INCOMPRESSIBLE NAVIER-STOKES-FOURIER-MAXWELL SYSTEM WITH OHM’S LAW LIMIT FROM VLASOV-MAXWELL-BOLTZMANN SYSTEM: HILBERT EXPANSION APPROACH

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Abstract. We prove a global-in-time classical solution limit from the two-species Vlasov-Maxwell-Boltzmann system to the two-fluid incompressible Navier-Stokes-Fourier-Maxwell system with Ohm’s law. Besides the techniques developed for the classical solutions to the Vlasov-Maxwell-Boltzmann equations in the past years, such as the nonlinear energy method and micro-macro decomposition are employed, key roles are played by the decay properties of both the electric field and the wave equation with linear damping of the divergence free magnetic field. This is a companion paper of [24] in which Hilbert expansion is not employed.

1. Introduction

1.1. Vlasov-Maxwell-Boltzmann system. Two-species Vlasov-Maxwell-Boltzmann system (in brief, VMB) describes the evolution of a gas of two species of oppositely charged particles (cations of charge \( q^+ > 0 \) and mass \( m^+ > 0 \), and anions of charge \( -q^- < 0 \) and mass \( m^- > 0 \)), subject to auto-induced electromagnetic forces. Such a gas of charged particles, under a global neutrality condition, is called a plasma. The particle number densities \( F^+(t, x, v) \geq 0 \) and \( F^-(t, x, v) \geq 0 \) represent the distributions of the positively charged ions (i.e. cations), and the negatively charged ions (i.e. anions) at time \( t \geq 0 \), position \( x \in \mathbb{T}^3 \), with velocity \( v \in \mathbb{R}^3 \), respectively. Precisely, VMB system consists the following equations:

\[
\begin{align*}
\partial_t F^+ + v \cdot \nabla_x F^+ + \frac{q^+}{m^+} (E + v \times B) \cdot \nabla_v F^+ &= Q(F^+, F^+) + Q(F^+, F^-), \\
\partial_t F^- + v \cdot \nabla_x F^- - \frac{q^-}{m^-} (E + v \times B) \cdot \nabla_v F^- &= Q(F^-, F^-) + Q(F^-, F^+), \\
\mu_0 \varepsilon_0 \partial_t E - \nabla_x \times B &= -\mu_0 \varepsilon_0 \int_{\mathbb{R}^3} (q^+ F^+ - q^- F^-) \, dv, \\
\partial_t B + \nabla_x \times E &= 0, \\
\text{div}_x E &= \frac{1}{\varepsilon_0} \int_{\mathbb{R}^3} (q^+ F^+ - q^- F^-) \, dv, \\
\text{div}_x B &= 0.
\end{align*}
\]

The evolutions of the densities \( F^\pm \) are governed by the Vlasov-Boltzmann equations, which are the first two lines in (1.1). They tell that the variations of the densities \( F^\pm \) along the trajectories of the particles are subject to the influence of a Lorentz force and inter-particle collisions in the gas. The Lorentz force acting on the gas is auto-induced. That is, the electric field \( E(t, x) \) and the magnetic field \( B(t, x) \) are generated by the motion of the particles in the plasma itself. Their motion is governed by the Maxwell’s equations, which are the remaining equations in (1.1), namely Ampère equation, Faraday’s equation and Gauss’ laws. In (1.1), the physical constants \( \mu_0, \varepsilon_0 > 0 \) are, respectively, the vacuum permeability (or magnetic constant) and the vacuum permittivity (or electric constant). Note that their relation to the speed of light is the formula \( c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \). For the sake of mathematical convenience, we make the simplification that both kinds of particles have the same mass \( m^\pm = m > 0 \) and charge \( q^\pm = q > 0 \).

The collision between particles is given by the standard Boltzmann collision operator \( Q(f, h) \) with hard potential. Let \( f(v), h(v) \) be two number density functions for two types of
particles with the same mass, then \( Q(f, h)(v) \) is defined as

\[
Q(f, h) = \int_{\mathbb{R}^3} \int_{S^2} (f'(h) - f h) b(v - v_s, \sigma) d\sigma dv_s ,
\]

where \( \sigma \in S^2 \) and

\[
b(v - v_s, \sigma) = |v - v_s|^{\gamma} \hat{b}(\cos \theta)
\]

for \( \gamma \in [0, 1] \). For convenience, we take \( \hat{b}(\cos \theta) \) such that \( \int_{S^2} \hat{b}(\cos \theta) d\sigma = 1 \). Here we have used the standard abbreviations

\[
f = f(v), \quad f' = f(v'), \quad h_s = h(v_s), \quad h'_s = h(v'_s)
\]

with \((v', v'_s)\) given by

\[
v' = v + [(v - v_s) \cdot \sigma] \sigma,
\]

\[
v'_s = v_s - [(v - v_s) \cdot \sigma] \sigma,
\]

which denote the velocities after a collision of particles having velocities \( v, v_s \) before the collision. We remark that (1.4) is derived from the conservation of momentum and energy during the collision process. The famous Boltzmann’s \( H \)-theorem indicates that the equilibriums of the collision operator \( Q \), i.e. the distributions \( f(v) \) so that \( Q(f, f) = 0 \) must have the form of Maxwellians:

\[
f(v) \equiv M(\rho, u, \theta) = \frac{\rho}{\sqrt{2\pi \theta}} \exp \left( -\frac{|v-u|^2}{2\theta} \right)
\]

for some \((\rho, u, \theta)\).

There have been extensive research on the well-posedness of the VMB. DiPerna-Lions developed a theory of global-in-time renormalized solutions with large initial data, in particular to the Boltzmann equation [8], Vlasov-Maxwell equations [7] and Vlasov-Poisson-Boltzmann equation [27, 28]. But for VMB there are severe difficulties, among which the major one is that the a priori bounds coming from physical laws are not enough to prove the existence of global solutions, even in the renormalized sense. Recently, Arsénio and Saint-Raymond [3, 2] eventually established global-in-time renormalized solutions with large initial data for VMB, both cut-off and non-cutoff collision kernels. We emphasize that by far renormalized solutions are still the only existing theory for solutions without any smallness requirements on initial data. On the other line, in the context of classical solutions, through a so-called nonlinear energy method, Guo [16] constructed a classical solution of VMB near the global Maxwellian. Guo’s work inspired many results on VMB with more general collision kernels among which we only mention results for the most general collision kernels with or without angular cutoff assumptions, see [9, 10, 11].

1.2. Hydrodynamic limits of Vlasov-Maxwell-Boltzmann system. One of the most important properties of kinetic equations is their connection to fluid equations in the regime where Knudsen number \( \varepsilon \) is small. Hydrodynamic limits from kinetic equations have been an active research field for decades. Among many research results in this field, the most successful program is the so-called BGL program (named after Bardos-Golse-Levermore [4]) which aimed to establish the limit between DiPerna-Lions solutions of the Boltzmann equation and Leray solutions of incompressible Navier-Stokes equations. The BGL program was completed by Golse and Saint-Raymond [14, 15], and for the domain with boundary, see [31, 25]. However, for the VMB, the corresponding hydrodynamic limits are much harder, even at the formal level, since it is coupled with Maxwell equations which are essentially hyperbolic. In a recent remarkable breakthrough [3], Arsénio and Saint-Raymond not only proved the existence of renormalized solutions of VMB, as mentioned above, more importantly, also justified various limits (depending on the scalings) towards incompressible viscous electro-magneto-hydrodynamics. Among these limits, the most singular one is from renormalized solutions of two-species VMB to dissipative solutions of the two-fluid incompressible
Navier-Stokes-Fourier-Maxwell (in brief, NSFM) system with Ohm’s law. More precisely, let the global Maxwellian distribution $M(v)$ be

$$M(v) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{|v|^2}{2} \right),$$

and let the two-species Vlasov-Maxwell-Boltzmann system be scaled as

$$\begin{align*}
\varepsilon \partial_t F_{\varepsilon}^\pm + v \cdot \nabla_x F_{\varepsilon}^\pm &\pm (\varepsilon E_\varepsilon + v \times B_\varepsilon) \cdot \nabla_v F_{\varepsilon}^\pm = \frac{1}{\varepsilon} Q(F_{\varepsilon}^\pm, F_{\varepsilon}^\mp) + \frac{1}{\varepsilon} Q(F_{\varepsilon}^\pm, F_{\varepsilon}^\mp), \\
F_{\varepsilon}^\pm &= M(1 + \varepsilon g_{\varepsilon}^\pm), \\
\partial_t E_\varepsilon - \nabla_x \times B_\varepsilon &= -\frac{1}{\varepsilon} \int_{\mathbb{R}^3} (g_\varepsilon^+ - g_\varepsilon^-) v Mdv, \\
\partial_t B_\varepsilon + \nabla_x \times E_\varepsilon &= 0, \\
\text{div}_x E_\varepsilon &= \int_{\mathbb{R}^3} (g_\varepsilon^+ - g_\varepsilon^-) Mdv, \\
\text{div}_x B_\varepsilon &= 0.
\end{align*}$$

(1.5)

They proved that a sequence of fluctuations $\{g_{\varepsilon}^\pm\}$ has a weak limit satisfies the following NSFM with Ohm’s law:

$$\begin{align*}
\partial_t u + u \cdot \nabla_x u - \mu \Delta_x u + \nabla_x p &= \frac{1}{2} n E + \frac{1}{2} j \times B, \\
\text{div}_x u &= 0, \\
\partial_t \theta + u \cdot \nabla_x \theta - \kappa \Delta_x \theta &= 0, \\
\partial_t E - \nabla_x \times B &= -j, \\
\partial_t B + \nabla_x \times E &= 0, \\
j &= nu + \sigma (-\frac{1}{2} \nabla_x n + E + u \times B), \\
\text{div}_x E &= n, \quad \text{div}_x B &= 0,
\end{align*}$$

(1.6)

where $\mu$, $\kappa$ and $\sigma$ are positive constants can be determined (see Section 2).

The proofs in [3] for justifying the weak limit from a sequence of solutions of VMB (1.5) to a dissipative solution of incompressible NSFM (1.6) are extremely hard. Part of the reasons are, besides many difficulties of the existence of renormalized solutions of VMB itself, our current understanding for the incompressible NSFM with Ohm’s law is far from complete. From the point view of mathematical analysis, NSFM have a behavior which is more similar to the much less understood incompressible Euler equations than to the Navier-Stokes equations. That is the reason in [3], they consider the so-called dissipative solutions of NSFM rather than the usual weak solutions. The dissipative solutions are were introduced by Lions for 3-dimensional incompressible Euler equations (see section 4.4 of [29]).

The studies of incompressible NSFM just started in recent years (for the introduction of physical background, see [5, 6]). For weak solutions, the existence of global in time Leray type weak solutions are completely open, even in 2-dimension. A first breakthrough comes from Masmoudi [30], who in 2-dimensional case proved the existence and uniqueness of global strong solutions of incompressible NSFM (in fact, the system he considered in [30] is little different with the NSFM in this paper, but the analytic analysis are basically the same) for the initial data $(u^{in}, E^{in}, B^{in}) \in L^2(\mathbb{R}^2) \times (H^s(\mathbb{R}^2))^2$ with $s > 0$. It is notable that in [30], the divergence-free condition of the magnetic field $B$ or the decay property of the linear part coming from Maxwell’s equations is not used. Ibrahim and Keraani [19] considered the data $(u^{in}, E^{in}, B^{in}) \in \dot{B}^1_{2,1}(\mathbb{R}^3) \times (\dot{H}^{1/2}(\mathbb{R}^3))^2$ for 3-dimension, and $(v_0, E_0, B_0) \in \dot{B}^{\frac{1}{2}}_{2,1}(\mathbb{R}^2) \times (L^2_{log}(\mathbb{R}^2))^2$ for 2-dimensional case. Later on, German, Ibrahim and Masmoudi [12] refines the previous results by running a fixed-point argument to obtain mild solutions, but taking the initial velocity field in the natural Navier-Stokes space $H^{1/2}$. In their results the regularity of the initial velocity and electromagnetic fields is lowered. Furthermore, they employed an $L^2L^\infty$-estimate on the velocity field, which significantly simplifies the fixed-point arguments used in [19]. For some other asymptotic problems related, say, the derivation of the MHD from the Navier-Stokes-Maxwell system in the context of weak solutions, see Arsenio-Ibrahim-Masmoudi [1]. Recently, in [23] the first named authors of the current paper proved the
global classical solutions of the incompressible NSFM with small intial data, by employing the decay properties of both the electric field and the wave equation with linear damping of the divergence free magnetic field. This key idea was already used in [12]. Thus, it is natural to construct a class of classical solution of VMB around the solutions of the incompressible NSFM with Ohm’s law established in [23]. This is the main concern of the current paper. We note that for the Boltzmann equation, Guo [17] proved the incompressible Navier-Stokes limit from the Boltzmann equation for classical solutions. For the Vlasov-Poisson-Boltzmann system, Guo-Jiang-Luo [18] proved the incompressible Navier-Stokes-Fourier-Poisson limit from the Vlasov-Poisson-Boltzmann system in the classical regime by employing the taking moments method. Fruthermore, Jiang-Zhang [26] proved the incompressible Navier-Stokes-Fourier-Poisson limit from the Vlasov-Poisson-Boltzmann system with uncertainty in the classical regime by employing the spectral analysis method. Our result in this paper is in the same spirit of [17], but for much more involved incompressible NSFM limit from two-species VMB.

For the simplicity of representations, we focus on the hard potential cut-off kernel cases, although more general cases could also be treated with some additional technical complexity. Working in the classical solutions has some mathematical convenience, comparing to renormalized solutions. We have relatively much better understandings for both VMB and incompressible NSFM, at least for classical solutions near equilibriums. Furthermore, we can employ some properties of the incompressible NSFM specific for classical solutions, for example, the decay properties of both the electric field and the wave equation with linear damping of the divergence free magnetic field. In fact, the analog of these properties in VMB will play an essential role in the current work. We mention that the only previous hydrodynamic limit result for the VMB for classical solutions belong to Jang [21]. In fact, in [21], it was taken a special scaling that the magnetic effect appears only at a higher order. As a consequence, it vanishes in the limit as the Knudsen number $\varepsilon \to 0$. So in the limiting equations derived in [21], there is no equation for the magnetic field.

The basic strategy of the current paper is to take a Hilbert expansion approach. The expansion is carefully designed, see (1.11) below. The leading order terms are governed by the two-fluid incompressible NSFM with Ohm’s law. For this limiting system, the first named authors proved established global-in-time classical solutions with small initial data. The remainder equations are less nonlinear micro-Vlasov-Voltzmann type equations coupled with macro-wave type equations. Besides the nonlinear energy method and micro-macro decomposition trick developed in [16] and [17], we found that there has a linear damping wave type structure in the micro part of the remainder equations. Based on this observation, we can employ the analog of the decay properties of both the electric field and the wave equation with linear damping of the divergence free magnetic field, and eventually prove the uniform estimates for the remainder equations. This is the most important step and the main novelty of this paper, see section 5.

1.3. Well-prepared initial data. It is well-known that classical solutions to the Vlasov-Maxwell-Boltzmann system (1.5) obey the following global conservation laws of mass, momentum and energy:

$$
\frac{d}{dt} \int_{\mathbb{R}^3} F_{\varepsilon}^\pm \, dv \, dx = 0,
$$

$$
\frac{d}{dt} \left( \int_{\mathbb{R}^3} v (F_{\varepsilon}^+ + F_{\varepsilon}^-) \, dv \, dx + \varepsilon \int_{\mathbb{R}^3} E_{\varepsilon} \times B_{\varepsilon} \, dx \right) = 0,
$$

$$
\frac{d}{dt} \left( \int_{\mathbb{R}^3} |v|^2 (F_{\varepsilon}^+ + F_{\varepsilon}^-) \, dv \, dx + \varepsilon^2 \int_{\mathbb{R}^3} |E_{\varepsilon}|^2 + |B_{\varepsilon}|^2 \, dx \right) = 0.
$$

(1.7)

Notice that from the Maxwell system and the periodic boundary condition of $E_{\varepsilon}(t,x)$,

$$
\frac{d}{dt} \int_{\mathbb{R}^3} B_{\varepsilon}(t,x) \, dx = 0.
$$

(1.8)

The initial data of the scaled VMB system (1.5) are imposed by

$$(F_{\varepsilon}^+ (0,x,v), E_{\varepsilon}(0,x), B_{\varepsilon}(0,x)) = (F_{\varepsilon}^{\pm, \text{in}}(x,v), E_{\varepsilon}^{\text{in}}(x), B_{\varepsilon}^{\text{in}}(x)) \in \mathbb{R}^\times \times \mathbb{R}^3 \times \mathbb{R}^2
$$

(1.9)
with the compatibility $\text{div}_x B_ε^∞ = 0$. Assuming that (1.9) has the same mass, total momentum and total energy as the global equilibrium $(M(v), 0, 0)$, we can then rewrite the conservation laws (1.7) and (1.8) in terms of $(F_±, E_ε, B_ε)$ as

$$
\begin{align*}
T^3 & \int \frac{1}{\varepsilon} F_± dx = \frac{1}{\varepsilon} \int M dv dx, \\
T^3 & \int v(F_± + F_±^*) dv dx + \varepsilon \int E_ε \times B_ε dx = 0, \\
T^3 & \int |v|^2 (F_± + F_±^*) dv dx + \varepsilon^2 \int |E_ε|^2 + |B_ε|^2 dx = 2 \int T^3 |v|^2 M dv dx, \\
T^3 & B_ε(t,x) dx = 0.
\end{align*}
$$

(1.10)

The goal of this paper is to establish the global-in-time solutions with the form

$$
\begin{align*}
F_±(t,x,v) &= M \{ 1 + \varepsilon|g_0^±(t,x,v) + \varepsilon g_1^±(t,x,v) + \varepsilon^2 g_2^±(t,x,v) | + \varepsilon^3 g_3^±(t,x,v) ] \}, \\
E_ε(t,x) &= E_0(t,x) + \varepsilon E_1(t,x) + \varepsilon E_ε(t,x), \\
B_ε(t,x) &= B_0(t,x) + \varepsilon B_1(t,x) + \varepsilon B_ε(t,x),
\end{align*}
$$

(1.11)

where the leading term $g_0^±$ is given by

$$
g_0^±(t,x,v) = \rho_0^±(t,x) + u_0(t,x) \cdot v + \theta_0(t,x)(\frac{|u|^2}{2} - \frac{3}{2}),
$$

(1.12)

and the functions $g_i^±(t,x,v)$ ($i = 1,2$) are of the form (2.48). We denote $\rho_0(t,x) = \frac{1}{\varepsilon}(\rho_0^±(t,x) + \rho_0^-(t,x)) = -\theta_0(t,x)$ and $n_0(t,x) = \rho_0^+(t,x) - \rho_0^-(t,x)$ Then the functions $(u_0, \theta_0, n_0, E_0, B_0)$ obey the Navier-Stokes-Fourier-Maxwell system (1.6). We impose the initial data of (1.6)

$$
(u_0(0,x), \theta_0(0,x), E_0(0,x), B_0(0,x)) = (u_0^∞(x), \theta_0^∞(x), E_0^∞(x), B_0^∞(x)),
$$

(1.13)

which subjects to the compatibility conditions $\text{div}_x u_0^∞ = 0$ and $\text{div}_x B_0^∞ = 0$, where $\theta_0^∞ \in \mathbb{R}$ and $u_0^∞(x), E_0^∞(x), B_0^∞(x) \in \mathbb{R}^3$. If the initial data (1.13) satisfies the conditions of Lemma 4.1, the functions $u_0, \theta_0, n_0, E_0$ and $B_0$ are all globally well-defined. Consequently, from (2.46), $\bar{u}_1(t,x), \bar{x}_1(t,x)$ and $\bar{\tau}_1(t,x)$ are also well-defined. One notices that the functions $\bar{E}_1(t,x), \bar{B}_1(t,x)$ and $\bar{\tau}_1(t,x)$ appeared on the definitions of $g_i^∞(t,x,v)$ in (2.48), satisfies the linear Maxwell-type system (2.47), namely

$$
\begin{align*}
\frac{\partial}{\partial t} \bar{E}_1 - \nabla_x \times \bar{B}_1 &= -j_1, \\
\frac{\partial}{\partial t} \bar{B}_1 + \nabla_x \times \bar{E}_1 &= 0, \\
\text{div}_x \bar{E}_1 &= \bar{n}_1, \\
\text{div}_x \bar{B}_1 &= 0, \\
j_1 &= \bar{n}_1(u_0 \cdot M + \theta_0 V) + u_1 + \sigma(-\frac{1}{2} \nabla_x \bar{n}_1 + \bar{E}_1 + u_0 \times \bar{B}_1 + u_1 \times B_0) + \sum \Gamma_0 U_0.
\end{align*}
$$

We give the initial data of the linear Maxwell-type system (2.47)

$$
(\bar{E}_1(0,x), \bar{B}_1(0,x)) = (\bar{E}_1^∞(x), \bar{B}_1^∞(x)) \in \mathbb{R}^3 \times \mathbb{R}^3,
$$

(1.14)

which satisfying the compatibility condition $\text{div}_x \bar{B}_1^∞ = 0$. We easily know that for the magnetic field $\bar{B}_1(t,x)$

$$
\frac{d}{dt} \int \bar{B}_1 dx = 0,
$$

which immediately means that $\int T^3 \bar{B}_1 dx = \int T^3 \bar{B}_1^∞ dx$. We further assume that the initial data $\bar{B}_1^∞(x)$ has zero mean value property, hence

$$
\int T^3 \bar{B}_1^∞ dx = 0.
$$

(1.15)

Thus, $\bar{B}_1(t,x)$ has also zero mean value property. If the initial conditions (1.13) and (1.14) subject to the conditions of Lemma 4.2, the functions $\bar{E}_1(t,x), \bar{B}_1(t,x)$ and $\bar{\tau}_1(t,x)$ are globally and uniquely determined. As a result, the functions $\bar{g}_i^∞(t,x,v)$ for $i = 1,2$ in (2.48) are the known.
By the formal analysis in Section 2, the remainder terms \((g_{R,\varepsilon}^+, E_{R,\varepsilon}, B_{R,\varepsilon})\) subject to the system

\[
\begin{align*}
\varepsilon \partial_t G_{R,\varepsilon} + v \cdot \nabla_x G_{R,\varepsilon} + T (v \times B_0) \cdot \nabla_v G_{R,\varepsilon} + T (v \times B_{R,\varepsilon}) \cdot \nabla_v G_0 - E_{R,\varepsilon} \cdot \nabla T_1 + \nabla L G_{R,\varepsilon} &= \varepsilon H_{R,\varepsilon}, \\
\partial_t E_{R,\varepsilon} - \nabla_x \times B_{R,\varepsilon} &= -\frac{1}{\varepsilon} \langle G_{R,\varepsilon} \cdot T_1 v \rangle, \\
\partial_t B_{R,\varepsilon} + \nabla_x \times E_{R,\varepsilon} &= 0, \\
\text{div}_x E_{R,\varepsilon} &= \langle G_{R,\varepsilon} \cdot T_1 \rangle, \quad \text{div}_x B_{R,\varepsilon} = 0,
\end{align*}
\]

(1.16)

with \(T_1 = (1, -1)^\top\), and the diagonal \(2 \times 2\) matrix \(T = \text{diag}(1, -1)\). Here \(G_{R,\varepsilon} = (g_{R,\varepsilon}^+, g_{R,\varepsilon}^-)\), \(G_0 = (g_0^+, g_0^-)\), and

\[
\begin{align*}
H_{R,\varepsilon} &= -\varepsilon \mathbf{T} E_1 \cdot \nabla_v G_{R,\varepsilon} + \varepsilon \mathbf{T} E_1 \cdot v G_{R,\varepsilon} - \varepsilon \mathbf{T} E_{R,\varepsilon} \cdot \nabla_v G_{R,\varepsilon} + \varepsilon \mathbf{T} E_{R,\varepsilon} \cdot v G_{R,\varepsilon} \\
&- \mathbf{T} E_0 \cdot \nabla_v G_{R,\varepsilon} - \mathbf{T} E_0 \cdot v G_{R,\varepsilon} - T (v \times \mathbf{B}_1) \cdot \nabla_v G_{R,\varepsilon} - T (v \times B_{R,\varepsilon}) \cdot \nabla_v G_{R,\varepsilon} \\
&- \mathbf{T} E_{R,\varepsilon} \cdot \nabla_v (G_0 + \varepsilon G_1 + \varepsilon^2 G_2) + \mathbf{T} E_{R,\varepsilon} \cdot v (G_0 + \varepsilon G_1 + \varepsilon^2 G_2) \\
&+ \frac{1}{\varepsilon} \Gamma_0 G_{R,\varepsilon} - \mathbf{T} (v \times B_{R,\varepsilon}) \cdot \nabla_v (G_1 + \varepsilon G_2) + \varepsilon \left( \frac{Q(\tilde{g}_2^+, g_{R,\varepsilon}^-) + g_{R,\varepsilon}^+)}{Q(\tilde{g}_2^+, g_{R,\varepsilon}^+ + g_{R,\varepsilon}^-)} \right) \\
&+ \left( \frac{Q(g_{R,\varepsilon}^+, g_{R,\varepsilon}^+ + g_{R,\varepsilon}^-)}{Q(g_{R,\varepsilon}^+, g_{R,\varepsilon}^+ + g_{R,\varepsilon}^-)} + \frac{Q(g_{R,\varepsilon}^+, g_{R,\varepsilon}^-)}{Q(g_{R,\varepsilon}^+, g_{R,\varepsilon}^-)} \right) + \varepsilon \left( \frac{\mathbf{R}^+}{\mathbf{R}^-} \right)
\end{align*}
\]

(1.17)
in which

\[
\Gamma_0 G_{R,\varepsilon} = \left( \begin{array}{c}
\frac{Q(\tilde{g}_0^+, g_{R,\varepsilon}^+) + Q(g_{R,\varepsilon}^+, g_{R,\varepsilon}^+)}{Q(\tilde{g}_0^+, g_{R,\varepsilon}^+) + Q(g_{R,\varepsilon}^+, g_{R,\varepsilon}^+)} \\
\frac{Q(g_{R,\varepsilon}^+, g_{R,\varepsilon}^+)}{Q(g_{R,\varepsilon}^+, g_{R,\varepsilon}^+)} + \frac{Q(g_{R,\varepsilon}^+, g_{R,\varepsilon}^+)}{Q(g_{R,\varepsilon}^+, g_{R,\varepsilon}^+)}
\end{array} \right),
\]

(1.18)

and the symbols \(\mathbf{R}^\pm\) are defined in (2.52). The initial data of the remainder system (1.16) is imposed on

\[
(G_{R,\varepsilon}(0, x, v), E_{R,\varepsilon}(0, x), B_{R,\varepsilon}(0, x)) = (G_{R,\varepsilon}^{in}(x, v), E_{R,\varepsilon}^{in}(x, v), B_{R,\varepsilon}^{in}(x, v)) \in \mathbb{R}^2 \times \mathbb{R}^3 \times \mathbb{R}^3
\]

(1.19)
satisfying the compatibility \(\text{div}_x B_{R,\varepsilon} \neq 0\), where \(G_{R,\varepsilon}^{in}(x, v) = (g_{R,\varepsilon}^{in, 1}(x, v), g_{R,\varepsilon}^{in, 2}(x, v), g_{R,\varepsilon}^{in, 3}(x, v))^\top\).

Noticing that we will find global-in-time classical solutions to (1.5) with the form (1.11), we naturally assume the initial data (1.9) has the same form as solutions. Thus, we let

\[
\begin{align*}
F_{\varepsilon}^{\pm, \text{in}}(x, v) &= M \{ 1 + \varepsilon |g_0^{\pm, \text{in}}(x, v) + \varepsilon g_1^{\pm, \text{in}}(x, v) + \varepsilon^2 g_2^{\pm, \text{in}}(x, v) + \varepsilon^3 g_3^{\pm, \text{in}}(x, v)| \}, \\
E_{\varepsilon}^{\pm, \text{in}}(x) &= E_0^{\pm, \text{in}}(x) + \varepsilon E_1^{\pm, \text{in}}(x) + \varepsilon^2 E_2^{\pm, \text{in}}(x) + \varepsilon^3 E_3^{\pm, \text{in}}(x) \}
\end{align*}
\]

(1.20)

where \(g_{R,\varepsilon}^{\pm, \text{in}}(x, v) \in \mathbb{R}, \ E_{R,\varepsilon}^{\text{in}}(x), B_{R,\varepsilon}^{\text{in}}(x) \in \mathbb{R}^3\) are some given functions, which will be regarded as the initial data of the remainder terms \((g_{R,\varepsilon}^\pm, E_{R,\varepsilon}, B_{R,\varepsilon})\), and

\[
g_0^{\pm, \text{in}}(x, v) = \pm \frac{1}{2} \text{div}_x E_0^{\pm, \text{in}}(x) - \theta_0^{\pm, \text{in}}(x) + u_0^{\pm, \text{in}}(x) \cdot v + \theta_0^{\pm, \text{in}}(x) |v|^2 - \frac{3}{2} \},
\]

and \(\mathbf{f}_k^{\pm, \text{in}}(x, v) \) (\(k = 1, 2\)) have the same forms as \(\mathbf{f}_k^{\pm}(t, x, v)\) defined in (2.48), just replacing the vectors \((u_0, \theta_0, E_0, B_0)\) and \((\mathbf{F}_1, \mathbf{F}_2)\) by \((u_0^{\pm, \text{in}}, \theta_0^{\pm, \text{in}}, E_0^{\pm, \text{in}}, B_0^{\pm, \text{in}})\) and \((\mathbf{F}_1^{\pm, \text{in}}, \mathbf{F}_2^{\pm, \text{in}})\) respectively. Moreover, the functions \(f_{\varepsilon}^{\pm, \text{in}}, E_{\varepsilon}^{\pm, \text{in}}\) and \(B_{\varepsilon}^{\pm, \text{in}}\) defined in (1.20) obey the conditions (??). We remark that the condition (1.15) implies that \(B_{R,\varepsilon}^{\pm, \text{in}}\) has zero mean value property, i.e.,

\[
\int_{\mathbb{T}^3} B_{R,\varepsilon}^{\pm, \text{in}} \, dx = 0.
\]
1.4. Notations. In order to state our results precisely, we now introduce the following notations. For notational simplicity, we shall use $\langle \cdot \rangle$ to denote the integral in $\mathbb{R}^3$ with the measure $Mdv$ for a function $f(v)$, hence

$$\langle f \rangle := \int_{\mathbb{R}^3} f(v)Mdv.$$ 

Let the multi-indices $m$ and $\beta$ be

$$m = [m_1, m_2, m_3], \quad \beta = [\beta_1, \beta_2, \beta_3],$$

and we define

$$\hat{\partial}_\beta^m := \partial_{x_1}^{m_1} \partial_{x_2}^{m_2} \partial_{x_3}^{m_3} \partial_{v_1}^{\beta_1} \partial_{v_2}^{\beta_2} \partial_{v_3}^{\beta_3},$$

where $m$ is related to the space derivatives, while $\beta$ is related to the velocity derivatives. If each component of $\theta$ is not greater than that of $\tilde{\theta}$’s, we denote by $\theta \leq \tilde{\theta}$. The symbol $\theta < \tilde{\theta}$ means $\theta \leq \tilde{\theta}$ and $|\theta| < |\tilde{\theta}|$, where $|\theta| = \theta_1 + \theta_2 + \theta_3$.

Now we introduce some basic properties of the linearized operators, which can be found in [3]. For the Boltzmann collision operator (1.2), define the collision frequency to be

$$\nu(v) := \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} |v - v_*|^3 \hat{b}(\cos \theta)d\sigma dv_* = \int_{\mathbb{R}^3} |v - v_*|^3 dv_*.$$ 

(1.21)

We also define the weight function $w(v)$ by

$$w(v) := \sqrt{1 + |v|^2}.$$ 

Moreover, for any $G = (g^+, g^-)^T$, the linearized collision operator $LG$ is given by

$$LG = \left( Lg^+ + Lg^-(g^+, g^-) \right),$$

(1.22)

where

$$Lg = -\frac{1}{M}\left[ Q(Mg, M) + Q(M, Mg) \right],$$

and

$$L(g, h) = -\frac{1}{M}\left[ Q(Mg, M) + Q(M, Mh) \right].$$

For notational simplicity, we denote by

$$Q(g, h) = \frac{1}{M}Q(Mg, Mh).$$

One easily know that the kernel of $L$ (denoted by ker$L$) is spanned by

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} v \\ v \end{pmatrix}, \begin{pmatrix} \frac{|v|^2}{2} - \frac{3}{2} \\ \frac{|v|^2}{2} - \frac{3}{2} \end{pmatrix}.$$ 

(1.23)

We now define the projection operator $\mathbb{P}$ from $L_0^2$ to ker$L$ as

$$\mathbb{P}G = \rho^+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \rho^- \begin{pmatrix} 0 \\ 1 \end{pmatrix} + u \cdot \begin{pmatrix} v \\ v \end{pmatrix} + \theta \left( \begin{pmatrix} \frac{|v|^2}{2} - \frac{3}{2} \\ \frac{|v|^2}{2} - \frac{3}{2} \end{pmatrix} \right),$$

(1.24)

where $\rho^\pm = \langle g^\pm \rangle$, $u = \langle v \frac{g^+ - g^-}{2} \rangle$ and $\theta = \langle \frac{|v|^2}{3} - 1 \rangle \frac{g^+ - g^-}{2}$.

We define the Hilbert space $L_2^2(\mathbb{T}^3)$ endowed with the norm

$$\|f\|_{L_2^2(\mathbb{T}^3)} := \left( \int_{\mathbb{T}^3} |f(x)|^2dx \right)^{\frac{1}{2}},$$

and we also give Sobolev space $H_x^s(\mathbb{T}^3)$ by $f \in H_x^s(\mathbb{T}^3)$ if and only if

$$\|f\|_{H_x^s(\mathbb{T}^3)} := \sum_{|m| \leq s} \int_{\mathbb{T}^3} |\partial^m f(x)|^2dx = \sum_{|m| \leq s} \|\partial^m f\|_{L_2^2(\mathbb{T}^3)}^2.$$
Next, for a function $a(v) > 0$, the weighted Hilbert space $L^2_v(a(v)dv)$ is endowed with the norm
\[ \|g\|_{L^2_v(a(v)dv)} := \left( \int_{\mathbb{R}^3} |g(v)|^2 a(v) M dv \right)^{\frac{1}{2}} < \infty. \]
Naturally, for the function $h(x, v)$, we can define a norm
\[ \|h\|_{L^2_v(T^3; L^2(a(v)dv))} := \left( \int_{T^3} \int_{\mathbb{R}^3} |h(x, v)|^2 a(v) d v d x \right)^{\frac{1}{2}} < \infty. \]
Furthermore, for a positive function $b(v)$ away from zero, we define a so-called weighted mixed Sobolev space $H^s_{x,v}(b(v)dvdx)$ by $h(x, v) \in H^s_{x,v}(b(v)dvdx)$ if and only if
\[ \|h\|_{H^s_{x,v}(b(v)dvdx)} := \sum_{|m| + |\beta| \leq s} \|h\|_{L^2(T^3; L^2(\nu M dv dx))} < \infty. \]

For notational simplicity, we denote by
\[
\begin{align*}
L^2_x := L^2(\mathbb{T}^3), & \quad H^s_x := H^s(\mathbb{T}^3), \\
L^2_v := L^2(\mathbb{R}^3; M dv), & \quad H^s_v := H^s(\mathbb{R}^3; \nu M dv), \\
L^2_{x,v} := L^2(\mathbb{T}^3; L^2_v(M dv)), & \quad H^s_{x,v} := H^s(\mathbb{T}^3; L^2_v(\nu M dv dx)), \\
& \quad H^s_{x,v}(w^\nu) := H^s_{x,v}(w^\nu M dv dx).
\end{align*}
\]
In what follows, we use a capital symbol $G$ denote by a column vector in $\mathbb{R}^2$ so that
\[ G = \left( \begin{array}{c} g^+ \\ g^- \end{array} \right). \]
For instance, in this paper, we use
\[ G_0 = \left( \begin{array}{c} g_0^+ \\ g_0^- \end{array} \right), \quad G_k = \left( \begin{array}{c} g_k^+ \\ g_k^- \end{array} \right) (k = 1, 2), \quad G_{R,\varepsilon} = \left( \begin{array}{c} g_{R,\varepsilon}^+ \\ g_{R,\varepsilon}^- \end{array} \right), \quad G_{R,\varepsilon}^{in} = \left( \begin{array}{c} g_{R,\varepsilon}^{+in} \\ g_{R,\varepsilon}^{-in} \end{array} \right). \]
We now define the instant energy functional and the dissipation rate (similar as in [17] which treated the Boltzmann equation).

**Definition 1.1 (Instant Energy).** For $N \geq 4$ and $l \geq 2\gamma + 1$, we give an energy functional $\overline{E}_{N,l}(G, E, B)$ by
\[
\overline{E}_{N,l}(G, E, B) = \|E\|_{H^N_{x,v}}^2 + \|B\|_{H_{x,v}^{N+1}}^2 + \sum_{|m| \leq N+1} \|\partial^m G\|_{L^2_{x,v}}^2 \\
+ \sum_{|m| + |\beta| \leq N} \|w^\beta \partial^m \mathbb{P} G\|_{L^2_{x,v}}^2 + \sum_{|m| + |\beta| \leq N+1} \|w^\beta \partial^m \mathbb{P} G\|_{L^2_{x,v}}^2 \tag{1.25}
\]
\[ + \mathcal{E}_{0,N+5}(t) + \mathcal{E}_{1,N+3}(t), \]
where the vector functions $G(t, x, v) = \left( \begin{array}{c} g^+(t, x, v) \\ g^-(t, x, v) \end{array} \right) \in \mathbb{R}^2$, $E(t, x)$, $B(t, x) \in \mathbb{R}^3$, the energy functionals $\mathcal{E}_{0,N+5}(t)$ and $\mathcal{E}_{1,N+3}(t)$ are given in (4.1) and (4.9) respectively, and the operator $\mathbb{P} = \mathbb{I} - \mathbb{P}$. Here $\mathbb{I}$ is the identity operator.

We call $\overline{E}_{N,l}(G, E, B)$ is an instant energy functional with respect to the energy functional $\overline{E}_{N,l}(G, E, B)$, if there is a constant $C > 0$ such that
\[ \frac{1}{C} \overline{E}_{N,l}(G, E, B) \leq \overline{E}_{N,l}(G, E, B) \leq C \overline{E}_{N,l}(G, E, B) \tag{1.26} \]
holds for all functions $G(t, x, v) \in \mathbb{R}^2$ and $E(t, x)$, $B(t, x) \in \mathbb{R}^3$. 
Definition 1.2 (Dissipation Rate). For $N \geq 4$ and $l \geq 2\gamma + 1$, the dissipation rate $\mathbb{D}_{N,l}(G, E, B)$ is defined as

$$
\mathbb{D}_{N,l}(G, E, B) = \|E\|^2_{H_{x,v}^{N-1}} + \|\nabla_x B\|^2_{H_{x,v}^{N-1}} + \|\partial_t B\|^2_{H_{x,v}^{N-1}} + \mathcal{D}_{0,N+5}(t) + \mathcal{D}_{1,N+3}(t) \\
+ \sum_{|m| \leq N+1} \|\partial^m \nabla G\|^2_{L_{x,v}^2} + \frac{1}{\varepsilon^2} \sum_{|m|+|\beta| \leq N+1} \|\partial^m \partial_\beta G\|^2_{L_{x,v}^2} \\
+ \frac{1}{\varepsilon^2} \sum_{|m| \leq N+1} \|\partial^m \nabla G\|^2_{L_{x,v}^2} + \frac{1}{\varepsilon^2} \sum_{|m|+|\beta| \leq N} \|\partial^m \partial_\beta G\|^2_{L_{x,v}^2},
$$

(1.27)

for all functions $G(t, x, v) \in \mathbb{R}^2$ and $E(t, x)$, $B(t, x) \in \mathbb{R}^3$, where $\mathcal{D}_{0,N+5}(t)$ and $\mathcal{D}_{1,N+3}(t)$ are given in (4.2) and (4.10) respectively.

1.5. Main results. Before stating our main results, we define an initial energy

$$
\mathcal{E}_{N,l}(G_{R,\varepsilon}^{in}, E_{R,\varepsilon}^{in}, B_{R,\varepsilon}^{in}) = \|E_{R,\varepsilon}^{in}\|^2_{H_{x,v}^{N+1}} + \|B_{R,\varepsilon}^{in}\|^2_{H_{x,v}^{N+1}} + \sum_{|m| \leq N+1} \|\partial^m \nabla G_{R,\varepsilon}^{in}\|^2_{L_{x,v}^2} \\
+ \sum_{|m|+|\beta| \leq N} \|\partial^m \partial_\beta G_{R,\varepsilon}^{in}\|^2_{L_{x,v}^2} + \sum_{|m|+|\beta| \leq N+1} \|\partial^m \partial_\beta G_{R,\varepsilon}^{in}\|^2_{L_{x,v}^2} \\
+ \mathcal{E}_{0,N+5}^{in} + \mathcal{E}_{1,N+3}^{in},
$$

where the quantities $\mathcal{E}_{0,N+5}^{in}$ and $\mathcal{E}_{1,N+3}^{in}$ are given in (4.3) and (4.11) respectively. Then, our main result is as follows.

Theorem 1.1. Let $N \geq 4$ and $l \geq 2\gamma + 1$. Given $u_0^{in}(x)$, $\theta_0^{in}(x)$, $E_0^{in}(x)$, $B_0^{in}(x)$, $\mathcal{E}_1^{in}(x)$, $E_{R,\varepsilon}^{in}(x)$, $B_{R,\varepsilon}^{in}(x)$ and $G_{R,\varepsilon}^{in}(x)$, let the initial data (1.9) of the Vlasov-Maxwell-Boltzmann system (1.5) satisfy the conditions (??), (1.15) and (1.20). If there are two small constants $\varepsilon_0, \eta_0 > 0$, depending only on $\mu$, $\sigma$, $\kappa$, $l$ and $N$, such that $\varepsilon \in (0, \varepsilon_0)$ and

$$
\mathcal{E}_{N,l}(G_{R,\varepsilon}^{in}, E_{R,\varepsilon}^{in}, B_{R,\varepsilon}^{in}) \leq \eta_0,
$$

(1.28)

then the system (1.5) with the initial data (1.9) admits a global-in-time classical solution $(F_{\varepsilon}^{\pm}(t, x, v), E_{\varepsilon}(t, x), B_{\varepsilon}(t, x))$ belonging to $L^\infty(\mathbb{R}^+; H_{x,v}^{N+1} \times H_{x,v}^{N+1} \times H_{x,v}^{N+1})$ with the form (1.11), in which $(u_0, \theta_0, n_0, E_0, B_0) \in$ is a unique solution the Navier-Stokes-Fourier-Maxwell system with Ohm’s law (1.6) with initial data $(u_0^{in}, \theta_0^{in}, \div x E_0^{in}, E_0^{in}, B_0^{in})$.

Moreover, there exists an instant energy functional $\mathcal{E}_{N,l}(R_{\varepsilon}, E_{R,\varepsilon}, B_{R,\varepsilon})$ such that

$$
\sup_{t \geq 0} \mathcal{E}_{N,l}(R_{\varepsilon}, E_{R,\varepsilon}, B_{R,\varepsilon})(t) \leq \mathcal{E}_{N,l}(R_{\varepsilon}, E_{R,\varepsilon}, B_{R,\varepsilon})(0) \leq C \mathcal{E}_{N,l}(R_{\varepsilon}, E_{R,\varepsilon}, B_{R,\varepsilon})
$$

(1.29)

holds for some constant $C > 0$, depending only on $\mu$, $\sigma$, $\kappa$, $l$ and $N$.

