A complete rewrite system and normal forms for \( (S)_{\text{reg}} \)

Jean-Camille Birget  
Dept. of Computer Science  
Rutgers University – Camden  
Camden, NJ 08102, USA  
birget@camden.rutgers.edu

Stuart W. Margolis*  
Dept. of Mathematics and Computer Science  
Bar Ilan University  
Ramat Gan 52900, Israel  
margolis@macs.biu.ac.il

Abstract. The \( (\cdot)_{\text{reg}} \) construction was introduced in order to make an arbitrary semigroup \( S \) divide a regular semigroup \( (S)_{\text{reg}} \) which shares some important properties with \( S \) (e.g., finiteness, subgroups, torsion bounds, \( J \)-order structure). We show that \( (S)_{\text{reg}} \) can be described by a rather simple complete string rewrite system, as a consequence of which we obtain a new proof of the normal form theorem for \( (S)_{\text{reg}} \). The new proof of the normal form theorem is conceptually simpler than the previous proofs.

1 Introduction

Regular semigroups have always played a special role in the structure theory of semigroups. Since, however, semigroups are in general not regular, it is interesting to connect arbitrary semigroups to regular ones. An obvious connection of this sort is the embedding of any semigroup \( S \) into a full transformation semigroup (which is always a regular semigroup). A much tighter connection was proved in [1, 2]: Any semigroup \( S \) divides a regular semigroup \( (\hat{S})_{\text{reg}} \); if \( S \) is finite, then \( (\hat{S})_{\text{reg}} \) is finite; every subgroup of \( (\hat{S})_{\text{reg}} \) divides a subgroup of \( S \); \( (\hat{S})_{\text{reg}} \) has the same regular \( J \)-order as \( S \), and shares many other properties with \( S \).

In more detail, the division of \( S \) into \( (\hat{S})_{\text{reg}} \) is done in two steps: First \( S \) is expanded to the left-right-iterated Rhodes expansion \( \hat{S} \); this yields an unambiguous semigroup, i.e., a semigroup whose \( L \)-order and \( R \)-order are forests [1]. Then \( \hat{S} \) is embedded into the regular semigroup \( (\hat{S})_{\text{reg}} \) by applying the \( (\cdot)_{\text{reg}} \) construction [2]. When \( S \) is any unambiguous semigroup then \( S \) is a subsemigroup of \( (S)_{\text{reg}} \); when \( S \) is not unambiguous then \( S \) is not a subsemigroup of \( (S)_{\text{reg}} \); in that case, the subsemigroup of \( (S)_{\text{reg}} \) generated by \( S \) is the Rees quotient of \( S \) over the ideal of ambiguous elements of \( S \) (by definition, an element \( s \in S \) is ambiguous iff the \( L \)-order and \( R \)-order above \( s \) are not both forests [2]).

As a consequence of this, every aperiodic (finite) semigroup divides a regular aperiodic (finite) semigroup. By definition, a semigroup is aperiodic iff it satisfies the identity \( x^n = x^{n+1} \) for some positive integer \( n \). More generally, an infinite torsion semigroup (or a bounded torsion semigroup, satisfying \( x^t = x^{t+c} \)) divides a regular torsion semigroup (respectively, a bounded

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torsion semigroup satisfying \( x^{t+1} = x^{(t+1)+c} \). Also, a semigroup whose subgroups belong to some variety (or pseudo-variety, or quasi-variety) \( G \) divides a regular semigroup whose subgroups belong to the same variety (resp., pseudo-variety, or quasi-variety) \( G \). So far, the above method is the only known proof of these results.

Another application of \((. \rangle_{\text{reg}}\) is to find an improved version of the Rhodes-Allen Synthesis theorem, a generalization of both the Rees theorem and the Krohn-Rhodes theorem (see [4] and [5] for background).

The \((. \rangle_{\text{reg}}\) construction itself has connections with two-way finite automata [8].

As we will see below, the \((. \rangle_{\text{reg}}\) construction is rather easy to describe, but it is not easy to prove the normal form theorem for the elements of \((S)_{\text{reg}}\). The normal form is important, because it is used to prove the main properties of \((S)_{\text{reg}}\). However, the fact that \( S \) is a subsemigroup of \((S)_{\text{reg}}\) when \( S \) is unambiguous, has a relatively simple direct proof – see [2]; pp. 73-75. All the known proofs of the normal form theorem are tedious. The original proof of the normal form theorem in [2] uses Van der Waerden’s method (letting \( S \) act faithfully on a set of normal forms). More recently, Grillet [6] introduced another method, based on congruences on non-associative structures. The present paper contains a third proof, based on string rewriting. Besides providing yet another proof, we show that \((S)_{\text{reg}}\) can be defined by a rather simple complete string rewrite system; this makes the normal forms of the elements of \((S)_{\text{reg}}\) obvious. Unfortunately, the catch is that the confluence of this rewrite system requires a rather tedious proof, though, conceptually, this proof is rather easy and looks almost like a verification by a machine.

We will assume from now on that \( S \) is unambiguous.

**Notation and definitions:**

By \( >_{\mathcal{L}}, \leq_{\mathcal{L}}, \equiv_{\mathcal{L}} \) we denote Green’s well known \( \mathcal{L} \)-relations, and similarly for the \( \mathcal{R} \)-relations. We also use the \( \mathcal{D} \)-equivalence \( \equiv_{\mathcal{D}} \). See e.g. [7] for background.

We will also need the \( \mathcal{L} \)-incomparability relation \( \not\leq_{\mathcal{L}} \) defined as follows: \( s \not\leq_{\mathcal{L}} t \) iff neither \( s \leq_{\mathcal{L}} t \) nor \( s \geq_{\mathcal{L}} t \). We also define \( \mathcal{L} \)-comparability: \( s \preceq_{\mathcal{L}} t \) iff either \( s \leq_{\mathcal{L}} t \) or \( s \geq_{\mathcal{L}} t \). A similar notation is used for \( \mathcal{R} \).

Following [1], [3], we call a semigroup \( S \) unambiguous iff for all \( s, t, u \in S - \{0\} : s \not\leq_{\mathcal{L}} u <_{\mathcal{L}} t \) implies \( s \preceq_{\mathcal{L}} t \) and \( s >_{\mathcal{R}} u <_{\mathcal{R}} t \) implies \( s \preceq_{\mathcal{R}} t \). This means that the \( \mathcal{L} \)-order on the \( \mathcal{L} \)-classes of \( S - \{0\} \) is a forest, and similarly for \( \mathcal{R} \). (Here, 0 is the zero of \( S \) if \( S \) has a zero; otherwise, \( S - \{0\} = S \).)

In order to avoid confusion between products of elements in a semigroup \( S \) and strings of elements of \( S \), we denote a string of length \( n \) as an \( n \)-tuple of the form \((s_1, s_2, \ldots, s_n)\). The product of these elements in \( S \) is denoted by \( s_1 s_2 \ldots s_n \) or \( s_1 \cdot s_2 \cdot \ldots \cdot s_n (\in S) \).

When \( S \) does not have an identity element, \( S^1 \) denotes the monoid obtained by adding a new identity element to \( S \); if \( S \) is already a monoid, \( S^1 \) is just \( S \).

We refer to [8] for background on rewrite systems.

### 2 The rewrite system

**Presentation of \((S)_{\text{reg}}\) by generators and relations:**
Let $S$ be a semigroup (possibly infinite). Let 0 be the zero of $S$, if $S$ has a zero; otherwise, let 0 be a new symbol not in $S$. Let $\emptyset$ be the zero of $S$, if $S$ has a zero; otherwise, let $\emptyset$ be a new symbol not in $S$. Let $S - \{0\} = \{s : s \in S - \{0\}\}$ be a set that is disjoint from $S \cup \{0\}$, where the map $x \in (S - \{0\}) \cup S - \{0\} \mapsto \overline{x} \in (S - \{0\}) \cup S - \{0\}$ is a bijection such that $\overline{\overline{x}} = x$. We also let $\emptyset = 0$; the symbol $\emptyset$ will never be used and will always automatically be replaced by 0.

Then, following [2], $(S)_{\text{reg}}$ is defined by the following presentation:

**Generators:**

$S \cup S - \{0\} \cup \{0\}$.

**Relations:**

1. $(s,t) = (st)$ for all $s,t \in S$
2. $(\overline{s},\overline{t}) = (\overline{ts})$ for all $s,t \in S - \{0\}$
3. $(0,0) = (0,s) = (s,0) = (0,\overline{s}) = (\overline{s},0)$ for all $s \in S$
4. $(s,\overline{t}) = (0)$ if $s \not\preceq_L t$, $s, t \in S - \{0\}$
5. $(\overline{s},t) = (0)$ if $s \not\preceq_R t$, $s, t \in S - \{0\}$
6. $(s,\overline{s},s) = (s)$ for all $s \in S - \{0\}$
7. $(\overline{s},s,\overline{s}) = (\overline{s})$ for all $s \in S - \{0\}$

It is proved in [2] (see also [3] and [4]) that $S$ is a subsemigroup of $(S)_{\text{reg}}$ if $S$ is unambiguous, and that $(S)_{\text{reg}}$ is a regular semigroup with involution (i.e., for all $x, y \in (S)_{\text{reg}}$: $\overline{x} = x$, $\overline{xy} = y\overline{x}$, $x\overline{x} = x$).

**Rewrite rules for $(S)_{\text{reg}}$:**

We now introduce a string rewrite system for $(S)_{\text{reg}}$. This rewrite system is finite iff $S$ is finite. The reduced words of this rewrite system are the normal forms of $(S)_{\text{reg}}$. In the next sections we will prove that this rewrite system is complete, when $S$ is unambiguous.

1. **Length-reducing rules:**

   The last two of the following set of rules make use of a partial function $B : S \times S \times S \rightarrow S$, that will be defined after the statement of all the rules.

   (1.1) $(s,t) \rightarrow (st)$, $(\overline{s},\overline{t}) \rightarrow (\overline{ts})$ for all $s,t \in S$
   (1.2) $(0,0) \rightarrow (0)$, $(0,s) \rightarrow (0)$, $(0,\overline{s}) \rightarrow (0)$, $(\overline{s},0) \rightarrow (0)$ for all $s \in S$
   (1.3) $(s,\overline{t}) \rightarrow (0)$ if $s \not\preceq_L t$, $s, t \in S - \{0\}$
   (1.4) $(\overline{s},t) \rightarrow (0)$ if $s \not\preceq_R t$, $s, t \in S - \{0\}$
   (1.5) $(u,\overline{v},w) \rightarrow (B(u,v,w))$ if $u \preceq_L v \geq_R w$, $u,v,w \in S - \{0\}$
   (1.6) $(\overline{u},v,\overline{w}) \rightarrow (\overline{B(u,v,w)})$ if $u \preceq_R v \geq_L w$, $u,v,w \in S - \{0\}$

2. **Length-preserving rules:**

   For these rules we choose one representative element in every $R$-class and in every $L$-class. We make these choices so that $D$-related representatives of $R$-classes are $L$-related, and $D$-related representatives of $L$-classes are $R$-related. Moreover, if two representatives (one
representing an \( \mathcal{L} \)-class and one representing an \( \mathcal{R} \)-class) are in the same \( \mathcal{H} \)-class they are chosen to be equal. Such a choice can always be made.

