On Bernstein algebras satisfying chain conditions II

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ABSTRACT

Following a previous work with Boudi, we continue to investigate Bernstein algebras satisfying chain conditions. First, it is shown that a Bernstein algebra $A$ with ascending or descending chain condition on sub-algebras is finite-dimensional. We also prove that $A$ is Noetherian (Artinian) if and only if its barideal $N = \ker(\alpha)$ is. Next, as a generalization of Jordan and nuclear Bernstein algebras, we study whether a Noetherian (Artinian) Bernstein algebra $A$ with a locally nilpotent barideal $N$ is finite-dimensional. The response is affirmative in the Noetherian case, unlike in the Artinian case. This question is closely related to a result by Zhevlakov on general locally nilpotent nonassociative algebras that are Noetherian, for which we give a new proof. In particular, we derive that a commutative nilalgebra of nilindex 3 which is Noetherian or Artinian is finite-dimensional. Finally, we improve and extend some results of Micali and Ouattara to the Noetherian and Artinian cases.

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0. Introduction

Bernstein algebras form a class of commutative nonassociative algebras whose origin lies in genetics. Historically, they have been introduced by Lyubich [16] and Holgate [13] as an algebraic formulation of the problem of classifying the stationary evolution operators in genetics. In this way, Bernstein algebras represent populations reaching the equilibrium after the first generation. Since then the theory has evolved into an independent branch of nonassociative algebras, and many researches have been done on the topic from various points of view (see, for instance, [4, 7, 8, 17, 19, 25, 26]).

One of the main questions on the structure of Bernstein algebras, posed by Lyubich and solved by Odoni and Stratton [20] as well as by Baeza [3] and Grishkov [9], says that the barideal of a finite-dimensional nuclear Bernstein algebra is nilpotent. The analogous question in the finitely generated case was proposed by Grishkov [9], and its affirmative solution was settled by Peresi [21] and Krapivin [14]. Consequently, finitely generated nuclear Bernstein algebras must be finite-dimensional, as was directly established by Suazo [24] using a different approach.

On the other hand, it is well-known that one of the most satisfactory developments of the theory of some varieties of algebras, as associative, Jordan, alternative and Lie algebras, is the structure theory of algebras with chain conditions, and there is presently a substantial bibliography in this subject. Concerning Bernstein algebras, a detailed treatment was done in [5] for Bernstein algebras satisfying chain conditions on ideals. Among many results in that paper, it was especially proved that for a Bernstein algebra which is Jordan or nuclear, each of the Noetherian and...
Artinian hypotheses implies finite-dimensionality of the algebra and so nilpotency of the barideal. Hence, the Lyubich conjecture is still valid in the Noetherian and Artinian cases.

In this present article we pursue the study of Bernstein algebras satisfying chain conditions initiated in [5]. After a first section devoted to preliminaries, we show in Section 2 that an arbitrary Bernstein algebra $A$ satisfying the ascending or descending chain condition on subalgebras is finite-dimensional. Then we prove in Section 3 that a Bernstein algebra $A$ is Noetherian (Artinian) if and only if its barideal $N = \ker(\omega)$ is, thus generalizing an early result by Krapivin [14] about the finitely generated case. Section 4 deals with Bernstein algebras having locally nilpotent barideals, as an extension of both Jordan and nuclear Bernstein algebras. Specifically, we study whether a Noetherian (Artinian) Bernstein algebra $A$ with a locally nilpotent barideal $N$ is finite-dimensional. The answer is positive in the Noetherian case and negative in the Artinian case. This question is connected with a result due to Zhevlakov [27] on general locally nilpotent nonassociative algebras for which we provide an independent proof. As a special case, we derive that a commutative nilalgebra of nilindex 3 which is Noetherian or Artinian is finite-dimensional. The answer is positive in the Noetherian case and negative in the Artinian case. This question is connected with a result due to Zhevlakov [27] on general locally nilpotent nonassociative algebras for which we provide an independent proof. As a special case, we derive that a commutative nilalgebra of nilindex 3 which is Noetherian or Artinian is finite-dimensional. This question is connected with a result due to Zhevlakov [27] on general locally nilpotent nonassociative algebras for which we provide an independent proof. As a special case, we derive that a commutative nilalgebra of nilindex 3 which is Noetherian or Artinian is finite-dimensional.

Various examples are presented along this work to serve as motivation and illustration for our results.

