EIGENVALUE COINCIDENCES AND MULTIPLICITY FREE SPHERICAL PAIRS

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Abstract. In our recent paper, we related the structure of subvarieties of \( n \times n \) complex matrices defined by eigenvalue coincidences to \( GL(n-1, \mathbb{C}) \)-orbits on the flag variety of \( \mathfrak{gl}(n, \mathbb{C}) \). In the first part of this paper, we extend these results to the complex orthogonal Lie algebra \( \mathfrak{g} = \mathfrak{so}(n, \mathbb{C}) \). In the second part of the paper, we use these results to study the geometry and invariant theory of the coisotropy representation of the spherical pairs \((G \times K, K_\Delta)\), where \( K = GL(n-1, \mathbb{C}), SO(n-1, \mathbb{C}), G = GL(n, \mathbb{C}), SO(n, \mathbb{C}) \), and \( K_\Delta \) is the diagonal copy of \( K \) in the product \( G \times K \). The coisotropy representation of \((G \times K, K_\Delta)\) is easily identified with the action of \( K \) on \( \mathfrak{g} \) by conjugation. We study the geometric quotient \( \mathfrak{g} \rightarrow \mathfrak{g}/K \), describing all of the closed \( K \)-orbits on \( \mathfrak{g} \) and the structure of the zero fibre. We also prove an analogue of Kostant’s theorem describing the variety of regular elements of \( \mathfrak{g} \). That is, we show that a \( K \)-orbit through an element \( x \in \mathfrak{g} \) has maximal dimension if and only if the algebraically independent generators of the invariant ring \( \mathbb{C}[\mathfrak{g}]^K \) are linearly independent at \( x \).

1. Introduction

This paper studies two related questions. In \([CEa]\), we related a subvariety of matrices of \( \mathfrak{gl}(n, \mathbb{C}) \) defined by eigenvalue coincidences to the \( GL(n-1, \mathbb{C}) \)-orbits on the flag variety of \( \mathfrak{gl}(n, \mathbb{C}) \). In the first part of the paper, we extend these results from \( \mathfrak{gl}(n, \mathbb{C}) \) to \( \mathfrak{so}(n, \mathbb{C}) \). In the second part of the paper, we use the results from \([CEa]\) and the first part of the paper in order to study \( K \)-orbits on \( \mathfrak{g} \) in the two cases \((K = GL(n-1, \mathbb{C}), \mathfrak{g} = \mathfrak{gl}(n, \mathbb{C}))\) and \((K = SO(n-1, \mathbb{C}), \mathfrak{g} = \mathfrak{so}(n, \mathbb{C}))\). In particular, we study the geometric invariant theory quotient \( \mathfrak{g} \rightarrow \mathfrak{g}/K \). By results of Knop \([Kno94]\), the algebra \( \mathbb{C}[\mathfrak{g}]^K = \mathbb{C}[\mathfrak{g}]^G \otimes \mathbb{C}[\mathfrak{t}]^K \) is a polynomial algebra, and it easily follows that the quotient morphism \( \mathfrak{g} \rightarrow \mathfrak{g}/K \) can be identified with a morphism \( \Phi_n \) from \( \mathfrak{g} \) to affine space, which is a partial version of a morphism considered by Kostant and Wallach \([KW06a, KW06b]\). We study this morphism, and as a consequence, we determine explicitly the closed \( K \)-orbits on \( \mathfrak{g} \), prove a variant of Kostant’s theorem using linear independence of differentials to characterize regular elements \([Kos63]\), and provide a new definition of the so-called strongly regular elements of \( \mathfrak{g} \), which Kostant and Wallach use to produce completely integrable systems on regular adjoint orbits on \( \mathfrak{g} \).

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In more detail, let \( \mathfrak{g} = \mathfrak{so}(n, \mathbb{C}) \) and let \( \mathfrak{k} \subset \mathfrak{g} \) be the symmetric subalgebra \( \mathfrak{k} = \mathfrak{so}(n - 1, \mathbb{C}) \) fixed by an involution \( \theta \) of \( \mathfrak{g} \). Let \( r_n \) and \( r_{n-1} \) be the ranks of \( \mathfrak{g} \) and \( \mathfrak{k} \) respectively. Recall that if \( a \) is an eigenvalue of \( x \in \mathfrak{g} \), then \(-a\) is also an eigenvalue of \( a \), and 0 is an eigenvalue of \( a \) provided \( n \) is odd. Let

\[
\sigma(x) = \{\pm b_1, \ldots, \pm b_{r_n}\}
\]

be the eigenvalues of \( x \), listed with multiplicity, except that if \( n \) is odd, we only list the eigenvalue 0 \( 2j \) times if it appears with multiplicity \( 2j + 1 \). We call \( \sigma(x) \) the spectrum of \( x \). For \( x \in \mathfrak{g} \), let \( x_{\mathfrak{k}} \) denote the projection of \( x \) onto \( \mathfrak{k} \) off of \( \mathfrak{g}^{-\theta} \), and let

\[
\sigma(x_{\mathfrak{k}}) = \{\pm a_1, \ldots, \pm a_{r_{n-1}}\}
\]

be the spectrum of \( x_{\mathfrak{k}} \). We consider the eigenvalue coincidence varieties

\[
\mathfrak{g}(\geq i) := \{x \in \mathfrak{g} : b_{jm} = \pm a_{km}, m = 1, \ldots, i, k_m \neq k_n\} \text{ for } i = 0, \ldots, r_{n-1}.
\]

We make use of the following notation throughout (cf \[CEa\]). For \( \mathfrak{b} \subset \mathfrak{g} \) a Borel subalgebra, we denote its \( \mathcal{K} \)-orbit by \( Q = \mathcal{K} \cdot \mathfrak{b} \subset \mathcal{B} \). We denote the \( \mathcal{K} \)-saturation of \( \mathfrak{b} \) in \( \mathfrak{g} \) by \( Y_Q := \mathrm{Ad}(\mathcal{K})\mathfrak{b} \). Note that the variety \( Y_Q \) depends only on the \( \mathcal{K} \)-orbit \( Q \) in \( \mathcal{B} \). We prove the following result.

**Theorem 1.1.** The irreducible component decomposition of the variety \( \mathfrak{g}(\geq i) \) is given by

\[
\mathfrak{g}(\geq i) = \bigcup_{\text{codim}(Q) = i} Y_Q.
\]

In particular, if \( \mathfrak{g} = \mathfrak{so}(2l, \mathbb{C}) \) is of type \( D \) then the varieties \( \mathfrak{g}(\geq i) \) are all irreducible. If \( \mathfrak{g} = \mathfrak{so}(2l + 1, \mathbb{C}) \) is of type \( B \) then \( \mathfrak{g}(\geq i) \) is irreducible for \( i = 0, \ldots, l - 1 \) and has exactly two irreducible components when \( i = l \).

Although this statement is similar to Theorem 1.1 of \[CEa\], the proof requires some significant new ideas, largely because of our inability to do computations analogous to ones for \( \mathfrak{gl}(n, \mathbb{C}) \).

We consider the pairs \( (\mathcal{G}, \mathcal{K}) \) given by \( \mathcal{G} = \mathcal{GL}(n, \mathbb{C}) \), \( \mathcal{K} = \mathcal{GL}(n - 1, \mathbb{C}) \) and \( G = \mathcal{SO}(n, \mathbb{C}) \), \( K = \mathcal{SO}(n - 1, \mathbb{C}) \), which are essentially the only multiplicity free symmetric pairs, and \( \mathfrak{g} \) and \( \mathfrak{k} \) be the corresponding Lie algebras. For each pair, let \( \tilde{G} = G \times K \) and let \( K_{\Delta} \) be the diagonal embedding of \( K \) in \( \tilde{G} \). It is well-known that \( K_{\Delta} \) is a spherical subgroup of \( \tilde{G} \), and we consider the coisotropy representation of \( K_{\Delta} \) on \( \tilde{\mathfrak{g}}/\mathfrak{k}_{\Delta} \), which coincides with the adjoint action of \( K \) on \( \mathfrak{g} \). We say that \( x \in \mathfrak{g} \) is \( K \)-regular if \( x \) is in a \( K \)-orbit of maximal dimension. We write the generators of \( \mathbb{C}[\mathfrak{g}]^K \) as \( J := f_{n-1,1}, \ldots, f_{n-1,r_{n-1}}, f_{n,1}, \ldots, f_{n,r_n} \), where for the case of \( \mathfrak{gl}(i, \mathbb{C}) \), \( r_i = i \). The following theorem generalizes a basic result of Kostant \[Kos63\].

**Theorem 1.2.** An element \( x \in \mathfrak{g} \) is \( K \)-regular if and only if the differentials \( \{df(x) : f \in J\} \) are linearly independent.

For the case of \( \mathfrak{gl}(n, \mathbb{C}) \), \( K \)-regular elements are called \( n \)-strongly regular in \[CEa\], and we call \( K \)-regular elements of \( \mathfrak{g} \) \( n \)-strongly regular also in the case of \( \mathfrak{so}(n, \mathbb{C}) \). We use the
above theorem to show that \( x \in g \) is \( n \)-strongly regular if and only if \( \mathfrak{z}_g(x) \cap \mathfrak{z}_k(x) = 0 \).

As a consequence, we provide a simplified criterion for the notion of strongly regular elements of \( \mathfrak{gl}(n, \mathbb{C}) \) and \( \mathfrak{so}(n, \mathbb{C}) \) from [KW06a, Co09]. We apply the above results to give explicit representatives for the closed \( K \)-orbits on \( g \), and to study the fibres of the morphism \( \Phi_n : g \to \mathbb{C}^r_{n-1} \times \mathbb{C}^r_n \). In particular, we show that in contrast to the case of \( \mathfrak{gl}(n, \mathbb{C}) \), for \( \mathfrak{so}(n, \mathbb{C}) \) the fibre \( \Phi_n^{-1}(0) \) contains no \( n \)-strongly regular elements. We expect this result to have important consequences for the study of integrable systems on regular adjoint orbits for \( \mathfrak{so}(n, \mathbb{C}) \).

This work is motivated by our interest in the Gelfand-Zeitlin system, which is a Poisson commutative family \( J_{GZ} \) in \( \mathbb{C}[g] \) given by using a family of subalgebras \( g_1 \subset g_2 \subset \cdots \subset g_{n-1} \subset g_n \), where \( g_i = \mathfrak{gl}(i, \mathbb{C}) \) in the general linear case, and \( g_i = \mathfrak{so}(i, \mathbb{C}) \) in the orthogonal case. In [CE12], we used the notion of \( n \)-strongly regular elements to study the collection of elements \( x \in \mathfrak{gl}(n, \mathbb{C}) \) such that \( f(x) = 0 \) for all \( f \in J_{GZ} \) and \( \{ df(x) : f \in J_{GZ} \} \) are linearly independent. We note that while the Gelfand-Zeitlin system is somewhat accessible to computation when \( g = \mathfrak{gl}(n, \mathbb{C}) \), this is not the case when \( g = \mathfrak{so}(n, \mathbb{C}) \). We hope that our methods will provide new results and a more conceptual understanding of the Gelfand-Zeitlin system for \( g = \mathfrak{gl}(n, \mathbb{C}) \), and will make the Gelfand-Zeitlin system more tractable to understand in the case of \( \mathfrak{so}(n, \mathbb{C}) \). We also have some preliminary results for applications of our study of the geometry of the Gelfand-Zeitlin system to the study of Gelfand-Zeitlin modules, initiated by Drozd, Futorny, and Ovsienko [DFO94], and this would be interesting to pursue further.

This paper is organized as follows. In Section 2, we establish a number of preliminary results. In Section 3, we prove Theorem 1.1. In Section 4, we prove Theorem 1.2, determine the closed \( K \)-orbits on \( g \), and discuss applications to strongly regular elements. In the appendix, we give an alternative simpler proof of a special case of a flatness result.

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2. Preliminaries

In this section, we recall some basic facts about the general linear and orthogonal Lie algebras. We give explicit descriptions of standard Cartan subalgebras and corresponding root systems of \( g \). We make use of the following notation throughout. We denote the standard basis of \( \mathbb{C}^n \) by \( \{ e_1, \ldots, e_n \} \). For \( 1 \leq i, j \leq n \), we let \( E_{i,j} \) be the matrix with 1 in the \( (i, j) \) position and zero elsewhere. Our exposition follows Chapters 1 and 2 of [GW98].

2.1. Root system of \( \mathfrak{gl}(n, \mathbb{C}) \). Let \( g = \mathfrak{gl}(n, \mathbb{C}) \) be the Lie algebra of \( n \times n \) complex matrices. Abusing, terminology we will refer to \( g \) as a Lie algebra of Type \( A \). We let \( h = \text{diag}[a_1, \ldots, a_n] \) with \( a_i \in \mathbb{C} \) be the standard Cartan subalgebra of diagonal matrices. Thus, \( \text{rank}(g) = n \). Let \( \epsilon_i \in h^* \) be the linear functional \( \epsilon_i(\text{diag}[a_1, \ldots, a_n]) = a_i \), and let
\(\Phi(g, h)\) be the set of roots of \(g\) with respect to \(h\). It is well-known that
\[
\Phi(g, h) := \{\epsilon_i - \epsilon_j : 1 \leq i \neq j \leq l\}.
\]
We take as our standard positive roots the set:
\[
\Phi^+(g, h) := \{\epsilon_i - \epsilon_j : 1 \leq i < j \leq l\}.
\]
with corresponding simple roots
\[
\Pi := \{\alpha_1, \ldots, \alpha_{l-1}, \alpha_l\} \text{ where } \alpha_i = \epsilon_i - \epsilon_{i+1}, i = 1, \ldots, n-1\}.
\]
The standard Borel subalgebra \(\mathfrak{b}_+ := \bigoplus_{\alpha \in \Phi^+(g, h)} g_\alpha\) is the set of \(n \times n\) upper triangular matrices.

2.2. Realization of Orthogonal Lie algebras.

In this section, we describe the explicit realization of the orthogonal Lie algebra \(\mathfrak{so}(n, \mathbb{C})\) that we will use throughout the paper. Let \(\beta\) be the non-degenerate, symmetric bilinear form on \(\mathbb{C}^n\) given by
\[
\beta(x, y) = x^T S_n y,
\]
where \(x, y\) are \(n \times 1\) column vectors and \(S_n\) is the \(n \times n\) matrix:
\[
S_n = \begin{bmatrix}
0 & \ldots & \ldots & 0 & 1 \\
\vdots & & & & 1 \\
\vdots & & & & \\
0 & 1 & \ldots & 0 \\
1 & 0 & \ldots & 0
\end{bmatrix}
\]
with ones down the skew diagonal and zeroes elsewhere. The orthogonal group
\[
SO(n, \mathbb{C}) := \{g \in SL(n, \mathbb{C}) : \beta(gx, gy) = \beta(x, y) \forall x, y \in \mathbb{C}^n\}
\]
Its Lie algebra is
\[
\mathfrak{so}(n, \mathbb{C}) = \{Z \in \mathfrak{gl}(n, \mathbb{C}) : \beta(Zx, y) = -\beta(x, Zy) \forall x, y \in \mathbb{C}^n\}
\]
For our purposes, it will be convenient to have explicit matrix descriptions of \(\mathfrak{so}(n, \mathbb{C})\). We consider the cases where \(n\) is odd and even separately.

2.2.1. Realization of \(\mathfrak{so}(2l, \mathbb{C})\). Let \(\mathfrak{g} = \mathfrak{so}(2l, \mathbb{C})\). We say that \(\mathfrak{g}\) is a Lie algebra of Type \(D\). For \(j = 1, \ldots, l\), we define \(e_{-j} := e_{2l+1-j}\). Then the basis \(\{e_{\pm 1}, \ldots, e_{\pm l}\}\) is an isotropic basis of \(\mathbb{C}^{2l}\) with respect to the form \(\beta\) in (2.4). Recall the matrix \(S_l\) in Equation (2.5).

Note that for any \(l \times l\) matrix \(X\), the matrix \(SX^tS\) is the skew-transpose of \(X\) about the skew-diagonal. The Lie algebra \(\mathfrak{so}(2l, \mathbb{C})\) consists of all \(2l \times 2l\) matrices \(A\) of the form
\[
A = \begin{bmatrix}
a & b \\
c & -s_1a^t s_l
\end{bmatrix},
\]
where \( a \in \mathfrak{gl}(l, \mathbb{C}) \), and \( b = -s_kb's_t, \ c = -s_lb'd's_t \), i.e. \( b \) and \( c \) are skew-symmetric about the skew-diagonal (see Corollary 1.27, [GW98]).

The subalgebra of diagonal matrices \( \mathfrak{h} := \text{diag}[a_1, \ldots, a_l, -a_l, \ldots, -a_1], \ a_i \in \mathbb{C} \) is a Cartan subalgebra of \( \mathfrak{g} \). We refer to \( \mathfrak{h} \) as the standard Cartan subalgebra. Let \( \epsilon_i \in \mathfrak{h}^* \) be the linear functional \( \epsilon_i(\text{diag}[a_1, \ldots, a_l, -a_l, \ldots, -a_1]) = a_i \), and let \( \Phi(\mathfrak{g}, \mathfrak{h}) \) be the roots of \( \mathfrak{g} \) with respect to \( \mathfrak{h} \). It is well-known that

\[
\Phi(\mathfrak{g}, \mathfrak{h}) := \{ \epsilon_i - \epsilon_j, \pm(\epsilon_i + \epsilon_j) : 1 \leq i \neq j \leq l \}.
\]

We take as our standard positive roots the set:

\[
\Phi^+(\mathfrak{g}, \mathfrak{h}) := \{ \epsilon_i - \epsilon_j, \epsilon_i + \epsilon_j : 1 \leq i < j \leq l \}.
\]

with corresponding simple roots

\[
\Pi := \{ \alpha_1, \ldots, \alpha_{l-1}, \alpha_l \} \text{ where } \alpha_i = \epsilon_i - \epsilon_{i+1}, \ i = 1, \ldots, l-1, \ \alpha_l = \epsilon_{l-1} + \epsilon_l.
\]

The root spaces are:

\[
\begin{align*}
\mathfrak{g}_{\epsilon_i + \epsilon_j} &= \mathbb{C}(E_{i,-j} - E_{j,-i}) \\
\mathfrak{g}_{\epsilon_i - \epsilon_j} &= \mathbb{C}(E_{-j,i} - E_{-i,j}) \\
\mathfrak{g}_{\epsilon_i - \epsilon_j} &= \mathbb{C}(E_{i,-j} - E_{j,-i}).
\end{align*}
\]

The standard Borel subalgebra \( \mathfrak{b}_+ := \bigoplus_{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{h})} \mathfrak{g}_\alpha \) is easily seen to be the set of upper triangular matrices of the form (2.6).

2.2.1. Realization of \( \mathfrak{so}(2l+1, \mathbb{C}) \). Let \( \mathfrak{g} = \mathfrak{so}(2l+1, \mathbb{C}) \). We say the \( \mathfrak{g} \) is a Lie algebra of Type B. For \( j = 1, \ldots, l \), we define \( e_{-j} := e_{2l-2-j} \) and \( e_0 = e_{l+1} \). Then the basis \( \{e_{\pm 1}, \ldots, e_{\pm l}, e_0\} \) is an isotropic basis of \( \mathbb{C}^{2l+1} \) with respect to the form \( \beta \) in (2.4). The Lie algebra \( \mathfrak{so}(2l+1, \mathbb{C}) \) consists of all \((2l+1) \times (2l+1)\) matrices \( A \) of the form

\[
A = \begin{bmatrix}
a & w & b \\
u & 0 & -w^ts_l \\
c & -slut & -slalts_l
\end{bmatrix},
\]

where \( a \in \mathfrak{gl}(l, \mathbb{C}), \ b = -s_kb's_l, \ c = -s_lb'd's_l \), i.e. \( b \) and \( c \) are skew-symmetric about the skew-diagonal, \( w \) is an \( l \times 1 \) column vector, and \( u \) is a \( 1 \times l \) row vector. (see Corollary 1.28, [GW98]).

