RIGIDITY OF ENTIRE SELF-SHRINKING SOLUTIONS TO KÄHLER-RICCI FLOW ON COMPLEX PLANE

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Abstract. We show that every entire self-shrinking solution on \( \mathbb{C}^1 \) to the Kähler-Ricci flow must be generated from a quadratic potential.

1. Introduction

In this short note, we prove the following result.

Theorem 1.1. Suppose that \( u(x) \) is an entire smooth subharmonic solution on \( \mathbb{R}^n \) to the equation

\[
\ln \Delta u = \frac{1}{2} x \cdot Du - u,
\]

then \( u \) is quadratic.

For \( n = 2 \), up to an additive constant, equation (1.1) is equivalent to the one-dimensional case of the complex Monge-Ampère equation

\[
\ln \det u_{i\bar{j}} = \frac{1}{2} x \cdot Du - u
\]

on \( \mathbb{C}^m \). Any entire solution to (1.2) leads to an entire self-shrinking solution

\[ v(x, t) = -tu \left( \frac{x}{\sqrt{-t}} \right) \]

to a parabolic complex Monge-Ampère equation

\[ v_t = \ln \det (v_{i\bar{j}}) \]

on \( \mathbb{C}^m \times (-\infty, 0) \), where \( z^i = x^i + \sqrt{-1}x^{m+i} \). Note that above equation of \( v \) is the potential equation of the Kähler-Ricci flow \( \partial_t g_{i\bar{j}} = -R_{i\bar{j}} \). In fact, the corresponding metric \( (u_{i\bar{j}}) \) is a shrinking Kähler-Ricci (non-gradient) soliton.

Assuming a certain decay of \( \Delta u \)–a specific completeness condition, Q. Ding and Y.L. Xin have proved Theorem (1.1) in [2]. Under the condition that the Kähler metric \( (u_{i\bar{j}}) \) is complete, rigidity theorem for equation (1.2) has been obtained by G. Drugan, P. Lu and Y. Yuan in [3]. Similar rigidity results for self-shrinking solutions to Lagrangian mean curvature flows in pseudo-Euclidean space were obtained in [1], [2], [4] and [5].

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Our contribution is removing extra assumptions for the rigidity of equation (1.1). As in [3] and [2], the idea of our argument is still to prove the phase–ln $\Delta u$ is constant. Then the homogeneity of the self-similar term on the right-hand side of equation (1.1) leads to the quadratic conclusion. However, it’s hard to construct a barrier function as in [3] or to find a suitable integral factor as in [2] without completeness assumption. Taking advantage of the conformality of the linearized equation (1.1), we establish a second order ordinary differential inequality for $M(r) = \max_{|x|=r} \ln \Delta u(x)$ in the sense of comparison function. Then we prove that $M(r)$ blows up in finite time by Osgood’s criterion unless $\ln \Delta u$ is constant.

2. Proof

Proof. Define the phase by
\[ \phi(x) = \frac{1}{2} x \cdot Du(x) - u(x). \]

Taking two derivatives and using (1.1), we have
\[ \Delta \phi = \frac{e^\phi}{2} x \cdot D\phi. \]

Define $M(r) : [0, +\infty) \to \mathbb{R}$ by
\[ M(r) = \max_{|x|=r} \phi(x). \]

Assuming $\phi(x)$ is not a constant, we prove that $M(r)$ blows up in finite time.

Since $M$ is locally Lipschitz, it is differentiable a.e. in $[0, +\infty)$. For all $r > 0$, there exists a corresponding angle $\theta_r \in S^{n-1}$ satisfying
\[ M(r) = \phi(r, \theta_r). \]

For $r' > 0$ small enough, we have $M(r + r') \geq \phi(r + r', \theta_r)$ and $M(r - r') \geq \phi(r - r', \theta_r)$. It follows that
\[ \frac{M(r + r') - M(r)}{r'} \geq \frac{\phi(r + r', \theta_r) - \phi(r, \theta_r)}{r'}, \]
and
\[ \frac{M(r) - M(r - r')}{r'} \leq \frac{\phi(r, \theta_r) - \phi(r - r', \theta_r)}{r'}. \]

Letting $r' \to 0$ in above two equations, we have
\[ \lim_{r'} M'(r) \leq \frac{\partial \phi}{\partial r}(r, \theta_r) \leq \lim_{r'} M'(r). \]

So if $r > 0$ is a differential point of $M$, we have
\[ M'(r) = \frac{\partial \phi}{\partial r}(r, \theta_r). \]

