PENTAGON RELATION IN QUANTUM CLUSTER SCATTERING DIAGRAMS

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Abstract. We formulate the pentagon relation for quantum dilogarithm elements in the structure group of a quantum cluster scattering diagram (QCSD). As an application, we establish the nonpositivity of a certain class of nonskew-symmetric QCSDs. Also, we explicitly present various consistency relations and their positivity for QCSDs of rank 2 completely or up to some degree, most of which are new in the literature.

1. Introduction

In [GHKK18] the scattering diagram method was employed to study cluster algebras [FZ02], where all information of a cluster algebra or a cluster pattern is encoded in the corresponding cluster scattering diagram (CSD, for short). It was clarified implicitly in [GHKK18] and more explicitly in [Nak23] that the dilogarithm elements $\Psi[n]$, where $n$ is its normal vector, and the pentagon relation among them play prominent roles in a CSD. Meanwhile, it has been known that cluster algebras admit natural quantizations [BZ05, Tra11, FG09a, FG09b], where the quantum dilogarithm plays a key role. Naturally, quantum cluster scattering diagrams (QCSD, for short), which are quantum analogs of CSDs, were introduced and studied in [Man21, DM21], where quantum dilogarithm elements play a primary role. Here and below, we distinguish the quantum dilogarithm $\Psi_q(x)$ and the quantum dilogarithm elements $\Psi_{a,b}[n]$. The former is a formal power series of an indeterminate $x$ with a parameter $q$ due to [Sch53, FK94, FG09a], while the latter are certain elements of the structure group $G$ of a QCSD. It is well known that the quantum dilogarithm satisfies the pentagon identity [FK94, CF99], which is a quantum analog of the pentagon identity (more often called Abel’s identity or the five-term relation) of the dilogarithm [Lew81].

Having this development in mind, in this paper we formulate the pentagon relation among the quantum dilogarithm elements (Theorem 2.7), which is the counterpart of the pentagon relation for the dilogarithm elements for a CSD in [Nak23]. The definition of QCSDs is in parallel with the one for CSDs thanks to a more general formulation of scattering diagrams by [KS14]. However, each quantum dilogarithm element $\Psi_{a,b}[n]$ carries some additional rational parameters (the quantum data) $a$ and $b$ compared with its classical counterpart $\Psi[n]$, and this is the source of the complication and the richness in the quantum case. (The situation is somewhat similar to the fusion of representations of quantum affine algebras with the spectral parameter.)

As an application, we study the positivity of QCSDs in Section 5. We first formulate the positivity of CSDs and QCSDs (Definitions 5.1 and 5.4) based on the structure of the walls therein. It is known that any CSD is positive [GHKK18] and it implies the
positivity of the theta functions for each CSD. It is also known that any QCSD with the skew-symmetric initial exchange matrix is positive \[DM21\]. On the other hand, there are examples of nonpositive theta functions for QCSDs \[LLZ12, CFMM24\]. This implies that QCSDs are not always positive in the nonskew-symmetric case. Using the pentagon relation, we clarify the origin of the nonpositivity in the combinatorics of quantum dilogarithm elements, and we establish the nonpositivity of a certain class of nonskew-symmetric QCSDs (Theorem \[5.10\] and Corollary \[5.12\]). This extends the results of \[LLRZ14, DM21, CFMM24\] systematically. Also, we explicitly present various consistency relations and their positivity for QCSDs of rank 2 completely (Section \[3.2\]) or up to degree 4 (Section \[5.4\]). Most of these formulas are new in the literature and exhibit the complexity of the positivity phenomenon.

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2. Quantum Dilogarithm Elements and Pentagon Relation

We introduce quantum dilogarithm elements and the pentagon relation among them.

2.1. Structure group \( G \). Here we introduce the underlying structure group \( G \) of a QCSD following \[KS14, Man21, DM21, CFMM24\]. See also \[GHKK18, Nak23\] for parallel notions for a CSD.

Let \( \Gamma \) be a fixed data consisting of the following:

- a lattice \( N \) of rank \( r \),
- a skew-symmetric bilinear form \( \langle \cdot, \cdot \rangle : N \times N \to \mathbb{Q} \),
- a sublattice \( N^0 \subset N \) of rank \( r \) such that \( \{ N^0, N \} \subset \mathbb{Z} \),
- positive integers \( \delta_1, \ldots, \delta_r \) such that there is a basis \( (e_1, \ldots, e_r) \) of \( N \), where \( \langle \delta_1 e_1, \ldots, \delta_r e_r \rangle \) is a basis of \( N^0 \),
- \( M = \text{Hom}(N, \mathbb{Z}) \) and \( M^0 = \text{Hom}(N^0, \mathbb{Z}) \).

Let \( M_B = M \otimes_{\mathbb{Z}} \mathbb{R} \). Let \( \langle n, m \rangle \) denote the canonical paring either for \( N^0 \times M^0 \) or for \( N \times M_B \). For \( n \in N \), \( n \neq 0 \), let \( n^\perp := \{ \xi \in M_B \mid \langle n, \xi \rangle = 0 \} \).

Let \( s = (e_1, \ldots, e_r) \) be a seed for \( \Gamma \), which is a basis of \( N \) such that \( \langle \delta_1 e_1, \ldots, \delta_r e_r \rangle \) is a basis of \( N^0 \). The dual bases of \( M \) and \( M^0 \) are given by \( \langle e_1^*, \ldots, e_r^* \rangle \) and \( \langle f_1, \ldots, f_r \rangle := \langle e_1^*/\delta_1, \ldots, e_r^*/\delta_r \rangle \), respectively. The initial exchange matrix \( B = (b_{ij}) \) of the corresponding quantum cluster algebra is given by

\[
(2.1) \quad b_{ij} = \{ \delta_i e_i, e_j \}.
\]

Let

\[
(2.2) \quad N^+ = \left\{ \sum_{i=1}^r a_i e_i \mid a_i \in \mathbb{Z}_{\geq 0}, \sum_{i=1}^r a_i > 0 \right\}
\]

be the positive vectors of \( N \) with respect to \( s \). Let \( N^+_p \) denote the set of primitive elements in \( N^+ \). The degree function \( \text{deg} : N^+ \to \mathbb{Z}_{>0} \) is defined by \( \text{deg}(\sum_{i=1}^r a_i e_i) := \sum_{i=1}^r a_i \).

For \( n \in N^+ \), let \( \delta(n) \) be the smallest positive rational number such that \( \delta(n) n \in N^0 \), which is called the normalization factor of \( n \). For example, \( \delta(e_i) = \delta_i \). We have \( \delta(n) \in \mathbb{Z}_{>0} \) and \( \delta(t n) = \delta(n)/t \) for any \( n \in N^+_p \) and \( t \in \mathbb{Z}_{>0} \). We set \( \delta_0 \) as the least common multiple of \( \delta_1, \ldots, \delta_r \). Then, \( \delta_0 n \in N^0 \) for any \( n \in N^+ \). It implies that \( \{ \cdot, \cdot \} \in (1/\delta_0) \mathbb{Z} \) and that \( \delta_0 / \delta(n) \in \mathbb{Z}_{>0} \).
Let $q$ be an indeterminate, and let $\mathbb{Q}(q^{1/\delta_0})$ be the rational function field of $q^{1/\delta_0}$. For any $\alpha \in (1/\delta_0)\mathbb{Z}$, let
\begin{equation}
[\alpha]_q := \frac{q^\alpha - q^{-\alpha}}{q - q^{-1}} \in \mathbb{Q}(q^{1/\delta_0})
\end{equation}
be the $q$-number, which has the limit
\begin{equation}
\lim_{q \to 1} [\alpha]_q = \alpha.
\end{equation}
Let $\mathfrak{g}$ be the $N^+$-graded Lie algebra over $\mathbb{Q}(q^{1/\delta_0})$ defined by
\begin{equation}
\mathfrak{g} = \bigoplus_{n \in N^+} \mathbb{Q}(q^{1/\delta_0})X_n, \quad [X_n, X_{n'}] := \{[n, n']\}_q X_{n+n'},
\end{equation}
where the Jacobi identity is easily verified. Let $\hat{\mathfrak{g}}$ be the completion of $\mathfrak{g}$ with respect to $\deg$, and let $G = \exp(\hat{\mathfrak{g}})$ be the exponential group of $\hat{\mathfrak{g}}$ whose product is defined by the Baker-Campbell-Hausdorff formula (e.g., [Jac79, §V.5]). We call $G$ the structure group for the forthcoming scattering diagrams.

Remark 2.1. So far, the only difference from the classical case is the Lie bracket in (2.5). It is a quantum analog of the Lie bracket
\begin{equation}
[X_n, X_{n'}] := \{[n, n']\} X_{n+n'}.
\end{equation}
for a CSD. In what follows all expressions involving $q$ converge to their classical counterparts in the limit $q \to 1$.

2.2. $y$-representation. Let $y$ be a symbol. Let $\mathbb{Q}(q^{1/\delta_0})[y]_q$ be the noncommutative and associative algebra over $\mathbb{Q}(q^{1/\delta_0})$ with generators $y^n (n \in N^+ \sqcup \{0\})$ and the $q$-commutative relations
\begin{equation}
y^ny^{n'} = q^{\{n, n'\}}y^{n+n'}.
\end{equation}
Let $\mathcal{R}_q(y)$ be the completion of $\mathbb{Q}(q^{1/\delta_0})[y]_q$ with respect to $\deg$. In other words, any element of $\mathcal{R}_q(y)$ is expressed as an infinite sum
\begin{equation}
\sum_{n \in N^+ \sqcup \{0\}} c_n y^n \quad (c_n \in \mathbb{Q}(q^{1/\delta_0})).
\end{equation}
Then, we define the action of $\hat{\mathfrak{g}}$ on $\mathcal{R}_q(y)$ by
\begin{equation}
X_n(y^{n'}) := \{[n, n']\}_q y^{n+n'} = q^{2\{n, n'\}} - 1 q - q^{-1} y^{n'} y^n.
\end{equation}
It is easy to check that this is indeed an action of $\hat{\mathfrak{g}}$ and also a derivation. Thus, it induces the action of $G$ on $\mathcal{R}_q(y)$ defined by
\begin{equation}
(\exp X)(y^n) = \sum_{j=0}^{\infty} \frac{1}{j!} X^j(y^n) \quad (X \in \hat{\mathfrak{g}}).
\end{equation}
Moreover, \( \exp X \) is an algebra automorphism \([Jac79, §1.2]\). We call the resulting representation \( \rho_y : G \to \text{Aut}(\mathcal{R}_q(y)) \) the \((\text{quantum}) y\)-representation of \( G \). It is faithful if and only if \( \{\cdot, \cdot\} \) is nondegenerate.

**Remark 2.2.** If we replace the relation (2.7) with the standard one \( y^n y'^n = y^{n+n'} \), the action (2.9) is still an action of \( \hat{g} \), but it is not a derivation. This explains the necessity of the relation (2.7).

### 2.3. Quantum dilogarithm elements and pentagon relation.

Let \( \Psi_q(x) \) be the quantum dilogarithm by [Sch53], [FV93], [FK94], [FG09a],

\[
\Psi_q(x) := \exp \left( \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j (q^j - q^{-j})} x^j \right) \in \mathbb{Q}(q)[[x]]
\]

\[(2.12)\]

\[
= \prod_{j=0}^{\infty} (1 + q^{2j+1} x)^{-1}.
\]

\[(2.13)\]

This convention is due to [FG09a]. The equality holds, because both expressions satisfy the property

\[
\Psi_q(q^2 x) = (1 + qx) \Psi_q(x), \quad \Psi_q(0) = 1,
\]

which uniquely determines \( \Psi_q(x) \). See, for example, [Kir95], [FG09a] for further information.

The following definition is similar to the first expression (2.12) of \( \Psi_q(x) \).

**Definition 2.3 (Quantum dilogarithm element).** For any \( n \in N^+ \), \( a \in (1/\delta_0)\mathbb{Z}_{>0} \), and \( b \in (1/\delta_0)\mathbb{Z} \), we define

\[
\Psi_{a,b}[n] := \exp \left( \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j [ja]_q} q^{jb} X_{jn} \right) \in G_{n_0}^{\parallel},
\]

where \( n_0 \in N^+_pr \) is the one such that \( n = jn_0 \) for some \( j \in \mathbb{N} \). We call it a \textit{quantum dilogarithm element}. We call the parameters \( a \) and \( b \) the quantum data of \( \Psi_{a,b}[n] \). They control the interval and the shift in the sum, respectively. For simplicity, we also write

\[
\Psi_a[n] := \Psi_{a,0}[n].
\]

**Remark 2.4.** The above \( \Psi_{a,b}[n] \) is a quantum analog of the dilogarithm element in the classical case [Nak23, §III.1.4]

\[
\Psi[n] = \exp \left( \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j^2} X_{jn} \right).
\]

\[(2.17)\]

We have

\[
\lim_{q \to 1} \Psi_{a,b}[n] = \Psi[n]^{1/a}.
\]

\[(2.18)\]

Note that the interval \( a \) appears in the RHS as the inverse.

**Proposition 2.5.** The group \( G \) is generated by \( \Psi_{a,b}[n]^{c} \) \((n \in N^+, \ a \in (1/\delta_0)\mathbb{Z}_{>0}, \ b \in (1/\delta_0)\mathbb{Z}, \ c \in \mathbb{Q}(q^{1/\delta_0})\) admitting the infinite product.