The subalgebra of diagonal matrices \( \mathfrak{h} := \text{diag}[a_1, \ldots, a_l, 0, -a_l, \ldots, -a_1], \ a_i \in \mathbb{C} \) is a Cartan subalgebra of \( \mathfrak{g} \). We again refer to \( \mathfrak{h} \) as the standard Cartan subalgebra. Let \( \epsilon_i \in \mathfrak{h}^* \) be the linear functional \( \epsilon_i(\text{diag}[a_1, \ldots, a_l, 0, -a_l, \ldots, -a_1]) = a_i \). In this case, we have

\[
\Phi(\mathfrak{g}, \mathfrak{h}) := \{ \epsilon_i - \epsilon_j, \pm(\epsilon_i + \epsilon_j) : 1 \leq i \neq j \leq l \} \cup \{ \pm \epsilon_k : 1 \leq k \leq l \}.
\]

We take as our standard positive roots the set:

\[
\Phi^+(\mathfrak{g}, \mathfrak{h}) := \{ \epsilon_i - \epsilon_j, \epsilon_i + \epsilon_j : 1 \leq i < j \leq l \} \cup \{ \epsilon_k : 1 \leq k \leq l \}.
\]
with corresponding simple roots

\[(2.14) \quad \Pi := \{\alpha_1, \ldots, \alpha_{l-1}, \alpha_l\} \text{ where } \alpha_i = \epsilon_i - \epsilon_{i+1}, \ i = 1, \ldots, l-1, \ \alpha_l = \epsilon_l.\]

(See page 77 of [GW98].) The root spaces are:

\[
\begin{align*}
\mathfrak{g}_{\epsilon_i + \epsilon_j} &= \mathbb{C}(E_{i-j} - E_{j-i}) \\
\mathfrak{g}_{-\epsilon_i - \epsilon_j} &= \mathbb{C}(E_{-j,i} - E_{-i,j}) \\
\mathfrak{g}_{\epsilon_i - \epsilon_j} &= \mathbb{C}(E_{i,j} - E_{-j,i}) \\
\mathfrak{g}_{\epsilon_i} &= \mathbb{C}(E_{i,0} - E_{0,-i}) \\
\mathfrak{g}_{-\epsilon_i} &= \mathbb{C}(E_{0,i} - E_{-i,0}).
\end{align*}
\]

The standard Borel subalgebra \( \mathfrak{b}_+ := \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha \) is easily seen to be the set of upper triangular matrices of the form \((2.11)\).

2.3. Real Rank 1 symmetric subalgebras. In this section, we recall the involutions realizing \( \mathfrak{gl}(n-1, \mathbb{C}) \oplus \mathfrak{gl}(1, \mathbb{C}) \) and \( \mathfrak{so}(n-1, \mathbb{C}) \) as symmetric subalgebras of \( \mathfrak{gl}(n, \mathbb{C}) \) and \( \mathfrak{so}(n-1, \mathbb{C}) \) respectively. For \( \mathfrak{g} = \mathfrak{gl}(n, \mathbb{C}) \), we let \( \theta = \text{Ad}(t) \), where \( t \in GL(n, \mathbb{C}) \) is the diagonal matrix, \( t = \text{diag}[1, \ldots, 1, -1] \). Note that \( \mathfrak{t} \) is the set of block diagonal matrices \( \mathfrak{t} = \mathfrak{gl}(n-1, \mathbb{C}) \oplus \mathfrak{gl}(1, \mathbb{C}) \). In the case \( \mathfrak{g} = \mathfrak{so}(2l+1, \mathbb{C}) \), we have \( \mathfrak{t} = \mathfrak{so}(2l, \mathbb{C}) = \mathfrak{g}^{\theta_{2l+1}} \) where \( \theta_{2l+1} = \text{Ad}(t) \), where \( t \) is an element of the Cartan subgroup with Lie algebra \( \mathfrak{h} \) with the property that \( \text{Ad}(t)|\mathfrak{g}_{\epsilon_i} = \text{id} \) for \( i = 1, \ldots, l-1 \) and \( \text{Ad}(t)|\mathfrak{g}_{-\epsilon_i} = -\text{id} \) (see [Kna02], p. 700). In the case \( \mathfrak{g} = \mathfrak{so}(2l, \mathbb{C}) \), \( \mathfrak{t} = \mathfrak{so}(2l-1, \mathbb{C}) = \mathfrak{g}^{\theta_{2l}} \), where \( \theta_{2l} \) is the involution induced by the diagram automorphism interchanging the simple roots \( \alpha_{l-1} \) and \( \alpha_l \) (see [Kna02], p. 703). Note that in this case, \( \theta_{2l}(\epsilon_i) = -\epsilon_i \) and \( \theta_{2l}(\epsilon_i) = \epsilon_i \) for \( i = 1, \ldots, l-1 \).

We also denote the corresponding involution on \( G = GL(n, \mathbb{C}) \) or \( G = SO(n, \mathbb{C}) \) by \( \theta \). For the case \( G = GL(n, \mathbb{C}) \), \( G^\theta \) consists of invertible block diagonal matrices \( G^\theta = GL(n-1, \mathbb{C}) \times GL(1, \mathbb{C}) \). For the case \( G = SO(n, \mathbb{C}) \), \( G^\theta = SO(n-1, \mathbb{C}) \times O(1, \mathbb{C}) \) is disconnected. We let \( K := (G^\theta)^0 \) be the identity component of \( G^\theta \). Then \( K = SO(n-1, \mathbb{C}) \), and \( \text{Lie}(K) = \mathfrak{k} = \mathfrak{g}^\theta \).

2.4. Notation. We now lay out some of the notation that we will use throughout the paper.

**Notation 2.1.**

(1) For the symmetric pair \((\mathfrak{gl}(n, \mathbb{C}), \mathfrak{gl}(n-1, \mathbb{C}) \oplus \mathfrak{gl}(1, \mathbb{C}))\), we will often replace the algebra \( \mathfrak{gl}(n-1, \mathbb{C}) \oplus \mathfrak{gl}(1, \mathbb{C}) \) by \( \mathfrak{gl}(n-1, \mathbb{C}) \), the upper left corner of \( \mathfrak{gl}(n, \mathbb{C}) \). Abusing notation, we often refer to this algebra as \( \mathfrak{t} \) and the corresponding algebraic subgroup of \( GL(n, \mathbb{C}) \) as \( K = GL(n-1, \mathbb{C}) \). We lose no information by doing this, since the orbits of \( GL(n-1, \mathbb{C}) \) on the flag variety \( \mathcal{B} \) of \( \mathfrak{g} \) are the same as the orbits of \( GL(n-1, \mathbb{C}) \times GL(1, \mathbb{C}) \). The two groups also have the same orbits on \( \mathfrak{g} \) under conjugation. This convention makes it easier to state results for the pairs \((\mathfrak{gl}(n, \mathbb{C}), \mathfrak{gl}(n-1, \mathbb{C})) \) and \((\mathfrak{so}(n, \mathbb{C}), \mathfrak{so}(n-1, \mathbb{C})) \) simultaneously.
Proposition 2.2. \(^{(1)}\) Recall the well-known fact that the fixed point algebra in invariant theory quotient. We define the partial Kostant-Wallach map to be quotient \(g\). Similarly, \(G = G_n = GL(n, \mathbb{C}), SO(n, \mathbb{C})\) and \(G_{n-1} = K = SO(n-1, \mathbb{C})\).

(2) We let \(r_n := \text{rank}(g)\) and \(r_{n-1} := \text{rank}(g_{n-1})\).

(3) For \(x \in \mathfrak{so}(n, \mathbb{C})\), we let \(x_{\mathfrak{f}}\) the projection of \(x\) onto \(\mathfrak{f}^\theta\), the \(-1\)-eigenspace of \(\theta\). For \(g = \mathfrak{gl}(n, \mathbb{C})\), we let \(x_{\mathfrak{f}}\) be the \((n-1) \times (n-1)\) upper left corner of \(x\).

(4) For any Lie algebra \(g\), we denote by \(\mathbb{C}[g]\), the ring of polynomial functions on \(g\) and by \(\mathbb{C}[g]^G\) the ring of adjoint invariant polynomial functions on \(g\).

2.5. The partial Kostant-Wallach map. For \(i = n-1, n\), let \(\chi_i : g_i \to \mathbb{C}^{r_i}\) be the invariant theory quotient. We define the partial Kostant-Wallach map to be

\[
\Phi_n : g \to \mathbb{C}^{r_{n-1}} \oplus \mathbb{C}^{r_n},
\]

\[
\Phi_n(x) = (\chi_{n-1}(x_{\mathfrak{f}}), \chi_n(x)) = (f_{n-1,1}(x_{\mathfrak{f}}), \ldots, f_{n-1,r_{n-1}}(x_{\mathfrak{f}}), f_{n,1}(x), \ldots, f_{n,r_n}(x)),
\]

where \(\mathbb{C}[g_i]^G = \mathbb{C}[f_{i,1}, \ldots, f_{i,r_i}]\).

Proposition 2.2. \(^{(1)}\) \(\mathbb{C}[g]^K = \mathbb{C}[g]^G \otimes \mathbb{C}[\mathfrak{f}]^K\).

(2) \(\Phi_n\) coincides with the invariant theory quotient morphism \(g \to g//K\). In particular, \(\Phi_n\) is surjective.

(3) The morphism \(\Phi_n\) is flat. In particular, its fibres are equidimensional varieties of dimension \(\dim g - r_n - r_{n-1}\).

Proof. Recall the well-known fact that the fixed point algebra \(U(g)^K\) of \(K\) in the enveloping algebra \(U(g)\) is commutative \((\text{Joh01})\). Hence, \(U(g)^K\) coincides with its center, \(Z(U(g)^K)\). In Theorem 10.1 of \([\text{Kno94}]\), Knop shows that \(Z(U(g)^K) \cong U(g)^G \otimes \mathbb{C} U(g)^K\).

The first assertion now follows by taking the associated graded algebra with respect to the usual filtration of \(U(g)\). By the first assertion, \(\Phi_n\) coincides with the invariant theory quotient \(g \to g//K\), which gives the second assertion. Note that if we embed \(\mathfrak{f}\) diagonally in \(g \times \mathfrak{f}\), then \(\mathfrak{f}^+ \cong g\), and this isomorphism is \(K\)-equivariant. Then the flatness of \(\Phi_n\) follows by Korollar 7.2 of \([\text{Kno90}]\), which gives a criterion for flatness of invariant theory quotients in the setting of spherical homogeneous spaces (see also \([\text{Pan90}]\).

Q.E.D.

For the case \(g = \mathfrak{gl}(n, \mathbb{C})\) and \(\mathfrak{f} = \mathfrak{gl}(n-1, \mathbb{C})\), the same proof works. In that case, we proved this result by a more elementary argument in \([\text{CEa}]\). In the appendix, we use conormal geometry to give a more elementary proof of Korollar 7.2 of \([\text{Kno90}]\) for a class of spherical varieties, which applies to our setting.

2.6. General Properties of eigenvalue coincidence varieties \(g(\geq i)\). In this section, we develop some fundamental facts about the eigenvalue coincidence varieties discussed in the introduction. For the case of \(g = \mathfrak{gl}(n, \mathbb{C})\) and \(\mathfrak{f} = \mathfrak{gl}(n-1, \mathbb{C})\), these properties
were developed in the beginning of Section 3 of \[\text{CEa}\]. However, that discussion is specific to \(\mathfrak{gl}(n, \mathbb{C})\). Here we develop the same results in a manner that covers both \(\mathfrak{gl}(n, \mathbb{C})\) and \(\mathfrak{so}(n, \mathbb{C})\).

Let \(\mathfrak{h}_n\) be the Cartan subalgebra of diagonal matrices in \(\mathfrak{g}\), and let \(\mathfrak{h}_{n-1} \subset \mathfrak{k}\) be the Cartan subalgebra of diagonal matrices in \(\mathfrak{k}\). We denote elements of \(\mathfrak{h}_{n-1} \times \mathfrak{h}_n\) by \((a, b)\), where \(a = (a_1, \ldots, a_{r_{n-1}})\) and \(b = (b_1, \ldots, b_r)\) represent the diagonal coordinates of \(a \in \mathfrak{h}_{n-1}\) and \(b \in \mathfrak{h}_n \subset \mathfrak{g}\) as in Section \[2\] above. Let \(W_n = W(\mathfrak{g}, \mathfrak{h})\) be the Weyl group of \(\mathfrak{g}\), and let \(W_{n-1} = W(\mathfrak{k}, \mathfrak{h})\) be the Weyl group of \(\mathfrak{k}\). For \(i = 1, \ldots, r_{n-1}\) define:

\[
(\mathfrak{h}_{n-1} \times \mathfrak{h}_n)(\geq i) := \{ (a, b) : \exists v \in W_{n-1}, u \in W_n \text{ such that } (v \cdot a)_j = (u \cdot b)_j, \ j = 1, \ldots, i \}.
\]

We note that \((\mathfrak{h}_{n-1} \times \mathfrak{h}_n)(\geq i)\) is a \(W_{n-1} \times W_n\)-invariant closed subvariety of \(\mathfrak{h}_{n-1} \times \mathfrak{h}_n\) and is equidimensional of codimension \(i\). Let \(p_i : \mathfrak{h}_i \rightarrow \mathfrak{h}_i/W_i\) for \(i = n - 1, n\) be the invariant theory quotient. Consider the finite morphism \(p := p_{n-1} \times p_n : (\mathfrak{h}_{n-1} \times \mathfrak{h}_n) \rightarrow (\mathfrak{h}_{n-1} \times \mathfrak{h}_n)/(W_{n-1} \times W_n)\). Let \(F_i : \mathfrak{h}_i/W_i \rightarrow \mathbb{C}^{r_i}\) be the Chevalley isomorphism, and let

\[
V^{r_{n-1},r_n} := \mathbb{C}^{r_{n-1}} \times \mathbb{C}^{r_n},
\]

so that \(F_{n-1} \times F_n : (\mathfrak{h}_{n-1} \times \mathfrak{h}_n)/(W_{n-1} \times W_n) \rightarrow V^{r_{n-1},r_n}\) is an isomorphism. The following varieties play a major role in our study of orthogonal eigenvalue coincidences.

**Definition 2.3.** For \(i = 0, \ldots, r_{n-1}\), we let

\[
V^{r_{n-1},r_n}(\geq i) := (F_{n-1} \times F_n)((\mathfrak{h}_{n-1} \times \mathfrak{h}_n)(\geq i))/(W_{n-1} \times W_n)),
\]

\[
V^{r_{n-1},r_n}(i) := V^{r_{n-1},r_n}(\geq i) \setminus V^{r_{n-1},r_n}(\geq i + 1).
\]

For convenience, we let \(V^{r_{n-1},r_n}(r_{n-1} + 1) = \emptyset\).

**Lemma 2.4.** The set \(V^{r_{n-1},r_n}(\geq i)\) is an irreducible closed subvariety of \(V^{r_{n-1},r_n}\) of dimension \(r_n + r_{n-1} - i\). Further, \(V^{r_{n-1},r_n}(i)\) is open and dense in \(V^{r_{n-1},r_n}(\geq i)\).

**Proof.** Indeed, the set

\[
Y := \{(a, b) \in \mathfrak{h}_{n-1} \times \mathfrak{h}_n : a_j = b_j \text{ for } j = 1, \ldots, i\}
\]

is closed and irreducible of dimension \(r_n + r_{n-1} - i\). The first assertion follows since \((F_{n-1} \times F_n) \circ p\) is a finite morphism and \((F_{n-1} \times F_n) \circ p(Y) = V^{r_{n-1},r_n}(\geq i)\). The last assertion of the lemma now follows from Equation \[2.18\].

Q.E.D.

We define

\[
\mathfrak{g}(\geq i) := \Phi_n^{-1}(V^{r_{n-1},r_n}(\geq i)),
\]

where \(\Phi_n : \mathfrak{g} \rightarrow \mathbb{C}^{r_{n-1}} \times \mathbb{C}^{r_n}\) is the orthogonal partial Kostant-Wallach map defined in Equation \[2.16\].

It is routine to check that

\[
\mathfrak{g}(i) := \mathfrak{g}(\geq i) \setminus \mathfrak{g}(\geq i + 1) = \Phi_n^{-1}(V^{r_{n-1},r_n}(i)).
\]
Remark 2.5. Let \( \mathfrak{g} = \mathfrak{so}(n, \mathbb{C}) \) and \( \mathfrak{k} = \mathfrak{so}(n-1, \mathbb{C}) \). The definition of \( \mathfrak{g}(\geq i) \) in (2.20) agrees with the one we gave in (1.2). We recall that \( \Phi_n := (\chi_{n-1}, \chi_n) \), where \( \chi_i : \mathfrak{so}(i, \mathbb{C}) \to \mathbb{C}^{r_i} \) is the adjoint quotient. For \( x \in \mathfrak{g} \), \( \chi_n(x) = F_n \circ p_n(b_1, \ldots, b_{r_{n-1}}) \), and \( \chi_{n-1}(x_t) = F_{n-1} \circ p_{n-1}(a_1, \ldots, a_{r_{n-1}}) \), where \( \{ \pm b_1, \ldots, \pm b_{r_{n-1}} \} = \sigma(x) \) is the spectrum of \( x \) and \( \{ \pm a_1, \ldots, \pm a_{r_{n-1}} \} = \sigma(x_t) \) is the spectrum of \( x_t \) as defined in Equation (1.2). Thus, if \( x \) is in the variety defined in Equation (1.2), \( \Phi_n(x) = (\chi_t(x_t), \chi(x)) \in V^{r_{n-1}, r_n}(\geq i) \).

Conversely, if \( x \in \Phi_n^{-1}(V^{r_{n-1}, r_n}(\geq i)) \), it follows from the definition of \( V^{r_{n-1}, r_n}(\geq i) \) in (2.17) and the definition of \( \Phi_n \) that \( x \) is in the variety defined in (1.3). Thus, the variety \( \mathfrak{g}(\geq i) \) defined in (2.20) consists exactly of the elements of \( \mathfrak{g} \) with at least \( 2i \) coincidences in the spectrum of \( x \) and \( x_t \).

We now use the flatness of the Kostant-Wallach morphism asserted in Proposition 2.2 to study the varieties \( \mathfrak{g}(\geq i) \).

Proposition 2.6. (1) The variety \( \mathfrak{g}(\geq i) \) is equidimensional of dimension \( \dim \mathfrak{g} - i \).

(2) \( \overline{\mathfrak{g}(i)} = \mathfrak{g}(\geq i) = \bigcup_{k \geq i} \mathfrak{g}(k) \).

Proof. By Proposition 2.2, the morphism \( \Phi_n \) is flat. By Proposition III.9.5 and Corollary III.9.6 of [Har77], the variety \( \mathfrak{g}(\geq i) \) is equidimensional of dimension \( \dim(V^{r_{n-1}, r_n}(\geq i)) + \dim \mathfrak{g} - r_n - r_{n-1} \), which gives the first assertion by Lemma 2.4. For the second assertion, by the flatness of \( \Phi_n \), Theorem VIII.4.1 of [Gro03], and Lemma 2.4,

\[
\overline{\mathfrak{g}(i)} = \Phi_n^{-1}(V^{r_{n-1}, r_n}(i)) = \Phi_n^{-1}(V^{r_{n-1}, r_n}(\geq i)) = \mathfrak{g}(i).
\]

The remaining equality follows since \( V^{r_{n-1}, r_n}(\geq i) = \bigcup_{k \geq i} V^{r_{n-1}, r_n}(k) \).

Q.E.D.

2.7. The varieties \( Y_Q \). We now study the geometry of the varieties \( Y_Q = \text{Ad}(K)b \) for a \( K \)-orbit \( Q = K \cdot b \in \mathcal{B} \). We begin by studying more general objects \( Y_{Q_T} \), where \( Q_T \) is a \( K \)-orbit in a partial flag variety. For a parabolic subgroup \( P \subset G \) with Lie algebra \( \mathfrak{p} \subset \mathfrak{g} \), consider the partial Grothendieck resolution \( \tilde{\mathfrak{g}}^\mathfrak{p} = \{(x, v) \in \mathfrak{g} \times G/P \mid x \in \mathfrak{t} \} \), as well as the morphisms \( \mu : \tilde{\mathfrak{g}}^\mathfrak{p} \to \mathfrak{g} \), \( \mu(x, v) = x \), and \( \pi : \tilde{\mathfrak{g}}^\mathfrak{p} \to G/P \), \( \pi(x, v) = v \). Then \( \pi \) is a smooth morphism of relative dimension \( \dim \mathfrak{p} \) (for \( G/P \), see Section 3.1 of [CG97] and Proposition III.10.4 of [Har77], and the general case of \( G/P \) follows by the same argument). For \( v \in G/P \), let \( Q_T = K \cdot v \subset G/P \). Then \( \pi^{-1}(Q_T) \) has dimension \( \dim(Q_T) + \dim(\mathfrak{t}) \). It is well-known that \( \mu \) is proper and its restriction to \( \pi^{-1}(Q_T) \) generically has finite fibres (Proposition 3.1.34 and Example 3.1.35 of [CG97] for the case of \( G/B \), and again the general case has a similar proof).

