Abstract

Dynamic connectivity is one of the most fundamental problems in dynamic graph algorithms. We present a new randomized dynamic connectivity structure with $O(\log n (\log \log n)^2)$ amortized expected update time and $O(\log n / \log \log \log n)$ query time, which comes within an $O((\log \log n)^2)$ factor of a lower bound due to Pătrașcu and Demaine. The new structure is based on a dynamic connectivity algorithm proposed by Thorup in an extended abstract at STOC 2000, which left out some important details.
1 Introduction

The dynamic connectivity problem is one of the most fundamental problems in dynamic graph algorithms. The goal is to support the following three operations on an undirected graph $G$ with $n$ vertices:

- **Insert**$(u,v)$: Insert a new edge $(u,v)$ into $G$.
- **Delete**$(u,v)$: Delete edge $(u,v)$ from $G$.
- **Conn?**$(u,v)$: Return true if and only if $u$ and $v$ are in the same connected component in $G$.

In this paper we prove the following bound on the complexity of dynamic connectivity.

**Theorem 1.1.** There exists a Las Vegas randomized dynamic connectivity data structure, that supports insertions and deletions of edges in amortized expected $O(\log n (\log \log n)^2)$ time, and answers connectivity queries in worst case $O(\log n / \log \log \log n)$ time.

**Previous Results.** The dynamic connectivity problem has been studied under both worst case and amortized measures of efficiency, and in deterministic, randomized Monte Carlo, and randomized Las Vegas models. We therefore have the opportunity to see six incomparably best algorithms! Luckily, there are currently only four. The best deterministic-worst case update time is $O(\sqrt{n} (\log \log n)^2 \log n)$ [10], improving on the longstanding $O(\sqrt{n})$ bound [3, 1], and the best deterministic-amortized update time is $O(\log^2 n / \log \log n)$ [15], improving on earlier $O(\log^2 n)$-time algorithms [8, 14] (see also [6, 7]). Kapron et al. [9] designed a worst case randomized Monte Carlo algorithm with $O(\log^5 n)$ update time, that is, there is some $1 / \text{poly}(n)$ probability of answering a connectivity query incorrectly. The update time was recently improved to $O(\log^4 n)$ [4]. In all dynamic connectivity algorithms the update time determines the query time [6]: $O(t(n) \log n)$ update implies $O(\log n / \log t(n))$ query time; see Theorem 2.1.

Thorup [14], in an extended abstract presented at STOC 2000, proposed a Las Vegas randomized-amortized algorithm with update time $O(\log n (\log \log n)^3)$, that is, queries must be answered correctly with probability 1, and the total update time for $m$ updates is a random variable, which is $m \cdot O(\log n (\log \log n)^3)$ in expectation. Unfortunately, the extended abstract [14] sketched or omitted a few critical data structural details. The problem of completing Thorup’s research program has, over the years, evolved into an important open research problem in the area of dynamic graph algorithms. A bound of $O(\log n \text{poly}(\log \log n))$ is substantially better than the best worst case and/or deterministic algorithms [10, 8, 9, 15], and comes within a tiny $\text{poly}(\log \log n)$ factor of known cell-probe lower bounds [11, 12].

Pătraşcu and Demaine [11] showed that for $t(n) = \Omega(1)$, update time $O(t(n) \log n)$ implies query time $\Omega(\log n / \log t(n))$ and Pătraşcu and Thorup [12] showed that there is no similar tradeoff in the reverse direction, that update time $o(\log n)$ implies $\Omega(n^{1-o(1)})$ query time. Whether there is a dynamic connectivity structure supporting all operations in $O(\log n)$ time (even amortized) is one of the main open questions in this area. This bound has only been achieved on forests [13] and embedded planar graphs [2].

**Our Contribution.** Thorup [14] proposed a dynamic connectivity structure based on four innovative ideas: (1) using a single, hierarchical representation of the graph, (2) imposing an overlay network of shortcuts on this representation in order to navigate between certain nodes in $O(\log \log n)$ time, (3) using random sampling (as in [7, 10]) to find replacement edges after Delete operations, and (4) maintaining a system of approximate counters to facilitate efficient random sampling of
edges. The interactions between these four elements is rather complex. Dynamic changes in the hierarchy (1) may require destroying and rebuilding the shortcuts in (2), and may invalidate the approximate counters in (4). In order for (3) to work correctly the approximate counters must be very accurate.

In this paper we use the same tools introduced by Thorup, but apply them differently in order to simplify parts of the algorithm, to accommodate a proof of correctness, and improve the expected amortized update time to $O(\log n(\log \log n)^2)$. Here is a summary of the technical differences.

- Thorup [14] (as in [8, 15]) assigns each edge a depth (aka level) between 1 and $\log n$ and maintains a spanning forest $F$. Depths are non-decreasing over time, so we can charge each depth promotion (from $i$ to $i + 1$) $(\log \log n)^2$ units of work. The depths of $F$-edges induce a hierarchy $H$, which is then refined into a binary hierarchy, $H^b$, by substituting “local trees” connecting each $H$-node to its $H$-children. One of the primitive operations supported by the hierarchy $H$ is to return an almost uniformly random depth-$i$ edge touching some component corresponding to an $H$-node. To implement this random sampling efficiently one needs a system of shortcuts and approximate counters. However, it is not obvious how to efficiently maintain approximate counters after edge promotions. Our data structure uses a more complicated classification of edges, which simplifies how approximate counters are implemented and analyzed. Each edge has a depth, as before, and each edge is either a witness (in $F$) or non-witness. The endpoints of a depth-$i$ non-witness edge can be either primary or secondary. We only keep approximate counters for $i$-primary endpoints, and only sample $i$-primary endpoints. When an edge is promoted from depth $i - 1$ to $i$, its endpoints are secondary, so there is no immediate need to update approximate counters for depth $i$. So long as good replacement edges can be found by sampling from the pool of $i$-primary endpoints we are happy, but if none can be found we are also happy to spend some time promoting depth-$i$ edges to depth-$(i + 1)$, and upgrading $i$-secondary endpoints to $i$-primary status. Since each edge’s endpoints can be upgraded at most $2\log n$ times over the lifetime of the edge, each upgrade can also be charged $(\log \log n)^2$ units of work. Whenever we upgrade $i$-secondary endpoints to $i$-primary status, we are guaranteed that the number of promotions/upgrades is large enough to completely rebuild the system of approximate counters for a pool of $i$-primary endpoints.

- One of Thorup’s [14] ideas was to maintain $\log n$ forests (one for each edge depth) on different subsets of the $H^b$-nodes, via a system of shortcuts. However, to be efficient it is important that these forests share shortcuts whenever possible. We provide a new method for storing and updating shortcuts, that allows us to find the right shortcut at a $H^b$-node in $O(1)$ time, and update information on all the shortcuts at a $H^b$-node in $O(\log \log n)$ time.

- We give a simpler random sampling procedure, which can be regarded as a two-stage version of the “provide or bound” routine of [6]. Our random sampling procedure is necessarily somewhat different than [14] because of the classification of non-witness edges into primary and secondary. The routine must either (i) provide a replacement edge with an $i$-primary endpoint, or (ii) determine that the fraction of such edges is less than a certain constant, with high probability. In case (ii) the procedure has found (statistical) evidence that there will be enough promotions/upgrades to pay for converting $i$-secondary endpoints to $i$-primary, promoting depth-$i$ edges to depth-$(i + 1)$, and rebuilding $i$-primary approximate counters.

- The structure of $H$ is uniquely determined by the depths of witness ($F$) edges, and $H^b$ is a binary refinement of $H$. In Thorup’s [14] system $H^b$ is only modified in response to structural...
changes in $\mathcal{H}$, due to promotions of witness edges in $\mathcal{F}$. A key invariant maintained by our data structure is that certain approximate counters, once initialized, are only subject to decrements, never increments. Thus, to preserve this invariant we actually update $\mathcal{H}^b$ in response to non-witness edge promotions/upgrades, which necessarily have no effect on $\mathcal{H}$.

**Organization of the Paper.** In Section 2 we review several fundamental concepts of dynamic connectivity algorithms. Section 3 gives a detailed overview of the data structure invariants and its three main components: maintaining a binary hierarchical representation of the graph, maintaining shortcuts for efficient navigation around the hierarchy, and maintaining a system of approximate counters to support $O(1)$-approximate random sampling. Each of these three main components is explained in great detail in Sections 4, 5, 6.

## 2 Preliminaries

In this section we review some basic concepts and invariants used in prior dynamic connectivity algorithms \cite{6, 5, 8, 14, 15}.

**Witness Edges, Witness Forests and Replacement Edges.** A common method for supporting connectivity queries is to maintain a spanning forest $\mathcal{F}$ of $G$ called the *witness forest*, together with a dynamic connectivity structure on $\mathcal{F}$. Each edge in the witness forest is called a *witness* edge and all others *non-witness* edges. Notice that deleting a non-witness edge does not change the connectivity. A dynamic connectivity data structure for $\mathcal{F}$ supports fast queries via Theorem 2.1.

**Theorem 2.1 (Henzinger and King \cite{5}).** For any function $t(n) = \Omega(1)$, there exists a dynamic connectivity data structure for forests with $O(t(n) \log n)$ update time and $O(\log n / \log t(n))$ query time.

The difficulty in maintaining a dynamic connectivity data structure is to find a *replacement* edge $e'$ when a witness edge $e \in \mathcal{F}$ is deleted, or determine that no replacement exists. To speed up the search for replacement edges we maintain Invariant 1 (below) governing edge *depths*.

**Edge Depths.** Each edge $e$ has a depth $d_e \in [1, d_{\text{max}}]$, where $d_{\text{max}} = \lceil \log n \rceil$. Let $E_i$ be the set of edges with depth $i$. All edges are inserted at depth 1 and depths are non-decreasing over time. Incrementing the depth of an edge is called a *promotion*. Since we are aiming for $O(\log n (\log \log n)^2)$ amortized time per update, if the actual time to promote an edge set $S$ is $O(|S| \cdot (\log \log n)^2)$, the amortized time per promotion is zero. Promotions are performed in order to maintain Invariant 1. There are other at most $O(\log n)$ status changes that an edge will undergo, each affording $O((\log \log n)^2)$ work. Define $G_i = (\hat{V}_i, \bigcup_{j \geq i} \hat{E}_j)$.

**Invariant 1 (The Depth Invariant).**

1. (Spanning Forest Property) $\mathcal{F}$ is a maximum spanning forest of $G$ with respect to the depths.
2. (Weight Property) For each $1 \leq i \leq d_{\text{max}}$, each connected component in the subgraph $G_i$ contains at most $n/2^{i-1}$ vertices.

**Hierarchy of connected components.** Define $\hat{V}_i$ to be in one-to-one correspondence with the connected components of $G_{i+1}$, which are called $(i+1)$-components. If $u \in V$, let $u^i \in \hat{V}_i$ be the unique $(i+1)$-component containing $u$. Define $\hat{G}_i = (\hat{V}_i, \hat{E}_i)$ to be the multigraph (including parallel edges and loops) obtained by contracting edges with depth above $i$ and discarding edges with depth below $i$, so $\hat{E}_i = \{(u^i, v^i) \mid (u, v) \in E_i\}$. The hierarchy $\mathcal{H}$ is composed of the undirected
3 Overview of the Data Structure

Following the key invariant in [8, 14, 15], the main goal is summarized as the following lemma:

**Lemma 3.1.** Invariant [4] is maintained throughout updates to $G$.

In the rest of this section, we provide an overview of the data structure. The underlined parts of the text refer to primitive data structure operations supported by Lemma 3.2, presented in Section 3.3.

The data structure. The hierarchy $H$ naturally defines a rooted forest (not to be confused with the spanning forest), which is called the *hierarchy forest*, and contains several *hierarchy trees*. We abuse notation and say that $H$ refers to this hierarchy forest, together with several auxiliary data structures supporting operations on the forest. The nodes in $H$ are the $i$-components for all $1 \leq i \leq d_{max}$. The roots of the hierarchy trees are nodes in $\tilde{V}_0$, representing 1-components. The set of nodes at depth $i$ is exactly $\tilde{V}_i$. The set of children of a node $v^i$ at depth $i$ is $\{u^{i+1} \in \tilde{V}_{i+1} \mid v^i = u^i\}$. The leaves are nodes in $\tilde{V}_{d_{max}} = V$. See Figure 1 for an example. The nodes in $H$ are called *$H$-nodes*, and the roots are called *$H$-roots*. Each non-leaf $H$-node $v$ is associated with a binary local tree, implicitly supporting operations between $v$ and its *$H$-children* (See Section 6).

![Figure 1: An illustration of a graph and the corresponding hierarchy forest $H$, where $n = 15$ and $d_{max} = 3$. All thick edges are witness edges and the thin edges are non-witness edges.](image)
3.1 Insertion

To execute an insert\((u, v)\) operation, where \(e = (u, v)\), the data structure first sets \(d_e = 1\). If \(e\) connects two distinct components in \(G\) (which is verified by a connectivity query on \(F\)), then the data structure accesses two \(H\)-roots \(v^0\) and \(v^0\), merges \(v^0\) and \(v^0\) and \(e\) is inserted into \(H\) (and \(F\)) as a 1-witness edge. Otherwise, \(e\) is inserted into \(H\) as a 1-non-witness edge.

3.2 Deletion

By the Spanning Forest Property of Invariant 1, the deletion of an edge \(e\) can only be replaced by edges of depth \(d_e\) or less. We always first look for a replacement edge at the same depth of the deleted edge. If we do not find a replacement edge at depth \(d_e\) then we demote \(e\) by setting \(d_e \leftarrow d_e - 1\), which preserves Invariant \(\square\) and continue looking for replacement edges at the new depth \(d_e\). Demotion is merely conceptual; the deletion algorithm does not actually update \(d_e\) in the course of deleting \(e\).

To execute a delete\((u, v)\) operation, where \(e = (u, v)\), the data structure first removes \(e\) from \(H\). If \(e\) is an i-non-witness edge, then the deletion process is done. If \(e\) is an i-witness edge, the deletion of \(e\) could split an i-component. Specifically, prior to the deletion, the edge \((u^i, v^i)\) connected two \((i + 1)\)-components, \(u^i\) and \(v^i\), which, possibly together with some additional i-witness edges and \((i + 1)\)-components, formed a single i-component \(u^{i-1} = v^{i-1}\) in \(G_i\). If no i-non-witness replacement edge exists, then deleting \((u,v)\) splits \(u^{i-1}\) into two i-components. In order to establish if this is the case, the data structure first accesses \(u^i, v^i\) and \(u^{i-1}\) in \(H\) and implicitly splits the i-component \(u^{i-1}\) into two connected components \(c_u\) and \(c_v\) in \(F_i = (V_i, \{(u^i, v^i) \mid (u,v) \in F_i\})\) where \(u^i \in c_u\) and \(v^i \in c_v\) (we define \(c_u\) and \(c_v\) but without context to the subscripts, see Figure 2\(a\)). The rest of the deletion process focuses on finding a replacement edge to reconnect \(c_u\) and \(c_v\) into one i-component. This process has two parts, explained in detail below: (1) establishing the two components \(c_u\) and \(c_v\), and (2) finding a replacement edge. Notice that \(c_u\) and \(c_v\) do not correspond to \(H\)-nodes.

3.2.1 Establishing Two Components

To establish the two components \(c_u\) and \(c_v\) created by the deletion of \(e\), the data structure executes in parallel two depth first searches (DFS) on \(F_i - \{(u^i, v^i)\}\), one DFS starting from \(u^i\) and one DFS starting from \(v^i\). To implement a DFS, the data structure repeatedly enumerates all i-witness edge endpoints touching an \((i + 1)\)-component. The DFSs are carried out in parallel until one of the connected components is fully scanned. By fully scanning one component, the weights of both components are determined (since \(w(u^{i-1}) = w(c_u) + w(c_v))\). Without lost of generality, assume that \(w(c_u) \leq w(c_v)\), and so by Invariant 1, \(w(c_u) \leq w(u^{i-1})/2 \leq n/2^i\).

