Wild ramification and the cotangent bundle

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Abstract

We define the characteristic cycle of a locally constant étale sheaf on a smooth variety in positive characteristic ramified along boundary as a cycle in the cotangent bundle of the variety, at least on a neighborhood of the generic point of the divisor on the boundary. The crucial ingredient in the definition is an additive structure on the boundary induced by the groupoid structure of multiple self products.

We prove a compatibility with pull-back and local acyclicity in non-characteristic situations. We also give a relation with the characteristic cohomology class under a certain condition and a concrete example where the intersection with the 0-section computes the Euler-Poincaré characteristic.

Let $k$ be a perfect field of characteristic $p > 0$, $X$ be a smooth scheme of dimension $d$ over $k$ and $D$ be a divisor of $X$ with simple normal crossings. Let $Λ$ be a ring finite over $\mathbb{Z}_\ell[ζ_p]$ or a finite extension of $\mathbb{Q}_\ell(ζ_p)$ for a prime $\ell \neq p$ and $\mathcal{F}$ be a locally constant constructible sheaf of free $Λ$-modules on the complement $U = X - D$. After shrinking $X$ to a neighborhood of the generic point of $ξ$, we define the characteristic cycle $\text{Char}(\mathcal{F})$ in Definition 3.5 as a $d$-dimensional cycle on the cotangent bundle $T^*X$. We show that the characteristic cycle has coefficients in $\mathbb{Z}[\frac{1}{p}]$, Proposition 3.10.

For a morphism $f : X' \to X$ of smooth schemes over $k$, we define the condition that $f$ is non-characteristic with respect to $\mathcal{F}$ in Definition 3.7 in terms of $\text{Char}(\mathcal{F})$. We prove that the construction of the characteristic cycles commutes with the pull-back by non-characteristic morphisms in Proposition 3.8. We deduce a characterization of the support of the characteristic cycle in terms of the restrictions to curves transversally meeting the boundary in Proposition 3.12.

The results in this article are non-logarithmic variants of the logarithmic version studied in [21] and [5] obtained by a different method of localization. The relation between the two methods is discussed in Section 2.6. An advantage of the non-logarithmic version is that it behaves better with the restriction to curves.

We also define the condition for a smooth morphism $f : X \to Y$ of smooth schemes to be non-characteristic with respect to $\mathcal{F}$ in Definition 3.13 in terms of $\text{Char}(\mathcal{F})$. We deduce the local acyclicity for a non-characteristic morphism in Proposition 3.15 from a result of Deligne-Laumon [19].

We give a relation with the characteristic class defined in [3] in Corollary 3.19 As an application, we give a concrete example of a computation of the Euler-Poincaré characteristic in Example 3.20.

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The crucial ingredient in the definition of the characteristic cycle is an additive structure on the boundary established in Corollary 2.15. Using a classification of a vector bundle in characteristic $p > 0$ by a finite étale group scheme of $\mathbf{F}_p$-vector spaces recalled in Section 1.4, we link the additive structure to the cotangent bundle in Definition 2.19. As an application of the link, we study the graded quotients of the filtration by non-logarithmic ramification groups of a local field of equal characteristic with imperfect residue field in Section 2.5.

The additive structure on the boundary is defined as the restriction of an extension constructed in Theorem 2.14 of the groupoid structure of multiple self products. Functoriality of multiple self products is interpreted as a groupoid structure using an equivalence of categories Proposition 1.4 stated in terms of an extra structure on simplicial objects introduced in Section 1.1.

More precisely speaking, the additive structure is defined on the boundary of the largest étale scheme inside the normalizations of some partial blow-ups, called dilatations, of the multiple self products in ramified coverings. Some functorial properties of the étale part of normalizations are established in Section 1.2. We study properties of dilatations abstractly in Section 1.3 and more concretely in Section 2.1.

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1 Preliminaries

1.1 Oversimplicial objects and groupoids

Definition 1.1. Let $\mathcal{C}$ be a category and let $\tilde{\Delta}$ denote the full subcategory of the category of sets consisting of the objects $[0, n] = \{0, 1, \ldots, n\}$ for integers $n \geq 0$.

1. We call a contravariant functor $P: \tilde{\Delta} \to \mathcal{C}$ an oversimplicial object of $\mathcal{C}$. For an oversimplicial object $P$ and an integer $n \geq 0$, we write $P_n$ for $P([0, n])$. For oversimplicial objects $P$ and $Q$ of $\mathcal{C}$, we call a morphism $f: P \to Q$ of functors a morphism of oversimplicial objects.

2. We call a cocartesian diagram

\[
\begin{array}{ccc}
[0, m] & \longrightarrow & [0, n] \\
\uparrow & & \uparrow \\
[0] & \longrightarrow & [0, l]
\end{array}
\]

(1.1)

in $\tilde{\Delta}$ an additive cocartesian diagram.

Assume that finite inverse images are defined in $\mathcal{C}$. We say that an oversimplicial object $P$ of $\mathcal{C}$ is multiplicative, if every additive cocartesian diagram (1.1) defines a cartesian diagram

\[
\begin{array}{ccc}
P_m & \longleftarrow & P_n \\
\downarrow & & \downarrow \\
P_0 & \longleftarrow & P_1.
\end{array}
\]

(1.2)

We say a multiplicative object of $P$ is strictly multiplicative if the two morphisms $P_1 \to P_0$ are equal.

We say that a morphism $Q \to P$ of oversimplicial objects of $\mathcal{C}$ is multiplicative, if, for every additive cocartesian diagram (1.1), the diagram

\[
\begin{array}{ccc}
Q_m \times_{P_m} P_n & \longleftarrow & Q_n \\
\downarrow & & \downarrow \\
P_n & \longleftarrow & Q_t \times_{P_t} P_n
\end{array}
\]

(1.3)

is cartesian.

The category $\tilde{\Delta}$ contains the subcategory $\Delta$ with the same underlying set and increasing morphisms. Consequently, an oversimplicial object defines a simplicial object by restriction.

In the rest of this subsection, let $\mathcal{C}$ denote a category where finite inverse images are defined. We will write an oversimplicial object $P$ as $P_*$ and a morphism of oversimplicial objects $f$ as $f_*$ in the following. For morphisms $Q_* \to P_*$ and $R_* \to P_*$ of oversimplicial objects, the fibered product $Q_* \times_{P_*} R_*$ is an oversimplicial objects.
Lemma 1.2. Let $P_\bullet$ be a multiplicative oversimplicial object. Then, for a morphism $f_\bullet: Q_\bullet \to P_\bullet$ of oversimplicial objects, the following conditions (1) and (2) are equivalent if $f_0: Q_0 \to P_0$ is injective.

1. $Q_\bullet$ is multiplicative.
2. $f_\bullet$ is multiplicative.

Proof. For an additive cocartesian diagram (1.1), if $P_n \to P_m \times_{P_0} P_1$ is an isomorphism, then the map

$$(Q_m \times_{P_m} P_n) \times_{P_n} (Q_I \times_{P_I} P_n) = Q_m \times_{P_m} P_n \times_{P_I} Q_I \to Q_m \times_{P_m} P_n \times_{P_0} P_I \times_{P_I} Q_I = Q_m \times_{P_0} Q_I$$

is also an isomorphism. If $Q_0 \to P_0$ is an injection, the last term is $Q_m \times_{Q_0} Q_I$. Hence the diagram (1.2) with $P$ replaced by $Q$ is cartesian if and only if the diagram (1.3) is cartesian. $\square$

For a pair of morphisms $s, t: P \to S$, we fix notation for fibered multi-products. For integers $n \geq 0$, we define $s_n, t_n: P^{\times n} = P \times_S P \cdots \times_S P \to S$ inductively as follows. For $n = 0$, we set $P^{\times 0} = S$ and $s_0 = t_0 = \text{id}_S$. For $n = 1$, we set $P^{\times 1} = P$ and $s_1 = s, t_1 = t$. For $n \geq 1$, we define $P^{\times n+1} = P^{\times n} \times_S P$ by the cartesian diagram

$$(1.4)$$

and set $s_{n+1} = s_n \circ \text{pr}_1, t_{n+1} = t \circ \text{pr}_2$. For integers $n = l + m$, a canonical isomorphism $P^{\times n} \to P^{\times l} \times_S P^{\times m}$ is defined by induction on $m$.

Definition 1.3. Let $\mathcal{C}$ be a category where finite inverse limits are defined. We say that a 7-ple $(P, S, s, t, e, \mu, \iota)$ is a groupoid in $\mathcal{C}$ if $P, S$ are objects of $\mathcal{C}$, $s, t: P \to S, e: S \to P, \mu: P^{\times 2} \to P^{\times 1}$ are morphisms of $\mathcal{C}$ such that the following diagrams are commutative:

We say a groupoid is a group if $s = t$.

We show that a multiplicative oversimplicial object defines a groupoid. Let $P_\bullet$ be a multiplicative oversimplicial object. We define $s, t: P_1 \to P_0, e: P_0 \to P_1, \mu: P_1 \to P_1$ to be the morphisms defined by the maps $[0] \to [0, 1]$ sending 0 to 0 and to 1 respectively, by the unique map $[0, 1] \to [0]$ and by the map $[0, 1] \to [0, 1]$ switching 0 and 1. We define
\( \mu: P_1 \times_{P_0} P_1 \to P_1 \) to be the composition of the inverse of the isomorphism \( P_2 \to P_1 \times_{P_0} P_1 \) defined by the cartesian diagram (1.2) corresponding to the additive cocartesian diagram

\[
\begin{array}{ccc}
[0, 1] & \xrightarrow{\text{inclusion}} & [0, 2] \\
\downarrow 0 \to 1 & & \downarrow 1 \\
[0] & \xrightarrow{\text{inclusion}} & [0, 1]
\end{array}
\]

with the map \( P_2 \to P_1 \) defined by the map \([0, 1] = \{0, 1\} \to [0, 2] = \{0, 1, 2\}\) sending 0 to 0 and 1 to 2.

**Proposition 1.4.** Let \( C \) be a category where finite inverse limits are defined.

1. Let \( P \) be a multiplicative oversimplicial object. Then \((P_1, P_0, s, t, e, \mu, \iota)\) defined above is a groupoid in \( C \).

2. Let \( G \) be the category of groupoids in \( C \) and \( M \) be the full subcategory of the category of oversimplicial objects in \( C \) consisting of multiplicative objects. Then, the functor

\[
M \to G
\]

defined by the construction in 1. is an equivalence of categories. The functor (1.5) induces an equivalence of categories on the full subcategories consisting of strictly multiplicative objects of \( C \) and of groups in \( C \).

**Proof.** 1. We identify \( P_n \) with \( P_{1 \times n} \) by the isomorphism defined by the morphisms \([0, 1] \to [0, n]\) sending \( 0 \mapsto i - 1 \) and \( 1 \mapsto i \) for \( i = 1, \ldots, n \). Then, the commutative diagrams in Definition 1.3 follow from the commutative diagrams

\[
\begin{array}{ccc}
[0, 3] & \xrightarrow{0 \to 0, 1 \to 2, 2 \to 3} & [0, 2] \\
\downarrow 0 \to 0, 1 \to 1, 2 \to 3 & & \downarrow 0 \to 0, 1 \to 2 \\
[0, 2] & \xrightarrow{0 \to 0, 1 \to 2} & [0, 1],
\end{array}
\]

\[
\begin{array}{ccc}
[0, 1] & \xrightarrow{0 \to 0} & [0] \\
\downarrow 0 \to 0, 1 \to 1, 2 \to 0 & & \downarrow 0 \to 0 \\
[0, 2] & \xrightarrow{0 \to 0, 1 \to 2} & [0, 1],
\end{array}
\]

in \( \tilde{\Delta} \).

2. We construct a quasi-inverse functor. Let \((P, S, s, t, e, \mu, \iota)\) be a groupoid in \( C \). We put \( P_n = P_{1 \times n} \) for integer \( n \geq 0 \). For a map \( f: [0, m] \to [0, n] \) in \( \tilde{\Delta} \), we define a morphism \( f^*: P_n \to P_m \) inductively on \( m \geq 0 \).

Assume \( m = 0 \) and \( i: [0] \to [0, m] \) be the map sending 0 to \( i \in [0, m] \). Then, we define \( i^*: P_n \to P_0 \) to be the composition

\[
P_n \xrightarrow{\text{can}} P_1 \times_{P_0} P_{n-1} \xrightarrow{i \times s_{n-1}} S \times S S = P_0.
\]

Next, we consider the case \( m = 1 \). We define \( \mu_0: P_n \to P_1 \) for \( n \geq 0 \) inductively as follows. We set \( \mu_0 = e, \mu_1 = \text{id} \) and \( \mu_2 = \mu \). For \( n \geq 1 \), we define \( \mu_{n+1} \) to be the composition

\[
P_{n+1} = P_n \times P_1 \xrightarrow{\mu_n \times \text{id}} P_1 \times P_1 \xrightarrow{\mu} P_1.
\]
If $0 \leq i \leq j \leq n$, for the map $(ij): [0, 1] \to [0, n]$ sending $0 \mapsto i$ and $1 \mapsto j$, we define $(ij)^*$ to be the composition

\[(1.8) \quad P_n \xrightarrow{\text{can}} P_i \times_S P_{j-i} \times_S P_{n-j} \xrightarrow{t_i \times \mu_{j-i} \times s_{n-j}} S \times S P_1 \times_S S = P_1\]

and define $(ji)^*$ to be the composition $i \circ (ij)^*$.

Let $f: [0, m + 1] \to [0, n]$ be a map. Let $g: [0, m] \to [0, n]$ be the restriction of $f$ and define $h: [0, 1] \to [0, n]$ by $h(0) = f(m)$ and $h(1) = f(m + 1)$. Then, we define $f^*: P_n \to P_{m+1}$ to be

\[(1.9) \quad P_n \xrightarrow{g^* \times h^*} P_m \times_{P_0} P_1 = P_{m+1} \]

For an integer $i \in [0, m]$ and a morphism $f: [0, m] \to [0, n]$, if $f_i: [0, i] \to [0, n]$ denotes the restriction of $f$ and $f'_i: [0, m - i] \to [0, n]$ be the composition of $+: [0, m - i] \to [0, m]$ with $f_i$. Then, by induction on $m - i$, one shows that $f^*: P_n \to P_m$ is the composition

\[(1.10) \quad P_n \xrightarrow{f_i^* \times f'_i^*} P_i \times_{P_0} P_{m-i} \xrightarrow{\text{can}} P_m.\]

We show that $P_\bullet$ defined above is a functor. It suffices to show that, for morphisms $f: [0, l] \to [0, m]$ and $g: [0, m] \to [0, n]$, we have $(g \circ f)^* = f^* \circ g^*$. By the inductive definition (1.9), it suffices to consider the cases $l = 0$ and $l = 1$. We show the case $l = 0$. Let $i: [0] \to [0, m]$ be the map sending $0 \mapsto i$. Then, by the decomposition (1.10) and by the definition (1.6), the composition $i^* \circ g^*: P_n \to P_0$ is the composition

\[(1.11) \quad P_n \xrightarrow{\text{can}} P_{f(i)} \times_{P_0} P_{n-f(i)} \xrightarrow{f_i^* \times f'_i^*} P_i \times_{P_0} P_{m-i} \xrightarrow{t_i \times s_{m-i}} S \times S S = P_0.\]

Since $t_i \circ f_i^* = t_{f(i)}$, $s_{m-i} \circ f'_i^* = s_{n-f(i)}$, it is equal to $f(i)^*: P_n \to P_0$ as required.

The case $l = 1$ is an immediate consequence of the following elementary fact on groupoid.

\[\square\]

Lemma 1.5. Let $(P, S, s, t, e, \mu, i)$ be a groupoid in the category of sets and $x_1, \ldots, x_n$ be elements of $P$ satisfying $s(x_i) = t(x_{i-1})$ for $i = 1, \ldots, n$. For $0 \leq i \leq j \leq n$, let $x_{ij}$ denote the product $x_{i+1} \cdots x_j$ and $x_{ji}$ the inverse $x_{ij}^{-1}$.

Then, for a morphism $f: [0, m] \to [0, n]$ and $0 \leq i \leq j \leq m$, we have

\[x_{f(i)f(j)} = x_{f(i)f(i+1)} \cdots x_{f(j-1)f(j)}.\]

Proof. It follows by induction on $j - i$. \[\square\]

We say that a multiplicative oversimplicial object $P_\bullet$ is associated to the groupoid $P_1$ over $P_0$. If $P_\bullet$ is associated to a groupoid $P$ over $S$, the groupoid $P$ over $S$ is determined by $P_0, P_1, P_2$ and the morphisms between them.

We give some examples of multiplicative oversimplicial objects. Let $X \to S$ be a morphism in $\mathcal{C}$. Then by setting $P_n = X^{X^{n+1}}_S$, we obtain a multiplicative oversimplicial object. Let $G$ be a group. We set $P_n = G^{X^{n+1}}/\Delta G$ to be the quotient by the diagonal action for integer $n \geq 0$. Then, they define a multiplicative oversimplicial object $P_\bullet$. Let $S$ be a scheme and $E$ be a vector bundle over $S$. Then by setting $P_n = \text{Coker} (\text{diag}: E \to E^{X^{n+1}}_S)$, we obtain a strictly multiplicative oversimplicial object.
1.2 Normalization and étaleness

Lemma 1.6. We consider a commutative diagram

\[
\begin{array}{ccc}
V & \leftarrow & V' \\
\downarrow & & \downarrow \\
X & \leftarrow^f & X'
\end{array}
\]

(1.12)

of morphisms of normal schemes where the vertical arrows are étale, separated and of finite type. Let \( Y \) and \( Y' \) be the normalizations of \( X \) and \( X' \) in \( V \) and \( V' \) and let \( W \subset Y \) and \( W' \subset Y' \) be the largest open subschemes étale over \( X \) and over \( X' \) respectively.

1. The map \( V' \to V \) is uniquely extended to a map \( g: Y' \to Y \).

2. Assume that the diagram (1.12) is cartesian. Assume either that \( X' \to X \) is component-wise dominant or that \( V \to X \) is factorized as the composition \( V \to U \to X \) of a finite étale map \( V \to U \) and an open immersion \( U \to X \). Then, we have \( g^{-1}(W) = W \times_X X' \subset W' \).

3. Assume that \( f: X' \to X \) is smooth. Then, we have \( g(W') \subset W \).

Proof. 1. Since the assertion is local on \( X \) and on \( X' \), we may assume \( X = \text{Spec} \, A, Y = \text{Spec} \, B \) and \( X' = \text{Spec} \, A' \) are affine and \( A' \) is an integral domain. Let \( K' \) be the fraction field of \( A' \). Then, \( Y' = \text{Spec} \, B' \) for the integral closure \( B' \) of \( A' \) in an étale \( K' \)-algebra \( L' = \Gamma(V' \times_X K', \mathcal{O}) \). The image of any element in \( B \) in \( L' \) is integral over \( A' \) and hence contained in \( B' \).

2. If the diagram (1.12) is cartesian, \( W \times_X X' \) is a normal scheme containing \( V' = V \times_X X' \) as an open subscheme. Further if \( X' \to X \) is componentwise dominant or \( V \to X \) is the composition of a finite étale map and an open immersion, \( V' = V \times_X X' \) is dense in \( W \times_X X' \). Hence \( Y' \) contains \( g^{-1}(W) = W \times_X X' \).

3. Since \( f: X' \to X \) is assumed smooth, the base change \( Y \times_X X' \) is normal. Hence, by replacing \( X \) and \( V \) by \( X' \) and \( V \times_X X' \), it is reduced to the case where \( X = X' \). Further we may assume \( X, Y \) and \( Y' \) are strictly local and then it is clear.

\[\square\]

Corollary 1.7. Let

\[
\begin{array}{ccc}
V_1 & \leftarrow^g & V_2 \\
\downarrow & & \downarrow \\
X_1 & \leftarrow^f & X_2
\end{array}
\]

(1.13)

be a commutative diagram of normal schemes where the vertical arrows are étale, separated and of finite type. Let \( Y,Y_1,Y_2 \) be the normalizations of \( X,X_1,X_2 \) in \( V,V_1,V_2 \) and let \( W,W_1,W_2 \) be the largest open subschemes of \( Y,Y_1,Y_2 \) étale over \( X,X_1,X_2 \) respectively.

Assume that \( f_1: X \to X_1 \) and \( f_2: X \to X_2 \) are smooth and that the diagram

\[
\begin{array}{ccc}
V_1 \times_{X_1} X & \leftarrow & V \\
\downarrow & & \downarrow \\
X & \leftarrow & V_2 \times_{X_2} X
\end{array}
\]

(1.14)
induced by (1.13) is cartesian. Then the diagram (1.14) is extended to a cartesian diagram

\[
\begin{array}{c}
W_1 \times_{X_1} X & \leftarrow & W \\
\downarrow & & \downarrow \\
X & \leftarrow & W_2 \times_{X_2} X.
\end{array}
\] (1.15)

**Proof.** By Lemma 1.6.1, the morphisms \( g_1 : V \to V_1 \) and \( g_2 : V \to V_2 \) are extended to \( Y \to Y_1 \) and \( Y \to Y_2 \). By the assumption that \( f_1 \) and \( f_2 \) are smooth and by Lemma 1.6.3, they induce \( W \to W_1 \) and \( W \to W_2 \). Hence, we obtain a commutative diagram (1.15).

We define an étale scheme \( W' \) over \( X \) by the cartesian diagram

\[
\begin{array}{c}
W_1 \times_{X_1} X & \leftarrow & W' \\
\downarrow & & \downarrow \\
X & \leftarrow & W_2 \times_{X_2} X.
\end{array}
\] (1.16)

The cartesian diagram (1.14) defines an open immersion \( V \to W' \). It induces an open immersion \( W' \to W \). Since the commutative diagram (1.15) and the cartesian diagram (1.16) define the inverse \( W \to W' \), the assertion follows.

We prove a key lemma for the proof of Theorem 2.14.

**Lemma 1.8.** Let \( V_\bullet \to X_\bullet \) be an étale, separated, of finite type and multiplicative morphism of oversimplicial normal schemes.

1. For integers \( n \geq 0 \), let \( Y_n \) be the normalization of \( X_n \). Then, the schemes \( Y_n \) form an oversimplicial normal scheme \( Y_\bullet \) containing \( V_\bullet \) as an oversimplicial subscheme.

2. For integers \( n \geq 0 \), let \( W_n \subset Y_n \) be the largest open subschemes étale over \( X_n \). Assume that the following conditions are satisfied:

   \begin{enumerate}
   \item For each injection \([0, m] \to [0, n]\), the morphism \( X_n \to X_m \) is smooth.
   \item The morphism \( V_0 \to V_1 \) is extended to \( W_0 \to W_1 \).
   \end{enumerate}

Then, the schemes \( W_n \) form an oversimplicial open subscheme \( W_\bullet \) of \( Y_\bullet \) and the restriction \( W_\bullet \to X_\bullet \) of the morphism \( Y_\bullet \to X_\bullet \) is multiplicative.

**Proof.** 1. It follows from Lemma 1.6.1.

2. We show that \( W_n \) form an oversimplicial subscheme \( W_\bullet \) of \( Y_\bullet \). It suffices to show that for each map \([0, n] \to [0, m]\), the morphism \( Y_m \to Y_n \) maps \( W_m \) to \( W_n \). We show this by induction on \( n \geq 0 \). If \([0, n] \to [0, m]\) is an injection, it follows from Lemma 1.6.3 by the assumption (1). Thus the case \( n = 0 \) is proved. We show the case \( n = 1 \). It suffices to consider the case \([0, 1] \to [0, m]\) is not an injection. In this case, it is factorized uniquely as \([0, 1] \to [0] \to [0, m]\). Hence, it follows from the case where \( n = 0 \) and the assumption (2).

We consider an additive cocartesian diagram (1.1). Let

\[
\begin{array}{c}
V_m & \leftarrow & V_n & \longrightarrow & V_l \\
\downarrow & & \downarrow & & \downarrow \\
X_m & \leftarrow & X_n & \longrightarrow & X_l
\end{array}
\] (1.17)
be the induced commutative diagram. Since $V_\bullet \to X_\bullet$ is assumed multiplicative, the diagram (1.17) defines a cartesian diagram

\[
\begin{array}{ccc}
V_m \times_{X_m} X_n & \leftarrow & V_n \\
\downarrow & & \downarrow \\
X_n & \leftarrow & V_l \times_{X_l} X_n
\end{array}
\] (1.18)

By Corollary [1.7] it is extended to a cartesian diagram

\[
\begin{array}{ccc}
W_m \times_{X_m} X_n & \leftarrow & W_n \\
\downarrow & & \downarrow \\
X_n & \leftarrow & W_l \times_{X_l} X_n
\end{array}
\] (1.19)

since the lower horizontal arrows in (1.17) are smooth by the assumption (1).

Apply the above consideration to the additive cocartesian diagram

\[
\begin{array}{ccc}
[0, n] & \xrightarrow{\text{inclusion}} & [0, n + 1] \\
\uparrow & & \uparrow \text{+n} \\
[0] & \xrightarrow{\text{inclusion}} & [0, 1].
\end{array}
\] (1.20)

Then, to extend $V_m \to V_{n+1}$ to $W_m \to W_{n+1}$, it suffices to extend the commutative diagram

\[
\begin{array}{ccc}
V_n & \leftarrow & V_m \longrightarrow V_1 \\
\downarrow & & \downarrow \\
X_n & \leftarrow & X_{n+1} \longrightarrow X_1
\end{array}
\] (1.21)

corresponding to $V_m \to V_{n+1}$ by the cartesian diagram (1.18) to a commutative diagram

\[
\begin{array}{ccc}
W_n & \leftarrow & W_m \longrightarrow W_1 \\
\downarrow & & \downarrow \\
X_n & \leftarrow & X_{n+1} \longrightarrow X_1
\end{array}
\] (1.22)

Thus, $V_m \to V_n$ is extended to $W_m \to W_n$ by induction on $n$.

By the cartesian diagram (1.19), the morphism $W_\bullet \to X_\bullet$ is multiplicative. \Box

Lemma 1.9 ([5, Lemma 2.7]). Let

\[
\begin{array}{ccc}
X' & \xleftarrow{2} & U' \leftarrow V' \\
\downarrow & & \downarrow \\
X & \xleftarrow{2} & U \leftarrow V
\end{array}
\]

be a cartesian diagram of normal schemes. Assume that the vertical arrows are quasi-finite and component-wise dominant, that the left horizontal arrows are open immersions and that the right horizontal arrows are finite and étale. Let $X \leftarrow Y$ and $X' \leftarrow Y'$ be the normalizations in $V$ and $V'$ respectively.

9
Let

\[
\begin{array}{ccc}
Z' & \xrightarrow{c} & X' \\
\downarrow & & \downarrow \\
Z & \xrightarrow{c} & X
\end{array}
\]

(1.23)

be a commutative diagram where the horizontal arrows are closed immersions. Assume that it is lifted to a commutative diagram

\[
\begin{array}{ccc}
Z' & \xrightarrow{s'} & Y' \\
\downarrow & & \downarrow \\
Z & \xrightarrow{s} & Y.
\end{array}
\]

Then, \( Y \to X \) is étale on a neighborhood of \( s(Z) \), if \( Y' \to X' \) is étale on a neighborhood of \( s'(Z') \) and if the following conditions are satisfied:

1. \( Z' \to Z \) is surjective.
2. The base change of the diagram (1.23) by \( U \to X \) is cartesian.
3. \( Z \cap U \) is dense in \( Z \).

For the reader’s convenience, we include the proof in [5] Lemma 2.7.

