Derivation languages, descriptional complexity measures and decision problems of a class of flat splicing systems

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Abstract

In this paper, we associate the idea of derivation languages with flat splicing systems and compare the families of derivation languages (Szilard and control languages) of these systems with the family of languages in Chomsky hierarchy. We show that the family of Szilard languages of labeled flat finite splicing systems of type $(m, n)$ (i.e., $SZLS_{m,FIN}^n$) and $REG$, $CF$ and $CS$ are incomparable. Also, it is decidable whether or not $SZLS_{m,FIN}^n(L_S) \subseteq R$ and $R \subseteq SZLS_{m,FIN}^n(L_S)$ for any regular language $R$ and labeled flat finite splicing system $L_S$. Also, any non-empty regular, non-empty context-free and recursively enumerable language can be obtained as homomorphic image of Szilard language of the labeled flat finite splicing systems of type $(1, 2)$, $(2, 2)$ and $(4, 2)$ respectively. We also introduce the idea of control languages for labeled flat finite splicing systems and show that any non-empty regular and context-free language can be obtained as a control language of labeled flat finite splicing systems of type $(1, 2)$ and $(2, 2)$ respectively. At the end, we show that any recursively enumerable language can be obtained as a control language of labeled flat finite splicing systems of type $(4, 2)$ when $\lambda$-labeled rules are allowed.

Keywords: Flat Splicing, Labeled flat splicing systems, Szilard language, Control language, Chomsky hierarchy

1 Introduction

Splicing systems mathematically formalize recombination behavior of DNA molecules under the presence of restriction enzymes and ligases. The restriction enzymes cut the DNA molecules in specific cites and ligases help the
molecules to recombine with each other to produce new molecules. After each splicing step new DNA molecules and sometimes the same molecules taking part in splicing are produced. Splicing systems are well-investigated topic in formal language theory [11]. Different variants of splicing systems and their computational capabilities already have been investigated in [5, 9, 10, 11]. Splicing systems containing finite set of axioms and rules can generate only regular languages [3]. But with different restrictions on the set of axioms and the rules, finite splicing systems can even characterize recursively enumerable languages [11]. In this work, we discuss a variant of splicing systems known as flat splicing systems. In flat splicing, if $x_0 = x'_0 u . v x''_0$ is spliced with $y_0 = y'_0 v_1 y''_0$ by the rule $r = \langle u \mid y'_0 - y_0 \mid v \rangle$, the string $x'_0 u . y'_0 v_1 y''_0 . v x''_0$ is generated. The idea of flat splicing was introduced by Berstel et. al in [1]. Also different variants of flat splicing systems and their computational power for linear as well as for circular words such as alphabetic flat splicing systems, pure alphabetic flat splicing systems, concatenation systems have been discussed in [1]. The language generating power of the alphabetic flat splicing P systems, its variants and matrix variant of flat splicing systems have been discussed in [13], [18] and [2] respectively. Some new model of picture generation using the flat splicing rules in arrays has been introduced in [19].

In this paper, we introduce the ideas of two types of derivation languages of labeled flat splicing systems and compared the families of these languages with the family of languages in Chomsky hierarchy. The first one is Szilard languages and the second one is control languages. Szilard languages are well-known concept in formal language theory. Szilard languages of Chomsky grammars, parallel communicating grammar systems, communicating distributive grammar systems etc. along with their closure properties, decidability aspects, complexity aspects have been investigated in [4]. Also, the idea of derivation languages for DNA computing and membrane computing models have been investigated in [6] and [7] respectively. In [6], derivation languages have been associated with splicing systems and in [7] the derivation languages have been introduced for splicing P systems. Also, the idea of control languages for spiking neural P systems, transition P systems and tissue P systems have been investigated in [15, 16, 17, 20].

In this paper, we show that the family of Szilard languages of labeled flat finite splicing systems and REG, CF and CS are incomparable. Moreover, any non-empty regular, non-empty context-free and recursively enumerable language can be obtained as homomorphic image of the Szilard language of labeled flat finite splicing systems of type $(1, 2), (2, 2)$ and $(4, 2)$ respectively. We also introduce the idea of control languages for labeled flat splicing
systems and show that unlike in the case of Szilard languages, any non-empty regular and context-free language can be obtained as control language by labeled flat finite splicing systems of type (1, 2) and (2, 2) respectively. Moreover, any recursively enumerable language can be obtained as control language of labeled flat finite splicing systems of type (4, 2) when the splicing rules can have λ-label.

The paper is organized as follows: in section 2 we recall the basic definitions required for this paper. We introduce Szilard and control languages for the labeled flat splicing systems in section 3 and in section 5 respectively. In section 4 and 6, we discuss the main results of this paper. In section 7, we discuss conclusion and open problems which can be investigated further.

2 Preliminaries

For the basic definitions and notions of formal language theory we refer to [14].

Chomsky normal form: For every context-free grammar \( G \), a grammar \( G' = (N, T, S, P) \) can be effectively constructed where the rules in \( P \) are of the form \( A \to BC \) and \( A \to a \), \( A, B, C \in N, a \in T \) such that \( L(G) \setminus \{\lambda\} = L(G') \setminus \{\lambda\} \).

Greibach normal form: A context-free grammar \( G = (N, T, S, P) \) is said to be in Greibach normal form if the rules in \( P \) are of the form \( A \to a\alpha, A \in N, a \in T, \alpha \in N^* \).

Type-0-grammar: A type-0-grammar is a construct of the form \( G = (N, T, S, P) \) where \( N \) is the non-terminal alphabet and \( T \) is the terminal alphabet such that \( N \cap T = \emptyset \). The starting symbol \( S \in N \) and the rules in \( P \) are ordered pairs \((u, v)\) where \( u \in (N \cup T)^*N(N \cup T)^* \) and \( v \in (N \cup T)^* \).

Kuroda normal form: Every type-0 grammar \( G = (N, T, S, P) \) is in Kuroda normal form if the rules of the grammar \( G \) are of the following forms:

\[
A \to BC, AB \to CD, A \to a, A \to \lambda \text{ for } A, B, C, D \in N \text{ and } a \in T.
\]

Homomorphism: A homomorphism is a mapping \( h \) from \( \Sigma^* \) to \( \Delta^* \) where \( \Sigma, \Delta \) are alphabets. Also, the mapping preserves concatenation, i.e., \( h(vw) = h(v).h(w), v, w \in \Sigma^* \).

Szilard languages [14]: Let \( G = (N, T, S, P) \) be Chomsky grammar and \( F \) be an alphabet such that the cardinality of the set \( F \) and \( P \) is same. Let \( f \) be a mapping from \( P \) to \( F \) such that for each \( p \in P \) a unique label \( f(p) \) is associated with \( p \) and is called the label of the rule \( p \). A derivation in \( G \) is called successful if a string over \( T \) is generated starting from \( S \). With each
successful derivation of $G$, a string over $F$ can be associated if the labels of the applied rules of any successful derivation are concatenated sequentially. The language generated in this manner is called Szilard language of the grammar $G$ and is denoted by $SZ(G)$.

Example 1. Let $G = (\{S\}, \{a,b\}, S, \{S \rightarrow aSb, S \rightarrow ab\})$ be a context-free grammar. The rules are labeled in the following manner: $f_1 : S \rightarrow aSb, f_2 : S \rightarrow ab$. Hence the Szilard language generated by the grammar $G$ is $SZ(G) = \{f_1^n f_2 \mid n \geq 0\}$.

The families of regular, context-free, context-sensitive and recursively enumerable languages are denoted by $REG, CF, CS$ and $RE$ respectively.

Flat splicing systems [1]: A flat splicing system is a construct of the form $S = (A, I, R)$ where $A$ is an alphabet, $I$ is a set of words over the alphabet $A$, $R$ is a finite set of splicing rules. The rules in flat splicing system are of the form $< \alpha | \gamma - \delta | \beta >$ where $\alpha, \beta, \gamma, \delta$ are words over the alphabet $A$. The strings $\alpha, \beta, \gamma, \delta$ are called the handles of the rule. If the rule $r = < \alpha | \gamma - \delta | \beta >$ is applied to the pair of strings $(u, v)$ where $u = x\alpha . \beta y$ and $v = \gamma z \delta$, then the string $v$ is inserted in the “.” location of $u$. Hence after application of the rule $r$, the string $w = x\alpha . \gamma z \delta . \beta y$ is obtained. The location “.” represents the location where the cutting and pasting operation take place. The flat splicing operation over the two words $u$ and $v$ can be represented as $(u, v) \vdash_r w$. Moreover, a flat splicing system is called finite, regular, context-free, context-sensitive if the corresponding initial set $I$ is finite, regular, context-free and context-sensitive respectively.

The language generated by the flat splicing system $S = (A, I, R)$ is denoted by $F(S)$ which is also the smallest language $L$ containing $I$. It is also closed under $R$, i.e., for $u, v \in L$ and $r \in R, (u, v) \vdash_r w \in L$.

In this paper, we introduce the notion of type with the rules in flat splicing systems. A flat splicing system $S = (A, I, R)$ is called of type $(m, n)$ where $m = \max\{|\alpha|, |\beta| < |\gamma - \delta| \beta \epsilon R\}$ and $n = \max\{|\gamma | - \delta| \beta \epsilon R\}$ where $u = x\alpha . \beta y, v = \gamma z \delta, (u, v) = (x\alpha . \beta y, \gamma z \delta) \vdash_r x\alpha . \gamma z \delta . \beta y$, and $x, y, \alpha, \beta, \gamma, \delta, z \epsilon A^*$, $|\gamma | \leq 1, |\delta| \leq 1$ and $|\gamma z \delta| \geq 1$. The parameters $m$ and $n$ represent the descriptive complexity measures of flat splicing systems.

Note that whenever a string $\gamma z \delta$ is inserted into another word $x\alpha . \beta y$, the inserted string is represented by $\gamma - \delta$, i.e., only by the end letters of the words. Furthermore, if the inserted word (i.e., $|\gamma z \delta| = 1$) is of length 1, then the word is denoted as $\epsilon - \gamma_1$ or $\gamma_1 - \epsilon$ where $\gamma_1 \epsilon A$. We have explained these notations with two examples in Example 2 and 3. Moreover, the language generated by the flat splicing system $S$ of type $(m, n)$ is denoted as $F(S)_m^n$.
and the families of languages generated by the flat splicing systems of type 
\((m, n)\) is denoted by \(FS^m_n\). When \(m\) and \(n\) are not specified, is replaced by 
“\(\ast\)”. Also the flat splicing of two strings is simply mentioned as splicing of 
two strings.

