SPACES OF CLOSED SUBGROUPS
OF LOCALLY COMPACT GROUPS

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Abstract. The set \( C(G) \) of closed subgroups of a locally compact group \( G \) has a natural topology which makes it a compact space. This topology has been defined in various contexts by Vietoris, Chabauty, Fell, Thurston, Gromov, Grigorchuk, and many others.

The purpose of the talk was to describe the space \( C(G) \) first for a few elementary examples, then for \( G \) the complex plane, in which case \( C(G) \) is a 4–sphere (a result of Hubbard and Pourezza), and finally for the 3–dimensional Heisenberg group \( H \), in which case \( C(H) \) is a 6–dimensional singular space recently investigated by Martin Bridson, Victor Kleptsyn and the author [BrHK].

These are slightly expanded notes prepared for a talk given at several places: the Kortrijk workshop on Discrete Groups and Geometric Structures, with Applications III, May 26–30, 2008; the Tripode 14, École Normale Supérieure de Lyon, June 13, 2008; and seminars at the EPFL, Lausanne, and in the Université de Rennes 1. The notes do not contain any other result than those in [BrHK], and are not intended for publication.

I. Mahler (1946)

Let \( n \) be a positive integer. A lattice in \( \mathbb{R}^n \) is a subgroup of \( \mathbb{R}^n \) generated by a basis. Two lattices \( L, L' \subset \mathbb{R}^n \) are close to each other if there exist basis \( \{e_1, \ldots, e_n\} \subset L, \{e'_1, \ldots, e'_n\} \subset L' \) with \( e_j \) and \( e'_j \) close to each other for each \( j \); this defines a topology on the space \( \mathcal{L}(\mathbb{R}^n) \) of lattices in \( \mathbb{R}^n \). It coincides with the natural topology on \( \mathcal{L}(\mathbb{R}^n) \) viewed as the homogeneous space \( GL_n(\mathbb{R})/GL_n(\mathbb{Z}) \). It is easy to check that the covolume function \( L \mapsto \text{vol}(\mathbb{R}^n/L) \) and the minimal norm \( L \mapsto \min_{L \neq 0} \|x\|^2 \) are continuous functions \( \mathcal{L}(\mathbb{R}^n) \to \mathbb{R}^*_+ \).

1. Mahler’s Criterion [Mahl–46]. For a subset \( \mathcal{M} \) of \( \mathcal{L}(\mathbb{R}^n) \), the two following properties are equivalent:
   (i) \( \mathcal{M} \) is relatively compact;
   (ii) there exist \( C, c > 0 \) such that \( \text{vol}(\mathbb{R}^n/L) \leq C \) and \( \min L \geq c \) for all \( L \in \mathcal{M} \).

For proofs of this criterion, see for example [Bore–69, corollaire 1.9] or [Ragh–72, Corollary 10.9].

An immediate use of the criterion is the proof that, in any dimension \( n \geq 1 \), the space \( \mathcal{L}^{\text{unimod}}(\mathbb{R}^n) \) of lattices of covolume 1 contains a lattice \( L_{\max} \) such that \( \min L_{\max} = \sup\{\min L \mid L \in \mathcal{L}^{\text{unimod}}(\mathbb{R}^n)\} \); equivalently, there exists a lattice \( L_{\max} \) with a maximal

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density for the associated ball packing. (It is a much more difficult problem to identify such a lattice $L_{\text{max}}$, and this problem is open unless $n \leq 8$ or $n = 24$.)

Let us quote one other application of Mahler’s Criterion. It is one of the ingredients of the proof of the following fact, special case for $SL_n(\mathbb{R})$ of a more general result conjectured by Godement and proved independently by Borel–Harish Chandra and by Mostow–Tamagawa [Ragh–72, Theorem 10.18]: an arithmetic lattice in $SL_n(\mathbb{R})$ is cocompact if and only if it does not contain any unipotent matrix.

II. Chabauty (1950) and Fell (1962)

For a topological space $X$, let $2^X$ denote the set of closed subsets of $X$. For a compact subset $K$ and a nonempty open subset $U$ of $X$, set

$$\mathcal{O}_K = \{ F \in 2^X \mid F \cap K = \emptyset \} \quad \text{and} \quad \mathcal{O}'_U = \{ F \in 2^X \mid F \cap U \neq \emptyset \}.$$  

The finite intersections $\mathcal{O}_{K_1} \cap \cdots \cap \mathcal{O}_{K_m} \cap \mathcal{O}'_{U_1} \cap \cdots \cap \mathcal{O}'_{U_n}, m, n \geq 0$, constitute a basis of the Chabauty topology on $2^X$; observe that $\mathcal{O}_{K_1} \cap \cdots \cap \mathcal{O}_{K_m} = \mathcal{O}_{K_1 \cup \cdots \cup K_m}$.

For example, consider the case $X = \mathbb{R}^n$, and two lattices $L_0, L$ in this space. If $K$ is a compact subset of $\mathbb{R}^n$ disjoint from $L_0$, then $L \in \mathcal{O}_K$ if and only if $L$ is also disjoint from $K$; if $U$ is an open neighbourhood of some point of $L_0$, then $L \in \mathcal{O}'_U$ if and only if $L$ has also one point (at least) in $U$. It follows that the Chabauty topology on $2^{\mathbb{R}^n}$ induces on $\mathcal{L}(\mathbb{R}^n)$ the same topology as that considered in Section I.

This topology was defined in [Chab–50] (written in French), as a tool to show a version of Mahler’s Criterion which is valid in a large class of topological groups (Proposition 3 below). Later, this topology was studied in greater detail by Fell for unrelated purposes, precisely for the study of the appropriate topology on the space of irreducible unitary representations of a locally compact group; Fell (who writes in English) does not quote Chabauty (see [Fell–62] and [Fell–64]). This Chabauty topology, or Chabauty–Fell topology, or $H$–topology (terminology of Fell), or geometric topology (terminology of Thurston), on $2^X$ should not be confused with other standard topologies on the same space, of which the study goes back to Hausdorff and Vietoris, and for which a canonical reference is [Mich–51] (however, for $X$ compact, the Chabauty topology coincides with the $F$–topology of Michael). The Chabauty topology is also useful in the study of low–dimensional manifolds; see [Thurs, Definition 9.1.1], as well as [CaEG–87, Chapter I.3], where Chabauty is quoted (but Fell is not). A pleasant account of the most basic properties of this topology can be found in [Paul–07].

