Derivation of Einstein’s Equation from a New Type of four Dimensional Superstring

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A new type of superstring in four dimension is proposed which has the central charge 26. The Neveu Schwarz and the Ramond vacua are both tachyonic. The self energy of the scalar tachyon cancels from the contribution of the fermionic loop of the Ramond sector. The NS tachyonic vacuum is used to construct a massless graviton. Coulombic vector excitations of zero mass, referred as a ‘Newtonian’ graviton are also shown to exist. The propagators are explicitly evaluated. Following the method of Weinberg, we deduce the Einstein’s field equation of general relativity.

The string theory has come to prominence due to appearance of the graviton in the mass spectrum of Nambu-Goto string. The researches on this line has been to develop gravity from a ten dimensional superstring by writing the 

where

or

The commutator of two supersymmetric transformation is a translation

with $e^j$ as the constant anticommutating spinor. $e^j$ and $e^k$ are the two unit component c-number row vectors with the property that $e^j e^j = \delta^j_j$ and $e^j e^k = \delta^j_k$, and it follows that $e^j e_j = 6$ and $e^j e_k = 5$.

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\begin{align}
\delta X^\mu &= \bar{\epsilon} \left( e^j \psi_{j}^{\mu} - e^k \phi_{k}^{\mu} \right), \\
\delta \psi^{\mu j} &= -i e^j \rho^\alpha \partial_\alpha X^\mu \, \epsilon, \\
\delta \phi^{\mu k} &= i e^k \rho^\alpha \partial_\alpha X^\mu \, \epsilon,
\end{align}

contributing eleven to the central charge like the super conformal Faddeev-Popov ghosts. They will be eliminated from the Fock space by the susidiary conditions. $\rho$-matrices are given in [1] and [2] along with other objects.

\begin{align}
\delta \psi^{\mu j} &= \psi^{(+)\mu j} + \psi^{(-)\mu j} \\
\delta \phi^{\mu k} &= \phi^{(+)\mu k} - \phi^{(-)\mu k}
\end{align}

where $+,-$ refer to the positive and the negative parts of the Majorana massless fermions. There are in all 13 metric ghosts, two bosonic and eleven fermionic. In the Lorentz metric, the fermionic sector light cone action is

\begin{align}
S_{L.c.} &= \frac{i}{2\pi} \int d^2 \sigma \sum_{\mu=0,3} \left( \bar{\psi}^{\mu j} \rho^\alpha \partial_\alpha \psi_{j}^{\mu} - \bar{\phi}^{\mu k} \rho^\alpha \partial_\alpha \phi_{k}^{\mu} \right)
\end{align}

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\delta \psi^{\mu j} &= -i e^j \rho^\alpha \partial_\alpha X^\mu \, \epsilon, \\
\delta \phi^{\mu k} &= i e^k \rho^\alpha \partial_\alpha X^\mu \, \epsilon,
\end{align}

with $\epsilon$ as the constant anticommutating spinor. $e^j$ and $e^k$ are the two unit component c-number row vectors with the property that $e^j e^j = \delta^j_j$ and $e^j e^k = \delta^j_k$, and it follows that $e^j e_j = 6$ and $e^j e_k = 5$.

The commutator of two supersymmetric transformation is a translation

\begin{align}
[\delta_1, \delta_2]X^\mu &= a^\alpha \partial_\alpha X^\mu,
\end{align}

where $a^\alpha = 2i \bar{\epsilon}_1 \rho^a \epsilon_2$. Similar are the results for the spinors $\psi^{\mu j} = e^j \Psi^\mu$ and $\phi^{\mu k} = e^k \Psi^\mu$ where

\begin{align}
\Psi^\mu = e^j \psi_{j}^{\mu} - e^k \phi_{k}^{\mu}.
\end{align}

It is easy to verify that $\delta X^\mu = \bar{\epsilon} \Psi^\mu$, $\delta \Psi^\mu = -i \rho^\alpha \partial_\alpha X^\mu \epsilon$ and $[\delta_1, \delta_2] \Psi^\mu = a^\alpha \partial_\alpha \Psi^\mu$. 

