Selection Principles and Games in Bitopological Function Spaces

Daniil Lyakhovets, Alexander V. Osipov

Abstract. For a Tychonoff space $X$, we denote by $(C(X), \tau_k, \tau_p)$ the bitopological space of all real-valued continuous functions on $X$, where $\tau_k$ is the compact-open topology and $\tau_p$ is the topology of pointwise convergence. In the papers [6, 7, 13] variations of selective separability and tightness in $(C(X), \tau_k, \tau_p)$ were investigated. In this paper we continue to study the selective properties and the corresponding topological games in the space $(C(X), \tau_k, \tau_p)$.

1. Introduction

In the papers [1, 3, 4, 9–12, 14] the authors investigated the selectors of dense subsets of the space $C(X)$ of all real-valued continuous functions on a Tychonoff space $X$ with the topology $\tau_p$ of pointwise convergence and with the compact-open topology $\tau_k$. For a Tychonoff space $X$, we denote by $(C(X), \tau_k, \tau_p)$ the bitopological space. In the articles [6, 7, 13] variations of selective separability and tightness in $(C(X), \tau_k, \tau_p)$ were investigated. In this paper, we continue to study the selective properties and the corresponding topological games in the space $(C(X), \tau_k, \tau_p)$. The following selection properties for $(C(X), \tau_k, \tau_p)$ are considered.

$$S_1(D^k, S^p) = S_{\text{fin}}(D^k, S^p) \Rightarrow S_1(D^k, D^p) \Rightarrow S_{\text{fin}}(D^k, D^p)$$

For example, a space $(C(X), \tau_k, \tau_p)$ satisfies $S_1(D^k, S^p)$ (resp., $S_{\text{fin}}(D^k, S^p)$) if whenever $(D_n : n \in \mathbb{N})$ is a sequence of dense subsets of $C(X)$, one can take points $f_n \in D_n$ (resp., finite $F_n \subset D_n$) such that $\{f_n : n \in \mathbb{N}\}$ (resp., $\bigcup F_n : n \in \mathbb{N}\}$) is sequentially dense in $C_p(X)$. There is a topological game, denoted by $G(A,B)$, corresponding to $S_1(A,B)$.

In this paper, we give characterizations for the bitopological space $(C(X), \tau_k, \tau_p)$ to satisfy the selection properties and the corresponding games.

2010 Mathematics Subject Classification. Primary 54C35; Secondary 54A20, 54E55, 91A05, 91A44

Keywords. Selection principles, compact-open topology, function space, bitopological space, topological games, separable space

Received: 19 March 2019; Revised: 01 June 2019; Accepted: 04 June 2019

Communicated by Ljubiša D.R. Kočinac

The work of the second author on this paper was supported by the Program for State Aid of Leading RF Universities (Agreement No. 02.A03.21.0006 between the Ministry of Education and Science of the Russian Federation and the Ural Federal University, 27.08.2013)

Email addresses: zoy01111@gmail.com (Daniil Lyakhovets), OAB01@gmail.com (Alexander V. Osipov)
2. Main Definitions and Notation

Let $\mathcal{A}$ and $\mathcal{B}$ be sets consisting of families of subsets of an infinite set $X$. Then many topological properties are characterized in terms of the following classical selection principles:

$S_1(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(b_n : n \in \mathbb{N})$ such that for each $n$, $b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

$S_{fin}(\mathcal{A}, \mathcal{B})$ is the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ of finite sets such that for each $n$, $B_n \subseteq A_n$, and $\bigcup_{n \in \mathbb{N}} B_n \in \mathcal{B}$.

The following prototype of many classical properties is called "$\mathcal{A}$ choose $\mathcal{B}$" in [15].

$(\mathcal{A}, \mathcal{B})$: For each $\mathcal{U} \in \mathcal{A}$ there exists $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathcal{V} \in \mathcal{B}$. In this paper we accept that $|V| = \aleph_0$.

Then $S_{fin}(\mathcal{A}, \mathcal{B})$ implies $(\mathcal{A}, \mathcal{B})$.

In this paper, by a cover we mean a nontrivial one, that is, $\mathcal{U}$ is a cover of $X$ if $X = \bigcup \mathcal{U}$ and $X \notin \mathcal{U}$.

