KATZ-RADON TRANSFORM OF $\ell$-ADIC REPRESENTATIONS

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Abstract. We prove a simple explicit formula for the local Katz-Radon transform of an $\ell$-adic representation of the Galois group of the fraction field of a strictly henselian discrete valuation ring with positive residual characteristic, which can be defined as the local additive convolution with a fixed tame character. The formula is similar to one proved by D. Arinkin in the $D$-module setting, and answers a question posed by N. Katz.

1. Introduction

In [10, 3.4.1], N. Katz defines some functors on the category of continuous $\ell$-adic representations of the inertia groups $I_0$ and $I_{\infty}$ of the projective line over $\bar{k}$ at 0 and infinity, where $\bar{k}$ is the algebraic closure of a finite field of characteristic $p$ and $\ell$ is a prime different from $p$. These functors arise during his study of middle convolution of sheaves on the affine line and, roughly speaking, correspond to locally convolving a representation with a fixed tame character $\mathcal{L}_\chi$ of $I_0$ or $I_{\infty}$. They are defined using G. Laumon’s local Fourier transform functors, and in fact correspond to taking the tensor product with the conjugate tame character $\mathcal{L}_{\bar{\chi}}$ on the other side of the equivalence of categories given by these functors. Katz asks [10, 3.4.1] whether there is a simple expression for the functors defined in this way.

Recently, D. Arinkin [1] has studied the analog of Katz’s functor in $D$-module theory: if $K$ is a field of characteristic 0, $K((x))$ is the field of Laurent series over $K$ and $D_x$ the ring of differential operators with coefficients in $K((x))$, the local Katz-Radon transform for a given $\lambda \in K - Z$ is an equivalence of categories $\rho_\lambda : D_x\text{-mod} \to D_x\text{-mod},$ originally defined in [5]. Arinkin proves the simple formula [1, Theorem C]

$$\rho_\lambda(F) \cong F \otimes \mathcal{K}^{(\alpha+1)}$$

for any $F \in D_x\text{-mod}$ with a single slope $\alpha$, where $\mathcal{K}^\mu$ is the Kummer $D_x$-module of rank 1 generated by $e$, on which the derivative acts by

$$\frac{d}{dx} e = \frac{\mu}{x} e.$$

In this article we will prove a similar formula in the $\ell$-adic case. More precisely, for a fixed tame $\ell$-adic character $\mathcal{L}_\chi$ and an $\ell$-adic representation $F$ of $I_0$, let

$$\rho_\chi(F) := FT^{-1}_{(0,\infty)} (\mathcal{L}_\chi \otimes FT^\psi_{(0,\infty)} F)$$

where $FT^\psi_{(0,\infty)}$ denotes Laumon’s local Fourier transform functor. If $F$ has a single slope $\alpha = c/d$ (with $c, d$ relatively prime positive integers), we will prove that there is an isomorphism of $I_0$-representations

$$\rho_\chi(F) \cong F \otimes \mathcal{L}_\chi^{(\alpha+1)}$$

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where $\mathcal{L}_{\chi}^{\otimes (a+1)}$ is any $d$-th root of the character $\mathcal{L}_{\chi}^{\otimes (c+d)}$.

For a large class of representations $\mathcal{F}$ of $I_0$ (in particular for many of those who appear in applications), the isomorphism can be proven via the explicit formulas for the local Fourier transforms given by L. Fu [3] and A. Abbes and T. Saito [2]. In this article we take a different approach that works for any $\mathcal{F}$, and is independent of any explicit expression for the local Fourier transforms.

2. The Katz-Radon transform

Fix a finite field $k$ of characteristic $p > 0$ and an algebraic closure $\bar{k}$. Let $\mathbb{P}_k^1$ be the projective line over $\bar{k}$ and, for every $t \in \mathbb{P}_k^1(\bar{k}) = \bar{k} \cup \{\infty\}$, denote by $I_t$ its inertia group at $t$: for $t \neq \infty$, if $x - t$ denotes a local coordinate at $t$, it is the Galois group of the fraction field of the henselization of the local ring $\bar{k}[x]_{(x-t)}$. We have an exact sequence [8, 1.0]

$$0 \to P_t \to I_t \to \prod_{\ell \neq p} \mathbb{Z}_\ell(1) \to 0$$

for every $t \in \mathbb{P}_k^1(\bar{k})$, where $P_t$ is the only $p$-Sylow subgroup of $I_t$. Moreover, there is a canonical filtration of $I_t$ by the higher ramification groups

$$I_t^{(r)} \supseteq I_t^{(s)}$$

for $0 \leq r < s \in \mathbb{R}$

which are normal in $I_t$.

Fix a prime $\ell \neq p$, and denote by $\mathcal{R}_t$ the abelian category of continuous $\ell$-adic representations of $I_t$ (i.e. continuous representations $\mathcal{F} : I_t \to \text{GL}_n(\mathbb{Q}_\ell)$, whose image is in $\text{GL}_n(E_\lambda)$ for some finite extension $E_\lambda$ of $\mathbb{Q}_\ell$). For every irreducible $\mathcal{F} \in \mathcal{R}_t$, the slope of $\mathcal{F}$ is $\inf\{r \geq 0 | \mathcal{F}_{I_t^{(r)}} \text{ is trivial}\}$. It is a non-negative rational number. In general, the slopes of $\mathcal{F}$ are the slopes of the irreducible components of $\mathcal{F}$. For every $\mathcal{F}$ there is a canonical direct sum decomposition [8, Lemma 1.8]

$$(1) \quad \mathcal{F} \cong \bigoplus_{r \geq 0} \mathcal{F}^r$$

with $\mathcal{F}^r$ having a single slope $r$. The slope 0 (tame) part will be denoted by $\mathcal{F}^t$. $\mathcal{F}$ is said to be tame (respectively totally wild) if $\mathcal{F} = \mathcal{F}^t$ (resp. $\mathcal{F}^t = 0$).

