LOG MINIMAL MODEL PROGRAM FOR THE MODULI SPACE OF STABLE CURVES OF GENUS THREE

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ABSTRACT. In this paper, we completely work out the log minimal model program for the moduli space of stable curves of genus three. We employ a rational multiple $\alpha \delta$ of the divisor $\delta$ of singular curves as the boundary divisor, construct the log canonical model for the pair $(\mathcal{M}_3, \alpha \delta)$ using geometric invariant theory as we vary $\alpha$ from one to zero, and give a modular interpretation of each log canonical model and the birational maps between them. By using the modular description, we are able to identify all but one log canonical models with existing compactifications of $\mathcal{M}_3$, some new and others classical, while the exception gives a new modular compactification of $\mathcal{M}_3$.

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1. Introduction and Preliminaries

In [Has05], Hassett gave an outline of a general program whose goal is to describe the canonical model
\[
\text{Proj } \bigoplus_{m \geq 0} \Gamma(M_g, mK_M^g)
\]
of the moduli space \(\overline{M}_g\) of stable curves of genus \(g\). The main idea of tackling the problem is to interpolate with the log canonical models
\[
\overline{M}_g(\alpha) := \text{Proj } \bigoplus_{m \geq 0} \Gamma(\overline{M}_g, mK_{\overline{M}_g}^g + \alpha\delta)
\]
where \(\delta\) is the divisor on the moduli stack of the singular curves. We decrease \(\alpha\) from 1 to 0 and try to describe the log canonical models and the relation between the models.

In this article, we carry out the program for \(\overline{M}_3\): We find all log canonical models and give a modular interpretation by realizing them as GIT quotients of suitable Hilbert scheme or Chow variety of curves. By using the modular interpretation, we are also able to identify all but one of the log canonical models with known moduli spaces. The sole exception is, to our knowledge, a new compactification of \(\overline{M}_3\).

By a theorem of Cornalba and Harris, \(K_{\overline{M}_g}^g + \alpha\delta\) is ample for \(9/11 < \alpha \leq 1\) and \(\overline{M}_g(\alpha)\) is isomorphic to \(\overline{M}_g\) for \(\alpha\) in that range. At \(\alpha = 9/11\), it is shown in [HH06] that the locus of elliptic tails gets contracted, resulting the moduli space \(\overline{M}_g^{ps}\) of pseudo-stable curves of Schubert.

Our first main theorem is:

**Theorem 1.**  
(1) There is a small contraction \(\Psi : \overline{M}_3^{ps} \to \overline{M}_3(7/10)\) contracting the locus of elliptic bridges, and \(\overline{M}_3(7/10)\) is isomorphic to the GIT quotient \(\text{Chow}_{3,2}/\!\!/\text{SL}(6)\) of the Chow variety of bicanonical curves;  
(2) There exists a flip \(\Psi^+: (\overline{M}_3^{ps})^+ \to \overline{M}_3(7/10)\), and \((\overline{M}_3^{ps})^+\) is isomorphic to \(\overline{M}_3(\alpha)\) for \(17/28 < \alpha < 7/10\). Moreover, \(\overline{M}_3(\alpha)\) for \(\alpha\) in that range is isomorphic to the GIT quotient \(\text{Hilb}_{3,2}/\!\!/\text{SL}(6)\) of the Hilbert scheme of bicanonical curves;  
(3) There is a divisorial contraction \(\Theta : \overline{M}_3^{hs} \to \overline{M}_3(17/28)\) that contracts the hyperelliptic locus, and the log canonical model is isomorphic to the compact moduli space \(\overline{Q} := \mathbb{P}(\Gamma(\mathcal{O}_{\mathbb{P}^2}(4)))/\!\!/\text{SL}(3)\) of plane quartics.
Here, an *elliptic bridge* of genus three is a curve consisting of two elliptic curves meeting each other in two nodes (Figure 1). Abusing terminology, an elliptic bridge shall sometimes mean an elliptic component meeting the rest of the curve in two nodes.

![Figure 1. A generic elliptic bridge](image)

The second main theorem is the description of the GIT quotient spaces as moduli spaces. We first introduce two new stability notions for curves: A complete curve $C$ is said to be *c-stable* if

1. $C$ has nodes, cusps and tacnodes as singularities;
2. $\omega_C$ is ample;
3. a genus one subcurve meets the rest of the curve in at least two points.

A *cusp* (resp. *tacnode*) is the singularity which is locally analytically isomorphic to $y^2 = x^3$ (resp. $y^2 = x^4$). If a c-stable curve is not an elliptic bridge, we call it *h-stable*.

**Theorem 2.** Let $C$ be a bicanonical curve of genus three.

1. $C$ is Hilbert semistable if and only if it is h-stable. It is Hilbert stable if and only if it is h-stable and has no tacnode.
2. $C$ is Chow semistable if and only if it is c-stable. It is Chow stable if and only if it is c-stable and has no tacnode or elliptic bridge. Moreover, all Chow strictly semistable curves are identified in the moduli space of c-stable curves.

Due to the theorem, the GIT quotient space $\text{Hilb}_{3,2}/\text{SL}(6)$ (resp. $\text{Chow}_{3,2}/\text{SL}(6)$) is the moduli spaces of h-stable curves (resp. c-stable curves), and we shall denote it by $\overline{M}_3^{hs}$ (resp. $\overline{M}_3^{cs}$). The semistability statement of this theorem is proved
in [HH07] for \( g \geq 4 \). Although some arguments used in the higher genera case go through, the GIT is quite different in the genus three case. For instance, for \( g \geq 4 \), there are many h-stable curves with tacnodes that are Hilbert stable, including all irreducible ones. Also, the fact that all tacnodal curves are identified in \( \overline{M}_3^{cs} \) allows us to identify \( \overline{M}_3^{cs} \) with the compact moduli space of plane quartics constructed by Kondo using the period domains of K3 surfaces (Proposition 21).

The following diagram gives a hawk’s eye view of the main theorems:

\[
\begin{align*}
\overline{M}_3(1) & \cong \overline{M}_3 \\
\overline{M}_3(\frac{9}{11}) & \cong \overline{M}_3^{ps} \quad \text{\( \tau \)} \downarrow \quad \overline{M}_3(\frac{7}{10} - \epsilon) & \cong \overline{M}_3^{hs} \\
\overline{M}_3(\frac{7}{10}) & \cong \overline{M}_3^{cs} \quad \psi \downarrow \\
\overline{M}_3(\frac{17}{28}) & \cong Q \quad \psi^+ \downarrow \\
\end{align*}
\]

This program was initiated by Brendan Hassett and Sean Keel, and the ideas were further developed in [HH06]. The genus two case was completely worked out by Hassett in [Has05] (see also [HL07]) and the first couple of steps of the program for the higher genera case were completed in [HH06] and [HH07].

We work over an algebraically closed field \( k \) of characteristic zero.

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2. GIT of bicanonical curves of genus three

Let \( V \) be a vector space of dimension 6. Let \( \text{Hilb} \) denote the Hilbert scheme parametrizing subschemes of \( \mathbb{P}(V) \) with Hilbert polynomial \( P(m) = 8m - 2 \), and let \( \text{Hilb}_{3,2} \) denote the closure in \( \text{Hilb} \) of the locus of the bicanonical images

\[
C \to \mathbb{P}(V)
\]

of smooth curves of genus three. We shall also consider the corresponding Chow variety \( \text{Chow}_{3,2} \) of bicanonical curves. Since \( \omega_C^{\otimes 2} \) is very ample if \( C \) is c-stable [HH07], \( \text{Hilb}_{3,2} \) and \( \text{Chow}_{3,2} \) include c-stable curves. We have a natural action of \( \text{SL}(V) \) on \( \text{Hilb}_{3,2} \) and \( \text{Chow}_{3,2} \), and the aim of this section is to construct the GIT quotients \( \text{Hilb}_{3,2}/\text{SL}(V) \) and \( \text{Chow}_{3,2}/\text{SL}(V) \). Of course we need to specify the line bundles that we use to linearize the group action: The Chow variety has a canonical polarization, as it is canonically a closed subscheme of \( \mathbb{P}(\bigotimes \text{Sym}^8 V^\ast) \); On the Hilbert scheme, we use the line bundle that comes from the embedding

\[
\phi_m : \text{Hilb}_{3,2} \hookrightarrow \text{Gr}(P(m), \text{Sym}^m V) \hookrightarrow \mathbb{P}(P(m) \bigwedge \text{Sym}^m V), \quad m \gg 0.
\]

We shall denote the Hilbert point in \( \text{Hilb} \) of \( C \) by \([C]\); Its image \( \phi_m([C]) \) will be denoted by \([C]_m \) and will be called the \( m \)th Hilbert point of \( C \).

**Definition 1.** (1) \( C \) is \( m \)-Hilbert (semi)stable if \([C]_m \) is (semi)stable with respect to the natural \( \text{SL}(V) \) action on \( \mathbb{P}(P(m) \bigwedge \text{Sym}^m V) \). It is said to be Hilbert (semi)stable if it is \( m \)-Hilbert (semi)stable for all \( m \gg 0 \).
(2) \( C \) is \textit{Chow (semi)stable} if the Chow point of \( C \) is (semi)stable with respect to the natural \( SL(V) \) action on \( \mathbb{P}(\bigotimes^2 Sym^8 V^*) \).

2.1. \textbf{Stability computation via Gröbner basis.} In this section, we give an overview of some results from [HHL07]. Proofs and detailed analysis as well as some computer algebra system implementations can be found in that paper.

Let \( X \subset \mathbb{P}^N \) be a projective variety and let \( I_X \) and \( P \) denote the homogeneous ideal and the Hilbert polynomial of \( X \). Let \( \rho : \mathbb{G}_m \to GL(N + 1) \) be a one parameter subgroup defined by \( \rho(\alpha).x_i = \alpha^{r_i}x_i \) where \( x_i \) are homogeneous coordinates. The associated one-parameter subgroup of \( SL(N + 1) \) with weights \( (N + 1)r_i - \sum r_i \) will be denote by \( \rho' \).

\textbf{Proposition 1.} The Hilbert-Mumford index of the \( m \)th Hilbert point of \( X \) with respect to \( \rho' \) is given by the following formula:

\[
\mu([X]_m, \rho') = -(N + 1) \sum_{i=1}^{P(m)} \text{wt}_\rho(x^{a(i)}) + m \cdot P(m) \cdot \sum_{i=0}^{N} r_i \tag{1}
\]

Here, \( x^{a(i)} \) are the degree \( m \) monomials not in the initial ideal of \( I_X \) with respect to the \( \rho \)-weighted graded lexicographic order.

From the functoriality of the Hilbert-Mumford index and a careful analysis of the tautological ring of the Hilbert scheme [KM76] along with the cohomology vanishing property of c-stable curves [HH07], one can deduce:

\textbf{Proposition 2.} Let \( C \subset \mathbb{P}(V) \) be a bicanonical c-stable curve and \( C^* \) denote the curve to which \( \rho(\alpha).C \) specializes. If \( C^* \) is also a bicanonical c-stable curve, then for all \( m \geq 2 \), \( C \) is

(1) \( m \)-Hilbert stable if and only if \( \mu([C]_3, \rho) \geq 2\mu([C]_2, \rho) > 0 \);
(2) \( m \)-Hilbert strictly semistable if and only if \( \mu([C]_3, \rho) = \mu([C]_2, \rho) = 0 \);
(3) \( m \)-Hilbert unstable if and only if \( \mu([C]_3, \rho) \leq 2\mu([C]_2, \rho) < 0 \).

2.2. \textbf{Unstable curves.} As with most GIT problems, it is much easier to classify the unstable bicanonical curves. We start with the curves that are easiest to destabilize:

\textbf{Proposition 3.} Let \( C \) be a bicanonical curve of genus \( \geq 3 \) which is not a curve consisting of two elliptic curves meeting in one tacnode. It is Chow unstable if it is not c-stable.
For non c-stable curves that is not the special curve excluded in the proposition, it is not very difficult to find a destabilizing one parameter subgroup. The proof given in [HH07] is indeed valid for all genus \( g \geq 2 \).