**Witness Edge Promotions.** The data structure promotes all i-witness edges touching nodes in \(c_u\) and merges all \((i + 1)\)-components contained in \(c_u\) into one \((i + 1)\)-component with weight \(w(c_u)\). This is permitted by Invariant \(\square\) since \(w(c_u) \leq w(u^{i-1})/2 \leq n/2^i\). The merged \((i + 1)\)-component has the node \(u^{i-1}\) as its parent in \(H\). See Figure 2\(b\).

To differentiate between versions of components before and after the merges, we use a convention where bold notation refers to the components after the merges take place. Thus, we denote the \((i + 1)\)-component contracted from all \((i + 1)\)-components inside \(c_u\) by \(u^i\). Similarly, the graph \(\hat{G}_i\) after merging some of its nodes is denoted by \(\hat{G}_i\).

Having contracted the \((i + 1)\)-components inside \(c_u\) into \(u^i\), we now turn our attention to identifying whether the deletion of \(e\) disconnects \(u^i\) from \(c_v\) in \(\hat{G}_i\). This task reduces to determining whether there exists an edge in \(\hat{G}_i\) that reconnects \(u^i\) to any \((i + 1)\)-components in \(u^{i-1}/u^i\).
Figure 2: Illustration of the hierarchy of components at depth $i - 1$ and $i$: (a) After identifying two components $c_u$ and $c_v$, it turns out that $c_u$ has smaller weight although it has more $(i + 1)$-components. (b) After merging all $(i + 1)$-components in the smaller weight component. (c) If no replacement edge is found, then $c_u$ and $c_v$ are two actual connected components in $\hat{G}_i$ and hence $u^i$ is split.

3.2.2 Finding a Replacement Edge

Notice that by definition of $\hat{G}_i$ and $u^{i-1}$, a depth $i$ edge is a replacement edge in $E$ if and only if it is an $i$-non-witness edge with exactly one endpoint $x \in V$ such that $x^i = u^i$. To find a replacement edge, the data structure executes one or both of the following two auxiliary procedures: the sampling procedure and the enumeration procedure.

**Intuition.** Consider these two situations. In Situation A at least a constant fraction of the $i$-non-witness edges touching $u^i$ have exactly one endpoint touching $u^i$, and are therefore eligible replacement edges. In Situation B a small $\epsilon$ fraction (maybe zero) of these edges have exactly one endpoint in $u^i$. If we magically knew which situation we were in and could sample $i$-non-witness endpoints uniformly at random then the problem would be easy. In Situation A we would iteratively sample an $i$-non-witness endpoint and test whether the other endpoint was in $u^i$; each test takes $O(\log n \log \log n)$ time. The expected number of samples required to find a replacement edge is $O(1)$ and this cost would be charged to the deletion operation. In Situation B we would enumerate and mark every $i$-non-witness endpoint touching $u^i$. Any edge with one mark is a replacement edge and any with two marks can be promoted to depth $i + 1$. Since a majority of the edges will end up being promoted, the amortized cost of the enumeration procedure is zero, so long as the enumeration and promotion cost is $O((\log \log n)^2)$ per endpoint.

There are two technical difficulties with implementing this idea. First, the set of $i$-non-witness edges incident to $u^i$ is a dynamically changing set, and supporting (almost-)uniformly random sampling on this set is a very tricky problem. Second, we do not know which situation, A or B, we are in. Note that it is insufficient to take $O(1)$ random samples and, if no replacement edges are found, deduce that we are in Situation B. Because the cost of enumeration is so high, we cannot afford to mistakenly think we are in Situation B unless the probability of error is inversely proportional to the cost of enumeration.

Thorup [14] addresses the first difficulty by maintaining a system of approximate counters and two layers of overlay networks and solves the second difficulty by using the “provide or bound”

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1The first overlay network, which we also use, supports navigation to the $H$-leaves incident to $i$-non-witness edges. The second overlay network, which is sketched in [14], is derived from a heavy-path decomposition of the first overlay network in order to guarantee some degree of balance. The second overlay network is used to facilitate dynamic updates to the approximate counters.
The data structure first upgrades all \( i \)-secondary endpoints touching \( u^i \) to \( i \)-primary endpoints, enumerates all \( i \)-primary endpoints touching \( u^i \) and establishes for each such edge how many of its endpoints touch \( u^i \) (either one or both). An edge is a replacement edge if and only if exactly one of its endpoints is enumerated. Each non-replacement edge encountered by the enumeration procedure has both endpoints in an \( (i + 1) \)-component, namely \( u^i \), and can therefore be promoted to be a depth \( (i + 1) \)-non-witness edge (making both endpoints secondary), without violating Invariant \([7]\). After all promotions and upgrades are completed, the sampling structure for \( i \)-primary endpoints touching \( u^i \) is rebuilt.

### 3.2.3 Iteration and Conclusion

If a replacement edge \( e' \) exists, then \( u^{i-1} \) is still an \( i \)-component and the data structure converts \( e' \) from an \( i \)-non-witness edge to an \( i \)-witness edge. Otherwise, \( c_u \) and \( c_v \) form two distinct
$i$-components in $\hat{G}_i$. In this case, depending on $i$, the data structure splits $u^{i-1}$ into two sibling nodes or two $H$-roots: a new node $u^{i-1}$ representing $c_u$ whose only child is $u^i$, and $v^{i-1}$ representing $c_v$ whose children are the rest of the $(i+1)$-components in $c_v$. Recall that while there may not be an $i$-non-witness replacement edge for $e$, there may be one at a lower depth, by the Spanning Forest Property. Therefore, if $i = 1$ then we are done. Otherwise, we set $i = i - 1$, conceptually demoting $e$, and repeat the procedure as if $e$ were deleted at depth $i - 1$.

![Figure 3](image1.png) After deletion of $(v_3, v_5)$ (See Figure [1]) By identifying $\{v_1, v_2, v_3\}$ to be the smaller weight component, the witness edge $(v_2, v_3)$ is promoted and the corresponding nodes in $\hat{V}_2$ is merged. The edge $(v_3, v_4)$ is the replacement edge.

![Figure 4](image2.png) After deletion of $(v_4, v_5)$: (1) Split the node in $\hat{V}_2$ associated with $v_4$ and $v_5$. (2) Identify that $\{v_5, v_6, v_7\}$ is the smaller weight component. (3) Merge nodes $v_5^2$ and $v_6^2 = v_7^2$. (4) Split the node $v_5^2$. (5) Found replacement edge $(v_1, v_6)$.

### 3.3 The Backbone of the Data Structure

Lemma 3.2 summarizes the primitive operations required to execute and Insert or Delete. Remember that the possible depths are integers in $[1, d_{max}]$, and that the possible endpoint types are WITNESS, PRIMARY and SECONDARY.

**Lemma 3.2.** There exists a data structure supporting the following operations on $H$ with the following amortized time complexities (in parentheses):

1. Add or remove an edge with a given edge depth and endpoint type $O(\log n (\log \log n)^2))$. 

\[ \hat{V}_0 \]
\[ \hat{V}_1 \]
\[ \hat{V}_2 \]
\[ v_1 \]
\[ v_2 \]
\[ v_3 \]
\[ v_4 \]
\[ v_5 \]
\[ v_6 \]
\[ v_7 \]
\[ v_8 \]
\[ v_9 \]
\[ v_{10} \]
\[ v_{11} \]
\[ v_{12} \]
\[ v_{13} \]
\[ v_{14} \]
\[ v_{15} \]
(2) Given a set $S$ of sibling $H$-nodes or $H$-roots, merge them into a single node $u^1$, and then promote all i-witness edges touching $u^1$ into $(i+1)$-witness edges. $(O(k(\log \log n)^2 + 1)$, where $k$ is the number of i-witness edges touching $u^1$).

(3) Given an $H$-node $v^i \in V_i$, upgrade all i-secondary endpoints associated with $v^i$ to i-primary endpoints $(O((p+s) (\log \log n)^2 + 1)$, where $p$ and $s$ denote the number of i-primary endpoints and i-secondary endpoints touching $v^i$ prior to the upgrade).

(4) Given an $H$-node $v^i \in V_i$ and a subset of i-primary endpoints associated with $v^i$, promote them to $(i+1)$-secondary endpoints. $(O(k(\log \log n)^2 + 1)$, where $k$ is the number of all i-primary endpoints associated with $v^i$).

(5) Convert a given i-non-witness edge into an i-witness edge $(O(\log n (\log \log n)^2))$.

(6) Given two $H$-nodes $u^{i-1}$ and $u^i$ where $u^i$ is an $H$-child of $u^{i-1}$, split $u^{i-1}$ into two sibling $H$-nodes: one with $u^i$ as a single $H$-child and the other with the rest of $u^{i-1}$’s former $H$-children as its $H$-children $(O(\log \log n))^2))$.

(7) Given an $H$-node $v^i \in V_i$ and a given endpoint type, enumerate all endpoints $(u,e)$ where $e$ is of the given endpoint type, $d_e = i$, and $u^i = v^i$. $(O(k \log \log n + 1)$, where $k$ is the number of enumerated endpoints).

(8) Given $v^i$, return $v^{i-1}$ $(O(\log(w(v^{i-1}) - \log(w(v^i)) + \log \log n))$.

(9) Given an $H$-node $v^i \in V_i$, return a $(1 + o(1))$-approximation to the number of i-primary endpoints touching $v^i$ $(O(1))$.

(10) (Batch Sampling Test) Given an $H$-node $v^i \in V_i$ and an integer $k$, independently sample $k$ i-primary endpoints touching $v^i$ $(1 + o(1))$-uniformly at random, and establish for each sampled endpoint whether the other endpoint is also in $v^i$. $(O(\min(k \log n \log \log n, k + (p+s) \log \log n))$, where $p$ and $s$ are the number of i-primary and i-secondary endpoints touching $v^i$, respectively).

Notice that each data structure operation stated in Lemma 3.2 on its own does not guarantee that the data structure maintains Property 1. However, given the use of Lemma 3.2 in the description of the algorithm above, the proof of Lemma 3.1 is straightforward.

### 3.4 The Main Modules of the Data Structure

To support Lemma 3.2, the data structure utilizes five main modules, some of which depend on each other: (1) the $H$-leaf data structure (2) the notion of induced $(i,t)$-forests (3) the shortcut infrastructure (4) approximate counters, and (5) local trees. The $H$-leaf data structure is fairly straightforward and is described in detail in Section 3.4.1. Then we define the notion of overlaying $O(\log n)$ forests on $H$ in Section 3.4.2. A brief overview of the other modules is described in Sections 3.4.3, 3.4.4, and 3.4.5. Sections 4, 5, and 6 provide a detailed explanation of each module.

The data structure also uses lookup tables in several modules. We describe in Section 3.5 a way to amortize the cost constructing the lookup tables. The general operations involving multiple modules, as well as the proof of Lemma 3.2 are described and analyzed in detail in Section 7.

#### 3.4.1 The $H$-Leaf Data Structure

The $H$-leaf data structure supports the following operations: (1) Given an endpoint with a specified edge depth and endpoint type, insert or delete an edge with an endpoint at the leaf. (2) Given a depth and type, enumerate all edge endpoints incident to the leaf with that depth and type. (3) Return a uniformly random endpoint among the set of edge endpoints with a given depth and type.
To support these operations, each leaf maintains a dynamic array of endpoints for each edge depth \(1 \leq i \leq d_{\text{max}}\) and each endpoint type \(t \in \{\text{WITNESS, PRIMARY, SECONDARY}\}\). Hence the three operations are supported in worst case \(O(1)\) time.

### 3.4.2 The Induced \((i,t)\)-forest

For a given edge depth \(i \in [1,d_{\text{max}}]\) and endpoint type \(t \in \{\text{WITNESS, PRIMARY, SECONDARY}\}\), an \(H\)-leaf \(v\) is an \((i,t)\)-leaf if \(v\) has an endpoint with depth \(i\) and type \(t\). An \(H\)-node \(v^i \in \hat{V}_i\) having an \((i,t)\)-leaf in its subtree is an \((i,t)\)-root. For each \((i,t)\) pair, consider the induced forest \(\mathcal{F}\) on \(H\) by taking the union of the paths from each \((i,t)\)-leaf to the corresponding \((i,t)\)-root. An \(H\)-node \(v\) in \(\mathcal{F}\) is an \((i,t)\)-node if

- \(v\) is an \((i,t)\)-leaf,
- \(v\) is an \((i,t)\)-root,
- \(v\) has more than one child in \(\mathcal{F}\). In this case we call \(v\) an \((i,t)\)-branching node, or
- \(v\) is an \(H\)-child of an \((i,t)\)-branching node but has only one \(H\)-child in \(\mathcal{F}\). In this case we call \(v\) an single-child \((i,t)\)-node.

Notice that an \((i,t)\)-root may or may not be an \((i,t)\)-branching node.

For each \((i,t)\)-node other than an \((i,t)\)-root, define its \((i,t)\)-parent to be the nearest ancestor on \(\mathcal{F}\) that is also an \((i,t)\)-node. An \((i,t)\)-child is defined accordingly. The \((i,t)\)-parent/child relation implicitly defines an \((i,t)\)-forest, which consists of \((i,t)\)-trees rooted at \(\hat{V}_i\) nodes. The single-child \((i,t)\)-nodes play a crucial role in the efficiency of traversing an \((i,t)\)-tree. An \(H\)-node \(v\) has an \((i,t)\)-status if \(v\) is an \((i,t)\)-node.

**Storing \((i,t)\)-status.** Each node in \(v \in H\) stores two bitmaps of size \(O(\log n)\) each, indicating whether \(v\) is an \((i,t)\)-node, and if so then indicating whether \(v\) is an \((i,t)\)-branching node or not.

**Operations on \((i,t)\)-forests.** A key idea introduced by Thorup [14] is that edges between an \((i,t)\)-node and its \((i,t)\)-parent or \((i,t)\)-children do not need to be maintained explicitly. The two components that simulate these edges are the shortcut infrastructure, and the local trees (which also use a relaxed version of the shortcut infrastructure). In particular, the shortcut infrastructure supports efficient traversals from a single-child \((i,t)\)-node to its unique \((i,t)\)-child, while the local trees support efficient enumeration of all \((i,t)\)-children of an \((i,t)\)-branching node. Lemma 3.3 summarizes the operations on \((i,t)\)-forests which are implemented via the shortcut infrastructure and local trees, together with their corresponding time cost. We emphasize that our implementation of the operations in Lemma 3.3 imply an \(O(\log \log n)\) factor improvement in time cost over the system of Thorup [14].

**Lemma 3.3.** There exists a data structure on \(H\) supporting the following operations:

- Given an \(H\)-leaf \(x\), make \(x\) an \((i,t)\)-leaf \((O(\log n(\log \log n)^2))\).
- Given an \((i,t)\)-leaf \(x\), remove the \((i,t)\)-leaf status from \(x\) \((O(\log n \log \log n))\).
- Given an \((i,t)\)-node \(v\), return the \((i,t)\)-parent of \(v\) \((O(\log \log n))\).
- Given an \((i,t)\)-node \(v\), enumerate the \((i,t)\)-children of \(v\) \((O(k \log \log n + 1))\) where \(k\) is the number of enumerated \((i,t)\)-children.
• Given an \((i, t)\)-tree \(T\) rooted at \(v\), an integer \(i \leq i' \leq d_{\text{max}}\), an endpoint type \(t'\), and two subsets of \((i, t)\)-leaves \(S^-\) and \(S^+\) (these subsets need not be disjoint), update \(H\) so that all of the leaves in \(S^-\) lose their \((i, t)\)-leaf status, and all leaves in \(S^+\) gain \((i', t')\)-leaf status (if they did not have it before) \((O(|T|(|\log \log n|^2 + 1)))\).

The proof of Lemma 3.3 is given in Section 7.2.

3.4.3 The Shortcut Infrastructure

The purpose of shortcuts is to simulate a traversal from a single-child \((i, t)\)-node to its only \(H\)-child. This traversal costs amortized \(O(\log \log n)\) time. The details and construction of shortcuts is described in Section 4. Nevertheless, there are two main conceptual components which we introduce that allow for simplification of the shortcut system, and the improved runtime in Lemma 3.3.