An open cover $\mathcal{U}$ of a space $X$ is called:

- an $\omega$-cover (a $k$-cover) if each finite (compact) subset $C$ of $X$ is contained in an element of $\mathcal{U}$.
- a $\gamma$-cover (a $\gamma_k$-cover) if $\mathcal{U}$ is infinite and for each finite (compact) subset $C$ of $X$ the set $\{U \in \mathcal{U} : C \notin U\}$ is finite.

For a topological space $X$ we denote:

- $\Phi$ — the family of all open covers of $X$;
- $\Phi_k$ — the family of all open $\gamma$-covers of $X$;
- $\Phi_k$ — the family of all open $\gamma_k$-covers of $X$;
- $\Phi_{\omega}$ — the family of all open $\omega$-covers of $X$;
- $\Theta$ — the family of all open $\omega$-covers of $X$;
- $\Theta_k$ — the family of all open $\gamma$-covers of $X$;
- $\Theta_k$ — the family of all open $\gamma_k$-covers of $X$;
- $\Theta_{\omega}$ — the family of all open $\omega$-covers of $X$;
- $\Theta_{\omega}^k$ — the family of all sequentially dense subsets of $C_k(X)$;
- $\Theta_{\omega}^\omega$ — the family of all non-empty compact subsets of $X$;
- $\Phi(X)$ — the family of all non-empty finite subsets of $X$.

A space $X$ is said to be a $\gamma_k$-set if each $k$-cover $\mathcal{U}$ of $X$ contains a countable set $\{U_n : n \in \mathbb{N}\}$ which is a $\gamma_k$-cover of $X$ [5].

If $X$ is a space and $A \subseteq X$, then the sequential closure of $A$, denoted by $[A]_{seq}$, is the set of all limits of sequences from $A$. A set $D \subseteq X$ is said to be sequentially dense if $X = [D]_{seq}$. A space $X$ is called sequentially separable if it has a countable sequentially dense set. Clearly, every sequentially separable space is separable.

Let $X$ be a topological space, and $x \in X$. A subset $A$ of $X$ converges to $x$, $x = \lim A$, if $A$ is infinite, $x \notin A$, and for each neighborhood $U$ of $x$, $X \setminus U$ is finite. Consider the following collections:

- $\Omega_k = \{A \subseteq X : x \in \overline{A} \setminus A\}$;
- $\Omega_{\omega} = \{A \subseteq X : x = \lim A\}$.

Note that if $A \in \Gamma_k$, then there exists $\{a_n\} \subset A$ converging to $x$. So, simply $\Gamma_k$ may be the set of non-trivial convergent sequences to $x$.

We write $\Pi(\mathcal{A}, \mathcal{B})$ without specifying $x$, we mean $(\forall x)\Pi(\mathcal{A}_x, \mathcal{B}_x)$.

So we have three types of topological properties of $(C(X), \tau_k, \tau_p)$ described through the selection principles of $X$ where the index $k$ means the compact-open topology and the index $p$ - the topology of pointwise convergence:

- local properties of the form $S_\omega(\Phi_k, \Psi_k^\omega)$;
- global properties of the form $S_\omega(\Phi_k, \Psi_k)$;
- semi-local properties of the form $S_\omega(\Phi_k, \Psi_k)$.

There is a game, denoted by $G_{fin}(\mathcal{A}, \mathcal{B})$, corresponding to $S_{fin}(\mathcal{A}, \mathcal{B})$; two players, ONE and TWO, play a round for each natural number $n$. In the $n$-th round ONE chooses a set $A_n \in \mathcal{A}$ and TWO responds with a finite subset $B_n$ of $A_n$. A play $A_1, B_1; \ldots; A_n, B_n; \ldots$ is won by TWO if $\bigcup_{n \in \mathbb{N}} B_n \in \mathcal{B}$; otherwise, ONE wins.
A strategy of a player is a function \( σ \) from the set of all finite sequences of moves of the opponent into the set of (legal) moves of the strategy owner.

