Extensions of Fiedler-Markham’s inequality and Thompson’s inequality

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Abstract

We present some new inequalities related to determinant and trace for positive semidefinite block matrices by using symmetric tensor product, which are extensions of Fiedler-Markham’s inequality and Thompson’s inequality.

Key words: Positive semidefinite matrices; Fiedler and Markham’s inequality; Thompson’s inequality.

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1 Introduction

Throughout the paper, we use the following standard notation. The set of $n \times n$ complex matrices is denoted by $\mathbb{M}_n(\mathbb{C})$, or simply by $\mathbb{M}_n$, and the identity matrix of order $n$ by $I_n$, or $I$ for short. In this paper, we are interested in complex block matrices. Let $\mathbb{M}_n(\mathbb{M}_k)$ be the set of complex matrices partitioned into $n \times n$ blocks with each block being a $k \times k$ matrix. The element of $\mathbb{M}_n(\mathbb{M}_k)$ is usually written as $H = [H_{ij}]_{i,j=1}^n$, where $H_{ij} \in \mathbb{M}_k$ for all $i, j$. By convention, if $X \in \mathbb{M}_n$ is positive semidefinite, we write $X \geq 0$. For two Hermitian matrices $A$ and $B$ of the same size, $A \geq B$ means $A - B \geq 0$.

Let $H = [H_{ij}]_{i,j=1}^n$ be positive semidefinite. It is well known that both $[\det H_{ij}]_{i,j=1}^n$ and $[\text{tr} H_{ij}]_{i,j=1}^n$ are positive semidefinite; see, e.g., [18]. Moreover, the renowned Fischer’s inequality (see [7] p. 506 or [19] p. 217) says that

$$\prod_{i=1}^n \det H_{ii} \geq \det H. \quad (1)$$

There are various extensions and generalizations of (1) in the literature, e.g., [2] [3] [4] [8] [11] [10]. In 1961, Thompson [17] generalized Fischer’s determinantal inequality as below [2] by an identity of Grassmann products; see [12] for a short proof.
Theorem 1.1 Let $H = [H_{ij}]_{i,j=1}^n \in M_n(M_k)$ be positive semidefinite. Then
\[
\det([\det H_{ij}]) \geq \det H.
\] (2)

Indeed, (2) is a generalization of Fischer’s result (1) since we can get by a special case of Fischer’s inequality that $\prod_{i=1}^n \det H_{ii} \geq \det([\det H_{ij}])$. In 1994, Fiedler and Markham [6] revisited Thompson’s result and proved the following inequality for trace.

Theorem 1.2 Let $H = [H_{ij}]_{i,j=1}^n \in M_n(M_k)$ be positive semidefinite. Then
\[
\left( \frac{\det([\text{tr} H_{ij}])}{k} \right)^k \geq \det H.
\] (3)

In fact, Lin [13, 14] pointed out that in their proof of Theorem 1.2, Fiedler and Markham used the superadditivity of determinant functional, which can be improved by Fan-Ky’s determinantal inequality (see [5] or [7, p. 488]), i.e., the log-concavity of the determinant over the cone of positive semidefinite matrices. Here we state the improved version (4) as follows; see [9, 10] for a short proof and extension to the class of sector matrices.

Theorem 1.3 Let $H = [H_{ij}]_{i,j=1}^n \in M_n(M_k)$ be positive semidefinite. Then
\[
\left( \frac{\det([\text{tr} H_{ij}])}{k^n} \right)^k \geq \det H.
\] (4)

The paper is organized as follows. In Section 2 for convenience, we briefly review some basic definitions and properties of symmetric tensor product in Multilinear Algebra Theory. In Section 3, we show two extensions of Fiedler-Markham’s inequality by using symmetric tensor product (Theorem 3.5 and Theorem 3.7). Additionally, some other determinantal inequalities of positive semidefinite block matrices are included. In Section 4, we give an extension of Thompson’s inequality (Theorem 4.1), which also yields a generalization of Fischer’s inequality (Corollary 4.2).