2. Proposition. Let $X$ be a topological space, and let $2^X$ be endowed with the Chabauty topology.

(i) The space $2^X$ is compact.

(ii) In case $X$ is discrete, $2^X$ is homeomorphic to the product space $\{0, 1\}^X$ with the compact Tychonoff product topology.

Suppose moreover that $X$ is a locally compact metric space.

(iii) The Chabauty topology on $2^X$ is induced by the metric defined by

$$d(F_1, F_2) = \inf \left\{ \varepsilon > 0 \mid F_1 \cup \left(X \setminus B(*, 1/\varepsilon)\right) \subset V_\varepsilon \left(F_2 \cup \left(X \setminus B(*, 1/\varepsilon)\right)\right), \quad F_2 \cup \left(X \setminus B(*, 1/\varepsilon)\right) \subset V_\varepsilon \left(F_1 \cup \left(X \setminus B(*, 1/\varepsilon)\right)\right) \right\}.$$  

Here, for a subset $S$ of $X$, we write $V_\varepsilon(S)$ for $\{ x \in X \mid d(x, S) < \varepsilon \}$; and $B(*, 1/\varepsilon)$ is the open ball of radius $1/\varepsilon$ around an arbitrarily chosen base point $*$ in $X$. 
(iv) A sequence \((F_j)_{j \geq 1}\) of closed subsets of \(X\) converges in \(2^X\) to a closed subset \(F\) if and only if the two following conditions hold:

- for all \(x \in F\), there exists for all \(i \geq 1\) a point \(x_i \in F_i\) such that \(x_i \to x\),
- for all strictly increasing sequence \((i_j)_{j \geq 1}\) and for all sequences \((x_{i_j})_{j \geq 1}\) such that \(x_{i_j} \in F_{i_j}\) and \(x_{i_j} \to x \in X\), we have \(x \in F\).

(v) If \(X' = X \cup \{\omega\}\) is the one-point compactification of \(X\), then \(F \mapsto F \cup \{\omega\}\) is a homeomorphism from \(2^X\) to the subspace of \(2^{X'}\) of closed sets containing \(\{\omega\}\).

Moreover, the subset \(\{\{x\} \in 2^X \mid x \in X\} \cup \emptyset\) of \(2^X\) and its image in \(2^{X'}\) are homeomorphic to \(X'\).

Suppose moreover that \(X = G\) is a locally compact group, and let \(\mathcal{C}(G)\) be the subset of \(2^G\) of closed subgroups.

(vi) The subspace \(\mathcal{C}(G)\) of \(2^G\) is closed, and therefore compact.

(vii) In \(\mathcal{C}(G)\), a basis of neighbourhoods of a closed subgroup \(C\) is given by

\[
\mathcal{N}_{K,U}(C) = \{D \in \mathcal{C}(G) \mid D \cap K \subset CU \text{ and } C \cap K \subset DU\}.
\]

Comments. (i) The proof appears in [Fell–62, Theorem 1]. In general, \(2^X\) need not be Hausdorff, even if \(X\) is metrisable; thus in a French–like terminology, \(2^X\) is quasi–compact. However, if \(X\) is locally compact and possibly non–Hausdorff, then \(2^X\) is a compact Hausdorff space. In the relevant context and in terms of the geometric topology (defined by the conditions of (iv)), the proof of (i) appears also in [Thurs, Proposition 9.1.6], where it is established that \(\mathcal{C}(G)\) is compact for a Lie group \(G\).

(ii) For a positive integer \(k\), let \(F_k\) denote the free group on \(k\) generators. The space \(\mathcal{N}(F_k)\) of normal subgroups of \(F_k\) is closed in \(2^{F_k} = \{0, 1\}^{F_k}\), and therefore compact. This space can be naturally identified with the space of marked groups on \(k\) generators, namely of groups \(\Gamma\) given together with a generating set \(\{s_1, \ldots, s_k\}\), or equivalently together with a quotient homomorphism \(F_k \twoheadrightarrow \Gamma\). This space has been intensively studied in recent years: see among others [Grom–81, final remarks], [Grig–84], [Cham–00], [ChGu–05], and [CoGP–07].

(iii) and (iv) See for example [Paul–07], Proposition 1.8, Page 60.
(v) See [Fell–62, Page 475], or [Bour–63], chapitre VIII, § 5, exercice 1.
(vi) See [Fell–62, Page 474].

From now on, \(\mathcal{C}(G)\) will denote the compact space of closed subgroups of a locally compact group \(G\), furnished with the Chabauty topology. It has several subspaces of interest, including:

- the space \(\mathcal{D}(G)\) of discrete subgroups of \(G\),
- the space \(\mathcal{L}(G)\) of lattices of \(G\),
- the space \(\mathcal{A}(G)\) of closed abelian subgroups of \(G\),
- the space \(\mathcal{N}(G)\) of closed normal subgroups of \(G\).

(Recall that a lattice in \(G\) is a discrete subgroup \(\Lambda\) such that \(G/\Lambda\) has a \(G\)-invariant probability measure.)

3. Proposition (Chabauty’s Mahler’s Criterion). Let \(G\) be a unimodular\(^1\) locally compact group satisfying some extra technical conditions, for example let \(G\) be a connected unimodular Lie group, and let \(M\) be a subset of \(\mathcal{L}(G)\). Then \(M\) is relatively compact if and only if

(i) there exists a constant \(C > 0\) such that \(\text{vol}(G/\Lambda) \leq C\) for all \(\Lambda \in M\),

(ii) there exists a neighbourhood \(U\) of \(e\) in \(G\) such that \(\Lambda \cap U = \{e\}\) for all \(\Lambda \in M\).

