NilCoxeter algebras categorify the Weyl algebra

Mikhail Khovanov

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1 Introduction

In this paper we present an example of elaborate categorical structures hidden in very simple algebraic objects. We look at the algebra of polynomial differential operators in one variable $x$, also known as the Weyl algebra, and its irreducible representation in the ring of polynomials $\mathbb{Q}[x]$. We construct an abelian category $\mathcal{C}$ whose Grothendieck group can be naturally identified with the ring of polynomials and define exact functors $F_X : \mathcal{C} \to \mathcal{C}$ and $F_D : \mathcal{C} \to \mathcal{C}$ such that

(a) on the Grothendieck group $K(\mathcal{C})$ of the category $\mathcal{C}$ functors $F_X$ and $F_D$ act as the multiplication by $x$ and differentiation, respectively, dimensional representations of the nilCoxeter algebra $A$

(b) there is a functor isomorphism $F_D F_X \cong F_X F_D \oplus \text{Id}$, which lifts the defining relation $\partial x = x \partial + 1$ of the Weyl algebra,

(c) functors $F_X$ and $F_D$ have nice adjointness properties: $F_X$ is left adjoint to $F_D$ and right adjoint to $F_D$, twisted by an automorphism of $\mathcal{C}$.

The category $\mathcal{C}$ is the direct sum of categories $\mathcal{C}_n$ over all $n \geq 0$, where $\mathcal{C}_n$ is the category of finite dimensional representations of the nilCoxeter algebra $A_n$, which is generated by $Y_i, 1 \leq i \leq n - 1$, subject to relations $Y_i^2 = 0$, $Y_i Y_j = Y_j Y_i$ for $|i - j| > 1$ and $Y_i Y_{i+1} Y_i = Y_{i+1} Y_i Y_{i+1}$. The nilCoxeter algebra is the algebra of divided difference operators (see Macdonald [M], Fomin and Stanley [FS])

$$Y_i f = \frac{f - s_i f}{x_{i+1} - x_i},$$

where $f$ is a polynomial in $x_1, \ldots, x_n$ and $s_i f(x_1, \ldots, x_n) = f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n)$. Functors $F_X$ and $F_D$ are induction and restriction functors associated to the inclusion of algebras $A_n \hookrightarrow A_{n+1}$. The following holds

(d) functors $F_X$ and $F_D$, restricted to each $\mathcal{C}_n$, are indecomposable.

It seems likely that ours is the only example (up to obvious modifications of base change, etc.) of an abelian category $\mathcal{C} = \bigoplus_{n \geq 0} \mathcal{C}_n$ and exact functors $F_X$ and $F_D$ satisfying conditions (a)-(d) above.

In Section 2 we study these and other properties of the category $\mathcal{C}$ and functors $F_X, F_D$. In Section 3 we equip $\mathcal{C}$ with a bialgebra-category structure. Specifically, inclusions of algebras $A_n \otimes A_m \hookrightarrow A_{n+m}$ give rise to induction and restriction functors, which, when summed over all $n, m \geq 0$, become functors $M : \mathcal{C}^{\otimes 2} \to \mathcal{C}$ and $\Delta : \mathcal{C} \to \mathcal{C}^{\otimes 2}$ between $\mathcal{C}$ and its second tensor power $\mathcal{C}^{\otimes 2}$. These functors are exact, boast neat adjointness properties and on the Grothendieck group descend to the multiplication and comultiplication in the commutative, cocommutative Hopf algebra $\mathbb{Q}[x]$ of polynomials in one variable. The associativity relation for the multiplication, coassociativity of the comultiplication and the consistency relation between the multiplication and comultiplication become isomorphisms of functors. We check that these isomorphisms satisfy the coherence relations of Crane and Frenkel [CF] for a bialgebra-category.

In Section 4.1 we sketch how working with graded modules and bimodules over the nilCoxeter algebra yields a categorification of the quantum Weyl algebra and of the quantum deformation of the Hopf algebra $H$. The grading shift automorphism in the category of graded modules descends to a map of Grothendieck groups which we interpret as the multiplication by $q$.

In Section 4.3 we present a simple generalization of our construction to cross-products and provide a short comment on the relation of our work to Ariki’s realization [A] of highest weight modules over affine Lie algebras.
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2 The Weyl algebra and bimodules over the nilCoxeter algebra

2.1 The Weyl algebra

The Weyl algebra $W$ is the algebra of differential operators with polynomial coefficients in one variable. For our purposes define $W$ as the algebra over $\mathbb{Z}$ with the unit 1, generators $x, \partial$ and defining relation $\partial x = x \partial + 1$. Let $R_Q$ be the $\mathbb{Q}$-vector space $R_Q$ spanned by $x^0, x^1, x^2, \ldots$. $W$ acts on $R_Q$ via $x \cdot x^i = x^{i+1}$ and $\partial \cdot x^i = i x^{i-1}$. Abelian subgroups $R$ and $R'$ of $R_Q$, generated by $\{x^i/i\}_{i=0}^\infty$ and $\{x^i\}_{i=0}^\infty$, respectively, are $W$-submodules of $R_Q$, i.e. the action of $x$ and $\partial$ has integral coefficients in each of these two bases.

The Weyl algebra has an antiinvolution $\tau : W \to W$ with

$$\tau(x) = \partial, \quad \tau(\partial) = x \quad \text{and} \quad \tau(ab) = \tau(b)\tau(a) \quad \text{for} \ a, b \in W. \tag{2}$$

Let $(,)$ be the symmetric bilinear form on $R_Q$ defined by $(x^i, x^j) = \delta_{i, j}!$. This form is $\tau$-invariant:

$$(ya, b) = (a, \tau(y)b) \quad \text{for} \ y \in W, \ a, b \in R_Q, \tag{3}$$

and it restricts to an integer valued bilinear product $(, ) : R' \times R \to \mathbb{Z}$.

