On Edge Dimension of a Graph

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Abstract

Given a connected graph $G(V, E)$, the edge dimension, denoted $\text{edim}(G)$, is the least size of a set $S \subseteq V$ that distinguishes every pair of edges of $G$, in the sense that the edges have pairwise different tuples of distances to the vertices of $S$. The notation was introduced by Kelenc, Tratnik, and Yero, and in their paper they asked several questions about some properties of $\text{edim}$. In this article we answer two of these questions: we classify the graphs on $n$ vertices for which $\text{edim}(G) = n-1$ and show that $\frac{\text{edim}(G)}{\text{dim}(G)}$ isn't bounded from above (here $\text{dim}(G)$ is the standard metric dimension of $G$). We also compute $\text{edim}(G \square P_m)$ and $\text{edim}(G + K_1)$.

1 Introduction

Let $G(V, E)$ be a simple unconnected graph. We define the distance between an edge $e = xy$ and vertex $v$ as:

$$d(e, v) = \min\{d(x, v), d(y, v)\}.$$ 

A vertex $v$ distinguishes two edges $e_1$ and $e_2$ if $d(e_1, v) \neq d(e_2, v)$. A set $S \subseteq V$ is an edge metric generator of a graph $G(V, E)$ if for any two distinct edges $e_1, e_2 \in E$ there is a vertex $s \in S$ such that $s$ distinguishes $e_1$ and $e_2$. An edge generating set with the smallest number of elements is called an edge basis of $G$, and the number of elements in an edge basis is the edge dimension of $G$ (denoted $\text{edim}(G)$).

This concept was introduced by Kelenc, Tratnik and Yero in [6] in analogy with the classical metric dimension $\text{dim}(G)$ defined as follows: a vertex $v \in V$
distinguishes \( v_1, v_2 \in V \) if \( d(v, v_1) \neq d(v, v_2) \). A set \( S \subseteq V \) is a **vertex generating set** of \( G \) if for any distinct \( v_1, v_2 \in V \) there is a vertex \( s \in S \) such that \( s \) distinguishes \( v_1 \) and \( v_2 \). A vertex generating set with the smallest number of elements is a **vertex basis** of \( G \), and the number of elements in a vertex basis is its **dimension** (denoted \( \text{dim}(G) \)).

Metric dimension was introduced by Slater in 1975 in [9], in connection with the problem of uniquely recognizing the location of an intruder in a network. The same concept was introduced independently by Harary and Melter in [4]. This graph invariant is helpful in areas such as robot navigation ([7]), chemistry ([2], [3], [5]) and problems of image processing and pattern recognition involving hierarchical data structures ([8]). Metric generators in graphs are also connected to coin weighing and the Mastermind game as discussed in [1].

In [6], Kelenc, Tratnic and Yero introduce \( \text{edim} \) and calculate it for various graphs, including paths, cycles, trees and grids. They give examples of graphs for which \( \text{edim}(G) < \text{dim}(G) \) (wheel graphs), \( \text{edim}(G) = \text{dim}(G) \) (trees) and \( \text{edim}(G) > \text{dim}(G) \) (\( C_r \sqcup C_t \) for integers \( r, t \)). They give examples of graphs with \( \frac{\text{edim}(G)}{\text{dim}(G)} \approx \frac{5}{2} \) and ask if the \( \frac{\text{edim}(G)}{\text{dim}(G)} \) ratio is bounded from above. They also ask for the classification of the graphs with \( \text{edim}(G) = |V| - 1 \). In this paper we answer both questions. We also calculate \( \text{edim}(G \sqcup P_m) \) and \( \text{edim}(G + K_1) \).

We will use the following notation:
Consider some vertex \( x \) of a graph. The **distance tuple** of \( x \) on \( S \subseteq V \), \( S = \{v_1, \ldots, v_k\} \) is the tuple
\[
d_S(x) = (d(x, v_1), d(x, v_2), \ldots, d(x, v_k)).
\]
It is easy to see that \( S \) is a vertex generator if and only if the distance tuples on \( S \) are different for all vertices of \( V(G) \).
We define the distance tuple identically if \( x \) is an edge. Similarly, \( S \) is an edge generator if and only if the distance tuples on \( S \) are different for all edges of \( E(G) \).

