Abstract—In this paper, we develop a direct method for the characterization of dark modes. The results can be used to construct a transformation that separates dark and bright modes, through the decomposition of system dynamics. We also study a synthesis problem by engineering the system-environment coupling and Hamiltonian engineering. We apply the theory to investigate an optomechanical dark mode.

Index Terms—Quantum linear systems, Dark mode, Optomechanical systems.

I. INTRODUCTION

A major obstacle in quantum information processing is the coherent manipulation of fragile quantum information in the presence of environmental noise. The coherence of quantum systems will be lost if the systems are perturbed by environmental noise. This process of losing quantum coherence is commonly called decoherence. One way to counteract the decoherence effect is by engineering a decoherence-free (DF) subsystem [1], [2], [3], [4], [5], [6], [7], [8], [9], [10]. According to the DF linear quantum subsystem theory developed in [7], the DF modes are defined as the uncontrollable and unobservable modes of a quantum linear system. The linear DF modes can be obtained using the standard uncontrollable and unobservable decomposition of a linear system [10], [2], [9].

In this paper, the definition of dark modes is as follows:

Definition 1: For quantum linear systems, the dark modes are uncontrollable and unobservable modes that are defined on an arbitrarily given Hilbert space which is associated with a subsystem.

According to the above definition, a dark mode $x_D$ can be characterized by the following dynamical equation

$$\dot{x}(t) = dx(t) = d \left( \begin{array}{c} x_D(t) \\ x_B(t) \end{array} \right) = \left( \begin{array}{cc} A_{11} & 0 \\ 0 & A_{22} \end{array} \right) x(t) dt + \left( \begin{array}{c} 0 \\ B_2 \end{array} \right) dW(t),$$

$$dW(t) = (0 | \hat{C}_2)x(t) dt + dW(t),$$

where $(x_D, x_B)$ are defined on the arbitrarily given Hilbert subspace $\mathcal{D}$ which is associated with a subsystem. For simplicity, we use the same notation $\mathcal{D}$ when referring to this subsystem. $x_d$ is defined on a Hilbert subspace $\mathcal{N}$ (subsystem $\mathcal{N}$). As a result, the quantum linear system is defined on $\mathcal{D} \otimes \mathcal{N}$. $W(t)$ is a noise process and $W_{out}(t)$ is an output process of the system. Based on [1], we have

$$dx_D = A_{11} x_D dt,$$

which implies that the dynamics of $x_D$ is decoupled from $x_B$, the environmental noise and the noisy subsystem $\mathcal{N}$. $x_B$ is called a bright mode if it interacts with the noisy subsystem $\mathcal{N}$ via $A_{22}$. The above definition of dark mode is consistent with the literature, e.g., see [12], [13], [14], in which the formation of a dark mode is used to achieve mediation between subsystems while being decoupled from a given noisy subsystem.

In this paper, we develop a direct method for the decomposition of the system dynamics as in [1], based on a suitable coordinate transformation. After preliminaries are presented in Section II, a direct method to characterize dark modes is developed in Section III. Section IV discusses the synthesis of dark modes. By engineering the system-environment coupling operator, we can remove the direct coupling of the dark modes to the noise and subsystem $\mathcal{N}$. Assisted by suitable Hamiltonian engineering, the indirect coupling can be eliminated as well and dark modes are generated. In order to illustrate the applications of the dark mode theory, in Section V we study an optomechanical system which relies on dark modes to function.

Notation: $A^T$ denotes the transpose of $A$. $A^\dagger$ is the Hermitian adjoint of $A$. $N(A) = \{ v | A v = 0 \}$ is the kernel of $A$, and $R(A) = \{ A v | v \}$ is the range of $A$. $R^\perp(A)$ is the orthogonal complement to $R(A)$. $A^+$ denotes the Moore-Penrose generalized inverse of $A$. $[X, Y] = XY - YX$. $n$ is an $n$-dimensional zero matrix, and $I_n$ is an $n$-dimensional identity matrix. $\emptyset$ is the empty set. $\Re(a), \Im(a)$ are the real and imaginary parts of a complex number $a$. $i$ is the imaginary unit. $\Sigma_n = \text{diag}\{\Sigma, \cdots, \Sigma\}$ is a block diagonal matrix containing $n$ two-dimensional matrices $\Sigma$ defined by $\Sigma = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$. $\delta_{ij} = 0$ if $i \neq j$, and $\delta_{ij} = 1$ if $i = j$. Also, we use $\delta(\cdot)$ to denote the Dirac delta function. $\rho$ denotes a quantum state which is a Hermitian operator satisfying $\text{trace}(\rho) = 1$ and $\rho \geq 0$. We set $h = 1$.