**Proof.** By (2.15), any element of \( G_{n_0}^{\parallel} \) is expressed as an infinite product of \( \Psi_{a,b}[jn_0]^{c} \)'s. \( \square \)
Remark 2.6. Let $\Psi$.

Theorem 2.7. Let $n_1, n_2 \in N^+$. The following relations hold in $G$.

(a) Suppose that $\{n_2, n_1\} = 0$. Then, for any $a_1, a_2, b_1, \text{ and } b_2$, we have

\begin{equation}
\Psi_{a_2, b_2}[n_2]\Psi_{a_1, b_1}[n_1] = \Psi_{a_1, b_1}[n_1]\Psi_{a_2, b_2}[n_2].
\end{equation}
(b. (Pentagon relation. Cf. [FK94 §2]). Suppose that \( \{ n_2, n_1 \} = c \in (1/\delta_0) \mathbb{Z}_{>0} \).

Then, for any \( b_1 \) and \( b_2 \), we have

\[
\Psi_{c,b_2}[n_2] \Psi_{c,b_1}[n_1] = \Psi_{c,b_1}[n_1] \Psi_{c,b_1+b_2}[n_1+n_2] \Psi_{c,b_2}[n_2].
\]

**Proof.** The relation (2.24) is clear by (2.5). Let us prove the relation (2.25). Since it only involves \( \Psi_{a,b}[n] \)'s for the rank 2 sublattice \( N' \) of \( N \) generated by \( n_1 \) and \( n_2 \), we concentrate on the subgroup \( G' \) of \( G \) corresponding to \( N' \). Accordingly, we consider the \( y \)-representation of \( G' \) acting on the subalgebra of \( \mathcal{R}_q(y) \) generated by \( y^{n_1} \) and \( y^{n_2} \). By the assumption \( \{ n_2, n_1 \} \neq 0 \), this is faithful. Thus, one can prove (2.25) with this representation.

First, we consider the action on \( y^{n_1} \). Note that \( \{ n_1 + n_2, n_1 \} = -\{ n_1 + n_2, n_2 \} = c > 0 \). Then, thanks to the formula (2.23), we have

\[
(\Psi_{c,b_2}[n_2] \Psi_{c,b_1}[n_1])(y^{n_1}) = \Psi_{c,b_2}[n_2](y^{n_1}) = y^{n_1}(1 + q^{c+b_2} y^{n_2}),
\]

and also

\[
(\Psi_{c,b_1}[n_1] \Psi_{c,b_1+b_2}[n_1+n_2] \Psi_{c,b_2}[n_2])(y^{n_1}) = (\Psi_{c,b_1}[n_1] \Psi_{c,b_1+b_2}[n_1+n_2])(y^{n_1}(1 + q^{c+b_2} y^{n_2})) \\
= \Psi_{c,b_1}[n_1](y^{n_1}(1 + q^{c+b_1+b_2} y^{n_1+n_2}) \\
\times (1 + q^{c+b_2} y^{n_2}(1 + q^{-c+b_1+b_2} y^{n_1+n_2})^{-1})) \\
= \Psi_{c,b_1}[n_1](y^{n_1}(1 + q^{c+b_2} y^{n_2} + q^{c+b_1+b_2} y^{n_1+n_2})) \\
= y^{n_1}(1 + q^{c+b_2} y^{n_2}(1 + q^{-c+b_1} y_1)^{-1} + q^{c+b_1+b_2} y^{n_1+n_2}(1 + q^{-c+b_1} y_1)^{-1}) \\
= y^{n_1}(1 + q^{-c+b_1} y_1 + q^{c+b_2} y^{n_2} + q^{b_1+b_2} y^{n_2} y_1)(1 + q^{-c+b_1} y_1)^{-1} \\
= y^{n_1}(1 + q^{c+b_2} y^{n_2}).
\]

Thus, they coincide. Next, we consider the action on \( y^{n_2} \). Similarly, we have

\[
(\Psi_{c,b_2}[n_2] \Psi_{c,b_1}[n_1])(y^{n_2}) = \Psi_{c,b_2}[n_2](y^{n_2}(1 + q^{-c+b_1} y^{n_1})^{-1}) = y^{n_2}(1 + q^{-c+b_1} y^{n_1}(1 + q^{c+b_2} y^{n_2})^{-1}) = y^{n_2}(1 + q^{-c+b_1} y^{n_1} + q^{b_1+b_2} y^{n_1} y^{n_2})^{-1},
\]

and also

\[
(\Psi_{c,b_1}[n_1] \Psi_{c,b_1+b_2}[n_1+n_2] \Psi_{c,b_2}[n_2])(y^{n_2}) = (\Psi_{c,b_1}[n_1] \Psi_{c,b_1+b_2}[n_1+n_2])(y^{n_2}) \\
= \Psi_{c,b_1}[n_1](y^{n_2}(1 + q^{-c+b_1+b_2} y^{n_1+n_2})^{-1}) \\
= y^{n_2}(1 + q^{-c+b_1} y^{n_1})^{-1}(1 + q^{-c+b_1+b_2} y^{n_1+n_2}(1 + q^{-c+b_1} y_1)^{-1})^{-1} \\
= y^{n_2}(1 + q^{-c+b_1} y^{n_1} + q^{b_1+b_2} y^{n_1} y^{n_2})^{-1}.
\]

Again, they coincide. \( \square \)

Also, the following operations to decompose and unify quantum dilogarithm elements are important for our purpose.
Proposition 2.8. Let \( p \) be a positive integer. The following equalities hold, where we assume \( a/p \in (1/\delta_0)\mathbb{Z}_{>0} \) in the second equality:

\[
(2.26) \quad \text{(fission)} \quad \Psi_{a,b}[n] = \prod_{t=1}^{p} \Psi_{pa,b+(2t-p-1)a}[n],
\]

\[
(2.27) \quad \text{(fusion)} \quad \prod_{t=1}^{p} \Psi_{a,b+(2t-p-1)a/p}[n] = \Psi_{a/p,b}[n].
\]

Proof. Two equalities (2.26) and (2.27) are equivalent. The equality (2.26) is shown as follows:

\[
\Psi_{a,b}[n] = \exp \left( \sum_{j=1}^{\infty} \frac{(-1)^{j+1} q^{ja} - q^{-ja} q^{bj} X_{jn}}{j^m} \right)
\]

\[
= \exp \left( \sum_{j=1}^{\infty} \frac{(-1)^{j+1} q^{ja} - q^{-ja} q^{bj} X_{jn}}{j^m} \right)
\]

\[
= \prod_{t=1}^{p} \Psi_{pa,b+(2t-p-1)a}[n].
\]

\[\square\]

3. Quantum cluster scattering diagrams

We introduce quantum cluster scattering diagrams. Examples in rank 2 are also given.

3.1. Quantum cluster scattering diagrams. The following definitions for (quantum) scattering diagrams are just the same as the classical one in [GHKK18], by replacing the structure group \( G \) with the one in Section 2.1.

A wall \( w = (d, g)_n \) for \( s \) is a triplet with \( n \in N \), a cone \( d \subset n^\perp \subset M \otimes \mathbb{R} \) of codimension 1, and \( g \in G_n^\parallel \). We call \( n, d, g \), the normal vector, the support, the wall element of \( w \), respectively. Let \( p^*: N \to M^\circ \subset M^\otimes \), \( n \mapsto \{ \cdot, n \} \). We say that a wall \( w = (d, g)_n \) is incoming if \( p^*(n) \in d \) holds.

Definition 3.1 (Scattering diagram). A scattering diagram \( \mathfrak{D} = \{ w_\lambda = (d_\lambda, g_\lambda)_n \}_{\lambda \in \Lambda} \) for \( s \) is a collection of walls for \( s \) satisfying the following finiteness condition: For any degree \( \ell \), there are only finitely many walls such that \( \pi_\ell(g_\lambda) \neq \text{id} \), where \( \pi_\ell: G \to G^{\leq \ell} \) is the canonical projection.

For a scattering diagram \( \mathfrak{D} \), we define

\[
(3.1) \quad \text{Supp}(\mathfrak{D}) = \bigcup_{\lambda \in \Lambda} d_\lambda, \quad \text{Sing}(\mathfrak{D}) = \bigcup_{\lambda \in \Lambda} \partial d_\lambda \cup \bigcup_{\lambda, \lambda' \in \Lambda, \dim d_\lambda \cap d_{\lambda'} = r-2} d_\lambda \cap d_{\lambda'}.
\]

A curve \( \gamma: [0, 1] \to M^\otimes \) is admissible for \( \mathfrak{D} \) if it satisfies the following properties:

(1) The endpoints of \( \gamma \) are in \( M^\otimes \setminus \text{Supp}(\mathfrak{D}) \).

(2) It is a smooth curve, and it intersects \( \text{Supp}(\mathfrak{D}) \) transversally.

(3) \( \gamma \) does not intersect \( \text{Sing}(\mathfrak{D}) \).
For any admissible curve \( \gamma \), the path-ordered product \( p_{\gamma, \mathcal{D}} \in G \) is defined as the product of the wall elements \( g_{\lambda}^{\gamma} \) of walls \( w_{\lambda} \) of \( \mathcal{D} \) intersected by \( \gamma \) in the order of intersection, where \( \epsilon_{\lambda} \) is the intersection sign defined by

\[
\epsilon_{\lambda} = \begin{cases} 
1 & \langle n_{\lambda}, \gamma' \rangle < 0, \\
-1 & \langle n_{\lambda}, \gamma' \rangle > 0,
\end{cases}
\]

and \( \gamma' \) is the velocity vector of \( \gamma \) at the wall \( w_{\lambda} \). The product \( p_{\gamma, \mathcal{D}} \) is infinite in general, and it is well-defined in \( G \) due to the finiteness condition. We say that a pair of scattering diagrams \( \mathcal{D} \) and \( \mathcal{D}' \) are equivalent if \( p_{\gamma, \mathcal{D}} = p_{\gamma, \mathcal{D}'} \) for any admissible curve \( \gamma \) for both \( \mathcal{D} \) and \( \mathcal{D}' \). We say that a scattering diagram \( \mathcal{D} \) is consistent if \( p_{\gamma, \mathcal{D}} = \text{id} \) for any admissible loop (i.e., closed curve) \( \gamma \) for \( \mathcal{D} \).

The following definition is due to [Man21].

**Definition 3.2** (Quantum cluster scattering diagram). A quantum cluster scattering diagram \( \mathcal{D}_s^q \) (QCSD, for short) for \( s \) is a consistent scattering diagram whose set of incoming walls are given by

\[
\text{In}_s := \{ (e_i^+, \Psi_{1/\delta_i} [e_i])_i | i = 1, \ldots, r \}.
\]

**Theorem 3.3** ([GHKK18, Theorems 1.12], [KS14, Theorem 2.1.6], [DN21, Theorem 2.13]). There exists a QCSD \( \mathcal{D}_s^q \) uniquely up to equivalence.

**Proof.** The proof of [GHKK18, Theorem 1.12], which originates in [KS14, §2], is also applicable to this case. See also [Nak23, §III.3], which is closer to the present setting. \( \square \)

Thanks to Proposition 2.5 we may assume that, up to equivalence, wall elements of \( \mathcal{D}_s^q \) are given by \( \mathbb{Q}(q^{1/\delta_0}) \)-powers of quantum dilogarithm elements \( \Psi_{a,b}[n]^c \) \( (n \in \mathbb{N}, c \in \mathbb{Q}(q^{1/\delta_0})) \).

**Remark 3.4.** To each admissible loop \( \gamma \), the consistency condition \( p_{\gamma, \mathcal{D}} = \text{id} \) gives a relation among quantum dilogarithm elements. Under the correspondence in Remark 2.6 this relation is translated as an identity for the quantum dilogarithm (quantum dilogarithm identity), which is a generalization of a quantum dilogarithm identity associated with a period of a quantum cluster pattern studied in [FG09b, Kel11, KN11, Nag11]. It is also the quantum counterpart of the (classical) dilogarithm identity for the Rogers dilogarithm in [Nak24].

### 3.2. Rank 2 examples

Let us demonstrate to construct the rank 2 QCSDs of finite and affine types based on the pentagon relation (2.25) in the spirit of [Nak23, §III.2.2, §III.3.5] for their classical counterparts.

#### 3.2.1. Finite type

Here we consider the case where \( \{ \cdot, \cdot \} \neq 0 \). Without loss of generality, we may assume that \( \{ e_2, e_1 \} = 1 \) and \( \delta_2 \geq \delta_1 \). Thus, the initial exchange matrix \( B \) in (2.1) is given by

\[
B = \begin{pmatrix} 
0 & -\delta_1 \\
\delta_2 & 0
\end{pmatrix}.
\]

Let \( e_1^*, e_2^* \in M \) be the dual basis of \( e_1, e_2 \in N \). Accordingly, let \( f_1 = e_1^*/\delta_1, f_2 = e_2^*/\delta_2 \in M^0 \) be the dual basis of \( \delta_1 e_1, \delta_2 e_2 \in N^0 \). Let \( e_1, e_2 \in \mathbb{Z}^2 \) be the unit vectors. We identify
Figure 1. Rank 2 QCSDs of finite type.

