Abstract—In this paper we consider strategies for MIMO interference channels which combine the notions of interference alignment and channel pre-inversion. For symmetric MIMO channels, we consider a variant on interference alignment wherein groups of users collaborate to share their data, and can thereby eliminate some of the interference by pre-inverting their channel matrices. For every possible grouping of $K = 4$ users each with $N = 5$ antennas, we completely classify the degrees of freedom available to each user, and construct explicit interference alignment strategies which maximize the sum capacity.

To improve the capacity of our schemes at finite SNR, we propose that the groups of users invert their subchannel using a regularized Tikhonov inverse. While the optimal Tikhonov parameter is already essentially known, we provide a new sleeker derivation of this result. We use random matrix theory to provide an explicit formula for the SINR as the number of users increases, which we believe is a new result. Lastly, for $K = 4$ and $N = 5$, we simulate our proposed hybrid schemes and compute the capacity of each experimentally.

Index Terms—Interference Alignment, Channel Pre-Inversion, MIMO Interference Channel, Tikhonov Regularization

I. INTRODUCTION

Beginning with the seminal paper [1] of Cadambe and Jafar, the promise of interference alignment to greatly increase capacity in the presence of interferers has made it a popular topic in recent years. For a $K$-user interference channel, as the number of parallel links and the SNR approach infinity, coding strategies exist which guarantee all users $K/2$ degrees of freedom. In the parlance of the interference alignment literature, “everyone gets half the cake”.

The situation is quite different for constant channel coefficients. In particular for a fully symmetric MIMO interference channel with $K$ users, each with $N$ antennas at the transmitter and receiver, and each user demanding $d$ degrees of freedom, a fundamental result by Bresler, Cartwright, and Tse [2] tells us that

$$d \leq 2N/(K + 1)$$  

which can place rather severe restrictions on the available degrees of freedom.

Interference alignment strategies generally assume each user has access to only their own data. At the other end of the spectrum from interference alignment is the notion of channel pre-inversion [3] in which all users share all of their data, allowing them to pre-multiply the data vector by the inverse of the channel matrix. In the symmetric MIMO interference channel, such a strategy would eliminate the interference to give every user the full $d = N$ degrees of freedom.

The fundamental question this paper begins to address is how the strategies of interference alignment and channel pre-inversion interact with each other. If all of the $K$ users share their data then interference alignment is unnecessary. However, if $K$ is large enough, such a data sharing scheme may become unrealistic, thus we consider situations where $K$ users are partitioned into smaller groups into which they share their data. Each group uses a channel pre-inversion strategy to clear interference within its own group, and interference alignment strategies are employed to minimize the interference from the other groups. Since we do not assume collaboration on the receive end, this problem is fundamentally different from simply considering fewer users each with more antennas.

The main contributions of this paper are the following:

- In the case of $K = 4$ users, the bound (1) tells us that the “minimal” case is when $d = 2$ and $N = 5$. For $K = 4$ and $N = 5$, we study every partition of the users into groups who share their data, and classify completely the increases in degrees of freedom. We restrict to cases in which every user has at least $2N/(K + 1)$ degrees of freedom, so that every user benefits from data sharing.
- We propose that individual groups invert their subchannels using a Tikhonov regularized inverse, and provide an apparently new derivation of the optimal Tikhonov parameter. Our Theorem 1 provides an explicit, very accurate estimate of the SINR when users employ Tikhonov inversion. We essentially show that for $K >> 0$ and large SNR, we have SINR (dB) $\approx$ SNR (dB)/2.
- We demonstrate Theorem 1 with simulation results, displayed in Fig. 1. We also simulate various hybrid channel pre-inversion and interference alignment strategies for $K = 4$ and $N = 5$, and display the results in Fig. 2.

II. INTERFERENCE ALIGNMENT FOR MIMO CHANNELS

Suppose we have $K$-user symmetric MIMO interference channel, in which each transmitter-receiver pair has $N$ an-
tennas at both ends. At a single time instance, such a channel can be modeled by the equation

\[ Y = HAX + Z \]  

(2)

in which \( X = [x_1, \ldots, x_K]^T \) with \( x_i \in \mathbb{C}^{1 \times N} \) is the data from transmitter \( i \) to receiver \( i \) for \( 1 \leq i \leq K \), \( A \) is a \( K \times K \) encoding matrix, \( Z \) is additive Gaussian noise, and the channel matrix is \( H = (H_{ij})_{1 \leq i, j \leq K} \) where \( H_{ij} \in M_N(\mathbb{C}) \) models the channel between transmitter \( j \) and receiver \( i \). We assume that the entries of the channel matrices are continuously distributed, so that, for example, \( H_{ij}^{-1} \) exists with probability 1.

