Boundaries of $\mathbb{Z}^n$-free groups

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Abstract

In this paper we study random walks on a f.g. group $G$ which has a free action on a $\mathbb{Z}^n$-tree. We show that if $\Gamma_G$ is a universal $\mathbb{Z}^n$-tree associated with $G$ and $\partial_n \Gamma_G$ is the set of all ends of $\mathbb{Z}^n$-type of $\Gamma_G$, then for every non-degenerate probability measure $\mu$ on $G$ and any $\mu$-stationary measure $\nu$, the space $(\partial_n \Gamma_G, \nu)$ is a $\mu$-boundary. Moreover, if $\mu$ has finite first moment with respect to a finite word metric on $G$, then the measure space $(\partial_n \Gamma_G, \nu)$ is the Poisson–Furstenberg boundary of $(G, \mu)$.

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1 Introduction

In this paper we study random walks on groups with free actions on $\mathbb{Z}^n$-trees ($\mathbb{Z}^n$-free groups) and the boundaries of these groups. This family of groups is
a particularly nice and well-studied subclass of groups acting freely on Λ-trees (Λ-free groups), where Λ is an arbitrary ordered abelian group.

The theory of group actions on Λ-trees goes back to the work of Lyndon who introduced in [25] abstract length functions on groups, axiomatizing Nielsen cancellation method; he initiated the study of groups with real valued length functions. Later in [7], Chiswell related such length functions with group actions on \(\mathbb{Z}\)- and \(\mathbb{R}\)-trees, providing a construction of the tree on which the group acts. At about the same time, Tits in [31] gave the first formal definition of \(\mathbb{R}\)-tree, which is a geodesic metric space with a tree-like structure. Eventually, in [27], Morgan and Shalen introduced Λ-trees for an arbitrary ordered abelian group Λ and the general form of Chiswell’s construction. A Λ-tree is a metric space whose metric takes values in Λ and is subject to certain tree axioms. The theory of group actions such objects was consistently developed by Alperin and Bass (see [1]), where authors state the fundamental problem: find the group theoretic information carried by a Λ-tree action, in particular, the structure of Λ-free groups. If the case of Archimedean free actions, that is, when Λ = \(\mathbb{R}\), is basically closed by the Rips’ Theorem that describes finitely generated \(\mathbb{R}\)-free groups (see [13, 3]), the general non-Archimedean case is still open, but a lot of progress was made (see [2, 16, 23, 22]) though.

Introduction of infinite Λ-words was one of the major recent developments in this theory. In [28] Myasnikov, Remeslennikov and Serbin showed that groups admitting faithful representations by Λ-words act freely on some Λ-trees, while Chiswell proved the converse [9]. This gives an equivalent approach to group actions on Λ-trees, in which one can work with group elements viewed as Λ-words in the same manner as with ordinary words in standard free groups. The method of infinite words becomes a very powerful tool in the case when Λ = \(\mathbb{Z}^n\) with the right lexicographic order due to the natural combinatorial structure of \(\mathbb{Z}^n\)-trees. This structure was exploited in [21] and [24] to obtain a description of finitely generated \(\mathbb{Z}^n\)-free groups in terms of free products with amalgamation and HNN-extensions of a particular type (see also [4]).

The class of \(\mathbb{Z}^n\)-free groups is a natural generalization of free groups which contains limit groups, \(\mathbb{R}\)-free groups, etc. and which is closed under taking subgroups, free products, and amalgamated free products along maximal cyclic subgroups (\(n\) is not preserved in general). All these groups are hyperbolic relative to non-cyclic maximal abelian subgroups (see [14, 15]) (hyperbolic if all maximal abelian subgroups are cyclic), coherent, with nice algorithmic properties.

In this paper we use the construction from [28] to represent elements of a \(\mathbb{Z}^n\)-free group by \(\mathbb{Z}^n\)-words which are certain (finite or infinite) configurations of letters from some alphabet on the \(\mathbb{Z}^n\)-lattice. To be more precise let us fix the right lexicographic order on \(\mathbb{Z}^n\), that is, given two n-tuples \((a_1, \ldots, a_n), (b_1, \ldots, b_n) \in \mathbb{Z}^n\), we have \((a_1, \ldots, a_n) < (b_1, \ldots, b_n)\) if there exists \(i \in [1, n]\) such that \(a_i < b_i\) and \(a_j = b_j, j \in [i + 1, n]\). Denote by \(\mathbb{Z}_+^n \subset \mathbb{Z}^n\) the subset of all positive elements in \(\mathbb{Z}^n\) with respect to the order just defined. For \(\alpha, \beta \in \mathbb{Z}^n\), such that \(\alpha \leq \beta\), the closed segment \([\alpha, \beta]\) is
the set \([\alpha, \beta] = \{\gamma \in \mathbb{Z}^n \mid \alpha \leq \gamma \leq \beta\}\) and we define \(\mathbb{Z}^n\)-words as functions on closed segments of the type \([1, \alpha]\), where \(\alpha \in \mathbb{Z}^n\) and 1 stands for the n-tuple \((1,0,\ldots,0) \in \mathbb{Z}^n\). To this end let \(X = \{x_i \mid i \in I\}\) be a set, and put \(X^{-1} = \{x_i^{-1} \mid i \in I\}\) and \(X^\pm = X \cup X^{-1}\). Now, a \(\mathbb{Z}^n\)-word (or just an infinite word) is a function of the type

\[w : [1, \alpha_w] \rightarrow X^\pm,\]

where \(\alpha_w \in \mathbb{Z}^n\), \(\alpha_w \geq 0\) is the natural length of \(w\), denoted \(|w|\). The set of all \(\mathbb{Z}^n\)-words over \(X\) is denoted by \(W(\mathbb{Z}^n, X)\). Concatenation, the trivial word \(\varepsilon\), and inversion are naturally defined as in the standard free group. An infinite word is reduced if it does not contain \(xx^{-1}\), \(x^{-1}x\) as a subword for \(x \in X\) and the set of all reduced \(\mathbb{Z}^n\)-words is denoted by \(R(\mathbb{Z}^n, X)\). Next, following [28], one can define multiplication “∗” on \(R(\mathbb{Z}^n, X)\) and the subset \(CDR(\mathbb{Z}^n, X) \subset R(\mathbb{Z}^n, X)\) of infinite words without torsion. From the results of [28] and [9] it follows that \(\mathbb{Z}^n\)-free groups are exactly subgroups of \(CDR(\mathbb{Z}^n, X)\) (subsets closed with respect to “∗” and inversion of infinite words), where \(X\) may vary (for all the details on infinite words see Subsection 2.2.2).

Next, using \(\mathbb{Z}^n\)-words representing elements of a \(\mathbb{Z}^n\)-free group \(G\) one can construct a universal \(\mathbb{Z}^n\)-tree \(\Gamma_G\) with a base-point \(\varepsilon\) equipped with a minimal free action (there is no \(G\)-invariant subtree) of \(G\) (see Subsection 2.3 for details). For each \(k \leq n\), by \(\partial_k \Gamma_G\) we denote the set of equivalence classes of isometries \(\gamma : [0, \infty)_{\mathbb{Z}^k} \rightarrow \Gamma_G\), which are called ends of \(\mathbb{Z}^k\)-type (see Subsection 2.2.1 for details), and by \(\partial \Gamma_G\) the set of all ends of \(\Gamma_G\), that is, \(\partial \Gamma_G = \bigcup_{k=1}^n \partial_k \Gamma_G\). The action of \(G\) on \(\Gamma_G\) induces an action on each \(\partial_k \Gamma_G\). In fact, each \(\partial_k \Gamma_G\) is a topological space, where the topology is generated by the “cones” \(\{U_x \mid x \in \Gamma_G\}\) and each \(U_x\) is the set of all geodesic rays \(\gamma : [0, \infty)_{\mathbb{Z}^k} \rightarrow \Gamma_G\) starting at \(\varepsilon\) and containing the segment \([\varepsilon, x]\). Observe that \(\partial_k \Gamma_G\) is not compact in general if \(k > 1\).

We are interested in the case when \(G\) is non-abelian and irreducible in \(CDR(\mathbb{Z}^n, X)\), that is, it is not conjugate in \(R(\mathbb{Z}^n, X)\) to a subgroup \(H\) of \(CDR(\mathbb{Z}^k, X)\), \(k < n\). In a sense, irreducibility means that \(G\) has the simplest possible representation by \(\mathbb{Z}^n\)-words, and hence, the associated \(\mathbb{Z}^n\)-tree \(\Gamma_G\) has the simplest possible structure. In this case \(\partial_n \Gamma_G\) is minimal, that is, it does not contain a \(G\)-invariant closed subset (see Theorem 3).

Our main result can be formulated as follows.

**Theorem 1.** Let \(G\) be a finitely generated irreducible and non-abelian \(\mathbb{Z}^n\)-free group for some \(n \in \mathbb{N}\) and \(\Gamma_G\) the associated universal \(\mathbb{Z}^n\)-tree on which \(G\) acts. Then \(\Gamma_G = \Gamma_G \cup \partial \Gamma_G\) is a compact metric \(G\)-space such that \(\partial_n \Gamma_G\) is \(G\)-invariant and minimal. Moreover, for every non-degenerate probability measure \(\mu\) on \(G\) and any \(\mu\)-stationary measure \(\nu\) on \(\Gamma_G\):

1. \(\nu\) is continuous and \(\nu(\partial_n \Gamma_G) = 1\),

2. for a. e. path \(\tau = \{\tau_i\}\) of the random \(\mu\)-walk, the sequence of measures \(\{\tau_i \cdot \nu\}\) converges to a random Dirac measure \(\delta_{\omega(\tau)}\) with \(\omega(\tau) \in \partial_n \Gamma_G\).
3. \((\Gamma_G, \nu)\) is a \(\mu\)-boundary, that is, \(\Gamma_G\) is \(\mu\)-proximal.

4. There is a unique \(\mu\)-stationary measure \(\nu\) on \(\Gamma_G\), it is concentrated on \(\partial_n \Gamma_G\), and \((\partial_n \Gamma_G, \nu_{\mu})\) is a \(\mu\)-boundary (hence \(\mu\)-proximal).

5. If \(\mu\) has finite first moment with respect to a finite word metric on \(G\), then the measure space \((\partial_n \Gamma_G, \nu)\) is the Poisson(-Furstenberg) boundary of \((G, \mu)\).

In the proof of the above theorem we use methods of [12] and [6] and the structure of the proof is as follows. At first, in Section 3, we introduce a metric \(d_{\Gamma_G}\) on \(\Gamma_G = \Gamma_G \cup \partial \Gamma_G\) so as to make \(\Gamma_G\) a compact metric space equipped with a continuous action of \(G\) (Theorem 7). Next, we prove that \((\Gamma_G, \nu)\) is a \(\mu\)-boundary for every non-degenerate measure \(\mu\) on \(G\) and any \(\mu\)-stationary measure \(\nu\) on \(\Gamma_G\) (see Proposition 3), which implies that the space \(\Gamma_G\) is \(\mu\)-proximal. Then, since every \(\mu\)-stationary measure \(\nu\) on \(\Gamma_G\) is concentrated on \(\partial_n \Gamma_G\) (see Proposition 1), we deduce that \((\partial_n \Gamma_G, \nu)\) is a \(\mu\)-boundary too, and hence \(\mu\)-proximal (see Corollary 5). Finally, we show that if \(\mu\) has finite first moment with respect to a finite word metric on \(G\) then \((\partial_n \Gamma_G, \nu)\) is a maximal \(\mu\)-boundary (Proposition 9).

Observe that there are other constructions of the Poisson boundaries of \(\mathbb{Z}^n\)-groups. First of all, since every \(\mathbb{Z}^n\)-group \(G\) is hyperbolic relative to its non-cyclic abelian subgroups (follows from results of [16] and [11]), one can study its Floyd boundary (see [15, 14]) which is non-trivial. Hence, from [13] it follows that the Floyd boundary of \(G\) is its Poisson(-Furstenberg) boundary. Another approach is to use the fact that \(G\) is \(\text{CAT}(0)\) with isolated flats (see [5]) then one can construct its Poisson boundary using results of [20] (see also [29]).

We would like to note that our construction is very natural and comes directly from the action of a \(\mathbb{Z}^n\)-group \(G\) on its universal \(\mathbb{Z}^n\)-tree without reference to general results about relatively hyperbolic groups and \(\text{CAT}(0)\)-groups.

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2 Preliminaries

2.1 Poisson–Furstenberg boundaries

Let \(G\) be a countable group and let \(\mu\) be a probability measure on \(G\) (we consider measures that are regular with respect to the discrete topology on the group). We say that \(\mu\) is non-degenerate if its support generates \(G\) as a semigroup. The right-hand random walk on \(G\) with distribution \(\mu\) (or, briefly, \(\mu\)-walk) is the time-homogeneous Markov chain whose state space is \(G\), the transition probabilities are given by \(P(g, h) = \mu(g^{-1}h)\), and the initial distribution is concentrated at the identity of the group. Realizations of this process are called paths of the random walk. We write \(P_\mu\) for the associated Markov measure on the path space \(G^{\mathbb{Z}_+}\).
An action of the group $G$ on a topological space $M$ is a homomorphism from $G$ to the group Homeo($M$) of all homeomorphisms of $M$. An action is said to be minimal if each orbit of the action is dense (or, equivalently, if the action has no proper closed invariant subsets). A space endowed with an action of the group $G$ is called a $G$-space. We denote the space of all (regular with respect to the discrete topology) probability measures on $G$ by $P(G)$ and the space of all regular Borel probability measures on $M$ by $P(M)$. We endow $P(M)$ with the weak topology (by duality with the space of all bounded continuous functions on $M$). Any action of $G$ on $M$ induces an action of $G$ on $P(M)$: $(g \cdot \theta)(E) = \theta(g^{-1} \cdot E)$. A measure is said to be a Dirac measure if it is concentrated at a point. We say that a measure is continuous if it takes zero value on each point.

Suppose that $\mu \in P(G)$. A measure $\nu \in P(M)$ is said to be $\mu$-stationary if

$$\mu \star \nu = \sum_{g \in G} (g \cdot \nu)\mu(g) = \nu.$$  

Assertion 1 (12). Let $G$ be a countable group acting on a compact metric space $M$ and let $\mu \in P(G)$ be an arbitrary measure. Then the set of $\mu$-stationary measures in $P(M)$ is non-empty.

Theorem 2 (14 18). Let $G$ be a countable group acting on a compact metric space $M$. Take $\mu \in P(G)$ and let $\nu \in P(M)$ be a $\mu$-stationary measure. Then for a.e. path $\tau = \{\tau_i\}_{i \in \mathbb{Z}_+}$ of the right random $\mu$-walk, the sequence $\{\tau_i \cdot \nu\}_{i \in \mathbb{Z}_+}$ converges to some measure $\lambda(\tau) \in P(M)$. We also have

$$\int \lambda(\tau) dP_\mu(\tau) = \nu.$$  

Lemma 1 (18). Let $G$ be a countable group acting on a space $M$, $\mu$ a non-degenerate measure on $G$ and $\nu \in P(M)$ be a $\mu$-stationary measure. Let $E \subset M$ be a measurable subset such that for every $g \in G$ we have either $g \cdot E = E$, or $(g \cdot E) \cap E = \emptyset$. Suppose further that there is an infinite family of pairwise-disjoint sets of the form $g \cdot E$, where $g \in G$. Then $\nu(E) = 0$. In particular, if the orbit $G \cdot x$ of every point $x \in M$ is infinite, then the $\mu$-stationary measure $\nu$ is continuous.

Lemma 2. Let $G$ be a countable group acting minimally on a topological space $M$, $\mu$ a non-degenerate measure on $G$ and $\nu \in P(M)$ be a $\mu$-stationary measure. Let $E \subset M$ be a non-empty open subset. Then $\nu(E) > 0$.

Proof. Observe that by definition, the $\mu$-stationary measure $\nu$ is $\mu^{*k}$-stationary for every $k \in \mathbb{N}$. Consequently, for every $k \in \mathbb{N}$ we have

$$\nu(E) = \sum_{g \in G} \mu^{*k}(g)(g \cdot \nu)(E) = \sum_{g \in G} \mu^{*k}(g)\nu(g^{-1} \cdot E). \quad (1)$$

Suppose that $\nu(E) = 0$. Then (1) implies that $\nu(g^{-1} \cdot E) = 0$ whenever there exists $k \in \mathbb{N}$ such that $\mu^{*k}(g) > 0$. Hence, $\nu(g^{-1} \cdot E) = 0$ for each $g \in G$ because


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\[ \mu \text{ is non-degenerate. On the other hand, since } G \text{ acts on } M \text{ minimally, and } E \text{ is open and non-empty, it follows that for each } x \in M \text{ there exists } g \in G \text{ such that } g \cdot x \in E. \text{ Hence, } \bigcup_{g \in G} g^{-1} \cdot E = \bigcup_{g \in G} g \cdot E = M. \text{ Since } G \text{ is countable, it follows that} \]

\[
1 = \nu(M) = \nu \left( \bigcup_{g \in G} g^{-1} \cdot E \right) \leq \sum_{g \in G} \nu(g^{-1} \cdot E) = 0.
\]

This contradiction proves that $\nu(E) > 0$. \(\square\)

Suppose that $M$ is a topological $G$-space, $\mu \in \mathcal{P}(G)$, and $\nu \in \mathcal{P}(M)$ is a $\mu$-stationary measure. The pair $(M, \nu)$ is called a $\mu$-boundary for $G$ if for a.e. path $\tau = \{\tau_i\}_{i \in \mathbb{Z}_+}$ of the right random $\mu$-walk, the sequence of measures $\{\tau_i \cdot \nu\}_{i \in \mathbb{Z}_+}$ converges to some Dirac measure $\delta_{\omega(\tau)}$, where $\omega(\tau) \in M$.

