On the Dimension of Unimodular Discrete Spaces
Part I: Definitions and Basic Properties

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Abstract

This work is focused on large scale properties of infinite graphs and discrete subsets of the Euclidean space. It presents two new notions of dimension, namely the unimodular Minkowski and Hausdorff dimensions, which are inspired by the classical Minkowski and Hausdorff dimensions. These dimensions are defined for unimodular discrete spaces, which are defined in this work as a class of random discrete metric spaces with a distinguished point called the origin. These spaces provide a common generalization to stationary point processes under their Palm version and unimodular random rooted graphs.

The main novelty is the use of unimodularity in the definitions where it suggests replacing the infinite sums pertaining to coverings by large balls by the expectation of certain random variables at the origin. In addition, the main manifestation of unimodularity, that is the mass transport principle, is the key element in the proofs and dimension evaluations.

The work is structured in three companion papers which are called Part I-III. Part I (the current paper) introduces unimodular discrete spaces, the new notions of dimensions, and some of their basic properties. Part II is focused on the connection between these dimensions and the growth rate of balls. In particular, it gives versions of the mass distribution principle, Billingsley’s lemma, and Frostman’s lemma for unimodular discrete spaces. Part III establishes connections with other notions of dimension. It also discusses ergodicity and the non-ergodic cases in more detail than the first two parts. Each part contains a comprehensive set of examples pertaining to the theory of point processes, unimodular random graphs, and self-similarity, where the dimensions in question are explicitly evaluated or conjectured.

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1 Introduction

1.1 Motivations

Point process theory and random graph theory feature various infinite discrete (random) structures, e.g., subsets of the support of a point process, or subsets of the set of vertices of a random graph. Some definitions are available to quantify how large such a subset is. For instance, the asymptotic density of a subset \( \Psi \subseteq \mathbb{R}^k \) (and similarly for \( \Psi \subseteq \mathbb{Z}^k \)) is defined as the limit when \( r \to \infty \) of the number of points in \( \Psi \cap B_r \) divided by the volume of \( B_r \), where \( B_r \) is the ball of radius \( r \) in \( \mathbb{R}^k \) centered at the origin (assuming the limit exists). If \( \Psi \) is random and stationary (i.e., its distribution is invariant under translations of \( \mathbb{R}^k \)), then the statistical homogeneity of \( \Psi \) can be leveraged to define the intensity of \( \Psi \) by \( \mathbb{E}[\text{card}(\Psi \cap B)] \), where \( B \) is any Borel set in \( \mathbb{R}^k \) with unit volume. Then, ergodic theorems imply that the asymptotic density of \( \Psi \) exists. In the ergodic case, the asymptotic density is equal to the intensity and in the general case, its expectation is equal to the intensity. The intensity can still be defined if the reference space \( (\mathbb{R}^k) \) and the set are replaced by any (random) discrete structure having a kind of statistical homogeneity. More precisely, the intensity can be defined for unimodular random graphs \[2\]. Such graphs cover (local weak) limits of finite graphs with a root chosen uniformly at random, Cayley graphs and (quasi-) transitive graphs with a unimodular automorphism group. More generally, the intensity is also defined for unimodular discrete spaces, introduced in this paper. These spaces provide a common generalization of unimodular random graphs, the Palm version of stationary point processes, point-stationary point processes (which are point processes satisfying the mass transport principle, with possibly a lower dimension than the space), unimodular random graphs with a distorted metric and subsets of unimodular graphs. The discrete space is assumed to have a distinguished point called the origin. Then the intensity of a subset \( \Psi \) of a unimodular discrete space \( \Phi \) is interpreted as the average number of points of \( \Psi \) per points of \( \Phi \) and is defined by the probability that \( \Psi \) contains the origin of \( \Phi \).

There are many examples of subsets having zero asymptotic density and whose sizes can yet be heuristically compared, for instance, \( \mathbb{Z} \) and \( \mathbb{Z}^2 \) as subsets of \( \mathbb{Z}^3 \). Also, notice that \( 2\mathbb{Z} \subseteq \mathbb{Z}^3 \) has the same nature as \( \mathbb{Z} \subseteq \mathbb{Z}^3 \), but heuristically occupies less space. As another motivating example, it seems natural to expect that all level sets of the symmetric simple random walk on \( \mathbb{Z} \) have
the same size and become smaller if they are scaled by a factor greater than one. This comparison is not captured by the asymptotic density. One may heuristically expect that subsets of zero asymptotic density have (generally) a lower dimension than the reference space. Also, it is natural to search for a quantification of the size of sets of equal dimension (note that the Hausdorff measure does this in the continuum).

The main contribution of the present paper is to define new notions of dimension for unimodular discrete spaces. The idea is to use the statistical homogeneity of unimodular discrete spaces to define discrete analogues of the Hausdorff and Minkowski dimensions. A discrete analog of the Hausdorff measure is also proposed, which is useful for comparing the sizes of unimodular spaces of equal dimension. Subsection 1.2 provides an introduction to the proposed notions. These definitions allow one to assess the dimensions in the situations mentioned above including the case where the asymptotic density cannot be used. The definitions depend on the metric only. In addition, they are defined for unimodular discrete spaces which are not necessarily embedded in the Euclidean space and are not necessarily graphs.

The powerful framework of unimodularity allows one to derive discrete analogues of many results known for the ordinary Hausdorff and Minkowski dimensions and to develop computational tools for the concrete analysis of the dimensions. However, although there are many similarities, special care is needed and it seems there is no automatic way to obtain such analogous results, neither the statements nor the proofs.

1.2 Introduction to the Definitions of Dimension

1.2.1 The Proposed Definitions for Unimodular Discrete Spaces

Recall that the ordinary Minkowski dimension of a compact metric space $X$ is defined using coverings of $X$ by balls of equal radii. If $n(X, \epsilon)$ is the minimum number of balls of radius $\epsilon$ needed to cover $X$, then the Minkowski dimension is the (polynomial) growth rate of $n(X, \epsilon)$ as $\epsilon$ tends to zero; i.e.,

$$\lim_{\epsilon \to 0} \frac{\log n(X, \epsilon)}{\log \epsilon}.$$

If the limit does not exist, one can replace it by $\limsup$ and $\liminf$ to define upper and lower Minkowski dimensions.

Now, consider a (unimodular) discrete space $D$ (it is useful to have in mind the example $D = \mathbb{Z}^k$ to see how the definitions work). It is convenient to consider coverings of $D$ by balls of equal but large radii. Of course, if $D$ is unbounded, then an infinite number of balls is needed to cover $D$. So one needs another measure to assess how many balls are used in a covering. Let $S \subseteq D$ be the set of centers of the balls in the covering. The idea pursued in this paper is that if $D$ is unimodular, then the intensity of $S$ is a measure of the average number of points of $S$ per points of $D$ (the distance to be defined, as discussed later). So by letting $\lambda_\epsilon(D)$ be the infimum intensity
of such coverings of $D$ by balls of radii $r$ for $r < \infty$, it is reasonable to regard the decay rate of $\lambda_r(D)$ as $r \to \infty$ as the discrete Minkowski dimension of $D$. More details will be given in the forthcoming sections.

The idea behind the definition of the unimodular Hausdorff dimension is similar. Recall that the $\alpha$-dimensional (ordinary) Hausdorff content of a compact metric space $X$ is defined by considering the infimum of $\sum_i R_i^\alpha$, where the $R_i$’s are the radii of a sequence of balls that cover $X$. Also, it is convenient to force an upper bound $\epsilon$ on the radii and let $\epsilon$ tend to zero to define the $\alpha$-dimensional Hausdorff measure of $X$. Then, the Hausdorff dimension of $X$ is the infimum value of $\alpha$ such that the above value is zero.

Now, consider a unimodular discrete space $D$ and a covering of $D$ by balls which may have different radii. Let $R(v)$ be the radius of the ball centered at $v$ and $R(v) = 0$ if no ball of the covering is centered at $v$. According to the above discussions, it is natural to consider a lower bound on the radii, say $R(v) \geq 1$ if $R(v) \neq 0$. Again, if $D$ is unbounded, then $\sum_v R(v)^\alpha$ is always infinite. The idea is to leverage the unimodularity of $D$ and consider the average of the values $R(\cdot)^\alpha \text{ per point}$ as a replacement of the sum. Under the unimodularity assumption, this can be defined by $\mathbb{E}[R(o)^\alpha]$, where $o$ stands for the distinguished origin of $D$.

To view this value in another way, it can be seen that if $D$ is a stationary point process, then this quantity (after conditioning $D$ to contain the origin and pointing it at the origin, i.e., considering the Palm version of $D$) is just a constant multiple of $\mathbb{E} \left[ \sum_{x \in D \cap B} R(x)^\alpha \right]$, where $B$ is any Borel set of unit volume and the latter can be interpreted as the average value of $R(\cdot)^\alpha \text{ per unit volume}$. With this definition, the infimum value of $\mathbb{E}[R(o)^\alpha]$ over such coverings is considered and the unimodular Hausdorff dimension of $D$ is defined similarly.

### 1.2.2 Earlier Notions of Discrete Dimension

The literature contains various definitions to study dimension for discrete structures. Here is a brief summary of those relevant in the present context. The connections of the earlier definitions with the proposed ones will be discussed in the next subsection. One is the growth rate of the cardinality of a large ball. Another is the discrete Hausdorff dimension [7] which uses the idea behind the definition of the Hausdorff dimension by considering coverings of $\Phi \subseteq \mathbb{R}^k$ by large balls (instead of small balls) and considering the cost $(\frac{r}{r+|x|})^\alpha$ for each ball in the covering, where $r$ and $x$ are the radius and the center of the ball and $\alpha$ is a constant (note that the definitions proposed in this paper differ from the discrete dimension. Notice the division by $r + |x|$). Other definitions are the spectral dimension of a graph (defined in terms of the return probabilities of the simple random walk), the typical displacement exponent of a graph (see [14] for both notions), the isoperimetric dimension of a graph [13], the stochastic dimension of a partition of $\mathbb{Z}^k$ [8], etc.

In statistical physics, one also assigns dimension and various exponents to finite models. Famous examples are self-avoiding walks and the boundaries of large percolation clusters. More on the matter is provided in Part III.
1.3 Organization of the Material and Summary of Results

The material is organized in three companion papers (the current paper, [4], and [5]), that will be referred to as Parts I-III respectively. The aim of this subsection is to give a brief summary of the main results and their localizations in the three parts.

Part I is centered on the framework, the definitions and the basic properties of unimodular dimensions. It also contains a comprehensive set of examples which will be continued in Part II. These examples stem from point process theory, random graph theory, random walk theory, self-similarity or from analogues in the continuum. In particular, unimodular self-similar discrete spaces are introduced, which are obtained by discretizing self-similar sets. Two examples are the unimodular discrete Cantor set and the unimodular discrete Koch snowflakes, which are necessarily random for being unimodular (a deterministic discrete Cantor set exists in the literature which is not unimodular).

In Part I, unimodular discrete spaces are first defined as random elements in the space of pointed discrete metric spaces which are boundedly finite; i.e., each ball should have finitely many points. To obtain a Polish probability space, a metric is presented whose topology extends that of the Benjamini-Schramm metric [11] (also known as the topology of local weak convergence) for rooted graphs. There is a need to allow each point or pair of points to have some mark (in the same way as networks are graphs equipped with marks [2]).

Second, the new notions of dimensions of such unimodular discrete spaces are defined. This includes the unimodular Minkowski and Hausdorff dimensions and the unimodular Hausdorff measure (Section 3.3). The strength of the new definitions is supported by a series of results throughout the three parts. In particular, it is shown that many of the properties in the continuum setting have counterparts in the unimodular discrete setting, e.g., the comparison of different notions of dimension, the dimension of subspaces, the dimension of the product of unimodular discrete spaces, and the other results listed below. However, there are important differences and it seems there is no automatic way to obtain such analogous results. Part I also discusses some basic properties of unimodular dimensions.

Part II discusses the connections between the proposed dimensions and the growth rate of the space. For example, if \( N_r(o) \) represents the ball of radius \( r \) centered at the origin, upper and lower bounds for the unimodular Hausdorff dimension are provided in terms of the upper and lower polynomial growth rates of \( \text{card}(N_r(o)) \) (i.e., \( \limsup \) and \( \liminf \) of \( \log(\text{card}(N_r(o)))/\log r \) as \( r \to \infty \)). This is an analogue of Billingsley’s lemma (see e.g., Section 1.3 of [12]). For upper bounds, a discrete analogue of the mass distribution principle (see e.g., Section 1.1 of [12]) is also provided. In the particular case of a point-stationary point process equipped with the Euclidean metric, it is also shown that the Minkowski dimension is bounded from above by the polynomial decay rate of \( E[1/\text{card}(N_r(o))] \). These bounds are very useful for calculating the unimodular dimensions in many examples. There are also weighted versions of these inequalities where a weight is assigned to each point. An important
result in the opposite direction is an analogue of Frostman’s lemma (see e.g., Section 3.1 of [12]). Roughly speaking, the lemma states that there is a weight function such that the upper bound in Billingsley’s lemma is sharp. This lemma is a powerful tool to study the unimodular Hausdorff dimension, in particular, to study the dimension of subspaces, product spaces, etc. It is also the basis of many of the results of Part III discussed in the next paragraph. In the Euclidean case, another proof of Frostman’s lemma is provided using a unimodular version of the max-flow min-cut theorem, which is of independent interest. Part II also contains a section about examples that completes the examples discussed in Part I and gives the proofs of the unimodular dimension results announced in Part I.

Part III discusses both the connections of the proposed notions of dimensions with other classical notions and further extensions. For the connections, it is first shown that the Hausdorff dimension is equal to a unimodular version of the capacity dimension which is widely used in potential theory in the continuum setting. Second, in the case of unimodular point processes in the Euclidean space, it is shown that the unimodular Hausdorff dimension is greater than or equal to the discrete dimension defined in [7] (see also Subsection 1.2.2 above). Third, the connection with the continuum Hausdorff dimension is also studied via scaling limits. Roughly speaking, it is conjectured that if the scaling of a unimodular discrete space converges to a (continuum) space as the scaling factor tends to zero, then the ordinary Hausdorff dimension of the latter is greater than or equal to the unimodular Hausdorff dimension of the former. This conjecture is proved under some extra assumptions. Finally, the connections between the unimodular dimensions defined here and the dimension of large finite models are discussed. Part III also contains a section on sample unimodular dimensions which refine the notions discussed in the three parts in the non-ergodic case.

2 Unimodular Discrete Spaces

The main objective of this section is the definition of unimodular discrete spaces as a common generalization of unimodular graphs, Palm probabilities and point-stationary point processes.

If the reader is only interested in unimodular graphs, at first reading he or she can jump to Subsection 2.6 directly, after looking at the notation in Subsections 2.1 and 2.4.

2.1 Notation

The following notation will be used throughout. The set of nonnegative real (resp. integer) numbers is denoted by $\mathbb{R}_{\geq 0}$ (resp. $\mathbb{Z}_{\geq 0}$). The minimum and maximum binary operators are denoted by $\land$ and $\lor$ respectively. The number of elements in a set $A$ is denoted by $|A|$, which is a number in $[0, \infty]$. If $P(x)$ is a property about $x$, the indicator $1_{\{P(x)\}}$ is equal to 1 if $P(x)$ is true and 0 otherwise.
Discrete metric spaces (discussed in details in Subsection 2.2) are denoted by \( D, D', \) etc. Graphs are an important class of discrete metric spaces. So the symbols and notations are mostly borrowed from graph theory. In the definitions, the reader can restrict attention to graphs for simplicity, but should keep in mind that the symbol \( G \) is also used for discrete metric spaces which are not necessarily graphs.

Following the graph terminology, points of the discrete spaces are mainly denoted by \( u, v, \ldots \). For \( r > 0 \), \( N_r(D,v) \) refers to the closed \( r \)-neighborhood of \( v \in D \); i.e., the set of points of \( D \) with distance less than or equal to \( r \) from \( v \). An exception is made for \( r = 0 \) (Subsection 3.3), where \( N_0(v) := \emptyset \). When there is no ambiguity about the underlying discrete space, \( N_r(D,v) \) will be denoted by \( N_r(v) \). The diameter of a subset \( A \) is denoted by \( \text{diam}(A) \).

**Definition 2.1.** For a function \( f : [1, \infty) \to \mathbb{R}_{\geq 0} \), define the polynomial growth rates and polynomial decay rates by the following formulas:

\[
\text{growth}(f) := -\text{decay}(f) := \lim_{r \to \infty} \frac{\log f(r)}{\log r},
\]

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\]

### 2.2 The Space of Discrete Spaces

Throughout the paper, the metric on any metric space is denoted by \( d \), except when explicitly mentioned. A metric space \( D \) is discrete if for every \( u \in D \), \( \inf\{d(u,v) : v \in D, v \neq u\} > 0 \). In this paper, it is always assumed that the discrete metric space is boundedly finite: i.e., every set included in a ball of finite radius in \( D \) is finite. The term discrete space will always refer to boundedly finite discrete metric space.

The following are two important classes of discrete spaces considered in this paper. A **pointed set** (or a **rooted set**), where \( D \) is a set and \( o \) a distinguished point of \( D \) called the origin (or the root) of \( D \). Similarly, a **doubly-pointed** set is a triple \((D,o_1,o_2)\), where \( o_1 \) and \( o_2 \) are two distinguished points of \( D \).

Let \( \Xi \) be a complete separable metric space called the mark space. A **marked set** is a pair \((D,m)\), where \( D \) is a set and \( m \) a function \( m : D \times D \to \Xi \). The mark of a single point \( x \) may also be defined by \( m(x) := m(x,x) \), where the same symbol \( m \) is used for simplicity. For a marked set \((D;m)\), \( D \) is called the underlying set. The notions of marked pointed sets and marked doubly-pointed sets are defined similarly and the same convention is applied for them; e.g., for a marked pointed set \((D,o;m)\), \((D,o)\) is called the underlying pointed set.

Consider a **marked discrete space** \((D;m)\), where \( D \) is a discrete space and \( m \) is a mark function on \( D \) as above. An **isomorphism** between two such
spaces, say \((D;m)\) and \((D';m')\), is a bijective function \(\rho: D \to D'\) which is an isometry between the metric spaces which respects the marks in the sense that \(\forall u, v \in D: m'(\rho(u), \rho(v)) = m(u, v)\). Similarly, an isomorphism (also called a pointed-isomorphism) between two pointed marked discrete spaces \((D, o; m)\) and \((D', o'; m')\) is an isomorphism between the un-pointed spaces such that \(\rho(o) = o'\) in addition. An isomorphism between doubly-pointed marked discrete spaces is defined similarly. Finally, an isomorphism from \(D\) to itself is called an automorphism.

Let \(D\) be the set of equivalence classes of discrete spaces under isomorphism. Similarly, let \(D_*\) (resp. \(D_{**}\)) be the set of equivalence classes of pointed (resp. doubly pointed) discrete spaces. Let \(D'_*, D'_*\) and \(D'_{**}\) be defined similarly for marked discrete spaces with mark space \(\Xi\) (which is usually given). The equivalence class containing \(D\), \((D, o)\), etc., is denoted by brackets \([D]\), \([D, o]\), etc.

**Example 2.2.** Let \(G\) be (the set of vertices of) a connected graph. If \(G\) is equipped with the graph-distance metric, then \(G\) is a discrete metric space. Also, \(G\) is boundedly finite if and only if the graph is locally finite; i.e., there is no vertex of infinite degree in \(G\). Moreover, locally finite connected (multi-) graphs can be regarded as elements of \(D'\) by letting \(d\) be the graph-distance metric, \(Z\) the mark space, and \(m(u, v)\) the number of edges between \(u\) and \(v\), which may be zero or positive.

Also, assume \(G\) is a simple graph and all vertices and edges in \(G\) have a mark (this is called a network in [2]). Let \(m(u, u)\) be the mark of \(u \in G\). For adjacent vertices \(u, v \in G\), let \(m(u, v)\) be the mark of the (directed) edge \(uv\). For non-adjacent pairs, let \(m(\cdot, \cdot)\) be an arbitrary fixed mark. Then \((G; m)\) is a marked discrete space and \([G; m] \in D'\). Similar arguments can be given for multi-graphs and networks. Therefore, the notion of marked discrete space generalizes that of graph and network.

**Example 2.3.** Assume \(k \geq 1\) and \(\varphi \subseteq \mathbb{R}^k\). If \(\varphi\) is equipped with the Euclidean metric and has no accumulation point, then it is a discrete space (similarly, \(\varphi\) can be equipped with other usual metrics on \(\mathbb{R}^k\)). Moreover, assume \(0 \in \varphi\) and assign marks to \(\varphi\) by the formula \(m(x, y) := y - x\). This way, the subset of \(\mathbb{R}^k\) can be recovered from the equivalence class \([\varphi, 0; m]\). Therefore, the set of discrete subsets of \(\mathbb{R}^k\) that contain the origin can be regarded as a subset of \(D'_*\).

Note that if \(G\) is a graph whose vertex set is a subset of \(\mathbb{R}^d\), there are two natural metrics on \(G\): the graph-distance metric and the Euclidean metric. These lead to two different discrete spaces which might have totally different large scale behavior (see, e.g., Subsection 4.3 and Example 3.46 below).

### 2.3 A Metric on the Space of Pointed Discrete Spaces

In what follows, an explicit metric is introduced on \(D_*\) (and also on \(D_{**}, D'_*\), and \(D'_{**}\)), which is the basis for defining random pointed discrete spaces in the next subsection. Heuristically, two elements of \(D_*\) are close if two large neighborhoods of the origins are almost isomorphic, where ‘large’ means containing...
a ball of large radius. This is made precise below. The reader can skip this subsection at first reading.

**Definition 2.4.** Let \((D, o)\) and \((D', o')\) be pointed discrete spaces. An \(r\)-embedding between \((D, o)\) and \((D', o')\), where \(r > 0\), is an injective function \(f : N_r(o) \to D'\) such that \(f(o) = o'\) and has distortion at most \(\frac{1}{r}\); i.e.,

\[
\forall x, y \in N_r(o) : |d(x, y) - d(f(x), f(y))| \leq \frac{1}{r}.
\] (2.1)

If such a function exists, then \((D, o)\) is \(r\)-embeddable in \((D', o')\). If each one of \((D, o)\) and \((D', o')\) is \(r\)-embeddable in the other, then they are called \(r\)-similar.

Note that the image of \(f\) is not necessarily contained in \(N_r(o')\). However, by (2.1), it is contained in \(N_{r+1/r}(o')\). Note also that if \((D, o)\) is \(r\)-embeddable in \((D', o')\) and \(0 < s < r\), then the former is also \(s\)-embeddable in the latter.

**Definition 2.5.** For two pointed discrete spaces \((D, o)\) and \((D', o')\), define

\[
\kappa((D, o), (D', o')) := 1 \lor \inf\{\epsilon > 0 : (D, o) \text{ and } (D', o') \text{ are } \frac{1}{\epsilon}\text{-similar}\}.
\]

The definition clearly depends only on the isomorphism classes of \((D, o)\) and \((D', o')\). So \(\kappa\) is well defined as a function on \(D_* \times D_*\).

**Theorem 2.6.** The function \(\kappa\) is a metric on \(D_*\). Moreover, there exists a Polish space which contains \(D_*\) as a Borel subset.