Note that this condition on the choice of representatives was not used, and not required, in \[2\] and \[4\]. A similar choice however is made in the Rees-Sushkevitch coordinatization, see e.g. \[3\]. Notation: For any \( s \in S \) the chosen representative of the \( \mathcal{R} \)-class (or \( \mathcal{L} \)-class) of \( s \) is \( r_s \) (respectively \( \ell_s \)).

The length-preserving rules make use of two partial functions, \( B_\mathcal{L} \) and \( B_\mathcal{R} : S \times S \to S \), that will be defined after the rules.

Note the unsymmetry between rules (2.1)-(2.2) and (2.3)-(2.4), which is needed for obtaining unique normal forms; see \[4\], \[3\] for more discussion on the normal forms.

\[
(2.1) \quad (s, t) \to (r_s, B_\mathcal{R}(s, t)) \quad \text{if} \quad s \geq \ell t \quad \text{and} \quad s \neq r_s \\
(2.2) \quad (s, t) \to (\ell_s, B_\mathcal{L}(t, s)) \quad \text{if} \quad s \geq \ell t \quad \text{and} \quad s \neq \ell_s \\
(2.3) \quad (t, s) \to (B_\mathcal{L}(t, s), \ell_s) \quad \text{if} \quad t \leq \ell s \quad \text{and} \quad s \neq \ell_s \\
(2.4) \quad (t, s) \to (B_\mathcal{R}(s, t), r_s) \quad \text{if} \quad t \leq \ell s \quad \text{and} \quad s \neq r_s
\]

**Definition of \( B \).** If \( u \leq \ell v \geq \ell w \), where \( u, v, w \in S - \{0\} \), then \( B(u, v, w) = uz \), where \( z \in S^1 \) is such that \( w = vz \).

This operation was used in \[2\], but was first explicitly defined in \[3\]. It is easy to see that if \( u \leq \ell v \geq \ell w \) then \( B(u, v, w) \) exists and is unique (i.e., it depends only on \( u, v, w \) and not on \( x \)); see Lemma \[3.3\] below. The main motivation for \( B \) is that in \( S_{\text{reg}} \), \( uz = B(u, v, w) \) if \( u \leq \ell v \geq \ell w \), as we will prove in Proposition 2.1 below.

**Definition of \( B_\mathcal{R} \) and \( B_\mathcal{L} \).** If \( u \geq \ell v \), where \( u, v \in S - \{0\} \), then \( B_\mathcal{R}(u, v) = xr_u \), where \( x \in S^1 \) is such that \( v = xu \). If \( v \leq \ell u \), where \( u, v \in S - \{0\} \), then \( B_\mathcal{L}(v, u) = \ell uy \), where \( y \in S^1 \) is such that \( v = uy \).

This operation was implicit in \[2\]. Again, it is easy to see that if \( u \geq \ell v \) (or \( v \leq \ell u \)) then \( B_\mathcal{R}(u, v) \) (resp. \( B_\mathcal{L}(v, u) \)) exists and is unique (i.e., it depends only on \( u \) and \( v \)). The main motivation for \( B_\mathcal{R} \) is that in \( S_{\text{reg}} \), \( \overline{uv} = B_\mathcal{R}(u, v) \overline{u} \) if \( u \geq \ell v \), as we will prove in Proposition 2.1 below. The motivation for \( B_\mathcal{L} \) is similar.

In the next section we will see another, pictorial motivation for \( B, B_\mathcal{R} \), and \( B_\mathcal{L} \).

Before proving the next proposition we need to recall a key property of \( (S)_{\text{reg}} \).

**Lemma 2.1** (Fact 2.5 in \[2\]). For all \( s, r, \ell \in S - \{0\} \): If \( s \equiv \ell r \) in \( S \) then \( s \overline{s} = r \overline{r} \) in \( (S)_{\text{reg}} \). If \( s \equiv \ell r \) in \( S \) then \( \overline{s} \overline{s} = \ell \overline{\ell} \) in \( (S)_{\text{reg}} \).

**Proof.** Let \( a, b \in S^1 \) be such that \( s = ra \), \( r = sb \); so, \( rab = r \). Then, by using the relations of the presentation of \( (S)_{\text{reg}} \) we have:

\[
s \overline{s} = ra \overline{r} = r \overline{r} a \overline{r} = r \overline{rab} ra \overline{r} = r \overline{r} \overline{r} \overline{r} a = r \overline{r} a = r \overline{r} = r \overline{r} = r \ell \overline{r} = r \.
\]

**Proposition 2.1** The rewrite system defines \( (S)_{\text{reg}} \).
Proof. The rewrite rules (when made symmetric) imply the relations of the presentation; to obtain the last two relations of the presentation, let \( u = v = w \) in rules (1.5) and (1.6).

Conversely it is straightforward to show that in \((S)_{\text{reg}}\) the relations corresponding to the rules (1.5), (1.6), (2.1)–(2.4) hold (see also [3]).

Let us derive rule (1.5). Since \( u \leq_v v \geq_R w \), let \( x, y \in S^1 \) be such that \( u = xv, w = vy \).
Then \( u\overline{v}w = x\overline{v}vy = xyv \), using \( v\overline{v} = v \) in \((S)_{\text{reg}}\). Moreover, \( xyv = uy = B(u, v, w) \) by the definition of \( B \). Thus, \( u\overline{v}w = B(u, v, v) \) in \((S)_{\text{reg}}\).

Let us derive rule (2.4). Since \( t \leq_v s \), let \( x \in S^1 \) be such that \( t = xs \). Then \( t\overline{s} = x s\overline{v} = x r_s\overline{v} \); the last equality follows from the last Lemma. And \( xr_s = B_{\mathcal{R}}(s, t) \), by the definition of \( B_{\mathcal{R}} \). Thus, \( t\overline{s} = B_{\mathcal{R}}(s, t)\overline{s} \) in \((S)_{\text{reg}}\).

The other rules can be derived in a very similar way. \( \Box \)

One of the main results of [2] is the following:

**Normal Form theorem for \((S)_{\text{reg}}\):** If \( S \) is unambiguous then \( S \) is a subsemigroup of \((S)_{\text{reg}}\), and (for any fixed choice of representatives of the \( \mathcal{L} \)- and \( \mathcal{R} \)-classes) every element of \((S)_{\text{reg}}\) can be written in a unique way in the normal form

\[
(0) \quad \text{or}
\]

\[
([r_1, \overline{r_2}], \ldots, r_{n-1}, \overline{r_n}, s, r_{m-1}' \overline{r_m}', \ldots, r_{l_2}' \overline{l_1}])
\]

where \([r_1 >_\mathcal{L} l_2 >_\mathcal{R} \ldots >_\mathcal{R} r_{n-1} >_\mathcal{L} l_n >_\mathcal{R} s \leq_\mathcal{L} l_m' <_\mathcal{R} l_{m-1}' <_\mathcal{L} \ldots <_\mathcal{L} r_{l_2}' [<_\mathcal{R} l_1'], \]

or in the form

\[
([r_1, \overline{r_2}], \ldots, r_{n-2}, \overline{r_n}, s, \overline{r_m}, r_{m-1}' \overline{r_m}', \ldots, r_{l_2}' \overline{l_1}])
\]

where \([r_1 >_\mathcal{L} l_2 >_\mathcal{R} \ldots >_\mathcal{R} r_{n-2} >_\mathcal{L} l_{n-1} >_\mathcal{R} r_n >_\mathcal{L} s \leq_\mathcal{L} l_m' <_\mathcal{R} l_{m-1}' <_\mathcal{L} \ldots <_\mathcal{L} r_{l_2}' [<_\mathcal{R} l_1']. \]

Here, every \( r_i, r_i', l_i, l_i' \) is a representative of an \( \mathcal{R} \)- or \( \mathcal{L} \)-class, and \( s \) is any element of \( S - \{0\} \). Elements in square brackets may be absent.

The normal form representation is the key to many structure properties of \((S)_{\text{reg}}\), e.g., the fact that \( S \) and \((S)_{\text{reg}}\) have the same \( J \)-class structure. The main result of this paper is:

**Theorem 2.1** The above rewrite system for \((S)_{\text{reg}}\) is complete (i.e., confluent and terminating). The normal forms of the rewrite systems are as given above.

The remainder of this paper consists of the proof of this theorem. In Section 3 we give some basic properties of \( B, B_{\mathcal{L}} \), and \( B_{\mathcal{R}} \), then in Section 4 we prove termination of the rewrite system, and finally in Section 5 we prove local confluence.

### 3 Properties of the functions \( B, B_{\mathcal{L}}, \text{ and } B_{\mathcal{R}} \)

In this section we collect all the basic properties of \( B, B_{\mathcal{L}}, \) and \( B_{\mathcal{R}} \) that we will need in order to prove that the rewrite system for \((S)_{\text{reg}}\) is terminating and locally confluent. The reader may skip this section, and come back to it while reading the proofs of termination and local confluence.
Below, when we write an expression like $B_R(x, y)$, $B_L(x, y)$, or $B(x, y, z)$, we always implicitly assume that these expressions are defined (i.e., we assume that $x \geq y$ when we use $B_R(x, y)$, etc.).

In all the proofs in this section it will be useful for the reader to represent $B$, $B_R$, and $B_L$, by the following diagrams, which are justified by the next few lemmas.

**Diagram of $B(u, v, w)$:**

If $u \leq v \geq w$, let $y, z \in S^1$ be any elements such that $u = yv$, $w = vz$. Then we have the commutative diagram:

```
    v
   /\  \\
  y \ >  z \\
 /   \     \\
u -- B(u, v, w) -- w
```

**Diagram of $B_R(u, v)$:**

If $s \geq t$, let $x \in S^1$ be any elements such that $t = xs$. Also, let $a, a' \in S^1$ be such that $ra = sa$ and $r_s = sa'$. Then we have the commutative diagram:

```
   /\  \\
  s \ >  a' \\
 /   \     \\
x -- B_R(s, t) -- x
```

The diagram for $B_L$ is similar to the diagram for $B_R$.

**Lemma 3.1**

(a) If $u = ra\alpha$ then $B_R(u, v) \cdot \alpha = v$. Similarly, if $v = \beta\ell_u$ then $\beta \cdot B_L(v, u) = u$.

(b) If $u = ra\alpha'$ then $B_R(u, v) = vo'$. Similarly, if $\ell_u = \beta'u$ then $\beta \cdot B_L(v, u) = \beta'u$.

