COMPACTIFICATIONS OF UNSTABLE NOBELING SPACES

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Abstract. We construct an embedding of a Nöbeling space $N^n_{n-2}$ of codimension 2 into a Menger space $M^n_{n-2}$ of codimension 2. This solves an open problem stated by R. Engelking in 1978 [5] in codimension 2.

1. Introduction

A Nöbeling space $N^n_m$ is a subset of $\mathbb{R}^n$ consisting of points with at most $m$ rational coordinates. If $n \geq 2m + 1$, then $N^n_m$ is homeomorphic to $\nu^n = N^n_{2n+1}$, the universal $n$-dimensional Nöbeling space. The space $\nu^n$ is considered to be an $n$-dimensional analogue of the Hilbert space $\ell^2$. In particular, it is characterized by the following theorem [3, 1, 2, 9, 11, 12], which is in direct analogy to the characterization of the Hilbert space given by Toruńczyk [14].

Theorem 1.1. An $n$-dimensional Polish space is a Nöbeling manifold if and only if it is an absolute neighborhood extensor in dimension $n$ that is strongly universal in dimension $n$.

For $n < 2m + 1$ we call $N^n_m$ an unstable Nöbeling space. A characterization of unstable Nöbeling spaces has been long sought [4]. The problem was stated again in a recent work of Gabai [7], where it was conjectured that boundaries at infinity of certain mapping class groups are homeomorphic to unstable Nöbeling spaces. The problem of characterizing unstable Nöbeling spaces is open.

In 1929 Menger proposed an axiomatic characterization of covering dimension $\text{dim}$ [10]. In [8] Hurewicz and Wallman described a problem whether these axioms characterize $\text{dim}$ as the most important open problem in dimension theory. Today, it is still not known whether $\text{dim}$ satisfies these axioms. By a difficult theorem of Štan’ko [13], the problem is now reduced to proving the following conjecture.

Conjecture 1.2. Every $m$-dimensional subset of $\mathbb{R}^n$ has an $m$-dimensional compactification that can be embedded into $\mathbb{R}^n$.

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Since $\nu^n \subset \mathbb{R}^{2n+1}$ is an universal space for $n$-dimensional separable metric spaces, Conjecture 1.2 is true for spaces of dimension $m$ such that $2m + 1 \leq n$. The problem is hard in unstable dimensions, i.e. dimensions $m$ such that $2m + 1 > n$. If $A$ is an $m$-dimensional subset of $\mathbb{R}^n$, then we say that $A$ is of codimension $n - m$. Conjecture 1.2 is trivial for codimension 0. In codimension 1, it is a theorem of Sierpiński [5]. The status of the conjecture is not known for codimension 2. Since for $n \leq 3$ codimension 2 is a stable case, the lowest dimensions where the conjecture is open are $m = 2$ and $n = 4$.

The main part of conjectured characterization of unstable Nöbeling spaces is the following conjecture [4].

**Conjecture 1.3.** Every $m$-dimensional subset of $\mathbb{R}^n$ can be embedded into $N^n_m$.

It raises the following question, which was stated by Engelking in 1978 [5]:

Does $N^n_m$ have $m$-dimensional compactification that can be embedded into $\mathbb{R}^n$?

In this paper we answer this question in affirmative in codimension 2. We prove the following theorem.

**Theorem 1.4.** For each $n$ the Nöbeling space $N^n_{n-2}$ has an $(n-2)$-dimensional compactification that embeds into $\mathbb{R}^n$.

2. Preliminaries

In this section we review some conditions for a set $A \subset \mathbb{R}^n$ to be of codimension $\geq 2$. Recall a necessary and sufficient condition for a subset $X \subset \mathbb{R}^n$ to be of codimension $\geq 1$.

**Lemma 2.1.** Let $X \subset \mathbb{R}^n$. We have $\dim X \leq n - 1$ if and only if for each open non-empty $U \subset \mathbb{R}^n$ the set $U \setminus X$ is non-empty.

There is an analogous condition for compact subsets of codimension $\geq 2$. This fails for non-compact subsets: recall the Sitnikov example [6] of a 2-dimensional subset $X$ of $\mathbb{R}^3$ with the property that for each open and connected $U \subset \mathbb{R}^3$ the set $U \setminus X$ is non-empty and connected.

**Lemma 2.2.** Let $X \subset \mathbb{R}^n$ be a compact subset. We have $\dim X \leq n - 2$ if and only if for each open non-empty connected $U \subset \mathbb{R}^n$ the set $U \setminus X$ is non-empty and connected.
Note that analogous condition fails in codimension 3: Antoine necklace is a 0-dimensional compact subset of $\mathbb{R}^3$ with non-simply connected complement.

In the proof we will use the following position property of $N^n_k$.