The proof of this theorem is given in Appendix A. It should be noted that \(D_*\) is not complete itself. Similar definitions and arguments can be proposed for \(D_{**}, D'_*\) and \(D'_{**}\). The latter is briefly explained below. Let \((D, o, p; m)\) and \((D', o', p'; m')\) be two doubly-pointed marked discrete spaces. Define \(r\)-embeddings similarly to Definition 2.4 by adding the constraints \(f(p) = p'\) (which implies that \(r \geq d(o, p)\)) and in addition,

\[
\forall x, y \in N_r(o) : d\left(m(x, y), m'(f(x), f(y))\right) \leq \frac{1}{r}.
\]

Being \(r\)-similar and \(\kappa\) are also defined in the same way. The result analogous to Theorem 2.6 is the following.

**Theorem 2.7.** The sets \(D_{**}, D'_*\) and \(D'_{**}\), equipped with the distance function \(\kappa\), are metric spaces and are Borel subsets of some Polish spaces.

It should be mentioned that the set \(D\) of non-pointed discrete spaces is not a Polish space (if the natural projection \(\pi : D_* \to D\) is required to be measurable).

**Remark 2.8.** It can be seen that \(\kappa\) extends the metric between rooted graphs used in [2] and called the Benjamini-Schramm metric [11]. Also, on the class of discrete subsets of \(\mathbb{R}^d\), its topology extends the classical one in the context of point processes, which is that of vague convergence (see e.g., Appendix A of [15]).
2.4 Random Pointed Discrete Spaces

Let $\hat{D}$ (resp. $\hat{D}'$) be the Polish space mentioned in Theorem 2.6 (resp. Theorem 2.7) which contains $D$ (resp. $D'$) as a Borel subset.

**Definition 2.9.** A random pointed discrete space is a random element in $D$; i.e., a measurable function from some probability space to $\hat{D}$ which takes values in $D$ a.s. and is denoted by bold symbols $[D,o]$. Here, $D$ and $o$ represent the discrete space and the origin respectively.

The probability space in the above definition is not referred to explicitly in this paper. Note that the whole symbol $[D,o]$ represents one random object, which is a random equivalence class of pointed discrete spaces. Therefore, any formula using $D$ and $o$ should be well defined for equivalence classes; i.e., should be invariant under pointed isomorphisms.

The following convention is helpful throughout.

**Convention 2.10.** In this paper, bold symbols are usually used in the random case or when extra randomness is used. For example, $[D,o]$ refers to a deterministic element of $D$, and $[D,o]$ refers to a random pointed discrete space.

Note that the distribution of a random pointed network $[D,o]$ is a probability measure on $D$ defined by $\mu(A) := P([D,o] \in A)$ for events $A \subseteq D$. Also, note that since $\hat{D}$ is a Polish space and contains $D$ as a Borel subset, the classical tools of probability theory regarding standard probability spaces can be used.

Similarly, one can allow marks for points as follows.

**Definition 2.11.** A random pointed marked discrete space is a random element in $D'$ and is denoted by bold symbols $[D,o;m]$. Here, $D$, $o$ and $m$ represent the discrete space, the origin and the mark function respectively.

**Example 2.12.** By Example 2.2, random rooted graphs and networks are special cases of random pointed (marked) discrete spaces. Another special case is a point-process in $\mathbb{R}^k$ (which is always regarded as a random discrete subset of $\mathbb{R}^k$ in this paper) that contains 0, where 0 is considered as the origin. See Example 2.3.

2.5 Unimodular Discrete Spaces

In the following definition, the symbol $g[D,o,v]$ is used as a short form of $g([D,o,v])$. Similarly, brackets $[\cdot]$ are used as a short form of $\langle \cdot \rangle$.

**Definition 2.13.** A unimodular discrete space is a random pointed discrete space, namely $[D,o]$, such that for all measurable functions $g : D \rightarrow \mathbb{R}_{\geq 0}$,

$$E \left[ \sum_{v \in D} g[D,o,v] \right] = E \left[ \sum_{v \in D} g[D,v,o] \right].$$

(2.2)
Similarly, a **unimodular marked discrete space** is a random pointed marked discrete space \([D, o; m]\) such that for all measurable functions \(g : D' \rightarrow \mathbb{R}^+\),

\[
\mathbb{E} \left[ \sum_{v \in D} g[D, o, v; m] \right] = \mathbb{E} \left[ \sum_{v \in D} g[D, v, o; m] \right].
\]  

(2.3)

Note that the expectations may be finite or infinite.

When there is no ambiguity, the term \(g[D, o, v]\) is also denoted by \(g_D(o, v)\) or simply \(g(o, v)\). The sum in the left (respectively right) side of (2.2) is called the **outgoing mass from** \(o\) (respectively **incoming mass into** \(o\)). The same notation can be used for the terms in (2.3). So (2.2) and (2.3) can be summarized by

\[
\mathbb{E} [g^+(o)] = \mathbb{E} [g^-(o)].
\]

This equation expresses some kind of conservation of mass in expectation. It is referred to as the **mass transport principle** in the literature. Another heuristic interpretation of this condition is that \(o\) is a point of \(D\) chosen randomly and uniformly\([2]\). This heuristic is of course meaningless when \(D\) is infinite, but is precise in the finite case (see Example 2.15 below).

**Remark 2.14.** If \([D, o; m]\) is a unimodular marked discrete space, then its unmarked version \([D, o]\) is also unimodular (consider the functions \(g\) in (2.3) that do not depend on the mark function).

**Example 2.15** (Unimodular Finite Spaces). Let \(D\) be a deterministic finite metric space. Choose a point \(o \in D\) randomly and uniformly. Then, \([D, o]\) is unimodular. Indeed, both sides of (2.2) are equal to \(\frac{1}{|D|} \sum_{u,v} g[D, u, v]\), where the sum is on \(u, v \in D\). Similarly, one can let \(D\) be a random finite metric space from the beginning. Moreover, it is not hard to see that every unimodular finite metric space is of this form.

**Example 2.16** (Lattices). It is well known that the lattice \([\mathbb{Z}^k, 0]\), pointed at the origin, is unimodular. To prove this, one can directly verify the mass transport principle (2.2) (see e.g., [20]). Similarly, the scaled lattice \([\delta\mathbb{Z}^k, 0]\) (for \(\delta > 0\)) is unimodular.

**Example 2.17** (Unimodular Random Graphs). In the case of random rooted graphs and networks, the concept of unimodularity in Definition 2.13 coincides with that of\([2]\) (see Examples 2.2 and 2.12 above). Therefore, unimodular random graphs and networks are special cases of unimodular (marked) discrete spaces. The interested reader is invited to see the examples in\([2]\), in particular Cayley graphs, the canopy tree and the unimodular Galton-Watson tree.

**Example 2.18** (Point-Stationary Point Processes). Let \(\Phi\) be a point process in \(\mathbb{R}^k\) (see Example 2.12) such that \(0 \in \Phi\) almost surely. \(\Phi\) is called **point-stationary** if

\[
\mathbb{E} \left[ \sum_{x \in \Phi} g(\Phi, 0, x) \right] = \mathbb{E} \left[ \sum_{x \in \Phi} g(\Phi, x, 0) \right],
\]

\[
\mathbb{E} \left[ \sum_{x \in \Phi} g(\Phi, 0, x) \right] = \mathbb{E} \left[ \sum_{x \in \Phi} g(\Phi, x, 0) \right],
\]

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for all non-negative measurable functions $g$ that are invariant under the translations of $\mathbb{R}^k$ (note that $g(\varphi, x, y)$ should be defined for all discrete subsets $\varphi \subseteq \mathbb{R}^k$ and $x, y \in \mathbb{R}^k$). See [2] and the references therein for more discussions. By regarding $[\Phi, 0; m]$ as a random pointed marked discrete space, as in Example 2.12 it follows that $\Phi$ is point-stationary if and only if $[\Phi, 0; m]$ is unimodular. Therefore, point-stationary point processes are a special case of unimodular marked discrete spaces.

Here are two classes of point-stationary point processes which will be used in Section 4. First, assume $\Psi$ is a stationary point process in $\mathbb{R}^k$ (resp. $\mathbb{Z}^k$); i.e., its distribution is invariant under all translations of $\mathbb{R}^k$ (resp. $\mathbb{Z}^k$). Assume the intensity of $\Psi$ (i.e., $E[\#\Psi \cap B]$, where $B$ is an arbitrary Borel set of $\mathbb{R}^k$ with unit volume) is finite. The Palm version of $\Psi$ is heuristically obtained by conditioning $\Psi$ to contain the origin (see e.g., Section 13 of [16] for the precise definition) and is point-stationary [2]. Second, assume $\Phi$ is a point process in $\mathbb{R}^k$ that contains the origin. Assume $\Phi$ can be written as $\Phi = \{X_n : n \in \mathbb{Z}\}$, where $X_n$ is a random point in $\mathbb{R}^k$ for each $n \in \mathbb{Z}$ and $X_0 = 0$. If the distribution of the sequence is invariant under the shift $X_n \rightarrow Y_n := X_{n+1} - X_1$; i.e., the sequence $(X_n)_n$ has stationary increments, then $\Phi$ is point-stationary (see e.g., Section 13.3 of [16]).

Example 2.19 (Weak Limits). Unimodularity is preserved under weak convergence, as observed in [11] for unimodular graphs. In particular, a weak limit of finite discrete spaces, where the origin is chosen uniformly, is unimodular. Even for unimodular graphs, the converse of this statement is an important conjecture. See e.g., Section 10 of [2].

## 2.6 Equivariant Process on a Unimodular Discrete Space

Unimodular marked discrete spaces were defined in the previous subsection. In many cases in this paper, an unmarked unimodular discrete space $[D, o]$ is given and various ways of assigning marks to $D$ are considered. Intuitively, an equivariant process on $D$ is an assignment of (random) marks to $D$ such that the new marked space is unimodular. Formally, it is

\[ a \text{ unimodular marked discrete space } [D', o'; m] \text{ such that the space } [D', o'], \text{ obtained by forgetting the marks, has the same distribution as } [D, o]. \]

In the following, another equivalent definition is proposed, which is more convenient in some examples despite of being more technical. Here, the mark space $\Xi$ is fixed as in Subsection 2.2.

**Definition 2.20.** Let $D$ be a deterministic discrete space which is boundedly-finite. A marking of $D$ is a function from $D \times D$ to $\Xi$; i.e., an element of $\Xi^{D \times D}$. So $\Xi^{D \times D}$ is the set of markings of $D$. By considering the product sigma-field on this set, a random marking of $D$ can be defined as a random element of $\Xi^{D \times D}$. 

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Note that if $X \in \Xi^{D \times D}$ is a marking of $D$ and $\rho : D \mapsto D'$ is an isomorphism, then $X \circ \rho^{-1} \in \Xi^{D' \times D'}$ is a marking of $D'$. Also, in the following, only the distribution of a random marking is important and no specific probability space is assumed.

In this paper, the notions of marking and random marking will only be used when the base space $D$ is deterministic. The notion of equivariant processes, defined below, extends the definition of marking and will be used when the base space is random.

**Definition 2.21.** An **equivariant process** $Z$ with values in $\Xi$ is a map that assigns to every deterministic discrete space $D$ a random marking $Z_D$ of $D$ satisfying the following properties:

(i) $Z$ is compatible with isomorphisms in the sense that for every isomorphism $\rho : D_1 \rightarrow D_2$, the random marking $Z_{D_1} \circ \rho^{-1}$ of $D_2$ has the same distribution as $Z_{D_2}$.

(ii) For every measurable subset $A \subseteq D_*$, the following function on $D_*$ is measurable:

$$[D,o] \mapsto \mathbb{P}(Z_D \in A).$$

**Convention 2.22.** For $v \in D$, if $D$ is clear from the context, $Z_D(v)$ is also denoted by $Z(v)$ for simplicity.

**Example 2.23 (Deterministic Process).** Let $z : D_* \rightarrow \Xi$ be a measurable function (e.g., $z[D,u,v] := d(u,v)$). For a deterministic discrete space $D$, define $Z_D \in \Xi^{D \times D}$ by $Z_D(u,v) := z[D,u,v]$. It can be seen that $Z$ is an equivariant process.

**Example 2.24 (i.i.d. Marks).** Let $\nu$ be a probability measure on $\Xi$. Let $D$ be a deterministic discrete space. For $x,y \in D$, define $Z_D(x,y)$ to be a random element of $\Xi$ chosen with distribution $\nu$ (the choice should be i.i.d. for different pairs of points $x,y \in D$). It can be seen that $Z$ is an equivariant process (see also Lemma 4.1 in [10]).

**Remark 2.25.** If $Z$ is an equivariant process and $[D,o]$ is a random pointed discrete space, then $[D,o; Z_D]$ makes sense as a random pointed marked discrete space with distribution $Q$ defined by

$$Q(A) := \int \int 1_A[D,o;m]d\mathcal{P}_D(m)d\mu([D,o]), \tag{2.4}$$

where $\mathcal{P}_D$ is the distribution of $Z_D$ (for every $D$) and $\mu$ is the distribution of $[D,o]$ (note that only the distribution of $Z_D$ is important and no common probability space is assumed for different $D$’s). It will be shown in the proof of the following lemma that $Q$ is indeed a probability measure on $D_*$. Hence, $[D,o; Z_D]$ is well defined.
**Definition 2.26.** Given a unimodular discrete space $[D, o]$, a map $Z$ satisfying the conditions of Definition 2.21 is also called an **equivariant process on $D$** with values in $\Xi$. Also, one can let $Z(\cdot)$ be undefined for a class of discrete spaces, as long as $Z_D$ is almost surely defined.

For instance, to define an equivariant process on $Z$, it is enough to define $Z_Z$ such that the conditions of Definition 2.21 are satisfied. Note also that in general, if $Z$ is an equivariant process on $D$, then $[D, o; Z_D]$ is well defined.

**Example 2.27** (Periodic Marking of $Z$). Let $N \in \mathbb{N}$ be given and $U \in \{0, 1, \ldots, n-1\}$ be chosen uniformly at random. For $x \in Z$, let

$$Z_Z(x) := Z_Z(x, x) := \begin{cases} 1, & \text{if } (x \mod N) = U, \\ 0, & \text{otherwise.} \end{cases}$$

Let $Z_D(x, y)$ be an arbitrary fixed constant for other inputs. The choice of $U$ being uniform ensures that Condition (i) of Definition 2.21 is satisfied. It can be seen that the other condition is also satisfied. So $Z$ is an equivariant process on $Z$.

Similarly, one can consider any stationary marking of $Z$; i.e., a random function $m \in \Xi^{Z \times Z}$ such that the distribution of $m$ is invariant under the shift $i \mapsto i - 1$. Then, $m$ is an equivariant process on $Z$.

**Lemma 2.28.** Let $[D, o]$ be a unimodular discrete space. If $Z$ is an equivariant process on $D$, then $[D, o; Z_D]$ is also unimodular.

The proof is given at the end of this subsection. A converse to this lemma is given in the appendix (see Remark 2.32 below). It is important that the distribution of the mark $Z_D$ does not depend on the origin (as in Definition 2.21).

The following describes a special case of equivariant processes.

**Definition 2.29** (Equivariant Subset). An **equivariant subset** $S$ is a map that assigns to each discrete space $D$, a random subset $S_D \subseteq D$ such that the (random) indicator function $Z_D(u) := 1_{S_D}(u)$ is an equivariant process. By equipping $S_D$ with the induced metric from $D$, one may regard $S$ as an **equivariant subspace** as well.

In addition, if $[D, o]$ is a unimodular discrete space, then the **intensity** of $S$ in $D$ is defined by $\rho_D(S) := \mathbb{P}[o \in S_D]$.

For example, $S_D := \{v \in D : \#N_1(v) = 4\}$ defines an equivariant subset. Also, let $D = Z$ and $S_D$ be the set of even numbers with probability $p$ and the set of odd numbers with probability $1 - p$. Then, $S$ is an equivariant subset of $Z$ if and only if $p = \frac{1}{2}$ (notice Condition (i) of Definition 2.21).

The following lemma is a generalization of Lemma 2.6 of [6] and Lemma 2.3 of [2].

**Lemma 2.30.** Let $[D, o]$ be a unimodular discrete space and $S$ an equivariant subset. Then $S_D \neq \emptyset$ with positive probability if and only if it has positive intensity. Equivalently, $S_D = D$ a.s. if and only if $\rho_D(S) = 1$. 

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Proof. The claim is implied by the mass transport principle (2.3) for the function 
\[ g[D, u, v; S] := 1_{\{v \in S\}} \] 
The details are left to the reader.

The following is another general class of equivariant processes.

**Example 2.31.** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and \(z : D_\ast \times \Omega \to \Xi\) be a measurable function. For all deterministic discrete spaces \(D\), points \(u, v \in D\) and \(\omega \in \Omega\), define \(Z_D(u, v) := Z_D(u, v, \omega) := z([D, u, v], w)\). Now, \(Z_D\) can be regarded as a random element in \(\Xi^{D \times D}\). It can be seen that \(Z\) is an equivariant process.

**Remark 2.32.** A converse to Lemma 2.28 is given in Proposition B.2 in the appendix. It states that equivariant processes on \(D\) are equivalent to unimodular marked discrete spaces \([D', o', m']\) such that \([D', o']\) has the same distribution as \([D, o]\). This enables one to discuss weak convergence of equivariant processes on \(D\) as probability measures on \(D'_\ast\). Then, it is proved in Lemma B.3 in the appendix that if the mark space \(\Xi\) is compact, then the set of equivariant processes on \(D\) is tight. These results are moved to the appendix to help to focus on the main thread of the paper.

**Remark 2.33.** Informally, the distribution of a unimodular discrete space \([D, o]\) is uniquely determined by that of the non-pointed space \([D]\) (see Theorem 3 of \[23\] for a rigorous statement and proof). This explains why the term ‘equivariant process on \(D\)’ is used instead of the more precise term ‘equivariant process on \([D, o]\)’.

**Remark 2.34.** One can easily extend the definition of equivariant process to allow the base space to be marked (this is not done here because the dimension does not depend on the marks).

For instance, an equivariant subgraph of a unimodular (multi-) graph can be defined this way, which is left to the reader (see Example 2.12). This includes equivariant random spanning trees and forests; e.g., the uniform spanning forest (see e.g., \[2\]), which will be discussed in Part III.

**Proof of Lemma 2.28** By the assumptions in the definition, the integrand in (2.4) is well defined and measurable. Hence, \(Q(A)\) in (2.4) is well defined. It can easily be seen that \(Q\) is indeed a probability measure on \(D'_\ast\). So \([D, o; Z_D]\) is well defined. One can also write (2.4) under the form \(Q(A) = \mathbb{P}([D, o; Z_D] \in A)\) or

\[ Q(A) := \mathbb{E} \left[ \int 1_A[D, o; m]dP_D(m) \right] \]

by keeping in mind that a realization of \([D, o]\) is considered in the term inside the expectation.

To prove unimodularity, let \(f : D'_\ast \to \mathbb{R}^{\geq 0}\) be a measurable function. For all deterministic discrete spaces \(D\) and \(x, y \in D\), let

\[ g(D, x, y) := \mathbb{E} [f(D, x, y; Z_D)] = \int f(D, x, y; m)dP_D(m). \]
One has

\[
E \left[ \sum_{x \in D} f[D, o, x; Z_D] \right] = E \left[ \int \sum_{x \in D} f[D, o, x; m] dP_D(m) \right] = E \left[ \sum_{x \in D} g[D, o, x] \right].
\]

One can similarly obtain that

\[
E \left[ \sum_{x \in D} f[D, x, o; Z_D] \right] = E \left[ \sum_{x \in D} g[D, x, o] \right].
\]

Therefore, the claim follows by (2.2) for \(g\).

\section{The Unimodular Minkowski and Hausdorff Dimensions}

This section presents new notions of dimension for unimodular discrete spaces. As mentioned in the introduction, the statistical homogeneity of unimodular discrete spaces is used to define discrete analogous of the Minkowski and Hausdorff dimensions. Also, basic properties of these definitions are discussed.

\subsection{The Unimodular Minkowski Dimension}

\begin{definition}
Let \([D, o]\) be a unimodular discrete space and \(r \geq 0\). An \textit{equivariant} \(r\)-\textit{covering} \(R\) of \(D\) is an equivariant subset of \(D\) (Definition 2.29) such that the set of balls \(\{N_r(v) : v \in R_D\}\) of radius \(r\) cover \(D\) almost surely. Here, the same symbol \(R\) is used for the following equivariant process (Definition 2.21):

\[R(v) := \begin{cases} r, & \text{there is a ball centered at } v \text{ in the covering}, \\ 0, & \text{otherwise}. \end{cases}\]

Define

\[\lambda_r := \lambda_r(D) := \inf \{\text{intensity of } R \text{ in } D\},\] (3.1)

where the infimum is over all equivariant \(r\)-coverings of \(D\) and the intensity is as in Definition 2.29. Note that \(\lambda_r\) is non-increasing in terms of \(r\).

A smaller \(\lambda_r\) heuristically means that \textit{a smaller number of balls per point} is needed to cover \(D\). So define

\begin{definition}
The \textit{upper and lower unimodular Minkowski dimensions} of \(D\) are defined by

\[
\overline{\text{udim}}_M(D) := \overline{\text{decay}}(\lambda_r),
\]

\[
\underline{\text{udim}}_M(D) := \underline{\text{decay}}(\lambda_r),
\]

as \(r \to \infty\). If the decay rate of \(\lambda_r\) exists, define the \textit{unimodular Minkowski dimension} of \(D\) by

\[\text{udim}_M(D) := \text{decay}(\lambda_r).\]
\end{definition}
One has

\[ 0 \leq \overline{\text{udim}_M}(D) \leq \underline{\text{udim}_M}(D) \leq \infty. \]

**Remark 3.3.** By monotonicity of \( \lambda_r \), the upper and lower decay rates can be obtained by knowing \( \lambda_r \) for \( r \in \{ r_i \}_{i=1}^{\infty} \), where \( r_1, r_2, \ldots \) is a suitable increasing sequence. It can be seen that it is enough to have \( \log n \rightarrow \infty \) as \( n \rightarrow \infty \).

**Remark 3.4.** It would be more precise to use the symbol \( \overline{\text{udim}_M}([D, o]) \) for the Minkowski dimension since it is defined using the distribution of \([D, o]\). However, according to Remark 2.33, the term \( \text{udim}_M(D) \) is used for simplicity. The same convention is used for the unimodular Hausdorff dimension of \( D \) defined in the next subsection.

**Remark 3.5.** It is essential that extra randomness be allowed in the definition of equivariant \( r \)-coverings (based on the definition of equivariant processes in Definition 2.21). In general, one may have to go beyond i.i.d. marks. See for instance Example 3.6 below.

The following are first illustrations of the definition.

**Example 3.6 (Lattices).** Example 2.16 shows that \( \mathbb{Z}^k, 0 \) is a unimodular discrete space. It will be proved that

\[ \text{udim}_M(\mathbb{Z}^k) = k. \]

Here, only the lower bound is proved. The proof of the upper bound is postponed to Subsection 3.2 (Example 3.15). To do this, a sequence of equivariant coverings is constructed. Here, the Euclidean metric is assumed on \( \mathbb{Z}^k \). Other equivalent metrics can be treated similarly.

