PRODUCTS OF LINEAR FORMS
AND
TUTTE POLYNOMIALS

ANDREW BERGET

Abstract. Let $\Delta$ be a finite sequence of $n$ vectors from a vector space over any field. We consider the subspace of $\text{Sym}(V)$ spanned by $\prod_{v \in S} v$, where $S$ is a subsequence of $\Delta$. A result of Orlik and Terao provides a doubly indexed direct sum decomposition of this space. The main theorem is that the resulting Hilbert series is the Tutte polynomial evaluation $T(\Delta; 1 + x, y)$. Results of Ardila and Postnikov, Orlik and Terao, Terao, and Wagner are obtained as corollaries.

1. Introduction and Statement of the Theorem

Let $V$ be a vector space of dimension $\ell$ over a field $K$ of arbitrary characteristic. Let $\Delta = (\alpha_1, \ldots, \alpha_n)$ be a sequence of elements spanning $V^*$, the dual space of $V$. We allow the possibility that some of $\alpha_i$ are the zero form, and for some $\alpha_i$’s to differ by scalars. This is to say, $\Delta$ is a realization of a rank $\ell$ matroid $M(\Delta)$ on ground set $E = \{1, 2, \ldots, n\}$. We assume that the reader is familiar with the basics of matroids and their Tutte polynomials (see [6, 17]).

Denote the symmetric algebra of $V^*$ by $\text{Sym}(V^*)$, which is thought of as the $K$-algebra of polynomial functions on $V$. Let $P(\Delta)$ be the $K$-subspace of $\text{Sym}(V^*)$ spanned by $\alpha_S = \prod_{i \in S} \alpha_i$, where $S \subset E$. Let $P(\Delta)_{j,k}$ be the $K$-span of those products $\alpha_S$ where $\{\alpha_i : i \in E - S\}$ spans a $j$-dimensional subspace of $V^*$ and $k = |E - S|$. It follows from a result of Orlik and Terao (see [12, 15] and Proposition 2.2) that there is a $K$-vector space decomposition

$$P(\Delta) = \bigoplus_{0 \leq j \leq k \leq n} P(\Delta)_{j,k}. \quad (1)$$

The main result of this paper is the following.

Theorem 1.1. The Tutte polynomial of $M(\Delta)$ is equal to

$$\sum_{0 \leq j \leq k \leq n} (x - 1)^{\ell - j} y^{k - j} \dim P(\Delta)_{j,k}.$$

This result was anticipated in the work of many authors. Following a suggestion of Aomoto, in [12] Orlik and Terao considered a vector space related to $P(\Delta)$. They studied an algebra whose underlying $K$-vector space was isomorphic to $\bigoplus_{j=0}^n P(\Delta)_{j,j}$. In [16], Wagner considered an algebra whose underlying $K$-vector space was isomorphic to $P(\Delta)$. In [2] Ardila and Postnikov investigate the spaces $P(\Delta)$ and $\bigoplus_{k=0}^n P(\Delta)_{j,k}$ from the point of view of power ideals. Other spaces related to $P(\Delta)$ were studied in [1 5 7 11 13 14] and [15]. There is also a vast
literature on box splines and their relationship to $P(\Delta)$ and its subspaces. In this area, Dahmen and Miccelli were considering related objects as early as 1983. A collection of relevant references to this area can be found in De Concini and Procesi [10].

It is worth noting that not all algebraic invariants of a matroid or vector configuration are specializations of the Tutte polynomial, and hence are not related to $P(\Delta)$. Two particular objects of interest to the author are the Whitney algebra of a matroid, defined by Crapo, Rota and Schmitt in [9] and the smallest general linear group representation containing a fixed decomposable tensor [3].

This paper is organized as follows: We start by setting up some notation and compute an example in Section 2. In Section 3 we list some corollaries of Theorem 1.1. Section 4 gives a proof of a formula, due to Terao [15], for the Hilbert series of the algebra generated by the reciprocals of linear forms. The benefit of this proof is that it works over an arbitrary field $K$, whereas Terao assumed that the characteristic of $K$ was zero. In Section 5 the following question is answered: For what $d$ does $P(\Delta)$ contain $\text{Sym}^d(V^*)$? In Section 6 we give the deletion-contraction proof of Theorem 1.1 while in Section 7 we give a short prove of the theorem using the seemingly weaker result stated in Corollary 3.2.

2. An Example

Before proceeding, we state a refinement of the decomposition (1), due to Orlik and Terao [12], and use this to give an example of Theorem 1.1. To do this we recall some notation. For a subset $S \subset E$, the dimension of the $K$-span of $\{\alpha_i : i \in S\}$ is called its rank (in $M(\Delta)$) and denoted $r(S)$. A set $S \subset E$ is said to be independent (in $M(\Delta)$) if $r(S) = |S|$ and dependent if $r(S) < |S|$. A flat of $\Delta$ is a set $X \subset E$ such that for all strict containments $X \subset Y$, $r(X) < r(Y)$. The collection of flats of $\Delta$, $L(\Delta)$, is a geometric lattice where the rank of $X$ is $r(X)$. The closure or span of a subset of $E$ is the smallest flat containing it.

For $X \in L(\Delta)$, let $P(\Delta)_{X}$ be the subspace of $P(\Delta)$ spanned by those $\alpha_S$ where $E - S$ has closure $X$.

Remark 2.1. When $\Delta$ does not contain the zero form, it is possible to avoid the somewhat backwards definition of $P(\Delta)_X$ by noting that $P(\Delta)_X$ is isomorphic to the vector space spanned by the rational functions $1/\alpha_S$ where the closure of $S$ is equal to $X$. This topic will be discussed further in Section 4.