Shared shortcuts and the local dictionary. Intuitively, a shortcut connects an \(H\)-node \(u\) and a descendant \(v\) of \(u\) in \(H\). We say that such a shortcut leaves \(u\) and enters \(v\). Since we are imposing \(O(\log n)\) independent \((i, t)\)-forests on \(H\), when \(H\)-nodes merge or split, an inefficient implementation may necessitate updating information for several \((i, t)\)-forests. However, notice that the paths between a single-child \((i, t)\)-node to its \((i, t)\)-child may overlap for several \((i, t)\) pairs. To improve efficiency, a shortcut is shared between several \((i, t)\)-forests, and is accessed through an \(O(\log n)\) size array \(\text{DOWN}_u\) with pointers to all shortcuts leaving \(u\). Moreover, we employ a local dictionary, which is an array \(\text{DOWNIDX}_u\) with a slot corresponding to each \((i, t)\)-forest. Each location in \(\text{DOWNIDX}_u\) stores an \(O(\log \log n)\) bit index of the location in \(\text{DOWN}_u\) containing the pointer to the shortcut for that specific \((i, t)\) pair. With the local dictionary, the data structure efficiently accesses the shortcut for any specific \((i, t)\) pair by two array lookups.

Lazy covers. One key aspect of shortcuts is that they do not cross (see Lemma 4.1), which means that if there is a shortcut between \(u\) and \(v\), then there is no shortcut between a node in the internal path between \(u\) and \(v\) (exclusive) and a node that is either a proper descendent or proper ancestor of both \(u\) and \(v\). Since shortcuts do not cross, they form a naturally partially ordered set (poset).

When structural changes take place in \(H\), all of the shortcuts that touch the nodes participating in these changes are removed. The cost for removing those shortcuts is amortized over the cost of creating them. However, once the structural changes are complete, we do not immediately return all the shortcuts back. Instead, the data structure partially recovers some of the shortcuts and employs a lazy approach in which shortcuts are only added when they are needed. We feel this method simplifies the description of the data structure.

3.4.4 Approximate Counters

Implementing the sampling operation in Lemma 3.2 reduces to being able to traverse from an \((i, \text{PRIMARY})\)-branching node to one of its \((i, \text{PRIMARY})\)-children \(v\), where the probability is almost proportional to the number of \(i\)-primary endpoints touching \(v\). The distribution over \((i, \text{PRIMARY})\)-children of an \((i, \text{PRIMARY})\)-branching node is supported by maintaining an approximate \(i\)-counter at each \((i, \text{PRIMARY})\)-node. Notice that an \(H\)-node could be an \((i, \text{PRIMARY})\)-node for several \(i\), so there are several \(i\)-counters maintained in an \(H\)-node. An approximate \(i\)-counter at such a node \(v\) stores an \((1 + o(1))\) approximation of the number of \(i\)-primary endpoints touching \(v\). This quality of approximation provides the guarantees needed for the sampling operation, as shown in Section 7.1.1. We emphasize that approximate \(i\)-counters are only stored for \(i\)-primary endpoints, not \(i\)-secondary endpoints.
Each approximate $i$-counter uses $O(\log \log n)$ bits, and its precision is relative to the depth and weight of the node. The $i$-counters are precisely maintained at the $(i, \text{primary})$-leaves. When the data structure sums $i$-counters together, the approximate $i$-counters may lose precision. However, this precision depends on the height of the arithmetic formula tree implicitly formed in the $(i, \text{primary})$-trees. The following property states the precision requirement in order to support accurate sampling:

**Invariant 2 (Precision of Approximate Counters).** Let $v$ be an $\mathcal{H}$-node and let $C_i(v)$ be the number of $i$-primary endpoints touching $v$. Let $j$ be the depth of $v$ and let

$$H(v) = (d_{\text{max}} - j) \cdot O(\log \log n) + |\log(w(v))|.$$  

If $v$ is an $(i, \text{primary})$-node then $v$ stores an approximate $i$-counter $\hat{C}_i(v)$, where

$$(1 - (\log^2 n))^{H(v) + 1} C_i(v) \leq \hat{C}_i(v) \leq C_i(v).$$

The shortcut infrastructure and local trees together allow us to efficiently guarantee that Invariant 2 holds. This is captured by the following lemma.

**Lemma 3.4.** There exists a data structure on $\mathcal{H}$ that maintains approximate $i$-counters and supports the following operations (the runtime is given in parenthesis):

- Update the approximate counters to support a change in the number of $(i, \text{primary})$-endpoints at a given $\mathcal{H}$-leaf. ($O(\log n(\log \log n)^2)$)
- Given an $(i, \text{primary})$-root $v^i$, update the approximate $i$-counters for all $(i, \text{primary})$-nodes in the $(i, \text{primary})$-tree of $v^i$ so that Invariant 2 holds for those nodes ($O(|T|(\log \log n)^2 + 1)$, where $T$ is the $(i, \text{primary})$-tree rooted at $v^i$).
- When merging two sibling $\mathcal{H}$-nodes, compute the approximate $i$-counters for all $i \in [1, d_{\text{max}}]$ at the merged node. ($O(\log \log n)$).
- When splitting an $\mathcal{H}$-node into two sibling $\mathcal{H}$-nodes, compute the approximate $i$-counters for all $i \in [1, d_{\text{max}}]$ at the two sibling nodes. ($O(\log \log n)$).

### 3.4.5 The Local Trees

The local tree is a specially constructed binary tree, where the root is associated with an $\mathcal{H}$-node $v$ and the leaves are the $\mathcal{H}$-children of $v$. The local trees support the following operations.

**Lemma 3.5.** There exists a data structure that supports the following operations between an $\mathcal{H}$-node $v$ and its $\mathcal{H}$-children.

- Add a new $\mathcal{H}$-child $x$ ($O((\log \log n)^2)$).
- Delete an $\mathcal{H}$-child $x$ ($O((\log \log n)^2)$).
- Merge two sibling $\mathcal{H}$-nodes $u$ and $v$. ($O((\log \log n)^2)$).
- Return the $\mathcal{H}$-parent $v^{i-1}$ of $\mathcal{H}$-node $v^i$ ($O(\log w(v^{i-1}) - \log w(v^i) + \log \log n)$).
- Enumerate all local tree leaves with an $(i, t)$-status ($O(\log \log n)$ per leaf).
- Add $(i, t)$-status to a local tree leaf. ($O((\log \log n)^2)$).
Given an \((i, \text{PRIMARY})\)-branching node \(w^j\) and an edge depth \(i\), sample an \((i, \text{PRIMARY})\)-child \(w^j\) with probability at most
\[
\hat{C}_i(w^j)(1 - \log^{-2} n)^{-\left(\log(w^j - 1) - \log(w^j) + O(\log \log n)\right)/\hat{C}_i(w^j - 1)}.
\]

Given an \(\mathcal{H}\)-node \(v\), test whether there is a unique \((i, t)\)-leaf in the local tree rooted at \(v\). If yes, return that \((i, t)\)-leaf \((O(\log \log n))\).

### 3.5 Lookup Tables.

There are several components of our data structure that use small lookup tables of size \(O(n^\epsilon)\) for a constant \(0 < \epsilon < 1\) for supporting fast operations on bit strings. By assuming that the initial graph is empty, the \(O(n^\epsilon)\) sized lookup tables are built on-the-fly and their cost is amortized through the operations as follows. As long as the number of graph updates is \(m \leq n\), all edge depths are at most \([\log m]\). Hence, for each \(0 \leq r \leq \log \log n\), after the \(m = 2^r\)-th graph update, the data structure rebuilds the lookup tables of size \(O(m^\epsilon)\). The time cost for building the lookup tables during the first \(m\) operations is bounded by
\[
\sum_{i=0}^{[\log \log m]} m^{\frac{i}{\epsilon}} = O(m^\epsilon).
\]

This is amortized \(o(1)\) per update.

### 4 Shortcut Infrastructure

**\(\mathcal{H}\)-shortcuts.** An \(\mathcal{H}\)-shortcut \(u \Leftarrow v\) is a data structure connecting an ancestor \(u\) and a descendant \(v\) in \(\mathcal{H}\). For a positive integer \(\ell\), define its least significant bit index, denoted by \(\text{LSBIndex}(\ell)\), to be the minimum integer \(b\) such that \(2^b\) divides \(\ell\) but \(2^{b+1}\) does not. The power of a pair of nodes \(u\) and \(v\) is defined as
\[
\mathcal{P}(u, v) = \min(\text{LSBIndex}(\text{depth}_{\mathcal{H}}(u) + 1), \text{LSBIndex}(\text{depth}_{\mathcal{H}}(v) + 1)).
\]

In order for an \(\mathcal{H}\)-shortcut to exist between \(u\) and \(v\), any node \(x\) on the path from \(u\) to \(v\) must have \(\text{LSBIndex}(\text{depth}_{\mathcal{H}}(x) + 1)) < \mathcal{P}(u, v)\). If \(v\) is the \(\mathcal{H}\)-child of \(u\) (and so \(\mathcal{P}(u, v) = 0\)) then we say that \(u \Leftarrow v\) is a fundamental \(\mathcal{H}\)-shortcut. The \(\mathcal{H}\)-shortcut \(u \Leftarrow v\) is always accessible to \(v\), but not necessarily to \(u\). From the perspective of \(v\), \(u \Leftarrow v\) is called an upward \(\mathcal{H}\)-shortcut, while from the perspective of \(u\), if \(u\) has access to \(u \Leftarrow v\) then \(u \Leftarrow v\) is called a downward \(\mathcal{H}\)-shortcut. We emphasize that our data structure does not store all of the possible \(\mathcal{H}\)-shortcuts, as will be evident from the description below.

The following lemma states that the set of \(\mathcal{H}\)-shortcuts on a ancestor-descendant path do not cross each other.

**Lemma 4.1.** For any four distinct \(\mathcal{H}\)-nodes \(x_1, x_2, x_3, x_4\) along a root-to-leaf path in \(\mathcal{H}\), it is impossible to have two \(\mathcal{H}\)-shortcuts \(x_1 \Leftarrow x_3\) and \(x_2 \Leftarrow x_4\).

**Proof.** For \(j \in \{1, 2, 3, 4\}\) let \(h_j\) be the depth of \(x_j\) in \(\mathcal{H}\). Then \(h_1 < h_2 < h_3 < h_4\). Assume the converse, there are two \(\mathcal{H}\)-shortcuts \(x_1 \Leftarrow x_3\) and \(x_2 \Leftarrow x_4\). By definition this implies \(\text{LSBIndex}(h_2 + 1) < \text{LSBIndex}(h_3 + 1)\) and \(\text{LSBIndex}(h_3 + 1) < \text{LSBIndex}(h_2 + 1)\), a contradiction.
The covering relationships of $\mathcal{H}$-shortcuts and the poset. We say that $a \equiv b$ covers $c \equiv d$ if $c$ and $d$ are on the path $P_{ab}$ in $\mathcal{H}$. Notice that a shortcut covers itself. By Lemma 4.1, we define a covering order of $\mathcal{H}$-shortcuts where an $\mathcal{H}$-shortcut $u \equiv v$ covers all other $\mathcal{H}$-shortcuts that lie on the path $P_{uv}$. Define $\succeq$ to be the covering partial order:

$$(a \equiv b) \succeq (c \equiv d) \text{ if } a \equiv b \text{ covers } c \equiv d.$$ 

For any $uv$-path $P_{uv}$ on $\mathcal{H}$, the largest covering $\mathcal{H}$-shortcuts of $P_{uv}$, denoted by $\text{LCS}^\mathcal{H}(u, v)$, is the set of maximal $\mathcal{H}$-shortcuts (with respect to $\succeq$) among all $\mathcal{H}$-shortcuts having both endpoints on $P_{uv}$.

![Figure 5: The figure above shows LCS$^\mathcal{H}(v_5, v_{14})$ as an example, where $v_i$ has depth$_\mathcal{H}(v_i) = i$. The dotted edges are the set of all possible shortcuts.](image)

The following lemma bounds the size of LCS$^\mathcal{H}(u, v)$.

**Lemma 4.2.** For any two nodes $u, v \in \mathcal{H}$ with $u$ an ancestor of $v$, all $\mathcal{H}$-shortcuts in LCS$^\mathcal{H}(u, v)$ form a path connecting $u$ and $v$, and $|\text{LCS}^\mathcal{H}(u, v)| = O(\log \log n)$.

**Proof.** All $\mathcal{H}$-shortcuts on $P_{uv}$ form a poset, and all fundamental $\mathcal{H}$-shortcuts on $P_{uv}$ form the path between $u$ and $v$. Thus, LCS$^\mathcal{H}(u, v)$ forms a path connecting $u$ and $v$.

The $\mathcal{H}$-shortcuts in LCS$^\mathcal{H}(u, v)$ can be partitioned into two sequences: one with strictly increasing powers and one with strictly decreasing powers. To see this, notice that for any sequence of consecutive integers, there is a unique largest $\text{LSBIndex}$ value among the sequence. For any $\mathcal{H}$-node let $p(x) = \text{LSBIndex}(\text{depth}_\mathcal{H}(x) + 1)$. Let $v^*$ be the unique $\mathcal{H}$-node on $P_{uv}$ such that $p(v^*) > p(x)$ for all $\mathcal{H}$-node $x \in P_{uv}, x \neq v^*$. It is straightforward to see that no $\mathcal{H}$-shortcut on $P_{uv}$ crosses $v^*$ and hence LCS$^\mathcal{H}(u, v) = \text{LCS}^\mathcal{H}(u, v^*) \cup \text{LCS}^\mathcal{H}(v^*, v)$.

Now we claim the following: let $P_{x', x''}$ be the ancestor-descendant path such that $p(x') > p(x)$ for all $x$ in $P_{x', x''}$ with $x \neq v'$. Then LCS$^\mathcal{H}(v', v)$ consists of $\mathcal{H}$-shortcuts with decreasing powers. We prove this claim by induction on the value of $p(v')$. For the base case $v' = v$, the largest covering $\mathcal{H}$-shortcuts is an empty set. Consider the case where $p(v') > p(v)$, let $v''$ be the unique node on the path $P_{v', v''}$ such that $p(v'') > p(x)$ for all $x \in P_{v', v''}$ with $x \notin \{v', v''\}$. The shortcut $v' \equiv v''$ must be in LCS$^\mathcal{H}(v', v)$ since the power of $v' \equiv v''$ is strictly greater than the power of any shortcut on $P_{v', v''}$. Then by the induction hypothesis on $P_{v', v''}$, the claim holds. Thus, all $\mathcal{H}$-shortcuts in LCS$^\mathcal{H}(v^*, v)$ have distinct (and decreasing) powers. By symmetry, all $\mathcal{H}$-shortcuts in LCS$^\mathcal{H}(u, v^*)$ also have distinct (and increasing) powers.

Finally, $|\text{LCS}^\mathcal{H}(u, v)| = O(\log \log n)$ since the largest possible power of an $\mathcal{H}$-shortcut is $\lceil \log \log n \rceil - 1$.

$(i, t)$-shortcuts. Let $u$ be a single-child $(i, t)$-node and let $v$ be the $(i, t)$-child of $u$ (which must be either an $(i, t)$-branching node or an $(i, t)$-leaf). Intuitively, the purpose of maintaining $\mathcal{H}$-shortcuts
is to allow to quickly move from $u$ to $v$. To do this, the data structure strives to be able to traverse all of the $\mathcal{H}$-shortcuts in $\text{LCS}^\mathcal{H}(u,v)$, thereby simulating the traversal from $u$ to $v$ by scanning $O(\log \log n)$ $\mathcal{H}$-shortcuts. However, due to simplicity considerations, the data structure does not necessarily store all of the $\mathcal{H}$-shortcuts in $\text{LCS}^\mathcal{H}(u,v)$. Instead, the data structure maintains a set of $(i,t)$-shortcuts that support the following invariant.

**Invariant 3** ($(i,t)$-Shortcuts). Let $u$ be a single-child $(i,t)$-node and let $v$ be the $(i,t)$-child of $u$. Then the $(i,t)$-shortcuts on $P_{uv}$ that are stored by the data structure form a path connecting $u$ and $v$.

