ON THE DYNAMICS OF CHARGED PARTICLES IN AN INCOMPRESSIBLE FLOW: FROM KINETIC-FLUID TO FLUID-FLUID MODELS

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Abstract. In this paper, we are interested in the dynamics of charged particles interacting with the incompressible viscous flow. More precisely, we consider the Vlasov–Poisson or Vlasov–Poisson–Fokker–Planck equation coupled with the incompressible Navier–Stokes system through the drag force. For the proposed kinetic-fluid model, we first establish the global-in-time existence of weak solutions. The proof relies on the moment, entropy, and $L^p$-estimates combined with weak and strong compactness arguments based on the velocity averaging and Aubin–Lions lemmas. We then study the asymptotic regime corresponding to strong local alignment and diffusion forces. Under suitable well-prepared initial data, we rigorously derive a coupled isothermal/pressureless Euler–Poisson system and incompressible Navier–Stokes system (in short, EPNS system). For this hydrodynamic limit, we employ the modulated kinetic energy estimate together with the relative entropy method and the bounded Lipschitz distance. We also construct a global-in-time strong solvability for the isothermal/pressureless EPNS system. In particular, this global solvability gives the estimates of hydrodynamic limit hold for all time. Finally, we provide the large-time behavior of the isothermal/pressureless EPNS system under suitable assumptions on the regularity of solutions. This shows the fluid velocities are aligned with each other and the fluid density converges to the background state exponentially fast as time tends to infinity.

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1. Introduction

In this work, we are interested in the dynamics of charged particles immersed in an incompressible viscous fluid. The motion of small solid charged particles (resp. affected by a white noise random force) can be described by the Vlasov–Poisson system (resp. Vlasov–Poisson–Fokker–Planck) and the incompressible viscous fluid can be modeled by the Navier–Stokes system [1, 2]. Specifically, let \( f = f(x, \xi, t) \) be the number density of charged particles at position \( x \in \mathbb{T}^d \) with velocity \( \xi \in \mathbb{R}^d \) at time \( t \in \mathbb{R}_+ \) and \( v = v(x, t) \) be the bulk velocity of the incompressible viscous fluid, respectively. Then, the dynamics of a pair \((f, v)\) is governed by the following Vlasov–Poisson–Navier–Stokes (in short, VPNS) system

\[
\begin{align*}
\partial_t f + \xi \cdot \nabla_x f + \nabla_\xi \cdot ((v - \xi) - \nabla_x U) f &= \nabla_\xi \cdot (\sigma \nabla_\xi f - (u - \xi)f), \quad (x, \xi) \in \mathbb{T}^d \times \mathbb{R}^d, \ t > 0, \\
- \Delta_x U &= \rho - c, \\
\partial_t v + (v \cdot \nabla_x) v + \nabla_x p - \Delta_x v &= -\int_{\mathbb{R}^d} (v - \xi) f \, d\xi, \\
\nabla_x \cdot v &= 0
\end{align*}
\]

subject to initial data:

\[
(f(x, \xi, 0), v(x, 0)) = (f_0(x, \xi), v_0(x)), \quad (x, \xi) \in \mathbb{T}^d \times \mathbb{R}^d,
\]

where \( \rho = \rho(x, t) \) and \( u = u(x, t) \) denote the local averaged particle density and velocity, respectively:

\[
\rho(x, t) := \int_{\mathbb{R}^d} f(x, \xi, t) \, d\xi \quad \text{and} \quad u(x, t) := \frac{1}{\rho(x, t)} \int_{\mathbb{R}^d} \xi f(x, \xi, t) \, d\xi.
\]

Here \( \sigma \geq 0 \) represents the strength of the diffusive force and \( c > 0 \) denotes the background state which is chosen as

\[
c := \int_{\mathbb{T}^d} \rho \, dx.
\]

Throughout this paper, without loss of generality, we may assume that \( c = 1 \) due to the conservation of mass.

The coupled kinetic-fluid models describe the motion of dispersed particles in an underlying gas. It has received considerable attention due to its wide range of applications in the modeling of chemical engineering, atmospheric pollution, aerosols, and sprays [3, 8, 11, 46]. We refer to [23, 41] for more physical backgrounds and modeling issues for the interactions between particles and fluids. Our main system (1.1) consists of the Vlasov–Poisson/Vlasov–Poisson–Fokker–Planck and the incompressible Navier–Stokes system, and these two systems are coupled through the drag force \( v - \xi \), which is also often called the friction force. If there is no coupling, i.e., without drag forces, the kinetic part in (1.1) becomes the nonlinear Vlasov–Poisson–Fokker–Planck system, see [15] for general discussion on the Vlasov–Fokker–Planck-type equations.

In the current work, we study a global-in-time existence of weak solutions, an asymptotic analysis for the VPNS system (1.1), and the global Cauchy problems for the isothermal/pressureless Euler–Poisson–Navier–Stokes system which can be rigorously derived from (1.1) under strong local alignment and diffusive force regimes. We present our main results in the following subsections.

1To be more precise, if there is no diffusion in the kinetic equation in (1.1), i.e., \( \sigma = 0 \), then the system (1.1) would be called the Vlasov–Poisson–Navier–Stokes system. On the other hand, if \( \sigma > 0 \), then the system (1.1) can be called the Vlasov–Poisson–Fokker–Planck–Navier–Stokes system. However, for simplicity, in the present work, we call (1.1) Vlasov–Poisson–Navier–Stokes system.
Global-in-time existence of weak solutions for the VPNS system. Our starting point is to discuss the global-in-time weak solvability for the VPNS system \((1.1)\). Before introducing a definition of a weak solution to the system \((1.1)\), we first define two function spaces:

\[
H := \{ v \in L^2(\mathbb{T}^d) \mid \nabla_x \cdot v = 0 \} \quad \text{and} \quad V := \{ v \in H^1(\mathbb{T}^d) \mid \nabla_x \cdot v = 0 \}.
\]

We also write \(V^\prime\) as the dual space of \(V\).

**Definition 1.1.** For \(T \in (0, +\infty)\), we say a pair \((f, v)\) is a weak solution to the system \((1.1)\) on the time interval \([0, T]\) if it satisfies the following:

1. \(f \in L^\infty(0, T; L^1_+ \cap L^\infty)(\mathbb{T}^d \times \mathbb{R}^d)\) \(\{\xi\} f \in L^\infty(0, T; L^1(\mathbb{T}^d \times \mathbb{R}^d))\),
2. \(v \in L^\infty(0, T; H) \cap L^2(0, T; V), \quad \partial_t v \in C([0, T]; V')\),
3. for every \(\Phi \in \mathcal{C}_c^\infty(\mathbb{T}^d \times \mathbb{R}^d \times [0, T])\) with \(\Phi(\cdot, \cdot, T) = 0\),
   \[
   \int_0^T \int_{\mathbb{T}^d \times \mathbb{R}^d} f \partial_t \Phi \, dx d\xi + \int_{\mathbb{T}^d \times \mathbb{R}^d} f_0 \Phi(\cdot, \cdot, 0) \, dx d\xi \\
   + \int_0^T \int_{\mathbb{T}^d \times \mathbb{R}^d} (\xi f) \cdot \nabla_x \Phi + ((u - \xi) f + (u - \xi) \Phi - (\nabla_x U) f - \sigma \nabla \xi f) \cdot \nabla_x \Phi \, dx d\xi dt = 0,
   \]
4. for every \(\phi \in H^1(\mathbb{T}^d)\) and a.e. \(t > 0\),
   \[
   \int_{\mathbb{T}^d} \nabla_x U \cdot \nabla_x \phi \, dx = \int_{\mathbb{T}^d} (\rho - 1) \phi \, dx,
   \]
5. for every \(\Psi \in \mathcal{C}_c^\infty(\mathbb{T}^d \times [0, T])\) with \(\nabla \cdot \Psi = 0\) and a.e. \(t > 0\),
   \[
   \int_{\mathbb{T}^d} v(x, t) \cdot \Psi(x, t) \, dx - \int_{\mathbb{T}^d} v_0(x) \cdot \Psi(x, 0) \, dx - \int_0^t \int_{\mathbb{T}^d} (v \partial_t \Psi + (v \cdot \nabla_x) \Psi \cdot v - \nabla_x v \cdot \nabla_x \Psi) \, dx ds \\
   = \int_0^t \int_{\mathbb{T}^d} \rho(u - v) \Psi \, dx ds.
   \]

We now state our first main result of the present work.

**Theorem 1.1.** For \(d = 2, 3\), suppose that the initial data \((f_0, v_0)\) satisfies the following conditions:

\(f_0 \in (L^1_+ \cap L^\infty)(\mathbb{T}^d \times \mathbb{R}^d)\) \quad \text{and} \quad v_0 \in H.

Furthermore, we assume that the initial free energy is bounded:

\[
\int_{\mathbb{T}^d \times \mathbb{R}^d} \left( \frac{\langle \xi \rangle^2}{2} + \sigma \log f_0 \right) \, f_0 \, dx d\xi + \frac{1}{2} \int_{\mathbb{T}^d} |\nabla_x U_0|^2 \, dx + \frac{1}{2} \int_{\mathbb{T}^d} |v_0|^2 \, dx < \infty.
\]

Then for every \(T > 0\), there exists at least one weak solution \((f, v)\) to the system \((1.1)\) on the time interval \([0, T]\) satisfying

\[
\int_{\mathbb{T}^d \times \mathbb{R}^d} \left( \frac{\langle \xi \rangle^2}{2} + \sigma \log f \right) \, f \, dx d\xi + \frac{1}{2} \int_{\mathbb{T}^d} |\nabla_x U|^2 \, dx + \frac{1}{2} \int_{\mathbb{T}^d} |v|^2 \, dx \\
+ \int_0^t \int_{\mathbb{T}^d \times \mathbb{R}^d} \left( \frac{1}{f} \langle \sigma \nabla \xi f - (u - \xi) f \rangle + \langle v - \xi \rangle f \right) \, dx d\xi ds + \int_0^t \int_{\mathbb{T}^d} |\nabla_x v|^2 \, dx ds \quad (1.2)
\]

\[
\leq \int_{\mathbb{T}^d \times \mathbb{R}^d} \left( \frac{\langle \xi \rangle^2}{2} + \sigma \log f_0 \right) \, f_0 \, dx d\xi + \frac{1}{2} \int_{\mathbb{T}^d} |\nabla_x U_0|^2 \, dx + \frac{1}{2} \int_{\mathbb{T}^d} |v_0|^2 \, dx + \sigma dt \|f_0\|_{L^1}.
\]

The existence theories for weak solutions of Vlasov–Navier–Stokes system and Vlasov–Fokker–Planck–Navier–Stokes system are well developed in \([7, 8]\) and \([11, 13, 22, 33]\), respectively. More recently, in \([24, 47]\), the global-in-time existence of weak solutions for Vlasov–Navier–Stokes-type system is also established when the collisional interactions for the particles are also taken into account. The Coulomb interactions between particles are dealt with in \([1, 2]\), and the global-in-time weak solutions for Vlasov–Poisson–Fokker–Planck–Navier–Stokes system are constructed. Compared to these works, we include the local alignment force \(u - \xi\) in the kinetic equation, and this introduces new difficulties. In particular, due to the lack of regularity of \(u = \int_{\mathbb{R}^d} \xi f \, d\xi / \int_{\mathbb{R}^d} f \, d\xi\), the method of characteristics can not be applied, nor is the weak compactness of the

\(^2L^1_+\) denotes the set of non-negative \(L^1\) functions.
product $uf$ obvious. This requires additional technical arguments, for instance, regularizations, weak/strong compactness arguments, velocity averaging lemma, and some uniform entropy inequalities. We want to emphasize that we also show that our weak solutions satisfy the entropy inequality \(\frac{\partial}{\partial t} \rho - \nabla \cdot (\rho u) = 0\), which is crucially used in the asymptotic analysis for the VPNS system \((1.1)\). The details on the existence theory for the system \((1.1)\) will be discussed in Section 2.

1.2. Hydrodynamic limit: from VPNS to EPNS equations. Next, we study the asymptotic analysis for the system \((1.1)\) under strong local alignment and diffusion regime. More precisely, for each $\varepsilon > 0$ we consider

$$
\partial_t f^\varepsilon + \xi \cdot \nabla_x f^\varepsilon + \nabla_\xi \cdot (v^\varepsilon - \xi) - \nabla_x U^\varepsilon(f^\varepsilon) = \frac{1}{\varepsilon} \nabla_\xi \cdot (\sigma \nabla_\xi f^\varepsilon - (u^\varepsilon - \xi)f^\varepsilon), \quad (x,\xi) \in \mathbb{T}^d \times \mathbb{R}^d, \ t > 0,
$$

$$
- \Delta_x U^\varepsilon = \rho^\varepsilon - 1,
$$

$$
\partial_t v^\varepsilon + (v^\varepsilon \cdot \nabla_x) v^\varepsilon + \nabla_x p^\varepsilon - \Delta_x v^\varepsilon = - \int_{\mathbb{R}^d} (v^\varepsilon - \xi) f^\varepsilon \, d\xi,
$$

$$
\nabla_x \cdot v^\varepsilon = 0.
$$

Formally, if we send the singular parameter $\varepsilon \to 0$, then the right hand side of the kinetic equation in \((1.3)\) becomes zero, and subsequently this yields

$$
f^\varepsilon(x,\xi,t) \simeq \begin{cases} \rho(x,t) \otimes \delta_{u(x,t)}(\xi) & \text{if } \sigma = 0, \\ \frac{\rho(x,t)}{(2\pi\sigma)^{d/2}} e^{-\frac{|x|^2}{2\sigma}} & \text{if } \sigma > 0. \end{cases}
$$

This formal observation together with estimating velocity moments of $f^\varepsilon$ gives that the system \((1.3)\) can be well approximated by the isothermal/pressureless Euler–Poisson system coupled with the incompressible Navier–Stokes system (in short, EPNS system):

$$
\begin{align*}
\partial_t\rho + \nabla_x \cdot (\rho u) &= 0, \quad x \in \mathbb{T}^d, \ t > 0, \\
\partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u) + \sigma \nabla_x \rho &= -\rho(u - v) - \rho \nabla_x U, \\
- \Delta_x U &= \rho - 1, \\
\partial_t v + (v \cdot \nabla_x) v + \nabla_x p - \Delta_x v &= \rho(u - v), \\
\nabla_x \cdot v &= 0.
\end{align*}
$$

Our second contribution is to make the above formal observation rigorous. More precisely, we will show that the unique strong solution to the system \((1.4)\) can be well approximated by weak solutions to the system \((1.1)\). Before stating the theorem precisely, let us introduce the notion of strong solutions to the system \((1.4)\).

**Definition 1.2** (Isothermal EPNS system). Let $d \geq 2$ and $s > d/2 + 1$. For a given $T \in (0, +\infty]$, we call $(\rho, u, v)$ a strong solution of \((1.4)\) with $\sigma > 0$ on the time interval $[0,T]$ if $(\rho, u, v)$ satisfy the following conditions:

(i) $(\rho, u) \in C([0,T]; H^s(\mathbb{T}^d)) \times C([0,T]; H^s(\mathbb{T}^d))$,

(ii) $v \in C([0,T]; H^s(\mathbb{T}^d)) \cap L^2(0,T; H^{s+1}(\mathbb{T}^d))$,

(iii) $(\rho, u, v)$ satisfies the system \((1.4)\) with $\sigma > 0$ in the sense of distributions.

**Definition 1.3** (Pressureless EPNS system). Let $d \geq 2$ and $s > d/2 + 1$. For a given $T \in (0, +\infty]$, we call $(\rho, u, v)$ a strong solution of \((1.4)\) with $\sigma = 0$ on the time interval $[0,T]$ if $(\rho, u, v)$ satisfy the following conditions:

(i) $(\rho, u) \in C([0,T]; H^s(\mathbb{T}^d)) \times C([0,T]; H^{s+1}(\mathbb{T}^d))$,

(ii) $v \in C([0,T]; H^{s+1}(\mathbb{T}^d)) \cap L^2(0,T; H^{s+2}(\mathbb{T}^d))$,

(iii) $(\rho, u, v)$ satisfies the system \((1.4)\) with $\sigma = 0$ in the sense of distributions.

**Remark 1.1.** As stated in Definition \((1.3)\), compared to the case with pressure the different regularities for $\rho$ and $u$ are taken due to the absence of pressure.

We also introduce our main assumptions for the hydrodynamic limit from the VPNS system \((1.3)\) to the EPNS system \((1.4)\) below.
Remark 1.5

The initial modulated energies of the systems (1.3) and (1.4) satisfy

\[ (H_2) \]

\( \int_{\mathbb{T}^d} \rho_0^ε |u_0^ε - u_0|^2 dx + \int_{\mathbb{T}^d} |v_0^ε - v_0|^2 dx + \int_{\mathbb{T}^d} \rho^ε \delta_u(x) dx + \int_{\mathbb{T}^d} |\nabla_x (U_0^ε - U_0)|^2 dx = O(\sqrt{ε}). \)

Theorem 1.2. For \( T > 0 \) and \( d \geq 2 \), let \((f^ε, v^ε)\) be weak solutions to the system (1.3) on the time interval \([0, T]\) corresponding to initial data \((f_0^ε, v_0^ε)\). Let \((ρ, u, v)\) be the unique strong solution to the system (1.4) on the time interval \([0, T]\) corresponding to initial data \((ρ_0, u_0, v_0)\). Assume that the initial data \((f^ε, v^ε)\) are well-prepared in the sense that \((H_1)-(H_2)\) hold.

(i) (Isothermal pressure case) If \( σ > 0 \), we have the following convergences:

\[ (ρ^ε, ρ^ε v^ε) \to (ρ, pu), \quad ρ^ε v^ε \to ρu \quad a.e. \quad \text{and} \quad L^∞(0, T; L^1(\mathbb{T}^d)), \]

\[ \int_{\mathbb{T}^d} f^ε \xi \otimes δ_u(x) dx \to ρu \quad a.e. \quad \text{and} \quad L^p(0, T; L^1(\mathbb{T}^d)) \quad \text{for} \quad 1 \leq p \leq 2, \quad \text{and} \]

\[ v^ε \to v \quad a.e. \quad \text{and} \quad L^∞(0, T; L^2(\mathbb{T}^d)) \]

as \( ε \to 0 \), where \( I_d \) is the \( d \times d \) identity matrix. Furthermore if the relative entropy between \( f_0^ε \) and \( M_{ρ_0, u_0} \) is well-prepared in the following sense:

\[ \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{f^ε - z}{z} dxdξdε \to 0 \]

as \( ε \to 0 \), we have

\[ f^ε \to M_{ρ, u} := \frac{ρ}{(2πσ)^{d/2}} e^{-\frac{|z - ρu|^2}{2σ}} \quad a.e. \quad \text{and} \quad L^∞(0, T; L^1(\mathbb{T}^d \times \mathbb{T}^d)). \]

as \( ε \to 0 \).

(ii) (Pressureless case) If \( σ = 0 \), we have

\[ (ρ^ε, ρ^ε v^ε) \to (ρ, pu), \quad ρ^ε v^ε \to ρu \quad \text{weakly in} \quad L^∞(0, T; M(\mathbb{T}^d)), \]

\[ \int_{\mathbb{T}^d} f^ε \xi \otimes δ_u(x) dx \to ρu \quad \text{weakly in} \quad L^1(0, T; M(\mathbb{T}^d)), \]

\[ f^ε \to ρ \otimes δ_u(ξ) \quad \text{weakly in} \quad L^p(0, T; M(\mathbb{T}^d \times \mathbb{T}^d)) \quad \text{for} \quad 1 \leq p \leq 2, \quad \text{and} \]

\[ v^ε \to v \quad a.e. \quad \text{and} \quad L^∞(0, T; L^2(\mathbb{T}^d)) \]

as \( ε \to 0 \), where \( M(\Omega) \) denotes the space of nonnegative Radon measures on \( Ω \).

Remark 1.2. The assumptions \((H_1)-(H_2)\) for the well-prepared initial data can be replaced by

\((H_1)′\)

\[ \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \left( |ξ|^2 \right) f_0^ε dxdξ - \int_{\mathbb{T}^d} \left( \frac{|u_0|^2}{2} + σ \log ρ_0 \right) dx = O(\sqrt{ε}). \]

\((H_2)′\)

\[ \|ρ_0 - ρ\|_{L^2(\mathbb{T}^d)} = O(\sqrt{ε}), \quad \|u_0 - u\|_{L^2(\mathbb{T}^d)} = O(\sqrt{ε}), \quad \text{and} \quad \|v_0 - v\|_{L^2(\mathbb{T}^d)} = O(\sqrt{ε}). \]

Indeed, we notice that

\[ \|∇_x (U^ε - U)\|_{L^2(\mathbb{T}^d)} \leq C \|ρ^ε - ρ\|_{H^{-1}(\mathbb{T}^d)} \leq C \|ρ^ε - ρ\|_{L^2(\mathbb{T}^d)}, \]

where \( C > 0 \) is independent of \( ε > 0 \). For the other replacements, we can use a similar argument as in [21] Remark 1.5].
exists a strong solution to the system (1.4).

established in Theorem 1.1, the estimate of the hydrodynamic limit in Theorem 1.2 holds as long as there
the limiting system (1.4). Since the global-in-time existence of weak solutions to the VPNS system (1.1) is
section is based on the weak-strong uniqueness principle, thus we need a strong regularity of solutions to
solutions to the EPNS system (1.4). As mentioned before, the asymptotic analysis stated in the previous
result, investigated in Section 4, is concerned with the global-in-time existence and uniqueness of strong
1.3.

enables us to have the hydrodynamic limit even in the absence of the diffusion in the kinetic equation in
our analyses do hold regardless of the presence of the diffusion force. We carefully analyze the Coulomb
of hydrodynamic limits of kinetic-fluid models. In those works, the presence of diffusion plays an important
system is derived under the same asymptotic regime as ours. We also refer to [21, 22, 30, 31] for other types
Discussion on that.

Global-in-time existence and uniqueness of strong solutions to EPNS system. Our third
result, investigated in Section 4, is concerned with the global-in-time existence and uniqueness of strong
to the EPNS system (1.4). As mentioned before, the asymptotic analysis stated in the previous
section is based on the weak-strong uniqueness principle, thus we need a strong regularity of solutions to
the limiting system (1.4). Since the global-in-time existence of weak solutions to the VPNS system (1.1) is
established in Theorem 1.1, the estimate of the hydrodynamic limit in Theorem 1.2 holds as long as there
exists a strong solution to the system (1.4).
The theorems below provide the global-in-time solvability of strong solutions to the system \((1.4)\) under suitable smoothness and smallness assumptions on the initial data. This together with Theorem 1.2 asserts that the hydrodynamic limit from \((1.3)\) to \((1.4)\) holds for all time \(t \geq 0\).

**Theorem 1.3.** Let \(d \geq 2\) and \(s > d/2 + 1\). Suppose that the initial data \((\rho_0, u_0, v_0)\) satisfy

(i) \(\inf_{x \in \mathbb{T}^d} \rho_0(x) > 0\) and

(ii) \((\rho_0, u_0, v_0) \in H^s(\mathbb{T}^d) \times H^s(\mathbb{T}^d) \times H^s(\mathbb{T}^d)\).

If

\[
\| \log \rho_0 \|_{H^s(\mathbb{T}^d)}^2 + \| u_0 \|_{H^s(\mathbb{T}^d)}^2 + \| v_0 \|_{H^s(\mathbb{T}^d)}^2 < \tilde{\varepsilon}_0^2,
\]

for some \(\tilde{\varepsilon}_0 > 0\) sufficiently small, then the system \((1.4)\) with \(\sigma > 0\) admits a global-in-time unique strong solution in the sense of Definition 1.2 with \(T = +\infty\) satisfying

\[
\sup_{t \geq 0} \left( \| \log \rho(\cdot, t) \|_{H^s(\mathbb{T}^d)}^2 + \| u(\cdot, t) \|_{H^s(\mathbb{T}^d)}^2 + \| v(\cdot, t) \|_{H^s(\mathbb{T}^d)}^2 \right) < \infty.
\]

**Theorem 1.4.** Let \(d \geq 2\) and \(s > d/2 + 1\). Suppose that the initial data \((\rho_0, u_0, v_0)\) satisfy

(i) \(\inf_{x \in \mathbb{T}^d} \rho_0(x) > 0\) and

(ii) \((\rho_0, u_0, v_0) \in H^s(\mathbb{T}^d) \times H^{s+1}(\mathbb{T}^d) \times H^{s+1}(\mathbb{T}^d)\).

If

\[
\| \rho_0 - 1 \|_{H^s(\mathbb{T}^d)}^2 + \| u_0 \|_{H^{s+1}(\mathbb{T}^d)}^2 + \| v_0 \|_{H^{s+1}(\mathbb{T}^d)}^2 < \tilde{\varepsilon}_0^2,
\]

for some \(\tilde{\varepsilon}_0 > 0\) sufficiently small, then the system \((1.4)\) with \(\sigma = 0\) admits a global-in-time unique strong solution in the sense of Definition 1.3 with \(T = +\infty\) satisfying

\[
\sup_{t \geq 0} \left( \| \rho(\cdot, t) - 1 \|_{H^s(\mathbb{T}^d)}^2 + \| u(\cdot, t) \|_{H^{s+1}(\mathbb{T}^d)}^2 + \| v(\cdot, t) \|_{H^{s+1}(\mathbb{T}^d)}^2 \right) < \infty.
\]

Note that the limiting system \((1.4)\) consists of the isothermal/pressureless Euler–Poisson system and the incompressible Navier–Stokes system, and it is well-known that the Euler–Poisson equations develop a formation of singularities in finite time no matter how smooth the initial data are \([12, 14, 27, 28, 36, 37, 38, 42]\). For that reason, it is not obvious to expect the global-in-time smooth regularity of solutions to the isothermal/pressureless EPNS system \((1.4)\). On the other hand, it is worth noticing that the incompressible Navier–Stokes system has a dissipative structure; it includes a smoothing effect of the viscous term. Thus it is required to use that to prevent the formation of finite-time singularities. Motivated from \([15, 16]\), we properly use the drag forcing term to transfer the smoothing effect in the viscous fluid to the Euler–Poisson system. More precisely, we consider the drag force in the Euler–Poisson system in \((1.4)\) as the relative damping to have a dissipation of the fluid velocity \(u\). Then we use the viscous term in the incompressible Navier–Stokes system in \((1.4)\) through the drag force to control the growth of the incompressible fluid velocity \(v\) in the Euler–Poisson system. Additionally, we carefully analyze the Poisson term to have some dissipative effect for the fluid density \(\rho\). We provide the details of proofs of Theorems 1.3 and 1.4 in Section 4.

**1.4. Large-time behavior of the isothermal/pressureless EPNS systems.** After establishing the global-in-time solvability of strong solutions to the isothermal/pressureless EPNS systems \((1.4)\), in Section 5 we analyze the large-time behavior of classical solutions for that equations, which is our final result.

For this, we first define averaged momentum and velocity:

\[
m_c = m_c(t) := \int_{\mathbb{T}^d} (\rho u)(x, t) \, dx \quad \text{and} \quad v_c = v_c(t) := \int_{\mathbb{T}^d} v(x, t) \, dx.
\]

We then employ a modulated total energy measuring the fluctuation of momentum and mass from the corresponding averages:

\[
\mathcal{L}(\rho, u, v) := \int_{\mathbb{T}^d} \rho |u - m_c|^2 \, dx + \int_{\mathbb{T}^d} |v - v_c|^2 \, dx + \sigma \int_{\mathbb{T}^d} (\rho - 1)^2 \, dx + \int_{\mathbb{T}^d} |\nabla_x U|^2 \, dx + |m_c - v_c|^2.
\]

We show the exponential convergence of the modulated energy \(\mathcal{L}\) as time goes to infinity, in particular which shows the velocity alignment between two fluid equations.

**Theorem 1.5.** Let \(d \geq 2\) and \((\rho, u, v)\) be a global-in-time solution to \((1.4)\) with sufficient regularity. Suppose that the density \(\rho\) is bounded from above by some positive constant \(\tilde{\rho}\), i.e., \(\rho(x, t) \in [0, \tilde{\rho}]\) for almost all \(x \in \mathbb{T}^d\) and all \(t \geq 0\). Furthermore, depending on \(\sigma \geq 0\), we assume the followings:
Depending on $\alpha, \beta$, total energy, which produces the dissipation terms for the fluid density to the case without pressure. To overcome that difficulty, we introduce a new perturbation of the modulated applied or not, due to the presence of Coulomb interactions. Furthermore, this strategy can not be applied employed, see [15, 16, 23]. However, in our case, it is not clear whether the Bogovkii’s argument can be Euler–Navier–Stokes-type systems, similar estimates for the large-time behavior of strong solutions can be derived dissipative term for the compressible fluid density using the Bogovskii’s argument [4, 17]. For domain, see also [32]. For the coupling with the compressible fluids, one of the important features is of Vlasov-type equation coupled with the incompressible Navier–Stokes equations in the spatial periodic exponentially fast.

1.5. Notation. We introduce several notations used throughout the current work. For functions, $f(x,v)$ and $g(x)$, $\|f\|_{L^p}$ and $\|g\|_{L^p}$ represent the usual $L^p(\mathbb{T}^d \times \mathbb{R}^d)$- and $L^p(\mathbb{T}^d)$-norms, respectively. We denote by $C$ a generic positive constant which may differ from line to line. $C = C(\alpha, \beta, \cdots)$ stands for a positive constant depending on $\alpha, \beta, \cdots$. For simplicity, we often omit $x$-dependence of differential operators, i.e. $\nabla f := \nabla_x f$ and $\Delta f := \Delta_x f$. $\nabla^k$ represents any partial derivative $\partial^\alpha$ with multi-index $\alpha$, $|\alpha| = k$. $C = C(\alpha, \beta, \cdots)$ stands for a positive constant depending on $\alpha, \beta, \cdots$. Finally, we write $f \lesssim g$ if there exists a constant $C > 0$ satisfying $f \leq Cg$. 2. Existence of weak solutions to VPNS system

In this section, we discuss the global-in-time existence of weak solutions to the system (1.1). We first notice that the potential $U$ can be represented by using the interaction potential $K$ which satisfies the following conditions (see [3] [14]):
(i) The potential $K$ is an even function explicitly written as

$$K(x) = \begin{cases} -c_0 \log |x| + G_0(x) & \text{if } d = 2, \\ c_1 |x|^{-1} + G_1(x) & \text{if } d = 3, \end{cases}$$

where $c_0 > 0$ and $c_1 > 0$ are normalization constants and $G_0$ and $G_1$ are smooth functions over $\mathbb{T}^2$ and $\mathbb{T}^3$, respectively.

(ii) For any $h \in L^2(\mathbb{T}^d)$ with $\int_{\mathbb{T}^d} h \, dx = 0$, $U := K \ast h \in H^1(\mathbb{T}^d)$ is the unique function that satisfies the following condition:

$$\int_{\mathbb{T}^d} U \, dx = 0 \quad \text{and} \quad \int_{\mathbb{T}^d} \nabla U \cdot \nabla \psi \, dx = \int_{\mathbb{T}^d} h \, \psi \, dx \quad \forall \psi \in H^1(\mathbb{T}^d),$$

i.e., $U$ is the unique weak solution to $-\Delta U = h$.

Thus, we rewrite the system (1.1) as

$$\begin{align*}
\partial_t f + \xi \cdot \nabla f + \nabla \xi \cdot (\{(v - \xi) - \nabla K \ast (\rho - 1)\}) &= \nabla \cdot (\sigma \nabla \xi f - (u - \xi) f), \\
\partial_t v + (v \cdot \nabla) v + \nabla p - \Delta v &= - \int_{\mathbb{R}^d} (v - \xi) f \, d\xi, \\
\nabla \cdot v &= 0.
\end{align*}$$

This reformulation is useful when we discuss the regularization of the interaction potential $U$, see (2.3) below.

In order to show the global-in-time existence of weak solutions to the system (2.1), we first regularize the system by employing a cut-off function for the fluid and local particle velocities in the kinetic equation and the drag force in the fluid equation. We also mollify the convection velocity of the fluid equation and remove the singularity in the local particle velocity $u$. Next, we show the existence of weak solutions for the regularized system by using a fixed point argument. We then use some weak and strong compactness arguments together with the velocity averaging lemma to pass to the limit when the regularization parameters tend to zero or infinity. Finally, we prove the limiting functions obtained from the previous step satisfy our VPNS system in the sense of Definition 1.1.