II. PRELIMINARIES

A. Heisenberg-picture Evolution and Quantum Stochastic Differential Equations

The dynamics of a quantum system can be characterized by the evolution of a quantum state defined as $\rho(t) = U(t, t_0) \rho(0) U(t, t_0)^\dagger$, where $t_0$ is the initial time and $U(t_0, t_0) = I$. $U(t, t_0)$ is the unitary operator which generates the quantum evolution [15], [16]. The expectation of a system

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with $G$ being a $2n \times 2n$ real symmetric matrix and $c_i$ being a column vector of $2n$ scalars. According to [4], the dynamics of $x(t)$ are described by the following linear system equation

$$dx(t) = Ax(t)dt + BdW(t).$$

Where $H \equiv (X_1, P_1, \ldots, X_m, P_m)^T$ and $\{X_i, P_i\}, i = 1, \ldots, m$ are defined by $X_i = (\tilde{B}_i + \tilde{B}_i^T)/\sqrt{2}, P_i = (\tilde{B}_i - \tilde{B}_i^T)/\sqrt{2}$. $W(t)$ is the noise process due to the environmental couplings. The coefficient matrices of (7) are given by

$$A = \Sigma_n(G + CT\Sigma_n C/2) \in \mathbb{R}^{2n \times 2n},$$

$$B = \Sigma_n C^T \Sigma_m \in \mathbb{R}^{2n \times 2m},$$

where the following definition is used [7]

$$C = \sqrt{2}(\Re(c_1), \Im(c_1), \ldots, \Re(c_m), \Im(c_m))^T \in \mathbb{R}^{2m \times 2n}. \quad (9)$$

Using (5), the input-output relation of the system can be written as

$$dW_{\text{out}}(t) = Cx(t)dt + dW(t). \quad (10)$$

A quantum linear system as expressed by (7) and (10) can be described by a triplet $(A, B, C)$.

A coordinate transformation $x = T x, T \in \mathbb{R}^{2n \times 2n}$ yields the following transformed system

$$dx(t) = T^{-1} A T x(t)dt + T^{-1} B dW(t),$$

$$dW_{\text{out}}(t) = C T x(t)dt + dW(t). \quad (11)$$

### C. Symplectic Matrices

$\Sigma$ is a symplectic since $\Sigma^T \Sigma \Sigma = -\Sigma^T = \Sigma$. Consider the commutator $[x_i, p_i]$ as a bilinear form which can be expressed as $(x_i, p_i) \Sigma(x_i, p_i)^T$. Since $\Sigma^T \Sigma \Sigma = \Sigma$, $\Sigma(x_i, p_i)^T$ is the symplectic transformation that preserves the commutation relation between the canonically conjugate operators of the harmonic oscillator. The same argument also applies to $\Sigma_n$. As a result, the symplectic matrix $\Sigma_n$ plays a fundamental role in the transformations of the physically realizable quantum systems, e.g. see [13, 19, 20]. The transfer function of the passive symplectic system [11] obeys the relation $G(i\omega)^\dagger G(i\omega) = G(i\omega)G(i\omega)^\dagger = I_m$ for all $\omega \in \mathbb{R}$ [9].

### III. Characterization of Dark Modes

We will refer to the following result from [7].

**Lemma 1:** $\Sigma_n v$ and $v$ are orthogonal column vectors, i.e. $(\Sigma_n v)^Tv = v^T \Sigma_n v = 0$. If $v$ is a normalized vector, i.e. $v^Tv = 1$, then $\Sigma_n v$ is a normalized vector as well.

Define a $P$-matrix as

$$P_{m,n}(X) = \begin{pmatrix} \Sigma_m X \Sigma_n \end{pmatrix}, \quad (12)$$

where $X$ is an arbitrary $2m \times 2n$ matrix. $P_{m,n}(X)$ is a $4m \times 2n$ matrix. We can prove the following lemma.

**Lemma 2:** Suppose a normalized vector $v$ is in $N(P_{m,n}(X))$, i.e.

$$P_{m,n}(X)v = \begin{pmatrix} \Sigma_m X \Sigma_n \end{pmatrix} v = 0. \quad (13)$$

Then $\Sigma_n v$ is a normalized vector in $N(P_{m,n}(X))$ as well.
Proof: $\Sigma_n v$ is normalized from Lemma 1. Using (13) we have $\Sigma_m \Sigma_n X \Sigma_n v = 0$. Hence, $-I_m X \Sigma_n v = 0$. Therefore, $\Sigma_m X \Sigma_n (\Sigma_n v) = -\Sigma_m X v = 0$ and $X (\Sigma_n v) = 0$ hold, which proves that $\Sigma_n v$ is also in the kernel of $P_{m,n}(X)$. ■

Lemma 2 implies that if a vector $v$ is in the intersection of $N(B^T)$ and $N(C)$ ($v$ is decoupled from the direct interaction with the input and output), then its symplectic transformation $\Sigma_n v$ is also in $N(B^T) \cap N(C)$. This will result in a symplectic coordinate transformation matrix $\mathcal{T}$ in generating the dark modes.

As explained in Sec. II-B, the vector $x(t)$ can be decomposed as $x(t) = (x_D^T \ x_N^T)^T$, where $x_D = (x_1 \ p_1 \ \cdots \ x_n \ p_n)^T$ is the collection of the system operators for the $n_1$ harmonic oscillators that constitute the subsystem $\mathcal{D}$, and $x_N = (x_{n+1} \ p_{n+1} \ \cdots \ x_n \ p_n)^T$ is the collection of the system operators for the $n - n_1$ harmonic oscillators that constitute the subsystem $\mathcal{N}$. Accordingly, $C$ is decomposed as $C = (C_1 \ C_2), C_1 \in \mathbb{R}^{2m \times 2n}, C_2 \in \mathbb{R}^{2m \times 2(n-n_1)}$.