**Proof.** Let \( x \) be a point of \( Z \). We show that \( Y \to X \) is étale at \( s(x) \). Let \( x' \) be a point of \( Z' \) above \( x \) and take a geometric point \( \bar{x} \) of \( Z \) above \( x \) and a geometric point \( \bar{x}' \) of \( Z' \) above \( x' \) and \( \bar{x} \). By replacing \( X, Y, X' \) by their strict localizations at \( \bar{x}, s(\bar{x}) \) and at \( \bar{x}', \), we may assume \( X, Y, X' \) are strictly local and \( X' \to X \) is finite and surjective. It suffices to prove that \( Y \to X \) is an isomorphism.

Since \( Y' \) is the normalization of \( X' \times_X Y \), it suffices to show that \( Y' \to X' \) is an isomorphism. Since \( Y' \) is the disjoint union of finitely many connected components, it suffices to show the following.

(a) There exists a unique component meeting \( s'(Z') \).
(b) A component meeting \( s'(Z') \) is isomorphic to \( X' \).
(c) Every component meets \( s'(Z') \).

Since \( Z' \) is a closed subscheme of \( X' \), its image \( s'(Z') \) is also a local scheme and (a) follows. Since the component of \( Y' \) containing \( s'(x') \) is the strict localization at \( s'(\bar{x}') \), it is isomorphic to \( X' \). Hence (b) is proved. We show (c). By the assumption (2), \( s(Z \cap U) \times_Y Y' \) is isomorphic to \( (Z \cap U) \times_X X' = Z' \cap U' \) and hence is equal to \( s'(Z' \cap U') \). By the assumption (3), \( s(Z \cap U) \subset Y \) is not empty. Let \( Y'_1 \) be a component of \( Y' \). Since \( Y'_1 \to Y \) is finite and dominant, it is surjective and hence \( s(Z \cap U) \times_Y Y'_1 = s'(Z' \cap U') \cap Y'_1 \subset s'(Z') \cap Y'_1 \) is not empty. Thus (c) is proved.

The following example shows that we can’t drop the assumption that \( X' \to X \) is quasi-finite even if \( U' = U \).

**Example 1.10.** Let \( k \) be an algebraically closed field of characteristic \( p > 0 \), \( E \) be an elliptic curves of degree \( > 1 \) over \( k \) and \( L \) be a very ample invertible \( \mathcal{O}_E \)-module e.g. \( \mathcal{O}(3 \cdot [0]) \). Let \( X = \text{Spec} \bigoplus \Gamma(E, \mathcal{L}^n) \) be the affine cone. The blow-up \( X' \) of \( X \) at the origin is the line bundle \( L \) over \( E \) defined by the symmetric algebra \( \bigoplus_{n \geq 0} \mathcal{L}^n \). The
complement $U \subset X$ of the origin is the complement $L \dashv E$ of the 0-section. Define a finite étale morphism $V \to U$ to be the pull-back of an étale isogeny $E' \to E$ of degree $> 1$.

The normalization $Y'$ of $X'$ in $V$ is the pull-back $E' \times_E X'$ and is étale over $X'$. The canonical map $Y' \to Y$ to the normalization $Y$ of $X$ in $V$ contracts the closed fiber $E'$ to a point and the morphism $Y \to X$ is not étale.

The inverse image $Z$ of the origin $0 \in E$ is a fiber of the line bundle $X' \to E$ and is isomorphic to $\mathbb{A}^1_k$. The composition $Z \to X' \to X$ is a closed immersion since $L$ is base point free. The inverse image of $0 \in E'$ defines closed immersions $Z \to X'$ and $Z \to X$. They satisfy the conditions (1)–(3) in Lemma 1.9.

1.3 Dilatations

Definition 1.11. Let

$$D \longrightarrow P \longleftarrow X$$

be closed immersions of schemes. Assume that $D$ is a Cartier divisor of $P$ and let $P' \to P$ denote the blow-up at the intersection $D \cap X = D \times_P X$. We call the complement

$$\tilde{P} = P^{(D \cdot X)} \subset P'$$

of the proper transform of $D$ the dilatation of $P$ with respect to $D$ and $X$.

If $P = \text{Spec} \ A$, if the Cartier divisor $D$ is defined by a non-zero divisor $t \in A$ and if the closed subscheme $X \subset P$ is defined by an ideal $I \subset A$, we have $P^{(D \cdot X)} = \text{Spec} \ \tilde{A}$ for the subring

$$\tilde{A} = A \left[ \frac{I}{t} \right] \subset A \left[ \frac{1}{t} \right].$$

The closed subscheme $\tilde{D}$ of $\tilde{P}$ defined by the cartesian diagram

$$\begin{array}{ccc}
\tilde{D} & \longrightarrow & \tilde{P} = P^{(D \cdot X)} \\
\downarrow & & \downarrow \\
D & \longrightarrow & P \longleftarrow X.
\end{array}$$

is a Cartier divisor and is a scheme over $D \cap X = D \times_P X$. The canonical map $\tilde{P} \to P$ induces an isomorphism $\tilde{P} \dashv \tilde{D} \to P \dashv D$. If $X = P$, the canonical map $\tilde{P} \to P$ is an isomorphism.

Example 1.12. 1. Let $Y$ be a closed subscheme of a scheme $X$. Set $P = \mathbb{A}^1_X$ regard the 0-section $X \to P$ as a Cartier divisor $D$ of $P$. Then, the dilatation $P^{(D \cdot X)}$ is the deformation to the normal cone of $Y$ in $X$.

2. Let $D$ be a Cartier divisor of $X$ and $Z$ be the 0-section of $P = \mathbb{A}^1_X$. Then, the dilatation $P^{(\mathbb{A}^1_X \cdot Z)}$ is the line bundle $L(D)$ over $X$ defined by the symmetric $O_X$-algebra $\bigoplus_{n \geq 0} T^n_D$.

The dilatation has the following universality, as an immediate consequence of that of blow-up. Let (1.24) and $\tilde{P} = P^{(D \cdot X)}$ be as in Definition 1.11. Let $\mathcal{I}_D \subset O_P$ and $\mathcal{I}_X \subset O_P$ be the ideals defining $D$ and $X$ respectively. Then, for a scheme $T$ over $P$, if the pull-back
\( \mathcal{I}_D \mathcal{O}_T \) is an invertible ideal of \( \mathcal{O}_T \) and contains \( \mathcal{I}_X \mathcal{O}_T \), then there exists a unique morphism \( T \to \mathcal{P} \) of schemes over \( P \).

For example, if \( D \cap X = D_X \) is a divisor of \( X \), then the immersion \( X \to P \) is uniquely lifted to \( X \to \mathcal{P} = P^{(D \cdot X)} \) by the universality of dilatation. If \( D = D_1 + D_2 \) is the sum of Cartier divisors, then we have a canonical morphism \( \mathcal{P} = P^{(D \cdot X)} \to \mathcal{P}_1 = P^{(D_1 \cdot X)} \) by the universality. Further assume that \( D_1 \cap X \) is a divisor of \( X \), regard \( X \) as a closed subscheme of \( \mathcal{P}_1 \) and let \( \mathcal{P}_2 \) denote the pull-back of \( D_2 \) to \( \mathcal{P}_1 \). Then, we have a canonical isomorphism \( \mathcal{P} \to \mathcal{P}_1^{(D_2 \cdot X)} \).

We study the structure of dilatations. For a regular immersion \( X \to P \) of schemes, the normal bundle \( T_X P \) is the vector bundle over \( X \) defined by the symmetric algebra \( S^* \mathcal{N}_{X/P} \) where the conormal sheaf \( \mathcal{N}_{X/P} = \mathcal{I}_X / \mathcal{I}_X^2 \) is defined by the ideal sheaf \( \mathcal{I}_X \subset \mathcal{O}_P \). If \( X \) and \( P \) are smooth over a scheme \( S \) and if the immersion \( X \to P \) is a morphism over \( S \), it fits in an exact sequence
\[
0 \to T_X \to T_P \times_P X \to T_X P \to 0
\]
of vector bundles on \( X \) where \( T_X \) and \( T_P \) denote the tangent bundles defined by the symmetric algebras of the locally free modules \( \Omega_X^1 \) and \( \mathcal{O}_P^1 \) respectively.

For vector bundles \( E \) and \( E' \) on a scheme \( S \), let \( E \otimes E' \) denote the tensor product. For a line bundle \( L \) on a scheme \( S \) and an integer \( n \), let \( L^\otimes n \) denote the \( n \)-th power. For a vector bundle \( E \) on a scheme \( S \) and a Cartier divisor \( D \) of \( S \), let \( E(D) = E \otimes L(D) \) denote the tensor product with the line bundle \( L(D) \) on \( S \). We have \( T_D S = L(D) \times_S D \).

**Lemma 1.13.** We consider a cartesian diagram
\[
\begin{array}{ccc}
\tilde{D} & \longrightarrow & \mathcal{P} = P^{(D \cdot X)} \\
\downarrow & & \downarrow \\
D & \longrightarrow & P \leftarrow X
\end{array}
\]
where the arrows in the lower line are closed immersions as in (1.24). Assume that \( X \to P \) is a regular immersion and that \( D \cap X = D_X \) is a Cartier divisor of \( X \).

1. We have a canonical isomorphism
\[
(1.25) \quad \tilde{D} \to T_X P(-D_X) \times_X D_X
\]
to the vector bundle on \( D_X \).

2. The immersion \( X \to \mathcal{P} \) is also a regular immersion and there is a canonical isomorphism
\[
(1.26) \quad T_X \mathcal{P} \to T_X P(-D_X)
\]
for the normal bundle.

**Proof.** 1. First, we consider locally on \( P \). We assume that \( P = \text{Spec} \ A \), that \( t_0, t_1, \ldots, t_m \) is a regular sequence of \( A \) and that \( D \) and \( X \) are defined by the ideals \( (t_0) \) and \( (t_1, \ldots, t_m) \) respectively. Then, we have \( P^{(D \cdot X)} = \text{Spec} \ \tilde{A} \) for \( \tilde{A} = A[T_1, \ldots, T_m] / (t_0 T_1 - t_1, \ldots, t_0 T_m - t_m) \). Hence, we have an isomorphism
\[
(1.27) \quad A / (t_0, t_1, \ldots, t_m) [T_1, \ldots, T_m] \to \tilde{A} / (t_0).
\]
Since $D_X = \text{Spec } A/(t_0, t_1, \ldots, t_m)$ and $t_0$ and $t_1, \ldots, t_m$ define linear coordinates of $T_P P$ and of $T_X P$ respectively, the isomorphism (1.27) defines an isomorphism (1.25). It is easily checked that the isomorphism (1.25) thus defined is independent of the choices and defined globally.

2. In the description in the proof of 1, $X \subset \tilde{P}$ is defined by the ideal of $\tilde{A}$ generated by the regular sequence $T_1, \ldots, T_m$. Hence, we obtain an isomorphism (1.26). It is easily checked that the isomorphism (1.26) thus defined is independent of the choices and defined globally.

Corollary 1.14. Assume that $P$ and $X$ are regular and that $D \subset P$ is a divisor with normal crossings meeting $X$ transversally. Then $\tilde{P} = P^{(D \cdot X)}$ is also regular and the pullback $\tilde{D} = D \times_P \tilde{P}$ is a divisor with normal crossings.

Proof. Since the assertion is étale local on $P$, we may assume that $D$ has simple normal crossings. Let $D_1, \ldots, D_h$ be the irreducible components of $D$. Then, by the isomorphism (1.25), the inverse images $\tilde{D}_i = \tilde{D} \times_P D_i$ are regular divisors of $\tilde{P}$ meeting transversally. Since the complement $\tilde{P} - \tilde{D}$ is isomorphic to $P - D$ and is regular, the scheme $\tilde{P}$ is regular.

We consider the functoriality of dilatations. Let

$$
E \longrightarrow Q \longleftarrow Y
$$

$$
\downarrow \quad f \downarrow \quad \downarrow
$$

$$
D \longrightarrow P \longleftarrow X
$$

be a commutative diagram of schemes satisfying the following properties: The horizontal arrows are closed immersions and $D$ and $E$ are Cartier divisors of $P$ and $Q$ respectively. Further, the left square is cartesian. Then, the map $Q \rightarrow P$ is uniquely lifted to a morphism $\tilde{f}: \tilde{Q} = Q^{(E \cdot Y)} \rightarrow \tilde{P} = P^{(D \cdot X)}$ by the universality of dilatations.

Lemma 1.15. We consider the commutative diagram (1.28) of schemes satisfying the conditions loc. cit. Assume that $f: Q \rightarrow P$ is flat.

1. If the diagram (1.28) is cartesian, the diagram

$$
Q \longleftarrow \tilde{Q} = Q^{(E \cdot Y)}
$$

$$
\downarrow \quad \downarrow
$$

$$
P \longleftarrow \tilde{P} = P^{(D \cdot X)}
$$

is cartesian and the vertical arrows are flat.

2. Assume that $D \cap X = D_X$ and $E \cap Y = E_Y$ are Cartier divisors of $X$ and of $Y$ respectively and that $E_Y \rightarrow D_X$ is flat. If $X \rightarrow P$ and $Y \rightarrow Q \times_P X$ are regular immersions, the morphism $\tilde{f}: \tilde{Q} = Q^{(E \cdot Y)} \rightarrow \tilde{P} = P^{(D \cdot X)}$ is flat.

Proof. 1. Since the question is local, we may assume $P = \text{Spec } A$ and $Q = \text{Spec } B$ are affine, the divisor $D$ is defined by a non-zero divisor $t \in A$ and $X$ is defined by an ideal $I \subset A$. Then, $\tilde{P} = \text{Spec } A[\frac{t}{I}]$ and $\tilde{Q} = \text{Spec } B[\frac{IB}{I}]$. Since $A \rightarrow B$ is flat, the injection $A[\frac{t}{I}] \rightarrow A[\frac{1}{I}]$ induces an injection $B \otimes_A A[\frac{t}{I}] \rightarrow B \otimes_A A[\frac{1}{I}] = B[\frac{1}{I}]$ and the assertion follows.
2. The restriction $\tilde{Q} \to \tilde{E} \to \tilde{P} \to \tilde{D}$ to the complements is flat by the assumption. Since the pull-backs $\tilde{D} = \tilde{P} \times_P D \subset \tilde{P}$ and $\tilde{E} = \tilde{Q} \times_Q E \subset \tilde{Q}$ are Cartier divisors and $f*\tilde{D} = \tilde{E}$, it suffices to show that $\tilde{E} \to \tilde{D}$ is flat.

Since $Q \to P$ is flat, the composition $Y \to Q \times_P X \to Q$ is a regular immersion. The isomorphisms (1.25) for $\tilde{D}$ and $\tilde{E}$ are functorial and make a commutative diagram

$$
\begin{align*}
\tilde{E} & \longrightarrow T_Y Q(-E_Y) \times_Y E_Y \\
& \downarrow \quad \quad \quad \downarrow \\
\tilde{D} & \longrightarrow T_X P(-D_X) \times_X D_X
\end{align*}
$$

(1.30)

Since $Q \to P$ is flat and the immersions $Y \to Q \times_P X$ and $X \to P$ are regular immersions, the linear map $T_Y Q \to T_X P \times_X Y$ of vector bundles is a surjection. By $f*\tilde{D} = \tilde{E}$, we have a canonical isomorphism $L(-E_Y) \to L(-D_X) \times_X Y$. Since $E_Y \to D_X$ is flat, the right vertical arrow is flat as required.

**Corollary 1.16.** 1. We consider the commutative diagram (1.28) of schemes satisfying the conditions loc. cit. Assume that $D \cap X = D_X$ and $E \cap Y = E_Y$ are Cartier divisors of $X$ and of $Y$ respectively and that the vertical arrows and $E_Y \to D_X$ are smooth. If $X \to P$ is a regular immersion, the morphism $Q^{(E_Y)} \to P^{(D_X)}$ is smooth.

2. We consider a cartesian diagram

$$
\begin{array}{ccc}
P_1 & \leftarrow & P_3 \\
\downarrow & & \downarrow \\
S & \leftarrow & P_2
\end{array}
$$

(1.31)

of flat separated morphisms of schemes. Let $X_1 \subset P_1$ and $X_2 \subset P_2$ be closed subschemes and define a closed subscheme $X_3 = X_1 \times_S X_2 \subset P_3$. Let $D$ be a Cartier divisor of $S$ and $D_1, D_2, D_3$ be the pull-back to $P_1, P_2, P_3$. Assume $X_1$ is flat over $S$ and the immersion $X_1 \to P_1$ is a regular immersion. Then, the diagram (1.31) induces a cartesian diagram

$$
\begin{array}{ccc}
P_1^{(D_1 \cdot X_1)} & \leftarrow & P_3^{(D_3 \cdot X_3)} \\
\downarrow & & \downarrow \\
S & \leftarrow & P_2^{(D_2 \cdot X_2)}
\end{array}
$$

(1.32)

**Proof.** 1. Since $Y \to X$ and $Q \to P$ are smooth, the immersion $Y \to Q \times_P X$ is a regular immersion. Hence, by Lemma 1.15.2, the morphism $Q \to P$ is flat. Further since $f*\tilde{D} = \tilde{E}$, it suffices to show that $\tilde{E} \to \tilde{D}$ is smooth. Since $E_Y \to D_X$ is smooth, the right vertical arrow in (1.30) is smooth.

2. Since the assertion is local, we may assume that $S = \text{Spec } A$ and $P_i = \text{Spec } A_i$ are affine, $D$ is defined by a non-zero divisor $t \in A$ and the closed subschemes $X_i \subset P_i$ are defined by ideals $I_i \subset A_i$. By Lemma 1.15.2, the dilatation $\tilde{P}_1 = P_1^{(D_1 \cdot X_1)}$ is flat over $S = S^{(D-S)}$. Therefore, the injection $A_2[\frac{I}{t}] \to A_2[\frac{I}{t}]$ induces an injection $A_1[\frac{I}{t}] \otimes_A A_2[\frac{I}{t}] \to A_1[\frac{I}{t}] \otimes_A A_2[\frac{I}{t}] = A_1 \otimes_A A_2[\frac{I}{t}]$. Hence the isomorphism $A_1 \otimes_A A_2 \to A_3$ induces an isomorphism $A_1[\frac{I}{t}] \otimes_A A_2[\frac{I}{t}] \to A_3[\frac{I}{t} + \frac{I}{t}] = A_3[\frac{I}{t}] \subset A_3[\frac{I}{t}]$. 

$\square$
We give a construction similar to the deformation to normal cone in Example 1.12. Let \( X \) be a scheme, \( D \) be a Cartier divisor and \( m \geq 1 \) be an integer. We consider the 0-section \( X \to \mathbb{A}^1_X \) as a Cartier divisor. Let it denoted by \( Z \) and consider closed immersions

\[
\mathbb{A}_D^1 \longrightarrow \mathbb{A}_X^1 \leftarrow mZ.
\]

Then, we define an open subscheme \( \tilde{\mathbb{X}}(mD) \subset \mathbb{A}_X^1(\mathbb{A}^1_D \cdot mZ) \) of the dilatation by further removing the proper transform of the zero-section \( Z \). If \( m = 1 \), we recover the construction in Example 1.12.

Let \( D_1, \ldots, D_h \) be Cartier divisors of \( X \) and \( M = m_1D_1 + \cdots + m_hD_h \) be a linear combination with integral coefficients \( m_1 \geq 1, \ldots, m_h \geq 1 \). Then, we define \( f: \tilde{X}(M) \to X \) to be the fibered product of \( \tilde{\mathbb{X}}(mD_i) \) over \( X \) for \( i = 1, \ldots, h \).

In the terminology of log product [18, Proposition 4.2.1], the construction of \( \tilde{X}(M) \) is described as follows. We regard \( X \) as a log scheme by the log structure defined by the Cartier divisors \( D_1, \ldots, D_h \). Let \( P \) and \( Q \) denote the monoids \( \mathbb{N}^h \) and let \( M: Q \to P \) be the map multiplying \( m_i \) on \( i \)-th component. The frame \( X \to [Q] \) defined by the Cartier divisors \( D_1, \ldots, D_h \) and the canonical frame \( \mathbb{A}^h_X \to P \) and \( \mathbb{A}^h_X \to [P + Q] \). Then \( \tilde{X}(M) \) is the log product \( \mathbb{A}^h_X \times_{[P + Q]} [P] \) with respect to the surjection \( \text{id}_P + M: P + Q \to P \) constructed in [18, Proposition 4.2.1].

Let \( U = X - D \) be the complement of the union \( D = D_1 \cup \cdots \cup D_h \). Then, the inverse image \( f^{-1}(U) \) is a trivial \( \mathbb{G}_m^h \)-torsor over \( U \). The action of \( \mathbb{G}_m^h \) on \( f^{-1}(U) \) is uniquely extended to an action of \( \mathbb{G}_m^h \) on \( \tilde{X}(M) \) over \( X \).

**Lemma 1.17.** Let \( X \) be a smooth scheme over a perfect field \( k \), \( D \) be a divisor of \( X \) with simple normal crossings and \( M = m_1D_1 + \cdots + m_hD_h \) be a linear combination with integral coefficients \( m_i \geq 1, \ldots, m_h \geq 1 \) of the irreducible components \( D_1, \ldots, D_h \).

Then, \( \tilde{X}(M) \) is smooth over \( k \) and the canonical morphism \( f: \tilde{X}(M) \to X \) is flat. Let \( \tilde{D}_i^{(M)} \) be the inverse image of \( Z \) by the composition \( \tilde{X}(M) \to \tilde{X}(mD_i) \to X \cdot \mathbb{A}^1_k \). Then, the sum \( \tilde{D}_i^{(M)} = \tilde{D}_i^{(M)} + \cdots + \tilde{D}_h^{(M)} \) is a divisor of \( \tilde{X}(M) \) with simple normal crossings. For \( i = 1, \ldots, h \), we have \( f^*D_i = m_i\tilde{D}_i^{(M)} \). If \( M = D \), the canonical morphism \( \tilde{X}(M) \to X \) is smooth.

**Proof.** Since the question is local on \( X \), we may assume that there is a smooth morphism \( X \to \mathbb{A}^h_k \) such that \( D_i \) is the inverse image of the \( i \)-th coordinate hyperplane for \( i = 1, \ldots, h \). Further we may assume that \( X = \mathbb{A}^h_k \) and \( h = 1 \). In this case, \( \tilde{X}(M) = \mathbb{A}^1_k(mZ) \to X = \mathbb{A}^1_k = \text{Spec} k[T] \) is given by \( \text{Spec} k[T, S, U^{\pm 1}]/(T - US^m) \) and the assertion follows.

If \( M = D \), the assertion follows from in Example 1.12.\(^2\) \( \square \)

We study the functoriality of the construction of \( \tilde{X}(M) \). Let \( f: Y \to X \) be a morphism of schemes, \( E_1, \ldots, E_k \) be Cartier divisors of \( Y \) and \( N = n_1E_1 + \cdots + n_kE_k \) be a linear combination with integral coefficients \( n_1 \geq 1, \ldots, n_k \geq 1 \). Assume that the pull-back \( f^*D_i = \sum_{j=1}^kn_{ij}E_j \) for integers \( e_{ij} \) and \( l_{ij} = e_{ij}n_{ij}/m_i \) is an integer for every \( j = 1, \ldots, k \) for each \( i = 1, \ldots, h \). Let \( (T_i) \) and \( (S_j) \) be the coordinates of \( \mathbb{A}^h_X \) and \( \mathbb{A}^k_Y \) respectively and we define a morphism \( \tilde{f}: \mathbb{A}^h_Y \to \mathbb{A}^k_X \) lifting \( f: Y \to X \) by sending \( T_i \) to \( S^l_{ij} \). Then, by the universality of dilatations, the morphism \( \tilde{f}: \mathbb{A}^h_Y \to \mathbb{A}^k_X \) is uniquely lifted to

\[
(1.33) \quad \tilde{f}: \mathbb{Y}^n \to \tilde{X}(M).
\]
If \( h = k \), if each \( E_i \) is the pull-back of \( D_i \) and if \( n_i = m_i \) for each \( i = 1, \ldots, h \), then the diagram

\[
\begin{array}{ccc}
\tilde{Y}^{(N)} & \longrightarrow & \tilde{X}^{(M)} \\
\downarrow & & \downarrow \\
Y & \longrightarrow & X
\end{array}
\]  

(1.34)

is cartesian.

We consider the case where \( f: Y \to X \) is the identity of \( X \). Let \( M' = m_1' D_1 + \cdots + m_h' D_h \) be another linear combination with integral coefficients \( m_1' \geq 1, \ldots, m_h' \geq 1 \) divisible by \( M \) in the sense that \( l_i = m_i'/m_i \) is an integer for each \( i = 1, \ldots, h \). Then, we have a canonical morphism

\[
\tilde{X}^{(M')} \to \tilde{X}^{(M)}
\]  

(1.35)

over \( X \). It is compatible with the actions of \( G_{m,X}^h \) with respect to the morphism \( G_{m,X}^h \to G_{m,X}^h \) defined by \( l_i \)-th power on \( i \)-th component.

### 1.4 Extensions of a vector bundle in characteristic \( p > 0 \)

We study extensions of a vector bundle by a finite étale group scheme on a scheme of characteristic \( p > 0 \).

**Lemma 1.18.** Let \( E \) be a vector bundle over a scheme \( S \) of characteristic \( p > 0 \) and let \( 1 \to G \to \tilde{E} \to E \to 1 \) be an extension of \( E \) by an étale group scheme \( G \) over \( S \). Assume that for every point \( s \) of \( S \), the fiber \( \tilde{E}_s \) is connected. Then, \( \tilde{E} \) and consequently \( G \) are commutative and killed by \( p \).

**Proof.** Since \( E \) is commutative, the morphism \( \tilde{E}_s \times_S \tilde{E} \to \tilde{E} \) defined by sending \((x, y)\) to the commutator \([x, y]\) induces a morphism of schemes to the kernel \( G \) of \( \tilde{E} \to E \). Since the fiber of \( \tilde{E}_s \times_S \tilde{E} \) is connected for every point of \( S \), the image is in the identity section \( S \subset G \) and it defines a morphism \( \tilde{E}_s \times_S \tilde{E} \to S \) of schemes. Hence \( \tilde{E} \) is commutative.

Similarly, the morphism \( p: \tilde{E} \to \tilde{E} \) defined by sending \( x \) to \( px \) induces a morphism of schemes to the kernel \( G \) of \( \tilde{E} \to E \) and to a morphism of schemes to \( S \). Hence \( \tilde{E} \) is killed by \( p \).

\( \square \)

Let \( E \) be a vector bundle over a scheme \( S \) of characteristic \( p > 0 \) and let \( G \) be a finite étale commutative group scheme over \( S \), killed by \( p \). Let \( E^\vee = \text{Hom}_S(E, \mathbb{A}^1) \) be the dual vector bundle of \( E \) and \( G^\vee = \text{Hom}_S(G, \mathbb{F}_p) \) be the dual finite étale scheme of \( G \). For commutative group schemes \( A \) and \( B \) over \( S \), let \( \text{Mor}_S(A, B) \) denote the abelian group of morphisms of group schemes.

