Minimum Complexity Pursuit
for Universal Compressed Sensing

Shirin Jalali, Arian Maleki, Richard G. Baraniuk

Abstract

The nascent field of compressed sensing is founded on the fact that high-dimensional signals with “simple structure” can be recovered accurately from just a small number of randomized samples. Several specific kinds of structures have been explored in the literature, from sparsity and group sparsity to low-rankedness. However, two fundamental questions have been left unanswered, namely: What are the general abstract meanings of “structure” and “simplicity”? And do there exist universal algorithms for recovering such simple structured objects from fewer samples than their ambient dimension? In this paper, we address these two questions. Using algorithmic information theory tools such as the Kolmogorov complexity, we provide a unified definition of structure and simplicity. Leveraging this new definition, we develop and analyze an abstract algorithm for signal recovery motivated by Occam’s Razor. Minimum complexity pursuit (MCP) requires just $O(\kappa \log n)$ randomized samples to recover a signal of complexity $\kappa$ and ambient dimension $n$. We also discuss the performance of MCP in the presence of measurement noise and with approximately simple signals.

I. INTRODUCTION

Compressed sensing (CS) refers to a body of techniques that undersample high-dimensional signals, and yet recover them accurately by exploiting their intrinsic “structure” or “compressibility” [1], [2]. This leads to more efficient sensing systems that have proved to be valuable in many applications, including cameras [3], magnetic resonance imaging (MRI) [4] and radar [5]–[7], to name a few. While the promise of compressed sensing has been to undersample “structured” signals, its premise is still limited to specific instances of “structure” such as sparsity and low-rankedness. While these notions are important in their own right, the notions of “structure” and “compressibility” are of course more general than these specific instances. The goal of this paper is to provide a general notion of structure and exploit it to recover signals from an undersampled set of linear measurements.

Towards this end, we use Kolmogorov complexity, which is a measure of complexity for finite-alphabet sequences introduced by Solomonoff [8] and Kolmogorov [9]. We define the Kolmogorov information dimension of a real-valued signal as the growth rate of the complexity of its quantized sequences as the quantization becomes finer.

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S. Jalali is with the Center for Mathematics of Information, California Institute of Technology, Pasadena, CA, shirin@caltech.edu
A. Maleki and R. G. Baraniuk are with the Digital Signal Processing group, Rice University, Houston, TX, {arian.maleki, richb}@rice.edu
We show that if the Kolmogorov information dimension of a signal is much smaller than its ambient dimension, then it can be recovered from fewer measurements than its ambient dimension. Based on Occam’s razor [10], we propose the \textit{minimum complexity pursuit} (MCP) recovery algorithm. MCP finds the simplest object (in the Kolmogorov complexity sense) that satisfies the measurement constraints. Roughly speaking, we prove that MCP is able to recover a signal with “complexity” \( \kappa \) using no more than \( O(\kappa \log n) \) measurements. Finally, we establish the robustness of MCP to noise on both the measurements and the signal.

Here is the structure of the paper. Section II describes the notation used in the paper and introduces the Kolmogorov information dimension. Section III summarizes our main contributions and their implications. Section IV bounds the Kolmogorov information dimension of several popular classes of signals in CS. Section V makes a comparison of our work with the related papers in the literature. Section VI provides the proofs of our main results. Finally, Section VII concludes the paper.

II. DEFINITIONS AND PROBLEM STATEMENT

A. Notation

Calligraphic letters such as \( \mathcal{A} \) and \( \mathcal{B} \) denote sets. For a set \( \mathcal{A} \), \( |\mathcal{A}| \) and \( \mathcal{A}^c \) denote its size and its complement, respectively. For a sample space \( \Omega \) and an event set \( \mathcal{A} \subseteq \Omega \), \( \mathbb{I}_{\mathcal{A}} \) denotes the indicator function of the event \( \mathcal{A} \).

Boldfaced letters denote vectors. For a vector \( \mathbf{x} \in \mathbb{R}^n \), \( x_i \), \( \| \mathbf{x} \|_p \triangleq (\sum_{i=1}^{n} |x_i|^p)^{1/p} \), and \( \| \mathbf{x} \|_{\infty} \triangleq \max_i |x_i| \) denote the \( i \)-th component, \( \ell_p \) norm and \( \ell_{\infty} \) norm of \( \mathbf{x} \), respectively. For \( 1 \leq i \leq j \leq n \), \( x^j_i \triangleq (x_i, x_{i+1}, \ldots, x_j) \).

Also, to simplify the notation, \( x^j \) denotes \( x^j_1 \). Uppercase letters are used for both matrices and random variables, and hence their usage will be clear from the context. For integer \( n \), \( I_n \) denotes the \( n \times n \) identity matrix.

Let \( \{0, 1\}^* \) denote the set of all finite-length binary sequences, i.e., \( \{0, 1\}^* \triangleq \cup_{n \geq 1} \{0, 1\}^n \). Similarly, \( \{0, 1\}^\infty \) denotes the set of infinite-length binary sequences.

For a real number \( x \in [0, 1] \), let \( [x]_m \) denote its \( m \)-bit approximation that results from taking the first \( m \) bits in the binary expansion of \( x \). In other words, if \( x = \sum_{i=1}^{\infty} 2^{-i}(x)_i \), where \( (x)_i \in \{0, 1\} \), then

\[
[x]_m \triangleq \sum_{i=1}^{m} 2^{-i}(x)_i.
\]

Similarly, for a vector \( \mathbf{x} \in [0, 1]^n \), define

\[
[x]_m \triangleq ([x_1]_m, \ldots, [x_n]_m).
\]

B. Kolmogorov complexity

The Kolmogorov complexity\(^1\) of a finite-length finite alphabet sequence \( \mathbf{x} \) with respect to a universal computer \( \mathcal{U} \) is defined as the minimum length over all programs that print \( \mathbf{x} \) and halt\(^2\).

\(^1\)See Appendix A or Chapters 2 and 3 of [11] for a general introduction to Kolmogorov complexity.

\(^2\)For some technical reasons in our definitions we are considering prefix Turing machines. See Appendix A or Chapters 2 and 3 of [11] for the definition.
For \( x \in \{0, 1\}^* \), let \( K_U(x) \) denote the Kolmogorov complexity of sequence \( x \) with respect to the universal computer \( U \). For a universal computer \( U \) and any computer \( A \), there exists a constant \( c_A \) such that \( K_U(x) \leq K_A(x) + c_A \), for all strings \( x \in \{0, 1\}^* \). This result is known as the invariance theorem in the field of algorithmic complexity. Note that the constant \( c_A \) is independent of the length of the sequence, \( n \), and hence can be neglected for sufficiently long \( x \). As suggested in [12], we drop the subscript \( U \), and let \( K(x) \) denote the Kolmogorov complexity of the binary string \( x \). For two finite alphabet sequences \( x \) and \( y \), \( K(x \mid y) \) is defined as the length of the shortest program that prints \( x \) and halts, given that the universal computer \( U \) has access to the sequence \( y \). Similarly, the Kolmogorov complexity of an integer \( n \in \mathbb{N} \), \( K(n) \), is defined as the Kolmogorov complexity of its binary representation. The following theorem summarizes some of the properties of the Kolmogorov complexity that will be used throughout the paper.

Define

\[
\log^* n \triangleq \left\lceil \log_2 n \right\rceil + 2 \log_2 \max(\left\lceil \log_2 n \right\rceil, 1).
\]

**Theorem 1:** Let \( x, y \) be binary strings of lengths \( \ell(x) \) and \( \ell(y) \), respectively. Furthermore, let \( m, n \in \mathbb{N} \). The Kolmogorov complexity satisfies the following properties:

i. \( K(x \mid \ell(x)) \leq \ell(x) + c \),

ii. \( K(x, y) \leq K(x) + K(y) + c \),

iii. \( K(x \mid y) \leq K(x) \),

iv. \( K(x) \leq K(x \mid \ell(x)) + K(\ell(x)) + c \),

v. \( K(n) \leq \log^* n + c \),

vi. \( K(n + m) \leq K(n) + K(m) + c \),

where \( c \) is a constant independent of \( x, y, n \) and \( m \), but might be different from one appearance to another.

The proofs of different parts of this theorem can be found in [11], [12]. However, for the sake of completeness, we overview the proofs in Appendix B.

Kolmogorov complexity provides a universal measure for compressibility of sequences. If an infinite length binary sequence \( x \) satisfies

\[
\lim_{n \to \infty} \frac{K(x_1, x_2, \ldots, x_n)}{n} = 1,
\]

then it satisfies all computable statistical tests for randomness ([12] Theorem 14.5.2). For instance, the number of zeros is “close to” the number of ones in this sequence. Furthermore, if the Kolmogorov complexity of \( x \) is smaller than the ambient dimension, then it means that we can compress \( x \) (represent it with fewer bits); the encoder returns the shortest program that has generated \( x \) and the decoder is the universal Turing machine that generates \( x \) from this short program.

\(^3\)Note that \( K(x \mid y) \) is often defined as \( K(x \mid y, p_y) \) where \( p_y \) is the shortest program that generates \( y \). This formulation provides symmetry in the definition of algorithmic mutual information. But we will not use this definition in this paper.
C. Problem statement

1) Compressed sensing versus compression: Algorithmic information theory is mainly concerned with the compression of binary sequences (or finite alphabet sequences). The objective of a compression algorithm is to represent these sequences using as few bits as possible. However, in this paper we are interested in the problem of CS, where the goal is to reconstruct a signal $x_o \in \mathbb{R}^n$ from its lower dimensional linear projections $y_o = Ax_o$, where $A \in \mathbb{R}^{d \times n}$ with $d < n$. Note two distinguishing features of this problem. First, since the system of equations is underdetermined, perfect reconstruction is not always possible. Therefore some knowledge of the structure of $x_o$ is required for recovering it from the measurements $y_o$. Second, the problem is different from the traditional problem of algorithmic information theory that considers the compression in terms of bits. Hence, this problem requires a new perspective on the Kolmogorov complexity of real-valued signals.

2) Kolmogorov information dimension: Let the components of $x_o \in [0,1]^n$ have terminating binary expansions. We define the Kolmogorov complexity of $x_o$ as the length of the shortest program that prints the binary expansion of $x_o$ and halts. By definition, the Kolmogorov complexity of sequences with nonterminating binary expansion is infinite. Following the ideas in algorithmic information theory, one can consider the “structure” of $x_o$ to be the shortest program that generates it [13]. The shorter the program, the more structured the signal. But this assumption is very restrictive, since most real-valued signals have infinite Kolmogorov complexity. The first step to remedy this issue is to calculate the Kolmogorov complexity of a “quantized” version of $x_o$. For $x = (x_1, x_2, \ldots, x_n) \in [0,1]^n$, define the Kolmogorov complexity of $x$ at resolution $m$ as

$$K[\cdot]^m(x) \triangleq \inf_{u \in [0,1]^n} \left\{ K(u | n, m) \mid \|x - u\|_\infty \leq 2^{-m} \right\}. \quad (1)$$

We can provide an upper bound for $K[\cdot]^m(x)$ by considering certain instances of $u$. For example,

$$K[\cdot]^m(x) \leq K([x]_m | m, n).$$

Note that $K[\cdot]^m(x)$ is defined as the Kolmogorov complexity of the “quantized” version of $x$ conditioned on $m$ and $n$, because it is natural to assume that the encoder and decoder have access to both the ambient dimension $n$ and the quantization level $m$. For most real valued signals this quantity goes to infinity as $m$ approaches infinity. But, the growth rate is proportional to $m$. Therefore, we consider a normalized version of the Kolmogorov complexity.

Definition 1: The Kolmogorov information dimension of $(x_1, x_2, \ldots, x_n) \in [0,1]^n$ at resolution $m$ is defined as

$$\kappa_{m,n}(x) \triangleq \frac{K[\cdot]^m(x_1, x_2, \ldots, x_n)}{m}.$$ 

In general the number of quantization levels $m$ may depend on the ambient dimension $n$. The division of $K[\cdot]^m(x)$ by the resolution level $m$ ensures that for a fixed value of $n$ this quantity is always finite.

