ON THE EVOLUTION OF FRACTIONAL DIFFUSIVE WAVES

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Abstract. In physics, phenomena of diffusion and wave propagation have great relevance; these physical processes are governed in the simplest cases by partial differential equations of order 1 and 2 in time, respectively. It is known that whereas the diffusion equation describes a process where the disturbance spreads infinitely fast, the propagation velocity of the disturbance is a constant for the wave equation. By replacing the time derivatives in the above standard equations with pseudo-differential operatorsinterpreted as derivatives of non integer order (nowadays misnamed as of fractional order) we are lead to generalized processes of diffusion that may be interpreted as slow diffusion and interpolating between diffusion and wave propagation. In mathematical physics, we may refer these interpolating processes to as fractional diffusion-wave phenomena. The use of the Laplace transform in the analysis of the Cauchy and Signalling problems leads to special functions of the Wright type.

In this work we analyze and simulate both the situations in which the input function is a Dirac delta generalized function and a box function, restricting ourselves to the Cauchy problem. In the first case we get the fundamental solutions (or Green functions) of the problem whereas in the latter case the solutions are obtained by a space convolution of the Green function with the input function. In order to clarify the matter for the non-specialist readers, we briefly recall the basic and essential notions of the fractional calculus (the mathematical theory that regards the integration and differentiation of non-integer order) with a look at the history of this discipline.

Keywords: Fractional calculus, time fractional derivatives, slow diffusion, transition from diffusion to wave propagation, Wright functions, Cauchy and Signaling Problems.

MSC Classification: 26A33, 33E12, 34A08, 65D20, 60J60, 74J05

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1. Introduction

It is known that mathematical models are usually governed by differential and/or integral equations of integer order. However, we can extend integration and differentiation to any order by using the tools of the so-called Fractional Calculus, that in recent years has gained considerable interest also because of its applications in different fields including Physics, Engineering, Biology, Economics and Finance, Geophysics, Computer Science and so on...

Fundamental phenomena of Physics, such as diffusion and wave propagation, are governed in their simplest form by the following partial differential equations (PDE) that in a standard notation read

- Diffusion equation:

\[ \frac{\partial u(r, t)}{\partial t} = D \nabla^2 u(r, t), \]
Wave equation:

\[ \frac{\partial^2 u(r, t)}{\partial t^2} = c^2 \nabla^2 u(r, t). \]

The above PDE's can be extended by replacing the time and/or spatial derivatives of integer order with some pseudo-differential operators interpreted as derivative of non integer order. In this paper we consider the generalizations of the above PDE's by replacing the time derivatives with derivatives of order \( \alpha \in (0, 2] \) and we restrict our attention to a single space dimension \( x \).

This work is organized as follows. In Section 2, we show the essential mathematical notions necessary to understand the integration and differentiation of non-integer order, that nowadays are ascribed to the mathematical discipline referred generally to as Fractional Calculus. In particular we point out the approach of Professor Caputo on which Fractional Calculus has grown and found several applications since the late 1960’s.

Then, in Sections 3 and 4 we provide some historical notes distinguishing two different eras with the advent of Caputo’s approach. Indeed, in Section 3 we recall the major mathematicians who in the last centuries have contributed in the birth of development of the Fractional Calculus whereas in Section 4 we outline how the Caputo approach was popularized.

The core of the paper is found in Section 5, where we consider from a mathematical viewpoint the evolution of the so called fractional diffusive waves generated by partial differential equations with time derivatives of non integer order.

Finally in Section 6, we provide some concluding remarks paying attention to work to be done in the next future.

