Statistics of sums of correlated variables described by a matrix product ansatz

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Abstract – We determine the asymptotic distribution of the sum of correlated variables described by a matrix product ansatz with finite matrices, considering variables with finite variances. In cases in which the correlation length is finite, the law of large numbers is obeyed, and the rescaled sum converges to a Gaussian distribution. In contrast, when the correlation extends over the system size, we observe either a breaking of the law of large numbers, with the onset of giant fluctuations, or a generalization of the central limit theorem with a family of non-standard limit distributions. The corresponding distributions are found as mixtures of delta functions for the generalized law of large numbers, and as mixtures of Gaussian distributions for the generalized central limit theorem. Connections with statistical physics models are emphasized.

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Introduction. – The law of large numbers and the central limit theorem are cornerstones of equilibrium statistical physics. Indeed, the very existence of deterministic values of macroscopic observables in large systems relies on the law of large numbers, while the Gaussian shape of tiny fluctuations around the mean value is described by the central limit theorem. The latter also bears strong connections with random walks [1]. Basic forms of these theorems are known for independent and identically distributed (i.i.d.) random variables [1–3]. Some generalizations with less restrictive assumptions are also known [3,4], and it is often assumed that these theorems remain valid as long as distributions of individual variables are not too broad and do not differ too much one from the other, and as long as correlations are weak enough. These assumptions are, however, not always valid in non-equilibrium systems, where the relevant observables may have an infinite mean (e.g., in aging [5,6] or laser cooling [7] phenomena), or may have long-range correlations (e.g., in boundary-driven [8,9] or active [10,11] systems), leading to the breakdown of the law of large numbers. Similarly, the standard central limit theorem breaks down in a number of cases. For broadly distributed variables with infinite variance, the generalized central limit theorem yields Lévy-stable laws [2], with many applications often related to anomalous diffusion [5]. Non-Gaussian distributions have also been found, for instance, in the context of $1/f^n$-noise [12] and related problems [13,14], where summed variables have very different statistics. For strongly correlated variables, generalizations of the central limit theorem have been derived for Gaussian processes [15,16]. However, in a statistical physics context, another class of strongly correlated variables, defined through a matrix product ansatz, has emerged from the exact solution of many types of one-dimensional non-equilibrium models, like the Asymmetric Simple Exclusion Process (ASEP) [9,17–23] and generalizations including several types of particles [20,24], as well as different types of reaction-diffusion processes [20,25–29], including, e.g., the branching-coalescing random walk and the asymmetric Glauber-Kawasaki process [30,31]. Note that such types of lattice models have proven useful in the study of intracellular motility [32] and of vehicular traffic [33,34]. The matrix product ansatz has also been used recently to solve coupled Kardar-Parisi-Zhang
(KPZ) equations [35]. Although infinite matrices may be needed, notably in the context of the ASEP model [18,20], many models can be solved using finite matrices, including reaction-diffusion models [20,25-31], the coupled KPZ equations [35] and the ASEP model for specific parameter values [9,19,36]. In spite of the increasing importance of this class of random variables in the description of correlated non-equilibrium systems, the corresponding generalizations of the law of large numbers and of the central limit theorem are presently not known. In this letter, we aim at providing such generalizations for variables described by a matrix product ansatz with finite matrices. Our study encompasses both discrete random variables as in the ASEP model and reaction-diffusion processes [18,20], and continuous ones as in signal processing [37,38] or in mass transport models [39]. For the sake of clarity, we restrict our presentation to specific, yet representative cases, deferring a full-length account of our results to a forthcoming publication [40].

**Matrix product representation.** — Following [37,38], we consider a set of random variables \(x_1, \ldots, x_N\) whose joint probability distribution can be described by a matrix product ansatz, namely

\[
P(x_1, \ldots, x_N) = \frac{1}{\mathcal{L}(E^N)} \mathcal{L}(\mathcal{R}(x_1) \mathcal{R}(x_2) \ldots \mathcal{R}(x_N)),
\]

where \(\mathcal{R}(x)\) is a \(D \times D\) matrix function \((D \geq 2)\) with real non-negative entries, \(E = \int_{-\infty}^{\infty} \mathcal{R}(x) \, dx\), and \(\mathcal{L}\) is a linear form defined as

\[
\mathcal{L}(M) = \text{tr}(A^T M),
\]

