Universality and crossover behaviors of single step growth model in one and two dimensions

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We study kinetic roughening of the (2+1)-dimensional single-step (SS) growth model with a tunable parameter \( p \). Using extensive numerical simulations, we show that there exist a slow crossover from an intermediate regime dominated by the Edwards-Wilkinson (EW) class to an asymptotic regime dominated by the Kardar-Parisi-Zhang (KPZ) class for any \( p < \frac{1}{2} \). Also we identify the crossover time, the nonlinear coupling constant, and some nonuniversal parameters in the KPZ equation as a function \( p \). The effective nonuniversal parameters are continuously decreasing with \( p \), but not in a linear fashion. Our results provide complete and conclusive evidences that the SS model for \( p \neq \frac{1}{2} \) belongs to the KPZ universality class in 2 + 1 dimensions.

I. INTRODUCTION

Understanding non-equilibrium evolution of growing surfaces and interfaces has attracted much interest from both theoretical and experimental points of view [1, 2]. Since four decades ago, a dynamic scaling approach was proposed to describe the morphological evolution of a growth front and various discrete models have been suggested to describe surface growth processes, for examples see [1, 3]. These discrete models can be described by some continuous Langevin equations. Two well known universality classes are the Edwards-Wilkinson (EW) [4] and the Kardar-Parisi-Zhang (KPZ) [5] equations. A large class of discrete growth models such as the ballistic deposition models (BD) [6], restricted solid on solid (RSOS) models [7], and directed polymers in random media [8] are believed to belong to the same universality class as the KPZ equation describing the growth interface fluctuations.

The KPZ equation describes the time evolution of a field \( h(x,t) \) that denotes its height at the position \( x \) and at time \( t \) on a \( d \)-dimensional substrate:

\[
\frac{\partial h(x,t)}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2} |\nabla h|^2 + \sqrt{D} \xi(x,t), \tag{1}
\]

where \( \xi(x,t) \) is an uncorrelated Gaussian white noise in both space and time with zero average i.e., \( \langle \xi(x,t) \rangle = 0 \) and \( \langle \xi(x,t) \xi(x',t') \rangle = \delta^d(x-x')\delta(t-t') \). The real constants \( \nu, \lambda \) and \( D \) take into account the surface relaxation intensity, the lateral growth and the amplitude of Gaussian white noise, respectively. One of the most important quantities that can be used to study and to classify different discrete or continuous growth models, like Eq. (1), is defined in terms of the scaling properties of the surface width \( w(L,t) = \sqrt{\langle |h(x,t) - \langle h(x,t) \rangle| \rangle^2} \) where \( \langle \cdot \rangle \) denotes spatial averaging. As a function of the system size \( L \), it is expected to have the scaling form [9] \( w(L,t) \sim L^\alpha f(t/L^2) \), where \( \alpha \) and \( z \) are two independent universal parameters known as roughness and dynamic exponents, respectively. The scaling function \( f \) usually has the asymptotic form \( f(x \to \infty) = \) constant

and \( f(x \to 0) \sim x^\beta \), where \( \beta \) is the growth exponent \( \beta = \alpha/z \). The particular behavior of \( f \) imply that \( w(L,t) \sim L^\alpha \) for \( t \gg L^2 \) and \( w(L,t) \sim t^\beta \) for \( t \ll L^2 \). The absence of the nonlinear term, i.e. Eq. (1) with \( \lambda = 0 \), results in another universality class known as the EW where the exact values of exponents are given by \( \alpha = (2-d)/2 \) and \( z = 2 \), in \( (d+1) \)-dimensions [4]. In the presence of \( \lambda \), although, due to the Galilean invariance, another scaling relation \( \alpha + z = 2 \) emerges [10], the exact solution only exists in \( 1d \) which gives \( \alpha = 1/2 \), and \( z = 3/2 \) [5]. In higher dimensions, the critical exponents are available only by various theoretical approaches [11] and numerical methods [12, 13].