![Figure 6: An example of an $(i,t)$-tree and its corresponding $(i,t)$-shortcuts: filled circles are $(i,t)$-nodes, and the curved line segments are $(i,t)$-shortcuts.](image)

An upper bound on the number of $\mathcal{H}$-shortcuts that need to be stored at each $\mathcal{H}$-node is captured by the following straightforward corollary.

**Corollary 4.1.** Assume Invariant 3 holds for all pairs of nodes in $\mathcal{H}$. Then for each node $v \in \mathcal{H}$, and each $(i,t)$ pair, there is at most one downward $(i,t)$-shortcut and at most one upward $(i,t)$-shortcut at $v$.

**Covering and uncovering.** Assume Invariant 3 holds. When the data structure traverses downward from a single-child $(i,t)$-node $u$ to its $(i,t)$-child $v$, the data structure covers several $(i,t)$-shortcuts of higher powers (see Section 4.1). After the traversal, the set of $(i,t)$-shortcuts between $u$ and $v$ is exactly $\text{LCS}^\mathcal{H}(u,v)$. Notice that, unless some of the $(i,t)$-shortcuts between $u$ and $v$ have been removed, subsequent traversals through the $(i,t)$-shortcuts between $u$ and $v$ will span only $|\text{LCS}^\mathcal{H}(u,v)| = O(\log \log n)$ $\mathcal{H}$-shortcuts.

To support structural changes in $\mathcal{H}$ or in $(i,t)$-forests, the data structure will at times uncover an $(i,t)$-shortcut of power $p$ by completely removing the $(i,t)$-shortcut and adding the two consecutive $(i,t)$-shortcuts of power $p - 1$ that were covered by the removed $(i,t)$-shortcut. In order to accommodate an efficient uncovering operation, during a covering operation the data structure continues to store the covered $\mathcal{H}$-shortcuts thereby having them readily available for potential uncovering operations. The $\mathcal{H}$-shortcuts stored by the data structure that are covered by some $(i,t)$-shortcuts
are called supporting $\mathcal{H}$-shortcuts. Supporting shortcuts are accessible from the lower descendant $\mathcal{H}$-node, but not necessarily the ancestor $\mathcal{H}$-node.

**Sharing shortcuts.** An $\mathcal{H}$-shortcut $u \leftrightarrow v$ that is an $(i,t)$-shortcut could also be an $(i',t')$-shortcut when $(i,t) \neq (i',t')$. For efficiency purposes, the data structure stores at most one copy of an $\mathcal{H}$-shortcut even if there are many $(i,t)$ pairs that use this shortcut. With this in mind, the maximum number of distinct $\mathcal{H}$-shortcuts touching a given ancestor-descendant path is bounded by the following lemma.

**Lemma 4.3.** Consider any node $v$ on $\mathcal{H}$. The total number of stored shortcuts having one endpoint at an ancestor of $v$ (including $v$) is $O(\log n \log \log n)$. In particular, the number of distinct fundamental $(i,t)$-shortcuts having one endpoint at an ancestor of $v$ is $O(\log n)$. Moreover, the number of $\mathcal{H}$-shortcuts having both endpoints being ancestors of $v$ (including $v$) is $O(\log n)$.

*Proof.* For a given path $P$, an $\mathcal{H}$-shortcut $u \leftrightarrow v$ is said to be a deviating shortcut if exactly one of its endpoints is on the path $P$.

Let $P$ be the path from $v \in \mathcal{H}$ to the corresponding $\mathcal{H}$-root. For each edge depth $i$ and endpoint type $t$, at most one $(i,t)$-shortcut is deviating from $P$, and each such shortcut has at most $O(\log \log n)$ supporting shortcuts having exactly one endpoint on $P$. In particular, for each $(i,t)$ pair, at most one fundamental $(i,t)$-shortcut deviates from $P$. All $\mathcal{H}$-shortcuts connecting $\mathcal{H}$-nodes on $P$ form a laminar set, and so there are at most $O(\log n)$ such $\mathcal{H}$-shortcuts. Thus, the total number of stored shortcuts with one endpoint in $P$ is $O(\log n \log \log n)$, and the total number of distinct fundamental $(i,t)$-shortcuts on $P$ is $O(\log n)$.

### 4.1 The $\mathcal{H}$-shortcut data structure

**Information stored at nodes.** Due to Corollary 4.1 every node in $\mathcal{H}$ has at most $3d_{\max} + 1 = O(\log n)$ downward $(i,t)$-shortcuts at any given time. Each node $u$ stores an array $\text{Down}_{u}$ of size at most $3d_{\max} + 1$ storing all downward $(i,t)$-shortcuts, together with a bitmap $b_u$ indicating which array slots of $\text{Down}_{u}$ are full. The size of $\text{Down}_{u}$ is chosen to be exactly enough for storing pointers to $(i,t)$-shortcuts for all possible $(i,t)$ pairs as well as one additional slot for temporary use. However, a single shortcut may be shared by many $(i,t)$ pairs. In order to support fast access from $u$ to its downward $(i,t)$-shortcut we make use of a local dictionary which is an array $\text{DownIdx}_{u}$ storing, for each $(i,t)$ pair, a $(\log \log n + 2)$-bit index to the location in $\text{Down}_{u}$ of the appropriate downward $\mathcal{H}$-shortcut, i.e.,

$$\text{Down}_{u}[\text{DownIdx}_{u}[i,t]]$$

points to an $(i,t)$-shortcut leaving $u$, if it exists.

Notice that for an $\mathcal{H}$-node and a power $p$, there is at most one upward $\mathcal{H}$-shortcut from $v$ with power $p$. Thus, each node $v$ maintains an array $\text{Up}_{v}$ of $O(\log \log n)$ pointers to shortcuts, sorted by power to the upwards supporting $\mathcal{H}$-shortcuts of $v$. Moreover, at each node $v$ the data structure stores a $(3d_{\max} + 1)$-length array $\text{UpIdx}_{v}$ of $\log \log n$ bit integers for each $(i,t)$ pair. Thus, the upward $(i,t)$-shortcut $x \leftrightarrow v$ is accessed via

$$\text{Up}_{v}[\text{UpIdx}_{v}[i,t]]$$

points to an $(i,t)$-shortcut entering $v$, if it exists.

The reason for employing two step accessing local dictionaries is that each entry in the $\text{DownIdx}_{u}$ and $\text{UpIdx}_{v}$ is represented with $O(\log \log n)$ bits, and there are $O(\log n)$ $(i,t)$ pairs. These entries

---

2 Notice that when the data structure allocates the array $\text{Down}_{u}$, it is not initialized and assumed to contain junk.
are packed into $O(\log \log n)$ memory words in a specific representation so that the data structure is able to update the entire array efficiently via lookup tables in $O(\log \log n)$ time.

The following straightforward lemma is proven (proof omitted) using bitwise operations or $O(n^\epsilon)$ lookup tables on $\text{DOWN}_u$, $\text{DOWNIDX}_u$, $b_u$, $\text{UPIDX}_x$, and $\text{UP}_v$.

**Lemma 4.4.** The following operations with the given runtimes are supported via shortcut information stored at nodes (runtimes are stated in parentheses):

- Given $u \equiv v$ and a bitmap $b$ of length $3d_{\max} + 1$, add $u \equiv v$ as an $(i,t)$-shortcut for all $(i,t)$ pairs indicated by $b$ ($O(\min(|b| + 1, \log \log n))$ where $|b|$ is the number of 1s in $b$).
- Given $u \equiv v$ and a bitmap $b$ of length $3d_{\max} + 1$, remove the $(i,t)$-shortcut status from $u \equiv v$ for all $(i,t)$ pairs indicated by $b$ ($O(\min(|b| + 1, \log \log n))$ where $|b|$ is the number of 1s in $b$).
- Given $u \in \mathcal{H}$ and an $(i,t)$ pair, return the $(i,t)$-downward $\mathcal{H}$-shortcut leaving $u$ or report that it does not exist ($O(1)$).
- Given $v \in \mathcal{H}$ and an $(i,t)$ pair, return the $(i,t)$-upward $\mathcal{H}$-shortcut leaving $v$ or report that it does not exist ($O(1)$).
- Given $u \in \mathcal{H}$ return an index of an empty slot in $\text{DOWN}_u$ ($O(1)$).
- Given $u \in \mathcal{H}$ enumerate all indices of used locations in $\text{DOWN}_u$ ($O(k + 1)$ where $k$ is the number of the enumerated indices).

**Information stored at shortcuts.** An $\mathcal{H}$-shortcut $u \equiv v$ is a small data structure storing the following information: (1) $\mathcal{P}(u,v)$: the power of the shortcut, (2) pointers to $u$ and $v$, (3) if $u \equiv v$ is stored at index $j$ in $\text{DOWN}_u$, then $u \equiv v$ stores $j$, (4) a $3d_{\max} + 1$ length bitmap $b_{u=v}$ containing one bit for each $(i,t)$ pair (called the $(i,t)$-bit) indicating whether the shortcut is an $(i,t)$-shortcut or not, and (5) if $\mathcal{P}(u,v) > 0$ then $u \equiv v$ stores two pointers to shortcuts that it covers with power $\mathcal{P}(u,v) - 1$.

**Lemma 4.5.** The $\mathcal{H}$-shortcut data structure supports the following operations (amortized runtime is given in parenthesis):

1. Given an $(i,t)$ pair and an $(i,t)$-shortcut that is not a fundamental $\mathcal{H}$-shortcut, uncover the $(i,t)$-shortcut ($O(1)$).
2. Given a single-child $(i,t)$-node $u$ whose $(i,t)$-child is $v$ such that Invariant 3 holds for the pair $u$ and $v$, traverse down the path of $(i,t)$-shortcuts from $u$ to $v$. After the path is traversed, the set of $(i,t)$-shortcuts on this path is exactly $\text{LCS}^H(u,v)$, and so Invariant 3 holds for the pair $u$ and $v$ after the traversal ($O(\log \log n)$).

**Proof.** Uncovering a given $(i,t)$-shortcut $u \equiv v$ that is not a fundamental $\mathcal{H}$-shortcut and has power $p > 0$ is executed by setting $b_{u=v}[i,t] = 0$, following the two pointers from $u \equiv v$ to its supported power $p - 1$ $\mathcal{H}$-shortcuts $u \equiv x$ and $x \equiv v$, and setting $b_{u=x}[i,t] = b_{x=v}[i,t] = 1$. We also update in a straightforward manner some local information in all affected nodes $\{u,v,x\}$ in $O(1)$ time.

**Traversing down an $(i,t)$-shortcut path.** The data structure begins at $u$ and repeatedly follows $(i,t)$-shortcuts going downwards until an $(i,t)$-node is reached. During this traversal, if there are two consecutive $(i,t)$-shortcuts $x \equiv y'$ and $y' \equiv y$ with the same power $p$ and $\text{LSBIndex}(\text{depth}_H(y') + 1)$ is strictly smaller than both $\text{LSBIndex}(\text{depth}_H(x) + 1)$ and $\text{LSBIndex}(\text{depth}_H(y) + 1)$...
1), then the data structure covers the two shortcuts with the $\mathcal{H}$-shortcut $x \equiv y$ having power $p + 1$. This is done as follows.

First the data structure verifies if $x \equiv y$ already exists in the data structure by checking whether $\text{Up}_y[P(x, y)]$ stores the pointer to a shortcut $x \equiv y$ or not. If this shortcut already exists it is accessed through $\text{Up}_y$, and if not then the shortcut is created and a pointer to $x \equiv y$ is added to $\text{Up}_y$.

Then, the data structure sets the $(i, t)$-bit in $b_{x\equiv y}$ to 1, sets the $(i, t)$-bit in $b_{x\equiv y'}$ and $b_{y'\equiv y}$ to 0. So it takes $O(1)$ time to cover an $(i, t)$-shortcut.

Next, the data structure uses $\text{Up}_x$ to access the upwards shortcut $x' \equiv x$ with power $p + 1$. If $x' \equiv x$ is also an $(i, t)$-shortcut then the data structure covers $x' \equiv x$ and $x \equiv y$ with $x' \equiv y$ and recursively follows the upwards shortcut from $x'$ with power $p + 2$.

It is straightforward to verify that at the end of the traversal the set of $(i, t)$-shortcuts connecting $u$ and $v$ is exactly $\text{LCS}^\mathcal{H}(u, v)$. The time cost of traversing down the path is $O(k + |\text{LCS}^\mathcal{H}(u, v)|) = O(k + \log \log n)$ where $k$ is the number of $(i, t)$-shortcuts being covered during the traversal. In Section 7.5 we show that by defining the potential function to be the number of all $(i, t)$-shortcuts that could be covered but are not covered yet, this operation has amortized cost $O(\log \log n)$.

4.2 Uncovering a path and covering

When an edge $(u, v)$ of depth $i$ is deleted, $\mathcal{H}$ goes through several structural changes by merging an ancestor of $u^i$ (or $v^i$) with its $\mathcal{H}$-siblings. So all affected $\mathcal{H}$-nodes (and their $\mathcal{H}$-siblings) ends up being on the paths originating at $u^i$ and $v^i$ and ending at their respective $\mathcal{H}$-roots (see Figure 7). Updating the shortcut information during these structural changes in an efficient manner seems to be a very difficult task. So instead we employ the following strategy.

First, we completely uncover and remove all $\mathcal{H}$-shortcuts that touch $\mathcal{H}$-nodes on the two paths. By Lemma 4.3 there are (1) $O(\log n)$ fundamental shortcuts, (2) $O(\log n)$ shortcuts with both endpoints on the path, and (3) $O(\log \log n)$ deviating shortcuts from each path for each $(i, t)$ pair.

Removing a fundamental shortcut is a local tree operation taking $O((\log \log n)^2)$ time, which we explain in detail in Section 7.3. Uncovering a shortcut with both endpoints on the path takes $O(\log \log n)$ time by Lemma 4.4. Uncovering a non-fundamental deviating $(i, t)$-shortcut takes $O(1)$ time.
Thus, the total cost of uncovering and removing all of the $H$-shortcuts on the affected paths is $O(\log n(\log \log n)^2)$.

After removing these $H$-shortcuts Invariant 3 no longer holds for pairs of $H$-nodes where at least one node is on the affected paths. However, during the deletion operations we never use such shortcuts, so removing them does not affect the other operations that take place during the edge deletion process.

In order to restore Invariant 3 for all pairs of $H$-nodes after the Delete operation, we guarantee that for all $(i, t)$ pairs, all of the fundamental $(i, t)$-shortcuts are added back, which suffices for the invariant. Since the process of removing all $H$-shortcuts on a path or adding the fundamental shortcuts back depend on local trees, we defer the detailed description of these operations to Section 7.3.

Covering all possible shortcuts connecting two nodes on the paths. In addition to adding all of the fundamental shortcuts, the data structure adds back all of the $H$-shortcuts on the paths from $u^i$ and $v^i$ to their corresponding $H$-roots. This is done by traversing the path $O(\log \log n)$ times. In the $p$-th traversal the data structure covers all possible $H$-shortcuts of power $p + 1$ that have both endpoints on the path. Each shortcut is covered in $O(\log \log n)$ time: to cover $x \leftrightarrow y$ from $x \leftrightarrow y'$ and $y' \leftrightarrow y$, the data structure first adds the shortcut $x \leftrightarrow y$ into $Up_y$. Then the data structure computes the bitwise AND of two bitmaps by setting $b_{x \leftrightarrow y} = b_{x \leftrightarrow y'} \land b_{y' \leftrightarrow y}$, and removes the bits in the covered shortcuts by setting $b_{x \leftrightarrow y'} = b_{x \leftrightarrow y'} \oplus b_{x \leftrightarrow y}$ and $b_{y' \leftrightarrow y} = b_{y' \leftrightarrow y} \oplus b_{x \leftrightarrow y}$.

Finally, the data structure updates $Up_{Idx_y}$, $Down_x$ and $Down_{Idx_x}$ according to $b_{x \leftrightarrow y}$. It is straightforward to see that, after $O(\log \log n)$ times of traversal along the path, if there is any $(i, t)$-shortcut with at least one endpoint on the path that could be covered, the other endpoint must be outside of the path and hence it is a deviating $(i, t)$-shortcut.