2. Essentials of Fractional Calculus: Fractional Derivatives and Integrals

The starting point for the Fractional Calculus is given by the Cauchy formula for repeated integration where \( a \in \mathbb{R} \). Taking as independent variable the time \( t \) and denoting by \( f(t) \) a function assumed to be sufficiently well-behaved for the next considerations, we recall the formula for the \( n \) repeated integral

\[
_aI^n f(t) := \int_a^t \cdots \int_a^t f(\tau) d\tau_{n-1} = \frac{1}{(n-1)!} \int_a^t (t-\tau)^{n-1} f(\tau) d\tau.
\]

The resulting function turns out to be the primitive of order \( n \) of \( f(t) \), that is vanishing in \( t = a \) along with its derivatives of order 1, \( \cdots \), \( n-1 \). The basic idea of the fractional calculus is in replacing the factorial term \( (n-1)! \) with the corresponding representation by the Gamma function and then replacing \( n \) with a positive real number \( \alpha \). So, writing \( (n-1)! = \Gamma(n) \) in Eq. (3), it follows the definition of fractional integral \( _aI^n t f(t) \) of order \( \alpha \) (\( \alpha > 0 \))

\[
_aI^n \alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau,
\]

where now \( \alpha \) is no longer restricted only to positive, integer values.

The fractional derivative of order \( \alpha \) is so defined (in the Riemann-Liouville sense) as the left inverse operator of the corresponding fractional integral and reads:

\[
_aD^n \alpha f(t) := \frac{d^n}{dt^n} aI^n_{-\alpha} f(t), \quad n = \lceil (\alpha) \rceil + 1.
\]
In view of our applications we take $a = 0$ so $t \geq 0$ and, from now on, we agree to delete $a$ in the notations for fractional integrals and derivatives. Henceforth, we consider the time fractional derivative in the Caputo sense, defined for $n - 1 < \alpha \leq n$ as:

$$D_C^\alpha f(t) := \int_t^0 \frac{f^{(n)}(\tau)}{\Gamma(n - \alpha)} (t - \tau)^{\alpha + 1 - n} \, d\tau.$$  

This definition provides us a regularization of the Riemann-Liouville fractional derivative at $t = 0$ as shown in 1997 by Gorenflo and Mainardi [11].

As a consequence of its definition, the Caputo derivative is vanishing when the derivative of integer order $f^{(n)}(t)$ is zero; in particular

$$D_C^\alpha 1 = 0, \quad \alpha > 0.$$  

The Caputo derivative appears suitable to be treated by the Laplace Transform technique for causal systems, that is quiescent for $t < 0$. In a standard notation we get

$$D_C^\alpha f(t) := s^\alpha \hat{f}(s) - \sum_{k=0}^{n-1} f^{(k)}(0^+) s^{\alpha - 1 - k}, \quad n - 1 < \alpha \leq n.$$  

It is requested the knowledge of the initial values of the function and of its integer derivatives of order $k = 1, 2, ..., n - 1$ as it is for $\alpha = n$.

### 3. Historical notes before Caputo’s era

Fractional Calculus may be considered born on September 30th, 1695 when L’Hôpital asked to Leibniz what would be if $n = 1/2$ in his notation for the $n^{th}$ derivative, $\frac{d^n}{dx^n}$. Leibniz answered: “An apparent paradox, from which one day useful consequences will be drawn.”

![Figure 1](LEFT: Gottfried Wilhelm (von) Leibniz; RIGHT: Guillaume François Antoine, Marquis de l’Hôpital.)

From that time we get a very long story involving eminent mathematicians who are pointed out in the next figure. More recently the term fractional calculus was used but it is a misnomer kept only for historical reasons related to the discussion between Marquis de l’Hôpital and Leibniz.

For more details see the historical notes by the late Bertram Ross, the organizer of the first conference devoted to fractional calculus in 1974 [28], and, more recently, by Machado and Kiryakova [17] from which the following picture is taken.
Figure 2. The major mathematicians in Fractional Calculus up to 1950’s.

4. HISTORICAL NOTES IN CAPUTO’S ERA

In this paper, attention is indeed devoted to the form (6) of fractional derivative alternative to that originally used by Liouville and Riemann expressed by Eq (5).

The form (6), where the orders of fractional integration and ordinary differentiation are interchanged, is nowadays known as the Caputo derivative. As a matter of fact, such a form is found in a paper by Liouville himself as noted by Butzer and Westphal [1] in the 2000 book edited by R. Hilfer, but Liouville, not recognizing its role, disregarded this notion. As far as we know, up to the middle of the twentieth century, most authors did not take notice of the difference between the two forms and of the possible use of the alternative form. Even in the classical book on Differential and Integral Calculus published in English in 1936 by the eminent mathematician R. Courant, the two forms of the fractional derivative were considered as equivalent, see [6], pp. 339-341.