Note that $x_D$ is directly coupled to the input $W(t)$ via $B_1 = \Sigma_n C^T \Sigma_m \in \mathbb{R}^{2n \times 2n}$, and to the output via $C_1$. The system Hamiltonian can be written as $H_0 = H_{int} + H_D + H_N$ with $H_{int} = x_D^T G_{int} x$ being the interaction Hamiltonian between the two subsystems, and the internal Hamiltonians of the two subsystems are written as $H_D = x_D^T G_D x$ and $H_N = x_D^T G_N x$, respectively. We decompose $G_{int} \in \mathbb{R}^{2 \times 2}$ as $G_{int} = (G_{1,1} \ G_{1,2}) \text{ with } G_{1,1} \in \mathbb{R}^{2n \times 2n}$. Furthermore, for simplicity we denote $\mathcal{P}_{m+n, n_1}(\begin{bmatrix} C_1 \\ G_{1,1} \end{bmatrix}) \in \mathbb{R}^{2(n+n_1) \times 2n}$ as $\mathcal{P}$. The rank defect of this $\mathcal{P}$ may indicate the existence of dark modes.

Lemma 3: If $\text{rank}(\mathcal{P}) = q < 2n_1$, then the system equations (7) and (10) can be transformed to

$$
\begin{align*}
&d \begin{pmatrix} x_D(t) \\ x_N(t) \end{pmatrix} \\
&= \begin{pmatrix} P_D^T \Sigma_n G_P \ \\
& \ \\
& \end{pmatrix} \begin{pmatrix} P_D^T \Sigma_n G_D P_2 \\ P_D^T \Sigma_n G_D P_1 \ \\
& \ \\
& \end{pmatrix} x(t)dt \\
&+ \begin{pmatrix} 0 \\ 0 \ \\
& \ \\
& \end{pmatrix} dW(t), \\
&dW_{out}(t) = (0 \ CP_2)x(t)dt + dW(t),
\end{align*}
$$

under a proper coordinate transformation $x = T x = (P_1 \ P_2)x$, with $x_D$ containing at least two modes.

\textbf{Proof:} We have $\text{rank}(N(\mathcal{P})) + \text{rank}(\mathcal{P}) = 2n_1$ due to the rank-nullity theorem. If $\text{rank}(\mathcal{P}) = q < 2n_1$, then $\text{rank}(N(\mathcal{P})) = 2n_1 - q > 0$. Then we can construct a transformation matrix $\mathcal{T} = (P_1 \ P_2)$ by letting

$$
P_1 = \begin{pmatrix} \cdots & v_i & \cdots \ \\
1 & \Sigma_n v_i & \vdots \ \\
0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{2n \times (2n_1-q)}
$$

with $\{v_i, \Sigma_n v_i, i = 1, 2, \cdots, (2n_1-q)/2, v_i \in \mathbb{R}^{2n_1 \times 1}\}$ being the basis vectors of $N(\mathcal{P})$. Here we have made use of Lemma 2. Note that the last 2$(n-n_1)$ rows of $P_1$ are set as 0. $v_i$ is chosen to be orthogonal to $v_j, \Sigma_n v_j$ for all $j < i$. By this construction, the column vectors of $P_1$ are mutually orthogonal. Next, we construct $P_2 \in \mathbb{R}^{2n \times (2n_2-n_1-q)}$ as $P_2 = (v_i/(2n_1-q)/2+1 \ \Sigma_n v_i/(2n_1-q)/2+1 \ \cdots \ v_n \ \Sigma_n v_n)$ which is composed of mutually-orthogonal normalized column vectors which are orthogonal to all the column vectors of $P_1$. Here we can choose $P_2$ to have this form based on Lemma 2. Then we can verify that $\mathcal{T} \in \text{Sp}(2n) \cap O(2n)$, where $\text{Sp}(2n)$ is the symplectic group of $2n \times 2n$ matrices and $O(2n)$ is the orthogonal group of $2n \times 2n$ matrices. The transformed coefficient matrices in (11) can be calculated using the relations $CP_1 = 0, B^T P_1 = 0, P_D^T G_{int} = 0$ as well as $P_D^T \Sigma_n G_{int} = 0$. Also note that $P_D^T G_N = 0$ and $P_D^T \Sigma_n G_N = 0$ are automatically satisfied because the last 2$(n-n_1)$ rows of $P_1$ are zero. Therefore, we have $P_D^T A P_1 = P_D^T \Sigma_n G_P P_2 = P_D^T \Sigma_n G_D P_2$ and $P_D^T A P_1 = P_D^T \Sigma_n G_D P_1$. The resulting system equations are thus given by (14).

The modes $x_D$ in (14) are not directly coupled to the noise. However, $x_D$ may be indirectly coupled to the noise via $x_B$ and $x_D$. Therefore, dark modes can only be generated after we remove the coupling between $x_D$ and $x_B$. This can be done by engineering the Hamiltonian $H_D$ of the subsystem $\mathcal{D}$. The following theorem provides a sufficient condition for the existence of dark modes.