The proof is trivial.

**Lemma 3.2** $B_R(r_u, v) = v$, and $B_L(v, \ell_u) = v$.
The proof is trivial.

**Lemma 3.3** \( B_\mathcal{R}(u, v) \equiv_\mathcal{R} v \), and \( B_\mathcal{L}(v, u) \equiv_\mathcal{L} v \).

**Proof.** If we multiply \( r_u \equiv_\mathcal{R} u \) on the left by \( x \) we obtain \( B_\mathcal{R}(u, v) = xr_u \equiv_\mathcal{R} xu = v \). For \( \mathcal{L} \) the proof is similar. \( \square \)

**Lemma 3.4** If \( s <_\mathcal{L} t \) then \( B_\mathcal{R}(t, s) <_\mathcal{L} r_t \) (and the same holds with \( <_\mathcal{L} \) replaced by \( \equiv_\mathcal{L} \) or \( \leq_\mathcal{L} \)). If \( t >_\mathcal{R} s \) then \( \ell_t >_\mathcal{R} B_\mathcal{L}(s, t) \) (and the same holds with \( >_\mathcal{R} \) replaced by \( \equiv_\mathcal{R} \) or \( \leq_\mathcal{R} \)).

**Proof.** We prove the first statement, the other ones having very similar proofs. Let \( a \) be such that \( ta = r_t \).

Since \( B_\mathcal{R}(t, s) = xr_t \) for some \( x \) such that \( xt = s \), we have \( B_\mathcal{R}(t, s) = xr_t \leq_\mathcal{L} r_t \). Actually we have \( B_\mathcal{R}(t, s) <_\mathcal{L} r_t \). Indeed, if we had \( xr_t \equiv_\mathcal{L} r_t \), then multiplying on the right by \( a \) yields \( s = xr_ta \equiv_\mathcal{L} r_ta = t \), i.e., \( s \equiv_\mathcal{L} t \), which contradicts the assumption. \( \square \)

**Lemma 3.5** If \( u \leq_\mathcal{L} v \geq_\mathcal{R} w \) then \( B(u, v, w) = yw = ux = yvx \), where \( x \) is such that \( w = vx \), and \( y \) is such that \( u = yv \). The value of \( B(u, v, w) \) does not depend on the \( x \) or \( y \) chosen.

**Proof.** By definition, \( B(u, v, w) = ux \) where \( x \) is such that \( w = vx \). Hence \( B(u, v, w) = ux = yvx = yw \).

To see that \( B(u, v, w) \) does not depend on the choice of \( x \) (provided that \( w = vx \)), let \( w = vx_1 = vx_2 \). Then \( B(u, v, w) = yvx_1 = yvx_2 \). Similarly, one sees that the choice of \( y \) does not matter (provided that \( u = yv \)). \( \square \)

**Lemma 3.6** If \( u \leq_\mathcal{L} v \geq_\mathcal{R} w \) and \( t \in S - \{0\} \) then \( B(tu, v, w) = t \cdot B(u, v, w) \) and \( B(u, v, wt) = B(u, v, w) \cdot t \).

**Proof.** Since \( B(u, v, w) = ux \) where \( x \) is such that \( w = vx \), we obtain \( t \cdot B(u, v, w) = txu \) with \( w = vx \). Hence by the definition of \( B(tu, v, w) \) we have \( B(tu, v, w) = t \cdot B(u, v, w) \).

The proof for \( B(u, v, wt) \) is similar, by using Lemma 3.3. \( \square \)

**Lemma 3.7**

1. If \( u \geq_\mathcal{L} su \geq_\mathcal{L} v \) then \( sr_u \geq_\mathcal{L} B_\mathcal{R}(u, v) \) and \( B_\mathcal{R}(su, v) = B_\mathcal{R}(sr_u, B_\mathcal{R}(u, v)) \).
2. If \( su \leq_\mathcal{L} v \leq_\mathcal{L} u \) then \( sr_u \leq_\mathcal{L} B_\mathcal{R}(u, v) \) and \( B_\mathcal{R}(v, su) = B_\mathcal{R}(B_\mathcal{R}(u, v), sru) \).
3. If \( su \leq_\mathcal{L} v \leq_\mathcal{L} u \) then \( sr_u \leq_\mathcal{L} B_\mathcal{R}(u, v) \).
4. If \( u \geq_\mathcal{L} v \) then \( B_\mathcal{R}(u, sv) = s \cdot B_\mathcal{R}(u, v) \).
5. Analogous properties hold for \( B_\mathcal{L} \).

**Proof.** (1) By definition of \( B_\mathcal{R} \) we have \( B_\mathcal{R}(u, v) = xr_u \) where \( x \) is such that \( v = xu \). But \( v = asu \) for some \( a \) since \( v \leq_\mathcal{L} su \), hence we can pick \( x = as \). So, \( B_\mathcal{R}(u, v) = asru \leq_\mathcal{L} sr_u \).

By definition of \( B_\mathcal{R} \) we have \( B_\mathcal{R}(sr_u, B_\mathcal{R}(u, v)) = xr_{sr_u} \), where \( x \) is any element of \( S \) such that \( B_\mathcal{R}(u, v) = xsr_u \).
Also, by definition of $B_R$ we have $B_R(u, v) = yr_{su}$, where $y$ is such that $v = yu$. By Lemma 3.1, multiplying $v = yu$ by $\alpha'$ we obtain $B_R(u, v) = v\alpha' = ysu' = yr_{su}$. Thus, $B_R(u, v) = ysr_u$, and since $x$ was any element such that $B_R(u, v) = xsr_u$, we can assume $x = y$. So, $B_R(su, v) = xr_{su}$. Moreover, $r_{sr_u} = r_{su}$ since $u \equiv_R r_u$. The result now follows.

(2) By definition of $B_R$ we have $B_R(u, v) = xr_u$ where $x$ is such that $v = xu$. Hence $B_R(u, v) = xru = v\alpha'$ where $\alpha'$ is such that $u\alpha' = r_u$. Moreover, $v \geq_L su$ (or $v >_L su$), thus $B_R(u, v) = v\alpha' \geq_L su\alpha' = sr_u$ (or $>_L su\alpha' = sr_u$).

By definition of $B_R$ we have $B_R(B_R(u, v), sr_u) = x'r_{B_R(u,v)}$, where $x$ is such that $sr_u = x \cdot B_R(u, v)$. By Lemma 3.1 if we multiply the last equality by $\alpha$ we obtain $su = xv$.

By definition we also have $B_R(v, su) = yr_v$, where $y$ is any element of $S$ such that $su = yv$. But we proved that $x$ also satisfies $su = xv$. Thus we can assume $x = y$.

So we have $B_R(B_R(u, v), sr_u) = yr_{B_R(u,v)}$. Moreover, since $B_R(u, v) \equiv_R v$ (by Lemma 3.3), we obtain the result.

(3) This follows directly from Lemma 3.1.

(4) By definition, $B_R(u, v) = yr_u$, where $yu = v$. Also $B_R(u, su) = xr_u$, where $x$ is any element of $S$ such that $xu = sv$. Since $yu = v$, we have $syu = sv$, hence we can pick $x$ to be $sy$. The result then follows. □

**Lemma 3.8** If $w \not\leq_L s$ then $B(u, v, w) \not\leq_L s$. Similarly, if $s \not\leq_R u$ then $s \not\leq_R B(u, v, w)$.

**Proof.** By contraposition, assume $B(u, v, w) \leq_L s$. By definition, $B(u, v, w) = ux$, where $x$ is such that $w = vx$. Since $B(u, v, w)$ exists, $u \leq_L v \geq_R w$; so $u = yv$ for some $y$.

Now we have $s \leq_L B(u, v, w) = ux = yvx = yw \leq_L w$.

In case $s \leq_L u$, the above implies $s \leq_L w$.

In case $s \geq_L ux$, the above implies $s \geq_L wx \leq_L w$, and hence, by unambiguity of the $L$-order, $s \leq_L w$.

In either case, $s \leq_L w$. □

**Lemma 3.9** (Lemma 1.1.(5) in [6].) If $u \leq_L v \geq_R w \leq_L s \geq_R t$, then $B(u, v, w) \leq_L s \geq_R t$, $u \leq_L v \geq_R B(w, s, t)$, and $B(B(u, v, w), s,t) = B(u, v, B(w, s,t))$.

**Proof.** We have $B(u, v, w) \leq_L w$ by the definition of $B$, and $w \leq_L s \geq_R t$, by assumption. Also, $u \leq_L v \geq_R w$ by assumption, and $w \geq_R B(w, s, t)$ by Lemma 3.5. So the claimed order relations hold.

By Lemma 3.5, $B(u, v, w) = yw$, where $u = yv$, and by definition, $B(w, s, t) = wx$, where $t = sx$. Then by definition $B(u, v, B(w, s, t)) = B(u, v, wx) = B(u, v, w) \cdot x$ (the latter equality holds by Lemma 3.6). This is equal to $yw \cdot x$. A similar reasoning shows that $B(B(u, v, w), s,t)$ is also equal to $yw$. □

**Lemma 3.10** Assume that $u \leq_L v \equiv_R w \geq_L s$ and $c \in S - \{0\}$. Then:

(a) $cs = B(u, v, w)$ iff $u = c \cdot B(s, w, v)$,

(b) $cu = B(s, w, v)$ iff $s = c \cdot B(u, v, w)$.
Proof of (a). By Lemma 3.3 there exist $x, y, x', y' \in S^1$ such that
\[ B(u, v, w) = ux = yw, \quad w = vx, \quad u = yv \quad \text{and} \quad B(s, w, v) = sx' = y'v, \quad v = wx', \quad s = y'w. \]

If the left side of the equivalence holds then $yw = B(u, v, w) = cs = cy'w$, so if we multiply by $x'$ we obtain $u = ywx' = cy'/v = c \cdot B(s, s, w, v)$.

If the right side of the equivalence holds then $u = c \cdot B(s, w, v) = cy'$, so if we multiply by $x$ we obtain $B(u, v, w) = ux = cy'vx = cy'w = cs$.

The proof of (b) is similar. \(\Box\)

Lemma 3.11 Assume that $u \leq \epsilon v \equiv \epsilon w \geq \epsilon s$. Then:
(1) $B(u, v, w) \leq \epsilon s$ iff $u \leq \epsilon B(s, w, v)$. The same holds with $\leq \epsilon$ replaced by $\geq \epsilon$ or $\not\equiv \epsilon$.\(\Box\)

(2. \leq) If $B(u, v, w) \leq \epsilon s$ then $r_s = r_{B(s,w,v)}$ and $B_R(s, B(u, v, w)) = B_R(B(s, w, v), u)$.
(2. \geq) If $B(u, v, w) \geq \epsilon s$ then $r_s = r_{B(u,v,w)}$ and $B_R(B(u, v, w), s) = B_R(u, B(s, w, v))$.

