Dynamically Stable Infinite-Width Limits of Neural Classifiers

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Abstract

Recent research has been focused on two different approaches to studying neural networks training in the limit of infinite width: (1) a mean-field (MF) and (2) a constant neural tangent kernel (NTK) approximations. These two approaches have different scaling of hyperparameters with a width of a network layer and as a result different infinite-width limit models. We propose a general framework to study how the limit behavior of neural models depends on the scaling of hyperparameters with a network width. Our framework allows us to derive scaling for existing MF and NTK limits, as well as an uncountable number of other scalings that lead to a dynamically stable limit behavior of corresponding models. However, only a finite number of distinct limit models are induced by these scalings. Each distinct limit model corresponds to a unique combination of such properties as boundedness of logits and tangent kernels at initialization or stationarity of tangent kernels. Existing MF and NTK limit models, as well as one novel limit model satisfy most of the properties demonstrated by finite-width models. We also propose a novel initialization-corrected mean-field limit that satisfies all properties noted above, and its corresponding model is a simple modification for a finite-width model. Source code to reproduce all the reported results is available on GitHub.

1 Introduction

For a couple of decades neural networks have proved to be useful in a variety of applications. However, their theoretical understanding is still lacking. Several recent works have tried to simplify the object of study by approximating a training dynamics of a finite-width neural network with its limit counterpart in the limit of a large number of hidden units; we refer it as an "infinite-width" limit. The exact type of the limit training dynamics depends on how hyperparameters of the training dynamics scale with width. In particular, two different types of limit models have been already extensively discussed in the literature: an NTK model [1] and a mean-field limit model [2, 3, 4, 5, 6, 7]. A recent work [8] attempted to provide a link between these two different types of limit models by building a framework for choosing a scaling of hyperparameters that lead to a "well-defined" limit model. Our work is the next step in this direction. Our contributions are following.

1. We develop a framework for reasoning about scaling of hyperparameters, which allows one to infer scaling parameters that allow for a dynamically stable model evolution in the limit of infinite width. This framework allows us to derive both mean-field and NTK limits that have been extensively studied in the literature, as well as the "intermediate limit" introduced in [8].

https://github.com/deepmipt/research/tree/master/Infinite_Width_Limits_of_Neural_Classifiers

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2. Our framework demonstrates that there are only 13 distinct stable model evolution equations in the limit of infinite width that can be induced by scaling hyperparameters of a finite-width model. Each distinct limit model corresponds to a region (two-, one-, or zero-dimensional) of a green band of the Figure 1 left.

3. We consider a list of properties that are satisfied by the evolution of finite-width models, but not generally are for its infinite-width limits. We demonstrate that mean-field and NTK limit models, as well as "sym-default" limit model which was not discussed in the literature previously, are special in the sense that they satisfy most of these properties among all limit models induced by hyperparameter scalings. We propose a model modification that allows for all of these properties in the limit of infinite width and call the corresponding limit "initialization-corrected mean-field limit (IC-MF)".

4. We discuss the ability of limit models to approximate the training dynamics of finite-width ones. We show that our proposed IC-MF limiting model is the best among all other possible limit models.

2 Training a one hidden layer net with SGD

Consider a one hidden layer network:

\[ f(x; a, W) = a^T \phi(W^T x) = \sum_{r=1}^{d} a_r \phi(w_r^T x), \tag{1} \]

where \( x \in \mathbb{R}^d \), \( W = [w_1, \ldots, w_d] \in \mathbb{R}^{d_x \times d} \), and \( a = [a_1, \ldots, a_d]^T \in \mathbb{R}^d \). We assume a nonlinearity to be real analytic and asymptotically linear: \( \phi(z) = \Theta_{z \to \infty}(z) \). Such a nonlinearity can be, e.g. "leaky softplus": \( \phi(z) = \ln(1 + e^z) - \alpha \ln(1 + e^{-z}) \) for \( \alpha > 0 \). This is a technical assumption introduced to ease proofs. For simplicity, we assume the loss function \( \ell(y, z) \) to be the standard binary cross-entropy loss: \( \ell(y, z) = \ln(1 + e^{-yz}) \), where labels \( y \in \{-1, 1\} \). The data distribution loss is defined as \( \mathcal{D}(a, W) = \mathbb{E}_{x, y \sim D} \ell(y, f(x; a, W)) \).
Weights are initialized with isotropic gaussians with zero means: \( w_r^{(0)} \sim \mathcal{N}(0, \sigma_w^2 I) \), \( a_r^{(0)} \sim \mathcal{N}(0, \sigma_a^2) \) \( \forall r = 1 \ldots d \). The evolution of weights is driven by the stochastic gradient descent (SGD):

\[
\Delta \theta^{(k)} = \theta^{(k+1)} - \theta^{(k)} = -\eta \frac{\partial \ell(y_{\theta}^{(k)}, f(x_{\theta}^{(k)}; a, W))}{\partial \theta}, \quad (x_{\theta}^{(k)}, y_{\theta}^{(k)}) \sim \mathcal{D},
\]

where \( \theta \) is either \( a \) or \( W \). We assume that gradients for \( a \) and \( W \) are estimated using independent data samples \( (x_a^{(k)}, y_a^{(k)}) \) and \( (x_w^{(k)}, y_w^{(k)}) \). We introduce this assumption for the ease of proofs. Note that corresponding stochastic gradients still give unbiased estimates for true gradients. Define:

\[
\hat{a}_r^{(k)} = \frac{a_r^{(k)}}{\sigma_a}, \quad \hat{w}_r^{(k)} = \frac{w_r^{(k)}}{\sigma_w}; \quad \hat{\eta}_a = \frac{\eta_a}{\sigma_a^2}, \quad \hat{\eta}_w = \frac{\eta_w}{\sigma_w^2}.
\]

Then the dynamics transforms to:

\[
\Delta \hat{\theta}_r^{(k)} = \hat{\eta}_r \frac{\partial \ell(y_{\theta}^{(k)}, f(x_{\theta}^{(k)}; \sigma_a \hat{a}, \sigma_w \hat{W}))}{\partial \hat{\theta}_r}, \quad (x_{\theta}^{(k)}, y_{\theta}^{(k)}) \sim \mathcal{D},
\]

while scaled initial conditions become: \( \hat{a}_r^{(0)} \sim \mathcal{N}(0, 1) \), \( \hat{w}_r^{(0)} \sim \mathcal{N}(0, I) \) \( \forall r = 1 \ldots d \).

By expanding gradients, we get the following:

\[
\Delta \hat{a}_r^{(k)} = -\hat{\eta}_a \sigma_a \nabla_{\hat{f}_d} \hat{f}_d(x_{a}^{(k)}, y_{a}^{(k)}) \phi(\sigma_w \hat{w}_r^{(k)} T x_a^{(k)}) \hat{a}_r^{(k)}, \quad \hat{a}_r^{(0)} \sim \mathcal{N}(0, 1),
\]

\[
\Delta \hat{w}_r^{(k)} = -\hat{\eta}_w \sigma_a \sigma_w \nabla_{\hat{f}_d} \hat{f}_d(x_{w}^{(k)}, y_{w}^{(k)}) \hat{a}_r^{(k)} \phi'(\ldots) x_{w}^{(k)}, \quad \hat{w}_r^{(0)} \sim \mathcal{N}(0, I),
\]

\[
\nabla_{\hat{f}_d} \hat{f}_d(x, y) = \left. \frac{\partial \ell(y, z)}{\partial z} \right|_{z=\hat{f}_d(x)} = \frac{-y}{1 + \exp(\hat{f}_d(x)y)}, \quad f_d^{(k)}(x) = \sigma_a \sum_{r=1}^{d} \hat{a}_r^{(k)} \phi(\sigma_w \hat{w}_r^{(k)} T x).
\]

Without loss of generality assume \( \sigma_w = 1 \) (we can rescale inputs \( x \) otherwise). We shall omit a subscript of \( \sigma_a \) from now on. Assume hyperparameters that drive the dynamics obey power-law dependence on \( d \):

\[
\sigma(d) = \sigma^* d^{\alpha}, \quad \hat{\eta}_a(d) = \hat{\eta}_a^* d^{\beta_a}, \quad \hat{\eta}_w(d) = \hat{\eta}_w^* d^{\beta_w}.
\]

This assumption is quite natural. Indeed, for He initialization \( [9] \) commonly used in practice \( \sigma \propto d^{-1/2} \), while we keep learning rates in the original parameterization constant while chancing the width by default: \( \eta_a/w = \text{const} \), which implies \( \hat{\eta}_a \propto d \) and \( \hat{\eta}_w \propto d^0 \). On the other hand, NTK scaling \( [10] \) requires scaled learning rates to be constants: \( \hat{\eta}_a/w \propto d^0 \). Here and then we write “\( a/w \)” meaning “\( a \) or \( w \)”.

Since \( \phi \) is smooth, we have:

\[
\Delta f_d^{(k)}(x) = f_d^{(k+1)}(x) - f_d^{(k)}(x) = \sum_{r=1}^{d} \frac{\partial f_d(x)}{\partial \theta_r} \Delta \hat{\theta}_r^{(k)} + O(\hat{\eta}_a^* \hat{\eta}_w^* + \hat{\eta}_w^*),
\]

\[
= -\hat{\eta}_a^* \nabla_{f_d} f_d(x_{a}^{(k)}, y_{a}^{(k)}) K_{a,d}(x, x_{a}^{(k)}) - \hat{\eta}_w^* \nabla_{f_d} f_d(x_{w}^{(k)}, y_{w}^{(k)}) K_{w,d}(x, x_{w}^{(k)}) + O(\hat{\eta}_a^* \hat{\eta}_w^* + \hat{\eta}_w^*),
\]

where we have defined kernels:

\[
K_{a,d}(x, x') = d^{\beta_a} \sigma^2 \sum_{r=1}^{d} \phi(\hat{w}_r^{(k)} T x) \phi(\hat{w}_r^{(k)} T x'),
\]

\[
K_{w,d}(x, x') = d^{\beta_w} \sigma^2 \sum_{r=1}^{d} \hat{a}_r^{(k)} \phi'(\hat{w}_r^{(k)} T x) \phi'(\hat{w}_r^{(k)} T x') x^T x'.
\]

Define the following quantity:

\[
\Delta f_d^{(k)}(x) = \frac{\partial \Delta f_d^{(k)}(x)}{\partial \hat{\eta}_d} \Bigg|_{\hat{\eta}_d=0} = -\nabla_d f_d(x_{a/w}^{(k)}, y_{a/w}^{(k)}) K_{a,d}(x, x_{a/w}^{(k)}).
\]

3
We use this quantity to rewrite the model increment:

$$\Delta f^{(k)}_d(x) = \tilde{\eta}^*_a \Delta f^{(k),r}_{d,a}(x) + \tilde{\eta}^*_w \Delta f^{(k),r}_{d,w}(x) + O(\tilde{\eta}^*_a \tilde{\eta}^*_w + \tilde{\eta}^*_w)^2).$$

Define $p_{err,d}^{(k)} = \mathcal{P}_{(y,x,y^a_\sigma,y^w_\sigma \in \mathcal{D}(k-1),x^{(k-1)},y^a_\sigma^{(k-1)},y^w_\sigma^{(k-1)})}(y f^{(k)}_d(x) < 0)$ — the probability of giving a wrong answer on the step $k$. Let $k_{term,d} \in \mathbb{N} \cup \{+\infty\}$ be a maximal $k$ such that $\forall k' < k$ $p_{err,d}^{(k')} > 0$. Generally, $k_{term,d}$ depends on hyperparameters, as well as on the data distribution $\mathcal{D}$.

Scaling exponents $(q_\sigma, \tilde{\eta}^*_a, \tilde{\eta}^*_w)$ together with proportionality factors $(\sigma^*, \tilde{\eta}^*_a, \tilde{\eta}^*_w)$ define a limit model $f^{(k)}_\infty(x) = \lim_{d \to \infty} f^{(k)}_d(x)$. We call a model "dynamically stable in the limit of large width" if it satisfies the following condition:

**Condition 1.** $\exists k_{balance} \in \mathbb{N} : \forall k \in [k_{balance}, k_{term,\infty}) \cap \mathbb{N}$ $\tilde{\eta}^*_a f^{(k)}_\infty(x^{(k)}_a) < 0$ and $\tilde{\eta}^*_w f^{(k)}_\infty(x^{(k)}_w) < 0$ imply $\Delta f^{(k),r}_{d,a/w}(x) = \Theta_{d \to \infty}(f^{(k, balance)}_d(x)) x$-a.e. $(\tilde{\eta}^*_a, x^{(k)}_a)$-a.s.

Roughly speaking, this condition states that the change of logits after a single step (when learning rates are sufficiently small) is comparable to logits themselves. This means that the model learns.

Note that this condition is weaker than the one used in [3], because it allows logits to vanish or diverge with width. Such situations are fine, because only logit signs matter for the binary classification.

This condition puts a constraint on exponents $(q_\sigma, \tilde{\eta}^*_a, \tilde{\eta}^*_w)$; this constraint generally depends on the train data distribution $\mathcal{D}$ and on proportionality factors $(\sigma^*, \tilde{\eta}^*_a, \tilde{\eta}^*_w)$. In order to obtain a data-independent hyperparameter-independent constraint, we need the condition above to hold for any value of $k_{term,\infty}$ and any values of $(\sigma^*, \tilde{\eta}^*_a, \tilde{\eta}^*_w)$. Without loss of generality we can assume $k_{term,\infty} = \infty$, which gives the following condition:

**Condition 2 (Dynamical stability).** Given $k_{term,\infty} = +\infty$, $\exists k_{balance} \in \mathbb{N} : \forall \sigma^* > 0$ $\forall \tilde{\eta}^*_a/w > 0$ $\forall k \geq k_{balance}$ $\tilde{\eta}^*_a f^{(k)}_\infty(x^{(k)}_a) < 0$ and $\tilde{\eta}^*_w f^{(k)}_\infty(x^{(k)}_w) < 0$ imply $\Delta f^{(k),r}_{d,a/w}(x) = \Theta_{d \to \infty}(f^{(k, balance)}_d(x)) x$-a.e. $(\tilde{\eta}^*_a, x^{(k)}_a)$-a.s.