$N \simeq \mathbb{Z}^2$, $e_i \mapsto e_i$, and $M_\mathbb{R} \simeq \mathbb{R}^2$, $f_i \mapsto e_i$. Then, we have $\{\mathbf{n}', \mathbf{n}\} = n'_1n_1 - n'_1n_2$. Also, the canonical paring $\langle n, z \rangle : N \times M_\mathbb{R} \to \mathbb{R}$ is given by the corresponding vectors $\mathbf{n}$ and $\mathbf{z}$ as

$$\langle n, z \rangle = n^T \begin{pmatrix} \delta^{-1}_1 & 0 \\ 0 & \delta^{-1}_2 \end{pmatrix} z.$$  

For $\mathbf{n} = (n_1, n_2)$, we write

$$\begin{bmatrix} n_1 \\ n_2_{a,b} \end{bmatrix} := \Psi_{a,b}[\mathbf{n}], \quad \begin{bmatrix} n_1 \\ n_2_{a} \end{bmatrix} := \Psi_{a}[\mathbf{n}].$$

(a). Type $A_2$. Let $(\delta_1, \delta_2) = (1, 1)$. Since $\{e_2, e_1\} = 1$, we apply the pentagon relation (2.25) with $c = 1$, and we have

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}_{1,0} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{1,0} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{1,0} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{1,0} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{1,0} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{1,0} \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{1,0}.$$  

This equality is naturally interpreted as a consistency relation of the QCSD of rank 2 in Figure 1 (a). Namely, it consists of three walls

$$\mathbb{Z} \leftarrow 1 \rightarrow 0 \leftarrow 1 \rightarrow 0 \leftarrow 1 \rightarrow 0 \leftarrow 1 \rightarrow 0.$$

The LHS of the equality (3.7) is the path-ordered product $p_{\gamma_1, \gamma_2}$ along $\gamma_1$, while the RHS is the one along $\gamma_2$ in Figure 1. (b). Type $B_2$. Let $(\delta_1, \delta_2) = (1, 2)$. To use the pentagon relation (2.25) with $c = 1$, we first apply the fission (2.26) for $\Psi_{1/2}(e_2)$ with $p = 2$. Then apply the pentagon relation repeatedly for adjacent pairs, and apply the fusion (2.27) in the end. We have

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}_{1,0} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{1,0} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{1,-\frac{1}{2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{1,\frac{1}{2}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{1,0} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{1,\frac{1}{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{1,0} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{1,\frac{1}{2}}$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{1,0} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{1,\frac{1}{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{1,\frac{1}{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{1,0} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{1,\frac{1}{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{1,0} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{1,\frac{1}{2}}$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{1,0} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{1,\frac{1}{2}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{1,0} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{1,\frac{1}{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{1,\frac{1}{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{1,0} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{1,\frac{1}{2}}.$$  

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{1,0} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{1,\frac{1}{2}} \begin{bmatrix} 1 \end{bmatrix}_{1,0} \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{1,\frac{1}{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{1,\frac{1}{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{1,0} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{1,\frac{1}{2}}.$$
This equality is naturally interpreted as a consistency relation of the QCSD in Figure 1 (b), which consists of four walls

\[(\mathbb{R}_{\geq 0}(1, -2), \Psi_{1/2}((1, 1))_{(1, 1)}), \quad (\mathbb{R}_{\geq 0}(1, -1), \Psi_1((1, 2))_{(1, 2)}).
\]

(c). Type $G_2$. Let $(\delta_1, \delta_2) = (1, 3)$. In the same way, we obtain

\[
\begin{bmatrix}
0 \\
1 \\
1 \\
0 \\
1 \\
0 \\
1 \\
1 \\
0 \\
1 \\
0 \\
0 \\
1 \\
0 \\
0 \\
1 \\
1 \\
0 \\
1 \\
0 \\
1 \\
0 \\
1 \\
0 \\
1 \\
1 \\
0 \\
1 \\
0 \\
1 \\
0
\end{bmatrix}
= \begin{bmatrix}
1 \\
0 \\
1 \\
0 \\
1 \\
0 \\
1 \\
0 \\
1 \\
0 \\
1 \\
0 \\
1 \\
0 \\
1 \\
0 \\
1 \\
0 \\
1 \\
0 \\
1 \\
0 \\
1 \\
0 \\
1 \\
0 \\
1 \\
0 \\
1 \\
0
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
2 \\
3 \\
1 \\
2 \\
3 \\
1 \\
2 \\
3 \\
1 \\
2 \\
3 \\
1 \\
2 \\
3 \\
1 \\
2 \\
3 \\
1 \\
2 \\
3 \\
1 \\
2 \\
3 \\
1 \\
2 \\
3 \\
1
\end{bmatrix}
\]

\[(\mathbb{R}_{\geq 0}(2, -3), \Psi_{1/3}((1, 2))_{(1, 2)}), \quad (\mathbb{R}_{\geq 0}(1, -1), \Psi_1((1, 3))_{(1, 3)}).
\]

Note that, in the RHSs of (3.7), (3.9), and (3.12), all factors have the form $\Psi_{1/\delta(n)}[n]$.

3.2.2. Affine type. Here we consider the affine type, where $\delta_1 \delta_2 = 4$ for $B$ in (2.1). There are two cases.

(a). Type $A_1^{(1)}$. Let $(\delta_1, \delta_2) = (2, 2)$. Note that, for any primitive $n \in \mathbb{N}_p$, we have $\delta(n) = 2$. The following description of a CSD $\mathcal{D}_5$ is known [GHKK18, Rei09, Rea20, Nak23, Mat21]: The walls of $\mathcal{D}_5$ are given by

\[(\mathbb{R}_{\geq 0}(1, -1), \Psi_2^2)((1, 1))_{n_0}\]

where $n_0 = (1, 1)$. Note that $\delta((p, p + 1)) = 2$ and $\delta(2^i n_0) = 2^{1 - i}$. See Figure 2 (a).
Again, we apply the fission and the pentagon relation to the element \( \Psi_{1/2}[e_2]\Psi_{1/2}[e_1] \) and obtain the equality

\[
\begin{bmatrix}
0 \\
\frac{1}{2},0 \\
0 \\
\frac{1}{2},0
\end{bmatrix} = \begin{bmatrix}
0 \\
\frac{1}{2},0 \\
0 \\
\frac{1}{2},0
\end{bmatrix} \cdot \begin{bmatrix}
0 \\
\frac{1}{2},0 \\
0 \\
\frac{1}{2},0
\end{bmatrix} = \begin{bmatrix}
1 \\
0,0 \\
0 \\
0
\end{bmatrix}.
\]
\[
(3.20)
\]

At this moment, the expression in the RHS is not yet ordered to be presented by a scattering diagram. Here, we say that the above product is ordered (resp. anti-ordered) if, for any adjacent pair \([n']_{a,b}[n]_{a,b}, \{n', n\} = n'_2 n_1 - n'_1 n_2 \leq 0\) (resp. \(n', n \geq 0\)) holds. Equivalently, if we view \([n]\) as a fraction \(n_1/n_2\), then, the numbers should be aligned in the decreasing order (resp. increasing order) form left to right. The LHS of \((3.20)\) is anti-ordered. To make the RHS of \((3.20)\) ordered, we need to interchange the factors for \([n'] = (1,2)\) and \(n = (1,0)\) in the middle, where \(\{n', n\} = 2\). As the lowest approximation, we consider modulo \(G^{>3}\). Then, \([1,2]_{1/2}\) commutes with other factors, and we have, modulo \(G^{>3}\),

\[
(3.21)
\]

To proceed to higher degree, we now apply the pentagon relation \((2.25)\) with \(c = 2\) to the pair \([1,2]_{1/2}\) and \([0,1]_{1/2}\) in \((3.20)\). Then, we have

\[
\begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
2 \\
1 \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
1 \\
2 \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix}.
\]
\[
(3.22)
\]

This is parallel to \((3.20)\) but with different quantum data. As the next approximation, we consider modulo \(G^{>7}\). Then, \([3,4]_{2,-3/2}\) commutes with other factors, and we have, modulo \(G^{>7}\),

\[
(3.23)
\]

Then, we plug it into \((3.20)\), and apply the pentagon relation, and we have, modulo \(G^{>7}\),

\[
\begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix} \times \begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix} \times \begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix}.
\]
\[
(3.24)
\]

By continuing the procedure modulo \(G^{>2^j - 1}\), one can naturally guess that the relation converges to the following one:

\[
\begin{bmatrix}
0 \\
\frac{1}{2},0 \\
0 \\
\frac{1}{2},0
\end{bmatrix} = \begin{bmatrix}
0 \\
\frac{1}{2},0 \\
0 \\
\frac{1}{2},0
\end{bmatrix} \times \begin{bmatrix}
0 \\
\frac{1}{2},0 \\
0 \\
\frac{1}{2},0
\end{bmatrix} \times \begin{bmatrix}
0 \\
\frac{1}{2},0 \\
0 \\
\frac{1}{2},0
\end{bmatrix} \times \cdots \left( \prod_{j=0}^{\infty} [2j] [2j] [2j] [2j] \right).
\]
\[
(3.25)
\]
Indeed, one can prove the relation (3.25) completely based on the pentagon relation by modifying the proof of [Mat21] for the classical case. The detail will be found in Appendix A. This is the simplest example of consistency relations involving infinite products. The formula appeared in [DGS11] Eq. (1.3)) without proof in an alternative setting of quantum dilogarithms. See also an alternative derivation in view of the quantum affine algebra of type $A^{(1)}_1$ [Sug23, Theorem 5.1].

(b) Type $A^{(2)}_2$. Let $(\delta_1, \delta_2) = (1, 4)$. The situation is in parallel with $A^{(1)}_1$ though a little more complicated. The following description of the CSD $\mathcal{D}_s$ is known [Rea20] [Nak23] [Mat21]: The walls of $\mathcal{D}_s$ are given by

\begin{equation}
(3.26) \quad (e^+_1, \Psi[e_1])_{e_1}, \quad (e^+_2, \Psi[e_2])_{e_2},
\end{equation}

\begin{equation}
(3.27) \quad (\mathbb{R} \geq 0(p, -2p, 1), \Psi[(2p + 1, 4p)]((2p + 1, 4p)) (p \in \mathbb{Z} > 0),
\end{equation}

\begin{equation}
(3.28) \quad (\mathbb{R} \geq 0(2p - 1, -4p), \Psi[(p, 2p - 1)]^4(p, 2p - 1) (p \in \mathbb{Z} > 0),
\end{equation}

\begin{equation}
(3.29) \quad (\mathbb{R} \geq 0(p, 2p + 1), \Psi[(2p - 1, 4p)]((2p - 1, 4p)) (p \in \mathbb{Z} > 0),
\end{equation}

\begin{equation}
(3.30) \quad (\mathbb{R} \geq 0(2p + 1, -4p), \Psi[(p, 2p + 1)]^4(p, 2p + 1) (p \in \mathbb{Z} > 0),
\end{equation}

\begin{equation}
(3.31) \quad (\mathbb{R} \geq 0(1, -2), \Psi[n_0]^{(6)}_{n_0}, \quad (\mathbb{R} \geq 0(1, -2), \Psi[2^j n_0]^{2^{2-j}}_{n_0} (j \in \mathbb{Z} > 0),
\end{equation}

where $n_0 = (1, 2)$. Note that $\delta(2^4 n_0) = 2^{1-j}$. See Figure 2 (b).