A rule of the flat splicing system \(\mathcal{S}\) is called alphabetic if for any rule 
\(r = \langle \alpha|\gamma - \delta|\beta \rangle\), either \(\alpha, \beta, \gamma, \delta\) are letters or empty strings. A flat
splicing system is called alphabetic if all the rules present in the system are 
alphabetic, i.e., alphabetic flat splicing systems are flat splicing systems of 
type \((1, 2)\).

Example 2. Let \(\mathcal{S} = (A, I, R)\) be a flat splicing system where 
\(A = \{a, b\}\), \(I = \{ab\}\) and \(R = \{< a|a - b|b >\}\). Since \(ab \in I\), at first \(ab\) splice with \(ab\) only and generate 
a\(^2\)b\(^2\). This process is continued and the strings of the form \(a^n b^n\) are spliced with the strings of the form \(a^n b^n\) only. Hence 
\(\mathcal{F}_{1}^{2}(\mathcal{S}) = \{a^n b^n|n \geq 1\}\).

In the previous example the strings inserted must be of length at least 
two, otherwise the rule cannot be applied. Now, we give example of a flat 
splicing system where strings of length one are inserted in the specified 
location.

Example 3. Let \(\mathcal{S} = (A, I, R)\) where \(A = \{X, Y, a\}\), \(I = \{XY, a\}\) and 
\(R = \{< X|\epsilon - a|Y >\}\). Hence \(\mathcal{F}_{1}^{1}(\mathcal{S}) = \{XaY\}\).

Example 4. Let \(\mathcal{S} = (A, I, R)\) be a flat splicing system where \(A = \{a, X, b, Z\}\), \(I = \{aXb, Z\}\), \(R = \{< \epsilon|\epsilon - Z|a >\}\). The application of \(< \epsilon|\epsilon - Z|a >\) to the words \(aXb, Z\) will generate the word \(ZaXb\). This flat splicing operation is a concatenation operation. Moreover, \(\mathcal{F}_{1}^{1}(\mathcal{S}) = \{ZaXb\}\).

In this work, we investigate two types of derivation languages of the 
labeled flat splicing systems. At first, we introduce the the idea of Szilard 
languages of labeled flat splicing systems. Then we introduce the idea of 
control languages of labeled flat splicing systems.

3 Labeled flat splicing system

Let \(\mathcal{S} = (A, I, R)\) be a flat splicing system. A labeled flat splicing system 
is a construct of the form \(\mathcal{L}\mathcal{S} = (A, I, R, Lab)\) such that \(A \cap Lab = \emptyset\). Each rule of the flat 
splicing system is labeled in one-to-one manner with the elements from \(Lab\). A derivation in flat splicing system is called terminal 
derivation if it follows the following pattern:

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\[(x_0, y_0) \vdash_{a_1} x_1, (x_1, y_1) \vdash_{a_2} x_2, (x_2, y_2) \vdash_{a_3} x_3, \ldots (x_{n-1}, y_{n-1}) \vdash_{a_n} x_n\]

where \(x_0, y_0, y_1, y_2, \ldots, y_{n-1} \in I, a_1, a_2, \ldots, a_n \in \text{Lab}\). Moreover, no rule from \(R\) is applicable to the word \(x_n \in A^+\) and the word \(a_1a_2 \ldots a_n\) is called a Szilard word. The language obtained by concatenating the labels of the applied rules in a terminal derivation is called Szilard language of the labeled flat splicing system \(\mathcal{L}\) and \(SZ^{m}_{n,FAM}(\mathcal{L})\) denotes the set of Szilard languages obtained by the labeled flat splicing system \(\mathcal{L}\) of type \((m,n)\) where the initial set \(I \in \text{FAM} = \{\text{FIN, REG, CF, CS}\}\). The families of Szilard languages of the labeled flat splicing systems of type \((m,n)\), is denoted as \(SZ_{LS}^{m}_{n,FAM}\). If \(m\) and \(n\) are not specified, they are replaced by “∗”.

**Example 5.** We give an example of an alphabetic labeled flat finite splicing system which can obtain a regular language as a Szilard language. Let \(\mathcal{L} = (A, I, R, \text{Lab})\) be a labeled flat splicing system where \(A = \{X, A_1, A', Y\}\), \(I = \{XA_1Y, A_1, A'\}\), \(R = \{a: < A_1|\epsilon - A_1|Y > ; c: < A_1|\epsilon - A'|Y > \}\).

The a-rule is applicable any number of times and it can splice the words \(XA^n_1Y, n \geq 1 \) and \(A_1\). But after application of the c-rule, the word \(XA^n_1A'Y, n \geq 1 \) is obtained. No rule is applicable to this word. Hence \(SZ_{1,FIN}(\mathcal{L}) = \{a^n c | n \geq 0\}\).

### 4 Results

In this section, we discuss the results related to the Szilard languages of the labeled flat splicing systems. The language \(\{aa\}\) cannot be obtained as Szilard language of any Chomsky grammar. But we prove that this language can be obtained as a Szilard language of labeled flat finite splicing system. Moreover, we show that some regular, non-regular and non-context free languages cannot be a Szilard language of any labeled flat finite splicing system. But any non-empty regular, non-empty context-free and recursively enumerable language can be obtained as homomorphic image of the Szilard language of labeled flat finite splicing systems of type \((1,2)\), \((2,2)\) and \((4,2)\) respectively.

**Theorem 1.** \(\{aa\} \in SZ_{LS}^{1}_{1,FIN}\).

**Proof.** Let \(\mathcal{L} = (A, I, R, \text{Lab})\) be a labeled alphabetic flat splicing system where \(A = \{S, X_n, Y\}\), \(I = \{SYSY, X_n\}\), \(R = \{a: < S|\epsilon - X_{n}|Y > \}\), \(\text{Lab} = \{a\}\). Hence \(SZ_{1,FIN}(\mathcal{L}) = \{aa\}\).
In the next two results, we show that the regular language \( \{a^n | n \geq 1\} \) cannot be a Szilard language of any labeled flat finite splicing system. But it can be obtained as a Szilard language if \( I \) is regular.

**Theorem 2.** \( \{a^n | n \geq 1\} \notin SZLS_{*,FIN}^* \).

*Proof.* Let us assume that there exists a labeled flat finite splicing system \( \mathcal{L} = (A, I, R, Lab) \) such that \( SZLS_{n,FIN}^*(\mathcal{L}) = \{a^n | n \geq 1\} \) where \( R = \{a : < u|v_1 - v_1|v >\} \) and \( x_i^1 u v x_i^2, u_1 \delta v_1 \in I, x_i^1, x_i^2, u, v, u_1, v_1, \delta \in \Sigma^* \), \( |u_1| \leq 1, |v_1| \leq 1 \). Now since \( a \in \{a^n | n \geq 1\} \), there exists a derivation

\[
(x_0^0, y_0^0) \vdash^{a} z_1^0 
\]

where \( x_0^0, y_0^0 \in I \) and \( a : < u|v_1 - v_1|v >\) is not applicable to \( z_1^0 \).

Similarly, the terminal derivation for \( a^2 \) is as follows:

\[
(x_0^1, y_0^1) \vdash^{a} z_1^1, \\
(z_1^1, y_1^1) \vdash^{a} z_2^1 
\]

where \( x_0^1, y_0^1, y_1^1 \in I \) and \( a : < u|v_1 - v_1|v >\) is not applicable to \( z_2^1 \).

Again, the terminal derivation of \( a^3 \) is as follows:

\[
(x_0^2, y_0^2) \vdash^{a} z_1^2, \\
(z_1^2, y_1^2) \vdash^{a} z_2^2, \\
(z_2^2, y_2^2) \vdash^{a} z_3^2 
\]

where \( x_0^2, y_0^2, y_1^2, y_2^2 \in I \) and \( a : < u|v_1 - v_1|v >\) is not applicable to \( z_3^2 \).

Since, the labeled flat splicing system \( \mathcal{L} \) contains only one \( a \)-rule, then from the above derivations it is clear that either after application of \( a \)-rule, the subword uv is again obtained or the pair of words \((x_i^0, y_0^0)\) are distinct (i.e., at least one of term of the pair is different from any other initial pair of words).

If after application of the \( a \)-rule again a subword \( uv \) is obtained, then no terminal derivation is obtained. Hence \( \{a^n | n \geq 1\} \) cannot be a Szilard language of the labeled flat finite splicing system \( \mathcal{L} \).

In the second case, to obtain \( \{a^n | n \geq 1\} \) as Szilard language, the pairs \((x_i^0, y_0^0)\) must be distinct, where \( x_i^0, y_0^0 \in I \). But \( I \) is finite. Hence \( \{a^n | n \geq 1\} \) cannot be a Szilard language of a labeled flat finite splicing systems.

\( \square \)

Now we show that \( \{a^n | n \geq 1\} \) can be a Szilard language of labeled flat regular splicing system.

**Theorem 3.** \( \{a^n | n \geq 1\} \) can be obtained as Szilard language of a labeled flat alphabetic regular splicing system, i.e., \( \{a^n | n \geq 1\} \in SZLS_{1,REG}^1 \).
Proof. Let $\mathcal{L} = (A, I, R, \text{Lab})$ be a labeled flat splicing system where $A = \{X, A_1, A', Y\}$, $I = \{XA^n_1Y|n \geq 2\} \cup \{A'\}$, $R = \{a : A_1\epsilon - A' A_1 >, b : A_1\epsilon - A_2 Y >, c : A_1\epsilon - A_2 A_2 >\}$. Hence $SZ_{L,\text{REG}}(\mathcal{L}) = \{a^n|n \geq 1\}$. \hfill $\square$

**Theorem 4.** $SZL_{1,\text{FIN}} \cap (\text{REG} \setminus \text{FIN}) \neq \emptyset$.

Proof. Follows from Example 5. \hfill $\square$

**Theorem 5.** $SZL_{2,\text{FIN}} \cap (\text{CF} \setminus \text{REG}) \neq \emptyset$.

Proof. Let $\mathcal{L} = (A, I, R, \text{Lab})$ be a labeled flat splicing system where $A = \{X, A_1, Y, A_2\}$, $I = \{XA_1Y, A_1, A_2\}$, $R = \{a : A_1\epsilon - A_1 Y >, b : A_1\epsilon - A_2 Y >, c : A_1\epsilon - A_2 A_2 >\}$. Hence $SZ_{2,\text{FIN}}(\mathcal{L}) = \{a^n b^n|n \geq 1\}$. \hfill $\square$

**Theorem 6.** $(CS \setminus \text{CF}) \cap SZL_{2,\text{FIN}} \neq \emptyset$.

Proof. Let $\mathcal{L} = (A, I, R, \text{Lab})$ be a labeled flat splicing system where $A = \{X, A, A', A'', Y\}$, $I = \{XAAY, A, A', A''\}$, $R = \{a : A\epsilon - A A >, b : A\epsilon - A' A >, c : A' A\epsilon - A'' A >\}$. Let $\text{Lab} = \{a, b, c\}$.

