Optimal strategies for patrolling fences

Bernhard Haeupler†  Fabian Kuhn‡
Carnegie Mellon University  University of Freiburg
Anders Martinsson§  Kalina Petrova¶  and Pascal Pfister∥
ETH Zürich

Abstract

A classical multi-agent fence patrolling problem asks: What is the maximum length \( L \) of a line that \( k \) agents with maximum speeds \( v_1, \ldots, v_k \) can patrol if each point on the line needs to be visited at least once every unit of time.

It is easy to see that \( L = \alpha \sum_{i=1}^{k} v_i \) for some efficiency \( \alpha \in [\frac{1}{2}, 1) \). After a series of works [3, 8–10] giving better and better efficiencies, it was conjectured that the best possible efficiency approaches \( \frac{2}{3} \). No upper bounds on the efficiency below 1 were known.

We prove the first such upper bounds and tightly bound the optimal efficiency in terms of the minimum ratio of speeds \( s = \frac{v_{\text{max}}}{v_{\text{min}}} \) and the number of agents \( k \). Our bounds of \( \alpha \leq \frac{1}{1 + \frac{1}{s}} \) and \( \alpha \leq 1 - \frac{1}{2\sqrt{k}} \) imply that in order to achieve efficiency \( 1 - \epsilon \), at least \( k \geq \Omega(\epsilon^{-2}) \) agents with a speed ratio of \( s \geq \Omega(\epsilon^{-1}) \) are necessary.

Guided by our upper bounds, we construct a scheme whose efficiency approaches 1, disproving the conjecture of Kawamura and Soejima. Our scheme asymptotically matches our upper bounds in terms of the maximal speed difference and the number of agents used, proving them to be asymptotically tight.

1 Introduction

Patrolling is a fundamental task in robotics, multi-agent systems, and security settings. Given some environment of interest, and a collection of mobile agents, the aim is to coordinate the movements of the agents in order to, for example, guard an area from intrusion by an enemy,
prevent accidents or failure of equipment, maintain up-to-date information of the environment, etc. For each of these tasks, ensuring that points in the environment get visited/monitored frequently is key. Performance of patrolling algorithms is consequently often measured in terms of idleness – roughly speaking, the time between two visits to a point in the environment.

Multi-agent patrolling has been extensively studied in the robotics literature since the early 2000’s, e.g., see [11] and the survey [12]. However, even for extremely clean and very simple models, determining optimal patrolling schemes poses many natural mathematical questions with interesting and surprisingly sophisticated answers [2–10].

1.1 Fence Patrolling

This paper studies a classical fence patrolling problem introduced by Czyzowicz et al. [3], which might be one of the cleanest and most natural patrolling problems: What is the maximum length $L$ of a line segment that $k$ agents $a_1, \ldots, a_k$ with maximum speeds $v_1, \ldots, v_k$ can patrol if each point needs to be visited at least once every unit of time.

It is easy to see that for any speeds the maximum length $L$ satisfies $L = \alpha \sum_{i=1}^{k} v_i$ for some efficiency $\alpha \in [\frac{1}{2}, 1)$. In particular, in one unit of time an agent $a_i$ can cover a length of at most $v_i$ and all agents can cover at most a total length of $\sum_{i=1}^{k} v_i$. An efficiency of exactly $\alpha = 1$ is furthermore never possible because agents have to turn around eventually. On the other hand, agent $a_i$ can easily patrol his/her own sub-segment of length $\frac{1}{2} v_i$ by going back and forth on this segment once every unit of time. Putting these segments together shows that an efficiency of $\alpha = \frac{1}{2}$, i.e., a fence length of $\frac{1}{2} \sum_{i=1}^{k} v_i$, is always achievable.

1.2 Prior Work on Fence Patrolling

Czyzowicz et al. [3] observed that the trivial scheme with efficiency $\frac{1}{2}$ is optimal if the paths of the agents never cross. To see this, note that the leftmost agent $a_i$ cannot walk away further than $\frac{1}{2} v_i$ from the leftmost point of the fence as it would take more than one unit of time between two visits of this point. By the same argument the agent $a_j$ to the right of agent $a_i$ cannot ever be further away than $\frac{1}{2} (v_i + v_j)$ from the leftmost point of the fence and induction shows that a total fence length of $\frac{1}{2} \sum_{i=1}^{k} v_i$ is best possible. For the special case of all agents having the same speed the assumption that the paths of the agents never cross is furthermore without loss of generality as one can equally well switch identities of agents at a crossing, making the agents bounce off each other instead of crossing. In the worst case an efficiency of $\frac{1}{2}$ is thus optimal and Czyzowicz et al. posited [3] that indeed no better efficiency can be achieved for any speeds.

Surprisingly, Kawamura and Kobayashi [9] disproved this by providing an explicit fence patrolling schedule for 6 agents with speeds $1, 1, 1, 1, \frac{7}{3}$, and $\frac{2}{3}$ for a fence of length $\frac{7}{2}$, thus achieving an efficiency of $\frac{26}{41} > \frac{1}{2}$. This was improved by Dumitresco, Ghosh and Toth [8], who proposed a family of patrolling schedules with efficiency approaching $\frac{25}{33}$, and finally by Kawamura and Soejima [10] which achieved an efficiency approaching $\frac{2}{3}$. Kawamura and Soejima furthermore explicitly conjecture that no efficiency better than $\frac{2}{3}$ is possible for any set of speeds [10, Conjecture 6, page 9].
On the other hand, except for the setting of equal speeds discussed above, no upper bounds on the efficiency below 1 have been provided in the literature \[3, 8–10\].