2. Formal analysis

Since the relation $F_{\varepsilon}^{\pm} = M(1 + \varepsilon g_{\varepsilon}^{\pm})$, one can rewrite the system (1.5) as

$$
\begin{cases}
\varepsilon \partial_t \left( \frac{g_{\varepsilon}^+}{g_{\varepsilon}^-} \right) + v \cdot \nabla_x \left( \frac{g_{\varepsilon}^+}{g_{\varepsilon}^-} \right) + (\varepsilon E_{\varepsilon} + v \times B_{\varepsilon}) \cdot \nabla_v \left( \frac{g_{\varepsilon}^+}{g_{\varepsilon}^-} \right) - E_{\varepsilon} \cdot v \left( \frac{1}{1 - \varepsilon g_{\varepsilon}^-} \right) \nabla_x (g_{\varepsilon}^+ - g_{\varepsilon}^-) v M dv, \\
- \frac{1}{\varepsilon} \left( L g_{\varepsilon}^+ + L g_{\varepsilon}^- + L g_{\varepsilon}^0 + L g_{\varepsilon}^0 \right) + \left( \mathcal{Q}(g_{\varepsilon}^+, g_{\varepsilon}^-) + \mathcal{Q}(g_{\varepsilon}^+, g_{\varepsilon}^-) + \mathcal{Q}(g_{\varepsilon}^-, g_{\varepsilon}^-) \right) \nabla_x (g_{\varepsilon}^+ - g_{\varepsilon}^-) v M dv, \\
\partial_t B_{\varepsilon} - \nabla_x E_{\varepsilon} = 0, \\
div E_{\varepsilon} = 0, \\
div B_{\varepsilon} = 0,
\end{cases}
$$

(2.1)
where we denote
\[ L_g = -\frac{1}{M} [Q(Mg, M) + Q(M, Mg)] , \]
and
\[ L(g, h) = -\frac{1}{M} [Q(Mg, M) + Q(M, Mh)] , \quad Q(g, h) = \frac{1}{M} Q(Mg, Mh) . \]

When \( \varepsilon \to 0 \), we take ansatz that the system (2.1) has a solution with the form
\[
\begin{cases}
  g^\pm_\varepsilon = g^\pm_0 + \varepsilon g^1_\varepsilon + \varepsilon^2 g^2_\varepsilon + \varepsilon^3 g^3_\varepsilon , \\
  E_\varepsilon = E_0 + \varepsilon E_1 + \varepsilon^2 E_2 , \\
  B_\varepsilon = B_0 + \varepsilon B_1 + \varepsilon^2 B_2 .
\end{cases}
\]

Plugging the relations (2.2) into the system (2.1), one gets
\[
\begin{aligned}
  &\varepsilon \partial_t \left( \frac{g^+_0 + \varepsilon g^+_1 + \varepsilon^2 g^+_2 + \varepsilon^3 g^+_3}{g^-_0 + \varepsilon g^-_1 + \varepsilon^2 g^-_2 + \varepsilon^3 g^-_3} \right) + v \cdot \nabla_x \left( \frac{g^+_0 + \varepsilon g^+_1 + \varepsilon^2 g^+_2 + \varepsilon^3 g^+_3}{g^-_0 + \varepsilon g^-_1 + \varepsilon^2 g^-_2 + \varepsilon^3 g^-_3} \right) \\
  &\quad + \left[ \varepsilon (E_0 + \varepsilon E_1 + \varepsilon^2 E_2) + v \times (B_0 + \varepsilon B_1 + \varepsilon^2 B_2) \right] \\
  &\quad - \varepsilon \partial_t \left( \frac{L(g^+_0 + \varepsilon g^+_1 + \varepsilon^2 g^+_2 + \varepsilon^3 g^+_3)}{L(g^-_0 + \varepsilon g^-_1 + \varepsilon^2 g^-_2 + \varepsilon^3 g^-_3)} \right) \\
  &\quad - \varepsilon \partial_t \left( Q(g^+_0 + \varepsilon g^+_1 + \varepsilon^2 g^+_2 + \varepsilon^3 g^+_3, g^-_0 + \varepsilon g^-_1 + \varepsilon^2 g^-_2 + \varepsilon^3 g^-_3) \right) \\
  &\quad + \left( Q(g^+_0 + \varepsilon g^+_1 + \varepsilon^2 g^+_2 + \varepsilon^3 g^+_3, g^-_0 + \varepsilon g^-_1 + \varepsilon^2 g^-_2 + \varepsilon^3 g^-_3) \right) \\
  &\quad + \left( Q(g^+_0 + \varepsilon g^+_1 + \varepsilon^2 g^+_2 + \varepsilon^3 g^+_3, g^-_0 + \varepsilon g^-_1 + \varepsilon^2 g^-_2 + \varepsilon^3 g^-_3) \right)
\end{aligned}
\]

For the order of \( \mathcal{O}(\varepsilon^2) \) in the expansion system (2.3), we have
\[
L \left( \frac{g^+_0}{g^-_0} \right) = \left( L(g^+_0, g^-_0) + Q(g^+_0, g^-_0) \right) = 0 ,
\]
and
\[
\int_{\mathbb{R}^3} (g^+_0 - g^-_0) v M dv = 0 .
\]

Noticing that the kernel \( \ker L \) is generated by
\[
\begin{pmatrix} 1 \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \begin{pmatrix} u_0 \\ v \end{pmatrix} , \begin{pmatrix} |u|^2 \\ |v|^2 \end{pmatrix} - \frac{3}{2} ,
\]
we derive from (2.4)
\[
g^+_0 = \rho^+_0 + u_0 \cdot v + \theta_0 \left( \frac{|v|^2}{2} - \frac{3}{2} \right) ,
\]
where \( \rho_0^+ = \langle g_0^+ \rangle \), \( u_0 = \langle g_0^+ \rangle = \langle g_0^- \rangle \) and \( \theta_0 = \langle g_0^+ (|v|^2 - \frac{1}{3}) \rangle = \langle g_0^- (|v|^2 - \frac{1}{3}) \rangle \). Here we denote the symbol \( \langle f \rangle \) by \( \langle f \rangle = \int_{\mathbb{R}^2} f \, dv \). Consequently, the relation (2.5) is identical.

The order of \( O(1) \) in (2.3) reads

\[
 v \cdot \nabla_x \left( \frac{g_0^+}{g_0^-} \right) + (v \times B_0) \cdot \nabla_v \left( \frac{g_0^+}{g_0^-} \right) - E_0 \cdot v \left( \begin{array}{c} 1 \\ -1 \end{array} \right)
\]

(2.7)

and

\[
\begin{cases}
\partial_t E_0 - \nabla_x \times B_0 = -j_0, \\
\partial_t B_0 + \nabla_x \times E_0 = 0, \\
\operatorname{div}_v E_0 = n_0, \quad \operatorname{div}_x B_0 = 0,
\end{cases}
\]

(2.8)

where we denote \( j_0 = u_1^+ - u_1^- = \int_{\mathbb{R}^3} (g_1^+ - g_1^-) v \, Mdv \) and \( n_0 = \rho_0^+ - \rho_0^- \).

The form (2.6) of \( g_0^\pm \) imply that

\[
(v \times B_0) \cdot \nabla_v g_0^\pm = -(u_0 \times B_0) \cdot v.
\]

(2.9)

If we let \( A(v) = v \otimes v - \frac{|v|^2}{3} I_d \) and \( B(v) = v \otimes |v|^2 - \frac{5}{2} \), which belong to the kernel orthogonal \( \ker^1 \mathcal{L} \) of the linearized Boltzmann operator \( \mathcal{L} \), then we can calculate

\[
v \cdot \nabla_x g_0^\pm = \operatorname{div}_x u_0 + \nabla x (\rho_0^+ + \theta_0) + v \cdot \frac{2}{3} \operatorname{div}_x u_0 \left( \frac{|v|^2}{2} - \frac{3}{2} \right) + A(v) : \nabla x u_0 + B(v) : \nabla x \theta_0.
\]

(2.10)

We denote by \( \rho_0 = \frac{\rho_0^+ + \rho_0^-}{2} \). Summing up for the two equations in (2.7), multiplying by \( \frac{1}{2} \) and combining the equalities (2.9) and (2.10) imply that

\[
\begin{align*}
\operatorname{div}_x u_0 + v \cdot \nabla_x (\rho_0 + \theta_0) + \frac{2}{3} \operatorname{div}_x u_0 \left( \frac{|v|^2}{2} - \frac{3}{2} \right) + A(v) : \nabla x u_0 + B(v) : \nabla x \theta_0 \\
= -2 \mathcal{L} \left( \frac{g_0^+ + g_0^-}{2} \right) + \mathcal{L}(A(v)) : u_0 \otimes u_0 + 2 \theta_0 u_0 \cdot B(v) + \theta_0^2 \mathcal{C}(v),
\end{align*}
\]

(2.11)

where \( \mathcal{C}(v) = \frac{1}{4} |v|^4 - \frac{5}{2} |v|^2 + \frac{15}{4} \in \ker^1 \mathcal{L} \). Here we make use of the following direct calculations:

\[
\begin{align*}
\mathcal{L} g_1^+ + \mathcal{L} g_1^- + \mathcal{L}^2 (g_1^+, g_1^-) + \mathcal{L}^2 (g_1^-, g_1^-) \\
= \mathcal{L}(g_1^+ + g_1^-) - Q(g_1^+ + g_1^-) - Q(1, g_1^+ + g_1^-) \\
= 2 \mathcal{L}(g_1^+ + g_1^-),
\end{align*}
\]

and by the fact that if \( g \in \ker \mathcal{L} \), \( Q(g, g) = \frac{1}{2} \mathcal{L}(g^2) \)

\[
\begin{align*}
\mathcal{C}(v) &= \frac{1}{4} |v|^4 - \frac{5}{2} |v|^2 + \frac{15}{4} \\
&= \frac{1}{4} \mathcal{L}(A(v)) : u_0 \otimes u_0 + 2 \theta_0 u_0 \cdot B(v) + \theta_0^2 \mathcal{C}(v),
\end{align*}
\]

where we utilize the relation

\[
\frac{1}{4} (g_0^+ + g_0^-)^2 = \rho_0^2 + 2 \rho_0 \theta_0 u_0 \cdot v + 2 \rho_0 \theta_0 (\frac{|v|^2}{2} - \frac{3}{2}) + \frac{|v|^2}{3} |u_0|^2 + A(v) : u_0 \otimes u_0 + 2 \theta_0 u_0 \cdot B(v) + \theta_0^2 \mathcal{C}(v)
\]

\( \in \ker \mathcal{L} + \ker^1 \mathcal{L} \).

It is easy to know that the projection \( \mathcal{P}_\mathcal{L} : L^2(Mdv) \rightarrow \ker \mathcal{L} \) is

\[
\mathcal{P}_\mathcal{L} g = \langle g \rangle + \langle g v \rangle : v + (g \frac{|v|^2}{2} - \frac{1}{3}) (\frac{|v|^2}{2} - \frac{3}{2}).
\]

Now we project the equation (2.11) into the kernel \( \ker \mathcal{L} \) of \( \mathcal{L} \) and then we have

\[
\operatorname{div}_x u_0 = 0, \quad \nabla_x (\rho_0 + \theta_0) = 0.
\]

Consequently,

\[
\operatorname{div}_x u_0 = 0, \quad \rho_0 + \theta_0 = 0.
\]

(2.12)
Thus the kernel orthogonal part $\ker^\perp L$ of (2.11) is
\[ L(\frac{g^+ + g^-}{2}) = L(\frac{1}{2}u_0 \otimes u_0 : A(v) + \theta_0 u_0 \cdot B(v) + \frac{1}{2} \theta_0^2 C(v) - \frac{1}{2} \nabla_x u_0 : \tilde{A}(v) - \frac{1}{2} \nabla_x \theta_0 \cdot \tilde{B}(v)), \]
where the terms $\tilde{A}(v), \tilde{B}(v) \in \ker^\perp L$ satisfy
\[ L\tilde{A}(v) = A(v), \text{ and } L\tilde{B}(v) = B(v), \]
which is uniquely determined in $\ker^\perp L$. Moreover, there exits functions $\varphi, \psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\tilde{A}(v) = \varphi(|v|)A(v)$ and $\tilde{B}(v) = \psi(|v|)B(v)$. Thus the equation (2.13) yields that for some functions $\rho_1(x,t), u_1(x,t)$ and $\theta_1(x,t)$
\[ \frac{g^+ + g^-}{2} = \rho_1 + u_1 \cdot v + \frac{1}{2} \theta_1(\frac{|v|^2}{2} - \frac{3}{2}) + \frac{1}{2} u_0 \otimes u_0 : A(v) + \theta_0 u_0 \cdot B(v) + \frac{1}{2} \theta_0^2 C(v) - \frac{1}{2} \nabla_x u_0 : \tilde{A}(v) - \frac{1}{2} \nabla_x \theta_0 \cdot \tilde{B}(v). \]
(2.14)

If subtracting the second equality from the first equality in (2.7), multiplying by $\frac{1}{2}$, we have
\[ \left( \frac{1}{2} \nabla_x n_0 + E_0 + u_0 \times B_0 \right) \cdot v \]
\[ = - \left( L - L^\dagger \right) \left( \frac{g^+ - g^-}{2} + \frac{\alpha_0}{2} \left( L + L^\dagger \right) (u_0 \cdot v + \theta_0 (\frac{|v|^2}{2} - \frac{3}{2})) \right), \]
(2.15)

where the linear operator $\mathcal{L}$ is defined as $\mathcal{L}g = -Q(g,1) - Q(1,-g)$ and $\ker \mathcal{L} = \text{span}\{1\}$. Here the following calculations are utilized:
\[ \mathcal{L}g^+ + \mathcal{L}^\dagger (g^+_1, g^-_1) - \mathcal{L}g^- - \mathcal{L}^\dagger (g^-_1, g^+_1) \]
\[ = \mathcal{L}(g^+ - g^-) - Q(g^+_1, 1) - Q(g^-_1, 1) + Q(g^+_1, 1) + Q(1, g^-_1) \]
\[ = (L + \mathcal{L})(g^+_1 - g^-_1), \]

and by the relation (2.6)
\[ Q(g^+_0, g^-_0) + Q(g^+_0, g^-_0) - Q(g^-_0, g^-_0) - Q(g^-_0, g^+_0) \]
\[ = Q(g^+_0 - g^-_0, g^+_0 + g^-_0) = 2n_0 Q(1, u_0 \cdot v + \theta_0 (\frac{|v|^2}{2} - \frac{3}{2})) \]
\[ = n_0 \left[ Q(u_0 \cdot v + \theta_0 (\frac{|v|^2}{2} - \frac{3}{2}), 1) + Q(1, u_0 \cdot v + \theta_0 (\frac{|v|^2}{2} - \frac{3}{2})) \right] \]
\[ - Q(u_0 \cdot v + \theta_0 (\frac{|v|^2}{2} - \frac{3}{2}), 1) + Q(1, u_0 \cdot v + \theta_0 (\frac{|v|^2}{2} - \frac{3}{2})) \]
\[ = - n_0 \mathcal{L}(u_0 \cdot v + \theta_0 (\frac{|v|^2}{2} - \frac{3}{2})) + n_0 \mathcal{L}(u_0 \cdot v + \theta_0 (\frac{|v|^2}{2} - \frac{3}{2})) \]
\[ = n_0 (L + \mathcal{L})(u_0 \cdot v + \theta_0 (\frac{|v|^2}{2} - \frac{3}{2})). \]

As shown in [3], for $\Phi(v) = v \in L^2(Mdv)$ and $\Psi(v) = \frac{|v|^2}{2} - \frac{3}{2} \in L^2(Mdv)$ there are functions $\tilde{\Phi}, \tilde{\Psi} \in \ker^\perp (L + \mathcal{L})$ such that
\[ (L + \mathcal{L})\tilde{\Phi} = \Phi \quad \text{and} \quad (L + \mathcal{L})\tilde{\Psi} = \Psi, \]
(2.16)
which can be uniquely determined in $\ker^\perp (L + \mathcal{L})$. Furthermore, there exist two scalar valued functions $\alpha, \beta : \mathbb{R}^+ \rightarrow \mathbb{R}$ such that
\[ \tilde{\Phi}(v) = \alpha(|v|)\Phi(v) \quad \text{and} \quad \tilde{\Psi}(v) = \alpha(|v|)\Psi(v), \]
which implies that
\[ \int_{\mathbb{R}^3} \Phi_i(v)\tilde{\Phi}_j(v)Mdv = \frac{1}{2} \sigma \delta_{ij}, \]
where $\sigma = \frac{2}{3} \int_{\mathbb{R}^3} \Phi \cdot \tilde{\Phi} Mdv$ defines the electrical conductivity $\sigma > 0$. One also define the energy conductivity $\lambda > 0$ by $\lambda = \int_{\mathbb{R}^3} \Psi \cdot \tilde{\Psi} Mdv$. As a consequence, the equation (2.15) implies that for some scalar-valued function $n_1(x,t)$
\[ \frac{g^+ + g^-}{2} = \frac{1}{2} n_1 + \frac{1}{2} n_0 u_0 \cdot v + \frac{1}{2} n_0 \theta_0 (\frac{|v|^2}{2} - \frac{3}{2}) + (- \frac{1}{2} \nabla_x n_0 + E_0 + u_0 \times B_0) \cdot \tilde{\Phi}(v). \]
(2.17)
Moreover, projecting the relation (2.17) to the kernel ker \( \mathcal{L} \) of \( \mathcal{L} \) reduces to

\[
\begin{align*}
\left\{
\begin{array}{l}
n_{1} = \langle g_{1}^{+} - g_{1}^{-} \rangle , \\
u_{1}^{+} - u_{1}^{-} = n_{0}u_{0} + \sigma (-\frac{1}{2} \nabla x n_{0} + E_{0} + u_{0} \times B_{0}), \\
\theta_{1}^{+} - \theta_{1}^{-} = n_{0}\theta_{0},
\end{array}
\right.
\end{align*}
\]

(2.18)

where \( u_{1}^{\pm} = \langle g_{1}^{\pm} v \rangle, \ \theta_{1}^{\pm} = \langle g_{1}^{\pm} (|v|^{2}/3 - 1) \rangle \).

Now we consider the order of \( \mathcal{O}(\varepsilon) \) in the expansion form (2.3), which reads that

\[
\begin{align*}
\partial_{t} \left( \frac{g_{0}^{+} \pm g_{0}^{-}}{g_{0}^{+} - g_{0}^{-}} \right) + v \cdot \nabla_{x} \left( \frac{g_{0}^{+} \pm g_{0}^{-}}{g_{0}^{+} - g_{0}^{-}} \right) + E_{0} \cdot \nabla_{v} \left( \frac{g_{0}^{+} - g_{0}^{-}}{2} \right) + (v \times B_{0}) \cdot \nabla_{v} \left( \frac{g_{0}^{+} - g_{0}^{-}}{2} \right) \\
\quad + (v \times B_{1}) \cdot \nabla_{v} \left( \frac{g_{0}^{+} - g_{0}^{-}}{2} \right) - E_{0} \cdot v \left( \frac{g_{0}^{+} - g_{0}^{-}}{2} \right) - E_{1} \cdot v \left( \frac{1}{1} \right)
\end{align*}
\]  

(2.19)

\[
\begin{align*}
\left\{
\begin{array}{l}
\partial_{t} E_{1} - \nabla_{x} \times B_{1} = -(u_{2}^{+} - u_{2}^{-}), \\
\partial_{t} B_{1} + \nabla_{x} \times E_{1} = 0, \\
\text{div}_{x} E_{1} = n_{1}, \text{ div}_{x} B_{1} = 0.
\end{array}
\right.
\end{align*}
\]

(2.20)

Taking the relations of the operators \( \mathcal{L} \) and \( \mathcal{Q} \) into consideration, the equations (2.19) are equivalent to

\[
\begin{align*}
\partial_{t} \left( \frac{g_{0}^{+} \pm g_{0}^{-}}{g_{0}^{+} - g_{0}^{-}} \right) + v \cdot \nabla_{x} \left( \frac{g_{0}^{+} \pm g_{0}^{-}}{g_{0}^{+} - g_{0}^{-}} \right) + E_{0} \cdot \nabla_{v} \left( \frac{g_{0}^{+} - g_{0}^{-}}{2} \right) + (v \times B_{0}) \cdot \nabla_{v} \left( \frac{g_{0}^{+} - g_{0}^{-}}{2} \right) \\
\quad + (v \times B_{1}) \cdot \nabla_{v} \left( \frac{g_{0}^{+} - g_{0}^{-}}{2} \right) - E_{0} \cdot v \left( \frac{g_{0}^{+} - g_{0}^{-}}{2} \right) - E_{1} \cdot v \left( \frac{0}{1} \right)
\end{align*}
\]  

(2.21)

Recalling the form (2.6) of \( g_{0}^{\pm} \) and the relations (2.12), direct calculations reduce to

\[
\begin{align*}
\partial_{t} \left( \frac{g_{0}^{+} \pm g_{0}^{-}}{g_{0}^{+} - g_{0}^{-}} \right) + E_{0} \cdot \nabla_{v} \left( \frac{g_{0}^{+} - g_{0}^{-}}{2} \right) + (v \times B_{1}) \cdot \nabla_{v} \left( \frac{g_{0}^{+} - g_{0}^{-}}{2} \right) \\
\quad - E_{0} \cdot v \left( \frac{g_{0}^{+} - g_{0}^{-}}{2} \right) - E_{1} \cdot v \left( \frac{0}{1} \right)
\end{align*}
\]  

(2.22)

\[
\begin{align*}
\left\{
\begin{array}{l}
\frac{1}{2} \partial_{t} \rho_{0} + (\partial_{t} u_{0} - \frac{1}{2} n_{0} E_{0}) \cdot v + \frac{1}{2} \partial_{t} \theta_{0} (|v|^{2}/2 - 3/2) \\
\frac{1}{2} \partial_{t} n_{0} + (\theta_{0} E_{0} - u_{0} \times B_{1} - E_{1}) \cdot v - \frac{2}{3} E_{0} \cdot u_{0} (|v|^{2}/2 - 3/2)
\end{array}
\right.
\end{align*}
\]

(2.22)

Now we compute the term \( v \cdot \nabla_{x} l_{1} (\frac{g_{0}^{+} + g_{0}^{-}}{2}) \). As in (2.14), we denote the kernel part and kernel orthogonal part of \( \frac{g_{0}^{+} + g_{0}^{-}}{2} \) by \( l_{1} \) and \( l_{1}^{\perp} \), respectively. We derive that

\[
\begin{align*}
\nabla_{x} l_{1} &= \text{div}_{x} u_{1} + \nabla_{x} (\rho_{1} + \theta_{1}) \cdot v + \frac{2}{3} \text{div}_{x} u_{1} (\frac{|v|^{2}}{2} - \frac{3}{2}) + \nabla_{x} u_{1} : \mathcal{A}(v) + \nabla_{x} \theta_{1} \cdot \mathcal{B}(v).
\end{align*}
\]  

(2.23)
Projecting the term $v \cdot \nabla_x l_1^-$ into the kernel $\ker \mathcal{L}$, we have
\[
\left(v \cdot \nabla_x l_1^\perp \left( \frac{1}{|v|} \right) \right) = \left( \begin{array}{c}
\text{div}_x(u_0 \otimes u_0) - \mu \Delta_x u_0 - \frac{1}{3} \nabla_x (|u_0|^2) \\
0 \\
\frac{1}{3} \text{div}_x(\theta_0 u_0) - \frac{2}{3} \kappa \Delta_x \theta_0
\end{array} \right),
\] (2.24)
where $\mu = \frac{1}{30} \int_{\mathbb{R}^3} \varphi(|v|)|v|^4 Mdv > 0$ and $\kappa = \frac{1}{30} \int_{\mathbb{R}^3} \tilde{B}(v) \cdot \mathcal{B}(v) Mdv > 0$. Since $\frac{g_1^+ - g_2^-}{2}$ is the form of (2.17), we can calculate that
\[
v \cdot \nabla_x \left( \frac{g_1^+ - g_2^-}{2} \right) = \frac{1}{2} \text{div}_x(u_0 u_0) + \frac{1}{2} \left( \nabla_x n_1 + \frac{5}{2} \nabla_x (n_0 \theta_0) \right) \cdot v + \frac{1}{3} \text{div}_x(n_0 u_0) \left( \frac{|u|^2}{2} - \frac{3}{2} \right)
+ \nabla_x \left( - \frac{1}{2} \nabla_x n_0 + E_0 + u_0 \times B_0 \right) : v \otimes \Phi(v)
+ \frac{1}{2} \nabla_x(n_0 u_0) : \mathcal{A}(v) + \frac{1}{2} \nabla_x(n_0 \theta_0) : \mathcal{B}(v).
\] (2.25)
Moreover, the relations (2.14) and (2.17) yield that
\[
(v \times B_0) \cdot \nabla_x \left( \frac{g_1^+ - g_2^-}{2} \right) = - \left( \frac{1}{2} u_0 n_0 \times B_0 \right) \cdot v - B_0 ( - \frac{1}{2} \nabla_x n_0 + E_0 + u_0 \times B_0 ) t \epsilon_{ijk} v_j \partial_{v_k} \Phi_1(v),
\] (2.26)
and
\[
(v \times B_0) \cdot \nabla_x \left( \frac{g_1^+ + g_2^-}{2} \right) = - (u_1 \times B_0) \cdot v - (B_0 \times u_0) \otimes u_0 : \mathcal{A}(v) - (n_0 u_0 \times B_0) \cdot \mathcal{B}(v)
+ \frac{1}{2} \partial_t u_0 m \epsilon_{ijk} v_j \partial_{v_k} \Phi_1(v)
+ \frac{1}{2} B_0 ( - \frac{1}{2} \nabla_x n_0 + E_0 + u_0 \times B_0 ) \cdot \mathcal{B}(v),
\] (2.27)
where $\epsilon_{ijk}$ ($1 \leq i, j, k \leq 3$) is zero, if there are two numbers identical among the $i, j, k$, is 1, if the order of $\{i, j, k\}$ is even, and is -1, if the order of $\{i, j, k\}$ is odd. One can easily derive that
\[
\mathcal{P}_\mathcal{L} \left( B_0 ( - \frac{1}{2} \nabla_x n_0 + E_0 + u_0 \times B_0 ) t \epsilon_{ijk} v_j \partial_{v_k} \Phi_1(v) \right)
= \left[ \frac{1}{2} \sigma \left( - \frac{1}{2} \nabla_x n_0 + E_0 + u_0 \times B_0 \right) \times B_0 \right] \cdot v,
\] (2.28)
and
\[
\mathcal{P}_{\mathcal{L} + \Sigma} \left( \frac{1}{2} \partial_t u_0 m \epsilon_{ijk} v_j \partial_{v_k} \Phi_1(v) \right)
= \mathcal{P}_{\mathcal{L} + \Sigma} \left( \frac{1}{2} B_0 \partial_t \theta_0 m \epsilon_{ijk} v_j \partial_{v_k} \tilde{B}_1(v) \right) = 0,
\] (2.29)
where the projection $\mathcal{P}_{\mathcal{L} + \Sigma} : L^2(Mdv) \rightarrow \ker (\mathcal{L} + \Sigma) \subset L^2(Mdv)$ is given by $\mathcal{P}_{\mathcal{L} + \Sigma}(g) = \langle g \rangle$. We notice that for all $g, h \in L^2(Mdv)$,
\[
\mathcal{Q}(g, h) + \mathcal{Q}(h, g) \in \ker \perp \mathcal{L}, \quad \text{and} \quad \mathcal{Q}(g, h) \in \ker \perp (\mathcal{L} + \Sigma).
\]
Consequently, projecting the first equation of (2.21) into the kernel $\ker \mathcal{L}$ and combining the equalities (2.22), (2.23), (2.24), (2.26) and (2.28) imply that
\[
\begin{cases}
\partial_t \rho_0 + \text{div}_x u_1 = 0, \\
\partial_t u_0 + \text{div}_x (u_0 \otimes u_0) - \mu \Delta_x u_0 + \nabla_x (\rho_1 + \theta_1 - \frac{1}{3} |u_0|^2) = \frac{1}{2} n_0 E_0 + \frac{1}{2} j_0 \times B_0, \\
\partial_t \theta_0 + \frac{1}{2} \text{div}_x (\theta_0 u_0) - \frac{2}{3} \kappa \Delta_x \theta_0 = 0
\end{cases}
\] (2.30)
which, by (2.8), (2.12) and (2.18), means that
\[
\begin{cases}
\partial_t u_0 + u_0 \cdot \nabla_x u_0 - \mu \Delta_x u_0 + \nabla_x p_0 = \frac{1}{2} n_0 E_0 + \frac{1}{2} j_0 \times B_0, \\
\partial_t \rho_0 = 0, \\
\theta_0 = - \frac{1}{2} \nabla_x \rho_0 - \kappa \Delta_x \theta_0 = 0, \\
\partial_t \rho_0 = 0, \\
\partial_t u_0 - \nabla_x \rho_0 = - j_0, \\
\partial_t B_0 - \nabla_x E_0 = 0, \\
j_0 = n_0 u_0 + \sigma \left( - \frac{1}{2} \nabla_x n_0 + E_0 + u_0 \times B_0 \right), \\
\text{div}_x E_0 = 0, \quad \text{div}_x B_0 = 0, \\
\rho_0 + \theta_0 = 0
\end{cases}
\] (2.31)
Therefore, the first equation of (2.21) projecting to $\ker^+ \mathcal{L}$ is
\[
\mathcal{L}\left(\frac{g^+ - g^-}{2}\right) = Q\left(\frac{g_{0}^{+} + g_{0}^{-}}{2}, \frac{g_{l}^{+} + g_{l}^{-}}{2}\right) + Q\left(\frac{g_{1}^{+} + g_{1}^{-}}{2}, \frac{g_{1}^{+} + g_{1}^{-}}{2}\right)
\]
\[\quad - \frac{1}{2} \nabla_x u_1 : \mathcal{A}(v) - \frac{1}{2} \nabla_x \theta_1 : \mathcal{B}(v) - \frac{1}{2} \mathcal{P}_\mathcal{L}^-(v \cdot \nabla l^1_1)\]
\[\quad + \frac{1}{2} B_0 (j_0 l - n_0 u_0 l) \epsilon_{ijk} \mathcal{P}_\mathcal{L}^+(v_j \partial_k \tilde{F}_1(v)),\]
where the orthogonal projection operator $\mathcal{P}_\mathcal{L}^\perp : L^2(\mathcal{M}dv) \to \ker^+ \mathcal{L}$ is given by $\mathcal{P}_\mathcal{L}^\perp = Id - \mathcal{P}_\mathcal{L}$, and
\[
l^1_1 = \frac{1}{2} u_0 \otimes u_0 : \mathcal{A}(v) + \theta_0 u_0 \cdot B(v) + \frac{1}{2} \theta_0^2 C(v) - \frac{1}{2} \nabla_x u_0 : \tilde{\mathcal{A}}(v) - \frac{1}{2} \nabla_x \theta_0 : \tilde{\mathcal{B}}(v).
\]
Since $Q(g, h) + Q(h, g) = \mathcal{L}(gh)$ holds for all $g, h \in \ker \mathcal{L}$ and $\frac{g^+ + g^-}{2} = l_1 + l^1_1$ with $l_1 \in \ker \mathcal{L}$, $l^1_1 \in \ker^+ \mathcal{L}$, we gain
\[
\begin{aligned}
Q\left(\frac{g_{0}^{+} + g_{0}^{-}}{2}, \frac{g_{1}^{+} + g_{1}^{-}}{2}\right) &+ Q\left(\frac{g_{1}^{+} + g_{1}^{-}}{2}, \frac{g_{1}^{+} + g_{1}^{-}}{2}\right) \\
&= Q\left(\frac{g_{0}^{+} + g_{0}^{-}}{2}, l_1\right) + Q\left(l_1, \frac{g_{0}^{+} + g_{0}^{-}}{2}\right) + Q\left(\frac{g_{1}^{+} + g_{1}^{-}}{2}, l^1_1\right) + Q\left(l^1_1, \frac{g_{0}^{+} + g_{0}^{-}}{2}\right)
\end{aligned}
\]
(2.33)
By direct calculation, one has
\[
\begin{aligned}
l_1 \frac{g_{0}^{+} + g_{0}^{-}}{2} &= (\rho_0 \rho_1 + u_0 \cdot u_1 + \frac{3}{2} \theta_0 \theta_1) + (\rho_0 u_1 + \rho_1 u_0 + \theta_0 u_1 + \theta_1 u_0) \cdot v \\
&\quad + \left(\frac{3}{2} u_0 \cdot u_1 + 2 \theta_0 \theta_1 + \rho_0 \theta_1 + \rho_1 \theta_0\right) (|v|^2 - \frac{3}{2}) \\
&\quad + (u_0 \otimes u_1 + u_1 \otimes u_0) : \mathcal{A}(v) + (\theta_0 u_1 + \theta_1 u_0) \cdot B(v) + \theta_0 \theta_1 C(v),
\end{aligned}
\]
which implies that
\[
\mathcal{L}(l_1 \frac{g_{0}^{+} + g_{0}^{-}}{2}) = \mathcal{L}(u_0 \otimes u_1 + u_1 \otimes u_0 : \mathcal{A}(v) + (\theta_0 u_1 + \theta_1 u_0) \cdot B(v) + \theta_0 \theta_1 C(v)).
\] (2.34)
Thus, by the structure of $l^1_1$ and plugging (2.6), (2.33) and (2.34) into (2.32), we gain that for some functions $\rho_2(x,t), u_2(x,t)$ and $\theta_2(x,t)$
\[
\frac{g_{1}^{+} + g_{1}^{-}}{2} = \rho_2 + u_2 \cdot v + \theta_2 \left(\frac{|v|^2}{2} - \frac{3}{2}\right)
\]
\[\quad + (u_0 \otimes u_1 + u_1 \otimes u_0) : \mathcal{A}(v) + (\theta_0 u_1 + \theta_1 u_0) \cdot B(v) + \theta_0 \theta_1 C(v)
\]
\[\quad - \frac{1}{2} \nabla_x u_1 : \tilde{\mathcal{A}}(v) - \frac{1}{2} \nabla_x \theta_1 : \tilde{\mathcal{B}}(v) + \sum \Gamma_0^+ \Upsilon^+(v).
\] (2.35)
Here the term $\sum \Gamma_0^+ \Upsilon^+(v)$ is the summation of the form $\Gamma_0^+ \Upsilon^+(v)$, where $\Gamma_0^+$ formally depends only on the $u_0, n_0, \theta_0, B_0, E_0$ and $\Upsilon^+(v) \in \ker^+ \mathcal{L}$ depends only on the variables $v \in \mathbb{R}^3$. More precisely,
\[
\begin{aligned}
\sum \Gamma_0^+ \Upsilon^+(v) &= -\frac{1}{2} \theta_0 u_0 \otimes u_0 : \tilde{\mathcal{A}}(v) - \frac{1}{2} \theta_0 \nabla_x \theta_0 : \tilde{\mathcal{B}}(v) + \frac{1}{2} \theta_0^2 u_0 \otimes u_0 : \mathcal{A}(v) + \frac{1}{2} \theta_0^2 u_0 \cdot B(v) \\
&\quad + \frac{1}{2} \theta_0^3 C(v) + \frac{1}{2} B_0 (j_0 l - n_0 u_0 l) \epsilon_{ijk} \mathcal{L}^{-1}[\mathcal{P}_\mathcal{L}^+(v_j \partial_k \tilde{F}_1(v))] \\
&\quad + \frac{1}{2} \theta_0 u_0 \otimes u_0 \otimes u_0 \cdot L^{-1}(Q(v, \mathcal{A}(v))) + Q(\mathcal{A}(v), v) \\
&\quad + \theta_0 u_0 \otimes u_0 : L^{-1}(Q(v, \mathcal{B}(v))) + Q(\mathcal{B}(v), v)) - \frac{1}{2} \nabla_x u_0 \otimes u_0 : L^{-1} \mathcal{P}_\mathcal{L}^- Q(\tilde{\mathcal{A}}(v), v) \\
&\quad + \frac{1}{2} \theta_0^2 u_0 \cdot L^{-1}(Q(v, C(v))) + Q(C(v), v)) - \frac{1}{2} \theta_0^2 u_0 \otimes u_0 \otimes u_0 \cdot L^{-1} \mathcal{P}_\mathcal{L}^- Q(\tilde{\mathcal{A}}(v), v) \\
&\quad - \frac{1}{2} \theta_0 u_0 \otimes u_0 : L^{-1} \mathcal{P}_\mathcal{L}^- Q(\tilde{\mathcal{B}}(v), v) - \frac{1}{2} u_0 \otimes \nabla_x \theta_0 : L^{-1} \mathcal{P}_\mathcal{L}^- Q(v, \tilde{\mathcal{A}}(v)) \\
&\quad + \frac{1}{2} \theta_0 u_0 \otimes u_0 : L^{-1}(Q(|v|^2, \mathcal{A}(v))) + Q(\mathcal{A}(v), |v|^2) \\
&\quad + \frac{1}{2} \theta_0^2 u_0 \cdot L^{-1}(Q(|v|^2, \mathcal{B}(v))) + Q(\mathcal{B}(v), |v|^2) \\
&\quad + \frac{1}{2} \theta_0^3 L^{-1}(Q(|v|^2, \mathcal{C}(v))) + Q(\mathcal{C}(v), |v|^2)) - \frac{1}{2} \nabla_x \theta_0^2 \cdot L^{-1} \mathcal{P}_\mathcal{L}^- (v C(v)) \\
&\quad - \frac{1}{4} \theta_0 \nabla_x u_0 : L^{-1}(Q(|v|^2, \tilde{\mathcal{A}}(v))) + Q(\tilde{\mathcal{A}}(v), |v|^2))
\end{aligned}
\] (2.36)
where the operator \( \mathcal{L}^{-1} \) represents \( (\mathcal{L}|_{\ker \mathcal{L}})^{-1} : \ker \mathcal{L} \to \ker \mathcal{L} \), which is a one to one and onto map.

If we project the second equation of (2.21) into the kernel \( \ker (\mathcal{L} + \mathcal{L}) \), together with (2.22), (2.25), (2.27) and (2.29), then we gain

\[
\partial_t n_0 + \text{div}_x j_0 = 0.
\]  

(2.37)

We emphasize that the equation (2.37) is implied in the limit system (2.30). More precisely, taking the divergence operator \( \text{div}_x \) on the forth equation of (2.30) and the relations \( \text{div}_x (\nabla_x \times B_0) = 0 \) and \( \text{div}_x E_0 = n_0 \) yield the equation (2.37). Consequently, the equation (2.21) projecting the kernel orthogonal \( \ker (\mathcal{L} + \mathcal{L}) \) is

\[
(\mathcal{L} + \mathcal{L})(\frac{g_1^+-g_1^-}{2}) = -\left( \frac{1}{2} \nabla_x n_1 - E_1 - u_0 \times B_1 - u_1 \times B_0 + \frac{5}{2} \nabla_x (n_0 \theta_0) + \theta_0 E_0 \right) \cdot v
\]

\[
- \frac{1}{2} \nabla_x (n_0 u_0) - 2E_0 \cdot u_0 \left( |v|^2 - \frac{1}{2} \right)
\]

\[
- \frac{1}{2} \nabla_x (n_0 u_0) - (E_0 + B_0 \times u_0) \cdot A(v)
\]

\[
- \frac{1}{2} \nabla_x (n_0 u_0 + \theta_0 E_0 - n_0 u_0 \times B_0) \cdot B(v) - \frac{1}{2} B_0 \partial_t \theta_0 \epsilon_{ijk} u_0 \partial_{xk} \tilde{\mathcal{B}}(v)
\]

\[
- \frac{1}{2} \nabla_x (j_0 - n_0 u_0) \cdot \mathcal{P}^{2}_{\mathcal{L}+\mathcal{L}}(v \otimes \tilde{\Phi}(v)) - \frac{1}{2} B_0 \partial_t \theta_0 \epsilon_{ijk} u_0 \partial_{xk} \tilde{\mathcal{A}}_m(v)
\]

\[
- n_0 \mathcal{L}(\frac{g_1^+-g_1^-}{2}) + \frac{1}{2} (\mathcal{L} + \mathcal{L}) \left( \frac{g_1^+-g_1^-}{2} \right) + \mathcal{Q} (g_1^+ - g_1^-, \frac{g_1^+-g_1^-}{2})
\]

where the operator \( \mathcal{P}^{2}_{\mathcal{L}+\mathcal{L}} \) is \( Id - \mathcal{P}_{\mathcal{L}+\mathcal{L}} \). By (2.6) and (2.17), direct calculation implies that

\[
\mathcal{Q} (g_1^+ - g_1^-, \frac{g_1^+-g_1^-}{2})
\]

\[
= (n_1 + n_0 \theta_0) u_0 \cdot \mathcal{Q}(1, v) + \frac{1}{2} (n_1 + n_0 \theta_0) \theta_0 \mathcal{Q}(1, |v|^2)
\]

\[
+ \frac{1}{2} n_0 \mathcal{P}_{\mathcal{L}+\mathcal{L}}(u_0 \otimes u_0 : A(v) + 2 \theta_0 u_0 \cdot B(v) + \theta_0^2 \mathcal{C}(v))
\]

\[
+ \frac{1}{2} \theta_0 (j_0 - n_0 u_0) \cdot \mathcal{Q}(\tilde{\Phi}(v), |v|^2 - 5) + \frac{1}{2} (j_0 - n_0 u_0) \otimes u_0 : \mathcal{Q}(\tilde{\Phi}(v), v)
\]

Then, we have

\[
- n_0 \mathcal{L}(\frac{g_1^+-g_1^-}{2}) + \mathcal{Q} (g_1^+ - g_1^-, \frac{g_1^+-g_1^-}{2})
\]

\[
= (n_1 + n_0 \theta_0) u_0 \cdot \mathcal{Q}(1, v) + \frac{1}{2} (n_1 + n_0 \theta_0) \theta_0 \mathcal{Q}(1, |v|^2)
\]

\[
+ \frac{1}{2} n_0 \nabla_x \theta_0 \cdot B(v) + \frac{1}{2} \theta_0 (j_0 - n_0 u_0) \cdot \mathcal{Q}(\tilde{\Phi}(v), |v|^2 - 5) + \frac{1}{2} (j_0 - n_0 u_0) \otimes u_0 : \mathcal{Q}(\tilde{\Phi}(v), v)
\]

(2.40)

As a result, for some function \( n_2(x, t) \),

\[
\frac{g_1^+-g_1^-}{2} = \frac{1}{2} n_2 - \left( \frac{1}{2} \nabla_x n_1 - E_1 - u_0 \times B_1 - u_1 \times B_0 \right) \cdot \tilde{\Phi}(v) + \frac{1}{2} u_1 \cdot v + \frac{1}{2} \theta_1 (|v|^2 - \frac{1}{2})
\]

\[
+ n_1 u_0 \cdot \mathcal{L}^{-1} \mathcal{Q}(1, v) + \frac{1}{2} n_1 \theta_0 (\mathcal{L} + \mathcal{L})^{-1} \mathcal{Q}(1, |v|^2) + \sum \Gamma_0 \mathcal{Y}^-(v).
\]

(2.41)

Here the term \( \sum \Gamma_0 \mathcal{Y}^-(v) \) is the summation of the form \( \Gamma_0 \mathcal{Y}^-(v) \), where \( \Gamma_0 \) formally depends only on the \( u_0, n_0, \theta_0, B_0, E_0 \) and \( \mathcal{Y}^-(v) \in \ker (\mathcal{L} + \mathcal{L}) \) depends only on the variables \( v \in \mathbb{R}^3 \). More precisely,

\[
\sum \Gamma_0 \mathcal{Y}^-(v) = - \left( \frac{5}{2} \nabla_x (n_0 u_0 + \theta_0 E_0) \cdot \tilde{\Phi}(v) - \frac{1}{3} \text{div}_x (n_0 u_0 - 2E_0 \cdot u_0) \tilde{\Phi}(v)
\]

\[
+ \frac{1}{2} u_0 \otimes u_0 : \mathcal{A}(v) + \frac{1}{2} \theta_0 u_0 \cdot B(v) + \frac{1}{4} \theta_0^2 \mathcal{C}(v) - \frac{1}{2} \nabla_x u_0 : \tilde{\mathcal{A}}(v) - \frac{1}{2} \nabla_x \theta_0 \cdot \tilde{\mathcal{B}}(v)
\]

\[
n_0 \theta_0 u_0 \cdot (\mathcal{L} + \mathcal{L})^{-1} \mathcal{Q}(1, v) + \frac{1}{2} n_0 \theta_0^2 (\mathcal{L} + \mathcal{L})^{-1} \mathcal{Q}(1, |v|^2)
\]

\[
- \left( \frac{1}{2} \nabla_x (n_0 u_0) - (E_0 + B_0 \times u_0) \otimes u_0 \right) : \mathcal{L}^{-1} \mathcal{A}(v)
\]
\[- \left( \frac{1}{\sigma} \nabla_x (n_0 \theta_0) + \theta_0 E_0 - n_0 u_0 \times B_0 \right) : (\mathcal{L} + \mathcal{M})^{-1} B(v) \]
\[- \frac{1}{2} B_0 \partial_t \theta_0 \epsilon_{ijk} (\mathcal{L} + \mathcal{M})^{-1} (v_j \partial_{x_k} \tilde{B}(v)) - \frac{1}{2} B_0 \partial_m \epsilon_{ijk} (\mathcal{L} + \mathcal{M})^{-1} (v_j \partial_{x_k} \tilde{A}_m(v)) \]
\[- \frac{1}{\sigma} \nabla_x \left( j_0 - n_0 u_0 \right) : (\mathcal{L} + \mathcal{M})^{-1} \rho_{\mathcal{E}} \bigotimes (\sigma \nabla_x \tilde{\Phi}(v)) \]
\[+ \frac{1}{\sigma} \theta_0 \epsilon_{ijk} (\mathcal{L} + \mathcal{M})^{-1} Q(\tilde{\Phi}(v), |v|^2 - 5) + \frac{n_0}{\sigma} \nabla_x u_0 : (\mathcal{L} + \mathcal{M})^{-1} A(v) \]
\[+ \frac{2}{\sigma} \left( j_0 - n_0 u_0 \right) \otimes u_0 : (\mathcal{L} + \mathcal{M})^{-1} Q(\tilde{\Phi}(v), v) + \frac{n_0}{\sigma} \nabla_x \theta_0 : (\mathcal{L} + \mathcal{M})^{-1} B(v) , \] 

where the symbol \((\mathcal{L} + \mathcal{M})^{-1}\) represents \(\left( (\mathcal{L} + \mathcal{M})|_{ker(\mathcal{L} + \mathcal{M})} \right)^{-1} : ker^+(\mathcal{L} + \mathcal{M}) \to ker^+(\mathcal{L} + \mathcal{M})\).

If we project the equality (2.41) into the kernel \(\text{ker}\mathcal{L}\), we gain
\[\begin{align*}
\left\{ \begin{array}{l}
n_2 = (g_{1}^{+} - g_{2}^{-}), \\
u_j^+ - u_j^- = n_1 (u_0 \cdot \mathbf{M} + \theta_0 \mathbf{V}) + u_1 + \sigma (-\frac{1}{3} \nabla x n_1 + E_1 + u_0 \times B_1 + u_1 \times B_0) + \sum \Gamma_0^{-} \mathbf{U}_T^{-} , \\
\theta_2^+ - \theta_2^- = \theta_0 + n_1 (\mathbf{V} \cdot u_0 + C \theta_0) + \sum C \Gamma_0^{-} ,
\end{array} \right.
\end{align*}\] 

and are of the form \((2.35), (2.41)\), respectively, in which the functions \(\rho_2, u_2, \theta_2\) and \(n_2\) do not impose any constraint condition. In summary, the functions \(\rho_1, u_1, \theta_1, n_1, E_1\) and \(B_1\) must obey the relations

\[\begin{align*}
\left\{ \begin{array}{l}
\text{div}_x u_1 = \partial_t \theta_0 , \\
\nabla_x (\rho_1 + \theta_1) = -\partial_t u_0 - u_0 \cdot \nabla_x u_0 + \mu \Delta_x u_0 + \frac{1}{3} \nabla_x (|u_0|^2) + \frac{1}{2} j_0 \times B_0 ,
\end{array} \right.
\end{align*}\]

and
\[\begin{align*}
\left\{ \begin{array}{l}
\partial_t E_1 - \nabla_x \times B_1 = - (u_2^+ - u_2^-) , \\
\partial_t B_1 + \nabla_x \times E_1 = 0 , \\
\text{div}_x E_1 = n_1 , \quad \text{div}_x B_1 = 0 , \\
u_2^+ - u_2^- = n_1 (u_0 \cdot \mathbf{M} + \theta_0 \mathbf{V}) + u_1 + \sigma (-\frac{1}{3} \nabla x n_1 + E_1 + u_0 \times B_1 + u_1 \times B_0) + \sum \Gamma_0^{-} \mathbf{U}_T^{-} ,
\end{array} \right.
\end{align*}\] 

We emphasize that there are many functions \(\rho_1, u_1\) and \(\theta_1\) satisfying the relations (2.44). However, if the functions \(u_0, \theta_0, E_0, B_0\) and \(u_1\) are the known, the equations (2.44) are a closed linear system. In other words, if the initial data of \(E_1\) and \(B_1\) are given, the functions \(E_1\) and \(B_1\) are uniquely determined, which exist globally if the known functions in the system (2.45) admit existence global in time.

