ANGLE SUMS OF SIMPLICIAL POLYTOPES

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ABSTRACT. The interior angle vector (\(\hat{\alpha}\)-vector) of a polytope is a metric analogue of the \(f\)-vector in which faces are weighted by their solid angle. For simplicial polytopes, Dehn-Sommerville-type relations on the \(\hat{\alpha}\)-vector were introduced by Sommerville (1927) and Höhn (1953). Camenga (2006) defined the \(\hat{\gamma}\)-vector, a linear transformation analogous to the \(h\)-vector and conjectured it to be non-negative. Using tools from geometric and algebraic combinatorics, we prove this conjecture and show that the \(\hat{\gamma}\)-vector increases in the first half and is flawless. In contrast to the \(h\)-vector, we construct a six-dimensional polytope with non-unimodal \(\hat{\gamma}\)-vector. More generally, all result remain valid when solid angles are replaced by simple and non-negative cone valuations.

1. INTRODUCTION

For a simplicial \(d\)-polytope \(P\), the entry \(f_i(P)\) of the \(f\)-vector \(f(P) = (f_{-1}(P), f_0(P), f_1(P), \ldots, f_{d-1}(P))\) is the number of \(i\)-dimensional faces of \(P\). The \(f\)-vector of a simplicial polytope has been a subject of investigation culminating in the \(g\)-theorem due to Billera-Lee [4], Stanley [18], and McMullen [14], which completely determines the whole set of possible \(f\)-vectors. The result is best expressed by a certain linear transformation of the \(f\)-vector called the \(h\)-vector \(h(P) = (h_0, h_1(P), \ldots, h_d(P))\), which can be concisely defined as an identity of polynomials in a single variable \(t\):

\[
\sum_{i=0}^{d} f_{i-1}(P) \cdot (t - 1)^{d-i} = \sum_{k=0}^{d} h_k(P) \cdot t^{d-k}.
\]

Note that one can go back and forth between \(f\) and \(h\)-vector, as one defines the other. In terms of the \(h\)-vector, the following linear relations and inequalities hold:

**Theorem 1.1** (\(h\)-vectors of simplicial polytopes). The \(h\)-vector \((h_0, \ldots, h_d)\) of a simplicial \(d\)-polytope satisfies:

1. **Dehn-Sommerville relations**: \(h_i = h_{d-i}\) for \(i = 0, \ldots, d\).
2. **Non-negativity**: \(0 \leq h_i\) for \(i = 0, \ldots, d\).
3. **Unimodality**: \(h_0 \leq h_1 \leq \cdots \leq h_m \geq h_{m+1} \geq \cdots \geq h_d\) for \(m := \left\lfloor \frac{d}{2} \right\rfloor\).

In this article, we focus on the interior angle-vector \(\hat{\nu}(P) = (\hat{\nu}_0(P), \ldots, \hat{\nu}_{d-1}(P))\) of a simplicial polytope. The \(\hat{\nu}\)-vector is a semi-discrete analogue of the \(f\)-vector and can be defined by the sum of solid angle measures of faces of fixed dimension.

Let \(P \subseteq \mathbb{R}^d\) be a \(d\)-polytope. For a face \(F\) of \(P\), define \(\hat{\nu}(F, P)\) to be the (solid) angle of \(F\) at \(P\):

\[
\hat{\nu}(F, P) := \lim_{\varepsilon \to 0} \frac{\text{vol}_d(P \cap B_\varepsilon(q))}{\text{vol}_d(B_\varepsilon(q))},
\]

where \(\text{vol}_d\) is the \(d\)-dimensional volume and \(B_\varepsilon(q)\) is the ball of radius \(\varepsilon\) around a point \(q\) in the relative interior of \(F\). This expression is independent of the choice of \(q\) and only depends on \(F\). Furthermore it is compatible with the intuitive planar notion of angle in Euclidean geometry, but with a normalization.

*Date:* July 15, 2020.

2020 Mathematics Subject Classification. 52B15, 52B45, 52B11, 52B05.

Key words and phrases. Solid angle, cone valuations, relative simplicial complexes, \(h\)-vector, Dehn-Sommerville relations.
such that a full circle has angle 1. Now the $i$-th entry of the interior angle vector is just defined as the sum of the interior angles at all $i$-faces:

$$\hat{\nu}_i(P) := \sum_{F \subseteq P, \dim F = i} \hat{\nu}(F, P).$$

In [2], the notion of interior angle vector was generalized to simple and normalized cone valuations $\alpha$, called cone angles, see [13] for a related notion. A cone valuation is a map $\alpha : \mathcal{C}^d \to \mathbb{R}$, subject to certain relations and we will give a precise definition of cone valuations and cone angles in given in Section 2. The notation $\hat{\alpha}_i(P)$ will be used to denote the entries of the interior angle vector of $P$ with respect to $\alpha$.

Cone valuations were used in [2], to determine the linear equations satisfied by the $\hat{\alpha}$-vector of all polytopes for any cone angle $\alpha$. In this note, we focus on simplicial polytopes and prove that similar linear inequalities as in Theorem 1.1 are true for their $\hat{\alpha}$-vectors – again, for any cone angle $\alpha$. As before, the expressions in terms of the $\hat{\alpha}$-vector will be more complicated, but using the same linear transformation, we derive the $\hat{\gamma}$-vector from the $\hat{\alpha}$-vector, as introduced by Camenga [6]. Being dependent on both $\alpha$ and $P$, we will denote the $\hat{\gamma}$-vector as $\hat{\gamma}(\alpha, P)$, but will omit the $\alpha$ when it is clear from the context. The $\hat{\gamma}$-vector $\hat{\gamma}(P) = \hat{\gamma}(\alpha, P)$ of a simplicial $d$-polytope $P \subseteq \mathbb{R}^d$ is the tuple $(\hat{\gamma}_0(P), \hat{\gamma}_1(P), \ldots, \hat{\gamma}_d(P))$ such that the following identity of polynomials holds:

$$\sum_{k=0}^d \hat{\gamma}_k(P) \cdot t^{d-k} := \sum_{i=0}^d \hat{\gamma}_{d-i}(P) \cdot (t - 1)^{d-i},$$

where we set $\hat{\gamma}_{-1} = 0$. It was shown for $\alpha = \nu$ by Hähn [10], that over the set of all simplicial polytopes, the linear relations satisfied by all $\hat{\alpha}$-vectors form a $\lfloor d/2 \rfloor$-dimensional linear subspace. By introducing the $\hat{\gamma}$-vector, Camenga rewrote this equalities into the following short equations reminiscent of the Dehn-Sommerville equations:

**Theorem 1.2** (Hähn [10], Camenga [6]). Let $P$ be a simplicial polytope and $\hat{\gamma}(P) = \hat{\gamma}(\nu, P)$ be the $\hat{\gamma}$-vector with respect to the standard cone angle $\nu$. Then for $i = 0, \ldots, d$:

$$\hat{\gamma}_i(P) + \hat{\gamma}_{d-i}(P) = h_i(P).$$

Furthermore, she conjectured that the entries of the $\hat{\gamma}$-vector are non-negative. In this paper we prove this conjecture and show the following:

**Theorem 1.3.** Let $\alpha$ be a non-negative cone angle, $P$ be a simplicial $d$-polytope and $\hat{\gamma}(P) = \hat{\gamma}(\alpha, P)$. Then the $\hat{\gamma}$-vector $\hat{\gamma}(P)$ satisfies:

1. Dehn-Sommerville equations: $\hat{\gamma}_i(P) + \hat{\gamma}_{d-i}(-P) = h_i(P)$ for $i = 0, \ldots, d$.
2. Non-negativity: $0 \leq \hat{\gamma}_i(P)$ for $i = 0, \ldots, d$.
3. Non-decreasing in the first half: $0 = \hat{\gamma}_0(P) \leq \hat{\gamma}_1(P) \cdots \leq \hat{\gamma}_m(P)$ for $m := \lfloor d/2 \rfloor$.
4. Flawless: $\hat{\gamma}_i(P) \leq \hat{\gamma}_{d-i}(P)$ for $0 \leq i \leq \lfloor d/2 \rfloor$.

