THE INVENTORY REPLENISHMENT POLICY IN AN UNCERTAIN PRODUCTION-INVENTORY-ROUTING SYSTEM

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Abstract. This study introduces an uncertain programming model for the integrated production routing problem (PRP) in an uncertain production-inventory-routing system. Based on uncertainty theory, an uncertain programming model is proposed firstly and then transformed into a deterministic and equivalent model. The study further probes into different types of replenishment policies under the condition of uncertain demands, mainly the uncertain maximum level (UML) policy and the uncertain order-up to level (UOU) policy. Some inequalities are put forward to define the UML policy and the UOU policy under the uncertain environments, and the influences brought by uncertain demands are highlighted. The overall costs with optimal solution of the uncertain decision model grow with the increase of the confidence levels. And they are simultaneously affected by the variances of uncertain variables but rely on the value of confidence levels. Results show that when the confidence levels are not less than 0.5, the cost difference between the two policies begins to narrow along with the increase of the confidence levels and the variances of uncertain variables, eventually being trending to zero. When there are higher confidence levels and relatively large uncertainty in realistic applications, in which the solution scale is escalated, being conducive to its efficiency advantage, the comprehensive advantages of the UOU policy is obvious.

1. Introduction. Efficient and sustainable use of resources can only be achieved by explicitly considering the interdependencies of integrated supply-chain services [42]. To promote the integrated efficiency of supply chain and the sustainable utilization

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of resources, a number of research achievements have been made by studies on the integration decision of the supply chain in the years since Chandra [15] first discussed the benefits of integrating production and distribution decisions in 1993. Bertazzi’s [12] research showed that with the level of integration becoming higher, and the integration competitive advantages becoming more and more powerful, the overall cost of the collaboration-supply chain, including production cost, inventory cost, and transportation cost, is decreasing. Which is more representative of the integrated PRP in a production-inventory-routing system, where one commodity is produced at a plant and shipped to several retailers (points of sale) by a fleet of vehicles over a finite time horizon. The goal of an integrated PRP is to determine the operation schedules needed to coordinate the production, inventory, distribution, and routing operations such that the retailers’ demands, the plant’s production constraints, the plant and retailers’ inventory constraints, and the transportation constraints are all satisfied while the overall operation cost (i.e., the sum of the production, inventory, and transportation costs) over a given planning horizon is minimized. The integrated PRP can be viewed as the integration of the vehicle scheduling problem (VRP) and the lot sizing problem (LSP), which are both classic scheduling optimization problems in the supply chain. Among all the integration patterns of the supply chain, the production-inventory-routing integration pattern of the PRP has a relatively high level of integration, having received more attention from scholars.

The integrated PRP was first put forward by Lei [27], who built a basic programming model and then designed an intelligence algorithm to solve the proposed model. He also pointed out that the production-inventory-routing integration pattern has the characteristic of higher level and significant competitive superiority of integration. Then, Nananukul [38], and later Bard and Nananukul [8, 9], put forward a more formidable standard model for solving the integrated PRP. Based on their groundbreaking research, many scholars have delved into this issue in the last decade. These scholars have conducted studies from different angles, such as proposing model realization solutions and effective algorithms to solve the basic model proposed by Nananukul [9], carrying out comparative studies on integrated and non-integrated situations, and considering rich PRP.

Compared to VRP or LSP, more constraints are introduced in the integrated PRP, which increases the difficulty of solving the optimization models. A large number of studies were conducted to better solve the PRP in the last few years, including introducing a wide variety of algorithms. Some intelligence algorithms, such as tabu search algorithm, two-stage iterative heuristic algorithm, and variable neighborhood search algorithm, were developed by Boutarfa [13] [1] and Qiu [40].

Some papers lay emphasis on analyzing the level of integration in the supply chain, its correlative factors, and the corresponding integration advantages [46] [11] [7] [17]. To validate the competitive advantages of the production-inventory-routing pattern in resource utilization, cost savings, and computing efficiency, some papers carried out comparative analyses between the non-integrated optimization process and the sequential decision, and the coordination optimization process [23] [2] [18].

On the topic of rich PRP, various scholars have developed a number of optimization models and algorithms, and taken the situations of multi-product, multi-producer, and multi-vehicle into consideration [14] [48] [10] [41] [43] [29] [22]. Besides, time windows, transshipment between retailers, and clusters for retailers have been
considered by some researchers [25][4][20][26][28][47][36][39]. With the concern of environmental protection, some scholars are interested in the integrated PRP of green supply chains. In some studies, reverse logistics and green remanufacture are introduced during model-building. The costs caused by carbon emissions and energy consumption are also taken into account in the overall cost consideration in some PRPs [37][19].

As research on the integrated PRP has progressed in recent years, some scientists have paid more attention to the integrated PRP under uncertain environments, such as those characterized by randomness and fuzziness, bringing the models closer to the real situation. Some researchers, such as Solyali [45], Adulyasak [3], Agra [5], Ghasemkhani [21], and Liu [34], studied collaborative optimization problems in a random environment, whereas Moon [37] focused on the PRP under a fuzzy environment. Probability theory is applicable for randomness when samples are available and it is also the common legitimate approach to deal with some randomness situation. However, there are many situations lacking historical data that can be referred to, such as a new product entering the market, or product promotion in a special period. Practically, experience data (belief degrees) are often used to estimate the distributions in cases without historical examples by experienced experts. Uncertainty theory, initiated by Liu [31] based on normality, duality, sub-additivity, and product axioms, can be introduced to deal with variables estimated by human belief degrees. In summary, the world is neither random nor fuzzy, but sometimes it can be analyzed by probability theory when samples are available, and sometimes by uncertainty theory when there’s lack of historical data. Uncertainty theory is mainly used to characterize human belief degree and deal with subjective uncertainty. It has been successfully used to solve many uncertain optimization decision-making problems in supply chains [16][44][24][35][49]. This paper focuses on the integrated PRP with multiple vehicles and analyzes different types of replenishment policies in this uncertain setting, which, to the best of the authors’ knowledge, has never before been considered.

In this paper, we consider the situation for the shipment from the plant to multiple retailers by heterogeneous vehicles and build an uncertain programming model. Not only does the model consider uncertain demands but it also takes series uncertain costs into consideration. Previous studies rarely accounted for both simultaneously. The objective of integrated decision planning in uncertain circumstances is to minimize the sum of the business costs, including production, inventory, and delivery costs, at a predefined confidence level by integrating various operation policies (i.e., the production policy, the transportation policy, and the replenishment policy).

Further, we focus on the replenishment policy under uncertain environments, which have huge and complicated impacts after introducing uncertain demand as compared to the traditionally determinate situation. It is important that the vendor managed inventory policy (VMI) is usually adopted in the integration decision of the supply chain, in which the retailer inventory is managed uniformly by the center supplier or manufacturer, and, on the basis of the inventory level, the follow-up production plans and replenishment plans are made accordingly. Archetti [6] specialized in the analysis of the common VMI policy in a certain production-distribution system, namely the ML policy, and the OU policy. Under the condition that the demand is uncertain, the level of the inventory at each retailer at the end of each period is no longer a determined variable but rather an uncertain variable,
which we cannot depict in a traditional manner. In order to improve the research of replenishment policy under uncertain environments, the concept of credibility initiated by Liu [31] is introduced, and two uncertain replenishment policies analogous to the ML and OU policy are defined. Furthermore, using mathematical deduction and logic inference, some analysis is obtained and then validated by experiments. 

