Resolved Branes and M-theory on Special Holonomy Spaces

Mirjam Cvetič, G.W. Gibbons, H. Lü and C.N. Pope

Abstract. We review the construction of regular $p$-brane solutions of M-theory and string theory with less than maximal supersymmetry whose transverse spaces have metrics with special holonomy, and where additional fluxes allow brane resolutions via transgression terms. We focus on the properties of resolved M2-branes and fractional D2-branes, whose transverse spaces are Ricci flat eight-dimensional and seven-dimensional spaces of special holonomy. We also review fractional M2-branes with transverse spaces corresponding to the new two-parameter $Spin(7)$ holonomy metrics, and their connection to fractional D2-branes with transverse spaces of $G_2$ holonomy.

1. Introduction

Regular supergravity solutions with less than maximal supersymmetry may provide viable gravity duals to strongly coupled field theories with less than maximal supersymmetry. In particular, the regularity of such solutions at small distances sheds light on confinement and chiral symmetry breaking in the infrared regime of the dual strongly coupled field theory \[1\].

In this contribution we shall review the construction of such regular supergravity solutions with emphasis on resolved M2-branes of 11-dimensional supergravity and fractional D2-branes of Type IIA supergravity, which provide viable gravity duals of strongly coupled three-dimensional theories with $\mathcal{N} = 2$ and $\mathcal{N} = 1$ supersymmetry.

This construction has been referred to as a “resolution via transgression” \[2\]. It involves the replacement of the standard flat transverse space by a smooth space of special holonomy, i.e. a Ricci-flat space with fewer covariantly constant spinors. Furthermore, additional field strength contributions are involved, which are provided by harmonic forms in the space of special holonomy. Transgression–Chern-Simons terms modify the equation of motion and/or Bianchi identity for the original $p$-brane field strength. The construction will be reviewed in Section 2 for the simpler prototype example of a self-dual string in $D = 6$ (0,1) supergravity, and then in Section 3 the results for resolved M2-branes and D2-branes are summarized.

The explicit construction of such solutions has led to some new mathematical developments, for example obtaining harmonic forms for a large class of special holonomy metrics. As a prototype example we shall review the construction of the metric and the middle-dimensional forms for the Stenzel manifolds in $D = 2n$ (with $n \geq 2$ integer) \[3, 4\] (Section 4a). We shall also discuss examples of known $G_2$ holonomy spaces and their associated harmonic forms (Section 4b). In Section 4c we also discuss the old as well as the new two-parameter metric with $Spin(7)$ holonomy \[5, 6\] and the associated harmonic forms (Section 4c).

We then turn to the physics implications for resolved M2-branes \[2, 4\] (Section 5a) and fractional D2-branes \[2, 7\] (Section 5b). In Section 5c we summarize the

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1Strings 2001 contribution, based on the talk given by M. Cvetič
properties of the fractional M2-brane whose transverse space is that of the new Spin(7) holonomy metrics, and the relationship to a fractional D2-brane obtained as a reduction on the $S^1$ fiber of the Spin(7) holonomy metric [5].

In Section 6 we summarize the key results and spell out directions for future work.

The work presented in this paper was initiated in [2] and further pursued in a series of papers that provide both new technical mathematical results and physics implications for resolved $p$-brane configurations [3, 4, 5, 6, 7].

2. Resolution via Transgression

2.1. Motivation. The $AdS_{D+1}/CFT_D$ correspondence [8, 9, 10] provides a quantitative insight into strongly coupled superconformal gauge theories in $D$ dimensions, by studying the dual supergravity solutions. The prototype supergravity dual is the D3-brane of Type IIB theory, with the classical solution

$$ds^2_{10} = H^{-1/2} dx \cdot dx + H^{1/2} (dr^2 + r^2 d\Omega^2_5),$$

$$F_{(5)} = d^4 x \wedge dH^{-1} + \hat{s}(d^4 x \wedge dH^{-1}),$$

$$\Box H = 0 \Rightarrow H = 1 + \frac{R^4}{r^4}. \tag{1}$$

In the decoupling limit $H = 1 + \frac{R^4}{r^4} \rightarrow \frac{R^4}{r^4}$ this reduces to $AdS_5 \times S^5$, which provides a gravitational dual of the strongly coupled $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory.

Of course, the ultimate goal of this program is to elucidate strongly coupled YM theory, such as QCD, that has no supersymmetry. But for the time being important steps have been taken to obtain viable (regular) gravitational duals of strongly coupled field theories with less than maximal supersymmetry. In particular, within this framework we shall shed light on gravity duals of field theories in $D = \{2, 3, 4\}$ with $\mathcal{N} = \{1, 2\}$ supersymmetry.

As a side comment, within $D = 5 \mathcal{N} = 2$ gauged supergravity progress has been made (see [11, 12] and references therein) in the explicit construction of domain wall solutions, both with vector-multiplets and hyper-multiplets, which lead to smooth solutions that provide viable gravity duals of $D = 4 \mathcal{N} = 1$ conformal field theories. However, the aim in this contribution is to discuss the higher dimensional embeddings and the interpretation of such gravity duals.

A typical way to obtain a supergravity solution with lesser supersymmetry is to replace the flat transverse 6-dimensional space $ds^2_6 = dr^2 + r^2 d\Omega^2_5$ of the D3-brane in (1) with a smooth non-compact Ricci-flat space with fewer Killing spinors. In this case the metric function $H$ still satisfies $\Box H = 0$, but now $\Box$ is the Laplacian in the new Ricci-flat transverse space. This procedure ensures one has a solution with reduced supersymmetry; however the solution for $H$ is singular at the inner boundary of the transverse space, signifying the appearance of the (distributed) D3-brane source there.

A resolution of the singularity (and the removal of the additional source) can take place if one turns on additional fluxes (“fractional” branes). Within the D3-brane context, the Chern-Simons term of type IIB supergravity modifies the equations of motion:

$$dF_{(5)} = d\ast F_{(5)} = F_{NS}^{(3)} \wedge F_{RR}^{(3)} = \frac{1}{2\pi} F_{(5)} \wedge \hat{F}_{(5)},$$

$$F_{(3)} = F_{RR}^{(3)} + i F_{NS}^{(3)} = m L_{(3)}, \tag{2}$$
where $L_{(3)}$ is a complex harmonic self-dual 3-form on the 6-dimensional Ricci-flat space. Depending on the properties of $L_3$, this mechanism may allow for a smooth and thus viable supergravity solution. This is precisely the mechanism employed by Klebanov and Strassler, which in the case of the deformed conifold yields a supergravity dual of $D = 4\,\mathcal{N} = 1$ SYM theory. (For related and follow up work see, for example, [13, 14, 15, 16, 17, 18, 19, 20, 21].) For earlier work see, for example, [22, 23, 24, 25].

In a general context the resolution via transgression [2] is a consequence of the Chern-Simons-type (transgression) terms that are ubiquitous in supergravity theories. Such terms modify the Bianchi identities and/or equations of motion when additional field strengths are turned on. $p$-brane configurations with $(n + 1)$-transverse dimensions, i.e. with “magnetic” field strength $F_{(n)}$, can have additional field strengths $F_{(p,q)}$ which, via transgression terms, modify the equations for $F_{(n)}$:

$$dF_{(n)} = F_{(p)} \wedge F_{(q)} ; \quad (p + q = n + 1).$$

If the $(n+1)$-dimensional transverse Ricci-flat space admits a harmonic $p$-form $L_{(p)}$ then the equations of motion are satisfied if one sets $F_{(p)} = mL_{(p)}$, and by duality $F_{(q)} \sim \mu \ast L_{(p)}$. Depending on the $L^2$ normalizability properties of $L_{(p)}$, one may be able to obtain resolved (non-singular) solutions. Let us now illustrate the principle explicitly, first for a somewhat simpler example than that of the D3-brane, namely the self-dual string.