By definition, if a non h-stable curve \( C \) is not an elliptic bridge, then it is not c-stable. By Proposition 3, C is Chow unstable if it is not the special curve excluded in Proposition 3. With this observation, we can rule out almost all non h-stable curves as Hilbert unstable, due to the following:

**Proposition 4.** (see, e.g. [HH07]) If a projective variety is Chow stable (resp. unstable) then it is \( m \)-Hilbert stable (resp. unstable) for \( m \gg 0 \).

The following corollary is immediate:

**Corollary 1.** [HHL07] If a bicanonical curve \( C \) is not h-stable and is not an elliptic bridge, then it is Hilbert unstable.

We are left with two cases: (a) Chow instability of a curve consisting of two elliptic curves meeting in one tacnode; (b) Hilbert instability of elliptic bridges.

**Proposition 5.** A bicanonical curve \( C \) consisting of two elliptic curves \( E_1 \) and \( E_2 \) meeting in one tacnode is Chow unstable.

**Proof.** Suppose that \( C \) is Chow semistable. The Deligne-Mumford stabilization \( D \) of \( C \) consists of an elliptic curve bridging \( E_1 \) and \( E_2 \). But \( D \) is also the stabilization of an irreducible curve \( C' \) with two cusps which is Hilbert stable ([Sch91], see also the proof of Proposition 12) and hence Chow stable by Proposition 4. Since \( C' \) and \( C \) are both semistable and have the same Deligne-Mumford stabilization, they must be identified in the quotient space \( \text{Chow}_{3,2}/\!\!/\text{SL}(V) \). This contradicts that \( C' \) is Chow stable.

The instability of elliptic bridges is rather nontrivial and we devote the rest of the section to its proof.

**Definition 2.** A snowman is a genus one curve consisting of two smooth rational curves meeting in one tacnode.

**Proposition 6.** Let \( C^* \) be the bicanonical curve consisting of two snowmen meeting in two nodes such that each rational component meets the rest of the curve in precisely one node and one tacnode (Figure 4). Then for every bicanonical elliptic bridge \( C \), there is a one-parameter subgroup \( \rho : \mathbb{G}_m \to \text{SL}(V) \) such that
(1) $\rho(\alpha).C$ specializes to $C^*$;
(2) $C^*$ is Hilbert unstable with respect to $\rho$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{snowmen.png}
\caption{The conjoined snowmen $C^*$}
\end{figure}

Proof. Let $C$ be an elliptic bridge consisting of elliptic curves $E_1$ and $E_2$ meeting in two nodes $p$ and $q$. Restricting the dualizing sheaf, we have $\omega_C|_{E_i} = \omega_{E_i}(p + q)$. This means that each elliptic component is embedded in $P^3$ by the $|2p + 2q|$ linear system. A generic elliptic curve embedded by $|2p + 2q|$ linear system is a complete intersection cut out by

\begin{align*}
\begin{cases}
x_0x_1 - x_2x_3 = 0 \\
x_1^2 + x_0x_3 + x_0x_2 + ax_0x_1 + bx_0^2 = 0
\end{cases}
\end{align*}

for some $a, b \in k$. Its $j$-invariant can be obtained by pulling back the equation (2) to $P^1 \times P^1$ and realizing it as a double cover of the $[x, y]$-line ramified over $\{(x^2 + axy)^2 - 4y^2(xy + bx^2) = 0\}$:

\[ j = \frac{-2^{8}3^{3}(a^2 - 12b)^3}{4(a^2 - 12b)^3 + 27(2a^3 - 72ab - 4233)).} \]

$C \subset P^5$ comprises of two elliptic curves embedded in such a fashion:

$E_1: \begin{cases}
x_4 = x_5 = 0 \\
x_0x_1 - x_2x_3 = 0 \\
x_1^2 + x_0x_3 + x_0x_2 + ax_0x_1 + bx_0^2 = 0
\end{cases}$

\footnote{We learned this computation from N. Nakayama.}
Let $\rho : \mathbb{G}_m \rightarrow \text{SL}(6)$ be the one parameter subgroup with weights $\{0, 1, 2, 2, 1, 0\}$. We find that the flat limit $\mathcal{C}^\rho$ of the family $\{\rho(\alpha), \mathcal{C}\}$ at $\alpha = 0$ does not depend on the j-invariants of $E_1$ and $E_2$, and is cut out by the ideal

\[ I_{\mathcal{C}^\rho} = \langle x_1x_5, x_0x_5, x_4^2 + x_2x_5 + x_4x_5, x_1x_4, x_0x_4, x_2x_3, x_3^2 + x_0x_2 + x_0x_3 \rangle \]

which is reduced and has the following associated primes

\[
\left\{ \langle x_2, x_1, x_0, x_4^2 + x_3x_5 \rangle, \langle x_3, x_1, x_0, x_4^2 + x_2x_5 \rangle, \langle x_5, x_4, x_3, x_4^2 + x_0x_2 \rangle, \langle x_5, x_4, x_2, x_3^2 + x_0x_3 \rangle \right\}.
\]

Note that these are rational conics and each component meets the rest of the curve in precisely one tacnode and one node. Hence (3) is the conjoined snowmen curve $\mathcal{C}^\rho$.

We compute the Hilbert-Mumford index of $\mathcal{C}^\rho$ with respect to $\rho$ by using Proposition [1]. For $m = 2$, the degree two monomials not in $I_{\mathcal{C}^\rho}$ are:

\[ x_0^2, x_0x_2, x_0x_3, x_1^2, x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_2^2, x_2x_4, x_3^2, x_3x_5, x_4^2, x_5^2 \]

of which $\rho$-weights sum up to 30. We have

\[ \mu([\mathcal{C}^\rho]_2, \rho) = -6 \cdot 30 + 2 \cdot P(2) \cdot (2 + 2 + 1 + 1) = -180 + 168 = -12. \]

For $m = 3$, the degree three monomials not in $I_{\mathcal{C}^\rho}$ are:

\[ x_0^3, x_0^2x_2, x_0^2x_3, x_0x_2^2, x_0x_3^2, x_1^3x_2, x_1^2x_3, x_1^2x_4, x_1x_2^2, x_1x_2x_4, x_1x_3^2, \\
 x_1x_3x_5, x_1x_4^2, x_1x_5^2, x_2^3, x_2^2x_4, x_2x_4^2, x_3^3, x_3^2x_5, x_3x_5^2, x_4^3, x_5^3 \]

of which $\rho$-weights sum up to 70. The Hilbert-Mumford index for the third Hilbert point is

\[ \mu([\mathcal{C}^\rho]_3, \rho) = -6 \cdot 70 + 2 \cdot P(3) \cdot (2 + 2 + 1 + 1) = -420 + 396 = -24. \]

Since $\mathcal{C}$ and $\mathcal{C}^\rho$ are c-stable and $\mu([\mathcal{C}^\rho]_3, \rho) = 2\mu([\mathcal{C}^\rho]_2, \rho) < 0$, it follows from Proposition [2] that $[\mathcal{C}^\rho]_m$ is unstable with respect to $\rho$ for all $m \geq 2$. \qed
2.3. **Semistability proof.** If an h-stable curve $C$ does not have a tacnode, then it is either Deligne-Mumford stable or pseudo-stable, and the Hilbert semistability of $C$ essentially follows from [Mum77], [Sch91].

To describe our strategy of treating the tacnodal ones, we first introduce the following two curves which are special among all h-stable curves:

**Definition 3.** (1) A *cat-eye* is a genus three curve consisting of two smooth rational curves $C_1$ and $C_2$ meeting in two tacnodes; 
(2) An *ox* is a genus three curve consisting of three smooth rational curves $C_1, C_2, C_3$ such that $C_1$ and $C_2$ meet each other in a node and meet $C_3$ in a tacnode.

**Definition 4.** Let $X \subset \mathbb{P}^N$ be a projective variety and $\rho : G \to \text{GL}(N + 1)$ be a one-parameter subgroup. The *basin of attraction* of a $\rho$-fixed point $x^* \in X$ is the set

$$A_\rho(x^*) := \{x \in X | \rho(\alpha).x \sim x^*\}.$$

It is easy to show that if $x^*$ is strictly semistable with respect to $\rho$, then $x^*$ is semistable if and only if every $x \in A_\rho(x^*)$ is semistable. We shall enumerate all possible h-stable replacements of a cat-eye (resp. the ox) by deformation theory, and show that those are in a basin of attraction of the particular cat-eye (resp. the ox) by direct flat limit computation.

**Proposition 7.** A genus three h-stable curve $C$ has infinite automorphisms if and only if it is a cat-eye or an ox (Figure 3).

**Proof.** An h-stable curve without a tacnode is Deligne-Mumford stable or pseudo-stable, and has finite automorphisms. Since an automorphism of $C$ lifts to an automorphism of its normalization, it is easy to see that among the tacnodal h-stable curves listed in Figure 3 only (d) and (i) have infinite automorphisms.

Let $C^{\text{cat}}$ be a cat-eye consisting of two smooth rational curves $C_1$ and $C_2$ meeting in two tacnodes $p$ and $q$, embedded in $\mathbb{P}^5$ by the bicanonical system $|\omega_{C^{\text{cat}}}^\otimes 2|$. There is a one parameter family $\{C^\text{cat}_\beta\}$ of cat-eyes where the parameter $\beta$ encodes the tangent space identifications. Since $\omega_{C^{\text{cat}}}|_{C_i} \simeq \omega_{C_i}(2p + 2q)$, each $C_i$ is the second Veronese image of a plane conic. Hence a cat-eye is the second Veronese
image of the plane quartic

\[(x_1^2 + x_0x_2)(\beta x_1^2 + x_0x_2) = 0, \quad \beta \in k \setminus \{0, 1\}.\]

which has tacnodes at \([1, 0, 0]\) and \([0, 0, 1]\). Note that \(C_1\) (resp. \(C_2\)) has a parametrization coming from \([s, t] \mapsto [s^2, st, -t^2]\) (resp., \([s, t, -\beta t^2]\)) and automorphisms \([s, t] \mapsto [\alpha s, t], \alpha \in \mathbb{G}_m\). This induces an action of the one-parameter subgroup \(\rho : \mathbb{G}_m \to \text{GL}(6)\) defined by \(\rho(\alpha) \cdot x_i = \alpha^{r_i}x_i\) with the weights \((r_0, \cdots, r_5) = (4, 2, 0, 3, 2, 1)\).

**Proposition 8.** A cat-eye is \(m\)-Hilbert strictly semistable with respect to \(\rho\) for all \(m \geq 2\).

**Proof.** The Gröbner basis of \(\mathcal{C}\) with respect to the weighted GLex is

\[
x_1x_2 - x_5^2, x_2x_3 - x_4x_5, x_1x_4 - x_3x_5, x_0x_2 - x_3^2, x_1^2\beta + x_3x_5\beta + x_3x_5 + x_4^3, \\
x_1x_5^2\beta + x_4x_5^2\beta + x_2x_4 + x_4^2x_5, x_2x_4x_5^2\beta + x_3^2\beta + x_2^2x_4^2 + x_2x_4x_5^2, x_0x_5 - x_3x_4, \\
x_1x_3x_5\beta + x_3x_4x_5 + x_3^4, x_3x_5^2\beta + x_2^2x_4^2 + x_2^2x_4^2 + x_4^2x_5^2, x_0x_1 - x_3^2, \\
x_3^2x_5^2\beta + x_3x_4x_5^2 + x_3x_4x_5 + x_4^4, x_3^2x_5^2\beta + x_3x_4x_5^2 + x_0x_3^3 + x_3x_4^2, \\
x_0x_3^2x_4 + x_3^3x_4 + x_0x_3^2x_4 + x_1x_3^2\beta + x_3^2x_4 + x_0x_3^2 + x_3^2x_4.
\]

Hence the degree two monomials not in the initial ideal are

\[
x_0^2, x_0x_3, x_0x_4, x_1x_3, x_1x_5, x_2^2, x_2x_4, x_2x_5, x_3^2, x_3x_4, x_3x_5, x_4^2, x_4x_5, x_5^2.
\]
and the sum of the weights of these monomials is 56. On the other hand, the average weight (the second term of the right hand side of (1) divided by $N+1 = 6$) is $\frac{2 \cdot 14 \cdot 12}{6} = 56$, and we have $\mu([C], \rho) = 0$ by Proposition 11.