The outline of the rest of the paper is as follows. In Sect. 2, we describe the problem and then build a programming model. Then, the process of crisp equilibrium transformation for the above model by the inverse criteria is performed in Sect. 3. Next, in Sect. 4, we define the replenishment policies under uncertain environments and make further analyses. Finally, computational results demonstrate the correctness of analyses in Sect. 5, and then draw some propositions and managerial insights. The paper is concluded in Sect. 6.

2. Problem description and model. We investigate the integrated PRP over a planning horizon discredited in time periods under uncertain environments. More specifically, in each period, a single commodity is made available at a single plant and then transported to multiple retailers (points of sale) by multiple vehicles. To better illustrate the problem, we can name the pattern single-commodity, single-plant, and multiple-retailer, multiple-vehicle SSMM. There exist some constraints in all business processes of SSMM of the PRP. For example, the production quantity must not be greater than the production capacity of the plant in each period, which is the producing ability limit. Regarding the transportation aspect, the vehicle starts and ends its route at the plant, and performs at most one route in each time period. A load of each vehicle cannot exceed its capacity. Besides, we also take storage constraints into consideration. A commodity can be stored at the plant and the retailers. The upper and lower limits reflect the amounts of inventory that can be stored at the plant and each retailer at the end of each period. Traditionally, the lower limit is usually set to zero to indicate that shortage is not permissible in a deterministic case where the demand of each retailer in each time period is predetermined. The upper limit represents the storage capacity.

The SSMM of the PRP is defined on a graph $G = (N_0, E)$, where $N_0$ is the set of nodes (the plant and retailer nodes), and $E$ is the arc set. We denote by 0 the plant and by $N$ the sets of retailer nodes. The set of time periods within the planning horizon is denoted by $T = \{1, 2, ..., \tau\}$. Over a finite set of time periods, a single product can be produced at the plant and delivered by a set of vehicles $K = \{1, 2, ..., m\}$ to the retailers. For ease of reference, the notation and parameters are summarized in Tables 1 and 2. The problem consists of deciding, for each time period, whether to produce at the plant, the quantity of production, how much to deliver to each retailer, and the route of each vehicle. The objective is to minimize the sum of production costs, inventory costs, and transportation costs at a certain confidence level. The decision variables are respectively summarized in Table 3.
Table 2. Parameters

| Parameter | Description |
|-----------|-------------|
| $\tilde{d}_{it}$ | Uncertain demand at retailer $i$ in period $t$. |
| $f$ | Uncertain fixed production setup cost. |
| $\tilde{u}$ | Uncertain unit production cost. |
| $\tilde{h}_i$ | Uncertain unit inventory holding cost at the plant or retailer. |
| $\tilde{c}_{ij}$ | Uncertain transportation cost from node $i$ to node $j$. |
| $C$ | Production capacity of the plant. |
| $m$ | The number of vehicles. |
| $Q_k$ | Capacity of vehicle $k$. |
| $B_i$ | Initial inventory at retailer $i$, where 0 corresponds to the plant. |
| $L_i$ | Maximum inventory level at the plant and retailers. |
| $\alpha$ | Confidence level about uncertain costs. |
| $\beta_i$ | Confidence level of node $i$ (satisfaction degree of uncertain demands). |
| $\gamma_i$ | Confidence level of node $i$ (satisfaction degree of uncertain demands). |

Table 3. Decision variables

| Decision variable | Description |
|-------------------|-------------|
| $z_t$ | Equal to 1 if there is production at the plant in period $t$, 0 otherwise. |
| $p_t$ | Production quantity in period $t$. |
| $x_{ijkt}$ | Equal to 1 if vehicle $k$ travels directly from node $i$ to node $j$ in period $t$, 0 otherwise. |
| $w_{ikt}$ | Load of vehicle $k$ immediately before making a delivery to retailer $i$ in period $t$. |
| $q_{ikt}$ | Quantity delivered to retailer $i$ by vehicle $k$ in period $t$. |
| $y_{ikt}$ | Equal to 1 if node $i$ is visited by vehicle $k$ in period $t$, 0 otherwise. |

Because expectations criteria cannot satisfy all of the needs in practical applications, decision makers may be more interested in the minimum cost $\bar{W}$ at the confidence level $\alpha$. What’s more, the confidence level $\beta_i$ reflects the level of need to the uncertain demand of retailer $i$. The confidence levels are between 0 and 1. The uncertain programming model is proposed as follows. The purpose of objective function (A1) with constraint (A2) is to minimize the overall cost, including production, setup, inventory, and routing costs, at the confidence level $\alpha$ under uncertain environments. Constraints (A3)-(A5) represent the lot-sizing part of the problem. Constraint (A3) are the inventory flow constraints at the plant, which reflect that stock-out is not allowed. The uncertain demand of retailer $i$ at time period $t$ must be at least satisfied at the confidence level $\beta_i$ in (A4). Constraint (A5) are the setup forcing and production capacity constraints. The constraint forces the setup variable to be one if production takes place in a given period and limit the production quantity to less than the production capacity. The remaining constraints, i.e., (A6)-(A11), are the vehicle loading and routing restrictions. Constraint (A6) limit the delivery quantities, which must be less than the capacity of the vehicle. Constraint (A7) reflect that each retailer can be visited by at most one vehicle in each period. Constraint (A8) builds a bridge between variables $\{x_{ijkt}\}$ and $\{y_{ikt}\}$. Constraint (A9) is used for vehicle flow conservation. Constraint (A10) is the vehicle loading restrictions and subtour-elimination constraints in the form...
of the Miller-Tucker-Zemlin inequalities. \( M \) is a maximum number. Finally, Constraint (A11) represents the load of vehicle \( k \) immediately before making a delivery to retailer \( r \) in period \( t \) must be less than its capacity.

\[
\begin{align*}
\min \quad & W \\
\text{s.t.} \quad & \mathcal{M} \left\{ \sum_{t \in T} \left( \tilde{f}_t + \alpha p_t + \tilde{h}_0 \right) \left( B_0 + \sum_{t=1}^{i=t} p_t - \sum_{k \in K} t \sum_{i=1}^{q_{ikt}} \right) + \sum_{i \in N} \tilde{h}_i \right. \\
& \left. \left( B_i + \sum_{k \in K} \sum_{t=1}^{i=t} q_{ikt} - \sum_{t=1}^{i=t} \tilde{d}_it \right) + \sum_{(i,j) \in E} \sum_{k \in K} c_{ij} x_{ijkt} \right\} \leq W, \quad (A2) \\
& B_0 + \sum_{t=1}^{i=t} \left( p_t - \sum_{i \in N} \sum_{k \in K} q_{ikt} \right) \geq 0, \forall t \in T, \quad (A3) \\
& \mathcal{M} \left\{ B_i + \sum_{k \in K} \sum_{t=1}^{i=t} q_{ikt} - \sum_{t=1}^{i=t} \tilde{d}_it \geq 0 \right\} \geq \beta_i, \forall i \in N, \forall t \in T, \quad (A4) \\
& p_t \leq Cz_t, \forall t \in T, \quad (A5) \\
& \sum_{i \in N} q_{ikt} \leq Q_k y_{ikt}, \forall i \in N, \forall t \in T, \quad (A6) \\
& \sum_{k \in K} y_{ikt} \leq 1, \forall i \in N, \forall t \in T \quad (A7) \\
& \sum_{i \in N, i \neq j} x_{ijkt} = y_{ikt}, \forall j \in N, \forall t \in T, \forall k \in K, \quad (A8) \\
& \sum_{i \in N, i \neq j} x_{ijkt} = \sum_{i \in N, i \neq j} x_{ijkt}, \forall j \in N, \forall t \in T, \forall k \in K, \quad (A9) \\
& w_{ikt} \leq w_{ikt} - q_{ikt} + M \left( 1 - x_{ijkt} \right), \\
& \forall i \in N, \forall j \in N_0, \forall k \in K, \forall t \in T, (A10) \\
& w_{ikt} \leq Q_k y_{ikt}, \forall i \in N, \forall k \in K, \forall t \in T, \quad (A11) \\
& p_t, q_{ikt}, w_{ikt} \geq 0, \forall i \in N, \forall k \in K, \forall t \in T, \quad (A12) \\
& z_t, x_{ijkt}, y_{ikt} \in \{0, 1\}, \forall (i, j) \in E, \forall i \in N_0, \forall k \in K, \forall t \in T. \quad (A13)
\end{align*}
\]