We use the notation \( N(v) \) for vertices adjacent to \( v \) (not including \( v \)). We use \( V(G) \) and \( E(G) \) to denote the vertices and edges of a graph \( G \). We say \( \text{diam}(G) = \max\{d(u, v) | u, v \in V(G)\} \) and denote the maximal degree of the vertices of \( G \) with \( \Delta(G) \). We use notation \( G_1 + G_2 \) for the sum of graphs \( G_1, G_2 \), which is constructed by connecting all the vertices of \( G_1 \) with all the vertices of \( G_2 \). We use \( P_m \) to denote a path of length \( m \). We use \( G_1 \sqcup G_2 \)
to denote the Cartesian product of $G_1$ and $G_2$. All the graphs are simple, connected and undirected.

2 Graphs for which edim $= |V| − 1$

For a graph $G(V, E)$ it is easy to see that if $|V| = n$, then edim $\leq n − 1$ as any $n − 1$ vertices form an edge generating set. We will now describe all the graphs for which edim $= n − 1$.

Definition 2.1. We call the set $(N(v_1) \cup N(v_2)) \setminus ((N(v_1) \cap N(v_2))$ the non-mutual neighbors of $v_1$ and $v_2$.

Theorem 2.2. Let $G(V, E)$ be a graph with $|V| = n$. Then edim($G$) $= n − 1$ if and only if for any distinct $v_1, v_2 \in V$ there exists $u \in V$ such that $v_1u \in E, v_2u \in E$ and $u$ is adjacent to all non-mutual neighbors of $v_1, v_2$.

Proof. Suppose edim($G$) $= n − 1$. Then for any distinct $v_1, v_2 \in V$, the set $V \setminus \{v_1, v_2\}$ doesn’t generate the edges of $G$. Fix some $v_1$ and $v_2$ and let $S = V \setminus \{v_1, v_2\}$. If $S$ doesn’t generate the edges of $G$, there must exist two edges that have the same distances to all elements of $S$. Call them $e_1, e_2$.

Claim 1. Let $e_1 \neq e_2$ and $d_S(e_1) = d_S(e_2)$. Then $e_1 = v_1u$ and $e_2 = v_2u$ for some $u \in V$.

Proof of claim 1. Suppose there is a vertex $v \in S$ such that $v$ is on exactly one of the two edges $e_1$ and $e_2$. Then $v$ distinguishes $e_1$ and $e_2$ since it has distance 0 to one of them and distance at least 1 to the other. Thus since we assumed $S$ doesn’t distinguish $e_1, e_2$, there can’t be such a vertex in $S$. This means all the non-mutual vertices of $e_1, e_2$ must not be in $S$ (so must be in $\{v_1, v_2\}$). This is only possible if $e_1 = v_1u, e_2 = v_2u$ for some $u \in V$. This proves the claim.

Notice this property restricts $G$ to having diam($G$) $\leq 2$, since we just showed for any choice of distinct $v_1, v_2 \in V$ there is a $u \in V$ such that $v_1u \in E$ and $v_2u \in E$. Thus, $v_1u$ and $v_2u$ have distances 1 or 2 to all vertices in $S \setminus \{u\}$.

Claim 2. Let $e_1 = v_1u, e_2 = v_2u$, and say $d_S(e_1) = d_S(e_2)$. Then $u$ is connected to all non-mutual neighbors of $e_1, e_2$. 3
Proof of claim 2. Consider a vertex $w \in S \setminus \{u\}$. Suppose $w$ is a non-mutual neighbor of $v_1, v_2$, so $wv_1 \in E, wv_2 \notin E$. Since $w \in S$, by assumption $d(e_2, w) = d(e_1, w)$. Thus since $d(v_1, w) = 1$ and $d(v_2, w) = 2$, we must have $d(u, w) = 1$ (so $uw \in E$). The same holds if we switch $v_1$ and $v_2$. Thus $u$ must be a neighbor of all non-mutual neighbors of $v_1$ and $v_2$.

This proves that the stated condition is necessary. It is also sufficient:

**Claim 3.** Let $e_1 = v_1u, e_2 = v_2u$ and say $u$ is connected to all non-mutual neighbors of $v_1$ and $v_2$. Then $e_1$ and $e_2$ are indistinguishable by all vertices of $S$.

**Proof of claim 3.** Consider $w \in S$. If $w = u$, $d(e_1, w) = 0 = d(e_2, w)$. Otherwise, $w$ has distance 1 or 2 to $e_1$ and $e_2$. Say $d(w, e_1) = 1$. There are two cases:

1. $d(w, u) = 1$. Then obviously $d(w, e_2) = 1$.

