Cosmology of the Jackiw-Teitelboim model

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Abstract

We investigate the cosmology of the two-dimensional Jackiw-Teitelboim model. Since the coupling of matter with gravitation is not defined uniquely, we consider two possible choices. The dilaton field plays an important role in the discussion of the properties of the solutions. In particular, the possibility of universes having a finite initial size emerges.

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1 Introduction

Models of gravity in two dimensions have been largely studied in recent years as toy models addressing issues that are too complex to be faced directly in four dimensions. However, the dynamics of two-dimensional gravity is rather different from its four-dimensional counterpart, since the Einstein-Hilbert action is a topological invariant in two dimensions and hence gives rise to trivial field equations. In order to derive the field equations from an action principle it is then necessary either to resort to higher-derivative theories \[1\] or to introduce an auxiliary scalar field $\eta$ (which in the following will be called dilaton) \[2\]. This field may be interpreted as the inverse of a spacetime-dependent gravitational coupling constant, and hence, as has been remarked in \[2\], cannot be ignored in the discussion of the spacetime structure. In particular, its zeroes should be regarded as true physical singularities, since the gravitational force blows up there.

One of the main topics in the theory of gravitation is the study of cosmological models. Recently this subject has received a special attention also in the context of the investigation of gravitational entropy bounds \[4\], and their relation to the holographic principle \[5\]. In \[6\] this problem has been addressed in a two-dimensional setting, using a Jackiw-Teitelboim action. It appears therefore useful to examine more closely the cosmology of two-dimensional models.

Actually, classical two-dimensional cosmology, which also emerges as a limiting case of string cosmology \[7\], has been studied in detail only in the context of a model where the curvature scalar is proportional to the trace of the energy-momentum tensor \[8\]. Although this model can be derived from a dilaton-gravity action \[10\], the role of the dilaton field has been neglected in these investigations. However, in our opinion its role is essential in the interpretation of the theory, since, as remarked above, in a cosmological context the dilaton can be considered as a time-varying Newton constant.

In the following we shall consider the simplest dilaton-gravity model in two dimensions, namely that of Jackiw and Teitelboim \[2, 11\]. Its specificity is that its action does not contain any kinetic energy term for the dilaton. Nevertheless, it can be related to a large class of equivalent models by conformal transformations \[12\].

\[1\] Curiously, more attention has been devoted to two-dimensional quantum cosmology \[9\].
The JT action for gravity coupled to matter reads:

\[ I = \int d^2x \sqrt{-g} \left( \eta \frac{\mathcal{R} - \Lambda}{16\pi G} + L_M \right), \]  

(1)

where \( \mathcal{R} \) is the curvature scalar, \( \Lambda \) is a cosmological constant and \( G \) is the gravitational coupling constant, which is dimensionless and may be absorbed in a redefinition of \( \eta \). \( L_M \) is the action of two-dimensional matter. In analogy with the four-dimensional case, this can be taken to be proportional to \( -\rho \), with \( \rho \) the mass density. However, the coupling of the matter with the dilaton \( \eta \) is not fixed a priori. In fact, one may choose \( L_M = -\eta^\alpha \rho \), for any \( \alpha \), giving rise to a large variety of inequivalent models. In the following we shall consider the two simplest possibilities: \( \alpha = 0 \) (minimal coupling) and \( \alpha = 1 \) (conformal coupling). These are the most interesting for physics since are the closest to those employed in higher dimensions.

2 Minimally coupled matter

In this section, we consider the case of minimally coupled matter. This is the most straightforward generalization of the higher dimensional formalism. However, as we shall see, the metric decouples from matter in this model.