On application of the $a$-rule, one $A$ is added between two $A$'s of the word $XAAY$. Similarly, application of the $b$-rule inserts $A'$ between the two $A$'s of the subword $AA$ and generates the subword $AA' A$. Similarly, when $c$-rule is applied, the string $AA'$ is replaced by $AA'' A'$. Hence terminal derivation can be obtained in $\mathcal{L}$ if $a$, $b$ and $c$-rules are applied same number of time in order and $SZ_{2,\text{FIN}}(\mathcal{L}) \cap a^* b^* c^* = \{a^n b^n c^n|n \geq 1\}$. Since $\{a^n b^n c^n|n \geq 1\}$ is a context-sensitive language and context-sensitive languages are closed under intersection with regular languages. The language $SZ_{2,\text{FIN}}(\mathcal{L})$ must be context-sensitive. \hfill $\square$

Now we show that the non-regular context-free language $\{a^n b^n|n \geq 1\}$ can be obtained as a Szilard language of labeled flat finite splicing system.

**Theorem 7.** $\{a^n b^n|n \geq 1\} \notin SZLS_{*,\text{FIN}}$.

Proof. Let $\mathcal{L} = (A, I, R, \text{Lab})$ be a labeled flat finite splicing system such that $SZ_n(\mathcal{L}) = \{a^n b^n|n \geq 1\}$ where $a : u|u|v >, b : u_2|u_3 - v_3|v_2 >, u_1 \delta_1 v_1, u_3 \delta_3 v_3 \in I$.

Since, $ab \in SZ_n(\mathcal{L})$, there exists a terminal derivation

\( (x_0, y_0) \vdash_a x_0 \)
\( (z_1, y_1) \vdash_b y_0 \)

where no rules of $R$ is applicable to the word $z_2^0$. 

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Similarly, \(a^2b^2 \in SZ^m_n(\mathcal{L})\). Hence there exists a derivation
\[
(x_1^0, y_0^1) \vdash z_1^1 \\
(z_1^1, y_1^1) \vdash z_2^1 \\
(z_2^1, y_2^1) \vdash b \cdot z_3^1 \\
(z_3^1, y_3^1) \vdash b \cdot z_4^1
\]
where \(x_1^0, y_1^1, y_2^1, y_3^1 \in I\) and no rule is applicable to \(z_4^1\).
Also, \(a^3b^3 \in SZ^m_n(\mathcal{L})\). Hence there exists a derivation
\[
(x_2^0, y_0^1) \vdash z_1^2 \\
(z_1^2, y_1^1) \vdash z_2^2 \\
(z_2^2, y_2^1) \vdash z_3^2 \\
(z_3^2, y_3^1) \vdash b \cdot z_4^2
\]
where \(x_2^0, y_1^1, y_2^1, y_3^1 \in I\) and no rule is applicable to \(z_6^1\).

\(R\) contains two rules \(a \cdot < u|u_1 - v_1|v > \) and \(b \cdot < u_2|u_3 - v_2|v_2 > \) where \(u, v, u_2, v_2 \in A^*\) and \(u_1, v_1, u_3, v_3 \in A\). Proceeding in the similar manner as in Theorem 2 we can say that to obtain \(a^n b^n | n \geq 1\) as a Szilard language of labeled flat finite splicing system \(\mathcal{L}\), either after application of \(a\)-rule and \(b\)-rule, the subwords \(uv\) and \(u_2v_2\) are obtained again in the newly generated words or at least one of the words in the initial pair \((x_0^i, y_0^i)\) is distinct from others pairs in terminal derivations.

If both \(a\)-rule and \(b\)-rule obtain the subwords \(uv\) and \(u_2v_2\) in the newly generated words, then no terminal derivation is obtained. Also if at least one word is distinct in the initial pair of words \((x_0^i, y_0^i)\) then \(I\) cannot be finite - a contradiction.

Hence \(a^n b^n | n \geq 1\) cannot be a Szilard language of any labeled flat finite splicing system.

\(\Box\)

**Theorem 8.** \(\{a^n b^n | n \geq 1\} \in SZLS^3_{1, REG}\).

**Proof.** Let \(\mathcal{L} = (A, I, R, Lab)\) be a labeled flat splicing system where \(A = \{X, A_1, B_1, Y\}\), \(I = \{X A_1 B_1 Y| n \geq 2\} \cup \{B_1, X\}\), \(R = \{a \cdot < A_1|\epsilon - B_1|A_1 B_1 >, b \cdot < X A_1 B_1|\epsilon - X|A_1 B_1 >\}\). Hence \(SZ^3_{1, REG}(\mathcal{L}) = \{a^n b^n | n \geq 1\}\).

\(\Box\)

**Theorem 9.** \(\{a^n b^n c^n | n \geq 1\} \notin SZLS^*_{\text{FIN}}\).

**Proof.** Follows from the proof of Theorem 7

\(\Box\)

**Theorem 10.** \(\{a^n b^n c^n | n \geq 1\} \in SZLS^4_{1, REG}\).
Proof. Let \( \mathcal{I} = (A, I, R, \text{Lab}) \) be a labeled flat splicing system where 
\[ A = \{ X, B_1, C_1, Y \}, \]
\[ I = \{ XB_1 C_1 Y, C_1, X, Y \}, \]
\[ R = \{ a : < B_1 | \epsilon - C_1 | B_1 C_1 >, \]
\[ b : < XB_1 C_1 | \epsilon - X | B_1 C_1 >, c : < B_1 C_1 | \epsilon - Y | XB_1 C_1 Y > \}. \]
Hence \( SZ_{1,\text{REG}}(\mathcal{I}) = \{ a^n b^n c^n | n \geq 1 \} \).

Theorem 11. The families \( SZ_{m,FIN} \) and \( REG \) (resp. \( CF, CS \)) are incomparable.

Proof. Follows from the Theorem 2, 4, 5, 6, 7 and 9.

Open problem: Can labeled flat regular splicing systems obtain any recursively enumerable language as Szilard language?

4.1 Decision problems

For any regular language \( R \) and labeled flat splicing system \( \mathcal{I} \) we can prove the following decision problems.

Theorem 12. \( R \subseteq SZ_{m,FIN}^{m}(\mathcal{I}) \) is decidable.

Proof. Let \( \mathcal{I} = (A, I, R, \text{Lab}) \) be a labeled flat finite splicing system of 
type \( (m, n) \). Let \( x = x_0 x_1 \ldots x_n \in R \). If starting from a word \( w \in I \), if 
the flat splicing rules with label \( x_0, x_1, \ldots, x_n \) are applied in order, a string 
\( w_1 \in A^+ \) is obtained. If no rule of \( \mathcal{I} \) is applicable to \( w_1 \), then the word 
\( x_0 x_1 \ldots x_n \in SZ_{m,FIN}^{m}(\mathcal{I}) \). Otherwise, \( x \notin SZ_{n,FIN}^{m}(\mathcal{I}) \). Hence it 
is decidable whether a regular language \( R \) is contained inside the Szilard language 
of a flat splicing system of type \( (m, n) \) with finite initial set.

Theorem 13. \( SZ_{n,FIN}^{m}(\mathcal{I}) \subseteq R \) is decidable.

Proof. Since \( R \) is a regular language, there exists a deterministic finite 
automata \( D \) accepting it. Moreover, the problem \( M = \{ < D, x > | D \)
accepts the string \( x \} \) is decidable. Let \( x_1 \in SZ_{n,FIN}^{m}(\mathcal{I}) \), then it is 
also decidable whether or not \( D \) accepts the string \( x_1 \). Hence the problem 
\( SZ_{n,FIN}^{m}(\mathcal{I}) \subseteq R \) is also decidable.

Open problem: Let \( \mathcal{I} \) be a labeled flat finite splicing system 
and \( R \) be a regular language, then \( SZ_{n,FIN}^{m}(\mathcal{I}) = R \) is decidable 
or not.

We have already proved that not all regular languages can be Szilard 
language of labeled flat finite splicing systems. But in the next result we 
show that any non-empty regular language can be obtained as homomorphic
image of the Szilard language of the labeled flat finite splicing systems of type (1, 2).

**Theorem 14.** Any non-empty regular language can be obtained as a homomorphic image of the Szilard language of the labeled flat finite splicing system of type (1, 2).

**Proof.** Let $L$ be a $\lambda$-free regular language. There exist a grammar $G = (N, T, S, P)$ which generates $L$, i.e., $L = L(G)$. We construct a labeled flat splicing system which simulates the rules in the grammar $G$. Initially, the non-terminals $N$ of $G$ are rewritten using the symbols $D_i$, $1 \leq i \leq n$, starting from $D_1 = S$ and the labeled flat splicing system $\mathscr{L}$ is constructed in such a manner that $L = L(G) = h(SZ_{2,FIN}(\mathscr{L}))$ where $h$ is a homomorphism and $SZ_{2,FIN}(\mathscr{L})$ denotes the Szilard language of the labeled flat splicing system of type (1, 2).

Now the rules in $P$ are of the form $D_i \rightarrow aD_i$, $D_i \rightarrow aD_j(i \neq j)$, and $D_i \rightarrow a$, where $D_i, D_j \in N$, and $a \in T$.

Let $\mathscr{L} = (A, I, R, Lab)$ be a labeled flat splicing system where:

- $A = \{X, Y, D_1, D_2, \ldots, D_n\} \cup \{Y_a | a \in T\}$;
- $I = \{XD_1Y\} \cup \{Y_aD_i | D_i \rightarrow aD_i \in P\} \cup \{Y_aD_j | D_i \rightarrow aD_j \in P\} \cup \{Y_a | D_i \rightarrow a \in P\}$;
- The rules in $R$ are of the form:

$$a_{D_i}^i \vdash D_i|Y_a - D_i|Y > (Y_a - D_i = Y_aD_i) \text{ for } D_i \rightarrow aD_i, D_i \in N, a \in T;$$

$$a_{D_j}^i \vdash D_i|Y_a - D_j|Y > (Y_a - D_j = Y_aD_j) \text{ for } D_i \rightarrow aD_j(i \neq j), D_i, D_j \in N, a \in T;$$

$$a^i \vdash D_i|\varepsilon - Y_a|Y > \text{ for } D_i \rightarrow a, a \in T;$$

- $Lab = \{a_{D_i}^i | D_i \rightarrow aD_i, D_i \in N, a \in T\} \cup \{a_{D_j}^i | D_i \rightarrow aD_j, D_i, D_j \in N, a \in T\} \cup \{a^i | D_i \rightarrow a, a \in T\}$.

The non-erasing homomorphism $h : (Lab)^* \rightarrow T^*$ is defined in the following manner: $h(a_{D_i}^i) = h(a_{D_j}^i) = a$ and $h(a^i) = a$ where $a_{D_i}^i, a_{D_j}^i, a^i \in Lab$.

