Quasi-Exactly Solvable Spin 1/2 Schrödinger Operators*

Federico Finkel
Artemio González-López
Miguel A. Rodríguez

Departamento de Física Teórica II
Universidad Complutense
28040 Madrid, SPAIN

August 3, 1995

Abstract

The algebraic structures underlying quasi-exact solvability for spin 1/2 Hamiltonians in one dimension are studied in detail. Necessary and sufficient conditions for a matrix second-order differential operator preserving a space of wave functions with polynomial components to be equivalent to a Schrödinger operator are found. Systematic simplifications of these conditions are analyzed, and are then applied to the construction of several new examples of multi-parameter QES spin 1/2 Hamiltonians in one dimension.

PACS numbers: 03.65.Ge, 11.30.Na, 03.65.Fd.

*Supported in part by DGICYT Grant PB92-0197.
1 Introduction

Symmetries have traditionally played an essential role in quantum mechanics. For a few remarkable Hamiltonians, the knowledge of enough symmetries leads to a complete characterization of the spectrum by algebraic methods, [14]. In general, however, the spectrum of an arbitrary Hamiltonian cannot be calculated analytically. During the last decade, a remarkable intermediate class of *quasi-exactly solvable* (QES) spectral problems was introduced, for which a finite part of the spectrum can be computed by purely algebraic methods, [18], [21], [16]. The key feature in the latter class of spectral problems is that the Hamiltonian $H$ is expressible as a quadratic combination of the generators of a finite-dimensional Lie algebra $g$ of first order differential operators preserving a finite-dimensional module of smooth functions $\mathcal{N}$. Thus, $H$ restricts to a linear transformation in the finite-dimensional vector space $\mathcal{N}$, and therefore part of its spectrum can be computed by matrix eigenvalue methods. Appropriate boundary conditions must be imposed so that the eigenfunctions thus obtained qualify as physical wave functions, as e.g. square integrability if they represent bound states of the system, [8].

These ideas, originally introduced for scalar Hamiltonians describing spinless particles, can be generalized to include particles with spin. The first step in this direction was taken by Shifman and Turbiner, [17], using the fact that a Hamiltonian for a spin $1/2$ particle in $d$ spatial dimensions can be constructed from a Lie superalgebra of first order differential operators in $d$ ordinary (commuting) variables and one Grassmann (anticommuting) variable. Alternatively, [1], $2 \times 2$ matrices (or $N \times N$ matrices for particles of arbitrary spin, [2]) can be used to represent the Grassmann variable. However, in stark contrast with the scalar case, very few examples of matrix QES Schrödinger operators have been found thus far, [17]. There are two important conceptual reasons for this fact. First, the algebraic structures underlying partial integrability in the matrix case are richer and less understood than in the scalar case. For one thing, as mentioned before, for matrix Hamiltonians Lie superalgebras of matrix differential operators naturally come into play, whereas in the scalar case only Lie algebras need be considered. Moreover, as we shall explain in Section 3, one even has to go beyond Lie superalgebras of matrix differential operators in order to explain quasi-exact solvability in the matrix case, [1], [2]. Secondly, [1], [3], every scalar second order differential operator in one dimension can be transformed into a Schrödinger operator of the form $-\partial_x^2 + V(x)$ by a suitable change of
the independent variable \( x \) and a local rescaling of the wave function. For matrix differential operators, the analogue of this result—\( V(x) \) being now a Hermitian matrix of smooth functions—is no longer true unless the operator satisfies quite stringent conditions, as we shall see in detail in Section 4.

The aim of this paper is to achieve a better theoretical understanding of quasi-exact solvability in the matrix case, which will enable us to construct new examples of matrix QES Schrödinger operators. To this end, in Sections 2 and 3 we study the algebraic properties of certain algebras of matrix QES operators, reviewing the literature on the subject and obtaining several new results as well. In particular, we give a self-contained proof of the characterization of the class of QES matrix differential operators preserving a finite-dimensional space of wave functions with polynomial components stated by Turbiner, [20], and Brihaye et al., [1], [2]. For the important particular case of spin \( 1/2 \) particles, we derive in Section 4 necessary and sufficient conditions for a QES operator to be equivalent to a non-trivial Schrödinger operator. These conditions turn out to be too complicated to be solved in full generality, and so in Sections 4 and 5 we introduce some key simplifications that will prove very useful in the task of finding explicit examples. Finally, the previous results are applied in Section 7 to the construction of several new examples of multi-parameter QES spin \( 1/2 \) Hamiltonians in one dimension.

## 2 Scalar QES Operators

We start with the scalar case, introducing the basic concepts and definitions and stating two theorems for the one-dimensional case which will play an important role in what follows. Since the results of this section are fairly standard, we will skip many details and all the proofs, referring the reader to the review articles [16] and [7] for an in-depth study.

Let \( M \) denote an open subset of \( \mathbb{R}^d \), and let \( \mathcal{D}^1(M) \) be the Lie algebra of first order differential operators

\[
X = \sum_{i=1}^{d} \xi^i(z) \frac{\partial}{\partial z^i} + \eta(z), \quad z = (z^1, \ldots, z^d) \in M,
\]

acting on \( C^\infty(M) \), the Lie bracket being defined as the usual commutator between operators:

\[
[X, Y] = XY - YX, \quad X, Y \in \mathcal{D}^1(M).
\]
Definition 2.1 A finite-dimensional Lie subalgebra \( \mathfrak{g} \) of \( \mathcal{D}(M) \) is called quasi-exactly solvable (QES) if it preserves a finite-dimensional module \( \mathcal{N} \subset C^\infty(M) \). A differential operator \( T \) is QES if it lies in the universal enveloping algebra \( \mathcal{U}(\mathfrak{g}) \) of a QES Lie algebra \( \mathfrak{g} \).

In general, quasi-exact solvability of a given differential operator \( T \) cannot be ascertained \textit{a priori}. Therefore, the procedure usually followed consists in classifying QES Lie algebras modulo a suitable equivalence relation, and then using the canonical forms in the classification thus obtained to construct QES operators.

Definition 2.2 Two differential operators \( T(z) \) and \( \bar{T}(\bar{z}) \) are equivalent if they are related by a change of the independent variables

\[
\bar{z} = \varphi(z)
\]  

and a local scale transformation by a non-vanishing function \( U(z) \), i.e.

\[
\bar{T}(\bar{z}) = U(z) T(z) U^{-1}(z).
\]

The corresponding notion of equivalence for QES algebras follows directly, i.e. two QES Lie algebras \( \mathfrak{g} \) and \( \bar{\mathfrak{g}} \) are equivalent if their elements can be mapped into each other by a \textit{fixed} transformation (2.1)–(2.2). Their associated finite-dimensional modules \( \mathcal{N} \) and \( \bar{\mathcal{N}} \) are then related by

\[
\bar{\mathcal{N}} = U \cdot \mathcal{N},
\]  

the functions being expressed in the appropriate coordinates. The local classification of finite-dimensional QES Lie algebras under the above notion of equivalence has already been completed for the case of one and two (real or complex) variables. Here we shall need only the one-dimensional case, [3], [8], [2], [5].

Theorem 2.3 Every (non-singular) QES Lie algebra in one (real or complex) variable is locally equivalent to a subalgebra of one of the Lie algebras

\[
\mathfrak{g}_n = \text{Span}\{\partial_z, z\partial_z, z^2\partial_z - nz, 1\},
\]  

where \( n \in \mathbb{N} \). The associated \( \mathfrak{g}_n \)-module is \( \mathcal{N}_n = \mathcal{P}_n \), the space of polynomials of degree at most \( n \).
The two-dimensional case, which is considerably more complicated but is not needed for the sequel, is discussed in [6], [5], and [9].

According to the previous theorem, every one-dimensional (scalar) QES differential operator $T$ is locally equivalent to an operator $T \in \mathcal{U}(g_n)$ preserving $P_n$ for a suitable $n$. A partial converse of the latter result follows from the following remarkable theorem due to Turbiner, [19]:

**Theorem 2.4** Let $T^{(k)}$ be a $k$-th order linear differential operator preserving $P_n$. We then have:

1) If $n \geq k$, then $T^{(k)}$ may be represented by a $k$-th degree polynomial in the operators

$$J^+_n = z^2 \partial_z - nz, \quad J^0_n = z \partial_z - \frac{n}{2}, \quad J^- = \partial_z, \quad (2.5)$$

2) If $k > n$, then $T^{(k)} = T \partial^{n+1}_z + \tilde{T}$, where $T$ is a linear differential operator of order $k - n - 1$, and $\tilde{T}$ is a linear differential operator of order at most $n$ satisfying i).

The operators $\{J^+_n, J^0_n, J^-\}$ defined above span a QES Lie algebra $\hat{g}_n$ isomorphic to $\mathfrak{sl}(2)$, and the Lie algebras $g_n$ in Theorem 2.3 are simply a central extension by the constant functions of the corresponding $\hat{g}_n$.

### 3 Algebraic Properties of PVSP Operators

In the last section we have seen that every scalar QES scalar differential operator in one variable is essentially (up to equivalence) a polynomial in the generators of a Lie algebra $\hat{g}_n$ preserving $P_n$ (for suitable $n$). When working with vector-valued wave functions, the natural generalization of $P_n$ is the polynomial vector space $P_{n_1,\ldots,n_N} = P_{n_1} \oplus \cdots \oplus P_{n_N}$, with elements $\Psi(z) = (\psi_1(z),\ldots,\psi_N(z))^t$ such that each component $\psi_i$ is a polynomial of degree at most $n_i$ with complex coefficients.

**Definition 3.1** A $N \times N$ matrix differential operator $T$ is called polynomial vector space preserving (PVSP) if it preserves $P_{n_1,\ldots,n_N} = P_{n_1} \oplus \cdots \oplus P_{n_N}$ for some non-negative integers $n_i, \ i = 1,\ldots,N$. 