Now, we choose functions \(\tilde{\rho}_1, \tilde{u}_1\) and \(\tilde{\theta}_1\) as follows, which satisfy the relations (2.44). If the vector \(\tilde{u}_1\) is of the gradient form, hence \(\tilde{u}_1 = \nabla_x \phi\) for some function \(\phi(x, t)\), then the first relation (2.44) implies that \(\Delta_x \phi = \partial_t \theta_0\). Assume \(\tilde{\rho}_1\) and \(\tilde{\theta}_1\) obey the second relation (2.44). Since \(\text{div}_x u_0 = 0\), we have
\[\Delta_x \tilde{\rho}_1 (t, x) = \Delta_x \tilde{\theta}_1 (t, x) = \frac{1}{\sigma} \Delta_x |u_0|^2 = - \frac{1}{2} \text{div}_x (u_0 \cdot \nabla_x u_0 - \frac{1}{2} j_0 \times B_0) .\] 

Then, if we take \(\tilde{\rho}_1 = \tilde{\theta}_1\), we have
\[\Delta_x \tilde{\rho}_1 (t, x) = \Delta_x \tilde{\theta}_1 (t, x) = \frac{1}{\sigma} \Delta_x |u_0|^2 - \frac{1}{2} \text{div}_x (u_0 \cdot \nabla_x u_0 - \frac{1}{2} j_0 \times B_0) .\]
If we replace the vector function \( u_1 \) occurring in the system (2.45) by the vector function \( \bar{u}_1 \) what we just choose, there exist the unique vector functions \( \bar{E}_1 \) and \( \bar{B}_1 \) solving the equations (2.45). In summary, the following functions

\[
\begin{align*}
\Delta_x \phi(t, x) &= \partial_t \theta_0(t, x), \\
\bar{u}_1(t, x) &= \nabla_x \phi(t, x), \\
\Delta_x \bar{\psi}_1(t, x) &= \frac{1}{6} \Delta_x |u_0|^2 - \frac{1}{2} \text{div}_x (u_0 \cdot \nabla_x u_0 - \frac{1}{2} j_0 \times B_0)
\end{align*}
\]

(2.46)

on \( x \in \mathbb{T}^3 \) and

\[
\begin{align*}
\partial_t \bar{E}_1 - \nabla_x \times \bar{B}_1 &= -j_1, \\
\partial_t \bar{B}_1 + \nabla_x \times \bar{E}_1 &= 0, \\
\text{div}_x \bar{E}_1 &= \bar{n}_1, \quad \text{div}_x \bar{B}_1 = 0,
\end{align*}
\]

(2.47)

obey the relations (2.44) and (2.45).

We define functions \( \bar{g}_1^+(x, v, t) \) and \( \bar{g}_2^+(x, v, t) \) with the similar form in (2.14), (2.17), (2.35) and (2.41). More precisely,

\[
\begin{align*}
\bar{g}_1^+ &= \bar{\rho}_1 + \bar{u}_1 \cdot v + \bar{\theta}_1 \left( \frac{|v|^2}{2} \right) + \frac{1}{2} u_0 \otimes u_0 : \mathcal{A}(v) + \theta_0 u_0 \cdot \mathcal{B}(v) + \frac{1}{2} \theta_0^2 \mathcal{C}(v), \\
\bar{g}_1^- &= -\frac{1}{2} \nabla_x u_0 : \mathcal{A}(v) - \frac{1}{2} \nabla_x \bar{u}_0 \cdot \mathcal{B}(v), \\
\bar{g}_2^+ &= \frac{1}{2} \bar{n}_1 + \frac{1}{2} n_0 u_0 v + \frac{1}{2} u_0 \bar{\theta}_0 \left( \frac{|v|^2}{2} \right) + \left( -\frac{1}{2} \nabla_x n_0 + E_0 + u_0 \times B_0 \right) \cdot \bar{\Phi}(v), \\
\bar{g}_2^- &= \left( u_0 \otimes \bar{u}_1 + \bar{u}_1 \otimes u_0 \right) : \mathcal{A}(v) + \left( \theta_0 \bar{u}_1 + \bar{\theta}_1 u_0 \right) \cdot \mathcal{B}(v) + \theta_0 \bar{\theta}_1 \mathcal{C}(v), \\
\bar{g}_3^- &= \left( -\frac{1}{2} \nabla_x \bar{u}_1 : \mathcal{A}(v) - \frac{1}{2} \nabla_x \bar{\theta}_1 \cdot \mathcal{B}(v) + \sum \Gamma_0^+ \mathcal{Y}^+(v), \\
+\bar{n}_1 u_0 \cdot (\mathcal{L} + \mathcal{Q})^{-1} \mathcal{Q}(1, v) + \frac{1}{2} \bar{n}_1 \theta_0 (\mathcal{L} + \mathcal{Q})^{-1} \mathcal{Q}(1, |v|^2) + \sum \Gamma_0^+ \mathcal{Y}^-(v),
\end{align*}
\]

where the functions \( \bar{\rho}_1, \bar{u}_1, \bar{\theta}_1, \bar{n}_1, \bar{E}_1 \) and \( \bar{B}_1 \) are defined in (2.47). Following the previous proceedings, one easily knows that the functions \( \bar{g}_1^+ \) and \( \bar{g}_2^+ \) satisfy the equalities (2.7) and (2.19).

Then, we revise the ansatz (2.2) as

\[
\begin{align*}
g_{\varepsilon}^\pm &= g_{0}^\pm + \varepsilon g_{1}^\pm + \varepsilon^2 g_{2}^\pm + \varepsilon g_{R, \varepsilon}^\pm, \\
E_{\varepsilon} &= E_0 + \varepsilon \bar{E}_1 + \varepsilon \bar{E}_{R, \varepsilon}, \\
B_{\varepsilon} &= B_0 + \varepsilon \bar{B}_1 + \varepsilon B_{R, \varepsilon},
\end{align*}
\]

(2.49)

where

\[
\begin{align*}
g_{R, \varepsilon}^\pm &= g_{1}^\pm - \bar{g}_{1}^\pm + \varepsilon (g_{2}^\pm - \bar{g}_{2}^\pm) + \varepsilon^2 g_{3, \varepsilon}^\pm, \\
E_{R, \varepsilon} &= E_1 - \bar{E}_1 + \varepsilon E_{2, \varepsilon}, \\
B_{R, \varepsilon} &= B_1 - \bar{B}_1 + \varepsilon B_{2, \varepsilon}.
\end{align*}
\]
Plugging the ansatz (2.49) into the system (2.1) reduces to

\[
\begin{align*}
\varepsilon \partial_t \left( g_0^+ + \varepsilon g_1^+ + \varepsilon^2 g_2^+ + \varepsilon g_{R,\varepsilon}^+ \right) + v \cdot \nabla_v \left( g_0^+ + \varepsilon g_1^+ + \varepsilon^2 g_2^+ + \varepsilon g_{R,\varepsilon}^+ \right) + v \cdot \nabla_x \left( g_0^+ + \varepsilon g_1^+ + \varepsilon^2 g_2^+ + \varepsilon g_{R,\varepsilon}^+ \right) + \varepsilon (E_0 + \varepsilon \tilde{E}_1 + \varepsilon E_{R,\varepsilon}) + v \times (B_0 + \varepsilon \tilde{B}_1 + \varepsilon B_{R,\varepsilon}) + \\
- (E_0 + \varepsilon \tilde{E}_1 + \varepsilon E_{R,\varepsilon}) \cdot v \left( 1 + \varepsilon (g_0^+ + \varepsilon g_1^+ + \varepsilon^2 g_2^+ + \varepsilon g_{R,\varepsilon}^+) \right) - 1 - \varepsilon (g_0^- + \varepsilon g_1^- + \varepsilon^2 g_2^- + \varepsilon g_{R,\varepsilon}^-) & = - \frac{1}{\varepsilon} \left( L(g_0^+ + \varepsilon g_1^+ + \varepsilon^2 g_2^+ + \varepsilon g_{R,\varepsilon}^+) \right) \left( L(g_0^- + \varepsilon g_1^- + \varepsilon^2 g_2^- + \varepsilon g_{R,\varepsilon}^-) \right) \\
& - \frac{1}{\varepsilon} \left( Q(g_0^+ + \varepsilon g_1^+ + \varepsilon^2 g_2^+ + \varepsilon g_{R,\varepsilon}^+) + Q(g_0^- + \varepsilon g_1^- + \varepsilon^2 g_2^- + \varepsilon g_{R,\varepsilon}^-) \right) \\
& + \left( g_0^+ + \varepsilon g_1^+ + \varepsilon^2 g_2^+ + \varepsilon g_{R,\varepsilon}^+ \right) + \left( g_0^- + \varepsilon g_1^- + \varepsilon^2 g_2^- + \varepsilon g_{R,\varepsilon}^- \right) \\
& + (Q(g_0^+ + \varepsilon g_1^+ + \varepsilon^2 g_2^+ + \varepsilon g_{R,\varepsilon}^+), g_0^- + \varepsilon g_1^- + \varepsilon^2 g_2^- + \varepsilon g_{R,\varepsilon}^-) \\
& + (Q(g_0^+ + \varepsilon g_1^+ + \varepsilon^2 g_2^+ + \varepsilon g_{R,\varepsilon}^+), g_0^- + \varepsilon g_1^- + \varepsilon^2 g_2^- + \varepsilon g_{R,\varepsilon}^-) , \\
& \partial_t (E_0 + \varepsilon \tilde{E}_1 + \varepsilon E_{R,\varepsilon}) - \nabla_x \cdot (B_0 + \varepsilon \tilde{B}_1 + \varepsilon B_{R,\varepsilon}) \\
& = - \frac{1}{\varepsilon} \int_{\mathbb{R}^3} \left[ (g_0^+ + \varepsilon g_1^+ + \varepsilon^2 g_2^+ + \varepsilon g_{R,\varepsilon}^+) - (g_0^- + \varepsilon g_1^- + \varepsilon^2 g_2^- + \varepsilon g_{R,\varepsilon}^-) \right] v^2 d\mathbf{v}, \\
& \partial_t (B_0 + \varepsilon \tilde{B}_1 + \varepsilon B_{R,\varepsilon}) + \nabla_x \cdot (E_0 + \varepsilon \tilde{E}_1 + \varepsilon E_{R,\varepsilon}) = 0 , \\
& \nabla_x \cdot (E_0 + \varepsilon \tilde{E}_1 + \varepsilon E_{R,\varepsilon}) \\
& = \int_{\mathbb{R}^3} \left[ (g_0^+ + \varepsilon g_1^+ + \varepsilon^2 g_2^+ + \varepsilon g_{R,\varepsilon}^+) - (g_0^- + \varepsilon g_1^- + \varepsilon^2 g_2^- + \varepsilon g_{R,\varepsilon}^-) \right] E_{R,\varepsilon} d\mathbf{v} , \\
& \nabla_x \cdot (B_0 + \varepsilon \tilde{B}_1 + \varepsilon B_{R,\varepsilon}) = 0 .
\end{align*}
\]

Consequently, the relations (2.4), (2.7), (2.8), (2.19) and (2.20) (since the functions \( \tilde{g}_1^\pm \) and \( \tilde{g}_2^\pm \) obey the relations (2.7), (2.19) and (2.20)) imply the remainder equations

\[
\begin{align*}
\varepsilon \partial_t (g_{R,\varepsilon}^+ \cdot \nabla_x \left( g_{R,\varepsilon}^+ \right) + E_{R,\varepsilon} \cdot \nabla_v \left( g_{R,\varepsilon}^+ \right) + v \cdot \nabla_x \left( v \cdot \nabla_v \left( g_{R,\varepsilon}^+ \right) + E_{R,\varepsilon} \cdot v \left( g_{R,\varepsilon}^+ \right) \right) + v \cdot \nabla_x \left( v \cdot \nabla_v \left( g_{R,\varepsilon}^+ \right) + E_{R,\varepsilon} \cdot v \left( g_{R,\varepsilon}^+ \right) \right) \\
- E_{R,\varepsilon} \cdot v \left( g_{R,\varepsilon}^+ \right) - E_{R,\varepsilon} \cdot v \left( g_{R,\varepsilon}^+ \right) \\
= - \frac{1}{\varepsilon} \left( L g_{R,\varepsilon}^+ + L^2(g_{R,\varepsilon}^+) \right) + \frac{1}{\varepsilon} \left( Q(g_0^+, g_{R,\varepsilon}^+) + Q(g_{R,\varepsilon}, g_0^+) \right) + \frac{1}{\varepsilon} \left( Q(g_1^+, g_{R,\varepsilon}^+) + Q(g_{R,\varepsilon}, g_1^+) \right) + \frac{1}{\varepsilon} \left( Q(g_2^+, g_{R,\varepsilon}^+) + Q(g_{R,\varepsilon}, g_2^+) \right) + \frac{1}{\varepsilon} \left( Q(g_{R,\varepsilon}, g_{R,\varepsilon}) \right) \\
+ \varepsilon \left( Q(g_{R,\varepsilon}, g_{R,\varepsilon}) \right) + \varepsilon \left( Q(g_{R,\varepsilon}, g_{R,\varepsilon}) \right) + \varepsilon \left( Q(g_{R,\varepsilon}, g_{R,\varepsilon}) \right) \\
\partial_t (E_{R,\varepsilon}) + \nabla_x \cdot B_{R,\varepsilon} = - \frac{1}{\varepsilon} \int_{\mathbb{R}^3} (g_{R,\varepsilon}^+ - g_{R,\varepsilon}^-) v^2 d\mathbf{v} , \\
\partial_t B_{R,\varepsilon} + \nabla_x \cdot E_{R,\varepsilon} = 0 , \nabla_x E_{R,\varepsilon} = \int_{\mathbb{R}^3} (g_{R,\varepsilon}^+ - g_{R,\varepsilon}^-) v^2 d\mathbf{v} , \nabla_x B_{R,\varepsilon} = 0 , \\
\end{align*}
\]

(2.51)
where the known terms $R^\pm$ are
\[
\begin{aligned}
\left(\begin{array}{c}
R^+
\end{array}\right) &= -\partial_t \left( \frac{g_1^+ + \varepsilon g_2^+}{g_1 + \varepsilon g_2} \right) - v \cdot \nabla_x \left( \frac{g_2^+}{g_2} \right) - E_1 \cdot \nabla_v \left( \frac{g_0^+ + \varepsilon g_1^+ + \varepsilon^2 g_2^+}{g_0 + \varepsilon g_1 + \varepsilon^2 g_2} \right) \\
&\quad - E_0 \cdot \nabla_v \left( \frac{g_1^+ + \varepsilon g_2^+}{g_1 + \varepsilon g_2} \right) - \langle \nabla \cdot B_0 \rangle \cdot \nabla_v \left( \frac{g_2^+}{g_2} \right) \\
&\quad + \langle \nabla \cdot B_1 \rangle \cdot \nabla_v \left( \frac{g_1^+ + \varepsilon g_2^+}{g_1 + \varepsilon g_2} \right) + E_1 \cdot v \left( \frac{g_0^+ + \varepsilon g_1^+ + \varepsilon^2 g_2^+}{g_0 + \varepsilon g_1 + \varepsilon^2 g_2} \right) \\
&\quad + E_0 \cdot v \left( \frac{g_1^+ + \varepsilon g_2^+}{g_1 + \varepsilon g_2} \right) + \left( Q(g_0^+, g_2^+, g_2^-) + Q(g_2^+, g_0^+, g_0^-) \right) \\
&\quad + \left( Q(g_1^+, g_2^+, g_1^- + \varepsilon g_2^-) + Q(g_1^+, g_2^+, g_1^- + \varepsilon g_2^-) \right).
\end{aligned}
\]

Consequently, the anisotropic system (2.51) can be rewritten as the brief system (1.16).

3. Basic estimates

We now study the linearized collision operator $L$. We can split
\[
L = 2\nu(v)I - \mathbb{K},
\]
where the integral operator $\mathbb{K}$ can be decomposed as $\mathbb{K} = \mathbb{K}_1 - \mathbb{K}_2$. More precisely, $\mathbb{K}_i$ $(i = 1, 2)$ are defined as
\[
\begin{aligned}
\mathbb{K}_1 \left( \frac{g}{h} \right) (v) &= \int_{\mathbb{R}^3} \left[ \frac{g}{h} \right] (v') \left( I + \hat{\mathcal{T}} \right) \left( \frac{g}{h} \right) (v') |v - v_*|^\gamma M(v_*) dv_* , \\
\mathbb{K}_2 \left( \frac{g}{h} \right) (v) &= \int_{\mathbb{R}^3} \left( I + \hat{\mathcal{T}} \right) \left( \frac{g}{h} \right) (v_*)(v - v_*) |v - v_*|^\gamma M(v_*) dv_* ,
\end{aligned}
\]
where $I$ is the $2 \times 2$ unit matrix $\text{diag}(1, 1)$, and $\hat{\mathcal{T}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. By following the analogous arguments in Chapter 3 of [13], we rewrite the above relations as
\[
\mathbb{K}_1 \left( \frac{g}{h} \right) (v) = \int_{\mathbb{R}^3} K_i(v, v_*) \left( \frac{g}{h} \right) (v_*) dv_*
\]
for $i = 1, 2$, where the integral kernels $K_1(v, v_*)$ and $K_2(v, v_*)$ are
\[
K_1(v, v_*) = \frac{3I + \hat{\mathcal{T}}}{|v - v_*|^2} \exp \left\{ - \frac{1}{4} |v_*|^2 + \frac{1}{4} |v|^2 - \frac{1}{8} |v - v_*|^2 - \frac{(v - v_*)^2}{8 |v - v_*|^2} \right\}
\times \int_{\eta \perp (v_* - v)} (|v - v_*|^2 + |\eta|^2)^2 M(\eta + \zeta) d\eta
\]
for $\zeta = \frac{1}{2} \left( (v + v_*) \cdot \frac{v - v_*}{|v - v_*|} \right) \frac{v - v_*}{|v - v_*|}$, and
\[
K_2(v, v_*) = (I + \hat{\mathcal{T}})|v - v_*|^\gamma M(v_*).
\]

Lemma 3.1. For any $G, H \in L^2_\nu \cap L^2_\nu(v)$, we have
\[
\langle L^{} G \cdot H \rangle = \langle G \cdot L^{} H \rangle
\]
and $L^{} G = 0$ if and only if $G = \mathbb{P}^{} G$. Moreover, there is $\Lambda > 0$ such that
\[
\langle L^{} G \cdot G \rangle \geq \Lambda \| \mathbb{P}^{} G \|^2_{L^2_\nu(v)}.
\]

Proof. The details of the proof can be referred in [3] and we omit it here. □
Lemma 3.2. The collision frequency $\nu(v)$ defined in (1.21) is smooth and there are positive constants $C_0$ and $C$ such that

$$C_0(1 + |v|)^\gamma \leq \nu(v) \leq C(1 + |v|)^\gamma \quad (3.6)$$

for every $v \in \mathbb{R}^3$. Moreover, for any multi-index $\beta \neq 0$,

$$\sup_{v \in \mathbb{R}^3} |\partial_\beta \nu(v)| < \infty. \quad (3.7)$$

Furthermore, if the velocities $v, v_*, v', v'_* \in \mathbb{R}^3$ satisfy

$$v + v_* = v' + v'_*, \quad |v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2, \quad (3.8)$$

then

$$\nu(v) + \nu(v_*) \leq C(\nu(v') + \nu(v'_*)) \quad (3.9)$$

holds for some generic positive constant $C > 0$.

Proof. We first prove the inequality (3.6). From the elementary bounds

$$|v - v_*|^2 \leq (1 + |v_*|^2)(1 + |v|^2) \leq (1 + |v_*|)^2(1 + |v|)^2 \quad (3.10)$$

for every $v \in \mathbb{R}^3$ and the fact $M(v_*) = \frac{1}{\sqrt{2\pi}} e^{-\frac{|v_*|^2}{2}} > 0$, we directly obtain the upper bound

$$\nu(v) = \int_{\mathbb{R}^3} |v - v_*|^\gamma M(v_*)dv_* \leq \int_{\mathbb{R}^3} (1 + |v_*|)^\gamma M(v_*)dv_* (1 + |v|)^\gamma,$$

which yields the upper bound of (3.6).

Next, the bound (3.10) and the fact $M(v_*) > 0$ imply that

$$(1 + |v|)^{-\gamma}|v - v_*|^\gamma M(v_*) \leq (1 + |v_*|)^\gamma M(v_*).$$

Then the Lebesgue Dominated Convergence Theorem implies that the positive function

$$v \mapsto \frac{\nu(v)}{(1 + |v|)^\gamma} = \frac{1}{(1 + |v|)^\gamma} \int_{\mathbb{R}^3} |v - v_*|^\gamma M(v_*)dv_* \quad (3.11)$$

is continuous over $\mathbb{R}^3$ and satisfies

$$\lim_{|v| \to \infty} \frac{\nu(v)}{(1 + |v|)^\gamma} = \lim_{|v| \to \infty} \frac{1}{(1 + |v|)^\gamma} \int_{\mathbb{R}^3} |v - v_*|^\gamma M(v_*)dv_*$$

$$= \int_{\mathbb{R}^3} M(v_*)dv_* = 1 > 0.$$

The function (3.11) is thereby bounded away from zero, thereby the lower bound of (3.6) follows.

We next derive the bound (3.7). One notices that

$$\nabla_v \nu(v) = \gamma \int_{\mathbb{R}^3} \frac{v - v_*}{|v - v_*|^\gamma} M(v_*)dv_*$$

$$= \gamma \int_{\mathbb{R}^3} \frac{u}{|u|^\gamma} M(v - u)du,$$

where the variables change $v_* \to u = v - v_*$ is utilized. Then for any $\beta' = [\beta'_1, \beta'_2, \beta'_3]$, $\gamma \beta'_\alpha M(v - u)du$

$$|\partial_{\beta'} \nabla_v \nu(v)| \leq \gamma \int_{\mathbb{R}^3} |u|^{-\gamma} |\partial_{\beta'} M(v - u)|du$$

$$\leq \gamma \int_{|u|^{-\gamma}} |\partial_{\beta'} M(v - u)|du. \quad (3.12)$$

By direct calculations, we know that there is a constant $C_{\beta'} > 0$ such that

$$|\partial_{\beta'} M(v - u)| \leq C_{\beta'} e^{-|u|^2/2},$$
which implies that by (3.12)
\[ |\partial_{\beta'} \nabla_w \nu(v)| \leq C_{\beta'} \gamma \int_{\mathbb{R}^3} \frac{1}{|u|^\gamma} e^{-\frac{|v-u|^2}{4}} \, du = \gamma C_{\beta'} \int_{\mathbb{R}^3} \frac{1}{(v-v_*)^\gamma} e^{-\frac{|v_*|^2}{4}} \, dv_* \]
\[ = \gamma C_{\beta'} \left\{ \int_{|v-v_*| \geq 1} + \int_{|v-v_*| < 1} \right\} \frac{1}{|v-v_*|^\gamma} e^{-\frac{|v_*|^2}{4}} \, dv_* \]
\[ \leq \gamma C_{\beta'} \int_{|v-v_*| \geq 1} e^{-\frac{|v_*|^2}{4}} \, dv_* + \gamma C_{\beta'} \int_0^1 \int_{\mathbb{S}^2} \frac{1}{r^{1+\gamma}} r^2 \omega \, dr \]
\[ \leq \gamma C_{\beta'} \int_{\mathbb{R}^3} e^{-\frac{|v_*|^2}{4}} \, dv_* + \frac{4\pi}{2\pi \gamma} C_{\beta'} < \infty. \]

Then the bound (3.7) holds.

Finally, we verify the inequality (3.9). By the elementary inequality
\[ a^2 + b^2 \leq 2(a + b)^2 \leq 4(a^2 + b^2) \]
for \(a, b \geq 0\), we derive from (3.8) that
\[ |v| + |v_*| \leq C(|v'| + |v_*'|). \tag{3.13} \]

From the inequality (3.6), we have
\[ \nu(v) + \nu(v_*) \leq \overline{C}\left[(1 + |v|)^\gamma + (1 + |v_*|)^\gamma\right] \]
\[ \leq 2\overline{C}\left[(1 + |v|) + (1 + |v_*|)\right]^\gamma \]
\[ \leq C(\gamma)\left[(1 + |v'|) + (1 + |v_*'|)\right]^\gamma \]
\[ \leq C(\gamma)\left[(1 + |v'|)^\gamma + (1 + |v_*'|)^\gamma\right] \]
\[ \leq \frac{C(\gamma)}{C}(\nu(v') + \nu(v_*')), \]
where the elementary inequality
\[(a + b)^\gamma \leq a^\gamma + b^\gamma \ (a, b \geq 0, \gamma \in [0, 1]) \]
is used. Then the proof of Lemma 3.2 is finished. \qed

We now summarize some mixed derivative estimates for the collision operators \(L\) and \(Q\).

**Lemma 3.3.** For the hard potential with \(\gamma \in [0, 1]\), there exist \(C|\beta|, C > 0\), such that
\[ \left\langle w^{2l} \partial_{\beta}^m L G \cdot \partial_{\beta}^m G \right\rangle \geq \frac{1}{2} \left\| w^{l} \partial_{\beta}^m G \right\|^2_{L^2(\nu)} - C|\beta| \left\| \partial_{\beta}^m G \right\|^2_{L^2(\nu)}, \tag{3.14} \]
\[ \left\langle w^{2l} \partial_{\beta}^m Q(g_1, g_2) \cdot \partial_{\beta}^m g_3 \right\rangle \leq C\left( \left\| w^{l} \partial_{\beta_1}^{m_1} g_1 \right\|_{L^2(\nu)} \left\| w^{l} \partial_{\beta_2}^{m_2} g_2 \right\|_{L^2} + \left\| w^{l} \partial_{\beta_1}^{m_1} g_1 \right\|_{L^2(\nu)} \left\| w^{l} \partial_{\beta_2}^{m_2} g_2 \right\|_{L^2(\nu)} \right) \left\| w^{l} \partial_{\beta}^m g_3 \right\|_{L^2(\nu)}, \tag{3.15} \]
and
\[ \left\| \partial_{\beta}^m Q(g_1, g_2) \right\|_{L^2} \leq C\left( \left\| \partial_{\beta_1}^{m_1} g_1 \right\|_{L^2} \left\| w^{2l} \partial_{\beta_2}^{m_2} g_2 \right\|_{L^2} + \left\| w^{2l} \partial_{\beta_1}^{m_1} g_1 \right\|_{L^2} \left\| \partial_{\beta_2}^{m_2} g_2 \right\|_{L^2} \right), \tag{3.16} \]
where \(l \geq 0\) and the summation is for \(|m| + |\beta| \leq N\) with \(\beta_1 + \beta_2 \leq \beta\) and \(m_1 + m_2 \leq m\) componentwise.
Proof. The first linear estimate (3.14) can be justified by the similar arguments in Lemma 3.3 of [17] and we omit details here. We now prove the last two nonlinear estimates (3.15) and (3.16). Without loss of generality, we only need to prove the case $m = 0$. Notice that

\[
\partial_\beta \mathcal{Q}(g_1, g_2) = \partial_\beta \int_{\mathbb{R}^3} (g_1(v')g_2(v'_* \cdot v) - g_1(v)g_2(v_*))|v - v_*|^\gamma M(v_*)dv_*
\]

\[
= \partial_\beta \int_{\mathbb{R}^3} g_1(v')g_2(v'_*)|u|^\gamma M(v - u)du - \partial_\beta \int_{\mathbb{R}^3} g_1(v)g_2(v_*)|u|^\gamma M(v - u)du
\]

\[
= \sum_{\beta_0 + \beta_1 + \beta_2 = \beta} \int_{\mathbb{R}^3} \partial_{\beta_1} g_1(v') \partial_{\beta_2} g_2(v'_*)|u|^\gamma \partial_{\beta_0} M(v - u)du
\]

\[
- \sum_{\beta_0 + \beta_1 + \beta_2 = \beta} \int_{\mathbb{R}^3} \partial_{\beta_1} g_1(v) \partial_{\beta_2} g_2(v_*)|u|^\gamma \partial_{\beta_0} M(v - u)du
\]

\[
:= I_{gain} + I_{loss}.
\]

Since for any $\eta \in (0, 1)$

\[
\partial_{\beta_0} M(v - u) \leq C M^n(v - u),
\]

the second term $I_{loss}$ in (3.17) is bounded by

\[
|I_{loss}| \leq C|\partial_{\beta_1} g_1(v)| \int_{\mathbb{R}^3} |u|^\gamma M^n(v - u)|\partial_{\beta_2} g_2(v - u)|du
\]

\[
\leq C|\partial_{\beta_1} g_1(v)| \left( \int_{\mathbb{R}^3} |u|^{2\gamma} M^{2n-1}(v - u)du \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} |\partial_{\beta_2} g_2(v - u)|^2 M(v - u)du \right)^{\frac{1}{2}}.
\]

Following the proof of the inequality (3.6) in Lemma 3.2, we deduce that for $\eta = \frac{3}{4}$ in (3.18)

\[
\left( \int_{\mathbb{R}^3} |u|^{2\gamma} M^{2n-1}(v - u)du \right)^{\frac{1}{2}} \leq C \nu(v).
\]

Then the inequalities (3.18) and (3.19) imply that

\[
|I_{loss}| \leq C|\partial_{\beta_1} g_1(v)||\nu(v)| \left( \int_{\mathbb{R}^3} |\partial_{\beta_2} g_2(v)|^2 Mdv \right)^{\frac{1}{2}}.
\]

By further multiplying with $w^2 \partial_{\beta_3} M$ in (3.20) and integrating over $\mathbb{R}^3$, we gain

\[
\langle I_{loss} \cdot w^2 \partial_{\beta_3} g_3 \rangle \leq C \|w^l \partial_{\beta_1} g_1\|_{L^2_v(v)} \|w^l \partial_{\beta_2} g_2\|_{L^2_v(v)} \|w^l \partial_{\beta_3} g_3\|_{L^2_v(v)}.
\]

For the first term $I_{gain}$ in (3.17), noticing the inequality (3.9) in Lemma 3.2, we estimate that by (3.19)

\[
|I_{gain}| \leq C \left( \int_{\mathbb{R}^3} |u|^{2\gamma} M^{\frac{1}{2}}(v - u)du \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} |\partial_{\beta_1} g_1(v')|^2 |\partial_{\beta_2} g_2(v'_*)|^2 M(v_*)dv_* \right)^{\frac{1}{2}}
\]

\[
\leq C \nu \left( v \right) \left( \int_{\mathbb{R}^3} [\nu \frac{1}{2}(v') + \nu \frac{1}{2}(v'_*)]|\partial_{\beta_1} g_1(v')|^2 |\partial_{\beta_2} g_2(v'_*)|^2 M(v_*)dv_* \right)^{\frac{1}{2}}.
\]

One easily derives from the inequality (3.6) that

\[
C_1 \nu(v) \leq w^\gamma(v) \leq C_2 \nu(v)
\]

for some constants $C_1, C_2 > 0$, where the elementary inequality

\[
a^2 + b^2 \leq 2(a + b)^2 \leq 4(a^2 + b^2)
\]
is utilized. Multiplying by $w^2 \partial_3 g_3 M$ in (3.22), integrating over $v \in \mathbb{R}^3$ and utilizing the relation (3.23), we have

$$\int_{\mathbb{R}^3} I_{\text{gain}} w^2 \partial_3 g_3(v) M(v) dv \leq C \int_{\mathbb{R}^3} \nu^{1/2} \partial_3 g_3(v) \left[ \int_{\mathbb{R}^3} \nu^{1/2}(v') + \nu^{1/2}(v'_*) \right] \times \left| \partial_3 g_1(v') \right|^2 ||\partial_3 g_2(v'_*)||^2 M(v_*') dv_*' \frac{1}{2} M^{1/2}(v') dv'$$

$$\leq C \left( \int_{\mathbb{R}^3} w^2(v') ||\partial_3 g_3(v')||^2 \nu(v) M(v) dv \right)^{1/2}$$

$$\times \left( \int_{\mathbb{R}^3} \left[ |v'(v)| \nu^{1/2}(v') + |v'_*(v)| \nu^{1/2}(v'_*) \right] \left| \partial_3 g_1(v') \right|^2 ||\partial_3 g_2(v'_*)||^2 M(v') M(v'_*) dv' dv'_* \right)^{1/2}$$

$$\leq C \left( \left( \int_{\mathbb{R}^3} |v'| \nu^{1/2}(v') + |v'_*(v)| \nu^{1/2}(v'_*) \right) \left| \partial_3 g_1(v') \right|^2 ||\partial_3 g_2(v'_*)||^2 M(v') M(v'_*) dv' dv'_* \right)^{1/2}$$

where the second inequality is implied by the facts $dv_*' dv' = dv' dv'_*$ and $M(v) M(v'_*) = M(v') M(v'_*)$. Consequently, combining the relations (3.17), (3.21) and (3.23) concludes the inequality (3.15).

It remains to prove the inequality (3.16). Together with (3.20) and (3.23), we deduce that

$$\int_{\mathbb{R}^3} |I_{\text{loss}}|^2 M(v) dv \leq C \left( \int_{\mathbb{R}^3} \left| \partial_3 g_2 \right|^2 ||\partial_3 g_1(v')||^2 \nu^2(v) M dv \right)^{1/2}$$

$$\leq C \left( \int_{\mathbb{R}^3} \left| \partial_3 g_2 \right|^2 \left( ||\partial_3 g_1||^2 + \||w^2 \partial_3 g_2||^2 \right) \right)^{1/2}.$$  

By making use of the relation (3.6) in Lemma 3.2, we derive from (3.22) and (3.23) that

$$\int_{\mathbb{R}^3} |I_{\text{gain}}|^2 M(v) dv \leq C \left( \int_{\mathbb{R}^3} \left| \partial_3 g_2 \right|^2 \left( ||\partial_3 g_1||^2 + \||w^2 \partial_3 g_2||^2 \right) \right)^{1/2}.$$

Then, the inequalities (3.25) and (3.26) imply the inequality (3.16). Consequently, the proof of Lemma 3.3 is completed. \hfill \Box

4. Energy bounds for NSFM system and linear Maxwell-type system

In this section, we will derive the energy estimates for the Navier-Stokes-Fourier-Maxwell (briefly, NSFM) system (2.30) and the linear Maxwell-type equations (2.46)-(2.47), which was employed in the process of deriving the uniform global-in-time estimates to the remainder system (2.51).

Firstly, we consider the NSFM system (2.30) and assume that the initial data $(u_0, E_0, B_0, \theta_0)|_{t=0} = (u_0^m, E_0^m, B_0^m, \theta_0^m)$ satisfies the compatibility condition $\text{div}_x u_0^m = \text{div}_x B_0^m = 0$. For any fixed integer $s \geq 2$, introduce respectively the energy and energy dissipation functionals $E_{0,s}(t)$ and $D_{0,s}(t)$ as follows,

$$E_{0,s}(t) = \left( u_0 \right)^2 H_x + \left( \theta_0 \right)^2 H_\theta + \frac{\kappa}{2} \left( E_0 \right)^2 H_x + \frac{\delta}{2} \left( \theta_0 \right)^2 H_\theta + \left( \frac{\gamma}{2} - \delta + \delta \sigma \right) \left( B_0 \right)^2 H_x$$

$$+ (1 - \delta) \left( \theta_0 B_0 \right)^2 H_x + \frac{\kappa}{2} \left( \theta_0 B_0 \right)^2 \sigma \text{div}_x B_0 + \frac{\kappa}{2} \left( \theta_0 \right)^2 H_\theta + \frac{\delta}{2} \left( \theta_0 \right)^2 \sigma \text{div}_x \theta_0 + \left( \frac{\gamma}{2} - \delta \right) \left( \theta_0 B_0 \right)^2 H_x$$

$$+ \delta \left( \text{div}_x \theta_0 \right)^2 H_x + \frac{\kappa}{2} \left( \text{div}_x \theta_0 \right)^2 \sigma \text{div}_x \theta_0 + \left( \frac{\gamma}{2} - \delta \right) \left( \theta_0 B_0 \right)^2 H_x$$

$$+ \frac{\delta}{2} \left( \theta_0 \right)^2 \sigma \text{div}_x \theta_0 + \left( \frac{\gamma}{2} - \delta \right) \left( \theta_0 B_0 \right)^2 H_x$$

(4.1)

$$D_{0,s}(t) = \mu \left( \text{div}_x u_0 \right)^2 H_x + \frac{\kappa}{2} \left( \theta_0 \right)^2 H_\theta + \sigma \left( \text{div}_x \theta_0 \right)^2 H_x + \frac{\sigma}{2} \left( \text{div}_x \theta_0 \right)^2 H_\theta + \frac{\delta}{2} \left( \text{div}_x \theta_0 \right)^2 \sigma \text{div}_x \theta_0$$

$$+ \delta \left( \theta_0 B_0 \right)^2 H_x + \frac{\kappa}{2} \left( \theta_0 B_0 \right)^2 \sigma \text{div}_x \theta_0 + \left( \frac{\gamma}{2} - \delta \right) \left( \theta_0 B_0 \right)^2 H_x$$

(4.2)
Moreover, the initial energy is denoted by
\[ E_{0,s}^{in} = \|u_0^{in}\|^2_{H^s_x} + \|\theta_0^{in}\|^2_{H^s_x} + \|E_0^{in}\|^2_{H^s_x} + \|\nabla_x E_0^{in}\|^2_{H^s_x} + \|\nabla \times E_0^{in}\|^2_{H^s_x} + \|B_0^{in}\|^2_{H^s_x}. \] (4.3)
It is an obvious fact that \( E_{0,s}(0) \leq CE_{0,s}^{in} \) for some positive constant \( C = C(\mu, \kappa, \sigma) > 0 \).

Besides, we point out that for \( 2 \leq s_1 \leq s_2 \), there hold the relations \( E_{0,s_1} \leq E_{0,s_2}, \mathcal{D}_{0,s_1} \leq \mathcal{D}_{0,s_2} \), and \( E_{0,s_1}^{in} \leq E_{0,s_2}^{in} \).

Now we employ the following lemma, which gives the uniform energy estimates for the NSFM system (2.30).

**Lemma 4.1.** Assume that the initial data \((u_0^{in}, \theta_0^{in}, E_0^{in}, B_0^{in}) \in H^s_x \times H^s_x \times H^{s+1}_x \times H^{s+1}_x \) with \( s \geq 2 \). If there is some small positive constant \( \lambda_0(s) = \lambda(s, \mu, \kappa, \sigma) \) depending only on \( s \), the viscous coefficient \( \mu \), the heat conductivity coefficient \( \kappa \), and the electric resistivity \( \sigma \), such that the initial energy satisfies \( E_{0,s}^{in} \leq \lambda_0(s) \). Then the NSMF system (2.30) admits a unique global-in-time solution \((u_1^{in}, \theta_1^{in}, E_1^{in}, B_1^{in})\) satisfying
\[ u_0, \theta_0 \in L^\infty(\mathbb{R}^+, H^s_x) \cap L^2(\mathbb{R}^+, H^{s+1}_x) \] (4.4)
\[ E_0 \in L^\infty(\mathbb{R}^+, H^s_x), \quad n_0(= \text{div}_x E_0) \in L^\infty(\mathbb{R}^+, H^s_x) \cap L^2(\mathbb{R}^+, H^{s+1}_x) \]
\[ B_0 \in L^\infty(\mathbb{R}^+, H^{s+1}_x), \quad \partial_t B_0 (= -\nabla_x \times B_0) \in L^\infty(\mathbb{R}^+, H^s_x). \]

Moreover, for any \( t \geq 0 \), there holds the following energy inequality
\[ \frac{d}{dt} E_{0,s}(t) + \mathcal{D}_{0,s}(t) \leq 0, \] (4.5)
and consequently, there exists some constant \( C = C(\mu, \kappa, \sigma) > 0 \), such that
\[ \sup_{t \geq 0} (\|u_0\|^2_{H^s_x} + \|\theta_0\|^2_{H^s_x} + \|E_0\|^2_{H^s_x} + \|n_0\|^2_{H^s_x} + \|B_0\|^2_{H^s_x} + \|\partial_t B_0\|^2_{H^s_x}) \] (4.6)
\[ + \int_0^\infty (\mu \|\nabla u_0\|^2_{H^s_x} + \kappa \|\nabla \theta_0\|^2_{H^s_x} + \sigma \|\nabla n_0\|^2_{H^s_x}) \, dt \leq C E_{0,s}^{in}. \]

**Proof.** In [23], the first two authors of the current paper have proved, under the assumptions of small initial data, that the global well-posedness of the following Navier-Stokes-Maxwell (NSM) system
\[
\begin{align*}
\partial_t u_0 + u_0 \cdot \nabla_x u_0 + \nabla_x P_0 &= \mu \Delta_x u_0 + \frac{1}{2}(n_0 E_0 + j_0 \times B_0), \\
\partial_t E_0 - \nabla_x \times B_0 &= -j_0, \\
\partial_t B_0 + \nabla_x \times E_0 &= 0, \\
j_0 - n_0 E_0 &= \sigma(-\frac{1}{2} \nabla_x n_0 + E_0 + u_0 \times B_0), \\
\text{div}_x E_0 &= n_0, \quad \text{div}_x B_0 = 0.
\end{align*}
\] (4.7)

We observe that the only difference between NSM and NSMF system is the equation of Fourier-law,
\[ \partial_t \theta_0 + u_0 \nabla_x \theta_0 = \kappa \Delta_x \theta_0, \]
which is a linear parabolic equation when the velocity field \( u_0 \) are known. As a consequence, by employing the arguments in [23], we can finish the proof of this lemma. The details are thus omitted here for simplicity. \( \square \)

Next, we turn to consider the linear Maxwell-type system (2.46)-(2.47) with the initial data \( (\bar{E}_1, \bar{B}_1)|_{t=0} = (E_1^{in}, B_1^{in}) \) satisfying the compatibility conditions \( \text{div}_x B_1^{in} = 0 \) and the zero mean property, i.e. \( \int_{\mathbb{T}^3} B_1^{in} \, dx = 0 \). Note that the zero mean property will be satisfied at any time under the same initial property. Indeed, we can check that by an integration by parts, the second equation of (2.47) can be written as
\[ \frac{d}{dt} \int_{\mathbb{T}^3} B_1 \, dx = 0. \]
Let the initial data 

\[ E(t) \geq C_1, \quad M_0, \text{ for any integer } M \geq 1, \]

that

\[ E_{1, M}(t) = \| E_1 \|^2_{H_{x}^{M+1}} + \| \tilde{n}_1 \|^2_{H_{x}^{M+1}} + (1 - \delta + \delta \sigma)\| \tilde{B}_1 \|^2_{H_{x}^{M}} + \| \nabla_x \tilde{B}_1 \|^2_{H_{x}^{M}}, \]

and

\[ D_{1, M}(t) = \frac{\sigma}{2} \| E_1 \|^2_{H_{x}^{M+1}} + \frac{3}{4} \| \tilde{n}_1 \|^2_{H_{x}^{M}} + \frac{1}{4} \sigma \| \nabla_x \tilde{n}_1 \|^2_{H_{x}^{M}} + \frac{\sigma}{2} \| \nabla_x \tilde{B}_1 \|^2_{H_{x}^{M}}, \]

with the constant \( \delta = \frac{1}{2} \min\{1, \sigma\} \in (0, \frac{1}{2}) \) ensuring that the all coefficients of the above two energy functionals are positive. We also define the initial energy as

\[ \mathcal{E}_{1, M}^{in} = \| E_1^{in} \|^2_{H_{x}^{M+1}} + \| \text{div} E_1^{in} \|^2_{H_{x}^{M+1}} + \| \nabla_x \times E_1^{in} \|^2_{H_{x}^{M}} + \| \tilde{B}_1^{in} \|^2_{H_{x}^{M}}. \]

It is easy to see that \( \mathcal{E}_{1, M}(0) \leq C \mathcal{E}_{1, M}^{in} \).

We now state the following lemma.

