Dynamics of a mean-reverting stochastic volatility equation with regime switching

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Abstract: In this paper, we consider a mean-reverting stochastic volatility equation with regime switching, and present some sufficient conditions for the existence of global positive solution, asymptotic boundedness in $p$th moment, positive recurrence and existence of stationary distribution of this equation. Some results obtained in this paper extend the ones in literature. Example is given to verify the results by simulation.

Keywords: Mean-reverting stochastic volatility equation; Global positive solution; Asymptotic boundedness in $p$th moment; Positive recurrence; Stationary distribution

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1 Introduction

Stochastic volatility means that volatility is not a constant, but a stochastic process. By assuming that the volatility of the underlying price is a stochastic process rather than a constant, it becomes possible to more accurately model derivatives. Mean-reverting stochastic volatility model is used in the fields of quantitative finance and financial engineering to evaluate derivative securities, such as options and swaps. The general mean-reverting stochastic volatility model can be expressed by the following equation:

$$dX_t = (a - bX_t)dt + \sigma X_t^\theta dB_t,$$

where $X_t$ presents the price or the variance of the price returns of a stock, $a, b$ are positive constants, $\sigma$ is the volatility rate, $\theta$ is a nonnegative constant, $B_t$ is a Brownian motion defined on a complete filtration probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, and the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfies the usual condition. Equation (1) has important application in economy, such as the variance of the price returns of stock and the option

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price (cf. 11). For example, when \( \theta = 0 \), it reduces to the mean-reverting Ornstein-Uhlenbeck model, Mao (1997) 2 discussed the limit distribution of its solution as time \( t \) goes to infinity; when \( \theta = 1/2 \), it is the square root stochastic variance model, which was used by Cox, Ingersoll and Ross (1985) 4 to express the dynamics of interest rate, and by Heston (1993) 4 to investigate the pricing of a European call option on an asset with stochastic volatility, and as the exchange rate processes by Bates (1996) 5 to investigate Deutsche mark options; while \( \theta = 1 \), it transforms to the Garch Diffusion model, and as \( \theta \geq 1/2 \) Model 1 reduces to the Constant Elasticity of Variance model, which provides a relatively simple illustration of many of the effects of volatility explosions; these two stochastic models were presented in Ghysels, Harvey and Renault (1996) 6 for studying the stochastic volatility in financial markets. For the general case of \( \theta \in [1/2, +\infty) \), Mao (1997) 2 shows its solution \( X_t > 0 \) for all \( t > 0 \) almost surely.

As we know that in the real world, the socio-economic environment is constantly changing, which means that the parameters \( a, b, \sigma \) and \( \theta \) in equation 1 will change as the social and economic environment changes. This type of noise caused by the change of economic environment is often called telegraph noise, which can be demonstrated as a switching between two or more regimes of states. The continuous time Markov chain models the regime switching well. Kazangey and Sworder (1971) 7 presented a switching system, where a macroeconomic model of the national economy was used to study the effect of federal housing removal policies on the stabilization of the housing sector. The term describing the influence of interest rates was modelled by a finite-state Markov chain to provide a quantitative measure of the effect of interest rate uncertainty on optimal policy. Readers can see Mao and Yuan (2006) 9, and Yin and Zhu (2009) 10 for more details on the theory of switching systems, which are two excellent references on this subject.

Motivated by these, in this paper we consider a general stochastic volatility equation with regime switching in the following form:

\[
\begin{align*}
\{ & dX_t = [a(r_t) - b(r_t)X_t]dt + \sigma(r_t)X_t^{\theta(r_t)} dB_t, \\
& (X_0, r_0) = (x_0, i_0),
\end{align*}
\]

where \( a(\cdot), b(\cdot), \sigma(\cdot) \) are positive constants, \( \theta(\cdot) \in [1/2, \infty) \), and \( (r_t)_{t \geq 0} \) is a continuous Markov chain taking values in a finite-state space \( \mathcal{M} = \{1, \cdots, m\} \), and with infinitesimal generator \( Q = (q_{ij}) \in \mathbb{R}^{m \times m} \). That is \( (r_t)_{t \geq 0} \) satisfies