Usually, it makes sense to consider compact $\mu$-boundaries only. In our case with $\mathbb{Z}^n$-free groups, we treat non-compact boundaries also. We give the following simple lemma here in order to shed some light on the non-compact case.

**Lemma 3.** Let $M$ be a compact metric $G$-space, $\mu \in \mathcal{P}(G)$, and $\nu \in \mathcal{P}(M)$ a $\mu$-stationary measure. Assume that $(M, \nu)$ is a $\mu$-boundary for $G$, and $M' \subset M$ is a measurable $G$-invariant subspace with $\nu(M') = 1$. Then $(M', \nu')$, where $\nu'$ is the restriction of $\nu$ on $M'$, is also a $\mu$-boundary for $G$.

**Proof.** Let $\Omega_1 \subset G^{\mathbb{Z}_+}$ be the set of sequences $\tau = \{\tau_i\}_{i \in \mathbb{Z}_+}$ for which the sequence of measures $\{\tau_i \cdot \nu\}_{i \in \mathbb{Z}_+}$ converges to some Dirac measure $\delta_{\omega(\tau)}$, $\omega(\tau) \in M$. Set

\[ \Omega_2 = \{\tau \in \Omega_1 \mid \omega(\tau) \in M'\}. \]

Since $(M, \nu)$ is a $\mu$-boundary for $G$, we have $P_\mu(\Omega_1) = 1$. Since $\nu(M') = 1$, it follows by Theorem 3 that $P_\mu(\Omega_2) = 1$.

Observe that in an arbitrary metric space $K$, a sequence $\{\lambda_i\}_{i \in \mathbb{Z}_+}$ of measures in $\mathcal{P}(K)$ converges to a Dirac measure $\delta_x$, $x \in K$ if and only if for each open subset $U$ containing $x$ we have $\lim_{i \to \infty} \lambda_i(U) = 1$. This clearly implies that for every $\tau = \{\tau_i\}_{i \in \mathbb{Z}_+} \in \Omega_2$, the sequence $\{\tau_i \cdot \nu'\}_{i \in \mathbb{Z}_+}$ weakly converges to $\delta_{\omega(\tau)}$, where $\omega(\tau) \in M'$. In other words, for a.e. path $\tau = \{\tau_i\}_{i \in \mathbb{Z}_+}$ of the right random $\mu$-walk (since a.e. path of the right random $\mu$-walk is in $\Omega_2$), the sequence $\{\tau_i \cdot \nu'\}_{i \in \mathbb{Z}_+}$ converges to some Dirac measure $\delta_{\omega(\tau)}$, where $\omega(\tau) \in M'$. This precisely means that $(M', \nu')$ is a $\mu$-boundary for $G$. \(\square\)

Given $\mu \in \mathcal{P}(G)$, we say that a topological $G$-space $M$ is $\mu$-proximal if for every $\mu$-stationary measure $\nu \in \mathcal{P}(M)$ the pair $(M, \nu)$ is a $\mu$-boundary.

**Assertion 2** ([24], Lemma 3.1). Let $G$ be a countable group acting on a compact metric space $M$ and let $\mu \in \mathcal{P}(G)$ be an arbitrary probability measure. Suppose that for every $\mu$-stationary measure $\nu \in \mathcal{P}(M)$ the pair $(M, \nu)$ is a $\mu$-boundary. Then the $\mu$-stationary measure on $M$ is unique.
In particular, if $M$ is a $\mu$-proximal compact metric $G$-space, then there is a unique $\mu$-stationary measure in $\mathcal{P}(M)$.

A $\mu$-boundary $(M, \nu)$ of the pair $(G, \mu)$ is called a Poisson(-Furstenberg) boundary if it is maximal, that is, if every $\mu$-boundary $(M_1, \nu_1)$ of the pair $(G, \mu)$ is a quotient space of $(M, \nu)$ (as a measure space endowed with a group action, disregarding the topology).

If $(M, \nu)$ is a $\mu$-boundary for $G$, it is natural to consider the partially defined mapping $\text{bnd} : G_{\mathbb{Z}^+} \to M$ that sends a path $\tau = \{\tau_i\}_{i \in \mathbb{Z}_+} \in G_{\mathbb{Z}^+}^G$ to a point $\delta_w$. The mapping $\text{bnd}$ is thus defined for $P_\mu$-a.e. path. Suppose that $E \subseteq M$ is a measurable subset and $\tau$ a path of random $\mu$-walk. Then Theorem 2 implies that $\text{bnd}(\tau)$ lies in $E$ with probability $\nu(E)$.

In order to establish the maximality of boundaries we use the Strip Criterion due to V. A. Kaimanovich. An increasing sequence $\mathcal{G} = (\mathcal{G}_k)$ of sets exhausts a countable group $G$ is called a gauge on $G$. By $| \cdot |_{\mathcal{G}} = \min \{k \mid g \in \mathcal{G}_k\}$ we denote the corresponding gauge function. If $\mu$ is a measure on a group $G$, we denote by $\hat{\mu}$ the reflected measure defined by $\hat{\mu}(g) = \mu(g^{-1})$.

**Theorem 3 (Strip Criterion [17]).** Let $\mu$ be a probability measure with finite entropy $H(\mu)$ on a countable group $G$, and let $(B_+, \lambda_+)$ and $(B_-, \lambda_-)$ be $\mu$- and $\hat{\mu}$-boundaries, respectively. If there exists a gauge $\mathcal{G} = (\mathcal{G}_k)$ on the group $G$ with gauge function $| \cdot |_{\mathcal{G}}$ and a measurable $G$-equivariant map $S$ assigning to pairs of points $(b_-, b_+) \in B_- \times B_+$ non-empty “strips” $S(b_-, b_+) \subset G$ such that for all $g \in G$ and $(\lambda_- \times \lambda_+)$-almost every $(b_-, b_+) \in B_- \times B_+$

$$\frac{1}{i} \log \left| S(b_-, b_+)g \cap G_{| \tau_i |_{\mathcal{G}}} \right| \underset{i \to \infty}{\longrightarrow} 0$$

in probability with respect to the measure $P_\mu$ in the space of sample paths $\tau = \{\tau_i\}_{i \in \mathbb{Z}_+}$, then $(B_+, \lambda_+)$ is a Poisson(-Furstenberg) boundary of the pair $(G, \mu)$.

**Corollary 1 ([17]).** Let $G$ be a finitely generated group with a finite word metric $| \cdot |$. Let $\mu$ be a probability measure on $G$ with finite first moment $\sum |g| \mu(g)$. Let $(B_+, \lambda_+)$ and $(B_-, \lambda_-)$ be $\mu$- and $\hat{\mu}$-boundaries, respectively. If there exists a measurable $G$-equivariant map $S$ assigning to pairs of points $(b_-, b_+) \in B_- \times B_+$ non-empty “strips” $S(b_-, b_+) \subset G$ such that for $(\lambda_- \times \lambda_+)$-almost every $(b_-, b_+) \in B_- \times B_+$ we have

$$\frac{1}{i} \log \left| \{g \in S(b_-, b_+) : |g| \leq i\} \right| \underset{i \to \infty}{\longrightarrow} 0,$$

then $(B_+, \lambda_+)$ is a Poisson(-Furstenberg) boundary of the pair $(G, \mu)$.

**Remark 1.** Note that, under the Strip Criterion, the “strips” $S(b_-, b_+)$ are required to be

(i) all non-empty,

(ii) $(\lambda_- \times \lambda_+)$-almost surely “thin”. 
Clearly, since the strips are allowed to meet the “thinness” requirement \((\lambda_- \times \lambda_+)-\text{almost surely} \) (not surely), we can handle the “non-emptiness” property in the same way. In other words, in order to use the Strip Criterion it suffices to construct a (measurable, equivariant) map \( S' : B_- \times B_+ \to 2^G \) with strips, which are

\begin{itemize}
  \item[(i')] \((\lambda_- \times \lambda_+)-\text{almost surely non-empty},
  \item[(ii')] \((\lambda_- \times \lambda_+)-\text{almost surely thin}.
\end{itemize}

This is clear, because we can pass from \( S' \) to a map \( S \) with the property (i) by setting \( S(b_-,b_+) = G \) if \( S'(b_-,b_+) = \emptyset \) and \( S(b_-,b_+) = S'(b_-,b_+) \) otherwise.

Note also that, having a map with the properties (i') and (ii'), we can replace all non-thin strips by empty ones and thus obtain a map that has the property (i) and the property

\begin{itemize}
  \item[(ii)] all strips are “thin”.
\end{itemize}

A real function \( \phi : G \to \mathbb{R} \) on a group \( G \) is called a seminorm if for every \( g, h \in G \): (a) \( \phi(g) \geq 0 \) and \( \phi(1) = 0 \), (b) \( \phi(gh) \leq \phi(g) + \phi(h) \), (c) \( \phi(g) = \phi(g^{-1}) \).

By virtue of Remark \( \ref{remark} \) the Strip Criterion as follows.

**Corollary 2.** Let \( G, | \cdot |, \mu, (B_+, \lambda_+), (B_-, \lambda_-) \) be as in Corollary \( \ref{corollary} \) and let \( h : G \to \mathbb{R} \) be a seminorm. Assume that there exists a measurable \( G \)-equivariant map \( S : B_- \times B_+ \to 2^G \) sending pairs of points \((b_-, b_+) \in B_- \times B_+ \) to “strips” \( S(b_-, b_+) \subseteq G \) such that for \((\lambda_- \times \lambda_+)-\text{almost every} \) pair \((b_-, b_+) \in B_- \times B_+ \) we have

\begin{enumerate}
  \item the strip \( S(b_-, b_+) \) is non-empty,
  \item \( \frac{1}{i} \log | \{ g \in S(b_-, b_+) : h(g) \leq i \} | \xrightarrow{i \to \infty} 0. \)
\end{enumerate}

Then \((B_+, \lambda_+)\) is a Poisson-(Furstenberg) boundary of the pair \((G, \mu)\).

**Proof.** Define the map \( S' : B_- \times B_+ \to 2^G \) by setting \( S'(b_-, b_+) = G \) if \( S(b_-, b_+) = \emptyset \) and \( S'(b_-, b_+) = S(b_-, b_+) \) otherwise. Then \( S' \) is still measurable and \( G \)-equivariant, and all its “strips” are non-empty. Observe also, that for \((\lambda_- \times \lambda_+)-\text{a.e.} \) pair \((b_-, b_+) \) we have

\[
\frac{1}{i} \log | \{ g \in S'(b_-, b_+) : h(g) \leq i \} | \xrightarrow{i \to \infty} 0,
\]

because \( S'(b_-, b_+) = S(b_-, b_+) \) for \((\lambda_- \times \lambda_+)-\text{a.e.} \) pair \((b_-, b_+) \). Now, let us prove that for \((\lambda_- \times \lambda_+)-\text{a.e.} \) pair \((b_-, b_+) \) the strip \( S'(b_-, b_+) \) meets the condition (ii) of Corollary \( \ref{corollary} \).

Recall that, for our metric \(| \cdot |\), there exists a (unique) subset \( Q \subseteq G \setminus \{1\} \) such that \(|q| = \min \{d \in \mathbb{N}_0 : g \in Q^d \} \) for all \( g \in G \) (the set \( Q \) is finite, since we consider “finite” word metrics). Since \( Q \) is finite, there exists \( k_0 \in \mathbb{N} \) such that \( h(q) \leq k_0 \cdot |q| = k_0 \) for each \( q \in Q \). Then for any \( g \in G \setminus \{e\} \) with \( g = q_1 \cdots q_{|g|} \), \( q_j \in Q \), we have (by the triangle inequality for seminorms)

\[
h(g) \leq h(q_1) + \cdots + h(q_{|g|}) \leq k_0 \cdot |q_1| + \cdots + k_0 \cdot |q_{|g|}| = k_0 \cdot |g|.
\]
Since $h(1) = |1| = 0$, it follows that $h(g) \leq k_0 \cdot |g|$ for all $g \in G$. Consequently, for any $i \in \mathbb{N}$ and $S \subset G$ we have

$$\{g \in S : |g| \leq i\} \subset \{g \in S : h(g) \leq k_0 i\},$$

whence

$$\frac{1}{i} \log |\{g \in S : |g| \leq i\}| \leq k_0 \frac{1}{k_0 i} \log |\{g \in S : h(g) \leq k_0 i\}|,$$

The last inequality clearly shows that every strip $S'(b_-, b_+)$ meeting the condition (3) above, also meets the condition (2) of Corollary 1.

Thus, the map $S$ satisfies all the requirements of Corollary 1 and hence $(B_+, \lambda_+)$ is a Poisson–Furstenberg boundary of the pair $(G, \mu).$ 

### 2.2 $\mathbb{Z}^n$-free groups

Here we give some basic definitions in the theory of $\Lambda$-trees.

#### 2.2.1 $\Lambda$-trees

A set $\Lambda$ equipped with addition “$+$” and a partial order “$\leq$” is called a partially ordered abelian group if

1. $\langle \Lambda, + \rangle$ is an abelian group,
2. $\langle \Lambda, \leq \rangle$ is a partially ordered set,
3. for all $\alpha, \beta, \gamma \in \Lambda$, $\alpha \leq \beta$ implies $\alpha + \gamma \leq \beta + \gamma$.

An abelian group $\Lambda$ is called orderable if there exists a linear order $\leq$ on $\Lambda$, satisfying the condition (3) above. In general, the ordering on $\Lambda$ is not unique.

An ordered abelian group $\Lambda$ is called discretely ordered if $\Lambda$ has a minimal positive element. With a slight abuse of notation we denote it by 1, but it is going to be clear from the context if 1 represents a natural number, or an element of $\Lambda$. In this event, for any $\alpha \in \Lambda$ the following hold:

1. $\alpha + 1 = \min\{\beta \mid \beta > \alpha\}$,
2. $\alpha - 1 = \max\{\beta \mid \beta < \alpha\}$.

For elements $\alpha, \beta \in \Lambda$ the closed segment $[\alpha, \beta]$ is defined by

$$[\alpha, \beta] = \{\gamma \in \Lambda \mid \alpha \leq \gamma \leq \beta\}.$$ 

Now a subset $C \subset \Lambda$ is called convex if for every $\alpha, \beta \in C$ the set $C$ contains $[\alpha, \beta]$. In particular, a subgroup $C$ of $\Lambda$ is convex if $[0, \beta] \subset C$ for every positive $\beta \in C$.

Let $X$ be a non-empty set and $\Lambda$ an ordered abelian group. A $\Lambda$-metric on $X$ is a mapping $p : X \times X \rightarrow \Lambda$ such that for all $x, y, z \in X$
Bounds of \( \mathbb{Z}^n \)-free groups

(M1) \( p(x, y) \geq 0 \),
(M2) \( p(x, y) = 0 \) if and only if \( x = y \),
(M3) \( p(x, y) = p(y, x) \),
(M4) \( p(x, y) \leq p(x, z) + p(y, z) \).

A \( \Lambda \)-metric space is a pair \((X, p)\), where \( X \) is a non-empty set and \( p \) is a \( \Lambda \)-metric on \( X \). If \((X, p)\) and \((X', p')\) are \( \Lambda \)-metric spaces, an isometry from \((X, p)\) to \((X', p')\) is a mapping \( f : X \to X' \) such that \( p(x, y) = p'(f(x), f(y)) \) for all \( x, y \in X \).

A segment in a \( \Lambda \)-metric space is the image of an isometry \( \alpha : [a, b]_\Lambda \to X \) for some \( a, b \in \Lambda \) and \([a, b]_\Lambda\) is a segment in \( \Lambda \). The endpoints of the segment are \( \alpha(a), \alpha(b) \).

We call a \( \Lambda \)-metric space \((X, p)\) geodesic if for all \( x, y \in X \), there is a segment in \( X \) with endpoints \( x, y \) and \((X, p)\) is geodesically linear if for all \( x, y \in X \), there is a unique segment in \( X \) whose set of endpoints is \( \{x, y\} \). We denote such a segment by \([x, y]\).

A \( \Lambda \)-tree is a \( \Lambda \)-metric space \((X, p)\) with \( X \neq \emptyset \) such that

(T1) \((X, p)\) is geodesically linear,
(T2) if \( x, y, z \in X \) then \([x, y] \cap [x, z] = [x, w]\) for some \( w \in X \); this \( w \) is unique and we write \( w = Y(y, x, z) \),
(T3) if \( x, y, z \in X \) then \([x, y] \cap [y, z] = \{y\}\) then \([x, y] \cup [y, z] = [x, z]\).

Let \( X \) be a \( \Lambda \)-tree. We call \( e \in X \) an end point of \( X \) if, whenever \( e \in [x, y] \subset X \) either \( e = x \) or \( e = y \). A linear subtree from \( x \in X \) or a geodesic ray \( x \in X \) is any linear subtree \( L \) of \( X \) having \( x \) as an end point. \( L \) carries a natural linear ordering with \( x \) as least element. If \( y \in L \) then \( L_y = \{z \in L \mid y \leq z\} \) is a linear subtree from \( y \).

A maximal linear subtree from \( x \) is an \( X \)-ray from \( x \). Observe that (Proposition 2.22) that if \( L, L' \) are \( X \)-rays from \( x, x' \) respectively such that \( L \cap L' \neq \emptyset \) then \( L \cap L' \) is either a closed segment, or \( L \cap L' = L_v \) for some \( v \in X \). In fact, we call \( X \)-rays \( L \) and \( L' \) equivalent if \( L \cap L' = L_v \) for some \( v \in X \). The equivalence classes of \( X \)-rays for this relation are called ends of \( X \).