Analogous properties hold for $B_{\epsilon'}$.

Proof. (1): For $\leq \epsilon$ this is an immediate consequence of Lemma 3.10 (a). The result (1) for $\geq \epsilon$ follows from Lemma 3.10 (b). Since $\geq \epsilon$ holds iff we have $\geq \epsilon$ and not $\leq \epsilon$, we also obtain (1) for $\geq \epsilon$. Also, since $\not\equiv \epsilon$ holds iff we have neither $\leq \epsilon$ nor $\geq \epsilon$, we obtain (1) for $\not\equiv \epsilon$.

(2. \leq): If $B(u, v, w) \leq \epsilon s$ then $B(s, w, v) = sx' \leq \epsilon s$, and $B(s, w, v) \geq \epsilon B(s, w, v) \cdot x = y'vx = y'w = s$, where $x', x, y'$ are as at the beginning of the proof of Lemma 3.10. Thus $s = R(s, B(s, w, v))$.

By definition, $B_R(s, B(u, v, w)) = x''r_s$, for any $x''$ such that $x''s = B(u, v, w)$.

And $B_R(B(s, w, v), u) = y''r_{B_B(s,w,v)} = y''r_s$, for any $y''$ such that $y'' \cdot B(s, w, v) = u$.

But by Lemma 3.10, $x''s = B(u, v, w)$ iff $u = x'' \cdot B(s, w, v)$. So we can choose $y''$ to be $x''$. Then the equality follows.

(2. \geq): The proof is very similar to that of (2. \leq). \(\Box\)

Lemma 3.12 Assume that $u \leq \epsilon v \geq \epsilon w \geq \epsilon s$, and let $c \in S$. Then:
(1) $B(u, v, w) = cs$ iff $B(u, v, r_w) = c \cdot B_R(w, s)$.
(2) $c \cdot B(u, v, w) = s$ iff $c \cdot B(u, v, r_w) = B_R(w, s)$.

Analogous properties hold for $B_{\epsilon'}$.

Proof. (1): Assume $B(u, v, w) = cs$, where (by Lemma 3.3) $B(u, v, w) = yw$ with $u = yv$. Multiplying $yw = cs$ on the right by $\alpha'$, where $\alpha'$ is such that $w\alpha' = r_w$, we obtain: $yr_w = csa'$.

The left side $yr_w$ is equal to $B(u, v, r_w)$ by Lemma 3.3, since $u = yv$. On the other hand, by the definition of $B_R$, we have, $B_R(w, s) = xw$ with $s = xw$. Since $w\alpha' = r_w$, we have $B_R(w, s) = x\omega' = sa'$, which when multiplied by $c$ yields the right side.

Conversely, if $B(u, v, r_w) = c \cdot B_R(w, s)$ we will have by Lemma 3.3 and by the definition of $B_{\epsilon'}$, in the above notation: $yr_w = csa'$.

Multiplying on the right by $\alpha$ (where $\alpha$ is such that $r_w\alpha = w$), we obtain: $yw = csa'\alpha = cs$.

We have $sa'\alpha = s$ because we assumed $w \geq \epsilon s$. Thus $B(u, v, w) = yw = csa'\alpha = cs$.

The proof of (2) is quite similar to the proof of (1). \(\Box\)
Lemma 3.13 Assume that \( u \leq_\ell v \geq_\mathfrak{R} w \geq_\ell s \). Then:

1. \( B(u, v, w) \leq_\ell s \iff B(u, v, r_w) \leq_\ell B(\mathfrak{R}(w, s)). \)

The same is true with \( \leq_\ell \) replaced by \( >_\ell \) or \( \not\leq_\ell \).

2. \( \leq \): If \( B(u, v, w) \leq_\ell s \) then \( s \equiv_\mathfrak{R} B(\mathfrak{R}(w, s)) \) and

\[
B(\mathfrak{R}(s, B(u, v, w))) = B(\mathfrak{R}(w, s), B(u, v, r_w)).
\]

2. \( > \): If \( B(u, v, w) >_\ell s \) then \( B(u, v, w) \equiv_\mathfrak{R} B(u, v, r_w) \) and

\[
B(\mathfrak{R}(B(u, v, w), s)) = B(\mathfrak{R}(B(u, v, r_w), B(\mathfrak{R}(w, s))).
\]

Analogous properties hold for \( B_\ell \):

1. If \( s \leq_\mathfrak{R} u \leq_\ell v \geq_\mathfrak{R} w \) then :

\[
B_\ell(s, u) \leq_\ell B(u, v, w).
\]

The same is true with \( \leq_\mathfrak{R} \) replaced by \( >_\mathfrak{R} \) or \( \not\leq_\mathfrak{R} \).

2. \( \leq \): By Lemma 3.3 we have \( r_s = r_{\mathfrak{R}(w, s)} \).

We will apply Lemma 3.7 (2), which we quote here with different parameters:

If \( s_o u_o \leq_\ell v_o \leq_\ell u_o \) then \( B_\ell(v_o, s_o u_o) = B_\ell(B(\mathfrak{R}(u_o, v_o), s_o r_{u_o})). \)

Let \( v_o = s \), \( u_o = y \), and \( s_o = w \), where (by Lemma 3.4) \( B(u, v, w) = yw \) and \( B(u, v, r_w) = y r_w \) with \( yv = u \). Then \( s_o u_o = B(u, v, w) \) and \( s_o r_{u_o} = B(u, v, r_w) \). By assumption, \( B(u, v, w) \leq_\ell s \leq_\ell w \), so \( s_o u_o \leq_\ell u_o \leq_\ell v_o \), hence Lemma 3.7 (2) is indeed applicable here. By substituting, the claimed result then follows immediately.

2. \( > \): By Lemma 3.3 we have \( B(u, v, w) = yw \) and \( B(u, v, r_w) = y r_w \), with \( u = yv \).

Since \( w \equiv_\mathfrak{R} r_w \) we obtain \( B(u, v, w) \equiv_\mathfrak{R} B(u, v, r_w) \).

We will apply Lemma 3.7 (1), which we quote here with different parameters:

If \( u_o \geq_\ell s_o u_o \geq_\ell v_o \) then \( B_\ell(s_o u_o, v_o) = B_\ell(s_o r_{u_o}, B_\ell(u_o, v_o)). \)

Let \( s_o = y \), \( u_o = w \), where \( B(u, v, w) = yw \) and \( B(u, v, r_w) = y r_w \), with \( yv = u \) (by Lemma 3.3). And let \( v_o = s \). Since by our assumptions \( w \equiv_\ell B(u, v, w) \geq_\ell s \), Lemma 3.7 (1) can be applied. The claimed result then follows immediately by substitution. \( \square \)

Lemma 3.14 Assume that \( u \leq_\ell v \geq_\mathfrak{R} w \). Then \( B(\mathfrak{R}(u, v, w)) \leq_\ell r_v \geq_\mathfrak{R} w \) and

\[
B(\mathfrak{R}(u, v, w)) = B(u, v, w).
\]

Analogous properties hold for \( B_\ell \):

If \( u \leq_\ell v \geq_\mathfrak{R} w \) then \( u \leq_\ell B_\ell(v, w) \) and

\[
B(u, v, w) = B(u, \ell_v B_\ell(w, v)).
\]

Proof. The fact that \( B(\mathfrak{R}(u, v, w)) \leq_\ell r_v \geq_\mathfrak{R} w \) is obvious from the definition of \( B(\mathfrak{R}) \).

By Lemma 3.3 \( B(u, v, w) = x_1 w \) for any \( x_1 \) such that \( u = x_1 v \). Also, by definition, \( B_\ell(v, u) = x_2 r_v \) for any \( x_2 \) such that \( u = x_2 v \); therefore we can choose \( x_2 = x_1 \).

Now \( B(B_\ell(v, u), r_v, w) = B_\ell(v, u) z \) with \( w = r_v z \), hence \( B(B_\ell(v, u), r_v, w) = x_1 r_v z = x_1 w \). This proves the result. \( \square \)
Lemma 3.15  Assume that \( u \leq_{L} v \geq_{R} w \leq_{L} s \). Then \( B_{R}(s, B(u, v, w)) = B(u, v, B_{R}(s, w)). \)

Analogous properties hold for \( B_{L} \):

If \( s \geq_{R} u \leq_{L} v \geq_{R} w \) then \( B_{L}(B(u, v, w), s) = B(B_{L}(u, s), v, w). \)

Proof. By definition, \( B_{R}(s, B(u, v, w)) = x_{1}r_{s} \) where \( x_{1} = B(u, v, w) = uz \), with (by definition of \( B \)) \( w = vz \). We also have:

\[
\begin{align*}
B(u, v, B_{R}(s, w)) &= y B_{R}(s, w) \\
&= yx_{2}r_{s} \\
&= yx_{2}s\alpha' \\
&= yw\alpha' \\
&= yvz\alpha' \\
&= uz\alpha' \\
&= B(u, v, w)\alpha' \\
&= x_{1}\alpha' \\
&= x_{1}r_{s} \\
&= B_{R}(s, B(u, v, w))
\end{align*}
\]

as we saw in the beginning of this proof. \( \square \)

Lemma 3.16  Assume that \( u \geq_{L} v \geq_{R} w \). Then

(1) \( B_{R}(u, v) \equiv_{L} B_{R}(u, \ell_{v}) \),

(2) \( B_{L}(w, B_{R}(u, v)) = B_{L}(B_{L}(w, u), B_{R}(u, \ell_{v})). \)

Proof. Property (1) follows easily from Lemma 3.7 (4).

(2): Let \( \beta \) and \( \beta' \) be such that \( v = \beta\ell_{v} \) and \( \ell_{v} = \beta'v \). By definition, \( B_{R}(u, v) = xr_{u} \), where \( xu = v \). Hence, by the definition of \( B_{R} \), we have \( B_{R}(u, \ell_{v}) = \beta'xr_{u} \) since \( \beta'x \) satisfies \( \beta'xu = \ell_{v} \).

Thus, \( B_{L}(w, B_{R}(u, v)) = B_{L}(w, xr_{u}) = \ell_{xr_{u}}y_{1} \), where \( y_{1} \) is such that \( w = xr_{u}y_{1} \).

On the other hand, \( B_{L}(B_{L}(w, u), B_{R}(u, \ell_{v})) = \ell_{B_{R}(u, \ell_{v})y_{2}} = \ell_{xr_{u}y_{2}} \), since \( B_{R}(u, v) \equiv_{L} B_{R}(u, \ell_{v}) \) (as we just proved in (1)). Here, by the definition of \( B_{L} \), \( y_{2} \) is any element of \( S \) such that \( B_{L}(w, u) = B_{R}(u, \ell_{v})y_{2} \). We saw that the latter is equal to \( \beta'xr_{u}y_{2} \). By the definition of \( B_{L} \) we also have \( B_{L}(w, u) = \ell_{v}y_{3} \) where \( y_{3} \) is such that \( w = vy_{3} \).