For every $r \geq 0$ let $\mathcal{R}_t^r$ denote the full subcategory of $\mathcal{R}_t$ consisting of representations with a single slope $r$. We have a decomposition

$$\mathcal{R}_t = \bigoplus_{r \geq 0} \mathcal{R}_t^r$$

in the sense that every $\mathcal{F} \in \mathcal{R}_t$ has a decomposition [1] and $\text{Hom}_{\mathcal{R}_t}(\mathcal{F}, \mathcal{G}) = 0$ if $\mathcal{F} \in \mathcal{R}_t^r$, $\mathcal{G} \in \mathcal{R}_t^s$ and $r \neq s$ [8, Proposition 1.1].

Let $k' \subseteq \bar{k}$ be a finite extension of $k$, and $\chi : k'^* \to \mathbb{Q}_\ell^*$ a multiplicative character. By [1, 1.4-1.8] there is an associated smooth Kummer sheaf $\mathcal{L}_\chi$ on $\mathcal{G}_{m,k}$, which is a tame character of $I_0$ (and of $I_\infty$) of the same order as $\chi$. If $k'' \subseteq k'$ is another extension, the sheaves defined by $\chi$ and $\chi \circ \text{N}_{k''/k} : k'^* \to \mathbb{Q}_\ell^*$ are isomorphic. Moreover, every tame character of $I_0$ (and of $I_\infty$) can be obtained in this way. Whenever we speak about a tame character of $I_0$, we will implicitly assume that we have made a choice of such a finite extension of $k$ and of a character.
Fix a non-trivial additive character \( \psi : k \to \mathbb{Q}_k^\times \). The local Fourier transform functors, defined by G. Laumon in [11], give equivalences of categories

\[
\text{FT}^\psi_{(0, \infty)} : \mathcal{R}_0 \to \mathcal{R}_{<1}^{<1},
\]

\[
\text{FT}^\psi_{(\infty, 0)} : \mathcal{R}_{>1}^>1 \to \mathcal{R}_{\infty}^>1
\]
and

\[
\text{FT}^\psi_{(\infty, 0)} : \mathcal{R}_{<1}^{<1} \to \mathcal{R}_0
\]

(where \( \mathcal{R}_{<1}^{<1} = \bigoplus_{r<1} \mathcal{R}_{r}^r \) and \( \mathcal{R}_{>1}^>1 = \bigoplus_{r>1} \mathcal{R}_{r}^r \)) that describe the relationship between the local monodromies of an \( \ell \)-adic sheaf on \( \mathbb{A}^1_k \) and its Fourier transform with respect to \( \psi \). The Katz-Radon transform is defined in terms of them.

**Definition 2.1.** Fix a tame character \( \chi \) of \( I_0 \). The (local) Katz-Radon transform (with respect to \( \chi \)) is the functor \( \rho_\chi : \mathcal{R}_0 \to \mathcal{R}_0 \) given by

\[
\rho_\chi(F) = \text{FT}^{\psi^{-1}}_{(0, \infty)}(\text{FT}^\psi_{(0, \infty)} \chi \otimes \text{FT}^\psi_{(0, \infty)} F) = \text{FT}^{\psi^{-1}}_{(0, \infty)} \chi \otimes \text{FT}^\psi_{(0, \infty)} F.
\]

The Katz-Radon transform is an auto-equivalence of the category \( \mathcal{R}_0 \) (since it is a composition of three equivalences of categories). It preserves dimensions and slopes, and for tame \( F \) it is given by \( \rho_\chi(F) = F \otimes \mathcal{L}_\chi \) [10, 3.4.1]. For totally wild \( F \), it can be interpreted as the “local additive convolution” of \( F \) and \( \mathcal{L}_\chi \) [10, 3.4.3]: if we extend \( F \) to a smooth sheaf on \( \mathbb{G}_{m, k} \), tamely ramified at infinity, then \( \rho_\chi(F) \) is the wild part of the local monodromy at 0 of \( F \otimes \mathcal{L}_\chi \), where

\[
F \otimes \mathcal{L}_\chi = R^1\sigma_!(F \boxtimes \mathcal{L}_\chi)
\]

and \( \sigma : \mathbb{A}^1_k \to \mathbb{A}^1_k \) denotes the addition map (in [10], the “middle convolution” is used instead, but that one differs from the one used here only by Artin-Shreier components, which are smooth at 0 and therefore do not affect the local monodromy). Notice that, in particular, \( \rho_\chi \) is independent of the choice of the additive character \( \psi \).

More intrinsically, it can be described in terms of vanishing cycles functors [11, 2.7.2]: if \( X = \mathbb{A}^2_{(0, 0)} \) (respectively \( S = \mathbb{A}^1_{(0)} \)) denotes the henselization of \( \mathbb{A}^2_k \) at \((0, 0)\) (resp. the henselization of \( \mathbb{A}^1_k \) at 0) then \( \rho_\chi(F) \cong R^1\Phi(\sigma, F \boxtimes \mathcal{L}_\chi)(0, 0) \), where \( R\Phi(\sigma, F \boxtimes \mathcal{L}_\chi) \) is the vanishing cycles complex for the addition map \( \sigma : X \to S \) with respect to the sheaf \( F \boxtimes \mathcal{L}_\chi \) on \( X \).