Let us make a few remarks. First, the observant reader may have noticed that our take on the Dehn-Sommerville equations has a minus sign in the second summand. We say that $P$ is $\alpha$-symmetric, if $\hat{\alpha}_i(P) = \hat{\alpha}_{d-i}(-P)$ for $i = 0, \ldots, d - 1$, which in turn implies that $\hat{\gamma}_i(P) = \hat{\gamma}_{d-i}(-P)$. While for $\nu$ every polytope $P$ is $\alpha$-symmetric, for most pairs $\alpha$ and $P$ this symmetry does not hold, and the minus sign in the statement is necessary in these cases.

Second, while the $\hat{\gamma}$-vector is non-decreasing in the first half, it is not unimodal. We show that unimodality only holds in low dimensions:

**Theorem 1.4.** Let $\alpha$ be a non-negative cone angle, $P$ be a simplicial $d$-polytope and $\hat{\gamma}(P) = \hat{\gamma}(\alpha, P)$. If $d \leq 3$, the $\hat{\gamma}$-vector $\hat{\gamma}(P)$ is unimodal. It is unimodal for all $\alpha$-symmetric $d$-polytopes if $d \leq 5$.

Furthermore we give counterexamples for dimensions 4 and 6, respectively. Finally, we investigate the $\hat{\gamma}$-vector of the simplest simplicial polytope:
Theorem 1.5. Let \( \alpha \) be a non-negative cone angle, \( \triangle_d \) be a \( d \)-simplex. Then its \( \hat{\gamma} \)-vector \((\hat{\gamma}_0, \hat{\gamma}_1, \ldots, \hat{\gamma}_d) = \hat{\gamma}(\alpha, \triangle_d) \) is non-decreasing:

\[
0 = \hat{\gamma}_0 \leq \hat{\gamma}_1 \leq \cdots \leq \hat{\gamma}_d.
\]

The converse is almost true: If \( \hat{\gamma}(P) = \hat{\gamma}(\alpha, P) \) is non-decreasing for a simplicial polytope \( P \), then \( P \) is either a simplex or a bipyramid. In the latter case, \( \hat{\gamma}(P) = (0, 1, 1, \ldots, 1, 1) \).

We refer to Section 5 for a more precise discussion of both cases. As a corollary, we will see that only one case occurs when we consider the standard cone angle \( \nu \), and we get the following nice result:

Corollary 1.6. Let \( P \) be a simplicial \( d \)-polytope and \( \hat{\gamma}(P) = \hat{\gamma}(\nu, P) \). Then \( \hat{\gamma}(P) \) is non-decreasing if and only if \( P \) is a \( d \)-simplex.

After an introduction to cone angles in the next section, we go on to prove the first half of Theorem 1.3 in Section 3, where we will introduce a method for transferring linear inequalities from the \( h \)-vector of certain relative simplicial complexes to inequalities of the \( \hat{\gamma} \)-vector. In Section 4, we will use methods from algebraic combinatorics to show the necessary inequalities of the relative simplicial complexes leading to the second half of Theorem 1.3. Combining the work of the previous two sections, we will be able two show Theorem 1.4 and Theorem 1.5 in Section 5.

Acknowledgments. The author would like to thank Raman Sanyal for insightful discussions.

2. Cone angles

We begin with the thorough introduction of cone angles and interior angle vectors. We call a map from the set \( C^d \) of all convex polyhedral cones in \( \mathbb{R}^d \) with apex at the origin to \( \mathbb{R} \) a valuation, if it satisfies the valuation property:

\[
\alpha(C \cup C') = \alpha(C) + \alpha(C') - \alpha(C \cap C')
\]

for all \( C, C' \in C^d \) such that \( C \cup C' \in C^d \). We call such a valuation simple, if \( \alpha(C) = 0 \) for all cones \( C \in C^d \) with \( \dim C < d \) and normalized, if \( \alpha(\mathbb{R}^d) = 1 \). If the valuation \( \alpha \) has both properties, we call \( \alpha \) a cone angle, cf. [2]. If \( \alpha(C) \geq 0 \) for all \( C \in C^d \), we say that \( \alpha \) is non-negative.

One can think of cone valuations and cone angles as a generalization of measures on the sphere. If \( \mu \) is a measure on (the Borel \( \sigma \)-algebra of) the sphere \( S^{d-1} \), we can define a valuation, which we will also denote with \( \mu \) by:

\[
\mu(C) := \mu(C \cap S^{d-1}).
\]

The valuation property follows from the stronger \( \sigma \)-additivity of measures. The most important example of a cone angle from a measure on the sphere is the (normalized) spherical volume or the standard cone angle \( \nu : C^d \to \mathbb{R} \):

\[
\nu(C) := \frac{\text{vol}_d(C \cap B_1(0))}{\text{vol}_d(B_1(0))}.
\]

While the most natural examples of cone angles arise from measures, it is worth mentioning that not all valuations are obtained through this construction. As an example, let \( q \in S^d \). Then

\[
(2) \quad \omega_q(C) := \lim_{\varepsilon \to 0} \frac{\text{vol}_d(B_{\varepsilon}(q) \cap C)}{\text{vol}_d(B_{\varepsilon}(q))}
\]

defines a cone angle. We note that \( \omega_q(C) = 0 \), if \( q \notin C \), \( \omega_q(C) = 1 \), if \( q \in \text{int } C \) and \( \omega_q(C) = \frac{1}{2} \), if \( q \) is in the interior of a facet of \( C \). Thus \( \omega_q \) is not continuous and therefore not a measure.

When evaluating a valuation, it is often necessary to subdivide a cone into smaller pieces. This can be done by the Principle of Inclusion-Exclusion:
Proposition 2.1 (Principle of Inclusion-Exclusion (IE)). Let \( \alpha : \mathbb{C}^d \to \mathbb{R} \) be any valuation and let \( C = \bigcup_{i=1}^n C_i \) for \( C, C_1, \ldots, C_n \in \mathbb{C}^d \). Then
\[
\alpha(C) = \sum_{\emptyset \neq I \subseteq [n]} (-1)^{|I|+1} \alpha\left( \bigcap_{i \in I} C_i \right),
\]
where the sum is over all non-empty subsets of \([n] := \{1, 2, \ldots, n\}\). If additionally \( \alpha \) is simple and \( \dim(C_i \cap C_j) < d \) for \( i \neq j \), then
\[
\alpha(C) = \sum_{i=1}^n \alpha(C_i).
\]

Let \( P \) be a \( d \)-polytope given as the finite intersection of \( n \) half-spaces \( H_k^\leq := \{x \in \mathbb{R}^d : \langle a_k, x \rangle \leq b_k\} \) for \( k = 1, \ldots, n \), each with a facet-defining hyperplane \( H_k := \{x \in \mathbb{R}^d : \langle a_k, x \rangle = b_k\} \). Let \( \mathcal{F}(P) \) be the face lattice of \( P \) and let \( \mathcal{F}_+(P) := \mathcal{F}(P) \setminus \{\emptyset\} \) be the set of non-empty faces. We will write \( \mathcal{F}_i(P) \) for the set of all \( i \)-faces of \( P \). For \( q \in P \), define the tangent cone \( T_q P \) to be
\[
T_q P := \{x \in \mathbb{R}^d : q + \epsilon x \in P \text{ for some } \epsilon > 0 \text{ small}\}.
\]

