The Perturbed Error-Correction Criterion and Rescaled Truncated Recovery

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Abstract

We revisit the notion of approximate error correction of finite dimension codes. We derive a computationally simple lower bound on the worst case entanglement fidelity of a quantum code, when the truncated recovery map of Leung et. al. is rescaled. As an application, we apply our bound to construct a family of multi-error correcting amplitude damping codes that are permutation-invariant. This demonstrates an explicit example where the specific structure of the noisy channel allows code design out of the stabilizer formalism via purely algebraic means.

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I. INTRODUCTION

Quantum information, when left unprotected, often decoheres because of its inevitable interaction with the environment. The field of quantum error correction arose from the need to combat decoherence in quantum systems, and treats the decoherence as a noisy quantum channel. An important problem in quantum error correction is that of determining the utility of a given code with respect to the noisy quantum channel. The quantum error correction conditions of Knill and Laflamme [1] are equations from which one can determine whether a quantum code is entirely robust against a given set of Kraus effects of the noisy channel. The Knill-Laflamme conditions lie at the foundations of Gottesman’s stabilizer formalism [12] from which quantum error correction codes are designed and studied.

In this paper, we revisit the approximate error correction of finite dimension codes via a perturbation of the Knill-Laflamme conditions. We derive a computationally simple lower bound on the worst case entanglement fidelity of a quantum code, when the truncated recovery map of Leung et. al. [2] is rescaled to guarantee its validity as a quantum operation. Our lower bound arises from repeated application of the Gershgorin circle theorem on the relevant matrices.

The simplicity of our bound comes at a price – we do not have the near-optimal guarantees that the methods of Barnum-Knill [3] and Tyson-Beny-Oreshkov [4–6] yield. However in this trade-off, we are able to construct a family of multi-error correcting amplitude damping qubit codes that are permutation-invariant. We thereby demonstrate an example where the specific structure of the noisy channel allows code design out of the stabilizer formalism via purely algebraic means, as opposed to optimization techniques [7–10] and other approaches [13–15]. Our qubit permutation-invariant codes also extend the existing theory of qubit permutation-invariant codes [11, 17–19]; while no qubit permutation-invariant code corrects arbitrary single qubit errors, there exist qubit permutation-invariant codes that correct multiple amplitude damping errors.

A. Organization

In Section [11] we introduce notation and concepts needed for this paper, including quantum channels, quantum codes and the entanglement fidelity of a code. In Section
we address the perturbed Knill-Laflamme conditions, revisit the Leung et. al. recovery map and determine using Lemma III.3 when the Leung et. al. recovery can be rescaled to a quantum operation. In Section III.C, we prove our algebraic lower bound on the worst case entanglement fidelity (Theorem III.4). Finally in Section IV, we apply our lower bound to construct a family of qubit permutation-invariant codes that correct multiple amplitude damping errors (Theorem IV.3).

II. PRELIMINARIES

A. Quantum channels

For a complex separable Hilbert space $\mathcal{H}$, let $\mathcal{B}(\mathcal{H})$ be the set of bounded linear operators mapping $\mathcal{H}$ to $\mathcal{H}$. Define the set of quantum states on Hilbert space $\mathcal{H}$ to be $\mathcal{D}(\mathcal{H})$ where $\mathcal{D}(\mathcal{H})$ is the set of all positive semi-definite and trace one operators in $\mathcal{B}(\mathcal{H})$. For any subspace $\mathcal{C} \subset \mathcal{H}$ with orthonormal basis $\mathcal{B}_\mathcal{C}$, define $\mathcal{D}(\mathcal{C})$ to be the set of all elements in $\mathcal{D}(\mathcal{H})$ that are invariant under conjugation by the projector $\Pi = \sum_{|\beta\rangle \in \mathcal{B}_\mathcal{C}} |\beta\rangle \langle \beta|$. A quantum operation $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}')$ is a linear map that is completely positive and trace non-increasing. A quantum channel $\Phi$ is a trace preserving quantum operation. In this paper, the Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ are always isomorphic. A quantum operation $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ can always be expressed in the Kraus representation [25]:

$$\Phi(\rho) = \sum_{A \in \mathcal{K}_\Phi} A \rho A^\dagger, \quad 1_{\mathcal{H}} \geq \sum_{A \in \mathcal{K}_\Phi} A^\dagger A$$

where $\rho \in \mathcal{B}(\mathcal{H})$, $\mathcal{K}_\Phi \subset \mathcal{B}(\mathcal{H})$ is a set of Kraus operators of quantum operation $\Phi$, and $1_{\mathcal{H}}$ is the identity operator on complex Hilbert space $\mathcal{H}$.