2 Our Results

This paper advances the understanding of the fence patrolling problem by giving tight upper and lower bounds on the optimal efficiency. To a large extent it concludes the main line of inquiry put forward in the works discussed above \[3, 8–10\].

We provide the first technique to prove general impossibility results for the fence patrolling problem. We explain our ideas in more detail in Section 4 and merely state our main upper bound here:

**Theorem 2.1.** Any fence patrol schedule with \(k\) agents with maximum speeds \(v_1, \ldots, v_k\) patrols a fence of length at most

\[
L \leq \sum_{i=1}^{k} \frac{v_i}{1 + \frac{v_i}{\max_j v_j}}.
\]

One way to interpret Theorem 2.1 is that the contribution of an agent \(a_i\) depends not only on his/her own speed \(v_i\) but also on how much slower he/she is than the fastest agent. In particular, instead of always contributing \(v_i\), as in the trivial upper bound, an agent contributes at most

\[
\frac{1}{1 + \frac{1}{s_i}} \cdot v_i
\]

given that the fastest agent patrolling is a factor of \(s_i\) faster than \(a_i\). That is, the “relative efficiency” of an agent \(a_i\) ranges anywhere between \(1/2\) and 1 depending on \(s_i\), which always constitutes an improvement over the trivial upper bound of \(\sum_i v_i\).

We also show that Theorem 2.1 can be used to prove an upper bound on the efficiency of a schedule solely in terms of the number of agents:

**Lemma 2.2.** Any fence patrolling schedule with \(k\) agents has an efficiency of at most

\[
1 - \frac{1}{2\sqrt{k}}.
\]

We note that our upper bounds are tight in several interesting special cases. Specifically, for the case of agents having identical speeds, Theorem 2.1 shows that the efficiency of the schedule (and indeed each agent) is at most \(\frac{1}{2}\), reproving the result of \[3\]. In contrast to the symmetry argument about non-crossing agents explained above, our arguments and upper bounds easily extend to near-identical speeds as well. Furthermore, Theorem 2.1 is tight when applied to the configuration of agents used by Kawamura and Soejima for their construction to obtain the conjectured to be optimal efficiency ratio of \(2/3 - o(1)\). They use \(L + n - 1\) agents of speed \(\frac{1}{2n-1}\), where \(1 \ll n \ll L\). From this speed profile alone Theorem 2.1 gives an upper bound of \(2/3 - o(1)\), i.e., it shows that any schedule using these specific agent speeds cannot be substantially better than the schedule provided by Kawamura and Soejima. Lastly, it is easy to check that the analogous statement is true for the speed profile used by Dumitrescu, Ghosh and Toth.

Our upper bounds do not exclude schedules with efficiency close to 1. They do however give important restrictions and clues about what an extremely efficient schedule, if it exists, has to look like. In particular, Lemma 2.2 implies that any schedule with efficiency \(1 - \epsilon\) has to have at
least \( (\frac{1}{12})^2 \), i.e., quadratically in \( \frac{1}{\epsilon} \) many agents. In the same manner, Theorem 2.1 implies that, with \( \epsilon \to 0 \), the ratio between the fastest and slowest agent has to be at least \( \Omega(\frac{1}{\epsilon}) \), i.e. grow unboundedly. Even more interestingly, the way the upper bound in Theorem 2.1 depends on \( \max_i v_i \) seems to indicate that even just a single very fast agent can raise the “relative efficiency” of slower agents from \( 1/2 \) to almost 1.

Equipped with this better understanding and guidance from our impossibility results we were, to our surprise, able to design schedules which achieve an efficiency arbitrarily close to 1, thus disproving the conjecture of [10]:

**Theorem 2.3.** For any sufficiently large \( k \), there exists a fence patrolling schedule with efficiency 
\[ 1 - \frac{3.5}{\sqrt{k}}. \] Such a schedule uses \( k-1 \) agents of speed one and one agent with maximum speed \( \Theta(\sqrt{k}) \).

Note that this theorem implies that for any \( \epsilon > 0 \) there exists a fence patrolling schedule with efficiency \( 1 - \epsilon \) using \( O(\frac{1}{\epsilon}) \) agents – one with speed \( \Theta(\frac{1}{\epsilon}) \) and all others with speed 1. In other words, the efficiency can be made arbitrarily close to 1 by choosing the appropriate number and maximum speeds of agents.

We remark that Theorem 2.3 also shows that both our upper bounds are asymptotically tight. In particular, the optimal efficiency for any schedule with \( k \) agents is indeed \( 1 - \Theta(\frac{1}{\sqrt{k}}) \). Furthermore, for any \( s \geq 1 \), there is a configuration (with \( k = \Theta(s^2) \) agents), where the maximum speeds of the agents differ by a factor \( s \) and for which the optimal efficiency is \( \frac{1}{1+\Theta(s)} = 1 - \Theta(\frac{1}{s}) \).

2.1 Organization

The rest of the paper is organized as follows: We first give a more formal model description of the fence patrolling problem as well as discuss some related models and works in Section 3. In Section 4 we explain and prove our upper bounds and Section 5 explains and proves our optimal fence patrolling schedule.