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Thus $\Psi^\mu$ is the supersymmetric partner of $X^\mu$. Introducing another supersymmetric pair, the zweibein $e_\alpha(\sigma, \tau)$ and the ‘gravitino’ $\chi^\alpha = \nabla_\alpha \epsilon$, the local 2d supersymmetric action is

$$S = -\frac{1}{2\pi} \int d^2\sigma \ e \left[ h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu - i \bar{\Psi}^\mu \rho^\alpha \nabla_\alpha \Psi_{\mu} + 2 \bar{\chi}^\alpha \rho^\beta \rho^\alpha \Psi^\mu \partial_\beta X_\mu + \frac{1}{2} \bar{\Psi}^\mu \Psi_{\mu} \chi^\beta \rho^\beta \rho^\alpha \chi_\alpha \right]. \quad (7)$$

The variation of this equation with respect to $\chi^\alpha$ and $e_\alpha^a$ leads to the equations for current and energy momentum tensors. In the gauge where the gravitino $\chi_\alpha$ will be zero, and $e_\alpha^a = \delta_\alpha^a$,

$$J_\alpha = \frac{\pi}{2e} \frac{\delta S}{\delta \chi^\alpha} = \frac{1}{2} \rho^\beta \rho_\alpha \bar{\Psi}^\mu \partial_\beta X_\mu - \frac{1}{2} \bar{\Psi}^\mu \Psi_{\mu} \rho_\alpha \rho^\beta \chi_\beta = 0, \quad (8)$$

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{i}{2} \bar{\Psi}^\mu \rho_{(\alpha} \partial_{\beta)} \Psi_\mu - (trace) = 0. \quad (9)$$

In the light cone basis $\tilde{\psi} = (\psi^+, \psi^-)$ and $\tilde{\phi} = (\phi^+, \phi^-)$, the above equations translate to

$$J_\pm = \partial_\pm X_\mu \Psi_\pm^\mu = 0, \quad (10)$$

and

$$T_{\pm\pm} = \partial_{\pm} X^\mu \partial_{\pm} X_\mu + \frac{i}{2} \psi_{\pm,j} \partial_{\pm} \psi_{\pm,j} - \frac{i}{2} \phi_{\pm,k} \partial_{\pm} \phi_{\pm,k} = 0 \quad (11)$$

where $\partial_{\pm} = \frac{1}{2} (\partial_\tau \pm \partial_\sigma)$. The component constraints are

$$\partial_{\pm} X_\mu \psi_{\pm,j} = \partial_{\pm} X_\mu \ e^j \Psi_\pm^\mu = 0, \quad j = 1, 2 \ldots 6. \quad (12)$$

$$\partial_{\pm} X_\mu \phi_{\pm,k} = \partial_{\pm} X_\mu \ e^k \Psi_\pm^\mu = 0, \quad k = 7, \ldots 11. \quad (13)$$

These eleven constraints are enough to eliminate the extra eleven Lorentz metric fermionic ghosts from physical spectrum. However, there will be one current constraint, equation (10), from which the eleven follow. So there will be eleven subsidiary current generators combinable to one and eleven pairs of $(\beta^1, \gamma^1), (\beta^2, \gamma^2)$ ghosts combinable to one pair $(\beta, \gamma)$ for the construction of nilpotent BRST charge.

We write the action $S_{\text{lc}}^\alpha$ in terms of light cone superpartner $\Psi_{\pm}$. The Hilbert space is supplemented by a space where particles obey Bose statistics and the light cone fermions are Grassmanians $\theta, \bar{\theta}$. In particular, $\bar{\Psi}^+(\theta) = \frac{1}{2} \bar{\theta}^\gamma$ and $\rho^a \Psi^- (\theta) = \frac{1}{2} \theta^a$. Integrating over $\theta \bar{\theta}$, we obtain $S_{\text{lc}}^\alpha = -\frac{1}{2\pi} \int d^2\sigma \gamma_\alpha \beta^\alpha$. Standard textbook results follow. For the two groups of fermions, the group invariant constraints are

$$\partial_{\pm} X_\mu \ e^k \phi_{\pm,k} = \partial_{\pm} X_\mu \ e^j \psi_{\pm,j}^\mu = 0. \quad (14)$$

In the covariant formulation, the total number of physical degrees of freedom is the total number fourteen minus the total number of constraints. Due to the above four constraints (14), there are 40 physical space time fermionic modes in the theory.

SO(3,1) has Dirac spinorial representation denoted by $\theta_{j, \delta}$ and $\theta_{k, \delta}$, where $\delta = 1, 2, 3, 4$. So we can construct a genuine space time spinor with four components

$$\theta_\delta = \sum_{j=1}^{6} e^j \theta_{j, \delta} - \sum_{k=7}^{11} e^k \theta_{k, \delta}. \quad (15)$$