We will denote by $\mathcal{P}^{(k)}_{n_1,\ldots,n_N}$ the complex vector space of linear PVSP operators of order at most $k$ preserving $\mathcal{P}_{n_1,\ldots,n_N}$. Following [1] and [2], we will restrict ourselves in this paper to studying matrix PVSP differential operators. As we will be mainly concerned with spin 1/2 particles, the case $N = 2$ deserves special attention.

### 3.1 Case $N = 2$

Let $n \geq \Delta$ be non-negative integers, and consider the following set of matrix differential operators

$$
T^+ = \begin{pmatrix} J^+_{n-\Delta} & 0 \\ 0 & J^+_n \end{pmatrix}, \quad T^0 = \begin{pmatrix} J^0_{n-\Delta} & 0 \\ 0 & J^0_n \end{pmatrix}, \quad T^- = \begin{pmatrix} J^- & 0 \\ 0 & J^- \end{pmatrix},
$$

$$
J = \frac{1}{2} \begin{pmatrix} n + \Delta & 0 \\ 0 & n \end{pmatrix},
$$

$$
Q_{\alpha} = z^\alpha \sigma^-, \quad \overline{Q}_{\alpha} = \overline{q}_{\alpha}(n, \Delta) \sigma^+, \quad \alpha = 0,\ldots,\Delta,
$$

with

$$
\overline{q}_{\alpha}(n, \Delta) = \prod_{k=1}^{\Delta-\alpha} (z \partial_z - n + \Delta - k) \partial_z^\alpha,
$$

where we have adopted the convention that a product with its lower limit greater than the upper one is automatically 1, and $\sigma^+ = (\sigma^-)^t = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

It can be easily checked that the $6 + 2\Delta$ operators in (3.1) (and also any polynomial thereof) preserve $\mathcal{P}_{n-\Delta,n}$. We now introduce a $\mathbb{Z}_2$-grading in the set of $2 \times 2$ matrix differential operators $\mathcal{D}_{2\times2}$ as follows: an operator

$$
T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathcal{D},
$$

is said to be even if $b = c = 0$, and odd if $a = d = 0$. Therefore, the $T$’s and $J$ are even and the $Q$’s and $\overline{Q}$’s odd. This grading, combined with the usual product (composition) of operators, endows $\mathcal{D}_{2\times2}$ with an associative
superalgebra structure. We can also construct a Lie superalgebra in \( \mathcal{D}_{2\times 2} \) by defining a generalized Lie product by
\[
[A, B]_s = AB - (-1)^{\text{deg} A \text{deg} B} BA. \tag{3.3}
\]

However, this product does not close within the vector space spanned by our operators (3.1), except for \( \Delta = 0, 1 \). The explicit commutation relations are as follows, \([1],[2]\):
\[
[T^+, T^-] = -2T^0, \quad [T^\pm, T^0] = \mp T^\pm,
\]
\[
[J, T^\epsilon] = 0, \quad [J, Q_\alpha] = -\frac{\Delta}{2} Q_\alpha, \quad [J, \overline{Q}_\alpha] = \frac{\Delta}{2} \overline{Q}_\alpha,
\]
\[
[Q_\alpha, T^\epsilon] = \left( -\alpha + \frac{\Delta}{2} (1 + \epsilon) \right) Q_{\alpha+\epsilon}, \quad [Q_\alpha, T^0] = \left( \alpha - \frac{\Delta}{2} (1 - \epsilon) \right) \overline{Q}_{\alpha-\epsilon},
\]
\[
\{Q_\alpha, Q_\beta\} = \begin{cases} M_{\alpha\beta} (T^-)^{\alpha-\beta}, & \alpha \geq \beta \\ (T^+)^{\beta-\alpha} M_{\beta\alpha}, & \beta \geq \alpha, \end{cases}
\]
\[
\{Q_\alpha, Q_\beta\} = \{\overline{Q}_\alpha, Q_\beta\} = 0, \tag{3.4}
\]

where \( \epsilon = +, 0, - \), and \( M_{\alpha\beta} \) is given, for \( \alpha \geq \beta \), by
\[
M_{\alpha\beta} = \prod_{j=0}^{\Delta-\alpha-1} (T^0 + J_c - j - \beta P_2) \prod_{k=0}^{\beta-1} (T^0 + J - k - (\Delta - \alpha) P_1)
\]
with \( J_c = \Delta - 1 - J \), and \( P_1 = 1 - P_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \).

As shown in \([1]\), \( M_{\alpha\beta} \) can be expressed in terms of \( T^0, J \), the identity, and the Casimir (for the even subalgebra)
\[
C = -\frac{1}{2} (T^+ T^- + T^- T^+) + T^0 T^0 = \frac{1}{4} \begin{pmatrix} m(m+2) & 0 \\ 0 & n(n+2) \end{pmatrix}, \tag{3.5}
\]
where \( m = n - \Delta \), independently from \( n \) and the projectors \( P_1 \) and \( P_2 \). It can be readily verified that \( \{Q_\alpha, Q_\beta\} \) gives a \( \Delta \)-th order even differential operator, so the vector space spanned by the operators in (3.1) is not closed.
under the Lie product (3.3) whenever $\Delta \geq 2$. Moreover, it is not difficult to show that the Lie superalgebra $s_\Delta$ generated by the operators (3.1) is in this case infinite dimensional. Indeed, if we commute $\{Q_\Delta, Q_0\} = (T^-)^\Delta$ with $\{Q_0, Q_\Delta\} = (T^+)^\Delta$ iteratively we obtain monomials in $T^+, T^0, T^-$ of increasingly higher order. For $\Delta = 1$ the underlying algebraic structure is the classical simple Lie superalgebra $osp(2, 2)$, [10], [20], whereas for $\Delta = 0$ it is $h_1 \oplus sl(2)$, where $h_1$ is the 3-dimensional Heisenberg superalgebra. As remarked in [1], in this latter case we can leave the grading aside and replace $J = 1$ by $\tilde{J} = \sigma_3$, ending up with the Lie algebra $sl(2) \oplus sl(2)$.

Our next objective is to prove the analogous of Theorem 2.4 for PVSP operators preserving $P_n - \Delta$ (first mentioned without proof by Turbiner for $\Delta = 1$ in [20], and subsequently by Brihaye and Kosinski for arbitrary $\Delta$, [1]). It turns out that the operators (3.1) play the same role for the matrix case as the $J$'s in (2.5) do for the scalar one. The proof relies on the next two Lemmas.

**Lemma 3.2** Let $T^{(k)} : P_n \to P_{n-\Delta}$ be a $k$-th order linear differential operator, with $n \geq \Delta$. We then have:

i) If $n \geq k \geq \Delta$, then $T^{(k)}$ may be represented as

$$T^{(k)} = \sum_{\alpha=0}^{\Delta} \varphi_{\alpha}(n, \Delta) p_{\alpha - \Delta}(J_0^0),$$

where $p_{\alpha - \Delta}(J_0^0)$ are polynomials of degree not higher than $k - \Delta$ in the operators $J_0^+, J_n^0$ and $J^-.$

ii) If $\Delta > k$, then $T^{(k)} = 0$.

iii) If $k > n$, then $T^{(k)} = T \partial_z^{n+1} + \tilde{T}$, where $T$ is a linear differential operator of order $k - n - 1$, and $\tilde{T}$ is a linear differential operator of order at most $n$ satisfying i) or ii).

**Proof.** i). The essential point of the argument is that, as $P_{n-\Delta} \subset P_n$, $T^{(k)}$ preserves also $P_n$, and so it must be a $k$-th degree polynomial in $J_0^0$ with suitable additional restrictions on its coefficients, according to Theorem 2.4.

We will prove that

$$T^{(k)} = \varphi_0(n, \Delta) p_{0 - \Delta}(J_0^+, J_n^0) + \sum_{\alpha=1}^{\Delta} \varphi_{\alpha}(n, \Delta) p_{\alpha - \Delta}(J_n^0)$$

$$+ \varphi_{\Delta}(n, \Delta) J^- p_{\Delta - 1}(J_n^0, J^-),$$

(3.7)
which clearly implies (3.6).

We will proceed by induction on $\Delta$. First assume $\Delta = 1$. Without any loss of generality (because of the scalar Casimir analogous to (3.5)) we may write

$$T_1^{(k)} = \sum_{r+s \leq k} c_{rs} (J_n^0)^r (J_n^+)^s + J^- p_{k-1}^1 (J_n^0) + (J^-)^2 p_{k-2} (J_n^0, J^-),$$

for some constants $c_{rs}$. As $J_n^+$ annihilates $z^n$, acting with $T_1^{(k)}$ on $z^{n-s}$, $s = 0, \ldots, k-1$, we find

$$T_1^{(k)} z^{n-s} = (-1)^s s! \sum_{r=0}^{k-s} \binom{n}{2}^r c_{rs} z^n + \text{lower order terms},$$

leading to $\sum_{r=0}^{k-s} \binom{n}{2}^r c_{rs} = 0$ for each $s$. It can be easily shown that the most general element in $\text{Span}\{(J_n^0)^r (J_n^+)^s\}_{r+s \leq k}$ satisfying this condition is

$$(J_n^0 - \frac{n}{2}) \sum_{r=0}^{k-s} \tilde{c}_{rs} (J_n^0)^r (J_n^+)^s.$$

Finally, no monomial $(J_n^+)^k$ may be present in $T_1^{(k)}$. Thus,

$$T_1^{(k)} = \sum_{\alpha=0}^{k} q_\alpha(n, 1) p_{k-1}^\alpha (J_n^+, J_n^0) + q_1(n, 1) \left(p_{k-1}^1 (J_n^0) + J^- p_{k-2} (J_n^0, J^-)\right),$$

completing the proof of (3.7) for $\Delta = 1$. Now assume that (3.7) holds for every $T_\Delta^{(k)} : \mathcal{P}_n \to \mathcal{P}_{n-\Delta}$. In particular, it is also true for an operator $T_\Delta^{(k)}$ mapping $\mathcal{P}_n$ into $\mathcal{P}_{n-\Delta-1}$. As $q_\alpha(n, \Delta)$ ($\alpha = 0, \ldots, \Delta - 1$) annihilates $z^{n-s}$ if $0 \leq s < \Delta - \alpha$, acting with $T_\Delta^{(k)}$ on $z^{n-s}$ ($s = 0, \ldots, k$) and reasoning as in the case $\Delta = 1$, we deduce that a factor $(J_n^0 - \frac{n-\Delta+\alpha}{2})$ must be present in front of each $p_{k-\Delta}^\alpha$ for $\alpha = 0, \ldots, \Delta$. If we move these factors before their respective $q_\alpha(n, \Delta)$ and group together the monomials in $p_{k-\Delta-1} (J_n^0, J^-)$ with no $J^-$, we find that $T_\Delta^{(k)}$ may be written in the form (3.7) with $\Delta$ replaced by $\Delta + 1$.