The rule $D_i \rightarrow aD_i$ in $G$ is simulated by the labeled splicing rule $a_{D_i}^i \vdash D_i|Y_a - D_i|Y >$ and the rule $D_i \rightarrow aD_j$ is simulated by the splicing rule $a_{D_j}^i \vdash D_i|Y_a - D_j|Y >$. Also, the terminating rule $D_i \rightarrow a$ is simulated by $a^i \vdash D_i|\varepsilon - Y_a|Y >$. Every $w \in L(G)$ can be generated after application of the rules in $G$ and there exist labeled splicing rules simulating the rules in $G$. The application of the labeled splicing rules starting from $XD_1Y$ in the
same sequence as in the derivation of $w \in T^*$, generates a string over $A_1$ such that no splicing rule is applicable to it. Also, concatenation of the labels of the splicing rules generate $w_1 \in (\text{Lab})^*$. Since, $h(a_{D_i}^i) = h(a_{D_i}^i) = a$ and $h(a^i) = a$, the homomorphic image of the string $w_1$ is $w$, i.e., $h(w_1) = w$. Hence $w \in h(SZ_{2,FIN}^{1}(\mathcal{L},\mathcal{I}))$. This imply, $L = L(G) \subseteq SZ_{2,FIN}^{1}(\mathcal{L},\mathcal{I})$.

It only remains to prove the inclusion $h(SZ_{2,FIN}^{1}(\mathcal{L},\mathcal{I})) \subseteq L(G)$. So, let $w \in h(SZ_{2,FIN}^{1}(\mathcal{L},\mathcal{I}))$. Hence there exists $w_1 \in SZ_{2,FIN}^{1}(\mathcal{L},\mathcal{I})$ such that $h(w_1) = w$. This inclusion can be proved by observing the one-to-one correspondence between the rules in $\mathcal{L},\mathcal{I}$ and $G$.

It was proved by Păun in [12] that some context-free languages cannot be represented as a homomorphic image of the Szilard language of any context-free language.

**Theorem 15.** [12] The families of context-free languages and homomorphic image of context-free languages are incomparable.

It was proved in [12] that $\{a^n b^n | n \geq 1\}$ cannot be obtained as a homomorphic image of the Szilard languages of the context-free languages. But we prove that any context-free language can be obtained as a homomorphic image of Szilard language of the labeled finite flat splicing systems of type $(2, 2)$.

**Theorem 16.** Any context-free language can be obtained as a homomorphic image of the Szilard language of the labeled flat finite splicing system of type $(2, 2)$.

**Proof.** Let $L$ be a non-empty context-free language and let $G = (N, T, S, P)$ be a grammar such that $L = L(G)$. Let the grammar $G$ is in Chomsky normal form and the rules in $P$ are of the form, $A_1 \rightarrow B_1 C_1$ and $A_1 \rightarrow a$, where $A_1, B_1, C_1 \in N, a \in T$. Each element of the language $L$ can be generated by initial application of the recursive rules and then by application of the terminal rules in the left-most manner. Also, each rule of $R$ is associated with unique label $r_i$.

We construct a labeled flat splicing systems $\mathcal{L},\mathcal{I} = (A, I, R, \text{Lab})$ such that $L = h(SZ_{2,FIN}^{2}(\mathcal{L},\mathcal{I}))$ where $h$ is a morphism from $(\text{Lab})^*$ to $T^*$ and $SZ_{2,FIN}^{2}(\mathcal{L},\mathcal{I})$ denotes the Szilard language of the labeled flat splicing system $\mathcal{L},\mathcal{I}$ of type $(2, 2)$.

Let $\mathcal{L},\mathcal{I} = (A, I, R, \text{Lab})$ be a labeled flat splicing system where:
• $A = \{X, Y, E\} \cup N \cup \Delta_1 \cup \Delta_2 \cup \{[r_k], [r_m]\}$ where $\Delta_1 = \{[r_i] \mid r_i : A_1 \rightarrow B_1C_1\}$, $\Delta_2 = \{[r_i] \mid r_i : A_1 \rightarrow a\};$

• $I = \{XSEY\} \cup \{[r_i]B_1C_1 \mid r_i : A_1 \rightarrow B_1C_1 \in P\} \cup \{[r_a] \mid r_a : A_1 \rightarrow a\} \cup \{[r_k], [r_m]\};$

• $R$ contains the following rules:
  
  for $r_i : A_1 \rightarrow B_1C_1$:
  
  $[r_i]^1 : \langle A_1\mid [r_i] - C_1\mid \alpha_1\beta_2 \rangle$ where $\alpha_1 \in N \cup \{E\}, \alpha_2 \in N \cup \{E, Y\} \cup \{[r_i]\} \in \Delta_1, \alpha_1\beta_2 \notin \{NY, EN\} \cup \{E[r_i]\} \in \Delta_1\}([r_i] - C_1 = [r_i]B_1C_1),$

  $[r_i]^{a} : \langle [r_m]\mid A_1\mid [r_a] - \alpha_3 \rangle, \alpha_3 \in N \cup \{E\} \text{ for } r_i : A_1 \rightarrow a.$

  $[r_k]^1 : \langle X\mid \epsilon - [r_m] \mid \alpha_4 \rangle, \alpha_4 \in N;$

  $[r_k]^2 : \langle [r_m]\mid \alpha_5 \mid \epsilon - [r_m] \mid \alpha_6 \rangle, \alpha_5 \in N \cup \Delta_1 \cup \Delta_2, \alpha_6 \in N \cup \Delta_1 \cup \Delta_2.$

• $Lab = \{[r_i]^1 \mid [r_i] \in \Delta_1\} \cup \{[r_i]^a \mid [r_i] \in \Delta_2\} \cup \{[r_k]^1, [r_k]^2\}.$

The morphism $h : (Lab)^* \rightarrow T^*$ is as follows: $h([r_i]) = h([r_k]) = h([r_k]^1) = h([r_k]^2) = \lambda, h([r_i]^a) = a$ where $[r_i]^1, [r_k]^1, [r_k]^2, [r_i]^a \in Lab$ and $a \in T.$

We first prove that $L(G) = L \subseteq h(SZ^{2}_{FIN}(\mathcal{L}\mathcal{S}))$. Let $w \in L(G)$ and it can be generated by application of the recursive rules and then by left-most application of the terminating rules. Each rule of $G$ is associated with a unique label and the rule $r_i : A_1 \rightarrow B_1C_1$ is simulated by the splicing rule $< A_1\mid [r_i] - C_1\mid \alpha_1\beta_2 >, \alpha_1 \in N \cup \{E\}, \alpha_2 \in N \cup \{E, Y\}, \alpha_1\beta_2 \notin \{EN, NY, EE\}. \text{ In fact, the string } Xw_1A_1w_2Y \text{ where } w_1, w_2 \in A^*$ is spliced with the string $[r_i]B_1C_1 \text{ and generate the string } Xw_1A_1[r_i]B_1C_1w_2Y.$

Each $r_i : A_1 \rightarrow a$ is applied in the left-most manner and is simulated when the string $Xw_1[r_m]A_1w_2Y$ is spliced with $[r_a]$ and generate the string $Xw_1[r_m]A[r_a]w_2Y.$ The symbol $[r_m]$ identifies the left-most non-terminal where the terminal rule can be applied. The $[r_k]^1$-rule splice the string $X\alpha w_3Y, w_3 \in A^*, \alpha \in N \text{ with } [r_k]$ and generate the string $X[r_k]\alpha w_3Y.$ The $[r_k]^2$ labeled rule insert $[r_m]$ in a specified location. The $[r_k]^1$ and $[r_k]^2$ labeled rules are constructed in such a manner that they help to identify the leftmost non-terminal where the terminating rules can be applied. If the corresponding labeled splicing rules are applied in $\mathcal{L}\mathcal{S}$ in the same sequence as in the derivation of $G$ generating $w$, a terminal derivation can be obtained in $\mathcal{L}\mathcal{S}, \text{ i.e., a string over } A \text{ is obtained where no rules can be applied.}$ Also, the concatenation of the labels of the applied splicing rules generate a string over $Lab$, say, $w_1.$ The morphism $h$ replaces each occurrence of $[r_i]^1, [r_k]^1$ and $[r_k]^2$ in $w_1$ by the empty string and $[r_i]^a$ by $a.$ Hence
each \( w \in L(G) \), can be represented as \( w = h(w_1) \in h(SZ^2_{2,FIN}(\mathcal{L}_S)) \) where \( w_1 \in SZ^2_{2,FIN}(\mathcal{L}_S) \).

Now to prove the other inclusion \( h(SZ^2_{2,FIN}(\mathcal{L}_S)) \subseteq L(G) = L \), let \( w \in h(SZ^2_{2,FIN}(\mathcal{L}_S)) \). Hence \( w = h(w_1) \) where \( w_1 \in SZ^2_{2,FIN}(\mathcal{L}_S) \). Since \( w_1 \in SZ^2_{2,FIN}(\mathcal{L}_S) \), i.e., concatenation of the labels of the applied splicing rules of a terminal derivation in \( \mathcal{L}_S \) generates \( w_1 \). The application of the rules in \( G \), starting from \( S \) in the same sequence as in the terminal derivation of \( \mathcal{L}_S \) generates \( w \in L(G) \). Hence \( h(SZ^2_{2,FIN}(\mathcal{L}_S)) \subseteq L(G) \).

\[ \Box \]

In the next result, we show that any recursively enumerable language can be characterized by the homomorphic image of the Szilard language of the labeled flat finite splicing system of type \((4,2)\).

**Theorem 17.** Each recursively enumerable language can be obtained as homomorphic image of the Szilard language of labeled finite flat splicing systems of type \((4,2)\).

**Proof.** Let \( L \in RE \) and we know that any \( RE \) language can be generated by a grammar \( G = (N,T,S,P) \) in Kuroda normal form, i.e., \( L(G) = L \). The rules of the grammar \( G \) are of the form \( A_1 \rightarrow B_1C_1, A_1B_1 \rightarrow C_1D_1, A_1 \rightarrow a, A_1 \rightarrow \lambda \) where \( A_1,B_1,C_1,D_1 \in N, a \in T \). Any element \( x \in L \) can be generated by application of the recursive rules and then by left-most application of the terminating rules \([8] \). In this proof, the labeled flat splicing system \( \mathcal{L}_S = (A,I,R,Lab) \) is constructed in such a manner that \( L = h(SZ^2_{2,FIN}(\mathcal{L}_S)) \) where \( A \cap Lab = \emptyset \). Also, the labeled splicing rules in \( \mathcal{L}_S \) are constructed by simulating the rules in \( P \) where each rule of \( P \) is associated with a unique label \( r_i (1 \leq i \leq n) \) if \( |P| = n \).