**Lemma 4.2.** Let the initial data \((E_1^{in}, \tilde{B}_1^{in}) \in H_{x}^{M+1} \cap H_{x}^{M+1} \) with \( M \geq 1 \). Assume that there exists some small constant \( \lambda_1(M+2) = \lambda_1(M, \mu, \kappa, \sigma) \in (0, \lambda_0(M+2)) \), such that \( \mathcal{E}_0^{in} \leq \lambda_1(M+2) \). Then smooth solution \((\rho_1, \tilde{u}_1, \tilde{\theta}_1, E_1, \tilde{B}_1)\) to the linear Maxwell-type equation (2.46)-(2.47) obey the following bounds

\[ \| \tilde{u}_1 \|^2_{H_{x}^{M+1}}(t) \leq C(1 + \mathcal{E}_{0, M+1}(t)) \mathcal{D}_{0, M+1}(t), \]

and

\[ \frac{d}{dt}[\mathcal{E}_{1, M}(t) + \tilde{C}_M \mathcal{E}_{0, M+2}(t)] + [D_{1, M}(t) + D_{0, M+2}(t)] \leq 0, \]

with the constants \( C = C(\mu, \kappa, \sigma) > 0 \) and \( \tilde{C}_M = \tilde{C}_M(\mu, \kappa, \sigma) > 1 \). Moreover, for any \( t \geq 0 \), the following energy bounds holds:

\[ \| \tilde{B}_1 \|^2_{H_{x}^{M}}(t) + \| \text{div} E_1 \|^2_{H_{x}^{M}}(t) + \| \nabla_x \times E_1 \|^2_{H_{x}^{M}}(t) + \| \tilde{B}_1 \|^2_{H_{x}^{M}}(t) \leq C(\mathcal{E}_{0, M+2}(t) + \mathcal{E}_{1, M}(t)). \]

**Proof of Lemma 4.2.**

**Step 1:** We first prove the bounds of \( \tilde{u}_1 \) (4.12). Recall the special chosen relations (2.46) that \( \tilde{u}_1(t, x) = \nabla_x \phi(t, x) \), where the potential function \( \phi(t, x) \) satisfies the Poisson equation \( \Delta_x \phi = \partial_t \theta_0 \) for any \( t \geq 0 \), then the standard elliptic theory implies that

\[ \| \phi(t, \cdot) \|_{H_{x}^{M+2}} \leq C(\| \partial_t \theta_0(t, \cdot) \|_{H_{x}^{M}}), \]

provided that \( \partial_t \theta_0(t, \cdot) \in H_{x}^{M} \). Hence, it follows from the third equation of the NSMF system (2.30) that, for \( M \geq 1 \),

\[ \| \tilde{u}_1 \|^2_{H_{x}^{M+1}} = \| \nabla_x \phi \|^2_{H_{x}^{M+1}}(t) \leq C(\| \partial_t \theta_0(t, \cdot) \|_{H_{x}^{M}}) = C(\| \kappa \Delta_x \theta_0 - u_0 \cdot \nabla_x \theta_0 \|_{H_{x}^{M}} \leq C(\| \nabla_x \theta_0 \|_{H_{x}^{M+1}} + \| u_0 \|_{H_{x}^{M+1}} \| \nabla_x \theta_0 \|_{H_{x}^{M}}). \]

Consequently the definitions of \( \mathcal{E}_{0, M+1} \) and \( \mathcal{D}_{0, M+1} \) enable us to get the inequality (4.12).

**Step 2:** We next derive the inequality (4.13). By applying the elliptic theory again to the third equation in the chosen relations (2.46), and combining with the last third equation of the NSMF system (2.30), we have

\[ \| \tilde{\theta}_1 \|^2_{H_{x}^{M+1}} \leq C(\frac{1}{6} \Delta_x |u_0|^2 - \frac{1}{2} \text{div}_x (u_0 \cdot \nabla_x u_0 - \frac{1}{2} j_0 \times B_0) \|_{H_{x}^{M-1}}. \]
\[\leq C\|u_0\|_{H^{M+1}_x}\|\nabla_x u_0\|_{H^{M+1}_x} + C\|\text{div}_x(j_0 \times B_0)\|_{H^{M+1}_x}\]
\[\leq C\|u_0\|_{H^{M+1}_x}\|\nabla_x u_0\|_{H^{M+1}_x} + C\|n_0\|_{H^{M+1}_x}\|u_0\|_{H^{M+1}_x}\|B_0\|_{H^{M+1}_x}\]
\[+ C\|\nabla_x n_0\|_{H^{M+1}_x}\|B_0\|_{H^{M+1}_x} + C\|E_0\|_{H^{M+1}_x}\|B_0\|_{H^{M+1}_x} + C\|\nabla_x u_0\|_{H^{M+1}_x}\|B_0\|_{H^{M+1}_x}^2\]
\[\leq C\left(\|\nabla_x u_0\|_{H^{M+1}_x} + \|\nabla_x n_0\|_{H^{M+1}_x} + \|E_0\|_{H^{M+1}_x}\right)\left(\||u_0\|_{H^{M+1}_x} + \|B_0\|_{H^{M+1}_x}\right)\]
\[+ C\left(\|\nabla_x u_0\|_{H^{M+1}_x} + \|n_0\|_{H^{M+1}_x}\right)\left(\||u_0\|_{H^{M+1}_x}^2 + \|B_0\|_{H^{M+1}_x}^2\right)\]

Thus, the inequality (4.13) follows by recalling again the definition of $\mathcal{E}_{0,M+1}$ and $\mathcal{D}_{0,M+1}$.

**Step 3:** It remains to derive the inequalities (4.14)-(4.15). The key point is to seek some essential decay structures, which will derive the global-in-time energy estimates (4.14)-(4.15).

Firstly, we observe that for the equation of $\tilde{E}_1$ in (2.47), the term $\sigma \tilde{E}_1$ in $j_1$ is the decay term. Under the relation $\text{div}_x \tilde{E}_1 = \tilde{n}_1$, the term $-\frac{1}{2}\sigma \nabla_x \tilde{n}_1$ in $j_1$ also supplies decay effect while carrying on the energy estimates. Hence, we get

\[
\begin{aligned}
\partial_t \tilde{E}_1 - \nabla_x \times \tilde{B}_1 + \sigma \tilde{E}_1 - \frac{1}{2}\sigma \nabla_x \tilde{n}_1 &= -\tilde{j}_1, \\
\partial_t \tilde{B}_1 + \nabla_x \times \tilde{E}_1 &= 0, \\
\text{div}_x \tilde{E}_1 &= \tilde{n}_1.
\end{aligned}
\]

(4.19)

where we have used one notation $\tilde{j}_1$ that

\[
\tilde{j}_1 = \tilde{n}_1(u_0 \cdot \mathbf{M} + \theta_0 \cdot \mathbf{V}) + \tilde{u}_1 + \sigma(u_0 \times \tilde{B}_1 + \tilde{u}_1 \times B_0) + \sum \Gamma_0 U_T^-. \tag{4.20}
\]

Furthermore, applying $\text{div}_x$ to the first equation of (4.19), and noticing the third equation of (4.19) that $\text{div}_x \tilde{E}_1 = \tilde{n}_1$, it follows that

\[
\partial_t \tilde{n}_1 - \frac{1}{2}\sigma \Delta_x \tilde{n}_1 + \sigma \tilde{n}_1 = -\text{div}_x \tilde{j}_1. \tag{4.21}
\]

On the other hand, we take derivative over time variable on the $\tilde{B}_1$-equation (the second equation of (2.47)), then we get from the first equation of (4.19) that

\[
\partial_t \tilde{B}_1 + \nabla_x \times (\nabla_x \times \tilde{B}_1) - \sigma \nabla_x \times \tilde{E}_1 = \nabla_x \times \tilde{j}_1. \tag{4.22}
\]

Noticing the fact that the divergence-free property of $\tilde{B}_1$ yields

\[
\nabla_x \times (\nabla_x \tilde{B}_1) = -\Delta_x \tilde{B}_1, \tag{4.23}
\]

which enables us to write that

\[
\partial_t \tilde{B}_1 - \Delta_x \tilde{B}_1 + \sigma \partial_t \tilde{B}_1 = \nabla_x \times \tilde{j}_1. \tag{4.24}
\]

We are now in the position to derive the energy estimates for the system (4.19), (4.21) and (4.24). For all $|m| \leq M$, applying the multi-derivative operator $\partial^m$ to the first and second equations of (4.19), and then taking $L^2_x$-inner product with $\partial^m \tilde{E}_1$ and $\partial^m \tilde{B}_1$, respectively, we can infer that

\[
\frac{1}{2} \frac{d}{dt} \left(\|\partial^m \tilde{E}_1\|_{L^2_x}^2 + \|\partial^m \tilde{B}_1\|_{L^2_x}^2\right) + \sigma \|\partial^m \tilde{E}_1\|_{L^2_x}^2 + \frac{1}{2} \sigma \|\partial^m \tilde{n}_1\|_{L^2_x}^2 = -\int_{T^3} \partial^m \tilde{j}_1 \cdot \partial^m \tilde{E}_1 dx, \tag{4.25}
\]

where we have used the third equation of (4.19) again, and the cancellation relation

\[
\int_{T^3} (\nabla_x \times \partial^m \tilde{E}_1) \cdot \partial^m \tilde{B}_1 dx = \int_{T^3} (\nabla_x \times \partial^m \tilde{B}_1) \cdot \partial^m \tilde{E}_1 dx.
\]

So, summing up for $|m| \leq M$ implies that

\[
\frac{1}{2} \frac{d}{dt} \left(\|\tilde{E}_1\|_{H^2_x}^2 + \|\tilde{B}_1\|_{H^2_x}^2\right) + \sigma \|\tilde{E}_1\|_{H^2_x}^2 + \frac{1}{2} \sigma \|\tilde{n}_1\|_{H^2_x}^2 = \sum_{|m| \leq M} \int_{T^3} \partial^m \tilde{j}_1 \cdot \partial^m \tilde{E}_1 dx. \tag{4.26}
\]

Perform the same scheme as above to equation (4.21), then we obtain that

\[
\frac{1}{2} \frac{d}{dt} \|\tilde{n}_1\|_{H^2_x}^2 + \sigma \|\tilde{n}_1\|_{H^2_x}^2 + \frac{1}{2} \sigma \|\nabla x \tilde{n}_1\|_{H^2_x}^2 = \sum_{|m| \leq M} \int_{T^3} \partial^m \tilde{j}_1 \cdot \nabla x \partial^m \tilde{n}_1 dx. \tag{4.27}
\]
For the wave equation (4.24), we apply the multi-derivative operator $\partial^m$, multiply by the function $\partial_t \partial^m \bar{B}_1$ and integrate over the spatial variables $\mathbb{T}^3$, then it follows that
\[
\frac{1}{2} \frac{d}{dt} \left( \| \partial_t \partial^m \bar{B}_1 \|_{L^2}^2 + \| \nabla_x \partial^m \bar{B}_1 \|_{L^2}^2 \right) + \sigma \| \partial_t \partial^m \bar{B}_1 \|_{L^2}^2 = \int_{\mathbb{T}^3} (\nabla_x \times \partial^m \tilde{j}_1) \cdot \partial_t \partial^m \bar{B}_1 dx.
\]
which results in by summing up for all $|m| \leq M$, that
\[
\frac{1}{2} \frac{d}{dt} \left( \| \partial_t \bar{B}_1 \|_{H^M}^2 + \| \nabla_x \bar{B}_1 \|_{H^M}^2 \right) + \sigma \| \partial_t \partial^m \bar{B}_1 \|_{H^M}^2 = \sum_{|m| \leq M} \int_{\mathbb{T}^3} (\nabla_x \times \partial^m \tilde{j}_1) \cdot \partial_t \partial^m \bar{B}_1 dx. \tag{4.28}
\]
We mention that in the above procedure, if we multiply by the function $\partial^m \bar{B}_1$ instead of $\partial_t \partial^m \bar{B}_1$, we can get easily that
\[
\int_{\mathbb{T}^3} \partial_t \partial^m \bar{B}_1 \cdot \partial^m \bar{B}_1 dx + \| \nabla_x \partial^m \bar{B}_1 \|_{L^2}^2 + \frac{1}{2} \frac{d}{dt} (\sigma \| \partial^m \bar{B}_1 \|_{L^2}^2)
= \int_{\mathbb{T}^3} (\nabla_x \times \partial^m \tilde{j}_1) \cdot \partial^m \bar{B}_1 dx. \tag{4.29}
\]
This equality, combined with the relation
\[
\int_{\mathbb{T}^3} \partial_t \partial^m \bar{B}_1 \cdot \partial^m \bar{B}_1 dx
= \frac{1}{2} \frac{d}{dt} \left( \| \partial_t \partial^m \bar{B}_1 + \partial^m \bar{B}_1 \|_{L^2}^2 - \| \partial_t \partial^m \bar{B}_1 \|_{L^2}^2 - \| \partial^m \bar{B}_1 \|_{L^2}^2 \right) - \| \partial_t \partial^m \bar{B}_1 \|_{L^2}^2, \tag{4.30}
\]
yields that
\[
\frac{1}{2} \frac{d}{dt} \left( \| \partial_t \partial^m \bar{B}_1 + \partial^m \bar{B}_1 \|_{L^2}^2 - \| \partial_t \partial^m \bar{B}_1 \|_{L^2}^2 - \| \partial^m \bar{B}_1 \|_{L^2}^2 \right) - \| \partial_t \partial^m \bar{B}_1 \|_{L^2}^2
= \int_{\mathbb{T}^3} \partial^m \tilde{j}_1 \cdot (\nabla_x \times \partial^m \bar{B}_1) dx,
\]
where we have used the equality
\[
(\nabla_x \times \partial^m \tilde{j}_1) \cdot \partial^m \bar{B}_1 = \text{div}_x (\partial^m \tilde{j}_1 \times \partial^m \bar{B}_1) + \partial^m \tilde{j}_1 \cdot (\nabla_x \times \partial^m \bar{B}_1).
\]
Then, we can get by summing up for all $|m| \leq M$ that
\[
\frac{1}{2} \frac{d}{dt} \left( \| \partial_t \bar{B}_1 + \bar{B}_1 \|_{H^M}^2 - \| \partial_t \bar{B}_1 \|_{H^M}^2 - (1 - \sigma)\| \bar{B}_1 \|_{H^M}^2 \right) - \| \partial_t \bar{B}_1 \|_{H^M}^2 + \| \nabla_x \bar{B}_1 \|_{H^M}^2 \tag{4.32}
= \sum_{|m| \leq M} \int_{\mathbb{T}^3} \partial^m \tilde{j}_1 \cdot (\nabla_x \times \partial^m \bar{B}_1) dx.
\]
Choose some positive constant $\delta = \frac{1}{2} \min \{1, \sigma \} \in (0, \frac{1}{4}]$, multiply equality (4.32) by the constant $\delta$, then take summation with equalities (4.26), (4.27) and (4.28), we can derive finally that
\[
\frac{1}{2} \frac{d}{dt} \left( \| \bar{E}_1 \|_{H^M}^2 + \| \bar{n}_1 \|_{H^M}^2 + (1 - \delta + \delta \sigma)\| \bar{B}_1 \|_{H^M}^2 + \| \nabla_x \bar{B}_1 \|_{H^M}^2 \right)
+ (1 - \delta)\| \partial_t \bar{B}_1 \|_{H^M}^2 + \delta\| \partial_t \bar{B}_1 + \bar{B}_1 \|_{H^M}^2
+ (\sigma - \delta)\| \partial_t \bar{B}_1 \|_{H^M}^2 + \delta\| \nabla_x \bar{B}_1 \|_{H^M}^2 + \sigma\| \bar{E}_1 \|_{H^M}^2 + \frac{1}{2} \sigma\| \nabla_x \bar{n}_1 \|_{H^M}^2 + \frac{3}{2} \sigma\| \bar{n}_1 \|_{H^M}^2
= \sum_{|m| \leq M} \int_{\mathbb{T}^3} \partial^m \tilde{j}_1 \cdot (\nabla_x \partial^m \bar{n}_1 - \delta \partial^m \bar{n}_1 + \delta \nabla_x \times \partial^m \bar{B}_1) + (\nabla_x \times \partial^m \tilde{j}_1) \cdot \partial_t \partial^m \bar{B}_1 dx
\triangleq I_M.
\]
The Hölder inequality yields that
\[
I_M \leq \frac{\alpha}{4} \| \tilde{n}_1 \|_{H^M}^2 + \frac{\alpha}{4} \| \nabla \tilde{n} \|_{H^M}^2 + \frac{\alpha}{4} \| \nabla \tilde{B}_1 \|_{H^M}^2 + \frac{\alpha-\delta}{2} \| \partial_t \tilde{B}_1 \|_{H^M}^2 + C \| \tilde{f}_1 \|_{H^M}^2. \tag{4.34}
\]
Recalling the expression of $\tilde{f}_1$ before, it follows from the Sobolev theory that
\[
\| \tilde{f}_1 \|_{H^{M+1}}^2 \leq \| \tilde{u}_1 \|_{H^M}^2 + C \| (\| u_0 \|_{H^M}^2 + \| \theta_0 \|_{H^M}^2 ) (\| n_0 \|_{H^M}^2 + \| \nabla \tilde{B}_1 \|_{H^M}^2 )
+ C \| B_0 \|_{H^M}^2 \| \tilde{u}_1 \|_{H^M}^2 + C \sum \| \Gamma_0 \|_{H^{M+1}}^2,
\tag{4.35}
\]
where we have used the Poincaré inequality $\| \tilde{B}_1 \|_{L^2} \leq C \| \nabla \tilde{B}_1 \|_{L^2}$.

By the definition of $\Gamma_0$ given in (2.42), one can derive from the Sobolev theory that
\[
\sum \| \Gamma_0 \|_{H^{M+1}}^2 \leq C \left\{ \| \nabla (n_0 u_0) \|_{H^M}^2 + \| \theta_0 E_0 \|_{H^M}^2 + \| \text{div}_x (n_0 u_0) \|_{H^M}^2 + \| u_0 \|_{H^M}^2 + \| \theta_0 u_0 \|_{H^M}^2 + \| \tilde{u}_1 \|_{H^M}^2
+ \| \nabla \theta \|_{H^M}^2 \right\},
\tag{4.36}
\]
where we have used the Ohm’s law in (2.30) that $\tilde{f}_1 = n_0 u_0 + \sigma (\nabla n_0 + E_0 + u_0 \times B_0)$.

Then the contribution of the term $\tilde{f}_1$ can be bounded by
\[
\| \tilde{f}_1 \|_{H^{M+1}}^2 \leq C \left( 1 + \mathcal{E}_{0,M+2}(t) \right) \| \tilde{u}_1 \|_{H^M}^2 \tag{4.37}
+ C \mathcal{E}_{0,M+2}(t) \left( \| \tilde{n}_1 \|_{H^M}^2 + \| \nabla \tilde{n} \|_{H^M}^2 + \| \nabla \tilde{B}_1 \|_{H^M}^2 \right)
+ C \left( 1 + \mathcal{E}_{0,M+2}(t) \right)^2 \mathcal{D}_{0,M+2}(t).
\]

Plugging the inequalities (4.34) and (4.37) into (4.33) implies that
\[
\frac{1}{2} \frac{d}{dt} \left( \| \tilde{E}_1 \|_{H^M}^2 + \| \tilde{n}_1 \|_{H^M}^2 + (1 - \delta + \sigma) \| \tilde{B}_1 \|_{H^M}^2 + \| \nabla \tilde{B}_1 \|_{H^M}^2 \right)
+ (1 - \delta) \| \partial_t \tilde{B}_1 \|_{H^M}^2 + \sigma \| \tilde{E}_1 \|_{H^M}^2 + \frac{\sigma}{2} \| \tilde{n}_1 \|_{H^M}^2
\leq C \left( 1 + \mathcal{E}_{0,M+2}(t) \right) \| \tilde{u}_1 \|_{H^M}^2 + C \left( 1 + \mathcal{E}_{0,M+2}(t) \right)^2 \mathcal{D}_{0,M+2}(t)
+ C \mathcal{E}_{0,M+2}(t) \left( \| \tilde{n}_1 \|_{H^M}^2 + \| \nabla \tilde{n} \|_{H^M}^2 + \| \nabla \tilde{B}_1 \|_{H^M}^2 \right).
\tag{4.38}
\]

Recalling the monotonicity of $\mathcal{E}_{0,s}(t)$ and $\mathcal{D}_{0,s}(t)$ with respect to the index $s \geq 0$, the inequality (4.12) can be recast as
\[
\| \tilde{u}_1 \|_{H^M}^2 \leq C \left( 1 + \mathcal{E}_{0,M+2}(t) \right) \mathcal{D}_{0,M+2}(t),
\tag{4.39}
\]
which, combined with the fact that $\mathcal{E}_{0,M+2}(t) \leq C\mathcal{E}_{0,M+2}^{\text{in}} \leq C\lambda_0(M + 2)$ due to Lemma 4.1, enables us to get
\begin{equation}
\frac{1}{2} \frac{d}{dt} \left( \|E_1\|^2_{H^2_x} + \|\bar{n}_1\|^2_{H^2_x} + (1 - \delta + \delta\sigma)\|\bar{B}_1\|^2_{H^2_x} + \|\nabla_x \bar{B}_1\|^2_{H^2_x} \right) + (1 - \delta)\|\partial_t \bar{B}_1\|^2_{H^2_x} + \delta\|\partial_t \bar{B}_1 + \bar{B}_1\|^2_{H^2_x} \\
+ (\sigma - \delta)\|\partial_t \bar{B}_1\|^2_{H^2_x} + \|\nabla_x \bar{B}_1\|^2_{H^2_x} + \sigma\|\bar{E}_1\|^2_{H^2_x} + \frac{3}{2}\|\nabla_x \bar{n}_1\|^2_{H^2_x} + \frac{3}{2}\|\bar{n}_1\|^2_{H^2_x} \\
\leq C(1 + \mathcal{E}_{0,M+2}^{\text{in}})^2 D_{0,M+2}(t) + C\mathcal{E}_{0,M+2}^{\text{in}} \left( \|\bar{n}_1\|^2_{H^{M+1}_x} + \|\nabla_x \bar{n}_1\|^2_{H^{M+1}_x} + \|\nabla_x \bar{B}_1\|^2_{H^{M+1}_x} \right).
\end{equation}

As a consequence, by the definitions of $\mathcal{E}_{1,M}(t)$ and $D_{1,M}(t)$, it follows that
\begin{equation}
\frac{1}{2} \frac{d}{dt} \mathcal{E}_{1,M}(t) + 2D_{1,M}(t) \leq C(1 + \mathcal{E}_{0,M+2}^{\text{in}})^2 D_{0,M+2}(t) + C\mathcal{E}_{0,M+2}^{\text{in}} D_{1,M}(t). 
\end{equation}

Applying the energy inequality (4.5) with $s = M + 2$ in Lemma 4.1 yields finally that
\begin{equation}
\frac{d}{dt} [\mathcal{E}_{1,M}(t) + \tilde{C}_M \mathcal{E}_{0,M+2}(t)] + [2D_{1,M}(t) + D_{0,M+2}(t)] \leq C\mathcal{E}_{0,M+2}^{\text{in}} D_{1,M}(t),
\end{equation}
with $\tilde{C}_M = 1 + C(1 + \lambda_0(M + 2))^2 \geq 1$.

Then, by choosing some positive constant $\lambda_1(M + 2) \in [\mathcal{E}_{0,M+2}^{\text{in}}, \lambda_0(M + 2)]$ such that $C\mathcal{E}_{0,M+2}^{\text{in}} \leq 1$, the above inequality will reduces to the desired energy inequality (4.14), and thus the bound (4.15) follows immediately. This completes the proof of Lemma 4.2.

5. Uniform spatial energy estimate for the remainder system

In this section, we shall establish a uniform spatial energy estimate for the remainder system (1.16). For notational simplicity, we drop the index $\varepsilon$ in (1.16), hence
\begin{equation}
\begin{aligned}
\varepsilon \partial_t G_R + v \cdot \nabla_x G_R + T(v \times B_0) \cdot \nabla_v G_R + T(v \times B_R) \cdot \nabla_v G_R \\
-E_R \cdot v \mathcal{T}_1 + \frac{1}{\varepsilon} \mathbb{L} G_R = \varepsilon H_R, \\
\partial_t E_R - \nabla_x \times B_R = -\frac{1}{\varepsilon} \langle G_R \cdot \mathcal{T}_1 v \rangle, \\
\partial_t B_R + \nabla_x \times E_R = 0, \\
\text{div}_x E_R = \langle G_R \cdot \mathcal{T}_1 \rangle, \quad \text{div}_x B_R = 0.
\end{aligned}
\end{equation}

We shall derive a uniform energy estimate for $(G_R, E_R, B_R)$ of (5.1) by two steps:

1) We first estimate the hydrodynamic part $\mathbb{P} G_R$ of $G_R$, which has the form
\begin{equation}
\mathbb{P} G_R = \rho_R^+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \rho_R^- \begin{pmatrix} 0 \\ 1 \end{pmatrix} + u_R \begin{pmatrix} v \\ 0 \end{pmatrix} + \theta_R \begin{pmatrix} \frac{|v|^2}{2} - \frac{3}{2} \\ -\frac{3}{2} \end{pmatrix}
\end{equation}
with $\rho_R^+ = \langle g_R^+ \rangle$, $u_R = \langle \frac{g_R^+ + g_R^-}{2} v \rangle$ and $\theta_R = \langle \frac{g_R^+ + g_R^-}{2} \frac{|v|^2}{3} - 1 \rangle$ by the definition of the projection operator $\mathbb{P}$ in (1.24). Our goal is to estimate $\rho_R^+(t,x), u_R(t,x)$ and $\theta_R(t,x)$ in terms of $\mathbb{P}^\perp G_R$.

2) We then estimate the electric field $E_R$ and the magnetic field $B_R$ by subtly finding some decay effects of $E_R$ and $B_R$, which will play an essential role in establishing the global energy estimates.

Before doing these, we introduce a basis of $L^2_w$
\begin{equation}
\mathfrak{B} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} v_1 \\ 0 \end{pmatrix}, \begin{pmatrix} v_1 \end{pmatrix}, \begin{pmatrix} v_1^2 \\ v_1 \end{pmatrix}, \begin{pmatrix} v_1 |v|^2 \\ v_1 |v|^2 \end{pmatrix}, \begin{pmatrix} v_1 v_k \\ v_j v_k \end{pmatrix} ; 1 \leq i \leq 3, 1 \leq j < k \leq 3 \right\},
\end{equation}
which will be utilized in deriving the uniform spatial estimates of the remainder system \((5.1)\). One easily justifies that \(\mathcal{B}\) is linearly independent in \(L^2_v\). Indeed, if

\[
k_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_- \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{i=1}^{3} k_{i+} \begin{pmatrix} v_i \\ 0 \end{pmatrix} + \sum_{i=1}^{3} k_{i-} \begin{pmatrix} 0 \\ v_i \end{pmatrix} + \sum_{i=1}^{3} k_i \begin{pmatrix} v_i^2 \\ v_i \end{pmatrix} + \sum_{1 \leq i < j \leq 3} k_{ij} \begin{pmatrix} v_i v_j \\ v_i \end{pmatrix} = 0,
\]

we take \(L^2_v\)-inner product in \((5.4)\) by multiplying each element in the set \(\mathcal{B}\), and then we gain

\[
\begin{cases}
k_+ + \sum_{i=1}^{3} k_i = 0, \\
k_i + 5k_i = 0 (1 \leq i \leq 3), \\
k_+ + k_- + 6k_i = 0 (1 \leq i \leq 3), \\
k_{i+} + k_{i-} + 2k_i = 0 (1 \leq i \leq 3), \\
k_{ij} = 0 (1 \leq i < j \leq 3),
\end{cases}
\]

where we make use of

\[
\int_{\mathbb{R}^3} M dv = 1, \quad \int_{\mathbb{R}^3} |v|^2 M dv = 3, \quad \int_{\mathbb{R}^3} |v|^4 M dv = 15.
\]

Straightforward calculations imply that the linear system \((5.5)\) admits only zero solution, namely

\[
k_\pm = k_{i\pm} = k_i = \bar{k}_i = k_{ij} = 0,
\]

which, consequently, means that \(\mathcal{B}\) is linearly independent. Furthermore, we define a projection

\[
\mathcal{P}_\mathcal{B} : L^2_v \to \text{span}\{\mathcal{B}\} \subset L^2_v
\]

by

\[
\mathcal{P}_\mathcal{B} f = \sum_{\zeta \in \mathcal{B}} \langle f \cdot \zeta \rangle (5.6)
\]

for any \(f \in L^2_v\).

Now we estimate the hydrodynamic part \(\mathbb{P}G_R\), hence \(\rho_R^\pm(t,x), u_R(t,x)\) and \(\theta_R(t,x)\).

**Lemma 5.1.** Assume that \((G_R, E_R, B_R)\) is a sufficiently smooth solution to \((5.1)\). Then there is a constant \(C_1 > 0\) such that

\[
\|u_R\|_{H^{N+1}_x}^2 + \|\theta_R\|_{H^{N+1}_x}^2 + \|\rho_R^+\|_{H^{N+1}_x}^2 + \|\rho_R^-\|_{H^{N+1}_x}^2 + \|\text{div}_x E_R\|_{H^2_x}^2 \\
\leq C_1 \varepsilon + \sum_{|m| \leq N+1} \|\partial^m \mathbb{P}^\perp G_R\|_{L^2_{x,v}}^2 \\
+ C_1 \varepsilon + \sum_{|m| \leq N} \|\mathcal{P}_\mathcal{B} \partial^m H_R\|_{L^2_{x,v}}^2 + C_1 \varepsilon \|B_R\|_{H^N_x}^2 \\
+ C_1 \left[ \left( \int_{T^3} u_R dx \right)^2 + \left( \int_{T^3} \theta_R dx \right)^2 + \left( \int_{T^3} \rho_R^+ dx \right)^2 + \left( \int_{T^3} \rho_R^- dx \right)^2 \right]
\]

\[(5.7)\]
where

$$\epsilon$$ is sufficiently small, where the quantity \( \tilde{A}_N(t) \) is defined as

$$\tilde{A}_N(t) = \sum_{|m| \leq N} \left\{ \sum_{i,j=1}^{3} \int_{\mathbb{T}^3} (\partial^m \mathbb{P}^\perp G_R \cdot \zeta_{ij}) \partial_i \partial^m u_R^i dx \right. - \frac{1}{4} \int_{\mathbb{T}^3} \partial^m u_R \cdot \nabla^m (\rho_R^+ + \rho_R^-) dx$$

$$+ \frac{1}{4} \sum_{i=1}^{3} \int_{\mathbb{T}^3} (\partial^m \mathbb{P}^\perp G_R \cdot \zeta_i) \partial_i \partial^m \theta_R dx$$

$$+ \frac{1}{4} \sum_{i=1}^{3} \int_{\mathbb{T}^3} (\partial^m \mathbb{P}^\perp G_R \cdot \zeta_{i+}) \partial_i \partial^m \rho_R^+ + (\partial^m \mathbb{P}^\perp G_R \cdot \zeta_{i-}) \partial_i \partial^m \rho_R^-) dx.$$  \hspace*{1.5cm} (5.8)

Here \( \zeta_{ij}(v), \zeta_i(v) \) and \( \zeta_{i\pm}(v) \) are some fixed linear combinations of the basis \( \mathfrak{B} \).

Proof. Noticing that \( G_R = \mathbb{P} G_R + \mathbb{P}^\perp G_R \), the first equation in (5.1) can be rewritten as

$$\epsilon \partial_t \mathbb{P} G_R + v \cdot \nabla_x \mathbb{P} G_R + \mathcal{T}(v \times B_0) \cdot \nabla_v \mathbb{P} G_R + \mathcal{T}(v \times B_R) \cdot \nabla_v G_0$$

$$- E_R \cdot v \mathcal{T}_1 = \Theta (\mathbb{P} G_R) + \epsilon H_R,$$

where

$$\Theta (\mathbb{P} G_R) = -(\epsilon \partial_t + v \cdot \nabla_x + \frac{1}{2} \mathbb{L} + (v \times B_0) \cdot \nabla_v) \mathbb{P}^\perp G_R.$$  \hspace*{1.5cm} (5.10)

Recalling the definition of the projection \( \mathbb{P} \) in (1.24), we directly derive that

$$\epsilon \partial_t \mathbb{P} G_R + v \cdot \nabla_x \mathbb{P} G_R - E_R \cdot v \mathcal{T}_1$$

$$= \epsilon \partial_t \left( \rho_R^+ - \frac{3}{2} \theta_R \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \epsilon \partial_t \left( \rho_R^- - \frac{3}{2} \theta_R \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$+ \sum_{i=1}^{3} \left[ \epsilon \partial_t u_R^i + \partial_t (\rho_R^+ - \frac{3}{2} \theta_R) - E_R^i \right] \begin{pmatrix} v_i \\ 0 \end{pmatrix}$$

$$+ \sum_{i=1}^{3} \left[ \epsilon \partial_t u_R^i + \partial_t (\rho_R^- - \frac{3}{2} \theta_R) + E_R^i \right] \begin{pmatrix} 0 \\ v_i \end{pmatrix}$$

$$+ \sum_{1 \leq i < j \leq 3} \left( \partial_i u_R^j + \partial_j u_R^i \right) \begin{pmatrix} \frac{v_i v_j}{v_i v_j} \\ \frac{v_i v_j}{v_i v_j} \end{pmatrix},$$

and

$$\mathcal{T}(v \times B_0) \cdot \nabla_v \mathbb{P} G_R + \mathcal{T}(v \times B_R) \cdot \nabla_v G_0$$

$$= - \sum_{i=1}^{3} ((u_0 \times B_R)^i + (u_R \times B_0)^i) \begin{pmatrix} v_i \\ 0 \end{pmatrix} + \sum_{i=1}^{3} ((u_0 \times B_0)^i + (u_R \times B_0)^i) \begin{pmatrix} 0 \\ v_i \end{pmatrix}.$$  \hspace*{1.5cm} (5.12)

In summary, the left-hand of the equation (5.9) is

$$\epsilon \partial_t \mathbb{P} G_R + v \cdot \nabla_x \mathbb{P} G_R - E_R \cdot v \mathcal{T}_1 + \mathcal{T}(v \times B_0) \cdot \nabla_v \mathbb{P} G_R + \mathcal{T}(v \times B_R) \cdot \nabla_v G_0$$

$$= \epsilon \partial_t \left( \rho_R^+ - \frac{3}{2} \theta_R \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \epsilon \partial_t \left( \rho_R^- - \frac{3}{2} \theta_R \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$+ \sum_{i=1}^{3} \left[ \epsilon \partial_t u_R^i + \partial_t (\rho_R^+ - \frac{3}{2} \theta_R) - E_R^i - (u_0 \times B_R)^i - (u_R \times B_0)^i \right] \begin{pmatrix} v_i \\ 0 \end{pmatrix}$$

$$+ \sum_{i=1}^{3} \left[ \epsilon \partial_t u_R^i + \partial_t (\rho_R^- - \frac{3}{2} \theta_R) + E_R^i + (u_0 \times B_R)^i + (u_R \times B_0)^i \right] \begin{pmatrix} v_i \\ 0 \end{pmatrix}.$$
\begin{align}
&+ \sum_{i=1}^{3} \left( \partial_t \rho_R^+ + \partial_i u_R^+ \right) \left( \frac{v_i v_j}{v_i^2} \right),
\end{align}

By the definition of the projection operator $\mathcal{P}_B$ in (5.6), we know that

\begin{align}
\mathcal{P}_B \Theta^\pm (t,x) &= \Theta^\pm_R (t,x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \Theta^+_R (t,x) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{i=1}^{3} \Theta^+_R (t,x) \begin{pmatrix} v_i \\ 0 \end{pmatrix} + \sum_{i=1}^{3} \Theta^-_R (t,x) \begin{pmatrix} 0 \\ v_i \end{pmatrix} \\
&+ \sum_{i=1}^{3} \Theta^+_R (t,x) \begin{pmatrix} v_i^2 \\ v_i \end{pmatrix} + \sum_{i=1}^{3} \Theta^-_R (t,x) \begin{pmatrix} v_i^2 \\ v_i \end{pmatrix} + \sum_{1 \leq i < j \leq 3} \Theta^{ij}_R (t,x) \begin{pmatrix} v_i v_j \\ v_i v_j \end{pmatrix},
\end{align}

and

\begin{align}
\mathcal{P}_B h_R &= h_R^+ (t,x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + h_R^- (t,x) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{i=1}^{3} h_R^+ (t,x) \begin{pmatrix} v_i \\ 0 \end{pmatrix} + \sum_{i=1}^{3} h_R^- (t,x) \begin{pmatrix} 0 \\ v_i \end{pmatrix} \\
&+ \sum_{i=1}^{3} h_R^+ (t,x) \begin{pmatrix} v_i^2 \\ v_i \end{pmatrix} + \sum_{i=1}^{3} h_R^- (t,x) \begin{pmatrix} v_i^2 \\ v_i \end{pmatrix} + \sum_{1 \leq i < j \leq 3} h_R^{ij} (t,x) \begin{pmatrix} v_i v_j \\ v_i v_j \end{pmatrix}.
\end{align}

As a consequence, projecting the relation (5.9) into $\text{span}\{\mathcal{B}\}$ and the equalities (5.13), (5.14) and (5.15) imply that

\begin{align}
\begin{cases}
\varepsilon \partial_t (\rho_R^+ - \frac{3}{2} \theta_R) = \Theta^+_R + \varepsilon h^+_R, \\
\varepsilon \partial_t u_R^+ + \partial_i (\rho_R^+ - \frac{3}{2} \theta_R) = \Theta^+_R + (u_0 \times B_R)^i + (u_0 \times B_0)^i = \Theta^+_R + \varepsilon h^+_R, \\
\frac{1}{2} \partial_t \theta_R = \Theta^+_R + \varepsilon h^+_R, \\
\frac{1}{2} \varepsilon \partial_t \theta_R + \partial_i u_R^+ = \Theta^+_R + \varepsilon h^+_R, \\
\partial_t u_R^+ + \partial_j u_R^+ = \Theta^+_R + \varepsilon h^+_R (i \neq j).
\end{cases}
\end{align}

Now by using the splitting $G_R = \mathbb{P} G_R + \mathbb{P}^\perp G_R$, we rewrite the equation of (5.1) with the form

\begin{align}
\varepsilon \partial_t G_R + v \cdot \nabla_x \mathbb{P} G_R + T (v \times B_0) \cdot \nabla_x \mathbb{P} G_R + T (v \times B_R) \cdot \nabla_x G_0 - E_R \cdot v T_1 + \frac{1}{\varepsilon} \mathbb{L} G_R \\
= -(v \cdot \nabla_x \mathbb{P}^\perp G_R + T (v \times B_0) \cdot \nabla_x \mathbb{P}^\perp G_R) + \varepsilon H_R,
\end{align}

which implies that by projecting it into the kernel $\ker \mathbb{L}$ of the operator $\mathbb{L}$

\begin{align}
\begin{cases}
\varepsilon \partial_t \rho_R^+ + \text{div}_x u_R = \varepsilon \left\langle H_R \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle, \\
\varepsilon \partial_t \rho_R^- + \text{div}_x u_R = \varepsilon \left\langle H_R \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle, \\
\varepsilon \partial_t u_R + \nabla_x \left( \frac{\rho_R^+ + \rho_R^-}{2} \right) + \nabla_x \theta_R = \frac{\varepsilon}{2} \left\langle H_R \cdot \begin{pmatrix} v \\ v \end{pmatrix} \right\rangle \\
- \frac{1}{2} \left\langle v \cdot \nabla_x \mathbb{P}^\perp G_R + T (v \times B_0) \cdot \nabla_x \mathbb{P}^\perp G_R \cdot \begin{pmatrix} v \\ v \end{pmatrix} \right\rangle, \\
\varepsilon \partial_t \theta_R + \frac{2}{3} \text{div}_x u_R = \frac{\varepsilon}{2} \left\langle H_R \cdot \begin{pmatrix} |v|^2 - 1 \\ |v|^2 - 1 \\ \frac{1}{3} \frac{|v|^2 - 1}{3} - 1 \end{pmatrix} \right\rangle - \frac{1}{2} \left\langle v \cdot \nabla_x \mathbb{P}^\perp G_R \cdot \begin{pmatrix} |v|^2 - 1 \\ |v|^2 - 1 \\ \frac{1}{3} \frac{|v|^2 - 1}{3} - 1 \end{pmatrix} \right\rangle.
\end{cases}
\end{align}
From the last two equations of (5.16), we derive

\[- \Delta_x \partial^m u_R^i = - \sum_{j=1}^{3} \partial_{jj} \partial^m u_R^i = - \sum_{j \neq i} \partial_{jj} \partial^m u_R^i - \partial_i \partial^m u_R^i \]

\[- \sum_{j \neq i} \partial_j \partial^m (- \partial_i u_R^i + \Theta^j_R + \varepsilon h^j_R) - \partial_i \partial^m (- \frac{1}{2} \varepsilon \partial_i \theta_R + \Theta^j_R + \varepsilon h^j_R) \]

\[= \sum_{j \neq i} \partial_j \partial^m (- \frac{1}{2} \varepsilon \partial_i \theta_R + \Theta^j_R + \varepsilon h^j_R) + \partial_i \partial^m (\frac{1}{2} \varepsilon \partial_i \theta_R - \Theta^j_R - \varepsilon h^j_R) - \sum_{j \neq i} \partial_j \partial^m (\Theta^j_R + \varepsilon h^j_R) \]

\[= - \frac{1}{2} \varepsilon \partial_i \partial^m \partial_i \theta_R - \partial_i \partial^m \Theta^j_R - \varepsilon \partial_i \partial^m h^i_R + \sum_{j \neq i} (\partial_i \partial^m \Theta^j_R + \varepsilon \partial_i \partial^m h^j_R - \partial_i \partial^m \Theta^j_R - \varepsilon \partial_j \partial^m h^j_R) \]

\[= \partial_i \partial^m u_R^i + \sum_{j \neq i} (\partial_i \partial^m \Theta^j_R - \partial_j \partial^m \Theta^j_R) - 2 \partial_i \partial^m \Theta^j_R \]

\[+ \varepsilon \left[ \sum_{j \neq i} (\partial_j \partial^m h^j_R - \partial_j \partial^m h^j_R) - 2 \partial_i \partial^m h^j_R \right]. \]

(5.19)

Noticing that there are certain linear combinations \( \zeta_{ij} \) of the basis \( \mathcal{B} \) such that

\[\sum_{j \neq i} (\partial_i \partial^m \Theta^j_R - \partial_j \partial^m \Theta^j_R) - 2 \partial_i \partial^m \Theta^j_R \]

\[= \sum_{j=1}^{3} \partial_j \partial^m \left\{ \langle - \varepsilon \partial_i \nabla_x \partial^m G_R - v \cdot \nabla_x \partial^m G_R - T(v \times B_0) \cdot \nabla_v \partial^m G_R \cdot \zeta_{ij} \rangle - \frac{1}{2} \langle \nabla \partial^m \partial^m G_R \cdot \zeta_{ij} \rangle \right\}, \]

(5.20)

and

\[\sum_{j \neq i} (\partial_i \partial^m h^j_R - \partial_j \partial^m h^j_R) - 2 \partial_i \partial^m h^j_R = \sum_{j=1}^{3} \partial_j \partial^m \langle H_R \cdot \zeta_{ij} \rangle, \]

(5.21)

we have

\[- \Delta_x \partial^m u_R^i - \partial_i \partial^m u_R^i = \sum_{j=1}^{3} \partial_j \left\{ - \varepsilon \partial_i \partial^m \partial^m G_R \cdot \zeta_{ij} \right\} - \langle v \cdot \nabla_x \partial^m \partial^m G_R \cdot \zeta_{ij} \rangle \]

\[\left. - \langle \partial^m [T(v \times B_0) \cdot \nabla_v \partial^m G_R \cdot \zeta_{ij}] \rangle - \frac{1}{2} \langle \nabla \partial^m \partial^m G_R \cdot \zeta_{ij} \rangle \right\}, \]

(5.22)

\[+ \varepsilon \sum_{j=1}^{3} \partial_j \partial^m \langle H_R \cdot \zeta_{ij} \rangle. \]

Multiplying by \( \partial^m u_R^i \) in (5.22), integrating by parts over \( x \in \mathbb{T}^3 \) and summing up for \( 1 \leq i \leq 3 \) imply that

\[\| \nabla_x \partial^m u_R \|^2_{L^2_x} + \| \text{div}_x \partial^m u_R \|^2_{L^2_x} \]

\[\leq \sum_{i,j=1}^{3} \int_{\mathbb{T}^3} -\langle \varepsilon \partial_i \partial^m \partial_j \partial^m G_R \cdot \zeta_{ij} \rangle \partial^m u_R^i dx \]

\[+ \frac{C}{\varepsilon} \left( \| \nabla_x \partial^m \partial^m G_R \|_{L^2_x} + \| \partial^m \partial^m G_R \|_{L^2_x} \right) \| \nabla_x \partial^m u_R \|_{L^2_x} \]

\[+ C \varepsilon \| \partial^m H_R \|_{L^2_x} \| \nabla_x \partial^m u_R \|_{L^2_x} \]

\[+ \sum_{i,j=1}^{3} \int_{\mathbb{T}^3} \langle \partial^m [T(v \times B_0) \cdot \nabla_v \partial^m G_R \cdot \zeta_{ij}] \rangle \partial_j \partial^m u_R^i dx. \]

(5.23)
The last term in (5.23) is estimated by
\[ \sum_{i,j=1}^{3} \int_{T^3} \langle \partial^m [T(v \times B_0) \cdot \nabla v^\perp G_R] \cdot \zeta_{ij} \rangle \partial_j \partial^m u^i_R dx \]
\[ = \sum_{i,j=1}^{3} \int_{T^3} \int_{R^3} \partial^m [T \varepsilon_{kpq} v_p B_0^q \partial_{vk}^\perp G_R] \cdot \zeta_{ij} M dv \partial_j \partial^m u^i_R dx \]
\[ = - \sum_{i,j=1}^{3} \int_{T^3} \int_{R^3} \varepsilon_{kpq} B_0^q T^\perp G_R \cdot \partial_{vk} (v_p \zeta_{ij} M) dv \partial_j \partial^m u^i_R dx \]
\[ = - \sum_{i,j=1}^{3} \int_{T^3} \int_{R^3} \varepsilon_{kpq} B_0^q T^\perp G_R \cdot v_p \partial_{vk} \zeta_{ij} M dv \partial_j \partial^m u^i_R dx \]
\[ = - \sum_{i,j=1}^{3} \int_{T^3} \int_{R^3} \varepsilon_{kpq} \partial^m B_0^q T^\perp \partial^m G_R \cdot (v_p \partial_{vk} \zeta_{ij}) M dv \partial_j \partial^m u^i_R dx \]
\[ \leq C \sum_{m' \leq m} \| \partial^m B_0 \|_{L^\infty} \| \partial^m G_R \|_{L^2 \cdot v} \| \nabla_x \partial^m u^i_R \|_{L^2}. \]