1. Preliminaries

In this section we briefly summarize notation, terminology and classical properties about Bernstein algebras and arbitrary nonassociative algebras. Throughout this paper, we will fix an infinite ground field $K$ of characteristic different from 2 and 3, and let $A$ be an algebra over $K$, not necessarily associative or finite-dimensional. If there exists a nonzero homomorphism of algebras $\omega : A \rightarrow K$, then the ordered pair $(A, \omega)$ is called a baric algebra and $\omega$ is its weight function. For every $e \in A$ with $\omega(e) \neq 0$, we have $A = Ke \oplus N$, where $N = \ker(\omega)$ is an ideal of $A$, called the barideal of $A$. A baric ideal of $A$ is an ideal $I$ of $A$ with $I \subseteq N$. Then the quotient algebra $A/I$ is a baric algebra with weight function $\bar{\omega}$ defined by $\bar{\omega}(x + I) = \omega(x)$.

A Bernstein algebra is a commutative baric algebra $(A, \omega)$ satisfying the identity $(x^3)^2 = (\omega(x))^2x^2$. A Bernstein algebra has a unique weight function $\omega$. If $x \in A$ and $\omega(x) = 1$, then $e = x^2$ is a nontrivial idempotent of $A$ which gives rise to the Peirce decomposition $A = Ke \oplus U \oplus V$, where $N = U \oplus V$, and

$$U = \left\{ u \in A \mid eu = \frac{1}{2} u \right\}, \quad V = \left\{ v \in A \mid ev = 0 \right\}. \quad (1)$$

Besides, the Peirce components multiply according to the relations

$$U^2 \subseteq V, \quad UV \subseteq U, \quad V^2 \subseteq U, \quad UV^2 = 0. \quad (2)$$

A Bernstein algebra $A = Ke \oplus U \oplus V$ is not unital unless in the trivial case $\dim(A) = 1$, and cannot be associative except when $U = 0$. However, Bernstein algebras may be power-associative, that is, if each element generates an associative subalgebra.

Recall that a commutative algebra $A$ is a Jordan algebra if the identity $x(\lambda y) = x^2(y)$ holds in $A$. Bernstein-Jordan algebras play a crucial role in the theory of Bernstein algebras. It is well-known that the following four conditions are equivalent for a Bernstein algebra $A = Ke \oplus U \oplus V$ (see, for instance, [8, 25]):

(a) $A$ is a Jordan algebra.
(b) $A$ is power-associative.
(c) $x^3 = \omega(x)x^2$ for all $x \in A$. 

\[(d) \quad V^2 = 0 \text{ and } (uv)v = 0 \text{ for all } u \in U \text{ and } v \in V.\]

Therefore, the elements of the barideal \( N = \ker(\omega) \) in a Bernstein-Jordan algebra \((A, \omega)\) satisfy \( x^2 = 0 \) and so the Jacobi identity
\[
(xy)z + (yz)x + (zx)y = 0. \tag{3}
\]

An important tool in Bernstein algebras is the ideal \( \operatorname{ann}_U(U) = \{u \in U/uU = 0\} \) of \( A \) which is independent of the selected idempotent \( e \) and satisfies \( \operatorname{ann}_U(U)(U \oplus U^2) = 0 \) and \( V^2 \subseteq \operatorname{ann}_U(U) \) (see, for instance, [17, Theorem 3.4.19]). It should be remarked the fundamental position played by this ideal \( \operatorname{ann}_U(U) \) in the connection between Bernstein algebras and Jordan algebras, since the quotient algebra \( A/\operatorname{ann}_U(U) \) is a Bernstein-Jordan algebra (see, e.g., [8, 11, 19]). A Bernstein algebra \( A \) is called nuclear if \( A^2 = A \), or equivalently, \( U^2 = V \) for an arbitrary idempotent \( e \); in this case, we have \( \operatorname{ann}_U(U)N = 0 \). Every Bernstein algebra \( A = Ke \oplus U \oplus V \) gives rise to a nuclear Bernstein subalgebra \( A^2 = Ke \oplus U \oplus U^2 \). Further information about algebraic properties of Bernstein algebras, as well as their possible genetic interpretation can be found in [17, 22, 26].

We let now \( A \) be an arbitrary algebra over \( K \). Following the notation of [28], we will consider the powers \( A^i \) and the right principal powers \( A^{<i>} \) of \( A \) defined recursively by \( A^1 = A^{<1>} = A, A^i = \sum_{r+s=i} A^r A^s \) and \( A^{<i>} = A^{<i-1>} A \). The algebra \( A \) is called nilpotent if \( A^n = 0 \) for some \( n \), and right nilpotent if \( A^{<n>} = 0 \). It is well known that these two notions of nilpotency are equivalent in the commutative case. We define the plenary powers \( A^{(i)} \) of \( A \) by setting \( A^{(1)} = A^2 \) and \( A^{(i)} = (A^{(i-1)})^2 \). The algebra \( A \) is said to be solvable when \( A^{(n)} = 0 \) for some \( n \).