If user \( i \) wants to transmit across \( d_i \leq N \) dimensions (i.e. user \( i \) has \( d_i \) degrees of freedom), they use linear encoding to write \( x_i = U_i \hat{x}_i \) for \( \hat{x}_i \in \mathbb{C}^{d_i} \) a vector of information symbols (for example QAM symbols), and a full-rank \( N \times d_i \) encoding matrix \( U_i \). Receiver \( i \) observes

\[ y_i = H_{ii}U_i \hat{x}_i + \sum_{j \neq i} H_{ij}U_j \hat{x}_j + z_i \]  

(3)

Let us define the **signal space** and **interference space** at receiver \( i \) to be, respectively,

\[ S_i := \text{colspan}(H_{ii}U_i), \quad \text{and} \quad I_i := \sum_{j \neq i} \text{colspan}(H_{ij}U_j). \]  

(4)

If \( V_i \) denotes projection onto \( (I_i)^\perp \), receiver \( i \) computes

\[ V_i y_i = V_i H_{ii}U_i \hat{x}_i + V_i z_i \]  

(5)

and can then reliably recover the desired signal \( \hat{x}_i \) as long as \( \dim(V_i S_i) = d_i \). With probability 1, we have \( \dim(S_i) = \dim(V_i S_i) = d_i \), provided 

\[ d_i + \dim(I_i) \leq N. \]  

(6)

The goal of interference alignment is to choose matrices \( U_i \) of rank \( d_i \) for all \( i = 1, \ldots, K \) to

\[
\text{maximize } C = \frac{1}{KN} \sum_{i=1}^{K} d_i \text{ subject to } (6).
\]  

We refer to \( C \) as the normalized sum capacity; multiplication by \( \frac{1}{KN} \) makes meaningful comparison possible over different numbers of users and antennas. We will call a choice of \( U_1, \ldots, U_K \) satisfying (6) an **interference alignment strategy**.

Existence results for IA strategies determine the feasibility of interference alignment and possible degrees of freedom, such as those found in [4], [5]. For our purposes these results have essentially been subsumed by the bound (1) of [2]. In addition to these fundamental limit, explicit IA strategies are often constructed using numerical optimization, as in [6], [7].

### III. LOCAL DATA SHARING

A partition of a positive integer \( K \) is a list \( p(K) = (k_1, \ldots, k_m) \) of positive integers such that \( \sum_{n=1}^{m} k_n = K \). Let us partition the \( K \) users into \( m \) groups, in which the \( n^{th} \) group has size \( k_n \). Within each group the users share their encoded data \( x_i \) at the transmit side, so that user 1, for example, has access to \( x_1, \ldots, x_{k_1} \). The \( n^{th} \) group then collaborates to encode their data with some \( k_n N \times k_n N \) matrix \( A_n \). If we let \( A \) be the block-diagonal matrix with the \( A_n \) along the diagonal, we can rewrite the channel equation as

\[ Y = \begin{bmatrix} H_1 & * & \cdots & * \\ * & \ddots & \cdots & * \\ * & \cdots & H_m \\ \end{bmatrix} \begin{bmatrix} A_1 & 0 & 0 \\ 0 & \cdots & 0 \\ 0 & 0 & A_m \\ \end{bmatrix} X + Z \]  

(8)

in which \( H_n \) denotes the channel matrix for the \( n^{th} \) group’s subchannel. The groups of users now set \( A_n = H_n^{-1} \) for all \( n = 1, \ldots, m \), resulting in an effective channel equation

\[ Y = \begin{bmatrix} I_{k_1N} & * & \cdots \\ * & \ddots & \cdots \\ * & \cdots & I_{k_mN} \\ \end{bmatrix} X + Z \]  

(9)

The net effect of this process, which we will refer to as **data sharing**, is that the users are able to eliminate all of the interference from the other members of their group. We will study how this process affects the normalized sum capacity when we construct IA strategies with the new effective channel matrix of [9].

The encoding matrices \( A_i \) we propose will not actually be \( H_i^{-1} \) but rather a regularized Tikhonov inverse, discussed below in Section IV. Thus the diagonal blocks in (9) are not actually identity matrices but rather perturbations of the identity matrix. However, the interference introduced by this perturbation is minuscule when compared with the signal power and thus for the purposes of constructing the IA strategy, we treat it as noise and simply assume the diagonal blocks of the effective channel matrix are identity matrices.