**ii).** If we apply i) for $\Delta = k$, we find $T^{(k)} = \sum_{\alpha=0}^{k} c_\alpha q_\alpha(n, k)$, and acting on $z^{n-s}$, $s = 0, \ldots, k$, we conclude that $c_\alpha = 0$ for all $\alpha$.

**iii).** This is obvious, since derivatives of order higher than $n$ annihilate $\mathcal{P}_n$. Q.E.D.

The corresponding case $T^{(k)} : \mathcal{P}_n \to \mathcal{P}_{n+\Delta}$ is treated next:
Lemma 3.3 Let $T^{(k)} : \mathcal{P}_n \to \mathcal{P}_{n+\Delta}$ be a $k$-th order linear differential operator. We then have:

i) If $n \geq k$, then $T^{(k)}$ may be represented as

$$T^{(k)} = \sum_{\alpha=0}^{\Delta} z^\alpha p_k^\alpha (J_n^+),$$

where each $p_k^\alpha$ is a polynomial of degree not higher than $k$ in the operators $J_n^+, J_n^0$ and $J^-$. 

ii) If $k > n$, then $T^{(k)} = T \partial_z^{n+1} + \bar{T}$, where $T$ is a linear differential operator of order $k - n - 1$, and $\bar{T}$ is a linear differential operator of order at most $n$ satisfying i).

The proof of this Lemma is similar to that of Theorem 2.4, and shall not be presented here. The Theorem analogous to 2.4 for a PVSP operator in $\mathcal{P}^{(k)}_{m,n}$ now follows applying Theorem 2.4 and the Lemmas above to each entry of the operator:

Theorem 3.4 Let $n \geq m$, and $\Delta = n - m$. Let $T^{(k)}$ be $k$-th order differential operator in $\mathcal{P}^{(k)}_{m,n}$. We then have:

i) If $m \geq k$, then $T^{(k)}$ is a polynomial in the operators (3.1). More explicitly, if $k \geq \Delta \geq 1$,

$$T^{(k)} = p_k(T^\epsilon) + \bar{J} \bar{p}_k(T^\epsilon) + \sum_{\alpha=0}^{\Delta} \bar{Q}_\alpha p_k^\alpha (T^\epsilon) + \sum_{\alpha=0}^{\Delta} Q_\alpha p_k^\alpha (T^\epsilon),$$

while if $\Delta = 0$ the same formula is valid with $J$ replaced by $\bar{J}$. If $k < \Delta$, every $\bar{p}_{k-\Delta}$ must be identically zero.

ii) If $n \geq k > m$, then $T^{(k)} = T \partial_z^{m+1} + \bar{T}$, where $T$ and $\bar{T}$ are matrix linear differential operators of the form

$$T = \begin{pmatrix} a^{(k-m-1)} & 0 \\ c^{(k-m-1)} & 0 \end{pmatrix}, \quad \bar{T} = \begin{pmatrix} \bar{a}^{(m)} & \bar{b}^{(k)} \\ \bar{c}^{(m)} & \bar{d}^{(k)} \end{pmatrix},$$

where the superscripts indicate the highest possible derivative in each entry, and $\bar{T}$ satisfies i).
iii) If \( k > n \), then \( T^{(k)} = T \partial_z^{k+1} + \tilde{T} \), where \( T \) is a \( 2 \times 2 \) matrix linear differential operator of order \( k - n - 1 \), and \( \tilde{T} : \mathcal{P}_{m,n} \to \mathcal{P}_{m,n} \) is a linear PVSP operator of order at most \( n \) verifying i) or ii).

The next issue to be addressed is to find out the number of parameters determining a generic \( k \)-th order linear differential operator preserving \( \mathcal{P}_{m,n} \), that is, the dimension of \( \mathcal{P}^{(k)}_{m,n} \). In the scalar case, any \( k \)-th degree polynomial in \( J^+ + J^- = 2J_0, J_0 = n \), and so \( \dim \mathcal{P}^{(k)}_{m,n} = (k + 1)^2 \) if \( n \leq k \), \([19]\).

Remarkably, in the matrix case we have \( \dim \mathcal{P}^{(k)}_{m,n} = 4(k + 1)^2 \) independently of \( m \) and \( n \), provided \( m \geq k \geq n - m \), \([1]\), as a consequence of the following Lemma:

**Lemma 3.5** The following set of monomials form a basis of the vector space of polynomials in the operators (3.1) of differential order at most \( k \):

\[
\{X(T^\pm)^r(J_0^0)^{s-r}\}_{r=0}^{s} \cup \{Q_\alpha(T^+)^s\}_{\alpha=1}^{\Delta}, \quad s = 0, \ldots, k,
\]

where \( X = 1, J \) (or \( \tilde{J} \), if \( \Delta = 0 \)), \( Q_0 \), along with, if \( \Delta \geq 1 \):

\[
\{\tilde{Q}_\alpha(T^\pm)^r(J_0^0)^{s-r}\}_{r=0}^{s} \cup \{\tilde{Q}_\alpha(T^-)^s\}_{\alpha=1}^{\Delta}, \quad s = 0, \ldots, k - \Delta.
\]

**Proof.** Linear independence of the monomials is straightforward from the definition of the operators. Completeness is a consequence of the following facts. In the first place, every \( J^s \) is a linear combination of \( \{1, J\} \) (and analogously for \( \tilde{J} \)). Secondly, \( JQ_\alpha \) is proportional to \( Q_\alpha \), and \( Q_\alpha Q_\beta = 0 \) (and the same for the \( \tilde{Q}_\alpha \)'s). Third, any product \( Q_\alpha \tilde{Q}_\beta \) is a diagonal PVSP operator, and thus expressible through the \( T^\alpha \)'s and \( J \) (or \( \tilde{J} \)). Finally, the formulas \( (\alpha \geq 1) \)

\[
Q_\alpha T^0 = Q_\alpha - 1 T^+ + \frac{n - \Delta}{2} Q_\alpha, \quad Q_\alpha T^- = Q_\alpha - 1 T^0 + \frac{n - \Delta}{2} Q_\alpha - 1,
\]

\[
\tilde{Q}_\alpha T^0 = \tilde{Q}_\alpha - 1 T^- + \left(\frac{n}{2} + 1\right) \tilde{Q}_\alpha, \quad \tilde{Q}_\alpha T^+ = \tilde{Q}_\alpha - 1 T^0 + \left(\frac{n}{2} + 1\right) \tilde{Q}_\alpha - 1,
\]

allow us to remove every \( T^0 \) and \( T^- \) (respectively \( T^+ \)) from the monomials with \( Q_\alpha \) (respectively \( \tilde{Q}_\alpha \)), \( \alpha = 1, \ldots, \Delta \). Q.E.D.
Corollary 3.6 Let \( n \geq m \geq k \), and \( \Delta = n - m \). We then have:

\[
\dim \mathcal{P}^{(k)}_{m,n} = \begin{cases} 
4(k + 1)^2, & k \geq \Delta \\
(k + 1)(3k + \Delta + 3), & \Delta > k.
\end{cases}
\]

If \( m < k \), \( \dim \mathcal{P}^{(k)}_{m,n} \) is no longer finite, as arbitrary differential operators are involved in this case.

3.2 Case \( N > 2 \)

We now examine briefly some aspects of the case \( N > 2 \). Let \( n_1 \leq \cdots \leq n_N \) be non-negative integers, and let \( \Delta_{ij} = n_j - n_i \), where \( j > i \). Consider the following set of \( N \times N \) matrix differential operators, \([2]\):

\[
T^\epsilon = \text{diag}(J^\epsilon_{n_1}, \ldots, J^\epsilon_{n_N}), \quad \epsilon = +, 0, -,
\]

\[
P_i = \text{diag}(0, \ldots, 0, 1, 0, \ldots, 0),
\]

\[
Q_\alpha(i, j) = z^\alpha \lambda_{ij}, \quad i > j, \quad \alpha = 0, \ldots, \Delta_{ji},
\]

\[
\overline{Q}_\alpha(i, j) = \overline{Q}_\alpha(n_j, \Delta_{ij}) \lambda_{ij}, \quad j > i, \quad \alpha = 0, \ldots, \Delta_{ij}, (3.9)
\]

where \((\lambda_{ij})_{pq} = \delta_{ip}\delta_{jq}\). It can be readily verified that the operators in (3.9) preserve \( \mathcal{P}_{n_1,\ldots,n_N} \). A complication arising when \( N > 2 \) is to define a suitable composition law between the latter operators. In the approach of Brihaye et al., \([4]\), this composition law is defined to be an anticommutator if both operators are off-diagonal and a commutator otherwise, but the algebra thus obtained is no longer a Lie superalgebra, since the anticommutator of two off-diagonal operators is not always a diagonal one. This reflects the fact that the \( \mathbb{Z}_2 \)-grading we introduced for \( N = 2 \) (i.e. classifying the operators in diagonal and off-diagonal) does not define an associative superalgebra in \( \mathcal{D}_{N\times N} \) when \( N > 2 \), for the usual product (composition) of two off-diagonal matrix differential operators is not necessarily diagonal. A possible generalization of this \( \mathbb{Z}_2 \)-grading, endowing \( \mathcal{D}_{N\times N} \) with an associative superalgebra structure can be defined as follows. An operator \( T = a \lambda_{ij} \), where \( a \in \mathcal{D} \), is said to be even (respectively odd) if \( i + j \) is even (respectively odd). Hence, any diagonal operator is even. We can likewise use this grading and the generalized Lie product (3.3) to construct a Lie superalgebra structure in \( \mathcal{D}_{N\times N} \).
It is not clear, however, whether this construction is really useful, and so it will not be further discussed.

