SHRINKAGE PRIORS ON COMPLEX-VALUED CIRCULAR-SYMMETRIC AUTOREGRESSIVE PROCESSES

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We investigate shrinkage priors on power spectral densities for complex-valued circular-symmetric autoregressive processes. We construct shrinkage predictive power spectral densities, which asymptotically dominate (i) the Bayesian predictive power spectral density based on the Jeffreys prior and (ii) the estimative power spectral density with the maximal likelihood estimator, where the Kullback–Leibler divergence from the true power spectral density to a predictive power spectral density is adopted as a risk. Furthermore, we propose general constructions of objective priors for Kähler parameter spaces, utilizing a positive continuous eigenfunction of the Laplace–Beltrami operator with a negative eigenvalue. We present numerical experiments on a complex-valued stationary autoregressive model of order 1.

1. Introduction. We investigate the time fluctuation of a single particle whose distribution is circular-symmetric in a two dimensional space. In this situation, the complex plane \( \mathbb{C} \) is often used for the representation of the process, and an observation of a single particle at different time points can be represented as a complex-valued vector \( z \in \mathbb{C}^N \). A complex-valued random vector \( Z \) is called circular-symmetric if for any constant \( \phi \in \mathbb{R} \), the distribution of \( e^{\sqrt{-1}\phi}Z \) equals the distribution of \( Z \).

We focus on complex-valued circular-symmetric discrete Gaussian processes, which are defined as complex-valued processes whose finite dimensional marginal distributions are complex normal distributions. The precise definitions of a complex normal distribution and a complex-valued Gaussian process are given in Section 2.

The circular-symmetry of complex normal distributions with mean \( 0 \in \mathbb{C} \) is of great importance in practical applications [13, 12]. Complex processes are commonly used for directional processes, such as wind, radar, and sonar signals. Also, complex-valued representations are widely used in diverse fields, such as electronics, physics, and biomedicine.

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We parametrize complex-valued Gaussian processes by $p$ complex variables $\theta = (\theta^1, \ldots, \theta^p) \in \Theta \subset \mathbb{C}^p$. For each parameter $\theta$, there is a corresponding power spectral density $S_\theta(\omega) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \gamma_h e^{-\sqrt{-1}h\omega}$ of a complex-valued Gaussian process. In other words, we regard the sequence $\{\gamma_h\}_{h \in \mathbb{Z}}$ of autocovariances of the process as functions as functions $\gamma_h(\theta)$ of the parameter $\theta$.

Suppose we observe a sample $z^{(N)} = (z_1, \ldots, z_N) \in \mathbb{C}^N$ of size $N$ from a complex-valued Gaussian process whose true parameter is $\theta_0 \in \Theta$. Let us consider the problem of constructing a power spectral density $\hat{S}^{(N)}$. The constructed power spectral density $\hat{S}^{(N)}$ is called a predictive power spectral density. More precisely, a predictive power spectral density $\hat{S}^{(N)}$ is defined as

\begin{equation}
R(\hat{S}^{(N)} \mid \theta_0) := E_{\theta_0} \left[ D_{\text{KL}} \left( S_{\theta_0} \mid \mid \hat{S}^{(N)} \right) \right] = \int_{\mathbb{C}^N} D_{\text{KL}} \left( S_{\theta_0} \mid \hat{S}^{(N)} \right) dP_{\theta_0}^{(N)}(z^{(N)}),
\end{equation}

where $P_{\theta_0}^{(N)}$ denotes the distribution of $z^{(N)}$ and the Kullback–Leibler divergence between two power spectral densities $S_1$ and $S_2$ is defined as

\begin{equation}
D_{\text{KL}} (S_1 \mid \mid S_2) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( - \log \frac{S_1(\omega)}{S_2(\omega)} - 1 + \frac{S_1(\omega)}{S_2(\omega)} \right) d\omega.
\end{equation}

The principal aim of this study is to construct a predictive power spectral density $\hat{S}^{(N)}$ with its risk $R(\hat{S}^{(N)} \mid \theta)$ as small as possible for most of $\theta \in \Theta$.

There are two basic constructions for predictive power spectral densities. The first construction, called the estimative method, is $\hat{S}^{(N)} := S_{\hat{\theta}^{(N)}}$, where $\hat{\theta}^{(N)} = \hat{\theta}^{(N)}(z^{(N)})$ is an estimator for the true parameter $\theta_0$. The second construction, called the Bayesian predictive method, is $\hat{S}^{(N)} := \int_{\Theta} S_\theta \pi \left( \theta \mid z^{(N)} \right) d\theta$ for a prior $\pi$, where $\pi \left( \theta \mid z^{(N)} \right)$ denotes the posterior based on $\pi$. The power spectral density $\hat{S}^{(N)}_\pi$ is called the Bayesian predictive power spectral density based on the prior $\pi$. If a prior $\pi$ is given and the risk $R(\hat{S}^{(N)} \mid \theta)$ is defined as (1.1), the Bayesian predictive power spectral density $\hat{S}^{(N)}_\pi$ minimizes the Bayes risk

\begin{equation}
r(\hat{S}^{(N)} \mid \pi) := \int_{\Theta} R(\hat{S}^{(N)} \mid \theta) \pi(\theta) d\theta
\end{equation}

among all the predictive power spectral densities $\hat{S}^{(N)}$ as long as the Bayes risk $r(\hat{S}^{(N)} \mid \pi)$ is finite. Therefore, the remaining problem is to determine and construct an appropriate prior $\pi$. 
Non-informative priors for time series models, such as the Jeffreys prior, which is usually improper, have been discussed in previous works [5, 22, 15]. We propose a proper prior \( \pi(-1) \) defined on the complex parameter space \( \Theta \subset \mathbb{C}^p \) for the complex-valued stationary autoregressive processes \( \text{AR}(p; \mathbb{C}) \) of order \( p \geq 1 \). The Bayesian predictive power spectral density \( \hat{S}^N_{\pi(-1)} \) based on the proposed prior \( \pi(-1) \) asymptotically dominates the estimative power spectral density \( S_{\hat{\theta}(N)} \) with the maximal likelihood estimator \( \hat{\theta}(N) \). Moreover, the proposed predictive power spectral density \( \hat{S}^N_{\pi(-1)} \) asymptotically dominates the Bayesian predictive power spectral density \( \hat{S}^N_{\pi J} \) based on the Jeffreys prior \( \pi_J \), and the \( O(N^{-2}) \) term of the risk improvement is constant regardless of \( \theta \in \Theta \):

\[
R(\hat{S}^N_{\pi_J} | \theta) - R(\hat{S}^N_{\pi(-1)} | \theta) = \frac{2p(p+1)}{N^2} + O(N^{-\frac{5}{2}}),
\]

which is summarized as the Main Theorem in Section 4.

An eigenfunction \( \phi \) of the Laplacian (Laplace–Beltrami operator) \( \Delta \) plays a crucial role in constructing the proposed prior \( \pi(-1) \). The importance of the super-harmonicity for the shrinkage effect of the estimation of the mean of a multivariate normal distribution is mentioned in [6, 17]. More generally, it is known that the super-harmonicity of the ratio of the proposed prior to the Jeffreys prior is the key to inducing the shrinkage effect [11].

Another important property lying behind the construction of the proposed prior \( \pi(-1) \) is the Kählerness, the generalization of the concept of exponential families, of the complex parameter space \( \Theta \). The parameter space of the complex-valued stationary autoregressive moving average processes \( \text{ARMA}(p, q; \mathbb{C}) \) is shown to be Kähler in [7].

We give a general construction of priors utilizing a positive continuous eigenfunction \( \phi > 0 \) of the Laplacian \( \Delta \) with a negative eigenvalue \( -K < 0 \), i.e., \( \Delta \phi = -K \phi < 0 \). We define a family of priors \( \{\pi(\alpha)\}_{\alpha \in \mathbb{R}} \), which are called \( \alpha \)-priors. We prove that if \( -1 \leq \alpha < 1 \), then \( \hat{S}^N_{\pi(\alpha)} \) asymptotically dominates \( \hat{S}^N_{\pi_J} \). To maximize the worst case of the risk improvement, we propose the \( \alpha \)-prior for \( \alpha = -1 \), which achieves the constant risk improvement. In Section 5, we explicitly give the construction of the positive continuous eigenfunction \( \phi \) with a negative eigenvalue \( -K = -p(p+1) \) on the Kähler parameter space for \( \text{AR}(p; \mathbb{C}) \). Generalization of the proposed prior \( \pi(-1) \) for the i.i.d. case is discussed in Section 6.

In Section 7, numerical experiments are reported for the value of the risk differences \( N^2(R(\hat{S}^N_{\pi_J} | \theta) - R(\hat{S}^N_{\pi(-1)} | \theta)) \) for \( \text{AR}(1; \mathbb{C}) \).
2. Bayesian predictive power spectral densities for complex-valued Gaussian processes. As explained in the Introduction, our aim is to construct a predictive power spectral density \( \hat{S}(N) \) after observing a sample \( z^{(N)} \in \mathbb{C}^N \) of size \( N \). In the present section, we provide the asymptotic expansion of Bayesian predictive power spectral densities \( \hat{S}(N) \) for complex-valued autoregressive moving average processes. This asymptotic expansion is a basic tool for assessing the performance of the choice of a prior \( \pi \).

Let \( \mu \in \mathbb{C}^N \) and \( \Sigma \) be an \( N \times N \) complex-valued positive definite Hermitian matrix. Note that the determinant \( |\Sigma| \) of the matrix \( \Sigma \) is positive.

An \( N \) dimensional complex normal distribution (complex-valued circular-symmetric multivariate normal distribution) with mean \( \mu \) and variance \( \Sigma \) is defined by its probability density function:

\[
p(z \mid \mu, \Sigma) := \frac{1}{\pi^N |\Sigma|} e^{-\frac{(z-\mu)^* \Sigma^{-1} (z-\mu)}{2}},
\]

where \( z = (z^1, \cdots, z^N) \in \mathbb{C}^N \) and \( z^* \) denotes the complex conjugate transpose of \( z \); see [13]. The circular-symmetry of a complex normal distribution with mean \( 0 \in \mathbb{C}^N \) is obvious from the definition (2.1). The complex normal distribution with mean \( 0 \in \mathbb{C}^N \) and its variance-covariance matrix, the identity matrix of size \( N \), is called the standard complex normal distribution of size \( N \).

If we let \( z^i = x^i + \sqrt{-1} y^i \) for \( i = 1, \cdots, N \), the \( 2N \) dimensional real-valued vector \( (x^1, \cdots, x^N, y^1, \cdots, y^N) \) follows the \( 2N \) dimensional real-valued multivariate normal distribution with mean \( (\Re(\mu), \Im(\mu)) \) and variance-covariance matrix

\[
\begin{bmatrix}
\frac{1}{2} \Re(\Sigma) & -\frac{1}{2} \Im(\Sigma) \\
\frac{1}{2} \Im(\Sigma) & \frac{1}{2} \Re(\Sigma)
\end{bmatrix}.
\]

Therefore, an \( N \) dimensional complex normal distribution is a special case of a \( 2N \) dimensional real normal distribution; however, the opposite is not the case.

A complex-valued discrete process \( \{Z_t\}_{t \in \mathbb{Z}} \) is called a Gaussian process (complex-valued circular-symmetric discrete Gaussian process) if the tuple \( (Z_{t_1}, Z_{t_2}, \cdots, Z_{t_N}) \) of size \( N \) follows a complex normal distribution for any \( N \) and any \( t_1, t_2, \cdots, t_N \in \mathbb{Z} \). A complex Gaussian white noise \( \{\varepsilon_t\}_{t \in \mathbb{Z}} \) with variance \( \sigma^2 \) is a Gaussian process, such that \( \frac{1}{N} \langle \varepsilon_{t_1}, \cdots, \varepsilon_{t_N} \rangle \) follows a standard complex normal distribution of size \( N \) for any \( N \) and any \( t_1, t_2, \cdots, t_N \in \mathbb{Z} \).

For a strongly stationary Gaussian process \( \{Z_t\}_{t \in \mathbb{Z}} \), we define the autocovariance \( \gamma_h \) of order \( h \) as the covariance of \( Z_{t+h} \) and \( \overline{Z_t} \). Note that the autocovariances \( \{\gamma_h\}_{h \in \mathbb{Z}} \) are complex-valued, and we have a relation \( \overline{\gamma_h} = \gamma_{-h} \).
for any $h$. The power spectral density $S$ of the process is defined as a Fourier transform

\begin{equation}
S(\omega) := \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \gamma_h e^{-\sqrt{-1}h\omega}
\end{equation}

of the autocovariances $\{\gamma_h\}_{h \in \mathbb{Z}}$, where $\omega \in [-\pi, \pi]$. Because we consider complex-valued processes, power spectral densities are not generally even functions on $[-\pi, \pi]$.