2 Preliminaries

Before starting our results, we first review some basic definitions and notations of multilinear algebra [15]. If $A = [a_{ij}]$ is of order $m \times n$ and $B$ is $s \times t$, the tensor product of $A, B$, denoted by $A \otimes B$, is an $ms \times nt$ matrix, partitioned into $m \times n$ block matrix with the $(i, j)$-block the $s \times t$ matrix $a_{ij}B$. Let $\otimes^r A := A \otimes \cdots \otimes A$ be the $r$-fold tensor power of $A$. Let $V$ be an $n$-dimensional Hilbert space and $\otimes^r V$ be the tensor product space of $r$ copies of $V$. The symmetric tensor product of vectors $v_1, v_2, \ldots, v_r$ in $V$ is defined as
\[
v_1 \vee v_2 \vee \cdots \vee v_r := \frac{1}{\sqrt{r!}} \sum_{\sigma} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(r)},\]
where $\sigma$ runs over all permutations of the $r$ indices. The linear span of all these vectors comprises the subspace $\vee^r V$ of $\otimes^r V$, this is called the $r$th symmetric tensor power of $V$. 


Let $A$ be a linear map on $V$, then $(\otimes^r A)(v_1 \vee \cdots \vee v_r) = Av_1 \vee \cdots \vee Av_r$ lies in $\vee^r V$ for all $v_1, \ldots, v_r$ in $V$. Therefore, the subspace $\vee^r V$ is invariant under the tensor operator $\otimes^r A$. The restriction of $\otimes^r A$ to this invariant subspace is denoted by $\vdash^r A$ and called the $r$th symmetric tensor power of $A$; see [11 pp. 16-19] and [15] for more details. We denote by $s_r(A)$ the $r$th complete symmetric polynomial of the eigenvalues of $A \in \mathbb{M}_n(\mathbb{C})$, i.e.,

$$s_r(A) := \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_r \leq n} \lambda_{i_1}(A)\lambda_{i_2}(A) \cdots \lambda_{i_r}(A).$$

Some basic properties of tensor product are summarised below.

**Proposition 2.1** Let $A, B, C$ be matrices of sizes $n \times n$. Then

1. $\otimes^r(AB) = (\otimes^r A)(\otimes^r B)$ and $\vdash^r(AB) = (\vdash^r A)(\vdash^r B)$.
2. $\text{tr}(\otimes^r A) = (\text{tr}A)^r$ and $\text{tr}(\vdash^r A) = s_r(A)$.
3. $\det(\otimes^r A) = (\det A)^{rn-r} \text{ and } \det(\vdash^r A) = (\det A)^{\frac{n}{r}(n+1)}$.

Furthermore, if $A, B, C$ are positive semidefinite matrices, then

4. $A \otimes B$ and $A \vdash B$ are positive semidefinite.
5. If $A \geq B$, then $A \otimes C \geq B \otimes C$ and $A \vdash C \geq B \vdash C$.
6. $\otimes^r(A + B) \geq \otimes^r A + \otimes^r B$ and $\vdash^r(A + B) \geq \vdash^r A + \vdash^r B$ for all positive integer $r$.

In this paper, we are mainly investigate positive semidefinite block matrices. For $H = [H_{ij}] \in \mathbb{M}_n(\mathbb{M}_k)$, we denote by $T^r_n(H) := [\otimes^r H_{ij}] \in \mathbb{M}_n(\mathbb{M}_{kr})$ and $Q^r_n(H) := [\vdash^r H_{ij}] \in \mathbb{M}_n(\mathbb{M}_{(k+1)^r})$.

### 3 Extensions of Fiedler-Markham’s inequality

In the section, we first prove some lemmas for latter use, and then we give two extensions of Fiedler-Markham’s inequality.

**Lemma 3.1** Let $H = [H_{ij}] \in \mathbb{M}_n(\mathbb{M}_k)$. Then $T^r_n(H)$ is a principal submatrix of $\otimes^r H$.

