Deformed Special Relativity as an effective theory of measurements on quantum gravitational backgrounds

R. Aloisio
INFN - Laboratori Nazionali del Gran Sasso, SS. 17bis, 67010 Assergi (L’Aquila) - Italy

A. Galante
INFN - Laboratori Nazionali del Gran Sasso, SS. 17bis, 67010 Assergi (L’Aquila) - Italy and Dipartimento di Fisica, Università di L’Aquila, Via Vetoio 67100 Coppito (L’Aquila) - Italy

A. Grillo
INFN - Laboratori Nazionali del Gran Sasso, SS. 17bis, 67010 Assergi (L’Aquila) - Italy

S. Liberati
International School for Advanced Studies and INFN, Via Beirut 2-4, 34014, Trieste - Italy

E. Luzio
Dipartimento di Fisica, Università di L’Aquila, Via Vetoio 67100 Coppito (L’Aquila) - Italy

F. Ménéd
INFN - Laboratori Nazionali del Gran Sasso, SS. 17bis, 67010 Assergi (L’Aquila) - Italy and Departamento de Fisica, Universidad de Santiago de Chile, Av. Ecuador 3493, Casilla 307 Stgo-2 - Chile

In this article we elaborate on a recently proposed interpretation of DSR as an effective measurement theory in the presence of non-negligible (albeit small) quantum gravitational fluctuations. We provide several heuristic arguments to explain how such a new theory can emerge and discuss the possible observational consequences of this framework.

PACS: 03.30+p, 04.60.-m
Keywords: Doubly special relativity; Planck scale; quantum gravity; Lorentz invariance

I. INTRODUCTION

Deformed or Doubly Special Relativity (DSR) \cite{1, 2} can be understood as a tentative to modify Special Relativity (SR) in order to incorporate a new invariant scale other than that provided by the speed of light c. The idea driving this attempt is that quantum gravity effects seems to introduce a new dimensional fundamental scale given by the Planck length ($\ell_{pl}$). This is possibly problematic for the relativity principle because the presence of a fundamental length scale might seem naively incompatible with boost invariance. However other ways to introduce such fundamental length scale could be envisaged which do not necessarily lead to a violation of the equivalence of inertial observers.

Concrete realizations of these ideas in the momentum space are known \cite{2, 3, 4}. In particular, deformed boosts transformations, deformed dispersion relations as well as composition laws have been widely investigated \cite{5}. On the other hand, the implementation of DSR in the spacetime is a more subtle subject and it is a theme of intense debate at present time \cite{6}.

DSR in momentum space can be intended as a deformation of the Poincaré algebra in the boost sector \cite{2, 3, 6}. Specifically the Lorentz commutators among rotations and boost are left unchanged but the action of boosts on momenta is changed in a non-trivial way (see e.g. \cite{2}) by corrections which are suppressed by some large quantum gravity scale $\kappa$. Most commonly such a scale is taken to be the Planck energy, $\kappa \approx \frac{1.22 \times 10^{19}}{\text{GeV}}$.

It was soon recognized \cite{2, 8} that such deformed boost algebra amounts to the assertion that physical energy and momentum of DSR can be always expressed as nonlinear functions of a fictitious pseudo-momentum $\pi$, whose components transform linearly under the action of the Lorentz group \cite{1}. More precisely one can assume the existence of an invertible map $\mathcal{F}$ between two momentum spaces: the classical space $\mathcal{P}$, with coordinates $\tau_{\mu}$ where the Lorentz group acts linearly and the physical space $\mathcal{P}$, with coordinates $p_{\mu}$, where the Lorentz group acts as the image of its action on $\mathcal{P}$. Also, $\mathcal{F}$ must be such that $\mathcal{F}: [\pi_0, \vec{\pi}] \to \kappa$ for all elements on $\mathcal{P}$ with $|\vec{\pi}| = \infty$ and/or $\pi_0 = \infty$.

The main open issues in this momentum formulation of DSR are the so called multiplicity and saturation problems. The first is related to the fact that in principle there are many possible deformations (an infinite num-

\footnote{Indeed this is automatically guaranteed if the realization of the Lorentz group on the physical energy-momentum space is one-to-one \cite{6}.}
ber, depending on the choice of an energy invariant scale, three-momentum scale or both \( \eta \). This seems to suggest that the set of linear transformations (that is SR) is the only one that have a physical sense (it is unique and linear, what else can we ask for?). Moreover the composition law for energy and momenta of DSR, being derived by imposing a standard composition law for the pseudo four-momenta \( p_\mu \), is characterized by a saturation at the Planck scale apparently in open contrast with the everyday life observation of classical objects with trans-planckian energies and momenta.

On the other hand, since DSR is not a formulation of QG, but gives a set of transformations with the typical QG scale, it is plausible to consider it as a low energy limit of QG, that is, as some effective theory. Indeed such point of view was taken in several works on the subject \( \text{[10, 11, 12].} \) We hence advocate here the point of view that the DSR transformations (as given in momentum space) are not fundamental law of transformations among different reference frames but, following the proposal of \( \text{[9]}. \) effective relations taking into account the first order corrections due to the quantum gravitational effects.

The main purpose of this work is to show how plausible effects due to the quantum nature of the space-time, once they are summed up, give rise to deformed dispersion relations of DSR-type. In order to show this, in the next sections (II and III) we will briefly review the proposal of \( \text{[8]}. \) formulated in terms of the metric and the tetrad fields. In section IV we show how this approach works through a very simple (albeit unphysical) example, but containing the main ideas of the proposal. In the next section (V), we present heuristic arguments for the outcome of the average, which will give rise to different kinds of DSR-type modifications. Section VI is devoted to the analysis of the operation consequences of the framework here discussed. In the final section we present the conclusions and also we discuss possible further developments on this topic.

II. DSR AS AN EFFECTIVE MEASUREMENT THEORY

Consider a four dimensional spacetime manifold with local coordinates \( x^\mu \) and a differential line element

\[
d s^2 = g_{\mu \nu} dx^\mu dx^\nu, \tag{1}
\]

where \( g_{\mu \nu}(x) \) is the spacetime metric.

A locally inertial frame can be defined in each point of the spacetime through the set of four covariant vector fields \( e_\mu^\alpha(x) \) (tetrad) defined through the relation

\[
g_{\mu \nu} = \eta_{\alpha \beta} e_\mu^\alpha e_\nu^\beta, \tag{2}
\]

with \( \eta_{\alpha \beta} = \text{diag}(-1, 1, 1, 1) \). The 1-forms \( e_\mu^\alpha = e_\mu^\alpha dx^\mu \), are vectors in the cotangent space transforming under the action of the (local) Lorentz group as \( (e_\mu^\alpha)' = \Lambda^\alpha_\beta e_\mu^\beta \).

