On stability manifolds of Calabi-Yau surfaces

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Abstract

We prove some general statements on stability conditions of Calabi-Yau surfaces and discuss the stability manifold of the cotangent bundle of $\mathbb{P}^1$. Our primary interest is in spherical objects.

1 Introduction

The notion of stability conditions on triangulated categories was formulated in [Br1]. It organizes certain bounded $t$-structures on a triangulated category into a complex manifold. In the case of Calabi-Yau spaces, this is expected to be an approximation of the stringy Kähler moduli of $X$.

Stability conditions have been studied for one-dimensional spaces in [Br1], [GKR], [Ok], [Ma1], and [BuK], higher-dimensional spaces in [Th], [Br2], [Br3], [Br4], [Br5], [Ma1], [Ma2], [AB], [To], [Hu], [Be], and [An], and $A_\infty$-categories in [Th], [Ta], [Wa], and [KST]. The stability manifold of the category $\mathcal{O}$ for $sl_2$ has been computed in [Ma]. Some general aspects have been studied in [AP] and [GKR]. The author recommends [Br6] and [Do1], [Do2], [Do3] for an introduction and the original physical motivation to this subject.

We begin with fundamental notions and properties of stability conditions. After preparation on spectral sequences and $n$-Calabi-Yau categories, we will concentrate on stability conditions on 2-Calabi-Yau categories. Our main result is the connectedness of the stability manifold of the cotangent bundle of $\mathbb{P}^1$.

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1.1 Definitions

Throughout this paper, $\mathcal{T}_0$ is the bounded derived category of an abelian category with enough injectives and $\mathcal{T}$ is a full triangulated subcategory of $\mathcal{T}_0$. In addition, $\mathcal{T}$ is assumed to be linear over $\mathbb{C}$ and of finite type; i.e., for objects $E, F \in \mathcal{T}$, $\text{Hom}_{\mathcal{T}}(E, F)$ is a vector space over $\mathbb{C}$ and the vector space $\bigoplus_i \text{Hom}_{\mathcal{T}}^i(E, F)$ is of finite dimension.

For example, $\mathcal{T}$ can be the bounded derived category $D(X)$ of coherent sheaves on a smooth projective variety $X$, and $\mathcal{T}_0$ the bounded derived category of quasi-coherent sheaves on $X$ ([BGI, Section II, Proposition 2.2.2]). Let $K(\mathcal{T})$ be the $K$-group of $\mathcal{T}$. For an object $E \in \mathcal{T}$, let $[E]$ be the class of $E$ in $K(\mathcal{T})$.

We will recall some notions from [Br1].

1.1.1 Stability conditions

A stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{T}$ consists of a group homomorphism $Z : K(\mathcal{T}) \to \mathbb{C}$, called the central charge, and a family $\mathcal{P}(\phi), \phi \in \mathbb{R}$, of full abelian subcategories of $\mathcal{T}$, called the slicing. These need to satisfy the following conditions. If for some $\phi \in \mathbb{R}$, $E$ is a nonzero object in $\mathcal{P}(\phi)$, then for some $m(E) \in \mathbb{R}_{>0}$, $Z(E) = m(E) \exp(i\pi \phi)$. For each $\phi \in \mathbb{R}$, $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$. For any object $E \in \mathcal{T}$, there exist real numbers $\phi_1 > \phi_2$ and objects $A_i \in \mathcal{P}(\phi_i)$, $\text{Hom}_{\mathcal{T}}(A_1, A_2) = 0$. For any object $E \in \mathcal{T}$, there exist real numbers $\phi_1 > \cdots > \phi_n$ and objects $H^0_{\phi_i}(E) \in \mathcal{P}(\phi_i)$ such that there exists a sequence of exact triangles $E_{i-1} \to E_i \to H^0_{\phi_i}(E)$ with $E_0 = 0$ and $E_n = E$. The sequence is called the Harder-Narasimhan filtration (or HN-filtration for short) of $E$. The HN-filtration of any object is unique up to isomorphisms.

1.1.2 Stability manifolds

For an interval $I \subset \mathbb{R}$, $\mathcal{P}(I)$ denotes the smallest full subcategory of $\mathcal{T}$ that contains $\mathcal{P}(\phi)$ for $\phi \in I$, it is closed under extension; i.e., if $E \to G \to F$ is an exact triangle in $\mathcal{T}$ and $E, F \in \mathcal{P}(I)$, then $G \in \mathcal{P}(I)$. If the length of $I$ is less than one, then $\mathcal{P}(I)$ is a quasi-abelian category (in particular, it is an exact category), whose exact sequences are triangles of $\mathcal{T}$ with vertices in $\mathcal{P}(I)$.

A stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{T}$ is called locally-finite, if for any $\phi \in \mathbb{R}$, there exists a real number $\eta > 0$ such that $\mathcal{P}(\phi - \eta, \phi + \eta)$ is of finite length. The set of all locally-finite stability conditions on $\mathcal{T}$ is called the stability manifold of $\mathcal{T}$ and denoted by $\text{Stab}(\mathcal{T})$. The stability manifold of $\mathcal{T}$ has a natural topology and each connected component is a manifold locally modeled on some topological vector subspace of $\text{Hom}_\mathbb{Z}(K(\mathcal{T}), \mathbb{C})$. 

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1.1.3 Some actions on stability manifolds

Any stability manifold has a natural action of the group $\tilde{\text{GL}}^+(2, \mathbb{R})$, the universal cover of orientation-preserving transformations of $\text{GL}(2, \mathbb{R})$. In particular, the group contains the following $\mathbb{C}$-action for rotation and rescaling of stability conditions; for $(Z, P) \in \text{Stab}(\mathcal{T})$ and $z = x + iy \in \mathbb{C}$, $z \ast (Z, P)$ is defined as $z \ast Z = e^z Z$ and $(z \ast P)(\phi) = P(\phi - y/\pi)$ (Ok, Definition 2.3)).

1.1.4 Hearts of stability conditions

For each $j \in \mathbb{R}$, $P((j-1,j])$ and $P([j-1,j))$ are hearts of bounded $t$-structures. By a heart of $\mathcal{T}$, we mean the heart of any bounded $t$-structure of $\mathcal{T}$. In particular, $P((0,1])$ is said to be the heart associated to a stability condition $\sigma = (Z, P) \in \text{Stab}(\mathcal{T})$. We will call all $c \ast P((0,1]), c \in \mathbb{C}$, “hearts of $\sigma$”.

1.1.5 Semistable objects and stable objects

For a nonzero object $E \in \mathcal{P}((j-1,j])$, the phase of $E$ is defined to be $\phi(E) = (1/\pi) \text{arg} Z(E) \in (j-1,j]$. We say a real number $k$ is a trivial phase of an object $E \in \mathcal{T}$, if $H^k(E) = 0$. For any $\phi \in \mathbb{R}$, nonzero objects in $\mathcal{P}(\phi)$ are called semistable objects. For each object $E \in \mathcal{T}$ and $k \in \mathbb{R}$, $H^k(E)$ is called the semistable factor of $E$ of the phase $k$. For each $k \in \mathbb{R}$, any object $E \in \mathcal{P}(k)$ has a Jordan-Hölder filtration in $\mathcal{P}(k)$. A nonzero object $E \in \mathcal{P}(k)$ is called stable if it has no nontrivial subobject in $\mathcal{P}(k)$.

1.1.6 Jordan-Hölder blocks

**Definition 1.1.** For an object $E \in \mathcal{T}$, $k \in \mathbb{R}$, and $\sigma \in \text{Stab}(\mathcal{T})$, we will choose (non-canonical) “Jordan-Hölder blocks” (or JH-blocks for short) of $E$ denoted by $J^k_\sigma(E)$. Let $A_0 = 0$ and $B_0 = H^k_\sigma(E)$. For $i > 0$, let $A_i$ be a maximal subobject of $B_{i-1}$ such that all stable factors of $A_i$ are isomorphic and let $B_i = B_{i-1}/A_i$. By the local-finiteness of $\sigma$, $B_n = 0$ for some large enough $n$. We let $J^k_\sigma(E) = \{A_1, \ldots , A_n\}$.