Thus, the inequalities (5.23) reduces to
\[ \| \nabla_x \partial^m u^i_R \|_{L^2}^2 + \| \text{div}_x \partial^m u^i_R \|_{L^2}^2 \]
\[ \leq \sum_{i,j=1}^{3} \int_{T^3} \langle -\varepsilon \partial_t \partial_j \partial^m G_R \cdot \zeta_{ij} \rangle \partial^m u^i_R dx + C\varepsilon \| \mathcal{P}_B \partial^m H_R \|_{L^2 \cdot v} \| \nabla_x \partial^m u^i_R \|_{L^2} \]
\[ + \xi \left( \| \nabla_x \partial^m G_R \|_{L^2 \cdot v} + \| \partial^m G_R \|_{L^2 \cdot v} \right) \| \nabla_x \partial^m u^i_R \|_{L^2} \]
\[ + C \sum_{m' \leq m} \| \partial^m B_0 \|_{L^2} \| \partial^m G_R \|_{L^2 \cdot v} \| \nabla_x \partial^m u^i_R \|_{L^2}. \]  

(5.24)

Now we estimate the first term in the right-hand side of (5.24). First of all, we have the following equality
\[ \sum_{i,j=1}^{3} \int_{T^3} \langle -\varepsilon \partial_t \partial_j \partial^m G_R \cdot \zeta_{ij} \rangle \partial^m u^i_R dx \]
\[ = \sum_{i,j=1}^{3} \frac{d}{dt} \int_{T^3} \varepsilon \langle -\partial_j \partial^m G_R \cdot \zeta_{ij} \rangle \partial^m u^i_R dx + \sum_{i,j=1}^{3} \int_{T^3} \langle \partial_j \partial^m G_R \cdot \zeta_{ij} \varepsilon \partial^m \partial_t u^i_R dx. \]

(5.25)

For the last term in (5.25), we derive from the third equation of (5.18) that
\[ \sum_{i,j=1}^{3} \int_{T^3} \langle \partial_j \partial^m G_R \cdot \zeta_{ij} \rangle \varepsilon \partial^m \partial_t u^i_R dx \]
\[ = \sum_{i,j=1}^{3} \int_{T^3} \langle \partial_j \partial^m G_R \cdot \zeta_{ij} \rangle \partial^m \left[ - \partial_l \left( \frac{v_l \cdot v_l}{2} \right) - \partial_t \theta_R + \frac{\varepsilon}{2} \left( H_R \cdot \left( \frac{v_i}{v_i} \right) \right) \right. \]
\[ - \frac{1}{2} \left( v \cdot \nabla_x \partial^m G_R \cdot \left( \frac{v_i}{v_i} \right) \right) - \frac{1}{2} \left( T(v \times B_0) \cdot \nabla_x \partial^m G_R \cdot \left( \frac{v_i}{v_i} \right) \right) \] \[ \right] dx \]
\[ \leq C \| \nabla_x \partial^m G_R \|_{L^2 \cdot v} \left\{ \| \nabla_x \partial^m u^\perp_R \|_{L^2} + \| \nabla_x \partial^m u^\perp_R \|_{L^2} + \| \nabla_x \partial^m \theta_R \|_{L^2} + \varepsilon \| \mathcal{P}_B \partial^m H_R \|_{L^2 \cdot v} \right\} \]
\[ + C \| \nabla_x \partial^m G_R \|_{L^2 \cdot v} + \sum_{i,j=1}^{3} \int_{T^3} \langle \partial_j \partial^m G_R \cdot \zeta_{ij} \rangle \partial^m \left( \varepsilon_{qnp} B_0^q T^\perp G_R \cdot \mathcal{T}_q \right) dx. \]
where \( T_2 = \left( \frac{1}{1} \right) \) and the last term in the above inequality is bounded by

\[
C \sum_{m' \leq m} \| \nabla_x \partial^m \mathbb{P}^\perp G_R \|_{L^2_{x,v}} \| \partial^{m'} \mathbb{P}^\perp G_R \|_{L^2_{x,v}} \| \partial^{m-m'} B_0 \|_{L^2_{x}}.
\]

Then, we have

\[
\sum_{i,j=1}^3 \int_{T^3} (\partial_j \partial^m \mathbb{P}^\perp G_R \cdot \xi_{ij}) \varepsilon \partial^m \partial_t u^i_R \, dx
\]

\[
\leq C \| \nabla_x \partial^m \mathbb{P}^\perp G_R \|_{L^2_{x,v}} \left( \| \nabla_x \partial^m p_R^\perp \|_{L^2_{x}} + \| \nabla_x \partial^m p_R \|_{L^2_{x}} + \| \nabla_x \partial^m \theta_R \|_{L^2_{x}} \right)
\]

\[
+ \| \nabla_x \partial^m \mathbb{P}^\perp G_R \|_{L^2_{x,v}} + \varepsilon \| \mathbb{P}^\perp \partial^m H_R \|_{L^2_{x,v}} \sum_{m' \leq m} \| \partial^{m-m'} B_0 \|_{L^2_{x}} \| \partial^{m'} \mathbb{P}^\perp G_R \|_{L^2_{x,v}}.
\]

(5.26)

Consequently, by combining the relations (5.24), (5.25), (5.26) and utilizing the Young’s inequality, we gain that for \( \varepsilon \in (0,1] \)

\[
\frac{1}{2} \| \nabla_x \partial^m u_R \|_{L^2_{x}}^2 + \| \text{div}_x \partial^m u_R \|_{L^2_{x}}^2
\]

\[
\leq \frac{d}{dt} \sum_{i,j=1}^3 \int_{T^3} \varepsilon (\partial^m \mathbb{P}^\perp G_R \cdot \zeta_{ij}) \partial_j \partial^m u^i_R \, dx + C \varepsilon^2 \| \mathbb{P}^\perp \partial^m H_R \|_{L^2_{x,v}}^2
\]

\[
+ \frac{C}{\varepsilon^2} \left( \| \nabla_x \partial^m \mathbb{P}^\perp G_R \|_{L^2_{x,v}}^2 + \| \partial^m \mathbb{P}^\perp G_R \|_{L^2_{x,v}}^2 \right) + C \sum_{m' \leq m} \| \partial^{m-m'} B_0 \|_{L^2_{x}} \| \partial^{m'} \mathbb{P}^\perp G_R \|_{L^2_{x,v}}^2
\]

\[
+ C \varepsilon^2 \left( \| \nabla_x \rho_R \|_{L^2_{x}}^2 + \| \nabla_x \rho_R \|_{L^2_{x}}^2 + \| \nabla_x \theta_R \|_{L^2_{x}}^2 \right).
\]

(5.27)

For the third equation of (5.16), i.e.,

\[
\frac{1}{2} \partial_t \theta_R = \bar{\Theta}_R^i + \varepsilon \bar{h}^i_R,
\]

we calculate that

\[
- \frac{1}{2} \Delta_x \partial^m \theta_R = - \frac{1}{2} \sum_{i=1}^3 \partial_i \partial^m \theta_R = \sum_{i=1}^3 \partial_i \partial^m (\frac{1}{2} \partial_i \theta_R) = - \sum_{i=1}^3 \partial_i \partial^m (\bar{\Theta}_R^i + \varepsilon \bar{h}^i_R).
\]

One observe that there are some certain linear combinations \( \zeta_i \) (\( 1 \leq i \leq 3 \)) of \( \mathfrak{S} \) such that

\[
- \sum_{i=1}^3 \partial_i \partial^m \bar{\Theta}_R^i = \sum_{i=1}^3 \partial_i \partial^m \left( \langle - (\varepsilon \partial_t \mathbb{P}^\perp G_R + v \cdot \nabla_x \mathbb{P}^\perp G_R) \cdot \zeta_i \rangle - \varepsilon \langle \mathcal{L}(\mathbb{P}^\perp G_R) \cdot \zeta_i \rangle \right)
\]

and

\[
- \sum_{i=1}^3 \partial_i \partial^m \bar{h}^i_R = \sum_{i=1}^3 \partial_i \partial^m \langle H_R \cdot \zeta_i \rangle.
\]

Then we have

\[
- \frac{1}{2} \Delta_x \partial^m \theta_R = \sum_{i=1}^3 \partial_i \partial^m \left( \langle - (\varepsilon \partial_t \mathbb{P}^\perp G_R + v \cdot \nabla_x \mathbb{P}^\perp G_R) \cdot \zeta_i \rangle - \varepsilon \langle \mathcal{L}(\mathbb{P}^\perp G_R) \cdot \zeta_i \rangle \right)
\]

\[
- \varepsilon \sum_{i=1}^3 \partial_i \partial^m \langle H_R \cdot \zeta_i \rangle.
\]

(5.28)
We take $L^2_x$-inner product by multiplying $\partial^m \theta_R$ in (5.28), integrating by parts over $x \in \mathbb{T}^3$ and then we derive that
\begin{align}
\frac{1}{2} \|\nabla_x \partial^m \theta_R\|_{L^2_x}^2 & \leq \sum_{i=1}^3 \int_{\mathbb{T}^3} \langle -\varepsilon \partial_t \mathbb{P}^\perp \partial_i \partial^m \rho_R \cdot \zeta_i \rangle \partial^m \theta_R dx + \frac{C}{\varepsilon} \| \partial^m \mathbb{P}^\perp G_R \|_{L^2_x} \| \nabla_x \partial^m \theta_R \|_{L^2_x} \\
& \quad + C \| \nabla_x \partial^m \mathbb{P}^\perp G_R \|_{L^2_x} \| \nabla_x \partial^m \theta_R \|_{L^2_x} + C \epsilon \| \mathcal{P} \partial^m H_R \|_{L^2_x} \| \nabla_x \partial^m \theta_R \|_{L^2_x} \\
& \quad + C \sum_{m' \leq m} \| \partial^m - m' B_0 \|_{L^\infty} \| \partial^m \mathbb{P}^\perp G_R \|_{L^2_x} \| \nabla_x \partial^m \theta_R \|_{L^2_x} .
\end{align}
(5.29)

For the first term in the right-hand side of (5.29), by using the forth relation in (5.18) we deduce that
\begin{align}
& \frac{d}{dt} \sum_{i=1}^3 \int_{\mathbb{T}^3} \langle -\varepsilon \partial_t \mathbb{P}^\perp \partial_i \partial^m \rho_R \cdot \zeta_i \rangle \partial^m \theta_R dx \\
& \quad \frac{d}{dt} \sum_{i=1}^3 \int_{\mathbb{T}^3} \langle \varepsilon \partial^m \mathbb{P}^\perp G_R \cdot \zeta_i \rangle \partial_i \partial^m \theta_R dx + \sum_{i=1}^3 \int_{\mathbb{T}^3} \langle \partial_i \partial^m \mathbb{P}^\perp G_R \cdot \zeta_i \rangle \varepsilon \partial^m \partial_i \theta_R dx \\
& \quad = \frac{d}{dt} \sum_{i=1}^3 \int_{\mathbb{T}^3} \langle \varepsilon \partial^m \mathbb{P}^\perp G_R \cdot \zeta_i \rangle \partial_i \partial^m \theta_R dx - \frac{3}{2} \sum_{i=1}^3 \int_{\mathbb{T}^3} \text{div}_x \partial^m u_R \langle \partial_i \partial^m \mathbb{P}^\perp \cdot \zeta_i \rangle dx \\
& \quad - \frac{1}{2} \sum_{i=1}^3 \int_{\mathbb{T}^3} \langle \partial_i \partial^m \mathbb{P}^\perp G_R \cdot \zeta_i \rangle \left( v \cdot \nabla_x \partial^m \mathbb{P}^\perp G_R \cdot \left( \frac{|v|^2}{2} - 1 \right) \right) dx \\
& \quad + \frac{\varepsilon}{3} \sum_{i=1}^3 \int_{\mathbb{T}^3} \langle \partial_i \partial^m \mathbb{P}^\perp G_R \cdot \zeta_i \rangle \left( \partial^m H_R \cdot \left( \frac{|v|^2}{2} - 1 \right) \right) dx \\
& \quad \leq \frac{d}{dt} \sum_{i=1}^3 \int_{\mathbb{T}^3} \langle \varepsilon \partial^m \mathbb{P}^\perp G_R \cdot \zeta_i \rangle \partial_i \partial^m \theta_R dx + C \| \nabla_x \partial^m \mathbb{P}^\perp G_R \|_{L^2_x}^2 \\
& \quad + C \| \nabla_x \partial^m \mathbb{P}^\perp G_R \|_{L^2_x} \| \nabla_x \partial^m u_R \|_{L^2_x} + C \varepsilon \| \nabla_x \partial^m \mathbb{P}^\perp G_R \|_{L^2_x} \| \mathcal{P} \partial^m H_R \|_{L^2_x} .
\end{align}
(5.30)

Plugging the inequality (5.30) into (5.29) and Young’s inequality yield that for $\varepsilon \in (0, 1]$
\begin{align}
\frac{1}{4} \| \nabla_x \partial^m \theta_R \|_{L^2_x}^2 & \leq \frac{d}{dt} \sum_{i=1}^3 \int_{\mathbb{T}^3} \langle \varepsilon \partial^m \mathbb{P}^\perp G_R \cdot \zeta_i \rangle \partial_i \partial^m \theta_R dx \\
& \quad + \frac{C}{\varepsilon^2} \left( \| \partial^m \mathbb{P}^\perp G_R \|_{L^2_x}^2 + \| \nabla_x \partial^m \mathbb{P}^\perp G_R \|_{L^2_x}^2 \right) \\
& \quad + C \varepsilon^2 \left( \| \nabla_x \partial^m u_R \|_{L^2_x}^2 + \| \mathcal{P} \partial^m H_R \|_{L^2_x}^2 \right) \\
& \quad + C \sum_{m' \leq m} \| \partial^m - m' B_0 \|_{L^\infty} \| \partial^m \mathbb{P}^\perp G_R \|_{L^2_x}^2 .
\end{align}
(5.31)

Now we estimate the quantities $\| \nabla_x \partial^m \rho_R \|_{L^2_x}^2$. From the second equation of (5.16), we derive that
\begin{align}
- \Delta_x \partial^m \rho_R^\pm & \pm \text{div}_x \partial^m E_R \pm \text{div}_x \partial^m (u_0 \times B_R) \pm \text{div}_x \partial^m (u_R \times B_0) \\
& \quad = \sum_{i=1}^3 \partial_i \partial^m \left( - \partial_i \rho_R^\pm \pm E_R^i \pm (u_0 \times B_R)^i \pm (u_R \times B_0)^i \right) \\
& \quad = \sum_{i=1}^3 \partial_i \partial^m \left( \varepsilon \partial_i u_R^i - \frac{3}{2} \partial_i \theta_R - \Theta_R^\pm - \varepsilon h_R^\pm \right) .
\end{align}
(5.32)
It is easy to know that there are some fixed linear combinations \( \zeta_{i\pm} \) \((1 \leq i \leq 3)\) of \( \mathcal{B} \) such that
\[
- \sum_{i=1}^{3} \partial_i \partial^m \Theta^\pm_{R} = \sum_{i=1}^{3} \partial_i \partial^m \left\{ \langle - (\varepsilon \partial_t \mathbb{P} G_R + v \cdot \nabla_x \mathbb{P} G_R) \cdot \zeta_{i\pm} \rangle - \frac{1}{\varepsilon} \langle L(\mathbb{P} G_R) \cdot \zeta_{i\pm} \rangle - \langle T(v \times B_0) \cdot \nabla_x \mathbb{P} G_R \cdot \zeta_{i\pm} \rangle \right\}
\]
and
\[
- \sum_{i=1}^{3} \partial_i \partial^m h_{R}^\pm = \sum_{i=1}^{3} \partial_i \partial^m \langle H_R \cdot \zeta_{i\pm} \rangle,
\]
which immediately lead to
\[
- \Delta_x \partial^m \rho_R^\pm \pm \text{div}_x \partial^m E_R
= \mp \text{div}_x \partial^m (u_0 \times B_R + u_R \times B_0)
+ \sum_{i=1}^{3} \varepsilon \partial_t \partial_i \partial^m u_i^R + \sum_{i=1}^{3} \langle - \varepsilon \partial_t \partial_i \partial^m \mathbb{P} G_R \cdot \zeta_{i\pm} \rangle \quad (5.33)
\]
\[
- \sum_{i=1}^{3} \langle v \cdot \nabla_x \partial_i \partial^m \mathbb{P} G_R \cdot \zeta_{i\pm} \rangle + \varepsilon \sum_{i=1}^{3} \partial_i \partial^m \langle H_R \cdot \zeta_{i\pm} \rangle
- \frac{1}{\varepsilon} \sum_{i=1}^{3} \langle L \partial_i \partial^m \mathbb{P} G_R \cdot \zeta_{i\pm} \rangle - \sum_{i=1}^{3} \partial_i \partial^m \langle T(v \times B_0) \cdot \nabla_x \mathbb{P} G_R \cdot \zeta_{i\pm} \rangle.
\]
Taking \( L_2^3 \)-inner product by dot with \( \partial^m \rho_R^\pm \), integrating by parts over \( x \in T^3 \) and Young’s inequality imply that for \( \varepsilon \in (0, 1] \)
\[
\frac{1}{2} \| \nabla_x \partial^m \rho_R^\pm \|^2_{L_2^3} \pm \int_{T^3} \text{div}_x \partial^m E_R \partial^m \rho_R^\pm dx
\leq \sum_{i=1}^{3} \int_{T^3} \varepsilon \partial_t \partial_i \partial^m u_i^R \cdot \partial^m \rho_R^\pm dx + \sum_{i=1}^{3} \int_{T^3} \langle - \varepsilon \partial_t \partial_i \partial^m \mathbb{P} G_R \cdot \zeta_{i\pm} \rangle \partial^m \rho_R^\pm dx
+ C \frac{\varepsilon^2}{\varepsilon^2} (\| \partial^m \mathbb{P} G_R \|_{L_2^3}^2 + \| \nabla_x \partial^m \mathbb{P} G_R \|_{L_2^3}^2) + C \varepsilon^2 \| \mathcal{P}_B \partial^m H_R \|_{L_2^3}^2
\]
\[
+ C \sum_{m' \leq m} \left( \| \partial^{m-m'} u_0 \|_{L_2^3}^2 + \| \partial^{m-m'} B_0 \|_{L_2^3}^2 \right) (\| \partial^m \mathbb{P} G_R \|_{L_2^3}^2 + \| \partial^m u_R \|_{L_2^3}^2) \quad (5.34)
\]
From the first two equations of (5.18), we can estimate the first two terms in the right-hand side of (5.34) as follows:
\[
\sum_{i=1}^{3} \int_{T^3} \varepsilon \partial_t \partial_i \partial^m u_i^R \cdot \partial^m \rho_R^\pm dx
\leq \frac{d}{dt} \int_{T^3} \| \text{div}_x \partial^m u_R \|^2_{L_2^3}
+ C \varepsilon \| \mathcal{P}_B \partial^m H_R \|_{L_2^3} \| \text{div}_x \partial^m u_R \|_{L_2^3}
\]
\[
(5.35)
\]
and
\[
\sum_{i=1}^{3} \int_{T^3} \langle - \varepsilon \partial_t \partial_i \partial^m \mathbb{P} G_R \cdot \zeta_{i\pm} \rangle \partial^m \rho_R^\pm dx
\leq \frac{d}{dt} \sum_{i=1}^{3} \int_{T^3} \langle \varepsilon \partial^m \mathbb{P} G_R \cdot \zeta_{i\pm} \rangle \cdot \partial_i \partial^m \rho_R^\pm dx
\]
\[
(5.36)
\]
Thus we have
\[
\frac{1}{2}\|\nabla_{x}\partial^{m}\rho_{R}^{\pm}\|_{L^{2}_{x,v}}^{2} + \int_{T^{3}} \text{div}_{x}\partial^{m}E_{R}\partial^{m}\rho_{R}^{\pm}dx
\]
\[
\leq \frac{d}{dt} \left\{ \int_{T^{3}} -\varepsilon\partial^{m}u_{R} \cdot \nabla_{x}\partial^{m}\rho_{R}^{\pm}dx + \sum_{i=1}^{3} \int_{T^{3}} \langle \varepsilon\partial^{m}L^{\perp}G_{R} \cdot \zeta_{i}\rangle \cdot \partial_{t}\partial^{m}\rho_{R}^{\pm}dx \right\}
\]
(5.37)
\[
+ 2\|\text{div}_{x}\partial^{m}E_{R}\|_{L^{2}_{x,v}}^{2} + \frac{C}{\varepsilon^{2}} \left( \|\partial^{m}L^{\perp}G_{R}\|_{L^{2}_{x,v}}^{2} + \|\nabla_{x}\partial^{m}L^{\perp}G_{R}\|_{L^{2}_{x,v}}^{2} \right) + C\varepsilon^{2}\|\mathcal{P}_{B}\partial^{m}H_{R}\|_{L^{2}_{x,v}}^{2}
\]
\[+ C \sum_{m' \leq m} \left( \|\partial^{m-m'}u_{0}\|_{L^{2}_{x,v}}^{2} + \|\partial^{m-m'}B_{0}\|_{L^{2}_{x,v}}^{2} \right) \left( \|\partial^{m}L^{\perp}G_{R}\|_{L^{2}_{x,v}}^{2} + \|\partial^{m}u_{R}\|_{L^{2}_{x,v}}^{2} + \|\partial^{m}B_{R}\|_{L^{2}_{x,v}}^{2} \right),
\]
which is derived from the inequalities (5.34), (5.35) and (5.36). We sum up for “±” in (5.37) and then by the last second equation of (5.1), i.e.,
\[
\text{div}_{x}E_{R} = \langle G_{R} \cdot T_{1} \rangle = \rho_{R}^{+} - \rho_{R}^{-},
\]
we obtain that for \( \varepsilon \in (0, 1] \)
\[
\frac{1}{2}\|\nabla_{x}\partial^{m}\rho_{R}^{\pm}\|_{L^{2}_{x,v}}^{2} + \frac{1}{2}\|\nabla_{x}\partial^{m}\rho_{R}^{-}\|_{L^{2}_{x,v}}^{2} + \|\text{div}_{x}\partial^{m}E_{R}\|_{L^{2}_{x,v}}^{2}
\]
\[
\leq \frac{d}{dt} \left\{ \int_{T^{3}} -\varepsilon\partial^{m}u_{R} \cdot \nabla_{x}\partial^{m}(\rho_{R}^{+} - \rho_{R}^{-})dx + \sum_{i=1}^{3} \int_{T^{3}} \langle \varepsilon\partial^{m}L^{\perp}G_{R} \cdot \zeta_{i}\rangle \partial_{t}\partial^{m}\rho_{R}^{-}dx \right\}
\]
(5.38)
\[
+ 4\|\text{div}_{x}\partial^{m}u_{R}\|_{L^{2}_{x,v}}^{2} + \frac{C}{\varepsilon^{2}} \left( \|\partial^{m}L^{\perp}G_{R}\|_{L^{2}_{x,v}}^{2} + \|\nabla_{x}\partial^{m}L^{\perp}G_{R}\|_{L^{2}_{x,v}}^{2} \right) + C\varepsilon^{2}\|\mathcal{P}_{B}\partial^{m}H_{R}\|_{L^{2}_{x,v}}^{2}
\]
\[+ C \sum_{m' \leq m} \left( \|\partial^{m-m'}u_{0}\|_{L^{2}_{x,v}}^{2} + \|\partial^{m-m'}B_{0}\|_{L^{2}_{x,v}}^{2} \right) \left( \|\partial^{m}L^{\perp}G_{R}\|_{L^{2}_{x,v}}^{2} + \|\partial^{m}u_{R}\|_{L^{2}_{x,v}}^{2} + \|\partial^{m}B_{R}\|_{L^{2}_{x,v}}^{2} \right).
\]
For any fixed integer \( N \geq 1 \), the Sobolev embedding \( H^{2}_{x}(T^{3}) \hookrightarrow L^{\infty}_{x}(T^{3}) \) implies that
\[
\|\partial^{m}u_{0}\|_{L^{\infty}_{x,v}}^{2} + \|\partial^{m}B_{0}\|_{L^{\infty}_{x,v}}^{2} \leq C(\|u_{0}\|_{H^{N+2}_{x}}^{2} + \|B_{0}\|_{H^{N+2}_{x}}^{2}) \leq C\varepsilon^{0,N+2} \leq \varepsilon^{0}(N + 2) \]
(5.39)
holds for all multi-index \( |m| \leq N \), where the last two inequalities are guaranteed by Lemma 4.1. Then, we add the inequalities (5.27), and (5.31) to the \( \frac{1}{4} \) times of (5.38) and sum up for \( |m| \leq N \). Then we deduce that by making use of the inequality (5.39)
\[
\frac{1}{2}\|\nabla_{x}u_{R}\|_{H^{2}_{x}}^{2} + \frac{1}{4}\|\nabla_{x}\partial^{m}L^{\perp}G_{R}\|_{H^{2}_{x,v}}^{2} + \frac{1}{8}\|\nabla_{x}\partial^{m}\rho_{R}^{\pm}\|_{H^{2}_{x,v}}^{2} + \frac{1}{8}\|\nabla_{x}\partial^{m}\rho_{R}^{-}\|_{H^{2}_{x,v}}^{2} + \frac{1}{4}\|\text{div}_{x}E_{R}\|_{H^{2}_{x}}^{2}
\]
\[
\leq \varepsilon\frac{d}{dt} \tilde{\mathcal{A}}(t) + C\varepsilon^{2}\left( \frac{1}{4}\|\nabla_{x}u_{R}\|_{H^{2}_{x}}^{2} + \frac{1}{8}\|\nabla_{x}\partial^{m}L^{\perp}G_{R}\|_{H^{2}_{x,v}}^{2} + \frac{1}{8}\|\nabla_{x}\partial^{m}\rho_{R}^{\pm}\|_{H^{2}_{x,v}}^{2} + \frac{1}{8}\|\nabla_{x}\partial^{m}\rho_{R}^{-}\|_{H^{2}_{x,v}}^{2} \right)
\]
(5.40)
\[
\leq \frac{C}{\varepsilon^{2}} \left( 1 + \varepsilon^{2}\mathcal{A}(t) \right) \left( \sum_{|m| \leq N} \|\partial^{m}L^{\perp}G_{R}\|_{L^{2}_{x,v}}^{2} \right) + C\varepsilon^{2} \sum_{|m| \leq N} \left( \|\partial^{m}L^{\perp}G_{R}\|_{L^{2}_{x,v}}^{2} \right) + C\varepsilon^{2} \sum_{|m| \leq N} \left( \|\partial^{m}u_{R}\|_{L^{2}_{x,v}}^{2} + \|\partial^{m}B_{R}\|_{L^{2}_{x,v}}^{2} \right)
\]
\[
+ C\varepsilon^{0,N+2}(\|u_{R}\|_{H^{N+2}_{x}}^{2} + \|B_{R}\|_{H^{N+2}_{x}}^{2})
\]
\[
+ C\varepsilon^{0,N+2}(\|u_{R}\|_{H^{N+2}_{x}}^{2} + \|B_{R}\|_{H^{N+2}_{x}}^{2})
\]
\[
+ C\varepsilon^{0,N+2}(\|u_{R}\|_{H^{N+2}_{x}}^{2} + \|B_{R}\|_{H^{N+2}_{x}}^{2})
\]
where the quantity \( \tilde{\mathcal{A}}(t) \) is defined in (5.8).

One notices that the Poincaré inequality
\[
\|f\|_{L^{2}_{x,v}}^{2} \leq C\|\nabla_{x}f\|_{L^{2}_{x,v}}^{2} + C\left( \int_{T^{3}} fdx \right)^{2}
\]
(5.41)
holds for all \( f(x) \in H^{2}_{x}(T^{3}) \). If we take \( \varepsilon \in (0, \min\{1, \frac{1}{\sqrt{2C}}\}) \) in (5.40) and take \( f = u_{R}, \theta_{R} \) and \( \rho_{R}^{\pm} \) in (5.41) respectively, then the relation (5.40) and the inequality (5.41) imply the inequality (5.7). Consequently, the proof of Lemma 5.1 is finished. \( \Box \)
We remark that the last three terms of (5.7) in Lemma 5.1 require to be estimated. The term $C_1 \varepsilon^2 \sum_{|m| \leq N} \| P_B \partial^m H_R \|^2_{L^2_\ast}$ will be controlled in estimating the mixed derivatives. The term $C_1 E_{0,N+2}^\infty \| B_R \|^2_{H^N}$ can be dominated by finding enough decay structures of the Maxwell equations on $E_R$, $B_R$ in the remainder system (5.1). By analyzing the conservation laws of mass, momentum and energy of the remainder system (5.1), one can give an estimation on the last term including the integral forms in (5.7).

We first estimate the last terms with the integral forms in (5.7). It is easy to be derived from the conservation laws (1.10) and the relations $f_\varepsilon^\pm = M(1 + \varepsilon g_\varepsilon^\pm)$ that

$$
\begin{align*}
\int_{T^3 \times R^3} g_\varepsilon^+ Mdvdx &= 0, \\
\int_{T^3 \times R^3} v(g_\varepsilon^+ + g_\varepsilon^-) Mdv + \int_{T^3} E_\varepsilon \times B_\varepsilon dx &= 0, \\
\int_{T^3 \times R^3} |v|^2 (g_\varepsilon^+ + g_\varepsilon^-) Mdvdx + \varepsilon \int_{T^3} |E_\varepsilon|^2 + |B_\varepsilon|^2 dx &= 0, \\
\int_{T^3} B_\varepsilon dx &= 0.
\end{align*}
$$

(5.42)

Recalling the expansion (2.49), namely

$$
\begin{align*}
g_\varepsilon^\pm &= g_0^\pm + \varepsilon g_1^\pm + \varepsilon^2 g_2^\pm + \varepsilon g_R^\pm, \\
E_\varepsilon &= E_0 + \varepsilon E_1 + \varepsilon E_R, \\
B_\varepsilon &= B_0 + \varepsilon B_1 + \varepsilon B_R,
\end{align*}
$$

(5.43)

the relations (5.42) imply the conservation laws of the leading term $(g_0^\pm, E_0, B_0)$

$$
\begin{align*}
\int_{T^3 \times R^3} g_0^+ Mdvdx &= 0, \\
\int_{T^3 \times R^3} v(g_0^+ + g_0^-) Mdvdx + \int_{T^3} E_0 \times B_0 dx &= 0, \\
\int_{T^3 \times R^3} |v|^2 (g_0^+ + g_0^-) Mdvdx &= 0, \\
\int_{T^3} B_0 dx &= 0,
\end{align*}
$$

(5.44)

and that of the remainder term $(g_R^\pm, E_R, B_R)$

$$
\begin{align*}
\int_{T^3 \times R^3} (g_R^+ + g_R^-) Mdvdx &= 0, \\
\int_{T^3 \times R^3} [v(g_R^+ + g_R^-) + v(\overline{g}_1^+ + \overline{g}_1^-)] Mdvdx \\
&\quad + \int_{T^3} [E_0 \times (\overline{B}_1 + B_R) + (\overline{E}_1 + E_R) \times B_0 + \varepsilon E_R \times B_R] Mdvdx = 0, \\
\int_{T^3 \times R^3} [|v|^2 (g_R^+ + g_R^-) + |v|^2 (\overline{g}_1^+ + \overline{g}_1^-)] Mdvdx \\
&\quad + \int_{T^3} [E_0 + \varepsilon \overline{E}_1 + \varepsilon E_R]^2 + |B_0 + \varepsilon \overline{B}_1 + B_R|^2 dx = 0 \\
\int_{T^3} B_R dx &= 0,
\end{align*}
$$

(5.45)

where we utilize the initial condition (1.15) and the facts

$$
\begin{align*}
\int_{R^3} \overline{g}_2^+ Mdv &= \int_{R^3} |v|^2 (\overline{g}_2^+ + \overline{g}_2^-) Mdv = 0, \\
\int_{R^3} v(\overline{g}_2^+ + \overline{g}_2^-) Mdv &= 0.
\end{align*}
$$

Consequently, combined with the relations

$$
\begin{align*}
\int_{R^3} g_R^+ Mdv &= \rho^+_R, \\
\int_{R^3} g_R^- Mdv &= u_R, \\
\int_{R^3} \frac{|v|^2}{3} - 1 g_R^+ Mdv &= \theta_R, \\
\int_{R^3} \overline{g}_2^+ Mdv &= \overline{\theta}_1 + \frac{1}{2} \overline{\rho}_1, \\
\int_{R^3} v(\overline{g}_1^+ + \overline{g}_1^-) Mdv &= 2 \overline{\rho}_1, \\
\int_{R^3} |v|^2 (\overline{g}_1^+ + \overline{g}_1^-) Mdv &= 12 \overline{\rho}_1,
\end{align*}
$$

The conservation laws (5.44) imply that

$$
\begin{align*}
\int_{T^3} (\rho^+_R + \overline{\theta}_1 + \frac{1}{2} \overline{\rho}_1) dx &= 0, \\
\int_{T^3} (u_R + \overline{\theta}_1) dx + \frac{1}{2} \int_{T^3} [E_0 \times (\overline{B}_1 + B_R) + (\overline{E}_1 + E_R) \times B_0 + \varepsilon E_R \times B_R] dx &= 0, \\
\int_{T^3} (\theta_R + 3 \overline{\theta}_1) dx + \int_{T^3} [E_0 + \varepsilon \overline{E}_1 + \varepsilon E_R]^2 + |B_0 + \varepsilon \overline{B}_1 + B_R|^2 dx &= 0, \\
\int_{T^3} B_R dx &= 0.
\end{align*}
$$
Based on the above conservation laws (5.45), we establish the following lemma to control the terms with integral forms of (5.7) in Lemma 5.1.

**Lemma 5.2.** Assume that \((G_R, E_R, B_R)\) is a sufficiently smooth solution to (5.1) with the initial condition (1.19). Then there is a constant \(C_2 > 0\) such that
\[
\|B_R\|_{L^2_t} \leq C_2 \|\nabla_x B_R\|_{L^2_t},
\]
and
\[
\left( \int_{\mathbb{T}^3} u_R dx \right)^2 + \left( \int_{\mathbb{T}^3} \theta_R dx \right)^2 + \left( \int_{\mathbb{T}^3} \rho_R^+ dx \right)^2 + \left( \int_{\mathbb{T}^3} \rho_R^- dx \right)^2 \\
\leq C_2 \left[ D_{0,2}(t) + D_{1,2}(t) + \varepsilon^2 \left( \|E_R\|^4_{L^2_x} + \|B_R\|^2_{L^2_t} \|\nabla_x B_R\|^2_{L^2_t} \right) \right]
\]
for \(\varepsilon\) small enough, where the quantities \(D_{0,2}(t)\) and \(D_{1,2}(t)\) are mentioned in Lemma 4.1 and 4.2 respectively.

**Proof.** From the zero mean value property of \(B_R\), namely, the last relation of (5.45), the inequality (5.46) is derived from the Poincaré inequality.

Now we justify the inequality (5.47). For the first equation of (5.45), we have
\[
\left( \int_{\mathbb{T}^3} \rho_R^+ dx \right)^2 \leq \left( \int_{\mathbb{T}^3} (\bar{v}_1 + \frac{1}{2} \bar{\pi}_1) dx \right)^2 \leq C(\|\bar{v}_1\|^2_{L^2_t} + \|\bar{\pi}_1\|^2_{L^2_t})
\]
\[\leq \varepsilon \varepsilon_0^{in}(1 + \varepsilon_0^{in}) D_{0,2}(t) + C \varepsilon_1^{in} D_{1,2}(t),\]
where the last inequality is implied by Lemma 4.1 and 4.2. Combined with the conclusions in Lemma 4.1 and 4.2, the second equality of (5.45) reduces to
\[
\left( \int_{\mathbb{T}^3} u_R dx \right)^2 \leq C \left( \int_{\mathbb{T}^3} \bar{\pi}_1 dx \right)^2 + C \left( \int_{\mathbb{T}^3} E_0 \times (\bar{B}_1 + B_R) dx \right)^2
\]
\[+ C \left( \int_{\mathbb{T}^3} (E_1 + E_R) \times B_0 dx \right)^2 + C \varepsilon^2 \left( \int_{\mathbb{T}^3} E_R \times B_R dx \right)^2
\]
\[\leq C \varepsilon_0^{in} \|\bar{v}_1\|^2_{L^2_t} + C \|E_0\|^2_{L^2_t} \left( \|\bar{v}_1\|^2_{L^2_t} + \|\bar{\pi}_1\|^2_{L^2_t} \right)
\[+ C \|B_0\|^2_{L^2_t} \left( \|\bar{v}_1\|^2_{L^2_t} + \|\bar{\pi}_1\|^2_{L^2_t} + \|E_R\|^2_{L^2_t} \right) + C \varepsilon^2 \|E_R\|^2_{L^2_t} \|B_R\|^2_{L^2_t}
\]
\[\leq C D_{1,2}(t) + C \varepsilon^2 \|E_R\|^2_{L^2_t} \|B_R\|^2_{L^2_t}
\[+ C \varepsilon_0^{in} D_{1,2}(t) + \|E_0\|^2_{L^2_t} + \|\nabla_x B_R\|^2_{L^2_t}
\]
\[\leq C(1 + \varepsilon_0^{in}) D_{1,2}(t) + C \varepsilon_0^{in} \|E_R\|^2_{L^2_t} \|B_R\|^2_{L^2_t} + C \varepsilon^2 \|E_R\|^2_{L^2_t} \|B_R\|^2_{L^2_t}
\]
where we utilize the inequality (5.46). One observe that from the third conservation law of energy in (5.45)
\[
\left( \int_{\mathbb{T}^3} \theta_R dx \right)^2 \leq C \left( \int_{\mathbb{T}^3} \bar{\theta}_1 dx \right)^2 + C \left( \|E_0\|^4_{L^2_t} + \|B_0\|^4_{L^2_t} \right)
\[+ C \varepsilon^2 \left( \|\bar{v}_1\|^2_{L^2_t} + \|\bar{\pi}_1\|^2_{L^2_t} + \|E_R\|^2_{L^2_t} + \|B_R\|^2_{L^2_t} \right)
\]
\[\leq C \varepsilon_0^{in} (1 + \varepsilon_0^{in}) D_{0,2}(t) + C \varepsilon^2 (\varepsilon_0^{in} + \varepsilon_1^{in}) D_{1,2}(t)
\[+ C \varepsilon^2 \left( \|E_R\|^2_{L^2_t} + \|B_R\|^2_{L^2_t} \|\nabla_x B_R\|^2_{L^2_t} \right),
\]
where the inequality (5.46), Lemma 4.1, 4.2 and the inequality \(\|B_0\|^4_{L^2_t} \leq C \|\nabla_x B_0\|^2_{L^2_t} \|B_0\|^2_{H^1}\) are utilized. Consequently, the inequalities (5.48), (5.49) and (5.50) imply the estimation (5.47), and the proof of Lemma 5.2 is completed. \(\square\)

Now we estimate the term \(C_1 \varepsilon_0^{in} \|B_R\|^2_{H^N}\). The key point is to find enough dissipation or decay properties on \(B_R\) by making use of the Maxwell equations, hence the last four equations
in \((5.1)\)
\[
\begin{cases}
    \partial_t E_R - \nabla_x \times B_R = -\frac{1}{\varepsilon}(G_R \cdot T_1 v) = -\frac{1}{\varepsilon}(\mathbb{P}^\perp G_R \cdot T_1 v), \\
    \partial_t B_R + \nabla_x \times E_R = 0, \\
    \text{div}_x E_R = \rho_R^+ - \rho_R^-, \quad \text{div}_x B_R = 0.
\end{cases}
\] (5.51)

It is noticed that the second Faraday’s law equation in \((5.51)\) does not have explicit dissipative term. If we take \(\partial_t\) on the evolution of \(B_R\) and combine with the evolution of \(E_R\), we have
\[
\partial_t B_R + \nabla_x \times (\nabla_x \times B_R) = \frac{1}{\varepsilon} \nabla \times (\mathbb{P}^\perp G_R \cdot T_1 v),
\]
which implies that
\[
\partial_t B_R - \Delta_x B_R = \frac{1}{\varepsilon} \nabla \times (\mathbb{P}^\perp G_R \cdot T_1 v)
\] (5.52)
by the equality \(\nabla_x \times (\nabla_x \times B_R) = -\Delta_x B_R\) under the divergence-free property \(\text{div}_x B_R = 0\).

However, the dissipation of \((5.52)\) is remain not enough. We try to derive the Ohm’s law from the microscopic equation of \(G_R\) in \((5.1)\), which will supply a decay term \(\partial_t B_R\). More precisely, we dot with \(T_1\) in the first equation of \((5.1)\), and then we gain
\[
\varepsilon \partial_t (G_R \cdot T_1) + v \cdot \nabla_x (G_R \cdot T_1) + (v \times B_0) \cdot \nabla_v (G_R \cdot T_2) - 2E_R \cdot v
\]
\[
+ (v \times B_R) \cdot \nabla_v (G_0 \cdot T_2) + \frac{1}{\varepsilon} (L + \mathcal{L})(G_R \cdot T_1) - \varepsilon H_R \cdot T_1 = 0,
\] (5.53)
where \(T_2 = \left(\begin{array}{c} 1 \\ 1 \end{array}\right)\) and we make use of the relation
\[
\mathbb{P} G_R \cdot T_1 = (L + \mathcal{L})(G_R \cdot T_1).
\]

Recalling the property \((2.16)\) of the operator \(L + \mathcal{L}\) in Section 2, we know that for \(\Phi(v) = v \in L_2^2\) there is a unique function \(\tilde{\Phi}(v) \in \ker^\perp(L + \mathcal{L})\) such that
\[
(L + \mathcal{L})\tilde{\Phi} = \Phi.
\]
We multiply by \(\tilde{\Phi}(v) M\) in \((5.53)\) and integrate over \(v \in \mathbb{R}^3\), then
\[
\frac{1}{\varepsilon} (G_R \cdot T_1 \tilde{\Phi}(v)) = -\varepsilon \partial_t (G_R \cdot T_1 \tilde{\Phi}(v)) - \langle v \cdot \nabla_x (G_R \cdot T_1 \tilde{\Phi}(v)) \rangle
\]
\[
+ \sigma E_R - \langle (v \times B_0) \cdot \nabla_v (G_R \cdot T_2) \tilde{\Phi}(v) \rangle
\]
\[
- \langle (v \times B_R) \cdot \nabla_v (G_0 \cdot T_2) \tilde{\Phi}(v) \rangle + \varepsilon \langle H_R \cdot T_1 \tilde{\Phi}(v) \rangle.
\] (5.54)

By the following equalities
\[
\langle G_R \cdot T_1 \tilde{\Phi}(v) \rangle = \langle \mathbb{P}^\perp G_R \cdot T_1 \tilde{\Phi}(v) \rangle,
\]
\[
\langle v \cdot \nabla_x (G_R \cdot T_1) \tilde{\Phi}(v) \rangle = \langle v \cdot \nabla_x (\mathbb{P}^\perp G_R \cdot T_1) \tilde{\Phi}(v) \rangle + \frac{1}{2} \sigma \nabla_x (\rho_R^+ - \rho_R^-),
\]
\[
\langle (v \times B_0) \cdot \nabla_v (G_R \cdot T_2) \tilde{\Phi}(v) \rangle = -\langle (v \times B_0) (\mathbb{P}^\perp G_R \cdot T_2) \cdot \nabla_v \tilde{\Phi}(v) \rangle - \sigma u_R \times B_0,
\]
\[
\langle (v \times B_R) \cdot \nabla_v (G_0 \cdot T_2) \tilde{\Phi}(v) \rangle = -\sigma u_0 \times B_R,
\]
we derive from \((5.54)\) that
\[
\frac{1}{\varepsilon} (G_R \cdot T_1 \tilde{\Phi}(v)) = -\varepsilon \partial_t (\mathbb{P}^\perp G_R \cdot T_1 \tilde{\Phi}(v)) - \langle v \cdot \nabla_x (\mathbb{P}^\perp G_R \cdot T_1) \tilde{\Phi}(v) \rangle
\]
\[
+ \langle (v \times B_0) (\mathbb{P}^\perp G_R \cdot T_2) \cdot \nabla_v \tilde{\Phi}(v) \rangle + \varepsilon \langle H_R \cdot T_1 \tilde{\Phi}(v) \rangle
\]
\[
+ \sigma E_R - \frac{1}{2} \sigma \nabla_x (\rho_R^+ - \rho_R^-) + \sigma (u_R \times B_0 + u_0 \times B_R)
\]
\[
:= \sigma E_R - \frac{1}{2} \sigma \nabla_x (\rho_R^+ - \rho_R^-) + \mathcal{K}(\mathbb{P}^\perp G_R).
\] (5.55)

Then, by substituting \((5.55)\) into \((5.52)\) we deduce that
\[
\partial_t B_R - \Delta_x B_R + \sigma \partial_t B_R = \nabla_x \times \mathcal{K}(\mathbb{P}^\perp G_R),
\] (5.56)
where we use the Faraday’s law equation $\partial_t B_R + \nabla_x \times E_R = 0$ and $\nabla_x \times (\nabla_x f) = 0$ for any function $f(x)$. So, we have found the decay term $\partial_t B_R$ of $B_R$-equation.

Based on the equation (5.56), we construct the following lemma.

** Lemma 5.3.** Assume that $(G_R, E_R, B_R)$ is a sufficiently smooth solution to (5.1) with the initial condition (1.19). If there is a small constant $\lambda_R(N+2) \in (0, \lambda_1(N+2)]$, depending only on $N$, $\sigma$, $\mu$, $\kappa$, such that

$$\mathcal{E}_0^{in} \leq \lambda_R(N+2),$$

then the inequality

$$\frac{d}{dt} \left( \|E_R\|_{H^{N-1}_x}^2 + (1 - \delta + \sigma \|) B_R \|_{H^{N-1}_x}^2 + \|\nabla_x B_R\|_{H^{N-1}_x}^2 + (1 - \delta) \|\partial_t B_R\|_{H^{N-1}_x}^2 \right)
\leq C_3 \varepsilon \frac{d}{dt} \sum_{|m| \leq N-1} \int_{\Omega_1} \langle \partial^m P \partial_1 F(v) \rangle \cdot (\partial^m E_R - \delta \nabla x \times \partial^m B_R) dx$$

holds for $\varepsilon$ sufficiently small and $C_3 > 0$, depending only on $N$, $\sigma$, $\mu$ and $\kappa$.