The Green Schwarz action [2] exhibiting local four dimensional N=1 supersymmetry is

$$S = \frac{1}{2\pi} \int d^2\sigma \left( \sqrt{g} g^{\alpha\beta} \Pi_\alpha \cdot \Pi_\beta + 2ie^{a\beta} \partial_\alpha X^\mu \bar{\theta}_\mu \partial_\beta \theta \right), \quad (16)$$

where $\Gamma_\mu$ are the Dirac gamma matrices and

$$\Pi_\alpha^\mu = \partial_\alpha X^\mu - i \bar{\theta} \Gamma_\mu \partial_\alpha \theta. \quad (17)$$
It is difficult to quantise this action, so we fall back on the Neveu-Schwarz \[3\] and the Ramond \[4\] formulations with the G.S.O \[5\] projection. The GSO operator is to project out the odd number of fermionic modes from the Hilbert space and is defined as

\[
G = \frac{1}{2} \left( 1 + (-)^F \right).
\]

where \(F\) is the fermion number. The forty fermionic modes can be placed in five identical groups, each group containing eight of them. The total partition function is that of a group of eight, raised to the power of five. It has been shown by Seiberg and Witten \[\text{[6]}\] that the partition function of eight fermions with Neveu-Schwarz and Ramond boundary conditions, vanish due to the famous Jacobi equality among the \(\Theta\)-functions. Thus the total product partition functions of the string states vanish. This is also the condition for a local supersymmetry.

The commutators and anticommutators between fields follow from the action of equation (1). The fields can be quantised in the usual way \[2\]. Let the bosonic quanta be denoted by \(\alpha_m\), the \((b^r_{m,j}, b^r_{m,k})\) with \(r\) half integral be the quanta of \((\psi^\mu, \phi^\mu)\) satisfying NS boundary condition and \((d^m_{n,j}, d^m_{n,k})\) with \(m\) integral, satisfying Ramond boundary condition. The nonvanishing commutation and anticommutation relations are

\[
[\alpha_m, \alpha_n] = m \delta_{m,-n} \gamma^\mu
\]

\[
\{ b^r_{m,j}, b^{r'}_{n,j} \} = g^{\mu\nu} \delta^{r,r'} \delta_{m,-n},
\]

\[
\{ d^m_{n,j}, d^{m'}_{n,j} \} = g^{\mu\nu} \delta^{m,m'} \delta_{n,-n},
\]

\[
\{ d^m_{n,j}, b^{r}_{m,k} \} = g^{\mu\nu} \delta^{m,m'} \delta_{n,-n}.
\]

The phase of the creation operator is such that for \(r, m > 0\), \(b^r_{m} = -b^r_{m}\) and \(d^m_{n} = -d^m_{n}\) It is necessary to identify the relations \(b^r_{m} = e^{iB^r_{m}}\), \(b^{r}_{m} = e^{kB^r_{m}}\) such that \(B^r_{m} = e^{j}b^r_{m,j} - e^{k}b^{r}_{m,k}\). Similarly for the \(d\), \(d\):

\[
D^m_{n} = e^{j}d^{m}_{m,j} - e^{k}d^{m}_{m,k}.
\]

The Virasoro generators

\[
L_m = \frac{1}{\pi} \int_0^\pi d\sigma \epsilon^{im\sigma} T_{++}
\]

\[
= \frac{1}{2} \sum_{r = \pm}^\infty : \alpha_{n} \cdot \alpha_{m+n} : + \frac{1}{2} \sum_{r = \pm}^\infty (r + \frac{1}{2} m) : (b_{m} \cdot b_{m+r} - b'_{m} \cdot b'_{m+r}) : \quad \text{NS}
\]

\[
= \frac{1}{2} \sum_{n = \pm}^\infty : \alpha_{n} \cdot \alpha_{m+n} : + \frac{1}{2} \sum_{n = \pm}^\infty (n + \frac{1}{2} m) : (d_{m} \cdot d_{m+n} - d'_{m} \cdot d'_{m+n}) : \quad \text{R}
\]

\[
G_r = \frac{\sqrt{3}}{\pi} \int_0^\pi d\sigma \epsilon^{i\sigma} J_+ = \sum_{n = \pm}^\infty \alpha_{n} \cdot B_{n+r}, \quad \text{NS}
\]

\[
F_m = \sum_{n = \pm}^\infty \alpha_{n} \cdot D_{n+m}, \quad \text{R}
\]

satisfy the super Virasoro algebra \[\text{[7]}\]

\[
[L_m, L_n] = (m - n) L_{m+n} + \frac{C}{12} (m^3 - m) \delta_{m,-n},
\]

\[
[L_m, G_r] = \left( \frac{1}{2} m - r \right) G_{m+r}, \quad \text{NS}
\]

\[
\{ G_r, G_s \} = 2 L_{s+r} + \frac{C}{3} (r^2 - \frac{1}{4}) \delta_{r,-s},
\]

\[
[L_m, F_n] = \left( \frac{1}{2} m - n \right) F_{m+n}, \quad \text{R}
\]

\[
\{ F_m, F_n \} = 2 L_{m+n} + \frac{C}{3} (m^2 - 1) \delta_{m,-n}, \quad m \neq 0.
\]
Since, from equation (11), \( <T_{\pm\pm}(z)T_{\pm\pm}(\omega)> = \frac{2\pi}{\omega}(z - \omega)^{-4} + \ldots \), the central charge \( C = 26 \). It is worth while to note that \( <T_{\pm\pm}(z)T_{\pm\pm}(\omega)> = \frac{\beta}{\omega}(z - \omega)^{-4} \) with central charge 11. The terms, containing the central charge \( C = 26 \), are the anomaly terms due to the normal ordering (2). As is well known, they are cancelled by the contribution from the conformal ghosts (h,c).