The field equations can be obtained by varying the action (1), with \( L_M = -\rho \). As in general relativity, in order to obtain the correct energy-momentum tensor the matter action must be subjected to a constrained variation [13]. The field equation then read

\[ \mathcal{R} = \Lambda, \]  

(2)

\[-(\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^2)\eta + \frac{\Lambda}{2} g_{\mu\nu} \eta = 8\pi G T_{\mu\nu}, \]  

(3)

with \( T_{\mu\nu} = pg_{\mu\nu} + (\rho + p)u_\mu u_\nu \). The conservation law for the energy-momentum tensor, \( \nabla^\mu T_{\mu\nu} = 0 \), can be obtained combining the covariant derivative of (3) with (2) and recalling that in two dimensions \( [\nabla^\mu, \nabla_\nu] A_\mu = \frac{1}{2} \mathcal{R} A_\nu \), for any \( A_\mu \).

It is evident from the field equations that in this model the only dependence on the matter content is through the dilaton, while the metric function is unaffected and depends only on the value of the cosmological constant. This is in accordance with our interpretation of the dilaton as a fundamental field of the theory.
We look for a solution of the form

$$ds^2 = -dt^2 + R^2(t)dr^2, \quad \eta = \eta(t),$$

(4)

The coordinate $r$ can be either compact, $0 \leq r \leq 2\pi$, or non-compact. This choice does not affect the field equations, that take the form

$$\frac{\dot{R}}{R} = \frac{\Lambda}{2},$$

(5)

$$\frac{\dot{R}}{R} \eta' = \frac{\Lambda}{2} \eta + 8\pi G \rho,$$

(6)

$$\ddot{\eta} = \frac{\Lambda}{2} \eta - 8\pi G p.$$ 

(7)

Eq. (5) admits the first integral

$$\dot{R}^2 - \frac{\Lambda}{2} R^2 = a,$$

(8)

that can be immediately integrated to yield $R$. Only two of the field equations are independent. In fact, differentiating (6) and combining with (5) and (7), one obtains the energy-momentum conservation law

$$\dot{\rho} = -(p + \rho) \dot{R}/R.$$ 

(9)

For a perfect fluid, the equation of state is $p = \gamma \rho$, with $0 \leq \gamma \leq 1$, where $\gamma = 0$ for dust and $\gamma = 1$ for radiation. Substituting in (5) and integrating, one obtains

$$\rho R^{1+\gamma} = M/2\pi,$$

(10)

with $M$ an integration constant. Substituting again in (6), one has

$$\dot{R} \dot{\eta} - \frac{\Lambda}{2} R \eta = 4GMR^{-\gamma},$$

(11)

from which one can easily determine $\eta$. We distinguish three cases: 1) $\Lambda = 0$, 2) $\Lambda < 0$, 3) $\Lambda > 0$. 

3
2.1 $\Lambda = 0$

There are two possible solutions: either both $R$ and $\eta$ are constant, or $R = At$, with $A = \sqrt{a}$ and $2$

$$\eta = \frac{4GM}{A^{1+\gamma}} \left( \frac{t^{1-\gamma}}{1-\gamma} - b \right) \quad \text{if } \gamma \neq 1,$$

$$\eta = \frac{4GM}{A^2} (\log t - b) \quad \text{if } \gamma = 1,$$

with $b$ an integration constant. In both cases the spacetime is flat. However, if $b > 0$, the time-dependent solutions have a zero of the dilaton at $t_0 = [(1 - \gamma)b]^{1/(1-\gamma)}$, which we interpret as an initial singularity. These solutions can therefore be viewed as expanding universes that begin at time $t_0$ with finite size $At_0$.  

2.2 $\Lambda < 0$

Integrating (8), one obtains

$$R = A \sin \lambda t,$$

where $\lambda = \sqrt{-\Lambda/2}$, $A = \sqrt{a}/\lambda$. From (11), one can write down $\eta$ in terms of hypergeometric functions,

$$\eta = \eta_0 \cos \lambda t + \frac{4GM}{A^{1+\gamma} \lambda^2} \mathbf{F}\left(-\frac{1}{2}, \frac{1+\gamma}{2}, \frac{1}{2}, \cos^2 \lambda t\right),$$

with $\eta_0$ an integration constant. In particular,

$$\eta = \eta_0 \cos \lambda t + \frac{4GM}{A^{1+\gamma}} \sin \lambda t \quad \text{if } \gamma = 0,$$

$$\eta = \eta_0 \cos \lambda t + \frac{4GM}{A^{1+\gamma}} \left(1 + \cos \lambda t \ \log \tan \frac{M}{2}\right) \quad \text{if } \gamma = 1.$$  