3 Fence Patrolling and Related Models

In this section we give a more detailed formal definition for the fence patrolling model/problem and briefly discuss related models and results.

The fence patrolling model as given by [3] is defined as follows:

- The environment \( \mathcal{E} \) to be patrolled is 1-dimensional and consists of a line segment of length \( L \). This line segment is also referred to as a fence.

- The fence patrolling problem consists of some finite number \( k \in \mathbb{N} \) of mobile agents \( a_1, a_2, \ldots, a_k \) to patrol the fence, each having a possibly distinct positive maximum speed \( v_1, v_2, \ldots, v_k \in \mathbb{R}_+ \).

- A schedule for the fence patrolling problem consists of a \( k \)-tuple of functions \( a_1, a_2, \ldots, a_k : [0, \infty) \to \mathcal{E} \) such that, for all \( i \in [k] \), \( t \geq 0 \) and \( \epsilon > 0 \),
\[ \text{dist}(a_i(t+\epsilon), a_i(t)) \leq \epsilon \cdot v_i. \]
That is, we assume patrolling starts at $t = 0$ and goes on indefinitely. Each agent follows a predetermined trajectory, in which he/she moves along $\mathcal{E}$ with at most his/her maximum speed.

- We say that a patrol schedule has *idle time* $T$ for some fixed positive parameter $T$ if for all $t \geq T$ and for all $x \in \mathcal{E}$, there is some agent that visits $x$ during $[t - T, t]$. Intuitively, this condition means that an intruder cannot remain undetected at a point for more than $T$ time.

- Given a patrol schedule, we say that a point $(x, t) \in \mathcal{E} \times [T, \infty)$ is *$T$-covered* if some agent $a_i$ visits the point $x \in \mathcal{E}$ on the fence during the time interval $[t - T, t]$. Note that in this model an agent patrols/monitors a point $x \in \mathcal{E}$ by visiting it. On the one hand, this means the agents are limited to zero line of sight. On the other hand, no additional operation (e.g. stop and look around) is necessary to patrol a point.

One can see that a schedule has idle time $T$ if and only if every point $(x, t) \in \mathcal{E} \times [T, \infty)$ is $T$-covered. It is easy to observe that any patrol schedule of a fence of length $L$ with idle time $T$ can be rescaled to a schedule of a fence of length $\alpha \cdot L$ with idle time $\frac{1}{\alpha} \cdot T$ for any $\alpha > 0$. Thus, to simplify terminology, we assume henceforth that $T = 1$ and we refer to 1-covered simply as covered.

Related models have been considered in the literature: where agents have positive line of sight [7], where agents have distinct walking and patrolling speeds [5], where some agents may be faulty [4], where only some regions of the environment need to be patrolled [2], where the environment is a geometric tree [6], or a cycle which can either be traversed in both directions or is restricted to only being traversed in one direction [3,8,10]. However, all of these models besides the cycle one feature identical agents and in particular do not allow for varying maximum speeds. Overall, the model given above is likely the cleanest and most natural model in which agents with different speeds can and have been studied. Despite the extreme simplicity of this model, this paper and prior works on the fence patrolling problem [3,8,10] show that very surprising and intricate phenomena occur when agents have different speeds and that these nontrivial consequences can be studied in the model defined above.

### 4 Impossibility Results: Proof of Theorem 2.1 and Lemma 2.2

In this section, we prove two upper bounds on the length of a fence patrolled by agents of maximum speeds $v_1, \ldots, v_k$. We here assume that $\mathcal{E} = [0, L]$.

The main idea of the proof is to consider the two-dimensional spacetime continuum $\mathcal{S} := [0, L] \times [0, \infty)$ and the trajectories of the agents along with the points they cover as geometric objects in it. To prove the upper bound on $L$ above, we consider starting with $\mathcal{S}$ and adding the agents one by one in a carefully chosen order. When we add an agent, some additional points are covered by him/her. We then examine what happens to the right border of what has been covered as we add more agents, where a point $p = (x_p, t_p)$ is on the right border if $x_p$ is the rightmost position which is covered at time $t_p$. Figure 1 shows this right border before and after a new agent is added.
Figure 1: If we have added the blue agent and the red agent so far, then the right border is shown in yellow. If we now also add the green agent, the new right border is shown in black.

It is clear that this right border can only move to the right as we add more agents. A geometrical argument furthermore shows that if we let \( v_{\text{max}} := \max_i v_i \) be the speed of the fastest agent, then the leftmost point on the right border in some fixed finite time interval can move to the right by at most \( \frac{1}{v_i} + \frac{1}{v_{\text{max}}} \) when adding an agent with speed \( v_i \), provided that adding this new agent covers the previous leftmost point on the right border and the points immediately to the right of it. We therefore add agents in the order in which they cover the leftmost point on the right border. By definition this leftmost point on the right border has space coordinate 0 at first. Its space coordinate has to be \( L \) after all agents are added because at the end all points in \([0,L] \times [0,\infty)\) have to be covered. The bound \( \frac{1}{v_i} + \frac{1}{v_{\text{max}}} \) can therefore be considered a meaningful measure on how useful and efficient the individual agent \( a_i \) is in terms of contributing to the progress of covering the fence (spacetime continuum).