The degree three monomials in the initial ideal are multiples of (6) together with $x_1x_5^2$, $x_1x_3x_5$, and $x_1x_3^2$. Hence the degree three monomials not in the initial ideal are

\[ x_3^2, x_3^2x_3, x_3^2x_4, x_0x_3^2, x_0x_3x_4, x_0x_4^2, x_3^2, x_2x_4, x_2x_5, x_2x_4x_5, x_2x_5^2, x_3x_4x_5, x_3x_5^2, x_4x_5, x_4x_5^2, x_5^3
\]

of which weights sum up to 132. The average weight is $\frac{2 \cdot 22 \cdot 12}{6} = 132$, and $C$ is 3-Hilbert strictly semistable. The assertion now follows from Proposition 2. □

Consider the irreducible h-stable tacnodal curves (a)~(c). These are obtained from a genus one curve by identifying two points and their tangent lines. Given a genus one curve $E$ and two distinct simple points $q$ and $r$, one can construct a one-parameter family $\{E_\beta\}$ of tacnodal curves by identifying $q, r$, and identifying the tangent lines by

\[ \frac{\partial}{\partial \sigma_q} = \beta \frac{\partial}{\partial \sigma_r}\]

where $\sigma_q$ and $\sigma_r$ denote the local parameters at $q$ and $r$.

**Proposition 9.** The flat limit of $\{\rho(\alpha).E_\beta\}$ at $\beta = 0$ is $C_\beta^{\text{at}}$.

**Proof.** We start with $E \subset \mathbb{P}^3$ embedded by $|2q + 2r|$. This is defined by

\[ x_0x_1 - x_2x_3 = x_1^2 + x_0x_3 + x_0x_2 + ax_0x_1 + bx_0^2 = 0 \]

where $a, b \in k$. Here $q = [0, 0, 1, 0]$ and $r = [0, 0, 0, 1]$. The following projection identifies $q$ and $r$:

\[ \text{pr}: [x_0, \ldots, x_3] \mapsto [x_0, x_1, x_2 + \beta x_3] \]

The order of vanishing of $x_0, \ldots, x_3$ at $q$ and $r$ are

|     | $x_0$ | $x_1$ | $x_2$ | $x_3$ |
|-----|------|------|------|------|
| $\text{ord}_q$ | 2    | 1    | 0    | 3    |
| $\text{ord}_r$ | 2    | 1    | 3    | 0    |
From this, we deduce that under \( pr \), the tangent lines are identified by the relation 
\[
\frac{\partial}{\partial q_i} = \beta \frac{\partial}{\partial \sigma_i}.
\]
The image of \( E \) under \( pr \) is the quartic curve defined by:
\[
(10) \quad f = x_0^2 x_1^2 a^2 \beta + 2 x_0^3 x_1 x_2 a \beta + 2 x_0 x_1^3 a \beta + x_0^2 x_1 x_2^2 a \beta + 2 x_0^2 x_1^3 b \beta + x_0 x_1 x_2^2 b \beta 
+ x_0^3 x_1^2 \beta^2 + x_0^2 x_1 x_2 a + x_0 x_1 x_2^2 - 2 x_0^3 x_1 \beta + x_1^4 \beta + x_0 x_1 x_2 \beta + x_0^2 x_1 + x_0 x_1 x_2 + x_0^2 x_2^2 = 0.
\]
The second Veronese image of \((10)\) is the bicanonical image of \( E_\beta \) of which ideal can be readily computed. We omit it as we do not need it.

We shall now prove that the irreducible tacnodal curve \( C \) degenerates to a cat-eye along the action of the one parameter subgroup with weights \( \{0, 2, 4, 1, 2, 3\} \)\(^2\). Since taking flat limits commutes with taking Veronese images, we shall work with the plane quartic \((10)\) and show that it degenerates to the cat-eye \((4)\) along the action of the one parameter subgroup with weights \( \{0, 1, 2\} \). By slightly abusing notations, we let \( C \) denote the quartic curve \((10)\) and \( \rho \), the one-parameter subgroup of \( GL(3) \) with weights \( \{0, 1, 2\} \). Then the flat projective closure of \( \{\rho(\alpha).C\} \) is defined by
\[
\alpha^4 f(x_0, \alpha^{-1} x_1, \alpha^{-2} x_2) = \alpha^2 x_0^2 x_1^2 a^2 \beta + \alpha^3 x_0^3 x_1 x_2 a \beta + \alpha^4 x_0^4 x_1^2 b \beta + \alpha^2 x_0^2 x_1 x_2 \beta + \alpha x_0 x_1 x_2^2 a + \alpha^2 x_0 x_1 x_2 b 
+ \alpha^3 x_0^3 x_1 x_2 \beta + \alpha^3 x_0^3 x_1 x_2 a + \alpha^2 x_0^2 x_1 x_2 b 
- \alpha^3 x_0^3 x_1 \beta + x_1^4 \beta + x_0 x_1^2 x_2 \beta + \alpha^3 x_0 x_1 x_2 + x_0^2 x_1 x_2 + x_0^2 x_2^2 = 0,
\]
and the flat limit at \( \alpha = 0 \) is defined by
\[
x_1^4 \beta + x_0 x_1^2 x_2 \beta + x_0 x_1 x_2^2 + x_0^2 x_2^2 = (x_1^2 + x_0 x_2)(x_1^2 \beta + x_0 x_2) = 0
\]
which is precisely the plane cat-eye \((4)\).

There is another curve that specializes to a cat-eye under the \( \rho \)-action: A curve of the form \((e)\) is the second Veronese image of the quartic curve
\[
D := \{(x_1^2 + x_0 x_2)(\beta x_1^2 + x_0 x_2 + \gamma x_2^2) = 0\}, \quad \gamma \in \mathbb{G}_m,
\]
where the parameter \( \gamma \) encodes the cross ratio of the three points of attachments. Since the \( \rho \)-weight of the extra term \( \gamma x_2^2 \) is zero, the flat limit of \( \{\rho(\alpha).D\} \) at \( \alpha = 0 \) is \( \{(x_1^2 + x_0 x_2)(\beta x_1^2 + x_0 x_2) = 0\} \).

From Propositions 8 and 9 it follows that

**Corollary 2.** A curve of the form \((a), (b), (c), (e)\) is Hilbert semistable if and only if the corresponding cat-eye is semistable.

\(^2\)This appears slightly different from the weights of \( \rho \) in Proposition 8 but this does not affect our argument as they are projectively equivalent.
Recall that an ox is a genus three curve consisting of three smooth rational curves $C_1, C_2, C_3$ such that $C_1$ and $C_2$ meets each other in a node $q$ and $C_3$, in tacnodes $p_1$ and $p_2$ respectively. Let $C_{\text{ox}}$ be an ox, and consider the restriction of $\omega_{C_{\text{ox}}}^{\otimes 2}$ to each component:

$$\omega_{C_{\text{ox}}}^{\otimes 2}|_{C_i} \simeq \omega_{C_i}^{\otimes 2}(4p_i + 2q), \quad i = 1, 2$$

$$\omega_{C_{\text{ox}}}^{\otimes 2}|_{C_3} \simeq \omega_{C_3}^{\otimes 2}(4p_1 + 4p_2).$$

This implies that the bicanonical image of $C_{\text{ox}}$ has two smooth conics $C_1$ and $C_2$ meeting each other in a node and meeting a smooth rational quartic curve in a tacnode. Since the second Veronese image $C$ of the plane quartic \( \{ x_0x_2(x_0x_2 - x_1^2) = 0 \} \) has precisely such components, and since ox is unique up to isomorphism, it follows that the bicanonical image of an ox is projectively equivalent to $C$.

The plane quartic \( \{ x_0x_2(x_0x_2 - x_1^2) = 0 \} \) admits automorphisms \([x_0, x_1, x_2] \mapsto [x_0, \alpha x_1, \alpha^2 x_2], \alpha \in \mathbb{G}_m\). The second Veronese image $C$ has associated automorphisms $x_i \mapsto \alpha^{r_i}x_i$ where $(r_0, \ldots, r_3) = (0, 2, 4, 1, 2, 3)$. Let $\rho$ be the one parameter subgroup with these weights.

**Proposition 10.** $C$ is $m$-Hilbert strictly semistable with respect to $\rho$ for all $m \geq 2$. 

---

**Figure 4.** Degeneration to the cat-eye and the ox
Proof. The ideal of $C$ is generated by
\[ x_3^2 - x_3 x_5, x_3 x_4 - x_0 x_5, x_1 x_4 - x_3 x_5, x_2 x_3 - x_4 x_5, \
\]  
\[ x_1 x_2 - x_5^2, x_0 x_2 - x_3 x_5, x_0 x_1 - x_3^2. \]

The degree two monomials not in the initial ideal are
\[ x_0^2, x_0 x_3, x_0 x_4, x_1^2, x_1 x_3, x_1 x_5, x_2^2, x_2 x_4, x_2 x_5, x_3^2, x_3 x_4, x_4 x_5, x_5^2. \]

The sum of the weights of these monomials is 56. This is equal to the average weight $\frac{2(2)(4+2+0+3+2+1)}{6} = \frac{214}{6}$. Therefore $\mu([C], \rho) = 0$ by Proposition 1.

Now we analyze the 3rd Hilbert point of $C$. The degree three monomials not in the initial ideal are
\[ x_2^3, x_0 x_3, x_0 x_4, x_1 x_3, x_1 x_5, x_2^3, x_1 x_3, x_1 x_5, x_3^3, x_2^3, x_3 x_4, x_6 x_5, x_4 x_5, x_5^3. \]

The weights of these sum up to 132, which is equal to the average weight $\frac{3(8^3 - 2)12}{8}$. Hence $\mu([C], \rho) = 0$. The assertion now follows from Proposition 2.

Let $F$ be a bicanonical curve of genus three consisting of a smooth elliptic curve $E$ meeting a smooth rational curve $R$ in a tacnode $p$ and a node $q$. We have $\omega_{F|E} = \omega_E(2p + q) \simeq O(2p + q)$ and $\omega_{F|R} = \omega_R(2p + q)$. Hence $F$ is the second Veronese image of
\[ F' = \{ f = x_0 (x_1^2 x_2 - x_0 (x_0 - x_2) (x_0 - \ell x_2)) = 0 \} \subset \mathbb{P}^2. \]

Proposition 11. The flat limit of $\{ \rho(\alpha). F \}$ is the ox.

Proof. It suffices to show that $F'$ degenerates to the canonical model of ox along the action of the one-parameter subgroup with weights $(0, 1, 2)$, which we abuse notation and denote by $\rho$. Then the flat projective closure of $\{ \rho(\alpha). F \}$ is defined by
\[ \alpha^4 f(\alpha^{-2} x_0, \alpha^{-1} x_1, x_2) = \alpha^4 x_0^4 - \alpha^2 x_0^3 x_2 \ell - \alpha^2 x_0^3 x_2 + x_0^2 x_2^2 \ell - x_0 x_1^2 x_2. \]

At $\alpha = 0$, this gives
\[ \{ x_0 x_2 (x_0 x_2 \ell - x_1^2) = 0 \} = \{ x_0 x_2 (x_0 x_2 - (x_1 / \sqrt{\ell})^2) = 0 \} \]
which is the ox.
It is easy to see that a curve of the form \((j)\) also belongs to the basin of attraction of the ox with respect to \(\rho\): Such a curve is the second Veronese image of the plane quartic

\[
D = \{x_0x_2(x_0x_2 + x_1^2 + \gamma x_1^2) = 0\}.
\]

Since the extra term \(\gamma x_0x_1^2\) has zero \(\rho\)-weight, the flat limit of the family \(\{\rho(\alpha).D\}\) is \(\{x_0x_2(x_0x_2 + x_1^2) = 0\}\).

In view of Propositions \([10]\) and \([11]\) we have

**Corollary 3.** Tacnodal curves of the form \((f), (g), (h), (j)\) are Hilbert semistable if and only if the ox is Hilbert semistable.