Notation 2.7. For a parabolic subalgebra \( \mathfrak{t} \) with \( K \)-orbit \( Q_T \subset G/P \), we consider the irreducible subset

\[
(2.23) \quad Y_T := \mu(\pi^{-1}(Q_T)) = \text{Ad}(K)\mathfrak{t}.
\]

\( Y_T \) depends only on \( Q_T \), and we will also denote this set as

\[
(2.24) \quad Y_{Q_T} := Y_T.
\]
It follows from generic finiteness of $\mu$ that $Y_{Q_{\mathfrak{r}}}$ contains an open subset of dimension
\begin{equation}
\dim(Y_{Q_{\mathfrak{r}}}) := \dim \pi^{-1}(Q_{\mathfrak{r}}) = \dim \mathfrak{r} + \dim(Q_{\mathfrak{r}}) = \dim \mathfrak{r} + \dim(\mathfrak{t}/\mathfrak{t} \cap \mathfrak{r}).
\end{equation}

**Remark 2.8.** Since $\mu$ is proper, the set $Y_{Q_{\mathfrak{r}}}$ is closed when $Q_{\mathfrak{r}} = K \cdot \mathfrak{r}$ is closed in $G/P$.

**Remark 2.9.** Note that
\[ \mathfrak{g} = \bigcup_{Q \subset G/P} Y_Q, \]
where the union is taken over the finitely many $K$-orbits in $G/P$.

**Lemma 2.10.** Let $Q \subset G/P$ be a $K$-orbit. Then
\begin{equation}
\overline{Y_Q} = \bigcup_{Q' \subset Q} Y_{Q'}.
\end{equation}

**Proof.** Since $\pi$ is a smooth morphism, it is flat by Theorem III.10.2 of [Har77]. Thus, by Theorem VIII.4.1 of [Gro03], $\pi^{-1}(Q) = \pi^{-1}(\overline{Q})$. The result follows since $\mu$ is proper. Q.E.D.

Using Proposition 2.6, we can easily prove:

**Proposition 2.11.** Let $Q$ be a $K$-orbit in $B$ with $\text{codim}(Q) = i$. Then
\begin{equation}
\dim Y_Q = \dim \mathfrak{g}(\geq i).
\end{equation}

**Proof.** By Equation (2.25), it follows that
\[ \dim Y_Q = \dim Q + \dim \mathfrak{b} = \dim(B) - i + \dim(\mathfrak{b}) = \dim(\mathfrak{g}) - i. \]
The assertion follows by Part (1) of Proposition 2.6. Q.E.D.

To see that $\overline{Y_Q}$ with $\text{codim}(Q) = i$ is an irreducible component of $\mathfrak{g}(\geq i)$, it remains to show that $\overline{Y_Q} \subset \mathfrak{g}(\geq i)$. For this, it is convenient to replace the orbit $K$-orbit $Q$ in $G/B$ with a $K$-orbit $Q_{\mathfrak{r}}$ of a $\theta$-stable parabolic subalgebra $\mathfrak{r} \supset \mathfrak{b}$ in a partial flag variety $G/P$. We show that $\mathfrak{r} \in G/P$ can be chosen so that $\overline{Y_Q} = Y_{Q_{\mathfrak{r}}}$, and $Y_{Q_{\mathfrak{r}}} \subset \mathfrak{g}(\geq i)$. The first step is to relate the geometry of $Q$ and $Q_{\mathfrak{r}}$ for a general $\theta$-stable $\mathfrak{r}$ and develop a necessary condition for $\overline{Y_Q} = Y_{Q_{\mathfrak{r}}}$.

Consider the canonical fibration:
\[ P/B \to G/B \xrightarrow{p} G/P. \]
This fibration induces a fibration on the $K$-orbit $Q$
\begin{equation}
Q \cap p^{-1}(Q_{\mathfrak{r}}) \to Q \to Q_{\mathfrak{r}}
\end{equation}
To study the fibration (2.28), we consider the $\theta$-stable parabolic subalgebra $\mathfrak{r}$ in more detail. It follows from Theorem 2 of [BH00] that $\mathfrak{r}$ has a $\theta$-stable Levi decomposition.
\( \mathfrak{r} = \mathfrak{l} \oplus \mathfrak{u} \). The Levi subalgebra decomposes as \( \mathfrak{l} = \mathfrak{z} \oplus \mathfrak{l}_{ss} \), with the centre \( \mathfrak{z} \) and the semisimple part \( \mathfrak{l}_{ss} = [\mathfrak{l}, \mathfrak{l}] \) both \( \theta \)-stable. Further, \( \mathfrak{k} \cap \mathfrak{r} \) is a parabolic subalgebra of \( \mathfrak{k} \) with Levi decomposition

\[ \mathfrak{k} \cap \mathfrak{r} = \mathfrak{k} \cap \mathfrak{l} \oplus \mathfrak{k} \cap \mathfrak{u} = \mathfrak{k} \cap \mathfrak{z} \oplus \mathfrak{k} \cap \mathfrak{l}_{ss} \oplus \mathfrak{k} \cap \mathfrak{u}. \]

Let \( R \) be the parabolic subgroup of \( G \) with Lie algebra \( \mathfrak{r} \). Then \( R \) is \( \theta \)-stable and \( K \cap R \) is a parabolic subalgebra of \( K \) with Levi decomposition \((K \cap \mathfrak{l}_{ss}) \cdot (K \cap \mathfrak{Z}) \cdot K \cap \mathfrak{U} \) (see Theorem 2.12). In particular, \( Q_{\mathfrak{r}} \cong K/(K \cap R) \) is closed. Recall that \( R/B \cong \mathcal{B}_{\mathfrak{l}_{ss}} \), where \( \mathcal{B}_{\mathfrak{l}_{ss}} \) denotes the flag variety of \( \mathfrak{l}_{ss} \). Thus, the fibration (2.28) gives the \( K \)-orbit \( Q \) on \( \mathcal{B} \) the structure of a \( K \)-homogeneous fibre bundle over the closed \( K \)-orbit \( Q_{\mathfrak{r}} = K \cdot \mathfrak{r} \) in \( G/P \) with fibre the \( K \cap \mathfrak{l}_{ss} \)-orbit of \( \mathfrak{b} \) in \( R/B \cong \mathcal{B}_{\mathfrak{l}_{ss}} \), i.e.:

\[ Q \cong K \times_{K \cap R} (K \cap \mathfrak{l}_{ss}) \cdot \mathfrak{b}. \]

**Proposition 2.12.** Suppose that the orbit \( K \cap \mathfrak{l}_{ss} \cdot \mathfrak{b} \) in (2.29) is open in \( R/B \cong \mathcal{B}_{\mathfrak{l}_{ss}} \). Then \( \dim Y_{\mathfrak{b}} = \dim Y_{\mathfrak{r}} \). Further, \( Y_{\mathfrak{r}} \) is a closed, irreducible subvariety of \( \mathfrak{g} \), so that \( Y_{\mathfrak{b}} = Y_{\mathfrak{r}} \).

**Proof.** Indeed,

\[
\dim Y_{\mathfrak{r}} = \dim Q_{\mathfrak{r}} + \dim \mathfrak{r} \quad \text{(by (2.25))}
\]

\[
= \dim Q - \dim \mathcal{B}_{\mathfrak{l}_{ss}} + \dim \mathfrak{r} \quad \text{(by (2.29))}
\]

\[
= \dim Q - \dim \mathcal{B}_{\mathfrak{l}_{ss}} + \dim \mathcal{B}_{\mathfrak{l}_{ss}} + \dim \mathfrak{b}
\]

\[
= \dim Q + \dim \mathfrak{b}
\]

\[
= \dim Y_{\mathfrak{b}} \quad \text{(by (2.25))}.
\]

It follows from definitions that \( Y_{\mathfrak{b}} \subset Y_{\mathfrak{r}} \). Since \( Q_{\mathfrak{r}} \) is closed, \( Y_{\mathfrak{r}} \) is closed by Remark 2.8. Thus, \( Y_{\mathfrak{b}} = Y_{\mathfrak{r}} \) since \( Y_{\mathfrak{r}} \) is irreducible.

Q.E.D.

In the case, where \( \mathfrak{g} = \mathfrak{so}(n, \mathbb{C}) \), \( \mathfrak{k} = \mathfrak{so}(n-1, \mathbb{C}) \), we show that for any Borel subalgebra \( \mathfrak{b} \subset \mathcal{B} \), we can find a \( \theta \)-stable parabolic subalgebra \( \mathfrak{r} \) with \( \mathfrak{b} \subset \mathfrak{r} \) such that the hypothesis of Proposition 2.12 is satisfied. To do this, we need to classify the \( K \)-orbits on \( \mathcal{B} \) and develop explicit descriptions of representatives of the \( K \)-orbits on \( \mathcal{B} \).

**Remark 2.13.** When \( \mathfrak{g} = \mathfrak{gl}(n, \mathbb{C}) \) and \( \mathfrak{k} = \mathfrak{gl}(n-1, \mathbb{C}) \oplus \mathfrak{gl}(1, \mathbb{C}) \) for any Borel subalgebra \( \mathfrak{b} \subset \mathfrak{g} \), we can find a \( \theta \)-stable parabolic subalgebra \( \mathfrak{r} \) with \( \mathfrak{b} \subset \mathfrak{r} \) so that \( (K \cap \mathfrak{l}_{ss}) \cdot \mathfrak{b} \) is open in \( \mathcal{B}_{\mathfrak{l}_{ss}} \). This is implicit in Lemma 3.5, [CE12] and in the computations of Proposition 2.15, [CEa].

### 2.8. Description of \( K \)-orbits on \( \mathcal{B} \) in the orthogonal case.

We classify the \( K \)-orbits on \( \mathcal{B} \) for \( \mathfrak{g} = \mathfrak{so}(n, \mathbb{C}) \) and \( \mathfrak{k} = \mathfrak{so}(n-1, \mathbb{C}) \). In particular, we explain how to recover the orbit diagrams from [Col85], Figure 4.3, but also give explicit representatives of each orbit for later use.
We recall some generalities regarding an involution \( \theta \) of a semisimple Lie algebra \( g \) and orbits of \( K = G^\theta \) on the flag variety \( \mathcal{B} \) of \( g \) (see [Mat79], [RS], [Vg], or [CEexp] for more details). Each Borel subalgebra \( b \) of \( g \) contains a \( \theta \)-stable Cartan subalgebra \( t \). Let \( \Phi(t, g) \) denote the roots of \( t \) in \( g \), and let \( \Phi^+_b \) denote the roots of \( t \) in \( b \), which we take to be the positive roots. Then the \( \theta \)-action on \( t \) induces an action on \( \Phi(t, g) \). Using this action, we define the type of a root \( \alpha \in \Phi^+_b \) as follows. A root \( \alpha \) is called real if \( \theta(\alpha) = -\alpha \), imaginary if \( \theta(\alpha) = \alpha \), and complex if \( \theta(\alpha) \neq \pm \alpha \). If \( \alpha \) is imaginary, then \( \alpha \) is called compact if \( \theta|_{g_\alpha} = \text{id} \) and non-compact if \( \theta|_{g_\alpha} = -\text{id} \). If \( \alpha \) is complex and positive, then \( \alpha \) is called complex \( \theta \)-stable if \( \theta(\alpha) \) is positive, and otherwise is called complex \( \theta \)-unstable. These notions do not depend on the choice of \( \theta \)-stable Cartan subalgebra \( t \subset b \).

**Example 2.14.** Let \( g = \mathfrak{so}(2l + 1, \mathbb{C}) \) and \( t = \mathfrak{so}(2l, \mathbb{C}) \), and let \( \theta = \text{Ad}(t) \) be as in Section 2.3. Let \( \Phi(g, h) \) be the set of standard roots of \( g \) as in (2.12). The roots \( \{\pm(\epsilon_i - \epsilon_j), \pm(\epsilon_i + \epsilon_j), 1 \leq i < j \leq l\} \) are compact imaginary, and the roots \( \{\pm\epsilon_i = 1, \ldots, l\} \) are non-compact imaginary.

Now let \( g = \mathfrak{so}(2l, \mathbb{C}) \) and \( t = \mathfrak{so}(2l - 1, \mathbb{C}) \) and \( \theta \) is as in Section 2.3. Then the simple roots \( \alpha_{l-1} = \epsilon_{l-1} - \epsilon_l \) and \( \alpha_l = \epsilon_{l-1} + \epsilon_l \) are complex \( \theta \)-stable with \( \theta(\alpha_{l-1}) = \alpha_l \). Note that we can choose as a representative for \( \theta \) a nontrivial element in the Weyl group of \( GL(\mathbb{C}e_l + \mathbb{C}e_{l-1}) \). Therefore, the roots \( \{\pm(\epsilon_i + \epsilon_j), 1 \leq i < j \leq l-1\} \) are complex imaginary, whereas the roots \( \{\pm(\epsilon_i + \epsilon_l), \pm(\epsilon_i - \epsilon_l), 1 \leq i \leq l-1\} \) are complex \( \theta \)-stable with \( \theta(\epsilon_i - \epsilon_l) = \epsilon_i + \epsilon_l \). Note that if \( \alpha \) is a complex \( \theta \)-stable root, then the subspace \( g_\alpha \oplus g_{\theta(\alpha)} \) is \( \theta \)-stable. Thus, \( g_\alpha \oplus g_{\theta(\alpha)} \) decomposes as \( g_\alpha \oplus g_{\theta(\alpha)} = (g_\alpha \oplus g_{\theta(\alpha)}) \cap t \oplus (g_\alpha \oplus g_{\theta(\alpha)}) \cap g^{-\theta} \).

Let \( Q = K \cdot b \) and suppose that \( \alpha \in \Phi^+_b \) is a positive root for \( b \). The type of the root \( \alpha \) with respect to \( \theta \) is used to construct a \( K \)-orbit \( Q' \) from \( Q \), which contains \( Q \) in its closure as a divisor. Let \( \alpha \) be a simple root, let \( P_\alpha \) be the variety of parabolic subalgebras of type \( \alpha \), and consider the projection \( \pi_\alpha : \mathcal{B} \to P_\alpha \). For a \( K \)-orbit \( Q = K \cdot b \), let \( m(s_\alpha) \cdot Q \) be the unique \( K \)-orbit of maximal dimension in \( \pi_\alpha^{-1} \pi_\alpha(Q) \). For each simple root \( \alpha \), choose root vectors \( e_\alpha \in g_\alpha, f_\alpha \in g_{-\alpha} \), and let \( h_\alpha = [e_\alpha, f_\alpha] \). Choose a Lie algebra homomorphism \( \phi_\alpha : \mathfrak{sl}(2, \mathbb{C}) \to g \) such that:

\[
(2.31) \quad \phi_\alpha : \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \to e_\alpha, \quad \phi_\alpha : \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \to f_\alpha, \quad \phi_\alpha : \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \to h_\alpha
\]

Let \( s_\alpha = \phi_\alpha(\mathfrak{sl}(2, \mathbb{C})) \). Also denote by \( \phi_\alpha : SL(2, \mathbb{C}) \to G \) the induced Lie group homomorphism, and let

\[
(2.32) \quad u_\alpha = \phi_\alpha \left( \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \right).
\]

We make use of the following notation throughout the paper.

**Notation 2.15.** Let \( T \) be the maximal torus with Lie algebra \( t \), and let \( W \) be the Weyl group with respect to \( T \). For an element \( w \in W \), let \( \bar{w} \in N_G(T) \) be a representative of \( w \). If \( t \subset b \), with \( b \in \mathcal{B} \), then \( \text{Ad}(\bar{w})b \) is independent of the choice of representative \( \bar{w} \) of \( w \), and we denote it by \( w(b) \).
Lemma 2.16. [CE6, Proposition 4.21]
Let $Q = K \cdot b$ be a $K$-orbit on $B$.

1. $m(s_\alpha) \cdot Q = Q$ unless $\alpha$ is either noncompact or $\theta$-stable, and when $m(s_\alpha) \cdot Q \neq Q$, then $\dim(m(s_\alpha) \cdot Q) = \dim(Q) + 1$.

2. If $\alpha$ is noncompact for $Q$, then $m(s_\alpha) \cdot Q = K \cdot \text{Ad}(u_\alpha)b$ and the $K$-orbits in $\pi^{-1}_\alpha(Q)$ are $Q, m(s_\alpha) \cdot Q$, and $K \cdot s_\alpha(b)$. Further, $m(s_\alpha) \cdot K \cdot s_\alpha(b) = m(s_\alpha) \cdot Q$.

3. If $\alpha$ is complex $\theta$-stable for $Q$, then $m(s_\alpha) \cdot b = K \cdot s_\alpha(b)$, and $\pi^{-1}_\alpha(Q)$ consists of $Q$ and $m(s_\alpha) \cdot Q$.

The action by operators $m(s_\alpha)$ on $K$-orbits is called the monoidal action. Starting with the closed $K$-orbits on $B$ and using the monoidal action, we can construct all $K$-orbits on $B$.

Lemma 2.17. Every $K$-orbit on $B$ is of the form $m(s_{\beta_1}) \cdots m(s_{\beta_k}) \cdot b_1$, where $K \cdot b_1$ is a closed $K$-orbit on $B$, $k \geq 0$, and $\beta_1, \ldots, \beta_k$ are simple roots.

We now briefly recall the classification of closed $K$-orbits on $B$ described in Section 4.3, [CE12]. Let $K \cdot b_0$ be a closed $K$-orbit with $b_0$ containing a $\theta$-stable Cartan subalgebra $\mathfrak{k}$ corresponding to a maximal torus $T$. Since $T$ is $\theta$-stable, $\theta$ acts naturally on the Weyl group $W = N_G(T)/T$. Further, the subgroup $T \cap K$ is a Cartan subgroup of $K$ (Lemma 5.1, [Ric82]), and the Weyl group $W_K = N_K(T \cap K)/(T \cap K)$ embeds into $W$ (Lemma 5.3, [Ric82]), and is contained in $W^\theta$, the fixed points of $\theta$ on $W$.

Lemma 2.18. [CE6, Theorem 4.10] The map $W^\theta/W_K \to K\mathcal{B}$ given by $wW_K \mapsto K \cdot w^{-1} \cdot (b_0)$ is a bijection to the closed $K$-orbits on $B$.

We now return to the case where $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$ and $\mathfrak{k} = \mathfrak{so}(n-1, \mathbb{C})$ and use Lemma 2.18 to determine the closed $K$-orbits on $B$. Before doing that, we state a result on the relation between $W$ and $W_K$, which we will also need later. Recall that when $\mathfrak{g} = \mathfrak{so}(2l+1, \mathbb{C})$, $W = S_l \rtimes U_l$, where $U_l$ is the group generated by sign changes $\tau_i, 1 \leq i \leq l$ in the root system, with $\tau_i(\epsilon_i) = -\epsilon_i$ and $\tau_i(\epsilon_j) = \epsilon_j$ for $j \neq i$. When $\mathfrak{g} = \mathfrak{so}(2l, \mathbb{C})$, $W = S_l \rtimes T_l$, where $T_l$ is the subgroup of $U_l$ generated by products $\tau_i \tau_j, 1 \leq i < j \leq l$ with $i \neq j$.

Proposition 2.19. (1) Let $\mathfrak{g} = \mathfrak{so}(2l+1, \mathbb{C})$ and let $\mathfrak{k} = \mathfrak{so}(2l, \mathbb{C})$. Then $W_K$ has index 2 in $W = W^\theta$ and $W/W_K = \{eW_K, s_\alpha W_K\}$, where $e$ denotes the identity element in $W$. Further, $W_K$ is the subgroup $S_l \rtimes T_l$ of $W$.

(2) Let $\mathfrak{g} = \mathfrak{so}(2l, \mathbb{C})$ and $\mathfrak{k} = \mathfrak{so}(2l-1, \mathbb{C})$. Then $W^\theta$ is the subgroup of $W$ generated by the elements $s_{\alpha_l}, \ldots, s_{\alpha_{l-2}}, s_{\alpha_{l-1}} \cdot s_{\alpha_l}$, and $W^\theta = W_K$.

Proof. In (1), $\theta = \text{Ad}(t)$ is inner, so it is easy to check that $W^\theta = W$. The assertion that $W_K$ has index two in $W$ follows by the above remarks on the Weyl groups, and since $\theta = \text{Ad}(t)$, one easily checks that the roots $\epsilon_i, \epsilon_j$ are exactly the roots of $\mathfrak{k}$ for the diagonal Cartan subalgebra of $\mathfrak{g}$, which is also a Cartan subalgebra of $\mathfrak{k}$ (see Example 2.14). The rest of (1) follows easily.
For (2), the first statement follows from 1.32(b) in [Ste68]. The second statement can be deduced from the proof of (5) in Theorem 8.2 of [Ste68], but we provide a more direct proof. An easy calculation shows that for \( \sigma \in S_l \), \( \theta \sigma \theta^{-1} = \sigma \cdot \tau_{\sigma^{-1}(1)} \cdot \tau_1 \). Note that \( T_l \) is commutative and \( \theta \) acts trivially on \( T_l \). It follows that \( W^\theta \) is identified with the semi-direct product of \( S_{l-1} \) with \( T_l \). Since \( W_K \subset W^\theta \), the second statement follows.

Q.E.D.