In the breakthrough theoretical approach [14], Johansson successfully computed a universal probability distribution function (PDF) for a discrete growth model, known as single step (SS) [15-18]. Most especially, the PDF of the height fluctuations is the Tracy-Widom (TW) distribution [19], which in the context of the random matrix theory, describes the typical fluctuations of the largest eigenvalue of random matrices belonging to the Gaussian Unitary Ensemble (GUE) [20]. In \( d = 1 + 1 \) dimensions, the surface (or interface) height in the KPZ systems asymptotically evolves according to the ansatz [14, 21, 22]

\[
h \approx v_\infty t + s_\lambda (\Gamma t)^\beta \chi, \tag{2}
\]

where \( \chi \) is a stochastic variable that carries universal information of the fluctuations, while the system-dependent parameters \( v_\infty, s_\lambda \), and \( \Gamma \) are the asymptotic interface velocity, the signal of \( \lambda \) in the KPZ equation Eq. (1), a non-universal constant associated to the amplitude of the interface fluctuations, respectively. Remarkably, there are a few non-Gaussian universal distributions that \( \chi \) selects one of them based on the global geometric shape of the initial condition \( h(x,t = 0) \) [23-26]. This geometry-dependent universality was tested and confirmed experimentally, in studies on growing interfaces of nematic liquid crystals [27]. Recent numerical simulations have shown that the KPZ ansatz, i.e. Eq. (2), can be generalized to two dimensions [28-30], but the ex-
act forms of the asymptotic distributions of $\chi$ are yet not known.

Although the first studies of the TW fluctuations was initially performed on the SS model in $(1 + 1)$ dimensions [14], numerical simulations in higher dimensions commonly fail to provide a reliable connection between this model and the KPZ class. Moreover, this model can be mapped onto some extensively studied models in equilibrium or nonequilibrium statistical mechanics, such as the kinetic Ising model [15, 16], the asymmetric simple exclusion process [31], and the six-vertex model [15, 32, 33]. Therefore some properties of the SS model can be acquired analytically from the exact results of these well-studied models [1, 16, 32].

In this paper we study SS model, which is defined in the following way: at any time $t$, we randomly select a site $i$ on the $d$—dimensional lattice, and we let the surface height $h_i$ at that site to increase by 2 with probability $p$ only if it is a local minimum, or to decrease by 2 with probability $q$ only if it is a local maximum. For simplicity, and without any loss of generality, we can impose $q = 1 - p$ condition. Since the height difference between two neighboring sites can only be two values (+1 or -1), the SS model is analytically more tractable [14, 32]. In one dimensions, it is known that this model can be exactly solved by mapping to the kinetic Ising model [1, 16], and belongs to EW (KPZ) universality class for $p = 0.5$ ($p \neq 0.5$) [16-18]. In contrast to the deep understanding of the SS model in $d = 1 + 1$, essentially conflicting results still exist regarding to the scaling behaviors of this model in $d = 2 + 1$. Although it is generally agreed upon that SS model belongs to the KPZ universality class for $p = 0$ and EW class for $p = 0.5$, the probability interval in which the model is consistent with the EW or KPZ classes is a matter of contention. In some reports, the nonlinearity coefficient $\lambda$ in the KPZ Eq. (1) have been considered as proportional to $p' \equiv (q - p)$ and concluded that the model asymptotically belongs to the KPZ universality class for all $p \neq 0$. [16, 17, 34-36]. However, some authors [18, 37] found that there exists a critical value $p_c$, around which for $p > p_c$, the model consistently resembles $p = 0.5$.

More recently, a geometrical investigation [37] reported a roughening transition around $p_c \approx 0.25$ from a rough phase in the KPZ universality to the smooth phase in the EW universality class, which, as we will see in the following, cannot occur in the hydrodynamic limits. In fact, reliable estimation of the universal parameters requires appropriate consideration of the crossover from the linear behavior of the surface fluctuations at early times to the nonlinear behavior at sufficient large times. We perform a detailed study of SS model on two-dimensional substrates and study numerically the crossover behavior from linear to nonlinear dynamics and estimate nonuniversal parameters of the KPZ equation. We concentrate on the SS model in 1+1 dimensions, since there the universal and the nonuniversal parameters, as well as asymptotic behavior of this model are well known and this therefore provides a convenient test for our numerical results.