2. $d(w, v_1) = 1$, $d(w, u) \neq 1$. We know $u$ has to be adjacent to all non-mutual neighbors of $e_1$ and $e_2$. We also know $u$ is not adjacent to $w$. This means $w$ can’t be a non-mutual neighbor, so since $w$ is adjacent to $v_1$, $w$ also has to be adjacent to $v_2$. Thus $d(v_2, w) = 1$ and hence $d(e_2, w) = 1$.

This means that if one of the edges has distance 1 to $w$, then so does the other. Since we already know the distances from these edges to elements of $S \setminus \{u\}$ can only be 1 or 2, this proves that $e_1$ and $e_2$ are equidistant from all elements of $S$.

This proves the theorem.

**Corollary 2.3.** Let $G$ be a graph on $n$ vertices. Suppose edim$(G) = n - 1$. Then diam$(G) \leq 2$ and every edge is in a cycle of length 3.

**Proof.** Theorem 2.2 implies that for any $v_1 \neq v_2$ there is a $u \in V$ such that $v_1u \in E$ and $v_2u \in E$, so diam$(G) \leq 2$. Moreover, for any $xy \in E$ there exists $u \in V$ such that $xu \in E$ and $yu \in E$. This means $xy$ is in a cycle $xuy$ of length 3.
3 The edim$(G)$ to dim$(G)$ ratio

A natural question that arises in the study of the edge dimension is how it is related to the dimension of the same graph.

**Question.** For what triples $(x, y, n)$ does there exist a graph $G$ with dim$(G) = x$, edim$(G) = y$ and $|V| = n$?

Kelenc, Tratnik and Yero give examples of graphs for which dim$(G) <$ edim$(G)$, dim$(G) =$ edim$(G)$, and dim$(G) >$ edim$(G)$. Moreover, they show that there exist graphs realizing all triples $(x, y, n)$ such that

$$1 < x \leq y \leq 2x \leq n - 2.$$

One of the questions they ask is whether $\frac{\text{edim}(G)}{\text{dim}(G)}$ is bounded from above. In this section we show it’s not.

**Theorem 3.1.** $\frac{\text{edim}(G)}{\text{dim}(G)}$ is not bounded from above.

We prove this theorem by finding a graph $F_k$ with edim$(F_k) = k + 2^k - 2$, and dim$(F_k) = k$. The graph $F_k$ is defined as follows:

**Definition 3.2.** For a positive integer $k$, let $F_k$ be the graph on vertex set $A \cup B$, where $B = \{b_1 \ldots b_k\}$ and $A = \{a_S | S \subseteq B\}$. Let $b_i, b_j$ be adjacent for all $b_i, b_j \in B$ with $b_i \neq b_j$, and let $a_S, a_T$ be adjacent for all $a_S, a_T \in A$ with $a_S \neq a_T$. For any $b_i \in B, a_S \in A$ let $b_i, a_S$ be adjacent if and only if $b_i \in S$. Notice $|V(F_k)| = k + 2^k$.

![Graph $F_2$.](image)

Figure 1: The graph $F_2$.

In order to determine some properties of $F_k$, we will use the following results.
Lemma 3.3 ([2]). Let $G(V, E)$ be a graph with diameter $D$ and $\dim(G) = k$. Then $|V| \leq k + D^k$.

We will prove this lemma to demonstrate the motivation for the construction of $F_k$.

Proof. Let $B = \{b_1, \ldots, b_k\}$ be a vertex basis for $F_k$ and $V = \{v_1, \ldots, v_n\}$ be the vertices. Consider the distance tuples $d_B(v_i)$ for all $v_i \in V$. There are $k$ basis vertex tuples with exactly one 0 in them (namely those of $b_1, \ldots, b_k$). All others tuples consist of $k$ numbers from 1 to $D$. This shows there can be no more than $k + D^k$ different distance tuples. But the distance tuples have to be different for all vertices in order for $B$ to be a vertex basis. Thus, there can be no more than $k + D^k$ vertices.

Below we prove an analogue of Lemma 3.3 for edge dimension (we won’t be using it for the proof of Theorem 3.1).