\[
P(r_{t+\delta} = j| r_t = i) = \begin{cases} 
q_{ij}\delta + o(\delta), & \text{if } i \neq j, \\
1 + q_{ii}\delta + o(\delta), & \text{if } i = j
\end{cases}
\]

as \( \delta \to 0^+ \),

where \( q_{ij} \geq 0 \) is the transition rate from \( i \) to \( j \) for \( i \neq j \), and \( q_{ii} = -\sum_{i \neq j} q_{ij} \), for each \( i \in \mathcal{M} \). We assume that the Brownian motion \( B_t \) is independent of the Markov chain \( (r_t)_{t \geq 0} \).

Our contributions in this paper are as follows.

- We show the existence of global almost surely positive solution to equation 2 for any initial value \( (x_0, i_0) \in \mathbb{R}^+ \times \mathcal{M} \), where \( \mathbb{R}^+ = (0, \infty) \), which extends the corresponding result in Mao et al. 11.
- We obtain the estimation of the \( p \)th moment, asymptotic boundedness in \( p \)th moment and the Lyapunov exponent of the solution \( X_t(x_0, i_0) \) to equation 2.
- We present some sufficient conditions for that the process \((X_t, r_t)\) determined by equation 2 is positive recurrence and admits a unique ergodic asymptotically invariant distribution.
For $V : \mathbb{R} \times \mathcal{M} \rightarrow \mathbb{R}^+$ such that $V(\cdot, i)$ is twice continuously differentiable with respect to the first variable for each $i \in \mathcal{M}$, we define the operator $L$ by

$$LV(x, i) = [a(i) - b(i)x] \frac{\partial V}{\partial x}(x, i) + \frac{1}{2} \sigma^2(i)x^{2\theta(i)} \frac{\partial^2 V}{\partial x^2}(x, i) + \sum_{k \in \mathcal{M}} q_{ik} V(x, k).$$

## 2 Global positive solution

**Theorem 2.1.** If for all $i \in \mathcal{M}$, $\theta(i) = 1/2$ and $2a(i) \geq \sigma^2(i)$; or $\theta(i) > 1/2$, then for any $(x_0, i_0) \in \mathbb{R}^+ \times \mathcal{M}$, there is a unique solution $X_i(x_0, i_0)$ to equation (2) on $t \geq 0$ and this solution also satisfies $\mathbb{P}(X_i(x_0, i_0) > 0, \forall t \geq 0) = 1$.

**Remark 2.1.** If $\#\mathcal{M} = 1(\mathcal{M}$ has only one state), that is, there is no regime switching in equation (2), Mao et al. [2] investigated the global existence of positive solution to it for the case of $\theta > 1/2$. For the regime switching equation (2) with $\theta(ri) \equiv \theta$, Mao et al. [11] proved the existence of global positive solution to it for the case of $\theta \in [1/2, 1]$. For the case of $\#\mathcal{M} > 1$, that is, the regime switching equation (2), Theorem 2.1 presents the same results for $\theta(ri) \in [1/2, 1]$. Moreover, one can see that Theorem 2.1 extends the existence results to the case of $\theta(ri) > 1$. For the case of $\theta(ri) \in (0, 1/2)$, pathwise uniqueness does not hold, see Girsanov [12]; and for $\theta(ri) \equiv 0$, the solution of equation (2) can be expressed explicitly, which shows that it can be negative.

**Proof.** For the case of $\theta \in [1/2, 1)$, the coefficients of equation (2) obey Hölder continuity and linear growth condition on $(0, +\infty)$; and for the case of $\theta \geq 1$, the coefficients of equation (2) are locally Lipschitz continuous on $(0, +\infty)$. Thus for any given initial value $(x_0, i_0) \in \mathbb{R}^+ \times \mathcal{M}$ there is a unique maximal local solution $X_i(x_0, i_0)$ on $[0, \tau_e)$, where $\tau_e$ is the explosion time, see Ikeda [5] for more details on the uniqueness of the solution to stochastic differential equation with Hölder continuous coefficients.