Any vector field \( V^\mu \) in the spacetime has components \( V^\alpha \) in the local inertial frame given by

\[
V^\alpha = e_\mu^\alpha V^\mu.
\]

Finally, the inverse tetrad, which will be denoted by \( e^\mu_\alpha \), is defined as the solution of \( e_\mu^\alpha e^\mu_\beta = \delta^\alpha_\beta \) and satisfies

\[
\eta_{\alpha \beta} = g_{\mu \nu} e^\mu_\alpha e^\nu_\beta. \tag{3}
\]

The use of a reference frame is crucial in order to extract, from the abstract tensors of any relativistic theory, scalar quantities that could be interpreted as measurement outcomes.

In particular, in the usual theory of measurement \( \text{[13]}, \) if a particle has four-momentum \( p_\mu \), its energy \( E \) and \( \i-th \) component of three-momentum \( p_i \), measured in the reference frame \( \{ e^\mu_\alpha \} \), are given by the expression

\[
p_\alpha = e^\mu_\alpha p_\mu, \tag{4}
\]

Note that now the \( p_\alpha \) are a set of four scalars (the actual, chart independent, measured quantities in the reference frame represented by the tetrad).

In flat spacetime one has \( e^\mu_\alpha \equiv \delta^\mu_\alpha \) so that the measured energy and momentum of the particle \( p_\alpha \), will be identical to the components of \( p_\mu \), i.e. \( p_\alpha \equiv \pi_\alpha \). In a curved spacetime, it is possible to perform the experiment in a locally flat space (that is, locally inertial) or in the accelerated frame. In any case, quantities as \( p_\alpha \) are what we generally associate to the outcome of a measure. Reduced to the bones the proposal of \( \text{[9]}. \) is that quantum gravitational effects can affect the measurement process and introduce Planck suppressed corrections to the measured momenta with respect to the component of the actual four momentum of the particle observed.

In this interpretation DSR is an effective theory taking into account in a semiclassical limit quantum gravity corrections to the process of measure. In particular one needs to postulate that these effects act in such a way that the relation between the measured scalars \( p_\alpha \) and the four momentum of the particle \( \pi_\mu \) is given by a non-linear function of the \( \pi_\mu \) and the quantum gravity scale \( \kappa \)

\[
p_\alpha = \mathcal{F}_\alpha[\pi_\beta, \kappa] = \mathcal{F}_\alpha[\pi_\mu e^\mu_\beta, \kappa]. \tag{5}
\]

For example the DSR formulation proposed by Magueijo

\[\text{[2].}\] Here the indices \( \mu, \nu, \ldots \) are standard tensor indices, associated with a choice of coordinates, while the indices \( \alpha, \beta, \ldots \) only label different vectors in the tetrad, and have nothing to do with any particular chart adopted.
and Smolin (DSR2) \[2\]

\[E = \frac{-\pi_0}{1 - \pi_0/\kappa}, \quad (6)\]

\[p_i = \frac{\pi_i}{1 - \pi_0/\kappa}, \quad (7)\]

in the measurement theory framework should actually be written as \[4\]

\[p_\alpha = \mathcal{F}_\alpha [\pi_\mu e^\mu_\beta; \kappa; \chi]. \quad (8)\]

with \(E = -p_0\), as usual.

Moreover a measurement of the energy-momentum of some composite object will generally depend also on many details of the internal structure of the composite object, its interaction with the detector, and the internal construction of the latter. Let us collectively denote these extra variables as \(X\), so that

\[p_\alpha = \mathcal{F}_\alpha [\pi_\mu e^\mu_\beta; \kappa; X]. \quad (9)\]

In particular, among the additional variables \(X\) one could include the spatial and temporal resolution of the detector as well as the number of particles involved in the measure. In general we shall define an ideal detector a device which will be able to provide a measured quantity which is independent of quantities intrinsic to the detector.

It is easy to see how now, the above cited problems of DSR in momentum space, can be understood in this framework: the multiplicity problem will be just a manifestation of the several possible measurement methods which will in turn determine different “measurement functions” \(\mathcal{F}_\alpha\): the saturation problem, conversely, can be interpreted as a constraint only on the size of measurable quantum objects (given that classical objects will never be able to probe quantum gravitational effects) \(9\) or it can be even removed if the dependence of \(\mathcal{F}_\alpha\) on the number of particles \(N\) for the composite object is not factorized out but is such that the composite measured momenta \(p_\alpha\) saturate e.g. at \(N\kappa\) rather than \(\kappa\). Interestingly this one of the few viable frameworks (together with that presented in \[12\]) where such a dependence on the number of particles of \(\mathcal{F}_\alpha\) (first postulated in \[2\]) would be natural.

III. MEASURES ON A FLUCTUATING BACKGROUND

In order to further explore the above interpretative framework for DSR one might start from the observation that the standard theory of measurement heavily relies on the notion of a classical metric structure as this is the fundamental prerequisite for introducing the tetrad vector fields via Eq. \[4\] which in turn characterize the local inertial reference frames. In a full quantum gravity theory these tetrad fields (or a suitable subset) are expected to behave as quantum fields which only in some appropriate limit will define the corresponding classical quantities. When dealing with measurements concerning particles at very high energies one may wonder if the quantum nature of the gravitational background can be always safely neglected. In particular our detectors are implicitly interpreting the results of their interactions with the observed particles using the classical measurement theories which is tantamount to say that when we perform a measurement we are implicitly performing some sort of averaging over the quantum gravitational fluctuations in order to recover a classical background. We shall argue here that such unavoidable averaging procedure will leave some energy dependent (and Planck suppressed) corrections to the classical, low energy, metric and hence to the tetrad fields. Such corrections could then lead to Planck suppressed corrections in the measured \(p_\alpha\) scalars defined by \(9\). Moreover the universal nature of gravitational interactions (added to the impossibility to screen them) leads us to conjecture that such corrections will have a universal functional form (when using ideal detectors) possibly with a dependence on the kind of particles detected and their numbers. In this sense the source of the non linearity in DSR is not some new dynamics for the quantum particles of momenta \(\pi_\mu\), but a first signal that our classical spacetime is emergent from some more fundamental quantum theory. Let us now try to give a more qualitative description of our ansatz.

If spacetime is an intrinsic emergent concept, the low energy by product of some yet unknown theory of quantum gravity, then at sub-Planckian energies it should be possible to split the quantum operator describing the spacetime causal structure \(\mathcal{G}\) in a classical (mean field) value \(\bar{g}\) plus some quantum fluctuations \(\hat{h}\)

\[\mathcal{G}_{\mu\nu} = \bar{g}_{\mu\nu} + \hat{h}_{\mu\nu} \quad (10)\]

where these fluctuations \(\hat{h}_{\mu\nu}\) can be intrinsic to the background metric as well as due to the presence of matter fields, but in any case characterized by the quantum gravity scale \(\kappa\) \(^3\).