Given \( n \geq 1 \), let \( U_n \) be a uniform point in \( \{ 0, 1, \ldots, n - 1 \}^k \). Let \( S_n := n\mathbb{Z}^k - U \) which is a sub-lattice of \( \mathbb{Z}^k \) shifted randomly. As in Example 2.27, it can be seen that \( S_n \) is an equivariant subset of \( \mathbb{Z}^k \). Also, it is clear that \( S_n \) gives an \( n \)-covering of \( \mathbb{Z}^k \). Therefore, \( \lambda_n \leq \mathbb{P}(0 \in S_n) = n^{-k} \). This implies that \( \text{udim}_M(\mathbb{Z}^k) \geq k. \)

**Proposition 3.7.** If \( D \) is finite with positive probability, then \( \text{udim}_M(D) = 0. \)

**Proof.** It is shown below that for any equivariant subset \( S \) such that \( S_D \) is nonempty a.s., the intensity of \( S \) in \( D \) is at least \( \mathbb{E}[1/\#D] \). In turn, this shows that \( \forall r : \lambda_r \geq \mathbb{E}[1/\#D] > 0 \), where \( \lambda_r \) is defined in (3.1). This implies the claim.

Before presenting the proof, here is a heuristic proof of the above claim in the case \( D \) is finite a.s.: By Example 2.15, one can assume that \([D]\) is a random finite non-pointed discrete space and \( o \) is a random point of \( D \) chosen uniformly at random. It follows that, given \( D \), the probability that \( o \in S_D \) is \( \#S_D/\#D \geq 1/\#D \). This implies the claim.

For the general case, let \( S \) be as above. For all discrete spaces \( D \) and \( u, v \in D \), let \( g(u, v) := 1/\#D \) if \( u \in S_D \) and let \( g(u, v) := 0 \) otherwise (where
1/∞ := 0 by convention). It can be seen that $g$ is an equivariant process. Then, $g^+(o) = 1$ if $o \in S_D$ and $D$ is finite. Also, $g^-(o) = \#S_D/\#D$ (where $\infty/\infty := 0$ by convention). Therefore, by Lemma 2.28 and the mass transport principle, one gets

$$P[o \in S_D] \geq P[o \in S_D, \#D < \infty] = E[g^+(o)] = E[g^-(o)] \geq E[1/\#D] =: c.$$ 

Therefore, the intensity of $S$ in $D$ is at least $c$. Now, the definition 3.1 of $\lambda_r$ implies that $\lambda_r > c$ for all $r$ and the claim is proved.

**Remark 3.8** (Bounding the Minkowski Dimension). In all examples in this work, lower bounds on the unimodular Minkowski dimension are obtained by constructing explicit examples of $r$-coverings, which lead to upper bounds for $\lambda_r$. For upper bounds, disjoint or bounded coverings are useful, as discussed in Subsection 3.2 below. Another method for providing upper bounds is by comparison with the unimodular Hausdorff dimension defined in Subsection 3.3 below (see Theorem 3.31).

### 3.2 Optimal Coverings for the Minkowski Dimension

**Definition 3.9.** Let $[D, o]$ be a unimodular discrete space and $r \geq 0$. If the infimum in the definition of $\lambda_r$ (3.1) is attained by an equivariant $r$-covering $S$; i.e., $P[o \in S_D] = \lambda_r$, then $S$ is called an **optimal** $r$-covering for $D$.

**Theorem 3.10.** Every unimodular discrete space has an optimal $r$-covering for every $r \geq 0$.

This theorem is proved in Appendix B by tightness arguments.

**Corollary 3.11.** For each $r \geq 0$, one has $\lambda_r > 0$.

**Proof.** By Lemma 2.30, any non-empty equivariant subset has positive intensity. So the claim follows by the existence of an optimal $r$-covering.

In general, finding an optimal covering is difficult. In some specific examples, the following is easier to study.

**Definition 3.12.** Let $K < \infty$ and $r \geq 0$. An $r$-covering of $D$ is **$K$-bounded** if each point of $D$ is covered at most $K$ times a.s. by the balls in the covering. A sequence $(R_n)_n$ of equivariant coverings of $D$ is called **uniformly bounded** if there is $K < \infty$ such that each $R_n$ is $K$-bounded.

**Lemma 3.13.** If $R$ is a $K$-bounded equivariant $r$-covering of $D$, then

$$\frac{1}{K}P[R(o) \neq 0] \leq \lambda_r \leq P[R(o) \neq 0].$$ 

(3.2)

So if $(R_n)_n$ is a sequence of equivariant coverings which is uniformly bounded, with $R_n$ an $n$-covering for each $n \geq 1$, then

$$\overline{\dim}_M(D) = \text{decay}(P[R_n(o) \neq 0]),$$ 

$$\underline{\dim}_M(D) = \text{decay}(P[R_n(o) \neq 0]).$$
Proof. The rightmost inequality in (3.2) is immediate from the definition of $\lambda_r$. Let $R'$ be another equivariant $r$-covering. Let $g(u,v) = 1$ if $R'(u) = R(v) = r$ and $d(u,v) \leq r$. Then $g^+(o) \leq K_1(R(o) \neq 0)$ and $g^-(o) \geq 1_{R(o) \neq 0}$. Hence by the mass transport principle (2.3), $\frac{1}{K_1} \mathbb{P}(R(o) \neq 0) \leq \mathbb{P}(R'(o) \neq 0) \leq \mathbb{P}(R(o) \neq 0)$ and the leftmost inequality in (3.2) then follows from the definition of $\lambda_r$. Now, the last two equalities follow directly from (3.2) and Definition 3.2 (see also Remark 3.3).

The first claim (3.2) of the lemma readily implies the following.

Corollary 3.14. If $R$ is an equivariant disjoint $r$-covering of $D$ (i.e., the balls used in the covering are pairwise disjoint a.s.), then it is an optimal $r$-covering for $D$.

Example 3.15. Consider the sequence of equivariant coverings of $\mathbb{Z}^k$ constructed in Example 3.6. Note that each point is covered at most $2^k$ times. So the sequence is uniformly bounded. Therefore, Lemma 3.13 completes the proof of $\text{udim}_M(\mathbb{Z}^k) = k$ announced in Example 3.6.

By a similar argument, one can construct a sequence of equivariant disjoint coverings for $\mathbb{Z}^k$ equipped with the $l_\infty$ norm (each ball under this metric is a cube). By Corollary 3.14 these coverings are optimal.

Example 3.16. Given $r > 0$, the following provides a 3-bounded $r$-covering for discrete subsets of $\mathbb{R}$. Let $\varphi$ be a discrete subset of $\mathbb{R}$. Let $U_r$ be a random number in the interval $[0, r)$ chosen uniformly. For each $n \in \mathbb{Z}$, put a ball of radius $r$ centered at the largest number of $\varphi \cap [nr + U_r, (n+1)r + U_r)$. These balls obviously cover $\varphi$. Denote this random covering of $\varphi$ by $R_\varphi$. One can see that if $\psi \subseteq \mathbb{R}^k$ is a translation of $\varphi$, then $R_\psi$ is obtained by the same translation of $R_\varphi$ (up to distribution). This implies that $R$ is an equivariant covering (verifying Condition (ii) of Definition 12.21 is skipped here). Moreover, note that each point is covered at most 3 times. Therefore, $R$ is 3-bounded (Definition 3.12).

Proposition 3.17. For any point-stationary point process $\Phi$ in $\mathbb{R}$ endowed with the Euclidean metric, one has

$$1 \wedge \text{decay}(\mathbb{P}(\Phi \cap (0,r) = \emptyset)) \geq \text{decay}\left(\frac{1}{r} \int_0^r \mathbb{P}(\Phi \cap (0,s) = \emptyset) \, ds\right)$$

$$= \text{udim}_M(\Phi)$$

$$\geq \text{udim}_M(\Phi)$$

$$= \text{decay}\left(\frac{1}{r} \int_0^r \mathbb{P}(\Phi \cap (0,s) = \emptyset) \, ds\right)$$

$$\geq 1 \wedge \text{decay}(\mathbb{P}(\Phi \cap (0,r) = \emptyset)) .$$

Proof. Let $r > 0$ and consider $U_r$ and $R$ as in Example 3.16. One has

$$\mathbb{P}(R(0) \neq 0) = \mathbb{P}(\Phi \cap (0,U_r) = \emptyset) = \frac{1}{r} \int_0^r \mathbb{P}(\Phi \cap (0,s) = \emptyset) \, ds.$$
So Lemma 3.13 implies both equalities in the claim. Let \( f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0} \) be an arbitrary non-increasing function. For \( \beta < \text{decay} (f(r)) \), one has \( f(r) < r^{-\beta} \) for large enough \( r \). If in addition \( \beta < 1 \), one gets that \( \frac{1}{r} \int_0^r f(s)ds \leq cr^{-\beta} \) for some constant \( c \). This implies that

\[
\text{decay} \left( \frac{1}{r} \int_0^r f(s)ds \right) \geq 1 \land \text{decay} (f(r)).
\]

This implies the last inequality in the proposition. Similarly, for \( \beta > \text{decay} (f(r)) \), one can show that \( \beta \geq \text{decay} \left( \frac{1}{r} \int_0^r f(s)ds \right) \). So

\[
\text{decay} \left( \frac{1}{r} \int_0^r f(s)ds \right) \leq \text{decay} (f(r)).
\]

In addition, for \( c := \int_0^1 f(s)ds \) and all \( r \geq 1 \), one has \( \forall r \geq 1 : \frac{1}{r} \int_0^r f(s)ds \geq \frac{c}{r} \). So \( \text{decay} \left( \frac{1}{r} \int_0^r f(s)ds \right) \leq 1 \). Therefore,

\[
\text{decay} \left( \frac{1}{r} \int_0^r f(s)ds \right) \leq 1 \land \text{decay} (f(r)).
\]

These imply the first inequality of the proposition and the proof is complete.

The last result shows that, for point-stationary point processes on the real line, the heavier the tail of the inter-arrival times, the smaller the Minkowski dimension.

In the following example, let \( T_k \) be the \( k \)-regular tree, which is an infinite tree when \( k \geq 2 \). It is well known that \( T_k \), rooted at an arbitrary vertex \( o \), is unimodular [9].

**Example 3.18.** An optimal covering for the \( k \)-regular tree \( T_k \) is explicitly constructed below, which is a disjoint covering. By Definition 3.9, the construction implies that \( \lambda_n = 1/N_n (o) \) which has exponential decay when \( k \geq 3 \). One hence gets

\[
\forall k \geq 3 : \text{udim}_M (T_k) = \infty.
\]

Here is the construction. The reader can verify that when no root is fixed, there is a unique (deterministic) disjoint \( r \)-covering of \( T_k \) up to isomorphisms. Denote this \( r \)-covering by \( m \). Also, for \( v \in T_k \), the isomorphism class of \( [T,v;m] \) depends only on the distance of \( v \) to the center of the unique ball that contains \( v \). Consider one of the balls in the covering and let \( o \) be a point in the ball chosen uniformly at random. It can be seen that \( [T_k,o;m] \) is unimodular (see Theorem 3.1 of [2]). Note that, by the heuristic in the beginning of Subsection 2.6, this can be regarded as an equivariant process on \( T_k \). Indeed, Proposition 3.2 allows one to obtain an equivariant process in the sense of Definition 2.21. So one gets an equivariant disjoint \( r \)-covering of \( T_k \) and the proof is completed.

**Problem 3.19.** Is there an algorithm to construct an optimal \( n \)-covering of an arbitrary unimodular discrete space?
Remark 3.20. In general, finding an optimal covering is difficult. Some examples where optimal coverings are known are lattices (with the $l_\infty$ distance), finite trees (Subsection 4.1.1), one-ended trees (Subsection 4.1.3), and regular trees (Example 3.18). A disjoint covering is optimal (Corollary 3.14), but does not necessarily exist; e.g., for $\mathbb{Z}^k$ with the $l_2$ distance (for large enough $r$). In some examples, it is easy to find a sequence of uniformly bounded coverings, which are intuitively nearly optimal (see Lemma 3.13).

3.3 The Unimodular Hausdorff Dimension

The definition of the unimodular Hausdorff dimension is based on coverings of the discrete space by balls of possibly different radii. Intuitively, a covering is just an (equivariant) assignment of marks to the points, where the mark of a point $v$ represents the radius of the ball centered at $v$. For reasons explained in the introduction (Subsection 1.2.1), the radii are assumed to be at least 1. Also, by convention, if there is no ball centered at $v$, the mark of $v$ is defined to be 0. In relation with this convention, the following notation is used for all discrete spaces $D$ and points $v \in D$:

$$N_r(v) := \begin{cases} \{ u \in D : d(v, u) \leq r \}, & r \geq 1, \\ \emptyset, & r = 0 \end{cases},$$

In words, $N_r(v)$ is the closed ball of radius $r$ centered at $v$, except when $r = 0$.\[1\]

Definition 3.21. Let $[D, o]$ be a unimodular discrete space. An equivariant (ball-) covering $R$ of $D$ is an equivariant process on $D$ (Definition 2.21) with values in $\Xi := \{0\} \cup [1, \infty)$ such that the family of balls $\{N_{R(v)}(v) : v \in D\}$ covers the points of $D$ almost surely. For simplicity, $N_{R(v)}(v)$ will also be denoted by $N_R(v)$.

The key point of assuming equivariance in the above definition is that by Lemma 2.28 $[D, o; R]$ is a unimodular marked discrete space. Note also that extra randomness is allowed in the definition of equivariant coverings.

For $0 \leq \alpha < \infty$ and $1 \leq M < \infty$, let

$$\mathcal{H}_M^\alpha(D) := \inf \{ \mathbb{E} [R(o)^\alpha] : R(v) \in \{0\} \cup [M, \infty), \forall v, \text{ a.s.} \}, \tag{3.3}$$

where the infimum is over all equivariant coverings $R$ such that almost surely, $\forall v \in D : R(v) \in \{0\} \cup [M, \infty)$, and, by convention, $0^0 := 0$. Note that $\mathcal{H}_M^\alpha(D)$ is a non-decreasing function of both $\alpha$ and $M$ (the condition $M \geq 1$ is necessary here).

Definition 3.22. Let $[D, o]$ be a unimodular discrete space. The number $\mathcal{H}_1^\alpha(D)$, defined in (3.3), is called the $\alpha$-dimensional Hausdorff content of $D$. The unimodular Hausdorff dimension of $D$ is defined by

$$\text{udim}_H(D) := \sup \{ \alpha \geq 0 : \mathcal{H}_1^\alpha(D) = 0 \}, \tag{3.4}$$
with the convention that $\sup \emptyset = 0$.

Note that
\[
0 \leq \mathcal{H}_1^\alpha (D) \leq 1,
\]
since for the covering by balls of radii 1, one has $\mathbb{E} [R(o)^\alpha] = 1$.

The following propositions and Example 3.27 provide simple illustrations of the Hausdorff dimension.

**Proposition 3.23.** If $D$ is finite with positive probability, then $\text{udim}_\mathcal{H}(D) = 0$.

**Proof.** Let $\alpha \geq 0$ and $R$ be an arbitrary equivariant covering of $D$. The proof of Proposition 3.7 shows that $\mathbb{P} [R(o)^\alpha > 0] \geq \mathbb{E} [1/\#D]$. Therefore, $\mathbb{E} [R(o)^\alpha] \geq \mathbb{E} [1/\#D] > 0$. This implies the claim. \qed

**Proposition 3.24.** One has $\text{udim}_\mathcal{H}(\mathbb{Z}^k) = k$.

**Proof.** Let $R$ be the equivariant covering constructed in Example 3.6. One has $\mathbb{E} [R(o)^\alpha] = n^{a-k}$. If $\alpha < k$, this value is arbitrarily small for large $n$. Therefore, $\mathcal{H}_1^\alpha (\mathbb{Z}^k) = 0$, which implies that $\text{udim}_\mathcal{H}(\mathbb{Z}^k) \geq \alpha$. So one gets $\mathcal{H}_1^\alpha (\mathbb{Z}^k) \geq \mathbb{E} [1/\#D] > 0$. This implies that $\text{udim}_\mathcal{H}(\mathbb{Z}^k) = k$.

**Lemma 3.25.** Let $[D, o]$ be a unimodular discrete space and $\alpha \geq 0$. If there exists $c \geq 0$ such that $\forall r \geq 1 : \#N_r(o) \leq cr^\alpha$ a.s., then $\text{udim}_\mathcal{H}(D) \leq \alpha$.

**Proof.** Let $R$ be an arbitrary equivariant covering. For all discrete spaces $D$ and $u, v \in D$, let $g_D(u, v) = 1$ if $d(u, v) \leq R_D(u)$ and 0 otherwise. One has $g^+(u) = \#N_R(u)$ and $g^-(u) \geq 1$ a.s. (since $R$ is a covering). By the assumption and the mass transport principle (2.3), one gets
\[
\mathbb{E} [R(o)^\alpha] \geq \frac{1}{c} \mathbb{E} [\#N_R(o)] = \frac{1}{c} \mathbb{E} [g^+(o)] = \frac{1}{c} \mathbb{E} [g^-(o)] \geq \frac{1}{c}.
\]
Since $R$ is arbitrary, one gets $\mathcal{H}_1^\alpha (D) \geq \frac{1}{c} > 0$. This implies that $\text{udim}_\mathcal{H}(D) \leq \alpha$. \qed

**Remark 3.26** (Bounding the Hausdorff Dimension). In most examples in this work, a lower bound on the unimodular Hausdorff dimension is provided, either by comparison with the Minkowski dimension (see Subsection 3.4 below), or by explicit construction of a sequence of equivariant coverings $R_1, R_2, \ldots$ such that $\mathbb{E} [R_n(o)^\alpha] \to 0$ as $n \to \infty$. Note that this gives $\mathcal{H}_1^\alpha (D) = 0$, which implies that $\text{udim}_\mathcal{H}(D) \geq \alpha$. Constructing coverings does not help to find upper bounds for the Hausdorff dimension. The derivation of upper bounds is mainly discussed in Part II. The main tools are the mass distribution principle (Theorem II.2.2), which is a stronger form of Lemma 3.25 above, and the unimodular Billingsley’s lemma (Theorem II.2.8).
Example 3.27. Let \([D, o]\) be \([\mathbb{Z}, 0]\) with probability \(\frac{1}{2}\) and \([\mathbb{Z}^2, 0]\) with probability \(\frac{1}{2}\). It is shown below that \(\operatorname{udim}_M(D) = \operatorname{udim}_H(D) = 1\).

For \(n \in \mathbb{N}\), the equivariant \(n\)-covering of Example 3.6 makes sense for \(D\) and is uniformly bounded. One has \(\mathbb{P}[R(0) > 0] = \frac{1}{2}(n^{-1} + n^{-2})\). This implies that \(\operatorname{udim}_M(D) = \text{decay} \left(\frac{1}{2}(n^{-1} + n^{-2})\right) = 1\) and also \(\operatorname{udim}_H(D) \geq 1\). On the other hand, for any equivariant covering \(S\), one has

\[
\mathbb{E}[S(o)] \geq \mathbb{E}[S(o)|D = \mathbb{Z}] \mathbb{P}[D = \mathbb{Z}] = \frac{1}{2} \mathbb{E}[S(o)|D = \mathbb{Z}].
\]

Let \(c > 2\). The proof of Lemma 3.25 for \([\mathbb{Z}, 0]\) implies that \(\mathbb{E}[S(o)|D = \mathbb{Z}] \geq \frac{1}{c}\). This implies that \(\mathcal{H}_1(D) \geq \frac{1}{2c} > 0\). So \(\operatorname{udim}_H(D) \leq 1\).

Remark 3.28. In Example 3.27 above, different samples of \(D\) have different natures heuristically. This is formalized by saying that \([D, o]\) is not ergodic; i.e., there is an event \(A \subseteq D\) such that the proposition \([D, o] \in A\) does not depend on the origin of \(D\) and \(0 < \mathbb{P}[[D, o] \in A] < 1\). The concept of ergodicity will be discussed in Part III. In this work, the focus is mainly on the ergodic case. However most definitions and results do not require ergodicity. In the non-ergodic cases, like in Example 3.27, it is desirable to assign a dimension to every sample of \(D\). This will be formalized as sample dimension in Part III.

Remark 3.29. In contrast to the Minkowski dimension, there might be no optimal covering for the Hausdorff content defined in (3.5); e.g., when \(\alpha < \operatorname{udim}_H(D)\).

Lemma 3.30. Let \([D, o]\) be a unimodular discrete space. Let \(\alpha > 0\) and \(M\) be a non-negative function of \([D, o]\) such that \(\mathbb{E}[M^\alpha] < \infty\). If \(\alpha < \operatorname{udim}_H(D)\), then \(\mathbb{E}[R(o)^\alpha]\) can be made arbitrarily small for equivariant coverings \(R\) such that, almost surely, for all \(v \in D\), \(R(v) \in \{0\} \cup \{M[D, v], \infty\}\).

The proof is left to the reader.

3.4 Comparison of Hausdorff and Minkowski Dimensions

Theorem 3.31 (Minkowski vs Hausdorff). One has

\[
\operatorname{udim}_M(D) \leq \operatorname{udim}_H(D) \leq \operatorname{udim}_H(D).
\]

Proof. The first inequality holds by the definition. For the second one, the definition of \(\lambda_r\) implies that for every \(\alpha \geq 0\) and \(r \geq 1\),

\[
\inf \{\mathbb{E}[R(o)^\alpha] : R\text{ is an equivariant }r\text{-covering}\} = r^\alpha \lambda_r.
\]

This readily implies that

\[
\mathcal{H}_1^r(D) \leq r^\alpha \lambda_r.
\]

Also, the definitions of the Minkowski dimensions and the decay rates (Definitions 3.2 and 2.1), easily imply that

\[
\operatorname{udim}_M(D) = \sup \{\alpha \geq 0 : \inf \{r^\alpha \lambda_r : r \geq 1\} = 0\}.
\]

The last two assertions imply the claim. \(\square\)
Remark 3.32. Two examples are the generalized canopy tree of Subsection 4.2.2 and the example in Subsection 4.4.

Example 3.33. By Theorem 3.31 and Example 3.18, one gets that the \( k \)-regular tree \((k \geq 3)\) satisfies \( \operatorname{udim}_H(T_k) = \infty \). Also, Example 3.15 implies that \( \operatorname{udim}_H(\mathbb{Z}^k) \geq k \), which is part of the claim of Proposition 3.24.

### 3.5 The Unimodular Hausdorff Measure

Consider the setting of Subsection 3.3. For \( 0 \leq \alpha < \infty \), let

\[
\mathcal{H}^\alpha_M(D) := \lim_{M \to \infty} \mathcal{H}^\alpha_M(D) \in [0, \infty],
\]

(3.5)

where \( \mathcal{H}^\alpha_M(D) \) is defined in (3.3). Note that the limit exists because of monotonicity.

**Definition 3.34.** The \( \alpha \)-dimensional Hausdorff measure of \( D \) is defined by

\[
M^{\alpha}(D) := (\mathcal{H}^\alpha_M(D))^{-1}.
\]

(3.6)

The term *measure* in the above definition will be justified in Part III. In fact, under some conditions, it gives rise to a measure on \( D_\ast \).