Now we show that the solution is globally existent and positive on $\mathbb{R}^+$ a.s. Let $k_0 > 0$ be sufficiently large such that $x_0 \in [1/k_0, k_0]$, and for each integer $k \geq k_0$ define the stopping time $\tau_k = \inf \{t \in [0, \tau_e) : X_i(x_0, i_0) \notin (1/k, k)\}$, where $\tau_\infty = \lim_{k \rightarrow \infty} \tau_k$. It is obvious that $\tau_\infty \leq \tau_e$. So we only need to show that $\tau_\infty = \infty$ a.s., which yields the positiveness of the solution almost surely and the existence of global solution of equation (2). If it is false, then there is a pair of constants $T > 0$ and $\epsilon \in (0, 1)$ such that $P\{\tau_\infty \leq T\} > \epsilon$, which yields that there exists an integer $k_1 \geq k_0$ such that

$$P\{\tau_k \leq T\} \geq \epsilon \text{ for all } k \geq k_1. \quad (3)$$

Let $p > 0$, and Define a $C^2$-function $V : \mathbb{R}^+ \times \mathcal{M} \rightarrow \mathbb{R}^+$ by

$$V(x, i) = x^p - 1 - p \log x,$$

then $V(x, i) \geq 0$ for all $x > 0$ and $i \in \mathcal{M}$. In fact, $u_1 = 1 - \ln u \geq 0$ for $u > 0$.

$$\frac{\partial V}{\partial x}(x, i) = px^{p-1} - px^{-1}; \quad \frac{\partial^2 V}{\partial x^2}(x, i) = p(p-1)x^{p-2} + px^{-2}.$$

By Itô formula, we have

$$V(x, i) = (px^{p-1} - px^{-1})[(a(i) - b(i)x)dt + \sigma x^{\theta(i)}dB_t] + 0.5p\sigma^2(i)(p-1)x^{p-2} + x^{-2}[x^{2\theta(i)}dt$$

$$= pF(x)dt + \sigma(i)p(x^{p+\theta(i)} - x^{\theta(i)-1})dB_t,$$
where \( F(x) = a(i)(x^{p-1} - x^{-1}) + b(i)(1 - x^p) + 0.5\sigma^2(i)[(p - 1)x^{p+2\theta(i)-2} + x^{2\theta(i)-2}], \) which is continuous on \( x \in \mathbb{R}^+. \)

Then we claim that the function \( F \) is bounded above on \( x \in \mathbb{R}^+. \) In fact, for the case of \( \theta(i) = 0.5, \) we take \( p > 1 \) and obtain
\[
F(x) = a(i)(x^{p-1} - x^{-1}) + b(i)(1 - x^p) + 0.5\sigma^2(i)[(p - 1)x^{p-1} + x^{-1}]
\]
and the higher power of \( x \) is \( p \) and the lower power of \( x \) is \( -1, \) which together with \( a(i) > 0.5\sigma^2(i) \) yields
\[
F(x) \sim -b(i)x^p \to -\infty \text{ as } x \to +\infty, \quad \text{and} \quad F(x) \sim -a(i)x^{-1} \to -\infty \text{ as } x \to 0^+ \text{ for } a(i) > 0.5\sigma^2(i)
\]
or \( F(x) \to b(i) \text{ as } x \to 0^+ \text{ while } a(i) = 0.5\sigma^2(i). \)

For the case of \( \theta(i) \in (0.5, 1). \) We note that the higher power and lower power of \( x \) in \( F(x) \) are \( p \) and \( -1, \) respectively; and the coefficients of these two terms are negative, which yield
\[
F(x) \sim -b(i)x^p \to -\infty \text{ as } x \to +\infty \text{ and } F(x) \sim -a(i)x^{-1} \to -\infty \text{ as } x \to 0^+.
\]

For the case of \( \theta(i) \in [1, +\infty), \) that is \( 2\theta(i) - 2 \geq 0. \) We see that the higher power and lower power of \( x \) in \( F(x) \) are \( p + \theta(i) - 2 \) and \( -1, \) respectively. Let \( p \in (0, 1), \) we get
\[
F(x) \sim -0.5\sigma^2(i)(1 - p)x^{p+2\theta(i)-2} \to -\infty \text{ as } x \to +\infty \text{ and } F(x) \sim -a(i)x^{-1} \to -\infty \text{ as } x \to 0^+.
\]