**Corollary 1.2.** With the notation in Definition 1.1, $\text{Hom}_\mathcal{T}(A_i, B_i) = 0$ for any $0 \leq i \leq n$ and $\sum_{i\leq i\leq n}[A_i] = [H^k_\sigma(E)]$.

2 Spectral sequences and $n$-Calabi-Yau categories

2.1 Spectral sequences

For complexes $E, F \in \mathcal{T}_0$ and a morphism of complexes $f : E \to F$, let $C(f)$ be the cone of $f$. Let us say that $f$ is injective and splitting if $f$ is injective and it splits in each degree.
Lemma 2.1. Let $n \in \mathbb{Z}_{>0}$. For each $0 \leq i \leq n$, let $F_i \in \mathcal{T}_0$ be a complex. For each $0 \leq i < n$, let $f_i$ be a morphism of complexes $f_i : F_i \to F_{i+1}$. Then there exist complexes of injective objects $\tilde{F}_i \in \mathcal{T}_0$ and morphisms of complexes $\tilde{f}_i : \tilde{F}_i \to \tilde{F}_{i+1}$ with the following properties: morphisms $F_i \xrightarrow{\tilde{f}_i} \tilde{F}_{i+1}$ and $\tilde{F}_i \xrightarrow{\tilde{f}_i} \tilde{F}_{i+1}$ are isomorphic: $\tilde{f}_i$ is injective and splitting.

Proof. For $F_0$ and $F_1$, choose quasi-isomorphic complexes of injective objects $\tilde{F}_0$ and $\tilde{F}_1$. Then $f_0 : F_0 \to F_1$ is isomorphic to a morphism of complexes $\tilde{f}_0 : \tilde{F}_0 \to \tilde{F}_1$. Let $\tilde{F}_0 = \tilde{F}_0$, $\tilde{F}_1 = \tilde{F}_1 \oplus C(f_0)$, and $\tilde{f}_0 = f_0 \oplus \alpha_0$, where $\alpha_0 : \tilde{F}_1 \to C(f_0)$ is the canonical morphism. Then $f_0, f_0, \tilde{f}_0$ are isomorphic in $\mathcal{T}_0$, and $\tilde{f}_0$ is injective in each degree, since $\alpha_0$ is injective in each degree. Moreover, $\tilde{f}_0$ splits in each degree, since $\tilde{F}_0$ and $\tilde{F}_1$ are injective objects. Now, proceed by induction. \[\square\]

For a heart $\mathcal{A}$ of $\mathcal{T}$, let $\tau^A$ denote the truncation functor. For any object $E \in \mathcal{T}$, by Lemma [21], the sequence of canonical morphisms from $\tau^A_{\leq i}(E)$ to $\tau^A_{\leq i}(E)$ can be realized as a sequence of injective and splitting morphisms of complexes $\tilde{\tau}^A_{\leq i}(E)$.

Definition 2.2. For an object $E \in \mathcal{T}$ and integers $i \leq j$, let $\tilde{\tau}^A_{i,j}(E) = \tilde{\tau}^A_{i,j}(E)/\tilde{\tau}^A_{i,j}(E)$, in particular, $\tilde{\tau}^A_{i-1,i}(E) \cong \tau^A_{i-1,i}(E)$ in $\mathcal{T}_0$. For an object $E \in \mathcal{T}$ and each $i \in \mathbb{Z}$, let $e^A_i(E)$ be the connecting morphism from $H^A_i(E)$ to $H^{i-1}_A(E)[2]$, in the exact triangle $\tilde{\tau}^A_{i-2,i-1}(E) \to \tilde{\tau}^A_{i-1,i}(E) \to \tilde{\tau}^A_{i-2,i-1}(E)$, here,

\[
H^{i-1}_A(E)[2-i] \cong \tilde{\tau}^A_{i-1,i}(E) \xrightarrow{e^A_i(E)} H^{i-1}_A(E)[2-i] \cong \tilde{\tau}^A_{i-2,i-1}(E)[1].
\]

Definition 2.3. For objects $E, F \in \mathcal{T}$, $p, q \in \mathbb{Z}$, and a heart $\mathcal{A}$ of $\mathcal{T}$, let

\[
E_{p,q}^A(E, F) = \oplus_{i \in \mathbb{Z}} \text{Hom}^A_{\mathcal{T}}(H^A_i(E), H^{i+q}_A(F)).
\]

For $\oplus f_i \oplus g_i \oplus h_i \oplus i \in \mathbb{Z}$,
\[
d^p_{A_{2-p},p}(E, F)(g_i) = \oplus_{i \in \mathbb{Z}} ((-1)^{p+q} f_{i-1} \circ c^A_i(E) - c^A_{i+1}(E) \circ f_i) \in E_{p,q}^A(E, F).
\]

Proposition 2.4. For any heart $\mathcal{A}$ of $\mathcal{T}$ and any objects $E, F \in \mathcal{T}$, there exists a spectral sequence converging to $\text{Hom}^A_{\mathcal{T}}(E, F)$ with its $(p, q)$-components and differentials on the second sheet given by $E_{p,q}^A(E, F)$ and $d_{A_{2-p},p}(E, F)$.

Proof. For $P = E$ or $P = F$, we define a decreasing finite splitting sequence of subcomplexes $F^0(P) = \tilde{\tau}^A_{\leq -1}(P)$. By [BBD] \[3.1.3.4\], applied to $\mathcal{T}_0$, there exists a spectral sequence $E_{p,q}^{A_{p}} = \oplus_{j \in \mathbb{Z}} \text{Hom}^{\tau^A_{i+j}(\text{Gr}^p(E)), \text{Gr}^p(E)}$ that converges to $\text{Hom}^{\tau^A_{i+j}(E), \text{Gr}^p(E)}$. Here, $\text{Gr}^p(E) = F^j(E)/F^{j+1}(E) \cong H_{\mathcal{A}}^j(E)[i]$. With new variables $q' = -p$, $p' = 2p + q$, $i' = -i$, and $j' = -j$, \[E_{p,q}^{A_{p}}\] reads $\oplus_{q' + i' = j'} \text{Hom}^{\tau^A_{i+j}(E), \text{Gr}^p(E)}$. For $n \in \mathbb{Z}_{>0}$, $(p, q) \mapsto (p + n, q - n + 1)$ translates into $(p', q') \mapsto (p' + n, q' - n)$. Observe that because of change of variables, term $E_n$ in the spectral sequence from [BBD] \[3.1.3.4\] is now viewed as $E_{n+1}$. \[\square\]
Lemma 2.5. For a heart $\mathcal{A}$ of $\mathcal{T}$ and an object $E \in \mathcal{T}$, let $id_i$ be the identity morphism of $H^{2}_{\mathcal{A}}(E)$. If $E$ is not a zero object, then $\operatorname{Ker} d_{2,\mathcal{A}}^{0,0}(E, E)$ contains the one-dimensional vector space $(\oplus i id_i) \otimes C$.

Proof. Here, $\oplus id_i \in E_{2,\mathcal{A}}^{0,0}(E, E)$ and $d_{2,\mathcal{A}}^{0,0}(E, E)(\oplus id_i) = 0$. $\square$

2.2 $n$-Calabi-Yau categories

The dual of a vector space $V$ will be written $V^*$. 