For \( y_0, y_1 \in X \) define \( X(y_0, y_1) \) to be a subtree of \( X \) spanned by all linear subtrees \( L \) from \( y \) such that \([y_0, y] \cap L = \{y\}\). In other words,

\[
X(y_0, y_1) = \{x \in X \mid y_1 \in [y_0, x]\}.
\]

We say that group \( G \) acts on a \( \Lambda \)-tree \( X \) if every element \( g \in G \) defines an isometry \( g : X \to X \).

Note, that every group has a trivial action on a \( \Lambda \)-tree, that is, all its elements act as identity.

Let a group \( G \) act as isometries on a \( \Lambda \)-tree \( X \). \( g \in G \) is called elliptic if it has a fixed point. \( g \in G \) is called an inversion if it does not have a fixed point,
but $g^2$ does. If $g$ is not elliptic and not an inversion then it is called hyperbolic.

For a hyperbolic element $g \in G$ define a characteristic set

$$\text{Axis}(g) = \{ p \in X \mid [g^{-1} \cdot p, p] \cap [p, g \cdot p] = \{ p \} \},$$

which is called the axis of $g$. $\text{Axis}(g)$ meets every $\langle g \rangle$-invariant subtree of $X$.

A group $G$ acts freely and without inversions on a $\Lambda$-tree $X$ if for all $1 \neq g \in G$, $g$ acts as a hyperbolic isometry. In this case we also say that $G$ is $\Lambda$-free. Observe that if $G$ is $\Lambda$-free then $[f, g] = 1$ implies $\text{Axis}(f) = \text{Axis}(g)$, hence we denote $\text{Axis}(g)$ by $\text{Axis}(C_G(g))$.

### 2.2.2 Length functions

Let $G$ be a group and $\Lambda$ be an ordered abelian group. Then a function $l : G \to \Lambda$ is called a (Lyndon) length function on $G$ if the following conditions hold

(L1) $\forall x \in G : l(x) \geq 0$ and $l(1) = 0$,

(L2) $\forall x \in G : l(x) = l(x^{-1})$,

(L3) $\forall x, y, z \in G : c(x, y) > c(x, z) \rightarrow c(x, z) = c(y, z)$,

where $c(x, y) = \frac{1}{2}(l(x) + l(y) - l(x^{-1}y))$.

It is not difficult to derive the following two properties of Lyndon length functions from the axioms (L1)–(L3)

- $\forall x, y \in G : l(xy) \leq l(x) + l(y)$,

- $\forall x, y \in G : 0 \leq c(x, y) \leq \min\{l(x), l(y)\}$.

The axiom below helps to describe the connection between $\Lambda$-valued Lyndon length functions and actions on $\Lambda$-trees.

(L4) $\forall x \in G : c(x, y) \in \Lambda$.

**Theorem 4** (7). Let $G$ be a group and $l : G \to \Lambda$ a Lyndon length function satisfying (L4). Then there are a $\Lambda$-tree $(X, p)$, an action of $G$ on $X$ and a point $x \in X$ such that $l = l_x$.

### 2.2.3 Infinite words

Let $\Lambda$ be an ordered abelian group, that is, there is a linear order “$\leq$” on $\Lambda$ such that for all $a, b, c \in \Lambda$, $a \leq b$ implies $a + c \leq b + c$. Recall that for elements $\alpha, \beta \in \Lambda$ the closed segment $[\alpha, \beta]$ is defined by

$$[\alpha, \beta] = \{ \gamma \in \Lambda \mid \alpha \leq \gamma \leq \beta \}.$$ 

Now a subset $C \subset \Lambda$ is called convex if for every $\alpha, \beta \in C$ the set $C$ contains $[\alpha, \beta]$. In particular, a subgroup $C$ of $\Lambda$ is convex if $[0, \beta] \subset C$ for every positive $\beta \in C$. 
An ordered abelian group $\Lambda$ is called *discretely ordered* if $\Lambda$ has a minimal positive element. With a slight abuse of notation we denote it by 1, but it is going to be clear from the context if 1 represents a natural number, or an element of $\Lambda$. In this event, for any $\alpha \in \Lambda$ the following hold:

(1) $\alpha + 1 = \min\{\beta \mid \beta > \alpha\}$,
(2) $\alpha - 1 = \max\{\beta \mid \beta < \alpha\}$.

Observe that if $\Lambda$ is any ordered abelian group then $\mathbb{Z} \oplus \Lambda$ is discretely ordered with respect to the right lexicographic order \((a_1, b_1) < (a_2, b_2) \iff b_1 < b_2\ or \ a_1 = b_2\ and \ a_1 < a_2\).

Let $\Lambda$ be a discretely ordered abelian group and let $X = \{x_i \mid i \in I\}$ be a set. Put $X^{-1} = \{x_i^{-1} \mid i \in I\}$ and $X^\pm = X \cup X^{-1}$. An $\Lambda$-word is a function of the type $w : [1, \alpha_w] \rightarrow X^\pm$, where $\alpha_w \in \Lambda, \alpha_w \geq 0$. The element $\alpha_w$ is called the length $|w|$ of $w$.

By $W(\Lambda, X)$ we denote the set of all $\Lambda$-words over $X^\pm$. Observe, that $W(\Lambda, X)$ contains an empty $\Lambda$-word which we denote by $\varepsilon$.

Concatenation $uv$ of two $\Lambda$-words $u, v \in W(\Lambda, X)$ is a $\Lambda$-word of length $|u| + |v|$ and such that:

\[(uv)(\alpha) = \begin{cases} u(\alpha) & \text{if } 1 \leq \alpha \leq |u| \\ v(\alpha - |u|) & \text{if } |u| < \alpha \leq |u| + |v| \end{cases} \]

Next, for any $\Lambda$-word $w$ we define an *inverse* $w^{-1}$ as an $\Lambda$-word of the length $|w|$ and such that

\[w^{-1}(\beta) = w(|w| + 1 - \beta)^{-1} \ (\beta \in [1, |w|]).\]

A $\Lambda$-word $w$ is *reduced* if $w(\beta + 1) \neq w(\beta)^{-1}$ for each $1 \leq \beta < |w|$. We denote by $R(\Lambda, X)$ the set of all reduced $\Lambda$-words. Clearly, $\varepsilon \in R(\Lambda, X)$. If the concatenation $uv$ of two reduced $\Lambda$-words $u$ and $v$ is also reduced then we write $uv = u \circ v$.

For $u \in W(\Lambda, X)$ and $\beta \in [1, |u|]$ by $u_\beta$ we denote the restriction of $u$ on $[1, \beta]$. If $u \in R(\Lambda, X)$ and $\beta \in [1, |u|]$ then

\[u = u_\beta \circ \tilde{u}_\beta,\]

for some uniquely defined $\tilde{u}_\beta$.

An element $\text{com}(u, v) \in R(\Lambda, X)$ is called the *longest* common initial segment of reduced $\Lambda$-words $u$ and $v$ if

\[u = \text{com}(u, v) \circ \tilde{u}, \ v = \text{com}(u, v) \circ \tilde{v}\]

for some (uniquely defined) $\Lambda$-words $\tilde{u}, \tilde{v}$ such that $\tilde{u}(1) \neq \tilde{v}(1)$. Note that $\text{com}(u, v)$ does not always exist, if it does then denote

\[c(u, v) = |\text{com}(u, v)|.\]
Now, we can define the product of two Λ-words. Let \( u, v \in R(\Lambda, X) \). If \( \com(u^{-1}, v) \) is defined then
\[
    u^{-1} = \com(u^{-1}, v) \circ \tilde{u}, \quad v = \com(u^{-1}, v) \circ \tilde{v},
\]
for some uniquely defined \( \tilde{u} \) and \( \tilde{v} \). In this event put
\[
    u \ast v = \tilde{u}^{-1} \circ \tilde{v}.
\]
The product \( \ast \) is a partial binary operation on \( R(\Lambda, X) \).

An element \( v \in R(\Lambda, X) \) is termed cyclically reduced if \( v(1) - 1 \neq v(|v|) \).

We say that an element \( v \in R(\Lambda, X) \) admits a cyclic decomposition if \( v = c^{-1} \circ u \circ c \), where \( c, u \in R(\Lambda, X) \) and \( u \) is cyclically reduced. Observe that a cyclic decomposition is unique (whenever it exists). We denote by \( \CDR(\Lambda, X) \) the set of all words from \( R(\Lambda, X) \) which admit a cyclic decomposition.

Below we refer to Λ-words as infinite words usually omitting Λ whenever it does not produce any ambiguity.

The following result establishes the connection between infinite words and length functions.

**Theorem 5** ([28]). Let \( \Lambda \) be a discretely ordered abelian group and \( X \) be a set. Then any subgroup \( G \) of \( \CDR(\Lambda, X) \) has a free Lyndon length function with values in \( \Lambda \) – the restriction \( L|_G \) on \( G \) of the standard length function \( L \) on \( \CDR(\Lambda, X) \).

The converse of the theorem above was obtained by Chiswell [9].

**Theorem 6** ([9]). Let \( G \) have a free Lyndon length function \( L : G \to A \), where \( \Lambda \) is a discretely ordered abelian group. Then there exists a set \( X \) and a length preserving embedding \( \phi : G \to \CDR(\Lambda, X) \), that is, \( |\phi(g)| = L(g) \) for any \( g \in G \).

**Corollary 3** ([9]). Let \( G \) have a free Lyndon length function \( L : G \to \Lambda \), where \( \Lambda \) is an arbitrary ordered abelian group. Then there exists an embedding \( \phi : G \to \CDR(N, X) \), where \( N = \mathbb{Z} \oplus \Lambda \) is discretely ordered with respect to the right lexicographic order and \( X \) is some set, such that, \( |\phi(g)| = (0, L(g)) \) for any \( g \in G \).

### 2.3 Universal trees

Let \( G \) be a subgroup of \( \CDR(\Lambda, X) \) for some discretely ordered abelian group \( \Lambda \) and a set \( X \). We assume \( G, \Lambda, \) and \( X \) to be fixed for the rest of this section.

Briefly recall (see [23] for details) how one can construct a universal \( \Lambda \)-tree \( \Gamma_G \) for \( G \). Every element \( g \in G \) is a function
\[
    g : [1, |g|] \to X^\pm,
\]
with the domain \([1, |g|]\) which is a closed segment in \( \Lambda \). Since \( \Lambda \) can be viewed as a \( \Lambda \)-metric space then \([1, |g|]\) is a geodesic connecting 1 and \(|g|\), and every \( \alpha \in [1, |g|] \) can be represented as a pair \((\alpha, g)\). Let
\[
    S_G = \{(\alpha, g) \mid g \in G, \alpha \in [0, |g|]\}.
\]
Since for every \( f, g \in G \) the word \( \text{com}(f, g) \) is defined, one can introduce an equivalence relation on \( S_G \) as follows: \((\alpha, f) \sim (\beta, g)\) if and only if \( \alpha = \beta \in [0, c(f, g)] \). Now, let \( \Gamma_G = S_G / \sim \) and \( \varepsilon = (0, 1) \), where \((\alpha, f)\) is the equivalence class of \((\alpha, f)\). It was shown in [\text{23}] that \( \Gamma_G \) is a \( \Lambda \)-tree with a designated vertex \( \varepsilon \) and a metric \( d : \Gamma_G \times \Gamma_G \to \Lambda \), on which \( G \) acts by \( \Lambda \)-isometries so that for every \( g \in G \) the distance \( d(\varepsilon, g \cdot \varepsilon) \) is exactly \(|g|\). Moreover, \( \Gamma_G \) is equipped with the labeling function \( \xi : (\Gamma_G \setminus \{\varepsilon\}) \to X^\pm \), where \( \xi(v) = g(\alpha) \) if \( v = (\alpha, g) \).

Next, for every \( v_0, v_1 \in \Gamma_G \) such that \( d(v_0, v_1) = 1 \) we call the ordered pair \((v_0, v_1)\) the edge from \( v_0 \) to \( v_1 \). Here, if \( e = (v_0, v_1) \) then denote \( v_0 = o(e) \), \( v_1 = t(e) \) which are respectively the origin and terminus of \( e \). There exists a natural orientation, with respect to \( \varepsilon \), of edges in \( \Gamma_G \), where an edge \((v_0, v_1)\) is positive if \( d(\varepsilon, v_1) = d(\varepsilon, v_0) + 1 \), and negative otherwise. Denote by \( E(G) \) the set of edges in \( \Gamma_G \). If \( e \in E(\Gamma_G) \) and \( e = (v_0, v_1) \) then the pair \((v_1, v_0)\) is also an edge and denote \( e^{-1} = (v_1, v_0) \). Obviously, \( o(e) = t(e^{-1}) \). Because of the orientation, we have a natural splitting

\[
E(\Gamma_G) = E(\Gamma_G)^+ \cup E(\Gamma_G)^- ,
\]

where \( E(\Gamma_G)^+ \) and \( E(\Gamma_G)^- \) denote respectively the sets of positive and negative edges. Now, we can define a function \( \sigma : E(\Gamma_G)^+ \to X^\pm \) as follows: if \( e = (v_0, v_1) \in E(\Gamma_G)^+ \) then \( \sigma(e) = \xi(v_1) \). Next, \( \sigma \) can be extended to \( E(\Gamma_G)^- \) (and hence to \( E(\Gamma_G) \)) by setting \( \sigma(f) = \sigma(f^{-1})^{-1} \) for every \( f \in E(\Gamma_G)^- \).

Example 1. Let \( F = F(X) \) be a free group on \( X \). Hence, \( F \) embeds into (coincides with) \( \text{CDR}(\mathbb{Z}, X) \) and \( \Gamma_F \) with the labeling \( \sigma \) defined above is just the Cayley graph of \( F \) with respect to \( X \). That is, \( \Gamma_F \) is a labeled simplicial tree.

The action of \( G \) on \( \Gamma_G \) induces the action on \( E(\Gamma_G) \) as follows \( g \cdot (v_0, v_1) = (g \cdot v_0, g \cdot v_1) \) for each \( g \in G \) and \((v_0, v_1) \in E(\Gamma_G) \). It is easy to see that \( E(\Gamma_G)^+ \) is not closed under the action of \( G \) but the labeling is equivariant as the following lemma shows.

Lemma 4. If \( e, e' \in E(\Gamma_G) \) belong to one \( G \)-orbit then \( \sigma(e) = \sigma(e') \).

Proof. Let \( e = (v_0, v_1) \in E(\Gamma_G)^+ \). Hence, there exists \( g \in G \) such that \( v_0 = o(g) \), \( v_1 = o(g \cdot 1) \). Let \( f \in G \) and consider the following cases.

Case 1. \( c(f^{-1}, g) = 0 \)

Then \( f \ast g = f \circ g \). Assume that \( \alpha = 0 \) (the case when \( \alpha \neq 0 \) is considered similarly). Then \( f \cdot v_0 = \langle |f|, f \rangle = \langle |f|, f \circ g \rangle \), and \( f \cdot v_1 = \langle |f| + 1, f \circ g \rangle \). Hence, \( f \cdot e \in E(\Gamma_G)^+ \) and \( \sigma(f \cdot e) = \xi(f \cdot v_1) = g(1) = \xi(v_1) = \sigma(e) \).

Case 2. \( c(f^{-1}, g) > 0 \)

(a) \( \alpha + 1 \leq c(f^{-1}, g) \)

Then \( f \cdot v_0 = \langle |f| + \alpha - 2 \alpha, f \rangle = \langle |f| - \alpha, f \rangle \) and \( f \cdot v_1 = \langle |f| - (\alpha + 1), f \rangle \).

So, \( d(\varepsilon, f \cdot v_1) < d(\varepsilon, f \cdot v_0) \) and \( f \cdot e \in E(\Gamma_G)^- \). Now,

\[
\sigma(f \cdot e) = \sigma((f \cdot e)^{-1}) = \sigma((f \cdot v_1, f \cdot v_0)^{-1})^{-1} = \xi(f \cdot v_0)^{-1} = f(|f| - \alpha)^{-1} = g(\alpha + 1) = \xi(v_1) = \sigma(e) .
\]
(b) \( \alpha = c(f^{-1}, g) \)

We have \( f \cdot v_0 = \langle |f| - \alpha, f \rangle \) and \( f \cdot v_1 = \langle |f| \rangle = \langle |f| + (\alpha + 1) - 2c(f^{-1}, g), f \rangle \). It follows that \( f \cdot e \in E(\Gamma_G)^+ \) and \( \sigma(f \cdot e) = \xi(f \cdot v_1) = (f * g)(\langle f \rangle - \alpha + 1) \). At the same time, \( f * g = f_1 \circ g_1 \), where \( |f_1| = |f| - c(f^{-1}, g) = |f| - \alpha, g = g_0 \circ g_1, |g_0| = \alpha \), so, \( (f * g)(\langle f \rangle - \alpha + 1) = g_1(1) = g(\alpha + 1) \) and \( \sigma(f \cdot e) = g(\alpha + 1) = \xi(\langle \alpha + 1, g \rangle) = \xi(v_1) = \sigma(e) \).

(c) \( \alpha > c(f^{-1}, g) \)

Hence, \( f \cdot v_0 = \langle |f| + \alpha - 2c(f^{-1}, g), f \rangle \) and \( f \cdot v_1 = \langle |f| \rangle = \langle |f| + \alpha + 1 - 2c(f^{-1}, g), f \rangle \). Obviously, \( f \cdot e \in E(\Gamma_G)^+ \) and

\[
\sigma(f \cdot e) = \xi(f \cdot v_1) = (f * g)(\langle f \rangle + \alpha + 1 - 2c(f^{-1}, g)) = g_1(\alpha + 1 - c(f^{-1}, g)) = g(\alpha + 1) = \xi(v_1) = \sigma(e),
\]

where \( f * g = f_1 \circ g_1, |f_1| = |f| - c(f^{-1}, g) = |f| - \alpha, g = g_0 \circ g_1, |g_0| = \alpha \).