Lemma 1: Let $x \in [0,1]^n$. Then we have

$$\kappa_{m,n}(x) \leq n + \frac{c}{m},$$
where $c$ is a positive constant independent of $m$, $n$, and $x$. In particular,

$$\limsup_{m \to \infty} \kappa_{m,n}(x) \leq n.$$  

**Proof:** We first note that

$$K^{[\cdot]}_m(x) = \inf_{u \in [0,1]^n} \left\{ K(u \mid n, m) \mid \|x - u\|_\infty \leq 2^{-m} \right\} \leq K([x]_m).$$

Now, we derive an upper bound on $K([x]_m)$ by providing a program that describes $[x]_m$ conditioned on knowing $m$ and $n$. Consider the program that first explains the structure of the sequence as consisting of $n$ $m$-bit subsequences and then identifies the bits. Since the computer has access to $m$ and $n$, a constant number of bits (independent of $m$ or $n$) is sufficient for specifying the structure, and it then requires $mn$ more bits to specify each component $[x_i]_m$. Therefore, overall

$$\kappa_{m,n}(x) \leq \frac{K([x_1]_m, [x_2]_m, \ldots, [x_n]_m \mid m, n)}{m} \leq \frac{nm + c}{m}.$$  

The second part of the theorem is a straightforward result of the first part. \hfill \square

**Remark 1:** Note that the existence of a finite upper bound on $K^{[\cdot]}_m(x)$ ensures that the infimum in (1) is achieved. This is due to the fact that the number of sequences $(u_1, u_2, \ldots, u_n)$ that have $K(u_1, u_2, \ldots, u_n) \leq mn + c$ is finite. In the rest of the paper we denote the minimizing vector by $\phi_m(x)$, i.e.,

$$\phi_m(x) \triangleq \arg \min_{u \in [0,1]^n} \left\{ K(u \mid n, m) \mid \|x - u\|_\infty \leq 2^{-m} \right\}.$$  

The following examples clarify some of the properties of the Kolmogorov information dimension.

**Example 1:** (Sparse signals) Consider a $k$-sparse signal $x \in [0,1]^n$. That is, $x$ has at most $k$ nonzero coefficients. For any given $\delta > 0$, the Kolmogorov information dimension of $x$ at resolution $m$, for large enough values of $m$, is upper bounded by $k + \delta$. See Section IV-A for the proof of this claim.

**Example 2:** (Low-rank matrices) Let $X$ denote a $M \times N$ real-valued matrix such that $\sigma_{\text{max}}(X) \leq 1$. For any given $\delta > 0$, the Kolmogorov information dimension of $X$ at resolution $m$ is upper bounded by $r(M + N + 1) + \delta$, for sufficiently large values of $m$. See Section IV-E for the proof of this claim.

Let $U[a,b]$ denote the uniform distribution between $a$ and $b$. Also, let $X \sim \text{Bern}(p)$ represent a Bernoulli random variable with $P(X = 1) = 1 - P(X = 0) = p$. The following proposition lets us construct the third example that represents an unstructured signal.

**Proposition 1:** Let $\{X_i\}_{i=1}^\infty \sim i.i.d. U[0,1]$. Then, for any $n \geq 1$,

$$\lim_{m \to \infty} \frac{1}{mn} K^{[\cdot]}_m(X_1, X_2, \ldots X_n) = 1.$$  

As long as all the singular values are upper bounded by a constant the statement of this example holds. For the notational simplicity we choose 1 as the upper bound for the singular values.
in probability.

**Proof:** For \( i \in \{1, 2, \ldots \} \), let \( X_i = \sum_{j=1}^{\infty} (X_i)_j 2^{-j} \), where \( (X_i)_j \in \{0, 1\} \). Then \( \{(X_i)_j\}_{j=1}^{\infty} \overset{i.i.d.}{\sim} \text{Bern}(1/2) \) [14].

Let \( U^n \triangleq \phi_m(X^n) \). Since \( |U_i - X_i| \leq 2^{-m} \), then, for \( j < m - 1 \), \( (X_i)_j = (U_i)_j \). Therefore,

\[
\frac{K(U^n \mid m, n)}{m} \geq \frac{K(\{(U_i)_1, \ldots, (U_i)_m\}_{i=1}^{n} \mid m, n) - c}{m} = \frac{K(\{(X_i)_1, \ldots, (X_i)_m\}_{i=1}^{n} \mid m, n) - c}{m}
\]

(2)

Theorem 14.5.3 in [12] states that the normalized Kolmogorov’s complexity of a sequence of i.i.d. Bern(1/2) bits converges to 1 in probability. In other words,

\[
\lim_{m \to \infty} \frac{K(\{(X_i)_1, (X_i)_2, \ldots, (X_i)_m\}_{i=1}^{n} \mid m, n)}{mn} = 1,
\]

(3)

in probability. Therefore, combining (2), Lemma [1] and (3) yields the desired result.

**Example 3:** If the random variables \( \{X_i\}_{i=1}^{n} \overset{i.i.d.}{\sim} U[0, 1] \), then

\[
\lim_{m \to \infty} \frac{K[\cdot]_{m}(X_1, X_2, \ldots, X_n)}{m} = n
\]

in probability. The proof follows directly from Proposition [1].

These examples demonstrate that, at least in cases where the ambient dimension is fixed and the quantization levels grow without bound, the Kolmogorov information dimension is much smaller than the ambient dimension for the two well-known structured signals in Examples [1] and [2], and is equal to the ambient dimension for the unstructured signal in Example [3]. We present more examples of structured signals and the corresponding upper bounds on their Kolmogorov information dimension in Section [IV].

3) **CS of real-valued signals:** Consider the problem of recovering a structured real-valued signal \( \mathbf{x}_o = (x_{o,1}, x_{o,2}, \ldots) \) with \( \kappa_{m,n}(\mathbf{x}_o^n) = O(n^{1-\alpha}) \), for some \( \alpha > 0 \) and proper choice of \( m \), from an underdetermined set of linear equations \( \mathbf{y}_o = A\mathbf{x}_o \), where \( \mathbf{y}_o \in \mathbb{R}^d \) and \( d < n \). We follow Occam’s Razor principle and among all the solutions of \( \mathbf{y}_o = A\mathbf{x}_o \), seek the solution that has the minimum complexity, i.e.,

\[
\arg \min_{\mathbf{x}} K[\cdot]_{m}(\mathbf{x})
\]

s.t. \( A\mathbf{x} = \mathbf{y}_o \).

(4)

We call this algorithm **minimum complexity pursuit** or MCP. MCP has a free parameter \( m \) whose effect on the performance of the algorithm will be discussed in detail later. We will show that MCP can recover \( \mathbf{x}_o \) from fewer measurements than the ambient dimension of the signal. This result extends the scope of CS from the class of sparse signals or the class of low-rank matrices to the class of all signal with small Kolmogorov information dimension.

In this paper we ignore the practical issues of approximating the MCP algorithm. In an independent work, [15], [16] have considered a practical version of this algorithm and provided promising results in that direction.
investigation of the practical issues is left for the future research.

III. OUR CONTRIBUTIONS

A. Recovery in the noiseless setting

Suppose that $A \in \mathbb{R}^{d \times n}$, where $A_{ij}$ are i.i.d. $\mathcal{N}(0,1)$, and assume that $y_o = Ax_o$. Let $\tilde{x}_o = \tilde{x}_o(x_o, A)$ denote the output of (4) to the inputs $y_o$ and $A$. For an infinite sequence $x_o = (x_{o,1}, x_{o,2}, \ldots)$, $x_o^n$ denotes the first $n$ elements of $x_o$, i.e., $x_o^n = (x_{o,1}, x_{o,2}, \ldots, x_{o,n})$.

**Theorem 2:** Let $x_o = (x_{o,1}, x_{o,2}, \ldots) \in [0,1]^\infty$. For integers $m$ and $n$, let $\kappa_{m,n}(x_o^n)$ denote the information dimension of $x_o^n$ at resolution $m$. Then, for any $\tau_n \leq 1$ and $t > 0$ we have

$$
P \left( \|x_o^n - \hat{x}_o^n\|_2 > (\tau_n^{-1} \sqrt{n/d + 1 + t} + 1) \sqrt{n2^{-2m+2}} \right) \leq 2 e^{\kappa_{m,n} \log n} \epsilon^{\frac{1}{2}} + e^{-\frac{t}{2}}.
$$

The proof is presented in Section VI-B. We consider several interesting corollaries of this theorem for high dimensional problems.

**Corollary 1:** Assume that $x_o = (x_{o,1}, x_{o,2}, \ldots) \in [0,1]^\infty$ and $m = m_n = [\log n]$. Let $\kappa_n \triangleq \kappa_{m,n}$. Then if $d_n = [\kappa_n \log n]$, for any $\epsilon > 0$, we have

$$
P (\|x_o^n - \hat{x}_o^n\|_2 > \epsilon) \to 0,
$$
as $n \to \infty$.

**Proof:** For $m = m_n = [\log n]$ and $d_n = [\kappa_n \log n]$, we have

$$
(\tau_n^{-1} \sqrt{n/d + 1} + 1) \sqrt{n2^{-2m+2}} \leq 2 \left( \tau_n^{-1} [\kappa_n \log n]^{-1} + (t + 1)n^{-1} + \sqrt{n^{-1}} \right).
$$

(5)

Hence, fixing $t > 0$ and setting $\tau_n = \tau = 0.04$, for any $\epsilon > 0$, if $n$ is large enough, then

$$
(\tau_n^{-1} \sqrt{n/d + 1} + 1) \sqrt{n2^{-2m+2}} \leq \epsilon.
$$

Therefore, for $n$ large enough,

$$
P (\|x_o^n - \hat{x}_o^n\|_2 > \epsilon) \leq 2 [\kappa_n \log n] e^{\frac{\kappa_{m,n} \log n}{2} (1 + 1 + \epsilon)} + e^{-\frac{t}{2} \epsilon^2} \leq e^{1.4 \kappa_n \log n} e^{-2.7 \kappa_n \log n} + e^{-\frac{t}{2} \epsilon^2},
$$

which shows that as $n \to \infty$, $P (\|x_o^n - \hat{x}_o^n\|_2 > \epsilon) \to 0$, as $n \to \infty$. $\square$

According to Corollary 1, if the complexity of the signal is less than $\kappa$, then the number of linear measurements required for its asymptotically perfect recovery is roughly speaking on the order of $\kappa \log n$. In other words, the number of measurements is proportional to the complexity of the signal and only logarithmically proportional to

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its ambient dimension.

**Corollary 2:** Assume that \( \mathbf{x}_o = (x_{o,1}, x_{o,2}, \ldots) \in [0,1]^{\infty} \) and \( m_n = \lceil \log n \rceil \). Let \( \kappa_{m,n} = \kappa_n \to \infty \) as \( n \to \infty \).

Then, if \( d_n = 3\kappa_n \), for \( \epsilon > 0 \) we have

\[
\mathbb{P}\left( \frac{1}{\sqrt{n}} \| x^n_o - \hat{x}^n_o \|_2 > \epsilon \right) \to 0
\]

as \( n \to \infty \).

**Proof:** Setting \( \tau_n = n^{-\frac{1}{2}} \), \( m = m_n = \lceil \log n \rceil \), and \( d = d_n = \lceil 3\kappa_n \rceil \) in Theorem 2 it follows that

\[
\mathbb{P}\left( \frac{1}{\sqrt{n}} \| x^n_o - \hat{x}^n_o \|_2 > 2\sqrt{d_n^{-1} + (t+1)n^{-1}} + 2\sqrt{n^{-1}} \right)
\]

\[
\leq 2^{2\kappa_n} n \log n e^{\frac{1}{2} \kappa_n (1 - n^{-1} - \log n)} + e^{-\frac{n}{2} t^2}
\]

\[
= e^{-\left(\frac{1}{2} - 2\log 2\right) \kappa_n \log n + \kappa_n (\frac{3}{2} - \frac{1}{2} n^{-1})} + e^{-\frac{n}{2} t^2}.
\]

Since \( \frac{3}{2} - 2\log 2 > 0 \), for any \( \epsilon > 0 \) and \( n \) large enough, we have

\[
2\sqrt{d_n^{-1} + (t+1)n^{-1}} + 2\sqrt{n^{-1}} < \epsilon.
\]

It follows that \( \mathbb{P}(\frac{1}{\sqrt{n}} \| x^n_o - \hat{x}^n_o \|_2 > \epsilon) \to 0 \), as \( n \to \infty \).

In other words, if we are interested in the normalized mean square error, \( \frac{1}{\sqrt{n}} \| x^n_o - \hat{x}^n_o \|_2^2 \) rather than the \( \ell_2 \)-norm error \( \| x^n_o - \hat{x}^n_o \|_2 \), then \( 3\kappa_n \) measurements are sufficient for asymptotically accurate recovery.

It is worth noting that, while \( m_n \) is set to \( \log n \) in Corollaries 1 and 2 it can be considered as a free parameter of the MCP algorithm. Theorem 2 describes the trade-off of the parameters. If we fix all the other parameters in Theorem 2 then increasing \( m \) is equivalent to decreasing the reconstruction mean square error. But also it decreases the probability of correct recovery.

**B. Recovery in the presence of Gaussian noise in measurements**

In the previous section, we considered the case of recovering low-complexity signals from their noise-free linear measurements. In this section, we extend those results to the case of noisy measurements, where \( y_o = A\mathbf{x}_o + \mathbf{w} \), with \( \mathbf{w} \sim \mathcal{N}(0,\sigma^2 I_d) \). Assuming that the complexity of the signal is known at the reconstruction stage, we consider the following reconstruction algorithm:

\[
\text{arg min}_{\mathbf{x}} \| A\mathbf{x} - y_o \|_2,
\]

s.t. \( K^{[1:m_n]}(\mathbf{x}) \leq \kappa_n m_n \).