With this intuition we can now proceed to prove our main upper bound result:

**Proof of Theorem 2.1.** Let \( S = [0,L] \times [0,\infty) \). Since, as noted in Section 3, a patrol schedule with agents \( a_1, \ldots, a_k \) has idle time 1 if and only if \( \forall x \in [0,L], \forall t \geq 1, (x,t) \) is covered by at least one agent \( a_j \), the theorem can be equivalently stated as that, for any patrol schedule such that all points \( (x,t) \in S \) such that \( t \geq 1 \) are covered by some agent, we have

\[
L \leq \sum_{i=1}^{k} \frac{1}{v_i + \frac{1}{v_{\text{max}}}}. \tag{4.1}
\]
In fact, we will show \(4.1\) under the weaker assumption that all points \((x,t) \in S\) such that \(t \in [1,2k]\) are covered by some agent.

Given a patrol schedule of \([0,L]\) with agents \(a_1, \ldots, a_k\) and a non-empty subset \(A \subseteq \{a_1, \ldots, a_k\}\), we define the right border of \(A\) as the function \(B^A : [1,\infty) \rightarrow [0,L]\) given by

\[
B^A(t) := \max\{x \in [0,L] : (x,t) \text{ is covered by some agent in } A\}.
\]

We show \(4.1\) by considering consecutively the collections of agents \(A_1, \ldots, A_q\), where \(A_1 = \{a_{i_1}\}, \forall j \in [q] \setminus \{1\}, A_j = A_{j-1} \cup \{a_{i_j}\}, i_1, \ldots, i_q\) is a sequence of distinct integers in \([k]\) to be specified later, and \(1 \leq q \leq k\) as we might not need to consider all agents. The intuition behind this is that we are starting with an empty set and adding more agents in a specific order until some termination condition is met. It is clear that \(\forall t \geq 0, \forall j \in [q-1], B^{A_j}(t) \leq B^{A_{j+1}}(t)\). The key idea of the proof is to consider what happens to the right border of \(A_j\) as \(j\) increases (that is, as more agents are added). An example of the right borders of \(A_j\) and \(A_{j+1}\) for some \(j\) is shown in Figure 1.

Consider a fixed patrol schedule of \([0,L]\) with agents \(a_1, \ldots, a_k\) and idle time 1. To pick the sequence \(i_1, \ldots, i_q\), consider the following procedure: initially, put \(l_0 = (x_0,t_0) = (0,k)\) and pick \(i_1 \in [k]\) such that agent \(a_{i_1}\) covers \(l_0\). For each consecutive \(j = 1, 2, \ldots\), we let \(l_j = (x_j,t_j)\) be such that

\[
t_j = \text{arg} \min_{t \in [k-j,k+j]} B^{A_j}(t)
\]

and \(x_j = B^{A_j}(t_j)\). Intuitively, \(l_j\) is the leftmost point of \(B^{A_j}\) between times \(k-j\) and \(k+j\). Now if \(x_j = L\), we stop adding agents and we set \(q := j\). Note that if \(j = k\), then \(x_j = L\) as agents \(a_1, \ldots, a_k\) cover all of \([1,2k]\). If \(x_j < L\), pick as agent \(a_{i_{j+1}}\) an agent that covers all the points with coordinates \((x_j + \nu, t_j)\) for \(\nu \in [0,\epsilon]\) for some small enough \(\epsilon > 0\). Such an agent should exist provided that \(x_j < L\) since all points \((x,t) \in [0,L] \times [1,2k]\) should be covered by agents \(\{a_1, \ldots, a_k\}\). To make sure \(l_j\) is always defined for any \(j \in \{0,1, \ldots, q\}\), if \(q = k\), set \(l_k := (L,k)\).

We note that \(x_0, x_1, \ldots, x_q\) is non-decreasing, \(x_0 = 0\) and \(x_q = L\) since we either stopped adding agents when \(q < k\) because \(x_q = L\) or we stopped when \(q = k\), in which case all of \([0,L] \times [1,2k]\) should be covered. Hence the theorem follows if we can show that \(\forall j \in \{0,1, \ldots, q-1\}, x_{j+1} - x_j \leq \frac{1}{\frac{1}{v_{l_j+1}} + \frac{1}{v_{\max}}}\).

In order to bound this difference, we investigate how the right border moves when agent \(a_{i_{j+1}}\) is added. Note that \(\forall t \in [0,\infty),\)

\[
B^{A_{j+1}}(t) = \max(B^{A_j}(t), B^{\{a_{j+1}\}}(t)).
\]

For any time \(t > t_j\), the rightmost point at time \(t\) that could be covered by any agent in \(A_j\) is \((x_j + (t - t_j)v_{\max}, t)\) since the speed of any agent is at most \(v_{\max}\). Similarly, for any time \(t < t_j\), the rightmost point at time \(t\) that could be covered is \((x_j + (t_j - t)v_{\max}, t)\). Thus, \(\forall t \in [0,\infty),\)

\[
B^{A_j}(t) \leq x_j + |t - t_j|v_{\max}.
\]
Denote by $u$ the ray $(x_j + (t_j - t)v_{\text{max}}, t)$ where $t \leq t_j$, and by $w$ the ray $(x_j + (t - t_j)v_{\text{max}}, t)$ where $t \geq t_j$.