Definition 2.6. [BoKa] Definition 3.1] A covariant additive functor $S : \mathcal{T} \to \mathcal{T}$ that commutes with shifts is called a Serre functor, if it is a category equivalence, and for any objects $E, F \in \mathcal{T}$, there exist bi-functorial isomorphisms $\psi_{E,F} : \operatorname{Hom}_{\mathcal{T}}(E, F) \cong \operatorname{Hom}_{\mathcal{T}}(F, S(E))^*$ such that the composite $(\psi_{S(E), S(F)}^{-1}) \circ \psi_{E,F}$ induces an isomorphism of functors $\{\phi_{E,F} \}_{F \in \mathcal{T}}$, the Serre duality of $\mathcal{T}$.

Definition 2.7. [Ko] A triangulated category $\mathcal{T}$ is called an $n$-Calabi-Yau category, if the shift $[n]$ is the Serre functor.

Definition 2.8. We define the $\mathcal{T}$-dimension of a heart $\mathcal{A}$ of $\mathcal{T}$ as the supremum of $n$ such that $\operatorname{Hom}^{\mathcal{T}}_{\mathcal{A}}(E, F) \neq 0$ for objects $E, F \in \mathcal{A}$.

Proposition 2.9. For any $n$-Calabi-Yau category $\mathcal{T}$, the $\mathcal{T}$-dimension of any heart of $\mathcal{T}$ is $n$.

Proof. For a non-zero object $E \in \mathcal{A}$, $\operatorname{Hom}^{\mathcal{T}}_{\mathcal{A}}(E, E) \cong \operatorname{Hom}^{\mathcal{T}}_{\mathcal{A}}(E, E)^* \neq 0$. For $m > n$ and any objects $E, F \in \mathcal{A}$, $\operatorname{Hom}^{\mathcal{T}}_{\mathcal{A}}(E, F) = \operatorname{Hom}^{\mathcal{T}}_{\mathcal{A}}(F, [m]) \cong \operatorname{Hom}^{\mathcal{T}}_{\mathcal{A}}(F[m], E[n])^* = \operatorname{Hom}^{\mathcal{T}}_{\mathcal{A}}(F[m], E[n])^* = 0$. $\square$

Corollary 2.10. For an $n$-Calabi-Yau category $\mathcal{T}$, a heart $\mathcal{A}$ of $\mathcal{T}$, and objects $E, F \in \mathcal{T}$, if $p < 0$ or $n < p$, then $E^{p,q}_{2,\mathcal{A}}(E, F) = 0$.

Proof. Here, $E^{p,q}_{2,\mathcal{A}}(E, F) = \oplus \operatorname{Hom}^{\mathcal{T}}_{\mathcal{A}}(H^i_\sigma(E), H^{i+q}_\sigma(F))$, which is zero when $p < 0$, since $\mathcal{A}$ is a heart of $\mathcal{T}$. When $p > n$, Proposition 2.9 applies. $\square$

Proposition 2.11. For an $n$-Calabi-Yau category $\mathcal{T}$, an object $E \in \mathcal{T}$, and $\sigma \in \operatorname{Stab}(\mathcal{T})$, let $k_1 > \cdots > k_m$ be all non-trivial phases of $E$. If for some $s \in \mathbb{Z}$, $k_{s+1} - k_s > n - 1$, then $E$ is decomposable.

Proof. Here, $E$ is an extension $E' \to E \to E''$ with $E'$ (resp. $E''$) being an extension of $H^i_\sigma(E)$ for $i < s$ (resp. $s \leq i$). For $i < s \leq j$, $\operatorname{Hom}^{\mathcal{T}}_{\mathcal{A}}(H^i_\sigma(E), H^j_\sigma(E)) = 0$. Hence, $\operatorname{Hom}^{\mathcal{T}}_{\mathcal{A}}(E'', E') = 0$. $\square$
For objects $E, F \in \mathcal{T}$ and $i \in \mathbb{Z}$, let $(E, F)^i = \dim \text{Hom}_\mathcal{T}^i(E, F)$. The Euler form on $K(\mathcal{T})$ is $\chi(E, F) = \sum_{i \in \mathbb{Z}} (-1)^i (E, F)^i$. The quotient $N(\mathcal{T}) = K(\mathcal{T})/K(\mathcal{T})^1$ is called the numerical Grothendieck group. For an $n$-Calabi-Yau category $\mathcal{T}$, the Euler form is (anti)symmetric depending on the parity of $n$ and factors through $N(\mathcal{T})$.

## 3 Stability conditions on 2-Calabi-Yau categories

From now on, $\mathcal{T}$ is assumed to be 2-Calabi-Yau.

**Lemma 3.1.** Let $A$ be a heart of $\mathcal{T}$. Then for any objects $E, F \in \mathcal{T}$ and any $q \in \mathbb{Z}$, $\text{Hom}^{1+q}_{\mathcal{T}}(E, F)$ has a filtration such that $\text{Ker} d_{2, A}^{0, q+1}(E, F)$, $\text{Coker} d_{2, A}^{0, q}(E, F)$, and $E_{2, A}^{1, q}(E, F)$ appear as distinct subquotients.

**Proof.** By Corollary 2.10, differentials on the third sheet are zero. Hence, again by Corollary 2.10, morphisms $\text{Hom}_{\mathcal{T}}(A, B)$, inequality $(\to)$ the exact sequence $0 \to E \to F \to 0$ is a short exact sequence in $\mathcal{T}$, $\text{Hom}(E, F)$ has a filtration such that $\text{Ker} d_{2, A}^{0, q+1}(E, F)$, $\text{Coker} d_{2, A}^{0, q}(E, F)$, and $E_{2, A}^{1, q}(E, F)$ appear as distinct subquotients. \hfill $\square$

**Lemma 3.2.** [BZ] Lemma 5.2 Suppose $A$ is a heart of $\mathcal{T}$ and $0 \to A \to B \to C \to 0$ is a short exact sequence in $\mathcal{A}$ with $(A, C)^0 = 0$. Then $(A, A)^1 + (C, C)^1 \leq (B, B)^1$.

**Proof.** The equation $\chi([B], [B]) = \chi([A] + [C], [A] + [C])$ reads $(B, B)^1 = (A, A)^1 + (C, C)^1 + 2((B, B)^0 + (A, C)^1 - ((A, A)^0 + (C, C)^0))$. The inequality $(B, B)^0 + (A, C)^1 - ((A, A)^0 + (C, C)^0) \geq 0$ follows from the exact sequence $0 \to \text{Hom}_{\mathcal{T}}(C, A) \to \text{End}_{\mathcal{T}}(B) \to \text{End}_{\mathcal{T}}(A) \oplus \text{End}_{\mathcal{T}}(C) \to \text{Hom}^{1}_{\mathcal{T}}(C, A)$, which is obtained from the condition $(A, C)^0 = 0$ and endomorphisms of the exact triangle $A \to B \to C$. \hfill $\square$

**Lemma 3.3.** For a heart $\mathcal{A}$ of $\sigma \in \text{Stab}(\mathcal{T})$ and an object $E \in \mathcal{T}$,

$$(E, E)^1 \geq \sum_{i \in \mathbb{Z}} (H^i_{\mathcal{A}}(E), H^i_{\mathcal{A}}(E))^1 \geq \sum_{k \in \mathbb{R}, \mathcal{S} \in J^k_{\mathcal{A}}} (S, S)^1.$$

**Proof.** By Lemma 3.1, $E_{2, A}^{1, 0}(E, E)$ is a subquotient of $\text{Hom}^{1}_{\mathcal{T}}(E, E)$. Hence, $(E, E)^1 \geq \dim E_{2, A}^{1, 0}(E, E) = \sum_{i \in \mathbb{Z}} (H^i_{\mathcal{A}}(E), H^i_{\mathcal{A}}(E))^1$.