The paper is organized as follows. The simulation details are presented in Sec. II. The scaling behaviors of surface width and related consequences are discussed in Sec. III. The interface velocities of SS model are estimated for different values of $p$ in Sec. IV, and in the following, the nonuniversal parameters in the KPZ ansatz given by Eq. (2) as function of the control parameter $p$ are determined in section V. Final discussions and conclusions of the SS model are presented in Sec.VI.

II. SIMULATION DETAILS

We performed extensive simulations of the SS model and simulated 2–dimensional lattices of size $L = 2^{n+3}$, $n = 1, 2, .., 7$ with periodic boundary conditions. The number of samples generated for each lattice size ranges from $10^5$ for the smallest lattice sizes till about 200 for the largest lattice sizes. Moreover, in order to inspect the scaling behavior of SS model near the $p \approx 0.25$, size $L = 2500$ was only used for observing the crossover behavior at $p = 0.25$. A checkerboard initial condition as described in [37] has been used. Moreover, to observe
the crossover behavior, and to check our algorithms with exact analytic results, we simulated 1d SS model up to size $10^{15}$. We also simulated the BD model and numerically obtained $v_{\infty}$, $\lambda$, and $\Gamma$ and finally checked them with more accurate results [38]. In numerical simulation of the SS model, we impose the condition $p+q=1$, so due to up/down symmetry, we just need to consider $p\leq0.5$.

The surface morphology grown by the KPZ equation is characterized by hills that are comparable to the lattice size, while the EW equation produces a very smooth surface and the size of the hills are negligible in comparison to the lattice size. In order to observe these morphological differences, we simulate a few samples on a lattice of size 1024, for different value of $p$. The surface morphologies of 2d SS model for various values of $p$ are shown in fig. 1. As expected, surface morphology decreases with $p$. At first glance, one would find a smooth surface on higher values of $p$ but, in principle, as we will see in the following, this can be described as a result of finite size effects.

III. SURFACE WIDTH AND CROSSING BEHAVIORS

It is known that in short time limit, the non-linear term in Eq. (1) is less important than the linear Laplacian term. In this limit the typical surface width is well described by the EW equation. In fact, in two dimensions and below, depending on the non-universal parameters, both discrete and continuous growth models present a crossover time $t_c$. As a matter of fact, in order to observe this crossover, the system size must be large enough so that saturation effects take place much later than the crossover time $t_c$. i.e. $L^2 \gg t_c$. Therefore, the minimum system size required to occur this crossover behavior, $l_c$, approximately scales as $t_c^{1/2}$.

In $d=1$, for simplicity we can work in rescaled units: $x \to l_c x$, $t \to t_c t$, and $h \to h_c h$ where from dimensional analysis these characteristic scales of space, time, and height can be obtained as [39, 40]

$$l_c = \frac{(2\nu)^3}{D\lambda^3}, \quad t_c = \frac{(2\nu)^5}{D^2\lambda^4}, \quad h_c = \frac{2\nu}{\lambda} \quad (3)$$

The crossover time $t_c$, and local surface height at crossing point $h_c$, as well as the crossing surface width $w_c$ scale as $v^3/(D^2\lambda)$, and $\nu/\lambda$ respectively.

For the 1d SS model, the exact analytic result for the coefficient of the nonlinear term in the KPZ equation is known as $\lambda = (q-p)$ [1, 21]. Additionally, the $D/\nu$ ratio in the Eq. (3) is related to the steady-state width of the interface, which scales with the finite system size $L$ via the relation $w_{sat} \sim \sqrt{LD/\nu}$ [21]. As shown in the inset of Fig. 2(a), the saturated surface width is independent of the value of the parameter $p$, consequently, we expect that the $D/\nu$ ratio is independent of $p$. Thus, the $\lambda$ parameter is responsible for the variation of the $t_c$ and $w_c$. As $p$ increases towards 0.5, based on naive scaling analysis of Eq. (3), the crossover time $t_c$ and the crossing surface width $w_c$ diverge as $p^{-4}$ and $p^{-1}$, respectively. However, to confirm this prediction, as shown in the Fig. 2(a), we plot the rescaled interface width $w_p$ as a function of the rescaled time $t_p^{1/4}$ which is in excellent agreement with the analytic predictions.