Thus, in all possible cases we get \( \sigma(f \cdot e) = \sigma(e) \) and the required statement follows.

Let \( v, w \) be two points of \( \Gamma_G \). Since \( \Gamma_G \) is a \( \Lambda \)-tree there exists a unique geodesic connecting \( v \) to \( w \), which can be viewed as a “path” in the following sense. A path from \( v \) to \( w \) is a sequence of edges \( p = \{ e_\alpha \}, \alpha \in [1, d(v, w)] \) such that \( o(e_1) = v, t(e_{d(v, w)}) = w \) and \( t(e_\alpha) = o(e_{\alpha + 1}) \) for every \( \alpha \in [1, d(v, w) - 1] \).

In other words, a path is an “edge” counter-part of a geodesic and usually, for the path from \( v \) to \( w \) (which is unique since \( \Gamma_G \) is a tree) we are going to use the same notation as for the geodesic between these points, that is, \( p = [v, w] \). In the case when \( v = w \) the path \( p \) is empty. The length of \( p \) we denote by \( |p| \) and set \( |p| = d(v, w) \). Now, the path label \( \sigma(p) \) for a path \( p = \{ e_\alpha \} \) is the function \( \sigma : \{ e_\alpha \} \rightarrow X^\pm \), where \( \sigma(e_\alpha) \) is the label of the edge \( e_\alpha \).

**Lemma 5.** Let \( v, w \) be points of \( \Gamma_G \) and \( p \) the path from \( v \) to \( w \). Then \( \sigma(p) \in R(\Lambda, X) \).

**Proof.** From the definition of \( \Gamma_G \) it follows that the statement is true when \( v = \varepsilon \).

Let \( v_0 = Y(\varepsilon, v, w) \) and let \( p_v \) and \( p_w \) be the paths from \( \varepsilon \) respectively to \( v \) and \( w \). Also, let \( p_1 \) and \( p_2 \) be the paths from \( v_0 \) to \( v \) and \( w \). Since \( \sigma(p_v), \sigma(p_w) \in R(\Lambda, X) \) then \( \sigma(p_1), \sigma(p_2) \in R(\Lambda, X) \) as subwords. Hence, \( \sigma(p) \notin R(\Lambda, X) \) implies that the first edges \( e_1 \) and \( e_2 \) correspondingly of \( p_1 \) and \( p_2 \) have the same label. But this contradicts the definition of \( \Gamma_G \) because in this case \( t(e_1) \sim t(e_2) \), but \( t(e_1) \neq t(e_2) \).

As usual, if \( p \) is a path from \( v \) to \( w \) then its inverse denoted \( p^{-1} \) is a path from \( w \) back to \( v \). In this case, the label of \( p^{-1} \) is \( \sigma(p)^{-1} \), which again an element of \( R(\Lambda, X) \).

Define \( V_G = \{ v \in \Gamma_G : \exists g \in G : v = \langle |g|, g \rangle \} \).
which is a subset of points in $\Gamma_G$ corresponding to the elements of $G$. Also, for every $v \in \Gamma_G$ let

\[ \text{path}_G(v) = \{ \sigma(p) \mid p = [v, w] \text{ where } w \in V_G \}. \]

The following lemma follows immediately.

**Lemma 6.** Let $v \in V_G$. Then $\text{path}_G(v) = G \subset CDR(\Lambda, X)$.

The action of $G$ on $E(\Gamma_G)$ extends to the action on all paths in $\Gamma_G$; hence, Lemma 4 extends to the case when $e$ and $f$ are two $G$-equivalent paths in $\Gamma_G$.

## 3 From $\mathbb{Z}^n$-trees to metric spaces

Let $G$ be a finitely generated $\mathbb{Z}^n$-free group and $\Gamma_G$ the associated universal $\mathbb{Z}^n$-tree described in Subsection 2.3. Observe that in this case every $\mathbb{Z}$-subtree of $\Gamma_G$ is locally finite. In this section we construct a metric space $\tilde{\Gamma}_G$, which is a compactification of $\Gamma_G$, equipped with an action of $G$ without a global fixed point such that the orbit every point is infinite.

### 3.1 Compactification of a simplicial tree

Let $T$ be a locally finite simplicial tree (that is, $\mathbb{Z}$-tree) and let $d_T$ be a usual metric on $T$. That is, if $x, y \in T$ then there exists a unique geodesic path $p(x, y)$ connecting them and hence $d_T(x, y) = \|p(x, y)\|$, where $\|p(x, y)\|$ is the length of the path $p(x, y)$. It is easy to see that $d_T$ is an ultrametric on $T$.

For $x, y, z \in T$ let $(x \cdot y)_z = \frac{1}{2}(d_T(x, z) + d_T(y, z) - d_T(x, y))$.

Let $x \in T$. Recall (see Subsection 2.2.1) that a $T$-ray, or a geodesic ray from $x$ is an isometry $r : \mathbb{Z}_+ \to T$ such that $r(1) = x$. Two geodesic rays $r_1, r_2$ are equivalent ($r_1 \sim r_2$) if the intersection $r_1 \cap r_2$ is also a geodesic ray. It is easy to check that “$\sim$” is an equivalence relation on the set of all geodesic rays in $T$; let $\partial T$ be the set of equivalent classes. For $x \in T$ and $a \in \partial T$ denote by $[x, a]$ the geodesic ray from $x$ to $a$.

Similarly, for $a, b \in \partial T$ a geodesic line from $a$ to $b$, denoted $(a, b)$, is an isometry $r : \mathbb{Z} \to T$ such that for every $n \in \mathbb{Z}$ the restriction $r|_{[-\infty, n]}$ is a geodesic ray from $r(n)$ to $a$, and the restriction $r|_{[n, \infty)}$ is a geodesic ray from $r(n)$ to $b$.

Let $w \in T$. It is easy to see that every $a \in \partial T$ can be put in correspondence with a unique infinite geodesic ray $w, w_1, w_2, \ldots$, $w_i \in T$ from $w$. Now, let $a, b \in \partial T$ corresponding to rays $w, x_1, x_2, \ldots$ and $w, y_1, y_2, \ldots$ and define

\[ (a \cdot b)_w = \lim_{n \to \infty} (x_1 \cdot y_i)_w. \]

Similarly, for $a \in \partial T$ represented by a geodesic ray $w, x_1, x_2, \ldots$ and $y \in T$ we define

\[ (a \cdot y)_w = \lim_{n \to \infty} (x_i \cdot y)_w. \]
Consider $\overline{T} = T \cup \partial T$ and define a function $d' : \overline{T} \times \overline{T} \to \mathbb{R}$ as follows
\[ d'(x, y) = e^{-(x-y)_w}, \ x, y \in \overline{T}. \]

Lemma 7. For every $x, y, z \in \overline{T}$, the function $d'$ satisfies the following properties

(a) $d'(x, y) \geq 0$,
(b) $d'(x, y) = 0 \iff x = y \in \partial T$,
(c) $d'(x, y) = d'(y, x)$,
(d) $d'(x, y) \leq \max \{d'(x, z), d'(y, z)\}$.

Proof. (a), (b) and (c) follow immediately from the definition.

To prove (d) notice that if $x \neq y \in \partial T$ are represented by the geodesic rays $w, x_1, x_2, \ldots$ and $w, y_1, y_2, \ldots$ then there exists $N > 0$ such that $(x \cdot y)_w = (x_n \cdot y_m)_w$ for every $n, m > N$ (otherwise $w, x_1, x_2, \ldots \sim w, y_1, y_2, \ldots$ and $x = y$). The same holds for $x \in \partial T$ and $y \in T$, that is, there exists $N > 0$ such that $(x \cdot y)_w = (x_n \cdot y)_w$ for every $n > N$. In other words it is enough to check (d) for $x, y, z \in T$, that is, we have to show that
\[ e^{-(x \cdot y)_w} \leq \max \{e^{-(x \cdot z)_w}, e^{-(y \cdot z)_w}\}. \]

In particular, the inequality holds if $(x \cdot y)_w = (x \cdot z)_w = (y \cdot z)_w$.

Since $T$ is a tree then the following holds for every $x, y, z \in T$
\[ (x \cdot y)_w > (x \cdot z)_w \Rightarrow (x \cdot z)_w = (y \cdot z)_w. \]

Thus we can consider the following cases

1. $(x \cdot y)_w > (x \cdot z)_w$ or $(x \cdot y)_w > (y \cdot z)_w$.

   Immediately $e^{-(x \cdot y)_w} < e^{-(x \cdot z)_w} = e^{-(y \cdot z)_w}$.

2. $(x \cdot z)_w > (x \cdot y)_w$ or $(x \cdot z)_w > (y \cdot z)_w$.

   Then $(x \cdot y)_w = (y \cdot z)_w$ and $e^{-(x \cdot y)_w} = e^{-(y \cdot z)_w} = \max \{e^{-(x \cdot z)_w}, e^{-(y \cdot z)_w}\}$.

3. $(y \cdot z)_w > (x \cdot y)_w$ or $(y \cdot z)_w > (x \cdot z)_w$.

   Then $(x \cdot y)_w = (x \cdot z)_w$ and $e^{-(x \cdot y)_w} = e^{-(x \cdot z)_w} = \max \{e^{-(x \cdot z)_w}, e^{-(y \cdot z)_w}\}$.

Eventually, define $d : \overline{T} \times \overline{T} \to \mathbb{R}$ as follows
\[ d(x, y) = \begin{cases} d'(x, y) & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases} \]

(4)

Lemma 8. $d$ is an ultrametric on $\overline{T}$.
Proof. Immediately follows from Lemma \[\text{[8]}\].

It is easy to see that \(d(x, y) \leq 1\) for every \(x, y \in \Omega\). In particular, \(d(x, y) = 1\) if either \(x \neq y \in \partial \Omega\) and \((x, y)\) is a geodesic line from \(x\) to \(y\) which passes through \(w\), or \(x, y \in \partial \Omega\) and the geodesic ray \([x, y]\) contains \(w\).

Let \(x \in \Omega\). It is easy to see that for every \(\delta > 0\) the ball \(B_\delta(x)\) of radius \(\delta\) centered at \(x\) is a subset of \(\Omega\) of the following type

\[B_\delta(x) = \{x\} \cup \{y \in \Omega \mid (x \cdot y)_w \geq -\ln \delta\}.
\]

Hence, if \(z\) is the closest to \(w\) vertex of the segment \([w, x]\) such that \((x \cdot z)_w = d(w, z) \geq -\ln \delta\) then \(B_\delta(x) = T(w, z) \cup \partial \Omega\). If such \(z\) does not exist, that is, if \(d(w, t) < -\ln \delta\) for every \(t \in [w, x]\) then \(B_\delta(x) = \{x\}\).

Similarly, for \(x \in \partial \Omega\). Here, for every \(\delta > 0\) we have

\[B_\delta(x) = \{y \in \Omega \mid (x \cdot y)_w \geq -\ln \delta\} = T(w, z) \cup \partial \Omega,
\]

where \(z\) is the closest to \(w\) vertex of the geodesic ray \([w, x]\) such that \((x \cdot z)_w = d(w, z) \geq -\ln \delta\). Observe that \(B_\delta(x)\) contains \(x\).

Using the above considerations we prove the following lemma.

**Lemma 9.** \(\Omega\) is compact in the topology induced by the metric \(d\).

Proof. It is enough to show that \(\Omega\) is totally bounded and complete.

Let \(\delta > 0\). Let \(K\) be a collection of points \(x \in \Omega\) such that \(d(w, x) \geq -\ln \delta\) and \(x\) is the closest to \(w\) point with this property. Then \(K\) is finite and \(T(w, x) \cup \partial \Omega\) is also finite and for every \(y \in T_0\) one can choose \(\delta_y < \delta\) such that \(B_{\delta_y}(y) = \{y\}\). It follows that \(\Omega\) can be covered by finitely many \(B_{\delta_y}(x), x \in K\) together with finitely many \(B_{\delta_y}(y), y \in T_0\). Hence, \(\Omega\) is totally bounded.

To prove completeness of \(d\) consider a Cauchy sequence \(\{x_n\} \subset \Omega\), that is, for every \(\delta > 0\) there exists \(N > 0\) such that for every \(n, m \geq N\) we have \(d(x_m, x_n) < \delta\). If \((x_m \cdot x_n)_w \geq -\ln \delta\) then follows that for every \(\delta > 0\) there exists \(N > 0\) such that for every \(n \geq N\) we have \(d(x_n, w) \geq -\ln \delta\). It follows that \(\{x_n\}_{n=1}^\infty \subset \Omega\) is a nested sequence of subsets of \(\Omega\). Hence, take \(a = \bigcap_{N=1}^\infty (T(w, x_N) \cup \partial \Omega)\) and it follows that \(x_n \to a\) as \(n \to \infty\). So, \(\Omega\) is complete and thus compact.

Observe that the metric \(d\) defined in \(\text{[8]}\) works for every simplicial tree.

### 3.2 General case

Let \(n \geq 2\) and \(\Gamma\) a \(Z^n\)-tree with a metric \(d_\Gamma : \Gamma \times \Gamma \to Z^n\). We assume that every \(Z\)-subtree of \(\Gamma\) is locally finite.

Below we use induction on \(n\) to construct a compactification of \(\Gamma\). The base of induction is the case when \(n = 1\) which was dealt with in Subsection 3.1.

In the inductive hypothesis we assume that for every \(Z^{n-1}\)-subtree \(T\) of \(\Gamma\) a compactification \(\overline{T}\) is constructed, which is a metric space with metric \(d_{\overline{T}}\) such
that $\overline{T}$ is compact in the topology induced by $d_{\overline{T}}$. Moreover, we assume that there exists a constant $C \in \mathbb{R}$ such that for every $\mathbb{Z}^{n-1}$-subtree $T$ of $\Gamma$ and every $x, y \in T$ we have $d_{\overline{T}}(x, y) \leq C$. For instance, if $n = 2$ then $C = 1$ (see Subsection 3.1).

We call $v, w \in \Gamma \mathbb{Z}^{n-1}$-equivalent ($v \sim_{\mathbb{Z}^{n-1}} w$) if $d_{\Gamma}(v, w) \in \mathbb{Z}^{n-1}$. “$\sim_{\mathbb{Z}^{n-1}}$” is an equivalence relation on $\Gamma$ and every equivalence class defines a $\mathbb{Z}^{n-1}$-subtree of $\Gamma$. For each $\mathbb{Z}^{n-1}$-subtree $T$ of $\Gamma$ and for each $k \leq n - 1$, by $\partial_k T$ (or by $\text{End}_2\mathbb{Z}_k^1(T)$) we denote the set of ends of $T$ of $\mathbb{Z}_k^1$-type, that is, equivalence classes of isometries $\gamma : [0, \infty)_{\mathbb{Z}_k^1} \to T$ (see Subsection 2.2.2 for details).

Observe that, $\Delta = \Gamma/\sim_{\mathbb{Z}^{n-1}}$ has a structure of a simplicial tree (not locally finite, in general) and denote by $d_{\Delta}$ the induced metric on $\Delta$. For a $\mathbb{Z}^{n-1}$-subtree $T$ of $\Gamma$ denote by $\nu_T$ the corresponding vertex in $\Delta$.

Fix a $\mathbb{Z}^{n-1}$-subtree $T_0$ of $\Gamma$. Then $T_0$ corresponds to a vertex $\nu_{T_0}$ in $\Delta$ and we can define the levels of $\mathbb{Z}^{n-1}$-subtrees of $\Gamma$ using $\Delta$ as follows. A $\mathbb{Z}^{n-1}$-subtree $T$ belongs to the $k$-level (with respect to $T_0$) of $\mathbb{Z}^{n-1}$-subtrees of $\Gamma$ if $d_{\Delta}(\nu_{T_0}, \nu_T) = k$. Also, a $\mathbb{Z}^{n-1}$-subtree $T'$ of $\Gamma$ is adjacent to $T$ if $d_{\Delta}(\nu_{T'}, \nu_T) = 1$.

Observe that the edge between $\nu_{T'}$ and $\nu_T$ in $\Delta$ corresponds to the pair of ends $(\alpha_{T'}, \alpha_T)$ of $\mathbb{Z}^{n-1}$-type in the adjacent trees $T'$ and $T$.

For each $k \leq n - 1$ define

$$
\partial_k \Gamma = \bigcup \{\partial_k T \mid T \text{ is a } \mathbb{Z}^{n-1} \text{-subtree of } \Gamma\},
$$

$$
\partial_n \Gamma = \{\text{maximal linear subtrees of } \Gamma\}/\sim,
$$

and

$$
\partial \Gamma = \bigcup_{k=1}^n \partial_k \Gamma.
$$

Now we construct a compactification $\overline{\Gamma}$ of $\Gamma$ as follows.

$$
\overline{\Gamma} = \Gamma \bigcup (\partial \Gamma/\sim),
$$

where $\alpha_{T'} \sim \alpha_T$ if $\alpha_{T'} \in \partial_{n-1}T'$ and $\alpha_T \in \partial_{n-1}T$ and $T', T$ are adjacent $\mathbb{Z}^{n-1}$-subtrees of $\Gamma$. In other words

$$
\overline{\Gamma} = \left(\bigcup \{T \mid T \text{ is a } \mathbb{Z}^{n-1} \text{-subtree of } \Gamma\}\right)/\sim.
$$

For each $\mathbb{Z}^{n-1}$-subtree $T$ of $\Gamma$, the compactification $\overline{T}$ embeds into $\overline{\Gamma}$ so we can identify $\overline{T}$ with its image in $\overline{\Gamma}$. Thus, if $T$ and $T'$ are adjacent $\mathbb{Z}^{n-1}$-subtrees then $\overline{T} \cap \overline{T'} = \partial_{n-1}T \cap \partial_{n-1}T' = \{a\}$, where $a$ is the identified $\alpha_{T'}$ and $\alpha_T$.