This is also known that all anomalies will cancel if the normal ordering constant of \( L_o \) is equal to one. We define the physical states as satisfying

\[
\begin{align*}
(L_o - 1)|\phi> &= 0, \quad L_m|\phi> = 0, \quad G_r|\phi> = 0 \quad \text{for} \quad r, m > 0, \quad : NS \quad Bosonic \\
L_m|\psi> &= F_m|\psi> = 0 \quad \text{for} \quad m > 0, \quad : R \quad Fermionic \\
(L_o - 1)|\psi>_{\alpha} &= (F_o^2 - 1)|\psi>_{\alpha} = 0.
\end{align*}
\]

So we have

\[
(F_o + 1)|\psi>_{+\alpha} = 0 \quad \text{and} \quad (F_o - 1)|\psi>_{-\alpha} = 0
\]

These conditions shall make the model ghost free.

It can be seen in a simple way. Applying \( L_o \) condition the state \( a_{0,1}^\alpha 0, k > \) is massless and the \( L_1 \) constraint gives the Lorentz condition \( k^\mu 0, k > 0 \) implying a transverse photon and Gupta Bleuler impose that \( a_{0,1}^\alpha 0, k > 0 \).

Applying \( L_2, \ L_3 \ldots \), constraints, one obtains \( a_{0,1}^\alpha 0, k > 0 \). Further, since \( [a_{0,1}^\alpha, G_{r+1}]|\phi> = 0, \ B_o^2|\phi> = 0 \) and \( b_r^2|\phi> = e_j b_r^2|\phi> = 0; b_r^2|\phi> = e_j b_r^2|\phi> = 0 \). All the time components are eliminated from Fock space.

From Fourier transforms and definition, the eleven subsidiary physical state conditions are

\[
\begin{align*}
G_r^j|\phi> &= e_j G_r|\phi> = 0, \quad G_r^j|\phi> = e^k G_r|\phi> = 0, \quad \text{for} \quad r > 0, \quad NS \\
F_m^j|\phi> &= e_j F_m|\phi> = 0, \quad F_m^j|\phi> = e^k F_m|\phi> = 0, \quad \text{for} \quad m > 0, \quad R.
\end{align*}
\]

We now proceed to write the nilpotent BRST charge. The part which comes from the usual conformal Lie algebra technique is

\[
(Q_1)^{NS,R} = \sum (L_{-m} c_m)^{NS,R} - \frac{1}{2} \sum (m - n) : c_{-m} c_{-n} b_{m+n} : - a_{01}; \quad Q_1^2 = 0 \quad \text{for} \quad a = 1.
\]

The eleven pairs of commuting ghost quanta \((\beta_r, \gamma_r)\) of ghost fields \((\beta(\tau), \gamma(\tau))\), satisfying \( d\gamma = d\beta = 0 \), needed for subsidiary current generator operator conditions (34) and (35), are related as

\[
\begin{align*}
\beta_r^j &= e^j \beta_r, \quad \beta_r^j = e^k \beta_r, \quad \beta_r = e^j \beta_r - e^k \beta_r^k, \\
\gamma_r^j &= e^j \gamma_r, \quad \gamma_r^j = e^k \gamma_r, \quad \gamma_r = e^j \gamma_r - e^k \gamma_r^k. \quad (NS)
\end{align*}
\]

There are identical ones for the Ramond sector with half integral ‘r’ replaced by integral ‘m’. The light cone ghost quanta satisfy \( \{b_{r+s}^+ \} = -\delta_{r+s} \delta_{r+s} \) and \( \{b_{r+s}^- \} = -\delta_{r-s} \delta_{r-s} \), where as \( [\beta_r^j, \beta_r^j] = \delta_{r-s} \delta_{r-s} \) and \( [\beta_r^j, \beta_r^j] = \delta_{r-s} \delta_{r-s} \) and it follows that \( [\gamma_r^j, \beta_r^j] = \delta_{r-s} \). All that we want to know is the conformal dimensions, \( \gamma \) with \(-\frac{3}{2}\), and \( \beta \) with \(-\frac{3}{2}\).