B. Quantum codes and Entanglement Fidelity

Define the minimum eigenvalue of a Hermitian matrix $\mathbf{H}$ restricted to subspace $\mathcal{C}$ to be

$$\lambda_{\text{min},\mathcal{C}}(\mathbf{H}) := \min_{|\beta\rangle \in \mathcal{C}} \frac{\langle \beta| \mathbf{H} |\beta\rangle}{\langle \beta|\beta\rangle = 1}.$$

The entanglement fidelity of a state $\rho$ with respect to the quantum channel $\mathcal{R} \circ \mathcal{A}$ is

$$F_e(\rho, \mathcal{R} \circ \mathcal{A}) = \sum_{\mathcal{B} \in \mathcal{K}_{R \circ A}} |\text{Tr}(\mathcal{B} \rho)|^2$$
where the set of Kraus operators of $\mathcal{R} \circ \mathcal{A}$ is $\mathcal{R}_{\mathcal{R} \circ \mathcal{A}}$ [26]. The entanglement fidelity of a state $\rho$ with respect to the quantum channel $\mathcal{R} \circ \mathcal{A}$ quantifies how well the entanglement consistent with state $\rho$ is preserved when the noisy channel is $\mathcal{A}$ and the recovery map is $\mathcal{R}$.

III. RESCALED TRUNCATED RECOVERY

In this section, we analyze the performance of the rescaled truncated recovery map. Firstly in Section III A, we study the diagonalization of the Knill-Laflamme conditions. We next analyze the rescaling of the truncated recovery of Leung et. al. [2] into a quantum operation. In Section III C, we prove Theorem III.4, the main result of this paper.

A. The perturbed Knill-Laflamme conditions

In this subsection, we apply the canonical procedure [26] to diagonalize the error correction perturbed Knill-Laflamme conditions. We also introduce notation that is used both in the statement and the proof of Theorem III.4. We start by introducing the following notation.

N1. Let the $M$-dimension code $\mathcal{C}$ be a subspace of the Hilbert space $\mathcal{H}$ with projector $\Pi$ and orthonormal basis $\mathcal{B}_C$. Let $\mathcal{A} : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H})$ be a quantum channel with truncated Kraus set $\Omega$.

Using the notation of N1 define an orthonormal basis $\{|E\rangle : E \in \Omega\} \subseteq \mathbb{C}^{|\Omega|}$ labeling the effects in $\Omega$. For all $A, B \in \Omega$ and $|\alpha\rangle, |\beta\rangle \in \mathcal{B}_C$, define

$$\epsilon(A, B, |\alpha\rangle, |\beta\rangle) := \langle \alpha | A^\dagger B | \beta \rangle - g_{A,B} \delta_{|\alpha\rangle,|\beta\rangle}.$$ (3)

to quantify the perturbation to the Knill-Laflamme condition, where

$$g_{A,B} := \frac{1}{M} \sum_{|\beta\rangle \in \mathcal{B}_C} \langle \beta | A^\dagger B | \beta \rangle.$$ (4)

Define the Hermitian matrix

$$G := \sum_{A, B \in \Omega} g_{A,B} |A\rangle \langle B|.$$ (5)
The hermiticity of $G$ implies the existence of a unitary matrix $V$ and diagonal matrix $D$ such that

$$V = \sum_{E,F \in \Omega} v_{E,F} |E\rangle \langle F|$$

(6)

$$D := VGV^\dagger = \sum_{E \in \Omega} d_E |E\rangle \langle E|$$

(7)

which implies that

$$\sum_{F, F' \in \Omega} v_{E,F}^* v_{E',F'}^* g_{F,F'} = d_E \delta_{E,E'}.$$ 

(8)

For all $A \in \Omega$, define the transformed Kraus operators

$$\tilde{A} := \sum_{F \in \Omega} v_{A,F} F.$$ 

(9)

Substituting (8) into (9) gives

$$\langle \alpha | \tilde{A}^\dagger \tilde{B} | \beta \rangle = d_A \delta_{A,B} \delta_{|\alpha\rangle,|\beta\rangle} + \tilde{\epsilon}(A,B,|\alpha\rangle,|\beta\rangle)$$

(10)

where

$$\tilde{\epsilon}(A,B,|\alpha\rangle,|\beta\rangle) := \sum_{F,F' \in \Omega} (v_{A,F}^* v_{B,F'}) \epsilon(F',F,|\alpha\rangle,|\beta\rangle).$$

(11)