Theorem 3.4. Let $G(V, E)$ be a simple connected graph with diameter $D$, $|V| = n$, and $\text{edim}(G) = k$. Then:

$$|E| \leq \binom{k}{2} + kD^{k-1} + D^k.$$ 

Proof. Let $S$ be an edge basis. Consider the distance tuples on $S$ of the edges of $G$. There are at most $\binom{k}{2}$ distance tuples with two zeros (corresponding to the edges between pairs of vertices of $S$), at most $kD^{k-1}$ tuples with one zero ($k$ ways to choose the position of the zero, $D^{k-1}$ options for the remaining places), and at most $D^k$ tuples with no zeros. Thus, since the tuples have to be different for all elements of $E$, we have $|E| \leq \binom{k}{2} + kD^{k-1} + D^k$.

Lemma 3.5 ([6]). Let $G(V, E)$ be a graph with $|V| = n$ and $\Delta(G) = n - 1$. Then:

$$\text{edim}(G) = n - 1 \text{ or } n - 2.$$ 

Lemma 3.6 ([6]). Let $G(V, E)$ be a graph with $|V| = n$ and $\Delta(G) = n - 1$. Suppose there are at least two vertices with degree $n - 1$. Then:

$$\text{edim}(G) = n - 1.$$ 

We will now use these lemmas to calculate $\dim(F_k)$ and $\text{edim}(F_k)$. 

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Theorem 3.7. For any positive integer $k$,

$$\dim(F_k) = k$$ and $$\text{edim}(F_k) = k + 2^k - 2.$$ 

Proof. Since $a_B = a_{\{b_1, \ldots, b_k\}}$ is connected to all the other vertices of $F_k$, $\text{diam}(F_k) = 2$. Since $|V(F_k)| = k + 2^k$, Lemma 3.3 guarantees $\dim(F_k) \geq k$. Moreover, $B$ is a vertex generating set since the distance tuples $d_B$ are different for all elements of $V(F_k)$ (this follows immediately from construction of $F_k$). Thus, 

$$\dim(F_k) = k.$$ 

Notice $a_B$ is connected to every vertex of $F_k$ by construction, so by Lemma 3.5 we know $\text{edim}(F_k)$ is either $|V(F_k)| - 1$ or $|V(F_k)| - 2$. Consider the vertices $a_\emptyset$ and $a_B$. By construction of $F_k$ we know $a_\emptyset$ is not connected to any elements of $B$, and $a_B$ is connected to all of them. This means all elements of $B$ are non-mutual connections of $a_\emptyset$ and $a_B$. Also, notice that $a_B$ is the only vertex adjacent to all elements of $B$. This shows the condition of Theorem 2.2 doesn’t hold for $F_k$, so 

$$\text{edim}(F_k) = |V(F_k)| - 2 = k + 2^k - 2.$$ 

Proof of Theorem 3.1. By Theorem 3.7, $F_k$ is a counterexample to the boundedness of the $\text{edim}(G)/\dim(G)$ ratio. 

Another related question we could ask is the following: Let $G(V, E)$ be a graph with $|V| = n$ and $\text{edim}(G) = n - 1$. How large can $\dim(G)$ be? Consider the following example:

**Definition 3.8.** For a positive integer $k$, define $H_k = F_k + K_1$.

We will preserve the notation for the vertices of the subgraph $F_k$ of $H_k$ and call the $K_1$ vertex $t$.

**Theorem 3.9.** For any positive integer $k$,

$$\dim(H_k) = k + 1$$ and $$\text{edim}(H_k) = k + 2^k = n - 1.$$
Proof. Due to Lemma 3.3, $\dim(H_k) \geq k + 1$. We claim equality holds, and $B \cup \{t\}$ is a vertex generating set. Indeed, consider any two vertices $x$ and $y$ in $V(H_k)$. If either of them is in $B \cup \{t\}$, it distinguishes them. Otherwise, both $x$ and $y$ are in $A$. By construction of $F_k$, the vertices of $A$ have pairwise different distance tuples on $B$ consisting of 1's and 2's. Notice that distance tuples of $A$ on $B$ are the same in $H_k$ as in $F_k$. Indeed, for $a \in A$ and $b \in B$, any path from $a$ to $b$ via $t$ will have length at least 2, so can’t be shorter than shorter than $d(a,b)$ in $F_k$. Hence, all pairs of vertices in $A$ are distinguished by $B$. This means $B \cup \{t\}$ is a vertex generation set as claimed, so $\dim(H) = k + 1$.