As mentioned before, we use a potential function to keep track of the number of potential $H$-shortcuts that could be added (by covering existing $H$-shortcuts) by the data structure without a structural change on $H$. By Lemma 4.3 there are $O(\log n \log \log n)$ deviating $H$-shortcuts and so we do not need to explicitly reconstruct all deviating shortcuts since the increase in potential is paid for by the Delete operation.

Lemma 4.6. The data structure supports the following operations on $H$:

- Given an $H$-node $v$, uncover and remove all $H$-shortcuts touching any node that is an ancestor of $v$ ($O(\log n(\log \log n)^2)$).
- Given an $H$-node $v$, add all fundamental $H$-shortcuts that are shared by at least one $(i, t)$ pair. These fundamental $H$-shortcuts have at least one endpoint being an ancestor of $v$ ($O(\log n \log \log n)$).
- Assume Invariant 3 holds. Given an $H$-node $v$, for all $(i, t)$ pair, cover all $(i, t)$-shortcuts having both endpoints at ancestors of $v$ ($O(\log n \log \log n)$).

5 Approximate Counters

In this section, we describe how approximate counters are implemented and how Invariant 2 is maintained after an update to $G$.

Without loss of generality we may assume that the input graph $G$ is a simple graph. Hence, all approximate counters are only required to represent $(1 + o(1))$-approximation of integers in $[0, n^2]$. 

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5.1 Approximate Counters

We are able to efficiently maintain approximate counters for nodes in \( H \) (with the required precision stated in Invariant 2) by updating approximate counters while only using addition operations, without ever using subtraction. Let \( \beta = 2 \) be a parameter that controls the quality of the approximation. Each approximate counter \( \hat{C} \) is defined by a pair \((m, e)\) composed of a mantissa \( m \in \{0, 1\}^{\beta \log \log n} \) and an exponent \( e \in \{0, 1\}^{\log \log n + 1} \). The floating point representation of \( \hat{C} \) concatenates \( m \) and \( e \) into a length \((\beta + 1)\log \log n + 1\) bitstring. The integer representation of \( \hat{C} \) is \( m2^e \), where we treat the mantissa part and the exponent part as unsigned integers. Notice that an approximate counter represents up to \( 2(\log n)^{\beta + 1} \) different integers.

From the definition above, every integer \( C \in [0, n^2] \) is maintained as \( \hat{C} = (m, e) \) in the data structure where \( m \) is the first \( \beta \log \log n \) bits of the binary representation of \( C \) and \( e \) is the number of truncated bits.

By means of adding two approximate counters \( a \) and \( b \), the result \( a + b \) is round down to the nearest possible approximate counter value. Notice that this kind of addition is not associative.

We denote the operation of adding two approximate counters by \( a \oplus b \). The precision guarantee of \( \oplus \) is summarized as follows:

**Corollary 5.1.** Let \( a \) and \( b \) be two approximate counters. Then
\[
(1 - \log^{-\beta} n)(a + b) \leq a \oplus b \leq a + b.
\]

Packing \( O(\log n) \) Approximate Counters. Each node in \( H \) stores \( d_{\text{max}} = \log n \) approximate counters. These counters are stored in \( O(\log \log n) \) words by packing \( O(\log n / \log \log n) \) counters in the floating pointer representation into each word. Thus, with the aid of lookup tables of size \( O(n^\epsilon) \), the following lemma is straightforward.

**Lemma 5.1.** The following operations are supported on approximate counters (runtimes are given in parenthesis):

1. Given a node \( v \in H \) and a depth \( i \), update/return the approximate \( i \)-counter stored at \( v \). \( (O(1)) \).
2. Given the floating point representation of an approximate counter, return its integer representation. \( (O(1)) \).
3. Given the integer representation of an approximate counter, return its floating point representation. \( (O(1)) \).
4. Given two approximate counters \( a \) and \( b \), return \( a \oplus b \). \( (O(1)) \).
5. Given two arrays of \( O(\log n) \) approximate counters packed into \( O(\log \log n) \) words, return an array with \( O(\log n) \) approximate counters packed into \( O(\log \log n) \) words with \( O(\log n) \) coordinate-wise summations of these counters. \( (O(\log \log n)) \).

Notice that the lookup tables are of size \( O(n^\epsilon) \) due to the fifth operation.

6 Local Trees

In this section, we follow the framework of Thorup \[14\]. A local tree \( L(v) \) for a node \( v \) is composed of a three layered binary tree and a special binary tree called the buffer tree. The three layered

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The comparison and the usual addition both act on the integer representation of the approximate counters.
binary tree is composed of a top layer, a middle layer and a bottom layer. The bottom layer is composed of bottom trees of height $O(\log \log n)$.

**H-node representatives.** Intuitively, a local tree $L(v)$ connects $v$ with all $H$-children of $v$. We emphasize that in our construction, for each $H$-child $x$ of $v$, there is a representative $\ell_x$ of $x$ maintained as the local tree leaf in $L(v)$. Technically, an $H$-node and its representative do not need to be stored separately in the data structure. However, they have very different meaning. For example, the approximate counters are only used locally in a local tree to support sampling operation from a branching node, so the approximate counters are not necessarily maintained in $H$-nodes. On the other hand, the $H$-shortcuts has nothing to do with the local trees, so in certain context when the data structure deletes an $H$-node representatives and reinserts it back to $L(v)$, there should be no structural change to the $H$-nodes and the $H$-shortcuts.

The algorithm adds new $H$-node representatives only to the buffer tree, while deletions of $H$-node representatives can take place in both buffer and bottom trees. As in Thorup’s system, groups of sibling $H$-nodes are deleted, merged, and reinserted to buffer trees in response to the promotion of $i$-witness edges. However, in this paper, sometimes the data structure deletes a bottom tree leaf and reinserts it into the buffer tree in the same local tree, even though the structure of $H$ does not change. This solves the efficiency issue caused by updating approximate counters in a long path in order to maintain Invariant 2.

The weight of a node $x$ in the local tree for $v$ denoted by $w(x)$ is defined as the sum of all weights of $H$-children of $v$ in the subtree of $x$. However, the weights are explicitly maintained only in the bottom trees.

**Local tree roots and local tree leaves.** The root of $L(v)$ has two children: the root of the buffer tree and the root of the top tree. The root of $L(v)$ also has a pointer pointing to $v$. Moreover, the local tree root maintains a bitmap indicating for which $(i,t)$ pairs $v$ is a $(i,t)$-branching node in $H$, approximate counters for these $(i,t)$-pairs.

For an $H$-child $x$ of $v$, there is a local tree leaf $\ell_x$ in $L(v)$. $\ell_x$ belongs to either a buffer tree or a bottom tree. The local tree leaf $\ell_x$ stores a pointer to $x$ and the weight of $x$, a parent pointer to a buffer tree node or a bottom tree node (depending on which tree this leaf is in), approximate counters and a bitmap maintaining the local $(i,t)$-status of the leaf, where an $(i,t)$-bit in the bitmap is set to 1 if and only if both $v$ and $x$ are $(i,t)$-nodes. Notice that if $x$ is an $(i,t)$-branching node but $v$ is not, then $x$ is said to have $(i,t)$-status but $\ell_x$ does not have local $(i,t)$-status.

**Buffer Trees.** A buffer tree has at most $2\log^\alpha n$ local tree leaves, where $\alpha$ is a constant to be determined in Section 6.2. Whenever its size exceeds $\log^\alpha n$ either from merging two $H$-nodes, or from inserting a new local tree leaf, this buffer tree becomes a bottom tree. Then the data structure adds this bottom tree into the bottom layer and creates a new empty buffer tree.

The buffer tree is implemented by any $O(\log \log n)$ height mergeable binary tree. Each buffer tree node stores approximate counters, and pointers to either a buffer tree parent or the local tree root. In addition, each buffer tree node maintains a bitmap indicating for each $(i,t)$ pair, whether there is a local tree leaf in its subtree with $(i,t)$-status or not.

**Lemma 6.1.** The data structure supports the following operations on the buffer trees:

- Add a buffer tree leaf. $(O(\log \log n)^2))$
- Delete a buffer tree leaf. $(O(\log \log n)^2))$
- Merge two buffer trees of two sibling $H$-nodes. $O((\log \log n)^2)$

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- Add local \((i, t)\)-status to a buffer tree leaf. \((O(\log \log n))\)
- Remove local \((i, t)\)-status from a buffer tree leaf. \((O(\log \log n))\)

**Bottom Trees.** A bottom tree is a dynamic binary tree subject only to leaf-deletions. When the bottom tree is formed from a buffer tree, it has \(O(\log \log n)\) height with size \(\Theta(\log^a n)\). Each bottom tree node maintains its weight (the weights are computed when the bottom tree is newly formed) and pointers to its parent and its children, the bitmap with the same definition as the bitmap stored in the buffer tree nodes.

**Invariant 4.** The only nodes that are allowed to gain new \((i, t)\)-status, or to increase their approximate \(i\)-counter values are buffer tree nodes and top tree nodes. Moreover, new local tree leaves can only be added into the buffer tree.

**Lemma 6.2.** The data structure supports the following operations on the bottom trees:
- Convert the buffer tree into a new bottom tree \((O(\log^a n))\).
- Delete a bottom tree leaf. \((O((\log \log n)^2))\)
- Remove local \((i, t)\)-status from a bottom tree leaf. \((O(\log \log n))\)

**Middle Trees.** An important feature of the middle trees is that they are maintained as weight-balanced binary trees. The path from any middle tree leaf \(x_B\) (which is also a bottom tree root) to the corresponding middle tree root \(x_M\) has length \(O(\log(w(x_M)) - \log(w(x_B)))\). This property is used to bound the number of local tree nodes being traversed from any \(\mathcal{H}\)-node to its \(\mathcal{H}\)-parent via a local tree.

Each middle tree node \(x\) stores its rank: \(\text{rank}(x) = \lfloor \log w(x) \rfloor\). For each middle tree node \(x\) which is not a middle tree leaf, both children of \(x\), denoted by \(x_L\) and \(x_R\), have the same rank, \(\text{rank}(x_L) = \text{rank}(x_R) = \text{rank}(x) - 1\).

The middle trees are allowed to join if their roots have the same rank. To join two middle trees, the data structure creates a new middle tree node as a middle tree root, and set the pointers to the two tree roots as its children.

Whenever there is a rank decrease at a bottom tree root \(x_B\) due to the removal of a bottom tree leaf, the path from \(x_B\) to the corresponding middle tree root \(x_M\) is removed, leaving \(O(\log(w(x_M)) - \log(w(x_B)))\) middle tree roots. These middle tree roots are re-inserted into the top tree.

Moreover, if any approximate \(i\)-counter is changed at a bottom tree root \(x_B\) with an \((i, \text{primary})\)-status, then the data structure updates the approximate \(i\)-counter values of all \((i, \text{primary})\)-nodes in the path from \(x_B\) to the corresponding middle tree root \(x_M\).

**Local Shortcuts.** Each of the middle trees maintains local shortcut infrastructure in much the same way that shortcuts are maintained in \(\mathcal{H}\). Let \(u\) and \(v\) be two nodes in the same middle tree such that \(u\) is a proper ancestor of \(v\). Then \(u \rightleftharpoons v\) is an eligible local shortcut if and only if for any internal node \(x\) on the path \(P_{uv}\), \(\text{LSBIndex}(\text{rank}(x) + 1) < \min(\text{LSBIndex}(\text{rank}(u) + 1), \text{LSBIndex}(\text{rank}(v) + 1))\). Notice that the \(\mathcal{H}\)-shortcuts are defined from the depths of \(\mathcal{H}\)-nodes which increase along the path from an \(\mathcal{H}\)-root to \(\mathcal{H}\)-leaves, while in the middle trees the ranks of middle tree nodes decrease from an middle tree root to middle tree leaves. However, the definition of power is symmetric between \(u\) and \(v\), so the incremental/decremental direction here makes no difference.

A local shortcut with power 0 is called a trivial shortcut. Trivial shortcuts are simply middle tree edges which are accessible from the middle tree nodes. Similar to the \(\mathcal{H}\)-shortcuts, two local
shortcuts on a path $P$ are not crossing. Similarly we define the power of local shortcuts, and define the covering relation between two local shortcuts on the same path. All eligible local shortcuts in each middle tree forms a poset, and the largest covering local shortcuts between an ancestor-descendant path $P_{uv}$ in a middle tree denoted by $\text{LCS}(u, v)$ is defined. Similar to Lemma 4.2 we have $|\text{LCS}(u, v)| = O(\log \log n)$.

**Local $(i, t)$-trees.** Similar to the definition of the $(i, t)$-forests, consider the induced tree $T$ on a local tree $L(v)$ by taking union of the paths from each local leaves with $(i, t)$-status to the local tree root.

For any local tree node in $T$ with both of its children in $T$ is called a local $(i, t)$-branching node. A local tree node $x$ is said to be a local $(i, t)$-node if

- $x$ is the local tree root.
- $x$ is a local tree leaf with local $(i, t)$-status.
- $x$ is a bottom tree node, a buffer tree node or a top tree node in $T$.
- $x$ is a middle tree root in $T$.
- $x$ is a middle tree node and is a local $(i, t)$-branching node.
- $x$ is a child of a local $(i, t)$-branching node in the middle tree, which we call $x$ single-child $(i, t)$-node.

We define the local $(i, t)$-tree similar to the $(i, t)$-forest defined on $\mathcal{H}$, and each local $(i, t)$-node is said to have $(i, t)$-status. Notice that a local $(i, t)$-tree is always a binary tree, this is important for maintaining the approximate $i$-counter precisions.

**Invariant 5.** Let $u$ be a single-child $(i, t)$-local node and let $v$ be the $(i, t)$-local child of $u$. Then the local $(i, t)$-shortcuts on $P_{uv}$ that are stored by the data structure form a path connecting $u$ and $v$.

**Lemma 6.3.** The data structure supports the following operations on the middle trees:

- Join two middle trees with the same rank. ($O(\log \log n)$)
- Given a middle tree leaf $x_B$, remove the entire path between $x_B$ and its corresponding middle tree root $x_M$. ($O(\log n \log \log n)$)
- Remove $(i, t)$-status from a middle tree leaf. ($O(\log \log n)$)
- Given an edge depth $i \in [1, d_{\text{max}}]$ and a middle tree leaf $x_B$, update all approximate $i$-counters from $x_B$ to its corresponding middle tree root $x_M$. ($O(\log n)$)

**Top Trees.** The top tree is a $O(\log \log n)$ height mergeable binary tree. All middle tree roots are top tree leaves. As a consequence of the middle tree merging procedure described below, each top tree consists of at most $2 \log n$ top tree leaves. Each top tree node $x$ maintains pointers to its parent (or the local tree root if $x$ is the top tree root) and children, approximate counters, a visiting bit and a bitmap of $(i, t)$ pairs indicating whether a local tree leaf with $(i, t)$-status appears in the subtree of $x$.

Whenever the middle tree joining procedure is performed, the entire top tree is immediately rebuilt.
Middle Tree Joining Procedure. There are at most \(\log n\) possible ranks for a middle tree node. If there are at least \(2\log n\) middle trees in a local tree, then the data structure invokes the middle tree joining procedure: repeatedly take two middle tree roots with the same rank, and merge the corresponding two middle trees.

**Lemma 6.4.** The data structure supports the following operations on the top trees:

- **Add a middle tree root into the top tree.** \(O((\log \log n)^2)\)
- **Remove a middle tree root from the top tree.** \(O((\log \log n)^2)\), this operation only comes after merging two \(H\)-nodes
- **Merge the top trees of two local trees.** \(O((\log \log n)^2)\)
- **Given a list \(S\) of all middle tree roots, rebuild the entire top tree.** \(O(|S| \log \log n)\)
- **Update approximate counters along the path from the given top tree leaf \(x_M\) to the top tree root \(x_T\).** \(O((\log \log n)^2)\)

From the construction of the local tree, it is straightforward to see that given an \(H\)-node \(x\), accessing its \(H\)-parent \(v\) takes \(O(\log(w(v)/w(x)) + \log \log n)\) time.