The following well-known observation shows that this expression depends only on the unique face \( F \) containing \( q \) in its relative interior:

Proposition 2.2. Let \( q \in \text{relint} F \), where \( F \) is a non-empty face of \( P \). Then
\[
T_q P = \{x \in \mathbb{R}^d : \langle a_k, x \rangle \leq 0 \text{ for all } k = 1, \ldots, n \text{ with } F \subseteq H_k \}.
\]

We therefore set \( T_F P := T_q P \) for some \( q \in \text{relint} F \) and we also set \( T_{\emptyset} P = \{0\} \). Evaluating the cone angle \( \alpha : \mathbb{C}^d \to \mathbb{R} \) at the tangent \( T_F P \) of a face \( F \) in a \( d \)-polytope \( P \) gives the interior angle of \( P \) at \( F \) with respect to \( \alpha \), which we will denote with a hat above \( \alpha \):
\[
\hat{\alpha}(F, P) := \alpha(T_F P).
\]

Summing these expressions over all \( i \)-faces of \( P \) gives the entries of the \( \hat{\alpha} \)-vector \( \hat{\alpha}(P) \), i.e. for \( 0 \leq i \leq d-1 \), let
\[
\hat{\alpha}_i(P) := \sum_{F \in \mathcal{F}_i(P)} \hat{\alpha}(F, P).
\]

Note that for \( \alpha = \nu \), this gives precisely the definition of \( \hat{\nu} \) in (1). We also remark that there is a dual notion of an exterior angle \( \check{\alpha}(F, P) \) and of an exterior angle vector \( \check{\alpha}(P) \), which comes from the evaluation of \( \alpha \) at the normal cone \( N_F P \):
\[
N_F P := \{c \in \mathbb{R}^d : \langle c, x \rangle \geq \langle c, y \rangle \text{ for all } x \in F, y \in P\} + \text{aff}(F - F).
\]

This definition of normal cone differs from the usual definition by the term \( \text{aff}(F - F) \), which forces \( N_F P \) to always be a full-dimensional cone, so that \( \check{\alpha}(F, P) := \alpha(N_F P) \) is not always zero for \( \dim F > 0 \). While being important in the overall theory of angles, see [2, 12], we will be only concerned with interior and not with external angles here.

Let \( H_k^\prime := \{x \in \mathbb{R}^d : \langle a_k, x \rangle = 0\} \) be the facet defining hyperplanes of \( P \) moved to the origin. We associate to \( P \) the hyperplane arrangement \( \mathcal{H}(P) \):
\[
\mathcal{H}(P) := \{H_1^\prime, H_2^\prime, \ldots, H_n^\prime\}.
\]

The closures of the connected components of \( \mathbb{R}^d \setminus \bigcup \mathcal{H}(P) \) are called the regions of \( \mathcal{H}(P) \) and we denote the set of all of them by \( \mathcal{R}(P) \). The following is an easy, but crucial observation for the rest of the paper:

Lemma 2.3. Let \( P \) be a \( d \)-polytope and \( F \in \mathcal{F}_+(P) \). Each tangent cone \( T_F P \) of some face \( F \) of \( P \) is a union of some regions in \( \mathcal{R}(P) \), i.e.
\[
T_F P = \bigcup_{R \in \mathcal{R}(P)} R \quad \text{whenever} \quad R \subseteq T_F P.
\]
Proposition 2.5. For all $r \in \mathbb{R}$, $\bar{\pi}$ is a common face of both $R,P$. Thus we will write $\alpha(P) = \sum_{F \subseteq \mathcal{F}(P)} \alpha(F) = 1$. Motivated by this expression, we find it useful to collect precisely those polytopal complexes $\Psi$ consisting of $(\Pi,\Gamma)$ a relative polytopal complex and think of $\Psi$ as $\Pi$ with the elements of $\Gamma$ being removed. Thus we will write $P \in \Psi$, if $P \in \Pi$, but $P \notin \Gamma$. The elements of $\Pi$ (respectively $\Psi$), are called their faces and the maximal faces under inclusion are called facets. A (relative) polytopal complex is a (relative) d-complex, if the maximal dimension of one of its facet is $d$ and it is called pure, if all its facets have the same dimension. Examples of pure polytopal complexes are the set of all faces of a polytope $P$, i.e. $\mathcal{F}(P)$, as well as their boundary complexes $\partial P := \mathcal{F}(P) \setminus \{P\}$.

Our main players will be (relative) polytopal complexes arising from projections. Let $r \in \text{int} R$ for some region $R \in \mathcal{R}(P)$ where $P \subseteq \mathbb{R}^d$ is a d-polytope. If we imagine that $r$ points at a light source “at infinity”, then $F$ is bright, if and only if $r \notin T_F P$ and dark otherwise. By Lemma 2.3, the set of facets which are bright/dark does only depend on $R$ and not on the choice of $r$, thus let $\bar{B}(R,P) := \{F \in \partial P : R \not\subseteq T_F P\}$ be the subcomplex of $\partial P$ formed by the bright faces and $D(R,P) = (\partial P, \bar{B}(R,P))$ be the relative subcomplex formed by the dark faces. Similarly, define

$$\bar{D}(R,P) := \{F \in \partial P : -R \not\subseteq T_F P\} = \bar{B}(-R,P),$$

$$B(R,P) := (\partial P, \bar{D}(R,P)) = D(-R,P).$$

Topologically, $\bar{B}(R,P)$ is just the closure of $B(R,P)$ and the same for $\bar{D}(R,P)$. Finally, let us define the set of faces right at the boundary from bright to dark:

$$\pi(R,P) := \{F \in \partial P : R,-R \not\subseteq T_F P\}.$$ 

Clearly, $\pi(R,P)$ is isomorphic to $\partial P'$, where $P'$ is the image of an orthogonal projection along a ray $r \in \text{int} R$.

Proposition 2.5. For all d-polytopes $P$ and $R \in \mathcal{R}(P)$, the sets $\bar{B}(R,P)$ and $\bar{D}(R,P)$ are pure $(d-1)$-complexes. The sets $B(R,P)$ and $D(R,P)$ are pure relative $(d-1)$-complexes, whereas the set $\pi(R,P)$ is a pure $(d-2)$-complex.
Proof. We only need to check that \( B(R, P) \) is a polytopal complex, the other statements follow immediately. Since \( B(R, P) \) is a subset of \( \partial P \), we only need to show that for faces \( F \subseteq G \) we have \( G \in B(R, P) \) implies \( F \in B(R, P) \). But for \( F \subseteq G \) we have \( T_F P \subseteq T_G P \), so if \( R \not\subseteq T_G P \), then \( R \not\subseteq T_F P \). \( \square \)

Example 2.6. Continuing Example 2.4, for a region \( R \) the following images depict the corresponding relative complexes \( D(R, P) \), \( \pi(R, P) \) and \( B(R, P) \).

\[
\begin{align*}
\mathcal{H}(P) & \quad D(R, P) & \quad \pi(R, P) & \quad B(R, P) \\
\end{align*}
\]

The \textit{f-vector} of a \( d \)-complex \( \Pi \) is the vector \((f_1(\Pi), \ldots, f_d(\Pi))\), where \( f_i(\Pi) \) counts the number of \( i \)-dimensional faces of \( \Pi \). With this notation at hand, we can rewrite the right hand side of (3) into the more concise form:

\[
\hat{\alpha}_i(P) = \sum_{R \in \mathcal{R}(P)} \alpha(R)f_i(D(R, P)).
\]

We think of \( f \)- and \( \hat{\alpha} \)-vectors, as well as the \( h \)- and \( \hat{\gamma} \)-vectors as real-valued sequences, so that we can freely do arithmetic with them. Any entry outside of the stated range is assumed to be zero.