In the late sixties of the past century the relevance of the alternative form was finally recognized. In fact, in 1968 the Soviet Scientists Dzherbashyan and Nersesyan [8] and then in 1989-90 Kochubei [13; 14] used the alternative form (6) in dealing with Cauchy problems for differential equations of fractional order. However, formerly since 1967, Caputo [2; 3] had introduced this form as given by Eq. (6) proving the corresponding rule in the Laplace transform domain, as Eq. (8). With his derivative Caputo was thus able to generalize the rule for the Laplace transform of a derivative of integer order and to solve some problems in Seismology in a proper way related to the dissipation function. Indeed, it was the corresponding rule in the Laplace transform the basic and original idea of the Caputo form of fractional derivative, not present in any previous paper. Soon later, this derivative was adopted by Caputo and Mainardi [4], [5] in the framework of the theory of Linear Viscoelasticity.

Since the seventies of the past century a number of authors have re-discovered and used the alternative form, recognizing its major utility for solving physical problems with standard initial conditions. Although several papers by different authors appeared where the alternative derivative was adopted, it was only in the late nineties, with the tutorial paper by Gorenflo and Mainardi in 1997 [11] and the 1999 book by Podlubny [26], that such
form was popularized. Nowadays the term *Caputo fractional derivative* is universally accepted in the literature. The reader, however, is alerted that in a very few papers the Caputo derivative is referred to as the Caputo–Dzherbashyan derivative. Note also the transliteration as Djrbashyan.

It should be noted that it was Mainardi, being initially an Italian researcher in Geophysics as a PhD student of Prof. Caputo, who pointed out the Caputo form (6) published in a Geophysical journal [2] and in an Italian book [3] to the attention of eminent colleagues in Mathematics including the late Rudolf Gorenflo [11], [10], Yuri Luchko [15], Igor Podlubny [26], the late Anatoly Kilbas [12]. Kai Diethelm [7]. Also Virginia Kiryakova, the Editor-in-Chief of the major journal devoted to Fractional Calculus, that is Fractional Calculus and Applied Analysis (FCAA), was alerted. Indeed, the papers by Caputo in 1967 [2] and by Caputo and Mainardi in 1971 [4] were reprinted in FCAA in 2007 on the occasion of the eighty birthday of Professor Caputo. Just in this volume there were the foreword by Mainardi [22] to the special issue of FCAA dedicated to the 40 years of Caputo derivative and to the 80th anniversary of Prof Caputo. This issue contains a tutorial survey by Mainardi and Gorenflo [25] and the biographical data of Prof. Caputo and his photo reported below. We enclose also the photo depicting Mainardi, Podlubny and Caputo at Ravello, where Mainardi and Podlubny delivered each a one-week course on fractional calculus and Caputo a seminar in the framework of the 37-th summer school on Mathematical Physics organized by the National Group of Mathematical Physics, held just in Ravello (17-29 September 2012) directed by Professors Rionero and Ruggeri.

5. Fractional calculus in diffusion-wave problems

The time fractional diffusion-wave equation reads:

\[
\frac{\partial^\alpha u(r,t)}{\partial t^\alpha} = D\nabla^2 u(r,t), \quad D > 0, \quad 0 < \alpha \leq 2,
\]
and is obtained from the diffusion equation (or from the wave equation) by replacing the
time derivative with a fractional derivative, in the Caputo sense.
This explicitly reads in the one-dimensional case with \((D = 1)\) taking \(u = u(x, t)\) and
\(\alpha \in (0, 2]\)
\[
\frac{1}{\Gamma(n - \alpha)} \int_0^t \frac{u^{(n)}(x, \tau)}{(t - \tau)^{\alpha + 1 - n}} \, d\tau = \frac{\partial^2}{\partial x^2} u(x, t),
\]
where the derivatives \(u^{(n)}\) are with respect time and \(n = 1\) if \(\alpha \in (0, 1]\) and \(n = 2\) if
\(\alpha \in (1, 2]\). We recognize the for \(\alpha \in (0, 1)\) we have a fractional diffusion equation and when
\(\alpha \in (1, 2]\) we have a transition from the standard diffusion equation to the standard wave
equation, that we refer to the fractional diffusion-wave equation. Of course, we find the
standard diffusion and wave equations when \(\alpha = 1\) and \(\alpha = 2\), respectively.