Again, we apply the fission and the pentagon relation to the element $\Psi_{1/4}[e_2] \Psi_1[e_1]$ and obtain the equality

\begin{equation}
(3.32) \quad \begin{bmatrix} 0 \\ 1 \\ \frac{1}{4} \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \frac{1}{1} \\ -\frac{1}{4} \\ 1 \\ -\frac{1}{4} \\ 1 \\ -\frac{1}{4} \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.
\end{equation}

Again, this is not yet ordered, and to order it, we need to interchange $[1, 3]_{1, 3/4}$ and $[1, 1]_{1, 3/4}$ in the middle. As the lowest approximation, we consider modulo $G^{25}$. Then, $
\Psi_{1, 3/4}[1, 3]$ commutes with other factors, and we have, modulo $G^{25}$,

\begin{equation}
(3.33) \quad \begin{bmatrix} 0 \\ 1 \\ \frac{1}{4} \\ 0 \\ 0 \\ 1 \end{bmatrix} \equiv \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ \frac{1}{4} \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ \frac{1}{4} \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ \frac{1}{4} \\ 0 \\ 1 \end{bmatrix}.
\end{equation}

To proceed to higher degree, we do in the same way as before. By applying the fission and the pentagon relation (2.25) with $c = 2$, we have

\begin{equation}
(3.34) \quad \begin{bmatrix} 1 \\ 3 \\ -\frac{1}{4} \\ 1 \\ 1 \\ 1 \\ \frac{3}{4} \\ -\frac{1}{4} \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \frac{1}{4} \\ -\frac{1}{4} \\ 3 \\ 2 \frac{1}{4} \\ -\frac{1}{4} \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \frac{1}{4} \\ -\frac{1}{4} \\ 3 \\ 2 \frac{1}{4} \end{bmatrix}.
\end{equation}

This parallel to (3.20), but with different quantum data. As the next approximation, we consider modulo $G^{11}$. Then, $[3, 7]_{2, -7/4}$ commutes with other factors, and we have, modulo
By continuing the procedure modulo $G^{>11}$,

$$
\begin{bmatrix}
1 \\
3 \\
1
\end{bmatrix}_{1,-\frac{3}{4}} \\
\begin{bmatrix}
1 \\
1
\end{bmatrix}_{1,\frac{3}{4}} \\
\begin{bmatrix}
1 \\
1
\end{bmatrix}_{1,\frac{3}{4}} \\
\begin{bmatrix}
2 \\
4 \\
1
\end{bmatrix}_{1,-1} \\
\begin{bmatrix}
2 \\
4 \\
1
\end{bmatrix}_{1,1} \\
\begin{bmatrix}
3 \\
7 \\
1
\end{bmatrix}_{1,-\frac{3}{4}} \\
\begin{bmatrix}
3 \\
7 \\
1
\end{bmatrix}_{1,\frac{3}{4}} \\
\begin{bmatrix}
1 \\
3 \\
1
\end{bmatrix}_{1,-\frac{3}{4}} \\
\begin{bmatrix}
1 \\
3 \\
1
\end{bmatrix}_{1,\frac{3}{4}}
\end{bmatrix}_{1,-\frac{3}{4}}.
$$

Then, we plug it into (3.32), and apply the pentagon relation, and we have, modulo $G^{>11}$,

$$
\begin{bmatrix}
0 \\
1
\end{bmatrix}_{\frac{1}{4},0} \\
\begin{bmatrix}
1 \\
0
\end{bmatrix}_{1,0} \\
\begin{bmatrix}
1 \\
0
\end{bmatrix}_{1,0} \\
\begin{bmatrix}
2 \\
3 \\
1
\end{bmatrix}_{\frac{1}{2},0} \\
\begin{bmatrix}
2 \\
3 \\
1
\end{bmatrix}_{\frac{1}{2},0} \\
\begin{bmatrix}
3 \\
4 \\
1
\end{bmatrix}_{\frac{1}{4},0} \\
\begin{bmatrix}
3 \\
4 \\
1
\end{bmatrix}_{\frac{1}{4},0} \\
\begin{bmatrix}
1 \\
2 \\
1
\end{bmatrix}_{\frac{1}{2},0} \\
\begin{bmatrix}
1 \\
2 \\
1
\end{bmatrix}_{\frac{1}{2},0} \\
\begin{bmatrix}
0 \\
1
\end{bmatrix}_{\frac{1}{4},0}
\end{bmatrix}_{\frac{1}{4},0}.
$$

By continuing the procedure modulo $G^{>2^j-1}$, one can naturally guess that the relation converges to the following one:

$$
\begin{bmatrix}
0 \\
1
\end{bmatrix}_{\frac{1}{4},0} \\
\begin{bmatrix}
1 \\
0
\end{bmatrix}_{1,0} \\
\begin{bmatrix}
1 \\
0
\end{bmatrix}_{1,0} \\
\begin{bmatrix}
2 \\
3 \\
1
\end{bmatrix}_{\frac{1}{2},0} \\
\begin{bmatrix}
2 \\
3 \\
1
\end{bmatrix}_{\frac{1}{2},0} \\
\begin{bmatrix}
3 \\
4 \\
1
\end{bmatrix}_{\frac{1}{4},0} \\
\begin{bmatrix}
3 \\
4 \\
1
\end{bmatrix}_{\frac{1}{4},0} \\
\begin{bmatrix}
1 \\
2 \\
1
\end{bmatrix}_{\frac{1}{2},0} \\
\begin{bmatrix}
1 \\
2 \\
1
\end{bmatrix}_{\frac{1}{2},0} \\
\begin{bmatrix}
0 \\
1
\end{bmatrix}_{\frac{1}{4},0}
\end{bmatrix}_{\frac{1}{4},0}
\times
\left(\prod_{j=0}^{\infty} \begin{bmatrix}
2j \\
2j+1
\end{bmatrix}_{2j-1,-2j-1} \begin{bmatrix}
2j+1 \\
2j
\end{bmatrix}_{2j-1,2j-1}
\right)
\times \cdots
\begin{bmatrix}
5 \\
12
\end{bmatrix}_{1,0} \\
\begin{bmatrix}
2 \\
3 \\
1
\end{bmatrix}_{\frac{1}{4},0} \\
\begin{bmatrix}
3 \\
4 \\
1
\end{bmatrix}_{\frac{1}{4},0} \\
\begin{bmatrix}
4 \\
5 \\
1
\end{bmatrix}_{1,0} \\
\begin{bmatrix}
0 \\
1
\end{bmatrix}_{\frac{1}{4},0}.
$$

Again, one can prove the relation (3.37) completely based on the pentagon relation by modifying the proof of [Mat21] for the classical case. The detail will be found in Appendix A.

Note that, in the RHSs of (3.25) and (3.37), all factors have the form $\Psi_{1/\delta(n),b}[n]$.

4. Principal x-representation

We introduce another representation of $G$ which we call the principal $x$-representation. It is closely related to the quantization of cluster variables ($x$-variables) in [BZ05]. Then, we obtain the reduction property of wall elements for a QCSD, which is important in our application of the pentagon relation to the positivity problem.

4.1. Principal $x$-representation. Following [GHKK18], we introduce

$$
\tilde{N} = N \oplus M^\circ, \quad \tilde{M}^\circ = M^\circ \oplus N,
$$

which are “dual” to each other twisted by $\delta_1, \ldots, \delta_r$. An element of $\tilde{N}$ is denoted by $\tilde{n} = (n, m)$ ($n \in N$, $m \in M^\circ$). Similarly, an element of $\tilde{M}^\circ$ is denoted by $\tilde{m} = (m, n)$ ($m \in M^\circ$, $n \in N$). Below we use the notation such as $\tilde{n}_1 = (n_1, m_1)$, $\tilde{m}_1 = (m', n')$ without explanation. Meanwhile, we also use the internal-sum notation. For example, for $n \in N$, we write $n = (n, 0) \in \tilde{N}$ and $n = (0, n) \in \tilde{M}^\circ$. Both spaces $\tilde{N}$ and $\tilde{M}^\circ$ are equipped with (essentially common) bilinear forms

$$
\begin{align*}
\{ \tilde{n}, \tilde{n}' \} \tilde{N} & := \{ n, n' \} + \langle n', m \rangle - \langle n, m' \rangle, \\
\{ \tilde{m}, \tilde{m}' \} \tilde{M}^\circ & := -\{ n, n' \} - \langle n', m \rangle + \langle n, m' \rangle.
\end{align*}
$$
where \( \langle \cdot, \cdot \rangle \) and \( \langle \cdot, \cdot \rangle \) in the RHSs are the ones in Section 2.1. We also have the canonical paring \( \langle \cdot, \cdot \rangle : \hat{N} \times M^o \rightarrow (1/\delta_0)\mathbb{Z} \) defined by
\[
\langle (n, m), (m', n') \rangle := \langle n, m' \rangle + \langle n', m \rangle.
\]

Let \( \tilde{p}^* \) be the group homomorphism defined by
\[
\tilde{p}^*: N \rightarrow \hat{M}^o, \quad \{n', m\}_{\hat{N}} = \{n', n\} + \langle n, m' \rangle.
\]

Note that the map \( \tilde{p}^* \) is injective. Let \( p^*: N \rightarrow M^o \) be the one in Section 3.1. Then, we have
\[
\tilde{p}^*(n) = (p^*(n), n) \in \hat{M}^o.
\]
The following duality relation holds:
\[
\{\tilde{p}^*(n), \tilde{m}\}_{\hat{M}^o} = \langle n, \tilde{m} \rangle.
\]

Indeed, by (4.6),
\[
\{\tilde{p}^*(n), \tilde{m}\}_{\hat{M}^o} = \{(p^*(n), n), (m', n')\}_{\hat{M}^o}
\]
\[
= -\{n, n'\} - \langle n', p^*(n) \rangle + \langle n, m' \rangle
\]
\[
= \langle n, m' \rangle = \langle n, \tilde{m} \rangle.
\]

**Remark 4.1.** The bilinear form \( \{\cdot, \cdot\}_{\hat{M}^o} \) in (4.3) has the representation matrix with respect to the basis \( f_1, \ldots, f_r, e_1, \ldots, e_r \) of \( M^o \) as
\[
\Lambda := \begin{pmatrix} O & -D \\ D & -DB \end{pmatrix}, \quad D = \text{diag}(\delta_1^{-1}, \ldots, \delta_r^{-1}).
\]

Let \( \tilde{B} \) be the principal extension of the matrix \( B \),
\[
\tilde{B} = \begin{pmatrix} B \\ I \end{pmatrix}.
\]
They satisfy the relation
\[
-\Lambda \tilde{B} = \begin{pmatrix} D \\ O \end{pmatrix},
\]
which is equivalent to the duality relation (4.7). Such a pair \( (\Lambda, \tilde{B}) \) is called a *compatible pair* in [BZ05].

Following [GHKK18], let \( \hat{P} \subset \hat{M}^o \) be a (not unique) monoid satisfying the following conditions:
(i). \( \hat{P} = \sigma \cap \hat{M}^o \), where \( \sigma \) is a 2r-dimensional strongly convex cone in \( \hat{M}_R^o \).
(ii). \( \tilde{p}^*(e_1), \ldots, \tilde{p}^*(e_r) \in \hat{P} \).
(iii). \( f_1, \ldots, f_r \in \hat{P} \).

For example, we take \( \sigma \) as the cone generated by \( f_1, \ldots, f_r, \tilde{p}^*(e_1), \ldots, \tilde{p}^*(e_r) \), which are a basis of \( \hat{M}^o \). (In [GHKK18], the condition (iii) was not assumed.)

Now we are ready to define the *(quantum) principal x-representation* of \( G \), which is defined in a parallel way to the \( y \)-representation.

Let \( x \) be a formal symbol. Let \( \mathbb{Q}(q^{1/\delta_0})[x]_q \) be the noncommutative and associative algebra over \( \mathbb{Q}(q^{1/\delta_0}) \) with generators \( x^{\tilde{m}} \) (\( \tilde{m} \in \hat{P} \)) and the \( q \)-commutative relations
\[
x^{\tilde{m}} x^{\tilde{m}'} = q^{\langle \tilde{m}, \tilde{m}' \rangle} x^{\tilde{m} + \tilde{m}'}.
\]
Let $R_q(x)$ be the completion of $\mathbb{Q}(q^{1/d_0})[x]_q$ with respect to the maximal ideal generated by $x^{\hat{m}}$s ($\hat{m} \in \hat{P} \setminus \{0\}$). Any element of $R_q(x)$ is expressed as an infinite sum

$$
\sum_{\hat{m} \in \hat{P}} c_{\hat{m}} x^{\hat{m}} \quad (c_{\hat{m}} \in \mathbb{Q}(q^{1/d_0})).
$$

We define the action of $\hat{g}$ on $R_q(x)$ by

$$
X_n(x^{\hat{m}}) := [(n, \hat{m})]_q x^{\hat{m}+\hat{p}'(n)}
$$

$$
= \frac{q^{2(n, \hat{m})} - 1}{q - q^{-1}} x^\hat{m} x^{\hat{p}'(n)},
$$

where in the second equality we used the duality relation (4.7). It is easy to check that this is an action of $\hat{g}$ and also a derivation. Thus, it induces the action of $G$ on $R_q(x)$ defined by

$$
(\exp X)(x^{\hat{m}}) = \sum_{j=0}^{\infty} \frac{1}{j!} X^j (x^{\hat{m}}) \quad (X \in \hat{g}).
$$

Moreover, $\exp X$ is an algebra automorphism. We call the resulting representation $\rho_x^{pr} : G \to \text{Aut}(R_q(x))$ the (quantum) principal $x$-representation of $G$. It is always a faithful representation, even when $\{\cdot, \cdot\}$ is degenerate.

### 4.2. Action of dilogarithm elements

Let us present an analogous result in Section 2.3 for the principal $x$-representation.