In [16], the following statement was proven in the case \( \nu = \alpha \) although it is hinted that it holds more generally for even cone angles.

Theorem 2.7. Let \( \alpha \) be a non-negative cone angle and \( P \subseteq \mathbb{R}^d \) be a \( d \)-polytope. Then

\[
\hat{\alpha}(P) + \hat{\alpha}(-P) \in \text{conv}\{f(\partial P) - f(\pi(R, P)) : R \in \mathcal{R}(P)\},
\]

where any entry of the \( \hat{\alpha} \)-vector outside its usual range is set to zero.

Proof. Note that \( \mathcal{H}(P) = \mathcal{H}(-P) \), so their regions coincide: \( \mathcal{R}(P) = \mathcal{R}(-P) \). By reflection in the origin, we can transform \( D(R, -P) \) into \( D(-R, P) = B(R, P) \), and we conclude \( f(D(R, -P)) = f(B(R, P)) \).

Furthermore by the decomposition of the boundary of \( P \), \( \partial P = D(R, P) \cup \pi(R, P) \cup B(R, P) \), one also has \( f(D(R, P)) + f(B(R, P)) = f(\partial P) - f(\pi(R, P)) \). Therefore

\[
\hat{\alpha}(P) + \hat{\alpha}(-P) = \sum_{R \in \mathcal{R}(P)} f(D(R, P)) \cdot \alpha(R) + \sum_{R \in \mathcal{R}(-P)} f(D(R, -P)) \cdot \alpha(R)
\]

\[
= \sum_{R \in \mathcal{R}(P)} (f(D(R, P)) + f(B(R, P))) \cdot \alpha(R)
\]

\[
= \sum_{R \in \mathcal{R}(P)} (f(\partial P) - f(\pi(R, P))) \cdot \alpha(R).
\]

But this is the desired convex combination since \( \sum_{R \in \mathcal{R}(P)} \alpha(R) = 1 \) and \( \alpha(R) \geq 0 \) for all \( R \in \mathcal{R}(P) \). \( \square \)

If \( \alpha \) is an even cone angle, then, as we recall from the introduction, \( P \) is \textit{\( \alpha \)-symmetric}, that is, \( \hat{\alpha}(P) = \hat{\alpha}(-P) \). In this case, the previous statement resolves to:

\[
2\hat{\alpha}(P) \in \text{conv}\{f(\partial P) - f(\pi(R, P)) : R \in \mathcal{R}(P)\}.
\]

Besides the case that \( \alpha \) is even, \( \alpha \)-symmetry does also occur, if \( P \) is \textit{centrally symmetric}, i.e. if \( P = -P \), in which case \( P \) is trivially \( \alpha \)-symmetric for all cone angles \( \alpha \). The next example shows that in general \( \hat{\alpha}(P) \neq \hat{\alpha}(-P) \) and that (4) does not generalize to cone angles which are not \( \alpha \)-symmetric:
Example 2.8. Let $P = \text{conv}(u, v, w)$ be any triangle in $\mathbb{R}^2$ with the origin in the interior and set $V = \{u, v, w\}$. We have $f(\partial P) = (1, 3, 3)$. $\mathcal{H}(P)$ is a hyperplane arrangement consisting of three lines through the origin and $\mathcal{R}(P) = \{T_u P, T_v P, T_w P, -T_u P, -T_v P, -T_w P\}$.

For $x \in V$, we have
\[ f(D(T_x P, P)) = (0, 1, 2) \quad \text{and} \quad f(D(-T_x P, P)) = (0, 0, 1). \]
Since $0 \in \text{int } P$, we have $-x \in \text{int } T_x P$. Thus $\omega_{-x}(R) = 1$ if and only if $R = T_x P$ and $= 0$ otherwise for all regions $R \in \mathcal{R}(P)$. Let
\[ \alpha = 1/4 \sum_{x \in V} \omega_{-x} + 1/12 \sum_{x \in V} \omega_x. \]
We calculate:
\[ \hat{\alpha}(P) = 1/4 \sum_{x \in V} \omega_{-x}(T_x(P)) \cdot f(D(T_x(P), P)) + 1/12 \sum_{x \in V} \omega_x(-T_x P) \cdot f(D(-T_x(P), P)) \]
\[ = 3/4 (0, 1, 2) + 3/12 (0, 0, 1) = (0, 3/2, 7/4). \]
Since the only combinatorial type of $\pi(R, P)$ is two points, we have $f(\pi(R, P)) = (1, 2, 0)$ for all $R \in \mathcal{R}(P)$, but
\[ 2\hat{\alpha}(P) = (0, 3, 3) \neq (0, 1, 3) = f(\partial P) - (1, 2, 0). \]

The previous example also showed a construction of cone angles with prescribed values on the regions of an arrangement $\mathcal{H}$. For future reference, let us make this precise:

Lemma 2.9. Let $\mathcal{H}$ be an arrangement of hyperplanes with regions $\{R_1, \ldots, R_n\}$ and let $\lambda_1, \ldots, \lambda_n \in [0, 1]$, such that $\sum_{i=1}^n \lambda_i = 1$. Then there exists a cone angle $\alpha$ such that $\alpha(R_i) = \lambda_i$.

Proof. Let $r_i \in \text{int } R_i$ be any ray. From the definition (2), we see
\[ \omega_{r_i}(R_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \]
and therefore $\alpha := \sum_{i=1}^n \lambda_i \omega_{r_i}$ has the desired properties. \qed

3. Simplicial Polytopes and the $\hat{\gamma}$-vector

In this section, we will introduce the $\hat{\gamma}$-vector and show its non-negativity.

A (geometrical) simplicial complex $\Delta$ is a polytopal complex where each face $\sigma \in \Delta$ is a simplex. Likewise, a relative simplicial complex is a relative polytopal complex $(\Delta, \Gamma)$, where $\Delta$ (and therefore $\Gamma$) is simplicial. Two relative simplicial complexes $(\Delta, \Gamma)$ and $(\Delta', \Gamma')$ are called isomorphic, $(\Delta, \Gamma) \cong (\Delta', \Gamma')$, if there exists a poset isomorphism $(\Delta \setminus \Gamma, \subseteq) \xrightarrow{\cong} (\Delta' \setminus \Gamma', \subseteq)$.

The $f$-vectors of (relative) simplicial complexes have been a topic of intensive study [8]. While unintuitive at first, rewriting the $f$-vector into the $h$-vector has been a fruitful approach. Recall from the introduction,
that the \( h\)-vector of a simplicial \( (d-1) \)-complex \( \Delta \) is a linear transformation of its \( f\)-vector, which can be compactly expressed as an equation of polynomials:

\[
\sum_{i=0}^{d} f_{i-1}(\Delta) \cdot (t-1)^{d-i} =: \sum_{k=0}^{d} h_{k}(\Delta) \cdot t^{d-k} =: h(\Delta, t).
\]

The right hand side is typically called the \( h\)-polynomial of \( \Delta \). As we want to think of \( \Psi = (\Delta, \Gamma) \) as \( \Delta \) with the faces in \( \Gamma \) removed, we define the \( f\) - and \( h\)-vectors of a relative simplicial \( d\)-complex \( \Psi \) with \( \dim \Delta = \dim \Gamma = d - 1 \) as follows:

\[
f_i(\Psi) := f_i(\Delta) - f_i(\Gamma), \quad h_i(\Psi) := h_i(\Delta) - h_i(\Gamma), \quad h(\Psi, t) := h(\Delta, t) - h(\Gamma, t).
\]