For simplicity assume $\tilde{q}_a = \tilde{q}_w = \tilde{q}$. We prove the following in SM [A.1]

**Proposition 1.** Suppose $\tilde{q}_a = \tilde{q}_w = \tilde{q}$ and $\mathcal{D}$ is a continuous distribution. Then Condition 2 requires $q_\sigma + \tilde{q} \in [-1/2, 0]$ to hold.

This statement gives a necessary condition for growth rates of $\sigma$ and $\tilde{\eta}$ to lead to a well-defined limit model evolution. This condition corresponds to a band in $(q_\sigma, \tilde{q})$-plane: see Figure 1 left. We refer it as a "band of dynamical stability".

Each point of this band corresponds to a dynamically stable limit model evolution. We present several conditions that separate the dynamical stability band into regions. We then show that each region corresponds to a single limit model evolution.

**Condition 3.** Following conditions separate the band of dynamical stability (Figure 7 left):

1. A limit model at initialization is finite: $f^{(0)}_d(x) = \Theta_{d \to \infty}(1)$ $x$-a.e.

2. Tangent kernels at initialization are finite: $K^{(0)}_{d,a/w}(x,x') = \Theta_{d \to \infty}(1)$ $(x,x')$-a.e.

3. Tangent kernels and a limit model are of the same order at initialization: $K^{(0)}_{d,a/w}(x,x') = \Theta_{d \to \infty}(f^{(0)}_d(x)) (x,x')$-a.e.

4. Tangent kernels start to evolve: $\Delta K^{(0),r}_{d,a/w}(x,x') = \Theta_{d \to \infty}(K^{(0)}_{d,a/w}(x,x')) (x,x')$-a.e. and $\Delta K^{(0),r}_{d,a/w}(x,x') = \Theta_{d \to \infty}(K^{(0)}_{d,a/w}(x,x')) (x,x')$-a.e.

Here kernel increments $\Delta K^{(k),r}_{d,a/w}$ are defined similarly to model increments $\Delta f^{(k),r}_{d,a/w}$:

$$\Delta K^{(k),r}_{d,a/w}(x,x') = \frac{\partial (K^{(k+1)}_{d,a/w}(x,x') - K^{(k)}_{d,a/w}(x,x'))}{\partial \tilde{\eta}^*_a/w} \mid \tilde{\eta}^*_a = 0, \tilde{\eta}^*_w = 0.$$
The other kernel increment $\Delta K_{a,w}^{(k)}$, is defined similarly; see SM A.2 for explicit formulae.

We prove the following in SM A.2:

**Proposition 2** (Separating conditions). Given Condition 1 reads as, point by point:

1. A limit model at initialization is finite: $q_\sigma + 1/2 = 0$.
2. Tangent kernels at initialization are finite: $2q_\sigma + \tilde{q} + 1 = 0$.
3. Tangent kernels and a limit model are of the same order at initialization: $q_\sigma + \tilde{q} + 1/2 = 0$.
4. Tangent kernels start to evolve: $q_\sigma + \tilde{q} = 0$.

We have also checked this Proposition numerically for limit models discussed below; see Figure 1 right. Each condition corresponds to a straight line in the $(q_\sigma, \tilde{q})$-plane: see Figure 1 left. These four lines divide the well-definiteness band into 13 regions: three are two-dimensional, seven are one-dimensional, and three are zero-dimensional. In SM B we show that each region corresponds to a single distinct limit model evolution; we also list corresponding evolution equations. Note that a segment (a one-dimensional region) that corresponds to the Condition 3 exactly coincides with a family of "intermediate scalings" introduced in [8].

### 3 Capturing the behavior of finite-width nets

Note that a typical finite-width model satisfies all four statements of Condition 3 (if we exclude the word "limit" from them). Indeed, neural nets are typically initialized with He initialization [9] that guarantees finite $f_d^{(0)}$ even for large width $d$. Since learning rates of finite nets are finite, the tangent kernels are finite as well. Nevertheless, a neural tangent kernel of a typical finite-width network evolves significantly: [11] have shown that freezing NTK of practical convolutional nets sufficiently reduces their generalization ability; [12] also noticed that evolution of NTK is sufficient for good performance.

Consequently, if we want a limit model to capture the dynamics of a finite-width net, we have to satisfy all four statements of Condition 3. However, as one can see from the Figure 1 we cannot satisfy all of them simultaneously. We say that one limit model captures the behavior of a finite-width one better than the other, if all statements of Conditions 3 satisfied by the latter are satisfied by the former too. If we say that the former dominates the latter in this case then one can easily notice that there are only three "non-dominated" limit models which we discuss in the upcoming section. After that, we will introduce a model modification that allows for a limit satisfying all four statements.

#### 3.1 "Non-dominated" limit models: MF, NTK and "sym-default"

Obviously, the three "non-dominated" limit models are exactly three zero-dimensional regions (points) in Figure 1 left. First suppose statements 1, 2 and 3 hold, hence tangent kernels are constant throughout training (see Figure 1 right). A corresponding point $q_\sigma = -1/2$, $\tilde{q} = 0$ reads as $\sigma \propto d^{-1/2}$ and $\tilde{q} = \text{const}$, which is the case considered in the seminal paper on NTK [1]. The limit dynamics is then given as (see SM B.1.1 and SM B for the general derivation):

$$ f_{ntk,\infty}^{(k+1)}(x) = f_{ntk,\infty}^{(k)}(x) - \eta^* \nabla f_{\infty}^{(k)}(x^{(k)}_a, y^{(k)}_a) K_{a,\infty}^{(0)}(x, x^{(k)}_a) - \eta_w \nabla f_{\infty}^{(k)}(x^{(k)}_w, y^{(k)}_w) K_{w,\infty}^{(0)}(x, x^{(k)}_w), $$

$$ f_{ntk,\infty}^{(0)}(x) \sim \mathcal{N}(0, \sigma^2 \sigma^{(0)}(x)^2)(x), $$

(13)

where $(x^{(k)}_a, y^{(k)}_a) \sim \mathcal{D}$ and limit tangent kernels $K_{a,w,\infty}^{(0)}$ and standard deviations at the initialization $\sigma^{(0)}(x)$ can be calculated along the same lines as in [10].

Next, suppose statements 2 and 4 hold. In this case $K_{a}^{(k)}$ does not coincide with $K_{\infty}^{(0)}$ (see Figure 1 right), hence the dynamics analogous to (13) is not closed. However, the limit dynamics can be expressed as an evolution of a weight-space measure (see [4, 6] for a similar dynamics for the gradient flow, SM B.2.1 and SM B for the general derivation):

$$ \mu_{\infty}^{(k+1)} = \mu_{\infty}^{(k)} + \text{div}(\mu_{\infty}^{(k)} \Delta \delta_{mf}^{(k)}), \quad \mu_{\infty}^{(0)} = \mathcal{N}(0, I_{1+d_w}), $$

(14)
\[ f^{(k)}_{mf,\infty}(x) = \sigma^* \int \hat{\alpha}(\hat{w}^T x) \mu^{(k)}_\infty(d\hat{u}, d\hat{w}), \]  

(15)

where the vector field \( \Delta \theta^{(k)}_{mf} \) is defined as follows:

\[ \Delta \theta^{(k)}_{mf}(\hat{a}, \hat{w}) = -[\nabla^{(k)}_{f_{mf}} f^T(x, a, y_a)^T \phi(\hat{w}^T x) + \nabla^{(k)}_{f_{mf}} f^{(k)}_{sym-def}(x, y_w a^T \phi(\hat{w}^T x x_k) \hat{x}_w^T x_k)^T]^T, \]  

(16)

where we write \( [u, v] \) meaning a concatenation of two row vectors \( u \) and \( v \). Here we have \( q_\sigma = -1 \), \( \tilde{q}_\sigma = 1 \), hence \( \sigma \propto d^{-1} \) and \( \tilde{\eta} \propto d \); this hyperparameter scaling were used in \([4, 6]\). Note that since a measure at the initialization \( \mu^{(0)}_\infty \) has a zero mean, a limit model vanishes at the initialization \( f^{(0)}_{mf,\infty} = 0 \) (see Figure 1 right) thus violating statements 1 and 3 of Condition 5.

Finally, consider a point for which statements 1 and 4 hold: \( q_\sigma = -1/2, \tilde{q}_\sigma = 1/2 \). This situation is very similar to what we call "default" scaling. Consider He initialization \([9]\), typically used in practice: \( \sigma_a \propto d^{-1/2} \) and \( \sigma_w \propto d_k^{-1/2} \). Assume learning rates (in original parameterization) are not modified with width: \( \eta_a = \text{const} \) and \( \eta_w = \text{const} \). This implies \( \eta_a \propto d \) and \( \eta_w \propto 1 \), or \( \tilde{q}_a = 1 \) and \( \tilde{q}_w = 0 \). We refer the scaling \( q_\sigma = -1/2, \tilde{q}_\sigma = 1/2 \) as "sym-default". A limit model evolution for the sym-default scaling looks as follows (see SM [B, 2.2] for an equivalent formulation and SM [D] for the general derivation):

\[ \mu^{(k+1)}_\infty = \mu^{(k)}_\infty + \text{div}(\mu^{(k)}_\infty \Delta \theta^{(k)}_{sym-def}), \quad \mu^{(0)}_\infty = \mathcal{N}(0, I_{1+d_a}), \]  

(17)

\[ f^{(0)}_{sym-def,\infty}(x) \sim \mathcal{N}(0, \sigma^* \sigma^{(0)} \sigma^{(0)}) \mathcal{N}(0, I_{1+d_a}), \]  

(18)

where the vector field \( \Delta \theta^{(k)}_{sym-def} \) is defined similarly to the MF case \([16]\):

\[ \Delta \theta^{(k)}_{sym-def}(\hat{a}, \hat{w}) = -[\nabla^{(k)}_{f_{sym-def}} f^T(x, a, y_a)^T \phi(\hat{w}^T x) + \nabla^{(k)}_{f_{sym-def}} f^{(k)}_{sym-def}(x, y_w a^T \phi(\hat{w}^T x x_k) \hat{x}_w^T x_k) \hat{x}_w^T x_k)^T]^T, \]

As we show in SM [C] the default scaling leads to an almost similar limit dynamics as the sym-default scaling. The quantity \( z^{(k)}_{sym-def,\infty} \) should be perceived as a sign of \( f^{(k)}_{sym-def,\infty} = \sigma^* \lim_{d \to \infty} \left. d^{q_\sigma+1} \int \hat{\alpha}(\hat{w}^T x) \mu^{(k)}_d(d\hat{u}, d\hat{w}) \right| \). The reason why we have to switch from logits to their signs is that the limit model diverges for \( k \geq 1 \): \( \lim_{d \to \infty} f^{(k)}_{d,\infty}(x) = \infty \). Nevertheless the gradient of the cross-entropy loss is well-defined even for infinite logits; it just degenerates into the gradient of a hinge-type loss: \( \lim_{d \to \infty} x z^{(k)}_{sym-def,\infty}(x) = -x \). For this reason, we redefine the loss gradient for \( k \geq 1 \) in terms of logit signs: \( \nabla^{(k)}_{f_{sym-def}} f(x, y) = -y z^{(k)}_{sym-def,\infty}(x) \). Note that besides of the fact that logits diverge in the limit of large width, the measure in the parameter space \( \mu^{(k)}_d \) stays well-defined.

