Optimal LP Rounding and Fast Combinatorial Algorithms for Clustering Edge-Colored Hypergraphs

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Abstract
We study the approximability of a recently introduced framework for clustering edge-colored hypergraphs, where goal is to color nodes in a way that minimizes the number of hyperedges containing a node with a different color than the hyperedge. This problem is closely related to chromatic correlation clustering and various generalized multiway cut problems. We first of all provide a min\left\{2 - \frac{2}{k}, \frac{2}{r + 1}\right\}-approximation by rounding a natural linear programming relaxation, where \(r\) is the maximum hyperedge size and \(k\) is the number of colors. This improves on the best previous rounding scheme that achieves an approximation of min\left\{2 - \frac{1}{k}, \frac{2}{r + 1}\right\}. We show our rounding scheme is optimal by proving a matching integrality gap. When \(r\) is large, our approximation matches a known 2(1 - \frac{1}{k})-approximation based on reducing to node-weighted multiway cut and rounding a different linear program. The exact relationship between the two linear programs was not previously known; we prove that the canonical relaxation is always at least as tight as the node-weighted multiway cut relaxation, and can be strictly tighter. We also show that when \(r\) and \(k\) are arbitrary, the edge-colored clustering objective is approximation equivalent to vertex cover. Explicitly reducing to a graph and applying existing vertex cover algorithms leads to runtimes that are worse than linear in terms of the hypergraph size. We improve on this by designing a linear-time 2-approximation algorithm that implicitly approximates the underlying vertex cover problem without explicitly iterating through all edges in the reduced graph.

1 Introduction
Graph clustering and partitioning are fundamental algorithmic problems that are broadly applied in scientific computing, machine learning, and data mining, and have also been studied extensively in theoretical computer science from the perspective of approximation algorithms and hardness results. Recently there has been a growing interest in clustering and partitioning problems defined over data structures that generalize graphs and capture rich information and metadata beyond just pairwise relationships. One direction has been to develop algorithms for clustering and partitioning hypergraphs [LM17, LM18, CC21, CC22, CLTZ18, Lou15, BCW22], which can model multiway relationships between data objects rather than just pairwise connections. Another recent trend has been to develop techniques for clustering edge-colored graphs [AVB20, ABK+16, BGG+15, AAEG15, KSZ+21], where edge colors represent the type or category of pairwise interaction modeled by the edge.

With both of these motivating directions in mind, we study the approximability of a recent framework for clustering edge-colored hypergraphs [AVB20]. Given a hypergraph...
Figure 1: An instance of Color-EC is given by an edge-colored hypergraph. The goal is to color nodes in a way that minimizes the number (or weight) of unsatisfied edges. In this example, the optimal node-coloring leaves two edges unsatisfied (dashed lines).

\[ H = (V, E) \text{ where each hyperedge } e \in E \text{ is associated with one of } k \text{ colors } [k] = \{1, 2, \ldots, k\}, \]\n
the goal of this framework is to assign colors to nodes in a way that minimizes hyperedge mistakes (see Figure 1). If all of the nodes in a hyperedge \( e \in E \) are given the same color as \( e \), then the hyperedge is satisfied, otherwise it is unsatisfied and we say the node color assignment has made a mistake at this hyperedge. We refer to this problem as Colored-Edge Clustering, or simply Color-EC.\(^1\) This problem is NP-hard in general but permits nontrivial approximation algorithms, and shares interesting connections to several other cut and clustering problems that have been extensively studied from the perspective of approximation algorithms and hardness results, including chromatic correlation clustering \([BGG^+15, AAEG15, KSZ^+21]\), generalized multiway cut problems \([GVY04, CE11b]\), and other related edge-colored clustering problems \([AK14, AK20, AK19, ABK^+16]\).

Amburg, Veldt, and Benson \([AVB20]\) proved that Color-EC is polynomial time solvable in the case of two colors \((k = 2)\), but showed it is NP-hard when \( k \geq 3 \). They showed how to round a natural linear programming (LP) relaxation of the objective to produce a \( \min\{2 - 1/k; 2 - 1/(r + 1)\}\)-approximation algorithm, where \( r \) is the rank of the hypergraph, i.e., the maximum hyperedge size. They also provided combinatorial approximation algorithms whose approximation factors scale linearly in terms of \( r \). Finally, they showed that the problem can be reduced in an approximation preserving way to an instance of the node-weighted multiway cut problem (NODE-MC) \([GVY04]\), which implies a \( 2(1 - 1/k) \) approximation by rounding the NODE-MC linear programming relaxation. In numerical experiments, the Color-EC objective was shown to perform well in synthetic edge-labeled community detection experiments, and was also applied to downstream clustering tasks such as detecting temporal communities in an online social media platform and identifying common food ingredients associated with different world cuisines \([AVB20]\). In follow-up work, an extension of this framework was designed for the task of diverse group discovery and team formation \([AVB22]\). The present paper focuses on improving the theoretical foundations for edge-colored hypergraph clustering, by developing new approximation algorithms and proving new hardness results for this problem, and by proving deeper connections to other well-studied combinatorial objectives.

\(^1\)Amburg, Veldt, and Benson \([AVB20]\) referred to this problem as Categorical Edge Clustering, but also used the term “color”, “category”, and “label” interchangeably. Here we will simply focus on the term “color” as this is general and intuitive, and also matches with independent work on the graph version of the objective \([ABK^+16]\).
1.1 Related Work

COLOR-EC is related to a number of other clustering, partitioning, and multiway cut objectives that have been analyzed from the perspective of approximation algorithms and hardness results. We outline these relationships before summarizing our motivation and contributions, as a number of our results pertain to these related objectives.

**Maximum \( k \)-edge-colored clustering.** Angel et al. [ABK+16] previously considered a maximization version of COLOR-EC in graphs, where the goal is to assign colors to nodes in a way that maximizes the weight of satisfied edges. At optimality, this is equivalent to applying the COLOR-EC when \( r = 2 \), though is different in terms of approximations. These authors showed the objective can be solved in polynomial time in the case of \( k = 2 \) but is NP-hard in the case of \( k \geq 3 \), though they focused only on the graph setting. They provided a \( \frac{1}{2} \)-approximation algorithm by rounding an LP relaxation for this maximization problem. In a sequence of follow-up papers [AK14, AK20, AK19], the best approximation factor was improved to \( \frac{4225}{11664} \approx 0.3622 \) [AK20]. The problem was also shown to be NP-hard to approximate above a factor \( \frac{241}{249} \approx 0.972 \) [AK19]. All of these results apply to the maximization version of the problem, and only apply to the graph setting.

**Chromatic correlation clustering.** The graph version of COLOR-EC is closely related to chromatic correlation clustering [BGG+15, AAEG15, KSZ+21], an edge-colored generalization of correlation clustering [BBC04]. Chromatic correlation clustering also takes an edge-colored graph as input and seeks to cluster nodes in a way that minimizes edge mistakes. Both of these objectives include penalties for (1) separating two nodes that share an edge, and (2) placing an edge of one color in a cluster of a different color. The key difference is that chromatic correlation clustering also includes a penalty for placing two non-adjacent nodes in the same cluster. In other words, chromatic correlation clustering interprets a non-edge as an indication that two nodes are dissimilar and should not be clustered together, whereas COLOR-EC treats non-edges simply as missing information and does not include this type of penalty. This results from a difference of interpretation: if two nodes do not share an edge, chromatic correlation clustering interprets this as an indication that the two nodes are dissimilar and should not be clustered together, whereas COLOR-EC essentially treats non-edges as missing information (e.g., the nodes could be related but an edge was simply not observed). As a result, a solution to chromatic correlation clustering may involve multiple different clusters of the same color, rather than one cluster per color. Unlike COLOR-EC, chromatic correlation clustering is known to be NP-hard even in the case of two colors. Various constant-factor approximation algorithms have been designed, culminating in a simple combinatorial 3-approximation [KSZ+21], but these apply only to the unweighted case. Another difference is that there are no results for chromatic correlation clustering in hypergraphs.

**Multiway cut and partition problems.** The reduction from COLOR-EC to a special case of NODE-MC situates COLOR-EC within a broad class of multiway cut and multiway partition problems [EVW13, GVY04, DJP+94, CKR00, CE11a, CE11b, CM16]. For NODE-MC [GVY04], one is given a node-weighted graph \( G = (V,E) \) with \( k \) special terminal nodes, and the goal is to remove a minimum weight set of nodes to disconnect all terminals. NODE-MC is approximation equivalent to the hypergraph multiway cut problem (HYPER-MC) [CE11b], where one is give a hypergraph \( H = (V,E) \) and \( k \) terminal nodes,
and the goal is to remove the minimum weight set of edges to separate terminals. These problems generalize the standard edge-weighted multiway cut problem in graphs (GRAPH-MC) [DJP+94], where the goal is to separate $k$ terminal nodes in a graph by removing a minimum weight set of edges. NODE-MC and HYPER-MC are also special cases of the more general submodular multiway partition problem (SUBMODULAR-MP), which has a best known approximation factor of $2(1 - 1/k)$ [EVW13]. This matches the best approximation guarantee for NODE-MC but requires solving and rounding a generalized Lovász relaxation [CE11b]. It is worth noting that when $k = 2$, COLOR-EC can be reduced to the minimum $s$-$t$ cut problem, i.e., the 2-terminal version of GRAPH-MC, but this does not generalize to $k > 2$. Amburg, Veldt, and Benson [AVB20] showed a way to approximate COLOR-EC by approximating a related instance of GRAPH-MC, but the objectives differ by a factor $(r + 1)/2$.

1.2 Motivating Questions and Contributions

We are motivated by several natural questions on the approximability of COLOR-EC and the relationship between this objective and other well-studied problems. Our first question concerns the best possible results obtainable via linear programming.

**Question 1.** What is the best approximation factor that can be obtained by rounding the canonical COLOR-EC LP relaxation?

The fact that a $2(1 - 1/k)$-approximation can be indirectly achieved by reducing to NODE-MC strongly suggests that it may be possible to improve the previous $(2 - 1/k)$-approximation when directly rounding the COLOR-EC LP relaxation. We would like to bridge this gap and better understand the relationship between COLOR-EC, NODE-MC, and their corresponding LP relaxations. More generally, we are interested in knowing the best approximation guarantee that can be achieved by rounding either relaxation. Improved approximations are particularly relevant for the case where $r$ is small, since in this case it may be possible to substantially improve upon approximations obtained by reducing to more general multiway cut objectives. Answering Question 1 additionally involves establishing integrality gap results for the COLOR-EC LP relaxation.

Linear programming relaxations often lead to good approximation factors but are computationally expensive even though polynomial-time solvable. This motivates another natural question.

**Question 2.** Can we obtain a combinatorial algorithm with a constant factor approximation that is independent of $r$ and $k$?

Amburg, Veldt, and Benson [AVB20] showed how to approximately reduce COLOR-EC to the standard multiway cut problem in graphs in a way that distorts approximation factors by $(r + 1)/2$. This leads to an $O(r)$ combinatorial approximation algorithm using the isolating cut heuristic for multiway cut [DJP+94]. They also showed that a very fast and simple method that places each node with its preferred color (i.e., the hyperedge color that a node participates in the most) provides an $r$-approximation for the unweighted case. Any combinatorial algorithm with an approximation factor that scales sublinearly in $r$ would be an improvement on these results.

Finally, although COLOR-EC was shown to be NP-hard, no refined hardness results for approximating the objective have been shown. This leads to our last motivating question.

**Question 3.** Can we prove refined hardness of approximation results for COLOR-EC?
Our contributions We answer all three of the previous motivating questions, and in the process prove new results on the relationship between COLOR-EC and other well-studied combinatorial problems. We first of all completely settle open Question 1, and we also clarify the relationship between the canonical LP relaxation and the relaxation obtained by first reducing to NODE-MC.

Result 1 (See Theorems 3.6, 4.1, 4.2). There is a rounding scheme for the canonical COLOR-EC LP relaxation that has an expected $2 \cdot \min\{1 - 1/k, 1 - 1/(r + 1)\}$ approximation factor. There exists an instance of the problem with a matching LP integrality gap.

Result 2 (See Theorem 5.1). The COLOR-EC LP relaxation is at least as tight as reducing to NODE-MC and using the NODE-MC LP relaxation, and in some cases is strictly tighter.

Fully proving Result 1 involves developing separate rounding schemes for different choices of $r$ and $k$. Proving a $\frac{1}{4}$-approximation for $r = 2$ is the most in-depth and challenging result in the paper. Result 2 confirms that although the previous best approximation factor for large $r$ was based on a reduction to NODE-MC, in general this strategy is not as strong as directly rounding the canonical LP relaxation. Although there are instances where both LP relaxations have an integrality gap of $2(1 - 1/k)$, the canonical relaxation is strictly tighter in some instances and also allows us to obtain much better approximations when $r$ is small.

In answer to Questions 2 and 3, we prove refined hardness results and design fast combinatorial approximation algorithms by proving that COLOR-EC is approximation equivalent to VERTEX COVER when $r$ and $k$ are assumed to be arbitrarily large. This follows from two reduction results.

Result 3 (See Theorem 6.1). An instance of VERTEX COVER on a graph with maximum degree $\Delta$ can be reduced in an approximation preserving way to an instance of COLOR-EC with maximum hyperedge size $\Delta$.

Result 4 (See Theorem 6.2). An instance of COLOR-EC with $m$ hyperedges can be reduced in an approximation preserving way to an instance of VERTEX COVER on a graph with $m$ nodes.