For some $j \in \mathbb{R}$, $\mathcal{A} = \mathcal{P}(j-1, j)$). So for each $i \in \mathbb{Z}$ and $k \in (i + j - 1, i + j]$, the HN-filtration of $H^k_{\mathcal{A}}(E)$ gives a short exact sequence $0 \to A \to H^k_{\mathcal{A}}(E) \to C \to 0$ in $\mathcal{A}$ such that $A$ and $B$ are extensions of $H^k_{\mathcal{A}}(E)$ for $k' > k$ and $k' \leq k$; in particular, $\text{Hom}_{\mathcal{T}}(A, B) = 0$. Hence, by Lemma 3.2, the second inequality follows.

By Lemma 3.2 and Corollary 1.2, the last inequality follows. \hfill $\square$

**Definition 3.4.** If an object $E \in \mathcal{T}$ satisfies $\sum_{i}(E, E)^i = 2$, then $E$ is called spherical ([ST] Definition 1.1]).
Lemma 3.5. Let $A$ be a heart of $\mathcal{T}$ and $E \in \mathcal{T}$ be a spherical object. If for some spherical object $S \in A$, every $H^i_A(E)$ is a multiple of $S$, then $E$ is a shift of $S$.

Proof. By taking a shift of $E$, for some $n \in \mathbb{Z}_{\geq 0}$, we may suppose $H^i_A(E)$ is nonzero only for $0 \leq i \leq n$. Then, $E_{2,i+1}^{0,n} = \oplus_{i \in \mathbb{Z}} \text{Hom}_\mathcal{T}(H^i_A(E), H^{i+n+1}_A(E)) = 0$. So, $\text{Coker} d^{0,n+1} = E_{2,i+1}(E, E) = \text{Hom}^2_\mathcal{T}(H^0_A(E), H^n_A(E))$, which is by Lemma 3.1 a subquotient of $\text{Hom}^2_\mathcal{T}(E, E)$. Since $E$ is spherical, $n = 0$. \hfill $\square$

Lemma 3.6. Let $\sigma \in \text{Stab}(\mathcal{T})$, $E \in \mathcal{T}$ be a non-semistable spherical object, and $k_1 > \cdots > k_n$ be all nontrivial phases of $E$. If every $H^k(E)$ is a multiple of a stable spherical object $S_i$, then $k_{s-1} - k_s < 1$ for some $s$.

Proof. Since $E$ is spherical, it is indecomposable. So by Proposition 2.11, there exists $k_{i-1} - k_i \leq 1$. If every $k_{i-1} - k_i = 1$, then since $E$ is not semistable, by Lemma 3.9 there exists $i$ such that $S_i \neq S$. Since $S_i \neq S$, and $S_i$ are non-isomorphic stable objects, $\text{Hom}_\mathcal{T}(S_i, S_i) = \text{Hom}_\mathcal{T}(S_i, S_i) = 0$. So $\text{Hom}^1_\mathcal{T}(S_i, S_i) = 0$. Then, since for any $p < i < q$, we have $\text{Hom}^1_\mathcal{T}(S_i, S_i) = 0$, $E$ would be decomposable. \hfill $\square$

Remark 3.7. If $A \in \mathcal{T}$ is stable for some stability condition, then $(A, A)^1$ is even; because, the skew-symmetric, non-degenerate pairing $\text{Hom}^1_\mathcal{T}(A, A) \times \text{Hom}^1_\mathcal{T}(A, A) \to \text{Hom}^1_\mathcal{T}(A, A) \cong \mathbb{C}$ implies $\text{Hom}^1_\mathcal{T}(A, A)$ is a symplectic vector space.

The pairing above is a simple case of the one in [RV Proposition I.1.4].

Definition 3.8. If an object $E \in \mathcal{T}$ satisfies $(E, E)^1 = 0$, then $E$ is called rigid (Mii Definition 3.1)).

Lemma 3.9. For $\sigma \in \text{Stab}(\mathcal{T})$, an object $E \in \mathcal{T}$, and a nontrivial phase $k \in \mathbb{R}$ of $E$, any rigid JH-block of $H^k_\sigma(E)$ is a multiple of a stable spherical object.

Proof. Let $S$ be a rigid JH-block of $H^k_\sigma(E)$. Then $[S] = n[A]$ for some stable object $A$ and $n > 0$. Since $S$ is semistable and rigid, $\chi(S, S) = 2(S, S)^0$. Since $A$ is stable, $0 < \chi(S, S) = n^2(2(A, A)^0 - (A, A)^1) = n^2(2 - (A, A)^1)$. So, by Remark 3.7, the Euler form forces $(A, A)^1 = 0$. \hfill $\square$

Lemma 3.10. For $\sigma \in \text{Stab}(\mathcal{T})$, a rigid object $E \in \mathcal{T}$, and a nontrivial phase $k$ of $E$, (a) any JH-block of $H^k_\sigma(E)$ is a multiple of a stable spherical object; and (b) if $J^k_\sigma(E)$ has more than one object, then there exist non-isomorphic stable spherical factors for $H^k_\sigma(E)$.

Proof. By Lemmas 3.3 and 3.9 (a) holds. For (b), not all stable factors of $H^k_\sigma(E)$ are isomorphic; otherwise, $J^k_\sigma(E)$ would have only one object. \hfill $\square$

Proposition 3.11. For any $\sigma \in \text{Stab}(\mathcal{T})$, if there exists a non-semistable spherical object, then in some heart of $\sigma$, there exist two non-isomorphic stable spherical objects.
Proof. Let $E \in \mathcal{T}$ be a non-semistable spherical object and $k_1 > \cdots > k_n$ be all nontrivial phases of $E$. Since $E$ is indecomposable, by Proposition 2.11 $k_{i-1} - k_i \leq 1$.

If some $J_{0i}^E(E)$ has more than one object, then the statement follows by Lemma 3.10 (b). Let us assume otherwise; by Lemma 3.10 (a), every $H_{\phi}^E(E)$ is a multiple of a stable spherical object. Since $E$ is not semistable, Lemma 3.6 applies.

### 3.1 Twist functors

**Definition 3.12.** For a spherical object $E \in \mathcal{T}$ and an object $F \in \mathcal{T}$, the cone of the evaluation map $R\text{Hom}_\mathcal{T}(E, F) \otimes E \to F$ is denoted by $T_E(F)$, the twist functor of $E$ (ST Section 1.1).

For $\sigma = (Z, \mathcal{P}) \in \text{Stab}(\mathcal{T})$, a spherical object $E \in \mathcal{T}$, and $T_E \in \text{Aut}(\mathcal{T})$, let $T_E\sigma = T_E(\sigma) = (T_E Z, T_E \mathcal{P}) = (Z \circ T_E^{-1}, T_E \circ \mathcal{P})$.

**Lemma 3.13.** For $\sigma = (Z, \mathcal{P}) \in \text{Stab}(\mathcal{T})$, let $E \in \mathcal{P}(0)$ be a stable spherical object and $F \in \mathcal{P}((0, 1))$ be an object such that $\text{Hom}_\mathcal{T}(E, F) = 0$. Then $F \in (T_E \mathcal{P})([0, 1])$.

**Proof.** By the assumption on $F$, $\text{Hom}_\mathcal{T}^i(E, F) = 0$ unless $i = 0$ or 1; so, $R\text{Hom}_\mathcal{T}(E, F) \otimes E$ is an extension of multiples of $E$ and $E[1]$. Since $E = T_E(E[1])$ lies in $(T_E \mathcal{P})(1)$, $R\text{Hom}_\mathcal{T}(E, F) \otimes E$ lies in $(T_E \mathcal{P})([0, 1])$. By the definition of $T_E\sigma$, $T_E(F) \in (T_E \mathcal{P})([0, 1])$. Hence, the statement follows by the exact triangle $R\text{Hom}_\mathcal{T}(E, F) \otimes E \to F \to T_E(F)$.