Similarly, it also has an interpretation as a “local multiplicative convolution” [12, Corollary 5.6]: if \( X = \mathbb{G}^2_{m,(1, 1)} \) (respectively \( S = \mathbb{G}_{m,(1)} \)) denotes the henselization of \( \mathbb{G}_{m,k} \) at \((1, 1)\) (resp. the henselization of \( \mathbb{G}_{m,k} \) at 1) then \( \rho_\chi(F) \cong R^1\Phi(\mu, F \boxtimes \mathcal{L}_\chi)(1, 1) \), where \( R\Phi(\mu, F \boxtimes \mathcal{L}_\chi) \) is the vanishing cycles complex for the multiplication map \( \mu : X \to S \) via the isomorphism \( I_0 \cong I_1 \) that maps the uniformizer \( x \) at 0 to the uniformizer \( x - 1 \) at 1.

The main result of this article is the following simple expression for \( \rho_\chi \):

**Theorem 2.2.** Let \( F \in \mathcal{R}_0 \) be totally wild with a single slope \( a > 0 \). Write \( a = c/d \), where \( c \) and \( d \) are relatively prime positive integers. Let \( \mathcal{L}_\eta \) be any tame character of \( I_0 \) such that \( \mathcal{L}_\eta^{\otimes d} = \mathcal{L}_\chi^{\otimes(c+d)} \). Then

\[
\rho_\chi(F) \cong F \otimes \mathcal{L}_\eta.
\]
In other words, we have the formula
\[ \rho_\chi(\mathcal{F}) \cong \mathcal{F} \otimes \mathcal{L}_\chi^{(a+1)} \]
where \( \mathcal{L}_\chi^{(a+1)} \) stands for “any character that can reasonably be called \( \mathcal{L}_\chi^{(a+1)} \).”

By the decomposition \( \mathcal{R}_0 = \bigoplus_{r \geq 0} \mathcal{R}_r^0 \), this determines \( \rho_\chi(\mathcal{F}) \) for any \( \mathcal{F} \in \mathcal{R}_0 \), thus answering the question posed by N. Katz in [10, 3.4.1].

A question that remains open is the following: in the article we prove that \( \rho_\chi(\mathcal{F}) \cong \mathcal{F} \otimes \mathcal{L}_\mu \), independently for any \( \mathcal{F} \) with slope \( a \). So the functors \( \mathcal{R}_0^a \to \mathcal{R}_0^0 \) given by \( \rho_\chi \) and \( (-) \otimes \mathcal{L}_0 \) map any \( \mathcal{F} \) to isomorphic objects. Is there an actual isomorphism of functors between them? In the affirmative case, is there a simple way to construct it?

3. PROOF OF THE MAIN THEOREM

In this section we will prove theorem [222]. We will start with the case where \( \mathcal{F} \in \mathcal{R}_0 \) is irreducible.

**Lemma 3.1.** Let \( \mathcal{F} \in \mathcal{R}_0 \). Then \( \mathcal{F}^i \neq 0 \) if and only if there exists \( \epsilon > 0 \) such that for every \( \mathcal{G} \in \mathcal{R}_0 \) with a single slope \( b \in (0, \epsilon) \) we have
\[ \text{Swan}(\mathcal{F} \otimes \mathcal{G}) > \text{Swan}(\mathcal{F}) \dim(\mathcal{G}). \]

**Proof.** Suppose that \( \mathcal{F}^i \neq 0 \), and let \( a_0 = 0 < a_1 < \cdots < a_r \) be the slopes of \( \mathcal{F} \), with multiplicities \( n_0, n_1, \ldots, n_r \). Then \( \text{Swan}(\mathcal{F}) = \sum n_i a_i \). Let \( \epsilon = a_1 \). Then for every \( \mathcal{G} \in \mathcal{R}_0 \) with a single slope \( b \in (0, \epsilon) \) the tensor product \( \mathcal{F} \otimes \mathcal{G} \) has slopes \( b < a_1 < \cdots < a_r \) with multiplicities \( n_0 m, n_1 m, \ldots, n_r m \) where \( m = \dim(\mathcal{G}) \) by [S] Lemma 1.3. Therefore
\[ \text{Swan}(\mathcal{F} \otimes \mathcal{G}) = n_0 m b + \sum_{i=1}^r n_i m a_i > \sum_{i=1}^r n_i m a_i = \text{Swan}(\mathcal{F}) \dim(\mathcal{G}). \]

Conversely, suppose that \( \mathcal{F}^i = 0 \), and let \( a_1 < \cdots < a_r \) be the slopes of \( \mathcal{F} \). Then for every \( \mathcal{G} \in \mathcal{R}_0 \) with a single slope \( b \in (0, a_1) \) the tensor product \( \mathcal{F} \otimes \mathcal{G} \) has the same slopes as \( \mathcal{F} \) by [S] Lemma 1.3, and in particular \( \text{Swan}(\mathcal{F} \otimes \mathcal{G}) = \text{Swan}(\mathcal{F}) \dim(\mathcal{G}) \). This proves the lemma, since for every \( \epsilon > 0 \) there exist representations in \( \mathcal{R}_0 \) with slope \( b \in (0, \epsilon) \) (for instance, one may take \( [n_i] \mathcal{H} \), where \( \mathcal{H} \in \mathcal{R}_0 \) has slope \( a > 0 \) and \( n \) is a prime to \( p \) integer greater than \( a/\epsilon \) [S] 1.13.2]). \( \square \)

For any two objects \( K, L \in \mathcal{D}_c^b(\mathbb{A}_k^1, \bar{\mathbb{Q}}_\ell) \), we will denote by \( K * L \in \mathcal{D}_c^b(\mathbb{A}_k^1, \bar{\mathbb{Q}}_\ell) \) their additive convolution:
\[ K * L = R\sigma_1(K \boxtimes L) \]
where \( \sigma : \mathbb{A}_k^2 \to \mathbb{A}_k^1 \) is the addition map.