Now it remains to describe the monoidal action. To determine the type of a root for a \( K \)-orbit, \( Q = K \cdot b \), it is convenient to replace the involution \( \theta \) by another involution \( \theta_Q \), which preserves the standard Cartan subalgebra of diagonal matrices \( h \) and thus acts on the standard root system \( \Phi(\mathfrak{g}, \mathfrak{h}) \). Suppose that \( b = \text{Ad}(v) b_+ \), where \( b_+ \) is the standard Borel subalgebra of upper triangular matrices. Then

\[
\theta_Q := \text{Ad}(v^{-1}) \circ \theta \circ \text{Ad}(v).
\]

It is easy to check that the action of \( \theta_Q \) on \( \Phi^+(\mathfrak{g}, \mathfrak{h}) \) is independent of the choice of representative for \( Q \), and that the type of a standard positive root \( \alpha \in \Phi^+(\mathfrak{g}, \mathfrak{h}) \) with respect to \( \theta_Q \) is the same as the type of the positive root \( \text{Ad}(v) \alpha = \alpha \circ \text{Ad}(v^{-1}) \) for \( b \) with respect to \( \theta \). (See Definition 4.6 and Proposition 4.7 of [CEb].) In Section 4.4, [CEb], we give an inductive method of constructing the involution \( \theta_{m(s_\alpha), Q} \) from the involution \( \theta_Q \) (Propositions 4.27 and 4.28).

**Proposition 2.20.** Let \( \mathfrak{g} = \mathfrak{so}(2l+1, \mathbb{C}) \) and \( \mathfrak{t} = \mathfrak{so}(2l, \mathbb{C}) \).

1. There are exactly \( l + 2 \) \( K \)-orbits on the flag variety \( \mathcal{B} \) of \( \mathfrak{g} \).
2. We let \( b_+ \) be the upper triangular matrices in \( \mathfrak{g} \), and let \( b_- = s_\alpha(b_+) \). Exactly two \( K \)-orbits on \( \mathcal{B} \) are closed, and they are \( Q_+ := K \cdot b_+ \) and \( Q_- = K \cdot b_- \). Further, \( m(s_\alpha) \cdot Q_+ = m(s_\alpha) \cdot Q_- = K \cdot \text{Ad}(u_\alpha)b_+ \).
3. The non-closed orbits are of the form

\[
Q_i := m(s_{\alpha_{i+1}}) \cdot m(s_{\alpha_{i+2}}) \cdots m(s_{\alpha_{i-1}}) \cdot m(s_\alpha) \cdot Q_+
\]

for \( i = 0, \ldots, l-1 \). Moreover, the codimension of \( Q_i \) in \( \mathcal{B} \) is \( i \). Further,

\[
b_i = \text{Ad}(u_\alpha) s_{\alpha_{i-1}} s_{\alpha_{i-2}} \cdots s_{\alpha_{i+1}} (b_+) \in Q_i.
\]

In particular, the unique open \( K \)-orbit contains the Borel subalgebra

\[
b_0 = \text{Ad}(u_\alpha) s_{\alpha_{l-1}} s_{\alpha_{l-2}} \cdots s_{\alpha_1}(b_+)
\]

**Proof.** For (2), since \( b_+ \) is \( \theta \)-stable, \( K \cdot b_+ \) is closed by Proposition 4.12, [CEb]. By Lemma 2.18 and Part (1) of Proposition 2.19, there are two closed orbits, and they are \( b_+ \) and \( b_- \). The assertion that \( b_{l-1} = \text{Ad}(u_\alpha)b_+ \in Q_{l-1} \) follows from Part (2) of Lemma 2.16 and Example 2.14.

For (3), first recall that by Proposition 4.27 of [CEb], the involution \( \theta_{Q_{l-1}} \) associated to \( Q_{l-1} = K \cdot b_{l-1} \) is \( \text{Ad}(s_{\alpha_l}^{-1}) \circ \theta \), so the involution on the standard root system is \( s_\alpha \), which is a sign change in the last variable. Since \( \theta = \text{Ad}(t) \), it follows that we can choose
the representative \( \dot{s}_{\alpha} \) so that \( \theta_{Q_{l-1}} \in GL(\mathbb{C}e_l + \mathbb{C}e_{-l}) \). It follows easily that \( \alpha_1, \ldots, \alpha_{l-2} \) are compact for \( \theta_{Q_{l-1}} \), while \( \alpha_{l-1} \) is complex \( \theta_{Q_{l-1}} \)-stable, and \( \alpha_l \) is real. Hence, the only monoidal action which gives us a new orbit is \( m(s_{\alpha_{l-1}}) \cdot Q_{l-1} = Q_{l-2} \). By Part (3) of Lemma 2.16, the Borel subalgebra \( \mathfrak{b}_{l-2} = Ad(\tilde{s}_{\alpha_{l-1}})(\mathfrak{b}_{l-1}) \), where \( \tilde{s}_{\alpha_{l-1}} \) is a representative of the simple reflection \( s_{\alpha_{l-1}} \) defined with respect to \( \mathfrak{b}_{l-1} \). Since \( \mathfrak{b}_{l-1} = Ad(u_{\alpha_l})(\mathfrak{b}_+) \), it follows that \( \tilde{s}_{\alpha_{l-1}} = u_{\alpha_l} \dot{s}_{\alpha_{l-1}} u_{\alpha_l}^{-1} \), where \( \dot{s}_{\alpha_{l-1}} \) is the representative of \( s_{\alpha_{l-1}} \) defined using \( \mathfrak{b}_+ \). Hence, \( \mathfrak{b}_{l-2} = Ad(u_{\alpha_l} \dot{s}_{\alpha_l} u_{\alpha_l}^{-1})(\mathfrak{b}_+) \), which verifies the last part of (3) for \( i = l - 2 \). By Proposition 4.28 of [CEB], the involution \( \theta_{Q_{l-2}} \) associated to \( Q_{l-2} \) is \( \text{Ad}(\tilde{s}_{\alpha_{l-2}})^{-1} \circ \theta_{Q_{l-1}} \circ \text{Ad}(\tilde{s}_{\alpha_{l-2}}) \). We can choose \( \dot{s}_{\alpha_{l-2}} \), so that \( \theta_{Q_{l-2}} \in GL(\mathbb{C}e_{l-1} + \mathbb{C}e_{-(l-1)}) \). Thus, \( \alpha_1, \ldots, \alpha_{l-3} \), and \( \alpha_l \) are compact, while \( \alpha_{l-2} \) is complex \( \theta_{Q_{l-2}} \)-stable and \( \alpha_{l-1} \) is complex \( \theta_{Q_{l-2}} \)-unstable. Now an inductive argument, which we leave to the reader, shows that if we define \( \mathfrak{b}_i \) as in (3) above, and let \( \theta_{Q_i} \) be the involution relative to \( Q_i \), the roots \( \alpha_1, \ldots, \alpha_{i-1} \) are compact for \( \theta_{Q_i} \), \( \alpha_i \) is \( \theta_{Q_i} \)-stable, \( \alpha_{i+1} \) is \( \theta_{Q_i} \)-unstable, and \( \alpha_{i+2}, \ldots, \alpha_l \) are compact for \( \theta_{Q_i} \). Hence, from \( Q_i \), the only monoidal action which gives a new orbit is \( m(s_{\alpha_i}) \cdot Q_i = Q_{i-1} \) and \( Q_{i-1} = K \cdot \mathfrak{b}_{i-1} \), by using the same argument as in the case \( i = l - 2 \). As a consequence, the codimension of \( Q_{i-1} \) in \( \mathcal{B} \) is \( i - 1 \). It now follows that \( Q_0, \ldots, Q_{l-1} \) are distinct orbits. The induction argument implies that no monoidal actions change \( Q_0 \), and it follows by Lemma 2.17 that \( Q_+, Q, Q_{l-1}, \ldots, Q_0 \) are all the \( K \)-orbits. This completes the proof of (3), and (1) is an easy consequence.

Q.E.D.

Proposition 2.21. Let \( \mathfrak{g} = so(2l, \mathbb{C}) \) and \( \mathfrak{k} = so(2l-1, \mathbb{C}) \).

1. There are exactly \( l \) \( K \)-orbits in the flag variety \( \mathcal{B} \) of \( \mathfrak{g} \).
2. Let \( \mathfrak{b}_+ \) be the set of upper triangular matrices in \( \mathfrak{g} \). Then \( Q_+ := K \cdot \mathfrak{b}_+ \) is the only closed \( K \)-orbit.
3. Let

\[ Q_i = m(s_{\alpha_i}) \cdots m(s_{\alpha_{i-1}}) \cdot \mathfrak{b}_+ \]

and let

\[ \mathfrak{b}_i := s_{\alpha_{i-1}} s_{\alpha_{i-2}} \cdots s_{\alpha_i}(\mathfrak{b}_+) \text{ for } i = 1, \ldots, l - 1 \]

Then \( Q_i = K \cdot \mathfrak{b}_i \) has codimension \( i - 1 \) in \( \mathcal{B} \). The distinct \( K \)-orbits are \( Q_+, Q_{l-1}, \ldots, Q_1 \).

In particular, the unique open orbit is \( Q_1 \) and contains the Borel subalgebra

\[ Q_1 = b_1 = s_{\alpha_{l-1}} s_{\alpha_{l-2}} \cdots s_{\alpha_1}(\mathfrak{b}_+) \]

(2.34)

Proof. For (2), since \( \mathfrak{b}_+ \) is preserved by \( \theta \), \( Q_0 = K \cdot \mathfrak{b}_+ \) is closed by Proposition 4.12 of [CEB]. Thus, \( Q_+ \) is the unique closed \( K \)-orbit by Part (2) of Proposition 2.19 and part (1) of Lemma 2.18.

For (3), we saw in Example 2.14 that \( \alpha_{l-1} \) and \( \alpha_l \) are complex \( \theta \)-stable, and that all other simple roots are compact. Hence, \( m(s_{\alpha_i}) \cdot Q_+ \) and \( m(s_{\alpha_{i-1}}) \cdot Q_+ \) are the orbits of dimension \( \text{dim} Q_+ + 1 \). We show that they coincide. Indeed, by Part (3) of Lemma 2.16, \( Q_+ = K \cdot s_{\alpha_1}(\mathfrak{b}_+) \) and \( m(s_{\alpha_{l-1}}) \cdot Q_+ = K \cdot s_{\alpha_{l-1}}(\mathfrak{b}_+) \). We may choose
the representatives for $s_{a_{i-1}}$ and $s_{a_i}$ in $W$ so that $\theta(\hat{s}_{a_{i-1}}) = \hat{s}_{a_i}$. Note that $s_{a_i}(b_+) = s_{a_i}s_{a_{i-1}}s_{a_{i-1}}(b_+)$. Since $s_{a_{i-1}}$ and $s_{a_i}$ commute, it follows that

$$\theta(\hat{s}_{a_i}s_{a_{i-1}}) = \hat{s}_{a_{i-1}}\hat{s}_{a_i} = \hat{s}_{a_i}s_{a_{i-1}}.$$ 

Thus, $\hat{s}_{a_i}s_{a_{i-1}} \in K$, and hence $K \cdot s_{a_{i-1}}(b_+) = K \cdot s_{a_i}(b_+)$. The orbit $Q_{l-1} = m(s_{a_i}) \cdot Q_+$ has involution $\theta_{Q_{l-1}} = \text{Ad}(\hat{s}_{a_{i-1}}^{-1}) \circ \theta \circ \text{Ad}(\hat{s}_{a_{i-1}})$, which changes the sign of the $l-1$ coordinate of $h$, and no other coordinates. It now follows that $a_{i-2}$ is complex $\theta_{Q_{l-1}}$-stable, while $a_1, \ldots, a_{l-3}$ are compact, and $a_{l-1}$ and $a_l$ are complex $\theta_{Q_{l-1}}$-unstable. The remainder of the argument follows by an easy induction similar to the proof of Part (3) of Proposition 2.20. Part (1) is an easy consequence of (2) and (3).

Q.E.D.

3. Orthogonal Eigenvalue Coincidence Varieties and $K$-orbits

We prove Theorem 1.1 in this section.

3.1. The varieties $Y_Q^i$ as irreducible components of $g(\geq i)$. Using our work in Sections 2.7 and 2.8 we can show that the Zariski closure of the varieties $Y_Q^i$ with codim($Q$) = $i$ are irreducible components of the eigenvalue coincidence varieties $g(\geq i)$. We consider the case where $g$ is of type $D$ and type $B$ separately. We first consider $Y_Q$, where the $K$-orbit $Q$ not closed.

Case I: $g = \mathfrak{so}(2l + 1, \mathbb{C})$, $\mathfrak{k} = \mathfrak{so}(2l, \mathbb{C})$

**Theorem 3.1.** Let $g(\geq i), i = 1, \ldots, l-1$ be the orthogonal eigenvalue coincidence variety defined in 2.20. Let $Q = K \cdot b \subset B$ be a $K$-orbit with codim($Q$) = $i$.

1. There exists a $\theta$-stable parabolic subalgebra $\mathfrak{r}$ with $b \subset \mathfrak{r}$ such that the hypothesis of Proposition 2.12 is satisfied. The parabolic subalgebra $\mathfrak{r}$ has $\theta$-stable Levi decomposition.

\[ \mathfrak{r} = \mathfrak{l} \oplus \mathfrak{u} \quad \text{with} \quad \mathfrak{l}_{ss} \cong \mathfrak{so}(2(l-i) + 1, \mathbb{C}) \quad \text{and} \quad \mathfrak{z} = \mathfrak{s}(\mathfrak{gl}(1, \mathbb{C}) \oplus \mathfrak{gl}(1, \mathbb{C}))^i, \]

Let $L_{ss} \cong SO(2(l-i) + 1, \mathbb{C}) \subset G$ be the connected algebraic subgroup with Lie algebra $L_{ss}$. The restriction $\theta|_{L_{ss}} = \theta_{2(l-i)+1}$ is the involution on $\mathfrak{so}(2(l-i) + 1, \mathbb{C})$ defining $\mathfrak{so}(2(l-i), \mathbb{C})$, so that

\[ \mathfrak{l}_{ss}^0 = \mathfrak{l}_{ss} \cap \mathfrak{k} \cong \mathfrak{so}(2(l-i), \mathbb{C}) \quad \text{and thus} \quad (L_{ss}^0)^0 = K \cap L_{ss} \cong SO(2(l-i), \mathbb{C}). \]

Furthermore, $SO(2(l-i), \mathbb{C}) \cdot (b \cap \mathfrak{l}_{ss})$ is open in $B_{\mathfrak{so}(2(l-i)+1, \mathbb{C})}$.

2. We have $\overline{Y_Q} = Y_{Q^\mathfrak{r}}$, and the variety $Y_{Q^\mathfrak{r}}$ is an irreducible component of $g(\geq i)$.

**Proof.** We first prove (1). Let $Q$ have codim($Q$) = $i$. By $K$-equivariance, it suffices to prove the statement for any representative $b$ of the $K$-orbit $Q$. By Part (3) of Proposition 2.20, we can take $b = b_i = \text{Ad}(u_{a_{i+1}})s_{a_{i-1}} \cdots s_{a_{i+1}}(b_+)$. Let $\mathfrak{r} \subset g$ be the standard parabolic subalgebra generated by $b_+$ and the negative simple root spaces $g_{-\alpha_i}, g_{-\alpha_{i-1}}, \ldots, g_{-\alpha_{i+1}}$. 


Note that \( r \) is \( \theta \)-stable with Levi decomposition (3.1) and also \( \theta|_{l_{ss}} = \theta_{2(l-i)+1} \). Equation (3.2) follows. To see that \( b_i \subset r \), note that for any representative \( \dot{s}_{\alpha_j} \) of \( s_{\alpha_j} \), we have \( \dot{s}_{\alpha_j} \in L_{ss} \) for \( j = i + 1, \ldots, l \), and \( u_{\alpha_j} \in L_{ss} \) by Equation (2.32). Thus, the element \( v = u_{\alpha_1} \dot{s}_{\alpha_{i-1}} \ldots \dot{s}_{\alpha_{i+1}} \in L_{ss} \subset R \).

Hence, \( b = \text{Ad}(v)b_+ \subset \text{Ad}(v)r = r \).

It remains to show that \( (K \cap L_{ss}) \cdot b \cap l_{ss} \) can be identified with the open \( SO(2(l-i), \mathbb{C}) \)-orbit in the flag variety \( B_{so(2(l-i)+1), \mathbb{C}} \). Note that \( b_+ \cap l_{ss} \) can be identified with the standard Borel subalgebra \( b_{+so(2(l-i)+1), \mathbb{C}} \) of upper triangular matrices in \( so(2(l-i)+1, \mathbb{C}) \). Since the element \( v \) in Equation (3.3) is in \( L_{ss} \), we have:

\[
b \cap l_{ss} = (\text{Ad}(v)b_+) \cap l_{ss} = \text{Ad}(v)(b_+ \cap l_{ss}) = \text{Ad}(v)b_{+so(2(l-i)+1), \mathbb{C}}.
\]

It follows from Equations (2.33) and (3.3) that \( \text{Ad}(v)b_{+so(2(l-i)+1), \mathbb{C}} \subset B_{so(2(l-i)+1), \mathbb{C}} \) is a representative of the open \( SO(2(l-i), \mathbb{C}) \)-orbit on \( b_{so(2(l-i)+1), \mathbb{C}} \).

We now prove (2). The first statement of (2) follows immediately from Part (1) and Proposition 2.12. By Proposition 2.11 to see that \( Y_\Phi \) is an irreducible component of \( g(\geq i) \), it suffices to show that \( Y_\Phi \subset g(\geq i) \).

Consider the partial Kostant-Wallach map \( \Phi_n \) defined in Equation (2.16). Let \( q \) be a parabolic subalgebra of \( g \) with \( q \subset Q_\Phi \), and let \( y \in q \). We need to show that \( \Phi_n(y) \in V^{r_{n-1}, \mathbb{R}_n}(\geq i) \). Since the map \( \Phi_n \) is \( K \)-invariant, it is enough to show that \( \Phi_n(x) \in V^{r_{n-1}, \mathbb{R}_n}(\geq i) \) for \( x \in r \). Recall that \( \Phi_n(x) = (\chi_{n-1}(x_\ell), \chi_n(x)) \), where \( \chi_i : so(i, \mathbb{C}) \rightarrow so(i, \mathbb{C})/SO(i, \mathbb{C}) \) is the adjoint quotient. For \( x \in r \), let \( x_\ell \) be the projection of \( x \) into \( l \) off of \( u \). It is well-known that \( \chi_{n-1}(x_\ell) = \chi_n(x_\ell) \). Using the decomposition in (3.1), we can write \( x_\ell \) as \( x_\ell = x_3 + x_{l_{ss}} \) with \( x_3 \in 3 = so(gl(1, \mathbb{C}) \oplus gl(1, \mathbb{C})) \) and \( x_{l_{ss}} \in l_{ss} = so(2(l-i)+1, \mathbb{C}) \).

It is easy to see that the coordinates of \( x_3 \) are in the spectrum of \( x \). Since \( r \) is \( \theta \)-stable, \( \ell \cap r \) is a parabolic subalgebra of \( \ell \) with Levi decomposition:

\[
\ell \cap r = \ell \cap l \oplus \ell \cap u \text{ and } \ell \cap l = 3 \cap \ell \oplus l_{ss} \cap \ell = 3 \oplus so(2(l-i), \mathbb{C}),
\]

where the last equality follows from (3.2) and the observation that \( 3 \subset h \subset \ell \), since \( \theta = \text{Ad}(t) \) with \( t \in H \). Since \( x_\ell \in \ell \cap r \), we know \( \chi_{n-1}(x_\ell) = \chi_{n-1}((x_\ell)_l \ell) \). Now, \((x_\ell)_l \ell = x_3 + x_{so(2(l-i), \mathbb{C})} \), and the coordinates of \( x_3 \) are in the spectrum of \((x_\ell)_l \ell \). Thus, Remark 2.3 implies that \( \Phi_n(x) = (\chi_{n-1}(x_\ell), \chi_n(x)) \in V^{r_{n-1}, \mathbb{R}_n}(\geq i) \), and it follows that \( Y_\Phi \subset g(\geq i) \).

Q.E.D.

Case II: \( g = so(2l, \mathbb{C}), \ell = so(2l-1, \mathbb{C}) \).