3. Crisp equivalent transformation for uncertain programming model.

Due to the complexity of settings with uncertainty, we transform the above uncertain model to equivalent crisp forms based on the inverse distribution criterion in uncertainty theory [31]. Let \( \tilde{f}, \tilde{u}, \tilde{h}_i, \tilde{c}_{ij} \) and \( \tilde{d}_it \) be independent uncertain variables with the inverse uncertainty distributions of \( \Phi_1^{-1}, \Phi_2^{-1}, \Psi_1^{-1}, \Psi_2^{-1}, \) and \( \Upsilon_1^{-1} \), respectively. Then, referring to uncertainty theory by Liu [32], the constraint (A2) can be converted into the following crisp form:

\[
C_1 + C_2 - C_3 \leq \tilde{W},
\]

where:

\[
C_1 = \sum_{t \in T} \left\{ \Phi_1^{-1}(\alpha)z_t + \Phi_2^{-1}(\alpha)p_t + \Psi_0^{-1}(\alpha) \left( B_0 + \sum_{t=1}^{i=t} p_t - \sum_{k \in K} t \sum_{i=1}^{q_{ikt}} \right) \right\}, \quad (2)
\]
\[ C_2 = \sum_{t \in T} \left\{ \sum_{i \in N} \Psi_i^{-1}(\alpha) \left( B_i + \sum_{k \in K} \sum_{t=1}^t q_{ikt} \right) + \sum_{(i,j) \in E} \sum_{k \in K} \Psi_{ij}^{-1}(\alpha)x_{ijkt} \right\}, \quad (3) \]

\[ C_3 = \sum_{t \in T} \left\{ \sum_{i \in N} \Psi_i^{-1}(1 - \alpha) \sum_{t=1}^t \Psi_{i}^{-1}(1 - \alpha) \right\}, \quad (4) \]

In the same way, the crisp form of constraints (A4) can be attained as follows:

\[ B_i + \sum_{k \in K} \sum_{t=1}^t q_{ikt} \geq \sum_{t=1}^t \Psi_{i}^{-1}(\beta_i). \quad (5) \]

3.1. **The linear uncertain situation.** Let uncertain variables satisfy the linear distributions respectively:

\[ \bar{a} \sim \mathcal{L}(a^U_i, b^U_i); \bar{f} \sim \mathcal{L}(a^F_i, b^F_i); \bar{h}_0 \sim \mathcal{L}(a^H_0, b^H_0); \]
\[ \bar{h}_t \sim \mathcal{L}(a^H_{it}, b^H_{it}); \bar{c}_{ij} \sim \mathcal{L}(a^C_{ij}, b^C_{ij}); \bar{d}_{it} \sim \mathcal{L}(a^D_{it}, b^D_{it}). \]

With the above assumptions and uncertainty theory by Liu [33], \( C_1, C_2 \) and \( C_3 \) can be deduced as follows:

\[ C_1 = \sum_{t \in T} \left\{ a_i^U z_t + a_i^F p_t + a_i^H \left( B_0 + \sum_{t=1}^t p_t - \sum_{k \in K} \sum_{t=1}^t q_{ikt} \right) \right\} \]
\[ + \sum_{t \in T} \left\{ b_i^U z_t + b_i^F p_t + b_i^H \left( B_0 + \sum_{t=1}^t p_t - \sum_{k \in K} \sum_{t=1}^t q_{ikt} \right) \right\}, \quad (6) \]

\[ C_2 = \sum_{t \in T} \left\{ \sum_{i \in N} a_i^H \left( B_i + \sum_{k \in K} \sum_{t=1}^t q_{ikt} \right) + \sum_{(i,j) \in E} \sum_{k \in K} a_{ij}^C x_{ijkt} \right\} \]
\[ + \sum_{t \in T} \left\{ \sum_{i \in N} b_i^H \left( B_i + \sum_{k \in K} \sum_{t=1}^t q_{ikt} \right) + \sum_{(i,j) \in E} \sum_{k \in K} b_{ij}^C x_{ijkt} \right\}, \quad (7) \]

\[ C_3 = \sum_{t \in T} \left\{ \sum_{i \in N} \left( \alpha a_i^H + (1 - \alpha) b_i^H \right) \sum_{t=1}^t \left( \alpha a_{it}^D + (1 - \alpha) b_{it}^D \right) \right\}. \quad (8) \]

Similarly, the constraints (A4) can be converted as follows:

\[ (1 - \beta_i) \sum_{t=1}^t a_{it}^d + \beta_i \sum_{t=1}^t b_{it}^d \leq B_i + \sum_{k \in K} \sum_{t=1}^t q_{ikt}. \quad (9) \]
3.2. The normal uncertain situation. Let uncertain variables satisfy the normal distribution respectively:

\[
\hat{u} \sim \mathcal{N}(e^{U}_1, \sigma^{U}_1); \hat{f} \sim \mathcal{N}(e^{F}_2, \sigma^{F}_2); \hat{h}_0 \sim \mathcal{N}(e^{H}_0, \sigma^{H}_0); \\
\hat{h}_i \sim \mathcal{N}(e^{H}_i, \sigma^{H}_i); \hat{c}_{ij} \sim \mathcal{N}(e^{C}_{ij}, \sigma^{C}_{ij}); \hat{d}_{it} \sim \mathcal{N}(e^{D}_{it}, \sigma^{D}_{it}).
\]

According to Liu [33], \( C_1, C_2, \) and \( C_3 \) can be deduced as follows:

\[
C_1 = \sum_{t \in T} \left\{ e^{U}_i z_t + e^{F}_2 p_t + e^{H}_0 \left( B_0 + \sum_{t=1}^{i} p_t - \sum_{k \in K} \sum_{i=1}^{t} q_{ikt} \right) \right\} + \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} 
\]

\[
\sum_{t \in T} \left\{ \sigma^{U}_1 z_t + \sigma^{F}_2 p_t + \sigma^{H}_0 \left( B_0 + \sum_{t=1}^{i} p_t - \sum_{k \in K} \sum_{i=1}^{t} q_{ikt} \right) \right\},
\]

\[
C_2 = \sum_{t \in T} \left\{ \sum_{i \in N} e^{H}_i \left( B_i + \sum_{k \in K} \sum_{t=1}^{i} q_{ikt} \right) + \sum_{(i,j) \in E} \sum_{k \in K} e^{C}_{ij} x_{ijkt} \right\} + \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} 
\]

\[
\sum_{t \in T} \left\{ \sum_{i \in N} \sigma^{H}_i \left( B_i + \sum_{k \in K} \sum_{t=1}^{i} q_{ikt} \right) + \sum_{(i,j) \in E} \sum_{k \in K} \sigma^{C}_{ij} x_{ijkt} \right\},
\]

\[
C_3 = \sum_{t \in T} \left\{ \sum_{i \in N} \left( e^{H}_i + \frac{\sqrt{3}}{\pi} \ln \frac{1-\alpha}{\alpha} \sigma^{H}_i \right) \sum_{t=1}^{i} \left( e^{d}_{it} + \frac{\sqrt{3}}{\pi} \ln \frac{1-\alpha}{\alpha} \sigma^{d}_{it} \right) \right\}.
\]