As remarked by Brihaye et al., [2], Theorem 3.4 can be easily generalized to arbitrary $N$, the operators (3.9) playing the same role as those in (3.1) for $N = 2$. Moreover, it is not difficult to show that $\dim \mathcal{P}_{n_1,\ldots,n_N}^{(k)}$ is still independent of the $n_i$’s if they are large enough and their differences are small enough. More precisely, if $n_1 \geq k$, we have:

$$\dim \mathcal{P}_{n_1,\ldots,n_N}^{(k)} = \begin{cases} N^2(k+1)^2, & k > \Delta_{1N} \\ \frac{N(N+1)}{2}(k+1)^2 + (k+1) \sum_{i<j} \theta_{ij}, & \Delta_{1N} > k, \end{cases}$$

where $\theta_{ij} = \Delta_{ij}$ if $\Delta_{ij} > k$, and $\theta_{ij} = k+1$ if $\Delta_{ij} \leq k$. If $k > n_1$, arbitrary differential operators are involved and thus $\mathcal{P}_{n_1,\ldots,n_N}^{(k)}$ is infinite-dimensional.

Although no attempt will be made here to give a formal definition of a QES algebra of matrix differential operators, it is clear that Definition 2.1 of a QES differential operator is too restrictive in the matrix case. Indeed, the results of this section suggest that in the matrix case one should include at least Lie superalgebras of differential operators—not necessarily finite-dimensional nor spanned by first order operators—preserving a finite-dimensional module of functions among the class of matrix QES algebras. In any case, it is intuitively clear that PVSP operators are just a particular class of QES operators.

### 4 Spin 1/2 Schrödinger Operators

From now on we will deal only with $2 \times 2$ matrix second order differential operators ($N = k = 2$ in the notation of the previous sections). We start by formally defining the class of matrix Schrödinger operators:

**Definition 4.1** A Schrödinger-like operator is a second order differential operator of the form $H = -\partial_x^2 + V(x)$, where $V$ is an arbitrary $2 \times 2$ (complex) matrix. A Schrödinger operator (or Hamiltonian) is a Hermitian Schrödinger-like operator, i.e. the matrix $V$ is of the form

$$V = \begin{pmatrix} v_1(x) & v^*(x) \\ v(x) & v_2(x) \end{pmatrix},$$

(4.1)
where \( v_1 \) and \( v_2 \) are real-valued functions and \( v \) is an arbitrary complex-valued function.

The notion of equivalence we shall use for matrix differential operators is the same as in the scalar case (see Definition 2.2), where now the gauge factor \( U(z) \) is an invertible complex 2 \( \times \) 2 matrix. We will be interested in constructing one-dimensional Schrödinger operators \( H \) equivalent to a second order differential operator \( T \) in \( P^{(2)}_{m,n} \), with \( \Delta = n - m \geq 0 \). This equivalence can then be used to construct \( m + n + 2 \) eigenfunctions of \( H \) from the corresponding ones of \( T \) obtained by diagonalization of the \( (m+n+2) \times (m+n+2) \) Hermitian matrix representing \( T \) in \( P_{m,n} \). We will assume that \( m \geq 2 \), and thus \( T \) is a polynomial in the operators (3.1), according to Theorem 3.4.

Let \( T : P_{m,n} \to P_{m,n} \) be a second-order PVSP operator. From Theorem 3.4 we have

\[
-T = A_2(z) \partial_z^2 + A_1(z) \partial_z + A_0(z),
\]

(4.2)

where the \( A_i \)'s are 2 \( \times \) 2 matrices with polynomial entries (in this section, capital letters will be reserved for matrices). Assume that \( T(z) \) is equivalent to a Schrödinger operator \( H(x) \) under a gauge transformation \( U(z) \) and a local change of variable by a real-valued function \( x = \varphi(z) \), i.e.

\[
-T = -U^{-1}(z)H(x)U(z)
\]

\[
= \partial_x^2 + 2A \partial_x - B,
\]

(4.3)

with

\[
A(x) = \bar{U}^{-1} \bar{U}_x, \quad B(x) = \bar{U}^{-1} V \bar{U} - A^2 - A_x, \quad \bar{U}(x) = U(\varphi^{-1}(x)).
\]

Here and in what follows, a subscripted \( x \) denotes derivation with respect to \( x \), while derivatives with respect to \( z \) will be denoted with a prime \( ' \). Expressing \( T(z) \) in the variable \( x \), we obtain the operator \( \tilde{T}(x) \) given by

\[
-\tilde{T}(x) = [A_2 \varphi'^2 \partial_x^2 + (A_1 \varphi' + A_2 \varphi'') \partial_x + A_0]_{\varphi^{-1}(x)},
\]

(4.4)

and comparing with (4.3) we conclude that \( A_2 \) must be a multiple of the identity. It then follows that only the first term in equation (3.8) of Theorem 3.4 contributes to \( A_2 \), and taking into account the explicit form of the \( T^* \)'s (see (3.1)), we conclude that \( A_2 \) is a 4-th degree polynomial \( p_4 \) times.
the identity matrix. (Unless otherwise stated, we will denote by $p_n(z)$ an arbitrary polynomial in $z$ of degree at most $n$ with complex coefficients.) We also deduce that $\varphi(z)$ satisfies the equation $p_4 \varphi^2 = 1$, or

$$x = \int^z \frac{1}{\sqrt{p_4(s)}} ds,$$

and thus the coefficients of $p_4$ must be real. Identifying the corresponding remaining terms in (4.3) and (4.4), we then get

$$A(x) = \frac{1}{2\sqrt{p_4}} \left( A_1 - \frac{1}{2} p'_4 \right) \varphi^{-1}(x), \quad B(x) = -A_0 \varphi^{-1}(x).$$

Thus, we have shown:

**Theorem 4.2** Let $T$ be a PVSP operator in $P_{m,n}^{(2)}$, with $n \geq m \geq 2$. Then $T$ is equivalent to a Schrödinger-like operator if and only if it is of the form

$$-T = p_4 \partial_x^2 + A_1 \partial_x + A_0 .$$

The operator $T$ is equivalent to a Schrödinger operator $-\partial^2_x + V(x)$ if and only if (4.7) holds, and in addition there is an invertible matrix $\tilde{U}$ satisfying the differential equation

$$\tilde{U}_x = \tilde{U} A$$

and such that

$$V = \tilde{U} \tilde{W} \tilde{U}^{-1}, \quad \text{where} \quad \tilde{W} = B + A^2 + A_x ,$$

is Hermitian (with $x, A$, and $B$ given by (4.5) and (4.6)).

The eigenfunctions of the Hamiltonian $H$ are of the form $\psi(x) = \tilde{U} \tilde{\Psi}$ with $\tilde{\Psi}(x) = \Psi(\varphi^{-1}(x))$, where $\Psi(z)$ is an eigenfunction of $T$ and $\psi$ must satisfy suitable boundary conditions to qualify as a physical wave function.

We note that once an invertible solution $\tilde{U}$ of equation (4.8) has been found, any other invertible solution is of the form $U_0 \tilde{U}$ for some $U_0$ in GL$(2, \mathbb{C})$. In fact, multiplying $\tilde{U}$ by such $U_0$ is equivalent to performing a further constant scale transformation by $U_0$. In the scalar case this additional freedom is absent, for differential operators are unaffected by scale
transformations by constant functions. Note also that a scale transformation by an arbitrary constant matrix $U_0$ will not map every Hamiltonian into another Hamiltonian, unless $U_0$ belongs to $\mathbb{R}^+ \times U(2)$.

A matrix differential operator will be called *uncoupled* if it is either upper or lower triangular. Since $A_2$ in (4.2) must be a multiple of the identity matrix, it follows that a PVSP operator of the form (4.7) will be automatically uncoupled whenever $\Delta > 1$, as no $Q$ can then be present in (3.8). Moreover, the following result shows that any Hamiltonian we may obtain from $T$ when $\Delta > 1$ will be essentially diagonal:

**Proposition 4.3** Every Hamiltonian $H$ obtained from an uncoupled PVSP operator $T$ of the form (4.7) is diagonal, up to equivalence.

*Proof.* If $T$ is uncoupled, the integration of equation (4.8) is straightforward. Multiplying $\tilde{U}$ from the left by an appropriate $U_0$ in $\mathbb{R}^+ \times U(2)$, we construct a new gauge factor uncoupled in the same way as $T$. Using this new gauge factor, we obtain a Hamiltonian $\hat{H}$ given by

$$\hat{H} = U_0HU_0^{-1} = U_0\tilde{U}\hat{T}(U_0\tilde{U})^{-1},$$

which is both diagonal and equivalent to the initial one. Q.E.D.

Consequently, we shall limit ourselves in what follows to the cases $\Delta = 0, 1$.