**Lemma 3.2.** Let \( K, L, M \in \mathcal{D}_c^b(\mathbb{A}_k^1, \bar{\mathbb{Q}}_\ell) \). Then
\[ R\Gamma_c(\mathbb{A}_k^1, (K * L) \otimes M) \cong R\Gamma_c(\mathbb{A}_k^1, K \otimes ((\tau_{-1})^* L) \otimes M) \]
where \( \tau_{-1} : \mathbb{A}_k^1 \to \mathbb{A}_k^1 \) is the additive inversion.

**Proof.** We have
\[ R\Gamma_c(\mathbb{A}_k^1, (K * L) \otimes M) = R\Gamma_c(\mathbb{A}_k^1, R\sigma_1(K \boxtimes L) \otimes M) = R\Gamma_c(\mathbb{A}_k^1, R\sigma((K \boxtimes L) \otimes \sigma^* M)) = R\Gamma_c(\mathbb{A}_k^1, (K \boxtimes L) \otimes \sigma^* M) \]
by the projection formula. If $\pi_1, \pi_2: \mathbb{A}_k^2 \to \mathbb{A}_k^1$ are the projections then
\[
\text{RG}_c(\mathbb{A}_k^2, (K \boxtimes L) \otimes \sigma^* M) = \text{RG}_c(\mathbb{A}_k^2, \pi_1^* K \otimes \pi_2^* L \otimes \sigma^* M).
\]
Consider the automorphism $\phi: \mathbb{A}_k^2 \to \mathbb{A}_k^2$ given by $(x, y) \mapsto (x + y, -y)$. Then $\sigma = \pi_1 \circ \phi$, $\pi_1 = \sigma \circ \phi$ and $\tau_{-1} \circ \pi_2 = \pi_2 \circ \phi$. It follows that
\[
\text{RG}_c(\mathbb{A}_k^2, \pi_1^* K \otimes \pi_2^* L \otimes \sigma^* M) \cong \text{RG}_c(\mathbb{A}_k^2, \phi^* \pi_1^* K \otimes \phi^* \pi_2^* L \otimes \phi^* \sigma^* M) = \\
\cong \text{RG}_c(\mathbb{A}_k^1, \phi^* \pi_1^* K \otimes \phi^* \pi_2^* L \otimes \phi^* \sigma^* M) = \\
\cong \text{RG}_c(\mathbb{A}_k^1, K \otimes \phi(\pi_2^* L \otimes \pi_1^* M)) = \text{RG}_c(\mathbb{A}_k^1, K \otimes (\pi_2^* L \otimes M)).
\]

If $\mathcal{F}$ is a smooth $\mathcal{O}_\mathcal{E}$-sheaf on $\mathbb{G}_{m, \overline{k}}$ which is totally wild at 0, then for every $t \in \overline{k}$ the sheaf $\mathcal{F} \otimes \mathcal{L}(t-x)$ (extended by zero to $\mathbb{A}_k^1$) is totally wild at 0 and has no punctual sections (where $\mathcal{L}(t-x)$ is the pull-back of the Kummer sheaf $\mathcal{L}_\chi$ under the map $x \mapsto t-x$), so its only non-zero cohomology group with compact support is $H^1$. We conclude that the only non-zero cohomology sheaf of $\mathcal{F}[0] \ast \mathcal{L}(0) \in D^b_c(\mathbb{A}_k^1, \mathcal{O}_\mathcal{E})$ is $H^1 = R^1\sigma(\mathcal{F} \otimes \mathcal{L}_\chi)$. We will denote this sheaf by $\mathcal{F} \ast \mathcal{L}_\chi$.

**Lemma 3.3.** Let $\mathcal{F}, \mathcal{G} \in \mathcal{R}_0$. Then
\[
\text{Swan}(\rho_\chi(\mathcal{F}) \otimes \mathcal{G}) = \text{Swan}(\mathcal{F} \otimes \mathcal{G}).
\]

**Proof.** By additivity of the Swan conductor, we may assume that $\mathcal{F}$ is irreducible, and in particular that it has a single slope $a \geq 0$. If $a = 0$ then $\rho_\chi(\mathcal{F}) \cong \mathcal{F} \otimes \mathcal{L}_\chi$, so the equality is clear. Suppose that $a > 0$. By [7] Theorem 1.5.6, $\mathcal{F}$ and $\mathcal{G}$ can be extended to smooth sheaves on $\mathbb{G}_{m, \overline{k}}$, tamely ramified at infinity, which we will also denote by $\mathcal{F}$ and $\mathcal{G}$. Let $\mathcal{F}$ and $\mathcal{G}$ be also their extensions by zero to $\mathbb{A}_k^1$.