\textbf{Theorem 1:} Suppose $\text{rank}(\mathcal{P}) = q < 2n_1$ and the transformed system is given by (14). If the condition

$$
P_D^T G_D P_2 = 0
$$

is satisfied, then $x_D$ are dark modes and the system equations become

$$
d \begin{pmatrix} x_D(t) \\ x_B(t) \\ x_D(t) \end{pmatrix} = \begin{pmatrix} P_D^T \Sigma_n G_P & 0 \\
& \ \\
& \end{pmatrix} \begin{pmatrix} 0 \\ P_D^T \Sigma_n G_D P_1 \\
& \ \\
& \end{pmatrix} x(t)dt \\
+ \begin{pmatrix} 0 \\ 0 \ \\
& \ \\
& \end{pmatrix} dW(t),
$$

$$
dW_{out}(t) = (0 \ CP_2)x(t)dt + dW(t). \tag{17}
$$

\textbf{Proof:} According to Lemma 3 the condition (16) can be explicitly written as

$$
(v_i^T 0) G_D v_i/(2n_1-q)/2+j = 0,
$$

$$
-(v_i^T 0) G_D v_i/(2n_1-q)/2+j = 0,
$$

$$
(v_i^T 0) G_D v_i/(2n_1-q)/2+j = 0,
$$

$$
-(v_i^T 0) G_D v_i/(2n_1-q)/2+j = 0, \tag{18}
$$

for $i = 1, \cdots, (2n_1-q)/2, j = 1, \cdots, (2n-2n_1+q)/2$. The elements of the matrix $P_D^T \Sigma_n G_D P_2$ are expressed as

$$
(v_i^T 0) G_D v_i/(2n_1-q)/2+j,
$$

$$
(v_i^T 0) G_D v_i/(2n_1-q)/2+j,
$$

$$
(v_i^T 0) G_D v_i/(2n_1-q)/2+j,
$$

$$
(v_i^T 0) G_D v_i/(2n_1-q)/2+j, \tag{19}
$$

for $i = 1, \cdots, (2n_1-q)/2, j = 1, \cdots, (2n-2n_1+q)/2$. Hence, we can conclude that $P_D^T \Sigma_n G_D P_2 = 0$ by (18). Similarly, we can prove $P_D^T \Sigma_n G_D P_1 = 0$. ■

Condition (16) proposes a Hamiltonian engineering problem. Additionally, it is straightforward to identify the bright modes using (17). $x_B$ contains $q$ modes which are linear combinations of the operators in $x_D$. If these modes are
coupled to the subsystem $N$ via the interaction terms in $P^T D A P_2$, then they are bright modes.

We can also consider the special case $C_1 = 0$. In this case, the sufficient conditions for the existence of dark modes are simplified as $\text{rank}(P_{n,n_1}(G^T_{1,int})) < 2n_1$ and $\{16\}$.

The dark modes are governed by the dynamical equation $\dot{x}_D = P^T_D \Sigma_N GP_1 x_D$. If $P^T_D \Sigma_N GP_1 = 0$, then $\dot{x}_D = 0$ and the dark modes are invariant. This fact can be summarized as the following theorem.

**Theorem 2:** Suppose $\text{rank}(P) = q < 2n_1$ and the transformed system is given by $\{14\}$. If the condition

\[
P^T_D G_D = 0
\]

is satisfied, then the dark modes $x_D$ are invariant.

**Remark 1:** The results of this section are closely related to the Popov-Belevitch-Hautus (PBH) controllabilityobservability criterion. Consider an equivalent statement of the PBH observability criterion: $(C, A)$ is unobservable if and only if there is a $v \neq 0$ with $Au = \lambda v$ and $Cv = 0$. Using $v \in N(C)$ we have $\Sigma_N Gv = \lambda v$, which leads to $v^T G = -v^T G \Sigma_N$. Similarly, if $v^T A = \mu v^T$, then $v^T B = 0$ for the same $v \neq 0$, then the unobservable mode is also uncontrollable and we have $-v^T \Sigma_N G = \mu v^T$ using $v^T B = 0$. Suppose $P_1, P_2$ are constructed using the same procedure as Lemma $\{3\}$. Then we have $v^T \Sigma_N GP_2 = -v^T \mu P_2$ and $P_2 \Sigma_N G = \lambda P_2 v$ since $v$ is orthogonal to the columns vectors of $P_2$. Using the coordinate transformation we can prove that the eigenvector $v$ corresponds to an uncontrollable and unobservable mode of the system. Here, $v \in N(B^T) \cap N(C)$ is equivalent to a rank-defect condition, and $\Sigma_N Gv = \lambda v, -v^T \Sigma_N G = \mu v^T$ are conditions on the system Hamiltonian. So the PBH conditions combined with the direct method of this paper can be used to characterize linear DF modes. Furthermore, imposing the additional requirement that the dark mode is in the subsystem $D$, then $C$ should be replaced with $C_1$ and the interaction between the dark mode and the subsystem $N$ should be eliminated. Using this approach we will arrive at the sufficient conditions that are similar to the ones of Theorem $\{1\}$.

This connection to PBH criterion also suggests that the direct method of this paper can be used to characterize and engineer a mode that is only uncontrollable and unobservable from some specific inputs and outputs. The details of this application is presented in the Appendix.

So far we have obtained a theory to characterize general dark modes. As shown in Theorem $\{1\}$ the existence of dark modes is conditioned in terms of the environmental couplings and the system Hamiltonian. In the next section, we consider the synthesis of dark modes through engineering the system-environment couplings followed by engineering the Hamiltonian.

**IV. Engineering the System-Environment Couplings and Hamiltonian**

Consider

\[
\mathcal{P}_{m,n}(C) = \left( \begin{array}{c} \Sigma m C \Sigma n \\ C \end{array} \right),
\]

where $C \in \mathbb{R}^{2m \times 2n}$ is the coefficient matrix for the environmental couplings associated with the subsystem $D$. We assume that $\{21\}$ is full column rank. As we have proven, no dark modes exist in this case.