\(^1\)Recall that, if \(G\) was not unimodular, it would not contain any lattice at all; in other terms, one would have \(\mathcal{L}(G) = \emptyset\). Even if \(G\) is unimodular, it may happen that \(\mathcal{L}(G) = \emptyset\); this happens for example with nilpotent Lie groups of which the Lie algebra \(\mathfrak{g}\) has no rational form; a rational form of a Lie algebra \(\mathfrak{g}\) is a Lie algebra \(\mathfrak{g}_0\) over \(\mathbb{Q}\) such that \(\mathfrak{g}_0 \otimes_{\mathbb{Q}} \mathbb{R}\) and \(\mathfrak{g}\) are isomorphic real Lie algebras.
The Chabauty topology provides some natural compactifications. More precisely, let $S$ be a space of which the points are in natural correspondence with closed subgroups of a given locally compact group $G$, in such a way that the corresponding injection $\varphi : S \to \mathcal{C}(G)$ is continuous. Then $\varphi(S)$ is a compactification of $S$. Examples to which this applies are:

\begin{itemize}
  \item Riemannian symmetric spaces of the non-compact type $S$, for which each point of $S$ corresponds to a maximal compact subgroup of the isometry group $S$;
  \item Bruhat–Tits buildings;
  \item the space of complete Riemannian manifolds of dimension $n$ and of constant sectional curvature $-1$, given together with a base point and an orthonormal basis of the tangent space at this base point; this is a space which can be identified with a space of discrete subgroups of the isometry group of the hyperbolic space $H^n$.
\end{itemize}

III. First examples

If $G = \mathbb{R}$, the space $\mathcal{C}(\mathbb{R})$ is homeomorphic to a compact interval $[0, \infty]$. The points $0$, $\lambda$ (with $0 < \lambda < \infty$), and $\infty$ correspond respectively to the subgroups $\{0\}$, $\frac{1}{\lambda}\mathbb{Z}$, and $\mathbb{R}$.

The space $\mathcal{C}(\mathbb{Z})$ is homeomorphic to the subspace $\{\frac{1}{n} \mid n \geq 1\} \cup \{0\}$ of $[0, 1]$, with $\frac{1}{n}$ corresponding to $n\mathbb{Z}$ and 0 to $\{0\}$. [Exercise: the spaces $\mathcal{C}(\mathbb{R}/\mathbb{Z})$ and $\mathcal{C}(\mathbb{Z})$ are homeomorphic.]

Even if the list of easily understandable spaces $\mathcal{C}(G)$ could be slightly extended (exercise: look at $SO(3)$ and at the affine group of the real line), it is essentially a very short list. This was our main motivation to understand two more cases : the additive group of $\mathbb{C}$ ($= \mathbb{R}^2$), see [HuPo–79] and Section IV below, and the Heisenberg group $H$, see [BrHK] and Sections V to VII.

In $\mathbb{R}^n$, $n \geq 1$, any closed subgroup is isomorphic to one of $\mathbb{R}^a \oplus \mathbb{Z}^b$, where $a, b$ are non-negative integers such that $0 \leq a + b \leq n$. For a given pair $(a, b)$, the subspace $\mathcal{C}_{\mathbb{R}^a \oplus \mathbb{Z}^b}(\mathbb{R}^n)$ of $\mathcal{C}(\mathbb{R}^n)$ is a homogeneous space of $GL_n(\mathbb{R})$, for example $\mathcal{C}_{\mathbb{R}^a}(\mathbb{R}^n) = \mathcal{L}(\mathbb{R}^n) = GL_n(\mathbb{R})/GL_n(\mathbb{Z})$, as already observed in Section I. But the way these different “strata” are glued to each other to form $\mathcal{C}(\mathbb{R}^n)$ is complicated, and we do not know of any helpful description if $n \geq 3$.

To describe the case $n = 2$, it will be convenient to think of $\mathbb{C}$ rather than of $\mathbb{R}^2$.

IV. Hubbard and Pourezza (1979)

We will describe $\mathcal{C}(\mathbb{C})$ in three steps: first the space $\mathcal{C}_{\mathbb{C}}(\mathbb{C}) = \mathcal{C}_{\{0\}, \mathbb{Z}, \mathbb{R}, \mathbb{R} \oplus \mathbb{Z}, \mathbb{C}}(\mathbb{C})$ of closed subgroups of $\mathbb{C}$ which are not lattices (easy step), then the space $\mathcal{L}(\mathbb{C})$ of lattices (classical step in complex analysis), and finally the way these two parts are glued to each other to form a 4–sphere (the contribution of Hubbard and Pourezza).

IV.1. The space $\mathcal{C}_{\mathbb{C}}(\mathbb{C})$ is a 2–sphere.

In $\mathcal{C}(\mathbb{C})$, the subset of closed subgroups isomorphic to $\mathbb{R}$ is the real projective line $\mathbb{P}^1$, namely a circle. Each infinite cyclic subgroup of $\mathbb{C}$ is contained in a unique real line, and the subgroup $\{0\}$ is contained in all of them. It follows that the space $\mathcal{C}_{\{0\}, \mathbb{Z}, \mathbb{R}}(\mathbb{C})$ is a cone over $\mathbb{P}^1$, namely a closed disc. We rather think of it as the closed lower hemisphere of a 2–sphere $S^2$.

2Let $G, A, \ldots, B$ be topological groups. We denote by $\mathcal{C}_{A, \ldots, B}(G)$ the subspace of $\mathcal{C}(G)$ of closed subgroups of $G$ topologically isomorphic to one of $A, \ldots, B$. 


A closed subgroup $C$ of $\mathbb{C}$ isomorphic to $\mathbb{R} \oplus \mathbb{Z}$ is uniquely determined by its connected component $C^0$, isomorphic to $\mathbb{R}$, and by the minimal norm $\min_{z \in C, z \notin C^0} |z|$. When this minimal norm is very large [respectively very small], $C$ is “near” a closed subgroup of $\mathbb{C}$ isomorphic to $\mathbb{R}$ [respectively is “near” $\mathbb{C}$]. It follows that the space $C_{\mathbb{R}, \mathbb{R} \oplus \mathbb{Z}}(\mathbb{C})$ is also a cone over $\mathbb{P}^1$, which can be identified to the closed upper hemisphere of $\mathbb{S}^2$.