Using the compatibility between Fourier transform with respect to $\psi$ and convolution [11 Proposition 1.2.2.7], we have
\[
\mathcal{F} \ast \mathcal{L}_\chi = \text{FT}(\text{FT}^\psi(\mathcal{F} \otimes \mathcal{L}_\chi)) = \text{FT}(\text{FT}^\psi(\mathcal{F} \otimes \mathcal{L}_\chi)),
\]
where $\text{FT}^\psi \mathcal{F}$ denotes the “naive” Fourier transform in the sense of [8, 8.2], that is, the $(-1)$-th cohomology sheaf of the Fourier transform of $\mathcal{F}[1] \in D^b_c(\mathbb{A}_k^1, \mathcal{O}_\mathcal{E})$ (which is its only non-zero cohomology sheaf, since $\mathcal{F}$ is totally wild at zero and therefore it is Fourier [8 Lemma 8.3.1]).

Let $n$ be the rank of $\mathcal{F}$, and denote by $\mathcal{F}(\infty) \in \mathcal{R}_\infty$ its local monodromy at infinity, which is a tame representation of $I_\infty$. By Ogg-Shafarevich [5 Expos´ e X, Corollaire 7.12], $\text{FT}^\psi \mathcal{F}$ is smooth on $\mathbb{G}_{m, \overline{k}}$ of rank $na + n = n(a+1)$. By Laumon’s local Fourier transform theory [9 Theorem 13], $\text{FT}^\psi \mathcal{F}$ has a single slope $\frac{a}{\pi i}$ at infinity, with multiplicity $n(a+1)$, and its monodromy at 0 has a trivial part of dimension $na$ and its quotient is the dual $\mathcal{F}(\infty)$ of $\mathcal{F}(\infty)$. Then $\text{FT}^\psi \mathcal{F} \otimes \mathcal{L}_\chi$ also has a single slope $\frac{a}{\pi i}$ at infinity with multiplicity $n(a+1)$, and its monodromy $\mathcal{M}$ at 0 sits in an exact sequence
\[
0 \to \mathcal{L}_\chi^{\oplus na} \to \mathcal{M} \to \mathcal{F}(\infty) \otimes \mathcal{L}_\chi \to 0.
\]
Its inverse Fourier transform, by Ogg-Shafarevich, is smooth of rank $n(a+1)$ on $\mathbb{G}_{m, \overline{k}}$, and by local Fourier transform its wild part at 0 has slope $a$ with multiplicity $n$. 

In fact, this wild part is simply \( \rho_\chi(\mathcal{F}) \) by the additive convolution interpretation of \( \rho_\chi \). Its monodromy at infinity sits in an exact sequence

\[
0 \to \mathcal{L}_\chi^{\otimes n_0} \to (\mathcal{F} \ast \mathcal{L}_\chi)(\infty) \to \mathcal{F}(\infty) \otimes \mathcal{L}_\chi \to 0
\]

obtained from (3) by local Fourier transform.

So \( \mathcal{F} \ast \mathcal{L}_\chi \) has rank \( n(a + 1) \) on \( \mathbb{G}_{m, k} \), and its monodromy at 0 is the direct sum of \( \rho_\chi(\mathcal{F}) \) and a constant part of dimension \( na = \text{Swan}(\mathcal{F}) \). So

\[
\text{Swan}_0((\mathcal{F} \ast \mathcal{L}_\chi) \otimes \mathcal{G}) = \text{Swan}(\rho_\chi(\mathcal{F}) \otimes \mathcal{G}) + \text{Swan}(\mathcal{F})\text{Swan}(\mathcal{G}).
\]

In particular, by Ogg-Shafarevic, the Euler characteristic of the sheaf \((\mathcal{F} \ast \mathcal{L}_\chi) \otimes \mathcal{G}\) (extended by zero to \( \mathcal{L}_\chi \)) is 

\[
\chi = (\mathcal{F} \ast \mathcal{L}_\chi) \otimes \mathcal{G}\bigg|_{\mathbb{G}_{m, k}} \equiv \chi_{\rho_\chi(\mathcal{F})} + \chi_{\text{Swan}(\mathcal{F})}.\]

Proof. Let \( \hat{\mathcal{F}} \) be the dual representation. We claim that the tame part of \( \rho_\chi(\mathcal{F}) \otimes \hat{\mathcal{F}} \) is non-zero. By lemma 3.1, it suffices to show that there is an \( \epsilon > 0 \) such that for any \( \mathcal{G} \in \mathcal{R}_0 \) with slope \( b \in (0, \epsilon) \), \( \text{Swan}(\rho_\chi(\mathcal{F}) \otimes \hat{\mathcal{F}} \otimes \mathcal{G}) > \text{Swan}(\rho_\chi(\mathcal{F}) \otimes \hat{\mathcal{F}}) \dim(\mathcal{G}) \).

But by lemma 3.3, we have

\[
\text{Swan}(\rho_\chi(\mathcal{F}) \otimes \hat{\mathcal{F}} \otimes \mathcal{G}) = \text{Swan}(\mathcal{F} \otimes \hat{\mathcal{F}} \otimes \mathcal{G})
\]

and

\[
\text{Swan}(\rho_\chi(\mathcal{F}) \otimes \hat{\mathcal{F}}) = \text{Swan}(\mathcal{F} \otimes \hat{\mathcal{F}})
\]

and, since \( \hat{\mathcal{F}} \) is the dual of \( \mathcal{F} \), the tensor product \( \mathcal{F} \otimes \hat{\mathcal{F}} \) has a trivial quotient and, in particular, has non-trivial tame part. By lemma 3.1 there exists \( \epsilon > 0 \) such that for any \( \mathcal{G} \in \mathcal{R}_0 \) with slope \( b \in (0, \epsilon) \), \( \text{Swan}(\mathcal{F} \otimes \hat{\mathcal{F}} \otimes \mathcal{G}) > \text{Swan}(\mathcal{F} \otimes \hat{\mathcal{F}}) \dim(\mathcal{G}) \).