If \( \text{ONE} \) does not have a winning strategy in the game \( G_*(\mathcal{A}, \mathcal{B}) \), then the selection hypothesis \( S_*(\mathcal{A}, \mathcal{B}) \) is true; it is easy to prove. The converse implication is not always true.

Similarly, one defines the game \( G_1(\mathcal{A}, \mathcal{B}) \), associated with \( S_1(\mathcal{A}, \mathcal{B}) \).

So we have three types of topological games on \((C(X), τ_1, τ_p)\) described through the selection principles (or topological games) of \( X \):

- local games of the form \( G_2(\Phi^1, Ψ^1) \);
- global games of the form \( G_2(Φ^1, Ψ^p) \);
- semi-local games of the form \( G_2(Φ^1, Ψ^p) \).

The symbol \( 0 \) denotes the constantly zero function in \( C(X) \). Since the compact-open topology coincides with the topology of uniform convergence on compact subsets of \( X \), we can represent a basic neighborhood of the point \( f \in C(X) \) as \( \langle f, A, ε \rangle \) where \( \langle f, A, ε \rangle := \{ g \in C(X) : |f(x) − g(x)| < ε \ \forall x \in A \} \), \( A \) is a compact subset of \( X \) and \( ε > 0 \).

3. \( S_1(\mathcal{D}^k, \mathcal{D}^p) \) and \( G_1(\mathcal{D}^k, \mathcal{D}^p) \)

**Theorem 3.1.** (\cite[Theorem 3.7]{noble}) For a space \( X \) the following are equivalent:

1. \((C(X), τ_1, τ_p)\) has the property \( S_1(Ω_0, Ω_0^1) \);
2. \( X \) has the property \( S_1(\mathcal{K}, \mathcal{Ω}) \).

Recall that the \( i \)-weight \( iv(X) \) of a space \( X \) is the smallest infinite cardinal number \( τ \) such that \( X \) can be mapped by a one-to-one continuous mapping onto a Tychonoff space of the weight not greater than \( τ \).

Note that a space \( X \) has a coarser second countable topology iff \( iv(X) = N_0 \).

**Theorem 3.2.** (Noble \cite{Noble}) A space \( C_λ(X) \) is separable if and only if \( iv(X) = N_0 \).

Recall that a subset \( A \) of a bitopological space \( (X, τ_1, τ_2) \) is bidense (double dense or short \( d \)-dense) in \( X \) if \( A \) is dense in both \((X, τ_1)\) and \((X, τ_2)\) (\cite{Lyakhovets2}). \((X, τ_1, τ_2)\) is \( d \)-separable if there is a countable set \( A \) which is \( d \)-dense in \( X \). Note that if \( iv(X) = N_0 \), then \((C(X), τ_1, τ_2)\) is \( d \)-separable.

**Theorem 3.3.** Let \( X \) be a space with a coarser second countable topology. The following assertions are equivalent:

1. \((C(X), τ_1, τ_p)\) has the property \( S_1(\mathcal{D}^k, \mathcal{D}^p) \);
2. \((C(X), τ_1, τ_p)\) has the property \( S_1(\mathcal{D}^k, \mathcal{Ω}_0^1) \);
3. \((C(X), τ_1, τ_p)\) has the property \( S_1(Ω_0^1, Ω_0^1) \);
4. \( X \) has the property \( S_1(\mathcal{K}, \mathcal{Ω}) \);
5. \((C(X), τ_1, τ_p)\) has the property \( S_1(\mathcal{D}^k, \mathcal{Ω}_0^1) \);
6. \( \text{ONE} \) has no winning strategy in the game \( G_1(\mathcal{K}, \mathcal{Ω}) \);
7. \( \text{ONE} \) has no winning strategy in the game \( G_1(\mathcal{D}^k, \mathcal{D}^p) \);
8. \( \text{ONE} \) has no winning strategy in the game \( G_1(Ω_0^1, Ω_0^1) \);
9. \( \text{ONE} \) has no winning strategy in the game \( G_1(\mathcal{D}^k, Ω_0^1) \).