(7)

Note that \( \kappa_n m_n \) is an upper bound on the Kolmogorov complexity of \( \mathbf{x}_o \) at resolution \( m_n \). We call this algorithm **low-complexity least squares** (LLS). Our quest in this section is to find the number of measurements required to make the LLS algorithm specified by (7) robust to noise.

**Theorem 3:** Assume that \( \mathbf{x}_o = (x_{o,1}, x_{o,2}, \ldots) \in [0,1]^{\infty} \). For integers \( m \) and \( n \), let \( \kappa_{m,n} \) denote the information dimension of \( \mathbf{x}_o \) at resolution \( m \). If \( m = m_n = \lceil \log n \rceil \), \( d = 8\rho \kappa_{m,n} m_n \), and \( \rho \triangleq (1 - \sqrt{r^{-1}})^2/2 \), where \( r > 1 \),
then
\[
P \left( \| \hat{x}_o^n - \hat{x}_o^n \|_2^2 > \frac{(2\kappa_{m,n}m)\sigma^2}{\rho d} \right) \to 0
\]  
(8)
as \ n \to \infty.

The proof is presented in Section VI-C. Note that, since the elements of the matrix \( A \) are i.i.d. \( \mathcal{N}(0, 1) \), as the ambient dimension \( n \) grows, so does the signal-to-noise ratio per measurement. In order to have fixed signal-to-noise ratio per measurement, one can draw the elements of \( A \) i.i.d. from \( \mathcal{N}(0, 1/n) \). In this case, it is not difficult to see that the normalized mean square error \( \| x_o^n - \hat{x}_o^n \|_2^2/n \leq \frac{(2\kappa_{m,n}m)\sigma^2}{\rho d} \) in probability.

### C. Recovery in the presence of deterministic noise

Consider again the measurement system we introduced in the last section: \( y_o = Ax_o + w \), where \( w \) represents measurement noise. Unlike the previous section, assume that the noise is deterministic and has bounded \( \ell_2 \)-norm, i.e., \( \| w \|_2 \leq e \). This type of noise provides a good model for quantization noise on the measurements, among other practical nonidealities. Note that unlike the case of stochastic noise, deterministic noise can be adversarial. We prove that the LLS algorithm considered in \( \text{(7)} \) provides a sufficiently accurate estimate of \( x_o \) even in the presence of such noise.

**Theorem 4:** Let \( x_o = (x_o, 1, \ldots, x_o, n) \in [0, 1]^n \). Let \( \kappa_{m,n} \) denote the information dimension of \( x_o \) at resolution \( m \). Then, for any \( \tau_n < 1 \) and \( t > 0 \),
\[
P \left( \| x_o - \hat{x}_o \|_2 > \frac{\sqrt{n/d + 1 + t} + 1}{\tau_n} \sqrt{n^2 - 2m^2} + \frac{e}{\tau_n \sqrt{d}} \right) 
\leq 2^{2\kappa_{m,n}m} \frac{d^{d/2}(1-\tau_n^2+2\log\tau_n)}{\tau_n \sqrt{d}} + e^{-d/2}.
\]

Since the proof of this theorem is very similar to the proof of Theorem 2, it is not included in the paper. Here the probability of accurate recovery is the same as in Theorem 2 and under similar conditions this probability converges to one. The reconstruction error has two terms. The first term, \( \frac{\sqrt{n/d + 1 + t} + 1}{\tau_n} \sqrt{n^2 - 2m^2} \), is again similar to Theorem 2 and under similar conditions converges to zero. The second term in the error, \( \frac{e}{\tau_n \sqrt{d}} \), is due to the noise in the measurements. As the number of measurements increases, \( \frac{e}{\tau_n \sqrt{d}} \) converges to zero. This is due to the fact that since \( A_{i,j} \sim \mathcal{N}(0, 1) \) as we increase the number of measurements, the energy of the signal per measurement is fixed. But since the total amount of energy of the noise is considered to be constant the average noise per measurement decreases by \( 1/\sqrt{d} \).

### D. Recovery of approximately low-complexity signals

In the last three sections, we considered recovering “low-complexity” signals from their linear (noisy or noise-free) projections. However, most applications feature signals that are not of exactly low-complexity but rather are “close” to low-complexity signals. In this section, we discuss this more general setting. Assume that the original signal \( x_o \) is not low-complexity but is close to the low-complexity signal \( \bar{x} \), i.e., \( \| x_o - \bar{x} \|_2 \leq \epsilon_n \) with \( \epsilon_n = o(1) \).
Again, let $y_o = Ax_o$. Consider the following reconstruction algorithm for recovering $x_o$ from its noisy linear measurements $y_o$:

$$\min \|y_o - Ax\|_2^2$$

s.t. $K^{[1]}(x) \leq \kappa_{m,n}m$. (9)

Assume that $A \in \mathbb{R}^{d \times n}$ and $A_{ij}$ are i.i.d. $\mathcal{N}(0,1)$. Let $\hat{x}_o = \hat{x}_o(y_o, A)$ denote the solution of (9).

**Theorem 5:** Assume that there exists $\tilde{x}_o \in \mathbb{R}^n$ such that $\|x_o - \tilde{x}_o\|_2 \leq \epsilon_n$, and $K^{[1]}(\tilde{x}_o) \leq \kappa_{m,n}m$. Let $y = Ax_o$, where $A$ is a $d \times n$ matrix with i.i.d. $\mathcal{N}(0,1)$ entries, and let $\hat{x}_o$ denote the minimizer of (9). Then, for any $0 < \tau_n < 1$,

$$\Pr\left(\|x_o - \hat{x}_o\|_2 > \frac{\sqrt{n/d + 1 + t + 1}}{\tau_n} \sqrt{\frac{n^2}{2m^2 + 2} + 2 \frac{n/d + 1 + t}{\tau_n} \epsilon_n}\right) \leq 2^{2\kappa_{n,n}}e^{\frac{1}{2}(1 - \tau_n^2 + 2\log \tau_n)} + e^{-\frac{d^2}{2}n^2}. \quad (10)$$

The proof is presented in Section VI-D. There are two error terms in (10). The first one, $\frac{\sqrt{n/d + 1 + t + 1}}{\tau_n} \sqrt{\frac{n^2}{2m^2 + 2}}$, is the reconstruction error due to the quantization performed in the calculation of Kolmogorov complexity. The second term is due to the fact that the signal $x_o$ is not of exactly low-complexity. The following corollary simplifies the statement of the theorem for some special useful cases.

**Corollary 3:** Assume $x_o = (x_{o,1}, x_{o,2}, \ldots) \in [0,1]^\infty$ satisfies all the conditions of Theorem 5 with $m_n = \lceil \log n \rceil$. Let $d = d_n = \lceil \kappa \log n \rceil$. If $d_n = \omega(n^{1/2})$, then for any $\epsilon > 0$

$$\Pr(\|x_o - \hat{x}_o\|_2^2 > \epsilon) \to 0$$
as $n \to \infty$.

**Proof** We showed in the proof of Corollary 1 that by fixing $t > 0$ and setting $\tau_n = 0.04$

$$2^{2\kappa_n \log n} e^{\frac{\kappa_n \log n}{1 - \tau_n^2 + 2\log \tau_n}} + e^{-\frac{d^2}{2}n^2} \to 0.$$ We also showed that in the same setting we have

$$\frac{\sqrt{n/d + 1 + t + 1}}{\tau_n} \sqrt{\frac{n^2}{2m^2 + 2}} \to 0.$$ It is straightforward to show that since $d_n = \omega(n^{1/2})$ we have

$$\frac{\sqrt{n/d + 1 + t}}{\tau_n} \epsilon_n \to 0,$$

and the result follows.
E. Other measurement matrices

For the sake of clarity, the results presented so far have focused on i.i.d. Gaussian measurement matrices. However, the results can be extended to the more general class of i.i.d. subgaussian matrices.

**Definition 2:** A random variable $X$ is called *subgaussian* if and only if there exist two constants $c_1, c_2 > 0$ such that

$$P(|X| > t) \leq c_1 e^{-c_2 t^2}.$$ 

Such a random variable is denoted by $SG(c_1, c_2)$.

Our goal in this section is to show how our results can be extended to the problem of CS with i.i.d. subgaussian measurement matrices. Our main conclusion is that the results presented for Gaussian matrices continue to hold for subgaussian matrices except for slight changes in the constants. However, as will be discussed later in Section VI-E, the proof techniques are different from those for Gaussian matrices. To show these differences we extend the result of Theorem 2 to subgaussian matrices. Similar arguments can be used for other extensions. As before we consider the problem of recovering $x_o$ from linear measurements $y_o = Ax_o$, where the elements of the matrix are i.i.d. $SG(c_1, c_2)$. Also assume that $E[A_{ij}] = 0$ and $E[A_{ij}^2] = 1$.

**Theorem 6:** Let $x_o = (x_{o,1}, x_{o,2}, \ldots) \in [0, 1]^\infty$. For integers $m$ and $n$, let $\kappa_{m,n}$ denote the Kolmogorov information dimension of $x_o$ at resolution $m$. Then, there exist three constants $c_1', c_2', c_3'$ depending only on $c_1$ and $c_2$ such that for any $1 - c_3' c_2' < \tau_n < 1$ and $t > 0$,

$$P(\|x_o - \hat{x}_o\|_2 > \tau_n^{-1} (c_2' + 1)n/d + 1 + 1) \sqrt{n^{2-2m+2}} \leq \frac{2^{2\kappa_{m,n}m} e^{-c'_3 n^{2-2m+2}}}{c_3' n^{2}} + e^{-c'_1 n}.$$ 

Theorem 6 shows that, by choosing $m = \lceil \log n \rceil$, $O(\kappa_{m,n} \log n)$ measurements remain sufficient for asymptotically accurate recovery. But, as expected, the constants might be different from those in Theorem 2.

F. Discussion

The LLS algorithms proposed in (7) and (9), corresponding to the cases when noise is present either in the signal or in the measurements, both assume the knowledge of an upper bound on the complexity of the signal. While such knowledge might be available or estimated in some applications, in many cases it is not straightforward to acquire it. In those cases, one might change the formulation of the MCP as follows:

$$\arg\min_x K^{|1|^m}(x)$$

s.t. $\|Ax - y_o\|_2 \leq z_n.$ \hspace{1cm} (11)

We call this new algorithm relaxed MCP or R-MCP. In this new optimization problem the challenge is to set parameter $z_n$ properly. The value of this parameter should be set according to the noise level present in the system. For instance, if we employ $z_n = (\sqrt{n} + (t+1)\sqrt{d})\epsilon_n$ and $z_n = e$ for the approximately low-complexity signals case
(corresponding to Section III-D) and exactly sparse signal in the presence of deterministic noise (corresponding to Section III-C), respectively, then we obtain results that are exactly the same as those stated in Theorems 4 and 5. Since the proofs are very similar to the proofs of Theorems 4 and 5, we skip them here. In the case of stochastic noise (corresponding to Section III-B) it is not clear if this new formulation provides a bound similar to Theorem 3. This problem is deferred for future research.

IV. KOLMOGOROV DIMENSION OF CERTAIN CLASSES OF FUNCTIONS

It is well known that the Kolmogorov complexity of a sequence is not computable ([12] Section 14.7). However, it is often possible to provide upper bounds on the Kolmogorov complexity. In this section, we consider several standard classes of functions and provide upper bounds on their Kolmogorov information dimension. Based on these upper bounds, one can use Theorems 2 and 5 to calculate the number of linear measurements required by the MCP to recover them. These examples demonstrate the connection between the results of Section III and the CS framework explained in the introduction.

A. Sparse signals

A class of signals that has played a starring role in compressed sensing is the class of $k$-sparse signals. The following proposition provides an upper bound on the Kolmogorov information dimension of such signals.

**Proposition 2:** Let the signal $x_o = (x_{o,1}, x_{o,2}, \ldots, x_{o,n})$ be $k$-sparse, i.e., $\|x_o\|_0 \leq k$. Then

$$\kappa_{m,n}(x_o) \leq k + \frac{k \log^* n + \log^* k + c}{m}.$$  

**Proof:** Consider the following program for describing $[x_o]_m$. First, use a program of constant length to describe the structure of the signal as “sparse” and the ordering of the rest of information, and the length of the sequence and the resolution. Next, code the sparsity level $k$ with $\log^* k$ bits, and spend $k \log^* n$ more bits to code the locations of the $k$ non-zero elements. Finally, use $km$ more bits to describe the quantized magnitudes of the non-zero coefficients. Therefore, we have

$$K^{[m]}(x_{o,1}, x_{o,2}, \ldots, x_{o,n}) \leq k + \frac{k \log^* n + \log^* k + c}{m},$$

where $c$ is a constant independent of $x_o, m$ and $n$. \qed

In most of our analysis in this paper we consider the case $m = \log n$. It is straightforward to confirm that in that case, for any $\delta > 0$, $\kappa_{m,n}(x_o) \leq 2k(1 + \delta)$, for sufficiently large $n$.