### 2.2. Self-dual string.

The self-dual string in $D = 6$, $(1,0)$ supergravity theory arises from a Lagrangian of the form:

$$\mathcal{L} = \sqrt{g}(R - \frac{1}{12} F_{(3)}^2), \quad F_{(3)} = *F_{(3)}.$$  \hfill (4)

The solution with a flat transverse space is

$$\begin{align*}
 ds_6^2 &= H^{-1} (-dt^2 + dx^2) + H (dr^2 + r^2 \Omega_3^2), \\
 F_{(3)} &= dt \wedge dx \wedge dH^{-1} + \text{dual}, \\
 \Box H &= 0 \Rightarrow H = 1 + \frac{R^2}{r^2}.
\end{align*}$$

(5)

In the decoupling limit we have $H = 1 + \frac{R^2}{r^2} \rightarrow \frac{R^2}{r^2}$, and the space-time becomes $AdS_3 \times S^3$.

A self-dual string with less supersymmetry is obtained by replacing the four-dimensional flat transverse space with the Eguchi-Hanson metric [22]:

$$ds_3^2 = W^{-1} dr^2 + \frac{1}{4} r^2 W (d\psi + \cos \theta \, d\phi)^2 + \frac{1}{4} r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2),$$

where $W = 1 - \frac{a^2}{r^2}$, and $r = \{a, \infty\}$ and the Vierbeine take the form:

$$\begin{align*}
 e^0 &= W^{-1/2} dt, \quad e^3 = \frac{1}{2} r W^{1/2} (d\psi + \cos \theta \, d\phi), \\
 e^1 &= \frac{1}{2} r \, d\theta, \quad e^2 = \frac{1}{2} r \sin \theta \, d\phi.
\end{align*}$$

(7)

This metric is a resolution of the Ricci-flat cone over $S^3/\mathbb{Z}_2$. As $r \rightarrow a$, the metric locally approaches $\mathbb{R}^2 \times S^2$, with $S^2$ being a minimal 2-surface or “bolt.”

The function $H$ is still harmonic, satisfying $\Box H = 0$, but now the Laplacian is calculated in the Eguchi-Hanson transverse space metric [8]:

$$(r^3 W H')' = 0 \Rightarrow H = 1 + c_1 \log \left( \frac{r^2 + a^2}{r^2 - a^2} \right).$$

(8)
As $r \to a$ the solution becomes singular, with the coefficient $c_1$ characterising the strength of a self-dual string source, distributed over the $S^2$ bolt.

The resolved self-dual string solution arises from the modification of the Bianchi identity and equations of motion via transgression terms (coming from a K3 reduction of the heterotic string):

\[ dF_{(3)} = d* F_{(3)} = F_{(2)}^i \wedge F_{(2)}^i. \]

Here $F_{(2)}^i (i = 1, \cdots, 16)$ are the gauge-field strengths in the Cartan subalgebra of $E_8 \times E_8$ or $Spin(32)$.

The modified Ansatz with a single $U(1)$-gauge field turned on is given by

\[ ds^2_{6} = H^{-1} (-dt^2 + dx^2) + H \, ds^2_{4}, \]
\[ F_{(3)} = dt \wedge dx \wedge dH^{-1} + \text{dual}, \]
\[ F_{(2)} = m L_{(2)}, \]
\[ H = -\frac{1}{2} L_{(2)}^2, \]

where $L_{(2)}$ is a self-dual harmonic 2-form in the Eguchi-Hanson metric:

\[ L_{(2)} = \frac{1}{r^4} (e^0 \wedge e^3 + e^1 \wedge e^2). \]

Here the $e^i$ are a Vierbeine (7) for Eguchi-Hanson metric. The harmonic 2-form $L_{(2)}$ is $L^2$-normalizable!

The modified solution is given by

\[ H = 1 + \frac{m^2 + a^4 b}{4a^6} \log \left( \frac{r^2 - a^2}{r^2 + a^2} \right) + \frac{m^2}{2a^4 r^2}. \]

By choosing the integration constant so that $b = -m^2/a^4$, the singular source at the inner boundary of the transverse space is eliminated, yielding a regular solution:

\[ H = 1 + \frac{m^2}{2a^4 r^2}. \]

The solution is supersymmetric, and because $L_{(2)}$ is in $L^2$ it is completely regular, with a well-defined ADM mass.

A reduction on the $\psi, \theta$ and $\phi$ coordinates of the Eguchi-Hanson metric (8) yields in the decoupling limit ($H \to R^2/r^2$) a regular domain wall solution in $D = 3$:

\[ ds^2_{3} = \frac{r^2}{R^2} W (-dt^2 + dx^2) + \frac{R^2}{r^2} \frac{dt^2}{r^2}, \]

which asymptotically approaches $AdS_3$ and provides a gravity dual of 2-dimensional CFT theory with 1/2 of maximal supersymmetry. A straightforward analysis in the IR (small $r$) regime reveals that the bound-state spectrum for minimally coupled scalars is discrete, thus indicating confinement.

3. Summary of Resolved M2-branes and D2-branes

3.1. M2-brane. The transgression term in the 4-form field equation in 11-dimensional supergravity is given by

\[ d* F_{(4)} = \frac{1}{2} F_{(4)} \wedge F_{(4)}, \]

and the modified M2-brane Ansatz takes the form

\[ ds^2_{11} = H^{-2/3} dx^\mu \, dx^\nu \, \eta_{\mu\nu} + H^{1/3} \, ds^2_8, \]
\[ F_{(4)} = d^3 x \wedge dH^{-1} + m \, L_{(4)}. \]

Here the $F_{(4)}^i$ are the gauge-field strengths in the Cartan subalgebra of $E_8 \times E_8$ or $Spin(32)$. The resolved self-dual string solution arises from the modification of the Bianchi identity and equations of motion via transgression terms (coming from a K3 reduction of the heterotic string):

\[ dF_{(3)} = d* F_{(3)} = F_{(2)}^i \wedge F_{(2)}^i. \]

Here $F_{(2)}^i (i = 1, \cdots, 16)$ are the gauge-field strengths in the Cartan subalgebra of $E_8 \times E_8$ or $Spin(32)$.
where $L_{(4)}$ is a harmonic self-dual 4-form in the 8-dimensional Ricci-flat transverse space. The equation for $H$ is then given by

$$\Box H = -\frac{1}{38}m^2 L_{(4)}^2.$$  

For related work see, for example, [17, 27, 28, 29, 30, 31].

### 3.2. D2-brane

The transgression modification in the 4-form field equation in type IIA supergravity is

$$d(e^{rac{1}{2}\phi} \ast F_4) = F_{(4)} \wedge F_{(3)},$$

and the modified D2-brane Ansatz takes the form:

$$\begin{align*}
\text{(18)} & \quad ds_{10}^2 = H^{-5/8} dx^\mu dx^\nu \eta_{\mu\nu} + H^{3/8} ds_7^2, \\
\text{(19)} & \quad F_{(4)} = d^8x \wedge dH^{-1} + m L_{(4)}, \quad F_{(3)} = m L_{(3)}, \quad \phi = \frac{1}{4} \log H,
\end{align*}$$

where $G_{(3)}$ is a harmonic 3-form in the Ricci-flat 7-metric $ds_7^2$, and $L_{(4)} = \ast L_{(3)}$, with $\ast$ the Hodge dual with respect to the metric $ds_7^2$. The function $H$ satisfies

$$\Box H = -\frac{1}{4}m^2 L_{(3)}^2,$$

where $\Box$ denotes the scalar Laplacian with respect to the transverse 7-metric $ds_7^2$. Thus the deformed D2-brane solution is completely determined by the choice of Ricci-flat 7-manifold, and the harmonic 3-form supported by it.