### 3.2 Initialization-corrected mean-field (IC-MF) limit

Here we propose a dynamics that satisfy all four statements of Condition 5. We then show how to modify the network training for the finite width in order to ensure that in the limit of the infinite width its training dynamics converge to the proposed limit one. Consider the following:

\[ \mu^{(k+1)}_\infty = \mu^{(k)}_\infty + \text{div}(\mu^{(k)}_\infty \Delta \theta^{(k)}_{icmf}), \quad \mu^{(0)}_\infty = \mathcal{N}(0, I_{1+d_a}), \]  

(19)

\[ f^{(k)}_{icmf,\infty}(x) = \sigma^* \int \hat{\alpha}(\hat{w}^T x) \mu^{(k)}_\infty(d\hat{u}, d\hat{w}) + f^{(0)}_{icmf,\infty}(x), \]  

(20)

where \( f^{(0)}_{icmf,\infty} \) is defined similarly to above:

\[ f^{(0)}_{icmf,\infty}(x) \sim \mathcal{N}(0, \sigma^* \sigma^{(0)} \sigma^{(0)}) \mathcal{N}(0, I_{1+d_a}), \]  

(21)

the vector field \( \Delta \theta^{(k)}_{icmf} \) is defined analogously to the mean-field case:

\[ \Delta \theta^{(k)}_{icmf}(\hat{a}, \hat{w}) = -[\nabla^{(k)}_{f_{icmf}} f^T(x, a, y_a)^T \phi(\hat{w}^T x) + \nabla^{(k)}_{f_{icmf}} f^{(k)}_{icmf}(x, y_w a^T \phi(\hat{w}^T x x_k) \hat{x}_w^T x_k) \hat{x}_w^T x_k)^T]^T, \]  

(22)
The only difference between this dynamics and the mean-field dynamics is a bias term \( f_{\text{ntk}, \infty}^{(0)} \) in the definition of logits. This bias term does not depend on \( k \) and stays finite for large \( d \); it ensures Condition 3-1 to hold. As for Condition 3-4, tangent kernels evolve with \( k \) simply because the measure \( \mu_{\infty}^{(k)} \) evolves with \( k \) similarly to the mean-field case (see Figure 4 right). Indeed,

\[
K_{w}^{(k)}(x', x) = \sigma^* \cdot 2 \int [\hat{a}^{(k)}]^{2} \phi'(\hat{w}^{(k)}, T) x \phi'(\hat{w}^{(k)}, T) x' \mu_{\infty}^{(k)}(d\hat{a}, d\hat{w}),
\]

and the limit of \( K_{a,d}^{(k)} \) is written in a similar way. Kernels at initialization \( K_{a,w}^{(0)} \) are finite due to the Law of Large Numbers (Condition 3-2); this, and the finiteness of \( f_{\text{ntk}}^{(0)} \) ensures Condition 3-3. As we show in SM [the dynamics] (24) is a limit for the GD dynamics of the following model with learning rates \( \tilde{\eta}_{a,w} = \tilde{\eta}_{a,w}^{*} d^{1} \):

\[
f_{\text{kmf}, d}(x; \hat{a}, \hat{W}) = \sigma^* d^{-1} \sum_{r=1}^{d} \tilde{a}_{r} \phi(\hat{w}_{r}^{(k)} x) + \sigma^* (d^{-1/2} - d^{-1}) \sum_{r=1}^{d} \tilde{a}_{r}^{(0)} \phi(\hat{w}_{r}^{(0)}(T) x).
\]

The reason for using a factor \((d^{-1/2} - d^{-1})\) before the second term instead of \(d^{-1/2}\) will be made clear in the next Section. Note that this does not alter the limit.

3.3 Experiments

Consider a network of width \( d^* \) initialized with a standard deviation \( \sigma^* \) and trained with learning rates \( \tilde{\eta}_{a,w}^{*} \). We call this model a "reference". Consider a family of models indexed by a width \( d \) initialized with a standard deviation \( \sigma(d) \) and trained with learning rates \( \tilde{\eta}_{a,w}(d) \) with following properties: (1) \( \sigma(d^*) = \sigma^* \), \( \tilde{\eta}_{a,w}(d^*) = \tilde{\eta}_{a,w}^{*} \), (2) \( \sigma(d) = \Theta_{d \to \infty}(d^*) \), \( \tilde{\eta}_{a,w}(d) = \Theta_{d \to \infty}(d^*) \) for some pre-defined scaling exponents \((q_{r}, \eta)\).

The first property ensures that a model of width \( d^* \) coincides with the reference model, while the second property ensures that a model converges to a limiting model defined by corresponding scaling parameters. Additionally assume \( \sigma(d) \propto d^{q_{r}}, \tilde{\eta}_{a,w}(d) \propto d^{\eta} \). This ensures that a model of the reference width at the initialization \( f_{d^*}^{(0)} \) provides an unbiased estimate for a limit model at the initialization \( f_{\text{ntk}}^{(0)} \), as well as kernels at the initialization \( K_{a,w,d^*}^{(0)} \) provide unbiased estimates for limit kernels \( K_{a,w,\infty}^{(0)} \). Given this, we slightly abuse the notation and consider \( \sigma(d) = \sigma^* (d/d^*)^{q_{r}} \) and \( \tilde{\eta}_{a,w}(d) = \tilde{\eta}_{a,w}^{*} (d/d^*)^{\eta} \). Here we note that the model (24) does not alter the limit behavior as \( d \to \infty \), at the same time ensuring that the model for \( d = d^* \) coincides with the reference model.

We train a reference network of width \( d^* = 128 \) for the binary classification with a cross-entropy loss on the CIFAR2 dataset (a subset of first two classes of CIFAR10). We track the divergence of a limit network from the reference one using the following quantity:

\[
D_{\text{logits}}(f_{\text{ntk}}^{(k)}(x) \parallel f_{\text{kmf}}^{(k)}(x)),
\]

where

\[
D_{\text{logits}}(\xi \parallel \xi^*) = \text{KL}(\mathcal{N}(\mathbb{E}\xi, \text{Var}\xi) \parallel \mathcal{N}(\mathbb{E}\xi^*, \text{Var}\xi^*)).
\]

Results are shown in Figure 2. As we see, the NTK limit tracks the reference network well only for the first 20 training steps; similar observation has been already made by [10]. At the same time, the mean-field limit starts with a high divergence (since the initial limit model is zero in this case), however, after the 80-th step, it becomes smaller than that of the NTK limit. This can be the implication of non-stationary kernels. As for default case, divergence of logits results in blow-up of the KL-divergence.

The best overall case is the proposed IC-MF limit, which retains the small KL-divergence related to the reference model throughout the training process.

4 Related work

A pioneering work of [11] have shown that a gradient descent training of a neural net can be viewed as a kernel gradient descent in the space of predictors. The corresponding kernel is called a neural
Figure 2: Initialization-corrected mean-field (IC-MF) limit captures the behavior of a given finite-width network best among other limit models. We plot a KL-divergence of logits of different infinite-width limits of a fixed finite-width reference model relative to logits of this reference model. Setup: we train a one hidden layer network with SGD on CIFAR2 dataset; see SM for details. KL-divergences are estimated using gaussian fits with 10 samples.

tangent kernel (NTK). Generally, NTK is random and non-stationary, however have shown that in the limit of infinite width it becomes constant given a network is parameterized appropriately. In this case the evolution of the model is determined by this constant kernel; see eq. (13). The training regime when NTK is hardly varying is coined as "lazy training", as opposed to the "rich" training regime, when NTK evolves significantly. While being theoretically appealing, "laziness" assumption turns out to have a number of limitations in explaining the success of deep learning.

Another line of works considers the evolution of weights as an evolution of a weight-space measure, similar to eq. (14). This weight-space measure becomes deterministic in the limit of infinite width, given the network is parameterized appropriately; the corresponding limit dynamics is called "mean-field". Note that the parameterization required here for the convergence to a limit dynamics differs from the one used in NTK literature.

Our framework for reasoning about scaling of hyperparameters is very similar in spirit to the one used in [8]. However, there are several crucial differences. First, we do not consider weight increments, as well as a model decomposition, and do not try to estimate exponents of the former and for terms of the latter, which arguably complicates the work of [8]. Instead, we present derivations in terms of the limit behavior of logits and kernels which appears to be simpler and clearer. Second, we do not assume constancy of $\nabla f\ell$. Third, our criterion of "dynamical stability" of scaling is weaker compared to the one of [8] and more suitable for classification problems, since it allows for diverging or vanishing logits, as long as they give meaningful classification responses. Note that "intermediate limits" investigated in [8] exactly correspond to limit models which satisfy Condition 3-2. Finally, both "sym-default" and IC-MF limit models we propose in the present work have not been discussed in the work of [8].

5 Conclusions

The current work follows a direction started in [8]: we study how one should scale hyperparameters of a neural network with a single hidden layer in order to converge to a "dynamically stable" limit training dynamics. A weaker dynamical stability condition leads us to a richer class of possible limit models as compared to [8]. In particular, the class of limit models we consider includes a "default" limit model that corresponds to a network with infinitely large number of nodes and finite learning rates in the original parameterization. This "default" limit model does not satisfy a "dynamical stability" condition in [8].

Moreover, we show that the class of limit models that can be achieved by scaling hyperparameters of finite-width nets is finite. The space of hyperparameter scalings is divided by regions with certain conditions on the training dynamics, and each region corresponds to a single limit model. All of these conditions are satisfied by finite-width networks, but cannot be satisfied by limit models all simultaneously. We propose a modification of a finite-width model; the limit of this modification corresponds to a limit model that satisfy all of the conditions mentioned above and tracks the dynamics of a "reference" finite-width net better than other limit models.
**Broader Impact**

This is a theoretical work. For this reason it does not present any foreseeable societal consequence.

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We formulate the following condition of dynamical stability:

\[ \phi \]

where we have defined kernels:

\[ \text{(27)} \]

The training dynamics is given as:

\[ \Delta \hat{a}^{(k)}_r = -\hat{\eta}_a \sigma \nabla_{f_d} \ell(x^{(k)}_a, y^{(k)}_a) \phi(w^{(k)}_r, T x^{(k)}_a), \quad \hat{a}^{(0)}_r \sim \mathcal{N}(0, 1), \]  

\[ \Delta w_r^{(k)} = -\hat{\eta}_w \sigma \nabla_{f_d} \ell(x^{(k)}_w, y^{(k)}_w) \tilde{a}^{(k)}_r \phi'(\tilde{w}_r, T x^{(k)}_w), \quad w_r^{(0)} \sim \mathcal{N}(0, 1) \quad \forall r \in [d], \]

\[ \nabla_{f_d} \ell(x, y) = \frac{\partial \ell(y, z)}{\partial z} \bigg|_{z=f_d^{(k)}(x)} = -\frac{y}{1 + \exp(f_d^{(k)}(x)y)}, \quad f_d^{(k)}(x) = \sigma \sum_{r=1}^d \hat{a}^{(k)}_r \phi'(\tilde{w}_r^{(k)}, T x), \]

where \((x^{(k)}_a, y^{(k)}_a) \sim \mathcal{D}\) for \(\mathcal{D}\) being the data distribution.

We assume hyperparameters to be scaled with width as power-laws:

\[ \sigma(d) = \sigma^* d^\sigma_a, \quad \hat{\eta}_a(d) = \hat{\eta}_a^* d^\hat{\sigma}_a, \quad \hat{\eta}_w(d) = \hat{\eta}_w^* d^\hat{\sigma}_w. \]

Since \(\phi\) is smooth, we have:

\[ \Delta f_d^{(k)}(x) = f_d^{(k+1)}(x) - f_d^{(k)}(x) = \sum_{r=1}^d \frac{\partial f_d(x)}{\partial \theta_r} \Delta \hat{\theta}_r^{(k)} + O(\hat{\eta}_a^* \hat{\eta}_w^* \sigma^* d^\sigma_a), \]

\[ = -\hat{\eta}_a^* \nabla_{f_d} \ell(x^{(k)}_a, y^{(k)}_a) K_{a,d}(x, x^{(k)}_a) - \hat{\eta}_w^* \nabla_{f_d} \ell(x^{(k)}_w, y^{(k)}_w) K_{w,d}(x, x^{(k)}_w) + O(\hat{\eta}_a^* \hat{\eta}_w^* \sigma^* d^\sigma_a), \]

where we have defined kernels:

\[ K_{a,d}(x, x') = d^2 a^2 \sum_{r=1}^d \hat{a}^{(k)}_r \phi'(\tilde{w}_r^{(k)}, T x) \phi'(\tilde{w}_r^{(k)}, T x'), \]

\[ K_{w,d}(x, x') = d^2 w^2 \sum_{r=1}^d \hat{a}^{(k)}_r \phi'(\tilde{w}_r^{(k)}, T x) \phi'(\tilde{w}_r^{(k)}, T x') x^T x'. \]

Define the following quantity:

\[ \Delta f_d^{(k),r}(x) = \frac{\partial \Delta f_d^{(k)}(x)}{\partial \hat{\theta}_r^{(k)}} \bigg|_{\hat{\theta}_r^{(k)}=0} = -\nabla_{f_d} \ell(x^{(k)}_a, y^{(k)}_a) K_{a,d}^{(k)}(x, x^{(k)}_a). \]

We use this quantity to rewrite the model increment:

\[ \Delta f_d^{(k)}(x) = \hat{\eta}_a^* \Delta f_d^{(k),r}(x) + \hat{\eta}_w^* \Delta f_d^{(k),w}(x) + O(\hat{\eta}_a^* \hat{\eta}_w^* \sigma^* d^\sigma_a). \]

Define \(p_{err,d}^{(k)} = \mathbb{P}_{y(x^{(k)}_a, y^{(k)}_a \neq y^{(k)}_a)}^{y}(x^{(k)}_a) \leq 0\) — the probability of giving a wrong answer on the step \(k\). Let \(k_{term,d} \in \mathbb{N} \cup \{+\infty\}\) be a maximal \(k\) such that \(\forall k' < k, p_{err,d}^{(k')} > 0\). Generally, \(k_{term,d}\) depends on hyperparameters, as well as on the data distribution \(\mathcal{D}\).