Thus the continuation of \( F \) in \( \mathbb{R}^+ \) implies that there must exist a positive constant \( K \) such that \( F(x) \leq K \) for all \( x \in \mathbb{R}^+. \) Therefore we obtain
\[
\forall(x, i) \leq pKdt + \sigma(i)p(x^{p+\theta(i)-1} - x^{\theta(i)-1})dB_t.
\]

Integrating both sides of the above inequality from 0 to \( T \wedge \tau_k, \) and taking expectations give
\[
EV(X_{T \wedge \tau_k}, r_{T \wedge \tau_k}) \leq V(x_0, i_0) + pKE(T \wedge \tau_k).
\]  

Set \( \Omega_k = \{ \tau_k \leq T \} \) for \( k \geq k_1, \) then it follows from (3) that \( P(\Omega_k) \geq \epsilon. \) Note that for every \( \omega \in \Omega_k, \) there is \( X_{\tau_k}(\omega) \) equals either \( k \) of \( 1/k, \) and hence
\[
V(X_{\tau_k}(\omega), r_{\tau_k}) \geq (k^p - 1 - p\log k) \wedge (k^{-p} - 1 + p\log k),
\]
which together with (3) yields
\[
\forall > V(x_0, i_0) + pKT \geq E[I_{\Omega_k}V(X_{\tau_k}(\omega), r_{\tau_k})] \geq (k^p - 1 - p\log k) \wedge (k^{-p} - 1 + p\log k).
\]

Then by letting \( k \to \infty, \) we get a contradiction. Thus \( \tau_\infty = \infty \text{ a.s.} \)

3 \ pth Moment Estimation

In this section, we will give the \( p \)th moment estimation of the solution to equation (2), and then present some sufficient conditions for asymptotic boundedness in \( p \)th moment of the solution \( X_t(x_0, i_0). \)
Definition 3.1. (Mao and Yuan (2006) [9])

A square matrix $A$ is called a nonsingular $M$-matrix if $A$ can be expressed in the form $A = sI - G$ with some $G \geq 0$ (that is each element of $G$ is non-negative) and $s > \rho(G)$, where $I$ is the identity matrix and $\rho(G)$ the spectral radius of $G$.

Corresponding to the infinitesimal generator $Q$ and $p \geq 1$, we define an $m \times m$ matrix

$$A(p) := p \text{diag}(b(1), \cdots, b(m)) - Q.$$  

Lemma 3.1. (Mao and Yuan (2006) [9])

If $A(p)$ is a nonsingular $M$-matrix, then there is a vector $(\beta_1, \cdots, \beta_m)^T$ with $\beta_i > 0$ such that $\mu_i := pb(i)\beta_i - \sum_{k \in \mathcal{M}} q_{ik}\beta_k > 0$ for all $i \in \mathcal{M}$.

Theorem 3.1. If one of the following conditions holds:

1) $A(1)$ is a nonsingular $M$-matrix;

2) $\forall i \in \mathcal{M}$, $\theta(i) \in [0, 1]$, and $A(p)$ is a nonsingular $M$-matrix with $p > \max\{1, 2\} \max(1 - \min_{i \in \mathcal{M}} \theta(i))$.

Then the $p$th moment of the solution to equation (2) has the following property,

$$\mathbb{E}[X_p^p(x_0, i_0)] \leq x_0^p e^{-\lambda_p t} + \frac{C_p}{\beta\lambda_p} (1 - e^{-\lambda_p t}) \quad \text{for any } (x_0, i_0) \in \mathbb{R}^+ \times \mathcal{M},$$

where $C_p = \max_{i \in \mathcal{M}} \left\{ \frac{1}{p}(p\beta_i a(i))^p \left( \frac{3}{\mu_i} \right)^{p-1} + \frac{2-2\theta(i)}{p} \left( \frac{1}{p}(p-1)\sigma^2(i)\beta_i \right) \frac{p/[2-2\theta(i)]}{[2-2\theta(i)]}, \frac{\beta_i a(i)}{\mu_i}, \frac{\beta_i a(i)}{\nu_i}, \frac{\beta_i a(i)}{\nu_i} \right\}$, $\beta = \min_{i \in \mathcal{M}} \beta_i$, and $\lambda_p = \frac{\min_{i \in \mathcal{M}} \beta_i}{\beta}$.