Equation (10) gives the ‘diagonalized’ form of the perturbed Knill-Laflamme conditions. The transformed error is quantified by (11). Let $\epsilon_{|\alpha\rangle,|\beta\rangle} := \max_{A,B \in \Omega} |\epsilon(A,B,|\alpha\rangle,|\beta\rangle)|$. Then the Cauchy-Schwarz inequality and normalization of the rows of $V$ implies that

$$|\tilde{\epsilon}(A,B,|\alpha\rangle,|\beta\rangle)| \leq \sum_{F,F' \in \Omega} |v_{A,F}^* v_{B,F'}| \epsilon_{|\alpha\rangle,|\beta\rangle} \leq |\Omega| \epsilon_{|\alpha\rangle,|\beta\rangle}.$$ 

(12)

B. The Leung et. al. truncated recovery

The truncated recovery of Leung et. al. [2] gives an algebraically simple lower bound on the entanglement fidelity of a code with respect to a noisy channel, under certain assumptions on the noisy channel and the quantum code. To understand this truncated recovery, we need the following notation.
We use the notation of \( N1 \). For all \( A \in \Omega \), define \( \Pi_A := U_A \Pi U_A^\dagger \) where \( U_A \) is the unitary in the polar decomposition \( A \Pi = U_A \sqrt{\Pi A^\dagger \Pi} \). Define the completely positive but not necessarily trace preserving linear operator \( R_{\Omega,C} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}) \) where for all \( \mu \in \mathcal{B}(\mathcal{H}) \),

\[
R_{\Omega,C}(\mu) := \sum_{A \in \Omega} R_A \mu R_A^\dagger, \quad R_A := U_A^\dagger \Pi A.
\] (13)

Without loss of generality, we pick \( \Omega \) such that \( \sqrt{\Pi A^\dagger \Pi} \) is never the zero operator.

We call the completely positive map \( R_{\Omega,C} \) a truncated recovery map, because \( \Omega \) is a truncated Kraus set of the quantum channel \( \mathcal{A} \). The truncated recovery map \( R_{\Omega,C} \) is also a quantum operation when the projectors \( \Pi_A \) in \( N2 \) are orthogonal, that is

\[
\Pi_A \Pi_B = \Pi_A \delta_{A,B} \quad \forall A, B \in \Omega.
\] (14)

**Lemma III.1** (Leung et. al. [2]). In the notation of \( N1 \) and \( N2 \), for any \( \rho \in \mathcal{D}(\mathcal{C}) \), the bound \( \sum_{A \in \Omega} |\text{Tr}(R_A A \rho)|^2 \geq \sum_{A \in \Omega} \lambda_{\text{min},C}(A^\dagger A) \) holds.

**Proof.** For \( A \in \Omega \), define \( \mu_A := \lambda_{\text{min},C}(A^\dagger A) \) and the positive semidefinite residue operator

\[
\pi_A := \sqrt{\Pi A^\dagger \Pi} - \sqrt{\mu_A} \Pi.
\]

Substituting \( \pi_A \) into the polar decomposition of \( A \Pi \) gives

\[
A \Pi = U_A \sqrt{\Pi A^\dagger \Pi}
\]

\[
= U_A (\sqrt{\Pi A^\dagger \Pi} - \sqrt{\mu_A} \Pi + \sqrt{\mu_A} \Pi)
\]

\[
= U_A (\pi_A + \sqrt{\mu_A} \Pi)
\]

\[
= U_A (\pi_A + \sqrt{\mu_A} \mathbb{1}_\mathcal{H} ) \Pi.
\] (15)

The spectral decomposition of \( \rho \) in the basis \( \mathcal{B}_C \) and equation (15) imply that

\[
\sum_{A \in \Omega} |\text{Tr}(R_A A \rho)|^2 \geq \left| \sum_{A \in \Omega} \left| \sum_{A \in \Omega} p_{|\beta\rangle} \langle \beta| R_A A |\beta\rangle \right|^2 \right|^2
\]

\[
= \sum_{A \in \Omega} \left| \sum_{|\beta\rangle \in \mathcal{B}_C} p_{|\beta\rangle} \langle \beta| U_A^\dagger (\mathbb{1}_\mathcal{H} + \pi_A) \Pi |\beta\rangle \right|^2.
\] (16)
Using $\Pi|\beta\rangle = |\beta\rangle$ and $U_A^\dagger U_A = 1_{\mathcal{H}}$, (16) becomes
\[
\sum_{A \in \Omega} \left( \sum_{|\beta\rangle \in \mathcal{B}_C} p_{|\beta\rangle} (|\beta\rangle \langle \beta| (\sqrt{\mu_A} 1_{\mathcal{H}} + \pi_A)|\beta\rangle)^2. 
\]
Moreover $\pi_A$ is positive semi-definite, hence the above expression is at least
\[
\sum_{A \in \Omega} \left( \sum_{|\beta\rangle \in \mathcal{B}_C} p_{|\beta\rangle} (|\beta\rangle \langle \beta| (\sqrt{\mu_A} 1_{\mathcal{H}})|\beta\rangle)^2 = \sum_{A \in \Omega} \mu_A. \]

Leung et. al. proved that when the orthogonality condition (14) holds, the truncated recovery map $R_{\Omega,C}$ is also quantum operation, and thus Lemma III.1 gives a lower bound on the worst case entanglement fidelity. We detail this in Lemma III.2.