**Proof.** By acting the operator $\partial^m$ on (5.56) for $|m| \leq N-1$, taking the $L^2_2$-inner product by dot with $\partial_t \partial^m B_R$ and integrating by parts over $x \in \Omega_1$, we deduce that

$$\frac{1}{2} \frac{d}{dt} \left( \|\partial_t \partial^m B_R\|_{L^2_2}^2 + \|\nabla_x \partial^m B_R\|_{L^2_2}^2 + \sigma \|\partial_t \partial^m B_R\|_{L^2_2}^2 \right)
= \int_{\Omega_1} \nabla_x \cdot \mathcal{K}(\partial^m G_R) \cdot \partial_t \partial^m B_R dx .$$

If we replace the multiplied vector $\partial_t \partial^m B_R$ with the vector $\partial^m B_R$ in the above procedure, we have

$$\frac{1}{2} \frac{d}{dt} \left( \|\partial_t \partial^m B_R + \partial^m B_R\|_{L^2_2}^2 - \|\partial_t \partial^m B_R\|_{L^2_2}^2 \right)
\leq \frac{1}{2} \frac{d}{dt} \left( \|\partial_t \partial^m B_R\|_{L^2_2}^2 + \|\nabla_x \partial^m B_R\|_{L^2_2}^2 \right)
= \int_{\Omega_1} \nabla_x \cdot \mathcal{K}(\partial^m G_R) \cdot \partial^m B_R dx .$$

where we utilize the relation

$$\int_{\Omega_1} \partial^m B_R \cdot \partial^m B_R dx$$

Now, we take $\partial^m$ on the first two equations of (5.51), dot with $\partial^m E_R$ and $\partial^m B_R$ respectively, and integrate by parts over $x \in \Omega_1$, where $|m| \leq N-1$. Then we derive that from the relation (5.55)

$$\frac{1}{2} \frac{d}{dt} (\|\partial^m E_R\|_{L^2_2}^2 + \|\partial^m B_R\|_{L^2_2}^2) = -\frac{1}{\varepsilon} \int_{\Omega_1} \partial^m (G_R \cdot \mathcal{T}_1 v) : \partial^m E_R dx$$

$$= -\sigma \|\partial^m E_R\|_{L^2_2}^2 + \frac{1}{2} \sigma \int_{\Omega_1} \nabla_x \partial^m (\rho_R^+ - \rho_R^-) \cdot \partial^m E_R dx - \int_{\Omega_1} \mu \partial^m \mathcal{K}(\partial^m G_R) : \partial^m E_R dx .$$
Recalling $\text{div}_x E_R = \rho_R^+ - \rho_R^-$, we have
\[
\int_T \nabla_x \partial^m (\rho_R^+ - \rho_R^-) \cdot \partial^m E_R \, dx = -\|\text{div}_x \partial^m E_R\|_{L^2}^2.
\]
Then, the equality (5.60) implies
\[
\frac{1}{2} \int T^3 \langle \partial^m E_R \|_{L^2}^2 + \|\partial^m B_R\|_{L^2}^2 \rangle + \sigma \|\partial^m E_R\|_{L^2}^2 + \frac{1}{2}\sigma \|\text{div}_x \partial^m E_R\|_{L^2}^2
\]
\[
= \int T^3 \partial^m K(\mathbb{P} \perp G_R) \cdot \partial^m E_R \, dx .
\]
(5.61)
We choose a number $\delta = \frac{1}{2} \min\{1, \sigma\} \in (0, \frac{1}{2}]$. Then the equality
\[
\frac{1}{2} \int T^3 \langle \|E_R\|_{H^{N-1}_x}^2 + (1 - \delta + \sigma \delta) \|B_R\|_{H^{N-1}_x}^2 + \|\nabla_x B_R\|_{H^{N-1}_x}^2 + (1 - \delta) \|\partial_t B_R\|_{H^{N-1}_x}^2 + \delta \|\partial_t B_R + B_R\|_{H^{N-1}_x}^2 \rangle
\]
\[
+ \sigma \langle \|E_R\|_{H^{N-1}_x}^2 + (\sigma - \delta) \|\partial_t B_R\|_{H^{N-1}_x}^2 + 1 \|\partial^m B_R\|_{H^{N-1}_x}^2 \rangle
\]
\[
\sum_{|m| \leq N-1} \int T^3 \nabla_x \times \partial^m K(\mathbb{P} \perp G_R) \cdot \partial_t \partial^m B_R \, dx
\]
\[
+ \sum_{|m| \leq N-1} \int T^3 \partial^m K(\mathbb{P} \perp G_R) \cdot (\delta \nabla_x \times \partial^m B_R - \partial^m E_R) \, dx
\]
\[
= I_{N-1} + II_{N-1}
\]
is derived from multiplying by $\delta$ in (5.59) and adding it and (5.61) to the equality (5.58).
For the term $I_{N-1}$ in (5.62), together with the relation (5.55), on calculates that
\[
I_{N-1} = -\varepsilon \sum_{|m| \leq N-1} \int T^3 \partial_t \nabla_x \times \langle \partial^m \mathbb{P} \perp G_R \cdot \partial_t \Phi(v) \rangle \cdot \partial_t \partial^m B_R \, dx
\]
\[
- \sum_{|m| \leq N-1} \int T^3 \nabla_x \times \langle v \cdot \nabla_x (\mathbb{P} \perp G_R \cdot \partial_t \Phi(v)) \rangle \cdot \partial_t \partial^m B_R \, dx
\]
\[
+ \sum_{|m| \leq N-1} \nabla_x \times \langle \partial^m (v \times B_0) \cdot \nabla_v \Phi(v) \mathbb{P} \perp G_R \cdot \partial_t \partial^m B_R \, dx
\]
\[
+ \varepsilon \sum_{|m| \leq N-1} \int T^3 \nabla_x \times \langle \partial^m H_R \cdot \partial_t \Phi(v) \rangle \cdot \partial_t \partial^m B_R \, dx
\]
\[
+ \sigma \sum_{|m| \leq N-1} \int T^3 \partial^m (u R \times B_0 + u_0 \times B_R) \cdot \partial_t \partial^m B_R \, dx
\]
\[
\leq -\varepsilon \sum_{|m| \leq N-1} \int T^3 \nabla_x \times \partial_t \langle \partial^m \mathbb{P} \perp G_R \cdot \partial_t \Phi(v) \rangle \cdot \partial_t \partial^m B_R \, dx
\]
\[
+ C \sum_{|m| \leq N+1} \|\partial^m \mathbb{P} \perp G_R\|_{L^2,v} \|\partial_t B_R\|_{H^{N-1}_x} + C \varepsilon \sum_{|m| \leq N} \|\partial^m H_R\|_{L^2,v} \|\partial_t B_R\|_{H^{N-1}_x}
\]
\[
+ C \sum_{|m| \leq N} \sum_{m' \leq m} \|\partial^m B_0\|_{L^\infty} \|\partial^m \mathbb{P} \perp G_R\|_{L^2,v} \|\partial_t B_R\|_{H^{N-1}_x}
\]
\[
+ C \sum_{|m| \leq N-1} \sum_{m' \leq m} \langle \|\partial^m u_0\|_{L^2} \|\partial^m B_0\|_{L^2} \|\partial^m u R\|_{L^2} \|\partial_t B_R\|_{H^{N-1}_x} \rangle
\]
\[
\leq -\varepsilon \sum_{|m| \leq N-1} \int T^3 \nabla_x \times \partial_t \langle \partial^m \mathbb{P} \perp G_R \cdot \partial_t \Phi(v) \rangle \cdot \partial_t \partial^m B_R \, dx
\]
(5.63)
\[ + C(1 + \|B_0\|_{H^{N+2}}) \sum_{|m| \leq N+1} \|\partial^m \mathbb{P}^\perp G_R\|_{L^2_{x,v}} \|\partial_t B_R\|_{H^{N-1}} \]
\[ + C\varepsilon \sum_{|m| \leq N} \|\partial^m H_R\|_{L^2_{x,v}} \|\partial_t B_R\|_{H^{N-1}} \]
\[ + C\varepsilon (\|u_0\|_{H^{N+1}} + \|B_0\|_{H^{N+2}}) (\|u_R\|_{H^{N-1}} + \|B_R\|_{H^{N-1}}) \|\partial_t B_R\|_{H^{N-1}}, \]

where we use the Sobolev embedding \( H^2(T^3) \hookrightarrow L^\infty(T^3) \).

We now estimate the first term of the right-hand side of the last inequality in (5.63). First of all, we derive from H"older inequality and integration by parts over \( x \in T^3 \) that
\[ - \varepsilon \sum_{|m| \leq N-1} \int_{T^3} \nabla_x \times \partial_t \langle \partial^m \mathbb{P}^\perp G_R \cdot \mathcal{T}_1 \tilde{\Phi}(v) \rangle \cdot \partial_t \partial^m B_R dx \]
\[ = - \frac{d}{dt} \sum_{|m| \leq N-1} \int_{T^3} \nabla_x \times \langle \partial^m \mathbb{P}^\perp G_R \cdot \mathcal{T}_1 \tilde{\Phi}(v) \rangle \cdot \partial_t \partial^m B_R dx \]
\[ + \varepsilon \sum_{|m| \leq N-1} \int_{T^3} \nabla_x \times \langle \partial^m \mathbb{P}^\perp G_R \cdot \mathcal{T}_1 \tilde{\Phi}(v) \rangle \]
\[ \cdot \|\partial_x \partial^m B_R - \sigma \partial_t \partial^m B_R + \nabla_x \times \partial^m \mathcal{K}(\mathbb{P}^\perp G_R)\| dx, \]

which will be estimated term by term. First of all, we derive from H"older inequality and integration by parts over \( x \in T^3 \) that
\[ \varepsilon \sum_{|m| \leq N-1} \int_{T^3} \nabla_x \times \langle \partial^m \mathbb{P}^\perp G_R \cdot \mathcal{T}_1 \tilde{\Phi}(v) \rangle \cdot \partial_x \partial^m B_R dx \]
\[ = - \varepsilon \sum_{|m| \leq N-1} \int_{T^3} \nabla_x (\nabla_x \times \langle \partial^m \mathbb{P}^\perp G_R \cdot \mathcal{T}_1 \tilde{\Phi}(v) \rangle) : \nabla_x \partial^m B_R dx \]
\[ \leq C\varepsilon \sum_{|m| \leq N+1} \|\partial^m \mathbb{P}^\perp G_R\|_{L^2_{x,v}} \|\nabla_x B_R\|_{H^{N-1}}, \]

and similarly
\[ \varepsilon \sum_{|m| \leq N-1} \int_{T^3} \nabla_x \times \langle \partial^m \mathbb{P}^\perp G_R \cdot \mathcal{T}_1 \tilde{\Phi}(v) \rangle \cdot (-\sigma \partial_t \partial^m B_R) dx \]
\[ \leq C\varepsilon \sum_{|m| \leq N} \|\partial^m \mathbb{P}^\perp G_R\|_{L^2_{x,v}} \|\partial_t B_R\|_{H^{N-1}}. \]

Finally, together with (5.55), the similar estimations in (5.63) reduces to
\[ \varepsilon \sum_{|m| \leq N-1} \int_{T^3} \nabla_x \times \langle \partial^m \mathbb{P}^\perp G_R \cdot \mathcal{T}_1 \tilde{\Phi}(v) \rangle \cdot \nabla_x \times \partial^m \mathcal{K}(\mathbb{P}^\perp G_R) dx \]
\[ \leq - \frac{\varepsilon^2}{2} \frac{d}{dt} \sum_{|m| \leq N-1} \|\nabla_x \times \langle \partial^m \mathbb{P}^\perp G_R \cdot \mathcal{T}_1 \tilde{\Phi}(v) \rangle\|^2_{L^2_x} \]
\[ + C\varepsilon (1 + \|B_0\|_{H^{N+2}}) \sum_{|m| \leq N+1} \|\partial^m \mathbb{P}^\perp G_R\|^2_{L^2_{x,v}} + C\varepsilon^4 \sum_{|m| \leq N} \|\partial^m H_R\|^2_{L^2_{x,v}} \]
\[ + C\varepsilon (\|u_0\|_{H^{N+2}} + \|B_0\|_{H^{N+2}}) (\|B_R\|_{H^{N}} + \|u_R\|_{H^{N}}). \]

Then, plugging the inequalities (5.65), (5.66) and (5.67) into (5.64), we deduce that
\[ - \varepsilon \sum_{|m| \leq N-1} \int_{T^3} \nabla_x \times \partial_t \langle \partial^m \mathbb{P}^\perp G_R \cdot \mathcal{T}_1 \tilde{\Phi}(v) \rangle \cdot \partial_t \partial^m B_R dx \]
\[ \leq -\frac{\varepsilon^2}{2} \frac{d}{dt} \sum_{|m| \leq N-1} \| \nabla_x \times \langle \partial^m \mathbb{P} \cdot \mathcal{T}_1 \bar{\Phi}(v) \rangle \|_{L_x^2}^2 \]

\[ + C\varepsilon(1 + \lambda_0^2(N + 2)) \sum_{|m| \leq N+1} \| \partial^m F_R \|_{L_x^2, v}^2 + C\varepsilon^4 \sum_{|m| \leq N} \| \partial^m H_R \|_{L_x^2, v}^2 \] (5.68)

\[ + C\varepsilon(\| \partial_1 B_R \|_{H_x^{N-1}}^2 + \| \nabla_2 B_R \|_{H_x^{N-1}}^2) + C\varepsilon \lambda_0(N + 2)(\| B_R \|_{H_x^N}^2 + \| u_R \|_{H_x^N}^2), \]

where we make use of the inequality (4.6) in Lemma 4.1. Therefore, substituting (5.68) into (5.63) reduces to

\[ I_{N-1} \leq -\frac{\varepsilon^2}{2} \frac{d}{dt} \sum_{|m| \leq N-1} \| \nabla_x \times \langle \partial^m \mathbb{P} \cdot \mathcal{T}_1 \bar{\Phi}(v) \rangle \|_{L_x^2}^2 \]

\[ + \frac{C}{\varepsilon} \sum_{|m| \leq N+1} \| \partial^m F_R \|_{L_x^2, v}^2 + C\varepsilon^4 \sum_{|m| \leq N} \| \partial^m H_R \|_{L_x^2, v}^2 \] (5.69)

\[ + C(\varepsilon + \xi + \sqrt{\varepsilon_{0, N+2}^m} (\| \partial_1 B_R \|_{H_x^{N-1}}^2 + \| \nabla_2 B_R \|_{H_x^{N-1}}^2) + C\varepsilon_{0, N+2} \| u_R \|_{H_x^N}, \]

where \( \xi > 0 \) is sufficiently small to be determined and we utilize the inequality (5.46) in Lemma 5.2.

It remains to estimate the term \( II_{N-1} \) in (5.62). By the relation (5.56), we calculate that by using Hölder inequality, Sobolev theory and Young's inequality

\[ II_{N-1} \leq -\varepsilon \sum_{|m| \leq N-1} \int_{T^3} \partial_t \langle \partial^m \mathbb{P} \cdot \mathcal{T}_1 \bar{\Phi}(v) \rangle \cdot (\delta \nabla_x \times \partial^m B_R - \partial^m E_R) dx \]

\[ - \sum_{|m| \leq N-1} \int_{T^3} \langle v \cdot \nabla_x \partial^m F_R \cdot \mathcal{T}_1 \bar{\Phi}(v) \rangle \cdot (\delta \nabla_x \times \partial^m B_R - \partial^m E_R) dx \]

\[ + \sum_{|m| \leq N-1} \int_{T^3} \langle (v \cdot B_0) \cdot \nabla \bar{\Phi}(v) \partial^m G_R \cdot \mathcal{T}_2 \rangle \cdot (\delta \nabla_x \times \partial^m B_R - \partial^m E_R) dx \]

\[ + \varepsilon \sum_{|m| \leq N-1} \int_{T^3} \langle \partial^m H_R \cdot \mathcal{T}_1 \bar{\Phi}(v) \rangle \cdot (\delta \nabla_x \times \partial^m B_R - \partial^m E_R) dx \]

\[ + \sigma \sum_{|m| \leq N-1} \int_{T^3} \partial^m (u_R \times B_0 + u_0 \times B_R) \cdot (\delta \nabla_x \times \partial^m B_R - \partial^m E_R) dx \]

\[ \leq -\varepsilon \sum_{|m| \leq N-1} \int_{T^3} \partial_t \langle \partial^m \mathbb{P} \cdot \mathcal{T}_1 \bar{\Phi}(v) \rangle \cdot (\delta \nabla_x \times \partial^m B_R - \partial^m E_R) dx \]

\[ + C \sum_{|m| \leq N} \| \partial^m \mathbb{P} \cdot \mathbb{P} \cdot G_R \|_{L_x^2, v} (\| \nabla_1 B_R \|_{H_x^{N-1}} + \| E_R \|_{H_x^{N-1}}) \]

\[ + C \sum_{|m| \leq N-1} \| \partial^m B_0 \|_{L_x^\infty} \| \partial^{m'} \mathbb{P} \cdot \mathbb{P} \cdot G_R \|_{L_x^2, v} (\| \nabla_1 B_R \|_{H_x^{N-1}} + \| E_R \|_{H_x^{N-1}}) \]

\[ + C\varepsilon \sum_{|m| \leq N-1} \| \partial^m H_R \|_{L_x^2, v} (\| \nabla_1 B_R \|_{H_x^{N-1}} + \| E_R \|_{H_x^{N-1}}) \]

\[ + C \sum_{|m| \leq N-1} \| \partial^m B_0 \|_{L_x^\infty} \| \partial^{m'} \mathbb{P} \cdot \mathbb{P} \cdot u_R \|_{L_x^2} + \| \partial^m B_0 \|_{L_x^\infty} \| \partial^{m'} \mathbb{P} \cdot B_R \|_{L_x^2} \]

\[ \times (\| \nabla_1 B_R \|_{H_x^{N-1}} + \| E_R \|_{H_x^{N-1}}) \]
where the last term in (5.70) is 

\[
\sum_{|m| \leq N-1} \int_{\mathbb{T}^3} \partial_t \langle \partial^m \mathbb{P} \cdot G_R \cdot T\Phi(v) \rangle \cdot (\delta \nabla_x \times \partial^m B_R - \partial^m E_R) dx 
\]

where the constant \(\xi > 0\) is sufficiently small to be determined.

For the first term of the right-hand side of the last inequality of (5.70), we have

\[
-\varepsilon \sum_{|m| \leq N-1} \int_{\mathbb{T}^3} \partial_t \langle \partial^m \mathbb{P} \cdot G_R \cdot T\Phi(v) \rangle \cdot (\delta \nabla_x \times \partial^m B_R - \partial^m E_R) dx 
\]

\[
= \varepsilon \frac{d}{dt} \sum_{|m| \leq N-1} \int_{\mathbb{T}^3} \langle \partial^m \mathbb{P} \cdot G_R \cdot T\Phi(v) \rangle \cdot (\delta \nabla_x \times \partial^m B_R - \partial^m E_R) dx 
\]

\[
+ \varepsilon \sum_{|m| \leq N-1} \int_{\mathbb{T}^3} \langle \partial^m \mathbb{P} \cdot G_R \cdot T\Phi(v) \rangle \cdot (\delta \nabla_x \times \partial_t \partial^m B_R - \partial_t \partial^m E_R) dx ,
\]

where the last term in (5.71) can be estimated as

\[
\varepsilon \sum_{|m| \leq N-1} \int_{\mathbb{T}^3} \langle \partial^m \mathbb{P} \cdot G_R \cdot T\Phi(v) \rangle \cdot (\delta \nabla_x \times \partial_t \partial^m B_R - \partial_t \partial^m E_R) dx 
\]

\[
= - \varepsilon \delta \sum_{|m| \leq N-1} \int_{\mathbb{T}^3} \nabla_x \times [\nabla_x \times \langle \partial^m \mathbb{P} \cdot G_R \cdot T\Phi(v) \rangle] \cdot \partial^m E_R dx 
\]

\[
- \varepsilon \sum_{|m| \leq N-1} \int_{\mathbb{T}^3} \langle \partial^m \mathbb{P} \cdot G_R \cdot T\Phi(v) \rangle \cdot \nabla_x \times \partial^m B_R dx 
\]

\[
+ \sum_{|m| \leq N-1} \int_{\mathbb{T}^3} \langle \partial^m \mathbb{P} \cdot G_R \cdot T\Phi(v) \rangle \cdot \langle \partial^m \mathbb{P} \cdot G_R \cdot T\Phi(v) \rangle dx 
\]

\[
\leq C \varepsilon \sum_{|m| \leq N} \|\partial^m \mathbb{P} \cdot G_R\|_{L^2_v} (\|\nabla_x B_R\|_{H^N_{x-1}} + \|E_R\|_{H^N_{x-1}}) 
\]

\[
+ C \sum_{|m| \leq N} \|\partial^m \mathbb{P} \cdot G_R\|_{L^2_v}^2 ,
\]

where we make use of the relation (5.51) and the equality

\[
\int_{\mathbb{T}^3} A \cdot \nabla_x \times (\nabla_x \times B) dx = \int_{\mathbb{T}^3} \nabla_x \times (\nabla_x \times A) \cdot B dx .
\]

Consequently, by collecting the inequalities (5.70), (5.71) and (5.72), we have

\[
\sum_{|m| \leq N-1} \int_{\mathbb{T}^3} \langle \partial^m \mathbb{P} \cdot G_R \cdot T\Phi(v) \rangle \cdot (\delta \nabla_x \times \partial^m B_R - \partial^m E_R) dx 
\]

\[
+ \frac{C}{\varepsilon^2} \sum_{|m| \leq N} \|\partial^m \mathbb{P} \cdot G_R\|_{L^2_v}^2 + C(\xi) \varepsilon^2 \sum_{|m| \leq N} \|\partial^m H_R\|_{L^2_v}^2 
\]

\[
+ (\varepsilon^2 + \xi + C\sqrt{\varepsilon_{0,N+2}(\|\nabla_x B_R\|_{H^N_{x-1}}^2 + \|E_R\|_{H^N_{x-1}}^2)} + C\sqrt{\varepsilon_{0,N+2}(\|u_R\|_{H^N_{x-1}}^2} .
\]

Together with (5.62), (5.69) and (5.73), we deduce that

\[
\frac{1}{2} \frac{d}{dt} \|E_R\|_{H^N_{x-1}}^2 + (1 - \delta + \sigma \delta) \|B_R\|_{H^N_{x-1}}^2 + \|\nabla_x B_R\|_{H^N_{x-1}}^2 + (1 - \delta) \|\partial_t B_R\|_{H^N_{x-1}}^2
\]
Lemma 5.4. Let \( N \geq 2 \) and \((G_R, E_R, B_R)\) be a sufficiently smooth solution to the remainder system (5.1) with the initial conditions (1.19). If there is a small constant \( \lambda_R(N+2) \in (0, \lambda_1(N + 2)] \), depending only on \( N \), \( \sigma \), \( \mu \), \( \kappa \), such that
\[
\mathcal{E}_{0,N+2}^{in} \leq \lambda_R(N + 2),
\]
then there exist some constants \( C_4 > 1 \), \( C_5, C_6 \), \( C_7 > 0 \), depending only on \( N \), \( \sigma \), \( \mu \), \( \kappa \), such that
\[
\begin{align*}
\frac{d}{dt} \left[ \|B_R\|_{H^N_x}^2 + (1 - \delta + \sigma \delta)\|B_R\|_{H^{N-1}_x}^2 + (1 - \delta)\|\partial_t B_R\|_{H^{N-1}_x}^2 + \delta\|\partial_t B_R + B_R\|_{H^{N-1}_x}^2 \right] - C_3\varepsilon A_N(t) + \epsilon^2 \sum_{|m| \leq N-1} \left\| \nabla_x \times \left( \partial^m \mathbb{P} \frac{1}{2} G_R \cdot \tilde{T}_1 \tilde{\Phi}(v) \right) \right\|_{L^2_x}^2 \\
+ C_4 \left( \|E_R\|_{H^N_x}^2 + \|B_R\|_{H^{N+1}_x}^2 + \sum_{|m| \leq N-1} \left\| \partial^m G_R \right\|_{L^2_{x,v}}^2 \right) + C_5 \left( \mathcal{E}_{1,N}(t) + \tilde{C}_N \mathcal{E}_{0,N+2}(t) \right) \\
+ C_6 \left( \|E_R\|_{H^N_x}^2 + \|\nabla_x B_R\|_{H^{N-1}_x}^2 + \|\partial_t B_R\|_{H^{N-1}_x}^2 + D_{0,N+1}(t) \right) \\
+ \|\text{div}_x E_R\|_{H^N_x}^2 + \sum_{|m| \leq N+1} \left\| \partial^m \mathbb{P} G_R \right\|_{L^2_{x,v}}^2 + \frac{1}{\epsilon^2} \sum_{|m| \leq N+1} \left\| \partial^m \mathbb{P} G_R \right\|_{L^2_{x,v}(\nu)}^2 \right) \\
\leq C_7 \left[ \epsilon^2 \sum_{|m| \leq N} \left\| \partial^m H_R \right\|_{L^2_{x,v}}^2 + \epsilon^2 \sum_{|m| \leq N} \left\| \mathbb{P}_\mathbb{B} \partial^m H_R \right\|_{L^2_{x,v}}^2 + \|B_R\|_{H^{N+1}_x}^2 D_{0,N+2}(t) \right]
\end{align*}
\]
\[ + \varepsilon_0^{m_{N+2}} \sum_{|m| \leq N} \| \nabla_v \partial^m \mathbb{P}^{-1} G_R \|_{L_2^x,v}^2 + \varepsilon^2 (\| E_R \|_{L_2^x}^4 + \| B_R \|_{L_2^x}^2 \| \nabla_x B_R \|_{L_2^x}^2) \]

\[ + \sum_{|m| \leq N+1} \int_{\mathbb{T}^3} \langle \partial^m H_R \cdot \partial^m G_R \rangle dx \]

for \( \varepsilon \) sufficiently small, where the constant \( C_3 > 0 \) is mentioned in Lemma 5.3 and the quantity \( A_N(t) \) is defined as

\[ A_N(t) = \tilde{A}_N(t) + \sum_{|m| \leq N-1} \int_{\mathbb{T}^3} (\partial^m G_R \cdot \mathcal{T}_1 \Phi(v)) \cdot (\partial^m E_R - \delta \nabla_x \times \partial^m B_R) . \]  

Here the quantity \( \tilde{A}_N(t) \) is given as (5.8) in Lemma 5.1.

Proof. For \( |m| \leq N + 1 \), we take \( \partial^m \) on the first equation of (5.1), and then we gain

\[ \partial_t \partial^m G_R + \frac{1}{\varepsilon} v \cdot \nabla_x \partial^m G_R - \frac{1}{\varepsilon} (\partial^m E_R \cdot v) \mathcal{T}_1 + \frac{1}{\varepsilon} \mathcal{T}(v \times B_0) \cdot \nabla_v \partial^m G_R + \frac{1}{\varepsilon^2} \mathcal{L} \partial^m G_R \]

\[ = \frac{1}{\varepsilon} \sum_{0 \neq m' \leq m} C^m_{m'} \mathcal{T}(v \times \partial^{m'} B_0) \cdot \nabla_v \partial^{m-m'} G_R \]

\[ - \frac{1}{\varepsilon} \partial^m [\mathcal{T}(v \times B_R) \cdot \nabla_v G_0] + \partial^m H_R . \]  

Taking \( L_2^2,v \)-inner product by dot with \( \partial^m G_R \) in (5.80) reduces to

\[ \frac{1}{\varepsilon^2} \frac{d}{dt} \| \partial^m G_R \|_{L_2^2,v}^2 - \int_{\mathbb{T}^3} \partial^m E_R \cdot \frac{1}{\varepsilon} (\partial^m G_R \cdot \mathcal{T}_1 v) dx + \frac{1}{\varepsilon^2} \int_{\mathbb{T}^3} (\mathcal{L} \partial^m G_R \cdot \partial^m G_R) dx \]

\[ = - \frac{1}{\varepsilon} \sum_{0 \neq m' \leq m} C^{m'}_m \int_{\mathbb{T}^3} \langle [\mathcal{T}(v \times \partial^{m'} B_0) \cdot \nabla_v \partial^{m-m'} G_R] \cdot \partial^m G_R \rangle dx \]

\[ - \frac{1}{\varepsilon} \int_{\mathbb{T}^3} \langle \partial^m [\mathcal{T}(v \times B_R) \cdot \nabla_v G_0] \cdot \partial^m G_R \rangle dx + \int_{\mathbb{T}^3} \langle \partial^m H_R \cdot \partial^m G_R \rangle dx . \]

Recalling the Maxwell equations of (5.1), we have

\[ - \int_{\mathbb{T}^3} \partial^m E_R \cdot \frac{1}{\varepsilon} (\partial^m G_R \cdot \mathcal{T}_1 v) dx \]

\[ = \int_{\mathbb{T}^3} \partial^m E_R \cdot (\partial_t \partial^m E_R - \nabla_x \times \partial^m B_R) dx \]

\[ = \frac{1}{\varepsilon^2} \frac{d}{dt} \| \partial^m E_R \|_{L_2^2}^2 - \int_{\mathbb{T}^3} (\nabla_x \times \partial^m E_R) \cdot \partial^m B_R dx \]

\[ = \frac{1}{\varepsilon^2} \frac{d}{dt} \| \partial^m E_R \|_{L_2^2}^2 + \int_{\mathbb{T}^3} \partial_t \partial^m B_R \cdot \partial^m B_R dx \]

\[ = \frac{1}{\varepsilon^2} \frac{d}{dt} (\| \partial^m E_R \|_{L_2^2}^2 + \| \partial^m B_R \|_{L_2^2}^2) , \]

where the second equality is implied by the relation

\[ \int_{\mathbb{T}^3} A \cdot (\nabla_x \times B) dx = \int_{\mathbb{T}^3} (\nabla_x \times A) \cdot B dx . \]

By the properties of the projection operator \( \mathcal{L} \) in Lemma 3.1, we know that there is a constant \( \Lambda > 0 \) such that

\[ \frac{1}{\varepsilon^2} \int_{\mathbb{T}^3} \langle \mathcal{L} \partial^m G_R \cdot \partial^m G_R \rangle dx \geq \frac{\Lambda}{\varepsilon^2} \| \partial^m \mathbb{P}^{-1} G_R \|_{L_2^2,v}^2 (v) . \]

(5.83)
Since \( G_0 = \left( \frac{g_0^+}{g_0^-} \right) \) with \( g_0^+ = \rho_0^+ + u_0 \cdot v + \theta_0 \left( \frac{|v|^2}{2} - \frac{3}{2} \right) \) and \( \mathbb{P} G_R \) is of the form (1.24), direct calculations imply that

\[
\left\langle \left[ \mathcal{T}(v \times \partial^{m-m'} B_R) \cdot \nabla_v \partial^{m'} G_0 \right] \cdot \partial^m \mathbb{P} G_R \right\rangle = 0. \tag{5.84}
\]

Splitting \( G_R = \mathbb{P} G_R + \mathbb{P} \perp G_R \) and using the above cancellation reduce to

\[
- \frac{1}{\varepsilon} \sum_{m' \leq m} C_m' \int_{T^3} \left| \partial^{m-m'} B_R \right| \left| \partial^{m'} u_0 \right| \left| \partial^m \mathbb{P} \perp G_R \right| |v| Mdvdx \\
\leq \frac{C}{\varepsilon} \sum_{m' \leq m} \left( \int_{T^3} \left| \partial^{m-m'} B_R \right| \left| \partial^{m'} u_0 \right|^2 dx \right)^{\frac{1}{2}} \left\| \partial^m \mathbb{P} \perp G_R \right\|_{L^2,v}.
\]  

(5.85)

where the last inequality is derived from the Sobolev embedding and the Hölder inequality.

Analogous to the cancellation (5.84), we also derive that

\[
\left\langle \left[ \mathcal{T}(v \times \partial^{m'} B_0) \cdot \nabla_v \partial^{m-m'} \mathbb{P} G_R \right] \cdot \partial^m \mathbb{P} G_R \right\rangle = 0. \tag{5.86}
\]

Then, by the relation \( G_R = \mathbb{P} G_R + \mathbb{P} \perp G_R \) and combining with (5.86), we can calculate that

\[
- \frac{1}{\varepsilon} \sum_{0 \neq m' \leq m} C_m' \int_{T^3} \left\langle \left[ \mathcal{T}(v \times \partial^{m'} B_0) \cdot \nabla_v \partial^{m-m'} G_R \right] \cdot \partial^m G_R \right\rangle dx = I_1 + I_2 + I_3.
\]  

(5.87)

By the definition of \( \mathbb{P} G_R \) in (1.24), we deduce that for \( |m| \leq N + 1 \)

\[
I_1 = - \frac{1}{\varepsilon} \sum_{0 \neq m' \leq m} C_m' \int_{T^3} \left\langle \left[ (v \times \partial^{m'} B_0) \cdot \partial^{m-m'} \left( \frac{u_R}{-u_R} \right) \right] \cdot \partial^m \mathbb{P} \perp G_R \right\rangle dx \\
\leq \frac{C}{\varepsilon} \sum_{0 \neq m' \leq m} \int_{T^3} \left| \partial^{m'} B_0 \right| \left| \partial^{m-m'} u_R \right| \left( \left| \partial^m \mathbb{P} \perp G_R \right|^2 \right)^{\frac{1}{2}} dx
\]  

(5.88)
\[
\begin{align*}
&\leq C \| B_0 \|_{H^{N+2}_x} \| u_R \|_{H^{N+1}_x} \| \partial^m P \partial^\perp G_R \|_{L^2_{x,v}}, \\
&\leq C \sqrt{\epsilon} \int_{T^3} \| [T(v \times \partial^m B_0) \cdot \nabla_v \partial^m P G_R] \cdot \partial^{m-m'} \partial^\perp G_R \| dx \\
&\leq C \sqrt{\epsilon} \int_{T^3} \| \partial^m \partial^\perp G_R \|_{L^2_{x,v}} \sum_{|m| \leq N+1} \| \partial^m G_R \|_{L^2_{x,v}}, \\
\end{align*}
\]

where we use the conclusions in Lemma 4.1 and the inequality (5.75). Similarly, we have

\[
I_2 = -\frac{1}{\epsilon} \sum_{0 \neq m' \leq m} C_m' \int_{T^3} \| [T(v \times \partial^{m'} B_0) \cdot \nabla_v \partial^{m-m'} \partial^\perp G_R] \cdot \partial^m \partial^\perp G_R \| dx \\
\leq C \sqrt{\epsilon} \int_{T^3} \| \partial^m \partial^\perp G_R \|_{L^2_{x,v}} \sum_{|m| \leq N} \| \partial^m G_R \|_{L^2_{x,v}}, \\
\]

which is derived from the integration by parts over \( v \in \mathbb{R}^3 \) and the symmetry of the integral on the \( v \)-variable. For the term \( I_3 \), Hölder inequality and Sobolev embedding \( H^2_2(T^3) \hookrightarrow L^\infty_x(T^3) \) reduce to

\[
I_3 \leq C \| B_0 \|_{H^{N+2}_x} \| \partial^m \partial^\perp G_R \|_{L^2_{x,v}} \sum_{|m| \leq N} \| \nabla_v \partial^m \partial^\perp G_R \|_{L^2_{x,v}} \\
\leq C \sqrt{\epsilon} \int_{T^3} \| \partial^m \partial^\perp G_R \|_{L^2_{x,v}} \sum_{|m| \leq N} \| \nabla_v \partial^m \partial^\perp G_R \|_{L^2_{x,v}}. \\
\]

Combining the relations (5.87), (5.88), (5.89) with (5.90), we gain

\[
-\frac{1}{\epsilon} \sum_{0 \neq m' \leq m} C_m' \int_{T^3} \| [T(v \times \partial^{m'} B_0) \cdot \nabla_v \partial^{m-m'} \partial^\perp G_R] \cdot \partial^m \partial^\perp G_R \| dx \\
 \leq C \sqrt{\epsilon} \int_{T^3} \| \partial^m \partial^\perp G_R \|_{L^2_{x,v}} \left( \sum_{|m| \leq N+1} \| \partial^m G_R \|_{L^2_{x,v}} + \sum_{|m| \leq N} \| \nabla_v \partial^m \partial^\perp G_R \|_{L^2_{x,v}} \right). \\
\]

We substitute the inequalities (5.82), (5.83) and (5.85) into (5.81), the we gain

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \left( \| \partial^m G_R \|_{L^2_{x,v}} + \| \partial^m E_R \|_{L^2_{x,v}} + \| \partial^m B_R \|_{L^2_{x,v}} \right) + \Lambda \| \partial^m \partial^\perp G_R \|_{L^2_{x,v}} \\
\leq C \| u_0 \|_{H^{N+2}_x} \| B_R \|_{H^{N+1}_x} \| \partial^m \partial^\perp G_R \|_{L^2_{x,v}} + \int_{T^3} \| \partial^m H_R \cdot \partial^m G_R \| dx \\
\leq C \| u_0 \|_{H^{N+2}_x} \| B_R \|_{H^{N+1}_x} \| \partial^m \partial^\perp G_R \|_{L^2_{x,v}} + \int_{T^3} \| \partial^m H_R \cdot \partial^m G_R \| dx. \\
\end{align*}
\]

Now we deal with the term \( C \| u_0 \|_{H^{N+2}_x} \| B_R \|_{H^{N+1}_x} \| \partial^m \partial^\perp G_R \|_{L^2_{x,v}} \) in the right-hand side of the above inequality (5.92). From the conservation laws (5.43) of the leading term \( (\theta_0^+, E_0, B_0) \), we derive that

\[
\begin{align*}
\left\{ \begin{array}{l}
\int_{T^3} \theta_0 dx = 0, \\
\int_{T^3} (2u_0 + E_0 \times B_0) dx = 0,
\end{array} \right.
\end{align*}
\]

which immediately implies that by the Poincaré inequality

\[
\| u_0 \|_{L^2_x} \leq C \| \nabla_x u_0 \|_{L^2_x} + C \left| \int_{T^3} u_0 dx \right| \\
\leq C \| \nabla_x u_0 \|_{L^2_x} + C \left| \int_{T^3} E_0 \times B_0 dx \right| \\
\leq C \| \nabla_x u_0 \|_{L^2_x} + C \| E_0 \|_{L^2_x} \| B_0 \|_{L^2_x}. \\
\]

Then, the above inequality (5.93) reduces to

\[
\| u_0 \|_{H^{N+2}_x} \leq C \| \nabla_x u_0 \|_{H^{N+1}_x} + C \| \nabla_x u_0 \|_{L^2_x} + C \| E_0 \|_{L^2_x} \| B_0 \|_{L^2_x} \\
\leq C \left( 1 + \sqrt{\epsilon} \right) D_{0,N+1}(t), \\
\]

where \( D_{0,N+1}(t) \) is the energy functional.
where the last inequality is implied by Lemma 4.1. Then, combining the inequality (5.94) and summing up for $|m| \leq N + 1$ in (5.92), we gain

$$\frac{1}{2} \frac{d}{dt} \left( \sum_{|m| \leq N + 1} \| \partial^m G_R \|^2_{L^2_{x,v}} + \| E_R \|^2_{H^2_x} + \| B_R \|^2_{H^2_x} \right) + \frac{\Lambda}{\varepsilon^2} \sum_{|m| \leq N + 1} \| \partial^m \nabla L^G_R \|^2_{L^2_{x,v}}$$

$$\leq C \left(1 + \sqrt{\mathcal{E}_{0,N+1}^{in}}\right) \mathcal{D}_{0,N+1}(t) \| B_R \|^2_{H^2_x} + \sum_{|m| \leq N + 1} \| \partial^m \nabla L^G_R \|^2_{L^2_{x,v}} + \int_{\mathbb{T}^3} \langle \partial^m H_R \cdot \partial^m G_R \rangle dx$$

$$+ \sum_{|m| \leq N + 1} \int_{\mathbb{T}^3} \langle \partial^m H_R \cdot \partial^m G_R \rangle dx,$$

which immediately means that by the Young’s inequality

$$\frac{d}{dt} \left( \sum_{|m| \leq N + 1} \| \partial^m G_R \|^2_{L^2_{x,v}} + \| E_R \|^2_{H^2_x} + \| B_R \|^2_{H^2_x} \right) + \frac{\Lambda}{\varepsilon^2} \sum_{|m| \leq N + 1} \| \partial^m \nabla L^G_R \|^2_{L^2_{x,v}}$$

$$\leq C \left(1 + \mathcal{E}_{0,N+1}^{in}\right) \mathcal{D}_{0,N+1}(t) \| B_R \|^2_{H^2_x} + \sum_{|m| \leq N + 1} \int_{\mathbb{T}^3} \langle \partial^m H_R \cdot \partial^m G_R \rangle dx$$

$$+ \mathcal{E}_{0,N+2} \left( \sum_{|m| \leq N + 1} \| \partial^m \nabla L^G_R \|^2_{L^2_{x,v}} + \sum_{|m| \leq N} \| \nabla \cdot \partial^m \nabla L^G_R \|^2_{L^2_{x,v}} \right).$$

Noticing that the inequality (5.46) in Lemma 5.2 implies

$$\| B_R \|^2_{H^2_x} = \| B_R \|^2_{L^2_x} + \| \nabla_x B_R \|^2_{L^2_x} \leq C_2 \| \nabla_x B_R \|^2_{H^2_x} \leq (1 + C_2) \| \nabla_x B_R \|^2_{H^2_x}$$

for $N \geq 1$, we derive from the condition (5.77), i.e., $\mathcal{E}_{0,N+2} \leq \lambda_R(N + 2)$ and the inequality (5.75) that

$$\left\{ \begin{array}{l}
C_3 \mathcal{E}_{0,N+2} \| B_R \|^2_{H^2_x} \leq C_3 (1 + C_2) \lambda_R(N + 2) \| \nabla_x B_R \|^2_{H^2_x}, \\
C_3 \sqrt{\mathcal{E}_{0,N+2}^{in}} \| u_R \|^2_{H^2_x} \leq \frac{C_3}{\sqrt{\sigma}} \lambda_R(N + 2) \sum_{|m| \leq N + 1} \| \partial^m \nabla L^G_R \|^2_{L^2_{x,v}}.
\end{array} \right.$$  (5.96)

If we choose $\lambda_R(N + 2) \in (0, \lambda_1(N + 2))$, depending only on $N$, $\mu$, $\sigma$, $\kappa$, such that

$$C_3 (1 + C_2) \lambda_R(N + 2) \leq \frac{\delta}{2}$$

and $\frac{C_3}{\sqrt{\sigma}} \lambda_R(N + 2) \leq \frac{C_3}{2C_1}$,

then the inequality (5.96) reduces to

$$\left\{ \begin{array}{l}
C_3 \mathcal{E}_{0,N+2} \| B_R \|^2_{H^2_x} \leq \frac{\delta}{2} \| \nabla_x B_R \|^2_{H^2_x}, \\
C_3 \sqrt{\mathcal{E}_{0,N+2}^{in}} \| u_R \|^2_{H^2_x} \leq \frac{C_3}{2C_1} \sum_{|m| \leq N + 1} \| \partial^m \nabla L^G_R \|^2_{L^2_{x,v}}.
\end{array} \right.$$  (5.97)

Therefore, together with inequalities (5.75), (5.76) and (5.97), the inequality

$$\frac{d}{dt} \left( \| E_R \|^2_{N^2_x} + (1 - \delta + \sigma \delta) \| B_R \|^2_{N^2_x} + \| \nabla_x B_R \|^2_{N^2_x} + (1 - \delta) \| \partial_t B_R \|^2_{N^2_x} \right)$$

$$\delta \| \partial_t B_R + B_R \|^2_{N^2_x} + \varepsilon^2 \sum_{|m| \leq N - 1} \| \nabla_x \times \langle \partial^m \nabla L^G_R \cdot \mathcal{T}_1 \Phi(v) \rangle \|^2_{L^2_{x,v}}$$

$$+ \sigma \| E_R \|^2_{N^2_x} + \frac{\delta}{2} \| \nabla_x B_R \|^2_{N^2_x} + (\sigma - \delta) \| \partial_t B_R \|^2_{N^2_x}$$

$$+ \frac{C_3 \varepsilon}{2C_1} \sum_{|m| \leq N + 1} \| \partial^m \nabla L^G_R \|^2_{L^2_{x,v}} + \frac{C_3 \varepsilon}{C_1} \| \text{div}_x E_R \|^2_{H^2_x}$$

$$\leq C_3 \varepsilon \frac{d}{dt} A_N(t) + \frac{2C_3}{\varepsilon} \sum_{|m| \leq N + 1} \| \partial^m \nabla L^G_R \|^2_{L^2_{x,v}} + C_3 \varepsilon^2 \sum_{|m| \leq N} \| \partial^m H_R \|^2_{L^2_{x,v}}.$$
\[ + C_3 \varepsilon^2 \sum_{|m| \leq N} \| \mathcal{P}_B \partial^m H_R \|_{L^2_x,v}^2 + C_2 C_3 (D_{0,2}(t) + D_{1,2}(t)) \] (5.98)

is derived from plugging the inequality (5.47) into (5.7), multiplying by \( \frac{C_A}{C_1} \) in (5.7) and adding them to the inequality (5.7), where the scalar functional \( A_N(t) \) is defined as in (5.79). If we multiply by \(\frac{\Lambda + 2C_3}{\Lambda} \) in (5.98) and \( C_2 C_3 \) in (4.14) in Lemma 4.2 for the case \( M = N \) respectively, and add them to (5.98), then we obtain

\[
\frac{d}{dt} \left[ \| E_R \|_{H^N_{x,v}}^2 + (1 - \delta + \sigma \delta) \| B_R \|_{H^N_{x,v}}^2 + \| \nabla_x B_R \|_{H^N_{x,v}}^2 + (1 - \delta) \| \partial_t B_R \|_{H^N_{x,v}}^2 \right] \]

\[
\delta \| \partial_t B_R + B_R \|_{H^N_{x,v}}^2 + \varepsilon^2 \sum_{|m| \leq N-1} \| \nabla_x \times (\partial^m \mathbb{P} \cdot G_R \cdot T_1 \tilde{\Phi}(v)) \|_{L^2_x}^2 - C_3 \varepsilon \frac{d}{dt} A_N(t) + C_2 C_3 (\mathcal{E}_{1,N}(t) + \tilde{C}_N \mathcal{E}_{0,N+2}(t))
\]

\[
\mathcal{E}_N \leq C_3 \varepsilon^2 \left( \sum_{|m| \leq N} \| \partial^m \mathbb{P} G_R \|_{L^2_x}^2 + \mathcal{P}_B \partial^m H_R \|_{L^2_x,v}^2 \right) + \frac{\Lambda + 2C_3}{\Lambda} \sum_{|m| \leq N} \left( \frac{\varepsilon}{|m|} + \mathcal{P}_B \partial^m \mathbb{P} \cdot G_R \|_{L^2_x,v}^2 \right)
\]

\[
\frac{\Lambda + 2C_3}{\Lambda} \left( 1 + \lambda_0 (N + 2) \right) \| B_R \|_{H^N_{x,v}}^2 \]

\[
+ C_2 C_3 \varepsilon^2 \left( \| E_R \|_{L^2_x}^2 + \| B_R \|_{L^2_x}^2 \| \nabla_x B_R \|_{L^2_x}^2 \right)
\] (5.99)

Now we let

\[
\begin{align*}
C_4 &= \frac{\Lambda + 2C_3}{\Lambda} > 1, \\
C_5 &= C_2 C_3 > 0, \\
C_6 &= \min \{ 1, \frac{4}{2}, \sigma - \delta, \frac{C_6 C_3}{C_1}, \frac{C_3}{C_1}, \Lambda \} > 0, \\
C_7 &= \max \{ C_3, \frac{\Lambda (\Lambda + 2C_3)}{\Lambda} (1 + \lambda_0 (N + 2)), \frac{\Lambda + 2C_3}{\Lambda}, C_2 C_3 \} > 0,
\end{align*}
\]

then the inequality (5.99) concludes the inequality (5.78). As a consequence, the proof of Lemma 5.4 is completed. \( \square \)

6. Estimates on the \((x,v)\)-derivatives

In this section, we are devoted to the energy estimates on the mixed derivatives of the remainder system (5.1). Firstly, we note that it holds for the hydrodynamic part \( \mathbb{P} G_R \), that

\[
\| w^l \partial^m \mathbb{P} G_R \|_{L^2_x,v} \leq C \| \partial^m \mathbb{P} G_R \|_{L^2_x,v},
\]

(6.1)

where we have used the notation \( w(v) = (1 + |v|^2)^{1/2} \). It suffices to estimate the remaining microscopic part \( w^l \partial^m \mathbb{P} \perp G_R \) with \(|m| + |\beta| \leq N\).