Then, the constraints (A4) can be converted as follows:

\[
\sum_{t=1}^{i} e^{d}_{it} + \frac{\sqrt{3}}{\pi} \ln \frac{1-\beta_i}{\beta_i} \sum_{t=1}^{i} \sigma^{d}_{it} \leq B_i + \sum_{k \in K} \sum_{t=1}^{i} q_{ikt}.
\]

4. The definition and analysis of uncertain inventory replenishment policy. In general, in order to describe the storage limits of the plants or retailers in certain situations, some storage policies are usually introduced in supply chain optimization problems. The ML policy and the OU policy are two well-known replenishment policies [6]. The ML policy is defined as the quantity delivered to retailer \( i \) such that the level of the inventory at retailer \( i \) at the end of each period is not greater than its maximum level \( L_i \), which is a predefined threshold. Correspondingly, the quantity shipped to the retailer \( i \) is such that the level of the inventory at retailer \( i \) at the end of each period must reach exactly its maximum level \( L_i \) in the OU policy. Owing to their convenience and practicality, the two replenishment policies are frequently used in supply chain systems. But in reality, the demands are usually uncertain and should be described as uncertain variables, and the storage qualities at the end of each period are also uncertain, so the traditional ML and OU policies no longer apply since we cannot directly compare an uncertain variable and the certain threshold value. Therefore, some definitions and the conclusions of analysis in certain situations are no longer applicable under uncertain demand circumstances.
4.1. **The definition of three replenishment policies.** In this section, three replenishment policies, named as the UML policy, the UMLI policy, and the UOU policy in an uncertain environment are defined. The UMLI policy can be viewed as a special case of the UML policy. The parameter $\gamma_i$ is a predetermined value between 0 and 1, which represents the confidence level and reflects the satisfaction degree of uncertain demands.

The UML policy is a concept analogous to the ML policy, in which the quantity delivered to retailer $i$ is such that the credibility (the possibility of the uncertain inventory at retailer $i$ at the end of each period is not greater than its maximum level $L_i$) is greater than $\gamma_i$. Based on the concept of credibility [31], the mathematical formula of the UML policy is defined as follows:

$$M \left\{ B_i + \sum_{k \in K} \sum_{t \leq 1} q_{ikt} - \sum_{t \leq 1} \tilde{d}_{it} \leq L_i \right\} \geq \gamma_i. \quad (14)$$

Similarly, the UOU policy integrates the traditional OU policy and the credibility concept, in which the quantity delivered to retailer $i$ is such that the credibility just reaches $\gamma_i$. The mathematical formula of the UOU policy is defined as follows:

$$M \left\{ B_i + \sum_{k \in K} \sum_{t \leq 1} q_{ikt} - \sum_{t \leq 1} \tilde{d}_{it} \leq L_i \right\} = \gamma_i. \quad (15)$$

Besides, one special case of the UML policy that shows up frequently is the infinity of the threshold. And specifically, if retailer $i$ is visited at time $t$, then the quantity delivered to retailer $i$ can be any nonnegative value. When the threshold is infinite, the constraints of the UML policy have been established. It is named as the UMLI policy and taken into consideration when we carry out a contrastive analysis in the future section.

In aggregate, the maximum level $L_i$ and the confidence level $\gamma_i$ of retailer $i$ reflect the capacity of the retailers' storage, viewed as the upper limit. Correspondingly, the constraint (A4) can be viewed as the lower limit of the storage level.

4.2. **The analysis of three replenishment policies.** In this section, we explore the relationship between the confidence levels, the variances of uncertain demand variables, and the lower and upper limits, and further analyze the change of the lower and upper limits in different replenishment policies.

4.2.1. **Analysis of the UMLI policy.** By means of mathematical deduction, the constraints with uncertainty (B4) can be transformed into linear mean-variance inequalities in uncertain environments. In linear uncertain situation, according to Liu [33], constraint (B4) can be converted as follows:

$$B_i + \sum_{k \in K} \sum_{t \leq 1} q_{ikt} - 0.5 \sum_{t \leq 1} (b_{it}^d + a_{it}^d) \geq 2\sqrt{3} (\beta_i - 0.5) \sum_{t \leq 1} \sqrt{ \left( b_{it}^d - a_{it}^d \right)^2 / 12}. \quad (16)$$

Similarly, the linear mean-variance inequality in normal uncertain situation can be attained as follows:

$$B_i + \sum_{k \in K} \sum_{t \leq 1} q_{ikt} - \sum_{t \leq 1} e_{it}^d \geq \frac{\sqrt{3}}{\pi} \ln \frac{\beta_i}{1 - \beta_i} \sum_{t \leq 1} \sigma_{it}^d. \quad (17)$$
Table 4. The lower and upper limit, and the interval in the MLI and UMLI policy

| Situation   | Policy | Lower Limit | Upper Limit | Interval               |
|-------------|--------|-------------|-------------|------------------------|
| Deterministic | MLI    | 0           | $+\infty$   | $[0, +\infty]$         |
| Linear      | UMLI   | $R_L(\beta_i)$ | $+\infty$   | $[R_L(\beta_i), +\infty]$ |
| Normal      | UMLI   | $R_N(\beta_i)$ | $+\infty$   | $[R_N(\beta_i), +\infty]$ |

For brevity, $R_L(x)$ and $R_N(x)$ can be defined as follows:

$$R_L(x) = 2\sqrt{3} (x - 0.5) \sum_{t=1}^{t} \frac{(b_{it}^d - a_{it}^d)^2}{12}, \quad (18)$$

$$R_N(x) = \frac{\sqrt{3}}{\pi} \ln \frac{x}{1 - x} \sum_{t=1}^{t} \sigma_{it}^d. \quad (19)$$

Then, we use $R_L(\beta_i)$ and $R_N(\beta_i)$ to represent the right sides of the above inequalities in linear and normal distributions, which are viewed as the lower limit and always 0 in the determined situation. As a comparison, the lower and upper limit, and the interval of the MLI policy in the determined situation, and the UMLI policy under uncertain environments are summarized in Table 4. The MLI policy is a special case of the ML policy in which case the upper limit is infinite.

The value of $R_L(\beta_i)$ and $R_N(\beta_i)$ is mainly affected by the confidence level of $\beta_i$ and the variances of uncertain demand variables, which are categorized into three cases.

- If $\beta_i < 0.5$, $R_L(\beta_i)$ and $R_N(\beta_i)$ are always less than 0, decrease as the variances of uncertain demand variables increase.
- If $\beta_i > 0.5$, $R_L(\beta_i)$ and $R_N(\beta_i)$ are always greater than 0, increase as the variances of uncertain demand variables.
- If $\beta_i = 0.5$, no matter how the variances of uncertain demand variables change, $R_L(\beta_i)$ and $R_N(\beta_i)$ are always equal to 0 as in the determined situation.