Moreover, the solution of equation (2) is asymptotic boundedness in $p$th moment with

$$\limsup_{t \to \infty} \mathbb{E}[X_p^p(x_0, i_0)] \leq \frac{C_p}{\beta\lambda_p}$$

and the Lyapunov exponent

$$\limsup_{t \to \infty} \frac{1}{t} \log \mathbb{E}[X_p^p(x_0, i_0)] \leq 0, \quad \text{for all } (x_0, i_0) \in \mathbb{R}^+ \times \mathcal{M}.$$

Proof. Lemma 3.1 yields that there is a vector $(\beta_1, \cdots, \beta_m)^T$ with $\beta_i > 0$ such that

$$\mu_i := pb(i)\beta_i - \sum_{k \in \mathcal{M}} q_{ik}\beta_k > 0 \quad \text{for all } i \in \mathcal{M}.$$ 

Define the $C^2$-function $V : \mathbb{R}^+ \times \mathcal{M} \to \mathbb{R}^+$ by the form:

$$V(x, i) = \beta_i x^p,$$

then $\frac{\partial V}{\partial x}(x, i) = p\beta_i x^{p-1}$ and $\frac{\partial^2 V}{\partial x^2}(x, i) = p(p-1)\beta_i x^{p-2}$, and we get

$$LV(x, i) = p\beta_i [a(i) - b(i)x]x^{p-1} + \frac{1}{2} p(p-1)\sigma^2(i)\beta_i x^{p+2\theta(i)-2} + \sum_{k \in \mathcal{M}} q_{ik}\beta_k x^p$$

$$= - \left( pb(i)\beta_i - \sum_{k \in \mathcal{M}} q_{ik}\beta_k \right)x^p + p\beta_i a(i)x^{p-1} + \frac{1}{2} p(p-1)\sigma^2(i)\beta_i x^{p+2\theta(i)-2}$$

$$\leq - \mu_i x^p + p\beta_i a(i)x^{p-1} + \frac{1}{2} p(p-1)\sigma^2(i)\beta_i x^{p+2\theta(i)-2} \quad \text{for all } (x, i) \in \mathbb{R}^+ \times \mathcal{M}.$$
By the elementary inequality
\[ a^{\gamma} b^{1-\gamma} \leq a \gamma + b(1-\gamma), \quad \forall a, b \geq 0, \gamma \in [0, 1], \]
we obtain
\[
p \beta_i a(i)x^{p-1} = p \beta_i a(i) \left( \frac{3}{\mu_i} \right)^{(p-1)/p} \left( \frac{\mu_i}{3} x^p \right)^{(p-1)/p}
= \left[ (p \beta_i a(i))^p \left( \frac{3}{\mu_i} \right)^{(p-1)/p} \right]^{1/p} \left( \frac{\mu_i}{3} x^p \right)^{(p-1)/p}
\leq \frac{1}{p} (p \beta_i a(i))^p \left( \frac{3}{\mu_i} \right)^{(p-1)} + p - \frac{1}{3p} \mu_i x^p.
\]