Given a partition \( p(K) \) of the users into groups who share their data, our goal is to

\[
\text{compute } C_p(K) = \max \left\{ \frac{1}{KN} \sum_{i=1}^{K} d_i \right\} \text{ subject to } d_i \geq 2N/(K+1) \text{ for all } i = 1, \ldots, K
\]  

where the max ranges over all possible interference alignment strategies. The bound of [2] limits the normalized sum capacity in the case of no data sharing, and we would like to improve on this bound if various combinations of users share their data. We impose the constraint \( d_i \geq 2N/(K+1) \) to limit us to cases in which no individual user is required to sacrifice any of the degrees of freedom they had in the case of no data sharing.

It is important to note that we do not assume any collaboration at the receive end. Thus, for example, the signal \( U_{2} \) at receiver 1 is still interference, whereas this was essentially data at transmitter 1. In particular, breaking the users into groups of, for example, size 2 does not reduce our problem to the study of symmetric MIMO interference channels with \( K/2 \) users each with \( 2N \) antennas.

### IV. DATA SHARING FOR \( K = 4 \) USERS

The bound of [2] gives us \( d \leq 2N/5 \), so let us consider the case \( d = 2, N = 5 \) as a base case. Note that for all \( i \) and \( j \) we have \( \max_{i \neq j} d_i \leq \dim(I_j) \). Hence by (6) we must have \( d_j +
max_{i \neq j} d_i \leq N$, from which it follows that max \( d_i \leq 3 \). Thus our constraints force $d_1 = 2$ or 3 for all $i$. Our results, which are derived in the following subsections, are summarized here:

| $p(4)$ | sizes of groups | $C_{p(4)}$ |
|--------|----------------|-------------|
| (1, 1, 1, 1) | (2, 2, 2, 2) | 2/3 |
| (2, 2, 1, 1) | (2, 2, 2, 2) | 2/3 |
| (2, 2) | (3, 3, 2, 2) | 1/2 |
| (3, 1) | (3, 3, 3, 2) | 11/20 |

### A. The Partition $p(4) = (1, 1, 1, 1)$

We start with the partition $p(4) = (1, 1, 1, 1)$ and outline an explicit interference alignment strategy achieving the upper bound $(4)$. We write $U_j = [U_j^{(1)} U_j^{(2)}]$ where $B_j^{(i)}$ denotes the $i$th column of a matrix $B_j$ and $U_j^{(1)}, U_j^{(2)} \in \mathbb{C}^{5 \times 1}$. To create the necessary $N - d = 3$-dimensional interference space $\mathcal{I}_j$, we construct bases $B_j$ of $\mathcal{I}_j$ of size 3. Explicitly, we choose

$$
B_1 = \{ H_{12}U_2^{(1)}, H_{12}U_2^{(2)}, H_{13}U_3^{(1)} \},
B_2 = \{ H_{23}U_3^{(1)}, H_{23}U_3^{(2)}, H_{24}U_4^{(1)} \},
B_3 = \{ H_{34}U_4^{(1)}, H_{34}U_4^{(2)}, H_{31}U_1^{(1)} \},
B_4 = \{ H_{41}U_1^{(1)}, H_{41}U_1^{(2)}, H_{42}U_2^{(1)} \},
$$

The signal $U_j^{(2)}$, for example, interferes at receivers 2 and 3, thus $H_{21}U_1^{(2)} \in \mathcal{I}_2$ and $H_{31}U_1^{(2)} \in \mathcal{I}_3$ which gives a non-trivial relationship between the basis vectors of $B_2$ and $B_3$. We continue in this manner, expressing each $U_j^{(i)}$ in two different ways, until we have 8 independent non-trivial linear combinations of the above basis vectors. We solve the resulting $8 \times 8$ linear system by first solving for $U_4^{(2)}$, substituting the result into the remaining equations, and continuing until we are left with an equation $AU_1^{(1)} = \lambda U_1^{(1)}$ for some generically invertible $A$ and some non-zero $\lambda$. We finish by picking $U_1^{(1)}$ to be an eigenvector of $A$ with the corresponding eigenvalue, and back-substitute to find the remaining $U_j^{(i)}$.