The set \( \Delta \) contains the labels of the rules in \( P \) and is divided into four parts such that \( \Delta = \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4 \), where

- \( \Delta_1 = \{ r_i \mid r_i : A_1 \rightarrow B_1C_1 \in P \} \);
- \( \Delta_2 = \{ r_i \mid r_i : A_1B_1 \rightarrow C_1D_1 \in P \} \);
- \( \Delta_3 = \{ r_i \mid r_i : A_1 \rightarrow a \in P \} \);
- \( \Delta_4 = \{ r_i \mid r_i : A_1 \rightarrow \lambda \in P \} \).

Let \( \mathcal{L}_S = (A,I,R,Lab) \) be a labeled flat splicing system, where:

- \( A = \{ X,Y \} \cup N \cup \{ r_i \mid r_i \in \Delta \} \cup \{ r_m \} \cup \{ k_a | r_i : A_1 \rightarrow a \} \cup \{ k_\lambda | r_i \} \);
- \( I = \{ XSY \} \cup \{ [r_i] \mid r_i : A_1 \rightarrow B_1C_1 \in P \} \cup \{ [r_i] \mid r_i : A_1B_1 \rightarrow C_1D_1 \in P \} \cup \{ [r_i] | r_i : A_1B_1 \rightarrow C_1D_1 \} \cup \{ k_a | r_i : A_1 \rightarrow a \in P \} \cup \{ k_\lambda | r_i : A_1 \rightarrow \lambda \} \in P \} \cup \{ r_m \} \);
• \( R \) contains the following rules:

(R_{11}) For \( r_i : A_1 \rightarrow B_1C_1 \):

\( r_i^1 : A_1[r_i] - C_1Y >, \)
\( r_i^2 : A_1[r_i] - C_1\alpha_1Y >, \alpha_1 \in N, \)
\( r_i^3 : A_1[r_i] - C_1\alpha_1\alpha_2Y >, \alpha_1, \alpha_2 \in N, \)
\( r_i^4 : A_1[r_i] - C_1\alpha_1\alpha_2\alpha_3Y >, \alpha_1, \alpha_2, \alpha_3 \in N, \)
\( r_i^5 : A_1[r_i] - C_1\alpha_1\alpha_2\alpha_3\alpha_4 >, \)
where \([r_i] - C_1 = [r_i]B_1C_1, \alpha_1 \in N, \alpha_2 \in N \cup \Delta_1, \alpha_3 \in N \cup \Delta_1 \cup \Delta_2, \alpha_4 \in N \cup \Delta_1 \cup \Delta_2, \alpha_2 \alpha_3 \notin (\Delta_1)(\Delta_1 \cup \Delta_2), \alpha_3 \alpha_4 \notin (\Delta_1 \cup \Delta_2)(\Delta_1 \cup \Delta_2), \)

\( r_i^6 : A_1[r_i] - C_1\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5 >, \)
where \([r_i] - C_1 = [r_i]B_1C_1, \alpha_1 \in N, \alpha_2 \in \Delta_2, \alpha_3 \in N, \alpha_4 \in \Delta_1 \) and \( \alpha_2 = \alpha_5. \)

(R_{12}) For \( r_i : A_1B_1 \rightarrow C_1D_1 \):

\( r_i^7 : A_1B_1[r_i] - D_1\alpha_1\alpha_2 >, \alpha_1 \in N, \alpha_2 \in N, \)
\( r_i^8 : A_1B_1[r_i] - D_1Y >, \)
\( r_i^9 : A_1B_1[r_i] - D_1\alpha_1Y >, \alpha_1 \in N \)
where \([r_i]C_1D_1 = [r_i] - D_1, \)

The application of the rules \( r_i^{10}, r_i^{11} \) and \( r_i^{12} \) in order also can simulate the rule \( r_i : A_1B_1 \rightarrow C_1D_1 \):

\( r_i^{10} : A_1[\epsilon - [r_i]]\alpha_1\alpha_2 >, \alpha_1 \in N, \alpha_2 \in \Delta_1, \)
\( r_i^{11} : [r_i]\alpha_1\beta_1[\epsilon - [r_i]]\alpha_2\alpha_3 >, \alpha_1 \in N, \alpha_2 \in N, \alpha_3 \in \Delta_1 \cup N, \alpha_4 \in \Delta_2, \beta_1 \in \Delta_1, \)
\( r_i^{12} : [\beta_1[r_i]B_1[r_i] - D_1\alpha_1\alpha_2 >, \alpha_1 \in N, \alpha_2 \in N \cup \{Y\} \cup \Delta_1, \alpha_4 \in \Delta_2, \beta_1 \in \Delta_1, \alpha_4 \in \Delta_1 \}
\)

and

\( r_i^{13} : A_1B_1[r_i] - D_1\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5 >, \)
where

\([r_i]C_1D_1 = [r_i] - D_1, \alpha_1 \in N, \alpha_2 \in \Delta_2, \alpha_3 \in N, \alpha_4 \in \Delta_1 \) and \( \alpha_2 = \alpha_5. \)

(R_{13}) For \( r_i : A_1 \rightarrow a \):

\( a_i^1 : XA_1[\epsilon - k_a^i]Y >, \)
\( a_i^2 : \langle [r_m]A_1[\epsilon - k_a^i]Y >, \)
\( a_i^3 : \langle [r_m]A_1[\epsilon - k_a^i]Y >, \alpha_1 \in N, \)
$a^4_i : \langle [r_m] A_1 | \epsilon - k^i_\lambda | a_1 a_2 Y >, a_1, a_2 \in N,$
$a^5_i : \langle [r_m] A_1 | \epsilon - k^i_\lambda | a_1 a_2 a_3 Y >, a_1, a_2, a_3 \in N,$
$a^6_i : \langle [r_m] A_1 | \epsilon - k^i_\lambda | a_1 a_2 a_3 a_4 >,$
where $a_1 \in N, a_2 \in N \cup \Delta_1, a_3 \in N \cup \Delta_1 \cup \Delta_2, a_4 \in N \cup \Delta_1 \cup \Delta_2,$
$a_2 a_3 \notin (\Delta_1)(\Delta_1 \cup \Delta_2), a_3 a_4 \notin (\Delta_1 \cup \Delta_2)(\Delta_1 \cup \Delta_2),$  
$a^7_i : \langle [r_m] A_1 | \epsilon - k^i_\lambda | a_1 a_2 a_3 a_4 a_5 >,$
where $a_1 \in N, a_2 \in \Delta_2, a_3 \in N, a_4 \in \Delta_1$ and $a_2 = a_5,$
$r^i_{m+1} : \langle [r_m] A_1 k^i_\lambda | \epsilon - [r_m] | a_1 a_2 >, a_1 \in N, a_2 \in \{Y\} \cup N \cup \Delta_1 \cup \Delta_2.$

(R_14) For $r_i : A \rightarrow \lambda$:
$r^1_i : \langle X A_1 | \epsilon - k^i_\lambda | Y >,$
$r^3_i : \langle [r_m] A_1 | \epsilon - k^i_\lambda | Y >,$
$r^5_i : \langle [r_m] A_1 | \epsilon - k^i_\lambda | a_1 Y >, a_1 \in N,$
$r^7_i : \langle [r_m] A_1 | \epsilon - k^i_\lambda | a_1 a_2 Y >, a_1, a_2 \in N,$
$r^9_i : \langle [r_m] A_1 | \epsilon - k^i_\lambda | a_1 a_2 a_3 Y >, a_1, a_2, a_3 \in N,$
where $a_1 \in N, a_2 \in N \cup \Delta_1, a_3 \in N \cup \Delta_1 \cup \Delta_2, a_4 \in N \cup \Delta_1 \cup \Delta_2,$
$a_2 a_3 \notin (\Delta_1)(\Delta_1 \cup \Delta_2), a_3 a_4 \notin (\Delta_1 \cup \Delta_2)(\Delta_1 \cup \Delta_2),$  
$r^{11}_i : \langle [r_m] A_1 | \epsilon - k^i_\lambda | a_1 a_2 a_3 a_4 >,$
where $a_1 \in N, a_2 \in \Delta_2, a_3 \in N, a_4 \in \Delta_1$ and $a_2 = a_5.$
$r^{13}_i : \langle [r_m] A_1 k^i_\lambda | \epsilon - [r_m] | a_1 a_2 >, a_1 \in N, a_2 \in \{Y\} \cup N \cup \Delta_1 \cup \Delta_2.$

(R_15) $r_m : \langle X a_1 \beta_1 | \epsilon - [r_m] | a_2 >, a_1, a_2 \in N, \beta_1 \in \Delta_1,$
$r_{m+1} : \langle [r_m] a_1 a_2 \beta_1 | \epsilon - [r_m] | a_3 >, a_1, a_2, a_3 \in N, \beta_1 \in \Delta_2,$
$r_{m+2} : \langle [r_m] a_1 \beta_1 | \epsilon - [r_m] | a_2 a_3 \beta_1 >, a_1, a_2 \in N, \beta_1 \in \Delta_2, \beta_2 \in \Delta_1,$
$r_{m+3} : \langle [r_m] a_1 a_2 \beta_1 | \epsilon - [r_m] | a_2 a_3 >, a_1 \in N, a_2 \in N, a_3 \in N \cup \Delta_1 \cup \Delta_2, \beta_1 \in \Delta_1,$
$r_{m+4} : \langle [r_m] a_1 a_2 \beta_1 | \epsilon - [r_m] | a_2 a_3 \beta_2 >, a_1, a_2 \in N, \beta_2 \in \Delta_2, \beta_1 \in \Delta_1,$
$r_{m+5} : \langle [r_m] a_1 a_2 \beta_1 | \epsilon - [r_m] | a_2 a_3 \beta_2 >, a_1 \in N, a_2 \in N, a_3 \in \Delta_1, \beta_1 \in \Delta_1, \beta_2 \in \Delta_2,$
$r_{m+6} : \langle [r_m] a_1 a_2 \beta_1 | \epsilon - [r_m] | a_2 a_3 >, a_1, a_2 \in N, a_3 \in N \cup \Delta_1 \cup \Delta_2, \beta_1 \in \Delta_1, \beta_2 \in \Delta_2.$
the same sequence generates $x = h$ in $L$. Similar, the same label can be assigned to multiple rules but one rule cannot have be labeled flat finite splicing systems. Unlike in the case of Szilard languages, represented as $w$ rules generate a word computation will be possible. The concatenation of the labels of the applied simulation of the rules has been discussed in the appendix.