**Theorem 3.2.** Let \( g(\geq i-1), i = 2, \ldots, l-1 \) be the orthogonal eigenvalue coincidence variety defined in (2.20). Let \( Q = K \cdot b \subset B \) be a \( K \)-orbit with length \( \text{codim}(Q) = i - 1 \).
(1) There exists a $\theta$-stable parabolic subalgebra $\mathfrak{r}$ with $\mathfrak{b} \subset \mathfrak{r}$ and $\mathfrak{r}$ satisfies the hypothesis of Proposition 3.1. The parabolic subalgebra $\mathfrak{r}$ has $\theta$-stable Levi decomposition

\[ \mathfrak{r} = \mathfrak{l} \oplus \mathfrak{u} \text{ with } \mathfrak{l}_{ss} \cong \mathfrak{so}(2(l - i) + 2, \mathbb{C}) \text{ and } \mathfrak{z} = \mathfrak{s}(\mathfrak{gl}(1, \mathbb{C}) \oplus \mathfrak{gl}(1, \mathbb{C}))^{l-1}. \]

Let $L_{ss} \cong SO(2(l - i) + 2, \mathbb{C}) \subset G$ be the connected algebraic subgroup with Lie algebra $I_{ss}$. Then $\theta|_{I_{ss}} = \theta_{2i-2i}$, so that

\[ I_{ss}^\theta = I_{ss} \cap \mathfrak{k} \cong \mathfrak{so}(2(l - i) + 1, \mathbb{C}) \text{ and thus } (I_{ss}^\theta)^0 = K \cap L_{ss} = SO(2(l - i) + 1, \mathbb{C}). \]

Furthermore, $SO(2(l - i) + 1, \mathbb{C}) \cdot \mathfrak{b} \cap \mathfrak{l}_{ss}$ is open in $\mathcal{B}_{\mathfrak{so}(2(l - i) + 2, \mathbb{C})}$. (2) We have $Y_Q = Y_Q^\theta$, and the variety $Y_Q^\theta$ is an irreducible component of $\mathfrak{g}(\geq i - 1)$.

Proof. The proof is very similar to the proof of Theorem 3.1.

We begin with the proof of part (1). Let $Q$ have codim($Q$) = $i - 1$. Again, by $K$-equivariance, it suffices to prove the statement for any representative $\mathfrak{b}$ of the $K$-orbit $Q$. By Part 3 of Proposition 2.21 we can take $\mathfrak{b} = \mathfrak{b}_1 = s_{\alpha_{l-1}} s_{\alpha_{l-2}} \cdots s_{\alpha_1}(\mathfrak{b}_+)$. Let $\mathfrak{r} \subset \mathfrak{g}$ be the standard parabolic subalgebra generated by $\mathfrak{b}_+$ and the negative simple root spaces $\mathfrak{g}_{-\alpha_1}, \mathfrak{g}_{-\alpha_{l-2}}, \ldots, \mathfrak{g}_{-\alpha_l}$. We claim that $\mathfrak{r}$ is $\theta$-stable. Indeed, we saw in Example 2.14 that the roots $\alpha_i$ are compact imaginary for $i = 1, \ldots, l - 2$ and that $\alpha_{l-1}$ and $\alpha_l$ are complex $\theta$-stable with $\theta(\alpha_{l-1}) = \alpha_l$. It then follows easily that $\mathfrak{r}$ has Levi decomposition (3.4) and that $\theta|_{I_{ss}} = \theta_{2(l - i) + 2}$, whence $I_{ss}^\theta = \mathfrak{k} \cap I_{ss} \cong \mathfrak{so}(2(l - i) + 1, \mathbb{C})$, and $(I_{ss}^\theta)^0 = K \cap L_{ss} \cong SO(2(l - i) + 1, \mathbb{C})$. The remainder of the proof proceeds exactly as in the proof of Part 1 of Theorem 3.1 using Equation (2.34) instead of Equation (2.33).

The proof of (2) is also analogous to the proof of Part 2 of Theorem 3.1. The key observation is that for $x \in \mathfrak{r}$ with $x_\mathfrak{b} = x_\mathfrak{z} \oplus x_{I_{ss}}$ the coordinates of $x_\mathfrak{z} \in \mathfrak{z} = \mathfrak{s}(\mathfrak{gl}(1, \mathbb{C}) \oplus \mathfrak{gl}(1, \mathbb{C}))^{l-1}$ are in the spectrum of both $x \in \mathfrak{g}$ and $x_\mathfrak{k} \in \mathfrak{k}$. We leave the remaining details to the reader.

Q.E.D.

We now consider the case where $Q$ is a closed $K$-orbit.

Theorem 3.3. Let $Q$ be a closed $K$-orbit on $\mathcal{B}$. Then $Y_Q$ is an irreducible component of $\mathfrak{g}(\geq r_{n-1}) = \mathfrak{g}(r_{n-1})$.

Proof. We show that for a closed $K$-orbit $Q = K \cdot \mathfrak{b}$, $Y_Q \subset \mathfrak{g}(r_{n-1})$. We can then argue as in the proof of Theorem 3.1 to show that $Y_Q$ is an irreducible component of $\mathfrak{g}(\geq r_{n-1})$. By $K$-equivariance, it suffices to show that $\mathfrak{b} \subset \mathfrak{g}(r_{n-1})$. If $\mathfrak{g}$ is of type $B$, then Part 2 of Proposition 2.20 implies that $\mathfrak{b} = \mathfrak{b}_+$ or $\mathfrak{b} = s_{\alpha_l}(\mathfrak{b}_+)$. In either case, $\mathfrak{b}$ contains the standard diagonal Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Since $\theta = \text{Ad}(t)$ with $t \in H$, it follows that $\mathfrak{h} \subset \mathfrak{k}$. Now by Proposition 4.12, $\mathfrak{CEb}$, $\mathfrak{b}$ is $\theta$-stable and $\mathfrak{b} \cap \mathfrak{k}$ is a Borel subalgebra of $\mathfrak{k}$ with Levi decomposition

\[ \mathfrak{b} \cap \mathfrak{k} = \mathfrak{h} \oplus (\mathfrak{n} \cap \mathfrak{k}), \]
where \( n = [b, b] \) is the nilradical of \( b \). Thus, for \( x \in b \) with \( x = x_h + x_n \), with \( x_h \in h \) and \( x_n \in n \), the coordinates of \( x_h \) are in the spectrum of both \( x \) and \( x_{\mathfrak{k}} \). It follows that \( b \subset g(r_{n-1}) \).

If \( g \) is of type \( D \), then Part (2) of Proposition 2.21 states that \( Q_+ = K \cdot b_+ \) is the only closed \( K \)-orbit. We recall that \( \theta(\epsilon_i) = -\epsilon_i \), and \( \theta(\epsilon_i) = \epsilon_i \) for all \( i \neq l \) (see Example 2.14). Therefore, \( h \cap \mathfrak{k} = \text{diag}(a_1, \ldots, a_{r_{n-1}}) \), and \( b \cap \mathfrak{k} \) is a Borel subalgebra of \( \mathfrak{k} \) with Levi decomposition \( b \cap \mathfrak{k} = h \cap \mathfrak{k} + n \cap \mathfrak{k} \). Thus, for any \( x \in b \), \( x = x_h + x_n \), and \( x_h = x_{h \cap \mathfrak{k}} + x_{h \cdot g^{-1}} \cdot \), with \( x_{\mathfrak{k}} = x_{\mathfrak{k} \cdot h} + x_{\mathfrak{k} \cap n} \in \mathfrak{k} \cap b \). Thus, \( (x_{\mathfrak{k}})_{h \cap \mathfrak{k}} \in h \) is in the spectrum of both \( x \) and \( x_{\mathfrak{k}} \). It follows that \( b_+ \subset g(\geq r_{n-1}) \).

Q.E.D.

**Remark 3.4.** Let \((g, \mathfrak{t})\) be a symmetric pair, and let \( Q = K \cdot b \subset B \) be an arbitrary \( K \)-orbit in the flag variety \( B \) of \( g \). Then if \( r \subset g \) is a \( \theta \)-stable parabolic subalgebra with \( b \subset r \), the \( K \)-orbit \( Q \) has the structure of a fibre bundle as in (2.22). However, \((K \cap L_{ss}) \cdot (b \cap L_{ss})\) need not be the open \( K \)-orbit in \( B_{ss} \).

As we noted in Remark 2.13, the hypothesis of Proposition 2.12 is true for the real rank one symmetric pair \((g = \mathfrak{gl}(n, \mathbb{C}), \mathfrak{t} = \mathfrak{gl}(n-1, \mathbb{C}) \oplus \mathfrak{gl}(1, \mathbb{C}))\) and for any \( K \)-orbit \( Q \). Of course, the analogue of Part 2 of Theorems 3.1 and 3.2 also holds in this setting (see Theorems 3.6 and 3.7, [CEa]).

For a negative example, consider the real rank one symmetric pair \((g = \mathfrak{sp}(2n, \mathbb{C}), \mathfrak{t} = \mathfrak{sp}(2n-2, \mathbb{C}) \oplus \mathfrak{sp}(2, \mathbb{C}) \cong \mathfrak{sp}(2n-2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}))\). We realize \( \mathfrak{t} \) as the fixed point set of the involution \( \theta(x) = xc^{-1}, x \in g \) and \( c \in \text{Sp}(2n, \mathbb{C}) \), \( c = \text{diag}(-1, 1, 1, \ldots, 1, -1) \). Let \( b_+ \subset g \) denote the standard Borel subalgebra of upper triangular matrices in \( g \). Let \( n = 3 \) and consider the \( K \)-orbit \( Q = K \cdot b \), where \( b = \text{Ad}(u_{\alpha_3} s_{\alpha_2} s_{\alpha_3}) \cdot b_+ \). Using Propositions 4.27 and 4.28 of [CEb], we compute that the minimal \( \theta \)-stable parabolic subalgebra containing \( b \) is \( g \). This follows easily from the fact that the involution associated to \( Q \), \( \theta_Q = s_{\alpha_3} s_{\alpha_2} s_{\alpha_1} \theta s_{\alpha_3} s_{\alpha_2} s_{\alpha_3} \) sends the simple root \( \alpha_3 \) to the lowest root \(-2\alpha_1 - 2\alpha_2 - \alpha_3 \). However, \( Q \) is not the open \( K \)-orbit in \( B \). In fact, \( \text{codim}(Q) = 1 \).

We can also define eigenvalue coincidence varieties for the symmetric pair \((g = \mathfrak{sp}(2n, \mathbb{C}), \mathfrak{t} = \mathfrak{sp}(2n-2, \mathbb{C}) \oplus \mathfrak{sp}(2, \mathbb{C}))\) in a similar fashion to (1.2). However, the partitions of \( g \), \( g = \bigcup g(\geq i) \) and \( g = \bigcup_{Q \subset B} Y_Q \) are not as clearly related as they are for the pairs \((g, \mathfrak{t})\), where \( g = \mathfrak{gl}(n, \mathbb{C}), \mathfrak{so}(n, \mathbb{C}) \) and \( \mathfrak{t} = \mathfrak{gl}(n-1, \mathbb{C}) \oplus \mathfrak{gl}(1, \mathbb{C}), \mathfrak{so}(n-1, \mathbb{C}) \). Let \( Q_+ = K \cdot b_+ \subset B \). Since \( b_+ \) is \( \theta \)-stable, \( Y_{Q_+} \) is closed. However, we claim that \( Y_{Q_+} \) is not an irreducible component of \( g(\geq n) \). Using elementary arguments, one can show that every irreducible component of the variety \( g(\geq n) \) must have dimension at least \( \text{dim} g - n = 2n^2 \). But we compute that \( \text{dim} Y_Q = \text{dim} B_{sp(2n, \mathbb{C}) \times sl(2, \mathbb{C})} + \text{dim} b = 2n^2 - n + 2 \). Thus, \( Y_{Q_+} \) is a proper, closed subvariety of \( g(\geq n) \).

### 3.2. Every irreducible component of \( g(\geq i) \) is of the form \( Y_Q \).

In this section, we complete the last step of the proof of Theorem 1.1. Consider the regular semisimple
elements $g^{rs}$ of $g$, and let $h^{reg} = g^{rs} \cap h$. For $x$ in $g$, consider the spectrum $\sigma(x) = \{ \pm b_1, \ldots, \pm b_n \}$ (see (11)). For type $D$, $x \in g^{rs}$ if and only if $b_i \neq \pm b_j$ for $i \neq j$. For type $B$, $x \in g^{rs}$ if and only if $b_i \neq \pm b_j$ for $i \neq j$, and all $b_i \neq 0$.

**Theorem 3.5.** Every irreducible component of the variety $\var{g}{\geq i}$, $i = 0, \ldots, r_n - 1$ is of the form $\overline{Y_Q}$ for some $K$-orbit $Q$ on $\mathcal{B}$ with $\text{codim}(Q) = i$.

**Proof.** Consider the non-empty Zariski open subset of $g$:

\begin{equation}
U := \{ x \in g : x_\mathfrak{t} \text{ is regular semisimple in } \mathfrak{t} \}.
\end{equation}

Let $U(\geq i) := U \cap \var{g}{\geq i}$. Since $U(\geq r_n - 1)$ is a non-empty Zariski open set of $\var{g}{\geq i}$, it follows that $U(\geq i)$ is a non-empty Zariski open set of $\var{g}{\geq i}$ by Proposition 2.6. By Proposition 2.2 and Exercise III.9.1 of [Har77], $\Phi_n(U) \subset V^{r_n - 1, r_n}$ is open. Thus, $V^{r_n - 1, r_n}(\geq i) \setminus \Phi_n(U)$ is a proper, closed subvariety of $V^{r_n - 1, r_n}(\geq i)$ and therefore has positive codimension by Lemma 2.4. It follows by Proposition 2.2 and Corollary III.9.6 of [Har77] that $\var{g}{\geq i} \setminus \var{U}{\geq i} = \Phi_n^{-1}(V^{r_n - 1, r_n}(\geq i) \setminus \Phi_n(U))$ is a proper, closed subvariety of $\var{g}{\geq i}$ of positive codimension. Since $\var{g}{\geq i}$ is equidimensional, it follows that $Z \cap U(\geq i)$ is nonempty for any irreducible component $Z$ of $\var{g}{\geq i}$. Thus, it suffices to show that

\begin{equation}
U(\geq i) \subset \bigcup_{\text{codim}(Q) = i} \overline{Y_Q}.
\end{equation}

To prove Equation (3.6), we consider the following subvariety of $U(\geq i)$:

\begin{equation}
\Xi = \{ x \in U(\geq i) : x_\mathfrak{t} = (a_1, \ldots, a_{r_n - 1}) \in h^{reg}_{r_n - 1}, \text{ and } \sigma(x_\mathfrak{t}) \cap \sigma(x) \supset \{ \pm a_1, \ldots, \pm a_i \} \}.
\end{equation}

It is easy to check that any element of $U(\geq i)$ is $K$-conjugate to an element in $\Xi$. Thus, by the $K$-equivariance of the varieties $\overline{Y_Q}$, it is enough to show that

\begin{equation}
\Xi \subset \bigcup_{\text{codim}(Q) = i} \overline{Y_Q}.
\end{equation}

We consider the cases where $g$ is of type $B$ and type $D$ separately. First, we assume that $g = \mathfrak{so}(2l + 1, \mathbb{C})$ is of type $B$. By Theorem 3.1 it suffices to show that

\begin{equation}
\Xi \subset Y_{Q_\mathfrak{t}} \text{ for } i < l,
\end{equation}

where $\mathfrak{t}$ is the parabolic subalgebra generated by $\mathfrak{b}_+$ and the negative simple root spaces $g_{-\alpha_{l+1}}, \ldots, g_{-\alpha_l}$. For $i = l$ we need to show that

\begin{equation}
\Xi \subset Y_{Q_+} \cup Y_{Q_-} \text{ for } i = l,
\end{equation}

where $Q_+ = K \cdot \mathfrak{b}_+$ and $Q_- = K \cdot \mathfrak{b}_-$ are the distinct closed $K$-orbits on $\mathcal{B}$ (see Part (2) of Proposition 2.20). To prove Equations (3.9) and (3.10), we need to describe the variety $\Xi$ in more detail. Recall from Example 2.14 that

\begin{equation}
g^{-\theta} = \bigoplus_{i=1}^l g_{\epsilon_i} \oplus \bigoplus_{i=1}^l g_{-\epsilon_i}.
\end{equation}
Let \( e_{\epsilon_j} = E_{j,0} - E_{0,-j} \in \mathfrak{g}_{\epsilon_j} \), and let \( e_{\epsilon_j} = E_{0,j} - E_{-j,0} \in \mathfrak{g}_{-\epsilon_j} \) (see (2.15)). Consider elements of the form:

\[
\mathfrak{a} \oplus u_je_{\epsilon_j} \oplus v_je_{-\epsilon_j},
\]

where \( \mathfrak{a} = \text{diag}[a_1, \ldots, a_l, 0, -a_l, \ldots, -a_1] \in \mathfrak{h}, a_i \neq \pm a_j, a_i \neq 0, u_j, v_j \in \mathbb{C} \). By definition \( \Xi \) consists of elements of the form (3.11) with eigenvalues \( \pm a_k \) for \( k = 1, \ldots, i \).

**Claim:** The variety \( \Xi \) consists of elements of the form (3.11) satisfying the equations:

\[
u_jv_j = 0 \text{ for } j = 1, \ldots, i.
\]

To verify the claim, we have to compute the characteristic polynomial of elements of the form (3.11). Such elements can be represented in matrix form as:

\[
\begin{pmatrix}
a_1 & \ldots & 0 & u_1 & 0 & \ldots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & a_l & u_l & 0 & \ldots & 0 \\
v_1 & \ldots & v_l & 0 & -u_l & \ldots & -u_1 \\
0 & \ldots & 0 & -v_l & -a_l & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & -v_1 & 0 & \ldots & -a_1
\end{pmatrix}
\]

Using the Schur complement formula for the determinant ([HJ85], pages 21-22), we compute the characteristic polynomial of (3.13) to be:

\[
-t \left[ \prod_{j=1}^l (a_j^2 - t^2) + 2 \sum_{m=1}^l \prod_{k=1,k\neq m}^l (a_k^2 - t^2)u_mv_m \right].
\]

Thus, \( \pm a_m \) is an eigenvalue of (3.11) if and only if \( u_mv_m = 0 \). The claim follows.

We can now describe the irreducible components of \( \Xi \) using (3.12). For \( k = 1, \ldots, i \), we define an index \( j_k \) which takes on two values \( j_k = U \) (\( U \) for upper) or \( j_k = L \) (\( L \) for lower). Consider the subvariety \( \Xi_{j_1, \ldots, j_i} \subset \Xi \) defined by:

\[
\Xi_{j_1, \ldots, j_i} := \{ x \in \Xi : v_k = 0 \text{ if } j_k = U, u_k = 0 \text{ if } j_k = L \}.
\]

Then

\[
\Xi = \bigcup_{j_k = U, L} \Xi_{j_1, \ldots, j_i}.
\]

is the irreducible component decomposition of \( \Xi \). Notice that in case \( j_k = U \) for all \( k = 1, \ldots, i \), then

\[
\Xi_{U, \ldots, U} \subset \mathfrak{r}.
\]

This follows from the observation that \( e_j = a_j + \cdots + a_l \) for any \( j = 1, \ldots, l \). Thus, for \( j = i + 1, \ldots, l \), \( \mathfrak{g}_{\epsilon_j} \subset \mathfrak{l}_{ss} \subset \mathfrak{r} \), where \( \mathfrak{l}_{ss} \) is the semisimple part of the Levi factor of \( \mathfrak{r} \), and \( \mathfrak{g}_{\epsilon_j} \subset \mathfrak{u} \) for \( j = 1, \ldots, i \) (see (3.14)). Observe also that if \( j_k = L \) for some \( k = 1, \ldots, i \), then

\[
\text{Ad}(\hat{s}_{\epsilon_k})\Xi_{j_1, \ldots, j_{k-1}, U, \ldots, j_i} = \Xi_{j_1, \ldots, j_{k-1}, U, \ldots, j_i}.
\]
This follows immediately from the fact that \( \text{Ad}(\dot{s}_{\epsilon_i})g_{\epsilon_j} = g_{\epsilon_j} \) for \( j \neq i \), and \( \text{Ad}(\dot{s}_{\epsilon_i})g_{\epsilon_i} = g_{-\epsilon_i} \).