### B. The Partition $p(4) = (2, 2, 1)$

After the groups pre-invert their subchannels, we reduce the channel matrix to the form

$$
H = \begin{bmatrix}
I_N & 0 & H_{13} & H_{14} \\
0 & I_N & H_{23} & H_{24} \\
H_{31} & H_{32} & I_N & H_{34} \\
H_{41} & H_{42} & H_{43} & I_N
\end{bmatrix}
$$

By symmetry, we can assume that $d_1 \geq d_2$ and that $d_3 \geq d_4$. If $d_1 = 3$ then $dim \mathcal{I}_i \geq 3$ for $i = 3, 4$, which forces $d_3 = d_4 = 2$. Suppose that the tuple $(d_1, d_2, d_3, d_4) = (3, 2, 2, 2)$ were achievable. Without loss of generality we can choose the following bases for the interference spaces:

$$
B_1 = \{ H_{13}U_3^{(1)}, H_{13}U_3^{(2)} \},
B_2 = \{ H_{23}U_3^{(1)}, H_{23}U_3^{(2)} \},
B_3 = \{ H_{31}U_1^{(1)}, H_{31}U_1^{(2)} \},
B_4 = \{ H_{41}U_1^{(1)}, H_{41}U_1^{(2)}, H_{41}U_1^{(3)} \},
$$

Successfully aligning the interference at the first two receivers would give us the following three equations:

$$
H_{14}U_4^{(1)} = a_1H_{13}U_3^{(1)} + a_2H_{13}U_3^{(2)},
H_{14}U_4^{(2)} = b_1H_{13}U_3^{(1)} + b_2H_{13}U_3^{(2)},
H_{24}U_4^{(2)} = c_1H_{23}U_3^{(1)} + c_2H_{23}U_3^{(2)} + c_3H_{24}U_4^{(2)}
$$

solving the above for $U_4^{(1)}$, $U_4^{(2)}$ and $U_3^{(3)}$ gives us expressions of the form

$$
U_3^{(3)} = G_1U_3^{(1)}, U_4^{(2)} = G_2U_3^{(1)}, U_4^{(2)} = G_3U_3^{(1)}
$$

At the third and fourth receivers we express all of the incoming interference in terms of the bases $B_3$ and $B_4$, respectively, and use the above expressions to arrive at the equations

$$
U_2^{(1)} = \sum_{i=1}^{3} d_iH_{32}^{-1}H_{31}U_1^{(i)} = \sum_{i=1}^{3} c_iH_{42}^{-1}H_{41}U_1^{(i)},
U_2^{(2)} = \sum_{i=1}^{3} f_iH_{32}^{-1}H_{31}U_1^{(i)} = \sum_{i=1}^{3} g_iH_{42}^{-1}H_{41}U_1^{(i)},
U_3^{(1)} = \sum_{i=1}^{3} h_iG_2H_{34}^{-1}H_{31}U_1^{(i)} = \sum_{i=1}^{3} k_iH_{43}^{-1}H_{41}U_1^{(i)},
U_3^{(1)} = \sum_{i=1}^{3} m_iG_3H_{34}^{-1}H_{31}U_1^{(i)} = \sum_{i=1}^{3} n_iH_{43}^{-1}H_{41}U_1^{(i)}
$$

Rearranging gives us four non-trivial equations of the form $\sum_{i=1}^{3} a_iU_1^{(i)} = 0$ which are all independent with probability 1. Such a system has no solution, contradicting our original assumption that $d_1 = 3$. A similar argument shows that we cannot have $d_3 = 3$, and we conclude that $C_{(2,1,1)} = 2/5$.

However, there is some benefit to users 1 and 2 inverting their subchannel, in that the interference alignment strategy is easier to construct. Explicitly, we first solve the two independent alignment chains

$$
\alpha_2H_{42}U_2^{(1)} = H_{43}U_3^{(1)}, \beta_2H_{42}U_2^{(2)} = H_{43}U_3^{(2)},
\alpha_3H_{23}U_3^{(1)} = H_{24}U_4^{(1)}, \beta_3H_{23}U_3^{(2)} = H_{14}U_4^{(2)},
\alpha_4H_{34}U_4^{(1)} = H_{32}U_2^{(1)}, \beta_4H_{34}U_4^{(2)} = H_{32}U_2^{(2)}
$$

which aligns the interference at receivers 1 and 2. We now need only to choose $U_1$ such that the two conditions

$$
\dim \mathcal{I}_3 = \dim \text{colspan}(H_{31}U_1, H_{32}U_2) = 3 \quad (11)
$$

$$
\dim \mathcal{I}_4 = \dim \text{colspan}(H_{41}U_1, H_{42}U_2) = 3 \quad (12)
$$

hold, which is easily done. One can choose, for example, $U_1^{(1)} = H_{31}^{-1}H_{32}U_2^{(2)}$ and $U_1^{(2)} = H_{41}^{-1}H_{42}U_2^{(1)}$.