The metric is that of anti-de Sitter spacetime and describes periodic solutions. However, for any $\gamma$, there is a range of values of $\eta_0$ for which the dilaton has a zero (corresponding to a physical singularity) at a finite time $t = t_0$. Such solutions can then be interpreted as universes which begin expanding with finite initial size at $t_0$ and then recollapse. Also solutions with both an initial and a final singularity of dilatonic type may occur for some values of the parameters.

\[\text{\footnotesize{\textsuperscript{2}} Here and in the following we choose the origin of time so that it simplifies the expression of the solutions.\textsuperscript{2}}}\]
2.3 $\Lambda > 0$

In this case, the integration of $[\text{8}]$ yields different results depending on the sign of $a$. Defining $\lambda = \sqrt{\Lambda / 2}$, the metric function can assume three qualitatively different forms:

- $a < 0$ \( R = A \cosh \lambda t \),
- $a = 0$ \( R = A e^{\lambda t} \),
- $a > 0$ \( R = A \sinh \lambda t \),

to which correspond dilaton solutions that are given, respectively, by

\[
\eta = \eta_0 \sinh \lambda t + \frac{4GM}{\lambda^2 A^{1+\gamma}} F \left( -\frac{1}{2}, \frac{1+\gamma}{2}, \frac{1}{2}, -\sinh^2 \lambda t \right),
\]

\[
\eta = \eta_0 e^{\lambda t} - \frac{4GM}{(2+\gamma)\lambda^2 A^{1+\gamma}} e^{-(1+\gamma)\lambda t},
\]

\[
\eta = \eta_0 \cosh \lambda t - \frac{4GM}{(2+\gamma)\lambda^2 (A \cosh \lambda t)^{1+\gamma}} F \left( 1 + \frac{\gamma}{2}, \frac{1+\gamma}{2}, 2 + \frac{\gamma}{2}, \frac{1}{\cosh^2 \lambda t} \right).
\]

In particular, for $\gamma = 0$, the solutions reduce to

\[
\eta = \eta_0 \sinh \lambda t - \frac{4GM}{\lambda^2 A} \cosh \lambda t,
\]

\[
\eta = \eta_0 e^{\lambda t} - \frac{2GM}{\lambda^2 A} e^{-\lambda t},
\]

\[
\eta = \eta_0 \cosh \lambda t + \frac{4GM}{\lambda^2 A} \sinh \lambda t.
\]

For $\gamma = 1$ one has instead

\[
\eta = \eta_0 \sinh \lambda t - \frac{4GM}{\lambda^2 A^2} (1 + \sinh \lambda t \arctan \sinh \lambda t),
\]

\[
\eta = \eta_0 e^{\lambda t} - \frac{4GM}{3\lambda^2 A^2} e^{-2\lambda t},
\]

\[
\eta = \eta_0 \cosh \lambda t + \frac{4GM}{\lambda^2 A^2} (1 + \cosh \lambda t \log \tanh \lambda t / 2).
\]

In all cases, the solutions are locally de Sitter, but have different global properties. In particular, $R$ has a zero at finite time if $a > 0$, but not in the other cases. Also for $\Lambda > 0$ there is a large range of values of $\eta_0$ for which all solutions have a zero of the dilaton at time $t_0$, where $R \neq 0$; the physical behaviour of all of them is similar, and corresponds to universes starting at time $t_0$ with a finite size and expanding forever.
3 Conformal coupling

We consider now the case in which the matter is linearly coupled to the dilaton, i.e. \( L_M = -\eta \rho \). This model is invariant under rescaling of the dilaton, which is fixed up to a multiplicative constant \( \eta_0 \). The field equations read

\[
\mathcal{R} = \Lambda + 16\pi G \rho, \quad \quad (12)
\]

\[-(\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^2) \eta + \frac{\Lambda}{2} g_{\mu\nu} \eta = 8\pi G \eta T_{\mu\nu}. \quad \quad (13)
\]

Contrary to the model studied in the previous section, one has now a direct coupling between the matter density and the curvature of spacetime.