**Proof.** (1) \( \Rightarrow \) (4). Let \((U^k_i : i \in N)\) be a sequence of \( k \)-covers of \( X \) and let \( D = \{ f_s : s \in N \} \) be a countable dense set in \( C_λ(X) \). Consider \( P_i := \{ h^i_{L,W,f} : h^i_{L,W,f} \in C(X) : h^i_{L,W,f} \uparrow L = f_s \uparrow L, L \in Κ(X), L \subset W, W \in U^k_i, h^i_{L,W,f} \uparrow (X \setminus W) = 1, f_s \in D \} \). Note \( P_i \) is a dense subset of \( C_λ(X) \) for each \( i \in N \). Indeed fix \( f \in C(X), K \in Κ(X), ε > 0 \). For \( (f, K, ε) \) there exists \( W_k \in U^k_i \) and \( f_s \in D \) such that \( K \subset W_k \) and \( f_s \in (f, K, ε) \). Take \( h^i_{L,W,f} \in (f, K, ε) \).

Since \((P_i : i \in N)\) is a countable set of dense sets of \( C_λ(X) \), by (1), there exists \( \{ p_i : i \in N \} \) such that \( p_i \in P_i \) and \( \{ p_i : i \in N \} \) is a dense subset of \( C_λ(X) \). For \( p_t = h^i_{L,W,f} : i \in N \), we have that \( \{ W_i : i \in N \} \) is an \( ω \)-cover of \( X \). Indeed, let \( M = \{ x_1, x_2, ..., x_k \} \in F(X) \). Consider \( U = (0, M, (−\frac{1}{2}, \frac{1}{2})) \), then there exists \( i' \) such that \( p_{i'} \in U \). It follows that \( M \subset W_{i'} \).
(3) $\Rightarrow$ (2) is immediate.
(4) $\Rightarrow$ (3). By Theorem 3.1.
(2) $\Rightarrow$ (1). Let $\{D_{ij} : i, j \in \mathbb{N}\}$ be a countable set of dense sets in $C_p(X)$. Let $D = \{d_i : i \in \mathbb{N}\}$ be a countable dense set in $C_p(X)$. By $S_1(D^k, \Omega^p_0)$ there exists $\{d_{ij} : j \in \mathbb{N}\}$ such that $\{d_{ij} : j \in \mathbb{N}\} \in D^p_{ij}$. Consider $M = \{d_{ij} : i, j \in \mathbb{N}\}$. The set $M$ is dense in $C_p(X)$. Fix $f \in C(X)$. Let $L = \{x_1, x_2, ..., x_n\} \in F(X)$ and $\epsilon > 0$. The set $\langle f, L, \epsilon \rangle$ is a neighborhood of $f$, then there is $d_r \in D$ such that $d_r \in \langle f, L, \epsilon \rangle$, hence $M \in D^p$.
(6) $\Rightarrow$ (4) is immediate.
(4) $\Rightarrow$ (6). Let $\sigma$ be a strategy for ONE in $G_1(K, \Omega)$ and let the first move of ONE be a $k$-cover $\sigma(\emptyset) = \{U_{(a^1)} : a^1 \in \Lambda^1\}$. Suppose that for each finite sequence $s$ of numbers $a^i \in \Lambda^i$ of length at most $m$, $U_s$ has been already defined. Then define $\{U_{(a^1, a^2, ..., a^m)} : a^i \in \Lambda_i\}$ to be the set $\sigma(U_{(a^1)}, U_{(a^2)}, ..., U_{(a^m)}) \setminus \{U_{(a^1)}, U_{(a^2)}, ..., U_{(a^m)}\}$. Because each compact subset of $X$ belongs to infinitely many elements of a $k$-cover, we have that, for each $s$, a finite sequence of numbers $a^i \in \Lambda^i$, the set $\{U_{(a^1, a^2, ..., a^m)} : a^i \in \Lambda^i\}$ is a $k$-cover. Apply (4) and, for each $s$, choose $a^i \in \Lambda^i$ such that $\{U_{(a^1, a^2, ..., a^m)} : s$ a finite sequence of numbers $a^i \in \Lambda^i, i \in \mathbb{N}\}$ is a $\omega$-cover of $X$. Then inductively define a sequence $a^1 = a^1, a^{k+1} = a^{a^1, ..., a^k}$ for $k \geq 1$. Then $U_{(a^1, a^2, ..., a^m, a^{a^1, ..., a^k}, ..., a^{a^1, ..., a^k})}$ is a $\omega$-cover, and because it is, in fact, a sequence of moves TWO in a play of game $G_2(K, \Omega)$, $\sigma$ is not a winning strategy for ONE.
Similarly to (4) $\Leftrightarrow$ (6) we have that (1) $\Leftrightarrow$ (7), (2) $\Leftrightarrow$ (9) and (3) $\Leftrightarrow$ (8).