2.2 The nilCoxeter algebra

Let $A_n$ be the unital algebra over $\mathbb{Q}$ generated by $Y_1, \ldots, Y_{n-1}$ with defining relations

$$Y_i^2 = 0 \tag{4}$$
$$Y_i Y_j = Y_j Y_i \quad |i-j| > 1$$
$$Y_i Y_{i+1} Y_i = Y_{i+1} Y_i Y_{i+1}.$$

Fomin and Stanley [FS] call $A_n$ the nilCoxeter algebra. It originally appeared in the work of Bernstein, Gelfand and Gelfand [BGG] on the cohomology of flag varieties and was later investigated and generalized in various ways by Lascoux and Schützenberger [LS], Macdonald [Mc], Kostant and Kumar [KK], Fomin and Stanley [FS] and others. Note that if we change the first relation in (4) to

$$Y_i = (3)$$
$$Y_i Y_{i+1} Y_i = Y_{i+1} Y_i Y_{i+1},$$

we obtain the group algebra of the symmetric group, which is, indeed, closely related to the nilCoxeter algebra:

**Proposition 1** The algebra $A_n$ is isomorphic to the algebra which is spanned over $\mathbb{Q}$ by $Y_w$, as $w$ ranges over elements of the symmetric group $S_n$, with the multiplication

$$Y_{w_1} Y_{w_2} = Y_{w_1 w_2} \quad \text{if} \ l(w_1 w_2) = l(w_1) + l(w_2),$$
$$Y_{w_1} Y_{w_2} = 0 \quad \text{otherwise}, \tag{5}$$

where $l(w)$ is the standard length function on the symmetric group, the number of inversions created by $w$. The isomorphism is given by sending the generator $Y_i$ of $A_n$ to $Y_s$, where $s_i = (i, i+1)$ is the transposition of $i$ and $i+1$.

In particular, $A_n$ has dimension $n!$. Note that $A_0 = A_1 = \mathbb{Q}$. Introduce a trace map $\text{tr}_n : A_n \to \mathbb{Q}$ by

$$\text{tr}_n(Y_{w_0}) = 1 \quad \text{where} \ w_0 \text{ is the longest permutation, } w_0(i) = n - i,$$
$$\text{tr}_n(Y_w) = 0 \quad \text{if} \ w \neq w_0. \tag{6}$$
**Proposition 2** The trace map $\text{tr}_n$ is nondegenerate and makes $A_n$ into a Frobenius algebra.

*Proof:* When we say that the trace map is nondegenerate we mean that for each $y \in A_n, y \neq 0$ we can find $y' \in A_n$ such that $\text{tr}_n(yy') = 1$. Algebras with a nondegenerate trace map are called Frobenius algebras. The basic properties of the length function in the symmetric group imply that $\text{tr}_n$ is nondegenerate. □

For more information about Frobenius algebras we refer the reader to Yamagata [Y] and references therein.

Let $B_1, B_2$ be finite-dimensional algebras over a field $k$ and $N$ a finite-dimensional $(B_1, B_2)$-bimodule. Then $N^\ast = \text{Hom}_k(N, k)$ is naturally a $(B_2, B_1)$-bimodule. The duality functor $\ast$ is a contravariant equivalence between categories of finite-dimensional $(B_1, B_2)$-bimodules and $(B_2, B_1)$-bimodules. When $B_2 = k$, the duality functor $\ast$ is a contravariant equivalence between categories of left and right $B_1$-modules.

If $B_1$ has an automorphism $\nu$, we can use it to twist the right action of $B_1$ on a bimodule $N$: for $y \in B_1$ and $t \in N$ the twisted left action of $B_1$ is $\nu(y)t$. We will denote the resulting bimodule by $\nu N$. An automorphism $\nu$ of $B_2$ allows to twist the right action of $B_2$ on $N$, we denote the resulting bimodule by $\nu N$.

Any algebra $B$ is a bimodule over itself in the obvious way. Denote by $\psi_n$ the involution of $A_n$ which takes $Y_i$ to $Y_{n-i}$ and by $A_n^\psi$ the algebra $A_n$ as a bimodule over itself with the right action twisted by $\psi_n$. Let $1^\psi_n \in A_n^\psi$ be the image of $1 \in A_n$ under the isomorphism $A_n \cong A_n^\psi$ of right $A_n$-modules, so that $1^\psi_n Y_i = Y_{n-i} 1^\psi_n$.

**Proposition 3** $A_n^\ast$-bimodules $A_n^\ast$ and $A_n^\psi$ are isomorphic.

*Proof:* For $w \in S_n$ let $Y_w^\ast \in A_n^\ast$ be the functional

$$Y_w^\ast(Y_\sigma) = \begin{cases} 1 & \text{if } \sigma w = w_0, \\ 0 & \text{otherwise} \end{cases} \quad (7)$$

It is easy to check that the map $A_n^\ast \to A_n^\psi$ given by $Y_w^\ast \mapsto Y_w 1^\psi_n$ is a bimodule isomorphism (use that $s_i w_0 = w_0 s_{n-i}$). □

**Corollary 1** The algebra $A_n$ is injective as a left and right $A_n$-module.

This follows from either of the last two propositions. □

Note that the trace $\text{tr}_n$ is quasi-symmetric w.r.t. $\psi_n$:

$$\text{tr}_n(ab) = \text{tr}_n(\psi_n(b)a) \quad \text{for } a, b \in A_n. \quad (8)$$

In the terminology of Frobenius algebras, $\psi_n$ is the Nakayama automorphism associated with $\text{tr}_n$ (see [Y], Section 2.1).

### 2.3 Bimodules

Denote by $\chi_n$ the algebra map $A_n \to A_{n+1}$ which sends each $Y_i$ to $Y_i$. Proposition [ii] implies that $\chi_n$ is injective. The inclusion $\chi_n : A_n \to A_{n+1}$ induces a left and right $A_n$-module structure on $A_{n+1}$. The left $A_n$ module structure on $A_{n+1}$ commutes with the right $A_{n+1}$-module structure on $A_{n+1}$, the latter coming from the right action of $A_{n+1}$ on itself. Thus, $A_{n+1}$ is an $(A_n, A_{n+1})$-bimodule in a natural way, and we denote this bimodule by $D_{n+1}$. Similary, we get an $(A_{n+1}, A_n)$-bimodule structure on $A_{n+1}$ and denote this bimodule by $X_n$.

