Some Polycubes Have No Edge-Unzipping

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Abstract

It is unknown whether or not every polycube has an edge-unfolding. A polycube is an object constructed by gluing cubes face-to-face. An edge-unfolding cuts edges on the surface and unfolds it to a net, a non-overlapping polygon in the plane. Here we explore the more restricted edge-unzippings where the cut edges form a path. We construct two different polycubes neither of which has an edge-unzipping.

1 Introduction

A polycube $P$ is an object constructed by gluing cubes whole-face to whole-face, such that its surface is a manifold. Thus the neighborhood of every surface point is a disk; so there are no edge-edge nor vertex-vertex nonmanifold surface touchings. Here we only consider polycubes of genus zero. The edges of a polycube are all the cube edges on the surface, even when those edges are shared between two coplanar faces. Similarly, the vertices of a polycube are all the cube vertices on the surface, even when those vertices are flat, incident to $2\pi$ face angles. Such polycube flat vertices are degree-4. It will be useful to distinguish these flat vertices from corner vertices, non-flat vertices with incident angles $\neq 2\pi$ (degree-3, -5, or -6). For a polycube $P$, let its 1-skeleton graph $G_P$ include every vertex and edge of $P$, with vertices marked as either corner or flat.

It is an open problem to determine whether every polycube has an edge-unfolding, a tree in the 1-skeleton that spans all corner vertices (but need not include flat vertices), which, when cut, unfolds the surface to a net, a planar, non-overlapping polygon [O’R19]. Here by non-overlapping is meant that no two points, each interior to a face, are mapped to same point in the plane. This

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allows two boundary edges to coincide in the net; so the polygon is “weakly simple.” The intent is that we want to be able to cut out the net and refold to $P$. Henceforth “edge-unfolding” will mean: an edge-unfolding to a net.

It would be remarkable if it were true that every polycube could be edge-unfolded, but no counterexample is known. There has been considerable exploration of orthogonal polyhedra, a more general type of object, for which there are examples that cannot be edge-unfolded \cite{BDD+98}. (See \cite{DFIS} for citations to earlier work.) But polycubes have more edges in their 1-skeleton graphs for the cut tree to follow than do orthogonal polyhedra, so it is conceivably easier to edge-unfold polycubes.

A restriction of edge-unfolding has been studied in \cite{DDL+10, OR10, DDU13}: edge-unzipping. This is an edge-unfolding whose cut tree is a path (so that the surface could be “unzipped”). It is apparently unknown if even this highly restricted edge-unzipping could unfold every polycube to a net. The result of this note is to settle this question in the negative: two different polycubes are constructed each of which has no edge-unzipping. They are shown in Figure 1 and will be described later.

Figure 1: Two polycubes that have no edge-unzipping.

2 Hamiltonian Paths

Shephard \cite{She75} introduced Hamiltonian unfoldings of convex polyhedra, what we are now calling edge-unzippings, following the terminology of \cite{DDL+10}. It is easy to see that not every convex polyhedron has an edge-unzipping, simply because the rhombic dodecahedron has no Hamiltonian path. This counterexample avoids confronting the difficult non-overlapping condition. We follow a

\[^1\] “Unzipping” is a slight variation on their “zipper unfoldings.”
similar strategy here, constructing a polycube with no Hamiltonian path. But
there is a difference in that a polycube edge-unzipping need not include flat
vertices, and so need not be a Hamiltonian path in $G_P$. Thus identifying a
polycube $P$ that has no Hamiltonian path does not immediately establish that
$P$ has no edge-unzipping, if $P$ has flat vertices.

So one approach is to construct a polycube $P$ that has no flat vertices—every
vertex is a corner vertex. Then if $P$ has no Hamiltonian path, then it has no
edge-unzipping. A natural candidate is the polycube object $P_6$ shown in Fig. 2.
However, the 1-skeleton of $P_6$ does admit Hamiltonian paths, and indeed we
found a path that unfolds $P_6$ to a net.

Let $\overline{G}_P$ be the dual graph of $P$: each cube is a node, and two nodes are
connected if they are glued face-to-face. A polycube tree is a polycube whose
dual graph is a tree. $P_6$ is a polycube tree. That it has a Hamiltonian path is
an instance of a more general claim:

**Lemma 1** The graph $G_P$ for any polycube tree $P$ has a Hamiltonian cycle.

**Proof:** It is easy to see by induction that every polycube tree can be built by
gluing cubes each of which touches just one face at the time of gluing: never is
there a need to glue a cube to more than one face of the previously built object.