In the same spirit, the \( \hat{\gamma}\)-vector \( \hat{\gamma}(\alpha, P) = (\hat{\gamma}_0(P), \ldots, \hat{\gamma}_d(P)) \) of a simplicial \( d\)-polytope with respect to a cone angle \( \alpha \) is defined by the equality of the following polynomials:

\[
\sum_{k=0}^{d} \hat{\gamma}_k(P) \cdot t^{d-k} := \sum_{i=0}^{d} \hat{\alpha}_{i-1}(P) \cdot (t-1)^{d-i}.
\]

We have seen that there is a connection between the \( f\)-vectors of the polytopal complexes \( D(R, P) \) for \( R \in \mathcal{R}(P) \) and the \( \alpha\)-vector of a polytope \( P \). In the case that \( P \) is simplicial, \( D(R, P) \) is a relative simplicial complex and this connection can be transferred further from the \( h\)-vectors of \( D(R, P) \) to the \( \hat{\gamma}\)-vector as defined in the introduction (see Section 1). More precisely:

\[
\sum_{k=0}^{d} \hat{\gamma}_k(P) \cdot t^{d-k} := \sum_{i=0}^{d} \hat{\alpha}_{i-1}(P) \cdot (t-1)^{d-i} = \sum_{i=0}^{d} \sum_{R \in \mathcal{R}(P)} f_{i-1}(D(R, P)) \cdot \alpha(R) \cdot (t-1)^{d-i}
\]

\[
= \sum_{R \in \mathcal{R}(P)} \alpha(R) \sum_{i=0}^{d} f_{i-1}(D(R, P)) \cdot (t-1)^{d-i} = \sum_{R \in \mathcal{R}(P)} \alpha(R) \sum_{k=0}^{d} h_k(D(R, P)) t^{d-k},
\]

or by comparing coefficients:

\[
\hat{\gamma}_k(P) = \sum_{R \in \mathcal{R}(P)} \alpha(R) \cdot h_k(D(R, P)).
\]

This insight allows us to rephrase (in-)equalities on the \( h\)-vector of the \( D(R, P) \)'s into (in-)equalities on the \( \hat{\gamma}\)-vector. Our first application is a generalization of Theorem 2.7. For its formulation we need to introduce the \( g\)-vector \( g(\Delta) = (g_0(\Delta), g_1(\Delta), \ldots, g_d(\Delta)) \) of a simplicial \( (d-2) \)-complex \( \Delta \). It is defined via:

\[
g(\Delta, t) := \sum_{k=0}^{d} g_k(\Delta) \cdot t^{d-k} := (t-1)h(\Delta, t).
\]

It immediately follows that \( g_0(\Delta) = h_0(\Delta) \) and \( g_k(\Delta) = h_k(\Delta) - h_{k-1}(\Delta) \) for \( 1 \leq k \leq d \). For the boundary complex of a simplicial \( (d-1) \)-polytope by the Dehn-Sommerville relations one has \( g_i(\Delta) = h_i(\Delta) - h_{i-1}(\Delta) = h_{d-i}(\Delta) - h_{d-i+1}(\Delta) = -g_{d-i}(\Delta) \) for \( 0 \leq i \leq \lceil \frac{d}{2} \rceil \) and for this reason typically the \( g\)-vector is defined to be only the first half of what we wrote above. We use the extended version, which makes it a bit easier to state our next proposition:

**Proposition 3.1.** Let \( \alpha \) be any cone angle, \( P \) be a simplicial \( d\)-polytope and \( \hat{\gamma}(P) = \hat{\gamma}(\alpha, P) \). Then

\[
\hat{\gamma}(P) + \hat{\gamma}(-P) = h(\partial P) - \sum_{R \in \mathcal{R}(P)} \alpha(R) \cdot g(\pi(R, P)).
\]
Proof. As in the proof of Theorem 2.7, we consider for each region $R \in \mathcal{R}(P)$ the decomposition of the boundary of $P$, $\partial P = D(R,P) \cup \pi(R,P) \cup B(R,P)$. We conclude:

$$h(D(R,P),t) + h(B(R,P),t) = \sum_{i=0}^{d} (f_{i-1}(D(R,P)) + f_{i-1}(B(R,P))) (t-1)^{d-i}$$

$$= \sum_{i=0}^{d} (f_{i-1}(\partial P) - f_{i-1}(\pi(R,P))) (t-1)^{d-i}$$

$$= h(\partial P,t) - (t-1)h(\pi(R,P),t) = h(\partial P,t) - g(\pi(R,P),t).$$

We highlight that we needed to use the $g$-vector here instead of the $h$-vector, since $\pi(R,P)$ is a $(d-2)$-complex, while $\partial P$, $D(R,P)$ and $B(R,P)$ are $(d-1)$-complexes. Noting that $B(R,P) = D(-R,P) \cong D(R,-P)$ and summing over all regions $R$ gives the statement:

$$\gamma(P) + \gamma(-P) = \sum_{R \in \mathcal{R}(P)} \alpha(R)(h(D(R,P),t) + h(D(R,-P),t)) = h(\partial P) - \sum_{R \in \mathcal{R}(P)} \alpha(R)g(\pi(R,P)).$$

The remainder of this section is devoted to the first half of Theorem 1.3. We will give an illustrative proof using shellings, but the statement is also a byproduct of our more thorough investigation of the algebraic combinatorics of these complexes in the next section. We will provide the basic definitions and facts, for proofs, we refer to [21, Chapter 8].

A shelling of a simplicial $(d-1)$-complex $\Delta$ is an ordering of the facets $F_1, \ldots, F_r$ of $\Delta$, such that for all $1 \leq k \leq r$,

$$\Theta^k := \{G \cap F_k \in \partial F_k : G \subseteq F_i \text{ for some } i = 1, \ldots, k-1\}$$

is a pure simplicial $(d-2)$-complex. If $\Delta$ has a shelling it is called **shellable**. For a shellable simplicial complex $\Delta$, let

$$\Delta^k := \{G \in \Delta : G \subseteq F_i \text{ for some } i = 1, \ldots, k\}.$$  

An important property of shellings is their relation to the $h$-vector. This can be most concretely seen when considering a single shelling step from $\Delta^{k-1}$ to $\Delta^k$: Then $h(\Delta^{k-1})$ and $h(\Delta^k)$ are almost identical, except that the $(d-i)$-th entry increased by one, where $i := f_{d-2}(\Theta^k)$ is the number of facets of $\Theta^k$.

A relative simplicial complex $\Psi = (\Delta, \Gamma)$ with pure $\Delta$ and $\Gamma$ and $\dim \Delta = \dim \Gamma$ is called **jointly shellable**, if there is a shelling $F_1, \ldots, F_r$ of the facets of $\Delta$, such that the facets $F_1, \ldots, F_s$ of $\Gamma$ are shelled first. Joint shellings are a special case of (relative) shellings, see [19, Chapter III.7] and [1], but are sufficient in our case.

Since the $h$-vector of a shellable simplicial complex increases in each shelling step, we can conclude that jointly shellable simplicial complex $\Psi = (\Delta, \Gamma)$ have

$$h(\Psi) = h(\Delta) - h(\Gamma) \geq 0.$$  

An important class of shellable simplicial complexes are the boundary complexes $\partial P$ of a simplicial polytopes $P$. In fact, there is an easy way to find a shelling of $\partial P$ via line shellings: Suppose that $0 \in \text{int} P$ and consider a “generic” line $\ell$ through $0$. We begin at one of the intersection points of $\ell$ with $\partial P$ and start to move on $\ell$ away from $P$. At each point on $\ell$, we consider all the visible facets of $P$. Only every so often a new facet will be show up, which was previously hidden, and by taking $\ell$ generically enough this will happen one facet at a time. Let us write down these facets in the order they will occur: $F_1, \ldots, F_s$. If we traveled $\ell$ long enough to reach “infinity”, we will start our process again from the other side of $\ell$ back to $P$. In this phase we will make a note whenever a facets disappears and is not visible anymore: $F_{s+1}, \ldots, F_r$. Then the whole sequence $F_1, \ldots, F_s, F_{s+1}, \ldots, F_r$ is a shelling of $P$, called a **line shelling** of $P$. If we start our tour on $\ell$ inside the cone $R$ of $\mathcal{R}(P)$, we can see that $(\partial P)_s = \cup_{i=1}^s F_i$ is precisely $B(R,P)$ and we have as a corollary:

**Corollary 3.2.** $D(R,P) = (\partial P, B(R,P))$ is a jointly shellable simplicial complex.
The following lemma is important for proving the Dehn-Sommerville relations of the \(\hat{\gamma}\)-vector. The statement is known as Dehn-Sommerville relations for simplicial manifolds with boundary, see [7, 15] for a more general treatment of Dehn-Sommerville relations for simplicial manifolds with boundary.