Since $d=2$ is the marginal dimension [1, 5, 34], we cannot follow the dimensional analysis approach. Based on RG analysis, it is known that the crossover length scale $l_c$ displays an exponential dependence on the value of the effective coupling constant $g \sim D^3/\nu^4$ [34]. In the absence
of exact analytical results for \( t_c \) and \( w_c \), as a function of \( p \) for the SS model, we can numerically estimate \( t_c \) and \( w_c \), by rescaling the time \( t \) and surface width \( w \) by arbitrary values for \( t_c \) and \( w_c \), respectively, in order to have a good data collapse, as shown in the Fig. 2(b). The obtained estimates for \( t_c \) and \( w_c \), are shown in the inset of same figure. The surface width cross from an intermediate regime dominated by the EW regime (logarithmic-law) to an asymptotic regime dominated by the KPZ regime (power-law). It is worth to mention that the same behavior reported in [35] for another growth model known as Hypercube-stacking (HCS). By increasing the value of \( p \), the crossover time \( t_c \), as well as crossover length \( l_c \), increase exponentially. For example, in order to observe the crossover behavior for \( p = 0.25 \), we need a lattice of size around 2500, which after a typical time \( 7 \times 10^4 \), it arise. Our observation is in agreement with the slow crossover scenario discussed in [34, 35]. It is also worth to mention, for the values \( p > 0.25 \), we are not able to observe the crossover behavior in a reasonable amount of computational time.

IV. THE INTERFACE VELOCITY

In some growth models, such as SS model, it is difficult to obtain reliable scaling exponents, due to complicated crossover and finite-size effects. An alternative method for identifying the universality class is to obtain direct evidence for the presence of different terms in the growth equation. The determination of the coefficient \( \lambda \) is of special interest since, if present, \( \lambda \) controls the scaling properties of the interface. The simplest method of obtaining information on the existence of the nonlinear terms affecting growth processes is based on the fact that the average interface velocity, \( v \equiv d \langle h \rangle / dt \), depends on both the interface orientation and finite size [1, 41, 42].

A central characteristic of KPZ class is the lateral growth that results in an excess interface velocity for a substrate with an overall tilt of slope \( m \equiv \langle \nabla h \rangle \). Based on this fact, the tilt method, as a powerful tool, was initially proposed by Krug [41, 42] to evaluate the non-linearity of associated equation for a discrete growth model. When \(|m| \ll 1\), there is a simple dependence between the interface velocity and slope \( m \) [1],

\[
v(m) = v(0) + \frac{\lambda}{2}m^2
\]

where \( v(0) \) is interface velocity for untilted lattice. The parameter \( \lambda \) in SS model can be determined using deposition on tilted large substrates with an overall slope \( m \). For this purpose, we can generate an overall slope \( m \) of the interface by tilting the surface. Operationally, this can be performed by applying the helical boundary conditions [1], i.e. \( h(L, t) = h(1, t) - m(L-1) \).

Based on an approach known as Krug-Meakin method [42], It is expected for the KPZ equation that the asymptotic velocity \( v_L \) for finite systems of size \( L \) is given by [43],

\[
\Delta v = v_L - v_\infty = -\frac{A\lambda}{2}L^{2\alpha-2}
\]

where \( A \sim D/\nu \) is the power-law coefficient of the second-order height-difference correlation as a function of the distance between columns. In the following, after a general description of the methods, we try to estimate the interface velocity as well as the nonlinear parameter associated to the KPZ equation for the SS model. For this purpose, we begin with the determination of the interface velocity. In \( d \)-dimensional substrate, we consider \( \mathcal{P}^+ (\mathcal{P}^-) \) as the probability of choosing a site eligible for growth (desorption). Since the interface height for each allowed growth (desorption) site increases (decreases) by 2, the interface velocity is given by the relation [1]

\[
v(t) = 2 [p\mathcal{P}^+(t) - q\mathcal{P}^-(t)],
\]