A single cube has a Hamiltonian cycle. Now assume that every polycube tree
of $\leq n$ cubes has a Hamiltonian cycle. For a tree $P$ of $n+1$ cubes, remove a $\overline{G}_P$
leaf-node cube $C$, and apply the induction hypothesis. The exposed square face
$f$ to which $C$ glues to make $P$ includes either 2 or 3 edges of the Hamiltonian
cycle (4 would close the cycle; 1 or 0 would imply the cycle misses some vertices
of $f$). It is then easy to extend the Hamiltonian cycle to include $C$, as shown
in Figure 3.

So to prove that a polycube tree has no edge-unzipping would require an argu-
ment that confronted non-overlap. This leads to an open question:

**Question 1** Does every polycube tree have an edge-unzipping?
To guarantee the non-existence of Hamiltonian paths, we can exploit the bipartiteness of \( G_P \), using Lemma 3 below.

**Lemma 2** A polycube graph \( G_P \) is 2-colorable, and therefore bipartite.

**Proof:** Label each lattice point \( p \) of \( \mathbb{Z}^3 \) with the \( \{0, 1\} \)-parity of the sum of the Cartesian coordinates of \( p \). A polycube \( P \)'s vertices are all lattice points of \( \mathbb{Z}^3 \). This provides a 2-coloring of \( G_P \); 2-colorable graphs are bipartite.

The **parity imbalance** in a 2-colored (bipartite) graph is the absolute value of the difference in the number of nodes of each color.

**Lemma 3** A bipartite graph \( G \) with a parity imbalance \( > 1 \) has no Hamiltonian path\(^2\).

**Proof:** The nodes in a Hamiltonian path alternate colors 010101.... Because by definition a Hamiltonian path includes every node, the parity imbalance in a bipartite graph with a Hamiltonian path is either 0 (if of even length) or 1 (if of odd length).

So if we can construct a polycube \( P \) that (a) has no flat vertices, and (b) has parity imbalance \( > 1 \), then we will have established that \( P \) has no Hamiltonian path, and therefore no edge-unzipping. We now show that the polycube \( P_{44} \), illustrated in Figure 4, meets these conditions.

**Lemma 4** The polycube \( P_{44} \)'s graph \( G_{P_{44}} \) has parity imbalance of 2.

**Proof:** Consider first the \( 2 \times 2 \times 2 \) cube that is the core of \( P_{44} \); call it \( P_{222} \). The front face \( F \) has an extra 0; see Fig. 5. It is clear that the 8 corners of \( P_{222} \) are all colored 0. The midpoint vertices of the 12 edges of \( P_{222} \) are colored 1. Finally the 6 face midpoints are colored 0. So 14 vertices are colored 0 and 12 colored 1.

\(^2\) Stated at [http://mathworld.wolfram.com/HamiltonianPath.html](http://mathworld.wolfram.com/HamiltonianPath.html)
Figure 4: The polycube $P_{44}$, consisting of 44 cubes, has no Hamiltonian path.

Figure 5: 2-coloring of one face of $P_{222}$. 
Next observe that attaching a cube $C$ to exactly one face of any polycube does not change the parity: the receiving face $f$ has colors 0101, and the opposite face of $C$ has colors 1010.

Now, $P_{44}$ can be constructed by attaching six copies of a 6-cube “cross,” call it $P_+$, which in isolation is a polycube tree and so can be built by attaching cubes each to exactly one face. And each $P_+$ attaches to one corner cube of $P_{222}$. Therefore $P_{44}$ retains $P_{222}$’s imbalance of 2.

The point of the $P_+$ attachments is to remove the flat vertices of $P_{222}$. Note that when attached to $P_{222}$, each $P_+$ has only corner vertices.

**Theorem 1** *There is no edge-unzipping of $P_{44}$.*

**Proof:** Although it takes some scrutiny of Figure 4 to verify, $P_{44}$ has no (degree-4) flat vertices. Thus an edge-unzipping must pass through every vertex, and so be a Hamiltonian path. Lemma 4 says that $G_{P_{44}}$ has imbalance 2, and Lemma 3 says it therefore cannot have a Hamiltonian path.