4. $S_{fin}(D^k, D^p)$ and $G_{fin}(D^k, D^p)$

Theorem 4.1. ([13, Theorem 3.9] for $\lambda = k$ and $\mu = p$) For a space $X$ the following are equivalent:
1. $(C(X), \tau_k, \tau_p)$ has the property $S_{fin}(\Omega^k, \Omega^p)$;
2. $X$ has the property $S_{fin}(K, \Omega)$.

Theorem 4.2. Let $X$ be a space with a coarser second countable topology. The following assertions are equivalent:
1. $(C(X), \tau_k, \tau_p)$ has the property $S_{fin}(D^k, D^p)$;
2. $(C(X), \tau_k, \tau_p)$ has the property $S_{fin}(D^k, \Omega^p_0)$;
3. $(C(X), \tau_k, \tau_p)$ has the property $S_{fin}(\Omega^k_0, \Omega^p_0)$;
4. $X$ satisfies the selection principle $S_{fin}(K, \Omega)$;
5. ONE has no winning strategy in the game $G_{fin}(K, \Omega)$;
6. ONE has no winning strategy in the game $G_{fin}(D^k, D^p)$;
7. ONE has no winning strategy in the game $G_{fin}(\Omega^k_0, \Omega^p_0)$;
8. ONE has no winning strategy in the game $G_{fin}(D^k, \Omega^p_0)$.

Proof. The implications are proved similarly to the proof of Theorem 3.3.

5. $S_{i}(D^k, S^p)$ and $G_{i}(D^k, S^p)$

Theorem 5.1. ([3, Theorem 15]) For a space $X$ the following are equivalent:
1. $(C(X), \tau_k, \tau_p)$ has the property $S_1(\Omega^k_0, \Gamma^p_0)$;
2. $X$ has the property $S_1(K, \Gamma)$.

Theorem 5.2. ([4, Theorem 10]) For a space $X$ the following are equivalent:
1. $X$ has the property $S_{fin}(K, \Gamma)$;
2. $X$ has the property $S_1(K, \Gamma)$;
3. ONE has no winning strategy in the game $G_1(K, \Gamma)$. 
Theorem 5.3. Let $X$ be a space with a coarser second countable topology. The following assertions are equivalent:

1. $(C(X), \tau_k, \tau_p)$ has the property $S_1(D^k, S^p)$;
2. $(C(X), \tau_k, \tau_p)$ has the property $(\mathcal{D}^p)$;
3. $X$ has the property $S_1(K, \Gamma)$;
4. $(C(X), \tau_k, \tau_p)$ has the property $S_{fn}(D^k, S^p)$;
5. $X$ has the property $S_{fn}(K, \Gamma)$;
6. Each finite power of $X$ has the property $S_1(K, \Gamma)$;
7. $(C(X), \tau_k, \tau_p)$ has the property $S_1(\Omega^p_0, \Gamma^p_0)$;
8. $(C(X), \tau_k, \tau_p)$ has the property $S_1(D^k, \Gamma^p_0)$;
9. $X$ has the property $(\mathcal{D}^p)$;
10. ONE has no winning strategy in the game $G_1(K, \Gamma)$;
11. ONE has no winning strategy in the game $G_1(D^k, S^p)$;
12. ONE has no winning strategy in the game $G_1(\Omega^p_0, \Gamma^p_0)$;
13. ONE has no winning strategy in the game $G_1(D^k, \Gamma^p_0)$.

Proof. By Theorem 5.1 ([3, Theorem 15]), $(3) \iff (7)$.

By Theorem 5.2 (Theorem 10 in [4]), $(3) \iff (5) \iff (10)$.