Firstly, we demonstrate that adding couplings alone cannot reduce the column rank of the matrix. With the additional couplings, the coefficient matrix becomes

\[
C' = \left( \begin{array}{c} C \\ C_e \end{array} \right),
\]

where $C_e \in \mathbb{R}^{2m' \times 2n}$ is associated with the $m'$ additional inputs. The updated $\mathcal{P}$-matrix for this system is thus $\mathcal{P}_{m+m',n}(C')$, the column rank of which is still $2n$ given that $\mathcal{P}_{m,n}(C)$ is full column rank.

For this reason, it is necessary to increase the dimension of the system. We consider three basic types of interconnections for increasing the dimension of the system, namely, cascade, direct coupling and coherent feedback $\{21, 22, 23, 24\}$.

**A. Cascade**

Suppose the original system and the additional system are defined by the triplets $(A_1, B_1, C_1)$ and $(A_2, B_2, C_2)$, respectively. Moreover, we assume that $C_1, C_2 \in \mathbb{R}^{2m \times 2n}$. To form the cascade, the output of the original system is taken as the input to the additional system, which can be modelled as

\[
\begin{align*}
\dot{x}_1(t) &= A_1 x_1(t) dt + B_1 dW_1(t), \\
\dot{x}_2(t) &= A_2 x_2(t) dt + B_2 dW_2(t),
\end{align*}
\]

and $\{23\}$

The above equations can be rewritten as

\[
\begin{align*}
\dot{x}_e(t) &= \left( \begin{array}{c} A_1 \\ B_2 C_1 \\ A_2 \\ B_1 \\ B_2 \\ B_2 \\ B_2 \\ B_2 \end{array} \right) x_e(t) dt + \left( \begin{array}{c} B_1 \\ B_2 \\ B_2 \end{array} \right) dW_1(t), \\
\dot{W}_{2,\text{out}} &= \left( C_1, C_2 \right) x_e(t) dt + dW_2(t),
\end{align*}
\]

where we have defined $x_e = (x_1 \ x_2)^T$. So we have

\[
\mathcal{P}_{m,2n}(\{C_1, C_2\}) = \left( \begin{array}{c} \Sigma m C_1 \Sigma n \\ \Sigma m C_2 \Sigma n \\ C_1 \\ C_2 \end{array} \right),
\]

whose column rank is $2n$ if we let $C_1 = C_2$. Therefore, it is possible to generate $4n - 2n = 2n$ dark modes which are not influenced by the couplings associated with $(C_1, C_2)$.

It is easy to see that a weaker sufficient condition for $\mathcal{P}_{m,2n}(\{C_1, C_2\})$ to be not full column rank is that at least one column vector of $\left( \begin{array}{c} \Sigma m C_2 \Sigma n \\ C_1 \\ C_2 \end{array} \right)$ lies in the column space of $\left( \begin{array}{c} \Sigma m C_1 \Sigma n \\ C_1 \end{array} \right)$.

**B. Direct coupling**

The original system and the additional system are defined by the triplets $(A_1, B_1, C_1)$ and $(A_2, B_2, C_2)$, with $C_1, C_2 \in \mathbb{R}^{2m \times 2n}$. The direct coupling is implemented by adding an interaction Hamiltonian $H_{\text{int}} = x^T G_{\text{int}} x$ between the two systems. If both $\mathcal{P}_{m,n}(C_1)$ and $\mathcal{P}_{m,n}(C_2)$ are full column rank, then we have

\[
\mathcal{P}_{2m,2n}(\left( \begin{array}{c} C_1 \\ 0 \\ 0 \\ C_2 \end{array} \right)) = \left( \begin{array}{c} \mathcal{D}_{m,n}(C_1) \\ 0 \\ \mathcal{D}_{m,n}(C_2) \end{array} \right),
\]

which is still full column rank. No dark modes exist in the augmented system.
C. Coherent feedback

We consider two types of coherent feedback. The first type is modelled as

\[
\begin{align*}
\dot{x}_1(t) &= A_1 x_1(t)dt + B_1 dW_1(t) + B_2 dW_2(t), \\
\dot{x}_2(t) &= A_2 \dot{x}_2(t)dt + B_3 dW_3(t), \\
\dot{W}_{1,\text{out}} &= C_i x_i(t)dt + dW_i(t), \quad i = 1, 2, \\
\dot{W}_{3,\text{out}} &= C_3 x_3(t)dt + dW_3(t),
\end{align*}
\]

(27)

where the original system and the additional system are defined by the triplets \((A_1, B_1, B_2), (C_1, C_2, t), A_1 \in \mathbb{R}^{2n \times 2n}, B_1, B_2 \in \mathbb{R}^{2n \times 2m}, C_1, C_2 \in \mathbb{R}^{2m \times 2n}\) and \((A_2, B_3, C_3), A_2 \in \mathbb{R}^{2n \times 2n}, B_3 \in \mathbb{R}^{2n \times 2m}, C_3 \in \mathbb{R}^{2m \times 2n}\), respectively. The additional system serves as the coherent controller, which processes the output \(dW_{1,\text{out}}(t)\) of the original system and feeds its output back to the original system. To close the loop we let \(dW_3 = dW_{1,\text{out}}\) and \(dW_2 = dW_{3,\text{out}}\).