\footnote{Note that in calculating the information dimension we assume that $n$ and $m$ are given to the universal computer. Otherwise we would need $\log^* n$ and $\log^* m$ bits to describe them to the machine.}
B. Power law compressible signals

While sparse signals have played an important role in the theory of compressed sensing, it is well-known that they rarely occur in practice. More accurate models assume that either the signal’s coefficients decay at a specified rate, or the signal belongs to an \(\ell_p\) ball with \(p < 1\) \([1]\), i.e., the signal belongs to the set

\[
B_p^\epsilon \triangleq \{x \in \mathbb{R}^n : \|x\|_p \leq 1\}.
\]

For \(x_o \in B_p^\epsilon\), let \((x_o(1), x_o(2), \ldots, x_o(n))\) denote the permuted version of \(x_o\) such that \(|x_o(1)| \geq |x_o(2)| \geq \cdots \geq |x_o(n)|\). It is straightforward to show that \(|x_o(i)| \leq \tau^{-\frac{1}{p}+\frac{1}{2}}\), i.e., it is power law compressible. Therefore, if we just keep the \(k\) largest coefficients of this signal and set the rest to zero, the resulting \(k\)-sparse vector \(\check{x}_o\) satisfies:

\[
\|x_o - \check{x}_o\|_2 \leq k^{-\frac{1}{p}+\frac{1}{2}}.
\]

In Section [IV-A], we derived an upper bound for the Kolmogorov information dimension of \(\check{x}_o\). Proposition 3 follows from this bound and Corollary 3.

**Proposition 3:** Let \(x_o \in B_p^\epsilon\). For \(n \in \mathbb{N}\), let \(y^n = Ax_o^n\), where \(A\) is a \(d_n \times n\) random matrix with i.i.d. \(\mathcal{N}(0,1)\) entries. Set \(d_n = \lceil n^p/2 \log n \rceil\). Let \(\hat{x}_o^n\) denote the minimizer of (9) with \(m = \lceil \log n \rceil\) and \(\kappa_{m,n} = 3n^p/2\). For any \(\epsilon_1 > 0\) and \(\epsilon_2 > 0\),

\[
P(\|x_o^n - \hat{x}_o^n\|_2 > \epsilon_1) \leq \epsilon_2,
\]

for sufficiently large \(n\).

**Proof:** Let \(\hat{x}_o^n\) denote the \(k\)-sparse approximation of \(x_o^n\) derived by keeping the \(k = n^p/2\) largest coefficients of \(x_o^n\), and setting the rest to zero. According to Proposition 2, for \(n\) large enough and any \(\delta > 0\), the Kolmogorov information dimension of \(\hat{x}_o^n\) at resolution \(m = \log n\) is upper bounded by \(2n^{\frac{p}{2}}(1 + \delta)\). By setting \(\delta = 0.5\) we obtain \(\kappa_{m,n}(\hat{x}_o^n) \leq 3n^{\frac{p}{2}}\). Therefore, according to Theorem 5, setting \(t = 1\) and \(\tau = 0.04\) yields

\[
P\left(\|x_o^n - \hat{x}_o^n\|_2 > \frac{n/d + 1 + t}{\tau} \frac{1}{\sqrt{n/2 - m + 1}} + \frac{\sqrt{n/d + 1 + t}}{\tau} \epsilon_n\right) \to 0,
\]

as \(n \to \infty\), where \(\epsilon_n\) is the \(\ell_2\) norm of the difference between \(\hat{x}_o^n\) and \(x_o^n\). This error is upper bounded by \(\|\hat{x}_o^n - x_o^n\|_2 \leq n^{\frac{1}{2} - \frac{p}{2}}\). Plugging in the values of \(t\), \(\tau\), and \(\epsilon_n\) in (12) completes the proof.

It is interesting to note that as the power \(p\) decreases, the number of measurements required for successful recovery decreases.

C. Piecewise polynomial functions

Let Poly\(_Q^N\) denote the class of piecewise polynomial functions \(f : [0,1] \to [0,1]\) with at most \(Q\) singularities\(^6\) and maximum degree of \(N\). For \(f \in \text{Poly}_Q^N\), let \((x_o,1, x_o,2, \ldots, x_o,n)\) be the samples of \(f\) at

\[
0, \frac{1}{n}, \ldots, \frac{n-1}{n}.
\]

\(^6\)A singularity is a point at which the function is not infinitely differentiable.
Let \( \{a_\ell^i\}_{i=0}^{N_\ell} \) denote the set of coefficients of the \( \ell \)th polynomial of \( f \), where \( N_\ell \leq N \) denotes its degree. For the notational simplicity, we assume that the coefficients of each polynomial belong to the \([0, 1]\) interval and that \( \sum_{i=0}^{N_\ell} a_\ell^i < 1 \), for every \( \ell \). Define

\[
\mathcal{P} \triangleq \left\{ x_o \in \mathbb{R}^n \mid x_o, i = f(i/n), f \in \text{Poly}_N^Q \right\}.
\]

**Proposition 4:** For every signal \( x_o \in \mathcal{P} \), we have

\[
k_{m,n}(x_o) \leq (Q + 1)(N + 1) + \frac{(Q + 1)(N + 1)[\log_2(N + 1)]}{m} + \frac{\log^* n + \log^* N + \log^* k + Q \log^* n + c_1 + c_2}{m}.
\]

**Proof:** Consider the following program for describing the quantized version of \( x_o \). The code first specifies the signal model as samples of a “piecewise polynomial” function with parameters \((n, Q, N)\). This requires \( \log^* N + \log^* Q + c \) bits. Then, for each singularity point, the code first specifies the largest sampling point \( i/n \) that is smaller than it. Since there are at most \( Q \) singularity points, describing this information requires at most \( Q \log^* n \) bits. The next step is to describe the coefficients of each polynomial. Using an \( m' \)-bit uniform quantizer for each coefficient, the induced error is bounded as

\[
\left| \sum_{i=0}^{N_\ell} a_\ell^i t^n - \sum_{i=0}^{N_\ell} [a_\ell^i]_{m'} t^n \right| \leq \sum_{i=0}^{N_\ell} |a_\ell^i - [a_\ell^i]_{m'}| \leq (N_\ell + 1)2^{-m'} \leq (N + 1)2^{-m'}.
\]

To ensure reconstructing the samples at resolution \( m \), we require \((N + 1)2^{-m'} < 2^{-m} \). Therefore, to describe the coefficients of the polynomials, at most, \((Q + 1)(N + 1)(m + \lfloor \log_2(N + 1) \rfloor) \) extra bits are required. Hence, overall, it follows that

\[
K_{[m]}(x_{o,1}, x_{o,2}, \ldots, x_{o,n}) \leq (Q + 1)(N + 1) + \frac{(Q + 1)(N + 1)[\log_2(N + 1)]}{m} + \frac{\log^* N + \log^* Q + Q \log^* n + c}{m}.
\]

It is straightforward to plug (14) in Corollary 1 and prove that, for large values of \( n \), \( O((Q + 1)(N + 2) \log n) \) measurements are sufficient for the successful recovery of the piecewise polynomial functions.

**D. Smooth functions**

Suppose that \( x_1, x_2, \ldots, x_n \) are equispaced samples of a smooth function \( f : [0, 1] \rightarrow [0, 1] \). Let \( \mathcal{S}_\beta \) represent the class of \( \beta + 1 \) times differentiable functions. For the notational simplicity we assume that \( |f^{(m)}(t)| \leq m! \) for every \( m \leq \beta + 1 \). This function is not necessarily a low-complexity signal, but it can be well-approximated by a piecewise polynomial function. To show this, consider partitioning the \([0, 1]\) interval into subintervals of size \( r_n \), and approximating the function \( f \) with a polynomial of degree \( \beta \) in each subinterval. Let \( \hat{f}_\beta(x) \) denote the resulting
piecewise polynomial function. It is straightforward to prove that \( \| f - \hat{f}_\beta \|_\infty \leq r_n^{\beta+1} \). Hence, if \( x_o \) and \( \hat{x}_o \) denote the vectors consisting of the equispaced samples of the original signal and its piecewise polynomial approximation, respectively, it follows that \( \| \hat{x}_o - x_o \|_2 \leq \sqrt{nr_n^{\beta+1}} \). We can summarize our discussion in the following proposition.

**Proposition 5:** For \( n \in \mathbb{N} \), let \( x^n_o \) denote the vector of \( n \) equispaced samples of \( f \in \mathcal{S}_\beta \). Let \( y^d = Ax^n_o \), where \( A \) is a \( d_n \times n \) random matrix with i.i.d. \( \mathcal{N}(0, 1) \) entries. Also, let \( \hat{x}^n \) denote the solution of low-complexity least square algorithm in (9), with \( m = \log n \) and \( \kappa_{m,n} = 2(2 + \beta)(n^{\beta+1} + 1) \). Then, for \( n \) large enough and \( d_n = \lceil \kappa_{m,n} \log n \rceil \) and any \( \epsilon_1, \epsilon_2 > 0 \), we have

\[
P(\| x^n_o - \hat{x}^n_o \|_2 > \epsilon_1) \leq \epsilon_2.
\]

**Proof:** Partition the \([0, 1]\) interval into subintervals of size \( r_n = n^{-\frac{1}{\beta+2}} \), and approximate the function \( f \) with a polynomial of degree \( \beta \) in each subinterval. Let \( \hat{f}_\beta \) denote the resulting piecewise polynomial function. According to Proposition 4 for \( n \) sufficiently large, the Kolmogorov information dimension of the samples of \( \hat{f}_\beta, \hat{x}^n_o \), at resolution \( m = \log n \), is less than \( (n^{\beta+1} + 1)(\beta + 2)(1 + \delta) \), for any \( \delta > 0 \). Set \( \delta = 1 \) and assume that \( n \) is large enough for this result to hold. By Theorem 5 if we set \( t = 1, \tau = 0.04, \) and \( d_n = \lceil \kappa_{m,n} \log n \rceil \), then it follows that

\[
P(\| x^n_o - \hat{x}^n_o \|_2 > \frac{\sqrt{n/d + 1 + t} + \sqrt{2m+1} + \sqrt{n/d + 1 + t} \epsilon_n}{\tau} ) \to 0,
\]
as \( n \to \infty \), where \( \epsilon_n \) is the \( \ell_2 \) of the difference between \( \hat{x}^n_o \) and \( x^n_o \). Furthermore, as described before, \( \epsilon_n = \| \hat{x}^n - x^n_o \|_2 \leq \sqrt{n}r_n^{\beta+1} = n^{-\frac{1}{\beta+2}} \). Plugging in \( t = 1, \tau = 0.04, \) and \( \epsilon_n = n^{-\frac{1}{\beta+2}} \) completes the proof. \( \square \)

**E. Low-rank matrices**

Let \( \mathcal{C}_r(M,N) \) be the class of \( M \times N \) real-valued rank-\( r \) matrices \( X \) with \( \sigma_{\text{max}}(X) \leq 1 \). The following theorem characterizes the Kolmogorov information dimension of a matrix in this class at resolution \( m \).