On the other hand, \( A \) being a commutative algebra, the linear mappings \( L_a : A \to A \) defined by \( L_a(x) = ax \) generate a subalgebra of \( \operatorname{End}_K(A) \), denoted by \( \mathcal{M}_a(A) \) and called the multiplication ideal of \( A \). The subalgebra of \( \operatorname{End}_K(A) \) generated by \( \mathcal{M}_a(A) \) and the identity endomorphism \( \text{id}_A \) will be denoted by \( \mathcal{M}(A) \), and will be called the multiplication algebra of \( A \). If \( B \) is a subalgebra of \( A \), we write \( \mathcal{M}_a^B(B) \) for the subalgebra of \( \mathcal{M}_a(A) \) generated by the operators \( L_{ab} \) where \( b \in B \). The unital algebra \( \mathcal{M}_a^B(B) \) is defined analogously.

For any subset \( S \subseteq A \), we will adopt the notations \( <S> \) and \( K(S) \), which mean respectively the subspace of \( A \) spanned by \( S \) and the free unital associative (noncommutative) algebra over \( K \) generated by \( S \). Since the ideal of \( A \) generated by \( S \) consists of finite sums of elements \( f(x) \), where \( f \in \mathcal{M}(A) \) and \( x \in S \), it is customary to denote it by \( \mathcal{M}(A)S \).

Returning to Bernstein algebras, recall that in a Bernstein algebra, the principal powers \( N^{<i>} \) are ideals [17, page 113]. Moreover, the barideal \( N \) satisfies the equation \( (x^2)^2 = 0 \), but is not in general nilpotent. However, \( N \) is always solvable, since \( N^{(3)} = 0 \) [4, Theorem 2.11] (see, also, [12]).

### 2. Chain conditions for subalgebras

In our preceding work [5] it has been proved that a Bernstein algebra \( A \) that is Jordan or nuclear is necessarily finite-dimensional whenever it is Noetherian or Artinian. In addition, a counter-example was given to show that the hypothesis that \( A \) be Jordan or nuclear is essential in this result. In the following we are going to relax the Jordan and nuclear assumptions in order to state a result for general Bernstein algebras satisfying the ascending (descending) condition for subalgebras instead of ideals. An arbitrary algebra \( A \) satisfies the ascending chain condition a.c.c. (descending chain condition d.c.c.) on subalgebras if it has no infinite strictly ascending (descending) chains of subalgebras. It is easily seen that the a.c.c. (d.c.c.) for subalgebras is equivalent to the maximal (minimal) condition for subalgebras, that is, every non-empty set of subalgebras has a maximal (minimal) element. Moreover, in an algebra satisfying the a.c.c. for subalgebras, all subalgebras are finitely generated. In the literature of general nonassociative algebras, there are
some results treating the maximal condition for subalgebras. For instance, Kubo constructed in [15] infinite-dimensional associative, Jordan and Lie algebras satisfying the maximal condition for subalgebras (see also [2]). In [28, Theorem 3, page 91] it is established that a Jordan nil-algebra satisfying the maximal condition for subalgebras is nilpotent, and therefore finite-dimensional. For Bernstein algebras, we may formulate the following result which is valid for both the a.c.c and d.c.c. conditions on subalgebras.

**Theorem 2.1.** For a Bernstein algebra $A$, the following conditions are equivalent:

(i) $A$ satisfies the a.c.c. (d.c.c.) condition for subalgebras;
(ii) $A$ satisfies the a.c.c. (d.c.c.) condition for subalgebras contained in $N = \ker(\omega)$;
(iii) $A$ is finite-dimensional.

**Proof.** It is enough to demonstrate that (ii) implies (iii). If (ii) holds, then $A$ satisfies a fortiori the a.c.c. (d.c.c.) condition for ideals contained in $\ker(\omega)$, and hence it is Noetherian (Artinian) in view of [5, Proposition 2.1]. It follows from [5, Proposition 3.1] that the Bernstein-Jordan algebra $A/ann_U(U)$ is finite-dimensional. Now, since $(ann_U(U))^2 = 0$, every subspace of $ann_U(U)$ is a subalgebra of $A$ contained in $\ker(\omega)$. It follows from the hypothesis that $ann_U(U)$ is finite-dimensional, which completes the proof.