Since the tame part of \( \rho_\chi(\mathcal{F}) \otimes \hat{\mathcal{F}} \) is non-zero and it is a direct summand, it contains a tame character \( \mathcal{L}_\eta \) of \( I_0 \) as a subrepresentation. Then

\[
\rho_\chi(\mathcal{F}) \otimes \hat{\mathcal{F}} \otimes \mathcal{L}_\eta = \rho_\chi(\mathcal{F}) \otimes \mathcal{F} \otimes \mathcal{L}_\eta = \text{Hom}(\mathcal{F} \otimes \mathcal{L}_\eta, \rho_\chi(\mathcal{F}))
\]

contains a trivial subrepresentation, so \( \text{Hom}_{I_0}(\mathcal{F} \otimes \mathcal{L}_\eta, \rho_\chi(\mathcal{F})) \neq 0 \). Since both \( \rho_\chi(\mathcal{F}) \) and \( \mathcal{F} \otimes \mathcal{L}_\eta \) are irreducible, any non-zero \( I_0 \)-equivariant map \( \mathcal{F} \otimes \mathcal{L}_\eta \to \rho_\chi(\mathcal{F}) \) must be an isomorphism.

**Proposition 3.5.** Let \( \mathcal{F} \in \mathcal{R}_0 \) be totally wild and irreducible of dimension \( n \) and slope \( a \), and let \( \mathcal{L}_\eta \) be a tame character of \( I_0 \) such that \( \rho_\chi(\mathcal{F}) \equiv \mathcal{F} \otimes \mathcal{L}_\eta \). Then \( \mathcal{L}_\eta^{\otimes n} \equiv \mathcal{L}_\chi^{\otimes n(a + 1)} \).
Proof. Extend \( F \) to a smooth \( \ell \)-adic sheaf on \( \mathbb{G}_{m, \overline{k}} \), tamely ramified at infinity, also denoted by \( F \). Let \( F \) also denote its extension by zero to \( \mathbb{A}^1 \). By the proof of lemma 3.3, the sheaf \( F \ast L \chi \) is smooth on \( \mathbb{G}_{m, \overline{k}} \), its monodromy at 0 is the direct sum of \( \rho(F) \cong F \otimes L \overline{\eta} \) and a trivial part of dimension \( na \), and its monodromy at infinity sits in the exact sequence (4). Its determinant is then a smooth sheaf of rank 1 on \( \mathbb{G}_{m, \overline{k}} \), whose monodromy at 0 is \( \det(F) \otimes L_{\overline{\eta}}^\otimes n \), and whose monodromy at \( \infty \) is \( \det(F(\infty)) \otimes L_{\chi}^{\otimes(n+1)} \).

Then \( \det(F) \otimes L_{\overline{\eta}}^\otimes n \otimes \det(F \ast L \chi) \) is a rank 1 smooth sheaf on \( \mathbb{G}_{m, \overline{k}} \), with trivial monodromy at 0 and tamely ramified at infinity. Since the tame fundamental group of \( \mathbb{A}^1 \) is trivial, we conclude that

\[
\det(F \ast L \chi) \cong \det(F) \otimes L_{\overline{\eta}}^\otimes n
\]
as sheaves on \( \mathbb{G}_{m, \overline{k}} \). Comparing their monodromies at infinity gives the desired isomorphism.

It remains to show that any such \( L_\eta \) works.

**Lemma 3.6.** Let \( F \in R_0 \) be irreducible of dimension \( n \), and let \( L_\eta \) be a tame character of \( I_0 \) such that \( L_{\overline{\eta}}^\otimes n \) is trivial. Then \( F \otimes L_\eta \cong F \).

**Proof.** Write \( n = n_0a^n \), where \( a \geq 0 \) and \( n_0 \) is prime to \( p \). Since the \( p \)-th power operation permutes the tame characters of \( I_0 \) preserving their order, \( L_{\overline{\eta}}^\otimes n_0 \) must be the trivial character. Now by \([5, 1.14.2]\), \( F \) is induced from a \( p^n \)-dimensional representation \( G \) of \( I_0(n_0) \), the unique open subgroup of \( I_0 \) of index \( n_0 \).

\[
F \otimes L_\eta = (\text{Ind}_{I_0(n_0)}^I G) \otimes L_\eta \cong \text{Ind}_{I_0(n_0)}^I (G \otimes \text{Res}_{I_0(n_0)}^I L_\eta) = \text{Ind}_{I_0(n_0)}^I (G) = F
\]
since the restriction of \( L_\eta \) to \( I_0(n_0) \) is trivial.

We can now finish the proof of theorem 2.2 for irreducible representations.