**Definition.** A space $X$ is $k$-connected if its homotopy groups of dimensions less than $k$ vanish.

**Lemma 2.3.** Let $U$ be an open $k$-connected subset of $\mathbb{R}^n$. Let $m = n - k$ and let $N^n_m$ be a Nöbeling space of codimension $k$. Let $l < k$.

1. $\pi_l(U \setminus N^n_m) = 0$.
2. If $2l + 1 < n$ and $\varphi: S^l \to U \setminus N^n_m$ is an embedding, then there exists a map $\Phi: B^{l+1} \to U \setminus N^n_m$ such that $\Phi|S^l = \varphi$.

In codimension 2 we have the following additional property.

3. The map $\Phi$ can be chosen to be an embedding.

**Definition.** We say that a set $A \subset \mathbb{R}^n$ is $k$-tame if it satisfies conditions (1), (2) and (3) of lemma 2.3.

### 3. A LIMIT THEOREM

We begin by stating a result that states sufficient conditions for a limit of a sequence of embeddings into $\mathbb{R}^n$ to be an embedding into $\mathbb{R}^n$. The key idea is to require that the inverses are uniformly continuous.

#### 3.1. The Game.

Let $X$ be a metric space and let $Y$ be a complete space. Let $f_0: X \to Y$ be an embedding such that $f_0^{-1}$ is uniformly continuous. Let $Y_0 = f_0(X)$. Consider the following infinite two-player game. Player I starts and the players alternate moves under the following rules for move number $k \geq 1$.

1. Player I moves by selecting $\varepsilon_k > 0$.
2. Player II moves by selecting $g_k: Y_{k-1} \to Y$ such that
   a. $g_k$ is an embedding that is $\varepsilon_k$-close to the identity
   b. $g_k^{-1}$ is uniformly continuous.

We let $f_k = g_k \circ f_{k-1}$ and $Y_k = g_k(Y_{k-1}) = f_k(X)$.

Player I wins if $\lim f_k: X \to Y$ exists and is an embedding of $X$ into $Y$.

**Theorem 3.1.** Player I has a winning strategy.

**Proof.** Fix $k \geq 1$. By the rules of the game, $f_{k-1}$ is an embedding such that $f_{k-1}^{-1}: X_{k-1} \to X$ is uniformly continuous. Let $\delta_k > 0$ such that if
y, y′ ∈ X_{k−1} and d_Y(y, y′) < δ_k, then d_X(f_{k−1}^{-1}(y), f_{k−1}^{-1}(y′) < \frac{1}{2^k}. We let

ε_k = \frac{1}{2^k} \min\{1, δ_k\}.

Since Y is complete and f_k is a Cauchy sequence, f = \lim f_k exists and is continuous.

Let x, x′ ∈ X and let k such that d_X(x, x′) > \frac{1}{2^k}. Let C = \prod_{m≥1}(1−\frac{1}{2^m}). We have C > 0. We will prove that

d_Y(f(x), f(x′)) ≥ C \cdot δ_k.

This inequality implies both that f is one-to-one and that f^{-1} is (uniformly) continuous.

By the choice of δ_k, we have d_Y(f_{k−1}(x), f_{k−1}(x′)) ≥ δ_k. Since g_k is \frac{δ_k}{2^k}-close to the identity,

d_Y(f_k(x), f_k(x′)) ≥ d_Y(f_{k−1}(x), f_{k−1}(x′))−\frac{2δ_k}{2^k} ≥ (1−\frac{1}{2^k})d_Y(f_{k−1}(x), f_{k−1}(x′)).

Therefore

d_Y(f(x), f(x′)) ≥ d_Y(f_{k−1}(x), f_{k−1}(x′)) \cdot \prod_{m≥k−1}(1−\frac{1}{2^m}) ≥ δ_k \cdot C.

3.2. The Moves. In this section we describe two constructions that we will use to play The Game as Player II. Since Player I wins the game, in the next section we’ll use these moves to construct an embedding that proves the Main Theorem of the paper.

3.2.1. Straightening Move.

Construction 3.2. Let A be an 1-tame subset of \mathbb{R}^n of finite diameter. Assume that n > 3. Let L ⊂ \mathbb{R}^n 1-dimensional hyperplane (a line). Let ε > 0. We construct a homeomorphism φ: \mathbb{R}^n → \mathbb{R}^n such that

(1) φ is the identity on a complement of a compact subset of \mathbb{R}^n.
(2) φ is ε-close to the identity.
(3) \text{im } φ \cap L = \emptyset.