Consequently, $C_{\mathbb{R}}(\mathbb{C})$ is homeomorphic to $\mathbb{S}^2$, with

- $\{0\}$ corresponding to the South Pole,
- $C_{\mathbb{Z}}(\mathbb{C})$ to the complement of the South Pole in the open lower hemisphere,
- $C_{\mathbb{R}}(\mathbb{C})$ to the equator,
- $C_{\mathbb{R} \oplus \mathbb{Z}}(\mathbb{C})$ to the complement of the North Pole in the open upper hemisphere,
- $C$ to the North Pole.

IV.2. The space $\mathcal{L}(\mathbb{C})$ is the product of an open interval with the complement of a trefoil knot in $\mathbb{S}^3$.

For a lattice $L$ in $\mathbb{C}$, set as usual

$$
g_2(L) = 60 \sum_{z \in L, z \neq 0} z^{-4}, \quad g_3(L) = 140 \sum_{z \in L, z \neq 0} z^{-6}, \quad \Delta(L) = g_2(L)^3 - 27g_3(L)^3.
$$

Let $\Sigma$ be the complex curve in $\mathbb{C}^2$ of equation $a^3 = 27b^2$. It is a classical fact that we have a homeomorphism

$$
g : \begin{cases} 
\mathcal{L}(\mathbb{C}) \longrightarrow \mathbb{C}^2 \setminus \Sigma \\
L \longmapsto (g_2(L), g_3(L))
\end{cases}
$$

(see for example [SaZy–65, § VIII.13]). By the same formulas, $g$ can be extended to a homeomorphism

$$
g' : \mathcal{L}(\{0\}, \mathbb{Z}, \mathbb{Z})(\mathbb{C}) \longrightarrow \mathbb{C}^2.
$$

Observe that $\Sigma$, viewed as a real surface in $\mathbb{C}^2 \simeq \mathbb{R}^4$, is smooth outside the origin, and that its intersection with the unit sphere $\mathbb{S}^3$ of equation $|a|^2 + |b|^2 = 1$ is a trefoil knot

$$
T = \{(a, b) \in \mathbb{S}^3 \mid a^3 = b^2 \}.
$$

(It follows that the origin is indeed a singular point of $\Sigma$.)

The multiplicative group $\mathbb{C}^*$ acts on $C(\mathbb{C})$ and on $\mathbb{C}^2$ by

$$
(s, C) \longmapsto \sqrt{s}C \quad \text{and} \quad (s, (a, b)) \longmapsto (s^{-3} a, s^{-3} b)
$$

(observe that $\sqrt{s}C$ is well defined, because $-C = C$). Moreover, the homeomorphism $g'$ is $\mathbb{C}^*$–equivariant.

Let us restrict these actions to the subgroup $\mathbb{R}^*_+$ of $\mathbb{C}^*$. The resulting action of $\mathbb{R}^*_+$ on $\mathcal{L}(\mathbb{C})$ is free, and its orbits are transverse to the subset $C_{\text{unimod}}(\mathbb{C})$ of unimodular lattices in $\mathbb{C}$. The resulting action of $\mathbb{R}^*_+$ on $\mathbb{C}^2 \setminus \Sigma$ is also free, and its orbits are transverse to the sphere $\mathbb{S}^3$. And the homeomorphism $g$ is $\mathbb{R}^*_+$–equivariant. [The restriction of the action to the subgroup of complex numbers of modulus one is also interesting, producing on $C_{\text{unimod}}(\mathbb{C})$ the structure of a Seifert manifold, but we will not expand this here.]

However, note that the transversals $C_{\text{unimod}}(\mathbb{C})$ and $\mathbb{S}^3 \setminus T$ do not correspond to each other by the homeomorphism $g$. Compare with the first comment following Theorem 4 below.
IV.3. The space $C(C)$ is a 4–sphere.

The covolume is usually defined on the set of lattices, but there is no difficulty to see it as a continuous mapping $C_{(0)} Z, Z^2(C) \rightarrow [0, \infty]$, cyclic subgroups being of infinite covolume. We can therefore consider the set $C_{\text{covol} \geq 1}(C)$ of closed subgroups of $C$ of covolume at least 1; it is a subspace of $C_{(0)} Z, Z^2(C)$ which contains $C_{(0)} Z, Z^2(C)$. We modify $L$ isomorphic to $R$ above. We modify $L$ isomorphic to $R$

For a precise definition of the function $\gamma$, see [HuPo–79] and [BrHK]. It can be shown that $f$ is a homeomorphism.

Moreover, $f$ can be extended to a mapping $C^2 \cup \{\infty\} \rightarrow C(C)$ by defining for $(a, b)$ outside $B^4$

$$f(a, b) = (f(\sigma(a, b)))^z$$

where $\sigma$ denotes the inversion $(a, b) \mapsto \frac{(a, b)}{|a|+|b|}$ of $C^2$ fixing $S^3$ and where the dual of a closed subgroup $C$ of $C$ is defined by

$$C^d = \{z \in C \mid \text{Im}(z c) \in Z \text{ for all } c \in C\}.$$  

(Observe that $C^d = C$ if and only if $C$ is either a unimodular lattice or a subgroup isomorphic to $R_1$.) The one–point compactification $C^2 \cup \{\infty\}$ of $C^2$ can be identified to the 4–sphere $S^4$, and we have:

4. Theorem [HuPo–79]. The mapping $f : S^4 \rightarrow C(C)$ is a homeomorphism.

Comments on Theorem 4. By the homeomorphisms $f$, the equator $S^3$ of $S^4$ corresponds to the union of the set $L_{\text{unimod}}(C)$ of unimodular lattices and the set $C_{(0)} Z, Z^2(C)$ of subgroups isomorphic to $R_1$, the latter corresponding to a trefoil knot $T \subset S^3$.