Let \( F: \mathbb{A}^1 \to \mathbb{A}^1 \) denote the Frobenius morphism defined by sending the coordinate \( t \) to \( t^p \). For \( E = \mathbb{A}^1 \) and \( G = \mathbb{F}_p \), the pull-back and the push-forward of the Artin-Schreier sequence

\[
0 \longrightarrow \mathbb{F}_p \longrightarrow \mathbb{A}^1 \overset{F-1}{\longrightarrow} \mathbb{A}^1 \longrightarrow 0
\]

(1.36)

define a morphism \( \mathbb{A}^1(S) = \text{Mor}_S(G^\vee, E^\vee) \to \text{Ext}_S(E, G) \) of abelian groups. Since a vector bundle \( E \) is locally isomorphic to a direct sum of \( \mathbb{A}^1 \) and a finite étale group scheme \( G \) of
F_p-vector spaces is étale locally isomorphic to a direct sum of F_p, we obtain a canonical morphism

\[(1.37) \quad \text{Mor}_S(G^\vee, E^\vee) \to \text{Ext}_S(E, G)\]
of abelian groups, by étale descent.

For an integer \( n \geq 1 \), let \( S_n \) denote the scheme \( S \) regarded as a scheme over \( S \) by the \( n \)-times iteration of the absolute Frobenius \( S \to S \). Since the étale site remains the same by a radicial surjective morphism, the pull-back map \( \text{Ext}_S(E, G) \to \text{Ext}_{S_n}(E \times_S S_n, G \times_S S_n) \) is an isomorphism. Hence, \((1.37)\) induces a morphism

\[(1.38) \quad \lim_{\longrightarrow} \text{Mor}_{S_n}(G^\vee, E^\vee) \to \text{Ext}_S(E, G).\]

**Proposition 1.19.** Let \( E \) be a vector bundle over a scheme \( S \) of characteristic \( p > 0 \) and let \( G \) be a finite étale commutative group scheme over \( S \), killed by \( p \). The morphism \((1.38)\) of abelian groups is an isomorphism if \( S \) is quasi-compact.

The following proof is a modification of that in the case \( S = \text{Spec} \ k \) for a perfect field \( k \) in \([12, \text{Lemme 3}]\).

**Proof.** Since a morphism \( E \to G \) of group schemes from a vector bundle to a finite étale group scheme is trivial, the presheaf \( U \mapsto \text{Ext}_U(E, G) \) is an étale sheaf on \( S \). Since \( S \) is assumed quasi-compact, we may assume \( S = \text{Spec} \ A \) is affine, \( E = A^n \) and \( G \) is constant. Further, we may assume \( E = A^1 \) and \( G = F_p \).

We identify the ring \( \text{End}_A(A^1) \) with the non-commutative ring \( A[F] = \bigoplus_n A^F_n \) defined by the relations \( F \cdot a = a^p F \) for \( a \in A \). Then, the boundary map for the Artin-Schreier sequence \((1.36)\) induces an injection

\[(1.39) \quad A[F]/(F - 1)A[F] \to \text{Ext}_A(A^1, F_p).\]

By the isomorphism defined by the inductive system

\[
\begin{align*}
A & \xrightarrow{a \mapsto a^{F^n+m}} A/F^{(F-1)}A[F] \\
& \uparrow F^n \\
& A \xrightarrow{a \mapsto a^{F^n}} A/F^{(F-1)}A[F]
\end{align*}
\]
of morphisms of abelian groups, we identify \( A[F]/(F - 1)A[F] \) with the additive group of the perfection \( A^{p^{-\infty}} = \lim_{\longrightarrow} F_n A \).

For \( S = \text{Spec} \ A \), \( E = A^1 \) and \( G = F_p \), the abelian group \( \text{Mor}_S(G^\vee, E^\vee) = \text{Mor}_A(F_p, A^1) \) is identified with the additive group \( A \) and the transition map \( A = \text{Mor}_{S_{n+1}}(G^\vee, E^\vee) \to A = \text{Mor}_{S_n}(G^\vee, E^\vee) \) is the absolute Frobenius. Hence the direct limit \( \lim_{\longrightarrow} \text{Mor}_{S_n}(G^\vee, E^\vee) \) is identified with the additive group of the perfection \( A^{p^{-\infty}} \) and to \( A[F]/(F - 1) \). Thus, the morphism \((1.38)\) is an injection by \((1.39)\).

We show the surjectivity. By replacing \( A \) by the perfection \( A^{p^{-\infty}} \), we may assume that the absolute Frobenius \( A \to A \) is a bijection. We consider a commutative diagram

\[
\begin{array}{ccc}
\text{Ext}_A(A^1, F_p) & \longrightarrow & H^1(A_A^1, F_p) \\
\downarrow & & \downarrow \\
A[F] & \longrightarrow & A[T] \\
\end{array}
\]

\[
\begin{array}{ccc}
H^1(A_A^1, F_p) & \longrightarrow & H^1(A_A^2, F_p) \\
\uparrow & & \uparrow \\
A[T] & \longrightarrow & A[T_1, T_2]
\end{array}
\]

\[(1.40)\]
where the left vertical arrow is induced by (1.39) and \(+, \text{pr}_1, \text{pr}_2: A_A^2 \rightarrow A_A^1\) denote the addition and the projections. The middle and the right vertical arrows are the surjections defined by the pull-back of the Artin-Schreier covering (1.36). The lower left horizontal arrow is the left \(A\)-linear map sending \(F^n\) to \(T^n\) for \(n \geq 0\).

Since the constant étale covering \(A_A^1 \times F_p\) of \(A_A^1\) has a unique structure of extension of \(A_A^1\) by \(F_p\), the upper left horizontal arrow \((1.39)\) and \(+, \text{pr}_1, \text{pr}_2: A_A^2 \rightarrow A_A^1\) denote the addition and the corresponding extension equivalent:

the quotient \(A_n^1, T_2\) where the second factors are free left \(A\)-modules. Since the absolute Frobenius on \(A\) is assumed to be a bijection, the inclusion induces an isomorphism \(A \rightarrow A[F]/(F-1)A[F]\). Thus, the sequence (1.41) is isomorphic to (1.42)\[A \rightarrow A/(F-1) \oplus A \cdot T^n \rightarrow A/(F-1) \oplus A \cdot T_1^n T_2^n.\]

The second map \(+^* - \text{pr}_1^* - \text{pr}_2^*\) sends \(T^n\) to

\[(T_1 + T_2)^n - T_1^n - T_2^n = \sum_{i=1}^{n-1} \binom{n}{i} T_1^{n-i} T_2^i\]

for an integer \(p \nmid n\). Since this is 0 if and only if \(n = 1\), the assertion follows.

\[\square\]

**Lemma 1.20.** Let \(S\) be a scheme over \(F_p\), \(E\) be a vector bundle over \(S\) and \(G\) be a finite étale group scheme of \(F_p\)-vector spaces over \(S\). For a morphism \(G^\vee \rightarrow E^\vee\) of the duals and for the corresponding extension \(0 \rightarrow G \rightarrow \tilde{E} \rightarrow E \rightarrow 0\), the following conditions are equivalent:

1. For every point \(s\) of \(S\), the fiber \(\tilde{E}_s\) is connected.
2. For every geometric point \(s\) of \(S\), the geometric fiber \(G_s^\vee \rightarrow E_s^\vee\) is an injection.
3. \(G^\vee \rightarrow E^\vee\) is a closed immersion.

**Proof.** (1)\(\Leftrightarrow\)(2) We may assume \(S = \text{Spec } k\) for a field \(k\). Further, we may assume \(k\) is algebraically closed.

Since the composition \(G \rightarrow \tilde{E} \rightarrow \pi_0(\tilde{E})\) is a surjection of \(F_p\)-vector spaces, it has a section. Hence, the restriction of \(G^\vee \rightarrow E^\vee\) to \(\pi_0(\tilde{E})^\vee\) regarded as a subgroup is trivial. Hence, if \(G^\vee \rightarrow E^\vee\) is injective, we have \(\pi_0(\tilde{E}) = 0\) and \(\tilde{E}\) is connected.

Let \(\chi: G \rightarrow F_p\) be a non-trivial character and \(f: E \rightarrow A^1\) be the image of \(\chi\). Then, the quotient \(\tilde{E}/\text{Ker } \chi\) is the extension defined by the commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & F_p & \longrightarrow & \tilde{E}/\text{Ker } \chi & \longrightarrow & E & \longrightarrow & 0 \\
\end{array}
\]

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & F_p & \longrightarrow & A^1 & \overset{F^{-1}}{\longrightarrow} & A^1 & \longrightarrow & 0.
\end{array}
\]
Hence \( f \neq 0 \) if and only if \( \tilde{E}/\text{Ker} \chi \) is connected. If \( \tilde{E} \) is connected, \( \tilde{E}/\text{Ker} \chi \) is connected for every \( \chi \in G^\vee \) and \( G^\vee \to E^\vee \) is injective.

(2)\( \Rightarrow \) (3) Since \( G^\vee \) is finite étale, the map \( G^\vee \to E^\vee \) is proper and unramified. The condition (2) is equivalent to that \( G^\vee \to E^\vee \) is radicial. Hence it follows from \( [34, \text{Corollaire (18.12.6)}}) \Rightarrow a\].

We study étale sheaves on \( E \) such that the restrictions on the geometric fibers of \( \tilde{E} \) are constant. First, we consider a more general setting. Let \( G \) be a finite étale commutative group scheme over a scheme \( S \) and \( n \geq 1 \) be an integer annihilating \( G \). Set \( \Lambda = \mathbb{Z}[\frac{1}{n}, \zeta_n] \).

The dual \( G^\vee = \mathcal{H}om(G, \mathbb{Z}/n\mathbb{Z}) \) of \( G \) is defined as a finite étale commutative group scheme over a scheme \( S \).

Let \( X \) be a scheme over \( S \) and \( \pi: E \to X \) be a \( G \)-torsor over \( X \). We define a locally constant sheaf \( \mathcal{L}_E \) of free \( \Lambda \)-modules of rank 1 on the scheme \( G^\vee \times_X S \) as follows. The push-forward \( \pi_*\Lambda \) is a locally constant sheaf of free \( \Lambda[G] \)-modules of rank 1. On \( G^\vee \), the tautological character \( \chi: G \to \mathbb{Z}/n\mathbb{Z} \) induces a character \( G \to \Lambda^\times \) by \( 1 \mapsto \zeta_n \) and hence defines a morphism \( \Lambda[G] \to \Lambda \) of \( \Lambda \)-algebras. We define \( \mathcal{L}_\chi \) as \( \pi_*\Lambda \otimes_{\Lambda[G]} \Lambda \).

**Lemma 1.21.** Let \( S \) be a scheme and \( n \geq 1 \) be an integer. Let \( G \) be a finite étale group scheme of \( \mathbb{Z}/n\mathbb{Z} \)-modules over \( S \) and \( \Lambda = \mathcal{H}om(G, \mathbb{Z}/n\mathbb{Z}) \) be the dual finite étale group scheme. Let \( \Lambda \) be the ring \( \mathbb{Z}[\frac{1}{n}, \zeta_n] \) and \( \mathcal{A} \) be an étale sheaf of \( \Lambda \)-algebras on \( S \). Let \( X \) be a scheme over \( S \) and \( \pi: E \to X \) be a \( G \)-torsor over \( X \). Let \( \mathcal{L}_E \) denote the locally constant sheaf of \( \Lambda \)-modules of rank 1 on \( X \times_S G^\vee \) defined above.

Let \( \mathcal{M} \) be an étale sheaf of \( \mathcal{A} \)-modules on \( X \) and \( \iota \in \Gamma(E, \mathcal{M}) \) be a section defining an isomorphism of \( \mathcal{A} \)-modules \( \mathcal{A} \to \mathcal{M} \) on \( E \). Let \( t: S \to E \) be a section and assume that the restriction map \( \iota^*: \Gamma(E, \mathcal{A}) \to \Gamma(S, \mathcal{A}) \) is an isomorphism.

Then, there exists an idempotent \( e_\mathcal{M} \in \Gamma(X \times_S G^\vee, \mathcal{A}) \) and an isomorphism

\[
\text{pr}_{1*}( (e_\mathcal{M} \cdot \mathcal{A}) \otimes_{\Lambda} \mathcal{L}_E ) \to \mathcal{M}
\]

of \( \mathcal{A} \)-modules on \( X \) where \( \text{pr}_1: X \times_S G^\vee \to X \) denotes the projection.

**Proof.** First, we assume \( G \) and hence \( G^\vee \) are constant. Since the restriction map \( \Gamma(E, \mathcal{A}) \to \Gamma(S, \mathcal{A}) \) is assumed an isomorphism, the action of \( G \) on \( \Gamma(E, \mathcal{M}) \) defines a character \( \alpha: G \to \Gamma(S, \mathcal{A}^\times) \) satisfying \( g(\iota) = \alpha(g) \cdot \iota \). The idempotents \( e_\chi = \frac{1}{|G|} \sum_g \chi^{-1}(g) \alpha(g) \) for characters \( \chi: G \to \Lambda^\times \) satisfy \( g(e_\chi \cdot \iota) = \chi(g)e_\chi \cdot \iota \) and \( \sum_\chi e_\chi = 1 \). The isomorphism \( \iota^*: \mathcal{A} \to \mathcal{M} \) induces an isomorphism \( e_\chi \mathcal{A} \otimes \mathcal{L}_\chi \to e_\chi \mathcal{M} \) for each \( \chi \). They define an idempotent \( e_\mathcal{M} = (e_\chi) \in \Gamma(X \times_S G^\vee, \mathcal{A}) = \bigoplus_\chi \Gamma(X, \mathcal{A}) \) and an isomorphism \( \text{pr}_{1*}( (e_\mathcal{M} \cdot \mathcal{A}) \otimes \mathcal{L}_E ) = \bigoplus_\chi e_\chi \mathcal{A} \otimes \mathcal{L}_\chi \to \bigoplus_\chi e_\chi \mathcal{M} = \mathcal{M} \).

In general, we obtain an idempotent \( e_\mathcal{M} \) and an isomorphism \((1.44)\)

by patching.

**Corollary 1.22.** Let \( S \) be a scheme over \( \mathbb{F}_p \) and let \( 0 \to G \to \tilde{E} \to E \to 0 \) be the extension of a vector bundle \( E \) over \( S \) by a finite étale group scheme \( G \) of \( \mathbb{F}_p \)-vector spaces over \( S \) corresponding to a morphism \( G^\vee \to E^\vee \) on the dual. Let \( \Lambda \) be the ring \( \mathbb{Z}[\frac{1}{p}, \zeta_p] \) and \( \mathcal{A} \) be an étale sheaf of \( \Lambda \)-algebras on \( S \).

1. The locally constant sheaf \( \mathcal{L}_E \) of free \( \Lambda \)-modules of rank 1 on \( E \times_S G^\vee \) defined above is the pull-back of the locally constant sheaf of free \( \Lambda \)-modules of rank 1 on \( E \times_S E^\vee \) defined by the Artin-Schreier equation \( t^p - t = \langle f, x \rangle \) by the map \( E \times_S G^\vee \to E \times_S E^\vee \).

2. Let \( \mathcal{M} \) be an étale sheaf of \( \mathcal{A} \)-modules on \( E \) and \( \iota \in \Gamma(\tilde{E}, \mathcal{M}) \) be a section defining an isomorphism of \( \mathcal{A} \)-modules \( \mathcal{A} \to \mathcal{M} \) on \( \tilde{E} \). Assume that the geometric fibers of \( \tilde{E} \to S \) are connected.

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Then, there exists an idempotent $e_M \in \Gamma(E \times_S G^\vee, A)$ and $\iota$ induces an isomorphism

$$\text{pr}_1^*((e_M \cdot A) \otimes_A \mathcal{L}_{\tilde{E}}) \rightarrow \mathcal{M}$$

of $A$-modules on $E$ where $\text{pr}_1: E \times_S G^\vee \rightarrow E$ denotes the projection.

**Proof.** The assertion 1 is clear from the definition of the extension $0 \rightarrow G \rightarrow \tilde{E} \rightarrow E \rightarrow 0$. The assumption that the geometric fibers of $\tilde{E} \rightarrow S$ are connected implies that the restriction $\Gamma(\tilde{E}, A) \rightarrow \Gamma(S, A)$ is an isomorphism. Hence, the assertion 2 is a special case of Lemma 1.21.  

2 Ramification

In this section, $k$ denotes a perfect field of characteristic $p > 0$ and $X$ denotes a smooth separated scheme over $k$. Let $D$ be a divisor of $X$ with simple normal crossings and $U = X - D$ be the complement. Let $D_1, \ldots, D_h$ be the irreducible components of $D$ and $R = r_1 D_1 + \cdots + r_h D_h$ be a linear combination with rational coefficients $r_i \geq 1$ for every $i = 1, \ldots, h$. Let $M = m_1 D_1 + \cdots + m_h D_h$ be a linear combination with integral coefficients $m_i \geq 1$ such that $m_i r_i$ is an integer for every $i = 1, \ldots, h$. Let $Z \subset D$ be the union of irreducible components $D_i$ such that $r_i > 1$.

2.1 Construction of dilatations

We define oversimplicial schemes

$$T_{(R,M)}^{(R,M)} \hookrightarrow P_{(R,M)}^{(R,M)} \xleftarrow{\sim} U^{*+1} \times_k G_m^h$$

in Lemma 2.3 below and study their structures. The superscript $h$ denotes the number of irreducible components of $D$. If $R$ has integral coefficients, we also define

$$T_{(R)}^{(R)} \hookrightarrow P_{(R)}^{(R)} \xleftarrow{\sim} U^{*+1}$$

directly without introducing an auxiliary divisor $M$ in Lemma 2.4.

To define (2.1), we construct a commutative diagram of oversimplicial schemes

$$T_{(R,M)}^{(R,M)} \hookrightarrow D_{(R,M)}^{(R,M)} \xleftarrow{\sim} P_{(R,M)}^{(R,M)} \xleftarrow{\sim} U^{*+1} \times_k G_m^h$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$D_{(D,M)}^{(D,M)} \hookrightarrow P_{(D,M)}^{(D,M)} \xleftarrow{\sim} U^{*+1} \times_k G_m^h$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$D_{(D)}^{(D)} \hookrightarrow P_{(D)}^{(D)} \xleftarrow{\sim} U^{*+1}$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$X^{*+1} = P_{(D)}^{(D)}$$

in Lemmas 2.1, 2.2, 2.3 below. The horizontal arrows in the right column are open immersions of the complements of the images of the closed immersions on the left. The second
Lemma 2.2. 1. Since \( P_{n,j}^{(D)} \) is the complement in the blow-up of \( X^{n+1} \) at \( D \subset X \), we define \( P_{n,j}^{(D)} \) to be the intersection of \( P_{n,k}^{(D)} \) for \( j = 0, \ldots, n \).

\[ \tag{2.4} \text{pr}_j^* D \longrightarrow X^{n+1} \leftarrow X. \]

The dilatation \( P_{n,j}^{(D)} = X^{n+1} (\text{pr}_j^* D \times X) \) is an open subscheme of the blow-up of \( X^{n+1} \) at \( D \subset X \). We define \( P_n^{(D)} \) to be the intersection of \( P_{n,j}^{(D)} \) for \( j = 0, \ldots, n \).

Lemma 2.1. 1. The inverse image \( D_n^{(D)} \) of \( D \subset X \) by the canonical map \( P_n^{(D)} \rightarrow X^{n+1} \) is the same as that by the composition \( P_n^{(D)} \rightarrow X^{n+1} \rightarrow X \) with \( n+1 \) projections. The complement \( P_n^{(D)} - D_n^{(D)} \) is \( U^{n+1} \).

2. The schemes \( P_n^{(D)} \) form an oversimplicial scheme \( P_\bullet^{(D)} \). The morphisms \( P_n^{(D)} \rightarrow X^{n+1} \) define a morphism \( P_\bullet^{(D)} \rightarrow P_\bullet = X^{n+1} \) of oversimplicial schemes.

3. For an injection \([0, m] \hookrightarrow [0, n] \), the induced morphism \( P_n^{(D)} \rightarrow P_m^{(D)} \) is smooth.

4. The scheme \( P_n^{(D)} \) is smooth over \( k \). The inverse image \( D_n^{(D)} \) is a divisor of \( P_n^{(D)} \) with simple normal crossings.

Proof. 1. Since \( P_{n,j}^{(D)} \) is the complement in the blow-up of \( X^{n+1} \) at \( D \subset X \), it follows from the definition of \( P_n^{(D)} \) as their intersection.

2. It follows from the assertion 1 and the universality of dilatations.

3. Let \( f : [0, m] \rightarrow [0, n] \) be an injection. By Corollary 1.16 applied to the commutative diagram

\[
\begin{array}{ccc}
\text{pr}_f^*(D) & \longrightarrow & X^{n+1} \leftarrow X \\
\downarrow & & \downarrow \\
D \times_k X^m & \longrightarrow & X^{m+1} \leftarrow X,
\end{array}
\]

the induced map \( P_n^{(D)} \rightarrow P_m^{(D)} \) is smooth.

4. Taking \( m = 0 \) in 3., we see that the scheme \( P_n^{(D)} \) is smooth over \( k \) since \( P_0^{(D)} = X \) is smooth over \( k \). The assertion on \( D_n^{(D)} \) follows from this and the assertion 1.

Next, we define schemes in the second line in (2.3). Let \( n \geq 0 \) be an integer. For \( i = 1, \ldots, h \), let \( D_{n,i}^{(D)} \subset P_n^{(D)} \) be the pull-back of \( D_i \). By Lemma 2.1 the sum \( D_n^{(D)} = D_{n,1}^{(D)} + \cdots + D_{n,h}^{(D)} \) is a divisor with simple normal crossings. We define \( P_n^{(D,M)} \) by applying the construction studied in Lemma 1.17 to the linear combination \( m_1 D_{n,1}^{(D)} + \cdots + m_h D_{n,h}^{(D)} \).

For \( i = 1, \ldots, h \), we define a divisor \( D_{n,i}^{(D,M)} \subset P_n^{(D,M)} \) as the pull-back of the zero-section by the \( i \)-th projection as loc. cit.

Lemma 2.2. 1. The scheme \( P_n^{(D,M)} \) is smooth over \( k \) and flat over \( P_n^{(D)} \). The union \( D_n^{(D,M)} = D_{n,1}^{(D,M)} \cup \cdots \cup D_{n,h}^{(D,M)} \) is a divisor with simple normal crossings. The pull-back of \( D_{n,i}^{(D)} \) by \( P_n^{(D,M)} \rightarrow P_n^{(D)} \) is \( m_i D_{n,i}^{(D,M)} \) for \( i = 1, \ldots, h \). The complement \( P_n^{(D,M)} - D_n^{(D,M)} \) is canonically isomorphic to \( U^{n+1} \times_k G^h_m \).
2. The schemes $P_n^{(D,M)}$ form an oversimplicial scheme $P_n^{(D,M)}$. The morphisms $P_n^{(D,M)} \to P_n^{(D)}$ define a morphism $P_n^{(D,M)} \to P_n^{(D)}$ of oversimplicial schemes. If $M = D$, the morphism $P_n^{(D,M)} \to P_n^{(D)}$ is smooth.

3. The diagram

\[
P_n^{(D,M)} \longrightarrow P_n^{(D)} \\
\downarrow \quad \quad \quad \quad \downarrow \\
P_m^{(D,M)} \longrightarrow P_m^{(D)}
\]

is cartesian for every $[0, m] \to [0, n]$. In particular $P_n^{(D,M)} \to P_m^{(D,M)}$ is smooth if $[0, m] \to [0, n]$ is an injection.

Proof. 1. It follows from Lemma 1.17

2. It follows from assertion 1 and the functoriality (1.33). The assertion in the case $M = D$ follows from the last assertion in Lemma 2.1.3.

3. The cartesian diagram (2.5) follows from the cartesian diagram (1.34). The last assertion on an injection follows from the cartesian diagram (2.5) and Lemma 2.1.3. \qed

We set

\[
\tilde{X}^{(M)} = P_0^{(D,M)} \supset \tilde{D}^{(M)} = D_0^{(D,M)}
\]

and define a subdivisor $\tilde{Z}^{(M)} \subset \tilde{D}^{(M)}$ to be the union of $\tilde{D}_i^{(M)}$ such that $r_i > 1$. Define Cartier divisors of $\tilde{X}^{(M)}$ by

\[
\tilde{R}^{(M)} = r_1m_1\tilde{D}_1^{(M)} + \cdots + r_hm_h\tilde{D}_h^{(M)}, \quad \tilde{M}^{(M)} = m_1\tilde{D}_1^{(M)} + \cdots + m_h\tilde{D}_h^{(M)}.
\]

By Lemma 2.2 the scheme $\tilde{X}^{(M)} = P_0^{(D,M)}$ is smooth over $k$ and its divisor $\tilde{D}^{(M)} = D_0^{(D,M)}$ has simple normal crossings. Further, the morphism $\tilde{X}^{(M)} = P_0^{(D,M)} \to X = P_0^{(D)}$ is flat.

Let $n \geq 0$ be an integer. We regard $\tilde{X}^{(M)} = P_0^{(D,M)}$ as a closed subscheme of $P_n^{(D,M)}$ by Lemma 2.2.2. The linear combination $\sum_{i=1}^h m_i(r_i - 1)D_{n,i}^{(D,M)}$ has integral coefficients and defines a Cartier divisor of $P_n^{(D,M)}$. We consider closed immersions

\[
\sum_{i=1}^h m_i(r_i - 1)D_{n,i}^{(D,M)} \longrightarrow P_n^{(D,M)} \longleftrightarrow \tilde{X}^{(M)}.
\]

We define $P_n^{(R,M)}$ to be the dilatation defined by the immersions. The canonical morphism $P_n^{(R,M)} \to P_n^{(D,M)}$ is an isomorphism outside the inverse image of $\tilde{Z}^{(M)}$.

Lemma 2.3. 1. The inverse images $T_n^{(R,M)} \subset D_n^{(R,M)}$ of $\tilde{Z}^{(M)} \subset \tilde{D}^{(M)} \subset \tilde{X}^{(M)}$ by the composition $P_n^{(R,M)} \to P_n^{(D,M)} \to \tilde{X}^{(M)}$ with $n + 1$ projections do not depend on the projection. The complement $P_n^{(R,M)} - D_n^{(R,M)}$ is $\mathbb{G}_m^h$.

2. The schemes $P_n^{(R,M)}$ form an oversimplicial scheme $P_n^{(R,M)}$. The morphisms $P_n^{(R,M)} \to P_n^{(D,M)}$ define a morphism $P_n^{(R,M)} \to P_n^{(D,M)}$ of oversimplicial schemes.

3. For an injection $[0, m] \to [0, n]$, the induced morphism $P_n^{(R,M)} \to P_m^{(R,M)}$ is smooth.

4. The scheme $P_n^{(R,M)}$ is smooth over $k$. The inverse images $T_n^{(R,M)} \subset D_n^{(R,M)}$ are divisors with simple normal crossings.
Proof. The proof is similar to that of Lemma 2.3.

1. It follows from the construction.

2. It follows from the assertion 1 and the universality of dilatations.

3. Let \( f : [0, m] \to [0, n] \) be an injection. By Corollary 1.16, applied to the commutative diagram

\[
\begin{array}{ccc}
\sum_{i=1}^{m} m_i (r_i - 1)D_{n,i}^{(D,M)} & \longrightarrow & P_n^{(D,M)} \leftarrow \tilde{X}^{(M)} \\
\downarrow & & \downarrow \\
\sum_{i=1}^{h} m_i (r_i - 1)D_{m,i}^{(D,M)} & \longrightarrow & P_m^{(D,M)} \leftarrow \tilde{X}^{(M)}
\end{array}
\]

and by Lemma 2.2, the induced morphism \( P_n^{(R,M)} \to P_m^{(R,M)} \) is smooth.

4. Taking \( m = 0 \) in 3., we see that the scheme \( P_n^{(R,M)} \) is smooth over \( k \) and that the inverse images \( T_n^{(R,M)} \subset D_n^{(R,M)} \) are divisors with simple normal crossings since \( P_0^{(R,M)} = P_0^{(D,M)} = \tilde{X}^{(M)} \) is smooth over \( k \) and that \( \tilde{X}^{(M)} \subset D^{(M)} \subset \tilde{X}^{(M)} \) are divisors with simple normal crossings.