In \( d = 1 \), there is a standard mapping between the height in the SS model and an kinetic Ising model [15, 16]. By using one essential property of the kinetic Ising model that in its steady state all spin configurations are equivalent, the exact value of the probability of choosing a site eligible for growth (desorption) in the steady state is presented in [21]

\[
\mathcal{P}_\infty^+ = \mathcal{P}_\infty^- = \frac{1}{4}(1 + \frac{1}{L-1})
\]

where \( \mathcal{P}_\infty^+ \) and \( \mathcal{P}_\infty^- \) are the steady state values of the \( \mathcal{P}^+ \), and \( \mathcal{P}^- \), respectively. After substitution of these values into Eq. (5), one obtains

\[
v_L = v_\infty + \frac{(p-q)}{2} \frac{1}{L}
\]

where \( v_\infty = \frac{1}{2}(p-q) \) is the asymptotic velocity of the interface. On the other hand, by tilting the substrate, the exact analytic result for the coefficient of the nonlinear term in the KPZ equation is known as \( \lambda = (q-p) \) [1, 21]. This relation expresses quantitatively the fact that only for \( p = q \), the nonlinear term vanishes, and the SS model belongs to the EW class which is in excellent agreement with our previous numerical observations in interface width. Comparing Eq. (8), and Eq. (5) with \( \lambda = (q-p) \) conclude to \( A = 1 \), independent of the value of \( p \). It should be noted that the exact values of \( A \) and \( \Gamma \) for the SS model at \( p = 0 \) are presented in [21], in this paper, we simply calculate these parameters for other values of \( p \). In Fig. 3, and Fig. 4, exact theoretical values (dashed line) and our numerical results (squares) are presented. The agreement between the theoretical and the numerical results is excellent for all values of \( p \).

In \( d = 2 \), the scenario is more complicated, although it is known that the SS interface can be mapped onto the
six-vertex model with equal vertex energies [15, 33], but, to our knowledge, this map has not provided any precise result about the universal and nonuniversal parameters of this model, yet. Therefore, we try to numerically obtain the probability of finding a site eligible for growth (deposition), i.e. \( P^+ (P^-) \), in the steady-state regime \( t \gg L^2 \) on a lattice of size 512. In table I we display the obtained values together with their statistical error of the \( P^+ \) and \( P^- \). As can be seen, these probabilities are numerically equal to each other only for the case \( p = q \), which, based on some symmetry principles, the model must be described by the EW equation. This finding is likely to be inconsistent with the claim that all possible configurations of the six-vertex model equally are weighted. We believe that the \( P^+ \) and \( P^- \) are obviously related to number of maxima and minima on the interface, as features of the local geometry, and consequently are related to the HD of the surface.

A matter of concern, when obtaining numerically the \( \lambda \) parameter for SS model as well as other growth models, is related to the lattice size, because, as mentioned before, we must perform our numerical simulations on a large lattice size. In order to reduce the finite size effects in our numerical results, and based on Eq. (4) and Eq. (5), we can estimate the asymptotic interface velocity for tilted substrates, i.e. \( m \neq 0 \), and then we can obtain the \( \lambda \) parameter for SS model. Consequently, we expect the following relation for the effective nonlinear parameter of a lattice of size \( L \)

\[
\lambda_{\text{eff}}(L) = \lambda + BL^{2a-2} \tag{9}
\]