4.2.2. Analysis of the UML policy. According to Liu [33], the crisp form of the uncertain formula of the UML policy can be obtained as follows:

$$B_i + \sum_{k \in K} \sum_{t=1}^{t} q_{ikt} \leq L_i + \sum_{t=1}^{t} Y_{it}^{-1} (1 - \gamma_i). \quad (20)$$

Furthermore, letting the uncertain variables satisfy the linear distribution, the above inequality can be converted as follows:

$$B_i + \sum_{k \in K} \sum_{t=1}^{t} q_{ikt} \leq L_i + \gamma_i \sum_{t=1}^{t} a_{it}^d + (1 - \gamma_i) \sum_{t=1}^{t} b_{it}^d. \quad (21)$$
According to mathematical deduction, the following inequality can be deduced:

\[
B_i + \sum_{k \in K} \sum_{t=1}^{t} q_{ikt} - 0.5 \sum_{t=1}^{t} (b_{it}^d + a_{it}^d) \leq L_i - 2\sqrt{3}(\gamma_i - 0.5) \sum_{t=1}^{t} \sqrt{\frac{(b_{it}^d - a_{it}^d)^2}{12}}. \tag{22}
\]

Letting the uncertain variables satisfy the normal distribution, the uncertain formula of the UML policy can be converted as follows:

\[
B_i + \sum_{k \in K} \sum_{t=1}^{t} q_{ikt} \leq L_i + \sum_{t=1}^{t} e_{it}^d + \frac{\sqrt{3}}{\pi} \ln \frac{1 - \gamma_i}{\gamma_i} \sum_{t=1}^{t} \sigma_{it}^d. \tag{23}
\]

Similarly the above inequality can be deduced as follows:

\[
B_i + \sum_{k \in K} \sum_{t=1}^{t} q_{ikt} - \sum_{t=1}^{t} e_{it}^d \leq L_i - \frac{\sqrt{3}}{\pi} \ln \frac{1 - \gamma_i}{\gamma_i} \sum_{t=1}^{t} \sigma_{it}^d. \tag{24}
\]

\(L_i - R_L(\gamma_i)\) and \(L_i - R_N(\gamma_i)\) represent the right sides of the above deduced inequalities in linear and normal distributions, respectively, which are viewed as the upper limit and equal to \(L_i\) as in the determined situation. As a comparison, the lower and upper limit, and the interval in the ML policy in the determined situation and the UML policy under uncertain environments are summarized in Table 5.

The values of \(L_i - R_L(\gamma_i)\) and \(L_i - R_N(\gamma_i)\) are affected by the confidence level of \(\gamma_i\) and the variances of uncertain demand variables simultaneously, which also can be categorized into three cases.

- If \(\gamma_i < 0.5\), the values of \(L_i - R_L(\gamma_i)\) and \(L_i - R_N(\gamma_i)\), being always greater than \(L_i\), increase as the variances of uncertain demand variables increase.
- If \(\gamma_i > 0.5\), the values of \(L_i - R_L(\gamma_i)\) and \(L_i - R_N(\gamma_i)\), being always less than \(L_i\), decrease as the variances of uncertain demand variables increase.
- If \(\gamma_i = 0.5\), the values of \(L_i - R_L(\gamma_i)\) and \(L_i - R_N(\gamma_i)\) are always equal to \(L_i\) as in the determined situation.

It is noted that the lower and upper limit are affected by the variances of uncertain demand variables simultaneously. For the better analysis of replenishment policies under uncertain environments, the cross effect brought by the lower and upper limit, and the interval are also considered simultaneously. \(G_L(\beta_i, \gamma_i)\) is introduced to describe the gap between the lower and upper limit in linear uncertain

| Situation | Policy | Lower Limit | Upper Limit | Interval |
|-----------|--------|-------------|-------------|----------|
| Deterministic | ML | 0 | \(L_i\) | \([0, L_i]\) |
| Linear | UML | \(R_L(\beta_i)\) | \(L_i - R_L(\gamma_i)\) | \([R_L(\beta_i), L_i - R_L(\gamma_i)]\) |
| Normal | UML | \(R_N(\beta_i)\) | \(L_i - R_N(\gamma_i)\) | \([R_N(\beta_i), L_i - R_N(\gamma_i)]\) |
situation. The mathematical expression of $G_L(\beta_i, \gamma_i)$ is obtained as follows:

$$G_L(\beta_i, \gamma_i) = L_i - 2\sqrt{3}(\beta_i + \gamma_i - 1) \sum_{t=1}^{t} \frac{\left( b_{it}^d - a_{it}^d \right)^2}{12}. \quad (25)$$

Similarly, $G_N(\beta_i, \gamma_i)$ is introduced to describe the gap between the lower and upper limit in a normal uncertain situation. The mathematical expression of $G_L$ is shown as follows:

$$G_N(\beta_i, \gamma_i) = L_i - \frac{\sqrt{3}}{\pi} \ln \frac{\beta_i \gamma_i}{(1 - \beta_i)(1 - \gamma_i)} \sum_{t=1}^{t} \sigma_{it}^d. \quad (26)$$

The values of $G_L(\beta_i, \gamma_i)$ and $G_N(\beta_i, \gamma_i)$ are affected by the sum of $\beta_i$ and $\gamma_i$, and the variances of uncertain demand variables simultaneously, which are similarly categorized into three cases.

- If $\beta_i + \gamma_i < 1$, the values of $G_L(\beta_i, \gamma_i)$ and $G_N(\beta_i, \gamma_i)$, being always greater than $L_i$, increase as the variances of uncertain demand variables increase.
- If $\beta_i + \gamma_i > 1$, the values of $G_L(\beta_i, \gamma_i)$ and $G_N(\beta_i, \gamma_i)$, being always less than $L_i$, decrease as the variances of uncertain demand variables increase.
- If $\beta_i + \gamma_i = 1$, the values of $G_L(\beta_i, \gamma_i)$ and $G_N(\beta_i, \gamma_i)$ are always equal to $L_i$ as in the determined situation.

4.2.3. Analysis of the UOU policy. According to Liu [33], the crisp form of the uncertain mathematical formula of the UOU policy can be obtained as follows:

$$B_i + \sum_{k \in K} \sum_{t=1}^{t} q_{ikt} = L_i + \sum_{t=1}^{t} \gamma_{it}^{-1} (1 - \gamma_i). \quad (27)$$

Further, letting the uncertain variables satisfy the linear distributions, the above inequalities can be converted as follows:

$$B_i + \sum_{k \in K} \sum_{t=1}^{t} q_{ikt} = L_i + \gamma_i \sum_{t=1}^{t} a_{it}^d + (1 - \gamma_i) \sum_{t=1}^{t} b_{it}^d. \quad (28)$$

According to mathematical deduction, the following inequality can be as follows:

$$B_i + \sum_{k \in K} \sum_{t=1}^{t} q_{ikt} - 0.5 \sum_{t=1}^{t} \left( b_{it}^d + a_{it}^d \right) = L_i - 2\sqrt{3}(\gamma_i - 0.5) \sum_{t=1}^{t} \frac{\left( b_{it}^d - a_{it}^d \right)^2}{12}. \quad (29)$$

Letting the uncertain variables satisfy the normal distributions, the uncertain mathematical formula of the UML policy can be converted as follows:

$$B_i + \sum_{k \in K} \sum_{t=1}^{t} q_{ikt} = L_i + \sum_{t=1}^{t} c_{it}^d + \frac{\sqrt{3}}{\pi} \ln \frac{1 - \gamma_i}{\gamma_i} \sum_{t=1}^{t} \sigma_{it}^d. \quad (30)$$

Similarly the above inequality can be as follows:

$$B_i + \sum_{k \in K} \sum_{t=1}^{t} q_{ikt} - \sum_{t=1}^{t} c_{it}^d = L_i - \frac{\sqrt{3}}{\pi} \ln \frac{\gamma_i}{1 - \gamma_i} \sum_{t=1}^{t} \sigma_{it}^d. \quad (31)$$
Table 6. The lower and upper limit, and the interval in the OU and UOU policies

| Situation   | Policy | Lower Limit | Upper Limit | Interval |
|-------------|--------|-------------|-------------|----------|
| Deterministic OU |        | 0           | $L[i]$      | $L[i]$   |
| Linear UOU |        | $R_L(\beta_i)$ | $L_i - R_L(\gamma_i)$ | $L_i - R_L(\gamma_i)$ |
| Normal UOU |        | $R_N(\beta_i)$ | $L_i - R_N(\gamma_i)$ | $L_i - R_N(\gamma_i)$ |

As a comparison, the lower and upper limit, and the interval of the OU policy in the determined situation and the UOU policy under uncertain environments are summarized in Table 6. The change trend of the value of the lower and upper limit is consistent with the UML policy, but the impacts of its change on the optimal results differ. In the UOU policy, the inventory at each retailer at the end of each period must be equal to the upper limit. When the lower limit is lower than the upper limit, the overall cost only is affected by the upper limit.