Result 3 implies that if $r$ and $k$ are arbitrary, it is NP-hard to approximate COLOR-EC to within a factor better than 1.3606 [DS05] and UGC-hard to approximate to within a factor that is asymptotically better than 2 [KR08]. The result also implies that if $k$ is arbitrary but $r$ is fixed, it is UGC-hard to approximate COLOR-EC to within a factor better than $2 - (2 + o_r(1))\frac{\ln \ln r}{\ln r}$ [AKS11]. By Result 4, we obtain fast combinatorial approximation algorithms for COLOR-EC by reducing to a graph and applying existing linear-time algorithms for VERTEX COVER. Explicitly constructing the instance of VERTEX COVER and applying existing methods leads to a 2-approximation for COLOR-EC that runs on a hypergraph $H = (V, E)$ in $O(\sum_{v \in V} d_v^2 + |E|^2)$ time, where $d_v$ is the degree of node $v$. Given that the reduced instance of VERTEX COVER can involve up to $O(|E|^2)$ edges, we cannot improve on this runtime bound if we rely on explicitly forming the VERTEX COVER instance. We overcome this issue by developing a more efficient way to implicitly implement a 2-approximation algorithm for VERTEX COVER in order to approximate COLOR-EC. Our approach avoids visiting all edges in the reduced instance of VERTEX COVER by taking advantage of certain structural properties that always hold for this reduced graph. The method runs in $O(\sum_{e \in E} |e|)$ time, i.e., linear in terms of the hypergraph size. This matches a natural runtime lower bound, as this is the time it takes to simply read the hypergraph input.
1.3 Overview of LP Rounding Techniques

Our LP-based approximation algorithms rely on combining and altering previous rounding schemes for the COLOR-EC LP and rounding schemes for relaxations of other multiway cut objectives such as the standard multiway cut problem (GRAPH-MC) [CKR00, DJP+94]. Despite differences between these problems (and their relaxations), the COLOR-EC LP relaxation and the Călinescu-Karloff-Rabani (CKR) LP relaxation for GRAPH-MC [CKR00] both involve a variable $x^i_v \in [0,1]$ for each node $v \in V$ and each $i \in \{1,2,\ldots,k\}$. This variable represents the “distance” between node $v$ and cluster $i$, where cluster $i$ either means the $i$th color or the $i$th terminal node depending on the problem. The standard rounding scheme for GRAPH-MC selects a distance threshold $\rho \in (0,1)$ uniformly at random and only places a node $v$ in cluster $i$ if $x^i_v < \rho$. Because $x^i_v < \rho$ is possible for more than one cluster at once, the clusters are assigned a random ordering that determines their priority when choosing nodes. The previous rounding scheme for COLOR-EC [AVB20] also selects a randomized ordering to determine cluster priority, but chooses a threshold $\rho \in (\frac{1}{2}, \frac{2}{3})$ deterministically based on $k$ and $r$. For both of these problems and their rounding techniques, the overall goal is to prove that the probability of incurring a cost at an arbitrary edge is bounded well enough to prove good approximation guarantees. One major difference between these objectives and their LP relaxations is that for GRAPH-MC, one can assume that if $u$ and $v$ are adjacent nodes, the variables $x^i_u$ and $x^i_v$ will differ for at most two choices of $i$ (see Lemma 3 in [CKR00]). This does not necessarily hold for COLOR-EC even in the graph case ($r = 2$).

This present paper combines aspects of both rounding strategies. For a given choice of $r$ and $k$, we begin by choosing an appropriate interval $[A, B]$ where $0 < A < B < 1$, then select a uniform random distance threshold $\rho \in [A, B]$. We again use a random permutation of clusters to determine their priority when rounding the LP relaxation for COLOR-EC. When $r$ is small, choosing $A$ too close to 0 or $B$ too close to 1 creates substantial issues for bounding the probability of making a mistake at an arbitrary edge (see Lemma 3.7). This is a notable deviation from previous rounding schemes for convex relaxations of related multiway cut problems that select a distance threshold $\rho \in (0,1)$ [CKR00] or $\rho \in (0,\frac{1}{2})$ [EVW13, CM16]. For COLOR-EC, we ultimately show that using a random threshold $\rho \in (\frac{1}{2}, \frac{2}{3})$ produces a $2(1 - 1/(r + 1))$ for $r \geq 3$ and arbitrary $k$, while choosing $\rho \in (\frac{1}{2}, \frac{2}{3})$ proves the $2(1 - 1/k)$ approximation for all $k \geq 2$ and arbitrary $r$. For the former case, the analysis can be slightly altered to provide a $\frac{3}{2}$-approximation for the graph version of the problem ($r = 2$). This improves on the previous $\frac{5}{3}$-approximation but does not match the integrality gap of $2(1 - 1/(r + 1)) = \frac{4}{3}$.

The most challenging and in-depth result in our paper is showing how to achieve a $\frac{4}{3}$-approximation when $r = 2$. This can be achieved using the same strategy with a rounding threshold $\rho \in (\frac{1}{2}, \frac{2}{3})$. However, proving our approximation guarantee requires checking a large number of cases, which correspond to different ways that the variables $(x^i_u, x^i_v)_{i \in [k]}$ (where $[k] = \{1,2,\ldots,k\}$) can affect the probability of making a mistake at an edge $(u, v) \in E$. We develop a strategy for efficiently checking all cases by first expressing the probability of making a mistake at $(u, v)$ in terms of the original LP variables for each case, and then upper bounding this probability by solving a small auxiliary linear program with at most 11 variables and 16 constraints. This small linear program is independent of the edge-colored graph that defines the instance of COLOR-EC under consideration, and simply serves as a proof technique for bounding probabilities. Solving or bounding a small LP once for each case is sufficient to prove our result. The challenge is that the proof requires checking 46
cases, each of which corresponds to a different small LP. Our key observation is that the dual variables for each small LP can be used to quickly define a linear combination of inequalities that bound the probability of making a mistake at the edge \( e = (u, v) \) for each case. Listing dual variables for each case in a table provides a proof of the probability bound needed for each case.

1.4 Overview of Vertex Cover Results

The connection between COLOR-EC and VERTEX COVER can be intuitively described by casting COLOR-EC as an edge-deletion problem. If two hyperedges in an edge-colored hypergraph overlap but have a different color, we define this to be a bad hyperedge pair, since we know that every node coloring will make a mistake at one or both of these hyperedges. The COLOR-EC objective is equivalent to deleting a minimum number of hyperedges in a way that destroys all bad hyperedge pairs. Analogously, VERTEX COVER is a node deletion problem where the goal is to remove the minimum number of nodes so that the remaining nodes are an independent set. Formalizing this analogy leads to a reduction from COLOR-EC to VERTEX COVER.

In order to obtain a linear-time 2-approximation algorithm for COLOR-EC, we design a method that corresponds to an implicit implementation of Pitt’s algorithm for VERTEX COVER [Pit85]. This method iteratively visits uncovered edges in an instance of VERTEX COVER and randomly samples one of the nodes to add to a cover. If we reduce an instance of COLOR-EC to VERTEX COVER, iterating over edges in the reduced graph is equivalent to iterating through bad hyperedge pairs in the original hypergraph \( H = (V, E) \). Because there could be up to \( O(|E|^2) \) bad hyperedge pairs, explicitly visiting all of them is inefficient and leads to suboptimal runtimes. In order to overcome this, we develop an approach for processing all bad hyperedge pairs containing a fixed node \( v \) in \( O(d_v) \) time. The key observation for accomplishing this is that whenever a hyperedge is deleted, we no longer need to consider other bad hyperedge pairs containing that hyperedge, since those pairs already contain a deleted hyperedge. Thus, although a node \( v \) can be contained in \( O(d_v^2) \) bad hyperedge pairs, we are able to iterate through pairs in a way that implicitly takes care of multiple bad hyperedge pairs in each step and takes \( O(d_v) \) time overall. This leads to a 2-approximation for COLOR-EC with a runtime of \( O(\sum_{v \in V} d_v) = O(\sum_{e \in E} |e|) \).

2 Linear Programming Framework

To more formally present the COLOR-EC objective, let \( H = (V, E, C, \ell) \) be a hypergraph where \( V \) is a set of nodes, \( E \) is a set of (hyper)edges, and \( C = \{1, 2, \ldots, k\} = [k] \) is a set of edge colors. We will typically use the word edge instead of hyperedge even when referring to hypergraphs, as this provides consistency between the graph and hypergraph versions of the problem. The function \( \ell : E \to C \) maps each edge to a color in \( C \). Let \( E_c \subseteq E \) denote the set of edges with color \( c \in C \). We also associate each edge \( e \in E \) with a nonnegative weight \( w_e \geq 0 \). Let \( r \) denote the maximum hyperedge size (also known as the rank or order).

The goal of COLOR-EC is to assign each node a color in a way that minimizes the weight of hyperedge mistakes. Formally, let \( Y : V \to C \) be the map from nodes to colors. For an edge \( e \in E \), if there is any node \( v \in e \) such that \( Y[v] \neq \ell(e) \), we have made a mistake at edge \( e \) and this incurs a penalty of \( w_e \). This is equivalent to partitioning nodes into \( k \) clusters, with each cluster corresponding to one of the colors, in a way that minimizes the weight of edges that are not completely contained in the cluster with a matching color. Given a node
color map \( Y \), let \( \mathcal{M}_Y \subseteq E \) denote the set of edges where \( Y \) makes a mistake. The objective is formally given by
\[
\min_{Y} \sum_{e \in E} w_e \mathbb{1}_{\mathcal{M}_Y}(e),
\]
where \( \mathbb{1}_{\mathcal{M}_Y} \) is the indicator function for edge mistakes and the minimization is over all valid node colorings \( Y \). The canonical linear programming relaxation for this objective is
\[
\min \sum_{e \in E} w_e x_e \\
\text{s.t.} \quad \sum_{i=1}^k x_v^i = k - 1 \quad \forall v \in V \\
\quad x_e = \max_{i \in [k]} x_v^i \\n\quad 0 \leq x_v^i \leq 1 \\n\quad 0 \leq x_e \leq 1 \\
\forall v \in V \text{ and } i \in [k] \\
\forall e \in E.
\]
The variable \( x_v^i \) can be interpreted as the distance between node \( u \) and color \( i \). Every node color map \( Y \) can be translated into a binary feasible solution for this LP by setting \( x_v^i = 0 \) if \( Y[v] = i \) and \( x_v^i = 1 \) otherwise, and by setting \( x_e = 1 \) when \( e \in \mathcal{M}_Y \) and \( x_e = 0 \) otherwise. In practice the constraint \( x_e = \max_{i \in [k]} x_v^i \) can be replaced with linear constraints \( x_e \leq x_v^c \) for each \( e \in \mathcal{E} \), which does not change the optimal solution. We leave this as \( x_e = \max_{i \in [k]} x_v^c \) in our formulation to explicitly highlight a useful relationship between variables that holds at optimality and which we will frequently use when proving our approximation guarantees.

2.1 Generic LP Rounding Algorithm

Algorithm 1 is a generic algorithm for rounding the LP relaxation of COLOR-EC that takes an interval \( I \subseteq [0, 1] \) as input. The algorithm generates a uniform random threshold \( \rho \in I \), and identifies the set of nodes \( v \) satisfying \( x_v^i < \rho \) for each cluster \( i \). If \( x_v^i < \rho \), we say that color \( i \) “wants” node \( v \) and that \( i \) is a candidate color for node \( v \). Because there may be more than one candidate color for each node, we define a uniform random permutation of \( \{1, 2, \ldots, k\} \) that defines the priority of each color. We then define \( Y[v] \) to be the color that has the highest priority among all nodes that want \( v \). In Algorithm 1 we update \( Y[v] \) for each color that wants \( v \), so this value may be overwritten many times until it is assigned to the candidate color with highest priority. With this implementation, a color has higher priority if it comes later in the permutation. Nodes that are not wanted by any color are assigned an arbitrary color.

The best previous rounding scheme for this LP corresponds to running Algorithm 1 for a fixed deterministic threshold \( \rho \) (i.e., \( I \) is a single point) whose value depends only on which of \( \{2 - 1/k, 2 - 1/(r + 1)\} \) is smaller [AVB20]. We will use a random threshold from a continuous interval \( I \). Our goal is to bound the probability that \( e \in \mathcal{M}_Y \) for an arbitrary edge \( e \in E \).

Observation 2.1. Let \( H = (V, E, C, \ell) \) be an instance of COLOR-EC and \( \{x_e, x_v^i : v \in V, e \in E, i \in [k]\} \) be a set of feasible variables for the canonical COLOR-EC LP. Let \( I \subseteq [0, 1] \) be an interval and \( Y \) be the node labeling returned by GenColorRound\((H, I)\). If \( \mathbb{P} [e \in \mathcal{M}_Y] \leq px_e \) for every edge \( e \in E \), then the node labeling \( Y \) has expected value at most \( p \sum_{e \in E} w_e x_e \).

This is simply the observation that the expected cost of Algorithm 1 is
\[
\mathbb{E} \left[ \sum_{e \in E} w_e \mathbb{1}_{\mathcal{M}_Y}(e) \right] = \sum_{e \in E} w_e \cdot \mathbb{P} [e \in \mathcal{M}_Y] \leq p \sum_{e \in E} w_e x_e.
\]
Algorithm 1 GenColorRound\((H, I)\)

**Input:** Hypergraph \(H = (V, E, C, \ell)\), interval \(I \subseteq [0, 1]\)

**Output:** Node label function \(Y : V \to C\)

Solve the LP-relaxation (2)

Generate \(\rho \in I\) uniformly at random

5: Generate uniform random permutation \(\pi\) of \(\{1, 2, \ldots, k\}\)

For each \(i \in \{1, 2, \ldots, k\}\) define \(S_i = \{v \in V : x_i^v < \rho\}\)

for \(i = 1\) to \(k\) do

\[\text{for } v \in S_{\pi(i)} \text{ do} \]

\[Y[v] = \pi(i)\]

10: end for

end for

Place all remaining nodes in an arbitrary cluster

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If we use the optimal LP solution, this immediately implies a \(p\)-approximation algorithm for Color-EC. This follows a standard strategy for rounding convex relaxations for multiway cut objectives [CKR00, EVW13, CM16]. Given this observation, we can now simply focus on bounding \(P[e \in M_Y]\) for different intervals \(I\) and under different assumptions about \(r\) and \(k\).