where \( \lambda \) and \( B \), respectively, are the nonlinear parameter of the associated KPZ equation in the thermodynamic limit, and a constant related to the \( A \) parameter. By plotting \( \lambda_{\text{eff}}(L) \) against \( L^{2a-2} \) with the value \( \alpha = 0.3869(4) \) which is adopted as the roughness exponent for the KPZ class in \( d = 2 + 1 \) [12], we determine \( \lambda \) as listed in table I. In contrast to 1d, the obtained results in 2d have not linear relationship with \( p \) (as shown in Fig. 3(a)). In order to demonstrate the accuracy and efficiency of Eq. (9), we also perform simulations on the BD model, and estimate the nonlinear parameter of this model (as shown in Fig. 3(c)). In a small amount of computational time, we obtain \( \lambda = 1.283(2) \), and 2.151(5) in one, and two dimensions, respectively, which are in unprecedented accuracy compared to reported values of 1.25 [44], 1.30 [21], and 1.34 [45] for \( d = 1 + 1 \), and 2.15(10) [38] for \( d = 2 + 1 \).

By using Eq. (8) and the obtained probabilities of \( P^+ \) and \( P^- \), we can directly calculate the interface velocity, but in order to reduce the finite size effects, we apply the Eq. (5) in our numerical simulations. Therefore, by plotting \( v_L \) against \( L^{2a-2} \), and by using the \( \lambda \) parameters, we determine \( v_{\infty} \) and \( A \) as listed in table I. Fig. 5 shows a nonlinear dependence on the parameter \( p \) for both \( v_{\infty} \) and \( A \).

FIG. 3: (Color online) (a) The nonlinear parameter of SS model vs. value of \( p \) in both one, and two dimensions. The dashed-line are plotted based on exact theoretical results in \( d = 1 \). In contrast to 1d, the obtained results in 2d have not linear relationship with \( p \). The plot of \( \lambda(L) \) against \( L^{2a-2} \), (b) for SS model \( p = 0 \), (c) for the BD model in 2d. (The error bars are smaller than the symbols)

FIG. 4: (Color online) (a) The average interface velocity \( v_{\infty} \), and (b) the parameter \( A \) of SS model vs. the value of \( p \) in both one, and two dimensions. The maximum lattice size was \( 2^{13} \) and \( 2^{10} \) for 1d and 2d, respectively. The \( v_{\infty} \) values and their statistical error are presented in table I

V. UNIVERSAL AND NON-UNIVERSAL PARAMETERS

The scaling analysis based on the KPZ ansatz, Eq. (2), requires precise estimates of both the universal and the nonuniversal parameters. In this section, we first estimate the non-universal parameter \( \Gamma \) in Eq. (2) which
TABLE I: Non-universal parameters for SS model in both one and two dimensions at different $p$ values which are shown in brackets. In the case of $p = 0$, ignoring the sign, the obtained $v_\infty$ value is in good agreement with 0.341368(3) reported in [28].

| $d$ | $|p|$ | $P_\infty$ | $P_\infty^{(p-q)}$ | $v_\infty$ | $\lambda$ | $\Gamma$ |
|-----|-------|------------|-------------------|------------|----------|---------|
| $1$ | $0.0$ | $0.19755(3)$ | $0.17069(4)$ | $0.34137(1)$ | $0.492(2)$ | $1.23(4)$ |
|     | $0.1$ | $0.20238(3)$ | $0.17825(3)$ | $0.28037(4)$ | $0.409(2)$ | $0.52(2)$ |
|     | $0.2$ | $0.20791(2)$ | $0.18718(2)$ | $0.21631(4)$ | $0.317(1)$ | $0.073(4)$ |
|     | $0.3$ | $0.21129(3)$ | $0.19563(2)$ | $0.14710(4)$ | $0.215(1)$ | $0.0054(9)$ |
|     | $0.4$ | $0.21106(3)$ | $0.20521(2)$ | $0.10733(12)$ | $0.0015(5)$ | $0.28(1)$ |
|     | $0.5$ | $0.20790(2)$ | $0.20790(2)$ | $0.073(1)$ | $0.0054(9)$ | $0.28(1)$ |

FIG. 5: (Color online) Amplitude fluctuation parameter estimated via KPZ ansatz for SS model in both one (a) and two (b) dimensions. The lattice size are $215$, and $210$ for $d = 1$, and $d = 2$ respectively. The dashed horizontal lines are at $\Gamma$ values given by exact value (1d) and extrapolation of $\Gamma_{eff}$ in the limit $t \to \infty$ (2d).