Any $\mathbb{Q}(q^{1/d_0})$-power of a dilogarithm element $\Psi_{a,b}[n]^c$ ($c \in \mathbb{Q}(q^{1/d_0})$) acts on $R_q(x)$ under $\rho_x^{pr}$ as

$$
\Psi_{a,b}[n]^c(x^{\hat{m}}) = \exp \left( c \sum_{j=1}^{\infty} \frac{(-1)^j}{j} q^j X_j^{\hat{m}} \right) (x^{\hat{m}})
$$

$$
= x^{\hat{m}} \exp \left( c \sum_{j=1}^{\infty} \frac{q^{2j(n, \hat{m})} - 1}{q^{2j} - 1} \frac{(-1)^j}{j} q^j a_q^j b x^{\hat{p}'(n)} \right).
$$

Let us consider a parallel situation to (2.20) and (2.21). Fix $n$ in (4.17), and let $a = 1/s\delta(n)$, where $s \in \mathbb{Z}_{>0}$ is a divisor of $\delta_0/\delta(n)$. Then, $\alpha := \langle s\delta(n)n, \hat{m} \rangle$ is an integer, and we have

$$
\frac{q^{2j(n, \hat{m})} - 1}{q^{2j} - 1} = \frac{q^{2\alpha j/s\delta(n)} - 1}{q^{2j} - 1} = \begin{cases} \sum_{p=0}^{\alpha-1} q^{2jp/s\delta(n)} & \alpha > 0, \\ 0 & \alpha = 0, \\ -\sum_{p=1}^{-\alpha} q^{-2jp/s\delta(n)} & \alpha < 0. \end{cases}
$$

Thus, we obtain

$$
\Psi_{1/s\delta(n),b}[n](x^{\hat{m}}) = \begin{cases} x^{\hat{m}} \prod_{p=1}^{\alpha} (1 + q^{(2p-1)/s\delta(n)} q^{b} x^{\hat{p}'(n)}) & \alpha > 0, \\ x^{\hat{m}} & \alpha = 0, \\ x^{\hat{m}} \prod_{p=1}^{-\alpha} (1 + q^{-(2p-1)/s\delta(n)} q^{-b} x^{\hat{p}'(n)} )^{-1} & \alpha < 0. \end{cases}
$$
Observe that this is indeed the automorphism part of the Fock-Goncharov decomposition of mutations of the quantum $x$-variables with principal coefficients in \cite{BZ05}. See also \cite{Man21, DM21, CFMM24} for an alternative approach, where the same representation is described through the adjoint action of the quantum dilogarithm on $\mathcal{R}_q(x)$ as in Remark 2.6

4.3. Reduction of wall elements in a QCSD. Let $S_q(x)$ be the subset of $\mathcal{R}_q(x)$ consisting of the elements such that the coefficients $c_{\tilde{m}}$ in \eqref{eq:4.13} are in $\mathbb{Z}[q^{\pm 1/\delta_0}]$. Then, $S_q(x)$ is an algebra over $\mathbb{Z}[q^{\pm 1/\delta_0}]$ by the same relation \eqref{eq:4.12}. The following lemma is a quantum analog of the one in \cite[Appendix C.3 Step II]{GHKK18}, \cite[Lemma III.5.8]{Nak23}.

Lemma 4.2. A $\mathbb{Q}(q^{1/\delta_0})$-power of a quantum dilogarithm element $\Psi_{a,b}[n]^c \ (c \in \mathbb{Q}(q^{1/\delta_0}))$ acts on $S_q(x)$ if and only if $a = 1/s\delta(n)$ for a divisor $s$ of $\delta_0/\delta(n)$ and $c \in \mathbb{Z}[q^{\pm 1/\delta_0}]$.

Proof. First, we prove the if-part. By the assumption $c \in \mathbb{Z}[q^{\pm 1/\delta_0}]$, $\Psi_{1/s\delta(n),b}[n]^c$ is decomposed into a product of $\Psi_{1/s\delta(n),b}[n]^c$ \ (for $\delta(n), b_i \in (1/\delta_0)\mathbb{Z}$, $c_i \in \mathbb{Z}$). By \eqref{eq:4.19}, each factor acts on $S_q(x)$. Thus, $\Psi_{a,b}[n]^c$ acts on $S_q(x)$. Next, we prove the only-if-part. Assume that $\Psi_{a,b}[n]^c$ acts on $S_q(x)$. Then, by \eqref{eq:1.17}, we have a necessary condition

\begin{equation}
\frac{c^q 2j(n,\tilde{m}) - 1 (-1)^{j+1}}{q^{2ja} - 1} q^ja^b \in \mathbb{Q}[q^{\pm 1/\delta_0}] \quad (j \in \mathbb{Z}_{>0}).
\end{equation}

Let $a = 1/s\delta(n)$, where $s$ is a positive rational number. Then, \eqref{eq:4.20} implies the condition $s\delta(n,\tilde{m}) \in \mathbb{Z}$ for any $\tilde{m} \in \tilde{P}$. By the assumption for $\tilde{P}$, $\tilde{P}$ contains a basis of $\tilde{M}$. It follows that $s\delta(n,\tilde{m}) \in \mathbb{Z}$ for any $\tilde{m} \in \tilde{M}$. Due to the definition of $\delta(n)$, there is some $\tilde{m} \in \tilde{M}$ such that $\delta(n,\tilde{m}) = 1$. Thus, we have $s \in \mathbb{Z}$. On the other hand, let $t = \delta_0/\delta(n) \in \mathbb{Z}_{>0}$. Then, $a = t/s\delta_0$. Since we require that $a \in (1/\delta_0)\mathbb{Z}_{>0}$, $s$ is a divisor of $t$. This finishes the condition for $a$. Meanwhile, the condition \eqref{eq:4.20} also implies $c \in \mathbb{Q}(q^{1/\delta_0})$. Then, $\Psi_{a,b}[n]^c$ is decomposed into a product of $\Psi_{1/s\delta(n),b}[n]^c$ \ (for $\tilde{m} \in (1/\delta_0)\mathbb{Z}$, $c_i \in \mathbb{Q}$). By \eqref{eq:4.19}, we have $c_i \in \mathbb{Z}$. Thus, we have $c \in \mathbb{Z}[q^{\pm 1/\delta_0}]$. \hfill $\square$

Below, we only use the if-part of Lemma 4.2.

Proposition 4.3. Any QCSD $\mathfrak{D}_\ell^q$ is equivalent to a consistent scattering diagram whose wall elements have the form

\begin{equation}
\Psi_{1/s\delta(n),b}[n]^c \quad (n \in N^+, \ b \in (1/\delta_0)\mathbb{Z}, \ c \in \mathbb{Z}).
\end{equation}

Proof. We follow Step 2 in the proof of \cite[Prop. C.13]{GHKK18}. This is proved inductively on the degree of $n$, by considering the consistency relations around the perpendicular joints. Here, we only describe the key point of the inductive step. Suppose that a scattering diagram $\mathfrak{D}_\ell$ has the walls with wall elements of the form \eqref{eq:4.21} with deg $n \leq \ell$ and that, for a small loop $\gamma$ around a perpendicular joint $j$, the consistency relation $p_\gamma,\mathfrak{D}_\ell = \text{id}$ holds modulo $G^{>\ell}$. By Lemma 4.2, $p_\gamma,\mathfrak{D}_\ell$ acts on $S_q(x)$. By adding walls around $j$ with wall elements $\Psi_{a,b}[n]^c$ \ (deg $n = \ell + 1$) to $\mathfrak{D}_\ell$, we have a scattering diagram $\mathfrak{D}_{\ell+1}$ such that $p_\gamma,\mathfrak{D}_{\ell+1} = \text{id}$ modulo $G^{>\ell+1}$. Since we consider the action of $\Psi_{a,b}[n]^c$ modulo $G^{>\ell+1}$, the second expression of \eqref{eq:4.17} is simplified as

\begin{equation}
x^\tilde{m} \left( 1 + \frac{c^q 2j(n,\tilde{m}) - 1 (-1)^{j+1}}{q^{2ja} - 1} q^ja^b x^\tilde{p}(n) \right).
\end{equation}
We may further concentrate on \( \tilde{m} \) with \( \langle \delta(n) \rangle_{n, \tilde{m}} = 1 \), because other cases are proportional to it with factors that are independent of \( \Psi_{1/\delta(n),0}[n]^c \). In particular, the action of \( \Psi_{1/\delta(n),0}[n]^c \) on such \( \tilde{m} \) is given by

\[
x^{\tilde{m}} \left( 1 + cq^a x^{\tilde{p}^*(n)} \right).
\]

This means that, for each \( n \in N^+ \) (deg \( n = \ell + 1 \)), the (possibly multiple) added wall elements \( \Psi_{a,b}[n]^{c_i} \) (\( i = 1, 2, \ldots \)) in \( D_{\ell+1} \) is replaced with a single wall element \( \Psi_{1/\delta(n),0}[n]^c \); moreover, we have \( c \in \mathbb{Z}[q^{\pm1/\delta_0}] \) by the induction assumption. Then, it is further decomposed into a product of \( \Psi_{1/\delta(n),0}[n]^c \)'s (\( c \in \mathbb{Z} \)).

\[ \square \]

5. Application to positivity and nonpositivity of QCSDs

In the rest of the paper we apply the pentagon relation, together with the combinatorics of quantum data, to study the positivity and nonpositivity of QCSDs.

5.1. Positivity of QCSDs. Let us briefly summarize known results concerning the positivity and nonpositivity of CSDs and QCSDs.

Let us introduce the following notion.

**Definition 5.1.** A CSD \( D_\alpha \) is a *positive realization* if the wall element of any wall has the form

\[
\Psi[n]^{\tilde{\delta}(n)} \quad (n \in N^+, b \in (1/\delta_0)\mathbb{Z}, c \in \mathbb{Z}_{>0}).
\]

We say that a CSD \( D_\alpha \) is *positive* if it is equivalent to a CSD that is a positive realization.

The following fact is one of the main results of \cite{GHKK18}.

**Theorem 5.2** (\cite[Theorem 1.13]{GHKK18}). Every CSD \( D_\alpha \) is positive.

For each CSD, a *theta function* is defined by broken lines \cite{GHKK18}. The definition of a broken line and Theorem 5.2 immediately imply the following positivity result.

**Corollary 5.3** (Positivity of theta functions \cite{GHKK18}). For any CSD \( D_\alpha \), every nonzero coefficients of any theta function is a positive integer.

Based on Proposition 4.3, we introduced a parallel notion for QCSDs.

**Definition 5.4.** A QCSD \( D_\alpha \) is a *positive realization* if the wall element of any wall has the form

\[
\Psi_{1/\delta(n),0}[n]^c \quad (n \in N^+, b \in (1/\delta_0)\mathbb{Z}, c \in \mathbb{Z}_{>0}).
\]

We say that a QCSD \( D_\alpha \) is *positive* if it is equivalent to a QCSD that is a positive realization.

**Example 5.5.** All examples of QCSDs in Section 3 are positive realizations.
Observing the parallelism between the CSDs and QCSDs so far, it is natural to expect that Theorem 5.2 also holds for QCSDs. However, this is not true in general. We say that a QCSD is skew-symmetric if its initial exchange matrix \( B \) is skew-symmetric. The following positivity result is known.

**Theorem 5.7** ([DM21, Theorem 2.15]). Any skew-symmetric QCSD \( \mathcal{D}_s^q \) is positive.

The proof of [DM21] is based on the Donaldson-Thomas theory on quiver-representations, which applies exclusively to the skew-symmetric case.

On the other hand, the following example suggests the abundance of the nonpositive QCSDs.

**Example 5.8** ([LLRZ14, CFMM24]). Consider a cluster algebra of rank 2 with the initial exchange matrix

\[
B = \begin{pmatrix}
0 & -\delta_1 \\
\delta_2 & 0
\end{pmatrix} \quad (\delta_1, \delta_2 \in \mathbb{Z}_{>0}).
\]

(5.4)

The (classical) greedy elements were introduced in [LLZ12]. Later they were identified with the theta functions [CGM+17]. Meanwhile, the quantum greedy elements were studied in [LLRZ14]. It turned out that the positivity of the coefficients of the quantum greedy elements fails for \((\delta_1, \delta_2) = (2, 3), (2, 5), (3, 4), (4, 6)\), for example. Motivated by this result, the nonpositivity of quantum theta functions for \((\delta_1, \delta_2) = (2, 3)\) was shown by [CFMM24]. Thus, the corresponding QCSD is nonpositive due to Proposition 5.6. Also, it was conjectured [LLRZ14, Conj. 14] that the positivity holds for the quantum greedy basis if \(\delta_1 | \delta_2\) or \(\delta_2 | \delta_1\).

Thus, there is an intriguing discrepancy of the positivity between CSDs and QCSDs especially in the nonskew-symmetric case. Then, we have the following natural questions.

Q1: Exactly when and why does the positivity of a nonskew-symmetric QCSD hold or fail?

Q2: When the positivity fails for a QCSD, how is the positivity restored in the classical limit \(q \to 1\)?

Below we study the above problems for the nonskew-symmetric QCSDs of rank 2 using the pentagon relation. Before that, let us present one immediate and important consequence of the positivity of QCSDs. We say that a CSD \( \mathcal{D}_s \) is with minimal support if \(\text{Supp}(\mathcal{D}_s)\) is minimal among all CSDs that are equivalent to \(\mathcal{D}_s\). If \(\mathcal{D}_s\) is a positive realization, then \(\mathcal{D}_s\) is with minimal support because it is impossible to cancel the wall elements of any walls due to the positivity.

**Proposition 5.9.** Suppose that a QCSD \( \mathcal{D}_s^q \) is a positive realization. Then, \(\text{Supp}(\mathcal{D}_s^q)\) coincides with \(\text{Supp}(\mathcal{D}_s)\), where \(\mathcal{D}_s\) is a CSD with minimal support.

**Proof.** By replacing each wall element \(\Psi_{1/\delta(n), 0}[n]^c\) in \(\mathcal{D}_s^q\) with \(\Psi[n]^{\delta(n)}\), we obtain a CSD \(\mathcal{D}_s\), which is a positive realization. \(\square\)

5.2. Rank 2 examples. Let us focus on the rank 2 case. As mentioned in Example 5.5, all examples with \(\delta_1\delta_2 \leq 4\) in Section 3 are positive. Let us consider examples with \(\delta_1\delta_2 \geq 5\), namely, QCSDs of nonaffine infinite type. We use the same conventions and notations in Section 3.

