Black Holes in Two Dimensions

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Abstract

Models of black holes in (1 + 1)-dimensions provide a theoretical laboratory for the study of semi-classical effects of realistic black holes in Einstein’s theory. Important examples of two-dimensional models are given by string theory motivated dilaton gravity, by ordinary general relativity in the case of spherical symmetry, and by Poincaré gauge gravity in two spacetime dimensions. In this paper, we present an introductory overview of the exact solutions of two-dimensional classical Poincaré gauge gravity (PGG). A general method is described with the help of which the gravitational field equations are solved for an arbitrary Lagrangian. The specific choice of a torsion-related coframe plays a central role in this approach. Complete integrability of the general PGG model is demonstrated in vacuum, and the structure of the black hole type solutions of the quadratic models with and without matter is analyzed in detail. Finally, the integrability of the general dilaton gravity model is established by recasting it into an effective PGG model.

1 Introduction

Standard General Relativity (GR) is trivial in two dimensions. Nevertheless, two–dimensional (2D) models of gravity which differ from GR have recently received considerable attention [2]-[35]. The interest in 2D gravity is strongly supported by the fact that usual four-dimensional GR, in the case of spherical symmetry, is described by an effective 2D gravitational model of the dilaton type. Such a dimensional reduction provides a technical tool for the study of long standing problems in black hole physics, including an understanding of the final state of a black hole with an account of the back reaction of the quantum evaporation process, see [4]-[8]. On the other hand, lower–dimensional black hole physics is discussed in the context of string theory motivated dilaton gravity (see, e.g., [9]-[13], [19]-[20], [25], [26], [27], [34]) and in the framework of PGG (“Poincaré gauge gravity”) [1]-[8]. The approaches [1]-[8] are attempts to construct string theories with a dynamical geometry [10]-[17]. In this paper we present an overview of the black hole solutions in classical two–dimensional PGG. At the same time, 2D gravity is of interest in itself
as a theoretical laboratory which offers a simple way to study difficult non-perturbative quantization problems [13].

In the studies of both, classical and quantized 2D models, it is of crucial importance to find exact solutions of the field equations. Here we describe an elegant method developed in [29]-[41] with the help of which one can explicitly integrate the field equations of classical PGG with and without matter sources. The central point is to use a specific coframe built up from the one–form of the torsion trace and its Hodge dual. The early proofs of the integrability of the quadratic PGG models in vacuum were based on the component approach and relied on specific gauge choices, like the conformal or the light-cone gauge [10]-[26]. The coupling to gauge, scalar, and spinor matter fields was shown to destroy the integrability in general. A peculiar but common feature of the standard matter sources (gauge, scalar, and spinor fields) in two dimensions is that for all of them the spin current vanishes. Thus, quite generally, the Lorentz connection is explicitly decoupled from 2D matter. Hence the material energy–momentum current is symmetric and covariantly conserved with respect to the Riemannian connection. The absence of the spin–connection coupling considerably facilitates the integration of the field equations.

The structure of the paper is as follows: Sec. 2 contains an introduction to 2D Riemann-Cartan geometry. In Secs. 3 and 4, we demonstrate the integrability of PGG with an arbitrary gravitational Lagrangian and prove the consistency of our method in general. As a particular application, a quadratic

\[ \begin{align*}
  x^0 & -axis \\
  x^1 & -axis
\end{align*} \]

\[ \Phi \]

\[ \omega \]

\[ v \]

\[ \Psi \]

\[ w \]

Fig. 1: Two-dimensional spacetime: A 0-form (scalar) $\Phi$ has one component, a 1-form $\Psi$ two components, and a 2-form $\omega$ one component. A vector $v$ has two components, a bivector $w$ one component.
Two-dimensional Riemann-Cartan spacetime

Fig. 2: Two-dimensional Riemann-Cartan spacetime: Coframe $\vartheta^\alpha$ (2 components), connection $\Gamma^{\alpha\beta} = -\Gamma^{\beta\alpha}$ (2 components), torsion $T^\alpha$ (2 components), curvature $R^{\alpha\beta} = -R^{\beta\alpha}$ (1 component). Here, $\vartheta^\alpha = \{\vartheta^0, \vartheta^1\}$ is a natural coframe, i.e., $\vartheta^0 = dx^0, \vartheta^1 = dx^1$.

model with an action containing squares of torsion and curvature is discussed in vacuum (Sec. 5) and in the presence of conformally invariant matter (Sec. 6). The properties of the exact solutions of black hole type are described in detail. Finally, in Sec. 7, we apply the general method to the (purely Riemannian) string theory motivated dilaton gravity models by rewriting them in form of an effective Poincaré-Brans-Dicke theory. The general solution of an arbitrary two-dimensional dilation gravity model is obtained in explicit form.

2 Two-dimensional Riemann-Cartan spacetime

The Riemann–Cartan geometry has rather remarkable properties in two dimensions, see Fig.1. In the PGG approach, the orthonormal coframe one–form $\vartheta^\alpha$ and the linear connection one–form $\Gamma^{\alpha\beta}$ are considered to be the translational and the Lorentz gauge potentials of the gravitational field, respectively. The corresponding field strengths are given by the torsion two–form $T^\alpha := D\vartheta^\alpha$ and the curvature two–form $R^{\alpha\beta} := d\Gamma^{\alpha\beta} - \Gamma^{\alpha\gamma} \wedge \Gamma^{\beta\gamma}$, see Fig.2. The frame $e_\alpha = e^i_\alpha \partial_i$ is dual to the coframe $\vartheta^\beta = e^i_\beta dx^i$, i.e., $e_\alpha |\vartheta^\beta = e_\alpha^i e^i_\beta = \delta_\alpha^\beta$. The spacetime manifold $M$ is equipped with a metric

$$g = g_{ij} \, dx^i \otimes dx^j. \quad (2.1)$$
Thus its coframe components satisfy
\[ o_{\alpha\beta} = e^i \epsilon^j g_{ij}, \quad (o_{\alpha\beta}) = \text{diag}(-1, +1). \] (2.2)

In an orthonormal frame, the curvature, like the connection, is antisymmetric in \( \alpha \) and \( \beta \).

| Object | Valuedness | Type | Components |
|--------|------------|------|-------------|
| \( T^\alpha \) | vector | 2 | \( n^2(n - 1)/2 \) |
| \( R^\alpha\beta \) | bivector | 2 | \( n^2(n - 1)^2/4 \) |
| \( \Sigma^\alpha \) | vector | \( n - 1 \) | \( n^2 \) |
| \( \tau^\alpha\beta \) | bivector | \( n - 1 \) | \( n^2(n - 1)/2 \) |
| \( \eta^\alpha \) | vector | \( n - 1 \) | \( n^2 \) |

For a 2D Riemann–Cartan space, as we can take from Table 1, we have two translation generators and one rotation generator. This allows us to introduce a Lie (or right) duality operation, that is, a duality with respect to the Lie–algebra indices, which maps a vector into a covector, and vice versa:

\[ \psi^* := \epsilon_{\alpha\beta} \psi^\beta, \quad \psi^\alpha = \epsilon^{\alpha\beta} \psi^* \] (2.3)

Here the completely antisymmetric tensor is defined by \( \epsilon_{\alpha\beta} := \epsilon_{\alpha\beta}^{\mu\nu} \epsilon_{\mu\nu} \), where \( \epsilon_{\alpha\beta} \) is the Levi–Civita symbol normalized to \( \epsilon_{01} = +1 \) (a circumflex on top of a number identifies the number as an anholonomic or frame index). For \( \psi^\beta = \vartheta^\beta \) we get \( \eta_{\alpha} := \epsilon_{\alpha\beta} \vartheta^\beta \), where \( * \) denotes the Hodge (or left) dual. Using the Lie (or right) duality in two dimensions, we can appreciably compactify the notation, see Table 2.