2.1. Regularized system. We regularize the system (2.1) and investigate the existence of solutions and associated entropy inequalities. For the regularization parameters $\lambda$ and $\varepsilon$, we consider

$$\begin{align*}
\partial_t f^{\lambda,\varepsilon} + \xi \cdot \nabla f^{\lambda,\varepsilon} + \nabla \xi \cdot (\{(\chi_\lambda (u^{\lambda,\varepsilon}) - \xi) - \nabla K^{\varepsilon} \ast (\rho^{\lambda,\varepsilon} - 1)\}) f^{\lambda,\varepsilon} &= \nabla \cdot (\sigma \nabla \xi f^{\lambda,\varepsilon} - (\chi_\lambda (u^{\lambda,\varepsilon}) - \xi) f^{\lambda,\varepsilon}), \\
\partial_t u^{\lambda,\varepsilon} + (\theta \ast u^{\lambda,\varepsilon}) \cdot \nabla u^{\lambda,\varepsilon} + \nabla p^{\lambda,\varepsilon} - \Delta u^{\lambda,\varepsilon} &= \rho^{\varepsilon} (u^{\lambda,\varepsilon} - v^{\lambda,\varepsilon}) \mathbb{1}_{\{|v^{\lambda,\varepsilon}| \leq \lambda\}}, \\
\nabla \cdot u^{\lambda,\varepsilon} &= 0
\end{align*}$$

subject to initial data:

$$f^{\lambda,\varepsilon}(x, \xi, 0) = f^\lambda(x, \xi, 0) \mathbb{1}_{\{|\xi| \leq \lambda\}}(x) \quad \text{and} \quad v^{\lambda,\varepsilon}(x, 0) = v^\varepsilon(x, 0) := (\theta \ast v_0)(x),$$

where $\chi_\lambda(v)$ is the truncation function given by

$$\chi_\lambda(v) = v \mathbb{1}_{\{|v| \leq \lambda\}}$$

and $u^{\lambda,\varepsilon}_\varepsilon$ is defined by

$$u^{\lambda,\varepsilon}_\varepsilon := \frac{\rho^{\varepsilon} u^{\lambda,\varepsilon}}{\rho^{\varepsilon} + \varepsilon}.$$  

Here the fluid velocity field $v$ is also regularized by using $\theta = (1/\varepsilon^d) \theta(x/\varepsilon)$, where $\theta$ is a standard mollifier satisfying

$$\theta \in C^\infty_0(\mathbb{T}^d), \quad \theta \geq 0, \quad \int_{\mathbb{T}^d} \theta(x) \, dx = 1, \quad \text{and} \quad \text{supp} \ \theta \subset B_1(0).$$

The regularized interaction potential $K^{\varepsilon}$ is given as

$$K^{\varepsilon}(x) = \begin{cases} -\frac{c_0}{2} \log(\varepsilon + |x|^2) + G_0(x) & \text{if } d = 2, \\ c_1 (\varepsilon + |x|^2)^{-1/2} + G_1(x) & \text{if } d = 3. \end{cases}$$
We use $\eta = (\lambda, \varepsilon)$ for notational simplicity whenever there is no confusion. Then, we partially linearize the system \((2.4)\) as follows:
\[
\begin{align*}
\partial_t \eta^0 + \xi \cdot \nabla \eta^0 + \nabla \xi \cdot ((\chi_\lambda(\tilde{u}) - \xi) - \nabla K^\varepsilon \ast (\rho^0 - 1)) f^0) &= \nabla \xi \cdot (\sigma \nabla \xi f^0 - (\chi_\lambda(\tilde{u}) - \xi) f^0), \\
\partial_t \nu^0 + ((\theta_\lambda \ast \nu^0) \cdot \nabla) \nu^0 + \nabla \rho^0 - \Delta \nu^0 &= \rho^0(u^0 - \tilde{v}) 1_{\{\varepsilon \leq \lambda\}}, \\
\nabla \cdot \nu^0 &= 0,
\end{align*}
\]
where \((\tilde{u}, \tilde{v})\) belong to \(S := L^2(T^d \times (0, T)) \times L^2(T^d \times (0, T))\). As mentioned before, our strategy is to apply the fixed point theory argument to the system \((2.4)\) to obtain the existence of weak solutions to system \((2.2)\) and associated entropy inequality. For this, we need to estimate \(L^p\)-norm and velocity-moments of solutions \(f^0\) to the kinetic equation in \((2.4)\). On the other hand, since \(\nabla K^\varepsilon\) is bounded and Lipschitz continuous for fixed \(\varepsilon > 0\), the global-in-time existence of weak solutions to the kinetic equation in \((2.4)\) can be found in \([20, 33]\). Furthermore, we get
\[
\frac{d}{dt} \int_{T^d \times \mathbb{R}^d} (f^0)^p \, dx \, d\xi = (p - 1) \int_{T^d \times \mathbb{R}^d} (f^0)^p \nabla \xi \cdot (2\xi + \nabla K^\varepsilon \ast (\rho^0 - 1) - \chi_\lambda(\tilde{u}) - \chi_\lambda(\tilde{v})) \, dx \, d\xi
\]
\[
\hspace{1cm} - \sigma p (p - 1) \int_{T^d \times \mathbb{R}^d} (f^0)^{p-2} |\nabla \xi f^0|^2 \, dx \, d\xi
\]
\[
= 2(p - 1) \int_{T^d \times \mathbb{R}^d} (f^0)^p \, dx \, d\xi - \frac{4\sigma (p - 1)}{p} \int_{T^d \times \mathbb{R}^d} |\nabla \xi (f^0)^{p/2}|^2 \, dx \, d\xi
\]
for \(p \in [1, \infty)\). We combine this with Grönwall’s lemma to obtain
\[
\|f^0(\cdot, t)\|_{L^p} + \frac{4\sigma (p - 1)}{p} \int_0^t e^{2d(p-1)(t-s)} \|\nabla \xi (f^0)^{p/2} (\cdot, s)\|^2_{L^2} \, ds \leq \|f^0\|_{L^p} e^{2dt}.
\]
In particular, we have
\[
\|f^0(\cdot, t)\|_{L^1} \leq \|f^0\|_{L^1} = 1 \quad \text{and} \quad \|f^0(\cdot, t)\|_{L^\infty} \leq \|f^0\|_{L^\infty} e^{2dt}\]
for \(t \in [0, T]\).

We next present the estimates for higher-order velocity moments and entropy inequality of solutions to the system \((2.4)\).

**Lemma 2.1.** For a weak solution \(f^0\) to the kinetic equation in \((2.4)\), its velocity moments satisfy the following bound:
\[
\sup_{t \in (0, T)} \int_{T^d \times \mathbb{R}^d} |\xi|^k f^0(x, \xi) \, dx \, d\xi \leq C(d, \eta, \kappa, \sigma, T) \quad \forall k \geq 0.
\]

**Proof.** First, we define a \(k\)-th moment of \(f\) in velocity by
\[
m_k(f) := \int_{T^d \times \mathbb{R}^d} |\xi|^k f \, dx \, d\xi.
\]
Then, for \(k \geq 2\), we estimate
\[
\frac{d}{dt} m_k(f^0) = -k \int_{T^d \times \mathbb{R}^d} (\xi - \chi_\lambda(\tilde{u}) + \nabla K^\varepsilon \ast (\rho^0 - 1)) \cdot \xi f^0 |\xi|^{k-2} \, dx \, d\xi
\]
\[
- k \int_{T^d \times \mathbb{R}^d} (\xi - \chi_\lambda(\tilde{u})) \cdot \xi f^0 |\xi|^{k-2} \, dx \, d\xi
\]
\[
= -2k m_k(f^0) - k \int_{T^d \times \mathbb{R}^d} \nabla K^\varepsilon \ast (\rho^0 - 1) \cdot \xi f^0 |\xi|^{k-2} \, dx \, d\xi
\]
\[
+ k \int_{T^d \times \mathbb{R}^d} (\chi_\lambda(\tilde{u}) + \chi_\lambda(\tilde{v})) \cdot \xi f^0 |\xi|^{k-2} \, dx \, d\xi
\]
\[
\leq -k m_k(f^0) + \frac{k}{2} \int_{T^d \times \mathbb{R}^d} |\nabla K^\varepsilon \ast (\rho^0 - 1)|^2 f^0 |\xi|^{k-2} \, dx \, d\xi
\]
\[
+ \frac{k}{2} \int_{T^d \times \mathbb{R}^d} (\chi_\lambda(\tilde{u}) + \chi_\lambda(\tilde{v}))^2 f^0 |\xi|^{k-2} \, dx \, d\xi
\]
\[
+ \sigma (k - 2 + d) m_{k-2}(f^0)
\]
\[
\leq \frac{k}{2} \int_{T^d \times \mathbb{R}^d} |\nabla K^\varepsilon * (\rho^n - 1)|^2 f^n |\xi|^{k-2} d\xi d\xi + \frac{k}{2} \int_{T^d \times \mathbb{R}^d} \left(\chi_\lambda(\tilde{u}) + \chi_\lambda(\tilde{v})\right)^2 f^n |\xi|^{k-2} d\xi d\xi + \sigma k(k - 2 + d) m_{k-2}(f^n)
\leq C m_{k-2}(f^n),
\]
where \( C = C(d, \eta, k, \sigma, T) \) is a positive constant and we used Young’s inequality and
\[
\|\nabla K^\varepsilon * (\rho^n - 1)\|_{L^\infty} \leq \|\nabla K^\varepsilon\|_{L^\infty} \|\rho^n - 1\|_{L^1} \leq C(\varepsilon).
\]
Since \( m_0(f^n) \) is just \( \|f^n(\cdot, \cdot, t)\|_{L^1} = \|f_0^n\|_{L^1} \) for \( t \geq 0 \), we combine Grönwall’s lemma and induction argument to yield
\[
\sup_{t \in (0, T)} \int_{T^d \times \mathbb{R}^d} |\xi|^k f^n d\xi d\xi \leq C(d, \eta, k, \sigma, T) \quad \forall k = 0, 2, 4, \ldots.
\]
Moreover, for \( k \in \mathbb{R}_+ \setminus \{0, 2, 4, \ldots\} \), we get
\[
\int_{T^d \times \mathbb{R}^d} |\xi|^k f^n d\xi d\xi \leq \left( \int_{T^d \times \mathbb{R}^d} |\xi|^{2|k|} f^n d\xi d\xi \right)^{1/2} \left( \int_{T^d \times \mathbb{R}^d} |\xi|^{2(k-|k|)} f^n d\xi d\xi \right)^{1/2},
\]
where \([k]\) denotes the greatest integer less than or equal to \( k \). Furthermore, we have
\[
\int_{T^d \times \mathbb{R}^d} |\xi|^{2(k-|k|)} f^n d\xi d\xi \leq \left( \int_{T^d \times \mathbb{R}^d} |\xi|^2 f^n d\xi d\xi \right)^{2(k-|k|)} \left( \int_{T^d \times \mathbb{R}^d} f^n d\xi d\xi \right)^{2-2(k-|k|)} \leq C(d, \eta, k, \sigma, T).
\]
This asserts our desired result. \( \square \)

In the proposition below, we provide the estimate of entropy inequality.

**Proposition 2.1.** For a weak solution \( f^n \) to the kinetic equation in \( [2, 4] \), we have the following relation:
\[
\int_{T^d \times \mathbb{R}^d} \left( \frac{|\xi|^2}{2} + \sigma \log f^n \right) f^n d\xi d\xi + \int_{T^d} (\rho^n - 1) K^\varepsilon * (\rho^n - 1) d\xi
\]
\[
+ \int_0^t \int_{T^d \times \mathbb{R}^d} \frac{1}{f^n} |\sigma \nabla f^n - (\chi_\lambda(\tilde{u}) - \xi) f^n|^2 d\xi d\xi d\sigma
\]
\[
\leq \int_{T^d \times \mathbb{R}^d} \left( \frac{|\xi|^2}{2} + \sigma \log f_0^n \right) f^\lambda d\xi d\xi + \int_{T^d} (\rho_0^\lambda - 1) K^\varepsilon * (\rho_0^\lambda - 1) d\xi
\]
\[
+ \int_0^t \int_{T^d \times \mathbb{R}^d} (\chi_\lambda(\tilde{u}) - \xi) \cdot \chi_\lambda(\tilde{u}) f^n d\xi d\xi ds + \int_0^t \int_{T^d \times \mathbb{R}^d} (\chi_\lambda(\tilde{v}) - \xi) \cdot \chi_\lambda(\tilde{v}) f^n d\xi d\xi ds + \sigma |f_0^n|_{L^1} t
\]
for almost all \( t \in [0, T] \).

**Proof.** First, it follows from Lemma \( 2.1 \) that
\[
\frac{1}{2} \frac{d}{dt} \left( \int_{T^d \times \mathbb{R}^d} |\xi|^2 f^n d\xi d\xi \right)
\]
\[
= - \int_{T^d \times \mathbb{R}^d} \left( \xi - \chi_\lambda(\tilde{v}) + \nabla K^\varepsilon * (\rho^n - 1) \right) : \xi f^n d\xi d\xi
\]
\[
- \int_{T^d \times \mathbb{R}^d} \left( \chi_\lambda(\tilde{u}) \right) \cdot \chi_\lambda(\tilde{v}) f^n d\xi d\xi - \sigma \int_{T^d \times \mathbb{R}^d} \xi \cdot \nabla f^n d\xi d\xi
\]
\[
= - \int_{T^d \times \mathbb{R}^d} \left( \xi - \chi_\lambda(\tilde{v}) + \nabla K^\varepsilon * (\rho^n - 1) \right) : \xi f^n d\xi d\xi - \int_{T^d \times \mathbb{R}^d} \left| \xi - \chi_\lambda(\tilde{u}) \right|^2 f^n d\xi d\xi
\]
\[
- \int_{T^d \times \mathbb{R}^d} \left( \xi - \chi_\lambda(\tilde{u}) \right) \cdot \chi_\lambda(\tilde{u}) f^n d\xi d\xi - \sigma \int_{T^d \times \mathbb{R}^d} \left( \xi - \chi_\lambda(\tilde{u}) \right) \cdot \nabla f^n d\xi d\xi.
\]
On the other hand, we get
\[
\frac{d}{dt} \left( \int_{T^d} (\rho^n - 1)K^\varepsilon \ast (\rho^n - 1) \, dx \right) = - \int_{T^d \times T^d} K^\varepsilon(x - y) \nabla \cdot (\rho^n u^n)(x)(\rho^n(y) - 1) \, dxdy \\
= \int_{T^d} \nabla K^\varepsilon(x - y)(\rho^n(y) - 1) \cdot (\rho^n u^n)(x) \, dxdy \\
= \int_{T^d} (\nabla K^\varepsilon \ast (\rho^n - 1)) \cdot (\rho^n u^n) \, dx \\
= \int_{T^d} (\nabla K^\varepsilon \ast (\rho^n - 1)) \cdot \xi f^n \, dxd\xi.
\]
Moreover, we estimate the entropy as
\[
\frac{d}{dt} \left( \int_{T^d \times \mathbb{R}^d} \sigma f^n \log f^n \, dxd\xi \right) = - \int_{T^d \times \mathbb{R}^d} \sigma \partial_t f^n \log f^n \, dxd\xi \\
= -\sigma \int_{T^d \times \mathbb{R}^d} (\xi - \chi_\lambda(\tilde{v})) + \nabla K^\varepsilon \ast (\rho^n - 1)) \cdot \nabla \xi f^n \, dxd\xi \\
- \sigma \int_{T^d \times \mathbb{R}^d} (\xi - \chi_\lambda(\tilde{u})) \cdot \nabla \xi f^n \, dxd\xi - \sigma^2 \int_{T^d \times \mathbb{R}^d} \frac{\nabla \xi f^n|^2}{f^n} \, dxd\xi \\
= \sigma d\|f^n\|_{L^1} - \sigma \int_{T^d \times \mathbb{R}^d} (\xi - \chi_\lambda(\tilde{u})) \cdot \nabla \xi f^n \, dxd\xi - \sigma^2 \int_{T^d \times \mathbb{R}^d} \frac{\nabla \xi f^n|^2}{f^n} \, dxd\xi.
\]
We then combine all the previous estimates to get the desired result. \(\square\)

For the fluid part, the following estimate is obvious.

**Proposition 2.2.** For a weak solution \(v^n\) to the Navier–Stokes system in (2.4), we have
\[
\frac{1}{2} \int_{T^d} |v^n|^2 \, dx + \int_0^t \int_{T^d} |\nabla v^n|^2 \, dxdy \leq \frac{1}{2} \int_{T^d} |v^0|^2 \, dx + \int_0^t \int_{T^d} \rho^n(u^n - \tilde{v}) \mathbb{1}_{\{|\tilde{v}| \leq \lambda\}} \cdot v^n \, dxdy
\]
for almost all \(t \in [0,T]\).

### 2.2. Existence of weak solutions to the regularized system.

Now, we proceed to the existence of weak solutions to system (2.2). For this, we define a map \(T : S \to S\) as
\[
(\tilde{u}, \tilde{v}) \mapsto T(\tilde{u}, \tilde{v}) := (u^n_\varepsilon, v^n).
\]

We first recall from [33 Lemma 2.4] or [39 Lemma 3.2] that the following lemma which provides some \(L^p\) bound estimates for \(\rho\) and \(pu\).

**Lemma 2.2.** Assume that \(f\) satisfies
\[
\|f\|_{L^\infty(T^d \times \mathbb{R}^d \times (0,T))} \leq M \text{ and } \sup_{0 \leq t \leq T} \int_{T^d \times \mathbb{R}^d} |\xi|^k f(x, \xi, t) \, dxd\xi \leq M \forall k \in [0,k^*]
\]
for some \(k^* > 1\). Then there exists a constant \(C = C(M) > 0\) such that
\[
\|\rho(\cdot, t)\|_{L^p} \leq C \forall p_1 \in [1,(k^* + d)/d] \text{ and } \|(pu)(\cdot, t)\|_{L^{p_2}} \leq C \forall p_2 \in [1,(k^* + d)/(d + 1)]
\]
for all \(t \in [0,T]\).

In the lemma below, we show that \(T\) is well-defined.

**Lemma 2.3.** There exists a constant \(C = C(d, \eta, \sigma, T)\) such that
\[
\|T(\tilde{u}, \tilde{v})\|_S \leq C \forall (\tilde{u}, \tilde{v}) \in S.
\]
Proof. For the kinetic part, we use (2.5) and Lemmas 2.1 and 2.2 to obtain
\[ \rho^0 \in L^p(T^d \times (0, T)), \quad \rho^0 u^0 \in L^p(T^d \times (0, T)) \quad \forall p \in [1, \infty), \]
and thus
\[ \|u^0\|_{L^2} \leq \frac{1}{\varepsilon} \|\rho^0 u^0\|_{L^2} \leq C. \]
For the fluid part, we use Cauchy–Schwarz inequality and Young’s inequality to get
\[
\left| \int_{T^d} \rho^0(u^0 - \tilde{v})1_{|\tilde{v}| \leq \lambda} \cdot v^0 \, dx \right| \leq \|\rho^0 u^0\|_{L^2}(\|v^0\|_{L^2} + \lambda\|\rho^0\|_{L^2}\|v^0\|_{L^2})
\leq \|v^0\|_{L^2}^2 + \frac{1}{2} \|\rho^0 u^0\|_{L^2}^2 + \frac{\lambda^2}{2} \|\rho^0\|_{L^2}^2 \leq \|v^0\|_{L^2}^2 + C.
\]
We then combine the above inequality with Proposition 2.2 and Grönwall’s lemma to conclude the proof. \( \square \)

Next, we show that \( T \) is compact. Here, we consider the velocity averaging lemma from [11, Lemma 3.2] (see also [43, Theorem 2] and [33, Lemma 2.7]) and state a modified version to be used in the following proof.

**Lemma 2.4.** Let \( T > 0, r > 1, q \in [1, (d + r)/(d + 1)), \) and \( \{G^m\} \) be bounded in \( L^q_{\text{loc}}(T^d \times \mathbb{R}^d \times (0, T)). \) Assume that
\[ f^m \text{ is bounded in } L^\infty(T^d \times \mathbb{R}^d \times (0, T)) \]
and
\[ |\xi|^r f^m \text{ is bounded in } L^\infty(0, T; L^1(T^d \times \mathbb{R}^d)). \]
If \( f^m \) and \( G^m \) satisfy
\[ \partial_t f^m + \xi \cdot \nabla f^m = \nabla^2 G^m, \quad f^m|_{t=0} = f_0 \in L^\infty(T^d \times \mathbb{R}^d) \]
for some multi-index, then for any \( \varphi(\xi) \), such that \( |\varphi(\xi)| \leq c|\xi| \) as \( |\xi| \to \infty, \) the sequence
\[ \left\{ \int_{\mathbb{R}^d} f^m \varphi(\xi) \, d\xi \right\} \]
is relatively compact in \( L^q(T^d \times (0, T)). \)

Now, we are ready to prove the compactness of \( T \).

**Lemma 2.5.** For a uniformly bounded sequence \( (\tilde{u}^m, \tilde{v}^m) \) in \( S \), the sequence \( \mathcal{T}(\tilde{u}^m, \tilde{v}^m) = ((u^0)_m, (v^0)_m) \) converges strongly in \( S \), up to a subsequence.

**Proof.** Since the convergence of \( \{(v^0)_m\} \) follows from the same argument as [11, Lemma 4.2], it suffices to show the convergence of \( \{(u^0)_m\} \). For the convergence of \( \{(u^0)_m\} \), we set
\[ f^m := (f^0)^m, \quad G^m := (\sigma \nabla \xi(f^0)_m + (2\xi + \nabla^2 \chi \rho^0 - \chi \rho^0)(f^0)_m), \]
then it is easy to see \( G^m \in L^2_{\text{loc}}(T^d \times \mathbb{R}^d \times (0, T)) \). Choose \( r \) appeared in (2.7) so that \( \frac{d + r}{d + 1} > 2 \). Then, we set \( \varphi(\xi) = 1 \) and \( \varphi(\xi) = \xi \) in Lemma 2.4 respectively, and obtain the following strong convergence up to a subsequence:
\[ (\rho^0)_m \to \rho^0 \quad \text{in} \quad L^2(T^d \times (0, T)), \quad \text{a.e.,} \]
\[ (\rho^0)_m(u^0)_m \to \rho^0 u^0 \quad \text{in} \quad L^2(T^d \times (0, T)). \]
Consequently, it gives the convergence of \( \{(u^0)_m\} \) up to a subsequence. \( \square \)

In conclusion, from Lemma 2.4 and Lemma 2.5, the operator \( \mathcal{T} \) is well-defined, continuous, and compact. Thus, we can use Schauder’s fixed point theorem to attain the existence of a fixed point of \( \mathcal{T} \), which asserts the existence of weak solutions to system (2.2). Then, we employ the fixed point argument with Propositions 2.1 and 2.2 to yield the following estimate.
Corollary 2.1. Let $T > 0$ and $(f^n, v^n)$ be a weak solution to the system (2.2) on the time interval $[0, T]$. Then, it satisfies the following entropy inequality:

$$
\int_{\mathbb{T}^d \times \mathbb{R}^d} \left( \frac{|\xi|^2}{2} + \sigma \log f^n \right) f^n \, dx \, d\xi + \int_{\mathbb{T}^d} (\rho^n - 1) K^\varepsilon \ast (\rho^n - 1) \, dx + \frac{1}{2} \int_{\mathbb{T}^d} |v^n|^2 \, dx \\
+ \int_0^t \int_{\mathbb{T}^d \times \mathbb{R}^d} \left( \frac{1}{f^n} |\sigma \nabla \xi f^n - (\chi(\lambda(u^n)) - \xi) f^n|^2 + |\chi(\lambda(v^n)) - \xi|^2 f^n \right) \, dx \, d\xi \, ds + \int_0^t \int_{\mathbb{T}^d} |\nabla v^n|^2 \, dx \, ds \\
\leq \int_{\mathbb{T}^d \times \mathbb{R}^d} \left( \frac{|\xi|^2}{2} + \sigma \log f_0^n \right) f_0^n \, dx \, d\xi + \int_{\mathbb{T}^d} (\rho_0^n - 1) K^\varepsilon \ast (\rho_0^n - 1) \, dx + \frac{1}{2} \int_{\mathbb{T}^d} |v_0^n|^2 \, dx \\
+ \sigma d\|f_0^n\|_{L^1} t + \int_0^t \int_{\mathbb{T}^d \times \mathbb{R}^d} (\chi(\lambda(u^n)) - \xi) \cdot \chi(\lambda(u^n)) f^n \, dx \, d\xi \, ds.
$$

for almost all $t \in [0, T]$.

Proof. From Propositions 2.1 and 2.2 we have

$$
\int_{\mathbb{T}^d \times \mathbb{R}^d} \left( \frac{|\xi|^2}{2} + \sigma \log f^n \right) f^n \, dx \, d\xi + \int_{\mathbb{T}^d} (\rho^n - 1) K^\varepsilon \ast (\rho^n - 1) \, dx + \frac{1}{2} \int_{\mathbb{T}^d} |v^n|^2 \, dx \\
+ \int_0^t \int_{\mathbb{T}^d \times \mathbb{R}^d} \left( \frac{1}{f^n} |\sigma \nabla \xi f^n - (\chi(\lambda(u^n)) - \xi) f^n|^2 + |\chi(\lambda(v^n)) - \xi|^2 f^n \right) \, dx \, d\xi \, ds + \int_0^t \int_{\mathbb{T}^d} |\nabla v^n|^2 \, dx \, ds \\
\leq \int_{\mathbb{T}^d \times \mathbb{R}^d} \left( \frac{|\xi|^2}{2} + \sigma \log f_0^n \right) f_0^n \, dx \, d\xi + \int_{\mathbb{T}^d} (\rho_0^n - 1) K^\varepsilon \ast (\rho_0^n - 1) \, dx + \frac{1}{2} \int_{\mathbb{T}^d} |v_0^n|^2 \, dx \\
+ \sigma d\|f_0^n\|_{L^1} t + \int_0^t \int_{\mathbb{T}^d \times \mathbb{R}^d} (\chi(\lambda(u^n)) - \xi) \cdot \chi(\lambda(u^n)) f^n \, dx \, d\xi \, ds.
$$

Here we obtain

$$
\int_{\mathbb{T}^d \times \mathbb{R}^d} (\chi(\lambda(u^n)) - \xi) \cdot \chi(\lambda(u^n)) f^n \, dx \, d\xi = \int_{\mathbb{T}^d} \rho^n \chi(\lambda(u^n)) \cdot (\chi(\lambda(u^n)) - u^n) \, dx \\
= \int_{\mathbb{T}^d} \rho^n u^n \left( \frac{\rho^n u^n}{\rho^n + \varepsilon} - u^n \right) \mathbb{1}_{\{|u^n| \leq \lambda\}} \, dx \\
= -\varepsilon \int_{\mathbb{T}^d} |\chi(\lambda(u^n))|^2 \, dx \leq 0,
$$

which implies our desired result. \qed

Before we proceed to the proof of the existence of a weak solution to system (1.1), we need two technical lemmas concerning the interaction potential energy whose proofs will be provided in Appendix A.

Lemma 2.6. For $d = 2, 3$, suppose that $\varrho \in L^\infty(0, T; \mathcal{L}^1(\mathbb{T}^d))$. Then we have

$$
\int_{\mathbb{T}^d} (K^\varepsilon \ast \varrho) \varrho \, dx \geq -C\|\varrho\|_{L^\infty(0, T; \mathcal{L}^1(\mathbb{T}^d))} (1 + \|\varrho\|_{L^\infty(0, T; \mathcal{L}^1(\mathbb{T}^d))}),
$$

where $C$ is a positive constant satisfying $C = O(1)$ as $\varepsilon \to 0$.

Lemma 2.7. For $d = 2, 3$, suppose that a sequence $\{\varrho^n\}$ is uniformly bounded in $L^\infty(0, T; \mathcal{L}^p(\mathbb{T}^d))$ for each $p \in [1, (d + 2)/d]$ and $\{\varrho^n\}$ converges to $\varrho$ almost everywhere. Then we have

$$
\lim_{n \to \infty} \int_{\mathbb{T}^d} (K^\varepsilon \ast \varrho^n) \varrho^n \, dx = \int_{\mathbb{T}^d} (K^\varepsilon \ast \varrho) \varrho \, dx.
$$

Moreover, if $\varepsilon \to 0$ as $n \to \infty$, then we have

$$
\lim_{n \to \infty} \int_{\mathbb{T}^d} (K^\varepsilon \ast \varrho^n) \varrho^n \, dx = \int_{\mathbb{T}^d} (K \ast \varrho) \varrho \, dx.
$$

2.3. Existence of a weak solution to VPNS. We are ready to present the existence of a weak solution to the system (1.1).
2.3.1. Convergence $\lambda \to \infty$. We will show the convergences of regularized solutions $(f^{\lambda,\varepsilon}, v^{\lambda,\varepsilon})$ as $\lambda$ tends to infinity.

- (Step A: Uniform bound estimates) First, we attain an upper bound estimate which is uniform in $\lambda$ using the entropy inequality from Corollary 2.1. Here, we recall that

$$\int_{T^d \times \mathbb{R}^d} f^{\lambda,\varepsilon} \log f^{\lambda,\varepsilon} \, dx \, d\xi \leq \frac{1}{4\sigma} \int_{T^d \times \mathbb{R}^d} f^{\lambda,\varepsilon} (1 + |\xi|^2) \, dx \, d\xi + C(\sigma),$$

where $\log_- g(x) := \max\{0, -\log g(x)\}$. We apply the above inequality and Lemma 2.6 to Corollary 2.1 and obtain

$$\int_{T^d \times \mathbb{R}^d} \left(\frac{|\xi|^2}{4} + \sigma |\log f^{\lambda,\varepsilon}|^2\right) f^{\lambda,\varepsilon} \, dx \, d\xi + \frac{1}{2} \int_0^t \int_{T^d} |v^{\lambda,\varepsilon}|^2 \, dx \, ds + \int_0^t \int_{T^d} \nabla v^{\lambda,\varepsilon} \, dx \, ds$$

$$\leq \int_{T^d \times \mathbb{R}^d} \left(\frac{|\xi|^2}{2} + \sigma |\log f^{\lambda,\varepsilon}_0|^2\right) f^{\lambda,\varepsilon}_0 \, dx \, d\xi + \int_{T^d \times \mathbb{R}^d} (\rho^\lambda_0 - 1) K^\varepsilon \ast (\rho^\lambda_0 - 1) \, dx + \frac{1}{2} \int_{T^d} |v^0|^2 \, dx$$

$$+ \sigma d \|f^\lambda_0\|_{L^1} + C,$$

where $C = C(T, \sigma)$ is a positive constant independent of $\lambda$ and $\varepsilon$, and it vanishes when $\sigma = 0$. Here, note that

$$\int_{T^d} |(\rho^\lambda_0 - 1) K^\varepsilon \ast (\rho^\lambda_0 - 1)| \, dx \leq \|K^\varepsilon\|_{L^p} \|\rho^\lambda_0 - 1\|^2_{L^p},$$

where $p$ and $q$ satisfies $2/p + 1/q = 2$ and we used Young’s convolution inequality:

$$\int_{T^d} (f \ast g) h \, dx \leq \|f\|_{L^p} \|g\|_{L^q} \|h\|_{L^r}, \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2.$$

Since the initial data $f^\lambda_0$ satisfy (2.6) with $k^* = 2$ uniformly in $\eta$, we can get

$$\|\rho^\lambda_0 - 1\|_{L^p} \leq C \quad \forall p \in [1, (d + 2)/d]$$

uniformly in $\lambda$, and this requires $\|K^\varepsilon\|_{L^p} < \infty$ uniformly in $\varepsilon$ for some $q > (d + 2)/4$. Since $d = 2$ or $3$, we can choose such $q$ and hence, we combine this argument with $\|f^\lambda_0\|_{L^1} \leq \|f_0\|_{L^1}$ and Lemma 2.6 and apply Grönwall’s lemma to yield, for $t \in [0, T]$,

$$\int_{T^d \times \mathbb{R}^d} |\xi|^2 f^{\lambda,\varepsilon} \, dx \, d\xi + \int_{T^d} |v^{\lambda,\varepsilon}|^2 \, dx + \int_0^t \int_{T^d} \nabla v^{\lambda,\varepsilon} \, dx \, ds \leq C. \quad (2.8)$$

Here $C = C(T, \sigma)$ is a positive constant independent of $\lambda$ and $\varepsilon$. Once we recall that

$$\|f^{\lambda,\varepsilon}(\cdot, \cdot, t)\|_{L^p} + \frac{4\sigma(p - 1)}{p} \int_0^t e^{2d(p - 1)(t - s)} \|\nabla f^{\lambda,\varepsilon}(\cdot, \cdot, s)\|_{L^2} \, ds \leq \|f^\lambda_0\|_{L^p} e^{2d(p - 1)t}$$

for every $p \in [1, \infty]$, we obtain the following uniform estimates:

$$\|f^{\lambda,\varepsilon}\|_{L^\infty(0, T; L^p(T^d \times \mathbb{R}^d))} + \|\rho^{\lambda,\varepsilon}\|_{L^\infty(0, T; L^q(T^d))} + \|\rho^{\lambda,\varepsilon} u^{\lambda,\varepsilon}\|_{L^\infty(0, T; L^r(T^d))} \leq C(T),$$

where $p \in [1, \infty)$, $q_1 \in [1, (d + 2)/d)$, $q_2 \in [1, (d + 2)/(d + 1)]$ and $C = C(T)$ is a positive constant independent of $\lambda$ and $\varepsilon$. Consequently, we can obtain the following weak convergence as $\lambda \to \infty$ up to a subsequence:

$$f^{\lambda,\varepsilon} \rightharpoonup f^\varepsilon \quad \text{in} \quad L^\infty(0, T; L^p(T^d \times \mathbb{R}^d)), \quad p \in [1, \infty),$$

$$\rho^{\lambda,\varepsilon} \rightharpoonup \rho^\varepsilon \quad \text{in} \quad L^\infty(0, T; L^p(T^d)), \quad p \in [1, (d + 2)/d),$$

$$\rho^{\lambda,\varepsilon} u^{\lambda,\varepsilon} \rightharpoonup \rho^\varepsilon u^\varepsilon \quad \text{in} \quad L^\infty(0, T; L^p(T^d)), \quad p \in [1, (d + 2)/(d + 1)],$$

$$v^{\lambda,\varepsilon} \rightharpoonup v^\varepsilon \quad \text{in} \quad L^\infty(0, T; H^1(T^d)) \cap L^2(0, T; H^1(T^d)).$$

For the proof, we need to obtain the strong convergence of the averaged quantities and fluid velocity. Let $p \in (1, (d + 2)/(d + 1))$ and write $G^{\lambda,\varepsilon}$ as

$$G^{\lambda,\varepsilon} := \sigma \nabla f^{\lambda,\varepsilon} + (2\xi + \nabla K^\varepsilon \ast (\rho^\lambda_0 - 1) - \chi_\lambda (u^\lambda - \chi_\lambda (v^\lambda)) f^{\lambda,\varepsilon},$$
then we can easily check that \( G^{\lambda,\varepsilon} \in L^p_{\text{loc}}(\mathbb{T}^d \times \mathbb{R}^d \times (0, T)) \). Thus, we set \( G^{\lambda,\varepsilon} \) as above, \( r = 2 \) in Lemma 2.4, to get, for \( p \in (1, (d + 2)/(d + 1)) \),

\[
\rho^{\lambda,\varepsilon} \to \rho^\varepsilon \quad \text{in} \quad L^p(\mathbb{T}^d \times (0, T)) \quad \text{and a.e.,}
\]

\[
\rho^{\lambda,\varepsilon}u^{\lambda,\varepsilon} \to \rho^\varepsilon u^\varepsilon \quad \text{in} \quad L^p(\mathbb{T}^d \times (0, T))
\]
as \( \lambda \to \infty \), up to a subsequence. For the fluid velocity, we first obtain from (2.8) that

\[
\|v^{\lambda,\varepsilon}\|_{L^\infty(0, T; L^2)} + \|\nabla v^{\lambda,\varepsilon}\|_{L^2(0, T; L^2)} \leq C,
\]

where \( C > 0 \) is independent of \( \lambda \) and \( \varepsilon \).