**Proposition 6:** Let \( X \in \mathcal{C}_r(M,N) \). Then

\[
\kappa_{m,n}(X) \leq r(M + N + 1) + \frac{\log^* r + r(M + N + 1)\log(3r) - r + c}{m}.
\]

Fig. 1. The representation of smooth function (solid black curve) and its piecewise polynomial approximation (dashed red). As \( r_n \) becomes smaller the approximation become more accurate.
Proof: Having access to the values of $M$, $N$ and the resolution level $m$, consider the program that describes $X$ through its singular value decomposition as follows. Denote the singular value decomposition of the matrix $X$ as $X = UΣVT$ where $U \in \mathbb{R}^{M \times r}$, $V \in \mathbb{R}^{N \times r}$ and $Σ \in \mathbb{R}^{r \times r}$ is a diagonal matrix. Note that $UTU = I_r$ and $VVT = I_r$. To describe $X$, first we use a constant number of bits to describe the structure of the data as a matrix of rank $r$, and also our coding strategy, which is describing the quantized versions of $U$, $Σ$, and $V$. To describe the rank $r$, the code uses $\log^* r$ bits. The next step is to describe the quantized versions of $U$, $Σ$ and $V$. Let $m_u$, $m_v$, and $m_σ$ denote the resolution levels used in the uniform quantization of the elements of $U$, $Σ$, and $V$, respectively. Hence, the quantized matrices can be described using $rMm_u + rNm_v + rm_σ$ bits. Let $\hat{U}$, $\hat{V}$ and $\hat{Σ}$ denote the quantized version of $U$, $V$ and $Σ$ at the specified resolutions, respectively. Let $\hat{X} \triangleq \hat{U} \hat{Σ} \hat{V}$. By the triangle inequality,

$$|X_{ij} - \hat{X}_{ij}| = |u_i^TΣv_j - \hat{u}_i^T\hat{Σ}\hat{v}_j| \leq |u_i^TΣv_j - \hat{u}_i^T\hat{Σ}\hat{v}_j| + |\hat{u}_i^T\hat{Σ}\hat{v}_j - \hat{u}_i^T\hat{Σ}\hat{v}_j|,$$

where $u_i^T$, $v_j^T$, $\hat{u}_i^T$, and $\hat{v}_j^T$ denote the $i$th rows of $U$, $V$, $\hat{U}$ and $\hat{V}$, respectively. Note that $|U_{ij}| \leq 1$, $|V_{ij}| \leq 1$, for all $i, j$. Also by assumption, $σ_{\max}(Σ) \leq 1$, and therefore $0 \leq Σ_{ii} < 1$, for $i = 1, \ldots, r$. Moreover, $|U_{ij} - \hat{U}_{ij}| < 2^{-m_u + 1}$, $|V_{ij} - \hat{V}_{ij}| < 2^{-m_v + 1}$, and finally $|Σ_{ii} - \hat{Σ}_{ii}| < 2^{-m_σ}$. Therefore,

$$|X_{i,j} - \hat{X}_{i,j}| \leq |u_i^TΣv_j - \hat{u}_i^T\hat{Σ}\hat{v}_j| + |\hat{u}_i^T\hat{Σ}\hat{v}_j - \hat{u}_i^T\hat{Σ}\hat{v}_j| \\ \leq \|u_i - \hat{u}_i\|_2\|\Sigma v_j\|_2 + \|\hat{u}_i\|_2\|\Sigma (v_j - \hat{v}_j)\|_2 \\ \leq \|u_i - \hat{u}_i\|_2σ_{\max}(Σ)\|v_j\|_2 + \|\hat{u}_i\|_2σ_{\max}(\hat{Σ})\|v_j - \hat{v}_j\|_2 \\ \leq \sqrt{r2^{-2m_u+1}r + r2^{-m_v}r} + \sqrt{r2^{-2m_σ+1}r + r2^{-m_v+1}} \\ \leq r2^{-m_u+1} + r2^{-m_v} + r2^{-m_σ+1}. \quad (16)$$

To ensure reconstructing the samples at resolution $m$, we have

$$r2^{-m_u+1} + r2^{-m_v} + r2^{-m_σ+1} \leq 2^{-m+1}.$$

Setting $m_u = m_v = m_σ + 1$, we obtain $m_σ \geq m + \log(3r) - 1$. Therefore, the Kolmogorov information dimension at resolution $m$ of $X$ is upper bounded as follows:

$$κ_{m,M,N} \leq \frac{\log^* r + rMm_u + rNm_v + rm_σ + c}{m} \leq \frac{\log^* r + rM(m + \log(3r)) + rN(m + \log(3r)) + r(m + \log(3r) - 1)}{m} \leq r(M + N + 1) + \frac{\log^* r + r(M + N + 1)\log(3r) - r + c}{m}.$$
V. RELATED WORK

A. Kolmogorov complexity and applications

This paper is inspired by [17] and [13]. [17] considers the well-studied problem of estimating \( \theta \in \mathbb{R}^n \) from its noisy observation \( s = \theta + z \), where \( z \) represents the noise in the system. It suggests using the minimum Kolmogorov complexity estimator (MKCE), and proves that if \( \{\theta_i\}_{i=1}^n \overset{i.i.d.}{\sim} \pi \), under several scenarios for the signal and noise, the average marginal distribution of the estimate derived by MKCE tends to the actual posterior distribution. [13] considers the problem of CS over real-valued sequences with finite Kolmogorov complexity and defines the Kolmogorov complexity of a real-valued sequence \( x = (x_1, \ldots, x_n) \) as the length of the program that prints the binary representation of \( x \) and halts. Consider the set of all real-valued sequences with Kolmogorov complexity less than or equal to \( k_0 \), i.e.,

\[
S(k_0) \triangleq \{ x : K(x) \leq k_0 \}
\]

Let \( A \) denote a \( d \times n \) binary matrix, \( x_o = (x_1, x_2, \ldots, x_n)^T \), \( y_o = Ax_o \). [13] proposes the following algorithm for recovering \( x_o \) from its linear measurements \( y_o \):

\[
\hat{x}(y_o, A) \triangleq \arg \min_{x \in \mathbb{R}^n} K(x).
\]

(17)

It proves that \( 2k \) random linear measurements are sufficient for recovering sequences in \( S(k_0) \) with high probability. This result does not consider any non-ideality in the signal or the measurements. Our paper settles both issues. Furthermore, note that \( S(k_0) \) covers none of the classes of signals of interest in compressed sensing, such as sparse vectors or low-rank matrices. Almost all such signals have infinite Kolmogorov complexity, and therefore are not covered by the framework proposed in [13]. Our generalizations require completely different proof techniques.

In an independent work, [15] and [16] have explored the performance of the MCP algorithm for CS problems. Replacing the Kolmogorov complexity with the empirical entropy, they propose a Markov chain Monte Carlo approach similar to [18]–[20] to solve the recovery problem. The empirical results provided in [16] are very promising. Our theoretical results explain why such algorithms perform well in practice.

Finally, we should mention that Kolmogorov complexity has proved to be useful in other applications such as similarity detection [21], [22], density estimation [23] and compression and denoising [24]. For more information on the progress in these areas, see [11].

B. Stochastic models

In this paper, we considered deterministic models for the signals. While deterministic signal models are the most popular models in CS, stochastic models have been also extensively explored in this field. See [25]–[35] and the references therein for more information. The most relevant to our work is [32]. It considers the problem of recovering a memoryless process from a linear set of measurements and proves a connection between the number of measurements required and the Rényi information dimension. The upper information dimension of a random
vector \((X_1, X_2, \ldots, X_n)\) is defined as

\[
d(X_1, \ldots, X_n) \triangleq \lim_{m \to \infty} \sup_{m} \frac{H([X_1]_m, \ldots, [X_n]_m)}{m}.
\]

There is a connection between the Kolmogorov information dimension of a sequence and its Rényi information dimension [12] (Theorem 14.3.1). In spite of such connections, there are several important differences between our work and the work of [32]. First, the results in [32] are asymptotic, and the amount of error and the probability of correct recovery for finite dimensional signals have not been established there. Second, the stochastic approach proposed in [32] considers a specific distribution that is assumed to be known in the recovery process while we are considering universal schemes in this paper.

C. Universal schemes and minimum entropy coder

Our work has some connections with the minimum entropy decoder proposed by Csiszar in [36]. He suggests a universal minimum entropy decoder for reconstructing an i.i.d. signal from its linear measurements at a rate determined by the entropy of the source. For more information, see [37], [38] and the references therein.

Finally we should emphasize that universal algorithms (that perform “optimally” without knowing the distribution of the data) have been explored extensively in information theory and are popular in many applications, including compression [19], [39], denoising [40], prediction [41], and many more. However, to the best of our knowledge our results provide the first universal approach for CS.

D. Signal models

As mentioned in the introduction, in this paper we address a central problem in the field of compressed sensing. Since the early days of CS, there have been many efforts to push the limits of the technique beyond sparsity. This line of work has resulted in a series of papers each of which either generalizes the signal model or reduces the required number of measurements by introducing more structure on the signal; see, for example, [42]–[48]. As proved in Section IV, some of these models can be considered as subclasses of the general model we consider here. However, it is worth noting that even though the MCP algorithm proposed here is universal, since Kolmogorov complexity is not computable, it is not immediately useful for practical purposes.

VI. Proofs of the main results

A. Useful lemmas

The following lemmas are frequently used in our proofs.

**Lemma 2 (\(\chi^2\) concentration):** Fix \(\tau > 0\), and let \(Z_i \sim \mathcal{N}(0, 1)\), \(i = 1, 2, \ldots, d\). Then,

\[
P\left( \sum_{i=1}^{d} Z_i^2 < d(1 - \tau) \right) \leq e^{\frac{d}{2} (\tau + \log(1 - \tau))}
\]
and

\[ P \left( \sum_{i=1}^{d} Z_i^2 > d(1 + \tau) \right) \leq e^{-\frac{d}{2}(\tau - \log(1+\tau))}. \]  

(18)

**Proof:** By the Markov inequality, for any \( \lambda > 0 \), we have

\[ P \left( \sum_{i=1}^{d} Z_i^2 - d < -d\tau \right) = P \left( -\sum_{i} Z_i^2 + d > d\tau \right) \leq e^{-\lambda d\tau} E \left[ e^{\lambda(d-\sum Z_i^2)} \right] \]

\[ = e^{-\lambda d\tau + \lambda d} \left( E[e^{-\lambda Z_i^2}] \right)^d \]

\[ = e^{-\lambda d\tau + \lambda d} (1 + 2\lambda)^{-d/2}. \]  

(19)

We optimize over \( \lambda \) to obtain

\[ \lambda^* = \frac{\tau}{2(1 - \tau)}. \]  

(20)

Plugging (20) into (19), we obtain (18).

**Lemma 3:** Let \( X \) and \( Y \) denote two independent Gaussian vectors of length \( n \) with i.i.d. elements. Further, assume that for \( i = 1, \ldots, n \), \( X_i \sim \mathcal{N}(0,1) \) and \( Y_i \sim \mathcal{N}(0,1) \). Then the distribution of \( X^T Y = \sum_{i=1}^{n} X_i Y_i \) is the same as the distribution of \( \|X\|_2 G \), where \( G \sim \mathcal{N}(0,1) \) is independent of \( \|X\|_2 \).

**Proof:** Note that

\[ \frac{X^T Y}{\|X\|_2} = \sum_{i=1}^{n} \frac{X_i}{\|X_i\|_2} Y_i. \]  

(21)

Given \( X/\|X\|_2 = a \),

\[ \sum_{i=1}^{n} \frac{X_i}{\|X_i\|_2} Y_i \sim \mathcal{N}(0,1), \]

because \( \|a\|_2^2 = 1 \). Therefore, since the distribution of \( X^T Y/\|X\|_2 \) given \( X/\|X\|_2 = a \) is independent of the value of \( a \), the unconditional distribution of \( X^T Y/\|X\|_2 \) is also \( \mathcal{N}(0,1) \). To prove independence, note that \( X/\|X\|_2 \) and \( Y \) are both independent of \( \|X\|_2 \).

The following lemma is adapted from [49] (Proposition 5.10).

**Lemma 4:** Let \( Z_1, Z_2, \ldots, Z_n \) be i.i.d. zero-mean SG\((c_1, c_2)\) random variables. Let \( a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n \) be a vector satisfying \( \|a\|_2^2 = 1 \). Then

\[ P \left( \sum_{i=1}^{n} a_i Z_i > t \right) \leq c_1 e^{-c_2 t^2}. \]

In other words \( \sum_{i=1}^{n} a_i Z_i \) is also SG\((c_1, c_2)\).

**Definition 3:** A random variable \( X \) is called subexponential, denoted by SE\((c_1, c_2)\), if and only if

\[ P(|X| > t) \leq c_1 e^{-c_2 t}. \]
Slightly modified versions of the proofs we provide in the rest of this section can be found in [49]. For the sake of clarity and uniformity we state these lemmas with their proofs here.

Lemma 5: Let $Z$ be a $\text{SE}(c_1, c_2)$ random variable. Then, it follows that
\[
\mathbb{E}[|Z|^p] \leq \frac{2c_1 p!}{c_2^p}.
\]

Proof: Here we prove this lemma for the case where $p$ is even. The other case follows the same approach. Let $F(z)$ denote the cumulative distribution function of the random variable $Z$

\[
\mathbb{E}[|Z|^p] = \int_{-\infty}^\infty z^p dF(z) = \int_{-\infty}^\infty p z^{p-1} \int_{z}^\infty dF(x) dz - \int_{-\infty}^0 p z^{p-1} \int_{-\infty}^z dF(x) dz
\]

\[
\leq \int_{0}^\infty p z^{p-1} c_1 e^{-cz^2} dz - \int_{-\infty}^0 p z^{p-1} c_1 e^{cz^2} dz = \frac{2c_1(p!)}{c_2^p}.
\]

Equality (a) is the result of integration by parts.