For the observation $z^{(N)} = (z_1, \ldots, z_N)$ of size $N$ from a Gaussian process with mean $0 \in \mathbb{C}$, let us denote its probability density by $p^{(N)}(z^{(N)})$. The probability density $p^{(N)}(z^{(N)})$ is explicitly calculated as (2.1) with mean $\mu = 0$ and its variance-covariance matrix

\begin{equation}
\Sigma^{(N)} := \begin{bmatrix}
\gamma_0 & \gamma_{-1} & \cdots & \gamma_{-N+1} \\
\gamma_1 & \gamma_0 & \cdots & \gamma_{-N+2} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_{N-1} & \gamma_{N-2} & \cdots & \gamma_0 
\end{bmatrix}.
\end{equation}

As a special case of a strongly stationary Gaussian process with mean $0 \in \mathbb{C}$, we introduce a complex-valued autoregressive moving average (ARMA) process. A complex-valued ARMA process of degree $(p,q)$ is a Gaussian process that satisfies the relation

\begin{equation}
Z_t = - \sum_{i=1}^{p} a_i Z_{t-i} + \varepsilon_t + \sum_{i=1}^{q} b_i \varepsilon_{t-i}
\end{equation}

for all $t$, where $a_1, \ldots, a_p, b_1, \ldots, b_q$ are complex-valued coefficients and $\varepsilon_t$ is a complex Gaussian white noise with variance $\sigma^2$. We denote the statistical model of complex-valued stationary ARMA processes by $\text{ARMA}(p,q;\mathbb{C})$ in the present paper. If $q = 0$, we call the model a complex-valued stationary autoregressive model, and denote it by $\text{AR}(p;\mathbb{C})$. We denote the model of real-valued stationary autoregressive processes of order $p$ by $\text{AR}(p;\mathbb{R})$.

The power spectral density of the ARMA model (2.5) is explicitly given by

\begin{equation}
S(\omega) = \frac{\sigma^2}{2\pi} \left| \frac{1 + \sum_{i=1}^{q} b_i e^{-i\sqrt{-1}\omega}}{1 + \sum_{i=1}^{p} a_i e^{-i\sqrt{-1}\omega}} \right|^2.
\end{equation}

Suppose a family of autocovariances $\{\gamma_h\}_{h \in \mathbb{Z}}$ of a Gaussian process are parameterized by complex parameters $\theta \in \Theta \subseteq \mathbb{C}^p$. For each $\theta \in \Theta$, we
 denote the corresponding power spectral density \((2.3)\) by \(S_\theta(\omega)\) or \(S(\omega \mid \theta)\) for \(\omega \in [-\pi, \pi]\).

Because we consider complex parameters \(\theta \in \Theta \subset \mathbb{C}^p\), we make use of Wirtinger calculus; see Appendix A. For the \(i\)-th complex parameter \(\theta^i\), there corresponds Wirtinger derivatives \(\partial_i\) and \(\partial_{\bar{i}}\). For simplicity of notation, for a power spectral density \(S = S_\theta\), we set

\[
S_{\alpha_1\ldots\alpha_a, \beta_1\ldots\beta_b, \ldots, \gamma_1\ldots\gamma_c} := S^{-1}(D_{\alpha_1\ldots\alpha_a}S) S^{-1}(D_{\beta_1\ldots\beta_b}S) \cdots S^{-1}(D_{\gamma_1\ldots\gamma_c}S)
\]

for indices \(\alpha_1, \ldots, \alpha_a, \beta_1 \cdots \beta_b, \ldots, \gamma_1 \cdots \gamma_c \in \{1, \ldots, p, \bar{1}, \ldots, \bar{p}\}\), where \(D_{\alpha_1\ldots\alpha_a} := \partial_{\alpha_1} \cdots \partial_{\alpha_a}\). For example, \(S_{11,2} = S^{-1}(\partial_1\partial_1 S) S^{-1}(\partial_2 S)\). We also set

\[
M_{\alpha_1\ldots\alpha_a, \beta_1\ldots\beta_b, \ldots, \gamma_1\ldots\gamma_c} := \frac{1}{2\pi} \int_{-\pi}^\pi S_{\alpha_1\ldots\alpha_a, \beta_1\ldots\beta_b, \ldots, \gamma_1\ldots\gamma_c}(\omega) \, d\omega
\]

and define the quantities \(g_{\alpha\beta}, T_{\alpha\beta\gamma}\), and \(\Gamma_{\alpha\beta\gamma}\) as

\[
g_{\alpha\beta} := M_{\alpha, \beta} = \frac{1}{2\pi} \int_{-\pi}^\pi (\partial_\alpha \log S(\omega \mid \theta))(\partial_\beta \log S(\omega \mid \theta)) \, d\omega,
\]

\[
T_{\alpha\beta\gamma} := 2M_{\alpha, \beta\gamma} = \frac{1}{\pi} \int_{-\pi}^\pi (\partial_\alpha \log S(\omega \mid \theta))(\partial_\beta \log S(\omega \mid \theta))(\partial_\gamma \log S(\omega \mid \theta)) \, d\omega,
\]

\[
\Gamma_{\alpha\beta\gamma} := M_{\alpha\beta, \gamma} = \frac{1}{2\pi} \int_{-\pi}^\pi (\partial_\alpha \log S(\omega \mid \theta))(\partial_\beta \log S(\omega \mid \theta))(\partial_\gamma \log S(\omega \mid \theta)) \, d\omega
\]

for \(\alpha, \beta, \gamma \in \{1, \ldots, p, \bar{1}, \ldots, \bar{p}\}\). Throughout the remainder of this paper, Einstein notation is assumed. Therefore, the summation is automatically taken over those indices that appear exactly twice, once as a superscript and once as a subscript. The symbols \(\alpha, \beta, \gamma, \ldots\) run through the indices \(\{1, \ldots, p, \bar{1}, \ldots, \bar{p}\}\).

The complex-valued \(2p \times 2p\) matrix \([g_{\alpha\beta}]\) is called the Fisher information matrix. The Fisher information matrix naturally induces the metric on the complex parameter space \(\Theta\). The inner product of two functions \(N_1 = N_1(\omega \mid \theta)\) and \(N_2 = N_2(\omega \mid \theta)\) defined on \([-\pi, \pi]\) at \(\theta \in \Theta\) is defined as

\[
\langle N_1, N_2 \rangle_\theta := \frac{1}{2\pi} \int_{-\pi}^\pi \frac{N_1(\omega \mid \theta) N_2(\omega \mid \theta)}{S(\omega \mid \theta)^2} \, d\omega
\]

and the norm \(||N||_\theta\) of a function \(N = N(\omega \mid \theta)\) at \(\theta \in \Theta\) is defined as \(||N||^2_\theta := \langle N, N \rangle_\theta\); see also [10].
This form (2.9) of the Fisher information matrix was introduced in [21] for real-valued time series analysis. For real-valued processes, the constant $4\pi$, rather than $2\pi$, is usually used for the denominator in (2.9); see [21, 4, 10, 19]. On the other hand, the constant $2\pi$ is used for the signal processing; see [2, 7]. For complex-valued processes, it is natural to use the constant $2\pi$ as in (2.9) because it gives $g_{\alpha\beta} = \frac{1}{N} E[-\partial_\alpha \partial_\beta l(N)] + O(N^{-1})$, where $l(N)$ denotes the log-likelihood (B.2); see Proposition D.2 in Appendix D.

Let us denote by $[g_{\alpha\beta}]$ the inverse matrix of the Fisher information matrix $[g_{\alpha\beta}]$, i.e., $g_{\alpha\gamma}g_{\gamma\beta} = \delta_{\alpha\beta}$ for the Kronecker delta $\delta_{\alpha\beta}$. The prior defined as the square root of the determinant of the $2p \times 2p$ complex-valued matrix $[g_{\alpha\beta}]$ is called the Jeffreys prior and denoted by $\pi_J$ in the present paper.

For a possibly improper prior $\pi$, we define the Bayesian predictive power spectral density $\hat{S}_\pi(N)$ by

$$
\hat{S}_\pi(N)(\omega) := \int_\Theta S(\omega \mid \theta) \pi(\theta \mid Z(N))\ d\theta
$$

for $\omega \in [-\pi, \pi]$, where

$$
\pi(\theta \mid Z(N)) := \frac{p_\theta(Z(N))\pi(\theta)}{\int_\Theta p_{\theta'}(Z(N))\pi(\theta') \ d\theta'}
$$

is the posterior distribution given an observation $Z(N) \in \mathbb{C}^N$.

Let us fix a possibly improper prior $\pi$ and assume that $\int_\Theta p_\theta(Z(N))\pi(\theta)\ d\theta$ is finite for any $Z(N) \in \mathbb{C}^N$ and the Bayesian predictive power spectral density (2.13) exists for any $\omega \in [-\pi, \pi]$. The asymptotic expansion of a Bayesian predictive power spectral density (2.13) of a complex-valued ARMA process around the maximal likelihood estimator $\hat{\theta}(N)$ is

$$
\hat{S}_\pi(N)(\omega) = S(\omega \mid \hat{\theta}) + \frac{1}{N} \left( G_\pi(N)(\omega \mid \hat{\theta}) + H(N)(\omega \mid \hat{\theta}) \right) + O_P(N^{-\frac{3}{2}}),
$$

where functions $G_\pi(N)$ and $H(N)$ represent the parallel and orthogonal parts of the quantity $N(S_\pi(N) - S_{\hat{\theta}(N)})$, respectively; see Appendix F.

Functions $G_\pi(N)$ and $H(N)$ are explicitly given by

$$
G_\pi(N)(\omega \mid \theta) := \frac{1}{2} g^{\alpha\beta}(\omega \mid \theta) \left( \partial_\alpha \log \frac{\pi}{\pi_J}(\theta) + \frac{1}{2} T_\alpha(\omega \mid \theta) \right) \partial_\beta S(\omega \mid \theta),
$$

(2.16)

$$
H(N)(\omega \mid \theta) := \frac{1}{2} g^{\alpha\beta}(\omega \mid \theta) \left( \partial_\alpha \partial_\beta S(\omega \mid \theta) - \Gamma_{\alpha\beta\gamma}(\theta) \partial_\gamma S(\omega \mid \theta) \right),
$$

(2.17)
where \( \Gamma_{\alpha\beta}^{\gamma}(\theta) := \Gamma_{\alpha\beta\delta}(\theta) g^{\delta\gamma}(\theta) \) and \( T_\alpha(\theta) := T_{\alpha\beta\gamma}(\theta) g^{\beta\gamma}(\theta) \). Note first that \( G^{(N)}_\pi \) and \( H^{(N)}_\pi \) are orthogonal in the sense that \( \langle G^{(N)}_\pi, H^{(N)}_\pi \rangle_\theta = 0 \) for any \( \theta \in \Theta \). Note also that, while the parallel part \( G^{(N)}_\pi \) may depend on the choice of prior \( \pi \), the orthogonal part \( H^{(N)}_\pi \) is independent of the choice; see \([11]\) for more detail. See Appendix F for the proof of the expansion (2.15).

The Bayesian predictive power spectral density \( \hat{S}^{(N)}_\pi \) minimizes the Bayes risk (1.3) among all the predictive power spectral densities \( \hat{S}^{(N)} \) as long as the Bayes risk of \( \hat{S}^{(N)}_\pi \) is finite; see Appendix B. Therefore, once we have a prior \( \pi \), we are able to calculate the best predictive power spectral density \( \hat{S}^{(N)}_\pi \) in the sense that it minimizes the Bayes risk (1.3). The only remaining problem is to find a reasonable prior \( \pi \).

### 3. Kähler parameter spaces for complex-valued autoregressive processes.

Let us consider a family \( \{S_\theta\}_{\theta \in \Theta} \) of power spectral densities of complex-valued stationary ARMA processes, where \( \Theta \subset \mathbb{C}^p \) is a complex parameter space. If the Fisher information matrix \( [g_{\alpha\beta}] \) of the process satisfies the relations \( g_{ij} = g_{i\bar{j}} = 0 \), \( g_{i\bar{j}} = g_{\bar{j}i} \) for all \( i, j = 1, \cdots, p \) and the relations \( \partial_i g_{jk} = \partial_j g_{ik} \), \( \partial_{\bar{i}} g_{j\bar{k}} = \partial_{\bar{k}} g_{j\bar{i}} \) for all \( i, j, k = 1, \cdots, p \), we say that the complex parameter space \( \Theta \) is Kähler; see also Appendix A. The Kählerness of the complex parameter space \( \Theta \) plays an important role in constructing priors.