with \(A\) a given \(D \times D\), non-zero matrix with real non-negative entries (an extension relaxing this positivity condition for \(A\) will be mentioned at the end of this letter). We further assume that for all \(N \geq 1\), \(\mathcal{L}(E^N) \neq 0\). Standard forms used in statistical physics for the linear form \(\mathcal{L}\) [20] are recovered either by taking \(A\) as the identity matrix, or by choosing \(A_{ij} = V_j W_j\) so that \(\mathcal{L}(M) = \text{tr}(V^T M W)\). Equation (1) is a natural generalization to correlated variables of the i.i.d. case, replacing the product of real functions by a product of matrix functions. It is useful to introduce the matrix of distributions \(P(x)\) through the relation

\[
\mathcal{R}_{ij}(x) = E_{ij} P_{ij}(x),
\]

so that \(P_{ij}(x)\) can be interpreted as a probability distribution, normalized to 1. The definition (1) is valid for any probability space; however, in the present letter, we restrict ourselves to real random variables. Moreover, we consider probabilities \(P_{ij}(x)\) with finite mean value \(m_{ij}\) and finite variance \(\sigma_{ij}^2\). Note that \(P_{ij}(x)\) is uniquely defined only when \(E_{ij} \neq 0\).

Using the product structure of eq. (1), the correlation can be computed as

\[
C_{kl} \equiv \langle x_k x_l \rangle - \langle x_k \rangle \langle x_l \rangle,
\]

\[
\langle x_k \rangle = \frac{\mathcal{L}(E^{k-1} M(1) E^{N-k-1})}{\mathcal{L}(E^N)},
\]

\[
\langle x_k x_l \rangle = \frac{\mathcal{L}(E^{k-1} M(1) E^{l-k-1} M(1) E^{N-l-1})}{\mathcal{L}(E^N)},
\]

where \(k < l\) and \(M(q)\) is the moment matrix \(M(q) = \int x^q \mathcal{R}(x) \, dx\). This expression of the correlation will be useful in the following.

**Hidden Markov chain representation.** — As shown in [37,38], the joint probability (1) can be reinterpreted using the concept of hidden Markov chain [41]. We introduce a Markov chain \(\Gamma \in \{1, \ldots, D\}^{N+1}\) such that

\[
\Pr(\Gamma_1 = i, \Gamma_{N+1} = f) = A_{jf} \left(\frac{E^{N+1}}{\mathcal{L}(E^N)}\right),
\]

\[
\Pr(\Gamma_{k+1} = j | \Gamma_k = i, \Gamma_{N+1} = f) = E_{ij} \left(\frac{E^{N-k}}{E^{N-k+1}}\right).\]

Note that this Markov chain is non-homogeneous and of a non-standard type, due to the dependence on the final state \(\Gamma_{N+1}\). In particular for \(k = N\), the transition rate \(\Pr(\Gamma_{N+1} = j | \Gamma_k = i, \Gamma_{N+1} = f)\) equals 1 if \(j = f\), and 0 otherwise. Combining eqs. (5) and (6), the global probability of a given chain \(\Gamma\) reads

\[
\kappa_{\Gamma} = \frac{A_{f\Gamma T \Gamma_N}}{\mathcal{L}(E^N)} \mathcal{E}_{\Gamma_1 \Gamma_2 \ldots \Gamma_N \Gamma_{N+1}}.
\]

For a given \(\Gamma\), the random variables \((x_1, \ldots, x_N)\) are independent but non-identically distributed, with a probability distribution depending on \(\Gamma\):

\[
\Pr(x_1, \ldots, x_N) = \prod_{k=1}^{N} \Pr_{\Gamma_k \Gamma_{k+1}}(x_k).
\]

This formulation using a hidden Markov chain \(\Gamma\) is equivalent to the definition given in eq. (1) using matrices [37,38]. This yields a procedure to simulate the correlated random variables described by eq. (1) [38]: i) \(\Gamma_1\) and \(\Gamma_{N+1}\) are chosen at random according to distribution (5); ii) the random chain \(\Gamma\) is obtained from transition rates (6); iii) the random variables \(x_k (k = 1, \ldots, N)\) are drawn randomly from the distributions \(\Pr_{\Gamma_k \Gamma_{k+1}}(x_k)\).