Because of the maximality of $\phi(r, \theta_r)$ among $\theta \in S^{n-1}$, we have $\Delta_{S^{n-1}} \phi(r, \theta_r) \leq 0$. Plug this inequality into (2.1), we obtain
\[ \frac{\partial^2 \phi}{\partial r^2}(r, \theta_r) + \frac{n - 1}{r} \frac{\partial \phi}{\partial r}(r, \theta_r) \geq \frac{r}{2} \exp[\phi(r, \theta_r)] \cdot \frac{\partial \phi}{\partial r}(r, \theta_r). \]

Fixing a positive $R_0$, for any $r \in [0, R_0]$, $\theta \in S^{n-1}$, $t \in [0, 1]$, we have the following Taylor’s expansion
\[ \phi(r + t, \theta) - \phi(r, \theta) \geq \frac{\partial \phi}{\partial r}(r, \theta_r) \cdot t + \frac{1}{2} \frac{\partial^2 \phi}{\partial r^2}(r, \theta_r) \cdot t^2 - Ct^3. \]
Combining (2.7) and (2.8), we obtain
\[ t \in a, b \]
Or equivalently,
\[ l \]
Then according to Hopf’s lemma, we know
\[ r \]
we have
\[ 1 \]
Choosing differentiable points \( a \) and \( b \) and letting \( t \to 0 \) in (2.6), we have
\[ M' (b) - M' (a) \geq \frac{1}{2} \int_a^b \left\{ \exp [M (r)] r - \frac{2(n - 1)}{r} \right\} M' (r) d r - C (b - a) t. \]

Since \( R_0 \) can be arbitrarily large, in fact (2.7) holds for all differentiable points \( a, b \in \mathbb{R}_+ \).

We claim there exists \( l_0 > 0 \) such that \( M' (r) > 0 \) at every differentiable point in \([l_0, +\infty)\). Otherwise, there exist an increasing sequence of differentiable points \( \{ r_k \} \subset \mathbb{R}_+ \), and a sequence of corresponding critical angles \( \{ \theta_k \} \subset \mathbb{S}^{n-1} \) such that
\[ M' (r_k) = \frac{\partial \phi}{\partial r} (r_k, \theta_k) \leq 0, \quad \text{and} \quad \lim_{k \to \infty} r_k = +\infty. \]

Then according to Hopf’s lemma, we know \( \phi(x) \) is constant in \( B_{r_k} (0) \). Since \( r_k \) can be arbitrarily large, \( \phi(x) \) is in fact a constant on the whole \( \mathbb{R}^n \), which contradicts our assumption.

So there exists a certain \( l_0 > 0 \), such that \( M' (r) > 0 \) holds a.e. in \([l_0, +\infty)\). Then \( M (r) \) monotonically increases on \([l_0, +\infty)\). When \( a \geq l_1 \triangleq l_0 + n + 2 \exp [-M (l_0)] \), we have
\[ \int_a^b \left\{ \exp [M (r)] r - \frac{2(n - 1)}{r} \right\} M' (r) d r > 2 \int_a^b \exp [M (r)] M' (r) d r = 2 \{ \exp [M (b)] - \exp [M (a)] \}. \]

Combining (2.7) and (2.8), we obtain
\[ M' (b) - M' (a) \geq \exp [M (b)] - \exp [M (a)]. \]

Above inequality holds for all differentiable points \( a, b \in [l_1, +\infty) \). Choosing a differentiable point \( l_2 \geq l_1 \), then \( M' (r) \geq M' (l_2) > 0 \) holds a.e. in \([l_2, +\infty)\). Thus \( M(r) \to +\infty \) as \( r \to +\infty \).
Then according to Osgood’s criterion, $M(r)$ blows up in finite time, which contradicts the assumption that $\phi(x)$ is entire. Hence, we conclude $\phi(x)$ is constant. Using $\phi(x) = \frac{1}{2} x \cdot Du(x) - u(x)$, we have

$$\frac{1}{2} x \cdot D [u(x) + \phi(0)] = u(x) + \phi(0).$$

Finally, it follows from Euler’s homogeneous function theorem that smooth $u(x) + \phi(0)$ is a homogeneous order 2 polynomial. $\square$

Remark 2.1. From the proof, it’s not hard to see that the theorem also holds for $\Delta u = f (x \cdot Du - 2u)$ if $f \in C^1(\mathbb{R})$ is convex, monotone increasing, and $f^{-1} \in L^1([d, +\infty))$ for a certain $d \in \mathbb{R}$. Integrability condition for $f^{-1}$ is necessary. Otherwise, we have such counterexample: $f(x) \equiv x$ and

$$u(x) = (x^2_1 - 1) \int_0^{x_1} \frac{1}{s^2} (\exp \frac{s^2}{2} - 1) \, ds - \frac{1}{x_1} (\exp \frac{x^2_1}{2} - 1) - x_1.$$

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