### 3.3. Other Examples

In general the transgression terms modify field equations or Bianchi identities as given in (3), thus allowing resolved branes with $(n+1)$ transverse dimensions for the following additional examples in M-theory and string theory:

- (i) D0-brane: $dF_{(3)} = \ast F_{(4)} \wedge F_{(3)}$,
- (ii) D1-brane: $dF_{(3)}^{\text{RR}} = F_{(5)} \wedge F_{(3)}$,
- (iii) D4-brane: $dF_{(4)} = F_{(3)} \wedge F_{(2)}$,
- (iv) IIA string: $dF_{(3)} = F_{(4)} \wedge F_{(4)}$,
- (v) IIB string: $dF_{(3)}^{\text{NS}} = F_{(5)} \wedge F_{(3)}^{\text{RR}}$,
- (vi) heterotic 5-brane: $dF_{(3)} = F_{(2)} \wedge F_{(2)}$.

In what follows, we shall focus on resolved M2-branes and fractional D2-branes. For details of other examples and their properties, see e.g., [2, 3, 32].

### 4. Mathematical Developments

The construction of resolved supergravity solutions necessarily involves the explicit form of the metric on the Ricci-flat special holonomy spaces. These spaces fall into the following classes:

- Kähler spaces in $D = 2n$ dimensions ($n$-integer) with $SU(n)$ holonomy, and two covariantly constant spinors. There are many examples, with the Stenzel metric on $T^*S^n$ providing a prototype. They are typically asymptotically conical (AC).
- Hyper-Kähler spaces in $D = 4n$ with $Sp(n)$ holonomy, and $n+1$ covariantly constant spinors. Subject to certain technical assumptions, Calabi’s metric on the co-tangent bundle of $\mathbb{CP}^n$ is the only complete irreducible cohomogeneity one example [33].
• In $D = 7$ there are exceptional $G_2$ holonomy spaces with one covariantly constant spinor. Until recently only three AC examples were known \[34, 35\], but new metrics have been recently constructed in \[37, 36\].

• In $D = 8$ there are exceptional $\text{Spin}(7)$ holonomy spaces with one covariantly constant spinor; until recently only one AC example was known \[34, 35\]. New metrics were recently constructed in \[5, 6\].

The focus is on a construction of cohomogeneity one spaces that are typically asymptotic to cones over Einstein spaces. Recent mathematical developments evolved along two directions: (i) construction of harmonic forms on known Ricci-flat spaces (see in particular \[3, 7\]), (ii) construction of new exceptional holonomy spaces \[5, 6, 36, 37\]. In the following two subsections we illustrate these developments by summarizing (i) results on the construction of harmonic forms on the Stenzel metric \[3\] and $G_2$ holonomy metrics \[2, 7\], and (ii) results for the construction of new $\text{Spin}(7)$ two-parameter metrics \[5, 6\], all serving as prototype examples.

4.1. Harmonic forms for the Stenzel metric. The Stenzel\[38\] construction provides a class of complete non-compact Ricci-flat Kähler manifolds, one for each even dimension, on the co-tangent bundle of the $(n + 1)$-sphere, $T^*S^{n+1}$. These are asymptotically conical, with principal orbits that are described by the coset space $\text{SO}(n+2)/\text{SO}(n)$, and they have real dimension $d = 2n + 2$.

4.1.1. Stenzel metric. In the following we summarize the relevant results for the construction of the Stenzel metric. (For more details see \[3\].) This construction \[3, 38\] of the Stenzel metric starts with $L_{AB}$, which are left-invariant 1-forms on the group manifold $\text{SO}(n+2)$. By splitting the index as $A = (1, 2, i)$, we have that $L_{ij}$ are the left-invariant 1-forms for the $\text{SO}(n)$ subgroup, and so the 1-forms in the coset $\text{SO}(n+2)/\text{SO}(n)$ will be

$$
\sigma_i \equiv L_{1i}, \quad \tilde{\sigma}_i \equiv L_{2i}, \quad \nu \equiv L_{12}.
$$

The metric Ansatz takes the form:

$$
ds^2 = dt^2 + a^2 \sigma_i^2 + b^2 \tilde{\sigma}_i^2 + c^2 \nu^2,
$$

where $a$, $b$ and $c$ are functions of the radial coordinate $t$. One defines Vielbeine

$$
e^0 = dt, \quad e^i = a \sigma_i, \quad \tilde{e}_i = b \tilde{\sigma}_i, \quad e^\tilde{0} = c \nu,
$$

for which one can introduce a holomorphic tangent-space basis of complex 1-forms $e^\alpha$:

$$
e^0 \equiv -e^0 + ie^\tilde{0}, \quad e^i = e^i + ie^\tilde{i}.
$$

Defining $a = e^\alpha$, $b = e^\tilde{\alpha}$, $c = e^\tau$, and introducing the new coordinate $\eta$ by $a^n b^\alpha c d\eta = dt$, one finds \[3\] that the Ricci-flat equations can be obtained from a Lagrangian $L = T - V$ which can be written as a “supersymmetric Lagrangian”:

$$
L = \frac{1}{2} g_{ij} (d\alpha^i / d\eta)(d\alpha^j / d\eta) - V,
$$

where the potential can be written in terms of a superpotential, as

$$
V = -\frac{1}{2} g^{ij} \frac{\partial W}{\partial \alpha^i} \frac{\partial W}{\partial \alpha^j}
$$

with

$$
W = \frac{1}{2} (ab)^{n-1} (a^2 + b^2 + c^2).
$$
Here \( \alpha = (\alpha, \beta, \gamma) \). The second-order equations for Ricci-flatness are satisfied if the first-order equations \( d\alpha_i / d\eta = \mp g^{ij} \partial_j W \) are satisfied. Thus we arrive at the first-order equations

\[
\dot{a} = \frac{1}{2} b^{-1} c^{-1} (b^2 + c^2 - a^2), \quad \dot{b} = \frac{1}{2} a^{-1} c^{-1} (a^2 + c^2 - b^2), \quad \dot{c} = \frac{1}{2} a^{-1} b^{-1} (a^2 + b^2 - c^2),
\]

where the dot again denotes the radial derivative \( d/dt \).

The explicit solution of (27) can be written in the form:

\[
a^2 \equiv e^{2\alpha} = R^{1/(n+1)} \coth r, \\
b^2 \equiv e^{2\beta} = R^{1/(n+1)} \tanh r, \\
h^2 = c^2 \equiv e^{2\gamma} = \frac{1}{n+1} R^{-n/(n+1)} (\sinh 2r)^n,
\]

where

\[
R(r) \equiv \int_0^r (\sinh 2u)^n \, du.
\]

Here the radial coordinate \( r \) is introduced as \( dt = h \, dr \).

For each \( n \) the result is expressible in relatively simple terms. For example, one finds

\[
n = 1 : R = \sinh^2 r; \quad n = 2 : R = \frac{1}{8} (\sinh 4r - 4r); \quad n = 3 : R = \frac{2}{5} (2 + \cosh 2r) \sinh^4 r.
\]

The case \( n = 1 \) is the Eguchi-Hanson metric [26], and for \( n = 2 \) it is the deformed conifold [39].

As \( r \) approaches zero, the metric takes the form

\[
ds^2 \sim dr^2 + r^2 \sigma_1^2 + \sigma_2^2 + \nu^2,
\]

which has the structure locally of the product \( \mathbb{R}^{n+1} \times S^{n+1} \), with \( S^{n+1} \) being a “bolt.” As \( r \) tends to infinity, the metric becomes

\[
ds^2 \sim dp^2 + p^2 \left\{ \frac{2}{(n+1)} \nu^2 + \frac{n}{2(n+1)} (\sigma_1^2 + \sigma_2^2) \right\},
\]

representing a cone over the Einstein space \( SO(n+2)/SO(n) \).

