When is a bottleneck a bottleneck?

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Abstract Bottlenecks, i.e. local reductions of capacity, are one of the most relevant scenarios of traffic systems. The asymmetric simple exclusion process (ASEP) with a defect is a minimal model for such a bottleneck scenario. One crucial question is "What is the critical strength of the defect that is required to create global effects, i.e. traffic jams localized at the defect position". Intuitively one would expect that already an arbitrarily small bottleneck strength leads to global effects in the system, e.g. a reduction of the maximal current. Therefore it came as a surprise when, based on computer simulations, it was claimed that the reaction of the system depends in non-continuous way on the defect strength and weak defects do not have a global influence on the system. Here we reconcile intuition and simulations by showing that indeed the critical defect strength is zero. We discuss the implications for the analysis of empirical and numerical data.

1 Introduction

One of the most important scenarios in any traffic system are bottlenecks, i.e. (local) flow limitations. Typical examples are a reduction in the number of lanes on a highway, local speed limits or narrowing corridors or exits in pedestrian dynamics. The identification of bottlenecks gives important information about the performance of the system. E.g. in evacuations, egress times are usually strongly determined by the relevant bottlenecks. Therefore a proper understanding of bottlenecks and their influence on properties like the flow is highly relevant.

One of the most natural questions is "When does a bottleneck lead to a traffic jam?" Does any bottleneck immediately lead to jam formation or is there a minimal bottleneck strength required? Intuitively one would say that even a small bottleneck

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strength leads to macroscopically observable effects, like a reduction of the maximal current or jams. However, other scenarios have been considered as well and have even been part of legal guidelines. One prime example in pedestrian dynamics is the dependence of the current on the width of a corridor \[1\,2\]. Originally it was believed that the current increases stepwise, i.e. non-continuously, with increasing bottleneck width. This increase was assumed to happen when the corridor width allows an additional lane of pedestrians to be formed (Fig. 1). Taking the corridor width as measure for the bottleneck strength (rather its inverse) this implies that an increasing bottleneck strength not necessarily leads to smaller current values or jam formation. In the meantime we know that this scenario is not correct and the current increases linearly with the width \[1\]. However it is still possible that there are situations where lane formation is relevant and this scenario is more adequate, e.g. in colloidal systems \[3\].

In the following we will take a theoretical physics point of view by considering a minimal model for bottlenecks. Experience shows that the results capture the generic nature of bottleneck transitions.

2 Bottlenecks in the ASEP

The Asymmetric Simple Exclusion Process (ASEP) is a paradigmatic model of of nonequilibrium physics (for reviews, see e.g. \[4\,5\,6\,7\,8\]) and arguably the simplest model that captures essential features of traffic systems, i.e. directed motion, volume exclusion and stochastic dynamics. It describes interacting (biased) random walks on a discrete lattice of \(N\) sites, where an exclusion rule forbids occupation of a site by more than one particle. A particle at site \(j\) moves to site \(j + 1\) with rate \(p\) if site \(j + 1\) is not occupied by another particle (Fig. 2). In the following we will mainly use a random-sequential update. If sites are updated synchronously (parallel update) the model is the \(v_{\text{max}} = 1\) limit of the Nagel-Schreckenberg model \[8\,9\]. Many exact results are known for the homogeneous case of the ASEP, e.g. the fundamental diagram and the phase diagram in case of open boundary conditions \[4\,5\,6\,7\,8\].

**Fig. 1** Three corridors of different widths \(w_j\). The bottleneck strength is inversely proportional to \(w_j\). Lane formation leads to a non-continuous dependence of the current on the bottleneck strength.
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For the ASEP, due to particle-hole symmetry, $\rho_1 = 1 - \rho_2$. 