**Proof.** Without loss of generality, we may write $H = X^*Y$, where $X, Y$ are $nk \times nk$. Now we partition $X = (X_1, X_2, \ldots, X_n)$ and $Y = (Y_1, Y_2, \ldots, Y_n)$ with each $X_i, Y_i$ is an $nk \times k$ complex matrix. Under this partition, we see that $H_{ij} = X_i^*Y_j$. Also we have $Y_j = YE_j$, where $E_j$ is a suitable $nk \times k$ matrix such that its $j$-th block is exactly $I_k$ and otherwise 0. By (1) of Proposition 2.1, we obtain

$$\otimes^r H_{ij} = \otimes^r(X_i^*Y_j) = \otimes^r(E_i^*X^*YE_j) = (\otimes^r E_i)^*(\otimes^r(X^*Y))(\otimes^r E_j).$$

In other words,

$$[\otimes^r H_{ij}]_{i,j=1}^n = E^*(\otimes^r A)E, \quad E = [\otimes^r E_1, \otimes^r E_2, \ldots, \otimes^r E_n].$$

It is easy to verify that $E$ is a permutation matrix with 1 only in diagonal entries. \hfill \blacksquare
Corollary 3.2 If $H \in M_n(M_k)$ is positive semidefinite, then so are $T^n_r(H)$ and $Q^n_r(H)$.

Proof. As $H$ is positive semidefinite, so are $\otimes^r H$ and $\vee^r H$. By Lemma 3.1 we can see that $T^n_r(H)$ and $Q^n_r(H)$ are positive semidefinite. ■

Lemma 3.3 Let $A, B \in M_n(M_k)$ be positive semidefinite. Then for $r \in \mathbb{N}^*$

$$T^n_r(A + B) \geq T^n_r(A) + T^n_r(B),$$

and

$$Q^n_r(A + B) \geq Q^n_r(A) + Q^n_r(B).$$

Proof. By the basic property of tensor power, Proposition 2.1, we have

$$\otimes^r (A + B) \geq \otimes^r A + \otimes^r B.$$

Since $[\otimes^r A_{ij}]_{i,j=1}^n$ is a principal submatrix of $\otimes^r A$, Lemma 3.1 it yields

$$[\otimes^r (A_{ij} + B_{ij})]_{i,j=1}^n \geq [\otimes^r A_{ij}]_{i,j=1}^n + [\otimes^r B_{ij}]_{i,j=1}^n.$$

By restricting above inequality to the symmetric tensors, we obtain

$$[\vee^r (A_{ij} + B_{ij})]_{i,j=1}^n \geq [\vee^r A_{ij}]_{i,j=1}^n + [\vee^r B_{ij}]_{i,j=1}^n.$$

This completes the proof. ■

The following Proposition 3.4 is a key step in proof of our extensions (Theorem 3.5), and it can be regarded as a Thompson-type determinantal inequality.

Proposition 3.4 Let $H = [H_{ij}] \in M_n(M_k)$ be positive definite. Then for $r \in \mathbb{N}^*$

$$\det T^n_r(H) \geq (\det H)^{rk^{r-1}}. \quad (5)$$

Proof. Since the determinant functional is continuous, we may assume without loss of generality that $H$ is positive definite by a standard perturbation argument. As $H$ is positive definite, we may further write $H = T^*T$ with $T = [T_{ij}] \in M_n(M_k)$ being block upper triangular matrix, see [7, p. 441]. Note that

$$\begin{align*}
(\det H)^{rk^{r-1}} &= (\det T^*T)^{rk^{r-1}} = \left(\prod_{i=1}^n \det T^*_i \right)^{rk^{r-1}} \\
&= \prod_{i=1}^n (\det T^*_i)^{rk^{r-1}} \cdot \prod_{i=1}^n (\det T^*_i)^{rk^{r-1}} \\
&= \prod_{i=1}^n \det(\otimes^r T^*_i) \prod_{i=1}^n \det(\otimes^r T^*_i),
\end{align*}$$

where the last equality is by Proposition 2.1. We next may assume $T_{ii} = I_k$ by pre- and post-multiplying both sides of (5) by $\prod_{i=1}^n \det(\otimes^r T^*_i)$ and $\prod_{i=1}^n \det(\otimes^r T^*_i)$, respectively. Thus, it suffices to show that