We now analyze the irreducible variety \( \Xi_{j_1, \ldots, j_l} \). Suppose that for the subsequence \( 1 \leq k_1 < \cdots < k_{m-1} \leq l \) we have \( j_{k_1} = j_{k_2} = \cdots = j_{k_{m-1}} = L \) and that for the complementary subsequence \( k_m < \cdots < k_l \) we have \( j_{k_m} = j_{k_{m+1}} = \cdots = j_{k_l} = U \). First, suppose that \( i < l \). Consider the element \( \sigma = s_{\epsilon_{k_1}}s_{\epsilon_{k_2}} \cdots s_{\epsilon_{k_{m-1}}} \in W \). It follows from Equations (3.17) and (3.18) that

\[
(3.19) \quad \text{Ad}(\dot{\sigma})\Xi_{j_1, \ldots, j_l} \subset r.
\]

Note that \( s_{\epsilon_j} \) acts on the coordinates of \( \mathfrak{b} \) by sign change in the \( j \)-th coordinate. Thus, if \( m-1 \) is even, it follows from Part 1 of Proposition 2.19 that \( \sigma \in W_K \), and we can choose its representative \( \dot{\sigma} \in K \). If \( m-1 \) is odd, then replace \( \sigma \) by \( \tau := s_{\epsilon_{k_1}}s_{\epsilon_{k_2}} \cdots s_{\epsilon_{k_{m-1}}} \). Then we can choose \( \dot{\tau} \in K \), and since \( \dot{s}_{\epsilon_i} \in L_{ss} \), Equation (3.19) implies

\[
\text{Ad}(\dot{\tau})\Xi_{j_1, \ldots, j_l} \subset r.
\]

In either case, the component \( \Xi_{j_1, \ldots, j_l} \) is \( W_K \)-conjugate to a subvariety of \( r \), and Equation (3.9) follows. Now consider the case where \( i = l \). It follows from (3.17) that \( \text{Ad}(\dot{\sigma})\Xi_{j_1, \ldots, j_l} \subset \mathfrak{b}_+ \). Now if \( m-1 \) is even, then \( \text{Ad}(\dot{\sigma})\Xi_{j_1, \ldots, j_l} \subset Y_{Q_+} = \text{Ad}(K)\mathfrak{b}_+ \). However, if \( m-1 \) is odd, then \( \text{Ad}(\dot{\tau})\Xi_{j_1, \ldots, j_l} \subset s_{\epsilon_l}(\mathfrak{b}_+) \), whence \( \Xi_{j_1, \ldots, j_l} \subset Y_{Q_-} = \text{Ad}(K)(s_{\epsilon_l}(\mathfrak{b}_+)) \). Thus, Equation (3.10) is proven.

We now prove (3.8) when \( \mathfrak{g} = \mathfrak{so}(2l, \mathbb{C}) \) is of type \( D \). By Theorem 3.2 it suffices to prove

\[
(3.20) \quad \Xi \subset Y_{Q_{\mathfrak{r}}},
\]

where \( \mathfrak{r} \) is the parabolic subalgebra generated by \( \mathfrak{b}_+ \) and the negative simple root spaces \( \mathfrak{g}_{-\alpha_{i+1}}, \ldots, \mathfrak{g}_{-\alpha_l} \) for \( i < l-1 \), and \( \mathfrak{r} = \mathfrak{b}_+ \) for \( i = l-1 \). Recall from Example 2.14 we have

\[
\mathfrak{g}^{-\theta} = \bigoplus_{j=1}^{l-1} (\mathfrak{g}_{\epsilon_{j-\epsilon_l}} \oplus \mathfrak{g}_{\epsilon_{j+\epsilon_l}})^{-\theta} \oplus \bigoplus_{j=1}^{l-1} (\mathfrak{g}_{-\epsilon_{j+\epsilon_l}} \oplus \mathfrak{g}_{-\epsilon_{j-\epsilon_l}})^{-\theta}.
\]

Let \( e_j = E_{j,l} - E_{l-1,-j} - E_{j,-l} + E_{l,j} \). Then \( e_j \) is a basis for \( (\mathfrak{g}_{\epsilon_{j-\epsilon_l}} \oplus \mathfrak{g}_{\epsilon_{j+\epsilon_l}})^{-\theta} \). Similarly, \( e'_j \) is a basis of \( (\mathfrak{g}_{-\epsilon_{j+\epsilon_l}} \oplus \mathfrak{g}_{-\epsilon_{j-\epsilon_l}})^{-\theta} \). Thus, the variety \( \Xi \) consists of elements of the form

\[
(3.21) \quad \mathfrak{g} \oplus u_j e_j \oplus v_j e_{-j},
\]

where \( a = \text{diag}[a_1, \ldots, a_l, -a_l, \ldots, -a_1] \), \( a_i \neq \pm a_j \in \mathfrak{h} \), and \( u_j, v_j \in \mathbb{C} \). Computing, the characteristic polynomial of elements of the form (3.21) as in the previous case, we see \( \Xi \) consists of elements of the form (3.21) satisfying

\[
(3.22) \quad u_j v_j = 0 \quad \text{for} \quad j = 1, \ldots, i.
\]

We define the varieties \( \Xi_{j_1, \ldots, j_l} \) with \( j_k = L, U \) analogously to (3.15). We have \( \Xi = \bigcup_{j_k = L, U} \Xi_{j_1, \ldots, j_l} \) (cf. (3.10)). Now we observe that if \( j_k = U \) for all \( k \), then

\[
(3.23) \quad \Xi_{U, \ldots, U} \subset r.
\]
This follows from the observation that $\epsilon_j - \epsilon_l = \alpha_j + \cdots + \alpha_{l-1}$ and $\epsilon_j + \epsilon_l = \alpha_j + \cdots + \alpha_{l-2} + \alpha_l$. Thus, for $j = i + 1, \ldots, l$, the root spaces $g_{\pm(\epsilon_j - \epsilon_l)}$ and $g_{\pm(\epsilon_j + \epsilon_l)}$ are in $t_{ss} \subset r$. Further, for $j = 1, \ldots, i$, the root spaces $g_{\epsilon_j - \epsilon_l}, g_{\epsilon_j + \epsilon_l} \subset u \subset r$ (see (3.4)). We now show that any $\Xi \subset Y_{Q_r}$. Recall from Part (2) of Proposition 2.19 that

$$W^\theta = W_K = \langle s_{\alpha_1}, \ldots, s_{\alpha_{l-2}}, s_{\alpha_{l-1}} \cdot s_{\alpha_l} \rangle.$$ 

For $j = 1, \ldots, i$, define $w_j := s_{\epsilon_j - \epsilon_{l-1}} s_{\epsilon_l} s_{\epsilon_j - \epsilon_{l-1}}$. Then $w_j \in W_K$, and $w_j$ has order 2. In fact, $w_j$ acts on $h$ via

$$w_j : (a_1, \ldots, a_j, \ldots, a_l) \rightarrow (a_1, \ldots, -a_j, \ldots, -a_l).$$ 

We claim that

$$\text{(3.24)} \quad w_j : (a_1, \ldots, a_j, \ldots, a_l) \rightarrow (a_1, \ldots, -a_j, \ldots, -a_l).$$

Indeed, (3.24) implies that

$$w_j \cdot (\epsilon_j + \epsilon_l) = - (\epsilon_j + \epsilon_l), \quad w_j \cdot (\epsilon_j - \epsilon_l) = -(\epsilon_j - \epsilon_l), \quad \text{and} \quad w_j \cdot (\epsilon_k + \epsilon_l) = \epsilon_k - \epsilon_l \text{ for } k \neq j.$$

Further, since $w_j \in W_K$,

$$\text{Ad}(w_j) : (g_{-(\epsilon_j - \epsilon_l)} \oplus g_{-(\epsilon_j + \epsilon_l)})^{-\theta} \rightarrow (g_{-(\epsilon_j - \epsilon_l)} \oplus g_{-(\epsilon_j + \epsilon_l)})^{-\theta} \text{ and } \text{Ad}(w_j) \text{ stabilizes } (g_{-(\epsilon_j - \epsilon_l)} \oplus g_{-(\epsilon_j + \epsilon_l)})^{-\theta}$$

for $k \neq j$. Equation (3.25) now follows from the definition of the varieties $\Xi_{j_1, \ldots, j_i}$. Thus, if we are given a variety $\Xi_{j_1, \ldots, j_i}$ with $j_{k_1} = j_{k_2} = \cdots = j_{k_{m-1}} = L$, it follows from (3.25) and (3.23) that

$$\text{Ad}(w_{k_1} \cdots w_{k_{m-1}}) \Xi_{j_1, \ldots, j_i} \subset r.$$ 

Since $\Xi = \bigcup_{j_k=L,U} \Xi_{j_1, \ldots, j_i}$, Equation (3.20) follows. This completes the proof.

Q.E.D.

Proof of Theorem 1.1. Equation (1.3) follows from Theorems 3.1 and 3.2 along with Theorem 3.5. The statement about the number of irreducible components of $g(\geq i)$ follows from Parts 2 and 3 of Propositions 2.20 and 2.21.

Q.E.D.

Corollary 3.6. Recall the variety $g(i)$ defined in Equation (2.27). The irreducible component decomposition of $g(i)$ is

$$\text{(3.26)} \quad g(i) = \bigcup_{\text{codim}(Q)=i} Y_Q \cap g(i).$$

Proof. Theorem 1.1 and Equation (1.3) imply that the irreducible component decomposition of the variety $g(i)$ is

$$\text{(3.27)} \quad g(i) = \bigcup_{\text{codim}(Q)=i} \overline{Y_Q} \cap g(i).$$
Note that since \( g(\geq i) \) is equidimensional, we have \( \overline{Y_Q} \cap g(i) \neq \emptyset \) for all \( Q \) with \( \text{codim}(Q) = i \). We claim that for each \( Q \) with \( \text{codim}(Q) = i \) that
\[
(3.28) \quad \overline{Y_Q} \cap g(i) = Y_Q \cap g(i).
\]
First, suppose that \( (3.28) \) is not an equality. Then since \( \overline{Y_Q} = \bigcup_{Q' \subseteq Q} Y_{Q'} \) by Lemma 2.10 there exists a \( K \)-orbit \( Q' \) with \( \text{codim}(Q') > \text{codim}(Q) \) such that \( Y_{Q'} \cap g(i) \neq \emptyset \). But this contradicts Theorem 1.1 which asserts that \( Y_{Q'} \subset g(\geq i + 1) \). Equation \((3.26)\) now follows from \((3.28)\) and \((3.27)\).

Q.E.D.

The following corollary will be useful in Section 4.3.

**Corollary 3.7.** For \( i = 1, \ldots, r_{n-1} - 1 \), the irreducible component decomposition of \( g(i) \) is
\[
(3.29) \quad g(i) = Y_Q \cap g(i),
\]
where \( r \) is the \( \theta \)-stable parabolic subalgebra of Theorems 3.1 and 3.2. For \( i = r_{n-1} \) and \( g = so(2l, \mathbb{C}) \),
\[
(3.30) \quad g(r_{n-1}) = g(\geq r_{n-1}) = Y_{Q_+},
\]
where \( Q_+ = K \cdot b_+ \) is the unique closed \( K \)-orbit on \( B \) (see Part (2) of Proposition 2.21). For \( g = so(2l + 1, \mathbb{C}) \) we have the irreducible component decomposition of \( g(r_{n-1}) \) is
\[
(3.31) \quad g(r_{n-1}) = g(\geq r_{n-1}) = Y_{Q_+} \cup Y_{Q_-},
\]
where \( Q_+ \) and \( Q_- \) are the distinct closed \( K \)-orbits on \( B \) (see Part (2) of Proposition 2.20).

**Proof.** The result follows immediately from Equation \((3.26)\) and Part (2) of Theorems 3.1 and 3.2.

Q.E.D.

4. The geometric invariant theory of multiplicity free spherical pairs

In this section, we study the \( K \)-action on \( g \) in the cases \((K, g) = (GL(n-1, \mathbb{C}), g(1, n, \mathbb{C})) \) and \((SO(n-1, \mathbb{C}), so(n, \mathbb{C})) \). By work of Knop, these are essentially the only two multiplicity-free spherical pairs. We extend a result of Kostant characterizing regular elements using differentials in Theorem 4.6. We then analyze the \( K \)-action on the subvariety \( g(0) \), and show that all the \( K \)-orbits in \( g(0) \) are closed. We use the above analysis to give representatives of the \( K \)-orbits in \( g \), and discuss some applications to strongly regular elements.

**Definition 4.1.** Let \( G \) be a reductive, algebraic group, and let \( H \subset G \) be a reductive algebraic subgroup. We say the pair \((G, H)\) is spherical if a Borel subgroup \( B_H \subset H \) acts on the flag variety \( B \) of \( g \) with finitely many orbits.
Remark 4.2. Let $V$ be a rational $G$-representation, and let $V^H$ be the set of $H$-fixed vectors in $V$. It is well-known that Definition 4.1 is equivalent to the statement that $\dim V^H \leq 1$ for every irreducible, rational $G$-representation $V$ (see [VK78], [Bri87]).

Let $(G,H)$ be a spherical pair. Let $g = \text{Lie}(G)$ and let $h = \text{Lie}(H)$. Let $\langle \cdot, \cdot \rangle$ denote the Killing form on $g$, and let $h^\perp$ be the annihilator of $h$ with respect to $\langle \cdot, \cdot \rangle$. Then the adjoint action of $G$ on $g$ restricts to an action of $H$ on $h^\perp$, which is referred to in the literature as the coisotropy representation of $H$ (see [Pan90]). Let $C[h^\perp]^H$ be the ring of $H$-invariant polynomials on $h^\perp$. Then it is well-known that $C[h^\perp]^H$ is a polynomial algebra (Kor 7.2 of [Kno90] or Corollary 5 of [Pan90]). Consider the geometric invariant theory quotient $\Psi : h^\perp \to h^\perp//H$. By Korollar 7.2 of [Kno90], $\Psi$ is flat. We give a different proof of this result in the appendix using conormal bundle geometry for spherical pairs satisfying:

\[(4.1) \quad \dim \mathcal{B} = \dim h^\perp - \dim h^\perp//H.\]

We now analyze further what the condition in Equation (4.1) means for the coisotropy representation. Recall that an element $x \in g$ is called regular if its $\text{Ad}(G)$-orbit is of maximal dimension, i.e., $\dim(\text{Ad}(G) \cdot x) = \dim(g) - \text{rank}(g)$. A basic result of Kostant (Theorem 9, [Kos63]) states if $C[g]^G = C[\psi_1, \ldots, \psi_r]$ is the ring of $\text{Ad}(G)$-invariant polynomials on $g$, then

\[(4.2) \quad x \in g_{\text{reg}} \text{ if and only if } d\psi_1(x) \wedge \cdots \wedge d\psi_r(x) \neq 0.\]

If $x \in g$ is regular, and we identify $T_x^*(g)$ with $g$ using the non-degenerate form on $g$, then

\[(4.3) \quad \text{span}\{d\psi_i(x) : i = 1, \ldots, r\} = \mathfrak{z}_g(x),\]

where $\mathfrak{z}_g(x)$ denotes the centralizer of $x$ in $g$. Similarly, we can define the set of regular elements of $h^\perp$ for the coisotropy representation of $H$ on $h^\perp$:

\[(4.4) \quad h^\perp_{\text{reg}} := \{x \in h^\perp : \dim H \cdot x \text{ is maximal}\}.

The following result relates the sets $h^\perp_{\text{reg}}$ and $g_{\text{reg}}$.

Theorem 4.3. Let $(G,H)$ be a spherical pair. Then the following conditions are equivalent.

1. Equation (4.1) holds.
2. We have $h^\perp_{\text{reg}} \subset g_{\text{reg}}$.

Proof. We first show that (1) implies (2). Let $x \in h^\perp_{\text{reg}}$. By Theorems 3 and 6 of [Pan90],

\[(4.5) \quad \dim H \cdot x = \dim \mathcal{B}.\]

for $x \in h^\perp_{\text{reg}}$. By (1),

\[(4.5) \quad \dim H \cdot x = \dim \mathcal{B}.\]
By Theorems 3 and 5 of Pan90, there is a dense open subset \( U \) of \( \mathfrak{h}^\perp_{\text{reg}} \) such that if \( y \in U \), then \( \dim(\text{Ad}(G)(y)) = 2 \dim(H \cdot y) \). By Proposition 1 of Pan90, \( \dim(\text{Ad}(G)(x)) \geq 2 \dim(H \cdot y) = 2 \dim(\mathcal{B}) \). It follows that \( x \in \mathfrak{g}_{\text{reg}} \). Conversely, let \( x \in U \subset \mathfrak{g}_{\text{reg}} \). Then \( \dim(H \cdot x) = \frac{1}{2} \dim(\text{Ad}(G)(x)) = \dim(\mathcal{B}) \). The assertion now follows by Theorems 3 and 6 of Pan90.

Q.E.D.

Recall that an involution \( \theta \) of \( \mathfrak{g} \) is quasi-split if there is a Borel subalgebra \( \mathfrak{b} \in \mathcal{B} \) such that \( \mathfrak{b} \cap \theta(\mathfrak{b}) \) is a Cartan subalgebra of \( \mathfrak{g} \). It is not difficult to show that the symmetric pair \((\mathfrak{g}, \mathfrak{g}^\theta)\) satisfies Equation \((4.1)\) if and only if \( \theta \) is quasi-split.

4.1. Kostant’s Theorem for the \( K \)-action on \( \mathfrak{g} \). We now apply Theorem 4.3 to the coisotropy representation of certain spherical pairs which we call multiplicity free. We consider the following situation. Let \( G \) be a connected, reductive algebraic group, and let \( K \) be a reductive, connected algebraic subgroup. We say that the branching from \( G \) to \( K \) is multiplicity free if for every irreducible, finite dimensional \( G \)-representation \( V \), and every irreducible, finite dimensional \( K \)-representation \( W \) we have \( \dim \text{Hom}_K(W, V) \leq 1 \). Now let \( \tilde{G} = G \times K \) and \( K_\Delta \subset \tilde{G} \) be the diagonal copy of \( K \) in \( G \times K \), i.e.

\[
K_\Delta := \{(g, g) : g \in K\}.
\]

Consider the pair \((\tilde{G}, K_\Delta)\).

**Proposition 4.4.** The pair \((\tilde{G}, K_\Delta)\) is spherical if and only if the branching rule from \( G \) to \( K \) is multiplicity free.

**Proof.** By Remark 4.2 the pair \((\tilde{G}, K_\Delta)\) is spherical if and only if \( \dim V^K_{\Delta} \leq 1 \) for every irreducible, rational \( \tilde{G} \)-representation \( V \). Let \( \tilde{G}, \tilde{K} \) index the sets of irreducible, rational \( G \) and \( K \)-representations respectively. Any irreducible \( \tilde{G} \)-representation is of the form \( V^\lambda \otimes W^\mu \) with \( \lambda \in \tilde{G} \) and irreducible \( G \)-representation an \( W^\mu \), \( \mu \in \tilde{K} \) an irreducible \( K \)-representation. Thus, \((\tilde{G}, K_\Delta)\) is spherical if and only if \( \dim(V^\lambda \otimes W^\mu)^{K_\Delta} \leq 1 \) for all \( \lambda \in \tilde{G} \) and for all \( \nu \in \tilde{K} \).

To compute, \( \dim(V^\lambda \otimes W^\mu)^{K_\Delta} \), we decompose \( V^\lambda \) into irreducible \( K \)-representations:

\[
V^\lambda \cong \bigoplus_{\nu^* \in \tilde{K}} m^\lambda_{\nu^*}(W^\nu)^*, \text{ where } m^\lambda_{\nu^*} := \dim \text{Hom}_K((W^\nu)^*, V^\lambda).
\]

Then we can write:

\[
(4.6) \quad V^\lambda \otimes W^\mu \cong \bigoplus_{\nu^* \in \tilde{K}} m^\lambda_{\nu^*}(W^\nu)^* \otimes W^\mu.
\]

We observe that \((W^\nu)^* \otimes W^\mu)^{K_\Delta} \cong \text{Hom}_K(W^\nu, W^\mu) \). Schur’s Lemma implies that \( \dim(\text{Hom}_K(W^\nu, W^\mu)) \leq 1 \). Equation \((4.6)\) then implies that \( \dim(V^\lambda \otimes W^\mu)^{K_\Delta} \leq 1 \) if and only if \( m^\lambda_{\nu^*} \leq 1 \) for all \( \nu^* \in \tilde{K} \), which is true if and only if \( m^\lambda_{\nu} = \dim \text{Hom}_K(W^\nu, V^\lambda) \leq 1 \) for all \( \nu \in \tilde{K} \). The result follows.
that the branching rule from $G_K = K$. By a result of Knop, there are essentially only two pairs $(G, K)$ such that the branching rule from $G$ to $K$ is multiplicity free. They are $G = GL(n, \mathbb{C})$, $K = GL(n-1, \mathbb{C})$ and $G = SO(n, \mathbb{C})$, $K = SO(n-1, \mathbb{C})$ (either $G$ or $K$ may be replaced by its derived subgroup or a covering group, see [Joh01]). From now on unless otherwise specified, we assume that $g$ by its derived subgroup or a covering group, see [Joh01]). From now on unless otherwise specified, we assume that $g = gl(n, \mathbb{C})$, $\mathfrak{so}(n, \mathbb{C})$, $\mathfrak{k} = gl(n-1, \mathbb{C})$, $\mathfrak{so}(n-1, \mathbb{C})$, and $G = GL(n, \mathbb{C})$, $SO(n, \mathbb{C})$ and $K = GL(n-1, \mathbb{C})$, $SO(n-1, \mathbb{C})$. (The results we prove will also be true for the pair $G = SL(n, \mathbb{C})$ and $K = SL(n-1, \mathbb{C})$.) Let $<< x, y >> = \text{Tr}(xy)$ be the trace form on $g$. It is easy to see that the restriction of $<< \cdot , \cdot >>$ to $\mathfrak{k}$ is non-degenerate. Equip $\tilde{g} = \text{Lie}(\tilde{G}) = g \oplus \mathfrak{k}$ with the non-degenerate invariant form $<< \cdot , \cdot >> = << \cdot , \cdot >> + << \cdot , \cdot >> |_{\mathfrak{k}}$. An easy calculation shows that

$$\mathfrak{k}^\perp_{\Delta} = \{(x, -x_{\mathfrak{k}}) : x \in g, x_{\mathfrak{k}} \in \mathfrak{k}\}.$$  

Note that $\mathfrak{k}^\perp_{\Delta} \cong g$ via the map $(x, -x_{\mathfrak{k}}) \mapsto x$. This isomorphism intertwines the coisotropy representation of $K_{\Delta}$ on $\mathfrak{k}^\perp_{\Delta}$ with the action of $K$ on $\mathfrak{g}$ via conjugation. We can now use the geometry of spherical varieties, in particular Theorem 4.3, to study the geometry of the $K$-conjugation on $\mathfrak{g}$ and the partial Kostant-Wallach map $\Phi_n$ (see (2.16)).