Let us compute in the lowest approximation, namely, modulo \(G^{>2}\).
(a). \((\delta_1, \delta_2) = (2,3)\). This is the case where the nonpositivity of a quantum greedy element [LLRZ14] and (quantum) theta function [CFMM24] was shown. We have

\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}_{\frac{1}{3},0} \odot \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}_{\frac{1}{2},0} = \begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix}_{\frac{1}{3},0} \odot \begin{bmatrix}
\frac{1}{3} & 0 \\
0 & \frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix}_{\frac{1}{2},0} \odot \begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix}_{\frac{1}{3},0} \equiv \begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix}_{\frac{1}{3},0} \odot \begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix}_{\frac{1}{2},0} \odot \begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix}_{\frac{1}{3},0} \equiv \begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix}_{\frac{1}{3},0}.
\]

(5.5)

Thus, we confirmed again that it is nonpositive in the sense of Definition 5.4. Moreover, in the classical limit, the negative power in the last expression is cancelled by the positive power, and we obtain \(\Psi(e_1)^2 \Psi((1,1))^6 \Psi(e_2)^3\). This clarifies the mechanism how the positivity is restored in the classical limit. Thus, we have a clear answer to Question 2 in Section 5.1. Moreover, comparing the computation in (5.5) with the much more complicated one in [CFMM24, Appendix B], we observe that the pentagon relation significantly simplifies the analysis (just in two lines in (5.5)).

(b). \((\delta_1, \delta_2) = (2,4)\). By a similar computation, we have

\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}_{\frac{1}{3},0} \odot \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}_{\frac{1}{2},0} = \begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix}_{\frac{1}{3},0} \odot \begin{bmatrix}
\frac{1}{3} & 0 \\
0 & \frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix}_{\frac{1}{2},0} \odot \begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix}_{\frac{1}{3},0} \equiv \begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix}_{\frac{1}{3},0} \odot \begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix}_{\frac{1}{2},0} \odot \begin{bmatrix}
1 & 1 \\
1 & 0
\end{batrix}_{\frac{1}{3},0} \equiv \begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix}_{\frac{1}{3},0}.
\]

(5.6)

This is positive up to \(G^{\leq 2}\).

(c). \((\delta_1, \delta_2) = (3,3)\). This is skew-symmetric, thus, positive by Theorem 5.7. Indeed, we have

\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}_{\frac{1}{3},0} \odot \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}_{\frac{1}{2},0} = \begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix}_{\frac{1}{3},0} \odot \begin{bmatrix}
\frac{1}{3} & 0 \\
0 & \frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix}_{\frac{1}{2},0} \odot \begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix}_{\frac{1}{3},0} \equiv \begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix}_{\frac{1}{3},0} \odot \begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix}_{\frac{1}{2},0} \odot \begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix}_{\frac{1}{3},0} \equiv \begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix}_{\frac{1}{3},0}.
\]

(5.7)

This is positive up to \(G^{\leq 2}\) as it should be.

(d). \((\delta_1, \delta_2) = (3,4)\). This is similar to Example (a). We have

\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}_{\frac{1}{3},0} \odot \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}_{\frac{1}{2},0} = \begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix}_{\frac{1}{3},0} \odot \begin{bmatrix}
\frac{1}{3} & 0 \\
0 & \frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix}_{\frac{1}{2},0} \odot \begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix}_{\frac{1}{3},0} \equiv \begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix}_{\frac{1}{3},0} \odot \begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix}_{\frac{1}{2},0} \odot \begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix}_{\frac{1}{3},0} \equiv \begin{bmatrix}
1 & 1 \\
1 & 0
\end{bmatrix}_{\frac{1}{3},0}.
\]

(5.8)

This is nonpositive.

5.3. Nonpositivity result. Consider a cluster algebra of rank 2 such that its exchange matrix \(B\) has the form in (5.4). As stated in Example 5.8, it was conjectured in [LLRZ14, Conj. 14] that the positivity of the quantum greedy basis holds if \(\delta_1|\delta_2\) or \(\delta_2|\delta_1\). The following result is motivated by the converse of this conjecture.

Theorem 5.10. Let \(\mathcal{D}_q^0\) be a QCSD of rank 2 such that \(b_{12} \neq 0\) for \(B\) in (2.11). Then, \(\mathcal{D}_q^0\) is nonpositive if \(\delta_1 \nmid \delta_2\) and \(\delta_2 \nmid \delta_1\).

Proof. Let us show that under the condition the nonpositivity already emerges at degree 2 just as above Examples (a) and (d). We continue to assume that \(\{e_2,e_1\} = 1\). By a similar
Consideration to the examples in Section 5.2, we have, modulo $G^{>2}$,

$$
\begin{bmatrix}
0 \\
\frac{1}{\delta_1} \frac{1}{\delta_2} 0 \\
\frac{1}{\delta_1} 0
\end{bmatrix}
= \left( \prod_{j=1}^{\delta_2} \begin{bmatrix} \frac{1}{\delta_1} 1 \\
1 & \frac{1}{\delta_1} \frac{1}{\delta_2 + 1 - 2j}
\end{bmatrix} \right)
\left( \prod_{i=1}^{\delta_1} \begin{bmatrix} \frac{1}{\delta_1} 1 \\
1 & \frac{1}{\delta_1 \delta_2 + 1 - 2i}
\end{bmatrix} \right)
$$

(5.9)

\[ \equiv \begin{bmatrix} 1 \\
\frac{1}{\delta_1} 0
\end{bmatrix}
\left( \prod_{i=1,j=1}^{\delta_1 \delta_2} \prod_{i,j}^{\delta_1 + 1} \begin{bmatrix} 1 \\
0 & 1, b(i,j)
\end{bmatrix} \right)
\begin{bmatrix} 0 \\
\frac{1}{\delta_2} 0
\end{bmatrix}, \]

(5.10)

$b(i, j) = (2\delta_1 \delta_2 + (1 - 2j)\delta_1 + (1 - 2i)\delta_2) / \delta_1 \delta_2$.

Let $c = \gcd(\delta_1, \delta_2)$, and let $\delta_1 = \delta'_1 c$, $\delta_2 = \delta'_2 c$. Then, $\delta((1, 1)) = \delta'_1 \delta'_2 c$, and we have

(5.11) $b(i, j) = (2\delta'_1 \delta'_2 c + \delta'_1 + \delta'_2 - 2\delta' j - 2\delta' i) / \delta((1, 1))$.

Let $m_0$ be the largest number of the numerator of $b(i, j)$, which is attained with $i = j = 1$. The exponents of $q$ in $\Psi_{1/\delta((1, 1)), b}[(1, 1)]$ align with the interval $2 / \delta((1, 1))$. Thus, if the positivity holds, we should also have the number $m_0 - 2$ in the numerator. This occurs only when $\delta'_1 = 1$ or $\delta'_2 = 1$. This means $\delta_1 | \delta_2$ or $\delta_2 | \delta_1$.

The above result implies the nonpositivity of (quantum) theta functions.

**Proposition 5.11.** Under the same condition of Theorem 5.10, there is a (quantum) theta function such that at least one of its coefficients is negative in $\mathbb{Z}[q^{\pm 1/b_0}]$.

**Proof.** Consider the theta function $\theta_{Q, \tilde{m}_0}$, where $Q$ is in the first quadrant and $m_0 = (\delta_1, -\delta_2 - 1)$. Then, there is a wall $w$ in $D^\circ_s$ whose wall element is a negative power of $\Psi_{1/\delta((1, 1)), b}[(1, 1)]$. Then, the broken line $\gamma$ bending only at $w$ contributing to $\theta_{Q, \tilde{m}_0}$ with a negative coefficient in $\mathbb{Z}[q^{\pm 1/b_0}]$. See Figure 3. Moreover, the contribution of $\gamma$ is not canceled by any other broken lines. Indeed, such broken lines should bend exactly once both at $e_1^+$ and $e_2^+$, but this is impossible as seen in the figure.

We have the following corollary of Theorem 5.10.

**Corollary 5.12.** Let $D^\circ_s$ be a QCSD of any rank, and let $B$ be the initial exchange matrix in (2.1). Then, $D^\circ_s$ is nonpositive if there is a pair $i \neq j$ with $b_{ij} \neq 0$ such that $\delta_i \not{|} \delta_j$ and $\delta_j \not{|} \delta_i$.

**Proof.** Suppose that $\delta_i \not{|} \delta_j$ holds for a pair $i \neq j$ with $b_{ij} \neq 0$. We consider a QCSD whose wall elements are in the form (4.21). Then, the proof of Theorem 5.10 tells that there is
some wall whose wall element is a negative power of $\Psi_{1/\delta(e_i+e_j), b[e_i+e_j]}$. Moreover, due to the construction of a CSD in \cite[Appendix C.1]{GHKK18}, other wall elements never cancel the above wall element.

\textbf{5.4. Positivity up to degree 4.} We are left with the problem to determine whether the converses of Theorem 5.10 and Corollary 5.12 hold or not. Here, we concentrate on the rank 2 case and examine the positivity up to degree 4.

First, let us give some examples.

\textbf{Example 5.13.} (a). Let $(\delta_1, \delta_2) = (1, 5)$. By a direct computation with the pentagon relation, we verify the following relation modulo $G^{>6}$:

\[
\begin{bmatrix}
0 \\
1\frac{1}{5}, 0 \\
1\frac{1}{5} , 1, 0
\end{bmatrix}
\equiv
\begin{bmatrix}
1 \\
0 \\
1\frac{1}{5} , 0
\end{bmatrix}
\times
\begin{bmatrix}
2 \\
2 \\
2
\end{bmatrix}
\times
\begin{bmatrix}
1 \\
1 \\
1\frac{1}{5} , 0
\end{bmatrix}
= (2\frac{1}{5} , 0)
\]
Then, we have the terms for \( n = (2,1) \) and \((1,2)\),

\[
\prod_{j=1}^{k\delta_1} \prod_{i=1}^{\delta_1-1} \prod_{t=1}^{i} \left[ \frac{2}{1}, \frac{2}{-k\delta_1+1-2j+2ki+2t} \right] = \prod_{i=1}^{\delta_1-1} \prod_{t=1}^{i} \left[ \frac{2}{1}, \frac{1}{k\delta_1}, \frac{-2\delta_1+2t}{k\delta_1} \right],
\]

\[
\prod_{j=1}^{k\delta_1-1} \prod_{i=1}^{\delta_1-j} \prod_{t=1}^{i} \left[ \frac{2}{1}, \frac{1}{k\delta_1+k-2j-2ki+2t} \right].
\]

\[
(5.15) = \begin{cases} 
\prod_{i=1}^{\delta_1} \prod_{t=1}^{\delta_1-1} \left[ \frac{2}{1}, \frac{1}{k\delta_1-2k+2t} \right] & \text{for } (k\delta_1 : \text{even}), \\
\prod_{i=1}^{\delta_1} \prod_{t=1}^{\delta_1-1} \left[ \frac{2}{1}, \frac{1}{k\delta_1-2k+2t} \right] & \text{for } (k\delta_1 : \text{odd}).
\end{cases}
\]

The equality (5.14) is straightforward, while the one (5.15) follows from the following re-
summation formula for any positive integer \( p \):

\[
\sum_{j=1}^{p-1} \sum_{t=1}^{p-j} x^{-2j+2t} = \frac{x^{-2p+4}(1-x^{2p-2})(1-x^{2p})}{(1-x^2)(1-x^4)}
\]

\[
(5.16) = \begin{cases} 
\sum_{j=1}^{p/2} \sum_{t=1}^{p-j} x^{-2j+2t} & \text{for } (p : \text{even}), \\
\sum_{j=1}^{p} \sum_{t=1}^{p-j} x^{-2j+4t} & \text{for } (p : \text{odd}).
\end{cases}
\]

To see the positivity, we need to resolve the discrepancy between (5.13) and (5.15) for
\( n = (1,2) \) when \( \delta_1 \) is even and \( k \) is odd, e.g., \((\delta_1, \delta_2) = (2,6)\). In fact, in this case we can
further rewrite the RHS of (5.15) (for even \( k\delta_1 \)) as

\[
(5.17) \prod_{i=1}^{\delta_1/2} \left( \prod_{t=1}^{(k+1)/2} \left[ \frac{2}{1}, \frac{1}{k\delta_1}, \frac{k-1-4k+4t}{k\delta_1} \right] \prod_{t=1}^{k\delta_1-k-2} \left[ \frac{2}{1}, \frac{1}{k\delta_1}, \frac{-2k+2t}{k\delta_1} \right] \right).
\]

(c). Degree 4. The results are similar to the degree 3 case. However, the formulas become
lengthy, so we put them in Appendix B. They also demonstrate the complexity of proving
the positivity in higher degrees by direct computation.

We summarize the above results in Example 5.14 as follows.

**Proposition 5.15.** The converse of Theorem 5.10 holds at least up to \( G^{\leq 4} \).

Encouraged by this partial but already nontrivial result, we give a conjecture in the spirit of [LLRZ14, Conj. 14].

**Conjecture 5.16** (Cf. [LLRZ14, Conj. 14]). The converse of Theorem 5.10 holds.

**Appendix A. Derivation of relations (3.25) and (3.37) by pentagon relation**

We faithfully follow the derivation of [Mat21] in the classical case. We ask the reader to consult [Mat21] for further details. It is important that in the derivation we only use the
pentagon and commutative relations in Theorem 2.7.
A.1. Derivation of relation \((3.25)\). Let us write \(\Psi_{a,b}[n]\) as \([n]_{a,b}\) for simplicity.