**Proposition 4** Bimodules $X_n$ and $D_n$ are left and right projective.

*Proof:* Bimodule $X_n$ is free of rank 1 as a left $A_{n+1}$-module and, thus, left projective. As a right $A_n$-module, it is free of rank $n + 1$ and spanned by $1, Y_n, Y_{n-1} Y_n, Y_{n-2} Y_{n-1} Y_n, \ldots, Y_1 Y_2 \ldots Y_n$ (since any element $s$ of the symmetric group $S_{n+1}$ admits a unique decomposition $s = s_i s_{i+1} \ldots s_n s'$ with $s' \in S_n$ and some $i, 1 \leq i \leq n + 1$). Hence, $X_n$ is projective as a right $A_n$-module. The same argument works for $D_n$. □
Proposition 5 For each $n$, there is an isomorphism of $A_n$-bimodules

$$D_{n+1} \otimes_{A_{n+1}} X_n \cong A_n \oplus (X_{n-1} \otimes_{A_{n-1}} D_n),$$

where $A_n$ is equipped with the standard bimodule structure.

Proof: The left hand side of (9) is isomorphic to $A_{n+1}$, considered as an $A_n$-bimodule via $\chi_n$. The right hand side is the direct sum of $A_n$ and $A_n \otimes_{A_{n-1}} A_n$. We have maps $m_1, m_2$ of $A_n$-bimodules

$$m_1 : A_n \to A_{n+1} \quad \text{and} \quad m_2 : A_n \otimes_{A_{n-1}} A_n \to A_{n+1}$$

which are uniquely determined by $m_1(1) = 1$ and $m_2(1 \otimes 1) = Y_n$. For $w \in S_{n+1}$, the element $Y_w$ of $A_{n+1}$ lies in $m_1(A_n)$ iff $w(n+1) = n+1$. If $w(n+1) \neq n+1$, we can write $w = yz$ for $y, z \in S_n$, so that $Y_w = m_2(Y_y \otimes Y_z)$. Therefore, $m_1$ and $m_2$ define an $A_n$-bimodule isomorphism

$$A_n \oplus (A_n \otimes_{A_{n-1}} A_n) \cong A_{n+1}. \quad \Box$$

Let $A = \bigoplus A_n$ be the direct sum of algebras $A_n$ over all $n$. Algebra $A$ does not have a unit, instead it has an infinite system of pairwise orthogonal idempotents $1 \in A_n, n \geq 0$.

An $(A_n, A_k)$-bimodule $N$ is naturally a bimodule over the algebra $A$. Namely, for $x \in A_i, i \neq n$ we set $xN = 0$ and let $Nx = 0$ if $x \in A_i, i \neq k$. In this way, bimodules $X_n$ and $D_n$ as we sum over all $n$, give rise to $A$-bimodules $X = \bigoplus X_n$ and $D = \bigoplus D_n$. We can reformulate Proposition 5 as

Proposition 6 There is a natural isomorphism of $A$-bimodules

$$D \otimes_A X \cong A \oplus (X \otimes_A D). \quad (12)$$

This is, of course, a bimodule version of the Weyl algebra relation $\partial x = x \partial + 1$. The generators $x$ and $\partial$ of the Weyl algebra become the bimodules $X$ and $D$, the product in the Weyl algebra becomes the tensor product of bimodules, addition becomes the direct sum and 1 becomes the identity bimodule $A$.

In the rest of this section we continue in the similar fashion, interpreting other structures of the Weyl algebra and its polynomial representation in the framework of nilCoxeter algebras.

2.4 Categories and functors

Let $C_n$ be the category of finite-dimensional unital left $A_n$-modules, and let $C = \bigoplus C_n$. The category $C$ can be viewed as the full subcategory of the category of finite-dimensional left $A$-modules, which consists of $A$-modules $N$ with $AN = N$ and $A_nN = 0$ for large enough $n$.

An $A$-bimodule $T$ is called small if it preserves the category $C$, i.e., for any $N \in C$, the module $T \otimes_A N$ is in $C$. Denote by $F_T$ the functor of tensoring with $T$. We can reformulate Proposition 5 as saying that there is a canonical isomorphism of functors

$$F_D F_X \cong F_X F_D \oplus \text{Id}_C \quad (13)$$

The Grothendieck group $K(\mathcal{U})$ of an abelian category $\mathcal{U}$ is the group generated by symbols $[N]$ for all objects $N$ of $\mathcal{U}$ subject to relations $[N_2] = [N_1] + [N_3]$ whenever there is a short exact sequence $0 \to N_1 \to N_2 \to N_3 \to 0$. The Grothendieck group of $C_n$ is isomorphic to $\mathbb{Z}$ and generated by $[L_n]$, where $L_n$ is the simple $A_n$-module ($L_n$ is uniquely defined, up to an isomorphism). We will identify $K(C_n)$ with the abelian subgroup of $R$ generated by $x^n/n!$, by sending $[L_n]$ to $x^n/n!$.

Since $K(C) = \bigoplus_{n \geq 0} K(C_n)$, the Grothendieck group of $C$ is canonically identified with the abelian group $R$, so that from now on we will consider $[N]$, for any object $N$ of $C$, as an element of $R$. The indecomposable projective module in $C_n$ (which we denote $P_n$) is mapped to $x^n$:

$$[P_n] = x^n, \quad [L_n] = \frac{x^n}{n!}. \quad (14)$$
We interpret the bilinear form $(\cdot): R' \times R \to \mathbb{Z}$ via the Hom bifunctor: if $P$ is a projective object of $\mathcal{C}$ and $N$ is any object, then
\[
\dim_{\mathbb{Q}}(\text{Hom}_{\mathcal{C}}(P, N)) = ([P], [N]) \tag{15}
\]
Note that we need $P$ to be projective, otherwise the dimension function on the left hand side will not be additive in $[N]$. Observe also that the form $(\cdot)$ takes values in $\mathbb{Z}$ when restricted to $R' \times R$, but is fractional when restricted to $R \times R$.