Proposition 2.2 (Orlik-Terao [12, Lemma 3.2]). There is a $K$-vector space direct sum decomposition

$$P(\Delta) = \bigoplus_{X \in L(\Delta)} P(\Delta)_X.$$ 

Since $P(\Delta)$ is spanned by homogeneous elements this sum may be refined by degree. Denote the degree $n - k$ subspace of $P(\Delta)_X$ by $P(\Delta)_{X,k}$. There is a $K$-vector space direct sum decomposition,

$$P(\Delta) = \bigoplus_{X \in L(\Delta), k \geq 0} P(\Delta)_{X,k}.$$ 

The decomposition (1) is obtained from the one above by taking the ranks of the flats of $M(\Delta)$. Note that, by definition, if $P(\Delta)_{X,k} \neq 0$ then $r(X) \leq k \leq |X|$. 

2
Example 2.3. Let $K = \mathbb{F}_2$ be the field with two elements and $\Delta = (\alpha_1, \ldots, \alpha_7)$ be the seven nonzero elements of the dual of $V = \mathbb{F}_2^3$. The matroid $M(\Delta)$ is known as the Fano matroid and it is the rank three matroid whose circuits of size three are the three point lines of the Figure 1.

![Figure 1. The Fano matroid.](image)

We will compute $T(\Delta; 1 + x, y)$ by finding $\dim P(\Delta)_{X,k}$ for all flats $X$ and all $k$ such that $r(X) \leq k \leq |X|$.

See that $P(\Delta)\emptyset,0$ is spanned by one nonzero element $\alpha_E = \prod_{i=1}^7 \alpha_i$ and hence has dimension one. The rank one flats of $\Delta$ are in bijection with the elements of $\Delta$. Hence $P(\Delta)_{\{i\},1}$ is spanned by the single element $\alpha_E/\alpha_i$ and $\dim P(\Delta)_{\{i\},1} = 1$.

There are seven rank two flats of $\Delta$. If $X$ is such a flat then it corresponds to a set of the form $\{\alpha_i, \alpha_j, \alpha_i + \alpha_j\}$. It follows that $P(\Delta)_{X,2}$ is spanned by the three elements $\alpha_i \alpha_{E-X}, \alpha_j \alpha_{E-X}, (\alpha_i + \alpha_j) \alpha_{E-X}$.

Adding these three terms up gives 0 and hence $\dim P(\Delta)_{X,2} \leq 2$. Since none of the $\alpha_i$ are parallel, $\dim P(\Delta)_{X,2} = 2$. Because $P(\Delta)_{X,3}$ is spanned by a single nonzero element, $\dim P(\Delta)_{X,3} = 1$.

The only rank 3 flat of $\Delta$ is the whole set $E$. The empty product spans $P(\Delta)_{E,7}$ and so it has dimension one. One finds that $P(\Delta)_{E,6}$ is the span of $\alpha_1, \ldots, \alpha_7$, so this space has dimension equal to the dimension of $V^*$, which is three. To compute $P(\Delta)_{E,5} \subset \text{Sym}^2(V^*)$, assume that $\alpha_1, \alpha_2$ and $\alpha_3$ are a basis for $V^*$. By considering leading terms (under any term order) we see that

$$
\alpha_1 \alpha_2, \alpha_1 \alpha_3, \alpha_2 \alpha_3, \alpha_1 (\alpha_1 + \alpha_2), \alpha_2 (\alpha_2 + \alpha_3), \alpha_3 (\alpha_1 + \alpha_3)
$$

forms a basis for $\text{Sym}^2(V^*)$. In a similar fashion we see that $P(\Delta)_{E,4}$ is equal to $\text{Sym}^3(V^*)$, which has dimension 10. We resort to a computer to find the dimension of $P(\Delta)_{E,3}$, which is spanned by 28 products. This space is contained in $\text{Sym}^4(V^*)$ which has dimension $\left(\frac{3+4-1}{2}\right) = 15$. One computes that $\dim P(\Delta)_{E,3} = 8$.

Adding all the terms up with the appropriate powers of $x-1$ and $y$, Theorem 1.1 says that

$$(x-1)^3 + 7(x-1)^2 + 14(x-1) + 7(x-1)y + y^4 + 3y^3 + 6y^2 + 10y + 8.$$

is the Tutte polynomial of the Fano matroid.

3. HILBERT SERIES

Recall that the polynomial algebra $\text{Sym}(V^*)$ has a grading

$$
\bigoplus_{d \geq 0} \text{Sym}^d(V^*)
$$
and that a vector subspace $P \subset \text{Sym}(V^*)$ is said to be graded if it is equal to the direct sum of its homogeneous pieces $P \cap \text{Sym}^d(V^*)$. The Hilbert series of $P$, $\text{Hilb}(P, t)$, is the generating function

$$\sum_{k \geq 0} \dim(P \cap \text{Sym}^k(V^*)) \; t^k.$$ 

Denote the Tutte polynomial of $M(\Delta)$ by $T(\Delta; x, y)$. Since $P(\Delta)$ is generated by products of linear forms it is a graded subspace of $\text{Sym}(V^*)$.

**Corollary 3.1** (Wagner [16, Proposition 3.1]). We have,

$$\text{Hilb}(P(\Delta), t) = t^{n-l}T(\Delta; 1 + t, 1/t).$$

The dimension of $P(\Delta)$ is the number of independent sets of the matroid $M(\Delta)$.

**Proof.** The first claim is found by setting $x = 1+t$ and $y = 1/t$ in Theorem 1.1. The second claim follows from the well-known Tutte polynomial valuation of $T(M; 2, 1)$ as the number of independent sets of the matroid $M$. □

A set $C \subset E$ is called a circuit of $M(\Delta)$ if $C$ is dependent but $C-e$ is independent for all $e \in C$. If $I \subset E$ is independent, define $e \in E = \{1, 2, \ldots, n\}$ to be externally active in $I$ if $e$ is the smallest element of a circuit in $I \cup e$. Denote the set of elements externally active in $I$ by $\text{ex}(I)$.

**Corollary 3.2** (Ardila-Postnikov [1, Theorem 4.2.2]). The Hilbert series of $P(\Delta)_E$, the subspace of $P(\Delta)$ spanned by those $\alpha_S$ with $r(E - S) = \ell$, is

$$\text{Hilb}(P(\Delta)_E, t) = t^{n-\ell}T(\Delta; 1, 1/t).$$

The dimension of $\dim P(\Delta)_E$ is the number of independent sets $I$ of $M(\Delta)$ with closure $X$ such that $|I \cup \text{ex}(I)| = k$.