**Lemma 4.5.** Consider the multiplicity-free spherical pairs $(\tilde{G}, K_{\Delta})$.

1. Equation (4.1) holds.
2. $\dim(K) = \dim(K \cdot x)$ if and only if $x \in (\mathfrak{k}^\perp_{\Delta})_{\text{reg}}$.
3. $(\mathfrak{k}^\perp_{\Delta})_{\text{reg}} \cong \{x \in g : \mathfrak{j}_k(x_{\mathfrak{k}}) \cap \mathfrak{j}_k(x) = 0\}$.

**Proof.** Equation (4.1) is equivalent to the routine identity $\dim(\mathcal{B}_{\mathfrak{g}}) + \dim(\mathcal{B}_{\mathfrak{k}}) = \dim(\mathfrak{g}) - \dim(\mathfrak{g} // K)$. The second assertion follows from the first assertion and Equation (4.3). The second assertion implies that $(x, -x_{\mathfrak{k}}) \in (\mathfrak{k}^\perp_{\Delta})_{\text{reg}}$ if and only if $\mathfrak{j}_k(x_{\mathfrak{k}}) \cap \mathfrak{j}_k(x) = 0$. The third assertion now follows since $\mathfrak{j}_k(x_{\mathfrak{k}}) \cap \mathfrak{j}_k(x) = \mathfrak{j}_k(x_{\mathfrak{k}}) \cap \mathfrak{j}_k(x)$.

Q.E.D.

We now prove an analogue of Kostant’s theorem describing the regular elements $\mathfrak{g}_{\text{reg}}$ (see (1.2)) for the regular elements $(\mathfrak{k}^\perp_{\Delta})_{\text{reg}}$ of the coisotropy representation of the spherical pairs $(\tilde{G}, K_{\Delta})$. Let $J := f_{n-1,1}, \ldots, f_{n-1,r_{n-1}}; f_{n,1}, \ldots, f_{n,r_n}$ be the generators of $\mathbb{C}[\mathfrak{g}]^K$. Let \[ \omega_{\mathfrak{g} // K} = df_{n-1,1} \wedge \cdots \wedge f_{n-1,r_{n-1}} \wedge f_{n,1} \wedge \cdots \wedge f_{n,r_n} \in \Omega^{r_{n-1} + r_n}(\mathfrak{g}). \]

**Theorem 4.6.**

$$x \in (\mathfrak{k}^\perp_{\Delta})_{\text{reg}}$$ if and only if $\omega_{\mathfrak{g} // K}(x) \neq 0$. 

Q.E.D.
Proof. We first suppose that $df_{n-1,1}(x) \wedge \cdots \wedge df_{n,r_n}(x) \neq 0$. By Equation (4.2), it follows that $x$ is regular in $\mathfrak{g}$ and $x_\mathfrak{k}$ is regular in $\mathfrak{k}$. Equation (4.3) then implies that $\mathfrak{j}_\mathfrak{k}(x_\mathfrak{k}) \cap \mathfrak{j}_\mathfrak{g}(x) = 0$, so $x \in (\mathfrak{t}_\Delta)^{\text{reg}}$ by Equation (4.7).

Conversely, suppose $x \in (\mathfrak{t}_\Delta)^{\text{reg}}$. Then by Theorem 4.3, $(x, -x_\mathfrak{k}) \in \tilde{\mathfrak{g}}^{\text{reg}}$. Thus, both $x \in \mathfrak{g}$ and $x_\mathfrak{k} \in \mathfrak{k}$ are regular. Hence by Equation (4.2),

\begin{equation}
(4.8) \quad df_{n-1,1}(x_\mathfrak{k}) \wedge \cdots \wedge df_{n-1,r_{n-1}}(x_\mathfrak{k}) \neq 0 \text{ and } df_{n,1}(x) \wedge \cdots \wedge df_{n,r_n}(x) \neq 0.
\end{equation}

Since $x \in (\mathfrak{t}_\Delta)^{\text{reg}}$, $\mathfrak{j}_\mathfrak{k}(x_\mathfrak{k}) \cap \mathfrak{j}_\mathfrak{g}(x) = 0$. It now follows from (4.8) and (4.3) that $\omega_{\mathfrak{g}/K}(x) \neq 0$.

Q.E.D.

Theorem 4.6 has an immediate corollary which is of interest in linear algebra.

Corollary 4.7. Let $x \in \mathfrak{g}$ and suppose that $\mathfrak{j}_\mathfrak{k}(x_\mathfrak{k}) \cap \mathfrak{j}_\mathfrak{g}(x) = 0$. Then $x \in \mathfrak{g}$ and $x_\mathfrak{k} \in \mathfrak{k}$ are both regular.

Proof. This follows by Equation (4.7) and Theorem 4.6.

Q.E.D.

Elements of $(\mathfrak{t}_\Delta)^{\text{reg}}$ regarded as elements of $\mathfrak{g}$ play a major role in our study of the $K$-action on $\mathfrak{g}$, so we give them a special name.

Definition 4.8. An element $x \in \mathfrak{g}$ such that $\mathfrak{j}_\mathfrak{k}(x_\mathfrak{k}) \cap \mathfrak{j}_\mathfrak{g}(x) = 0$ is said to be $n$-strongly regular. We denote the set of $n$-strongly regular elements by $\mathfrak{g}_{\text{nsreg}}$.

Remark 4.9. In [CEa], we defined the set of $n$-strongly regular elements for $\mathfrak{g}$ to be the set of elements for which $df_{n-1,1}(x) \wedge \cdots \wedge df_{n,n}(x) \neq 0$. It follows from Theorem 4.6 that our new definition is consistent with the previous one.

4.2. The $K$-orbit structure of $\mathfrak{g}(0)$. We now study the $K$-orbit structure of the Zariski open subset

$$\mathfrak{g}(0) := \{x \in \mathfrak{g} : \sigma(x_\mathfrak{k}) \cap \sigma(x) \neq \emptyset\}.$$ 

We show that $\mathfrak{g}(0) \subset \mathfrak{g}_{\text{nsreg}}$, and that the $K$-orbit through an element of $\mathfrak{g}(0)$ is closed in $\mathfrak{g}$. The fact that $\mathfrak{g}(0) \subset \mathfrak{g}_{\text{nsreg}}$ follows from the following result in linear algebra.

Lemma 4.10. Let $V$ be a finite dimensional complex vector space. Suppose we are given a direct sum decomposition of $V$

$$V = V_1 \oplus V_2.$$ 

Let $X \in \text{End}(V)$, and let $Y \neq 0 \in \text{End}(V)$ such that $Y : V_1 \to V_1$ and $Y|_{V_2} = 0$. Suppose that $[Y, X] = 0$. Then $X$ has a nonzero eigenvector $u \in V_1$.

Proof. The assumptions imply that the image $\text{Im}(Y)$ of $Y$ is nonzero, contained in $V_1$, and stable under the action of $X$. The result follows.

Q.E.D.
The following consequence plays a crucial role in our study of $g(0)$.

**Proposition 4.11.** Let $X$, $Y$, and $V = V_1 \oplus V_2$ be as in the statement of Lemma 4.10. Define $X_1 : V_1 \to V_1$ by $X_1 := \pi_{V_1} \circ X|_{V_1}$, where $\pi_{V_1} : V \to V_1$ is the projection onto $V_1$ off of $V_2$. Then $\sigma(X_1) \cap \sigma(X) \neq \emptyset$.

**Proof.** Let $u \in V_1$ be an eigenvector of $X$ of eigenvalue $\lambda$. It follows from definitions that

$$X_1 u = \pi_{V_1} (X u) = \pi_{V_1} (\lambda u) = \lambda u$$

Thus, $\lambda \in \sigma(X_1) \cap \sigma(X)$.

Q.E.D.

**Example 4.12.** Let $V = \mathbb{C}^n$, and let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{C}^n$. Let $V_1 = \text{span}\{e_1, \ldots, e_k\}$, and let $V_2 = \text{span}\{e_{k+1}, \ldots, e_n\}$. Let $x \in gl(n, \mathbb{C})$, and let $x_k$ be the $k \times k$ submatrix in the upper lefthand corner of $x$. We embed $gl(k, \mathbb{C})$ in $gl(n, \mathbb{C})$ in the upper left corner. Suppose there exists nonzero $Y \in gl(k, \mathbb{C})$ with $[Y, x] = 0$. Then Proposition 4.11 implies that $\sigma(x_k) \cap \sigma(x) \neq 0$.

We now return to the pairs $(gl(n, \mathbb{C}), gl(n-1, \mathbb{C}))$ and $(so(n, \mathbb{C}), so(n - 1, \mathbb{C})$. Using Proposition 4.11 we can prove a fundamental result regarding the structure of $g(0)$.

**Theorem 4.13.** Let $x \in g(0)$. Then $x \in g_{\text{nsreg}}$.

**Proof.** Let $x \in g$ and suppose that $\mathfrak{z}_g(x) \cap \mathfrak{z}_t(x) \neq 0$. We show that $x \in g(\geq 1)$. We proceed on a case-by-case basis: First, suppose that $g$ is type $A$. Then decompose $V = \mathbb{C}^n$ as $V = V_1 \oplus V_2$ where $V_1 = \text{span}\{e_1, \ldots, e_{n-1}\}$, and $V_2 = \text{span}\{e_n\}$. Now apply Example 4.12. Similarly, when $g = so(2l, \mathbb{C})$ is of type $D$, we decompose $\mathbb{C}^{2l}$ as $V = V_1 \oplus V_2$, where $V_1 = \text{span}\{e_{\pm 1}, \ldots, e_{\pm (l-1)}, e_l + e_{-l}\}$, and $V_2 = \{e_l - e_{-l}\}$. We compute that $t$ annihilates $V_2$. Since the involution $\theta$ acts on $e_{\pm i}$ via $\theta(e_{\pm i}) = e_{\mp i}$ for $i \neq l$ and $\theta(e_l) = e_{-l}$ (see Section 2.3), we have $\theta$ acts on $V_1$ as the identity and on $V_2$ as minus the identity. Therefore $x_t : V_1 \to V_1$ and $x_{g \circ \theta} : V_1 \to V_2$, and it follows that $x_1 = x_t$. The result now follows from Proposition 4.11. The type $B$ case is similar to the previous cases.

Q.E.D.

Let $c = (c_{r_n-1}, c_{r_n}) \in V^{r_n-1, r_n}$ and write $c_{r_i} = (c_{i,1}, \ldots, c_{i,r_i}) \in \mathbb{C}^{r_i}$ for $i = n - 1, n$. Let $I_{n,c}$ be the ideal of $\mathbb{C}[g]$ generated by the functions $f_{i,j} - c_{i,j}$ for $i = n - 1, n$ and $j = 1, \ldots, r_i$.

**Corollary 4.14.** Let $c = (c_{r_n-1}, c_{r_n}) \in V^{r_n-1, r_n}(0)$, so that $\Phi_n^{-1}(c) \subset g(0)$.

1. Then $I_{n,c}$ is radical, so that $I_{n,c}$ is the ideal of the fibre $\Phi_n^{-1}(c)$. Further, the variety $\Phi_n^{-1}(c)$ is smooth.
2. The fibre $\Phi_n^{-1}(c)$ is a single closed $K$-orbit.
Theorem 4.15. (4.9) \( \lim_{t \to -\infty} \text{Ad}(\exp tz) x \in I \cap \overline{K \cdot x} \).

Proof. By Theorem 4.13 every element of the fibre \( \Phi_n^{-1}(c) \) is \( n \)-strongly regular. It follows from Theorem 4.16 that the differentials \( \{ df_{ij}(x) : i = n - 1, n, j = 1, \ldots, r_i \} \) are independent for all \( x \in \Phi_n^{-1}(c) \). By Theorem 18.15 (a) of [Eis95], the ideal \( I_{n,c} \) is radical, so \( I_{n,c} \) is the ideal of \( \Phi_n^{-1}(c) \). The smoothness of \( \Phi_n^{-1}(c) \) now follows since the differentials of the generators of \( I_{n,c} \) are independent at every point of \( \Phi_n^{-1}(c) \). For the second assertion, note first that \( \dim(K) = \dim(\Phi_n^{-1}(c)) \). By Lemma 4.15 \( \dim(K \cdot x) = \dim(K) \) for all \( x \in \Phi_n^{-1}(c) \). By Proposition 2.2 (2), each fibre \( \Phi_n^{-1}(c) \) has a unique closed \( K \)-orbit, which implies the assertion.

Q.E.D.

4.3. Classification of closed \( K \)-orbits on \( g \). In Section 4.2, we showed that \( K \)-orbits in \( g(0) \) are closed. In this section, we describe the other closed \( K \)-orbits in \( g \). Our main tool is Theorem 4.1 when \( g = \mathfrak{so}(n, \mathbb{C}) \), \( K = SO(n - 1, \mathbb{C}) \) and Theorem 3.7 [CEa] when \( g = \mathfrak{gl}(n, \mathbb{C}) \), \( K = GL(n - 1, \mathbb{C}) \times GL(1, \mathbb{C}) \). Recall the varieties \( g(i) = g(\geq i) \setminus g(\geq i + 1) \) defined in (2.21). (The definition is analogous when \( g \) is of Type A. See Equation (3.3) of [CEa].)

Theorem 4.15. Let \( x \in g(i) \). Then \( K \cdot x \) is closed if and only if \( K \cdot x \cap I(i) \neq \emptyset \), where \( I(i) := g(i) \cap \mathfrak{l} \) and \( \mathfrak{l} \) is a \( \theta \)-stable Levi subalgebra of the following form:

1. If \( g = \mathfrak{gl}(n, \mathbb{C}) \), then \( \mathfrak{l} \) is the Levi subalgebra of block diagonal matrices

\[ \mathfrak{l} = \mathfrak{gl}(1, \mathbb{C})^i \oplus \mathfrak{gl}(n - i, \mathbb{C}). \]

2. If \( g = \mathfrak{so}(2l + 1, \mathbb{C}) \), then

\[ \mathfrak{l} = \mathfrak{s}(\mathfrak{gl}(1, \mathbb{C}) \oplus \mathfrak{gl}(1, \mathbb{C}))^i \oplus \mathfrak{so}(2l - i + 1, \mathbb{C}) \]

(see (3.1)).

3. If \( g = \mathfrak{so}(2l, \mathbb{C}) \), then

\[ \mathfrak{l} = \mathfrak{s}(\mathfrak{gl}(1, \mathbb{C}) \oplus \mathfrak{gl}(1, \mathbb{C}))^{i-1} \oplus \mathfrak{so}(2l - i + 2, \mathbb{C}) \]

(see (3.4)), or \( \mathfrak{l} = \mathfrak{h} \) is the standard Cartan subalgebra of diagonal matrices.

Theorem 4.15 will follow from two lemmas.

Lemma 4.16. Let \( x \in g(i) \), and let \( \mathfrak{l} \) be one of the corresponding Levi subalgebras in Theorem 4.15. Then \( K \cdot x \) contains an element of \( \mathfrak{l}(i) \).

Proof. Any element \( x \in g \) is \( K \)-conjugate to an element in a \( \theta \)-stable parabolic subalgebra \( \mathfrak{r} \) with Levi factor \( \mathfrak{l} \). This follows from Corollary 3.7 when \( g = \mathfrak{so}(n, \mathbb{C}) \) and from Theorem 3.7, [CEa] when \( g = \mathfrak{gl}(n, \mathbb{C}) \). So we can assume that \( x \in \mathfrak{r} \). If \( g \) is of type A or B, then \( \mathfrak{h} \subset \mathfrak{k} \). Consider an element \( z \in \mathfrak{h} \subset \mathfrak{k} \) such that and \( \alpha_i(z) > 0 \) for all simple roots \( \alpha_i \in \Phi(g, \mathfrak{h}) \) (see (2.14)). Then

\[ (4.9) \lim_{t \to -\infty} \text{Ad}(\exp tz) x \in I \cap \overline{K \cdot x}. \]
If $\mathfrak{g} = \mathfrak{so}(2l, \mathbb{C})$, then $\mathfrak{r}$ is the parabolic subalgebra of Theorem 3.2 and $\mathfrak{h} \cap \mathfrak{k} = \text{diag}[a_1, \ldots, a_{l-1}, 0, 0, -a_{l-1}, \ldots, -a_1]$. Let $x = \text{diag}[(l-1, l-2, \ldots, 1, 0, 0, -1, \ldots, -(l-1)]$, then $\alpha(z) > 0$ for all simple roots $\alpha$ (see Equation (2.9)). Therefore Equation (4.9) also holds in this setting.

Q.E.D.

We now study the $K$-orbits of elements in $I$.

**Lemma 4.17.** Let $I$ be one of the Levi subalgebras in Theorem 4.15. Any two elements in $I(i)$ which lie in the same fibre of the partial Kostant-Wallach map $\Phi_n$ are $K$-conjugate.

**Proof.** Suppose that $x, y \in \Phi_n^{-1}(c)$, with $c \in V^{r_{n-1}, r_n}(0)$. Then Corollary 4.14 implies that $x, y$ are $K$-conjugate.

Now suppose that $x, y \in \Phi_n^{-1}(c) \cap I$ with $c \in V^{r_{n-1}, r_n}(i)$ with $i > 0$. Decompose $x$ and $y$ as $x = x_3 + x_{ss}$ and $y = y_3 + y_{ss}$, with $x_3, y_3 \in \mathfrak{z}$ and $x_{ss}, y_{ss} \in I_{ss}$. Then $\sigma(x) \cap \sigma(x_{ss})$ are the coordinates of $x_3$ and similarly for $y$. Since $\Phi_n(x) = \Phi_n(y) \in V^{r_{n-1}, r_n}(i)$, we have $\sigma(x) \cap \sigma(x_{ss}) = \sigma(y) \cap \sigma(y_{ss})\cdotKI$. It follows that there exists a $w \in N_K(L \cap K)/(L \cap K)$ such that $\text{Ad}(w) x_3 = y_3$. So without loss of generality, we may assume that $x_3 = y_3$. Since $x, y \in \mathfrak{g}(i) \cap I$, $x_{ss}, y_{ss} \in I_{ss}(0)$, where

$$I_{ss}(0) := \{z \in I_{ss} : \sigma(z) \cap \sigma(z_{ss}) = \emptyset\}.$$ 

Let $\Phi_{I_{ss}} : I_{ss} \to \mathbb{C}^{\operatorname{rk}(I_{ss})} \times \mathbb{C}^{\operatorname{rk}(I_{ss})}$ be the partial Kostant-Wallach map for $I_{ss}$. Then $\Phi_n(x) = \Phi_n(y)$ implies that $\Phi_{I_{ss}}(x_{ss}) = \Phi_{I_{ss}}(y_{ss})$. But then Corollary 4.14 applied to $I_{ss}$ forces $x_{ss}$ and $y_{ss}$ to lie in the same $K \cap I_{ss}$-orbit. This completes the proof.

Q.E.D.

Using Lemmas 4.16 and 4.17 we can prove Theorem 4.15.

**Proof of Theorem 4.15.** Suppose that $x \in \mathfrak{g}$ with $\text{Ad}(K) \cdot x$ closed. Then by Lemma 4.16, there exists an element $z \in I \cap \text{Ad}(K) \cdot x$. But since $\text{Ad}(K) \cdot x$ is closed, we have $\text{Ad}(K) \cdot x = \text{Ad}(K) \cdot z$.

Conversely, suppose that $x \in I$ and consider $\text{Ad}(K) \cdot x$. By Lemma 4.16 there exists a $z \in I$ with $K \cdot z$ closed and $K \cdot z \subset \text{Ad}(K) \cdot x$. But then $z$ and $x$ are in the same fibre of the partial Kostant-Wallach map. Therefore $\text{Ad}(K) \cdot x = \text{Ad}(K) \cdot z$ by Lemma 4.17. Thus, $\text{Ad}(K) \cdot x$ is closed.