5.1. Cauchy and Signaling Problems. The two basics problems for the Fractional
PDE are the Cauchy and Signaling ones. Denoting by \(f(x)\) and \(h(t)\) two given, well behaved functions:
\[
\text{Cauchy Problem : } \begin{cases} 
  u(x, 0^+) = f(x), & -\infty < x < +\infty \\
  u(\pm \infty, t) = 0, & t > 0.
\end{cases}
\]
\[
\text{Signaling Problem : } \begin{cases} 
  u(x, 0^+) = 0, & 0 < x < +\infty \\
  u(0^+, t) = h(t), & u(+\infty, t) = 0, \ t > 0.
\end{cases}
\]
When \(1 < \alpha \leq 2\) an extra initial condition is needed for both problems: \(u_t(x, 0^+) = g(x)\).
The solutions turn out to be expressed by proper convolutions between the source function
and a (two) characteristic function(s), the so called Green functions or fundamental solutions \(G(x, t)\) of the problem.
The function \(G_C(x, t)\) for the Cauchy problem represents the solution for \(f(x) = \delta(x)\), and the function \(G_S(x, t)\) for the Signaling problem represents the solution for \(h(t) = \delta_+(t)\).
The Green functions are connected by the Reciprocity Relation:
\[
2\nu x G_C(x, t; \nu) = t G_S(x, t; \nu) = \nu z M_\nu(z),
\]
with \(\nu = \alpha/2\) and \(z = x/t^{\nu}\) being the similarity variable. Here \(M_\nu(z)\) is a function of
the Wright type, entire for \(0 \leq \nu < 1\), referred to as \(M\)-Wright function briefly discussed below. For more details see e.g the survey paper by Mainardi, Luchko and Pagnini [24]
based on the previous papers by Mainardi [19; 20; 21].

5.2. The Fundamental Solution of the Cauchy problem. According to Eq. (13)
the fundamental solution of the Cauchy problem is thus provided by
\[
G_C(x, t; \nu) = \frac{1}{2\nu} M_\nu \left( \frac{x}{\nu t} \right),
\]
where the function \(M_\nu(z)\) reads in its series and integral representations as
\[
M_\nu(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n! \Gamma(-\nu n + (1 - \nu))} = \frac{1}{2\pi i} \int_{Ha} e^{\sigma - z\sigma} \frac{d\sigma}{\sigma^{1-\nu}},
\]
where \(Ha\) denotes the Hankel path. \(M_\nu(z)\) results to be a particular case of the Wright function \(W_{\lambda, \mu}(z)\):
\[
W_{\lambda, \mu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{n! \Gamma(\lambda n + \mu)}, \quad \lambda > -1, \ \mu \in \mathbb{C}, \ z \in \mathbb{C}
\]
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5.3. Plots of Solutions for the Cauchy Problem. We start this graphical sub-section by reporting the plots of the evolution in time of fundamental solutions (that is $f(x) = \delta(x)$ and $g(x) = 0$) taking $\nu = 0.65$, $\nu = 0.75$ and $\nu = 0.85$. The general solutions for the Cauchy problem are obtained through suitable convolution in space as shown hereafter:

\begin{equation}
\label{eq:5.9}
u(x, t; \nu) = \int_{-\infty}^{+\infty} G_C(\xi, t; \nu)f(x - \xi)d\xi, \quad 0 < \nu \leq 1/2,
\end{equation}

\begin{equation}
\label{eq:5.10}
u(x, t; \nu) = \int_{-\infty}^{+\infty} \left[ G_C^{(1)}(\xi, t; \nu)f(x - \xi) + G_C^{(2)}(\xi, t; \nu)g(x - \xi) \right]d\xi, \quad 1/2 < \nu \leq 1,
\end{equation}

where $G_C^{(2)}$ is the primitive in time of $G_C^{(1)}$. 