This cannot be seen using only the homeomorphism $g'$ of Subsection IV.1. Indeed, whenever a point $(a, b) \in C^2 \setminus \Sigma$ converges towards a point $(a_{\text{lim}}, b_{\text{lim}}) \in \Sigma$, the corresponding $(g')^{-1}(a, b)$ has a volume which tends to infinite. When $(a, b)$ is rescaled in such a way that the corresponding lattice is unimodular, then $(a, b)$ escapes any compact subset of $C^2$, the minimal norm of the corresponding lattice tends to 0, and the lattice itself tends inside $C(C)$ to a subgroup isomorphic to $R_1$.

If we view $S^4$ as the suspension of its equator $S^3$, the two–dimensional sphere $S^2$ of Subsection IV.1, which corresponds to the complement of $L(C)$ in $C(C)$, corresponds to the suspension of $T$. A pole has a typical neighbourhood inside $S^2$ which is a cone over
the knot $T$ (other points of $S^2$ have typical neighbourhoods in $S^2$ which are cones over unknotted closed curves inside $S^3$); in particular, the embedding of this $S^2$ in the total space $S^3$ is not locally flat at the South and North Poles.

We know that the space $\mathcal{L}_{\text{unimod}}(C)$ is not simply connected; indeed, it is an aspherical space with fundamental group the inverse image of $SL_2(\mathbb{Z})$ in the universal covering group of $SL_2(\mathbb{R})$. An oriented closed curve $\ell$ in $\mathcal{L}_{\text{unimod}}(C)$ can be viewed as inside $S^3$ and disjoint from the trefoil knot $T$ (oriented in some way), so that there is an associated linking number $\text{link}(\ell, T)$. An interesting particular case is that of a periodic orbit of the geodesic flow on $SL_2(\mathbb{R})/SL_2(\mathbb{Z})$, viewed as the unit tangent bundle of the modular surface $D^2/PSL_2(\mathbb{Z})$, as explained in [Ghys–07].

V. On the Heisenberg group

The Heisenberg group is a 3–dimensional nilpotent Lie group, which is connected and simply connected. We use the model

$$H = C \times \mathbb{R} \quad \text{with} \quad (z_1, t_1)(z_2, t_2) = (z_1 + z_2, t_1 + t_2 + \frac{1}{2}\text{Im}(z_1 \overline{z_2})), $$

we denote by $p : H \rightarrow C$ the projection $(z, t) \mapsto z$, and we identify $\mathbb{R}$ to $\{0\} \times \mathbb{R} \subset H$.

The following is a collection of easily verified properties.

5. Proposition. (i) The subset $\mathbb{R}$ of $H$ is both the commutator subgroup and the centre $Z(H)$ of the Heisenberg group; moreover, commutators are given by

$$[(z_1, t_1), (z_2, t_2)] = (0, \text{Im}(\overline{z_1}z_2)).$$

(ii) Maximal abelian subgroups of $H$ are of the form $\mathbb{R}z_0 \times \mathbb{R}$, with $z_0 \in C^*$.

(iii) For any $C \in \mathcal{C}(H)$ with $C \cap Z(H) \neq \emptyset$, the projection $p(C)$ is a closed subgroup of $C$ (this applies in particular to non–abelian closed subgroups of $H$).

(iv) For any non–abelian $C \in \mathcal{C}(H)$, either $Z(H) \subset C$ or $C \in \mathcal{L}(H)$.

(v) For any lattice $\Lambda$ in $H$, the commutator subgroup $[\Lambda, \Lambda]$ is a subgroup of finite index in the centre $Z(\Lambda)$.

(vi) The automorphism group of $H$ is a semi–direct product $(H/Z(H)) \rtimes GL_2(\mathbb{R})$, with $H/Z(H)$ the group of inner automorphisms and with $GL_2(\mathbb{R})$ acting on $H$ by

$$\left(\begin{array}{cc} a & b \\ c & d \end{array}\right), \ (x + iy, t) \quad \mapsto \quad (ax + by) + i(cx + dy), \ (ad - bc)^2t).$$

For any positive integer $n$, we denote by $\mathcal{L}_n(H)$ the subspace of lattices $\Lambda \in \mathcal{L}(H)$ with $[\Lambda, \Lambda]$ of index $n$ in $Z(\Lambda)$. We denote by $\mathcal{L}_\infty(H)$ the subspace of $\mathcal{C}(H)$ of subgroups of the form $p^{-1}(L)$, with $L \in \mathcal{L}(C)$.

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3An element $(x+iy, t)$ corresponds to a matrix

$$\begin{pmatrix} 1 & x & t + \frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

in the matrix picture

$$\begin{pmatrix} 1 & R & R \\ 0 & 1 & R \\ 0 & 0 & 1 \end{pmatrix}$$

of the Heisenberg group.
6. Examples. For any \( n \geq 1 \), the subgroup \( \Lambda_n \) of \( H \) generated by \((1, 0), (i, 0), \) and \((0, 1/n)\) is in \( L_n(H) \). Moreover,

\[
\Lambda_n = \mathbb{Z}[i] \times \frac{1}{n} \mathbb{Z}
\]

if \( n \) is even and

\[
\Lambda_n = \left\{ \left(x + iy, \frac{t}{2n}\right) \in \mathbb{Z}[i] \times \frac{1}{2n} \mathbb{Z} \mid xy \equiv t \pmod{2} \right\}
\]

if \( n \) is odd. In the matrix picture for \( H \), we have \( \Lambda_n = \begin{pmatrix} 1 & \frac{1}{n} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix} \) for all \( n \).

The subgroup of \( H \) generated by \((1, 0), (i, 0), \) and \( Z(H) \) is in \( L_{\infty}(H) \).

VI. Three subspaces of \( C(H) \)

The space \( C(H) \) consists of several parts which, as it is the case for \( C(C) \), are relatively easy to describe, and their gluing is harder. From Proposition 5, the space \( C(H) \) consists of three parts:

\[
\rightarrow \text{the space } A(H) \text{ of closed abelian subgroups},
\]

\[
\rightarrow \text{the space } C_{\geq Z}(H) \text{ of closed subgroups of } H \text{ containing } Z(H),
\]

\[
\rightarrow \text{the space of lattices } L(H) = \bigsqcup_{n=1}^{\infty} L_n(H).
\]

Moreover, \( L(H) \) is disjoint from \( A(H) \cup C_{\geq Z}(H) \).