Local Lorentz transformations are defined by the \( 2 \times 2 \) matrices \( \Lambda_{\beta\alpha}(x) \in SO(1, 1) \),
\[ \Lambda_{\beta\alpha} = \delta_{\alpha}^\beta \cosh \omega + \eta^\alpha_{\beta} \sinh \omega. \] (2.4)

| n=2 | Valuedness | p–form | Components |
|------|------------|--------|-------------|
| \( \Gamma^* := (1/2) \eta_{\alpha\beta} \Gamma^{\alpha\beta} \) | scalar | 1 | 2 |
| \( t^{\alpha} := *T^{\alpha} \) | vector | 0 | 2 |
| \( T := e_{\alpha} t^{\alpha} \) | scalar | 1 | 2 |
| \( t^{2} := o_{\alpha\beta} t^{\alpha} t^{\beta} \) | scalar | 0 | 1 |
| \( R^{*} = d\Gamma^{*} \) | scalar | 2 | 1 |
| \( R := e_{\alpha} [e_{\beta}] R^{\beta\alpha} \) | scalar | 0 | 1 |
The gauge transformations of the basic gravitational field variables read
\[
\vartheta'_{\alpha} = (\Lambda^{-1})_{\alpha}^{\beta} \vartheta^\beta = \vartheta^\alpha \cosh \omega - \eta^\alpha \sinh \omega, \quad (2.5)
\]
\[
\Gamma'^{\alpha}_{\beta} = \Lambda_{\alpha}^{\gamma} \Gamma_{\gamma}^{\delta} (\Lambda^{-1})_{\delta}^{\beta} - \Lambda_{\alpha}^{\gamma} d(\Lambda^{-1})_{\gamma}^{\beta} = \Gamma^{\beta}_{\alpha} + \eta_{\alpha}^{\beta} d\omega, \quad (2.6)
\]
or
\[
\Gamma^{*'} = \Gamma^* - d\omega. \quad (2.7)
\]

The curvature 2–form has only one irreducible component, and it can be expressed in terms of the curvature scalar
\[
R_{\alpha\beta} = -\frac{1}{2} R \vartheta^\alpha \wedge \vartheta^\beta. \quad (2.8)
\]

In two dimensions torsion is irreducible and reduces to its vector piece
\[
T^\alpha = -t^\alpha \eta, \quad (2.9)
\]
where the vector–valued torsion zero–form \( t^\alpha \) is defined via the Hodge dual
\[
t^\alpha := *T^\alpha. \quad (2.10)
\]

When the torsion square is not identically zero, i.e. \( t^2 := t^\alpha t^\alpha \neq 0 \), we call the corresponding manifold \( M \) a non-degenerate Riemann–Cartan spacetime. In this case, using the scalar-valued torsion one–form \( T := e_\alpha T^\alpha \), we can write a coframe as
\[
\vartheta^\alpha = -\frac{1}{t^2} (T \eta^\alpha \beta t_\beta + *T \cdot t^\alpha) = -\frac{1}{t^2} (T \cdot t^\alpha + *T \cdot t^\alpha). \quad (2.11)
\]

Thus, the torsion one–form \( T \) and its dual \(*T\) specify a coframe with respect to which one can expand all the 2D geometrical objects. When \( t^2 \neq 0 \), this coframe is non-degenerate, hence the terminology of a non-degenerate Riemann–Cartan space. In this case, the volume two–form can be calculated, in the non-degenerate case, as an exterior square of the torsion one–form \( T \):
\[
\eta := \frac{1}{2} \eta_{\alpha\beta} \vartheta^\alpha \wedge \vartheta^\beta = \frac{1}{t^2} *T \wedge T. \quad (2.12)
\]

Defining a coframe of a 2D Riemann-Cartan spacetime in terms of the torsion one–form turns out to be extremely convenient, and, in fact, underlies the integrability of the 2D gravity models with and without matter.

3 The field equations of PGG: invariant formulation

The total action of the interacting matter field \( \Psi \) and the PGG fields in two dimensions reads
\[
W = \int \left[ L(\vartheta^\alpha, \Psi, D\Psi) + V(\vartheta^\alpha, T^\alpha, R^{\alpha\beta}) \right], \quad (3.13)
\]
where the matter Lagrangian two–form \( L \) will be specified later.
One can prove that torsion and curvature can enter a general gravitational Lagrangian \( V \) only in form of the scalars \( t^2 := o_{\alpha\beta} t^\alpha t^\beta \) and \( R \). The gravitational Lagrangian density is denoted by \( V := *V \). Then the general gauge invariant PGG Lagrangian reads

\[
V(\vartheta^\alpha, T^\alpha, R^\alpha\beta) = V(\vartheta^\alpha, t^2, R) = -\mathcal{V}(t^2, R) \eta. \quad (3.14)
\]

The partial derivatives

\[
P := -2 \left( \frac{\partial V}{\partial t^2} \right), \quad \kappa := 2 \left( \frac{\partial V}{\partial R} \right), \quad (3.15)
\]

i.e. the generalized gravitational field momenta (‘excitations’), define two functions \( P = P(t^2, R) \) and \( \kappa = \kappa(t^2, R) \) which are assumed to be smooth and \textit{nontrivial}: \( P \neq 0, \kappa \neq 0 \).

![Energy-momentum current \( \Sigma^\alpha \) (4 components) and spin current \( \tau_{\alpha\beta} \) (2 components) in two-dimensional spacetime.](image)

The variational derivatives

\[
\Sigma_\alpha := \frac{\delta L}{\delta \vartheta^\alpha}, \quad \tau_{\alpha\beta} := \frac{\delta L}{\delta \Gamma^\alpha_{\beta \gamma}}, \quad (3.16)
\]

yield the energy–momentum and the spin one–forms of matter, respectively, see Fig.3. Using the right duality, one straightforwardly replaces the bivector–valued spin by the scalar–valued one–form \( \tau^* := \frac{1}{2} \eta_{\alpha\beta} \tau^{\alpha\beta} \). Similarly, instead of using the vector-valued energy-momentum one-form \( \Sigma_\alpha \), it turns out to be more convenient to introduce two scalar-valued one-forms

\[
S := t^\alpha \Sigma_\alpha, \quad S^* := t_\alpha \eta^{\alpha\beta} \Sigma_\beta = t^* \alpha \Sigma_\alpha. \quad (3.17)
\]
3 The field equations of PGG: invariant formulation

Analogously to (2.11), which expresses the coframe in terms of $T$ and $\star T$, one can rewrite the energy–momentum current in terms of $S$ and $S^\star$:

$$\Sigma_\alpha = \frac{1}{t^2} \left( t_\alpha S + \eta_{\alpha\beta} t^\beta S^\star \right) = \frac{1}{t^2} \left( t_\alpha S + t^\star_\alpha S^\star \right).$$

(3.18)

If we use the Hodge star, we find by straightforward algebra

$$S^\star + \star S = \star(\vartheta^\alpha \wedge \Sigma_\alpha) T - \star(\eta^\alpha \wedge \Sigma_\alpha).$$

(3.19)

Let us recall that $\vartheta^\alpha \wedge \Sigma_\alpha$ describes the trace of the energy-momentum whereas $\eta^\alpha \wedge \Sigma_\alpha$ represents its antisymmetric part. The latter is related to the spin via the second Noether identity:

$$2 d\tau^\star = \eta^\alpha \wedge \Sigma_\alpha.$$

(3.20)

The general field equations of PGG arise from independently varying (3.13) with respect to the coframe $\vartheta^\alpha$ and the connection $\Gamma^{\alpha\beta}$. Remarkably, these equations can be rewritten in a completely coordinate and gauge invariant form

$$d(P^2 t^2) = 2 P(\bar{V} T + S),$$

(3.21)

$$d(P \star T) = (P t^2 - 2 \bar{V}) \eta + \vartheta^\alpha \wedge \Sigma_\alpha,$$

(3.22)

$$d\kappa = - P T + 2 \tau^\star,$$

(3.23)

$$Pt^2(\Gamma^\star + du) = \bar{V} \star T + S^\star,$$

(3.24)

where

$$\bar{V} := V + P t^2 - \frac{1}{2} \kappa R$$

(3.25)

is the so–called modified Lagrangian function and

$$t^2 du := \eta_{\alpha\beta} t^\alpha dt^\beta.$$

(3.26)

The term $du$ in (3.24) is physically irrelevant, since a 2D Lorentz transformation can create such an Abelian shift, see (2.7). In (3.21)-(3.24), the source terms of the matter field $\Psi$ are represented by $S$ and $S^\star$, by the energy–momentum trace $\vartheta^\alpha \wedge \Sigma_\alpha$, and by the spin $\tau^\star$. We marked them in the field equations by letters in boldface. Besides the gravitational field equations, we have the matter field equation. For matter described by a $p$–form field $\Psi$ it reads

$$\frac{\delta L}{\delta \Psi} = \frac{\partial L}{\partial \Psi} - (-1)^p D \frac{\partial L}{\partial D \Psi} = 0.$$