We state the following lemma.

Lemma 6.1. Let \( N \geq 2 \) and \( l \geq 0 \). Assume that \( (G_R, E_R, B_R) \) is a smooth solution of the remainder system (5.1). Then there exists some constant \( C > 0 \) such that for sufficiently small \( \varepsilon > 0 \), it holds that

\[
\frac{d}{dt} \sum_{|m| + |\beta| \leq N} \| w^l \partial^m \mathbb{P} \perp G_R \|_{L^2_x,v}^2 + \sum_{|m| + |\beta| \leq N} \| w^l \partial^m \mathbb{P} \perp G_R \|_{L^2_x,v(v)}^2
\] (6.2)
\[
\leq \frac{C}{\varepsilon^2} \sum_{|m| \leq N+1} \|\partial^m \mathbb{P} G_R\|_{L^2_{x,v}}^2(v) + C(\|E_R\|_{H^{2\varepsilon}}^2 + \|\nabla x B R\|_{H^{2\varepsilon}}^2 + \sum_{|m| \leq N+1} \|\partial^m \mathbb{P} G_R\|_{L^2_{x,v}}^2)
+ \sum_{|m|+|\beta| \leq N} \int_{\mathbb{T}^3} \langle \partial^m \beta H_R \cdot w^2 \partial^m \mathbb{P} G_R \rangle dx.
\]

**Proof.** We apply the mixed multi-derivative operator \(\partial^m \beta\) to the first equation of the remainder system (5.1), and perform the decomposition \(G_R = \mathbb{P} G_R + \mathbb{P}^\perp G_R\) to get
\[
[\partial_t + \frac{1}{\varepsilon} v \cdot \nabla x + \frac{1}{\varepsilon} T(v \times B_0) \cdot \nabla v] \partial^m \mathbb{P} G_R + \frac{1}{\varepsilon^2} \partial^m \mathbb{P} \mathbb{P}^\perp G_R = - \langle \partial_t + \frac{1}{\varepsilon} v \cdot \nabla x + \frac{1}{\varepsilon} T(v \times B_0) \cdot \nabla v \rangle \partial^m \mathbb{P} G_R - \frac{1}{\varepsilon^2} \partial^m \mathbb{P} \mathbb{P}^\perp G_R
\]
\[= \sum_{|m| \leq 1} C_{m1} \partial_{\beta} l_1 \cdot \nabla x \partial_{\beta_{m-1}} G_R - \frac{1}{\varepsilon^2} \partial^m [\mathcal{T}(v \times B_0) \cdot \nabla v G_0] + \frac{1}{\varepsilon} \partial^m E_R \cdot \partial_{\beta} v \mathcal{T}_1
\]
\[- \frac{1}{\varepsilon} \sum_{|\beta| \leq 1} C_{m1} \partial_{\beta} l_1 \cdot \nabla v \partial_{\beta_{m-1}} G_R + \partial_{\beta} l_1 H_R.
\]

Taking the \(L^2_{x,v}\) inner product with \(w^2 \partial^m \mathbb{P} G_R\), we have
\[
\frac{1}{\varepsilon^2} \int_{\mathbb{T}^3} \partial^m \mathbb{P} G_R \cdot \partial_{\beta} l_1 \cdot \nabla v \partial_{\beta_{m-1}} G_R \|_{L^2_{x,v}}^2 + \frac{1}{\varepsilon} \int_{\mathbb{T}^3} \langle w^2 \partial^m \mathbb{P} G_R \cdot \partial^m \mathbb{P} G_R \rangle dx
\]
\[- \frac{1}{\varepsilon} \int_{\mathbb{T}^3} \partial^m E_R \cdot \partial_{\beta} l_1 \cdot \nabla v \partial_{\beta_{m-1}} G_R \|_{L^2_{x,v}}^2 + \frac{1}{\varepsilon} \int_{\mathbb{T}^3} \langle \mathcal{T}(v \times B_0) \cdot \nabla v \partial^m \mathbb{P} G_R \rangle dx
\]
\[- \frac{1}{\varepsilon} \sum_{|\beta| \leq 1} C_{m1} \partial_{\beta} l_1 \cdot \nabla v \partial_{\beta_{m-1}} G_R \|_{L^2_{x,v}}^2 + \frac{1}{\varepsilon} \int_{\mathbb{T}^3} \langle \mathcal{T}(v \times B_0) \cdot \nabla v \partial^m \mathbb{P} G_R \rangle dx
\]
\[- \frac{1}{\varepsilon} \sum_{|\beta| \leq 1} C_{m1} \partial_{\beta} l_1 \cdot \nabla v \partial_{\beta_{m-1}} G_R \|_{L^2_{x,v}}^2 + \frac{1}{\varepsilon} \int_{\mathbb{T}^3} \langle \mathcal{T}(v \times B_0) \cdot \nabla v \partial^m \mathbb{P} G_R \rangle dx
\]
\[\triangleq \sum_{1 \leq i \leq 7} I_i.
\]

By the inequality (3.14) in Lemma 3.3, we obtain
\[
\frac{1}{\varepsilon} \int_{\mathbb{T}^3} \langle w^2 \partial^m \mathbb{P} G_R \cdot \partial^m \mathbb{P} G_R \rangle dx
\]
\[\geq \frac{1}{\varepsilon^2} \|w^2 \partial^m \mathbb{P} G_R\|_{L^2_{x,v}}^2(v) - \frac{C}{\varepsilon} \|w^2 \partial^m \mathbb{P} G_R\|_{L^2_{x,v}}^2(v). \quad (6.5)
\]

We now estimate the terms \(I_i\) for \(1 \leq i \leq 6\) in the right-hand side of the equality (6.4), while estimating the last term \(I_7\) is postponed behind.

For the term \(I_1\), we observe that \(I_1 = 0\) for the case \(|\beta| \geq 2\). So, we only need to consider the case \(\beta = 0\) and \(|\beta| = 1\). In the case of \(\beta = 0\), we have for \(|m| + |\beta| \leq N\) that
\[
I_1 = \frac{1}{\varepsilon} \int_{\mathbb{T}^3} \partial^m \mathbb{P} G_R \int_{\mathbb{R}^3} v \mathcal{T}_1 \cdot w^2 \partial^m \mathbb{P} G_R M dv dx
\]
\[= - \frac{1}{\varepsilon} \sum_{|\beta| \leq 1} \int_{\mathbb{T}^3} \partial^m \mathbb{P} G_R \int_{\mathbb{R}^3} v \mathcal{T}_1 \cdot w^2 \partial^m \mathbb{P} G_R M dv dx,
\]
then the Hölder inequality yields that

\[
I_1 \leq \frac{C_\varepsilon}{\varepsilon} \sum_{|\xi|=1} \int_{\mathbb{T}^3} |\partial^{m-\varepsilon} E_R| \left\{ \int_{\mathbb{R}^3} |v|^2 w^l M dv \right\} \frac{1}{2} \left\{ \int_{\mathbb{R}^3} |\partial^{m+\varepsilon} \mathbb{P} G_R|^2 M dv \right\} \frac{1}{2} dx \tag{6.7}
\]

\[
\leq \frac{C_\varepsilon}{\varepsilon} \sum_{|\xi|=1} \|\partial^{m-\varepsilon} E_R\|_{L^2_x} \|\partial^{m+\varepsilon} \mathbb{P} G_R\|_{L^2_{x,v}}.
\]

\[
\leq \frac{C_\varepsilon}{\varepsilon} \|E_R\|_{H^\varepsilon_x^{-1}} \sum_{|m|\leq N+1} \|\partial^m \mathbb{P} G_R\|_{L^2_{x,v}}.
\]

In summary, the term \(I_1\) can be bounded by

\[
I_1 \leq \frac{C_\varepsilon}{\varepsilon} \|E_R\|_{H^\varepsilon_x^{-1}} (\|w^l \partial^m_\beta \mathbb{P} G_R\|_{L^2_{x,v}} + \sum_{|m|\leq N+1} \|\partial^m \mathbb{P} G_R\|_{L^2_{x,v}}). \tag{6.9}
\]

Notice that \(I_2 = 0\) for \(l = 0\), then it suffices to estimate \(I_2\) for the case \(l > 0\). By the simple facts \(|v| \leq w(v)\) and \(|\nabla v w^l(v)| = |w^{l-1}(v)| \leq w^{l-1}(v)\), we derive that

\[
I_2 \leq \frac{C_\varepsilon}{\varepsilon} \int_{\mathbb{T}^3 \times \mathbb{R}^3} |B_0| |v| w^{l-1} |\partial^m_\beta \mathbb{P} G_R|^2 w^l M dx \tag{6.10}
\]

\[
\leq \frac{C_\varepsilon}{\varepsilon} \|B_0\|_{L^\infty_x} \sum_{|m|\leq N+1} \|\partial^m_\beta \mathbb{P} G_R\|_{L^2_{x,v}}.
\]

where we have used the Sobolev embedding inequality \(H^2_x(\mathbb{T}^3) \hookrightarrow L^\infty_x(\mathbb{T}^3)\) and the result \((4.6)\) in Lemma 4.1.

We next turn to the estimate on \(I_3\). Recalling the hydrodynamic part \(\mathbb{P} G_R\) defined in \((5.2)\) that

\[
\mathbb{P} G_R = \rho^+ R \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right) + \rho^- R \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right) + u_R \cdot \left( \begin{array}{c} v \\ 0 \\ 0 \end{array} \right) + \theta_R \left( \begin{array}{c} |v|^2 \\ 3 |v|^2 - 3 \frac{|v|^2}{2} \end{array} \right),
\]

we deduce that

\[
I_3 \leq C (\|\partial \partial^m \rho^+_R\|_{L^2_x} + \|\partial \partial^m \rho^-_R\|_{L^2_x} + \|\partial \partial^m u_R\|_{L^2_x} + \|\partial \partial^m \theta_R\|_{L^2_x}) \|w^l \partial^m_\beta \mathbb{P} G_R\|_{L^2_{x,v}}
\]

\[
+ \frac{C_\varepsilon}{\varepsilon} (\|\nabla^m \rho^+_R\|_{L^2_x} + \|\nabla^m \rho^-_R\|_{L^2_x} + \|\nabla^m u_R\|_{L^2_x} + \|\nabla^m \theta_R\|_{L^2_x}) \|w^l \partial^m_\beta \mathbb{P} G_R\|_{L^2_{x,v}}
\]

\[
\leq C (\|\partial \partial^m \rho^+_R\|_{L^2_x} + \|\partial \partial^m \rho^-_R\|_{L^2_x} + \|\partial \partial^m u_R\|_{L^2_x} + \|\partial \partial^m \theta_R\|_{L^2_x}) \|w^l \partial^m_\beta \mathbb{P} G_R\|_{L^2_{x,v}}
\]

\[
+ \frac{C_\varepsilon}{\varepsilon} (1 + \|B_0\|_{H^2_x}) ( \sum_{|m|\leq N+1} \|\partial^m \mathbb{P} G_R\|_{L^2_{x,v}} ) \|w^l \partial^m_\beta \mathbb{P} G_R\|_{L^2_{x,v}}, \tag{6.11}
\]

where \(\varepsilon\) has used the Sobolev embedding inequality again \(H^2_x(\mathbb{T}^3) \hookrightarrow L^\infty_x(\mathbb{T}^3)\), and the equivalent relation \(C_1 \|\partial^m \mathbb{P} G_R\|_{L^2_{x,v}} \leq \|\partial^m \rho^+_R\|_{L^2_x} + \|\partial^m \rho^-_R\|_{L^2_x} + \|\partial^m u_R\|_{L^2_x} + \|\partial^m \theta_R\|_{L^2_x} \leq C_2 \|\partial^m \mathbb{P} G_R\|_{L^2_{x,v}}\) for some constant \(C_1, C_2 > 0\).
Recalling the relation (5.18), one can easily derive from the Hölder inequality that for $|m| \leq N$,

$$
\|\partial_I \partial^m \rho_{R} \|_{L^2} + \|\partial_I \partial^m \rho_R^\perp \|_{L^2} + \|\partial_I \partial^m u_R \|_{L^2} + \|\partial_I \partial^m \theta_R \|_{L^2} \leq \frac{C}{\varepsilon} (\|\nabla_x \partial^m \rho_R^\perp \|_{L^2} + \|\nabla_x \partial^m \rho_R \|_{L^2} + \|\nabla_x \partial^m u_R \|_{L^2} + \|\nabla_x \partial^m \theta_R \|_{L^2})$

(6.12)

$$
+ \frac{C}{\varepsilon} (\int \int \int \partial^m \partial_I \partial^m \rho_R \|_{L^2} \|\partial^m \partial_I \partial^m \rho_R \|_{L^2} \|\partial^m \partial_I \partial^m u_R \|_{L^2} \|\partial^m \partial_I \partial^m \theta_R \|_{L^2} \|\partial^m \partial_I \partial^m G_R \|_{L^2}) + C\|\partial^m H_R \|_{L^2}.
$$

(6.13)

Then, combining the above two inequalities (6.11) and (6.12) gives the estimate on $I_3$, i.e.

$$
I_3 \leq \frac{C}{\varepsilon} (1 + \lambda_0^2 (N + 2))(\sum_{|m| \leq N+1} \|\partial^m \partial_I \partial^m G_R \|_{L^2}) \|w^J \partial^m \partial_I \partial^m G_R \|_{L^2},
$$

where we have used the result (4.6) again.

As for the term $I_4$, we write that

$$
I_4 = \frac{1}{\varepsilon} \sum_{|\beta| = 1} C_{\beta} \int T_{3} \langle \partial_{\beta_1} v \cdot \nabla_x \partial^m \partial_{\beta_1} (\mathbb{P} G_R + \mathbb{P} \nabla \mathbb{P} G_R) \cdot w^J \partial^m \partial_{\beta_1} \mathbb{P} \nabla \mathbb{P} G_R \rangle dx
$$

(6.14)

$$
\leq \frac{C}{\varepsilon} (\|\nabla_x \partial^m \rho_R^\perp \|_{L^2} + \|\nabla_x \partial^m \rho_R \|_{L^2} + \|\nabla_x \partial^m u_R \|_{L^2} + \|\nabla_x \partial^m \theta_R \|_{L^2}) \|w^J \partial^m \partial_I \partial^m G_R \|_{L^2},
$$

$$
+ \frac{C}{\varepsilon} \sum_{|m| + |\beta| \leq N} \|w^J \partial^m \partial_I \partial^m G_R \|_{L^2},
$$

$$
\leq \frac{C}{\varepsilon} (\sum_{|m| \leq N+1} \|\partial^m \partial_I \partial^m G_R \|_{L^2} + \sum_{|m| + |\beta| \leq N} \|w^J \partial^m \partial_I \partial^m G_R \|_{L^2} \|w^J \partial^m \partial_I \partial^m G_R \|_{L^2}).
$$

We now estimate the term $I_5$. Recalling that

$$
G_0 = \rho_0^+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \rho_0^- \begin{pmatrix} 0 \\ 1 \end{pmatrix} + u_0 \cdot \begin{pmatrix} v \\ 0 \end{pmatrix} + \theta_0 \begin{pmatrix} \frac{|v|^2}{2} - \frac{3}{2} \\ \frac{|v|^2}{2} - \frac{3}{2} \end{pmatrix}
$$

with the relations $-\theta_0 = \rho_0 = \frac{\rho_0^+ + \rho_0^-}{2}$ and $n_0 = \rho_0^+ - \rho_0^-$, we can infer that

$$
I_5 = \frac{1}{\varepsilon} \int T_{3} \langle \partial^m \nabla (B_R \cdot u_0) \cdot w^J \partial^m \mathbb{P} \nabla \mathbb{P} G_R \rangle dx
$$

(6.15)

$$
\leq \frac{C}{\varepsilon} \int T_{3} \|u_0 \|_{H^n + 2} \|B_R \|_{H^n} \|w^J \partial^m \partial_I \partial^m G_R \|_{L^2},
$$

$$
\leq \frac{C}{\varepsilon} \lambda_0^2 (N + 2) \|\nabla x B_R \|_{H^n} \|u_0 \|_{H^n} \|w^J \partial^m \partial_I \partial^m G_R \|_{L^2},
$$

where we have used the Sobolev embedding inequality and the result (4.6) again, and the bound (5.46) in Lemma 5.2.
By the similar arguments in estimating the term $I_4$ and $I_5$, we give directly the estimation of the term $I_6$ as

$$ I_6 \leq \frac{C}{\varepsilon} \| B_0 \|_{H^{n+2}_x} \left( \sum_{|m| \leq N+1} \| \partial^m \mathbb{P} G_R \|_{L^2_{x,v}} + \sum_{|m|+|\beta| \leq N} \| w^l \partial^m \mathbb{P} \partial_\beta G_R \|_{L^2_{x,v}} \right) $$

$$ \leq \frac{C}{\varepsilon} \left( N + 2 \right) \left( \sum_{|m| \leq N+1} \| \partial^m \mathbb{P} G_R \|_{L^2_{x,v}} + \sum_{|m|+|\beta| \leq N} \| w^l \partial^m \mathbb{P} \partial_\beta G_R \|_{L^2_{x,v}} \right) .$$

Combining with the above estimations for $I_i$ with $1 \leq i \leq 6$, namely, equations (6.9), (6.10), (6.13), (6.14), (6.15) and (6.16), gives that

$$ \frac{1}{2} \frac{d}{dt} \| w^l \partial^m \mathbb{P} \partial_\beta G_R \|_{L^2_{x,v}}^2 + \frac{1}{2\varepsilon} \| w^l \partial^m \mathbb{P} \partial_\beta G_R \|_{L^2_{x,v}(\nu)}^2 $$

$$ \leq \frac{C}{\varepsilon} \| \partial^m \mathbb{P} \|_{L^2_{x,v}(\nu)} + \frac{C}{\varepsilon} \| E_R \|_{H^{n-1}_x} \left( \sum_{|m| \leq N+1} \| \partial^m \mathbb{P} \partial_\beta G_R \|_{L^2_{x,v}(\nu)} \right) $$

$$ + \frac{C}{\varepsilon} \| \partial^m H R \|_{L^2_{x,v}} \left( \sum_{|m|+|\beta| \leq N} \left( \| \partial^m \mathbb{P} \partial_\beta G_R \|_{L^2_{x,v}(\nu)} \right) + \sum_{|m|+|\beta| \leq N} \int_{\mathbb{T}^3} \langle \partial^m \partial_\beta H R \cdot w^l \partial^m \mathbb{P} G_R \rangle dx , $$

where we have used the fact $\| \cdot \|_{L^2_{x,v}} \leq \| \cdot \|_{L^2_{x,v}(\nu)}$. By summing up for $|m| + |\beta| \leq N$ and using the Hölder inequality, this inequality concludes the inequality (6.2), and thus completes the proof of Lemma 6.1.

**Remark 6.1.** We remark that Lemma 6.1 is constructed to deal with the term $\sum_{|m| \leq N} \| \partial^m H R \|_{L^2_{x,v}}^2$ arising in the right-hand side of inequality (5.78) in Lemma 5.4, since it includes the form $\| \partial^m Q(g_1, g_2) \|_{L^2_{x,v}}^2$, which will be controlled by some weighted norms with the form $\| w^l \partial^m \|_{L^2_{x,v}}^2$ (see Lemma 3.3). However, this method does not work on controlling the term $\sum_{|m| \leq N} \| \nabla \partial^m \mathbb{P} \partial_\beta G_R \|_{L^2_{x,v}}^2$ in (5.78), which arises also in the process of dealing with $\sum_{|m| \leq N} \| \partial^m H R \|_{L^2_{x,v}}^2$.

**Lemma 6.2.** Let $N \geq 2$ and $l \geq 0$. Assume that $(G_R, E_R, B_R)$ is a smooth solution of the remainder system (5.1). Then there exists some constant $C > 0$ such that for sufficiently small $\varepsilon > 0$, it holds that

$$ \frac{d}{dt} \sum_{|m|+|\beta| \leq N+1 \atop \beta \neq 0} \| w^l \partial^m \mathbb{P} \partial_\beta G_R \|_{L^2_{x,v}(\nu)}^2 + \frac{1}{2\varepsilon} \sum_{|m|+|\beta| \leq N+1 \atop \beta \neq 0} \| w^l \partial^m \mathbb{P} \partial_\beta G_R \|_{L^2_{x,v}(\nu)}^2 $$

$$ \leq \frac{C}{\varepsilon^2} \sum_{|m| \leq N+1} \| \partial^m \mathbb{P} G_R \|_{L^2_{x,v}(\nu)}^2 + C \left( \| E_R \|_{H^{n-1}_x}^2 + \| \nabla x B R \|_{H^{n-1}_x}^2 + \sum_{|m| \leq N+1} \| \partial^m \mathbb{P} G_R \|_{L^2_{x,v}}^2 \right) $$

$$ + \frac{C}{\varepsilon^2} \sum_{|m| \leq N} \| \partial^m H R \|_{L^2_{x,v}}^2 + \sum_{|m|+|\beta| \leq N+1 \atop \beta \neq 0} \int_{\mathbb{T}^3} \langle \partial^m \partial_\beta H R \cdot w^l \partial^m \mathbb{P} G_R \rangle dx . $$

**Proof.** The process is similar as the proof of Lemma 6.1, with some necessary refinements to match the case $|m| + |\beta| \leq N + 1, \beta \neq 0$. Note that in this case, the index $m$ satisfies $|m| \leq N$, which ensures it is still possible to control the higher-order derivatives with respect to the spatial variables.
We start from the weak formulation (6.4). For term $I_1$, it suffices to deal with the case $|\beta| = 1$, so we have

$$I_1 = \frac{1}{\varepsilon} \int_{\mathbb{T}^3} \partial^m E_R \int_{\mathbb{R}^3} \partial_\beta v T_1 \cdot w^{2l} \partial^m_\beta \mathbb{P}^\perp G_R M d\nu dx$$

(6.19)

$$= \frac{1}{\varepsilon} \sum_{|\epsilon| = 1} \int_{\mathbb{T}^3} \partial^{m-\epsilon} E_R \int_{\mathbb{R}^3} \partial_\epsilon (\partial_\beta v T_1 \cdot w^{2l} M) \partial^{m+\epsilon}_\beta \mathbb{P}^\perp G_R d\nu dx$$

$$\leq \frac{C}{\varepsilon} \sum_{|\epsilon| = 1} \int_{\mathbb{T}^3} |\partial^{m-\epsilon} E_R| \left\{ \left( \int_{\mathbb{R}^3} (w^{l-1} + w^{l+1})^2 M d\nu \right)^{\frac{1}{2}} \right\} \left\{ \int_{\mathbb{R}^3} |w^l \partial^{m+\epsilon}_\beta \mathbb{P}^\perp G_R|^2 M d\nu \right\}^{\frac{1}{2}} dx$$

$$\leq \frac{C}{\varepsilon} \|E_R\|_{H^N_x} \sum_{|m| \leq N+1} \|w^l \partial^m \mathbb{P}^\perp G_R\|_{L^2_{x,v}}.$$  

where we have used the fact $|\nabla_v (w^{2l} M)| \leq C (w^{2l-1} + w^{2l+1}) M$.

On the other hand, the estimation on $I_3$ can be refined as

$$I_3 \leq \frac{C}{\varepsilon} \sum_{|m| \leq N+1} \left( \|\partial^m \mathbb{P} G_R\|_{L^2_{x,v}} + \|\partial^m \mathbb{P}^\perp G_R\|_{L^2_{x,v}} \right) \|w^l \partial^m \mathbb{P}^\perp G_R\|_{L^2_{x,v}(\nu)}$$

(6.20)

$$+ C \|\partial^m H R\|_{L^2_{x,v}} \|w^l \partial^m \mathbb{P}^\perp G_R\|_{L^2_{x,v}(\nu)}.$$  

Furthermore, performing analogous arguments yields that

$$I_4 + I_5 + I_6 \leq \frac{C}{\varepsilon} \left( \|\nabla_x B_R\|_{H^N_x} + \sum_{|m| \leq N+1} \|\partial^m \mathbb{P} G_R\|_{L^2_{x,v}} + \sum_{|m| + |\beta| \leq N+1} \|w^l \partial^m \mathbb{P}^\perp G_R\|_{L^2_{x,v}(\nu)} \right)$$

$$\cdot \|w^l \partial^m \mathbb{P}^\perp G_R\|_{L^2_{x,v}(\nu)}.$$  

(6.21)

Combining the above inequalities together gives for $|m| + |\beta| \leq N + 1$, $\beta \neq 0$, that

$$\frac{d}{dt} \sum_{|m| + |\beta| \leq N+1} \|w^l \partial^m \mathbb{P}^\perp G_R\|_{L^2_{x,v}}^2 + \frac{1}{\varepsilon^2} \sum_{|m| + |\beta| \leq N+1} \|w^l \partial^m \mathbb{P}^\perp G_R\|_{L^2_{x,v}(\nu)}^2$$

$$- \frac{C}{\varepsilon^2} \sum_{|m| \leq N} \|\partial^m \mathbb{P}^\perp G_R\|_{L^2_{x,v}(\nu)}^2$$

$$\leq \frac{C}{\varepsilon} \left\{ \|\nabla_x B_R\|_{H^N_x} + \sum_{|m| \leq N+1} \left( \|\partial^m \mathbb{P} G_R\|_{L^2_{x,v}} + \|\partial^m \mathbb{P}^\perp G_R\|_{L^2_{x,v}} \right) \right\}$$

$$\cdot \sum_{|m| + |\beta| \leq N+1} \|w^l \partial^m \mathbb{P}^\perp G_R\|_{L^2_{x,v}(\nu)}^2$$

$$+ \frac{C}{\varepsilon} \sum_{|m| + |\beta| \leq N+1} \|w^l \partial^m \mathbb{P}^\perp G_R\|_{L^2_{x,v}(\nu)}^2 + C \|E_R\|_{H^N_x} \sum_{|m| \leq N+1} \|w^l \partial^m \mathbb{P}^\perp G_R\|_{L^2_{x,v}}$$

$$+ C \sum_{|m| \leq N} \|\partial^m H R\|_{L^2_{x,v}} \sum_{|m| + |\beta| \leq N+1} \|w^l \partial^m \mathbb{P}^\perp G_R\|_{L^2_{x,v}(\nu)}$$

$$+ \sum_{|m| + |\beta| \leq N+1} \int_{\mathbb{T}^3} (\partial^m H R \cdot w^{2l} \partial^m \mathbb{P}^\perp G_R) d\nu,$$

which, combined with the Hölder inequality, implies immediately the desired estimate (6.18).

The proof of Lemma 6.2 is thus completed. \qed
We notice that the terms
\[
\varepsilon^2 \sum_{|m| \leq N} \left\| \partial^m H_R \right\|_{L^2_x}^2, \quad \varepsilon^2 \sum_{|m| \leq N} \left\| P_2 \partial^m H_R \right\|_{L^2_x}^2, \quad \sum_{|m| \leq N+1} \int_{\mathbb{T}^3} \langle \partial^m H_R \cdot \partial^m G_R \rangle \, dx
\]
in the inequality (5.78) and
\[
\sum_{|m|+|\beta| \leq N} \int_{\mathbb{T}^3} \langle \partial^m H_R \cdot w^2 \partial^m P \perp G_R \rangle \, dx, \quad \sum_{|m|+|\beta| \leq N+1} \int_{\mathbb{T}^3} \langle \partial^m H_R \cdot w^2 \partial^m P \perp G_R \rangle \, dx
\]
in the inequalities (6.2) and (6.18), respectively, are all uncontrolled, where \( H_R \) is defined in (1.17) (dropping the index \( \varepsilon \)). In this section, we thereby estimate these terms. Noticing
\[
\varepsilon^2 \sum_{|m| \leq N} \left\| P_2 \partial^m H_R \right\|_{L^2_x}^2 \leq C \varepsilon^2 \sum_{|m| \leq N} \left\| \partial^m H_R \right\|_{L^2_x}^2
\]
by the definition of the operator \( P_2 \) in (5.6), we only need to estimate the other four terms by constructing the following lemmas.

**Lemma 7.1.** Let \( N \geq 4 \). There exists a constant \( C_* > 0 \), depending only on \( \mathcal{E}^{m}_{0,N+5}, \mathcal{E}^{m}_{1,N+3}, \mu, \kappa, \sigma \) and \( N \), such that
\[
\varepsilon^2 \sum_{|m| \leq N} \left\| \partial^m H_R \right\|_{L^2_x}^2 \leq C \mathcal{E}^{m}_{0,N+5} \sum_{|m| \leq N} \left( \left\| \partial^m P G_R \right\|_{L^2_x}^2 + \left\| w^{2+1} \partial^m P \perp G_R \right\|_{L^2_x}^2 \right) + C_* \varepsilon^2 \left[ \left\| \nabla x B_R \right\|_{H^{N-1}_x}^2 + \sum_{|m| \leq N} \left( \left\| \partial^m P G_R \right\|_{L^2_x}^2 + \left\| w^{2+1} \partial^m P \perp G_R \right\|_{L^2_x}^2 \right) \right] + C_* (1 + \| E_R \|_{H^N_x}^2) (D_{0,N+5}(t) + D_{1,N+3}(t))
\]
\[
\sum_{|m| \leq N} \left( \left\| \partial^m P G_R \right\|_{L^2_x}^2 + \left\| w^{2+1} \partial^m P \perp G_R \right\|_{L^2_x}^2 \right) \times \left( \left\| E_R \right\|_{H^N_x}^2 + \left\| B_R \right\|_{H^N_x}^2 + \sum_{|m| \leq N} \left( \left\| \partial^m P G_R \right\|_{L^2_x}^2 + \left\| w^{2+1} \partial^m P \perp G_R \right\|_{L^2_x}^2 \right) \right)
\]
holds for some constant \( C > 0 \), depending only on \( \mu, \kappa, \sigma \) and \( N \).

**Proof.** We first decompose \( H_R \) as
\[
H_R = \frac{1}{\varepsilon} \Gamma_0 G_R + H_R^{(1)} + H_R^{(2)} + H_R^{(3)},
\]
where \( \Gamma_0 G_R \) is given in (1.18), the symbol \( H_R^{(1)} \) is
\[
H_R^{(1)} = \varepsilon \left( \mathcal{Q}(g_2^+, g_R^+ + g_R^-) + \mathcal{Q}(g_R^+, g_2^+ + g_2^-) + \mathcal{Q}(g_R^+, g_2^+ + g_2^-) \right) + \mathcal{Q}(g_R^+, g_2^+ + g_2^-)
\]
\[
+ \mathcal{Q}(g_2^+, g_R^+ + g_R^-) + \mathcal{Q}(g_2^+, g_2^+ + g_2^-)
\]
the \( H_R^{(2)} \) is
\[
H_R^{(2)} = -\varepsilon \nabla E_1 \cdot \nabla v G_R + \varepsilon \nabla E_1 \cdot v G_R - \varepsilon T E_R \cdot \nabla v G_R + \varepsilon T R \cdot \nabla G_R
\]
\[
- \nabla E_0 \cdot \nabla v G_R + \nabla E_0 \cdot v G_R - \mathcal{T}(v \times B_1) \cdot \nabla v G_R - \mathcal{T}(v \times B_R) \cdot \nabla v G_R
\]
\[
- \mathcal{T} E_R \cdot \nabla v (G_0 + \varepsilon G_1 + \varepsilon^2 G_2) + \mathcal{T} E_R \cdot v (G_0 + \varepsilon G_1 + \varepsilon^2 G_2)
\]
\[
- \mathcal{T}(v \times B_R) \cdot \nabla v (G_1 + \varepsilon G_2),
\]
and

\[ H_R^{(3)} = \left( \frac{\mathcal{R}^+}{\mathcal{R}^-} \right), \]  

(7.6)

where \( \mathcal{R}^\pm \) are defined in (2.52).

We notice that for \( |m| \leq N \)

\[
|\partial^m \Gamma_0 G_R|^2 \leq C \left( |\partial^m Q(g_0^+, g_R^+ + g_R^-)|^2 + |\partial^m Q(g_0^+, g_R^+ + g_R^-)|^2 \right. \\
+ \left. |\partial^m Q(g_0^-, g_R^+ + g_R^-)|^2 + |\partial^m Q(g_0^-, g_R^+ + g_R^-)|^2 \right),
\]

which implies by (3.16) in Lemma 3.3

\[
\|\partial^m \Gamma_0 G_R\|_{L_\alpha^2} \leq C \sum_{m_1 + m_2 = m} \left( \|\partial^{m_1} g_0^+\|_{L_\alpha^2}^2 + \|\partial^{m_1} g_0^-\|_{L_\alpha^2}^2 + \|u^{2\gamma} \partial^{m_1} g_0^+\|_{L_\alpha^2}^2 + \|u^{2\gamma} \partial^{m_1} g_0^-\|_{L_\alpha^2}^2 \right) \\
\times \left( \|\partial^{m_2} g_R^+\|_{L_\alpha^2}^2 + \|\partial^{m_2} g_R^-\|_{L_\alpha^2}^2 + \|u^{2\gamma} \partial^{m_2} g_R^+\|_{L_\alpha^2}^2 + \|u^{2\gamma} \partial^{m_2} g_R^-\|_{L_\alpha^2}^2 \right) \\
\leq C \sum_{m_1 + m_2 = m} \left( \|\partial^{m_1} G_0\|_{L_\alpha^2}^2 + \|u^{2\gamma} \partial^{m_1} G_0\|_{L_\alpha^2}^2 \right) \left( \|\partial^{m_2} G_R\|_{L_\alpha^2}^2 + \|u^{2\gamma} \partial^{m_2} G_R\|_{L_\alpha^2}^2 \right).
\]

(7.7)

Recalling that

\[
G_0 = \rho_0^+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \rho_0^- \begin{pmatrix} 0 \\ 1 \end{pmatrix} + u_0 \cdot \begin{pmatrix} v \\ v \end{pmatrix} + \theta_0 \left( \frac{|v|^2}{2} - \frac{3}{2} \right)
\]

and \(-\theta_0 = \rho_0^- \rho_0^+ + n_0\), \(n_0 = \rho_0^+ - \rho_0^-\), we deduce that

\[
\|\partial^{m_1} G_0\|_{L_\alpha^2}^2 + \|u^{2\gamma} \partial^{m_1} G_0\|_{L_\alpha^2}^2 \leq C \left( \|\partial^{m_1} u_0\|_{L_\alpha^\infty}^2 + \|\partial^{m_1} \theta_0\|_{L_\alpha^\infty}^2 + \|\partial^{m_1} n_0\|_{L_\alpha^\infty}^2 \right).
\]

(7.8)

Then the inequalities (7.7) and (7.8) reduce to

\[
\|\partial^m \Gamma_0 G_R\|^2_{L_\alpha^2} \leq C \sum_{m_2 \leq m} \left( \|\partial^{m_2} u_0\|_{L_\alpha^\infty}^2 + \|\partial^{m_2} \theta_0\|_{L_\alpha^\infty}^2 + \|\partial^{m_2} n_0\|_{L_\alpha^\infty}^2 \right) \\
\times \left( \|\partial^{m_2} G_R\|^2_{L_\alpha^2} + \|u^{2\gamma} \partial^{m_2} G_R\|^2_{L_\alpha^2} \right) \\
\leq C \sum_{m_2 \leq m} \left( \|\partial^{m_2} u_0\|^2_{L_\alpha^2} + \|\partial^{m_2} \theta_0\|^2_{L_\alpha^\infty} + \|\partial^{m_2} n_0\|^2_{L_\alpha^\infty} \right) \|u^{2\gamma} \partial^{m_2} G_R\|^2_{L_\alpha^2}.
\]

(7.9)

By splitting \(G_R = \mathcal{P} G_R + \mathcal{P}^\perp G_R\) and combining the inequality (6.1), we know

\[
\|u^{2\gamma} \partial^{m_2} G_R\|^2_{L_\alpha^2} \leq C \left( \|u^{2\gamma} \partial^{m_2} \mathcal{P} G_R\|^2_{L_\alpha^2} + \|u^{2\gamma} \partial^{m_2} \mathcal{P}^\perp G_R\|^2_{L_\alpha^2} \right) \\
\leq C \left( \|\partial^{m_2} \mathcal{P} G_R\|^2_{L_\alpha^2} + \|u^{2\gamma} \partial^{m_2} \mathcal{P}^\perp G_R\|^2_{L_\alpha^2} \right).
\]

(7.10)

Together with (7.9) and (7.10), we deduce that

\[
\sum_{|m| \leq N} \|\partial^m \Gamma_0 G_R\|^2_{L_\alpha^2} \leq C \left( \|u_0\|^2_{H_x^{N+2}} + \|\theta_0\|^2_{H_x^{N+2}} + \|n_0\|^2_{H_x^{N+2}} \right) \\
\times \sum_{|m| \leq N} \left( \|\partial^m \mathcal{P} G_R\|^2_{L_\alpha^2} + \|u^{2\gamma} \partial^m \mathcal{P}^\perp G_R\|^2_{L_\alpha^2} \right),
\]

(7.11)

where we utilize the Sobolev embedding \(H_x^2(T^3) \hookrightarrow L_\alpha^\infty(T^3)\) and the conclusions of Lemma 4.1.

By the similar estimations of (7.7), we gain

\[
\|\partial^m H_R^{(1)}\|^2_{L_\alpha^2} \leq C e^2 \sum_{m_1 \leq m} \|u^{2\gamma} \partial^{m_1} G_0\|^2_{L_\alpha^2} \|u^{2\gamma} \partial^{m_1} G_R\|^2_{L_\alpha^2}
\]
\[ + C \sum_{m_1 \leq m} \|w^{2\gamma} \partial^{m-m'} G_1\|_{L^2_v}^2 \|w^{2\gamma} \partial^{m_1} G_R\|_{L^2_v}^2 \\
+ C \sum_{m_1 \leq m} \|w^{2\gamma} \partial^{m-m'} G_R\|_{L^2_v}^2 \|w^{2\gamma} \partial^{m_1} G_R\|_{L^2_v}^2 \]  
(7.12)

Recalling the definition of \( \tilde{g}^+ \) in (2.48), we have that by the Sobolev embedding \( H^2_x(T^3) \hookrightarrow L^\infty_x(T^3) \) and the conclusions of Lemma 4.1, 4.2

\[ \sum_{|m| \leq N} \|\partial^m G_1\|_{L^2_v}^2 \leq C \left( \|((\bar{u}_1, \bar{\theta}_1, \bar{n}_1))\|_{H^{N+2}_x}^2 + \|((u_0, \theta_0, n_0, E_0))\|_{H^{N+3}}^2 + \|((u_0, \theta_0, n_0, B_0))\|_{H^{N+3}}^4 \right) \]

\[ \leq C \|((\bar{u}_1, \bar{\theta}_1, \bar{n}_1))\|_{H^{N+2}_x}^2 + C \varepsilon_{0,N+3}^m (1 + \varepsilon_{0,N+3}^m) \]

\[ \leq C (1 + \varepsilon_{0,N+2}^m) D_{0,N+2}(t) + C (\varepsilon_{0,N+4}^m + \varepsilon_{1,N+2}^m) + C \varepsilon_{0,N+3}^m (1 + \varepsilon_{0,N+3}^m) \]

\[ \leq C (\varepsilon_{0,N+4}^m, \varepsilon_{1,N+2}^m) < \infty, \]  
(7.13)

where the fact \( D_{0,N+2}(t) \leq C \varepsilon_{0,N+3}(t) \) is utilized, and the notation \( \|(X_1, \ldots, X_p)\|_M^2 = \sum_{i=1}^p \|X_i\|_M^2 \) is employed. Similarly, we can deduce that

\[ \sum_{|m| \leq N} \|\partial^m G_2\|_{L^2_v}^2 \leq C (\varepsilon_{0,N+5}^m, \varepsilon_{1,N+3}^m) < \infty. \]  
(7.14)

Consequently, the inequalities (7.10), (7.12), (7.13) and (7.14) reduce to

\[ \int_{T^3} (III_1 + III_2) dx \leq C (\varepsilon_{0,N+5}^m, \varepsilon_{1,N+3}^m) \sum_{|m| \leq N} \left( \|\partial^m p G_R\|_{L^2_v}^2 + \|w^{2\gamma} \partial^m p \|_{L^2_v}^2 \right). \]  
(7.15)

Next we estimate the integral \( \int_{T^3} III_3 dx \). By using the Sobolev embedding \( H^2_x(T^3) \hookrightarrow L^\infty_x(T^3) \) and the H"older inequality, we know that if \( |m_1| \leq 2 \),

\[ \int_{T^3} \|w^{2\gamma} \partial^{m-m_1} G_R\|_{L^2_v}^2 \|w^{2\gamma} \partial^{m_1} G_R\|_{L^2_v}^2 dx \]

\[ \leq \sum_{x \in T^3} \|w^{2\gamma} \partial^{m_1} G_R\|_{L^2_v}^2 \int_{T^3} \|w^{2\gamma} \partial^{m-m_1} G_R\|_{L^2_v}^2 dx \]  
(7.16)

\[ \leq C \sum_{|m_1| \leq 4} \|w^{2\gamma} \partial^{m_1} G_R\|_{L^2_v}^2 \sum_{|m| \leq N} \|w^{2\gamma} \partial^m G_R\|_{L^2_v}^2 \]

\[ \leq C \left( \sum_{|m| \leq N} \|w^{2\gamma} \partial^m G_R\|_{L^2_v}^2 \right) \sum_{|m| \leq N} \|w^{2\gamma} \partial^m G_R\|_{L^2_v(v)}^2 \]

for \( |m| \leq N \ (N \geq 4) \) and \( m_1 \leq m \), and if \( |m_1| > 2 \),

\[ \int_{T^3} \|w^{2\gamma} \partial^{m-m_1} G_R\|_{L^2_v}^2 \|w^{2\gamma} \partial^{m_1} G_R\|_{L^2_v}^2 dx \]

\[ \leq \sum_{x \in T^3} \|w^{2\gamma} \partial^{m-m_1} G_R\|_{L^2_v}^2 \int_{T^3} \|w^{2\gamma} \partial^{m_1} G_R\|_{L^2_v}^2 dx \]  
(7.17)

\[ \leq C \left( \sum_{|m| \leq N} \|w^{2\gamma} \partial^m G_R\|_{L^2_v}^2 \right) \sum_{|m| \leq N} \|w^{2\gamma} \partial^m G_R\|_{L^2_v(v)}^2 \].