Therefore \( \ell_{v}y_{3} = \beta'xr_{u}y_{2} \). Multiplying on the left by \( \beta'y \) yields \( w = vy_{3} = xr_{u}y_{2} \), i.e., \( y_{2} \) satisfies \( w = xr_{u}y_{2} \), which is the defining property of \( y_{1} \).

Hence, \( y_{2} \) can be chosen above so that \( y_{2} = y_{1} \). \( \square \)

Lemma 3.17  Assume that \( u' \geq_{L} v \leq_{R} w \). Then \( B_{L}(B_{R}(u, v), w) = B_{R}(u, B_{L}(v, w)). \)

Proof. By the definition of \( B_{R} \) and \( B_{L} \), \( B_{R}(u, v) = xr_{u} \), where \( v = xu \), and \( B_{L}(v, w) = \ell_{w}y \), where \( v = wy \). Let \( \alpha, \alpha', \beta \) and \( \beta' \) be such that \( r_{u}\alpha = u, w\alpha' = r_{u}, \beta\ell_{w} = w \), and \( \beta'w = \ell_{w} \).

Then \( B_{L}(B_{R}(u, v), w) = \ell_{w}y_{1} \), where \( y_{1} \) is such that \( (xr_{u}) = B_{R}(u, v) = wy_{1} \).

Also, \( B_{R}(u, B_{L}(v, w)) = x_{1}r_{u} \), where \( x_{1} \) is such that \( (\ell_{w}y) = B_{L}(v, w) = x_{1}u \). By multiplying the latter equalities by \( \beta \) we obtain:

\( (*) \)

\[ wy = \beta x_{1}u \]

We need to show that \( \ell_{w}y_{1} = x_{1}r_{u} \).
We saw that \( v = xu = xr_\alpha = B_\R(u,v)\alpha \) (by the choice of \( x \) and of \( \alpha \), and by the definition of \( B_\R \)). Thus
\[
B_\R(u,v)\alpha = v.
\]
In this equation we replace \( v \) by \( wy \) (see the definition of \( B_\L(v,w) \)), and we replace \( B_\R(u,v) \) by \( wy_1 \) (see the expression for \( B_\L(B_\R(u,v), w) \)). Thus,
\[
w_{y_1}\alpha = wy.
\]
By (*) we can replace \( wy \) by \( \beta x_1u \). So,
\[
w_{y_1}\alpha = \beta x_1u.
\]
Multiplying this by \( \alpha' \) (on the left) and by \( \beta' \) (on the right) yields \( \ell w_{y_1}\alpha = x_{1r_u} \), which is what we wanted. \( \square \)

**Lemma 3.18** Assume that \( u \leq_\L v \geq_\R w \). Then \( B(B_\R(v,u), r_v, w) = B(u, \ell_v, B_\L(w, v)) \).

**Proof.** By the definition of \( B_\R \) and \( B_\L \), we have:
\[
B(B_\R(v,u), r_v, w) = B(xr_v, r_v, w), \text{ where } u = xv, \text{ and}
\]
\[
B(u, \ell_v, B_\L(w, v)) = B(u, \ell_v, \ell_vy), \text{ where } w = vy.
\]

By the definition of \( B \), \( B(xr_v, r_v, w) = xr_vz_1 \), where \( w = r_vz_1 \). Hence, \( B(xr_v, r_v, w) = xw \).

Similarly, \( B(u, \ell_v, \ell_vy) = uz_2 \), where \( z_2 \) is any element of \( S \) satisfying \( \ell_vy = \ell_vz_2 \); hence we can pick \( z_2 \) to be \( y \). Then we have \( B(u, \ell_v, \ell_vy) = uy = xvy \) (since \( u = xv \)), and \( xvy = xw \) (since \( vy = w \)). Thus \( B(u, \ell_v, \ell_vy) = xw \), which is equal to \( B(xr_v, r_v, w) \), as we saw. \( \square \)

## 4 Termination

In this section we prove that the rewrite system for \((S)_{\text{reg}}\) is terminating.

**Lemma 4.1** If the sub-system consisting of the rules (2.1)–(2.4) is terminating then the whole rewrite system is terminating.

**Proof.** Imagine, by contraposition, that the whole rewrite system allows an infinite rewrite chain. Since the first group of rules is strictly length-reducing, the chain contains only rules of the form (2.1)–(2.4), from some point on. Hence the rules (2.1)–(2.4) do not form a terminating system. \( \square \)

The rest of this section deals with the proof that the sub-system consisting of the rules (2.1)–(2.4) terminates. In the remainder of this section, rewriting means applying the rules (2.1)–(2.4).

Since the rules (2.1)–(2.4) are length-preserving, the notion of *position* in a string is invariant under rewriting. More precisely, a string \( x = (x_1, \ldots, x_n) \) of length \( n \) over the generators of \((S)_{\text{reg}}\) has positions \( 1, 2, \ldots, n \), and when a rule of type (2.1)–(2.4) is applied, the new string still has positions \( 1, 2, \ldots, n \).

Our first step is to find factorizations of strings that are preserved under rewriting. See [3] for more background on preserved factorization schemes; here we do not need exact definitions since the context will make everything clear.
Lemma 4.2  In a string, a position occupied by 0 is invariant under rewriting. Similarly, the fact that a position is occupied by an element of \( S - \{0\} \) (respectively by an element of \( \overline{S - \{0\}} \)) is invariant under rewriting.

Proof. Since the rules (2.1)–(2.4) do not use the symbol 0, a position occupied by 0 will never change, and a non-0 symbol never turns into 0. Similarly, a position occupied by an element \( s \in S - \{0\} \) will always remain occupied by an element of \( S - \{0\} \), although the value of \( s \) can change. Similarly for \( \overline{S - \{0\}} \).  

Lemma 4.3 (Preservation of \( \preceq_L \), \( \equiv_L \), \( >_L \), and \( \overline{>}_L \), and similarly for \( \mathcal{R} \)).

In a string, a pair of positions occupied by elements \((s, \overline{t})\) \(\in S \times \overline{S} \) with \( s <_L t \) (or \( \equiv_L \) or \( >_L \) or \( \overline{>}_L \)) will always remain occupied by some pair of in \( S \times \overline{S} \) related by \( \preceq_L \) (respectively \( \equiv_L \) or \( >_L \) or \( \overline{>}_L \)). Similarly, for a pair in \( \overline{S} \times S \) related by \( \preceq_R \) (or \( \equiv_R \) or \( >_R \) or \( \overline{>}_R \)), this relation is preserved between these two positions.

Proof. Let us look at the four ways \( s \) or \( \overline{t} \) could be changed when a rule is applied just to the left or right of \((s, \overline{t})\).

If the symbol to the left of \((s, \overline{t})\) is \( \overline{u} \), with \( u >_R s \), then (2.2) can change \((\overline{u}, s, \overline{t})\) into \((\ell_u, B_L(s, u), \overline{t})\). Since \( B_L(s, u) \equiv_L s \) (by Lemma 3.3), we still have \( B_L(s, u) <_L t \) at this pair of positions.

If the symbol to the left of \((s, \overline{t})\) is \( \overline{u} \), with \( u \leq_R s \), then (2.3) can change \((\overline{u}, s, \overline{t})\) into \((B_L(u, s), \ell_s, \overline{t})\). Since \( \ell_s \equiv_L s \) we still have \( \ell_s <_L t \) at this pair of positions.

If the symbol to the right of \((s, \overline{t})\) is \( v \) with \( t >_R v \) (or \( t \leq_R v \)) then the reasoning is similar.  

As a consequence of these preservation lemmas we can factor any string into maximal subsegments, defined by the following properties:

- 0 does not occur in a subsegment, unless the subsegment consists of only 0;
- neighboring positions in a subsegment are occupied by pairs in \( S \times \overline{S} \) or \( \overline{S} \times S \);
- the incomparability relation \( \overline{>}_L \) (for \( \mathcal{L} \) or \( \mathcal{R} \)) does not occur inside a subsegment.

We call such subsegments continuous strings, i.e., we view the break between two maximal such subsegments as a discontinuity. The rewrite rules (2.1)–(2.4) preserve this factorization; no rewrite rule applies to two positions that are in different maximal subsegments.

A string is called continuous iff it consists of just one maximal subsegment. For a continuous string \( x = (x_1, \ldots, x_n) \) over the generators of \( (S)_{\text{reg}} \) and a position \( i \) (\( 1 \leq i < n \)), we write \( x_i > x_{i+1} \) (or \( <, \leq, \geq \)) iff the corresponding \( \mathcal{R} \)- or \( \mathcal{L} \)-relation holds in \( S \) according to the above Lemma.

**Definition.** Let \( x = (x_1, \ldots, x_n) \) be a continuous string of length \( n \). We call a position \( i \) (\( 1 \leq i \leq n \)) in \( x \) maximal iff

- \( i = 1 \) and \( x_1 > x_2 \), or
- \( i = n \) and \( x_{n-1} \leq x_n \), or
- \( 1 < i < n \) and \( x_{i-1} \leq x_i > x_{i+1} \).

By Lemma 4.3, maximal positions remain maximal during rewriting.
Lemma 4.4 (Maximal positions).
During the rewriting of a continuous string using rules (2.1)–(2.4), an element of $S \cup \overline{S}$ at a
maximal position is rewritten at most twice. From then on, the symbol at the maximal position never changes.

Proof. Suppose that a maximal position is occupied by an element $s \in S$ (the case of an element of $\overline{S}$ is similar). Let $\overline{u}, s, \overline{v}$ be the neighboring elements in the continuous string, with $u \leq s > v$. The element $\overline{u}$ or the element $\overline{v}$ may be absent. If (2.3) is applied, $(\overline{u}, s)$ will be rewritten to $(\ldots, \ell_s)$. If (2.1) is applied, $(s, \overline{v})$ will be rewritten to $(r_s, \ldots)$. If (2.3) is now applied (or (2.1) is applied to the previous alternative), the element at the maximal position is rewritten to $\ell_r s$ (respectively $r \ell_s$). Further rewriting with rules (2.1), (2.3) cannot change the element at the maximal position because $r \ell_s = \ell r_s$ and $\ell r_s = r \ell_s$. This follows from the special choice of the representatives of the $L$- and $R$-classes; recall that $\equiv_H$-related representatives are equal.

Note that the above Lemma (and the termination property itself) is not true if the representatives of the $L$- and $R$-classes are chosen differently than we did (except in trivial cases, e.g., when $S - \{0\}$ has no strict $> R$ and $> L$ chains).

Lemma 4.5 (Chains $\ldots > \cdot > \ldots$ and chains $\ldots \leq \cdot \leq \ldots$ stabilize).
If $s \in S$ occurs in a continuous string, with $\ldots > L s > R \ldots$ or $\ldots \leq R s \leq L \ldots$ in this string, then after a finite number of applications of the rules (2.1)–(2.4) to the string, the symbol at the position of $s$ will not change any more.