The Insets show rescaled $\Gamma_{eff}(t)/\Gamma$ vs. $t/t_\infty$.

is controlling the amplitude of fluctuations in the KPZ ansatz. Then we investigate the universal properties of $\chi$ in both the growth and the stationary regimes.

According to an approach which is commonly called Krug-Meakin method [42], and based on the definitions adopted in past studies, the parameter $\Gamma$ is given by $\Gamma = (1/2)|\lambda|A^2$ for one dimensions and $\Gamma = |\lambda|A^{1/\alpha}$ for two dimensions [21, 42]. The parameter $A$ can be obtained from the foregoing expression of the asymptotic velocity $v_\infty$, i.e. Eq. (5). In $d=1$, accepting $A = 1$, and $\lambda = (q-p)$ result in $\Gamma = (q-p)/2$. For 2d SS model, we numerically determine the parameter of $\Gamma$ for different values of $p$. These estimated values of $\Gamma$ for different values of $p$ are shown in Table I.

Moreover, in order to obtain $\Gamma$, there are another method which is directly related to the KPZ ansatz, Eq. (2), $\Gamma$ can also be obtained using

$$\Gamma = \lim_{t \to \infty} \left[ \frac{\langle h^2 \rangle_c}{t^{1/\beta} \langle \chi^2 \rangle_c} \right]^{1/2\beta},$$

where we use $\langle \chi^2 \rangle_c = 0.63805$ in one dimensions [22], and $\langle \chi^2 \rangle_c = 0.235$ in two dimensions [29, 30]. We also adopt $\beta = 0.2398(3)$ as the KPZ growth exponent in $d = 2 + 1$, which is extracted from [12] by using $\beta = \alpha/(2 - \alpha)$ relation.

In order to considering the finite time effects on $\Gamma$, from Eq. (2) we define

$$\Gamma_{eff}(t) \equiv \left[ \frac{\langle h^2 \rangle_c}{\langle \chi^2 \rangle_c} \right]^{1/2\beta} = \Gamma + ct^{-2\beta} + \cdots.$$ (11)

The Fig. 5 shows $\Gamma_{eff}(t)$ as a function of time for the SS model, in both one and two dimensions, on a lattice of size $215$ in 1$d$ and $210$ in 2$d$. For large value of $p$, the linear regime expected in the KPZ ansatz is observed only for very long times. As shown in the insets of Fig. 5, after the crossover time scale, the KPZ clearly dominates in the growing regime with TW distributions. The asymptotic $\Gamma$ values obtained using this approach are the same, inside the error bars, as those found using the Krug-Meakin analysis shown in Table I.

Although so far, we have estimated all the parameters of Eq. (2) which is valid in the limit of $t \to \infty$, but in the finite time scale, some other nonuniversal parameters are also required to be added to that equation. Particularly, it has been reported that the first cumulant of the scaled height $h \equiv (h - v_\infty t)/(s_\lambda (\Gamma t)^{1/3})$ approaches the theoretical value of TW distributions as a power law $t^{-\beta}$, i.e. $\langle h \rangle - \langle \chi \rangle \sim t^{-\beta}$ (for example see [26, 28, 38, 46–48]). By adding a model-dependent stochastic quantity such as $\eta$ which is responsible by a shift in the mean of the scaled height $h$, to the Eq. (2), a modified KPZ ansatz in the finite-time regime can be obtained. Interestingly, one can obtain the exact analytical form of KPZ ansatz, Eq (2), for 1d SS model in the KPZ-regime:

$$h(t) \approx \frac{(p-q)}{2} t + s_\lambda \left( \frac{q-p}{2} \right)^{1/3} t^{1/3} \chi + \eta,$$ (12)

where $s_\lambda = Sgn(q-p)$ is the sign of the $\lambda$. This exact analytical expression can be used to verify different
FIG. 6: (Color online) The variation of the $\langle \tilde{h} \rangle - \langle \chi \rangle$ vs. rescaled time $tp^{4-\beta}$ for a lattice size $2^{15}$ for different values of $p$ in 1d SS model. The inset shows the same data in a log-log plot.