5. **Experimental results and analysis.** The crisp deterministic model is implemented in C++ using ILOG Concert and CPLEX 12.8. The goals of this experimental campaign are to solve the crisp equivalent model after crisp equivalent conversion from the uncertain model proposed in Sect. 2, to simultaneously verify the analysis in Sect. 4, to discuss how the confidence levels and the variance of uncertain variables influence the computational results in more depth, and to compare the UML and UOU policies in uncertain environments.

Since much of the previous research on the PRP took only deterministic circumstances into account, uncertainties may have gone undetected, so previously proposed approaches are unsuitable for researching uncertain PRPs. In order to solve the uncertain decision model, the uncertain instances are proposed based on the classic instance from Archetti [6], which has been considered as a classical instance for solving PRPs with exact solution algorithms and heuristics.

These uncertain variables are generated in Table 7, including uncertain demands $\tilde{d}_{it}$, uncertain production setup cost $\tilde{f}$, uncertain unit production cost $\tilde{p}$, uncertain inventory cost $\tilde{h}_0$ for the plant and $\tilde{h}_i$ for retailer $i$, and uncertain translation cost $\tilde{c}_{ij}$ between nodes $i$ and $j$. The numerical experiments take a linear uncertain circumstance as an example. The series parameters, including $\epsilon^{ld}$, $\epsilon^{lf}$, $\epsilon^{lp}$, $\epsilon^{lh}$, and $\epsilon^{lc}$, are used for linear uncertain variables and take values between 0 and 1. The values of these parameters have an important impact on the upper and lower bounds of linear uncertain variables, reflecting the value of the variance of uncertain variables. Moreover, the predefined series confidence levels $\alpha$, $\beta_i$, and $\gamma_i$ always vary between 0 and 1. Compared to the deterministic instance from Archetti [6], the major difference is that the variables are uncertain, and the corresponding confidence levels and correlation parameters are introduced. To better focus on the effect of uncertain demand variables, $\epsilon^{ld}$ is set as a random number in three intervals, such as $[0, 0.3],[0.3, 0.6],[0.6, 0.9]$, reflecting that the value of the uncertain demand variables’ variances decreases gradually.

5.1. **The UMLI policy.** Firstly, we consider the experiment in the UMLI policy, which also may be considered as no storage constraints. The series parameter values of $\epsilon^{lf}$, $\epsilon^{lp}$, $\epsilon^{lh}$ and $\epsilon^{lc}$ about uncertain costs are set to 0.2. The confidential level
Table 7. The uncertain variables of the PRP in linear uncertain environment

| Parameters | Values |
|------------|--------|
| $d_{it}$   | $\mathcal{L}(d_{it}(1 - \epsilon^d), d_{it}(1 + \epsilon^d))$ |
| $\hat{f}$  | $\mathcal{L}(f(1 - \epsilon^f), f(1 + \epsilon^f))$ |
| $\hat{p}$  | $\mathcal{L}(p(1 - \epsilon^p), p(1 + \epsilon^p))$ |
| $\tilde{h}_0$ | $\mathcal{L}(h_0(1 - \epsilon^h), h_0(1 + \epsilon^h))$ |
| $\tilde{h}_i$ | $\mathcal{L}(h_i(1 - \epsilon^h), h_i(1 + \epsilon^h))$ |
| $c_{ij}$   | $\mathcal{L}(c_{ij}(1 - \epsilon^c), c_{ij}(1 + \epsilon^c))$ |

$\alpha$ is also set to 0.5. The experiments are performed to examine the effects of the confidence level $\beta_i$ and the variance of uncertain demand variables.

Figure 1. The changes of cost in the UMLI policy

As a general rule, the higher the values of the lower limit, then the higher the overall cost with the optimal solution of the uncertain programming model. Therefore, the confidence level and the variances of uncertain demand variables have a significant effect on the overall cost by affecting the lower limit. The experimental results in Fig.1 verify the validity of the analysis of the UMLI policy about the lower limit. The overall cost strictly increases with the confidence level of $\beta_i$. The effect of the variances of uncertain demand variables on the overall costs mostly depends on the value of the confidence level of $\beta_i$. The overall cost increases with the variances of uncertain demand variables when the confidence level of $\beta_i$ is greater than 0.5,
but decreases when the confidence level of $\beta_i$ is smaller than 0.5. In the special case where $\beta_i$ is equal to 0.5, they have no effect on the optimal solution or the overall cost.

We can draw Propositions 1 and 2 about the UMLI policy as follows.

**Proposition 1.** The lower limit grows linearly with the increase of the confidence level of $\beta_i$ in the UMLI policy. As the confidence level of $\beta_i$ increases, the lower limit in UMLI policy rises, resulting in the optimal solution with a higher cost.

**Proposition 2.** The lower limit, the optimal solution of the uncertain decision model, and its overall cost are affected by the variances of uncertain demand variables but rely on the confidence level of $\beta_i$, specially described as follows.

- If $\beta_i < 0.5$, as the variances of uncertain demand variables increase, the lower limit in the UMLI policy is smaller, bringing about the optimal solution with lower cost.
- If $\beta_i > 0.5$, as the variances of uncertain demand variables are increased, the lower limit in the UMLI policy is bigger, bringing about the optimal solution with higher cost.
- If $\beta_i = 0.5$, no matter how the variances of uncertain demand variables change, they have no effect on the lower limit and the overall cost.

5.2. The UML policy. Secondly, the effect of the UML policy is taken into account. The series parameter values of $\epsilon_l^f$, $\epsilon_l^p$, $\epsilon_l^h$ and $\epsilon_l^c$ take the same value. The confidence levels $\alpha$ and $\beta_i$ are also set to 0.5. The experiments are performed to examine the effects of the confidence level $\gamma_i$ and the variance of uncertain demand variables.

Obviously, the higher the values of the upper limit, then the lower the overall cost with the optimal solution of the uncertain programming model. So the confidence level and the variances of uncertain demand variables also influence the overall cost by affecting the upper limit. The experimental results in Fig.2 confirm the analysis of the UML policy about the upper limit. The overall cost might increase with the confidence level of $\gamma_i$. The effect of the variances of uncertain demand variables on the overall cost mostly depends on the value of the confidence level of $\gamma_i$. When $\gamma_i$ is equal to 0.5, they have no effect on the optimal solution or the overall cost. The overall cost might increase with the variances of uncertain demand variables when the confidence level of $\gamma_i$ is greater than 0.5. When the variances of uncertain demand variables and the confidence level of $\gamma_i$ are large enough, such as when $\epsilon_l^{rd}$ is in the interval $[0.6, 0.9)$ and $\gamma_i$ is greater than 0.8, there’s no solution.

We draw Propositions 3 and 4 about the UML policy as follows.

**Proposition 3.** The upper limit decreases linearly with the increase of the confidence level of $\gamma_i$ in the UML policy. As the confidence level of $\gamma_i$ increases, the upper limit in UML policy decreases, resulting in the optimal solution with a higher cost.