Starting from $XSY$ inclusions of the Szilard language of the labeled finite flat splicing system. Moreover, no other word can be obtained as homomorphic to prove that any element of the recursively enumerable language $L$ can be obtained as homomorphic image of a Szilard language of a labeled finite flat splicing system. Moreover, no other word can be obtained as homomorphic image of the Szilard language of the labeled finite flat splicing system $\mathcal{L}_\mathcal{P}$ except the words from $L$.

To prove the inclusion $L = L(G) \subseteq h(SZ_{2,FIN}^4(\mathcal{L}_\mathcal{P}))$, let us assume that $w \in L = L(G)$. Now, $w \in L$ can be generated at first by application of the recursive rules and then by left-most application of the terminal rules. Starting from $XSY$ if the rules in labeled flat splicing system $\mathcal{L}_\mathcal{P}$ are applied to in the same sequence, a word over $A$ is generated and no further computation will be possible. The concatenation of the labels of the applied rules generate a word $w' \in (Lab)^*$. Since, the morphism $h$ only maps the labels $a_1^i$, $a_2^i$, $a_3^i$, $a_4^i$, $a_5^i$, $a_6^i$, $a_7^i$ to $a$ and others to $\lambda$, the string $w$ can be represented as $w = h(w')$.

Hence $L = L(G) \subseteq h(SZ_{2,FIN}^4(\mathcal{L}_\mathcal{P}))$.

To prove the second inclusion $h(SZ_{2,FIN}^4(\mathcal{L}_\mathcal{P})) \subseteq L(G) = L$, let $x \in h(SZ_{2,FIN}^4(\mathcal{L}_\mathcal{P}))$. Hence there exists a $x_1 \in SZ_{2,FIN}^4(\mathcal{L}_\mathcal{P})$ such that $x = h(x_1)$. The word $x_1$ is obtained by concatenating the labels of a terminal derivation of $\mathcal{L}_\mathcal{P}$. Since, the flat splicing rules are simulated from the rules in $G$ and no extra derivation is possible, the application of the rules in $G$ in the same sequence generates $x \in T^*$. Hence $h(SZ_{2,FIN}^4(\mathcal{L}_\mathcal{P})) \subseteq L(G) = L$. This will imply, $h(SZ_{2,FIN}^4(\mathcal{L}_\mathcal{P})) = L(G) = L$. □

In the next section, we associate the idea of control languages with labeled flat finite splicing systems. Unlike in the case of Szilard languages, the same label can be assigned to multiple rules but one rule cannot have
multiple labels. We show that although there exists regular languages which cannot be Szilard language by any labeled flat finite splicing systems, any non-empty regular language can be obtained as control language of labeled flat finite splicing systems. Also, any non-empty context-free language can be obtained as control language by these systems and any recursively enumerable language can be obtained as control language when some rules are associated with label $\lambda$.

5 Control languages of labeled flat splicing systems

Let $\mathcal{S} = (A, I, R)$ be a flat splicing system. A labeled flat splicing system is a construct of the form $L\mathcal{S} = (A, I, R, \text{Lab})$ such that $A \cap \text{Lab} = \emptyset$. Also, each rule of the flat splicing system are associated with a label from the set Lab. Unlike in the case of Szilard languages of labeled flat splicing systems, if the rules in $R$ are not labeled in one-to-one manner (multiple rules can have the same label but one rule cannot have multiple labels) and also the rules can have empty label (i.e., $\lambda$-label), then the language obtained by concatenating the labels of the applied rules of any terminal derivation of the labeled flat splicing system is called as control language. The control languages of the labeled flat splicing system $L\mathcal{S}$ of type $(m, n)$ is denoted by $\text{CL}_{m,\text{FAM}}^n(L\mathcal{S})$. The families of control languages of the labeled flat splicing systems of type $(m, n)$, is denoted as $\text{CLLS}_{m,\text{FAM}}^{n}$.

If the rules of $L\mathcal{S}$ are associated with label $\lambda$ (empty) label, then the control language of the labeled flat splicing system $L\mathcal{S}$ of type $(m, n)$ is denoted by $\text{CL}_{n,\text{FAM}}^m(L\mathcal{S})$. The families of control languages of the labeled flat splicing systems of type $(m, n)$ with $\lambda$-labeled rules is denoted by $\text{CLLS}_{n,\lambda,\text{FAM}}^{m}$. When $m$ and $n$ are not specified, they are replaced by “*”.

Now we give examples of labeled flat finite splicing systems which can obtain non context-free and non regular languages as control languages.

Example 6. Let $L\mathcal{S} = (A, I, R, \text{Lab})$ be a labeled flat finite splicing system where $A = \{X, Y, A_1, A'\}$, $I = \{XY, A_1, A'\}$, $R = \{a : < X \mid A_1 - \epsilon \mid Y >, a : < A_1 \mid A_1 - \epsilon \mid Y >, b : < A_1 \mid A' - \epsilon \mid Y >, b : < A_1 \mid A' - \epsilon \mid A_1A' >\}$. On application of the $a$- rules, one $A_1$ is added between the markers $X$ and $Y$. Similarly, the first $b$- rule insert $A'$ between $A$ and $Y$ and the second $b$- rule inserts $A'$ between $A_1$ and $A_1A'$. Also, the $b$-rules are applicable after application of $a$-rules and only after application of same number of $a$-
and $b$-rules a string over $A$ is generated where no rule is applicable further. Hence $CL_{1,FIN}^2(\mathcal{L}) = \{a^nb^n \mid n \geq 1\}$.

**Example 7.** Let $\mathcal{L}$ be a labeled flat finite splicing system where $A = \{X, Y, A', B', A_1\}$, $I = \{X, Y, A', B', A_1\}$, $R = \{a :< X \mid A_1 - \epsilon \mid Y >, a :< A_1 \mid A_1 - \epsilon \mid Y >, b :< A_1 \mid A' - \epsilon \mid Y >, b :< A_1 \mid A' - \epsilon \mid A_1 A' >, c :< X A_1 A'_1 \mid B' - \epsilon \mid A_1 >, c :< B' A_1 A'_1 \mid B' - \epsilon \mid A_1 >, c :< B' A_1 A'_1 \mid B' - \epsilon \mid Y >\}$. Hence $CL_{1,FIN}^3(\mathcal{L}) = \{a^nb^nc^n \mid n \geq 1\}$.

**6 Results**

In the previous section, we proved that there exists some regular languages which cannot be obtained as Szilard language by any labeled flat finite splicing systems. But in the next result, we prove that any non-empty regular language can be obtained as control language of labeled flat finite splicing systems of type $(1, 2)$.

**Theorem 18.** $(\text{REG} \setminus \{\lambda\}) \subseteq CLLS_{2,FIN}^1$.

Proof. Let $L$ be a $\lambda$-free regular language and there exists a grammar $G = (N, T, S, P)$ such that $L = L(G)$. The non-terminals $N$ of $G$ are renamed as $D_i, 1 \leq i \leq n$, starting from $D_1 = S$. Now, the rules in $P$ are of the form $D_i \rightarrow aD_i$, $D_i \rightarrow aD_j (i \neq j)$, and $D_i \rightarrow a$, $D_i, D_j \in N$, and $a \in T$. In this proof we construct a labeled flat splicing systems $\mathcal{L}$ such that $L = L(G) = CL_{2,FIN}^1(\mathcal{L})$.

Let $\mathcal{L} = (A, I, R, \text{Lab})$ be a labeled flat splicing system where:

- $A = \{X, Y, D_1, D_2, \ldots, D_n\} \cup \{Y_a\}$;
- $I = \{XD_1Y\} \cup \{Y_aD_i|D_i \rightarrow aD_i \in P\} \cup \{Y_aD_j|D_i \rightarrow aD_j \in P\} \cup \{Y_a|D_i \rightarrow a \in P\}$;
- The rules in $R$ are of the following form:
  
  $a :< D_i|Y_a - D_i|Y > \text{ for } D_i \rightarrow aD_i, D_i \in N, a \in T$;
  
  $a :< D_i|Y_a - D_j|Y > \text{ for } D_i \rightarrow aD_j, D_j \in N, a \in T$;
  
  $a :< D_i|\epsilon - Y_a|Y > \text{ for } D_i \rightarrow a, a \in T$;
- $\text{Lab} = \{a \mid D_i \rightarrow aD_i, D_i \in N, a \in T\} \cup \{a \mid D_i \rightarrow aD_j, D_j \in N, a \in T\}$.

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Recursive rules $D_i \rightarrow aD_i, D_i \rightarrow aD_j$ in $G$ are simulated by $a$-rules $a :< D_i | Y_a - D_i | Y >$ and $a :< D_i | e - Y_a | Y >$ respectively. The terminal rule $D_i \rightarrow a$ is simulated by $a :< D_i | Y_a | Y >$. Hence $L \subseteq CL_{2,FIN}^1(\mathcal{L})$. Again, from the one-to-one correspondence between the rules in $P$ and labeled flat splicing rules we have the inclusion $CL_{2,FIN}^1(\mathcal{L}) \subseteq L(G)$.

Now we show that any non-empty context-free language can be obtained as a control language of the labeled flat finite splicing systems of type $(2,*)$.

**Theorem 19.** $(CF \setminus \{\lambda\}) \subseteq CLLS_{2,FIN}^2$.

**Proof.** Let $L$ be a non-empty context-free language and $G = (N, T, S, P)$ be a grammar in Greibach normal form such that $L = L(G)$. The rules in $P$ are of the form, $A_1 \rightarrow a\alpha$ and $A_1 \rightarrow a$, where $A_1 \in N, \alpha \in N^+, a \in T$.

The main idea of the proof is to construct a labeled flat splicing system $\mathcal{L} = (A, I, R, Lab)$ such that $L = CL_{2,FIN}^2(\mathcal{L})$ where $CL_{2,FIN}^2(\mathcal{L})$ denotes the control language of the labeled flat splicing systems $\mathcal{L}$ of type $(2,2)$.

At first the rules in $G$ are rewritten in the following manner:

1. If the terminal symbol at the starting of the right hand side of any two rules are same, i.e., say $a \in T$ is the starting symbol of any of the two rules. Then the $a \in T$ in each rule is replaced with distinct $a_i, i \in \mathbb{N}$. More precisely, for each distinct pair of rules $A_1 \rightarrow a\alpha$ and $B_1 \rightarrow a\beta$ where $A_1, B_1 \in N, \alpha, \beta \in N^+$, are rewritten as: $A_1 \rightarrow a_k\alpha$ and $B_1 \rightarrow a_l\beta$ where $k \neq l, k, l \in \mathbb{N}$.

2. Each distinct pair of rules of the form $A_1 \rightarrow a$ and $B_1 \rightarrow a$ where $A_1, B_1 \in N$, are rewritten as $A_1 \rightarrow a_k$ and $B_1 \rightarrow a_l$ where $k \neq l, k, l \in \mathbb{N}$.