Next, consider $B^{\{a_{j+1}\}}(t)$. Since agent $a_{j+1}$ covers $l_j$, this means that he/she visits $x_j$ at some time between $t_j - 1$ and $t_j$, say at point $(x_j, t_{\text{visit}})$. Under the restriction that the trajectory of agent $a_{i,j+1}$ should go through $(x_j, t_{\text{visit}})$, it is clear that $B^{\{a_{i,j+1}\}}(t)$ is maximized if agent $a_{i,j+1}$ comes from the right at maximum speed, hits $(x_j, t_{\text{visit}})$ and then turns around and moves to the right at maximum speed, in which case equality is achieved in

$$\forall t \in [0, \infty), B^{\{a_{j+1}\}}(t) \leq \min(x' + |t - t'|v_{ij+1}, L),$$

where $(x', t') = (x_j + \frac{v_{ij+1}}{2}, t_{\text{visit}} + \frac{1}{2})$. Denote by $h$ the ray $(x' + (t' - t)v_{ij+1}, t)$ where $t \leq t'$ and by $g$ the ray $(x' + (t - t')v_{ij+1}, t)$ where $t \geq t'$.

Figure 2: Suppose $A_{j+1} = A_j \cup \{\text{the gray agent}\}$. If $l_j = L_{\text{old}}$ and $L_{\text{new}} = (x_{\text{new}}, t_{\text{new}})$, then $l_{j+1} = (x_{j+1}, t_{j+1})$, where $x_{j+1} \leq x_{\text{new}}$. In red and green is the rightmost bound of the right border of $A_j$. Note that in gray we have only a bound to the right of the trajectory of the gray agent; the important thing is that this agent covers $L_{\text{old}}$ and the points immediately to the right of it.

We thus get $\forall t \in [0, \infty)$,

$$B^{A_{j+1}}(t) = \max(B^{A_j}(t), B^{\{a_{j+1}\}}(t)) \leq f(t),$$

where $f(t) = \max(x_j + |t - t_j|v_{\text{max}}, x' + |t - t'|v_{ij+1})$. Consider $t_{\text{new}} = \arg \min_{t \in [1, \infty)} f(t)$. Let $L_{\text{new}} = (x_{\text{new}} := f(t_{\text{new}}), t_{\text{new}})$ be the leftmost point on the aforementioned upper bound on $B^{A_{j+1}}(t)$. Notice that $L_{\text{new}}$ is either the intersection of $h$ and $w$, or the intersection of $g$ and $u$, or the intersection of $g$ and $h$. These three cases are illustrated in Figure 2.

We
have $u$ in red and $w$ in green. Consider the upper bound on $B^{(a_{j+1})}(t)$ mentioned above. The trajectory of agent $a_{j+1}$ that would correspond to matching this upper bound is given in black and the points $a_{j+1}$ would cover if this was his/her trajectory are given in gray. We have that $L_{old} = (x_{old, t_{old}}) := l_j$. It can be seen by inspection of the three cases in Figure 2 that $|t_{new} - t_j| \leq 1$. Then $t_{new} \in [k - (j + 1), k + (j + 1)]$, which makes $L_{new}$ a candidate for $l_{j+1}$, therefore $l_{j+1} = (x_{j+1, t_{j+1}})$ will have $x_{j+1} \leq x_{new}$. Thus it is enough to show that $x_{new} - x_j \leq \frac{1}{v_{j+1} + v_{max}}$, where $v_{j+1}$.

We need an upper bound on $d = x_{new} - x_{old}$. We consider the points $M = (x_M, t_M)$ and $N = (x_N, t_N)$ as illustrated in Figure 2 such that in all three cases $x_M = x_{new}$ and $|t_M - t_{new}| = 1$. In Case 1, we consider the segment $MN$ of slope $\frac{1}{v_{j+1}}$ and $x_M - x_N = d$, and the segment $L_{old}L_{new}$ of $w$ of slope $-\frac{1}{v_{max}}$ and $x_{new} - x_{old} = d$. This gives us

$$\frac{d}{v_{j+1}} + \frac{d}{v_{max}} \leq 1 \Rightarrow d \leq \frac{1}{v_{j+1}} + \frac{1}{v_{max}}.$$ (4.2)

In Case 2, we consider the segment $L_{new}L_{old}$ of $u$ of slope $\frac{1}{v_{j+1}}$ and $x_{new} - x_{old} = d$, and the segment $NM$ of slope $-\frac{1}{v_{j+1}}$ and $x_M - x_N = d$. This implies Equation (4.2) for Case 2 as well.

In Case 3, we consider the segment $MN$ of slope $\frac{1}{v_{j+1}}$ and $x_M - x_N = d$, and the segment $NL_{new}$ of slope $-\frac{1}{v_{j+1}}$ and $x_{new} - x_N = d$. This means that

$$\frac{2d}{v_{j+1}} = 1 \Rightarrow d = \frac{v_{j+1}}{2} \leq \frac{1}{v_{j+1}} + \frac{1}{v_{max}}.$$ 

Therefore, $x_{new} - x_{old} \leq \frac{1}{v_{j+1} + v_{max}}$ as desired.