**Lemma III.2** (Leung et.al. [2]). *We use the notation of $N_1$ and $N_2$. Suppose that (14) holds. Then the truncated recovery map $R_{\Omega,C}$ is a quantum operation and
\[
\min_{\rho \in \mathcal{D}(\mathcal{C})} F_e(\rho, R_{\Omega,C} \circ A) \geq \sum_{A \in \Omega} \lambda_{\min,C}(A^\dagger A). \quad (17)
\]
*Proof.* Observe that
\[
\sum_{A \in \Omega} R_A^\dagger R_A = \sum_{A \in \Omega} \Pi_A U_A U_A^\dagger \Pi_A = \sum_{A \in \Omega} \Pi_A. \quad (18)
\]
Orthogonality of the projectors $\Pi_A$ implies that
\[
\left( \sum_{A \in \Omega} \Pi_A \right) \left( \sum_{A' \in \Omega} \Pi_{A'} \right) = \sum_{A, A' \in \Omega} \Pi_A \delta_{A, A'} = \sum_{A \in \Omega} \Pi_A. \quad (19)
\]
Hence $\sum_{A \in \Omega} R_A^\dagger R_A$ is also a projector. The map $R_{\Omega,C}$ is also completely positive, that is $\sum_{A \in \Omega} R_A \rho R_A^\dagger \geq 0$, because $U_A^\dagger \rho U_A \geq 0$ which implies that $R_A \rho R_A^\dagger = U_A^\dagger \rho U_A \Pi \geq 0$. Obmitting non-negative terms in the sum pertaining to the entanglement fidelity, we get
\[
F_e(\rho, R_{\Omega,C} \circ A) \geq \sum_{A \in \Omega} |\text{Tr}(R_A A \rho)|^2. \quad (20)
\]
Applying Lemma III.1 gives the result. □

Rescaled maps have been used in the study of near-optimal quantum recovery operations, including the Barnum-Knill recovery map [3] and the Tyson-Beny-Oreshkov quadratic recovery map [4, 5]. In the notation of $N_1$ and $N_2$, the completely positive map $R_{\Omega,C}$ might increase trace and hence not be a quantum operation. Fortunately a bounded $R_{\Omega,C}$ can be rescaled to the quantum operation $R_{\Omega,C,\eta} := (1 + \eta)^{-1} R_{\Omega,C}$. 

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Lemma III.3. Using the notation of $N_1$ and $N_2$ let $\eta \geq |\Omega|^2 \max_{A \neq B \in \Omega} \| \Pi U_A^\dagger U_B \Pi \|_2$. Then $R_{\Omega, C, \eta}$ is a quantum operation.

Proof. It suffices to show that $\left\| \sum_{A \in \Omega} R_A^\dagger R_A \right\|_2 \leq 1 + \eta$. First observe that
\[
\sum_{A \in \Omega} R_A^\dagger R_A = \sum_{A \in \Omega} \Pi_A U_A U_A^\dagger \Pi_A = \sum_{A \in \Omega} \Pi_A.
\]
The projectors $\Pi_A$ may not be orthogonal, so
\[
\left( \sum_{A \in \Omega} R_A^\dagger R_A \right)^2 = \left( \sum_{A \in \Omega} \Pi_A \right) \left( \sum_{B \in \Omega} \Pi_B \right) = \sum_{A, B \in \Omega} U_A \Pi_A U_A^\dagger U_B \Pi_B U_B^\dagger
\]
and hence
\[
\left\| \left( \sum_{A \in \Omega} R_A^\dagger R_A \right)^2 \right\|_2 = \left\| \sum_{A \in \Omega} R_A^\dagger R_A \right\|_2 \leq \left\| \sum_{A \in \Omega} R_A^\dagger R_A \right\|_2 + |\Omega|^2 \max_{A \neq B \in \Omega} \| \Pi U_A^\dagger U_B \Pi \|_2.
\]
Let $\epsilon = \left\| \sum_{A \in \Omega} R_A^\dagger R_A \right\|_2 - 1 \geq 0$. Then substituting $\epsilon$ and $\eta$ into the above inequality gives
\[
1 + 2\epsilon + \epsilon^2 \leq 1 + \epsilon + \eta,
\]
which implies that $\epsilon \leq \eta$. Hence $\left\| \sum_{A \in \Omega} R_A^\dagger R_A \right\|_2 \leq 1 + \eta$, and the completely positive map $R_{\Omega, C, \eta}$ is a quantum operation.