4.1.2. Harmonic middle-dimension \((p,q)\) forms. An Ansatz compatible with the symmetries of the Stenzel metric is of the form:

\[
L_{(p,q)} = f_1 \epsilon_{i_1 \cdots i_p-j_1 \cdots j_p} \epsilon^0 \wedge \tilde{e}^{i_1} \wedge \cdots \wedge \tilde{e}^{q-1} \wedge \epsilon^{j_1} \wedge \cdots \wedge \epsilon^{j_p}
\]

\[
+ f_2 \epsilon_{i_1 \cdots i_p-j_1 \cdots j_p} \epsilon^0 \wedge \epsilon^{i_1} \wedge \cdots \wedge \epsilon^{i_{p-1}} \wedge \tilde{e}^{j_1} \wedge \cdots \wedge \tilde{e}^{j_q},
\]

with \( f_1, f_2 \) being functions of \( r \), only. The harmonicity condition becomes \( dL_{(p,q)} = 0 \), since \( *L_{(p,q)} = 1^{p-q} L_{(p,q)} \). The functions \( f_1, f_2 \) are solutions of coupled first-order homogeneous differential equations, yielding a solution that is finite as \( r \to 0 \):

\[
f_1 = q_2 F_1 \left[ \frac{1}{2} p, \frac{1}{2} (q+1), \frac{1}{2} (p+q) + 1; -(\sinh 2r)^2 \right],
\]

\[
f_2 = -p_2 F_1 \left[ \frac{1}{2} q, \frac{1}{2} (p+1), \frac{1}{2} (p+q) + 1; -(\sinh 2r)^2 \right].
\]

For any specific integers \((p,q)\), these are elementary functions of \( r \).

For the two special cases of greatest interest, they have the following properties:

- \((p,p)\)-forms in \(4p\)-dimensions: \( f_1 = -f_2 = \frac{1}{(\cosh r)^p} \) with \( |L_{(p,p)}|^2 = \frac{\text{const}}{(\cosh r)^{2p}} \),

  falls-off fast enough as \( r \to \infty \). This turns out to be the only \( L^2 \) normalizable form!
\( (p+1,p) \)-forms in \((4p+2)\)-dimensions. As \( r \to \infty \): \( |L_{(p+1,p)}|^2 \sim \frac{1}{\sinh(2r)^2} \) which is marginally \( L^2 \)-non-normalizable.

As for physics implications, the case in \( 2(n+1) = 4 \) dimensions with an \( L^2 \)-normalizable \( L_{(1,1)} \)-form is precisely the example of the resolved self-dual string discussed in detail in Section 2.2.

In \( 2(n+1) = 6 \) dimensions, the \( L_{(2,1)} \)-form was constructed in [34, 35], and provides a resolution of the D3-brane. Since \( L_{(2,1)} \) is only marginally non-normalizable as \( r \to \infty \), the decoupling limit of the space-time does not give an AdS\(_5\), but instead there is a logarithmic modification. In particular, this modification accounts for a renormalization group running of the difference of the inverse-squares of the two gauge group couplings in the dual \( SU(N) \times SU(N+M) \) SYM [24].

On the other hand in \( 2(n+1) = 8 \) dimensions the \( L^2 \) normalizable \( L_{(2,2)} \)-form supports additional fluxes that resolve the original \( M2 \)-brane. The details of the explicit solution, as well as general properties of \( M2 \)-branes, will be given in Section 5.1.

It turns out that one can construct regular resolved \( M2 \)-branes for many other examples of 8-dimensional special holonomy transverse spaces. In particular, examples with the original \( Spin(7) \) holonomy transverse space [2], a number of new Kähler spaces [2, 7], and hyper-Kähler spaces [4], can all support \( L^2 \) normalizable 4-forms and therefore can give rise to resolved supersymmetric \( M2 \)-brane solutions.

### 4.2. Old \( G_2 \) holonomy metrics and their harmonic forms.

#### 4.2.1. Resolved cones over \( S^2 \times S^4 \) and \( S^2 \times \mathbb{CP}^2 \).

The first type of complete Ricci-flat 7-dimensional metrics of \( G_2 \) holonomy, obtained in [34, 35], correspond to \( R^3 \) bundles over four-dimensional quaternionic-Kähler Einstein base manifolds \( M \). These spaces are of cohomogeneity one, with level surfaces that are \( S^2 \) bundles over \( M \) (also known as “twistor spaces” over \( M \)). There are two cases that arise, with \( M \) being \( S^4 \) or \( \mathbb{CP}^2 \). Thus the two surfaces have level surfaces that are \( \mathbb{CP}^1 \) \((S^2 \) bundle over \( S^4 \)) or the flag manifold \( SU(3)/(U(1) \times U(1)) \) \((S^2 \) bundle over \( \mathbb{CP}^2 \)), respectively. These two manifolds are the bundles of self-dual 2-forms over \( S^4 \) or \( \mathbb{CP}^2 \) respectively. They approach \( R^3 \times S^4 \) or \( R^3 \times \mathbb{CP}^2 \) locally near the origin. (The calculations for the two cases, with the principal orbits being \( S^2 \) bundles either over \( S^4 \) or over \( \mathbb{CP}^2 \), proceed essentially identically.)

The complete Ricci-flat 7-metric on the bundle of self-dual 2-forms over \( S^4 \) was constructed in [34, 35]. In the notation of [35], it is given by

\[
\tilde{d}s^2 = h^2 \, dv^2 + a^2 \, (D\mu^i)^2 + b^2 \, d\Omega^2,
\]

where \( \mu^i \) are coordinates on \( R^3 \) subject to \( \mu^i \mu^i = 1 \); \( d\Omega^2 \) is the metric on the unit 4-sphere, with \( SU(2) \) Yang-Mills instanton potentials \( A^i \), and

\[
D\mu^i \equiv d\mu^i + \epsilon_{ijk} A^j \mu^k.
\]

The field strengths \( J^i \equiv dA^i + \frac{1}{4} \epsilon_{ijk} A^j \wedge A^k \) satisfy the algebra of the unit quaternions, \( J^i_{\alpha \gamma} J^j_{\beta \delta} = -\delta_{ij} \delta_{\alpha \beta} + \epsilon_{ijk} J^k_{\alpha \beta} \). A convenient orthonormal basis is

\[
\bar{e}^0 = h \, dr, \quad \bar{e}^i = a \, D\mu^i, \quad \bar{e}^\alpha = b \, e^\alpha.
\]

(Note that although \( i \) runs over 3 values, there are really only two independent Vielbeine on the 2-sphere, because of the constraint \( \mu^i \mu^i = 1 \).)
Constructing fractional D2-branes requires finding a suitable harmonic 3-form, which is square-integrable at short distance and whose dual 4-form has a non-vanishing flux integral at infinity. In fact a fully $L^2$-normalizable harmonic 3-form in this example exit was obtained in [4] and is given by

$$L_{(3)} = \frac{1}{2} u_1 \mu^i \epsilon_{ijk} \tilde{e}^0 \wedge \tilde{e}^j \wedge \tilde{e}^k + \frac{1}{2} u_2 \mu^i J_{i\alpha\beta} \tilde{e}^0 \wedge \tilde{e}^\alpha \wedge \tilde{e}^\beta + \frac{1}{2} J_{i\alpha\beta} u_3 \tilde{e}^i \wedge \tilde{e}^\alpha \wedge \tilde{e}^\beta,$$

where the harmonicity conditions $dG_{(3)} = 0$ and $d\ast G_{(3)} = 0$ impose first-order equations on the functions $u_{1,2,3}$. There is a regular solution, given by

$$u_1 = \frac{1}{r^4} + \frac{P(r)}{r^3 (r^4 - 1)^{1/2}},$$

$$u_2 = -\frac{1}{2(r^4 - 1)} + \frac{P(r)}{r (r^4 - 1)^{3/2}},$$

$$u_3 = \frac{1}{4r^4 (r^4 - 1)} - \frac{(3r^4 - 1) P(r)}{4r^5 (r^4 - 1)^{3/2}},$$

where

$$P(r) = F\left(\frac{1}{2} \pi \right) - F(\arcsin(r^{-1})| - 1) = \int_{\arcsin(r^{-1})}^{\arcsin(1)} \frac{d\theta}{\sqrt{1 + \sin^2 \theta}}.$$ 

Note that the functions $u_i$ satisfy $u_1 + 2u_2 + 4u_3 = 0$, which turns out to be a key condition required for compatibility with the supersymmetry constraints for fractional D2-branes. The asymptotic form of (39) reveals that the origin it approaches locally $\mathbb{R}^4 \times S^3$. (Topologically, it is in fact just $\mathbb{R}^4 \times S^3$, since the bundle is topologically trivial.)