We next show that \( \|\partial_t v^{\lambda,\varepsilon}\|_{L^q(0, T; V')} \leq C \) for some \( q \in (1, \infty) \). For \( p \in (1, (d + 2)/(d + 1)) \) and \( \varphi \in V \), we get

\[
\left| \int_0^T \int_{T^d} \partial_t v^{\lambda,\varepsilon} \cdot \varphi \, dx \, ds \right|
\]

\[
\leq \|\theta^\varepsilon \ast v^{\lambda,\varepsilon}\|_{L^1(T^d \times (0, T))} \|\nabla v^{\lambda,\varepsilon}\|_{L^2(T^d \times (0, T))} \|\varphi\|_{L^p(T^d \times (0, T))} + \|\nabla v^{\lambda,\varepsilon}\|_{L^2(T^d \times (0, T))} \|\varphi\|_{L^2(T^d \times (0, T))} + \|\rho^{\lambda,\varepsilon} u^{\lambda,\varepsilon}\|_{L^p(T^d \times (0, T))} \|\varphi\|_{L^p(T^d \times (0, T))},
\]

where \( p' \) is the Hölder conjugate of \( p \). Here, we use Gagliardo–Nirenberg interpolation inequality to get

\[
\|v^{\lambda,\varepsilon}\|_{L^3(T^d)} \leq \begin{cases} 
C \|\nabla v^{\lambda,\varepsilon}\|_{L^2}^{1/3} \|v^{\lambda,\varepsilon}\|_{L^2}^{2/3} & \text{if} \quad d = 2, \\
C \|\nabla v^{\lambda,\varepsilon}\|_{L^2}^{1/2} \|v^{\lambda,\varepsilon}\|_{L^2}^{1/2} & \text{if} \quad d = 3,
\end{cases}
\]

where \( C \) only depends on \( d \). Thus, if \( d = 2 \),

\[
\|\theta^\varepsilon \ast v^{\lambda,\varepsilon}\|_{L^3(T^d \times (0, T))} \leq \|v^{\lambda,\varepsilon}\|_{L^3(T^d \times (0, T))} \leq C \|v^{\lambda,\varepsilon}\|_{L^6(0, T; L^2(T^d))} \|\nabla v^{\lambda,\varepsilon}\|_{L^2(0, T; L^2(T^d))},
\]

and if \( d = 3 \),

\[
\|\theta^\varepsilon \ast v^{\lambda,\varepsilon}\|_{L^3(T^d \times (0, T))} \leq \|v^{\lambda,\varepsilon}\|_{L^3(T^d \times (0, T))} \leq C \|v^{\lambda,\varepsilon}\|_{L^6(0, T; L^2(T^d))} \|\nabla v^{\lambda,\varepsilon}\|_{L^2(0, T; L^2(T^d))},
\]

where \( C \) only depends on \( d \). On the other hand, the local moment can be estimated as

\[
\|\rho^{\lambda,\varepsilon} u^{\lambda,\varepsilon}\|_{L^2(0, T; L^p(T^d))} \leq \|\rho^{\lambda,\varepsilon}\|_{L^2(0, T; L^{\frac{d+2}{d-2p}}(T^d))} \|u^{\lambda,\varepsilon}\|_{L^2(0, T; L^{\frac{d+2}{d-2p}}(T^d))} \leq \|\rho^{\lambda,\varepsilon}\|_{L^2(0, T; L^{\frac{d+2}{d-2p}}(T^d))} \|u^{\lambda,\varepsilon}\|_{L^2(0, T; H^1(T^d))},
\]

which is finite since \( p \in (1, (d + 2)/(d + 1)) \) implies \( (d + 2)p/(d + 2 - p) < (d + 2)/d \). Thus, taking \( p = (d + 3)/(d + 2) \) gives \( \|\varphi\|_{L^p} \leq C \|\varphi\|_{H^1} \), and thus we obtain

\[
\partial_t v^{\lambda,\varepsilon} \in L^{\frac{d+3}{d+2}}(0, T; V'), \quad \text{uniformly in} \quad \lambda \quad \text{and} \quad \varepsilon.
\]

Combining this with (2.9), we can use Aubin–Lions lemma to get

\[
v^{\lambda,\varepsilon} \to v^\varepsilon \quad \text{in} \quad L^2(T^d \times (0, T)) \quad \text{and a.e.,}
\]

up to a subsequence.

• (Step B: Existence of weak solutions and entropy inequality) Now, we show that the limit \( (f^\varepsilon, v^\varepsilon) \) satisfies the following system in distributional sense:

\[
\partial_t f^\varepsilon + v \cdot \nabla f^\varepsilon - \nabla \xi \cdot ((\xi - v^\varepsilon + \nabla K^\varepsilon \ast (\rho^\varepsilon - 1)) f^\varepsilon) = \nabla \xi \cdot (\sigma \nabla f^\varepsilon - (u^\varepsilon - \xi) f^\varepsilon),
\]

\[
\partial_t v^\varepsilon + ((\theta^\varepsilon \ast v^\varepsilon) \cdot \nabla) v^\varepsilon + \nabla p^\varepsilon - \Delta u^\varepsilon = -\rho^\varepsilon (v^\varepsilon - u^\varepsilon),
\]

\[
\nabla \cdot v^\varepsilon = 0,
\]

(2.10)
and the entropy inequality:
\[
\begin{aligned}
&\int_{\mathbb{T}^d \times \mathbb{R}^d} \left( \frac{|\xi|^2}{2} + \sigma \log f^\varepsilon \right) f^\varepsilon \, dx d\xi + \int_{\mathbb{T}^d} (\rho^\varepsilon - 1) K^\varepsilon * (\rho^\varepsilon - 1) \, dx + \frac{1}{2} \int_{\mathbb{T}^d} |v^\varepsilon|^2 \, dx \\
&+ \int_0^t \int_{\mathbb{T}^d \times \mathbb{R}^d} \left( \frac{1}{2} |\nabla \xi f^\varepsilon - (u^\varepsilon - \xi) f^\varepsilon|^2 + |v^\varepsilon - \xi|^2 f^\varepsilon \right) \, dx d\xi ds + \int_0^t \int_{\mathbb{T}^d} |\nabla v^\varepsilon|^2 \, dx ds \\
&\leq \int_{\mathbb{T}^d \times \mathbb{R}^d} \left( \frac{|\xi|^2}{2} + \sigma \log f_0 \right) f_0 \, dx d\xi + \int_{\mathbb{T}^d} (\rho_0 - 1) K^\varepsilon * (\rho_0 - 1) \, dx + \frac{1}{2} \int_{\mathbb{T}^d} |v_0^\varepsilon|^2 \, dx + \sigma dt.
\end{aligned}
\]
\tag{2.11}
\]

For the weak solution \((f^\varepsilon , v^\varepsilon )\) to \((2.10)\), it suffices to show the following convergence in distributional sense since the others are linear:
\[
\begin{aligned}
&\begin{cases}
(i) \quad (\nabla K^\varepsilon \ast \rho^\varepsilon ) f^\lambda \varepsilon \\ = (\nabla K^\varepsilon \ast \rho^\varepsilon ) f^\varepsilon ,
\end{cases} \\
&\begin{cases}
(ii) \quad \chi \lambda (u^\lambda \varepsilon \ast f^\lambda \varepsilon - u^\varepsilon f^\varepsilon .
\end{cases}
\end{aligned}
\]
\tag{2.12}

Although the proof is similar to that in [10], we present the detail for the completeness of our work. For the proof, we arbitrarily choose \(\Phi \in \mathcal{C}_c^\infty (\mathbb{T}^d \times \mathbb{R}^d \times [0,T])\) with \(\Phi(\cdot,\cdot, T) = 0\).

\(\diamond\) (Step B-1: Convergence of \((2.12)\) (i)) We write
\[
\rho^\lambda \varepsilon (x,t) := \int_{\mathbb{R}^d} f^\lambda \varepsilon (x,\xi,t) \Phi(x,\xi,t) \, d\xi.
\]

Then, we estimate
\[
\begin{aligned}
&\int_0^T \int_{\mathbb{T}^d \times \mathbb{R}^d} \left[ (\nabla K^\varepsilon \ast \rho^\lambda \varepsilon ) f^\lambda \varepsilon - (\nabla K^\varepsilon \ast \rho^\varepsilon ) f^\varepsilon \right] \Phi \, dx d\xi dt \\
&= \int_0^T \int_{\mathbb{T}^d \times \mathbb{R}^d} \nabla K^\varepsilon \ast (\rho^\lambda \varepsilon - \rho^\varepsilon ) \rho^\lambda \varepsilon \, dx dt + \int_0^T \int_{\mathbb{T}^d \times \mathbb{R}^d} (\nabla K^\varepsilon \ast \rho^\varepsilon ) (f^\lambda \varepsilon - f^\varepsilon ) \Phi \, dx d\xi dt \\
&=: J_1^\lambda \varepsilon + J_2^\lambda \varepsilon.
\end{aligned}
\]

For \(J_1^\lambda \varepsilon\), we use the uniform bound of \(f^\lambda \varepsilon\) in \(L^\infty (\mathbb{T}^d \times \mathbb{R}^d \times (0,T))\) and the compact support of \(\Phi\) to yield
\[
\rho^\lambda \varepsilon \in L^\infty (0,T; L^p(\mathbb{T}^d)) \quad \text{for any} \quad p \in [1, \infty]
\]
uniformly in \(\lambda\) and \(\varepsilon\). This gives
\[
\begin{aligned}
J_1^\lambda \varepsilon \leq \|\nabla K^\varepsilon\|_{L^p(\mathbb{T}^d)} \|\rho^\lambda \varepsilon - \rho^\varepsilon\|_{L^p(\mathbb{T}^d \times (0,T))} \|\rho^\lambda \varepsilon\|_{L^{p'}(\mathbb{T}^d \times (0,T))},
\end{aligned}
\]
where \(p \in (1, (d+2)/(d+1))\), \(p'\) is the Hölder conjugate of \(p\), and we used Young’s convolution inequality. Since \(\|\nabla K^\varepsilon\| \leq C(1 + |x|^{-d+1})\), we use the strong convergence of \(\rho^\lambda \varepsilon\) to get \(J_1^\lambda \varepsilon \to 0\) as \(\lambda \to \infty\). For \(J_2^\lambda \varepsilon\), we notice
\[
\begin{aligned}
(\nabla K^\varepsilon \ast \rho^\varepsilon ) \Phi \in L^1 (0,T; L^p(\mathbb{T}^d)) \quad \text{for some} \quad p \in (1, \infty).
\end{aligned}
\]

Thus, the weak convergence of \(f^\lambda \varepsilon\) asserts that \(J_2^\lambda \varepsilon \to 0\) as \(\lambda \to \infty\).

\(\diamond\) (Step B-2: Convergence of \((2.12)\) (ii)) We estimate
\[
\begin{aligned}
&\int_0^T \int_{\mathbb{T}^d \times \mathbb{R}^d} (\chi \lambda (u^\lambda \varepsilon \ast f^\lambda \varepsilon - u^\varepsilon f^\varepsilon )) \Phi \, dx d\xi dt \\
&= \int_0^T \int_{\mathbb{T}^d} (u^\lambda \varepsilon - u^\varepsilon ) \rho^\lambda \varepsilon \, dx ds + \int_0^T \int_{\mathbb{T}^d \times \mathbb{R}^d} u^\varepsilon (f^\lambda \varepsilon - f^\varepsilon ) \Phi \, dx d\xi dt \\
&\quad + \int_0^T \int_{\mathbb{T}^d \times \mathbb{R}^d} u^\varepsilon f^\lambda \varepsilon \mathbf{1}_{\{|u^\lambda \varepsilon | > \lambda\}} \Phi \, dx d\xi dt \\
&=: J_1^\lambda \varepsilon + J_2^\lambda \varepsilon + J_3^\lambda \varepsilon.
\end{aligned}
\]

For \(J_1^\lambda \varepsilon\), we find
\[
\begin{aligned}
J_1^\lambda \varepsilon &= \int_0^T \int_{\mathbb{T}^d} \left( \frac{1}{\rho^\lambda \varepsilon + \varepsilon} - \frac{1}{\rho^\varepsilon + \varepsilon} \right) (\rho^\varepsilon u^\varepsilon) + \frac{1}{\rho^\lambda \varepsilon + \varepsilon} (\rho^\lambda \varepsilon u^\lambda \varepsilon - \rho^\varepsilon u^\varepsilon) \rho^\lambda \varepsilon \, dx dt.
\end{aligned}
\]
Since $\rho^{\lambda,\varepsilon} \to \rho^x$ a.e. and $\rho^{\lambda,\varepsilon}_0 \in L^\infty(\mathbb{T}^d \times (0,T))$ uniformly in $\lambda$ and $\varepsilon$, we get
\[
\left(\frac{1}{\rho^{\lambda,\varepsilon} + \varepsilon} - \frac{1}{\rho^x + \varepsilon}\right) (\rho^x u^x) \rho^{\lambda,\varepsilon}_0 \to 0 \quad \text{a.e.}
\]
as $\lambda \to \infty$, and
\[
\left|\left(\frac{1}{\rho^{\lambda,\varepsilon} + \varepsilon} - \frac{1}{\rho^x + \varepsilon}\right) (\rho^x u^x) \rho^{\lambda,\varepsilon}_0\right| \leq \frac{2\|\rho^{\lambda,\varepsilon}_0\|_{L^\infty}}{\varepsilon} |\rho^x u^x| \leq C(\varepsilon)|\rho^x u^x|,
\]
where $C = C(\varepsilon)$ is a positive constant independent of $\lambda$. Then, we use the dominated convergence theorem to get
\[
\int_0^T \int_{\mathbb{T}^d} \left(\frac{1}{\rho^{\lambda,\varepsilon} + \varepsilon} - \frac{1}{\rho^x + \varepsilon}\right) (\rho^x u^x) \rho^{\lambda,\varepsilon}_0 \, dx \, dt \to 0
\]
as $\lambda \to \infty$. Moreover, we obtain
\[
\int_0^T \int_{\mathbb{T}^d} \left(\frac{1}{\rho^{\lambda,\varepsilon} + \varepsilon} (\rho^{\lambda,\varepsilon} u^{\lambda,\varepsilon} - \rho^x u^x)\right) \rho^{\lambda,\varepsilon}_0 \, dx \, dt \leq \frac{1}{\varepsilon} \|\rho^{\lambda,\varepsilon} u^{\lambda,\varepsilon} - \rho^x u^x\|_{L^p} \|\rho^{\lambda,\varepsilon}\|_{L^p},
\]
where $p \in (1, (d+2)/(d+1))$, and this implies $\mathcal{J}_{3}^{\lambda,\varepsilon} \to 0$ as $\lambda \to \infty$. For $\mathcal{J}_{2}^{\lambda,\varepsilon}$, it is clear that $u_x^2 \Phi \in L^1(0,T; L^p(\mathbb{T}^d \times \mathbb{R}^d))$ for some $p \in (1, \infty)$. Thus, the weak convergence of $f^{\lambda,\varepsilon}$ gives $\mathcal{J}_{2}^{\lambda,\varepsilon} \to 0$ as $\lambda \to \infty$. Finally for $\mathcal{J}_{3}^{\lambda,\varepsilon}$, we use
\[
|u^{\lambda,\varepsilon}| \leq \left(\frac{\int_{\mathbb{T}^d} |\xi|^2 f^{\lambda,\varepsilon} d\xi}{\rho^{\lambda,\varepsilon}}\right)^{1/2}
\]
to get
\[
\mathcal{J}_{3}^{\lambda,\varepsilon} \leq \frac{1}{\lambda} \int_0^T \int_{\mathbb{T}^d \times \mathbb{R}^d} |u^{\lambda,\varepsilon}|^2 f^{\lambda,\varepsilon} \Phi \, dx \, d\xi \, dt \leq \frac{\|\Phi\|_{L^\infty}}{\lambda} \int_0^T \int_{\mathbb{T}^d \times \mathbb{R}^d} |\xi|^2 f^{\lambda,\varepsilon} d\xi \, dt \to 0
\]
as $\lambda \to \infty$. This concludes that $(f^x, v^x)$ is a weak solution to the system \ref{sys2.10}.

\dagger (Step B-3: Entropy inequality) For the entropy inequality \ref{sys2.11}, we first take the liminf on the left hand side of Corollary \ref{cor2.11} convexity of the entropy and use $\|f^{\lambda}_0\|_{L^1} \leq \|f_0\|_{L^1} = 1$ and Lemma \ref{lem2.10} to get
\[
\int_{\mathbb{T}^d \times \mathbb{R}^d} \left(\frac{|\xi|^2}{2} + \sigma \log f^x \right) f^x \, dx \, d\xi + \int_{\mathbb{T}^d} (\rho^x - 1)K^x \ast (\rho^x - 1) \, dx + \frac{1}{2} \int_{\mathbb{T}^d} |v^x|^2 \, dx
\]
\[
+ \int_0^t \int_{\mathbb{T}^d \times \mathbb{R}^d} \left(\frac{1}{f^x} |\sigma \nabla \xi f^x - (u^x - \xi)|^2 + |v^x - \xi|^2 f^x\right) \, dx \, d\xi \, ds + \int_0^t \int_{\mathbb{T}^d} |v^x|^2 \, dx \, ds
\]
\[
\leq \liminf_{\lambda \to \infty} \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} \left(\frac{|\xi|^2}{2} + \sigma \log f^0 \right) f^0 \, dx \, d\xi + \int_{\mathbb{T}^d} (\rho^0 - 1)K^x \ast (\rho^0 - 1) \, dx + \frac{1}{2} \int_{\mathbb{T}^d} |v^0|^2 \, dx\right) + \sigma dt.
\]
We employ the reverse Fatou’s lemma and almost everywhere convergences $\rho_0^\lambda \to \rho_0$ and $f_0^\lambda \to f_0$ as $\lambda \to \infty$ to have
\[
\liminf_{\lambda \to \infty} \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} \left(\frac{|\xi|^2}{2} + \sigma \log f^0 \right) f^0 \, dx \, d\xi + \int_{\mathbb{T}^d} (\rho^0 - 1)K^x \ast (\rho^0 - 1) \, dx + \frac{1}{2} \int_{\mathbb{T}^d} |v^0|^2 \, dx\right)
\]
\[
\leq \limsup_{\lambda \to \infty} \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} \left(\frac{|\xi|^2}{2} + \sigma \log f^0 \right) f^0 \, dx \, d\xi + \int_{\mathbb{T}^d} (\rho^0 - 1)K^x \ast (\rho^0 - 1) \, dx + \frac{1}{2} \int_{\mathbb{T}^d} |v^0|^2 \, dx\right)
\]
\[
\leq \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} \left(\frac{|\xi|^2}{2} + \sigma \log f^0 \right) f^0 \, dx \, d\xi + \int_{\mathbb{T}^d} (\rho_0 - 1)K^x \ast (\rho_0 - 1) \, dx + \frac{1}{2} \int_{\mathbb{T}^d} |v_0|^2 \, dx\right),
\]
which implies the desired estimate.
2.3.2. Convergence $\varepsilon \to 0$. Now, it remains to show the convergence as $\varepsilon \to 0$. First, the weak convergence $f^n \rightharpoonup f^\varepsilon$ implies the following uniform upper bound estimate:

$$
\|f^\varepsilon\|_{L^\infty(0,T;L^p(\mathbb{T}^d \times \mathbb{R}^d))} + \frac{4\sigma(p-1)}{p} \int_0^T \|\nabla \xi(f^\varepsilon)^{p/2}(\cdot, s)\|^2_{L^2} ds \leq \|f_0\|_{L^p} e^{2(p-1)t}
$$

(2.13)

for $p \in [1, \infty)$ and

$$
\|f^\varepsilon\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d \times \mathbb{R}^d))} \leq C\|f_0\|_{L^\infty}
$$

(2.14)

for some $C > 0$ independent of $\varepsilon > 0$. Hence, the inequalities (2.13) and (2.14) together with the entropy inequality (2.11) yield the following weak convergence up to a subsequence as $\varepsilon \to 0$:

$$
f^\varepsilon \rightharpoonup f \quad \text{in} \quad L^\infty(0,T;L^p(\mathbb{T}^d \times \mathbb{R}^d)), \quad p \in [1, \infty],
$$

$$
\rho^\varepsilon \rightharpoonup \rho \quad \text{in} \quad L^\infty(0,T;L^p(\mathbb{T}^d)), \quad p \in [1, (d+2)/d),
$$

$$
\rho^\varepsilon u^\varepsilon \rightharpoonup \rho u \quad \text{in} \quad L^\infty(0,T;L^p(\mathbb{T}^d)), \quad p \in [1, (d+2)/(d+1)),
$$

$$
v^\varepsilon \rightharpoonup v \quad \text{in} \quad L^\infty(0,T;L^2(\mathbb{T}^d)) \cap L^2(0,T;H^1(\mathbb{T}^d)).
$$

Once again, we apply Lemma 2.3 to obtain the strong convergence up to a subsequence for $p \in (1, (d+2)/(d+1))$:

$$
\rho^\varepsilon \to \rho \quad \text{in} \quad L^p(\mathbb{T}^d \times (0,T)) \quad \text{and a.e.,}
$$

$$
\rho^\varepsilon u^\varepsilon \to \rho u \quad \text{in} \quad L^p(\mathbb{T}^d \times (0,T))
$$

as $\varepsilon \to 0$. In order to show that $(f, v)$ is a weak solution of (1.1), it suffices to show the following convergences in distribution sense, since the other convergences can be deduced from the previous argument [11]:

$$
\begin{align*}
& (i) \quad (\nabla K^\varepsilon \ast \rho^\varepsilon) f^\varepsilon \Rightarrow (\nabla K \ast \rho) f, \\
& (ii) \quad u^\varepsilon f^\varepsilon \Rightarrow uf.
\end{align*}
$$

(2.15)

However, the convergence estimates for (2.15) can be found in [10], thus it remains to show that the weak solution obtained up to now satisfies the entropy inequality:

$$
\int_{\mathbb{T}^d \times \mathbb{R}^d} \left( \frac{\|f\|^2}{2} + \sigma \log f \right) f \, dx d\xi + \int_{\mathbb{T}^d} (\rho - 1)K \ast (\rho - 1) \, dx + \frac{1}{2} \int_{\mathbb{T}^d} |v|^2 \, dx \\
+ \int_0^t \int_{\mathbb{T}^d \times \mathbb{R}^d} \left( \frac{1}{2} f \sigma \nabla \xi f - (u - \xi) f^2 \right) dx d\xi ds + \int_0^t \int_{\mathbb{T}^d} |v|^2 \, dx ds \\
\leq \int_{\mathbb{T}^d \times \mathbb{R}^d} \left( \frac{\|f_0\|^2}{2} + \sigma \log f_0 \right) f_0 \, dx d\xi + \int_{\mathbb{T}^d} (\rho_0 - 1)K \ast (\rho_0 - 1) \, dx + \frac{1}{2} \int_{\mathbb{T}^d} |v_0|^2 \, dx + \sigma dt \|f_0\|_{L^1}.
$$

With the entropy inequality (2.11), the strong convergences of $\rho^\varepsilon$ and $\rho^\varepsilon u^\varepsilon$ thanks to the velocity averaging lemma, and Lemma 2.3, we employ a similar argument as in the previous step to conclude the desired result.

3. Hydrodynamic limit: from VPNS to EPNS

In this section, we present the rigorous derivation of the EPNS system (1.4) from the VPNS (1.1) with $\sigma \geq 0$. As mentioned before, our proof is based on the modulated kinetic energy estimate and the relative entropy method. For this, we first provide the entropy inequality estimate. Set

$$
\mathcal{F}(f, v) := \int_{\mathbb{T}^d \times \mathbb{R}^d} \left( \frac{\|f\|^2}{2} + \sigma \log f \right) f \, dx d\xi + \frac{1}{2} \int_{\mathbb{T}^d} |\nabla K \ast (\rho - 1)|^2 \, dx + \int_{\mathbb{T}^d} \frac{1}{2} |v|^2 \, dx,
$$

$$
D_1(f) := \int_{\mathbb{T}^d \times \mathbb{R}^d} \frac{1}{f} \sigma \nabla \xi f - (u - \xi) f^2 \, dx d\xi, \quad \text{and}
$$

$$
D_2(f, v) := \int_{\mathbb{T}^d \times \mathbb{R}^d} |v - \xi|^2 f \, dx d\xi + \int_{\mathbb{T}^d} |\nabla v|^2 \, dx.
$$

Recall from the previous section that $\mathcal{F}(f^\varepsilon, v^\varepsilon)$ satisfies

$$
\mathcal{F}(f^\varepsilon, v^\varepsilon) + \frac{1}{\varepsilon} \int_0^t D_1(f^\varepsilon) \, ds + \int_0^t D_2(f^\varepsilon, v^\varepsilon) \, ds \leq \mathcal{F}(f_0^\varepsilon, v_0^\varepsilon) + \sigma dt \|f_0^\varepsilon\|_{L^1}.
$$

(3.1)

Without loss of generality, we assume $\|f_0\|_{L^1} = 1$. In the lemma below, we show the uniform bound for the system (1.3).
Lemma 3.1. For $T > 0$, let $(f^\varepsilon, v^\varepsilon)$ be a weak solution to the system (1.3) on the time interval $[0, T]$ corresponding to initial data $(f_0^\varepsilon, v_0^\varepsilon)$. Then we have

$$\int_{T^d \times \mathbb{R}^d} \left( \frac{|x|^2}{4} + \frac{|\xi|^2}{4} + \sigma \log f^\varepsilon \right) f^\varepsilon \, dx \, d\xi + \frac{1}{2} \int_{T^d} |\nabla K \ast (\rho^\varepsilon - 1)|^2 \, dx + \int_{T^d} \frac{1}{2} |v^\varepsilon|^2 \, dx$$

$$+ \frac{1}{\varepsilon} \int_0^T D_1(f^\varepsilon) \, ds + \int_0^T D_2(f^\varepsilon, v^\varepsilon) \, ds \leq C(F(f_0^\varepsilon, v_0^\varepsilon) + 1)e^{CT},$$

where $C > 0$ is a constant independent of $\varepsilon$.

Proof. First, we note that

$$\int_{T^d \times \mathbb{R}^d} f^\varepsilon |\nabla \log f^\varepsilon| \, dx \, d\xi \leq \frac{1}{4\sigma} \int_{T^d \times \mathbb{R}^d} f^\varepsilon (1 + |\xi|^2) \, dx \, d\xi + C,$$

where $C = C(\sigma)$ is independent of $\varepsilon$. Then, we combine the above estimate with (3.1) and Grönwall’s lemma to get the desired estimate.

Next, we investigate a modified entropy inequality.

Lemma 3.2. For $T > 0$, let $(f^\varepsilon, v^\varepsilon)$ be a weak solution to (1.3) on the time interval $[0, T]$ corresponding to initial data $(f_0^\varepsilon, v_0^\varepsilon)$. Then we have

$$\mathcal{F}(f^\varepsilon, v^\varepsilon) + \int_0^T \int_{T^d} |\nabla v^\varepsilon|^2 \, dx \, ds + \int_0^T \int_{T^d} \rho^\varepsilon |u^\varepsilon - v^\varepsilon|^2 \, dx \, ds + \frac{1}{2\varepsilon} \int_0^T D_1(f^\varepsilon) \, ds \leq \mathcal{F}(f_0^\varepsilon, v_0^\varepsilon) + C\varepsilon,$$

where $C > 0$ is a constant independent of $\varepsilon$.