### 2.2 Linear Program Integrality Gap

Before proving new approximation guarantees, we establish our integrality gap result, which is related to integrality gap instances for Node-MC [GVY04] and Hyper-MC [EVW13]. For these multiway cut objectives, an integrality gap was previously given only in terms of the number of clusters \(k\). For Color-EC, we can establish the desired integrality gap in terms of both \(k\) and \(r\) simultaneously by considering an edge colored hypergraph where \(k = r + 1\).

**Figure 2:** For every integer \(k \geq 3\) there is an edge-colored hypergraph with \(k = r + 1\) for which the integrality gap of the COLOR-EC linear program is \(2(1 - 1/k) = 2(1 - 1/(r + 1))\). For \(k = 3\) the instance is a triangle with three different edge colors. For general \(k\), the construction involves \(k\) hyperedges of different colors, and for each pair of distinct hyperedges there is one node in the intersection. The construction is easy to visualize using a bipartite representation: there is a hyperedge-node (squares) for each color index, and each standard node (circles) is defined by pairs of hyperedge-nodes.

**Lemma 2.1.** For every integer \(k \geq 3\), there exists an instance \(H = (V, E, C, \ell)\) with \(k = r + 1\) that has an optimal COLOR-EC solution with \(k - 1 = r\) mistakes, and for
which the LP relaxation has a value of $k/2 = (r + 1)/2$. Thus, the integrality gap is $2(1-1/k) = 2(1-1/(r+1))$.

Proof. We will construct a hypergraph $H$ that has exactly one hyperedge $e_c$ for each color $c \in \{1,2,\ldots,k\}$, and a total of $\left(\binom{k}{2}\right)$ nodes, each of which participates in exactly two hyperedges. To define the hypergraph structure precisely, we index each node by a pair of color indices $(i,j) \in C \times C$ with $i \neq j$. We place node $(i,j)$ in the hyperedge $e_i$ and the hyperedge $e_j$. This means that each hyperedge contains exactly $r = k - 1$ nodes.

The optimal COLOR-EC solution will make a mistake at all but one of the hyperedges. To see why, observe that if we do not make a mistake at hyperedge $e_c$, this means that every node in $e_c$ is given label $c$. For each color $i \neq c$, the node with index $(i,c)$ is in $e_c$ and $e_i$, which means we will make a mistake at $e_i$ since this node was given label $c$. Meanwhile, the optimal solution to the LP relaxation is $k/2$: for node $v$ corresponding to color pair $(i,j)$, we set $x_{i,v} = x_{j,v} = 1/2$ and $x_{c,v} = 1$ for every $c \notin \{i,j\}$. This means $x_e = 1/2$ for each hyperedge $e \in E$, for an overall LP objective of $k/2$. \hfill \Box

3 Optimal LP Rounding for the Graph Objective

When $r = 2$, the integrality gap for the COLOR-EC LP relaxation is $4/3$, and the instance in Lemma 2.1 corresponds to a single triangle where each edge has a different color. In this section we prove that when $I = \left(\frac{1}{2}, \frac{7}{8}\right)$, Algorithm 1 provides a matching $4/3$-approximation guarantee.

3.1 Proof Setup, Overview, and Challenges

Throughout this section we consider a fixed arbitrary edge $e = (u,v)$ with color $c = \ell(e)$ and LP variable $x_e = \max\{x^c_u, x^c_v\}$. We will show that for every feasible solution to the LP relaxation (2),

$$\Pr[e \in M_Y] \leq \frac{4}{3} x_e,$$

where $Y$ is the node label function returned by Algorithm 1. The $4/3$-approximation guarantee will then follow immediately from Observation 2.1.

Recall that we say color $i$ “wants” node $v$ when $x^i_v < \rho$ for the randomly chosen threshold $\rho$. In order to guarantee we do not make a mistake at $e$, we must first of all know that color $c$ wants both $u$ and $v$, which is true if and only if $\rho > x_e$. This means that

$$\Pr[e \in M_Y \mid \rho \leq x_e] = 1.$$

Even if $\rho > x_e$, there is a chance of making a mistake at $e$ if there are any other colors that want $u$ or $v$. The challenge in proving inequality (3) is that the probability of making a mistake depends heavily on the relationship among LP variables $x_e$ and $\{x^i_u, x^i_v : i \in [k]\}$, and more specifically the relative ordering of these variables. Especially given that $k$ can be arbitrarily large, there are many possible orderings to consider, which may require separate analyses. If we are willing to use loose bounds, it is not challenging to prove that $\Pr[e \in M_Y] \leq px_e$ for constant factors $p > 4/3$ for many different orderings and cases at once. However, as we shall see, proving an approximation guarantee that tightly matches the integrality gap lower bound will require a careful and in-depth cases analysis.

We first prove the result for extreme values of $x_e$, where the result is easy to show.
Lemma 3.1. If \( x_e \not\in (\frac{1}{5}, \frac{3}{4}) \), for Algorithm 1 with \( I = (\frac{1}{5}, \frac{7}{8}) \) we have \( \mathbb{P}[e \in \mathcal{M}_Y] \leq \frac{1}{8}x_e \).

Proof. If \( x_e \geq \frac{3}{4} \), the inequality is trivial since \( \mathbb{P}[e \in \mathcal{M}_Y] \leq 1 \leq \frac{1}{4}x_e \). If \( x_e \leq \frac{1}{4} \), then for every \( \rho \in (\frac{1}{5}, \frac{7}{8}) \), color \( c = \ell(e) \) wants both \( u \) and \( v \) and in fact no other color wants either node. If color \( i \neq c \) wants \( u \), this would mean that \( x_i^u < \rho < \frac{7}{8} \). Since \( x_i^c \leq x_e \leq \frac{1}{8} \), this would imply that \( x_i^u + x_i^v < 1 \), contradicting the constraint \( \sum_{j=1}^k x_i^j = k - 1 \). Therefore, if \( x_e \leq \frac{1}{8} \), then \( c \) is the only color that wants \( e \) and we are guaranteed to avoid making a mistake at this edge. \( \square \)

The following subsections cover the cases where \( x_e \in (\frac{1}{5}, \frac{1}{2}) \) and \( x_e \in [\frac{1}{2}, \frac{3}{4}) \), which are far more challenging. For these cases, it is possible for many colors to want nodes \( u \) and \( v \), and we must account for a large number of different configurations of the LP variables and how they can affect bounds for \( \mathbb{P}[e \in \mathcal{M}_Y] \). In order to provide a systematic proof for all cases, we first introduce a new set of variables \( \{w_i\}_{i \in [k-1]} \) that can be viewed as a convenient rearrangement of the node-color distance variables \( \{x_i^u, x_i^v : i \in [k]\} \), and which indicate the points at which there is a change in the number of distinct colors that “want” a node in \( e \). We prove an important lemma on the relationship between \( x_e \) and the \( \{w_i\}_{i \in [k-1]} \) variables, and then show how to express \( \mathbb{P}[e \in \mathcal{M}_Y] \) in terms of these variables for different possible feasible LP solutions. We finally prove \( \mathbb{P}[e \in \mathcal{M}_Y] \leq \frac{1}{8}x_e \) for different possible conditions by solving a small auxiliary linear program, whose optimal dual variables automatically determine a sequence of inequalities that bound \( \mathbb{P}[e \in \mathcal{M}_Y] \).

Throughout the proof, we highlight several reasons why it is challenging to develop a simpler proof that avoids this type of case analysis. At the end of the section, we also give examples of possible values for the LP variables \( x_e \) and \( \{x_i^u, x_i^v : i \in [k]\} \) that indicate why it is challenging (or in some cases impossible) to prove inequality (3) using a different interval \( I \) other than \((\frac{1}{2}, \frac{7}{8})\).

### 3.2 The Color Threshold Lemma

When \( x_e \in (\frac{1}{5}, \frac{3}{4}) \), it is possible for many different colors to want one or both of the nodes in \( e = (u, v) \), and deriving tight bounds on \( \mathbb{P}[e \in \mathcal{M}_Y] \) will depend on when different colors start to want \( u \) or \( v \). We keep track of the number of distinct colors that want a node in \( e \) by defining a color threshold \( w_i \) for each \( i \in \{1, 2, \ldots, k-1\} \).

**Definition 3.1.** For \( i \in \{0, 1, 2, \ldots, k-1\} \), the \textbf{color threshold} \( w_i \) is the smallest value such that for every \( \rho > w_i \), there are \( i \) distinct colors not equal to \( c = \ell(e) \) that want a node in \( e \).

The definition tells us that the color threshold values are monotonically increasing:

\[
0 = w_0 \leq w_1 \leq w_2 \leq \cdots \leq w_{k-1} \leq 1.
\]

(5)

The value \( w_i \) is defined so that we can easily express the probability of making a mistake at edge \( e \) if we know \( \rho > x_e \) and we know the value of \( \rho \) relative to the color thresholds values. Formally:

\[
\mathbb{P}[e \in \mathcal{M}_Y | \rho > x_e \text{ and } w_i < \rho \leq w_{i+1}] = \frac{i}{i+1}.
\]

(6)

One helpful way to interpret the color threshold values is as a rearrangement of the node-color variables \( \{x_i^u, x_i^v : i \in [k]\} \). To gain intuition for the meaning of these variables, note
that \( w_1 = \min_{i \in \{u,v\}, i \neq c} x_i^j \) is the first point at which another color \( i \) begins to “want” a node from \( e = (u, v) \). We prove a relationship between the variable \( x_e \) and color threshold values that will be key to proving inequality \( (3) \).

**Lemma 3.2.** For every integer \( t \leq k/2, t \leq x_e + w_t + w_{t+1} + \cdots + w_{2t-1} \).

**Proof.** Let \( m \) be an arbitrary odd integer less than \( k \). When \( \rho > w_m \), the definition of \( w_m \) tells us that there are \( m \) colors (which are distinct from each other and not equal to \( c = \ell(e) \)) that want at least one of the nodes in \( e = (u, v) \). By the pigeonhole principle, at least \( h = \lceil \frac{m}{2} \rceil = \frac{m+1}{2} \) of these colors want the same node. Without loss of generality, let \( u \) be this node and \( \{c_1, c_2, \ldots, c_h\} \) be these \( h \) colors, arranged so that

\[
x_{u}^{c_1} \leq x_{u}^{c_2} \leq \cdots \leq x_{u}^{c_h} \leq w_m.
\]

Without loss of generality we can assume these \( h \) colors are the colors (not including \( c \)) that want node \( u \) the most, in the sense that \( x_{u}^{c_j} \geq x_{u}^{c_k} \) for every color \( j \notin \{c, c_1, c_2, \ldots, c_h\} \).

Meanwhile, for every \( \rho \leq w_m \), there can be at most \( m - h = \frac{m-1}{2} \) other colors not in \( \{c, c_1, c_2, \ldots, c_h\} \) that want node \( v \). We can show then that for \( j \in \{1, 2, \ldots, h\} \),

\[
x_{u}^{c_j} \leq w_{j+m-h}. \tag{7}
\]

If this were not true and we instead assume \( x_{u}^{c_j} > w_{j+m-h} \), this would mean that for any \( \rho \in (w_{j+m-h}, x_{u}^{c_j}) \), there are \( j + m - h \) distinct colors (not equal to \( c \)) that want at least one of the two nodes in \( e = (u, v) \). However, since \( \rho < x_{u}^{c_j} \), we know that \( \{c_1, c_2, \ldots, c_{j-1}\} \) are the only colors (not counting \( c \)) that want node \( u \). Furthermore, since \( \rho < w_m \), there are at most \( m - h \) distinct colors (again not counting \( c \)) that want node \( v \). Thus, \( \rho < x_{u}^{c_j} \) would imply there are at most \( j + m - h - 1 \) colors not equal to \( c \) that want either \( u \) or \( v \). This is a contradiction, so \( (7) \) must hold. Combining this inequality with the fact that \( \sum_{i=1}^{k} x_i^j = k - 1 \), we obtain

\[
h \leq x_{u}^{c_j} + \sum_{j=1}^{h} x_{u}^{c_j} \leq x_e + \sum_{j=1}^{h} w_{j+m-h} = x_e + w_{1+m-h} + w_{2+m-h} + \cdots + w_m.
\]

Since we assumed that \( m \) is odd, we know \( h = 1 + m - h \) and \( m = 2h - 1 \), so setting \( t = h \) yields the inequality in the statement of the lemma. \( \square \)

### 3.3 Bounding Probabilities Using Auxiliary Linear Programs

If we know the exact relationship between \( x_e \), the color thresholds \( \{w_i\}_{i \in [k-1]} \), and the endpoints of the interval \( \left( \frac{1}{2}, \frac{5}{8} \right) \), we can derive an expression for \( P[c \in M_Y] \) in terms of these values when applying Algorithm 1 with \( I = (\frac{1}{2}, \frac{5}{8}) \). The goal would then be to apply a sequence of carefully chosen inequalities to prove that this expression is bounded above by \( \frac{5}{8} x_e \). In this subsection we will provide an overarching strategy for accomplishing this task when we already know the relationships between these variables. In the next subsection we will apply this strategy to guarantee the result holds for all possible relationships among \( x_e, \{w_i\}_{i \in [k-1]} \), and \( \{\frac{1}{2}, \frac{5}{8}\} \).