**Proposition 4.** The upper limit, the optimal solution of the uncertain decision model, and its overall cost are affected by the variances of uncertain demand variables simultaneously but rely on the value of the confidence level $\gamma_i$ as follows.

- If $\gamma_i < 0.5$, as the variances of uncertain demand variables increase, the upper limit in the UML policy is bigger, bringing about the optimal solution with a smaller cost.
• If $\gamma_i > 0.5$, as the variances of uncertain demand variables increase, the upper limit in the UML policy is smaller, bringing about the optimal solution with higher cost. In the special case, when the confidence level $\gamma_i$ and the variances of uncertain demand variables are great in a certain degree, there may be no solution for the decision model.

• If $\gamma_i = 0.5$, no matter how the variances of uncertain demand variables change, they have no effect on the upper limit and the overall cost.

Figure 2. The changes of cost in the UML policy

Based on the above propositions, and Propositions 1 and 2, the change trend of the lower and upper limit, the interval and the cost in the UML policy as the variances of uncertain demand variables change are described as in Table 8.

It is important that when $G_L(\beta_i, \gamma_i)$ and $G_N(\beta_i, \gamma_i)$ are less than 0, there must be no solution in both the UML and UOU policy. Therefore, the below analysis is based on the case that $G_L(\beta_i, \gamma_i)$ and $G_N(\beta_i, \gamma_i)$ are greater than 0. The change of the upper and lower limit, and its gap may influence the optimal solution of the uncertain decision model, and its overall cost. They are mainly affected by the confidence levels $\beta_i$ and $\gamma_i$, and the variances of uncertain demand variables. For the most part, the effect on the total cost can be analyzed according to the values of $G_L(\beta_i, \gamma_i)$ and $G_N(\beta_i, \gamma_i)$ in the UML policy. However, it is important that when the lower and upper limit increase or decrease simultaneously, the effect on the cost is arguable, being described as $\ast$.

5.3. The UOU policy. Accordingly the UOU policy, the higher the values of the upper limit, then the higher the overall cost with the optimal solution of the uncertain programming model. Similarly, the confidence level and the variances of
Table 8. The change trend of lower and upper limit, the interval, and the cost in the UML policy

| $(\beta_i, \gamma_i)$ | Lower Limit | Upper Limit | Gap | Cost_M |
|-----------------------|-------------|-------------|-----|--------|
| $< 0.5, < 0.5$        | ↓           | ↑           | ↑   | ↓      |
| $< 0.5, = 0.5$        | ↓           | =           | ↓   | ↓      |
| $< 0.5, > 0.5$        | ↓           | ↓           | *   | *      |
| $= 0.5, < 0.5$        | =           | ↑           | ↑   | ↓      |
| $= 0.5, = 0.5$        | =           | =           | =   | =      |
| $= 0.5, > 0.5$        | =           | ↓           | ↓   | ↑      |
| $> 0.5, < 0.5$        | ↑           | ↑           | *   | *      |
| $> 0.5, = 0.5$        | ↑           | =           | ↓   | ↑      |
| $> 0.5, > 0.5$        | ↑           | ↓           | ↓   | ↑      |

uncertain demand variables also influence the overall cost by affecting the upper limit. The experimental results in Fig.3 confirm the analysis of the UOU policy about the upper limit. Contrary to the UML policy, the overall cost strictly decreases with the confidence level of $\gamma_i$. The effect of the variances of uncertain demand variables on the overall cost mostly depends on the value of the confidence level of $\gamma_i$. The overall cost decrease with the variances of uncertain demand variables when the confidence level of $\gamma_i$ is greater than 0.5. In the special case where $\gamma_i$ is equal to 0.5, they have no effect on the optimal solution or overall cost. One similarity with the UML policy is that if the variances of uncertain demand variables and the confidence level of $\gamma_i$ are large enough, such as when $\epsilon_{ld}$ is in the interval $[0.6, 0.9)$ and $\gamma_i$ is greater 0.8, there’s no solution.

We draw Propositions 5 and 6 about the UOU policy as follows.

**Proposition 5.** The upper limit decreases linearly with the increase of the confidence level of $\gamma_i$ in the UOU policy. As the confidence level of $\gamma_i$ increases, the upper limit in UOU policy decreases, resulting in the optimal solution with lower cost.

**Proposition 6.** The upper limit, the optimal solution of the uncertain decision model, and its overall cost are affected by the variances of uncertain demand variables simultaneously but rely on the value of the confidence level $\gamma_i$ as follows.

- If $\gamma_i < 0.5$, as the variances of uncertain demand variables increase, the upper limit in the UOU policy is bigger, bringing about the optimal solution with higher cost.
- If $\gamma_i > 0.5$, as the variances of uncertain demand variables increase, the upper limit in the UOU policy is smaller, bringing about the optimal solution with lower cost. In extreme cases, when the critical values of the upper limit increase to a certain degree as the confidence level $\gamma_i$ and the variances of uncertain demand variables increase, there may be no solution for the decision model.
- If $\gamma_i = 0.5$, no matter how the variances of uncertain demand variables change, they have no effect on the upper limit and the overall cost.
Same with the UML policy, the cost is affected by the lower limit simultaneously in the UOU policy. Based on the above propositions, and Propositions 1 and 2, the change trend of the lower and upper limit, the interval and the cost in the UOU policy as the variances of uncertain demand variables change are described as in Table 9.

### Table 9. The change trend of lower and upper limit, the interval, and the cost in the UOU policies

| $(\beta_i, \gamma_i)$ | Lower Limit | Upper limit | Gap | $Cost_U$ |
|-----------------------|-------------|-------------|-----|----------|
| $(< 0.5, < 0.5)$      | ↓           | ↑           | ↑   | ↑        |
| $(< 0.5, = 0.5)$      | ↓           | =           | ↓   | =        |
| $(< 0.5, > 0.5)$      | ↓           | ↓           | *   | ↓        |
| $(= 0.5, < 0.5)$      | =           | ↑           | ↑   | ↑        |
| $(= 0.5, = 0.5)$      | =           | =           | =   | =        |
| $(= 0.5, > 0.5)$      | =           | ↓           | ↓   | ↓        |
| $(> 0.5, < 0.5)$      | ↑           | ↑           | *   | ↑        |
| $(> 0.5, = 0.5)$      | ↑           | =           | ↓   | =        |
| $(> 0.5, > 0.5)$      | ↑           | ↓           | ↓   | ↓        |

5.4. **Comparative analysis about the UML and UOU policies.** Referring to the methods of the profit difference ratios [30], we define the cost difference ratios
and then compare the costs obtained by the UML policy with the ones obtained by the UOU policy as the variances of uncertain demand variables change. Based on the above, the values of the confidence levels of $\beta_i$ and $\gamma_i$ are adjusted, and the threshold of storage limit is loosened. We do a comparative experiment about the UML and UOU policies in uncertain environments. The value of $L_x$ is a multiple of the initial threshold of the storage limit $L_i$, reflecting the storage capacity of the retailers.

The experimental results in Tables 10, 11, and 12 also verify the comparative analysis conclusion about the UML and UOU policy. Results show that the cost relative difference between the UML and UOU policies begins to narrow along with the increase of the variances of uncertain variables when the confidence levels $\beta_i$ and $\gamma_i$ are both not less than 0.5. Decision-makers are usually most interested in higher confidence levels in realistic applications, the conclusion is very meaningful to the supply chain decision-making. It is noted that the situation that $\circ$ represents no solution in both the UML and UOU policy.

Therefore, based on the analysis of the UML and UOU policy, the change trend of the cost in the UML and UOU policy, and their cost relative difference \( \left( \frac{\text{Cost}_U - \text{Cost}_M}{\text{Cost}_M} \right) \) are described as in Table 13. We draw Propositions 7 as follows.