There are two main difficulties associated with the method just outlined for constructing QES spin 1/2 Hamiltonians. In the first place, one needs to invert the elliptic integral (4.5) in order to compute $z$ as a function of $x$, which is no easy task. Secondly, the differential equation (4.8) cannot in general be solved in closed form, thus preventing us from verifying the Hermiticity of $V$. The former complication can be overcome, as we shall see in the next section. The latter is more difficult to handle, although imposing further constraints on the initial PVSP operator will contribute to simplify the problem, as shown in Section 6.

We shall finish this section with a few remarks on the physical significance of matrix Schrödinger operators. First of all, one-dimensional $2 \times 2$ matrix Schrödinger operators can be obtained by separation of variables from the three-dimensional Pauli Hamiltonian describing a spin 1/2 charged particle in non-relativistic quantum mechanics. Consider, indeed, the Pauli Hamiltonian

$$H_{\text{Pauli}} = (i\nabla + eA)^2 + e\phi - e\sigma \cdot B,$$
where $\phi$ and $A = (A^1, A^2, A^3)$ are respectively the scalar and vector potential of the external electromagnetic field, $B = \nabla \times A$ is the magnetic field, the Pauli matrices $\sigma = (\sigma^1, \sigma^2, \sigma^3)$ are given by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$e$ is the electric charge, and physical units have been chosen so that $\hbar = c = 2m = 1$. If, for example, the vector and scalar potentials depend only on the $x$ coordinate (and we take, without loss of generality, $A^1 = 0$) then $H_{\text{Pauli}}$ obviously commutes with the $y$ and $z$ components of the linear momentum. The eigenfunctions of $H_{\text{Pauli}}$ can then be sought in the form

$$e^{i(p_y y + p_z z)} \psi(x),$$

where $p_y, p_z \in \mathbb{R}$ are the values of the $y$ and $z$ components of the linear momentum, and the two-component spinor $\psi(x)$ is an eigenfunction of the one-dimensional matrix Schrödinger operator with potential (4.1) given by

$$v_j(x) = e \phi + (e A^2 - p_y)^2 + (e A^3 - p_z)^2 + (-1)^j e \frac{dA^2}{dx}, \quad j = 1, 2,$$

$$v(x) = i e \frac{dA^3}{dx}.$$

More surprisingly, one-dimensional $2 \times 2$ matrix operators are also directly related to Dirac’s relativistic equation for a spin 1/2 charged particle in an external electromagnetic field. To see this, let us write the latter equation as

$$(i\partial - eA - m)\Psi(x) = 0,$$ (4.10)

where $\phi = \gamma^\mu a_\mu$, the $\gamma^\mu$’s are $4 \times 4$ matrices satisfying

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu},$$

the metric tensor $(g_{\mu\nu}) = \text{diag}(1, -1, -1, -1)$ is used to raise and lower indices, $\partial_\mu = \partial/\partial x^\mu$, $x^0 = t$, $A^0 = \phi$ and $m$ is the particle’s mass. Multiplying Dirac’s equation by the operator $i\partial - eA + m$ we easily arrive at the second-order equation

$$\left[ (i\partial - eA)^2 - m^2 + \frac{e}{2} F_{\mu\nu} \sigma^{\mu\nu} \right] \Psi = 0,$$ (4.11)

17
where $\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]$ and \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the electromagnetic field strength tensor. Conversely, if $\Phi$ is a solution of (4.11) then either $(i\partial - eA + m)\Phi$ or $\gamma_5 \Phi \equiv i\gamma^0 \gamma^1 \gamma^2 \gamma^3 \Phi$ (but not both simultaneously!) is a non-trivial solution of Dirac’s equation (4.10). In the chiral representation of the gamma matrices, we have

\[
\sigma^{0j} = i \begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix}, \quad \sigma^{jk} = \epsilon_{jkl} \begin{pmatrix} 0 & \sigma^l \\ \sigma^l & 0 \end{pmatrix}
\]

$(j, k, l = 1, 2, 3,$ and summation over $l$ is understood), and therefore (4.11) decouples into two independent equations for the upper and lower components $\Psi_\pm(x)$ of $\Psi(x)$, namely (cf. [3])

\[
\left[ (i\partial - eA)^2 - m^2 + e \sigma \cdot (B \mp iE) \right] \Psi_\pm = 0.
\]

If the electromagnetic four-potential $A^\mu$ is time-independent, and we look for solutions of Dirac’s equation with well-defined energy $E$, i.e. we set $\Psi_\pm = e^{-iEt}\psi_\pm(x, y, z)$, we obtain the following equation for $\psi_\pm$:

\[
\left[ (i\nabla + eA)^2 - (E - eA^0)^2 + m^2 + e \sigma \cdot (B \mp iE) \right] \psi_\pm = 0 , \quad (4.12)
\]

which has the same structure as Pauli’s non-relativistic equation. Just as was the case with Pauli’s equation, separation of variables in (4.12) often leads to the eigenvalue problem for a one-dimensional matrix Schrödinger operator. For instance, suppose that $A^0 = 0$ and that $A$ has cylindrical symmetry, that is

\[
A = A(\rho) \mathbf{e}_z
\]

in cylindrical coordinates $(\rho, \varphi, z)$. The left-hand side of (4.12) then commutes with the $z$ components of the linear momentum $(-i\partial_z)$ and the total angular momentum $(-i\partial_\varphi + \frac{1}{2}\sigma^3)$, which allows us to look for solutions of (4.12) of the form

\[
\psi(\rho, \varphi, z) = e^{ip_z z} \begin{pmatrix} R_1(\rho)e^{ij_z^{(j_z-1/2)}\varphi} \\ R_2(\rho)e^{ij_z^{(j_z+1/2)}\varphi} \end{pmatrix} . \quad (4.13)
\]

Here we have dropped the subscript $\pm$, since $\psi_+$ and $\psi_-$ satisfy the same equation, $p_z \in \mathbb{R}$ is the value of the $z$ component of the linear momentum,
and \( j_z \in \mathbb{N} + \frac{1}{2} \) is the value of the \( z \) component of the total angular momentum. Substituting (4.13) into (4.12) we arrive at the following equation for \((R_1 R_2)^t\):

\[
\begin{pmatrix}
-\partial_{\rho}^2 - \frac{1}{\rho} \partial_{\rho} + (p_z - eA)^2 + m^2 - E^2 \\
+ \frac{1}{\rho^2} \begin{pmatrix}
(j_z - 1/2)^2 & 0 \\
0 & (j_z + 1/2)^2 
\end{pmatrix} + e \frac{dA}{d\rho} \sigma^2
\end{pmatrix} \begin{pmatrix}
R_1 \\
R_2
\end{pmatrix} = 0.
\]

Defining
\[ f_i(x) = x^{1/2} R_i(x), \quad i = 1, 2, \]
we finally obtain that the two-component spinor \( \left( f_1(x) \ f_2(x) \right)^t \) is an eigenfunction of the one-dimensional \( 2 \times 2 \) matrix Schrödinger operator with potential
\[ V(x) = \left( p_z - eA(x) \right)^2 + e \frac{dA}{dx}(x) \sigma^2 + \frac{j_z x^2}{2}(j_z - \sigma^3), \quad x \in (0, \infty), \]
with eigenvalue \( E^2 - m^2 \) and boundary condition \( f_1(0) = f_2(0) = 0. \)

5 GL(2) Action and Canonical Forms

In this section we will study how the GL(2) action on the projective line \( \mathbb{R}P^1 \) induces an automorphism in the superalgebras \( \mathfrak{s}_\Delta \) generated by the operators (3.1). This will allow us to reduce the polynomial \( p_4(z) \) to some simple canonical forms, facilitating the evaluation of the integral (4.5). These ideas were first applied in the context of QES systems to analyze the normalizability of the wave functions of scalar QES Hamiltonians, [8]. We introduce some definitions and results in the scalar case, and then show how to extend these concepts to the matrix superalgebras \( \mathfrak{s}_\Delta \).

The action of \( \text{GL}(2) = \text{GL}(2, \mathbb{R}) \) on \( \mathbb{R}P^1 \) via linear fractional or Möbius transformations,
\[ z \mapsto w = \frac{\alpha z + \beta}{\gamma z + \delta}, \quad C = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad |C| = \alpha \delta - \beta \gamma \neq 0, \]
induces an action on \( \mathcal{P}_n \), mapping a polynomial \( p(w) \) to the polynomial \( \bar{p}(z) \) given by
\[ \bar{p}(z) = (\gamma z + \delta)^n p \left( \frac{\alpha z + \beta}{\gamma z + \delta} \right). \]
This defines an irreducible multiplier representation of GL(2) in $P_n$, which will be denoted by $\rho_{n,0}$. Since the infinitesimal generators of this multiplier representation coincide with the generators of $g_n$, cf. (2.4), it follows that the representation $\rho_{n,0}$ induces an automorphism of the Lie algebra $\hat{g}_n$ spanned by the $J^\epsilon_n$’s in (2.5). Performing the explicit scale transformation and change of variable,

$$J^\epsilon_n(w) \mapsto (\gamma z + \delta)^n J^\epsilon_n\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right) (\gamma z + \delta)^{-n}, \quad (5.3)$$

we obtain:

$$\begin{pmatrix} J^+_n \\ J^0_n \\ J^-_n \end{pmatrix} \mapsto \frac{1}{|C|} \begin{pmatrix} \alpha^2 & 2\alpha\beta & \beta^2 \\ \alpha\gamma & \alpha\delta + \beta\gamma & \beta\delta \\ \gamma^2 & 2\gamma\delta & \delta^2 \end{pmatrix} \begin{pmatrix} J^+_n \\ J^0_n \\ J^-_n \end{pmatrix}.$$

Therefore, the $J^\epsilon_n$’s transform according to the representation $\rho_{2,-1} = \rho_{2,0} \otimes \text{det}^{-1}$ of GL(2), where $\text{det}^{-1}$ is the reciprocal of the representation $\text{det} : C \mapsto |C|$. It is convenient at this stage to introduce a larger class of representations of GL(2):