For notational ease, let \( M \) be the event \( e \in M_Y \), i.e., the event of making a mistake at edge \( e \) when rounding the LP relaxation. In the next two lemmas, we show how to bound
Figure 3: This linear program provides an upper bound for \( \mathbb{P}[e \in \mathcal{M}_Y] / x_e \) when \( x_e \leq \frac{1}{2} \leq w_1 \leq w_q \leq \frac{7}{8} \) for integers \( q \leq 6 \). We explicitly name each constraint as it will be convenient to reference individual constraints in later parts of the proof.

\[
\begin{align*}
\text{maximize} & \quad \frac{q}{q+1} \frac{7}{8} \chi - \sum_{j=1}^{q} \frac{1}{j(j+1)} \omega_j \\
\text{subject to} & \quad (\text{A}1) \quad \omega_i - \omega_{i+1} \leq 0 \text{ for } i = 1, 2, \ldots, 5 \\
& \quad (\text{A}6) \quad \chi - \omega_1 \leq 1 \\
& \quad (\text{A}7) \quad 2 \chi - \omega_2 - \omega_1 \leq 1 \\
& \quad (\text{A}8) \quad 3 \chi - 3 \omega_3 \leq 1 \\
& \quad (\text{A}9) \quad - \chi \leq -2 \\
& \quad (\text{A}10) \quad \omega_q - \frac{q}{7} \chi \leq 0.
\end{align*}
\]

The optimal solution to linear program in Figure 3.

Lemma 3.3. If \( x_e \in (\frac{1}{8}, \frac{7}{8}] \) and \( w_{p-1} \leq \frac{1}{2} \leq w_p \leq w_q \leq \frac{7}{8} \leq w_{q+1} \) for integers \( p \leq q \), then \( p = 1, q \leq 6 \), and \( \mathbb{P}[M] \leq \frac{8}{3} A_q x_e \) where \( A_q \) is the optimal solution to linear program in Figure 3.

Proof. When \( x_e < \frac{1}{2} \), we know that for any \( \rho \in (\frac{1}{2}, \frac{7}{8}] \), the color \( c = \ell(e) \) will want both nodes in \( e = (u, v) \). If we use Lemma 3.2 with \( t = 4 \) and the inequalities in (5), we can see that

\[
4 \leq x_e + w_4 + w_5 + w_6 + w_7 \leq x_e + 4w_7 \implies 1 - \frac{x_e}{4} \leq w_7 \implies w_7 > \frac{7}{8}.
\]

This implies that for a random \( \rho \in (\frac{1}{2}, \frac{7}{8}] \), at most 6 colors other than \( c \) will want a node from \( e = (u, v) \). Thus, if \( w_{p-1} \leq \frac{1}{2} \leq w_p \leq w_q \leq \frac{7}{8} \leq w_{q+1} \), then \( q \leq 6 \). Similarly, we can see that \( 1 \leq x_e + w_1 \implies w_1 > \frac{1}{2} \), so we must have \( p = 1 \). Next, we use (8) and (9) to provide a convenient expression for \( \mathbb{P}[M] \):

\[
\mathbb{P}[M] = \sum_{j=1}^{q-1} \mathbb{P}[M \mid \rho \in (w_j, w_{j+1})] \mathbb{P}[\rho \in (w_j, w_{j+1})] + \mathbb{P}[M \mid \rho \in (w_q, 7/8)] \mathbb{P}[\rho \in (w_q, 7/8)]
\]

\[
= \frac{8}{3} \left( \frac{q}{q+1} \left( \frac{7}{8} - w_q \right) + \sum_{j=1}^{q-1} \frac{j}{j+1} (w_{j+1} - w_j) \right) = \frac{8}{3} \left( \frac{q}{q+1} \frac{7}{8} - \sum_{j=1}^{q} \frac{1}{j(j+1)} w_j \right).
\]

Our goal is to upper bound \( \mathbb{P}[M] / x_e \). To do so we have the following inequalities and case-specific assumptions at our disposal: (1) the monotonicity of color thresholds: \( w_i \leq w_{i+1} \) for \( i \in \{0, 1, \ldots, k\} \), (2) the relationship between \( x_e \) and color thresholds given in Lemma 3.2,
maximize \[ \frac{1}{p} + \left( \frac{q}{q + 18} - \frac{1}{2} \right) \chi - \sum_{j=p}^{q} \frac{1}{j(j+1)} \omega_j \]

subject to

(B1) \( \omega_i - \omega_{i+1} \leq 0 \) for \( i = 1, 2, \ldots, 9 \)
(B10) \( \chi - \omega_1 \leq 1 \)
(B11) \( 2\chi - \omega_2 - \omega_3 \leq 1 \)
(B12) \( 3\chi - \omega_3 - \omega_4 - \omega_5 \leq 1 \)
(B13) \( 4\chi - 4\omega_7 \leq 1 \)
(B14) \( \omega_{p-1} \leq 1, \) (B15) \( -\omega_p \leq -1 \) (B16) \( \omega_q - \frac{7}{8} \chi \leq 0. \)

Figure 4: This linear program provides an upper bound for \( \mathbb{P}[e \in \mathcal{M}_Y] / x_e \) when \( x_e \in \left[ \frac{1}{2}, \frac{3}{4} \right] \) and \( w_{p-1} \leq x_e \leq w_p \leq w_q \leq \frac{7}{8} \leq w_{q+1} \) for integers \( p \leq 5 \) and \( q \leq 10 \) satisfying \( p \leq q \). We explicitly name each constraint as it will be convenient to reference individual constraints in later parts of the proof. We could add additional constraints based on the assumptions in Lemma 3.4 and the relationship between variables given in Lemma 3.2, but it suffices to consider a subset of these constraints to bound \( \mathbb{P}[e \in \mathcal{M}_Y] / x_e \), and this also simplifies the analysis.

and (3) our assumptions that \( x_e \leq \frac{1}{2} \) and \( w_q \leq \frac{7}{8} \leq w_{q+1} \). Since at most 6 colors other than \( c \) can want a node in \( e = (u, v) \), we can extract a set of 10 convenient inequalities from these three categories that apply to \( x_e \) and the first 6 color threshold values:

Monotonicity Constraints

(A1) \( \frac{w_1}{x_e} \leq \frac{w_2}{x_e} \), (A2) \( \frac{w_2}{x_e} \leq \frac{w_3}{x_e} \), (A3) \( \frac{w_3}{x_e} \leq \frac{w_4}{x_e} \), (A4) \( \frac{w_4}{x_e} \leq \frac{w_5}{x_e} \), (A5) \( \frac{w_5}{x_e} \leq \frac{w_6}{x_e} \)

Edge-Node Relationship Constraints (Lemma 3.2)

(A6) \( \frac{1}{x_e} \leq 1 + \frac{w_1}{x_e} \), (A7) \( \frac{2}{x_e} \leq 1 + \frac{w_2}{x_e} + \frac{w_3}{x_e} \), (A8) \( \frac{3}{x_e} \leq 1 + \frac{3w_5}{x_e} \)

Boundary Assumption Constraints

(A9) \( 1 \leq \frac{1}{2} \frac{1}{x_e} \), (A10) \( \frac{w_q}{x_e} \leq \frac{7}{8} \frac{1}{x_e} \).

If the maximum value of \( \mathbb{P}[M] / x_e \) over all values of \( \{x_e, w_1, w_2, w_3, w_4, w_5, w_6\} \) that respect these inequalities is at most \( P \), then this guarantees \( \mathbb{P}[M] \leq P x_e \). Finally, note that \( \mathbb{P}[M] / x_e \) and the inequalities above can be given as linear expressions in terms of \( 1 / x_e \) and \( \{w_i / x_e : i \in \{1, 2, 3, 4, 5, 6\}\} \). We can therefore maximize \( \frac{3}{8} \mathbb{P}[M] / x_e \) subject to these inequalities by setting \( \chi = \frac{1}{x_e} \) and \( \omega_i = \frac{w_i}{x_e} \) and solving the linear program in Figure 3. Note that we choose to maximize \( \frac{3}{8} \mathbb{P}[M] / x_e \), rather than simply maximizing \( \mathbb{P}[M] / x_e \), only because this will simplify some of our analysis later.

We derive an analogous result for the case where \( x \in \left[ \frac{1}{2}, \frac{3}{4} \right) \).

Lemma 3.4. If \( x_e \in \left[ \frac{1}{2}, \frac{3}{4} \right) \) and \( w_{p-1} \leq x_e \leq w_p \leq w_q \leq \frac{7}{8} \leq w_{q+1} \) for integers \( p \leq q \), then \( p \leq 5 \), \( q \leq 10 \), and \( \mathbb{P}[M] \leq \frac{5}{3} B_{p,q} x_e \) where \( B_{p,q} \) is the optimal solution to linear program in Figure 4.
We again use (8) and (9) to derive a useful expression for $P$. Section 3.5 we will discuss why it is challenging to avoid a lengthy case analysis.

In order to prove that $P[M] \leq \frac{4}{7} x_e$, it remains to show that for all valid choices of $p$ and $q$, the optimal solutions to the linear programs in Figure 3 (for $x_e \in \left(\frac{1}{8}, \frac{3}{4}\right)$) and Figure 4 (for $x_e \in \left[\frac{1}{2}, \frac{3}{4}\right]$) are bounded above by $\frac{1}{2}$. The result will then follow by Lemmas 3.3 and 3.4. The difficulty is that there are many valid choices of $p$ and $q$ to check, each of which requires a different linear program. When $x_e \in \left(\frac{1}{8}, \frac{1}{2}\right)$, Lemma 3.3 indicates that $p = 1$ and $q \in \{1, 2, 3, 4, 5, 6\}$, so we much consider six cases. The situation is considerably worse for $x_e \in \left[\frac{1}{2}, \frac{3}{4}\right]$: Lemma 3.4 shows that any pair $(p, q)$ satisfying $p \leq 5$, $q \leq 10$, and $p \leq q$ is a valid case, and there are 40 such pairs. If we are content with simply computing numerical solutions for each case, we can quickly confirm that the optimal solution for each of the 46 linear programs is bounded above by $\frac{1}{2}$. In the next subsection, we will show how to obtain a complete proof by using LP duality theory to quickly extract a set of inequalities for each case that will prove an upper bound on the solution to each linear program. In Section 3.5 we will discuss why it is challenging to avoid a lengthy case analysis.
Table 1: Optimal dual variables (one for each constraint in the primal LP) for the linear program in Figure 3, for each valid integer $q$. Multiplying dual variables by the left and right hand sides of each constraint and adding the left and right hand sides together proves the upper bound (last column) on the optimal solution to the primal LP. The bound is always $\frac{1}{2}$ or smaller.

| $q$ | (A1) | (A2) | (A3) | (A4) | (A5) | (A6) | (A7) | (A8) | (A9) | (A10) | Bound |
|-----|------|------|------|------|------|------|------|------|------|-------|-------|
| 1   | 0    | 0    | 0    | 0    | 0    | $\frac{1}{2}$ | 0    | 0    | $\frac{1}{12}$ | 0    | $\frac{3}{5}$ |
| 2   | $\frac{1}{5}$ | 0    | 0    | 0    | 0    | $\frac{2}{3}$ | 0    | 0    | $\frac{1}{12}$ | 0    | $\frac{1}{2}$ |
| 3   | 0    | 0    | 0    | 0    | 0    | $\frac{1}{2}$ | $\frac{1}{2}$ | 0    | $\frac{5}{48}$ | $\frac{1}{12}$ | $\frac{11}{24}$ |
| 4   | 0    | 0    | $\frac{1}{12}$ | 0    | 0    | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{6}$ | 0    | $\frac{5}{48}$ | $\frac{1}{30}$ | $\frac{11}{24}$ |
| 5   | 0    | 0    | $\frac{1}{12}$ | $\frac{1}{30}$ | 0    | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{6}$ | 0    | $\frac{5}{48}$ | 0    | $\frac{11}{24}$ |
| 6   | 0    | 0    | $\frac{1}{12}$ | $\frac{1}{30}$ | $\frac{1}{12}$ | $\frac{1}{2}$ | $\frac{1}{6}$ | $\frac{1}{126}$ | $\frac{3}{25}$ | 0    | $\frac{29}{63}$ |

Before moving on, we remark that there are many other constraints that we could include in linear programs (10) and (11), but choose to omit. For example, the constraint $3\chi - 4\omega_5 \leq 1$ in linear program (10) is a weaker version of the constraint that can be obtained by applying Lemma 3.2 with $t = 3$. For this linear program, we also do not enforce the constraint $4\chi - \omega_4 - \omega_5 - \omega_6 \leq 1$, even though this is a valid constraint that results from applying Lemma 3.2 with $t = 4$. We omit some constraints and simplify others in order to obtain the smallest and simplest constraint set that suffices to prove $\mathbb{P}[e \in M_Y] / x_e \leq \frac{3}{5}$. Adding more constraints makes the analysis more cumbersome, without making the worst case upper bound any smaller.

3.4 Case Analysis and Probability Bounds via LP Duality

We need to show that for all valid choices of $p$ and $q$, the optimal solutions to the linear programs in Figures 3 and 4 are at most $\frac{1}{2}$. Here we illustrate our strategy for LP (10); the same basic steps also work for LP (11). The dual of LP (10) is another linear program with a variable $y_i$ for each constraint $A\{i\}$. By LP duality theory, every feasible solution to the dual provides an upper bound on the primal LP objective. Thus, in order to show that the solution is at most $\frac{1}{2}$ in all cases, we simply need to produce an appropriate set of dual variables for each choice of $q$. This means providing a set of 10 dual variables for each $q$, one variable for each constraint $A\{i\}$ for $i = 1, 2, \ldots, 10$. Table 1 lists dual variables for each valid choice of $q$. The desired bound on the LP solution, and hence a bound on $\frac{\mathbb{P}[e \in M_Y]}{3x_e}$, is obtained simply by multiplying primal LP constraints by dual variables.