Proof. From a direct computation, we get

$$\frac{1}{2} \int_{T^d \times \mathbb{R}^d} \rho^\varepsilon(x) \rho^\varepsilon(y) |u^\varepsilon(y) - u^\varepsilon(x)|^2 \, dx \, dy$$

$$= \frac{1}{2} \int_{T^d \times \mathbb{R}^d} \rho^\varepsilon(x) \rho^\varepsilon(y) (u^\varepsilon(x) - u^\varepsilon(y)) \cdot u^\varepsilon(x) \, dx \, dy$$

$$= \int_{T^d \times \mathbb{R}^d} f^\varepsilon(x, \xi) f^\varepsilon(y, \xi) (\xi - \xi_*) \cdot u^\varepsilon(x) \, dx \, dy \, d\xi d\xi_*$$

$$= \int_{T^d \times \mathbb{R}^d} f^\varepsilon(y, \xi_*) f^\varepsilon(x, \xi) (\xi - \xi_*) \cdot \xi \, dx \, dy \, d\xi d\xi_*$$

$$+ \int_{T^d \times \mathbb{R}^d} f^\varepsilon(y, \xi_*) (\xi - \xi_*) \cdot (f^\varepsilon(x, \xi) (u^\varepsilon(x) - \xi) - \sigma \nabla \xi f(x, \xi)) \, dx \, dy \, d\xi d\xi_*$$

$$+ \int_{T^d \times \mathbb{R}^d} f^\varepsilon(y, \xi_*) (\xi - \xi_*) \cdot \sigma \nabla f(x, \xi) \, dx \, dy \, d\xi d\xi_*$$

$$=: \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.$$

For $\mathcal{I}_1$, we use the change of variables $(x, \xi) \leftrightarrow (y, \xi_*)$ to get

$$\mathcal{I}_1 = \frac{1}{2} \int_{T^d \times \mathbb{R}^d} f^\varepsilon(y, \xi_*) f^\varepsilon(x, \xi) (\xi - \xi_*|^2 \, dx \, dy \, d\xi d\xi_*.$$

For the estimate of $\mathcal{I}_2$, similarly to (3.4), we set

$$V^\varepsilon(x, \xi) := \frac{1}{\sqrt{f^\varepsilon(x, \xi)}} (f^\varepsilon(x, \xi) (u^\varepsilon(x) - \xi) - \sigma \nabla \xi f(x, \xi)).$$
lated energy associated to the systems (1.3) and (1.4). For this purpose, we define

### 3.1. Modulated energy estimates.

Then we obtain

\[
\mathcal{I}_2 = \int_{\mathbb{T}^d \times \mathbb{T}^d \times \mathbb{R}^d} \sqrt{f^\varepsilon(x, \xi)} \rho^\varepsilon(y) (\xi - u^\varepsilon(y)) \cdot V^\varepsilon(x, \xi) \, dx dy d\xi
\]

\[
= \left( \int_{\mathbb{T}^d} \rho^\varepsilon(y) \, dy \right) \int_{\mathbb{T}^d \times \mathbb{R}^d} \xi \sqrt{f^\varepsilon(x, \xi)} \cdot V^\varepsilon(x, \xi) \, dx d\xi
\]

\[
- \left( \int_{\mathbb{T}^d} (\rho^\varepsilon u^\varepsilon)(y) \, dy \right) \int_{\mathbb{T}^d \times \mathbb{R}^d} \sqrt{f^\varepsilon(x, \xi)} \cdot V^\varepsilon(x, \xi) \, dx d\xi
\]

\[
\leq 2 \| f^\varepsilon \|_{L^1} \left( \int_{\mathbb{T}^d \times \mathbb{R}^d} |\xi|^2 f^\varepsilon \, dx d\xi \right)^{1/2} \left( \int_{\mathbb{T}^d \times \mathbb{R}^d} (V^\varepsilon(x, \xi))^2 \, dx d\xi \right)^{1/2}
\]

\[
\leq \frac{1}{2\varepsilon} D_1(f^\varepsilon) + C\varepsilon,
\]

where \( C > 0 \) is independent of \( \varepsilon \), and we used Cauchy–Schwarz inequality and

\[
\rho^\varepsilon |u^\varepsilon|^2 \leq \int_{\mathbb{R}^d} |\xi|^2 \, d\xi.
\]

For \( \mathcal{I}_3 \), we directly estimate

\[
\mathcal{I}_3 = -\sigma d \int_{\mathbb{T}^d \times \mathbb{R}^d} f^\varepsilon(y, \xi_*) f^\varepsilon(x, \xi) \, dx dy d\xi d\xi_* = -\sigma d.
\]

We combine the estimates for \( \mathcal{I}_i, i = 1, 2, 3 \), and put it into (3.2) to complete the proof. \( \square \)

#### 3.1. Modulated energy estimates.

In this subsection, we provide several estimates regarding the modulated energy associated to the systems (1.3) and (1.4). For this purpose, we define

\[
U := \begin{pmatrix} \rho & m \\ m & v \end{pmatrix} \quad \text{and} \quad A(U) := \begin{pmatrix} \frac{m \cdot \varepsilon m}{\rho} & 0 \\ \sigma \rho \beta_d & 0 \\ 0 & v \odot v \end{pmatrix},
\]

where \( m = \rho u \). Then, the system (1.4) can be recast in the form of hyperbolic conservation law:

\[
\partial_t U + \nabla \cdot A(U) = F(U),
\]

where \( F = F(U) \) is given by

\[
F(U) := \begin{pmatrix} 0 \\ -\rho(u - v) - \rho \nabla K \ast (\rho - 1) \\ -\nabla p + \Delta v + \rho(u - v) \end{pmatrix}.
\]

Then, the associated total energy can be written as

\[
E(U) := \frac{|m|^2}{2\rho} + \frac{|v|^2}{2} + \sigma \rho \log \rho.
\]

With these newly defined macroscopic variables, we define a modulated energy functional as follows:

\[
\mathcal{E}(\bar{U}|U) = E(\bar{U}) - E(U) - DE(U)(\bar{U} - U), \quad \bar{U} := \begin{pmatrix} \bar{\rho} \\ \bar{m} \\ \bar{v} \end{pmatrix},
\]

which can be rewritten as

\[
\mathcal{E}(\bar{U}|U) = \frac{\bar{\rho}}{2} |\bar{u} - u|^2 + \frac{1}{2} |\bar{v} - v|^2 + \sigma \mathcal{H}(\bar{\rho}|\rho),
\]

where the relative entropy \( \mathcal{H}(x|y) \) is given by

\[
\mathcal{H}(x|y) := x \log x - y \log y - (1 + \log y)(x - y) \geq \frac{1}{2} \min\left\{ \frac{1}{x}, \frac{1}{y} \right\} |x - y|^2.
\]

Note that if \( \sigma = 0 \), then the modulate energy \( \mathcal{E} \) does not include the relative entropy \( \mathcal{H} \), and thus it is not convex with respect to the fluid density \( \rho \). As mentioned in Introduction, this makes a huge problem with the estimate of a term that appears due to the drag force. Regarding this issue, we do not employ the relative entropy \( \mathcal{H} \) to handle that term but use the Coulomb interactions to overcome this difficulty. See Remark 3.1 for more details.
Now, we provide the estimate for the modulated energy functional.

**Lemma 3.3.** The modulated energy functional $E$ satisfies the following differential equality:

\[
\frac{d}{dt} \int_{\mathbb{T}^d} E(\bar{U}|U) \, dx + \int_{\mathbb{T}^d} |\nabla (\bar{v} - v)|^2 \, dx + \int_{\mathbb{T}^d} \bar{p}(\bar{u} - \bar{v}) - (u - v)^2 \, dx \\
= \int_{\mathbb{T}^d} \partial_t E(U) \, dx + \int_{\mathbb{T}^d} \bar{p}|\bar{u} - u|^2 \, dx + \int_{\mathbb{T}^d} |\nabla \bar{v}|^2 \, dx + \int_{\mathbb{T}^d} \bar{p}u \cdot \nabla K * (\bar{\rho} - 1) \, dx \\
- \int_{\mathbb{T}^d} \nabla D E(U) : A(\bar{U}|U) \, dx + \int_{\mathbb{T}^d} D E(U) (\bar{U} + \nabla \cdot A(\bar{U}) - F(\bar{U})) \, dx \\
+ \int_{\mathbb{T}^d} (\bar{\rho} - \rho)(\bar{u} - v)(u - v) \, dx - \int_{\mathbb{T}^d} \bar{p}(\bar{u} - u) \cdot \nabla K * (\bar{\rho} - \rho) \, dx,
\]

where $A(\bar{U}|U) := A(\bar{U}) - A(U) - DA(\bar{U})(\bar{U} - U)$ is the modulated flux functional.

**Proof.** A straightforward computation yields

\[
\frac{d}{dt} \int_{\mathbb{T}^d} E(\bar{U}|U) \, dx = \int_{\mathbb{T}^d} \partial_t E(U) \, dx - \int_{\mathbb{T}^d} \nabla D E(U) : A(\bar{U}|U) - F(\bar{U}) \, dx \\
+ \int_{\mathbb{T}^d} D^2 E(U) \nabla \cdot A(U)(\bar{U} - U) + \int_{\mathbb{T}^d} D E(U) \nabla \cdot A(\bar{U}) \, dx \\
- \int_{\mathbb{T}^d} D^2 E(U) F(U)(\bar{U} - U) + \int_{\mathbb{T}^d} D E(U) F(\bar{U}) \, dx
\]

For $J_3$, we use the same estimate in [11] Appendix A to get

\[
J_3 = - \int_{\mathbb{T}^d} \nabla D E(U) : A(\bar{U}|U) \, dx.
\]

For $J_4$, we note that

\[
D E(U) = \begin{pmatrix} -m^2/(2 \rho^2) + \sigma(\log \rho + 1) \\
\frac{m}{\rho} \\
v \end{pmatrix} \quad \text{and} \quad D^2 E(U) = \begin{pmatrix} * & -m/\rho^2 \\ * & 1/\rho \end{pmatrix}.
\]

Then we find

\[
D^2 E(U) F(U)(\bar{U} - U) = (\bar{\rho} - \rho)u \cdot \nabla K * (\bar{\rho} - 1) - u \cdot (v - u)(\bar{\rho} - \rho) \\
- (\bar{m} - m) \cdot \nabla K * (\bar{\rho} - 1) + (\bar{m} - m) \cdot (v - u) \\
- \nabla p \cdot (\bar{v} - v) + \Delta v \cdot (\bar{v} - v) + \rho(u - v) \cdot (\bar{v} - v)
\]

and

\[
D E(U) F(\bar{U}) = -\bar{\rho}u \cdot \nabla K * (\bar{\rho} - 1) + \bar{\rho}u \cdot (\bar{v} - \bar{u}) - \nabla \bar{p} \cdot \bar{v} + \Delta \bar{v} \cdot \bar{v} + \bar{\rho}v \cdot (\bar{\rho} - \bar{v}).
\]

Thus we obtain

\[
\int_{\mathbb{T}^d} D^2 E(U) F(U)(\bar{U} - U) + \int_{\mathbb{T}^d} D E(U) F(\bar{U}) \, dx
\]

\[
= \int_{\mathbb{T}^d} \bar{\rho}(u - \bar{u}) \cdot \nabla K * (\bar{\rho} - \rho) - \bar{\rho}u \cdot \nabla K * (\bar{\rho} - 1) \, dx + \int_{\mathbb{T}^d} \bar{\rho}(u^2 - uv + 2uv - 2\bar{u}u + \bar{u}\bar{v} - \bar{v}v) \, dx
\]

\[
+ \int_{\mathbb{T}^d} \rho(u\bar{v} - \bar{v}u - uv + u^2) \, dx + \int_{\mathbb{T}^d} \Delta v \cdot (\bar{v} - v) + \Delta \bar{v} \cdot \bar{v} \, dx.
\]

This together with [11] Appendix A yields

\[
J_4 = - \int_{\mathbb{T}^d} \bar{\rho}(u - \bar{u}) \cdot \nabla K * (\bar{\rho} - \rho) \, dx + \int_{\mathbb{T}^d} \bar{\rho}u \cdot \nabla K * (\bar{\rho} - 1) \, dx
\]

\[
+ \int_{\mathbb{T}^d} \bar{\rho}|\bar{u} - u|^2 \, dx + \int_{\mathbb{T}^d} |\nabla \bar{v}|^2 \, dx - \int_{\mathbb{T}^d} \bar{\rho}((\bar{u} - \bar{v}) - (u - v))^2 \, dx
\]
\[ -\int_{\mathbb{T}^d} |\nabla (\bar{v} - v)|^2 \, dx + \int_{\mathbb{T}^d} (\bar{\rho} - \rho)(\bar{v} - v)(u - v) \, dx. \]

This completes the proof.

In the proposition below, we present the estimate of the modulated energy associated to the systems (1.3) and (1.4).

**Proposition 3.1.** For \( T > 0 \) and \( d \geq 2 \), let \((f^\varepsilon, u^\varepsilon)\) be weak solutions to the system (1.3) on the interval \([0, T]\) corresponding to initial data \((f_0^\varepsilon, v_0^\varepsilon)\). Let \((\rho, u, v)\) be the unique strong solution to the system (1.4) on the interval \([0, T]\) corresponding to initial data \((\rho_0, u_0, v_0)\). Assume that the initial data \((f^\varepsilon, v^\varepsilon)\) satisfy the assumptions (H1)-(H2) on Theorem 1.2. Then we have

\[
\begin{aligned}
&\int_{\mathbb{T}^d} \rho^\varepsilon |u^\varepsilon - u|^2 \, dx + \int_{\mathbb{T}^d} |v^\varepsilon - v|^2 \, dx + \sigma \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{\rho^\varepsilon - z}{z} \, dxds + \int_{\mathbb{T}^d} |\nabla K * (\rho^\varepsilon - \rho)|^2 \, dx \\
&\quad + \int_0^T \int_{\mathbb{T}^d} |\nabla (v^\varepsilon - v)|^2 \, dxds + \int_0^T \int_{\mathbb{T}^d} \rho^\varepsilon (u^\varepsilon - v^\varepsilon) - (u - v)^2 \, dxds \\
&\leq C \sqrt{\varepsilon}
\end{aligned}
\]

for almost every \( t \in [0, T] \), where \( C \) is independent of \( \varepsilon \).

**Proof.** First, we set

\[
U := \begin{pmatrix} \rho u \\ \rho^\varepsilon v^\varepsilon \end{pmatrix} \quad \text{and} \quad U^\varepsilon := \begin{pmatrix} \rho^\varepsilon u^\varepsilon \\ v^\varepsilon \end{pmatrix},
\]

where

\[
\rho^\varepsilon := \int_{\mathbb{R}^d} f^\varepsilon \, d\xi \quad \text{and} \quad \rho^\varepsilon u^\varepsilon := \int_{\mathbb{R}^d} \xi f^\varepsilon \, d\xi.
\]

We replace \( \bar{U} \) with \( U^\varepsilon \) in Lemma 3.3 to get

\[
\begin{aligned}
&\int_{\mathbb{T}^d} \mathcal{E}(U^\varepsilon | U) \, dx + \int_0^T \int_{\mathbb{T}^d} |\nabla (v^\varepsilon - v)|^2 \, dxds + \int_0^T \int_{\mathbb{T}^d} \rho^\varepsilon |(u^\varepsilon - v^\varepsilon) - (u - v)|^2 \, dxds \\
&= \int_{\mathbb{T}^d} \mathcal{E}(U_0^\varepsilon | U_0) \, dx + \int_0^T \int_{\mathbb{T}^d} (\partial_t \mathcal{E}(U^\varepsilon) + \rho^\varepsilon |u^\varepsilon - u|^2 + |\nabla v^\varepsilon|^2 + \rho^\varepsilon u^\varepsilon \cdot \nabla K * (\rho^\varepsilon - 1)) \, dxds \\
&\quad - \int_0^T \int_{\mathbb{T}^d} \nabla D\mathcal{E}(U) : A(U^\varepsilon | U) \, dxds - \int_0^T \int_{\mathbb{T}^d} D\mathcal{E}(U^\varepsilon) \mathcal{A}(U^\varepsilon - F(U^\varepsilon)) \, dxds \\
&\quad + \int_0^T \int_{\mathbb{T}^d} (\rho^\varepsilon - \rho)(v^\varepsilon - v)(u - v) \, dxds - \int_0^T \int_{\mathbb{T}^d} \rho^\varepsilon (u^\varepsilon - u) \cdot \nabla K * (\rho^\varepsilon - \rho) \, dxds \\
&=: \sum_{i=1}^6 K_i.
\end{aligned}
\]

From now on, we separately estimate \( K_i, i = 2, \ldots, 6 \) as follows:

\(\diamond\) (Estimates for \( K_2 \)): Note that

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^d} |\nabla K * (\rho^\varepsilon - 1)|^2 \, dx = \int_{\mathbb{T}^d} \rho^\varepsilon u^\varepsilon \cdot \nabla K * (\rho^\varepsilon - 1) \, dx.
\]

This observation entails

\[
\begin{aligned}
K_2 &= \mathcal{F}(f^\varepsilon, v^\varepsilon) + \int_0^T \int_{\mathbb{T}^d} \rho^\varepsilon |u^\varepsilon - u|^2 \, dxds + \int_0^T \int_{\mathbb{T}^d} |\nabla v^\varepsilon|^2 \, dxds \\
&\quad - \int_{\mathbb{T}^d} \left( E(U_0^\varepsilon) + \frac{1}{2} |\nabla K * (\rho_0^\varepsilon - 1)|^2 \right) \, dx + \int_{\mathbb{T}^d} \rho^\varepsilon \frac{|u^\varepsilon|^2}{2} \, dxds - \int_{\mathbb{T}^d} \frac{|\xi|^2}{2} f^\varepsilon \, dxd\xi \\
&\leq C \varepsilon + \mathcal{F}(f_0^\varepsilon, v_0^\varepsilon) - \int_{\mathbb{T}^d} \left( E(U_0^\varepsilon) + \frac{1}{2} |\nabla K * (\rho_0^\varepsilon - 1)|^2 \right) \, dx \\
&\leq \mathcal{O}(\sqrt{\varepsilon}),
\end{aligned}
\]
where we used the fact $\rho^e|u^e|^2 \leq \int_{\mathbb{R}^d} |\xi|^2 f^e d\xi$, (H1), and Lemma 3.2

$\diamond$ (Estimates for $K_3$): By the definition of $A(U^\varepsilon|U)$, we get

$$A(U^\varepsilon|U) = \begin{pmatrix} 0 & \rho^e (u^\varepsilon - u) \otimes (u^\varepsilon - u) & 0 & 0 \\ \rho^e (u^\varepsilon - u) \otimes (u^\varepsilon - u) & 0 & 0 & 0 \\ 0 & 0 & (v^\varepsilon - v) \otimes (v^\varepsilon - v) \end{pmatrix}.$$  

This together with

$$DE(U) = \begin{pmatrix} * \\ u \\ v \end{pmatrix}$$

yields

$$K_3 \leq \|\nabla (u, v)\|_{L^\infty} \int_0^t \int_{\mathbb{T}^d} \rho^e |u^e - u|^2 + |v^\varepsilon - v|^2 \, dx \leq C \int_0^t \int_{\mathbb{T}^d} E(U^\varepsilon|U) \, dx \, ds,$$

where $C > 0$ is a positive constant independent of $\varepsilon$.

$\diamond$ (Estimates for $K_4$): We integrate the kinetic equation in (13) with respect to $\xi$ to find

$$\partial_t \rho^e + \nabla \cdot (\rho^e u^e) = 0,$$

$$\partial_t (\rho^e u^e) + \nabla \cdot (\rho^e (u^e \otimes u^e) + \sigma \nabla \rho^e + \rho^e ((u^e - v^\varepsilon) + \nabla K * (\rho^e - 1))$$

$$= \nabla \cdot \left( \int_{\mathbb{R}^d} (u^e \otimes u^e - \xi \otimes \xi + \sigma \mathbb{I}_d) f^e \, d\xi \right),$$

$$\partial_t v^e + (v^e \cdot \nabla) v^e + \nabla p^e - \Delta v^e = \rho^e (u^e - v^\varepsilon),$$

$$\nabla \cdot v^e = 0$$

in the sense of distributions. Thus we use Cauchy–Schwarz inequality to get

$$K_4 = -\int_0^t \int_{\mathbb{T}^d} DE(U)(U^e + \nabla \cdot A(U^\varepsilon) - F(U^\varepsilon)) \, dx \, ds$$

$$= \int_0^t \int_{\mathbb{T}^d} \nabla u : \left( \int_{\mathbb{R}^d} (u^e \otimes u^e - \xi \otimes \xi + \sigma \mathbb{I}_d) f^e \, d\xi \right) \, dx \, ds.$$

Then, by [34] Lemma 4.4, we estimate

$$\int_{\mathbb{R}^d} (u^e \otimes u^e - \xi \otimes \xi + \sigma \mathbb{I}_d) f^e \, d\xi$$

$$= \int_{\mathbb{R}^d} (u^e \otimes (u^e - \xi) + (u^e - \xi) \otimes \xi + \sigma \mathbb{I}_d) f^e \, d\xi$$

$$= \int_{\mathbb{R}^d} \left[ (u^e \sqrt{f^e} \otimes ((u^e - \xi) \sqrt{f^e} - 2\sigma \nabla \xi \sqrt{f^e}) + (u^e \otimes \nabla f^e) + ((u^e - \xi) \sqrt{f^e}) \otimes \xi \sqrt{f^e} + \sigma \nabla f^e \otimes \xi + \sigma f^e \mathbb{I}_d \right] \, d\xi$$

$$= \int_{\mathbb{R}^d} \left[ (u^e \sqrt{f^e} \otimes ((u^e - \xi) \sqrt{f^e} - 2\sigma \nabla \xi \sqrt{f^e}) + (u^e - \xi) \sqrt{f^e} - 2\sigma \nabla \xi \sqrt{f^e}) \otimes \xi \sqrt{f^e} + \sigma \nabla f^e \otimes \xi + \sigma f^e \mathbb{I}_d \right] \, d\xi$$

$$\leq 2 \left( \int_{\mathbb{R}^d} |\xi|^2 f^e \, d\xi \right)^{1/2} \left( \int_{\mathbb{R}^d} \frac{1}{f^e} \sigma \nabla f^e \otimes (u^e - \xi) \sqrt{f^e} \, d\xi \right),$$

and this yields

$$K_4 \leq C \left( \int_0^t \int_{\mathbb{T}^d \times \mathbb{R}^d} |\xi|^2 f^e \, dx \, d\xi \, ds \right)^{1/2} \left( \int_0^t D_1 (f^e) \, ds \right)^{1/2} \leq C \sqrt{\varepsilon}.$$  

Here $C = C(T, \|\nabla u\|_{L^\infty}) > 0$ is a constant independent of $\varepsilon$ and we used Lemma 3.4.
Now, we combine all the previous estimates to yield

\[ \int_{T^d} \mathcal{E}(U^\varepsilon | U) \, dx + \frac{1}{2} \int_{T^d} |\nabla K \star (\rho^\varepsilon - \rho)|^2 \, dx \]

\[ + \int_0^t \int_{T^d} |\nabla (v^\varepsilon - v)|^2 \, dx ds + \int_0^t \int_{T^d} \rho^\varepsilon (u^\varepsilon - v^\varepsilon) - (u - v)^2 \, dx ds \]
\[ \leq C \sqrt{\varepsilon} + \int_{T_d} \mathcal{E}(U^\varepsilon_0|U_0) \, dx + \frac{1}{2} \int_{T_d} |\nabla K \ast (\rho^\varepsilon_0 - \rho_0)|^2 \, dx + C \int_0^t \int_{T_d} \mathcal{E}(U^\varepsilon|U) \, dx ds \]
\[ + C \int_0^t \int_{T_d} |\nabla K \ast (\rho^\varepsilon - \rho)|^2 \, dx ds, \]
and thus, we use (H2) and Grönwall’s lemma to get
\[ \int_{T_d} \mathcal{E}(U^\varepsilon|U) \, dx + \int_{T_d} |\nabla K \ast (\rho^\varepsilon - \rho)|^2 \, dx \]
\[ + \int_0^t \int_{T_d} |\nabla (v^\varepsilon - v)|^2 \, dx ds + \int_0^t \int_{T_d} \rho^\varepsilon |(u^\varepsilon - v^\varepsilon) - (u - v)|^2 \, dx ds \]
\[ \leq C \sqrt{\varepsilon}, \]
where \(C > 0\) is independent of \(\varepsilon\). □

**Remark 3.1.** In the proof of Proposition 3.1, the most delicate term is \(K_5\) which appears due to the coupling, drag force, between particles and fluid. In the case with diffusion, i.e., the modulated energy is convex with respect to \(\rho\), by using the same argument as in the proof of [11, Proposition 5.2], we can also estimate \(K_5\) as
\[ K_5 \leq C \int_0^t \int_{T_d} \mathcal{E}(U^\varepsilon|U) \, dx ds + \frac{1}{2} \int_0^t \int_{T_d} \rho^\varepsilon |(u^\varepsilon - v^\varepsilon) - (u - v)|^2 \, dx ds, \]
where \(C > 0\) is independent of \(\varepsilon\). Here the relative entropy is used to control the difference between \(\rho^\varepsilon\) and \(\rho\). This estimate also provides the same result as in Proposition 3.1. However, this strategy only works in the presence of diffusion. As mentioned in Introduction, we find that the Coulomb interaction can be used to control \(\rho^\varepsilon - \rho\), and this works regardless of the presence of diffusion.

**Remark 3.2.** Proposition 3.1 does not require the boundedness and periodicity of the domain. The estimates in Proposition 3.1 hold in the whole space as long as there exist the weak solutions \((f^\varepsilon, v^\varepsilon)\) to the system (1.4) and the strong solutions \((\rho, u, v)\) to the system (1.3) in the whole space with desired regularities, at least locally in time.

### 3.2. Proof of Theorem 1.2

In this subsection, we provide the details of the proof of Theorem 1.2 showing the EPNS system (1.4) can be well approximated by the VPNS system (1.3) for \(\varepsilon > 0\) small enough.

#### 3.2.1. Isothermal pressure case

Let us first show the convergences in the isothermal pressure case. For this, we only show the following convergence:
\[ f^\varepsilon \rightarrow M_{\rho,u} := \frac{\rho}{(2\pi \sigma^2)^{d/2}} e^{-\frac{|x - \xi|^2}{2\sigma^2}} \quad \text{a.e. and } L^\infty(0,T; L^1(\mathbb{R}^d \times \mathbb{R}^d)), \]

since the other convergences can be directly obtained by using the modulated energy estimated in Proposition 5.1, see [10, Corollary 2.1] or [21, Corollary 1.1]. For simplicity, we set \(\sigma = 1\) and consider
\[ \mathcal{H}(f^\varepsilon | M_{\rho,u}) = f^\varepsilon \log f^\varepsilon - M_{\rho,u} \log M_{\rho,u} + (f^\varepsilon - M_{\rho,u})(1 + \log M_{\rho,u}) \]
\[ = f^\varepsilon (\log f^\varepsilon - \log M_{\rho,u}) + (f^\varepsilon - M_{\rho,u}). \]

Since
\[ \log M_{\rho,u} = \log \rho - \frac{|u - \xi|^2}{2} - \frac{d}{2} \log(2\pi), \]
we find
\[ \int_{\mathbb{T}^d \times \mathbb{R}^d} \mathcal{H}(f^\varepsilon | M_{\rho,u}) \, dx d\xi = \int_{\mathbb{T}^d \times \mathbb{R}^d} f^\varepsilon \log f^\varepsilon \, dx d\xi - \int_{\mathbb{T}^d} \rho^\varepsilon \log \rho \, dx + \frac{1}{2} \int_{\mathbb{T}^d \times \mathbb{R}^d} |u - \xi|^2 f^\varepsilon \, dx d\xi + \frac{d}{2} \log(2\pi). \]

Note that
\[ \|f^\varepsilon - M_{\rho,u}\|_{L^1}^2 \leq 4 \int_{\mathbb{T}^d \times \mathbb{R}^d} \mathcal{H}(f^\varepsilon | M_{\rho,u}) \, dx d\xi. \]

In the proposition below, we estimate the above integral. Although its proof is similar to [10] or [21], for the completeness of our work we provide the details in Appendix 3.
Proposition 3.2. Let \((f^\varepsilon, v^\varepsilon)\) be a global weak solution to the system (1.3) and \((\rho, u, v)\) be a strong solution to the system (1.4) on the time interval \([0, T]\). Then we have

\[
\int_{T^d \times \mathbb{R}^d} \mathcal{H}(f^\varepsilon | M_{\rho,u}) \, dx \, d\xi + \frac{1}{2} \int_{T^d} \left| \nabla K \ast (\rho^\varepsilon - \rho) \right|^2 \, dx + \frac{1}{2 \varepsilon} \int_0^t \int_{T^d \times \mathbb{R}^d} \frac{1}{\varepsilon} |\nabla \xi f^\varepsilon - (u^\varepsilon - \xi) f^\varepsilon|^2 \, dx \, d\xi \, ds
\]

\[
\leq \int_{T^d \times \mathbb{R}^d} \mathcal{H}(f^\varepsilon_0 | M_{\rho_0,u_0}) \, dx \, d\xi + \frac{1}{2} \int_{T^d} \left| \nabla K \ast (\rho^\varepsilon_0 - \rho_0) \right|^2 \, dx
\]

\[
+ C \int_0^t \left( \min \left\{ 1, \int_{T^d} |\rho^\varepsilon| |u - u^\varepsilon|^2 \, dx \right\} \right)^{1/2} \, ds + \frac{1}{\varepsilon^{1/2}} \int_0^t \int_{T^d} |u^\varepsilon - u|^2 \rho^\varepsilon \, dx \, ds
\]

\[
+ C \varepsilon \int_0^t \int_{T^d \times \mathbb{R}^d} |\nabla \xi f^\varepsilon|^2 \, dx \, d\xi \, ds + C \varepsilon^{1/4} \int_0^t \int_{T^d} |u^\varepsilon - u|^2 \rho^\varepsilon \, dx \, ds,
\]

where \(C > 0\) is independent of \(\varepsilon > 0\).

As a direct consequence of Proposition 3.2, we obtain from the modulated energy estimates in Theorem 1.2 and the entropy estimate in Lemma 3.1 that

\[
\int_{T^d \times \mathbb{R}^d} \mathcal{H}(f^\varepsilon | M_{\rho,u}) \, dx \, d\xi \leq \int_{T^d \times \mathbb{R}^d} \mathcal{H}(f^\varepsilon_0 | M_{\rho_0,u_0}) \, dx \, d\xi + C \varepsilon^{1/4}.
\]

3.2.2. Pressureless case. For the pressureless case, the convergence of \(\rho^\varepsilon\) towards \(\rho\) is not clear from Proposition 3.1. In order to get the desired convergence of \(\rho^\varepsilon\) towards \(\rho\), we can use the previous result [10, Lemma 4.1], see also [9, Lemma 2.2], [18, Proposition 3.1], and [29, Lemma 5.2], which asserts that the bounded Lipschitz distance \(d_{BL}\) between local densities can be bounded from above by the modulated kinetic energy. Let \(\mu, \nu \in \mathcal{M}(T^d)\) be two Radon measures, then the bounded Lipschitz distance, which is denoted by \(d_{BL} : \mathcal{M}(T^d) \times \mathcal{M}(T^d) \to \mathbb{R}_+\), between \(\mu\) and \(\nu\) is defined by

\[
d_{BL}(\mu, \nu) := \sup_{\phi \in \Omega} \left| \int_{T^d} \phi(x)(\mu(dx) - \nu(dx)) \right|,
\]

where the admissible set \(\Omega\) of test functions are given by

\[
\Omega := \left\{ \phi : T^d \to \mathbb{R} : \|\phi\|_{L^\infty} \leq 1, \text{Lip}(\phi) := \sup_{x \neq y} \frac{|\phi(x) - \phi(y)|}{|x - y|} \leq 1 \right\}.
\]

Then by [10, Lemma 4.1] we have the following estimate which gives the convergence \(\rho^\varepsilon \rightharpoonup \rho\) weakly in \(L^\infty(0,T; \mathcal{M}(T^d))\).

Lemma 3.4. Let \((f^\varepsilon, v^\varepsilon)\) be a global weak solution to the system (1.3) and \((\rho, u, v)\) be a strong solution to the system (1.4) on the time interval \([0, T]\). Then we have

\[
d_{BL}(\rho(t), \rho^\varepsilon(t)) \leq C d_{BL}(\rho_0, \rho^\varepsilon_0) + C \left( \int_0^t \int_{T^d} \rho^\varepsilon |u^\varepsilon - u|^2 \, dx \, ds \right)^{1/2}
\]

for \(0 \leq t \leq T\), where \(C > 0\) is independent of \(\varepsilon > 0\).

This together with Proposition 3.1 asserts

\[
d_{BL}(\rho(t), \rho^\varepsilon(t)) \leq C d_{BL}(\rho_0, \rho^\varepsilon_0) + C \varepsilon^{1/4}.
\]

For the other convergence estimates, we refer to [10, Corollary 2.3]. This completes the proof of Theorem 1.2.

Remark 3.3. Lemma 3.4 holds in the whole space. This together with Remark 3.2 implies that the estimates of hydrodynamic limit in Theorem 1.2 also hold when we add the assumption that \(d_{BL}(\rho_0, \rho^\varepsilon_0) = \mathcal{O}(\varepsilon^{1/4})\) to Theorem 1.3. On the other hand, in the spatial periodic domain \(T^d\), we can bound the bounded Lipschitz between \(\rho^\varepsilon\) and \(\rho\) by \(H^{-1}(T^d)\)-norm between them. Indeed, we find

\[
\int_{T^d} (\rho^\varepsilon - \rho) \phi \, dx \leq \|\rho^\varepsilon - \rho\|_{H^{-1}} \|\phi\|_{H^1} \leq \|\rho^\varepsilon - \rho\|_{H^{-1}} \|\phi\|_{W^{1,\infty}} \leq \|\rho^\varepsilon - \rho\|_{H^{-1}},
\]
for \( \phi \in W^{1,\infty}(\mathbb{T}^d) \) with \( \|\phi\|_{W^{1,\infty}} \leq 1 \). This implies that we do not need to add additional assumption on the initial densities \( \rho_0^\ast \) and \( \rho_0 \) to get the convergence estimates in Theorem 1.2.

4. Global-in-time strong solvability for the isothermal/pressureless EPNS system

In this section, we study the global-in-time existence and uniqueness of strong solutions to the system 1.4.

4.1. Local solvability. We discuss the local-in-time strong solvability for the isothermal and pressureless cases separately.

4.1.1. Isothermal EPNS system. Consider the following isothermal Euler–Poisson system coupled with Navier–Stokes system:

\[
\begin{align*}
\partial_t \rho + \nabla \cdot (\rho u) &= 0, \quad (x, t) \in \mathbb{T}^d \times \mathbb{R}_+, \\
\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla \rho &= -\rho(u - v + \nabla K \ast (\rho - 1)), \\
\partial_t v + (v \cdot \nabla) v + \nabla p - \Delta v &= \rho(u - v), \\
\nabla \cdot v &= 0.
\end{align*}
\]

Here we set \( \sigma = 1 \) without loss of generality. Then, we reformulate the system (4.1) by letting \( g := \log \rho \) as follows:

\[
\begin{align*}
\partial_t g + \nabla g \cdot u + \nabla u &= 0, \quad (x, t) \in \mathbb{T}^d \times \mathbb{R}_+, \\
\partial_t u + (u \cdot \nabla) u + \nabla g &= -(u - v + \nabla K \ast (e^g - 1)), \\
\partial_t v + (v \cdot \nabla) v + \nabla p - \Delta v &= e^g(u - v), \\
\nabla \cdot v &= 0
\end{align*}
\]

subject to initial data:

\[
(g(x, 0), u(x, 0), v(x, 0)) =: (g_0(x), u_0(x), v_0(x)), \quad x \in \mathbb{T}^d.
\]

Now, we state the result on the local well-posedness of the system 1.4.