**Lemma 3.3.** \(h_i(D(R, P)) = h_{d-i}(\overline{D}(R, P))\) and \(h_i(B(R, P)) = h_{d-i}(\overline{B}(R, P))\) for all \(0 \leq i \leq d\).

**Proof.** Since \(B(R, P) = D(-R, P)\), we only need to show one of the equations. Consider a line shelling of \(P\) giving a joint shelling \(F_1, \ldots, F_s, \ldots, F_r\) of \(\partial P\), which shells the facets \(F_1, \ldots, F_s\) of \(\Gamma := B(R, P) = \Delta^s\) first. The reverse shelling \(F_r, \ldots, F_s, \ldots, F_1\) is a shelling too, see [21, Lemma 8.10], so we define

\[
\tilde{\Theta}^k := \{ G \cap F_k \in \partial F_k : G \subsetneq F_i \text{ for some } i = k + 1, \ldots, d \}
\]

and the contribution of \(F_k\) to the \(h\)-vector is at the entry \(d - f_{d-2}(\Theta^k)\) in the forward and at the entry \(f_{d-2}(\tilde{\Theta}^k)\) in the reverse shelling. Considering the contributions of \(F_k\) for \(k > s\) both ways we obtain the first equality in:

\[
h_i(\Psi) = h_i(\Delta) - h_i(\Delta^s) = h_{d-i}(\tilde{\Theta}^s) = h_{d-i}(D(R, P)).
\]

With this, we are able to prove the first half of Theorem 1.3:

**Proposition 3.4.** Let \(P\) be a simplicial polytope and let \(\hat{\gamma}(P) = \hat{\gamma}(\alpha, P)\), where \(\alpha\) is a non-negative cone angle. The \(\hat{\gamma}\)-vector \(\hat{\gamma}(P)\) satisfies

1. Dehn-Sommerville: \(\hat{\gamma}_i(P) + \hat{\gamma}_{d-i}(-P) = h_i(P)\),
2. Non-negativity: \(\hat{\gamma}_i(P) \geq 0\).

**Proof.** As outlined, we prove the properties by considering \(h_i(D(R, P))\). The non-negativity follows readily from (5) and the fact that both \(\alpha(R) \geq 0\) and \(\Psi := D(R, P)\) satisfies (6).

For the Dehn-Sommerville relations, we note that \(D(R, -P) = B(-R, -P) \cong B(R, P)\) by reflection at the origin, and

\[
\hat{\gamma}_i(P) + \hat{\gamma}_{d-i}(-P) = \sum_{R \in R(P)} \alpha(R) \cdot (h_i(D(R, P)) + h_{d-i}(D(R, -P)))
\]

\[
= \sum_{R \in R(P)} \alpha(R) \cdot (h_i(D(R, P)) + h_{d-i}(B(R, P)))
\]

\[
= \sum_{R \in R(P)} \alpha(R) \cdot (h_i(D(R, P)) + h_i(B(R, P))) = h_i(P). \quad \square
\]

Remarkably, we can deduce that for \(\alpha\)-symmetric polytopes the middle entry of the \(\hat{\gamma}\)-vector is combinatorial in even dimensions.

**Corollary 3.5.** Let \(P\) be a simplicial and \(\alpha\)-symmetric 2m-polytope. Then \(2\hat{\gamma}_m(P) = h_m(P)\).

Again, we close with an example, in fact with the same example as before. It shows, that the minus sign in our version of the Dehn-Sommerville equations is necessary:

**Example 3.6.** Let \(P\) and \(\alpha\) be as in Example 2.8. We have

\[
h(D(T_x P, P)) = (0, 1, 1) \quad \text{and} \quad h(D(-T_x P, P)) = (0, 0, 1)
\]

and therefore \(\hat{\gamma}(P) = (0, \frac{3}{4}, 1)\). But since \(h(P) = (1, 1, 1)\), the Dehn-Sommerville relations do not hold when neglecting the minus sign: \(2\hat{\gamma}_1(P) \neq h_1(P)\). On the other hand, \(\hat{\gamma}(-P) = (0, \frac{1}{4}, 1)\) and we see, that \(\hat{\gamma}_1(P) + \hat{\gamma}_{2-i}(-P) = 1 = h_i\) as required.
4. Relative Stanley-Reisner-Theory

In this section we will review some of the concepts of relative Stanley-Reisner-Theory and use it to prove inequalities on the $h$-vectors of $D(R, P)$. These inequalities are already known, still, for completeness we give proofs for them. We refer to [1] and [19, Chapter III.7] for details and further applications.

Let $\Delta$ be a simplicial complex of dimension $d$ on $n$ vertices and identify the vertices with $[n] := \{1, 2, \ldots, n\}$. For each face $F \in \Delta$, let $V(F)$ be the subset of $[n]$, which corresponds to the vertices of $F$. For an infinite field $\mathbb{k}$, let $S := \mathbb{k}[x_1, \ldots, x_n]$ and let $N(\Delta) := 2^n \setminus \{V(F) : F \in \Delta\}$ be the set of non-faces of $\Delta$. We define the Stanley-Reisner-Ideal $I_\Delta \subseteq S$ of $\Delta$ as follows:

$$I_\Delta := \langle x^I : I \in N(\Delta) \rangle,$$

where $x^I := \prod_{i \in I} x_i$, and the Stanley-Reisner-Ring $k[\Delta] := S/I_\Delta$. A simplicial complex $\Delta$ is said to be a Cohen-Macaulay complex (over $\mathbb{k}$), if $k[\Delta]$ is Cohen-Macaulay. For a relative simplicial complex $\Psi = (\Delta, \Gamma)$, we similarly define the Stanley-Reisner-Module

$$M[\Psi] := I_\Gamma/I_\Delta.$$

We can also think of $M[\Psi]$ as the kernel of the surjection $k[\Delta] \rightarrow k[\Gamma]$, i.e. there exists a short exact sequence of graded $S$-modules:

$$0 \longrightarrow M[\Psi] \longrightarrow k[\Delta] \longrightarrow k[\Gamma] \longrightarrow 0.$$  \hspace{1cm} (7)

Assume that both $\Delta$ and $\Gamma$ are $d$-dimensional Cohen-Macaulay complexes. Since we assumed $\mathbb{k}$ to be infinite, by the Kind-Kleinschmidt criterion [11] there exists a regular sequence $(\theta_1, \ldots, \theta_d)$ of $k[\Delta]$, which is also a regular sequence of $k[\Gamma]$. We can recover the $h$-vector of the $\Delta$ and $\Gamma$ by taking the quotient with $\Theta := \theta_1 S + \cdots + \theta_d S$:

$$\dim(k[\Delta]/\Theta)_i = h_i(\Delta), \quad \dim(k[\Gamma]/\Theta)_i = h_i(\Gamma).$$

Then, from Lemma 1.1.4 in [5] we also have an exact sequence of graded $S$-modules when we quotient (7) by $\Theta$:

$$0 \longrightarrow M[\Psi]/\Theta \longrightarrow k[\Delta]/\Theta \longrightarrow k[\Gamma]/\Theta \longrightarrow 0.$$  

We immediately see that $\dim(M[\Psi]/\Theta)_i = h_i(\Delta) - h_i(\Gamma) = h_i(\Psi) \geq 0$. A linear form $\omega \in k[\Delta]_1$, is called a Lefschetz element for $k[\Delta]$, if $(k[\Delta]/\Theta)_i \xrightarrow{\omega} (k[\Delta]/\Theta)_{d-i}$ is an isomorphism. These linear forms played an important role in Stanley’s proof of the $g$-theorem [18], see also [9]. There, the following is established:

**Theorem 4.1.** Let $P$ be a simplicial $d$-polytope and let $(\theta_1, \ldots, \theta_d)$ be a regular sequence of $k[\partial P]$. Then $k[\partial P]/\Theta$ has a Lefschetz element.