**Definition 5.1** Let $n \geq 0$, $i$ be integers. The (irreducible) multiplier representation $\rho_{n,i}$ of GL(2) on $P_n$ is defined by

$$p(w) \mapsto \tilde{p}(z) = (\alpha\delta - \beta\gamma)^i (\gamma z + \delta)^n p\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right).$$

We note the isomorphism between $\rho_{n,i}$ and $\rho_{n,0} \otimes \text{det}^i$. As shown in [3], a second-degree polynomial in the $J^\epsilon_n$’s (in fact, any operator in $P(2)$ if $n \geq 2$, or any QES operator on the line modulo equivalence, according to Theorems 2.3 and 2.4) may be written as

$$p_2(J^\epsilon_n) = p \partial_z^2 + \left(q - \frac{n - 1}{2} p' \right) \partial_z + r - \frac{n}{2} q' + \frac{n(n - 1)}{12} p'' , \quad (5.4)$$

where $p$, $q$, and $r$ are polynomials in $z$ of degrees 4, 2, and 0 respectively. The transformation of $p_2(J^\epsilon_n)$ under the action (5.3) is easily described in terms of the triple $(p(w), q(w), r)$:

**Lemma 5.2** Let $p_2$ be a second-degree polynomial in the operators $J^\epsilon_n(w)$, determined by the triple $(p(w), q(w), r)$. Then, the transformed polynomial
$\bar{p}_2$ under the GL(2) action (5.3) is determined by the triple $(\bar{p}(z), \bar{q}(z), \bar{r})$ given by

$$
\bar{p}(z) = \frac{(\gamma z + \delta)^4}{|C|^2} p\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right), \quad \bar{q}(z) = \frac{(\gamma z + \delta)^2}{|C|} q\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right), \quad \bar{r} = r.
$$

Therefore, a second-degree polynomial $p_2$ in the $J^{n}$'s transforms according to the direct sum representation $\rho_{4, -2} \oplus \rho_{2, -1} \oplus \rho_{0, 0}$ under the GL(2) action (5.3). One can choose a particularly simple representative of the GL(2) orbit generated by $p_2$ by placing the polynomial $p$ (assumed to be real) in its associated triple $(p, q, r)$ in canonical form, \[10\]:

**Theorem 5.3** Every non-zero quartic real polynomial $p(z)$ transforming under the representation $\rho_{4, -2}$ of GL(2) is equivalent to one of the following canonical forms:

1) $\nu(z^4 + \tau z^2 + 1), \tau \neq \pm 2,$
2) $\nu(z^4 + \tau z^2 - 1),
3) $\nu(z^2 + 1)^2,
4) $\nu(z^2 + 1),
5) $\nu(z^2 - 1),
6) $\nu z^2,
7) $z,$
8) $1,$

(5.5)

where $\nu \neq 0$ and $\tau$ are real numbers.

We now generalize these results to the matrix case. The induced action of GL(2) on $P_{n-\Delta, n}$ analogous to (5.2) is:

$$
\begin{pmatrix}
p^1(w) \\
p^2(w)
\end{pmatrix} \mapsto \begin{pmatrix}
\bar{p}^1(z) \\
\bar{p}^2(z)
\end{pmatrix} = \hat{U}(z) \begin{pmatrix}
p^1(w(z)) \\
p^2(w(z))
\end{pmatrix},
$$

where

$$
\hat{U}(z) = \text{diag}\left( (\gamma z + \delta)^{n-\Delta}, (\gamma z + \delta)^{n}\right),
$$

and $w(z)$ is given by (5.1). This representation of GL(2) in $P_{n-\Delta, n}$ is obviously isomorphic to $\rho_{n-\Delta, 0} \oplus \rho_{n, 0}$. The following Lemma describes the induced action on $s_\Delta$:

**Lemma 5.4** The action of GL(2) on $s_\Delta$ given by

$$
X(w) \mapsto \overline{X}(z) = \hat{U}(z) X\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right) \hat{U}^{-1}(z), \quad X \in s_\Delta,
$$

(5.6)
defines a Lie superalgebra automorphism. The generators of \( s\Delta \), cf. (3.1), transform according to the following irreducible representations:

\[
\begin{align*}
\{T^\epsilon\} &\rightarrow \rho_{2,-1}, & \{J\} &\rightarrow \rho_{0,0}, \\
\{Q_\alpha\} &\rightarrow \rho_{\Delta,0}, & \{\overline{Q}_\alpha\} &\rightarrow \rho_{\Delta,-\Delta}.
\end{align*}
\]

A straightforward generalization of equation (5.4) and of Lemma 5.2 to a second-degree polynomial in the \( T^\epsilon \)'s shows that the (real) polynomial \( p_4 \) in (4.7) transforms according to the representation \( \rho_{4,-2} \) under the \( \text{GL}(2) \) action (5.6). We will henceforth assume that \( p_4 \) is one of the canonical forms given in Theorem 5.3. The integral (4.5) and the inverse \( z = \varphi^{-1}(x) \) can then be easily computed for each of these canonical forms.

Before finishing this section, let us point out that equations (4.8) and (4.9) adopt a simpler form in the variable \( z \), because in that case only rational functions appear. Explicitly, the equation for the gauge factor reads:

\[
U' = U\hat{A}, \quad \text{with} \quad \hat{A}(z) = \frac{A|\varphi(z)}{\sqrt{p_4}} = \frac{1}{2} p_4 \left( A_1 - \frac{1}{2} p_4' \right),
\]

(5.7)

while \( \tilde{U} \) and \( \tilde{W} \) in equation (4.9) are substituted by \( U \) and \( W \), where

\[
W = -A_0 + p_4\hat{A}' + p_4\hat{A}^2 + \frac{1}{2} p_4'\hat{A}.
\]

(5.8)

We now use Theorem 3.4, Lemma 3.5 and the explicit form of the operators (3.1) to compute \( \hat{A} \) and \( A_0 \) for the most general operator in \( P_{n-\Delta,n}^{(2)} \) of the form (4.7), in the cases \( \Delta = 0, 1 \). We denote by \( p_n^\alpha \) the polynomial \( \sum_{i=0}^n \alpha_i z^i \), where \( \alpha_i \) are arbitrary complex numbers. If the polynomial \( p_4 \) is not one of the first three canonical forms, we obtain:

Case \( \Delta = 0 \):

\[
p_4\hat{A} = \begin{pmatrix} p_2^a & p_2^b \\ p_2^c & p_2^d \end{pmatrix}, \quad A_0 = \begin{pmatrix} \hat{a}_0 - 2na_2z & \hat{b}_0 - 2nb_2z \\ \hat{c}_0 - 2nc_2z & \hat{d}_0 - 2nd_2z \end{pmatrix}.
\]

(5.9)

Case \( \Delta = 1 \):

\[
p_4\hat{A} = \begin{pmatrix} p_3^a & p_3^b \\ p_3^c & p_2^d \end{pmatrix}, \quad A_0 = \begin{pmatrix} \hat{a}_0 - 2(n-1)a_2z & -2nb_1 \\ \hat{c}_1z - 2(n-1)c_3z^2 & \hat{d}_0 - 2nd_2z \end{pmatrix}.
\]

(5.10)
where $\hat{a}_0, \hat{b}_0, \hat{c}_0, \hat{c}_1,$ and $\hat{d}_0$ are arbitrary complex numbers. If $\hat{p}_4$ is one of the first three canonical forms, the following extra terms are present in $\hat{p}_4 \hat{A}$ and $A_0$:

$$
\hat{p}_4 \hat{A} \rightarrow -\text{diag}\left( (n - \Delta)nu^2, nnu^2 \right), \quad (5.11)
\hat{A}_0 \rightarrow +\text{diag}\left( (n - \Delta)(n - \Delta - 1)nu^2, n(n - 1)nu^2 \right). \quad (5.12)
$$

### 6 The Gauge Factor

In this section we will deal with the differential equation (5.7) for the gauge factor $U(z)$. As remarked in Section 4, this equation cannot be solved in closed form for every $\hat{A}$. Equation (5.7) admits the formal iterative solution

$$
U(z) = U_0 \left( 1 + \sum_{n=1}^{\infty} \int_z^s ds_1 \int_{s_1}^{s_2} ds_2 \cdots \int_{s_{n-1}}^{s_n} ds_n \hat{A}(s_n) \cdots \hat{A}(s_2) \hat{A}(s_1) \right),
$$

but this is of little practical use in checking the Hermiticity of $V$. We will impose a certain condition on $\hat{A}$ so that (5.7) can be explicitly solved, thus reducing the number of parameters defining $\hat{A}$, but still leaving us with room to construct relevant examples of QES Schrödinger operators.

Denoting

$$
U = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}, \quad \hat{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},
$$

we may write equation (5.7) as

$$
\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad (6.1)
$$

with the same system for $(u_3, u_4)^t$. Unfortunately, the associated scalar second order differential equations for $u_1$ or $u_2$ are not any of the standard equations of Mathematical Physics. We also note that the quotient $q = u_1/u_2$ satisfies the following Riccati equation:

$$
q' = -bq^2 + (a - d)q + c. \quad (6.2)
$$

In fact, solving (6.1) is equivalent to solving (6.2). Moreover, if we try to uncouple $\hat{A}$ by performing the linear transformation

$$
\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix},
$$

23
we find that the transformed of $\hat{A}$ will be lower triangular whenever $q$ satisfies
\[
qs' - sq' + cs^2 - bq^2 + qs(a - d) = 0, \tag{6.3}
\]
with a similar condition to change $\hat{A}$ into upper triangular form. If $s$ is non-zero we can take $s = 1$ without any loss of generality, and equation (6.3) reduces to (6.2). Unfortunately, none of these equations can be solved without further assumptions.