and
\[
\frac{1}{2} p(p-1) \sigma^2(i) \beta_i x^{p+2\theta(i)-2}
= \frac{1}{2} p(p-1) \sigma^2(i) \beta_i \left( \frac{3}{\mu_i} \right)^{(p-2+2\theta(i))/p} \left( \frac{\mu_i}{3} x^p \right)^{(p-2+2\theta(i))/p}
= \left[ \left( \frac{1}{2} p(p-1) \sigma^2(i) \beta_i \right)^p \left( \frac{3}{\mu_i} \right)^{(p-2+2\theta(i))/p} \right]^{p-2+2\theta(i)} \left( \frac{\mu_i}{3} x^p \right)^{(p-2+2\theta(i))/p}
\leq \frac{2 - 2\theta(i)}{p} \left( \frac{1}{2} p(p-1) \sigma^2(i) \beta_i \right)^p \left( \frac{3}{\mu_i} \right)^{(p-2+2\theta(i))/p} \left( \frac{\mu_i}{3} x^p \right)^{(p-2+2\theta(i))/p} + p - 2 + 2\theta(i) \mu_i x^p.
\]
Substituting (6) and (7) into (5) gives
\[
LV(x, i) \leq - \mu_i x^p + p - \frac{1}{3p} \mu_i x^p + p - 2 + 2\theta(i) \mu_i x^p + \frac{1}{p} (p \beta_i a(i))^p \left( \frac{3}{\mu_i} \right)^{(p-1)}
+ \frac{2 - 2\theta(i)}{p} \left( \frac{1}{2} p(p-1) \sigma^2(i) \beta_i \right)^p \left( \frac{3}{\mu_i} \right)^{(p-2+2\theta(i))/p} \left( \frac{\mu_i}{3} x^p \right)^{(p-2+2\theta(i))/p}
\leq - \lambda_p V(x, i) + C_p \text{ for all } (x, i) \in \mathbb{R}^+ \times \mathcal{M},
\]
Defining the stopping time \( \tau_n = \inf\{ t \geq 0 \mid X_t \geq n \} \), it is obvious that \( \tau_n \to \infty \) as \( n \to \infty \). By applying Itô’s formula, we have
\[
\mathbb{E}[e^{\lambda_p t_n} V(X_{t_n}, r_{t_n})] = V(x_0, i_0) + \mathbb{E} \int_0^{t_n} e^{\lambda_p s}LV(X_s, r_s)ds + \lambda_p \mathbb{E} \int_0^{t_n} e^{\lambda_p s}V(X_s, r_s)ds,
\]
where \( t_n = \tau_n \wedge t \). It follows from \( \mathbb{E} \) that
\[
\mathbb{E}[e^{\lambda_p t_n} V(X_{t_n}, r_{t_n})] \leq V(x_0, i_0) + C_p \int_0^{t_n} e^{\lambda_p s}ds \leq V(x_0, i_0) + C_p \left( e^{\lambda_p t_n} - 1 \right),
\]
and thus
\[
\beta \mathbb{E}[X_{t_n}^p (x_0, i_0)] \leq \beta_i x_0^p e^{-\lambda_p t_n} + C_p \left( 1 - e^{-\lambda_p t_n} \right).
\]
Letting \( n \to \infty \), we get
\[
\mathbb{E}[X_{t_n}^p (x_0, i_0)] \leq x_0^p e^{-\lambda_p t} + \frac{C_p}{\beta \lambda_p} (1 - e^{-\lambda_p t}), \text{ for any } (x_0, i_0) \in \mathbb{R}^+ \times \mathcal{M}.
\]
Thus \( \limsup_{t \to \infty} \mathbb{E}[X_{t}^p (x_0, i_0)] \leq \frac{C_p}{\beta \lambda_p} \), which gives
\[
\limsup_{t \to \infty} \frac{1}{t} \log \mathbb{E}[X_{t}^p (x_0, i_0)] \leq 0 \text{ for all } (x_0, i_0) \in \mathbb{R}^+ \times \mathcal{M}.
\]
4 Stationary distribution

In this section, we assume that $q_{ij} > 0$ for $i \neq j$, and the discrete component $(r_{ij})_{i \geq 0}$ in equation (2) is an irreducible continuous-time Markov chain with an invariant distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_m)$.