We will be only concerned with $\Delta = \partial P$ for some simplicial polytope $P$, so assume that there is in fact a Lefschetz element $\omega$ of $k[\Delta]/\Theta$ and let $i < \frac{d}{2}$. Then the following diagram commutes:

$$\begin{array}{ccc}
(M[\Psi]/\Theta)_i & \longrightarrow & (k[\Delta]/\Theta)_i \\
\downarrow \omega & & \downarrow \omega \\
(M[\Psi]/\Theta)_{i+1} & \longrightarrow & (k[\Delta]/\Theta)_{i+1}
\end{array}$$

and since all maps except for the left downwards arrow are injective, it has to be injective as well. By looking at the dimensions we see

$$h_i(\Psi) = \dim(M[\Psi]/\Theta)_i \leq \dim(M[\Psi]/\Theta)_{i+1} = h_{i+1}(\Psi).$$
If we instead consider the isomorphism $\omega^{d-2i}$, we get the commutative diagram

$$(M[\Psi]/\Theta)_i \xrightarrow{\cdot \omega^{d-2i}} (k[\Delta]/\Theta)_i \xrightarrow{\cdot \omega^{d-2i}} (M[\Psi]/\Theta)_{d-i} \xrightarrow{\cdot \omega^{d-2i}} (k[\Delta]/\Theta)_{d-i}$$

and the same argument shows that the left downwards arrow is again injective. Comparing the dimension gives us the second inequality:

$$h_i(\Psi) = \dim(M[\Psi]/\Theta)_i \leq \dim(M[\Psi]/\Theta)_{d-i} = h_{d-i}(\Psi).$$

We have therefore shown:

**Proposition 4.2.** Let $\Psi = (\Delta, \Gamma)$ be a relative simplicial complex such that both $\Delta$ and $\Gamma$ are $d$-dimensional Cohen-Macaulay complexes and such that $\Delta$ has a Lefschetz element. Then $0 \leq h_{i-1}(\Psi) \leq h_i(\Psi)$ and $h_i(\Psi) \leq h_{d-i}(\Psi)$ for all $0 \leq i \leq m := \lceil \frac{d}{2} \rceil$.

As a corollary, we can prove the second half of Theorem 1.3:

**Corollary 4.3.** Let $P$ be a simplicial polytope and $\hat{\gamma}(P) = \hat{\gamma}(\alpha, P)$, where $\alpha$ is a non-negative cone angle.

1. $\hat{\gamma}(P)$ increases in the first half: $\hat{\gamma}_0(P) \leq \hat{\gamma}_1(P) \leq \cdots \leq \hat{\gamma}_m(P)$, where $m := \lceil \frac{d}{2} \rceil$, and
2. $\hat{\gamma}(P)$ is flawless: $\hat{\gamma}_i(P) \leq \hat{\gamma}_{d-i}(P)$ for all $0 \leq i \leq \lceil \frac{d}{2} \rceil$.

**Proof.** As we have seen, $D(R, P)$ is jointly shellable for each $R \in \mathcal{R}(P)$. Therefore, both $\Delta = \partial P$ and $\Gamma = \partial P$ are Cohen-Macaulay complexes. By Theorem 4.1, $\partial P$ has a Lefschetz element. Together with Proposition 4.2 the statement is now an immediate consequence of (5).

5. **The $\hat{\gamma}$-vector of a Simplex and Unimodality**

Our inequalities give us a somewhat complete picture of the $\hat{\gamma}$-vector of a $d$-polytope $P$ in its first half. But besides being flawless, we cannot say much about what happens in the second half. In this section, we will look at the question of unimodality. A sequence $(a_0, a_1, \ldots, a_n)$ is called **unimodal**, if there exists $0 \leq p \leq n$ such that $a_0 \leq a_1 \leq \cdots \leq a_p \geq \cdots \geq a_n$. We show that we can only hope for unimodality of the $\hat{\gamma}$-vector in small dimensions. Unless otherwise stated, let $P$ be an simplicial polytope and $\hat{\gamma}(P) = \hat{\gamma}(\alpha, P)$ for some non-negative cone angle $\alpha$.

From our previous results, we have the following observation:

**Proposition 5.1.** Let $P$ be a simplicial $d$-polytope, $d \leq 3$. Then $\hat{\gamma}(P)$ is unimodal.

**Proof.** By Theorem 1.3 (3), the $\hat{\gamma}$-vector of $P$ is non-decreasing until $\hat{\gamma}_{d-1}$.

Nonetheless, we can also see that unimodality fails for simplicial polytopes in dimension 4.

**Example 5.2.** There exist 4-dimensional simplicial polytopes with non-unimodal $\hat{\gamma}$-vector, even if $\alpha$ is non-negative. For this take the cross-polytope $\Diamond_4$ and transform it projectively such that a orthogonal projection along some $u \in \mathbb{R}^4$ has a single simplex as image. Let $R \in \mathcal{R}(P)$ be the region containing $u$. By appealing to Lemma 2.9 we set $\alpha(R) = \frac{1}{6}$ and $\alpha(-R) = \frac{5}{6}$ and get

$$\hat{\gamma}(\Diamond_4) = \frac{5}{6}(0, 0, 0, 0, 1) + \frac{1}{6}(0, 4, 6, 4, 1) = (0, \frac{2}{3}, 1, \frac{2}{3}, 1).$$

More can be said, if $P$ is a simplex. We want to denote by $\Delta_d$ any $d$-simplex in $\mathbb{R}^d$. 
Lemma 5.3. $\hat{\gamma}(\triangle_d)$ is non-decreasing.

Proof. Let $\Psi_k = (\triangle_d, \Gamma_k)$, where $\Gamma$ is formed by any subset of $k$ facets of $\Psi$, $1 \leq k \leq d$. Since any ordering of the facets is a shelling, we see that

$$h(\Gamma_k) = (1,1,\ldots,1,0,0,\ldots,0)$$

and therefore

$$h(\Psi_k) = (0,0,\ldots,0,1,1,\ldots,1).$$

For every region $R \in \mathcal{R}(\triangle_d)$, we have $D(R, P) \cong \Psi_k$, where $k$ is the number of facets of $\hat{B}(R, K) \cong \Gamma_k$. Since the $h$-vector of $D(R, P)$ is non-decreasing for any $R$, the $\hat{\gamma}$-vector is too, by (5).

Interestingly, the converse is also almost true for $\alpha$-symmetric polytopes. A bipyramid over a simplex, or simply a bipyramid, is any polytope combinatorially isomorphic to the convex hull of $\triangle_{d-1}$ together with two points $v, v'$ on different sides of the hyperplane $\text{aff}(\triangle_{d-1})$ such that $\text{conv}(v, v') \cap \triangle_{d-1} \neq \emptyset$.