Substituting the ansatz (4) into the field equation, one obtains

\[
\frac{\ddot{R}}{R} = \frac{\Lambda}{2} + 8\pi G \rho, \quad \quad (14)
\]

\[
\frac{\dot{R}}{R} \dot{\eta} = \frac{\Lambda}{2} \eta + 8\pi G \eta \rho, \quad \quad (15)
\]

\[
\ddot{\eta} = \frac{\Lambda}{2} \eta - 8\pi G \eta \rho. \quad \quad (16)
\]

Combining (14-16), one can check that the conservation law (9) is still valid and hence also (10) holds. Eq. (15) can then be written as

\[
\dot{R} \dot{\eta} = \left( \frac{\Lambda}{2} R + 4GM \gamma R^{-\gamma} \right) \eta. \quad \quad (17)
\]

Moreover, using (11), eq. (14) can be integrated once to read

\[
\dot{R}^2 - \frac{\Lambda}{2} R^2 - \frac{8GM}{1-\gamma} R^{1-\gamma} = a, \quad \quad \text{if } \gamma \neq 1,
\]

\[
\dot{R}^2 - \frac{4}{3} R^2 - 8GM \log R = a, \quad \quad \text{if } \gamma = 1, \quad \quad (18)
\]

with \( a \) an integration constant. The equations above can be integrated in terms of elementary functions only when \( \gamma = 0 \) or when \( a = 0 \) and \( \gamma \neq 1 \), so in the following we shall limit our considerations to these cases. Again, we must distinguish three possibilities according to the sign of the cosmological constant.
3.1 $\Lambda = 0$

In this case the integration of (18) for $\gamma = 0$ gives

$$R = 2GMt^2 - b,$$

with $b = a/8GM$. Integration of (17) gives for the dilaton

$$\eta = \eta_0 t.$$

The dilaton is singular at $t = 0$. Moreover, if $b < 0$, the metric is always regular, while, if $b \geq 0$, a curvature singularity is located at $t_0 = \sqrt{b/2GM}$. Depending on the value of $b$, the universe begins at $t = 0$ with finite size, or at $t = t_0$ with zero size, and expands forever.

A special solution can be obtained also for $0 < \gamma < 1$ if the integration constant $a$ vanishes. In that case,

$$R = \left(\frac{2GM(1 + \gamma)^2}{(1 - \gamma)}\right)^{1/(\gamma+1)} |t|^{2/(\gamma+1)}, \quad \eta = \eta_0 |t|^{(1+\gamma)/(1-\gamma)}.$$

Both the metric function $R$ and the dilaton have a zero at $t = 0$, corresponding to a physical singularity, and grow monotonically with time.

3.2 $\Lambda < 0$

If $\gamma = 0$, the solution is

$$R = A \sin \lambda t + 4GM/\lambda^2,$$

$$\eta = \eta_0 \cos \lambda t,$$

where $A = \lambda^{-1} \sqrt{a + (4GM/\lambda)^2}$. A dilaton singularity occurs at $t = 0$ and a curvature singularity at $t_0 = \text{arccos}(4GM/\lambda^2)$, if $A < 4GM/\lambda^2$. The universe begins expanding at $t = 0$ or $t = t_0$ and then recollapses.

If $\gamma \neq 0$ one can find also static solutions. These have positive $\rho$, but negative pressure, namely $\rho = -p = -\Lambda/16G\gamma$.