To show how the dual variables in Table 1 prove the desired bound, we illustrate this procedure for LP (10) when $q = 1$. The primal LP objective for this case is $\frac{7}{16}\chi - \frac{1}{2}\omega_1$, and we must prove this can be at most $\frac{1}{2}$. From Table 1, we have a dual variable value of
\[ \frac{1}{2} \text{ for constraint (A6): } \chi - \omega_1 \leq 1, \text{ and a dual variable value of } \frac{1}{16} \text{ for constraint (A9): } -\chi \leq -2. \] All other dual variables are zero. Multiplying constraints by the dual variables, adding, and comparing left and right hand sides, we obtain the desired bound:

\[
\left( \frac{1}{2} \text{(A6)} + \frac{1}{16} \text{(A9)} \right) : \quad \frac{1}{2}(\chi - \omega_1) + \frac{1}{16}(-\chi) \leq \frac{1}{2}(1) + \frac{1}{16}(-2) \implies \frac{7}{16} \chi - \frac{1}{2} \omega_1 \leq \frac{3}{8} \leq \frac{1}{2}.
\]

It is not hard to check that the dual variables in Table 1 provide corresponding bounds of \( \frac{1}{2} \) (or better) for all other values of \( q \). We use the same strategy to confirm that linear program (11) always has an optimal solution of \( \frac{1}{2} \) or smaller, for all valid choices of \( p \) and \( q \) given in Lemma 3.4. This leads to the following Lemma.

**Lemma 3.5.** For any integer \( q \in \{1, 2, \ldots, 6\} \), the optimal solution to the linear program in Figure 3 is at most \( \frac{1}{2} \), and for any pair of integers \((p, q)\) satisfying \( 1 \leq p \leq 5 \) and \( p \leq q \leq 10 \), the optimal solution to the linear program in Figure 4 is at most \( \frac{1}{2} \). Hence, for every \( x_e \in \left( \frac{1}{8}, \frac{3}{4} \right) \), \( \mathbb{P} \left[ e \in \mathcal{M}_Y \right] \leq \frac{4}{3} x_e \) where \( Y \) is the node coloring returned by Algorithm 1 when \( I = \left( \frac{1}{2}, \frac{7}{8} \right) \).

The proof of this result for the LP in Figure 3 is encoded in Table 1. Multiplying the left hand side of each constraint by the dual variables and summing produces the LP objective function, while multiplying the right hand side by dual variables and summing gives the upper bound. In the appendix, we provide further details for how to prove the bounds in the last column of Table 1 hold by simply checking a few matrix-vector products. We also provide a list of dual variables for the LP in Figure 4, for all 40 valid choices of \((p, q)\), which can similarly be used to prove the desired upper bound. Combining all of these pieces yields our desired approximation guarantee.

**Theorem 3.6.** Algorithm 1 with \( I = \left( \frac{1}{2}, \frac{7}{8} \right) \) is a \( \frac{4}{3} \)-approximation algorithm for COLOR-EC.

**Proof.** If we select an arbitrary edge \( e = (u, v) \), Lemmas 3.1 and 3.5 guarantee that \( \mathbb{P} \left[ e \in \mathcal{M}_Y \right] \leq \frac{4}{3} \), regardless of the value of \( x_e \). By Observation 2.1, Algorithm 3 is a \( \frac{4}{3} \)-approximation algorithm. \( \square \)

### 3.5 Challenges in Avoiding Case Analysis

It is natural to wonder whether we can rule out or simultaneously handle a large number of cases for Lemmas 3.3 and 3.4, rather than check a different set of dual variables for 46 small linear programs. However, there are several challenges in escaping from this lengthy case analysis. First of all, the 46 linear programs we have considered all have feasible solutions, so we cannot rule any out on the basis that they correspond to impossible cases. The optimal solution for many cases is exactly \( \frac{1}{2} \), so the inequalities we apply to bound \( \mathbb{P} \left[ \mathcal{M} \right] / x_e \) for these cases must be tight. For many of the other cases, the optimal solution is very close to \( \frac{1}{2} \). We attempted to use looser bounds to prove a \( \frac{1}{2} \) bound for multiple cases at once, but were unsuccessful. Each case seems to require a slightly different argument in order to prove the needed bound. Adding more constraints to the linear programs also did not lead to a simpler analysis.

A second natural strategy is to try to first identify and prove which values of \( x_e, p, \) and \( q \) correspond to the worst cases, and then simply confirm that the probability bound holds for these cases. However, the worst case values for \( x_e, p, \) and \( q \) are not obvious, and are in fact somewhat counterintuitive. At first glance it may appear that the worse case is
when \( q \) is as large as possible and \( p \) is as small as possible. This maximizes the expected number of colors that want a node in \( e \) for a randomly chosen \( \rho \in (\frac{1}{2}, \frac{7}{8}) \), leading to a higher probability of making a mistake at \( e \). However it is important to recall that the goal is to bound \( \mathbb{P}[M]/x_e \), and not just \( \mathbb{P}[M] \), so this line of reasoning does not apply. As it turns out, the worst case scenario (i.e., largest optimal LP solution) for \( x_e \in [\frac{1}{2}, \frac{7}{8}) \) is when \( q \in \{2, 4\} \), but we do not have a proof (other than by checking all cases) for why this case leads to the largest value of \( \mathbb{P}[M]/x_e \).

Finally, another natural question is whether we can avoid a lengthy case analysis by choosing an interval other than \( I = (\frac{1}{2}, \frac{7}{8}) \) when applying Algorithm 1. It is especially tempting to use an interval \((a, b)\) with \( b < \frac{7}{8} \), since this would potentially decrease the maximum number of colors that want a node in \( e = (u, v) \). This is desirable because if fewer colors want a node in \( e \), then we have fewer color threshold variables to worry about and fewer cases to consider. The following result shows why this approach will fail.

**Lemma 3.7.** Let \( Y \) denote the node color map returned by running Algorithm 1 with \( I = (a, b) \subset (0, 1) \) on an edge-colored graph.

- If \( a > \frac{1}{2} \), there exists a feasible LP solution such that \( \mathbb{P}[e \in \mathcal{M}_Y] > \frac{4}{3} x_e \).
- If \( a \leq \frac{1}{2} \) and \( b < \frac{7}{8} \), there exists a feasible LP solution such that \( \mathbb{P}[e \in \mathcal{M}_Y] > \frac{4}{3} x_e \).

**Proof.** If \( a > \frac{1}{2} \), then there exists some \( \varepsilon > 0 \) such that \( a = \frac{1+\varepsilon}{2} \). Consider an edge \( e = (u, v) \) with color \( c = \ell(e) \) whose LP variables satisfy \( x_e = x^c_u = x^c_v = \frac{1-\varepsilon}{2} \) and where \( x^i_u = x^j_v = \frac{1+\varepsilon}{2} = a \) for two colors \( i \neq j \) that are both distinct from \( c \). This is feasible as long all as \( x^t_u = 1 \) for \( t \notin \{c, i\} \) and \( x^t_v = 1 \) for \( t \notin \{c, j\} \). For every \( \rho \in (a, b) \), color \( i \) will want node \( u \), color \( j \) will want node \( v \), and color \( c \) will want both of them. Therefore, the probability of making a mistake at \( e \) is exactly

\[
\mathbb{P}[e \in \mathcal{M}_Y] = \frac{2}{3} = \frac{2}{3} \cdot \frac{1}{x_e} x_e = \frac{2}{3} \cdot \frac{2}{1-\varepsilon} x_e > \frac{4}{3} x_e.
\]

If instead \( a \leq \frac{1}{2} \), consider the feasible solution where \( x_e = x^c_u = x^c_v = x^i_u = x^i_v = x^j_u = x^j_v = \frac{2}{3} \) where \( \{c, g, h, i, j\} \) are all distinct colors. If \( b < \frac{2}{3} \) we are guaranteed to make a mistake at \( x_e \), so assume that \( \frac{2}{3} \leq b < \frac{7}{8} \). Then we can calculate that

\[
\frac{\mathbb{P}[e \in \mathcal{M}_Y]}{x_e} = \frac{1}{b-a} \left( \frac{2}{3} - a + \frac{4}{5} \left( b - \frac{2}{3} \right) \right) \cdot \frac{3}{2} = \frac{2-3a}{10(b-a)} + \frac{6}{5} > \frac{2-3 \cdot \frac{1}{2}}{10 \left( \frac{7}{8} - \frac{1}{2} \right)} + \frac{6}{5} = \frac{4}{3}.
\]

The first case in this lemma indicates that we should not use an interval \((a, b)\) where \( a > \frac{1}{2} \). The second case rules out the hope of choosing some \( b < \frac{7}{8} \) in order to make the analysis easier. The interval \( I = (\frac{1}{2}, \frac{7}{8}) \) is chosen to avoid these two issues and be as simple to analyze as possible.

Lemma 3.7 and the other challenges highlighted above do not imply that it is impossible to design a \( \frac{1}{4} \)-approximation algorithm with a simpler approximation guarantee proof. Indeed, a simpler proof of a tight approximation is a compelling direction for future research. Nevertheless, this discussion highlights why it is challenging to escape from using a length case analysis when proving that \( \mathbb{P}[e \in \mathcal{M}_Y] \leq \frac{4}{3} x_e \) for Algorithm 1.
4 Optimal LP Rounding for the Hypergraph Objective

Finding an interval \( I = (a, b) \) for Algorithm 1 that provides a provably optimal rounding strategy turns out to be easier for COLOR-EC when \( r \geq 3 \), i.e., the hypergraph variant of the objective. Unlike when \( r = 2 \), it suffices to consider smaller intervals where \( b < \frac{2}{3} \), which makes it possible to guarantee that for every random threshold \( \rho \in (a, b) \) and every edge \( e \), not many colors will want \( e \). Mirroring our definitions in Section 3.2, for an arbitrary hyperedge \( e \) with color \( c = \ell(e) \) we define \( w_1(e) \) to be the smallest point at which some color other than \( c \) begins to want a node in \( e \). Formally,

\[
w_1(e) = \min_{v \in e, i \neq c} x^i_v.
\]

This satisfies the following inequality, which is simply a special case of Lemma 3.2:

\[
1 - w_1(e) \leq x_e. \tag{13}
\]

This can be shown by applying the fact that \( \sum_{i=1}^k x^i_v = k - 1 \) and \( x_e \geq x^c_v \) for each \( v \in e \).

For the proofs in this section, we do not need to consider any higher color threshold \( w_i(e) \) for \( i > 1 \).

**Theorem 4.1.** Algorithm 1 with \( I = (\frac{1}{2}, \frac{3}{7}) \) is a \( 2 \left( 1 - \frac{1}{7} \right) \)-approximation for COLOR-EC.

**Proof.** If \( k = 2 \), COLOR-EC is known to be polynomial time solvable. In this case, the constraint matrix for the LP relaxation is totally unimodular. This means that for every basic feasible solution, the LP variables will be binary, and our rounding procedure will find the optimal solution. Assume for the rest of the proof that \( k \geq 3 \). By Observation 2.1, it suffices to show that for an arbitrary (hyper)edge \( e \) with color \( c = \ell(e) \), \( \mathbb{P}[e \in \mathcal{M}_Y] \leq 2 \left( 1 - \frac{1}{k} \right) x_e. \) If \( x_e \geq \frac{3}{2} \), then \( \mathbb{P}[e \in \mathcal{M}_Y] \leq 1 \leq \frac{4}{3} x_e \leq 2 \left( 1 - \frac{1}{k} \right) x_e \) for \( k \geq 3 \). We break up the remainder of the proof into two cases.

**Case 1:** \( x_e < 1/2. \) For every \( \rho \in (\frac{1}{2}, \frac{3}{7}) \), color \( c \) will want all nodes in \( e \) because \( x_e < \frac{1}{2} \).

Inequality (13) implies that \( w_1(e) > \frac{1}{2} \). If \( w_1(e) \geq \frac{3}{4} \), then no other color \( i \neq c \) will want a node from \( e \), meaning that \( \mathbb{P}[e \in \mathcal{M}_Y] = 0 \). If \( \frac{1}{2} < w_1(e) \leq \frac{3}{4} \), it is possible to make a mistake at \( e \) only if \( \rho > w_1(e) \), and even then the probability of making a mistake is bounded by \( \frac{k-1}{k} \).

Using inequality (13) we see that

\[
\mathbb{P}[e \in \mathcal{M}_Y] = \mathbb{P}[\rho > w_1(e)] \mathbb{P}[e \in \mathcal{M}_Y | \rho > w_1(e)] \leq \frac{3 - w_1(e)}{3 - \frac{1}{2}} \cdot \frac{k - 1}{k} = \frac{k - 1}{k} \left( 1 - w_1(e) - \frac{1}{4} \right) \leq \frac{3 - w_1(e)}{3 - \frac{1}{2}} \cdot \frac{k - 1}{k} \left( x_e - \frac{x_e}{2} \right) = 2 \left( 1 - \frac{1}{k} \right) x_e.
\]

**Case 2:** \( x_e \in [\frac{1}{2}, \frac{3}{4}) \). We apply a similar set of steps to see that

\[
\mathbb{P}[e \in \mathcal{M}_Y] = \mathbb{P}[\rho \leq x_e] \mathbb{P}[e \in \mathcal{M}_Y | \rho \leq x_e] + \mathbb{P}[\rho > x_e] \mathbb{P}[e \in \mathcal{M}_Y | \rho > x_e] = \frac{x_e - \frac{1}{2}}{\frac{3}{4} - \frac{1}{2}} \cdot \frac{k - 1}{k} \left( x_e - \frac{1}{2} + \frac{3(k - 1)}{4k} - \frac{k - 1}{k} x_e \right) = 4 \left( x_e + \frac{k - 3}{4k} \right) \leq \frac{1}{k + \frac{k - 3}{2k}} x_e = 2 \left( 1 - \frac{1}{k} \right) x_e.
\]

\[\square\]
Theorem 4.2. Algorithm 1 with \( I = (\frac{1}{2}, \frac{2}{3}) \) is a \( 2 \left( 1 - \frac{1}{r+1} \right) \)-approximation for COLOR-EC.