3. Also, if only one rule $A_1 \rightarrow a\alpha, A_1 \in N, \alpha \in N^+, a \in T$ is present in $G$, then the rule is rewritten as $A_1 \rightarrow a_1\alpha$. Similarly, if there exists only one rule $A_1 \rightarrow a, A_1 \in N, a \in T$, then it is rewritten as $A_1 \rightarrow a_1$.

Now, we construct a labeled flat splicing system which simulates the newly transformed rules.

Let $\mathcal{L} = (A, I, R, Lab)$ be a labeled flat splicing system where:

- $A = \{X, Y\} \cup N \cup \{Y_{a_i} | A_1 \rightarrow a_i\alpha \in P\} \cup \{Y_{a_i} | A_1 \rightarrow a_i \in P\}$;
- $I = \{XSX\} \cup \{Y_{a_i}\alpha | A_1 \rightarrow a_i\alpha \in P\} \cup \{Y_{a_i} | A_1 \rightarrow a_i \in P\}$;
- $R$ contains the following rules:
  - For $A_1 \rightarrow a_i\alpha$:
a :< XS|Y_{a_{i}} - \beta|Y > where Y_{a_{i}} - \beta = Y_{a_{i}}\alpha \in I, \beta \in N, a_{i} \in T;

a :< Y_{a_{j}}|Y_{a_{i}} - \beta|\alpha_{2} >, where \alpha_{1} \in N, \alpha_{2} \in N \cup \{Y\}, Y_{a_{i}} - \beta = Y_{a_{i}}\alpha \in I, \beta \in N, a_{i}, a_{j} \in T.

For A_{1} \rightarrow a_{i}:

a :< XS|\epsilon - Y_{a_{i}}|Y >, a_{i} \in T;

a :< Y_{a_{j}}|\epsilon - Y_{a_{i}}|\alpha_{2} >, \alpha_{1} \in N, \alpha_{2} \in N \cup \{Y\}, a_{i}, a_{j} \in T where i, j \in N.

Lab = \{a|A_{1} \rightarrow a_{i}\alpha\} \cup \{a|A_{1} \rightarrow a_{i}\}.

We first prove that L(G) = L \subseteq CL_{2,FIN}^{2}(\mathcal{L}). Any element x \in L can be generated after sequential application of the rules in P. If the flat splicing rules are applied in the same sequence starting from XSY, a string over A_{1} is generated where no rule can be applied further, i.e., a terminal derivation is obtained. Also, concatenation of the labels of the flat splicing rules generates the string x. Hence L(G) = L \subseteq CL_{2,FIN}^{2}(\mathcal{L}).

In the similar manner, we can prove the other inclusion CL_{2,FIN}^{2}(\mathcal{L}) \subseteq L(G) = L. Let w_{1} \in CL_{2,FIN}^{2}(\mathcal{L}), i.e., there exists a terminal derivation in \mathcal{L} such that the concatenation of the labels of the splicing rules generate w_{1}. If the rules in G are applied in the same sequence as in the terminal derivation of \mathcal{L} generating w_{1}, the string w_{1} is generated. This will imply CL_{2,FIN}^{2}(\mathcal{L}) \subseteq L.

In the next theorem, we show that if the rules in \mathcal{L} are labeled with \lambda, then any recursively enumerable language can be obtained as a control language by the labeled flat finite splicing systems of type (4, 2).

**Theorem 20.** RE = CL_{\lambda}LS_{2,FIN}^{4}.

**Proof.** The inclusion CL_{\lambda}LS_{2,FIN}^{4} \subseteq RE follows from the Church-Turing thesis. It only remains to prove the inclusion RE \subseteq CL_{\lambda}LS_{2,FIN}^{4}. The proof of this inclusion follows from the proof of the Theorem 12. If all the labels of the rules of Theorem 12 except a_{1}^{4}, a_{2}^{4}, a_{4}^{4}, a_{6}^{4} and a_{7}^{4} are replaced with \lambda and the a_{1}^{3}, a_{2}^{3}, a_{3}^{3}, a_{4}^{3}, a_{5}^{3}, a_{6}^{3} and a_{7}^{3} labeled rules are replaced by a, then by following the same procedure as in Theorem 12 we can prove that RE \subseteq CL_{\lambda}LS_{2,FIN}^{4}.

\[\square\]
7 Conclusion

In this work, we compared the Szilard and control languages of labeled flat splicing systems with the family of languages in the Chomsky hierarchy. It has been shown that Szilard language of labeled flat finite splicing systems and family of regular, context-free and context-sensitive languages are incomparable. Also, any non-empty regular and context-free language can be obtained as Szilard language of these systems when a homomorphism is applied. Also any recursively enumerable language can be obtained as homomorphic image of Szilard language of labeled flat finite splicing system of type (4, 2). We also proved that any non-empty regular and context-free language can be obtained as control language by labeled flat finite splicing systems and any recursively enumerable language can be obtained as control language if some of the rules can be labeled with the empty label (i.e., $\lambda$). It remains to be investigated whether the bounds mentioned in this paper are optimal.

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9 References

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**Appendix**

**Extended proof of Theorem [17]**

**Simulation of** $r_i : A_1 \rightarrow B_1C_1$ using $r_1^1, r_2^2, r_3^3, r_4^4, r_5^5$-rule:

For each $r_i : A_1 \rightarrow B_1C_1$ there exists $r_1^1, r_2^2, r_3^3, r_4^4$ and $r_5^5$-rule. These rules are applicable to the word $Xw_1A_1w_2Y$ where $w_1, w_2 \in A^*$. Moreover, after application of the rules depending on the contexts, the words $Xw_1A_1w_2Y$ and $[r_i]B_1C_1$ are spliced together and the word $Xw_1A_1[r_i]B_1C_1w_2Y$ is obtained.

$$(Xw_1A_1w_2Y, [r_i]B_1C_1) \vdash r_1^1, r_2^2, r_3^3, r_4^4, r_5^5 \ Xw_1A_1[r_i]B_1C_1w_2Y \quad \ldots (1)$$

No rule of $R$ is applicable to the subword $A_1[r_i]$. But rules from (R15) are applicable to the subword $[r_m]A[r_i]$ obtained during identification of the leftmost non-terminal symbol for simulation of terminating rules.

**Simulation of** $r_i : A_1B_1 \rightarrow C_1D_1$ using $r_7^7, r_8^8, r_9^9$-rule:

For each $r_i : A_1B_1 \rightarrow C_1D_1$ there exists $r_7^7, r_8^8$ and $r_9^9$-rule. These rules splice the strings $Xw_1A_1B_1A_1\alpha_1\alpha_2w_2Y$ and $[r_i]C_1D_1$ where $\alpha_1, \alpha_2 \in N, w_1, w_2 \in A^*$.

$$(Xw_1A_1B_1A_1\alpha_1\alpha_2w_2Y, [r_i]C_1D_1) \vdash r_7^7 \ Xw_1A_1B_1[r_i]C_1D_1\alpha_1\alpha_2w_2Y \quad \ldots (2)$$

$$(Xw_1A_1B_1A_1\alpha_1Y, [r_i]C_1D_1) \vdash r_8^8 \ Xw_1A_1B_1[r_i]C_1D_1\alpha_1Y \quad \ldots (3)$$

$$(Xw_1A_1B_1Y, [r_i]C_1D_1) \vdash r_9^9 \ Xw_1A_1B_1[r_i]C_1D_1Y \quad \ldots (4)$$

No rule in $R$ is applicable to the subword $A_1B_1[r_i]$ of the words obtained in (2), (3) and (4). But the rules in (R15) is applicable to the subword $[r_m]A_1B_1[r_i]$ is obtained during the computation.

**Simulation of** $r_i : A_1B_1 \rightarrow C_1D_1$ in a word of the form $Xw_1A_1\alpha_1[r_i] \alpha_2[r_2] \ldots \alpha_n[r_n]B_1w_2Y$ where $\alpha_1, \ldots, \alpha_n \in N, [r_1], \ldots, [r_n] \in \Delta_1, w_1, w_2 \in A^*$ using $r_{10}^{10}, r_{11}^{11}$ and $r_{12}^{12}$-rule:

$$
\begin{align*}
(Xw_1A_1\alpha_1[r_1] \alpha_2[r_2] \ldots \alpha_n[r_n]B_1w_2Y, [r_i]) & \vdash r_{10}^{10} Xw_1A_1[r_i] \alpha_1[r_1] \alpha_2[r_2] \ldots \alpha_n[r_n]B_1w_2Y \\
(Xw_1A_1[r_1] \alpha_1[r_1] \alpha_2[r_2] \ldots \alpha_n[r_n]B_1w_2Y, [r_i]) & \vdash r_{11}^{11} Xw_1A_1[r_i] \alpha_1[r_1] \alpha_2[r_2] \ldots \alpha_n[r_n]B_1w_2Y \\
& \vdots
\end{align*}
$$

Application of $a_i^1$-rule and $r_{j14}$-rule:

The $a_i^1$-rule for each $r_i : A_1 \rightarrow a$ and $r_{j14}$-rule for each $r_j : A_2 \rightarrow \lambda$ is applicable to the word $X A_1 Y$ and $X A_2 Y$.

$$X A_1 Y, k_a^i \vdash a^1_i \cdot X A_1 k_a^i Y$$  \hspace{1cm} \ldots (6)

$$X A_2 Y, k_a^j \vdash r_{j14} \cdot X A_2 k_a^j Y$$  \hspace{1cm} \ldots (7)

Simulation of $r_i : A_1 \rightarrow a$ using the $a_i^2, a_i^3, a_i^4, a_i^5, a_i^6$-rule:

The flat splicing rules simulating the terminal rules $r_i : A_1 \rightarrow a$ and $r_j : A_2 \rightarrow \lambda$ are constructed in such a manner that the left-most non-terminal is identified by the marker $X$ and the symbol $[r_m]$ and then the splicing is performed. This process is performed by application of the $r_i^1_{m+1}, r_i^2_{m+2}$ - rule and the rules in ($R_{15}$). At first we discuss the simulation of the rules $r_i : A_1 \rightarrow a$ and $r_j : A_2 \rightarrow \lambda$.