4.2.2. *Resolved cone over $S^3 \times S^3$.* The remaining complete 7-dimensional manifold of $G_2$ holonomy obtained in [34, 35] is again of cohomogeneity one, with principal orbits are topologically $S^3 \times S^3$. The manifold is the spin bundle of $S^3$; near the origin it approaches locally $\mathbb{R}^4 \times S^3$. The radial coordinate runs from $r = a$ to $r = \infty$. We define an orthonormal frame by

$$e^0 = \alpha dr, \quad e^i = \gamma \Sigma_i, \quad e^j = \beta u_i,$$

where $\nu_i \equiv \sigma_i - \frac{1}{2} \Sigma_i$.

The metric [42] admits the following regular harmonic 3-form [2]:

$$L_{(3)} = v_1 e^0 \wedge e^j \wedge e^i + v_2 \epsilon_{ijk} e^j \wedge e^k \wedge e^i + \frac{1}{6} v_3 \epsilon_{ijk} e^j \wedge e^k \wedge e^i,$$
where the functions $v_1$, $v_2$ and $v_3$ are given by

$$v_1 = -\frac{(3r^2 + 2r + 1)}{r^4(r^2 + r + 1)^2}, \quad v_2 = \frac{(r^2 + 2r + 3)}{r(r^2 + r + 1)^2}, \quad v_3 = \frac{3(r + 1)(r^2 + 1)}{r^4(r^2 + r + 1)}.$$  

The functions $v_{1,2,3}$ satisfy the relation $3v_1 - 3v_2 + v_3 = 0$, which is crucial for establishing that $L_{(3)}$ is compatible with supersymmetry of the associated deformed D2-brane. The harmonic 3-form is square-integrable at short distance, but gives a linearly divergent integral at large distance. As shown in [2], the short-distance square-integrability is enough to give a perfectly regular deformed D2-brane solution, even though $L_{(3)}$ is not $L^2$-normalizable.

### 4.3. New Spin(7) holonomy metrics.

#### 4.3.1. The old metric and harmonic 4-forms.

Until recently only one explicit example of a complete non-compact metric on a Spin(7) holonomy space was known [34, 35]. The principal orbits are $S^7$, viewed as an $S^3$ bundle over $S^4$. The solution (47) is asymptotic to a cone over the “squashed” Einstein 7-sphere, and it approaches $\mathbb{R}^4 \times S^4$ locally at short distance (i.e. $r \approx \ell$). The metric is of the form:

$$ds_8^2 = \left(1 - \frac{f^{10/3}}{r^{10/3}}\right)^{-1} dr^2 + \frac{9}{100} r^2 \left(1 - \frac{f^{10/3}}{r^{10/3}}\right) h_i^2 + \frac{9}{20} r^2 d\Omega_4^2,$$

where $h_i \equiv \sigma_i - A^i_{(1)}$, and the $\sigma_i$ are left-invariant 1-forms on $SU(2)$, $d\Omega_4^2$ is the metric on the unit 4-sphere, and $A^i_{(1)}$ is the $SU(2)$ Yang-Mills instanton on $S^4$. The $\sigma_i$ can be written in terms of Euler angles as

$$\sigma_1 = \cos \psi \, d\theta + \sin \psi \, \sin \theta \, d\varphi, \quad \sigma_2 = -\sin \psi \, d\theta + \cos \psi \, \sin \theta \, d\varphi, \quad \sigma_3 = d\psi + \cos \theta \, d\varphi.$$  

A regular harmonic 4-form in this metric was obtained in [2] and is given in terms of a 3-form potential $B_{(3)}$ by $L_{(4)} = dB_{(3)}$. The 3-form $B_{(3)}$ is given by

$$B_{(3)} = f \, h_1 \wedge h_2 \wedge h_3 + g \, h_i \wedge F^i,$$

where

$$f = \frac{1}{3} \left[ \left( \frac{f}{\ell} \right)^{2/3} - 1 \right], \quad g = \left( \frac{f}{\ell} \right)^{2/3},$$

yielding an $L^2$-normalizable 4-form.

#### 4.3.2. New Spin(7) holonomy metric.

The generalization that we shall consider involves allowing the $S^3$ fibers of the previous construction themselves to be “squashed.” Namely, the $S^3$ bundle is itself written as a $U(1)$ bundle over $S^2$ leading to the following “twice squashed” Ansatz:

$$ds_8^2 = dt^2 + a^2 \mu_1^2 + b^2 \mu_2^2 + c^2 \, d\Omega_4^2,$$

where $a$, $b$ and $c$ are functions of the radial variable $t$. (The previous Spin(7) example has $a = b$.) Here

$$\mu_1 = \sin \theta \, \sin \psi, \quad \mu_2 = \sin \theta \, \cos \psi, \quad \mu_3 = \cos \theta,$$

are the $S^2$ coordinates, subject to the constraint $\mu_i \mu_i = 1$, and

$$D \mu^i \equiv d\mu^i + \epsilon_{ijk} \sigma_j A^i_{(1)} \mu_k, \quad \sigma \equiv d\varphi + A^i_{(1)}, \quad A^i_{(1)} \equiv \cos \theta \, d\psi - \mu^i A^i_{(1)},$$

where the field strength $F_{(2)}$ of the $U(1)$ potential $A_{(1)}$ turns out to be given by:

$$F_{(2)} = \frac{1}{2} \epsilon_{ijk} \mu^k D \mu^i \wedge D \mu^j - \mu^i F^i_{(2)}.$$
The Ricci-flatness conditions can be satisfied by solving the first-order equations of a supersymmetric Lagrangian, yielding the following special solution (for details see [3, 4]):

$$ds^2_8 = \frac{(r-\ell)^2 dt^2}{(r-3\ell)(r+\ell)} + \ell^2 \frac{(r-3\ell)(r+\ell)}{(r-\ell)^2} \sigma^2 +$$

$$\frac{1}{2} (r-3\ell)(r+\ell) (D\mu^i)^2 + \frac{1}{2} (r^2 - \ell^2) d\Omega^2_4,$$

(54)

The quantity $\frac{1}{2} \sigma^2 + (D\mu^i)^2$ is the metric on the unit 3-sphere, and so in this case we find that the metric smoothly approaches $\mathbb{R}^4 \times S^4$ locally, at small distance ($r \to 3\ell$), i.e. it has the same topology as the old $Spin(7)$ holonomy space. On the other hand, it locally approaches $\mathcal{M}_7 \times S^1$ at large distance. Here $\mathcal{M}_7$ denotes the 7-manifold of $G_2$ holonomy that is the $\mathbb{R}^4$ bundle over $S^3$. Asymptotically the new metric behaves like a circle bundle over an asymptotically conical manifold in which the length of the $U(1)$ fibers tends to a constant; in other words, it is ALC.

If one takes $r$ to be negative, or instead analytically continues the solution so that $\ell \to -\ell$ (keeping $r$ positive), one gets a different complete manifold. Thus instead of (54), the quantity $\frac{1}{2} \sigma^2 + (D\mu^i)^2$ is precisely the metric on the unit 7-sphere, and so as $r$ approaches $\ell$ the metric $ds^2_8$ smoothly approaches $\mathbb{R}^8$. At large $r$ the function $b$, which is the radius in the $U(1)$ direction, approaches a constant, and so the metric tends to an $S^1$ bundle over a 7-metric of the form of a cone over $\mathbb{C}P^3$; it has the same asymptotic form as (54). The manifold in this case is topologically $\mathbb{R}^8$.