A specific complex parameter space \( \Theta \subset \mathbb{C}^{p+q} \) for complex-valued stationary autoregressive moving average processes ARMA(\( p, q; \mathbb{C} \)) was shown to be Kähler in \([7]\). We focus this specific Kähler parameter space \( \Theta \) for complex-valued stationary autoregressive processes AR(\( p; \mathbb{C} \)).

We examine the power spectral densities of AR(\( p; \mathbb{C} \)) of the form

\[
S(\omega) = \frac{1}{2\pi} \left| \prod_{i=1}^{p} (1 - \xi^i e^{-\sqrt{-1}\omega}) \right|^2,
\]

where complex parameters \( \xi = (\xi^1, \cdots, \xi^p) \) are roots of the polynomial \( z^p(1 + \sum_{i=1}^{p} a_i z^{-i}) \) of the formal variable \( z \), and \( \sigma^2 = 1 \) is assumed. From the stationarity condition, we assume that \( |\xi^i| < 1 \) for any \( i = 1, \cdots, p \).

We define the parameter space \( \Theta_1 \subset \mathbb{C}^p \) as

\[
\Theta_1 := \{ (\xi^1, \cdots, \xi^p) \in \mathbb{C}^p \mid |\xi^i| < 1 \text{ for any } i = 1, \cdots, p \} = U \times U \times \cdots \times U,
\]

where \( U \) is the open unit disk in the complex plane \( \mathbb{C} \). In this specific parameterization \( \xi = (\xi^1, \cdots, \xi^p) \), the center \( 0 = (0, \cdots, 0) \in \mathbb{C}^p \) corresponds to the white noise process.
Because we want to ignore the measure zero subset, where the denominator of (3.1) has multiple roots, we restrict our attention to the dense subset
(3.2) \[ \Theta_1 := \left\{ (\xi^1, \cdots, \xi^p) \in \tilde{\Theta}_1 \mid \xi^i \neq \xi^j \text{ for any } i, j = 1, \cdots, p \right\} \]
of the original parameter space \( \tilde{\Theta}_1 \). The parameter space \( \Theta_1 \) is a complex manifold of complex dimension \( p \), and the space \( \Theta_1 \) is open as a topological space with a boundary \( \partial \Theta_1 \). In particular, \( \Theta_1 \) is relatively compact but not compact.

For the specific parameterization \( \xi = (\xi^1, \cdots, \xi^p) \) defined in (3.1) for AR(\( p, \mathbb{C} \)), the Fisher information matrix is explicitly given by
(3.3) \[ g_{ij} = g_{\bar{i} \bar{j}} = 0, \quad g_{i \bar{j}} = g_{\bar{j} i} = \frac{1}{1 - \xi^i \bar{\xi}^j} \]
for \( i, j = 1, \cdots, p \). Therefore, the complex parameter space \( \Theta_1 \) is Kähler. This is a very important property of the complex parameter space \( \Theta_1 \) for analyzing the super-harmonicity of priors.

For a Kähler parameter space, the Jeffreys prior is the determinant of the \( p \times p \) complex-valued Hermitian matrix \( [g_{ij}] \); see Appendix A. For the specific parameterization \( \xi = (\xi^1, \cdots, \xi^p) \) defined in (3.1) for AR(\( p, \mathbb{C} \)), the Jeffreys prior \( \pi_J \) is explicitly given by
(3.4) \[ \pi_J(\xi) = \frac{\prod_{1 \leq i < j \leq q} |\xi^i - \xi^j|^2}{\prod_{i=1}^{p} \prod_{j=1}^{p} (1 - \xi^i \bar{\xi}^j)}. \]
The Jeffreys prior (3.4) for AR(\( p, \mathbb{C} \)) is continuous in the parameter space \( \tilde{\Theta}_1 = U \times \cdots \times U \). The Jeffreys prior vanishes if and only if the denominator of (3.1) has multiple roots. Thus, the Jeffreys prior is strictly positive on the parameter space \( \Theta_1 \). Moreover, the Jeffreys prior diverges at the boundary \( \partial \tilde{\Theta}_1 \) of the parameter space \( \tilde{\Theta}_1 \), and defines an improper prior on \( \tilde{\Theta}_1 \).

4. Main Theorem. Let us consider a family \( \{S_\theta\}_{\theta \in \Theta} \) of power spectral densities of complex-valued stationary ARMA processes, where \( \Theta \subset \mathbb{C}^p \) is a complex parameter space. Our objective is to construct a predictive power spectral density \( \hat{S}^{(N)} \) whose risk \( R(\hat{S}^{(N)}) \) is kept as small as possible. We say that a predictive power spectral density \( \hat{S}_1^{(N)} \) dominates a predictive power spectral density \( \hat{S}_2^{(N)} \) if \( R(\hat{S}_1^{(N)} | \theta) \leq R(\hat{S}_2^{(N)} | \theta) \) for any \( \theta \in \Theta \) and the strict inequality holds for some \( \theta \).

Suppose that the parameter space \( \Theta \) is Kähler and there exists a positive continuous eigenfunction \( \phi \) of the Laplacian (Laplace–Beltrami operator \( A.12 \)) with a negative eigenvalue \( -K \) globally defined on \( \Theta \). We define
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a family of priors \( \{ \pi^{(\alpha)} \}_{\alpha \in \mathbb{R}} \), called \( \alpha \)-priors, as \( \pi^{(\alpha)} := \phi^{-\alpha+1} \pi_J \), where \( \pi_J \) denotes the Jeffreys prior. We state that, with a suitable choice of \( \alpha \in \mathbb{R} \), the Bayesian predictive power spectral density \( \hat{S}^{(N)}_{\pi^{(\alpha)}} \) based on the proposed prior \( \pi^{(\alpha)} \) asymptotically dominates the Bayesian predictive power spectral density \( \hat{S}^{(N)}_{\pi_J} \) based on the Jeffreys prior \( \pi_J \).

Throughout this section, we use the Einstein notation, where the symbols \( i, j, k, \cdots \) run through the indices \( \{ 1, \cdots, p \} \); see Appendix A. For example, the quantity \( g^{ij} \left( \frac{\partial}{\partial \theta_i} \log \phi \right) \left( \frac{\partial}{\partial \bar{\theta}_j} \log \phi \right) \) represents a non-negative function \( \sum_{i=1}^{p} \sum_{j=1}^{p} g^{ij} \left( \frac{\partial}{\partial \theta_i} \log \phi \right) \left( \frac{\partial}{\partial \bar{\theta}_j} \log \phi \right) \) defined on the parameter space \( \Theta \).

We have proved the following theorem for \( \alpha \)-priors for complex-valued ARMA processes \( \{ S_\theta \}_{\theta \in \Theta} \), where the Kählerness of the complex parameter space \( \Theta \subset \mathbb{C}^p \) and the existence of a positive continuous eigenfunction \( \phi \) of the Laplacian with a negative eigenvalue \( -K \) defined globally on the parameter space \( \Theta \) are assumed. Recall that the Fisher information matrix \([g_{ij}]\) of the model is defined as (2.9).

**Theorem 4.1 (Main Theorem).** Let \( \pi_1 := \pi^{(\alpha_1)} \) and \( \pi_2 := \pi^{(\alpha_2)} \) be two \( \alpha \)-priors for \( \alpha_1, \alpha_2 \in \mathbb{R} \), and assume that Bayesian predictive power spectral densities \( \hat{S}^{(N)}_{\pi^{(\alpha)}} \) exist for \( \alpha = \alpha_1, \alpha_2 \). Then, we have

\[
R \left( \hat{S}^{(N)}_{\pi_1} \mid \theta \right) - R \left( \hat{S}^{(N)}_{\pi_2} \mid \theta \right) = \frac{(\alpha_1 - \alpha_2)K + (\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2)g^{ij} \left( \frac{\partial}{\partial \theta_i} \log \phi \right) \left( \frac{\partial}{\partial \bar{\theta}_j} \log \phi \right)}{N^2} + O \left( N^{-\frac{5}{2}} \right)
\]

for \( \theta \in \Theta \).

The proof of the Main Theorem is largely aided by the form (2.15) of the asymptotic expansion of the Bayesian predictive power spectral density \( \hat{S}^{(N)}_{\pi} \) around the maximal likelihood estimator \( \hat{\theta}^{(N)} \). The eventual proof of the Main Theorem is given in Appendix E. This theorem is a very general theorem that always holds as long as the parameter space \( \Theta \) is Kähler. The importance of the Kählerness of parameter spaces in statistics and the generalization of the Main Theorem for the i.i.d. case are discussed in Section 6. The metric \( g_{ij} \) and its inverse \( g^{ij} \) of a Kähler manifold is explained in Appendix A.

Setting \( \pi_1 := \pi^{(+1)} = \pi_J \) in Theorem 4.1, we can easily see that if \( -1 \leq \alpha < 1 \), then \( \hat{S}^{(N)}_{\pi^{(\alpha)}} \) asymptotically dominates \( \hat{S}^{(N)}_{\pi_J} \).
Corollary 4.1. Let $\pi := \pi^{(\alpha)}$ be an $\alpha$-prior on a Kähler parameter space $\Theta \subset \mathbb{C}^p$. We have

$$R(S^{(N)}_{\pi_j} \mid \theta) - R(S^{(N)}_{\pi^{(\alpha)}} \mid \theta) = \frac{(1 - \alpha)K + (1 - \alpha^2)g^{\tilde{\beta}}(\partial_{\beta} \log \phi)(\partial_{\bar{\beta}} \log \phi)}{N^2} + O(N^{-\frac{5}{2}}).$$

Therefore, if $-1 \leq \alpha < 1$, then $\hat{S}^{(N)}_{\pi_j}$ asymptotically dominates $\hat{S}^{(N)}_{\pi^{(\alpha)}}$.

Recall that $g^{\tilde{\beta}}(\partial_{\beta} \log \phi)(\partial_{\bar{\beta}} \log \phi) \geq 0$. Therefore, to maximize the worst case of the risk improvement, we propose the $\alpha$-prior for $\alpha = -1$. When $\alpha = -1$, the Bayesian predictive power spectral density $\hat{S}^{(N)}_{\pi_{-1}}$ achieves constant risk improvement $R(\hat{S}^{(N)}_{\pi_{-1}} \mid \theta) = 2K/N^2 + O(N^{-\frac{5}{2}})$. The formula (1.4) is a special case of Corollary 4.1 for AR($p; \mathbb{C}$) and $\alpha = -1$, where $K = p(p + 1)$. The existence of the positive continuous eigenfunction $\phi$ with a negative eigenvalue $-K = -p(p + 1)$ on the parameter space $\Theta_1$ for AR($p; \mathbb{C}$) is given in Section 5.

5. Super-harmonic priors on AR($p; \mathbb{C}$). In the present section, we prove the existence of the positive continuous eigenfunction $\phi$ of the Laplacian $\Delta$ with eigenvalue $-K = -p(p + 1)$ for AR($p; \mathbb{C}$). Furthermore, for the AR($p; \mathbb{C}$) model, we show that the Bayesian predictive power spectral density $\hat{S}^{(N)}_{\pi_{-1}}$ based on the $(-1)$-prior $\pi^{(-1)}$ asymptotically dominates the estimative power spectral density $S_{\hat{\theta}^{(N)}}$ with the maximal likelihood estimator $\hat{\theta}^{(N)}$. This is another reason why we propose the $(-1)$-prior $\pi^{(-1)}$ for the case of AR($p; \mathbb{C}$). Throughout this section, we use the Einstein notation, where the symbols $i, j, k, \cdots$ run through the indices $\{1, \cdots, p\}$.