(3.27)

As we have seen, the system (3.21)-(3.24) involves the energy–momentum one–forms $S$ and $S^\star$ as sources. They satisfy the following equations which can be derived from the Noether identities:

$$d(PS) = T \wedge P(S - R\tau^\star) + \frac{1}{2} \bar{V} d\tau^\star,$$

(3.28)

$$d(PS^\star) = T \wedge P(S^\star + R\star \tau^\star) + \frac{2}{t^2} S \wedge S^\star + \left( \bar{V} - Pt^2 \right) \vartheta^\alpha \wedge \Sigma_\alpha.$$
In 2D, the specific feature of standard matter (scalar, spinor, Abelian and non–Abelian gauge fields) is that the spin current is zero:

$$\tau_{\alpha\beta} = 0 \quad (\text{standard matter}).$$  \hspace{1cm} (3.30)

Thus only the canonical energy–momentum one–form $\Sigma_\alpha$ enters the gravitational field equations as a source. Moreover, in view of (3.20), it becomes symmetric: $\eta^\alpha \wedge \Sigma_\alpha = 0$. In this paper, we limit ourselves to the discussion of massless, conformally invariant matter models. Then we have

$$\vartheta^\alpha \wedge \Sigma_\alpha = 0 \quad (\text{massless, conformally invariant matter}),$$  \hspace{1cm} (3.31)

and the corresponding terms drop out from the field equation (3.22) and the Noether identity (3.29). Consequently, only $S$ is left as a source. From now on, we will specialize to this physically most interesting case obeying (3.30) and (3.31). Under these conditions, (3.19) simply reduces to

$$S^* + \ast S = 0.$$  \hspace{1cm} (3.32)

Accordingly, the complete system (3.21)-(3.24) and (3.28)-(3.29), with (3.30) and (3.31), should be jointly solved with the matter field equation (3.27).

**Consistency check of the invariant formulation**

As it is clearly suggested by the field equation (3.23), the function $\kappa$ of the Riemann-Cartan curvature $R$ (and, in general, of $t^2$) can be conveniently treated as one of the local coordinates on a two–dimensional manifold $M$. However, one has always to check the consistency of the scheme by explicitly calculating the curvature from the connection which itself is obtained from the field equations. This was done for the vacuum solutions of the general PGG model in [29] and for non-vacuum solutions of the quadratic models in [31]. Here we will demonstrate consistency in general, for arbitrary matter sources and arbitrary gravitational Lagrangian. We consider the non-trivial non-degenerate case with $t^2 \neq 0$.

Eq.(3.24) yields the general solution for the Lorentz connection. Starting from the definitions (3.15) and using (3.23), it is straightforward to compute the differential of the modified Lagrangian:

$$d\tilde{V} = \frac{1}{2} \left( \frac{1}{P} d(P^2t^2) + RPT \right) - R \tau^*.$$  \hspace{1cm} (3.33)

With the help of this relation and eqs.(3.21), (3.22), (3.29), (3.19), one finds

$$d[P(\tilde{V} + \ast S)] = \frac{1}{P^2t^2} d(P^2t^2) \wedge [P(\tilde{V} + \ast S)] + \frac{1}{2} R P^2 T \wedge \ast T.$$  \hspace{1cm} (3.34)

The consistency proof is completed by taking the exterior differential of the left- and right-hand sides of equation (3.24). With the help of (3.33) and (3.34), this yields

$$d\Gamma^* = - \frac{1}{2} R \eta.$$
4 Exact solutions of PGG with arbitrary gravitational Lagrangian

The two cases of two-dimensional PGG should be treated separately, the degenerate case with $t^2 = 0$ and the non-degenerate one with $t^2 \neq 0$. We will formulate our answers for matter sources obeying (3.30) and (3.31).

4.1 Degenerate torsion solutions

If $t^2 = 0$, the torsion one-form is either self- or anti-self-dual,

$$T = \pm \ast T.$$  \hspace{1cm} (4.35)

Then (3.22) and (3.23) yield $\tilde{V} = 0$. This in turn, with (3.25) and (3.15), yields

$$f(R) := V - R \frac{\partial V}{\partial R} = 0.$$  \hspace{1cm} (4.36)

For a given Lagrangian $V = V(R)$, the $t^2$-dependence drops out because of the degeneracy, the solutions of $f(R) = 0$ determine some $R = R_1$, $R = R_2$, ... Therefore the curvature is constant, $R = \text{const}$ and, by implication, also the $R$-dependent Lorentz field momentum, $\kappa = \text{const}$. Then, from (3.23), one finds $T = 0$. Finally, an analysis of the matter field equation and of the Noether identities shows that only trivial matter configurations are allowed: A constant field in the case of a zero-form $\Psi$, e.g..

Summarizing, we see that the degenerate solutions of PGG reduce to the torsionless de Sitter geometry,

$$T^\alpha = 0, \quad R = \text{const}, \quad \Psi = \text{const},$$  \hspace{1cm} (4.37)

where the constant values of the curvature are roots of equation (4.36). Incidentally, the same turns out also to be true for some conformally non-invariant matter, for a massive scalar field with arbitrary self-interaction, e.g.. In the rest of the paper we will mainly consider the non-degenerate case with $t^2 \neq 0$.

4.2 Non-degenerate vacuum solutions

Let us now specialize to the vacuum field equations. Accordingly, in (3.21)-(3.23) we have to put $S = 0$, $\vartheta^\alpha \wedge \Sigma_\alpha = 0$, and $\tau^* = 0$. The formal general solution is obtained as follows: Let us introduce a coordinate system $(\kappa, \lambda)$ which is related to the torsion 1-form basis $(T, \ast T)$ via

$$PT = -d\kappa, \quad P \ast T = B d\lambda,$$  \hspace{1cm} (4.38)

with some function $B(\kappa, \lambda)$. Consequently, the volume 2-form is given by

$$\eta = \frac{B}{P^2 t^2} d\kappa \wedge d\lambda,$$  \hspace{1cm} (4.39)
5 Exact vacuum solutions of PGG with quadratic gravitational Lagrangian

The first equation in (4.38) is simply the field equations (3.23). Substitution of the ansatz (4.38) into (3.21) and (3.22) results in

\[
\frac{\partial}{\partial \kappa} (P^2 t^2) = -2 \dot{V}, \quad \frac{\partial}{\partial \lambda} (P^2 t^2) = 0,
\]

(4.40)

\[
\frac{\partial}{\partial \kappa} \ln B = \frac{\partial}{\partial \kappa} \ln (P^2 t^2) + \frac{1}{P}.
\]

(4.41)

Formal integration of (4.41) yields the solution

\[
B = B_0(\lambda) P^2 t^2 \exp \left( \int \frac{d \kappa}{P} \right),
\]

(4.42)

where \(B_0(\lambda)\) is an arbitrary function of \(\lambda\) only.

Provided the gravitational Lagrangian \(V\), and hence \(P\), is smooth, there always exists a solution of the first order ordinary differential equations (4.40). This describes \(P^2 t^2\) as a function of \(\kappa\) and \(\lambda\), thus completing our formal demonstration of the integrability of the general two-dimensional vacuum PGG. The complete non-degenerate vacuum solution is evidently of the black hole type with the metric

\[
g = -\frac{d \kappa^2}{P^2 t^2} + P^2 t^2 \exp \left( 2 \int \frac{d \kappa}{P} \right) d \lambda^2.
\]

(4.43)

Here, without restricting generality, we put \(B_0 = 1\). Torsion and curvature for our solution are obtained by inverting the relations \(P = P(t^2, R), \kappa = \kappa(t^2, R) \rightarrow t^2 = t^2(P, \kappa), R = R(P, \kappa)\). For the solution to be unique, one must assume the relevant Hessian \(\left( \frac{\partial^2 V}{\partial \kappa^2}, \frac{\partial^2 V}{\partial \lambda^2} \right)\) to be non-degenerate. It is straightforward to derive from (3.24) the curvature scalar of the general solution:

\[
R = \frac{P^2 t^2}{B} \frac{\partial}{\partial \kappa} \left( \frac{B}{P^2 t^2} \frac{\partial}{\partial \kappa} (P^2 t^2) \right).
\]

(4.44)

The position of the horizon(s) is evidently determined by the zeros of the metric coefficient \(g_{\lambda \lambda} = P^2 t^2 \exp (2 \int d \kappa/P)\). It is impossible to say more without explicitly specializing the gravitational Lagrangian. An important particular case is represented by quadratic PGG which will be discussed in the next two sections.