We formulate the following condition of dynamical stability:

**Condition 4 (Condition 3 restated).** Given \(k_{term,d} = +\infty, k_{balance} \in \mathbb{N} : \forall \sigma^* > 0 \forall \hat{\eta}_a^* > 0 \forall k \geq k_{balance} y_a, f_\infty(x_a^{(k)}) < 0 \text{ and } y_w, f_\infty(x_w^{(k)}) < 0 \text{ imply } \Delta f_d^{(k),r}(x) = \Theta_{d \to \infty}(f_d^{(k),r}(x)) x^T a.e. (y_a^{(k)}, x_a^{(k)}) a.s.
A.1 Proof of Proposition 1

Define:
\[ q_\theta^{(k)} = \inf \{ q : \theta_1^{(k)} = O_{d \to \infty}(d^r) \}, \quad q_{\Delta \theta}^{(k)} = \inf \{ q : \Delta \theta_1^{(k)} = O_{d \to \infty}(d^r) \}, \]

where \( \theta \) should be substituted with \( a \) or \( w \). We define \( \inf(\theta) = +\infty \). We introduce similar definitions for other quantities:

\[ q_f^{(k)}(x) = \inf \{ q : f_d^{(k)}(x) = O_{d \to \infty}(d^r) \}, \quad q_{\nabla f_d}^{(k)}(x, y) = \inf \{ q : \nabla f_d (x, y) = O_{d \to \infty}(d^r) \}, \]

\[ q_{\Delta f_d}^{(k)}(x) = \inf \{ q : \Delta f_d^{(k)}(x) = O_{d \to \infty}(d^r) \}, \quad q_{\Delta f_d,a/w}^{(k)}(x) = \inf \{ q : \Delta f_d^{(k),a/w}(x) = O_{d \to \infty}(d^r) \}. \]

Lemma 1. Assume \( D \) is a continuous distribution. Then following hold:

1. \( \forall k \geq 0 \forall x, y \ q_x^{(k)}(x, y) \leq 0 \), while \( |y f_d^{(k)}(x) < 0 | \) implies \( q_x^{(k)}(x, y) = 0 \).
2. \( q_{a/w}^{(0)} = 0, q_f^{(0)}(x) = q_a + \frac{1}{2} x \)-a.e.
3. \( \forall k \geq 0 q_{\Delta a/\Delta w}^{(k)} = \tilde{q}_{a/w} + q_a + q_{\nabla \theta}^{(k)}(\theta(k), y(k)) (x^{(k)}(x^{(k)}(x^{(k)}(x^{(k)}(y(k)))) - a.s.
4. \( \forall k \geq 0 q_{\Delta f_d,a/w}^{(k)}(x) = 2q_a + 1 + \tilde{q}_{a/w} + 2q_\sigma(a/w) + q_{\nabla f_d}^{(k)}(x^{(k)}(x^{(k)}(x^{(k)}(x)) (x^{(k)}(x^{(k)}(x^{(k)}(x)))) - a.e. \)
5. \( \forall k \geq 0 q_{a/w} + \tilde{q}_{a/w} + q_a(k) \leq 0 \) implies that for sufficiently small \( \tilde{q}_{a/w}^{(k)} \) and \( \tilde{q}_{a/w}^{(k)} \) \( q_{\Delta f_d}^{(k)}(x) = \max(q_{\Delta a/\Delta w}^{(k)}(x), q_{\Delta f_d,a/w}^{(k)}(x)) x-a.e. \)
6. \( \forall k \geq 0 q_{a/w}^{(k+1)} = \max(q_{a/w}^{(k)}, q_{\Delta a/\Delta w}^{(k)}(x), q_{\Delta f_d,a/w}^{(k)}(x)) x-a.e. \)

Proof. (1) follows from the fact that \( \partial\ell(y, z)/\partial z \) is bounded \( \forall y, \) while \( |\partial\ell(y, z)/\partial z| \in [1/2, 1] \) when \( y \leq 0 \). \( \hat{a}_r^{(0)} \sim N(0, 1) \) which is not zero and does not depend on \( d \), hence \( q_{a}^{(0)} = 0 \); similar holds for \( w \). For \( x \neq 0 \) we have \( f_d^{(k)}(x) = \sigma \sum_{r=1}^{d} \hat{a}_r^{(0)} \phi(w_r^{(0)}, T x_a^{(k)}) = \Theta_{d \to \infty}(d^{1/2+q_a}) \) due to the Central Limit Theorem. Hence (2) holds.

Since \( D \) is a.c. wrt Lebesgue measure on \( \mathbb{R}^{1+d_s} \), and \( \phi \) is real analytic and non-zero, \( \phi(w_r^{(k)}, T x_a^{(k)}) \neq 0 \) and \( \phi(w_r^{(0)}, T x_a^{(k)}) \) is well-defined \( (x_a^{(k)}, y_a^{(k)}) \)-a.s. This implies that \( q_{\Delta a/\Delta w}^{(k)} = \tilde{q}_{a/w} + q_a + q_{\nabla \theta}^{(k)}(\theta(k), y(k)) (x_a^{(k)}, y_a^{(k)}) - a.s. \), which is exactly (3).

Consider \( \Delta f_d,a/w \):

\[ \Delta f_d,a/w(x) = -\nabla f_d (x, a/w) K_{a/d}(x, a/w) = -\nabla f_d (x, a/w) d^{\alpha_2} \sum_{r=1}^{d} \phi(w_r^{(k)}, T x_a^{(k)}) \phi(w_r^{(k)}, T x_a^{(k)}). \]

For the same reason as discussed above \( \phi(w_r^{(k)}, T x_a^{(k)}) \neq 0 \) \( (x_a^{(k)}, y_a^{(k)}) \)-a.s., and \( \phi(w_r^{(k)}, T x_a^{(k)}) \neq 0 \) \( x \)-a.e. Since the summands are distributed identically and are generally non-zero, the sum introduces a factor of \( d \) by the law of large numbers. Since \( \phi \) is asymptotically linear, each \( \phi \)-term scales as \( d^{\alpha_2} \). Collecting all terms together, we obtain \( q_{\Delta f_d,a/w}^{(k)}(x) = 2q_a + 1 + \tilde{q}_a + 2q_\sigma + q_{\nabla \theta}^{(k)}(\theta(k), y(k)) \) \( x \)-a.e. \( (x_a^{(k)}, y_a^{(k)}) \)-a.s. Following the same steps for \( \Delta f_d,a/w \), we get (4).
Let us overview $\Delta f_d^{(k)}(x)$ in detail:

$$
\Delta f_d^{(k)}(x) = \sum_{r=1}^{d} \left( \sum_{j=1}^{\infty} \frac{1}{j!} \frac{\partial^j f_d(x)}{\partial \tilde{w}^r_1 \ldots \partial \tilde{w}^r_j} \bigg|_{\tilde{w}_r = \tilde{w}_r^{(k)}, \tilde{\alpha}_r = \tilde{\alpha}_r^{(k)}} \Delta \tilde{w}_r^{(k),ij} \ldots \Delta \tilde{w}_r^{(k),ij} + \sum_{r=1}^{\infty} \frac{1}{j!} \frac{\partial^j f_d(x)}{\partial \tilde{w}_r \partial \tilde{w}^r_1 \ldots \partial \tilde{w}^r_j} \bigg|_{\tilde{w}_r = \tilde{w}_r^{(k)}, \tilde{\alpha}_r = \tilde{\alpha}_r^{(k)}} \Delta \tilde{\alpha}_r \Delta \tilde{w}_r^{(k),ij} \ldots \Delta \tilde{w}_r^{(k),ij} \right) = 
$$

$$
= \sum_{r=1}^{d} \left( \sum_{j=1}^{\infty} \frac{1}{j!} (-1)^j \tilde{\eta}_0^{(k)} \sigma^{j+1} \left( \nabla_{f_d}^{(k)}(x_{w}, y_{w}) \right)^j \times \left( \tilde{\alpha}_r^{(k)} \right)^{j+1} (\phi'(\tilde{w}_r^{(k)}, T x_{w}) \phi(\tilde{w}_r^{(k)}, T x_{w}))^j \times (\phi'(\tilde{w}_r^{(k)}, T x_{w}))^{j-1} \phi(j-1) (\phi'(\tilde{w}_r^{(k)}, T x_{w}))^{j-1} \phi(\tilde{w}_r^{(k)}, T x_{w}) (x_{w}, y_{w})^j - 1 \times \right) \right) \right)
$$

(37)

Assumption $q_{\sigma} + \tilde{q}_{u} + q_{\alpha}^{(k)} \leq 0$ implies $\tilde{\eta}_0^{(k)} \sigma^{j+1} (\tilde{\alpha}_r^{(k)})^{j+1} = O_{d \to \infty}(\tilde{\eta}_0^{(k)} \sigma^{2} (\tilde{\alpha}_r^{(k)})^{2})$ and $\tilde{\eta}_0^{(k)} \sigma^{j+1} (\tilde{\alpha}_r^{(k)})^{j+1} = O_{d \to \infty}(\tilde{\eta}_0^{(k)} \sigma^{2})$.

Since $q_{\sigma}^{(k)}(x, y) \leq 0 \forall x, y$ due to (1), $\left( \nabla_{f_d}^{(k)}(x_{w}, y_{w}) \right)^j = O_{d \to \infty}(\nabla_{f_d}^{(k)}(x_{w}, y_{w}))$ and $\nabla_{f_d}^{(k)}(x_{w}, y_{w})/(\nabla_{f_d}^{(k)}(x_{w}, y_{w}))^{j-1} = O_{d \to \infty}((\nabla_{f_d}^{(k)}(x_{w}, y_{w}))^{j-1})$.

Since $\phi(z) = \Theta_{z \to \infty}(z)$, $\phi'(\tilde{w}_r^{(k)}, T x_{w}) = O_{d \to \infty}(1)$ and $(\phi'(\tilde{w}_r^{(k)}, T x_{w}))^{j} = O_{d \to \infty}(\phi'(\tilde{w}_r^{(k)}, T x_{w}))$ for $j \geq 1$.

Hence for small enough $\tilde{\eta}_0^{(k)}$ and $\tilde{\eta}_0^{(k)}$, the first term of each sum which corresponds to $j = 1$ dominates all others, even in the limit of infinite $d$:

$$
\Delta f_d^{(k)}(x) = - \sum_{r=1}^{d} \left( \tilde{\eta}_0^{(k)} \sigma^{2} \nabla_{f_d}^{(k)}(x_{w}, y_{w})^{2} (\tilde{\alpha}_r^{(k)})^{2} \phi'(\tilde{w}_r^{(k)}, T x_{w}) \phi(\tilde{w}_r^{(k)}, T x_{w}) x_{w}, T x_{w} + \right.

+ \left. \tilde{\eta}_0^{(k)} \sigma^{2} \nabla_{f_d}^{(k)}(x_{w}, y_{w})^{2} (\tilde{\alpha}_r^{(k)})^{2} \phi'(\tilde{w}_r^{(k)}, T x_{w}) \phi(\tilde{w}_r^{(k)}, T x_{w}) x_{w}, T x_{w} \right.

+ \left. o_{d \to \infty} \left( \left( \nabla_{f_d}^{(k)}(x_{w}, y_{w})^{2} (\tilde{\alpha}_r^{(k)})^{2} \sigma^{2} \right) \right) \right)
$$

(38)

Note that two summands depend on $(x_{w}, y_{w})$ and $(x_{w}, y_{w})$, which do not depend on each other. Hence $q_{\Delta f_d^{(k)}(x)} = \max(q_{\Delta f_d^{(k)}(x)}, q_{\Delta f_d^{(k)}(x)})$ x-a.e. $(x_{w}, y_{w})$ a.s., which is (5).

Note that the $o$-term does not alter the exponent. Indeed,

$$
\left( \nabla_{f_d}^{(k)}(x_{w}, y_{w}) \right)^2 = O_{d \to \infty} \left( \left( \nabla_{f_d}^{(k)}(x_{w}, y_{w}) \right)^2 \right)
$$

(39)

One before the last equality holds, because $q_{\Delta f_d^{(k)}(x)}^{(k)} \geq 0$ due to (2) and (6), while the last equality holds due to (4).
By definition we have $\hat{a}_r^{(k+1)} = \hat{a}_r^{(k)} + \Delta a_r^{(k)}$. Since the second term depends on $(x_a^{(k)}, y_a^{(k)})$, while the first term does not, we get $q_a^{(k+1)} = \max(q_a^{(k)}, q_\Delta a^{(k)})$. Similar holds for $\hat{w}_r$ and $f_d^{(k)}(x)$ a.e., which gives (6).

\begin{proof}
Lemma 2. Assume $D$ is a continuous distribution, $k_{\text{term}} = +\infty$ and $\hat{q}_a = \hat{q}_w = \hat{q}$. Then

1. If $q_\sigma + \hat{q} \leq 0$ then for each $k \geq 0$ we have $q_a^{(k)} = 0$ and $q_w^{(k)} = 0$.

2. If $q_\sigma + \hat{q} > 0$ then for each $k \geq 0$ we have $q_a^{(k)} = k(q_\sigma + \hat{q})$ with positive probability wrt $(x_a^{(k)}, y_a^{(k)}, x_w^{(k)}, y_w^{(k)})$.

Next, we will prove an exact result for $q_\Delta$ and $\Delta x$.

Proof. Here and in subsequent proofs we will write "almost surely" meaning "almost surely with a term $w(k)$ for appropriate $k$; we apply a similar shortening for "with positive probability wrt $(x_a^{(k)}, y_a^{(k)}, x_w^{(k)}, y_w^{(k)})$".

If $q_\sigma + \hat{q} \leq 0$ then statements 1, 2, and 3 of Lemma 2 imply $\forall k \geq 0$ $q_a^{(k)} = 0$ a.s.

Assume $q_\sigma + \hat{q} > 0$. We will prove that $\forall k \geq 0$ $q_a^{(k)} = \max(0, k(q_\sigma + \hat{q}))$ with positive probability by induction. Induction base is given by Lemma 2.

Combining the induction assumption and Lemma 3 we get $q_\Delta^{(k)}(x, y) = (k + 1)(q_\sigma + \hat{q})$ with positive probability wrt $(x_a^{(k)}, y_a^{(k)}, x_w^{(k)}, y_w^{(k)})$.

Finally, Lemma 6 concludes the proof of the induction step.

Lemma 3. Assume $D$ is a continuous distribution, $k_{\text{term}} = +\infty$ and $\hat{q}_a = \hat{q}_w = \hat{q}$ and $q_\sigma + \hat{q} \leq 0$. Then $\forall k \geq 0$

1. $y_a^{(k)} f_{d_{r/a}}^{(k)}(x_a^{(k)}) \leq 0$ implies $q_\Delta^{(k)}(x, y) = 2q_\sigma + 1 + \hat{q}$ a.e. wrt $(x_a^{(k)}, y_a^{(k)}, x_w^{(k)}, y_w^{(k)})$.