In performing a measurement, as in eq. \(4\) or \(5\), we are always using a classical tetrad, and hence a classical local structure of spacetime, which will be generally the outcome of some averaging process over the quantum gravitational metric. In order to recover Eq. \(9\) we have then to postulate that the outcome of this average should depend on the characteristic energy of the measurement process as well as on \(\kappa\) and the \(X\) variables \(^4\). In particular we stress that even if measures are performed in

\(^3\) We stress here the strong analogy of this ansatz to those typical of the so called analog models of gravity where an emergent geometry is observed in condensed matter systems like e.g. Bose-Einstein condensates \(14, 15\) (see also \(16\) for an extensive review on analog models).

\(^4\) See also \(17\) for a similar point of view.
macroscopic detectors the actual measurement of a particle of some energy $E$ requires from the point of view of the measurement theory a characterization of the local inertial frame and hence of the metric on scales at least of order $1/E$. Hence when performing a measurement we always introduce a classical metric which is supposed to hold at the microscopic scales over which the particle and our detector interact

$$\langle \hat{g}_{\mu \nu} \rangle = \tilde{g}_{\mu \nu} + \langle \hat{h}_{\mu \nu} \rangle_{E,\kappa,X} \quad (11)$$

Compatibility with low energy physics will be tantamount to say that the quantum fluctuations $\hat{h}_{\mu \nu}$ will average to zero when the averaging is done over scales much larger than the Planck energy. Note however that when the averaged quantum fluctuations are non negligible “we pay” the use of a classical measurement theory with the introduction of an energy dependent “effective classical metric”\(^5\)

$$\langle \hat{g}_{\mu \nu} \rangle = \tilde{g}_{\mu \nu} + \langle \hat{h}_{\mu \nu} \rangle_{E,\kappa,X} = \tilde{g}_{\mu \nu}^{\text{eff}}(E, \kappa, X) \quad (12)$$

Following this idea let us then rewrite Eq. (6) as an averaged quantity

$$p_\alpha = F_\alpha[\pi_\beta, \kappa] = \langle \pi_\alpha, \kappa \rangle = \pi_\mu \langle e^\mu_\alpha(x) \rangle_{E,\kappa,X} \quad (13)$$

where the last line tells again that non-linearity comes from averaging over the gravitational degrees of freedom weighted via the energy of the particle involved $E, \kappa$ and the variables $X$. It is interesting to note that now if we want to preserve the relativistic principle we not only need to have $\kappa$ as an invariant energy scale but also we shall need to postulate that the form of the quantum gravitational fluctuations $\hat{h}$ is universal, in the sense of being the same in any inertial system of reference.

We now need to link the averaged tetrad field in (13) to the average of the quantum fluctuations of the metric $\hat{h}_{\mu \nu}$. To do so we can again split a mean value detected at very low energies (with respect to Planck) plus the weighted average of the fluctuating part.

$$\langle e^\mu_\alpha \rangle = \bar{e}^\mu_\alpha + \delta^\mu_\alpha f^\beta_\alpha, \quad (14)$$

where $f^\beta_\alpha$ is a matrix in the tangent spacetime and contains all the information about the average; $\bar{e}^\mu_\alpha$ is the mean (very low energy) value of the tetrad (that in flat spacetime has the simple form $\delta^\mu_\alpha$).

The inverse tetrad $e^\alpha_\mu$, is defined as the solution of $e^\beta_\mu e^\mu_\alpha = \delta^\beta_\alpha$. However, since by hypothesis, only averaged quantities are available to the observer, this definition cannot be used and the observer can only compute $\langle e^\beta_\mu \rangle$ which satisfies $\langle e^\beta_\mu \rangle \langle e^\mu_\alpha \rangle = \delta^\beta_\alpha$. It is straightforward to show that

$$\langle e^\alpha_\mu \rangle = \bar{e}^\alpha_\mu - f^\alpha_\beta \bar{e}^\beta_\mu. \quad (15)$$

In order to match the averaged quantum fluctuations $\langle \hat{h}_{\mu \nu} \rangle_{E,\kappa,X}$ with the part of the tetrad dependent on $f$ we can just impose that

$$\langle \hat{g}_{\mu \nu} \rangle = \tilde{g}_{\mu \nu} + \langle \hat{h}_{\mu \nu} \rangle_{E,\kappa,X} = \eta_{\alpha \beta} \langle e^\alpha_\mu e^\beta_\nu \rangle \quad (16)$$

where we are making use of Eq. (2).

As we did for the evaluation of the inverse tetrad, here we will assume that $\langle e^\alpha_\mu e^\beta_\nu \rangle = \langle e^\beta_\mu \rangle \langle e^\alpha_\nu \rangle$. Then, the correction to the metric is

$$\langle \hat{h}_{\mu \nu} \rangle_{E,\kappa,X} = -(f_{\alpha \beta} + f_{\beta \alpha}) \langle e^\alpha_\mu, e^\beta_\nu \rangle \quad (17)$$

Not surprisingly in Eq. (17) the fluctuations of the metric determine only the symmetric part of the function $f_{\alpha \beta}$. In what follows, as a simplification of the model, we shall assume a symmetric $f$.

So in the end the momentum measured for a particle propagating in spacetime can be read from Eq. (13) using (14) and (17)

$$p_\alpha = \pi_\mu \left( \bar{e}^\mu_\alpha - \frac{1}{2} \langle \hat{h}_{\mu \alpha} \rangle_{E,\kappa,X} \bar{e}^\beta_\alpha \right),$$

$$= \pi_\mu \bar{e}^\mu_\alpha - \frac{1}{2} \pi^\tau \langle \hat{h}_{\tau \alpha} \rangle_{E,\kappa,X} \bar{e}^\beta_\alpha, \quad (18)$$

with $\pi^\tau = \tilde{g}^{\tau \pi} \pi_\pi$. If one considers an ideal detector then one expects the variables $X$ to include at most the number of particles involved in the measurement process and other quantities intrinsic of the observed object (not of the detector). In what follows we shall make the simplifying assumption that such an ideal detector is used and, for the moment, we shall consider measures of a single particle per time.

IV. A SIMPLE ANALOGY

We now try to clarify the above proposal with a simple, albeit quite unphysical, example. Let us consider a space-time containing a particle and a Planck mass Schwarzschild black hole. We assume that, in the frame in which the black hole is at rest, the particle has an energy much smaller than Planck energy and we neglect the gravitational perturbation induced by the particle. Clearly the particle will feel the gravitational potential of the black hole and a global frame attached to the particle will not be inertial. However it is of course always possible to attach to the particle a local inertial frame, and the job is properly done through the tetrad connected to the black hole metric.