Bimodules $X$ and $D$ are right projective, so that the functors of tensoring with them are exact and induce maps $x$ and $\partial$ of the Grothendieck group $R = K(\mathcal{C})$:
\[
[X \otimes_A N] = x[N], \quad [D \otimes_A N] = \partial[N] \quad \text{for} \quad N \in \text{Ob} \, \mathcal{C}. \tag{16}
\]

The functor of tensoring with $X_n$ is the induction functor from $A_n$-modules to $A_{n+1}$-modules, while tensoring with $D_{n+1}$ is the restriction functor from $A_{n+1}$-modules to $A_n$-modules. Since the induction is left adjoint to the restriction, we conclude that $F_{X_n}$ is left adjoint to $F_{D_{n+1}}$ and $F_X$ is left adjoint to $F_D$, i.e., there is a bifunctor isomorphism
\[
\text{Hom}_{\mathcal{C}}(F_X ?, ?) \cong \text{Hom}_{\mathcal{C}}(?, F_D ?). \tag{17}
\]

This isomorphism can be interpreted as the lift of the equality $(xa, b) = (a, \partial b)$ for $a, b \in R_\mathbb{Q}$, since we just established that the Hom-bifunctor lifts the bilinear form $(\cdot)$ (formula (13)).

Note that $(\partial a, b) = (a, zb)$, so that a natural guess says that $F_X$ is not only left but also right adjoint to $F_D$. This is false, but not far from the truth. Consider the bimodule $A_n^\psi$, which was defined in Section 2.2. Denote by $A^\psi$ the $A$-bimodule which is the direct sum of $A_n^\psi$ over all $n$, and by $\Psi: \mathcal{C} \to \mathcal{C}$ the functor $F_{A^\psi}$ of tensoring with $A^\psi$. Since $\psi_n$ is an involution, $\Psi^2 \cong \text{Id}_{\mathcal{C}}$.

**Proposition 7** The functor $F_X$ is right adjoint to $\Psi F_D \Psi$.

**Proof:** We use the following

**Lemma 1** Let $B_1, B_2$ be Frobenius algebras over a field $k$ and $\nu_1, \nu_2$ be Nakayama automorphisms of $B_1, B_2$. Suppose that $N$ is a finite-dimensional $(B_1, B_2)$-bimodule which is projective as a left $B_1$-module and as a right $B_2$-module. Then the functor
\[
N \otimes_{B_2} \, ? : B_2\text{-mod} \to B_1\text{-mod} \tag{18}
\]
of tensoring with $N$ has the right adjoint functor
\[
(N^*)_{\nu_1} \otimes_{B_1} \, ? : B_1\text{-mod} \to B_2\text{-mod} \tag{19}
\]
(here $(N^*)_{\nu_1}$ is the dual of $N$, with the right $B_1$-action twisted by $\nu_1$) and the left adjoint functor
\[
\nu_2^{-1}(N^*) \otimes_{B_1} \, ? : B_1\text{-mod} \to B_2\text{-mod} \tag{20}
\]

In the case when $B_1$ and $B_2$ are symmetric algebras (i.e. $\nu_1, \nu_2$ are identity maps), rather than just Frobenius algebras, this lemma is proved in Rickard [R], Corollary 9.2.4. The same proof works for Frobenius algebras. □

Applying this lemma to the $(A_{n+1}, A_n)$-bimodule $X_n$ proves the proposition (the Nakayama automorphism of $A_n$ is $\psi_n$, hence the conjugation by $\Psi$ in the second adjointness isomorphism). □

The algebra $W$ has a $\mathbb{Q}$-vector space basis $\{x^m \partial^n\}$ for $n, m \geq 0$. We will call this basis the canonical basis of $W$. A product of two elements of the canonical basis decomposes as a linear combination of canonical basis vectors with nonnegative integral coefficients. This basis can be interpreted in terms of indecomposable bimodules. Namely, the $(A, A)$-bimodule $X^\otimes m \otimes D^\otimes n$ (all tensor products are over $A$), which in our theory is naturally associated to $x^m \partial^n$, is the direct sum of $(A_{m+k-n}, A_k)$-bimodules $A_{m+k-n} \otimes_{A_{k-n}} A_k$, over all $k \geq n$, and we have

**Proposition 8** The $(A_{m+k-n}, A_k)$-bimodule $A_{m+k-n} \otimes_{A_{k-n}} A_k$ is indecomposable.

**Proof:** An exercise. □
2.5 Contravariant duality

Denote by \( C_n \) the category of finite-dimensional right \( A_n \)-modules and by \( C^r \) the direct sum of categories \( C_n \) over all \( n \geq 0 \). The duality functor \( \mathcal{N}^* = \text{Hom}(N,k) \), defined in Section 2.2 for bimodules, will be considered in this section as a contravariant functor from \( C \) to \( C^r \).

Let \( u \) be the antiinvolution of \( A_n \) which takes \( Y_i \) to \( Y_i \). It induces an equivalence of categories of left and right \( A_n \)-modules. As we sum over all \( n \), we obtain an equivalence of categories \( U : C^r \to C \).

The functor \( \Omega = U^* \) is a contravariant equivalence \( C \to C \).

**Proposition 9** There are natural isomorphisms of functors

\[
\begin{align*}
\Omega^2 & \cong \text{Id}_C \\
\Omega\Phi & \cong \Phi\Omega \\
\Omega\Phi F_X & \cong F_X\Phi \\
\Omega F_D & \cong F_D\Omega
\end{align*}
\]

**Proof** \( \Omega^2 \cong \text{Id}_C \), since \( u \) is an antiinvolution, and \( \Omega\Phi \cong \Phi\Omega \), since \( u\psi_n = \psi_n u \). Isomorphism \( \Phi \) is a corollary of

**Lemma 2**

1. There is an isomorphism of bimodules

\[
X^*_n \cong D_{n+1} \otimes A_{n+1} A^\psi_{n+1}
\]

2. There are isomorphisms, functorial in \( N \in C_n \),

\[
\begin{align*}
(A^\psi_n \otimes A_n N)^* & \cong N^* \otimes A_n A^\psi_n \\
(X_n \otimes A_n N)^* & \cong N^* \otimes A_n \tilde{D}_{n+1},
\end{align*}
\]

where \( \tilde{D}_{n+1} \) is the \( (A_n, A_{n+1}) \)-bimodule \( A^\psi_n \otimes A_n D_{n+1} \otimes A_{n+1} A^\psi_n \).