As seen in eq. (8), for a fixed \(\Gamma\), the random variables \((x_1, \ldots, x_N)\) are independent. Correlations, when present, thus emerge from the correlation of the hidden chain \(\Gamma\). Consequently, the matrix \(\mathcal{E}\) plays a key role in the statistical properties of the Markov chain \(\Gamma\), which, in turn, determine correlations between the \(x_k\)’s. In particular, the short-range or long-range nature of the correlations depends, respectively, on the ergodic or non-ergodic nature of the Markov chain \(\Gamma\). This ergodic nature is determined by the matrix \(\mathcal{E}\) (see examples in [38] where \(\mathcal{E}\) is...
either diagonalizable or given by the sum of the identity and a nilpotent matrix). This characteristic of the chain \( \Gamma \) is key to the classification of the representative examples used below.

**Statistics of the sum.** – We now focus on the study of the statistical properties of the sum \( S = \sum_{k=1}^{N} x_k \), in the limit \( N \to \infty \). We are specifically interested in the validity of the law of large numbers and of the central limit theorem. Introducing the variable \( s = S/N \), the law of large numbers breaks down if the limit distribution \( \Psi(s) \) for \( N \to \infty \) does not reduce to a delta function. When the law of large numbers holds, namely \( \Psi(s) = \delta(s-m) \), one can investigate the fluctuations of \( S \) around its mean value \( Nm \) on a scale \( \sim \sqrt{N} \), through the rescaled variable

\[
z = \frac{S - Nm}{\sqrt{N}}.
\]  

When correlations are weak enough, the central limit theorem should hold, and the distribution \( \Phi_N(z) \) should converge to a Gaussian when \( N \to \infty \).

As the only dependence on \( \Gamma \) is through \( \nu \), the average over \( \Gamma \) can be replaced by an average over \( \nu \). The limit distribution \( \Psi(s) \) is thus obtained by averaging \( \Psi(s|\nu) \) over the asymptotic \( (N \to \infty) \) distribution of \( \nu \), denoted as \( Q(\nu) \):

\[
\Psi(s) = \int \prod_{i,j=1}^{D} d\nu_{ij} Q(\nu) \delta \left( s - \sum_{i,j=1}^{D} \nu_{ij} m_{ij} \right). 
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\]
chain is quasi-homogeneous far from its end point and its
homogeneous transition matrix is also irreducible. From
standard results of the homogeneous Markov chain theory
[42], the chain converges to a unique stationary state
described by a probability \( q_i \) to occupy state \( i \). The
convergence towards the stationary state implies that,
for \( N \to \infty \), \( \nu_{ij} \) converges (almost surely) to the
average value \( \bar{\nu}_{ij} = q_i \text{Pr}(\Gamma_{N+1} = j|\Gamma_k = i) \). The
empirical frequencies \( \nu_{ij} \) thus do not fluctuate and both
the law of large numbers and the central limit theorem hold.
This can be seen formally by plugging the distribution
\( Q(\nu) = \prod_j \delta (\nu_{ij} - \bar{\nu}_{ij}) \) into eqs. (14) and (16). Interestingly,
this result is independent of the form of the matrix
\( \mathcal{A} \). Note also that the above result is consistent with the
fact that correlations, computed from eq. (4), exponentially
decrease when the matrix \( \mathcal{E} \) is irreducible, due to
the presence of a dominant eigenvalue \( \lambda_1 \) of multiplicity 1:
\[
C_{kl} \approx \sum_{i=2}^{D} K_i \left( \frac{1}{\lambda_1} \right)^{|k-l|}
\]  
with \( K_i \) real constants [38].