The eigenfunction $\phi$ for AR($p; \mathbb{C}$) is defined as

$$\phi(\xi) := \prod_{i=1}^{p} \prod_{j=1}^{p} (1 - \xi^i \xi^j)$$

for $\xi = (\xi^1, \cdots, \xi^p) \in \tilde{\Theta}_1 = U \times \cdots \times U$. The function $\phi$ is the inverse of the determinant $|\Sigma^{(N)}|$ of the variance-covariance matrix $\Sigma^{(N)}$ of size $N \geq p$ for AR($p; \mathbb{C}$); see Appendix G. The function $\phi$ is a real-valued continuous function defined globally on the parameter space $\tilde{\Theta}_1$. Moreover, it is positive on $\tilde{\Theta}_1$ and is 0 at the boundary $\partial \tilde{\Theta}_1$ of $\tilde{\Theta}_1$. Note also that the function $\phi$ has its maximum at the white noise process.
The \(\alpha\)-prior \(\pi^{(\alpha)}\) for \(\text{AR}(p; \mathbb{C})\) is

\[
(5.2) \quad \pi^{(\alpha)} := \phi^{-\alpha+1} \pi_J = \left( \prod_{i=1}^{p} \prod_{j=1}^{p} (1 - \xi^i \bar{\xi}^j) \right)^{-\alpha} \prod_{1 \leq i < j \leq q} |\xi^i - \xi^j|^2,
\]

where \(\pi_J\) is the Jeffreys prior \((3.4)\) for \(\text{AR}(p; \mathbb{C})\). The \(\alpha\)-prior \(\pi^{(\alpha)}\) for \(\text{AR}(p; \mathbb{C})\) is proper if \(\alpha < 1\) and improper if \(\alpha \geq 1\) on the parameter space \(\Theta_1\); see Appendix G. In particular, the Jeffreys prior \(\pi_J = \pi^{(+1)}\) is improper on the parameter space \(\Theta_1\).

The Bayesian predictive power spectral densities \(\hat{\varphi}^{(N)}\) for \(\text{AR}(p; \mathbb{C})\) based on the \(\alpha\)-prior \(\pi^{(\alpha)}\) exists if \(\alpha < 2\) and \(N \geq p\); see Appendix G.

Before proving \(\Delta \phi = -p(p+1)\phi\), we introduce a useful lemma.

**Lemma 5.1.**

\[
(5.3) \quad \partial_i \log \phi = \sum_{j=1}^{p} \frac{-\xi^j}{1 - \xi^i \bar{\xi}^j} = -g_{ij} \bar{\xi}^j, \quad \partial_j \log \phi = \sum_{i=1}^{p} \frac{-\xi^i}{1 - \xi^i \bar{\xi}^j} = -g_{ij} \xi^i.
\]

\[
(5.4) \quad \xi^i \partial_i \log \pi_J = \bar{\xi}^j \partial_j \log \pi_J = \frac{1}{2} p(p-1) + \xi^i \bar{\xi}^j g_{ij}.
\]

**Proof.** Use the identity \(\sum_{i=1}^{p} \sum_{j\neq i}^{p} \frac{\xi^i}{1 - \xi^i \bar{\xi}^j} = \frac{1}{2} p(p-1)\). \(\square\)

Using Lemma 5.1, we see that \(\phi\) is in fact an eigenfunction of the Laplacian with eigenvalue \(-K = -p(p+1)\).

**Proposition 5.1.**

\[
(5.5) \quad \Delta \phi = -p(p+1)\phi.
\]

**Proof.** Because the parameter space is Kähler, we can use the formula \((A.12)\) for its definition of the Laplacian. The direct computation shows

\[
\Delta \phi = 2g^{ij} \partial_i \partial_j \phi = 2g^{ij} \partial_i \left(-g_{kj} \xi^k \phi \right)
= 2g^{ij} \left(- (\partial_i g_{kj}) \xi^k \phi - g_{kj} (\partial_i \xi^k) \phi - g_{kj} \xi^k (\partial_i \phi) \right)
= 2g^{ij} \left(- (\partial_k g_{ij}) \xi^k \phi - g_{ij} \phi + g_{kj} g_{l}\xi^k \bar{\xi}^l \phi \right)
= -2 \xi^k (\partial_k \log \pi_J) \phi - 2g^{ij} g_{ij} \phi + 2g_{kl} \xi^k \bar{\xi}^l \phi
= -p(p-1) \phi - 2p\phi = -p(p+1) \phi,
\]

where we have used the Kählerness \((A.11)\), the Jacobi formula \((A.10)\), and \(g^{ij} g_{ij} = p\). \(\square\)
As stated in Corollary 4.1, $\hat{S}^{(N)}_{\pi^{(\alpha)}}$ asymptotically dominates $\hat{S}^{(N)}_{\pi^{(1+\alpha)}}$ if $-1 \leq \alpha < 1$. For AR(2; $\mathbb{C}$), the specific prior $\psi := (1 - \xi^1 \bar{\xi}^2)(1 - \xi^2 \bar{\xi}^1)(1 - |\xi^1|^2)(1 - |\xi^2|^2)$ is introduced as a super-harmonic prior in [7]. This prior $\psi$ is the special case of Corollary 4.1 for $(p, \alpha) = (2, 0)$. For AR($p; \mathbb{R}$) with $p \geq 2$, a similar but slightly different prior is presented in [18]. This prior corresponds to the $\alpha = 0$ case for a positive eigenfunction of the Laplacian with eigenvalue $-K = -p(p - 1)$.

Let us fix the true parameter $\theta_0 \in \Theta$, and denote the maximal likelihood estimator by $\hat{\theta}^{(N)} := \hat{\theta}^{(N)}(z^{(N)})$ for the observation $z^{(N)} \in \mathbb{C}^N$. According to [10], if we fortunately find a prior $\pi$ such that $G^{(N)}_{\pi} = 0$, then we have

$$R(S_{\hat{\theta}^{(N)}} \mid \theta_0) - R(\hat{S}^{(N)}_{\pi} \mid \theta_0) = \frac{1}{2N^2}\|H^{(N)}\|^2_{\hat{\theta}_0} + O(N^{-\frac{3}{2}}).$$

For the specific parametrization $\xi \in \Theta_1 \subset \mathbb{C}^p$ for AR($p; \mathbb{C}$) defined in (3.1), the direct computation shows

$$G^{(N)}_{\pi^{(\alpha)}} = 2g^j\left(\partial_i \log \frac{\pi^{(\alpha)}_{\pi}}{\pi_j} + \frac{1}{2}T_i\right)\partial_j S = -2(\alpha + 1)\xi^j \partial_j S$$

for the $\alpha$-prior $\pi^{(\alpha)} := \phi^{-\alpha+1}\pi_j$. Thus, if $\alpha = -1$, then $G^{(N)}_{\pi} = 0$. Therefore, $\hat{S}^{(N)}_{\pi^{(1+\alpha)}}$ asymptotically dominates the estimative power spectral density $S_{\hat{\theta}^{(N)}}$ with the maximal likelihood estimator $\hat{\theta}^{(N)}$.

6. Generalization of the Main Theorem. Although the present paper mainly focuses on the complex Gaussian process, the Main Theorem is valid for the i.i.d. case as long as the complex parameter space is Kähler.

Consider first a family of the probability density functions $\{p_\theta\}_{\theta \in \Theta}$ of the exponential family parameterized by real parameters $\theta = (\theta^1, \cdots, \theta^p) \in \Theta \subset \mathbb{R}^p$. We may assume the probability density function is of the form $p_\theta(x) = \exp(\theta^1 x_1 - \Psi(\theta))$. We know that the Fisher information matrix is given by $g_{ij} = \partial_i \partial_j \Psi$.

The Kähler parameter space $\Theta$ is the generalization of the exponential family in the sense that there exists, at least locally, a function $K$ on $\Theta$, called a Kähler potential, such that $g_{ij} = \partial_i \partial_j K$. If there exists a Kähler potential $K$ on $\Theta$, then it is easy to see that the definition of Kählerness (A.11) holds. The converse is also true; see [7, 14].

Suppose we have a family of probability density functions $\{p_\theta\}_{\theta \in \Theta}$ parameterized by complex parameters $\theta = (\theta^1, \cdots, \theta^p) \in \Theta \subset \mathbb{C}^p$, where $\Theta$ is Kähler. Denote the sample space of this model by $\mathcal{Z}$; the sample space $\mathcal{Z}$ may be any subset of $\mathbb{R}^r$ or $\mathbb{C}^r$. Let $\pi$ be a possibly improper prior for this
model. Suppose we have an i.i.d. sample \( z^{(N)} = (z_1, \ldots, z_N) \in \mathbb{Z}^N \) of size \( N \) from the distribution at \( \theta \in \Theta \). The predictive distribution \( \hat{p}_\pi^{(N)}(z) := \int_\theta p_\theta(z) \pi(\theta \mid z^{(N)}) \, d\theta \) for \( z \in \mathbb{Z} \) is called the Bayesian predictive distribution based on prior \( \pi \). The risk of \( \hat{p}_\pi^{(N)} \) is defined as

\[
R(\hat{p}_\pi^{(N)} \mid \theta) := E_\theta \left[ D_{KL} \left( p_\theta \parallel \hat{p}_\pi^{(N)} \right) \right] = \int_{\mathbb{Z}^N} D_{KL} \left( p_\theta \parallel \hat{p}_\pi^{(N)} \right) dP_\theta^{(N)}(z^{(N)}),
\]

where \( dP_\theta^{(N)}(z^{(N)}) := \prod_{i=1}^N p_\theta(z_i) \, dz_i \). We omit the details due to space limitations, but we have a similar asymptotic expansion of \( \hat{p}_\pi^{(N)} \) to (2.15); see [9]. Suppose we have a positive continuous eigenfunction \( \phi \) of the Laplacian \( \Delta \) with negative eigenvalue \( -K \), i.e., \( \Delta \phi = -K \phi < 0 \). Then, we can construct the \( \alpha \)-prior by \( \pi^{(\alpha)} := \phi^{-\alpha-1} \pi_J \) for \( \alpha \in \mathbb{R} \), where \( \pi_J \) is the Jeffreys prior of this model.

Theorem 4.1 holds for the Kähler parameter space \( \Theta \); for two \( \alpha \)-priors \( \pi_1 := \pi^{(\alpha_1)} \) and \( \pi_2 := \pi^{(\alpha_2)} \),

\[
R(\hat{p}_{\pi_1}^{(N)} \mid \theta) - R(\hat{p}_{\pi_2}^{(N)} \mid \theta) = \frac{(\alpha_1 - \alpha_2)K + (\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2)g^{ij}(\partial_i \log \phi)(\partial_j \log \phi)}{N^2} + O_P(N^{-\frac{3}{2}})
\]

for \( \theta \in \Theta \). Therefore, our proposal is to use the prior \( \pi^{(-1)} := \phi^{-2} \pi_J \) where \( \phi \) is an eigenfunction of the Laplacian with the smallest negative eigenvalue.

7. Numerical experiments for risk differences for AR(1; \( \mathbb{C} \)). We consider the AR(1; \( \mathbb{C} \)) case of \( z^t = \xi z^{t-1} + \varepsilon \) in the present section. The parameter space is the open unit disk \( U = \{ \xi \in \mathbb{C} \mid ||\xi|| < 1 \} \). The \( \alpha \)-prior for AR(1; \( \mathbb{C} \)) is \( \pi^{(\alpha)}(\xi) := (1 - ||\xi||^2)^{-\alpha} \) for \( \xi \in U \). Recall that \( \alpha = +1 \) corresponds to the improper Jeffreys prior \( \pi_J = (1 - ||\xi||^2)^{-1} \), which is mentioned as a reference prior in [5] for AR(1; \( \mathbb{R} \)). On the other hand, \( \alpha = -1 \) corresponds to the proposed proper prior \( \pi^{(-1)} = (1 - ||\xi||^2) \), which is the inverse of the Jeffreys prior, and is also mentioned in [22] for the AR(1; \( \mathbb{R} \)) case. In fact, the inverse of the Jeffreys prior for AR(1; \( \mathbb{R} \)) is a maximal data information prior (MDIP) for AR(1; \( \mathbb{R} \)); see [22]. Note that for \( p \geq 2 \), the proposed proper prior \( \pi^{(-1)} \) is not the inverse of the Jeffreys prior \( \pi^{(+1)} \) for AR(1; \( \mathbb{C} \)).

Corollary 4.1 for AR(1; \( \mathbb{C} \)) now becomes

\[
R(\hat{S}_{\pi_J}^{(N)} \mid \xi) - R(\hat{S}_{\pi^{(\alpha)}}^{(N)} \mid \xi) = \frac{1}{N^2}Q^{(\alpha)}(\xi) + O(N^{-\frac{3}{2}}),
\]

where
where

\begin{equation}
Q^{(\alpha)}(\xi) := 2(1 - \alpha) + (1 - \alpha^2) \frac{|\xi|^2}{1 - |\xi|^2}
\end{equation}

is the expected pointwise limit of the risk difference

\begin{equation}
N^2 (R^{(N)}_j - R^{(N)}_\alpha) := N^2 \left( R \left( \hat{S}^{(N)}_\pi \mid \xi \right) - R \left( \hat{S}^{(N)}_{\pi(\alpha)} \mid \xi \right) \right),
\end{equation}

which is displayed in Figure 1. As stated in Corollary 4.1, if \(-1 \leq \alpha \leq 1\), then \(Q^{(\alpha)}(\xi) \geq 0\). In particular, if \(\alpha = -1\), then the risk difference \(N^2 (R^{(N)}_j - R^{(N)}_\alpha)\) asymptotically achieves the constant \(Q^{(-1)}(\xi) = 2K = 4\).