\[
\begin{align*}
[L_m, \gamma_r^{(j,k)}] &= -(\frac{3}{2} m + n) \gamma_r^{(j,k)}, \quad [L_m, \beta_r^{(j,k)}] = (\frac{1}{2} m - n) \beta_r^{(j,k)}.
\end{align*}
\]

Using the Graded Lie algebra, we get the additional BRST charge.

\[
\begin{align*}
Q_{NS} &= \sum G_{-r} \gamma_r - \sum \gamma_r \gamma_s b_{r+s}, \\
Q_R &= \sum F_{-r} \gamma_r - \sum \gamma_r \gamma_s b_{r+s}.
\end{align*}
\]

It is to be noted that the products

\[
G_{-r} \gamma_{-r} = G_{-r} \gamma_{-r} - G_{-r} \gamma_{-r},
\]

so that all the eleven pairs of ghosts are present in the charge. As constructed, the BRST charge

\[
Q_{BRST} = Q_1 + Q',
\]

is such that \( Q_{BRST}^2 = 0 \) in both NS and R sector (1). In proving \( \{Q', Q'\} + 2\{Q_1, Q'\} = 0 \), we have used the fourier transforms, wave equations and integration by parts to show that \( \sum \sum r^2 \gamma_r \gamma_s \delta_{r-s} = \sum_r \sum_s \gamma_r \gamma_s \delta_{r-s} = 0 \). The theory is unitary and ghost free.
The mass spectrum is
\[ NS : \quad \alpha' M^2 = -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \]
\[ R : \quad \alpha' M^2 = -1, 0, 1, 2, 3, \ldots. \]  
(42)

The GSO projection eliminates the half integral masses. The scalar bosonic vacuum energy \( < 0 \left( L^N_{\alpha} \right)^{-1} |0 > \) is cancelled by the fermionic energy \(- < 0 \left( F_{\alpha} - 1 \right)^{-1} \left( F_{\alpha} + 1 \right)^{-1} |0 > \), the negative sign arising due to the normal ordering of the fermions. In both the sectors, we have the Regge trajectories \( \alpha' M^2 = -1, 0, 1, 2, 3, \ldots. \).

Thus, satisfying ourselves that we have an anomaly free, ghost free and harmless but useful tachyons, we attempt to tackle the problem of gravitational field theory from the above string theory. Let us construct tensors like \( b^{\mu}_{\frac{1}{2}}, b^{\mu}_{-\frac{1}{2}} \).

For simplicity, we drop the suffix \( -\frac{1}{2} \). Consider the string state
\[ a^{\mu \nu}(p) = \sum_{i,j} C^{ij}(b^{\mu}_{i} b^{\nu}_{j} + b^{\nu}_{j} b^{\mu}_{i} - 2 \eta^{\mu \nu} b^{\lambda}_{i} b^{\lambda}_{j}) |0, p > . \]  
(43)

This is symmetric and traceless. Further \( L_{\alpha} \) will be taken as the free Hamiltonian \( H_{\alpha} \) in the interaction representation. Operating on this state, one gets \( L_{\alpha} a^{\mu \nu}(p) = 0 \). So this is massless. Further \( p_{\mu} a^{\mu \nu} = p_{\nu} a^{\mu \nu} = 0 \) if \( C^{ij} = 0 \).

If \( C^{ij} \) is symmetric, a little algebra shows that \( G_{\frac{1}{2}} a^{\mu \nu}(p) = [G_{\frac{1}{2}}, a^{\mu \nu}(p)] = 0 \).

Thus \( a^{\mu \nu}(p) \) satisfy all the physical state conditions due to Virasoro algebraic relations. This is the graviton. The commutator is
\[ [a^{\mu \nu}(p), a^{\lambda \sigma}(q)] = 2 \pi |C| \delta^{4}(p - q). \]  
(45)

To switch over to the quantum field theory, we define the gravitational field by space time fourier transform of \( a^{\mu \nu}(p) \)
\[ a^{\mu \nu}(x) = \frac{1}{(2\pi)^{2}} \int \frac{d^{4}p}{2p^{0}} \left[ a^{\mu \nu}(p) e^{ipx} + a^{\nu \mu}(p) e^{-ipx} \right], \]  
(46)

with the commutator
\[ \left[ a^{\mu \nu}(x), a^{\lambda \sigma}(y) \right] = \frac{1}{(2\pi)^{2}} \int \frac{d^{4}p}{2p^{0}} \left[ e^{ip(x-y)} - e^{-ip(x-y)} \right] f^{\mu \nu, \lambda \sigma}, \]  
(47)

where
\[ f^{\mu \nu, \lambda \sigma} = g^{\mu \lambda} g^{\nu \sigma} + g^{\mu \sigma} g^{\nu \lambda} - g^{\mu \nu} g^{\lambda \sigma}. \]  
(48)