2. $y_a^{(k)} f_{d_{r/a}}^{(k)}(x_a^{(k)}) < 0$ and $y_w^{(k)} f_{d_{r/a}}^{(k)}(x_w^{(k)}) < 0$ implies $q_\Delta^{(k)}(x, y) = 2q_\sigma + 1 + \hat{q}$ a.e. wrt $(x_a^{(k)}, y_a^{(k)}, x_w^{(k)}, y_w^{(k)})$ for sufficiently small $\hat{q}_a$ and $\hat{q}_w$.

Proof. By Lemma 2 $\forall k \geq 0$ $q_a^{(k)} = 0$ a.s. Since $y_a^{(k)} f_{d_{r/a}}^{(k)}(x_a^{(k)}) < 0$, $q_\Delta^{(k)} = 0$ due to Lemma 1.

Given this, Lemma 4 implies $\forall k \geq 0$ $q_\Delta^{(k)}(x) = 2q_\sigma + 1 + \hat{q}$ a.e. a.s. Hence by virtue of Lemma 5 $\forall k \geq 0$ $q_\Delta^{(k)}(x) = 2q_\sigma + 1 + \hat{q}$ a.e. a.s. for sufficiently small $\hat{q}_a$ and $\hat{q}_w$.

Proposition 3. Suppose $\hat{q}_a = \hat{q}_w = \hat{q}$ and $D$ is a continuous distribution. Then Condition $D$ requires $q_\sigma + \hat{q} \in [-1/2, 0]$ to hold.

Proof. By Lemma 2 if $q_\sigma + \hat{q} > 0$ then $q_a^{(k)} = k(q_\sigma + \hat{q})$ with positive probability. At the same time by virtue of Lemma 1 $k_{\text{term}} = +\infty$ implies $q_\sigma^{(k)} = 0$ with positive probability. Given this, Lemma 4 implies $q_\Delta^{(k)}(x) = q_\sigma + 1 + (2k+1)(q_\sigma + \hat{q})$ a.e. with positive probability. This means that the last quantity cannot be almost surely equal to $q_f^{(k_{\text{balance}})}(x)$ for any $k_{\text{balance}}$ independent on $k$. Since $\Delta f_{d_{r/a}}^{(k)}(x) = \Theta_{d_{r/a}} f_{d_{r/a}}^{(k_{\text{balance}})}(x)$ requires $q_\Delta^{(k)}(x) = q_f^{(k_{\text{balance}})}(x)$, we conclude that Condition $D$ cannot be satisfied if $q_\sigma + \hat{q} > 0$. 

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Hence \( q_\sigma + \tilde{q} \leq 0 \). Then by Lemma 2, \( \forall k \geq 0 \), \( y_a^{(k)} f_\infty^{(k)}(x_a^{(k)}) < 0 \) and \( y_w^{(k)} f_\infty^{(k)}(x_w^{(k)}) < 0 \) imply \( \Delta f_d^{(k)}(x) = 2q_\sigma + 1 + \tilde{q} x \text{-a.e.} \). \( x_a^{(k-1)}, y_a^{(k-1)}, x_w^{(k-1)}, y_w^{(k-1)} \)-a.s. for sufficiently small \( \tilde{n}_w^* \) and \( \tilde{n}_w^* \). We will show that Condition 4 requires \( q_\sigma + \tilde{q} \in [-1/2, 0] \) to hold already for these sufficiently small \( \tilde{n}_w^* \) and \( \tilde{n}_w^* \).

Suppose \( y_a^{(k)} f_\infty^{(k)}(x_a^{(k)}) < 0 \) and \( y_w^{(k)} f_\infty^{(k)}(x_w^{(k)}) < 0 \). Given this, points 1 and 6 of Lemma 1 imply \( \forall k \text{balance} \geq 1 \) \( q_{f_{\text{balance}}}(x) = \max(q_{f_0}(x), 2q_\sigma + 1 + \tilde{q}) = \max(q_\sigma + \frac{1}{2}, 2q_\sigma + 1 + \tilde{q}) \) x-a.e. a.s. Hence \( q_{f_{\text{balance}}}(x) = q_{f_0}(x) \text{-a.e. a.s. if and only if} q_\sigma + \tilde{q} \geq -1/2; \) we can take \( \text{balance} = 1 \) without loss of generality. Having \( q_{\Delta f_d^{(k)}}(x) = q_{f_{\text{balance}}}(x) \) is necessary to have \( \Delta f_{d,a/w}^{(k)}(x) = \Theta_{d \to \infty}(f_{d,a/w}^{(k_{\text{balance}})}(x)) \).

Summing all together, Condition 4 requires \( q_\sigma + \tilde{q} \in [-1/2, 0] \) to hold. \( \square \)

A.2 Proof of Proposition 2

Proposition 4. Let Condition 4 holds; then

1. \( f_d^{(0)}(x) = \Theta_{d \to \infty}(1) \) x-a.e. is equivalent to \( q_\sigma + 1/2 = 0 \).
2. \( K_{d,a/w}^{(0)}(x, x') = \Theta_{d \to \infty}(1) \) \( (x, x') \)-a.e. is equivalent to \( 2q_\sigma + \tilde{q} + 1 = 0 \).
3. \( K_{d,a/w}^{(0)}(x, x') = \Theta_{d \to \infty}(f_d^{(0)}(x)) \) \( (x, x') \)-a.e. is equivalent to \( q_\sigma + \tilde{q} + 1/2 = 0 \).
4. \( \Delta K_{d,a/w}^{(0)}(x, x') = \Theta_{d \to \infty}(K_{d,a}^{(0)}(x, x')) \) \( (x, x') \)-a.e. and \( \Delta K_{d,a/w}^{(0)}(x, x') = \Theta_{d \to \infty}(K_{d,a}^{(0)}(x, x')) \) \( (x, x') \)-a.e. is equivalent to \( q_\sigma + \tilde{q} = 0 \).

Proof. Statement (1) directly follows from Lemma 2:

\[
\frac{d}{d_\sigma^2} \sum_{r=1}^{d} \phi(\tilde{w}_r^{(0)},T) x = \Theta_{d \to \infty}(d_\sigma + 1/2)
\]

Statement (2) follows from the definition of kernels and the Law of Large Numbers:

\[
K_{d,a}^{(0)}(x, x') = 2 \tilde{q} \sigma^2 \sum_{r=1}^{d} \phi(\tilde{w}_r^{(0)},T) x = \Theta_{d \to \infty}(2\tilde{q} \sigma + 1/2)
\]

\( (x, x') \)-a.e.; the same logic holds for the other kernel: \( K_{d,a}^{(0)}(x, x') = \Theta_{d \to \infty}(d_\sigma + 2\tilde{q} \sigma + 1) \) \( (x, x') \)-a.e. Combining derivations of the two previous statements, we get the statement (3). Now we proceed to the last statement. Consider again the kernel \( K_{d,a}^{(0)}; \) its increment is given by:

\[
\Delta K_{d,a}^{(0)}(x, x') = -\tilde{n}_w^* q_\sigma^2 \sigma^3 \sum_{r=1}^{d} \phi(\tilde{w}_r^{(0)},T) x \phi(\tilde{w}_r^{(0)},T) x + \phi(\tilde{w}_r^{(0)},T) x \phi(\tilde{w}_r^{(0)},T) x' \times
\]

\[
\times \nabla T f_d^{(0)}(x, y_w^{(0)}, T) \tilde{n}_w^* q_\sigma^2 \sigma^3 \phi(\tilde{w}_r^{(0)},T) x + \phi(\tilde{w}_r^{(0)},T) x' T x_w^{(0)} + \Theta_{d \to \infty}(\tilde{n}_w^* q_\sigma^2 \sigma^3 \phi(\tilde{w}_r^{(0)},T) x + \phi(\tilde{w}_r^{(0)},T) x' T x_w^{(0)}).
\]

Consider a linear part of this increment with respect to proportionality factors of learning rates:

\[
\Delta K_{d,a}^{(0)}(x, x') = \left. \frac{\partial \Delta K_{d,a}^{(0)}(x, x')}{\partial \tilde{n}_w^*} \right|_{\tilde{n}_w^* = 0} =
\]

\[
= -d_\sigma^2 \sigma^3 \sum_{r=1}^{d} \phi(\tilde{w}_r^{(0)},T) x \phi(\tilde{w}_r^{(0)},T) x + \phi(\tilde{w}_r^{(0)},T) x \phi(\tilde{w}_r^{(0)},T) x' \times
\]

\[
\times \nabla T f_d^{(0)}(x, y_w^{(0)}, T) \tilde{n}_w^* q_\sigma^2 \sigma^3 \phi(\tilde{w}_r^{(0)},T) x + \phi(\tilde{w}_r^{(0)},T) x' T x_w^{(0)},
\]

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Hence $\Delta K_{aw,d}^{(0)},r = \Theta_{d \to \infty}(K_{aw,d}^{(0)})$ is equivalent to $q_r + \tilde{q} = 0$. Considering the second kernel $K_{w,d}^{(k)}$ and its increment is equivalent to the same condition.

### B The number of distinct limit models is finite

It is easy to see that due to the Proposition 3 Condition 3 divides the well-definiteness band into 13 regions. We now show that when proportionality factors $\sigma^*$ and $\tilde{\eta}_{aw}$ are fixed, choosing a limit model evolution is equivalent to picking a single region from these 13.

Indeed, for any width $d$ a model evolution can be written as follows:

$$\Delta f_d^{(k)}(x) = -\tilde{\eta}_{aw}^* \nabla f_d \ell(x(k), y(k)) \left(K_{w,d}(x, x(k)) + O_{\tilde{\eta}_{aw} \to 0} \tilde{\eta}_{aw}^* \Delta K_{w,d}^{(k),r} + \tilde{\eta}_{aw}^* \Delta K_{aw,d}^{(k),r} \right) -$$

$$-\tilde{\eta}_{aw}^* \nabla f_d \ell(x(k), y_d(k)) \left(K_{aw,d}(x, x(k)) + O_{\tilde{\eta}_{aw} \to 0} \tilde{\eta}_{aw}^* \Delta K_{aw,d}^{(k),r} \right).$$

(44)

$$f_d^{(k+1)}(x) = f_d^{(k)}(x) + \Delta f_d^{(k)}(x), \quad \nabla f_d \ell(x, y) = \frac{-y}{1 + \exp(f_d^{(k)}(x)y)},$$

(45)

$$f_d^{(0)}(x) = \sigma^* d^{**} \sum_{r=1}^d \tilde{a}_r^{(0)} \phi(\tilde{w}_r^{(0)}, T x), \quad (\tilde{a}_r^{(0)}, \tilde{w}_r^{(0)}) \sim \mathcal{N}(0, I_{1+d_n}).$$

(46)

Now we introduce normalized kernels:

$$\tilde{K}_{aw,d}^{(k)}(x, x') = d^{-1-\tilde{q}-2q_r} K_{aw,d}^{(k)}(x, x') = \sigma^* d^{-1} \sum_{r=1}^d \phi(\tilde{w}_r^{(k), T} x) \phi(\tilde{w}_r^{(k), T} x'),$$

(47)

$$\tilde{K}_{w,d}^{(k)}(x, x') = d^{-1-\tilde{q}-2q_r} K_{w,d}^{(k)}(x, x') = \sigma^* d^{-1} \sum_{r=1}^d |a_r^{(k)}|^2 \phi(\tilde{w}_r^{(k), T} x) \phi(\tilde{w}_r^{(k), T} x') x^T x'.$$

(48)

Note that after normalization kernels stay finite in the limit of large width due to the Law of Large Numbers. Similarize, we normalize logits, as well as kernel and logit increments:

$$\Delta \tilde{K}_{aw,d}^{(k),r} = d^{-1-\tilde{q}-2q_r} \Delta K_{aw,d}^{(k),r}, \quad \Delta \tilde{f}_d^{(k)} = d^{-1-\tilde{q}-2q_r} \Delta f_d^{(k)}, \quad \tilde{f}_d^{(k)} = d^{-1-\tilde{q}-2q_r} f_d^{(k)}.$$ 

(49)

We then rewrite the model evolution as:

$$\Delta \tilde{f}_d^{(k)}(x) = -\tilde{\eta}_{aw}^* \nabla \tilde{f}_d \ell(x(k), y_d(k)) \left(\tilde{K}_{aw,d}(x, x(k)) + O_{\tilde{\eta}_{aw} \to 0} \tilde{\eta}_{aw}^* \Delta \tilde{K}_{aw,d}^{(k),r} + \tilde{\eta}_{aw}^* \Delta \tilde{K}_{aw,d}^{(k),r} \right) -$$

$$-\tilde{\eta}_{aw}^* \nabla \tilde{f}_d \ell(x(k), y_d(k)) \left(\tilde{K}_{aw,d}(x, x(k)) + O_{\tilde{\eta}_{aw} \to 0} \tilde{\eta}_{aw}^* \Delta \tilde{K}_{aw,d}^{(k),r} \right).$$

(50)

$$\tilde{f}_d^{(k+1)}(x) = \tilde{f}_d^{(k)}(x) + \Delta \tilde{f}_d^{(k)}(x), \quad \forall k \geq 0,$$

(51)

$$\tilde{f}_d^{(0)}(x) = \sigma^* d^{-1-\tilde{q}-q_r} \sum_{r=1}^d \tilde{a}_r^{(0)} \phi(\tilde{w}_r^{(0), T} x), \quad (\tilde{a}_r^{(0)}, \tilde{w}_r^{(0)}) \sim \mathcal{N}(0, I_{1+d_n}),$$