VI.1. The space \( A(H) \) of abelian subgroups.

From Proposition 5.ii, the space \( C_{\mathbb{R}^2}(H) \) of maximal abelian subgroups of \( H \) is homeomorphic to the projective line \( C_{\mathbb{R}^2}(C) \approx P^1 \) of lines in the space \( H/Z(H) \approx \mathbb{R}^2 \). For a closed abelian subgroup \( A \) of \( H \), there are two cases to distinguish:

(i) if \( A \nsubseteq Z(H) \), then \( A \) is contained in an unique maximal abelian subgroup of \( H \),

(ii) if \( A \subseteq Z(H) \), then \( A \) is contained in all maximal abelian subgroups of \( H \).

If one could identify coherently each maximal abelian subgroup of \( H \) to a copy of \( \mathbb{R}^2 \), the space \( A(H) \) would simply be the quotient of \( C_{\mathbb{R}^2}(H) \times C(C) \approx S^1 \times S^1 \) by the relation identifying each of the circle \( (A_{\text{max}}, C) \) to a point (for \( A_{\text{max}} \) in the circle \( C_{\mathbb{R}^2}(H) \), and for \( C \subseteq Z(H) \) fixed). This identification is possible locally, but not globally; indeed, even once \( H \) is furnished with some Riemannian structure, each \( A_{\text{max}} \) can be identified to \( C \) in two ways, corresponding to the two orientations of \( A_{\text{max}} \). However, the following can be shown (Proposition 6.1.i of [BrHK]):

7. Proposition. There exists in \( S^1 \) a tame closed interval \( I \), corresponding in \( C(C) \) to the subspace of closed subgroups contained in \( \mathbb{R} \), such that

\[
A(H) \approx \left( S^1 \times S^4 \right)/\left( (\varphi, x) \sim (\varphi', x), \text{ for all } \varphi, \varphi' \in S^1 \text{ and } x \in I \right).
\]

In particular, \( A(H) \) is a space of dimension 5.

VI.2. The space \( C_{\geq Z}(H) \) of groups containing the centre \( Z(H) \).

As a straightforward consequence of the results of Section IV, we have:
8. Proposition. The space $C_{\geq z}(H)$ is homeomorphic to $C(C)$, namely to a 4–dimensional sphere. Moreover, the intersection $A(H) \cap C_{\geq z}(H)$ is embedded in $C_{\geq z}(H)$ as a closed 2–disc.

More precisely, this intersection $A(H) \cap C_{\geq z}(H)$ is the lower hemisphere of the 2–sphere described in the comments which follow Theorem 4. Observe that the complement of this 2–sphere in $C_{\geq z}(H)$ is precisely the space $L_{\infty}(H)$ of subgroups of the form $p^{-1}(L)$ for $L \in L(C)$, as defined just before Example 6.

VI.3. The space $L(H)$ of lattices in $H$.

Consider an integer $n \geq 1$, a lattice $\Lambda_0 \in L_n(H)$, and denote by $L_0 = p(\Lambda_0) \in L(C)$ its projection. Choose a positively oriented basis $(z_0, z'_0)$ of $L_0$ and two points $(z_0, t_0) \in \Lambda_0 \cap p^{-1}(z_0), (z'_0, t'_0) \in \Lambda_0 \cap p^{-1}(z'_0)$. The infinite cyclic subgroup $[\Lambda_0, \Lambda_0]$ is generated by the commutator

\[ [(z_0, t_0), (z'_0, t'_0)] = (0, \text{Im}(\overline{z}_0 z'_0)) = (0, \text{vol}(C/L_0)), \]

and $\Lambda_0$ itself is generated by $(z_0, t_0), (z'_0, t'_0)$, and $(0, \frac{1}{n} \text{vol}(C/L_0))$.

Moreover, the group of inner automorphisms operates transitively on the set of possible choices for $t, t'$. We have essentially proved the first two claims of the following proposition; for details and for the last claim, see [BrHK, Section 7].

9. Proposition. For each $n \geq 1$, the space $L_n(H)$ is both

(i) a torus bundle with base space $L(C)$,

(ii) a homogeneous space of the 6–dimensional Lie group $\text{Aut}(H)$ by a discrete subgroup.

For $n, n' \geq 1$, the spaces $L_n(H), L_{n'}(H)$ are homeomorphic to each other; moreover, the torus bundles $L_n(H) \longrightarrow L(C)$ and $L_{n'}(H) \longrightarrow L(C)$ are isomorphic.

VI.4. Summing up for $A(H)$, $C_{\geq z}(H)$, and $L(H)$.

Let $(\varphi_s)_{s \geq 0}$ be the one–parameter subgroup of $\text{Aut}(H)$ defined by $\varphi_s(z, t) = (sz, s^2 t)$. We have $\lim_{s \to -\infty} \varphi_s(D) = \{e\}$ for any discrete subgroup $D$ of $H$ and $\lim_{s \to 0} \varphi_s(\Lambda) = H$ for any $\Lambda \in L(H)$. Since $A(H)$ and $C_{\geq z}(H)$ are clearly arc–connected (Propositions 7 and 8), and contain respectively $\{e\}$ and $H$, it follows that $C(H)$ is arc–connected. This is the very first part of the following theorem, one of the two theorems which summarise the results in [BrHK]; the space $C(H)$ is not locally connected because any neighbourhood of a point in $L_{\infty}(H)$ is disconnected, containing points from $L_n(H)$ for $n$ large enough.

10. Theorem (Theorem 1.3 in [BrHK]). The compact space $C(H)$ is arc–connected but not locally connected. It can be expressed as the union of the following three subspaces.

(i) $L(H)$, which is open and dense in $C(H)$; this has countably many connected components $L_n(H)$, each of which is homeomorphic to a fixed aspherical 6–manifold that is a 2–torus bundle over $L(C) \approx GL_2(R)/GL_2(Z)$.
(ii) \( \mathcal{A}(H) \), which is homeomorphic to the space obtained from \( S^4 \times \mathbb{P}^1 \) by fixing a tame arc are \( I \subset S^4 \) and collapsing each of the circles \( \{i\} \times \mathbb{P}^1 : i \in I \) to a point.