### 6.1 Local Tree Operations

In this section, we prove Lemmas 6.1, 6.2, 6.3, and 6.4, as well as the main local tree Lemma 3.5. Since we have packed the information in a local tree node into \(O(\log \log n)\) words, the operations updating the information inside individual local tree node takes \(O(\log \log n)\) time. Moreover, the height of the top tree, bottom trees and the buffer tree is always \(O(\log \log n)\). Thus, any local tree operation involved in updating a leaf-to-root path inside the top tree, bottom trees and the buffer tree takes \(O((\log \log n)^2)\) time and are instantly paid for each operation. Other operations that involves reconstructing/updating a path of length \(O(\log n)\) in the middle tree, their cost are charged to the updates in some bottom tree that invokes these middle tree operations. Here the constant \(\alpha\) comes in and the polylog overhead cost is amortized through the creation of this polylog sized bottom tree. Hence the amortized cost per these operations is at most a constant.

#### 6.1.1 Buffer Tree Operations — Proof of Lemma 6.1

A buffer tree is implemented by an off-the-shelf mergeable binary tree with \(O(\log \log n)\) amortized time for insertion, deletion, and merging. However, in order to support updates of approximate counters, a \(O(\log \log n)\) overhead is applied to each of the insertion, deletion, and merging operations. Hence the amortized time cost for insertion, deletion, and merging is \(O((\log \log n)^2)\).

To add \((i, t)\)-status to a buffer tree leaf \(x\), the data structure climbs up the buffer tree and sets the \((i, t)\)-bit to 1 to all ancestors of \(x\) in the buffer tree. To remove \((i, t)\)-status from a leaf \(x\), the data structure updates all \((i, t)\)-bits to ancestors of \(x\).

#### 6.1.2 Bottom Tree Operations — Proof of Lemma 6.2

When the buffer tree is converted to a bottom tree, the data structure computes the weight of each bottom tree node, and inserts the bottom tree root/middle tree root (a single node middle tree) into the top tree. If the top tree has at least \(2\log n\) top tree leaves, then a middle tree joining procedure is performed and the top tree is rebuilt.

To delete a bottom tree leaf \(x\), the data structure updates the weights, the approximate counters, and the bitmaps at all ancestors of \(x\) in the bottom tree. For each \((i, t)\) pair such that the \((i, t)\)-bit
is set to 0 at the bottom tree root, the removing \((i, t)\)-status operation is invoked in the middle tree. Furthermore, for each edge depth \(i\), if the \((i, \text{PRIMARY})\)-bit is 1 at the bottom tree root and the approximate \(i\)-counter value is changed, then the data structure updates all approximate \(i\)-counters from the bottom tree root \(x_B\) all the way up to the middle tree root \(x_M\), as well as the ancestors in the top tree.

To remove \((i, t)\)-status from a bottom tree leaf \(x\), the data structure updates the \((i, t)\)-bit to all ancestors of \(x\) in the bottom tree accordingly. If the bottom tree root loses \((i, t)\)-status, then the corresponding middle tree operation is invoked.

### 6.1.3 Middle Tree Operations — Proof of Lemma 6.3

When a middle tree node \(x\) is created, it could be that (1) a new bottom tree is formed from a buffer tree (2) two middle trees rooted at \(x_L\) and \(x_R\) are joined by a new parent. In the former case, the bottom tree root is already maintaining the approximate counters and the bitmaps. In the latter case, the data structure first computes the approximate counters by adding two arrays of \(O(\log n)\) approximate counters in \(O(\log \log n)\) time. Then the data structure sets the bitmap of \(x\) to be the bitwise OR of bitmaps stored in \(x_L\) and \(x_R\). In order to maintain Invariant 5 the data structure adds trivial \((i, t)\)-shortcuts whenever \(x\) has an \((i, t)\)-bit set to 1 but exactly one of \(x_L\) or \(x_R\) has its \((i, t)\)-bit set to 1. This is done in \(O(1)\) time using bitwise operations.

Given a middle tree leaf \(x_B\), to remove the entire path between \(x_B\) and its corresponding middle tree root \(x_M\), the data structure first uncovers all local shortcuts touching the path from \(x_M\) to \(x_B\), by the same algorithm described in Section 4.2. Then the entire path is removed. Notice that, after the path is removed, each middle tree root \(x'\) stores a bitmap indicating which \((i, t)\) pairs appear in at least one middle tree leaf in the middle tree rooted at \(x'\). Then, the data structure enumerates all middle trees roots (as top tree leaves) from the top tree, removes all internal top tree nodes, performs the middle tree joining procedure, and then rebuilds the entire top tree from these middle trees.

**Removing \((i, t)\)-status from a middle tree leaf.** Similar to the \((i, t)\)-forests, in the local \((i, t)\)-tree the middle tree edges between a local \((i, t)\)-branching node \(x\) and its \((i, t)\)-children are not considered as a trivial \((i, t)\)-shortcut. If one of its middle tree children loses its \((i, t)\)-status, a trivial \((i, t)\)-shortcut is added from \(x\) to the other middle tree child \(y\) (both \(x\) and \(y\) lose the \((i, t)\)-status unless \(x\) is the middle tree root.)

To remove \((i, t)\)-status from a bottom tree root/middle tree leaf \(x\), the data structure follows the \((i, t)\)-local upward shortcuts and/or its middle tree parent to find the local \((i, t)\)-parent of \(x\). Next, the data structure adds a trivial shortcut from the \((i, t)\)-parent to the middle tree children that is not an ancestor of \(x\). Then the data structure removes all \((i, t)\)-shortcuts between \(x\) and \((i, t)\)-parent.

It is straightforward to update all approximate \(i\)-counters along the path from a given middle tree leaf \(x_B\) to its corresponding middle tree root \(x_M\).

### 6.1.4 Top Tree Operations — Proof of Lemma 6.4

Similar on implementing the buffer tree, the top tree implements the insertion, deletion and merging in \(O((\log \log n)^2)\) time. The rebuild takes time proportional to the number of middle trees with \(O(\log \log n)\) overhead for updating approximate counters at each top tree node in the post order of an traversal of the top tree.
6.1.5 Local Tree Operations — Proof of Lemma 3.5

Add a new $H$-child. The local tree leaf is created and inserted into the buffer tree.

Delete an $H$-child $x$. Depending on whether the local tree representative of $x$ is in a bottom tree or in the buffer tree, invoke the corresponding operation.

Merge two sibling $H$-nodes $u$ and $v$. To merge two local trees rooted at $u$ and $v$, the data structure merges both their buffer trees and top trees. If the merged buffer tree has at least $\log^\alpha n$ leaves, then a new bottom tree is formed (which creates a single node middle tree root and is inserted into the top tree). Next, if the top tree has at least $2\log n$ leaves, the middle tree joining procedure is invoked and the top tree is rebuilt.

Enumerate all local tree leaves with an $(i,t)$-status. The data structure performs a DFS from the local tree root. If the data structure encounters a top tree, a bottom tree, or a buffer tree node, the bitmaps in both children indicate whether the child contains a local tree leaf with an $(i,t)$-status or not. If the data structure encounters a middle tree node $x$, then the data structure checks in $\text{Down}[\text{DownIdx}[i,t]]$ whether there is a downward local $(i,t)$-shortcut leaving $x$ or not. If no downward local $(i,t)$-shortcut leaving $x$, then $x$ is a local $(i,t)$-branching node and the data structure recursively performs the search on both middle tree children. Otherwise, the data structure navigates downward from a local $(i,t)$-non-branching node $u$ to its highest descendant $(i,t)$-node $v$. The same navigation algorithm described in Section 4.1 is performed so that after the navigation all $(i,t)$-shortcuts on the path $P_{uv}$ are exactly local shortcuts in $\text{LCS}(u,v)$.

Hence, moving from a single-child local $(i,t)$-branching node to its local $(i,t)$-child takes amortized $O(\log \log n)$ time per local tree leaf with an $(i,t)$-status.

Add $(i',t')$-status to a subset of $(i,t)$-leaves. The data structure enumerates all the $(i,t)$-leaves in the subset, and moves each such leaf from the bottom tree to the buffer tree and adds $(i',t')$-status to that buffer tree leaf. If the buffer tree has $\log^\alpha n$ leaves, then the data structure immediately converts the buffer tree into a bottom tree. Each movement of the leaf from a bottom tree to the buffer tree cost $O((\log \log n)^2)$ time (for updating the approximate counters).

Add the $(i,t)$-status to a local tree leaf. This operation is invoked when a fundamental shortcut gets uncovered because an $H$-node becomes an $(i,t)$-branching node. The data structure moves the local tree leaf from the bottom tree into the buffer tree, and adds $(i,t)$-status at the buffer tree. The movement takes $O((\log \log n)^2)$ time.

6.2 Cost Analysis

Each operation involving a bottom tree, buffer tree, or top tree costs $O((\log \log n)^2)$.

For each local tree leaf $x$ being inserted into the buffer tree we store $O(1)$ credits on $x$. Hence when a bottom tree is created we have $O(\log^\alpha n)$ credits. For a bottom tree root $x_B$, by Invariant [1] the following three types of events changes the information stored in $x_B$: (1) its weight is non-increasing so at most $\log n$ rank changes happen to $x_B$; (2) the data structure only removes $(i,t)$-status from $x_B$, so at most $3d_{\text{max}} = O(\log n)$ removals take place; (3) for each edge depth $i$, if $x_B$ has $(i,\text{primary})$-status, then the approximate $i$-counter on $x_B$ is non-increasing. Each approximate $i$-counter represents up to $O(\log^{\beta+1} n)$ values (see Section 5) so at most $O(\log^{\beta+2} n)$ changes occur at $x_B$. 26
Each of the above events triggers a deletion or an update to the entire path from \( x_R \) to the corresponding middle root \( x_M \), and costs \( O(\log n \log \log n) \) time for (1), \( O(\log n) \) for both (2) and (3). Since we set \( \beta = 2 \), the events of the third type is the bottleneck of the cost, which can be amortized to \( O(1) \) as long as \( \alpha \geq \beta + 3 = 5 \).

The removal of the local shortcuts are paid by the creation of the local shortcuts. The local shortcuts are created through (1) creation of a middle tree node (joining two middle trees) and (2) lazy covering. The cost of the first case is paid by the deletion of that middle tree node, which is paid by the creation of the bottom tree. The cost of the second case is due to the removal of \((i, t)\)-status at the corresponding middle tree leaf with an \((i, t)\)-status, which is charged into the second type of the events.

### 6.3 Maintaining Precision when Sampling

Recall that in Invariant 2 we defined \( H(x^j) = (d_{\text{max}} - j) \cdot O(\log \log n) + [\log (w(x^j))] \) for all \( x^j \in \hat{V}_j \). Define similarly for every local tree node \( v \in \mathcal{L}(x^j) \),

\[
H_L(v) = (d_{\text{max}} - j - 1) \cdot O(\log \log n) + [\log (w(v))] + h_{\text{bot/top}}(v)
\]

where \( h_{\text{bot/top}}(v) \) is precisely the number of top, bottom, and buffer trees nodes on a path from \( v \) to a descendant local tree leaf. With this definition, it is straightforward to see that when \( v_L, v_R \) are the children of \( v \), that

\[
H_L(v) \geq \max(H_L(v_L), H_L(v_R)) + 1
\]

We first prove that all nodes in a local tree have the correct precision in terms of \( H(v) \).

#### 6.3.1 Maintaining Invariant 2

Fix an edge depth \( i \) and an \((i, \text{primary})\)-branching node \( x \). From the description of Section 6.1.5, for every local \((i, \text{primary})\)-branching node \( v \in \mathcal{L}(x) \), we have \( \hat{C}_i(v) = \hat{C}_i(v_L) \square \hat{C}_i(v_R) \) where \( v_L \) and \( v_R \) are the two local \((i, \text{primary})\)-children of \( v \).

Assume that for every local \((i, \text{primary})\)-leaf \( \ell_y \) in \( \mathcal{L}(x) \) representing the \((i, \text{primary})\)-child \( y \) of \( x \), \( \hat{C}_i(\ell_y) = \hat{C}_i(y) \) satisfies the Invariant 2.

By induction on \( H_L(v) \),

\[
\hat{C}_i(v) \geq (1 - \log^{-\beta} n) \left( \hat{C}_i(v_L) + \hat{C}_i(v_R) \right) \\
\geq (1 - \log^{-\beta} n) \max(H_L(v_L), H_L(v_R))^{+1} \left( C_i(v_L) + C_i(v_R) \right) \\
\geq (1 - \log^{-\beta} n)^{H_L(v)} C_i(v).
\]

On the other hand, \( \hat{C}_i(v) \leq \hat{C}_i(v_L) + \hat{C}_i(v_R) \leq C_i(v_L) + C_i(v_R) = C_i(v) \). In addition, for any local single-child \((i, \text{primary})\)-node \( u \), the \( i \)-counter \( \hat{C}_i(u) \) is identical to the \( i \)-counter value from its \((i, \text{primary})\)-child \( v \). Since \( H_L(u) \geq H_L(v) \), the precision requirement still holds.

Let \( x \) be the root of \( \mathcal{L}(x^j) \). Then \( H_L(x) \leq H(x^j) \) and Invariant 2 follows, provided the constants in the asymptotic notation in the definitions of \( H_L \) and \( H \) are set appropriately.

#### 6.3.2 Sample an \((i, \text{primary})\)-child

Given an \((i, \text{primary})\)-branching node \( w^{j-1} \). To sample an \((i, \text{primary})\)-child, the data structure starts navigating down the local \((i, \text{primary})\)-tree from the local tree root in \( \mathcal{L}(w^{j-1}) \). For each local
(i, PRIMARY)-branching node \(x\), let \(x_L\) and \(x_R\) be the \((i, PRIMARY)\)-children of \(x\). The data structure then randomly chooses a child with probability proportional to \(\hat{C}(x_L)\) and \(\hat{C}(x_R)\) respectively, and navigates downward using \((i, PRIMARY)\)-shortcuts to find the next local \((i, PRIMARY)\)-branching child or a local tree leaf with \((i, PRIMARY)\)-status.

Let \(x_0\) be the root of \(L(u^{j-1})\), and the sequence \(x_1, x_2, \ldots, x_k\) be all local \((i, PRIMARY)\)-branching nodes which are on the path between \(x_0\) and \(x_{k+1} = \ell_{u^j}\). For all \(1 \leq t \leq k\), let \(x'_t\) and \(x''_t\) be the two local \((i, PRIMARY)\)-children of \(x_t\), and \(x'_t\) is an ancestor of \(\ell_{u^j}\). Then we have for all \(1 \leq t \leq k\), \(\hat{C}_i(x'_t) = \hat{C}_i(x_{t+1})\), and the probability that an \((i, PRIMARY)\)-child \(u^j\) is sampled is at most

\[
\prod_{t=1}^{k} \frac{\hat{C}_i(x'_t)}{\hat{C}_i(x'_t) + \hat{C}_i(x''_t)} \leq \prod_{t=1}^{k} \left[ \frac{\hat{C}_i(x'_t)}{\hat{C}_i(x'_t) \oplus \hat{C}_i(x''_t)} (1 - \log^{-\beta} n)^{-1} \right] = \prod_{t=1}^{k} \left[ \frac{\hat{C}_i(x_{t+1})}{\hat{C}_i(x_t)} (1 - \log^{-\beta} n)^{-1} \right] = \hat{C}_i(x_{k+1})(1 - \log^{-\beta} n)^{-k} / \hat{C}_i(x_1) \leq \hat{C}_i(u^j)(1 - \log^{-\beta} n)^{-(\log(w(u^{j-1})/w(u^j))) + O(\log \log n)}/\hat{C}_i(u^{j-1}).
\]

7 General Operations

7.1 The Batch Sampling Test

In this section we show how the data structure performs the batch sampling test among \(i\)-primary endpoints touching \(u^1\), where \(u^1\) is the \(\mathcal{H}\)-node merged from Section 3.2.1. Let \(s\) be the number of \(i\)-primary endpoints touching \(u^1\), \(s_2\) be the number of \(i\)-secondary endpoints touching \(u^1\), and \(\hat{s}\) be the \((1 + o(1))\)-approximation of \(s\) obtained via Lemma 3.2 (Operation 9).