**Theorem 4.1.** Let \( d \geq 2 \) and \( s > d/2 + 1 \). Suppose that the initial data satisfies

\[
(g_0, u_0, v_0) \in H^s(\mathbb{T}^d) \times H^s(\mathbb{T}^d) \times H^s(\mathbb{T}^d) \quad \text{and} \quad e^{g_0} > 0.
\]

Then for any positive constants \( \epsilon_0 < M_0 \), there exists a positive constant \( T^* \) such that if

\[
\|g_0\|_{H^s}^2 + \|u_0\|_{H^s}^2 + \|v_0\|_{H^s}^2 < \epsilon_0,
\]

then the system 1.4 admits a unique solution

\[
(g, u, v) \in C([0, T^*]; H^s(\mathbb{T}^d)) \times C([0, T^*]; H^s(\mathbb{T}^d)) \times C([0, T^*]; H^s(\mathbb{T}^d))
\]

satisfying

\[
\sup_{0 \leq t \leq T^*} \left( \|g(\cdot, t)\|_{H^s}^2 + \|u(\cdot, t)\|_{H^s}^2 + \|v(\cdot, t)\|_{H^s}^2 \right) \leq M_0.
\]

**Proof.** Since the proof is rather lengthy and technical, we leave it in Appendix C. \( \square \)

4.1.2. Pressureless EPNS system. For the pressureless case \( \sigma = 0 \), we set \( h := \rho - 1 \) and reformulate the system 1.4 as

\[
\begin{align*}
\partial_t h + \nabla \cdot ((1 + h)u) &= 0, \quad (x, t) \in \mathbb{T}^d \times \mathbb{R}_+, \\
\partial_t u + (u \cdot \nabla) u &= -(u - v + \nabla K \ast h), \\
\partial_t v + (v \cdot \nabla) v + \nabla p - \Delta v &= (1 + h)(u - v), \\
\nabla \cdot v &= 0.
\end{align*}
\]
For the local-in-time strong solvability of the above system, we use a similar strategy to that for Theorem 4.1; see Appendix C. We construct a sequence of approximate solutions to the reformulated system:

\[
\partial_t h^{n+1} + \nabla \cdot ((1 + h^{n+1})u^n) = 0, \quad (x, t) \in \mathbb{T}^d \times \mathbb{R}_+,
\]

\[
\partial_t u^{n+1} + (u^n \cdot \nabla) u^{n+1} = -(u^{n+1} - u^n + \nabla K \ast h^n),
\]

\[
\partial_t v^{n+1} + (v^n \cdot \nabla) v^{n+1} + \nabla p - \Delta v^{n+1} = (1 + h^n)(u^n - v^{n+1}),
\]

\[
\nabla \cdot v^{n+1} = 0.
\]

In this case, we use the following estimate for the Coulomb interaction term:

\[
\int_{\mathbb{T}^d} \nabla (\nabla K \ast h^n) \cdot \nabla u^{n+1} \, dx = \sum_{i,j=1}^d \int_{\mathbb{T}^d} \partial_{x_i}(\partial_{x_j}(K \ast h^n)) \partial_{x_i} u^{n+1} \, dx
\]

\[
= \sum_{i,j=1}^d \int_{\mathbb{T}^d} \partial_{x_i}(\partial_{x_j}(K \ast h^n)) \partial_{x_i} u^{n+1} \, dx
\]

\[
= \int_{\mathbb{T}^d} \Delta K \ast h^n \nabla \cdot u^{n+1} \, dx
\]

\[
= -\int_{\mathbb{T}^d} h^n \nabla \cdot u^{n+1} \, dx.
\]

Thus, the above observation enables us to have $H^{s+1}$-estimates for $u$ and thus for $v$, i.e., for any $M > N$, if

\[
\|h_0\|_{H^s}^2 + \|u_0\|_{H^{s+1}}^2 + \|v_0\|_{H^{s+1}}^2 < N,
\]

then there exists $T^* > 0$ such that

\[
\sup_{0 \leq t \leq T^*} \left( \|h^n(\cdot, t)\|_{H^s}^2 + \|u^n(\cdot, t)\|_{H^{s+1}}^2 + \|v^n(\cdot, t)\|_{H^{s+1}}^2 \right) < M \quad \forall n \in \mathbb{N}.
\]

Thus, we can employ the similar argument to Theorem 4.1 to obtain the local-in-time well-posedness of strong solutions to the system (4.1) with $\sigma = 0$.

**Theorem 4.2.** Let $d \geq 2$ and $s > d/2 + 1$. Suppose that the initial data satisfies

\[
(h_0, u_0, v_0) \in H^s(\mathbb{T}^d) \times H^{s+1}(\mathbb{T}^d) \times H^{s+1}(\mathbb{T}^d), \quad \text{and} \quad 1 + h_0 > 0.
\]

Then for any positive constants $\epsilon_0 < M_0$, there exists a positive constant $T^*$ such that if

\[
\|h_0\|_{H^s}^2 + \|u_0\|_{H^{s+1}}^2 + \|v_0\|_{H^{s+1}}^2 < \epsilon_0,
\]

then the system (4.1) admits a unique solution

\[
(h, u, v) \in C([0, T^*]; H^s(\mathbb{T}^d)) \times C([0, T^*]; H^{s+1}(\mathbb{T}^d)) \times C([0, T^*]; H^{s+1}(\mathbb{T}^d))
\]

satisfying

\[
\sup_{0 \leq t \leq T^*} \left( \|h(\cdot, t)\|_{H^s}^2 + \|u(\cdot, t)\|_{H^{s+1}}^2 + \|v(\cdot, t)\|_{H^{s+1}}^2 \right) \leq M_0.
\]

**4.2 Proof of Theorem 4.3.** In this subsection, we take further steps to obtain the global-in-time existence of strong solutions to the system (4.1). First, we define

\[
X(T; k) := \sup_{0 \leq t \leq T} \left( \|g(\cdot, t)\|_{H^s}^2 + \|u(\cdot, t)\|_{H^{s+1}}^2 + \|v(\cdot, t)\|_{H^{s+1}}^2 \right) \quad \text{and} \quad X_0(k) = \|g_0\|_{H^s}^2 + \|u_0\|_{H^{s+1}}^2 + \|v_0\|_{H^{s+1}}^2.
\]

We then temporarily move back to the original system (4.1) and provide the energy estimate.

**Proposition 4.1.** Let $T > 0$, and suppose that $(\rho, u, v)$ is a strong solution to (4.1) on the time interval $[0, T]$ corresponding to the initial data $(\rho_0, u_0, v_0)$. Then, we have

\[
\frac{1}{2} \int_{\mathbb{T}^d} \rho |u|^2 \, dx + \int_{\mathbb{T}^d} |\nabla K \ast (\rho - 1)|^2 \, dx + \int_{\mathbb{T}^d} |v|^2 \, dx + 2\sigma \int_{\mathbb{T}^d} \rho \log \rho \, dx
\]

\[
+ \int_{\mathbb{T}^d} |\nabla v|^2 \, dx + \int_{\mathbb{T}^d} \rho |u - v|^2 \, dx = 0.
\]
Proof. We first easily find
\[
\frac{1}{2} \frac{d}{dt} \int_{T^d} \rho |u|^2 \, dx = -\sigma \int_{T^d} \nabla \rho \cdot u \, dx - \int_{T^d} \rho u \cdot \nabla K \ast (\rho - 1) \, dx - \int_{T^d} \rho (u - v) \cdot u \, dx.
\]
On the other hand, the first two terms on the right hand side of the above can be estimated as
\[
\frac{1}{2} \frac{d}{dt} \int_{T^d} |\nabla K \ast (\rho - 1)|^2 \, dx = \int_{T^d} \rho u \cdot \nabla K \ast (\rho - 1) \, dx
\]
and
\[
\frac{d}{dt} \int_{T^d} \rho \log \rho \, dx = \int_{T^d} u \cdot \nabla \rho \, dx.
\]
For the incompressible Navier–Stokes equations, we get
\[
\text{Proof. Lemma 4.1.}
\]
\[
\text{Lemma 4.2.}
\]
\[
\text{Proof.}
\]
\[
\text{Note that}
\]
\[
\text{we begin by estimating the higher-order derivatives of } (f, u), \text{ and for this, the following Moser-type inequalities will be significantly used.}
\]
\[
\text{Lemma 4.1.}
\]
\[
\text{Furthermore, if } \nabla f \in L^\infty(T^d), \text{ we have}
\]
\[
\text{Lemma 4.2.}
\]
\[
\text{Proof. Since the proof is rather technical, we postpone it to Appendix D for the smooth flow of reading.}
\]
It is worth noting that Lemma 4.2 does not provide the dissipation rate for $\nabla^k(\nabla g)$. In order to have it, inspired by [10], we estimate the mixing term in the lemma below.

**Lemma 4.3.** Let $d \geq 2$, $s > d/2 + 1$, and $T > 0$ be given. Suppose that $\mathcal{X}(T; s) \leq \varepsilon^2 \ll 1$. Then we have
\[
\frac{d}{dt} \int_{T_d} \nabla^k(\nabla g) \cdot \nabla^k u \, dx + \frac{1}{2} \|\nabla^k(\nabla g)\|_{L^2}^2 \\
\leq C \varepsilon_1 (\|\nabla^k(\nabla g)\|_{L^2}^2 + \|\nabla^k(\nabla u)\|_{L^2}^2) + C (\|g\|_{H^k}^2 + \|\nabla^k u\|_{L^2}^2) + \|\nabla^k(\nabla u)\|_{L^2}^2
\]
for $0 \leq k \leq s - 1$, where $C$ is a positive constant independent of $T$.

**Proof.** Direct computations and applying Lemma 3.1 yield
\[
\frac{d}{dt} \int_{T_d} \nabla^k(\nabla g) \cdot \nabla^k u \, dx \\
= -\frac{d}{dt} \int_{T_d} (\nabla^k g) \cdot \nabla^k (\nabla \cdot u) \, dx \\
= \int_{T_d} \nabla^k (\nabla g \cdot u + \nabla \cdot u) \nabla^k (\nabla u) \, dx + \int_{T_d} \nabla^k (\nabla \cdot (u \cdot \nabla u)) \, dx + \int_{T_d} \nabla^k (\nabla u \cdot \nabla u) \, dx \\
= \int_{T_d} \nabla^k (\nabla g \cdot u) \nabla^k (\nabla u) \, dx + \int_{T_d} \nabla^k (\nabla \cdot u) \nabla^k (\nabla u) \, dx \\
+ \|\nabla^k(\nabla \cdot u)\|_{L^2}^2 - \int_{T_d} (u \cdot \nabla^k(\nabla u)) \cdot \nabla^k(\nabla g) \, dx - \int_{T_d} [\nabla^k(\nabla \cdot u) - u \cdot \nabla^k(\nabla u)] \cdot \nabla^k(\nabla g) \, dx \\
- \|\nabla^k(\nabla g)\|_{L^2}^2 + \int_{T_d} \nabla^k (e^g - 1) \nabla^k g \, dx - \int_{T_d} \nabla^k u \cdot \nabla^k (\nabla g) \, dx \\
\leq 2\|u\|_{L^\infty} (\|\nabla^k(\nabla g)\|_{L^2} + \|\nabla^k(\nabla u)\|_{L^2} + C \|\nabla^k(\nabla u)\|_{L^2}) (\|\nabla^k u\|_{L^2} + \|\nabla g\|_{L^\infty} \|\nabla^k u\|_{L^2}) \\
+ \|\nabla^k(\nabla \cdot u)\|_{L^2}^2 + C \|\nabla u\|_{L^\infty} \|\nabla^k(\nabla g)\|_{L^2} \|\nabla^k u\|_{L^2} - \frac{1}{2} \|\nabla^k(\nabla g)\|_{L^2}^2 - \frac{1}{2} \|\nabla^k u\|_{L^2}^2 + C \|g\|_{H^k}^2 \\
\leq C \varepsilon_1 (\|\nabla^k(\nabla g)\|_{L^2}^2 + \|\nabla^k(\nabla u)\|_{L^2}^2) + C (\|g\|_{H^k}^2 + \|\nabla^k u\|_{L^2}^2) + \|\nabla^k(\nabla u)\|_{L^2}^2 - \frac{1}{2} \|\nabla^k(\nabla g)\|_{L^2}^2.
\]
This completes the proof. \hfill \Box

Now, we provide the higher-order estimates for the incompressible Navier–Stokes equations.

**Lemma 4.4.** Let $d \geq 2$, $s > d/2 + 1$, and $T > 0$ be given. Suppose that $\mathcal{X}(T; s) \leq \varepsilon^2 \ll 1$ so that
\[
\sup_{0 \leq t \leq T} \|g(\cdot, t)\|_{L^\infty} \leq \log 2.
\]

Then we have
\[
\frac{d}{dt} \|\nabla^k v\|_{L^2}^2 + \frac{1}{2} \|\nabla^k v\|_{L^2}^2 + \|\nabla^k(\nabla v)\|_{L^2}^2 \\
\leq C \varepsilon_1 \|\nabla^k v\|_{L^2}^2 + C (\|\nabla^k u\|_{L^2}^2 + \|\nabla^{k-1} v\|_{L^2}^2 + \|\nabla^{k-1} u\|_{L^2}^2 + \|g\|_{H^{k-1}}^2) (1 - \delta_{k,0})
\]
for $0 \leq k \leq s - 1$, where $C$ is a positive constant independent of $T$. Here $\delta_{k,0}$ denotes the Kronecker delta, i.e., the terms with $(k - 1)$-th order do not appear when $k = 0$.

**Proof.** It follows from 4.4 that
\[
\frac{1}{2} \frac{d}{dt} \|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 = -\int_{T_d} (v \cdot \nabla v) \cdot v \, dx + \int_{T_d} \nabla ^k (v - u) \cdot v \, dx \\
\leq -\frac{1}{2} \int_{T_d} e^g |v|^2 \, dx + \frac{1}{2} \int_{T_d} e^g |u|^2 \, dx \\
\leq -\frac{1}{4} \|v\|_{L^2}^2 + \|u\|_{L^2}^2.
\]
For $k \geq 1$, we estimate
\[
\frac{1}{2} \frac{d}{dt} \|\nabla^k v\|_{L^2}^2 + \|\nabla^k(\nabla v)\|_{L^2}^2
\]
First, Lemmas 4.2 and 4.3 deduce that there exists a positive constant

where

The proof is based on the inductive argument. Since it suffices to show the induction step, we suppose the following inequalities:

Then we have

where

Thus, we obtain

This concludes the desired result.

Now, we can investigate the uniform energy estimates using Lemmas 4.2–4.4.

Corollary 4.1. Let \( s > d/2 + 1, T > 0 \) be given and suppose that \( \mathcal{X}(T; s) \leq \varepsilon_1^2 \ll 1 \) so that

\[
\sup_{0 \leq t \leq T} \|g(\cdot, t)\|_{L^\infty} \leq \log 2.
\]

Then we have

\[
\mathcal{X}(T; s) \leq C_1 \mathcal{X}_0(s),
\]

where \( C_1 \) is independent of \( T \).

Proof. The proof is based on the inductive argument. Since it suffices to show the induction step, we suppose that there exists a positive constant \( C > 0 \) that is independent of \( T \) such that

\[
\mathcal{X}(T; m) \leq C \mathcal{X}_0(m) \quad \text{for} \quad 0 \leq m \leq k.
\]

First, Lemmas 4.2 and 4.3 deduce

\[
\frac{d}{dt} \left( \|\nabla^k(\nabla g)\|_{L^2}^2 + \|\nabla^k(\nabla u)\|_{L^2}^2 + \int_{T^d} \nabla^k(\nabla g) \cdot \nabla^k u \, dx \right)
\]

\[
\leq \left( -\frac{1}{2} + C \varepsilon_1 \right) \left( \|\nabla^k(\nabla g)\|_{L^2}^2 + \|\nabla^k(\nabla u)\|_{L^2}^2 \right) + C \mathcal{X}(T; k) + \frac{3}{4} \|
abla^k(\nabla g)\|_{L^2}^2 + \|
abla^k(\nabla v)\|_{L^2}^2,
\]

where \( C \) is independent of \( T \). Moreover, the estimates in Lemma 4.2 imply

\[
\frac{1}{2} \frac{d}{dt} \|\nabla^k u\|_{L^2}^2 \leq C \varepsilon_1 \|\nabla^k u\|_{L^2}^2 - \int_{T^d} \nabla^k(\nabla g) \cdot \nabla^k u \, dx
\]

\[
+ 2 \|\nabla K \ast \nabla^k(e^g - 1)\|_{L^2}^2 - \frac{3}{4} \|\nabla^k u\|_{L^2}^2 + 2 \|\nabla^k v\|_{L^2}^2
\]

\[
\leq C \varepsilon_1 \|\nabla^k u\|_{L^2}^2 - \frac{1}{12} \|\nabla^k u\|_{L^2}^2 + \frac{3}{8} \|
abla^k(\nabla g)\|_{L^2}^2 + C \mathcal{X}(T; k).
\]

Here \( C \) is independent of \( T \). Thus, we obtain

\[
\frac{d}{dt} \left( \|\nabla^k(\nabla g)\|_{L^2}^2 + \|\nabla^k(\nabla u)\|_{L^2}^2 + \int_{T^d} \nabla^k(\nabla g) \cdot \nabla^k u \, dx + \frac{5}{8} \|\nabla^k u\|_{L^2}^2 \right)
\]

\[
\leq \left( -\frac{1}{32} + C \varepsilon_1 \right) \left( \|\nabla^k(\nabla g)\|_{L^2}^2 + \|\nabla^k(\nabla u)\|_{L^2}^2 + \|\nabla^k u\|_{L^2}^2 \right) + C \mathcal{X}(T; k) + 4 \|
abla^k(\nabla v)\|_{L^2}^2.
\]
Next, we combine the previous relation with Lemma 4.4 to get
\[
\frac{d}{dt} \left( \|\nabla^k (\nabla g)\|_{L^2}^2 + \|\nabla^k (\nabla u)\|_{L^2}^2 + \int_{\mathbb{T}^d} \nabla^k (\nabla g) \cdot \nabla^k u \, dx + \frac{5}{8} \|\nabla^k u\|_{L^2}^2 + 4\|\nabla^k v\|_{L^2}^2 \right)
\leq \left( -\frac{1}{32} + C\varepsilon \right) \left( \|\nabla^k (\nabla g)\|_{L^2}^2 + \|\nabla^k (\nabla u)\|_{L^2}^2 + \|\nabla^k u\|_{L^2}^2 + \|\nabla^k v\|_{L^2}^2 \right) + C\mathcal{X}(T; k)
\]
where \( C \) is independent of \( T \). Since \( \varepsilon_1 \) is sufficiently small, we have \(-1/32 + C\varepsilon < 0\). Since Young’s inequality gives
\[
\frac{1}{4} \left( \|\nabla^k (\nabla g)\|_{L^2}^2 + \|\nabla^k (\nabla u)\|_{L^2}^2 + \|\nabla^k u\|_{L^2}^2 + \|\nabla^k v\|_{L^2}^2 \right)
\leq \|\nabla^k (\nabla g)\|_{L^2}^2 + \|\nabla^k (\nabla u)\|_{L^2}^2 + \int_{\mathbb{T}^d} \nabla^k (\nabla g) \cdot \nabla^k u \, dx + \frac{5}{8} \|\nabla^k u\|_{L^2}^2 + 4\|\nabla^k v\|_{L^2}^2
\leq 4 \left( \|\nabla^k (\nabla g)\|_{L^2}^2 + \|\nabla^k (\nabla u)\|_{L^2}^2 + \|\nabla^k u\|_{L^2}^2 + \|\nabla^k v\|_{L^2}^2 \right),
\]
the previous relation combined with Grönwall’s lemma yields
\[
\sup_{0 \leq t \leq T} \left( \|\nabla^k (\nabla g)\|_{L^2}^2 + \|\nabla^k (\nabla u)\|_{L^2}^2 \right) \leq C\mathcal{X}_0(k + 1), \tag{4.5}
\]
where \( C \) is independent of \( T \). Finally, we obtain from Lemma 4.4 that
\[
\frac{d}{dt} \|\nabla^{k+1} v\|_{L^2}^2 + \|\nabla^{k+1} (\nabla v)\|_{L^2}^2
\leq \left( -\frac{1}{2} + C\varepsilon \right) \|\nabla^{k+1} v\|_{L^2}^2 + C \left( \|\nabla^{k+1} u\|_{L^2}^2 + \|\nabla^{k+1} v\|_{L^2}^2 + \|\nabla^k u\|_{L^2}^2 + \|\nabla^k v\|_{L^2}^2 \right)
\leq \left( -\frac{1}{2} + C\varepsilon \right) \|\nabla^{k+1} v\|_{L^2}^2 + C\mathcal{X}_0(k + 1).
\]
Here we used (4.5) and \( C \) is independent of \( T \). Thus, we again use Grönwall’s lemma to get
\[
\sup_{0 \leq t \leq T} \|\nabla^{k+1} v\|_{L^2}^2 \leq C\mathcal{X}_0(k + 1). \tag{4.6}
\]
Hence, we gather the estimates (4.5) and (4.6) to complete the proof.

Based on the \textit{a priori} estimates in Corollary 4.4, we can extend our local-in-time existence result in Theorem 4.2 to the global-in-time estimate.

\textbf{Proof of Theorem 4.3} First, we choose \( \varepsilon_1 < 1 \) sufficiently small so that the conditions in Corollary 4.4 hold. Set
\[
\bar{\varepsilon}_0^2 := \frac{\varepsilon_1^2}{2(1 + C_1)},
\]
where \( C_1 > 0 \) is appeared in Corollary 4.4 and assume that the initial data \((g_0, u_0, v_0)\) satisfy
\[
\|g_0\|_{\dot{H}^s}^2 + \|u_0\|_{\dot{H}^s}^2 + \|v_0\|_{\dot{H}^s}^2 < \bar{\varepsilon}_0^2.
\]
Now, we define the lifespan of the solutions \((g, u, v)\) to the system (4.1)–(4.3) as
\[
\bar{T} := \sup\{ t \geq 0 \mid \sup_{0 \leq s \leq t} \left( \|g(\cdot, s)\|_{\dot{H}^s}^2 + \|u(\cdot, s)\|_{\dot{H}^s}^2 + \|v(\cdot, s)\|_{\dot{H}^s}^2 \right) < \varepsilon_1^2 \}. \tag{4.7}
\]
Since \( \bar{\varepsilon}_0 < \varepsilon_1 \), Theorem 4.2 implies \( \bar{T} > 0 \). Suppose \( \bar{T} < +\infty \). Then, by definition we have
\[
\sup_{0 \leq s \leq \bar{T}} \left( \|g(\cdot, s)\|_{\dot{H}^s}^2 + \|u(\cdot, s)\|_{\dot{H}^s}^2 + \|v(\cdot, s)\|_{\dot{H}^s}^2 \right) = \varepsilon_1^2.
\]
However, Corollary 4.4 implies
\[
\sup_{0 \leq s \leq \bar{T}} \left( \|g(\cdot, s)\|_{\dot{H}^s}^2 + \|u(\cdot, s)\|_{\dot{H}^s}^2 + \|v(\cdot, s)\|_{\dot{H}^s}^2 \right) \leq C_1 \bar{\varepsilon}_0^2 \leq \frac{\varepsilon_1^2}{2} < \varepsilon_1^2,
\]
which contradicts our assumption (4.7). Therefore, the lifespan $\tilde{T} = \infty$, i.e., the strong solution exists globally in time. \hfill \square

4.3. **Proof of Theorem 1.4** In this part, we investigate the global-in-time existence of strong solutions to the system (1.4). Similarly to the isothermal pressure case, we define

$$\mathcal{Y}(T; k) := \sup_{0 \leq t \leq T} \left( \|h\|_{H^{k-1}}^2 + \|u\|_{H^k}^2 + \|v\|_{H^k}^2 \right) \quad \text{and} \quad \mathcal{Y}_0(k) := \|h_0\|_{H^{k-1}}^2 + \|u_0\|_{H^k}^2 + \|v_0\|_{H^k}^2.$$

**Remark 4.1.** Note that

$$\|h\|_{H^{-1}} \leq \|\nabla K \ast (\rho - 1)\|_{L^2} \leq C \|h\|_{H^{-1}}.$$  

Thus, once we choose a sufficiently small $\varepsilon_1 > 0$ such that

$$\mathcal{Y}(T; s + 1) \leq \varepsilon_1^2 \ll 1 \quad \text{so that} \quad \sup_{0 \leq t \leq T} \|h(\cdot, t)\|_{L^\infty} \leq \frac{1}{2},$$

then by Proposition [4.3] with $\sigma = 0$ we have

$$\mathcal{Y}(T, 0) \leq 2 \left( \int_{\mathbb{R}^d} |\rho| u|^2 dx + \int_{\mathbb{R}^d} |\nabla K \ast (\rho - 1)|^2 dx + \int_{\mathbb{R}^d} |v|^2 dx \right)$$

$$\leq 2 \left( \int_{\mathbb{R}^d} \rho_0 |u_0|^2 dx + \int_{\mathbb{R}^d} |\nabla K \ast (\rho_0 - 1)|^2 dx + \int_{\mathbb{R}^d} |v_0|^2 dx \right)$$

$$\leq C \mathcal{Y}_0(0).$$

We next present some relevant estimates similar to the isothermal pressure case.

**Lemma 4.5.** Let $s > d/2 + 1$, $T > 0$ be given and suppose that $\mathcal{Y}(T; s + 1) \leq \varepsilon_1^2 \ll 1$. Then we have

$$\frac{d}{dt}\left( \|\nabla^k h\|_{L^2}^2 + \|\nabla^k (\nabla u)\|_{L^2}^2 \right) + \frac{3}{2} \|\nabla^k (\nabla u)\|_{L^2}^2$$

$$\leq C \varepsilon_1 \left( \|\nabla^k h\|_{L^2}^2 + \|\nabla^k (\nabla u)\|_{L^2}^2 + \|\nabla^k u\|_{L^2}^2 \right) + 2 \|\nabla^k (\nabla v)\|_{L^2}^2$$

for $0 \leq k \leq s$, where $C = C(d, k)$ is independent of $T$.

**Proof.** Since zero-order estimates are analogous, we only consider higher-order estimates. For $1 \leq k \leq s$, it follows from (4.3) that

$$\frac{1}{2} \frac{d}{dt} \|\nabla^k h\|_{L^2}^2 = - \int_{\mathbb{R}^d} \nabla^k (\nabla \cdot u) \nabla^k h dx - \int_{\mathbb{R}^d} \nabla^k ((1 + h) \nabla \cdot u) \nabla^k h dx$$

$$= \frac{1}{2} \int_{\mathbb{R}^d} (\nabla \cdot u) |\nabla^k h|^2 dx - \int_{\mathbb{R}^d} |\nabla^k (\nabla \cdot u) - \nabla^k (\nabla h) \cdot u| \nabla^k h dx$$

$$- \int_{\mathbb{R}^d} (h \nabla^k (\nabla \cdot u) - \nabla^k (\nabla h) \cdot u) \nabla^k h dx$$

$$- \int_{\mathbb{R}^d} |\nabla^k (h \nabla \cdot u) - h \nabla^k (\nabla \cdot u)| \nabla^k h dx$$

$$\leq \frac{(\|\nabla \cdot u\|_{L^\infty})}{2} \|\nabla^k h\|_{L^2}^2 + C \|\nabla^k h\|_{L^2} (\|\nabla u\|_{L^\infty} \|\nabla^k h\|_{L^2} + \|\nabla h\|_{L^\infty} \|\nabla^k u\|_{L^2})$$

$$+ \|h\|_{L^\infty} \|\nabla^k (\nabla h)\|_{L^2} \|\nabla^k h\|_{L^2} - \int_{\mathbb{R}^d} \nabla^k h \nabla^k (\nabla \cdot u) dx$$

$$+ C \|\nabla^k h\|_{L^2} (\|\nabla h\|_{L^\infty} \|\nabla^k (\nabla \cdot u)\|_{L^2} + \|\nabla^k h\|_{L^2} \|\nabla \cdot u\|_{L^\infty})$$

$$\leq C \varepsilon_1 (\|\nabla^k h\|_{L^2}^2 + \|\nabla^k (\nabla u)\|_{L^2}^2) + C \varepsilon_1 \|\nabla^k u\|_{L^2}^2 - \int_{\mathbb{R}^d} \nabla^k h \nabla^k (\nabla \cdot u) dx,$$
Thus, we combine the previous two estimates to get the desired result.

\[ -\int_{\mathbb{T}^d} \nabla^k(\nabla K \ast h) : \nabla^k(\nabla u) \, dx = \int_{\mathbb{T}^d} \nabla^k(\Delta K \ast h) : \nabla^k(\nabla u) \, dx = \int_{\mathbb{T}^d} \nabla^k h \nabla^k(\nabla \cdot u) \, dx. \]

Here we used

\[ -\int_{\mathbb{T}^d} \nabla^k(\nabla K \ast h) : \nabla^k(\nabla u) \, dx = -\int_{\mathbb{T}^d} \nabla^k(\Delta K \ast h) : \nabla^k(\nabla u) \, dx = \int_{\mathbb{T}^d} \nabla^k h \nabla^k(\nabla \cdot u) \, dx. \]

Thus, we combine the previous two estimates to get the desired result.

Similarly as before, in the lemma below, we present the estimate which gives the dissipation rate for \( \nabla^k h \).

**Lemma 4.6.** Let \( s > d/2 + 1 \), \( T > 0 \) be given and suppose that \( \mathcal{F}(T; s+1) \leq \varepsilon^2 \ll 1 \). Then we have

\[ -\frac{d}{dt} \int_{\mathbb{T}^d} \nabla^k h \nabla^k(\nabla \cdot u) \, dx + \frac{1}{4} \| \nabla^k h \|_{L^2}^2 \leq C \varepsilon_1 \left( \| \nabla^k h \|_{L^2}^2 + \| \nabla^k(\nabla u) \|_{L^2}^2 + \| \nabla^{k-1} h \|_{L^2}^2 + \| \nabla^k u \|_{L^2}^2 \right) + \frac{4}{3} \| \nabla^k(\nabla u) \|_{L^2}^2 \]

for \( 0 \leq k \leq s \), where \( C = C(d, k) \) is independent of \( T \).