In 2.3, the multiplication of \( G^h_m \) on the second factors \( G^h_m \) in the right terms \( U^{r+1} \times_k \)

\( G^h_m \) on the top and the second lines are extended to \( G^h_m \)-actions on \( P_1^{(D,M)} \) and on \( P_2^{(R,M)} \).

They induce \( G^h_m \)-actions on \( D_1^{(D,M)} \) and on \( D_2^{(R,M)} \). The left horizontal arrows are closed immersions of Cartier divisors and the right horizontal arrows are the open immersions of the complements of the union. The right squares of the diagram (2.3) are cartesian. The pull-backs of the divisor \( D_n^{(D)} \subset P_n^{(D)} \) are \( M_n^{(D,M)} = \sum_{i=1}^{h} m_i D_{n,i}^{(D,M)} \subset P_n^{(D,M)} \) and \( M_n^{(R,M)} = \sum_{i=1}^{h} m_i D_{n,i}^{(R,M)} \subset P_n^{(R,M)} \) respectively.

If \( R = r_1 D_1 + \cdots + r_h D_h \) has integral coefficients, we define \( P_n^{(R)} \) directly as the dilatation \( (P_n^{(D)})^{(\sum_{i=1}^{h} (r_i - 1)D_{n,i}^{(D,M)})} \) without introducing an auxiliary divisor \( M \). The canonical map \( P_n^{(R)} \to P_n^{(D)} \) is an isomorphism outside the inverse image of \( Z = \bigcup_{r_i > 1} D_i \).

Lemma 2.4. Assume that \( R \) has integral coefficients.

1. The schemes \( P_n^{(R)} \) form an oversimplicial scheme \( P_n^{(D)} \). The morphisms \( P_n^{(R)} \to P_n^{(D)} \) define a morphism \( P_n^{(R)} \to P_n^{(D)} \) of oversimplicial schemes.

2. For an injection \([0, m] \to [0, n]\), the morphism \( P_n^{(R)} \to P_m^{(R)} \) is smooth.

3. The scheme \( P_n^{(R)} \) is smooth over \( k \). The inverse images \( T_n^{(R)} \subset D_n^{(R)} \) of \( Z \subset D \subset X \) by the composition \( P_n^{(R)} \to P_n^{(D)} \to X \) with \( n+1 \) projections are the same and are divisors with simple normal crossings. The complement \( P_n^{(R)} \) is \( U^{n+1} \).

4. For a divisor \( M = m_1 D_1 + \cdots m_h D_h \) with integral coefficients \( m_i \geq 1 \), the diagram

\[
\begin{array}{ccc}
P^{(D,M)}_n & \longrightarrow & P^{(R,M)}_n \\
\downarrow & & \downarrow \\
P^{(D)}_n & \longrightarrow & P^{(R)}_n
\end{array}
\]

is cartesian and the vertical arrows are flat. If \( M = D \), the vertical arrows are smooth.

Proof. 1.-3. Similar to Lemma 2.3.

4. For integers \( n \geq 0 \), the diagram

\[
\begin{array}{ccc}
\sum_{i=1}^{m} m_i (r_i - 1)D_{n,i}^{(D,M)} & \longrightarrow & P_n^{(D,M)} \leftarrow \tilde{X}^{(M)} \\
\downarrow & & \downarrow \\
\sum_{i=1}^{m} (r_i - 1)D_{n,i}^{(D)} & \longrightarrow & P_n^{(D)} \leftarrow X
\end{array}
\]

is cartesian.
Lemma 2.6. \[ \text{Let } R \text{ be the canonical map } P \text{ section induced by the canonical map } \tilde{D} \text{ pull-backs of alternative description. We regard Lemma 2.5.}\]

1. If \( M = D \), the morphism \( \tilde{X}^{(M)} \to X \) is smooth. 

We show that the oversimplicial schemes in the diagram (2.3) are multiplicative.

**Lemma 2.5.** 1. The oversimplicial schemes \( P_\bullet, P^{(D)}_\bullet, P^{(D,M)}_\bullet \) and \( P^{(R,M)}_\bullet \) are multiplicative. If \( R \) has integral coefficients, \( P^{(R)}_\bullet \) is also multiplicative.

2. The oversimplicial schemes \( D^{(D)}_\bullet, D^{(D,M)}_\bullet, D^{(R,M)}_\bullet \) and \( T^{(R,M)}_\bullet \) are strictly multiplicative. If \( R \) has integral coefficients, \( D^{(R)}_\bullet \) and \( T^{(R)}_\bullet \) are also strictly multiplicative.

**Proof.** 1. Since \( P_\bullet = X^{\bullet + 1} \) is multiplicative, the oversimplicial scheme \( P^{(D)}_\bullet \) is also multiplicative by Corollary 1.16.2. By Lemma 2.2.3, \( P^{(D,M)}_\bullet \) is multiplicative. Further by Corollary 1.16.2, \( P^{(R,M)}_\bullet \) and \( P^{(R)}_\bullet \) in the case \( R \) has integral coefficients are also multiplicative.

2. Since \( P^{(D)}_\bullet, P^{(D,M)}_\bullet, P^{(R,M)}_\bullet \) are multiplicative and since the diagrams

\[
\begin{array}{cccccc}
D^{(D)}_m & \longrightarrow & P^{(D)}_m & \longrightarrow & P^{(D,M)}_m & \longrightarrow & P^{(R,M)}_m \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
D^{(D)}_n & \longrightarrow & P^{(D,M)}_n & \longrightarrow & P^{(R,M)}_n \\
\end{array}
\]

are cartesian for \([0,n] \to [0,m] \), the oversimplicial schemes \( D^{(D)}_\bullet, D^{(D,M)}_\bullet, D^{(R,M)}_\bullet \) and \( T^{(R,M)}_\bullet \) are multiplicative. Since the morphisms \( D^{(D)}_m \to D^{(D)}_0 = D, D^{(D,M)}_n \to D^{(D,M)}_0 = \tilde{D}^{(M)}, D^{(R,M)}_n \to D^{(R,M)}_0 = \tilde{D}^{(M)} \) and \( T^{(R,M)}_n \to T^{(R,M)}_0 = \tilde{Z}^{(M)} \) are independent of \([0] \to [0,n] \), the multiplicative oversimplicial schemes \( D^{(D)}_\bullet, D^{(D,M)}_\bullet, D^{(R,M)}_\bullet \) and \( T^{(R,M)}_\bullet \) are strictly multiplicative. Similarly, \( D^{(R)}_\bullet \) and \( T^{(R)}_\bullet \) are strictly multiplicative in the case \( R \) has integral coefficients. 

\[ \square \]

### 2.2 Properties of the dilatations

In order to study further properties of \( P^{(R,M)}_\bullet \) including the functoriality, we give an alternative description. We regard \( \tilde{X}^{(M)} \) as a closed subscheme of \( \tilde{X}^{(M)} \times_k X^n \) by the section induced by the canonical map \( \tilde{X}^{(M)} \to X \) and the identity of \( \tilde{X}^{(M)} \). The pull-back \( \tilde{R}^{(M)} = \sum_{i=1}^h m_i r_i \tilde{D}_i^{(M)} \) of \( R \) by \( \tilde{X}^{(M)} \to X \) is a Cartier divisor with integral coefficients.

**Lemma 2.6.** 1. Let the notation be as above. We regard \( \tilde{X}^{(M)} \) as a closed subscheme of \( \tilde{X}^{(M)} \times_k X^n \) embedded as the graph of the product \( \tilde{X}^{(M)} \to X^n \) of the canonical map and consider the closed immersions

\[
(2.9) \quad \text{pr}_0^* \tilde{R}^{(M)} \longrightarrow \tilde{X}^{(M)} \times_k X^n \longrightarrow \tilde{X}^{(M)}.
\]

Let \( \tilde{D}^{(R)} \to \tilde{X}^{(M)} \times_k X^n \) denote the dilatation defined by the immersions (2.9). Then the canonical map \( P^{(R,M)}_n \to \tilde{P}^{(R)}_0 \) \( \times_k X^n = \tilde{X}^{(M)} \times_k X^n \) is uniquely lifted to an open immersion \( P^{(R,M)}_n \to \tilde{P}^{(R)}_0 \). The image is the complement of the proper transforms of the pull-backs of \( D \) by the compositions \( \tilde{X}^{(M)} \times_k X^n \to X^n \to X \) with the \( n \) projections.

2. Assume further that \( R \) has integral coefficients and consider the closed immersions

\[
(2.10) \quad \text{pr}_0^* R \longrightarrow X^{n+1} \longrightarrow X
\]
Let $P \to X^{n+1}$ denote the dilatation defined by the immersions. Then the canonical map $P_{n}^{(R)} \to X^{n+1}$ is uniquely lifted to an open immersion $P_{n}^{(R)} \to P$. The image is the complement of the proper transforms of the pull-backs of $D$ by the $n$ projections $X^{n+1} \to X$ different from the $0$-th one.

Proof. 1. First, we consider the case where $R = D$. Let $P_{n,0}^{(D)} \to X^{n+1}$ denote the dilatation defined by the immersions (2.4) for $j = 0$. Then, since $\tilde{X}^{(M)} \to X$ is flat, we have a cartesian diagram

$$
\begin{array}{c}
P_{n,0}^{(D)} \leftarrow \tilde{P}^{(D)} \\
\downarrow \quad \downarrow \\
X^{n+1} \leftarrow \tilde{X}^{(M)} \times_k X^n.
\end{array}
$$

(2.11)

The open subscheme $P_{n}^{(D)} \subset P_{n,0}^{(D)}$ is obtained by removing the proper transforms of the pull-backs of $D$ by the $n$ projections $X^{n+1} \to X$ different from the $0$-th projection. Since $P_{n}^{(D,M)}$ is defined by the cartesian diagram (2.11), with $P_{n,0}^{(D)}$ replaced by $P_{n}^{(D)}$, the assertion in this case $R = D$ follows.

We show the general case. Let $p: \tilde{P}^{(D)} \to \tilde{X}^{(M)}$ denote the projection and regard $\tilde{X}^{(M)}$ as a closed subscheme of $\tilde{P}^{(D)}$ by the section lifting $\tilde{X}^{(M)} \to \tilde{X}^{(M)} \times_k X^n$. Then, the dilatation $\tilde{P}^{(R)}$ is canonically identified with the dilatation defined by the immersions

$$
p^*(\tilde{R}^{(M)} \cdot \tilde{M}^{(M)}) \longrightarrow \tilde{P}^{(D)} \leftarrow \tilde{X}^{(M)}.
$$

Thus the assertion for $R$ follows from that for $D$.

2. Similar to and easier than that of 1. □

We study the functoriality of the diagram (2.3). Let $f: X' \to X$ be a morphism of smooth schemes over $k$ such that the inverse image $U' = f^{-1}(U)$ is the complement of a divisor $D'$ of $X'$ with simple normal crossings. Let $D'_1, \ldots, D'_{h'}$ be the irreducible components of $D'$ and set $f^*D_i = \sum_{j=1}^{h'} e_{ij}D'_j$ for $i = 1, \ldots, h$.

We set $R' = f^*(R) = \sum_{i=1}^{h} r_i \sum_{j=1}^{h'} e_{ij}D'_j = \sum_{j=1}^{h'} r'_j D'_j$. Let $M' = \sum_{j=1}^{h'} m'_j D'_j$ be a divisor with integral coefficients $m'_j \geq 1$. Then, the diagram (2.3) is defined for $X', D', R'$ and $M'$. We denote the schemes constructed for $X', D', R'$ and $M'$ by putting $'$ as $P_{n}^{(R',M'), T_{n}^{(R,M')})}$ etc.

We assume that $l_{ij} = e_{ij}m'_j/m_i$ is an integer for every $i = 1, \ldots, h$ and $j = 1, \ldots, h'$. Then, by the cartesian diagram (1.34), a canonical morphism $\tilde{X}^{(M')} \to \tilde{X}^{(M)}$ is defined. Since $m'_j r'_j = \sum_{i=1}^{h} l_{ij} m_i r_i$ is an integer for every $j = 1, \ldots, h'$, further by Lemma 2.6 and the functoriality of dilatation, a canonical morphism

$$
P_{n}^{(R',M')} \to P_{n}^{(R,M)}
$$

(2.12)

of oversimplicial schemes is defined. If $R$ and hence $R'$ have integral coefficients, a canonical morphism $P_{n}^{(R')} \to P_{n}^{(R)}$ is defined similarly.

We study the schemes in (2.3) more in detail. We compute some normal bundles. For a line bundle $L$ on a scheme $S$, let $L^\times$ denote the complement $L \cdot S$ of the 0-section.
Lemma 2.7. For the normal bundles, we have canonical isomorphisms

\begin{align}
(2.13) \quad & T_X P_n \to TX^{n+1}/\Delta TX, \\
(2.14) \quad & T_X P_n^{(D)} \to (TX^{n+1}/\Delta TX)(-D), \\
(2.15) \quad & T_{\tilde{X}(M)} P_n^{(D,M)} \to \left((TX^{n+1}/\Delta TX) \times_X \tilde{X}(M)\right)(-\tilde{M}(M)), \\
(2.16) \quad & T_{\tilde{X}(M)} P_n^{(R,M)} \to \left((TX^{n+1}/\Delta TX) \times_X \tilde{X}(M)\right)(-\tilde{R}(M)).
\end{align}

If $R$ has integral coefficients, we have a canonical isomorphism

\begin{equation}
(2.17) \quad T_X P_n^{(R)} \to (TX^{n+1}/\Delta TX)(-R).
\end{equation}

Proof. The diagonal map $X \to X^{n+1}$ defines an exact sequence

$$0 \to N_{X/X^{n+1}} \to \Omega_{X^{n+1}/k}^{1} \otimes_{O_{X^{n+1}}} O_X \to \Omega_{X/k}^{1} \to 0.$$ 

Since $\Omega_{X^{n+1}/k}^{1} \otimes_{O_{X^{n+1}}} O_X = \Omega_{X/k}^{1}$, we obtain an isomorphism \((2.13)\).

In the notation of proof of Lemma 2.6, the scheme $P_n^{(D)}$ is an open subscheme of the dilatation $P_{n,0} \to X^{n+1}$. Hence, the isomorphism \((1.26)\) gives an isomorphism $T_X P_n^{(D)} \to T_X P_n \otimes (TDX)^{\otimes -1}$. Thus \((2.14)\) follows from \((2.13)\).

The diagram

$$\begin{array}{ccc}
\tilde{X}(M) & \longrightarrow & P_n^{(D,M)} \\
\downarrow & & \downarrow \\
X & \longrightarrow & P_n^{(D)}
\end{array}$$

is cartesian and the vertical arrows are flat by Lemma 2.2. Since the pull-back of $D$ to $\tilde{X}(M)$ is $\tilde{M}(M)$ by Lemma 2.2, \((2.14)\) implies \((2.15)\).

In the notation of proof of Lemma 2.6, the scheme $P_n^{(R,M)}$ is an open subscheme of the dilatation $P_{n,0}^{(R)} \to X^{n+1}$. Hence, the isomorphism \((1.26)\) gives an isomorphism $T_X P_n^{(R,M)} \to (T_{\tilde{X}(M)}(\tilde{X}(M) \times_k X^n)) \otimes (T_{\tilde{R}(M)}\tilde{X}(M)^{\otimes -1})$. Since $\tilde{X}(M) \to X$ is flat, the normal bundle $T_{\tilde{X}(M)}(\tilde{X}(M) \times_k X^n)$ is the pull-back of $T_X X^{n+1} = T_X P_n$ and is canonically isomorphic to $TX^{n+1}/\Delta TX \times_X \tilde{X}(M)$. Thus \((2.16)\) follows. The isomorphism \((2.17)\) is proved similarly and more easily.

For a subset $I \subset \{1, \ldots, h\}$, we set $D_I = \nu_{i \in I}D_i$ and $D^o_I = D_I - \nu_{i \notin I}D_{\nu(i)}$. Similarly, set $\tilde{D}^{(M)}_I = \nu_{i \in I}\tilde{D}^{(M)}_i$ and $\tilde{D}^{(M)\circ}_I = \tilde{D}^{(M)}_I - \nu_{i \notin I}\tilde{D}^{(M)\circ}_{\nu(i)}$.

Lemma 2.8. 1. The normal bundles $T_{\tilde{D}^{(M)}_i}\tilde{X}(M)$ is a trivial line bundle over $\tilde{D}^{(M)}_i$. The action of $G^h_m$ is the diagonal action of that on the base $\tilde{D}^{(M)}_i$ and the multiplication by the $i$-th component.

2. For a subset $I \subset \{1, \ldots, h\}$, the open subscheme $\tilde{D}^{(M)\circ}_I \subset \tilde{D}^{(M)}_I$ is a $G^h_m$-torsor over $D^o_I$ canonically isomorphic to $\prod_{i \in I}(TD_iX^i \times_{D_i} D^o_i) \times_{D^o_I} \prod_{i \notin I} G_m D^o_{\nu(i)}$. The $G^h_m$-action on the latter is by multiplication componentwise.

3. We have a canonical isomorphism

\begin{equation}
(2.18) \quad D_n^{(D,M)} \to \tilde{D}^{(M)}_I \times_D D_n^{(D)}
\end{equation}

over $\tilde{D}^{(M)}_I$ compatible with the $G^h_m$-action defined on the base $\tilde{D}^{(M)}_I$ on the right hand side.
4. We have a canonical isomorphism

\[
T_n^{(R,M)} \to ((TX^{n+1}/\Delta TX) \times_X \wZ^{(M)}) (-\wR^{(M)})
\]

over \(\wZ^{(M)}\). The action of \(G_m^h\) is compatible with the diagonal action on the base scheme \(\wZ^{(M)}\) and multiplication of the product of \(m, r_i\)-th powers of \(i\)-th components on the fiber.

If \(R\) has integral coefficients, we have a canonical isomorphism

\[
T_n^{(R)} \to ((TX^{n+1}/\Delta TX) (-R) \times_X Z).
\]

**Proof.** 1. and 2. Clear from the construction of \(\wX^{(M)}\).

3. It follows from the cartesian diagram (2.5).

4. Similarly as in the proof of (2.16), the isomorphism (1.25) gives a canonical isomorphism

\[
T_n^{(R,M)} \to (T_{\wX^{(M)}}(\wX^{(M)} \times X^n)) \times_{\wX^{(M)}} \wZ^{(M)} \otimes (T_{\wR^{(M)}} \wX^{(M)})^{\otimes -1}.
\]

Since the normal bundle \(T_{\wX^{(M)}}(\wX^{(M)} \times X^n)\) is \((TX^{n+1}/\Delta TX) \times_X \wX^{(M)}\), we obtain an isomorphism (2.19). The compatibility on the \(G_m^h\)-actions follows from the assertion 1.

If \(R\) has integral coefficients, an isomorphism (2.20) is defined similarly and more easily.

**Corollary 2.9.** The strictly multiplicative smooth oversimplicial schemes \(T_{\bullet}^{(R,M)}\) is an oversimplicial scheme associated to the vector bundle

\[
(TX \times_X \wZ^{(M)}) (-\wR^{(M)})
\]

over \(\wZ^{(M)}\).

If \(R\) has integral coefficients, \(T_{\bullet}^{(R)}\) is an oversimplicial scheme associated to the vector bundle \(TX (-R) \times_X Z\) over \(Z\).

**Proof.** By Proposition 1.1 and Lemma 2.8, the strictly multiplicative oversimplicial scheme \(T_{\bullet}^{(R,M)}\) is associated to a group scheme \(T_1^{(R,M)} = ((TX^2/\Delta TX) \times_X \wZ^{(M)}) (-\wR^{(M)}) = (TX \times_X \wZ^{(M)}) (-\wR^{(M)})\) over \(T_0^{(R,M)} = \wZ^{(M)}\). It is easily checked that the group structure of \(T_1^{(R,M)}\) is defined by the addition on the vector bundle \((TX \times_X \wZ^{(M)}) (-\wR^{(M)})\).

In the case \(R\) has integral coefficients, the assertion on \(T_{\bullet}^{(R)}\) is proved in the same way.

The schemes \(P_{\bullet}^{(D,M)}\) and \(P_{\bullet}^{(R,M)}\) depend on \(M\) as follows.

**Lemma 2.10.** 1. If \(M = D\), the scheme \(P_{n}^{(D,M)}\) is canonically isomorphic to the \(G_m\)-torsor \(\prod_{l=1}^{h} L(D_{n,i})\) over \(P_{n}^{(D)}\).

2. Let \(m'_i = l_im_i \geq 1\) be integers for \(i = 1, \ldots, h\) and set \(M = m_1D_1 + \cdots + m_hD_h\) and \(M' = m'_1D_1 + \cdots + m'_hD_h\). Then, the diagram

\[
\begin{array}{ccc}
P_{n}^{(R,M')} & \longrightarrow & A^h \\
\downarrow & & \downarrow (t_i) \rightarrow (t'_i) \\
P_{n}^{(R,M)} & \longrightarrow & A^h
\end{array}
\]

(2.22)
is cartesian and the vertical arrows are finite flat. The vertical arrows are compatible with the $G^h_m$-actions and the morphism $G^h_m \to G^h_m$ sending $(t_i)$ to $(t_i^l)$.

If $R = D$, the isomorphisms (2.18) for $M$ and $M'$ make a commutative diagram

$$D_n^{(D,M')} \longrightarrow \tilde{D}^{(M')} \times_D D_n^{(D)}$$

(2.23)

$$D_n^{(D,M)} \longrightarrow \tilde{D}^{(M)} \times_D D_n^{(D)}.$$  

The isomorphisms (2.19) for $M$ and $M'$ make a commutative diagram

$$T_n^{(R,M')} \longrightarrow ((TX^{n+1}/\Delta TX) \times_X \tilde{Z}^{(M')})(-\tilde{R}^{(M')})$$

(2.24)

$$T_n^{(R,M)} \longrightarrow ((TX^{n+1}/\Delta TX) \times_X \tilde{Z}^{(M)})(-\tilde{R}^{(M)}).$$

Proof. 1. It follows from the last assertion in Lemma 1.17.

2. For $n = 0$ and $R = D$, the cartesian diagram (2.22) follows from the construction of $X^{(M)} = P_0^{(D,M)}$ and $\tilde{X}^{(M')} = P_0^{(D,M')}$. The rest follows from this and Lemma 2.6.

2.3 Ramification of Galois covering and an additive structure

We keep the notation fixed at the beginning of the section. Let $G$ be a finite group and $V \to U = X \rightrightarrows D$ be a $G$-torsor. The quotients $V^{n+1}/\Delta G$ of the $(n+1)$-fold fibered products $V^{n+1}$ over $k$ by the diagonal action of $G$ for integers $n \geq 0$ define a finite étale morphism $V^{\bullet+1}/\Delta G \to U^{\bullet+1}$ of oversimplicial schemes.

Lemma 2.11. The morphism $V^{\bullet+1}/\Delta G \to U^{\bullet+1}$ is a multiplicative morphism of oversimplicial schemes.

Proof. It suffices to show that an additive cocartesian diagram (1.1) defines a cartesian diagram

$$V^{m+1}/\Delta G \times_{U^{m+1}} U^{n+1} \longleftarrow V^{n+1}/\Delta G$$

$$U^{n+1} \longleftarrow V^{l+1}/\Delta G \times_{U^{l+1}} U^{n+1}.$$

The fibered product $V^{m+1}/\Delta G \times_{U^{m+1}} U^{n+1}$ is canonically identified with the quotient of $V^{n+1}$ by $\Delta G \times G^{l+1}$. Similarly $V^{l+1}/\Delta G \times_{U^{l+1}} U^{n+1}$ is canonically identified with the quotient of $V^{n+1}$ by $G^{m+1} \times \Delta G$. Hence it follows from $(\Delta G \times G^{l+1}) \cap (G^{m+1} \times \Delta G) = \Delta G$ in $G^{n+1}$.

We consider the morphisms

$$P_{(R,M)}^{\bullet} \leftarrow U^{\bullet+1} \times G^h_m \leftarrow V^{\bullet+1}/\Delta G \times G^h_m$$

of oversimplicial schemes over $k$. For an integer $n \geq 0$, let $Q_n^{(R,M)}$ denote the normalization of $P_n^{(R,M)}$ in the finite étale covering $V^{n+1}/\Delta G \times G^h_m \to U^{n+1} \times G^h_m$. Then, by Lemma
1.8.1, the schemes $Q_n^{(R,M)}$ for integers $n \geq 0$ form an oversimplicial scheme $Q^{(R,M)}_\bullet$ over $k$ and we obtain a cartesian diagram

$$
Q^{(R,M)}_\bullet \leftarrow V^{*+1}/\Delta G \times G^h_m
$$

(2.25)

$$
P^{(R,M)}_\bullet \leftarrow U^{*+1} \times G^h_m
$$

of oversimplicial schemes. For $n \geq 0$, the canonical map $Q^{(R,M)}_0 = \tilde{X}(M) \to Q^{(R,M)}_n$ is a lifting of the lifting $P^{(R,M)}_0 = \tilde{X}(M) \to P^{(R,M)}_n$ of the diagonal map $X \to X^{n+1}$.

**Definition 2.12.** Let $V \to U = X - D$ be a $G$-torsor for a finite group $G$. Let $R = r_1D_1 + \cdots + r_hD_h$ and $M = m_1D_1 + \cdots + m_hD_h$ be linear combinations with rational coefficients $r_i \geq 1$ and integral coefficients $m_i \geq 1$ such that $m_i r_i$ is an integer for every irreducible components $D_1, \ldots, D_h$ of $D$. We say that the ramification of $V$ over $U$ along $D$ (resp. at a point $x$ of $D$) is bounded by $R+$, if the finite morphism

$$
Q^{(R,M)}_1 \to P^{(R,M)}_1
$$

is étale on a neighborhood of the image of $Q^{(R,M)}_0 = \tilde{X}(M) \to Q^{(R,M)}_1$ (resp. of the image by $Q^{(R,M)}_0 = \tilde{X}(M) \to Q^{(R,M)}_1$ of the inverse image of $x$ by $\tilde{X}(M) \to X$).

We show that the condition in Definition 2.12 is independent of the choice of a divisor $M$.

**Lemma 2.13.** Let $M' = m'_1D_1 + \cdots + m'_hD_h$ be linear combinations with integral coefficients $m_i \geq 1$ such that $m'_i = l_im_i$ is divisible by $m_i$ for every $1, \ldots, h$.

1. The cartesian diagram

$$
U^{*+1} \times G^h_m \leftarrow V^{*+1}/\Delta G \times G^h_m
$$

$$
U^{*+1} \times G^h_m \leftarrow V^{*+1}/\Delta G \times G^h_m
$$

of oversimplicial schemes over $k$ where the vertical arrows are defined by the map on the second factor $G^h_m$ sending $(t_i)$ to $(t^h_i)$ is extended to a commutative diagram

$$
P^{(R,M')}_\bullet \leftarrow Q^{(R,M')}_\bullet
$$

(2.26)

$$
P^{(R,M)}_\bullet \leftarrow Q^{(R,M)}_\bullet
$$

of oversimplicial schemes over $k$.

2. Let $x$ be a point of $X(M) = Q^{(R,M)}_0$ and $x'$ be a point of $X(M') = Q^{(R,M')}_0$ above $x$ by the map defined in 1. Then, $Q^{(R,M)}_1 \to P^{(R,M)}_1$ is étale on the image of $x$ by $Q^{(R,M)}_0 \to Q^{(R,M)}_1$ if and only if $Q^{(R,M')}_1 \to P^{(R,M')}_1$ is étale on the image of $x'$ by $Q^{(R,M')}_0 \to Q^{(R,M')}_1$.