### C. The Partition $p(4) = (2, 2)$

The two groups separately invert their subchannels to obtain

$$
H = \begin{bmatrix}
I_{2N} & * \\
* & I_{2N}
\end{bmatrix}
$$

We are free to assume that $d_1 \geq d_2$ and that $d_3 \geq d_4$. We then align the interference at receivers 1 and 2 so that

$$
H_{14}U_4 < H_{13}U_3, \quad H_{24}U_4 < H_{23}U_3
$$

\(\square\)
and similarly at receivers 3 and 4. This leads to the single constraint equation $d_1 + d_3 = 5$. It is easy to see an optimal choice is now given by $d_1 = d_2 = 3$, and $d_3 = d_4 = 2$, resulting in $C_{(2,2)} = 1/2$.

D. The Partition $p(4) = (3,1)$

Subchannel pre-inversion gives the effective channel matrix

$$H = \begin{bmatrix} I_{3N} & 0 \\ 0 & I_{N} \end{bmatrix}$$ (15)

Suppose without loss of generality that $d_1 \geq d_2 \geq d_3$. It is easy to align the interference at receiver 4 by picking any $U_2$ and $U_3$ satisfying

$$U_2 \times H_{42}^{-1} H_{41} U_1 \quad \text{and} \quad U_3 \times H_{43}^{-1} H_{42} U_2$$ (16)

Hence we are free to pick $d_1 = d_2 = d_3 = 5 - d_4$ without restriction. The maximum of $C_{(3,1)} = (3(5 - d_4) + d_4)/20$ is achieved when $d_4$ is minimized, thus $C_{(3,1)}$ obtains its maximum value of $11/20$ when $d_4 = 2$ and $d_1 = d_2 = d_3 = 3$.

V. GLOBAL DATA SHARING - FULL CHANNEL PRE-INVERSION USING THE TIKHONOV INVERSE

In this section we treat the numerics of the partition $p(K) = (K)$, when all channel state information and all data is available to all users. Firstly, it is clear that we can reduce to the case where $N = 1$, with the channel matrix as (2). Here the signal $X = [x_1, \ldots, x_K]^T \in \mathbb{C}^K$ satisfies $E(|x_i|^2) = 1$ for all $i$, the entries $z_i$ of the noise vector $Z$ are assumed i.i.d. zero-mean Gaussian with variance $\sigma^2$ per complex dimension, and the users employ a $K \times K$ precoding matrix $A$ after they have shared their data.

A. Tikhonov Regularization

Completely inverting the channel as in (2) requires us to set $A = H^{-1}$. However, if one of the singular values $s$ of $H$ is close to zero, then the corresponding singular value $s^{-1}$ of $H^{-1}$ will cause the average energy per transmitter to be enormous. We therefore consider for any matrix $B$ the Tikhonov regularization, an approximate inverse defined by

$$B_\alpha := B^\dagger (\alpha I + BB^\dagger)^{-1}$$ (17)

where $\alpha > 0$ is some fixed constant. One can show directly from the above definition that if

$$B = U \Sigma V^\dagger, \quad \Sigma := \text{diag}(s_i(B))$$ (18)

is a singular value decomposition of a $K \times K$ matrix $B$, then

$$B_\alpha = V \Sigma_\alpha U^\dagger, \quad \Sigma_\alpha := \text{diag}(s_i(B)/(s_i(B)^2 + \alpha))$$ (19)

is a singular value decomposition of $B_\alpha$. Hence for ill-conditioned $B$, the Tikhonov regularization $B_\alpha$ damps the effect of badly-behaved singular values.

Now let $G := H/\sqrt{K}$ and $A := G_\alpha/\sqrt{K}$. A simple computation shows that our channel equation becomes

$$Y = GG_\alpha X + Z, \quad \text{for} \quad G = H/\sqrt{K}.$$ (20)

This normalization has some advantages: the optimal $\alpha$ is independent of $K$, and the asymptotics of the SINR as $K \rightarrow \infty$ becomes easier to study using random matrix theory.

B. The Optimal Tikhonov Parameter

We now address the issue of choosing the optimal $\alpha$, which we consider to be the one which maximizes the signal-to-interference-plus-noise ratio, or SINR. This question was of course addressed in [3], however we present an apparently new and sleeker derivation of the optimal Tikhonov parameter.