**3. The barideal of a Bernstein algebra**

Krapivin established in [14] that a Bernstein algebra $(A, \omega)$ is finitely generated if and only if its barideal $N = \ker(\omega)$ is finitely generated (as an algebra). Hence, it is legitimate to ask the analogous question for the Noetherian and Artinian cases. An arbitrary algebra is said to be Noetherian (Artinian) if it satisfies the ascending chain condition a.c.c. (descending chain condition d.c.c.) on ideals, that is, every ascending (descending) sequence of ideals is stationary. Before embarking in this direction, we require some preparation. The key ingredient is the deep link exhibited in [5] between Bernstein algebras and modules over associative (noncommutative) algebras. In details, let $A = K \oplus U \oplus V$ be a Bernstein algebra, and consider the free unital associative (noncommutative) algebra $K[V]$ generated by the set $V$. Then the ideal $ann_U(U)$ becomes a left module over $K[V]$ by setting

$$(v_1 \ast \ldots \ast v_k).u = v_1(\ldots(v_ku)\ldots), \text{ for all } v_1, \ldots, v_k \in V \text{ and } u \in ann_U(U).$$

This $K[V]$-module $ann_U(U)$ contains many information on the Bernstein algebra $A$. For instance, the submodules of this module are just the ideals of $A$ contained in $ann_U(U)$. Moreover, the finiteness behavior of the Bernstein algebra $A$ was studied with much benefit in terms of its attached $K[V]$-module $ann_U(U)$. In particular, it was established that the Bernstein algebra $A$ is finitely generated (resp. Noetherian, Artinian) if and only if $A/ann_U(U)$ is finite-dimensional and the $K[V]$-module $ann_U(U)$ is finitely generated (resp. Noetherian, Artinian).

Now, we are in a position to prove the following result which extends the Krapivin theorem [14] to both the Noetherian and Artinian contexts.

**Theorem 3.1.** Let $A = K \oplus U \oplus V$ be a Bernstein algebra with barideal $N = U \oplus V$. Then the following conditions are equivalent:

(i) $A$ is Noetherian (Artinian);
(ii) $N$ is Noetherian (Artinian).

**Proof.** The implication (ii) ⇒ (i) is trivial, since a Bernstein algebra $A$ is Noetherian (Artinian) if and only if it satisfies a.c.c. (d.c.c.) on baric ideals of $A$ [5, Proposition 2.1].
(i) ⇒ (ii) : Assume that A is Noetherian. Then by [5, Proposition 3.4], \( A/\text{ann}_U(U) \) is finite-dimensional and the \( K(V) \)-module \( \text{ann}_U(U) \) is Noetherian. If \( I \) is an ideal of \( N \), then \( I \cap \text{ann}_U(U) \) is an ideal of \( A \) contained in \( \text{ann}_U(U) \), because \( ex = \frac{1}{2} x \) for all \( x \in I \cap \text{ann}_U(U) \subseteq U \). Hence, \( I \cap \text{ann}_U(U) \) is an submodule of the \( K(V) \)-module \( \text{ann}_U(U) \).

Now, let \((I_n)\) be an increasing sequence of ideals of \( N \). Then \((I_n \cap \text{ann}_U(U))\) is an increasing sequence of submodules of the \( K(V) \)-module \( \text{ann}_U(U) \), and \((I_n + \text{ann}_U(U))/\text{ann}_U(U)\) is an increasing sequence of subspaces of the quotient space \( A/\text{ann}_U(U) \). Then both chains must stabilize, and by a standard argument, the sequence \((I_n)\) is stationary.

The Artinian case is treated analogously.

\[ \square \]

4. Bernstein algebras and locally nilpotent nonassociative algebras

Jordan and nuclear Bernstein algebras are important types of Bernstein algebras. The barideal \( N = \ker(\omega) \) in a Bernstein-Jordan algebra \((A, \omega)\) satisfies the identity \( x^3 = 0 \), hence by [28, page 114] \( N \) is locally nilpotent, that is, every finitely generated subalgebra of \( N \) is nilpotent (see, also, [7]). Let us explain the similar fact for nuclear Bernstein algebras:

**Proposition 4.1.** Let \( A \) be a nuclear Bernstein algebra. Then the barideal \( N \) of \( A \) is locally nilpotent.