**Proof.** Direct computation yields

\[ -\frac{d}{dt} \int_{\mathbb{T}^d} \nabla^k h \nabla^k(\nabla \cdot u) \, dx = \int_{\mathbb{T}^d} \nabla^k(\nabla \cdot (1 + h) u) \nabla^k(\nabla \cdot u) \, dx \]

\[ + \int_{\mathbb{T}^d} \nabla^k(\nabla \cdot (u \cdot \nabla u) - h \nabla \cdot u) \nabla^k h \, dx =: \mathcal{I}_1 + \mathcal{I}_2. \]

For \( \mathcal{I}_1 \), we use Lemma 4.1 to get

\[ \mathcal{I}_1 = -\int_{\mathbb{T}^d} \nabla^k((1 + h) u) \cdot \nabla(\nabla^k(\nabla \cdot u)) \, dx \]

\[ = -\int_{\mathbb{T}^d} (\nabla^k h) u \cdot \nabla(\nabla^k(\nabla \cdot u)) \, dx + \int_{\mathbb{T}^d} \nabla \cdot [\nabla^k((1 + h) u) - u \nabla^k h] \nabla^k(\nabla \cdot u) \, dx \]

\[ = -\int_{\mathbb{T}^d} (\nabla^k h) u \cdot \nabla(\nabla^k(\nabla \cdot u)) \, dx + \int_{\mathbb{T}^d} [\nabla^k(\nabla h \cdot u) - \nabla^k(\nabla h) \cdot u] \nabla^k(\nabla \cdot u) \, dx \]

\[ + \int_{\mathbb{T}^d} [\nabla^k(\nabla \cdot u) - \nabla^k h \nabla \cdot u] \nabla^k(\nabla \cdot u) \, dx + \int_{\mathbb{T}^d} |\nabla^k(\nabla \cdot u)|^2 \, dx \]

\[ \leq -\int_{\mathbb{T}^d} (\nabla^k h) u \cdot \nabla(\nabla^k(\nabla \cdot u)) \, dx + C \| \nabla^k(\nabla \cdot u) \|_{L^2} (\| \nabla u \|_{L^\infty} \| \nabla^k h \|_{L^2} + \| \nabla h \|_{L^\infty} \| \nabla^k u \|_{L^2}) \]

\[ + C \| \nabla^k(\nabla \cdot u) \|_{L^2} (\| \nabla^2 u \|_{L^\infty} \| \nabla^{k-1} h \|_{L^2} + \| \nabla h \|_{L^\infty} \| \nabla^k(\nabla \cdot u) \|_{L^2}) + \| \nabla^k(\nabla \cdot u) \|_{L^2}^2 \]

\[ \leq -\int_{\mathbb{T}^d} (\nabla^k h) u \cdot \nabla(\nabla^k(\nabla \cdot u)) \, dx \]

\[ + C \varepsilon_1 (\| \nabla^k h \|_{L^2}^2 + \| \nabla^k(\nabla u) \|_{L^2}^2 + \| \nabla^{k-1} h \|_{L^2}^2 + \| \nabla^k u \|_{L^2}^2) + \| \nabla^k(\nabla u) \|_{L^2}^2, \]

where \( C \) only depends on \( d \) and \( k \). Here we used

\[ \| \nabla u \|_{W^{1, \infty}} \leq C \| u \|_{H^{s+1}} \quad \text{and} \quad \| h \|_{W^{1, \infty}} \leq C \| h \|_{H^s} \]

due to \( s > d/2 + 1 \).
Next, we estimate $I_2$ as
\[
I_2 = \int_{T^d} u \cdot \nabla (\nabla^k (\nabla \cdot u)) \nabla^k h \, dx + \int_{T^d} (\nabla^k (u \cdot \nabla (\nabla \cdot u)) - u \cdot \nabla^k (\nabla (\nabla \cdot u))) \nabla^k h \, dx \\
+ \int_{T^d} \nabla^k (\nabla (u \cdot \nabla u) - u \cdot \nabla (\nabla \cdot u)) \nabla^k h \, dx - \|\nabla^k h\|_{L^2}^2 + \int_{T^d} \nabla^k h \nabla^k (\nabla \cdot u) \, dx \\
\leq \int_{T^d} u \cdot \nabla (\nabla^k (\nabla \cdot u)) \nabla^k h \, dx + C\|\nabla^k h\|_{L^2} (\|\nabla u\|_{L^\infty} \|\nabla^k (\nabla u)\|_{L^2} + \|\nabla^2 u\|_{L^\infty} \|\nabla^k u\|_{L^2}) \\
- \frac{1}{4} \|\nabla^k h\|_{L^2}^2 + \frac{1}{3} \|\nabla^k (\nabla u)\|_{L^2}^2,
\]
where we used Lemma 4.1 and $C$ only depends on $d$ and $k$. Finally, we combine the estimates for $I_1$ and $I_2$ to get the desired result. \qed

**Remark 4.2.** Since
\[
\left| \int_{T^d} \nabla^k h \nabla^k (\nabla \cdot u) \, dx \right| \leq \frac{1}{2} \|\nabla^k h\|_{L^2}^2 + \frac{1}{2} \|\nabla^k (\nabla u)\|_{L^2}^2,
\]
the following relation is direct:
\[
\frac{1}{2} \left( \|\nabla^k h\|_{L^2}^2 + \|\nabla^k (\nabla u)\|_{L^2}^2 \right) \leq \|\nabla^k h\|_{L^2}^2 + \|\nabla^k (\nabla u)\|_{L^2}^2 - \int_{T^d} \nabla^k h \nabla^k (\nabla \cdot u) \, dx \\
\leq \frac{3}{2} \left( \|\nabla^k h\|_{L^2}^2 + \|\nabla^k (\nabla u)\|_{L^2}^2 \right).
\]

Now, we consider the estimates for the Navier-Stokes part in (4.4).

**Lemma 4.7.** Let $s > d/2 + 1$, $T > 0$ be given and suppose that $\mathcal{Q}(T; s + 1) \leq \varepsilon^2 \ll 1$. Then we have
\[
\frac{d}{dt}\|\nabla^k v\|_{L^2}^2 + \frac{3}{2} \|\nabla^k (\nabla v)\|_{L^2}^2 \\
\leq C \varepsilon_1 (\|\nabla^k v\|_{L^2}^2 + (\|\nabla^{k-1} v\|_{L^2}^2 + \|\nabla^{k-1} u\|_{L^2}^2 + \|\nabla^{k-1} h\|_{L^2}^2) (1 - \delta_{k,0})) + 2 \|\nabla^k u\|_{L^2}^2
\]
for $0 \leq k \leq s + 1$, where $C = C(d, k)$ is independent of $T$.

**Proof.** For $k = 0$, we estimate
\[
\frac{1}{2} \frac{d}{dt}\|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 = - \int_{T^d} (v \cdot \nabla v) \cdot v \, dx - \int_{T^d} (1 + h) (v - u) \cdot v \, dx \\
\leq - \frac{3}{4} \|v\|_{L^2}^2 + \|u\|_{L^2}^2 + \|h\|_{L^\infty} \|u\|_{L^2} \|v\|_{L^2} \\
\leq C \varepsilon_1 (\|v\|_{L^2}^2 + \|u\|_{L^2}^2) - \frac{3}{4} \|v\|_{L^2}^2 + \|u\|_{L^2}^2,
\]
where $C$ is independent of $T$. For $k \geq 1$, we deduce from (4.4) that
\[
\frac{1}{2} \frac{d}{dt}\|\nabla^k v\|_{L^2}^2 + \|\nabla^k (\nabla v)\|_{L^2}^2 \\
= - \int_{T^d} \nabla^k (v \cdot \nabla v) \cdot \nabla^k v \, dx - \int_{T^d} \nabla^k ((1 + h) (v - u)) \nabla^k v \, dx \\
= - \int_{T^d} \nabla^k (v \cdot \nabla v) - v \cdot \nabla (\nabla^k v) \cdot \nabla^k v \, dx \\
+ \int_{T^d} \nabla^{k-1} (h (v - u)) \nabla^{k+1} v \, dx - \int_{T^d} \nabla^k (v - u) \cdot \nabla^k v \, dx \\
\leq C \|\nabla v\|_{L^\infty} \|\nabla^k v\|_{L^2}^2 + \|\nabla^{k-1} (h (v - u))\|_{L^2} \|\nabla^k (\nabla v)\|_{L^2} - \frac{3}{4} \|\nabla^k v\|_{L^2}^2 + \|\nabla^k u\|_{L^2}^2 \\
\leq C \varepsilon_1 \|\nabla^k v\|_{L^2}^2 + C \|\nabla^k (\nabla v)\|_{L^2} \left( \|h\|_{L^\infty} \|\nabla^{k-1} (v - u)\|_{L^2} + \|\nabla^{k-1} h\|_{L^2} \|u - v\|_{L^\infty} \right) \\
- \frac{3}{4} \|\nabla^k v\|_{L^2}^2 + \|\nabla^k u\|_{L^2}^2 \\
\leq C \varepsilon_1 (\|\nabla^k v\|_{L^2}^2 + \|\nabla^{k-1} v\|_{L^2}^2 + \|\nabla^{k-1} u\|_{L^2}^2 + \|\nabla^{k-1} h\|_{L^2}^2)$
Here we used Lemma \ref{lem:ind-hyp} and $C$ only depends on $d$ and $k$. This implies our desired estimate.

Now, we are ready to present the uniform energy estimates based on Lemmas \ref{lem:key}, \ref{lem:upper-bound}.}

\begin{corollary}
Let $s > d/2 + 1$, $T > 0$ be given and suppose that $\mathfrak{F}(T; s + 1) \leq \varepsilon^2 \ll 1$. Then we have

$$\mathfrak{F}(T; s + 1) \leq C \mathfrak{F}_0(s + 1),$$

where $C_2$ is independent of $T$.
\end{corollary}

\begin{proof}
The proof is similar to that of Corollary \ref{cor:uniform-energy-estimate}. By Remark \ref{rem:induction-step}, it suffices to show the induction step. Suppose that there exists a positive constant $C > 0$ that is independent of $T$ such that

$$\mathfrak{F}(T; m) \leq C \mathfrak{F}_0(m) \quad \text{for} \quad 0 \leq m \leq k.$$

First, we combine Lemmas \ref{lem:upper-bound} and \ref{lem:lower-bound} to obtain

$$\frac{d}{dt} \left( \|\nabla^k h\|_{L^2}^2 + \|\nabla^k (\nabla u)\|_{L^2}^2 - \int_{\mathbb{T}^d} \nabla^k h \nabla^k (\nabla \cdot u) \, dx \right) \leq -\frac{1}{6} + C \varepsilon_1 \left( \|\nabla^k h\|_{L^2}^2 + \|\nabla^k (\nabla u)\|_{L^2}^2 + C \mathfrak{F}_0(T; k) + 2 \|\nabla^k (\nabla v)\|_{L^2}^2 \right),$$

where $C$ is independent of $T$. Next, we combine the previous relation with Lemma \ref{lem:inequality} to get

$$\frac{d}{dt} \left( \|\nabla^k h\|_{L^2}^2 + \|\nabla^k (\nabla u)\|_{L^2}^2 - \int_{\mathbb{T}^d} \nabla^k h \nabla^k (\nabla \cdot u) \, dx + 2 \|\nabla^k v\|_{L^2}^2 \right) \leq \left( -\frac{1}{6} + C \varepsilon_1 \right) \left( \|\nabla^k h\|_{L^2}^2 + \|\nabla^k (\nabla u)\|_{L^2}^2 + \|\nabla^k v\|_{L^2}^2 \right) + C \mathfrak{F}_0(T; k).$$

Here we used the induction hypothesis and $C$ is independent of $T$. Since $\varepsilon_1$ is sufficiently small, one knows $-1/6 + C \varepsilon_1 < 0$. Thus, we use the estimates in Remark \ref{rem:inequality} and Grönwall’s lemma to have

$$\sup_{0 \leq t \leq T} \left( \|\nabla^k h\|_{L^2}^2 + \|\nabla^k (\nabla u)\|_{L^2}^2 \right) \leq C \mathfrak{F}_0(k + 1),$$

where $C$ is independent of $T$. Next, Lemma \ref{lem:inequality} implies

$$\frac{d}{dt} \|\nabla^k v\|_{L^2}^2 + \|\nabla^k (\nabla v)\|_{L^2}^2 \leq \left( -\frac{3}{2} + C \varepsilon_1 \right) \left( \|\nabla^k v\|_{L^2}^2 + \|\nabla^k u\|_{L^2}^2 + \|\nabla^k u\|_{L^2}^2 \right) + \frac{3}{2} \|\nabla^k v\|_{L^2}^2 + \|\nabla^k (\nabla v)\|_{L^2}^2 \right) \leq \frac{3}{2} + C \varepsilon_1 \left( \|\nabla^k v\|_{L^2}^2 + \|\nabla^k (\nabla v)\|_{L^2}^2 \right) + C \mathfrak{F}_0(k + 1),$$

where we used the induction hypothesis and \ref{eq:inequality}. Here $C$ is independent of $T$. Thus, we again use Grönwall’s lemma to have

$$\sup_{0 \leq t \leq T} \|\nabla^k v\|_{L^2}^2 \leq C \mathfrak{F}_0(k + 1).$$

Thus we combine \ref{eq:inequality} and \ref{eq:inequality-2} to prove the induction step and hence, we conclude the proof.
\end{proof}

\begin{proof}[Proof of Theorem \ref{thm:uniform-energy-estimate}]
By using Corollary \ref{cor:uniform-energy-estimate} and almost the same argument as in the proof of Theorem \ref{thm:uniform-energy-estimate} we can extend the local-in-time strong solutions to the global-in-time ones.
\end{proof}

5. Large-time behavior of the isothermal/pressureless EPNS equations

In this section, we investigate \textit{a priori} large-time behavior estimates of solutions to the system \eqref{eq:EPNS}. Let us first recall our Lyapunov functional $\mathcal{L}(\rho, u, v)$:

$$\mathcal{L}(\rho, u, v) = \int_{\mathbb{T}^d} \rho |u - m_c|^2 \, dx + \int_{\mathbb{T}^d} |v - v_c|^2 \, dx + \sigma \int_{\mathbb{T}^d} (\rho - 1)^2 \, dx + \int_{\mathbb{T}^d} |\nabla K \ast (\rho - 1)|^2 \, dx + |m_c - v_c|^2.$$
5.1. Variations of $\mathcal{L}$. Before estimating the functional $\mathcal{L}$ directly, we estimate a variation functional $\mathcal{E}$ defined as

$$
\mathcal{E}(\rho, u, v) := \frac{1}{2} \int_{\mathbb{T}^d} \rho |u - m_c|^2 \, dx + \frac{1}{2} \int_{\mathbb{T}^d} |v - v_c|^2 \, dx + \sigma \int_{\mathbb{T}^d} \rho \int_1^\rho \frac{s - 1}{s^2} \, ds dx \\
+ \frac{1}{2} \int_{\mathbb{T}^d} |\nabla K * (\rho - 1)|^2 \, dx + \frac{1}{4} |m_c - v_c|^2.
$$

We notice from [16, Lemma 2.2] that

$$
\int_{\mathbb{T}^d} \rho \int_1^\rho \frac{s - 1}{s^2} \, ds dx = \int_{\mathbb{T}^d} \rho \log \rho \, dx
$$

and there exists a $C > 0$ which depends only on $\bar{\rho}$ such that

$$
\frac{1}{C} \int_{\mathbb{T}^d} (\rho - 1)^2 \, dx \leq \int_{\mathbb{T}^d} \rho \int_1^\rho \frac{s - 1}{s^2} \, ds dx \leq C \int_{\mathbb{T}^d} (\rho - 1)^2 \, dx.
$$

This together with Proposition 4.1 gives

$$
\mathcal{E}(t) := \int_{\mathbb{T}^d} \rho |u|^2 \, dx + \sigma \int_{\mathbb{T}^d} (\rho - 1)^2 \, dx + \int_{\mathbb{T}^d} |v|^2 \, dx + \int_{\mathbb{T}^d} |\nabla K * (\rho - 1)|^2 \, dx \leq C \mathcal{E}(0)
$$

for some $C > 0$ independent of $t$. Moreover, we have that the functional $\mathcal{E}$ is equivalent to $\mathcal{L}$ in the following sense

$$
\frac{1}{C} \mathcal{L}(\rho, u, v) \leq \mathcal{E}(\rho, u, v) \leq C \mathcal{L}(\rho, u, v)
$$

for some $C > 0$ independent of $t$. Thus the exponential decay of $\mathcal{E}$ implies the exponential decay of $\mathcal{L}$. Let us denote this equivalence relation by $\mathcal{E} \approx \mathcal{L}$, and we also denote $\mathcal{L} := \mathcal{L}(\rho, u, v)$ and $\mathcal{E} := \mathcal{E}(\rho, u, v)$ for notational simplicity.

In the lemma below, we show that the functional $\mathcal{E}$ is not increasing in time, and it also have the same dissipation rate for the total energy, see Proposition 4.1.

**Lemma 5.1.** Let $(\rho, u, v)$ be a global-in-time solution to the system (4.1) with sufficient regularity. Then we have

$$
\frac{d}{dt} \mathcal{E}(t) + \mathcal{D}(t) = 0,
$$

where the dissipation $\mathcal{D} := \mathcal{D}(\rho, u, v)$ is given by

$$
\mathcal{D}(t) := \int_{\mathbb{T}^d} \rho |u - v|^2 \, dx + \int_{\mathbb{T}^d} |\nabla v|^2 \, dx.
$$

**Proof.** By using similar arguments as in Proposition 4.1, we find

$$
\frac{d}{dt} \left( \frac{1}{2} \int_{\mathbb{T}^d} \rho |u - m_c|^2 \, dx + \int_{\mathbb{T}^d} \rho \log \rho \, dx \right)
$$

$$
= - \int_{\mathbb{T}^d} \rho (u - m_c) \cdot \nabla K * (\rho - 1) \, dx - \int_{\mathbb{T}^d} \rho (u - m_c) \cdot (u - v) \, dx
$$

and

$$
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^d} |v - v_c|^2 \, dx + \int_{\mathbb{T}^d} |\nabla v|^2 \, dx = \int_{\mathbb{T}^d} \rho (v - v_c) \cdot (u - v) \, dx.
$$

Note that

$$
\int_{\mathbb{T}^d} \rho \nabla K * (\rho - 1) \, dx = \int_{\mathbb{T}^d} (\rho - 1) \nabla K * (\rho - 1) \, dx = 0
$$

due to $\nabla K(-x) = -\nabla K(x)$. Since

$$
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^d} |\nabla K * (\rho - 1)|^2 \, dx = \int_{\mathbb{T}^d} \rho u \cdot \nabla K * (\rho - 1) \, dx,
$$

we have

$$
\frac{d}{dt} \left( \frac{1}{2} \int_{\mathbb{T}^d} \rho |u - m_c|^2 \, dx + \int_{\mathbb{T}^d} \rho \log \rho \, dx + \frac{1}{2} \int_{\mathbb{T}^d} |v - v_c|^2 \, dx + \frac{1}{2} \int_{\mathbb{T}^d} |\nabla K * (\rho - 1)|^2 \, dx \right)
$$

$$
= - \int_{\mathbb{T}^d} |\nabla v|^2 \, dx - \int_{\mathbb{T}^d} \rho |u - v|^2 \, dx + (m_c - v_c) \cdot \int_{\mathbb{T}^d} \rho (u - v) \, dx.
$$
On the other hand, we easily get
\[
\frac{d}{dt} m_c(t) = - \int_{\mathbb{T}^d} \rho(u - v) \, dx \quad \text{and} \quad \frac{d}{dt} v_c(t) = \int_{\mathbb{T}^d} \rho(u - v) \, dx,
\]
and thus we obtain
\[
\frac{1}{4} \frac{d}{dt} |m_c(t) - v_c(t)|^2 = \frac{1}{2} (m_c(t) - v_c(t)) \cdot \frac{d}{dt} (m_c(t) - v_c(t)) = -(m_c - v_c) \cdot \int_{\mathbb{T}^d} \rho(u - v) \, dx.
\]
This combined with (5.3) and (5.1) asserts the desired result.

We next present an auxiliary lemma concerning the averages \( m_c \) and \( v_c \).

**Lemma 5.2.** The averages \( m_c \) and \( v_c \) satisfy the followings:
\[
|m_c(t)|^2 + |v_c(t)|^2 \leq C \tilde{E}(0) \quad \text{and} \quad |m'_c(t)|^2 + |v'_c(t)|^2 \leq C \int_{\mathbb{T}^d} \rho|u - v|^2 \, dx
\]
for all \( t \geq 0 \), where \( C > 0 \) is independent of \( t \).

**Proof.** Using the Hölder’s inequality, we easily get
\[
|m_c|^2 \leq \int_{\mathbb{T}^d} \rho|u|^2 \, dx \leq C \tilde{E}(0)
\]
due to (5.2). The other estimates can also be obtained similarly.

In order to show the exponential decay of \( \mathcal{L}^\ast \), or equivalently \( \tilde{\mathcal{E}} \), we need to show that the functional \( \mathcal{L}^\ast \) can be bounded from above by the dissipation rate \( \mathcal{D} \). For this, as an intermediate step, in the lemma below, we provide that some terms can be controlled by \( \mathcal{D} \).

**Lemma 5.3.** There exists a constant \( C > 0 \) independent of \( t \) such that
\[
\mathcal{L}^\ast(t) \leq C \mathcal{D}(t),
\]
where
\[
\mathcal{L}^\ast(t) := \mathcal{L}(t) - \sigma \int_{\mathbb{T}^d} (\rho - 1)^2 \, dx - \int_{\mathbb{T}^d} |\nabla K \ast (\rho - 1)|^2 \, dx.
\]

**Proof.** First, we have
\[
\int_{\mathbb{T}^d} \rho|u - v|^2 \, dx = \int_{\mathbb{T}^d} \rho|u - m_c + m_c - v_c + v_c - v|^2 \, dx
\]
\[
= \int_{\mathbb{T}^d} \rho|u - m_c|^2 \, dx + \int_{\mathbb{T}^d} \rho|v_c - v|^2 \, dx + |m_c - v_c|^2 + 2 \int_{\mathbb{T}^d} \rho(u - v_c) \cdot (v_c - v) \, dx.
\]
Here, we use Young’s inequality to get
\[
-4 \int_{\mathbb{T}^d} \rho(u - v_c) \cdot (v_c - v) \, dx \leq 4 \int_{\mathbb{T}^d} \rho|v - v_c|^2 \, dx + \int_{\mathbb{T}^d} \rho|u - v_c|^2 \, dx
\]
\[
= 4 \int_{\mathbb{T}^d} \rho|v - v_c|^2 \, dx + \int_{\mathbb{T}^d} \rho|u - m_c|^2 \, dx + |m_c - v_c|^2,
\]
and this results in
\[
\frac{1}{2} |m_c - v_c|^2 + \frac{1}{2} \int_{\mathbb{T}^d} \rho|u - m_c|^2 \, dx \leq \int_{\mathbb{T}^d} \rho|v - v_c|^2 \, dx + \int_{\mathbb{T}^d} \rho|u - v|^2 \, dx.
\]
Thus, we obtain
\[
\mathcal{L}^\ast(t) \leq 2 \int_{\mathbb{T}^d} \rho|v - v_c|^2 \, dx + 2 \int_{\mathbb{T}^d} |v - v_c|^2 \, dx + 2 \int_{\mathbb{T}^d} \rho|u - v|^2 \, dx
\]
\[
\leq 2(\|\rho\|_{L^\infty} + 1) \int_{\mathbb{T}^d} |v - v_c|^2 \, dx + 2 \int_{\mathbb{T}^d} \rho|u - v|^2 \, dx
\]
\[
\leq C \left( \int_{\mathbb{T}^d} |\nabla v|^2 \, dx + \int_{\mathbb{T}^d} \rho|u - v|^2 \, dx \right),
\]
where we used Poincaré inequality and hence, the proof completes.
As we observed in the previous lemma, we still need to control the other terms in \( \mathcal{L} \) by the dissipation \( \mathcal{D} \). Similar issue appears in \([15, 16, 19, 20]\) where the large-time behavior of isothermal Euler-type equations are studied. In those works, as mentioned before, the Bogovskii’s estimate is used to have a proper dissipation term. Since we also take into account the pressureless Euler equations and Poisson’s equation, it is not clear how to apply those previous arguments to our case. Instead of that, we consider a perturbation \( \mathcal{E}^\lambda \) of the energy functional \( \mathcal{E} \) defined by

\[
\mathcal{E}^\lambda := \mathcal{E}(t) + \lambda \int_{\mathbb{T}^d} \rho(u - m_c) \cdot \nabla K \star (\rho - 1) \, dx,
\]

where \( \lambda > 0 \) will be determined later. Using Young’s inequality and bounded assumption of \( \rho \), we easily show that

\[
\mathcal{E}^\lambda(t) \approx \mathcal{E}(t) \approx \mathcal{L}(t)
\]

for \( \lambda > 0 \) small enough.

We next show that the perturbation \( \mathcal{E}^\lambda \) produces proper dissipations for \( \mathcal{L} - \mathcal{L}^- \).

**Lemma 5.4.** The perturbed energy functional \( \mathcal{E}^\lambda \) satisfies the following relation:

\[
\frac{d}{dt} \mathcal{E}^\lambda(t) + \mathcal{D}^\lambda(t) = 0,
\]

where \( \mathcal{D}^\lambda(t) \) is given by

\[
\mathcal{D}^\lambda(t) := \mathcal{D} + \left[ -\int_{\mathbb{T}^d} \rho u \otimes u : \nabla^2 (K \star (\rho - 1)) \, dx + \int_{\mathbb{T}^d} \rho |\nabla K \star (\rho - 1)|^2 \, dx \\
+ \sigma \int_{\mathbb{T}^d} (\rho - 1)^2 \, dx + \int_{\mathbb{T}^d} \rho (u - v) \cdot \nabla K \star (\rho - 1) \, dx \\
- \int_{\mathbb{T}^d} \rho (u - m_c) \cdot \nabla K \star (\partial_t \rho) \, dx + \int_{\mathbb{T}^d} \partial_t (\rho m_c) \cdot \nabla K \star (\rho - 1) \, dx \right].
\]

**Proof.** It suffices to estimate

\[
\frac{d}{dt} \int_{\mathbb{T}^d} \rho(u - m_c) \cdot \nabla K \star (\rho - 1) \, dx \\
= \int_{\mathbb{T}^d} \partial_t (\rho u) \cdot \nabla K \star (\rho - 1) \, dx + \int_{\mathbb{T}^d} \rho (u - m_c) \cdot \nabla K \star (\partial_t \rho) \, dx - \int_{\mathbb{T}^d} \partial_t (\rho m_c) \cdot \nabla K \star (\rho - 1) \, dx \\
=: I_1 + I_2 + I_3.
\]

For \( I_1 \), we have

\[
I_1 = -\int_{\mathbb{T}^d} \left( \nabla \cdot (\rho u \otimes u) + \sigma \nabla \rho + \rho \nabla K \star (\rho - 1) + \rho (u - v) \right) \cdot \nabla K \star (\rho - 1) \, dx \\
= \int_{\mathbb{T}^d} \rho u \otimes u : \nabla^2 (K \star (\rho - 1)) \, dx - \sigma \int_{\mathbb{T}^d} (\rho - 1)^2 \, dx - \int_{\mathbb{T}^d} \rho |\nabla K \star (\rho - 1)|^2 \, dx \\
- \int_{\mathbb{T}^d} \rho (u - v) \cdot \nabla K \star (\rho - 1) \, dx,
\]

and this implies the desired relation. \( \square \)

**5.2. Proof of Theorem 1.5.** For sufficiently small \( \lambda > 0 \), we claim

\[
\mathcal{L}(t) \leq C \mathcal{D}^\lambda(t),
\]
where $C > 0$ is independent of $t$. Let us estimate $\mathcal{D}^\lambda$ term by term as follows:

$$\mathcal{D}^\lambda(t) = \int_{\mathbb{T}^d} |\nabla v|^2 \, dx + \int_{\mathbb{T}^d} \rho|u - v|^2 \, dx - \lambda \int_{\mathbb{T}^d} \rho \nabla \cdot u : \nabla^2 (K \ast (\rho - 1)) \, dx$$

$$+ \lambda \int_{\mathbb{T}^d} \rho \nabla (K \ast (\rho - 1))^2 \, dx + \lambda \int_{\mathbb{T}^d} (\rho(u - v) \cdot \nabla) (\rho - 1) \, dx$$

$$+ \lambda \int_{\mathbb{T}^d} (u - m_c) \cdot \nabla (\rho - 1) \, dx + \lambda \int_{\mathbb{T}^d} \partial_t (\rho m_c) \cdot \nabla (\rho - 1) \, dx$$

$$=: \sum_{i=1}^8 J_i.$$

- (Estimate for $J_6$): We use Young’s inequality to get

$$J_6 \geq -\lambda^{1/2} \int_{\mathbb{T}^d} \rho|u - v|^2 \, dx - \lambda^{3/2} \int_{\mathbb{T}^d} \rho \nabla (\rho - 1)^2 \, dx.$$

- (Estimate for $J_7$): We split $J_7$ into two terms:

$$J_7 = \lambda \int_{\mathbb{T}^d} (u - m_c) \cdot \nabla (\rho - 1) \, dx + \lambda \int_{\mathbb{T}^d} (u - m_c) \cdot \nabla (\rho - 1) \, dx.$$

Note that the second term on the right hand side of the above can be rewritten as

$$\int_{\mathbb{T}^d} (u - m_c) \cdot \nabla (\rho - 1) \, dx$$

$$= -\int_{\mathbb{T}^d} \nabla \cdot (u - m_c)(x) \left( \int_{\mathbb{T}^d} K(x - y) \nabla_y \cdot ((\rho - 1)(y) \, m_c) \, dy \right) \, dx$$

$$= \int_{\mathbb{T}^d} \nabla \cdot (u - m_c)(x) \left( \int_{\mathbb{T}^d} \nabla_y (K(x - y)) \cdot ((\rho - 1)(y) \, m_c) \, dy \right) \, dx$$

$$= -\int_{\mathbb{T}^d} \nabla \cdot (u - m_c)(x) \, m_c \cdot \left( \int_{\mathbb{T}^d} (\nabla K)(x - y) \cdot ((\rho - 1)(y) \, m_c) \, dx \right)$$

$$= \int_{\mathbb{T}^d} \rho (u - m_c) \otimes m_c : \nabla^2 (\rho - 1) \, dx.$$

Together with this, we estimate

$$J_7 \geq -\lambda \|\rho(u - m_c)\|_{L^2} \|\nabla (\rho - 1)\|_{L^2} + \lambda \int_{\mathbb{T}^d} \rho(u - m_c) \otimes m_c : \nabla^2 (\rho - 1) \, dx$$

$$\geq -C\lambda \|\rho(u - m_c)\|_{L^2} \|\nabla (\rho - 1)\|_{H^{-1}} + \lambda \int_{\mathbb{T}^d} \rho(u - m_c) \otimes m_c : \nabla^2 (\rho - 1) \, dx$$

$$\geq -C\lambda \|\rho(u - m_c)\|_{L^2}^2 + \lambda \int_{\mathbb{T}^d} \rho(u - m_c) \otimes m_c : \nabla^2 (\rho - 1) \, dx$$

$$\geq -C\lambda \|\rho\|_{L^\infty} \int_{\mathbb{T}^d} |u - m_c|^2 \, dx + \lambda \int_{\mathbb{T}^d} \rho(u - m_c) \otimes m_c : \nabla^2 (\rho - 1) \, dx,$$

where we used

$$\|\nabla K \ast h\|_{L^2} \leq C \|h\|_{H^{-1}} \quad \forall h \in L^2(\mathbb{T}^d) \quad \text{with} \quad \int_{\mathbb{T}^d} h \, dx = 0.$$

- (Estimate for $J_8$): With the following inequality in mind

$$\partial_t (\rho m_c) = -\nabla \cdot (\rho u) m_c + \rho m'_c,$$

we obtain

$$J_8 = -\lambda \int_{\mathbb{T}^d} \nabla \cdot (\rho u) m_c \cdot \nabla (\rho - 1) \, dx + \lambda \int_{\mathbb{T}^d} \rho m'_c \cdot \nabla (\rho - 1) \, dx$$

$$= -\lambda \int_{\mathbb{T}^d} \nabla \cdot (\rho u \otimes m_c) \cdot \nabla (\rho - 1) \, dx + \lambda \int_{\mathbb{T}^d} \rho m'_c \cdot \nabla (\rho - 1) \, dx$$
\[
\begin{aligned}
&= \lambda \int_{\mathcal{B}_o} \rho u \otimes m_c : \nabla^2 K \ast (\rho - 1) \, dx + \lambda \int_{\mathcal{B}_o} \rho m_c' \cdot \nabla K \ast (\rho - 1) \, dx \\
&\geq \lambda \int_{\mathcal{B}_o} \rho u \otimes m_c : \nabla^2 K \ast (\rho - 1) \, dx - \lambda \|\rho\|_{L^\infty} \frac{1}{2} \|m_c'\| \left( \int_{\mathcal{B}_o} \rho \|\nabla K \ast (\rho - 1)\|^2 \, dx \right)^{1/2} \\
&\geq \lambda \int_{\mathcal{B}_o} \rho u \otimes m_c : \nabla^2 K \ast (\rho - 1) \, dx - C \lambda^{1/2} \int_{\mathcal{B}_o} \rho \|u - v\|^2 \, dx - C \lambda^{1/2} \int_{\mathcal{B}_o} \rho \|\nabla K \ast (\rho - 1)\|^2 \, dx 
\end{aligned}
\]
due to Lemma 5.2, where \(C = C(\|\rho\|_{L^\infty})\) is a constant independent of \(t\) and \(\lambda\).

We now combine the estimates for \(\mathcal{J}_7\) and \(\mathcal{J}_8\) to yield

\[
\begin{aligned}
\mathcal{J}_3 + \mathcal{J}_7 + \mathcal{J}_8 &\geq \lambda \int_{\mathcal{B}_o} \rho u \otimes (m_c - u) : \nabla^2 K \ast (\rho - 1) \, dx + \lambda \int_{\mathcal{B}_o} \rho (u - m_c) \otimes m_c : \nabla^2 K \ast (\rho - 1) \, dx \\
&\quad - C \lambda \int_{\mathcal{B}_o} \rho |u - m_c|^2 \, dx - C \lambda^{1/2} \int_{\mathcal{B}_o} \rho \|u - v\|^2 \, dx - C \lambda^{1/2} \int_{\mathcal{B}_o} \rho \|\nabla K \ast (\rho - 1)\|^2 \, dx \\
&= -\lambda \int_{\mathcal{B}_o} \rho (u - m_c) \otimes (u - m_c) : \nabla^2 K \ast (\rho - 1) \, dx - C \lambda \int_{\mathcal{B}_o} \rho |u - m_c|^2 \, dx \\
&\quad - C \lambda^{1/2} \int_{\mathcal{B}_o} \rho \|u - v\|^2 \, dx - C \lambda^{1/2} \int_{\mathcal{B}_o} \rho \|\nabla K \ast (\rho - 1)\|^2 \, dx,
\end{aligned}
\]

where \(C = C(\|\rho\|_{L^\infty}) > 0\) is independent of \(t\) and \(\lambda\) and we used the symmetry of \(\nabla^2 K \ast (\rho - 1)\) to get

\((u - m_c) \otimes m_c : \nabla^2 K \ast (\rho - 1) = m_c \otimes (u - m_c) : \nabla^2 K \ast (\rho - 1)\).