The above choice of normalized Tikhonov inverse reduces us to studying the SINR for the channel (20). The transmitters need to rescale by the Frobenius norm of the encoding matrix $G_\alpha/\sqrt{K}$ before transmission to achieve unit average energy per user. The ideal signal and interference powers will hence be scaled by the same constant. Before rescaling, the ideal signal has expected power $K$, the expected noise power is $K\sigma^2$, and we measure interference to be the failure of $GG_\alpha$ to be the identity matrix. In particular, we count as interference the failure of the path gains along the diagonal of the effective channel matrix to be ones. Rescaling the signal and interference powers by dividing by $\frac{20}{K}||G_\alpha||_F^2$ gives us

$$\text{SINR} = \frac{K}{||G_\alpha||_F^2 \sigma^2 + ||GG_\alpha - I_K||_F^2}$$ (21)

One can deduce the equality $||GG_\alpha - I_K||_F^2 = ||G_\alpha G - I_K||_F^2$ by considering singular value decompositions as in [18] and [19] and using the invariance of the Frobenius norm under orthogonal transformation. Choosing the resulting optimal Tikhonov parameter $\alpha$ to maximize the above SINR is therefore equivalent to solving the optimization problem

$$\alpha_{\text{opt}} = \arg \min \left( ||G_\alpha||_F^2 \sigma^2 + ||G_\alpha G - I_K||_F^2 \right)$$ (22)

which is simply an MMSE optimization problem with the well-known solution

$$\alpha_{\text{opt}} = \alpha_{\text{MMSE}} = \sigma^2.$$ (23)

Remark 1: An easy computation shows that $GG_\alpha = HH_\alpha$, hence our derivation of the optimal Tikhonov parameter results in the same as in [3], since they conclude that the optimal Tikhonov inverse is $H_\alpha$. In fact, what is essentially done in [3] is that the authors work with the definition

$$\text{SINR} = \frac{||\text{diag}(GG_\alpha)||_F^2}{K\sigma^2 + ||GG_\alpha - \text{diag}(GG_\alpha)||_F^2}$$ (24)

from which one can make the approximation $\alpha_{\text{opt}}$ to be diagonal, rather than to be the identity matrix as in (21). This difference simply represents what the receiver identifies as the “true” signal.

C. Behavior of SINR as $K \rightarrow \infty$

Suppose from now on that the entries of our channel matrix $H$ are i.i.d. zero-mean Gaussian with variance 1 per complex dimension. The following theorem and proof use random matrix theory to study the growth of the expression (21), providing a way to compute the asymptotic SINR explicitly to within some error introduced by using Jensen’s Inequality.


**Theorem 1**: If $K$ users encode their data using the matrix $G_{\alpha}/\sqrt{K}$ for a fixed constant $\alpha > 0$, then

$$
\lim_{K \to \infty} \mathbb{E}_G \left( \frac{1}{\text{SINR}} \right) = \frac{\alpha + (\alpha + 1)\sigma^2}{\sqrt{\alpha(\alpha + 4)}} - \frac{\sigma^2}{2}
$$

(25)

**Proof:** First we summarize the necessary ideas from random matrix theory, for which our main reference is \cite{11}. Let $f : [0, \infty) \to \mathbb{R}$ be a bounded, continuous function. If $B$ is a $K \times K$ matrix such that $B^* = B$ with necessarily real, positive eigenvalues $\lambda_i$, we define

$$
\text{tr}_K(f(B)) := \frac{1}{K} \sum_{i=1}^{K} f(\lambda_i)
$$

(26)

The key theorem we will use is Corollary 7.8 of \cite{11}:

$$
\lim_{K \to \infty} \mathbb{E}_G(\text{tr}_K(f(G^\dagger G))) = \frac{1}{2\pi} \int_{0}^{4} f(x) \sqrt{\frac{x(4-x)}{x}} dx
$$

Let us rewrite $1/$SINR as

$$
\frac{1}{\text{SINR}} = \frac{1}{K} \left( ||G_{\alpha}||_F^2 + ||GG_{\alpha} - I_K||_F^2 \right).
$$

(27)

We use the above theorem to evaluate the asymptotic expectation with respect to $G$ of this expression.

Let $\lambda_i$ for $i = 1, \ldots, K$ be the eigenvalues of $G^\dagger G$. It follows from plugging the expressions (18) and (19) for the singular value decompositions of $G$ and $G_{\alpha}$ into (27) that

$$
\frac{1}{\text{SINR}} = \frac{1}{K} \left( \sum_{i=1}^{K} \frac{\lambda_i}{(\lambda_i + \alpha)^2} \sigma^2 + \sum_{i=1}^{K} \frac{\alpha^2}{(\lambda_i + \alpha)^2} \right)
$$

(28)

$$
\text{tr}_K(f(G^\dagger G))
$$

(29)

where $f : [0, \infty) \to \mathbb{R}$ is the function

$$
f(x) = \frac{x}{(x + \alpha)^2} \sigma^2 + \frac{\alpha^2}{(x + \alpha)^2}
$$

(30)

Using Mathematica to compute the relevant integral yields

$$
\frac{1}{2\pi} \int_{0}^{4} f(x) \sqrt{\frac{x(4-x)}{x}} dx = \frac{\alpha + (\alpha + 1)\sigma^2}{\sqrt{\alpha(\alpha + 4)}} - \frac{\sigma^2}{2}
$$

as desired.