The following results gather some elementary properties of the function \( \mathcal{H}^\alpha_M \) and the Hausdorff measure.

**Lemma 3.35.** One has

(i) \( \mathcal{H}^\alpha_1(D) \leq \mathcal{H}^\alpha_M(D) \leq M^{\alpha} \mathcal{H}^\alpha_1(D) \).

(ii) If \( \mathcal{H}^\alpha_1(D) = 0 \), then \( \mathcal{H}^\alpha_\infty(D) = 0 \); i.e., \( M^{\alpha}(D) = \infty \).

(iii) If \( \alpha \geq \beta \), then \( \mathcal{H}^\alpha_M(D) \geq M^{\alpha-\beta} \mathcal{H}^\beta_M(D) \).

**Proof.**

(i). If \( R \) is an equivariant covering, then \( M \) is also an equivariant covering and satisfies \( \forall v \in D : M \mathcal{R}(v) \in \{0\} \cup [M, \infty) \) a.s.

(ii). The claim is implied by part (i).

(iii). If \( R \) is an equivariant covering such that \( \forall v \in D : R(v) \in \{0\} \cup [M, \infty) \) a.s., then \( R(o)^\alpha \geq M^{\alpha-\beta} R(o)^\beta \) a.s. \( \square \)

**Lemma 3.36.** One has

\[
\forall \alpha < \operatorname{udim}_H(D), \quad M^{\alpha}(D) = \infty,
\]

\[
\forall \alpha > \operatorname{udim}_H(D), \quad M^{\alpha}(D) = 0.
\]

**Proof.** For \( \alpha < \operatorname{udim}_H(D) \), one has \( \mathcal{H}^\alpha_1(D) = 0 \). So part (iii) of Lemma 3.35 implies that \( M^{\alpha}(D) = \infty \). For \( \alpha > \operatorname{udim}_H(D) \), there exists \( \beta \) such that \( \alpha > \beta > \operatorname{udim}_H(D) \). For this \( \beta \), one has \( \mathcal{H}^\beta_1(D) > 0 \) and part (iii) of the same lemma implies that \( \mathcal{H}^\alpha_M(D) \geq M^{\alpha-\beta} \mathcal{H}^\beta_M(D) \geq M^{\alpha-\beta} \mathcal{H}^\beta_1(D) \). This implies that \( \mathcal{H}^\alpha_\infty(D) = \infty \), which proves the claim. \( \square \)
For the case $\alpha = \udim_H(D)$, see Example 3.39 below.

The following propositions provide examples of computation of the Hausdorff measure.

**Proposition 3.37** (0-dimensional Hausdorff Measure). One has
\[
\mathcal{M}^0(D) = \left( \mathbb{E}\left[ \frac{1}{\#D} \right] \right)^{-1}.
\]

In particular, if $\#D$ is almost surely constant, then $\mathcal{M}^0(D) = \#D$ a.s.

**Proof.** As in the proof of Proposition 3.23, one gets $\mathcal{H}_M^0(D) \geq \mathbb{E}[1/\#D]$. It is enough to show that equality holds.

First, assume $D$ is finite a.s. Put a ball of radius $M \vee \text{diam}(D)$ centered at a point of $D$ chosen uniformly at random. It can be seen that the conditions of Definition 2.21 hold and that an equivariant covering, denoted by $R$, is obtained. One has $\mathbb{E}[R(o)^0] = \mathbb{E}[1/\#D]$, which shows the claim.

Second, assume $D$ is infinite a.s. It is enough to construct an equivariant covering $R$ such that $\mathbb{P}[\text{R(o) > 0}]$ is arbitrarily small. Let $p > 0$ be arbitrary and $S$ be the Bernoulli equivariant subset obtained by selecting each point with probability $p$ in an i.i.d. manner. For all infinite discrete spaces $D$ and $v \in D$, let $\tau_D(v)$ be the closest point of $S_D$ to $v$ (if there is a tie, choose one of them uniformly at random independently). It can be seen that $\tau_D^{-1}(u)$ is finite almost surely (use the mass transport principle for $g(x, y) := 1_{\{y = \tau_D(x)\}}$ and Lemma 2.28). For $u \in S_D$, let $R(u) := \text{diam}(\tau_D^{-1}(u))$ be the diameter of the Voronoi cell of $u$. It is clear that $R$ is a covering, and in fact, an equivariant covering. One has $\mathbb{P}[\text{R(o) > 0}] = \mathbb{P}[o \in S_D] = p$, which is arbitrarily small. So the claim is proved.

Finally, assume $D$ is finite with probability $q$ and infinite with probability $1-q$. Let $p > 0$ be arbitrary. For finite discrete spaces, consider the construction in the first case. For infinite discrete spaces, consider the second construction. This gives an equivariant covering $R$. It satisfies $\mathbb{P}[\text{R(o) > 0}] = \mathbb{E}[1/\#D] + p$. Since $p$ is arbitrary, the claim is proved. \hfill $\square$

**Proposition 3.38.** The $k$-dimensional Hausdorff measure of the scaled lattice $[\delta \mathbb{Z}^k, 0]$, equipped with the $l_\infty$ metric, is given by
\[
\mathcal{M}^k(\delta \mathbb{Z}^k) = \left( \frac{2}{\delta} \right)^k.
\]

**Proof.** Let $U_n \in [-n, n]^k \cap \mathbb{Z}^k$ be chosen uniformly at random. Let $S_n$ be the scaled and shifted lattice $(2n + 1)\delta \mathbb{Z}^k - \delta U_n$. As in Example 3.38, $S_n$ is an equivariant subset of $\delta \mathbb{Z}^k$. Since the $l_\infty$ metric is assumed, $S_n$ can be regarded as an equivariant $n\delta$-covering. Notice that $\mathbb{E}[S_n(o)^k] = (n\delta)^k/(2n + 1)^k$. This easily implies that $\mathcal{H}_\infty^k(\delta \mathbb{Z}^k) \leq (\delta/2)^k$.

On the other hand, let $R$ be any equivariant covering. The proof of Lemma 3.25 shows that $\mathcal{H}_\infty^k(\delta \mathbb{Z}^k) \geq c\delta^k$, where $c$ is any constant such that $\forall r \geq 1 : r^k \geq c\#N_r(0)$. It follows that $\mathcal{H}_\infty^k(\delta \mathbb{Z}^k) \geq (\delta/2)^k$. This implies the claim. \hfill $\square$
Example 3.39. For \( \alpha := \operatorname{udim}_H(D) \), the \( \alpha \)-dimensional Hausdorff measure of \( D \) can be zero, finite or infinite. The lattice \( \mathbb{Z}^k \) (Proposition 3.38) is a case where \( \mathcal{M}^\alpha(D) \) is positive and finite. Examples II.3.11 and II.3.12 provide examples of the infinite and zero cases respectively.

3.6 The Effect of Changing the Metric

In some examples of discrete spaces, several natural metrics can be considered. Intuitively, one expects that the dimension of a unimodular discrete space depends on the choice of metric. This is formalized in the following.

To avoid confusion between the metrics, a pointed discrete space is denoted by \( (D,d,o) \) here, where \( d \) is the metric on \( D \) and \( o \) is the origin. Note that if \( d' \) is another metric on \( D \), then \( d' \in \mathbb{R}^{D \times D} \). So \( d' \) can be considered as a marking of \( D \) in the sense of Definition 2.20 and \( (D,d,o);d' \) is a pointed marked discrete space.

Definition 3.40. An equivariant (boundedly finite) metric is an \( \mathbb{R} \)-valued equivariant process \( d' \) (Definition 2.21) such that, for all discrete spaces \( D \), \( d'_D \) is almost surely (w.r.t. the extra randomness) a metric on \( D \) and \( (D,d'_D) \) is a boundedly-finite metric space.

If in addition, \( [(D,d),o] \) is a unimodular discrete space, then \( [(D,d),o;d'] \) is a unimodular marked discrete space by Lemma 2.28. Now, the measurability result of Lemma [A.4] in the appendix implies that \( [(D,d'),o,d] \), obtained by swapping the metrics, makes sense as a random pointed marked discrete space. It is also unimodular as shown in the following theorem, which is the main result of this subsection. The statements in the theorem and its corollaries are valid for both the Hausdorff and the (upper and lower) Minkowski dimensions.

Theorem 3.41 (Change of Metric). Let \( [(D,d),o] \) be a unimodular discrete space and \( d' \) be an equivariant metric. Then,

(i) \( [(D,d'),o;d] \) is a unimodular marked discrete space.

(ii) If \( d'_D \leq cd_D + a \) a.s., with \( c \) and \( a \) constants, then the dimension of \( (D,d') \) is larger than or equal to that of \( (D,d) \). Moreover, for every \( \alpha \geq 0 \),

\[
\mathcal{M}^\alpha(D,d') \geq c^{-\alpha} \mathcal{M}^\alpha(D,d).
\]

Proof. The first claim can be easily proved by Lemma 2.28 and directly verifying the mass transport principle (2.3) (see also Lemma [A.4]).

The second part can also be proved easily using the fact that the ball \( N_r((D,d'),v) \) contains the ball \( N_{cr+a}((D,d),v) \).

Corollary 3.42. If for some constants \( c_1, c_2 > 0 \) and \( a \geq 0 \), one has

\[
c_1 d_D - a \leq d'_D \leq c_2 d_D + a, \quad \text{a.s.,}
\]

then the dimension of \( (D,d') \) is equal to the dimension of \( (D,d) \).
Corollary 3.43 (Scaling). If (3.7) holds with \( c_1 = c_2 = c \), then for every \( \alpha \geq 0 \), \( M^\alpha(D, d') = c^{-\alpha} M^\alpha(D, d) \). In particular, \( cD \) has the same dimension as \( D \) and \( M^\alpha(cD) = c^{-\alpha} M^\alpha(D) \).

As an example, this gives \( M^\alpha(\delta Z^k) = \delta^{-\alpha} M^\alpha(Z^k) \), which is consistent with Proposition 3.38.

**Corollary 3.44.** For point-stationary point processes in \( \mathbb{R}^k \), choosing any of the usual metrics on \( \mathbb{R}^k \) (as long as it is equivalent to the Euclidean metric) does not affect the Hausdorff and Minkowski dimensions.

**Example 3.45** (Equivariant Edge Lengths). Assume \( l \) is an equivariant process which assigns a positive number to the edges of every deterministic graph. For a simple path \( v_1, \ldots, v_k \), call \( \sum_i l(v_i, v_{i+1}) \) the weight of the path. For all trees \( T \), one readily obtains a metric \( d'_T \) on \( T \) by letting \( d'_T(u, v) \) be the weight of the unique path connecting \( u \) and \( v \). For all graphs \( G \), one can let \( d'_G(u, v) \) be the infimum of the weights of the simple paths connecting \( u \) and \( v \). Assuming that for every graph \( G \), \( d'_G \) is boundedly finite a.s., then \( d' \) is an equivariant metric and will be said to be generated by equivariant edge lengths in this paper.

An instance of such metrics appears in the definitions the Poisson-weighted infinite tree in [3], which is discussed in Subsection II.3.2.3.

**Example 3.46.** Let \( [G, o] \) be a unimodular graph. Let \( H \) be an equivariant subgraph (see Remark 2.34) such that \( H_G \) is a spanning subgraph of \( G \) and is connected a.s. By Lemma 2.28, \( [H_G, o] \) is unimodular. Now, Theorem 3.41 implies that, under the graph-distance metric, the dimension of \( H_G \) is less than or equal to that of \( G \). For instance, see the drainage network model of Subsection 4.5 below.

More examples of metric change are provided in Subsections 4.1.4 and II.3.3.2.

### 3.7 Dimension of Subspaces

Let \( [D, o] \) be a unimodular discrete space and \( S \) be an equivariant subset which is almost surely nonempty. Lemma 2.30 implies that \( \mathbb{P}[o \in S_D] > 0 \). So one can consider \( [S_D, o] \) conditioned on \( o \in S_D \). By directly verifying the mass transport principle (2.2), it is easy to see that \( [S_D, o] \) conditioned on \( o \in S_D \) is unimodular (see the similar claim for unimodular graphs in [6]).

**Convention 3.47.** For an equivariant subset \( S \) as above, the unimodular Hausdorff dimension of \( [S_D, o] \) (conditioned on \( o \in S_D \)) is denoted by \( \text{udim}_H(S_D) \) (see also Remark 2.33). The same convention is used for the Minkowski dimension, the Hausdorff measure, etc.

**Theorem 3.48.** Let \( [D, o] \) be a unimodular discrete space and \( S \) an equivariant subset such that \( S_D \) is nonempty a.s. Then,

1. The Hausdorff dimension of \( S_D \) satisfies
   \[
   \text{udim}_H(S_D) = \text{udim}_H(D).
   \]
(ii) The Minkowski dimension of $S_D$ satisfies

$$\overline{\text{udim}}_M(S_D) \geq \overline{\text{udim}}_M(D),$$

$$\underline{\text{udim}}_M(S_D) \geq \underline{\text{udim}}_M(D).$$

(iii) For every $\alpha \geq 0$, the $\alpha$-dimensional Hausdorff measure of $S_D$ satisfies

$$\mathcal{M}^{\alpha}(S_D) = \rho_D(S)\mathcal{M}^{\alpha}(D),$$

where $\rho_D(S) = \mathbb{P}[\omega \in S_D]$ is the intensity of $S$ in $D$.

Proof. [4] Let $0 < \alpha < \text{udim}_H(S_D)$. Lemma [3.35] implies that $H^{\alpha}_n(S_D) = 0$ for every $n \in \mathbb{N}$. Therefore, one can find a sequence $R_n$ of equivariant coverings of $S_D$ such that $\mathbb{E}[R_n(o)^\alpha | o \in S_D] \to 0$ as $n \to \infty$ and for all $v$, $R_n(v) \in \{0\} \cup [n, \infty)$ a.s. One may extend $R_n$ to be defined on $D$ by letting $R_n(v) := 0$ for $v \in D \setminus S_D$. Hence, one gets $\mathbb{E}[R_n(o)^\alpha] \to 0$. Let $\epsilon > 0$ be any constant; e.g., $\epsilon := 1$ (a small enough $\epsilon$ will be needed in part (iii) below). Let $B_n \subseteq D$ be the union of $N_{(1+\epsilon)R_n}(v)$ for all $v \in D$. Let

$$R'_n(u) := \begin{cases} (1+\epsilon)R_n(u), & u \in B_n, \\ 1, & u \notin B_n. \end{cases}$$

It is clear that $R'_n$ is an equivariant covering of $D$. Also,

$$\mathbb{E} [R'_n(o)^\alpha] = (1+\epsilon)^n \mathbb{E} [R_n(o)^\alpha] + \mathbb{P} [o \notin B_n].$$

(3.8)

Since the radii of the balls in $R_n$ are at least $n$, one gets that $\mathbb{P} [o \notin B_n] \leq \mathbb{P} [N_{n}(o) \cap S_D = \emptyset]$. Since $S_D$ is nonempty a.s., this implies that $\mathbb{P} [o \notin B_n] \to 0$ (note that the events $N_{n}(o) \cap S_D = \emptyset$ are nested and converge to the event $S_D = \emptyset$). From this and (3.8), one gets $\mathbb{E} [R'_n(o)^\alpha] \to 0$. This shows that $\text{udim}_H(D) \geq \alpha$. Thus, $\text{udim}_H(D) \geq \text{udim}_H(S_D)$.

Conversely, let $0 < \alpha < \text{udim}_H(D)$ and let $R_n$ be an equivariant covering of $D$ such that $\mathbb{E}[R_n(o)^\alpha] \to 0$. For any $v \in D$, let $\tau_n(v)$ be an element of $N_{R_n}(v) \cap S_D$ chosen uniformly at random in this set, and be undefined if the intersection is empty (do this independently for all points using extra randomness). For $w \in S_D$, let

$$R'_n(w) := 2\max\{R_n(v) : v \in \tau_{n}^{-1}(w)\}$$

(3.9)

with the convention $\max\emptyset := 0$. For any $v \in \tau_{n}^{-1}(w)$, one has $N_{R'_n(w)}(w) \supseteq N_{R_n}(v)$. So $R'_n$ is a covering of $S_D$. It can be seen that $R'_n$ is an equivariant covering. In addition, one has

$$\mathbb{E} [R'_n(o)^\alpha] \leq \mathbb{E} \left[ \sum_v (2R_n(v))^{\alpha} 1_{\{v \in \tau_{n}^{-1}(o)\}} \right]$$

$$= \mathbb{E} \left[ \sum_v (2R_n(o))^{\alpha} 1_{\{o \in \tau_{n}^{-1}(v)\}} \right]$$

$$\leq 2^\alpha \mathbb{E} [R_n(o)^\alpha]$$

$$\to 0,$$
where the mass transport principle is used in the equality. It follows that $\mathbb{E}(R_n(o)^\alpha | o \in S_D) \to 0$. Therefore, $\text{udim}_H(S_D) \geq \alpha$, which implies that $\text{udim}_H(S_D) \geq \text{udim}_H(D)$. So the claim is proved.

(ii) The proof is similar to that of part (i). Let $R$ be an arbitrary equivariant $r$-covering of $D$. Define $R'$ similarly to (3.9), which provides a $2r$-covering of $S_D$. The calculations in part (i) show that $\mathbb{E}(R(o)^\alpha) \leq 2\alpha \mathbb{E}(R(o)^\alpha)$. This implies that

$$\rho_D(S) \lambda_{2r}(S_D) \leq 2\alpha \lambda_r(D).$$

(3.10)

The proof of the claim is now easily concluded.

(iii) The idea of the proof is similar to that of the previous parts. First, let $R_n$ be a sequence of equivariant coverings of $S_D$ for $n = 1, 2, \ldots$ as in the proof of part (i). Let $\epsilon > 0$ be arbitrary and define $R_n$ similarly. Equation (3.3) implies that

$$\liminf_{n \to \infty} \mathbb{E}(R_n(o)^\alpha) \leq (1 + \epsilon^\alpha \liminf_{n \to \infty} \mathbb{E}(R_n(o)^\alpha).$$

Note that $\mathbb{E}(R_n(o)^\alpha) = P(o \in S_D) \mathbb{E}(R_n(o)^\alpha | o \in S_D)$. Therefore, by choosing the sequence $R_n$ suitably, one obtains $H_\infty^\alpha(D) \leq (1 + \epsilon^\alpha) \rho_D(S) H_\infty^\alpha(S_D)$. Since $\epsilon$ is arbitrary, one gets $H_\infty^\alpha(D) \leq \rho_D(S) H_\infty^\alpha(S_D)$, i.e., $\mathcal{M}^\alpha(S_D) \leq \rho_D(S) \mathcal{M}^\alpha(D)$.

Conversely, let $R_M$ be a sequence of equivariant coverings of $D$ for $M = 1, 2, \ldots$ such that $R_M(\cdot) \in \{0\} \cup |M, \infty|$ a.s. and $\mathbb{E}(R_M(o)^\alpha) \mathcal{H}_\infty^\alpha(D)$. Let $\epsilon > 0$ be arbitrary. Fix $M$ in the following. Let $A := A_D := \{v : N_{R_M(v)\cap S_D} = \emptyset\}$ and $B := B_D := \{v : N_{R_M(v)\cap S_D} \neq \emptyset\}$. For each $v \in A$, let $\tau_M(v)$ be an element chosen uniformly at random in $N_{R_M(v)} \cap S_D$. For each $v \in B \setminus A$, let $\tau_M(v)$ be an element chosen uniformly at random in $N_{R_M(v)} \cap S_D$. For $v \not\in B$, let $\tau_M(v)$ be undefined. For $w \in S_D$, let

$$R_M'(w) := \begin{cases} 1 + \epsilon \max\{R_M(v) : v \in \tau_M^{-1}(w) \cap A\}, & \tau_M^{-1}(w) \cap A \neq \emptyset, \\ 2 \max\{R_M(v) : v \in \tau_M^{-1}(w)\}, & \tau_M^{-1}(w) \cap A = \emptyset. \end{cases}$$

It can be seen that $R_M'$ is an equivariant covering of $S_D$. One has

$$\mathbb{E}(R_M(o)^\alpha) \leq \mathbb{E} \left[ \sum_v ((1 + \epsilon) R_M(v))^\alpha 1_{v \in \tau_M^{-1}(o) \cap A} + (2 R_M(v))^\alpha 1_{v \in \tau_M^{-1}(o) \setminus A} \right]$$

$$= \mathbb{E} \left[ \sum_v ((1 + \epsilon) R_M(o))^\alpha 1_{v \in \tau_M^{-1}(o) \cap A} + (2 R_M(o))^\alpha 1_{v \in \tau_M^{-1}(o) \setminus A} \right]$$

$$\leq \mathbb{E} \left[ ((1 + \epsilon) R_M(o))^\alpha 1_{o \in A} + (2 R_M(o))^\alpha 1_{o \not\in A} \right]$$

$$\to (1 + \epsilon)^\alpha \lim \mathbb{E}(R_M(o)^\alpha)$$

$$= (1 + \epsilon^\alpha \mathcal{H}_\infty^\alpha(D),$$

where the convergence holds because, as $M$ tends to infinity, the events $o \in A$ increase to the whole space mod 0 (note that the claim is valid in both cases.
where $\mathbb{E}[\mathbf{R}_M(o)^\alpha]$ tends to infinity or a finite value). It follows that

$$\rho_D(S) \lim_{M \to \infty} \mathbb{E}[\mathbf{R}'_M(o)^\alpha | o \in S_D] \leq (1 + \epsilon)^\alpha \mathcal{H}_\infty^\alpha(D).$$

So $\rho_D(S) \mathcal{H}_\infty^\alpha(S_D) \leq (1 + \epsilon)^\alpha H_\infty^\alpha(D)$. Since $\epsilon$ is arbitrary, one gets $\mathcal{M}_\alpha^\ast(S_D) \geq \rho_D(S) \mathcal{M}_\alpha^\ast(D)$ and the claim is proved.

**Remark 3.49.** In the setting of Theorem 3.48, $\text{udim}_\alpha(S_D)$ can be strictly larger than $\text{udim}_\alpha(D)$ (see e.g., the example in Subsection 4.4). Also, it can be seen that, in order to have the equalities for the Minkowski dimension in part (ii) of Theorem 3.48, it is enough that $S_D$ gives a $\mathcal{M}_\alpha^\ast$-covering of $D$ for some constant $M$.

### 3.8 Covering By Other Sets

Motivated by the continuum setting, it is natural to think of coverings by subsets which are not necessarily balls. One also expects that the notions of the Minkowski and Hausdorff dimensions do not change. This idea will be used in Subsection 4.1.3. The main challenge in the general case is to define such coverings such that the mass transport principle (2.2) holds. Again, this is done by means of the equivariant processes of Subsection 2.6.