We will need the notion of

**Proof.** Set $x = 1$ and $y = 1/t$ in Theorem 1.1 to obtain the Hilbert series. When $X = E$ the second claim follows from the well known expression for the Tutte polynomial of a matroid $M$ on an ordered set $E$ as

$$T(M; x, y) = \sum_{B \text{ a base of } M} x^{|\text{in}(B)|} y^{|\text{ex}(B)|},$$

Here $\text{in}(B)$ is the set of elements of $e \in B$ that are the smallest elements of a bond of $E - (B - e)$; these are the internally active elements of $B$.

In case $X \neq E$ write $\Delta_X$ for the sequence obtained from $\Delta$ by deleting the $\alpha_i$ with $i \notin X$. It follows that $P(\Delta)_X$ is isomorphic to $P(\Delta_X)_{X,k}$, the isomorphism being multiplication by $\alpha_{E-X}$. In this way we reduce to the case when $X = E$. □

A circuit $C$ with its smallest element deleted is called a broken circuit. A set $S \subset E$ is said to have no broken circuits, or be nbc, if it does not contain any broken circuits. This implies that $S$ does not contain any circuit, and hence, is independent. In terms of external activity, $I$ is nbc if it is independent and $\text{ex}(I) = \emptyset$.

**Corollary 3.3** (Orlik-Terao [12, Theorem 4.3]). The Hilbert series of

$$\bigoplus_{X \in \mathcal{L}(\Delta)} P(\Delta)_{X,r(X)}$$
is the equal to $t^{n-t}T(\Delta; 1 + t, 0)$. The dimension of $P(\Delta)_{X, r(X)}$ is the number of nbc sets of $M(\Delta)$ with closure $X$.

In the case that $\Delta$ is the collection of linear forms defining a complex hyperplane arrangement $\mathcal{A}$, there is a natural isomorphism of graded vector spaces between this vector space and the complexified cohomology of the complement $V - \bigcup_i \ker(\alpha_i)$. This map takes $\alpha_{E-I}$ to $(\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_k})/\alpha_I$, where $I = \{i_1 < \cdots < i_k\}$.

**Proof.** Set $x = 1 + 1/t$ and $y = 0$ in Theorem 1.1 to obtain the Hilbert series. The second claim follows from Corollary 3.2, since the dimension of $P(\Delta)_{X, r(X)}$ is the number of bases $I$ of $X$ with $\text{ex}(I) = \emptyset$. □

**4. Algebras Generated by Reciprocals of Linear Forms**

Let $\text{Sym}(V^*)_\{0\}$ be the $K$-algebra of rational functions on $V$. Assume that $\alpha_i \neq 0$ for all $i$. In this section we will investigate the $K$-algebra of non-constant rational functions without zeros whose poles are contained in the hyperplane arrangement $\bigcup_{i \in E} \ker \alpha_i$. This was studied earlier by Brion and Vergne [5], and Terao [15].

Let $C(\Delta)$ be the $K$-algebra generated by the rational functions $\{1/\alpha_i : i \in E\}$. Define the degree of $1/\alpha_i$ to be 1, so that $C(\Delta)$ is a graded algebra. If $X \in L(\Delta)$, define $C(\Delta)_X$ to be the space spanned by products $(\alpha_{i_1} \cdots \alpha_{i_n})^{-1}$ where the set $\{i : d_i > 0\} \subset E$ has closure $X$ in $M(\Delta)$. Let the degree $k$ piece of $C(\Delta)_X$ be $C(\Delta)_{X,k}$. Let $m\Delta$ be the sequence $\Delta$ repeated $m$ times. For example, if $\Delta = (\alpha, \beta)$ then $4\Delta = (\alpha, \beta, \alpha, \beta, \alpha, \beta, \alpha, \beta)$. Since adding parallel elements does not change the span of a set, we can identify the flats of $m\Delta$ with those of $\Delta$.

**Proposition 4.1.** For each $k \geq 1$ there is an isomorphism of vector spaces $C(\Delta)_{X,k} \to P(k\Delta)_{X,k}$.

**Proof.** The isomorphism is simply multiplication by $(\alpha_{i_1} \cdots \alpha_{i_n})^k$. □

Proposition 2.2 allows us to conclude the following result of Terao.

**Proposition 4.2** (Terao [15, Proposition 2.1]). There is a $K$-vector space direct sum decomposition $C(\Delta) = \bigoplus_{X \in L(\Delta), k \geq 0} C(\Delta)_{X,k}$.

In [5], Brion and Vergne first studied $C(\Delta)$ and its subalgebra $C(\Delta)_E$. They viewed the latter as a module for the algebra $\partial(V)$ of constant coefficient derivations on $V$. Assuming that $\text{char}(K) = 0$, one of their main results was that $C(\Delta)_E$ is a free $\partial(V)$-module. In [15] Terao, using Brion and Vergne’s result, derived a formula for the Hilbert series of $C(\Delta)$ in terms of the Poincaré polynomial of $\Delta$. The goal of this section is to derive Terao’s formula with no assumption on the characteristic of the field $K$.

**Theorem 4.3** (Terao [15, Theorem 1.2] if $\text{char}(K) = 0$). We have,

$$\text{Hilb}(C(\Delta), t) = \left(\frac{y}{1 - y}\right)^\ell T(\Delta; 1/y, 0).$$
To prove this we will need the following easy result, which appears as Lemma 6.3.24 in [6]. It allows us to determine the Tutte polynomial of \( m\Delta \) in terms of the Tutte polynomial of \( \Delta \).