\textbf{Chain \( \Gamma \) with disconnected ergodic components.}

Non-standard distributions are expected to emerge
when the chain \( \Gamma \) is non-ergodic. A potential source of
non-ergodicity is the presence of disconnected ergodic
components, that is, subdomains of configuration space
between which no transitions are possible. The simplest
case appears when \( \mathcal{E} \) is equal to the identity matrix \( I \) (a
proportionality factor can be scaled out by a redefinition of
\( \mathcal{R}(x) \)). From eq. (6), one sees that the only possible chains
are the constant chains \( \Gamma^{(i)} = (i, i, \ldots, i) \), for \( i = 1, \ldots, D \).
The chain \( \Gamma \) is therefore trapped inside state \( i \) and
cannot explore the whole domain. For a given chain \( \Gamma^{(i)} \), one
thus has \( \nu_{ii} = 1 \) and \( \nu_{ij} = 0 \) for \( k, l \neq (i, i) \). In addition, eq. (7) implies that the chain \( \Gamma^{(i)} \) appears with probability
\( q_i = \mathcal{A}_{ii}/\sum_j \mathcal{A}_{jj} \). As a result, one finds
\[
Q(\nu) = \frac{1}{D} \sum_{i=1}^{D} q_i \delta (\nu_{ij} - 1) \prod_{(k,l) \neq (i,i)} \delta (\nu_{kl}) ,
\]  
which, from eq. (14), leads to a generalization of the law of
large numbers,
\[
\Psi(s) = \frac{1}{D} \sum_{i=1}^{D} q_i \delta (s - \nu_{ii}) .
\]  
In other words, in a large sample the scaled sample mean
can take several distinct values. If all the \( \nu_{ii} \)'s are
equal, the standard law of large numbers holds, and using
eq. (16), the central limit theorem is generalized into
\[
\Phi(z) = \frac{1}{D} \sum_{i=1}^{D} \frac{q_i}{\sigma_i^2} e^{-z^2/2 \sigma_i^2} ,
\]  
Another characteristic of this form of non-ergodicity is
the presence of a non-zero constant correlation \( C_{kl} \) which
reads, using eq. (4) with \( \mathcal{E} = I \),
\[
C_{kl} = \frac{\mathcal{L}(M(1)^2) - \mathcal{L}(M(1))^2}{\mathcal{L}(I)}
\]  
In terms of Markovs, this constant correlation can be
interpreted as resulting from the permanent trapping of
the chain \( \Gamma \) inside one of the ergodic components [38].

\textbf{Non-ergodic chain \( \Gamma \) with irreversible transitions.}

Another major potential source of non-ergodicity
in the chain \( \Gamma \) is the presence of irreversible transitions:
some subdomains \( D_1, \ldots, D_p \) of the configuration space of
the chain are such that the chain \( \Gamma \) can never go from \( D_k \)
to \( D_l \) if \( k > l \). A simple and representative example of
this case consists in the matrix
\[
\mathcal{E} = I + U, \quad U = \begin{pmatrix}
0 & b & 0 \\
\vdots & \ddots & \ddots \\
0 & 0 & b
\end{pmatrix}
\]  
with \( b \) an arbitrary strictly positive real. From eq. (6),
chains \( \Gamma \) having non-zero probability contain only
transitions from \( i \) to \( i+1 \) and from \( i \) to \( i \), and thus take the
form
\[
\Gamma = (i_1, \ldots, i_1, i_2, \ldots, i_p, i_p, \ldots, i_p), \quad i_1 < i_2 < \cdots < i_p .
\]
Using the form (22) of the matrix \( \mathcal{E} \), eq. (7) implies that
all the chains sharing the same start and end points are
equiprobable. Moreover, using the expansion
\[
\mathcal{E}^N = \sum_{n=0}^{D-1} b^n \binom{N}{n} U^n , \quad N \geq D - 1 ,
\]
in eq. (5), the probability for a chain to start at \( k \) and end
at \( l \) reads
\[
\text{Pr}(\Gamma_1 = k, \Gamma_{N+1} = l) = \frac{b^{l-k} \binom{N}{\nu_{kl}} \mathcal{A}_{kl}}{\sum_{k'<l'} b^{l'-k'} \binom{N}{\nu_{k'l'}} \mathcal{A}_{k'l'}}
\]
for \( k \leq l \), and zero for \( k > l \). Since \( \binom{N}{r} \) grows as \( N^r \)
for \( N \to \infty \), one finds in this limit that, with probability 1,
the chain starts from \( \Gamma_1 = 1 \) and ends at \( \Gamma_{N+1} = d \), on
condition that \( \mathcal{A}_{1,n} \) is non-zero. Given that for \( i \neq j \), the
chains \( \Gamma \) have at most one transition from \( i \) to \( j \), the corre-
responding empirical frequencies \( \nu_{ij} \) converge to zero for
\( N \to \infty \). Only the frequencies \( \nu_{ii} \) are non-zero, and the
equiprobability of chains implies that they are uniformly
distributed, under the constraint \( \sum_i \nu_{ii} = 1 \). Hence, at
odds with previous cases, empirical frequencies have a con-
tinuous distribution in the limit \( N \to \infty \),
\[
Q(\nu) = \frac{1}{(D-1)!} \delta ( \sum_{i=1}^{D} \nu_{ii} - 1 ) \prod_{k \neq l} \delta (\nu_{kl}) .
\]
From eq. (14), the generalized law of large numbers reads

