Dissipative Quantum Systems in ThermoField Dynamics

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We investigate a class of microscopic systems in interaction with a macroscopic system in thermal equilibrium, following the construction of Dalibard, Dupont-Roc and Cohen-Tannoudji (DDC). By considering self-adjoint operators as elements of Schwinger’s Measurement Algebra (SMA), we construct statistical mean values of the relevant observables as matrix elements in a suitable operator basis, which correspond to the vacuum states of ThermoField Dynamics (TFD).

1. Schwinger Operators and TFD

In SMA [1] an operator is defined as

\[ X = \sum_{a',a''} \langle a' \mid X \mid a'' \rangle M(a',a''), \]  

(1)

where \( \langle a' \mid X \mid a'' \rangle \) represents its matrix element in the \( a \) representation:

\[ \langle a' \mid X \mid a'' \rangle = \text{Tr} X M(a'',a'). \]  

(2)

The product of two general measurement symbols, referring to distinct sets of compatible observables \( A, B, C \) and \( D \), satisfies the following composition rule:

\[ M(a',b')M(c',d') = \langle b' \mid c' \rangle M(c',d'), \]  

(3)

where the number \( \langle b' \mid c' \rangle \) defines the statistical relation between the corresponding sets.

The expectation value of a given property \( A \) in the \( b' \) basis is

\[ \langle A \rangle_{b'} = \sum_{a'} a' p(a',b') = \text{Tr} A M(b') = \langle b' \mid A \mid b' \rangle , \]  

(4)

where we introduce the operator

\[ A = \sum_{a'} a' M(a') = \sum_{a',a''} a' \delta(a',a'') M(a',a''), \]  

(5)

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in the basis $M(a') \equiv M(a', a')$. This implies that $\langle a' | A | a'' \rangle = a'\delta(a', a'').$

Now we define the statistical mean

$$\langle A \rangle = \sum_{b'} \Pi(b') \langle A \rangle_{b'} = \sum_{a'} \langle a' | \rho A | a' \rangle = Tr(\rho A),$$

(6)

with $\Pi(b') \geq 0$ and $\sum_{b'} \Pi(b') = 1$, where

$$\rho \equiv \sum_{b'} \Pi(b') M(b') = \sum_{b', b''} \Pi(b') \delta(b', b'') M(b', b'').$$

(7)

Hence $\langle b' | \rho | b'' \rangle = \Pi(b') \delta(b', b'')$, with $Tr\rho = \sum_{b'} \Pi(b') = 1$. In thermal equilibrium, by considering the basis $M(n, m)$ associated to the number operator $N$ in which its matrix elements are diagonal,

$$N = \sum_{n,m} \langle n | N | m \rangle \delta_{nm} M(n, m)$$

$$= \sum_{n} \langle n | N | n \rangle M(n),$$

(8)

we have

$$\rho = \sum_{n} \Pi(E_n) M(n),$$

(9)

where

$$\Pi(E_n) = Z^{-1} e^{-\beta E_n},$$

(10)

or

$$\rho = \sum_{n,m} \Pi(E_n) \delta_{nm} M(n, m).$$

(11)

Introducing an auxiliary operator basis where a new fictitious operator $\tilde{N}$, corresponding to $N$, is diagonal, we can write $\delta_{nm} M(n, m)$ as

$$\langle \tilde{n} | \tilde{m} \rangle M(n, m) = M(n, \tilde{n}) M(\tilde{m}, n),$$

(12)

so that

$$\rho = \sum_{n,m} \sqrt{\Pi(E_n) \Pi(E_m)} M(n, \tilde{n}) M(\tilde{m}, n)$$

$$= \left[ \sum_{n} \sqrt{\Pi(E_n)} M(n, \tilde{n}) \right] \left[ \sum_{m} \sqrt{\Pi(E_m)} M(\tilde{m}, m) \right]$$

$$\equiv |0(\beta)) \langle 0(\beta)|.$$

(13)
We may identify the measurement symbols inside the square brackets as composite states of the thermal vacuum of TFD [2]:

\[ \left| 0(\beta) \right\rangle \equiv \sum_n \sqrt{\Pi(E_n)} \left| n, \tilde{n} \right\rangle, \tag{14} \]