The closed-loop system is expressed as

\[
\begin{align*}
\dot{x}_c(t) &= \left( \begin{array}{ccc}
A_1 + B_2 C_1 & B_2 C_3 \\
B_3 C_1 & A_2
\end{array} \right) x_c(t)dt \\
&\quad + \left( \begin{array}{cc}
B_1 + B_2 \\
B_3
\end{array} \right) dW_1(t), \\
\dot{W}_{2,\text{out}} &= \left( \begin{array}{c}
C_1 + C_2 \\
C_3
\end{array} \right) x_c(t)dt + dW_1(t).
\end{align*}
\]

(28)

Similar to the cascade case, if we let \(C_3 = C_1 + C_2\), then the matrix \(P_{m,2n}((C_1 + C_2 \ C_3))\) for the closed-loop system does not have full column rank.

The second type of closed-loop system is the cross feedback between two systems. In this case, the additional system is defined by \((A_2, B_3, C_4, (C_3, C_4)), B_3, B_4 \in \mathbb{R}^{2n \times 2m}, C_3, C_4 \in \mathbb{R}^{2m \times 2n}\) with two inputs and two outputs. The system equations are given by

\[
\begin{align*}
\dot{x}_1(t) &= A_1 x_1(t)dt + B_1 dW_1(t) + B_2 dW_2(t), \\
\dot{x}_2(t) &= A_2 x_2(t)dt + B_3 dW_3(t) + B_4 dW_4(t), \\
\dot{W}_{1,\text{out}} &= C_i x_i(t)dt + dW_i(t), \quad i = 1, 2, \\
\dot{W}_{3,\text{out}} &= C_3 x_3(t)dt + dW_3(t), \\
\end{align*}
\]

(29)

The cross feedback is realized by letting \(dW_3 = dW_{1,\text{out}}\) and \(dW_2 = dW_{3,\text{out}}\), which transform the system equations to

\[
\begin{align*}
\dot{x}_c(t) &= \left( \begin{array}{cc}
A_1 & B_2 C_4 \\
B_3 C_1 & A_2
\end{array} \right) x_c(t)dt \\
&\quad + \left( \begin{array}{cc}
B_1 & B_2 \\
B_3 & B_4
\end{array} \right) \left( \begin{array}{c}
\dot{W}_1(t) \\
\dot{W}_4(t)
\end{array} \right), \\
\dot{W}_{2,\text{out}} &= \left( \begin{array}{c}
C_2 \\
C_4
\end{array} \right) x_c(t)dt + dW_4(t), \\
\dot{W}_{3,\text{out}} &= \left( \begin{array}{c}
C_1 \\
C_3
\end{array} \right) x_c(t)dt + dW_1(t).
\end{align*}
\]

(30)

By (30), if we let \(C_3 = C_1\) and \(C_4 = C_2\), the rank-defect condition of Theorem (36) is satisfied and dark modes may exist.

Example 1: In this example we consider the cross feedback design using two linear systems. The system operators are denoted as \((x_i, p_i), \ i = 1, 2\), where \(x_i, p_i\) are position and momentum operators of the harmonic oscillators, respectively. Each system has two inputs and two outputs. The four coupling operators \(L_i = \sqrt{\kappa}(x_1 + ip_1)/\sqrt{2}, \ i = 1, 2, L_j = \sqrt{\kappa}(x_2 + ip_2)/\sqrt{2}, \ j = 3, 4\) have equal coupling strength. We can obtain the following coefficient matrices for (29):

\[
\begin{align*}
B_1 &= B_2 = B_3 = B_4 = -\sqrt{\kappa} \left( \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right), \\
C_1 &= C_2 = C_3 = C_4 = -B_1^T.
\end{align*}
\]

(31)

Using the closed-loop equation (30), it is straightforward to verify that \(x_1 - x_2\) and \(p_1 - p_2\) are dark modes which are decoupled from \(L_1\) and \(L_4\) if

\[
P_1^T \left( \begin{array}{cc}
A_1 \\
B_2 C_1 \\
A_2
\end{array} \right) P_2 = 0
\]

(32)

holds for

\[
P_1 = \frac{1}{\sqrt{2}} \left( \begin{array}{c}
I_2 \\
-I_2
\end{array} \right), \quad P_2 = \frac{1}{\sqrt{2}} \left( \begin{array}{c}
I_2 \\
I_2
\end{array} \right).
\]

(33)

This condition leads to

\[
A_1 = A_2,
\]

(34)

or \(G_1 = G_2\). Therefore, a sufficient condition for the existence of dark modes is that the Hamiltonians of the two systems are the same. Eq. (34) shows that cross feedback provides a robust realization for dark modes. The structures and parameters of the two linear systems can be uncertain, and dark modes can still be fabricated for cross feedback. Also, it is worth mentioning that in this case we have generated the dark modes under the condition that each system is coupled to full-rank noises \(\{B_i\}\).

D. Hamiltonian engineering

We consider the solution \(G\) to the Hamiltonian engineering problems \(P_1^T G P_2 = 0\) and \(P_1^T G = 0\) for the generation of dark modes and invariant modes.

First, \(G = I\) and \(G = 0\) are special solutions to \(P_1^T G P_2 = 0\), and \(G = 0\) is a solution to \(P_1^T G = 0\). Also we have the following result.