5 Exact vacuum solutions of PGG with quadratic gravitational Lagrangian

Let us now analyze two-dimensional PGG with a gravitational Lagrangian quadratic in torsion and curvature,

\[
V = -\left( \frac{a}{2} T_\alpha * T^\alpha + \frac{1}{2} R^\alpha_\beta \eta_{\alpha \beta} + \frac{b}{2} R^\alpha_\beta * R^\alpha_\beta \right) - \Lambda \eta.
\]

(5.45)

Here \(a, b, \) and \(\Lambda\) are the coupling constants. Using (5.45) in (3.17), we find:

\[
P = a, \quad \kappa = b R - 1.
\]

(5.46)
The modified Lagrangian function reads
\[ \tilde{V} = \frac{a}{2} t^2 - \frac{b}{4} R^2 + \Lambda. \] (5.47)

We will concentrate here on vacuum solutions. The general scheme for an arbitrary Lagrangian \( V \) was given in the previous section. The degenerate solutions (4.37) are de Sitter spacetimes:
\[ T^\alpha = 0, \quad R = \pm R_{\text{DS}}, \quad R_{\text{DS}} := 2\sqrt{\frac{\Lambda}{b}}. \] (5.48)

Also the non-degenerate solutions can be easily obtained. Substituting (5.46)-(5.47) into (4.40), we explicitly find for the scalar torsion square
\[ -t^2 = 2M_0 e^{-bR/a} - \frac{b}{2a} R^2 + R + \frac{2\Lambda}{a} - \frac{a}{b}. \] (5.49)

The integration constant \( M_0 \) has the physical meaning of the mass of a point-like source. This can be derived from the existence of the timelike Killing vector field \( \zeta = \partial_t \) which yields the conserved energy–momentum 1–form
\[ \varepsilon_{\text{RC}} := \zeta^\alpha \Sigma_{\alpha} = -e^{bR/a} \left( \frac{1}{2} d(t^2) + \frac{1}{a^2} \tilde{V} ds \right). \] (5.50)

This 1–form is strongly conserved, i.e. \( d\varepsilon_{\text{RC}} = 0 \) even when the field equations are not fulfilled. One can verify that
\[ \varepsilon_{\text{RC}} = dM, \quad M := -\frac{e^{bR/a}}{2} \left( t^2 - \frac{b}{2a} R^2 + R + \frac{2\Lambda}{a} - \frac{a}{b} \right). \] (5.51)

When the vacuum field equations are satisfied, \( \varepsilon_{\text{RC}} = 0 \), and thus \( M = M_0 \).

**Black hole structure of non-degenerate spacetimes**

The degenerate solutions are torsionless de Sitter spacetimes with constant Riemannian curvature \( R_{\text{DS}} \), see (5.48). The properties of the non-degenerate solutions are obviously defined by the values of the coupling constants \( (a, b, \Lambda) \) and by the value of the first integral \( M_0 \). Let us denote
\[ M_{\pm} := \frac{e^{\mp bR_{\text{DS}}/a}}{2} \left( \frac{a}{b} \pm R_{\text{DS}} \right). \] (5.52)

We recognize that always \( M_- \leq M_+ \), equality is achieved only for vanishing cosmological constant: \( M_- = M_+ = a/(2b) \) for \( R_{\text{DS}} = \Lambda = 0 \). For sufficiently large \( \Lambda \), namely when \( \Lambda > a^2/(4b) \) (or, equivalently, \( R_{\text{DS}} > a/b \)) one finds negative \( M_- \), otherwise \( M_{\pm} \geq 0 \). The special case \( \Lambda = a^2/(4b) \) in a de Sitter gauge gravity model was discussed in [36, 38] (then \( R_{\text{DS}} = a/b \) and \( M_- = 0 \)).

As we already know, the spacetime metric is given by the line element (4.43) with (5.46) and (5.48) inserted. Clearly, instead of \( \kappa \), we can use the scalar
curvature $R$ as a spatial coordinate. Another convenient choice is a “radial” coordinate
\[ r := \exp \left( \frac{b R}{a} \right). \tag{5.53} \]

The meaning of the quantities (5.52) becomes clear when we analyze the metric coefficient
\[ g_{\lambda \lambda} = -2M_0 e^{bR/a} + e^{2bR/a} \left( \frac{b}{2a} R^2 - R - \frac{2\Lambda}{a} + \frac{a}{b} \right). \tag{5.54} \]

Its zeros define the positions of the horizons $R_h$,
\[ g_{\lambda \lambda}(R_h) = 0. \tag{5.55} \]

At $R = -\infty$, we have $g_{\lambda \lambda} = 0$. However, this point is not a horizon but a true singularity with infinite curvature. This corresponds to $r = 0$ which one can consider as the position of a central point-like source mass. Such a singularity is, in general, hidden by the horizons.

The position, the number, and the type of horizons are completely determined by the total mass $M_0$. We can distinguish five qualitatively different configurations:

(i) $M_0 < M_-$: one horizon at $R_h < -R_{dS}$. \tag{5.56}
(ii) \( M_0 = M_- \): two horizons at

\[
R_{h_1} < -R_{dS}, \quad R_{h_2} = R_{dS}.
\]  

(5.57)

(iii) \( M_- < M_0 < M_+ \): three regular horizons at

\[
R_{h_1} < -R_{dS} < R_{h_2} < R_{dS}, \quad R_{h_c} > R_{dS}.
\]  

(5.58)

(iv) \( M_0 = M_+ \): two horizons at

\[
R_{h_1} = -R_{dS}, \quad R_{h_2} > R_{dS}.
\]  

(5.59)

(v) \( M_0 > M_+ \): one regular horizon at

\[
R_h > R_{dS}.
\]  

(5.60)

Here we have assumed \( M_0 > 0 \) which is physically reasonable. For zero or negative values of \( M_0 \), the smallest horizons disappear for the cases (i), (ii), and (iii). In Fig. 4, the function (5.54) is depicted for \( \Lambda = \frac{a^2}{16} \) and positive values of \( M_0 \).

The geometry of these solutions is obtained by an appropriate gluing of a charged black hole to a de Sitter spacetime. In (iii), the largest \( R \), namely \( R_{h_c} \), describes the cosmological event horizon which hides de Sitter singularity at \( R = \infty \) \((r = \infty)\) from an observer at \( R < R_{h_c} \). Other horizons, with \( R_{h_1,2} \), describe the charged black hole. The cases (ii) and (iv) correspond to the extremal Reissner-Nordström black hole.

The Hawking temperature of the black holes is related to their surface gravity. The latter can be straightforwardly calculated from the knowledge of the timelike Killing vector \( \zeta = \partial_\lambda \):

\[
\sqrt{- \frac{1}{2} \left( \nabla_i \zeta_j \right) \left( \nabla_i \zeta_j \right)} \bigg|_{R = R_h} = \frac{b}{2a^2} (R_{h_c}^2 - R_{dS}^2).
\]  

(5.61)

The temperature vanishes for the extremal black hole configurations (ii) and (iv).

It seems worthwhile to mention that \( g_{\lambda\lambda} \) reaches a local maximum at \( R_{h_1} \) [case (iv)], and a local minimum at \( R_{h_2} \) [case (ii)], when \( M_- \neq M_+ \). For \( M_- = M_+ \), the three cases (ii), (iii), and (iv) degenerate to a configuration with a horizon at \( R_{h_1} = R_{h_2} = 0 \), which is an inflexion point of \( g_{\lambda\lambda} \). The qualitative behavior is given in Fig. 5 for positive values of \( M_0 \).