It is easy to deduce from the defining equation \((4)\) that the ox is the flat limit of the family \(\{C_{\beta}^{\text{cat}}\}\) at zero and infinity.

Now we are ready to complete the proof of the first part of Theorem \([2]\)

**Proposition 12.** An \(h\)-stable bicanonical curve is Hilbert semistable. Furthermore, an \(h\)-stable bicanonical curve without a tacnode is Hilbert stable.

*Proof.* Let \(C\) be an \(h\)-stable curve. By a theorem of Mumford (\([\text{Mum77}], 4.15\)), a nonsingular curve is Chow stable and hence Hilbert stable by Proposition \([4]\). For singular \(h\)-stable curves \(C\), we apply a standard degeneration argument. If \(C\) does not have a tacnode and is not an elliptic bridge, then it does not belong to any basin of attraction. Hence \(C\) is the only possibly semistable replacement (other than its Deligne-Mumford stable model \(D\) which is unstable unless \(D = C\)). It follows that \(C\) is Hilbert stable.

It remains to show that \(h\)-stable curves with tacnodes are Hilbert semistable. In view of Propositions \([9]\) and \([11]\) it suffices to establish the semistability for the cat-eyes and the ox. We first note that a generic cat-eye must be semistable: If not, all cat-eyes and the ox would be unstable since the ox is in the closure of the locus of cat-eyes, and consequently bicanonical elliptic bridges would not have Hilbert semistable stabilizations.

Consider the cat-eye \(C_{\beta}^{\text{cat}}\) and a smoothing \(\pi : C \to B\) of it. By the semistable replacement theorem, there is a family \(\pi' : C' \to B\) whose generic fibre \(C'_n\) is isomorphic to the generic fibre \(C_n\) of \(C\) and the special fibre \(C'_s\) is Hilbert semistable. Since the cat-eyes and the ox are the only ones that have elliptic bridges as the
Deligne-Mumford stabilization, $C'$ must be $C_{\beta'}^{st}$ for some $\beta'$. Here $C_{0}^{st} = C_{\infty}^{st}$ means the ox.

By choosing a basis of $\pi_{*}(\omega_{C/B})$ (resp. $\pi'_{*}(\omega_{C'/B})$), we obtain an embedding $C \hookrightarrow \mathbb{P}^{2} \times B$ (resp. $C' \hookrightarrow \mathbb{P}^{2} \times B$). These in turn lead to morphisms $f, f' : B \rightarrow \overline{Q} := \mathbb{P}(\Gamma(O_{\mathbb{P}^{2}}(+4)))/\text{PGL}(3)$. Since the generic fibres of $C'$ and $C$ are isomorphic, $f(0) = f'(0)$. But since the cat-eyes and the ox are separated in the moduli space $\overline{Q}$ of plane quartic curves (Section 3.4), it follows that $C_{s} = C'_{s}$ and $C_{\beta}^{st}$ is semistable. \square

We let $\overline{M}_{3}^{hs}$ denote the GIT quotient $\text{Hilb}_{3,2}//\text{SL}(6)$ and call it the moduli space of $h$-stable curves of genus three. Since tacnodal curves are identified with suitable cat-eyes and all cat-eyes are separated in $\overline{M}_{3}^{hs}$,

**Corollary 4.** The locus of tacnodal curves in $\overline{M}_{3}^{hs}$ is a smooth rational curve.

It remains to show that a c-stable curve is Chow semistable. Since a c-stable curve that is not an elliptic bridge is $h$-stable, we deduce from Propositions 4 and 12 that

**Proposition 13.** A bicanonical c-stable curve that is not an elliptic bridge is Chow semistable. Moreover, such a curve is Chow stable if and only if it has no tacnode.

To show that elliptic bridges are Chow semistable, we will need the following degeneration results.

**Proposition 14.** There are two one-parameter subgroups $\rho_{1}$ and $\rho_{2}$ of $\text{GL}(6)$ such that the following holds (Figure 5):

1. Let $C$ be an irreducible tacnodal curve of genus three. The flat limit $C^{\flat}$ of $\{\rho_{1}(\alpha).C\}$ at $\alpha = 0$ consists of an elliptic curve meeting a snowman in two nodes: one in the head and the other in the body.
2. The flat limit of $\{\rho_{2}(\alpha).C^{\flat}\}$ at $\alpha = 0$ is the conjoined snowmen $C^{*}$ (Proposition 0).

**Proof.** (a) We retain the homogeneous coordinates $x_{0}, x_{1}, x_{2}$ and equations for $(E, q, r)$ and $E_{\beta}$ from the proof of Proposition 0. The second Veronese embedding takes $[x_{0}, x_{1}, x_{2}]$ to $[x_{0}^{2}, x_{1}^{2}, x_{2}^{2}, x_{0}x_{1}, x_{0}x_{2}, x_{1}x_{2}]$. Let $\rho_{1}$ be the one-parameter
subgroup of $\text{GL}(6)$ with weights $(0, 0, 2, 0, 0, 1)$. The ideal of the flat limit of \{\rho_1(\alpha), C\} at $\alpha = 0$ is generated by:

$$
(11)
\begin{align*}
&z_3z_5, z_0z_5, z_2z_3, z_1z_2 - z_3^2, z_0z_1 - z_3^2, z_1z_3^2\beta + z_4z_3^2\beta + z_2z_4^2 + z_4z_5^2, \\
&z_2z_4z_3^2\beta + z_4^2\beta + z_2^2z_4^2 + z_2z_4z_5^2, z_4^2z_3^3\beta + z_1z_4z_5^2 + z_1z_4z_5 + z_2z_5^2, \\
&z_3^2a^2\beta + 2z_0z_3ab\beta + z_0^2b^2\beta + 2z_1z_3a\beta + z_3z_4a\beta + 2z_3^2b\beta + z_0z_4b\beta + z_0z_3\beta^2 \\
&+ z_3z_4a + z_0z_4b + z_1\beta - 2z_0z_3\beta + z_1z_4\beta + z_0z_3 + z_1z_4 + z_3^2,
\end{align*}
$$

This has the following associated primes:

- $C_1 := (z_3, z_0, z_1 + z_4, z_2z_4 + z_3^2)$;
- $C_2 := (z_3, z_0, z_1\beta + z_4, z_1z_2 - z_3^2, z_2^2\beta + z_2z_4)$;
- $E := (z_5, z_2, z_0z_1 - z_3^2, z_0^2b^2\beta + 2z_0z_3b\beta a + z_0z_3^2a^2 + 2z_2^2b\beta + z_0z_4b\beta + z_0z_3\beta^2 \\
+ 2z_1z_3\beta a + z_3z_4\beta a + z_0z_4b + z_1z_4\beta - 2z_0z_3\beta + z_1z_4\beta + z_2z_4a + z_0z_3 + z_1z_4 + z_3^2, \\
z_0z_3^2b^2\beta + 2z_3^2b\beta a + z_1z_3^2\beta a^2 + 2z_1z_3^2b\beta + z_3^2z_4\beta + z_3^2\beta^2 + 2z_4^2z_4\beta a + z_1z_4z_4\beta a \\
+ z_3^2z_4b + z_1\beta - 2z_3^2\beta + z_1z_4\beta + z_1z_4z_4a + z_3^2 + z_1z_4 + z_1z_4^2, z_3^2b^2\beta + 2z_1z_3^2b\beta a \\
+ z_1z_3^2\beta a^2 + 2z_1z_3^2b\beta + z_1z_3^2z_4\beta + z_1z_3^2\beta^2 + 2z_3^2z_4\beta a + z_1z_3^2z_4b + z_1\beta \\
- 2z_1z_3^2\beta + z_3^2z_4\beta + z_3^2z_4z_4a + z_1z_3^2 + z_3^2z_4 + z_3^2z_4^2).$

The first two components are conics and the third is an elliptic curve isomorphic to $E$. The two conics meet in a single tacnode at $[0,0,1,0,0,0]$ and the elliptic

![Figure 5](image_url)

**Figure 5.** Degeneration of an elliptic bridge to the conjoined snowmen.
component intersects $C_1$ and $C_2$ in nodes at $[0,1,0,0,-1,0]$ and $[0,1,0,0,-\beta,0]$, respectively.

(2) Let $\rho_2$ be the one-parameter subgroup with weights $(0,2,2,1,2,2)$. The flat limit of $\{\rho_2(\alpha).C^2\}$ at $\alpha = 0$ is defined by the ideal

$$I^* : = \langle x_3 x_5, x_0 x_5, x_2 x_3, x_1 x_2 - x_3^2, x_0 x_2, x_0 x_1 - x_3^2, x_1^2 \beta + x_1 x_4 \beta + x_1 x_4 + x_4^2, \rangle$$

which is reduced and has the following four associated primes:

$$H_1 : = \langle x_5, x_2, x_1 + x_4, x_3^2 + x_0 x_4 \rangle, ~ B_1 : = \langle x_5, x_2, x_1 \beta + x_4, x_0 x_1 - x_3^2, x_3^2 \beta + x_0 x_4 \rangle,$$

$$H_2 : = \langle x_3, x_0, x_1 + x_4, x_2 x_4 + x_3^2 \rangle, ~ B_2 : = \langle x_3, x_0, x_1 \beta + x_4, x_1 x_2 - x_3^2, x_3^2 \beta + x_2 x_4 \rangle.$$

A quick observation reveals that

(1) $H_1$ and $B_1$ intersect in a tacnode at $[1,0,0,0,0,0]$;
(2) $H_2$ and $B_2$ intersect in a tacnode at $[0,0,1,0,0,0]$;
(3) $H_1$ and $H_2$ intersect in a node at $[0,1,0,0,-1,0]$;
(4) $B_1$ and $B_2$ intersect in a node at $[0,1,0,0,-\beta,0]$;
(5) $H_i$ and $B_j$ do not intersect if $i \neq j$.

We conclude that $I^*$ defines a curve isomorphic to the conjoined snowmen curve $C^\circ$.

\[ \blacksquare \]

**Proposition 15.** $C^\circ$ is Chow (strictly) semistable.

**Proof.** We first show that $C$ is Chow strictly semistable with respect to $\rho_1$. Consider the normalization $\nu : E \to C$, $\nu^{-1}(p) = \langle q, r \rangle$. We have $\text{ord}_q(\nu^* x_i) + r_i \geq 2$ and $\text{ord}_r(\nu^* x_i) + r_i \geq 2$ for all $i$ where $r_i$'s denote the weights of $\rho_1$, and it follows from Lemma 1.4 of [Sch91] that

$$e_{\rho_1}(C) = e_{\rho_1}(E)_q + e_{\rho_1}(E)_r \geq 2^2 + 2^2 = 8$$

where $e_{\rho_1}(C)$ is the Hilbert-Samuel multiplicity of $C$ with respect to $\rho_1$. On the other hand, the right hand side of the inequality in Theorem 1.1 of [Sch91] is $\frac{2}{3} \cdot 8 \cdot (2 + 1) = 8$. Hence $C$ is either Chow strictly semistable or unstable with respect to $\rho_1$. It cannot be the latter since $C$ is Hilbert semistable, and we conclude that $C$ (and hence $C^\circ$) is Chow strictly semistable with respect to $\rho_1$. 
By Proposition \[4\] \( C \) is Chow semistable. Since \( C \) is in the basin of attraction of \( C^\circ \) (with respect to \( \rho_1 \)), it follows that \( C^\circ \) is also Chow semistable and \( C \) and \( C^\circ \) are identified in the GIT quotient space. \( \square \)