**Proof of Lemma 2.2** We show how $L \leq \left(1 - \frac{1}{2\sqrt{k}}\right)\sum_{i=1}^{k} v_i$ follows from Theorem 2.1 First note that $L \leq \sum_{i=1}^{k} v_i - \frac{v_{max}}{2}$ as each agent $a_i$ contributes at most $v_i \cdot \frac{1}{1 + \frac{v_{max}}{2}} \leq v_i$ while the agent with maximum speed contributes exactly $\frac{v_{max}}{2}$. Therefore, if $v_{max} \geq \frac{1}{\sqrt{k}} \sum_{i=1}^{k} v_i$ the desired upper bound for $L$ follows immediately. It remains to deal with the case $v_{max} < \frac{1}{\sqrt{k}} \sum_{i=1}^{k} v_i$. For this we first note that $x \cdot \frac{1}{x + \frac{v_{max}}{2}} = \frac{2}{x + \frac{v_{max}}{2}}$ is a concave function in $x$ for $0 \leq x \leq v_{max}$, since the second derivative $-\frac{2}{(x + \frac{v_{max}}{2})^2v_{max}}$ is always negative. This allows us to apply Jensen’s inequality and thus we have

$$L \leq \sum_{i=1}^{k} \frac{1}{v_{i}} \cdot \frac{1}{v_{avg} + \frac{v_{max}}{2}} \leq k \frac{1}{v_{avg} + \frac{v_{max}}{2}} = \frac{1}{1 + \frac{1}{k} \sum_{i=1}^{k} v_i} \sum_{i=1}^{k} v_i \leq \left(1 - \frac{1}{k + 1}\right) \sum_{i=1}^{k} v_i \leq \left(1 - \frac{1}{2\sqrt{k}}\right) \sum_{i=1}^{k} v_i,$$

where $v_{avg} = \frac{1}{k} \sum_{i=1}^{k} v_i$. This concludes the proof of Lemma 2.2.
In this section, we will prove that for any $k$ agents, there exist speeds $v_1, \ldots, v_k$ and a scheme for these agents to patrol a fence of length

\[
L = \left(1 - \frac{3.5}{\sqrt{k}} + O(1/k)\right) \sum_{i=1}^{k} v_i.
\]

This improves the result from [10] and therewith falsifies the corresponding conjecture stated in this paper.

**Proof of Theorem 2.3.** Assume $k$ is sufficiently large, and, for ease of notation, define $n := k - 2$. Let $L = n - 3/2\sqrt{n}$. We construct a schedule that patrols $E = [0, L]$ with idle time 1, using $n + 1$ agents with maximum speed 1 and 1 agent with maximum speed $2\sqrt{n} - 1$. Thus we have a total speed of $V = \sum_{i=1}^{k} v_i = n + 2\sqrt{n}$. As the ratio between $L$ and $V$ approaches $1 - \frac{3.5}{\sqrt{k}} + O(1/k)$, Theorem 2.3 follows.

To simplify presentation of the patrol schedule, we will allow agents to occasionally “step out of the fence $[0, L]$”, i.e. we allow an agent $a_i$ to assume positions $a_i(t) < 0$ and $a_i(t) > L$ (to avoid this, we could also modify the schedule so that they stay at the respective end of the fence for a while). To keep the notation as clean as possible, we henceforth assume that $n$ is a square number. Our schedule works as follows (see figure Figure 5 for a graphical representation):

**Slow agents $a_1, \ldots, a_n$:** For each $i \in \{0, \ldots, n\}$, agent $a_i$ starts at time 0 at position $x = i - i/\sqrt{n}$ and moves $i/(2\sqrt{n})$ time units to the right. Then he or she repeats:

- move to the left for $\sqrt{n}$ time units.
- move to the right for $\sqrt{n}$ time units.

**Fast agent $a_{n+1}$:** The fast agent $a_{n+1}$ starts at time 0 and repeats the following four steps:

1. Move from position 0 to position $L + 1/2$ with speed $2\sqrt{n} - 1$.
2. Move from position $L + 1/2$ to position $-1/2$ during the next $\sqrt{n}/2 + 1$ time units (e.g. with constant speed $(L + 1)/((\sqrt{n}/2 + 1) = 2\sqrt{n} - 7 + (16)/(\sqrt{n} + 2))$).
3. Move from position $-1/2$ to position $L$ with speed $2\sqrt{n} - 1$.
4. Move from position $L$ to position 0 in the next $\sqrt{n}/2$ time units (e.g. with constant speed $(L)/((\sqrt{n}/2) = 2\sqrt{n} - 3)$.

The idea behind our patrol schedule is to initially place the agents with maximum speed 1 equidistantly along the fence with gaps of length slightly smaller than 1, similar to the schedule for the fast agents in [10]. In contrast to their schedule, this is performed slightly out of phase between the agents. This will cover most of the points on the fence. The only problem appears whenever the agents turn around, as then the points right next to these turning points are
not visited for more than one time unit, hence creating uncovered triangles in the “spacetime” diagram (white triangles in Figure 5). By timing the turning times of the agents appropriately, we ensure that these uncovered triangles are placed such that they can all be cleaned up by the last fast agent. This will be described in further detail in the paragraphs that follow. Figure 3 gives a complete illustration of our schedule.

Figure 3: The described schedule for $n = 16$. The dark grey area describes the points $(x,t)$ which are covered by the slow agents $a_0, \ldots, a_n$ while the light grey shaded area describes the points $(x,t)$ which are covered by the fast agent in steps (1) and (3) of his protocol.

We will show that the above schedule indeed has idle time 1. As mentioned in Section 3 this is equivalent to showing that every $(x,t) \in [0,L] \times [1,\infty)$ is covered.