3. There are functorial in \( N \in C_n^r \) isomorphisms

\[
U(N \otimes A_n D_{n+1}) \cong X_n \otimes A_n U(N), \quad U(N \otimes A_n A^\psi_n) \cong A^\psi_n \otimes A_n U(N).
\]

Statement 1 of the lemma follows from Proposition 3. Let us now prove (27). If \( N = P_n \), the indecomposable projective \( A_n \)-module, the isomorphism (27) follows from (22). Moreover, (25) also implies that (27) is functorial relative to \( A_n \)-module maps \( P_n \to P_n \) (here \( N = P_n \)). For an arbitrary \( N \), represent \( N \) as the cokernel of a map of projective modules: \( P_n^{\otimes a} \to P_n^{\otimes b} \to N \to 0 \). Applying the functors on the left and right hand sides of (27) to each term of this exact sequence, and using the exactness of tensoring with \( X_n \) and \( \tilde{D}_{n+1} \), we conclude that (27) holds for any \( N \), functorially in \( N \). Other statements of the lemma can be proved in a similar or easier fashion. \( \square \)

Armed with Lemma 2, we compute, for \( N \in C \),

\[
\Omega\Phi F_X(N) = \Psi\Omega(X \otimes N) = \Psi U(N^* \otimes \tilde{D}) = \Psi^2 X \otimes (\Psi U(N^*)) = F_X\Psi \Omega(N).
\]

Isomorphism (24) is adjoint to (23). \( \square \)

2.6 The integral

We can next ask about the meaning of the indefinite integral in our model. The formula

\[
\int x^n = \frac{x^{n+1}}{n+1}
\]

suggests to look for an exact functor from \( C_n \) to \( C_{n+1} \) which takes the projective module \( P_n \) to a module which is \( n + 1 \) times “smaller” than the projective module \( P_{n+1} \) (since in our correspondence the image of the projective module \( P_i \) in the ring of polynomials is \( x^i \)). Let \( I_n \) be the \((A_{n+1}, A_n)\)-bimodule, which is
isomorphic to \( A_n \) as the right \( A_n \)-module, and the left \( A_{n+1} \)-action is via the homomorphism of algebras \( t_{n+1} : A_{n+1} \to A_n, t_{n+1}(Y_i) = Y_i \) for \( i < n \) and \( t_{n+1}(Y_n) = 0 \). Since \( I_n \) is projective as a right \( A_n \)-module, the functor \( F_{I_n} : C_n \to C_{n+1} \) of tensoring an \( A_n \)-module with \( I_n \) is exact. Moreover, \( F_{I_n} \) takes the indecomposable projective module \( P_n \) to a module of dimension \( nl \), while the projective generator \( P_{n+1} \) of \( C_{n+1} \) has dimension \((n + 1)! \), so that the desired relation holds: \( [I_n \otimes_{A_n} N] = \int [N] \) for \( N \in \text{Ob}(C_n) \). To formulate this relation without the index \( n \), we form \( I = \bigoplus_{n=0}^{\infty} I_n \), the \( A \)-bimodule which is the direct sum of \( I_n \) over all \( n \). Then we have

\[
[I \otimes_A N] = \int [N] \quad \text{for all} \quad N \in \text{Ob}C. \tag{31}
\]

The following result is obvious:

**Proposition 10**  
There are bimodule isomorphisms \( D_{n+1} \otimes_{A_{n+1}} I_n \cong A_n \) and \( D \otimes_A I \cong A \).

This isomorphism can be considered as a categorification of the formula \( d \int f(x) = f(x) \), for a polynomial function \( f(x) \). On the other hand, we don’t get to categorify the formula \( \int df(x) = f(x) \), for \( I \otimes_A D \) is not isomorphic to \( A \) as an \( A \)-bimodule.

### 2.7 Multiplication and the Leibniz rule

Let \( \gamma_{n,m} \) be the algebra homomorphism \( A_n \otimes A_m \to A_{n+m} \) given by

\[
\gamma_{n,m}(Y_i \otimes 1) = Y_i, \quad \gamma_{n,m}(1 \otimes Y_i) = Y_{n+i}. \tag{32}
\]

\( \gamma_{n,m} \) is injective and induces a bifunctor, denoted \( J_{n,m} \), from the product \( C_n \times C_m \) of categories \( C_n \) and \( C_m \) to \( C_{n+m} \):

\[
J_{n,m}(N_1, N_2) = A_{n+m} \otimes_{(A_n \otimes A_m)} (N_1 \otimes N_2) \quad \text{for} \quad N_1 \in C_n, N_2 \in C_m. \tag{33}
\]

Denote by \( J \) the bifunctor \( C \times C \to C \), which is the direct sum of \( J_{n,m} \) over all \( n, m \geq 0 \).

**Proposition 11**  
1. Bifunctor \( J \) is biexact.

2. There is a functorial isomorphism \( F_D \circ J(N_1, N_2) \cong J(N_1, F_D N_2) \otimes J(F_D N_1, N_2) \), satisfying the natural consistency relation for the decomposition of \( F_D \circ J(N_1, N_2, N_3) \).

We omit the proof. \( \square \)

Since \( J \) is biexact, it induces a map of Grothendieck groups \( K(C) \times K(C) \to K(C) \), which is just the multiplication in the ring of polynomials. Part 2 of the proposition is a functor version of the Leibniz rule \( \partial(ab) = (\partial a)b + a(\partial b) \).