**Proposition 3.7.** Let \( F \in R_0 \) be irreducible of slope \( a > 0 \). Write \( a = c/d \), where \( c \) and \( d \) are relatively prime positive integers. Let \( L_\eta \) be any tame character of \( I_0 \) such that \( L_{\overline{\eta}}^\otimes d = L_{\chi}^{\otimes(c+d)} \). Then

\[
\rho_\chi(F) \cong F \otimes L_\eta.
\]

**Proof.** Let \( n \) be the dimension of \( F \). By propositions 3.4 and 3.5, there exists a tame character \( L_{\eta'} \) of \( I_0 \) such that \( \rho_\chi(F) \cong F \otimes L_{\eta'} \), and \( L_{\overline{\eta'}}^\otimes n \cong L_{\chi}^{\otimes n(a+1)} \). Since the Swan conductor \( na = nc/d \) of \( F \) is an integer, \( n \) must be divisible by \( d \). Then

\[
(L_{\overline{\eta'}} \otimes L_\eta)^\otimes n = L_{\overline{\eta'}}^{\otimes n} \otimes L_\eta^{\otimes d(n/d)} = L_{\chi}^{\otimes n(a+1)} \otimes L_{\chi}^{\otimes(c+d)n/d} = L_{\chi}^{\otimes n(a+1)} \otimes L_{\chi}^{\otimes n(a+1)} = 1
\]

so, by lemma 3.6

\[
\rho_\chi(F) \cong F \otimes L_{\eta'} \cong (F \otimes L_{\eta'}) \otimes (L_{\overline{\eta'}} \otimes L_\eta) = F \otimes L_\eta.
\]

**Proof of theorem 2.2.** The functors \( R_0^0 \to R_0^0 \) given by \( F \mapsto \rho_\chi(F) \) and \( F \mapsto F \otimes L_\eta \) are equivalences of categories, so they preserve direct sums. It is enough then to prove the isomorphism for indecomposable representations.
So let $F \in \mathcal{R}_0^a$ be indecomposable of length $m$. Then by [10] Lemma 3.1.6, Lemma 3.1.7(3)] there exist an irreducible $F_0 \in \mathcal{R}_0^a$ and a (necessarily tame) indecomposable unipotent $U_m \in \mathcal{R}_0$ of dimension $m$ such that $F = F_0 \otimes U_m$. Since $F$ is a succesive extension of $m$ copies of $F_0$, by exactness $\rho_\chi(F)$ is a succesive extension of $m$ copies of $\rho_\chi(F_0) \cong F_0 \otimes \mathcal{L}_\eta$, which is irreducible. By [10] Lemma 3.1.7(2), there is a unipotent $U \in \mathcal{R}_0$ of dimension $m$ such that $\rho_\chi(F) \cong F_0 \otimes \mathcal{L}_\eta \otimes U$.

Since $\rho_\chi$ is an equivalence of categories, $\rho_\chi(F)$ must be indecomposable, so $U$ itself must be indecomposable. Therefore $U \cong U_m$ and

$$\rho_\chi(F) \cong F_0 \otimes \mathcal{L}_\eta \otimes U_m \cong F \otimes \mathcal{L}_\eta.$$  

\[\square\]

4. Some variants

We will consider now representations of the inertia group $I_\infty$ at infinity. For any $F \in \mathcal{R}_\infty$ of slope $> 1$, we can take its local Fourier transform $\text{FT}^{\psi}_{(\infty, \infty)} F$, which is again in the same category. In [10] 3.4.4, N. Katz asks about a simple formula for

$$\rho'_\chi(F) := \text{FT}^{\psi}_{(\infty, \infty)}(\mathcal{L}_\chi \otimes \text{FT}^{\psi}_{(\infty, \infty)} F),$$

which is an auto-equivalence of the category of continuous $\ell$-adic representations of $\mathcal{R}_\infty$ with slopes $> 1$. It can be interpreted as the wild part of the monodromy at infinity of the (additive) convolution $F \ast \mathcal{L}_\chi$ [11] 3.4.6], where $F$ is any extension of the representation $F$ to a smooth sheaf on $\mathbb{G}_{m, \bar{k}}$ tamely ramified at $0$. In this section we will prove

**Theorem 4.1.** Let $F \in \mathcal{R}_\infty$ be totally wild with a single slope $a > 1$. Write $a = c/d$, where $c$ and $d$ are relatively prime positive integers. Let $\mathcal{L}_\eta$ be any tame character of $I_\infty$ such that $\mathcal{L}_\eta^{\otimes d} = \mathcal{L}_\xi^{\otimes (c-d)}$. Then

$$\rho'_\chi(F) \cong F \otimes \mathcal{L}_\eta.$$

In other words, we have the formula

$$(5) \quad \rho'_\chi(F) \cong F \otimes \mathcal{L}_\xi^{\otimes (a-1)}$$

where $\mathcal{L}_\xi^{\otimes (a-1)}$ stands for “any character that can reasonably be called $\mathcal{L}_\xi^{\otimes (a-1)n}$”.

The proof is very similar to the one for $\rho_\chi$. Since every representation in $\mathcal{R}_\infty$ is a direct sum of representations with single slopes, we can assume that $F$ has a single slope $a$.

**Lemma 4.2.** Let $F, G \in \mathcal{R}_\infty$ be totally wild, with $F$ having all slopes $> 1$. Then

$$\text{Swan}(\rho'_\chi(F) \otimes G) = \text{Swan}(F \otimes G).$$

**Proof.** We can assume that $F$ has a single slope $a > 1$. Extend $F$ and $G$ to smooth sheaves on $\mathbb{G}_{m, \bar{k}}$, tamely ramified at $0$, which we will also denote by $F$ and $G$ (as well as their extensions by zero to $\mathbb{A}^1_k$).