11
Observe that in our schedule all agents have periodicity \(2\sqrt{n}\) (agent \(a_{n+1}\) walks \(\sqrt{n}/2 - 1/2\) time units in steps (1) and (3), \(\sqrt{n}/2 + 1\) time units in step (2) and \(\sqrt{n}/2\) time units in step (4)). Thus, any times \(t\) below should be interpreted as \(t \mod 2\sqrt{n}\) (for clarity of representation we will usually omit the \(\mod 2\sqrt{n}\) term and hope that this will not cause confusion). Moreover, due to this \(2\sqrt{n}\)-periodicity, we could easily extend our patrol schedule to a schedule for all times \(t \in (-\infty, \infty)\), i.e. we need not worry about times \(t\) being smaller than 1 in our arguments below.

Denote by \(L^i = (x^i_L, t^i_L)\) and \(R^i = (x^i_R, t^i_R)\) the left respectively right turning point of agent \(a_i\), \(\forall i \in \{0, \ldots, n\}\), i.e. agent \(a_i\) walks along the fence from \(x^i_L\) to \(x^i_R\) and back, turning around at times \(t^i_L\) and \(t^i_R\) respectively. It follows directly from the protocol of the slow agents that

\[
R^i = \left( i \left( 1 - \frac{1}{2\sqrt{n}} \right), \frac{i}{2\sqrt{n}} \right) \quad \text{and} \quad L^i = \left( i \left( 1 - \frac{1}{2\sqrt{n}} \right) - \sqrt{n}, \frac{i}{2\sqrt{n}} + \sqrt{n} \right)
\]

Let \(x\) be a fixed point along the fence. We will now argue that the point \(x\) gets visited at least every 1 time unit. To do this, we will write down a \(2\sqrt{n}\)-periodic sequence of visiting times, (i.e. times \(x\) gets visited by an agent \(a_i\)) such that any two neighboring visiting times differ by at most 1 time unit.

First, we note that agent \(a_i\) visits \(x\) if \(i(1-1/(2\sqrt{n})) - \sqrt{n} \leq x \leq i(1-1/(2\sqrt{n}))\) or equivalently if

\[
\frac{2\sqrt{n}x}{2\sqrt{n}-1} \leq i \leq \frac{2\sqrt{n}x + 2n}{2\sqrt{n}-1}.
\]

Denote by \(j_{\text{min}} := \left\lceil \frac{2\sqrt{n}x}{2\sqrt{n}-1} \right\rceil = 0\) and by \(j_{\text{max}} := \left\lfloor \frac{2\sqrt{n}x + 2n}{2\sqrt{n}-1} \right\rfloor = n\) the indices such that agents \(a_{j_{\text{min}}}, \ldots, a_{j_{\text{max}}}\) are exactly the agents visiting the point \(x\).

For each \(i \in \{j_{\text{min}}, \ldots, j_{\text{max}}\}\) the agent \(a_i\) visits \(x\) at times

\[
s_i = \frac{i}{2\sqrt{n}} - \left( i \left( 1 - \frac{1}{2\sqrt{n}} \right) - x \right) = x - i \left( 1 - \frac{1}{\sqrt{n}} \right)
\]

walking from left to right and

\[
t_i = \frac{i}{2\sqrt{n}} + \left( i \left( 1 - \frac{1}{2\sqrt{n}} \right) - x \right) = i - x
\]

walking from right to left. Furthermore, we observe that the fast agent \(a_{n+1}\) visits \(x\) in steps (1) and (3) of his protocol at times

\[
f_1 = \frac{x}{2\sqrt{n} - 1} \quad \text{and} \quad f_3 = \frac{x + 1/2}{2\sqrt{n} - 1} + \sqrt{n} + \frac{1}{2} = \frac{x + 2n}{2\sqrt{n} - 1}.
\]

We claim that

\[
s_{j_{\text{max}}} \ldots, s_{j_{\text{min}}}, f_1, t_{j_{\text{min}}} \ldots, t_{j_{\text{max}}}, f_3
\]
is a sequence of visiting times of the point \( x \) with all adjacent visiting times differing in at most 1 time unit. It is obvious that the differences \( s_{i-1} - s_i \) and \( t_i - t_{i-1} \) are not greater than 1 for all \( i = j_{\min} + 1, \ldots j_{\max} \). Thus it remains to check whether this is also true for the remaining four gaps between \( s_{j_{\min}} \) and \( f_1 \), \( f_1 \) and \( t_{j_{\min}} \), \( t_{j_{\max}} \) and \( f_3 \) and \( f_3 \) and \( s_{j_{\max}} + 2\sqrt{n} \).

As \( j_{\min} \leq \frac{2\sqrt{n}x}{2\sqrt{n} - 1} + 1 \) we have

\[
 f_1 - s_{j_{\min}} = \frac{x}{2\sqrt{n} - 1} - \left( x - j_{\min} \left( 1 - \frac{1}{\sqrt{n}} \right) \right)
\]

\[
 = \frac{x}{2\sqrt{n} - 1} - x + \left( \frac{2\sqrt{n}x}{2\sqrt{n} - 1} + 1 \right) \left( 1 - \frac{1}{\sqrt{n}} \right)
\]

\[
 = \frac{x}{2\sqrt{n} - 1} - \frac{2\sqrt{n}x}{2\sqrt{n} - 1} + \frac{2\sqrt{n}x}{2\sqrt{n} - 1} - \frac{x}{2\sqrt{n} - 1} + 1 - \frac{1}{\sqrt{n}} = 1 - \frac{1}{\sqrt{n}}
\]

and

\[
 t_{j_{\min}} - f_1 = j_{\min} - x - \frac{x}{2\sqrt{n} - 1}
\]

\[
 \leq \frac{2\sqrt{n}x}{2\sqrt{n} - 1} + 1 - x - \frac{x}{2\sqrt{n} - 1}
\]

\[
 = \frac{2\sqrt{n}x}{2\sqrt{n} - 1} - \frac{2\sqrt{n}x - x}{2\sqrt{n} - 1} - \frac{x}{2\sqrt{n} - 1} + 1 = 1
\]