**Lemma A.1** (cf. [Mat21], Lemma 1). If \(\{n', n\} = c > 0\), the following formulas hold:

(A.1) \([n']_{c/2,b} [n]_{c,b} = [n]_{c,b} [n + n']_{c/2,b+b'} [n + 2n']_{c,b+2b'} [n']_{c/2,b'}\),

(A.2) \([n']_{c,b'} [n]_{c/2,b} = [n]_{c/2,b} [n + n']_{c,2b+b'} [n + n']_{c/2,b+b'} [n']_{c,b}\).

**Proof.** This is proved by the fission/fusion and the pentagon relation in the same way as \([3.9]\).

**Lemma A.2** (cf. [Mat21], Lemma 2). Let \(\ell\) be any nonnegative integer. If \(\{n', n\} = c > 0\), the following formula holds:

\[
[n']_{c/2,b'} \left( \prod_{0 \leq p \leq \ell} [n + 2pn']_{c,b+2b' \pm pc} \right)
= [n]_{c,b} \left( \prod_{1 \leq p \leq 2\ell+1} [n + pn']_{c/2,b+pb' \pm (p-1)c/2} \right) [n + (2\ell + 2)n']_{c,b+(2\ell+2)b' \pm \ell c} [n']_{c,b},
\]

\[
\left( \prod_{0 \leq p \leq \ell} [2pn + n']_{c,2pb+b' \pm pc} \right) [n]_{c/2,b} = [n]_{c/2,b} \left( \prod_{1 \leq p \leq 2\ell+1} [pn + n']_{c,2pb+b' \pm (p-1)c/2} \right) [n']_{c,b}.\]

**Proof.** For \(\ell = 0\), they coincide with the relations in Lemma A.1. Then, they are proved by the induction on \(\ell\) with the pentagon relation and Lemma A.1 in the same way as [Mat21], Lemma 2.

**Lemma A.3** (cf. [Mat21], Lemma 3). Let \(\ell\) be any nonnegative integer. If \(\{n', n\} = c > 0\), the following formulas hold:

(A.5) \([n']_{c/2,b'} \left( \prod_{p \geq 0} [n + 2pn']_{c,b+2b' \pm pc} \right) = [n]_{c,b} \left( \prod_{p \geq 1} [n + pn']_{c/2,b+pb' \pm (p-1)c/2} \right) [n']_{c/2,b'},\)

(A.6) \(\left( \prod_{p \geq 0} [2pn + n']_{c,2pb+b' \pm pc} \right) [n]_{c/2,b} = [n]_{c/2,b} \left( \prod_{p \geq 1} [pn + n']_{c,2pb+b' \pm (p-1)c/2} \right) [n']_{c,b'}\).

**Proof.** This is obtained by taking the limit \(\ell \to \infty\) of the relations in Lemma A.2.

**Theorem A.4** (cf. [Mat21], Theorem 2). If \(\{n', n\} = c > 0\), the following formula holds:

\[
[n']_{c/2,b'} [n]_{c/2,b} = \left( \prod_{p \geq 0} [(p + 1)n + pn']_{c/2,(p+1)b+b'} \right) \times \left( \prod_{p \geq 0} [2^p(n + n')]_{2p-1,c,2p(b+b')-2p-1,c} [2^p(n + n')]_{2p-1,c,2p(b+b')} [2^p(n + n')]_{2p-1,c,2p(b+b') + 2p-1,c} \times \left( \prod_{p \geq 0} [pn + (p + 1)n']_{c/2,2pb+(p+1)b'} \right).\]

In particular, if we set \(n = e_1\), \(n' = e_2\), \(c = 1\), and \(b = b' = 0\), we obtain the relation \((3.25)\).
Proof. It is enough to prove that, for a given positive integer $k$, the relation holds modulo $G^\ell$ with $\ell = k \deg(n + n') - 1$. We prove it by the induction on $k$. For $k = 1$, the RHS reduced to

$$[n]_{c/2, b} [n']_{c/2, b'} \mod G^\ell.$$  

Then, the relation certainly holds because $[n]_{c/2, b} = [n]_{c, b - c/2} [n]_{c, b + c/2}$ and $[n']_{c/2, b'} = [n']_{c, b' - c/2} [n']_{c, b' + c/2}$ commute modulo $G^\ell$ by the pentagon relation. Suppose that the claim holds for $k - 1$ for some $k \geq 2$. First, we consider the relation

$$[n']_{c/2, b'} [n]_{c/2, b} = ([n']_{c, b' - c/2} [n]_{c, b + c/2}) \mod G^\ell.$$  

We need to calculate the anti-ordered product $[n + 2n']_{c, b + 2b' - c/2} [n]_{c, b + c/2}$ in the last expression. Note that $\{n + 2n', n\} = 2c$. Then, by the induction hypothesis, we have, modulo $G^\ell$ with $\ell' = (k - 1) \deg(n + (n + 2n'))$,

$$[n + 2n']_{c, b + 2b' - c/2} [n]_{c, b + c/2} \equiv \left( \prod_{p \geq 0} [(2p + 1)n + 2pn']_{c, (2p + 1)b + pb' + c/2} \right) \times \left( \prod_{p \geq 0} [2^{p+1}(n + n')]_{2p, (2p + 1)(b + b') - 2p} \right) \times \left( \prod_{p \geq 0} [(2p + 1)n + (2p + 2)n']_{c, (2p + 1)b + (2p + 2)b' - c/2} \right).$$  

Meanwhile, we have the inequality

$$(k - 1) \deg(n + (n + 2n')) = 2(k - 1) \deg(n + n') \geq k \deg(n + n'),$$  

where the second inequality holds due to the assumption $k \geq 2$. Therefore, the relation holds also modulo $G^\ell$ with $\ell = k \deg(n + n')$. We put (A.10) into the last expression of (A.9) and apply Lemma [A.3] to terms therein as

$$[n + n']_{c, b + b' - c/2} \left( \prod_{p \geq 0} [(2p + 1)n + 2pn']_{c, (2p + 1)b + 2pb' + c/2} \right)$$  

$$= [n]_{c, b + c/2} \left( \prod_{p \geq 1} [(p + 1)n + pn']_{c, (p + 1)b + pb'} \right) [n + n']_{c, b + b' - c/2}.$$  

(A.12)

$$\left( \prod_{p \geq 0} [(2p + 1)n + (2p + 2)n']_{c, (2p + 1)b + (2p + 2)b' - c/2} \right) [n + n']_{c, b + b' + c/2}$$  

$$= [n + n']_{c, b + b' + c/2} \left( \prod_{p \geq 1} [(p + 1)n + (p + 2)n']_{c, (p + 1)b + (p + 2)b'} \right) [n + 2n']_{c, b + 2b' - c/2}.$$  

(A.13)
Then, after a little manipulation, we obtain the desired expression in the RHS of (A.1). □

A.2. Derivation of relation (3.37).

**Lemma A.5** (cf. [Mat21, Lemma 4]). Let \( \ell \) be any nonnegative integer. If \( \{n', n\} = c > 0 \), the following formula holds:

\[
[n']_{c,b} \left( \prod_{0 \leq p \leq \ell} [n + pn']_{c/2,b+p' \pm pc/2} \right) = [n]_{c/2,b} \left( \prod_{1 \leq p \leq \ell} [2n + (2p - 1)n']_{c,2b+(2p-1)b' \pm (p-1)c} \prod_{1 \leq p \leq \ell} [n + (\ell + 1)n']_{c/2,b+(\ell+1)b' \pm \ell c/2} \right)
\]

(A.14)

**Proof.** For \( \ell = 0 \), they coincide with the relations in Lemma A.1. Then, they are proved by the induction on \( \ell \) with the pentagon relation and Lemma A.1 in the same way as [Mat21, Lemma 4]. □

**Lemma A.6** (cf. [Mat21, Lemma 3]). Let \( \ell \) be any nonnegative integer. If \( \{n', n\} = c > 0 \), the following formulas hold:

\[
[n']_{c,b} \left( \prod_{p \geq 0} [n + pn']_{c/2,b+p' \pm pc/2} \right) = [n]_{c/2,b} \left( \prod_{p \geq 1} [2n + (2p - 1)n']_{c,2b+(2p-1)b' \pm (p-1)c} \prod_{p \geq 1} [n + (2p - 1)n']_{c,2b+(2p-1)b' \pm (p-1)c} \right)
\]

(A.16)

\[
\left( \prod_{p \geq 0} [pn + n']_{c/2,b+p' \pm pc/2} \right) [n]_{c,b} = [n]_{c,b} \left( \prod_{p \geq 1} [pn + n']_{c/4,b+p' \pm (p-1)c} \prod_{p \geq 1} [2n + (2p - 1)n']_{c,2b+(2p-1)b' \pm (p-1)c} \right)
\]

(A.17)

**Proof.** This is obtained by taking the limit \( \ell \to \infty \) of the relations in Lemma A.5. □
Theorem A.7 (cf. [Mat21, Theorem 3]). If \( \{n', n\} = c > 0 \), the following formula holds:

\[
(n')_{c/4,b'}[n]_{c,b} = \left( \prod_{p \geq 0} \left( (2p+1)n + 4pn' \right)_{c,(p+1)b+4pb'} \right) \times [n+2n']_{c/2,b+2b'}
\]

\[
\times \left( \prod_{p \geq 0} \left[ 2^p(n + 2n') \right]_{2^p-1} \times [2^p(n + 2n')]_{2^p-1} \right)
\]

\[
\times \left( \prod_{p \geq 0} \left[ pm + (2p + 1)n' \right]_{c, p(b+2b')} \right).
\]

In particular, if we set \( n = e_1 \), \( n' = e_2 \), \( c = 1 \), and \( b = b' = 0 \), we obtain the relation (3.25).

Proof. In contrast to the proof of Theorem [A.4] we do not use the induction; rather, we use Theorem [A.4]. First, we consider the relation

\[
(n')_{c/2,b'-c/4, b+2b'}[n]_{c,b} = \left( \prod_{p \geq 0} \left( (2p+1)n + 4pn' \right)_{c,(p+1)b+2b'} \right) \times [n+2n']_{c/2,b+2b'}
\]

\[
\times \left( \prod_{p \geq 0} \left[ 2^p(n + 2n') \right]_{2^p-1} \times [2^p(n + 2n')]_{2^p-1} \right)
\]

\[
\times \left( \prod_{p \geq 0} \left[ pm + (2p + 1)n' \right]_{c, p+2b+2b'} \right).
\]

We put (A.20) into the last expression of (A.19) and apply Lemma [A.5] to terms therein as

\[
[n+2n']_{c,b+2b'-c/2} \left( \prod_{p \geq 0} \left( (p+1)n + (2p + 1)n' \right)_{c,(p+1)b+2b'} \right)
\]

\[
\times [n+2n']_{c/2,b+2b'-c/2}.
\]
\[
\left( \prod_{p \geq 0} \left[ pn + (2p + 1)n' \right] c_{,p} b_{+(2p+1)b'-c/4} \right) \left[ n + 2n' \right] c_{,b+2b'+c/2} \\
= \left[ n + 2n' \right] c_{,b+2b'+c/2} \\
\times \left( \prod_{p \geq 1} \left[ pn + (2p + 1)n' \right] c_{,p} b_{+(2p+1)b'-c/4} \right) \\
\times \left[ n' \right] c_{/2,b'-c/4}.
\]

(A.22)

Then, after a little manipulation, we obtain the desired expression in the RHS of (A.18). □

Appendix B. Positivity at degree 4

This is a continuation of Example 5.14 for degree 4. The purpose of writing down the explicit expressions is to demonstrate how the problem becomes complicated even for such a small degree. We have

\[
\delta((3, 1)) = k\delta_1,
\]

(B.1)

\[
\delta((2, 2)) = \begin{cases} 
  k\delta_1/2 & (k\delta_1: \text{even}), \\
  k\delta_1 & (k\delta_1: \text{odd}),
\end{cases}
\]

(B.2)

\[
\delta((1, 3)) = \begin{cases} 
  k\delta_1/3 & (k \equiv 0 \mod 3), \\
  k\delta_1 & (k \not\equiv 0 \mod 3).
\end{cases}
\]

(B.3)

Firstly, we have the term for \( n = (3, 1) \),

\[
\prod_{j=1}^{k\delta_1} \delta_1 \prod_{i=1}^{j-2} \delta_1 \prod_{t=1}^{i} 1 \prod_{s=1}^{t} \left[ \frac{3}{1} \right] \left[ 1, \frac{-2k\delta_1+3k+1-2j+2hi+2kt+2ks}{s\delta_1} \right]
\]

(B.4)

Next, we have the term for \( n = (2, 2) \),

\[
\prod_{j=1}^{k\delta_1} \prod_{s=1}^{j-1} \prod_{i=1}^{s-1} \prod_{t=1}^{i} \left[ \frac{2}{1} \right] \left[ 1, \frac{-2j-2s+2hi+2kt+2}{k\delta_1} \right] - 1
\]