**Proof.** 1. It follows from the functoriality of normalizations, Lemma 1.6.1.
2. The commutative diagram (2.26) defines a commutative diagram

\[
\begin{align*}
\tilde{X}(M') &= P_0^{(R,M')} &\longrightarrow & P_1^{(R,M')} \\
\downarrow & &\downarrow & \\
\tilde{X}(M) &= P_0^{(R,M)} &\longrightarrow & P_1^{(R,M)}
\end{align*}
\]

and its lifting

\[
\begin{align*}
\tilde{X}(M') &= Q_0^{(R,M')} &\longrightarrow & Q_1^{(R,M')} \\
\downarrow & &\downarrow & \\
\tilde{X}(M) &= Q_0^{(R,M)} &\longrightarrow & Q_1^{(R,M)}
\end{align*}
\]

If \(Q_1^{(R,M)} \to P_1^{(R,M)}\) is étale on the image of \(x\), then \(Q_1^{(R,M')} \to P_1^{(R,M')}\) is étale on the image of \(x'\) by Lemma 1.8.2. Since the diagrams above satisfy the conditions (1)–(3) of Lemma 1.9 if \(Q_1^{(R,M')} \to P_1^{(R,M')}\) is étale on the image of \(x'\), then \(Q_1^{(R,M)} \to P_1^{(R,M)}\) is étale on the image of \(x\) by Lemma 1.9.

**Theorem 2.14.** Let \(X\) be a smooth separated scheme over a perfect field \(k\) and \(D\) be a divisor with normal crossings. Let \(V \to U\) be a \(G\)-torsor on the complement \(U = X \setminus D\) for a finite group \(G\). Let \(R = r_1D_1 + \cdots + r_hD_h\) and \(M = m_1D_1 + \cdots + m_hD_h\) be linear combinations with rational coefficients \(r_i \geq 1\) and integral coefficients \(m_i \geq 1\) such that \(m_ir_i\) is an integer for every irreducible components \(D_1, \ldots, D_h\) of \(D\). We consider the commutative diagram

\[
\begin{align*}
Q_\bullet^{(R,M)} &\longleftarrow V^{\bullet+1}/\Delta G \times G^h_m \\
\downarrow & &\downarrow \\
P_\bullet^{(R,M)} &\longleftarrow U^{\bullet+1} \times G^h_m
\end{align*}
\]

of oversimplicial schemes over \(k\).

Assume that the ramification of \(V\) over \(U\) along \(D\) is bounded by \(R^+\). For each integer \(n \geq 0\), let \(W_n^{(R,M)}\) be the largest open subscheme of \(Q_0^{(R,M)}\) étale over \(P_0^{(R,M)}\). Then \(W_n^{(R,M)}\) form an oversimplicial open subscheme \(W_\bullet^{(R,M)}\) of \(Q_\bullet^{(R,M)}\). Further, the morphism \(W_\bullet^{(R,M)} \to P_\bullet^{(R,M)}\) is multiplicative and the oversimplicial scheme \(W_\bullet^{(R,M)}\) is multiplicative.

**Proof.** We apply Lemma 1.8.2 to the étale and multiplicative morphism of oversimplicial schemes \(V^{\bullet+1}/\Delta G \times G^h_m \to P_\bullet^{(R,M)}\). By Lemma 2.3, the assumption (1) in Lemma 1.8.2 that \(P_n^{(R,M)} \to P_m^{(R,M)}\) are smooth for injections \([0,m] \to [0,n]\) is satisfied.

Since \(V/G \to U\) is an isomorphism, the morphism \(Q_0^{(R,M)} \to P_0^{(R,M)} = \tilde{X}(M)\) is an isomorphism. The assumption that the ramification is bounded by \(R^+\) means that the assumption (2) in Lemma 1.8.2 that the morphism \(V/G \times G^h_m \to (V \times V)/\Delta G \times G^h_m\) is extended to \(\tilde{X}(M) = W_0^{(R,M)} \to W_1^{(R,M)}\) is also satisfied.

Thus by Lemma 1.8.2, \(W_n^{(R,M)}\) form an oversimplicial open subscheme \(W_\bullet^{(R,M)}\) of \(Q_\bullet^{(R,M)}\) and the morphism \(W_\bullet^{(R,M)} \to P_\bullet^{(R,M)}\) is multiplicative. Since \(P_\bullet^{(R,M)}\) is multiplicative by Lemma 2.5, the oversimplicial scheme \(W_\bullet^{(R,M)}\) is multiplicative by Lemma 1.2 (2)⇒(1).
Corollary 2.15. Define an oversimplicial scheme $E^{(R,M)}_*$ by the cartesian diagram

$$
\begin{array}{ccc}
E^{(R,M)}_* & \longrightarrow & W^{(R,M)}_* \\
\downarrow & & \downarrow \\
T^{(R,M)}_* & \longrightarrow & P^{(R,M)}_*. 
\end{array}
$$

Then, the oversimplicial scheme $E^{(R,M)}_*$ is strictly multiplicative and the left vertical arrow $E^{(R,M)}_* \to T^{(R,M)}_*$ of oversimplicial schemes is étale and multiplicative.

The multiplicative oversimplicial scheme $E^{(R,M)}_*$ is associated to a smooth group scheme $E_1^{(R,M)}$ over $E_0^{(R,M)} = \tilde{Z}^{(M)}$. Further, the multiplicative morphism $E^{(R,M)}_* \to T^{(R,M)}_*$ is associated to an étale morphism

$$
E_1^{(R,M)} = E^{(R,M)} \to T_1^{(R,M)} = T^{(R,M)}
$$

of smooth group schemes over $\tilde{Z}^{(M)}$.

Proof. Since the morphism $W^{(R,M)}_* \to P^{(R,M)}_*$ is an étale multiplicative morphism of multiplicative oversimplicial schemes by Theorem 2.14, its base change $E^{(R,M)}_* \to T^{(R,M)}_*$ is also an étale multiplicative morphism of multiplicative oversimplicial smooth schemes. Since $T^{(R,M)}_*$ is strictly multiplicative by Lemma 2.5.2, $E^{(R,M)}_*$ is also strictly multiplicative. Hence by Proposition 2.11, the oversimplicial scheme $E^{(R,M)}_*$ is associated to a smooth group scheme $E_1^{(R,M)}$ over $E_0^{(R,M)} = \tilde{Z}^{(M)}$ and the morphism $E^{(R,M)}_* \to T^{(R,M)}_*$ is associated to an étale morphism $E_1^{(R,M)} \to T_1^{(R,M)}$ of group schemes over $\tilde{Z}^{(M)}$.

Proposition 2.16. 1. There exists a unique open subgroup scheme

$$E^{(R,M),0} \subset E^{(R,M)}$$

such that for every point of $x \in \tilde{Z}^{(M)}$, the fiber $E^{(R,M),0} \times_{\tilde{Z}^{(M)}} x$ is the connected component of $E^{(R,M)} \times_{\tilde{Z}^{(M)}} x$ containing the unit section. The group scheme $E^{(R,M),0}$ over $\tilde{Z}^{(M)}$ is commutative and killed by $p$. The restriction of (2.27)

$$
E^{(R,M),0} \to T^{(R,M)} = (TX \times_X \tilde{Z}^{(M)})(-\tilde{R}^{(M)})
$$

is an étale and surjective morphism of commutative group schemes. The kernel of (2.28) is an étale commutative group scheme annihilated by $p$.

2. Let $M'$ be as in Lemma 2.13. Then the diagram

$$
\begin{array}{ccc}
E^{(R,M')0} & \longrightarrow & T^{(R,M')} \\
\downarrow & & \downarrow \\
E^{(R,M)0} & \longrightarrow & T^{(R,M)} 
\end{array}
$$

is cartesian. The morphism $E^{(R,M),0} \to T^{(R,M)}$ is finite if and only if $E^{(R,M'),0} \to T^{(R,M')}$ is finite.

Proof. 1. Since $E^{(R,M)}_*$ is a smooth group scheme, the open subgroup scheme $E^{(R,M),0}$ exists by [7, Theorem 3.10 (i) $\Rightarrow$ (iv)]. By Lemma 2.18, $E^{(R,M),0}$ is commutative and killed by $p$. Consequently, the kernel $\text{Ker}(E^{(R,M),0} \to T^{(R,M)})$ of an étale morphism is an
étale commutative group scheme annihilated by \( p \). Since \( E^{(R,M)0} \) is an open subscheme of \( E^{(R,M)} \), the morphism (2.28) is étale. Since the fibers of the vector bundle \( T^{(R,M)} \) are connected, it is surjective. The description of \( T^{(R,M)} \) is given in (2.19).

2. The commutative diagram (2.26) defines a commutative diagram

\[
\begin{array}{ccc}
E^{(R,M')} & \longrightarrow & \overline{Z}^{(M')}
\end{array}
\]

(2.30)

\[
\begin{array}{ccc}
E^{(R,M)} & \longrightarrow & \overline{Z}^{(M)}
\end{array}
\]

without 0. The right square is cartesian by Lemma 2.10.2 and hence the left vertical arrow induces an open immersion \( E^{(R,M')} \to E^{(R,M)} \times D^{(M')} \). Thus, the left square in (2.29) is also cartesian.

Since (2.29) is cartesian, the finiteness of the morphism \( E^{(R,M)0} \to T^{(R,M)} \) implies the finiteness of \( E^{(R,M')}0 \to T^{(R,M')} \). Since \( \overline{Z}^{(M')} \to \overline{Z}^{(M)} \) is finite surjective, the converse holds.

If \( R \) has integral coefficients, we can work with \( P^{(R)} \) without introducing an auxiliary divisor \( M \), which is much easier. In fact, in this case we may take \( M = D \) and then the vertical arrows in the cartesian diagram (2.8) are smooth.

**Definition 2.17.** We say that the ramification of \( V \) over \( U \) is non-degenerate along \( D \) at multiplicity \( R \) if the étale surjective morphism (2.28) is finite.

Proposition 2.16.2 implies that Definition 2.17 is independent of the choice of \( M \) such that \( m_i r_i \) is an integer for every \( i = 1, \ldots, h \). If \( D \) is smooth and if \( r = r_1 > 1 \), the right vertical arrow \( \overline{Z}^{(M')} \to \overline{Z}^{(M)} \) of (2.29) is an isomorphism by Lemma 2.10.2, and \( \overline{Z}^{(M)} \) is canonically identified with \( T_D X^\times \) by Lemma 2.10.1.

**Example 2.18.** Let \( X \) be the affine plane \( A_k^2 = \text{Spec} \ k[x, y] \) and \( U \) be the complement of the smooth divisor \( D \) defined by \( x = 0 \). Let \( r > 1 \) be an integer and set \( R = r D \). Then, the dilatation \( P_1^{(R)} = \text{Spec} \ k[x, y, x', y'] \left[ \frac{1}{1 + u x^{r-1}} \right] \) by \( u \mapsto \frac{x'-x}{x^r}, v \mapsto \frac{y'-y}{x^r} \) is isomorphic to \( \text{Spec} \ k[x, y, u, v] \left[ \frac{1}{1 + u x^{r-1}} \right] \) by \( u \mapsto \frac{x'-x}{x^r}, v \mapsto \frac{y'-y}{x^r} \). Set \( G = F_p \).

1. Let \( n \geq 1 \) be an integer prime to \( p \) and \( V \) be the \( G \)-torsor over \( U \) defined by the Artin-Schreier equation \( t^p - t = \frac{1}{x^n} \). Then, \( V \times V/\Delta G \) is the \( G \)-torsor over \( U \times U \) defined by the Artin-Schreier equation \( t^p - t = \frac{1}{x^m} - \frac{1}{x^n} = \frac{1}{x^n} \left( \frac{1}{1 + u x^{r-1}} \right)^n - 1 \). For \( r = n + 1 \), the right hand side is a regular function on \( P_1^{(R)} \) and hence the ramification of \( V \) over \( U \) along \( D \) is bounded by \( R+ \). Further, the right hand side is congruent to \( -nu \) modulo \( x \). Hence, \( E^{(R)} \to T^{(R)} \) is the \( G \)-torsor defined by the Artin-Schreier equation \( t^p - t = -nu \) and the ramification of \( V \) over \( U \) along \( D \) is non-degenerate. By identifying \( T^{(R)} \) with \( TX(-R) \times_X D \), the linear form \( -nu \) is identified with the differential form \( -nu \frac{dx}{x^{n+1}} = \frac{1}{x^n} \).

2. Let \( n \geq 1 \) be an integer divisible \( p \) and \( V \) be the \( G \)-torsor over \( U \) defined by the Artin-Schreier equation \( t^p - t = \frac{y}{x^n} \). Then, \( V \times V/\Delta G \) is the \( G \)-torsor over \( U \times U \) defined
by the Artin-Schreier equation

\[(2.31)\quad t^p - t = \frac{y'}{x^n} - \frac{y}{x^n} = \frac{y'}{x^n} \left(\frac{1}{(1 + ux^{r-1})} - 1\right) - \frac{ux^r}{x^n}.
\]

For \(r = n\), the right hand side is a regular function on \(P_1^{(R)}\) and hence the ramification of \(V\) over \(U\) along \(D\) is bounded by \(R^+\).

We put \(n = n'n''\) where \(n'\) is the prime-to-\(p\) part of \(n\) and \(n''\) is a power of \(p\). Then, we have \((1 + ux^{n-1})^n \equiv 1 + n'n''x^{(n-1)n''} \mod x^{2(n-1)n''}\). Since \(n \leq (n-1)n''\) and the equality is equivalent to \(n = n'' = 2\), the right hand side of \(2.31\) is congruent to \(-v\) modulo \(x\) if \(n \neq 2\) and to \(yu^2 - v\) modulo \(x\) if \(n = 2\). Hence, \(E^{(R)} \to T^{(R)}\) is the \(G\)-torsor defined by the Artin-Schreier equation \(t^p - t = -v\) if \(n \neq 2\) and \(t^p - t = \sqrt{yu} - v\) defined over the radicial covering of degree 2 of \(D\) if \(n = 2\). Consequently the ramification of \(V\) over \(U\) along \(D\) is non-degenerate. By identifying \(\tilde{T}^{(R)}\) with \(\tilde{TX}(-R) \times_X D\), the linear form \(-v\) is identified with the differential form \(-\frac{dy}{x^n} = \frac{dy}{x^n}\).

### 2.4 Ramification and the cotangent bundle

We keep the notation \(V \to U = X - D\) over \(k\) in the previous subsection. We relate ramification to the cotangent bundle using the finite étale surjective morphism \((2.32)\), assuming that the ramification is non-degenerate along \(D\). Assume that the ramification of \(V \to U\) along \(D\) is bounded by \(R^+\) and is non-degenerate at the multiplicity \(R\). In this subsection, we fix \(M\) and write the finite étale morphism \((2.28)\) of smooth group schemes over \(\tilde{Z} = \tilde{Z}^{(M)}\) as

\[(2.32)\quad E^{(R)0} \to T^{(R)} = (TX \times_X \tilde{Z})(-\tilde{R})
\]

by abuse of notation.

The kernel \(\tilde{G}^{(R)}\) of the finite étale morphism \((2.32)\) is a commutative finite étale group scheme killed by \(p\). By Proposition \(1.19\) and Lemma \(1.20\) the extension

\[(2.33)\quad 0 \longrightarrow \tilde{G}^{(R)} \longrightarrow E^{(R)0} \longrightarrow T^{(R)} \longrightarrow 0
\]

defines a closed immersion

\[(2.34)\quad \tilde{G}^{(R)\vee} \to T^{(R)\vee}
\]

of a finite étale \(F_p\)-vector space scheme to the dual vector bundle defined over a finite radicial covering \(F^n:\tilde{Z}^{(p^{-n})} \to \tilde{Z}\). Here and in the following, for a scheme over \(S\) over \(k\), we define a scheme \(S^{(p^{-n})}\) over \(k\) and a radicial covering \(F^n:S^{(p^{-n})} \to S\) by the diagram

\[
\begin{array}{ccc}
S^{(p^{-n})} & \longrightarrow & S \\
\downarrow & & \downarrow \\
k & \longrightarrow & k \\
F_k^{-n} & \longrightarrow & F_k^n
\end{array}
\]

as follows. The right square is the usual commutative diagram with the \(n\)-th powers of the absolute Frobeniusses \(F_k\) and \(F_S\). The right square is the base change by the inverse of \(F_k^n\). The composition of the lower line is the identity and that of the upper line defines \(F^n:S^{(p^{-n})} \to S\).
**Definition 2.19.** We call the injection (2.34) of commutative group schemes defined over $F^m: \tilde{Z}(p^{-n}) \to \tilde{Z}$ the characteristic form of $V$ over $U$ at multiplicity $R$ and write

$$\text{Char}_R(V/U): \tilde{G}^{(R)}_\bullet \to T^{(R)}_\bullet = (T^*X \times_Z \tilde{Z})(\tilde{R}).$$

The logarithmic variant of the characteristic form for an abelian covering is defined and studied by Kato in [16] and [17] and called the refined Swan conductor.

We study the functoriality of characteristic form. Let $X'$ be another smooth scheme over $k$ and $f: X' \to X$ be a morphism over $k$. Assume that $f^{-1}(U)$ is the complement $U' = X' - D'$ of a divisor $D' \subset X'$ with normal crossings. Let $D'_1, \ldots, D'_{h'}$ be the irreducible components of $D'$ and set $f^*D_i = \sum_{j=1}^{h'} e_{ij} D'_j$ for $i = 1, \ldots, h$.

We set $R' = f^*(R) = \sum_{i=1}^h r_i \sum_{j=1}^{h'} e_{ij} D'_j = \sum_{j=1}^{h'} r'_j D'_j$. Let $M' = \sum_{j=1}^{h'} m'_j D'_j$ be a divisor with integral coefficients $m'_j \geq 1$. We assume that $l_{ij} = e_{ij}m'_j/m_i$ is an integer for every $i = 1, \ldots, h$ and $j = 1, \ldots, h'$. Then a canonical morphism $F^{(R',M')}_{\bullet} \to P^{(R,M)}_\bullet$ (2.12) is defined. We also fix $M'$ and we write $\tilde{Z}' = \tilde{Z}'(M')$ etc. by abuse of notation.

**Definition 2.20.** Assume that the ramification of $V$ over $U$ along $D$ is bounded by $R+$ and is non-degenerate at multiplicity $R$. Let $f: X' \to X$ be a morphism of smooth schemes over $k$ such that $f^{-1}(U)$ is the complement $U' = X' - D'$ of a divisor $D' \subset X'$ with simple normal crossings.

We say that $f: X' \to X$ is non-characteristic with respect to the ramification of $V \to U$ along $D$ at multiplicity $R$, if the composition

$$\tilde{G}^{(R)}_\bullet \times_{\tilde{Z}} \tilde{Z}' \longrightarrow T^{(R)}_\bullet \times_{\tilde{Z}} \tilde{Z}' \longrightarrow T^{(R)}_\bullet$$

(2.36) defined over $F^m: \tilde{Z}(p^{-n}) \to \tilde{Z}'$ is injective.

**Lemma 2.21.** Assume that the ramification of $V \to U$ along $D$ is bounded by $R+$ and is non-degenerate at multiplicity $R$ and let $f: X' \to X$ be a morphism of smooth schemes over $k$ as above. Then, the following conditions are equivalent:

1. $f: X' \to X$ is non-characteristic with respect to the ramification of $V \to U$ along $D$ at multiplicity $R$.
2. For every point $x$ of $\tilde{Z}'$, the fiber $\left( E^{(R)}_{0} \times_{T^{(R)}} T^{(R)} \right) \times_{\tilde{Z}} x$ is connected.

**Proof.** By the definition of characteristic form, the composition $\tilde{G}^{(R)}_\bullet \times_{\tilde{Z}} \tilde{Z}' \to T^{(R)}_\bullet \times_{\tilde{Z}} \tilde{Z}' \to T^{(R)}_\bullet$ corresponds to the extension $E^{(R)}_{0} \times_{T^{(R)}} T^{(R)}$ of $T^{(R)}$ by $G^{(R)}_\bullet \times_{\tilde{Z}} \tilde{Z}'$ obtained as the pull-back by $T^{(R)} \to T^{(R)} \times_{\tilde{Z}} \tilde{Z}'$ of the base change of (2.33) by $\tilde{Z}' \to \tilde{Z}$. Thus, the injectivity of the composition $\tilde{G}^{(R)}_\bullet \times_{\tilde{Z}} \tilde{Z}' \to T^{(R)}_\bullet$ is equivalent to the condition (2) by Lemma 1.20. \qed

We define a finite étale $G$-torsor $V' \to U'$ by the cartesian diagram

$$\begin{array}{ccc}
X' & \to & U' \to & V' \\
\downarrow & & \downarrow & & \downarrow \\
X & \to & U & \to & V
\end{array}$$
and set $R' = f^*(R)$. Then, by the functoriality of dilatations (2.12), we obtain a cartesian diagram

$$
\begin{array}{ccc}
P_{\bullet}^{(R',M')} & \xleftarrow{\cdot} & U'^{\bullet+1} \times G_{m'}^{h'} \\
\downarrow & & \downarrow \phi_{i+1} \times f_{ij} \\
P_{\bullet}^{(R,M)} & \xleftarrow{\cdot} & U^{\bullet+1} \times G_{m}^{h}
\end{array}
$$

(2.37)

**Proposition 2.22.** Assume that the ramification of $V \rightarrow U$ along $D$ is bounded by $R^+$. Let $f: X' \rightarrow X$ be a morphism of smooth schemes over $k$ and $M$ and $M'$ be as above.

1. The ramification of $V'$ over $U'$ along $D'$ is bounded by $R'^+$ and we have an open immersion

$$
W_{\bullet}^{(R,M)} \times_{P_{\bullet}^{(R,M)}} P_{\bullet}^{(R',M')} \longrightarrow W_{\bullet}^{(R',M')}
$$

(2.38)

of multiplicative oversimplicial schemes étale over $P_{\bullet}^{(R',M')}$. Further assume that the ramification of $f$ is non-degenerate at multiplicity $R$ and we have an open immersion

2. The open immersion (2.38) induces an isomorphism

$$
E'^{(R)0} \longrightarrow E^{(R)0} \times_{T^{(R)}} T'^{(R)}.
$$

(2.39)

3. The ramification of $V'$ over $U'$ along $D'$ is non-degenerate at multiplicity $R'$ and there exists an isomorphism $\tilde{G}^{(R)\vee} \times_{\tilde{Z}} \tilde{Z}' \longrightarrow \tilde{G}'^{(R')\vee}$ that makes a commutative diagram

$$
\begin{array}{ccc}
\tilde{G}^{(R)\vee} \times_{\tilde{Z}} \tilde{Z}' & \xrightarrow{\text{Char}^{(V'/U')}} & T'^{(R')\vee} \\
\uparrow & & \uparrow \\
\tilde{G}'^{(R')\vee} \times_{\tilde{Z}} \tilde{Z}' & \xrightarrow{\text{Char}^{(V/U)}} & T^{(R)\vee} \times_{\tilde{Z}} \tilde{Z}'.
\end{array}
$$

(2.40)

**Proof.** 1. It suffices to apply the functoriality Lemma 1.6.2 to the diagram (2.37).

2. The open immersion (2.38) induces an open immersion $E^{(R,M)} \times_{T^{(R,M)}} T^{(R,M')} \rightarrow E^{(R,M')}$ and hence an open immersion (2.39).

By Lemma 2.21, the condition (1) is equivalent to that, for every point $x'$ of $\tilde{Z}'$ the fiber $(E^{(R)0} \times_{T^{(R)}} T^{(R')}) \times_{\tilde{Z}} x'$ is connected. Hence, the condition (1) is equivalent to the condition (2).

An isomorphism $\tilde{G}^{(R)\vee} \times_{\tilde{Z}} \tilde{Z}' \rightarrow \tilde{G}'^{(R')\vee}$ making the diagram (2.40) is equivalent to an isomorphism

$$
\begin{array}{cccc}
0 & \longrightarrow & \tilde{G}^{(R)} & \longrightarrow & E^{(R)0} & \longrightarrow & T^{(R')} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \text{(2.39)} & & \| & & \downarrow \\
0 & \longrightarrow & \tilde{G}^{(R)} \times_{\tilde{Z}} \tilde{Z}' & \longrightarrow & E^{(R)0} \times_{T^{(R)}} T^{(R')} & \longrightarrow & T^{(R')} & \longrightarrow & 0
\end{array}
$$

of extensions over $\tilde{Z}'$. Hence the condition (3) is equivalent to that the étale morphism $E^{(R)0} \rightarrow T^{(R)}$ is finite and that (2.39) is an isomorphism. By the assumption that the ramification of $V$ over $U$ is non-degenerate along $D$ at multiplicity $R$, the étale morphism $E^{(R)0} \rightarrow T^{(R)}$ is finite and the target of (2.39) is finite over $T^{(R)}$. Thus, the condition (3) is also equivalent to the condition (2).
Corollary 2.23. Assume that the ramification of $V \to U$ along $D$ is bounded by $R+$ and is non-degenerate at multiplicity $R$ as above. Let $X'' \to X' \to X$ be morphisms of smooth schemes over $k$ such that the inverse images of $U$ are the complement of divisors of $X'$ and of $X''$ with simple normal crossings. Then, the following conditions are equivalent.

1) The composition $X'' \to X$ is non-characteristic with respect to the ramification of $V$ over $U$ along $D$ at multiplicity $R$.

2) $X' \to X$ is non-characteristic with respect to the ramification of $V$ over $U$ along $D$ at multiplicity $R$ on a neighborhood of the image of $X'' \to X'$ and $X'' \to X'$ is non-characteristic with respect to the ramification of $V'$ over $U'$ along $D'$ at multiplicity $R'$.

Proof. We consider the open immersions
\[(2.41) \quad E^{\nu(R')_0} \longrightarrow E^{\nu(R)_0} \times_{T^{\nu(R)/R}} T^{\nu(R')_0} \longrightarrow E^{\nu(R)_0} \times_{T^{\nu(R)/R}} T^{\nu(R')}
\]
over $\tilde{Z}''$ in (2.39) for $X'' \to X' \to X$. By Proposition 2.22.2, the condition (1) is equivalent to that the composition of (2.41) is an isomorphism. Since the morphisms in (2.41) are open immersions, it is equivalent to that the first map is an isomorphism and that (2.39) is an isomorphism on the image of $X'' \to X'$. Hence the assertion follows.

We study the restrictions to curves. Let $\Sigma \subset TX$ denote the union of the images of the hyperplane bundles defined as the zero locus of the image by the injection $\text{Char}_R(V/U): G(R)^{\nu} \to T(R)^{\nu}$ of non-zero sections defined over $F^n: \tilde{Z}^{(\nu,n)} \to \tilde{Z}$.

Lemma 2.24. Assume that the ramification of $V \to U$ along $D$ is bounded by $R+$ and is non-degenerate at multiplicity $R$ as above. Let $C$ be a smooth curve and $f: C \to X$ be a morphism over $k$ such that the pull-back $f^*D$ is a divisor of $C$. Then, for $x \in f^*(D)$, the following conditions are equivalent.

1) $f: C \to X$ is non-characteristic with respect to the ramification of $V$ over $U$ along $D$ at multiplicity $R$ on a neighborhood of $x$.

2) The image of the map $f_*: T_xC \to T_{f(x)}X$ on the tangent space is not contained in $\Sigma$ defined above.

Proof. Clear from the definition of non-characteristicity.