Then the inequalities (7.16) and (7.17) imply that

\[ \int_{T^3} III_3 dx \leq C \left( \sum_{|m| \leq N} \|w^{2\gamma} \partial^m G_R\|_{L^2_v}^2 \right) \sum_{|m| \leq N} \|w^{2\gamma} \partial^m G_R\|_{L^2_v(v)}^2 \]
\[ \sum_{|m| \leq N} \left( \| \partial^m P_G R \|_{L^2_{x,v}}^2 + \| w^{2\gamma} \partial^m P_G R \|_{L^2_{x,v}}^2 \right) \leq C \sum_{|m| \leq N} \left( \| \partial^m P_G R \|_{L^2_{x,v}}^2 + \| w^{2\gamma} \partial^m P_G R \|_{L^2_{x,v}}^2 \right) \times \sum_{|m| \leq N} \left( \| \partial^m P_G R \|_{L^2_{x,v}}^2 + \| w^{2\gamma} \partial^m P_G R \|_{L^2_{x,v}}^2 \right), \tag{7.18} \]

where the last inequality is derived from the inequality (6.1). Consequently, combining with (7.12), (7.15) and (7.18), we know

\[ \sum_{|m| \leq N} \| \partial^m H^{(1)}_{L^2_{x,v}} \|_{L^2_{x,v}}^2 \leq C \left( \epsilon_{0, N+1}, \epsilon_{1, N+3}^3 \right) \sum_{|m| \leq N} \left( \| \partial^m P_G R \|_{L^2_{x,v}}^2 + \| w^{2\gamma} \partial^m P_G R \|_{L^2_{x,v}}^2 \right) + C \sum_{|m| \leq N} \left( \| \partial^m P_G R \|_{L^2_{x,v}}^2 + \| w^{2\gamma} \partial^m P_G R \|_{L^2_{x,v}}^2 \right) \times \sum_{|m| \leq N} \left( \| \partial^m P_G R \|_{L^2_{x,v}}^2 + \| w^{2\gamma} \partial^m P_G R \|_{L^2_{x,v}}^2 \right). \tag{7.19} \]

We then estimate the quantity \( \sum_{|m| \leq N} \| \partial^m H^{(2)}_{L^2_{x,v}} \|_{L^2_{x,v}}^2 \). By the similar arguments in (7.18), we deduce that for any \( f(x), g(x) \in H^2_{L^2} (N \geq 4) \)

\[ \sum_{|m| \leq N} \sum_{m_1 \leq m} \int_{T^3} |\partial^{m-m_1} f|^2 |\partial^{m_1} g|^2 \, dx \leq C \| f \|_{H^2_{L^2}}^2 \| g \|_{H^2_{L^2}}^2. \tag{7.20} \]

Then, by the inequality (7.20), we have

\[ \sum_{|m| \leq N} \int_{T^3} \int_{\mathbb{R}^3} \left[ -\epsilon \partial^m (T E_1 \cdot \nabla_v G_R) \right]^2 M \, d\nu \, dx \leq C \epsilon^2 \sum_{|m| \leq N} \sum_{m_1 \leq m} \int_{T^3} |\partial^{m-m_1} E_1|^2 \| \partial^{m_1} \nabla_v G_R \|_{L^2_{x,v}}^2 \, dx \tag{7.21} \]

Similarly, we derive that

\[ \sum_{|m| \leq N} \int_{T^3} \int_{\mathbb{R}^3} |\epsilon \partial^m (T E_1 \cdot v G_R)|^2 M \, d\nu \, dx \leq C \epsilon^2 \| E_1 \|_{H^2_{L^2}}^2 \sum_{|m| \leq N} \| w \partial^m G_R \|_{L^2_{x,v}}^2, \tag{7.22} \]

and

\[ \sum_{|m| \leq N} \int_{T^3} \int_{\mathbb{R}^3} |\epsilon \partial^m (-T E_R \cdot \nabla_v G_R + T E_R \cdot v G_R)|^2 M \, d\nu \, dx \leq C \epsilon^2 \| E_R \|_{H^2_{L^2}}^2 \sum_{|m| \leq N} \left( \| \nabla_v \partial^m G_R \|_{L^2_{x,v}}^2 + \| w \partial^m G_R \|_{L^2_{x,v}}^2 \right), \tag{7.23} \]

and

\[ \sum_{|m| \leq N} \int_{T^3} \int_{\mathbb{R}^3} |\partial^m (-T E_0 \cdot \nabla_v G_R + T E_0 \cdot v G_R)|^2 M \, d\nu \, dx \leq C \| E_0 \|_{H^2_{L^2}}^2 \sum_{|m| \leq N} \left( \| \nabla_v \partial^m G_R \|_{L^2_{x,v}}^2 + \| w \partial^m G_R \|_{L^2_{x,v}}^2 \right), \tag{7.24} \]
\[
\sum_{|m| \leq N} \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |\partial^m [-\mathcal{T} E_R \cdot \nabla_v (G_0 + \varepsilon G_1 + \varepsilon^2 G_2) + \mathcal{T} E_R \cdot v (G_0 + \varepsilon G_1 + \varepsilon^2 G_2)]|^2 M dvdx \\
\leq C \|E_R\|_{H_N}^2 \sum_{|m| \leq N} (\|\nabla_v \partial^m (G_0 + \varepsilon G_1 + \varepsilon^2 G_2)\|_{L^{2,\infty}_x}^2 + \|\partial^m (G_0 + \varepsilon G_1 + \varepsilon^2 G_2)\|_{L^{2,\infty}_x}^2) .
\]

(7.25)

Recalling that
\[
G_0 = \rho_0^+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \rho_0^- \begin{pmatrix} 0 \\ 1 \end{pmatrix} + u_0 \begin{pmatrix} v \\ v \end{pmatrix} + \theta_0 \begin{pmatrix} |\varepsilon^2| \| \nabla \| \frac{3}{2} \left( \frac{3}{2} - \frac{3}{2} \right) \end{pmatrix} ,
\]
\[
G_1 = \left( \begin{pmatrix} \tilde{g}_1^+ \\ \tilde{g}_1^- \end{pmatrix} \right) , G_2 = \left( \begin{pmatrix} \tilde{g}_2^+ \\ \tilde{g}_2^- \end{pmatrix} \right) ,
\]
where \( \tilde{g}_i^\pm (i = 1, 2) \) are defined in (2.48), one deduce that by Sobolev embedding theory, Hölder inequality and the conclusions of Lemma 4.1 and Lemma 4.2
\[
\sum_{|m| \leq N} (\|\nabla_v \partial^m (G_0 + \varepsilon G_1 + \varepsilon^2 G_2)\|_{L^{2,\infty}_x}^2 + \|\partial^m (G_0 + \varepsilon G_1 + \varepsilon^2 G_2)\|_{L^{2,\infty}_x}^2) \\
\leq C (1 + \mathcal{E}_{0,N+3}^{in} (D_{0,N+2}(t) + D_{1,N+2}(t))) ,
\]
which means that by (7.25)
\[
\sum_{|m| \leq N} \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |\partial^m [-\mathcal{T} E_R \cdot \nabla_v (G_0 + \varepsilon G_1 + \varepsilon^2 G_2) + \mathcal{T} E_R \cdot v (G_0 + \varepsilon G_1 + \varepsilon^2 G_2)]|^2 M dvdx \\
\leq C \|E_R\|_{H_N}^2 (D_{0,N+2}(t) + D_{1,N+2}(t))
\]
for \( N \geq 4 \). Furthermore, if \( N \geq 4 \), the following inequalities are derived from the Sobolev embedding and the Hölder inequality:
\[
\sum_{|m| \leq N} \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |\partial^m [-\mathcal{T} (v \times B_R) \cdot \nabla_v G_R]|^2 M dvdx
\]
\[
\leq C \|B_R\|_{H_N}^2 \sum_{|m| \leq N} \|v \nabla_v \partial^m G_R\|_{L^{2,\infty}_x}^2 ,
\]
and additionally by Lemma 4.2
\[
\sum_{|m| \leq N} \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |\partial^m [-\mathcal{T} (v \times B_1) \cdot \nabla_v G_R]|^2 M dvdx
\]
\[
\leq C \|B_1\|_{H_N}^2 \sum_{|m| \leq N} \|v \nabla_v \partial^m G_R\|_{L^{2,\infty}_x}^2
\]
\[
\leq C (\mathcal{E}_{0,N+2}^{in} + \mathcal{E}_{1,N+2}^{in}) \sum_{|m| \leq N} \|v \nabla_v \partial^m G_R\|_{L^{2,\infty}_x}^2 ,
\]
and
\[
\sum_{|m| \leq N} \int_{\mathbb{T}^3} \int_{\mathbb{R}^3} |\partial^m [-\mathcal{T} (v \times B_R) \cdot \nabla_v (G_1 + \varepsilon G_2)]|^2 M dvdx
\]
\[
\leq C \|B_R\|_{H_N}^2 \sum_{|m| \leq N} \|v \nabla_v \partial^m (G_1 + \varepsilon G_2)\|_{L^{2,\infty}_x}^2
\]
\[
\leq C (\mathcal{E}_{0,N+5}^{in} + \mathcal{E}_{1,N+3}^{in}) \|\nabla_v B_R\|_{H_{N-1}^2}^2 ,
\]
where the last inequality is implied by the inequalities (5.46), (7.13) and (7.14).
In summary, from the inequalities (7.21), (7.22), (7.23), (7.24), (7.27), (7.28), (7.29) and (7.30), we derive that

\[
\sum_{|m| \leq N} \|\partial^m H^{(2)}_R\|_{L^2_{t,v}}^2 \leq C(\varepsilon^2 \|E_1\|_{H^N}^2 + \|E_0\|_{H^N}^2 \sum_{|m| \leq N} (\|\nabla_v \partial^m G_R\|_{L^2_{t,v}}^2 + \|w \partial^m G_R\|_{L^2_{t,v}}^2)) + C(E_0, N, +) \varepsilon \sum_{|m| \leq N} (\|\nabla_v \partial^m G_R\|_{L^2_{t,v}}^2 + \|w \partial^m G_R\|_{L^2_{t,v}}^2) + C\|E_R\|_{H^N}^2 (D_{0,2}(t) + D_{1,2}(t)) + C(E_0, N, +) \varepsilon \sum_{|m| \leq N} (\|\nabla_v \partial^m G_R\|_{L^2_{t,v}}^2 + \|w \partial^m G_R\|_{L^2_{t,v}}^2) + C\|E_R\|_{H^N}^2 (D_{0,2}(t) + D_{1,2}(t)),
\]

(7.31)

where the last inequality is derived from the inequality (6.1) and Lemma 4.1, 4.2.

Finally, we estimate the quantity \( \sum_{|m| \leq N} \|\partial^m H^{(3)}_R\|_{L^2_{t,v}}^2 \). Noticing that \( H^{(3)}_R = \left( R^+_{1, T} \right) \) is the known vector valued function given in (2.52), one easily derives from Lemma 4.1, 4.2, the Sobolev embedding theory and the Hölder inequality that

\[
\sum_{|m| \leq N} \|\partial^m H^{(3)}_R\|_{L^2_{t,v}}^2 \leq C(E_0, N, +, \varepsilon) \sum_{|m| \leq N} \left( \|\nabla_x B_R\|_{H^{N-1}}^2 + \|\nabla_v \partial^m G_R\|_{L^2_{t,v}}^2 + \|w \partial^m G_R\|_{L^2_{t,v}}^2 \right) + C\|E_R\|_{H^N}^2 (D_{0,2}(t) + D_{1,2}(t)).
\]

(7.32)

Although the method of proof of (7.32) is simple, the details of the estimations will occupy to much space of this paper. So, we omit the details here. Then, together with the relations (7.11), (7.19), (7.31) and (7.32), we conclude the inequality (7.2), and then we complete the proof of Lemma 7.1.

**Lemma 7.2.** Let \( N \geq 4 \). Then there is a constant \( C > 0 \) such that

\[
\left| \sum_{|m| + |\beta| \leq N} \int_{T^3} (H_R \cdot w^2 \partial^m_{\beta} \partial_{\beta} G_R) dx \right| + \left| \sum_{|m| + |\beta| \leq N + 1} \int_{T^3} (H_R \cdot w^2 \partial^m_{\beta} \partial_{\beta} G_R) dx \right| + \left| \sum_{|m| \leq N + 1} \int_{T^3} (\partial^m H_R \cdot \partial^m G_R) dx \right| \leq C(\varepsilon + \sqrt{E_0, N, +, \varepsilon}) \mathbb{D}_{N, I}(G_R, E_R, B_R) + C\varepsilon \mathbb{D}_{N, I}(G_R, E_R, B_R) \mathbb{D}_{N, I}(G_R, E_R, B_R),
\]

(7.33)

where the functionals \( \mathbb{E}_{N, I}(G_R, E_R, B_R) \) and \( \mathbb{D}_{N, I}(G_R, E_R, B_R) \) are defined in (1.25) and (1.27), respectively.

**Proof.** For convenience on notations, we denote \( \mathbb{P}^\perp = \left( \frac{p^+_R}{p^-_R} \right) \). For \( |m| + |\beta| \leq N + 1 \) and \( \beta \neq 0 \), we derive from (3.15) in Lemma 3.3 that

\[
\int_{T^3} (\frac{1}{\varepsilon} \partial^m G_R \cdot w^2 \partial^m_{\beta} \partial_{\beta} G_R) dx = \sum_{\tau = \pm} \int_{T^3} \frac{1}{\varepsilon} (\partial^m_{\beta} Q(\tilde{g}_R + \tilde{g}_R^+ + \tilde{g}_R^-) + \partial^m_{\beta} Q(\tilde{g}_R^+ + \tilde{g}_R^-)) \cdot w^2 \partial^m_{\beta} p^\perp_R dx.
\]
\[
\begin{align*}
&\leq C \varepsilon \sum_{\tau = \pm 1} \int_{\mathbb{T}^3} \left[ \| w^l \partial_{\beta_1}^{m_1} g_0^T \|_{L^2_z(\nu)} \| w^l \partial_{\beta_2}^{m_2} (g_R^T + g_R) \|_{L^2_z} + \| w^l \partial_{\beta_1}^{m_1} g_0^T \|_{L^2_z(\nu)} \| w^l \partial_{\beta_2}^{m_2} (g_R^T + g_R) \|_{L^2_z(\nu)} \right. \\
&\quad + \left. \| w^l \partial_{\beta_1}^{m_1} g_0^R \|_{L^2_z(\nu)} \| w^l \partial_{\beta_2}^{m_2} (g_R^0 + g_R^0) \|_{L^2_z} \right] \| w^l \partial_{\beta_3}^m \mathcal{P} R \|_{L^2_z(\nu)} \\
&\leq C \varepsilon \int_{\mathbb{T}^3} \left( \| w^l \partial_{\beta_1}^{m_1} G_0 \|_{L^2_z(\nu)} \| w^l \partial_{\beta_2}^{m_2} G_R \|_{L^2_z} \right. \\
&\quad + \left. \| w^l \partial_{\beta_1}^{m_1} G_0 \|_{L^2_z(\nu)} \| w^l \partial_{\beta_2}^{m_2} G_R \|_{L^2_z(\nu)} \right) \| w^l \partial_{\beta_3}^m \mathcal{P} \cdot \mathcal{P} G_R \|_{L^2_z(\nu)} dx \\
&\leq C \varepsilon \sum_{m_1 + m_2 = m} \sup_{\beta_1 + \beta_2 = \beta} \sum_{x \in \mathbb{T}^3} \| w^l \partial_{\beta_1}^{m_1} G_0 \|_{L^2_z(\nu)} \int_{\mathbb{T}^3} \left( \| w^l \partial_{\beta_2}^{m_2} G_R \|_{L^2_z(\nu)} \| w^l \partial_{\beta_3}^m \mathcal{P} \cdot \mathcal{P} G_R \|_{L^2_z(\nu)} \right) dx \\
&\leq C \varepsilon \sum_{|m| \leq N} \sum_{|m| + |\beta| \leq N+1} \sum_{\beta \neq 0} \| \partial^m \mathcal{P} G_R \|_{L^2_z(v)} \end{align*}
\]

where the inequality (6.1) and Lemma 4.1 are also utilized. Similar calculations on (7.34) imply that for \(|m| + |\beta| \leq N + 1\) and \(\beta \neq 0\),

\[
\int_{\mathbb{T}^3} \langle \partial_{\beta}^{m} H_R^{(1)} \rangle \cdot w^l \partial_{\beta}^{m} \mathcal{P} \cdot \mathcal{P} G_R dx 
\]

\[
\leq C \sum_{m_1 + m_2 = m} \sum_{\beta_1 + \beta_2 = \beta} \int_{\mathbb{T}^3} \left( \| w^l \partial_{\beta_1}^{m_1} G_R \|_{L^2_z} \| w^l \partial_{\beta_2}^{m_2} G_R \|_{L^2_z(\nu)} \right) \| w^l \partial_{\beta_3}^m \mathcal{P} \cdot \mathcal{P} G_R \|_{L^2_z(\nu)} dx \\
+ C \sum_{m_1 + m_2 = m} \sum_{\beta_1 + \beta_2 = \beta} \int_{\mathbb{T}^3} \left( \| w^l \partial_{\beta_1}^{m_1} G_1 \|_{L^2_z} \| w^l \partial_{\beta_2}^{m_2} G_R \|_{L^2_z(\nu)} \right) \| w^l \partial_{\beta_3}^m \mathcal{P} \cdot \mathcal{P} G_R \|_{L^2_z(\nu)} dx \\
\leq C \varepsilon \left( 1 + \sum_{|m| \leq N} \| \partial^m \mathcal{P} G_R \|_{L^2_z(v)} \right) + \sum_{|m| + |\beta| \leq N} \| w^l \partial_{\beta}^{m} \mathcal{P} \cdot \mathcal{P} G_R \|_{L^2_z(v)} dx \\
+ \sum_{|m| + |\beta| \leq N+1} \| w^l \partial_{\beta}^{m} \mathcal{P} \cdot \mathcal{P} G_R \|_{L^2_z(v)} \end{align*}
\]

We next estimate the quantity \(\sum_{|m| + |\beta| \leq N+1} \int_{\mathbb{T}^3} \langle \partial_{\beta}^{m} H_R^{(2)} \rangle \cdot w^l \partial_{\beta}^{m} \mathcal{P} \cdot \mathcal{P} G_R dx\). First of all, we have

\[
\begin{align*}
&- \varepsilon \langle \partial_{\beta}^{m} (TE_R \cdot \nabla_v G_R) \cdot w^l \partial_{\beta}^{m} \mathcal{P} \mathcal{P} G_R \rangle \\
&= - \varepsilon \langle \partial_{\beta}^{m} (TE_R \cdot \nabla_v \partial_{\beta}^{m} \mathcal{P} G_R) \cdot w^l \partial_{\beta}^{m} \mathcal{P} \mathcal{P} G_R \rangle \\
&- \varepsilon \langle (TE_R \cdot \nabla_v \partial_{\beta}^{m} \mathcal{P} \mathcal{P} G_R) \cdot w^l \partial_{\beta}^{m} \mathcal{P} \mathcal{P} G_R \rangle
\end{align*}
\]
\[-\varepsilon \sum_{0 \neq m_{1} \leq m} \langle (T \partial^{m_{1}} E_{R} \cdot \nabla_{v} \partial_{\beta}^{m_{1} - m_{1}} P \perp G_{R}) \cdot w^{2l} \partial_{\beta}^{m \perp} G_{R} \rangle \quad (7.36)\]
\[:= A_{1} + A_{2} + A_{3}.\]

For the term $A_{1}$, we derive from the definition of the projection operator $P$ in (1.24) that for $|m| + |\beta| \leq N + 1$ and $\beta \neq 0$
\[
\int_{T_{3}} A_{1} dx \leq C \varepsilon \sum_{m_{1} \leq m} \int_{T_{3}} |\partial^{m_{1}} E_{R}| \langle |\partial_{m_{1}} \theta_{R}| \| w^{l} \partial_{\beta}^{m \perp} G_{R} \|_{L_{x}^{2}} \rangle dx
\]
\[\leq C \varepsilon \| E_{R} \|_{H_{x}^{N}} \| \theta_{R} \|_{H_{x}^{N}} \| w^{l} \partial_{\beta}^{m \perp} G_{R} \|_{L_{x}^{2}, v} \]
\[\leq C \varepsilon \| E_{R} \|_{H_{x}^{N}} \sum_{|m| \leq N} \| \partial^{m} P G_{R} \|_{L_{x}^{2}, v} \sum_{|m| + |\beta| \leq N + 1} \| w^{l} \partial_{\beta}^{m \perp} G_{R} \|_{L_{x}^{2}, v}^{2},\]
where we make use of the inequality (6.1) and (7.20) for $N \geq 4$. For the term $A_{2}$, by straightforward calculations we know
\[
\int_{T_{3}} A_{2} dx = -\frac{\varepsilon}{2} \int_{T_{3}} \langle E_{R} \cdot \nabla_{v} [T \partial^{m_{1}} P \perp G_{R} \cdot \partial_{\beta}^{m \perp} G_{R}] w^{2l}(v) \rangle dx
\]
\[\leq C \varepsilon \| E_{R} \|_{L_{x}^{2}} \int_{T_{3}} \langle |v| w^{2l}(v) \rangle \left(1 - \frac{2l}{\varepsilon^{2}(v)} \right) |\partial_{\beta}^{m \perp} G_{R}|^{2} \rangle dx
\]
\[\leq C \varepsilon \| E_{R} \|_{H_{x}^{N}} \| w^{l} \partial_{\beta}^{m \perp} G_{R} \|_{L_{x}^{2}, v}^{2},\]
where we utilize the fact $|1 - \frac{2l}{\varepsilon^{2}(v)}| \leq 2l + 1$ for every $v \in \mathbb{R}^{3}$ and the the Sobolev embedding $H_{x}^{2}(\mathbb{T}^{3}) \hookrightarrow L_{x}^{\infty}(\mathbb{T}^{3})$. For the term $A_{3}$, the estimation
\[
\int_{T_{3}} A_{3} dx \leq C \varepsilon \sum_{0 \neq m_{1} \leq m} \left( \int_{T_{3}} |\partial^{m_{1}} E_{R}|^{2} \| w^{l} \nabla_{v} \partial_{\beta}^{m_{1} - m_{1}} P \perp G_{R} \|_{L_{x}^{2}} dx \right)^{\frac{2}{l}} \| w^{l} \partial_{\beta}^{m \perp} G_{R} \|_{L_{x}^{2}, v}^{2}
\]
\[\leq C \varepsilon \| E_{R} \|_{H_{x}^{N}} \sum_{|m| + |\beta| \leq N + 1} \| w^{l} \partial_{\beta}^{m \perp} G_{R} \|_{L_{x}^{2}, v}^{2},\]
(7.39)
is derived from (7.20) and the Hölder inequality. We summarize the inequalities (7.36), (7.37), (7.38) and (7.39) that
\[
\int_{T_{3}} -\varepsilon \langle \partial_{\beta}^{m} (T E_{R} \cdot \nabla_{v} G_{R}) \cdot w^{2l} \partial_{\beta}^{m \perp} G_{R} \rangle dx
\]
\[\leq C \varepsilon^{2} \| E_{R} \|_{H_{x}^{N}} \left( \sum_{|m| \leq N} \| \partial^{m} P G_{R} \|_{L_{x}^{2}, v}^{2} + \frac{1}{\varepsilon^{2}} \sum_{|m| + |\beta| \leq N + 1} \| w^{l} \partial_{\beta}^{m \perp} G_{R} \|_{L_{x}^{2}, v}^{2} \right).\]
(7.40)

For the term $-T (v \times B_{R}) \cdot \nabla_{v} G_{R}$ in (7.5), we deduce that for $|m| + |\beta| \leq N + 1$ and $\beta \neq 0$
\[
\int_{T_{3}} \langle \partial_{\beta}^{m} (\varepsilon T E_{R} \cdot v G_{R}) \cdot w^{2l} \partial_{\beta}^{m \perp} G_{R} \rangle dx
\]
\[= \sum_{m_{1} \leq m} \int_{T_{3}} \langle (\varepsilon T \partial^{m_{1}} E_{R} \cdot v \partial_{\beta}^{m_{1}} (P G_{R} + P \perp G_{R})) \cdot w^{2l} \partial_{\beta}^{m \perp} G_{R} \rangle dx
\]
\[+ \sum_{m_{1} \leq m} \int_{T_{3}} \langle (\varepsilon T \partial^{m_{1}} E_{R} \cdot \partial_{\beta_{1}} v \partial_{\beta_{-1}}^{m_{1}} (P G_{R} + P \perp G_{R})) \cdot w^{2l} \partial_{\beta}^{m \perp} G_{R} \rangle dx
\]
\[\leq C \varepsilon \sum_{m_{1} \leq m} \int_{T_{3}} |\partial^{m_{1}} E_{R}| \langle |\partial^{m_{1}} P G_{R} \|_{L_{x}^{2}} + \| w^{l} \partial_{\beta_{1}}^{m_{1}} P \perp G_{R} \|_{L_{x}^{2}} \rangle \| w^{l} \partial_{\beta_{-1}}^{m_{1}} P \perp G_{R} \|_{L_{x}^{2}} d x
\]

\[ + C \varepsilon \sum_{m_1 \leq m} \frac{1}{\varepsilon^2} \sum_{|m|+|\beta| \leq N+1} \| w^l \partial^{m_1}_{\beta_1} \|_{L^2}^2 \leq C \varepsilon^2 \| E_R \|_{H^N_x} \left( \sum_{|m|+|\beta| \leq N+1} \| w^l \partial^{m_1}_{\beta_1} \|_{L^2}^2 \right) \leq C \varepsilon^2 \| E_R \|_{H^N_x} \left( \sum_{|m|+|\beta| \leq N+1} \| w^l \partial^{m_1}_{\beta_1} \|_{L^2}^2 \right) \] (7.41)

By the same arguments of the inequalities (7.40) and (7.41), we estimate that for \(|m|+|\beta| \leq N+1\) and \(\beta \neq 0\)

\[ \int_{T^3} \langle \partial^m (-\varepsilon T \vec{E}_1 \cdot \nabla_v G_R + \varepsilon T \vec{E}_1 \cdot v G_R) \cdot w^2 \partial^m_{\beta} \| G_R \|_{L^2}^2 \rangle \leq C \varepsilon^2 \int_{T^3} \langle \partial^m (-\varepsilon T \vec{E}_1 \cdot \nabla_v G_R + \varepsilon T \vec{E}_1 \cdot v G_R) \cdot w^2 \partial^m_{\beta} \| G_R \|_{L^2}^2 \rangle \] (7.42)

and

\[ \int_{T^3} \langle \partial^m (-\varepsilon T \vec{E}_0 \cdot \nabla_v G_R + \varepsilon T \vec{E}_0 \cdot v G_R) \cdot w^2 \partial^m_{\beta} \| G_R \|_{L^2}^2 \rangle \leq C \varepsilon^2 \int_{T^3} \langle \partial^m (-\varepsilon T \vec{E}_0 \cdot \nabla_v G_R + \varepsilon T \vec{E}_0 \cdot v G_R) \cdot w^2 \partial^m_{\beta} \| G_R \|_{L^2}^2 \rangle \] (7.43)

where we also make use of Lemma 4.2 and Lemma 4.1 in (7.42) and (7.43) respectively.

For the term \(-T (v \times B_R) \cdot \nabla_v G_R\) in (7.5), we deduce that for \(|m|+|\beta| \leq N+1\) and \(\beta \neq 0\)

\[ \int_{T^3} \langle \partial^m (-T (v \times B_R) \cdot \nabla_v G_R) \cdot w^2 \partial^m_{\beta} \| G_R \|_{L^2}^2 \rangle \leq C \varepsilon^2 \int_{T^3} \langle \partial^m (-T (v \times B_R) \cdot \nabla_v G_R) \cdot w^2 \partial^m_{\beta} \| G_R \|_{L^2}^2 \rangle \]
+ \int_{\mathbb{T}^3} \langle -\mathcal{T}(v \times B_R) \cdot \nabla v \partial^m_{\beta} \mathbb{P} G_R \rangle \cdot w^{2l} \partial^m_{\beta} \mathbb{P} G_R \rangle dx \\
\leq C \|B_R\|_{H^N_x} \sum_{|m| \leq N} \|\partial^m \mathbb{P} G_R\|_{L^2_{x,v}} \sum_{|m|+|\beta| \leq N+1} \|\partial^m_{\beta} \mathbb{P} G_R\|_{L^2_{x,v}(\nu)} \\
+ C \|B_R\|_{H^N_x} \sum_{|m|+|\beta| \leq N+1} \|\partial^m_{\beta} \mathbb{P} G_R\|_{L^2_{x,v}(\nu)}^2 \tag{7.44}
\\
\leq C \varepsilon^2 \|B_R\|_{H^N_x} \left( \sum_{|m| \leq N} \|\partial^m \mathbb{P} G_R\|_{L^2_{x,v}}^2 + \frac{1}{\varepsilon^2} \sum_{|m|+|\beta| \leq N+1} \|\partial^m_{\beta} \mathbb{P} G_R\|_{L^2_{x,v}(\nu)}^2 \right),

where we make use of the inequality (7.20) and the cancelation

\langle -\mathcal{T}(v \times B_R) \cdot \nabla v \partial^m_{\beta} \mathbb{P} G_R \rangle \cdot w^{2l} \partial^m_{\beta} \mathbb{P} G_R \rangle \\
= \frac{1}{\varepsilon^2} \langle (v \times B_R) \cdot (2l w^{2l-2}(v) - 1) v \rangle \mathcal{T} \partial^m_{\beta} \mathbb{P} G_R \cdot \partial^m_{\beta} \mathbb{P} G_R \rangle = 0.

Similarly, we have

\sum_{|m|+|\beta| \leq N+1} \sum_{\beta \neq 0} \int_{\mathbb{T}^3} \langle \partial^m_{\beta} (-\mathcal{T}(v \times B_1) \cdot \nabla v G_R) \cdot w^{2l} \partial^m_{\beta} \mathbb{P} G_R \rangle dx \\
\leq C \varepsilon^2 \sqrt{\varepsilon_{0,N+2}} \left( \sum_{|m| \leq N} \|\partial^m \mathbb{P} G_R\|_{L^2_{x,v}}^2 + \frac{1}{\varepsilon^2} \sum_{|m|+|\beta| \leq N+1} \|\partial^m_{\beta} \mathbb{P} G_R\|_{L^2_{x,v}(\nu)}^2 \right), \tag{7.45}

and

\sum_{|m|+|\beta| \leq N+1} \sum_{\beta \neq 0} \int_{\mathbb{T}^3} \langle \partial^m_{\beta} (-\mathcal{T}(v \times B_R) \cdot \nabla v (G_1 + \varepsilon G_2)) \cdot w^{2l} \partial^m_{\beta} \mathbb{P} G_R \rangle dx \\
\leq C \varepsilon^2 \left( \|\nabla v B_R\|_{H^N_x}^2 + \frac{1}{\varepsilon^2} \sum_{|m|+|\beta| \leq N+1} \|\partial^m_{\beta} \mathbb{P} G_R\|_{L^2_{x,v}(\nu)}^2 \right). \tag{7.46}

One notices that $G_0$, $G_1$ and $G_2$ are the known vector valued functions, which can be controlled by applying Lemma 4.1 and 4.2. Then, we can easily deduce that

\sum_{|m|+|\beta| \leq N+1} \int_{\mathbb{T}^3} \langle \partial^m_{\beta} [-\mathcal{T} E_R \cdot \nabla v (G_0 + \varepsilon G_1 + \varepsilon^2 G_2) \\
+ \mathcal{T} E_R \cdot v (G_0 + \varepsilon G_1 + \varepsilon^2 G_2)] \cdot w^{2l} \partial^m_{\beta} \mathbb{P} G_R \rangle dx \\
\leq C \varepsilon^2 \|E_R\|_{H^N_x} \left( D_{0,N+5}(t) + D_{1,N+3}(t) + \frac{1}{\varepsilon^2} \sum_{|m|+|\beta| \leq N+1} \|\partial^m_{\beta} \mathbb{P} G_R\|_{L^2_{x,v}(\nu)}^2 \right). \tag{7.47}
Together with the inequalities (7.40), (7.41), (7.42), (7.43), (7.44), (7.45), (7.46) and (7.47), we obtain

\[
\sum_{|m|+|\beta| \leq N+1} \int_{\mathbb{T}^3} \langle \partial^m H_R^{(2)} \cdot w^{2l} \partial^m_{\beta} \mathbb{P} G_R \rangle dx \\
\leq C \varepsilon^2 \left( 1 + \| E_R \|_{L^2} + \| B_R \|_{L^2} \right) \left( D_{0,R}^{N+5} + D_{1,R}^{N+3} + \sum_{|m| + |\beta| \leq N} \| \partial^m \mathbb{P} G_R \|_{L^2} \right) \\
+ \| \nabla_x B_R \|_{L^2}^{N-1} + \frac{1}{\varepsilon^2} \sum_{|m| + |\beta| \leq N} \| w^l \partial^m \mathbb{P} G_R \|_{L^2_{x,v}} + \frac{1}{\varepsilon^2} \sum_{|m| + |\beta| \leq N+1} \| w^l \partial^m \mathbb{P} G_R \|_{L^2_{x,v}}. 
\]

(7.48)

Finally, for the term \( H_R^{(3)} \) in (7.6), one easily derives from Lemma 4.1 and 4.2 that

\[
\sum_{|m|+|\beta| \leq N+1} \int_{\mathbb{T}^3} \langle \partial^m H_R^{(3)} \cdot w^{2l} \partial^m_{\beta} \mathbb{P} G_R \rangle dx \\
\leq C \varepsilon^2 \left( D_{0,R}^{N+5} + D_{1,R}^{N+3} + \sum_{|m| + |\beta| \leq N} \| w^l \partial^m \mathbb{P} G_R \|_{L^2_{x,v}} \right). 
\]

(7.49)

Consequently, combining the inequalities (7.35), (7.48) and (7.49) reduces to

\[
\left| \sum_{|m|+|\beta| \leq N+1} \int_{\mathbb{T}^3} \langle H_R \cdot w^{2l} \partial^m_{\beta} \mathbb{P} G_R \rangle dx \right| \\
\leq C \varepsilon^2 \left( 1 + \| E_R \|_{L^2} + \| B_R \|_{L^2} + \sum_{|m| + |\beta| \leq N} \| \partial^m \mathbb{P} G_R \|_{L^2} \right) \\
+ \sum_{|m| \leq N} \| \partial^m \mathbb{P} G_R \|_{L^2_{x,v}} + \sum_{|m| + |\beta| \leq N+1} \| w^l \partial^m \mathbb{P} G_R \|_{L^2_{x,v}} \\
\times \left( D_{0,R}^{N+5} + D_{1,R}^{N+3} + \| \nabla_x B_R \|_{L^2}^{N-1} + \frac{1}{\varepsilon^2} \sum_{|m| + |\beta| \leq N} \| w^l \partial^m \mathbb{P} G_R \|_{L^2_{x,v}} \right) \right) \\
+ \sum_{|m| \leq N} \| \mathbb{P} G_R \|_{L^2_{x,v}}^{2} + \frac{1}{\varepsilon^2} \sum_{|m| + |\beta| \leq N+1} \| w^l \partial^m \mathbb{P} G_R \|_{L^2_{x,v}}^{2}. 
\]

(7.50)

By the similar arguments of the proof of (7.50), we deduce that

\[
\left| \sum_{|m|+|\beta| \leq N} \int_{\mathbb{T}^3} \langle H_R \cdot w^{2l} \partial^m_{\beta} \mathbb{P} G_R \rangle dx \right| \\
\leq C \varepsilon^2 \left( 1 + \| E_R \|_{L^2} + \| B_R \|_{L^2} + \sum_{|m| \leq N} \| \partial^m \mathbb{P} G_R \|_{L^2_{x,v}} \right) \\
+ \sum_{|m| + |\beta| \leq N} \| \partial^m \mathbb{P} G_R \|_{L^2_{x,v}} \left( D_{0,R}^{N+5} + D_{1,R}^{N+3} + \| \nabla_x B_R \|_{L^2}^{N-1} \right) \\
+ \sum_{|m| \leq N} \| \mathbb{P} G_R \|_{L^2_{x,v}}^{2} + \frac{1}{\varepsilon^2} \sum_{|m| + |\beta| \leq N} \| w^l \partial^m \mathbb{P} G_R \|_{L^2_{x,v}}^{2}. 
\]

(7.51)

One notices that

\[
\langle \Gamma_0 G_R \cdot \mathbb{P} G_R \rangle = \langle H_R^{(1)} \cdot \mathbb{P} G_R \rangle = 0, 
\]

(7.52)
where the terms $\Gamma_0 G_R$ and $B_R^{(1)}$ are given in (1.18) and (7.4) respectively. Based on the relations (7.52), we derive from following the derivation of the inequality (7.50) that

$$\left| \sum_{|m| \leq N+1} \int_{T^3} \langle \partial^m H_R \cdot \partial^m G_R \rangle dx \right| \leq C \varepsilon^2 \left( 1 + \| E_R \|_{H_N^{s+1}} + \| B_R \|_{H_N^{s+1}} + \sum_{|m| \leq N+1} \| \partial^m \Psi G_R \|_{L^2_{x,v}} \right)$$

$$\times \left( D_{0,N+5}(t) + \sum_{|m| \leq N+1} \| \partial^m \Psi G_R \|_{L^2_{x,v}}^2 \right)$$

$$+ D_{1,N+3}(t) + \frac{1}{\varepsilon^2} \sum_{|m| \leq N+1} \| \partial^m \Psi G_R \|_{L^2_{x,v}(v)}^2 \right)$$

(7.53)

Therefore, recalling that the definitions of the functionals $\tilde{E}_{N,I}(G_R, E_R, B_R)$ and $\mathbb{D}_{N,I}(G_R, E_R, B_R)$ in (1.25) and (1.27) respectively, and combining the inequalities (7.50), (7.51) and (7.53), we can establish the relation (7.33), and the proof of Lemma 7.2 is completed. □

8. Proof of the main theorem

Based on the previous lemmas, we will give the proof of Theorem 1.1 in this section. We first figure out that if the number $\eta_0$ is sufficiently small, the initial assumption (1.28) will imply the initial conditions of Lemma 4.1 with the case $s = N + 5$ and Lemma 4.2 with the case $M = N + 3$. Thus, all the previous lemmas are validity. We multiply by $\rho$ on the inequalities (6.2) in Lemma 6.1 and (6.18) in Lemma 6.2, and then add them and the $\Upsilon$ times of the inequality (4.14) for the case $M = N + 3$ in Lemma 4.2 to the inequality (5.78) in Lemma 5.4, where the constant $\rho > 0$ is small and $\Upsilon > 0$ is large to be determined. We thereby obtain

$$\frac{d}{dt} \left( E_p(t) + \rho \sum_{|m|+|\beta| \leq N} \| w^l \partial^m_{\beta} \Psi G_R \|_{L^2_{x,v}}^2 \right)$$

$$+ C_0 \mathbb{D}_p(t) + \frac{\rho}{2\varepsilon^2} \sum_{|m|+|\beta| \leq N} \| w^l \partial^m_{\beta} \Psi G_R \|_{L^2_{x,v}(v)}^2 \right)$$

$$+ C \sum_{|m| \leq N+1} \int_{T^3} \langle \partial^m H_R \cdot \partial^m G_R \rangle dx$$

(8.1)

$$+ C \sum_{|m| \leq N+1} \int_{T^3} \langle \partial^m H_R \cdot \partial^m G_R \rangle dx + C \sum_{|m|+|\beta| \leq N+1} \int_{T^3} \langle \partial^m H_R \cdot \partial^m G_R \rangle dx$$

where the functionals $\tilde{E}_{N,I}(G_R, E_R, B_R)$ and $\mathbb{D}_{N,I}(G_R, E_R, B_R)$ are defined in (1.25) and (1.27) respectively, and the energy $E_p(t)$ and $\mathbb{D}_p(t)$ are defined as

$$E_p(t) = \| E_R \|_{H_N^{s+1}}^2 + (1 - \delta + \sigma \delta) \| B_R \|_{H_N^{s+1}}^2 + (1 - \delta) \| \partial_t B_R \|_{H_N^{s+1}}^2 + \delta \| \partial_t B_R + B_R \|_{H_N^{s+1}}^2$$

$$- C_3 \varepsilon \mathcal{A}(t) + \varepsilon^2 \sum_{|m| \leq N-1} \langle \Delta \psi \cdot \tilde{\mathcal{F}}(t) \rangle_{L^2_{x,v}}^2 + \Upsilon (E_{1,N+3} + \widetilde{\mathcal{C}}_{N+3} E_{0,N+5}(t))$$

(8.2)
and
\[ D_p(t) = \|E_R\|_{H^2}^2 + \|\nabla_x B_R\|_{H^2}^2 + \|\partial_t B_R\|_{H^2}^2 + \frac{1}{C_6}(D_{1,N+3}(t) + D_{0,N+5}(t)) \]
\[ + \|\text{div}_x E_R\|_{H^2}^2 + \sum_{|m| \leq N+1} \|\partial_m^{\perp} G_R\|_{L^2}^2 + \frac{1}{\varepsilon^2} \sum_{|m| \leq N+1} \|\partial_m^{\perp} G_R\|_{L^2}^2 \quad \text{(8.3)} \]

If we choose \( \rho > 0 \) small such that
\[ 2C_\rho \leq \frac{1}{2} C_6 , \]
then the term \( 2C_\rho D_p(t) \) in the right-hand side of (8.1) can be absorbed by the left-hand term \( C_0 D_p(t) \). Furthermore, by the definitions (1.25) and (1.27), the bound (7.2) implies that there is a constant \( C_0 > 0 \) such that the term \( C\varepsilon^2 \sum_{|m| \leq N} \|\partial_m^{\perp} G_R\|_{L^2}^2 \) in (8.1) is bounded by
\[ C\varepsilon^2 \sum_{|m| \leq N} \|\partial_m^{\perp} G_R\|_{L^2}^2 \leq C_0(D_{0,N+5}(t) + D_{1,N+3}(t)) \quad \text{(8.4)} \]

We take \( C_0 = C_0 + C_0 > 0 \) such that the term \( C_0(D_{0,N+5}(t) + D_{1,N+3}(t)) \) in the right-hand side of (8.4) can be absorbed by \( C_0 D_p(t) \) in (8.1). We figure out that we require the condition \( l \geq 2\gamma + 1 \) such that the other terms in (8.4) can be controlled. More precisely,
\[ \mathbb{E}_{N,2\gamma+1}(G_R, E_R, B_R) \leq \mathbb{E}_{N,1}(G_R, E_R, B_R) , \]
\[ \mathbb{D}_{N,2\gamma+1}(G_R, E_R, B_R) \leq \mathbb{D}_{N,1}(G_R, E_R, B_R) . \]

So, we have constructed an energy functional
\[ E_{N,l}(G_R, E_R, B_R) = E_p(t) + \rho \sum_{|m|+|\beta| \leq N} \|w^{\dagger} \partial_\beta^{\perp} G_R\|_{L^2}^2 + \rho \sum_{|m|+|\beta| \leq N+1} \|w^{\dagger} \partial_\beta^{\perp} G_R\|_{L^2}^2 \quad \text{(8.5)} \]
for \( l \geq 2\gamma + 1 \). One easily verifies that the energy functional \( E_{N[l]}(G_R, E_R, B_R) \) is an instant energy functional with respect to the given energy functional \( \mathbb{E}_{N,1}(G_R, E_R, B_R) \) for \( \varepsilon \) sufficiently small.

We emphasize that the term \( C\varepsilon_{0,N+2} \sum_{|m| \leq N} \|\nabla_x \partial_m^{\perp} G_R\|_{L^2}^2 \) in the right-hand side of (8.1) can be dominated by
\[ C\varepsilon_{0,N+2} \sum_{|m|+|\beta| \leq N+1} \|w^{\dagger} \partial_\beta^{\perp} G_R\|_{L^2}^2 \leq C\varepsilon_{0,N+2} \sum_{|m|+|\beta| \leq N+1} \|w^{\dagger} \partial_\beta^{\perp} G_R\|_{L^2}^2 \quad \text{(8.6)} \]
which means the this term can be absorbed by the term \( C'\mathbb{D}_{N,1}(G_R, E_R, B_R) \), where the constant \( C' = \frac{1}{4} \min\{C_6, \rho\} > 0 \).

Combining the initial condition (1.28) and the inequality (7.33) in Lemma 7.2, we summarize the above statements that
\[ \frac{d}{dt} \mathbb{E}_{N,l}(G_R, E_R, B_R) + C'\mathbb{D}_{N,l}(G_R, E_R, B_R) \]
\[ \leq \tilde{C}(\varepsilon + \varepsilon^2 + \sqrt{\eta_0 + \eta_0}) \mathbb{D}_{N,1}(G_R, E_R, B_R) \]
\[ + \tilde{C}\mathbb{E}_{N,l}(G_R, E_R, B_R) \mathbb{D}_{N,l}(G_R, E_R, B_R) \quad \text{(8.6)} \]
holds for \( l \geq 2\gamma + 1 \) and some constant \( \tilde{C} > 0 \). We thereby take some small \( \varepsilon_0 \) and \( \eta_0 \), depending only on \( \mu, \sigma, \kappa \) and \( N \), such that for all \( \varepsilon \in (0, \varepsilon_0) \)
\[ \tilde{C}(\varepsilon + \varepsilon^2 + \sqrt{\eta_0 + \eta_0}) \leq \frac{1}{2} C' . \]
Then, the inequality (8.6) reduces to
\[
\frac{d}{dt} \mathbb{E}_{N,l}(R, E, B) + \frac{1}{2} C' \mathbb{D}_{N,l}(R, E, B) \leq \tilde{C} C_0 \mathbb{E}_{N,l}(R, E, B) \mathbb{D}_{N,l}(R, E, B) \quad (8.7)
\]
for \( l \geq 2\gamma + 1 \) and \( N \geq 4 \), where we utilize the definition of the instant energy functional with respect to the energy \( \mathbb{E}_{N,l}(R, E, B) \).

Moreover, we have
\[
\mathbb{E}_{N,l}(R, E, B)(0) \leq \overline{C} \mathbb{E}_{N,l}(R^\infty, E^\infty, B^\infty)(0) \leq \overline{C} \eta_0 \quad (8.8)
\]
for some constant \( \overline{C} > 0 \). We further take \( \eta_0 \leq \frac{C'}{8SC_0} \) such that
\[
\tilde{C} C_0 \mathbb{E}_{N,l}(R, E, B)(0) \leq \frac{1}{8} C'.
\]
We now define
\[
T := \sup \left\{ \tau \geq 0; \sup_{t \in [0, \tau]} \tilde{C} C_0 \mathbb{E}_{N,l}(R, E, B)(t) \leq \frac{1}{8} C' \right\}.
\]
By the continuity of the instant energy functional \( \mathbb{E}_{N,l}(R, E, B)(t) \), we know that \( T > 0 \).

We assert that \( T = \infty \). Indeed, if \( T < \infty \), the inequality (8.7) with the definition of \( T \) implies that for all \( t \in [0, T] \)
\[
\frac{d}{dt} \mathbb{E}_{N,l}(R, E, B) + \frac{1}{2} C' \mathbb{D}_{N,l}(R, E, B) \leq 0.
\]
Integrating the above inequality on \([0, t]\), we have
\[
\mathbb{E}_{N,l}(R, E, B)(t) \leq \mathbb{E}_{N,l}(R, E, B)(0) \leq \frac{C'}{8SC_0},
\]
hence for all \( t \in [0, T] \)
\[
\tilde{C} C_0 \mathbb{E}_{N,l}(R, E, B)(t) \leq \frac{1}{8} C' < \frac{1}{4} C',
\]
which contradicts to the definition of \( T \) by the continuity of the instant energy functional \( \mathbb{E}_{N,l}(R, E, B)(t) \). Thus \( T = \infty \) and the proof of Theorem 1.1 is completed.

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