(iii) \( C_{\geq Z}(H) \), from which there is a natural homeomorphism to \( S^4 \); the complement of \( \mathcal{L}_\infty(H) \) in \( C_{\geq Z}(H) \) is a 2–sphere \( \Sigma^2 \subset S^4 \) (which fails to be locally flat at two points).

The union \( \mathcal{A}(H) \cup C_{\geq Z}(H) \) is the complement of \( \mathcal{L}(H) \) in \( \mathcal{C}(H) \). The intersection

\[
\mathcal{A}(H) \cap C_{\geq Z}(H) = \{ C \in C_{\geq Z}(H) \mid p(C) \subset C_{[0, \infty]}(\mathbb{Z}, \mathbb{R}(\mathbb{C})) \}
\]

is a closed 2–disc in \( \Sigma^2 \). The space \( \mathcal{L}(H) \cup \mathcal{L}_\infty(H) \) is precisely \( \{ C \in \mathcal{C}(H) \mid p(C) \in \mathcal{L}(\mathbb{C}) \} \).

\( C_{\geq Z}(H) \) is a weak retract of \( \mathcal{C}(H) \): there exists a continuous map \( f : \mathcal{C}(H) \rightarrow S^4 \), constant on \( \mathcal{A}(H) \), such that \( f \circ j \simeq \text{id}_{S^4} \), where \( j : S^4 \rightarrow C_{\geq Z}(H) \) is the homeomorphism of (iii). In particular, \( \pi_4(\mathcal{C}(H)) \) surjects onto \( \mathbb{Z} \).

The subspace \( \mathcal{N}(H) \) of normal closed subgroups of \( H \) is the union of \( C_{\geq Z}(H) \) (which is homeomorphic to \( \mathcal{C}(\mathbb{C}) \simeq S^4 \)) and the closed interval \( \{ C \in \mathcal{C}(H) \mid C \subset Z(H) \} \), attached to the sphere \( C_{\geq Z}(H) \) by one of its endpoints.

We would like to know more generally when \( \mathcal{L}(G) \) is dense in \( \mathcal{C}(G) \), say for a unimodular connected Lie group \( G \).

VII. The space \( \mathcal{C}(H) \)

As an illustration of the density of \( \mathcal{L}(H) \) in \( \mathcal{C}(H) \), let us describe the following simple

11. Example. Let \( n \geq 1 \) be fixed. For any integer \( k \geq 1 \), let \( \Lambda_k \) denote the subgroup of \( \mathbb{R} \times \mathbb{R} \) generated by \((1,0)\) and \((-\frac{1}{k},1)\); let \( A \) denote the subgroup \( \mathbb{Z}^2 \) of \( \mathbb{R} \times \mathbb{R} \); and let \( \Lambda_k \) denote the subgroup of \( H \) generated by \( A_k \) and \((-ik^2n,0)\).

Then \( \Lambda_k \in \mathcal{L}_n(H) \) for all \( k \geq 1 \) and \( \lim_{k \to \infty} \Lambda_k = \lim_{k \to \infty} A_k = A \) in \( \mathcal{C}(H) \).

Remark, and consequence of this example. Here, \( \mathbb{R} \times \mathbb{R} \) is viewed as a subgroup of \( \mathbb{C} \times \mathbb{C} = H \); in particular, it should not be confused with \( \mathbb{C} \). (Inside \( H = \mathbb{C} \times \mathbb{R} \), note that \( \mathbb{C} \) is not a subgroup.) Observe that \( p(A_k) \) is a lattice in \( \mathbb{C} \) for each \( k \geq 1 \), but the projection of the limit, \( p(A) \), is a cyclic subgroup of \( \mathbb{C} \).

It follows from Example 11 that the frontier of \( \mathcal{L}_n(H) \) in \( \mathcal{C}(H) \) contains the closure of the \( \text{Aut}(H) \)–orbit of \( A \), and it is a fact that this closure coincides with \( \mathcal{A}(H) \) (Proposition 6.1.ii in [BrHK]). Moreover, it can be seen that the frontier of \( \mathcal{L}_n(H) \) contains \( C_{\geq Z}(H) \).

The previous argument shows part of the following theorem, again copied from [BrHK], by which we will end this account. Recall that we denote here by \( \Sigma^2 \) the topologically embedded sphere in \( C_{\geq Z}(H) \) which corresponds to the \( S^2 \subset S^4 \) in the comments on Theorem 4. The symbols \( \mathbb{P}^2 \) and \( \mathbb{K} \) stand respectively for a real projective plane and a Klein bottle.

12. Theorem (Theorem 1.4 in [BrHK]). The spaces \( \mathcal{L}_n(H) \) are homeomorphic to a common aspherical homogeneous space, namely the quotient of the 6–dimensional automorphism group \( \text{Aut}(H) \cong \mathbb{R}^2 \rtimes GL_2(\mathbb{R}) \) by the discrete subgroup \( \mathbb{Z}^2 \rtimes GL_2(\mathbb{Z}) \).

The frontier of \( \mathcal{L}_n(H) \), which is independent of \( n \), consists of the following subspaces:

(i) the trivial group \( \{e\} \);

\(^4\text{Here, } \simeq \text{ denotes homotopy equivalence.}\)
(ii) $C_{R}(H) \approx \mathbb{P}^{2}$;
(iii) $C_{Z}(H) \approx \mathbb{P}^{2} \times [0, \infty]$;
(iv) $C_{R^{2}}(H) \approx \mathbb{P}^{1}$;
(v) $C_{R,K}(H) \approx \mathbb{K} \times [0, \infty]$, which is a $(\mathbb{P}^{1} \times [0, \infty])$–bundle over $\mathbb{P}^{1}$;
(vi) $C_{Z}(H)$, which is a $(\mathbb{S}^{4} \setminus \Sigma^{2})$–bundle over $\mathbb{P}^{1}$;
(vii) $p^{-1}_{\ast}(C_{R,K}(\mathbb{C}))$;
(viii) the full group $H$.