In [3, 4] the general solution to the first-order system of equations is obtained, leading to additional families of regular metrics of $Spin(7)$ holonomy, which are complete on manifolds $\mathbb{B}^8_{\pm}$ that are similar to $\mathbb{B}_{8}$. These additional metrics have a non-trivial integration constant which parameterizes inequivalent solutions. (For details see [3] and Appendix A of [5]).

### 4.3.3. $L^2$-normalized harmonic 4-forms in the new $Spin(7)$ metrics.

$L^2$-normalized harmonic 4-forms for the new $Spin(7)$ 8-manifolds were obtained in [3]. The starting point is the following Ansatz:

$$L_{(4)} = u_1 (h a^2 b dr \wedge \sigma \wedge X_{(2)} \pm c^4 \Omega_{(4)}) + u_2 (h b c^2 dr \wedge \sigma \wedge Y_{(2)} \pm a^2 c^2 X_{(2)} \wedge Y_{(2)}) + u_3 (h a c^2 dr \wedge Y_{(3)} \mp b a c^2 \sigma \wedge X_{(3)}),$$

(55)

where $\Omega_{(4)}$ is the volume form of the unit $S^4$, and

$$X_{(2)} \equiv \frac{1}{2} \epsilon_{ijk} \mu^i D\mu^j \wedge D\mu^k, \quad X_{(3)} \equiv D\mu^i \wedge F^i_{(2)},$$

$$Y_{(2)} \equiv \mu^i F^i_{(2)}, \quad Y_{(3)} \equiv \epsilon_{ijk} \mu^i D\mu^j \wedge F^k_{(2)}.$$

(56)

The upper and lower sign choices in (55) correspond to self-dual and anti-self-dual 4-forms respectively. Here a radial coordinate $r$ is introduced, which is related to $t$ by $dt = h dr$.

The harmonicity $dL_{(4)} = 0$ of $L_{(4)}$ implies first-order coupled differential equations for the functions $u_1$, $u_2$ and $u_3$ (for details see [3]). For the metric (54), the regular solution is given by

$$u_1 = \frac{2(r^4 + 8r^3 + 34r^2 - 48r + 21)}{(r-1)^3(r+1)^5} \quad u_2 = \frac{r^4 + 4r^3 - 18r^2 + 52r - 23}{(r-1)^3(r+1)^5},$$

$$u_3 = \frac{2(r^2 + 14r - 11)}{(r-1)^2(r+1)^5}.$$

(57)
(For the sake of simplicity we set \( \ell = 1 \).) It can be seen that \( L(4) \), whose norm is \(|L(4)|^2 = 48(u_1^2 + 2u_2^2 + 4u_3^2)\), is in \( L^2 \).

For the manifold corresponding to the analytic continuation to \( \ell \to -\ell \) the resulting anti-selfdual 4-form is specified by the following functions:

\[
(58) \quad u_1 = \frac{2}{(r+1)^3(r+3)}, \quad u_2 = \frac{r^2 + 10r + 13}{(r+1)^3(r+3)^3}, \quad u_3 = -\frac{2}{(r+1)^2(r+3)^3}.
\]

(Again for the sake of simplicity we set \( \ell = 1 \).) The resulting 4-form is again \( L^2 \)-normalizable.

Both of the above harmonic anti-self-dual 4-forms (58) and (57) satisfy the linear relation

\[
\quad u_1 + 2u_2 - 4u_3 = 0,
\]

which ensures the supersymmetry of the resolved brane configurations. (Explicit expressions for the self-dual forms have also been obtained in [5], but these do not yield supersymmetric resolved brane configurations.)

5. Applications

5.1. Resolved M2-branes.

5.1.1. Resolved M2-brane with transverse Stenzel space. In this subsection, we shall summarize the construction of a deformed M2-brane using the 8-dimensional Stenzel metric for the transverse \( ds_8^2 \). In this case, the index \( i \) on \( \sigma_i \) and \( \tilde{\sigma}_i \) in the metric (22) runs over 3 values. The Ricci-flat solution coming from the first-order equations (27) is given by

\[
\quad a^2 = \frac{1}{3}(2 + \cosh 2r)^{1/4} \cosh r, \quad b^2 = \frac{1}{3}(2 + \cosh 2r)^{1/4} \sinh r \tanh r, \quad h^2 = c^2 = (2 + \cosh 2r)^{-3/4} \cosh^3 r,
\]

with the metric then given by (22). The radial coordinate runs from \( r = 0 \) to \( r = \infty \), and the metric lives on a smooth complete non-compact manifold.

The \( L^2 \)-normalizable self-dual harmonic 4-form (of type (2, 2)) is a special case of (33) and is given by:

\[
L(2, 2) = \frac{3}{\cosh^3 r} [e^\tilde{0} \wedge e^1 \wedge e^2 \wedge e^3 + e^0 \wedge e^\tilde{1} \wedge e^2 \wedge e^3] + \frac{1}{2 \cosh r} \epsilon_{ijk} [e^0 \wedge e^i \wedge e^j \wedge e^k + e^{\tilde{0}} \wedge e^i \wedge e^{\tilde{j}} \wedge e^k],
\]

implying that \(|L(2, 2)|^2 = \frac{360}{\cosh r}\).

The 8-dimensional Stenzel manifold can be used as the transverse space for constructing a deformed M2-brane, by employing the explicit construction in Subsection 3.1 and the explicit form (60). The solution of (16) and (17) can be obtained by making a coordinate redefinition, \( 2 + \cosh 2r = y^4 \), in terms of which \( H \) is given by

\[
H = c_0 - \frac{5m^2}{4\sqrt{2}} \left( 5y^5 - 7y \right) + \frac{25m^2}{4\sqrt{2}} F(\arcsin(\frac{1}{y})) - 1,
\]

where \( c_0 \) is an integration constant, and \( F(\phi|m) \equiv \int_0^{\phi} (1 - m \sin^2 \theta)^{-1/2} d\theta \) is the incomplete elliptic integral of the first kind.
The function $H$ is regular for $r$ running from 0 to infinity. Near $r = 0$, $H$ approaches a constant, and at large $r$ we have

$$H = c_0 + \frac{640m^2}{2187\rho^6} - \frac{20480 \, 2^{1/3} m^2}{28431 \, 3^{2/3} \rho^{26/3}} + \cdots,$$

(62)

where $\rho$ is the proper distance, defined by $h \, dr = d\rho$. Thus the M2-brane has no singularity, and it has a well-defined ADM mass.

5.1.2. Properties of Resolved M2-branes. (i) Supersymmetry For the supersymmetry of the deformed solution to be preserved, the harmonic 4-form must satisfy

$$F_{abcd} \Gamma_{bcd} \eta = 0.$$

(63)

where $\eta$ is a covariantly constant spinor in the Ricci flat 8-space. For $L(2, 2)$ in the Stenzel metric, one can show that this is indeed satisfied for each such spinor. In other words, turning on the deforming flux from the harmonic 4-form $G^{(4)}$ does not lead to any further breaking of supersymmetry, and so the resolved M2-brane preserves 1/4 of the original supersymmetry.

For other examples of the Kähler, hyper-Kähler, and old $\text{Spin}(7)$ holonomy transverse spaces, $L^2$-normalizable 4-forms were also constructed, that lead to supersymmetric M2-branes with the same numbers of preserved supersymmetries as the M2-branes without the additional field strengths.