**Proposition 16.** \( C^* \) is Chow semistable.

**Proof.** We first show that \( C^* \) is Chow strictly semistable with respect to \( \rho_2 \). We start with the ideal (11) of \( C^\flat \). Let \( \rho_2 \) be the one-parameter subgroups with weights \((0, 2, 2, 1, 2, 2)\). We shall use Proposition 1 to show that \( C^\flat \) is Hilbert unstable with respect to \( \rho_2 \): The degree two monomials not in the initial ideal are

\[
\begin{align*}
x_0^2, x_0 x_3, x_0 x_4, x_1 x_3, x_1 x_4, x_1 x_5, x_2^2, x_2 x_4, x_2 x_5, x_3^2, x_3 x_4, x_4^2, x_4 x_5, x_5^2
\end{align*}
\]

whose \( \rho_2 \) weights sum up to 43. On the other hand, the average weight is \( \frac{2 \cdot p(2)}{6} \cdot (2 + 2 + 1 + 2 + 2) = 42 \). Hence the Hilbert-Mumford index \( \mu_2 \) of the second Hilbert point of \( C^\flat \) with respect to \( \rho_2 \) is \( 6 \cdot (42 - 43) = -6 \). The degree three monomials not in the initial ideal are

\[
\begin{align*}
x_0^3, x_0^2 x_3, x_0^2 x_4, x_0 x_3 x_4, x_1 x_3^2, x_1 x_3 x_4, x_1 x_4^2, x_1 x_4 x_5, x_2^3, \\
x_2^2 x_4, x_2^2 x_5, x_2 x_4 x_5, x_2 x_5^2, x_3^3, x_3^2 x_4, x_3^2 x_5, x_4^3, x_4^2 x_5, x_4 x_5^2, x_5^3
\end{align*}
\]

whose \( \rho_2 \) weights sum up to 101, and the average weight is \( \frac{3 \cdot p(3)}{6} \cdot (2 + 2 + 1 + 2 + 2) = 99 \). Hence we have \( \mu_3 := \mu([C^\flat]_3, \rho_2) = 6 \cdot (99 - 101) = -12 \). Since \( \mu_3 = 2 \mu_2 < 0 \), it follows from Proposition 2 that \( C^\flat \) is m-Hilbert unstable with respect to \( \rho_2 \) for all \( m \geq 2 \). By Proposition \[4\] we conclude that \( C^\circ \) is Chow strictly semistable or unstable with respect to \( \rho_2 \). But the latter cannot be the case since \( C^\circ \) is Chow semistable.

Since \( C^\circ \) is in the basin of attraction of \( C^* \) (with respect to \( \rho_2 \)), it follows that \( C^* \) is also Chow semistable and \( C^\circ \) and \( C^* \) are identified in the GIT quotient space. \( \square \)

Since elliptic bridges and tacnodal curves belong to the basin of attraction of the conjoined snowmen (Proposition \[14\]), we have

**Corollary 5.** All elliptic bridges and tacnodal curves are identified in \( \overline{\mathcal{M}}_3^{cs} \).

This completes the proof of Chow semistability of c-stable curves, the second part of Theorem \[2\].
3. Modular interpretation of the log canonical models

3.1. The first divisorial contraction. In this section, we summarize the result of [HH06]. In his thesis [Sch91], David Schubert considered the Chow stability of tri-canonical curves and proved that a tri-canonical curve of genus $\geq 3$ has a GIT stable Chow point if and only if it is pseudostable. The corresponding statement was proved for genus two curves by the authors in [HL07]. A complete curve is pseudostable if

1. it is connected, reduced, and has only nodes and cusps as singularities;
2. every subcurve of genus one meets the rest of the curve in at least two points;
3. the canonical sheaf of the curve is ample.

The last condition means that each subcurve of genus zero meets the rest of the curve in at least three points.

Schubert also proved that there is no strictly semistable points and constructed the moduli space $\overline{M}_3^{ps}$ of pseudostable curves. Our minimal model program is guided by the log canonical divisor $K_{M_3} + \alpha \delta$ which is ample for $9/11 < \alpha \leq 1$ but contracts a unique extremal ray consisting of elliptic tails at $\alpha = 9/11$. In [HH06] it is shown that the result of this contraction, the log canonical model $\overline{M}_g(9/11)$, is isomorphic to the moduli space $\overline{M}_g^{ps}$ for $g \geq 3$. For $g = 2$, this was established by the authors in [HL07]. More precisely,

**Theorem.** (1) There is a natural morphism of stacks $\mathcal{T} : \overline{M}_g \to \overline{M}_g^{ps}$ that replaces each elliptic tail with a cusp, which descends to a birational contraction $\mathcal{T} : \overline{M}_g \to \overline{M}_g^{ps}$ with exceptional locus $\Delta_1$;
(2) The log canonical model $\overline{M}_g(9/11)$ is isomorphic to $\overline{M}_g^{ps}$.

Let $\delta^{ps}$ denote the divisor of singular curves in $\overline{M}_3^{ps}$ and let $\lambda^{ps}$ denote the divisor class of $\overline{M}_3^{ps}$ that agrees with the Hodge class $\lambda$ on the open substack $\mathcal{M}_3$ of smooth curves. Since the locus of cuspidal curves is of codimension two, $\lambda^{ps}$ is uniquely determined. Also, since $\mathcal{T}$ is a divisorial contraction, $K_{\overline{M}_3^{ps}}$ is $\mathbb{Q}$-Cartier and the equation $K_{\overline{M}_3} = 13\lambda - 2\delta$ on the open substack $\mathcal{M}_3$ extends to

$$K_{\overline{M}_3^{ps}} = 13\lambda^{ps} - 2\delta^{ps}$$

on $\overline{M}_3^{ps}$.
Fix the $j$-invariant and vary $p$ on $D$

Fix the $j$-invariants and $q$ and vary $p$

\textbf{Figure 6.} $F_{1}$ and $F_{EE}$

\textbf{Lemma 1.} $K^{ps}(\alpha) := K_{\overline{M}^{ps}_{3}} + \alpha\delta^{ps}$ is ample and $\overline{M}_{3}(\alpha) \simeq \overline{M}_{3}(9/11)$ for $7/10 < \alpha \leq 9/11$.

\textit{Proof.} Since $\overline{M}_{3}^{ps} \simeq \overline{M}_{3}(9/11)$, $K^{ps}(9/11)$ is ample on $\overline{M}_{3}^{ps}$. Also, $K^{ps}(7/10)$ is nef on $\overline{M}_{3}^{ps}$ due to Faber \cite{Fab90}: The cone $\text{Nef}^{1}(\overline{M}_{3})$ of nef divisors is generated by $\lambda$, $12\lambda - \delta_{0}$ and $10\lambda - \delta_{0} - 2\delta_{1}$, and $T^{*}(K_{\overline{M}^{ps}_{3}} + 7/10 \delta^{ps}) = 13(10\lambda - \delta_{0} - 2\delta_{1})$.

We can write $K^{ps}(\alpha)$ as a linear combination of $K^{ps}(9/11)$ and $K^{ps}(7/10)$:

\[ K^{ps}(\alpha) = \frac{11}{13}(10\alpha - 7)K^{ps}(9/11) + \frac{10}{13}(9 - 11\alpha)K^{ps}(7/10). \]

Since the first coefficient is positive and the second is nonnegative for $7/10 < \alpha \leq 9/11$, the ampleness follows. \qed

From Faber’s result, one also readily deduces that the nef cone of $\overline{M}_{3}^{ps}$ is generated by $10\lambda^{ps} - \delta^{ps}$ and $12\lambda^{ps} - \delta^{ps}$. Since $\overline{M}_{3}^{ps}$ is of Picard number two, it suffices to show that these are nef and extremal. But these divisors pull back to the nef generators $10\lambda - \delta_{0} - 2\delta_{1}$ and $12\lambda - \delta_{0}$, and contract the loci $T(F_{EE})$ and $T(F_{1})$ (Figure 6), respectively.

\textbf{3.2. The small contraction.} In this section, we prove the part (1) of Theorem 1.
**Theorem 1.** (1) \( \overline{M}_3(7/10) \) is isomorphic to the moduli space \( \overline{M}_3^{cs} \) of c-stable curves of genus three.

The moduli space \( \overline{M}_3^{cs} \) of c-stable curves (resp. \( \overline{M}_3^{hs} \) of h-stable curves) was constructed in §2 as a GIT quotient of the Chow variety \( \text{Chow}_{3,2} \) (resp. the Hilbert scheme \( \text{Hilb}_{3,2} \)) of the bicanonical curves of genus three. We shall employ the following strategy to prove Theorem 1.(1): We first show the existence of the log canonical model \( \overline{M}_3(7/10) \) by applying Kawamata-Shokurov base point freeness theorem on \( K_{\overline{M}_3^{ps}} + \frac{7}{10} \delta_{ps} \). Then we’ll show that the polarizations on \( \overline{M}_3(7/10) \) and \( \overline{M}_3^{cs} \) agree when pulled back to \( \overline{M}_3^{ps} \). Since the two varieties are normal and isomorphic away from loci of codimension \( \geq 2 \), the theorem follows by Hartog’s Lemma.

Recall the base point freeness theorem:

**Theorem.** (see, e.g. [KMM87], [KM98]) Let \((X, \Delta)\) be a proper klt pair with \( \Delta \) effective. Let \( D \) be a Cartier divisor such that \( aD - K_X - \Delta \) is nef and big for some \( a > 0 \). Then \(|bD|\) has no basepoint for all \( b \gg 0 \).

Let us apply this theorem to the pair \((X, \Delta) = (\overline{M}_3^{ps}, \alpha \delta_{ps})\) for some \( \alpha < 7/10 \), \( D = K_{\overline{M}_3^{ps}} + \frac{7}{10} \delta_{ps} \) and \( a = 2 \).

**Proposition 17.** The linear system \(|m(K_{\overline{M}_3^{ps}} + \frac{7}{10} \delta_{ps})|\) is base point free for sufficiently large and divisible \( m \), and the associated morphism \( \Psi : \overline{M}_3^{ps} \to \overline{M}_3(7/10) \) is a small contraction.

**Proof.** Due to the base point freeness theorem, it suffices to establish that

(A) \( (\overline{M}_3^{ps}, \alpha \delta_{ps}) \) is klt for \( \alpha < 1 \);

(B) \( 2(K_{\overline{M}_3^{ps}} + \frac{7}{10} \delta_{ps}) - (K_{\overline{M}_3^{ps}} + \alpha \delta_{ps}) \) is nef and big for \( \alpha < 7/10 \).

(B) follows immediately from that \( K_{\overline{M}_3^{ps}} + \beta \delta_{ps} \) is ample for \( 7/10 < \beta \leq 9/11 \). For (A), recall the log discrepancy formula

\[ K_{\overline{M}_3^{ps}} + \alpha \delta = T^*(K_{\overline{M}_3^{ps}} + \alpha \delta_{ps}) + (9 - 11\alpha) \delta_1. \]

Since \( \overline{M}_3 \) is smooth, (12) implies that the stacky pair \( (\overline{M}_3^{ps}, \alpha \delta_{ps}) \) is klt for \( 9 - 11\alpha \geq -1 \) and \( \alpha < 1 \). That \( (\overline{M}_3^{ps}, \alpha \delta_{ps}) \) is klt now follows from [KM98], 5.20. (See also [HH06], A.13.)
We now prove that $\Psi$ is a birational morphism. Let $C \subset \overline{M}_3$ be a curve that is not contained in the boundary. The Moriwaki divisor

$$A := 28\lambda - 3\delta_0 - 8\delta_1$$

intersects with any such curve non-negatively [Mor98]. But $K_{\overline{M}_3^{ps}} + 7/10\delta^{ps}$ pulls back to a line bundle proportional to $10\lambda - \delta_0 - 2\delta_1$, which in turn is a positive rational multiple of $A + 2\lambda + 2\delta_1$ and

$$(A + 2\lambda + 2\delta_1).C \geq 2\lambda.C > 0$$

where the last inequality follows since $\lambda$ gives rise to the Torelli map.

It remains to show that $\Psi$ is a small contraction. If $\Psi$ is a divisorial contraction, it contracts an irreducible divisor. Since (13) implies that $\Psi$ does not contract the hyperelliptic divisor, it must contract $\delta^{ps}$. This would force $T^*(K_{\overline{M}_3^{ps}} + 7/10\delta^{ps})$ to contract both $\delta_0$ and $\delta_1$, but a divisor contracting $\delta_0$ and $\delta_1$ must be proportional to $\lambda$.