By Theorem 14 in [3], $(3) \iff (9)$.

$(3) \iff (6)$ (Proposition 13 and Theorem 10 in [4]).

$(1) \Rightarrow (4)$ is immediate.

$(7) \Rightarrow (8)$ is immediate.

Similarly to $(3) \Rightarrow (10)$ (the implication $(2) \Rightarrow (3)$ in Theorem 10 in [4]) we have that $(1) \Rightarrow (11), (7) \Rightarrow (12)$ and $(8) \Rightarrow (13)$.

$(4) \Rightarrow (2)$. Let $D$ be a dense subset of $C_0(X)$. By the property $S_{fn}(D^k, S^p)$, for sequence $(D_i : D_i = D$ and $i \in \mathbb{N})$ there is a sequence $(K_i : i \in \mathbb{N})$ such that for each $i$, $K_i$ is finite, $K_i \subset D_i$, and $\bigcup_{i \in \mathbb{N}} K_i$ is a countable sequentially dense subset of $C_0(X)$.

$(2) \Rightarrow (9)$. Let $U$ be an open $k$-cover of $X$. Note that the set $D := \{f \in C(X) : f \upharpoonright (X \setminus U) \equiv 1$ for some $U \in \mathcal{U}\}$ is dense in $C_0(X)$ and, hence, $D$ contains a countable sequentially dense set $A$ in $C_0(X)$. Take $\{f_n : n \in \mathbb{N}\} \subset A$ such that $f_n \rightarrow 0$ ($n \rightarrow \infty$) in $C_0(X)$. Let $f_i \upharpoonright (X \setminus U_i) \equiv 1$ for some $U_i \in \mathcal{U}$. Then $\{U_n : n \in \mathbb{N}\}$ is a $\gamma$-subcover of $\mathcal{U}$, because of $f_n \rightarrow 0$. Hence, $X$ satisfies $(\mathcal{D}^p)$.

$(3) \Rightarrow (1)$. Let $(D_{i,j} : i, j \in \mathbb{N})$ be a sequence of dense subsets of $C_0(X)$ and let $D = \{f_i : i \in \mathbb{N}\}$ be a countable dense subset of $C_0(X)$.

For every $f_i \in D$ and $j \in \mathbb{N}$ consider $U_{i,j} = \{U_{h,i,j} : U_{h,i,j} = (f_i - h)_{\upharpoonright (-\frac{1}{j}, \frac{1}{j}]} \} \in C_0(X)$ for each $i, j \in \mathbb{N}$. Since $X$ satisfies $S_1(K, \Gamma)$, there is a sequence $(U_{h,i,j} : i, j \in \mathbb{N})$ such that $U_{h,i,j} \in \mathcal{U}_{i,j}$, and $\phi := [U_{h,i,j} : i, j \in \mathbb{N}]$ is an element of $\Gamma$.

We claim that $\{h(i, j) : i, j \in \mathbb{N}\}$ is a sequentially dense subset of $C_0(X)$.

Fix $g \in C(X)$. There exists $(f_{i,k} : k \in \mathbb{N})$ such that $f_i \rightarrow g (k \rightarrow \infty)$ in $\tau_p$. Then $(g - f_i) \rightarrow 0$ in $\tau_p$. Show that $h(i,k) \rightarrow g$ in $\tau_p$. Let $W = (g, A, \epsilon)$ be a base neighborhood of $g$ in $C_0(X)$, where $A \in \mathcal{F}(X)$ and $\epsilon > 0$. Since $\phi$ is a $\gamma$-cover of $X$, then $\{U_{h(i,k),j} : j \in \mathbb{N}\}$ is a $\gamma$-cover of $X$, too. There exists $k', j'$ such that $\frac{1}{k'} > \frac{1}{j}$ and for every $k > k', j > j'$ the following statements are true: $(g - f_i)_{\upharpoonright (X \setminus U_{i,j})} \subset (-\frac{1}{j}, \frac{1}{j})$ and $(f_i - h(i,k))_{\upharpoonright (X \setminus U_{i,j})} \subset (-\frac{1}{j}, \frac{1}{j})$. Notice, that $((g - f_i) + (f_i - h(i,k)))_{\upharpoonright (X \setminus U_{i,j})} \subset (-\epsilon, \epsilon)$. Then $h(i,k) \in W$ for every $k > k', j > j'$.