(52)

$$\tilde{f}_d^{(k)}(x) = d^{1+\tilde{q}+2q_r} \tilde{f}_d^{(k)}(x), \quad \nabla \tilde{f}_d \ell(x, y) = \frac{-y}{1 + \exp(\tilde{f}_d^{(k)}(x)y)}, \quad \forall k \geq 0.$$ 

(53)
B.1 Constant normalized kernels case

Kernels $\tilde{K}^{(k)}_{a/w,d}$ are either constants (hence $\Delta \tilde{K}^{(k)}_{a/w,d} \to 0$ as $d \to \infty$) or evolve with $k$ in the limit of large $d$. First assume they are constants; in this case $q_\sigma + \tilde{q} < 0$ due to Proposition 4-4, and

$$\Delta \tilde{f}^{(k)}_d (x) = -\tilde{\eta}_w \nabla f_d \ell(x^{(k)}_w, y^{(k)}_w) \left( \tilde{K}^{(0)}_{w,d}(x, x^{(k)}_w) + o_{d \to \infty}(1) \right) -$$
$$- \tilde{\eta}_a \nabla f_d \ell(x^{(k)}_a, y^{(k)}_a) \left( \tilde{K}^{(0)}_{a,d}(x, x^{(k)}_a) + o_{d \to \infty}(1) \right).$$

(54)

Since normalized kernels $\tilde{K}^{(0)}_{a/w,d}$ converge to non-zero limit kernels $\tilde{K}^{(0)}_{a/w,\infty}$, we can rewrite the formula above as:

$$\Delta \tilde{f}^{(k)}_d (x) = -\tilde{\eta}_w \nabla f_d \ell(x^{(k)}_w, y^{(k)}_w) \left( \tilde{K}^{(0)}_{w,\infty}(x, x^{(k)}_w) + o_{d \to \infty}(1) \right) -$$
$$- \tilde{\eta}_a \nabla f_d \ell(x^{(k)}_a, y^{(k)}_a) \left( \tilde{K}^{(0)}_{a,\infty}(x, x^{(k)}_a) + o_{d \to \infty}(1) \right).$$

(55)

$$\tilde{f}^{(0)}_d (x) = \sigma^* d^{-1/2} - \tilde{q} \sigma (\mathcal{N}(0, \sigma^{(0)}, 2)(x)) + o_{d \to \infty}(1),$$

(56)

where $\sigma^{(0)}(x)$ can be calculated in the same manner as in [10]. As required by Proposition 3, $1/2 + \tilde{q} + q_\sigma \geq 0$, hence $\tilde{f}^{(0)}_d (x) = O_{d \to \infty}(1)$. This implies the following:

$$\nabla \tilde{f}^{(0)}_d (x, y) = \lim_{d \to \infty} \nabla f_d \ell(x, y) =$$

$$= \lim_{d \to \infty} -\frac{\exp(d^{1+\tilde{q} + 2q_\sigma} \tilde{f}^{(0)}_d (x,y))}{1 + \exp(d^{1+\tilde{q} + 2q_\sigma} \tilde{f}^{(0)}_d (x,y))} \left\{ \begin{array}{ll} -y \mathcal{N}(0, \sigma^{(0)}, 2)(x) < 0 \quad & \text{for } 1/2 + q_\sigma > 0; \\
-y & \text{for } 1/2 + q_\sigma = 0; \\
-y/2 & \text{for } 1/2 + q_\sigma < 0. \end{array} \right.$$  

(57)

On the other hand, $\Delta \tilde{f}^{(0)}_d (x) = \Theta_{d \to \infty}(1)$ with positive probability over $(x^{(0)}_a, y^{(0)}_a)$. Hence $\tilde{f}^{(0)}_d = O_{d \to \infty}(\Delta \tilde{f}^{(0)}_d)$ and $\tilde{f}^{(1)}_d = \tilde{f}^{(0)}_d + \Delta \tilde{f}^{(0)}_d = \Theta_{d \to \infty}(1)$. For the same reason, $\tilde{f}^{(k+1)}_d = \tilde{f}^{(k)}_d + \Delta \tilde{f}^{(k)}_d = \Theta_{d \to \infty}(1) \forall k \geq 0$.

This implies the following:

$$\forall k \geq 0 \quad \nabla f^{(k+1)}_d (x, y) = \lim_{d \to \infty} \nabla f^{(k+1)}_d (x, y) = \lim_{d \to \infty} \frac{-y \mathcal{N}(0, \sigma^{(0)}, 2)(x) < 0}{1 + \exp(d^{1+\tilde{q} + 2q_\sigma} \tilde{f}^{(k+1)}_d (x,y))} =$$

$$= \lim_{d \to \infty} \frac{-y \mathcal{N}(0, \sigma^{(0)}, 2)(x) < 0}{1 + \exp(d^{1+\tilde{q} + 2q_\sigma} \tilde{f}^{(k+1)}_d (x,y))} \left\{ \begin{array}{ll} -y \mathcal{N}(0, \sigma^{(0)}, 2)(x) < 0 \quad & \text{for } 1 + \tilde{q} + 2q_\sigma > 0; \\
-y & \text{for } 1 + \tilde{q} + 2q_\sigma = 0; \\
-y/2 & \text{for } 1 + \tilde{q} + 2q_\sigma < 0. \end{array} \right.$$  

(58)

If we define $f^{(k)}_d (x) = \lim_{d \to \infty} \tilde{f}^{(k)}_d (x)$, we get the following limit dynamics:

$$\Delta f^{(k)}_d (x) = -\tilde{\eta}_w \nabla f \ell(x^{(k)}_w, y^{(k)}_w) K^{(0)}_{w,\infty}(x, x^{(k)}_w) - \tilde{\eta}_a \nabla f \ell(x^{(k)}_a, y^{(k)}_a) K^{(0)}_{a,\infty}(x, x^{(k)}_a),$$

(59)

$$K^{(0)}_{a,\infty}(x, x') = \sigma^{*2} E_{w_\sim \mathcal{N}(0, I_{d \times d})} (\phi' (\tilde{w}^T x)) \phi(\tilde{w}^T x'),$$

(60)

$$K^{(0)}_{w,\infty}(x, x') = \sigma^{*2} E_{(\tilde{w}, w) \sim \mathcal{N}(0, I_{1+d \times d})} |\tilde{u}|^2 \phi' (\tilde{w}^T x) \phi(\tilde{w}^T x') x^T x',$$

(61)

$$f^{(k+1)}_d (x) = f^{(k)}_d (x) + \Delta f^{(k)}_d (x), \quad f^{(0)}_d (x) = \left\{ \begin{array}{ll} \mathcal{N}(0, \sigma^{(0)}, 2)(x) \quad & \text{for } 1/2 + \tilde{q} + q_\sigma = 0; \\
0 & \text{for } 1/2 + \tilde{q} + q_\sigma > 0; \end{array} \right.$$  

(62)
If we follow the lines of the previous section, we will get a limit dynamics which is not closed:

\[
\eta \text{ evolve in the limit of large width (at least, for sufficiently small } q\).
\]

Suppose now matters. Since given those mentioned in Proposition 4, points 1, 2 and 3. One can easily notice from Figure 1 (left) that of exponents is equivalent to choosing a limit model. Note that these exponents exactly correspond to those mentioned in Proposition 3 points 1, 2 and 3. One can easily notice from Figure 1 (left) that giving \( q_\alpha + \hat{q} < 0 \), there are 8 distinct sign configurations.

Note also that since we are interested in binary classification problems, only the sign of logits matters. Since \( f^{(k)}_d = d^{1+\hat{q}+2q_\alpha} \hat{f}^{(k)}_d \), signs of \( f^{(k)}_d \) and of \( \hat{f}^{(k)}_d \) are the same for all \( d \). Hence \( \forall x, y \lim_{d \to \infty} \text{sign}(f^{(k)}_d(x)) = \lim_{d \to \infty} \text{sign}(\hat{f}^{(k)}_d(x)) = \text{sign}(f^{(0)}_\infty(x)) \).

### B.1.1 NTK limit model

We state here a special case of the NTK scaling (\( q_\alpha = -1/2, \hat{q} = 0 \), see (1)) explicitly. Since in this case \( 1 + \hat{q} + 2q_\alpha \), we can omit tildas everywhere. This results in the following limit dynamics:

\[
\Delta f^{(k)}_\infty(x) = -\hat{\eta}_\alpha \nabla f^{(k)} \ell(x^{(k)}_\alpha, y^{(k)}_\alpha)K^{(0)}_{\infty,\infty}(x, x^{(k)}_\alpha) - \hat{\eta}_\alpha \nabla f^{(k)} \ell(x^{(k)}_\alpha, y^{(k)}_\alpha)K^{(0)}_{\alpha,\infty}(x, x^{(k)}_\alpha),
\]

\[
K^{(0)}_{\alpha,\infty}(x, x') = \sigma^{+,-2}E_{w \sim N(0, I_d)}(\hat{w}^T x)\hat{w}^T x',
\]

\[
K^{(0)}_{\infty,\infty}(x, x') = \sigma^{+,-2}E_{(\tilde{a}, w) \sim N(0, I_{1+d}a)}(\tilde{a})^2\phi'(\hat{w}^T x)\phi'(\hat{w}^T x')x^T x',
\]

\[
f^{(k+1)}_\infty(x) = f^{(k)}_\infty(x) + \Delta f^{(k)}_\infty(x), \quad f^{(0)}_\infty(x) = \sigma^{+}N(0, \sigma^{(0),2}(x)),
\]

\[
\nabla f^{(k)}_\infty \ell(x, y) = \frac{-y}{1 + \exp(f^{(k)}_\infty(x)y)} \forall k \geq 0.
\]

### B.2 Non-stationary normalized kernels case

Suppose now \( q_\alpha + \hat{q} = 0 \). In this case \( \Delta K^{(0),r}_{\alpha,\infty}(x, x') = \Theta_{d \to \infty}(K^{(0)}_{\alpha,\infty}(x, x')) \) (x, x')-a.e. and \( \Delta K^{(0),r}_{d,\alpha w}(x, x') = \Theta_{d \to \infty}(K^{(0)}_{d,\alpha w}(x, x')) \) (x, x')-a.e. by virtue of the Proposition 4. Hence kernels evolve in the limit of large width (at least, for sufficiently small \( \eta^{(0)}_w \)).

If we follow the lines of the previous section, we will get a limit dynamics which is not closed:

\[
\Delta \hat{f}^{(k)}_\infty(x) = -\hat{\eta}_\alpha \nabla f^{(k)} \ell(x^{(k)}_\alpha, y^{(k)}_\alpha)(\hat{K}^{(k)}_{\infty,\infty}(x, x^{(k)}_\alpha) + \\
+ O_{\hat{\eta}_\alpha \to 0}(\hat{\eta}_\alpha \Delta \hat{K}^{(0),r}_{\infty,\infty}(x, x^{(k)}_\alpha) + \hat{\eta}_\alpha \Delta \hat{K}^{(k),r}_{\alpha,\infty}(x, x^{(k)}_\alpha))) - \\
- \hat{\eta}_\alpha \nabla f^{(k)} \ell(x^{(k)}_\alpha, y^{(k)}_\alpha)(\hat{K}^{(k)}_{\alpha,\infty}(x, x^{(k)}_\alpha) + O_{\hat{\eta}_\alpha \to 0}(\hat{\eta}_\alpha \Delta \hat{K}^{(k),r}_{\alpha,\infty}(x, x^{(k)}_\alpha)))).
\]

\[
\hat{f}^{(k+1)}_\infty(x) = \hat{f}^{(k)}_\infty(x) + \Delta \hat{f}^{(k)}_\infty(x), \quad \hat{f}^{(0)}_\infty(x) = 0,
\]

\[
\nabla \hat{f}^{(k)}_\infty \ell(x, y) = \frac{-y}{1 + \exp(\sigma^{+}N(0, \sigma^{(0),2}(x))y)} \forall k \geq 0.
\]
\[ \nabla_{f_{\infty}}^{(k+1)} f(x, y) = \begin{cases} -y \frac{f_{\infty}^{(k+1)}(x) y < 0}{1 + \exp(f_{\infty}^{(k+1)}(x) y)} & \text{for } 1 + q_\sigma > 0; \\ -y/2 & \text{for } 1 + q_\sigma = 0; \\ 1 - \exp(f_{\infty}^{(k+1)}(x) y) & \text{for } 1 + q_\sigma < 0; \end{cases} \] (73)

The reason for this is non-stationarity of kernels. As a workaround we consider a measure in the weight space:

\[ \mu_d^{(k)} = \frac{1}{d} \sum_{r=1}^{d} \delta_{\hat{a}_r^{(k)}} \otimes \delta_{\hat{w}_r^{(k)}}. \] (74)

Recall the stochastic gradient descent dynamics:

\[ \Delta \hat{a}_r^{(k)} = -\hat{\eta}_w \sigma^* \nabla_{f_d} f(x_a^{(k)}, y_a) \phi(\hat{w}_r^{(k)T} x_a^{(k)}), \quad \hat{a}_r^{(0)} \sim \mathcal{N}(0, 1), \] (75)

\[ \Delta \hat{w}_r^{(k)} = -\hat{\eta}_w \sigma^* \nabla_{f_d} f(x_w^{(k)}, y_w) \hat{a}_r^{(k)} \phi'(\hat{w}_r^{(k)T} x_w^{(k)}) x_r^{(k)}, \quad \hat{w}_r^{(0)} \sim \mathcal{N}(0, I_{d_w}). \] (76)