Corollary 2.25. Assume that the ramification of $V \to U$ along $D$ is bounded by $R+$ and is non-degenerate at multiplicity $R$ as above. Let $f: X' \to X$ be a morphism of smooth schemes such that the inverse image $U'$ of $U$ is the complement of a divisor $D'$ of $X'$ with simple normal crossings. Then, the following conditions are equivalent.

1) $f: X' \to X$ is non-characteristic with respect to the ramification of $V$ over $U$ along $D$ at multiplicity $R$.

2) For every closed point $x'$ of $D'$, there exists a smooth curve $C$ defined on a neighborhood of $x'$ in $X'$ meeting each component of $D'$ transversally at $x'$ such that the composition $C' \to X' \to X$ is non-characteristic with respect to the ramification of $V$ over $U$ along $D$ at multiplicity $R$.

Proof. (1)⇒(2): By Proposition 2.22.2, the ramification of the pull-back $V' = V \times_U U'$ along $D'$ is non-degenerate at multiplicity $R'$. By Lemma 2.24 (2)⇒(1), for every closed point $x'$ of $D'$, there exists a smooth curve $C$ defined on a neighborhood of $x'$ in $X'$ meeting each component of $D'$ transversally at $x'$ such that the composition $C' \to X'$ is non-characteristic with respect to $V'$ over $U'$. By Corollary 2.23 (2)⇒(1), the composition $C' \to X$ is non-characteristic with respect to $V$ over $U$.

(2)⇒(1): It follows from Corollary 2.23 (1)⇒(2).
2.5 Ramification groups

We briefly recall from [1], [2] and [21] the definition and basic properties of the filtration by ramification groups of the absolute Galois group of a local field with not necessarily perfect residue field. Let $K$ be a complete discrete valuation field and $\mathcal{O}_K$ be the valuation ring. We fix a separable closure $\bar{K}$ of $K$. Its residue field $\bar{F}$ is an algebraic closure of the residue field $F$. Let $L$ be a finite étale algebra over $K$ and let $T$ denote the normalization of $S = \text{Spec } \mathcal{O}_K$ in $L$. We consider a cartesian diagram

\[ Q \leftarrow T \quad \text{down} \quad \text{down} \quad \text{down} \quad \text{down} \quad \text{down} \]

\[ P \leftarrow S \]

of schemes over $S$ satisfying the condition:

(P) The horizontal arrows are closed immersions, the vertical arrows are finite flat and $P$ and $Q$ are smooth over $S$.

Let $r > 0$ be a rational number. Let $K'$ be a finite separable extension of $K$ contained in $\bar{K}$ such that $e(r)$ is an integer for the ramification index $e$ of $K' / K$. Let $S'$ denote the normalization of $S$ in $K'$ and $F'$ be the residue field of $K'$. Let $D \subset P$ and $D' = P \times_S \text{Spec } F' \subset P_{S'} = P \times_S S'$ denote the closed fibers and define the dilatation $P_{S'}^{(\text{er})}$ to be $P_{S'}^{(\text{er})}$. Let $Q_{S'}^{(\text{er})}$ denote the normalization of the base change $Q \times_P P_{S'}^{(\text{er})}$. As a consequence of Epp’s theorem, there exists a finite separable extension $K' \subset \bar{K}$ such that the geometric closed fiber $Q_{S'}^{(\text{er})} \times_{S'} \text{Spec } \bar{F}$ is reduced. Further the finite morphism

\[ Q_F^{(r)} = Q_{S'}^{(\text{er})} \times_{S'} \text{Spec } \bar{F} \rightarrow P_F^{(r)} = P_{S'}^{(\text{er})} \times_{S'} \text{Spec } \bar{F} \]

on the geometric closed fibers is independent of $K'$.

**Definition 2.26.** Let $K$ be a complete discrete valuation field, $L$ be a finite étale $K$-algebra and $r > 0$ be a rational number. Then, we say that the ramification of $L$ over $K$ is bounded by $r+$ (resp. $r$) if there exists a cartesian diagram (2.42) satisfying the condition (P) above and a finite separable extension $K'$ over $K$ such that $Q_F^{(r)} \rightarrow P_F^{(r)}$ (2.43) is a finite étale covering (resp. a totally decomposed finite étale covering).

We recall main results from [1] and [2]. The condition in **Definition 2.26** holds for one cartesian diagram (2.42) satisfying the condition (P) for an extension $K'$ if and only if it holds for every such diagram for every finite separable extension over $K'$. The full subcategory of the category of finite étale $K$-algebras consisting of those such that the ramification is bounded by $r$ (resp. by $r+$) form a Galois subcategory and its fundamental group is the quotient $G_K / G_K^r$ (resp. $G_K^r / G_K^{+r}$) by the ramification group $G_K^r$ (resp. $G_K^{+r} = \bigcup_{s > r} G_K^s$) of the absolute Galois group $G_K = \text{Gal}(\bar{K} / K)$. The subgroup $G_K^r$ is the inertia subgroup $I_K = \text{Gal}(K^{ur} / K)$ and $G_K^{+r}$ is its $p$-Sylow subgroup $P_K$.

The diagram (2.42) induces a commutative diagram

\[ Q_{S'}^{(\text{er})} \leftarrow T_{S'} \quad \text{down} \quad \text{down} \quad \text{down} \quad \text{down} \quad \text{down} \]

\[ P_{S'}^{(\text{er})} \leftarrow S' \]
where the normalization $\tilde{T}_S$ of $T \times_S S'$ is the disjoint union of copies of $S'$ indexed by embeddings $L \to \tilde{K}$ over $K$ if $K'$ contains the Galois closure of $L$ over $K$. We call the closed point of $P_F^{(r)}$ defined by the bottom horizontal arrow of (2.44) the origin and let it denoted by $0$. Then, if the ramification of $L$ is bounded by $r+$, the diagram (2.44) induces a cartesian diagram

$$
\begin{array}{c}
Q_F^{(r)} & \leftarrow & \text{Mor}_K(L, \tilde{K}) \\
\downarrow & & \downarrow \\
P_F^{(r)} & \leftarrow & 0.
\end{array}
$$

(2.45)

Assume $L$ is a Galois extension of $K$ such that the ramification is bounded by $r+$. We fix an embedding $L \to \tilde{K}$ and identify the Galois group $G = \text{Gal}(L/K)$ with the set $\text{Mor}_K(L, \tilde{K})$ of embeddings. Let $G^{(r)} \subset G$ denote the ramification subgroup defined as the image of $G_K^{(r)} \subset G_K$ by the surjection $G_K \to G = \text{Gal}(L/K)$. Then, $G^{(r)} \subset G$ is identified with the intersection $Q_F^{(r)0} \cap G$ with the connected component $Q_F^{(r)0}$ containing the image of the fixed embedding $1 \in \text{Gal}(L/K) = \text{Mor}_K(L, \tilde{K})$. Further $Q_F^{(r)0}$ is a $G^{(r)}$-torsor over $P_F^{(r)}$ compatible with its canonical action on the subset $G^{(r)}$. Thus, we obtain a canonical surjection

$$
\pi_1(P_F^{(r)}, 0) \to G^{(r)}.
$$

(2.46)

**Proposition 2.27.** Let $X$ be a smooth scheme over a perfect field $k$ of characteristic $p > 0$ and $D$ be a smooth irreducible divisor of $X$. Let $\xi$ be the generic point of $D$ and $K$ be the fraction field of the completion $\hat{O}_{X, \xi}$ of the discrete valuation ring. Let $G$ be a finite group and $V \to U$ be a $G$-torsor. Then for the finite étale $K$-algebra $L = \Gamma(V \times_U K, \mathcal{O})$ and a rational number $r > 1$, the following conditions are equivalent:

1. The ramification of $V$ over $U$ at $\xi$ is bounded by $R+ = rD+$.
2. The ramification of $L$ over $K$ is bounded by $r+$.

**Proof.** Set $S = \text{Spec} \mathcal{O}_K$ and let $Y$ be the normalization of $X$ in $V$. Since $T = \text{Spec} \mathcal{O}_L \to Y \times_X S$ is an isomorphism, the cartesian diagram

$$
\begin{array}{c}
Q = Y \times_k S & \leftarrow & T \\
\downarrow & & \downarrow \\
P = X \times_k S & \leftarrow & S
\end{array}
$$

satisfies the condition (P) for (2.42). Let $K'$ be a finite separable extension of $K$ of ramification index $e_{K'/K} = e$ such that $er$ is an integer, that $K'$ contains $L$ as a subfield and that the geometric closed fiber of $Q_{S'}^{(er)}$ is reduced.

Set $M = eD$. In order to link Definitions 2.12 and 2.26 we construct a diagram

$$
\begin{array}{c}
Q_1^{(R,M)} & \leftarrow & Q_{S'}^{(er)} & \leftarrow & Q_{S'}^{(e,R)} \\
\downarrow & & \downarrow & & \downarrow \\
P_1^{(R,M)} & \leftarrow & P_{S'}^{(er)} & \leftarrow & P_{S'}^{(e,R)} \\
\downarrow & & \downarrow & & \downarrow \\
X & \leftarrow & S'
\end{array}
$$

(2.47)
such that the base change by $U \to X$ is

\[(2.48) \quad (V \times_k V)/\Delta G \times_k \mathbb{G}_m \leftarrow V \times_k \text{Spec } K' \times_k \mathbb{G}_m \longrightarrow Y \times_k K'\]

as follows. Set $P_{S'} = P \times_S S' = X \times_k S'$ and let $D' = P_{S'} \times_S \text{Spec } F'$ be the closed fiber. The scheme $P_{S'}^{(e)}$ for $r = 1$ is the dilatation $P_{S'}^{(eD',S')}$ with respect to the Cartier divisor $eD'$ and the transpose $S' \subset P_{S'} = X \times_k S'$ of the graph of the composition $S' \to S \to X$. Defined $P_{S'}^{(D)} \subset P_{S'}^{(e)} = P_{S'}^{(eD',S')}$ to be the complement of the proper transform of the inverse image of $D$ by the projection $P_{S'} \to X$. The canonical map $P_{S'}^{(D)} \to S'$ is smooth and we have a natural map $P_{1}^{(D)} \leftarrow P_{1}^{(D)} \times_X S \leftarrow P_{S'}^{(D)}$ by the functoriality of dilatation. Let $D_{S'} = P_{S'}^{(D)} \times_S \text{Spec } F'$ denote the closed fiber.

Similarly to the definition of $P_{1}^{(D,M)}$ and $P_{1}^{(R,M)}$, we define $P_{S'}^{(e,D)}$ and $P_{S'}^{(e,R)}$ as follows. We consider the dilatation of $P_{S'}^{(D)} \times \mathbb{A}^1$ with respect to $e$-times the 0-section regarded as a Cartier divisor and the closed subscheme $D_{S'}^{(D)} \times \mathbb{A}^1$. We define $P_{S'}^{(e)}$ by further removing the proper transform of $D_{S'}^{(D)} \times \mathbb{A}^1$. The scheme $P_{S'}^{(e,D)}$ is a $\mathbb{G}_m$-bundle over $P_{S'}^{(D)}$ and hence the canonical map $P_{S'}^{(e,D)} \to S'$ is smooth. Let $T_{F'}^{(e,D)}$ denote the closed fiber and let $\tilde{S}' = S' \times_{P_{S'}^{(D)}} P_{S'}^{(e,D)}$ be the $\mathbb{G}_m$-bundle over $S'$. Similarly, we define $\tilde{S}'$ as an open subscheme of the dilatation of $(S' \times \mathbb{A}^1)^{(e(0)-(D' \times \mathbb{A}^1))}$ obtained by removing the proper transform of $D' \times \mathbb{A}^1$ and regard it as a closed subscheme of $P_{S'}^{(e,D)}$.

Let $P_{S'}^{(e,R)}$ be the dilatation of $P_{S'}^{(e,D)}$ with respect to the Cartier divisor $e(r - 1)T_{F'}^{(e,D)}$ and the closed subscheme $\tilde{S}' \subset P_{S'}^{(e,D)}$. By the functoriality of the dilatation and by the canonical isomorphism $(P_{S'}^{(e)})^{(e(r-1)D',S')/} \to P_{S'}^{(e,r)}$, we obtain the lower part of the commutative diagram (2.47).

By the assumption that $K'$ contains $L$ as a subfield, we obtain the upper left square of the diagram (2.48) as a cartesian square. By taking the normalizations of the schemes on the middle line of (2.47) in those on the top line of (2.48), we complete the construction of the commutative diagram (2.47). The scheme $P_{S'}^{(e,R)}$ is a $\mathbb{G}_m$-bundle over the open subscheme $P_{S'}^{(e,R)} \times_{P_{S'}^{(e,D)}} P_{S'}^{(e,D)}$ of $P_{S'}^{(e,R)}$ and hence the canonical map $P_{S'}^{(e,R)} \to P_{S'}^{(e,R)}$ is smooth. The closed immersion $\tilde{S}' \to P_{S'}^{(e,D)}$ is canonically lifted to a closed immersion $\tilde{S}' \to P_{S'}^{(e,R)}$.

We show that the conditions (1) and (2) are equivalent. We consider the diagram

\[(2.49) \quad \tilde{X}^{(M)} = Q_0^{(R,M)} \leftarrow \tilde{S}' \longrightarrow S' \]

\[
Q_1^{(R,M)} \leftarrow Q_{S'}^{(e,R)} \longrightarrow Q_{S'}^{(e,r)} \]

\[
P_1^{(R,M)} \leftarrow P_{S'}^{(e,R)} \longrightarrow P_{S'}^{(e,R)}
\]
where the lower half is the same as the upper half of (2.47) and the upper vertical arrows are the canonical sections. The condition (1) is equivalent to that the lower left vertical arrow in (2.49) is étale on a neighborhood of the image of the generic point $\xi$ of $D$ by $\tilde{X}^{(M)} \to X$. This is equivalent to the same condition for the base change of the left column of (2.49) by $X \leftarrow S$. The condition (2) is equivalent to that the lower right vertical arrow of (2.49) is étale on a neighborhood of the image of the closed point $\xi'$ of $S'$. Since the bottom right horizontal arrow $P^{(e,R)}_{S'} \to P^{(e,r)}_{S'}$ is smooth, the right half of the diagram (2.49) is cartesian. Hence the condition (2) is equivalent to that the lower middle vertical arrow in (2.49) is étale on a neighborhood of the inverse image of the generic point $\xi'$ of $S'$. Since the bottom horizontal arrow in the diagram

$$Q^{(R,M)}_1 \times_X S \leftarrow Q^{(e,R)}_{S'}$$

is finite, it suffices to apply Lemma 1.9 to this diagram.

Corollary 2.28. Let $K$ be a complete discrete valuation field of characteristic $p > 0$. Let $L$ be a finite Galois extension of $K$ and $G = \text{Gal}(L/K)$ be the Galois group. Let $r > 1$ be a rational number such that the ramification of $L$ over $K$ is bounded by $r+$ and define an $\bar{F}$-vector space $N^{(r)} = m^r_K / m^{r+}_K$ of dimension 1 by $m^r_K = \{a \in \bar{K} \mid \text{ord}_K a \geq r\}$, $m^{r+}_K = \{a \in \bar{K} \mid \text{ord}_K a > r\}$.

1. The last graded piece $G^{(r)}$ is an abelian group killed by $p$.

2. Assume that the residue field $F$ is finitely generated over a perfect subfield $k$. Then, there is a canonical injection

$$\text{Hom}(G^{(r)}, F_p) \to \text{Hom}_{\bar{F}}(N^r, \Omega^1_{(\mathcal{O}_K/m^r_K)/k} \otimes F \bar{F}).$$

Proof. 1. By a standard limit argument, we may assume that the residue field $F$ is finitely generated over a perfect subfield $k$. There exist a $G$-torsor $V \to U = X \leftarrow D$ and an isomorphism $\mathcal{O}_{X,\xi} \to \mathcal{O}_K$ as in Proposition 2.27. Then, by Proposition 2.27 after replacing $X$ by a neighborhood of $\xi$ if necessary, the ramification of $V$ over $U$ along $D$ is bounded by $R+$. Further, by the definition of the ramification group recalled above, we have an isomorphism $G^{(r)}_{\xi} \to G^{(r)}$. Hence the assertion follows by Proposition 2.16 1.

2. It suffices to take a uniformizer and the stalk of the characteristic form (2.35) at the corresponding $\bar{F}$-point of $D$.

Let $X$ be a smooth scheme over a perfect field $k$ and $D$ be a smooth irreducible divisor of $X$. Let $V \to U = X \leftarrow D$ be a finite étale Galois covering wildly ramified along $D$. Then by Proposition 2.27 after shrinking $D$, there exists a rational number $r > 1$ such that, for $R = rD$, the ramification of $V$ over $U$ along $D$ is bounded by $R+$ and is non-degenerate at multiplicity $R$. By Lemma 2.24 for every closed point $x$, there exists a curve $C \subset X$ meeting $D$ transversally at $x$ such that the pull-back of $V$ to $C$ is wildly ramified at $x$.

### 2.6 Logarithmic variant

In [21] and [5], a logarithmic variant is studied by a different method. We indicate how to treat the logarithmic variant by the method of this article and how to translate the results there.
We keep the notation set at the beginning of this section. We consider a commutative diagram of oversimplicial schemes

\[
\begin{array}{cccc}
T^{(R-D,M)}_{\log D} & \subset & P^{(R-D,M)}_{\log D} & \leftarrow \subset U^{\bullet+1} \times_k G^h_m \\
\downarrow & & \downarrow & \\
T^{(M)}_{\log D} & \subset & P^{(M)}_{\log D} & \leftarrow \subset U^{\bullet+1} \times_k G^h_m \\
\downarrow & & \downarrow & \\
D_{\log D} & \subset & P_{\log D} & \leftarrow \subset U^{\bullet+1}
\end{array}
\]

(2.50)

constructed as follows. For integer \( n \geq 0 \), let \( P'_{n \log D} \) be the blow-up of \( P_n = X^{n+1} \) at \( D^1_{1+1}, \ldots, D^{n+1}_h \) and define \( P_{n \log D} \) to be the complement of the proper transforms of the inverse image of \( D \) by the \( n + 1 \) projections. By the universality of blow-up, the schemes \( P_{n \log D} \) form an oversimplicial scheme \( P_{\log D} \). By replacing \( P^{(D)} \) by \( P_{\log D} \) in the construction of (2.3), we define (2.50). By the functoriality of dilatation, we have canonical morphisms \( P_{\log D} \leftarrow P^{(D)} \) and \( P_{\log D}^{(R-D,M)} \leftarrow P^{(R-M)}_{\log D} \).

We define the condition that the logarithmic ramification of a \( G \)-torsor \( V \) over \( U \) is bounded by \( (R-D)+ \) as in Definition 7.3 by replacing \( P^{(R-M)}_{1 \log D} \) by \( P^{(R-D,M)}_{1 \log D} \). If the ramification is bounded by \( R+ \), the logarithmic ramification is bounded by \( R+ \). If the logarithmic ramification is bounded by \( (R-D)+ \), the ramification is bounded by \( R+ \). The latter conditions are equivalent if \( X \) is a curve since the canonical map \( P_{\log D} \leftarrow P^{(D)} \) is an isomorphism in this case.

We verify that the definition of bounding log ramification here is equivalent to [5] Definition 7.3 as follows. We take a log smooth faithfully morphism \( f: X' \rightarrow X \) such that the inverse image \( U' = f^{-1}(U) \) is the complement of a divisor \( D' \) with simple normal crossings and that \( f^*R = R' \) has integral coefficient as in [5] Proposition 7.7. As in the proof of Proposition 2.27 we set \( M = m_1D_1 + \cdots + m_hD_h \) such that \( f^*(m_1^{-1}D_1 + \cdots + m_h^{-1}D_h) = D' \) and consider morphisms

\[
\begin{align*}
P^{(R,M)}_{1 \log D} & \leftarrow P^{(R',D')}_{1 \log D'} & \rightarrow P^{(R')}_{1 \log D'}
\end{align*}
\]

where \( P^{(R')}_{1 \log D'} \) is defined as \( P^{(R')}_1 \) without introducing an auxiliary divisor \( M' \). The smoothness of the morphisms (2.51) implies that the definition using \( P^{(R,M)}_{1 \log D} \) is equivalent to the criterion [5] Proposition 7.7 using \( P^{(R')}_{1 \log D'} \).

If we assume a strong form of resolution of singularities, for a Galois covering \( V \rightarrow U \), we can conclude an existence of \( R \) such that the ramification of \( V \rightarrow U \) is bounded by \( R+ \). More precisely, we consider the following condition on a smooth separated scheme \( U \) of finite type over \( k \):

(RS) If \( X \) is a proper scheme over \( k \) containing \( U \) as a dense open subscheme, there exist a proper smooth scheme \( X' \) over \( k \) and a morphism \( f: X' \rightarrow X \) such that \( U' = f^{-1}(U) \rightarrow U \) is an isomorphism and \( U' \) is the complement of a divisor with simple normal crossings.
Proposition 2.29. Let $U$ be a separated smooth scheme of finite type over $k$ and $V \to U$ be a finite étale Galois covering. Then, there exists a proper scheme over $k$ there exist a proper smooth scheme $X$ over $k$ containing $U$ as the complement of a divisor $D$ with simple normal crossings and a linear combination $R = r_1D_1 + \cdots + r_hD_h$ of irreducible components of $D$ with rational coefficients $r_i \geq 1$ such that the ramification of $V$ over $U$ along $D$ is bounded by $R^+$.  

Proof. By [2, Proposition 7.22], there exist $X$ and $R'$ such that the log ramification of $V$ over $U$ along $D$ is bounded by $R'^+$. It suffices to set $R = R' + D$.  

3 Characteristic cycles

We keep the notation in the previous section. Namely, $k$ denotes a perfect field of characteristic $p > 0$ and $X$ denotes a smooth separated scheme over $k$. Let $D$ be a divisor of $X$ with simple normal crossings and $U = X - D$ be the complement. Let $D_1, \ldots, D_h$ be the irreducible components of $D$ and $R = r_1D_1 + \cdots + r_hD_h$ be a linear combination with rational coefficients $r_i \geq 1$ for every $i = 1, \ldots, h$. Let $M = m_1D_1 + \cdots + m_hD_h$ be a linear combination with integral coefficients $m_i \geq 1$ such that $m_ir_i$ is an integer for every $i = 1, \ldots, h$. Let $Z \subset D$ be the union of irreducible components $D_i$ such that $r_i > 1$.

3.1 Definition of characteristic cycles

Let $\Lambda$ be a local ring over $\mathbb{Z}[\frac{1}{p}, \zeta_p]$ and $\mathcal{F}$ be a locally constant sheaf of free $\Lambda$-modules of finite rank on $U = X - D$. Set $d = \dim X$. We will define the characteristic cycle of $\mathcal{F}$ as a cycle of dimension $d$ of the cotangent bundle $T^*X$ under a certain non-degenerate hypothesis.

We prepare some terminology on a representation of the absolute Galois group $G_K$ of a local field $K$ as in [2.5]. Let $M$ be a representation of $G_K$ on a free $\Lambda$-module of finite rank. For a rational number $r > 1$, we say that $M$ is isoclinic of slope $r$ if $G_K^r$ acts trivially on $M$ and if the $G_K^r$-fixed part is 0. For $r = 1$, we say $M$ is isoclinic of slope 1 if the wild inertia group $P_K = G_K^{+}$ acts trivially on $M$ or equivalently it is tamely ramified. Since $P_K$ is a pro-$p$ group, there exists a unique decomposition $M = \bigoplus_{r \geq 1} M^{(r)}$, called the slope decomposition such that $M^{(r)}$ is isoclinic of slope $r$.

First, we consider the isoclinic case. In this case, the characteristic class will be defined by (3.13) using the image of the locally constant function (3.6) by the map (3.11).

Definition 3.1. Let $\Lambda$ be a local ring over $\mathbb{Z}[\frac{1}{p}]$ and $\mathcal{F}$ be a locally constant sheaf on $U$ of free $\Lambda$-modules of finite rank. Let $D_1, \ldots, D_h$ be the irreducible components of the divisor $D$ with simple normal crossings, let $\xi_i$ be the generic point of $D_i$ and $\bar{\eta}_i$ be the geometric point of $U$ defined by a separable closure $\bar{K}_i$ of the local field $K_i = \text{Frac} \mathcal{O}_{X, \xi_i}$ for $i = 1, \ldots, h$.

1. We say that $\mathcal{F}$ is isoclinic of slope $R = r_1D_1 + \cdots + r_hD_h$ if the representation $\mathcal{F}_{\bar{\eta}_i}$ of the absolute Galois group $G_{K_i}$ is isoclinic of slope $r_i$ for every $i = 1, \ldots, h$. We say that the ramification of an isoclinic sheaf $\mathcal{F}$ of slope $R$ is non-degenerate along $D$ if there is a finite étale Galois torsor $V \to U$ such that the pull-back of $\mathcal{F}$ on $V$ is constant and that the ramification of $V$ over $U$ along $D$ is bounded by $R^+$ and is non-degenerate along $D$ at multiplicity $R$.  

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2. We say that \( \mathcal{F} \) is polyisoclinic if there exists a finite number of divisors \( R_j \geq D \) with rational coefficients and a decomposition \( \mathcal{F} = \bigoplus_j \mathcal{F}_j \) such that each \( \mathcal{F}_j \) is isoclinic of slope \( R_j \). We call such a decomposition an isoclinic decomposition.

We say that the ramification of a polyisoclinic sheaf \( \mathcal{F} \) is non-degenerate along \( D \) if there exists an isoclinic decomposition \( \mathcal{F} = \bigoplus_j \mathcal{F}_j \) such that the ramification of each isoclinic sheaf \( \mathcal{F}_j \) is non-degenerate along \( D \) at multiplicity \( R_j \).

3. We say that the ramification of \( \mathcal{F} \) is non-degenerate along \( D \) if it is étale locally polyisoclinic and the ramification is non-degenerate along \( D \).

4. We say that \( \mathcal{F} \) is totally wildly ramified along \( D \), if for every irreducible component \( D_i \) of \( D \), the slopes of the representations of the absolute Galois group of the local field \( K_i \) are strictly greater than 1.

A sheaf \( \mathcal{F} \neq 0 \) is tamely ramified along \( D \) if and only if \( \mathcal{F} \) is isoclinic of slope \( D \).

**Lemma 3.2.** Any locally constant sheaf \( \mathcal{F} \) on \( U \) is polyisoclinic on an étale neighborhood of a geometric point above the generic point of each irreducible component of \( D \).