$$\det T^n_r(T^*T) \geq 1. \quad (6)$$
We now prove (6) by induction. When \( n = 2 \),
\[
\det (T_2^n(T^*T)) = \det \begin{bmatrix} \otimes^r I_k & \otimes^r T_{12} \\ \otimes^r T_{12} & \otimes^r (I_k + T_{12}T_{12}) \end{bmatrix} = \det \begin{bmatrix} I_k & \otimes^r T_{12} \\ \otimes^r T_{12} & \otimes^r (I_k + T_{12}T_{12}) \end{bmatrix} = \det \begin{bmatrix} I_k & \otimes^r (I_k + T_{12}T_{12}) \\ 0 & \otimes^r (I_k + T_{12}T_{12}) - \otimes^r T_{12}\otimes^r T_{12} \end{bmatrix} = \det (\otimes^r (I_k + T_{12}T_{12}) - \otimes^r (T_{12}T_{12})) \geq \det(\otimes^r I_k) = 1,
\]
in which the first inequality is by Proposition\textsuperscript{2,1}.
Suppose now (6) is true for \( n = m \), and then consider the case \( n = m + 1 \). For notational convenience, we denote
\[
T = \begin{bmatrix} I_k & V \\ 0 & \hat{T} \end{bmatrix}, \quad \text{where} \quad V = [T_{12} \cdots T_{1n}] \quad \text{and} \quad \hat{T} = [T_{i+1,j+1}]_{i,j=1}^m.
\]
Let \( \hat{V} = [\otimes^r T_{12} \cdots \otimes^r T_{1n}] \). Clearly, by Proposition\textsuperscript{2,1} \( \hat{V}^* \hat{V} = T_m^r(V^*V) \).
Now computing
\[
T^*T = \begin{bmatrix} I_k & V^* \\ 0 & \hat{T} \end{bmatrix} \begin{bmatrix} I_k & V \\ 0 & \hat{T} \end{bmatrix} = \begin{bmatrix} I_k & V \\ V^* & \hat{T}^* \hat{T} + V^*V \end{bmatrix}.
\]
Then
\[
\det (T_n^n(T^*T)) = \det \begin{bmatrix} \otimes^r I_k & \hat{V} \\ \hat{V}^* & T_m^r(\hat{T}^* \hat{T} + V^*V) \end{bmatrix} = \det (T_m^r(\hat{T}^* \hat{T} + V^*V) - \hat{V}^* \hat{V}) = \det (T_m^r(\hat{T}^* \hat{T} + V^*V) - T_m^r(V^*V)) \geq \det (T_m^r(\hat{T}^* \hat{T}) + T_m^r(V^*V) - T_m^r(V^*V)) = \det (T_m^r(\hat{T}^* \hat{T})) \geq 1,
\]
in which the first inequality is by Lemma\textsuperscript{3,3} while the second one is by the induction hypothesis. Thus, (6) holds for \( n = m + 1 \), so the proof of the induction step is complete. Hence we complete the proof of the proposition. \( \blacksquare \)

We now give the first extension of Fiedler-Markham’s inequality\textsuperscript{3} and\textsuperscript{4}.

**Theorem 3.5** Let \( H = [H_{ij}] \in M_n(M_k) \) be positive semidefinite. Then for \( r \in \mathbb{N}^* \)
\[
\left( \frac{\det(\text{tr}H_{ij})}{k^{rn}} \right)^k \geq (\det H)^r. \quad (7)
\]

**Proof.** The proof is a combination of Theorem\textsuperscript{1,3} and Proposition\textsuperscript{3,4}. By Corollary\textsuperscript{3,2} \( T_n^r(H) \in M_n(M_{kr}) \) is positive semidefinite, then by (4) of Theorem\textsuperscript{1,3} we have
\[
\left( \frac{\det(\text{tr}H_{ij})}{k^{rn}} \right)^k = \left( \frac{\det(\otimes^r H_{ij})}{k^{rn}} \right)^k \geq \det T_n^r(H),
\]
which together with Proposition 3.4 leads to the following

\[
\left( \frac{\text{det}([\text{tr}H_{ij}]^r)}{kr^n} \right)^k \geq (\text{det} H)^r k^{kr-1}.
\]

Hence, the desired result (7) follows. ■

Obviously, when \( r = 1 \), (7) reduces to Fiedler and Markham’s result (4). Using the same idea in the proof of Proposition 3.4, one could also get the following determinantal inequality for \( Q^r_n(H) \). We omit the proof and leave the details for the interested reader.