so that \( \rho \) acquires the status of a projection operator, as well as \( M(n, \tilde{n}) \equiv \left| \tilde{n} \right\rangle \left\langle n \right| \) and \( M(\tilde{m}, m) \equiv \left| \tilde{m} \right\rangle \left\langle m \right| \). Therefore, for a given observable \( F \),

\[
\text{Tr} (\rho F) = \langle 0(\beta) | F | 0(\beta) \rangle = \text{Tr} \left( \sum_{n,m} \sqrt{\Pi(E_n)\Pi(E_m)} | \tilde{m} \rangle \left\langle m | F | n \right\rangle \langle \tilde{n} | \right) \\
= \sum_{n,m} \sqrt{\Pi(E_n)\Pi(E_m)} \langle \tilde{n}|\tilde{m}\rangle \langle m|F|n \rangle \\
= \sum_n \Pi(E_n) \langle n|F|n \rangle. \tag{15} \]

### 2. Radiation Considered as a Reservoir

Let us consider the problem of a small system \( A \) interacting with a large reservoir \( R \). Following DDC construction [3], let

\[ H = H_A + H_R + V, \tag{16} \]

be the Hamiltonian of global system \( A + R \), where \( H_A \) and \( H_R \) are, respectively, the Hamiltonians of \( A \) and \( R \), whilst \( V \) stands for the interaction between them. In the interaction picture, the density operator of the global system obeys the evolution equation,

\[ \frac{d}{dt} \rho(t) = \frac{1}{i\hbar} [V(t), \rho(t)], \tag{17} \]

from which we obtain

\[
\rho(t + \Delta t) = \rho(t) + \frac{1}{i\hbar} \int_t^{t+\Delta t} [V(t'), \rho(t)]dt' + \\
+ \left( \frac{1}{i\hbar} \right)^2 \int_t^{t+\Delta t} dt' \int_t^{t'} [V(t'), [V(t''), \rho(t'')]]dt''. \tag{18} \]

By taking the trace with respect to \( R \), we arrive at

\[
\Delta \sigma(t) \equiv \sigma(t + \Delta t) - \sigma(t) = \frac{1}{i\hbar} \int_t^{t+\Delta t} \text{Tr}_R [V(t'), \rho(t)]dt' + \\
+ \left( \frac{1}{i\hbar} \right)^2 \int_t^{t+\Delta t} dt' \int_t^{t'} \text{Tr}_R [V(t'), [V(t''), \rho(t'')]]dt'', \tag{19} \]

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where
\[ \sigma(t) = Tr_R \rho(t). \] (20)
The interaction \( V \) between \( A \) e \( R \) will be taken as a product of an observable \( A \) of \( A \) and an observable \( R \) of \( R \):
\[ V = -AR. \] (21)
Since the average value in \( \sigma_R \) of the coupling \( V(t) \) is zero, the leading contribution in (19) stems from the two-time average
\[ g(t', t'') = Tr_R[\sigma_R R(t') R(t'')]. \] (22)
If \( V \) is sufficiently small, and \( \Delta t \) sufficiently short compared with the evolution time \( t_R \) of \( \sigma \), \( \rho(t) \) can be written in the form
\[ \rho(t) = Tr_R \rho(t) \otimes Tr_A \rho(t), \] (23)
where the contributions of the correlations between \( A \) and \( R \) in the time \( t \) were neglected. The general idea is that the initial correlations between \( A \) and \( R \) at time \( t \) disappear after a collision time \( \tau_c \ll t_R \). Thus, there exist two very different time scales, such that
\[ \tau_c \ll \Delta t \ll t_R, \] (24)
bringing us to the ‘coarse-grained’ rate of variation for the system \( A \).

3. Master Equation for a Damped Harmonic Oscillator

3.1. The Physical System

We are interested in the case where the small system \( A \) is a one-dimensional harmonic oscillator of frequency \( \omega_0 \) whose Hamiltonian is
\[ H_A = \hbar \omega_0 (b^\dagger b + \frac{1}{2}), \] (25)
where \( b^\dagger \) and \( b \) are the rising and lowering operators of this oscillator. The reservoir \( R \) consists of an infinite number of one-dimensional harmonic oscillators, of frequency \( \omega_i \), with ladder operators \( a_i^\dagger \) and \( a_i \), so that the Hamiltonian \( H_R \) for \( R \) is written as
\[ H_R = \sum_i \hbar \omega_i (a_i^\dagger a + \frac{1}{2}). \] (26)
We take a sesquilinear interaction between $A$ and $R$ of the form

$$V = V^\dagger = \sum_i (\eta_i a_i^\dagger + \eta_i^* b_i^\dagger),$$  \hspace{1cm} (27)

where $\eta_i$ is the coupling constant between $A$ and the $i$-th oscillator of $R$.