**Corollary 3.14.** In addition to the assumptions in Lemma 3.13, assume further that $\phi(F) \in (0, 1)$ and $\text{Hom}_\mathcal{T}(E, F) \neq 0$. Then $F$ is not semistable in $T_E\sigma$.

**Proof.** In this setting, $Z(F)$ is in the open upper-half plane of $\mathbb{C}$. The same is then true for $(T_E Z)(F) = Z(F) - \chi(F, E)Z(E)$, since $Z(E) \in \mathbb{R}$. So if $F$ were semistable in $T_E\sigma$, then $F \in (T_E \mathcal{P})([0, 1])$ would imply that $F \in (T_E \mathcal{P})([0, 1])$. However, $\text{Hom}_\mathcal{T}(E, F) \neq 0$ would contradict $(T_E \phi)(F) < (T_E \phi)(E)$.

**Remark 3.15.** For $\sigma \in \text{Stab}(\mathcal{T})$, we will refer to the following conditions: (a) there exist non-isomorphic stable spherical objects $E$ and $F$ in $\sigma$ such that $\text{Hom}_\mathcal{T}(E, F) \neq 0$ and $0 = \phi(E) < \phi(F) < 1$; (b) any two stable spherical objects of the same phases are isomorphic.

**Corollary 3.16.** In addition to the assumptions in Corollary 3.14, assume that $F$ is spherical and the condition (b) in Remark 3.15. Then, $H^0_{T_E\sigma}(F) = 0$.

**Proof.** By the condition (b) in Remark 3.15, $E[-1] = T_E(E)$ is the only stable spherical object in $T_E\sigma$ of the phase zero. So by Lemma 3.10 (a), $H^0_{T_E\sigma}(F)$ is a multiple of $E[-1]$. Here, by Lemma 3.13, $F \in (T_E \mathcal{P})([0, 1])$. So if $H^0_{T_E\sigma}(F)$ were not zero, then $\text{Hom}_\mathcal{T}(F, E[-1])$ would not be zero; however, since $E$ and $F$ are in some heart of $\sigma$, $\text{Hom}_\mathcal{T}(F, E[-1]) = 0$. □
We have some Lemma 4.1.

4 Cotangent bundle of \(\mathbb{P}^1\)

Let \(Z = \mathbb{P}^1\), \(X\) the cotangent bundle of \(Z\), \(\text{Coh}_Z(X)\) the category of the coherent sheaves of \(X\) supported by \(Z\), and \(\mathcal{T}\) the full subcategory of \(D(X)\) consisting of objects supported on \(Z\). The space \(X\) is the minimal resolution of the Kleinian singularity \(\mathbb{C}^2/\mathbb{Z}_2\). Let us prove the connectedness of \(\text{Stab}(\mathcal{T})\).

4.1 Pairs of stable spherical objects

Lemma 4.1. For spherical objects \(E, F \in \mathcal{T}\), we have the following: (a) for some \(s_F(E) = \pm 1\) and \(p_F(E) \in \mathbb{Z}\), \([E] = s_F(E)[F] + p_F(E)[\mathcal{O}_Z]\); (b) \(p_E(F) = p_E(T_E(F))\).

Proof. We have \(\mathcal{N}(\mathcal{T}) \cong \mathbb{Z} \cdot Z(\mathcal{O}_Z)\), so \(\chi(\mathcal{O}_Z, \mathcal{O}_Z) = 2 = \chi(F, F)\) implies that \([F]\) is also a basis of \(\mathcal{N}(\mathcal{T})\). Since \(\mathbb{Z}[\mathcal{O}_Z] = \text{Ker}[K(\mathcal{T}) \to \mathcal{N}(\mathcal{T})]\), \([F]\) and \([\mathcal{O}_Z]\) is a basis of \(K(\mathcal{T})\). Now, \(s_F(E) = \pm 1\), since \([E] = s_F(E)[F] \in \mathcal{N}(\mathcal{T})\). For the latter part, since \(T_E[F] = [F] - \chi(E, F)[E] = (s_E - \chi(E, F))[E] + p_E(F)[\mathcal{O}_Z]\), \(p_E(F) = p_E(T_E(F))\).

Definition 4.2. For a spherical object \(E \in \mathcal{T}\), we will call the sign \(s_{\mathcal{O}_Z}(E)\), the sign of \(E\).

Lemma 4.3. If \(E, F, S\) are spherical objects in \(\mathcal{T}\) such that \(E\) and \(F\) have different signs and \(\phi(E[-1]) < \phi(S) < \phi(F) < \phi(E)\), then \(F\) and \(S\) have the same signs.

Proof. Since \(\phi(F) - \phi(S) \notin \mathbb{Z}\), \(Z(F \oplus S) \neq 0\). So if \(F\) and \(S\) had different signs, then by Lemma 4.1 (a), \(Z(F \oplus S) = (p_E(F) + p_E(S))Z(\mathcal{O}_Z) \neq 0\). Hence, \(\phi(S) < \arg((p_E(F) + p_E(S))Z(\mathcal{O}_Z))/\pi < \phi(F)\). However, since \(\phi(F) - \phi(S) \notin \mathbb{Z}\), again by Lemma 4.1 (a), \(Z(F \oplus F) = p_E(F)Z(\mathcal{O}_Z) \neq 0\), which implies \(\phi(F) < \arg(p_E(F)Z(\mathcal{O}_Z))/\pi < \phi(E)\).

By [ST] Theorem 1.2, twist functors restrict to autoequivalences of \(\mathcal{T}\).

Lemma 4.4. Suppose \(\sigma \in \text{Stab}(\mathcal{T})\) satisfies the conditions (a) and (b) in Remark 3.15. Then for \(E\) and \(F\) in the condition (a) in Remark 3.15, if \(E\) and \(F\) have different signs, then there exist stable spherical objects \(E'\) and \(F'\) such that \(E'\) and \(F'\) have different signs, \(\phi(E') < \phi(F') < \phi(E'[1])\), and 

\[0 < |(Z(E') + Z(F'))/Z(\mathcal{O}_Z)| < |(Z(E) + Z(F))/Z(\mathcal{O}_Z)|.\]
Proof. By the conditions (a) and (b) in Remark 3.15 and Proposition 3.17, $F$ is in $(T_E P)((0, 1])$ and not semistable in $T_E \sigma$. Let $1 = k_1 > \cdots > k_n > 0$ be all nontrivial phases of $F$ in $T_E \sigma$. By the condition (b) in Remark 3.15 and Lemma 4.10, each semistable factor of $F$ of the phase $k_i$ is a multiple of a stable spherical object $S_i$.

Since $E[-1]$ and $F$ have the same signs and $(T_E \phi)(E[-1]) < (T_E \phi)(S_n) < (T_E \phi)(F) < (T_E \phi)(E)$, by Lemma 4.3, $E[-1]$ and $S_n$ have the same signs. So $E' = T_E^{-1}(E)$ and $F' = T_E^{-1}(S_n)$ have different signs.

Since all $E[-1], S_n$, and $F$ have the same signs, $0 = (T_E \phi)(E[-1]) < (T_E \phi)(S_n) < (T_E \phi)(F)$ reads $0 = \text{arg}(Z[E]) < \text{arg}(Z[E] + p_E(F'|O_x)) < \text{arg}(Z(E) + p_E(T_E^{-1}(F))|O_x))$. Hence, $0 < |p_E(F')| < |p_E(T_E^{-1}(F))|$. By Lemma 4.1 (b), $p_E(T_E^{-1}(F)) = p_E(F)$. So $0 < |p_E(F')| < |p_E(F)|$.