Proof of Theorem 1, part (1). Let $Z^{cs} \subset \overline{M}_3^{cs}$ (resp. $Z^+ \subset \overline{M}_3^{hs}$) denote the locus consisting of tacnodal c-stable curves and elliptic bridges (resp. h-stable curves with tacnodes). $Z^{cs}$ is a point by Corollary 5. By definition, a c-stable curve without a tacnode or an elliptic bridge is pseudo-stable; Similarly, an h-stable curve without a tacnode is pseudo-stable. Hence there are isomorphisms

$$\overline{M}_3^{ps} \setminus Z \simeq \overline{M}_3^{cs} \setminus Z^{cs} \simeq \overline{M}_3^{hs} \setminus Z^+$$

Note that the loci $Z$, $Z^{cs}$ and $Z^+$ are of codimension $\geq 2$.

Let $\pi : \text{Hilb}_{3,2} \to \overline{M}_3^{hs}$ denote the quotient map. The projective structure on $\overline{M}_3^{hs}$ is given by the invariant sections of $\mathcal{O}_{\text{Hilb}_{3,2}}(+m)$ on $\text{Hilb}_{3,2}$. Hence $\pi^*(\mathcal{O}_{\overline{M}_3^{hs}}(1)) = \mathcal{O}_{\text{Hilb}_{3,2}}(1)$ which is proportional to ([Vie89], [HH07])

$$\left(10 - \frac{3}{2m}\right)\lambda - \delta.$$

Hence for large enough $m$, $\left(10 - \frac{3}{2m}\right)\lambda - \delta$ descends to an ample $\mathbb{Q}$-divisor on $\overline{M}_3^{hs}$. It follows that $\lambda$ (and consequently, $\delta$) descends to a Cartier divisor since it is proportional to the difference of $\left(10 - \frac{3}{2m}\right)\lambda - \delta$ and $\left(10 - \frac{3}{2(m+1)}\right)\lambda - \delta$.

Let $\lambda^{hs}$ and $\delta^{hs}$ denote the Cartier divisors on $\overline{M}_3^{hs}$ that pull back to $\lambda$ and $\delta$. 

The cycle map \( \varpi : \text{Hilb}_{3,2} \to \text{Chow}_{3,2} \) is \( \text{SL}(6) \)-equivariant and descends to
\[
\Psi^+ : \overline{M}^{\text{hs}}_3 \to \overline{M}^{\text{cs}}_3.
\]
By a theorem of Mumford \cite{Mum77}, the polarization on \( \overline{M}^{\text{cs}}_3 \) pulls back by \( \Psi^+ \) to
\[
10\lambda - \delta_{\text{hs}} = T^*(10\lambda_{\text{ps}} - \delta_{\text{ps}})
\]
over \( \overline{M}^{\text{ps}}_3 \setminus Z \), via the isomorphism \((16)\). The varieties are normal, and a section of a line bundle over a normal variety defined except over a codimension \( \geq 2 \) locus extends uniquely to a global section. We conclude that
\[
\Gamma(\overline{M}^{\text{ps}}_3, m(K_{\overline{M}^{\text{ps}}_3} + (7/10)\delta_{\text{ps}})) \simeq \Gamma(\overline{M}^{\text{cs}}_3, \mathcal{O}_{\overline{M}^{\text{cs}}_3}(m))
\]
which leads to
\[
\overline{M}_3(7/10) \simeq \text{Proj} \oplus_{m \in \mathbb{Z}^+} \Gamma(\overline{M}^{\text{ps}}_3, m(K_{\overline{M}^{\text{ps}}_3} + (7/10)\delta_{\text{ps}})) \simeq \text{Proj} \oplus_{m \in \mathbb{Z}^+} \Gamma(\overline{M}^{\text{cs}}_3, \mathcal{O}_{\overline{M}^{\text{cs}}_3}(m)) \simeq \overline{M}^{\text{cs}}_3
\]
where the first isomorphism follows from the projection formula.

It remains to prove that the exceptional locus of \( \Psi \) is \( Z \). Recall the test curve \( F_{\text{EE}} \) from Section 3.1 (Figure 6). Since there is a \( T(F_{\text{EE}}) \)-curve that passes through a general point of \( Z \) and \( 10\lambda - \delta_0 - 2\delta_1 = T^*(10\lambda_{\text{ps}} - \delta_{\text{ps}}) \) contracts \( F_{\text{EE}} \), it follows that the exceptional loci of \( \Psi \) contains \( Z \). We showed that \( K_{\overline{M}^{\text{ps}}_3} + (7/10)\delta_{\text{ps}} \) defines a small contraction from \( \overline{M}^{\text{ps}}_3 \) to \( \overline{M}^{\text{cs}}_3 \) (Proposition 17), and observed above that under the isomorphism \( \overline{M}^{\text{cs}}_3 \setminus Z \simeq \overline{M}^{\text{ps}}_3 \setminus Z \), the polarization on \( \overline{M}^{\text{cs}}_3 \) is proportional to \( K_{M^{\text{ps}}_3} + (7/10)\delta_{\text{ps}} \) on \( \overline{M}^{\text{ps}}_3 \setminus Z \). It follows that \( \Psi \) is an isomorphism on the open subset \( \overline{M}^{\text{ps}}_3 \setminus Z \). Hence \( \text{Excep}(\Psi) = Z \), and this completes the proof of the first part of Theorem 1.

Now that we have modular interpretations of \( \overline{M}_3(9/11) \) and \( \overline{M}_3(7/10) \), we can describe the small contraction \( \Psi \) induced by \( K_{\overline{M}^{\text{cs}}_3} + 7/10 \delta_{\text{ps}} \) in a concrete manner. It is an isomorphism over the \( \overline{M}^{\text{ps}}_3 \setminus Z \). What does \( \Psi \) do to curves in \( Z \)? That is, if \( C \) is an elliptic bridge, what is the c-stable curve that corresponds to \( \Psi(C) \)? Since \( \overline{M}_3(7/10) \) is the moduli space of c-stable curves and \( C \) is c-stable, the semistable replacement theorem implies that \( \Psi(C) \) is [\( C \)] itself. Moreover, we proved in \( \S \) that elliptic bridges and tacnodal c-stable curves are identified with the two snowmen curve \( C^* \) (Proposition 6) in the GIT quotient. We conclude that

**Proposition 18.** \( \Psi \) collapses \( Z \) to the point in \( \overline{M}^{\text{cs}}_3 \) that represents tacnodal curves.
3.3. The flip. In this section, we shall prove

**Theorem 1.** (2) For \( \alpha \in (17/28, 7/10) \), the log canonical model \( \overline{M}_3(\alpha) \) is isomorphic to \( \overline{M}_3^{hs} \cong \text{Hilb}_{3,2}/\text{SL}(6) \). Moreover, the following diagram is a flip in the sense of Mori theory:

\[
\begin{array}{ccc}
\overline{M}_3^{ps} & \xrightarrow{\Psi} & \overline{M}_3^{hs} \\
\downarrow \Psi & & \downarrow \Psi^+ \\
\overline{M}_3^{cs} & \xleftarrow{\Psi^+} & \overline{M}_3^{cs}
\end{array}
\]

*Proof.* We first prove the assertion for \( \alpha \in (7/10 - \epsilon, 7/10) \) for small enough \( \epsilon \). It will be extended to all \( \alpha \in (17/28, 7/10) \) in Corollaries 6 and 7.

The ampleness of (15) implies that

\[
\overline{M}_3^{hs} \cong \text{Proj} \bigoplus_{s \in \mathbb{Z}_+} \Gamma \left( \overline{M}_3^{hs}, s \left( \left( 10 - \frac{3}{2m} \right) \lambda - \delta \right) \right), \quad m \gg 0.
\]

Recall from §3.2 that \( \overline{M}_3^{hs} \cong \overline{M}_3^{ps} \setminus \mathbb{Z}_+ \). On \( \overline{M}_3^{ps} \), \((10 - \frac{3}{2m})\lambda - \delta\) is proportional to \( K_{\overline{M}_3^{ps}} + \frac{14m - 6}{2m - 3} \delta_{ps} \). Since \( \overline{M}_3^{hs} \) and \( \overline{M}_3^{ps} \) are normal and isomorphic away from the codimension \( \geq 2 \) loci \( Z \) and \( Z^+ \), we have:

\[
\bigoplus_{s \in \mathbb{Z}_+} \Gamma \left( \overline{M}_3^{hs}, s \left( \left( 10 - \frac{3}{2m} \right) \lambda - \delta \right) \right) = \bigoplus_{s \in \mathbb{Z}_+} \Gamma \left( \overline{M}_3^{ps}, s \left( (K_{\overline{M}_3^{ps}} + (7/10 - \epsilon(m))\delta_{ps}) \right) \right)
\]

where \( \epsilon(m) := 39/(200m - 30) \). From the log discrepancy formula \( K_{\overline{M}_3^{ps}} + \alpha \delta = T^*(K_{\overline{M}_3^{ps}} + \alpha \delta_{ps}) + (9 - 11\alpha)\delta_1 \) for \( T : \overline{M}_3 \to \overline{M}_3^{ps} \), we get:

\[
\Gamma(\overline{M}_3, s(K_{\overline{M}_3^{ps}} + \alpha \delta)) = \Gamma(\overline{M}_3, s(T^*(K_{\overline{M}_3^{ps}} + \alpha \delta_{ps}) + (9 - 11\alpha)\delta_{ps}))
\]

for all \( \alpha \). Combining (18) - (20) gives the desired isomorphism

\[
\overline{M}_3^{hs} \cong \text{Proj} \bigoplus_{s \in \mathbb{Z}_+} \Gamma(\overline{M}_3, s(K_{\overline{M}_3^{ps}} + (7/10 - \epsilon(m))\delta))
\]

Recall that \( \Psi \) is the birational contraction with exceptional locus \( Z \) which is the extremal face for the divisor \( K_{\overline{M}_3^{ps}} + (7/10 - \epsilon)\delta_{ps} \) for a small enough \( \epsilon \). Hence to show that (17) is a Mori flip, we need to establish that

(1) \( \Psi^+ \) is a small contraction;
(2) The strict transformation of $K_{\mathcal{M}_3} + (7/10 - \epsilon)\delta_{ps}$ is $\mathbb{Q}$-Cartier and $\Psi^+$-ample.

The first item follows since $\Psi^+$ has exceptional locus $Z^+$ is of dimension one (Corollary 4). The strict transformation of $K_{\mathcal{M}_3} + (7/10 - \epsilon)\delta_{ps}$ is $K_{\mathcal{M}_3} + (7/10 - \epsilon)\delta_{hs}$. Note that $K_{\mathcal{M}_3} + (7/10 - \epsilon)\delta_{hs}$ is $Q$-Cartier and $\Psi^+$-ample.

Proposition 19. $K_{\mathcal{M}_3} + \frac{17}{28}\delta_{hs}$ is nef and big on $\mathcal{M}_3$ and has a unique extremal ray generated by hyperelliptic curves.

Proof. Since $K_{\mathcal{M}_3} = 13\lambda_{hs} - 2\delta_{hs}$, $K_{\mathcal{M}_3} + \frac{17}{28}\delta_{hs}$ is proportional to $28\lambda_{hs} - 3\delta_{hs}$. On the other hand, the Moriwaki divisor $A = 28\lambda - 3\delta_0 - 8\delta_1$ properly transforms to $28\lambda_{hs} - 3\delta_{hs}$ on $\mathcal{M}_3$. Therefore, $K_{\mathcal{M}_3} + \frac{17}{28}\delta_{hs}$ is big. Since the hyperelliptic locus is equal to $9\lambda - \delta_0 - 3\delta_1$ on $\mathcal{M}_3$, $h = 9\lambda_{hs} - \delta_{hs}$ on $\mathcal{M}_3$ and we have

$$28\lambda_{hs} - 3\delta_{hs} = (10\lambda_{hs} - \delta_{hs}) + 2(9\lambda_{hs} - \delta_{hs}) = (10\lambda_{hs} - \delta_{hs}) + 2h.$$  

The divisor $10\lambda_{hs} - \delta_{hs}$ is nef, since it is proportional to $K_{\mathcal{M}_3} + \frac{7}{10}\delta_{hs}$ which is a limit of ample divisors. It follows that to show the nefness of $28\lambda_{hs} - 3\delta_{hs}$, it suffices to prove that the divisor non-negatively intersects with curves in $h$. In fact, we claim that $28\lambda_{hs} - 3\delta_{hs}$ is trivial on $h$.