numerical algorithms. The mean $\langle \eta \rangle$ can be determined using the height scaled in terms of exact values of the parameters $v_\infty$ and $\Gamma$ as $\langle \tilde{h} \rangle - \langle \chi \rangle = \frac{(q-p)^{\eta}}{\lambda v_\infty} t^{-\beta}$. Here we use $\langle \chi \rangle = -0.76007$ in one dimensions [22]. Fig. 6 shows that the power law $t^{-\beta}$ describes very precisely the shift. So, using the prefactor of the power law $t^{-\beta}$, we can determine $\langle \eta \rangle$ as a function of $p$. In order to obtain a good data collapse, both of time $t$ and $\Gamma$ should scale with $p'$ for other values of $p$ respect to the case of $p = 0$. The former and the later need to scale with $p'^4$, and $p'^3$, consequently the time in the prefactor need to scale with a factor of $p'^{4-\beta}$ Therefore, by applying this appropriate scaling, we expect a good data collapse as shown in Fig. 6. Using the prefactor of the power law $t^{\beta}$, Finally we can estimate the mean value of $\eta$,

$$\langle \eta \rangle \approx \frac{\Omega}{8} (q-p)^{\frac{\eta}{3}}$$

where $\Omega \approx \frac{5}{2}$ is a constant which is theoretically unknown at the present, but can be estimated from the slope of the fitted curve in the main Fig. 6.

In order to study the universal properties of $\chi$ of 2d SS model, as a quick check, we calculate the dimensionless cumulant ratio skewness $S = (h^3)_c/(h^2)_c^{1.5}$, in both growth and stationary regimes, where $(X^n)_c$ represents the nth cumulant of $X$. In particular for $p = 0.15$, the obtained skewness values in growth and saturated regime are $0.40(3), 0.26(2)$, respectively. The obtained data are in good agreement with $0.428(5)$ [28] in the growth regime and, $0.2657(4)$ [12] and $0.270(5)$ [13] in stationary regime. Therefore, we show that SS model in two dimensions obey the KPZ ansatz with the expected universal stochastic term $\chi$, which would be practically impossible with the currently computer resources if the strong finite-time corrections were not explicitly taken into account in the analysis.

VI. CONCLUSIONS

In this paper, we study kinetic roughening of the SS model for surface growth in one and two dimensions. The results of extensive simulations as well as our careful finite-size scaling analysis clearly indicate that: (i) In contrast to recent studies of this model [37], we show that there exist a slow crossover from an intermediate regime dominated by the EW class to an asymptotic regime dominated by the KPZ class for any $p < 0.5$. Therefore, our results rule out any roughening transition in two dimensions. In fact, the presence of long crossover time for large value of $p$ leads to failure of observation of hydrodynamic limit behaviors in numerical simulations on small lattices, (ii) As shown in Figures 3 and 4, the effective nonuniversal parameters of $\lambda, v_\infty$, and $\Gamma$ continuously decrease with $p$, but not in a linear fashion, (iii) The universal and the nonuniversal properties of height distributions of SS model also show a good agreement with the KPZ ansatz. Therefore, in the hydrodynamic limit, one expects that the growth dynamics of SS model is described by the KPZ equation for $p \neq 0.5$.

Our study can open a new theoretical challenge in the field and can also shed light on the controversial relation between SS model and some extensively studied models in equilibrium or nonequilibrium statistical mechanics, such as the six-vertex model. We believe that because of having smallest step sized within all discreet growth models, it is expected the SS model can be described better the asymptotic, long wave scaling behavior [13].

We also believe that the Eq. 9, and the Eq. 12 should be useful in numerical studies of growth models, helping to estimate with a good accuracy the non-universal parameters, and to verify the numerical recipes with an exact theoretical result, respectively.

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