\[ \Psi(s) = \int \frac{\prod_{i=1}^{D} d\nu_{i}}{(D-1)!} \delta \left( \sum_{i} \nu_{i} - 1 \right) \delta \left( s - \sum_{i} \nu_{i} m_{i} \right). \]  

(27)

The distribution \( \Psi(s) \) is piecewise polynomial, with support \([\min\{m_{i}\}, \max\{m_{i}\}]\) (see below for an explicit example). If all the \( m_{i} \)'s are equal, the standard law of large numbers holds, and a generalized form of the central limit theorem is obtained from eq. (16) as

\[ \Phi(z) = \int \frac{\prod_{i} d\nu_{i}}{(D-1)!} \delta \left( \sum_{i} \nu_{i} - 1 \right) e^{-z^{2}/(2\sqrt{2\pi}\nu_{i} \delta_{i})} \gamma \left( \frac{z}{\sqrt{2\nu_{i}} \delta_{i}} \right). \]  

(28)

As an example, we consider eqs. (27) and (28) in the case \( D = 2 \). Then, \( \Psi(s) \) is simply a uniform distribution on the interval \([\min\{m_{i}\}, \max\{m_{i}\}]\), and if \( m_{11} = m_{22} \),

\[ \Phi(z) = \sqrt{2 \pi} \int_{\sigma_{22} - \sigma_{11}}^{\sigma_{22}} \frac{e^{-z^{2}/2\sigma^{2}}}{\sigma^{2}} d\sigma. \]  

(29)

This distribution is illustrated in fig. 1 for different values of \( \sigma_{22}/\sigma_{11} \), keeping the variance of \( \Phi(z) \) fixed to 1. Increasing \( \sigma_{22}/\sigma_{11} \) makes the central peak of the distribution sharper, while the tails remain essentially Gaussian.

Another consequence of the form (22) of the matrix \( \mathcal{E} \) is that the correlation is long-range and becomes a function of \( k/N \) and \( l/N \). Using eq. (4) together with the expansion (24), one finds in the limit \( N \to +\infty \), keeping \( k/N \) and \( l/N \) fixed,

\[ \langle x_k \rangle \approx \sum_{r+s=D-1} a_{r,s} \left( \frac{k}{N} \right)^{r} \left( \frac{l}{N} \right)^{s}, \]

\[ \langle x_k x_l \rangle \approx \sum_{r+s+t=D-1} c_{r,s,t} \left( \frac{k}{N} \right)^{r} \left( \frac{l-k}{N} \right)^{s} \left( 1 - \frac{l}{N} \right)^{t}. \]  

(30)

with \( a_{r,s} \) and \( c_{r,s,t} \) real constants [38].

More generally, for an arbitrary matrix \( \mathcal{E} \), an analysis can be performed in terms of decomposition of \( \mathcal{E} \) into irreducible blocks. This general case can be interpreted as a combination of the different cases above, involving partial equilibration of the chain in some domains, irreversible transitions between domains, and disconnected domains. The final result for the distribution \( \Psi(s) \) (or when applicable, for \( \Phi(z) \)) is generically a complicated mixture, both continuous and discrete, of standard laws. Details will be given in a forthcoming publication [40].