We then estimate the first term on the right hand side of (5.5) by dividing two cases: pressureless \(\sigma = 0\) and isothermal pressure law \(\sigma > 0\). When \(\sigma = 0\), we estimate

\[
-\lambda \int_{\mathcal{B}_o} \rho (u - m_c) \otimes (u - m_c) : \nabla^2 K \ast (\rho - 1) \, dx \geq -\lambda \|\nabla^2 K \ast (\rho - 1)\|_{L^\infty} \int_{\mathcal{B}_o} \rho |u - m_c|^2 \, dx \\
\geq -\lambda \|\nabla K\|_{L^1} \|\nabla\rho\|_{L^\infty} \int_{\mathcal{B}_o} \rho |u - m_c|^2 \, dx \\
\geq -C \lambda \int_{\mathcal{B}_o} \rho |u - m_c|^2 \, dx,
\]

where \(C = C(\|\nabla K\|_{L^1}, \|\nabla\rho\|_{L^\infty})\) is a positive constant independent of \(t\) and \(\lambda\). On the other hand, when \(\sigma > 0\), we have

\[
-\lambda \int_{\mathcal{B}_o} \rho (u - m_c) \otimes (u - m_c) : \nabla^2 K \ast (\rho - 1) \, dx \geq -\lambda (\|\nabla^2 K\|_{L^\infty} + |m_c|) \|\rho (u - m_c)\|_{L^2} \|\nabla^2 K \ast (\rho - 1)\|_{L^2} \\
\geq -C \lambda^{1/2} \int_{\mathcal{B}_o} \rho |u - m_c|^2 \, dx - C \lambda^{3/2} \int_{\mathcal{B}_o} (\rho - 1)^2 \, dx.
\]

Here \(C = C(\|\rho\|_{L^\infty}, \|\rho\|_{L^\infty}, \tilde{E}_0)\) is a positive constant independent of \(t\) and \(\lambda\) and we used

\(\|\nabla^2 (K \ast (\rho - 1))\|_{L^2} = \|\Delta (K \ast (\rho - 1))\|_{L^2} = \|\rho - 1\|_{L^2}\).

Thus, we choose \(\lambda > 0\) sufficiently small, use Lemma 5.3 and \(\rho \geq \rho_0 > 0\) when \(\sigma = 0\), and combine all the above estimates to get

\[
\mathcal{D}^\lambda \geq \int_{\mathcal{B}_o} |\nabla v|^2 \, dx + (1 - C \lambda^{1/2}) \int_{\mathcal{B}_o} \rho |u - v|^2 \, dx + C \sigma \int_{\mathcal{B}_o} (\rho - 1)^2 \, dx \\
+ \lambda (1 - C \lambda^{1/2}) \int_{\mathcal{B}_o} \rho |\nabla K \ast (\rho - 1)|^2 \, dx - C \lambda^{1/2} (1 + \lambda^{1/2}) \int_{\mathcal{B}_o} \rho |u - m_c|^2 \, dx \\
\geq C_1 \left( \int_{\mathcal{B}_o} |\nabla v|^2 \, dx + \int_{\mathcal{B}_o} \rho |u - v|^2 \, dx \right) + C_2 \sigma \int_{\mathcal{B}_o} (\rho - 1)^2 \, dx + C_3 \int_{\mathcal{B}_o} |\nabla K \ast (\rho - 1)|^2 \, dx \\
\geq C_4 \mathcal{L}(t) + C_2 \sigma \int_{\mathcal{B}_o} (\rho - 1)^2 \, dx + C_3 \int_{\mathcal{B}_o} |\nabla K \ast (\rho - 1)|^2 \, dx \\
\geq \min\{C_2, C_3, C_4\} \mathcal{L}(t),
\]
where we used the fact that
\[ \int_{\mathbb{T}^d} (\rho - 1)^2 \, dx \geq \|\rho - 1\|_{\mathcal{H}^{-1}} \geq C \|\nabla K \ast (\rho - 1)\|_{L^2}, \]
and thus the dissipation term \( \int_{\mathbb{T}^d} \rho |\nabla K \ast (\rho - 1)|^2 \, dx \) is not needed when \( \sigma > 0 \). This is why the strictly positive lower bound assumption on \( \rho \) is not required for the case \( \sigma > 0 \). Here \( C_2, C_3, \) and \( C_4 \) are positive constants which depend on \[
\begin{aligned}
\|\rho\|_{W^{1,\infty}} \quad &\text{and} \quad \|\nabla K\|_{L^1} \quad \text{when} \quad \sigma = 0, \\
\|\rho\|_{L^\infty} \quad &\text{and} \quad \|u\|_{L^\infty}, \quad \text{and} \quad \tilde{E}_0 \quad \text{when} \quad \sigma > 0.
\end{aligned}
\]
This together with (5.4) and Lemma 5.4 implies
\[
\frac{d}{dt} \sigma^A(t) + C \sigma^A(t) \leq 0 \quad \forall \ t \geq 0
\]
for some \( C > 0 \) independent of \( t \). Applying Grönwall’s lemma to the above concludes the proof of Theorem 1.5.

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APPENDIX A. PROOFS FOR LEMMAS 2.6 AND 2.7

In this appendix, we present the proofs for Lemmas 2.6 and 2.7.

A.1. Proof of Lemma 2.6

When \( d = 3 \), it is obvious since
\[
\int_{\mathbb{T}^3 \times \mathbb{T}^3} K^\varepsilon(x - y) \varphi(x) \varphi(y) \, dx \, dy \geq \int_{\mathbb{T}^3 \times \mathbb{T}^3} G_0(x - y) \varphi(x) \varphi(y) \, dx \, dy.
\]
Thus, it suffices to show the case \( d = 2 \). First, we use the following inequality:
\[
\varepsilon + |x - y|^2 \leq (1 + \varepsilon)(1 + |x - y|^2) \leq (1 + \varepsilon)(1 + 2|x|^2 + 2|y|^2) \leq 2(1 + \varepsilon)(1 + |x|^2)(1 + |y|^2),
\]
which subsequently gives
\[
\log(\varepsilon + |x - y|^2) \leq \log (2(1 + \varepsilon)) + \log(1 + |x|^2) + \log(1 + |y|^2).
\]
We combine the previous inequality with \( \log(1 + x) \leq x \) on \( x \geq 0 \) to get
\[
\int_{\mathbb{T}^2 \times \mathbb{T}^2} K^\varepsilon(x - y) \varphi(x) \varphi(y) \, dx \, dy
\]
\[
= -\frac{c_1}{2} \int_{\mathbb{T}^2 \times \mathbb{T}^2} \left( \log(\varepsilon + |x - y|^2) + G_1(x - y) \right) \varphi(x) \varphi(y) \, dx \, dy
\]
\[
\geq -\frac{c_1}{2} \int_{\mathbb{T}^2 \times \mathbb{T}^2} \left[ \log (2(1 + \varepsilon)) + \log(1 + |x|^2) + \log(1 + |y|^2) + G_1(x - y) \right] \varphi(x) \varphi(y) \, dx \, dy
\]
\[
\geq -\frac{c_1}{2} \left( \log (2(1 + \varepsilon)) + \|G_1\|_{L^\infty} \|\varphi\|_{L^\infty(0, T; L^1)}^2 \right)
\]
\[
- \frac{c_1}{2} \left( \log(1 + |x|^2) + \log(1 + |y|^2) \right) \varphi(x) \varphi(y) \, dx \, dy
\]
\[
\geq -\frac{c_1}{2} \left( \log (2(1 + \varepsilon)) + \|G_1\|_{L^\infty} \|\varphi\|_{L^\infty(0, T; L^1)}^2 - c_1 \|\varphi\|_{L^\infty(0, T; L^1)} \int_{\mathbb{T}^2} \varphi \log(1 + |x|^2) \, dx \right)
\]
\[
\geq -\frac{c_1}{2} \left( \log (2(1 + \varepsilon)) + \|G_1\|_{L^\infty} \|\varphi\|_{L^\infty(0, T; L^1)}^2 - c_1 \|\varphi\|_{L^\infty(0, T; L^1)} \int_{\mathbb{T}^2} |x|^2 \varphi \, dx \right).
\]
This completes the proof.
A.2. Proof of Lemma 2.7

Since the proof for \( d = 3 \) is analogous, we only consider the case \( d = 2 \). First, we recall a partial result from [10, Theorem 5.2].

**Lemma A.1.** A sequence \( \{h_n\} \) in \( L^1(\mathbb{T}^d) \) converges to \( h \in L^1(\mathbb{T}^d) \) in \( L^1(\mathbb{T}^d) \) if the following three conditions hold:

(i) \( h_n \) converges to \( h \) almost everywhere.

(ii) for every \( \varepsilon > 0 \), there exists \( \delta = \delta(\varepsilon) > 0 \) such that whenever \( m(E) < \delta \),

\[
\sup_{n \in \mathbb{N}} \int_E |h_n| \, dx < \varepsilon.
\]

Without loss of generality, we may assume

\[
\sup_{n \in \mathbb{N}} \|q^n\|_{L^\infty(0,T;L^1(\mathbb{T}^2))} \leq 1.
\]

First, we show the pointwise convergence of \( K^\varepsilon \ast q^n \) to \( K^\varepsilon \ast q \). Since \( q^n \) converges to \( q \) pointwisely, \( K^\varepsilon(x - \cdot)q^n(\cdot) \) also pointwisely converges to \( K^\varepsilon(x - \cdot)q(\cdot) \) for each \( x \). When \( d = 2 \), we use the fact

\[
|\log x| \leq \max \{x, x^{-1}\} \quad \forall x > 0,
\]

and choose a measurable set \( E \) and \( p \in (1,2) \) to obtain

\[
\int_E K^\varepsilon(x - y)q^n(y) \, dy
\]

\[
\leq 2c_1 \int_E \max \{(\varepsilon + |x - y|^2)^{1/4},(\varepsilon + |x - y|^2)^{-1/4}\} q^n(y) \, dy + \int_E G_1(x - y)q^n(y) \, dy
\]

\[
\leq C \int_E \max \{(\varepsilon + |x - y|^2)^{1/4},(\varepsilon + |x - y|^2)^{-1/4}\} q^n(y) \, dy + \int_E (\varepsilon + |x - y|^2)^{-1/4} q^n(y) \, dy
\]

\[
\leq C \left( |x| + \sqrt{\varepsilon} \right) m(E)^{1/(2p')} + \|\cdot|^{-1/2}1_{\{|\cdot| \leq 1\}}\|_{L^{7/2}}\|q^n\|_{L^{7/4}} m(E)^{1/7} + \|G_1\|_{L^\infty} \|q^n\|_{L^p} m(E)^{1/p'},
\]

where \( p' \) is the Hölder conjugate of \( p \). This guarantees the condition (ii) in Lemma A.1. Thus, Lemma A.1 gives the pointwise convergence of \( K^\varepsilon \ast q^n \) to \( K^\varepsilon \ast q \), and subsequently this asserts \( (K^\varepsilon \ast q)_\varepsilon \) almost everywhere. To prove the desired \( L^1 \)-convergence, we notice from the uniform boundedness of \( q^n \) in \( L^\infty(0,T;L^p(\mathbb{T}^2)) \) for \( p \in [1,2] \) that

\[
|K^\varepsilon \ast q^n| \leq C(1 + |x|),
\]

where \( C \) is a constant independent of \( n \) and \( C = \mathcal{O}(1) \) as \( \varepsilon \to 0 \). Indeed, we have

\[
\int_{\mathbb{T}^2} K^\varepsilon(x - y)q^n(y) \, dy
\]

\[
\leq 2c_1 \int_{\mathbb{T}^2} \max \{(\varepsilon + |x - y|^2)^{1/4},(\varepsilon + |x - y|^2)^{-1/4}\} q^n(y) \, dy + \int_{\mathbb{T}^2} G_1(x - y)q^n(y) \, dy
\]

\[
\leq C \int_{\{y \in \mathbb{T}^2 : |x - y| \geq 1 - \varepsilon\}} \sqrt{\varepsilon + |x - y|^2} q^n(y) \, dy + C \int_{\{y \in \mathbb{T}^2 : |x - y| < 1 - \varepsilon\}} (\varepsilon + |x - y|^2)^{-1/4} q^n(y) \, dy
\]

\[
\leq C(1 + \sqrt{\varepsilon} + |x|) + \int_{\{y \in \mathbb{T}^2 : |x - y| < 1\}} |x - y|^{-1/2} q^n(y) \, dy
\]

\[
\leq C(1 + |x|) + \|\cdot|^{-1/2}1_{\{|\cdot| \leq 1\}}\|_{L^{7/2}}\|q^n\|_{L^{7/5}}
\]

\[
\leq C(1 + |x|).
\]

Thus, for a measurable set \( E \subset \mathbb{T}^2 \) we use the above estimate to show

\[
\int_E (K^\varepsilon \ast q^n) \, dx \leq C \int_E (1 + |x|) q^n \, dx \leq C\|q^n\|_{L^p} m(E)^{1/p'}.
\]
where \( p \in (1, 2) \). This together with Lemma A.1 yields
\[
\lim_{n \to \infty} \int_E (K^\varepsilon \ast \varrho^n) \varrho^n \, dx = \int_E (K^\varepsilon \ast \varrho) \varrho \, dx.
\]
Since the estimates can be made independently of \( \varepsilon \), the similar analysis also gives
\[
\lim_{n \to \infty} \int_E (K^\varepsilon \ast \varrho^n) \varrho^n \, dx = \int_E (K \ast \varrho) \varrho \, dx,
\]
if \( \varepsilon \to 0 \) as \( n \to 0 \). This concludes the proof of Lemma 2.7.

**Appendix B. Convergence of \( f^\varepsilon \) towards \( M_{\rho,u} \) in \( L^\infty(0, T; L^1(\mathbb{T}^d \times \mathbb{R}^d)) \)**

In this appendix, we provide the details of proof for Proposition 5.2. Let us first recall the relative entropy:
\[
\mathcal{H}(f^\varepsilon | M_{\rho,u}) = f^\varepsilon (\log f^\varepsilon - \log M_{\rho,u}) + (f^\varepsilon - M_{\rho,u}).
\]
In the rest, we separately estimate the terms on the right hand side of the above equality.

A direct computation gives
\[
\frac{d}{dt} \int_{\mathbb{T}^d \times \mathbb{R}^d} f^\varepsilon \log f^\varepsilon \, dx d\xi = \int_{\mathbb{T}^d \times \mathbb{R}^d} \nabla f^\varepsilon \cdot (\varepsilon - \xi - \nabla K \ast (\rho^\varepsilon - 1)) \, dx d\xi - \frac{1}{\varepsilon} \int_{\mathbb{T}^d \times \mathbb{R}^d} \nabla f^\varepsilon \cdot (\nabla f^\varepsilon - (u^\varepsilon - \xi) f^\varepsilon) \, dx d\xi
\]
\[
= \int_{\mathbb{T}^d \times \mathbb{R}^d} \nabla f^\varepsilon \cdot (\varepsilon - \xi) \, dx d\xi - \frac{1}{\varepsilon} \int_{\mathbb{T}^d \times \mathbb{R}^d} \frac{1}{2} |\nabla f^\varepsilon - (u^\varepsilon - \xi) f^\varepsilon|^2 \, dx d\xi
\]
\[
- \frac{1}{\varepsilon} \int_{\mathbb{T}^d \times \mathbb{R}^d} (u - \xi) \cdot (\nabla f^\varepsilon - (u^\varepsilon - \xi) f^\varepsilon) \, dx d\xi,
\]
where we used
\[
\int_{\mathbb{T}^d \times \mathbb{R}^d} (u^\varepsilon - u) \cdot (\nabla f^\varepsilon - (u^\varepsilon - \xi) f^\varepsilon) \, dx d\xi = 0
\]
for the estimate of the last term on the right hand side of the above. We next use the continuity equations of \( \rho^\varepsilon \) and \( \rho \) to obtain
\[
- \frac{d}{dt} \int_{\mathbb{T}^d} \rho^\varepsilon \log \rho = - \int_{\mathbb{T}^d} \rho^\varepsilon (u^\varepsilon - u) \cdot \frac{\nabla \rho}{\rho} \, dx - \int_{\mathbb{T}^d} u \cdot \nabla \rho^\varepsilon \, dx.
\]
We next estimate
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^d \times \mathbb{R}^d} |u - \xi|^2 f^\varepsilon \, dx d\xi = \int_{\mathbb{T}^d \times \mathbb{R}^d} (u - \xi) f^\varepsilon \cdot \partial_t u \, dx d\xi + \frac{1}{2} \int_{\mathbb{T}^d \times \mathbb{R}^d} |u - \xi|^2 \partial_t f^\varepsilon \, dx d\xi
\]
\[
= : \mathcal{L}_1 + \mathcal{L}_2,
\]
where we use the momentum equation in (1.4) to get
\[
\mathcal{L}_1 = \int_{\mathbb{T}^d} (u - u^\varepsilon) \rho^\varepsilon \cdot \partial_t u \, dx
\]
\[
= - \int_{\mathbb{T}^d} (u - u^\varepsilon) \rho^\varepsilon \cdot (u \cdot \nabla u + (u - v)) \, dx - \int_{\mathbb{T}^d} \frac{\nabla \rho}{\rho} \cdot (u - u^\varepsilon) \rho^\varepsilon \, dx - \int_{\mathbb{T}^d} \nabla K \ast (\rho^\varepsilon - 1) \cdot (u - u^\varepsilon) \rho^\varepsilon \, dx
\]
\[
\leq C \left( \int_{\mathbb{T}^d} \rho^\varepsilon |u - u^\varepsilon|^2 \, dx \right)^{1/2} - \int_{\mathbb{T}^d} \frac{\nabla \rho}{\rho} \cdot (u - u^\varepsilon) \rho^\varepsilon \, dx - \int_{\mathbb{T}^d} \nabla K \ast (\rho^\varepsilon - 1) \cdot (u - u^\varepsilon) \rho^\varepsilon \, dx.
\]
Here \( C > 0 \) is independent of \( \varepsilon > 0 \). For the estimate of \( \mathcal{L}_2 \), we use the kinetic equation in (1.3) to find
\[
\mathcal{L}_2 = \int_{\mathbb{T}^d \times \mathbb{R}^d} \xi f^\varepsilon \otimes (u - \xi) \cdot \nabla u \, dx d\xi - \int_{\mathbb{T}^d \times \mathbb{R}^d} (v^\varepsilon - \xi) \cdot (u - \xi) f^\varepsilon \, dx d\xi
\]
\[
+ \int_{\mathbb{T}^d} \nabla K \ast (\rho^\varepsilon - 1) \cdot (u - u^\varepsilon) \rho^\varepsilon \, dx + \frac{1}{\varepsilon} \int_{\mathbb{T}^d \times \mathbb{R}^d} (u - \xi) \cdot (\nabla f^\varepsilon - (u^\varepsilon - \xi) f^\varepsilon) \, dx d\xi.
\]
On the other hand, the first term on the right hand side can be estimated as

\[
\int_{\mathbb{T}^d \times \mathbb{R}^d} \xi f^\varepsilon \otimes (u - \xi) \cdot \nabla u \, dx \, d\xi
\]

\[
= - \int_{\mathbb{T}^d \times \mathbb{R}^d} (u - \xi) f^\varepsilon \otimes (u - \xi) \cdot \nabla u \, dx \, d\xi + \int_{\mathbb{T}^d \times \mathbb{R}^d} u \otimes (u - \xi) f^\varepsilon \cdot \nabla u \, dx \, d\xi
\]

\[
= - \int_{\mathbb{T}^d \times \mathbb{R}^d} ((u - u^\varepsilon) \otimes (u - u^\varepsilon) + (u^\varepsilon - \xi) \otimes (u^\varepsilon - \xi)) f^\varepsilon \cdot \nabla u \, dx \, d\xi
\]

\[
+ \int_{\mathbb{T}^d \times \mathbb{R}^d} u \otimes (u - \xi) f^\varepsilon \cdot \nabla u \, dx \, d\xi
\]

\[
\leq \|\nabla u\|_{L^\infty} \int_{\mathbb{T}^d} \rho^\varepsilon |u - u^\varepsilon|^2 \, dx + \|u\|_{L^\infty} \|\nabla u\|_{L^\infty} \int_{\mathbb{T}^d} \rho^\varepsilon |u - u^\varepsilon| \, dx
\]

\[
- \int_{\mathbb{T}^d \times \mathbb{R}^d} ((u^\varepsilon - \xi) \sqrt{f^\varepsilon} - 2\nabla \xi \sqrt{f^\varepsilon}) \otimes (u^\varepsilon - \xi) \sqrt{f^\varepsilon} \cdot \nabla u \, dx \, d\xi
\]

\[
- 2 \int_{\mathbb{T}^d \times \mathbb{R}^d} \nabla \xi \sqrt{f^\varepsilon} \otimes (u^\varepsilon - \xi) \sqrt{f^\varepsilon} \cdot \nabla u \, dx \, d\xi
\]

\[
\leq C \int_{\mathbb{T}^d} \rho^\varepsilon |u - u^\varepsilon|^2 \, dx + C \int_{\mathbb{T}^d} \rho^\varepsilon |u - u^\varepsilon| \, dx - \int_{\mathbb{T}^d \times \mathbb{R}^d} \nabla \xi f^\varepsilon \otimes (u^\varepsilon - \xi) \cdot \nabla u \, dx \, d\xi
\]

\[
+ C \left( \int_{\mathbb{T}^d \times \mathbb{R}^d} |u^\varepsilon - \xi|^2 f^\varepsilon \, dx \, d\xi \right)^{1/2} \left( \int_{\mathbb{T}^d \times \mathbb{R}^d} \frac{1}{\rho^\varepsilon} |\nabla \xi f^\varepsilon - (u^\varepsilon - \xi) f^\varepsilon|^2 \, dx \, d\xi \right)^{1/2}
\]

\[
\leq C \int_{\mathbb{T}^d} \rho^\varepsilon |u - u^\varepsilon|^2 \, dx + C \left( \int_{\mathbb{T}^d} \rho^\varepsilon |u - u^\varepsilon|^2 \, dx \right)^{1/2} + \int_{\mathbb{T}^d} \nabla \rho^\varepsilon \cdot u \, dx + C \varepsilon \int_{\mathbb{T}^d \times \mathbb{R}^d} |\xi|^2 f^\varepsilon \, dx \, d\xi
\]

\[
+ \frac{1}{4\varepsilon} \int_{\mathbb{T}^d \times \mathbb{R}^d} \frac{1}{f^\varepsilon} |\nabla \xi f^\varepsilon - (u^\varepsilon - \xi) f^\varepsilon|^2 \, dx \, d\xi,
\]

where we used

\[
\int_{\mathbb{T}^d \times \mathbb{R}^d} |u^\varepsilon - \xi|^2 f^\varepsilon \, dx \, d\xi \leq \int_{\mathbb{T}^d \times \mathbb{R}^d} |\xi|^2 f^\varepsilon \, dx \, d\xi - \int_{\mathbb{T}^d} \rho^\varepsilon |u|^2 \, dx \leq \int_{\mathbb{T}^d \times \mathbb{R}^d} |\xi|^2 f^\varepsilon \, dx \, d\xi.
\]

Combining these estimates implies

\[
\frac{d}{dt} \int_{\mathbb{T}^d \times \mathbb{R}^d} |u - \xi|^2 f^\varepsilon \, dx \, d\xi
\]

\[
\leq C \left( \min \left\{ 1, \int_{\mathbb{T}^d} \rho^\varepsilon |u - u^\varepsilon|^2 \, dx \right\} \right)^{1/2} - \int_{\mathbb{T}^d} \nabla \rho^\varepsilon \cdot (u - u^\varepsilon) \rho^\varepsilon \, dx + \int_{\mathbb{T}^d} \nabla K \ast (\rho^\varepsilon - \rho) \cdot (u - u^\varepsilon) \rho^\varepsilon \, dx
\]

\[
- \int_{\mathbb{T}^d \times \mathbb{R}^d} (\dot{v}^\varepsilon - \xi) \cdot (u - \xi) f^\varepsilon \, dx \, d\xi + \frac{1}{\varepsilon} \int_{\mathbb{T}^d \times \mathbb{R}^d} (u - \xi) \cdot (\nabla \xi f^\varepsilon - (u^\varepsilon - \xi) f^\varepsilon) \, dx \, d\xi
\]

\[
+ C \varepsilon \int_{\mathbb{T}^d \times \mathbb{R}^d} |\xi|^2 f^\varepsilon \, dx \, d\xi + \frac{1}{4\varepsilon} \int_{\mathbb{T}^d \times \mathbb{R}^d} \frac{1}{f^\varepsilon} |\nabla \xi f^\varepsilon - (u^\varepsilon - \xi) f^\varepsilon|^2 \, dx \, d\xi.
\]

We now combine this with (B.1) and (B.2) to have

\[
\frac{d}{dt} \int_{\mathbb{T}^d \times \mathbb{R}^d} \mathcal{H}(f^\varepsilon | M_{\rho, u}) \, dx \, d\xi + \frac{1}{\varepsilon} \int_{\mathbb{T}^d \times \mathbb{R}^d} \frac{1}{f^\varepsilon} |\nabla \xi f^\varepsilon - (u^\varepsilon - \xi) f^\varepsilon|^2 \, dx \, d\xi
\]

\[
\leq C \left( \min \left\{ 1, \int_{\mathbb{T}^d} \rho^\varepsilon |u - u^\varepsilon|^2 \, dx \right\} \right)^{1/2} + \int_{\mathbb{T}^d} \nabla K \ast (\rho^\varepsilon - \rho) \cdot (u - u^\varepsilon) \rho^\varepsilon \, dx
\]

\[
+ C \varepsilon \int_{\mathbb{T}^d \times \mathbb{R}^d} |\xi|^2 f^\varepsilon \, dx \, d\xi + \frac{1}{4\varepsilon} \int_{\mathbb{T}^d \times \mathbb{R}^d} \frac{1}{f^\varepsilon} |\nabla \xi f^\varepsilon - (u^\varepsilon - \xi) f^\varepsilon|^2 \, dx \, d\xi
\]

\[
+ \int_{\mathbb{T}^d \times \mathbb{R}^d} \nabla \xi f^\varepsilon \cdot (\dot{v}^\varepsilon - \xi) \, dx \, d\xi - \int_{\mathbb{T}^d \times \mathbb{R}^d} (\dot{v}^\varepsilon - \xi) \cdot (u - \xi) f^\varepsilon \, dx \, d\xi.
\]
We further estimate the last two terms on the right hand side as

$$\int_{T^d \times \mathbb{R}^d} \nabla \xi f^\varepsilon \cdot (v^\varepsilon - \xi) \, dx \, d\xi - \int_{T^d \times \mathbb{R}^d} (v^\varepsilon - \xi) \cdot (u - \xi) f^\varepsilon \, dx \, d\xi$$

$$= \int_{T^d \times \mathbb{R}^d} (\nabla \xi f^\varepsilon - (u^\varepsilon - \xi) f^\varepsilon) \cdot (v^\varepsilon - \xi) \, dx \, d\xi + \int_{T^d \times \mathbb{R}^d} (v^\varepsilon - \xi) \cdot (u^\varepsilon - u) f^\varepsilon \, dx \, d\xi$$

$$\leq \left( \int_{T^d \times \mathbb{R}^d} \frac{1}{f^\varepsilon} |\nabla \xi f^\varepsilon - (u^\varepsilon - \xi) f^\varepsilon|^2 \, dx \, d\xi \right)^{1/2} \left( \int_{T^d \times \mathbb{R}^d} |v^\varepsilon - \xi|^2 f^\varepsilon \, dx \, d\xi \right)^{1/2}$$

$$+ \left( \int_{T^d} \left| v^\varepsilon - u^\varepsilon \right|^2 \rho^\varepsilon \, dx \right)^{1/2} \left( \int_{T^d} |u^\varepsilon - u|^2 \rho^\varepsilon \, dx \right)^{1/2}$$

$$\leq C \varepsilon \int_{T^d \times \mathbb{R}^d} |v^\varepsilon - \xi|^2 f^\varepsilon \, dx \, d\xi + \frac{C}{4 \varepsilon} \int_{T^d \times \mathbb{R}^d} \frac{1}{f^\varepsilon} |\nabla \xi f^\varepsilon - (u^\varepsilon - \xi) f^\varepsilon|^2 \, dx \, d\xi$$

$$+ \frac{C}{\varepsilon^{1/4}} \int_{T^d \times \mathbb{R}^d} |v^\varepsilon - \xi|^2 f^\varepsilon \, dx \, d\xi + \frac{1}{\varepsilon^{1/4}} \int_{T^d} |u^\varepsilon - u|^2 \rho^\varepsilon \, dx.$$

Here we used

$$\int_{T^d} |v^\varepsilon - u|^2 \rho \, dx \leq \int_{T^d \times \mathbb{R}^d} |v^\varepsilon - \xi|^2 f^\varepsilon \, dx \, d\xi.$$

For the second term on the right hand side of (15.3), we recall the estimates in the proof of Proposition 2.1

$$\frac{1}{2} \frac{d}{dt} \int_{T^d} |\nabla K * (\rho^\varepsilon - \rho)|^2 \, dx = \int_{T^d} \nabla K * (\rho^\varepsilon - \rho) \cdot (\rho^\varepsilon \, u^\varepsilon - \rho \, u) \, dx$$

$$= \int_{T^d} \nabla K * (\rho^\varepsilon - \rho) \cdot (u^\varepsilon - u) \rho^\varepsilon \, dx + \int_{T^d} \nabla K * (\rho^\varepsilon - \rho) \cdot u(\rho^\varepsilon - \rho) \, dx$$

and

$$\int_{T^d} \nabla K * (\rho^\varepsilon - \rho) \cdot u(\rho^\varepsilon - \rho) \, dx$$

$$= -\frac{1}{2} \int_{T^d} |\nabla K * (\rho^\varepsilon - \rho)|^2 \nabla \cdot u \, dx + \int_{T^d} \nabla K * (\rho^\varepsilon - \rho) \otimes \nabla K * (\rho^\varepsilon - \rho) : \nabla u \, dx$$

$$\leq C \int_{T^d} |\nabla K * (\rho^\varepsilon - \rho)|^2 \, dx,$$

where $C > 0$ is independent of $\varepsilon > 0$. Hence we have

$$\frac{d}{dt} \left( \int_{T^d \times \mathbb{R}^d} \mathcal{H}(f^\varepsilon | M_{\rho,u}) \, dx \, d\xi + \frac{1}{2} \int_{T^d} |\nabla K * (\rho^\varepsilon - \rho)|^2 \, dx \right) + \frac{1}{2 \varepsilon} \int_{T^d \times \mathbb{R}^d} \frac{1}{f^\varepsilon} |\nabla \xi f^\varepsilon - (u^\varepsilon - \xi) f^\varepsilon|^2 \, dx \, d\xi$$

$$\leq C \left( \min \left\{ 1, \left( \int_{T^d} \rho^\varepsilon |u - u^\varepsilon|^2 \, dx \right) \right\} \right)^{1/2} + \frac{1}{\varepsilon^{1/4}} \int_{T^d} |u^\varepsilon - u|^2 \rho^\varepsilon \, dx + C \int_{T^d} |\nabla K * (\rho^\varepsilon - \rho)|^2 \, dx$$

$$+ C \varepsilon \int_{T^d \times \mathbb{R}^d} |\xi|^2 f^\varepsilon \, dx \, d\xi + C \varepsilon \int_{T^d \times \mathbb{R}^d} |v^\varepsilon - \xi|^2 f^\varepsilon \, dx \, d\xi + C \varepsilon^{1/4} \int_{T^d \times \mathbb{R}^d} |v^\varepsilon - \xi|^2 f^\varepsilon \, dx \, d\xi.$$

This completes the proof.

**APPENDIX C. LOCAL SOLVABILITY FOR THE EPNS SYSTEM**

In this appendix, we present the details of the proof for Theorem 4.1

C.1. **Solvability for the linearized system.** First, we linearize the system (12.2) and investigate its local-in-time estimates. To be specific, for a given triplet

$$(\tilde{g}, \tilde{u}, \tilde{v}) \in C([0,T]; H^s(T^d)) \times C([0,T]; H^s(T^d)) \times C([0,T]; H^s(T^d)) \quad \text{with} \quad \nabla \cdot \tilde{v} = 0,$$
we consider the following linearized system:

\begin{align*}
\partial_t g + \tilde{u} \cdot \nabla g + \nabla \cdot u &= 0, \quad (x, t) \in \mathbb{T}^d \times \mathbb{R}_+,
\partial_t u + (\tilde{u} \cdot \nabla) u + \nabla g &= -(u - \tilde{v} + \nabla K \ast (e^\tilde{g} - 1)),
\partial_t v + (\tilde{v} \cdot \nabla) v + \nabla p - \Delta v &= e^{\tilde{g}}(\tilde{u} - v),
\nabla \cdot v &= 0,
\end{align*}

subject to the initial data \((g_0, u_0, v_0) \in H^s(\mathbb{T}^d) \times H^s(\mathbb{T}^d) \times H^s(\mathbb{T}^d)\).