The same is true for an occurrence of $\overline{s} \in \overline{S}$ in a continuous string, with $\ldots > R s > L \ldots$ or $\ldots \leq L s \leq R \ldots$.

Proof. Let us consider a continuous string $(\ldots, s, \ldots)$ with $s \in S$ and $\ldots > L s > R \ldots$. By the previous lemma, we know that the element at the maximal position towards the left of $s$ will eventually stabilize. By induction, suppose that all elements in the descending alternating $> L > R$ chain to the left of $s$ have stabilized. No rule among (2.1)–(2.4) can be applied to the left of $s$ in this chain anymore (otherwise the element just left of $s$ would change again, since $u \neq r_u$, resp. $u \neq \ell_u$ in the rules). On the other hand, if a rule is applied to $s$ and the element just right of $s$ (in that case it would be rule (2.2)), then $s$ is replaced by $r_s$ and after this, no rule can be applied anymore at this position.

Let us also consider the case of a continuous string $(\ldots, s, \ldots)$ with $s \in S$ and $\ldots \leq R s \leq L \ldots$. As before, let us assume that all maximal positions have stabilized, and let us assume by induction that all elements in the ascending alternating $\leq L \leq R$ chain to the right of $s$ have stabilized. Again, no rule will be applied to the right of $s$ anymore. On the other hand, if a rule is applied to $s$ and the element just left of $s$ (in that case it will be rule (2.3)), then $s$ is replaced by $\ell_s$, and after this, no rule can be applied anymore at this position.

The reasoning is similar in the other cases.

Definition. Let $x = (x_1, \ldots, x_n)$ be a continuous string of length $n$. We call a position $i$ $(1 \leq i \leq n)$ minimal iff
- $i = 1$ and $x_1 \leq x_2$, or
- $i = n$ and $x_{n-1} > x_n$, or
- $1 < i < n$ and $x_{i-1} > x_i \leq x_{i+1}$.
By Lemma 4.3, minimal positions remain minimal during rewriting.

**Lemma 4.6** (Minimal positions stabilize).

*After a finite number of applications of the rules (2.1)–(2.4) to a continuous string the symbols at the minimal positions do not change anymore.*

*Proof.* Consider the case of a minimal position occupied by an element \( v \in S - \{0\} \), occurring in a context \((\ldots, u, v, w, \ldots)\), with \( u >_{\mathbb{R}} v \leq_{\mathcal{L}} w \). By the previous Lemma we assume that \( u \) and \( w \) will not change anymore. Then no rule can be applied to \( v \), otherwise \( u \) or \( w \) would change again, since \( s \neq r_s \), resp. \( s \neq \ell_s \) in the rules. \(\Box\)

The Lemmas imply that all positions in a string eventually stabilize for the rewrite rules (2.1)–(2.4).

### 5 Local confluence

This section contains the proof that the rewrite system for \((S)_{\text{reg}}\) is locally confluent. We have to look at all the overlap cases (see [7]), which is tedious but straightforward in each case. Each case is either trivial or it is resolved by using the properties of \(B, B_{\mathcal{L}}\) and \(B_{\mathcal{R}}\) proved in Section 3.

**Overlap 1.1–1.1:** \((st, u) \xleftarrow{1.1} (s, t, u) \xrightarrow{1.1} (s, tu)\).

Then \((st, u) \xrightarrow{1.1} (stu) \xleftarrow{1.1} (s, tu)\), where we also use associativity of the multiplication in \(S\).

The overlap for the \(S\)-form of rule 1.1 has the form \((ts, u) \xleftarrow{1.1} (s, t, u) \xrightarrow{1.1} s, tu\).

Confluence follows easily as above.

**Overlaps with 1.2:** In all overlaps with rule 1.2 one easily shows confluence to \((0)\).

**Overlap 1.1–1.3:**

**Case 1.** \(S\)-form of rule 1.1.

\[(tu, \overline{v}) \xleftarrow{1.1} (t, u, \overline{v}) \xrightarrow{1.3} (t, 0) \text{ where } u \not\preceq_{\mathcal{L}} v.\]

Then \((t, 0) \xrightarrow{1.2} (0) \xleftarrow{1.3} (tu, \overline{v})\). The last application of rule 1.3 is justified by the following.

*Claim:* If \( u \not\preceq_{\mathcal{L}} v \) then \( tu \not\preceq_{\mathcal{L}} v \).

*Proof of the Claim:* By contraposition, if \( u \preceq_{\mathcal{L}} tu \preceq_{\mathcal{L}} v \) then obviously \( u \preceq_{\mathcal{L}} v \). And if \( u \preceq_{\mathcal{L}} tu \leq_{\mathcal{L}} v \) then \( u \preceq_{\mathcal{L}} v \), by unambiguity of \(S\). This proves the Claim.

**Case 2.** \(\overline{S}\)-form of rule 1.1.

\[(0, \overline{v}) \xleftrightharpoons{1.3} (t, u, \overline{v}) \xrightarrow{1.1} (t, \overline{vu}) \text{ where } t \not\preceq_{\mathcal{L}} u.\]

Confluence is proved in the same way as above.

**Overlap 1.1–1.4:** Similar to the previous case.

**Overlap 1.1–1.5:**
Case 1. \((tu, \overline{\nu}, w) \xleftarrow{11} (t, u, \overline{\nu}, w) \xrightarrow{15} (t, B(u, v, w)), \) where \(u \leq \mathcal{L} v \geq \mathcal{R} w.\)

Then \((tu, \overline{\nu}, w) \xrightarrow{15} B(tu, v, w), \) and \(t \cdot B(u, v, w) \xleftarrow{11} (t, B(u, v, w)).\)

But by Lemma 3.6, \(B(tu, v, w) = t \cdot B(u, v, w), \) so we have confluence.

Case 2. \((u, \overline{\nu}, wt) \xleftarrow{11} (u, \overline{\nu}, w, t) \xrightarrow{15} (B(u, v, w) \cdot t) \) where \(u \leq \mathcal{L} v \geq \mathcal{R} w.\)

As in the previous case, we have confluence by Lemma 3.6.

Here we only considered the \(S\)-form of rule 1.1; the \(\overline{S}\)-form does not overlap with 1.5.

Overlap 1.1–1.6: Only the \(\overline{S}\)-form of 1.1 overlaps with 1.6. Confluence is proved in a similar way as in 1.1–1.5.

Overlap 1.1(\(S\)-form) – 2.1: \((su, \overline{\nu}) \xleftarrow{11} (s, u, \overline{\nu}) \xrightarrow{21} (s, r_{u, B_{\mathcal{R}}(u, v)}), \) where \(u \geq \mathcal{L} v.\)

Case 1. \(su > \mathcal{L} v.\)

Then \((su, \overline{\nu}) \xrightarrow{21} (r_{su, B_{\mathcal{R}}(su, v)}), \) since \(su > \mathcal{L} v.\)

Moreover, \((s, r_{u, B_{\mathcal{R}}(u, v)}) \xrightarrow{11} (sr_{u, B_{\mathcal{R}}(u, v)})) \xrightarrow{21} (r_{sr_{u, B_{\mathcal{R}}(sr_{u, B_{\mathcal{R}}(u, v)}}))\), where the latter application of rule 2.1 is justified since \(sr_{u} > \mathcal{L} B_{\mathcal{R}}(u, v)\) (indeed we assumed \(su > \mathcal{L} v,\) so by Lemma 3.7, \(sr_{u} = su\).

To have confluence we need \(rs_{u} = r_{sr_{u}}\) \(\) (which easily follows from \(u \equiv \mathcal{R} r_{u},\) and \(B_{\mathcal{R}}(su, B_{\mathcal{R}}(u, v))\) \(\) \(\) (which is proved in Lemma 3.7 (1)).

Case 2. \(su \leq \mathcal{L} v.\)

Then \((su, \overline{\nu}) \xrightarrow{24} (B_{\mathcal{R}}(v, su), \overline{\nu}).\)

Moreover, \((s, r_{u, B_{\mathcal{R}}(u, v)}) \xrightarrow{11} (sr_{u, B_{\mathcal{R}}(u, v)})) \xrightarrow{24} (B_{\mathcal{R}}(B_{\mathcal{R}}(u, v), sr_{u}, r_{B_{\mathcal{R}}(u, v)}))\). The latter application of rule 2.4 is justified since \(sr_{u} \leq \mathcal{L} B_{\mathcal{R}}(u, v),\) which follows from the assumption \(su \leq \mathcal{L} v\) and from Lemma 3.1.

In order to have confluence we need \(B_{\mathcal{R}}(B_{\mathcal{R}}(u, v), sr_{u}) = B_{\mathcal{R}}(v, su)\) (which was proved in Lemma 3.7 (2)), and \(r_{B_{\mathcal{R}}(u, v)} = r_{v}\) (which follows from Lemma 3.3).

Case 3. \(su \nmid \mathcal{L} v.\)

Then \((su, \overline{\nu}) \xrightarrow{13} (0).\)

Moreover, \((s, r_{u, B_{\mathcal{R}}(u, v)}) \xrightarrow{11} (sr_{u, B_{\mathcal{R}}(u, v)})).\) By Lemma 3.7 (3), \(sr_{u} \nmid \mathcal{L} B_{\mathcal{R}}(u, v),\) so we can now apply rule 1.3, thus obtaining confluence to (0).

Overlap 1.1(\(S\)-form) – 2.1: \((r_{u, B_{\mathcal{R}}(u, v), \overline{s}}) \xleftarrow{11} (u, \overline{\nu}, \overline{s}) \xrightarrow{21} (u, \overline{s}),\) where \(u \geq \mathcal{L} v.\)

Then \((r_{u, B_{\mathcal{R}}(u, v), \overline{s}}) \xrightarrow{11} (r_{u, B_{\mathcal{R}}(u, v)}),\) and \((u, \overline{s}) \xrightarrow{21} (r_{u, B_{\mathcal{R}}(u, sv)});\) 2.1 was applicable since \(u > \mathcal{L} v \geq \mathcal{S} v.\) Confluence then follows directly from Lemma 3.7 (4).

Overlap 1.1–2.2: This is similar to the overlap 1.1–2.1.

Overlap 1.1–2.3: This is similar to the overlap 1.1–2.4, which we consider next.