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A simple but generic model for a bottleneck is obtained by replacing one of the hopping probabilities $p$ by a defect, or slow bond, with hopping probability $r < p$ (Fig 2). Many properties of this defect system have been obtained in a seminal paper by Janowsky and Lebowitz [10]. They have shown that the shape of the fundamental diagram can be understood by a simple mean-field theory. In the stationary state the current can be obtained by matching the current $J_{\text{hom}}$ in the homogeneous system with the current $J_{\text{def}}$ at the defect. Neglecting correlations at the defect site one finds that the defect has no influence on the system for low densities $\rho < \rho_1$ and large densities $\rho > \rho_2$. The density remains uniform throughout the whole system and the current is identical to that of the homogeneous system (Fig 3).

For densities $\rho_1 < \rho < \rho_2$, on the other hand, the fundamental diagram exhibits a plateau where the current is independent of the density (Fig 3). The plateau value $J_{\text{plat}}$ corresponds to the maximal current that is supported by the defect. In this density regime the stationary state is no longer characterized by a uniform density. Instead phase separation into a high and a low density region is observed. The high
density region corresponds to a jam that is formed at the defect position (Fig. 4). For periodic boundary conditions the length of jam shows characteristic fluctuations (Fig. 4 left) [10].

Fig. 4 Phase separation in the plateau regime. Left: Periodic boundary conditions. Right: Open boundary conditions.

For the ASEP with periodic boundary conditions, random-sequential update and a defect $r$ mean-field theory makes quantitative predictions for the phase separated regime [10]. The value of the current in the plateau region is given by

$$J_{\text{plat}} = \frac{pr}{(p + r)^2}$$

and the densities in the low and high density region by

$$\rho_l = \frac{r}{p + r}, \quad \text{and} \quad \rho_h = \frac{p}{p + r}$$

The critical densities $\rho_1, \rho_2$ which determine the plateau regime $\rho_1 < \rho < \rho_2$ are simply

$$\rho_1 = \rho_l \quad \text{and} \quad \rho_2 = \rho_h.$$ 

The mean-field results are supported by systematic series expansions [11]. Fig. 5 shows the resulting phase diagram. For any defect $r < p$ only currents up to the plateau value $J_{\text{plat}}$ can be realized in the system which then phase separates into a high density region pinned at the defect and a low density regime. For currents $J < J_{\text{plat}}$ the density is uniform. The important point is that $J_{\text{plat}} < J_{\text{max}}$ for any $r < p$ where $J_{\text{max}}$ is the maximal current in the homogeneous system. In other words: any bottleneck leads to a reduction of the current and a phase separated state (at intermediate densities).
3 What is the critical bottleneck strength?

Mean-field theory predicts that any bottleneck $r < p$ leads to the formation of a plateau in the fundamental diagram and the associated phase-separated state \cite{10}. Defining the bottleneck strength by

$$\Delta p = \frac{p - r}{p}$$

\hspace{1cm} (4)

this implies that the critical bottleneck strength $(\Delta p)_c$ at which the defect has global influence on the system (e.g. its current or the density) is predicted to be

$$(\Delta p)_c = 0, \quad \text{i.e.} \quad r_c = p.$$ \hspace{1cm} (5)

As mentioned in the Introduction this is what is intuitively expected. Therefore it came as quite a surprise when it was claimed \cite{12}, based on extensive computer simulations, that $r_c \approx 0.8$, i.e.

$$\left(\Delta p\right)_{c(Ha)} \approx 0.2.$$ \hspace{1cm} (6)

The corresponding phase diagram is shown in Fig.5. In contrast to Fig.5 for defects $r > r_c$ all currents up to $J_{\text{max}}$ can be realized and there is no phase separation at any density for weak defects! In this case the bottleneck has only local effects which can be observed near the defect, but not in the whole system.