We can apply Jensen’s Inequality to the convex function $1/x$, which tells us that

$$
\lim_{K \to \infty} \mathbb{E}_G(\text{SINR}) \geq \lim_{K \to \infty} 1/\mathbb{E}_G(1/$$SINR)

(31)

thus Theorem 1 allows us to predict the asymptotic SINR to within the error introduced by Jensen’s Inequality. As shown in Fig. 1., our simulations predict that this error is negligible, being $\approx 0.2$ dB at SINR $= 10$ dB and less at higher SINR. Notice that (31) and Theorem 1 also serve as a lower bound on the asymptotic SINR.

A simple computation shows that for sufficiently large SNR, one can use Theorem 1 to deduce the following approximation for $\alpha = \alpha_{\text{opt}} = \sigma^2$:

$$
\lim_{K \to \infty} \mathbb{E}_G(1/$$SINR) $\approx \sqrt{\sigma^2}.

(32)

When combined with the estimate (31) given by Jensen’s Inequality, this provides us with the convenient expression for the SINR in terms of the SNR, for a large number of users $K$ and sufficiently large SNR:

$$
\text{SINR (dB)} \approx \frac{\text{SNR (dB)}}{2}
$$

(33)

One can also observe this experimentally from Fig. 1.

**Remark 2:** If one wishes to work instead with the expression (24) for the SINR, then the error between the asymptotic experimental SINR and the estimate provided by Theorem 1 is slightly greater, being $\approx 1$ dB at SNR $= 10$ dB. However, preliminary simulations suggest that this error decreases to zero for a sufficiently large number of users and sufficiently large SNR. We hope to make more precise the relationship between Theorem 1 and the expression (24) in future work.

**VI. Simulation Results for $K = 4$ and $N = 5$**

In this section we study the capacity of some hybrid channel pre-inversion and interference alignment schemes for $K = 4$ users each with $N = 5$ antennas, to empirically demonstrate the benefits of using the Tikhonov regularization in concert with an interference alignment strategy.

For $p(4) = (2, 2)$, that is, dividing four users into two groups of two users, we use the example IA scheme from Section IV-C in which $d_1 = d_2 = 3$ and $d_3 = d_4 = 2$. Let us write out completely the resulting channel equations when both Tikhonov inversion and the interference alignment strategy of Section IV-C is used. For users 1 and 2 we have

$$
\begin{bmatrix}
V_1 y_1 \\
V_2 y_2
\end{bmatrix} = \begin{bmatrix}
V_1 & 0 \\
0 & V_2
\end{bmatrix} G_1 G_{1\alpha} \begin{bmatrix}
U_1 & 0 \\
0 & U_2
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
V_1 z_1 \\
V_2 z_2
\end{bmatrix}
$$

(34)

and similarly for users 3 and 4. Here $G_1$ is the normalized subchannel matrix for users 1 and 2, and $G_{1\alpha}$ is its Tikhonov regularization.
inverse. We define the SINR to be
\[
\text{SINR} = \frac{S_{12}/E_{12} + S_{34}/E_{34}}{N_{12} + N_{34} + I_{12}/E_{12} + I_{34}/E_{34}} \quad (35)
\]
where
\[
S_{12} = \|V_1 U_1\|_F^2 + \|V_2 U_2\|_F^2, \quad N_{12} = (\|V_1\|_F^2 + \|V_2\|_F^2)\sigma^2
\]
\[
E_{12} = \frac{1}{2N} \left\| G_{1\alpha} \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} \right\|_F^2
\]
\[
I_{12} = \left\| \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} (G_{1\alpha} G_{1\alpha} - I_{2N}) \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} \right\|_F^2
\]
and similarly for the analogous quantities for users 3 and 4. The quantity \(S_{12}\) represents the unnormalized ideal signal power at users 1 and 2, \(E_{12}\) the constant transmitters 1 and 2 need to scale by to use unit power per antenna, and \(I_{12}\) the unnormalized interference power between users 1 and 2. Note that if \(\alpha = 0\), then we are using full channel inversion and \(I_{12} = I_{34} = 0\).