Overlap 1.1(\(S\)-form) – 2.4: \((sv, \overline{\nu}) \xleftarrow{11} (s, v, \overline{\nu}) \xrightarrow{24} (s, B_{\mathcal{R}}(u, v), \overline{r_{u}}),\) where \(v \leq \mathcal{L} u.\)

Then \((sv, \overline{\nu}) \xrightarrow{24} (B_{\mathcal{R}}(u, sv), \overline{r_{u}}).\)

Moreover, \((s, B_{\mathcal{R}}(u, v), \overline{r_{u}}) \xrightarrow{11} (s \cdot B_{\mathcal{R}}(u, v), \overline{r_{u}}).\)

Confluence then follows from Lemma 3.7 (4).
Overlap 1.1 (S-form) – 2.4:  \((B_R(u, s), \overline{r}, \overline{v}) \xrightarrow{2.4} (s, \overline{w}, \overline{v}) \xrightarrow{1.1} (s, \overline{w}u), \) where \(s \leq \ell u.\)

**Case 1.** \(s \leq \ell vu \leq \ell u.\)

Then \((s, \overline{w}u) \xrightarrow{2.4} (B_R(vu, s), \overline{v}u).\)

On the other hand, \((B_R(u, s), \overline{r}, \overline{v}) \xrightarrow{1.1} (B_R(u, s), \overline{v}r_u) \xrightarrow{2.4} (B_R(vu, B_R(u, s)), r_{vu}).\)

The last application of rule 2.4 is justified by Lemma 3.7 (1).

To check confluence we observe that \(vu \equiv_R vr_u,\) and that \(B_R(vu, s) = B_R(vu, B_R(u, s))\) by Lemma 3.7 (1).

**Case 2.** \(vu < \ell s \leq \ell u.\)

Then \((s, \overline{w}u) \xrightarrow{2.1} (r, \overline{B_R(s, vu)}).\)

On the other hand, \((B_R(u, s), \overline{r}, \overline{v}) \xrightarrow{1.1} (B_R(u, s), \overline{v}r_u) \xrightarrow{2.1} (r_{B_R(u, s)}, \overline{B_R(B_R(u, s), vr_u)}).\)

The last application of rule 2.1 is justified by Lemma 3.7 (2).

Confluence now follows from Lemma 3.7 (2), and from the fact that \(s \equiv_R B_R(u, s)\) (Lemma 3.2).

**Case 3.** \(vu \not\equiv_{\ell} s.\)

Then \((s, \overline{w}u) \xrightarrow{1.3} (0).\) On the other hand, \((B_R(u, s), \overline{r}, \overline{v}) \xrightarrow{1.1} (B_R(u, s), \overline{v}r_u) \xrightarrow{1.3} (0).\)

We used Lemma 3.7 (3) to justify the last application of rule 1.3.

So far we have considered all overlaps involving the rule 1.1. We mentioned already that the rule 1.2 always leads to confluence to \((0).\) Let us now look at all the overlaps that involve rule 1.3 (other than with rule 1.1, seen already).

There is no overlap of 1.3 with itself.

**Overlap 1.3–1.4:** \((0, s) \xleftarrow{1.3} (u, \overline{v}, s) \xrightarrow{1.4} (u, 0), \) where \(u \not\equiv_{\ell} v\) and \(v \not\equiv_{\ell} s.\)

Then we obviously have confluence to \((0).\)

The case of \((\overline{u}, v, \overline{s})\), where \(u \not\equiv_{\ell} v\) and \(v \not\equiv_{\ell} s,\) is handled in a similar way.

**Overlap 1.3–1.5:** \((B(u, v, w), \overline{s}) \xrightarrow{1.5} (u, \overline{v}, w, \overline{s}) \xrightarrow{1.3} (u, \overline{v}, 0),\)

where \(u \leq \ell v \geq_R w\) and \(w \not\equiv_{\ell} s.\)

Then \((u, \overline{v}, 0) \rightarrow (0)\) by two applications of rule 1.2. Moreover, since \(B(u, v, w) \not\equiv_{\ell} s\) if \(w \not\equiv_{\ell} s\) (by Lemma 3.8), we also have \((B(u, v, w), \overline{s}) \xrightarrow{1.3} (0).\)

**Overlap 1.3–1.6:** This is similar to 1.3–1.5.

There are no overlaps 1.3–2.1, 1.3–2.4, nor 1.4–1.4, 1.4–2.2, 1.4–2.3. The overlaps 1.4–1.5 and 1.4–1.6 are similar to the case 1.3–1.5.

**Overlaps 1.3–2.2, 1.3–2.3, or 1.4–2.1:** This is very similar to the case considered next.

**Overlap 1.4–2.4:** \((B_R(u, v), \overline{u}, w) \xleftarrow{2.4} (v, \overline{u}, w) \xrightarrow{1.4} (v, 0), \) where \(v \leq \ell u \not\equiv_{\ell} w.\)

Then \((v, 0) \rightarrow (0)\) by rule 1.2. Moreover, since \(r_u \equiv_R u \not\equiv_{\ell} w\) we have \((B_R(u, v), \overline{u}, w) \rightarrow (B_R(u, v), 0)\) by rule 1.4; this then leads to \((0)\) by 1.2.

**Overlap 1.5–1.5:** \((B(u, v, w), \overline{s}, t) \xleftarrow{1.5} (u, \overline{v}, w, \overline{s}, t) \xrightarrow{1.5} (u, \overline{v}, B(w, s, t)),\)

where \(u \leq \ell v \geq_R w \leq \ell s \geq_R t.\)
Then \( (B(u, v, w), \overline{s}, t) \xrightarrow{1.5} (B(B(u, v, w), s, t)) \); rule 1.5 was applicable here by Lemma 3.9. Also, \((u, \overline{v}, B(w, s, t)) \xrightarrow{1.5} (B(u, v, B(w, s, t))\); rule 1.5 was applicable here by Lemma 3.9. Confluence then follows from Lemma 3.9.

**Overlap 1.5–1.6:** \( (B(u, v, w), \overline{s}) \xleftarrow{1.5} (u, \overline{v}, w, \overline{s}) \xrightarrow{1.6} (u, \overline{B(s, w, v)}) \), where \( u \leq_{\mathcal{L}} v \equiv_{\mathcal{R}} w \geq_{\mathcal{L}} s \).

**Case 1.** \( B(u, v, w) \leq_{\mathcal{L}} s \).

In this case rule 2.4 applies and \( (B(u, v, w), \overline{s}) \xrightarrow{2.4} (B_{\mathcal{R}}(s, B(u, v, w)), \overline{r_s}) \). By Lemma 3.11 (1), rule 2.4 then also applies to \((u, B(s, w, v))\), thus producing \((B_{\mathcal{R}}(B(s, w, v), u), \overline{r_{B(s, w, v)}})\). Lemma 3.11 (2.\( \leq \)) then shows confluence.

**Case 2.** \( B(u, v, w) >_{\mathcal{L}} s \).

In this case \((B(u, v, w), \overline{s}) \xrightarrow{2.1} (r_{B(u,v,w)}, B_{\mathcal{R}}(B(u, v, w), s))\). By Lemma 3.11 (1), rule 2.1 then also applies to \((u, B(s, w, v))\), and this yields \((B_{\mathcal{R}}(B(s, w, v), u), \overline{r_{B(s, w, v)}})\). Lemma 3.11 (2.\( > \)) then shows confluence.

**Case 3.** \( B(u, v, w) \not\equiv_{\mathcal{L}} s \).

Then \((B(u, v, w), \overline{s}) \xrightarrow{1.3} (0)\). Moreover, by Lemma 3.11 (1), in this case we also have \( u \not\equiv_{\mathcal{L}} B(s, w, v)\), hence rule 1.3 also applies to \((u, B(s, w, v))\) and produces \((0)\).

The overlap case \( \xleftarrow{1.6} (u, \overline{v}, \overline{w}, s) \xrightarrow{1.5} \) is similar to the case above.

**Overlap 1.5–2.1:** \( (B(u, v, w), \overline{s}) \xleftarrow{1.5} (u, \overline{v}, w, \overline{s}) \xrightarrow{2.1} (u, \overline{v}, r_w, B_{\mathcal{R}}(w, s)), \) where \( u \leq_{\mathcal{L}} v \equiv_{\mathcal{R}} w >_{\mathcal{L}} s \).

**Case 1.** \( B(u, v, w) \leq_{\mathcal{L}} s \).

Then \( (B(u, v, w), \overline{s}) \xrightarrow{2.4} (B_{\mathcal{R}}(s, B(u, v, w)), \overline{r_s}) \). Moreover, \((u, \overline{v}, r_w, B_{\mathcal{R}}(w, s)) \xrightarrow{1.5} (B(u, v, r_w), B_{\mathcal{R}}(w, s)) \). The last application of rule 2.4 is justified by Lemma 3.13 (1). Confluence then follows immediately from Lemma 3.13 (2.\( \leq \)).

**Case 2.** \( B(u, v, w) >_{\mathcal{L}} s \).

Then \( (B(u, v, w), \overline{s}) \xrightarrow{2.1} (r_{B(u,v,w)}, B_{\mathcal{R}}(B(u, v, w), s)) \). Moreover, \((u, \overline{v}, r_w, B_{\mathcal{R}}(w, s)) \xrightarrow{1.5} (B(u, v, r_w), B_{\mathcal{R}}(w, s)) \). The last application of rule 2.1 is justified by Lemma 3.13 (1). Confluence then follows immediately from Lemma 3.13 (2.\( < \)).

**Case 3.** \( B(u, v, w) \not\equiv_{\mathcal{L}} s \).

Then \( (B(u, v, w), \overline{s}) \xrightarrow{1.3} (0)\). Moreover, \((u, \overline{v}, r_w, B_{\mathcal{R}}(w, s)) \xrightarrow{1.5} (B(u, v, r_w), B_{\mathcal{R}}(w, s)) \xrightarrow{1.3} (0)\). The last application of rule 1.3 is justified by Lemma 3.13 (1).

**Overlap 1.5–2.2:**

**Case 1.** \( u \leq_{\mathcal{L}} v \equiv_{\mathcal{R}} w \) and \((B(u, v, w)) \xleftarrow{1.5} (u, \overline{v}, B_{\mathcal{L}}(w, v)) \xrightarrow{1.5} (B(u, \overline{v}, B_{\mathcal{L}}(w, v)))\).

Confluence then follows from the \(B_{\mathcal{L}}\)-version of Lemma 3.14.
Case 2. \( s > \underline{v}, u \leq \underline{v} \geq \overline{v} \) and
\( (\overline{s}, B_{\overline{\ell}}(u, s)), \overline{v}, w ) \xrightarrow{2.2} (\overline{s}, u, \overline{v}, w ) \xrightarrow{1.5} (\overline{s}, B(u, v, w)). \)

Then \( (\overline{s}, B_{\overline{\ell}}(u, s)), \overline{v}, w ) \xrightarrow{1.5} (\overline{s}, B(B_{\overline{\ell}}(u, s), v, w)) \); rule 1.5 is applicable here since by Lemma \( \textit{3.3} \). \( B_{\overline{\ell}}(u, s) \equiv \underline{u} \leq \underline{v} \geq \overline{v} \) \( w \).