**Lemma 4.4.** Let \( M \) be a matroid and \( m \) be a positive integer. If \( mM \) is the matroid obtained from \( M \) by replacing every element of \( M \) by \( m \) parallel elements then

\[
T(mM; x, y) = \left( 1 - \frac{y^m}{1 - y} \right)^{r(M)} T\left( M; \frac{x y - x - y + y^m}{y^m - 1}, y^m \right).
\]

**Proof of Theorem 4.3.** For a flat \( X \in L(\Delta) \), let \( \Delta_X \) be the sequence obtained from \( \Delta \) by deleting those \( \alpha_i \) where \( i \notin X \). By Proposition 4.1 the dimension of \( C(\Delta)_{X,k} \) is the dimension of \( P(k\Delta)_{X,k} \). Multiplication by \((\alpha E - X)^k\) gives rise to a \( K \)-vector space isomorphism

\[
P(k\Delta_X)_{X,k} \rightarrow P(k\Delta)_{X,k}
\]
and so it suffices to compute \( \dim P(k\Delta_X)_{X,k} \) to find \( \dim C(\Delta)_{X,k} \).

If \( f \) is a polynomial in \( y \) then \([y^j]f\) is the coefficient of \( y^j \) in \( f \). By Theorem 1.1 and Lemma 4.4 the dimension of \( P(kX)_{X,k} \) can be written as

\[
[y^{k - r(X)}]T(k\Delta_X; 1, y) = [y^k] \left( \frac{1 - y^k}{1 - y} \right)^{r(X)} T(\Delta_X; 1, y^k)
\]

\[
= [y^k] \left( \frac{y}{1 - y} \right)^{r(X)} T(\Delta_X; 1, 0)
\]

The second equality follows since the powers of \( y \) that appear in \( T(\Delta_X; 1, y^k) \) are multiples of \( k \). We know that \( T(\Delta_X; 1, 0) \) is the number of nbc bases of \( X \), that is, the number of nbc sets of \( M(\Delta) \) with closure \( X \), and it follows that

\[
\text{Hilb}(C(\Delta)_X, t) = \left( \frac{t}{1 - t} \right)^{r(X)}
\]

Summing over all flats \( X \in L(\Delta) \),

\[
\text{Hilb}(C(\Delta), y) = \sum_{X \in L(\Delta)} \left( \frac{y}{1 - y} \right)^{r(X)}
\]

\[
= \left( \frac{y}{1 - y} \right)^{\ell} T(1/y, 0).
\]

The second equality can be verified using, e.g., the Tutte polynomial interpretation in Corollary 3.3. \( \Box \)

5. **Spanning Homogeneous Pieces of Symmetric Powers**

In this section we investigate the following problem: Given a sequence of linear forms \( \Delta = (\alpha_1, \ldots, \alpha_n) \) spanning the dual of a \( K \)-vector space \( V \), determine those \( d \) such that every \( d \)-form on \( V \) can be written in the form

\[
\sum_{S \in \binom{[n]}{d}} c_{S \alpha S}.
\]
Phrased differently, determine those $d$ such that $\Sym^d(V^*) \subset P(\Delta)$. The answer to this question is phrased in terms of the cocircuits. Recall that collection of cocircuits of $\Delta$ is

$$\{ E - H : H \in L(\Delta), r(H) = \ell - 1 \}.$$  

It is possible to reconstruct the matroid $M(\Delta)$ from its collection of cocircuits.

**Theorem 5.1.** There is a containment $\Sym^d(V^*) \subset P(\Delta)$ if and only if $d$ is less than or equal to the size of the smallest cocircuit of $\Delta$.

**Example 5.2.** For $q$ a prime power, consider all of the $[\ell]_q := (q^\ell - 1)/(q - 1)$ hyperplanes in $V = \mathbb{P}^d_q$. Let $\Delta$ be a sequence of linear forms, one defining each of these hyperplanes. It is a fact that every cocircuit of $\Delta$ has size $[\ell]_q - [\ell - 1]_q = q^{\ell-1}$. Theorem 5.1 states that $P(\Delta)$ contains $\Sym^d(V^*)$ if and only if $d \leq q^{\ell-1}$.

In the case that $q = 2$ and $\ell = 3$ then $\Delta$ is the collection from Example 2.3. Using the Tutte polynomial calculation there,

$$\text{Hilb}(P(\Delta), t) = 1 + 3t + 6t^2 + 10t^3 + 15t^4 + 14t^5 + 7t^6 + t^7.$$  

For $d \leq 2^{3-1} = 4$ the coefficient of $t^d$ can be written as $\binom{3 + d - 1}{d}$. For these $d$, $P(\Delta) \cap \Sym^d(V^*)$ and $\Sym^d(V^*)$ have equal dimensions, so they are equal.

**Proof of Theorem 5.1** To start, we investigate the Hilbert series of $P(\Delta)$ and determine the first coefficient not of the form $\binom{\ell + d - 1}{d}$. By Corollary 5.1

$$\text{Hilb}(P(\Delta), t) = t^{n-\ell}T(\Delta; 1 + t, t^{-1}).$$  

Rewrite this using the formula,

$$T(M; x, y) = \frac{1}{(y-1)^{r(M)}} \sum_{X \in L(M)} y^{|X|} \chi(M/X; (x-1)(y-1), 0).$$

Here $\chi(M; \lambda)$ is the characteristic polynomial of $M$, obtained from $T(M; x, y)$ by the rule $\chi(M; \lambda) = (-1)^{r(M)}T(1-\lambda, 0)$. This formula can be found in the discussion of the coboundary polynomial in [6]. Doing the stated substitutions allows us to write

$$\text{Hilb}(P(\Delta), t) = \frac{1}{(1-t)^\ell} \sum_{X \in L(\Delta)} t^{n-|X|}(-1)^{\ell - r(X)}T(M(\Delta)/X; t, 0)$$

Denote the polynomial on the right by $h(t)$. If $h(t)$ is of the form $1 - at^{N+1} + \cdots$, where $a$ is nonzero and the ellipsis denotes higher degree terms, then

$$\text{Hilb}(P(\Delta), t) = \sum_{k=0}^{N} \binom{\ell + k - 1}{k} t^k + \binom{\ell + N}{N + 1} - a \ t^{N+1} + \cdots$$