The Feynman propagator is
\[ \Delta^{\mu \nu, \lambda \sigma}(x - y) = < 0 | T(a^{\mu \nu}(x) a^{\lambda \sigma}(y)) |0 > = \frac{1}{(2\pi)^{2}} \int d^{4}p \, e^{i p \cdot (x-y)} \Delta^{\mu \nu, \lambda \sigma}(p), \]
\[ \Delta^{\mu \nu, \lambda \sigma}(p) = \frac{1}{2} f^{\mu \nu, \lambda \sigma} \frac{1}{p^{2} - i \epsilon}. \]

This is the propagator of the graviton in the interaction representation.

As already noted in this superstring theory there is also a massless vector following from the \( L_{0} \) condition acting on \( \alpha' p^{\mu} |0, p > \) with Lorentz relation \( p^{\mu} |0, p > = 0 \), due to the \( L_{1} \) condition. With the help of a time like vector \( n^{\mu} \), we can construct another traceless second rank ‘Newtonian’(N) tensor
\[ a^{\mu \nu}_{N,1} = (n^{\mu} \alpha^{\nu}_{-1} + n^{\nu} \alpha^{\mu}_{-1} - g^{\mu \nu} (n \cdot \alpha_{-1}) ) |0, p >_{NS}. \]  
(50)

There are several points which are conflicting. While forming the commutator like equation(47), we will arrive at a term with \( g^{\mu \nu} g^{\lambda \sigma} \) which is already there in the graviton propagator. So there will be over counting. Secondly,
We now proceed to construct the propagators, remembering that this is local and there is no pole at \(|\mathbf{p}|^2 = p_o^2\) except in the \(g^{\mu\nu} g^{\lambda\sigma}\) term

\[
\Delta_{N,1}^{\mu\nu,\lambda\sigma} = \frac{1}{2} \left( \frac{f_N^{\mu\nu\lambda\sigma}}{|\mathbf{p}|^2} - \frac{g^{\mu\nu} n^\lambda n^\sigma + g^{\lambda\sigma} n^\mu n^\nu - g^{\mu\nu} g^{\lambda\sigma}}{|\mathbf{p}|^2} \right),
\]

(51)

where

\[
f_N^{\mu\nu\lambda\sigma} = g^{\mu\lambda} n^{\nu\sigma} + g^{\mu\sigma} n^{\nu\lambda} + g^{\nu\lambda} n^{\mu\sigma} + g^{\nu\sigma} n^{\mu\lambda} - g^{\mu\nu} n^\lambda n^\sigma - g^{\lambda\sigma} n^\mu n^\nu.
\]

(52)

We have still the traceless tensor \(\Pi^{\mu\nu}(p)\) [8] and the creation operator \(L^{-1}|o,\mathbf{p}>\). Since \(<o,\mathbf{p}|L^1 L^{-1}|o,\mathbf{p}>\rangle = 2\), we construct a third traceless tensor,

\[
a_{N,2}^{\mu\nu} = \frac{1}{\sqrt{2}} \Pi^{\mu\nu} L^{-1} |o,\mathbf{p}>.
\]

(53)

The propagator is simply

\[
\Delta_{N,2}^{\mu\nu,\lambda\sigma} = \frac{1}{2(p^2 - i\epsilon)} \Pi^{\mu\nu} \Pi^{\lambda\sigma}.
\]

(54)

The gradient terms do not contribute when contracted with conserved energy momentum tensor. Ignoring the gradient terms

\[
\Pi^{\mu\nu} \rightarrow g^{\mu\nu} + \frac{p^2}{|\mathbf{p}|^2} n^\mu n^\nu,
\]

leading to

\[
\Delta_{N,2}^{\mu\nu,\lambda\sigma} = \frac{1}{2(p^2 - i\epsilon)} \left( g^{\mu\nu} g^{\lambda\sigma} + (g^{\mu\nu} n^\lambda n^\sigma + n^\mu n^\nu g^{\lambda\sigma}) \frac{p^2}{|\mathbf{p}|^2} + n^\mu n^\nu n^\lambda n^\sigma \frac{(p^2)^2}{|\mathbf{p}|^4} \right).
\]

(55)

The total 'Newtonian' graviton propagator is

\[
\Delta_N^{\mu\nu,\lambda\sigma} = \frac{1}{2} \frac{f_N^{\mu\nu\lambda\sigma}}{|\mathbf{p}|^2} + \frac{1}{2} n^\mu n^\nu n^\lambda n^\sigma \frac{(p^2)}{|\mathbf{p}|^4}.
\]