The numerical experiment results of (7.3) for \(N = 30\) and \(N = 120\) are displayed in Figure 2 and Figure 3 respectively, where the Monte Carlo method is used for evaluating the value of (2.13). From Figure 2 and Figure 3, we see that the risk difference (7.3) asymptotically achieves \(Q^{(\alpha)}(\xi)\), but the rate of convergence may depend on \(\xi \in U\). It appears that the convergence is not uniform on \(U\).
Fig 2. Numerical experiment for the risk difference $N^2(R_J^{(N)} - R_0^{(N)})$ for AR(1; C) with $N = 30$, where $\xi \in (-1, 1) \subset U$. The proposed $\hat{S}_{\pi(-1)}^{(N)}$ dominates the baseline $\hat{S}_{\pi J}^{(N)}$ for $\xi \in [-0.825, 0.825]$.

Fig 3. Numerical experiment for the risk difference $N^2(R_J^{(N)} - R_0^{(N)})$ for AR(1; C) with $N = 120$, where $\xi \in (-1, 1) \subset U$. The proposed $\hat{S}_{\pi(-1)}^{(N)}$ dominates the baseline $\hat{S}_{\pi J}^{(N)}$ for $\xi \in [-0.9, -0.85] \cup [-0.7, 0.925]$. 
APPENDIX A: WIRTINGER CALCULUS

In this section, we introduce an elegant equivalent formulation of usual differential calculus, called Wirtinger calculus [12]. Let us consider a complex-valued function $f$ defined on $\mathbb{C}^p$. A function defined on the domain $\mathbb{C}^p$ is always regarded as a function defined on the domain $\mathbb{R}^{2p}$. For the $i$-th complex coordinate $z^i = x^i + \sqrt{-1} y^i$ in $\mathbb{C}^p$, define the Wirtinger derivatives $\partial_i$ and $\partial_{\bar{i}}$ as the linear partial differential operators of first order

$$\partial_i := \frac{1}{2} \left( \frac{\partial}{\partial x^i} - \sqrt{-1} \frac{\partial}{\partial y^i} \right), \quad \partial_{\bar{i}} := \frac{1}{2} \left( \frac{\partial}{\partial x^i} + \sqrt{-1} \frac{\partial}{\partial y^i} \right)$$

where $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial y^i}$ denote the usual partial differential operators on $\mathbb{R}^{2p}$. The symbol $\partial_{\bar{i}}$ is sometimes denoted by $\overline{\partial_i}$.

We should mention that although the variables $z^i$ and $\overline{z^i}$ are not independent, the derivatives $\partial_i$ and $\partial_{\bar{i}}$ are independent as differential operators in the complexified tangent space of $\mathbb{C}^p = \mathbb{R}^{2p}$. In fact, the direct computation shows that the set $\{\partial_1, \ldots, \partial_p, \partial_{\bar{1}}, \ldots, \partial_{\bar{p}}\}$ forms a basis of the complexified tangent space of $\mathbb{C}^p = \mathbb{R}^{2p}$.

The Wirtinger derivatives are not the partial derivatives in usual differential calculus; however, Wirtinger calculus inherits most of the properties that usual differential calculus has.

The most fascinating property of Wirtinger calculus, which is inherited from the usual differential calculus, is its chain rule property,

$$\partial_i (f \circ g) = \sum_{j=1}^{q} (\partial_i g^j)(\partial_j f \circ g) + \sum_{j=1}^{q} (\partial_{\bar{i}} g^j)(\partial_j f \circ g),$$

$$\partial_{\bar{i}} (f \circ g) = \sum_{j=1}^{q} (\partial_{\bar{i}} g^j)(\partial_j f \circ g) + \sum_{j=1}^{q} (\partial_{\bar{i}} \bar{g}^j)(\partial_j f \circ g)$$

for $i = 1, \ldots, p$, where $f: \mathbb{C}^q \to \mathbb{C}$ and $g = (g^1, \ldots, g^q): \mathbb{C}^p \to \mathbb{C}^q$.

Another important property of Wirtinger calculus is its summation rule. For a complex vector $\lambda = (\lambda^1, \ldots, \lambda^p) \in \mathbb{C}^p$ and a complex-valued function $f$ defined on $\mathbb{C}^p$, we can easily see that

$$\sum_{i=1}^{p} \lambda^i \partial_i f + \sum_{j=1}^{p} \overline{\lambda^j} \partial_{\bar{j}} f = \sum_{k=1}^{p} \Re(\lambda^k) \frac{\partial f}{\partial x^k} + \sum_{k=1}^{p} \Im(\lambda^k) \frac{\partial f}{\partial y^k},$$

where the function $f$ is regarded as a function defined on $\mathbb{R}^{2p}$ on the right hand side of the equation.
Throughout the present paper, Einstein notation is assumed. Therefore, the summation is automatically taken over those indices that appear exactly twice once as a superscript and once as a subscript. Therefore, when the Einstein notation is used, the left hand side of (A.4) is denoted by $\lambda^\alpha \partial_\alpha f$ if $\alpha$ runs through the indices $\{1, \ldots, p, \bar{1}, \ldots, \bar{p}\}$, or sometimes by $\lambda^i \partial_i f + \lambda^\bar{j} \partial_{\bar{j}} f$ if $i, \bar{j}$ run through the indices $\{1, \ldots, p\}$, where $\lambda^\bar{j}$ represents the complex conjugate of $\lambda^i$. In the present paper, we try to use the symbols $\alpha, \beta, \gamma, \cdots$ when they run through the indices $\{1, \ldots, p, \bar{1}, \ldots, \bar{p}\}$ and to use the symbols $i, j, k, \cdots$ when they run through the indices $\{1, \ldots, p\}$.

Suppose we have a positive definite metric $g_{\alpha\beta}$ on $\mathbb{C}^p$, i.e., $g_{\alpha\beta} = g_{\beta\alpha}$ for any $\alpha, \beta \in \{1, \ldots, p, \bar{1}, \ldots, \bar{p}\}$, and

(A.5) \[ g_{\alpha\beta} \lambda^\alpha \lambda^\beta = g_{ij} \lambda^i \lambda^j + g_{i\bar{j}} \lambda^i \lambda^{\bar{j}} + g_{\bar{i}j} \lambda^{\bar{i}} \lambda^j + g_{\bar{i}\bar{j}} \lambda^{\bar{i}} \lambda^{\bar{j}} > 0 \]

for any $\lambda \in \mathbb{C}^p \setminus \{0\}$. Let us denote by $[g^{\alpha\beta}]$ the inverse matrix of the $2p \times 2p$ matrix $[g_{\alpha\beta}]$, i.e., $g_{\alpha\gamma} g^{\gamma\beta} = \delta^\beta_\alpha$ for the Kronecker delta $\delta^\beta_\alpha$.

For the $i$-th complex coordinate $z^i = x^i + \sqrt{-1} y^i$ in $\mathbb{C}^p$, define the contravariant derivatives $\partial^i$ and $\partial^\bar{i}$ of the covariant derivatives $\partial_i$ and $\partial_{\bar{i}}$ by

(A.6) \[ \partial^i := g^{ij} \partial_j + g^{i\bar{j}} \partial_{\bar{j}} \quad \partial^{\bar{i}} := g^{\bar{i}j} \partial_j + g^{\bar{i}\bar{j}} \partial_{\bar{j}}. \]

The contravariant derivatives are also simply defined by $\partial^\alpha := g^{\alpha\beta} \partial_\beta$ for $\alpha \in \{1, \ldots, p, \bar{1}, \ldots, \bar{p}\}$. The symbol $\partial^i$ is sometimes denoted by $\partial^i$. For later use, let us define the differential operators $D_{\alpha_1 \cdots \alpha_a}$ and $D^{\alpha_1 \cdots \alpha_a}$ by

(A.7) \[ D_{\alpha_1 \cdots \alpha_a} := \partial_{\alpha_1} \cdots \partial_{\alpha_a} \quad D^{\alpha_1 \cdots \alpha_a} := g^{\alpha_1 \beta_1} \cdots g^{\alpha_a \beta_a} \partial_{\beta_1} \cdots \partial_{\beta_a}, \]

respectively, for $\alpha_1, \cdots, \alpha_a \in \{1, \cdots, p, \bar{1}, \cdots, \bar{p}\}$. In particular, $D_\alpha = \partial_\alpha$ and $D^\alpha = \partial^\alpha$.

A metric $g_{\alpha\beta}$ is called Hermitian, if

(A.8) \[ g_{ij} = g_{\bar{i}\bar{j}} = 0 \quad g_{ij} = g_{ji} = g_{\bar{j}i} = g_{\bar{i}j} \]

for all $i, j = 1, \ldots, p$; see Section 8.4 in [14]. If the metric $g_{\alpha\beta}$ is Hermitian, the square of the distance of the infinitesimal complex vector $ds$ is given by

(A.9) \[ ds^2 = g_{\alpha\beta} ds^\alpha ds^\beta = 2 g_{ij} ds^i ds^\bar{j}. \]

If the metric $g_{\alpha\beta}$ is Hermitian, we only need to consider one-fourth of the $2p \times 2p$ complex-valued matrix $[g_{\alpha\beta}]$, namely the $p \times p$ Hermitian matrix $[g_{ij}]$. A complex manifold with a Hermitian metric is called a Hermitian manifold. The Jacobi formula for the Hermitian manifold is

(A.10) \[ g^{ij} \partial_k g_{ij} = \partial_k \log \pi_j \quad g^{\bar{i}\bar{j}} \partial_k g_{\bar{i}\bar{j}} = \partial_k \log \pi_j, \]
where \( \pi J \) is the determinant of the \( p \times p \) Hermitian matrix \( [g_{ij}] \), i.e., the square root of the determinant of \( 2p \times 2p \) matrix \( [g_{\alpha \beta}] \); see Section 8.4 in [14].

A Hermitian manifold with a metric \( g_{ij} \) is called a Kähler manifold, if

\[
\partial_i g_{jk} = \partial_j g_{ik}, \quad \partial_i \bar{g}_{jk} = \partial_k g_{ji}
\]

for all \( i, j, k = 1, \cdots, p \); see Section 8.5 in [14].

The Laplacian (Laplace–Beltrami operator) on a Kähler manifold is

\[
\Delta = \partial^2 \alpha \partial_\alpha = g_{\alpha \beta} \partial_\alpha \partial_\beta = 2g_{ij} \partial_i \log \phi \partial_j \log \phi,
\]

which does not hold in general for the usual Riemannian manifold. By A.12, we have

\[
\frac{\Delta \phi^\kappa}{\phi^\kappa} = \kappa \frac{\Delta \phi}{\phi} + 2\kappa(\kappa - 1)g_{ij} \partial_i \log \phi \partial_j \log \phi,
\]

which is a useful formula for calculating \( \Delta \phi^\kappa \).

**APPENDIX B: KULLBACK–LEIBLER DIVERGENCE BETWEEN POWER SPECTRAL DENSITIES**

In the present section, the derivation and justification of the form (1.1) of risk \( R(\hat{S} \mid \theta) \) for a predictive power spectral density \( \hat{S} \) for a complex-valued Gaussian process is explained, and it is explained why the Bayesian predictive power spectral density \( \hat{S}_k^{(N)} \) minimizes the Bayes risk (1.3) given an observation \( z^{(N)} \in \mathbb{C}^N \) and a prior \( \pi \).