The next step is to introduce a metric $d_{\overline{T}}$ on $\overline{\Gamma}$. By definition, $\overline{\Gamma}$ is a union of compactifications $\overline{T}$ of all its $\mathbb{Z}^{n-1}$-subtrees $T$. Recall that all $\mathbb{Z}^{n-1}$-subtrees of $\Gamma$ are split in levels around a fixed $\mathbb{Z}^{n-1}$-subtree $T_0$. Denote by $\mathcal{T}_k$ be the set of all $\mathbb{Z}^{n-1}$-subtrees of the $k$-level. Each $\mathcal{T}_k$ is countable (as a countable union of countable sets) and we can enumerate its elements in an arbitrary way. Thus we set

$$
d_{\overline{T}}(x, y) = d_{\overline{T_0}}(x, y), \ x, y \in \overline{T_0}
$$
and 
\[ d_T(x, y) = \frac{1}{2^{k+n}} d_{T_n}(x, y), \ x, y \in T_n \]
for each \( T_n \in T_k \).

Thus having defined \( d_T \) on each \( T \) we extend it to \( \Gamma \) as follows. Let \( x, y \in \Gamma \)
and we consider three cases.

(i) If \( x, y \in \Gamma \setminus \partial \Gamma \) then there exists a sequence \( T_1, \ldots, T_k \) of \( \mathbb{Z}^n \)-subtrees of \( \Gamma \) such that \( x \in T_1 \) and \( T_i \) is adjacent to \( T_{i+1} \) for each \( i \in [1, k-1] \). Denote \( a_i = T_i \cap T_{i+1}, \ i \in [1, k-1] \). Observe that the sequence \( T_1, \ldots, T_k \) corresponds to a geodesic \( v_{T_1}, \ldots, v_{T_k} \) in \( \Delta \). Now set
\[ d_\Gamma(x, y) = d_{T_1}(x, a_1) + \sum_{i=1}^{k-2} d_{T_{i+1}}(a_i, a_{i+1}) + d_{T_k}(a_{k-1}, y). \]

(ii) If \( x \in \Gamma \setminus \partial \Gamma \), \( y \in \partial \Gamma \) then there exists an infinite sequence \( T_1, \ldots, T_k, \ldots \) of \( \mathbb{Z}^n \)-subtrees of \( \Gamma \) with \( T_i \) adjacent to \( T_{i+1} \) for each \( i \geq 1 \) such that \( x \in T_1 \) and the sequence of linear subtrees \( [x, a_k] \) converges to the linear subtree \( [x, y] \) as \( k \to \infty \) (here \( a_i = T_i \cap T_{i+1}, \ i \geq 1 \)). Now set
\[ d_\Gamma(x, y) = d_{T_1}(x, a_1) + \sum_{i=1}^{\infty} d_{T_{i+1}}(a_i, a_{i+1}). \]

(iii) If \( x, y \in \partial \Gamma \) then there exists an infinite sequence \( \ldots, S_{-k}, \ldots, S_0, \ldots, S_k, \ldots \) of \( \mathbb{Z}^n \)-subtrees of \( \Gamma \) with \( S_i \) is adjacent to \( S_{i+1} \) for each \( i \) such that the sequence of linear subtrees \( (a_{-k}, a_k) \) converges to the linear subtree \( (x, y) \) as \( k \to \infty \) (here \( a_i = S_i \cap S_{i+1}, \ i \in \mathbb{N} \)). Now set
\[ d_\Gamma(x, y) = \sum_{i=-\infty}^{\infty} d_{S_{i+1}}(a_i, a_{i+1}). \]

Observe that \( d_\Gamma(x, y) \leq 2C \) for every \( x, y \in \Gamma \), it follows from the construction (recall that \( C \) bounds original metric on the compactification of every \( \mathbb{Z}^n \)-tree of \( \Gamma \)). Indeed, the maximal possible value of \( d_\Gamma(x, y) \) is attained when \( x, y \in \partial \Gamma \) and the infinite sequence of \( \mathbb{Z}^n \)-subtrees of \( \Gamma \) connecting \( x \) and \( y \) contains \( T_0 \) - in this case \( d_\Gamma(x, y) \leq C \sum_{n=0}^{\infty} \frac{1}{2^n} = 2C \).

**Lemma 10.** \( d_\Gamma \) is a metric on \( \Gamma \).

**Proof.** The properties
(a) \( d_\Gamma(x, y) \geq 0 \),
(b) \( d_\Gamma(x, y) = 0 \iff x = y \),
(c) \( d_\Gamma(x, y) = d_\Gamma(y, x) \),
follow immediately from the definition. Let us prove

\[(d)\quad d_\Gamma(x, y) \leq d_\Gamma(x, z) + d_\Gamma(y, z).\]

Let \(x, y, z \in \Gamma\). Then each of these points either belongs to the compactification of some \(\mathbb{Z}^{n-1}\)-subtree of \(\Gamma\), or it is an element of \(\partial_1 \Gamma\), and \(x, y, z\) are pairwise connected by linear subtrees in \(\Gamma\). The corresponding vertices (or ends) in \(\Delta\) form a tripod in \(\Delta\) whose sides are \(K_{x,y}, K_{x,z}, K_{y,z}\). If \(q = K_{x,y} \cap K_{x,z} \cap K_{y,z}\), then there exists a \(\mathbb{Z}^{n-1}\)-subtree \(T\) of \(\Gamma\) such that \(q = v_T\). It follows that

\[d_\Gamma(x, y) = A + d_\Gamma(a, b) + B; \quad d_\Gamma(x, z) = A + d_\Gamma(a, c) + C,\]

where \(a, b, c \in T\), \(A = d_\Gamma(x, a)\), \(B = d_\Gamma(y, b)\), \(C = d_\Gamma(z, c)\), and the triangle inequality for the triple \(x, y, z\) reduces to one for the triple \(a, b, c\) which holds since \(d_\Gamma\) is the original metric on \(T\) rescaled by a constant factor.

\[\square\]

**Lemma 11.** \(\Gamma\) is compact in the topology induced by the metric \(d_\Gamma\).

**Proof.** It is enough to show that \(\Gamma\) is totally bounded and complete.

Let \(\delta > 0\). \(\Gamma_0\) is compact so it can be covered by finitely many balls of radius \(\frac{\delta}{2}\) and centered at some \(y_1, \ldots, y_k \in \Gamma_0\). Hence, all points of intersection of \(\Gamma_0\) with the compactifications of the adjacent trees are covered too. Let \(T_N\) be adjacent to \(T_0\) so that \(d_\Gamma(y_i, y_j) < \frac{\delta}{2}\) and let \(a_N = T_0 \cap T_N \in B_{\delta/2}(y_i)\). Then \(\Gamma(\varepsilon, a_N) \cup \partial \Gamma(\varepsilon, a_N) \subseteq B_{\delta/2}(y_i)\). Indeed, for each \(y \in \Gamma(\varepsilon, a_N) \cup \partial \Gamma(\varepsilon, a_N)\) we have \(d_\Gamma(y, x) < \sum_{n=0}^{\infty} \frac{C}{2^n} = \frac{C}{2^n} < \frac{\delta}{2}\) and we get the required from the triangle inequality. It follows that for all \(x\) adjacent to \(T_0\), the set \(\Gamma(\varepsilon, a_N) \cup \partial \Gamma(\varepsilon, a_N)\) is in one of finitely many balls of radius \(\delta\). If \(S_1, \ldots, S_m\) is the list of \(\mathbb{Z}^{n-1}\)-subtrees adjacent to \(T_0\) (all of which belong to the 1-level) then the procedure above can be repeated for each of them producing another finite list of the 2-level subtrees which are not covered etc. But with the growth of level the diameters of these subtrees decrease and eventually become smaller than \(\delta\). Hence, if \(S\) is such a \(\mathbb{Z}^{n-1}\)-subtree of radius \(\frac{C}{2^n}\) then for every point \(x \in S\) we have \(S \cup (\Gamma(\varepsilon, x) \cup \partial \Gamma(\varepsilon, x)) \subseteq B_\delta(x)\). Thus, all but finitely many \(\mathbb{Z}^{n-1}\)-subtrees of \(\Gamma\) can be covered by finitely many balls of radius \(\delta\). Now we are done since each of the balls left is compact, that is, \(\Gamma\) is totally bounded.

To prove completeness of \(\Gamma\) let \(\{x_n\} \subseteq \Gamma\) be a Cauchy sequence, that is, for every \(\delta > 0\) there exists \(N > 0\) such that for every \(n, m \geq N\) we have \(d_\Gamma(x_m, x_n) < \delta\). Let \(\Gamma_0\) be the subtree of \(\Gamma\) spanned by \(\{x_n\} \cup \{x\}\), where \(x \in T_0\) and consider the following cases.

- **(a)** There exists a \(\mathbb{Z}^{n-1}\)-subtree \(T\) of \(\Gamma\) such that \(\Gamma_0 \cap \partial_{n-1}T\) is infinite.

In this case there exists a subsequence \(\{y_k\}\) of \(\{x_n\}\) such that every segment \([x, y_k]\) intersects \(\partial_{n-1}T\). If \(\{a_k\} \subseteq \partial_{n-1}T\) then \(\{a_k\}\) is a Cauchy sequence, hence it converges to some \(b \in \partial_{n-1}T\). But \(d_\Gamma(y_k, a_k) \to 0\) as
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$k \to \infty$, so $d_T(y_k, b) \to 0$ as $k \to \infty$. It follows that \( \{y_k\} \) converges to $b \in \overline{\Gamma}$ and since \( \{x_n\} \) is a Cauchy sequence then \( \{x_n\} \) also converges to $b$.

(b) The intersection $\Gamma_0 \cap \partial_{n-1}T$ is finite for every $\Z^{n-1}$-subtree $T$ of $\Gamma$.

It follows that $\Gamma_0$ is locally finite and infinite, so there exists an infinite branch which converges to some $b \in \partial \Gamma$. Here, since the level of $S_n$ such that $x_n \in S_n$ grows to infinity, it follows that $d_{\Gamma_n}(x_n, b) \to 0$ as $n \to \infty$ and \( \{x_n\} \) also converges to $b$.

So, $\overline{T}$ is complete and thus compact. 

Remark 2. In the process of defining $d_T$ we rescaled the original metric on the compactification of every $\Z^{n-1}$-subtree of $\Gamma$, and the rescaling constants were chosen to decrease exponentially $\approx 2^{-n}$ with respect to the order number $n$ of the $\Z^{n-1}$-subtree. In principle, one can choose these constants differently - either exponentially decreasing $\approx \rho^{-n}$ for any given constant $\rho > 1$, or polynomially decreasing $\approx \frac{1}{n^r}$ for every $r > 1$ - and any choice of the constants will define a metric generating the same topology as $d_T$ defined above.

Remark 3. Let a group $G$ act freely on a $\Z^n$-tree $\Gamma$ and let $x \in \overline{\Gamma}$. Then $G$ can be put in correspondence with the subset $\{G \cdot x\}$ of $\Gamma$. Now, restriction of the metric $d_T$ to $\Gamma$ produces a metric $d' : X \to \mathbb{R}$ on the Cayley graph $X$ of $G$ with respect to some generating set by setting $d'(f, g) = d_T(f \cdot x, g \cdot x)$ (metric axioms for $d'$ follow from metric axioms for $d_T$). Observe that this rescaling $d \to d'$ of the word metric $d$ on $X$ is similar to the rescaling applied in the construction of Floyd type boundaries, where edges from $X$ are assigned lengths coming from values of a conformal factor $f$ which is a monotonically decreasing summable function $f : \mathbb{N} \to \mathbb{R}$.

3.3 Action of $G$ on $\overline{\Gamma}_G$

Let $G$ be a finitely generated $\Z^n$-free group and let $\Gamma_G$ be the corresponding minimal tree with the base point $\varepsilon$. Let $\overline{\Gamma}_G$ be the compactification of $\Gamma_G$ with respect to the metric $d_{\overline{\Gamma}_G}$ constructed in Subsection 3.2.

Theorem 7. Let $G$ be a finitely generated non-abelian $\Z^n$-free group.

(i) The action of $G$ on $\Gamma_G$ induces an action of $G$ on $\overline{\Gamma}_G$ without a global fixed point. Moreover, the orbit of every point of $\overline{\Gamma}_G$ is infinite.

(ii) The action of $G$ on $\overline{\Gamma}_G$ is continuous with respect to the topology generated by metric $d_{\overline{\Gamma}_G}$.
Proof. (i) The action of $G$ on $\Gamma_G$ is minimal and free. So, $G$ does not have a fixed point in $\Gamma_G$ and the orbit of every point of $\Gamma_G$ is infinite.

Since every end of $\mathbb{Z}^k$-type of $\Gamma_G$, where $k \leq n$, belongs to some geodesic ray from $\varepsilon$ then the action of $G$ on $\Gamma_G$ considered as a collection of geodesic rays from $\varepsilon$ naturally extends to the action on $\overline{\Gamma}_G$. Let $\alpha \in \partial \Gamma_G$. By Lemma 3.1.9 [8], if $\alpha$ is fixed by $g$ then $\alpha$ is an end of $\text{Axis}(g)$, which is the maximal linear $g$-invariant subtree of $\Gamma_G$. If the same $\alpha$ is fixed by another element $h \in G$ then $\alpha \in \partial \text{Axis}(h)$ and then $\text{Axis}(g) = \text{Axis}(h)$, from which it follows that $[g, h] = 1$. Thus, no end of $\Gamma_G$ and hence no point of $\overline{\Gamma}_G$ is fixed by every element of $G$ unless $G$ is abelian and $\Gamma_G$ is a linear subtree, which is impossible by the assumption.

Finally, let $x \in \overline{\Gamma}_G$ be such that $\{G \cdot x\}$ is finite. It follows that $|G : \text{Stab}_G(x)| < \infty$. Now, since $x \in \partial \text{Axis}(g)$ for every $g \in \text{Stab}_G(x)$ then $\text{Stab}_G(x)$ is abelian, and $G$ is virtually abelian. But $G$ is $\mathbb{Z}^n$-free, so commutation is transitive and it follows that $G$ is abelian — a contradiction.

(ii) Let $g \in G$ and let $U$ be an open subset of $\overline{\Gamma}_G$. Without loss of generality we can assume that $U$ is a base element of the topology generated by $\overline{\text{Stab}}_G$, that is, $U$ is an open ball around some point $x$ in $\overline{\Gamma}_G$. We have to show that $g^{-1} \cdot U$ is open, that is, for every $y \in g^{-1} \cdot U$ there exists $\delta > 0$ such that $g \cdot B_\delta(y) \subseteq U$.

If $y \in \Gamma_G$ then it is always possible to choose $\delta$ small enough to guarantee that $B_\delta(y) = \{y\}$, and then the required follows.

Let $y \in \overline{\Gamma}_G \setminus \Gamma_G$. Hence, $g \cdot y \in \overline{\Gamma}_G \setminus \Gamma_G$ too. Since $U$ is open, there exists $\epsilon > 0$ such that $B_\epsilon(g \cdot y) \subseteq U$. Observe that the action of $g$ corresponds to rescaling the metric and there exists a constant $C = C(g, \epsilon)$ such that for every $z \in g^{-1} \cdot B_\epsilon(g \cdot y)$ we have $d_{\overline{\Gamma}_G}(g \cdot y, g \cdot z) \leq C d_{\overline{\Gamma}_G}(y, z)$. Choose $\delta$ to be such that $\delta \leq \frac{\epsilon}{C}$. Hence, for every $z \in B_\delta(y)$ we have $d_{\overline{\Gamma}_G}(g \cdot y, g \cdot z) \leq C d_{\overline{\Gamma}_G}(y, z) \leq C \delta \leq \epsilon$, that is, $g \cdot z \in B_\epsilon(g \cdot y) \subseteq U$. In other words, $g \cdot B_\delta(y) \subseteq U$, as required.

Theorem 8. Let $G$ be a finitely generated irreducible non-abelian $\mathbb{Z}^n$-free group. Then the action of $G$ on $\partial_n \Gamma_G$ is minimal.

Proof. Let $p \in \partial_n \Gamma_G$. We are going to show that the closure of $G \cdot p$ coincides with $\partial_n \Gamma_G$. Since $p$ is an end of $\mathbb{Z}^n$-type of $\Gamma_G$, it follows that there exists a sequence of elements $\{g_i\}$ from $G$ such that $Y(\varepsilon, g_i \cdot \varepsilon, g_{i+1} \cdot \varepsilon) \to p$ as $i \to \infty$.

In fact, for every $i$, the point $Y(\varepsilon, g_i \cdot \varepsilon, g_{i+1} \cdot \varepsilon)$ belongs to the geodesic ray $[\varepsilon, p]$ in $\Gamma_G$.