Theorem 3: If \(R(P_2)\) is invariant under \(G\), then \(G\) is a solution to \(P_1^T G P_2 = 0\). The general solution to \(P_1^T G = 0\) is given by

\[
G = (I - P_1^T) Z (I - P_1^T)
\]

(35)

with \(Z\) being an arbitrary matrix.

Proof: The fact that \(R(P_2)\) is invariant under \(G\) implies \(GP_2 = P_2 M\) for a matrix \(M\). Then \(P_1^T G P_2 = 0\) follows since \(P_1^T P_2 = 0\).

The general solution to \(P_1^T G = 0\) is given by

\[
G = (I - P_1^T)^+ Z (I - (P_1^T)^+ P_1^T)
\]

(36)

with \(Z\) being an arbitrary matrix. It is easy to verify that \((P_1^T)^+ = P_1\).

V. APPLICATION TO THE DARK MODES OF A QUANTUM OPTOMECHANICAL SYSTEM

In this section, we apply our theoretical results to the analysis of optomechanical dark modes and bright modes. Optomechanical systems are conventionally used for the fundamental study of light-matter interaction. Recently, they have
Hence, we have the bright mode is used to capture the photons, while the dark coupling is to switch between dark and bright modes. Here, an effective coupling between the optical modes and at the cavity of different optical modes. Therefore, the formation of stable one way to solve this problem is to exploit the optomechanical oscillator. However, the mechanical damping will undermine the mediated coupling and cause losses to the optical modes.

This optomechanical setup is proposed to mediate the coupling between the two optical modes using the mechanical oscillator. However, the mechanical damping will undermine the mediated coupling and cause losses to the optical modes. One way to solve this problem is to exploit the optomechanical dark modes [12]. The dark modes are decoupled from the mechanical oscillator. The dark modes are the superposition of different optical modes. Therefore, the formation of stable dark modes is the result of the interaction and energy transfer between the optical fields. This process can be used to mediate an effective coupling between the optical modes and at the same time minimize the losses [12]. Another way to mediate coupling is to switch between dark and bright modes. Here, the bright mode is used to capture the photons, while the dark mode is used to store them.

For the proposed optomechanical system, we have

\[ C = √κ(0_2 \ 0_2 \ I_2). \]  

(37)

Hence, we have \( C_1 = √κ(0_2 \ 0_2) \) for the engineering of dark modes that consist of optical modes only. The interaction Hamiltonian between the cavity and the mechanical oscillator is given by \( H_{int} = λ_1a_1^d a_1 x_3 + λ_2a_2^d a_2 x_3 \), with \( a_i = (x_i + ip_i)/√2 \), \( i = 1, 2 \). Applying a standard linearization procedure [26] to \( H_{int} \), we can obtain \( G_{int} \) as

\[
\begin{pmatrix}
0_2 & 0_2 & γ_1 0 0 \\
0_2 & 0_2 & 0_0 & 0_2 \\
γ_1 0 & γ_2 0 & 0 0 & 0 0 \\
0 0 & 0 0 & 0 0 & 0 0 \\
\end{pmatrix},
\]  

(38)

where \( γ_i, i = 1, 2 \) is determined by \( κ \) and \( λ_i \). The \( P \)-matrix as defined in Theorem 1 is given by

\[
\begin{pmatrix}
0_4 \\
γ_1 0 & γ_2 0 \\
0 0 & 0 0 \\
0 0 & 0 0 \\
0 0 & 0 0 \\
\end{pmatrix},
\]  

(39)

which is not full column rank. Therefore we can obtain

\[
P_1 = \begin{pmatrix} a & 0 \\ 0 & a \\ b & 0 \\ 0 & b \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} eI_2 & 0 \\ fI_2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},
\]  

(40)

with the condition \( aγ_1 + bγ_2 = 0 \). \( (e \ f)^T \) and \( (a \ b)^T \) are required to be orthogonal. We then choose \( G_D \) such that \( P_1^T G_D P_2 = 0 \). Finally, we can obtain the optomechanical dark modes as \( a x_1 + b x_2 = \frac{γ_1}{√(γ_1^2 + γ_2^2)} x_1 - \frac{γ_2}{√(γ_1^2 + γ_2^2)} x_2 \) and its canonical conjugate mode \( a p_1 + b p_2 = \frac{γ_1}{√(γ_1^2 + γ_2^2)} p_1 - \frac{γ_2}{√(γ_1^2 + γ_2^2)} p_2 \). It has been experimentally verified in [12] that these two modes are decoupled from the mechanical dissipation.

Note that any \( G_D \) satisfying \( P_1^T G_D P_2 = 0 \) will generate the above dark modes. Here we assume a specific realization of the Hamiltonian \( H_D \) of the optical harmonic oscillators as \( H_D = \frac{p_1^2}{m_1} + m_1ω_1^2 x_1^2 + \frac{p_2^2}{m_2} + m_2ω_2^2 x_2^2 \). It is straightforward to verify that \( P_1^T G_D P_2 = 0 \) if and only if \( m_1 = m_2 \) and \( ω_1 = ω_2 \). In other words, the dark modes exist if the two optical modes have the same energy. Experimentally, this can be realized by driving the optical modes with different frequencies \( ω_{l1} = ω_{c1} - ω_m \) and \( ω_{l2} = ω_{c2} - ω_m \), where \( \{ω_{c1}, i = 1, 2\} \) are the cavity resonance frequencies.