Now let us introduce an observer in the plot and assume that it is:

1. idealized: it does not disturb space-time in any way,

\(^5\) See also \[17\] for similar ideas about linking DSR to an energy dependent metric structure

\(^6\) Note that all manipulations are done assuming no corrections on $\tilde{g}$ because we are working only at first order.
2. coarse-grained: it cannot observe scales smaller than some scale $\gg$ Planck scale.

3. not particularly clever.

Such an observer could be tempted to define, in the frame in which the black hole is at rest, a “dispersion relation” for a particle of mass $m$ using the Schwarzschild metric for a Planckian black hole. In particular for a particle located at a distance $d$ much larger than Planck length $\ell_{pl}$ from the black hole one would get

$$E^2 - p^2 \approx m^2 + \frac{\ell_{pl}}{d} E^2. \quad (19)$$

A clever observer, even if he/she could not “see” the black hole (coarse-grained measures), would easily infer its presence (i.e. the fact that the spacetime is curved) by the position and time dependence of Eq. (19); a not so clever one would instead insist that spacetime is flat and would attach to the particle the above non-trivial dispersion relation.

This situation mimics to some extent that of DSR: on one hand the dispersion relation is modified, on the other hand frame invariance of physics is preserved. In fact, for instance it is possible (although in general very difficult) to find transformations that would relate the frame in which the black hole is at rest to that in which the particle is. An observer at rest with the particle would in fact observe a time-varying mass related to the time varying potential of the black hole at the particle location. Note also that, for this last observer, there is “new”, physics: e.g. if the particle is charged, then it will emit photons in its “vacuum”; this “new” physics would be, however, frame independent.

Indeed in the case of the framework we are envisaging here the situation is worse than in this example. We don’t have a “quantum spacetime theory of measurement” so we are obliged to average over the gravitational fluctuations and hence to deal with deformed dispersion relations similar of the form of Eq. (19).

Of course it is not possible to push the just presented analogy too far at this stage. We can however complicate it a little bit to try to better mimic the real world. For example let us now consider a quantum mechanical description of the particle so that it will naturally have a “size” associated to its De Broglie wave-length $\lambda = 1/E$. If we now assume that $\lambda$ is the characteristic scale over which the observer “measures” the particle, then he/she will infer a dispersion relation:

$$E^2 - p^2 \approx m^2 + 4 E^3 / \kappa. \quad (20)$$

which is again of DSR-like form (here $\kappa = 1/\ell_{pl}$).

As a final remark about the limits of this analogy we can stress that a single black hole does not seem to be an appropriate approximation for the QG vacuum; to go a step further we have to consider some ansatz for the fluctuations of the QG vacuum metric. This is beyond the scope of the present paper and it will be better discussed in [18]. However it would seem that, in order to preserve frame invariance à la DSR, the metric fluctuations have to preserve some form of coherence in different frames. In particular it is probable that a necessary (although possibly not sufficient) condition is that if the fluctuations have a characteristic scale, this be the same in all frames.

V. HEURISTIC ARGUMENTS FOR THE OUTCOME OF THE AVERAGE

In the previous section we exposed our operative framework and linked the averaged fluctuations of the metric to the effective (energy dependent) tetrad fields used in the standard theory of measurement. So doing we have seen that the application of a classical measurement theory on a fluctuating background might lead to the non-linearities characteristic of DSR theories. In this section we shall further support our hypothesis by considering some simple heuristic arguments.

A. Dimensional arguments

Compatibility with low energy physics implies that the corrective term $f$ in [14], must be nearly zero when the measurement probes/averages the spacetime over distances large enough compared with the Planck scale, but becomes larger and larger once the distances explored become smaller and smaller. Therefore, $f$ must be some power of $\ell_{pl}/d$ where $d$ is the typical distance over which the spacetime is probed/averaged.

Let us assume that the typical scale over which one probes the spacetime is fixed by the wavelength (inverse of the energy) of the particle detected. Then, we can argue that $f$ should be proportional to $(E/\kappa)^n$, where $E = -\pi_0$ is again the intrinsic energy of the particle.

The tensor structure of $f$ is what we need now. This tensorial character can be constructed with $\eta_{\alpha\beta}$ and $\pi_{\alpha} \pi_{\beta}$, if we insist on covariance in the tangent space. The last option will be discarded because all the dependence on momenta has been decoupled by the average. Then, we will assume:

$$f_{\alpha\beta} = \sigma (E/\kappa)^n \eta_{\alpha\beta}, \quad (21)$$

with $\sigma$ a dimensionless quantity (that can include also a sign) of the order 1. The physical momentum can be read now from [14] and it turn out to be:

$$p_{\alpha} = \pi_{\mu} e^\mu_{\alpha} (1 + \sigma (E/\kappa)^n). \quad (22)$$

This is the DSR2 relation [2] (at first order in $\ell_{pl}$) if $\alpha = 1, \sigma = -1$. Using the fact that $\eta^{\mu\nu} \pi_{\mu} \pi_{\nu} = \mu^2$ is an invariant, we find the dispersion relation:

$$E^2 - p^2 = \mu^2 (1 + \sigma (E/\kappa)^n)^2. \quad (23)$$

However the above is not the most general choice, since in principle, releasing the request of covariance in tangent
space, we should take into account all the possible contributions coming from $f_{00}, f_{0i}, f_{ij}$. But it is not hard to see that contributions from $f_{00}$ can be always absorbed in the other terms, since it is always possible to write $|\pi| \rangle$ in terms of $E$ from the dispersion relation $\pi_n \pi^\mu = \mu^2$. However a more general tensorial structure could be given by

$$f_{00} = \sigma_0 (E/\kappa)^n, \quad f_{ij} = \sigma_1 (E/\kappa)^n \eta_{ij},$$

(24)

which in this case could accommodate, as the special case $\sigma_0 = -1, \sigma_1 = 1/2$ and $n = 1$, the so called DSR1 \[1\] implementation of deformed special relativity.

In the above derivation we have guessed the result of the average over quantum gravitational fluctuations on the base of a dimensional analysis. However we shall show in what follows that general assumptions about the nature of such fluctuations can also lead to the same qualitative result.

**B. Average**

Let us review the average process with a little more detail. We are interested in a definition of the right hand of (14). We can formally rewrite the average of the fluctuations of the metric as

$$\langle h_{\mu\nu} \rangle_{E, \kappa} = \int dx \int DG \tilde{h}_{\mu\nu} \mathcal{P}_{E, \kappa}(G),$$

(25)

where $\tilde{h}$ is the part of the metric that fluctuates (all the effects of quantum gravity are included here), and the mean is performed in the space of metrics with weight $\mathcal{P}_{E, \kappa}$. The integration on $x$ takes into account these effects at some characteristic distance.