Let $q \in \partial_n \Gamma_G$. Again, there exists a sequence of elements $\{h_j\}$ from $G$ such that $Y(\varepsilon, h_j \cdot \varepsilon, h_{j+1} \cdot \varepsilon) \to q$ as $j \to \infty$. To prove minimality of $\partial_n \Gamma_G$ we have to find a sequence $\{f_k\}$ of elements from $G$ such that $f_k \cdot p \to q$ with respect to the metric $d_{\overline{\Gamma}_G}$. By definition of $d_{\overline{\Gamma}_G}$, it is enough to show that for each given $\alpha \in \mathbb{Z}^n$ there exists $N \in \mathbb{N}$ such that for every $k > N$ the length of the intersection $[\varepsilon, f_k \cdot p) \cap [\varepsilon, q]$ in $\Gamma_G$ is at least $\alpha$. By Theorem 8 the action of $G$ on $\partial_n \Gamma_G$ is continuous, so for each $k$ we have $(f_k g_i) \cdot \varepsilon \to f_k \cdot p$ as $i \to \infty$. Hence, it is enough to have a sequence $\{f_k\}$ of elements from $G$ such that for every given $\alpha \in \mathbb{Z}^n$
there exists $N \in \mathbb{N}$ such that for every $k > N$ the length of the intersection $[\varepsilon, (f_k g_i) \cdot \varepsilon] \cap [\varepsilon, q]$ is greater than $\alpha$ for all big enough $i$.

Suppose there is no such sequence $\{f_k\}$. That is, for each $f \in G$ and each $i$, the length of the intersection $[\varepsilon, (f g_i) \cdot \varepsilon] \cap [\varepsilon, q]$ is bounded by some $\beta \in \mathbb{Z}^n$. Then it holds when $f = h_j^{-1}$, $j \in \mathbb{N}$ and it follows that for every $j$ there exists $i^*_j$ such that the length of the common initial subword $\text{com}(h_j^{-1}, g_{i^*_j})$ grows to infinity with the growth of $j$. In other words, we can replace the sequence $\{g_i\}_i$ converging to $p$ by its subsequence $\{g_{i^*_j}\}_j$ converging to $p$ as well, and then we can replace $\{g_{i^*_j}\}_j$ by $\{h_j^{-1}\}_j$ which also converges to $p$. So, it turns out that $h_j \cdot \varepsilon \rightarrow q$ and $h_j \cdot \varepsilon \rightarrow p$ as $j \rightarrow \infty$.

Without loss of generality we can assume that $|h_1| \in \mathbb{Z}^n \setminus \mathbb{Z}^{n-1}$ and consider a terminal subword $c$ of $h_1$ such that $|c| \in \mathbb{Z}^n \setminus \mathbb{Z}^{n-1}$. Observe that we can assume that $c^{-1}$ is an initial subword of each $h_j^{-1}$ (otherwise we can replace $h_i$ with $h_i r$ for big enough $r$). Now there exists $w \in G$ such that neither $c$, nor $c^{-1}$ cancel completely in the product $c w c^{-1}$. Observe that otherwise $G$ is either reducible, or abelian. It follows that for every $j$ and $k$ we have $h_k w h_j^{-1} = h_k' \circ (c w c^{-1}) \circ h_j^{-1}$, where $h_k = h_k' \circ c$, $h_j = h_j' \circ c$ and the length of $h_k'$ infinitely grows as $k \rightarrow \infty$. Hence, the length of the common initial subword $\text{com}(h_k w h_j^{-1}, h_k)$ infinitely grows as $k \rightarrow \infty$ and if we choose $f_k = h_k w$ for each $k$. It follows that the length of the intersection $[\varepsilon, (f_k h_j^{-1}) \cdot \varepsilon] \cap [\varepsilon, q]$, which is actually $[\varepsilon, (h_k w h_j^{-1}) \cdot \varepsilon] \cap [\varepsilon, q]$, infinitely grows as $k \rightarrow \infty$ -- a contradiction.

\[ \square \]

Remark 4. (i) The action of $G$ on $\overline{\Gamma}_G \setminus \Gamma_G = \bigcup_{k=1}^n \partial_k \Gamma_G$ may not be minimal. That is the case if $\text{clos}(\partial_n \Gamma_G) \neq \overline{\partial_n \Gamma_G} \setminus \Gamma_G$. For example, let $G = \langle a, b, u^5 \rangle$ be a subgroup of $R(\mathbb{Z}^2, X)$, where $X = \{a, b\}$ and $|u| = (0, 1) \in \mathbb{Z}^2$ is such that $u(\alpha) = a$ if $\alpha \in [1, |u|]$ (observe that $u \notin G$).

Obviously, $G$ is a f.g. irreducible non-abelian $\mathbb{Z}^2$-free group. At the same time, the $\mathbb{Z}$-type end defined by the sequence $\{u^a\}$ is in $\partial_1 \Gamma_G$, but the corresponding point does not belong to $\text{clos}(\partial_2 \Gamma_G)$ (nor to $\text{clos}(G \cdot \varepsilon)$).

(ii) If $n \geq 2$ then $\partial_n \Gamma_G$ may be neither open, nor closed in $\Gamma_G$.

(iii) $\partial_n \Gamma_G$ is Borel-measurable in $\Gamma_G$.

For any $a, b \in \overline{\Gamma}_G$ denote by $[a, b]$ the geodesic in $\Gamma_G$ connecting $a$ and $b$. Observe that if $a, b \in \Gamma_G$ then $[a, b]$ is a geodesic in the usual sense, that is, an isometry from a closed segment in $\mathbb{Z}^n$ into $\Gamma_G$, but if either $a$, or $b$, or both belong to $\overline{\Gamma}_G \setminus \Gamma_G$ then $[a, b]$ is the image of an isometric map from an open (or half-open) segment in $\mathbb{Z}^n$ into $\Gamma_G$.

If $a \neq b$ then we denote by $[a, b]$ the closure of $[a, b]$ in the space $\overline{\Gamma}_G$. We also set $[a, a] = \{a\}$ for every $a \in \overline{\Gamma}_G$.

Lemma 12. Let $a, b \in \overline{\Gamma}_G$. Then there exists $c \in \Gamma_G$ such that $[\varepsilon, a] \cap [\varepsilon, b] = [\varepsilon, c]$. Moreover, if either $a, b \in \Gamma_G$, or $a \notin [\varepsilon, b]$ and $b \notin [\varepsilon, a]$, then $c \in \Gamma_G$. 
Lemma 13. Let $\mathcal{G}$ be the set of all $\mathbb{Z}^n$-subtrees of $\Gamma_G$ and let

$$\mathcal{G} = \{S \subset \Gamma_G \mid S \in \mathcal{G}\}$$

be the set of closures of these subtrees in $\Gamma_G$. Observe that in the case when $n = 1$, the sets $\mathcal{G}$ and $\mathcal{G}$ both coincide with the set of vertices of the simplicial tree $\Gamma_G$.

**Definition 1.** For $a, b \in \Gamma_G$ such that $a \neq b$, define

$$h_n(a, b) = \#\{S \in \mathcal{G} \mid S \cap [a, b] \neq \emptyset\} + \frac{\#\{\mathcal{G} \cap \{a, b\}\}}{2} - 1,$$

where $\#\{S \in \mathcal{G} \mid S \cap [a, b] \neq \emptyset\}$ is the number of distinct $\mathbb{Z}^n$-subtrees of $\Gamma_G$ intersecting with $[a, b]$ and $\frac{\#\{\mathcal{G} \cap \{a, b\}\}}{2}$ is a number from $\{0, \frac{1}{2}, 1\}$.

If either $a$, or $b$ belongs to $\partial_n \Gamma_G$ then $h_n(a, b) = \infty$. Also set $h_n(a, a) = 0$ for each $a \in \Gamma_G$.

For $p \in \Gamma_G$, we set $h_n(p) = h_n(\varepsilon, p)$ and for $g \in G$, we set $h_n(g) = h_n(g \cdot \varepsilon) = h_n(\varepsilon, g \cdot \varepsilon)$.

**Remark 5.** Consider the simplicial tree $\Delta = \Gamma_G / \sim_{\mathbb{Z}^n}$ defined in Subsection 2.2, where two points are $\mathbb{Z}^n$-equivalent if they belong to the same $\mathbb{Z}^n$-subtree of $\Gamma_G$. Then each geodesic segment $[a, b] \subset \Gamma_G$, where $a \neq b \in \Gamma_G$, naturally projects to $\Delta$ forming a path there. The value $h_n(a, b)$ equals to the length of this path.

If $v \in \Gamma_G$ lies in the $\mathbb{Z}^n$-subtree of the $k$-level (the $\mathbb{Z}^n$-subtree $T_0$ containing $\varepsilon$ has level 0) then $h_n(v) = k$.

If $g \in G$ and $|g| = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ then $h_n(g) = a_n$. In particular, $h_n : G \to \mathbb{Z}$ is a seminorm on $G$.

**Lemma 13.** Let $a, b \in \Gamma_G$ and $p \in [a, b]$, then

$$h_n(a, b) = h_n(a, p) + h_n(p, b).$$

**Proof.** Immediately follows from the definition of $h_n$.  

**Lemma 14.** Let $\{a_i\}$ and $\{b_i\}$ be two sequences of points in $\Gamma_G$. Suppose that for each vertex $v \in \Gamma_G$, the set $\{i \in \mathbb{N} \mid v \in [a_i, b_i]\}$ is at most finite. Then $d_{\Gamma_G}(a_i, b_i) \to 0$ as $i \to \infty$. 


Proof. Suppose that $d_{\overline{\Gamma}_G}(a_i, b_i) \to 0$ as $i \to \infty$. Then there exists $\delta > 0$ such that for every $N \in \mathbb{N}$ there exists $i > N$ such that $d_{\overline{\Gamma}_G}(a_i, b_i) > \delta$. In other words, the number of indices $i$ such that $d_{\overline{\Gamma}_G}(a_i, b_i) > \delta$ is infinite and without loss of generality we can assume that $d_{\overline{\Gamma}_G}(a_i, b_i) > \delta$ holds for every $i$.

Since $\overline{\Gamma}_G$ is compact, it follows that there exist subsequences $\{a_{i_j}\}$ and $\{b_{i_k}\}$ which converge in $\overline{\Gamma}_G$ respectively to $p$ and $q$. Let $z = Y(\varepsilon, p, q)$ and let $z_q \in \Gamma_G$ be such that the edge $(z, z_q)$ belongs to the geodesic $[z, \varepsilon]$, and let $z_p \in \Gamma_G$ be such that the edge $(z, z_p)$ belongs to the geodesic $[z, q]$. Then from the convergence $\{a_{i_j}\} \to p$ it follows that for infinitely many indices $i_j$ the point $a_{i_j}$ belongs to the subtree $\Gamma_G(\varepsilon, z_p)$ and from the convergence $\{b_{i_k}\} \to q$ it follows that for infinitely many indices $i_k$ the point $b_{i_k}$ belongs to the subtree $\Gamma_G(\varepsilon, z_q)$. Observe that the subtrees $\Gamma_G(\varepsilon, z_p)$ and $\Gamma_G(\varepsilon, z_q)$ are connected by the geodesic $[z_p, z_q]$ which contains $z$, and the set $\{i_m \in \mathbb{N} \mid z \in [a_{i_m}, b_{i_m}]\}$ is infinite – a contradiction. □

Lemma 15. Let $\{g_i\}$ be a sequence of elements from $G$ and let $y \in \partial_n \Gamma_G$ be an end of $\mathbb{Z}^n$-type. Assume that there exists a vertex $v_0 \in \Gamma_G$ such that the sequence $\{g_i \cdot v_0\}$ converges to $y$. Then for every $v \in \Gamma_G$, the sequence $\{g_i \cdot v\}$ converges to $y$.

Proof. Consider the sequence of numbers $\{H_i\} = \{h_n(g_i \cdot v_0)\}$. Since $\{g_i \cdot v_0\}$ converges to an end of $\mathbb{Z}^n$-type, it follows that $\{H_i\}$ tends to infinity. Let $H = h_n(v_0, v)$. Observe that $h_n(g \cdot v_0, g \cdot v) = h_n(v_0, v) = H$ for every $g \in G$. Consequently, for any $i \in \mathbb{N}$ and $z \in [g_i \cdot v_0, g_i \cdot v]$ we have $h_n(z) \geq H - H$. Since the sequence $\{H_i - H\}$ tends to infinity, it follows that for each vertex $v \in \Gamma_G$ the set $\{i \in \mathbb{N} \mid z \in [g_i \cdot v_0, g_i \cdot v]\}$ is at most finite. The result then follows by Lemma [4]. □

Corollary 4. Let $\{a_i\}$ and $\{b_i\}$ be two sequences of points in $\overline{\Gamma}_G$. Suppose that $h_n(Y(\varepsilon, a_i, b_i)) \to \infty$ as $i \to \infty$ and $d_{\overline{\Gamma}_G}(a_i, b_i) \to 0$ as $i \to \infty$. Then $d_{\overline{\Gamma}_G}(a_i, b_i) \to 0$ as $i \to \infty$. In particular, $\{a_i\}$ converges to a point $\omega \in \overline{\Gamma}_G$ if and only if $\{b_i\}$ converges to $\omega$.

Proof. Observe that if $x \in \overline{\Gamma}_G$ and $y \in [\varepsilon, x]$, then $h_n(y) \leq h_n(z)$ for each $z \in [\varepsilon, x]$ (by Lemma [3]). If $a, b \in \overline{\Gamma}_G$ and $c = Y(\varepsilon, a, b)$ then we have $c \in [\varepsilon, a]$, $c \in [\varepsilon, b]$, and $[a, b] = [c, a] \cup [c, b]$. Then it follows that

$$\min_{z \in [a, b]} h_n(z) = h_n(Y(\varepsilon, a, b)).$$

Therefore, we have $\min_{z \in [a, b]} h_n(z) = h_n(Y(\varepsilon, a, b))$ for all $i \in \mathbb{N}$. Since we assume $h_n(Y(\varepsilon, a_i, b_i)) \to \infty$ as $i \to \infty$, it follows that for each $v \in \Gamma_G$ the set $\{i \in \mathbb{N} \mid v \in [a_i, b_i]\}$ is at most finite (because we have $v \notin [a_i, b_i]$ whenever $\min_{z \in [a_i, b_i]} h_n(z) > h_n(v)$). Then the result follows from Lemma [14]. □

Recall that for every $v_1, v_2 \in \Gamma_G$, the set $\Gamma_G(v_1, v_2) = \{x \in \Gamma_G \mid v_2 \in [v_1, x]\}$ is a subtree of $\Gamma_G$ spanned by all geodesic rays containing $[v_1, v_2]$. We can extend this definition to $\overline{\Gamma}_G$, that is, for every $v_1, v_2 \in \Gamma_G$ we define $\overline{\Gamma}_G(v_1, v_2) = \{x \in \overline{\Gamma}_G \mid v_2 \in [v_1, x]\}$.
Similarly, for \( v_1, v_2 \in \Gamma_G \) we define
\[
\partial_n \Gamma_G(v_1, v_2) = \{ x \in \partial_n \Gamma_G \mid v_2 \in [v_1, x] \}.
\]
Observe that \( \partial_n \Gamma_G(v_1, v_2) = \Gamma_G(v_1, v_2) \cap \partial_n \Gamma_G \).

**Lemma 16.** For every \( v_1, v_2 \in \Gamma_G \)

(a) the set \( \Gamma_G(v_1, v_2) \) is open and closed in \( \Gamma_G \).

(b) the set \( \partial_n \Gamma_G(v_1, v_2) \) is open and closed in \( \partial_n \Gamma_G \).

**Proof.** (a) If \( v_1 = v_2 \) then \( \Gamma_G(v_1, v_2) = \Gamma_G \).
Suppose \( v_1 \neq v_2 \). Let \( v_0 \) be a vertex of \( \Gamma_G \) lying on the geodesic \([v_1, v_2]\) and adjacent to \( v_2 \). Then \( \Gamma_G(v_1, v_2) = \Gamma_G(v_0, v_2) \). Next, observe that
\[
\Gamma_G \setminus \Gamma_G(v_0, v_2) = \Gamma_G(v_2, v_0)
\]
and that for any \( a \in \Gamma_G(v_2, v_0) \) and \( b \in \Gamma_G(v_0, v_2) \), the geodesic \([a, b]\) contains the edge \((v_0, v_2)\). Consequently, there exists \( \delta > 0 \) such that \( d_{\Gamma_G}(a, b) > \delta \) whenever \( a \in \Gamma_G(v_2, v_0) = \Gamma_G \setminus \Gamma_G(v_0, v_2) \) and \( b \in \Gamma_G(v_0, v_2) \). Therefore,
\[
\bigcup_{p \in \Gamma_G(v_1, v_2)} \{ x \in \Gamma_G \mid d_{\Gamma_G}(x, p) < \delta \} = \Gamma_G(v_1, v_2),
\]
which means that \( \Gamma_G(v_1, v_2) \) is open. Similar argument shows that \( \Gamma_G \setminus \Gamma_G(v_1, v_2) = \Gamma_G(v_2, v_1) \) is open, hence \( \Gamma_G(v_1, v_2) \) is closed.