**Lemma C.1.** Let \(T > 0\) and \(s > d/2 + 1\). For any positive constants \(N < M\), if

\begin{align*}
\|g_0\|_{H^s}^2 + \|u_0\|_{H^s}^2 + \|v_0\|_{H^s}^2 < N, \quad (C.2)
\end{align*}

and

\begin{align*}
\sup_{0 \leq t \leq T} \left( \|\tilde{g}(\cdot, t)\|_{H^s}^2 + \|\tilde{u}(\cdot, t)\|_{H^s}^2 + \|\tilde{v}(\cdot, t)\|_{H^s}^2 \right) < M,
\end{align*}

then the system \((C.1)\) admits a unique classical solution

\begin{align*}
(g, u, v) \in C([0, T]; H^s(\mathbb{T}^d)) \times C([0, T]; H^s(\mathbb{T}^d)) \times C([0, T]; H^s(\mathbb{T}^d))
\end{align*}

satisfying

\begin{align*}
\sup_{0 \leq t \leq T^*} \left( \|g(\cdot, t)\|_{H^s}^2 + \|u(\cdot, t)\|_{H^s}^2 + \|v(\cdot, t)\|_{H^s}^2 \right) < M
\end{align*}

for some \(T^* \leq T\).

**Proof.** A standard theory of linear PDEs would assure the existence and uniqueness of solutions to the system \((C.1)\). Thus, it suffices to prove bound estimates for \(g\), \(u\), and \(v\). A straightforward computation gives

\begin{align*}
\frac{1}{2} \frac{d}{dt} \|g\|_{L^2}^2 &\leq \frac{\|\nabla \cdot \tilde{u}\|_{L^\infty}}{2} \|g\|_{L^2}^2 - \int_{\mathbb{T}^d} g \nabla \cdot u \, dx
\end{align*}

and

\begin{align*}
\frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 &\leq \frac{\|\nabla \cdot \tilde{u}\|_{L^\infty}}{2} \|u\|_{L^2}^2 - \int_{\mathbb{T}^d} \nabla g \cdot u \, dx + \|\nabla K \ast (e^{\tilde{g}} - 1)\|_{L^2} \|u\|_{L^2}^2 + \|u\|_{L^2} \|\tilde{v}\|_{L^2}
\end{align*}

\begin{align*}
&\leq \frac{\|\nabla \cdot \tilde{u}\|_{L^\infty}}{2} \|u\|_{L^2}^2 - \int_{\mathbb{T}^d} \nabla g \cdot u \, dx + \|\nabla K\|_{L^1} \|\tilde{g}\|_{L^\infty} \|\tilde{g}\|_{L^2} \|u\|_{L^2}^2 + \|u\|_{L^2} \|\tilde{v}\|_{L^2}.
\end{align*}

For \(v\), we have

\begin{align*}
\frac{1}{2} \frac{d}{dt} \|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 &\leq e^{\|\tilde{g}\|_{L^\infty}} \|\tilde{u}\|_{L^2} \|v\|_{L^2}.
\end{align*}

Then we use Sobolev inequality and Young’s inequality to get

\begin{align*}
\frac{d}{dt} \left( \|g\|_{L^2}^2 + \|u\|_{L^2}^2 + \|v\|_{L^2}^2 \right) &\leq C e^{CM} (1 + M) \left( \|g\|_{L^2}^2 + \|u\|_{L^2}^2 + \|v\|_{L^2}^2 \right) + C e^{CM} (1 + M), \quad (C.3)
\end{align*}

where \(C > 0\) only depends on \(s\), \(d\), and \(\|\nabla K\|_{L^1}\).

For \(1 \leq k \leq s\), we first estimate \(\nabla^k g\) as follows:

\begin{align*}
\frac{1}{2} \frac{d}{dt} \|\nabla^k g\|_{L^2}^2 &= - \int_{\mathbb{T}^d} \nabla(\nabla^k g) \cdot \tilde{u} \, dx - \int_{\mathbb{T}^d} (\nabla^k (\nabla g) - \nabla(\nabla^k g) \cdot \tilde{u}) \, dx
\end{align*}

\begin{align*}
&\leq \int_{\mathbb{T}^d} \left( \|\nabla \cdot (\nabla^k u)\|_{L^2} \|\nabla^k g\|_{L^2} + C \|\nabla^k g\|_{L^2} \left( \|\nabla \tilde{u}\|_{L^\infty} \|\nabla(\nabla^{k-1} g)\|_{L^2} + \|\nabla g\|_{L^\infty} \|\tilde{u}\|_{L^2} \right) \right)
\end{align*}

\begin{align*}
&\leq CM \|\nabla^k g\|_{L^2} + CM \|\nabla^k g\|_{L^2} \|g\|_{H^s} \int_{\mathbb{T}^d} (\nabla \cdot (\nabla^k u)) \|\nabla^k g\| \, dx,
\end{align*}
where $C > 0$ only depends on $s$ and $d$. Similarly, $\nabla^k u$ can be estimated as

$$
\frac{1}{2} \frac{d}{dt} \|\nabla^k u\|^2_{L^2} = -\int_{\mathbb{T}^d} (\ddot{u} \cdot \nabla(\nabla^k u)) \cdot \nabla^k u \, dx - \int_{\mathbb{T}^d} (\nabla^k (\ddot{u} \cdot \nabla u) - \ddot{u} \cdot \nabla(\nabla^k u)) \cdot \nabla^k u \, dx \\
- \int_{\mathbb{T}^d} \nabla(\nabla^k \dot{g}) \cdot \nabla^k u \, dx - \int_{\mathbb{T}^d} \nabla_k (\nabla K \ast (e^{\vec{g}} - 1)) \cdot \nabla^k u \, dx - \int_{\mathbb{T}^d} \nabla_k (u - \ddot{v}) \cdot \nabla^k u \, dx \\
\leq \frac{1}{2} \|\nabla \cdot \ddot{u}\|_{L^\infty} \|\nabla^k u\|_{L^2} + C \|\nabla^k u\|_{L^2} \left\|\nabla(\nabla^{k-1} u)\right\|_{L^2} + \|\nabla u\|_{L^\infty} \|\nabla^k \dot{u}\|_{L^2} \\
+ \|\nabla K \ast (\nabla^k e^{\vec{g}})\|_{L^2} \|\nabla^k u\|_{L^2} - \int_{\mathbb{T}^d} \nabla(\nabla^k \dot{g}) \cdot \nabla^k u \, dx + \|\nabla^k u\|_{L^2} \|\nabla v\|_{L^2} \\
\leq CM \|\nabla^k u\|^2_{L^2} + CM \|\nabla^k u\|_{L^2} \|u\|_{H^s} + C e^{CM} (1 + M) \|\nabla^k u\|_{L^2} \\
+ C \|\nabla^k (e^{\vec{g}})\|_{L^2} \|\nabla^k u\|_{L^2} - \int_{\mathbb{T}^d} \nabla(\nabla^k \dot{g}) \cdot \nabla^k u \, dx.
$$

Here $C > 0$ only depends on $d$, $s$, and $\|\nabla K\|_{L^1}$. For the estimate of the Poisson interaction term, we follow the same argument in [10] as follows: we let $a_k := \|\nabla^k (e^{\vec{g}})\|_{L^2}$. Obviously, we have $a_0 \leq e^{\|\vec{g}\|_{L^\infty}} \leq C e^{CM}$.

Then, the usage of Lemma [11] and Sobolev inequality yields

$$
a_k = \|\nabla^{k-1} (e^{\vec{g}} \nabla \vec{g})\|_{L^2} \\
\leq \|e^{\vec{g}} \nabla^k \vec{g}\|_{L^2} + \|\nabla^{k-1} (e^{\vec{g}} \nabla \vec{g})\|_{L^2} \\
\leq M C e^{CM} + C (\|\nabla e^{\vec{g}}\|_{L^\infty} \|\nabla^{k-1} \vec{g}\|_{L^2} + \|\nabla^{k-1} (e^{\vec{g}})\|_{L^2} \|\nabla \vec{g}\|_{L^\infty}) \\
\leq C M a_{k-1} + C M e^{CM} (1 + M),
$$

where $C > 0$ only depends on $d$ and $k$, and inductively, we get

$$
a_k \leq C M^k a_0 + C M^{k-1} e^{CM} (1 + M) \leq C e^{CM}.
$$

Here $C > 0$ only depends on $d$ and $k$ and this gives

$$
\frac{d}{dt} \|\nabla^k u\|^2_{L^2} \leq CM \|\nabla^k u\|^2_{L^2} + CM \|\nabla^k u\|_{L^2} \|u\|_{H^s} + C e^{CM} (1 + M) \|\nabla^k u\|_{L^2} - \int_{\mathbb{T}^d} \nabla(\nabla^k \dot{g}) \cdot \nabla^k u \, dx.
$$

Finally, we estimate $\nabla^k v$ as

$$
\frac{1}{2} \frac{d}{dt} \|\nabla^k v\|^2_{L^2} = -\int_{\mathbb{T}^d} (\ddot{v} \cdot \nabla(\nabla^k v)) \cdot \nabla^k v \, dx - \int_{\mathbb{T}^d} (\nabla^k (\ddot{v} \cdot \nabla v) - \ddot{v} \cdot \nabla(\nabla^k v)) \cdot \nabla^k v \\
- \int_{\mathbb{T}^d} \|\nabla(\nabla^k \dot{v})\|^2 \, dx - \int_{\mathbb{T}^d} e^{\vec{g}} \nabla^k (v - \ddot{u}) \cdot \nabla^k v \, dx \\
- \int_{\mathbb{T}^d} \|\nabla^k (e^{\vec{g}} (v - \ddot{u})) - e^{\vec{g}} \nabla^k (v - \ddot{u})\| \cdot \nabla^k v \, dx \\
\leq C \|\nabla^k v\|_{L^2} \|\nabla v\|_{L^\infty} \|\nabla(\nabla^{k-1} v)\|_{L^2} + \|\nabla v\|_{L^\infty} \|\nabla^k \dot{v}\|_{L^2} + e^{\|\vec{g}\|_{L^\infty}} \|\nabla^k v\|_{L^2} \|\nabla^k \ddot{u}\|_{L^2} \\
+ C \|\nabla^k v\|_{L^2} \left(\|e^{\vec{g}} \nabla \vec{g}\|_{L^\infty} \|\nabla^{k-1} (v - \ddot{u})\|_{L^2} + \|\nabla^k (e^{\vec{g}})\|_{L^2} \|\ddot{u} - v\|_{L^\infty}\right) \\
\leq C M \|\nabla^k v\|^2_{L^2} + C e^{CM} (1 + M) \|\nabla^k v\|_{L^2} (1 + \|v\|_{H^s}).
$$

Thus, we combine the estimates for $\nabla^k g$, $\nabla^k u$ and $\nabla^k v$ to obtain

$$
\frac{d}{dt} \left(\|\nabla^k g\|^2_{L^2} + \|\nabla^k u\|^2_{L^2} + \|\nabla^k v\|^2_{L^2}\right) \\
\leq CM \left(\|\nabla^k g\|^2_{L^2} + \|\nabla^k u\|^2_{L^2} + \|\nabla^k v\|^2_{L^2}\right) \\
+ C e^{CM} \left(\|\nabla^k g\|_{L^2} \|g\|_{H^s} + \|\nabla^k u\|_{L^2} \|u\|_{H^s} + \|\nabla^k v\|_{L^2} \|v\|_{H^s}\right) \\
+ C e^{CM} (1 + M) (\|\nabla^k u\|_{L^2} + \|\nabla^k v\|_{L^2}).
$$

(C.A)

Now we sum the relation (C.A) over $1 \leq k \leq s$ and combine this with zeroth-order estimate (C.X) to yield

$$
\frac{d}{dt} \left(\|g\|^2_{H^s} + \|u\|^2_{H^s} + \|v\|^2_{H^s}\right) \leq C e^{CM} (1 + M) \left(\|g\|^2_{H^s} + \|u\|^2_{H^s} + \|v\|^2_{H^s}\right) + C e^{CM} (1 + M).
$$
We set \( b(M) := Ce^{CM}(1 + M) \) and use Grönwall’s lemma to obtain
\[
\|g\|_{H^s}^2 + \|u\|_{H^s}^2 + \|v\|_{H^s}^2 \leq (\|g_0\|_{H^s}^2 + \|u_0\|_{H^s}^2 + \|v_0\|_{H^s}^2)e^{b(M)t} + e^{b(M)t} \left(1 - e^{-b(M)t}\right)
\]
\[
\leq Ne^{b(M)t} + e^{b(M)t} \left(1 - e^{-b(M)t}\right) = N + (N + 1) (e^{b(M)t} - 1).
\]
Since \( N < M \) and \( e^{b(M)t} - 1 \) can be arbitrary small if \( t \ll 1 \), we can find \( T^* > 0 \) satisfying
\[
N + (N + 1) (e^{b(M)T^*} - 1) < M.
\]
This asserts the desired result.

\[\Box\]

C.2. Construction of approximate solutions. Based on the estimates for the linearized system (C.1), we construct a sequence of approximate solutions to (1.2) and present the uniform estimates for the sequence. Specifically, we consider a sequence \( (g^n, u^n, v^n) \) which uniquely solves the following system:
\[
\begin{align*}
\partial_t g^{n+1} + \nabla g^{n+1} \cdot w + \nabla \cdot u^{n+1} &= 0, \quad (x, t) \in T^d \times \mathbb{R}_+, \\
\partial_t u^{n+1} + (u^n \cdot \nabla) u^{n+1} + \nabla g^{n+1} &= -(u^{n+1} - u^n + \nabla K \ast (e^{g^n} - 1)), \\
\partial_t v^{n+1} + (v^n \cdot \nabla) v^{n+1} + \nabla p^{n+1} - \Delta v^{n+1} &= e^{g^n}(u^n - v^{n+1}),
\end{align*}
\]
with the initial step and initial data given by
\[
(g^0(x, t), u^0(x, t), v^0(x, t)) = (g_0(x), u_0(x), v_0(x)), \quad (x, t) \in T^d \times \mathbb{R}_+
\]
and
\[
(g^n(x, 0), u^n(x, 0), v^n(x, 0)) = (g_0(x), u_0(x), v_0(x)) \quad \forall n \in \mathbb{N}, \quad x \in T^d,
\]
respectively. Here, Lemma C.1 assures that the sequence \( (g^n, u^n, v^n) \) is well-defined. Furthermore, Lemma C.1 gives the following uniform-in-\( n \) bound estimates to the approximation sequence.

**Corollary C.1.** Let \( s > d/2 + 1 \). For any \( M > N \), there exists \( T^* > 0 \) such that if the initial data \( (g_0, u_0, v_0) \) satisfy (C.2), then for each \( n \in \mathbb{N} \)
\[
(g^n, u^n, v^n) \in \mathcal{C}([0, T^*]; H^s(T^d)) \times \mathcal{C}([0, T^*]; H^s(T^d)) \times \mathcal{C}([0, T^*]; H^s(T^d)),
\]
and
\[
\sup_{0 \leq t \leq T^*} \left(\|g^n(\cdot, t)\|_{H^s}^2 + \|u^n(\cdot, t)\|_{H^s}^2 + \|v^n(\cdot, t)\|_{H^s}^2\right) < M \quad \forall n \in \mathbb{N} \cup \{0\}.
\]

**Proof.** The proof employs the inductive argument. Since the initial step \( (n = 0) \) is trivial, it suffices to check the induction step. First, Lemma C.1 asserts that
\[
Ne^{b(M)T^*} + e^{b(M)T^*} (1 - e^{-b(M)T^*}) < M
\]
for some \( T^* > 0 \). Then, by the induction hypothesis, we get
\[
\sup_{0 \leq t \leq T^*} \left(\|g^n(\cdot, t)\|_{H^s}^2 + \|u^n(\cdot, t)\|_{H^s}^2 + \|v^n(\cdot, t)\|_{H^s}^2\right) < M.
\]
This combined with the same estimates in Lemma C.1 deduces
\[
\|g^{n+1}\|_{H^s}^2 + \|u^{n+1}\|_{H^s}^2 + \|v^{n+1}\|_{H^s}^2 \leq (\|g_0\|_{H^s}^2 + \|u_0\|_{H^s}^2 + \|v_0\|_{H^s}^2)e^{b(M)t} + e^{b(M)t} \left(1 - e^{-b(M)t}\right)
\]
\[
\leq Ne^{b(M)t} + e^{b(M)t} \left(1 - e^{-b(M)t}\right) < M
\]
for \( 0 \leq t \leq T^* \). This concludes the proof. \[\Box\]

Next, we show that the sequence \( (g^n, u^n, v^n) \) is a Cauchy sequence in \( \mathcal{C}([0, T^*]; L^2(T^d)) \times \mathcal{C}([0, T^*]; L^2(T^d)) \times \mathcal{C}([0, T^*]; L^2(T^d)) \).
Lemma C.2. Let \((g^n, u^n, v^n)\) be a sequence of the approximate solutions with the initial data \((g_0, u_0, v_0)\) satisfying (C.2). Then we have

\[
\|(g^{n+1} - g^n)(\cdot, t)||_2^2 + ||(u^{n+1} - u^n)(\cdot, t)||_2^2 + ||(v^{n+1} - v^n)(\cdot, t)||_2^2 \\
\leq C\int_0^t \left( ||(g^{n+1} - g^n)(\cdot, s)||_2^2 + ||(u^{n+1} - u^n)(\cdot, s)||_2^2 + ||(v^{n+1} - v^n)(\cdot, s)||_2^2 \right) ds
\]

for \(0 \leq t \leq T^*\) and \(n \in \mathbb{N}\), where \(C > 0\) is independent of \(n\).

Proof. First, we deduce from (C.4) that

\[
\frac{1}{2} \frac{d}{dt} ||g^{n+1} - g^n||_2^2 \\
= -\int_{\mathbb{T}^d} u^n \cdot \nabla (g^{n+1} - g^n)(g^{n+1} - g^n) dx - \int_{\mathbb{T}^d} (u^n - u^{n-1}) \cdot \nabla g^n (g^{n+1} - g^n) dx \\
- \int_{\mathbb{T}^d} \nabla \cdot (u^{n+1} - u^n)(g^{n+1} - g^n) dx \\
\leq \frac{\|\nabla \cdot u^n\|_{L^\infty}}{2} ||g^{n+1} - g^n||_2^2 + \|\nabla g^n\|_{L^\infty} \|u^n - u^{n-1}\|_2 \|g^{n+1} - g^n\|_{L^2} \\
- \int_{\mathbb{T}^d} \nabla \cdot (u^{n+1} - u^n)(g^{n+1} - g^n) dx \\
\leq C \left( ||g^{n+1} - g^n||_2^2 + \|u^n - u^{n-1}\|_2^2 \right) - \int_{\mathbb{T}^d} \nabla \cdot (u^{n+1} - u^n)(g^{n+1} - g^n) dx.
\]

Next, we estimate \((u^{n+1} - u^n)\) as

\[
\frac{1}{2} \frac{d}{dt} ||u^{n+1} - u^n||_2^2 \\
= -\int_{\mathbb{T}^d} u^n \cdot \nabla (u^{n+1} - u^n) \cdot (u^{n+1} - u^n) dx - \int_{\mathbb{T}^d} (u^n - u^{n-1}) \cdot \nabla u^n \cdot (u^{n+1} - u^n) dx \\
- \int_{\mathbb{T}^d} \nabla (g^{n+1} - g^n) \cdot (u^{n+1} - u^n) dx - \|u^{n+1} - u^n\|_2^2 \\
- \int_{\mathbb{T}^d} \nabla K \ast (e^{g^n} - e^{g^{n-1}}) \cdot (u^{n+1} - u^n) dx + \int_{\mathbb{T}^d} (u^n - v^{n-1}) \cdot (u^{n+1} - u^n) dx \\
\leq \frac{\|\nabla \cdot u^n\|_{L^\infty}}{2} \|u^{n+1} - u^n\|_2^2 + \|\nabla u^n\|_{L^\infty} \|u^{n+1} - u^n\|_2 \|u^n - u^{n-1}\|_2 \\
- \int_{\mathbb{T}^d} \nabla (g^{n+1} - g^n) \cdot (u^{n+1} - u^n) dx + \|\nabla K\|_{L^1} \|e^{g^n} - e^{g^{n-1}}\|_{L^2} \|u^{n+1} - u^n\|_2 \\
+ \|u^n - v^{n-1}\|_2 \|u^{n+1} - u^n\|_2 \\
\leq C \left( \|u^{n+1} - u^n\|_2^2 + \|u^n - u^{n-1}\|_2^2 + \|g^n - g^{n-1}\|_2^2 + \|v^n - v^{n-1}\|_2^2 \right) \\
- \int_{\mathbb{T}^d} \nabla (g^{n+1} - g^n) \cdot (u^{n+1} - u^n) dx,
\]

where we used the mean value theorem to obtain

\[
\|e^{g^n} - e^{g^{n-1}}\|_{L^2} \leq \exp \left( \max \{ \|g^n\|_{L^\infty}, \|g^{n-1}\|_{L^\infty} \} \right) \|g^n - g^{n-1}\|_{L^2} \leq C \|g^n - g^{n-1}\|_{L^2}.
\]
Finally, for \( (v^{n+1} - v^n) \), we have
\[
\frac{1}{2} \frac{d}{dt} \|v^{n+1} - v^n\|_{L^2}^2 = \int_{T^d} v^n \cdot \nabla (v^{n+1} - v^n) \cdot (v^{n+1} - v^n) \, dx - \int_{T^d} (v^n - v^{n-1}) \cdot \nabla v^n \cdot (v^{n+1} - v^n) \, dx \\
+ \int_{T^d} \Delta (v^{n+1} - v^n) \cdot (v^{n+1} - v^n) \, dx - \int_{T^d} (g^n - g^{n-1})(v^{n+1} - u^n) \cdot (v^{n+1} - v^n) \, dx \\
- \int_{T^d} e^{g^{n-1}} |v^{n+1} - v^n|^2 \, dx + \int_{T^d} e^{g^{n-1}} (u^n - u^{n-1}) \cdot (v^{n+1} - v^n) \, dx \\
\leq \|\nabla v^n\|_{L^\infty} \|v^n - v^{n-1}\|_{L^2} \|v^{n+1} - v^n\|_{L^2} + \|e^{g^n} - e^{g^{n-1}}\|_{L^2} (\|v^{n+1}\|_{L^2} + \|u^n\|_{L^2}) \|v^{n+1} - v^n\|_{L^2} \\
+ e^{\|g^{n-1}\|_{L^\infty}} \|u^n - u^{n-1}\|_{L^2} \|v^{n+1} - v^n\|_{L^2}
\]
for \( 0 \leq t \leq T^* \), where \( C > 0 \) is independent of \( n \). Therefore, we can apply Grönwall’s lemma to get the desired result.

C.3. Proof of Lemma 4.2

Now, we prove the well-posedness of strong solutions to (1.2). First, Lemma (C.2) implies
\[
g^n \to g, \ u^n \to u, \ and \ v^n \to v \ \text{in} \ C([0,T];L^2(\mathbb{T}^d))
\]
as \( n \to \infty \). Moreover, we can extend the convergence in \( C([0,T];L^2(\mathbb{T}^d)) \) to that in \( C([0,T];H^{s-1}(\mathbb{T}^d)) \) by interpolating this with the uniform bound in \( C([0,T];H^s(\mathbb{T}^d)) \) from Corollary (C.1)
\[
g^n \to g \ \text{in} \ C([0,T];H^{s-1}(\mathbb{T}^d)) \ \text{and} \ u^n \to u, \ v^n \to v \ \text{in} \ C([0,T];H^{s-1}(\mathbb{T}^d))
\]
as \( n \to \infty \). Concerning the \( H^s \)-regularity of \( (g,u,v) \), we can use a standard argument from functional analysis. We refer to [12] for details.

For the uniqueness, let \( (g,u,v) \) and \( (\hat{g},\hat{u},\hat{v}) \) be two solutions with the same initial data \( (g_0,u_0,v_0) \). Then, the Cauchy estimate in Lemma (C.2) implies
\[
\|(g - \hat{g})(\cdot, t)\|_{L^2}^2 + \|(u - \hat{u})(\cdot, t)\|_{L^2}^2 + \|(v - \hat{v})(\cdot, t)\|_{L^2}^2 \\
\leq C \int_0^t \left( \|(g - \hat{g})(\cdot, s)\|_{L^2}^2 + \|(u - \hat{u})(\cdot, s)\|_{L^2}^2 + \|(v - \hat{v})(\cdot, s)\|_{L^2}^2 \right) \, ds
\]
for \( t \leq T^* \). Thus, we apply Grönwall’s lemma to the above to conclude the uniqueness of solutions.

Appendix D. Proof of Lemma 4.2

We separately estimate the case \( k = 0 \) and \( k > 0 \) as follows:

• (Step A: First-order estimate) For \( \nabla g \), we have
\[
\frac{1}{2} \frac{d}{dt} \|\nabla g\|_{L^2}^2 = -\int_{T^d} \nabla (\nabla g \cdot u) \cdot \nabla g \, dx - \int_{T^d} \nabla g \cdot \nabla (\nabla \cdot u) \, dx \\
= \frac{1}{2} \int_{T^d} (\nabla \cdot u) \|\nabla g\|^2 \, dx - \int_{T^d} (\nabla g \cdot \nabla) u \cdot \nabla g \, dx - \int_{T^d} \nabla g \cdot \nabla (\nabla \cdot u) \, dx
\]
\[
\leq \frac{3}{2} \|\nabla u\|_{L^\infty} \|\nabla g\|_{L^2}^2 - \int_{T^d} \nabla g \cdot \nabla (\nabla \cdot u) \, dx
\]
\[
\leq C\varepsilon_1 \|\nabla g\|_{L^2}^2 - \int_{T^d} \nabla g \cdot \nabla (\nabla \cdot u) \, dx
\]
Then, for $\nabla u$, we obtain
\[
\frac{1}{2} \frac{d}{dt} \| \nabla u \|_{L^2}^2 = - \int_{T^d} \nabla (u \cdot \nabla u) : \nabla u \, dx - \int_{T^d} \nabla^2 g : \nabla u \, dx
\]
\[
- \int_{T^d} \nabla (\nabla K * (e^g - 1)) : \nabla u \, dx - \int_{T^d} \nabla (u - v) : \nabla u \, dx
\]
\[
\leq C \| \nabla u \|_{L^\infty} \| \nabla u \|_{L^2} - \int_{T^d} \nabla^2 g : \nabla u \, dx + \int_{T^d} (e^g - 1)(\nabla \cdot u) \, dx - \frac{7}{8} \| \nabla u \|_{L^2}^2 + 2 \| \nabla v \|_{L^2}^2
\]
\[
\leq C \varepsilon_1 \| \nabla u \|_{L^2}^2 - \int_{T^d} \nabla^2 g : \nabla u \, dx + C \| g \|_{L^2}^2 - \frac{3}{4} \| \nabla u \|_{L^2}^2 + 2 \| \nabla v \|_{L^2}^2,
\]
where $C$ depends on $k$ and $d$ and we used the smallness of $\varepsilon_1$ to get
\[
\| e^g - 1 \|_{L^2} \leq e^{\| g \|_{L^\infty}} \| g \|_{L^2} \leq e^{C\varepsilon_1} \| g \|_{L^2} \leq C \| g \|_{L^2}.
\]
So we combine two estimates to yield the desired result when $k = 0$.

- (Step B: Higher-order estimate) For $1 \leq k \leq s - 1$, we estimate $\nabla^k(\nabla g)$ as
\[
\frac{1}{2} \frac{d}{dt} \| \nabla^k(\nabla g) \|_{L^2}^2
\]
\[
= - \int_{T^d} \nabla^k(u \cdot \nabla(\nabla g)) \cdot \nabla^k(\nabla g) \, dx - \int_{T^d} \left[ \nabla^k(\nabla(g \cdot u)) - u \cdot \nabla(\nabla^k(\nabla g)) \right] \nabla^k(\nabla g) \, dx
\]
\[
- \int_{T^d} \nabla^k(\nabla \cdot u) \cdot \nabla^k(\nabla g) \, dx
\]
\[
\leq \frac{\| \nabla \cdot u \|_{L^\infty}}{2} \| \nabla^k(\nabla g) \|_{L^2}^2 + C \| \nabla^k(\nabla g) \|_{L^2} \left( \| \nabla u \|_{L^\infty} \| \nabla^k(\nabla g) \|_{L^2} + \| \nabla g \|_{L^\infty} \| \nabla^k(\nabla u) \|_{L^2} \right)
\]
\[
- \int_{T^d} \nabla^k(\nabla \cdot u) \cdot \nabla^k(\nabla g) \, dx
\]
\[
\leq C \varepsilon_1 \left( \| \nabla^k(\nabla g) \|_{L^2}^2 + \| \nabla^k(\nabla u) \|_{L^2}^2 \right) - \int_{T^d} \nabla^k(\nabla \cdot u) \cdot \nabla^k(\nabla g) \, dx,
\]
where $C$ only depends on $k$ and $d$. For $\nabla^k(\nabla u)$, we get
\[
\frac{1}{2} \frac{d}{dt} \| \nabla^k(\nabla u) \|_{L^2}^2
\]
\[
= - \int_{T^d} u \cdot \nabla(\nabla^k(\nabla u)) : \nabla^k(\nabla u) \, dx - \int_{T^d} \left[ \nabla^k(\nabla(u \cdot \nabla u)) - u \cdot \nabla(\nabla^k(\nabla u)) \right] : \nabla^k(\nabla u) \, dx
\]
\[
- \int_{T^d} \nabla^k(\nabla^2 g) : \nabla^k(\nabla u) \, dx
\]
\[
- \int_{T^d} \nabla^k(\nabla^2 K * (e^g - 1)) : \nabla^k(\nabla u) \, dx - \int_{T^d} \nabla^k(\nabla(u - v)) : \nabla^k(\nabla u) \, dx
\]
\[
\leq C \| \nabla u \|_{L^\infty} \| \nabla^k(\nabla u) \|_{L^2}^2 - \int_{T^d} \nabla^k(\nabla^2 g) : \nabla^k(\nabla u) \, dx
\]
\[
+ \int_{T^d} \nabla^k(e^g - 1)\nabla^k(\nabla \cdot u) : \nabla^k(\nabla u) \, dx - \frac{7}{8} \| \nabla^k(\nabla u) \|_{L^2}^2 + 2 \| \nabla^k(\nabla v) \|_{L^2}^2
\]
\[
\leq C \varepsilon_1 \| \nabla^k(\nabla u) \|_{L^2}^2 - \int_{T^d} \nabla^k(\nabla^2 g) : \nabla^k(\nabla u) \, dx + 2 \| \nabla^k(e^g - 1) \|_{L^2}^2 - \frac{3}{4} \| \nabla^k(\nabla u) \|_{L^2}^2 + 2 \| \nabla^k(\nabla v) \|_{L^2}^2.
\]
Here, for the estimate of $\| \nabla^k(e^g) \|_{L^2}$, we use the same argument in Lemma C.1, let $a_k := \| \nabla^k(e^g - 1) \|_{L^2}$. First, one obtains
\[
a_0 \leq e^{\| g \|_{L^\infty}} \| g \|_{L^2} \leq C e^{C\varepsilon_1} \| g \|_{L^2}.
\]
Then, we use Lemma 4.1 to have
\[
\begin{aligned}
  a_k &\leq \|\nabla^{k-1} (e^a \nabla g)\|_{L^2} \\
  &\leq C (\|e\|_{L^\infty} \|\nabla^k g\|_{L^2} + a_{k-1} \|\nabla g\|_{L^\infty}) \\
  &\leq C (C_{\varepsilon_1} \|\nabla^k g\|_{L^2} + \varepsilon_1 a_{k-1}) \\
  &\leq C (C_{\varepsilon_1} \|\nabla^k g\|_{L^2} + \varepsilon_1 (C_{\varepsilon_1} \|\nabla^{k-1} g\|_{L^2} + \varepsilon_1 a_{k-2})) \\
  &\leq C e^{C_{\varepsilon_1}} \sum_{\ell=0}^{k} \varepsilon_1^{\ell} \|\nabla^{k-\ell} g\|_{L^2},
\end{aligned}
\]
where $C$ only depends on $k$ and $d$. Since $\varepsilon_1$ is sufficiently small, we obtain
\[
a_k \leq C \|g\|_{H^k}.
\]
Combining all of the above estimates, we complete the proof.

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