This is to say, $P(\Delta)$ contains $\Sym^d(V^*)$ for $d \leq N$ and $P(\Delta)$ does not contain all of $\Sym^{N+1}(V^*)$. To find the smallest power of $t$ appearing in $h(t)$ consider the smallest power of $t$ appearing in each of its summands. If $X \neq E$ then $T(M(\Delta)/X; t, 0)$ has no constant term and hence the smallest power of $t$ in each summand is at least $n - |X| + 1$. The summand corresponding to $X = E$ is the constant polynomial 1. Let $X_0$ denote a flat of largest size which is not $E$. For any such flat $M(\Delta)/X_0$ is a rank one matroid with no loops. This implies that $T(M(\Delta)/X_0; t, 0) = t$ and hence

$$h(t) = 1 - at^{n-|X_0|+1} + \cdots$$
where \( a \) is the number of flats of \( \Delta \) with size \(|X_0|\). To summarize: If \( d \leq |E - X_0| \) then \( P(\Delta) \) contains \( \text{Sym}^d(V^*) \). Further, \( P(\Delta) \) does not contain all of \( \text{Sym}^{X_0+1}(V^*) \).

Suppose there was some \( d > |E - X_0| + 1 \) such that \( P(\Delta) \) contained \( \text{Sym}^d(V^*) \). If \( \alpha_1, \ldots, \alpha_\ell \) are a basis for \( V^* \) then for all sequences \( (\sigma(1), \ldots, \sigma(\ell)) \) such that \( \sigma(1) + \cdots + \sigma(\ell) = d \),

\[
\alpha^{\sigma(1)}_1 \cdots \alpha^{\sigma(\ell)}_{\ell} \in P(\Delta)
\]

We will prove in Lemma 6.2 that this implies

\[
\alpha^{\tau(1)}_1 \cdots \alpha^{\tau(\ell)}_{\ell} \in P(\Delta)
\]

for any \( \tau \) such that \( \tau(i) \leq \sigma(i) \). It follows that \( P(\Delta) \) contains \( \text{Sym}^{d'}(V^*) \) for any \( d' \leq d \). This cannot be, since we known that \( P(\Delta) \) does not contain \( \text{Sym}^{d'}(V^*) \) when \( d' = |E - X_0| + 1 < d \). It follows that \( P(\Delta) \) contains \( \text{Sym}^d(V^*) \) if and only if \( d \leq |E - X_0| \). By definition, \( E - X_0 \) is a cocircuit of smallest size, so the theorem follows.

\[\Box\]

### 6. Deletion-Contraction Proof of Theorem 1.1

Define \( H(\Delta; x, y) \) by the rule

\[
H(\Delta; x, y) = \sum_{X \in L(\Delta), k \geq 0} x^{r(M(\Delta)) - r(X)} y^{k - r(X)} \dim P(\Delta)_{X,k}.
\]

Here \( r(M(\Delta)) \) is the rank of the matroid of \( M(\Delta) \), which is \( \ell \). Theorem 1.1 can be stated as

\[
H(\Delta; x, y) = T(\Delta; 1 + x, y).
\]

To prove this we will apply the following fundamental theorem on the Tutte polynomial (see [6, Theorem 6.2.2]).

**Theorem 6.1.** Let \( \mathcal{M} \) be the class of isomorphism classes of matroids. There is a unique function, called the Tutte polynomial, \( T : \mathcal{M} \rightarrow \mathbb{Z}[x,y] \) which satisfies

(a) The Tutte polynomial of the one element isthmus is \( x \) and that of the one element loop is \( y \).

(b) The Tutte polynomial is multiplicative in the sense that \( T(M \oplus N) = T(M)T(N) \), where \( M \oplus N \) is the direct sum of matroids.

(c) If \( M \) is a matroid on \( E \) and \( e \in E \) is neither a loop nor an isthmus then

\[
T(M) = T(M - e) + T(M/e).
\]

Here \( M - e \) is deletion of \( e \) from \( M \) and \( M/e \) is contraction of \( M \) by \( e \).

To prove that \( H(\Delta; x, y) \) equals \( T(\Delta; 1 + x, y) \) we need to check that it satisfies properties (a), (b) and (c) in Theorem 6.1.

---

\(^1\)If \( \text{char}(K) = 0 \) it is easy to see this without the lemma. Indeed, \( P(\Delta) \) is a module for \( \partial(V^*) \), the \( K \)-algebra of constant coefficient differential operators on \( V^* \). For \( 1 \leq i \leq \ell \), there is a unique differential operator on \( V^* \) taking \( \alpha_i \) to 1 and \( \{\alpha_1, \ldots, \alpha_\ell\} - K \cdot \alpha_i \) to zero. Thus, if \( \alpha_1^{\tau(1)} \cdots \alpha_\ell^{\tau(\ell)} \) is in \( P(\Delta) \) then by applying these differential operators we get that an integer multiple of \( \alpha_1^{\tau(1)} \cdots \alpha_\ell^{\tau(\ell)} \) is in \( P(m\Delta) \) whenever \( \tau(i) \leq \sigma(i) \).

---

8
Verification of property (a). It must be checked that
\[H(\{1\}; x, y) = T(\{1\}; 1 + x, y) = 1 + x,
H(\{0\}; x, y) = T(\{0\}; 1 + x, y) = y.\]
This easy task is left to the reader. □