**Proof.** Consider the Bernstein-Jordan algebra \( \tilde{A} = A/\text{ann}_U(U) \) and let \( \pi : A \to A/\text{ann}_U(U) \) be the canonical surjection. We know that the barideal \( N/\text{ann}_U(U) \) of the Bernstein-Jordan algebra \( A/\text{ann}_U(U) \) is locally nilpotent. Let the subalgebra \( S \) of \( A \) generated by the elements \( a_1, \ldots, a_n \). Then the subalgebra \( T = \pi(S) \) of \( A/\text{ann}_U(U) \) generated by \( \pi(a_1), \ldots, \pi(a_n) \) is nilpotent, say \( T^{<k>} = 0 \). It follows that \( S^{<k>} \subseteq \text{ann}_U(U) \). Hence, \( S^{<k+1>} = S^{<k>} S \subseteq \text{ann}_U(U) S \subseteq \text{ann}_U(U) \)

\[ N = 0, \] which means that \( S \) is nilpotent. It follows that \( N \) is locally nilpotent.

\[ \square \]

**Remark 4.2.** Let \( A = K e \oplus U \oplus V \) be an arbitrary Bernstein algebra. Then the subspace \( U \oplus U^2 \) is an ideal of \( A \) which is locally nilpotent. To be convinced, it suffices to consider the nuclear Bernstein subalgebra \( A^2 = K e \oplus U \oplus U^2 \) whose barideal is \( U \oplus U^2 \).

In virtue of [5, Theorem 2.3], for a Bernstein algebra which is Jordan or nuclear, each of the Noetherian and Artinian conditions implies finite-dimensionality of the algebra and so nilpotency of the barideal. This result suggests us to raise the following more general question: Let \((A, \omega)\) be a Bernstein algebra such that its barideal \( N = \ker(\omega) \) is locally nilpotent. If \( A \) is Noetherian or Artinian, is it finite-dimensional? In the light of our Theorem 3.1, the above question has a closed link with the following question on general locally nilpotent nonassociative algebras, which seems to be of independent interest: Is a locally nilpotent algebra which is Noetherian or Artinian finite-dimensional? Searching in the wide literature of general nonassociative algebras, we found a noteworthy article [27] published in 1972 by the eminent algebraist Zhevlakov (1939–1972) after his death, which gives a positive answer to the Noetherian case and constructs a counter-example to the Artinian case. For the sake of completeness, we provide below an alternative proof of this result which is substantially different from the proof of Zhevlakov mentioned in [27, Note 1], and that we have done before discovering Zhevlakov’s paper.

**Theorem 4.3.** Let \( N \) be a nonassociative (possibly noncommutative) algebra over a field \( K \). Assume that \( N \) satisfies the ascending chain condition on ideals. If \( N \) is locally nilpotent, then \( N \) is finite-dimensional.

**Proof.** Let the ideal \( N \) be generated by the elements \( e_1, \ldots, e_r : N = M(N)e_1 + \ldots + M(N)e_r \). We denote by \( F \) the subalgebra generated by \( e_1, \ldots, e_r \). Then \( F \) is nilpotent, and by a straightforward argument, \( F \) is finite-dimensional. Choose \( m \geq 2 \) such that the power \( F^m = 0 \). Clearly,
$N = F + N^2$, and by a simple induction one may show that

$$N^i \subseteq F^i + N^{i+1} \text{ for each } i \geq 1$$

Indeed, if the inclusion (4) is true for each $j \leq i$, let us show that $N^{i+1} \subseteq F^{i+1} + N^{i+2}$. We have:

$$N^{i+1} = \sum_{j_1 + j_2 = i+1} N^{j_1} N^{j_2} \subseteq \sum_{j_1 + j_2 = i+1} (F^{j_1} + N^{j_1+1})(F^{j_2} + N^{j_2+1}).$$

Now, the following relations are easy to verify:

$$F^{j_1} F^{j_2} \subseteq F^{j_1 + j_2} = F^{i+1}, \quad F^{j_1} N^{j_2+1} \subseteq N^{j_1} N^{j_2+1} \subseteq N^{j_1 + j_2+1} = F^{i+2},$$

$$N^{j_1+1} F^{j_2} \subseteq N^{j_1+1} N^{j_2} \subseteq N^{j_1+1 + j_2} = N^{i+2}, \quad N^{j_1+1} N^{j_2+1} \subseteq N^{j_1+1 + j_2+1} = N^{i+3} \subseteq N^{i+2},$$

from which we get the desired inclusion $N^{i+1} \subseteq F^{i+1} + N^{i+2}$.

As a consequence of (4), it follows that $N^i \subseteq F + N^{i+1}$, for each $i \geq 1$.