Here we have replaced \( \hat{\eta}_a/w \) with \( \hat{\eta}_a/w \sigma^* \), because \( q_\sigma + \hat{q} = 0 \). Similar to [4][6], this dynamics can be expressed in terms of the measure defined above:

\[ \mu_d^{(k+1)} = \mu_d^{(k)} + \text{div}(\mu_d^{(k)} \Delta \theta_d^{(k)}), \quad \mu_d^{(0)} = \frac{1}{d} \sum_{r=1}^{d} \delta_{\hat{a}_r^{(0)}}, \quad \hat{\theta}_r^{(0)} \sim \mathcal{N}(0, I_{d_w}), \quad \forall r \in [d], \] (77)

\[ \Delta \theta_d^{(k)}(\hat{a}, \hat{w}) = -[\hat{\eta}_a \sigma^* \nabla_{f_d} f(x_a^{(k)}, y_a) \phi(\hat{w}_r^{(k)T} x_a^{(k)}), \quad \hat{\eta}_w \sigma^* \nabla_{f_d} f(x_w^{(k)}, y_w) \hat{a}_r^{(k)} \phi'(\hat{w}_r^{(k)T} x_w^{(k)}) x_r^{(k)}] \] (78)

\[ f_d^{(k)}(x) = \sigma^* d^{1-q_\sigma} \int \hat{a} \phi(\hat{w}_r^{(k)T} x) \mu_d^{(k)}(d\hat{a}, d\hat{w}), \quad \nabla_{f_d} f(x, y) = \frac{-y}{1 + \exp(f_d^{(k)}(x) y)} \quad \forall k \geq 0. \] (79)

We rewrite the last equation in terms of \( \tilde{f}_d^{(k)}(x) = d^{-1-q_\sigma} f_d^{(k)}(x) \):

\[ \tilde{f}_d^{(k)}(x) = \sigma^* \int \hat{a} \phi(\hat{w}_r^{(k)T} x) \mu_d^{(k)}(d\hat{a}, d\hat{w}), \quad \nabla_{f_d} \tilde{f}_d^{(k)}(x, y) = \frac{-y}{1 + \exp(d^{1-q_\sigma} \tilde{f}_d^{(k)}(x) y)} \quad \forall k \geq 0. \] (80)

This dynamics is closed. Taking the limit \( d \to \infty \) yields:

\[ \mu_{\infty}^{(k+1)} = \mu_{\infty}^{(k)} + \text{div}(\mu_{\infty}^{(k)} \Delta \theta_{\infty}^{(k)}), \quad \mu_{\infty}^{(0)} = \mathcal{N}(0, I_{d_w}), \] (81)

\[ \Delta \theta_{\infty}^{(k)}(\hat{a}, \hat{w}) = -[\hat{\eta}_a \sigma^* \nabla_{f_d} f(x_a^{(k)}, y_a) \phi(\hat{w}_r^{(k)T} x_a^{(k)}), \quad \hat{\eta}_w \sigma^* \nabla_{f_d} f(x_w^{(k)}, y_w) \hat{a}_r^{(k)} \phi'(\hat{w}_r^{(k)T} x_w^{(k)}) x_r^{(k)}] \] (82)

\[ \tilde{f}_{\infty}^{(k)}(x) = \sigma^* \int \hat{a} \phi(\hat{w}_r^{(k)T} x) \mu_d^{(k)}(d\hat{a}, d\hat{w}), \] (83)

\[ \nabla_{f_{\infty}}^{(0)} f(x, y) = \begin{cases} -y \frac{f_{\infty}^{(0)}(x) y < 0}{1 + \exp(f_{\infty}^{(0)}(x) y)} & \text{for } 1/2 + q_\sigma > 0; \\ -y/2 & \text{for } 1/2 + q_\sigma = 0; \end{cases} \] (84)

\[ \nabla_{f_{\infty}}^{(k+1)} f(x, y) = \begin{cases} -y \frac{f_{\infty}^{(k+1)}(x) y < 0}{1 + \exp(f_{\infty}^{(k+1)}(x) y)} & \text{for } 1 + q_\sigma > 0; \\ -y/2 & \text{for } 1 + q_\sigma = 0; \end{cases} \] (85)

Since proportionality factors \( \sigma^* \) and \( \hat{\eta}_w \) are assumed to be fixed, choosing \( q_\sigma \) is sufficient to define the dynamics. Signs of exponents \( 1/2 + q_\sigma \) and \( 1 + q_\sigma \) give 5 distinct limit dynamics. Together with 8 limit dynamics for constant normalized kernels case, this gives 13 distinct limit dynamics, each corresponding to a region in the band of a dynamical stability (Figure left).

As was noted earlier, only the sign of logits matters, and our \( \tilde{f}_d^{(k)} \) preserve the sign for any \( d \): \( \forall x \lim_{d \to \infty} \text{sign}(\tilde{f}_d^{(k)}(x)) = \lim_{d \to \infty} \text{sign}(\tilde{f}_d^{(k)}(x)) = \text{sign}(\tilde{f}_d^{(k)}(x)) \).
B.2.1 MF limit model

We state here a special case of the mean-field scaling ($q_\sigma = -1$, $\tilde{q} = 1$, see [11] or [6]) explicitly. Similar to NTK case, since $1 + q_\sigma = 0$ we can omit tildas. This results in the following limit dynamics:

$$
\mu^{(k+1)}_{\infty} = \mu^{(k)}_{\infty} + \text{div}(\mu^{(k)}_{\infty} \Delta \theta^{(k)}_{\infty}), \quad \mu^{(0)}_{\infty} = \mathcal{N}(0, I_{1+d_k}),
$$

(86)

$$
\Delta \theta^{(k)}_{\infty}(\hat{a}, \hat{w}) = 
= -[\tilde{\eta}_a^s \sigma^* \nabla^{(k)}_{f_{\infty}} f(x_a^{(k)}, y_a^{(k)}) \phi(\hat{w}^T x_a^{(k)}), \quad \eta_w^s \sigma^* \nabla^{(k)}_{f_{\infty}} f(x_w^{(k)}, y_w^{(k)}) \phi(\hat{w}^T x_w^{(k)})] \hat{a},
$$

(87)

$$
\hat{f}^{(k)}_{\infty}(x) = \sigma^* \int \hat{a} \phi(\hat{w}^T x) \mu^{(k)}_{\infty}(d\hat{a}, d\hat{w}),
$$

(90)

$$
\nabla^{(k)}_{f_{\infty}} f(x, y) = -\frac{y}{1 + \exp(f^{(k)}_{\infty}(x)y)}, \quad \forall k \geq 0.
$$

(92)

B.2.2 Sym-default limit model

Another special case which deserves explicit formulation is what we have called a "sym-default" limit model. The corresponding scaling is: $q_\sigma = -1/2$, $\tilde{q} = 1/2$. The resulting limit dynamics is the following:

$$
\mu^{(k+1)}_{\infty} = \mu^{(k)}_{\infty} + \text{div}(\mu^{(k)}_{\infty} \Delta \theta^{(k)}_{\infty}), \quad \mu^{(0)}_{\infty} = \mathcal{N}(0, I_{1+d_k}),
$$

(89)

$$
\Delta \theta^{(k)}_{\infty}(\hat{a}, \hat{w}) = 
= -[\tilde{\eta}_a^s \sigma^* \nabla^{(k)}_{f_{\infty}} f(x_a^{(k)}, y_a^{(k)}) \phi(\hat{w}_x^{T} x_a^{(k)}, \eta_w^s \sigma^* \nabla^{(k)}_{f_{\infty}} f(x_w^{(k)}, y_w^{(k)}) \phi(\hat{w}_x^{T} x_w^{(k)})] \hat{a},
$$

(93)

C Default scaling

Consider the special case of the default scaling: $q_\sigma = -1/2$, $\tilde{q}_a = 1$, $\tilde{q}_w = 0$. Then corresponding dynamics can be written as follows:

$$
\Delta \hat{a}_r^{(k)} = -\tilde{\eta}_a^s \sigma^* d^{1/2} \nabla_{f_a}^{(k)} f(x_a^{(k)}, y_a^{(k)}) \phi(\hat{w}_r^{T} x_a^{(k)}), \quad \hat{a}_r^{(0)} \sim \mathcal{N}(0, 1),
$$

(94)

$$
\Delta \hat{w}_r^{(k)} = -\tilde{\eta}_w^s \sigma^* d^{-1/2} \nabla_{f_w}^{(k)} f(x_w^{(k)}, y_w^{(k)}) \hat{a}_r^{(k)} \phi(\hat{w}_r^{T} x_w^{(k)}) \hat{w}_r^{(k)}, \quad \hat{w}_r^{(0)} \sim \mathcal{N}(0, I_{d_k}),
$$

(95)

$$
\hat{f}_d^{(k)}(x) = \sigma^* d^{-1/2} \sum_{r=1}^{d} \hat{a}_r^{(k)} \phi(\hat{w}_r^{T} x), \quad \nabla_{f_d}^{(k)} f(x, y) = -\frac{y}{1 + \exp(f_d^{(k)}(x)y)}, \quad \forall k \geq 0.
$$

(96)

As one can see, increments of output layer weights $\Delta \hat{a}_r^{(k)}$ diverge with $d$. We introduce their normalized versions: $\Delta \hat{a}_r^{(k)} = d^{-1/2} \Delta \hat{a}_r^{(k)}$. Similarly, we normalize output layer weights themselves: $\hat{a}_r^{(k)} = d^{-1/2} \hat{a}_r^{(k)}$. Then the dynamics transforms to:

$$
\Delta \hat{a}_r^{(k)} = -\tilde{\eta}_a^s \sigma^* \nabla_{f_a}^{(k)} f(x_a^{(k)}, y_a^{(k)}) \phi(\hat{w}_r^{T} x_a^{(k)}), \quad \hat{a}_r^{(0)} \sim \mathcal{N}(0, d^{-1}),
$$

(97)

$$
\Delta \hat{w}_r^{(k)} = -\tilde{\eta}_w^s \sigma^* \nabla_{f_w}^{(k)} f(x_w^{(k)}, y_w^{(k)}) \hat{a}_r^{(k)} \phi(\hat{w}_r^{T} x_w^{(k)}) \hat{w}_r^{(k)}, \quad \hat{w}_r^{(0)} \sim \mathcal{N}(0, I_{d_k}),
$$

(98)
\[ f_d^{(k)}(x) = \sigma^* \sum_{r=1}^{d} \hat{a}^{(k)}_r \phi(\hat{w}^{(k)}_r \cdot x), \quad \nabla f_d^{(k)}(x,y) = \frac{-y}{1 + \exp(f_d^{(k)}(x)y)} \quad \forall k \geq 0. \]  

Similar to SM B.2 we have to introduce a weight-space measure in order to take a limit of \( d \to \infty \):

\[ \mu_d^{(k)} = \frac{1}{d} \sum_{r=1}^{d} \delta_{\hat{a}^{(k)}_r} \otimes \delta_{\hat{w}^{(k)}_r}, \]  

In terms of the measure the dynamics is expressed then as follows:

\[ \mu_d^{(k+1)} = \mu_d^{(k)} + \text{div} (\mu_d^{(k)} \Delta \theta_d^{(k)}), \]  

\[ \Delta \theta_d^{(k)}(\hat{a}, \hat{w}) = \]  

\[ -[\hat{\eta}_a^* \sigma^* \nabla f_d^{(k)}(x^{(k)}, y^{(k)}) \phi(\hat{w}^T x^{(k)}), \hat{\eta}_w^* \sigma^* \nabla f_d^{(k)}(x^{(k)}, y^{(k)}) \phi(\hat{w}^T x^{(k)}) x^{(k)}, T]^T, \]  

\[ f_d^{(k)}(x) = \sigma^* \int \hat{a} \phi(\hat{w}^T x) \mu_d^{(k)}(d \hat{a}, d \hat{w}), \quad \nabla f_d^{(k)}(x,y) = \frac{-y}{1 + \exp(f_d^{(k)}(x)y)} \quad \forall k \geq 0. \]  

We rewrite the last equation in terms of \( \tilde{f}_d^{(k)}(x) = d^{-1} f_d^{(k)}(x) \):

\[ \tilde{f}_d^{(k)}(x) = \sigma^* \int \hat{a} \phi(\hat{w}^T x) \mu_d^{(k)}(d \hat{a}, d \hat{w}), \quad \nabla f_d^{(k)}(x,y) = \frac{-y}{1 + \exp(d f_d^{(k)}(x)y)} \quad \forall k \geq 0. \]  

A limit dynamics then takes the following form:

\[ \mu^{(k+1)}_\infty = \mu^{(k)}_\infty + \text{div} (\mu^{(k)}_\infty \Delta \theta^{(k)}_\infty), \quad \mu^{(0)}_\infty = \delta \otimes \mathcal{N}(0, I_{d_{\alpha}}) \]  

\[ \Delta \theta^{(k)}_\infty(\hat{a}, \hat{w}) = \]  

\[ -[\hat{\eta}_a^* \sigma^* \nabla f_d^{(k)}(x^{(k)}, y^{(k)}) \phi(\hat{w}^T x^{(k)}), \hat{\eta}_w^* \sigma^* \nabla f_d^{(k)}(x^{(k)}, y^{(k)}) \phi(\hat{w}^T x^{(k)}) x^{(k)}, T]^T, \]  

\[ \nabla f^{(k)}(x,y) = \frac{-y}{1 + \exp(\sigma^* \mathcal{N}(0, \sigma^{(0)} x)^2 y)}, \]  

\[ \tilde{f}^{(k)}_\infty(x) = \sigma^* \int \hat{a} \phi(\hat{w}^T x) \mu^{(k)}_\infty(d \hat{a}, d \hat{w}), \quad \nabla \tilde{f}^{(k+1)}_\infty(x,y) = -y[\tilde{f}^{(k+1)}_\infty(x)y < 0] \quad \forall k \geq 0. \]  

As one can notice, the only difference between this limit dynamics and the limit dynamics of sym-default scaling (SM B.2.2) is the initial measure.