Let \( W^{(N)} \) be a circulant matrix and \( D^{(N)} \) be a diagonal matrix defined by

\[
W^{(N)} := \begin{pmatrix}
0 & 1 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
1 & 0 & \cdots & 0 & 0
\end{pmatrix},
\quad D^{(N)} := \begin{pmatrix}
e^{2\pi i \frac{1}{N}} & 0 & \cdots & 0 \\
0 & e^{2\pi i \frac{2}{N}} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & e^{2\pi i \frac{N}{N}}
\end{pmatrix},
\]

respectively. We have a relation \( W^{(N)} U^{(N)} = U^{(N)} D^{(N)} \), where \( U^{(N)} := [U^{(N)}_{st}] \) is a unitary matrix defined by \( U^{(N)}_{st} := \frac{1}{\sqrt{N}} e^{2\pi i \frac{st}{N}} \). Set \( \Lambda(z) := \sum_{h=-\infty}^{\infty} \gamma_h z^{-h} \), where \( z \) is a formal variable. Note that \( S(\omega) = \frac{1}{2\pi} \Lambda(e^{i\omega}) \) and \( \Lambda(W^{(N)}) = U^{(N)} \Lambda(D^{(N)}) U^{(N)^*} \). Suppose \( \Lambda(z) \) has a Laurent expansion on \( |z| = 1 \). Then, \( (\Lambda(z))^{-1} = 1/\Lambda(z) \) is defined on the neighborhood of \( |z| = 1 \).
If the autocovariance $\gamma_t$ decreases exponentially, we have $\Lambda(W^{(N)}) \approx \Sigma^{(N)}$ for large $N$ because the $(s,t)$-th element of the matrix $\Lambda(W^{(N)})$ is approximated as

\[(B.1) \quad [\Lambda(W^{(N)})]_{st} = \sum_{k=-\infty}^{\infty} \gamma(s-t+kN) \approx \gamma(s-t) = [\Sigma^{(N)}]_{st}.
\]

With this approximation, the log-likelihood

\[(B.2) \quad l^{(N)}(z^{(N)}) = -N \log \pi - \log |\Sigma^{(N)}| - z^* (\Sigma^{(N)})^{-1} z
\]

of the observation $z = z^{(N)}$ from a complex-valued Gaussian process with mean $0 \in \mathbb{C}$ is approximated as

\[(B.3) \quad l^{(N)}(z^{(N)}) \approx -N \log \pi - \log |\Lambda(W)| - z^* (\Lambda(W))^{-1} z
\]

\[
= NC - \sum_{n=1}^{N} \log S \left(\frac{2\pi n}{N}\right) - \sum_{n=1}^{N} \frac{I(2\pi \frac{n}{N})}{S(2\pi \frac{n}{N})},
\]

where $I$ denotes the empirical power spectral density (periodogram) defined by $I(2\pi \frac{n}{N}) := \frac{1}{2\pi} |\tilde{z}_n|^2$ with $\tilde{z}_n := (U^* z)_n = \frac{1}{\sqrt{N}} \sum_{s=1}^{N} e^{-2\pi i s \frac{n}{N}} z_s$, and $C$ is a constant independent of $N$ and $S$. See [21] for more explanation of (B.3) for real-valued stationary processes and [4] for real-valued ARMA processes.

Suppose the variance-covariance matrix (2.4) is parameterized by complex parameters $\theta \in \Theta \subset \mathbb{C}^p$, i.e., the autocovariances $\{\gamma_h\}_{h \in \mathbb{Z}}$ are parametrized by $\theta \in \Theta$. Its power spectral density (2.3) is denoted by $S_\theta(\omega)$ or $S(\omega \mid \theta)$ for $\theta \in \Theta$. For $\theta \in \Theta$, denote the corresponding probability distribution, probability density function, and log-likelihood of the observation $z^{(N)} \in \mathbb{C}^N$ by $P_{\theta_1}^{(N)}$, $P_{\theta_2}^{(N)}$, and $l_{\theta_1}^{(N)}$, respectively.

For $\theta_1, \theta_2 \in \Theta$, the KL-divergence $D_{KL}(P_{\theta_1} \parallel P_{\theta_2})$ of the distributions $P_{\theta_2}$ from the distribution $P_{\theta_1}$ is approximated as

\[(B.4) \quad D_{KL}(P_{\theta_1} \parallel P_{\theta_2}) = \int_{\mathbb{C}^p} \left(l_{\theta_1}^{(N)}(z^{(N)}) - l_{\theta_2}^{(N)}(z^{(N)})\right) dP_{\theta_1}(z^{(N)})
\]

\[
\approx E_{\theta_1} \left[ -\sum_{n=1}^{N} \log \frac{S_{\theta_1}(2\pi \frac{n}{N})}{S_{\theta_2}(2\pi \frac{n}{N})} - \sum_{n=1}^{N} \left( \frac{I(2\pi \frac{n}{N})}{S_{\theta_1}(2\pi \frac{n}{N})} - \frac{I(2\pi \frac{n}{N})}{S_{\theta_2}(2\pi \frac{n}{N})} \right) \right]
\]

\[
\approx \frac{N}{2\pi} \int_{-\pi}^{\pi} \left( -\log \frac{S_{\theta_1}(\omega)}{S_{\theta_2}(\omega)} - 1 + \frac{S_{\theta_1}(\omega)}{S_{\theta_2}(\omega)} \right) d\omega = ND_{KL}(S_{\theta_1} \parallel S_{\theta_2}),
\]

where $D_{KL}(S_{\theta_1} \parallel S_{\theta_2})$ is the KL-divergence (1.2) between power spectral densities, which is also discussed in the literature [2, 7] of signal processing.
On the other hand, in the literature \cite{10, 19} of real-valued process, \(4\pi\) instead of \(2\pi\) is used for the denominator in the definition in (1.2). However, as we have explained, the constant in the denominator in the definition in (1.2) for the complex-valued process should be \(2\pi\). Note also that for any power spectral densities \(S_1\) and \(S_2\), because \(-\log x - 1 + x \geq 0\) for any \(x \geq 0\), we have \(D_{KL} \left( S_1 \| S_2 \right) \geq 0\) in general, and \(D_{KL} \left( S_1 \| S_2 \right) = 0\) if and only if \(S_1(\omega) = S_2(\omega)\) for \(\omega \in [-\pi, \pi]\) almost everywhere. The asymptotic expansion

\[
-B.5 \quad -\log \frac{1}{1 + x} - 1 + \frac{1}{1 + x} = \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{3}{4}x^4 - \frac{4}{5}x^5 + O(x^6)
\]

is a useful formula for calculating the value of \(D_{KL} \left( S \| S + dS \right)\).

For a possibly improper prior \(\pi\), the Bayesian predictive power spectral density \(\hat{S}^{(N)}_\pi\) minimizes the Bayes risk (1.3) among all the predictive power spectral densities \(\hat{S}^{(N)}\) if \(r(\hat{S}^{(N)}_\pi \mid \pi) < +\infty\); see \cite{1}. In fact,

\[
B.6 \quad r(\hat{S}^{(N)}_\pi \mid \pi) - r(\hat{S}^{(N)}_\pi \mid \pi) = \int_{\Theta} \int_{C^p} \left( D_{KL} \left( S_\theta \| \hat{S}^{(N)}_\pi \right) - D_{KL} \left( S_\theta \| \hat{S}^{(N)}_\pi \right) \right) dP^{(N)}_\theta d\Theta \\
= \int_{\Theta} \int_{C^p} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( -\log \frac{\hat{S}^{(N)}_\pi}{S^{(N)}} - \frac{S_\theta}{\hat{S}^{(N)}_\pi} + \frac{S_\theta}{S^{(N)}} \right) d\omega \right) dP^{(N)}_\theta d\Theta \\
= \int_{C^p} D_{KL} \left( \hat{S}^{(N)}_\pi \| \hat{S}^{(N)} \right) m^{(N)}_\pi(\hat{z}^{(N)}) d\hat{z}^{(N)} \geq 0
\]

for any predictive power spectral density \(\hat{S}^{(N)}_\pi\), where \(dP^{(N)}_\theta := p^{(N)}_\theta(\hat{z}^{(N)}) d\hat{z}^{(N)}, d\Theta := \pi(\theta) d\theta\), and \(m^{(N)}_\pi(\hat{z}^{(N)}) := \int_{C^p} p^{(N)}_\theta(\hat{z}^{(N)}) \pi(\theta) d\theta\) is the marginal distribution of the observation \(\hat{z}^{(N)} \in C^N\) based on the prior \(\pi\).

**APPENDIX C: TENSORIAL HERMITE POLYNOMIALS**

Tensorial Hermite polynomials, as introduced in \cite{3}, are very useful tools for calculating Edgeworth expansions. We present the complexified version of tensorial Hermite polynomials to calculate (F.8).

Suppose we have a metric \(g_{\alpha \beta}\) on \(C^p\). Define a complex-valued function \(\phi\) on \(C^p\) by \(\phi(\lambda) := \frac{1}{C} e^{-\frac{1}{2}g_{\alpha \beta}\lambda^\alpha \lambda^\beta}\) for \(\lambda = (\lambda^1, \ldots, \lambda^p) \in C^p\), where \(\alpha, \beta\) run through the indices \\{1, \ldots, p, \bar{1}, \ldots, \bar{p}\}, and \(G := \int_{C^p} e^{-\frac{1}{2}g_{\alpha \beta}\lambda^\alpha \lambda^\beta} d\lambda\) is the normalization factor to have \(\int_{C^p} \phi(\lambda) d\lambda = 1\). We assume positive definiteness of the metric \(g_{\alpha \beta}\) so that \(\lim_{|\lambda| \to \infty} |\phi(\lambda)| = 0\).

If the hermiticity of the metric \(g_{\alpha \beta}\) holds, the normalization factor \(G\) reduces to the product of \(\pi^p\) and the determinant of the \(p \times p\) Hermitian matrix
However, to obtain a general result, we do not assume hermiticity of the metric $g_{\alpha\beta}$ in this section.

We define the complex-valued tensorial Hermite polynomial $h_{\alpha_1 \cdots \alpha_a}$ for $\alpha_1, \ldots, \alpha_a \in \{1, \ldots, p, \bar{1}, \ldots, \bar{p}\}$ by the identity

$$(-1)^k D_{\alpha_1 \cdots \alpha_a} \phi(\lambda) = h_{\alpha_1 \cdots \alpha_a}(\lambda) \phi(\lambda),$$

where the differential operator $D_{\alpha_1 \cdots \alpha_a}$ is defined by (A.7). For example, $h_{\alpha_1}(\lambda) = \lambda^{\alpha_1}$ and $h_{\alpha_1 \alpha_2}(\lambda) = \lambda^{\alpha_1} \lambda^{\alpha_2} - g_{\alpha_1 \alpha_2}$.

Following the similar procedure in [3], we have

$$\int_{\mathbb{C}^p} h_{\alpha_1 \cdots \alpha_a}(\lambda) h_{\beta_1 \cdots \beta_b}(\lambda) \phi(\lambda) d\lambda = \begin{cases} a! g_{\alpha_1 \beta_1} \cdots g_{\alpha_a \beta_b} & \text{if } a = b \\ 0 & \text{otherwise} \end{cases}$$

for $\alpha_1, \ldots, \alpha_a, \beta_1, \ldots, \beta_b \in \{1, \ldots, p, \bar{1}, \ldots, \bar{p}\}$, where the symmetrization () is taken over the indices $\alpha_1, \ldots, \alpha_a$ only. For example,

$$\int_{\mathbb{C}^p} \lambda^{\alpha_1} \lambda^{\alpha_2} \phi(\lambda) d\lambda = \int_{\mathbb{C}^p} (h_{\alpha_1 \alpha_2} + g_{\alpha_1 \alpha_2}) \phi(\lambda) d\lambda = g_{\alpha_1 \alpha_2},$$

and

$$\int_{\mathbb{C}^p} \lambda^{\alpha_1} \lambda^{\alpha_2} \lambda^{\alpha_3} \phi(\lambda) d\lambda$$

$$= \int_{\mathbb{C}^p} (h_{\alpha_1 \alpha_2 \alpha_3} + \lambda^{\alpha_1} \lambda^{\alpha_2} g_{\beta_1 \beta_2} + g_{\alpha_1 \alpha_2} \lambda^{\beta_1} \lambda^{\beta_2} - g_{\alpha_1 \alpha_2} g_{\beta_1 \beta_2}) \phi(\lambda) d\lambda$$

$$= g_{\alpha_1 \beta_1} g_{\alpha_2 \beta_2} + g_{\alpha_2 \beta_1} g_{\alpha_1 \beta_2} + g_{\alpha_1 \alpha_2} g_{\beta_1 \beta_2}.$$
For example, $\Sigma_{11,2} = \Sigma^{-1}(\partial_1 \partial_1 \Sigma) \Sigma^{-1}(\partial_2 \Sigma)$. Direct computation shows that

$$\partial_\alpha l = z^*(\Sigma_\alpha) \Sigma^{-1} z - \text{tr} (\Sigma_\alpha),$$
$$\partial_\alpha \partial_\beta l = -z^* (\Sigma_{\alpha,\beta} + \Sigma_{\beta,\alpha}) \Sigma^{-1} z + z^* (\Sigma_{\alpha,\beta}) \Sigma^{-1} z - \text{tr} (-\Sigma_{\alpha,\beta} + \Sigma_{\alpha,\beta}),$$
$$\partial_\alpha \partial_\beta \partial_\gamma l = z^* (\Sigma_{\alpha,\beta,\gamma} + \Sigma_{\alpha,\gamma,\beta} + \Sigma_{\beta,\alpha,\gamma} + \Sigma_{\beta,\gamma,\alpha} + \Sigma_{\gamma,\alpha,\beta} + \Sigma_{\gamma,\beta,\alpha}) \Sigma^{-1} z$$
$$- z^* (\Sigma_{\alpha,\beta,\gamma} + \Sigma_{\beta,\alpha,\gamma} + \Sigma_{\beta,\gamma,\alpha} + \Sigma_{\gamma,\alpha,\beta} + \Sigma_{\gamma,\beta,\alpha}) \Sigma^{-1} z$$
$$+ z^* (\Sigma_{\alpha,\beta,\gamma}) \Sigma^{-1} z$$
$$- \text{tr} (\Sigma_{\alpha,\beta,\gamma} + \Sigma_{\alpha,\beta,\gamma} - \Sigma_{\alpha,\beta,\gamma} - \Sigma_{\gamma,\alpha,\beta} + \Sigma_{\alpha,\beta,\gamma}),$$

where $l = l_\theta^{(N)}(z^{(N)})$ is the log-likelihood (B.2) of the observation $z^{(N)} \in \mathbb{C}^N$ from the complex-valued Gaussian process at the parameter $\theta \in \Theta \subset \mathbb{C}^p$.