Hence, $N = F + N^2 = F + N^3 = \ldots = F + N^m$. Now, consider $a_1, \ldots, a_t$ in $N^m$ such that $N^m = \mathcal{M}(N)a_1 + \ldots + \mathcal{M}(N)a_t$. Without loss of generality, we can assume that each $a_i$ is a non-associative product $x_1 \ldots x_m$ (with some distribution of parentheses) of $m$ factors $x_i \in N$. Writing $x_k = u_k + v_k$ ($k = 1, \ldots, m$), where $u_k \in F, v_k \in N^m$ and using the fact that $F^m = 0$, we obtain that each $a_i$ is a sum of elements of the form $y_1 \ldots y_m$ (with some distribution of parentheses), where $y_1, \ldots, y_m \in N$ such that at least one of them, say $y_d$, belongs to $N^m$. Decomposing $y_d$ into $N^m = \mathcal{M}(N)a_1 + \ldots + \mathcal{M}(N)a_t$, we get that $y_1 \ldots y_d \ldots y_m$ is a sum of products $h_1 \ldots h_{s-1} a_i h_{s+1} \ldots h_p$ (with some distribution of parentheses), where $h_1, \ldots, h_{s-1}, h_{s+1}, \ldots, h_p \in N$ and $i \in \{1, \ldots, t\}$. As a consequence, each $a_j$ can be expressible in the form

$$a_j = \sum h_1^{j_1} \ldots h_{s-1}^{j_{s-1}} a_i h_{s+1}^{j_{s+1}} \ldots h_p^{j_p},$$

where $h_k^{j_k} \in N, \ p_{i,j} \geq s_i, \ j \geq 1$.

The subalgebra $H$ generated by the finite set $\{a_1, \ldots, a_t\} \cup \{h_k\}$ is nilpotent. Let $H^l = 0$ ($l \geq 2$). We may now replace $a_j$ by its expression ($l$ times) in (5) to conclude that $a_j = 0$. This implies that $N^m = 0$, and therefore $N = F$, which completes the proof. \hfill $\Box$

A special case of locally nilpotent algebras are commutative nilalgebras of nilindex at most 3, whose natural examples are barideals of Bernstein-Jordan algebras and train algebras of rank 3 (see, for instance, [29]). They satisfy the identity $x^3 = 0$ and so also the Jacobi identity $(xy)z + (yz)x + (zx)y = 0$. It is well known that they are Jordan algebras (see [3, Lemma 1], [6, Lemma 2.2], [10, 28, page 114]). They appear in the literature as Jacobi-Jordan algebras in [1, 6] and as mock-Lie algebras in [30]. In addition, any such an algebra $N$ is solvable and $N^{(4)} = 0$ [29, Lemma 3.1]. On the other hand, since such algebras are locally nilpotent, we infer the following immediate consequence of Theorem 4.3.

**Corollary 4.4.** Let $N$ be a commutative algebra satisfying the identity $x^3 = 0$. If $N$ satisfies the ascending chain condition on ideals, then $N$ is finite-dimensional.

The above corollary is certainly not new, and it is quite easy to prove it directly. Indeed, since $N$ is a Noetherian solvable Jordan algebra, it follows from a result of Medvedev and Zelmanov [18] that $N$ is nilpotent. Hence, by a simple reasoning, one may prove that $N$ is finite-dimensional.

The analog of Theorem 4.3 for the descending chain condition is false, as already shown by Zhevlakov [27]:

**Example 4.5.** (Zhevlakov) Let $N$ be a countably-dimensional algebra with basis $\{e_n\}_{n \in \mathbb{N}^*}$ and nonzero products

$$e_i e_j = e_{\min(i,j)-1} \text{ for } i, j \geq 2.$$
By [27, Note 2], $N$ is a locally nilpotent commutative algebra which is Artinian and satisfies $N^2 = N$.

We offer another counter-example below:

**Example 4.6.** Let $N$ be an infinite dimensional commutative algebra with basis $\{e_n\}_{n \in \mathbb{N}}$ and non-zero multiplication table given by $e^2_n = e_{n-1}$ for $n \geq 2$.

Let $S$ be a nonzero subalgebra of $N$, and let $x = x_1 e_1 + \ldots + x_k e_k \in S$, with $x_k \neq 0$. By considering the plenary powers $x^{(r)} (1 \leq r \leq k)$ defined by $x^{(1)} = x$ and $x^{(r)} = (x^{(r-1)})^2$, a simple calculation gives $x^{(2)} = x_2 e_1 + \ldots + x_k e_{k-1} + x_k e_2, x^{(k)} = x_k e_1$. It follows that $e_1, \ldots, e_k \in S$. If $n = \max \{k/x \neq 0\}$ is finite, then $S = \langle e_1, \ldots, e_n \rangle$, which is also an ideal of $N$. In the opposite case, we get $S = N$. Therefore, each proper subalgebra (ideal) of $N$ coincides with a subspace $\langle e_1, \ldots, e_n \rangle$ for some $n \geq 1$. Clearly, no infinite sequence of ideals of $N$ can exist, and so $N$ is Artinian. On the other hand, it is not hard to observe that $N$ is locally nilpotent, since every finitely generated subalgebra $\langle e_1, \ldots, e_n \rangle$ is nilpotent (of index $n+1$). Obviously, $N$ is not Noetherian because the ascending chain condition $(\langle e_1, \ldots, e_n \rangle)_{n \geq 1}$ of ideals does not break off.