C. Algebraic lower bounds on the worst case entanglement fidelity

Here, we prove algebraic lower bounds on the worst case entanglement fidelity of a code using a rescaled truncated recovery map, given partial knowledge of the noisy quantum channel. Our main technical result is the following:

Theorem III.4. Let the $M$-dimension code $C$ with an orthonormal basis $B_C$ be a subspace of the Hilbert space $H$. Let $A : \mathcal{B}(H) \to \mathcal{B}(H)$ be a quantum channel with truncated Kraus set $\Omega$. Define $G = \sum_{A, B \in \Omega} g_{A, B}$ where $g_{A, B} := \frac{1}{M} \sum_{|\beta\rangle \in B_C} \langle \beta | A^\dagger B | \beta \rangle$, and suppose that $\lambda_{\min}(G) > 0$. Suppose that for all Kraus effects $A, B$ in the truncated Kraus set $\Omega$ and for all distinct orthonormal basis vectors $|\alpha\rangle, |\beta\rangle \in B_C$,
\[
|g_{A, B} - \langle \alpha | A^\dagger B | \alpha \rangle| \leq \epsilon, \quad |\langle \alpha | A^\dagger B | \beta \rangle| \leq \epsilon.
\]

(21)
Then the minimum entanglement fidelity of our code $C$ with respect to the noisy channel $A$ is at least
\[
(\text{Tr } G - M|\Omega|^2\epsilon) \left( 1 + \frac{M|\Omega|^3\epsilon}{\lambda_{\min}(G)} \right)^{-1}.
\]

The proof of Theorem III.4 follows from the direct application of Lemma III.3 which gives us the important properties of our rescaled truncated recovery, and the repeated application of the Gershgorin circle theorem.

**Proof of Theorem III.4.** From (3), we have
\[
\frac{1}{M} \sum_{|\alpha\rangle \in B_C} \epsilon(A, B, |\alpha\rangle, |\alpha\rangle) = 0
\]
which implies that
\[
\frac{1}{M} \sum_{|\alpha\rangle \in B_C} \tilde{\epsilon}(A, B, |\alpha\rangle, |\alpha\rangle) = 0.
\]
Hence $\lambda_{\max}(\Pi \tilde{A}^\dagger \tilde{A} \Pi) \geq \frac{1}{M} \sum_{|\alpha\rangle \in B_C} \langle \alpha | \tilde{A}^\dagger \tilde{A} | \alpha \rangle \geq d_A$ which is at least $\lambda_{\min}(G)$.

Applying the Gershgorin circle theorem on the matrix $\Pi \tilde{A}^\dagger \tilde{B} \Pi$ with the error estimate (12), we get
\[
\lambda_{\max}(\Pi \tilde{A}^\dagger \tilde{B} \Pi) \leq M|\Omega|\epsilon \text{ for distinct } A, B \in \Omega.
\]

For distinct $A, B \in \Omega$, let $\tilde{A} \Pi$ and $\tilde{B} \Pi$ have polar decompositions $\tilde{A} \Pi = U_{\tilde{A}} \sqrt{\Pi \tilde{A}^\dagger \tilde{A} \Pi}$ and $\tilde{B} \Pi = U_{\tilde{B}} \sqrt{\Pi \tilde{B}^\dagger \tilde{B} \Pi}$ respectively. Then
\[
\Pi \tilde{A}^\dagger \tilde{B} \Pi = \sqrt{\Pi \tilde{A}^\dagger \tilde{A} \Pi U_{\tilde{A}}^\dagger U_{\tilde{B}} \sqrt{\Pi \tilde{B}^\dagger \tilde{B} \Pi}} = \sqrt{\Pi \tilde{A}^\dagger \tilde{A} \Pi (\Pi U_{\tilde{A}}^\dagger U_{\tilde{B}} \Pi) \sqrt{\Pi \tilde{B}^\dagger \tilde{B} \Pi}}.
\]
Hence by the sub-multiplicative property for norms, $\|\Pi U_{\tilde{A}}^\dagger U_{\tilde{B}} \Pi\|_2$ is at most
\[
\left\| \sqrt{\Pi \tilde{A}^\dagger \tilde{A} \Pi} \right\|_2^{-1} \left\| \sqrt{\Pi \tilde{B}^\dagger \tilde{B} \Pi} \right\|_2^{-1} \leq \frac{\|\Pi \tilde{A}^\dagger \tilde{B} \Pi\|_2}{\min_{F \in \{A, B\}} \lambda_{\max}(\Pi F^\dagger F \Pi)} \leq \frac{M|\Omega|\epsilon}{\lambda_{\min}(G)}.
\]
Hence by Lemma III.3, the map $R_{\Omega, C, \eta}$ is a quantum operation whenever $\eta \geq \frac{M|\Omega|^3\epsilon}{\lambda_{\min}(G)}$. By the Gershgorin circle theorem, $\lambda_{\min,C}(\tilde{A}^\dagger \tilde{A})$ is at least $d_A - M|\Omega|\epsilon$, and hence
\[
\sum_{A \in \Omega} \lambda_{\min,C}(\tilde{A}^\dagger \tilde{A}) \geq \sum_{A \in \Omega} (d_A - M|\Omega|\epsilon) = \text{Tr } D - M|\Omega|^2\epsilon = \text{Tr } G - M|\Omega|^2\epsilon.
\]
Use of Lemma III.1 then gives the result.