Verification of property (b). Suppose that
\[\Delta = (\alpha_1, \ldots, \alpha_n), \quad \Delta' = (\beta_1, \ldots, \beta_m)\]
are sequences of linear forms on two vector spaces \(V\) and \(W\) over \(K\). Let \(\Delta \oplus \Delta'\) denote the concatenation of the sequences, \((\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m)\), viewed as vectors in \(V^* \oplus W^*\). This agrees with the direct sum of matroids since the matroid of \(\Delta \oplus \Delta'\) is the direct sum of the matroids of \(\Delta\) and \(\Delta'\). Recall that there is a natural isomorphism of graded \(K\)-algebras,
\[\text{Sym}(V^* \oplus W^*) \approx \text{Sym}(V^*) \otimes \text{Sym}(W^*).\]
(2) The flats of \(\Delta \oplus \Delta'\) are in bijection with \(L(\Delta) \times L(\Delta')\). We claim that if \(X \in L(\Delta)\) and \(Y \in L(\Delta')\) then, as graded vector spaces,
\[P(\Delta \oplus \Delta')(X, Y) \approx P(\Delta)_X \otimes P(\Delta')_Y.\]
Indeed, the isomorphism (2) maps \(\alpha_S \otimes \beta_T\) to \(\alpha_S \beta_T\) which is in \(P(\Delta \oplus \Delta')(X, Y)\). Since every monomial defining \(P(\Delta \oplus \Delta')(X, Y)\) is of this form, we have the needed isomorphism. Lastly, since the rank of the flat corresponding to \((X, Y)\) is sum of the ranks of \(X\) and \(Y\) and the isomorphism (2) is of graded algebras,
\[H(\Delta \oplus \Delta'; x, y) = H(\Delta; x, y)H(\Delta'; x, y).\]
□

To set up our verification of property (c), note that for any linear form \(\alpha \in V^*\) there is a complex of graded vector spaces
\[
0 \rightarrow \text{Sym}(V^*) \hookrightarrow \text{Sym}(V^*) \rightarrow \text{Sym}(\ker(\alpha)) \rightarrow 0.
\]
(3) The second map is induced by restricting linear forms on \(V\) to \(\ker(\alpha)\). When \(\alpha \neq 0\) this complex is exact by definition. Let \(\Delta - \alpha_i\) be obtained from \(\Delta\) by deleting the element in the \(i\)-th position and \(\Delta/\alpha_i\) be the restriction of the forms in \(\Delta - \alpha_i\) to \(\ker(\alpha_i)\). These definitions are made in such a way that they coincide with deletion and contraction of matroids:
\[M(\Delta - \alpha_i) = M(\Delta) - i, \quad M(\Delta/\alpha_i) = M/i\]
where \(M(\Delta), M(\Delta - \alpha_i)\) and \(M(\Delta/\alpha_i)\) denote, respectively, the matroids of \(\Delta, \Delta - \alpha_i\) and \(\Delta/\alpha_i\).

For any \(\alpha_i\), the complex (3) is seen to restrict to the complex
\[
0 \rightarrow P(\Delta - \alpha_i) \overset{\partial_i}{\rightarrow} P(\Delta) \rightarrow P(\Delta/\alpha_i) \rightarrow 0.
\]
(4) Lemma 6.2. If \(i \in E\) is neither a loop nor an isthmus of \(M(\Delta)\) then this complex is exact.

The assumption that \(i\) is not an isthmus is strictly for technical reasons: We need to have that \(\Delta - \alpha_i\) spans \(V^*\), which will not happen if \(i\) is an isthmus.
Proof. The existence of the complex guarantees that
\[ \dim P(\Delta) \geq \dim P(\Delta - \alpha_i) + \dim P(\Delta/\alpha_i). \]
We may assume, by induction, that if \( \Delta' \) has fewer elements than \( \Delta \) then \( \dim P(\Delta') \) is the number of independent sets of the matroid \( M(\Delta') \). It follows, from our induction hypothesis, that \( \dim P(\Delta) \) is at least the sum of the number of independent subsets of \( M(\Delta - \alpha_i) \) and \( M(\Delta/\alpha_i) \). This is the number of independent sets of \( M(\Delta) \), so we need only show that \( P(\Delta) \) is spanned by this many elements to conclude exactness.

We claim that the elements \( \{ p_{E-(I \cup \text{ex}(I))} : I \text{ is independent in } M(\Delta) \} \) span \( P(\Delta) \), where \( \text{ex}(I) \) is the set of elements which are externally active in \( I \) (cf. Section 3). The proof of the claim is by what Las Vergnas and Forge call lexicographic compression and variants of this proof abound (see e.g., Ardila’s thesis [1] nearly identical claim. This claim also appears in [2]). Let \( S \) be the lexicographically least subset of \( E \) such that \( \alpha_{E-S} \) is not in the span of these elements. We can uniquely write \( S = I \cup J \) where \( J \subset \text{ex}(I) \). To see this we take \( I \subset S \) to be the lexicographically largest independent set spanning \( S \), and \( J \) to be the complement of \( I \) in \( S \). One immediately sees that \( J \subset \text{ex}(I) \). We will be done if we can show that \( J = \text{ex}(I) \). Suppose, on the contrary, that \( f \in \text{ex}(I) - J \) and write \( \alpha_f = \sum_{e \in I} c_e \alpha_e \).

We have \( \alpha_{E-S} = \alpha_{E-(S \cup f)} \alpha_f \) and hence
\[
\alpha_{E-S} = \sum_{e \in I} c_e \alpha_{E-(I \cup J \cup f)} \alpha_e = \sum_{e \in I} c_e \alpha_{E-(I-e) \cup (J \cup f)}
\]
Since \( \alpha_{E-S} \) is not in the span of \( \{ p_{E-(I \cup \text{ex}(I))} : I \in I(M(\Delta)) \} \) we know there is some element on the right which is not in the span of these elements. For each \( e \in I \) we see that \( (I-e) \cup (J \cup f) \) is lexicographically smaller than \( I \cup J \), which is a contradiction.

\[ \square \]

Remark 6.3. The lemma proves Corollary 3.2 by exhibiting a basis for \( P(\Delta)_{X,k} \) with the cardinality stated there.

There are two ways to proceed from here. The first is to use the combinatorial basis just constructed to verify that the deletion-contraction recurrence holds. The second is show that the exact sequence above can be refined to consider a flat and a degree. In the linear-algebraic spirit of this paper, we take the latter route.