**Proposition 3.6** Let \( H = [H_{ij}] \in \mathbb{M}_n(\mathbb{M}_k) \) be positive definite. Then for \( r \in \mathbb{N}^* \)

\[
\text{det} Q^r_n(H) \geq (\text{det} H)^{rk} k^{kr-1}.
\]

We next show another extension of Fiedler-Markham’s inequality similarly.

**Theorem 3.7** Let \( H = [H_{ij}] \in \mathbb{M}_n(\mathbb{M}_k) \) be positive semidefinite. Then for \( r \in \mathbb{N}^* \)

\[
\left( \frac{\text{det} [s_r(H_{ij})]}{(k+r-1)^r} \right)^{(k+r-1)r} \geq (\text{det} H)^r.
\]

**Proof.** By Corollary 3.2 and Theorem 1.3 we obtain

\[
\left( \frac{\text{det} [s_r(H_{ij})]}{(k+r-1)^r} \right)^{(k+r-1)r} = \left( \frac{\text{det} [\text{tr} \bigvee^r H_{ij}]}{(k+r-1)^r} \right)^{(k+r-1)r} \geq \text{det} Q^r_n(H),
\]

which together with Proposition 3.6 yields the following

\[
\left( \frac{\text{det} [s_r(H_{ij})]}{(k+r-1)^r} \right)^{(k+r-1)r} \geq (\text{det} H)^r k^{kr-1},
\]

Thus, the desired result (8) follows. ■

Clearly, when \( r = 1 \), (8) reduces to Fiedler and Markham’s result (4).

### 4 Extensions of Thompson’s inequality

Motivated by Theorem 3.5 and Theorem 3.7, we apply Theorem 1.4 to matrices \( T^r_n(H) \) and \( Q^r_n(H) \), respectively, and then combining with Proposition 3.4 and Proposition 3.6, we have

\[
\text{det} \left[ \text{det} \bigotimes^r H_{ij} \right] \geq \text{det} T^r_n(H) \geq (\text{det} H)^r k^{kr-1},
\]

and

\[
\text{det} \left[ \text{det} \bigvee^r H_{ij} \right] \geq \text{det} Q^r_n(H) \geq (\text{det} H)^r k^{kr-1}.
\]

By Proposition 2.1, we get the following extensions of Thompson’s result (2),

\[
\text{det} \left[ (\text{det} H_{ij})^{rk-1} \right] \geq (\text{det} H)^r k^{kr-1},
\]

(9)
and
\[ \det \left[ (\det H_{ij})^{(k+r-1)} \right] \geq (\det H)^{(k+r-1)}. \]  

(10)

At the end of this paper, we present a more general setting of (9) and (10), i.e., we will relax the restriction of exponent, which also can be viewed as an extension of Thompson’s inequality [2]. Let \( A \) and \( B \) be complex matrix with the same size, we denote by \( A \circ B \) the Hadamard product of \( A, B \) and denote by \( \circ^r A \) the \( r \)-fold Hadamard power of \( A \).

**Theorem 4.1** Let \( H = [H_{ij}] \in M_n(M_k) \) be positive semidefinite. Then for \( r \in \mathbb{N}^* \)
\[ \det \left[ (\det H_{ij})^r \right] \geq (\det H)^r. \]  

(11)

**Proof.** By Oppenheim’s inequality [7, p. 509], we obtain
\[ \det \left[ (\det H_{ij})^r \right] = \det (\circ^r [\det H_{ij}]) \geq (\det [\det H_{ij}])^r. \]

By (2) of Theorem 1.1 we get
\[ (\det [\det H_{ij}])^r \geq (\det H)^r. \]

This completes the proof. \( \blacksquare \)

By taking the special case \( n = 2 \) in (11), we can easily get the following Corollary 4.2, which is a generalization of Fischer’s inequality [11].

**Corollary 4.2** Let \( H = [H_{ij}] \in M_2(M_k) \) be positive semidefinite. Then for \( r \in \mathbb{N}^* \)
\[ (\det H_{11} \det H_{22})^r - (\det H_{21} \det H_{12})^r \geq (\det H)^r. \]

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