Let $B_i \subset \mathcal{M}_3$ denote the locus of curves obtained by taking the stabilization of the admissible cover of $C \in \mathcal{M}_{0,8}$ consisting of smooth rational curves $C_1$ with $i$ marked points and $C_2$ with $8 - i$ marked points meeting in one node. Abusing notation, let $h$ denote the locus in $\mathcal{M}_3$ of hyperelliptic curves. For $g = 3$, we have three boundary divisors of $h$ which generate the rational Picard group:

1. $B_2$ consists of irreducible curves with one node;
2. $B_3$ consists of elliptic tails;
3. $B_4$ consists of elliptic bridges.

Among these, $B_3$ is contracted by $T$ and $B_4$, by $\Psi$. Hence the rational Picard group of the hyperelliptic locus $h^{cs}$ of $\mathcal{M}_3^{cs}$ is generated by the image of $B_2$. The small contraction $\Psi^+: \mathcal{M}_3^{hs} \to \mathcal{M}_3^{cs}$ restricts to a small contraction on $h$, and
\( \Psi^+ h \) induces an isomorphism \( \text{Pic}(h) \otimes \mathbb{Q} \simeq \text{Pic}(h^{\text{cs}}) \otimes \mathbb{Q} \) of the Picard groups. We conclude that \( \text{Pic}(h) \otimes \mathbb{Q} \) is generated by the image of \( B_2 \).

We summarize some results from §2.1 of [Rul01]: Any smooth hyperelliptic curve of genus three is a divisor of type \((2,4)\) in \( \mathbb{P}^1 \times \mathbb{P}^1 \). Let \( F_h \) denote a pencil of these divisors, which is equivalent to twice the curve class \( F'_h \) in \( B_2 \) obtained by letting one of the six marked points move. Since \( F'_h \) is in \( \overline{M}_3 \setminus (\delta_1 \cup \{\text{elliptic bridges}\}) \), the following intersection computation on \( \overline{M}_3 \) carries over to \( \overline{M}^{\text{hs}}_3 \):

\[
F'_h \cdot \delta_0 = 14, F'_h \cdot \delta_1 = 0, F'_h \cdot \lambda = \frac{3}{2}.
\]

It follows that \((28\lambda^{\text{hs}} - 3\delta^{\text{hs}}) \cdot F'_h = 0\). Since \( \text{Pic}(h) \otimes \mathbb{Q} \) is generated by the image of \( B_2 \), the divisor \( 28\lambda^{\text{hs}} - 3\delta^{\text{hs}} \) is nef and it contracts \( h \) to a point.

Since the Mori cone of \( \overline{M}^{\text{hs}}_3 \) is of dimension two and \( K_{\overline{M}_3} + \frac{17}{28} \delta^{\text{hs}} \) is big, the unicity of the extremal ray is obvious.

\[\square\]

**Corollary 6.** \( K_{\overline{M}_3} + \alpha \delta^{\text{hs}} \) is ample if \( \alpha \in (\frac{17}{28}, \frac{7}{10}) \cap \mathbb{Q} \).

**Proof.** Given \( \alpha \in (\frac{17}{28}, \frac{7}{10}) \cap \mathbb{Q} \) and small \( \epsilon \), \( K_{\overline{M}_3} + \alpha \delta^{\text{hs}} \) is a positive multiple of the linear combination

\[ (\alpha - \frac{17}{28}) \left( K_{\overline{M}_3} + (\frac{7}{10} - \epsilon) \delta^{\text{hs}} \right) + (\frac{7}{10} - \alpha - \epsilon) \left( K_{\overline{M}_3} + (\frac{17}{28}) \delta^{\text{hs}} \right). \]

Since the divisor \( K_{\overline{M}_3} + (\frac{7}{10} - \epsilon) \delta^{\text{hs}} \) is ample for small enough \( \epsilon \) and \( K_{\overline{M}_3} + (\frac{17}{28}) \delta^{\text{hs}} \) is nef, \( K_{\overline{M}_3} + \alpha \delta^{\text{hs}} \) is ample. \[\square\]

We have established that

**Corollary 7.** For \( \alpha \in (\frac{17}{28}, \frac{7}{10}) \), \( \overline{M}_3(\alpha) \) is isomorphic to \( \overline{M}_3^{\text{hs}} \).

This completes the proof of the second part of Theorem 1.

### 3.4. The second divisorial contraction.

This is the last (nontrivial) step in the Mori program for \( \overline{M}_3 \). We have shown in Proposition 19 that \( K_{\overline{M}_3} + (\frac{17}{28}) \delta^{\text{hs}} \) contracts the hyperelliptic locus, and we aim to describe the resulting log canonical model \( \overline{M}_3(17/28) \).

We first need to show that \( \overline{M}_3(17/28) \) exists, and we use the base point freeness theorem as we did in Proposition 17. In general, if \((X, \Delta)\) is klt then so is...
(X, aΔ) for any 0 < a < 1. Since \((\overline{M}_3^{hs}, a\delta^{hs})\) is klt for any \(\alpha \in (17/28, 7/10)\),

\((\overline{M}_3^{hs}, a\delta^{hs})\) is klt for any 0 < a < 7/10.

Choose \(\epsilon\) such that \(7/10 - 17/28 < \epsilon < 2(7/10 - 17/28)\). We have

\[
2(K_{\overline{M}_3^{hs}} + (17/28)\delta^{hs}) - (K_{\overline{M}_3^{hs}} + (7/10 - \epsilon)\delta^{hs}) = \frac{1}{2}K_{\overline{M}_3^{hs}} + \beta\delta^{hs}
\]

where \(\beta := 2 \cdot 17/28 - 7/10 + \epsilon\). The log canonical divisor \((21)\) is nef and big since \(17/28 < \beta < 7/10\). In fact, it is ample. The base point freeness theorem implies that \(|b(K_{\overline{M}_3^{hs}} + (17/28)\delta^{hs})|\) is base point free for \(b \gg 0\). Hence

**Lemma 2.** \(\overline{M}_3(17/28)\) exists as a projective variety, and there is a divisorial contraction

\[\Theta: \overline{M}_3^{hs} \to \overline{M}_3(17/28)\]

with exceptional locus \(h\).

We recall some classical results on the GIT of plane quartics [Mum65], [Art05]. Consider the natural action of \(\text{PGL}(6)\) on the space \(\mathbb{P}(\Gamma(\mathcal{O}_{\mathbb{P}^2}(4)))\) of plane quartics. With respect to this action, a plane quartic curve \(C\) is

(1) stable if and only if it has ordinary nodes and cusps as singularity;

(2) strictly semistable if it is a double conic or has a tacnode. Moreover, \(C\)

belongs to a minimal orbit if and only if it is either a double conic or the

union of two tangent conics (where at least one is smooth).

The minimal orbit statement in (2) implies that in the GIT quotient space \(\overline{Q}\),

an irreducible tacnodal curve is identified with the corresponding cat-eye, as in \(\overline{M}_3^{hs}\).

**Theorem 1.** (3) \(\overline{M}_3(17/28)\) is isomorphic to the compact space of plane quartics \(\overline{Q} := \mathbb{P}(\Gamma(\mathcal{O}_{\mathbb{P}^2}(4)))//\text{PGL}(6)\).

**Proof.** Consider the universal quartic curve \(\mathcal{X}\) over \(\mathbb{P} := \mathbb{P}(\Gamma(\mathcal{O}_{\mathbb{P}^2}(4)))^{ss}\):

\[
\mathcal{X} \quad \overset{\pi}{\longrightarrow} \quad \mathbb{P} \times \mathbb{P}^5 \quad \overset{\nu_2}{\longrightarrow} \quad \mathbb{P} \times \nu_2(\mathbb{P}^2) \subset \mathbb{P} \times \mathbb{P}^5
\]

where \(\nu_2\) denotes the second Veronese embedding. By abusing notation, let \(\mathcal{X}\) denote the image of \(\mathcal{X}\) in \(\mathbb{P} \times \mathbb{P}^5\). Let \(y \in \overline{Q}\) denote the point corresponding to the double conic. Away from the orbit of the double conic \(\mathcal{X}_y\), \(\mathcal{X} \hookrightarrow \mathbb{P} \times \mathbb{P}^5\) is
a family of bicanonical $h$-semistable curves, and induces a map from $\mathbb{P} \setminus \mathcal{O}(\mathcal{X}_y)$ to $\text{Hilb}^{hs}_{3,2}$ and subsequently to the quotient $\text{Hilb}_{3,2}/\text{SL}$. This map is $\text{PGL}(3)$ invariant, and it descends to give $f : \mathcal{Q} \setminus \{y\} \to \overline{M}_3^{hs}$. Let $Z^q \subset \mathcal{Q}$ denote the locus of tacnodal curves. Over the stable locus, we have isomorphisms

$$\mathcal{Q} \setminus (\{y\} \cup Z^q) \simeq \overline{M}_3^{hs} \setminus (Z^+ \cup h) \simeq \overline{M}_3 \setminus (\delta_1 \cup \{\text{elliptic bridges}\} \cup h)$$

where $f$ induces the first isomorphism. The loci $Z^q$ and $Z^+$ are of codimension $\geq 2$, and $f|_{Z^q}$ is bijective (see item (2) above and the subsequent remark). It follows that the inverse rational map $f^{-1}$ is regular on $Z^+$, giving an isomorphism

$$\mathcal{Q} \setminus \{y\} \simeq \overline{M}_3^{hs} \setminus h \simeq \overline{M}_3(17/28) \setminus \{\Theta(h)\}.$$ 

The assertion now follows by applying Hartog’s Lemma again. □

Let $\lambda_{\mathcal{Q}}$ and $\delta_{\mathcal{Q}}$ denote the unique divisor extending $\lambda^{hs}$ and $\delta^{hs}$ on the open set $\mathcal{Q} \setminus \{y\} \simeq \overline{M}_3^{hs} \setminus Z^+$.

**Lemma 3.** $K_{\mathcal{Q}} + \alpha \delta_{\mathcal{Q}}$ is ample on $\mathcal{Q}$ for $\alpha \in (5/9, 17/28]$ and $K_{\mathcal{Q}} + (5/9) \delta_{\mathcal{Q}}$ is trivial.

**Proof.** The identities $K_{\overline{M}_3^{hs}} = 13\lambda^{hs} - 2\delta^{hs}$ and $h = 9\lambda^{hs} - \delta^{hs}$ carry over to $\mathcal{Q}$ and give

$$K_{\mathcal{Q}} + 5/9 \delta_{\mathcal{Q}} = 13\lambda_{\mathcal{Q}} - 2\delta_{\mathcal{Q}} + 5/9 \delta_{\mathcal{Q}} = (13/9 - 2 + 5/9) \delta_{\mathcal{Q}} = 0.$$ 

Hence $K_{\mathcal{Q}} + 5/9 \delta_{\mathcal{Q}}$ is trivial. From this follows that $K_{\mathcal{Q}} + \alpha \delta_{\mathcal{Q}}$ is ample for $\alpha \in (5/9, 17/28]$ since it is a linear combination

$$a \left( K_{\mathcal{Q}} + 5/9 \delta_{\mathcal{Q}} \right) + b \left( K_{\mathcal{Q}} + 17/28 \delta_{\mathcal{Q}} \right)$$

for some positive rational numbers $a, b$ (determined by $\alpha$). □

**Corollary 8.** $\overline{M}_3(5/9)$ is a point.

4. **Relation to other moduli spaces: Work of Hassett, Hacking and Kondo**

There are various constructions of compact moduli spaces of plane quartics [Has99, Hac04, Kon00]. How do these moduli spaces fit in our minimal model program? In this section we give a brief sketch of these moduli spaces and show that they are indeed log canonical models for $\overline{M}_3$. 
4.1. **Hassett’s moduli space** $\overline{P}_4$. In [Has99], B. Hassett constructed a compact moduli space $\overline{P}_4$ by taking the closure of the $\mathcal{P}_4$ in the connected moduli scheme of *smoothable* stable log surfaces, where $\mathcal{P}_4$ is the quasi-projective GIT moduli space of smooth plane quartics. A stable log surface is a pair consisting of a surface $S$ and a curve $C \subset S$ such that $(S, C)$ has semi-log canonical singularities and $K_S + C$ is ample. It is said to be smoothable if there is a one parameter family of deformations $(S, C)$ whose general fiber is a smooth pair consisting of $\mathbb{P}^2$ with a smooth plane quartic curve and both $\mathbb{Q}$-divisors $K_S + C$ and $C$ are $\mathbb{Q}$-Cartier.