Using [17], we have \( dx_D(t) = P_1^T Σ_n GP_1 x_D(t) dt \) and

\[
d \begin{pmatrix} x_B(t) \\ x_d(t) \end{pmatrix} = \begin{pmatrix} 0 & P_2^T AP_2 x(t) dt \\ P_2^T Σ_n C^T Σ_m dW(t) \end{pmatrix},
\]  

(41)

with

\[
P_2^T AP_2 = \begin{pmatrix}
0 & 1 \frac{1}{m_1} & 0 & 0 \\
0 & eγ_1 - fγ_2 & 0 & 0 \\
eγ_1 - fγ_2 & 0 & -m_3ω_3^2 & 0 \\
m_3ω_3^2 & 0 & -\frac{e}{m_2} & -\frac{f}{m_2} \end{pmatrix},
\]  

(42)
where we have $e = \frac{\gamma_1}{\sqrt{\gamma_1^2 + \gamma_2^2}}$ and $f = \frac{\gamma_2}{\sqrt{\gamma_1^2 + \gamma_2^2}}$. According to (42), the bright mode $e x_1 + f x_2$ is directly coupled to the mechanical mode, while $e x_1 - f x_2$ is an indirect-coupled bright optical mode.

VI. CONCLUSION

We have developed a direct method for the characterization and synthesis of dark modes. The key is to ensure that the interaction Hamiltonian between the subsystems does not affect the dark modes. Sufficient conditions are derived in terms of environmental couplings and the system Hamiltonian, which provides a straightforward and tractable way to engineer dark modes.

APPENDIX

UNCONTROLLABLE AND UNOBSERVABLE MODES FROM A SPECIFIC INPUT AND OUTPUT

In order to decouple modes from a specific input and output, we may decompose $C$ as $C = (C_1 T \ C_2 T)^T$. Here $C_1 \in \mathbb{R}^{2n_1 \times 2n}$ is associated with $n_1$ coupling operators from $n_1$ inputs, and $C_2 \in \mathbb{R}^{(m-n_1) \times 2n}$ is associated with the other $m - n_1$ inputs. The modes are required to decouple from the given $n_1$ inputs and outputs. If $\text{rank}(\sum_{n_1} C_1 \Sigma_n) = q < 2n$, we can construct a transformation matrix $T = (P_1 \ P_2) \in \mathbb{R}^{2n \times 2n}$, where $P_1 \in \mathbb{R}^{2n \times 2n_q}$ is chosen as $P_1 = (v_1 \Sigma_n v_1 \cdots v_{2n_q-1} / 2 \Sigma_n v_{2n_q-2} / 2)$ with $\{v_1, \Sigma_n v_1\}$ being mutually-orthogonal basis vectors of $N(\sum_{n_1} C_1 \Sigma_n)$. The $q$ column vectors of $P_2 \in \mathbb{R}^{2n \times q}$ are chosen to be mutually-orthogonal normalized vectors which are orthogonal to the column vectors of $P_1$. The coordinate transformation yields

$$
\begin{align*}
\dot{d} & \left( \begin{array}{c} x_{d1} \\ x_{d2} \end{array} \right) = \begin{pmatrix} P_1^T A_1 P_1 & P_1^T A_2 P_2 \\ P_2^T A_1 P_1 & P_2^T A_2 P_2 \end{pmatrix} \left( \begin{array}{c} x_{d1} \\ x_{d2} \end{array} \right) dt \\
& + \begin{pmatrix} P_1^T \Sigma_1 C_1 \Sigma_n \Sigma_n C_1 T \Sigma_m-n_1 \\ P_1^T \Sigma_1 C_1 \Sigma_n \Sigma_n C_1 T \Sigma_m-n_1 \end{pmatrix} \left( \begin{array}{c} x_{d1} \\ x_{d2} \end{array} \right) dt + dW, \\
\end{align*}
\right.
\]

and

$$
dW_{out} = \begin{pmatrix} 0 \\ C_1 P_2 \\ C_2 P_2 \end{pmatrix} \left( \begin{array}{c} x_{d1} \\ x_{d2} \end{array} \right) dt + dW. \tag{43}
$$

Theorem 4: A sufficient condition for the modes $x_{d1}$ to be decoupled from the $n_1$ inputs is $P_1^T A_2 P_2 = 0$.

Proof: First, we decompose $dW$ and $dW_{out}$ as $dW_{d1}$, $dW_{d2}$, $dW_{d1,out}$, and $dW_{d2,out}$, where $dW_{d1}$ and $dW_{d1,out}$ are associated with $C_1$. Using this decomposition we have

$$
\begin{align*}
& \dot{d}x_{d1} = P_1^T A_1 P_1 \dot{x}_{d1} dt + P_1^T \Sigma_1 C_1 T \Sigma_m-n_1 dW_{d2}, \\
& \dot{dW}_{d1,out} = C_1 P_2 \dot{x}_{d2} dt + dW_{d1}, \tag{44}
\end{align*}
$$

given that $P_1^T A_2 P_2 = 0$ is satisfied. According to (44), $x_{d1}$ are decoupled from the input $W_{d1}$, and the corresponding output $W_{d1,out}$ is decoupled from $x_{d1}$ as well. Thus we have proven that $x_{d1}$ is neither controllable nor observable from the $n_1$ inputs.

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