Let $n$ be the rank of $F$, and denote by $F(0)$ its local monodromy at $0$, which is a tame representation of $I_0$. Since all slopes of $F$ at infinity are $> 1$, it is a Fourier sheaf [3] Lemma 8.3.1], so its Fourier transform is a single sheaf that we will denote by $\text{FT}^{\psi} F$. By Ogg-Shafarevic, $\text{FT}^{\psi} F$ is smooth on $\mathbb{G}_{m, \bar{k}}$ of rank $na$. By Laumon’s local Fourier transform theory [9] Remark 9], it has a single positive slope $\frac{a}{a-1}$ at infinity with multiplicity $n(a-1)$ and tame part isomorphic to $\widehat{F(0)}$, where $\widehat{F(0)}$...
and it is unramified at 0. Then \( FT^\psi F \otimes L_\chi \) also has a single slope \( \frac{n(a-1)}{a-1} \) at infinity with multiplicity \( n(a-1) \), tame part isomorphic to \( L_\chi \otimes \hat{F}_{(0)} \), and its monodromy at 0 is a direct sum of \( na \) copies of \( L_\chi \).

Its inverse Fourier transform, by Ogg-Shafarevic, is smooth of rank \( n(a-1)\frac{a}{a-1} + n = n(a+1) \) on \( G_{m,k} \), and by local Fourier transform its monodromy at infinity is the direct sum of \( \rho_\chi(F) \) and \( na = \text{Swan}(F) \) copies of \( L_\chi \). At 0 is trivial part of rank \( na \), with quotient isomorphic to \( L_\chi \otimes F_{(0)} \). So

\[
\text{Swan}_\infty((F \otimes L_\chi) \otimes G) = \text{Swan}(\rho_\chi(F) \otimes G) + \text{Swan}(F)\text{Swan}(G).
\]

We conclude exactly as in lemma 3.3.

Using lemma 3.1 as in proposition 3.2, we deduce

**Proposition 4.3.** Let \( F \in R_\infty \) be irreducible with slope \( > 1 \). Then there exists a tame character \( L_\eta \) of \( I_\infty \) such that \( \rho_\chi(F) \cong F \otimes L_\eta \).

**Proposition 4.4.** Let \( F \in R_\infty \) be irreducible of dimension \( n \) and slope \( a > 1 \), and let \( L_\eta \) be a tame character of \( I_\infty \) such that \( \rho_\chi(F) \cong F \otimes L_\eta \). Then \( L_\eta \cong L_\chi \otimes \hat{F}_{(0)} \).

**Proof.** Extend \( F \) to a smooth \( \ell \)-adic sheaf on \( G_{m,k} \), tamely ramified at 0, also denoted by \( F \), and let \( F \) also denote its extension by zero to \( A^1 \). By the proof of lemma 4.2, the sheaf \( F \otimes L_\chi \) is smooth on \( G_{m,k} \), its monodromy at infinity is the direct sum of \( \rho_\chi(F) \cong F \otimes L_\eta \) and \( na \) copies of \( L_\chi \), and its monodromy at 0 has trivial part of dimension \( na \) with quotient isomorphic to \( L_\chi \otimes F_{(0)} \). Its determinant is then a smooth sheaf of rank 1 on \( G_{m,k} \), whose monodromy at infinity is \( \det(\hat{F}_{(0)}) \otimes L_\chi^{\otimes na} \), and whose monodromy at 0 is \( \det(F_{(0)}) \otimes L_\chi^{\otimes n} \).

We conclude, as in proposition 3.3, that

\[
\det(F \otimes L_\chi) \cong \det(F) \otimes L_\chi^{\otimes n} \otimes L_\chi^{\otimes na}
\]

as sheaves on \( G_{m,k} \). Comparing their monodromies at 0 gives the desired isomorphism.

The remainder of the proof of theorem 4.1 is identical to the one for \( \rho_\chi \).

We have a third variant, for representations \( F \in R_\infty \) with slopes \( < 1 \):

\[
\rho_\chi''(F) := FT^\psi_{(0)} (L_\chi \otimes FT^\psi_{(0)}(F)),
\]

which is again an auto-equivalence of the category of continuous \( \ell \)-adic representations of \( R_\infty \) with slopes \( < 1 \). As in the \( \rho_\chi \) case we have \( \rho_\chi''(F) \cong F \otimes L_\chi \) for \( F \) tame. The corresponding formula for wild \( F \) is

**Theorem 4.5.** Let \( F \in R_\infty \) be totally wild with a single slope \( a < 1 \). Write \( a = c/d \), where \( c \) and \( d \) are relatively prime positive integers. Let \( L_\eta \) be any tame character of \( I_\infty \) such that \( L_\eta^{\otimes d} = L_\chi^{\otimes (d-c)} \). Then

\[
\rho_\chi''(F) \cong F \otimes L_\eta.
\]

**Proof.** Let \( G := FT^\psi_{(0)}(F) \in R_0 \), which has slope \( \frac{a}{d} = \frac{c}{d} \) \[9, Theorem 13\]. The statement is then equivalent to

\[
FT^\psi_{(0)} (L_\eta \otimes G) \cong L_\eta \otimes FT^\psi_{(0)}(G)
\]
or

\[
FT^\psi_{(0)} (L_\eta \otimes FT^\psi_{(0)}(G)) \cong G \otimes L_\chi.
\]
But the left hand side is just $\rho_\eta(\mathcal{G})$, since the inverse of $\text{FT}_\psi^{\psi}(\infty,0)$ is $\text{FT}^{\overline{\psi}}(0,\infty)$ with respect to the conjugate additive character, and $\rho_\chi$ does not depend on the choice of the non-trivial additive character $\psi$. So the isomorphism follows from theorem 2.2.

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