Lemma 6: Let $Z$ be a zero-mean $\text{SE}(c_1, c_2)$ random variable. Then we have
\[
\mathbb{E}[e^{\lambda Z}] \leq e^{4c_1 \lambda^2 / c_2^2}, \quad \forall \lambda < c_2 / 2.
\]

Proof: We prove this theorem by expanding the exponential function $e^{\lambda Z}$ and bounding the moments using Lemma 5 as follows:
\[
\mathbb{E}[e^{\lambda Z}] = \mathbb{E}\left[1 + X + \sum_{k=2}^{\infty} \frac{\lambda^k X^k}{K!}\right] = \mathbb{E}\left(1 + \sum_{k=2}^{\infty} \frac{\lambda^k X^k}{K!}\right)
\]

\[
\leq 1 + 2c_1 \left(\frac{\lambda}{c_2}\right)^2 + \left(\frac{\lambda}{c_2}\right)^3 + \ldots \leq 1 + 2c_1 \left(\frac{\lambda}{c_2}\right)^2 \left(\frac{1}{1 - \lambda/c_2}\right).
\]

Assuming that $\frac{\lambda}{c_2} < \frac{1}{2}$, we obtain
\[
\mathbb{E}[e^{\lambda Z}] \leq 1 + 4c_1 \left(\frac{\lambda}{c_2}\right)^2 \leq e^{4c_1 \lambda^2 / c_2^2},
\]
where the last inequality is due to the fact that $1 + x \leq e^x$ for $x \geq 0$.

Lemma 7: Let $Z_1, Z_2, \ldots, Z_n$ be i.i.d. $\text{SG}(c_1, c_2)$ random variables with mean zero and variance 1. Then we have
\[
\mathbb{P}\left(\left|\sum_{i=1}^n (Z_i^2 - 1)\right| > nt\right) \leq 2e^{-n c_3 t^2 / 16c_3}, \quad \text{for } t \in (0, c_3 / c_2),
\]
where $c_3 \triangleq \max(e^{c_2}, c_1 e^{-c_2})$.

Proof: Define $X_i \triangleq Z_i^2 - 1$. It is straightforward to confirm that for all $t > 1$,
\[
\mathbb{P}(|X_i| > t) \leq c_1 e^{-c_2(t+1)}.
\]
Define \( c_3 \triangleq \max(c_1, c_2) \). If we combine the fact that \( P(|X_i| > t) \leq 1 \) for \( 0 \leq t \leq 1 \) with (23), we obtain
\[
P(|X_i| > t) \leq c_3 e^{-c_2 t}.
\]
We have
\[
P \left( \sum_{i=1}^n X_i > n t \right) = P \left( e^{\lambda \sum_{i=1}^n X_i} > e^{\lambda n t} \right) \leq e^{-\lambda n t} \left( E \left[ e^{\lambda X_1} \right] \right)^n \leq e^{-\lambda n t + 4n c_3 \lambda^2 / c_2^2},
\]
where the last inequality is the result of Lemma 6. Assuming \( t < c_3^3 c_2^2 \) and setting \( \lambda = tc_2^2 / (8c_3) \), we obtain
\[
P \left( \sum_{i=1}^n X_i > n t \right) \leq e^{-n \left( c_2^2 t \right)^2 / 16c_3}.
\]
Using the same argument we find a similar upper bound for \( P(\sum_{i=1}^n X_i < -nt) \).

**Lemma 8:** Let \( A \) be a \( d \times n \) matrix with i.i.d. \( \text{SG}(c_1, c_2) \) elements, and suppose that the elements satisfy \( E(A_{ij}) = 0 \) and \( E(A_{ij}^2) = 1 \). Then there exist two constants \( c_1', c_2' \) depending only on \( c_1 \) and \( c_2 \) such that with probability at least \( 1 - e^{-c_2' t^2} \),
\[
\sigma_{\text{max}}(A) \leq \sqrt{d} + c_1' \sqrt{n} + t.
\]

**Proof:** See Theorem 5.39 in [49] for more information on the proof and the constants that are involved.

**B. Proof of Theorem 2**

Let \( \hat{x}_o \) denote the solution of MCP. Also, let \( q_m \triangleq x_o - \phi_m(x_o) \) and \( \hat{q}_m \triangleq \hat{x}_o - \phi_m(\hat{x}_o) \) denote the quantization errors of the original and the reconstructed signals at resolution \( m \), respectively, where for \( x \in [0, 1]^n \), \( \phi_m(x) \) is defined in Remark 1.

Since both \( Ax_o = y_o \) and \( A\hat{x}_o = y_o \), it follows that
\[
A(\phi_m(x_o) + q_m) = A(\phi_m(\hat{x}_o) + \hat{q}_m)
\]
and
\[
A(\phi_m(x_o) - \phi_m(\hat{x}_o)) = A(\hat{q}_m - q_m).
\]
On the other hand, by our definition in (1), it follows that
\[
\| \hat{q}_m - q_m \|_2 \leq n 2^{-2m+2}.
\]
Hence,
\[
\| A(\phi_m(x_o) - \phi_m(\hat{x}_o)) \|_2 = \| A(\hat{q}_m - q_m) \|_2 \\
\leq \sigma_{\text{max}}(A) \sqrt{n 2^{-2m+2}},
\]

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where $\sigma_{\text{max}}(A)$ is the maximum singular value of matrix $A$. By definition, $K^{[1]m}(x_o) \leq \kappa_{m,n}m$, and since $\hat{x}_o$ is the solution of (4), we have

$$K^{[1]m}(\hat{x}_o) \leq K^{[1]m}(x_o) \leq \kappa_{m,n}m. \quad (27)$$

Define the set $S_o$ as

$$S_o \triangleq \{ h : h = \phi_m(\hat{x}_o) - \phi_m(x_o), \hat{x}_o, x_o \in [0, 1]^n, K(\phi_m(\hat{x}_o)) \leq \kappa_{m,n}m, K(\phi_m(x_o)) \leq \kappa_{m,n}m \}.$$  

Define the event $\mathcal{E}_1^{(n)}$ as

$$\mathcal{E}_1^{(n)} \triangleq \{ \# h \in S ; \| A(h) \|_2 < \tau \sqrt{d} \| h \|_2 \}, \quad (28)$$

and, for $t > 0$, the event $\mathcal{E}_2^{(n)}$ as

$$\mathcal{E}_2^{(n)} \triangleq \{ \sigma_{\text{max}}(A) - \sqrt{d} - \sqrt{n} < t \sqrt{d} \}. \quad (29)$$

Let $\epsilon \triangleq (\tau^{-1}\sqrt{nd^{-1}} + 1 + t) \sqrt{n}2^{-2m+2}$. Using these definitions and the union bound, we have

$$P (\| x_o - \hat{x}_o \|_2 > \epsilon) = P (\| x_o - \hat{x}_o \|_2 > \epsilon, \mathcal{E}_1^{(n)} \cap \mathcal{E}_2^{(n)}) + P (\| x_o - \hat{x}_o \|_2 > \epsilon, (\mathcal{E}_1^{(n)} \cap \mathcal{E}_2^{(n)})^c)
\leq P (\| x_o - \hat{x}_o \|_2 > \epsilon, \mathcal{E}_1^{(n)} \cap \mathcal{E}_2^{(n)})
+ P (\mathcal{E}_1^{(n)} \cap \mathcal{E}_2^{(n)}) + P (\mathcal{E}_1^{(n)},c) + P (\mathcal{E}_2^{(n)},c) \quad (30)$$

But if $A \in \mathcal{E}_1^{(n)} \cap \mathcal{E}_2^{(n)}$, then it follows that

$$\| x_o - \hat{x}_o \|_2 = \| \phi_m(x_o) + q_m - \phi_m(\hat{x}_o) - \hat{q}_m \|_2
\leq \| \phi_m(x_o) - \phi_m(\hat{x}_o) \|_2 + \| q_m - \hat{q}_m \|_2
\overset{(a)}{\leq} (\tau \sqrt{d})^{-1} \| A(\phi_m(x_o) - \phi_m(\hat{x}_o)) \|_2 + \| q_m - \hat{q}_m \|_2
\overset{(b)}{\leq} (\tau \sqrt{d})^{-1} \sigma_{\text{max}}(A) \sqrt{n}2^{-2m+2} + \sqrt{n}2^{-2m+2}
\leq \left( (\tau \sqrt{d})^{-1} \sigma_{\text{max}}(A) + 1 \right) \sqrt{n}2^{-2m+2}
\overset{(c)}{\leq} \left( \frac{1}{\tau} \left( \sqrt{nd^{-1}} + 1 + t \right) + 1 \right) \sqrt{n}2^{-2m+2}. \quad (31)$$

Inequality (a) holds due to the assumption that $A \in \mathcal{E}_1^{(n)}$, and therefore $\| A(\phi_m(x_o) - \phi_m(\hat{x}_o)) \|_2 \geq \tau \sqrt{d} \| (\phi_m(x_o) - \phi_m(\hat{x}_o)) \|_2$. 

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\[ \phi_m(\mathbf{x}_o) \|_2. \] Inequality (b) is the result of (26), and Inequality (c) is due to the fact that \( A \in E^{(n)}_2 \). Hence,

\[
P \left( \| \mathbf{x}_o - \mathbf{\hat{x}}_o \|_2 > \epsilon, E^{(n)}_1 \cap E^{(n)}_2 \right) = 0.
\] (32)

On the other hand, by Lemma 2, for any sequence \( \mathbf{x} \in \mathbb{R}^n \),

\[
P \left( \| A \mathbf{x} \|_2 \leq \tau \sqrt{d} \| \mathbf{x} \|_2 \right) = P \left( \frac{\| A \mathbf{x} \|_2}{\| \mathbf{x} \|_2} \leq \tau \sqrt{d} \right) = P \left( \sum_{i=1}^{d} Z_i^2 \leq \tau^2 d \right) \leq e^{\frac{d}{2}(1 - \tau^2 + 2 \log \tau)},
\]
where, for \( i = 1, \ldots, d \), \( Z_i = \| \mathbf{x} \|_2^{-1} \sum_{j=1}^{n} A_{i,j} x_j \). Using the union bound, it follows that

\[
P \left( E^{(n)}_1 \cap E^{(n)}_2 \right) \leq e^{\frac{d}{2}(1 - \tau^2 + 2 \log \tau)}. \] (33)

Finally, using the results on the concentration of Lipschitz functions of a Gaussian random vector \[50\], we obtain

\[
P \left( E^{(n)}_2 \right) = P \left( \sigma_{\max}(A) - \sqrt{d} - \sqrt{n} > t \sqrt{d} \right) \leq e^{-dt^2/2}. \] (34)

Plugging (32), (33), and (34) into (30) completes the proof. \( \square \)

C. Proof of Theorem 3

Let \( \mathbf{\hat{x}}_o \) denote the solution of

\[
\begin{align*}
\arg \min & \quad \| A \mathbf{x} - \mathbf{y}_o \|_2, \\
\text{s.t.} & \quad K^{(1)m}(\mathbf{x}) \leq k_n m_n,
\end{align*}
\] (35)

Since, by the assumption of the theorem, \( K^{(1)m}(\mathbf{x}_o) \leq k_n m_n \), \( \mathbf{x}_o \) is also a feasible point in (35). Therefore,

\[
\| A \mathbf{\hat{x}}_o - \mathbf{y}_o \|_2^2 \leq \| A \mathbf{x}_o - \mathbf{y}_o \|_2^2 \leq \| A \mathbf{x}_o - A \mathbf{x}_o - \mathbf{w} \|_2^2 = \| \mathbf{w} \|_2^2.
\] (36)

Expanding \( \| A \mathbf{\hat{x}}_o - \mathbf{y}_o \|_2^2 = \| A \mathbf{x}_o - A \mathbf{x}_o - \mathbf{w} \|_2^2 \) in (36), it follows that

\[
\| A (\mathbf{\hat{x}}_o - \mathbf{x}_o) \|_2^2 + \| \mathbf{w} \|_2^2 - 2 \mathbf{w}^T A (\mathbf{\hat{x}}_o - \mathbf{x}_o) \leq \| \mathbf{w} \|_2^2.
\] (37)
Cancelling $\|w\|_2^2$ from both sides of (37), we obtain
\[
\|A(\hat{x}_o - x_o)\|_2^2 \leq 2w^T A(\hat{x}_o - x_o) \\
\leq 2 \|w^T A(\hat{x}_o - x_o)\|.
\]
Let $q_m \triangleq x_o - \phi_m(x_o)$ and $\hat{q}_m \triangleq \hat{x}_o - \phi_m(\hat{x}_o)$ denote the quantization errors of the original and the reconstructed signals, respectively. On the one hand using these definitions and the Cauchy-Schwartz inequality, we find a lower bound for $\|A(\hat{x}_o - x_o)\|_2^2$ as
\[
\|A(\hat{x}_o - x_o)\|_2^2 \\
= \|A(\phi_m(\hat{x}_o) + \hat{q}_m - \phi_m(x_o) - q_m)\|_2^2 \\
= \|A(\phi_m(\hat{x}_o) - \phi_m(x_o)) + A(\hat{q}_m - q_m)\|_2^2 \\
\geq \|A(\phi_m(\hat{x}_o) - \phi_m(x_o))\|_2^2 \\
- 2 \|A(\hat{q}_m - q_m)^T A(\phi_m(\hat{x}_o) - \phi_m(x_o))\| \\
\geq \|A(\phi_m(\hat{x}_o) - \phi_m(x_o))\|_2^2 \\
- 2 \|A(\hat{q}_m - q_m)\|_2 \|A(\phi_m(\hat{x}_o) - \phi_m(x_o))\|_2.
\] (38)