Figure 5. Plots of the functions $M_\nu(|x|)$ for $|x| \leq 5$ at $t = 1$; LEFT: for $\nu = 0, 1/4, 3/8, 1/2$. RIGHT: for $\nu = 1/2, 5/8, 3/4, 1$.

when $\lambda = -\nu$ and $\mu = 1 - \nu$. We note that for convenience and for historical reasons the Wright functions may be classified in two kinds: the first kind with $\lambda \geq 0$ and the second kind with $-1 < \lambda < 0$ so following the Appendix F of Mainardi’s book [23], where the reader can find also some historical notes. For $\nu = 1/2$, we find back the diffusion equation and indeed the $M_\nu(|x|)$ function becomes the Gaussian function known as fundamental solution of the diffusion equation for the Cauchy Problem whereas, for $\nu \to 1$, we find back the wave equation and indeed the $M_\nu(|x|)$ function tends to two Dirac delta functions ad fundamental solutions of the Cauchy Problem, centered in $x = \pm 1$. It should be noted that in the 1993 book by Prüss [27] we find a figure quite similar to our Fig. 5 reporting the $M$-Wright function in for $\nu \in [1/2, 1)$. It was derived from inverting the Fourier transform expressed in terms of the Mittag-Leffler function following the approach by Fujita [9] for the fundamental solution of the Cauchy problem for the diffusion-wave equation, fractional in time. Both approaches were carried out without any relation to the Wright function, presumably unknown to Fujita and Prüss. However, our plot must be considered independent from that of Prüss because Mainardi used the Laplace transform in his former paper presented at WASCOM, Bologna, October 1993 [18] (and published later in a number of papers and in his 2010 book [23]) so he was aware of the book by Prüss only later. Furthermore, we invite readers to look at the simulation of the fundamental solution $M_\nu(x, t)$ at $t = 1$ for $x \in [-5, +5]$ at varying $\nu$ available in YOUTUBE: https://www.youtube.com/watch?v=uf_4aB1COPg carried out by Consiglio with Matlab.
Figure 6. Time evolution of the fundamental solution for $\nu = 0.65$ in the Cauchy problem

Figure 7. Time evolution of the fundamental solution for $\nu = 0.75$ in the Cauchy problem
We now consider for the Time-Fractional Diffusion Wave equation the evolution of an initial centered box-signal $f(x) = 1$ for $-1 \leq x \leq +1$ and zero otherwise, assuming $g(x) = 0$. Based on Eqs. (17)-(18) we simulate the cases $\nu = 0.50$ (standard diffusion), $\nu = 0.75$ and $\nu = 1$ (standard wave), as shown in the next figures where the initial box function is denoted by dotted lines.

**Figure 8.** Time evolution of the fundamental solution for $\nu = 0.85$ in the Cauchy problem

**Figure 9.** Time evolution of an initial box-signal for $\nu = 0.50$, seen at $t = 0.50$ (left) and $t = 1.00$ (right) in $0 \leq x \leq 3.5$. The problem is symmetric on the negative axis.
Figure 10. Time evolution of an initial box-signal for $\nu = 0.75$, seen at $t = 0.50$ (left) and $t = 1.00$ (right) in $0 \leq x \leq 3.5$. The problem is symmetric on the negative axis.

Figure 11. Time evolution of an initial box-signal for $\nu = 1$, seen at $t = 0.50$ (left) and $t = 1.00$ (right) in $0 \leq x \leq 3$. The problem is symmetric on the negative axis.

6. Concluding remarks

We have considered the so-called time fractional diffusion wave equation with particular attention when this equation is interpolating the processes of diffusion and the wave propagation. In these cases we speak about fractional diffusive waves using a term incorporating both diffusion and wave phenomena.

We have analyzed and simulated both the situations in which the input function is a Dirac delta generalized function and a box function, restricting ourselves to the Cauchy problem. In the first case we get the fundamental solutions (or Green function) of the problem whereas in the latter case the solutions are obtained by a space convolution of the Green function with the input function.

In the next future we plan to consider the signaling problem in order to complete the topic of fractional diffusive waves.

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