\begin{align*}
\text{(Cut)}^{p}_{\mu, R^{n}} & \frac{p}{\Delta \vdash ?, \Delta} \\
\text{(!W*)} & \frac{\Delta', !A \vdash ?^{n+1}, T'}{\Delta', !A \vdash T^{n+1}}
\end{align*}
\]
• (?W, !W)-Cut\(_\mu\). A proof of the form

\[
\begin{array}{c}
\frac{p}{(\text{Cut}\_\mu^{W}L^{n+1})} \\
\quad \frac{p'}{(\text{Cut}\_\mu^{W}L^{n+1})}
\end{array}
\]

\[
\frac{\Delta \vdash T}{\Delta \vdash A, ?T} \\
\frac{\Delta', !A^{n} \vdash ?T'}{!\Delta^{n+1}, !\Delta' \vdash ?T^{n+1}, T'}
\]

is transformed into

\[
\begin{array}{c}
\frac{p}{(\text{Cut}\_\mu^{W}L^{n})} \\
\quad \frac{p'}{(\text{Cut}\_\mu^{W}L^{n})}
\end{array}
\]

\[
\frac{\Delta \vdash T}{\Delta \vdash A, ?T} \\
\frac{\Delta', !A^{n} \vdash ?T'}{!\Delta^{n+1}, !\Delta' \vdash ?T^{n+1}, T'}
\]

Moreover, a proof of the form

\[
\begin{array}{c}
\frac{p'}{(\text{Cut}\_\mu^{W}R^{n+1})} \\
\quad \frac{p}{(\text{Cut}\_\mu^{W}R^{n+1})}
\end{array}
\]

\[
\frac{\Delta' \vdash A^{n}, ?T'}{\Delta' \vdash A^{n+1}, ?T'} \\
\frac{\Delta' \vdash A^{n+1} \vdash ?T', ?T^{n+1}}{!\Delta', !\Delta^{n+1} \vdash ?T', ?T^{n+1}}
\]

is transformed into

\[
\begin{array}{c}
\frac{p'}{(\text{Cut}\_\mu^{W}R^{n})} \\
\quad \frac{p}{(\text{Cut}\_\mu^{W}R^{n})}
\end{array}
\]

\[
\frac{\Delta' \vdash A^{n}, ?T'}{\Delta' \vdash A^{n+1}, ?T'} \\
\frac{\Delta' \vdash A^{n+1} \vdash ?T', ?T^{n+1}}{!\Delta', !\Delta^{n+1} \vdash ?T', ?T^{n+1}}
\]

**Remark.** If \(n = 0\), then there is no canonical choice between the left or the right multiple Cut on the (?W, !W)-Cut\(_\mu\), which is what makes the cut-elimination on Cut\(_\mu\) undirected.

• (?C, !C)-Cut\(_\mu\). A proof of the form

\[
\begin{array}{c}
\frac{p}{(\text{Cut}\_\mu^{C}L^{n+1})} \\
\quad \frac{p'}{(\text{Cut}\_\mu^{C}L^{n+1})}
\end{array}
\]

\[
\frac{\Delta \vdash A, ?T}{\Delta \vdash A, ?T} \\
\frac{\Delta' \vdash A^{n} \vdash ?T'}{!\Delta^{n+1}, !\Delta' \vdash ?T^{n+1}, ?T'}
\]

is transformed into

\[
\begin{array}{c}
\frac{p}{(\text{Cut}\_\mu^{C}L^{n+2})} \\
\quad \frac{p'}{(\text{Cut}\_\mu^{C}L^{n+2})}
\end{array}
\]

\[
\frac{\Delta \vdash A, ?T}{\Delta \vdash A, ?T} \\
\frac{\Delta' \vdash A^{n} \vdash ?T'}{!\Delta^{n+1}, !\Delta' \vdash ?T^{n+2}, ?T'}
\]

The case of the right multiple Cut\(_\mu\) is just symmetric, and thus we omit it.
\[ \text{(C, !D)-Cut}_{\mu}^*: \] A proof of the form
\[
\begin{array}{c}
\text{(!C)} \quad \frac{\Delta \vdash \exists A, \exists A, ?T}{!\Delta' \vdash \exists A, ??\Gamma} \\
\text{(!D)} \quad \frac{\Delta' \vdash \exists A, ??\Gamma}{!\Delta' \vdash \exists A, ?T'}
\end{array}
\]

is transformed into
\[
\begin{array}{c}
\text{(C, !D)-Cut}_{\mu}^*: \] A proof of the form
\[
\begin{array}{c}
\text{(!C)} \quad \frac{\exists A \vdash \exists A, ?A, ?T}{!\exists A \vdash ?A, ??\Gamma} \\
\text{(!D)} \quad \frac{\exists A', \exists A, ?T}{!\exists A', \exists A, ?T'}
\end{array}
\]

Moreover, a proof of the form
\[
\begin{array}{c}
\text{(C, !W)-Cut}_{\mu}^*: \] A proof of the form
\[
\begin{array}{c}
\text{(!C)} \quad \frac{\exists A \vdash \exists A, ?A, ?A, ?T'}{!\exists A \vdash ?A, ?T', ??\Gamma} \\
\text{(!W)} \quad \frac{\exists A' \vdash ?T'}{!\exists A', ?T'}
\end{array}
\]

is transformed into
\[
\begin{array}{c}
\text{(C, !W)-Cut}_{\mu}^*: \] A proof of the form
\[
\begin{array}{c}
\text{(!C)} \quad \frac{\exists A \vdash \exists A, ?A, ?A, ?T'}{!\exists A \vdash ?A, ?T', ??\Gamma} \\
\text{(!W)} \quad \frac{\exists A' \vdash ?T'}{!\exists A', ?T'}
\end{array}
\]
Moreover, a proof of the form

\[
\frac{\vdash A, A, A, A_n, T'}{\vdash A, A, A_n, T'} \quad \frac{\vdash A, A_n, T'}{\vdash A, A_n, T'}
\]

is transformed into

\[
\frac{\vdash A, A_n, T'}{\vdash A, A_n, T'} \quad \frac{\vdash A, A_n, T'}{\vdash A, A_n, T'}
\]

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