Let $Z$ be an equivariant process with values in $\{0, 1\}$ (Definition 2.21). Consider the equivariant subset $S$ defined by $S_D := \{u \in D : Z_D(u) = 1\}$ for all deterministic discrete spaces $D$. Also, for $u \in D$, let $U_D(u) := \{v \in D : Z_D(u, v) = 1\}$. Then, consider the random family of subsets $C_D := \{U_D(u) : u \in S_D\}$ of $D$. If, for all discrete spaces $D$, the family $C_D$ covers $D$, then $C$ is called a **generalized equivariant covering**. This indeed generalizes equivariant coverings as follows: if $R$ is an equivariant covering, one can let $Z_D(u) := 1_{[R(u) > 0]}$ and $Z_D(u, v) := 1_{[v \in N_R(u)]}$ for $v \neq u$. Let $C$ be the family of all generalized equivariant coverings. Let $C_r \subseteq C$ consist of all generalized equivariant coverings with subsets of diameter at most $2r$; i.e., for all $D$ and all $u \in D$, one has $\text{diam}(U_D(u)) \leq 2r$ a.s.

Let $[D, o]$ be a unimodular discrete space. For $r \geq 0$, define

$$\lambda'_r := \inf_{C_r} \{\text{intensity of } S \text{ in } D\}.$$ 

Also, for $0 \leq \alpha < \infty$ and $1 \leq M < \infty$, let

$$\mathcal{H}'_{\alpha, M}(D) := \inf_{C_r} \left\{ \mathbb{E} \left[ (M \lor \frac{1}{2diam(U_o)})^\alpha 1_{S_D(o)} \right] \right\}.$$ 

Taking the maximum with $M$ is **similar** to the condition that the subsets have diameter at least $2M$. Note however that a ball of radius $M$ might have diameter strictly less than $2M$.

**Lemma 3.50.** One has

$$\lambda_{2r} \leq \lambda'_r \leq \lambda_r,$$

$$\mathcal{H}_{2M}^\alpha(D) \leq \mathcal{H}'_{\alpha, M}(D) \leq \mathcal{H}_M^\alpha(D).$$
Proof. Both right-most inequalities are obtained by regarding equivariant ball-coverings as special cases of generalized equivariant coverings. For the left-most inequalities, it is enough to see that for a generalized equivariant covering as above, by letting \( R_D(u) := \text{diam}(U_D(u))1_{S_D}(u) \), an equivariant ball-covering is obtained. \( \square \)

This lemma readily implies the following.

**Theorem 3.51.** For all unimodular discrete spaces \([D, o]\),

\[
\overline{\text{udim}}_M(D) = \overline{\text{decay}}(\lambda'_r), \\
\overline{\text{udim}}_M'(D) = \overline{\text{decay}}(\lambda'_r)
\]

and

\[
\text{udim}_H(D) = \sup\{\alpha \geq 0 : \mathcal{H}^\alpha(D) = 0\}.
\]

**Remark 3.52.** Note that in the above definition of generalized equivariant coverings, each subset \( U_D(u) \) in the covering has a center, which is \( u \) itself. Indeed, it is true that *changing the centers of the subsets* (in an equivariant way) does not affect anything. This can be made rigorous as follows: Let \([D, o]\) be a unimodular discrete space and \( Z \) be an equivariant process with values in \( \{0, 1\}^2 \). Define \( S^{(1)}, U^{(1)} \) and \( C^{(1)} \) as above by the first coordinates of the values of \( Z \) and similarly for the second coordinates. Assume the families \( C^{(1)} \) and \( C^{(2)} \) of subsets are identical almost surely (by counting multiplicities). Then, \( S^{(1)} \) and \( S^{(2)} \) have the same intensity in \( D \) and

\[
\mathbb{E}\left[f(U^{(1)}_o)1_{S^{(1)}_D}(o)\right] = \mathbb{E}\left[f(U^{(2)}_o)1_{S^{(2)}_D}(o)\right],
\]

for all good function \( f \). The proof is left to the reader.

## 4 Examples

This section presents a comprehensive set of examples of unimodular discrete spaces together with discussions about their dimensions. The main tools for bounding the dimensions can be described as follows. Lower bounds are mainly obtained by explicit constructions of equivariant coverings and some general results based on explicit coverings (e.g., Theorem 1). Upper bounds for the Minkowski dimensions are obtained by constructing either optimal coverings or uniformly bounded coverings. The tools for obtaining upper bounds for the Hausdorff dimension will be discussed in Part II. These are the mass distribution principle and the unimodular Billingsley lemma. So the upper bounds for most of the examples are completed therein.

### 4.1 General Unimodular Trees

In this subsection, general results are presented regarding the dimension of unimodular trees with the graph-distance metric. Specific instances are presented
later in the section. It turns out that the number of ends of the tree plays an important role (an end in a tree is an equivalence class of simple paths in the tree, where two such paths are equivalent if their symmetric difference is finite). It is well known that the number of ends in a unimodular tree belongs to \( \{0, 1, 2, \infty\} \) \(^2\). In this subsection, each case is treated separately. It turns out that the richest case is the one-ended case.

### 4.1.1 Unimodular Finite Trees

Trees with no end are precisely finite trees. Therefore, if \([T, o]\) is a unimodular tree with no end, then \(\text{udim}_M(T) = \text{udim}_H(T) = 0\) (Proposition 3.23).

Also, it is easy to see that an optimal \(n\)-covering is obtained by considering the minimum \(n\)-covering (by choosing uniformly at random among the ties) in each sample of \(T\). The same holds for all unimodular finite spaces. In the case of trees, the minimum \(n\)-covering can be constructed easily by the following greedy algorithm (the algorithm is well known in the case \(n = 1\)). A similar idea will be used for one-ended trees (Subsection 4.1.3).

Before stating the algorithm, the height of vertices is defined as follows. Let \(T_0 := T\) and for every \(i \geq 0\), \(T_{i+1}\) be obtained by deleting the leaves of \(T_i\). For \(v \in T\), let the height of \(v\) be the maximum \(i\) such that \(v \in T_i\).

**Algorithm 1:** Greedy algorithm for minimum \(n\)-coverings of finite trees.

**Proposition 4.1.** For every deterministic tree \(T\) and \(n \in \mathbb{N}\), the result of Algorithm 1 is a minimum covering of \(T\). Moreover, for every unimodular finite tree \([T, o]\), it gives an optimal covering of \(T\).

**Proof.** The proof is only sketched here and the details are left to the reader. The first part is by induction on the number of vertices of \(T\). If \(T\) has no vertex of height \(n\), then it can be covered by only one ball and the claim holds. So assume there are vertices of height \(n\), namely \(u_1, \ldots, u_k\). It can be seen that there exist \(k\) leaves \(v_1, \ldots, v_k\) of \(T\) with pairwise distance at least \(2n + 1\). It follows that at least \(k\) balls are needed to cover \(v_1, \ldots, v_k\). Also, the union of such balls is always included in \(\bigcup_i N_n(u_i)\). Then, one can delete \(\bigcup_i N_n(u_i)\) from the tree and use the induction hypothesis to prove the claim.
For the second part, it is straightforward that the result of the algorithm is an equivariant $n$-covering (Definition 3.21). It is left to the reader to use the arguments of Example 2.15 to show that it is an optimal $n$-covering.

### 4.1.2 Unimodular Two-Ended Trees

If $T$ is a tree with two ends, then there is a unique bi-infinite path in $T$ called its trunk. Moreover, each connected component of the complement of the trunk is finite.

**Theorem 4.2.** For all unimodular two-ended trees $[T, o]$ endowed with the graph-distance metric, one has

$$\udim_M(T) = \udim_H(T) = 1.$$  

Moreover, the one-dimensional Hausdorff measure of $T$ is twice the inverse of the intensity of the trunk of $T$.

**Proof.** For a two-ended tree $T$, let $S_T$ be the trunk of $T$. Then, $S$ is an equivariant subset (Definition 2.29). Therefore, part (i) of Theorem 3.48 implies that $\udim_H(T) = \udim_H(S_T)$. Note that the trunk is isometric to $\mathbb{Z}$ as a metric space. So Proposition 3.24 implies that $\udim_H(T) = 1$. In addition, part (iii) of Theorem 3.48 and Proposition 3.38 imply that $M^1(T) = (\rho_T(S))^{-1}M^1(\mathbb{Z}) = 2(\rho_T(S))^{-1}$.

The claim concerning the Minkowski dimension is implied by Proposition II.2.14 of Part II, which shows that any unimodular infinite graph satisfies $\overline{\text{udim}}_M(G) \geq 1$ (this theorem will not be used throughout).

### 4.1.3 Unimodular One-Ended Trees

Unimodular one-ended trees are the most important class of unimodular trees in the literature. They arise naturally in many examples (see [2]). In particular, the (local weak) limit of many interesting sequences of finite trees/graphs are one-ended ([3, 2]). In terms of unimodular dimensions, it will be shown that unimodular one-ended trees are the richest class of unimodular trees.

First, the following notation is borrowed from [6]. Every one-ended tree $T$ can be regarded as a family tree as follows. For every vertex $v \in T$, there is a unique infinite simple path starting from $v$. Denote by $F(v)$ the next vertex in this path and call it the parent of $v$. By deleting $F(v)$, the connected component containing $v$ is finite. This set is denoted by $D(v)$ and its elements are called the descendants of $v$. The maximum distance of $v$ to its descendants is called the height of $v$ and is denoted by $h(v)$.

**Theorem 4.3.** If $[T, o]$ is a unimodular one-ended tree endowed with the graph-distance metric, then

$$\overline{\text{udim}}_M(T) = 1 + \overline{\text{decay}}(\mathbb{P}[h(o) \geq n]),$$  

$$\text{udim}_M(T) = 1 + \text{decay}(\mathbb{P}[h(o) \geq n]).$$  

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In addition,
\[
\text{udim}_M(T) \leq \text{decay}(\mathbb{P}[h(o) = n]) \leq \text{udim}_H(T). \tag{4.3}
\]

Note that in the formulas for the decay rates mentioned in the theorem, \(\mathbb{P}[h(o) \geq n]\) and \(\mathbb{P}[h(o) = n]\) are regarded as functions of \(n\). Note also that in general, \(\text{decay}(\mathbb{P}[h(o) = n])\) can be strictly larger than \(1 + \text{decay}(\mathbb{P}[h(o) \geq n])\) (see e.g., Subsection 4.2.2).

To prove Theorem 4.3, especially for establishing upper bounds on the Minkowski dimensions, an optimal \(n\)-covering is constructed by the following algorithm. This algorithm resembles Algorithm 1 for finite trees (Subsection 4.1.1).

\textbf{Data:} A unimodular one-ended tree \([T, o]\) and \(n \in \mathbb{N}\);

\textbf{Result:} An optimal \(n\)-covering of \(T\);

\begin{verbatim}
S := ∅;
while true do
  for each connected component \(C\) of \(T\), do
    if \(C\) has some vertices of height \(n\) then
      Add the vertices of height \(n\) in \(C\) to \(S\);
    else
      Add the vertex of \(C\) with the largest height to \(S\);
    end
  end
end
\end{verbatim}

\textbf{Algorithm 2:} Greedy algorithm for optimal coverings of unimodular one-ended trees.

\textbf{Lemma 4.4.} Given a unimodular one-ended tree \([T, o]\), the output \(S\) of Algorithm 2 is an optimal equivariant \(n\)-covering of \(T\).

The proof is omitted since it is similar to that of the next lemma. Because of the appearance of multiple connected components during the algorithm, a variant of this algorithm is easier to analyze if one considers coverings by \textit{cones} rather than balls. Nevertheless, it will be shown below that the Minkowski dimensions do not change.

The \textit{cone} with height \(n\) at \(v \in T\) is defined by \(C_n(v) := N_n(v) \cap D(v)\); i.e., the first \(n\) generations of the descendants of \(v\), including \(v\) itself. Let \(\lambda''_n\) be the infimum intensity of equivariant coverings by cones of height \(n\). The claim is that
\[
\lambda''_{2n} \leq \lambda_n \leq \lambda''_n. \tag{4.4}
\]
This immediately implies that
\[
\text{udim}_M(T) = \text{decay}(\lambda''_n), \quad \text{udim}_M(T) = \text{decay}(\lambda''_n). \tag{4.5}
\]
To prove (4.4), note that any covering by cones of height \(n\) is also a covering by balls of radii \(n\). This implies that \(\lambda_n \leq \lambda''_n\). Also, if \(S\) is a covering by balls of radii \(n\), then \(\{F^n(v) : v \in S\}\) is a covering by cones of height \(2n\). By the mass
transport principle $2.2$, one can show that the intensity of the latter is not greater than the intensity of $S$. This implies that $\lambda_n'' \leq \lambda_n$. So (4.4) is proved.

**Data:** A unimodular one-ended tree $[T, o]$ and $n \in \mathbb{N}$;

**Result:** An optimal covering of $T$ by cones of height $n$;

**Proof.** Let $A$ be any equivariant covering of $T$ by cones of height $n$. Consider a realization $(T; A)$ of $[T, A]$. Let $v$ be a vertex such that $h(v) = n$. Since $A$ is a covering by cones of height $n$, $A$ should have at least one vertex in $D(v)$ (to see this, consider the farthest leaf from $v$ in $D(v)$). Now, for all such vertices $v$, delete the vertices in $A \cap D(v)$ from $A$ and then add $v$ to $A$. Let $A_1$ be the subset of $T$ obtained by doing this operation for all vertices $v$ of height $n$. So $A_1$ is also a covering of $T$ by cones of height $n$. Now, remove all vertices $\{v : h(v) = n\}$ and their descendants from $T$ to obtain a new one-ended tree. The same procedure for the remaining tree and its intersection with $A$. Inductively, one obtains a sequence of subsets $A = A_0, A_1, \ldots$ of $T$ such that, for each $i$, $A_i$ is a covering of $T$ by cones of height $n$ and agrees with $S_T$ on the set of vertices that are removed from the tree up to step $i$.

By letting $(T; A)$ be random, the above induction gives a sequence of equivariant subsets $A = A_0, A_1, \ldots$ on $T$. It can be seen that the intensity of $A_1$ is at most that of $A$ (this can be verified by the mass transport principle $2.2$). It is left to the reader to obtain inductively that $\mathbb{P}[o \in A_{i+1}] \leq \mathbb{P}[o \in A_i]$. Also, $\lim_{i \to \infty} A_i = S$ as equivariant subsets of $T$. This implies that $\mathbb{P}[o \in A] \geq \mathbb{P}[o \in S_T$], hence, $S$ is an optimal covering by cones of height $n$. \hfill \Box

**Lemma 4.6.** Under the above setting, one has

$$\mathbb{P}[h(o) \mod (n + 1) = -1] \leq \lambda_n'' \leq \mathbb{P}[h(o) \mod \frac{n}{2} = -1]. \quad (4.6)$$

**Proof.** An equivariant covering will be constructed to prove the second inequality in (4.6). Let

$$A_n := \{v \in T : h(v) \mod n = -1\},$$

$$A'_n := \{F^{n-1}(v) : v \in A_n\}.$$

The claim is that $A'_n$ is a covering of $T$ by cones of height $2n - 2$. Let $v \in T$ be an arbitrary vertex. Let $k$ be such that $(k - 1)n - 1 < h(v) \leq kn - 1$. Let $j$ be the first nonnegative integer such that $h(F^j(v)) \geq kn - 1$ and let $w := F^j(v)$.
One has \(0 \leq j \leq n - 1\). By considering the longest path in \(D(w)\) from \(w\) to the leaves, one finds \(z \in D(w)\) such that \(h(z) \mod n = -1\) and \(0 \leq d(w, z) \leq n - 1\) (see Figure 4.1.3). Therefore \(w\) (and hence \(v\)) is a descendant of \(F^{n-1}(z)\). Also, \(d(w, F^{n-1}(z)) \leq n - 1\). It follows that \(d(v, F^{n-1}(z)) \leq 2n - 2\). So \(v\) is covered by the cone of height \(2n - 2\) at \(F^{n-1}(z)\). Since \(F^{n-1}(z) \in A'_n\), it is proved that \(A'_n\) gives a \((2n - 2)\)-covering by cones. This implies the second inequality in (4.6).

Figure 1: The relative position of the vertices in the proof of Lemma 4.6 in a special case. Here, \(n = 4\) and the path in the left is the longest path in \(D(w)\) from \(w\) to the leaves.

To prove the first inequality in (4.6), let \(S\) be the optimal covering by cones of height \(n\) given by Algorithm 3. Send unit mass from each vertex \(v \in S\) to the first vertex in \(v, F(v), \ldots, F^n(v)\) which lies in \(A_{n+1}\) (if there is any). So the outgoing mass from \(v\) is at most \(1_{\{v \in S\}}\). In the next paragraph, it is proved that the incoming mass to each \(w \in A_{n+1}\) is at least 1. This in turn (by the mass transport principle) implies that \(P\{o \in S\} \geq P\{o \in A_{n+1}\}\), which proves the first inequality in (4.6).

The final step consists in proving that the incoming mass to each \(w \in A_{n+1}\) is at least 1. If \(h(w) = n\), then \(w \in S\) and the claim is proved. So assume \(h(w) > n\). By considering the longest path in \(D(w)\) from \(w\), one can find a vertex \(z\) such that \(w = F^{n+1}(z)\) and \(h(z) = h(w) - (n + 1)\). This implies that no vertex in \(\{F(z), \ldots, F^n(z)\}\) is in \(A_{n+1}\). So to prove the claim, it suffices to show that at least one of these vertices or \(w\) itself lies in \(S\). Note that in the algorithm in Lemma 4.5 at each step, the height of \(w\) decreases by a value at least 1 and at most \(n + 1\) until \(w\) is removed from the tree. So in the last step before \(w\) is removed, the height of \(w\) is in \(\{0, 1, \ldots, n\}\). This is possible only if in the same step of the algorithm, an element of \(\{F(z), \ldots, F^n(z), w\}\) is added to \(S\). This implies the claim and the lemma is proved.

Now, the tools needed to prove the main results are available.
Proof of Theorem 4.3. Lemma 4.6 and (4.5) imply that the upper and lower Minkowski dimensions of $T$ are exactly the upper and lower decay rates of $P[h(o) \mod n = -1]$ respectively. So one should prove that these rates are equal to the upper and lower decay rates of $P[h(o) \geq n]$ plus 1.

The first step consists in showing that $P[h(o) = n]$ is non-increasing in $n$. To see this, send unit mass from each vertex $v$ to $F(v)$ if $h(v) = n$ and $h(F(v)) = n + 1$. Then the outgoing mass is at most $1_{\{h(v) = n\}}$ and the incoming mass is at least $1_{\{h(v) = n + 1\}}$. The result is then followed by the mass transport principle.

This implies that

$$n \cdot P[h(o) \mod n = -1] \geq P[h(o) \geq n - 1].$$

Similarly, by monotonicity,

$$\frac{n}{2} P[h(o) \mod n = -1] \leq P[h(o) \mod n \in \{-1, -2, \ldots, -\left\lfloor \frac{n}{2} \right\rfloor\}] \leq P[h(o) \geq \left\lfloor \frac{n}{2} \right\rfloor].$$

These inequalities conclude the proof of (4.1) and (4.2).

We now prove the second inequality. Fix $0 < \epsilon < \alpha < \text{dec}(P[h(o) = n])$. So there is a sequence $n_0 < n_2 < \cdots$ such that $P[h(o) = n_i] \leq n_i^{-\alpha}$ for each $i$. One may assume the sequence is such that $n_i \geq 2^i$ for each $i$. Now, for each $k \in \mathbb{N}$, consider the following covering of $T$:

$$R_k(v) := \begin{cases} 2(n_i - n_{i-1}), & \text{if } h(v) = n_i \text{ and } i > k, \\ 2n_k, & \text{if } h(v) = n_k, \\ 0, & \text{otherwise}. \end{cases}$$

By arguments similar to Lemma 4.6, it can be seen that $R_k$ is indeed a covering. It is claimed that $E[R_k(o)^{\alpha-\epsilon}] \to 0$ as $k \to \infty$. If the claim is proved, then $\text{udim}_H(T) \geq \alpha - \epsilon$ and the proof of (4.3) is concluded. Let $c := 2^{\alpha-\epsilon}$. One has

$$E[R_k(o)^{\alpha-\epsilon}] = cn_k^{\alpha-\epsilon} P[h(o) = n_k] + c \sum_{i=k+1}^{\infty} (n_i - n_{i-1})^{\alpha-\epsilon} P[h(o) = n_i]$$

$$\leq cn_k^{\alpha-\epsilon} + c \sum_{i=k+1}^{\infty} (n_i - n_{i-1})^{\alpha-\epsilon} n_i^{-\alpha}.$$  

Therefore, it is enough to prove that

$$\sum_{i=1}^{\infty} (n_i - n_{i-1})^{\alpha-\epsilon} n_i^{-\alpha} < \infty.$$  

(4.7)

It is easy to see that the maximum of the function $(x - n_{i-1})^{\alpha-\epsilon} x^{-\alpha}$ over $x \geq n_{i-1}$ happens at $\frac{n_{i-1}}{2}$ and the maximum value is $c'n_{i-1}^{\alpha-\epsilon}$, where $c' = (\frac{2}{\alpha} - 1)^{\alpha-\epsilon}$ is a constant. So the left hand side of (4.7) is at most $c' \sum_{i=0}^{\infty} n_i^{-\epsilon}$, which is finite by the assumption $n_i \geq 2^i$. So (4.7) is proved and the proof is completed. \qed
4.1.4 Unimodular Trees with Infinitely Many Ends

The following is the main conjecture of this subsection. It is shown below to be implied by Conjecture 4.8 below. More discussion and also proofs in some special cases will be provided in Subsection II.3.1.2 of Part II.

**Conjecture 4.7.** For all unimodular trees \([T, o]\) with infinitely many ends and endowed with the graph-distance metric, one has \(\text{udim}_H(T) = \infty\).

**Conjecture 4.8.** For every equivariant metric \(d'\) on the 3-regular tree \(T_3\) (Definition 3.40), one has \(\text{udim}_H(T_3, d') = \infty\).

By Theorem 3.41 it is enough to consider only the equivariant metrics that are generated by equivariant edge lengths (Example 3.45); i.e., to assume that for every simple path \(v_0 v_1 \cdots v_k\), one has \(d'(v_0, v_k) = \sum_i d'(v_i, v_{i+1})\).

**Proposition 4.9.** Conjecture 4.8 implies Conjecture 4.7.

**Proof.** For a deterministic tree \(T\), the trunk of \(T\) is the set of vertices \(v\) of \(T\) such that by deleting \(v\), at least two infinite connected components appear. Also, call \(v\) a branching point if by deleting \(v\), at least 3 infinite connected components appear.

Assume Conjecture 4.8 holds and let \([T, o]\) be a unimodular tree with infinitely many ends. The claim is that \(\text{udim}_H(T) = \infty\). The proof is given in the following steps.

**Step 1.** It is enough to assume that \(T\) has no leaves a.s. To show this, let \(S_T\) be the subtree consisting of the trunk of \(T\). Since \(T\) has infinitely many ends, \(S_T\) is non-empty a.s. So Theorem 3.48 implies that \(\text{udim}_H(S_T) = \text{udim}_H(T)\). So it is enough to prove the claim for \(S_T\), which has infinitely many ends and no leaves.