### 4 Construction of $P_{14}$

It turns out that the smaller polycube $P_{14}$ shown in Figure 6 also has no edge-unzipping, even though it has flat vertices. To establish this, we still need an imbalance $> 1$, which easily follows just as in Lemma 3.
Lemma 5  The polycube $P_{14}$’s graph $G_{P_{14}}$ has parity imbalance of 2.

But notice that $P_{14}$ has three flat vertices: $a$, $b$, and $c$.

Theorem 2  There is no edge-unzipping of $P_{14}$.

Proof: An edge-unzipping need not pass through the three flat vertices, $a$, $b$, and $c$, but it could pass through one, two, or all three. We show that in all cases, an appropriately modified subgraph of $G_{P_{14}}$ has no Hamiltonian path. Let $\rho$ be a hypothetical edge-unzipping cut path. We consider four exhaustive possibilities, and show that each leads to a contradiction.

(0) $\rho$ includes $a$, $b$, $c$. So $\rho$ is a Hamiltonian path in $G_{P_{14}}$. But Lemma 5 says that $G_{P_{14}}$ has imbalance 2, and Lemma 3 says that no such graph has a Hamiltonian path.

(1) $\rho$ excludes one flat vertex $a$ and includes $b$, $c$. (Because of the symmetry of $P_{14}$, it is no loss of generality to assume that it is $a$ that is excluded.) If $\rho$ excludes $a$, then it does not travel over any of the four edges incident to $a$. Thus we can delete $a$ from $G_{P_{14}}$; say that $G_{-a} = G_{P_{14}} \setminus a$. This graph is shown in Fig. 7. Following the coloring in Fig. 5 all corners of $P_{222}$ are colored 0, so each of the edge midpoints $a$, $b$, $c$ is colored 1. The parity imbalance of $P_{14}$ is 2 extra 0’s. Deleting $a$ maintains bipartiteness and increases the parity imbalance of $G_{-a}$ to 3. Therefore by Lemma 3, $G_{-a}$ has no Hamiltonian path, and such a $\rho$ cannot exist.

(2) $\rho$ includes just one flat vertex $c$, and excludes $a$, $b$. (Again symmetry ensures there is no loss of generality in assuming the one included flat vertex is $c$.) $\rho$ must include corner $x$, which is only accessible in $G_{P_{14}}$ through the three flat vertices. If $\rho$ excludes $a$, $b$, then it must include the edge $cx$. Let $G_{-ab} = G_{P_{14}} \setminus \{a,b\}$. In $G_{-ab}$, $x$ has degree 1, so $\rho$ terminates there. It must be that $\rho$ is a Hamiltonian path in $G_{-ab}$, but the deletion of $a$, $b$ increases the parity imbalance to 4, and so again such a Hamiltonian path cannot exist.

(3) $\rho$ excludes $a$, $b$, $c$. Because corner $x$ is only accessible through one of these flat vertices, $\rho$ never reaches $x$ and so cannot be an edge-unzipping

Thus the assumption that there is an edge-unzipping path $\rho$ for $P_{14}$ reaches a contradiction in all four cases. Therefore, there is no edge-unzipping path for $P_{14}$.  

5  Edge-unfoldings of $P_{14}$ and $P_{44}$

Now that it is known that $P_{14}$ and $P_{44}$ each have no edge-unzipping, it is natural to wonder if either settles the edge-unfolding open problem: Can they
Figure 7: Schlegel diagram of $G_{-a}$. We follow [DF18] in labeling the faces of a cube as $F, K, R, L, T, B$ for Front, Back, Right, Left, Top, Bottom respectively. The corners of $P_{222}$ are labeled 0, 1, 2, 3 around the bottom face $B$, and 4, 5, 6, 7 around the top face $T$. $m$ is the vertex in the middle of $B$. The edges deleted by removing $a$ are shown dashed.
be edge-unfolded? Indeed both can: see Figures 8 and 9. The colors in these layouts are those used by Origami Simulator [GDG18]. Figure 10 shows a partial folding of $P_{44}$, and animations are at [http://cs.smith.edu/~jorourke/Unf/NoEdgeUnzip.html](http://cs.smith.edu/~jorourke/Unf/NoEdgeUnzip.html).

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\[^3\] Just to verify this conclusion, we constructed these graphs in Mathematica and `FindHamiltonianPath[]` returned {} for each.
Figure 9: Edge-unfolding of $P_{44}$. Colors: green=cut, red=mountain, blue=valley, yellow=flat.
Figure 10: Partial folding of the layout in Fig. 9. Compare Fig. 4.
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