Since $p_E(E') = p_E(E) = 0$, $(Z(E') + Z(F'))/Z(O_x) = p_E(F')$ and $(Z(E) + Z(F))/Z(O_x) = p_E(F)$. \(\square\)

**Proposition 4.5.** For $\sigma \in \text{Stab}(T)$, if there exists a non-semistable spherical object, then there exist two stable spherical objects in some heart of $\sigma$ such that they have no morphisms between them.

Proof. Since there exists a non-semistable spherical object, by Proposition 4.11, some heart of $\sigma$ contains two non-isomorphic stable spherical objects $E$ and $F$.

Since any pair of non-isomorphic stable spherical objects of the same phases satisfies the conclusion, we may assume otherwise; i.e., we may assume the condition (b) in Remark 3.15. In particular, $E$ and $F$ have different phases.

By taking a shift of $E$ or $F$ if necessarily, we may assume $E$ and $F$ have different signs. By using rotation and switching of $E$ and $F$, we can assume $0 = \phi(E) < \phi(F)$. Now, if $\text{Hom}_T(E, F) = 0$, then again $E$ and $F$ satisfy the conclusion. So let $\text{Hom}_T(E, F) \neq 0$, so for $E$ and $F$, the condition (a) in Remark 3.15 is satisfied.

For our convenience, let $E_0 = E$ and $F_0 = F$. By Lemma 4.4, there exist stable spherical objects $E_1$ and $F_1$ such that $E_1$ and $F_1$ have different signs, $\phi(F_1) < \phi(E_1) < \phi(F_1[1])$, and $0 < |(Z(E_1) + Z(F_1))/Z(O_x)| < |(Z(E_0) + Z(F_0))/Z(O_x)|$.

So if we keep assuming $\text{Hom}_T(E_i, F_i) \neq 0$, then we would obtain an infinite sequence of strictly decreasing positive integers $|p_E(F_i)|$. Hence, for some $E_i$ and $F_i$, $\text{Hom}_T(E_i, F_i) = \text{Hom}_T(F_i, E_i) = 0$. \(\square\)

### 4.2 Pairs of stable spherical objects and autoequivalences

**Lemma 4.6.** For $s, t, i \in \mathbb{Z}$, $\text{Hom}^*_T(O_Z(s), O_Z(t)) = 0$ if and only if (a) $i = 0$ and $s - t > 0$, (b) $i = 1$ and $|s - t| < 2$, or (c) $i = 2$ and $s - t < 0$.

Proof. For (a), $\text{Hom}_T(O_Z(s), O_Z(t)) = \text{Hom}_T(O_Z(s - t), O_Z) = 0$ if and only if $s - t > 0$; by the Serre duality, (c) follows.

For (b), if $|s - t| < 2$ and $k \neq 0$, then $\text{Hom}^*_D(O_Z, O(s - t)) = \text{Hom}^*_D(O_Z(s - t), O) = 0$. So by [13, *Lemma 4.6*], $\text{Hom}^*_T(O_Z(s - t), O_Z) = \text{Hom}^*_D(O_Z(s - t), O_Z) \oplus \text{Hom}^*_D(O_Z, O_Z(s - t))[-2]$; in particular, $\text{Hom}^*_T(O_Z(s - t), O_Z) = 0.$
For $|t-s| \geq 2$, let us recall that for the canonical bundle $\mathcal{O}(-2)$ of $Z$, the spectral sequence in the proof of [Br3] Lemma 4.6 reads $E_2^{p,q} = \text{Hom}_F^p(\mathcal{O}(t-s) \otimes \wedge^q \mathcal{O}(-2))$. Since the homological dimension of $\text{Coh}(Z)$ is one, $E_2^{0,q} = 0$ for $p > 1$; also, since $\text{Coh}(Z)$ is a heart of $D(\text{Coh} Z)$, $E_2^{0,q} = 0$ for $p < 0$.

If $t-s \geq 2$, then since $E_2^{0,1} = \text{Hom}_Z(\mathcal{O}, \mathcal{O}(t-s-2)) \neq 0$, $\text{Hom}_T^1(\mathcal{O}_Z, \mathcal{O}_Z(t-s))$ has a nonzero subquotient. If $t-s \leq -2$, then the Serre duality applies.

For objects $E, F \in T$, we will write $H^i(E)$ and $E_2^{p,q}(E, F)$ for $H^i_{\text{Coh}_Z(X)}(E)$ and $E_2^{p,q}_{\text{Coh}_Z(X)}(E, F)$ (see Definition [22]).

**Lemma 4.7.** For objects $E, F \in T$ and $s, t, q \in \mathbb{Z}$, let $\text{Hom}_T^i(E, F) = 0$ unless $i = 1$, $\mathcal{O}_Z(s)$ is a summand of $H^q(E)$, and $F = \mathcal{O}_Z(t)$. Then, (a) $q \neq 0$ implies $|s-t| < 2$, (b) $q = 0$ and $\text{Hom}_T(H^{-1}(E), F) = 0$ implies $s - t < 0$, and (c) $q = 0$ and $\text{Hom}_T(\mathcal{O}_Z(H^1(E), F) = 0$ implies $s - t > 0$.

**Proof.** For (a), if $|s-t| \geq 2$, then by Lemma [14], $\text{Hom}_T^1(\mathcal{O}_Z(H^q(E), F)) \neq 0$; i.e., $E_2^{2,-q}(E, F) \neq 0$. Hence, by Lemma [5.1] $\text{Hom}_T^{-1-q}(E, F) \neq 0$. So $1 = q - 1$.

For (b), by $E_2^{0,1}(E, F) = \text{Hom}_T(H^{-1}(E), F) = 0$. $\text{Coker} d_{-1}^{0,1}(E, F)$ is isomorphic to $E_2^{2,0}(E, F)$. Since by Lemma [3.1] $\text{Coker} d_{-1}^{0,1}(E, F)$ is a subquotient of $\text{Hom}_T^2(E, F) = 0$, $E_2^{2,0}(E, F) = \text{Hom}_T^2(H^0(E), F) = 0$. So Lemma [4.6] (c) applies.

For (c), by $E_2^{-1}(E, F) = \text{Hom}_T(\mathcal{O}_Z(H^1(E), F) = 0$, $\text{Coker} d^{-1,0}(E, F)$ is isomorphic to $E_2^{2,0}(E, F)$. Since by Lemma [3.1] $\text{Coker} d^{-1,0}(E, F)$ is a subquotient of $\text{Hom}_T(E, F) = 0$, $E_2^{0,0}(E, F) = \text{Hom}_T(H^0(E), F) = 0$. So Lemma [4.6] (a) applies.

**Remark 4.8.** By [IU] Section 5, for any spherical object $E \in T$ and any $q \in \mathbb{Z}$, we have some $v \in \mathbb{Z}$ and $f_q, g_q \in \mathbb{Z}_{\geq 0}$ such that each $H^q(E)$ is isomorphic to $\mathcal{O}_Z(v)^{f_q} \oplus \mathcal{O}_Z(v-1)^{g_q}$, here, $l(E)$ is defined as $\sum_q (f_q + g_q)$.

Let $S_Z(X)$ be the subgroup of $\text{Aut}(T)$ generated by twists and shift functors on $T$.

**Lemma 4.9.** For spherical objects $E, F \in T$ and some $t \in \mathbb{Z}$, if $l(E) > 1$, $F$ is a shift of $\mathcal{O}_Z(t)$, and for every irreducible summand $\mathcal{O}_Z(s)$ of $H^*(E)$ we have $|s-t| < 2$, then for some $\Psi \in S_Z(X)$ we have $l(\Psi(E)) < l(E)$ and $l(\Psi(F)) = l(F)$.