On the other hand, \( (\overline{s}, B(u, v, w)) \) \( \xrightarrow{2.2} (\overline{s}, B_{\overline{\ell}}(B(u, v, w), s)) \); rule 2.2 is applicable here since \( s > \underline{u} \geq \overline{u} \) \( x = B(u, v, w) \) (where the last equality holds by Lemma \( \textit{3.3} \)).

Confluence then follows from the \( B_{\overline{\ell}} \)-version of Lemma \( \textit{3.13} \).

Overlap 1.5–2.3:
Case A. \( \xrightarrow{1.5} (u, \overline{v}, w) \xrightarrow{2.3} , \) where \( u \leq \underline{v} \geq \overline{v} \).

This is similar to Case A of the overlap 1.5–2.4, treated below.

Case B. \( (B_{\overline{\ell}}(s, u), \ell_u, \overline{v}, w ) \xrightarrow{2.3} (\overline{s}, u, \overline{v}, w ) \xrightarrow{1.5} (\overline{s}, B(u, v, w)), \) where \( s \leq \underline{u} \leq \underline{v} \geq \overline{v} \) \( w \).

Then \( (B_{\overline{\ell}}(s, u), \ell_u, \overline{v}, w ) \xrightarrow{1.5} (B_{\overline{\ell}}(s, u), B(\ell_u, v, w)). \)

Case B.1 \( s \leq \underline{u} \leq \underline{v} \geq \overline{v} \).

Then \( (\overline{s}, B(u, v, w)) \xrightarrow{2.3} (B_{\overline{\ell}}(s, B(u, v, w)), \ell B(u, v, w)). \)

On the other hand, \( (B_{\overline{\ell}}(s, u), B(\ell_u, v, w)) \xrightarrow{2.3} (B_{\overline{\ell}}(B_{\overline{\ell}}(s, u), B(\ell_u, v, w)), \ell B(\ell_u, v, w)). \) Rule 2.3 was applicable here by the \( \mathcal{R} \)-version of Lemma \( \textit{3.13} \) (1).

Confluence then follows from the \( \mathcal{R} \)-version of Lemma \( \textit{3.13} \) (2, \( \preceq \)).

Case B.2 \( s > \underline{u} \leq \underline{v} \geq \overline{v} \).

Then \( (\overline{s}, B(u, v, w)) \xrightarrow{2.2} (\overline{s}, B_{\overline{\ell}}(B(u, v, w), s)), \) and
\( (B_{\overline{\ell}}(s, u), B(\ell_u, v, w)) \xrightarrow{2.2} (\overline{s}, B_{\overline{\ell}}(B(\ell_u, v, w), B_{\overline{\ell}}(s, u))). \) Rule 2.2 was applicable here by the \( \mathcal{R} \)-version of Lemma \( \textit{3.13} \) (1).

Confluence then follows from the \( \mathcal{R} \)-version of Lemma \( \textit{3.13} \) (2, \( < \)).

Case B.3 \( s \xrightarrow{1.4} \leq \underline{v} \geq \overline{v} \).

Then \( (\overline{s}, B(u, v, w)) \xrightarrow{1.4} (0) \) and
\( (B_{\overline{\ell}}(s, u), B(\ell_u, v, w)) \xrightarrow{1.4} (0), \) where the application of rule 1.4 is justified by the \( \mathcal{R} \)-version of Lemma \( \textit{3.13} \) (1).

Overlap 1.5–2.4:
Case A. \( (B_{\overline{\ell}}(v, u), \overline{w}, w ) \xrightarrow{2.4} (u, \overline{v}, w ) \xrightarrow{1.5} (B(u, v, w)), \) where \( u \leq \underline{v} \geq \overline{v} \).

Then rule 1.5 is applicable to \( (B_{\overline{\ell}}(v, u), \overline{w}, w ) \) because \( u \leq \underline{v} \geq \overline{v} \) \( w \) implies by Lemma \( \textit{3.4} \). \( B_{\overline{\ell}}(v, u) \leq \underline{v} \leq \overline{v} \geq \overline{w} \). Applying 1.5 then yields \( (B(B_{\overline{\ell}}(v, u), r_v, w)) \). Thus by Lemma \( \textit{3.14} \) we have confluence.

Case B. \( (B(u, v, w), \overline{s}) \xrightarrow{1.5} (u, \overline{v}, w, \overline{s}) \xrightarrow{2.4} (u, \overline{v}, B_{\overline{\ell}}(s, u), \overline{s}), \) where \( u \leq \underline{v} \geq \overline{v} \) \( w \leq \underline{s} \).

Then rule 2.4 is applicable to \( (u, \overline{v}, B_{\overline{\ell}}(s, w), \overline{s}) \) because by Lemma \( \textit{3.3} \) \( B(u, v, w) = yw \leq \underline{v} \geq \overline{w} \leq \underline{s} \leq \overline{v} \). Then 2.4 yields \( (B_{\overline{\ell}}(s, B(u, v, w)), \overline{s}) \).

On the other hand, rule 1.5 is applicable to \( (u, \overline{v}, B_{\overline{\ell}}(s, w), \overline{s}) \) because \( v \geq \overline{w} \equiv \overline{w} \) \( B_{\overline{\ell}}(s, w) \) (the latter by Lemma \( \textit{3.3} \)). Then 1.5 yields \( (B(u, v, B_{\overline{\ell}}(s, w)), \overline{s}). \)
By Lemma 3.15 we have confluence.

The overlaps of rule (1.6) with rules (1.6), (2.1)–(2.4) are handled in a similar way as the overlaps of (1.5) with rules (1.5), (2.1)–(2.4).

We now come to the overlaps of the rules 2.1–2.3. Obviously, 2.1 cannot overlap with itself nor with 2.4.

**Overlap 2.1–2.2:** \((r_u, B_R(u, v), w) \xrightarrow{2.1} (u, v, w) \xrightarrow{2.2} (u, v, w, B_\ell(w, v))\), where \(u >_\ell v \geq w\).

Then \((r_u, B_R(u, v), w) \xrightarrow{2.2} (r_u, v, B_R(u, v), w)\). Rule 2.2 was applicable here since by Lemma 3.3, \(B_R(u, v) \equiv v >_w w\).

On the other hand, \((u, v, B_\ell(w, v)) \xrightarrow{2.1} (u, v, v, B_\ell(w, v))\). Rule 2.1 was applicable here since \(u >_\ell v \equiv \ell_v v\).

Next, applying rule 2.2 to this yields \((r_u, v, B_R(u, v), w)\). Rule 2.2 was indeed applicable here since by Lemma 3.3, \(B_R(u, v) \equiv v \geq \ell_v \geq \ell_v y = B_\ell(w, v)\) where \(uv = v\); moreover, \(u \geq \ell_v v = u\), which contradicts an assumption.

Lemma 3.16 immediately shows confluence now.

The other overlap case for rules 2.1 and 2.2 is of the form
\((\ell_v, B_\ell(v, w), w) \xrightarrow{2.2} (u, v, w) \xrightarrow{2.1} (u, v, v, B_\ell(v, w))\),
where \(u >_\ell v \geq w\).

This case is similar to the case above.

**Overlap 2.1–2.3:** \((r_u, B_R(u, v), w) \xrightarrow{2.1} (u, v, w) \xrightarrow{2.3} (u, B_\ell(v, w), w)\),
where \(u >_\ell v \leq w\).

Then \((r_u, B_R(u, v), w) \xrightarrow{2.3} (r_u, B_\ell(v, w), w)\). Rule 2.3 was applicable here since \(B_R(u, v) \equiv v\).

On the other hand, \((u, B_\ell(v, w), w) \xrightarrow{2.1} (u, v, v, B_\ell(v, w))\). Rule 2.1 was applicable here since \(B_\ell(v, w) \equiv v\).

Confluence now follows from Lemma 3.17.

The other overlap case for the rules 2.1 and 2.3 is of the form
\((\ell_v, B_\ell(v, w)) \xrightarrow{2.3} (u, v, w) \xrightarrow{2.1} (u, v, v, B_\ell(v, w))\),
where \(u \leq v \geq _\ell w\).

This is similar to the overlap case of 2.2–2.4 that we will study next.

**Rule 2.2** has no overlap with itself nor with 2.3.

**Overlap 2.2–2.4:** \((B_R(v, u), w) \xrightarrow{2.4} (u, v, w) \xrightarrow{2.2} (u, v, B_\ell(w, v), w)\),
where \(u \leq v \geq w\).

Rule 1.5 is applicable to \((B_R(v, u), v, w)\) since \(B_R(v, u) = v \leq v \equiv v > w\). This yields \((B(B_R(v, u), v, w))\).

Rule 1.5 is also applicable to \((u, v, B_\ell(w, v))\) since \(u \leq v \equiv \ell_v \geq \ell_v y = B_\ell(w, v)\).

This yields \((B(u, v, B_\ell(w, v)))\).
Lemma 3.18 immediately implies confluence.

The other overlap case for the rules 2.2 and 2.4 is of the form

\((\overline{\ell u}, B_L(v, u), w) \xrightarrow{2.2} (\overline{\ell v, w}) \xrightarrow{2.4} (\overline{\ell, B_R(w, v), r_w})\), where \(u >_R v \leq_L w\).

This is very similar to the overlap case of 2.1–2.3 that we studied explicitly.

Overlap 2.3–2.4: \((B_R(v, u), r_v, w) \xrightarrow{2.4} (u, r_w, B_L(v, w), \ell_w)\), where \(u \leq_L v \geq_R w\).

Then \((B_R(v, u), r_v, w) \xrightarrow{2.3} (B_R(v, u), B_L(r_v, w), \ell_w) \xrightarrow{2.4} (B_R(B_L(r_v, w), B_R(v, u)), r_{B_R(r_v, w)}, \ell_w)\); the last application of rule 2.4 was justified since \(B_R(v, u) = xr_v \leq_L r_v \equiv_L B_L(r_v, w)\) (the last \(L\)-equivalence follows from Lemma 3.3).

On the other hand, \((u, B_L(v, w), \ell_w) \xrightarrow{2.3} (B_R(v, w), r_w, \ell_w) \xrightarrow{2.4} (B_R(B_L(v, w), u), r_{B_L(v, w)}), \ell_w)\); the application of rule 2.4 was justified since \(u \leq_L v \equiv_L B_L(v, w)\) (where the last \(L\)-equivalence follows from Lemma 3.3).

Confluence now follows immediately from the \(L - R\) dual of Lemma 3.16.

The other overlap case for the rules 2.3 and 2.4 is of the form

\((B_L(u, v), \ell_v, w) \xrightarrow{2.3} (u, v, w) \xrightarrow{2.4} (\overline{\ell, B_R(w, v), r_v})\),

where \(u \geq_R v \leq_L w\).

This is similar to the above case.

This completes the exhaustive analysis of all overlap cases, and shows that the rewrite system for \((S)_{reg}\) is locally confluent.

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