Likewise, as \( j_{\max} \geq \frac{2\sqrt{n}x + 2n}{2\sqrt{n} - 1} - 1, \) we have

\[
 f_3 - t_{j_{\max}} = \frac{x + 2n}{2\sqrt{n} - 1} - (j_{\max} - x)
\]

\[
 \leq \frac{x + 2n}{2\sqrt{n} - 1} - \frac{2\sqrt{n}x + 2n}{2\sqrt{n} - 1} + 1 + x
\]

\[
 = \frac{x + 2n}{2\sqrt{n} - 1} - \frac{2\sqrt{n}x + 2n}{2\sqrt{n} - 1} + \frac{2\sqrt{n}x - x}{2\sqrt{n} - 1} + 1 = 1
\]

and

\[
 s_{j_{\max}} + 2\sqrt{n} - f_3 = x - j_{\max} \left( 1 - \frac{1}{\sqrt{n}} \right) + 2\sqrt{n} - \frac{x + 2n}{2\sqrt{n} - 1}
\]

\[
 \leq x - \left( \frac{2\sqrt{n}x + 2n}{2\sqrt{n} - 1} - 1 \right) \left( 1 - \frac{1}{\sqrt{n}} \right) + 2\sqrt{n} - \frac{x + 2n}{2\sqrt{n} - 1}
\]

\[
 = \frac{2\sqrt{n}x - x}{2\sqrt{n} - 1} - \frac{2\sqrt{n}x + 2n}{2\sqrt{n} - 1} + \frac{2x + 2\sqrt{n}}{2\sqrt{n} - 1} + 1 - \frac{1}{\sqrt{n}} + \frac{4n - 2\sqrt{n}}{2\sqrt{n} - 1} - \frac{x + 2n}{2\sqrt{n} - 1}
\]

\[
 = 1 - \frac{1}{\sqrt{n}}
\]

Thus, \( s_{j_{\min}}, \ldots, s_{j_{\min}}, f_1, t_{j_{\min}}, \ldots, t_{j_{\max}}, f_3 \) is a sequence of visiting times of the point \( x \) with all adjacent visiting times differing in at most 1 time unit, concluding the proof. \( \square \)
References

[1] Yann Chevaleyre, *Theoretical analysis of the multi-agent patrolling problem*, Proceedings. ieee/wic/acm international conference on intelligent agent technology, 2004. (iat 2004), 2004, pp. 302–308.

[2] Andrew Collins, Jurek Czyzowicz, Leszek Gąsieniec, Adrian Kosowski, Evangelos Kranakis, Danny Krizanc, Russell Martin, and Oscar Morales Ponce, *Optimal patrolling of fragmented boundaries*, SPAA, 2013, pp. 241–250.

[3] Jurek Czyzowicz, Leszek Gąsieniec, Adrian Kosowski, and Evangelos Kranakis, *Boundary patrolling by mobile agents with distinct maximal speeds*, European symposium on algorithms, 2011, pp. 701–712.

[4] Jurek Czyzowicz, Leszek Gąsieniec, Adrian Kosowski, Evangelos Kranakis, Danny Krizanc, and Najmehe Taleb, *When patrolmen become corrupted: Monitoring a graph using faulty mobile robots*, Algorithmica 79 (2017), no. 3, 925–940.

[5] Jurek Czyzowicz, Konstantinos Georgiou, Evangelos Kranakis, Fraser MacQuarrie, and Dominik Pajak, *Distributed patrolling with two-speed robots (and an application to transportation)*, Operations research and enterprise systems, 2017, pp. 71–95.

[6] Jurek Czyzowicz, Adrian Kosowski, Evangelos Kranakis, and Najmehe Taleb, *Patrolling trees with mobile robots*, Foundations and practice of security, 2017, pp. 331–344.

[7] Jurek Czyzowicz, Evangelos Kranakis, Dominik Pajak, and Najmehe Taleb, *Patrolling by robots equipped with visibility*, Structural information and communication complexity, 2014, pp. 224–234.

[8] Adrian Dumitrescu, Anirban Ghosh, and Csaba D Tóth, *On fence patrolling by mobile agents*, The Electronic Journal of Combinatorics 21 (2014), no. 3, P3–4.

[9] Akitoshi Kawamura and Yusuke Kobayashi, *Fence patrolling by mobile agents with distinct speeds*, Distributed Computing 28 (2015), no. 2, 147–154.

[10] Akitoshi Kawamura and Makoto Soejima, *Simple strategies versus optimal schedules in multi-agent patrolling*, International conference on algorithms and complexity, 2015, pp. 261–273.

[11] Aydano Machado, Geber Ramalho, Jean-Daniel Zucker, and Alexis Drogoul, *Multi-agent patrolling: An empirical analysis of alternative architectures*, Multi-agent-based simulation ii, 2003, pp. 155–170.

[12] David Portugal and Rui Rocha, *A survey on multi-robot patrolling algorithms*, Technological innovation for sustainability, 2011, pp. 139–146.