**Proof.** By Remark [1.3] for some $f_q, g_q \in \mathbb{Z}_{\geq 0}$, either every $H^q(E) \cong \mathcal{O}_Z(t+1)^{f_q} \oplus \mathcal{O}_Z(t)^{g_q}$, or every $H^q(E) \cong \mathcal{O}_Z(t)^f \oplus \mathcal{O}_Z(t-1)^{g_q}$. By [IU] Claim 5.2, for the former case, $l(T_{\mathcal{O}_Z(t)}(E)) < l(E)$ and for the latter case, $l(T_{\mathcal{O}_Z(t-1)}(E)) < l(E)$. Also for $F_i$, by [IU] Lemma 4.15 (i)(1)], $T_{\mathcal{O}_Z(t)}(\mathcal{O}_Z(t)) = \mathcal{O}_Z(t)[-1]$ and $T_{\mathcal{O}_Z(t-1)}(\mathcal{O}_Z(t)) = \mathcal{O}_Z(t-2)[-1]$.

**Lemma 4.10.** For any $v \in \mathbb{Z}$, we have $\Psi \in S_Z(X)$ such that $\{\Psi(\mathcal{O}_Z(v)), \Psi(\mathcal{O}_Z(v-1)[1])\} = \{\mathcal{O}_Z, \mathcal{O}_Z(-1)[1]\}$.
Proof. By Lemma 4.15 (i)(2)], $T_{O_2(v)} \circ T_{O_2(-1)}$ acts as tensoring with $O_Z(-2)$. So up to $S_Z(X)$, $v$ is zero or one. For the latter case, by Lemma 4.15 (i)(1)], $T_{O_2(O_Z[1])} = O_Z$ and $T_{O_2(O_Z(1))} = O_Z(-1)[1]$.

**Proposition 4.11.** For spherical objects $E, F \in T$ with $\text{Hom}_{T}^{i}(E, F) = 0$ unless $i = 1$, we have $\Psi \in S_Z(X)$ such that $\{\Psi(E), \Psi(F)\} = \{O_Z, O_Z(-1)[1]\}$.

Proof. By Proposition 5.1, for some $\Psi_1 \in S_Z(X)$ and $t \in \mathbb{Z}$, $\Psi_1(F) = O_Z(t)$. We will prove that for every summand $O_Z(s)$ of $H^s(\Psi_1(E))$, $|s - t| < 2$.

For some $q \neq 0$, if $O_Z(s)$ is a summand of $H^s(\Psi_1(E))$, then by Lemma 4.7 (a), $|s - t| < 2$. Consider the case $q = 0$. If for a summand $O_Z(s)$ of $H^s(\Psi_1(E))$, $s - t \geq 2$, then by Remark 4.8 any irreducible summand of $H^{s}(\Psi_1(E))$ is $O_Z(r)$ of some $r - t > 0$. So by Lemma 4.6 (a), $\text{Hom}_{T}(H^{-1}(\Psi_1(E)), \Psi_1(F)) = 0$.

Here, by Lemma 4.7 (b), $s - t < 0$ in contradiction to $s - t \geq 2$. If for a summand $O_Z(s)$ of $H^0(\Psi_1(E))$, $s - t \leq -2$, then by Remark 4.8 any irreducible summand of $H^{s}(\Psi_1(E))$ is $O_Z(r)$ of some $r - t < 0$. So by Lemma 4.6 (c), $\text{Hom}_{T}(H^{s}(\Psi_1(E)), \Psi_1(F)) = 0$.

Hence by Lemma 4.9 for some $\Psi_2$ and $l, m, n \in \mathbb{Z}$, $\Psi_2 \circ \Psi_1(E) = O_Z(m)[l]$ and $\Psi_2 \circ \Psi_1(F) = O_Z(n)$. By the argument above on $\Psi_2 \circ \Psi_1(E)$ and $\Psi_2 \circ \Psi_1(F)$, $|m - n| < 2$. Furthermore, we will prove that either $m = n - 1$ and $l = 1$, or $n = m - 1$ and $l = -1$.

If $m = n - 1$, then by Lemma 4.6 (a), $\text{Hom}_{T}(\Phi_2 \circ \Phi_1(E), \Phi_2 \circ \Phi_1(F)) \neq 0$, which by the assumption, implies $l = 1$. If $n = m - 1$, then by Lemma 4.6 (c), $\text{Hom}_{T}(\Phi_2 \circ \Phi_1(E), \Phi_2 \circ \Phi_1(F)) \neq 0$, which by the assumption, implies $l + 2 = 1$. If $m$ were equal to $n$, then by Lemmas 4.6 (a) and 4.6 (c), both $\text{Hom}_{T}(\Phi_2 \circ \Phi_1(E), \Phi_2 \circ \Phi_1(F))$ and $\text{Hom}_{T}(\Phi_2 \circ \Phi_1(E), \Phi_2 \circ \Phi_1(F))$ would be nonzero, which contradicts the assumption.

So, when $l = 1$, $\Phi_2 \circ \Phi_1(E) = O_Z(m - 1)[1]$ and $\Phi_2 \circ \Phi_1(F) = O_Z(n)$; also, when $l = -1$, $\Phi_2 \circ \Phi_1(E)[1] = O_Z(m)$ and $\Phi_2 \circ \Phi_1(F)[1] = O_Z(m - 1)[1]$. Now, Lemma 4.10 applies.

### 4.3 Connectedness

By Theorem 1.3, a connected component $\text{Stab}_0(T) \subset \text{Stab}(T)$ is invariant under $S_Z(X)$. Let $\mathcal{A} \subset T$ be the smallest extension-closed full subcategory containing $O_Z$ and $O_Z(-1)[1]$, and $U$ be a subset of $\text{Stab}(T)$ consisting of stability conditions $(Z, P)$ such that $\mathcal{P}((0, 1)) = \mathcal{A}$, and $\text{Im} Z(O_Z)$, $\text{Im} Z(O_Z(-1)[1]) > 0$ (Lemma 3.1)]. By Lemma 3.6], for any $\sigma \in \text{Stab}_0(T)$, there exists $\Psi \in S_Z(X)$ such that some rotation of $\Psi(\sigma)$ lies in the closure of $U$.

Hence, to show the connectedness of $\text{Stab}(T)$, we will prove that for any $\sigma \in \text{Stab}(T)$, there exists $\Psi \in S_Z(X)$ such that $\mathcal{A}$ is a heart of $\Psi(\sigma)$.

**Theorem 4.12.** $\text{Stab}(T)$ is connected.

Proof. Let $\sigma \in \text{Stab}(T)$. Suppose all spherical objects are semistable. Then, by Lemma 4.6 (a), for any $v \in \mathbb{Z}$, $\phi(O_Z(v - 1)) \leq \phi(O_Z(v))$. We will prove that for some $w \in \mathbb{Z}$, $\phi(O_Z(w - 1)) < \phi(O_Z(w))$. 

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Let us assume otherwise; i.e., for any \( v \in \mathbb{Z} \), \( \phi(\mathcal{O}_Z(v-1)) = \phi(\mathcal{O}_Z(v)) \). Then, since by Lemma 4.6, \( R\mathrm{Hom}_{T}(\mathcal{O}_Z(1), \mathcal{O}_Z) \otimes \mathcal{O}_Z(1) \) is a multiple of \( \mathcal{O}_Z[1][2] \), the exact triangle \( \mathcal{O}_Z \to T_{\mathcal{O}_Z(1)}(\mathcal{O}_Z) \to R\mathrm{Hom}_{T}(\mathcal{O}_Z(1), \mathcal{O}_Z) \otimes \mathcal{O}_Z[1][1] \) would be the nontrivial \( \mathrm{HN} \)-filtration of the semistable spherical object \( T_{\mathcal{O}_Z(1)}(\mathcal{O}_Z) \).