**Step 2.** It is enough to assume that every two branch points have graph-distance at least 3. To show this, split all edges in \(T\) into 3 equidistant parts by adding two points on each edge without changing the metric (the new points have distance \(\frac{1}{3}\) to the set of the original points). The resulting random pointed metric space, namely \([T', o]\), is not necessary unimodular, but, by a suitable biasing (Definition B.1) and changing the origin, one can obtain a unimodular discrete space (see Example 9.8 of [2] for the precise formula) which contains \(T\) as an equivariant subset (see also Proposition 6 of [23]). So by Theorem 3.48 its Hausdorff dimension is equal to that of \(T\). Finally, by multiplying the metric by 3 and using Theorem 3.41 one obtains a tree in which the distance of every two branch points is at least 3.

**Step 3.** It is enough to assume that the degree of each vertex is either 2 or 3. To show this, assume \(T\) satisfies the assumptions in the previous steps. So each vertex is a neighbor of at most one branching point. For every branching point \(v\), add some edges as follows. Let \(v_1, \ldots, v_k\) be the neighbors of \(v\). Let \(\pi\) be a random permutation of \(\{1, \cdots, k\}\) chosen uniformly. Then add an edge between \(v_{\pi_i}\) and \(v_{\pi_{i+1}}\) for every \(1 \leq i \leq k - 1\). Do this for all branching points independently. One can see that this construction fits in the context of
equivariant processes (see Remark 2.34). It can be shown that the new graph-distance is an equivariant metric. Also, by using Theorem 3.41, one can see that the Hausdorff dimension does not increase by adding these edges. Now, delete all of the branching points mentioned above. The assumptions imply that the result is a tree in which the degree of each vertex is either 2 or 3. Also, the Hausdorff dimension does not change by Theorem 3.48.

Step 4. Finally, assume the degree of each vertex of $T$ is either 2 or 3. Let $S$ be the equivariant subset consisting of the vertices with degree 3. Note that $S_T$ has a natural tree structure: connect $x \in S_T$ to $y \in S_T$ whenever the path connecting $x$ to $y$ in $T$ does not contain any other element of $S_T$. Moreover, unimodularity of $[T,o]$ implies that $T$ has no isolated ends a.s. (see Theorem 6.10 of [2]); i.e., by deleting finitely many vertices, every infinite connected component has infinitely many ends (and hence has a branching point). This implies that the degree of each vertex in the tree structure of $S_T$ is precisely three. Therefore, $S_T$ is obtained by an equivariant metric on a the 3-regular tree (see Proposition B.2). So Conjecture 4.8 (which is assumed) gives that $\text{udim}_H(S_T) = \infty$. Therefore, $\text{udim}_H(T) = \infty$ by Theorem 3.48. So the claim is proved.

For the converse of the above proposition, one might need stronger conditions. For instance, let $[T,o]$ and $d'$ be as assumed in Conjecture 4.8 and assume $d'$ is generated by equivariant edge lengths (Example 3.45). If the random variable $\sum_{v \sim o} d'(o, v)$ has finite mean and Conjecture 4.7 holds, one can show that $\text{udim}_H(T) = \infty$ (one should add vertices inside each edge and change the root to make it unimodular as in Example 9.8 of [2]).

4.2 Instances of Unimodular Trees

This subsection discusses the dimension of some explicit unimodular trees. More examples are given in Subsection 4.3 below, in Part II (e.g., the unimodular Galton-Watson tree and the Poisson weighted infinite tree), and also in Part III (e.g., uniform spanning forests).

4.2.1 The Canopy Tree

The canopy tree $C_k$ with offspring distribution $k$ is constructed as follows. Its vertex set is partitioned in levels $L_0, L_1, \ldots$. Each vertex in level $n$ is connected to $k$ vertices in level $n-1$ (if $n \neq 0$) and one vertex (its parent) in level $n+1$. Let $o$ be a random vertex of $C_k$ such that $P[o \in L_n]$ is proportional to $k^{-n}$. Then, $[C_k,o]$ is a unimodular random tree.

Below, three types of metrics are considered on $C_k$.

First, consider the graph-distance metric. Given $n \in \mathbb{N}$, let $S := \{v \in C_k : h(v) \geq n\}$, where $h(v)$ is the height of $v$ defined in Subsection 4.1.3. The set $S$ gives an equivariant $n$-covering and $P[o \in S]$ is exponentially small as $n \to \infty$. So $\text{udim}_M(C_k) = \text{udim}_H(C_k) = \infty$. 

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Second, for each $n$, let the length of each edge between $L_n$ and $L_{n+1}$ be $a^n$, where $a > 1$ is constant. Let $d_1$ be the resulting metric on $C_k$. Given $r > 0$, let $S_1$ be the set of vertices having distance at least $r/a$ to $L_0$ (under $d_1$). One can show that $S_1$ is an $r$-covering of $(C_k, d_1)$ and decay $(\mathbb{P}[o \in S_1]) = \log k / \log a$. Therefore, $\text{udim}_M(C_k, d_1) \geq \log k / \log a$. On the other hand, one can see that the ball of radius $a^n$ centered at $o$ (under $d_1$) has cardinality of order $k^n$. One can then use Lemma 3.25 to show that $\text{udim}_H(C_k, d_1) \leq \log k / \log a$. This implies that $\text{udim}_M(C_k, d_2) = \text{udim}_H(C_k, d_2) = 0$.

4.2.2 The Generalized Canopy Tree

This example generalizes the canopy tree of Subsection 4.2.1. The goal is to provide an example where the lower Minkowski dimension, the upper Minkowski dimension and the Hausdorff dimension are all different when suitable parameters are chosen.

Fix $p_0, p_1, \ldots > 0$ such that $\sum p_i = 1$. Let $U_0, U_1, \ldots$ be an i.i.d. sequence of random number in $[0,1]$ with the uniform distribution. For each $n \geq 0$, let $\Phi_n := (\frac{1}{p_n}(Z+U_n)) \times \{n\}$, which is a point process on the horizontal line $y = n$ in the plane. Let $o_n := (\frac{1}{p_n}U_n, n) \in \Phi_n$ and $\Phi := \cup \Phi_i$. Then, $\Phi$ is a point process in the plane which is stationary under horizontal translations. Choose $m$ independent of the sequence $(U_i)$, such that $\mathbb{P}[m = n] = p_n$ for each $n$. Then, let $o := o_m$.

Construct a graph $T$ on $\Phi$ as follows: For each $n$, connect each $x \in \Phi_n$ to its closest point (or closest point on its right) in $\Phi_{n+1}$. Note that $T$ is a forest. Moreover, the next lemma shows that $[T, o]$ is a unimodular tree.

**Definition 4.10.** The generalized canopy tree with parameters $p_0, p_1, \ldots$ is the unimodular tree $[T, o]$ constructed above.

Note that in the case where $p_n$ is proportional to $k^{-n}$ for $k$ fixed, $[T, o]$ is just the ordinary canopy tree $C_k$ of Subsection 4.2.1.

**Lemma 4.11.** One has

(i) $[\Phi, o]$, endowed with the Euclidean metric, is a unimodular discrete space.

(ii) $T$ is a tree a.s. and $[T, o]$ is unimodular.

**Remark 4.12.** Part (i) of the lemma means that $\Phi - o$ is a point-stationary point process in the plane. The proof of the lemma, given below, is similar to that of the formula for the Palm version of the superposition of stationary point processes, e.g., in [27].
Proof. Let \( g = g(\varphi, x, y) \) be a (measurable) function that assigns a real number to every discrete subset \( \varphi \subseteq \mathbb{R}^2 \) and each \( x, y \in \varphi \) which is invariant under (joint) translations of \( \varphi, x \) and \( y \). The claim is that

\[
E \left[ \sum_{x \in \Phi} g(\Phi, o, x) \right] = E \left[ \sum_{x \in \Phi} g(\Phi, x, o) \right]. \tag{4.8}
\]

By additivity, it is enough to assume that \( g(\Phi, x, y) \) is zero except when \( x \in \Phi_i \) and \( y \in \Phi_j \) for some fixed \( i \) and \( j \). So (4.8) is equivalent to

\[
p_i E \left[ \sum_{x \in \Phi_j} g(\Phi, o, x) \right] = p_j E \left[ \sum_{x \in \Phi_i} g(\Phi, x, o) \right]. \tag{4.9}
\]

Define \( h : \mathbb{Z}^2 \to \mathbb{R} \) by \( h(k, l) := E \left[ \sum \sum g(\Phi, x, y) \right] \), where the sum is over all pairs of points \( x, y \in \Phi \) such that \( x \in [k, k+1) \times \{i\}, \ y \in [l, l+1) \times \{j\} \).

It can be seen that (4.9) can be written as

\[
\sum_{i \in \mathbb{Z}} h(0, l) = \sum_{k \in \mathbb{Z}} h(k, 0). \tag{4.10}
\]

Note that the coefficient \( p_i \) and \( p_j \) disappears in this formula since \( P \left[ o \in [0, 1) \times \{i\} \right] = p_i \). Now, the invariance of \( g \) under translations and the stationarity of \( \Phi \) under horizontal translations imply that \( h(0, k) = h(-k, 0) \). This proves (4.10). So (4.8) is also proved and hence, \( \Phi - o \) is a point-stationary point process. Therefore, by Example 2.18, \( [\Phi, o] \) is a unimodular discrete space.

To prove (ii), note that \( T \) can be realized as an equivariant process on \( \Phi \) (see Definition 2.21 and Remark 2.34). Therefore, by Lemma 2.28 and Theorem 3.41, it is enough to prove that \( T \) is connected a.s. Nevertheless, the same lemma implies that the connected component \( T' \) of \( T \) containing \( o \) is a unimodular tree. Since it is one-ended, Theorem 3.9 of [6] implies that the foils \( T' \cap \Phi_i \) are infinite a.s. By noting that the edges do not cross (as segments in the plane), one obtains that \( T' \cap \Phi_i \) should be the whole \( \Phi_i \); hence, \( T' = T \). Therefore, \( T \) is connected a.s. and the claim is proved. \( \square \)

Proposition 4.13. The sequence \( (p_n)_{n} \) can be chosen such that

\[
\udim_M(T) < \udim_M(T) < \udim_H(T),
\]

where \( T \) is endowed with the graph-distance metric. Moreover, for any \( 0 \leq \alpha \leq \beta \leq \gamma \leq \infty \), the sequence \( (p_n)_{n} \) can be chosen such that

\[
\udim_M(T) \leq \alpha, \ \ \udim_M(T) = \beta, \ \ \udim_H(T) \geq \gamma.
\]

For example, it is possible to have \( \udim_M(T) = 0 \) and \( \udim_H(T) = \infty \) simultaneously.
Proof. \( T \) is a one-ended tree (see Subsection 4.1.3). Assume the sequence \((p_n)\) is non-increasing. So the construction implies that there is no leaf of the tree in \( \Phi_n \) for all \( n > 0 \). Therefore, for all \( n \geq 0 \), the height of every vertex in \( \Phi_n \) is precisely \( n \). So by letting \( q_n := \sum_{i \geq n} p_i \), Theorem 4.3 implies that

\[
\udim_H(T) \geq \text{decay}(p_n), \\
\udim_M(T) = 1 + \text{decay}(q_n), \\
\udim_M(T) = 1 + \text{decay}(q_n).
\]

For simplicity, assume \( 0 < \alpha \) and \( \gamma < \infty \) (the other cases can be treated similarly). Define \( n_0, n_1, \ldots \) recursively as follows. Let \( n_0 := 0 \). Given that \( n_i \) is defined, let \( n_{i+1} \) be large enough such that the line connecting points \((n_i, n_i^{-\beta})\) and \((n_{i+1}, n_{i+1}^{-\beta})\) intersects the graph of the function \( x^{-\alpha} \) and has slope larger than \(-n^{-\gamma}\). Now, let \( q_n := n_i^{-\beta} \) for each \( i \) and define \( q_n \) linearly in the interval \([n_i, n_{i+1}]\). Let \( p_n := q_n - q_{n+1} \). It can be seen that \( p_n \) is non-increasing, \( \text{decay}(q_n) \leq \alpha \), \( \text{decay}(q_n) = \beta \) and \( \text{decay}(p_n) \geq \gamma \).

Remark 4.14. One can extend the definition of the generalized canopy tree in the following ways. First, \( \Phi_n \) can be chosen on the line \( y = y_n \), where \( y_0 < y_1 < \cdots \) is a fixed sequence. Second, one can let \( \Phi_n \) be any stationary (under horizontal translations) point process on this line. Then, given \( m \) as before, condition to the event \((y_m, m) \in \Phi_m \) (which is defined by Palm distributions) and let \( o := (y_m, m) \). The only requirement is that the joint distribution of \( \Phi_0, \Phi_1, \ldots \) is stationary under horizontal translations. The claim of Lemma 4.11 is still valid for this more general setting.

4.2.3 Unimodular Eternal Galton-Watson Trees

Eternal Galton-Watson (EGW) trees are defined in [6]. Unimodular EGW trees (in the nontrivial case) can be characterized as unimodular one-ended trees in which the descendants of the root constitute a Galton-Watson tree. Also, the latter Galton-Watson tree is necessarily critical. Here, the trivial case that each vertex has exactly one offspring is excluded (where the corresponding EGW tree is a bi-infinite path). In particular, the Poisson skeleton tree [3] is an eternal Galton-Watson tree.

Recall that the offspring distribution of a Galton-Watson tree is the probability measure \((p_0, p_1, \ldots)\) on \( \mathbb{Z}^+ \) where \( p_n \) is the probability that the root has \( n \) offsprings.

Theorem 4.15. Let \([T, o]\) be a unimodular eternal Galton-Watson tree. If the offspring distribution has finite variance, then

\[
\udim_M(T) = 2.
\]

Proof. By Kesten’s theorem [22] for the Galton-Watson tree formed by the descendants of the root, \( \lim_n n \mathbb{P}[h(o) \geq n] \) exists and is positive. It follows that \( \mathbb{P}[h(o) \geq n] = 1 \). So the claim is implied by Theorem 4.3.
In fact, the same result holds for the Hausdorff dimension of $T$, which will be proved in Theorem II.3.7.

**Conjecture 4.16.** Let $[T, o]$ be a unimodular eternal Galton-Watson tree. If the offspring distribution is in the domain of attraction of an $\alpha$-stable distribution, where $\alpha \in [1, 2]$, then

$$\text{udim}_M(T) = \text{udim}_H(T) = \frac{\alpha}{\alpha - 1}.$$  

**4.3 Examples associated with Random Walks**

Let $\mu$ be a probability measure on $\mathbb{R}^k$. Consider the simple random walk $(S_n)_{n \in \mathbb{Z}}$, where $S_0 = 0$ and the jumps $S_n - S_{n-1}$ are i.i.d. with distribution $\mu$. In this subsection, unimodular discrete spaces are constructed based on the image and the zero set of this random walk and their dimensions are studied in some special cases. The graph of the simple random walk will be studied in Subsection II.3.3.2.

**4.3.1 The Image of the Simple Random Walk**

Assume the random walk is transient; i.e., visits every given ball only finitely many times. It follows that the image $\Phi = \{S_n\}_{n \in \mathbb{Z}}$ is a random discrete subset of $\mathbb{R}^d$. If no point is visited more than once a.s. (as in the following theorem), then $\Phi$ is a point-stationary point process (see the arguments at the end of Example 2.18), hence, $[\Phi, 0]$ is a unimodular discrete space. In the general case, by similar arguments, one should bias the distribution of $[\Phi, 0]$ (Definition B.1) by the inverse of the multiplicity of the origin; i.e., by $1/\#\{n : S_n = 0\}$, to obtain a unimodular discrete space. This claim can be proved similarly to Example 2.18 by direct verification of the mass transport principle.

**Theorem 4.17.** Let $\Phi := \{S_n\}_{n \in \mathbb{Z}}$ be the image of a simple random walk $S$ in $\mathbb{R}$, where $S_0 := 0$. Assume the jumps $S_n - S_{n-1}$ are positive a.s.

(i) $\text{udim}_M(\Phi) \geq 1 \wedge \text{decay}(\mathbb{P}[S_1 > r])$.

(ii) $\text{udim}_M(\Phi) \leq 1 \wedge \text{decay}(\mathbb{P}[S_1 > r])$.

(iii) If $\mathbb{P}[S_1 > r] \sim r^{-\beta}$ in the sense that $\beta := \text{decay}(\mathbb{P}[S_1 > r])$ exists, then

$$\text{udim}_M(\Phi) = 1 \wedge \beta.$$  

In fact, the same claims are valid for the Hausdorff dimension as well. This will be shown in Theorem II.3.9.

**Proof.** For every $r > 0$, one has $\mathbb{P}[\Phi \cap (0, r) = \emptyset] = \mathbb{P}[S_1 \geq r]$. So the claims are direct consequences of Proposition 3.17.

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4.3.2 Zeros of the Simple Random Walk

**Theorem 4.18.** Let $\Psi$ be the zero set of the symmetric simple random walk on $\mathbb{Z}$ with uniform jumps in $\{\pm 1\}$. Then,

$$\text{udim}_M(\Psi) = \frac{1}{2}.$$

In fact, the same result holds for the Hausdorff dimension of $\Psi$, which will be proved in Theorem II.3.10.

**Proof.** Represent $\Psi$ uniquely as $\Psi := \{S_n : n \in \mathbb{Z}\}$ such that $S_0 := 0$ and $S_n < S_{n+1}$ for each $n$. Then, $(S_n)_n$ is another simple random walk and $\Psi$ is its image. The distribution of the jump $S_1$ is explicitly computed in the classical literature on random walks (using the reflection principle). In particular, there exist $c_1, c_2 > 0$ such that $c_1 r^{-\frac{1}{2}} < \mathbb{P}[S_1 > r] < c_2 r^{-\frac{1}{2}}$ for every $r \geq 1$. Therefore, the claim is implied by Theorem 4.17. \qed

4.4 A Subspace with Larger Minkowski Dimension

Let $\Phi \subseteq \mathbb{R}$ be an arbitrary point-stationary point process and $0 < \alpha < 1$. Let $S_1$ be the first point of $\Phi$ on the right of the origin. Assume $\beta := \text{decay}(\mathbb{P}[S_1 > r])$ exists (e.g., the case in Theorem 4.17) and $\alpha < \beta < 1$. Then, Proposition 3.17 gives that $\text{udim}_M(\Phi) = \beta$.

Consider the intervals divided by consecutive points of $\Phi$. In each such interval, namely $(a, b)$, add $\lceil (b - a)^\alpha \rceil - 1$ points to split the interval into $\lceil (b - a)^\alpha \rceil$ equal parts. Let $\Phi'$ denote the resulting point process. By the assumption $\alpha < \beta$, one can show that $\mathbb{E}[S_1^\alpha] < \infty$. Now, by biasing the distribution of $\Phi'$ (Definition 3.11) by $[S_1^\alpha]$ and changing the origin to a point of $\Phi' \cap [0, S_1]$ chosen uniformly at random, one obtains a point-stationary point process $\Psi$ (see Theorem 5 in [23] and also the examples in [2]). The distribution of $\Psi$ is determined by the following equation (where $h$ is any measurable nonnegative function).

$$\mathbb{E}[h(\Psi)] = \frac{1}{\mathbb{E}[[S_1^\alpha]]} \mathbb{E} \left[ \sum_{x \in \Phi' \cap (0, S_1)} h(\Phi' - x) \right]. \quad (4.11)$$

**Proposition 4.19.** Let $\Phi$ and $\Psi$ be as above. Then, $\Phi$ has the same distribution as an equivariant subspace of $\Psi$ (conditioned on having the root) and

$$\text{udim}_M(\Phi) = \beta > \frac{\beta - \alpha}{1 - \alpha} = \text{udim}_M(\Psi).$$

Before presenting the proof, note that Theorem 3.48 implies that $\text{udim}_H(\Phi) = \text{udim}_H(\Psi)$. Therefore, $\text{udim}_M(\Psi) < \text{udim}_H(\Psi)$.

**Proof.** Let $A$ be the set of newly-added points in $\Psi$, which can be defined by adding marks from the beginning and is an equivariant subset of $\Psi$. By (4.11),
one can verify that $\Psi \setminus A$ conditioned on $0 \not\in A$ has the same distribution as $\Phi$ (see also Proposition 6 in [23]).

Also, by letting $c := E \left[ \lceil S_1^a \rceil \right]$, (4.11) gives

\[
\mathbb{P}[\Psi \cap (0, r) = 0] = \frac{1}{c} E \left[ \sum_{x \in \Phi \cap (0, S_1)} 1_{\{(\Phi' - x) \cap (0, r) = \emptyset\}} \right]
= \frac{1}{c} E \left[ [S_1^a] 1_{\{\Phi' \cap (0, r) = \emptyset\}} \right]
= \frac{1}{c} E \left[ [S_1^a] 1_{\{S_1 / [S_1^a] > r\}} \right].
\]

One can easily deduce that

\[
\text{decay} \left( \mathbb{P}[\Psi \cap (0, r) = 0] \right) = \frac{\beta - \alpha}{1 - \alpha}.
\]

Therefore, Proposition 4.17 gives the claim. \hfill \Box

**Remark 4.20.** The fact that $\Psi$ has a smaller Minkowski dimension than $\Phi$ means that the tail of the distribution of the inter-arrivals of $\Psi$ is heavier than that of the inter-arrivals of $\Phi$. This may look surprising as the inter-arrival times of $\Psi$ are obtained by subdividing those of $\Phi$ into smaller sub-intervals. The explanation of this apparent contradiction is of the same nature as that of Feller’s paradox (Section I.4 of [18]). It comes from the renormalization of *size-biased sampling*: the typical inter-arrival of $\Psi$ has more chance to be found in a larger inter-arrival of $\Phi$, and this length-biasing dominates the effect of the subdivision.

This remark can be rephrased in terms of the following paradox:

**Paradox 4.21.** Assume $X_1, X_2, \ldots$ are the lengths of a sequence of chopsticks whose lengths are i.i.d. with a heavy-tailed distribution. Split each chopstick as above (a chopstick of length $l$ is split into $[l^a]$ equal parts). Then, after splitting, the tail of the length distribution of the typical chopstick is heavier than that of the original chopsticks (the length of the typical chopstick is the random variable which is the weak limit of $Y_1, Y_2, \ldots$, where $Y_n$ is one of $X_1, \ldots, X_n$ chosen uniformly at random and independently).

### 4.5 A Drainage Network Model

Practical observations show that large river basins have a fractal structure. For example, [19] discovered a power law relating the area and the width of river basins. There are various ways to model river basins and their fractal properties in the literature. In particular, [20] formalizes and proves a power law with exponent $3/2$ for a specific model called *Howard’s model*. Below, the simpler model of [24] is studied. One can ask similar questions for Howard’s model or other drainage network models.
Connect each \((x, y)\) in the even lattice \(\{(x, y) \in \mathbb{Z}^2 : x + y \mod 2 = 0\}\) to either \((x - 1, y - 1)\) or \((x + 1, y - 1)\) with equal probability in an i.i.d. manner to obtain a directed graph \(T\). Note that the downward path starting at a given vertex is the rotated graph of a simple random walk. It is known that \(T\) is connected and is a one-ended tree (see e.g., [26]). Also, by Lemma 2.28, \([T, 0]\) is unimodular.

Note that by considering the Euclidean metric on \(T\), Theorem 3.48 implies that the Hausdorff dimension of \(T\) is 2. In the following, the graph-distance metric is considered on \(T\).