**Proof.** Since the wild inertia subgroup \( P_{K_i} \) is a pro-\( p \) group, any representation of \( G_{K_i} \) admits a slope decomposition. Hence the assertion follows. \( \square \)

Let \( \Lambda \) be a local ring over \( \mathbb{Z}[\frac{1}{p}, \zeta_p] \) and \( \mathcal{F} \) be a locally constant sheaf on \( U \) of free \( \Lambda \)-modules of finite rank. Let \( V \) be a \( G \)-torsor over \( U \) for a finite group \( G \) such that the pull-back of \( \mathcal{F} \) on \( V \) is constant and let \( M \) be a representation of \( G \) on a free \( \Lambda \)-module such that \( \mathcal{F} \) is corresponding to \( M \). Define a locally constant sheaf \( \mathcal{H} \) on \( U \times_k U \) by

\[
\mathcal{H} = \mathcal{H}om(pr_2^* \mathcal{F}, pr_1^* \mathcal{F}).
\]

The locally constant sheaf \( \mathcal{H} \) on \( U \times_k U \) is trivialized by the \( G \times G \)-torsor \( V \times_k V \) over \( U \times_k U \) and corresponds to the representation \( \text{End}(M) \) of \( G \times G \). We consider the diagram

\[
\begin{array}{cccc}
E^{(R)0} \subset E^{(R)} & \xrightarrow{i} & W_1^{(R,M)} & \leftarrow \xrightarrow{j} & V \times_k V/\Delta G \times G_m^h \\
\pi & \downarrow & \downarrow &  & \downarrow \\
T^{(R)} & \xrightarrow{i} & P_1^{(R,M)} & \leftarrow & U \times U \times G_m^h \\
pr_2 & \downarrow & \downarrow &  & \downarrow \\
\widetilde{\mathcal{Z}} & \xrightarrow{i} & \widetilde{X}^{(M)} & \leftarrow & U \times G_m^h \\
\end{array}
\]

(3.1)

where the closed immersions and the open immersions are denoted abusively by \( i \) and \( j \) respectively. To describe the sheaf \( i^*j_* \mathcal{H} \) on \( T^{(R)} \), we introduce some notation. Let \( \mathcal{L}_\psi \) be the locally constant sheaf of rank 1 on \( T^{(R)} \times \widetilde{\mathcal{Z}} T^{(R)\vee} \) defined by the Artin-Schreier covering \( t^p - t = \langle x, f \rangle \) and the injection \( \psi: F_p \rightarrow \Lambda^x \) sending 1 to \( \zeta_p \). Let \( \mathcal{L} = \mathcal{L}_\psi \tilde{G}^{(R)\vee} \) be the pull-back on \( T^{(R)} \times \widetilde{\mathcal{Z}} \tilde{G}^{(R)\vee} \) by the characteristic form \( \tilde{G}^{(R)\vee} \rightarrow T^{(R)\vee} \).

**Lemma 3.3.** Let \( \mathcal{F} \) be a locally constant sheaf on \( U = X - D \) isoclinic of slope \( R \) along \( D \). Let \( V \rightarrow U \) be a \( G \)-torsor such that the pull-back of \( \mathcal{F} \) on \( V \) is constant and that the ramification of \( V \) over \( U \) along \( D \) is bounded by \( R^+ \) and is non-degenerate along \( D \) at multiplicity \( R \). Let \( M \) be a representation of \( G \) corresponding to \( \mathcal{F} \).
There exists an idempotent \( e_\mathcal{F} \in \Gamma(T^{(R)} \times \tilde{G}^{(R)\vee}, i^* j_* \mathcal{E}_\mathcal{F}(\mathcal{F})) \) and a canonical isomorphism
\[
(3.2) \quad \text{pr}_1^* ((e_\mathcal{F} \cdot i^* j_* \mathcal{E}_\mathcal{F}(\mathcal{F})) \otimes \mathcal{L}) \to i^* j_* \mathcal{H}
\]
of \( i^* j_* \mathcal{E}_\mathcal{F}(\mathcal{F}) \)-modules.

**Proof.** Let
\[
i \in \Gamma(V \times_k V/\Delta G, \mathcal{H})
\]
denote the image of the identity \( 1 \in \text{End}_G(M) \) by the canonical map
\[
(3.3) \quad \text{End}_G(M) \to \Gamma(V \times_k V, \mathcal{H} \text{om}(\text{pr}_2^* \mathcal{F}, \text{pr}_1^* \mathcal{F}))^G = \Gamma(V \times_k V/\Delta G, \mathcal{H}).
\]
The diagram \((3.1)\) defines canonical maps
\[
(3.4) \quad \Gamma(V \times_k V/\Delta G, \mathcal{H}) \to \Gamma(V \times_k V/\Delta G \times G_m, \mathcal{H}) = \Gamma(W_1^{(R,M)}, j_* \mathcal{H})
\]
\[
\to \Gamma(E^{(R)_0}, i^* j_* \mathcal{H}).
\]
The section \( i \) defines an isomorphism \( j_* \text{pr}_2^* \mathcal{F} \to j_* \text{pr}_1^* \mathcal{F} \) on \( W_1^{(R,M)} \) and hence its image in \( \Gamma(E^{(R)_0}, i^* j_* \mathcal{H}) \) defines an isomorphism
\[
i^* j_* \mathcal{E}_\mathcal{F}(\mathcal{F}) \to i^* j_* \mathcal{H}
\]
of \( i^* j_* \mathcal{E}_\mathcal{F}(\mathcal{F}) \)-modules on the extension \( E^{(R)_0} \) of the vector bundle \( T^{(R)} \) by the finite étale group scheme \( \tilde{G}^{(R)} \). It suffices apply Corollary \([1.22]\) to the extension \( 0 \to \tilde{G}^{(R)} \to E^{(R)_0} \to T^{(R)} \to 0 \) and to the \( \mathcal{A} = i^* j_* \mathcal{E}_\mathcal{F}(\mathcal{F}) \)-module \( \mathcal{M} = i^* j_* \mathcal{H} \).

Let \( \bar{x} \to \bar{Z} \) be a geometric point and \( \chi \in \tilde{G}^{(R)\vee}_{\bar{x}} \) be the geometric point corresponding to a character \( \tilde{G}^{(R)}_{\bar{x}} \to \Lambda^x \). Then, the stalk \( e_\chi \in (j_* \mathcal{E}_\mathcal{F}(\mathcal{F}))_\chi \subset \text{End}(M) \) of \( e_\mathcal{F} \) at \( \chi \) is the projector to the \( \chi \)-part \( M_\chi = \{ a \in M \mid \sigma a = \chi(\sigma)a \text{ for } \sigma \in \tilde{G}^{(R)}_{\bar{x}} \} \). In an \( \ell \)-adic setting, we may replace the construction in the proof of Lemma \([3.3]\) by the Fourier transform as in \([21]\).

The non-additive map \( \text{rank} : \mathcal{E}_\mathcal{F}(\mathcal{F}) \to \mathbb{Z} \) induces a map rank: \( i^* j_* \mathcal{E}_\mathcal{F}(\mathcal{F}) \to \mathbb{Z} \) and
\[
(3.5) \quad \Gamma(T^{(R)} \times \tilde{Z}, i^* j_* \mathcal{E}_\mathcal{F}(\mathcal{F})) \to \Gamma(T^{(R)} \times \tilde{Z}, \tilde{G}^{(R)\vee}, \mathbb{Z}) \xrightarrow{\cong} \Gamma(\tilde{G}^{(R)\vee}, \mathbb{Z})
\]
where the second arrow is an isomorphism since \( T^{(R)} \) is a vector bundle over \( \tilde{Z} \). As the image of the idempotent \( e_\mathcal{F} \) by the map \((3.5)\) we define a locally constant function
\[
rk_\mathcal{F} : \tilde{G}^{(R)\vee} \to \mathbb{Z}.
\]

We define the characteristic cycle of an isoclinic sheaf out of the locally constant function \( \text{rk}_\mathcal{F} \) in \((3.6)\).

**Lemma 3.4.** Let \( S \) be a scheme and \( T \) be a \( G_m^h \)-torsor over \( S \). Let \( X \) be a scheme over \( S \) with a \( G_m^h \) action and \( \pi : X \to T \) be a finite étale morphism over \( S \). Let \( m_1, \ldots, m_n \) be integers and assume that the morphism \( \pi : X \to T \) and the map \( G_m^h \to G_m^h \) sending \( (t_i) \) to \((t_i^{m_i})\) are compatible with the \( G_m^h \)-actions.

Then, the quotient \( X/G_m^h \to T/G_m^h = S \) is finite étale.
Proof. By induction on \( h \), we may assume \( h = 1 \). Since the assertion is local on \( S \), we may assume \( S = \text{Spec } A \) is affine, \( T = \text{Spec } A[t^{\pm 1}] \) is the trivial \( G_m \)-torsor and \( X = \text{Spec } B \) is affine. Then, the \( G_m \)-action on \( X \) defines a grading \( B = \bigoplus_{n \in \mathbb{Z}} B_n \) on a finite étale \( A[t^{\pm 1}] \)-algebra \( B \). The grading on \( B \) is compatible with the natural one on \( A[t^{\pm 1}] \) multiplied by \( m = m_1 \).

We show that the quotient \( X/G_m = \text{Spec } B_0 \) is finite étale over \( A \). Since \( B \) is finite étale over \( A[t^{\pm 1}] \), its direct summand \( B_0 \) is finite flat over \( A \). To show it is étale, we may assume \( A \) is an algebraically closed field. Then, \( B_0 \) is an \( A \)-subalgebra of the étale \( A \)-algebra \( B \otimes_{A[t^{\pm 1}]} A = B_0 \oplus \cdots \oplus B_{q-1} \) defined by \( t \mapsto 1 \), it is also étale over \( A \). \( \square \)

Let \( \tilde{G}^{(R)\vee} \rightarrow \tilde{Z} \) denote the complement of the 0-section. By Lemma \ref{lem:3.3}, the quotient

\[
PG^{(R)\vee} = (\tilde{G}^{(R)\vee} \rightarrow \tilde{Z})/G_m
\]

is a finite étale scheme over \( Z = \tilde{Z}/G_m^h \). Then, since the injection \( \tilde{G}^{(R)\vee} \rightarrow T^{(R)\vee} = (T^*X \times_X \tilde{Z})(\tilde{R}) \) defined over a radicial covering \( F^n: \tilde{Z}(p_n^{-n}) \rightarrow \tilde{Z} \) is compatible with the \( G_m^h \)-actions, it induces a morphism

\[
(3.7) \quad PG^{(R)\vee} = (\tilde{G}^{(R)\vee} \rightarrow \tilde{Z})/G_m^h \rightarrow (T^{(R)\vee} \rightarrow \tilde{Z})/G_m^h = P(T^{(R)\vee}) \rightarrow P(T^*X)
\]

on the quotients defined over a radicial covering \( F^n: Z(p_n^{-n}) \rightarrow Z \). Since the fibers of the map \( \tilde{G}^{(R)\vee} \rightarrow \tilde{Z} \rightarrow PG^{(R)\vee} \) are connected, the pull-back

\[
(3.8) \quad \Gamma(PG^{(R)\vee}, Z) \rightarrow \Gamma(\tilde{G}^{(R)\vee} \rightarrow \tilde{Z}, Z)
\]

is an isomorphism.

Let \( L(1) \) denote the tautological line bundle on the projective space bundle \( P(T^*X) \). We consider the commutative diagram

\[
\begin{array}{ccc}
L(1) \times_{P(T^*X)} PG^{(R)\vee} & \longrightarrow & T^*X \\
\downarrow & & \downarrow \\
PG^{(R)\vee} & \longrightarrow & X
\end{array}
\]

(3.9)

where the top vertical arrow is the composition of the first projection with the blow-up \( L(1) \rightarrow T^*X \) at the 0-section. Then, the pull-back by the flat map \( L(1) \times_{P(T^*X)} PG^{(R)\vee} \rightarrow PG^{(R)\vee} \) and the push-forward by the proper map \( L(1) \times_{P(T^*X)} PG^{\vee} \rightarrow T^*X \) define maps

\[
(3.10) \quad \Gamma(PG^{(R)\vee}, Z) \rightarrow \Gamma(L(1) \times_{P(T^*X)} PG^{\vee}, Z) \rightarrow Z_d(T^*X)
\]

to the free abelian group of \( d \)-dimensional cycles of \( T^*X \). Let

\[
(3.11) \quad L: \Gamma(\tilde{G}^{(R)\vee} \rightarrow \tilde{Z}, Z) \rightarrow Z_d(T^*X) \otimes_{\mathbb{Z}} Q = Z_d(T^*X)Q
\]

denote the composition of the inverse of (3.8) and (3.10) divided by the degree \( p_{n(d-1)} \) of the radicial covering \( F^n: Z(p_n^{-n}) \rightarrow Z \) on which \( \tilde{G}^{(R)\vee} \rightarrow T^{(R)\vee} \) is defined. It is independent of the integer \( n \geq 0 \) such that \( 3.7 \) is defined over \( F^n: Z(p_n^{-n}) \rightarrow Z \). The morphism

\[
L: \Gamma(\tilde{G}^{(R)\vee} \rightarrow \tilde{Z}, Z) \rightarrow Z_d(T^*X)Q
\]

divides by \( p_{n(d-1)} \) and produces the sum of the compositions of the restriction maps with

\[
(3.12) \quad L_i: \Gamma(\tilde{G}^{(R)\vee} \times \tilde{Z} D_i \rightarrow \tilde{Z}, Z) \rightarrow Z_d(T^*X)Q
\]

for \( i = 1, \ldots, h \) such that \( r_i > 1 \).
Definition 3.5. Let $\mathcal{F}$ be a locally constant constructible sheaf of free $\Lambda$-modules on $U$. Assume that the ramification of $\mathcal{F}$ is non-degenerate along $D$. We define the characteristic cycle $\text{Char}(\mathcal{F})$ of $\mathcal{F}$ as an effective $d$-cycle with rational coefficients on the cotangent bundle $T^*X$ and the total dimension divisor $DT(\mathcal{F})$ of $\mathcal{F}$ as a multiple of $D$ with rational coefficient as follows.

1. If $\mathcal{F}$ is isoclinic of slope $R = r_1D_1 + \cdots + r_hD_h$, define $\text{Char}(\mathcal{F})$ by

$$\text{Char}(\mathcal{F}) = (-1)^d \cdot \left( \text{rank } \mathcal{F} \cdot \sum_{i \in I, r_i = 1} [T^*_D, X] + \sum_{r_i > 1} r_i \cdot L_i(\text{rk } \mathcal{F}) \right).$$

If $\mathcal{F}$ is polyisoclinic and if $\mathcal{F} = \bigoplus_j \mathcal{F}_j$ is an isoclinic decomposition, define $\text{Char}(\mathcal{F})$ by

$$\text{Char}(\mathcal{F}) = \sum_j \text{Char}(\mathcal{F}_j).$$

In general, define $\text{Char}(\mathcal{F})$ by étale descent.

2. If $\mathcal{F}$ is isoclinic of slope $R$, define $DT(\mathcal{F})$ by

$$DT(\mathcal{F}) = \sum_i r_i \cdot \text{rank } \mathcal{F} \cdot D_i$$

If $\mathcal{F}$ is polyisoclinic and if $\mathcal{F} = \bigoplus_j \mathcal{F}_j$ is an isoclinic decomposition, define $\text{Char}(\mathcal{F})$ by

$$DT(\mathcal{F}) = \sum_j DT(\mathcal{F}_j).$$

In general, define $DT(\mathcal{F})$ by étale descent.

We expect that $\text{Char}(\mathcal{F})$ has integral coefficients (Conjecture 3.11). We will show that $DT(\mathcal{F})$ has integral coefficients in Proposition 3.10.

If $\mathcal{F}$ is tamely ramified along $D$, we have $R = D$ and the characteristic cycle $\text{Char}(\mathcal{F})$ is involutive. However, it is not involutive in general (see Example 3.6.2 below). In general, the total dimension divisor $DT(\mathcal{F})$ is much less precise than the characteristic cycle $\text{Char}(\mathcal{F})$. However for curves, they are equivalent.

Example 3.6. 1. Assume $\dim X = 1$ and $D = \{x\}$. Recall that, for a representation $V$ of $G_K$, the sum $\dim V + Sw V$ of the rank and the Swan conductor is called the total dimension and denoted $\dim \text{tot} V$. We have

$$\text{Char}(\mathcal{F}) = - \left( \text{rank } \mathcal{F} \cdot [T^*_X, X] + \dim \text{tot}_x(\mathcal{F}) \cdot [T^*_D, X] \right),$$

$$DT(\mathcal{F}) = \dim \text{tot}_x(\mathcal{F}) \cdot D.$$

The minus sign in the formula for $\text{Char}(\mathcal{F})$ comes from that a curve has odd dimension. In particular, $\text{Char}(\mathcal{F})$ and $DT(\mathcal{F})$ have integral coefficients by the classical theorem of Hasse-Arf [22].

2. Let $X = \mathbb{A}^2_k = \text{Spec } k[x, y]$ and let $U$ be the complement of the divisor $D$ defined by $x = 0$. Then by Example 2.18 for the locally constant sheaf $\mathcal{F}$ of rank 1 defined by the Artin-Schreier covering $t^n - t = \frac{1}{x^n}$ where $p \nmid n$, we have $\text{Char}(\mathcal{F}) = [T^*_X, X] + (n+1) \cdot [T^*_D, X]$.

For $\mathcal{F}$ defined by $t^n - t = \frac{y}{x^n}$ where $p \nmid n$, we have $\text{Char}(\mathcal{F}) = [T^*_X, X] + n \cdot [L]$ where $L$ is the sub line bundle of the restriction $T^*X \times X D$ generated by the section $dy$ unless $p = n = 2$. If $p = n = 2$, we have $\text{Char}(\mathcal{F}) = [T^*_X, X] + F_*(L)$ where $L$ is the sub line bundle of the pull-back $T^*X \times X D^{(2-1)}$ by $F : D^{(2-1)} \to D \to X$ generated by the section $\sqrt{y} dx + dy$. 46
3.2 Pull-back and local acyclicity

We define a condition for a morphism \( f : X' \to X \) of smooth schemes to be non-characteristic with respect to a locally constant sheaf on the complement of a divisor on \( X \).

**Definition 3.7.** Let \( X \) and \( X' \) be schemes smooth of dimension \( d \) and \( d' \) over \( k \) respectively and \( U \subset X \) and \( U' \subset X' \) be the complements of divisors \( D \subset X \) and \( D' \subset X' \) with simple normal crossings. Let \( f : X' \to X \) be a morphism over \( k \) such that \( f^{-1}(U) = U' \).

1. Let \( L = \sum_{i} l_i L_i \in Z_d(T^*X)_Q \) be a conic cycle with rational coefficients \( l_i > 0 \).

We say that \( f : X' \to X \) is non-characteristic with respect to \( L \) if the intersection of the inverse image of \( L_i \) by \( f \) is contained in the \( 0 \)-section of \( T^*X' \) for every \( i \).

If \( f : X' \to X \) is non-characteristic with respect to \( L \), then we define \( f^*(L) \in Z_{d'}(T^*X')_Q \) as the linear combination \( \sum_i m_i [f^*L_i] \) of the images of \( [L_i \times_X X'] \in Z_d(T^*X \times_X X') \).

2. Let \( F \) be a locally constant constructible sheaf of \( \Lambda \)-modules on \( U \) such that the ramification of \( F \) is non-degenerate along \( D \). We say \( f : X' \to X \) is non-characteristic with respect to \( F \) if \( f : X' \to X \) is non-characteristic with respect to \( \text{Char}(F) \in Z_d(T^*X)_Q \).

We prove a compatibility of the characteristic cycle with the pull-back by a non-characteristic morphism \( f : X \to Y \).

**Proposition 3.8.** Let \( X \) be a smooth scheme over \( k \) and \( F \) be a locally constant sheaf of free \( \Lambda \)-modules of finite rank on the complement \( U = X - D \) of a divisor \( D \) with normal crossings. Let \( f : X' \to X \) be a morphism of smooth schemes over \( k \) such that \( U' = f^{-1}(U) \) is the complement of a divisor \( D' \subset X' \) with normal crossings.

Assume that either of the following conditions is satisfied.

- \( F \) is totally wildly ramified along \( D \) and the ramification of \( F \) along \( D \) is non-degenerate.

- \( F \) is tamely ramified along \( D \).

Then, if \( f : X' \to X \) is non-characteristic with respect to \( F \), the ramification of the pull-back \( f'^*F \) is non-degenerate along \( D' \) and

\[
(-1)^{\dim X'} \text{Char}(f'^*F) = (-1)^{\dim X} f^* \text{Char}(F), \quad DT(f'^*F) = f^* DT(F).
\]

**Proof.** (s) Since the assertion is étale local, we may assume that \( F \) is isoclinic of slope \( R \). Let \( V \to U \) be a finite étale Galois covering trivializing \( F \) such that the ramification along \( D \) is bounded by \( R^+ \) and is non-degenerate at multiplicity \( R \). Then, by Proposition 2.22, the ramification along \( D' \) of the étale Galois covering \( V' = V \times_U U' \to U' \) is bounded by \( R'^+ \) and is non-degenerate at multiplicity \( R' \). Further, the characteristic form \( \text{Char}_R(V'/U') \) is the pull-back of \( \text{Char}_R(V/U) \). Thus, the assertion follows.

(t) Let \( I \) be a subset of \( \{1, \ldots, h\} \). The conormal bundle \( T^*_D X \) is locally spanned by \( dt_i \) for \( i \in I \) such that \( t_i \) defines \( D_i \). Then by the condition that \( X' \to X \) is non-characteristic to \( T^*_D X \), it follows that the inverse images \( D'_i = f^*D_i \) are smooth and meet transversally each other. Hence, the pull-back \( f^*(\sum_{i=1}^h D_i) = \sum_{i=1}^h D'_i \) and the assertion on \( DT(F) \) follows. Further, if we put \( D'_i = \bigcap_{i \in I} D'_i \), then the pull-back of \( [T^*_D X] \) is \( [T^*_{D'} X'] \) and the assertion on \( \text{Char}(F) \) also follows.

**Corollary 3.9.** Let \( X \) be a smooth scheme over \( k \) and \( F \) be a locally constant sheaf of free \( \Lambda \)-modules of finite rank on the complement \( U = X - D \) of a divisor \( D \) with simple normal crossings. Assume that one of the conditions (s) and (t) in Proposition 3.8 is satisfied.
1. Let C be a smooth curve and \( f : C \to X \) be a morphism over \( k \). Then, we have

\[
\text{dim tot}_x \mathcal{F} \leq (DT_D(\mathcal{F}), C)_x. \tag{3.17}
\]

2. Let \( \Sigma \subset TX \) be the union of the images of the hyperplane bundles defined as zero-locus of non-vanishing sections of the characteristic cycle \( \text{Char}(\mathcal{F}) \). Let C be a smooth curve and \( f : C \to X \) be a morphism over \( k \). For a point \( x \in f^{-1}(D) \) of the inverse image, the following conditions are equivalent.

(1) \( f : C \to X \) is non-characteristic at \( x \) with respect to \( \mathcal{F} \).
(2) We have an equality in (3.17).
(3) The image of the map \( f_* : T_x C \to T_{f(x)}X \) on the tangent space is not contained in \( \Sigma \) defined above.

3. Let \( f : X' \to X \) be a morphism of smooth schemes over \( k \) such that \( f^{-1}(U) \) is the complement of a divisor \( D' \) with simple normal crossings. Then, the following conditions are equivalent.

(1) \( f : X' \to X \) is non-characteristic with respect to \( \mathcal{F} \).
(2) For every point closed point \( x' \) of \( D' \), there exists a curve \( C \) on \( X' \) meeting \( D' \) transversally at \( x' \) such that the composition \( C \to X' \to X \) is non-characteristic with respect to \( \mathcal{F} \).

Proof. 1. Since the assertion is étale local, we may assume that \( \mathcal{F} \) is isoclinic of slope \( R \). Then, the inequality (3.17) follows from Proposition 2.22.1.

2. The equivalence (1) \( \iff \) (3) is clear from the definition of non-characteristicity. If \( f : C \to X \) is non-characteristic with respect to \( \mathcal{F} \), then (3.17) is an equality by Proposition 3.8. Hence, we have (1) \( \Rightarrow \) (2).

We show (2) \( \Rightarrow \) (1) or equivalently (2) \( \Rightarrow \) (3). By the inequality (3.17), we may assume \( \mathcal{F} \) is isoclinic of slope \( R \). Let \( V \) be a \( G \)-torsor over \( U \) for a finite group \( G \) such that the ramification of \( V \) is bounded by \( R' + \) and that \( \mathcal{F} \) is corresponding to a faithful representation \( M \) of \( G \).

If \( \mathcal{F} \) is tamely ramified along \( D \), the equality (3.17) is equivalent to that \( C \) meets \( D \) transversally. Thus in this case (2) implies (1).

Assume that the condition (s) in Proposition 3.8 is satisfied. Then, since the ramification of \( V' = V \times_X C \) over \( U' = U \times_X C \) is bounded by \( R' + \), the equality \( \text{dim tot}_x \mathcal{F} = (DT_D(\mathcal{F}), C)_x \) is equivalent to that every character \( \chi \) of \( G^{(R)} \) such that \( \text{rk}_\mathcal{F}(\chi) \neq 0 \) induces a non-trivial character of \( G^{(R)} \). This means that a non-zero differential form \( \text{Char}_R(\chi) \) in the line \( L_\chi \) for such a character \( \chi \) does not vanish on the tangent line \( T_x C \) and the assertion follows.

3. By 2.(3) \( \Rightarrow \) (1), the condition (1) implies (2). Conversely, by 2.(1) \( \Rightarrow \) (3), the condition (2) implies (1).

On the integrality of the coefficients of the characteristic cycle and of the total dimension divisor, we deduce the following from the classical Hasse-Arf theorem.

Proposition 3.10. 1. The characteristic cycle \( \text{Char}(\mathcal{F}) \) has coefficients in \( \mathbb{Z}_{[\frac{1}{p}]} \).
2. The total dimension divisor \( DT(\mathcal{F}) \) has an integral coefficient.

For the total dimension divisor \( DT(\mathcal{F}) \), the integrality is proved by Xiao Liang [23] by using \( p \)-adic differential equations.
Proof. We may assume that $D$ is irreducible. Since the question is local on an étale neighborhood of the generic point $\xi$ of $D$, we may assume that $\mathcal{F}$ is isoclinic of slope $rD$ and $r > 1$. Further we may assume that the support of the characteristic cycle consists of the 0-section and the image of one line bundle defined over the radicial covering $F^n: D^{(p^{-n})} \to D$ for an integer $n \geq 0$.

1. Let $m$ be the prime-to-$p$ part of the denominator of $r$. Then, the subgroup $\mu_m \subset G_m$ stabilizes the line bundle and acts faithfully on the characteristic form. Hence the rank of $F$ is divisible by $m$ and the assertion follows.

2. It follows from Corollary 3.9.2. \hfill \Box

Conjecture 3.11. The characteristic cycle $\text{Char}(\mathcal{F})$ has integral coefficients.

We will later state Conjecture 3.16 stronger than Conjecture 3.11.

Corollary 3.9 immediately implies the following characterization on the support of the characteristic cycle and the total dimension divisor.

Proposition 3.12. Keep the assumptions in Corollary 3.9.

1. There exists a unique linear combination $L = \sum_{i=1}^{l} l_i D_i$ with integral coefficients $l_i \geq 0$ satisfying the following property: For every irreducible curve in $X$ not contained in $D$ and every point $x \in D \cap C$, we have

\begin{equation}
\dim \text{tot}_x F \leq (L, C)_x.
\end{equation}

Further, for each irreducible component $D_i$ of $D$, there exists a smooth curve $C$ in $X$ meeting $D_i$ transversally and meeting no other irreducible components of $D$ such that we have an equality in (3.18).

The unique linear combination $L$ above is the total dimension divisor $DT(F)$.

2. There exists a closed subset $\Sigma \subset TX$ characterized by the following property: For a point $x \in D$ and a smooth curve $C$ in $X$ meeting every irreducible component of $D$ transversally at $x \in C$, an equality in (3.17) is equivalent to that the image of the map $f_*: T_x C \to T_{f(x)} X$ on the tangent space is not contained in $\Sigma$.

3. The closed subset $\Sigma \subset TX$ in 2. is the union of the following two parts:

   (1) Finitely many hyperplane bundles $D_1, \ldots, D_l$ defined over finite étale schemes of the radicial covering $F^n: Z^{(p^{-n})} \to Z$ for an integer $n \geq 0$.

   (2) The linear span of the union of the tangent bundles $TD_i \subset TX$ for $r_i = 1$.