On the other hand, again using our definitions plus the Cauchy-Schwartz inequality, we find an upper bound on $|w^T A(\hat{x}_o - x_o)|$ as
\[
|w^T A(\hat{x}_o - x_o)| \\
= |(\phi_m(\hat{x}_o) - \phi_m(x_o) + \hat{q}_m - q_m)^T A^T w| \\
\leq |(\phi_m(\hat{x}_o) - \phi_m(x_o))^T A^T w| + |(\hat{q}_m - q_m)^T A^T w| \\
\leq |(\phi_m(\hat{x}_o) - \phi_m(x_o))^T A^T w| + \|\hat{q}_m - q_m\|_2 \|A^T w\|_2.
\] (39)

For any $z \in [0, 1]$, $0 \leq z - [z]_m < 2^{-m}$. Therefore,
\[
\|\hat{q}_m - q_m\|_2 \leq \sqrt{m2^{-2m+2}}.
\] (40)

Let $\Delta_m \triangleq \phi_m(\hat{x}_o) - \phi_m(x_o)$. Define the set $S$ as follows
\[
S \triangleq \{h : h = \phi_m(\hat{x}_o) - \phi_m(x_o), \hat{x}_o, x_o \in [0, 1]^n, K(\phi_m(\hat{x}_o)) \leq \kappa_{m,n,m}, K(\phi_m(x_o)) \leq \kappa_{m,n,m}\}.
\]

Note that $|S| \leq 2^{2\kappa_{m,n,m}}$. For $t_1 > 0$, define the event $E_1$ as
\[
E_1^{(n)} \triangleq \{\forall h \in S : \|w^T Ah\|_2 \leq t_1 \|h\|_2\}.
\]
For any \( h \in \mathbb{R}^n \), \( Ah \) is a vector of length \( d \) with i.i.d. entries distributed as \( \mathcal{N}(0, \|h\|_2^2) \). Assuming that \( \|h\|_2 = 1 \) and applying Lemma 3, it follows that

\[
\begin{align*}
P \left( |w^T Ah| \geq t_1 \right) &= P \left( \|w\|_2 G \geq t_1, \|w\|_2 \geq \sqrt{d} \sigma (1 + t_2) \right) \\
&\quad + P \left( \|w\|_2 G \geq t_1, \|w\|_2 < \sqrt{d} \sigma (1 + t_2) \right) \\
&\leq P \left( \|w\|_2 \geq \sqrt{d} \sigma (1 + t_2) \right) \\
&\quad + P \left( G \geq t_1 (\sqrt{d} \sigma (1 + t_2))^{-1} \right) \\
&\leq e^{-dt_2^2/2} + e^{-\frac{t_1^2}{2d \sigma^2 (1 + t_2)}}. 
\end{align*}
\]

(41)

Hence, by the union bound and the fact that \( |S| \leq 2^{2k_m m} \), we obtain

\[
\begin{align*}
P(\mathcal{E}_1^{(n),c}) &\leq 2^{2k_m m} \left( e^{-dt_2^2/2} + e^{-\frac{t_1^2}{2d \sigma^2 (1 + t_2)}} \right). 
\end{align*}
\]

(42)

Note that

\[
\|A(\hat{q}_m - q_m)\|_2 \leq \sigma_{\text{max}}(A)\|\hat{q}_m - q_m\|_2.
\]

For \( t_3 > 0 \), define event \( \mathcal{E}_2 \) as

\[
\mathcal{E}_2^{(n)} \triangleq \left\{ \sigma_{\text{max}}(A) < (1 + t_3)\sqrt{d} + \sqrt{n} \right\}.
\]

It can be proved that [50]

\[
P \left( \mathcal{E}_2^{(n),c} \right) \leq e^{-dt_2^2/2}.
\]

(43)

But if \( \sigma_{\text{max}}(A) < (1 + t_3)\sqrt{d} + \sqrt{n} \), then from (40)

\[
\|A(\hat{q}_m - q_m)\|_2 \leq \left( 1 + (1 + t_3)\sqrt{\frac{d}{n}} \right) 2^{-m+1} n.
\]

For \( 0 < t_4 < 1 \), define the event \( \mathcal{E}_3^{(n)} \) as

\[
\mathcal{E}_3^{(n)} \triangleq \{ \forall h \in S : \|Ah\|_2^2 > (1 - t_4)d\|h\|_2^2 \},
\]

By the union bound and Lemma 2 (similar to the proof of (33) in the proof of Theorem 2), it follows that

\[
P \left( \mathcal{E}_3^{(n),c} \right) \leq 2^{2k_m m} e^{\frac{d}{2} (t_4 + \log(1 - t_4))}.
\]

(44)

For \( t_5 > 0 \), define the event \( \mathcal{E}_4^{(n)} \) as

\[
\mathcal{E}_4^{(n)} \triangleq \{ \forall h \in S : \|Ah\|_2^2 < (1 + t_5)d\|h\|_2^2 \}.
\]

(44)
Again by the union bound and Lemma 2, it follows that
\[
P \left( \mathcal{E}_4^{(n)} \right) \leq 2^{2\kappa_{m,n} m} e^{-\frac{t_5}{2} (t_5 - \log(1 - t_5))}. \tag{45}
\]

Finally, for \( t_6 > 0 \), define
\[
\mathcal{E}_5^{(n)} \triangleq \{ \| A^T w \|_2^2 \leq nd(1 + t_6) \}.
\]

Given \( w \), \( A^T w \) is an \( n \)-dimensional i.i.d. Gaussian random vector with variance \( \| w \|_2^2 \). Hence, by Lemma 2,
\[
P \left( \| A^T w \|_2^2 \geq n\gamma^2 (1 + t_7) \mid \| w \|_2^2 = \gamma^2 \right) \leq e^{-\frac{t_7}{2} (t_7 - \log(1 + t_7))}.
\]

On the other hand, again by Lemma 2,
\[
P(\| w \|_2^2 < d(1 - t_8)) \leq e^{\frac{t_8}{2} (t_8 + \log(1 - t_8))}.
\]

Choosing \( t_6, t_7, t_8 > 0 \) such that \( t_6 < t_7 \) and \( 1 + t_6 = (1 - t_8)(1 + t_7) \), it follows that
\[
P \left( \| A^T w \|_2^2 \geq nd(1 + t_6) \right)
= P \left( \| A^T w \|_2^2 \geq nd(1 + t_6), \| w \|_2^2 > d(1 - t_8) \right)
+ P \left( \| A^T w \|_2^2 \geq nd(1 + t_6), \| w \|_2^2 < d(1 - t_8) \right)
\leq e^{-\frac{t_7}{2} (t_7 - \log(1 + t_7))} + e^{\frac{t_8}{2} (t_8 + \log(1 - t_8))}. \tag{46}
\]

Combining (38) and (39) and the upper and lower bounds derived on the corresponding terms of (38) and (39) and choosing \( t_1 = 2\sigma \sqrt{d(1 + t_2) 2\kappa_{m,n} m} \), with probability \( P(\mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3 \cap \mathcal{E}_4 \cap \mathcal{E}_5) \), the following inequality holds:
\[
(1 - t_4)\sqrt{d}\| \Delta_m \|_2^2 \leq
2 \left( \sqrt{1 + t_5} 2^{-m+1} \sqrt{m} ((1 + t_3)\sqrt{d} + \sqrt{m}) \right) \| \Delta_m \|_2
+ 2 (\sigma \sqrt{1 + t_2} 2^{\kappa_{m,n} m} + c) \| \Delta_m \|_2
+ 2^{-m+1} \sqrt{1 + t_6} n. \tag{47}
\]

Inequality (47) involves a quadratic equation in \( \| \Delta_m \|_2 \). Finding the roots of this quadratic equation, and using \( \sqrt{1 + x} \leq 1 + x/2 \), we obtain
\[
\| \Delta_m \|_2 \leq \sigma \gamma_3 \sqrt{2\kappa_{m,n} m d^{-1}}
+ 2^{-m+1} (\gamma_1 \sqrt{n} + \gamma_2 n \sqrt{d^{-1}})
+ n 2^{-m} \sqrt{d^{-1} \gamma_4}, \tag{48}
\]

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where \(\gamma_1 = \sqrt{1 + t_5(1 + t_3)(1 - t_4)^{-1}}, \gamma_2 = \sqrt{1 + t_5(1 - t_4)^{-1}}, \gamma_3 = \sqrt{1 + t_2(1 - t_4)^{-1}}\) and \(\gamma_4 = \sqrt{1 + t_6(1 - t_4)^{-1}}\). Note that the terms \(\sigma \gamma_3 \sqrt{(2 \kappa_{m,n} m) n^{-1}}\) and \(2^{-m+1}(\gamma_1 \sqrt{n} + \gamma_2 n \sqrt{d^{-1}}) + n^{2-m} \sqrt{d^{-1}} \gamma_4\) are the variance and bias of the estimator given in (33), respectively. Suppose that \(d\) is fixed and \(\kappa_{m,n}\) is independent of \(m\). Then as \(m\) increases the bias of the estimator decreases since we are applying a very fine quantization to the signal, but the variance increases. This is the well-known bias-variance trade-off. By the union bound,

\[
\mathbb{P}

\left(\left(\mathcal{E}^{(n)}_1 \cap \mathcal{E}^{(n)}_2 \cap \mathcal{E}^{(n)}_3 \cap \mathcal{E}^{(n)}_4 \cap \mathcal{E}^{(n)}_5\right)^c\right)

\leq \mathbb{P}(\mathcal{E}^{(n)}_1 \cup \mathcal{E}^{(n)}_2 \cup \mathcal{E}^{(n)}_3 \cup \mathcal{E}^{(n)}_4 \cup \mathcal{E}^{(n)}_5).
\]

(49)

Setting \(m = \lceil \log n \rceil\) and \(d = 8 \kappa_{m,n} m\) as proposed in Theorem 3, it is straightforward to confirm that as \(n\) grows to infinity, for all \(i \in \{1, 2, 3, 4, 5\}\),

\[
\mathbb{P}(\mathcal{E}^{(n)}_i) \to 0.
\]

Furthermore, except for \(\sigma \gamma_3 \sqrt{(2 \kappa_{m,n} m) n^{-1}}\), as \(n\) grows to infinity, all the terms in (48) also converge to zero. This completes the proof of Theorem 3.

\[\square\]

D. Proof of Theorem 5

Since the proof of this theorem is similar to the proof of Theorem 2, we skip most of the steps and only emphasize the main differences. Let \(\hat{x}_o\) denote the solution of (9). Define \(q_m, \tilde{q}_m,\) and \(\hat{q}_m\) as the quantization error vectors of \(x_o, \tilde{x}_o,\) and \(\hat{x}_o\), respectively, i.e., \(q_m \triangleq x_o - \phi_m(x_o), \tilde{q}_m \triangleq \tilde{x}_o - \phi_m(\tilde{x}_o),\) and \(\hat{q}_m \triangleq \hat{x}_o - \phi_m(\hat{x}_o).\) Since \(\|A\tilde{x}_o - y_o\|_2 = \|A(\hat{x}_o - x_o)\|_2 \leq \sigma_{max}(A) \epsilon_n\) and \(\hat{x}_o\) is the minimizer of (9), it follows that \(\|A\hat{x}_o - y_o\| \leq \sigma_{max}(A) \epsilon_n.\) Therefore,

\[
\|A\hat{x}_o - A\hat{x}_o\|_2 = \|A\hat{x}_o - y_o - (A\hat{x}_o - y_o)\|_2 \\
\leq 2 \sigma_{max}(A) \epsilon_n.
\]

(50)

Again, by the triangle inequality,

\[
\|A\tilde{x}_o - A\hat{x}_o\|_2 \\
= \|A(\phi_m(\tilde{x}_o) + \tilde{q}_m) - A(\phi_m(\hat{x}_o) + \hat{q}_m)\|_2 \\
\geq \|A(\phi_m(\tilde{x}_o) - \phi_m(\hat{x}_o))\|_2 - \|A(\tilde{q}_m - \hat{q}_m)\|_2 \\
\geq \|A(\phi_m(\tilde{x}_o) - \phi_m(\hat{x}_o))\|_2 - \sigma_{max}(A)\|\tilde{q}_m - \hat{q}_m\|_2 \\
\geq \|A(\phi_m(\tilde{x}_o) - \phi_m(\hat{x}_o))\|_2 - \sigma_{max}(A) \sqrt{n^2 - 2m + 1}.
\]

(51)
Combining (50) and (51), it follows that
\[ \|A(\phi_m(\hat{x}_o) - \phi_m(\bar{x}_o))\|_2 \leq \sigma_{\text{max}}(A) \sqrt{n2^{-2m+1}} + 2\sigma_{\text{max}}(A)\epsilon_n. \] (52)

We also have: \( K[::m](\hat{x}_o) \leq m\kappa_{m,n} \) and \( K[::m](\bar{x}_o) \leq m\kappa_{m,n} \).