Simulation of $r_i : A_1 \rightarrow \lambda$ using the $a_i^2, a_i^3, a_i^4, a_i^5, a_i^6$-rule:

For each $r_i : A_1 \rightarrow a$ there exists rules $a_i^1, a_i^2, a_i^3, a_i^4, a_i^5$, and $a_i^6$-rule which splice the words $X w_1 A_1 w_2 Y$ and $k_a^i$.

$$X w_1[r_m] A_1 Y, k_a^i \vdash a^2_i \cdot X w_1 A_1 k_a^i Y$$

$$X w_1[r_m] A_1 \alpha_1 Y, k_a^i \vdash a^3_i \cdot X w_1 A_1 k_a^i \alpha_1 Y, \alpha_1 \in N$$

$$X w_1[r_m] A_1 \alpha_1 \alpha_2 Y, k_a^i \vdash a^4_i \cdot X w_1 A_1 k_a^i \alpha_1 \alpha_2 Y, \alpha_1, \alpha_2 \in N$$

$$X w_1[r_m] A_1 \alpha_1 \alpha_2 \alpha_3 Y, k_a^i \vdash a^5_i \cdot X w_1 A_1 k_a^i \alpha_1 \alpha_2 \alpha_3 Y, \alpha_1, \alpha_2, \alpha_3 \in N$$

$$X w_1[r_m] A_1 \alpha_1 \alpha_2 \alpha_3 \alpha_4 Y, k_a^i \vdash a^6_i \cdot X w_1 A_1 k_a^i \alpha_1 \alpha_2 \alpha_3 \alpha_4 Y$$  \hspace{1cm} \ldots (8)

Similarly, the rule $r_i : A_1 \rightarrow \lambda$ can be simulated by application of the $r_i^{15}, r_i^{16}, r_i^{17}, r_i^{18}$, and $r_i^{19}$-rule.

Application of $r_i^{15}$-rule for $r_i : A_1 \rightarrow a$ and $r_i^{14}$-rule for $r_j : A_2 \rightarrow \lambda$:

After simulating the rules $r_i : A_1 \rightarrow a$ and $r_j : A_2 \rightarrow \lambda$ the words $[r_m] A_1 k_a^i$ and $[r_m] A_2 k_a^j$ are obtained after application of the rules $a_i^2, \ldots, a_i^6$ and $r_i^{15}, \ldots, r_i^{19}$ respectively. To proceed the computation further $r_i^{15}$-rule and $r_{j14}$-rule are applied. These rules are applicable to the words $X w_1[r_m] A_1 k_a^i \alpha_1 \alpha_2 w_2 Y$ and $X w_1[r_m] A_2 k_a^j \alpha_1 \alpha_2 w_2 Y$ respectively where $w_1, w_2 \in A^*, \alpha_1 \in N, \alpha_2 \in \{Y\} \cup N \cup \Delta_1 \cup \Delta_2$. This computation can be done by splicing the words $X w_1[r_m] A_1 k_a^i \alpha_1 \alpha_2 w_2 Y$ and $X w_1[r_m] A_2 k_a^j \alpha_1 \alpha_2 w_2 Y$ with $[r_m]$.

$$X w_1[r_m] A_1 k_a^i \alpha_1 \alpha_2 w_2 Y, [r_m] \vdash r_i^{15} \cdot X w_1[r_m] A_1 k_a^i [r_m] \alpha_1 \alpha_2 w_2 Y$$  \hspace{1cm} \ldots (9)

$$X w_1[r_m] A_2 k_a^j \alpha_1 \alpha_2 w_2 Y, [r_m] \vdash r_{j14} \cdot X w_1[r_m] A_2 k_a^j [r_m] \alpha_1 \alpha_2 w_2 Y$$  \hspace{1cm} \ldots (10)
The symbol \([r_m]\) helps the system to identify the leftmost non terminal where the rules \(a^2, \ldots, a^6\) and \(r_i^{15}, \ldots, r_i^{19}\) can be applied. The identification process is performed by the \(r_m, r_{m+1}, r_{m+2}, r_{m+3}, r_{m+4}\) and \(r_{m+5}\)-rule.

**Application of \(r_m, r_{m+1}, r_{m+2}, r_{m+3}, r_{m+4}, r_{m+5}\)-rule:**

Any computation of \(L\) starts with \(XSY\) and a rule \(r_i : A_1 \rightarrow B_1C_1\) is applied and the word \(XA_1[r_i]B_1C_1Y\) is obtained. During the computation since no words can be removed from the word \(XwY, w \in A^+\), the word present in the system will be of the form \(XA_1[r_i]wY\) where \(\alpha \in N, w_2 \in A^+\). Moreover, the terminal rules are applied in leftmost manner and the corresponding flat splicing rules are constructed in such a manner such that the rules are applied to the leftmost non terminal. So, to identify the leftmost non terminal \(r_m\)-rule is applied first.

\[
(XA[r_i]\alpha w_2Y, [r_m]) \vdash^{r_m} XA[r_i][r_m]\alpha w_2Y
\]

Moreover, after simulation of the rule \(r_i : A_1B_1 \rightarrow C_1D_1\) using the \(r_i^7\) and \(r_i^8\)-rule we have a word \(Xw_1A_1B_1[r_i]w_2Y, w_1, w_2 \in A^+\). The subword \(A_1[r_i]B_1[r_i]\) becomes inactive, i.e., no rule is applicable to the subword. But it becomes active again when the subword \([r_m]A_1[r_i]\) is obtained.

\[
(Xw_1[r_m]\alpha_k[r_k]A_1B_1[r_i]w_2Y, [r_m])
\]

\[
l \vdash^{r_m+3} Xw_1[r_m]s_w[r_k][r_m]A_1B_1[r_i]w_2Y
\]

\[
Xw_1[r_m]\alpha_k[r_k][r_m]A_1B_1[r_i]w_2Y
\]

\[
l \vdash^{r_m+3} Xw_1[r_m]\alpha_k[r_k][r_m]A_1B_1[r_i]w_2Y
\]

After the simulation of \(r_i : A_1B_1 \rightarrow C_1D_1\) in the word \(Xw_1\alpha[r_i]\alpha_2[r_2] \ldots \alpha_n[r_n]w_2Y\), to further proceed the computation the following rules are applied. Moreover, after simulation of the rule \(r_i : A_1B_1 \rightarrow C_1D_1\), the word \(Xw_1A_1[r_i]\alpha [r_1][r_i]\alpha_2[r_2][r_i] \ldots [r_i]\alpha_n[r_n][r_i]B_1[r_i]C_1D_1w_2Y\) where \(w_1, w_2 \in A^+\) is obtained. So, the computation proceeds in the following manner when a word \(Xw_1[r_m] A_1[r_i] \alpha [r_1][r_i]\alpha_2[r_2][r_i] \ldots [r_i]\alpha_n[r_n][r_i]B_1[r_i]w_2Y\) is obtained.

\[
(Xw_1[r_m]A_1[r_i]\alpha_1[r_1][r_i]\alpha_2[r_2][r_i] \ldots [r_i]\alpha_n[r_n][r_i]B_1[r_i]w_2Y, [r_m])
\]

\[
l \vdash^{r_m+2} Xw_1[r_m]A_1[r_i][r_m]\alpha_1[r_1][r_i]\alpha_2[r_2][r_i] \ldots [r_i]\alpha_n[r_n][r_i]B_1[r_i]w_2Y
\]

\[
Xw_1[r_m]A_1[r_i][r_m]\alpha_1[r_1][r_i]\alpha_2[r_2][r_i] \ldots [r_i]\alpha_n[r_n][r_i]B_1[r_i]w_2Y
\]

\[
l \vdash^{r_m+3} Xw_1[r_m]A_1[r_i][r_m]\alpha_1[r_1][r_i]\alpha_2[r_2][r_i] \ldots [r_i]\alpha_n[r_n][r_i]B_1[r_i]w_2Y
\]

So repeated application of the \(r_{m+3}\)-rule the word \(Xw_1[r_m]A_1[r_i][r_m]\alpha_1[r_1][r_i]\alpha_2[r_2][r_i] \ldots [r_i]\alpha_n[r_n][r_i]B_1[r_i]w_2Y\) is obtained.

\[
(Xw_1[r_m]A_1[r_i][r_m]\alpha_1[r_1][r_i]\alpha_2[r_2][r_i] \ldots [r_i]r_n\alpha_n[r_n][r_i]B_1[r_i]w_2Y,
\]

\[
[r_m])
\]

\[
l \vdash^{r_m+4} Xw_1[r_m]A_1[r_i][r_m]\alpha_1[r_1][r_i]\alpha_2[r_2][r_i] \ldots [r_i]r_m\alpha_n[r_n][r_i]B_1[r_i]w_2Y;
\]

\[
(Xw_1[r_m]A_1[r_i][r_m]\alpha_1[r_1][r_i]\alpha_2[r_2][r_i] \ldots [r_i][r_m]\alpha_n[r_n][r_i]B_1[r_i]w_2Y,
\]

\[
[r_m])
\]

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\[ \vdash \text{for } r_i : A_1 B_1 \rightarrow C_1 D_1: \]

After the simulation of \( r_i : A_1 B_1 \rightarrow C_1 D_1 \), a word \( X w_1 A_1[r_1]\alpha_1[r_1] \ldots \alpha_n[r_n][r_1]B_1[r_1]w_2 Y[r_j] \)

\[ \vdash \alpha_i^6 X w_1' A_2[r_1][r_1] \ldots \alpha_n[r_n][r_1]B_1[r_1]w_2 Y[r_j] \] \quad \ldots (13)

**Application of the rule \( r_i^6 \)** for \( r_i : A_1 B_1 \rightarrow C_1 D_1 \):

Similarly, the application of the \( a_j^5 \)-rule for \( a_j : A \rightarrow a \) and \( r_i^{20} \)-rule for \( r_i : A \rightarrow \lambda \):

From the above derivations we know that after simulating the rules, subwords of the form \( A_1[r_1], A_1 B_1[r_1], A_1 k_2 \) and \( A_1 k_3 \) are obtained. Moreover, these subwords become active again when the subwords \( [r_m] A_1[r_1], [r_m] A_1 B_1[r_1], [r_m] A_1 k_2 \) and \( [r_m] A_1 k_3 \). So the flat splicing rules are constructed in such a manner that no extra derivation is possible. Again, any \( x \in L \) can be generated by application of the recursive rules and then by leftmost application of the terminating rules. Now, if the flat splicing rules simulating the rules in \( P \) are applied in the same order, then there exists a terminal derivation such that the concatenation of the labels of the applied flat splicing rules obtain a string \( x_1 \) where \( h(x_1) = x \). Hence \( x \in SZ_2^4(\mathcal{L} \mathcal{S}) \). So, \( L \subseteq SZ_2^4(\mathcal{L} \mathcal{S}) \).

Again, let \( x_1 \in SZ_2^4(\mathcal{L} \mathcal{S}) \). Then \( x_1 \) can be obtained by concatenating the labels of the flat splicing rules in a terminal derivation. Since no extra derivation is possible, if the rules in \( P \) are applied in the same order, a string over \( T \) is obtained. In fact, the rules are constructed in such a way that, the string \( x \in T^+ \) is obtained where \( h(x_1) = x \). Hence \( SZ_2^4(\mathcal{L} \mathcal{S}) \subseteq L \).