Since we do not know how to calculate the integral for the metrics, let us write it as a general function $\Phi_{\mu\nu}(x)$, and therefore the average has the shape

$$\langle \tilde{h}_{\mu\nu} \rangle = \int dx \Phi_{\mu\nu}(x, \kappa, E).$$

(26)

Missing a definitive theory of quantum gravity we shall try to see what can be said using a minimal set of assumptions about the typical fluctuations of spacetime. We want to show that under very general assumption the above discussed average \[25\] will lead to a non linear relation between measured and classical momenta.

To start with, let us make few important assumptions about the nature of the spacetime fluctuations.

1. The spacetime is characterized by a classical background over which are imposed universal quantum, Planck scale, fluctuations. In order to preserve the relativity principle these fluctuations should have the same form in any inertial system of reference.

2. The spacetime fluctuations have an average value that tend to $\pm 1$ (in Planck units) for $E \to \kappa$ where $E$ is the energy of the probing particle. This implies that in this limit one cannot recover a classical spacetime. Conversely for $E \ll \kappa$ we expect to recover the classical metric so that the average process should give vanishing fluctuations in this limit.

3. Any measure done with particles of energy $E$ probes a large number of such Planck scale fluctuations. This is a kind of coarse graining over a scale inversely proportional to $1/E$.

Given the above assumptions, we can then expect that the fluctuations are such that their average goes to zero with some power law of $N$ where $N = L/\ell_P = \kappa/E$ is the number of fluctuations contained in an interval of length $L = 1/E$. Henceforth

$$\langle \hat{h}_{\mu\nu} \rangle = \int_0^{1/E} d^4x \sqrt{-\eta} \Phi_{\mu\nu}(x, \kappa) \approx \frac{1}{N^{4n}} = \left(\frac{E}{\kappa}\right)^{4n},$$

(27)

and it is easy to see that in this case the modification of the dispersion relation \[18\] looks like

$$E^2 - p^2 = \mu^2 + \frac{p^{4n+2}}{\kappa^{4n}},$$

(28)

which is just \[23\] written in a different form.

So for $n = 1/4$ one gets cubic deformations. For $n = 1/2$ (Poissonian fluctuations) one gets quartic deformations and so on. Actually it would be interesting to catalog which kind of fluctuations give the various plausible values of $n$. In this way, if we shall find out that actually some form of modified relation is realized in nature, we could able to deduce which class of gravitational fluctuations should be recovered in the low energy limit of the full QG theory. Alternatively ruling out some specific form of dispersion relation (within the DSR framework) will rule out some class of fluctuations of the classical metric. Note that in principle we could be even more general and assume that the fluctuations are described by some function $f$ which goes to zero for $N \to \infty$. This more general formulation should allow to recover all the possible modified dispersion relations associated to DSR.

**C. Mechanical view**

An alternative approach to the evaluation of the outcome of the averaging over the spacetime quantum fluctuations is suggested by the split of the metric \[10\] which we took as a basic assumption in our framework. This splitting is in fact reminiscent of the linearized Einstein equations and the analogy is strengthened by the fact that we are primarily considering $\tilde{g}$ as the flat metric.

In this case, however, one could argue that Einstein equations are not fundamental in the sense that they are not valid once we approach to distances of the order of
the Planck scale and moreover it could seem inconsistent to apply them to a quantum object like the Planck scale and moreover it could seem inconsistent of the standard treatment of this systems.

In fact a dilute Bose gas could be described through a quantum field $\hat{\Psi}$ satisfying a non-linear Schrödinger equation. Similarly to what we did for the metric, it is possible to split the quantum field into a macroscopic (classical) condensate and a fluctuation: $\hat{\Psi} = \psi + \hat{\varphi}$, with $\langle \hat{\varphi} \rangle = 0$. One then obtains two equations respectively for the classical background and its quantum excitations. The equation for the classical wave function of the condensate is closed only when the back-reaction effect due to the fluctuations are neglected. This is the approximation contemplated by the Gross–Pitaevskii equation.

The interesting point is that when the back-reaction effects are neglected the equations for the quantum perturbations are formally identical to what one would have get from considering linear perturbations of the classical background equations (see e.g. [19]). In strict analogy we shall here conjecture that the equations for the quantum gravitational fluctuations are identical to those for the linear perturbations of the classical metric, i.e. the linearized Einstein equations, when their back-reaction is negligible. In this analogy, therefore, it has sense to consider the contribution to the flat metric of the quantum fluctuations as a solution of the linearized Einstein equations which in the Lorentz gauge take the form [21]

$$\partial_\nu \partial^\alpha h_{\mu\nu} = -16\pi G_N T_{\mu\nu} \tag{29}$$

Note also that this approach is formally similar to what is the “averaged Einstein equations” introduced in order to consider the inhomogeneities at cosmological level [21], but this is just at formal level and in fact, the main departure with this approach is the definition of the average.

At this point we must consider the two cases, namely, presence or absence of matter. They can be interpreted as the modifications due to the presence of matter and modifications originated just by fluctuations of the spacetime. Let us review first the case of vacuum.

In the radiation gauge, the formal solution of the linearized Einstein equations is

$$h_{\mu\nu} = e_{\mu\nu} e^{ikx} + e^{*}_{\mu\nu} e^{-ikx}, \tag{30}$$

where $e_{\mu\nu}$ is the polarization tensor and $^*$ denotes the complex conjugate (to render real the previous solution). In this coordinate system (harmonic), the relation $2k_\nu e^{ik_\mu} = k_\nu e^{ik_\mu}$ must be fulfilled as well as the condition $k^2 = 0$.

Here we are interested on the result of the average of the previous solution, and as we pointed out in the introduction, a few physical assumptions should permit us to give a general form for it. Firstly, we assume that this process is ergodic so that the time dependent average can be replaced by a mean on ensemble. On the other hand, since the temporal part has been decoupled, the only relevant piece of the wave number is the space-like part, which we will assume to be of the order of $1/\kappa$. Note that here, we are not speaking about a wave that propagates in a flat spacetime, instead we are talking about the space time itself which oscillates with spatial amplitude of the order of the invariant scale.

The average of the previous solution, therefore, will have the shape

$$\langle h_{\mu\nu} \rangle = \gamma_{\mu\nu} F[x/\kappa] + \gamma^*_{\mu\nu} G[x/\kappa], \tag{31}$$

where $\gamma_{\mu\nu}$ is the result of the average of the polarization tensor and $F,G$ the results of the average of the exponential functions of the solution.

Consider now the case of Eq. (29) with sources. The formal solution of the Einstein equation is

$$h_{\mu\nu}(x) = 4G_N \int_V d^3y \frac{T_{\mu\nu}(y)}{|x-y|}, \tag{32}$$

where $V$ is the past light cone of $x$.