Lemma 4.1. (See Zhu and Yin (2007) [13])

If there is a bounded open subset $D \subset \mathbb{R}^+$ and for each $i \in \mathcal{M}$ there exists a nonnegative function $V(\cdot, i) : D^c \to \mathbb{R}$ such that $V(\cdot, i)$ is twice continuously differentiable and for some $\xi > 0$, $LV(x, i) \leq -\xi$ for any $(x, i) \in D^c \times \mathcal{M}$, then equation (2) is positive recurrence. Moreover, the process $X_t(x, i)$ has a unique ergodic stationary distribution $\nu$. That is, if $f$ is a function integrable with respect to the measure $\nu$, then

$$P\left( \lim_{t \to \infty} \frac{1}{t} \int_0^t f(x(s))ds = \int_0^\infty f(x)\nu(dx) \right) = 1.$$  

Theorem 4.1. If $2\theta(i) \in (1, 3]$ for all $i \in \mathcal{M}$, then for any $(x_0, i_0) \in \mathbb{R}^+ \times \mathcal{M}$, equation (2) is positive recurrence and admits a unique ergodic stationary distribution $\nu$.

Proof. Define a $C^2$-function $V : \mathbb{R}^+ \times \mathcal{M} \to \mathbb{R}^+$ by the form:

$$V(x, i) = (\gamma - p\xi_i)x^{-p} + x,$$

where $p, \gamma$ are positive constants satisfying $\min_{i \in \mathcal{M}} \{\gamma - p\xi_i\} > 0$ where $\xi = (\xi_1, \cdots, \xi_m)^T$ is a solution of the following Poisson system,

$$Q\xi = -\mu + \sum_{i \in \mathcal{M}} \pi_i u(i) 1,$$

where $\mu = (\mu(1), \mu(2), \ldots, \mu(m))$. By applying Ito’s formula, we obtain

$$LV(x, i) = -pa(i)(\gamma - p\xi_i)x^{-p-1} + pb(i)(\gamma - p\xi_i)x^{-p} + a(i) - b(i)x + 0.5p(1 + p)(\gamma - p\xi_i)\sigma^2(i)x^{2\theta(i)-p-2} - px^{-p} \sum_{k \in \mathcal{M}} q_{ik}\xi_k.$$

Then, it follows from $1 < 2\theta(i) \leq 3$ that

$$LV(x, i) \sim_{x \to +\infty} -b(i)x \quad \text{and} \quad LV(x, i) \sim_{x \to 0+} -pa(i)(\gamma - p\xi_i)x^{-p-1}.$$

Set $D = (1/N, N) \subset \mathbb{R}^+$, then for sufficiently large $N$ we get

$$LV(x, i) \leq -1, \quad \text{for all} \quad (x, i) \in D^c \times \mathcal{M}.$$

Then, Lemma 4.1 shows that equation (2) admits a stationary distribution with nowhere-zero density in $\mathbb{R}^+$.

Theorem 4.2. If $\theta(i) \equiv 1/2$ and $\mu(i) := a(i) - 0.5\sigma^2(i) > 0$ for all $i \in \mathcal{M}$, then, for any $(x_0, i_0) \in \mathbb{R}^+ \times \mathcal{M}$, the solution $X_t(x_0, i_0)$ of equation (2) is positive recurrence and admits a unique ergodic stationary distribution $\nu$. 

\[\square\]
Proof. Define the $C^2$-function $V : \mathbb{R}^+ \times \mathcal{M} \to \mathbb{R}^+$ by the following form

$$V(x, i) = (\gamma - p\xi_i)x^{-p} + x,$$

where $p, \gamma$ are positive constants satisfying $\min_{i \in \mathcal{M}} \{\gamma - p\xi_i\} > 0$, and $\xi = (\xi_1, \cdots, \xi_m)^T$ is a solution of the Poisson system \ref{poisson_system}. By using Itô’s formula, we obtain

$$LV(x, i) = - p(\gamma - p\xi_i)x^{-p-1}[\mu(i) - 0.5p\sigma^2(i)] + pb(i)(\gamma - p\xi_i)x^{-p} + a(i)x - b(i)x - px^{-p}\sum_{k \in \mathcal{M}} q_{ik}\xi_k.$$  

Meanwhile, $\mu(i) > 0$ yields one can choose sufficient small $p$ such that $\mu(i) - 0.5p\sigma^2(i) > 0$ for all $i \in \mathcal{M}$, then

$$LV(x, i) \sim_{x \to +\infty} -b(i)x \quad \text{and} \quad LV(x, i) \sim_{x \to 0^+} -p(\gamma - p\xi_i)[\mu(i) - 0.5p\sigma^2(i)]x^{-p-1},$$

which together with Lemma \ref{lemma} give the result immediately. \hfill \Box

5 Example

In this section, we will give an example to verify the theorems obtained in previous sections by simulation. The numerical method used here is Milsteins Higher Order Method, see Higham (2001) \ref{Higham} for more details.