Proof. Let \( e \) be an arbitrary hyper(edge) with color \( c = \ell(e) \). By Observation 2.1, it suffices to show that \( \mathbb{P}[e \in M_Y] \leq 2 \left( 1 - \frac{1}{r+1} \right) x_e \). If \( x_e \geq \frac{2}{3} \), this is trivial since \( \mathbb{P}[e \in M_Y] \leq 1 \leq \frac{3}{2} x_e \leq 2 \left( 1 - \frac{1}{r+1} \right) x_e \) as long as \( r \geq 3 \). For the rest of the proof we assume that \( x_e < \frac{2}{3} \). Let \( w_1 = w_1(e) \).

Case 1: \( x_e < \frac{1}{2} \). For every \( \rho \in (\frac{1}{2}, \frac{2}{3}) \), color \( c \) wants all nodes in \( e \), and inequality (13) implies that \( w_1 > 1/2 \). Therefore, the algorithm can only make a mistake at \( e \) if \( \rho \) falls between \( w_1 \) and \( \frac{2}{3} \). This never happens if \( w_1 \geq \frac{2}{3} \), so assume that \( w_1 < \frac{2}{3} \). Because \( \rho < \frac{2}{3} \), for every \( v \in e \) it is possible for two different colors to want \( v \), but never three colors. Since color \( c \) is guaranteed to want every \( v \in e \), the worst case scenario is when \( \rho > w_1 \) and each \( v \in e \) is wanted by a different color that is unique to that node. In this case, the probability of making a mistake at \( e \) is \( \frac{r}{r+1} \). Putting all of the pieces together we have the following bound:

\[
\mathbb{P}[e \in M_Y] = \mathbb{P}[\rho > w_1] \mathbb{P}[e \in M_Y | \rho > w_1] = \frac{2}{3} - w_1 \cdot \frac{r}{r+1} \\
= \frac{6r}{r+1} \left( 1 - w_1 - \frac{1}{3} \right) \leq \frac{6r}{r+1} \left( x_e - \frac{2}{3} x_e \right) = 2 \left( 1 - \frac{1}{r+1} \right) x_e.
\]

Case 2: \( x_e \in [\frac{1}{2}, \frac{2}{3}) \). If \( \rho \leq x_e \), we are guaranteed to make a mistake at \( e \), but this happens with bounded probability. If \( \rho > x_e \), we can argue as in Case 1 that the probability of making a mistake at \( e \) is at most \( \frac{r}{r+1} \). We therefore have:

\[
\mathbb{P}[e \in M] = \mathbb{P}[\rho \leq x_e] \mathbb{P}[e \in M_Y | \rho \leq x_e] + \mathbb{P}[\rho > x_e] \mathbb{P}[e \in M_Y | \rho > x_e] \\
= \frac{x_e - \frac{1}{2}}{\frac{2}{3} - \frac{1}{2}} \cdot \frac{\frac{2}{3} - x_e}{\frac{3}{2} - \frac{1}{2}} \cdot \frac{r}{r+1} = 6 \left( x_e - \frac{1}{2} + \frac{2r}{3r+3} - \frac{x_e r}{r+1} \right) \\
= 6 \left( \frac{x_e}{r+1} + \frac{r - 3}{6r + 6} \right) \leq 6 \left( \frac{3x_e}{3r+3} + \frac{(r - 3)x_e}{3r+3} \right) = 2 \left( 1 - \frac{1}{r+1} \right) x_e.
\]

The rounding strategies in Theorems 4.1 and 4.2 are optimal, as they match the integrality gap lower bound given in Lemma 2.1.

5 Relation to Multiway Cut Objectives

In this section we prove that the canonical COLOR-EC LP relaxation is always at least as tight as the NODE-MC LP relaxation, and that it can be strictly tighter in some cases. Therefore, although these approaches lead to a matching worst case guarantee when \( r \) is arbitrarily large, applying the canonical relaxation leads to better results in many cases. We will also highlight why our approximation guarantees for small values of \( r \) are much better than any guarantees we obtain by reducing to HYPER-MC.
5.1 Reductions from Color-EC to Node-MC and Hyper-MC

As shown previously [AVB20], an instance $H = (V, E, C, \ell)$ of Color-EC can be reduced to an instance of Node-MC on a graph $G$ via the following steps:

- For every node $v \in E$, add a corresponding node $v$ to $G$ with weight $\infty$.
- For each color $i \in \{1, 2, \ldots, k\}$, place a terminal node $t_i$ in $G$.
- For each hyperedge $e \in E$ with weight $w_e$, add a node $v_e$ to $G$ with weight $w_e$ and place an edge $(v, v_e)$ for each $v \in e$.

The Color-EC objective on $H$ is then approximation equivalent to Node-MC on $G$. Node-MC is in turn approximation equivalent to Hyper-MC [CE11b]. Although not previously shown explicitly, there is a particularly simple reduction from Color-EC to Hyper-MC: introduce a terminal node $t_i$ for each color $i$, and then for a hyperedge $e \in E$ of color $i$, add terminal node $t_i$ to that hyperedge and then ignore the hyperedge color. In this way, Color-EC can be viewed as a special case of Hyper-MC where every hyperedge contains one of the terminal nodes. Figure 5 illustrates the procedure of reducing an instance of Color-EC to Node-MC and Hyper-MC.

Approximation guarantees for Color-EC via reductions

There are a few known approximation guarantees for Hyper-MC in terms of the maximum hyperedge size $r$, but these do not imply any useful results for Color-EC. When $r = 2$, Hyper-MC is the well-studied graph multiway cut objective [DJP+94, CKR00], but this has no direct bearing on Color-EC since an instance of Color-EC with maximum hyperedge size $r \geq 2$ corresponds to an instance of Hyper-MC with maximum hyperedge size $r + 1 \geq 3$. Chekuri and Ene [CE11b] gave an $H_r$ approximation for Hyper-MC where $H_i$ is the $i$th harmonic number. However, this only implies a 1.833-approximation for Color-EC when $r = 3$, and is worse than a 2-approximation when $r \geq 4$. The approximation guarantees we obtain for small values of $r$ by rounding the Color-EC LP relaxation are therefore significantly
**Node-MC LP**: path constraints version

$$\begin{align*}
\text{min} & \quad \sum_{v \in V - T} w_v d_v \\
\text{s.t.} & \quad \sum_{v \in p} d_v \geq 1 \quad \forall p \in \mathcal{P} \\
& \quad d_v = 0 \quad \forall v \in T \\
& \quad d_v \geq 0 \quad \forall v \in V - T
\end{align*}$$

(14)

**Node-MC LP**: polynomial constraints version

$$\begin{align*}
\text{min} & \quad \sum_{v \in V - T} w_v d_v \\
\text{s.t.} & \quad y_i^u \leq y_i^u + d_v \quad \forall (u, v) \in E, \forall i \in [k] \\
& \quad y_{t_i}^i = 0 \quad \forall i \in [k] \\
& \quad y_{t_j}^j \geq 1 \quad \forall i, j \in [k], j \neq i \\
& \quad d_v \geq 0 \quad \forall v \in V - T
\end{align*}$$

(15)

Figure 6: Two equivalent ways to write the distance-based LP relaxation for node-weighted multiway cut [GVY04]. \(\mathcal{P}\) is the set of all paths between terminal nodes.

better and than results obtained by reducing to generalized multiway cut objectives. In contrast, when \(r\) is arbitrarily large, our approximation guarantee of \(2(1 - \frac{1}{k})\) for rounding the COLOR-EC LP relaxation matches the guarantee obtained by first reducing to NODE-MC and rounding the LP relaxation for this objective [GVY04]. We would like to better understand how these linear programs and their corresponding rounding strategies are related.

### 5.2 The Color-EC LP is Tighter than the Node-MC LP

Let \(G = (V, E)\) be a graph with terminal nodes \(T = \{t_1, t_2, \ldots, t_k\}\) and a weight \(w_v \geq 0\) for each node \(v \in V\). The distance-based LP relaxation for the NODE-MC objective on \(G\) is shown in Figure 6, where \(\mathcal{P}\) represents the set of all paths between pairs of terminal nodes. The formulation in (14) uses a single variable for each node, with the constraint that the distance between every pair of terminal nodes is at least 1. This requires an exponential number of path constraints. The bottom of Figure 6 shows an alternative way to write the LP using more variables but only polynomially many constraints and variables. The two LPs are known to be equivalent [GVY04]. The variable \(d_u\) provides an indication for how strongly we wish to delete node \(u\), and the variable \(y_i^u\) is interpreted as the distance between node \(u\) and cluster \(i\).

Consider now an instance of COLOR-EC encoded by an edge-colored hypergraph \(H = (V, E, C, \ell)\) that we reduce to a node-weighted graph \(G = (\hat{V}, \hat{E})\) using the strategy in Section 5.1. The node set \(\hat{V} = T \cup V \cup V_E\) is made up of terminal nodes \(T = \{t_i: i = 1, 2, \ldots, k\}\), the original node set \(V\) from hypergraph \(H\), and the node set \(V_E = \{v_e: e \in E\}\). In Figure 5, these are represented by square nodes, numbered circular nodes, and irregular shaped hyperedge-nodes, respectively. The edge set \(\hat{E} = E_H \cup E_T\) has two parts, defined
Let Proof. 

\[ E_H = \{(v, v_e) : v \in e \text{ in } H\} \]  
\[ E_T = \{(t_i, v_e) : \ell(e) = i \text{ in } H\}. \] 

(16)  
(17) 

The node weight for each \( u \in V \cup T \) is given by \( w_u = \infty \), and the node weight for \( v_e \in V_E \) is \( w_e \). Focusing specifically on the formulation shown in (15), we see that the NODE-MC LP shares some similarities with the canonical COLOR-EC relaxation. For example, both linear programs involve one variable for each node-color pair \((u, i) \in V \times C\). However, in terms of the hypergraph \( H = (V, E, C, \ell) \), the NODE-MC relaxation involves \( O(k|V|+k|E|) \) variables and \( O(k\sum_{e \in E} |e|) \) constraints overall, while the COLOR-EC relaxation has \( O(k|V|+|E|) \) variables and \( O(k|V|+\sum_{e \in E} |e|) \) constraints. Although the two linear programs both have an integrality gap of \( 2(1-\frac{1}{k}) \), the following theorem proves that the COLOR-EC relaxation will always be at least as tight as the lower bound obtained via the NODE-MC relaxation.

**Theorem 5.1.** The value of the NODE-MC relaxation on \( G = (\hat{V}, \hat{E}) \) is at most the value of the COLOR-EC relaxation on \( H = (V, E, C, \ell) \).

**Proof.** Let \( X = \{x_v^i, x_e : v \in V, e \in E, i \in [k]\} \) denote an arbitrary set of variables for the COLOR-EC LP relaxation for \( H \). Our goal is to use these to construct a set of feasible variables for the NODE-MC LP relaxation on graph \( G = (\hat{V}, \hat{E}) \) with the same objective score. This means we have to define \( y_v^j \) and \( d_v \) for each \( v \in \hat{V} \) and \( i \in [k] \). Since there are multiple different types of nodes in \( \hat{V} \), in order to simplify notation we will let \( y_v^j = y_v^{j_e} \) and \( d_v = d_v \) when we are considering a node \( v_e \in V_E \) that is associated with a hyperedge \( e \in E \). Define the following set of variables for the NODE-MC LP:

- Let \( d_v = 0 \) for \( v \in V \cup T \).
- For \( v_e \in V_E \), define \( y_v^j = \begin{cases} x_v & \text{if } \ell(e) = j \text{ in } H \\ 1 & \text{otherwise.} \end{cases} \)
- For \( v_e \in V_E \), define \( d_v = y_v^{j_e} \) where \( c = \ell(e) \in H \).
- For \( v \in V \), define \( y_v^i = x_v^i \) for every \( i \in [k] \).
- For \( t_i \in T \) where \( i \in [k] \), define \( y_v^{t_i} = 0 \), and \( y_v^{t_i} = \) 1 whenever \( i \neq j \).

The objective for the NODE-MC LP relaxation is then \( \sum_{v \in V} w_v d_v = \sum_{e \in E} w_e x_e \), which matches the COLOR-EC relaxation value. It only remains to check that the following distance constraints hold for all different types of edges \((u, v) \in \hat{E}\) and every color \( j \in [k] \):

\[ y_v^j \leq y_u^j + d_v \]  
\[ y_u^j \leq y_v^j + d_u. \] 

(18)  
(19) 

For \((v, v_e) \in E_H \), if \( \ell(e) = c \) then \( y_v^c = x_v^c \leq y_v^j = y_v^c + d_v \), since \( d_v = 0 \) for \( v \in V \). Therefore, constraint (18) holds in this case. Also, when \( \ell(e) = c \), constraint (19) holds because \( y_v^c = x_v^c = d_v \leq y_v^j = y_v^c + d_v \). If \((v, v_e) \in E_H \) but we consider a color \( i \neq c = \ell(e) \), then constraint (18) is satisfied because \( y_v^j = x_v^c \leq 1 = y_v^c = y_v^c + d_v \). Constraint (19) is satisfied as well since \( y_v^c = 1 \leq x_v^c \leq x_v^c \leq x_v^c + x_e = y_v^j + d_v \). Here we have used the fact that \( \sum_{i=1}^k x_v^i = 1 \) implies \( 1 \leq x_v^c + x_e \). Finally, for an edge \((v_e, t_e) \in E_T \) where \( c = \ell(e) \), constraints (18) and (19) become \( y_v^j \leq y_v^j + d_e \) and \( y_v^{t_e} \leq y_v^j \). If \( j \neq c \), then both of these hold because \( y_v^j = y_v^j = 1 \). Meanwhile, if \( j = c \), then they follow from \( y_v^j = d_e \) and \( y_v^{t_e} = 0 \).
Figure 7: Color-EC can be reduced to Vertex Cover by replacing each hyperedge with a node and adding an edge between nodes that represent overlapping hyperedges of different colors. A Vertex Cover problem on a graph $G$ can be reduced to an instance of Color-EC where all hyperedges have different colors by introducing a node for each edge in $G$, and a hyperedge for each node-neighborhood in $G$. When reducing from Color-EC to Vertex Cover and then back ($H \rightarrow G \rightarrow \hat{H}$), the starting hypergraph $H$ and ending hypergraph $\hat{H}$ are typically not the same.