(56)

In all

\[
\Delta_F^{\mu\nu,\lambda\sigma} = \Delta_N^{\mu\nu,\lambda\sigma} + \Delta_{N,2}^{\mu\nu,\lambda\sigma}.
\]

(57)

In space time, the fourier transformed propagator is

\[
\Delta_N^{\mu\nu,\lambda\sigma}(x - y) = \frac{1}{(2\pi)^4} \int d^4 p \ e^{i\mathbf{p} \cdot (x - y)} \Delta_N^{\mu\nu,\lambda\sigma}(p)
\]

\[
= \frac{1}{2} \left[ \left( f_N^{\mu\nu\lambda\sigma} + n^\mu n^\nu n^\lambda n^\sigma \right) \delta(x^o - y^o) D(x - y) + n^\mu n^\nu n^\lambda n^\sigma \delta(x^o - y^o) E(x - y) \right],
\]

(58)

where

\[
E(x) = \frac{1}{(2\pi)^3} \int d^3 q e^{i\mathbf{q} \cdot \mathbf{x}} \frac{1}{|\mathbf{q}|} = E(o) - \frac{|x|}{4\pi},
\]

(59)

and

\[
D(x) = \frac{1}{(2\pi)^4} \int d^4 q e^{i\mathbf{q} \cdot \mathbf{x}} \frac{1}{|\mathbf{q}|^2} = \frac{1}{4\pi |x|}.
\]

(60)
Thus we have obtained the quantum field theoretic result of Weinberg \[^{[8]}\] from this new string theory. $\Delta_{\mu \nu \lambda \sigma}^{\mu \nu \lambda \sigma} (x - y)$ is highly divergent and to cancel this we must add to the free Hamiltonian $H_0$, an interaction Hamiltonian $H'_N(t)$ as specified below

\[
H'_N(t) = \frac{1}{2} \int \int d^3x \, d^3y \, \bigg( 2 \theta^\mu_\nu(x,t) \theta_{\mu \nu}(y,t) - \frac{1}{2} \theta^\mu_\nu(x,t) \theta_{\mu \nu}(y,t) \bigg) + \frac{1}{2} \int \int d^3x \, d^3y \, \theta_{\mu \nu}(x,t) \theta_{\mu \nu}(y,t) \cdot D(x - y) + \frac{1}{2} \int \int d^3x \, d^3y \, \theta_{\mu \nu}(x,t) \theta_{\mu \nu}(y,t) \cdot E(x - y). \quad (61)
\]

As Weinberg puts it, ‘this term ugly as it seems, is precisely what is needed to generate Einstein Field Equations when we pass to Heisenberg representation’. $\theta_{\mu \nu}$ is the symmetric energy momentum tensor with vanishing divergence and includes the self energy of the gravitational field, the matter field and the Pauli term. In the Heisenberg representation, the spatial components are defined as

\[
a^{ij}_H(x) - \frac{1}{3} \delta^{ij} \delta^{kl} a_H^{kl}(x) = U(x^o) a^{ij}(x) U^{-1}(x^o),
\]

where

\[
U(x^o) = e^{i H_o t} \quad \text{and} \quad \partial_\mu \partial^\nu a^{ij}(x) = 0, \quad \partial_i a^{ij}(x) = 0.
\]

The only nonvanishing commutator is

\[
[a^{ij}(x), \; a^{kl}(y)] = i D^{ij, kl}(x - y).
\]

To evaluate the D’s from the derived propagation function, we introduce a four vector $\hat{p}^\mu = (1, \hat{p})$ and note that

\[
\hat{p}^\mu = \frac{p^\mu + n^\mu (|p| - p_o)}{|p|}.
\]

The terms containing $p^\mu$ when contracted with conserved currents vanish. So we have the effective equality

\[
\frac{n^\mu n^\nu}{|p|^2} = \frac{1}{p^2} [\hat{p}^\mu n^\nu + \hat{p}^\nu n^\mu - \hat{p}^\mu \hat{p}^\nu] + \text{gradient terms}.
\]

With the poles at $p^2$, the Green’s functions are easily calculated. After some algebra, retaining the spatial parts, we get

\[
D^{ij, kl}(x - y) = \frac{1}{2} \left[ \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk} - \delta^{ij} \delta^{kl} \right] \delta^{(3)}(x - y) + \frac{1}{2} \left[ \partial_i \partial^j \delta^{kl} + \partial^i \partial^j \delta^{kl} - \partial^i \partial^j \delta^{kl} - \partial^i \partial^j \delta^{ij} \right] D(x - y)
\]

\[
+ \frac{1}{2} \partial^i \partial^j \partial^k \partial^l E(x - y). \quad (66)
\]