Proposition 5.4. Let $\alpha$ be a non-negative cone angle and let $P$ be a simplicial and $\alpha$-symmetric $d$-polytope, $d \geq 2$. Let $\hat{\gamma}(P) := \hat{\gamma}(\alpha, P)$ be non-decreasing. Then $P$ is either a simplex or a bipyramid. In the latter case, $\hat{\gamma}(P) = (0, 1, \ldots, 1)$ and $\alpha(R) = 0$ for all regions $R \in \mathcal{R}(P)$ with $\pi(R, P) \not\sim \partial \triangle_{d-1}$.

Proof. Since $\hat{\gamma}(P)$ is non-decreasing and $\hat{\gamma}_d(P) = 1$, we have by the Dehn-Sommerville-relations, Proposition 3.4, for all $k \leq \frac{d}{2}$,

$$h_k(\partial P) = \hat{\gamma}_k(P) + \hat{\gamma}_{d-k}(-P) = \hat{\gamma}_k(P) + \hat{\gamma}_{d-k}(P) \leq 2\hat{\gamma}_d(P) = 2.$$

If $h_k(P) = 1$ for $k \geq 1$, then $1 \leq h_1(\partial P) \leq h_k(\partial P) = 1$ and $P$ is a simplex. Otherwise, the only possible $h$-vector of $\partial P$ is $h(\partial P) = (1, 2, 2, \ldots, 2, 2, 1)$, and from, for example, the lower bound theorem [3], we see that $P$ is a bipyramid over a simplex.

We are left to examine what happens in this second case. For $1 \leq k \leq \frac{d}{2}$, we see

$$2 = h_k(P) = \hat{\gamma}_k(P) + \hat{\gamma}_{d-k}(P) \leq 2\hat{\gamma}_{d-k}(P) \leq 2\hat{\gamma}_d(P) = 2,$$

so $\hat{\gamma}_k(P) = 2 - \hat{\gamma}_{d-k}(P) = 1$ and $\hat{\gamma}(P) = (0, 1, 1, \ldots, 1, 1)$. With Proposition 3.1, we have

$$0 = h_k(\partial P) - 2\hat{\gamma}_k(P) = \sum_{R \in \mathcal{R}(P)} \alpha(R) g_k(\pi(R, P)).$$

For any region $R \in \mathcal{R}(P)$, $g_k(\pi(R, P)) \geq 0$ by the characterization of the $h$-vectors of simplicial polytopes, Theorem 1.1(3), since $\pi(R, P)$ is isomorphic to the boundary complex of a projection of $P$. It follows that $g(\pi(R, P)) = (1, 0, 0, \ldots, 0, 0, -1)$ and that $\pi(R, P)$ is the boundary of a simplex for all regions $R$ such that $\alpha(R) > 0$.

Since the standard angle is positive for all full-dimensional cones $C$, we have the following:

Corollary 5.5. Let $P$ be a simplicial $d$-polytope and let $\hat{\gamma}(P) := \hat{\gamma}(\nu, P)$. If $P$ has a non-decreasing $\hat{\gamma}$-vector, then $P$ is a simplex.

In the remainder we want to examine the unimodality of $\alpha$-symmetric polytopes. First we need the following lemma:

Lemma 5.6. Either a simplicial and $\alpha$-symmetric $d$-polytope $P$ is a simplex or $\hat{\gamma}_{d-1}(P) \geq \hat{\gamma}_d(P) = 1$.

Proof. Suppose that $P$ is not a simplex, then, $h_1(P) \geq 2$ and

$$2\hat{\gamma}_{d-1}(P) \geq \hat{\gamma}_1(-P) + \hat{\gamma}_{d-1}(P) = \hat{\gamma}_1(P) + \hat{\gamma}_{d-1}(P) = h_1(P) \geq 2.$$
Proposition 5.7. All simplicial and \(\alpha\)-symmetric \(d\)-polytopes, \(d \leq 5\), have an unimodal \(\hat{\gamma}\)-vector.

Proof. If \(P\) is a simplex, then we are done by Lemma 5.3. Otherwise, for \(d = 4, 5\), we know that \(\hat{\gamma}(P)\) is increasing until \(\lceil \frac{d}{2} \rceil = d - 2\) and that \(\hat{\gamma}_{d-1}(P) \geq \hat{\gamma}_d(P) = 1\). Thus either \(\hat{\gamma}_{d-2}(P) \geq \hat{\gamma}_{d-1}(P)\) or \(\hat{\gamma}_{d-2}(P) < \hat{\gamma}_{d-1}(P)\), but both possibilities give an unimodal \(\hat{\gamma}\)-vector. \(\square\)

Giving an counter example of an \(\alpha\)-symmetric simplicial polytope with non-unimodal \(\hat{\gamma}\)-vector is not as easy as in the non-\(\alpha\)-symmetric case. The following example was found by brute-force using a SAGE [20] script to generate random polytopes and projections:

Example 5.8. Let \(P = \text{conv}(V) \subseteq \mathbb{R}^6\) for \(V\) being given as the following nine vertices.

\[
\begin{pmatrix}
-48 & -8 & 16 & -10 & 6 & -12 \\
-12 & -6 & -8 & 4 & 2 & 2 \\
36 & 0 & -8 & 1 & -1 & -7 \\
-23 & -2 & -4 & 6 & 2 & 8 \\
3 & 0 & 2 & -2 & 3 & 1 \\
50 & -9 & 4 & -10 & 6 & -4 \\
-20 & -8 & 2 & -8 & 9 & 5 \\
22 & -12 & 2 & -9 & 6 & -3 \\
56 & 6 & 4 & -2 & 0 & -2
\end{pmatrix}
\]

Projecting \(P\) orthogonally along the standard basis vector \(e_1\) gives \(Q \subseteq \mathbb{R}^5\). Then \(\partial Q \cong \pi(R, P)\) for \(R \in \mathcal{R}(P)\) with \(e_1 \in \text{int } R\). Set \(\alpha := \frac{1}{2}(\omega_{e_1} + \omega_{-e_1})\) and \(\hat{\gamma}(P) = \hat{\gamma}(\alpha, P)\). Note that \(\alpha\) is even, thus \(P\) is \(\alpha\)-symmetric. SAGE tells us that

\[
\begin{align*}
\hat{h}(\partial P) &= (1, 3, 4, 5, 4, 3, 1), \\
\hat{g}(\partial Q) &= (1, 3, 0, 0, 0, -3, -1)
\end{align*}
\]

and therefore

\[
2\hat{\gamma}(P) = \hat{h}(\partial P) - (\alpha(R) + \alpha(-R))\hat{g}(\pi(R, P))
\]

\[
= (1, 3, 4, 5, 4, 3, 1) - (1, 3, 0, 0, 0, -3, -1) = (0, 0, 4, 5, 4, 6, 2),
\]

which is not unimodal. If we flatten \(P\) in direction \(e_1\) with the following linear transformation for small enough \(\varepsilon > 0\)

\[
C_{\varepsilon} := \begin{pmatrix}
\varepsilon & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]

we see that all angles \(\nu(F, P)\) for \(F \in D(R, P)\) or \(F \in B(R, P)\) tend to \(\frac{1}{2}\), while those for \(F \in \pi(R, P)\) tend to 0. Thus we have

\[
\lim_{\varepsilon \to 0} \hat{\gamma}(\nu, C_{\varepsilon}P) = (0, 0, 4, 5, 4, 6, 2),
\]

so there are non-unimodal \(\hat{\gamma}\)-vectors even for the standard cone angle.

While it is plausible to be false, it seems hard to find a counterexample of a simplicial \(d\)-polytope with non-unimodal \(\hat{\gamma}\)-vector for an even cone angle and \(d \geq 7\).

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