Under stronger assumptions on the set of conditions the truncated set of our noisy channel must satisfy, we can obtain the following corollary.
Corollary III.5. Let $t$ be a positive integer, and $\gamma > 0$ be an error parameter, and assume that the requirements of Theorem III.4 hold with $\epsilon = \gamma^{2t+1}$. Suppose that every $A \in \Omega$ is of order $O(\gamma^{s_A})$ for some non-negative integer $s_A \leq t$, and all Kraus effects not in $\Omega$ are of order $O(\gamma^{t+1})$. Then the minimum entanglement fidelity of the code $C$ with respect to the noisy channel $A$ is at least $1 - O(\gamma^{t+1})$.

Proof. Note that $\text{Tr} G = 1 - O(\gamma^{t+1})$, and $\lambda_{\text{min}}(G) = O(\gamma^{t}) \neq 0$. Substitution of these parameters into Theorem III.4 gives the result. \hfill \Box

IV. APPLICATION: PERMUTATION-INVARIND AMPLITUDE DAMPING CODES

In this section, we apply Theorem III.4 to prove the existence of family of multiple amplitude error correcting permutation-invariant codes.

First we define permutation-invariant codes encoding a single qubit. Define $P_{m,a}$ to be the set of all length $m$ binary vectors of weight $a$, and the corresponding permutation-invariant qubit states to be $|P_{m,a}\rangle := \sum_{x \in P_{m,a}} |x\rangle \left(\begin{array}{c} m \\ a \end{array}\right)^{-1/2}$. The permutation-invariant codes we consider have the logical codewords $|j_L\rangle := \sum_{a=0}^{m} \sqrt{\lambda_{j,a}} |P_{m,a}\rangle$ where $j \in \{0,1\}$ and $|j_L\rangle$ are linearly independent. Moreover the non-negative weights $\lambda_{j,a}$ are normalized so that $\sum_{a=0}^{m} \lambda_{j,a} = 1$.

Now we introduce notation relevant to the amplitude damping channel. The amplitude damping channel $A_\gamma$ has Kraus operators $A_0 = |0\rangle\langle 0| + \sqrt{1-\gamma}|1\rangle\langle 1|$ and $A_1 = \sqrt{\gamma}|0\rangle\langle 1|$, where the non-negative parameter $\gamma \leq 1$ quantifies the amount of amplitude damping of $A_\gamma$. We denote the Kraus operators of $A_\gamma^\otimes m$ by $A_k := A_{k_1} \otimes \ldots \otimes A_{k_m}$, with $k = (k_1, \ldots, k_m)$ is a binary vector. For positive integers $t$, we say that a code $C$ is a $t$-amplitude damping code (or a $t$-AD code) if there exists a quantum operation $R$ such that

$$\max_{\rho \in \mathcal{D}(C)} (1 - F_e(\rho, R \circ A_\gamma^\otimes m)) = o(\gamma^t). \quad (22)$$

The recovery operation that we use in this section is the rescaled recovery map of Leung et. al.
Let \( k \) and \( k' \) be length \( m \) binary vectors with weights \( k \) and \( k' \) respectively. Then

\[
\langle j'_{L} | A_{k}^{\dagger} A_{k'} | j_{L} \rangle = \sum_{b=0}^{m} \frac{\lambda_{j',b+k'-k} \lambda_{j,b}}{m \choose b} \sqrt{\gamma^{k+k'}(1-\gamma)^{b-k}} \left( \frac{m}{b-k} \right) \left( m \left( b - k \right) \right) \]

(23)

where \( \tau \) is cardinality of the union of supports of \( k \) and \( k' \). The expression (23) allows us to deduce algebraically sufficient conditions on \( \lambda_{j,a} \) for our code to correct \( t \)-AD errors.