**Single Sample Test.** To \((1 + o(1))\)-uniformly sample one \(i\)-primary endpoint touching \(u^1\), the data structure sets \(x = v^i\) and recursively performs the following: if \(x\) is an \((i, PRIMARY)\)-leaf, then randomly pick an \(i\)-primary endpoint at \(x\) uniformly at random. If \(x\) is an \((i, PRIMARY)\)-branching node, then the data structure samples an \((i, PRIMARY)\)-child \(x'\) of \(x\) through the local tree \(L(x)\) with probability at most \(\hat{C}_i(x')(1 + (1/\log^3 n))^{\log(w(x)/w(x')) + O(\log \log n)}/\hat{C}_i(x)\). If \(x'\) is a single-child \((i, PRIMARY)\)-node, the data structure repeatedly follows the \((i, PRIMARY)\)-shortcuts leaving \(x'\) to its \((i, PRIMARY)\)-child \(x''\) and recurses by setting \(x = x''\). Otherwise, \(x'\) is either an \((i, PRIMARY)\)-branching node or an \(\mathcal{H}\)-leaf. In this case the data structure sets \(x = x'\) and recurses.

Notice that with accurate counters this procedure picks a perfectly uniformly random \(i\)-primary endpoint. Let \(\langle x, (x, y) \rangle\) be the sampled endpoint and \(x_0 = x, x_1, \ldots, x_k = u^1\) be all \((i, PRIMARY)\)-branching nodes which are ancestors of \(x\) on \(\mathcal{H}\) so that for each \(0 \leq j \leq k\), \(x_{j+1}\) is an ancestor of
Then, the probability of \((x, (x, y))\) being picked is at most
\[
\frac{1}{C_1(x)} \prod_{j=0}^{k-1} \left[ \frac{\tilde{C}_i(x_j)}{\tilde{C}_i(x_{j+1})} (1 - \log^{-\beta} n)^{-\log(w(x_j)/w(x_{j+1})) + O(\log \log n)} \right]
\]
\[
= \frac{1}{C_1(x)} \prod_{j=0}^{k-1} \tilde{C}_i(x_j) (1 - \log^{-\beta} n)^{-\log(w(x_j)/w(x_{j+1})) + O(\log \log n)}
\]
\[
= \frac{1}{C_1(x)} \frac{\tilde{C}_i(x)}{\tilde{C}_i(u^i)} (1 - \log^{-\beta} n)^{-O(\log n \log \log n)}
\]
\[
= (1 + o(1)) \frac{1}{C_i(u^i)} (1 - \log^{-\beta} n)^{-O(\log n \log \log n)} = (1 + o(1)) \frac{1}{C_i(u^i)}.
\]

Notice that in the summation, the \(\log(w(x_j)/w(x_{j+1}))\) terms telescope to \(\log n\) and the \(\log \log n\) terms sum to at most \(\log n \log \log n\).

To check whether \((x, y)\) is a replacement edge or not, it suffices to check whether \(y^i = u^i\). Notice that the endpoint \((y, (x, y))\) is either primary or secondary. Then the status of this edge is confirmed by repeatedly accessing \((i, t)\)-parents at most \(\log n\) times where \(t\) is the endpoint type of \((y, (x, y))\). The time cost telescopes into \(O(\log n \log \log n)\) by summing over the local tree operations accessing \(\mathcal{H}\)-parent from each encountered single-child \((i, t)\)-node. Hence, performing a single sample test costs \(O(\log n \log \log n))\) time.

**The Preprocessing Method.** Notice that another way to sample \(i\)-primary endpoints is to first enumerate all \(i\)-primary endpoints and all \(i\)-secondary endpoints touching \(u^i\) in \(O((s + s_2) \log \log n)\) time, mark all enumerated endpoints and store all \(i\)-primary endpoints in an array. Then the data structure samples an \(i\)-primary endpoint uniformly at random from all enumerated \(i\)-primary endpoints and checks whether the other endpoint is marked in \(O(1)\) time.

**Batch Sampling Test on \(k\) Samples.** The data structure runs the single sample test \(k\) times, and runs the preprocessing method in parallel, and halts whenever one of them finishes its computation. Hence, the time cost for the batch sampling test on \(k\) samples is
\[
O(\min((s + s_2) \log \log n + k, k \log n \log \log n)).
\]

### 7.1.1 Cost Analysis for Sampling Procedure

The sampling procedure either returns a replacement edge, or invokes the enumeration procedure. Once the enumeration procedure is invoked, the data structure upgrades all enumerated \(i\)-secondary endpoints touching \(u^i\) to \(i\)-primary endpoints, and all \(i\)-primary endpoints touching \(u^i\) associated with non-replacement edges are promoted to \((i + 1)\)-secondary endpoints. The first batch sampling test costs
\[
T_1 = O(\min((s + s_2) \log \log n, \log \log s \log n \log \log n)).
\]

The second batch sampling test, if invoked, costs
\[
T_2 = O(\min((s + s_2) \log \log n, \log s \log n \log \log n)).
\]

The enumeration procedure, if invoked, costs
\[
T_E = O((s + s_2)(\log \log n)^2)
\]


time, where \(s_2\) is the number of \(i\)-secondary endpoints touching \(u^i\). Let \(\rho\) be the fraction of the \(i\)-primary endpoints touching \(u^i\) associated with replacement edges before the execution of the sampling procedure. The rest of the analysis is separated into two cases:

**Case 1.** If \(\rho \geq 3/4\), the probability that the first batch sampling test returns with a replacement edge is at least \(1 - (1/4 + o(1))^{O(\log \log s)} > 1 - 1/\log s\). The second batch sampling test, if invoked, returns a replacement edge if at least half the \(O(\log s)\) endpoints sampled belong to replacement edges. By a standard Chernoff bound, the probability that the second batch *fails* to return a replacement edge and halt is \(\exp(-\Omega(\log s)) < 1/s\).

The expected time cost is therefore
\[
T_1 + (1/\log s)T_2 + (1/s)T_E = O((\log n + (s + s_2)/s)(\log \log n)^2) = O((\log n + s_2)(\log \log n)^2)
\]
and is charged to the \texttt{Delete} operation and the upgrade of \(i\)-secondary endpoints to \(i\)-primary status.

**Case 2.** Otherwise, \(\rho < 3/4\). If the enumeration procedure is ultimately invoked, a \(1 - \rho = \Omega(1)\) fraction of the \(i\)-primary endpoints touching \(u^i\) belong to non-replacement edges, which are promoted to depth \(i + 1\), and all \(s_2\) \(i\)-secondary endpoints are upgraded to either \(i\)-primary or \((i + 1)\)-secondary status. In this case the time cost is
\[
T_1 + T_2 + T_E = O((s + s_2)(\log \log n)^2),
\]
which is charged to the promoted/upgraded endpoints. We need to prove that the probability of terminating after the second batch sampling test is sufficiently small. If \(\rho \geq 1/4\) then the probability of the first batch sampling test *not* returning a replacement edge is at most \((3/4 + o(1))^{O(\log \log s)} < 1/\log s\). In this case the expected cost is
\[
T_1 + (1/\log s)T_2 = O(\log n(\log \log n)).
\]
If \(\rho < 1/4\) then, by a Chernoff bound, the probability that at least half the sampled endpoints belong to replacement edges is \(\exp(-\Omega(\log s)) < 1/s\). Therefore the expected cost when the enumeration procedure is not invoked with \(\rho < 1/4\) is at most
\[
(1/s)(T_1 + T_2) = O(\log n(\log \log n)),
\]
which is charged to the \texttt{Delete} operation.

### 7.2 Maintaining \((i,t)\)-forests — Proof of Lemma 3.3

**Add \((i,t)\)-status to an \(\mathcal{H}\)-leaf.** Let \(x\) be the \(\mathcal{H}\)-leaf. In order to identify the \((i,t)\)-branching parent of \(x\), the data structure climbs up \(\mathcal{H}\) and finds the first \(\mathcal{H}\)-node \(x'\) that is either an \((i,t)\)-node or has a downward \((i,t)\)-shortcut \(x' \equiv x''\). If \(x'\) is an \((i,t)\)-branching node, then since the \(\mathcal{H}\)-child of \(x'\) that is also an ancestor of \(x\) is not an \((i,t)\)-node, \(x'\) is the \((i,t)\)-branching parent of \(x\). Otherwise, the data structure performs a binary search on the path \(P_{x'x''}\) to find \((i,t)\)-branching parent as follows:

If \(x' \equiv x''\) is not a fundamental \((i,t)\)-shortcut, the data structure uncover \(x' \equiv x''\) into \(x' \equiv y\) and \(y \equiv x''\) and recurses to one of the two subpaths depending on whether \(y\) is an ancestor of \(x\) or not. Otherwise, \(x' \equiv x''\) is fundamental, then \(x'\) is the \((i,t)\)-branching node. In this case, the data structure uncover the fundamental \((i,t)\)-shortcut \(x' \equiv x''\) (see Section 7.3).

After the \((i,t)\)-branching node is found, the data structure also identifies the single-child \((i,t)\)-node \(z\) and covers fundamental shortcuts along the path \(P_{zx}\) on \(\mathcal{H}\). The cost for walking up these local trees telescopes to \(O(\log n \log \log n)\).

---

\[4\] It is \(1/4 + o(1)\) because the sampling procedure is only \((1 + o(1))\)-approximate.
Remove \((i, t)\)-status from an \((i, t)\)-leaf. Let \(x\) be the \(H\)-leaf. The data structure navigates up from \(x\) by upward \((i, t)\)-shortcuts until it reaches a single-child \((i, t)\)-node \(q\). The intermediate \((i, t)\)-shortcuts are removed by setting the \((i, t)\)-bits to 0.

Then the data structure removes the \((i, t)\)-status of the local tree leaf corresponding to \(q\). If now the \((i, t)\)-parent \(p\) (which is the \(H\)-parent of \(q\)) has only one \((i, t)\)-child \(q'\), \(p\) is no longer an \((i, t)\)-branching node. The data structure removes the \((i, t)\)-status of that leaf in the local tree, removes the \((i, t)\)-branching status of \(p\) and covers the fundamental \((i, t)\)-shortcut from \(p\) to \(q'\).

Notice that this operation is equivalent to first performing the lazy covering on the \((i, t)\)-shortcuts from \(x\) to its \((i, t)\)-parent then removing it. Hence, the time cost for removing \((i, t)\)-status from \(x\) is amortized \(O(\log \log n)\).

Remove \((i, t)\)-status from a set of \((i, t)\)-leaves from a \((i, t)\)-tree \(T\). For each \((i, t)\)-leaf \(x\), the data structure removes the \((i, t)\)-status from \(x\) using the above procedure. The time cost is amortized \(O(\log \log n)\) per leaf.

Given an \((i, t)\)-tree \(T\) and a set of leaves \(S^+\) in \(T\), add \((i', t')\)-status to the leaves in \(S^+\), where \(i' = i\) or \(i + 1\). First of all, the data structure creates a dummy tree induced from the set of the leaves \(S^+\) and the root of \(T\), by first copying the entire \((i, t)\)-tree \(T\), enumerating all its leaves and removing all the leaves that do not belong to \(S^+\).

Hence, we now assume \(S^+\) is the entire leaf set of \(T\). Notice that, after adding \((i', t')\)-status to the leaves in \(S^+\), every \((i, t)\)-branching node in \(T\) is also an \((i', t')\)-branching node. Moreover, for each such \((i, t)\)-branching node, adding \((i', t')\)-status to the node creates at most one new \((i', t')\)-branching node.

Adding \((i', t')\)-status to every \((i, t)\)-branching node is straightforward, by a traversal through \(T\). To identify all newly created \((i', t')\)-branching nodes, consider a path \(P_{uv}\) between a single-child \((i, t)\)-node \(u\) and its \((i, t)\)-child \(v\). If \(u\) is on the \((i', t')\)-tree, then a binary search method is applied to \(P\) so that the data structure identifies the correct \((i', t')\)-branching parent of \(v\) in amortized \(O(\log \log n)\) time. This is proved in Lemma 7.1.

If \(i' = i\), then an \((i, t)\)-root is also a \((i', t')\)-root. Hence, by traversing \(T\), for each such path \(P_{uv}\) between a single-child \((i, t)\)-node \(u\) and its \((i, t)\)-child \(v\), the prerequisite of Lemma 7.1 holds so each \((i', t')\)-branching node is created in amortized \(O(\log \log n)\) time.

Otherwise, \(i' = i + 1\). The data structures identifies all depth-\((i + 1\) \(H\)-nodes that have at least one \((i, t)\)-leaf. Then for each such \(H\)-node it is an \((i', t')\)-root so the prerequisite of Lemma 7.1 holds and the rest part is the same as the \(i' = i\) case. To identify all depth-\((i + 1\) \(H\)-nodes, if the root of \(T\) is an \((i, t)\)-branching node then we enumerate all \((i, t)\)-children through local tree operation. Otherwise, the data structure repeatedly uncovers the root’s downward \((i, t)\)-shortcut until it becomes a fundamental \((i, t)\)-shortcut; following this shortcut leads to a depth-\((i + 1\) \(H\)-node, which is the \((i', t')\)-tree root.

After an \((i', t')\)-branching node \(x\) is created, the data structure uncovers the fundamental \((i', t')\)-shortcut \(x \equiv y\) and adds the \((i', t')\)-status to the local tree leaf representing \(y\) in \(L(x)\). Since at most \(2|T|\) branching nodes are created, the entire operation takes \(O(|T|(|\log \log n|)^2)\) time.

Lemma 7.1. Let \(u\) be a single-child \((i, t)\)-node and \(v\) be the \((i, t)\)-child of \(u\), and suppose the \((i, t)\)-shortcuts connect \(u\) and \(v\) form \(\text{LCS}^H(u, v)\). If \(u\) is also an \((i', t')\)-node, then the data structure identifies the \((i', t')\)-branching parent of \(v\), after \(v\) gains \((i', t')\)-status, in worst case \(O(\log \log n)\) time.

Proof. Since there is a sequence of downward \((i, t)\)-shortcuts connecting from \(u\) to \(v\), the data structure first traverses the \((i, t)\)-shortcuts \(u(= u_0) \equiv u_1, u_1 \equiv u_2, \ldots, u_{k-1} \equiv v(= u_k)\) in the set.
LCS^H(u,v) and finds the last \( H \)-node \( u_\ell \) such that either \( u_\ell \) has a downward \((i',t')\)-shortcut or \( u_\ell \) is an \((i',t')\)-branching node.

By the non-crossing property of the shortcuts, the following straightforward fact is useful to our binary search approach:

**Fact.** If there are an \((i,t)\)-shortcut \( x \rightarrow y \) and an \((i',t')\)-shortcut \( x \rightarrow y' \) with \( y \neq y' \), then \( y \) do not have any \((i',t')\)-leaf in its subtree in \( H \) and the \((i,t)\)-branching node is on the path \( P_{xy} \).

Set \( x = u_\ell \). Then the binary search method is described in detailed as the following:

If there exists an \((i,t)\)-shortcut \( x \rightarrow y \) and an \((i',t')\)-shortcut \( x \rightarrow z \), then the data structure repeatedly uncovers the one with the larger power (or any one if both powers are the same) until either (1) \( z \) is an ancestor of \( v \) and there is a downward \((i,t)\)-shortcut leaving \( z \) (this case can be tested through marking the \( H \)-nodes \( u_1, \ldots, u_k \) first); or (2) both \( x = y \) and \( x = z \) are fundamental. In the first case, the data structure sets \( x = z \) and repeat the above procedure until \( x = v \). In the second case, the data structure sets \( x \) to be an \((i',t')\)-branching node by uncovering the fundamental \((i',t')\)-shortcut and adds the \((i',t')\)-status to the corresponding local tree leaf.

If there exists an \((i,t)\)-shortcut but does not exist any \((i',t')\)-shortcut leaving \( x \), the data structure repeatedly uncovers the \((i,t)\)-shortcut until it is a fundamental shortcut. Then the data structure adds the \((i',t')\)-status to the corresponding local tree leaf.