![Fig. 2. Comparison of hybrid pre-inversion and interference alignment schemes for \(K = 4\) users with \(N = 5\) antennas for various partitions of the users, using the Tikhonov inverse with \(\alpha = \sigma^2\) and the full subchannel inverses. For every value of SNR, \(10^4\) random Gaussian channel matrices were generated, and the average SINRs were calculated using (35).](image)

We proceed similarly for the partitions \(p(4) = (2, 1, 1)\), \((3, 1)\), defining signal, interference, and noise power by groups as above and scaling the former two by the Frobenius norms of the appropriate encoding matrices. The definition (35) agrees with and generalizes the previous definition (21) in the sense that when using the full Tikhonov regularized inverse (i.e. \(p(4) = (4)\)), there is only one group and the preceding and projection matrices are both the identity, reducing (35) to (21).

Following [3], we approximate the normalized capacity for a partition \(p(K)\) by
\[
C_{\text{norm}} \approx C_{p(K)} \log_2 (1 + \text{SINR}) \quad (36)
\]
and plot this as a function of SNR in Fig. 2., for the partitions \(p(4) = (2, 1, 1), (2, 2), (3, 1), \) and \(4)\). We include results for both the Tikhonov inverse with \(\alpha = \sigma^2\) and for fully inverting the channel submatrices, to verify empirically the validity of using the Tikhonov inverse simultaneously with an interference alignment strategy. Our simulations suggest that while using the full Tikhonov inverse provides the highest capacity as SNR \(\to \infty\), for low- and mid-range SNR, the best strategies appear to employ both channel pre-inversion and interference alignment. We omit the partition \(p(4) = (1, 1, 1, 1)\) (i.e. no data sharing) in order to ease presentation, and since its asymptotic slope is identical to that of \(p(4) = (2, 1, 1)\).

VII. CONCLUSION

We have proposed a hybrid interference alignment and channel pre-inversion scheme in which groups of users in an interference channel collaboratively share their data, allowing for greater degrees of freedom. For the case of \(K = 4\) users each with \(N = 5\) antennas, we have completely classified the available degrees of freedom for every partition of the users into data-sharing groups. The groups each invert their subchannel using a regularized Tikhonov inverse. We have provided an explicit formula which predicts the asymptotic behavior of the SINR as \(K \to \infty\). Lastly, we have provided simulations which measure the benefits of the hybrid channel pre-inversion and interference alignment strategies we have constructed, showing that for low- and mid-range SNR, it appears to be optimal to use both strategies in concert.

Future work abounds. Obvious questions include designing joint interference alignment and channel pre-inversion strategies for different values of \(K\) and \(N\), as well as computing the optimal Tikhonov parameter for the definition (35) of the SINR.

VIII. ACKNOWLEDGEMENTS

The first author is supported by Academy of Finland grant 268364.

REFERENCES

[1] V.R. Cadambe and S.A. Jafar, “Interference alignment and degrees of freedom of the k-user interference channel”, IEEE Transactions on Information Theory, vol. 54, no. 8, August 2008.
[2] G. Bresler, D. Cartwright, and D. Tse, “Interference alignment for the mimo interference channel”, 2014, preprint available at http://arxiv.org/abs/1303.5678.
[3] C. Peel, B. Hochwal, and A. Swindlehurst, “A vector-perturbation technique for near-capacity multi-antenna multi-user communication part i: channel inversion and regularization”, IEEE Transactions on Communications, vol. 53, no. 1, pp. 195–202, January 2005.
[4] C.M. Yetis, T. Gou, S.A. Jafar, and A.H. Kayran, “On feasibility of interference alignment in mimo interference networks”, IEEE Transactions on Signal Processing, vol. 58, no. 9, pp. 4771–4782, September 2010.
[5] A. Ghasemi, A.S. Motari, and A.K. Khandani, “Interference alignment for the k-user mimo interference channel”, in Proceedings of the IEEE International Symposium on Information Theory (ISIT), 2010.
[6] S. Peters and R. Heath, “Cooperative algorithms for mimo interference channels”, IEEE Transactions on Vehicular Technology, vol. 59, no. 1, pp. 206–218, January 2011.
[7] O. El Ayach and R. Heath, “Grassmannian differential limited feedback for interference alignment”, IEEE Transactions on Signal Processing, vol. 60, no. 12, pp. 6481–6494, December 2012.
[8] U. Haagerup and S. Thorbjørnsen, “Random matrices with complex gaussian entries”, Expositiones Math., vol. 21, pp. 293–337, 2003.