6 Quadratic PGG with massless matter

6.1 Massless fermions

Dirac spinors in two dimensions have two (complex) components,

\[
\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.
\]  

(6.62)
The spinor space at each point of the spacetime manifold is related to the tangent space at this point via the spin-tensor objects. In the approach with Clifford–algebra valued forms, the central object is the Dirac one–form
\[ \gamma = \gamma_\alpha \theta^\alpha, \tag{6.63} \]
which satisfies
\[ \gamma \otimes \gamma = g, \quad \gamma \wedge \gamma = -2\gamma_5 \eta. \tag{6.64} \]
The \( \gamma_5 \) matrix is implicitly defined by
\[ ^* \gamma = \gamma_5 \gamma. \tag{6.65} \]
We are using the following explicit realization of the Dirac one–form:
\[ \gamma := \begin{pmatrix} 0 \\ (\tilde{\theta}^0 + \tilde{\theta}^1) \\ -(\tilde{\theta}^0 - \tilde{\theta}^1) \\ 0 \end{pmatrix}. \tag{6.66} \]
The Dirac matrices \( \gamma^\alpha \) satisfy the usual identity \( \gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = 2g^{\alpha\beta} \).

The gauge and coordinate invariant Lagrangian two–form for the massless Dirac spinor field can be written in the form
\[ L = \frac{i}{2} (\bar{\psi} \gamma \wedge d\psi + d\bar{\psi} \wedge \gamma \psi). \tag{6.67} \]
It is well known that in two dimensions there is no interaction of spinors with the Lorentz connection, and the above Lagrangian contains ordinary exterior differentials and not the covariant ones. Nevertheless, the theory is explicitly invariant under local Lorentz rotations.

The (Dirac) field equation is obtained from the variation of $L$ with respect to $\bar{\psi}$ and reads

$$\gamma \wedge d\psi - \frac{1}{2} (d\gamma) \psi = 0. \quad (6.68)$$

The degenerate case was described in Sec. 4.1. Here we assume that $t^2 \neq 0$. Thus the one–forms $T$ and $*T$ can be treated as the coframe basis in a two–dimensional Riemann–Cartan spacetime. In the explicitly gauge invariant approach, it is convenient to define, instead of the original spinor (6.62), two complex (four real) Lorentz invariant functions:

$$\varphi_1 = u_1 + i u_2 := \sqrt{(\hat{t}^0 + \hat{t}^1)} \psi_1, \quad \varphi_2 = v_1 + i v_2 := \sqrt{(\hat{t}^0 - \hat{t}^1)} \psi_2. \quad (6.69)$$

The Dirac equation (6.68) yields for the real variables $u_A, v_A, A = 1, 2$ (using (2.11))

$$d \left[ \frac{u_A u_B}{t^2} (T - *T) \right] = 0, \quad d \left[ \frac{v_A v_B}{t^2} (T + *T) \right] = 0. \quad (6.70)$$

This can be immediately integrated. In particular, the Poincaré lemma (locally) guarantees the existence of two real functions. We denote them by $x$ and $y$, such that

$$\frac{|\varphi_1|^2}{t^2} (T - *T) = dx, \quad \frac{|\varphi_2|^2}{t^2} (T + *T) = dy. \quad (6.71)$$

This evidently provides local null coordinates $(x, y)$ for the spacetime manifold. Introducing the phases of the complex spinor components explicitly,

$$\varphi_1 = |\varphi_1| e^{i\alpha}, \quad \varphi_2 = |\varphi_2| e^{i\beta}, \quad (6.72)$$

we find, using (6.70), that these phases depend only on one of the above variables,

$$\alpha = \alpha(x), \quad \beta = \beta(y). \quad (6.73)$$

This construction gives the general exact solution of the massless Dirac equation in an arbitrary two–dimensional Riemann–Cartan spacetime.

The energy–momentum one–form is straightforwardly obtained,

$$S = A_1 (T - *T) + A_2 (T + *T), \quad (6.74)$$

where we denoted

$$A_1 := - \frac{d\alpha}{dx} \frac{|\varphi_1|^4}{t^2}, \quad A_2 := - \frac{d\beta}{dy} \frac{|\varphi_2|^4}{t^2}. \quad (6.75)$$

In (6.74), we have $S^* = - *S$, well in accordance with (3.32).
6.2 Massless bosons

Let us now turn to a gravitationally coupled massless scalar field $\phi$ with the Lagrangian two–form

$$L = -\frac{1}{2} d\phi \wedge *d\phi.$$  \hfill (6.76)

Variation with respect to $\phi$ yields the Klein-Gordon equation:

$$*d*d\phi = 0.$$  \hfill (6.77)

In non-degenerate spacetimes with $t^2 \neq 0$, we can use the torsion coframe basis and can write, in the most general case,

$$d\phi = \Phi_1(T - *T) + \Phi_2(T + *T),$$  \hfill (6.78)

with some functions $\Phi_{1,2}$. We substitute (6.78) into the Klein-Gordon equation (6.77). Then it turns out that locally there exists such a scalar function $z$ that

$$\Phi_1(T - *T) - \Phi_2(T + *T) = dz.$$  \hfill (6.79)

This describes a general solution of the Klein–Gordon equation.

For the energy-momentum one–form $S$ we get, similarly to (6.74),

$$S = A_1(T - *T) + A_2(T + *T),$$  \hfill (6.80)

where now

$$A_1 = -t^2\Phi_1^2, \quad A_2 = -t^2\Phi_2^2.$$  \hfill (6.81)

6.3 Chiral solutions

Both, the massless Dirac equation and the massless Klein–Gordon equation, admit chiral solutions. For fermions chirality means that only one component of the spinor field is nontrivial. For bosons chirality can be formulated in terms of self- or anti-self-duality of the “velocity” one–form $d\phi$. In both cases, the field equations describe right- or left-moving configurations. In this section we describe the corresponding gravitational field for the quadratic PGG model (5.45).

Let us assume that $\varphi_2 = \psi_2 = 0$ for the fermion and $\Phi_2 = 0$ for the boson field. Then $A_2 = 0$ in (6.75) as well as in (6.81). Hence the energy–momentum one–form $S$ is anti-self-dual $S^* = S$. These are chiral configurations.

The integrals (6.71) and (6.79),

$$T - *T = \frac{t^2}{|\varphi_1|^2} \, dx \quad \text{(spinor),} \quad T - *T = \frac{1}{\Phi_1} \, dz \quad \text{(scalar),}$$  \hfill (6.82)

together with the equations (3.23), (5.46), suggest a natural interpretation of the variables $R$ and $x$ (or $R$ and $z$, respectively,) as two local spacetime coordinates. Clearly, $x$ and $z$ are different in each case, but we can unify the two problems
without risk of confusion. For the torsion coframe the equations (6.82) and (3.23) explicitly yield

\[ *T = - \left( \frac{b}{a} dR + B dx \right), \quad T = - \frac{b}{a} dR, \]  

(6.83)

for the vacuum case, see (4.38). Hence the volume two–form reads

\[ \eta = \frac{1}{t^2} *T \wedge T = \frac{bB}{at^2} dx \wedge dR, \]  

(6.84)

and the spacetime metric is given by

\[ g = \frac{1}{t^2} \left[ \left( B dx + \frac{b}{a} dR \right)^2 - \frac{b^2}{a^2} dR^2 \right]. \]  

(6.85)

Here we use the unifying notation

\[ B = \frac{t^2}{|\varphi_1|^2}. \]  

(6.86)

for fermions, while for bosons this function relates the two coordinate systems via \( dz = \Phi_1 B dx \).

The spacetime geometry is completely described when one solves the field equations (3.21)-(3.23), (3.28)-(3.29), thus finding the functions \( t^2 \) and \( B \) explicitly. By means of (6.74) and (6.80), the energy–momentum one–form turns out to be

\[ S = A dx, \]  

(6.87)

where

\[ A := -|\varphi_1|^2 \frac{d\alpha}{dx} \text{ (spinor)}, \quad A := -t^2 B\Phi_1^2 \text{ (scalar)}. \]  

(6.88)

Substituting (6.87), (6.83) into (3.21)-(3.23) and (3.28), one finds

\[ \frac{\partial t^2}{\partial R} = - \frac{2b}{a^2} V, \quad \frac{\partial t^2}{\partial x} = \frac{2}{a} A, \]  

(6.89)

\[ \frac{1}{b} \frac{\partial \ln B}{\partial R} + \frac{1}{a} - \frac{2}{a^2 t^2} V = 0, \]  

(6.90)

\[ \frac{\partial A}{\partial R} + \frac{b}{a} A = 0. \]  

(6.91)

The equation (3.29) is redundant. This can be compared with the vacuum case (4.40), (4.41).