**Lemma IV.1.** Let \( m \) and \( t \) be positive integers, with \( m > t \). Let \( \lambda_{j,b} = 0 \) for all integers \( b \in (m - t, m] \) and \( j \in \{0, 1\} \). Further suppose that for all non-negative integers \( c \leq t \) and \( \ell \leq 2t \), we have

\[
\sum_{b=0}^{m} \lambda_{0,b} \binom{m-b}{c} \binom{b}{\ell} = \sum_{b=0}^{m} \lambda_{1,b} \binom{m-b}{c} \binom{b}{\ell}.
\]

Let \( k \) and \( k' \) be binary vectors of equal weight \( k \) where \( k \leq t \). Then

\[
\langle 0_{L} | A_{k}^{\dagger} A_{k'} | 0_{L} \rangle = \langle 1_{L} | A_{k}^{\dagger} A_{k'} | 1_{L} \rangle + O(\gamma^{2t+1}).
\]

**Proof.** For non-negative integers \( m, k, b, c \) such that \( b \leq m - c \) and \( c \leq k \), we have

\[
\frac{m-k-c}{b-k} \binom{m}{b} = \frac{b}{k} \binom{m-b}{c} \binom{k}{c} (c!)^2.
\]

(24)

Using the identity (24) with (23), we get

\[
\langle j_{L} | A_{k}^{\dagger} A_{k'} | j_{L} \rangle = \sum_{b=0}^{m} \lambda_{j,b} \gamma^{k}(1-\gamma)^{b-k} \left( \frac{b}{k} \right) \left( \frac{m-b}{c} \right) \binom{k}{c} (c!)^2
\]

\[
= \sum_{b=0}^{m} \lambda_{j,b} \gamma^{k} \sum_{\beta=0}^{b-k} \frac{b-k}{\beta} \gamma^{\beta}(-1)^{\beta} \left( \frac{b}{k} \right) \left( \frac{m-b}{c} \right) \binom{k}{c} (c!)^2
\]

\[
= \left( \frac{k}{c} \right) (c!)^2 \left( \frac{m}{k-c} \right)^{-1} \gamma^{k} \sum_{\beta=0}^{b-k} \frac{b(k)(b-k)(\beta)}{k!\beta!} \gamma^{\beta}(-1)^{\beta}
\]

\[
= \left( \frac{k}{c} \right) (c!)^2 \left( \frac{m}{k-c} \right)^{-1} \gamma^{k} \sum_{\beta=0}^{b-k} \lambda_{j,b} \left( \frac{m-b}{c} \right) \binom{k}{c} \binom{b}{k+\beta} \gamma^{\beta}(-1)^{\beta}.
\]

(25)
Hence the coefficient of $\gamma^{\beta+k}$ in the expression above is

$$
\binom{k}{c} (c!)^2 \left( \frac{m}{k-c} \right)^{-1} \left( \sum_{b=0}^{m} \lambda_{j,b} \binom{m-b}{c} \binom{b}{k+\beta} \right) \binom{k+\beta}{k} (-1)^{\beta}.
$$

By the assumption of the lemma, the desired result follows.

We also need the following combinatorial lemma:

**Lemma IV.2.** Let $t$ be a positive integer. Then for any non-negative $\alpha \leq t + 1$ we have

$$
\sum_{i=0}^{t+1} \binom{t+1}{i} i^\alpha (-1)^i = 0.
$$

**Proof.** Let $g(x) := (1 - x)^{t+1}$ so that $g^{(\alpha)}(x) = (t+1)_{(\alpha)}(1-x)^{t+1-\alpha}$ and $g^{(\alpha)}(1) = 0$. Substituting $x = 1$ into the expansion

$$
g^{(\alpha)}(x) = \sum_{i=0}^{t+1} \binom{t+1}{i} (-1)^i i^{(\alpha)} x^{i-\alpha},
$$

we get the binomial identity

$$
\sum_{i=0}^{t+1} \binom{t+1}{i} (-1)^i i^{(\alpha)} = 0.
$$

We prove our lemma by inducting on $\alpha$. The binomial identity (26) implies that our lemma is true for the base cases $\alpha = 0, 1$. The identity (26) also implies that

$$
\sum_{i=0}^{t+1} \binom{t+1}{i} (-1)^i i^{(\alpha'+1)} = 0.
$$

Expanding the falling factorial $i^{(\alpha'+1)}$ into a sum of monomials in $i$ when $\alpha' \leq t$, we get

$$
\sum_{i=0}^{t+1} \binom{t+1}{i} i^{\alpha'+1} (-1)^i = \sum_{\beta=0}^{\alpha'} b_{\beta} \left( \sum_{i=0}^{t+1} \binom{t+1}{i} \right) \beta (-1)^i
$$

for some choice of constants $b_{\beta} \in \mathbb{R}$. Note that the bracketed term in the equation above is zero by the hypothesis that our lemma is true for $\alpha = \alpha'$, where $1 \leq \alpha' \leq t$. Hence the lemma also holds for $\alpha = \alpha' + 1$.