(b) The required statement follows from the fact that
\[
\partial_n \Gamma_G(v_1, v_2) = \Gamma_G(v_1, v_2) \cap \partial_n \Gamma_G.
\]
Again, if \( v_1 \neq v_2 \) then there exists \( \delta > 0 \) such that
\[
\bigcup_{p \in \partial_n \Gamma_G(v_1, v_2)} \{ x \in \partial_n \Gamma_G \mid d_{\Gamma_G}(x, p) < \delta \} = \partial_n \Gamma_G(v_1, v_2),
\]
which shows that \( \partial_n \Gamma_G(v_1, v_2) \) is open since \( \{ x \in \partial_n \Gamma_G \mid d_{\Gamma_G}(x, p) < \delta \} \subset \partial_n \Gamma_G \) is the open ball of radius \( \delta \) centered at \( p \). Similarly, it follows that \( \partial_n \Gamma_G \setminus \partial_n \Gamma_G(v_1, v_2) = \partial_n \Gamma_G(v_2, v_1) \) is open, hence \( \partial_n \Gamma_G(v_1, v_2) \) is closed in \( \partial_n \Gamma_G \). \( \square \)

### 4 Proof of Theorem 1

In this section, we prove Theorem 1 according to the scheme outlined in the introduction. For the rest of this section we fix \( n \in \mathbb{N} \), a finitely generated irreducible non-abelian \( \mathbb{Z}^n \)-free group \( G \), the associated universal \( \mathbb{Z}^n \)-tree \( \Gamma_G \) with the base point \( \epsilon \) described in Subsection \( \text{23} \) and the compactification \( \overline{\Gamma_G} \) of \( \Gamma_G \) constructed in Section \( \text{3} \). Recall that \( \partial_n \Gamma_G \) is the set of ends of \( \Gamma_G \) of \( \mathbb{Z}^n \)-type. We also fix a non-degenerate probability measure \( \mu \) on \( G \).
4.1 $\mu$-proximallity of $\overline{\Gamma}_G$ and $\partial_n \Gamma_G$

**Proposition 1.** If $\nu$ is a $\mu$-stationary measure on $\overline{\Gamma}_G$ then

(i) $\nu$ is continuous,

(ii) $\nu(\partial_n \Gamma_G) = 1$,

(iii) $\nu(E) > 0$ for every non-empty open set $E \subset \overline{\Gamma}_G$ with $E \cap \partial_n \Gamma_G \neq \emptyset$.

**Proof.** (i) Continuity of $\nu$ follows from Lemma 1 because the action of $G$ on $\overline{\Gamma}_G$ has no finite orbits (see Theorem 7).

(ii) In the case when $n = 1$ the set $\overline{\Gamma}_G \setminus \partial_n \Gamma_G = \Gamma_G$ is countable. Then, since $\nu$ is continuous, we have $\nu(\Gamma_G) = 0$. Hence, $\nu(\partial_n \Gamma_G) = 1$.

Assume that $n \geq 2$. Recall that $\Xi_{n-1}$ is the set of all $\mathbb{Z}^{n-1}$-subtrees of $\Gamma_G$ and $\hat{\Xi}_{n-1}$ is the set of closures of these subtrees in $\overline{\Gamma}_G$ (see Subsection 3.3). The action of $G$ on $\overline{\Gamma}_G$ induces an action of $G$ on $\hat{\Xi}_{n-1}$, and since $\hat{\Xi}_{n-1}$ is countable then the orbits of this action are countable.

Let

$$K = \bigcup_{S \neq T \in \hat{\Xi}_{n-1}} (S \cap T).$$

Then $K$ is a countable subset of $\overline{\Gamma}_G$ because $\hat{\Xi}_{n-1}$ is countable and for any $S \neq T \in \hat{\Xi}_{n-1}$, the set $S \cap T$ consists of at most one point. Consequently, $\nu(K) = 0$ (since $\nu$ is continuous).

Now, let $\hat{\Xi}'_{n-1}$ be the family of subsets in $\overline{\Gamma}_G$ of the form $S \setminus K$, where $S \in \hat{\Xi}_{n-1}$. By construction, we obviously have:

(1) $\hat{\Xi}'_{n-1}$ is countable,

(2) the action of $G$ on $\overline{\Gamma}_G$ induces an action of $G$ on $\hat{\Xi}'_{n-1}$, and that the orbits of the latter action are countable,

(3) for any $S \in \hat{\Xi}_{n-1}$, the set $S \setminus K \in \hat{\Xi}'_{n-1}$ is a measurable subset of $\overline{\Gamma}_G$, because $S$ is closed and $K$ is countable,

(4) for any $g \in G$ and $S' \in \hat{\Xi}'_{n-1}$, either $g \cdot S' = S'$, or $(g \cdot S') \cap S' = \emptyset$.

It then follows from Lemma 1 that $\nu(S') = 0$ for each $S' \in \hat{\Xi}'_{n-1}$, whence

$$\nu\left( \bigcup_{\hat{\Xi}'_{n-1}} S' \right) = 0.$$

Thus, we have

$$\nu\left( \bigcup_{\hat{\Xi}_{n-1}} S \right) = \nu\left( \bigcup_{\hat{\Xi}'_{n-1}} S' \right) + \nu(K) = 0.$$
Consequently
\[
\nu(\partial_n \Gamma_G) = \nu(\Gamma_G) - \nu \left( \bigcup_{\mathbb{Z}_{n-1}} S \right) = 1 - 0 = 1.
\]

(iii) This follows from (ii) by Lemma 3 because \( G \) is countable and acts on \( \partial_n \Gamma_G \) minimally (see Theorem 3).

Define \( \mathcal{S} \) to be the subset in \( G^{\mathbb{Z}_+} \) consisting of all sequences \( \{g_i\} \) with the following three properties:

(i) \( \{g_i \cdot \varepsilon\} \) converges to a point from \( \Gamma_G \setminus \Gamma_G \),
(ii) \( \{g_i^{-1} \cdot \varepsilon\} \) converges to a point from \( \Gamma_G \setminus \Gamma_G \),
(iii) \( \{h_n(g_i)\} \) tends to infinity as \( i \to \infty \) (recall Definition 4).

**Proposition 2.** A.e. path \( \{\tau_i\} \) of the random \( \mu \)-walk contains a subsequence \( \{\tau_i'\} \in \mathcal{S} \).

**Proof.** First, we prove that a.e. path \( \{\tau_i\} \) of the random \( \mu \)-walk contains a subsequence with the property (iii) from the definition of \( \mathcal{S} \). Let \( d \in \mathbb{N} \). Since \( G \) is an irreducible \( \mathbb{Z}^n \)-free group, it follows that there exists an element \( g_{2d} \in G \) such that \( h_n(g_{2d}) > 2d \). Since \( \mu \) is a non-degenerate measure, there exists \( k \in \mathbb{N} \) such that \( \mu^k(g_{2d}) > 0 \). It follows that for a.e. path \( \{\tau_i\} \) of the random \( \mu \)-walk there exists \( m \in \mathbb{N} \) such that \( \tau_{m+k} = \tau_m g_{2d} \) (to see this, use ergodicity of the Bernoulli shift in the space of sequences of \( \mu^k \)-equidistributed random variables \( \{\tau_{jk}^{-1} \tau_{jk+k}\}_{j \in \mathbb{Z}_+} \), or observe that the measure \( P_{\mu} \) of the set of all \( \mu \)-walk paths with \( \tau_{jk+k} \neq \tau_{jk} g_{2d} \) for each \( j \in \mathbb{N} \) equals \( \lim_{s \to \infty} (1 - \delta)^s = 0 \), where \( \delta = \mu^k(g_{2d}) > 0 \). Then

\[
h_n(\tau_{m}^{-1} \tau_{m+k}) = h_n(g_{2d}) > 2d.
\]

Since \( h_n : G \to \mathbb{Z} \) is a seminorm (see Remark 4), in particular, it satisfies the triangle inequality, and \( h_n(h) = h_n(h^{-1}) \) for each \( h \in G \), it follows that

\[
\min\{h_n(\tau_{m+k}), h_n(\tau_m)\} > d.
\]

We have thus showed that for any \( d \in \mathbb{N} \), the path \( \{\tau_i\} \) a.s. has an element \( \tau_i \) such that \( h_n(\tau_i) > d \). It obviously follows that \( \{\tau_i\} \) a.s. has a subsequence \( \{\tau_i'\} \) such that \( \{h_n(\tau_i')\} \) tends to infinity (the property (iii)).

Since \( \Gamma_G \) is compact, \( \{G \cdot \varepsilon\} \) is discrete, and \( \{\tau_i'\} \) has no infinite constant subsequences, it follows that \( \{\tau_i'\} \) has a subsequence \( \{\tau_i''\} \) such that \( \{\tau_i'' \cdot \varepsilon\} \) converges to a point of \( \Gamma_G \setminus \Gamma_G \) (the property (i)). Similar argument shows that \( \{\tau_i''\} \) has a subsequence \( \{\tau_i'''\} \) such that \( \{(\tau_i''')^{-1} \cdot \varepsilon\} \) converges to a point of \( \Gamma_G \setminus \Gamma_G \) (the property (ii)). Therefore, \( \{\tau_i'''\} \) is a subsequence of \( \{\tau_i\} \) and \( \{\tau_i'\} \in \mathcal{S} \).
Lemma 17. Let \( \{g_i\} \) be a sequence from \( \mathcal{G} \). Let \( \omega \) and \( \bar{\omega} \) be points from \( \overline{\Gamma_G} \) which the sequences \( \{g_i, \varepsilon\} \) and \( \{g_i^{-1}, \varepsilon\} \) converge to. Then for each \( p \in \overline{\Gamma_G} \setminus \omega \), the sequence \( \{g_i \cdot p\} \) converges to \( \omega \).

Proof. In order to prove the lemma, we will show that for every \( p \in \overline{\Gamma_G} \setminus \omega \) we have
\[
h_n(p_i) \xrightarrow[i \to \infty]{} \infty,
\] where \( p_i = Y(\varepsilon, g_i \cdot \varepsilon, g_i \cdot p) \), and then apply Corollary 4.

Observe that for each \( i \) we have \( g_i^{-1} \cdot p_i = Y(\varepsilon, g_i^{-1} \cdot \varepsilon, p) \) because
\[
g_i^{-1} \cdot p_i = g_i^{-1} \cdot Y(\varepsilon, g_i \cdot \varepsilon, g_i \cdot p) = Y(g_i^{-1} \cdot \varepsilon, g_i \cdot p) = Y(\varepsilon, g_i^{-1} \cdot \varepsilon, p).
\]

Let us prove that the sequence \( \{h_n(Y(\varepsilon, g_i^{-1} \cdot \varepsilon, p))\} \) is bounded. Assume that it is not. Then it has a subsequence \( \{h_n(Y(\varepsilon, g_i^{-1} \cdot \varepsilon, p))\} \) which tends to infinity. Then, by Corollary 4, \( \{g_i^{-1} \cdot \varepsilon\} \) converges to \( p \), which contradicts the assumption that \( g_i^{-1} \cdot \varepsilon \to \bar{\omega} \neq p \). Thus, \( \{h_n(Y(\varepsilon, g_i^{-1} \cdot \varepsilon, p))\} \) is bounded, that is, there exists \( N \in \mathbb{N} \) such that
\[
h_n(Y(\varepsilon, g_i^{-1} \cdot \varepsilon, p)) < N \quad \text{for each} \quad i \in \mathbb{N}.
\]
Therefore, for each \( i \in \mathbb{N} \) we have
\[
h_n(g_i \cdot \varepsilon, p_i) = h_n(\varepsilon, g_i^{-1} \cdot p_i) \overset{\text{def}}{=} h_n(g_i^{-1} \cdot p_i) = h_n(Y(\varepsilon, g_i^{-1} \cdot \varepsilon, p)) < N.
\]
Now, \( \{g_i\} \in \mathcal{G} \), so
\[
h_n(\varepsilon, g_i \cdot \varepsilon) \overset{\text{def}}{=} h_n(g_i \cdot \varepsilon) \overset{\text{def}}{=} h_n(g_i)
\]
tends to \( \infty \) (the property (iii)), while \( p_i \in [\varepsilon, g_i \cdot \varepsilon] \) and hence from
\[
h_n(\varepsilon, p_i) + h_n(p_i, g_i \cdot \varepsilon) = h_n(\varepsilon, g_i \cdot \varepsilon),
\]
it follows that \( h_n(p_i) \to \infty \). Finally, by Corollary 4, this implies that \( \{g_i \cdot p\} \) converges to \( \omega \) since \( \{g_i \cdot \varepsilon\} \) does. \( \square \)

Lemma 17 obviously implies the following corollary.

Corollary 5. Let \( \{g_i\} \) be a sequence from \( \mathcal{G} \) with \( \{g_i \cdot \varepsilon\} \) converging to a point \( \omega \in \overline{\Gamma_G} \). Then for any continuous Borel probability measure \( \lambda \) on \( \overline{\Gamma_G} \), the sequence \( \{g_i \cdot \lambda\} \) converges to the Dirac measure \( \delta_\omega \).

Now, we can draw the main result of this subsection from the results above.

Proposition 3. There exists a unique \( \mu \)-stationary measure \( \nu_\mu \) on \( \overline{\Gamma_G} \) such that

(i) \( \nu_\mu \) is continuous, \( \nu_\mu(\partial_\mu \Gamma_G) = 1 \), and \( \nu_\mu(E) > 0 \) for every open non-empty subset \( E \subset \overline{\Gamma_G} \) with \( E \cap \partial_\mu \Gamma_G \neq \emptyset \).
(ii) the pair \((\Gamma_G, \nu_\delta)\) is a \(\mu\)-boundary of \(G\), that is, for a.e. path \(\tau = \{\tau_i\}\) of the random \(\mu\)-walk, the sequence of measures \(\{\tau_i \cdot \nu_\delta\}\) converges to a random Dirac measure \(\delta_{\omega(\tau)}\), where \(\omega(\tau) \in \Gamma_G\), moreover, with probability 1 \(\nu_\delta(\partial_n \Gamma_G) = 1\).

Proof. Observe that uniqueness of a \(\mu\)-stationary measure \(\nu_\delta\) follows from Assertion 3 (the last one is applicable, because \(\Gamma_G\) is compact and metrizable). Next, (i) follows from Proposition 3.

Let us prove (ii). We have to show that a.e. path \(\{\tau_i\}\) of the random \(\mu\)-walk contains a subsequence \(\{\tau_{i_j}\}\) such that the sequence \(\{\tau_{i_j} \cdot \nu_\delta\}\) converges to some Dirac measure. By combining this with Theorem 5, we see that for a.e. path \(\{\tau_i\}\) the sequence \(\{\tau_i \cdot \nu_\delta\}\) converges to some measure, and at the same time has a subsequence converging to a Dirac measure. It follows that the limit of \(\{\tau_i \cdot \nu_\delta\}\) is this Dirac measure, and the proof of (ii) is completed (see the definition of \(\mu\)-boundary in Subsection 2.1).

Let \(\{\tau_i\}\) be a path of the random \(\mu\)-walk. Then by Theorem 5, the sequence \(\{\tau_i \cdot \nu_\delta\}\) a.s. converges to some limit. By Proposition 3, \(\{\tau_i\}\) a.s. contains a subsequence \(\{\tau_{i_j}\}\) \(\in \mathcal{S}\). By Corollary 6, the subsequence \(\{\tau_{i_j} \cdot \nu_\delta\}\) converges to a Dirac measure (since \(\nu_\delta\) is continuous, see Proposition 3). Thus, the limit of \(\{\tau_i \cdot \nu_\delta\}\) is a.s. the same Dirac measure. \(\square\)

Remark 6. Observe that in many cases, in order to apply the pattern from the proof of Proposition 3, it suffices to define \(\mathcal{S}\) with the property (i) only. In fact, the property (ii) is redundant in our case too, we include it due to the local tactic of our proof. However, the property (iii) (or some analogue of it) is indispensable in our case. Note that the property (iii) does not imply that the ends which \(\{g_i \cdot \varepsilon\}\) and \(\{g_i^{-1} \cdot \varepsilon\}\) converge to are ends of \(\mathbb{Z}^n\)-type.

Corollary 6. There exists a unique \(\mu\)-stationary measure \(\nu_\delta\) on \(\partial_n \Gamma_G\) which is continuous and the pair \((\partial_n \Gamma_G, \nu_\delta)\) is a \(\mu\)-boundary of \(G\).

Proof. By Proposition 3, there is a unique \(\mu\)-stationary measure \(\nu_\delta'\) on \(\Gamma_G\), the pair \((\Gamma_G, \nu_\delta')\) is a \(\mu\)-boundary for \(G\), and \(\nu_\delta'\) is concentrated on \(\partial_n \Gamma_G\). Since \(\nu_\delta'(\partial_n \Gamma_G) = 1\) and \(\partial_n \Gamma_G\) is a measurable \(G\)-invariant subset in \(\Gamma_G\), it then follows that the pair \((\partial_n \Gamma_G, \nu_\delta)\), where \(\nu_\delta\) is the measure in \(\mathcal{P}(\partial_n \Gamma_G)\) associated with the measure \(\nu_\delta' \in \mathcal{P}(\Gamma_G)\), is also a \(\mu\)-boundary of \(G\) (see Lemma 3). \(\square\)

4.2 Stability of paths in \(\mathbb{Z}^n\)-free groups

Proposition 4. Let \(\omega \in \partial_n \Gamma_G\) be an end of \(\mathbb{Z}^n\)-type and let \(\{g_i\}\) be a sequence of elements from \(G\). Assume that \(\{g_i \cdot \nu_\delta\}\) converges to the Dirac measure \(\delta_\omega\). Then, for each \(x \in \Gamma_G\), the sequence \(\{g_i \cdot x\}\) converges to \(\omega\).

Proof. Take two distinct points \(\omega_1, \omega_2 \in \partial_n \Gamma_G\), then take two distinct vertices \(v_1, v_2 \in \Gamma_G\) on the geodesic \([\omega_1, \omega_2]\), and consider the sets \(B_1 = \Gamma_G(v_1, v_2)\) and \(B_2 = \Gamma_G(v_2, v_1)\). Observe that \(B_1\) and \(B_2\) are open subsets of \(\Gamma_G\) (by Lemma 4), and that the intersections \(B_1 \cap \partial_n \Gamma_G\) and \(B_1 \cap \partial_n \Gamma_G\) are non-empty (because
our definition of $B_1$ and $B_2$ implies that $B_1$ contains one of the points $\omega_1$, $\omega_2$, while $B_2$ contains the other one). It then follows from Proposition 1 (iii) that $\nu_\mu(B_1) > 0$ and $\nu_\mu(B_2) > 0$.