The support of the characteristic cycle $\text{Char}(\mathcal{F})$ is the union of the following two parts:

   (1) The union of the annihilators $L_1, \ldots, L_l$ of $D_1, \ldots, D_l$.

   (2) The linear span of the union of the conormal bundles $T^*_D X \subset T^*_X$ for $r_i = 1$.

We introduce a condition for a smooth morphism $f: X \to Y$ of smooth schemes to be non-characteristic with respect to a locally constant sheaf on the complement of a divisor on $X$. In the rest of this section, we further assume that $\Lambda$ is a noetherian ring annihilated by an integer invertible in $k$.

Definition 3.13. Let $X$ be a smooth scheme over $k$ and let $\mathcal{F}$ be a locally constant sheaf of free $\Lambda$-modules of finite rank on the complement $U = X - D$ of a divisor $D$ with simple normal crossings. Assume that the ramification of $\mathcal{F}$ along $D$ is non-degenerate. Let $f: X \to Y$ be a smooth morphism of smooth schemes over $k$.

We say that $f$ is non-characteristic with respect to the ramification of $\mathcal{F}$ along $D$ if the inverse image of $\text{Char}(\mathcal{F})$ by $T^*Y \times_Y X \to T^*X$ is contained in the 0-section.
The proposition below answers a question raised by Deligne [11]. We recall the definition of local acyclicity [10, Définition 2.12]. Let \( f : X \to S \) be a morphism of schemes and \( K \) be a complex of sheaves on the étale site of \( X \). Let \( \bar{x} \to X \) be a geometric point of \( X \) and let \( \bar{s} \to S \) denote its image. Then, we say that \( f : X \to S \) is locally acyclic at \( \bar{x} \) relatively to \( K \) if, for every geometric point \( \bar{t} \) of the strict localization \( S_{\bar{x}} \), the canonical map \( K_{\bar{x}} \to RF(X_{\bar{t}} \times_{S_{\bar{t}}} \bar{t}, K) \) is a quasi-isomorphism. We say that \( f : X \to S \) is locally acyclic relatively to \( K \) if it is locally acyclic at every geometric point \( \bar{x} \to X \). It is universally locally acyclic relatively to \( K \) if every base change is locally acyclic.

The following well-known fact is a consequence of the local acyclicity of smooth morphisms.

**Lemma 3.14.** Let \( f : X \to S \) be a smooth morphism of schemes, \( D \) be a divisor of \( S \) with simple normal crossings relatively to \( f : X \to S \) and \( j : U \to X \) be the open immersion of the complement \( U = X - D \). Let \( F \) be a locally constant constructible sheaf of \( \Lambda \)-modules tamely ramified along \( D \). Then, \( f : X \to S \) is universally locally acyclic relatively to \( j_*F \) and to \( Rj_*F \).

For the sake of convenience of the reader, we sketch a proof, similar to that of [14, 1.3.3 (i)].

**Proof.** Since the assertion is local on \( X \), we may assume that \( D \) has simple normal crossings relatively to \( X \to S \) and that there is a smooth map \( X \to \mathbb{A}^n_S \) defined by functions \( t_1, \ldots, t_n \) on \( X \) such that \( D \) is defined by \( t_1 \cdots t_n \). Further, we may assume that the pullback of \( F \) to the inverse image \( U' \subset X' \) of \( U \) by the finite flat covering \( \pi : X' \to X \) defined by \( s_1^m = t_1, \ldots, s_n^m = t_n \) for an integer \( m \geq 1 \) invertible on \( S \) is constant.

The canonical map \( F \to \pi_*\pi^*F \) is injective and the cokernel is locally constant and tamely ramified along \( D \). Further the morphism \( X' \to \mathbb{A}^n_S \) defined by \( s_1, \ldots, s_n \) is smooth and hence \( X' \to S \) is smooth and \( U' \) is the complement of a divisor \( D' \) of \( X' \) with simple normal crossings relatively to \( X' \to S \). Thus \( F \) admits a resolution by the direct images of constant sheaves as above and the assertion is reduced to the case where \( F = \Lambda \) is constant.

Let \( D_1, \ldots, D_n \) be the irreducible components of \( D \) and, for a subset \( I \subset \{1, \ldots, n\} \), set \( D_I = \bigcap_{i \in I} D_i \). Then, we have an exact sequence \( 0 \to j_i! \Lambda \to \Lambda_X \to \bigoplus_{i=1}^n \Lambda_{D_i} \to \cdots \to \bigoplus_{|I|=q} \Lambda_{D_I} \to \cdots \). By the relative purity, we also have an isomorphism \( \bigoplus_{|I|=q} \Lambda_{D_I}(-q) \to R^qj_*\Lambda \). Since the intersections \( D_I \) are smooth over \( S \), it follows from this and the local acyclicity of smooth morphism [10].

**Proposition 3.15.** Let \( X \) be a smooth scheme over \( k \) and let \( F \) be a locally constant sheaf of free \( \Lambda \)-modules of finite rank on the complement \( U = X - D \) of a divisor \( D \) with simple normal crossings. Let \( j : U = X - D \to X \) denote the open immersion and let \( f : X \to Y \) be a smooth morphism of smooth schemes over \( k \). Assume that \( f : X \to Y \) is non-characteristic with respect to the ramification of \( F \) along \( D \) and that either of the following conditions is satisfied.

(s) \( F \) is totally wildly ramified along \( D \) and the ramification of \( F \) along \( D \) is non-degenerate. Further the restriction \( D \to Y \) of \( f \) is flat.

(t) \( F \) is tamely ramified along \( D \).

Then \( f : X \to Y \) is universally locally acyclic relatively to \( j_*F \). In the case (t), \( f : X \to Y \) is universally locally acyclic relatively to \( Rj_*F \).
By the proper base change theorem, if we further assume that $f$ is proper, Proposition implies that $R^q g_*\mathcal{F}$ and $R^q g_*\mathcal{F}$ are locally constant for $g = f \circ j$ under the assumptions.

**Proof.** Since $f: X \to Y$ is assumed smooth, the locally constant sheaf $\mathcal{F}$ on $U$ is universally locally acyclic by the local acyclicity of smooth morphism \cite{6}. Thus, it suffices to prove the assertion for each point on $D$.

Assume (t) is satisfied. Then, by the definition of the characteristic cycle and of non-characteristic morphism, the divisor $D$ has simple normal crossings relatively to $f: X \to Y$. Further, the sheaf $\mathcal{F}$ is tamely ramified along $D$. Hence, it follows from Lemma 3.14 in this case.

Assume (s) is satisfied. First we prove the case where $f: X \to Y$ is of relative dimension 1. Since $D$ is flat over $Y$, it is quasi-finite over $Y$. For each closed point $y$ of $Y$, the fiber $X_y$ is a smooth scheme over $k$ and the local acyclicity of smooth morphism \cite{3.15}. Hence, by Corollary 3.9.2, the total dimension $\dim_{\text{tot}}(\mathcal{F}|_{X_{j(y)}})$ at a closed point $x$ of $D$ of the restriction on the fiber is equal to the intersection number $(DT^*(\mathcal{F}), X_y)_x$. Thus, in the notation of \cite{19}, the function $\varphi$ is locally constant after shrinking $X$ and $S$ if necessary and the assertion follows from \cite[Théorème 2.1.1]{19}.

We prove the general case (s) by reducing it to the case of relative dimension 1. It suffices to show that, for every closed point $x$ of $D$, there exists an open neighborhood of $x$ such that the restriction of $f: X \to Y$ is universally locally acyclic relatively to $j_i \mathcal{F}$. Let $x \in D$ be a closed point of $D$ and $y \in Y$ be the image. Since the assertion is étale local, we may assume that $k$ is algebraically closed. The fiber of the support of the characteristic cycle $\text{Char} \mathcal{F}$ is a union of lines $L_i$ defined over finite extensions of $k$.

By the assumption that $f: X \to Y$ is non-characteristic, their intersections with the image of the injection $T^*_y Y \to T^*_x X$ reduce to 0. Therefore, as $k$ is algebraically closed, there exist functions $t_1, \ldots, t_n$ defined on a neighborhood of $X$ such that $dt_1, \ldots, dt_n$ form a basis of the cokernel $T^*_x X/T^*_y Y$ and that the intersections of $L_j$ with $T^*_y Y \oplus \langle dt_1, \ldots, dt_{n-1} \rangle \subset T^*_x X$ reduce to 0. After shrinking $X$, we define a morphism $g: X \to P = \mathbb{A}^{n-1} \times_k Y$ by $t_1, \ldots, t_{n-1}$. Then, $g: X \to P$ is smooth of relative dimension 1 and non-characteristic with respect to the ramification of $\mathcal{F}$ along $D$. By modifying $t_i$ if necessary without changing $dt_i$, we may assume that the intersection of $D$ with the fiber $g^{-1}(g(x))$ reduced to the point $x$. Hence, after shrinking $X$ if necessary, the morphism $g: X \to P$ is universally locally acyclic relatively to $j_i \mathcal{F}$. Since $P \to Y$ is smooth, the assertion follows from \cite[Corollaire 2.7]{14} and the local acyclicity of smooth morphism.

The following statement is conjectured by Deligne in \cite{11}, in a more general setting.

**Conjecture 3.16.** Let $X$ be a smooth scheme over $k$ and let $\mathcal{F}$ be a locally constant sheaf of free $\Lambda$-modules of finite rank on the complement $U = X - D$ of a divisor $D$ with simple normal crossings. Let $j: U = X - D \to X$ denote the open immersion and let $f: X \to Y$ be a smooth morphism of smooth schemes over $k$.

Let $f: X \to \mathbb{A}^1$ be a flat morphism satisfying either of the conditions (s) and (t) in Proposition 3.13 and let $f \in \Gamma(X, \mathcal{O}_X)$ also denote the function defining the morphism $f: X \to \mathbb{A}^1$. Assume that the intersection of $\text{Char}(\mathcal{F})$ and the section $df: X \to T^*_X X$ is supported in the fiber $T^*_x X$ of a closed point $x$ of $X$. Then, the total dimension of the space of vanishing cycles is given by the intersection number:

\begin{equation}
\dim_{\text{tot}} \phi_x(f, j_i \mathcal{F}) = -(\text{Char}(\mathcal{F}), df(x))_{T^*_x X}.
\end{equation}
Conjecture \[3.16\] implies the integrality of the coefficients of \(\text{Char}(\mathcal{F})\) (Conjecture \[3.11\]). Proposition \[3.15\] implies that Conjecture \[3.16\] holds if the intersection of \(\text{Char}(j_*\mathcal{F})\) and the section \(df: X \rightarrow T^*X\) is empty. If \(x\) is in \(U\), it is a consequence of the Milnor formula proved in [9]. In the case where \(X\) is a surface, Conjecture \[3.16\] is proved under a certain assumption in [20].

### 3.3 Characteristic cycle and characteristic class

We briefly recall the definition of the characteristic class \[3\] specialized to the following situation. Let \(k\) be a field and \(X\) be a smooth separated scheme of finite type of dimension \(d\) over \(k\). Let \(\ell\) be a prime number invertible in \(k\) and \(\Lambda\) be a local ring finite over \(\mathbb{Z}_\ell[\zeta_\ell]\) or a finite extension of \(\mathbb{Q}_\ell(\zeta_\ell)\). Let \(j: U \rightarrow X\) be an open immersion and \(\mathcal{F}\) be a smooth sheaf of flat \(\Lambda\)-modules on \(U\).

Let \(j_1: U \times_k X \rightarrow X \times_k X\) and \(j_2: U \times_k U \rightarrow U \times_k X\) denote the open immersions. Let \(\mathcal{H}_0\) denote the smooth sheaf \(\text{Hom}(pr_2^*\mathcal{F}, pr_1^*\mathcal{F})\) on \(U \times_k U\) and set \(\mathcal{H} = j_1^! j_2^! \mathcal{H}_0\) on \(X \times_k X\). We consider the commutative diagram

\[
\begin{array}{ccc}
X & \xleftarrow{j} & U \\
\delta \downarrow & & \delta_U \\
X \times_k X & \xleftarrow{j \times j} & U \times_k U
\end{array}
\]

where the vertical arrows are the diagonal immersions and regard \(X\) and \(U\) as closed subschemes of \(X \times_k X\) and \(U \times_k U\) respectively. Then, it is shown in \[3\] that the restriction map

\[(3.20) \quad H^{2d}_X(X \times_k X, \mathcal{H}(d)) \longrightarrow H^{2d}_U(U \times_k U, \mathcal{H}_0(d)) = H^0(U, \text{End}(\mathcal{F})) = \text{End}_U(\mathcal{F})\]

is an isomorphism. The pull-back to the diagonal and the trace map define

\[(3.21) \quad H^{2d}_X(X \times_k X, \mathcal{H}(d)) \longrightarrow H^{2d}(X, j_*\text{End}(\mathcal{F})(d)) \longrightarrow H^{2d}(X, \Lambda(d)).\]

The characteristic class

\[C(j_*\mathcal{F}) \in H^{2d}(X, \Lambda(d))\]

is defined as the image by \(3.21\) of the inverse image of the identity \(1_\mathcal{F} \in \text{End}_U(\mathcal{F})\) by the isomorphism \(3.20\). If \(X\) is proper over \(k\), the Lefschetz trace formula implies that the Euler number \(\chi_c(U_k, \mathcal{F}) = \sum q (-1)^q \dim H^q(U_k, \mathcal{F})\) is equal to the image of the characteristic class \(C(j_*\mathcal{F})\) by the trace map \(\text{Tr}: H^{2d}(X, \Lambda(d)) \rightarrow \Lambda\).

We compute the characteristic class assuming that the ramification is isoclinic and non-degenerate. Let \(U = X - D\) be the complement of a divisor with simple normal crossings. To treat the case where \(R = r_1D_1 + \cdots + r_hD_h\) has non integral coefficients, we use the cohomology of classifying space which we recall briefly. Let \(M = m_1D_1 + \cdots + m_hD_h\) be a linear combination with integral coefficients \(m_i \geq 1\). Recall that \(\mathcal{P}^{(R,M)}\) carries a canonical \(G^h_m\)-action. The canonical \(G^h_m\)-equivariant map \(\tilde{X}^{(M)} = P_0^{(R,M)} \rightarrow X\) induces a morphism \([\tilde{X}^{(M)}/G^h_m] \rightarrow X\) of algebraic stacks.

The cohomology of the quotient stack \([\tilde{X}^{(M)}/G^h_m]\) is computed as that of the simplicial scheme \([\tilde{X}^{(M)}/G^h_m]_\bullet\) \[8\] (6.1.2.1). The canonical map \([\tilde{X}^{(M)}/G^h_m] \rightarrow X\) of stacks is interpreted as an augmentation morphism \([\tilde{X}^{(M)}/G^h_m]_\bullet \rightarrow X\) to the constant simplicial scheme.
Lemma 3.17. There exists a constant \( c \geq 1 \) independent of \( M \) or \( \Lambda \) such that the kernel and the cokernel of the pull-back

\[
H^{2d}(X, \Lambda(d)) \to H^{2d}([\tilde{X}(M)/G_m^h], \Lambda(d))
\]

is killed by \((m_1 \cdots m_h)^c\).

Proof. By the stratifications \((D^\circ_i)\) of \( X \) and \((\tilde{D}^\circ_i)\) of \( \tilde{X}(M) \), it suffices to show the assertion for \( H^0(D^\circ_i, \Lambda(r)) \) \( \to H^0([\tilde{D}^\circ_i/G_m^h], \Lambda(r)) \). For \( M = D \), \( \tilde{X}(D) \) is a \( G_m^h \)-torsor over \( X \) and the morphisms \([\tilde{D}^\circ_i/G_m^h] \to D^\circ_i \) are isomorphisms. Hence (3.22) is an isomorphism for \( M = D \).

We show the general case. It suffices to show the assertion for the map on the cohomology defined by \([\tilde{D}^\circ_i/G_m^h] \to [\tilde{D}^\circ_i/G_m^h]\). The canonical map \( \tilde{D}^\circ_i \to \tilde{D}^\circ_i \) fits in a cartesian diagram

\[
\begin{array}{ccc}
\tilde{D}^\circ_i & \longrightarrow & \prod_{i, \notin I} G_m \\
\downarrow & & \downarrow (t_i) \mapsto (t_i^{m_i}) \\
\tilde{D}^\circ_i & \longrightarrow & \prod_{i, \notin I} G_m \\
\end{array}
\]

and is compatible with the map \( G_m^h \to G_m^h \) sending \( (t_i) \) to \( (t_i^{m_i}) \) by Lemma 2.10.1. By computing the cohomology using the simplicial schemes \([\tilde{D}^\circ_i/G_m^h] \) and \([\tilde{D}^\circ_i/G_m^h] \), it suffices to apply the Künneth formula.

We compute the pull-back of the characteristic class \( C(j_1, F) \) in \( H^{2d}([\tilde{X}(M)/G_m^h], \Lambda(d)) \) using the Chern class of the tangent bundle.

Proposition 3.18. Assume that the ramification of \( F \) along \( D \) is isoclinic of slope \( R = r_1 D_1 + \cdots + r_h D_h \) is bounded by \( R^+ \) and is non-degenerate at multiplicity \( R \). Assume that \( F \) is totally wildly ramified along \( D \). Then the pull-back of the characteristic class \( C(j_1, F) \) in \( H^{2d}([\tilde{X}(M)/G_m^h], \Lambda(d)) \) equals

\[
\operatorname{rank}(F) \cdot c_d((TX \times_X \tilde{X}(M))(-R(M))).
\]

If the denominators of the coefficients in \( R \) are invertible in \( \Lambda \), it is further equal to the pull-back of the cycle class

\[
\operatorname{rank}(F) \cdot ((X, X)_{T^*X} + (c(\Omega^1_{X/k})(1 - R)^{-1}[R])_{\dim 0}).
\]

Proof. By the assumption on \( R \), we have \( Z = D \) and \( T^{(R,M)}_1 \) is equal to the complement \( D^{(R,M)}_1 = P^{(R,M)}_1 - (U \times_k U \times_k G_m^h) \). We consider the cartesian diagram

\[
\begin{array}{ccc}
[T^{(R,M)}_1/G_m^h] & \longrightarrow & [P^{(R,M)}_1/G_m^h] \\
\uparrow & & \uparrow \delta \\
[\tilde{D}^{(M)}/G_m^h] & \longrightarrow & [\tilde{X}^{(M)}/G_m^h] \\
\end{array}
\]

where the vertical arrows are regular closed immersions and the right horizontal arrows are the open immersions of complements of the images of the left horizontal arrows. The cycle class of the middle vertical arrow is defined as a cohomology class

\[
[\tilde{X}^{(M)}/G_m^h] \in H^{2d}_{[\tilde{X}^{(M)}/G_m^h]}([P^{(R,M)}_1/G_m^h], \Lambda(d)).
\]
Set $\widetilde{H} = j_*\mathcal{H}_0$ on $[P_1^{\{R,M\}}/G^h_m]$. Since $\mathcal{H}$ is isomorphic to the pull-back of $\text{End}(\mathcal{F})$ on $(V \times_k V)/\Delta G$ and $W_1^{\{R,M\}} \to P_1^{\{R,M\}}$ is étale, the base change map

$$\delta^*\widetilde{H} = \delta^*j_*\mathcal{H}_0 \to j_*^{(M)}\delta_U\mathcal{H}_0 = j_*^{(M)}\text{End}(\mathcal{F})$$

is an isomorphism. Hence the restriction map

$$H^0([\tilde{X}^{(M)}/G^h_m], \delta^*\widetilde{H}) \to H^0(U, \text{End}(\mathcal{F})) = \text{End}_U(\mathcal{F})$$

is an isomorphism and the identity of $\mathcal{F}$ defines a section

$$1_{\mathcal{F}} \in H^0([\tilde{X}^{(M)}/G^h_m], \delta^*\widetilde{H}).$$

The cup product of \eqref{eq:3.26} and \eqref{eq:3.27} defines a cohomology class

\begin{equation}
\label{eq:3.28}
[\tilde{X}^{(M)}/G^h_m] \cdot 1_{\mathcal{F}} \in H^2d([\tilde{X}^{(M)}/G^h_m], [P_1^{\{R,M\}}/G^h_m], \widetilde{H}(d)).
\end{equation}

Let $\varphi: [P_1^{\{R,M\}}/G^h_m] \to X \times X$ be the canonical map. We show that the pull-back

$$\varphi^*: H^2d(X \times X, \mathcal{H}(d)) \longrightarrow H^2d(\varphi^{-1}(X) ([P_1^{\{R,M\}}/G^h_m], \widetilde{H}(d))$$

of the identity $1_{\mathcal{F}} \in H^2d(X \times X, \mathcal{H}(d))$ has the same image as the cup product \eqref{eq:3.28}. The top line in \eqref{eq:3.25} defines an exact sequence

$$H^2d_{[T_1^{\{R,M\}}/G^h_m]}([P_1^{\{R,M\}}/G^h_m], \widetilde{H}(d)) \to H^2d(\varphi^{-1}(X) ([P_1^{\{R,M\}}/G^h_m], \widetilde{H}(d)) \to H^2d(U \times U, \mathcal{H}_0(d)).$$

Since both the identity $1_{\mathcal{F}} \in H^2d(X \times X, \mathcal{H}(d))$ and the cup product \eqref{eq:3.28} have the identity $1_{\mathcal{F}}$ as the image in $H^2d(U \times U, \mathcal{H}_0(d)) = \text{End}_U(\mathcal{F})$, it is reduced to showing $H^2d_{[T_1^{\{R,M\}}/G^h_m]}([P_1^{\{R,M\}}/G^h_m], \widetilde{H}(d)) = 0$. Let $p: [T_1^{\{R,M\}}/G^h_m] \to \tilde{D}^{(M)}/G^h_m$ denote the canonical map. Then, it is further reduced to showing

$$R_{\pi^*}\pi^!\widetilde{H} = 0.$$  

Similarly to Lemma 3.3, we have a canonical isomorphism

$$p^* \bigoplus_{\chi \neq 0} R_{\pi^*}e_{\chi}\text{End}(\mathcal{F}) \otimes \mathcal{L}_\chi \to R_{\pi^*}\widetilde{H}.$$  

Since $R_{\pi^*}\mathcal{L}_\chi = 0$ for $\chi \neq 0$, it induces an isomorphism $R_{\pi^*}\pi^!\widetilde{H} \to \bigoplus_{\chi \neq 0} R_{\pi^*}e_{\chi}\text{End}(\mathcal{F}) \otimes R_{\pi^*}\mathcal{L}_\chi = 0$ and \eqref{eq:3.29} is proved.

Thus, by the definition of the characteristic cycle $C(j, \mathcal{F})$ recalled above, its image in $H^2d([\tilde{X}^{(M)}/G^h_m], \Lambda(d))$ is equal to the image of the cup-product \eqref{eq:3.28} by the variant of \eqref{eq:3.21}. The trace map sends the identity to the rank and the pull-back of the cycle class $[\tilde{X}^{(M)}/G^h_m]$ by the lifting of the diagonal map is the self-intersection product of $[\tilde{X}^{(M)}/G^h_m]$ in $[P_1^{\{R,M\}}/G^h_m]$ and is the top Chern class $c_d(TX \times_X \tilde{X}^{(M)})(-qr\tilde{D}^{(M)})$ of the normal bundle. Hence we obtain \eqref{eq:3.23}.
To deduce (3.24) from (3.23), it suffices to apply the formula
\[
c_d(E \otimes L^{-1}) = (-1)^d \sum_{i=0}^d c_i(E^\vee)c_1(L)^{d-i}
\]
for a vector bundle $E$ of rank $d$, the dual $E^\vee$ and a line bundle $L$.

**Corollary 3.19.** Assume that the ramification of $\mathcal{F}$ along $D$ is isoclinic of slope $R = r_1D_1 + \cdots + r_hD_h$, is bounded by $R^+$ and is non-degenerate at multiplicity $R$. Assume that we have $r_i > 1$ for every $i = 1, \ldots, h$. If the denominators of $r_i$ are invertible in $\Lambda$, for the pull-back of the characteristic class, we have an equality
\[
(3.31) \quad C(j_! \mathcal{F}) = [\text{Char}(\mathcal{F})]
\]
in $H^{2d}([\overline{X}(M)/\mathbb{G}_m^h], \Lambda)$.

**Proof.** By the definition of the characteristic cycle (3.13) and by (3.24), it suffices to show
\[
(X, r \cdot L(\text{rk} \mathcal{F}))_{T^*X} = \text{rank}(\mathcal{F})\left(c(\Omega^1_{X/k})(1 - R)^{-1}[R]\right)_{\text{dim }0}.
\]
By applying the excess intersection formula to the cartesian diagram (3.9), we obtain
\[
(X, L(1) \times \mathcal{P}(T^*X))_{T^*X} = \left(c(\Omega^1_{X/k})c(L(1))^{-1}[PV]\right)_{\text{dim }0}.
\]
Since the pull-back of the line bundle $L(1) \times \mathcal{P}(T^*X)$ to $\overline{G} = \overline{D}$ is $L(-qr\overline{D})$, the assertion follows. \qed

In the case where $\mathcal{F}$ is tamely ramified along $D$ namely $R = D$, the equality (3.31) is an immediate consequence of [21, Corollary 3.2]. In fact, in this case, $[\text{Char} \mathcal{F}]$ is rank $\mathcal{F}$-times the pull-back of the class of the $0$-section of the logarithmic cotangent bundle $T^*X(\log D)$ by the canonical map $T^*X \to T^*X(\log D)$.

**Example 3.20.** Let $k$ be a perfect field of characteristic 2, set $X = \mathbb{P}^2_k \supset U = \mathbb{A}^2_k = \text{Spec } k[x, y]$ and let $D = \mathbb{P}^1_k$ be the line at infinity. Define a finite étale morphism $V \to U$ by the Artin-Schreier equation $t^2 - t = xy$. Then, the ramification of $V$ over $U$ is bounded by $2D^+$ and non-degenerate along $D$ at multiplicity $R = 2D$.

By Example 2.18, the characteristic form $\text{Char}_R(V/U) : \mathbb{Z}/2\mathbb{Z} \to F^*(T^*X(2D) \times X D)$ sends 1 to the section
\[
(3.32) \quad d(xy) + \sqrt{\frac{y}{x}}dx
\]
of $\Omega^1_{X/k}(2D)$ defined on the radial double covering $F : D^{(2-1)} \to D$. The characteristic cycle of the locally constant sheaf $\mathcal{L}$ on $U$ of rank 1 corresponding to the injection $\text{Gal}(V/U) = F_2 \to \{\pm 1\} \subset \Lambda^*$ is the sum of the $0$-section $T^*_X X$ of $T^*X$ and the image of a sub line bundle $L \to F^*(T^*X \times_X D)$ defined by (3.32) defined over the radial double covering $F : D^{(2-1)} \to D$. The composition $L \to F^*(T^*D \times_D D)$ with the the map induced by the canonical surjection $T^*X \times_X D \to T^*D \times_D D$ is an isomorphism.

By Proposition 3.18 the characteristic class $C(j_! \mathcal{L})$ is $(X, X)_{T^*X} + (K_X + 2D) \cdot 2D$ where the canonical divisor $K_X$ of the projective plane $X = \mathbb{P}^2_k$ is $-3D$. Hence, the Euler number is $\chi_c(U_k, \mathcal{L}) = \chi(X, \Lambda) - 2 = 1$. The normalization $Y$ of $X$ in $V$ is a smooth quadric, isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and has the Euler number $\chi(Y, \Lambda) = \chi(X, \Lambda) + \chi_c(U_k, \mathcal{L}) = 3 + 1 = 4.$
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