The formal average $\langle \cdot \rangle$ of the solution is

$$\langle h_{\mu\nu}(x) \rangle = 4G_N \int d^3y \frac{T_{\mu\nu}(y)}{|x-y|}. \tag{33}$$

Note that the general solution admits another piece which is the solution of the homogeneous equation. According to our interpretation, this means that the fluctuations of the spacetime contributes with an additional term which in principle could be added to (33), but which we are discarding just to simplify the analysis.

As before, the physical information will be put in this solution through conditions on the average. A first assumption will be that the average on time is equivalent to space average over a large number of copies of the system (ensembles). Therefore the time evolution of the system will be replaced by the statically description averaged over ensembles.

A second assumption is the independence of the averaged energy momentum tensor from coordinates. That is, we will assume that $\langle T_{\mu\nu}(x) \rangle$ do not depend on the coordinates. It only depends on the characteristic length and energy of the particle which is probing the space.

With this two assumptions is clear that the correction to the metric has the same form as the Newtonian gravitational potential, but generated by $\langle T_{\mu\nu} \rangle$ of the particle. That is, the correction to the flat metric has the shape

$$\langle h_{\mu\nu} \rangle \sim G_N V \frac{T_{\mu\nu}}{d}, \tag{34}$$

where $d$ is the distance explored by the particle and $V$ is the volume occupied by the particle.
Note that, at the end, all the content of QG effects is now codified in the mean value of $T_{\mu \nu}$ — which also could include standard quantum mechanics fluctuations — and since these effects are linked to the metric fluctuations, we have to demand that a self consistence between (34) and the definition of the average.

In order to get an explicit expression for $\langle h_{\mu \nu} \rangle$ and derive the modified dispersion relation let us build a (naive and highly unrealistic, at this stage) model of the particle as a set of free, independent, particles of size $l_P$ distributed on a region of the order of $1/E$, then
\[
T_{\mu \nu} = \sum_n \pi_0^{(n)} \nu_\nu^{(n)} \delta^3(x - x^{(n)}),
\]
where $\nu^\mu = dx^\mu/dt$ and $x^0 = t$.

It is possible to introduce the fluctuations as a correction in the velocities due to a random walk process (see e.g., [22]). That is, since we are considering the particle composed by non interacting pieces with a mean size of the order of the Planck scale, we assume that the velocity of every piece does not depend on the other pieces and evolves randomly. Then
\[
\langle T_{\mu \nu} \rangle = \sum_n \pi_0^{(n)} \nu_\nu^{(n)} \delta^3(x - x^{(n)}).
\]
The effect of the average in velocities can be written as $\langle \nu_\mu^{(n)} \rangle = \tilde{\nu}_\mu^{(n)} + \omega_\mu^{(n)}$, where $\tilde{\nu}_\mu^{(n)}$ is the mean value, while $\omega_\mu^{(n)}$, the variance, which for example in the random walk process considered in [22] turns out to be $\omega_\mu^{(n)} \sim \sqrt{\kappa/E(n)}\delta_{0\mu}$.

Therefore, by introducing this fluctuation we find
\[
V\langle T_{\mu \nu} \rangle = \pi_0 \left( \pi_\mu^{\nu} \pi_0 + \omega_\mu + \omega_\nu \right).
\]
The contribution to the metric $h$ turns out to be
\[
\langle h_{\mu \nu} \rangle = \frac{\pi_\mu \pi_\nu}{\kappa^2} + \frac{\pi_0 \pi_{\mu \nu}}{\kappa^2},
\]
where we have used the approximation $\pi_0 d \sim 1$ and the signs () around the indices mean a symmetrized sum.

The first term in the RHS of the previous equation, gives rise to a redefinition of the mass, as is expected from a potential that is purely Newtonian. The final result for $p$ is straightforward to evaluate
\[
p_\alpha = \pi_\alpha \left( 1 - \frac{\mu^2}{2\kappa^2} - \frac{\pi_0 \pi \cdot \omega}{2\kappa^2} \right) - \omega_\alpha \frac{\pi_0 \mu^2}{2\kappa^2},
\]
where again $\mu^2 = \eta_{\mu \nu} \pi_\mu \pi_\nu$ and $\pi \cdot \omega = \pi_\alpha \omega^\alpha = \pi_0 e_\mu^\alpha \omega^\mu$.

Note also that energy and momenta have different corrections, depending, in part, in the choice of $\omega$ and that our last expression turn out to be quadratic in momenta.

Assume now, as an extra hypothesis, that the linear term in [40] gives only the redefinition of mass $\mu$, that is $\pi_\alpha \omega^\alpha = 0$. Then it is possible to write $\omega_0$ in terms of $\omega = |\omega|$ and $\pi = |\pi|$ as follow $\omega_0 = \sigma \omega (\pi/\pi_0)$, with $\sigma \in [-1, 1]$. Under these assumptions we obtain for the energy and momentum
\[
E = \pi_0 \left( 1 - \frac{\mu^2}{2\kappa^2} - \omega \pi \pi_0 \mu^2/2\kappa^2 \right),
\]
\[
p = \pi \left( 1 - \frac{\mu^2}{2\kappa^2} \right) - \omega \pi_0 \mu^2/2\kappa^2.
\]
and the first order dispersion relation can be deduced evaluating $E^2 - p^2$, as was done in [22]
\[
E^2 - p^2 = \mu^2 + \frac{\mu^2}{\kappa^2} (1 - \sigma) \pi_0 \pi + O(1/\kappa^2).
\]

A DSR1 type of dispersion relation can be obtained by setting $\omega$ proportional to $p$ — equivalently, proportional to $\pi$ since we are working at first order in $\kappa^{-1}$ — but we also require that this term, which is a fluctuation, must depend on $\kappa^{-1}$. Considering that $\omega$ is a dimensionless quantity, we see that, in order to obtain a DSR-like dispersion relation, we can make the (minimal) choice $\omega \sim p/\kappa \sim \pi/\kappa$.

VI. QUANTUM GRAVITY PHENOMENOLOGY

We want now to discuss the operational consequences of the framework presented here from the point of view of the quantum gravity phenomenology tests extensively considered in the literature [23]. In fact it might seem that our proposal makes DSR a by product of a direct measurement which would imply that some of the processes considered in quantum gravity phenomenology would be unaffected.

A. Time of flight

The time of flight test of modified Lorentz symmetry is based on the cumulative effect of the energy dependence of the group velocity of photons in the presence of dispersion relations like [23]. By looking at the dispersion in the time of arrival for photons which are supposed to be emitted simultaneously one can cast a proper bound on the magnitude of the anomalous terms in the dispersion relation.