To compute the expectation of the derivative of the log-likelihood, we make use of the theorem proved in [20], which was originally proved for real-valued processes but is still valid for complex-valued processes. We introduce the space $\mathcal{D}$ of power spectral densities of complex-valued processes defined on $[-\pi, \pi]$:

$$\mathcal{D} := \{ S \mid S(\omega) = \sum_{h=-\infty}^{\infty} \gamma_h e^{-\sqrt{-1} h \omega}, |\gamma_h| = 1, \sum_{h=-\infty}^{\infty} |h||\gamma_h| < \infty \}.$$
Proposition D.2. For a complex-valued stationary ARMA process parameterized by $\theta \in \Theta \subset \mathbb{C}^p$,

\begin{align}
\text{(D.1)} & \quad \frac{1}{N} E_{\theta} \left[ \partial_{\alpha} l_{\theta}^{(N)} \right] = 0, \\
\text{(D.2)} & \quad \frac{1}{N} E_{\theta} \left[ \partial_{\alpha} \partial_{\beta} l_{\theta}^{(N)} \right] = -g_{\alpha \beta} + O(N^{-1}), \\
\text{(D.3)} & \quad \frac{1}{N} E_{\theta} \left[ \partial_{\alpha} \partial_{\beta} \partial_{\gamma} l_{\theta}^{(N)} \right] = 2T_{\alpha \beta \gamma} - \Gamma_{\alpha \beta \gamma} - \Gamma_{\beta \gamma \alpha} - \Gamma_{\gamma \alpha \beta} + O(N^{-1})
\end{align}

for $\alpha, \beta, \gamma \in \{1, \cdots, p, \bar{1}, \cdots, \bar{p}\}$, where $l_{\theta}^{(N)}$ denotes the log-likelihood (B.2) of the observation $z = z^{(N)} \in \mathbb{C}^N$ of size $N$, and $E_{\theta}$ denotes the expectation over the distribution $P_{\theta}^{(N)}$ of the observation $z^{(N)} \in \mathbb{C}^N$ at the parameter $\theta \in \Theta$.

Proof. Use $E[zz^*] = \Sigma$ and Proposition D.1. \qed

Appendix E: Asymptotic Expansion of Estimative Power Spectral Densities

We provide the asymptotic expansion of the risk of the estimative power spectral density with the maximal likelihood estimator. We see that the risk is approximately $pN$ for most predictive power spectral densities with asymptotically efficient estimators when we use $p$ complex parameters.

Let us fix the true parameter $\theta_0 \in \Theta$ and denote the maximal likelihood estimator by $\hat{\theta}^{(N)}$ for a while, and set $S_0 := S_{\theta_0}$ and $\hat{S}^{(N)} := \hat{S}^{(N)}_{\theta_0}$. For $\lambda := \sqrt{N}(\hat{\theta}^{(N)} - \theta_0) = O_P(1)$, utilizing the formula (B.5) and the Taylor expansion of $\hat{S}^{(N)}$ around $S_0$, we have

\[
D_{\text{KL}} \left( S_0 \left\| \hat{S}^{(N)} \right\| \right) = \frac{1}{2N} g_{\alpha \beta} \lambda^\alpha \lambda^\beta + \frac{1}{N \sqrt{N}} \left( \frac{1}{2} \Gamma_{\alpha \beta \gamma} - \frac{1}{3} T_{\alpha \beta \gamma} \right) \lambda^\alpha \lambda^\beta \lambda^\gamma + O(N^{-2}),
\]

where the quantities $M_{\alpha_1 \cdots \alpha_p, \beta_1 \cdots \beta_p, \gamma_1 \cdots \gamma_p}$ appearing on the right hand side are all evaluated at $\theta_0$. In particular, the asymptotic expansion of the risk of the maximal likelihood estimator evaluated at the true parameter $\theta_0$ is given by

\[
R(\hat{S}^{(N)} | \theta_0) = E_{\theta_0} \left[ D_{\text{KL}} \left( S_0 \left\| \hat{S}^{(N)} \right\| \right) \right] = \frac{p}{N} + O(N^{-2})
\]

because

\[
E_{\theta_0} [\lambda^\alpha \lambda^\beta] = g^{\alpha \beta} + O(N^{-1}), \quad E_{\theta_0} [\lambda^\alpha \lambda^\beta \lambda^\gamma] = O(N^{-\frac{3}{2}}) \quad \text{and} \quad g^{\alpha \beta} g_{\alpha \beta} = 2p.
\]
APPENDIX F: ASYMPTOTIC EXPANSION OF BAYESIAN PREDICTIVE POWER SPECTRAL DENSITIES

We give an asymptotic expansion of the Bayesian predictive power spectral density $\tilde{S}_\pi(N)$ of a complex-valued ARMA process around the maximal likelihood estimator $\hat{\theta}(N)$. This is the very first step to obtaining the asymptotic expansion of the risk differences needed in the proof of the Main Theorem.

We follow the original proof [19] for the real-valued ARMA process. However, because we consider complex-valued processes, the definitions of some quantities must be slightly modified. Throughout this section, we use the Einstein notation, where the symbols $\alpha, \beta, \gamma, \delta, \ldots$ run through the indices $\{1, \ldots, p, \bar{1}, \ldots, \bar{p}\}$.

For the maximal likelihood estimator $\hat{\theta} = \hat{\theta}(N) (z(N)) = \theta_0 + O_P(N^{-\frac{3}{2}})$ for the observation $z(N) \in \mathbb{C}^N$ of size $N$ from the complex-valued Gaussian process whose true parameter is $\theta_0 \in \Theta \subset \mathbb{C}^p$, the Bayesian predictive power spectral density is expanded as

\begin{equation}
\tilde{S}_\pi(N)(\omega) = S_\theta(\omega) + \frac{1}{\sqrt{N}} \left( \partial_\alpha S_\theta(\omega) \right) E_\pi[\tilde{\lambda}^\alpha] + \frac{1}{2N} \left( \partial_\alpha \partial_\beta S_\theta(\omega) \right) E_\pi[\tilde{\lambda}^\alpha \tilde{\lambda}^\beta] + O_P(N^{-\frac{5}{2}})
\end{equation}

around the maximal likelihood estimator $\hat{\theta}$, where

\begin{equation}
E_\pi[\tilde{\lambda}^\alpha_1 \cdots \tilde{\lambda}^\alpha_a] := \int_\Theta \tilde{\lambda}^\alpha_1 \cdots \tilde{\lambda}^\alpha_a \pi(\theta | z(N))
\end{equation}

for $\tilde{\lambda} = \sqrt{N}(\theta - \hat{\theta})$ and $\alpha_1, \ldots, \alpha_a \in \{1, \ldots, p, \bar{1}, \ldots, \bar{p}\}$. To complete the asymptotic expansion of (F.1) around the maximal likelihood estimator $\hat{\theta}$, we require the asymptotic expansions of (F.2) for $a = 1, 2$.

Let us fix, for a while, the observation $z(N) \in \mathbb{C}^N$, and denote the maximal likelihood estimator by $\hat{\theta} = \hat{\theta}(N)(z(N))$. For any $\theta \in \Theta \subset \mathbb{C}^p$ such that $\tilde{\lambda} = \sqrt{N}(\theta - \hat{\theta}) = O(1)$, the asymptotic expansion of $\tilde{l}_\theta^N (z(N)) := l_\theta^N (z(N)) + \log \pi(\theta)$ around the maximal likelihood estimator $\hat{\theta}$ is calculated as

\begin{equation}
\tilde{l}_\theta^N = \tilde{l}_\theta^N(\hat{\theta}) - \frac{1}{2} J_{\alpha \beta}(\hat{\theta}) \tilde{\lambda}^\alpha \tilde{\lambda}^\beta + \frac{A(N)(\tilde{\lambda})}{\sqrt{N}} + O\left(N^{-1}\right),
\end{equation}

where $J_{\alpha \beta}(\hat{\theta}) := -\frac{1}{2} \partial_\alpha \partial_\beta l_\theta^N$ and

\begin{equation}
A(N)(\tilde{\lambda}) := \frac{1}{6} \left( \frac{1}{N} \partial_\alpha \partial_\beta \partial_\gamma l_\theta^N \right) \tilde{\lambda}^\alpha \tilde{\lambda}^\beta \tilde{\lambda}^\gamma + \left( \partial_\alpha \log \pi(\hat{\theta}) \right) \tilde{\lambda}^\alpha.
\end{equation}
By referring to (F.3) and the procedure in [15], we can expand (F.2) as,

$$E^\pi \left[ \tilde{\lambda}^{\alpha_1 \cdots \alpha_a} \right] = \int_{\sqrt{N}(\theta - \hat{\theta})} \tilde{\lambda}^{\alpha_1 \cdots \alpha_a} \frac{1}{G(N)} e^{-\frac{1}{2} j_{\alpha\beta}^{(N)} \tilde{\lambda}^{\alpha} \tilde{\lambda}^{\beta}} \times \left( 1 + \frac{A^{(N)}(\tilde{\lambda})}{\sqrt{N}} + O_P(N^{-1}) \right) d\bar{\lambda}$$

for $\tilde{\lambda} = \sqrt{N}(\theta - \hat{\theta})$ and $\alpha_1, \cdots, \alpha_a \in \{1, \cdots, p, 1, \cdots, \bar{p}\}$, where $G(N) := \int_{\sqrt{N}(\theta - \hat{\theta})} e^{-\frac{1}{2} j_{\alpha\beta}^{(N)} \tilde{\lambda}^{\alpha} \tilde{\lambda}^{\beta}} d\bar{\lambda}$ is a normalization constant.

The terms (F.2) for $a = 1, 2$ are expanded as

(F.5) \( E^\pi \left[ \tilde{\lambda}^{a} \right] = \frac{1}{6 \sqrt{N}} L_{\beta\gamma\delta}^{a} \pi_{\beta\gamma\delta}^{a} + \frac{1}{\sqrt{N}} \left( \partial_{\beta} \log \pi(\hat{\theta}) \right) \pi_{\beta}^{a} + O_P(N^{-1}), \)

(F.6) \( E^\pi \left[ \tilde{\lambda}^{\alpha} \tilde{\lambda}^{\beta} \right] = I^{\alpha\beta} + O_P(N^{-1}), \)

where $\alpha, \beta, \gamma, \delta$ run through the indices $\{1, \cdots, p, 1, \cdots, \bar{p}\}$, and

(F.7) \( L_{\alpha_1 \cdots \alpha_a} := \frac{1}{N} \partial_{\alpha_1} \cdots \partial_{\alpha_a} t^{(N)}_{\theta}, \)

(F.8) \( I^{\alpha_1 \cdots \alpha_a} := \int_{\sqrt{N}(\theta - \hat{\theta})} \tilde{\lambda}^{\alpha_1} \cdots \tilde{\lambda}^{\alpha_a} \frac{1}{G(N)} e^{-\frac{1}{2} j_{\alpha\beta}^{(N)} \tilde{\lambda}^{\alpha} \tilde{\lambda}^{\beta}} d\bar{\lambda} \)

for $\alpha_1, \cdots, \alpha_a \in \{1, \cdots, p, 1, \cdots, \bar{p}\}$. Note that $L_{\alpha_1 \cdots \alpha_a}$ and $I^{\alpha_1 \cdots \alpha_a}$ are complex-valued random variables, because they depend on the realization of the observation $z^{(N)} \in \mathbb{C}^N$ from the process.