Set the state space of Markov chain $(r_t)_{t \geq 0}$ as $\mathcal{M} = \{1, 2, 3, 4\}$, and its generator $Q$ as follows

$$Q = \begin{pmatrix}
-7 & 3 & 2 & 2 \\
3 & -9 & 3 & 3 \\
3 & 2 & -8 & 3 \\
2 & 3 & 4 & -9
\end{pmatrix},$$

Then its stationary distribution is $\pi = (0.2773, 0.2277, 0.2681, 0.2269)$. Its one step transition probability matrix $P(\delta)$ with $\delta = 10^{-4}$ is

$$P(\delta) = \exp(\delta Q) = \begin{pmatrix}
0.9993 & 0.0003 & 0.0002 & 0.0002 \\
0.0003 & 0.9991 & 0.0003 & 0.0003 \\
0.0003 & 0.0002 & 0.9992 & 0.0003 \\
0.0002 & 0.0003 & 0.0004 & 0.9991
\end{pmatrix}.$$  

Let $a = (4, 1.5, 0.8, 0.55)$, $b = (2, 2, 1, 1)$, $\sigma = (2, 2, 1, 0.4)$, $\theta = (0.5, 0.5, 0.5, 0.5)$ and $p = 2$, then the conditions $2a(i) \geq \sigma^2(i), (i = 1, 2, 3, 4)$ hold.

$$A(p) = p \text{diag}(b_1, b_2, b_3, b_4)^t - Q = \begin{pmatrix}
11 & -3 & -2 & -2 \\
-3 & 11 & -3 & -3 \\
-3 & -2 & 11 & -3 \\
-2 & -3 & -4 & 11
\end{pmatrix} = sI - G, \quad \text{where} \ s = 11, \ G = \begin{pmatrix}
0 & 3 & 2 & 2 \\
3 & 0 & 3 & 3 \\
3 & 2 & 1 & 3 \\
2 & 3 & 4 & 0
\end{pmatrix}$$

and $\rho(G) = 8.5208$. Thus from Theorem \ref{main_theorem} we see that $X_t(x_0, i_0) > 0$ for $t \geq 0$ a.s.. Meanwhile, $A(2)$ is a nonsingular $\mathcal{M}$-matrix, then it follows from Theorem \ref{lyapunov_exponent} that $X_t(x_0, i_0)$ of equation \ref{equation} is asymptotic boundedness in mean-square, and the Lyapunov exponent of $X_t(x_0, i_0)$ is no more than $0$. Moreover, according to Theorem \ref{ergodic_stationary_distribution} the solution $X_t(x_0, i_0)$ of equation \ref{equation} is positive recurrence and admits a unique ergodic stationary distribution $\nu$. These claims are supported by Figures 1, 2 and 3, respectively.
Figure 1. Sample path of $X_t$ (in blue line) along the Markov chain $(r_t)_{t \geq 0}$ (in red line) with initial value $(x_0, r) = (0.2, 3)$.

Figure 2. Mean-square value of $X_t$ and $\frac{1}{t} \log E[X_t^2(x_0, r)]$ with initial value $(x_0, r) = (0.2, 3)$. 
6 Conclusion

In this paper, we investigate a general stochastic volatility equation with regime switching, and show that for any initial value \((x_0, i_0) \in \mathbb{R}^+ \times \mathcal{M}\) there is a unique global almost surely positive solution \(X_t(x_0, i_0)\) to this equation; and give the pth moment estimation of the solution by using the properties of nonsingular \(M\)-matrix, and present some simple sufficient conditions for the existence of a unique ergodic stationary distribution of the equation.

It will be more interesting if the Markov chain \((r_t)_{t \geq 0}\) is state dependent or infinity, also the equation with jumps will be better than the one without jumps in describing some complicated dynamics behaviors in the real world.

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