We now check the Condition 3. First of all, by the Central Limit Theorem, \( \tilde{f}^{(0)}_d(x) = \Theta_{d \to \infty}(1) \), hence the first point of Condition 3 holds. As for kernels, we have:

\[ \tilde{K}^{(k)}_{a,d}(x,x') = \sigma^* 2 \sum_{r=1}^{d} \phi(\tilde{w}^{(k)}_r \cdot x) \phi(\tilde{w}^{(k)}_r \cdot x'), \]  

\[ \tilde{K}^{(k)}_{w,d}(x,x') = \sigma^* 2 d^{-1} \sum_{r=1}^{d} |\hat{a}^{(k)}_r|^2 \phi(\tilde{w}^{(k)}_r \cdot x) \phi(\tilde{w}^{(k)}_r \cdot x') x^T x'. \]
We see that while $K^{(0)}_{w,d}$ converges to a constant due to the Law of Large Numbers, $K^{(0)}_{a,d}$ diverges as $d \to \infty$. This violates the second statement of Condition 3 and the third as well, since $f^{(0)}_{\infty}$ is finite. Consider now kernel increments:

$$\Delta K_{w,d}^{(k),t}(x, x') = -\sigma^* d^{-1/2} \sum_{r=1}^{d} \left( \phi'((\hat{w}^{(k)}_r)^T x') \phi((\hat{w}^{(k)}_r)^T x) + \phi''((\hat{w}^{(k)}_r)^T x') \right) \times \nabla \ell_f^{(k)}(x', y_{w,d}) (\hat{w}^{(k)}_r)^T(\hat{x}_w, \hat{x}_d) \Delta \hat{w}_r^{(k)}(x) \phi((\hat{w}^{(k)}_r)^T x) (x + x')^T x_k, \quad (112)$$

$$\Delta K_{w,w}^{(k),t}(x, x') = -\sigma^* d^{-3/2} \sum_{r=1}^{d} a_r^{(k)} \phi'((\hat{w}^{(k)}_r)^T x) \phi''((\hat{w}^{(k)}_r)^T x') \times \nabla \ell_f^{(k)}(x, y_{w,w}) (\hat{w}^{(k)}_r)^T(\hat{x}_w, \hat{x}_d) \Delta \hat{w}_r^{(k)}(x) \phi((\hat{w}^{(k)}_r)^T x) (x + x')^T x_k, \quad (113)$$

$$\Delta K_{a,d}^{(k),t}(x, x') = -\sigma^* d^{-1/2} \sum_{r=1}^{d} \Delta \hat{x}_r^{(k)} \phi'((\hat{w}^{(k)}_r)^T x) \phi((\hat{w}^{(k)}_r)^T x) \times \nabla \ell_f^{(k)}(x, y_{w,d}) (\hat{w}^{(k)}_r)^T(\hat{x}_w, \hat{x}_d) \Delta \hat{w}_r^{(k)}(x) \phi((\hat{w}^{(k)}_r)^T x) (x + x')^T x_k. \quad (114)$$

For $k = 0$ terms inside sums of each increment have zero expectations. Hence the Central Limit Theorem can be used here. We get: $\Delta K_{a,d}^{(0),t} = \Theta_{d \to \infty}(1)$, $\Delta K_{w,w}^{(0),t} = \Theta_{d \to \infty}(d^{-1})$, $\Delta K_{a,d}^{(0),t} = \Theta_{d \to \infty}(1)$. Since $K^{(0)}_{a,d} = \Theta_{d \to \infty}(d)$, $K^{(0)}_{w,d} = \Theta_{d \to \infty}(1)$, the last statement of Condition 3 is violated as well.

### D Initialization-corrected mean-field (IC-MF) limit

Here we consider the same training dynamics as for the mean-field scaling (see SM B.2), but with a modified model definition:

$$\Delta \hat{a}_r^{(k)} = -\hat{\eta}_r^{*} \sigma^* \nabla \ell_f^{(k)}(x_r^{(k)}, y_{w,d}) \phi((\hat{w}^{(k)}_r)^T x_r^{(k)}), \quad \hat{a}_r^{(0)} \sim \mathcal{N}(0, 1), \quad (115)$$

$$\Delta \hat{w}_r^{(k)} = -\hat{\eta}_r^{*} \sigma^* \nabla \ell_f^{(k)}(x_w, y_{w,w}) \hat{a}_r^{(k)} \phi'((\hat{w}^{(k)}_r)^T x_r^{(k)}), \quad \hat{w}_r^{(0)} \sim \mathcal{N}(0, I_{d_w}). \quad (116)$$

$$f_d^{(k)}(x) = \sigma^* d^{-1} \sum_{r=1}^{d} \Delta \hat{x}_r^{(k)} \phi((\hat{w}^{(k)}_r)^T x) + \sigma^* d^{-1/2} \sum_{r=1}^{d} \Delta a_r^{(0)} \phi((\hat{w}^{(0)}_r)^T x), \quad (117)$$

$$\nabla \ell_f^{(k)}(x, y) = \frac{-y}{1 + \exp(f_d^{(k)}(x)y)} \quad \forall k \geq 0. \quad (118)$$

Similar to the mean-field case (SM B.2), we rewrite the dynamics above in terms of the weight-space measure:

$$\mu_d^{(k+1)} = \mu_d^{(k)} + \text{div}(\mu_d^{(k)} \Delta \theta_d^{(k)}), \quad \mu_d^{(0)} = 1 \sum_{r=1}^{d} \delta_{\theta_r^{(0)}}, \quad \hat{\theta}_r^{(0)} \sim \mathcal{N}(0, I_{1+d_w}) \quad \forall r \in [d], \quad (119)$$

$$\Delta \theta_d^{(k)}(\hat{a}, \hat{w}) = -[\hat{\eta}_r \sigma^* \nabla \ell_f^{(k)}(x_r^{(k)}, y_{w,d}) \phi((\hat{w}^{(k)}_r)^T x_r^{(k)}), \hat{\eta}_w \sigma^* \nabla \ell_f^{(k)}(x_w, y_{w,w}) \Delta \hat{w}_r^{(k)} \phi'((\hat{w}^{(k)}_r)^T x_r^{(k)}), \Delta \hat{a}_r^{(k)} \phi((\hat{w}^{(k)}_r)^T x_r^{(k)}), \Delta \hat{a}_r^{(k)} \phi((\hat{w}^{(k)}_r)^T x_r^{(k)})], \quad (120)$$

$$f_d^{(k)}(x) = \sigma^* \int \hat{a} \phi(\hat{w}^T x) \mu_d^{(k)}(d\hat{a}, d\hat{w}) + \sigma^* d^{1/2} \int \hat{a} \phi(\hat{w}^T x) \mu_d^{(0)}(d\hat{a}, d\hat{w}), \quad (121)$$
\[ \nabla_{f_d}^{(k)} \ell(x, y) = \frac{-y}{1 + \exp(f_d^{(k)}(x))y} \quad \forall k \geq 0. \]  

(122)

Note that here \( f_d^{(k)} \) stays finite in the limit of \( d \to \infty \) for any \( k \geq 0 \). Hence taking the limit \( d \to \infty \) yields:

\[ \mu^{(k+1)}_\infty = \mu^{(k)}_\infty + \text{div}(\mu^{(k)}_\infty \Delta \theta^{(k)}_\infty), \quad \mu^{(0)}_\infty = N(0, I_{1 + d_a}), \]  

(123)

\[ \Delta \theta^{(k)}_\infty(\hat{\theta}, \hat{w}) = \]

\[ = -[\hat{\eta}^*_a \sigma^* \nabla_{f_\infty}^{(k)} \ell(x_\infty^{(k)}, y_\infty^{(k)}) \phi(\hat{w}^T x_\infty^{(k)}), \hat{\eta}^*_a \sigma^* \nabla_{f_\infty}^{(k)} \ell(x_\infty^{(k)}, y_\infty^{(k)}) \hat{\phi}'(\hat{w}^T x_\infty^{(k)}) x_\infty^{(k)} T], \]  

(124)

\[ f_\infty^{(k)}(x) = \sigma^* \int \hat{\phi}(\hat{w}^T x) \mu_\infty^{(k)}(d\hat{\theta}, d\hat{w}) + \sigma^* N(0, \sigma^{(0)}^2)(x), \]  

(125)

\[ \nabla_{f_\infty}^{(k)} \ell(x, y) = \frac{-y}{1 + \exp(f_\infty^{(k)}(x))y} \quad \forall k \geq 0. \]  

(126)

E Experimental details

We perform our experiments on a feed-forward fully-connected network with a single hidden layer with no biases. We learn our network as a binary classifier on a subset of the CIFAR2 dataset (which is a dataset of first two classes of CIFAR10) of size 1024. We report results using a test set from the same dataset of size 2000. We do not do a hyperparameter search, for this reason we do not use a validation set.

We train our network for 2000 training steps to minimize the binary cross-entropy loss. We use a full-batch GD as an optimization algorithm. We repeat our experiments for 10 random seeds and report mean and deviations in plots for logits and kernels (e.g. Figure 1, left). For plots of the KL-divergence, we use logits from these 10 random seeds to fit a single gaussian. Where necessary, we estimate data expectations (e.g. \( \mathbb{E}_{x \sim D}[f(x)] \)) using 10 samples from the test dataset.

We experiment with other setups (i.e. using a mini-batch gradient estimation instead of exact one, a larger train dataset, a multi-class classification) in SM. All experiments were conducted on a single NVIDIA GeForce GTX 1080 Ti GPU using the PyTorch framework. Our code is available online: [https://github.com/deepmipt/research/tree/master/Infinite_Width_Limits_of_Neural_Classifiers](https://github.com/deepmipt/research/tree/master/Infinite_Width_Limits_of_Neural_Classifiers)

Although our analysis assumes initializing variables with samples from a gaussian, nothing changes if we sample \( \sigma \xi \) instead, where \( \xi \) can be any symmetric random variable with a distribution independent on hyperparameters.

In our experiments, we took a network of width \( d^* = 2^7 = 128 \) and apply the Kaiming He uniform initialization [9] to its layers; we call this network a reference network. According to the Kaiming He initialization strategy, initial weights have a zero mean and a standard deviation \( \sigma^* \propto (d^*)^{-1/2} \) for the output layer, while the standard deviation of the input layer does not depend on the reference width \( d^* \). For this network we take learning rates in the original parameterization \( \eta^*_a = \eta^*_{a/w} = 0.02 \). After that, we scale its initial weights and learning rates with width \( d \) according to a scaling at hand:

\[ \sigma = \sigma^* \left( \frac{d}{d^*} \right)^{q_a}, \quad \hat{\eta}^*_{a/w} = \hat{\eta}^*_a \left( \frac{d}{d^*} \right)^{q_a/w}. \]

Note that we have assumed \( \sigma_w = 1 \). By definition, \( \hat{\eta}^*_a/w = \eta^*_{a/w}/\sigma^2_{a/w} \); this implies:

\[ \eta_a = \eta^*_a \left( \frac{\sigma}{\sigma^*} \right)^2 \left( \frac{d}{d^*} \right)^{q_a^{2q_a}} = \eta^*_a \left( \frac{d}{d^*} \right)^{q_a + 2q_a}, \quad \eta_w = \eta^* \left( \frac{d}{d^*} \right)^{q_w}. \]

\(^2\text{CIFAR10 can be downloaded at [https://www.cs.toronto.edu/~kriz/cifar.html](https://www.cs.toronto.edu/~kriz/cifar.html)}\)
F Experiments for other setups

Although plots provided in the main body represent the full-batch GD on a subset of CIFAR2, we have experimented with other setups as well. In particular, we have varied the batch size and the size of the train dataset. Results are shown in Figures 3-7. Differences are quantitative and marginal.

Figure 3: Test accuracy of different limit models, as well as of the reference model. Setup: We train a one hidden layer network on subsets of the CIFAR2 dataset of different sizes with SGD with varying batch sizes.
Figure 4: Mean kernel diagonals $E_{x \sim D}(\hat{\eta}^* K_{a,d}(x,x) + \hat{\eta}^* w K_{w,d}(x,x))$ of different limit models, as well as of the reference model. **Setup:** We train a one hidden layer network on subsets of the CIFAR2 dataset of different sizes with SGD with varying batch sizes. Data expectations are estimated with 10 test data samples.

Figure 5: Mean absolute logits $E_{x \sim D}|f(x)|$ of different limit models, as well as of the reference model. **Setup:** We train a one hidden layer network on subsets of the CIFAR2 dataset of different sizes with SGD with varying batch sizes. Data expectations are estimated with 10 test data samples.
Figure 6: Mean absolute logits relative to kernel diagonals $\mathbb{E}_{x \sim D}|f_d(x)/(\hat{\eta}^* K_{a,d}(x,x) + \hat{\eta}^* w K_{w,d}(x,x))|$ of different limit models, as well as of the reference model. **Setup:** We train a one hidden layer network on subsets of the CIFAR2 dataset of different sizes with SGD with varying batch sizes. Data expectations are estimated with 10 test data samples.

Figure 7: KL-divergence of different limit models relative to a reference model. **Setup:** We train a one hidden layer network on subsets of the CIFAR2 dataset of different sizes with SGD with varying batch sizes.