Q.E.D.

4.4. The nilfibre of the partial Kostant-Wallach map in the orthogonal case.

Though there are many similarities between the $GL(n - 1, \mathbb{C})$-action on $\mathfrak{g}(n, \mathbb{C})$ and the $SO(n - 1, \mathbb{C})$-action on $\mathfrak{so}(n, \mathbb{C})$, in this subsection we show that their $n$-strongly regular sets are different. In the case of $\mathfrak{g}(n, \mathbb{C})$, every fibre of the partial Kostant-Wallach map contains $n$-strongly regular elements. This follows easily from Theorem 2.3 of [KW06a].
However, this is not the case for \( \mathfrak{so}(n, \mathbb{C}) \). To see this, we need to study the nilfibre \( \Phi_n^{-1}(0, 0) \) of the orthogonal partial Kostant-Wallach map in more detail using Theorems 1.1 and 4.6.

**Theorem 4.18.** Let \( \Phi_n : \mathfrak{so}(n, \mathbb{C}) \to \mathbb{C}^{n-1} \oplus \mathbb{C}^n \) be the orthogonal partial Kostant-Wallach map \( \Phi_n \) defined in Equation (2.76).

Case I: Suppose \( n = 2l \). Then \( \Phi_n^{-1}(0, 0) = K \cdot n_+ \) is irreducible, where \( n_+ = [b_+, b_+] \) and \( Q_+ = K \cdot b_+ \) is the unique closed \( K \)-orbit in \( B \) (see Part (2) of Proposition 2.21).

Case II: Suppose \( n = 2l + 1 \). Then \( \Phi_n^{-1}(0, 0) = K \cdot n_+ \cup K \cdot n_- \) has two irreducible components, where \( n_+ = [b_+, b_+] \) and \( Q_+ = K \cdot b_+ \), are the two closed \( K \)-orbits in \( B \) (see Part (2) of Proposition 2.21).

**Proof.** Let \( Q = K \cdot b \) be a closed \( K \)-orbit. Let \( n = [b, b] \) be the nilradical of \( b \). We first show that \( \text{Ad}(K) \cdot n \) is an irreducible component of \( \Phi_n^{-1}(0, 0) \). Since \( Q \) is closed, \( b \) is \( \theta \)-stable by Corollary 1 of [BH00]. Thus, \( b \cap \mathfrak{k} \) is a Borel subalgebra of \( \mathfrak{t} \) with nilradical \( \mathfrak{n} \cap \mathfrak{k} \). It follows that for any \( x \in n \), we have \( \Phi_n(x) = (0, 0) \). By the \( K \)-equivariance of \( \Phi_n \), \( \text{Ad}(K) \cdot n \subset \Phi_n^{-1}(0, 0) \).

Recall the Grothendieck resolution \( \tilde{\mathfrak{g}} = \{(x, b) : x \in \mathfrak{b} \} \subset \mathfrak{g} \times B \) and the morphisms \( \pi : \tilde{\mathfrak{g}} \to B, \pi(x, b) = b \) and \( \mu : \tilde{\mathfrak{g}} \to \mathfrak{g} \), \( \mu(x, b) = x \). Corollary 3.1.33 of [CG97] gives a \( G \)-equivariant isomorphism \( \tilde{\mathfrak{g}} \cong G \times_B \mathfrak{b} \). Under this isomorphism \( \pi^{-1}(Q) \) is identified with the closed subvariety \( K \times_{K \cap B} \mathfrak{b} \subset G \times_B \mathfrak{b} \). The closed subvariety \( K \times_{K \cap B} \mathfrak{n} \subset K \times_{K \cap B} \mathfrak{b} \) maps surjectively under \( \mu \) to \( \text{Ad}(K) \cdot \mathfrak{n} \). Since \( \mu \) is proper, \( \text{Ad}(K) \cdot \mathfrak{n} \) is closed and irreducible.

We also note that the restriction of \( \mu \) to \( K \times_{K \cap B} \mathfrak{n} \) generically has finite fibres (Proposition 3.2.14 of [CG97]). Thus, the same reasoning that we used in Equation (2.25) along with Propositions 2.11 and 2.6 shows that

\[
\dim \text{Ad}(K) \cdot n = \dim (K \times_{K \cap B} \mathfrak{n}) = \dim (Y_Q) - r_n = \dim (\mathfrak{g} \cap \mathfrak{n}) - r_n = \dim (\mathfrak{g}) - r_{n-1} - r_n = \dim \Phi_n^{-1}(0, 0).
\]

Thus, \( \text{Ad}(K) \cdot n \) is an irreducible component of \( \Phi_n^{-1}(0, 0) \).

We now show that every irreducible component of \( \Phi_n^{-1}(0, 0) \) is of the form \( \text{Ad}(K) \cdot n \). It follows from definitions that \( \Phi_n^{-1}(0, 0) \subset \mathfrak{g}(r_{n-1}) \cap \mathcal{N} \), where \( \mathcal{N} \subset \mathfrak{g} \) is the nilpotent cone in \( \mathfrak{g} \). Thus, if \( \mathfrak{X} \) is an irreducible component of \( \Phi_n^{-1}(0, 0) \), then \( \mathfrak{X} \subset \text{Ad}(K) \cdot n \) by Theorem 1.1 with \( i = r_{n-1} \). But then \( \mathfrak{X} = \text{Ad}(K) \cdot n \) by Proposition 2.2.

**Q.E.D.**

We use Theorem 4.18 to study \( \Phi_n^{-1}(0, 0) \) in more detail. In [CEa], we studied the interaction between the set of \( n \)-strongly regular elements for the pair \( (\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C}), \mathfrak{t} = \mathfrak{gl}(n - 1, \mathbb{C})) \) and the nilfibre of the corresponding partial Kostant-Wallach map. The
geometry of the nilfibre appears to be very different in the orthogonal case. We now show that unlike in the case of \( \mathfrak{gl}(n, \mathbb{C}) \), there are no \( n \)-strongly regular elements in the nilfibre of the partial Kostant-Wallach map for the orthogonal Lie algebra \( \mathfrak{so}(n, \mathbb{C}) \). The key observation is the following proposition, which can be viewed as a generalization of Proposition 3.8 in [CE12].

**Proposition 4.19.** Let \( n > 3 \), let \( \mathfrak{g} = \mathfrak{so}(n, \mathbb{C}) \), and let \( K = SO(n-1, \mathbb{C}) \). Let \( \mathfrak{b} \subset \mathfrak{g} \) be a Borel subalgebra and suppose that the \( K \)-orbit \( K \cdot \mathfrak{b} \) is closed in \( B \). Let \( \mathfrak{n} = [\mathfrak{b}, \mathfrak{b}] \) be the nilradical of \( \mathfrak{b} \). Then

\[
\mathfrak{z}(\mathfrak{n} \cap \mathfrak{t}) \cap \mathfrak{z}(\mathfrak{n}) \neq 0.
\]

**Proof.** Consider a closed \( K \)-orbit \( Q \) in \( B \). By \( K \)-equivariance, it suffices to show Equation (4.11) for a representative \( \mathfrak{b} \) of \( Q \). By Part (2) of Propositions 2.20 and 2.21, we can assume that the standard diagonal Cartan subalgebra \( \mathfrak{h} \) is in \( \mathfrak{b} \). Let \( \phi \in \Phi^+(\mathfrak{g}, \mathfrak{h}) \) be the highest root of \( \mathfrak{b} \). We claim for \( n > 4 \) that \( \phi \) is compact imaginary. It then follows that the root space \( \mathfrak{g}_\phi \in \mathfrak{z}(\mathfrak{n}_k) \cap \mathfrak{z}(\mathfrak{n}) \).

Suppose first that \( \mathfrak{g} = \mathfrak{so}(2l, \mathbb{C}) \). By Part (2) of Proposition 2.21 we can assume that \( \mathfrak{b} = \mathfrak{b}_+ \). The highest root is then \( \epsilon_1 + \epsilon_2 \), which is compact imaginary for \( l > 2 \) (Example 2.14). If \( \mathfrak{g} = \mathfrak{so}(2l + 1, \mathbb{C}) \), then by Part (2) of Proposition 2.20 we can assume that \( \mathfrak{b} = \mathfrak{b}_+ \) or \( \mathfrak{b} = \mathfrak{b}_- = s_{\alpha_l}(\mathfrak{b}_+) \). In both cases, the highest root is \( \epsilon_1 + \epsilon_2 \), which is compact imaginary (Example 2.14).

If \( \mathfrak{g} = \mathfrak{so}(4, \mathbb{C}) \), then \( \phi = \epsilon_1 + \epsilon_2 \) is complex \( \theta \)-stable. We compute directly that \( (\mathfrak{g}_\phi \oplus \mathfrak{g}_{\theta(\phi)})^\theta \subset \mathfrak{z}(\mathfrak{n}_k) \cap \mathfrak{z}(\mathfrak{n}) \).

Q.E.D.

**Corollary 4.20.** Let \( n > 3 \), and let \( \Phi_n : \mathfrak{so}(n, \mathbb{C}) \rightarrow \mathbb{C}^r \oplus \mathbb{C}^r \) be the orthogonal partial Kostant-Wallach map. Then \( \Phi_n^{-1}(0, 0) \) contains no \( n \)-strongly regular elements.

**Proof.** Suppose \( x \in \Phi_n^{-1}(0, 0) \), so by Theorem 4.18 \( x \) is contained in \( \mathfrak{n} \), the nilradical of a Borel subalgebra \( \mathfrak{b} \) with \( K \cdot \mathfrak{b} \) closed. By Proposition 4.19 there is a nonzero element \( y \) of \( \mathfrak{z}(\mathfrak{n} \cap \mathfrak{t}) \cap \mathfrak{z}(\mathfrak{n}) \). Then \( y \in \mathfrak{z}(x) \cap \mathfrak{z}(x_\mathfrak{F}) \), so \( x \) is not \( n \)-strongly regular.

Q.E.D.

For \( n = 3 \) there are strongly regular elements in \( \Phi_n^{-1}(0, 0) \), essentially because \( \mathfrak{so}(3, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C}) \).

The following result is analogous to Proposition 3.11 of [CE1a]. We let \( I_n \) be the ideal of \( \mathbb{C}[\mathfrak{so}(n, \mathbb{C})] \) generated by elements of \( \mathbb{C}[\mathfrak{so}(n, \mathbb{C})]^{SO(n, \mathbb{C})} \) of positive degree.

**Corollary 4.21.** The ideal \( I_n \) is radical if and only if \( n \leq 3 \).
Proof. By Theorem 18.15 (a) of [Eis95], the ideal \( I_n \) is radical if and only if the set of differentials \( \{ df_i(x) : j = 1, \ldots, r_i, i = n - 1, n \} \) is linearly independent on an open, dense subset of each irreducible component of \( \Phi_n^{-1}(0,0) \). It follows from Definition 4.8 and Theorem 4.3 that \( I_n \) is radical if and only if each irreducible component of \( \Phi_n^{-1}(0,0) \) contains \( n \)-strongly regular elements. But it follows from Corollary 4.20 and the case of \( \mathfrak{so}(3, \mathbb{C}) \) that each irreducible component of \( \Phi_n^{-1}(0,0) \) contains \( n \)-strongly regular elements if and only if \( n = 3 \).

Q.E.D.

Note that we have derived results concerning the \( n \)-strongly regular set without using a slice, in contrast to the case of \( \mathfrak{gl}(n, \mathbb{C}) \) studied by Kostant and Wallach [KW06a].

5. Appendix

In the appendix, we prove a general result which implies Proposition 2.2. The proof is an adaptation of the proof of Proposition 2.3 from [CEa].

Theorem 5.1. Let \((G,H)\) be a spherical pair such that

\[
\dim \mathcal{B} = \dim \mathfrak{h}^\perp - \dim \mathfrak{h}^\perp//H.
\]

(cf Equation (4.1)). Then \( \Psi : \mathfrak{h}^\perp \to \mathfrak{h}^\perp//H \) is flat.

Proof. We first show that \( \Psi^{-1}(0) \) is equi-dimensional, of dimension \( \dim \mathfrak{h}^\perp - \dim \mathfrak{h}^\perp//H \). Let \( C \) be an irreducible component of \( \Psi^{-1}(0) \). By standard results, \( \dim(C) \geq \dim \mathfrak{h}^\perp - \dim \mathfrak{h}^\perp//H \). Label the finite number of \( H \)-orbits on \( \mathcal{B} \) by \( Q_1, \ldots, Q_s \). Let \( Z = \bigcup_{i=1}^s T_{Q_i}(\mathcal{B}) \) and note that the irreducible components of \( Z \) are the subvarieties \( Z_i := T_{Q_i}(\mathcal{B}) \), and also that \( \dim Z_i = \dim \mathcal{B} \) for \( i = 1, \ldots, s \). Recall the standard identification \( T^*\mathcal{B} = \{ (b, x) \in \mathcal{B} \times \mathfrak{g} : x \in \mathfrak{b}, b \} \) and let \( \mu : T^*\mathcal{B} \to \mathfrak{g} \) be the moment map, \( \mu(b, x) = x \).

We claim that \( \Psi^{-1}(0) \subset \mu(Z) \). Indeed, by Theorem 6 of [Pan90], \( \mathbb{C}[\mathfrak{h}^\perp]^H = \mathbb{C}[g_1, \ldots, g_k] \) is a polynomial ring in \( k \) generators, which can be taken to be homogeneous. Further, the morphism \( \Psi : \mathfrak{h}^\perp \to \mathfrak{h}^\perp//H \) may be identified with \( (g_1, \ldots, g_k) : \mathfrak{h}^\perp \to \mathbb{C}^k \). For \( f \in \mathbb{C}[\mathfrak{g}]^G \) of positive degree, note that \( f|_{\mathfrak{h}^\perp} \in \mathbb{C}[\mathfrak{h}^\perp]^H \), so \( f|_{\mathfrak{h}^\perp} \) is a polynomial of strictly positive degree in the variables \( g_1, \ldots, g_k \). By the above identification, \( g_i(x) = 0 \) for each \( x \in \Psi^{-1}(0) \), so \( f(x) = 0 \). By Proposition 16 of [Kos63], it follows that \( x \) is nilpotent, and hence lies in \( \mathfrak{n} := [\mathfrak{b}, \mathfrak{b}] \), the nilradical of a Borel subalgebra \( \mathfrak{b} \). Thus, if \( Q_i = H \cdot b \), then \( (b, x) \in Z_i \), and \( x = \mu(b, x) \in \mu(Z) \).

Since \( \mu \) is proper, it follows that \( C \subset \mu(Z_i) \) for some \( i \). Hence, \( \dim C \leq \dim Z_i = \dim \mathcal{B} = \dim \mathfrak{h}^\perp - \dim \mathfrak{h}^\perp//H \), so \( \dim C = \dim \mathfrak{h}^\perp - \dim \mathfrak{h}^\perp//H \).

To prove the proposition, for \( x \in \mathfrak{h}^\perp \), let \( d_x \) be the maximum of the dimension of the irreducible components of \( \Psi^{-1}(\Psi(x)) \). Since the functions \( g_1, \ldots, g_k \) are homogeneous, scalar multiplication by \( \lambda \in \mathbb{C}^* \) induces an isomorphism \( \Psi^{-1}(\Psi(x)) \cong \Psi^{-1}(\Psi(\lambda x)) \), so
By upper semi-continuity of dimension, the set \( \{ y \in h^\perp : d_y \geq d \} \) is closed for each integer \( d \) (Proposition 4.4 of [Hum75]). Hence, \( d_y \leq d_0 = \dim h^\perp - \dim h^\perp // H \) for all \( y \in h^\perp \). It follows that \( d_y = \dim h^\perp - \dim h^\perp // H \) for all \( y \in h^\perp \). Hence, \( \Psi \) is flat by the corollary to Theorem 23.1 of [Mat86].

Q.E.D.

References

[BH00] Michel Brion and Aloysius G. Helminck, *On orbit closures of symmetric subgroups in flag varieties*, Canad. J. Math. **52** (2000), no. 2, 265–292.

[Bri87] M. Brion, *Classification des espaces homogènes sphériques*, Compositio Math. **63** (1987), no. 2, 189–208.

[CEa] Mark Colarusso and Sam Evens, *Eigenvalue coincidences and K-orbits, I*, to appear in the Journal of Algebra, 2014, 22 pages.

[CEb] Mark Colarusso and Sam Evens, *The Gelfand-Zeitlin integrable system and K-orbits on the flag variety*, to appear in: “Symmetry: Representation Theory and its Applications: ,” Progr. Math. Birkhäuser, Boston, 2014, 36 pages.

[CE12] Mark Colarusso and Sam Evens, *K-orbits on the flag variety and strongly regular nilpotent matrices*, Selecta Math. (N.S.) **18** (2012), no. 1, 159–177.

[CG97] Neil Chriss and Victor Ginzburg, *Representation theory and complex geometry*, Birkhäuser Boston Inc., Boston, MA, 1997.

[Col85] David H. Collingwood, *Representations of rank one Lie groups*, Research Notes in Mathematics, vol. 137, Pitman (Advanced Publishing Program), Boston, MA, 1985.

[Col09] Mark Colarusso, *The Gelfand-Zeitlin integrable system and its action on generic elements of gl(n) and so(n)*, New Developments in Lie Theory and Geometry (Cruz Chica, Córdoba, Argentina, 2007), Contemp. Math., vol. 491, Amer. Math. Soc., Providence, RI, 2009, pp. 255–281.

[DFO94] Yu. A. Drozd, V. M. Futorny, and S. A. Ovsienko, *Harish-Chandra subalgebras and Gelfand-Zetlin modules*, Finite-dimensional algebras and related topics (Ottawa, ON, 1992), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 424, Kluwer Acad. Publ., Dordrecht, 1994, pp. 79–93.

[Eis95] David Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995, With a view toward algebraic geometry.

[Gro03] Alexander Grothendieck, *Revêtements Étales et groupe fondamental (SGA 1)*, Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 3, Société Mathématique de France, Paris, 2003, Séminaire de Géométrie Algébrique du Bois Marie, 1960-1961, Augmenté de deux exposés de Michèle Raynaud. [With two exposés by Michèle Raynaud].

[GW98] Roe Goodman and Nolan R. Wallach, *Representations and invariants of the classical groups*, Encyclopedia of Mathematics and its Applications, vol. 68, Cambridge University Press, Cambridge, 1998.

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52.

[HJ85] Roger A. Horn and Charles R. Johnson, *Matrix analysis*, Cambridge University Press, Cambridge, 1985.

[Hum75] James E. Humphreys, *Linear algebraic groups*, Springer-Verlag, New York, 1975, Graduate Texts in Mathematics, No. 21.

[Joh01] Kenneth D. Johnson, *A note on branching theorems*, Proc. Amer. Math. Soc. **129** (2001), no. 2, 351–353.

[Kna02] Anthony W. Knapp, *Lie groups beyond an introduction*, second ed., Progress in Mathematics, vol. 140, Birkhäuser Boston Inc., Boston, MA, 2002.
[Kno90] Friedrich Knop, *Weylgruppe und Momentabbildung*, Invent. Math. **99** (1990), no. 1, 1–23.

[Kno94] Friedrich Knop, *A Harish-Chandra homomorphism for reductive group actions*, Ann. of Math. (2) **140** (1994), no. 2, 253–288.

[Kos63] Bertram Kostant, *Lie group representations on polynomial rings*, Amer. J. Math. **85** (1963), 327–404.

[KW06a] Bertram Kostant and Nolan Wallach, *Gelfand-Zeitlin theory from the perspective of classical mechanics. I*, Studies in Lie theory, Progr. Math., vol. 243, Birkhäuser Boston, Boston, MA, 2006, pp. 319–364.

[KW06b] Bertram Kostant and Nolan Wallach, *Gelfand-Zeitlin theory from the perspective of classical mechanics. II*, The unity of mathematics, Progr. Math., vol. 244, Birkhäuser Boston, Boston, MA, 2006, pp. 387–420.

[Mat86] Hideyuki Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1986, Translated from the Japanese by M. Reid.

[Pan90] D. I. Panyushev, *Complexity and rank of homogeneous spaces*, Geom. Dedicata **34** (1990), no. 3, 249–269.

[Ric82] R. W. Richardson, *Orbits, invariants, and representations associated to involutions of reductive groups*, Invent. Math. **66** (1982), no. 2, 287–312.

[Ste68] Robert Steinberg, *Endomorphisms of linear algebraic groups*, Memoirs of the American Mathematical Society, No. 80, American Mathematical Society, Providence, R.I., 1968.

[VK78] É. A. Vinberg and B. N. Kimel’fel’d, *Homogeneous domains on flag manifolds and spherical subsets of semisimple Lie groups*, Funktsional. Anal. i Prilozhen. **12** (1978), no. 3, 12–19, 96.

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