It might seem at first sight that such a test is lost in our framework as light is probed only at its arrival on Earth and travels undisturbed on long distances. Let us start by considering that strictly speaking a time of flight constraint is obtained with two measurements. The first one being the observations of a simultaneous emission the second the detection of the upper bound on the time lag between two photons of different energy. Normally we replace the first measurement by a proper assumption, based on our knowledge of the emitting object, about the simultaneity of the emission. The second measure will of course involve the detection of photons of comparable energies, say $\mathcal{E}$. Such measurement hence involves
in our framework an extrapolation of the local metric structure of spacetime via an average over metric fluctuations on scales of order $1/\mathcal{E}$. When we do so, we can argue that we are actually doing something more than simply measuring the four momentum components $p_\alpha$ of the photons arriving on Earth. What we are implicitly doing is also detecting the average metric that the photon will have experienced during its travel. More correctly if we assume that the quantum fluctuations of the metric are universal (have the same form at any time and in any place) then we can effectively ignore the quantum interactions of gravitons with photons and simply say that the average effect of the propagation of the photon on a fluctuating background can be described as a photon which has propagated classically on an energy dependent background as determined by the measurement made on Earth. Of course there is a caveat in this reasoning. We know that photons are red shifted in the travel over cosmological distances, hence the photons strictly speaking will not have probed the same effective metric say at the start w.r.t. the end of their trip. In principle a correct calculation would involve taking into account the integrated effect of averaging over the quantum fluctuations over different energies.

### B. Thresholds

Thresholds interactions have also been extensively used \cite{25}. In this case the fact that the $p_\alpha$ scalars are the only relevant quantities is much more clear as any interaction can be though as a measurement process. Moreover it is now obvious in our framework that energy and momentum conservation will have to be imposed on the classical momenta $\pi_\mu$ as these are actually involved in the interaction. However such interaction will now take place on an energy dependent background that will lead to a modified kinematics for the observed momenta $\pi_\alpha$. Studies about threshold reactions in DSR have been carried out (see e.g. \cite{24}) and the main conclusions are that

1. Forbidden reactions like gamma decay or vacuum threshold effect are not allowed in DSR

2. Standard threshold reactions are very mildly modified

We want however to comment about the impossibility in our framework to allow in DSR momenta space reactions usually kinematically forbidden in the “Platonic” ones. This result is related to the fact that if energy momentum conservation is not satisfied in the $\pi$ variables it will not be satisfied in the $p$ ones \footnote{In other words, every solution for threshold equation in the classical space ($\pi$ variables) can be mapped to the real space ($p$) and since the threshold equations in the physical space are maps e.g. of a photon decay would be automatically associated with a preferred system of reference.}. Moreover the possibility

The interesting point is that if some energy could be exchanged between the particle and the gravitational fluctuations then this could off set the energy balance and indeed allow for the reaction to happen even in the $\pi$ and consequently it would introduce a preferred system of reference!

Hence it seems that a crucial requirement for our Ansatz is that energy exchange between gravitational and matter degrees of freedom should be absent or negligible at the energies so far tested. Probably this is the same to say that particles should not be so high energy that they have a back reaction on the background metric.

### C. Synchrotron effect

Synchrotron radiation has also been used \cite{25} to provide constraints. The electrons responsible for such an emission would probe a spacetime averaged on their typical energies. We then expect them to probe similarly an effective, energy dependent, background.

### VII. CONCLUSIONS

In this paper we have explored concrete realizations for the proposal of \cite{9}, on a new interpretation of DSR as a new theory of the measurement. According to this new interpretation, the measured momenta of a particle acquires corrections with respect to the actual four momenta of the particle, due to quantum gravitational effects. Deformed dispersion relations, as DSR ones, appear as a result of this modifications in the measurements \footnote{Note however that, given that we always work at first order, we cannot reconstruct the explicit form of $\mathcal{F}$ and its saturation properties}.

The previous idea is implemented by assuming

- The space time emerges as a consequence of an underlying quantum theory of gravity.
- Once a measurement is performed, the outcome will be the average on the quantum gravitational structure.
- The average shall depend on the characteristic energy of the process measured, as well as on the quantum gravity scale $\kappa$ and other possible properties (detector included) which we denote by $X$.
- The classical structure of the spacetime is recovered when the energies involved satisfies $\mathcal{E} \ll \kappa$. 

from threshold equations in the classical space, it is possible to see that every solution in the classical space is mapped one to one to the physical space, therefore, there are no new reactions.
With this minimal set of assumptions we have showed that for a wide range of models for the effective structure of the underlying spacetime, dispersion relations DSR-like can be obtained.

In all cases studied we have also assumed, as a simplicity criteria, that a) detectors are ideal in the sense that the variable $X$ do not depend on intrinsic characteristics of apparatus and b) we analyze only the one-particle case.

The first model considered is based just on dimensional analysis. The previous requirements enforced us to consider corrections to the flat spacetime metric with the shape $(\mathcal{E} \ell_{pl})$ and from here it is possible to show that the relation between the measured momenta and the particle momenta are DSR2-type. On the base of this result is not only the dependence on $(\mathcal{E} \ell_{pl})$, the tensorial structure of the correction — which was assumed here proportional to $\eta_{\alpha\beta}$ — is crucial for this result.

The natural question is if it is possible to make some statements model independent about the result of the average on quantum contributions. Indeed this is the case as we have shown in section \ref{sec:average} under just a few assumptions. In fact it is possible to give a characterization of the average which, under our assumptions, is strictly related with the kind of fluctuations one considers. This result suggest that it could be possible to identify a specific model of fluctuations with a specific family of DSR dispersion relations, and then, a dispersion relation ruled out by experiments would force to discard a specific model of quantum fluctuations for the gravitational field (at least under the assumptions discussed in this paper).

In the last model proposed, the basic assumption is that the correction to the metric can be extracted from the linearized Einstein equations. The main point in this approach is that, in spite of the fact that Einstein equations cannot be assumed as fundamental at the Planck scale, they provide a starting point to construct the perturbations on the metric. The additional information — what makes this approach different from \cite{21} — comes from the average which, at this point, is constructed making a few (reasonable) physical assumptions. In any case, this can not be considered as a problem since our goal is not to obtain a precise definition of the average from first principles but to show how the proposal of \cite{9} could work.

In fact, as we pointed out in that section, the sense of the equation \cite{21} is not clear unless an explicit model for fluctuations is given through, for example, a distribution function. However, what are we saying is that if we assume an ergodic condition — which is an assumption on the nature of these quantum fluctuations — and also that the average gives an effective value for the energy momentum tensor, then we can approximate the solution of the equation by a Newtonian type potential.

In spite of the limitations of the model, we already see how the fluctuations of the spacetime itself appear, independently of the content of matter and all the dependence on the energy of the particle that probes the space time, comes from the definition of average. The final result is a metric that depends on this energy.

In conclusion, these examples show a concrete implementation of the ideas of \cite{9} with just a few assumptions on the structure of the averages. Note however, that in all of them, what we have had to do is to argue on the result of the average and not on the structure itself of the spacetime. In other words, these examples show the compatibility of this approach with the interpretation of DSR proposed. Explicit models of geometry fluctuations leading to DSR will be discussed in a forthcoming paper \cite{18}.