The existence of permutation-invariant amplitude damping codes is given by the following theorem:
Theorem IV.3. Let \( t \) be any positive integer and \( m = 9t^2 + 4t \). For all \( j \in \{0, 1\} \) and integers \( b \leq m \) divisible by \( 3t \) let

\[
\lambda_{j,b} = \left( \frac{3t + 1}{b/(3t)} \right)^{2 - \frac{3t}{1} + \frac{(-1)^{b+j}}{2}} \quad (27)
\]

and all other values of \( \lambda_{j,b} \) be zero. Then the span of \( |0_L\rangle = \sum_{b=0}^{m} \sqrt{\lambda_{0,b}} |P_{m,b}\rangle \) and \( |1_L\rangle = \sum_{b=0}^{m} \sqrt{\lambda_{1,b}} |P_{m,b}\rangle \) is a \( t \)-AD permutation-invariant code.

Proof. By definition, \( \lambda_{j,b} \) satisfy the normalization condition \( \sum_{b=0}^{m} \lambda_{j,b} = 1 \), so \( \lambda_{j,b} \) define valid logical codewords \( |j_L\rangle \). Moreover, \( |0_L\rangle \) and \( |1_L\rangle \) have distinct support, and are hence linearly independent spanning a two-dimension codespace.

Now define the truncated Kraus set of amplitude damping effects \( \Omega := \{ A_k : k \in \mathbb{Z}_2^m, \| k \|_1 \leq t \} \), so that \( \Omega \) satisfies the order constraints of Corollary III.5. Define the matrix \( G_j \) to be \( \sum_{E,F \in \Omega} \langle j_L|E|F\rangle|E\rangle\langle F| \) and \( G := \frac{G_0 + G_1}{2} \). To use Corollary III.5 we first have to prove that the matrix \( G \) is positive definite.

Let \( \tilde{A} = \sum_{F \in \Omega} v_{A,F} F \) and \( V := \sum_{E,F \in \Omega} v_{E,F} |E\rangle\langle F| \) be as defined in (9) and \( \vec{0} \) respectively. and correspondingly define the vector \( |\Psi_A\rangle := \sum_{F \in \Omega} v_{A,F} |F\rangle \). Observe that \( \lambda_{\min}(G) = \lambda_{\min}(VGV^\dagger) = \min_{A \in \Omega} \langle A|VGV^\dagger|A\rangle = \min_{A \in \Omega} \langle \Psi_A|G|\Psi_A\rangle = \min_{A \in \Omega} \frac{1}{2} \langle \Psi_A|(G_0 + G_1)|\Psi_A\rangle = \min_{A \in \Omega} \frac{1}{2} \sum_{j=0}^{1} \langle j_L|\tilde{A}^\dagger \tilde{A}|j_L\rangle = \min_{A \in \Omega} \frac{1}{2} \sum_{j=0}^{1} \| \tilde{A}|j_L\rangle \parallel^2 \).

Now the Kraus elements in \( \Omega \) annihilate at most \( t \) excitations, but the logical states \( |j_L\rangle \) are permutation-invariant with support containing at least \( 3t \) excitations. Hence \( \| \tilde{A}|j_L\rangle \parallel^2 > 0 \) which implies that \( \lambda_{\min}(G) > 0 \).

Our choice of \( \lambda_{j,b} \) implies that \( \langle j_L|A_k^\dagger A_{k'}|j_L\rangle = 0 \) and \( \langle 0_L|A_k^\dagger A_{k'}|1_L\rangle = 0 \) when \( k, k' \in \mathbb{Z}_2^m \) such that \( \| k \|_1 = \| k' \|_1 \leq t \). Now we set \( t \) to \( 3t \) in Lemma IV.2 and note that the coefficient of \( \lambda_{j,b} \) in the assumption of Lemma IV.1 is a polynomial of order no more than \( 3t \) in the variable \( b \). Then applying Lemma IV.2 to Lemma IV.1 shows that the conditions of Theorem III.4 are satisfied with \( \epsilon = \mathcal{O}(\gamma^{2t}) \). \( \square \)
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