There is a forgetting morphism $F : \overline{P}_4 \to \overline{M}_3$ defined by $F((S, C)) = C$ which in fact, is isomorphism: For each curve $C$ in $\overline{M}_3$, Hassett explicitly constructs the unique corresponding surface $S$ with $(S, C) \in \overline{P}_4$. Then he shows that the morphism $F$ is proper, birational, and locally an isomorphism.

The cone of effective divisors $\text{NE}^1(\overline{M}_3)$ is generated by $\delta_0, \delta_1$ and $h$. A general element in each of these divisors corresponds to the following stable log surface in $\overline{P}_4$:

1. $C$ in $\delta_0 \iff (\mathbb{P}^2, C)$;
2. $C = C_1 \cup_p C_2$ in $\delta_1$ where $C_1$ and $C_2$ are irreducible curves of genus two and one respectively $\iff (S_1 \cup_B S_2, C)$ where $S_1$ is the toroidal blowup of $\mathbb{P}^2$ and $S_2 = \mathbb{P}(1, 2, 3)$. The curve $C_1$ in $S_1$ does not pass through the two singular points of type $\frac{1}{2}(1, 1)$ and $\frac{1}{3}(1, 1)$ of the surface $S_1$. Also the curve $C_2$ in $S_2$ does not pass through the two singular points $\frac{1}{2}(1, 1)$ and $\frac{1}{3}(1, 2)$ of the surface $S_2$. The curve $C = C_1 \cup_p C_2$ meets the double curve $B$ at the point $p$.
3. $C$ in $h \iff (S, C)$ where $S = \mathbb{P}(1, 1, 4)$. Since $C$ is a smooth hyperelliptic curve of genus three, $C$ is regarded as a bisection of the rational surface $\mathbb{F}_4 = \mathbb{P}(O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(4))$. $S$ is obtained by the contraction of the zero section. The curve $C$ does not pass through the singular point of type $\frac{1}{4}(1, 1)$.

4.2. **Hacking’s moduli space** $\overline{P}_4'$. Hacking gave an alternate compactification of the moduli space of plane curves of degree $d$ [Hac04] by employing a method similar to [Has99] but allowing worse singularities. We denote his moduli space for $d = 4$ by $\overline{P}_4$ (His original notation $\mathcal{M}_3$ is unfortunately reserved for the moduli stack of smooth curves of genus three.) A *stable pair* of degree 4 is a pair consisting of a surface $S$ and a curve $C \subset S$ such that $(S, \beta C)$ has semi-log
canonical singularities and $K_S + \beta C$ is ample, where $\beta$ is a rational number $\frac{3}{4} + \epsilon$ for a sufficient small positive number $\epsilon$. We also impose the condition that there be a one parameter family of deformations $(S, C)$ whose general fiber is a smooth pair $(\mathbb{P}^2, \text{smooth quartic curve})$. Both $\mathbb{Q}$-divisors $K_S + \beta C$ and $C$ are $\mathbb{Q}$-Cartier.

From his classification of stable surfaces of degree 4, $\mathcal{P}_4' = Z_0 \cup Z_1 \cup Z_2$ where $Z_1$ has codimension 1 and $Z_2$ has codimension 2 such that:

1. Any element in $Z_0$ is a pair $(\mathbb{P}^2, C)$ where $C$ is a pseudo-stable plane curve of degree 4;
2. Any element in $Z_1$ is a pair $(S, C)$ where $S = \mathbb{P}(1, 1, 4)$ and $C$ is a (degenerating) hyperelliptic curve of genus 3. The curve $C$ does not pass through the singular point of type $\frac{1}{4}(1, 1)$;
3. Any element in $Z_2$ is a pair $(S, C)$ where $S = S_1 \cup_B S_2$ is the union of two $\mathbb{P}(1, 1, 2)$s and $C$ is a (degenerating) elliptic bridge. Both irreducible components $S_1$ and $S_2$ have cyclic quotient singularities of type $\frac{1}{2}(1, 1)$ on the double curve $B = \mathbb{P}^1$. The curve $C$ does not pass through the singular points of $S_1$ and $S_2$.

If $(S, C)$ is a stable pair in $\mathcal{P}_4'$ then $C$ has nodes and cusps as singularities, and there is a forgetting morphism $F' : \mathcal{P}_4' \to \overline{\mathcal{M}}_3^{\text{ps}}$ defined by $F'((S, C)) = C$.

4.3. $\mathcal{P}_4$ and $\mathcal{P}_4'$ as log canonical models. Hassett proved that $\mathcal{P}_4$ is isomorphic to the moduli space $\overline{\mathcal{M}}_3$ of stable curves. Implicit in Hacking’s work is that

**Proposition 20.** $\mathcal{P}_4'$ is isomorphic to $\overline{\mathcal{M}}_3(9/11)$.

**Proof.** It is remarked in Hassett’s paper that for $\beta > 5/6$, $K_S + \beta C$ is ample for any pair $(S, C) \in \mathcal{P}_4$. Therefore $\mathcal{P}_4(\beta) \simeq \mathcal{P}_4$ for $\beta > 5/6$, where $\mathcal{P}_4(\beta)$ denotes the moduli space of pairs constructed in §4.2 using the prescribed value of $\beta$ whereas $3/4 + \epsilon$ was used for $\beta$ in §4.2.

We consider what happens at $\beta = 5/6$. First, there is a morphism $T' : \mathcal{P}_4 \to \mathcal{P}_4(5/6)$ that associates to $(S, C)$ the pair $(S', C')$ constructed as follows: By assumption $(S, C)$ is smoothable and there is a one-parameter family $(S, C)$, $\pi : S \to \text{Spec}(k[[t]])$ whose special fibre is $(S, C)$ and whose generic fibre is smooth. $S'$ is then the special fibre of the relative log canonical model $S' := \text{Proj} \oplus_{m \geq 0} \pi_*(m(K_S + 5/6 C))$ and $C'$, the scheme theoretic image $f(C')$ where $f$ is the canonical fibration from $S$ to $S'$. Regardless of the choice of the smoothing,
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\((S', C')\) is always the log canonical model of \((S, C)\) (If \( S \) has more than one component, then \( S' \) is the union of the log canonical models of the components with the restriction of the log canonical divisor as boundary divisor.)

Since the divisor \( K_S + 5/6 C \) is ample for all pairs \((S, C)\) \( \in \overline{P}_4 \) except the ones with \( S = (\text{toroidal blowup of } \mathbb{P}^2 \cup \mathbb{P}(1, 2, 3)) \) and \( C \) an elliptic tail, \( T' \) is an isomorphism away from the locus \( \mathcal{D}_1 \subset \overline{P}_4 \) of elliptic tails. For elliptic tails \((S_1 \cup_B S_2, C)\), restricting \( K_S + 5/6 C \) on \( S_2 \), we find that

\[
(K_S + \frac{5}{6} C)|_{S_2} = K_{S_2} + \frac{5}{6} C + B = \mathcal{O}_{S_2}(-6 + 5 + 1) = \mathcal{O}_{S_2}.
\]

This means that \( f : S \to S' \) contracts the elliptic component: In fact, Hassett proves that \( f : C \to C' \) replaces the elliptic tail by an ordinary cusp. Also, the log canonical models of \((S, C)\) \( \in \overline{P}_4 \) with respect to \( K_S + 5/6 C \) are precisely the stable pairs in Hacking’s moduli space. All in all, we have a birational contraction

\[
T' : \overline{M}_3 \to \overline{P}_4'
\]

such that \( T'(\text{curve}) = \text{point} \) if and only if \( \text{curve} \subset \overline{M}_3 \) is a curve in \( \delta_1 \) obtained by varying the \( j \)-invariant of the elliptic tail. Hence the forgetful map \( F' : \overline{P}_4' \to \overline{M}_3^{ps} \) is a bijective birational morphism between normal varieties. By Zariski’s main theorem, \( F' \) is an isomorphism. \( \square \)

4.4. Kondo’s compact moduli space. Kondo constructed a compact moduli space of plane quartic curves by using the period domains of K3 surfaces \[Kon00\]. Let \( C \) be a smooth plane quartic curve. Then the cyclic \( \mathbb{Z}_4 \)-cover of \( \mathbb{P}^2 \) branched along \( C \) is a K3 surface. The period domains of such K3 surfaces correspond to an arithmetic quotient of a bounded symmetric domain \( \mathcal{D} \) minus two hyperplanes. He extends this correspondence to the whole \( \mathcal{D} \) by allowing hyperelliptic curves of genus 3 and singular pseudo-stable plane curves of genus 3 \[Kon00\]. By using the Baily-Borel compactification of period domains, he constructs a compact moduli space which is normal and whose boundary is one point. Details can be found in \[Art05\] and \[Kon00\]. Let us denote Kondo’s compact moduli space by \( \overline{K} \), and denote the unique point in the boundary by \( q \).

**Proposition 21.** \( \overline{K} \simeq \overline{M}_3(\frac{7}{10}) \).

**Proof.** Recall from \[3.2\] that \( \text{Chow}_{3,2}/\text{SL} \) is isomorphic to \( \overline{M}_3(\frac{7}{10}) \). Considering Kondo’s construction and the classification of curves in \( \text{Chow}_{3,2}/\text{SL} \) reveals that there is a forgetful morphism \( F_K : \overline{K} \to \text{Chow}_{3,2}/\text{SL} \) mapping \((S, C)\) to \( C \).
We claim that there is a birational map $\mathcal{P}_4' \to \overline{K}$ that induces an isomorphism
\begin{equation}
\mathcal{P}_4' \setminus Z_2 \simeq \overline{K} \setminus \{q\}.
\end{equation}
Recall that $\mathcal{P}_4' = Z_0 \cup Z_1 \cup Z_2$. Let $(\mathbb{P}^2_{Z_0}, \mathcal{C})$ denote the universal pair over $Z_0$. The universal pair has $\mathbb{P}^2$ as the constant surface part and is parametrized by the curve part that walks through all pseudostable plane curves. By taking the cyclic $\mathbb{Z}_4$-cover of $\mathbb{P}^2$ branched along the curve, we obtain a universal pair $(\mathcal{X}, \mathcal{C})$ of K3 surfaces paired with pseudo-stable plane curves. This induces a morphism from $Z_0$ to $\overline{K}$. It is an isomorphism onto its image since it is a bijective morphism of normal varieties. A similar construction gives an isomorphism from $Z_1$ to its image in $\overline{K}$. Considering the description of curves in $Z_0$ and $Z_1$, we obtain the desired isomorphism $\mathcal{P}_4' \setminus Z_2 \simeq \overline{K} \setminus \{q\}$.

Retain notations from §3.2. The upshot of the isomorphism (22) is that, since $\mathcal{P}_4' \setminus Z_2 \simeq \overline{M}^{ps}_3 \setminus Z$ which is isomorphic under $\Psi$ to $\overline{M}^{cs}_3 \setminus Z^{cs}$, we have $\overline{K} \setminus \{q\} \simeq (\text{Chow}_{3,2}/\text{SL}) \setminus \{\text{strictly semistable point}\}$.

By Hartog’s theorem this birational map defined away from a locus of codimension $> 2$ is extended to an isomorphism since both varieties are normal. \hfill \Box

Remark 1. To our knowledge, the moduli space $\overline{M}^{hs}_3$ of h-stable curves is a new modular compactification of $M_3$. But we note that h-stable curves are precisely the curves that appear in the semistable pairs of degree four [Hac04]. Although Hacking defined the notion, the corresponding moduli space was not constructed.

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