**Discussion.** It is of interest to try to make a connection between the formalism presented here and non-equilibrium stochastic models which can be solved using a matrix product ansatz. An immediate difficulty is that the condition that all coefficients of matrices \( \mathcal{A} \) and \( \mathcal{R}(x) \) should be non-negative is violated in most known examples with finite matrices [20]. However, in some cases including, for instance, the ASEP model [19] and the coagulation-decoagulation model [25], only matrix \( \mathcal{A} \) contains negative coefficients, and our approach can be generalized by separating positive and negative terms in the operator \( \mathcal{L}(M) \) in eq. (1), writing \( \mathcal{L}(M) = \mathcal{L}_{+}(M) - \mathcal{L}_{-}(M) \), which yields \( P(x_{1}, \ldots, x_{N}) = P_{+}(x_{1}, \ldots, x_{N}) - P_{-}(x_{1}, \ldots, x_{N}) \). The Markov chain reformulation can then be applied to each part \( P_{+} \) and \( P_{-} \) separately (after proper normalization), and the distribution of the sum \( S = \sum_{k=1}^{N} x_k \) can be recomposed from the two distributions of \( S \) obtained from \( P_{+} \) and \( P_{-} \).

In addition, let us discuss a tentative physical interpretation of the chain \( \Gamma \), in the case of a non-ergodic chain with irreversible transitions. Considering, for instance, two-dimensional matrices, a typical chain has the form \( \Gamma = (1, \ldots, 1, 2, \ldots, 2) \), with a single transition from state \( 1 \) to state \( 2 \). From eq. (8), the probability distribution \( P_{1} \equiv P_{1}(x_{1}, \ldots, x_{N}) \) associated to \( \Gamma \) reads

\[ P_{1} = P_{11}(x_{1}) \ldots P_{11}(x_{k}) P_{22}(x_{k+1}) \ldots P_{22}(x_{N}), \]  

(31)

and precisely corresponds to a Bernoulli shock measure, known to be the building block of the dynamics in the ASEP and in lattice reaction-diffusion models [43] (\( x_{k} = 0 \) or \( 1 \) characterizes the occupancy of site \( k \)). These shocks are not purely formal objects, but rather describe the typically observed configurations on the coexistence line of the ASEP [44,45] and related models [30]. The fact that the position of the shock is uniformly distributed over the system [44,46] reflects the random transition in the chain \( \Gamma \), and results in a uniform distribution \( \Psi(s) \) on an interval \([\min, \max]\), of the scaled number of particles. Such models thus provide explicit realizations of systems where the scaled sample mean can take several values (here a continuum of values), thus breaking the law of large numbers. Note that the observation of breakings of the central limit theorem (in situations where the law of large numbers is valid) cannot be observed in such models with binary variables \( x_k \), where the random variables \( x_k \) take only two values 0 and 1. This would require that all first
moments $m_{mk}$ are equal, which for binary variables implies that the distributions $P_{ij}(x)$ are identical. The introduction of models with continuous variables [39] should thus be helpful in identifying physical situations where the generalized central limit theorem obtained in eq. (28) is valid.

In conclusion, we have derived generalizations of the law of large numbers and of the central limit theorem for strongly correlated variables described by a matrix product ansatz, a class of random variables ubiquitous in the description of one-dimensional non-equilibrium systems. We have shown that the type of distribution found is related both to the ergodicity properties of the associated Markov chain, and to the short-range or long-range nature of the correlation. We believe that beyond their applicability to exactly solvable models, our results also provide families of reference distributions, like the one given in eq. (29), which can be useful to describe experimental or numerical data in more general correlated non-equilibrium systems. As concerns future work, it would be interesting to try to extend these results to matrices with coefficients of arbitrary signs, as well as to infinite matrices. Providing a more physical interpretation of the Markov chain in terms of dynamics of the underlying stochastic model would also be valuable.