**Theorem 4.22.** Under the graph-distance metric, one has
\[\dim M(T) = \frac{3}{2}.\]

Before presenting the proof, it is worthwhile mentioning that the same result is valid for the Hausdorff dimension of \(T\), which will be proved in Theorem II.3.15.

**Proof.** The idea is to use Theorem 1.3. Following [26], there are two backward paths (going upward) in the odd lattice that surround the descendants \(D(o)\) of the origin. These two paths have exactly the same distribution as (rotated) graphs of independent simple random walks starting at \((-1, 0)\) and \((1, 0)\), respectively, until they hit for the first time. In this setting, \(h(o)\) is exactly the hitting time of these random walks. So classical results on random walks imply that \(P[h(o) \geq n]\) is bounded between two constant multiples of \(n^{-\frac{1}{2}}\) for all \(n\). So Theorem 1.3 implies that \(\dim M(T) = \frac{3}{2}\). \(\square\)

4.6 Self Similar Unimodular Discrete Spaces

This section provides a class of examples obtained by discretizing self-similar sets. Before going into the general definition (Subsection 4.6.3), two special cases are provided in Subsections 4.6.1 and 4.6.2 to help illustrating the idea.

4.6.1 Unimodular Discrete Cantor Set

Let \(K_n\) be the set in the \(n\)-th step of the definition of the Cantor set; i.e., the set of rational numbers of the form \(\frac{m}{3^n} \in [0, 1]\) such that the expansion of \(m\) in base 3 contains only digits 0 and 2. Let \(o_n\) be a random point of \(K_n\) chosen uniformly and \(\Psi_n := 3^n(K_n - o_n)\); i.e., the random set obtained by centering \(K_n\) at \(o_n\) and scaling it by \(3^n\). It can be seen that \(\Psi_n\) tends weakly to a random discrete space \(\Psi \subseteq \mathbb{Z}\), which can be explicitly constructed as follows: Let \(U = \{\cdots a_2 a_1 a_0\}\) be an i.i.d. sequence in \(\{0, 2\}\) considered as a symbolic infinite expansion in base 3, and \(\Psi\) be the set of \(i \in \mathbb{Z}\) such that \(i + U\) (a symbolic summation, which is just the 3-adic summation) has only digits 0 and 2.

Since \(o_n\) is chosen uniformly at random, \([\Psi_n, 0]\) is unimodular (Example 2.15). Therefore, \([\Psi, 0]\) is also unimodular (see Example 2.19); i.e., \(\Psi\) is a point-stationary subset of \(\mathbb{Z}\). In this paper, \(\Psi\) is called the unimodular discrete
Figure 2: Four ways to attach 3 isometric copies to $T_n$ in the construction of the unimodular discrete Koch snowflake, where each copy is a rotated/translated version of $T_n$ (relative to $A_n$ and $B_n$). Here, $T_n$ is shown in black and the segments connecting the points are added for clearer visualization.

**Cantor set.** The results of Subsections I.6.3 and II.3.5 imply that

$$\text{udim}_M(\Psi) = \text{udim}_H(\Psi) = \frac{\log 2}{\log 3}.$$ 

Another construction of the unimodular discrete Cantor set is $\Psi := \cup_n T_n$, where $T_n$ is defined by letting $T_0 := \{0\}$ and $T_{n+1} := T_n \cup (T_n \pm 2 \times 3^n)$, where the $+$ or $-$ sign is chosen i.i.d., each sign with probability $1/2$. Note that each $T_n$ has the same distribution as $\Psi_n$, but the sequence $T_n$ is nested.

### 4.6.2 Unimodular Discrete Koch Snowflake

Consider the usual triangular lattice of the plane and build a Koch snowflake on one of the triangles. Let $K_n$ be the set of points in the $n$-th step of the construction. Let $\mathbf{K}_n$ be a random point of $K_n$ chosen uniformly and $\Phi_n := 3^n(K_n - \mathbf{o}_n)$. The scaling ensures that $\Phi_n$ is a (random) subset of the triangular lattice. It can be seen that $\Phi_n$ tends weakly to a random discrete subset $\Phi$ of the triangular lattice which is almost surely a bi-infinite path (note that the cycle disappears in the limit).

The limit $\Phi$ can also be constructed by $\Phi := \cup_n T_n$, where $T_n$ is a random finite path in the triangular lattice equipped with distinguished end points $A_n$ and $B_n$ defined inductively as follows: Let $T_1 := \{A_1, B_1\}$, where $A_1$ is the
origin and \( B_1 \) is a neighbor of the origin in the triangular lattice chosen uniformly at random. For each \( n \geq 1 \), given \((T_n, A_n, B_n)\), let \((T_{n+1}, A_{n+1}, B_{n+1})\) be obtained by attaching to \( T_n \) three isometric copies of itself as shown in Figure 2. There are 4 ways to attach the copies and one of them should be chosen at random with equal probability (the copies should be attached to \( T_n \) relative to the position of \( A_n \) and \( B_n \)). It can be seen that no points overlap.

Since \( o_n \) is chosen uniformly at random, \([\Phi_n, 0]\) is unimodular (Example 2.15). Therefore, \([\Phi, 0]\) is also unimodular; i.e., \( \Phi \) is a point-stationary subset of \( \mathbb{R}^2 \). In this paper, \( \Phi \) is called the unimodular discrete Koch snowflake. The results of Subsections 4.6.3 and II.3.5 imply that

\[
\udim_M(\Phi) = \udim_H(\Phi) = \frac{\log 4}{\log 3}.
\]

### 4.6.3 The General Setting

First, a brief summary of self-similar sets is provided. In the following, for \( x \in \mathbb{R}^k \), \( B_r(x) \) represents the closed ball of radius \( r \) centered at \( x \) in \( \mathbb{R}^k \). A similitude of \( \mathbb{R}^k \) with similarity ratio \( r \) is a function \( f : \mathbb{R}^k \to \mathbb{R}^k \) such that \( \forall x, y : d(f(x), f(y)) = rd(x, y) \), where \( d \) is the Euclidean metric.

Let \( l \geq 1 \) and \( f_1, \ldots, f_l \) be similitudes of \( \mathbb{R}^k \) with similarity ratios \( r_1, \ldots, r_l \) respectively. Assume \( r_i < 1 \) for each \( i \). It is known that there is a unique compact subset \( K \) of \( \mathbb{R}^k \) such that \( K = \bigcup_j f_i(K) \), which is called the attractor of \( f_1, \ldots, f_l \) (see Section 2.1 of [12]). The set \( K \) is called self-similar and its dimension can be studied. In particular, if the \( f_i \)'s satisfy the open set condition; i.e., there is a bounded open set \( V \subseteq \mathbb{R}^k \) such that \( f_i(V) \subseteq V \) and \( f_i(V) \cap f_j(V) = \emptyset \) for each \( i, j \), then the Minkowski and Hausdorff dimensions of \( K \) are equal to the similitude dimension, which is the unique \( \alpha \geq 0 \) such that \( \sum r_i^\alpha = 1 \).

In the following, for every \( n \geq 0 \) and every string \( \sigma = (j_1, \ldots, j_n) \in \{1, \ldots, l\}^n \), let \( f_\sigma := f_{j_1} \cdots f_{j_n} \). Also let \( |\sigma| := n \). In addition, for a metric space \( Y \) and \( s > 0 \), \( sY \) denotes the same space as \( Y \) with the metric scaled by factor \( s \).

This section introduces a discrete analogue of self-similar sets. Below, it is assumed that \( r_i = r \) for each \( i \) and that the maps \( f_1, \ldots, f_l \) satisfy the open set condition. Fix a point \( o \in \mathbb{R}^k \). Let \( K_0 := \{o\} \) and \( K_{n+1} := \bigcup_j f_j(K_n) \) for each \( n \geq 0 \). Equivalently,

\[
K_n = \{f_\sigma(o) : |\sigma| = n\}. \tag{4.12}
\]

Using contraction arguments (e.g., in the proof of Theorem 1.1 of Section 2.1 of [12]), one can show that \( K_n \) tends to \( K \) under the Hausdorff metric.

**Proposition 4.23.** Let \( o_n \) be a point of \( K_n \) chosen uniformly at random, where \( K_n \) is defined in \((4.12)\). Then, the distribution of \([r^{-n}K_n, o_n]\) converges to some unimodular discrete space.

**Proof.** The claim is directly implied by Proposition 4.27 below. \( \Box \)

**Definition 4.24.** The unimodular discrete space in Proposition 4.23 is called a self-similar unimodular discrete space.
It should be noted that it can also be constructed directly by Algorithm 4 below.

**Example 4.25.** The lattice $\mathbb{Z}^k$ and the triangular lattice in the plane are self similar. The corresponding explicit constructions are left to the reader. The reader is also invited to construct a unimodular discrete version of the Sierpinski carpet.

**Remark 4.26.** If the $r_i$'s are not all equal, the guess is that there is no scaling of the sequence $[K_n, o_n]$ that converges to a nontrivial discrete space (which is not a single point). This has been verified by the authors in the case $o \in V$. In this case, by letting $a_n$ be the distance of $o_n$ to its closest point in $K_n$, it is shown that for any $\epsilon > 0$, $P[a_n/\bar{r}^n < \epsilon] \to \frac{1}{2}$ and $P[a_n/\bar{r}^n > \frac{1}{2}] \to \frac{1}{2}$, where $\bar{r}$ is the geometric mean of $r_1, \ldots, r_l$. This implies the claim.

To find the limiting object in the last proposition, the following construction will be used. Let $u_1, u_2, \ldots$ be i.i.d. uniform random numbers in $\{1, \ldots, l\}$ and $\delta_n := (u_n, \ldots, u_1)$. Let $o' := f_{\delta_n}(o)$. Let $K_n := f_{\delta_n}^{-1}K_n = f_{u_1}^{-1} \cdots f_{u_n}^{-1}K_n$. The chosen order of the indices in $\delta_n$ ensures that $K_n \subseteq K_{n+1}$, $\forall n$.

Note that, in contrast, $K_n$ is not necessarily contained in $K_{n+1}$, unless $o$ is a fixed point of some $f_i$. It is easy to see that $[K_n, o]$ has the same distribution as $[r^{-n}K_n, o']$. For $v \in K_n$, let

$$w_n(v) := \#\{\sigma : |\sigma| = n, f_\sigma(o) = f_{\delta_n}(v)\}.$$ 

One has $w_n(v) \leq w_{n+1}(v)$. In the case $o \in V$, $w_n(\cdot) = 1$ and the arguments are much simpler. The reader can assume this at first reading.

**Proposition 4.27.** Let $\hat{K} := \cup_n K_n$ and $w(\cdot) := \lim_n w_n(\cdot)$ for $v \in \hat{K}$.

(i) $w(\cdot)$ is uniformly bounded.

(ii) Almost surely, $\hat{K}$ is a discrete set.

(iii) The distribution of $[\hat{K}, o]$, biased by $1/w(o)$ (Definition B.1), is the limiting distribution alluded to in Proposition 4.25.

**Proof.** (i). Assume $f_{\sigma_k}(o) = \cdots = f_{\sigma_1}(o)$ and $|\sigma_j| = n$ for each $j \leq k$. Let $D$ be a fixed number such that $V$ intersects $B_D(o)$. Now, the sets $f_{\sigma_j}(V)$ for $1 \leq j \leq k$ are disjoint and intersect a common ball of radius $Dr^n$. Moreover, each of them contains a ball of radius $ar^n$ and each is contained in a ball of radius $br^n$ (for some fixed $a, b > 0$). Therefore, Lemma 1.10 of Chapter 2 of [12] implies that $k \leq (\frac{b}{a+2b})^k =: C$. This implies that $w_n(\cdot) \leq C$ a.s., hence $w(\cdot) \leq C$ a.s.

(ii). Let $D$ be arbitrary as in the previous part. Assume $f_{\delta_n}^{-1}f_{\sigma_j}(o) \in B_D(o)$ for $j = 1, \ldots, k$. Now, for $j = 1, \ldots, k$, the sets $f_{\sigma_j}(V)$ are disjoint and intersect
a common ball of radius $2Dr^n$. As in the previous part, one obtains $k \leq (2D+2k)^k$. Therefore, $\#N_D(o) \leq (2D+2k)^k$ a.s. Since this holds for all large enough $D$, one obtains that $\hat{K}$ is a discrete set a.s.

(iii). Note that the distribution of $o'_n$ is just the distribution of $o_n$ biased by the multiplicities of the points in $K_n$. It follows that biasing the distribution of $[K_n,o]$ by $1/w_n(o)$ gives just the distribution of $[r^{-n}K_n,o_n]$. The latter is unimodular since $o_n$ is uniform in $K_n$. By Example 2.19, the distribution of $[K,o]$ biased by $1/w(o)$ is also unimodular and satisfies the claim of Proposition 4.23.

\textbf{Theorem 4.28.} The self similar unimodular discrete space defined above satisfies

$$\text{udim}_M(\hat{K}) = \text{udim}_M(K) = \frac{\log l}{|\log r|}.$$  

Moreover, for $\alpha := \log l/|\log r|$, $\hat{K}$ has positive and finite $\alpha$-dimensional Hausdorff measure.

In this theorem, with an abuse of notation, the dimension of $\hat{K}$ means the dimension of the unimodular space obtained by biasing the distribution of $K$ by $1/w(o)$ (see Proposition 4.27).

\textbf{Proof.} The proof is based on the construction of a sequence of equivariant coverings of $K$. Let $D > \text{diam}(K)$ be given, where $K$ is the attractor of $f_1, \ldots, f_l$. Let $m > 0$ be large enough so that $\text{diam}(K_m) < D$. Note that each element in $K$ can be written as $f_{\delta_n}^{-1}f_{\sigma}(o)$ for some $n$ and some string $\sigma$ of length $n$. Let $\gamma_m$ be a string of length $m$ chosen uniformly at random and independently of other variables. For an arbitrary $n$ and a string $\sigma$ of length $n$, let

$${U}_{\sigma} := f_{\delta_{n+m}}^{-1}f_{\sigma}(K_m),$$

$${z}_{\sigma} := f_{\delta_{n+m}}^{-1}f_{\sigma}f_{\gamma_m}(o).$$

Note that $U_\sigma \subseteq \hat{K}$ is always a scaling of $K_m$ with ratio $r^{-m}$ and $z_\sigma \in U_\sigma$.

Now, define the following covering of $\hat{K}$:

$$R_m(v) := \begin{cases} Dr^{-m}, & \text{if } v = z_\sigma \text{ for some } \sigma, \\ 0, & \text{otherwise.} \end{cases}$$

It can be seen that $R_m$ gives an equivariant covering. Also, note that $R_m(o) > 0$ if and only if $f_{\sigma}f_{\gamma_m}(o) = f_{\delta_{n+m}}(o)$ for some $n$ and some string $\sigma$ of length $n$. Let $A_{n,m}(o)$ be the set of possible outcomes for $\gamma_m$ such that there exists a string $\sigma$ of length $n$ such that the last equation holds. One can see that this set is increasing with $n$ and deduce that $w_m(o) \leq \#A_{n,m}(o) \leq w_{n+m}(o)$. By letting $w'_m(o) := \# \cup_n A_{n,m}(o)$, it follows that $w_m(o) \leq w'_m(o) \leq w(o)$. According to the above discussion, $R_m(o) > 0$ if and only if $\gamma_m \in \cup_n A_{n,m}(o)$. So

$$\mathbb{P}[R_m(o) > 0 | u_0, u_1, \ldots] = w'_m(o)r^{\alpha}.$$
Therefore, by considering the biasing that makes \( \hat{K} \) unimodular, one gets

\[
E \left[ \frac{1}{w(o)} 1_{\{R_m(o) > 0\}} \right] = E \left[ \frac{w'(o)^{\alpha} m^\alpha}{w(o)} \right] \leq r^m. \tag{4.13}
\]

Since the balls in the covering have radius \( Dr^{-m} \), one gets \( \text{udim}_M(\hat{K}) \geq \alpha \).

On the other hand, by (4.13) and monotone convergence, one finds that

\[
E \left[ \frac{1}{w(o)} 1_{\{R_m(o) > 0\}} \right] \geq \frac{1}{2} r^m,
\]

for large enough \( m \). Similar to the proof of part (i) of Proposition 4.27, one can show that the sequence of coverings \( R_m \) (for \( m = 1, 2, \ldots \)) is uniformly bounded. Therefore, Lemma 3.13 implies that \( \text{udim}_M(\hat{K}) = \alpha \) (see also Remark 3.3).

Moreover, since \( E [R_m(o)^\alpha / w(o)] \) is bounded (by \( D^\alpha \)), one can get that \( M^\alpha(\hat{K}) > 0 \).

Lemma 3.25 will be used to bound the Hausdorff dimension. Let \( D > 1 \) be arbitrary. Choose \( m \) such that \( r^{-m} \leq D < r^{-m-1} \). By Proposition 4.27 there are finitely many points in \( \hat{K} \cap B_D(o) \). Therefore, one finds \( n \) such that \( \hat{K} \cap B_D(o) = \hat{K}_{n+m} \cap B_D(o) \). It follows that the sets \( \{U_\sigma : |\sigma| = n\} \) cover \( \hat{K}_{n+m} \). Now, assume \( \sigma_1, \ldots, \sigma_k \) are strings of length \( n \) such that \( U_\sigma \) are distinct and intersects \( B_D(o) \). One obtains that

\[
\#B_D(o) \cap \hat{K} \leq \sum_{j=1}^k \#B_D(o) \cap U_{\sigma_j} \leq kl^m = kr^{-m} \leq kD^\alpha. \tag{4.14}
\]

Consider the sets \( V_{\sigma_j} := \int_{\delta_{\sigma_j}} f_{\sigma_j}(V) \) which are disjoint (since \( \sigma_j \)'s have the same length). Note that if \( \epsilon > \text{diam}(V \cup \{o\}) \) is fixed, then the \( \epsilon \)-neighborhood of \( V \) contains \( K_m \). Therefore, all \( V_{\sigma_j} \)'s intersect a common ball of radius \( D + cr^{-m} \leq (1 + \epsilon)D \). Moreover, each of them contains a ball of radius \( ar^{-m} \geq aD \) and is contained in a ball of radius \( br^{-m} \leq bD \) (for some \( a, b > 0 \) not depending on \( D \)). Therefore, Lemma 1.10 of Chapter 2 of [12] implies that \( k \leq \left( \frac{(1+\epsilon)^2}{ar} \right)^k \). Therefore, (4.14) implies that

\[
\#B_D(o) \cap \hat{K} \leq CD^\alpha, \quad \text{a.s.}
\]

Therefore, Lemma 3.25 implies that \( \text{udim}_H(\hat{K}) \leq \alpha \). Moreover, the proof of the lemma shows that \( M^\alpha(\hat{K}) < \infty \). This completes the proof.

In explicit examples, the following algorithm is easier to construct the nested sets \( \hat{K}_n \) and the limiting set \( \hat{K} \). This algorithm is a generalization of the explicit constructions in Subsections 4.6.1 and 4.6.2.

Remark 4.29. The sets \( \hat{K}_n \) can be constructed by Algorithm 4 below.
\[ K_0 := \{ o \}; \]
Let \( g_0 \) be the identity map;
Choose i.i.d. random numbers \( i_1, i_2, \ldots \) uniformly in \( \{1, \ldots, l\} \);
for \( n = 1, 2, \ldots \) do
\[
\hat{K}_n := \bigcup_{j=1}^{l} g_{n-1} f_{i_n}^{-1} f_{j} g_{n-1}^{-1}(\hat{K}_{n-1});
\]
end

Algorithm 4: Another inductive construction of the sets \( \hat{K}_n \).

Remark 4.30. The limit \( [\hat{K}, \hat{o}] \) highly depends on the choice of \( o \), but its scaling limit does not (see below). To see this, note that even \( \hat{K}_1 \) depends on the choice of \( o \). See the examples.

Remark 4.31. One can similarly start with any finite subset of \( \mathbb{R}^k \) instead of a single point.

4.7 Stationary and Point-Stationary Point Processes

The following results are proved in Part II. Let \( \Psi \) be a stationary point process in \( \mathbb{R}^k \) and \( \Phi \) be its Palm version (Example 2.18). It will be proved that \( u\dim_M(\Phi) = u\dim_H(\Phi) = k \). Moreover, the \( k \)-dimensional Hausdorff measure of \( \Phi \) is proportional to the intensity of \( \Psi \) in \( \mathbb{R}^k \) (Proposition II.2.18). Also, if \( \Phi' \) is a point-stationary point process in \( \mathbb{R}^k \) which is not the Palm version of any stationary point process, then \( u\dim_H(\Phi') \leq k \) (Proposition II.2.17). It is also conjectured that \( \Phi' \) has zero \( k \)-dimensional Hausdorff measure, which is proved in the case \( k = 1 \) (Proposition II.2.20).

4.8 Cayley Graphs

Cayley graphs are an important class of unimodular graphs [2]. Let \( H \) be a finitely generated group and \( S \subseteq H \) be a generating set which is symmetric (i.e., if \( s \in S \), then \( s^{-1} \in S \)). By considering the set of edges \( \{(x, sx) : x \in H, s \in S\} \), the Cayley graph \( \Gamma(H, S) \) is obtained.

Lemma 4.32. The Minkowski and Hausdorff dimensions of the Cayley graph \( \Gamma(H, S) \) do not depend on the generating set \( S \) of \( H \).

By this lemma, one can define
\[
\begin{align*}
\text{udim}_H(H) & := u\dim_H(\Gamma(H, S)), \\
\text{udim}_M(H) & := u\dim_M(\Gamma(H, S)).
\end{align*}
\]
Proof of Lemma 4.32. Let $S'$ be another generating set for $H$. Denote the graph-distance metric on $\Gamma(H, S')$ by $d'$, which is a metric on $H$. Let $M$ be large enough such that each element of $S'$ can be represented by the product of at most $M$ elements of $S$ and vice versa. It easily follows that $d' \le Md$ and $d \le Md'$. Now, the claim is implied by Theorem 3.41.$\blacksquare$

In fact, it will be proved in Part II that the Minkowski and Hausdorff dimensions of $H$ are equal to the polynomial growth rate of $H$. See Subsection II.3.6 for more discussion.

### Appendices

#### Appendix A  More on the Metric $\kappa$

First, Theorem 2.6 is proved. According to the discussion at the end of Subsection 2.3, Theorem 2.7 is proved with the same arguments and its proof is skipped.

Before proving the result, the completion of $\mathcal{D}$ is directly defined below. As a motivation, it can be seen that the sequence $\mathcal{D}_n := \{0, \frac{1}{n}\} \subseteq \mathbb{R}$, endowed with the Euclidean metric on $\mathcal{D}_n$, is a Cauchy sequence in $\mathcal{D}$ and is not convergent. In fact, intuitively, the limit should be a single point with multiplicity two. Based on this idea, one should generalize pointed discrete spaces by allowing each point to have a multiplicity (the same issue exists in the study of simple point processes, where considering non-simple point processes solves the problem - see e.g., Section 9 of [16]). In what follows, roughly speaking, a point with multiplicity $n$ is represented by $n$ points with zero distance. This is formalized by the notion of pseudo metric.