In particular, the frontier of $\mathcal{L}_{n}(H)$ is the union of $\mathcal{A}(H)$ and the complement $\Sigma^{2}$ of $\mathcal{L}_{\infty}(H)$ in $\mathcal{C}_{\geq Z}(H)$; the part $\mathcal{A}(H)$ is itself the union of the subspaces (i) to (vi), and $\Sigma^{2} \setminus (\Sigma^{2} \cap \mathcal{A}(H))$ is itself the union of the subspaces (vii) and (viii). The frontier of $\bigcup_{n=1}^{\infty} \mathcal{L}_{n}(H)$ further contains
(ix) $\mathcal{L}_{\infty}(H)$.

Each of these spaces, except (vi), consists of finitely many $\text{Aut}(H)$–orbits.

Observe that, as $\mathcal{L}(H)$ is open dense, the spaces (i) to (ix) of Theorem 1.4 together with the spaces $\mathcal{L}_{n}(H)$ for $n \geq 1$ constitute a partition of $\mathcal{C}(H)$.

References

Bore–69. A. Borel, Introduction aux groupes arithmétiques, Hermann, 1969.
Bour–63. N. Bourbaki, Intégration, chapitres 7 et 8, Hermann, 1963.
BrHK–88. M. Bridson, P. de la Harpe, and V. Kleptsyn, The Chabauty space of closed subgroups of the three-dimensional Heisenberg group, Preprint (2007), posted on arXiv.
CaEG–87. R.D. Canary, D.B.A. Epstein, and P.L. Green, Notes on notes of Thurston, in “Analytical and geometrical aspects of hyperbolic spaces”, D.B.A. Epstein Editor, London Math. Soc. Lecture Notes Series 111 (Cambridge Univ. Press, 1987), 3–92 [reprinted with a new foreword by R.D. Canary in “Fundamental of hyperbolic manifolds: selected expositions”, R. Canary, D. Epstein, and A. Marden Editors, London Math. Soc. Lecture Notes Series 328 (Cambridge Univ. Press 2006), pages 1–115].
Chab–50. C. Chabauty, Limite d’ensembles et géométrie des nombres, Bull. Soc. Math. France 78 (1950), 143–151.
Cham–00. C. Champetier, L’espace des groupes de type fini, Topology 39 (2000), 657–680.
ChGu–05. C. Champetier and V. Guirardel, Limit groups as limits of free groups: compactifying the set of free groups, Israel J. Math. 146 (2005), 1–76.
CoGP–07. Y. de Cornulier, L. Guyot, and W. Pitsch, On the isolated points in the space of groups, J. of Algebra 307 (2007), 254–277.
Fell–62. J.M.G. Fell, A Hausdorff topology for the closed subsets of a locally compact non Hausdorff space, Proc. Amer. Math. Soc. 13 (1962), 472–476.
Fell–64. J.M.G. Fell, Weak containment and induced representations of groups. II, Trans. Amer. Math. Soc. 110 (1964), 424–447.
Ghys–07. E. Ghys, Knots and dynamics, Proceedings ICM, Madrid, 2006, Volume I, Plenary lectures and ceremonies, European Math. Soc. (2007), 247–277.
Grig–84. R. Grigorchuk, Degrees of growth of finitely generated groups and the theory of invariant means, Math. USSR Izv. 25 (1985), 259–300.
Grom–81. M. Gromov, Groups of polynomial growth and expanding maps, Publ. Inst. Hautes Études Scient. 53 (1981), 53–73.
HuPo–79. J. Hubbard and I. Pomezzza, The space of closed subgroups of $\mathbb{R}^{2}$, Topology 18 (1979), 143–146.
Mich–51. E. Michael, Topologies on spaces of subsets, Trans. Amer. Math. Soc. 71 (1951), 151–182.
Mahl–46. K. Mahler, On lattice points in $n$–dimensional star bodies : I. Existence theorems, Proc. Roy. Soc. London A, 187 (1946), 151–187.
Pau–07. F. Paulin, De la géométrie et de la dynamique de $\text{SL}_{n}(\mathbb{R})$ et $\text{SL}_{n}(\mathbb{Z})$, in “Sur la dynamique des groupes de matrices et applications arithmétiques”, N. Berline, A. Plagne & C. Sabbah Editeurs, Éditions de l’École Polytechnique (2007), 47–110.
Ragh–72. M.S. Raghunathan, *Discrete subgroups of Lie groups*, Springer, 1972.
SaZy–65. S. Saks and A. Zygmund, *Analytic functions*, Second Edition, Polish Scientific Publishers, 1965.
Thurs. W.P. Thurston, *The geometry and topology of 3–manifolds*, Circulated notes from a course in Princeton University, 1978–1980.

**Other references related to Chabauty’s paper**

**Math Reviews citations**

Grom–81. M. Gromov, *Hyperbolic manifolds (according to Thurston and Jørgensen)*, *Bourbaki Seminar, Vol. 1979/80*, Springer Lecture Notes in Math. **842** (1981), 40–53.
Mumf–71. D. Mumford, *A remark on Mahler’s compactness theorem*, Proc. Amer. Math. Soc. **28** (1971), 289–294.
Sant–55. L.A. Santaló, *On geometry of numbers*, J. Math. Soc. Japan **7** (1955), 208–213.
Wang–68. S.P. Wang, *Limit of lattices in a Lie group*, Trans. Amer. Math. Soc. **133** (1968), 519–526.
Wang–69. S.P. Wang, *On a conjecture by Chabauty*, Proc. Amer. Math. Soc. **23** (1969), 569–572.
Wang–70a. S.P. Wang, *On the limit of subgroups in a group*, Amer. J. Math. **92** (1970), 708–724.
Wang–70b. S.P. Wang, *Some properties of lattices in a Lie group*, Illinois J. Math. **14** (1970), 35–39.
Wang–75. S.P. Wang, *On isolated points in the dual spaces of locally compact groups*, Math. Ann. **218** (1975), 19–34.

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