It is straightforward to see that the power of uncovered \((i',t')\)-shortcuts throughout the algorithm is always decreasing, and that after the procedure is done and the \((i,t)\)-branching node \( u' \) is found, the \((i,t)\)-shortcuts on the path are exactly LCS^H(u,u') and LCS^H(u'',v) where \( u'' \) is the \((i,t)\)-child of \( u' \).

### 7.3 Covering Fundamental Shortcuts — Proof of Lemma 4.6

Let \( P \) be a path from the given \( H \)-node \( u^i \) to the corresponding \( H \)-root \( u^0 \).

**Uncover and remove all \( H \)-shortcuts touching \( P \).** For each \( H \)-node \( x \) iterated from \( u^0 \) to \( u^i \), the data structure first enumerates all downward \( H \)-shortcuts in \( \text{DOWN}_x \). Then the data structure repeatedly uncovers the \( H \)-shortcut with the largest power \( > 0 \) until every \( H \)-shortcut leaving \( x \) is a fundamental \( H \)-shortcut. It is straightforward to see that for each distinct \( H \)-shortcut leaving \( x \), this \( H \)-shortcut is examined and uncovered at most once.

Then the data structure uncovers each fundamental \( H \)-shortcut leaving \( x \) by the following.

To uncover(remove) a fundamental \( H \)-shortcut \( x \rightarrow y \), the data structure first deletes the local leaf \( \ell_y \) in \( L(x) \) representing \( y \) and then inserts \( \ell_y \) into the buffer tree. Notice that this operation does not alter the structure of \( H \), so any \( H \)-shortcut leaving \( y \) is not affected. Then the data structure adds \((i,t)\)-status to \( \ell_y \) for all \((i,t)\) pairs indicated from \( b_{x=y} \). By the local tree operation listed in Lemmas 6.1 and 6.2, the time cost for uncovering(removeing) a fundamental \( H \)-shortcut is amortized \( O((\log \log n)^2) \).

**Adding all fundamental \( H \)-shortcuts touching \( P \) shared by some \((i,t)\) pairs.** There are two types of fundamental \( H \)-shortcut touching \( P \): (1) having both endpoints on \( P \), and (2) deviating from \( P \).

To add all fundamental \( H \)-shortcuts touching \( P \), the data structure checks for edge depth \( j \) iterated from \( i \) to 1 whether to add the fundamental shortcut \( u^{i-1} \rightarrow u^i \) or not. To check this, the data structure first obtains a bitmap \( b \) stored in \( u^i \) indicating which \((i,t)\) pairs have an \((i,t)\)-status at \( u^i \) then accesses the path in the local tree \( L(u^{i-1}) \) from \( u^i \) to the local tree root of \( L(u^{i-1}) \). Throughout the traversal in the local tree, for any node that is a local \((i,t)\)-branching node, the
(i, t)-bit is removed from b. After the data structure reaches $u^{j-1}$, if there is any bit set to 1 in b, then the data structure creates the fundamental $\mathcal{H}$-shortcut $u^{j-1} \equiv u^j$ with $b_{u^{j-1}} \equiv u^j = b$. Also for each (i, t)-bit set to 1 in b, the data structure removes (i, t)-status from the local tree leaf representing $u^i$.

To handle the second case, notice that by Lemma 4.3, for each (i, t) pair there is at most one fundamental (i, t)-shortcut deviating from P. In particular, for an (i, t) pair, at most one deviating fundamental (i, t)-shortcut is added touching the unique $\mathcal{H}$-node $u^i$ such that $u^i$ belongs to an (i, t)-forest but $u^{j+1}$ does not.

In the implementation, the data structure iterates j from $i - 1$ down to 1, and finds the set difference $\text{diff}$ of the bitmap on $u^j$ and the bitmap stored on $u^{j+1}$. For every (i, t) pair indicated from $\text{diff}$, the data structure invokes the local tree operation test whether there is a unique (i, t)-local-leaf or not. If there is a unique (i, t)-local-leaf $\ell_y$ representing an $\mathcal{H}$-node $y$, then the data structure creates the fundamental $\mathcal{H}$-shortcut $u^{j-1} \equiv y$ and removes (i, t)-status from $\ell_y$.

To analyze the time cost, for (1) at most $O(\log n)$ $\mathcal{H}$-shortcuts are covered, and each covering involves multiple (i, t) pairs so each covering can be done in $O(\log(w(u^j)/w(u^{j+1}))) + \log \log n)$ time, which telescopes to $O(\log n \log \log n)$. Moreover, removing (i, t)-status on local tree leaves is charged into the bottom tree updates so it takes amortized $O(\log \log n)$ time. For (2), the amortized $O(\log \log n)$ time is used for each (i, t) pair so the time cost for creating and covering all deviating (i, t)-shortcuts is amortized $O(\log n \log \log n)$.

Cover all (i, t)-shortcuts having both endpoints on P. See Section 4.2.

7.4 Approximate Counters Operations — Proof of Lemma 3.4

Update ancestor approximate i-counters. The data structure updates the approximate i-counters from a given $\mathcal{H}$-leaf $x$ to the corresponding $\mathcal{H}$-root. For every encountered single-child (i, PRIMARY)-node, the data structure copies the approximate i-counter value from its (i, PRIMARY)-child. For every encountered (i, PRIMARY)-branching node $u$, let $v$ be the (i, PRIMARY)-child of $u$ which is an ancestor of $x$. Let $\ell_v$ be the local tree leaf represents $v$. The data structure deletes $\ell_v$ from the local tree and reinserts $\ell_v$ into the buffer tree and then updates its approximate i-counter. The time cost is by updating approximate i-counters in each local tree rooted at an ancestor of $x$, which is $O(\log n (\log \log n)^2)$.

Update approximate i-counters in (i, PRIMARY)-tree $\mathcal{T}$ rooted at $u^i$. The data structure traverses $\mathcal{T}$, and updates the approximate counters in the post-order manner: for any (i, PRIMARY)-node $x$, updates all counters in (i, PRIMARY)-children first, then update the approximate counters at $x$.

To update an approximate i-counter at an (i, PRIMARY)-branching node $x$, the data structure enumerates all its (i, PRIMARY)-children of $x$, deletes and reinserts all of the corresponding local leaves from bottom tree to the buffer tree in $\mathcal{L}(x)$. Then, the data structure update the approximate i-counter $\hat{C}_i(x) = \hat{C}_i(x_L) \oplus \hat{C}_i(x_R)$ where $x_L$ and $x_R$ are the roots of the buffer tree and the top tree of $\mathcal{L}(x)$.

Notice that for all other edge depths $i' \neq i$, Invariant 2 still holds for $i'$ when the local tree leaves are deleted and reinserted into the buffer tree in $\mathcal{L}(x)$. It is not guaranteed that for any given local tree root $x$, $\hat{C}_{i'}(x) = \hat{C}_{i'}(x_L) \oplus \hat{C}_{i'}(x_R)$ where $x_L$ and $x_R$ are the buffer tree root and the top tree root in $\mathcal{L}(x)$.

The time cost is $O(|\mathcal{T}| \log \log n)$. 

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Update approximate counters at a merged/split \( \mathcal{H} \)-node \( x \). The data structure obtains the indices set \( I \) from the bitmap maintained on \( x \) where \( i \in I \) if and only if \( x \) is an \((i, \text{primary})\)-branching node. Then the data structure creates a bitmask of \( O(\log \log n) \) words from \( I \), and copies in parallel the approximate \( i \)-counters for all \( i \in I \) from the local tree root in \( \mathcal{L}(x) \) to \( x \). This can be done by table lookups and bitwise operations in \( O(\log \log n) \) time.

7.5 Lazy Covering: Cost Analysis

The cost of deleting shortcuts is charged to the creation of those shortcuts, and can thus be ignored in an amortized analysis. Since we are using a lazy covering method, the amortized analysis focuses on the supporting potential shortcuts defined as follows:

**Definition 7.1.** For an edge depth \( i \) and an endpoint type \( t \), let \( u \) be a single-child \((i, t)\)-node and \( v \) be the \((i, t)\)-child of \( u \). Then the potential \((i, t)\)-shortcuts are the maximal shortcuts (with respect to the covering relation) that lie on the path \( P_{uv} \). The supporting potential shortcuts are the \( \mathcal{H} \)-shortcuts that support some potential shortcuts.

Notice that the \( \succeq \) definition of the potential shortcuts is in contrast to the actual \((i, t)\)-shortcuts that are stored in the data structure, since the stored \((i, t)\)-shortcuts are not required to be maximal. However, the set of all supporting potential shortcuts is a superset of the set of all stored shortcuts.

For any supporting potential shortcut \( u \equiv v \), let \( k \) be the number of \((i, t)\) pairs such that \( u \equiv v \) is covered by a potential \((i, t)\)-shortcut. Define function \( f \) on \( u \equiv v \) by the following:

\[
f(u \equiv v) = k, \text{ if } u \equiv v \text{ is not a fundamental shortcut, and } 0 \text{ otherwise.}
\]

Let \( C \) be the set of all supporting potential shortcuts and \( C' \) be the set of all stored shortcuts. Then

\[
\Phi = f(C) - f(C') = \sum_{u \equiv v \in C \setminus C'} f(u \equiv v)
\]

is the potential function used by the data structure to amortize the cost of creating \((i, t)\)-shortcut pointers, when they are needed.

**Time Costs for Lazy Covering.** The lazy covering method only covers non-fundamental shortcuts, so each covering takes \( O(1) \) time.

**Potential Changes.** In the beginning of the deletion operation, the data structure spent \( O(\log \log n) \) time to uncover each \( \mathcal{H} \)-shortcut with both endpoints on the path from \( w^i \) (or \( v^i \)) to the corresponding \( \mathcal{H} \)-root. These \( \mathcal{H} \)-shortcuts may be shared by \( \Omega(\log \log n) \) \((i, t)\)-pairs. As mentioned in Section 4.2, the data structure covers every possible potential \((i, t)\)-shortcut on these paths and adds all necessary fundamental \( \mathcal{H} \)-shortcuts after each deletion operation. Hence the increase of \( \Phi \) only depends on the number of deviating shortcuts, which is \( O(\log n \log \log n) \) by Lemma 4.3.

The only other case in which \( \Phi \) increases is due to local changes to \((i, t)\)-trees. These local changes pay for themselves as follows:

- Adding a \((i, t)\)-status to a leaf increases \( \Phi \) by \( O(\log n) \) since all new support maximal \((i, t)\)-shortcuts lie on the path from the leaf to its \((i, t)\)-parent.
- Removing \((i, t)\)-status from a leaf \( x \) increases \( \Phi \) by \( O(\log \log n) \), due to the fact that only the \((i, t)\)-parent of \( x \) loses its \((i, t)\)-status, so all new support maximal \((i, t)\)-shortcuts \( u \equiv v \) have distinct powers. Thus, increases in the corresponding \( f \)-values sum up to \( \log \log n \).
• Merging a dummy tree $T$ induced from an $(i, t)$-tree into $(i', t')$-tree does not increase $\Phi$ because a potential $(i', t')$-shortcuts after merge is always a supporting shortcut for some potential $(i', t')$-shortcut before the merge, or an actual $T$-shortcut. Notice that before the merge the lazy covering method has been performed on $T$ so the actual $T$-shortcuts are the potential $T$-shortcuts.

• Removing $(i, t)$-status from a subset of leaves in a $(i, t)$-tree $T$ increases $\Phi$ by $O(|T| \log \log n)$. Since $O(|T|)$ leaves are removed from a $(i, t)$-tree, and each removal increases $\Phi$ by $O(\log \log n)$.

7.6 Main Operations — Proof of Lemma 3.2

Operation 1 — Add or remove an edge with depth $i$ and endpoint type $t$. The data structure first adds the given edge to the $H$-leaf data structure according to the $(i, t)$ pair. Then the data structure adds $(i, t)$-status to both endpoints of the edge. By Lemma 3.3 the time cost is $O(\log(\log \log n)^2)$.

Operation 2 — Merge a subset of $H$-siblings into $u^1$ and promote all $i$-witness edges touching $u^1$. This is combined from enumerating all $(i, \text{WITNESS})$-children from an $(i, \text{WITNESS})$-branching node in the local tree and navigating downward along the $(i, \text{WITNESS})$-shortcut path. By Lemmas 3.5 and 4.5 the time cost is $O(k(\log \log n)^2)$ where $k$ is the number of promoted $i$-witness edges.

Operation 4 — Promote a subset of $i$-primary endpoints touching $u^1$. This is combined from enumerating all $(i, \text{PRIMARY})$-children from the local tree of an $(i, \text{PRIMARY})$-branching node, and navigating downward along the $(i, \text{PRIMARY})$-shortcut path. Then adds new $(i+1, \text{SECONDARY})$-status to the subset of $(i, \text{PRIMARY})$-leaves. By Lemmas 3.5 and 4.5 the time cost is $O(k(\log \log n)^2)$ where $k$ is the number of promoted $i$-primary endpoints.

Operation 3 — Upgrade all $i$-secondary endpoints touching $u^1$. This is combined from enumerating all $(i, \text{SECONDARY})$-children from the local tree of an $(i, \text{SECONDARY})$-branching node, and navigating downward along the $(i, \text{SECONDARY})$-shortcut path. Then adds new $(i, \text{PRIMARY})$-status to the subset of $(i, \text{SECONDARY})$-leaves. In addition, all approximate counters in the entire $(i, \text{PRIMARY})$-tree is updated via Lemma 3.4 The time cost is $O((p+s)(\log \log n)^2)$ where $p$ is the number of $i$-primary edges touching $u^1$ and $s$ is the number of $i$-secondary edges touching $u^1$.

Operation 5 — Convert an $i$-non-witness edge to an $i$-witness edge. The data structure removes the endpoints of the given edge from the $H$-leaf data structures. If any $i$-primary endpoint is removed, the data structure updates approximate $i$-counters at all $(i, \text{PRIMARY})$-ancestors.

Next, if necessary, the data structure removes $(i, \text{PRIMARY})$-status and/or $(i, \text{SECONDARY})$-status from the leaves that touch the given edge.

Then the data structure adds $(i, \text{WITNESS})$-status to these $H$-leaves. By Lemmas 3.3 and 3.4 the time cost is $O(\log n(\log \log n)^2)$.

Operation 6 — Split an $H$-node $u^i$ with a single child $u^{i}$. The data structure first creates a new $H$-node $x$. Then the local tree leaf representative $\ell_{u^{i}}$ is deleted from $\mathcal{L}(u^{i-1})$. Next, the data structure inserts $\ell_{u^{i}}$ into $\mathcal{L}(x)$. If $i = 1$, then $x$ is an $H$-root and we are done. Otherwise $i > 1$, then the data structure creates a local tree leaf representative $\ell_{x}$ of $x$, accesses $u^{i-2}$ from
through local tree operation, and inserts $\ell_x$ into the local tree $L(u^{i-2})$. The time cost is $O(\log(w(u^{i-2})/w(u^{i-1})) + (\log \log n)^2)$.

**Operation 7** — Enumerate all $(i, t)$ endpoints in the $(i, t)$-tree rooted at $u^i$. The data structure performs a traversal on the implicitly maintained $(i, t)$-tree. For each traversed $(i, t)$-leaf, the data structure enumerates all the endpoints of depth $i$ and type $t$ from the $H$-leaf data structure. By Lemmas 3.5 and 4.5, the time cost is $O(k \log \log n)$, where $k$ is the number of enumerated endpoints.

**Operation 8** — Accessing $H$-parent $v^{i-1}$ from $v^i$. This is a local tree operation. Lemma 3.5, the time cost is $O(\log(w(v^{i-1})/w(v^i)) + \log \log n)$.

**Operation 9** — Accessing an approximate $i$-counter. Accessing the approximate $i$-counter. This can be done by table lookups in $O(1)$ time.

**Operation 10** — Batch Sampling Test. From Section 7.1, the batch sampling test on $k$ samples has time cost $O(\min((p + s) \log \log n + k, k \log n \log \log n))$ where $p$ is the number of $i$-primary edges touching $u^i$ and $s$ is the number of $i$-secondary edges touching $u^i$.

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