(ii) Flux Integrals Interestingly, there are no additional conserved charges\textsuperscript{3} for the resolved M2-brane. In particular, the resolved M2-brane with the transverse Stenzel metric has $\int_{S^4} F_4 \sim m L_{(2, 2)} \sim (\cosh r)^{-1} \int_{S^4} \nu \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3 \to 0$ thus leading to zero “fractional” charges. Here $S^4$ is the bolt in the 8-dimensional Stenzel metric, and had the above integral been non-zero this would have implied non-zero fractional charge for M5-branes wrapped around a three-cycle dual to $S^4$ within $M_7$. The electric flux integral of the original M2-brane is determined by $\int_{M_7} \ast F^{(4)} \sim m^2$, leading to a non-zero charge proportional to $m^2$. Here $M_7$ is the transverse seven-dimensional base over which the resolved cone (of the Ricci flat space) is defined.

All the known examples on resolved M2-branes with transverse spaces that are asymptotically conical turn out to have zero fractional charge.

(iii) Dual Field Theory The $L^2$-normalizability of the 4-forms supporting additional fields-strengths of the resolved M2-branes imply that in the decoupling limit the large distance space-time is still of the form $AdS_4 \times M_7$. This indicates that in the ultraviolet (UV) we shall get $D = 3$ CFT, with an $\mathcal{N} = 2$ field theory ($\mathcal{N} = 1$ for the original $\text{Spin}(7)$ space and an $\mathcal{N} = 3$ theory for the hyper-Kähler space). On the other hand, the additional field strengths imply a deformation by relevant operators associated with the pseudo-scalar fields in a dual field theory. This property is conjectured in\textsuperscript{3} to extend to all resolved M2-branes whose Ricci flat transverse spaces are asymptotically conical.

5.2. Properties of fractional D2-branes. Fractional D2-branes involve $G_2$ holonomy metrics, and the examples are summarized for each of the original $G_2$

\textsuperscript{3} The issue of brane charges, and their relation to the notion of brane wrapping, is a subtle one, which we shall not pursue in detail here. Thus, we shall just follow some standard terminology for now.
holonomy metrics in Subsection 4.2. In the following subsections we summarize the properties of fractional branes for these examples.

5.2.1. Cones over $S^2 \times S^4$ and $S^2 \times \mathbb{CP}^2$. We can substitute the explicit solution for $L_{(3)}$ into (19) and obtain the solution for the resolved fractional D2-brane. It is difficult to give a fully explicit expression for $H$, since the harmonic 3-form itself has a rather complicated expression. However, the large and small distance analysis of the solution is straightforward. In these limits we have

$$
\text{Large distance : } H = c_0 + \frac{Q}{r^3} - \frac{m^2}{4r^6} + \cdots,
$$

$$
\text{Small distance : } H = c_1 - \frac{11m^2 (r - 1)}{24} + \cdots,
$$

where

$$
Q = \frac{m^2}{30} \int_1^{\infty} dr \, r^4 \sqrt{r^4 - 1} |L_{(3)}|^2.
$$

The solution is asymptotically of the same form as the D2-brane without fractional charges, except that the D2-brane charge $Q$ is now proportional to $m^2$. At short distance, on the other hand, the solution is regular. Examining the flux associated with additional field strength $F_{(4)}$ yields a finite integer $L_{(4)} \equiv *L_{(3)} = \frac{1}{4} u_1 r^4 \Omega_{(4)} + \cdots$, (where $\Omega_{(4)}$ is the volume form of the unit 4-sphere and $u_1$ is given by (40)) over the 4-sphere at infinity:

$$
P_4 \equiv \frac{1}{V(S^4)} \int L_{(4)} = \frac{1}{7}.
$$

This result implies that our D2-brane solution carries additional wrapped D4-brane charge.

This is a completely regular supersymmetric solution describing the usual D2-brane together with a fractional D2-brane coming from the wrapping of D4-branes around a 2-cycle. (There is an analogous fractional D2-brane in which the Ricci-flat 7-manifold is replaced by the related example where the principal orbits are $S^2$ bundles over $\mathbb{CP}^2$ instead of $S^4$.)

To summarise, because of the existence of a normalizable harmonic 3-form on these Ricci-flat 7-manifolds with the topology of $\mathbb{R}^3$ bundles over $S^4$ or $\mathbb{CP}^2$, the corresponding fractional D2-brane solutions are regular both at small distances as well as at large distances. Thus, in the decoupling limit they have the same asymptotic large-distance behaviour as regular D2-branes with Euclidean transverse spaces. As a consequence, the dual asymptotic field theory is that of a regular D2-brane (whose original charge $Q \sim N$ determines the SYM factor $SU(N)$), but now with a different overall charge $Q \propto m^2 \sim M$, which is related to the contribution of the additional fluxes of the wrapped D4-branes, and thus indicating a single SYM factor $SU(M)$. These deformed solutions preserve $1/16$ of the original supersymmetry and so they describe a dual three-dimensional $\mathcal{N} = 1$ field theory.

---

\footnote{In the context of M-theory on G2 holonomy spaces the properties of harmonic three forms play an important role as axion moduli in $D = 4 \mathcal{N} = 1$ field theory. For details see [40].}
5.2.2. Resolved cone over $S^3 \times S^3$. The solution for the function $H$ in the corresponding deformed D2-brane solution (19) was shown to be given by [2]:

$$H = c_0 + \frac{m^2 (r+1)(16r^7 + 24r^6 + 48r^5 + 47r^4 + 54r^3 + 36r^2 + 18r + 9)}{108r^3 (r^2 + r + 1)^3} + \frac{8m^2}{27\sqrt{3}} \arctan \left[ \frac{2r + 1}{\sqrt{3}} \right].$$

(67)

The function $H$ is non-singular for $r$ running from the origin at $r = 1$ to infinity. (Note also that the small distance behaviour of the above solution is qualitatively the same as that of the fractional D2-brane discussed in the previous subsection, thus indicating the same universal infrared behaviour of the dual field theories for both types of solution.)

The fractional flux is supported by an harmonic 3-form that is not $L^2$ nor-
malizable, and the solution, while regular at small distances, corresponds at large
distances to the resolution of the D2-brane whose “charge”-$Q$ is now given in terms
of the parameter $m$ by $Q = -4m^2/15r$. The deformed D2-brane solution also has a
non-vanishing flux for the NS-NS 5-brane, which wraps around the $S^3$. Specifically,
at large $r$ we find that $L_3 = \frac{1}{3!} v_3^3 \Omega_3 + \cdots$, (where $\Omega_3$ is the volume form and
$v_3$ is defined in (46)) and this leads to a non-vanishing, finite $F_3$ flux, integrated
over the 3-sphere associated with the metric $\Sigma_3^2$ at infinity,

$$P_3 \equiv \frac{1}{V(S^3)} \int L_3 = \frac{1}{\sqrt{3}}.$$

(68)

This implies that the D2-brane carries additional wrapped NS-NS 5-brane charge and the solution can be viewed as a fractional NS-NS 2-brane, together with the usual D2-brane supported by the 4-form.

The linear growth of the “charge” with distance accounts for the asymptotic
renormalization group running of $g_1^{-2} - g_2^{-2}$, linearly with the energy scale, where
$g_1$ and $g_2$ are the gauge couplings of two SYM factors. The solution preserves
1/16 of the original supersymmetry, thus describing a regular supergravity dual of
a three-dimensional $\mathcal{N} = 1$ field theory.

5.3. Fractional M2-branes with new Spin(7) holonomy space. One of the main motivations for constructing new Spin(7) holonomy metrics was to try
to relate them to 7-metrics of $G_2$ holonomy, and thus in turn to find interesting
relations between fractional M2-branes and D2-branes. Indeed the Spin(7) holono-
my metrics, described in Subsection 4.3, asymptotically approach $S^3 \times M_7$ locally,
where the size of the circle is finite and $M_7$ is asymptotically the $G_2$ holonomy space,
whose principal orbits are those of the $S^2$ bundle over $S^4$.