### 3.2. The Master Equation

We write the coarse-grained rate of variation,

$$\frac{\Delta \sigma}{\Delta t} = -\frac{1}{\hbar^2} \int_0^{\infty} dt' \int_t^{t+\Delta t} dt'' \text{tr}_R[V(t'), [V(t''), \sigma_A(t) \otimes \sigma_R]],$$  \hspace{1cm} (28)

with $V$ given by (27). Changing to the Schrödinger representation, we obtain the following operator form for the master equation:

$$\frac{d\sigma}{dt} = -\frac{\Gamma}{2} [\sigma, b^\dagger b]_+ - \Gamma' [\sigma, b^\dagger b] - \Gamma' \sigma$$

$$- i(\omega_0 + \Delta) [b^\dagger b, \sigma] + \Gamma b \sigma b^\dagger + \Gamma' (b^\dagger \sigma b + \sigma b^\dagger).$$  \hspace{1cm} (29)

In this equation $[.,.]_+$ represents the anticommutator and we have made the following definitions

$$\Gamma = \frac{2\pi}{\hbar} \sum_i |\eta_i|^2 \delta(h\omega_0 - \omega_i), \quad \Gamma' = \frac{2\pi}{\hbar} \sum_i |\eta_i|^2 \langle n_i \rangle \delta(h\omega_0 - \omega_i),$$  \hspace{1cm} (30)

and

$$\hbar \Delta = V.P. \left( \sum_i \frac{|\eta_i|^2}{\hbar\omega_0 - \hbar\omega_i} \right), \quad \hbar \Delta' = V.P. \left( \sum_i \frac{|\eta_i|^2 \langle n_i \rangle}{\hbar\omega_0 - \hbar\omega_i} \right).$$  \hspace{1cm} (31)

Here $\langle n_i \rangle$ is the average number of excitation quanta of the oscillator $i$. If this number depends only on the energy of this oscillator, due to delta function in the second equation in (30), we have

$$\Gamma' = \langle n(\omega_0) \rangle \Gamma,$$  \hspace{1cm} (32)

where $\langle n(\omega_0) \rangle$ is the average number of quanta in the reservoir oscillators, having the same frequency $\omega_0$ of oscillator $A$. If, moreover, $R$ is in thermodynamic equilibrium, $\langle n(\omega_0) \rangle$ is equal to $[\exp \hbar\omega_0/k_B T - 1]^{-1}$.  

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Alternatively, temperature effects can be incorporated from the very beginning, replacing the trace in (28) by the thermal expectation value (15), leading to

\[
\frac{\Delta \sigma}{\Delta t} = -\frac{1}{\hbar^2} \frac{1}{\Delta t} \int_0^\infty d(t' - t'') \int_t^{t+\Delta t} dt'' \langle 0(\beta) | [V(t'), [V(t''), \sigma_A(t)]] | 0(\beta) \rangle,
\]

(33)

and, consequently, to the master equation (29). In this case, the average number \( \langle n_i \rangle \) in (30) and (31) is given by a mean value in the TFD vacuum state, defined in (14),

\[
\langle n_i \rangle = \langle 0(\beta) | a_i^\dagger a_i | 0(\beta) \rangle,
\]

(34)

after working out the thermal Green function

\[
g(\tau) = \text{Tr} \left[ R(t) R(t - \tau) | 0(\beta) \rangle \langle 0(\beta) | \right] = \langle 0(\beta) | R(t) R(t - \tau) | 0(\beta) \rangle,
\]

(35)

instead of the two-time average (22), with \( \tau \equiv t' - t'' \).

As a straightforward application of the formalism just presented, equation (33) can be employed to derive the corresponding master equation (29), in order to study a brownian particle interacting with a macroscopic system in the framework of equilibrium TFD [4].

References

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