So, since by Lemma 4.6 (b), \( \phi(\mathcal{O}_Z(w+1)[-1]) \leq \phi(\mathcal{O}_Z(w-1)) \), \( \phi(\mathcal{O}_Z(w+1)[-1]) \leq \phi(\mathcal{O}_Z(w)) \leq \phi(\mathcal{O}_Z(w+1)) \). Hence, \( \mathcal{O}_Z(w) \) and \( \mathcal{O}_Z(w-1)[1] \) are in a heart of \( \sigma \). Now, Lemma 4.10 applies.

If not all spherical objects are semistable, then by Proposition 4.5, there exist non-isomorphic stable spherical objects \( E \) and \( F \) in some heart of \( \sigma \) such that \( \mathrm{Hom}_{T}(E, F) = \mathrm{Hom}_{T}(F, E) = 0 \). By the Serre duality, \( \mathrm{Hom}^{2}_{\mathcal{O}_Z}(E, F) = 0 \). Since \( E \) and \( F \) are in some heart of \( \sigma \), for \( i < 0 \), \( \mathrm{Hom}^{i}_{\mathcal{O}_Z}(E, F) = 0 \). So by Proposition 2.9, \( \mathrm{Hom}^{i}_{\mathcal{O}_Z}(E, F) = 0 \) unless \( i = 1 \). Now, Proposition 4.11 applies.

References

[An] R. Anno, *Spherical functors*, Talk at University of Massachusetts Amherst Quantum Field Theory Seminar (2006).

[AP] D. Abramovich and A. Polishchuk, *Sheaves of \( t \)-structures and valuative criteria for stable complexes*, J. Reine Angew. Math. 590 (2006), 89–130; MR2208130, also [math.AG/0309435](https://arxiv.org/abs/math.AG/0309435).

[AB] D. Arcara and A. Bertram, *New moduli spaces for K3 surfaces*, Talk at the 2005 Seattle AMS Summer Institute.

[BGI] P. Barthelot, A. Grothendieck, and L. Illusie, *Théorie des intersections et théorème de Riemann-Roch*, Lecture Notes in Math., 225, Springer, Berlin, 1971; MR0354655, also [math.AG/0309435](https://arxiv.org/abs/math.AG/0309435).

[BBD] A. A. Beilinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, in Analysis and topology on singular spaces, I (Luminy, 1981), 5–171, Astérisque, 100, Soc. Math. France, Paris, 1982; MR0751966 (86g:32015).

[Be] R. Bezrukavnikov, *Noncommutative counterparts of the Springer resolution*, in Proceedings of the International Congress of Mathematicians, Vol II, (Madrid 2006), 1119–1144, European Mathematical Society Publishing House, also [math.RT/0604445](https://arxiv.org/abs/math.RT/0604445).

[BoKa] A. I. Bondal and M. M. Kapranov, *Representable functors, Serre functors, and reconstructions*, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 53 (1989), no. 6, 1183–1205, 1337; translation in Math. USSR-Izv. 35 (1990), no. 3, 519–541; MR1039961 (91b:14013).

[Br1] T. Bridgeland, *Stability conditions on triangulated categories*, to appear in Ann. of Math. (2), [math.AG/0212237](https://arxiv.org/abs/math.AG/0212237).

[Br2] T. Bridgeland, *Stability conditions on K3 surfaces*, [math.AG/0307164](https://arxiv.org/abs/math.AG/0307164).
[Br3] T. Bridgeland, *t-structures on some local Calabi-Yau varieties*, J. Algebra **289** (2005), no. 2, 453–483; MR2142382 (2006a:14067).

[Br4] T. Bridgeland, *Stability conditions and Kleinian singularities*, math.AG/0508257

[Br5] T. Bridgeland, *Stability conditions on a non-compact Calabi-Yau threefold*, Comm. Math. Phys. **266** (2006), no. 3, 715–733, also math.AG/0509048

[Br6] T. Bridgeland, *Derived categories of coherent sheaves*, in Proceedings of the International Congress of Mathematicians, Vol II, (Madrid 2006), 563–582, European Mathematical Society Publishing House, also math.AG/0602129

[BuKr] I. Burban and B. Kreussler, *Derived categories of irreducible projective curves of arithmetic genus one*, to appear in Compos. Math., math.AG/0503496

[Do1] M. R. Douglas, *D-branes on Calabi-Yau manifolds*, in European Congress of Mathematics, Vol. II (Barcelona, 2000), 449–466, Progr. Math., **202**, Birkhäuser, Basel, 2001; MR1909947 (2004d:81090), also math.AG/0009209.

[Do2] M. R. Douglas, *D-branes, categories and $N = 1$ supersymmetry*, J. Math. Phys. **42** (2001), no. 7, 2818–2843; MR1840318 (2003b:81158), also hep-th/0011017

[Do3] M. R. Douglas, *Dirichlet branes, homological mirror symmetry, and stability*, in Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002), 395–408, Higher Ed. Press, Beijing, 2002; MR1957548 (2004c:81200), also math.AG/0207021.

[IU] A. Ishii and H. Uehara, *Autoequivalences of derived categories on the minimal resolutions of $A_n$-singularities on surfaces*, J. Differential Geom. **71** (2005), no. 3, 385–435; MR2198807 (2006k:14024).

[GM] S. I. Gelfand and Y. I. Manin, *Methods of homological algebra*, Second edition, Springer, Berlin, 2003; MR1950475 (2003m:18001).

[GKR] A. L. Gorodentsev, S. A. Kuleshov and A. N. Rudakov, *t-stabilities and t-structures on triangulated categories*, (Russian) Izv. Ross. Akad. Nauk Ser. Mat. **68** (2004), no. 4, 117–150; translation in Izv. Math. **68** (2004), no. 4, 749–781 ; MR2084563 (2005j:18008), also math.AG/0312442

[Hu] D. Huybrechts, *Derived and abelian equivalence of K3 surfaces*, math.AG/0604150

[KST] H. Kajiura, K. Saito, and A. Takahashi, *Matrix factorizations and representations of quivers II: type ADE case*, YITP-05-65, RIMS-1521, math.AG/0511155
[Ko] M. Kontsevich, 1998 lectures at the École Normale Supérieure.

[Ma1] E. Macrì, Some examples of moduli spaces of stability conditions on derived categories, math.AG/0411613

[Ma2] E. Macrì, Some examples of stability conditions on Fano manifolds, Talk at University of Massachusetts Amherst Quantum Field Theory Seminar (2005).

[Mi] I. Mirković, Examples of stability moduli, Talk at University of Massachusetts Amherst Representation Theory Seminar (2004).

[Mu] S. Mukai, On the moduli space of bundles on $K3$ surfaces. I, in Vector bundles on algebraic varieties (Bombay, 1984), 341–413, Tata Inst. Fund. Res., Bombay, 1987; MR0893604 (88i:14036).

[Ok] S. Okada, Stability manifold of $P^1$, J. Algebraic Geom. 15 (2006), no. 3, 487–505; MR2219846.

[RV] I. Reiten and M. Van den Bergh, Noetherian hereditary abelian categories satisfying Serre duality, J. Amer. Math. Soc. 15 (2002), no. 2, 295–366; MR1887637 (2003a:18011).

[ST] P. Seidel and R. Thomas, Braid group actions on derived categories of coherent sheaves, Duke Math. J. 108 (2001), no. 1, 37–108; MR1831820 (2002e:14030).

[Ta] A. Takahashi, Matrix factorizations and representations of quivers I, RIMS-1503, math.AG/0506347

[Th] R. Thomas, Stability conditions and the braid group, Comm. Anal. Geom. 14 (2006), no. 1, 135–161; MR2230573, also math.AG/0212214.

[To] Y. Toda, Stability conditions and crepant small resolutions, math.AG/0512648

[Wa] J. Walcher, Stability of Landau-Ginzburg branes, J. Math. Phys. 46 (2005), no. 8, 082305, 29 pp.; MR2165838 (2006k:81324), also hep-th/0412274