The normalizable, supersymmetric 4-form implies a resolved M2-branes solution. For the harmonic form, the function $H$ is given by

$$H = 1 + \frac{m^2 (1323r^6 + 9786r^5 + 32937r^4 + 64428r^3 + 52237r^2 - 136934r + 29983)}{1680(r+1)^3(r-1)^2}.$$

(69)

(Again for the sake of simplicity we set $\ell = 1$.) The solution is regular everywhere.
For the manifold obtained by the analytic continuation $\ell \rightarrow -\ell$, the solution is given by

$$
H = 1 + \frac{m^2(3r^2 + 26r + 63)}{20(r + 1)^2(r + 3)^5}.
$$

(70)

The solution is smooth everywhere, and it interpolates between eleven-dimensional Minkowski spacetime at small distance and $M_5 \times S^1 \times M_7$ locally at large distance.

Note that this brane does have a fractional charge: the 4-form $F_{(4)}$ carries a magnetic M5-brane charge proportional to $m$ in addition to the electric M2-brane charge proportional $m^2$. The magnetic charge is given by $Q_m = \frac{1}{\omega_4} \int F_{(4)} = q m$, where $\omega_4$ is the volume of the unit 4-sphere, and $q = \frac{1}{2}$ for the two cases discussed above. Thus, in contrast to the resolved M2-branes with transverse asymptotically flat spaces, these resolved M2-brane solutions, describe fractional magnetic M2-branes as wrapped M5-branes, together with the usual electric M2-brane. This is because the new $Spin(7)$ holonomy spaces are not asymptotically conical, but instead have an asymptotically local conical structure with an $S^1$ whose radius tends to a constant at infinity.

The fractional D2-brane obtained from the wrapping of a D4-brane around the $S^2$ in a manifold of $G_2$ holonomy was conjectured in [4] to be related to a resolved M2-brane with a transverse 8-dimensional space of $Spin(7)$ holonomy. The examples found in [4] and summarized above provide a concrete realization of this conjecture: reducing the solution on the $U(1)$ fiber in $Spin(7)$ holonomy space yields a regular D2-brane charge proportional to $m^2$ (coming from the double dimensional reduction of the M2-brane), and a fractional D2-brane charge proportional to $m$ arising from D4-branes wrapped on 2-cycles (coming from the vertical reduction of the M5-brane), as well as a wrapped D6-brane charge proportional to $\ell$. The D6-brane charge is a consequence of the Kaluza-Klein reduction on the $U(1)$ fiber in the original Ricci-flat space, corresponding to a reduction from $D = 11$ M-theory to $D = 10$ Type IIA supergravity [41, 42, 43].

6. Conclusions and open avenues

In this contribution we have presented a summary of some recent developments in the construction of regular $p$-brane configurations with less than maximal supersymmetry. In particular, the method involves the introduction of complete non-compact special holonomy metrics and additional fluxes, supported by harmonic-forms in special holonomy spaces, which modify the original $p$-brane solutions via Chern-Simons (transgression) terms.

The work led to a number of important mathematical developments which we have also summarized. Firstly, the construction of harmonic forms for special holonomy spaces in diverse dimensions was reviewed, and the explicit construction of harmonic forms for Stenzel metrics was summarized. In particular, only middle dimension harmonic forms of the type $(n, n)$ in $D = 4n$ dimensions are $L^2$ normalizable, while those of type $(n + 1, n)$ in $D = 4n + 2$ are marginally non-$L^2$ normalizable. Secondly, a construction of new two-parameter $Spin(7)$ holonomy spaces was discussed. These have the property that they interpolate asymptotically between a local $S^1 \times M_7$, where the length of the circle is finite and $M_7$ is the $G_2$ holonomy space with the topology of the $S^2$ bundle over $S^4$, while at small distance they approach the “old” $Spin(7)$ holonomy space with the topology of the chiral spin bundle over $S^4$. 
The mathematical developments also led to a number of important physics implications, relevant for the properties of the resolved p-brane solutions. In particular, in this contribution we focused on the properties of resolved M2-branes with 8-dimensional special holonomy transverse spaces (e.g., Stenzel, hyper-Kähler and Spin(7) holonomy spaces) and the fractional D2-branes with three 7-dimensional $G_2$ holonomy transverse spaces. Resolved M2-branes are always supported by $L^2$-normalizable harmonic forms and thus they are regular at short distance and have decoupling limits at large distance that yield $AdS_4$. They have no conserved additional (fractional) charges, but only the usual M2-brane electric charge (which is now proportional to $m^2$, where $m$ is the strength of the additional flux). The dual 3-dimensional dual field theory is therefore a superconformal field theory (with $\mathcal{N} = 1$ or $\mathcal{N} = 2$ supersymmetry) which is in turn perturbed by marginal operators associated with pseudo-scalar fields [31]. On the other hand the fractional D2-branes are either supported by $L^2$ normalizable harmonic forms (in the case of $G_2$ holonomy spaces which are $\mathbb{R}^3$ bundles over $S^4$ or $\mathbb{CP}^2$) or by a harmonic form that has a linearly divergent integrated norm (in the case of the $\mathbb{R}^4$ bundle over $S^3$). In all these cases there are conserved fractional charges corresponding to D4-branes wrapping the 2-cycles dual to $S^4$ ($\mathbb{CP}^2$) in $M_7$, or to NS-NS 5-branes wrapping the 3-cycle dual to $S^3$ in $M_7$, respectively.

An interesting implication involves the properties of fractional M2-branes using the new Spin(7) holonomy spaces. After reduction on $S^1$ these give fractional D2-branes where, in addition to the original D2-brane charge, there is fractional magnetic charge for D4-branes wrapping 2-cycles and for D6-branes wrapping 4-cycles, arising as a consequence of the Kaluza-Klein reduction. The fact that the resolved M2-brane on the new spin Spin(7) holonomy space has non-zero fractional charge is a consequence of the asymptotically locally conical structure of the new Spin(7) holonomy space.

There are a number of open avenues in the exploration of further properties of such solutions. In particular, it is of importance to study the properties of the dual three dimensional field theories in greater detail.

Another important direction involves the study of novel special holonomy spaces in M-theory. In particular, novel constructions of $G_2$ holonomy spaces [36, 37] are of importance in the study of $\mathcal{N} = 1$ D=4 field theory aspects of M-theory [40, 41, 42] and the study of M-theory on spaces of $G_2$ holonomy has recently attracted considerable attention. Specifically, it has been proposed that M-theory compactified on a certain singular seven-dimensional space with $G_2$ holonomy might be related to an $\mathcal{N} = 1$, $D = 4$ gauge theory [10, 43, 44] that has no conformal symmetry. The quantum aspects of M-theory dynamics on spaces of $G_2$ holonomy can provide insights into non-perturbative aspects of four-dimensional $\mathcal{N} = 1$ field theories, such as the preservation of global symmetries and phase transitions. For example, Ref. [40] provides an elegant exposition and study of these phenomena for the three manifolds of $G_2$ holonomy that were obtained in [34, 35].

A new exciting development in this direction is the discovery of M3-brane configurations [15, 36] which have a flat 4-dimensional worldvolume and the transverse space that is a deformation of the $G_2$ along with the 4-form field strength turned on. These configurations turn out to have zero charge and ADM mass (leading to naked singularities at small distances). A study of their properties is the subject of further research.
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Department of Physics and Astronomy, University of Pennsylvania, Philadelphia, PA 19104, USA, and, Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106-4030
E-mail address: cvetic@cvetic.hep.upenn.edu

DAMTP, Centre for Mathematical Sciences, Cambridge University, Wilberforce Road, Cambridge CB3 0WA, UK
E-mail address: G.W.Gibbons@damtp.cam.ac.uk

Michigan Center for Theoretical Physics, University of Michigan, Ann Arbor, MI 48109, USA
E-mail address: honglu@umich.edu

Center for Theoretical Physics, Texas A&M University, College Station, TX 77843, USA
E-mail address: pope@physics.tamu.edu