Proof. Triangulate L into segments and two half-lines in such a way that 0-skeleton of L misses A. Let τ(L) denote the triangulation. Because diameter of A is finite, we may assume that both half lines are disjoint from the closure of A. Also we may assume that segments in τ(K) have lengths smaller than 1/2ε. For each segment S ∈ τ(L) let C_S denote a cylindrical neighborhood of Int S in \mathbb{R}^n such that diam C_S < ε and S connects centers of opposite bases of cylinder C_S. Let S′ denote an embedded arc in C_S that connects centers of opposite bases of C_S and that is disjoint from A. It exists, because
A is 1-tame. Since $n > 3$ we may unknot $S'$ in $C_S$ by a homeomorphism $h : C_S \to C_S$ that is the identity on $\partial C_S$ and that maps $S'$ to onto $S$. We define $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ by a formula $\varphi_{|C_S} = h_S$ on each $C_S$ and let it be the identity on the complement. Because diameter of each $C_S$ is smaller than $\varepsilon$, $\varphi$ is $\varepsilon$-close to the identity. Because support of $\varphi$ is compact, $\varphi^{-1}$ is uniformly continuous. By the assumption that $2m + 1 \leq n$ we can unknot the image of $\varphi$. We are done. □

3.2.2. Push Away Move.

Construction 3.3. Let $L$ be an $m$-dimensional hyperplane in $\mathbb{R}^n$. Let $\varepsilon > 0$. We construct an embedding $\psi : \mathbb{R}^n \setminus L \to \mathbb{R}^n \setminus L$ such that

1. $\psi^{-1}$ is uniformly continuous.
2. $\psi$ is $\varepsilon$-close to the identity.
3. $\text{Cl}(\text{im} \psi) \cap L = \emptyset$.

Proof. Let $\mathbb{R}^n = L \times N$, where $N$ is $(n - m)$-dimensional hyperplane that is normal to $L$. Define $\psi : L \times (N \setminus \{0\}) \to L \times N$ by the formula

$$\psi(l, n) = (l, \xi(||n||) \cdot n),$$

where $\xi : (0, \infty) \to (0, \infty)$ is given by the formula

$$\xi(r) = \begin{cases} \frac{1}{2}(\varepsilon + r) & r \in (0, \varepsilon), \\ r & r \in [\varepsilon, \infty). \end{cases}$$

Since $\xi^{-1}$ is 2-Lipschitz, $\psi^{-1}$ is uniformly continuous. □

4. Main Theorem

Using the tools developed in previous sections we prove the Main Theorem of the paper.

Theorem 4.1. If $A$ is a 2-tame subset of $\mathbb{R}^n$ of codimension 2 and $\text{diam} \ A < \infty$, then the identity on $A$ can be arbitrarily closely approximated by embeddings into compact subsets of $N_{n-2}^n$.

Proof. Let $l_1, l_2, \ldots$ be a sequence of 1-dimensional hyperplanes such that $\bigcup_{i=1}^{\infty} l_i = \mathbb{R}^n \setminus N_{n-2}^n$. Fix $\varepsilon > 0$.

We construct a sequence of embeddings $\varphi_i : A \to \mathbb{R}^n$ such that $\varphi_i$ is $\varepsilon_i$-close to $\varphi_{i-1}$ and $\varphi_{i-1} \circ \varphi_i^{-1}$ is uniformly continuous. The $\varepsilon_i$'s are selected according to the winning strategy of Player I from Theorem 3.1. This guarantees that the limit $\varphi = \lim_{i \to \infty} \varphi_i$ exists and is an embedding of $A$ into $\mathbb{R}^n$. Additionally, we select $\varepsilon_i$'s so that $\sum_{i=1}^{\infty} \varepsilon_i < \varepsilon$. Then $\varphi$ is $\varepsilon$-close to the identity on $A$. 

An embedding \( \varphi_i \) is constructed so that

\[
\text{Cl}(\text{im } \varphi_i) \cap l_i = \emptyset.
\]

The construction is done in two steps. In the first step we use the Straightening Move to perturb \( \varphi_{i-1} \) to an embedding \( \varphi'_{i-1} \) such that \( \text{im } \varphi_{i-1} \cap l_i = \emptyset \). In the second step we compose \( \varphi'_{i-1} \) with a Push Away move to construct the embedding \( \varphi_i \) from \( \varphi'_{i-1} \).

Observe that \( \text{im } \varphi_i \) has finite diameter, so \( \text{Cl im } \varphi_i \) is compact. Hence \( \text{dist}(\text{im } \varphi_i, l_i) > 0 \). Hence if subsequent \( \varepsilon_i \)'s are sufficiently close to each other, then the limit \( \varphi \) satisfies \( \text{dist}(\text{im } \varphi, l_i) > 0 \). Then \( \text{Cl im } \varphi \) is compact and disjoint from the complement of \( N^n_{n-2} \) and therefore it is an embedding of \( A \) into a compact subset of \( N^n_{n-2} \). □

By lemma 2.3 an unstable Nöbeling space \( N^n_{n-2} \) is an 1-tame 2-codimensional subset of \( \mathbb{R}^n \). It is homeomorphic to \( N^2 \cap (0,1)^n \), which is of finite diameter. By theorem 4.1 we can remebed it into a compact subset of \( N^n_{n-2} \). The closure of the image of this embedding will be \( (n-2) \)-dimensional and compact. Therefore proof of Theorem 1.4 is completed.

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