**Strictly tighter instance.** Theorem 5.1 shows that the Color-EC relaxation is at least as tight of a lower bound as using the Node-MC relaxation. Although they have the same integrality gap, we can additionally show that the Color-EC relaxation can be strictly tighter. Consider color set $C = \{1, 2, 3\}$ and let $H = (V,E)$ be a star graph with center node $v_0$ and three leaf nodes $\{v_1, v_2, v_3\}$, where edge $e_i = (v_0, v_i)$ has color $i$ for $i = 1, 2, 3$. The optimal Color-EC solution will satisfy one of the edges and make mistake at the two other edges. The Color-EC LP relaxation will also have an optimal value of 2 by setting $x_{e_i}^i = 0$ for $i = 1, 2, 3$, and choosing either $x_{v_0}^i = \frac{2}{3}$ for every $i = 1, 2, 3$, or $x_{v_0}^1 = 1$ for exactly one $i \in \{1, 2, 3\}$. Meanwhile, if we reduce to an instance of Node-MC, each terminal node participates in one edge $(v_i, t_i)$ for $i \in \{1, 2, 3\}$. If we specifically consider the path constraints formulation of the Node-MC LP in (14), we can see that it is feasible to set $d_{e_i} = \frac{1}{2}$, leading to an optimal LP solution of $\frac{3}{2}$. This matches the worst case integrality gap.

6 Vertex Cover Equivalence and Linear-Time Algorithm

When $k$ and $r$ are both arbitrarily, Color-EC is approximation equivalent to Vertex Cover. This immediately leads to new hardness results as well as faster combinatorial approximation algorithms. For our results in this section, it is useful to view Color-EC as an edge deletion problem. Given an edge-colored hypergraph $H = (V,E,C,\ell)$, we say that two hyperedges $\{e,f\} \subseteq E$ are a bad hyperedge pair if $|e \cap f| > 0$ and $\ell(e) \neq \ell(f)$. Every node labeling $Y$ will make a mistake at at least one of these two edges. If we ensure that at least one hyperedge in each bad hyperedge pair is deleted, then we can color nodes in such a way that all remaining edges are satisfied. This view of Color-EC effectively describes a reduction to Vertex Cover: instead of choosing nodes to cover edges in a graph, we must delete edges to “cover” bad hyperedge pairs in a hypergraph. Figure 7 illustrates the reduction procedure from Color-EC to Vertex Cover, and then a reduction from Vertex Cover to Color-EC. We provide more details for each in the rest of the section.
6.1 Reduction from Vertex Cover and Hardness Results

Garg and Vazirani \[GVY04\] previously showed a straightforward approximation-preserving reduction from Vertex Cover to Node-MC. We can use a similar reduction to Color-EC that additionally provides refined hardness results in terms of the hyperedge size $r$.

**Theorem 6.1.** Let $G = (V, E)$ be a graph with maximum degree $\Delta$. There exists an edge-colored hypergraph $H$ with maximum degree $\Delta$ such that the Vertex Cover objective on $G$ is approximation-equivalent to the Color-EC objective on $H$.

**Proof.** See Figure 7. To construct the edge-colored hypergraph $H$, introduce a node $v_{ij}$ for each $(i, j) \in E$, and for each $u \in V$ define a hyperedge $e_u = \{v_{uj} : j \text{ where } (u, j) \in E\}$. Let each hyperedge be associated with its own unique color. Each node in $G$ corresponds to a hyperedge in $H$ and deleting a hyperedge in $H$ is equivalent to covering a node in $G$. Because all hyperedges have different colors, for every pair of overlapping hyperedges $(e_u, e_v)$ we know that either $e_u$ or $e_v$ must be deleted. This is equivalent to covering node $u$ or node $v$ when $(u, v) \in E$. By construction, the degree distribution in $G$ is exactly the hyperedge size distribution in $H$.

This result implies that an $\alpha$-approximation for Color-EC implies an $\alpha$-approximation for Vertex Cover. As immediate corollaries, it is NP-hard to approximate Color-EC to within a factor better than 1.3606 \[DS05\], and UGC-hard to approximate it to within a factor that is asymptotically better than 2 \[KR08\]. The latter result implies that our best approximation guarantees for Color-EC are asymptotically the best possible, assuming the unique games conjecture. It is worth noting that these hardness results apply when we treat $r$ and $k$ as arbitrary. This does not mean we cannot obtain improved guarantees for specific (e.g., small) values of $r$ and $k$, as we indeed already have in this paper.

Theorem 6.1 also tells us that an $\alpha$-approximation for rank-$r$ Color-EC directly implies an $\alpha$-approximation for Vertex Cover. By applying the results of Austrin, Khot, and Safra \[AKS11\], we know that it is UGC-hard to obtain a $2 - (2 + o_r(1))\frac{\ln \ln r}{\ln r}$ approximation for rank-$r$ Color-EC that applies for arbitrary $k$. This reduction does not imply any improved approximation algorithms for Vertex Cover in graph with maximum degree $\Delta$. Applying our LP-based algorithm for Color-EC would only lead to a $2 \left(1 - \frac{1 - 1}{\Delta + 1}\right)$-approximation for Vertex Cover with max degree $\Delta$, which is not as good as the $2(1 - \frac{1}{\Delta})$-approximation of Hochbaum \[Hoc83\], let alone the best known approximation factor of $2 - (1 - o(1))\frac{2\ln \ln \Delta}{\ln \Delta}$ by Halperin \[Hal02\].

6.2 Reduction to Vertex Cover and Combinatorial Algorithms

While there is an easy reduction from Vertex Cover to Node-MC, no reduction in the opposite direction has been shown. We show that a reduction in the other direction exists for Color-EC.

**Theorem 6.2.** Let $H = (V, E, C, \ell)$ be an edge-colored hypergraph. There exists a graph $G$ with $|E|$ nodes such that the Color-EC objective on $H$ is approximation-equivalent to Vertex Cover on $G$.

**Proof.** See Figure 7. To construct the graph $G$, first introduce a node $v_e$ for every hyperedge $e \in E$. If two hyperedges $e \in E$ and $f \in E$ define a bad hyperedge pair ($|e \cap f| > 0$ and $\ell(e) \neq \ell(f)$), then we add an edge $(v_e, v_f)$ in the graph $G$. If $e \in E$ has weight $w_e$, we
give node \(v_e\) weight \(w_e\) in \(G\). Deleting a minimum weight set of hyperedges in \(H\) to ensure remaining hyperedges of different colors do not overlap is equivalent to deleting a minimum weight set of nodes in \(G\) so that the remaining nodes form an independent set (i.e., the Vertex Cover problem).

Unlike in Theorem 6.1, the reduction in Theorem 6.2 does not imply any relationship between the node degrees in the instance of Vertex Cover and the hyperedge sizes in the instance of Color-EC. Therefore, this theorem does not imply anything about approximation algorithms or hardness results for rank-\(r\) Color-EC or Vertex Cover with maximum degree \(\Delta\). However, this reduction immediately allows us to obtain fast combinatorial 2-approximation algorithms for Color-EC. A naive approach to obtaining a 2-approximation is to use the reduction in Theorem 6.2 to explicitly convert the edge-colored hypergraph \(H = (V, E)\) into a graph \(G = (V_G, E_G)\) based on bad hyperedge pairs in \(H\), and then apply an existing combinatorial Vertex Cover algorithm as a black-box. Visiting each node \(v \in V\) and then iterating through all pairs of hyperedges incident to \(v\) in \(H\) takes \(O(\sum_{v \in V} d_v^2)\) time where \(d_v\) is the degree of node \(v\). The fastest algorithms for Vertex Cover, both in the weighted and unweighted case, take linear-time in terms of the number of edges in the graph \cite{Pit85, BYE81, HLL18, BYE85}. Because there could be up to \(O(|E|^2)\) edges in the reduced graph \(G\), this basic approach has a runtime of \(O(|E|^2 + \sum_{v \in V} d_v^2)\).

### 6.3 Linear-Time 2-Approximation via Implicit Pitt’s Algorithm

We would ideally like to develop an algorithm for Color-EC that is not just linear in terms of the number of edges in some reduced graph, but is linear in terms of the original hypergraph size, i.e., \(O(\sum_{e \in E} |e|)\). This may seem inherently challenging, given that existing Vertex Cover algorithms rely on iterating through all edges in the Vertex Cover instance. However, this can be accomplished using a careful implicit implementation of a Vertex Cover algorithm. Pitt’s algorithm \cite{Pit85} (Algorithm 2) is a simple randomized 2-approximation for weighted Vertex Cover that iteratively visits edges in a graph \(G = (V_G, E_G)\). Each time it encounters an uncovered edge, it randomly samples one of the two endpoints to include in the cover. The edges in \(G\) can be visited in an arbitrary order. If \(G = (V_G, E_G)\) is obtained by reducing an instance of Color-EC \(H = (V, E, C, \ell)\) to Vertex Cover, then every node in \(G\) corresponds to an edge in \(H\), and an edge in \(G\) corresponds to a bad hyperedge pair in \(H\). We will specifically design an implicit implementation of Pitt’s algorithm that can be applied directly to \(H\) without explicitly forming \(G\).

#### Additional notation.

In order to provide a clear description and analysis, assume a fixed ordering of edges \(E = \{e_1, e_2, \ldots, e_{|E|}\}\). For notational simplicity, let \(w(i) = w_{e_i}\) and \(\ell(i) = \ell(e_i)\) denote the weight and color of the \(i\)th edge, respectively. For each node \(v \in V\), using a slight abuse of notation let \(v(j)\) denote the index of the \(j\)th edge that is adjacent to \(v\). These edge indices will be stored in a list \(L_E(v) = [v(1), v(2), \ldots, v(d_v)]\) where \(d_v\) is the degree of \(v\). For example, if node \(v\) is in three hyperedges \(\{e_2, e_5, e_9\}\), then \(L_E(v) = [2, 5, 9]\). One crucial assumption that we make is that edges are ordered by color, so that in \(L_E(v)\), all of the indices for edges of color 1 come first, then edges with color 2, etc. If the hypergraph is not stored in this format, it can easily be re-arranged in \(O(\sum_{e \in E} |e|)\) time to satisfy this property. Because Color-EC is equivalent to an edge-deletion problem, we maintain a set \(D\) of hyperedge indices to delete. This corresponds to a vertex cover in
Algorithm 2 PittVertexCover(G)

Input: $G = (V_G, E_G)$ with node weight $w_v \geq 0$ for each $v \in V_G$
Output: Vertex cover $C \subseteq V$.
$C \leftarrow \emptyset$ // Initialize empty vertex cover
for $(u, v) \in E_G$ do
  if $u \notin C$ and $v \notin C$ then
    Generate uniform random $\rho \in (0, 1)$
    if $\rho < \frac{w_u}{w_u + w_v}$ then
      $C \leftarrow C \cup \{u\}$
    else
      $C \leftarrow C \cup \{v\}$
    end if
  end if
end for
Return $C$

Once we have implicitly stored an approximate vertex cover in the form of an edge set $D$, we will iterate through all edges in $H$, and for each $e \in E - D$ we assign all nodes in $e$ to have color $\ell(e)$. This final procedure takes $O(\sum_{e \in E} \mid e \mid)$ time, so our goal is to efficiently determine a set $D$ encoding a 2-approximate vertex cover in $G$.

Implementing Pitt’s algorithm implicitly. Iterating through edges in $G$ is equivalent to iterating through bad hyperedge pairs in $H$. A bad hyperedge pair is “covered” if one of the hyperedges is added to $D$. Because the ordering of edges $E_G$ in Algorithm 2 is arbitrary, we can traverse bad hyperedge pairs in any way that is convenient. We choose to iterate through nodes, and for each node $v \in V$ we will consider all bad hyperedge pairs that contain $v$ in their intersection. Explicitly visiting all pairs of hyperedges incident to $v$ would take $O(d_v^2)$ time. However, deleting an edge will cover multiple bad hyperedge pairs at once, so we will be able to “skip” many pairs to speed up the process. To accomplish this, we maintain pointers to the front ($f$) and back ($b$) of $L_E(v) = [v(1), v(2), \cdots, v(d_v)]$ which we initialize to $f = 1$ and $b = d_v$. Unless $v$ is only adjacent to nodes of one color, $(e_{v(f)}, e_{v(b)})$ will be a bad hyperedge pair that needs to be covered. Using Pitt’s technique, we randomly sample one edge to delete based on its weight. If we delete $e_{v(f)}$, then we no longer need to consider any other bad hyperedge pairs involving this edge, so we update $f \leftarrow f + 1$ and consider the next edge in the list. Otherwise, we delete edge $e_{v(b)}$ and decrement the back pointer: $b \leftarrow b - 1$. If at any point we encounter an edge that was added to $D$ previously, we move past it by incrementing $f$ or decrementing $b$. We stop this procedure when $\ell(v(f)) = \ell(v(b))$. Even if $f \neq b$, our assumption that edges are ordered by color means that there are no more bad hyperedge pairs containing $v$ to consider. Figure 8 is an illustration of this process. Since we update either $f$ or $b$ in each step, and each step involves $O(1)$ operations, covering all bad hyperedge pairs containing $v$ takes $O(d_v)$ time. The full algorithm is given in Algorithm 3, and we end with a summarizing theorem.

Theorem 6.3. Algorithm 3 is a randomized 2-approximation algorithm for weighted COLOR-EC that runs in $O(\sum_{v \in V} d_v) = O(\sum_{e \in E} \mid e \mid)$ time.