Due to this apparent contradiction with expectations we have revisited the ASEP defect problem in \cite{13} based on highly accurate Monte Carlo simulations. Similar to \cite{12} we have simulated the ASEP with open boundary conditions, random-sequential dynamics (with $p = 1$) and a defect in the middle of the system (Fig.2). However, choosing $\alpha = \beta = \frac{1}{2}$ as in \cite{12}, corresponds exactly to the phase boundary of the high, low and maximal current phase \cite{5,6,8}. Fluctuations in finite-size systems will systematically underestimate the defect current $J(r)$ \cite{13}. We have
Fig. 6 Phase diagram of ASEP with defect according to [12]. Defects with \( r_c < r \leq p \) have no influence on the current \( J \).

therefore choose \( \alpha = \beta = 1 \) well inside the maximal current phase which allows to obtain a much better statistics.

To determine rather subtle bottleneck effects, very good statistics and advanced Monte Carlo techniques are required. To minimize errors induced by pseudo-random number generators we have used the Mersenne Twister [13].

Measurements of bottleneck effects for small defect strengths are easily hidden by fluctuations. Instead of using independent measurements for each defect strength \( r \) the systems are evolved in parallel, i.e. with the same protocol and the same set of random numbers, which leads to a strong suppression of fluctuations [13].

In order to minimize finite-size corrections, system lengths of up to \( N = 200,000 \) were considered (Fig. 7) which is two orders of magnitude larger than the systems considered in [12].

Fig. 7 Finite-size corrections to the current. The exactly known current in the infinite homogeneous system is \( J(N = \infty, r = 1) = 1/4 \).
To estimate the global effects of the defect we first considered the finite-size current \( J(N,r) \) through a system of length \( N \) and with a defect \( r \). Due to the fact that finite size corrections lead to an enhanced current, i.e. \( J(r,N) > J(r,N = \infty) \), one finds a lower bound for the critical hopping rate by satisfying \( J(N,r_c) - J(N = \infty, r = 1) < 0 \). However, in this way we only could derive a lower bound \( r_c \geq 0.86 \) for the critical hopping rate (Fig. 7). Assuming the existence of an essential singularity at \( r_c = 1 \), i.e. \( j(1) - j(r) \sim \exp(-a/(1-r)) \) \( \text{[11]} \), further improvement of the lower bound for the critical defect \( r_c \) by increasing the system length is a hopeless enterprise: e.g. a numerical proof of \( r_c > 0.9 \), \( r_c > 0.95 \), \( r_c > 0.99 \) would require \( N > 10^{10} \), \( N > 10^{22} \), \( N > 10^{147} \), respectively.

A much better quantity to determine the global influence of the defect (see e.g. Fig. 4, right) is the density profile or rather the difference between the density profile of the defect system with a corresponding homogeneous system (Fig. 8). Using the approach of parallel evolving systems we could clearly show a nonlocal influence on the density profile for defect strengths up to \( r = 0.99 \) (Fig. 8). This strongly supports the mean-field prediction \( r_c = 1 \).

**Fig. 8** Arbitrary defects \( r \) have a non-local effect on the density profile. The figure shows the difference between the density profile \( \rho_r(x) \) with and that without defect \( \rho_{r=1}(x) \) where \( x = j/N \) is the rescaled position.

4 Discussion and relevance for empirical results

Despite its relevance for applications some fundamental aspects of bottlenecks are not fully understood. Even for a minimal model like the ASEP with a defect the influence of weak bottlenecks is rather subtle and can be easily lost in fluctuations.

We have shown how to reconcile computer simulations with the intuition that even small defects have a global influence on the system. These effects are not easily seen in a reduction of the current which presumably shows a non-analytic
dependence on the bottleneck strength. Bottlenecks are better identified by their effects on the density profile which spreads throughout the whole system.

Based on a careful statistical analysis of Monte Carlo simulations we have found strong evidence that an arbitrarily weak defect $\Delta p \rightarrow 0$ in the ASEP has a global influence on the system. Meanwhile a mathematical proof of $(\Delta p)_c = 0$ has been announced in [14].

These results are believed to be generic for bottleneck systems. As a consequence the identification of weak bottlenecks in noisy empirical data is extremely difficult. Even for computer simulations very good statistics is required. Since the effect on the current is rather small, the density profile might be a better indicator for the presence of weak bottlenecks.

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