Since $a_B$ and $t$ are connected to all the other vertices of $H_k$, by Lemma 3.3, $\text{edim}(H_k) = |V(H_k)| - 1 = k + 2^k$. \hfill $\Box$

Recall that for a graph $G(V,E)$ with diameter 2, Lemma 3.3 implies that $
 + 2^{\text{dim}(G)} \geq |V|$. In particular, in the case $|V| = k + 2^k + 1$, this means that we can’t make $\text{dim}(G)$ smaller than $k+1$. Since we showed in section 1 that graphs $G(V,E)$ with edge dimension $|V| - 1$ have to have diameter 2, this means we cannot further decrease the dimension if we want the edge dimension to be maximal.

4 edim for $G + K_1$ and $G \square P_m$

In this section we characterize how the edge dimension changes upon taking a Cartesian product with a path, or upon adding a vertex adjacent to all the original vertices.

Theorem 4.1. Let $G(V,E)$ be a graph with $|V| = n$. Suppose for any vertex $x \in V$ there is another vertex $u \in V$ such that $V \setminus N(x) \subseteq N(u)$. Then $\text{edim}(G + K_1) = n$. Otherwise, $\text{edim}(G + K_1) = n - 1$.

Proof. Denote the $K_1$ graph vertex $t$. Since $t$ is connected to all the other vertices of $G + K_1$, by Lemma 3.5 $\text{edim}(G + K_1)$ is either $n$ or $n - 1$. We will use Theorem 2.2 to see when each case holds. Consider $x,y \in V$. Whatever their non-mutual connections are, $t$ is connected to all of them and to $x$ and $y$, so the hypothesis of Theorem 2.2 holds for this vertex pair. Now consider a pair $t$ and $x \in V$. Their non-mutual neighbors are precisely $V \setminus N(x)$. This means the condition stated in Theorem 2.2 holds for $x,t$ if and only if there
exists \( u \in V \) such that \( V \setminus N(x) \subseteq N(u) \). Thus \( \text{edim}(G + K_1) = n \) if and only if this is true for any \( x \in V \), which is what we were to prove. \( \square \)

**Theorem 4.2.** Let \( G(V, E) \) be a graph and \( P_m \) a path of length \( m \geq 2 \). Let \( B_E \subseteq 2^V \) be the set of all the edge bases of \( G \), let \( B_V \subseteq 2^V \) be the set of all vertex bases of \( G \). Let \( k \) be the smallest possible cardinality of a union of an edge and a vertex basis, that is,

\[
k = \min \left\{ |S \cup T| \mid S \in B_V, T \in B_E \right\}.
\]

Then:

\[
k \leq \text{edim}(G \Box P_m) \leq k + 1.
\]

**Proof.** Let \( M = S \cup T \) with \( S \in B_V, T \in B_E \) be a set for which the minimum cardinality is achieved, that is \( |M| = k \).

The graph \( G \Box P_m \) can be constructed the following way: First, take \( m \) copies of \( G \). Denote the \( i \)th copy \( G(i) \). Denote the vertices of \( G(i) \) with \( v(i) \) for all \( v \in V \). Then, connect \( v(i) \) and \( v(i + 1) \) for all \( v \in V, i \in \{1, \ldots, m - 1\} \).

![Graph G and the described construction of the graph G \( \Box P_4 \).](image)

Figure 2: A graph \( G \) and the described construction of the graph \( G \Box P_4 \).
**Lower bound:** Suppose $B$ is an edge basis of $G \Box P_m$. Let $B_1$ be the projection of $B$ on $G(1)$ (where we "project" $v(i)$ to $v(1)$). Consider $e \in E(G(1))$. Notice that

$$d(e, v(i)) = d(e, v(1)) + i - 1.$$ 

Thus, $e_1, e_2 \in E(G(1))$ are distinguished by $v(1)$ if and only if they are distinguished by $v(i)$. Thus, if $B$ is an edge generating set of $G \Box P_m$, then $B_1$ is an edge generating set of $G$.

Consider an edge $e = v(1)v(2)$. Notice that for $i \geq 2$ we have

$$d(e, w(i)) = d(v(2), w(2)) + (i - 2) = d(v(1), w(1)) + (i - 2),$$

and for $i = 1$,

$$d(e, w(i)) = d(v(1), w(1)).$$

These differ by a constant only dependent on $i$. This means that if we consider two edges $x = v(1)v(2)$ and $y = u(1)u(2)$, then:

$$w(i) \text{ distinguishes } x \text{ and } y \iff w(1) \text{ distinguishes } x \text{ and } y.$$ 

Moreover, $w(1)$ distinguishes $x$ and $y$ if and only if $w(1)$ distinguishes $v(1)$ and $u(1)$. Thus, $B_1$ is a vertex generating set of $G(1)$ as well. This shows that $B_1$ is both an edge generating set and a vertex generating set, so $|B_1| \geq |M| = k$.