The solutions to the Heisenberg field equations are easily worked out and are given by Weinberg \[^{[8]}\]. With $\theta_{H, kl}$ as the energy momentum tensor in the Heisenberg representation

\[
\partial^\mu \partial_\mu \left[ a_H^{ij}(x, t) - \frac{1}{3} \delta^{ij} \delta_{kl} a_H^{kl}(x, t) \right] = - \int d^3y \, D^{ij, kl}(x - y) \theta_{H, kl}(y, t). \quad (67)
\]

With Weinberg we also invent the time derivatives from the direct contact term

\[
H'_N(t) = \frac{1}{2} \int \int d^3x d^3y \left( \left[ 2 \theta^\mu_\nu(x, t) \theta_{\mu \nu}(y, t) - \frac{1}{2} \theta^\mu_\nu(x, t) \theta_{\mu \nu}(y, t) \right] D(x - y) \right. \\
- \left. \frac{1}{2} \theta_{\mu \nu}(x, t) \theta^\mu_\nu(y, t) - \frac{1}{2} \theta_{\mu \nu}(x, t) \theta_{\mu \nu}(y, t) \right) \bigg) D(x - y) + \theta_{\mu \nu}(x, t) \theta_{\mu \nu}(y, t) E(x - y). \quad (68)
\]
and define
\begin{align*}
a^{\alpha}_{iH}(x,t) &= \int d^3 y \, \theta^{\alpha}_{iH}(y,t) D(x-y); \quad \nabla^2 a^{\alpha}_{iH}(x) = -\theta^{\alpha}_{iH}(x), \\
a^{i}_{iH}(x,t) &= \frac{3}{2} \int d^3 y \, \theta^{i}_{iH}(y,t) D(x-y); \quad \nabla^2 a^{i}_{iH}(x) = -\frac{3}{2} \theta^{i}_{iH}(x), \\
a^{\alpha}_{i0}(x,t) &= \frac{1}{2} \int d^3 y \, \left[ \theta^{\alpha}_{i0}(y,t) + \theta^{\alpha}_{iH}(y,t) \right] D(x-y) - \frac{1}{2} \int d^3 y \, \theta^{\alpha}_{i0}(y,t) E(x-y), \\
\nabla^2 a^{\alpha}_{i0}(x) &= -\frac{1}{2} \theta^{\alpha}_{iH}(x) - \frac{1}{2} \theta^{\alpha}_{i0}(x) + \frac{1}{3} \ddot{a}^{i}_{iH}(x). \quad (69)
\end{align*}

Using tracelessness condition and current conservation condition
\begin{align*}
\partial_{i} a^{ij}_{iH}(x) &= \frac{1}{2} \partial^{j} \theta^{i}_{iH}(x), \\
\partial_{i} a^{i}_{i0}(x) &= -\frac{2}{3} \partial_{i} \theta^{i}_{iH}(x). \quad (70)
\end{align*}

and after some algebra we arrive at the result obtained by Weinberg
\begin{align*}
R_{\mu \nu}^{\alpha H}(x) &= -\theta^{\mu \nu}(x) + \frac{1}{2} g^{\mu \nu} \theta^{\lambda}_{\lambda H}(x), \quad (71)
\end{align*}

where
\begin{align*}
R_{\mu \nu}^{\alpha H}(x) &= \partial^{\nu} \partial_{\mu} a^{\mu \nu}_{iH}(x) - \partial^{\mu} \partial_{\lambda} a^{\lambda \nu}_{iH}(x) - \partial^{\nu} \partial_{\lambda} a^{\lambda \mu}_{iH}(x) + \partial^{\mu} \partial^{\nu} a_{iH}^{\lambda \lambda}(x). \quad (72)
\end{align*}

This equation can be put in the Einsteinian form
\begin{align*}
R_{\mu \nu}^{\alpha H}(x) - \frac{1}{2} g_{\mu \nu} R_{\alpha H}^{\lambda} = -\theta^{\mu \nu}_{H}(x). \quad (73)
\end{align*}

Thus the presence of the tachyons in superstring theory has provided the massless states which have led to the construction of graviton and Newtonian graviton and finally enabled us to deduce the Einstein’s field equations following Weinberg. This was first deduced by S.N Gupta [9]. We have made a direct contact from spin 2 quanta string amplitude to the field of the graviton with the help of the tachyonic vacuum. To our knowledge, this is the first direct derivation of the Einstein’s field equation from the superstring theory following Weinberg [8]. Since superstring theory is renormalisable, we hope that our research will help in probing further into the subtleties of Quantum Gravity.

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