The system (6.89)-(6.91) is solved by

\[ A = f(x) e^{-bR/a}, \]  

(6.92)

\[ B = B_0(x) t^2 e^{bR/a}, \]  

(6.93)

\[ -t^2 = 2 M(x) e^{-bR/a} - \frac{b}{2a} R^2 + R + \frac{2\Lambda}{a} - \frac{a}{b}. \]  

(6.94)
where
\[ M(x) := -\frac{1}{a} \int f(x) dx, \quad (6.95) \]
with the arbitrary functions \( f(x) \) and \( B_0(x) \). Without loss of generality we can put \( B_0 = 1 \) since a redefinition of \( x \) is always possible. For completeness, let us write down the Lorentz connection. Inserting (6.92)-(6.94) into (3.24), we find
\[ \Gamma^* = d\tilde{u} + \frac{ebR/a}{2} \left( R - \frac{a}{b} \right) dx, \quad (6.96) \]
where \( \tilde{u} \) is a pure gauge contribution.

The gravitational field defined by (6.92)-(6.94) has the same form for chiral fermionic and bosonic sources. However, the function \( f(x) \) is different for each particular physical source.

For fermions combining (6.93), (6.86), (6.92), and (6.88), we find
\[ |\varphi_1|^2 = e^{-bR/a}, \quad f(x) = -\frac{d\alpha}{dx}. \quad (6.97) \]
Hence the solution for the chiral fermion field, in terms of its invariant complex component, reads
\[ \varphi_1 = \exp \left( -\frac{b}{2a} R + i\alpha(x) \right), \quad (6.98) \]
whereas the metric is described by (6.85) with \( B \) as specified in (6.93) and
\[ M(x) = \frac{\alpha(x)}{a} + M_0, \quad (6.99) \]
where \( M_0 \) is an arbitrary integration constant.

For bosons, combining (6.94) with (6.88) and (6.78), one finds
\[ f(x) = -(\Phi_1 B)^2 = -\left( \frac{d\phi}{dx} \right)^2, \quad (6.100) \]
and the scalar field \( \phi(x) \) remains an arbitrary function of \( x \).

The physical meaning of the solutions obtained is clear. In Sec. 5 it was shown that the structure of a static black hole in vacuum is determined by the value of the total mass \( M_0 \) entering the torsion square (5.49). The massless chiral (fermionic and bosonic) matter contributes a variable “mass” \( M(x) \) to the torsion square (6.94). As a result, the black hole in general becomes non-static (6.85).

One can illustrate this process of a restructuring of a black hole by matter falling into it \([9, 41]\). Let us consider \( f(x) = -m \delta(\frac{x - x_0}{a}) \). In view of (6.87) and (6.92), this function describes a point-like “impulse” of matter: the field energy is zero everywhere except for a single moving point (recall that \( x \) is a null coordinate). Then, for (6.93), one obtains
\[ M(x) = M_0 + m \theta(x - x_0). \quad (6.101) \]
In the region \( x < x_0 \) we have a static black hole with mass \( M_0 \), whereas for \( x > x_0 \) its mass increases to \( M_0 + m \).
6.4 Conformally non-invariant matter

The above results are restricted to the chiral case and the conformally invariant massless matter sources. Some remarks are necessary for the more general cases. The non–chiral solutions were obtained in [31] for fermionic and in [33] for bosonic matter. In general, the resulting system cannot be integrated analytically, and a numeric analysis is needed. It is possible, though, to obtain exact analytic solutions for certain models with a complicated matter content: a nonlinear spinor field interacting with scalars, e.g., see the discussion of the instanton type solutions in [2]-[4].

The lack of conformal invariance does not always lead to serious difficulties. Let us consider Yang-Mills theory, for example, with an arbitrary in general non-Abelian gauge group. The dynamical variable is the gauge potential or, equivalently, the Lie algebra-valued connection one–form $A^B$. The Yang-Mills Lagrangian is constructed from the corresponding gauge field strength two–form $F^B$:

$$L_{YM} = -\frac{1}{2}F^B \wedge *F_B.$$  (6.102)

The energy–momentum current attached to (6.102) reads

$$\Sigma_\alpha := e_\alpha [L_{YM} + (e_\alpha F^B) \wedge *F_B] = -\frac{1}{2}f^2 \eta_\alpha,$$  (6.103)

where $f^2 := f^B f_B$, $f_B = *F_B$. As in the previous cases, the spin current vanishes, $\tau_{\alpha\beta} = 0$, since the Lorentz connection does not couple to the Yang–Mills potential.

Observe that the energy–momentum trace, in contrast to four dimensions, does not vanish:

$$\nabla^\alpha \wedge \Sigma_\alpha = -f^2 \eta \neq 0.$$  (6.104)

The results described in Sec. 3 refer only to the conformally invariant case. Thus one needs a proper generalization for the case (6.104). Quite fortunately, the situation is greatly simplified due to the constancy of $f^2$. Although, of course, the Lie algebra–valued scalar field $f_A$ is not constant in view of the nonlinear nature of the Yang-Mills equations

$$D *F_A = df_A + c_{ABC} A^B f^C = 0,$$  (6.105)

obviously its square is conserved: $f^2 = const$.

As a result, the gravitational field equations (3.21)-(3.23) are modified by a following simple shift of the cosmological constant:

$$\Lambda \rightarrow \bar{\Lambda} = \Lambda + \frac{1}{2} f^2.$$  (6.106)

In particular, the complete integrability of the vacuum system is not disturbed by Yang–Mills matter, and one again recovers the static black hole solutions described in Sec. [3].
7 Black hole solution for general dilaton gravity

In this section we demonstrate that our method also successfully works in the string motivated dilaton models, cf. \[9, 5, 28, 30, 27, 18, 19, 34, 35\].

Let us denote the purely Riemannian curvature scalar by a tilde: \(\tilde{R}\). In general, the same notation is used for all geometrical objects and operations which are defined by the torsion-free Riemannian connection \(\tilde{\Gamma}^{\alpha\beta}_{\gamma}\) (Christoffel symbols). Let be given, in two dimensions, the gravitational potential \(\vartheta^a\) on one side and a scalar field \(\Phi\) and a Yang–Mills potential \(A^B = A^B dx^i\) on the other, the matter side. These fields are interacting with each other. A corresponding general Lagrangian two–form reads

\[
V_{\text{dil}} = \eta \left( F(\Phi) \tilde{R} + G(\Phi)(\partial_\alpha \Phi)^2 + U(\Phi) + J(\Phi)(F^B_{ij})^2 \right). \tag{7.107}
\]

Here the kinetic terms are constructed from \(\partial_\alpha \Phi := e_\alpha^i d\Phi\) and \(F^B = \frac{1}{2} F^B_{ij} dx^i \wedge dx^j\), respectively. For the string motivated dilaton models, the coefficient functions read:

\[
F(\Phi) = e^{-2\Phi}, \quad G(\Phi) = \gamma e^{-2\Phi}, \quad U(\Phi) = e^{-2\Phi} U(\Phi), \quad J(\Phi) = \frac{-e^{(\epsilon-2)\Phi}}{4}, \tag{7.108}
\]

where \(\gamma = 4\) and \(U(\Phi) = c\) in the tree approximation of string theory. A number of physically interesting models correspond to different values of \(\gamma, \epsilon, \) and \(U(\Phi)\).

7.1 Main result

Locally one can always treat the scalar function \(\Phi\) as a coordinate on a two-dimensional spacetime manifold. Denote the second coordinate by \(\lambda\).

In terms of the local coordinates \((\lambda, \Phi)\), the metric of the general solution of the gravitational field equation of the model (7.107) reads

\[
ds^2 = -4 h(\Phi) e^{-2\nu(\Phi)} d\lambda^2 + \frac{(F')^2}{h(\Phi)} d\Phi^2, \tag{7.109}
\]

where

\[
h(\Phi) = \nu(\Phi) \left( M_0^2 + \int d\varphi \left( U(\varphi) + \frac{Q_0^2}{2J(\varphi)} \right) F'(\varphi) e^{-\nu(\varphi)} \right), \tag{7.110}
\]

\[
\nu(\Phi) = \int d\varphi \frac{G(\varphi)}{F'(\varphi)}. \tag{7.111}
\]

Here \(M_0\) and \(Q_0^2\) are the integration constants which are related to the total mass and the (squared) charge of a solution. For completeness, let us give the solution for the Yang–Mills field:

\[
F^B = f^B \eta, \quad f^B f_B = \left( \frac{Q_0}{J(\Phi)} \right)^2. \tag{7.112}
\]
The proof of this result, see below, is obtained by the method developed for two-dimensional PGG.