**Proposition 7.** When the confidence levels $\beta_i$ and $\gamma_i$ are both not less than 0.5, the cost relative difference between the UML and UOU policies begins to narrow along with the increase of the variances of uncertain variables.

### 5.5. Effects of uncertain costs

Finally, an experimental study on the effects of the confidence level $\alpha$ and the variance of uncertain cost variables is performed.
Table 11. The cost relative difference value between the UML and UOU policies with $L_x = 2$

| $\epsilon^{ld}$ | $\beta_i$ | $\gamma_i = 0.1$ | $\gamma_i = 0.3$ | $\gamma_i = 0.5$ | $\gamma_i = 0.7$ | $\gamma_i = 0.9$ |
|-----------------|----------|------------------|------------------|------------------|------------------|------------------|
| [0.0,0.3)       | 0.1      | 1.448            | 1.369            | 1.278            | 1.199            | 1.120            |
|                 | 0.3      | 1.268            | 1.195            | 1.215            | 1.048            | 0.975            |
|                 | 0.5      | 1.128            | 1.060            | 0.991            | 0.923            | 0.854            |
|                 | 0.7      | 0.991            | 0.927            | 0.862            | 0.798            | 0.734            |
|                 | 0.9      | 0.870            | 0.809            | 0.749            | 0.689            | 0.629            |
| [0.3,0.6)       | 0.1      | 2.843            | 2.531            | 2.219            | 1.906            | 1.570            |
|                 | 0.3      | 1.959            | 1.718            | 1.478            | 1.194            | 0.938            |
|                 | 0.5      | 1.378            | 1.184            | 0.991            | 0.798            | 0.590            |
|                 | 0.7      | 0.989            | 0.828            | 0.666            | 0.487            | 0.299            |
|                 | 0.9      | 0.710            | 0.571            | 0.405            | 0.267            | 0.000            |
| [0.6,0.9)       | 0.1      | 5.495            | 4.709            | 3.923            | 3.118            | 2.253            |
|                 | 0.3      | 2.850            | 2.384            | 1.918            | 1.441            | 0.928            |
|                 | 0.5      | 1.627            | 1.309            | 0.991            | 0.666            | 0.278            |
|                 | 0.7      | 0.989            | 0.748            | 0.507            | 0.234            | 0.000            |
|                 | 0.9      | 0.599            | 0.385            | 0.178            |                  |                  |

Table 12. The cost relative difference value between the UML and UOU policies with $L_x = 4$

| $\epsilon^{ld}$ | $\beta_i$ | $\gamma_i = 0.1$ | $\gamma_i = 0.3$ | $\gamma_i = 0.5$ | $\gamma_i = 0.7$ | $\gamma_i = 0.9$ |
|-----------------|----------|------------------|------------------|------------------|------------------|------------------|
| [0.0,0.3)       | 0.1      | 2.482            | 2.402            | 2.322            | 2.243            | 2.163            |
|                 | 0.3      | 2.224            | 2.150            | 2.076            | 2.002            | 1.928            |
|                 | 0.5      | 2.000            | 1.932            | 1.863            | 1.794            | 1.725            |
|                 | 0.7      | 1.799            | 1.735            | 1.671            | 1.606            | 1.542            |
|                 | 0.9      | 1.628            | 1.568            | 1.507            | 1.447            | 1.387            |
| [0.3,0.6)       | 0.1      | 4.233            | 3.921            | 3.608            | 3.296            | 2.960            |
|                 | 0.3      | 3.029            | 2.789            | 2.548            | 2.308            | 2.049            |
|                 | 0.5      | 2.251            | 2.057            | 1.863            | 1.669            | 1.460            |
|                 | 0.7      | 1.711            | 1.550            | 1.388            | 1.226            | 1.052            |
|                 | 0.9      | 1.329            | 1.190            | 1.051            | 0.912            | 0.762            |
| [0.6,0.9)       | 0.1      | 7.621            | 6.835            | 6.049            | 5.244            | 4.379            |
|                 | 0.3      | 4.110            | 3.644            | 3.178            | 2.701            | 2.188            |
|                 | 0.5      | 2.501            | 2.182            | 1.863            | 1.536            | 1.185            |
|                 | 0.7      | 1.641            | 1.400            | 1.159            | 0.913            | 0.648            |
|                 | 0.9      | 1.123            | 0.930            | 0.736            | 0.538            | 0.313            |
The series parameter values of $\epsilon_{lf}, \epsilon_{lp}, \epsilon_{lh}$ and $\epsilon_{lc}$ are adjusted for the variations of uncertain cost variables. The confidence levels $\beta_i$ and $\gamma_i$ are also set to 0.5. The experimental results in Fig.4 focus on the effect of the confidence level $\alpha$ and

\begin{table}[h]
\centering
\caption{The change trend of the cost relative difference between the UML and UOU policies}
\begin{tabular}{llll}
\hline
$\beta_i\gamma_i$ & $<0.5$ & $=0.5$ & $>0.5$ \\
\hline
$<0.5$ & $\uparrow (\uparrow, \downarrow)$ & $\uparrow (=, \downarrow)$ & $*(\downarrow, \ast)$ \\
$=0.5$ & $\uparrow (\uparrow, \downarrow)$ & $=(=, =)$ & $\downarrow (\downarrow, \uparrow)$ \\
$>0.5$ & $*(\uparrow, \ast)$ & $\downarrow (=, \uparrow)$ & $\downarrow (\downarrow, \uparrow)$ \\
\hline
\end{tabular}
\end{table}

Figure 4. The change of the total cost with cost uncertainty

The variances of uncertain cost variables, and indicate that the overall cost strictly increases with the confidence level $\alpha$. The effect of the variances of uncertain cost variables on the overall cost mostly depends on the value of the confidence level $\alpha$. The overall cost increases with the variances of uncertain cost variables when the confidence level $\alpha$ is greater than 0.5, but decreases when the confidence level $\alpha$ is smaller than 0.5. In the special case where $\alpha$ is equal to 0.5, they have no effect on the optimal solution, the overall cost being constant. These results accord closely with our predictions.

6. Conclusions. In this paper, we study the integrated PRP with multiple delivery routes in uncertain environments and highlight different types of replenishment
policies under the condition of uncertain demands. An uncertain programming model is proposed to solve the integrated PRP with heterogeneous vehicles in uncertain environments. Then the UML policy and the UOU policy are defined. We make a comparative analysis about the UOU and UML policies in uncertain environments. The lower and upper limit of inventory in the UML and UOU policies, and the total cost are influenced by the confidence levels and the variances of uncertain variables. The conclusion shows that the optimal solution of the uncertain decision model and its overall cost grow with the increase of the confidence levels. And they are simultaneously affected by the variances of uncertain variables but rely on the value of confidence levels. We then compare the results obtained by the UML policy with the ones obtained by the UOU policy. Similar to that in the certain situation, the UML policy guarantees smaller costs. However, when the confidence levels are not less than 0.5, the cost difference between the two policies begins to narrow along with the increase of the confidence levels and the variances of uncertain variables. Specifically, the overall cost increases with the increasing of the variances of uncertain variables in both the UML or UOU policy. But under the situation of high-leveled confidence levels, the cost in the UOU policy has less of an increase with the growth of the variances of uncertain variables. Eventually, as the variances of uncertain variables grow to a certain degree, the cost difference between the two policies will be close to 0. Decision-makers are usually most interested in higher confidence levels in realistic applications, in which the solution scale is escalated, being conducive to the efficiency advantage of the UOU policy. Therefore, the research presented in this paper is of great significance.

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