Define the events \( E_1^{(n)} \) and \( E_2^{(n)} \) as done in (28) and (29) in the proof of Theorem 2. Then, applying the argument used there, it follows that
\[
P(\|x_o - \hat{x}_o\|_2 > \epsilon) \leq P\left(\|x_o - \bar{x}_o\|_2 > \epsilon, E_1^{(n)} \cap E_2^{(n)}\right) + P\left(E_1^{(n)}\right) + P\left(E_2^{(n)}\right).
\] (53)

The rest of the proof is exactly the same as that for Theorem 2. \(\square\)

**E. Proof of Theorem 6**

Let \( \hat{x}_o \) be the solution of the MCP algorithm and \( q_m \triangleq x_o - \phi_m(x_o) \) and \( \bar{q}_m \triangleq \hat{x}_o - \phi_m(\hat{x}_o) \) denote the quantization errors of the original and the reconstructed signals at resolution \( m \), respectively. Following exactly the same steps as the proof of Theorem 2, we obtain
\[
K[::m](\hat{x}_o) \leq K[::m](x_o) \leq \kappa_{m,n} m
\] (54)

and
\[
\|A(\phi_m(x_o) - \phi_m(\bar{x}_o))\| = \sigma_{\text{max}}(A) \sqrt{n2^{-2m+2}}.
\] (55)

Since we are dealing with subgaussian random matrices, we define slightly different events here. Let the set \( S_o \) as
\[
S_o \triangleq \{ h : h = \phi_m(\hat{x}_o) - \phi_m(x_o), \hat{x}_o, x_o \in [0, 1]^n, K(\phi_m(\hat{x}_o)) \leq \kappa_{m,n} m, K(\phi_m(x_o)) \leq \kappa_{m,n} m \},
\]
and define
\[
E_1^{(n)} \triangleq \{ h \in S_o : \|A(h)\|_2 < \tau\sqrt{d}\|h\|_2 \},
\] (56)
\[
E_2^{(n)} \triangleq \left\{ \sigma_{\text{max}}(A) < \sqrt{d} + (\epsilon'_2 + 1)\sqrt{n} \right\},
\] (57)

where \( \epsilon'_2 \) is the constant introduced in Lemma 8. \( P(\|x_o - \hat{x}_o\| > \epsilon) \) can be upper bounded by
\[
P(\|x_o - \hat{x}_o\| > \epsilon) \leq P(\|x_o - \bar{x}_o\| > \epsilon, E_1^{(n)} \cap E_2^{(n)}) + P(E_1^{(n)}) + P(E_2^{(n)}).
\]

If \( A \in E_1^{(n)} \cap E_2^{(n)} \), then similar to (31) we can prove
\[
\|x_o - \hat{x}_o\|_2 \leq \left(\tau^{-1} \left(\sqrt{(\epsilon'_2 + 1)n^{-1}d^{-1} + 1}\right) + 1\right) \sqrt{n2^{-2m+2}}.
\]
Hence,
\[ P(\|x_0 - \hat{x}_0\|_2 > \epsilon, \mathcal{E}_1^{(n)} \cap \mathcal{E}_2^{(n)}) = 0. \]

Also, according to Lemma 8, \( P(\mathcal{E}^{(n)}_2) \leq e^{-c_1 n} \). Therefore, the main difference is in the calculation of \( P(\mathcal{E}^{(n)}_1) \):

\[
P(\|Ax\|_2 \leq \tau \sqrt{d}\|x\|_2) = P\left(\left\|A \frac{x}{\|x\|_2}\right\|_2 \leq \tau^2 d\right)
= P\left(\sum_{i=1}^{d} Z_i^2 \leq \tau^2 d\right),
\]

where for \( i = 1, 2, \ldots, d \), \( Z_i = \|x\|_2^{-1} \sum_j A_{ij} x_j \). Therefore, by Lemma 4 we obtain

\[
P(|Z_i| > t) \leq c_1 e^{-c_2 t}.
\]

According to Lemma 7 we have

\[
P\left(\sum_{i=1}^{d} Z_i^2 \leq \tau^2 d\right) \leq e^{-\frac{c_3 (\tau^2 - 1)^2}{16 c_2^2}},
\]

where \( c_3 \triangleq \max(c_1 e^{-c_2}, e^{c_2}) \) and \( 1 - \tau^2 < c_3 / c_2 \). Finally, the union bound proves that

\[
P(\mathcal{E}_1^{(n)}) \leq 2^{c_3} m e^{-\frac{c_3 (\tau^2 - 1)^2}{16 c_2^2}},
\]

which completes the proof.

\[ \square \]

VII. Conclusions

In this paper, we have considered the problem of recovering structured signals from their linear measurements. We have used the Komogorov complexity of the quantized signal as a universal measure of complexity to both cover many of the examples explored in the CS literature and also provide a framework to analyze future structured signal models. We have shown that, if we consider low-complexity signals, then the minimum complexity pursuit (MCP) scheme inspired by Occam’s razor recovers the simplest solution of a set of random linear measurements. In fact, we have proved that the number of measurements required is proportional to the complexity and logarithmically to the ambient dimension of the signal. We have also considered more practical scenarios where the signal is not exactly low complexity but rather is “close” to a low complexity signal. We have shown that, even in such cases, the MCP algorithm provides a good estimate of the signal from much fewer samples than the ambient dimension of the signal.

As mentioned above, Kolmogorov complexity of a sequence is not computable. However, currently we are working on deriving implementable schemes by replacing the Kolmogorov complexity by computable measures such as minimum description length [51].
A. Review of Kolmogorov Complexity

1) Universal machine: In an effort to formalize the concept of computability of functions, Turing introduced the notion of Turing machine. A Turing machine is a device that has a finite number of states and a memory that is in the form of a tape. The tape consists of small blocks, each of which can store one of the three symbols \( \mathcal{I} = \{0, 1, B\} \) where \( B \) represents a blank. The Turing machine also has a head that at each point in time points to one of the blocks on the tape. The machine works in discrete time steps. At every time instance, it reads one symbol from the tape (at the location the head is pointing to), and based on its current state and the acquired information from the tape, it performs the following actions:

1) Moves either to a new state or remains at the current state.
2) Writes one symbol from \( \mathcal{I} \) onto the tape at the location of the head.
3) Moves the head one block either left or right.

The process continues until the machine enters the halting state. The output of a Turing machine \( M \) given a finite bit string \( s \) is defined as follows. Write the string \( s \) on the input tape of \( M \) and assume that except for these bits all other blocks are occupied by symbol \( B \). Set the head of machine to the left-most block of \( s \) and run the machine \( M \) until it halts. If the machine does not halt on \( s \), then \( M(s) \) is not defined. If \( M \) halts, then the tape contains a binary string that is surrounded by blanks. This binary string is defined as \( M(s) \), i.e., the output of \( M \) given \( s \). If the output string contains blanks between the binary symbols, then they are replaced by zeros to make the output a binary sequence.

A partial function \( \phi : \{0, 1\}^* \to \{0, 1\}^* \cup \{0, 1\}^\infty \) is called computable if there exists a Turing machine \( M \) such that for every \( s \in \{0, 1\}^* \) that \( M(s) \) is defined, \( \phi(s) = M(s) \). According to Church-Turing thesis (conjecture), if a function is computable in any intuitive sense, then it is computable in the above sense [52]. This thesis simplifies the general problem of computability to the computability on the Turing machine.

One of the most fundamental results in the algorithmic information theory is the existence of a universal Turing machine. A universal Turing machine \( U \) is a machine that is able to imitate the behavior of any other Turing machine \( M \) on any input string. The existence of Universal Turing machines is a result of the fact that any Turing machine can be uniquely specified with a finite number of bits. We call these bits the simulator program and for a Turing Machine (\( M \)) denote this bit stream by \( P_M \). See Chapter 1 of [11], [53] for more information on the universal Turing Machines.

2) Kolmogorov complexity: There exist several notions of Kolmogorov complexity. In this paper, we use the one based on prefix recursive functions.

Let \( \phi \) be a partial recursive (partial computable) function. Define the complexity of string \( s \) for function \( \phi \) as

\[
\bar{K}_\phi(s) \triangleq \min_{p \in \{0, 1\}^* : \phi(p) = s} \ell(p),
\]

where \( \ell(p) \) denotes the length of the binary string \( p \). As mentioned earlier, each computable function \( \phi \) corresponds...
to a Turing machine. Therefore, \( p = \phi^{-1}(s) \) provides a uniquely decodable code for \( s \).

The notion of complexity defined in (58) depends on machine \( \phi \). However, the invariance theorem unifies all these definitions.

**Theorem 7:** There exists a partial recursive function \( U \) such that for any given partial recursive function \( \phi \), we have

\[
\tilde{K}_U(s) \leq \tilde{K}_\phi(s) + c_\phi,
\]

for all \( s \in \{0,1\}^* \).

One of the main issues with this definition of Kolmogorov complexity is that the programs that generate different outputs may be prefixes of one another. For instance, suppose that our goal is to print two strings \( s_1 \) and \( s_2 \) with this machine. Let \( p_1 \) and \( p_2 \) denote the programs that generate \( s_1 \) and \( s_2 \), respectively. We expect that the following strategy results in a code that generates \( s_1 s_2 \): i) Use a constant number of bits (independent of \( s_1 \) or \( s_2 \)) to describe the goal of generating the concatenation of two strings. ii) Provide \( p_1 \) and \( p_2 \). However, this program does not print \((s_1, s_2)\). In fact, since either \( p_1 \) or \( p_2 \) could be the prefix of the other one, the machine cannot decide where to stop and print the first number. Therefore, the machine also requires the length of the first program to be able to determine where to stop. This means that \( \tilde{K}_U(s_1, s_2) \leq \tilde{K}_U(s_1) + \tilde{K}_U(s_2) + \log^*(\tilde{K}_U(s_1)) + c \). This \( \log^* \) factor will appear in many other places, such as calculation of mutual information. A simple remedy for this problem is to define prefix Kolmogorov complexity described in the next section.

3) **Prefix Kolmogorov complexity:**

**Definition 4:** A partial recursive function \( \phi : \{0,1\}^* \rightarrow \{0,1\}^* \cup \{0,1\}^\infty \) is called a partial recursive prefix function if and only if \( \phi(p) < \infty \) and \( \phi(q) < \infty \) imply that neither \( p \) nor \( q \) is a prefix of the other.

Let \( \phi \) be a partial recursive prefix function. Define the complexity of string \( s \) for function \( \phi \) as

\[
K_\phi(s) \triangleq \min_{p: s = \phi(p)} \ell(p).
\]

As before, an invariance theorem holds for \( K_\phi(s) \) [11] (Section 3.1). See [11] (Chapter 3) for several other appealing properties of the prefix Kolmogorov complexity.

**B. Proof of Theorem 7**

i. The following program prints \( x \): Print the following bit sequence \( x_1, x_2, \ldots, x_\ell(x) \). The first part that explains the structure has a constant length, \( c \), and then the bits themselves require \( \ell(x) \) bits. Therefore, the length of the program is less than \( \ell(x) + c \).

ii. Let \( p_x \) and \( p_y \) denote the shortest programs that print \( x \) and \( y \) respectively. The following program prints \( (x,y) \): Print a concatenation of two numbers and the programs for these two numbers are \( p_x \) and \( p_y \). Note that since the programs are assumed to be prefix free, after the explanation, “Print a concatenation of two numbers”, the machines continues until it goes into the halting state. At this point it has already printed \( x \). But since it knows that we expect another number, it again starts to read the bits and therefore will print
y as well.

iii. The proof of this part is also straightforward since we can ignore our information on y and code x as if we do not know y.

iv. We use the same program that we used in Part 1. Notice that since the machine does not know $\ell(x)$ we should spend $K(\ell(x))$ bits to describe this number as well. Hence, overall we require $K(x) + K(\ell(x)) + c$ bits.

v. First note that the length of the binary representation of $n$ which is denoted by $\ell(n)$ is $\log n$. According to Part iv we have

$$K(n) \leq K(n | \ell(n)) + K(\ell(n)) + c \leq \ell(n) + 2 \max(\log(\log n), 1) + c'$$

(59)

vi. The proof is very similar to the proof of Part ii, and hence we skip it.

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