$(8) \Rightarrow (3)$. Let $\{U_i : i \in \mathbb{N}\}$ be a countable dense subset of $C_0(X)$.

Consider $D_i = \{f_{k,l,i,j} : C(X)\}$ such that $f_{k,l,i,j} \upharpoonright (X \setminus U_i) \equiv 1$ where $K \subset U \in \mathcal{U}$ for every $i \in \mathbb{N}$. Since $D_i$ is a dense subset of $C_0(X)$, then $D_i$ is a dense subset of $C_0(X)$ for every $i \in \mathbb{N}$. By (8), there is a set $\{f_{k,l,i,j} : i \in \mathbb{N}\}$ such that $f_{k,l,i,j} \in D_i$ and $(f_{k,l,i,j} : i \in \mathbb{N}) : i \in \mathbb{N}$ in $\Gamma^p_0$.

Claim that the set $\{U(i) : i \in \mathbb{N}\} \in \Gamma$. Let $K \subset \mathcal{F}(X)$. Define $U(0, K, \frac{1}{2})$ be a base neighborhood of $0$. Since $\{f_{k,l,i,j} : i \in \mathbb{N}\} : i \in \mathbb{N}$ there is $i' \in \mathbb{N}$ such that $f_{k,l,i,j} \in W$ for every $i > i'$. It follows that $K \subset U(i)$ for every $i > i'$ and hence $\{U(i) : i \in \mathbb{N}\} \in \Gamma$. □
We can summarize the relationships between considered notions in next diagrams.

\[ G_1(\mathcal{D}, \Gamma^p_0) \iff G_{fin}(\mathcal{D}, \Gamma^p_0) \iff G_1(\mathcal{D}, \Omega^p_0) \iff G_{fin}(\mathcal{D}, \Omega^p_0) \]

\[ G_1(\mathcal{D}, \Omega^p_0) \iff G_{fin}(\mathcal{D}, \Omega^p_0) \iff G_1(\mathcal{D}, \Omega^p_0) \iff G_{fin}(\mathcal{D}, \Omega^p_0) \]

\[ G_1(\mathcal{D}, \mathcal{S}^p) \iff G_{fin}(\mathcal{D}, \mathcal{S}^p) \iff G_1(\mathcal{D}, \mathcal{D}^p) \iff G_{fin}(\mathcal{D}, \mathcal{D}^p) \]

\[ S_1(\mathcal{D}, \mathcal{S}^p) \iff S_{fin}(\mathcal{D}, \mathcal{S}^p) \iff S_1(\mathcal{D}, \mathcal{D}^p) \iff S_{fin}(\mathcal{D}, \mathcal{D}^p) \]

\[ S_1(\mathcal{D}, \Gamma^p_0) \iff S_{fin}(\mathcal{D}, \Gamma^p_0) \iff S_1(\mathcal{D}, \Omega^p_0) \iff S_{fin}(\mathcal{D}, \Omega^p_0) \]

\[ S_1(\mathcal{D}, \Omega^p_0) \iff S_{fin}(\mathcal{D}, \Omega^p_0) \iff S_1(\mathcal{D}, \Omega^p_0) \iff S_{fin}(\mathcal{D}, \Omega^p_0) \]

Fig. 1. The Diagram of games and selectors of \((C(X), \tau_k, \tau_p)\).

\[ G_1(\mathcal{K}, \Gamma) \iff G_{fin}(\mathcal{K}, \Gamma) \iff G_1(\mathcal{K}, \Omega) \iff G_{fin}(\mathcal{K}, \Omega) \]

\[ S_1(\mathcal{K}, \Gamma) \iff S_{fin}(\mathcal{K}, \Gamma) \iff S_1(\mathcal{K}, \Omega) \iff S_{fin}(\mathcal{K}, \Omega) \]

Fig. 2. The Diagram of games and selection principles for a space \(X\) with \(iuv(X) = \aleph_0\) corresponding to selectors of \((C(X), \tau_k, \tau_p)\).

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