(B.5)

which is written as

\[
\prod_{s=1}^{(k\delta_1-1)} \prod_{i=1}^{s-1} \prod_{t=1}^{i} \left[ \frac{2}{k\delta_1} \right] \left[ 1, \frac{-3k\delta_1+2s+2hi+2kt}{k\delta_1} \right] - 1
\]

(B.6)

\[
\prod_{s=1}^{(k\delta_1-1)/2} \prod_{i=1}^{s-1} \prod_{t=1}^{i} \left[ \frac{2}{k\delta_1} \right] \left[ 1, \frac{-3k\delta_1+1+4s+2hi+2kt+2}{k\delta_1} \right] + 1
\]

(B.7)

\( (k\delta_1: \text{even}) \)

\( (k\delta_1: \text{odd}) \).
Here we used the following resummation formula for any positive integer \( p \):

\[
\sum_{j=1}^{p} \sum_{s=1}^{j-1} x^{-2j-2s} = \frac{x^{-4p+2}(1-x^{2p-2})(1-x^{2p})}{(1-x^2)(1-x^4)}
\]

\( (B.8) \)

\[
= \begin{cases} 
\sum_{j=1}^{p/2} \sum_{s=1}^{p-1} x^{-2p-4j+2s} & (p : \text{even}), \\
\sum_{j=1}^{p} \sum_{s=1}^{(p-1)/2} x^{-2p-2j+4s} & (p : \text{odd}).
\end{cases}
\]

Finally, we have the term for \( n = (1, 3) \),

\[
\prod_{j=1}^{k_1-2} \prod_{i=1}^{\delta_1} \prod_{t=1}^{i} \prod_{s=1}^{t} \left[ \begin{array}{c} 1 \\ \frac{3}{k_1} \end{array} \right] \frac{1}{x_{i,s}^{k_1+k+2-2k_i+2t+2s}} \text{ (} k_1 \equiv 0 \mod 3),
\]

\( (B.9) \)

which is written as

\[
\prod_{j=1}^{k_1-4} \prod_{i=1}^{(k_1-1)/6} \prod_{t=1}^{i} \prod_{s=1}^{t} \left[ \begin{array}{c} 1 \\ \frac{3}{k_1} \end{array} \right] \frac{1}{x_{i,s}^{k_1+k+4-2k_i+12t}} \text{ (} k_1 \equiv 1 \mod 6),
\]

\( (B.10) \)

\[
\prod_{j=1}^{k_1-3} \prod_{i=1}^{(k_1-2)/6} \prod_{t=1}^{i} \prod_{s=1}^{t} \left[ \begin{array}{c} 1 \\ \frac{3}{k_1} \end{array} \right] \frac{1}{x_{i,s}^{k_1+k+4-2k_i+12t}} \text{ (} k_1 \equiv 2 \mod 6),
\]

\( (B.11) \)

Here we used the following resummation formula for any positive integer \( p \): The Laurent polynomial

\[
\sum_{j=1}^{p-2} \sum_{t=1}^{p-j-1} \sum_{s=1}^{t} x^{-2j+2t+2s} = \frac{x^{-2p+8}(1-x^{2p-4})(1-x^{2p-2})(1-x^{2p})}{(1-x^2)(1-x^4)(1-x^6)}
\]

\( (B.15) \)

equals to

\[
\sum_{j=1}^{p/3} \sum_{t=1}^{p-2} \sum_{s=1}^{t} x^{4-6j+2t+2s} \text{ (} p \equiv 0 \mod 3),
\]

\( (B.16) \)
First, we note that, due to the translation invariance of the expression (B.10) with respect to the index $i$, it is enough to concentrate on the case $\delta_1 = 3$. Let
\[
A_k(x) := \sum_{i=1}^{3} \sum_{t=1}^{3k-2} \sum_{s=1}^{t} x^{-2ki+2t+2s+6k-4}
\]
\[
= \frac{(1 - x^{6k-4})(1 - x^{6k-2})(1 - x^{6k})}{(1 - x^{2})(1 - x^{4})(1 - x^{2k})}
\]
be the (normalized) generating polynomial of the quantum data of the product (B.10) for $\delta_1 = 3$. Let
\[
B_k(x) := \frac{A_k(x)}{1 + x^2 + x^4} = \frac{(1 - x^{6k-4})(1 - x^{6k-2})(1 - x^{6k})}{(1 - x^{2})(1 - x^{4})(1 - x^{2k})}. 
\]
Then, the positivity property of the product (B.10) is equivalent to the fact that $B_k(x)$ is a polynomial in $x$ with nonnegative integer coefficients for $k \equiv 0 \mod 3$. This is not obvious from the expression (B.22). However, one can find and prove such a positive expression with the help of a computer algebra software. (We used SageMath 9.4.) We separate it into the cases $k \equiv 1, 2, 4, 5 \mod 6$.

(i). Let $k = 6p + 1 \ (p \in \mathbb{Z}_{\geq 0})$. Then, the polynomial $B_k(x)$ has the following positive expression.

\[
B_{6p+1}(x) = \sum_{t=1}^{p} \sum_{s=1}^{3} t(x^{12t+4s-16} + x^{-12t-4s+16+96p})
\]
\[
+ \sum_{t=1}^{p} \left\{ \sum_{s=1}^{2} (p + 2t - 1)(x^{12t+4s-16+12p} + x^{-12t-4s+16+84p})
\right.
\]
\[
+ (p + 2t)(x^{12t-4+12p} + x^{-12t+4+84p}) \right\}
\]
\[
+ \sum_{t=1}^{3p} (3p + t)(x^{4t-4+24p} + x^{-4t+4+72p})
\]
\[
+ \sum_{t=1}^{p} \sum_{s=1}^{3} (6p + t)(x^{12t+4s-16+36p} + x^{-12t-4s+16+60p})
\]
+ (7p + 1)x^{48p}
+ \sum_{t=1}^{p-1} \sum_{s=1}^{3} t(x^{12t+4s-10} + x^{-12t-4s+10+96p})
+ \sum_{t=1}^{p+1} \bigg\{ \sum_{s=1}^{2} (p + 2t - 2)(x^{12t+4s-22+12p} + x^{-12t-4s+22+84p})
+ (p + 2t - 1)(x^{12t-10+12p} + x^{-12t+10+84p}) \bigg\}
+ 3p-2 \sum_{t=1}^{3p+1} (3p + t + 1)(x^{4t+2+24p} + x^{-4t-2+72p})
+ 6p(x^{36p-2} + x^{36p+2} + x^{60p-2} + x^{60p+2})
+ \sum_{t=1}^{p-1} \sum_{s=1}^{3} (6p + t)(x^{12t+4s-10+36p} + x^{-12t-4s+10+60p})
+ 7p(x^{48p-6} + x^{48p-2} + x^{48p+2} + x^{48p+6}).

(ii). Let \( k = 6p + 2 \) (\( p \in \mathbb{Z}_{\geq 0} \)). Then, the polynomial \( B_k(x) \) has the following positive expression.

\[
B_{6p+2}(x) = \sum_{t=1}^{p} \sum_{s=1}^{3} t(x^{12t+4s-16} + x^{-12t-4s+32+96p})
+ \sum_{t=1}^{p} \bigg\{ \sum_{s=1}^{2} (p + 2t)(x^{12t+4s-12+12p} + x^{-12t-4s+28+84p})
+ (p + 2t - 1)(x^{12t-12+12p} + x^{-12t+28+84p}) \bigg\}
+ 3p+1 \sum_{t=1}^{3p+t} (3p + t)(x^{4t-4+24p} + x^{-4t+20+72p})
+ (6p + 2)(x^{36p+4} + x^{36p+8} + x^{60p+8} + x^{60p+12})
+ \sum_{t=1}^{p-1} \sum_{s=1}^{3} (6p + t + 2)(x^{12t+4s-4+36p} + x^{-12t-4s+20+60p})
+ (7p + 2)(x^{48p} + x^{48p+4} + x^{48p+8} + x^{48p+12} + x^{48p+16})
+ \sum_{t=1}^{p} \sum_{s=1}^{3} t(x^{12t+4s-10} + x^{-12t-4s+26+96p})
+ \sum_{t=1}^{p} \bigg\{ \sum_{s=1}^{2} (p + 2t)(x^{12t+4s-6+12p} + x^{-12t-4s+22+84p})
+ (p + 2t - 1)(x^{12t-6+12p} + x^{-12t+22+84p}) \bigg\}
+ 3p \sum_{t=1}^{3p+t} (3p + t)(x^{4t+2+24p} + x^{-4t+14+72p})
+ \sum_{t=1}^{p} \sum_{s=1}^{3} (6p + t)(x^{12t+4s-10+36p} + x^{-12t-4s+26+60p})
(iii). Let $k = 6p + 4 \ (p \in \mathbb{Z}_{\geq 0})$. Then, the polynomial $B_k(x)$ has the following positive expression.

$$B_{6p+4}(x) = \sum_{t=1}^{p+1} \sum_{s=1}^{2} t(x^{12t+4s-16} + x^{-12t-4s+64+96p})$$
$$+ \sum_{t=1}^{p+1} \sum_{s=1}^{2} (p + 2t - 1)(x^{12t+4s-16+12p} + x^{-12t-4s+64+84p})$$
$$+ (p + 2t)(x^{12t-4+12p} + x^{-12t+52+84p})$$
$$+ \sum_{t=1}^{3p+1} (3p + t + 2)(x^{4t+8+24p} + x^{-4t+40+72p})$$
$$+ (6p + 4)(x^{36p+16} + x^{36p+20} + x^{60p+28} + x^{60p+32})$$
$$+ \sum_{t=1}^{p} \sum_{s=1}^{3} (6p + t + 4)(x^{12t+4s+8+36p} + x^{-12t-4s+40+60p})$$
$$+ (7p + 5)x^{48p+24}$$
$$+ \sum_{t=1}^{p} \sum_{s=1}^{3} t(x^{12t+4s-10} + x^{-12t-4s+58+96p})$$
$$+ \sum_{t=1}^{p+1} \sum_{s=1}^{2} (p + 2t - 1)(x^{12t+4s-10+12p} + x^{-12t-4s+58+84p})$$
$$+ (p + 2t)(x^{12t+2+12p} + x^{-12t+46+84p})$$
$$+ \sum_{t=1}^{3p} (3p + t + 2)(x^{4t+14+24p} + x^{-4t+34+72p})$$
$$+ \sum_{t=1}^{p} \sum_{s=1}^{3} (6p + t + 2)(x^{12t+4s+2+36p} + x^{-12t-4s+46+60p})$$
$$+ (7p + 3)(x^{48p+18} + x^{48p+22} + x^{48p+26} + x^{48p+30}).$$

(iv). Let $k = 6p + 5 \ (p \in \mathbb{Z}_{\geq 0})$. Then, the polynomial $B_k(x)$ has the following positive expression.

$$B_{6p+5}(x) = \sum_{t=1}^{p+1} \sum_{s=1}^{3} t(x^{12t+4s-16} + x^{-12t-4s+80+96p})$$
$$+ \sum_{t=1}^{p} \sum_{s=1}^{2} (p + 2t + 1)(x^{12t+4s+12p} + x^{-12t-4s+64+84p})$$
$$+ (p + 2t)(x^{12t+12p} + x^{-12t+64+84p})$$
$$+ \sum_{t=1}^{3p+3} (3p + t + 1)(x^{4t+8+24p} + x^{-4t+56+72p})$$

$$+ \sum_{t=1}^{p} \sum_{s=1}^{3} t(x^{12t+4s-0} + x^{-12t-4s+58+96p})$$
$$+ \sum_{t=1}^{p+1} \sum_{s=1}^{2} (p + 2t - 1)(x^{12t+4s-0+12p} + x^{-12t-4s+58+84p})$$
$$+ (p + 2t)(x^{12t+2+12p} + x^{-12t+46+84p})$$
$$+ \sum_{t=1}^{3p} (3p + t + 2)(x^{4t+14+24p} + x^{-4t+34+72p})$$
$$+ \sum_{t=1}^{p} \sum_{s=1}^{3} (6p + t + 2)(x^{12t+4s+2+36p} + x^{-12t-4s+46+60p})$$
$$+ (7p + 3)(x^{48p+18} + x^{48p+22} + x^{48p+26} + x^{48p+30}).$$
\[
+ \sum_{t=1}^{p} \sum_{s=1}^{3} (6p + t + 4)(x^{12t+4s+8+36p} + x^{-12t-4s+56+60p})
+ (7p + 5)(x^{48p+24} + x^{48p+28} + x^{48p+32} + x^{48p+36} + x^{48p+40})
+ \sum_{t=1}^{p} \sum_{s=1}^{3} t(x^{12t+4s-10} + x^{-12t-4s+74+96p})
+ \sum_{t=1}^{p+1} \sum_{s=1}^{2} (p + 2t)(x^{12t+4s-6+12p} + x^{-12t-4s+70+84p})
+ (p + 2t - 1)(x^{12t-6+12p} + x^{-12t+70+84p})
+ \sum_{t=1}^{3p+2} (3p + t + 2)(x^{4t+14+24p} + x^{-4t+50+72p})
+ (6p + 4)(x^{36p+26} + x^{60p+38})
+ \sum_{t=1}^{p} \sum_{s=1}^{3} (6p + t + 4)(x^{12t+4s+14+36p} + x^{-12t-4s+50+60p})
+ (7p + 5)(x^{48p+30} + x^{48p+34}).
\]

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