### 3 The bialgebra-category structure of \( C \)

#### 3.1 Multiplication and comultiplication

The algebra of polynomials \( R_Q = \mathbb{Q}[x] \) has a comultiplication \( c(x) = x \otimes 1 + 1 \otimes x \) which makes \( R_Q \) into a bialgebra. The subring \( R \) of \( R_Q \) is stable under the comultiplication and has a structure of a bialgebra over \( \mathbb{Z} \). We will explains in detail how to lift the bialgebra structure from \( R \) to the category \( C \).

Let \( n = (n_1, \ldots, n_i) \) be an ordered \( i \)-tuple of nonnegative integers. Let \( A_n = A_{n_1} \otimes \cdots \otimes A_{n_i} \) and denote by \( C_n \) the category of finite dimensional left \( A_n \)-modules. Let \( C^{\otimes i} \) the direct sum of categories \( C_n \) over all \( i \)-tuples \( n \).

Algebra homomorphisms \( \gamma_{n,m} : A_n \otimes A_m \to A_{n+m} \), summed over all \( n \) and \( m \), define induction and restriction functors:

\[
M : C^{\otimes 2} \to C, \quad \Delta : C \to C^{\otimes 2}. \tag{34}
\]
Note that the Grothendieck group of $C^{\otimes i}$ is naturally isomorphic to the $i$-th tensor power of $K(C)$. The symmetric group $S_i$ acts on the set of $i$-tuples by permutations of terms. This action induces an action of $S_i$ on the category $C^{\otimes i}$. We denote by $S_{j,j+1}$ the action of the transposition $(j,j+1)$.

If a functor $G_j : C^{\otimes k} \to C^{\otimes k}$, $k = 1, 2$ is given by tensoring with a bimodule $W_k$, denote by $G_1 \otimes G_2$ the functor $C^{\otimes (1+1+2)} \to C^{\otimes (1+2+2)}$ of tensoring with the bimodule $W_1 \otimes Q W_2$.

**Proposition 12**

1. $M$ is left adjoint to $\Delta$ and right adjoint to $S_{12} \Delta$.

2. There are functor isomorphisms

$$MS_{12} \cong \Psi M \Psi \otimes^2$$

$$S_{12} \Delta \cong \Psi^2 \Delta \Psi$$

3. Functors $M$ and $\Delta$ are exact and on the Grothendieck group descend to the multiplication and comultiplication in the bialgebra $K(C)$.

**Proof** Part 2 of the proposition follows from an obvious computation with bimodules. Next, $M$ is induction and $\Delta$ is restriction, thus, $M$ is left adjoint to $\Delta$. $A_{n+m}$ is projective as left or right $A_n \otimes A_m$-module, so we can apply Lemma 3 and conclude that $M$ is right adjoint to $\Psi \otimes^2 \Delta \Psi$. Together with the isomorphism (36), this implies that $M$ is right adjoint to $S_{12} \Delta$. Since $M$ and $\Delta$ each have left and right adjoints, they are exact. □

**Proposition 13**

1. There are functor isomorphisms

$$M(M \otimes \text{Id}) \cong M(\text{Id} \otimes M),$$

$$(\Delta \otimes \text{Id}) \Delta \cong (\text{Id} \otimes \Delta) \Delta,$$

$$\Delta M \cong M \otimes^2 S_{23} \Delta \otimes^2.$$

**Proof** Functors $M(M \otimes \text{Id})$ and $M(\text{Id} \otimes M)$, restricted to $C_n \otimes C_m \otimes C_k$, are canonically isomorphic to the functor of tensoring with $A_{n+m+k}$, considered as a left $A_{n+m+k}$-module and a right $A_n \otimes A_m \otimes A_k$-module. Hence the functor isomorphism (37). The same argument works for (38). To prove (39), note that both sides of it decompose as direct sums of functors $C_n \otimes C_m \to C_k \otimes C_l$, over all quadruples $(n, m, k, l)$ such that $n + m = k + l$. The left hand side of (39), as a functor $C_n \otimes C_m \to C_k \otimes C_l$, is naturally isomorphic to the functor of tensoring with $A_{n+m}$, considered as a left $A_k \otimes A_l$ and a right $A_n \otimes A_m$-module, via algebra homomorphisms $\gamma_{k,l}$ and $\gamma_{n,m}$.

**Lemma 3** Let $w_1, \ldots, w_p$, where $p = \min(n, m, k, l)$, be minimal length representatives of double cosets $S_k \times S_l \setminus S_{k+l} \cap S_n \times S_m$. Then $A_{n+m}$ is isomorphic, as an $(A_k \otimes A_l, A_n \otimes A_m)$-bimodule, to the direct sum of subbimodules of $A_{n+m}$, spanned by $Y_{w_1}, \ldots, Y_{w_p}$.

We omit the proof of the lemma. □

The right hand side of (39), as a functor $C_n \otimes C_m \to C_k \otimes C_l$, is isomorphic to the direct sum (over all admissible $r$) of the following functors: restrict from $A_n \otimes A_m$ to $A_r \otimes A_{n-r} \otimes A_{k-r} \otimes A_{l+r-n}$ and then induce to $A_k \otimes A_l$. Denote the corresponding $(A_k \otimes A_l, A_n \otimes A_m)$-bimodule by $B_r$, it has a canonical generator that we will call $g_r$. To $r$ there is associated a minimal length representative, $w(r)$, of the double cosets $S_k \times S_l \setminus S_{k+l} \cap S_n \times S_m$. Namely, $w(r)$ is the permutation that preserves the first $r$ elements of the set $\{1, 2, \ldots, n+m\}$, shifts the next $n-r$ elements by $k-r$ to the right, shift the following $k-r$ elements by $n-r$ to the left and preserves the last $l+r-n$ elements. Sending the generator $g_r$ of this bimodule to $Y_{w(r)} \in A_{n+m}$ and summing over all admissible $r$ gives us an isomorphism of bimodules $\oplus B_r \cong A_{n+m}$.