### 7.2 Dilaton models and PGG

In two dimensions, torsion is represented by its trace one-form \( T := e_\alpha T^\alpha \), see Table 2. Then the Riemann-Cartan connection decomposes into the Riemannian and post-Riemannian pieces as follows:

\[
\Gamma_{\alpha\beta} = \tilde{\Gamma}_{\alpha\beta} - \vartheta^\alpha e_\beta |T + \vartheta_\beta e_\alpha |T = \tilde{\Gamma}_{\alpha\beta} - \eta_{\alpha\beta} \ast T. \tag{7.113}
\]

By differentiation we find a corresponding decomposition of the Riemann-Cartan curvature. For the Hilbert-Einstein two-form this yields

\[
R_{\alpha\beta} \eta_{\alpha\beta} = \tilde{R}_{\alpha\beta} \eta_{\alpha\beta} + 2 d \ast T. \tag{7.114}
\]

Let us consider the “scalar-tensor” type PGG model with the Lagrangian

\[
V_0 = -\frac{1}{2} \left[ \xi(\Phi) T_\alpha \wedge \ast T^\alpha + \omega(\Phi) R^\alpha\beta \eta_{\alpha\beta} \right]. \tag{7.115}
\]

Here \( \xi(\Phi) \) and \( \omega(\Phi) \) are the scalar functions which describe a variable gravitational “constant” à la Jordan and Brans-Dicke. Variation of (7.115) with respect to the connection yields the field equation

\[
\xi(\Phi) T = \omega'(\Phi) d \Phi. \tag{7.116}
\]

Hereafter the prime denotes derivative with respect to \( \Phi \).

As we see, the torsion trace turns out to be an exact one-form, and the scalar dilaton field plays a role of its generalized potential. Substituting (7.116) into (7.114) and (7.113), one finds (dropping an exact form):

\[
V_0 = -\frac{1}{2} \left[ \omega \tilde{R} + \left( \frac{\omega'}{\xi} \right)^2 (\partial_\alpha \Phi)^2 \right]. \tag{7.117}
\]

This is evidently equivalent to the dilaton model (7.107) with

\[
F(\Phi) = \frac{\omega(\Phi)}{2}, \quad G(\Phi) = \frac{(\omega'(\Phi))^2}{2\xi(\Phi)}. \tag{7.119}
\]

The equivalence of (7.115) and (7.118) can be also verified by comparing the relevant sets of field equations.

The scalar-tensor PGG model (7.113) can be straightforwardly generalized in order to include the potential for the dilation field \( \Phi \) and a possible interaction with the matter field \( \Psi \):

\[
L_{\text{tot}} = V + L_{\text{mat}}(\Psi, D\Psi, \Phi), \tag{7.120}
\]

\[
V = -\frac{1}{2} \xi(\Phi) T_\alpha \wedge \ast T^\alpha - \frac{1}{2} \omega(\Phi) R^\alpha\beta \eta_{\alpha\beta} + U(\Phi) \eta. \tag{7.121}
\]
As we know, for standard matter in two dimensions (scalar, spinor, Abelian and non–Abelian gauge fields), the spin current is zero, $\tau_{\alpha\beta} = 0$, and hence the “second” field equation (7.116), which results from varying the connection, remains the same for the generalized model (7.121). This fact is the basis of the equivalence of the scalar-tensor model (7.121) and the general dilaton type model (7.107) with the same identifications of the coefficient functions (7.119). The dilaton field potential $U(\Phi)$ is taken from (7.107), whereas

$$L_{\text{mat}} = 2J(\Phi) F_B \wedge *F^B = J(\Phi) (F^B_{ij})^2 \eta$$

represents a specific matter field $\Psi$, with $\Psi = A^B$. In general, we may have a larger set of matter fields.

### 7.3 Proof

The explicit construction of the general solution for dilaton gravity, (7.109), (7.110), and (7.111), is obtained by the same machinery as that developed for two-dimensional PGG.

Again, we have a Lagrangian which depends on the torsion square $t^2$ and the curvature scalar $R$. The gravitational field momenta (3.15) read

$$P = \xi(\Phi), \quad \kappa = -\omega(\Phi).$$

The Lagrangian two–form (7.121) can be transformed into the corresponding Lagrangian density in (3.14):

$$V = -\frac{1}{2}(\xi t^2 + \omega R) - U.$$

Hence (3.25) yields

$$\tilde{V} = \frac{1}{2}\xi t^2 - U.$$

In the absence of matter, which can be enforced in (7.122) by putting $J(\Phi) = 0$, we immediately obtain the gravitational field equations (3.21)-(3.23) in coordinate and gauge invariant form:

$$d(\xi^2 t^2) = 2 \left( U(\Phi) - \frac{1}{2} \xi(\Phi) t^2 \right) d\kappa,$$

$$d(\xi * T) = 2 U(\Phi) \eta,$$

$$d\kappa = -\xi T.$$

If a Yang–Mills field is present, its contribution manifests itself in a simple “deformation” of the potential function $U(\Phi)$, similar to the shift of the cosmological constant in (6.106). Indeed, taking into account the Yang-Mills field equation for $\tilde{J}(\Phi)$,

$$D(\tilde{J}(\Phi) * F_A) = d(\tilde{J}(\Phi)f_A) + c_{ABC}A^B \tilde{J}(\Phi)f^C = 0,$$

where $f_A := *F_A$, we straightforwardly obtain the first integral

$$f^B f_B (\tilde{J}(\Phi))^2 =: Q_0^2.$$ 

With the help of this result, one can prove that the field equations (7.124),(7.125) are formally the same, if the potential $U(\Phi)$ is replaced by

$$U(\Phi) \longrightarrow U(\Phi) + \frac{1}{2} Q_0^2 / \tilde{J}(\Phi).$$

(7.129)
In order to simplify the notation, we will treat both cases simultaneously, considering the system (7.124)-(7.126) with $U(\Phi)$ properly defined.

The integration of (7.124)-(7.126) is straightforward. At first, after substituting (7.123), we immediately obtain a linear equation for $\xi^2 t^2$,

$$(\xi^2 t^2)' = \frac{\omega'}{\xi} \xi^2 t^2 - 2U\omega', \quad (7.130)$$

where the prime denotes a derivative with respect to $\Phi$. Formal integration yields

$$-\xi^2 t^2 = 2e^{\nu(\Phi)} \left( M_0 + \int d\varphi U(\varphi) \omega'(\varphi) e^{-\nu(\varphi)} \right), \quad (7.131)$$

$$\nu(\Phi) = \int d\varphi \frac{\omega'(\varphi)}{\xi(\varphi)}. \quad (7.132)$$

The construction of the metric can be completed along the lines of our general method. Namely, since $t^2 \neq 0$, one can construct a zweibein from the torsion one-form $T$ and its dual $*T$. According to (7.126), we may consider either $\kappa = -\omega(\Phi)$ or the the field $\Phi$ itself as a first local coordinate. The second (“time”) coordinate, say $\lambda$, is then naturally associated with another leg of the zweibein,

$$\xi*T := B d\lambda, \quad (7.133)$$

with some function $B = B(\lambda, \kappa(\Phi))$. Substituting the latter into (7.123) and taking into account the volume 2–form $\eta = \frac{\eta}{\xi^2} d\kappa \wedge d\lambda$, see (2.12), we obtain an equation for $B$:

$$\frac{\partial \ln B}{\partial \kappa} = \frac{2U}{\xi^2 t^2}. \quad (7.134)$$

On integration

$$B = B_0(\lambda) \xi^2 t^2 e^{\int \frac{d\kappa}{\xi}} = B_0(\lambda) \xi^2 t^2 e^{-\nu(\Phi)}, \quad (7.135)$$

and, again without loss of generality, one can put the function $B_0(\lambda) = 1$.

Accordingly, the metric finally reads:

$$g = -\frac{d\kappa^2}{\xi^2 t^2} + \xi^2 t^2 \exp \left( 2 \int \frac{d\kappa}{\xi} \right) d\lambda^2 = \xi^2 t^2 e^{-2\nu(\Phi)} d\lambda^2 - \frac{(\omega')^2}{\xi^2 t^2} d\Phi^2. \quad (7.136)$$

Recalling the identifications (7.119), which establish the relation between the dilaton and PGG, we arrive at the result (7.109) by putting $h(\Phi) := -\xi^2 t^2 / 4$. Then the equations (7.131) and (7.132) reduce to (7.110) and (7.111), respectively.
7.4 Concluding remarks

The general solution described in Sec. 7.1 contains all the exact black hole configurations reported earlier in the literature as particular cases which correspond to specific choices of the coefficient functions $\mathcal{F}(\Phi), \mathcal{G}(\Phi), \mathcal{U}(\Phi), \mathcal{J}(\Phi)$, cf. [9, 28, 30, 27, 34], e.g..