However, if we restrict ourselves to matrices $\hat{A}$ satisfying the equation
\[
[\hat{A}(z), \int_{z_0}^{z} \hat{A}(s) \, ds] = 0, \tag{6.4}
\]
for some $z_0 \in \mathbb{R}$, we can readily integrate the gauge equation (5.7). Recall that this condition on $\hat{A}$ was indeed verified by the QES Schrödinger operator found by Shifman and Turbiner, [17]. If (6.4) is satisfied, we shall say that we are in the commuting case. In this case, the general solution of the gauge equation is given by:
\[
U(z) = U_0 \exp \int_{z_0}^{z} \hat{A}(s) \, ds, \tag{6.5}
\]
where $U_0$ is in $\text{GL}(2, \mathbb{C})$. It is worth mentioning in passing that we can use
\[
\begin{pmatrix}
1 & 0 \\
\xi e^{i\phi} & \eta
\end{pmatrix}, \quad \xi \geq 0, \quad \eta > 0, \quad \phi \in [0, 2\pi) \tag{6.6}
\]
to parametrize the orbits of $\mathbb{R}^+ \times U(2)$ acting on $\text{GL}(2, \mathbb{C})$ by left multiplication. Since matrices in the same orbit lead to equivalent Schrödinger operators, we can take $U_0$ in the form (6.6).

We shall be mainly concerned with the commuting case. We will make use of the following elementary Lemma to describe the most general form of $\hat{A}$ in the commuting case:

**Lemma 6.1** Let $M(z)$ be a $2 \times 2$ matrix satisfying the equation (6.4). Then $M$ is of the form:
\[
M = f(z) M_0 + g(z), \tag{6.7}
\]
where $f$ and $g$ are scalar functions, and $M_0$ is a $2 \times 2$ constant matrix.
With this Lemma in mind, looking at the expressions for \( \hat{A} (\Delta = 0, 1) \) that we obtained in Section 5, cf. (5.9)–(5.11), we find that the most general \( \hat{A} \) satisfying (6.4) is of the form

\[
p_4 \hat{A} = \hat{p}_2(z) + \hat{A}(z),
\]

where, as usual, \( \hat{p}_2 \) denotes a second-degree polynomial in \( z \) with complex coefficients, and the matrix \( \hat{A} \) is given in Table 1.

| \( \Delta = 0 \) | \( p_4 \) in canonical forms 1–3 | \( p_4 \) in canonical forms 4–8 |
|-----------------|---------------------------------|---------------------------------|
| \( p_2 \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) - n\nu z^3 \) | \( p_2 \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) \) |
| \( (p_2 - \nu z^3) \left( \begin{array}{cc} n - 1 & 0 \\ \gamma & n \end{array} \right) \) | \( p_1 \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right), p_2 \left( \begin{array}{cc} \alpha & 0 \\ \gamma & \delta \end{array} \right), p_3 \left( \begin{array}{cc} 0 & 0 \\ \gamma & 0 \end{array} \right) \) |

Table 1: Matrix \( \hat{A}(z) \) for the canonical forms 1–8 (\( \alpha, \beta, \gamma, \delta \in \mathbb{C} \)).

Note that if \( \Delta = 1 \) and \( p_4 \) is one of the first three canonical forms, every Hamiltonian we can possibly obtain will be diagonal, modulo equivalence.

Finally, let us remark that in the non-commuting case (that is, when \( [\hat{A}, \int_0^z \hat{A}] \neq 0 \)), we may still be able to integrate (5.7) explicitly by imposing other constraints on \( \hat{A} \), as e.g. assuming it is uncoupled. Alternatively, if \( p_4 \) is not any of the first three canonical forms, and we assume that the columns (or rows) of \( \hat{A} \) are proportional to each other (the ratio of the respective entries being a constant), we can also reduce (5.7) to quadratures. Unfortunately, we have not been able to find any interesting examples of QES Hamiltonians in the non-commuting case.

### 7 Examples

In this final section we exhibit some new examples of spin \( 1/2 \) Schrödinger operators equivalent to a PVSP operator of the form (1.7). In the previous section we have seen how, by restricting ourselves to the commuting case, we were able to integrate equation (5.7) explicitly. This is not, however, the end
of the problem, for we must still check that the matrix

\[ V = U W U^{-1} \big|_{\varphi^{-1}(x)}, \tag{7.1} \]

with \( W \) given by (5.8), is self-adjoint. Again, if we start with the most general PVSP operator of the form (4.7), the algebraic constraints imposed by the condition \( V = V^\dagger \) are too complicated to be solved in full generality, even if we limit ourselves to the commuting case. This situation is completely analogous to what we find when trying to construct scalar QES Schrödinger operators in more than one spatial dimension, [16], [7]. We have thus no choice but to look for particular examples. We now present some relevant examples of QES spin 1/2 Hamiltonians for the commuting case. In the first two examples we take \( \Delta = 0 \), while in the remaining ones we assume that \( \Delta = 1 \). Finally, let us point out that many otherwise interesting examples are often reduced to trivial ones after we impose the square integrability condition on the eigenfunctions.

**Example 1.** Consider the six-parameter PVSP operator \( T \) given by

\[-T = T_0 - (a_2 + d_2) T^+ + (a_2 - d_2) \tilde{J} T^+ + 2a_1 T_0 + \left( 2a_0 + \frac{1}{2}(n-1) \right) T^- + (Q_0 + \overline{Q}_0)(2b_2 T^+ + b_0),\]

where all the parameters are real numbers coinciding with those appearing in equation (5.9), and \( b_2 \neq 0 \). Note that we are in case 7 of Theorem 5.3 (\( p_4 = z \)), and so \( z = x^2/4 \). The gauge factor \( U(z) \) is given by

\[ U(z) = f(z) \begin{pmatrix} \Lambda \cosh u + (a_2 - d_2) \sinh u & 2b_2 \sinh u \\ 2b_2 \sinh u & \Lambda \cosh u + (d_2 - a_2) \sinh u \end{pmatrix}, \]

with

\[ f(z) = z^{a_0} \exp \left( a_1 z + \frac{1}{4} (a_2 + d_2) z^2 \right), \quad u = \frac{\Lambda}{4} z^2, \quad \Lambda^2 = (a_2 - d_2)^2 + 4b_2^2, \]

where we have taken \( U_0 = \Lambda \) in equation (6.3). Using (7.1), we obtain a potential \( V(x) \) with entries given by (see (4.1)):

\begin{align*}
v_1 &= \alpha_{-2} x^{-2} + \alpha_0 + \alpha_2 x^2 + \alpha_4 x^4 + \alpha_6 x^6, \\
v_2 &= \delta_{-2} x^{-2} + \delta_0 + \delta_2 x^2 + \delta_4 x^4 + \delta_6 x^6, \\
v &= \beta_0 + \beta_2 x^2 + \beta_4 x^4 + \beta_6 x^6,
\end{align*}

26
where
\[ \alpha_{-2} = 2a_0(2a_0 - 1), \quad \alpha_0 = -\frac{a_2 b_0}{b_2}, \quad \alpha_2 = \frac{a_1^2}{4} + \frac{a_2(a_0 + n + \frac{3}{4})}{2}, \]
\[ \alpha_4 = \frac{a_1 a_2}{8}, \quad \alpha_6 = \frac{a_2^2 + b_2^2}{64}, \]
\[ \beta_0 = -\lambda_0, \quad \beta_2 = \frac{b_2}{2}(a_0 + n + \frac{3}{4}), \quad \beta_4 = \frac{a_1 b_2}{8}, \quad \beta_6 = \frac{b_2(a_2 + d_2)}{64}, \]

while each \( \delta_n \) is given by the corresponding \( \alpha_n \) replacing \( a_2 \) by \( d_2 \). We have ignored a constant multiple of the identity in \( V \), which is equivalent to fixing a new origin in the energy scale (this will be done for the subsequent examples without further notice).

We note that the potential \( V \) is singular at \( x = 0 \) unless \( a_0 \) is either 0 or \( 1/2 \). Let us introduce the parameter \( \lambda = 2a_0 - 1 \), in terms of which we have \( \alpha_{-2} = \lambda(\lambda + 1) \). If \( \lambda \) is a non-negative integer \( l \), we may regard
\[
(-\partial_x^2 + V(x) - E)\psi(x) = 0, \quad 0 < x < \infty,
\]
Eq. (7.2)
as the radial equation obtained after separating variables in the three-dimensional Schrödinger equation with a spherically symmetric Hamiltonian given by
\[ \hat{H} = -\Delta + U(r), \quad \text{with} \quad U(x) = V(x) - \frac{l(l+1)}{x^2}, \]
where \( \Delta \) denotes the usual flat Laplacian. Given a non-negative integer \( l \) and a spherical harmonic \( Y_{lm}(\theta, \phi) \), if \( \psi \) is an eigenfunction for the equation (7.2) satisfying
\[
\lim_{x \to 0^+} \psi(x) = 0,
\]
Eq. (7.3)
then
\[ \hat{\Psi}(r, \theta, \phi) = \frac{\psi(r)}{r} Y_{lm}(\theta, \phi) \]
will be an eigenfunction for \( \hat{H} \) with angular momentum \( l \). If \( \lambda \) is not a non-negative integer, we shall consider (7.2) as the radial equation for the singular potential \( U(r) = V(r) \) at zero angular momentum. The potential \( U(r) \) is physically meaningful, in the sense that the Hamiltonian \( \hat{H} \) admits self-adjoint extensions and its spectrum is bounded from below, whenever \( \lambda \neq -1/2, 1 \). The boundary condition (7.3) must be satisfied in the
singular case for all values of $\lambda$. This boundary condition is verified if and only if $a_0 > 0$. Finally, the additional conditions

$$a_2 d_2 > b_2^2, \quad a_2 < 0,$$

ensure that $\psi$ lies in $L^2(I) \oplus L^2(I)$, where $I = [0, \infty)$ in the singular case or at zero angular momentum, or $I = \mathbb{R}$ in the non-singular one-dimensional case.