Our strategy for iterating through bad hyperedge pairs can also be used in conjunction with other 2-approximation algorithms for VERTEX COVER. For example, if edges are
Figure 8: An illustration of our procedure for efficiently covering bad hyperedges containing a single node $v$. (A) Node $v$ is contained in 5 hyperedges and 8 bad hyperedge pairs. (B) We consider edges at opposite ends of $v$’s edge list $L_E(v)$, ordered by color. Edges $e_{v(1)}$ and $e_{v(5)}$ define a bad hyperedge pair. (C) Given a bad hyperedge pair, we sample one to delete. (D) Once one edge is deleted, all bad hyperedge pairs involving the edge are covered, so we no longer need to consider it. In the example, after deleting $e_{v(1)}$, we can advance the pointer $f$ to consider the next hyperedge in $L_E(v)$.

unweighted, every time we visit an uncovered bad hyperedge pair in the list $L_E(v)$ we can instead delete both edges and update both pointers $f$ and $b$. This implicitly implements the standard strategy of finding a maximal matching and adding all nodes in the matching to a vertex cover. This does not apply to edge-weighted COLOR-EC but provides a deterministic 2-approximation for the unweighted version.

7 Conclusion and Discussion

This paper settles a number of open questions on the approximability of the Colored-Edge Clustering objective in hypergraphs. We not only have improved the best approximation factors for this objective based on linear programming, but have also shown that our approximations are tight in various ways. The integrality gap result shows that it is impossible to improve on the given $\min\{2−2/k, 2−2/(r+1)\}$-approximation using the canonical linear programming relaxation. The reduction from VERTEX-COVER shows that it is impossible to obtain an approximation factor that is asymptotically better than 2, assuming the unique games conjecture.

One open question that remains is to determine whether alternative technique could lead to improved approximations for fixed values of $r$ or $k$. Previous research has shown that the optimal approximation factor for node-weighted multiway cut and hypergraph multiway cut coincides with the integrality gap of their LP relaxation, assuming the unique games conjecture [EVW13]. In other words, it is UGC-hard to improve on the best approximation factor of $2(1−1/k)$ for these problems. An analogous UGC-hardness result has been shown for the graph multiway cut objective [MNRS08]. One interesting question is whether a similar result can also be shown for COLOR-EC and its canonical LP relaxation. On a related note, another open direction is to better understand the relationship between the COLOR-EC canonical LP and other relaxations for more general objectives, such as the Lovász relaxation for submodular multiway partition [CE11b] or the basic LP for minimum constraint satisfaction problems [EVW13]. Finally, although our $\frac{4}{3}$-approximation algorithm for the $r = 2$ case matches the integrality gap and is therefore an optimal rounding strategy, the proof of our approximation guarantee is quite involved and relies on a lengthy case analysis. Section 3.5 highlights why this case analysis is challenging to overcome, but
Algorithm 3 PittColorEC(H)

Input: Edge-colored hypergraph $H = (V, E, C, \ell)$ with weight $w(e) \geq 0$ for each $e \in E$

Output: Set $D \subseteq E$ so that $H = (V, E - D, C, \ell)$ has no bad hyperedge pairs

$D \leftarrow \emptyset$

for $v \in V$ do

$L_E(v) = [v(1) \ v(2) \ \cdots \ v(d_v)]$ // Indices for $v$’s edges, ordered by color

$f = 1, b = d_v$ // Pointers to front/back of index array

while $\ell(v(f)) \neq \ell(v(b))$ do

Generate $\rho \in (0, 1)$

if $\rho < \frac{w(v(f))}{w(v(f)) + w(v(b))}$ then

$D \leftarrow D \cup \{e_v(b)\}$

while $e_v(b) \in D$ and $b > f$ do

// Ignore any deleted edges

$b \leftarrow b - 1$

end while

else

$D \leftarrow D \cup \{e_v(f)\}$

while $e_v(f) \in D$ and $b > f$ do

// Ignore any deleted edges

$f \leftarrow f + 1$

end while

end if

end while

end for

Return $D$

this does not rule out the possibility of obtaining a $\frac{4}{3}$-approximation with a simpler proof using an alternative rounding strategy or a different approach for proving an approximation guarantee.

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A Dual Variables and LP Bounds for $r = 2$

In Section 3.4 of the main text we proved that when $q = 1$, the solution to the linear program in Figure 3 is bounded above by $\frac{1}{2}$. We accomplished this by multiplying constraints in this LP by dual variables presented in Table 1. For the sake of completeness, here we provide additional details for how to prove an upper bound on the LP solution for all values of $q \in \{1, 2, 3, 4, 5, 6\}$, without having to explicitly write down and sum a linear combination of constraints. The upper bound can be shown more succinctly by checking a few matrix-vector products, which is the approach we take here. We also provide dual variables for Figure 4 for all valid choices of $p$ and $q$, which can be used to prove the desired upper bound on this LP in the same fashion.

A.1 Overview of LP bounding strategy

The linear programs in Figures 3 and 4 can both be written in the following format:

$$\max \quad c^T x$$
$$\text{s.t.} \quad Ax \leq b.$$  \hspace{1cm} \text{(20)}

The dual of this LP is given by

$$\min \quad b^T y$$
$$\text{s.t.} \quad A^T y = c$$
$$y \geq 0.$$  \hspace{1cm} \text{(21)}

We want to prove that $c^T x^* \leq \frac{1}{2}$ where $x^*$ is the optimal solution to the primal LP (20). By weak duality theory, it is sufficient to find a vector $y$ of dual variables satisfying the constraints in the dual LP (21), such that $b^T y \leq \frac{1}{2}$. Note that our goal is actually to prove that this can be done for multiple slight variations of the objective function vector $c$ and constraint matrix $A$. For all cases we have the same right hand side vector $b$. If
\{c_1, c_2, \ldots, c_j\} denotes the set of objective function vectors, \{A_1, A_2, \ldots, A_j\} is the set of corresponding constraint matrices, and \{y_1, y_2, \ldots, y_j\} is an appropriately chosen set of dual vectors, then for i = 1, 2, \ldots, j, we need to perform one matrix-vector product and one vector inner product to confirm that

\[
A_j^T y_j = c_j
\]

\[
b^T y_j \leq \frac{1}{2}.
\]

A.2 Matrix computations for bounding LP in Figure 3

For the linear program in Figure 3, the variable vector is

\[
x^T = [\omega_1 \omega_2 \omega_3 \omega_4 \omega_5 \omega_6 \chi]
\]

and the right hand size vector is

\[
b^T = [0 0 0 0 0 1 1 1 -2 0].
\]

The dual variables we will use are given in Table 1. For each \(q \in \{1, 2, \ldots, 6\}\), let \(y_q\) denote the vector of dual variables (which corresponds to the \(q\)th row in Table 1), and let

\[
Y = \begin{bmatrix} y_1 & y_2 & \cdots & y_6 \end{bmatrix},
\]

so \(Y\) is the matrix of dual variables. In order to check that \(b^T y_q \leq \frac{1}{2}\) for each choice of \(q\) we just need to compute \(Yb\), which is made easier by the fact that most entries of \(b\) are zero.

\[
Yb = \begin{bmatrix}
\frac{1}{2} & 0 & 0 & 0 & \frac{1}{16} & 1
\frac{1}{2} & 0 & 0 & \frac{5}{12} & \frac{1}{2}
0 & \frac{1}{12} & 0 & 0 & \frac{5}{48} & \frac{1}{2}
0 & 0 & \frac{1}{12} & \frac{1}{30} & 0 & \frac{5}{48} & \frac{1}{24}
0 & 0 & \frac{1}{12} & \frac{1}{30} & \frac{1}{2} & 0 & \frac{5}{48} & \frac{1}{24}
0 & 0 & \frac{1}{12} & \frac{1}{30} & \frac{1}{2} & 0 & \frac{5}{48} & \frac{1}{24}
\end{bmatrix} \begin{bmatrix} 1 \ 1 \ -2 \end{bmatrix} = \begin{bmatrix} \frac{3}{8} \\
\frac{1}{2} \\
\frac{11}{24} \\
\frac{11}{24} \\
\frac{11}{24} \\
\frac{29}{54} \end{bmatrix}
\]

This confirms that as long as these dual variables are feasible for the dual linear program, the primal LP is always bounded above by \(\frac{1}{2}\). So, we just need to confirm that \(A_q^T y_q = c_q\) for each \(q\).

The constraint matrix \(A_q\) is nearly the same for all choices of \(q\). Only the last constraint \((\omega_q - \frac{7}{8} \chi \leq 0)\) is dependent on \(q\). The first nine rows of the constraint matrix are always
given by

\[
A_q(1:9,:) = \begin{bmatrix}
1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & -1 & -1 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & -3 & 0 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}.
\]

The last row has two entries: \(A_q(10,7) = -\frac{7}{8}\) and \(A_q(10,q) = 1\). The objective function is

\[
c_q^T x = \frac{7q}{8(q+1)} x - \sum_{j=1}^{q} \frac{1}{j(j+1)} \omega_j,
\]

where \(c_q\) is the objective function vector for \(q \in \{1,2,\ldots,6\}\). The matrix \(C = \begin{bmatrix} c_1 & c_2 & \cdots & c_6 \end{bmatrix}\) of objective function vectors is given by:

\[
C = \begin{bmatrix}
-\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
0 & -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} \\
0 & 0 & -\frac{1}{12} & -\frac{1}{12} & -\frac{1}{12} & -\frac{1}{12} \\
0 & 0 & 0 & -\frac{1}{20} & -\frac{1}{20} & -\frac{1}{20} \\
0 & 0 & 0 & 0 & -\frac{1}{30} & -\frac{1}{30} \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{42} \\
\frac{7}{16} & \frac{7}{12} & \frac{21}{32} & \frac{7}{19} & \frac{35}{38} & \frac{3}{4}
\end{bmatrix}
\]

Given the matrices and vectors listed above, a few simple computations confirm that \(A_q^T y_q = c_q\) indeed holds for every \(q \in \{1,2,\ldots,6\}\).

A.3 Dual variables for LP in Figure 4

Tables 2 and 3 give optimal dual variables for the linear program in Figure 4 for all possible values of \(p\) and \(q\). We also list the optimal solution to each linear program for each of the 40 cases, which is always less than or equal to \(\frac{1}{2}\). We highlight again that our proof does not rely on proving that this value is optimal—it suffices to use the dual variables to get a linear combination of inequalities such that summing the left hand sides produces the LP objective function, and summing the right hand sides yields the LP upper bound given in the last column of each row. We can use the strategy outlined in the previous subsection to check these dual variables and prove the upper bound by computing a matrix-vector product and a vector inner product for each case. We omit detailed calculations as they are straightforward but would take up a significant amount of space.
Table 2: Optimal dual variables, one for each of the 16 constraints, for the linear program in Figure 4, for each valid pair \((p, q)\) where \(p \in \{1, 2\}\). We omit the column for constraint number 6, \(\omega_6 - \omega_7 \leq 0\), since the dual variable for this constraint is always zero for these cases. The last column is the bound on the LP objective (it is in fact the optimal LP solution value) that we can prove for each case using the dual variables. It is always \(\frac{1}{2}\) or less.

| Constraints | LP bound |
|-------------|----------|
| \(p, q\)   |          |
| 1, 1        | 3/7      |
| 2           | 1/2      |
| 3           | 31/64    |
| 4           | 1/2      |
| 5           | 37/75    |
| 6           | 125/252  |
| 7           | 2003/4032 |
| 8           | 94/189   |
| 9           | 2509/5040 |
| 10          | 1381/2772 |
| 2, 2        |          |
| 3           | 31/64    |
| 4           | 1/2      |
| 5           | 37/75    |
| 6           | 125/252  |
| 7           | 2003/4032 |
| 8           | 94/189   |
| 9           | 2509/5040 |
| 10          | 1381/2772 |
Table 3: Optimal dual variables, one for each of the 16 constraints, for the linear program in Figure 4, for each pair \((p,q)\) when \(p \in \{3,4,5\}\). We omit columns for constraint 1 \((\omega_1 - \omega_2 \leq 0)\) and constraint 10 \((\chi - \omega_2 \leq 1)\), as dual variables for these are always zero for these cases.

| Constraints | \(p\), \(q\) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | LP bound |
|-------------|--------------|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|-----------|
| \(\omega_1 \leq 1)\) | 3, 3 | 0 | 0 | 5 | 64 | 0 | 0 | 5 | 64 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 31 64 |
| \(\omega_2 \leq 2)\) | 4 | 0 | 1/20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1/10 | 0 | 0 | 0 | 0 | 0 | 0 | 37 75 |
| \(\omega_2 \leq 1)\) | 5 | 0 | 1/30 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 2 |
| \(\omega_2 \leq 1)\) | 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 125 252 |
| \(\omega_2 \leq 1)\) | 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2003 4032 |
| \(\omega_2 \leq 1)\) | 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 94 188 |
| \(\omega_2 \leq 1)\) | 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2599 5030 |
| \(\omega_2 \leq 1)\) | 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1381 2772 |
| \(\omega_1 \leq 1)\) | 4, 4 | 1/10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1/10 | 0 | 0 | 0 | 0 | 0 | 0 | 1/2 |
| \(\omega_1 \leq 1)\) | 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 157 320 |
| \(\omega_1 \leq 1)\) | 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 55 112 |
| \(\omega_1 \leq 1)\) | 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 63 128 |
| \(\omega_1 \leq 1)\) | 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 71 144 |
| \(\omega_1 \leq 1)\) | 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 79 160 |
| \(\omega_1 \leq 1)\) | 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 87 176 |
| \(\omega_1 \leq 1)\) | 5, 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 31 64 |
| \(\omega_1 \leq 1)\) | 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 22 45 |
| \(\omega_1 \leq 1)\) | 7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 41 85 |
| \(\omega_1 \leq 1)\) | 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 84 168 |
| \(\omega_1 \leq 1)\) | 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 309 640 |
| \(\omega_1 \leq 1)\) | 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 851 1700 |

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