Although the similar result of Theorem 4.3 fails in the Artinian case, we may prove the following Artinian version of Corollary 4.4, whose proof also holds in the Noetherian case.

**Corollary 4.7.** Let $N$ be a commutative algebra satisfying the identity $x^3 = 0$. If $N$ satisfies the descending chain condition on ideals, then $N$ is finite-dimensional.

**Proof.** When $N^2 = 0$, each subspace of $N$ is an ideal of $N$, and therefore $N$ must be finite-dimensional by the Artinian hypothesis. Now, if $N^2 \neq 0$, then the former case shows that $N/N^2$ is finite-dimensional. Finally, we apply [29, Lemma 3.2] stating that any commutative algebra $N$ satisfying the identity $x^3 = 0$ and such that $N/N^2$ is finite-dimensional must be finite-dimensional.

At present, let us return to Bernstein algebras with locally nilpotent barideals. Applying Theorem 3.1 together with Theorem 4.3, we deduce immediately the next consequence:

**Corollary 4.8.** Let $A$ be a Bernstein algebra with locally nilpotent barideal $N$. If $A$ is Noetherian, then $A$ is finite-dimensional.

As attempted, we will give in the sequel a counter-example to the Artinian version of Corollary 4.8. We emphasize that the locally nilpotent algebras $N$ treated in Examples 4.5 and 4.6 cannot be embedded in a Bernstein algebra $(A, \omega)$ as its barideal $\ker(\omega)$, since in either cases the identity $(x^2)^2 = 0$ is not valid in $N$. For this reason, we make appeal to the following example taken from [5, Example 3.10]:

**Example 4.9.** Let $A$ be the Bernstein algebra with denumerable basis $\{e, v_1, u_1, u_2, u_3, \ldots\}$ and nonzero products

\[ e^2 = e, \quad eu_i = \frac{1}{2} u_i \quad (i \geq 1), \quad u_i v_1 = u_{i-1} \quad (i \geq 2) \]

The weight function $\omega : A \to K$ is defined by $\omega(e) = 1, \omega(v_1) = \omega(u_i) = 0 (i \geq 1)$ and the Peirce components are $U = \langle u_1, u_2, \ldots \rangle$ and $V = \langle v_1 \rangle$. We know from [5, Example 3.10] that $A$ is Artinian. Actually, it is clear that the barideal $N = U \oplus V$ is locally nilpotent. We point out that the Bernstein algebra $A$ is not Noetherian, and moreover, it is neither nuclear nor Jordan.

## 5. On the nilpotence

In this final section we shall proceed to revisit some results of Micali and Ouattara [19] in the aim to improve and generalize them to the Noetherian and Artinian situations.
First, we start with the following result which was proved in [19, Lemmas 4.3 and 4.4] when the Bernstein algebra $A$ was assumed to be finitely generated.

**Lemma 5.1.** Let $A = Ke \oplus U \oplus V$ be a Bernstein algebra which is Noetherian or Artinian, and let $N = U \oplus V$ be its barideal. Let $I$ be a subspace of $A$.

(i) If $NI = I$, then $I \subseteq \text{ann}_U(U)$ and $I$ is an ideal of $A$.

(ii) $NI = I$ if and only if $VI = I$.

**Proof.** In view of [5, Proposition 3.1], since $A$ is Noetherian or Artinian, the Bernstein-Jordan algebra $\frac{A}{\text{ann}_U(U)}$ is finite-dimensional. Hence, its barideal $\frac{N}{\text{ann}_U(U)}$ is nilpotent, so $N^k \subseteq \text{ann}_U(U)$ for some integer $k$. The remainder of the proof follows as in [19, Lemmas 4.3 and 4.4]. More precisely:

(i) Evidently, $I = NI \subseteq N$. Now, $I = NI = N(NI) = \ldots \subseteq N^k$, yielding $I \subseteq \text{ann}_U(U)$. Furthermore, since $I \subseteq \text{ann}_U(U) \subseteq U$, we have $eI = I$, and by the condition $NI = I$, we deduce that $I$ is an ideal of $A$.

(ii) If $NI = I$, then the above assertion gives $I \subseteq \text{ann}_U(U)$. It follows that $I = NI = (U \oplus V)I = VI$, because $UI = 0$.