Finally, summing over all $(n, m, k, l)$ with $n + m = k + l$, we get a functor isomorphism (39). □

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3.2 Coherence relations

Bialgebra-categories first appeared in the work of Crane and Frenkel [CF]. Crane and Frenkel argued that, while Hopf algebras produce invariants of 3-manifolds, quantum invariants of 4-manifolds will be governed by Hopf categories. In Hopf categories multiplication and comultiplication operations become functors, functor isomorphisms take place of (co)associativity of (co)multiplication and of the consistency relation between multiplication and comultiplication. Crane and Frenkel imposed 4 coherence relations on these functor isomorphisms. These relations pop up in our simple example:

Proposition 14 Isomorphisms (37), (38) and (39) satisfy the coherence relations of Crane and Frenkel for bialgebra-categories.

The coherence relation for the multiplication can be viewed as a cube, depicted in Figure 1.

![Figure 1](image)

In the vertices of the cube we have placed categories, arrows are functors and 6 square facets of the cube are functor isomorphisms. For simplicity we write $1$ for the identity functor $\text{Id}$. Any path leading from $C^\otimes 4$ to $C$ defines a functor, and any square facet defines an isomorphism of functors. Starting with the functor corresponding to a path, we can apply all 6 isomorphisms and return to the functor we started with. The coherence relation requires this natural transformation of functors to be the identity. This relation is obvious in our case. Note that the coherence relation for the multiplication is just the coherence relation for the tensor product functor in the monoidal categories, also known as the pentagon associativity (see Mac Lane [M], for instance).

The coherence relation for the comultiplication is obtained from Figure 1 by reversing all arrows and changing all appearances of $M$ into $\Delta$. This coherence holds in our category for obvious reasons too. Moreover, if we start from the multiplication coherence relation and change all functors and functor isomorphisms to their right adjoints, we get the coherence relation for the comultiplication. Or, if we start from the multiplication coherence relation and pass to left adjoints, we again get the comultiplication coherence relation (after canceling out all appearances of permutations in left adjoints, since the left adjoint of $M$ is $S_{12}\Delta$).

There are two coherence relations that contain both multiplication and comultiplication. One of them is depicted in Figure 2. To get the other one, change Figure 2 in the following way: exchange $M$ with $\Delta$ everywhere, reverse the direction of all arrows and invert the order of all compositions of functors, i.e., $M^2 \circ S_{23}$ should become $S_{23} \circ \Delta^2$. 

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In our category $C$ functors $M$ and $\Delta$ have nice adjointness properties, and these two coherence relations are equivalent via adjointness. Figure 2 coherence cube in the category $C$ follows from a simple computation with double cosets $S_k \times S_l \setminus S_{n+m+p}/S_m \times S_p \times S_p$ for $k + l = n + m + p$. We omit the details. □

3.3 Other structures

Commutativity and cocommutativity: The bialgebra $R$ is commutative and cocommutative. The bialgebra-category $C$ is not commutative or cocommutative, in the sense that $MS_{12}$ is not isomorphic to $M$ and $S_{12}\Delta$ is not isomorphic to $\Delta$. Instead, we have isomorphisms (35) and (36), which say that $MS_{12}$, resp. $S_{12}\Delta$ is isomorphic to $M$, resp. $\Delta$, twisted by the involution functor $\Psi$. We will refer to these properties of $M$ and $\Delta$ as quasi-commutativity and quasi-cocommutativity, respectively. What are the coherence relations for quasi-commutativity and quasi-cocommutativity? First of all, the usual coherence cube for the associativity and commutativity constraints in symmetric monoidal categories can be twisted by $\Psi$ into the one, depicted in Figure 3 (where $\Psi^2$ denotes $\Psi \otimes \Psi$, etc.)
To obtain the coherence relation between the quasi-cocommutativity and coassociativity isomorphisms, change the direction of all arrows in Figure 3, substitute $\Delta$ for $M$ and invert the order of all compositions. Finally, the Figure 4 below shows a coherence cube for the “mixed” quasi-(co)commutativity.

![Figure 4]

**Proposition 15** These 3 coherence relations hold in the category $C$.

**Unit and counit:** Let $\mathbb{Q}$-vect be the category of finite-dimensional $\mathbb{Q}$-vector spaces. The functor $\iota : \mathbb{Q}$-vect $\to C$ which takes a vector space $V$ to itself, considered as a module over $A_0 = \mathbb{Q}$, plays the role of the unit in the bialgebra-category $C$. The functor $\epsilon : C \to \mathbb{Q}$-vect which takes $C_n$ to 0 for $n > 0$ and $V \in C_0$ to $V \in \mathbb{Q}$-vect is the counit functor.

**Antipode:** So far we referred to $C$ as a bialgebra-category, rather than a Hopf category, and did not say a word about the antipode. The antipode $s$ in the Hopf algebra $R$ is given by $s(x) = -x$. Clearly, the antipode cannot be lifted to any exact functor in $C$, since it does not have positive coefficients in the basis of simple modules. This negativity is not a serious obstacle, though. We can pass to the bounded derived category $D^b(C)$ of $C$, derive the functors $M$ and $\Delta$ and define the antipode functor $T : D^b(C) \to D^b(C)$ as the composition of a shift by $[n]$ (for $C_n$) in the derived category and $\Psi$,

$$T(N) = \Psi N[n], \text{ for } N \in D^b(C_n).$$

(40)

On the Grothendieck group level $\Psi$ does nothing, but it enables us to lift the identity $s(ab) = s(b)s(a)$ to the isomorphism of functors $TM \cong MT \otimes S_{12}$. But we are in for a bigger trouble: there is no functor isomorphism

$$M(T \otimes \text{Id})\Delta \cong \iota \epsilon,$$

(41)

which any self-respecting Hopf category must have. No easy way to save the day by modifying the antipode functor is in sight. The problem lies with our childish definition of tensor powers of $C$, as the direct sum of many little blocks. One conjectural remedy would be to glue these little pieces into a more sophisticated construct, which should retain all the nice bialgebra-category properties of $C$ and should also have an antipode functor with the isomorphism $[\Pi]$ and coherence relations for it.
4 Miscellaneous

4.1 Graded bimodules and a categorification of the quantum Weyl algebra

The algebra $A_n$ is graded, with each $Y_i$ in degree 1, and the Poincare polynomial of $A_n$ is $[n]!$ where $[n]! = [1] \cdots [n]$ and $[i] = 1 + q + \cdots + q^{i-1}$. Let $\mathcal{C}_n$ be the category of finite-dimensional graded left $A_n$-modules and $\mathcal{C} = \bigoplus_{n \geq 0} \mathcal{C}_n$. Let $\{i\}$ be the functor of shifting the grading up by $i$. Bimodules $X$ and $D$ over $A$ are graded, tensoring with these bimodules over $\mathbb{A}$ give us functors, denoted $F_X, F_D$, in the category $\mathcal{C}$.