Example 2. As our second example, we take

$$-T = (T^-)^2 + (a_1 + d_1) T^0 + (a_1 - d_1) \tilde{J} T^0 + 2a_0 T^- + (b_1 Q_0 + b_1^* \tilde{Q}_0)(2 T^0 + n - \mu) + \frac{1}{2}(n - \mu)(a_1 - d_1) \tilde{J},$$

where the coefficients are real, excepting $b_1 \neq 0$ which is complex. In this example $p_4 = 1$ and thus $z = x$ (case 8 of Theorem 5.3). We choose the following gauge factor:

$$U(z) = f(z) \begin{pmatrix} \Lambda \cosh u + (a_1 - d_1) \sinh u & 2 b_1^* \sinh u \\ 2 b_1 \sinh u & \Lambda \cosh u + (d_1 - a_1) \sinh u \end{pmatrix},$$

with

$$f(z) = \exp \left( a_0 z + \frac{1}{4} (a_1 + d_1) z^2 \right), \quad u = \frac{\Lambda}{4} z^2, \quad \Lambda^2 = (a_1 - d_1)^2 + 4 |b_1|^2.$$

We obtain the potential with entries given by

$$v_1 = a_1 (\mu + 1) + 2a_0 a_1 x + (a_1^2 + |b_1|^2) x^2,$$

$$v_2 = d_1 (\mu + 1) + 2a_0 d_1 x + (d_1^2 + |b_1|^2) x^2,$$

$$v = b_1 \left( \mu + 1 + 2a_0 x + (a_1 + d_1) x^2 \right).$$

The eigenfunctions $\psi(x)$ are square-integrable provided

$$a_1 d_1 > |b_1|^2, \quad a_1 < 0.$$

Since the parameter $n$ does not appear in the gauge factor or in the potential, the Hamiltonian preserves $U \cdot \mathcal{P}_{n,n}$ for arbitrary $n$. (In the literature, an operator with this property is usually referred to as exactly solvable, [19].)
Note that if \( \mu = 2n \) we could also obtain \( T \) from the PVSP operators (3.1) with \( \Delta = 1 \), but in this case the Hamiltonian would preserve \( U \cdot P_{m-1,m} \) only for \( m = n \).

**Example 3.** From now on we take \( \Delta = 1 \). Let \( T \) be the four-parameter PVSP operator given by

\[
-T = (T^0)^2 + 2a_2 T^++ 2(n + 1)T^0 - 2J T^0 + 2b_1 \overline{Q}_0 + 2b_0 \overline{Q}_1 - 2b_1 Q_0 T^0
-2b_0 Q_0 T^- - \left( 4a_2 b_0 + (3n + 1) b_1 \right) Q_0 - 4a_2 b_1 Q_1 - (2 \hat{d}_0 + n + \frac{1}{2}) J,
\]

with all the parameters real. Since \( p_4 = z^2 \) we are in case 6 of Theorem 5.3. Solving equation (4.5) for \( z \), we obtain \( z = e^x \). The gauge factor reads:

\[
U(z) = \sqrt{z} e^{a_2 z} \begin{pmatrix} \cos u & \sin u \\ -\sin u & \cos u \end{pmatrix}, \quad \text{where} \quad u = -\frac{b_0}{z} + b_1 \log z.
\]

The potential is given by

\[
v = -b_0^2 e^{-2x} - 2b_0 b_1 e^{-x} + (2n + 1) a_2 e^x + a_2^2 e^{2x}
+ (-1)^j \left( \alpha(x) \cos 2\bar{u} - \beta(x) \sin 2\bar{u} \right), \quad j = 1, 2
\]

\[
v_j = \alpha(x) \sin 2\bar{u} + \beta(x) \cos 2\bar{u},
\]

where \( \bar{u} = b_1 x - b_0 e^{-x} \), and

\[
\alpha(x) = -\frac{\hat{d}_0}{2} + a_2 e^x, \quad \beta(x) = (2n + 1) b_1 + 2a_2 (b_0 + b_1 e^x).
\]

It may be easily verified that the expected value of the potential is bounded from below, i.e.

\[
\langle \psi, V \psi \rangle \geq c \| \psi \|^2, \quad \text{with} \quad \psi \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R}), \quad (7.4)
\]

for some \( c \in \mathbb{R} \), if and only if \( b_0 = 0 \). (Note, however, that even in this case the amplitude of the oscillations of \( v(x) \) tends to infinity as \( x \to +\infty \).) Finally, the condition \( a_2 < 0 \) is necessary and sufficient to ensure the square integrability of the eigenfunctions \( \psi(x) \).
Example 4. As our last example, we consider:

\[-T = T^- T^0 + 2a_1T^0 + (2a_0 + n - \frac{1}{2})T^- - JT^- + 2b_1 \overline{Q}_0 - 2b_1 Q_0 T^0 - b_1 (4a_0 + 3n + 1) Q_0 - 4a_1 b_1 Q_1 + 2(2a_0 - a_1)J,\]

where all the coefficients are real, and \(b_1 \neq 0\). Since \(p_4 = z\) (case 7), we have \(z = x^2/4\). The gauge factor is chosen as follows:

\[U(z) = z^{a_0} e^{a_1 z} \begin{pmatrix} \cos b_1 z & \sin b_1 z \\ -\sin b_1 z & \cos b_1 z \end{pmatrix}.\]

The entries of the potential \(V(x)\) are given by

\[v_j = \frac{2a_0(2a_0 - 1)}{x^2} + \frac{1}{4} (a_1^2 - b_1^2) x^2 + (-1)^j \left( \hat{a}_0 \cos \frac{b_1 x^2}{2} - \alpha(x) \sin \frac{b_1 x^2}{2} \right),\]

\[v = \hat{a}_0 \sin \frac{b_1 x^2}{2} + \alpha(x) \cos \frac{b_1 x^2}{2},\]

with \(j = 1, 2\), and \(\alpha(x)\) is defined by

\[\alpha(x) = \frac{b_1}{2} (4a_0 + 4n + 1 + a_1 x^2).\]

We first note that the potential is singular at the origin unless \(a_0 = 0, 1/2\). This situation is completely analogous to that of Example 1, so it will not be further discussed. The expected value of the potential is bounded from below, that is, equation (7.4) holds, if and only if

\[\left| \frac{a_1}{b_1} \right| > 1 + \sqrt{2}.\]

Finally, the conditions

\[a_0 \geq 0, \quad a_1 < 0,\]

guarantee the square integrability of the eigenfunctions \(\psi(x)\).
References

[1] Y. Brihaye and P. Kosinski, Quasi exactly solvable $2 \times 2$ matrix equations, *J. Math. Phys.* **35** (1994), 3089–3098.

[2] Y. Brihaye, S. Giller, C. Gonera, and P. Kosinski, The algebraic structures of quasi exactly solvable systems, preprint, 1994.

[3] R.P. Feynman and M. Gell-Mann, Theory of the Fermi interaction, *Phys. Rev.* **109** (1958), 193–198.

[4] A. Galindo and P. Pascual, *Quantum Mechanics I*, Springer-Verlag, Berlin, 1990.

[5] A. González–López, N. Kamran, and P.J. Olver, Quasi-exactly solvable Lie algebras of first order differential operators in two complex variables, *J. Phys.* **A24** (1991), 3995–4008.

[6] A. González–López, N. Kamran, and P.J. Olver, Lie algebras of differential operators in two complex variables, *American J. Math.* **114** (1992), 1163–1185.

[7] A. González–López, N. Kamran, and P.J. Olver, Quasi-exact solvability, *Contemp. Math.* **160** (1993), 113–140.

[8] A. González–López, N. Kamran, and P.J. Olver, Normalizability of one-dimensional quasi-exactly solvable Schrödinger operators, *Commun. Math. Phys.* **153** (1993), 117–146.

[9] A. González–López, N. Kamran, and P.J. Olver, Real Lie algebras of differential operators and quasi-exactly solvable potentials, preprint, 1995.

[10] G.B. Gurevich, *Foundations of the Theory of Algebraic Invariants*, P. Noordhoff, Groningen, Holland, 1964.

[11] C. Itzykson and J.-B. Zuber, *Quantum Field Theory*, McGraw Hill, New York, 1980.

[12] N. Kamran and P.J. Olver, Lie algebras of differential operators and Lie-algebraic potentials, *J. Math. Anal. Appl.* **145** (1990), 342–356.
[13] W. Miller, Jr., *Lie Theory and Special Functions*, Academic Press, New York, 1968.

[14] M.A. Olshanetskii and A.M. Perelomov, Quantum integrable systems related to Lie algebras, *Phys. Rep.* 94 (1983), 313–404.

[15] P.J. Olver, *Equivalence, Invariants and Symmetry*, Cambridge University Press, Cambridge, U.K., 1995.

[16] M.A. Shifman, New findings in quantum mechanics (partial algebraization of the spectral problem), *Int. J. Mod. Phys.* A126 (1989), 2897–2952.

[17] M.A. Shifman and A.V. Turbiner, Quantal problems with partial algebraization of the spectrum, *Commun. Math. Phys.* 126 (1989), 347–365.

[18] A.V. Turbiner, Quasi-exactly solvable problems and \(\mathfrak{sl}(2)\) algebra, *Commun. Math. Phys.* 118 (1988), 467–474.

[19] A.V. Turbiner, Lie algebraic approach to the theory of polynomial solutions. I. Ordinary differential equations and finite-difference equations in one variable, preprint, CERN, CPT-92/P.2679-REV.

[20] A.V. Turbiner, Lie algebraic approach to the theory of polynomial solutions. II. Differential equations in one real and one Grassmann variables and \(2 \times 2\) matrix differential equations, ETH-TH/92-21.

[21] A.G. Ushveridze, Quasi-exactly solvable models in quantum mechanics, *Sov. J. Part. Nucl.* 20 (1989), 504–528.