Finally, the phenomenological issue is addressed. Since we have a link between DSR intended as a deformation of the outcomes of a measurements due to quantum gravitational effects, one could argue that all these modifications arise only when a measurement is done and therefore are extremely local and completely decoupled from the entire history of the particle. We have however argued (for example in the case of the “time of flight” observations) that a careful analysis is needed before to draw any conclusion. In particular if one assumes a universal character for the fluctuations then our assumption that the average depend on the characteristic energy of the process — that is, the distance scale for the average is of the order of $1/E$ — is everywhere valid. If, for the contrary, the fluctuations are not universal, the average should contain this information also. At the end this is then equivalent to replace the metric of the space time with an energy dependent metric, which plays the role of the effective metric probed by the particle. In this sense most of the constraints already discussed on DSR theories can be recovered also in our framework.

Acknowledgements

SL would like to thank S. Sonego for illuminating discussions. FM thanks INFN for postdoctoral fellowship and MECESUP, grant USA108.

\begin{thebibliography}{99}
\bibitem{1} G. Amelino-Camelia, Int. J. Mod. Phys. D11, 35–59 (2002) \texttt{arXiv:gr-qc/0012051}; Int. J. Mod. Phys. D 11, 35–59 (2002) \texttt{arXiv:gr-qc/0012238};
\bibitem{2} J. Magueijo and L. Smolin, “Lorentz invariance with fundamental interaction theories”, “Karpacz 2001, New developments in fundamental interaction theories”, 137 (2001) \texttt{arXiv:hep-th/0110004}.
\end{thebibliography}
“Generalized Lorentz invariance with an invariant energy scale,” Phys. Rev. D 67, 044017 (2003) [arXiv:gr-qc/0207085].

G. Amelino-Camelia, Nature 418, 34 (2002).

D.V. Ahluwalia-Khalilova, Int. J. Mod. Phys. D13, 335 (2004).

N. R. Bruno, G. Amelino-Camelia and J. Kowalski-Glikman, Phys. Lett. B522, 133 (2001).

A. A. Deriglazov, Phys. Lett. B603, 124 (2004); S. Mignemi, [arXiv: gr-qc/0403038]; S. Gao and Xiao-ning Wu, [arXiv: gr-qc/0311009]; D. Kimberly, J. Magueijo and J. Medeiros, Phys. Rev. D70, 084007 (2004); R. Aloisio, A. Galante, A. Grillo, E. Luzio and F. Méndez, Phys. Lett. B610, 101 (2005); F. Hinterleitner, Phys. Rev. D71, 025016 (2005).

J. Lukierski and A. Nowicki, Int. J. Mod. Phys. A18, 7 (2003).

S. Judes and M. Visser, “Conservation laws in “doubly special relativity”, Phys. Rev. D 68, 045001 (2003) [arXiv:gr-qc/0205067].

S. Liberati, S. Sonego and M. Visser, “Interpreting doubly special relativity as a modified theory of measurement,” Phys. Rev. D 71, 045001 (2005) [arXiv:gr-qc/0410113].

L. Freidel, J. Kowalski-Glikman and Lee Smolin, Phys. Rev. D69, 044001 (2004); J. Kowalski-Glikman, 3rd International Sakharov Conference on Physics, Moscow, Russia, 24-29 Jun 2002 hep-th/0209264.

J. Magueijo and L. Smolin, “Gravity’s Rainbow,” Class. Quant. Grav. 21, 1725 (2004) [arXiv:gr-qc/0305055].

F. Girelli, E. R. Livine and D. Oriiti, “Deformed special relativity as an effective flat limit of quantum gravity,” Nucl. Phys. B 708, 411 (2005) [arXiv:gr-qc/0406100]; F. Girelli and E. R. Livine, “Physics of deformed special relativity: Relativity principle revisited,” arXiv:gr-qc/0412004; ibidem “Physics of Deformed Special Relativity,” Braz. J. Phys. 35, 432 (2005) [arXiv:gr-qc/0412079].

F. de Felice and C. J. S. Clarke, Relativity on Curved Manifolds (Cambridge, Cambridge University Press, 1990).

C. Barcelo, S. Liberati and M. Visser, “Analog gravity from Bose-Einstein condensates,” Class. Quant. Grav. 18, 1137 (2001) [arXiv:gr-qc/0011026].

C. J. Pethick and H. Smith, Bose-Einstein condensation in dilute gases, Cambridge University Press, (2001).

C. Barcelo, S. Liberati and M. Visser, “Analog gravity,” [arXiv:gr-qc/0505065].

L. Smolin, “Falsifiable predictions from semiclassical quantum gravity,” arXiv:hep-th/0501091.

R. Aloisio, A. Galante, A. Grillo, E. Luzio and F. Méndez, in preparation.

C. Barcelo, S. Liberati and M. Visser, “Probing semiclassical analogue gravity in Bose–Einstein condensates with widely tunable interactions,” Phys. Rev. A 68, 053613 (2003) [arXiv:cond-mat/0307491].

R. M. Wald, General Relativity, The University of Chicago Press. (1984).

R. Isaacson, Phys. Rev. 166, 1263 (1968); T. W. Noonan, Gen. Rel. Grav. 16, 1103 (1984); ibid, 17 (1985); N. Zotov and W. Stoeger, Class. Quant. Grav. 9, 1023 (1992); W. Stoeger, A. Helmi and D. Torres, arXiv:gr-qc/ 9904020.

S. Basu and D. Mattingly, “Constraints from cosmic rays on non-systematic Lorentz violation,” arXiv:astro-ph/0501425.

R. Aloisio, P. Blasi, P. L. Ghia and A. F. Grillo, “Probing the structure of space-time with cosmic rays,” Phys. Rev. D 62, 053010 (2000) [arXiv:astro-ph/0001258]; T. Jacobson, S. Liberati and D. Mattingly, “Threshold effects and Planck scale Lorentz violation: Combined constraints from high energy astrophysics,” Phys. Rev. D 67, 124011 (2003) [arXiv:hep-ph/0209264].

T. Jacobson, S. Liberati and D. Mattingly, “Lorentz violation at high energy: Concepts, phenomena and astrophysical constraints,” arXiv:astro-ph/0505267; Dan Heyman, Franz Hinteleitner and Seth Major, Phys. Rev. D69, 105016 (2004).

D. Heyman, F. Hinteletner and S. Major, “On reaction thresholds in doubly special relativity,” Phys. Rev. D 69, 105016 (2004) [arXiv:gr-qc/0312089].

T. Jacobson, S. Liberati and D. Mattingly, “Lorentz violation and Crab synchrotron emission: A new constraint far beyond the Planck scale,” Nature 424, 1019 (2003) [arXiv:astro-ph/0212190].