**Proposition 16** There is a functor isomorphism

$$F_{n}E_{X} \cong E_{X}E_{D}(1) \oplus \text{Id}$$

We define the quantum Weyl algebra as the algebra over $\mathbb{Z}[q,q^{-1}]$, generated by $x$ and $\partial$, with relation $\partial x = qx\partial + 1$. Let $R$ be the module over the quantum Weyl algebra, spanned over $\mathbb{Z}[q,q^{-1}]$ by $[n]!$, with the action $x \cdot x^i = x^{i+1}, \partial x^i = [i]x^{i-1}$.

The Grothendieck group $K(\mathcal{C})$ of $\mathcal{C}$ is a free $\mathbb{Z}[q,q^{-1}]$-module, spanned by the images of simple modules $L_n$. The $\mathbb{Z}[q,q^{-1}]$-module structure comes from the grading, $[N\{i\}] = q^i[N]$, for a graded module $N$. Thus,

$$[L_n] = \frac{x^n}{[n]!}, \quad [P_n] = x^n.$$

As a result, $K(\mathcal{C})$ can be naturally identified with $\overline{R}$. All other structures described in Section 3 have their graded versions. We skip the details.

The product $M : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ is again defined as the induction functor, while the coproduct $\Delta$, considered as a functor from $\mathcal{C}_n$ to $\bigoplus_{0 \leq k \leq n} \mathcal{C}_k \otimes \mathcal{C}_{n-k}$, is the restriction from $A_n$ to $A_k \otimes A_{n-k}$, composed with the shift in the grading up by $n-k$. On the Grothendieck group, the coproduct functor acts as the comultiplication $\Delta(x) = x \otimes 1 + q \otimes x$. Functor isomorphisms of Proposition 16 hold in the graded case as well and all results of Section 3 generalize easily to the graded case.

4.2 Representations of symmetric groups

The bialgebra-category $\mathcal{C}$ is reminiscent of a similar structure for symmetric groups, discovered by Geissinger [G], who observed that induction and restriction functors associated to inclusions of symmetric groups $S_n \times S_m \hookrightarrow S_{n+m}$ induce a bialgebra structure on the direct sum of Grothendieck groups of the categories of $S_n$-modules, over all $n$. Geissinger [G] and Zelevinsky [Z] consistently derived many classical results on representations of symmetric groups from this Hopf algebra structure. Zelevinsky also generalized this construction from symmetric groups to wreath products of symmetric groups with finite groups and to $GL(n, \mathbb{F})$, for a finite field $\mathbb{F}$. Although Geissinger and Zelevinsky work mostly with Grothendieck groups, their results can be immediately reformulated in terms of categories. In particular, induction and restriction define a bialgebra-category structure on the category $\bigoplus_{n \geq 0} k[S_n]$-mod, where $k$ is a field and $k[S_n]$-mod the category of finite-dimensional modules over the group algebra of $S_n$. There are several other interesting examples of bialgebra-categories that naturally appear in representation theory. We will discuss them elsewhere.

4.3 Nil wreath products

Let $B$ be an algebra over $\mathbb{Q}$. By $A_n(B)$ we denote the semidirect product of $A_n$ and $B^\otimes n$, with the multiplication $Y_w b_1 \otimes \cdots \otimes b_n = b_{w(1)} \otimes \cdots \otimes b_{w(n)} Y_w$ for $w \in S_n$ and $b_i \in B$.

**Proposition 17** If $B$ is a Frobenius algebra then $A_n(B)$ is also Frobenius.

If $B = \mathbb{Q}[z]/\{z^k = 0\}$, denote $A_n(B)$ by $A_n(k)$. Let $\mathcal{C}_n(k)$ be the category of finite dimensional $A_n(k)$-modules and $\mathcal{C}(k) = \bigoplus_{n \geq 0} \mathcal{C}_n(k)$. Inclusions $A_n(k) \hookrightarrow A_{n+1}(k)$ induce induction and restriction functors between categories $\mathcal{C}_n(k)$ and $\mathcal{C}_{n+1}(k)$. Denote by $F_{X,k}$, resp. $F_{D,k}$, the direct sum of these induction, resp. restriction functors over all $n$. 

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Proposition 18 There is a functor isomorphism $F_{D,k}F_{X,k} \cong F_{X,k}F_{D,k} \oplus (\text{Id}^\otimes k)$.

Various constructions and results of previous sections, including adjointness isomorphisms and bialgebra-category structures, can be generalized to algebras $A_n(k)$. These algebras are nilpotent counterparts of the wreath products of symmetric groups with cyclic groups and of Ariki-Koike cyclotomic Hecke algebras [AK]. For instance, the nilCoxeter and Hecke algebras belong to a two-parameter family of algebras with generators $T_1, \ldots, T_{n-1}$ and relations $T_i^2 = aT_i + b, T_iT_j = T_jT_i$ for $|i - j| > 1$ and $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$ (specializing $a = b = 0$, resp. $a = 1 - q, b = q$, gets us the nilCoxeter algebra, resp. the Hecke algebra).

From this point of view, our categorification of the Weyl algebra action on polynomials is a toy degeneration of Ariki’s magnificent realization [A] of irreducible highest weight modules over the affine Lie algebra $\hat{\mathfrak{sl}}_n$ as Grothendieck groups of categories of modules over cyclotomic Hecke algebras.

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