Let us show that the sequences $\{g_i \cdot v_1\}$ and $\{g_i \cdot v_2\}$ converge to $\omega$. To do this it suffices to prove that for each vertex $v \in [\varepsilon, \omega]$, the set $B_v = \Gamma_G(\varepsilon, v)$ contains all but finitely many elements of our two sequences. Since $B_v \ni \omega$, the condition that $\{g_i \cdot \nu_\mu\}$ converges to $\delta_\omega$ implies that the sequence $\{(g_i \cdot \nu_\mu)(B_v)\}$ converges to $\delta_\omega(B_v) = 1$. Hence, there exists $N_v \in \mathbb{N}$ such that $(g_j \cdot \nu_\mu)(B_v) > 1 - \min\{\nu_\mu(B_1), \nu_\mu(B_2)\}$ whenever $j > N_v$. In particular, since $(g_j \cdot \nu_\mu)(B_v) = \nu_\mu(g_j^{-1} \cdot B_v)$, for each $j > N_v$, we have $(g_j^{-1} \cdot B_v) \cap B_1 \neq \emptyset$ and $(g_j^{-1} \cdot B_v) \cap B_2 \neq \emptyset$, whence $B_v \cap (g_j \cdot B_1) \neq \emptyset$ and $B_v \cap (g_j \cdot B_2) \neq \emptyset$. Therefore, for each $j > N_v$, there exist points $p^{(j)}(1) \in B_v \cap (g_j \cdot B_1)$ and $p^{(j)}(2) \in B_v \cap (g_j \cdot B_2)$. Now, on the one hand, we have $[p^{(j)}(1), p^{(j)}(2)] \subset B_v$ (because $p^{(j)}(1), p^{(j)}(2) \in B_v$, while $B_v$ is geodesically convex by definition). On the other hand, we have $[p^{(j)}(1), p^{(j)}(2)] \supset [g_j \cdot v_1, g_j \cdot v_2]$ (because $p^{(j)}(1) \in g_j \cdot B_1 = \Gamma_G(g_j \cdot v_1, g_j \cdot v_2)$ and $p^{(j)}(2) \in g_j \cdot B_2 = \Gamma_G(g_j \cdot v_2, g_j \cdot v_1)$ (see the definition of the subtree $X(a, b)$ of a $\Lambda$-tree in Subsection 2.2 and [1] Ch. 2, Lemma 1.5)). It follows that $g_j \cdot v_1, g_j \cdot v_2 \in B_v$ (for each $j > N_v$). We have thus proved that the sequences $\{g_i \cdot v_1\}$ and $\{g_i \cdot v_2\}$ converge to $\omega$.

Since $\omega$ is an end of $\mathbb{Z}^n$-type, it then follows from Lemma 6 that $\{g_i \cdot x\}$ converges to $\omega$ for every $x \in \Gamma_G$.

4.3 Maximality of the boundary $\partial_n \Gamma_G$

According to Corollary 6, the pair $(\partial_n \Gamma_G, \nu_\mu)$ is a $\mu$-boundary of $G$. Now we would like to show that it is a maximal $\mu$-boundary.

Let $X$ be a measurable $G$-space with a (quasi-invariant) measure $\nu$. Recall that the action of $G$ on $X$ is called ergodic if $\nu(Y) \in \{0, 1\}$ for each $G$-invariant measurable subset $Y \subset X$.

**Lemma 18** ([1]). Let $G$ be a countable group, $\mu \in \mathcal{P}(G)$ and $\tilde{\mu} \in \mathcal{P}(G)$ the reflected measure defined by $\tilde{\mu}(g) = \mu(g^{-1})$. Let $(M_+, \nu_+)$ and $(M_-, \nu_-)$ be respectively a $\mu$-boundary and a $\tilde{\mu}$-boundary of $G$. Then the action of $G$ on the product $(M_+ \times M_-, \nu_+ \times \nu_-)$ is ergodic.

**Remark 7.** In [1], Lemma 18 is stated and proved in the case when $(M_+, \nu_+)$ and $(M_-, \nu_-)$ are maximal, respectively, $\mu$- and $\tilde{\mu}$-boundaries. But the proof works verbatim for the general case.

**Theorem 9.** If $\mu$ has finite first moment with respect to a finite word metric on $G$ and $\nu_\mu$ is a $\mu$-stationary measure then the measure space $(\partial_n \Gamma_G, \nu_\mu)$ is a maximal $\mu$-boundary of $G$.

**Proof.** In order to prove the theorem, we are going to construct a map

$$S : \partial_n \Gamma_G \times \partial_n \Gamma_G \rightarrow 2^G$$

and then show that it satisfies all the requirements of the Strip Criterion (see Theorem 5 as well as Corollary 6 and Corollary 8).
Let $a, b \in \partial_n \Gamma_G$. There exists a unique bi-infinite $\mathbb{Z}^n$-geodesic $(a, b)$ in $\Gamma_G$ from $a$ to $b$. Hence, define

$$S(a, b) = \{ g \in G \mid g \cdot \varepsilon \in (a, b) \}.$$ 

Obviously, $S(a, b) \subset G$ is measurable and $G$-equivariant.

**Claim 1.** Let $\nu_\varepsilon$ be a $\mu$-stationary measure on $\partial_n \Gamma_G$ and $\nu_-$ be a $\tilde{\mu}$-stationary measure on $\partial_n \Gamma_G$, where $\tilde{\mu}$ is the reflected measure of $\mu$ defined by $\tilde{\mu}(g) = \mu(g^{-1})$. Then for $(\nu_- \times \nu_\varepsilon)$-a.e. pair $(b_-, b_+) \in \partial_n \Gamma_G \times \partial_n \Gamma_G$, the set $S(b_-, b_+)$ is non-empty.

First of all, observe that since $G$ is non-abelian, there exist at least two elements $g, h \in G$ such that $\text{Axis}(g) \neq \text{Axis}(h)$. Moreover, since $G$ is irreducible, we can assume that $h_n(g), h_n(h) > 0$, hence, the ends of $\text{Axis}(g)$ and $\text{Axis}(h)$ belong to $\partial_n \Gamma_G$. It follows that there exist $\omega_1, \omega_2 \in \partial_n \Gamma_G$ such that the bi-infinite $\mathbb{Z}^n$-geodesic $(\omega_1, \omega_2)$ contains $\varepsilon$.

Take distinct points $v_1, v_2$ on the geodesic $(\omega_1, \omega_2)$ such that $\varepsilon \in [v_1, v_2]$ and define the map

$$S_{v_1, v_2} : \partial_n \Gamma_G \times \partial_n \Gamma_G \to 2^G$$

as follows: for the pair $(a, b) \in \partial_n \Gamma_G \times \partial_n \Gamma_G$ set

$$S_{v_1, v_2}(a, b) = \{ g \in G \mid g \cdot [v_1, v_2] \subset (a, b) \quad \text{and} \quad g \cdot v_1 \in (a, g \cdot v_2) \}.$$ 

Here, the condition $g \cdot [v_1, v_2] \subset (a, b)$ is equivalent to the conditions $g \cdot v_1 \in (a, b)$ and $g \cdot v_2 \in (a, b)$, while the second condition $g \cdot v_1 \in (a, g \cdot v_2)$ just means that $g$ maps $[v_1, v_2]$ into the geodesic $(a, b)$ preserving the orientation. Observe that since $\varepsilon \in [v_1, v_2]$, for any $a, b \in \partial_n \Gamma_G$ we have

$$S_{v_1, v_2}(a, b) \subseteq S(a, b).$$

Now, we are going to prove that for $(\nu_- \times \nu_\varepsilon)$-a.e. pair $(b_-, b_+) \in \partial_n \Gamma_G \times \partial_n \Gamma_G$, the set $S_{v_1, v_2}(b_-, b_+)$ is non-empty. Hence, it is going to imply Claim 1.

Corollary 1 implies that $(\partial_n \Gamma_G, \nu_\varepsilon)$ is a $\mu$-boundary of $G$ and $(\partial_n \Gamma_G, \nu_\varepsilon)$ is a $\tilde{\mu}$-boundary of $G$. In particular, the action of $G$ on the space $\partial_n \Gamma_G \times \partial_n \Gamma_G$ is ergodic with respect to the product measure $\nu_- \times \nu_\varepsilon$ (see Lemma 18).

Let $B_1 = \partial_n \Gamma_G(v_2, v_1)$ and $B_2 = \partial_n \Gamma_G(v_1, v_2)$, and consider the set

$$\Omega_{v_1, v_2} = B_1 \times B_2 \subset \partial_n \Gamma_G \times \partial_n \Gamma_G.$$

Our definitions immediately imply that

$$S_{v_1, v_2}(b_-, b_+)=\{ g \in G \mid g^{-1} \cdot (b_-, b_+) \in \Omega_{v_1, v_2} \}.$$ 

Observe that by Lemma 10, $B_1$ and $B_2$ are open subsets of $\partial_n \Gamma_G$. Next, both $B_1$ and $B_2$ are non-empty since $\omega_1 \in B_1$ and $\omega_2 \in B_2$. It follows from Lemma 3 that $\nu_-(B_1) > 0$ and $\nu_+(B_2) > 0$. Hence,

$$(\nu_- \times \nu_\varepsilon)(\Omega_{v_1, v_2}) = (\nu_- \times \nu_\varepsilon)(B_1 \times B_2) = \nu_-(B_1) \cdot \nu_+(B_2) > 0.$$
Since the action of $G$ on the space $(\partial_n \Gamma_G \times \partial_n \Gamma_G, \nu_- \times \nu_+)$ is ergodic and $(\nu_- \times \nu_+)(\Omega_{\nu_1, \nu_2}) > 0$, it follows that

$$(\nu_- \times \nu_+)(G \cdot \Omega_{\nu_1, \nu_2}) = 1.$$  

In particular, for $(\nu_- \times \nu_+)$-a.e. pair $(b_-, b_+) \in \partial_n \Gamma_G \times \partial_n \Gamma_G$, the set $S_{\nu_1, \nu_2}(b_-, b_+)$ is non-empty and Claim 1 follows.

Now, recall that $G$ is a finitely generated irreducible non-abelian $\mathbb{Z}^n$-free group. 

Recall that $\Xi_{n-1}$ is the set of all $\mathbb{Z}^{n-1}$-subtrees of $\Gamma_G$. By contracting elements of $\Xi_{n-1}$ to points we obtain from $\Gamma_G$ a $\mathbb{Z}$-tree $\Delta_G$ with an action of $G$ inherited from $\Gamma_G$. From Bass-Serre Theory it follows that $\Psi_G = \Delta_G/G$ is a graph in which vertices and edges correspond to $G$-orbits of vertices and edges in $\Delta_G$. Moreover, since $G$ is finitely generated, $\Psi_G$ is finite (see, for example, [24]) and $G$ is isomorphic to the fundamental group of the graph of groups associated with $\Psi_G$. In other words, $G$ can be obtained from finitely many $\mathbb{Z}^{n-1}$-free groups $G_v, v \in \mathcal{V}(\Psi_G)$, which are $G$-stabilizers of some elements $T_v \in \Xi_{n-1}$, by means of amalgamated free products and HNN extensions along free abelian groups $G_e, e \in \mathcal{E}(\Psi_G)$ of rank not greater than $n - 1$. Moreover, since $G$ and all $G_v$ are finitely generated, and $\Psi_G$ is finite, it follows that all $G_v$ are also finitely generated (see, [10, Proposition 29, Proposition 35]). Using induction on $n$ one can similarly show that the $G$-stabilizer of any $\mathbb{Z}^m$-subtree of $\Gamma_G$, where $m \in \{1, \ldots, n-1\}$, is finitely generated.

To finish the proof of the theorem we are going to prove the following claim.

Claim 2. There exists $C(m) \in \mathbb{N}$ such that for any $m \in \{1, \ldots, n\}$, any $\mathbb{Z}^m$-subtree $T$ of $\Gamma_G$, and any distinct $a, b \in \partial_m T$ we have

$$|\{g \in G \mid g \cdot \varepsilon \in (a, b)\} \cap B_G(k)| \leq C(m)k^m,$$

where $B_G(k) = \{g \in G \mid |g|_G \leq k\}$ is the ball of radius $k$ in the Cayley graph of $G$ with respect to some (finite) generating set.

We use induction on $m$. If $m = 1$ then $T$ is a $\mathbb{Z}$-tree and $(a, b)$ is isometric to $\mathbb{Z}$. Suppose there exists $g_T \in G$ such that $g_T \cdot \varepsilon \in (a, b)$. If there exists another $h \in G$ such that $h \cdot \varepsilon \in (a, b)$ then $h g_T^{-1} \in Stab_G(T)$, that is, $h$ belongs to the coset $Stab_G(T)g_T$. Consider two cases.

(a) If $\varepsilon \in T$ then $g_T \in Stab_G(T)$ and

$$\{g \in G \mid g \cdot \varepsilon \in (a, b)\} = \{g \in Stab_G(T) \mid g \cdot \varepsilon \in (a, b)\}.$$

Since $Stab_G(T)$ is a finitely generated free group whose Cayley graph is a subtree of $T$, the required inequality

$$|\{g \in G \mid g \cdot \varepsilon \in (a, b)\} \cap B_G(k)| \leq C(T)k$$

holds for some constant $C(T)$ (here, $| \cdot |_G$ can be considered as a seminorm on $Stab_G(T)$).
(b) If $\varepsilon \notin T$ then for every $u \in Stab_G(T)$ we have $ht(g_T) > ht(u)$, that is, $g_T$ is infinitely long with respect to $u$, and $ht(g_T^{-1}ug_T) < ht(g_T)$ since $g_T^{-1}ug_T$ is the label of a finite segment of $(a, b)$. If there exist at least two distinct elements $u_1, u_2 \in Stab_G(T)$ such that $(u_1g) \cdot \varepsilon, (u_2g) \cdot \varepsilon \in (a, b)$ (the case, when there exists only one such element is trivial) then by [21, Lemma 8] we have that $[u_1, u_2] = 1$. Hence, $\{g \in G \mid g \cdot \varepsilon \in (a, b)\}$ is a subset of a cyclic subgroup $A$ of $Stab_G(T)$. It is a general fact that a cyclic subgroup of a finitely generated abelian group is quasi-isometrically embedded. Thus, $A$ is quasi-isometrically embedded into a maximal abelian subgroup $K$ of $G$, where $K$ is a finitely generated centralizer in $G$. Next, since $G$ is bi-automatic (see [20]), $K$ is quasi-isometrically embedded into $G$ (see [4]).

It follows that there exists a constant $D$ such that for every $g \in A$ we have $|g|_A \leq D|g|_G$.

Finally, we have

$$
|\{g \in G \mid g \cdot \varepsilon \in (a, b)\} \cap B_G(k)| = |\{g \in A \mid g \cdot \varepsilon \in (a, b)\} \cap B_G(k)|
\leq |\{g \in A \mid g \cdot \varepsilon \in (a, b)\} \cap B_A(Dk)| \leq C(T)k
$$

for some $C(T) \in \mathbb{N}$.

Finally, notice that there are only finitely many orbits of $\mathbb{Z}$-subtrees of $\Gamma_G$ with non-trivial $G$-stabilizers. Hence, one can choose $C(1)$ to be the maximal $C(T)$, where $T$ ranges through finitely many representatives of orbits of $\mathbb{Z}$-subtrees of $\Gamma_G$.

Suppose Claim 2 holds for $m = n - 1$, that is, there exists $C(n - 1) \in \mathbb{N}$ such that for any $\mathbb{Z}^{n-1}$-subtree $T$ of $\Gamma_G$, and any distinct $a_T, b_T \in \partial_{n-1}T$ we have

$$
|\{g \in G \mid g \cdot \varepsilon \in (a_T, b_T)\} \cap B_G(k)| \leq C(n - 1)k^{n-1}.
$$

Let $m = n$ and denote by $T_\varepsilon$ the $\mathbb{Z}^{n-1}$-subtree of $\Gamma_G$ which contains $\varepsilon$.

Observe that $(a, b)$ visits infinitely many $\mathbb{Z}^{n-1}$-subtrees of $\Gamma_G$, which we can enumerate by integers

$$
\ldots, T_{-n}, \ldots, T_{-1}, T_0, T_1, \ldots, T_n, \ldots,
$$

where $T_0$ is the “closest” one to $T_\varepsilon$ in terms of the function $h_n$. There exists a constant $M > 0$ such that for any $g \in G$ with $|g|_G \leq k$ we have $h_n(g) \leq Mk$ (the constant depends on the values of $h_n$ for the generating set of $G$). Denote by $T_k \subset \{T_i\}$ the subset of $\mathbb{Z}^{n-1}$-subtrees with the property $h_n(T, T_\varepsilon) \leq Mk$, $T \in T_k$. Obviously, $|T_k| \leq 2Mk + 1$ and for any $g \in B_G(k)$ such that $g \cdot \varepsilon \in (a, b)$ there exists $T \in T_k$ such that $g \cdot \varepsilon \in (a, b) \cap T = (a_T, b_T)$, where $a_T, b_T \in \partial_{n-1}T \subset \partial_{n-1}\Gamma_G$. Thus, we have

$$
|\{g \in G \mid g \cdot \varepsilon \in (a, b)\} \cap B_G(k)| \leq \sum_{T \in T_k} |\{g \in G \mid g \cdot \varepsilon \in (a_T, b_T)\} \cap B_G(k)|
\leq (2Mk + 1)(C(n - 1)k^{n-1}) \leq C(n)k^n,
$$

where $C(n)$ is a constant which does not depend on $a$ and $b$.  

$\square$
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