Verification of property (c). Assume that \( i \in E \) is neither a loop nor an isthmus of \( M(\Delta) \), that is, \( r(i) = 1 \) and \( r(E-i) = r(E) \). Recall (see [17, Chapter 7]) that the flats of the deletion \( \Delta - \alpha_i \) are in bijection with the flats of \( X \) of \( \Delta \) such that \( r(X-i) = r(X) \). Also, the flats of \( \Delta/\alpha_i \) are bijection with the flats of \( \Delta \) containing \( i \). We wish to refine the exact sequence (4) to consider a flat \( X \) of \( \Delta \) and a degree. To do this, consider three cases depending on whether or not \( i \in X \), and if \( i \in X \), then whether \( i \) is an isthmus of \( X \).

First suppose that \( i \notin X \). It follows that \( X \) is a flat of \( \Delta - \alpha_i \). The complex (4) restricts to the exact complex
\[ 0 \to P((\Delta - \alpha_i)_{X,k}) \to P(\Delta)_{X,k} \to 0. \]
Every product \( \alpha_S \in P(\Delta)_{X,k} \) is of the form \( \alpha_S \) where \( i \in S \), hence \( \alpha_{S-i} \) lies in \( P(\Delta - \alpha_i)_{X,k} \) and the map is surjective. Since the map is the restriction of an injection it is an isomorphism.
Suppose that \( i \in X \) and \( i \) is not an isthmus of \( X \). In this case \( X - i \) is a flat of both \( \Delta - \alpha_i \) and \( \Delta/\alpha_i \) and we claim that (4) restricts to the exact complex

\[
0 \to P(\Delta - \alpha_i)_{X-i,k} \xrightarrow{\alpha_i} P(\Delta)_{X,k} \to P(\Delta/\alpha_i)_{X-i,k-1} \to 0.
\]

To see this we pick some \( \alpha_S \in P(\Delta - \alpha_i)_{X-i,k} \) and see that \( \alpha_S \cup i \) is in \( P(\Delta)_{X,k} \). This is because \( (E - i) - S = E - (S \cup i) \) has closure \( X \) in \( M(\Delta) \). If the closure were the smaller set \( X - i \), then \( i \) would have been an isthmus of the flat \( X \). The map on the left in (4) is the restriction of an injection, hence we have exactness on the left.

If \( \alpha_S \in P(\Delta)_{X,k} \) and \( i \notin S \) then the closure in \( M(\Delta/\alpha_i) \) of \( (E - i) - S \) is \( X - i \). The degree of \( \alpha_S \) is unchanged under \( \alpha_i \mapsto 0 \) hence \( P(\Delta/\alpha_i)_{X,k} \) has image in \( P(\Delta/\alpha_i)_{X-i,k-1} \). That every monomial spanning \( P(\Delta/\alpha_i)_{X-i,k-1} \) is in the image of \( P(\Delta)_{X,k} \) follows from the definition of \( \Delta/\alpha_i \) as the restriction of the elements of \( \Delta - \alpha_i \) to \( \ker(\alpha_i) \). The exactness on the right of (4) follows.

We now prove exactness in the middle of (4). If an element of \( P(\Delta)_{X,k} \) restricts to zero on \( \ker(\alpha_i) \) then, by Lemma 6.2, it can be written as \( \alpha_i \) times some linear combination \( \sum c_S \alpha_S \) where \( \alpha_S \in P(\Delta - \alpha_i) \). For these \( S \) we have \( \alpha_i \alpha_S \in P(\Delta)_{X,k} \) and, since \( i \) was not an isthmus of \( X \), we have \( \alpha_S \) is in \( P(\Delta - \alpha_i)_{X-i,k} \).

In the case that \( i \in X \) and \( i \) is an isthmus of \( X \) it follows that (4) restricts to the exact complex

\[
0 \to P(\Delta)_{X,k} \to P(\Delta/\alpha_i)_{X-i,k-1} \to 0.
\]

The surjectivity here is clear. If an element is in the kernel of this map we may, as before, write it as a linear combination of terms \( \alpha_i \alpha_S \in P(\Delta)_{X,k} \). It follows that \( E - (S \cup i) \) has closure \( X \) in \( M(\Delta) \), which cannot be since \( i \notin E - (S \cup i) \) and \( i \) is in every base of \( X \). We conclude that the kernel is zero and (4) is exact.

Finally, we can verify the deletion-contraction recurrence. To do so, break up the defining sum for \( H(\Delta; x, y) \) according to the three cases we just considered. Let \( L_1 \subset L(\Delta) \) be the set of flats of \( \Delta \) not containing \( i \), \( L_2 \subset L(\Delta) \) be the set of flats containing \( i \) as an isthmus, and let \( L_3 \subset L(\Delta) \) be the remaining flats. If \( \text{cr}(X) \) denotes \( r(M(\Delta)) - r(X) \), the corank of \( X \), we see that the exact complexes (5), (6) and (7) imply that we can write

\[
H(\Delta; x, y) = \sum_{X \in L_1 \cup L_3} x^{\text{cr}(X)} y^{k-r(X)} \dim P(\Delta - \alpha_i)_{X-i,k} + \sum_{X \in L_2 \cup L_3} x^{\text{cr}(X)} y^{k-r(X)} \dim P(\Delta/\alpha_i)_{X-i,k-1}.
\]

The first sum is (5) and the left of (7), while the second sum is (6) and the right of (7). We claim that the first sum here is \( H(\Delta - \alpha_i) \) and the second is \( H(\Delta/\alpha_i) \). Since the flats of \( \Delta - \alpha_i \) are in bijection with \( \{ X - i : X \in L_1 \cup L_3 \} \) and the flats of \( \Delta/\alpha_i \) are in bijection with \( \{ X - i : X \in L_2 \cup L_3 \} \), we only need to check that the exponents of \( x \) and \( y \) in each sum are correct. If \( X \in L_1 \cup L_3 \), then the rank of \( X - i \) in the matroid \( M(\Delta - \alpha_i) \) is equal to the rank of \( X \) in \( M(\Delta) \). If \( X \in L_2 \cup L_3 \) then the rank of \( X - i \) in \( M(\Delta/\alpha_i) \) is one less than its rank in \( M(\Delta) \). It follows that the exponents of \( x \) and \( y \) are correct and so

\[
H(\Delta; x, y) = H(\Delta - \alpha_i; x, y) + H(\Delta/\alpha_i; x, y),
\]

which is what we wanted to show. \( \square \)
7. A Second Proof of Theorem 1.1

It has been observed by an anonymous referee that Theorem 1.1 may be proved directly from Ardila and Postnikov’s result in Corollary 3.2. This is the case, and in this section we give this proof. Note that the proof of Corollary 3.2 in [1] is characteristic independent, but the proof in [2] is not.