Also, clearly, $|B_1| \leq |B|$. This gives us the lower bound.

**Upper bound:** Let $M \subseteq V$ be a set defined in the statement of the theorem with $|M| = k$, and let $t \in M$. Set

$$B = \{v(1) \mid v \in M\} \cup t(m).$$

We will prove $B$ is an edge generating set of $G \Box P_m$. There are five cases of pairs of edges.

1. $e(i), f(i) \in E(G(i))$.
   By definition of $M$, some $v \in M$ distinguishes $e(1), f(1)$. Since it’s clear that
   $$d(v, z(i)) = d(v, z(1)) + i - 1$$ for any $z \in E$,
   $v$ also distinguishes $e(i)$ and $f(i)$.
2. $x(i)x(i+1): y(i)y(i+1)$ for $x, y \in V$.
   By definition of $M$, some $v \in M$ distinguishes $x(1), y(1)$. Since
   
   $$d(v, z(i)) = d(v, z(1)) + i - 1$$
   for any $z \in V(G)$,

   $v$ also distinguishes $x(i)$ and $y(i)$. Thus
   
   $$d(v, x(i)x(i+1)) = d(v, x(i)) \neq d(v, y(i)) = d(v, y(i)y(i+1)).$$

3. $x(i)x(i+1), y(j)y(j+1)$ for $x, y \in V$ and $i \neq j$.
   Notice that
   
   $$d(x(i)x(i+1), t(1)) = d(x(1), t(1)) + i - 1$$
   and
   
   $$d(x(i)x(i+1), t(m)) = d(x(m), t(m)) + m - i - 1 = d(x(1), t(1)) + m - i - 1,$$
   so
   
   $$d(x(i)x(i+1), t(1)) = d(x(i)x(i+1), t(m)) + m - 2i.$$

   Thus, since we assumed $i \neq j$, we conclude that if $t(1)$ doesn’t distinguish $x(i)x(i+1), y(j)y(j+1)$, then $t(m)$ does.

4. $e(i), f(j)$ for $e, f \in E, i \neq j$.
   Similarly to case 3, we can see
   
   $$d(e(i), t(1)) = i - 1 + d(e(1), t(1)),$$
   and
   
   $$d(e(i), t(m)) = m - i + d(e(m), t(m)) = m - i + d(e(1), t(1)) = d(e(i), t(1)) + m - 2i + 1.$$

   Thus, if $t(1)$ does not distinguish $e(1)$ and $f(j)$, then $t(m)$ does.

5. $e(i), y(j)y(j+1)$ for $e \in E, y \in V$.
   Suppose these two edges aren’t distinguished by $t(1)$, so
   
   $$d(e(i), t(1)) = d(y(j)y(j+1), t(1)) = d.$$

   As we have noted, then
   
   $$d(e(i), t(m)) = d + m - 2i + 1$$
   and
   
   $$d(y(j)y(j+1), t(m)) = d + m - 2j.$$

   These can not be equal since they have different parity.

Since $|B| = |M| + 1 = k + 1$, this concludes the proof.
5 Conclusion and Open Problems

We have shown $\frac{\text{edim}(G)}{\text{dim}(G)}$ isn’t bounded from above in section 3. More questions can be asked about the relationship between edim$(G)$ and dim$(G)$. For instance,

- Are there graphs $G$ for which $\text{edim}(G) \gg 2^{\text{dim}(G)}$?

- For what triples $x, y, n$ does there exist a graph $G$ with $|V| = n$, $\text{dim}(G) = x$ and $\text{edim}(G) = y$?

Another approach that could be taken to understand how dim$(G)$ and edim$(G)$ compare to each other is deriving some more properties of edim analogues to the known properties of dim, as we did in the last sections 2 and 4. For example:

- For which graphs $G(V, E)$ is $\text{edim}(G) = |V| - 2$?

- For which graphs $G(V, E)$ is $\text{edim}(G) = 2$?

- For a graph $G$ and a positive integer $n$, bound $\text{edim}(G \Box C_n)$ in terms of some function of $G$.

- For graphs $G_1, G_2$, bound $\text{edim}(G_1 \Box G_2)$ in terms of some function of $G_1$ and $G_2$.

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