Other exact solutions can be found if $a = 0$ and $\gamma < 1$. They read

$$R = \left|\frac{8GM}{(1 - \gamma)\lambda} \sin \frac{(1 + \gamma)\lambda t}{2}\right|^{2/(1+\gamma)},$$

7
\[ \eta = \eta_0 \left| \cos \left( \frac{(1 + \gamma)\lambda t}{2} \right) \right|^{2/(1+\gamma)} \left| \sin \left( \frac{(1 + \gamma)\lambda t}{2} \right) \right|^{(1-\gamma)/(1+\gamma)}. \]

At \( t = 0 \) both the metric and the dilaton are singular. At \( t = \pi/(1 + \gamma)\lambda \), where the universe reaches its maximum expansion, the dilaton has a zero, and hence a physical singularity occurs.

### 3.3 \( \Lambda > 0 \)

For positive cosmological constant and \( \gamma = 0 \), one can obtain three different solutions depending on the value of \( a \) being lower, equal or greater than \((4GM/\lambda)^2\). Defining \( A = \lambda^{-1} \sqrt{|a - (4GM/\lambda)^2|} \), one has, respectively,

\[
R = A \cosh \lambda t - 4GM/\lambda^2, \\
\eta = \eta_0 \sinh \lambda t.
\]

\[
R = \lambda^{-2}(e^\lambda - 4GM), \\
\eta = \eta_0 e^{\lambda t}.
\]

\[
R = A \sinh \lambda t - 4GM/\lambda^2, \\
\eta = \eta_0 \cosh \lambda t.
\]

The first case is similar to the previous ones: a curvature singularity is present if \( A < 4GM/\lambda^2 \) (i.e. \( a < 0 \)), while the dilaton is always singular at \( t = 0 \). In the remaining cases, a curvature singularity occurs at the zero of \( R \), while \( \eta \) is regular everywhere. In all cases the universe expands forever.

If one requires positive \( \rho \), no static solutions exist for \( \Lambda > 0 \). Special solutions can be found for \( a = 0, \gamma \neq 1 \). They read

\[
R = \left| \frac{8GM}{(1 - \gamma)\lambda} \sinh \left( \frac{(1 + \gamma)\lambda t}{2} \right) \right|^{2/(1+\gamma)}, \\
\eta = \eta_0 \left| \cosh \left( \frac{(1 + \gamma)\lambda t}{2} \right) \right|^{2/(1+\gamma)} \left| \sinh \left( \frac{(1 + \gamma)\lambda t}{2} \right) \right|^{(1-\gamma)/(1+\gamma)}. \]
These solutions are qualitatively similar to those occurring for $\Lambda = 0, a = 0$. A curvature and a dilaton singularity occur at $t = 0$, after which $R$ and $\eta$ grow monotonically.

4 Particle horizons

An important property of cosmological models is the existence of particle horizons, defined as the location of the most distant place from which a light ray can have reached us since the beginning of the universe. It is easy to see that its distance is proportional to the integral

$$\int_{t_0}^{t} \frac{dt'}{R(t')}$$

where $t_0$ is the initial time. If this integral is infinite, no particle horizon is present.

For our models, one must distinguish two cases: when the initial singularity is a dilatonic one, the metric is regular at $t_0$ and therefore the integral (19) cannot diverge, and a particle horizon always exists. When one has an initial curvature singularity, instead, a computation of the integral (19) shows that in all cases, except the de Sitter solutions $\Lambda > 0, a < 0$ of section 2.3, it diverges at the initial singularity, and hence no particle horizon is present.

5 Final remarks

We have studied two-dimensional cosmologies in the context of the Jackiw-Teitelboim model, in the case of minimally coupled and conformally coupled matter. All solutions present initial singularities. However, these can be either of metric or dilatonic nature. In the latter case, the universe can have a finite size at its beginning. The universe expands forever or recollapses depending on the value of the cosmological constant. In general, a particle horizon only exists in the case of dilatonic singularities.
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