The new results are most useful for the investigation of the dynamical picture of a gravitational collapse for nontrivial matter sources. Of particular interest is the case of a non-minimally coupled scalar field which describes the semi-classical correction to the Lagrangian due to Hawking radiation.

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References

[1] A. Achucarro, *Lineal gravity from planar gravity*, Phys. Rev. Lett. **70** (1993) 1037-1040.

[2] K.G. Akdeniz, A. Kizilersü, and E. Rizaoğlu, *Instanton and eigen-modes in a two-dimensional theory of gravity with torsion*, Phys. Lett. **B215** (1988) 81-83.

[3] K.G. Akdeniz, A. Kizilersü, and E. Rizaoğlu, *Fermions in two-dimensional theory of gravity with dynamical metric and torsion*, Lett. Math. Phys. **17** (1989) 315-320.

[4] K.G. Akdeniz, O.F. Dayi, and A. Kizilersü, *Canonical description of a two-dimensional gravity*, Mod. Phys. Lett. **A7** (1992) 1757-1764.

[5] T. Banks, A. Dabholkar, M.R. Douglas, and M. O’Loughlin, *Are horned particles the end point of Hawking evaporation?*, Phys. Rev. **D45** (1992) 3607-3616.

[6] D. Cagnemi, *One formulation for both lineal gravities through a dimensional reduction*, Phys. Lett. **B297** (1992) 261-265.

[7] D. Cagnemi and R. Jackiw, *Geometric gravitational force on particles moving in a line*, Phys. Lett. **299** (1993) 24-29.

[8] D. Cagnemi and R. Jackiw, *Poincaré gauge theory for gravitational forces in (1+1) dimensions*, Ann. Phys. (N.Y.) **225** (1993) 229-263.

[9] C.G. Callan, S.B. Giddings, J.A. Harvey, and A. Strominger, *Evanescent black holes*, Phys. Rev. **D45** (1992) R1005-R1009.

[10] M.O. Katanaev, *Complete integrability of two dimensional gravity with dynamical torsion*, J. Math. Phys. **31** (1990) 882-891.
[11] M.O. Katanaev, *Conformal invariance, extremals, and geodesics in two-dimensional gravitation with torsion*, J. Math. Phys. 32 (1991) 2483–2496.

[12] M.O. Katanaev, *All universal coverings of two-dimensional gravity with torsion*, J. Math. Phys. 34 (1993) 700-736.

[13] M.O. Katanaev, *Canonical quantization of the string with dynamical geometry and anomaly free nontrivial string in two dimensions*, Nucl. Phys. B416 (1994) 563-605.

[14] M.O. Katanaev, *Euclidean two–dimensional gravity with torsion*, J. Math. Phys. 38 (1997) 946-980.

[15] M.O. Katanaev, W. Kummer, and H. Liebl, *Geometric interpretation and classification of global solutions in generalized dilaton gravity*, Phys. Rev. D53 (1996) 5609-5618.

[16] M.O. Katanaev and I.V. Volovich, *Two-dimensional gravity with dynamical torsion and strings*, Ann. Phys. (N.Y.) 197 (1990) 1-32.

[17] M.O. Katanaev and I.V. Volovich, *Theory of defects in solids and three-dimensional gravity*, Ann. Phys. (N.Y.) 216 (1992) 1-28.

[18] T. Klösch and T. Strobl, *Classical and quantum gravity in (1 + 1)-dimensions. Part 1: A unifying approach*, Class. Quantum Grav. 13 (1996) 965-984.

[19] T. Klösch and T. Strobl, *Classical and quantum gravity in (1 + 1)-dimensions. Part 2: The universal coverings*, Class. Quantum Grav. 13 (1996) 2395-2422.

[20] W. Kummer, *Deformed iso(2, 1) symmetry and non-Einsteinian 2d-gravity with matter*, in: Proc. of the Conf. “Hadron Structure 1992”, Stará Lesná (Slovakia), 6-11 Sept. 1992, D. Brunko and J. Urban, eds. (Košice University Publications, Košice 1992) 48-56.

[21] W. Kummer and D.J. Schwarz, *Renormalization of R^2 gravity with dynamical torsion in d=2*, Nucl. Phys. B382 (1992) 171-186.

[22] W. Kummer and D.J. Schwarz, *General analytic solution of R^2 gravity with dynamical torsion in two dimensions*, Phys. Rev. D45 (1992) 3628-3635.

[23] W. Kummer and D.J. Schwarz, *Two-dimensional R^2-gravity with torsion*, Class. Quantum Grav. Suppl. 10 (1993) S235-S238.

[24] W. Kummer and D.J. Schwarz, *Comment on “Canonical description of a two–dimensional gravity”*, Mod. Phys. Lett. A8 (1993) 2903.

[25] W. Kummer and P. Widerin, *Non-Einsteinian gravity in d = 2: symmetry and current algebra*, Mod. Phys. Lett. A9 (1994) 1407-1414.
[26] W. Kummer and P. Widerin, *Conserved quasilocal quantities and general covariant theories in two dimensions*, Phys. Rev. D52 (1995) 6965-6975.

[27] O. Lechtenfeld and C.R. Nappi, *Dilaton gravity and no-hair theorem in two dimensions*, Phys. Lett. B288 (1992) 72-76.

[28] M. D. McGuigan, C.R. Nappi, and S.A. Yost, *Charged black holes in two-dimensional string theory*, Nucl. Phys. B375 (1992) 421-450.

[29] E.W. Mielke, F. Gronwald, Yu.N. Obukhov, R. Tresguerres, and F.W. Hehl, *Towards complete integrability of two-dimensional Poincaré gauge gravity*, Phys. Rev. D48 (1993) 3648-3662.

[30] C.R. Nappi and A. Pasquinucci, *Thermodynamics of two-dimensional black-holes*, Mod. Phys. Lett. A7 (1992) 3337-3346.

[31] Yu.N. Obukhov, *Two-dimensional Poincaré gauge gravity with matter*, Phys. Rev. D50 (1994) 5072-5086.

[32] Yu.N. Obukhov and S.N. Solodukhin, *Dynamical gravity and conformal and Lorentz anomalies in two dimensions*, Class. Quantum Grav. 7 (1990) 2045-2054.

[33] Yu.N. Obukhov, S.N. Solodukhin, and E.W. Mielke, *Coupling of lineal Poincaré gauge gravity to scalar fields*, Class. Quantum Grav. 11 (1994) 3069-3079.

[34] J.G. Russo and A.A. Tseytlin, *Scalar-tensor quantum gravity in two dimensions*, Nucl. Phys. B382 (1992) 259-275.

[35] J.G. Russo, L. Susskind, and L. Thorlacius, *End point of Hawking radiation*, Phys. Rev. D46 (1992) 3444-3449.

[36] S.N. Solodukhin, *Topological 2D Riemann–Cartan–Weyl gravity*, Class. Quantum Grav. 10 (1993) 1011-1021.

[37] S.N. Solodukhin, *Black-hole solution in 2D gravity with torsion*, Pis’ma ZhETF 57 (1993) 317-322; JETP Lett. 57 (1993) 329-334.

[38] S.N. Solodukhin, *Two-dimensional black hole with torsion*, Phys. Lett. B319 (1993) 87-95.

[39] S.N. Solodukhin, *Cosmological solutions in 2D Poincaré gravity*, Int. J. Mod. Phys. D3 (1994) 269-272.

[40] S.N. Solodukhin, *On exact integrability of 2-D gravity*, Mod. Phys. Lett. A9 (1994) 2817-2823.

[41] S.N. Solodukhin, *Exact solution of two-dimensional Poincaré gravity coupled to fermion matter*, Phys. Rev. D51 (1995) 603-608.

[42] T. Strobl, *All symmetries of non-Einsteinian gravity in d=2*, Int. J. Mod. Phys. A8 (1993) 1383-1397.