Recall that Corollary 3.2 states, among other things, that

$$\dim P(\Delta)_{X,k}$$

is the number of independent sets $I$ of $M(\Delta)$ with closure $X$ such that $|I \cup ex(I)| = k$.

Also, the Tutte polynomial of an arbitrary matroid $M$ can be written as

$$T(M; x, y) = \sum_B x^{|in(B)|} y^{|ex(B)|}$$

the sum over bases $B$ of $M$. Here $in(B)$ is the set elements that are internally active in $B$ and $ex(B)$ is the set of elements that are externally active in $B$. Recall that $e \in B$ is internally active in $B$ if $e$ is the smallest element of a bond of $D \subset E - (B - e)$, i.e., $e$ is the smallest element $f$ of a (necessarily unique) set $D \subset E - (B - e)$ that is minimal with the property that

$$r(E - D) < r(E).$$

It follows from that we may write

$$T(M; 1 + x, y) = \sum_B \sum_{I \subset in(B)} x^{|B - I|} y^{ex(B)}.$$

We now need a combinatorial result of Crapo.

Lemma 7.1 (Crapo [8, Lemmas 6,8,9]). Let $M$ be an arbitrary matroid on an ordered set $E$. Every subset $S \subset E$ can be written uniquely as $S = B - I \cup J$ where $I \subset in(I)$ and $J \subset ex(B)$.

Further, if $I \subset in(B)$ then $ex(B - I) = ex(B)$.

It follows that given an independent set $I$, we may write this set uniquely as $I = B - I_0$, where $I_0 \subset in(B)$. From this, we may write the Tutte polynomial as

$$T(M; 1 + x, y) = \sum_B \sum_{I \subset in(B)} x^{|B - I|} y^{ex(B - I)}$$

$$= \sum_I x^{r(M) - r(I)} y^{r(I)}$$

the second sum over all independent sets $I$ of $M$. From this it follows at once that

$$T(M; 1 + x, y) = \sum_{X \in L(M), k \geq 0} x^{r(M) - r(X)} y^{k - r(X)} \cdot \# \left\{ I \subset E : \begin{array}{l} I \text{ independent,} \\ \cl(I) = X, \\ |I \cup ex(I)| = k. \end{array} \right\}$$

Theorem 1.1 now follows from this expression for $T(M; 1 + x, y)$ and Corollary 3.2.
8. Acknowledgements

The author thanks Victor Reiner for many helpful conversations and suggesting the problem considered in Section 5. Thanks to Hiroaki Terao for prompting me to go back to his paper [14], and several anonymous referees for their many helpful suggestions. The author was partially supported through NSF grant DMS 0601010.

References

[1] Federico Ardila. Enumerative and algebraic aspects of matroids and hyperplane arrangements. PhD thesis, Massachusetts Institute of Technology, February 2003.
[2] Federico Ardila and Alexander Postnikov. Combinatorics and geometry of power ideals. To appear in Trans. Amer. Math. Soc., 2010.
[3] Andrew Berget. A short proof of Gamas’s theorem. Linear Algebra Appl., 430(2-3):791–794, 2009.
[4] Anders Björner. The homology and shellability of matroids and geometric lattices. In Matroid applications, volume 40 of Encyclopedia Math. Appl., pages 226–283. Cambridge Univ. Press, Cambridge, 1992.
[5] Michel Brion and Michèle Vergne. Arrangement of hyperplanes. I. Rational functions and Jeffrey-Kirwan residue. Ann. Sci. École Norm. Sup. (4), 32(5):715–741, 1999.
[6] Thomas Brylawski and James Oxley. The Tutte polynomial and its applications. In Matroid applications, volume 40 of Encyclopedia Math. Appl., pages 123–225. Cambridge Univ. Press, Cambridge, 1992.
[7] Raul Cordovil. A commutative algebra for oriented matroids. Discrete Comput. Geom., 27(1):73–84, 2002. Geometric combinatorics (San Francisco, CA/Davis, CA, 2000).
[8] Henry H. Crapo. The Tutte polynomial. Aequationes Math., 3:211–229, 1969.
[9] Henry Crapo and William Schmitt. The Whitney algebra of a matroid. J. Combin. Theory Ser. A, 91(1-2):215–263, 2000. In memory of Gian-Carlo Rota.
[10] Corrado De Concini and Claudio Procesi. Hyperplane arrangements and box splines. Michigan Math. J., 57:1–26, 2008. With an appendix by Anders Björner.
[11] David Forge and Michel Las Vergnas. Orlik-Solomon type algebras. European J. Combin., 22(5):699–704, 2001. Combinatorial geometries (Luminy, 1999).
[12] Peter Orlik and Hiroaki Terao. Commutative algebras for arrangements. Nagoya Math. J., 134:65–73, 1994.
[13] Nicholas Proudfoot and David Speyer. A broken circuit ring. Beiträge Algebra Geom., 47(1):161–166, 2006.
[14] Hal Schenck and Stefan Tohaneanu. Orlik-Terao algebra and 2-formality. arXiv:0901.0253, 2009.
[15] Hiroaki Terao. Algebras generated by reciprocals of linear forms. J. Algebra, 250(2):549–558, 2002.
[16] David G. Wagner. Algebras related to matroids represented in characteristic zero. European J. Combin., 20(7):701–711, 1999.
[17] Neil White, editor. Theory of matroids, volume 26 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1986.
[18] Neil White, editor. Matroid applications, volume 40 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1992.