Conversely, if $VI = I$, then $I = V(V(\ldots V(VI)\ldots)) \subseteq N^k \subseteq \text{ann}_U(U)$. Therefore, $UI = 0$ and so $NI = (U \oplus V)I = VI = I$. \hfill \Box

Actually, after proving Lemma 5.1, the result of [19, Théorème 4.7] can be ameliorated by deleting the superfluous hypothesis that $A$ be finitely generated. Namely:

**Theorem 5.2.** Let $A = Ke \oplus U \oplus V$ be an Artinian Bernstein algebra. Then the following conditions are equivalent:

(i) The ideal $N = U \oplus V$ is nilpotent;

(ii) The associative algebra $\mathcal{M}_e^N(V)$ is nilpotent;

(iii) $I = 0$ is the unique subspace of $A$ satisfying $VI = I$.

**Proof.** As in the proof of [19, Théorème 4.7], the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are always true even if $A$ is not Artinian. And the implication (iii) $\Rightarrow$ (i) follows the same path as in [19, Théorème 4.7], by applying our Lemma 5.1 instead of [19, Lemma 4.3]. In details:

(i) $\Rightarrow$ (ii) : Since $N$ is nilpotent, then also is the multiplication ideal $\mathcal{M}_e(N)$ (see [23, Chapitre II, Théorème 2.3]). In particular, the subalgebra $\mathcal{M}_e^N(V)$ is nilpotent.

(ii) $\Rightarrow$ (iii) : Let $I$ be a subspace of $A$ with $VI = I$, so that $I \subseteq N$ by Lemma 5.1. Since $I = V(V(\ldots V(VI)\ldots))$ and $\mathcal{M}_e^N(V)$ is nilpotent, then $I = 0$.

(iii) $\Rightarrow$ (i) : Since $A$ is Artinian, the descending chain of ideals $(N^r)_{r \geq 1}$ of $A$ must stabilize. Hence, $N^r = N^{r+1}$ for some integer $r \geq 1$, or equivalently $N^r = NN^r$. It follows from Lemma 5.1(ii) that $N^r = VN^r$, implying $N^r = 0$ by hypothesis. \hfill \Box

**Remark 5.3.** We point out that the implication (ii) $\Rightarrow$ (i) of Theorem 5.2 is already true when the Bernstein algebra $A$ is finitely generated [19, Lemma 4.1], so it holds automatically in the Noetherian case, since Noetherian Bernstein algebras are finitely generated [5, Corollary 3.6]. Nevertheless, the implication (iii) $\Rightarrow$ (i) fails when $A$ is not Artinian, even if it is finitely generated or Noetherian, as the following example will illustrate.

**Example 5.4.** Let $A$ be the infinite-dimensional Bernstein algebra considered in [5, Example 3.11], with basis $\{e, v_2, u_1, u_2, u_3, \ldots\}$ and nonzero products
\[ e^2 = e, \quad eu_i = \frac{1}{2} u_i (i \geq 1), \quad u_i v_2 = u_{i+1} (i \geq 1), \quad (i \geq 2). \]

Then \( A = Ke \oplus U \oplus V \), where \( U = \langle u_1, u_2, \ldots \rangle \) and \( V = \langle v_2 \rangle \).

Let \( I \) be a subspace of \( A \) such that \( VI = I \). Assume that \( I \neq 0 \) and choose an element \( a = (x_1 u_1 + \cdots + x_p u_p) + x v_2 \in I \), with \( p \) minimal such that \( x \neq 0 \). Since \( a \in VI \), there exists \( b = (\beta_1 u_1 + \cdots + \beta_q u_q) + \beta v_2 \in I \) with \( \beta q \neq 0 \) and \( a = v_2 b = \beta_1 u_2 + \cdots + \beta_q u_{q+1} \). Then the contradiction \( q + 1 = p < q \) yields \( I = 0 \). However, the ideal \( N = U \oplus V \) is not nilpotent. In fact, this Bernstein algebra \( A \) is finitely generated, Noetherian but not Artinian \([5, \text{Example 3.11}]\).

We close our paper by making the following comment. The Grishkov conjecture \([9]\) asserts that if \( A = Ke \oplus U \oplus V \) is a finitely generated Bernstein algebra that is nuclear, then the barideal \( N = U \oplus V \) is nilpotent. This question has been shown affirmatively by Peresi \([21]\) and Krapivin \([14]\). Nevertheless, the proof of this result presented in \([19, \text{Théorème 4.10}]\) is not correct, because it relies on \([19, \text{Théorème 4.7}]\) which requires the additional assumption that \( A \) be Artinian.

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