REFERENCES

[1] Feller W., An Introduction to Probability Theory and its Applications, Vol. I (Wiley, New York) 1966.
[2] Gnedenko B. V. and Kolmogorov A. N., Limit Distributions for Sums of Independent Random Variables (Addison-Wesley, Cambridge, USA) 1954.
[3] Feller W., An Introduction to Probability Theory and its Applications, Vol. II (Wiley, New York) 1966.
[4] Petrov V. V., Limit Theorems of Probability Theory (Oxford University Press, Oxford) 1995.
[5] Bouchaud J.-P. and Georges A., Phys. Rep., 195 (1990) 127.
[6] Bouchaud J.-P., J. Phys. I, 2 (1992) 1705.
[7] Bardou F., Bouchaud J.-P., Aspect A. and Cohen-Tannoudji C., Lévy Statistics and Laser Cooling (Cambridge University Press, Cambridge) 2002.
[8] Derrida B. and Evans M. R., J. Phys. I, 3 (1993) 311.
[9] Essler F. H. L. and Rittenberg V., J. Phys. A: Math. Gen., 29 (1996) 3375.
[10] Ramaswamy S., Simha R. A. and Toner J., Europhys. Lett., 62 (2003) 196.
[11] Toner J., Tu Y. and Ramaswamy S., Ann. Phys. (N.Y.), 318 (2005) 170.
[12] Antal T., Droz M., Györgyi G. and Rácz Z., Phys. Rev. E, 65 (2002) 046140.
[13] Bramwell S. T., Fortin J.-Y., Holdsworth P. C. W., Peysson S., Pinton J.-F., Portelli B. and Sellitto M., Phys. Rev. E, 63 (2001) 041106.
[14] Clusel M. and Bertin E., Int. J. Mod. Phys. B, 22 (2008) 3311.
[15] Taqqu M. S., Z. Wahrscheinlichkeitstheor. Verw. Geb., 50 (1979) 53.
[16] Rosenblatt M., Z. Wahrscheinlichkeitstheor. Verw. Geb., 55 (1981) 123.
[17] Hakim V. and Nadal J. P., J. Phys. A: Math. Gen., 16 (1983) L213.
[18] Derrida B., Evans M. R., Hakim V. and Pasquier V., J. Phys. A: Math. Gen., 26 (1993) 1493.
[19] Mallick K. and Sandow S., J. Phys. A: Math. Gen., 30 (1997) 4513.
[20] Blythe R. A. and Evans M. R., J. Phys. A: Math. Theor., 40 (2007) R333.
[21] Crampe N., Ragoüët E. and Simon D., J. Phys. A: Math. Theor., 44 (2011) 405003.
[22] Lazarescu A. and Mallick K., J. Phys. A: Math. Theor., 44 (2011) 315001.
[23] Lazarescu A., J. Phys. A: Math. Theor., 46 (2013) 145003.
[24] Prolla S., Evans M. R. and Mallick K., J. Phys. A: Math. Theor., 42 (2009) 165004.
[25] Hinrichsen H., Sandow S. and Peschel I., J. Phys. A: Math. Gen., 29 (1996) 2643.
[26] Hinrichsen H., Adv. Phys., 49 (2000) 815.
[27] Hieida Y. and Sasamoto T., J. Phys. A: Math. Gen., 37 (2004) 9873.
[28] Busu U. and Mohanty P. K., Phys. Rev. E, 79 (2009) 041143.
[29] Zeraati S., Jafarpour F. H. and Hinrichsen H., Phys. Rev. E, 87 (2013) 062120.
[30] Jafarpour F. H., J. Phys. A: Math. Gen., 36 (2003) 7497.
[31] Jafarpour F. H., Physica A, 339 (2004) 369.
[32] Parmeggiani A., Franosch T. and Frey E., Phys. Rev. Lett., 90 (2003) 086001.
[33] Reichenbach T., Frey E. and Franosch T., New J. Phys., 9 (2007) 159.
[34] Appert-Rolland C., Hilhorst H. J. and Schehr G., J. Stat. Mech. (2010) P08024.
[35] Ferrari P. L., Sasamoto T. and Spohn H., J. Stat. Phys., 153 (2013) 377.
[36] Speer E. R., J. Stat. Phys., 89 (1997) 169.
[37] Angeletti F., Bertin E. and Abry P., in Proceedings of the IEEE ICASSP Conference 2012 (IEEE) 2012, p. 3897.
[38] Angeletti F., Bertin E. and Abry P., IEEE Trans. Signal Process., 61 (2013) 5389.
[39] Bertin E., in preparation.
[40] Angeletti F., Bertin E. and Abry P., in preparation.
[41] Cappe O., Moulines E. and Rydén T., Inference in Hidden Markov Models, Springer Ser. Stat. Springer, New York) 2005.
[42] Seneta E., Non-negative Matrices and Markov Chains, Springer Ser. Stat. Springer, New York) 2006.
[43] Krebs K., Jafarpour F. H. and Schütz G. M., New J. Phys., 5 (2003) 145.
[44] Schütz G. and Domany E., J. Stat. Phys., 72 (1993) 277.
[45] Santen L. and Appert C., J. Stat. Phys., 106 (2002) 187.
[46] Depken M. and Stinchcombe R., Phys. Rev. Lett., 93 (2004) 040602.