A set $D$ equipped with a function $d : D \times D \to \mathbb{R}^{\geq 0}$ is called a **pseudo metric space**, if $d$ has the properties of a metric except that, $d(x, y) = 0$ does not necessarily imply $x = y$. As before, the pseudo metric is always denoted by $d$ except when explicitly mentioned. The balls $N_r(v)$ are defined in the usual way. In this paper, it is always assumed that the pseudo metric is **boundedly finite**; i.e., every subset of $D$ included in a ball of finite radius is finite. For the sake of simplicity, the term **discrete pseudo metric space** (abbreviated as DPMS) will be used to refer to boundedly finite pseudo metric spaces.

Note that $\{(u, v) \in D : d(u, v) = 0\}$ is an equivalence relation on $D$. By the assumption of boundedly finiteness, each equivalence class is finite. One can regard an equivalence class with $n$ elements as a point with multiplicity $n$.

Pointed DPMSs and isomorphisms are defined in the usual way (being injective is important for pseudo metric spaces in this definition). Let $\hat{\mathcal{D}}$ be the set of equivalence classes of pointed DPMSs under isomorphism. Note that $\mathcal{D} \subseteq \hat{\mathcal{D}}$. Moreover, it is easy to see that $\mathcal{D}$ is dense in $\hat{\mathcal{D}}$.

Now, $r$-embeddability, $r$-similarity and the function $\kappa$ can be defined in exactly the same way as in Definitions 2.4 and 2.5.
Lemma A.1. The function $\kappa$ is a metric on $\hat{D}_s$.

Proof. The definition readily implies that $\kappa$ is well defined on $\hat{D}_s$ and is symmetric. Also, it is clear that for all $[D,o] \in D_s$, one has $\kappa([D,o],[D,o]) = 0$.

Conversely, assume $(D,o)$ and $(D',o')$ are pointed DPMSs satisfying the condition $\kappa((D,o),(D',o')) = 0$. The first claim is that for all $r > 0$, there is a pointed-isomorphism between $N_r(o)$ and $N_r(o')$. Let $r > 1$ be given. Let $s \geq r$ be large enough, which will be determined later. Since $\kappa((D,o),(D',o')) = 0 < \frac{1}{s}$, the definition of $\kappa$ implies that there is an injective function $f : N_r(o) \rightarrow D'$ such that $f(o) = o'$ and the distortion of $f$ is at most $\frac{1}{s}$. By (2.1), the image of $f$ is contained in $N_{r+1/s}(o')$. Assume $s$ is large enough to ensure that $N_{r+1/s}(o') = N_r(o')$ (which is possible since $N_r(o')$ is a closed ball). So the range of $f$ is contained in $N_r(o')$. Being injective implies that $\#N_r(o) \leq \#N_r(o')$. By switching the roles of the two spaces, one similarly proves the other direction of the inequality; Hence $\#N_r(o) = \#N_r(o')$. This implies that $f : N_r(o) \rightarrow N_r(o')$ is also surjective. Now consider the union $A$ of the sets $\{d(u,v) : u,v \in N_r(o)\}$ and $\{d(u,v) : u,v \in N_r(o')\}$. Let $\epsilon$ be the minimum distance of the pairs of distinct numbers in $A$. From the beginning, assume $s > \frac{1}{\epsilon}$. Now, (2.1) implies that $d(x,y) = d(f(x),f(y))$ for all $x,y \in N_r(o)$. In other words, $f$ is an isometry between $N_r(o)$ and $N_r(o')$ and the above claim is proved.

For each integer $n \geq 1$, let $A_n$ be the set of pointed-isomorphisms from $N_n(o)$ to $N_n(o')$ which is already shown to be nonempty. Since each ball is finite, $A_n$ is also finite. It is clear that for $n \geq 2$ and $f \in A_n$, the restriction of $f$ to $N_{n-1}(o)$ belongs to $A_{n-1}$. Therefore, by König’s infinity lemma, there is a sequence of isomorphisms $f_n \in A_n$ such that the restriction of $f_n$ to $N_{n-1}(o)$ is equal to $f_{n-1}$ for each $n$. Now, one can safely define $\rho(v) := \lim f_n(v)$ for each $v \in D$. It is easy to see that $\rho$ is an isomorphism between $(D,o)$ and $(D',o')$. So $[D,o] = [D',o']$.

It remains to prove the triangle inequality for $\kappa$. Let $(D_i,o_i), 1 \leq i \leq 3$ be pointed DPMSs. Let $\kappa_{ij} := \kappa((D_i,o_i),(D_j,o_j))$. One has to prove $\kappa_{13} \leq \kappa_{12} + \kappa_{23}$. If $\kappa_{12} + \kappa_{23} \geq 1$, the claim is clear. So assume $\kappa_{12} + \kappa_{23} < 1$. Let $\epsilon > \kappa_{12}$ and $\delta > \kappa_{23}$ be arbitrary such that $\epsilon + \delta < 1$. Below, it is proved that $\kappa_{13} \leq \epsilon + \delta$. Since $\epsilon$ and $\delta$ are arbitrary, the claim follows.

Since $\kappa_{12} < 1$, $\kappa_{12} < \epsilon$ and $\epsilon + \delta > \epsilon$, by Definition 2.2 there is an injective function $f : N_{1/(\epsilon+\delta)}(o_1) \rightarrow D_2$ with distortion at most $\epsilon$ such that $f(o_1) = o_2$. Similarly, there is an injective function $g : N_{1/\delta}(o_2) \rightarrow D_3$ with distortion at most $\delta$ such that $f(o_2) = o_3$. The image of $f$ is contained in $N_{1/(\epsilon+\delta)+\epsilon}(o_2)$. It is straightforward that $\epsilon + \delta < 1$ implies $1/(\epsilon+\delta) + \epsilon < 1/\delta$ (use the mean value theorem for the function $1/x$ in the interval $(0,1)$). Therefore, $g \circ f : N_{1/(\epsilon+\delta)}(o_1) \rightarrow D_3$ is well defined. By the definition of distortion in (2.1), one readily gets that the distortion of $g \circ f$ is at most $\epsilon + \delta$. This means that $(D_1,o_1)$ is $(\epsilon + \delta)^{-1}$-embeddable in $(D_3,o_3)$. By swapping the roles of the two spaces, one gets that they are $(\epsilon + \delta)^{-1}$-similar. This means that $\kappa_{13} \leq \epsilon + \delta$ and the claim is proved.

Lemma A.2. The metric space $\hat{D}_s$ is separable.
Proof. Let $Q$ denote the subset of $\mathcal{D}_*$ including all finite DPMSs for which the pseudo metric is rational valued; i.e.

$$Q = \left\{ [D, o] \in \mathcal{D}_* : \#D < \infty, \forall x, y \in D : d(x, y) \in \mathbb{Q} \right\}.$$  

It is enough to show that $Q$ is countable and dense with respect to $\kappa$. Note that for each element of $Q$, say with points $u_1, \ldots, u_n$, the distance matrix $a_{ij} := d(u_i, u_j)$ is a $n \times n$ rational matrix. This easily implies that $Q$ is countable. In order to prove it is dense, let $(D, o)$ and $\epsilon > 0$ be given. Let $r = \frac{1}{\epsilon}$. By the definition of $\mathcal{D}_*$, $N_r(o)$ is finite. For each pair of distinct points $x, y \in N_r(o)$, let $d'(x, y)$ be an arbitrary rational number such that

$$d(x, y) + \epsilon \leq d'(x, y) \leq d(x, y) + 2\epsilon,$$  

(A.1)

and let $d'(x, x) := 0$. It is easy to check that $d'$ is a metric on $N_r(o)$. Denote by $D'$ the DPMS with the same underlying set $N_r(o)$ and equipped with the metric $d'$. Now by defining $f$ as the identity function on $N_r(o)$, (A.1) shows $(D, o)$ and $(D', o)$ are $r$-similar and hence $\kappa((D, o), (D', o)) \leq \frac{r}{\epsilon} = 2\epsilon$. But one has $[D', o] \in Q$ and hence $Q$ is dense in $\mathcal{D}_*$.  

□

Lemma A.3. The metric space $\mathcal{D}_*$ is complete.

Proof. Assume $\{[D_n, o_n]\}_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{D}_*$ under the metric $\kappa$. One needs to show that it is convergent.

The first claim is that, for all given $r > 0$, $\{\#N_r(o_n)\}_{n=1}^\infty$ is bounded. Assume this is not the case. So for each $m$, there is $n > m$ such that $\#N_r(o_n) > \#N_{r+1}(o_m)$. The reader can verify that this implies that $\kappa((D_n, o_n), (D_m, o_m)) \geq \frac{1}{r}$, which contradicts being a Cauchy sequence. So this claim is proved.

The second claim is that for $r > 0$ given, the sequence of balls $N_r(o_n)_{n=1}^\infty$ has a convergent subsequence under the metric $\kappa$ (it should be noted that the whole sequence of balls is not necessarily convergent). By the above argument and passing to a subsequence, one may assume $\#N_r(o_n) = l$ for each $n$, where $l$ is constant. For each $n$, consider an arbitrary order on the points of $N_r(o_n)$ but let $o_n$ be the first in this order. Let $A^{(n)}$ be the distance matrix of $N_r(o_n)$ (i.e., $A_{i,j}^{(n)}$ is the distance from the $i$-th point to the $j$-th point). Note that the entries of these matrices are in $[0, 2r]$. Therefore, one can find a convergent subsequence of these matrices, say converging to matrix $A$. It is easy to see that $(i, j) \mapsto A_{i,j}$ is a pseudo metric on $\{1, \ldots, l\}$. Let $D'_r$ be the resulting DPMS with origin $o'_r := 1$. Now, it is easy to see that $\kappa(N_r(o_n), (D'_r, o'_r)) \to 0$ as $n \to \infty$.

Third, by a diagonal argument using the second claim, one can assume that for each $m \in \mathbb{N}$, $\lim_n N_m(o_n)$ exists (in $\mathcal{D}_*$), say $(D'_m, o'_m)$. Fix $m$ and let $\epsilon > 0$ be arbitrary. So for large enough $n$, one has $\kappa(N_m(o_n), (D'_m, o'_m)) < \epsilon$ and $\kappa(N_{m-1}(o_n), (D'_{m-1}, o'_{m-1})) < \epsilon$. If $\epsilon < 1/m$, there exist injective functions $f : D'_m \to N_m(o_n)$ and $g : N_{m-1}(o_n) \to D'_{m-1}$ with distortion less than $\epsilon$ and respecting the origins. Therefore, $g \circ f$ is well-defined and injective on
The proof of the first claim is similar to that of Theorem 2.6: For $\hat{N}_{m-1}(o'_m)$ and has distortion less than $2\epsilon$. By letting $\epsilon$ tend to zero while $m$ is fixed, one finds an isometric embedding of $N_m(o'_m)$ into $D_m^{\epsilon}$, where $N_m^\epsilon(o)$ is the open ball $\{u \in D : d(o,u) < r\}$. By considering $\epsilon$-embeddings in the other side, one also finds an isometric embedding of $N_m^{\epsilon'}(o_m^{\epsilon})$ into $D_m^{\epsilon'}$.

Therefore, $N_m^{\epsilon}(o_m^{\epsilon})$ is isomorphic to $N_m^{\epsilon}(o_m^{\epsilon})$. It follows that the sequence $(D_m', o'_n)$ of pointed biactions can be paste together to form a pseudo metric space, namely $(D, o)$, which is discrete and boundedly finite. Also, $N_m(o)$ is isometric to $N_m(o')$ for each $m$. It follows easily that $\kappa((D_n, o_n), (D, o)) \to 0$. In other words, $[D_n, o_n] \to [D, o]$.

Finally, note that a specific subsequence is taken in the beginning of the third step and its convergence is proved. Being a Cauchy sequence implies the convergence of the whole sequence. So the claim is proved.

**Proof of Theorem 2.6**. It is proved in the above lemmas that $\hat{D}_o$ is a complete separable metric space. So it remains to prove the last claim.

For integers $m, n \geq 1$, let $A_{m,n}$ be the set of elements $[D, o] \in D_o$ such that

$$\forall x, y \in N_m^o(o) : \text{ if } x \neq y, \text{ then } d(x, y) \geq \frac{1}{n},$$

where $N_m^o(o)$ is the open ball defined above. It is not hard to show that $A_{m,n}$ is a closed subset of $D_o$ and also $D_o = \cap_m \cup_n A_{m,n}$. This proves the claim.

The following lemma is needed in Subsection 3.6.

**Lemma A.4.** Let $N \subseteq D_o'$ be the set of pointed marked discrete spaces $[(D, d), o; d']$ with mark space $\mathbb{R}$ such that $d'$ is a boundedly-finite metric on $D$. Then, $N$ is a Borel subset of $D_o'$ and the map $\rho : N \to N$ defined by $\rho[(D, d), o; d'] := [(D, d'), o; d]$ is Borel measurable.

**Proof.** The proof of the first claim is similar to that of Theorem 2.6. For $n \in \mathbb{N}$ and $\epsilon > 0$, let $N_{n,\epsilon}$ be the set of $[(D, d), o; d'] \in D_o'$ such that the restriction of $d'$ to $N_n^o(o)$ is a metric and $\forall u \neq v \in N_n^o(o) : d'(u, v) \geq \epsilon$. It is not hard to see that $N_{n,\epsilon}$ is closed in $D_o'$. Also, one has $N' = \cap_n \cup_n N_{n,\epsilon}$, which implies that $N'$ is Borel subset of $D_o'$.

For the second claim, it is enough to prove that the inverse image of any open ball in $N$ under $\rho$ is measurable. Let $\xi_0 := [(D_0, d_0), o_0; d_0'] \in N$ and consider the open ball $U := B_r(\xi_0) := \{\xi \in N : \kappa(\xi, \xi_0) < \epsilon\}$ in $N$. Let $r > 1/\epsilon$ be an arbitrary rational number and $M \in \mathbb{N}$. Let $A_{r,M}$ be the set of $[(D, d), o; d'] \in N$ such that $\rho(N_M^o(o))$ and $\xi_0$ are $r$-similar (Definition 2.4), where $N_M^o(o)$ is equipped with the origin and metrics induced from $[(D, d'), o; d']$.

It is not hard to show that $A_{r,M}$ is a closed subset of $N$. Also, one has $\rho^{-1}(U) = \cup_{r > 1/\epsilon} \cap_M A_{r,M}$. This implies that $\rho^{-1}(U)$ is a Borel subset of $N$ and the claim is proved.
Appendix B  Tightness and Other Lemmas

**Definition B.1.** Let $\mu$ be a probability measure on a measurable space $X$ and $w : X \to \mathbb{R}_{\geq 0}$ be a measurable function. Assume $0 < c := \int_X w(x) d\mu(x) < \infty$. By **biasing** $\mu$ by $w$ we mean the measure $\nu$ on $X$ defined by

$$\nu(A) := \frac{1}{c} \int_A w(x) d\mu(x).$$

The choice of $c$ implies that $\nu$ is a probability measure (in some literature, the normalizing factor $\frac{1}{c}$ is dropped in the definition). It is the unique probability measure on $X$ whose Radon-Nikodym derivative w.r.t. $\mu$ is proportional to $w$.

In particular, let $X := D^*$ and $\mu$ be the distribution of a random pointed discrete space $[D, o]$. Denote by $[D', o']$ the random pointed discrete space with distribution $\nu$ defined above. For every measurable function $f : D^* \to \mathbb{R}_{\geq 0}$, one gets

$$\mathbb{E} [f[D', o']] = \frac{1}{\mathbb{E} [w[D, o]]} \mathbb{E} [f[D, o] w[D, o]].$$

It can be seen that biasing $\mu$ by $w$ is equal to $\mu$ if and only if $w$ is almost surely constant w.r.t. $\mu$; i.e., for some constant $k \in \mathbb{R}$ one has $w = k$, $\mu$-a.s.

The following proposition is a converse to Lemma 2.28.

**Proposition B.2.** Let $[D, o]$ be a unimodular discrete space. If $[D', o']$ is a unimodular marked discrete space such that $[D', o']$ (obtained by forgetting the marks) has the same distribution as $[D, o]$, then there is an equivariant process $Z$ such that $[D, o; Z]$ has the same distribution as $[D', o]; m']$.

**Proof.** The reader can skip the proof at first reading. Here is a sketch. The random mark functions are obtained by the disintegration theorem for the natural map $\pi : D^* \to D_*$ (some care needs to be taken since only equivalence classes of discrete spaces are considered). The harder part is to use the crucial assumption of unimodularity to deduce that the distribution of the marks do not depend on the origin. This is similar to the invariant disintegration theorem [21]. To reduce it to the invariant disintegration theorem, an action of a countable group is needed. The latter is given by a result of Feldman and Moore [17] as discussed below. This theorem is in the context of Borel equivalence relations, which we refrain from introducing here.

To prove the claim, consider the following equivalence relation on $D^*$:

$$\forall D : \forall u, v \in D : [D, u; m] \sim [D, v; m].$$

The equivalence class of $[D, o; m]$ is always countable and if $D$ has no automorphisms, has a natural bijection with the points of $D$. It can be seen that it is a Borel equivalence relation [17]. By Theorem 1 of [17], there is a countable group $H$ acting measurably on $D^*$ such that for every $[D, o; m]$, its equivalence class is just \{ $h \cdot [D, o; m] : h \in H$ \}. In particular, the action is compatible with the projection $\pi$. It can be seen that if $D$ has no nontrivial automorphisms, this defines a natural map $h_{(D, m)} : D \to D$ for each $h \in H$ and this map is bijective.
For simplicity, assume \( D \) has no automorphisms almost surely. At the end of the proof, it is mentioned how to treat the general case. Therefore, so does \( D' \). By the above discussion, \( h_{(D', m')} \) is bijective almost surely. Therefore, by using the mass transport principle \( [23] \), one can show that \( h \cdot [D', o'; m'] \) has the same distribution as \( [D', o'; m'] \) itself (this is analogous to Mecke’s theorem. See also \([24]\) or Proposition 3.6 of \([9]\)). Similarly, one can get that the joint distribution of \( ([D', o'; m'], [D', o']) \) is invariant under the action of the group \( H \). Therefore, the invariant disintegration theorem \([21]\) gives a kernel \( t \) from \( D_* \) to \( D'_* \) that is invariant under the action of \( H \) and such that \( t([D, o], \cdot) \) is supported on \( \pi^{-1}[D, o] \) and \( t \) pushes the distribution of \( [D', o'] \) to the distribution of \( [D', o'; m'] \). By the assumption of no automorphisms, this easily gives a random function in \( Z_{(D, o)} \in \Xi_{D \times D} \) for every deterministic \( (D, o) \). It can be shown from the invariance of \( t \) that \( Z_{(D, h \circ o)} \) has the same distribution as \( Z_{(D, o)} \) for every \( h \in H \). Hence, \( Z_{(D, v)} \) has the same distribution as \( Z_{(D, o)} \) for every \( v \in D \). So one can finally write \( Z_D \) instead of \( Z_{(D, o)} \). It can be seen that \( T \) satisfies the claim.

Now, consider the general case when \( D \) may have nontrivial automorphisms. Let \( U \) be the equivariant process obtained by adding i.i.d. marks to the points with the uniform distribution in \([0, 1]\) (Example \([2.24]\)). Then, \( [D, o; U] \) is unimodal and has no nontrivial automorphisms a.s. Also, by Remark \([2.34]\) \( [D', o'; (U, m')] \) is an equivariant process on \([D, o; U]\). Now, one can repeat the above arguments line by line, which result in a random element in \( \Xi_{D \times D} \) for every \( D \) that is equipped with marks in \([0, 1]\). By considering the latter marks to be random, this gives the desired \( Z_D \) for non-marked \( D \), which satisfies the claim. \( \square \)

**Lemma B.3.** Let \( [D, o] \) be a unimodal discrete space. If \( \Xi \) is a compact metric space, then the set of \( \Xi \)-valued equivariant processes on \( D \) is tight and compact (see Remark \([2.32]\)).

**Proof.** Let \( M \) be the set of unimodal marked discrete spaces \([D', o'; m']\) with marks in \( \Xi \) such that \( [D', o'] \) has the same distribution as \([D, o]\). By Proposition \([3.2]\) it is enough to prove that \( M \) is tight and compact (see also Remark \([2.32]\)). It is easy to see that \( M \) is closed (under weak convergence). So it is enough to show that it is tight. Let \( \epsilon > 0 \) and \( \pi : D'_* \to D_* \) be the projection of forgetting the marks. By Prokhorov’s theorem, a single probability measure on \( D_* \) is tight. So there is a compact set \( K \subseteq D_* \) such that \( \mathbb{P} \left[ [D, o] \in K \right] > 1 - \epsilon \). So for any equivariant process \([D', o']\) on \([D, o]\) with values in \( \Xi' \), \( \mathbb{P} \left[ [D', o'] \in \pi^{-1}(K) \right] > 1 - \epsilon \). It is shown below that \( \pi^{-1}(K) \) is compact. This implies that \( M \) is tight and the claim is proved.

It remains to show that \( \pi^{-1}(K) \) is compact. Let \( [D_n, o_n; m_n] \) \((n \in \mathbb{N})\) be an arbitrary sequence in \( \pi^{-1}(K) \). One has \( [D_n, o_n] \in K \). So the latter has a convergent subsequence. Thus, from the beginning, one may assume \([D_n, o_n]\) is convergent. Let \( r > 0 \) be given. According to the proof of Lemma \([\Lambda 3] \), the sequence \( \#N_r(o_n) \) is bounded. Now, the proof of the claim that \([D_n, o_n; m_n]\) has a convergent subsequence is similar to that of Lemma \([\Lambda 3] \) and is left to the
reader (one should first show that the sequence of balls \([N_r(o_n), o_n; m_n]\) has a convergent subsequence and then, deduce the claim by a diagonal argument).

**Proof of Theorem 3.10.** Let \(S_1, S_2, \ldots\) be a sequence of \(r\)-coverings of \(D\) such that \(\mathbb{P}[o \in S_n] \to \lambda_r\). By Lemma B.3 and choosing a subsequence if necessary, one may assume from the beginning that the equivariant subsets \(S_n\) converge weakly to an equivariant subset \(S\) of \(D\). Since each \(S_n\) is an \(r\)-covering, \(\mathbb{P}[S_n \cap N_r(o) = \emptyset] = 0\). By the assumption of weak convergence, one can obtain \(\mathbb{P}[S \cap N_r(o) = \emptyset] = 0\) (let \(\epsilon > 0\) be arbitrary and \(h : \mathbb{R}_{\geq 0} \to [0, 1]\) be a continuous function that is identical to one on \([0, r]\) and zero on \([r+\epsilon, \infty)\). It can be seen that \(h'(D, o; S) := 1 \wedge \sum_{v \in S} h(d(o, v))\) is a continuous bounded function on \(D'_o\). By weak convergence, one gets \(\mathbb{E}[h'(D, o; S)] = \lim_n \mathbb{E}[h'(D, o; S_n)] = 1\). Now, the claim follows by letting \(\epsilon\) tend to zero. So by putting balls of radius \(r\) on the points of \(S, o\) is covered a.s. This implies that every point is covered a.s. (Lemma 2.30), which shows that \(S\) is an \(r\)-covering. Also, by weak convergence, \(\mathbb{P}[o \in S] = \lim_n \mathbb{P}[o \in S_n] = \lambda_r\). This implies that \(S\) is an optimal \(r\)-covering. □

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