The expansion of (F.1) now becomes

(F.9) \( S^{(N)}(\omega) = S(\omega | \hat{\theta}) + \frac{1}{N} B^{(N)}_{\pi} (\omega | \hat{\theta}) + O_P(N^{-\frac{1}{2}}), \)

where $B^{(N)}_{\pi}$ is the $O_P(1)$ term defined as

(F.10) \( B^{(N)}_{\pi} (\omega | \theta) := \frac{1}{6} L_{\beta\gamma\delta}^{a} I^{\alpha\beta\gamma\delta} (\partial_{\alpha} S(\omega | \theta)) + I^{\alpha\beta} (\partial_{\beta} \log \pi(\theta) \partial_{\alpha} S(\omega | \theta)) + \frac{1}{2} I^{\alpha\beta} (\partial_{\alpha} \partial_{\beta} S(\omega | \theta)). \)

Making use of the complex-valued tensorial Hermite polynomials defined in Appendix C and by Proposition D.2 in Appendix D, we have an asymptotic expansion $B^{(N)}_{\pi} (\omega | \theta) = G^{(N)}_{\pi} (\omega | \theta) + H^{(N)} (\omega | \theta) + O_P(N^{-\frac{1}{2}})$, which yields the asymptotic expansion (2.15).

Functions $G^{(N)}_{\pi}$ and $H^{(N)}$ represent the parallel and orthogonal parts of the quantity $N (S^{(N)} - S^{(N)}_{\theta})$, respectively; see also [11, 19].
APPENDIX G: EXISTENCE OF BAYESIAN PREDICTIVE POWER SPECTRAL DENSITIES FOR AR($p; \mathbb{C}$)

First, we prove that the $\alpha$-prior (5.2) for $AR(p; \mathbb{C})$ is proper on $\Theta_1 = U \times \cdots \times U$ if $\alpha < 1$ and is improper if $\alpha \geq 1$. Because if $\alpha \leq 0$ then the $\alpha$-prior $\pi^{(\alpha)}$ is certainly integrable on $\Theta_1$, we may assume $\alpha > 0$. Because $|1 - \xi^i \xi^j|^2 - |\xi^i - \xi^j|^2 = (1 - |\xi^i|^2)(1 - |\xi^j|^2) \geq 0$, we have $|\xi^i - \xi^j|^2 / |1 - \xi^i \xi^j|^2 \leq 1$. Thus, for $0 < \alpha < 1$,

$$
\int_{\tilde{\Theta}_1} \pi^{(\alpha)}(\xi) \, d\xi = \int_{\tilde{\Theta}_1} \frac{1}{\prod_{i=1}^p \prod_{j=1}^p (1 - |\xi^i \xi^j|^2)} \prod_{1 \leq i < j \leq p} |\xi^i - \xi^j|^2 \, d\xi
$$

$$
= \int_{\tilde{\Theta}_1} \prod_{i=1}^p (1 - |\xi^i|^2)^{\alpha} \left(\prod_{1 \leq i < j \leq p} \frac{|\xi^i - \xi^j|^2}{1 - |\xi^i \xi^j|^2}\right) \prod_{1 \leq i < j \leq p} |\xi^i - \xi^j|^2(1 - \alpha) \, d\xi
$$

$$
\leq 2^{p(p-1)}(1 - \alpha) \prod_{i=1}^p \left(\int_U \frac{1}{(1 - |\xi^i|^2)^{\alpha}} \, d\xi^i\right) = 2^{p(p-1)}(1 - \alpha) \left(\frac{\pi}{1 - \alpha}\right)^p,
$$

since $\int_U (1 - |\xi^i|^2)^{-\alpha} \, d\xi^i = \int_0^1 \int_{\pi}^{\pi} r^{-\alpha} \, dr \, d\theta = \frac{\pi}{1 - \alpha}$. Therefore, the $\alpha$-prior $\pi^{(\alpha)}$ is integrable on $\tilde{\Theta}_1$ if $\alpha < 1$.

Set $m := \min_{\xi \in \Xi} \prod_{1 \leq i < j \leq p} |\xi^i - \xi^j|^2(1 - |\xi^i \xi^j|^2)^{\alpha} > 0$, where $\Xi := V_1 \times \cdots \times V_p \subset \tilde{\Theta}_1$ and $V_i := \{\xi \in U \mid \frac{1}{2} < |\xi| < 1, \frac{2\pi}{N}(i - 1/2) < \arg \xi < \frac{2\pi}{N}\}$. We see that $\pi^{(\alpha)}$ is not integrable on $\tilde{\Theta}_1$ if $\alpha \geq 1$, because $\int_{\tilde{\Theta}_1} \pi^{(\alpha)}(\xi) \, d\xi \geq \int_{\Xi} \pi^{(\alpha)}(\xi) \, d\xi \geq m \prod_{i=1}^p \int_{V_i} (1 - |\xi|^2)^{-\alpha} \, d\xi = +\infty$.

Next, we prove that a function $p^{(N)}(z^{(N)} | \xi) \pi^{(\alpha)}(\xi)$ of $\xi$ is integrable on the parameter space $\tilde{\Theta}_1$ if $\alpha < 2$. The explicit form of the determinant $|\Sigma^{(N)}|$ of the variance-covariance matrix $\Sigma^{(N)}$ of $AR(p; \mathbb{C})$ of the form (3.1) is $\gamma_0 = |\Sigma^{(1)}| \leq \cdots \leq |\Sigma^{(p)}| = |\Sigma^{(p+1)}| = \cdots = \prod_{i=1}^p \prod_{j=1}^p (1 - \xi^i \xi^j)^{-1}$; see Section 5.5 (b), (c) and (d) in [8], or Theorem 3.1 in [16]. Thus, if $N \geq p$, we have

$$
p^{(N)}(z^{(N)} | \xi) \pi^{(\alpha)}(\xi)
$$

$$
= \pi^{-N} |\Sigma^{(N)}|^{-1} e^{-\sum_{i=1}^p \sum_{j=1}^p |\xi^i - \xi^j|^2} \prod_{1 \leq i < j \leq p} \frac{|\xi^i - \xi^j|^2}{\prod_{i=1}^p \prod_{j=1}^p (1 - \xi^i \xi^j)^{\alpha-1}}
$$

$$
\leq \pi^{-N} \prod_{1 \leq i < j \leq q} |\xi^i - \xi^j|^2 \prod_{i=1}^p \prod_{j=1}^p (1 - \xi^i \xi^j)^{-\alpha-1} = \pi^{-N} (\pi^{(\alpha-1)}(\xi))
$$

for $z^{(N)} \in \mathbb{C}^N$. If $\alpha < 2$ and $N \geq p$, then $\int_{\tilde{\Theta}_1} p^{(N)}(z^{(N)} | \xi) \pi^{(\alpha)}(\xi) \, d\xi$ is bounded, regardless of a sample $z^{(N)} \in \mathbb{C}^N$. Therefore, the Bayesian predictive power spectral densities $\hat{S}^{(N)}_{\pi^{(\alpha)}}$ for $AR(p; \mathbb{C})$ based on the $\alpha$-prior $\pi^{(\alpha)}$ exists if $\alpha < 2$ and $N \geq p$. 
APPENDIX H: PROOF OF THE MAIN THEOREM

We provide a proof of the Main Theorem for complex-valued ARMA processes. The generalization, including the i.i.d. case, for this theorem is discussed in Section 6. Except Proposition H.3, most of the propositions presented in this section are merely complexified versions of previous works; see [11, 19].

We first give the asymptotic expansion of the risk $R(\hat{S}_\pi(N) \mid \theta)$ for a Bayesian predictive power spectral density $\hat{S}_\pi(N)$. The original proof of Proposition H.1 for the i.i.d. case was introduced in [11], and the proof for real-valued processes is available in [19].

The covariant derivative $(e) \nabla^\alpha$ of the vector field $V^\beta$ is defined as

$$(e) \nabla^\alpha V^\beta := \partial^\alpha V^\beta + (e) \Gamma^\alpha_{\gamma\beta} V^\gamma,$$

where $(e) \Gamma^\alpha_{\gamma\beta} := (m) \Gamma^\alpha_{\gamma\beta} - T^\alpha_{\gamma\delta} g^\delta\beta$.

**Proposition H.1.** For a complex parameter space $\Theta$ and a possibly improper prior $\pi$, we have

$$R(\hat{S}_\pi(N) \mid \theta) = \frac{1}{2N^2} g^{\alpha\beta} \left( \partial_\alpha \log \frac{\pi}{\pi_J} + \frac{1}{2} T_\alpha \right) \left( \partial_\beta \log \frac{\pi}{\pi_J} + \frac{1}{2} T_\beta \right) + \frac{1}{N^2} \nabla^\alpha \left( g^{\alpha\beta} \left( \partial_\beta \log \frac{\pi}{\pi_J} + T_\beta \right) \right) + C + O(N^{-\frac{1}{2}}),$$

where $C$ is the term independent of the prior $\pi$.

**Proof.** Recall that the asymptotic expansion of a Bayesian predictive power spectral density $\hat{S}_\pi(N)$ is given by (2.15). Follow the procedure in [19], replacing its summation rule $i, j, k = 1, \cdots, p$ with $\alpha, \beta, \gamma = 1, \cdots, p, \bar{1}, \cdots, \bar{p}$.

The asymptotic expansion of the risk difference $R(\hat{S}_{\pi_1}(N) \mid \theta) - R(\hat{S}_{\pi_2}(N) \mid \theta)$ is calculated as follows; see also [11, 19].

**Proposition H.2.** For a complex parameter space $\Theta$ and a possibly improper prior $\pi$, we have

$$N^2 \left( R(\hat{S}_{\pi_j}(N) \mid \theta) - R(\hat{S}_{\pi}(N) \mid \theta) \right)$$

$$= g^{\alpha\beta} \left( \partial_\alpha \log \frac{\pi}{\pi_J} \right) \left( \partial_\beta \log \frac{\pi}{\pi_J} \right) - \left( \frac{\pi}{\pi_J} \right)^{-1} \Delta \left( \frac{\pi}{\pi_J} \right) + O(N^{-\frac{1}{2}})$$

$$= -2 \left( \frac{\pi}{\pi_J} \right)^{-\frac{1}{2}} \Delta \left( \frac{\pi}{\pi_J} \right)^{\frac{1}{2}} + O(N^{-\frac{1}{2}})$$
for $\theta \in \Theta$.

**Proof.** Use (H.1) and follow [19].

**Proposition H.3.** Let $\phi$ be a positive continuous function on a Kähler parameter space $\Theta$, and define a family $\pi^{(\alpha)} := \phi^{-\alpha+1}\pi_J$ of priors for $\alpha \in \mathbb{R}$. Then, we have

\[
N^2 \left( R(\hat{\theta}^{(N)}_{\pi_1} \mid \theta) - R(\hat{\theta}^{(N)}_{\pi_2} \mid \theta) \right) \\
= -(\alpha_1 - \alpha_2) \frac{\Delta \phi}{\phi} + (\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2)g^{ij}(\partial_i \log \phi)(\partial_j \log \phi) + O(N^{-\frac{1}{2}})
\]

for $\theta \in \Theta$, where $\pi_1 := \pi^{(\alpha_1)}$ and $\pi_2 := \pi^{(\alpha_2)}$.

**Proof.** Using (A.13) for $\kappa = -\frac{\alpha+1}{2}$, we have

\[
N^2 \left( R(\hat{\theta}^{(N)}_{\pi_2} \mid \theta) - R(\hat{\theta}^{(N)}_{\pi_1} \mid \theta) \right) = -2 \frac{\Delta \phi^{-\alpha+1}}{\phi^{-\alpha+1}} + O(N^{-\frac{1}{2}}) \\
= (-\alpha + 1) \frac{\Delta \phi}{\phi} + (\alpha^2 - 1)g^{ij}(\partial_i \log \phi)(\partial_j \log \phi) + O(N^{-\frac{1}{2}}). 
\]

If there exists a positive continuous eigenfunction $\phi > 0$ of the Laplacian $\Delta$ with a negative eigenvalue $-K < 0$, then $\frac{\Delta \phi}{\phi} = -K$, which yields the proof of Theorem 4.1.

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