Solitary wave solutions of the 2+1 and 3+1 dimensional nonlinear Dirac equation constrained to planar and space curves

Fred Cooper 1,2 , Avinash Khare 3 and Avadh Saxena 3

1 Santa Fe Institute, Santa Fe, NM 87501, United States of America
2 Theoretical Division and Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, United States of America
3 Physics Department, Savitribai Phule Pune University, Pune 411007, India

Author to whom any correspondence should be addressed.

E-mail: cooper@santafe.edu, khare@physics.unipune.ac.in and avadh@lanl.gov

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Abstract

We study the effect of curvature and torsion on the solitons of the nonlinear Dirac equation considered on planar and space curves. Since the spin connection is zero for the curves considered here, the arc variable provides a natural setting to understand the role of curvature. We obtain for various curves in two and three dimensions the transformation of the Dirac equation to the 1 + 1 dimensional flat space Dirac equation written in terms of the arc length. Depending on the curvature, the soliton profile either narrows or expands. Our results may be applicable to yet-to-be-synthesized curved quasi-one dimensional Bose condensates.

1. Introduction

Recently there has been renewed interest in the nonlinear Dirac equation (NLDE) because it arises in Bose condensation in honeycomb optical lattices where confining potentials allow for quasi-one dimensional (Q1D) confinement [1, 2]. Over the past decade there has been significant amount of research on solitons and related aspects in NLDE including in its massless limit, i.e. the nonlinear Weyl equation [3]. Double humped [4] and multi-humped [5] soliton solutions have been found for this equation. NLDE has also been studied in the presence of an external field [6], an external spinor force and damping [7] as well as when it is driven parametrically [8]. Coupled NLD equations have been considered both in one-dimension [9] and in two dimensions [10]. Two-dimensional discrete analog of the NLDE has been explored [11] as well. There is also a study of three-dimensional NLD [12] as well as its Lax pair [13] and the two-spinor form [14].

Here we show that one can find exact solitary wave solutions for the nonlinear Dirac equation confined to various space curves by using either the arc variable representation or the vierbein formalism of Weyl [15]. The latter is elucidated in the appendix. We find that since the curves are only in the spatial part of the metric, one can transform the NLDE on the curved surface to another flat space NLDE by a coordinate transformation. This allows us to obtain the solutions of the nonlinear Dirac equation analytically for the conic sections such as a hyperbola or a parabola or an ellipse as well as helical (space) curves in 3 + 1 dimensions in terms of the spatial arc length parameter $s$. One can furthermore analytically obtain the connection between the parametric description of the curve and the spatial arc length variable $s$.

However, to our knowledge there are no studies of NLD solitons in curved geometries either experimentally or theoretically. Nonetheless, one could envision constructing curved Q1D Bose condensates, which serves as one of our motivations. In contrast, solitons and breathers have been studied on curves in the context of the nonlinear Schrödinger (NLS) equation [16, 17] with an interesting interplay of curvature and nonlinearity through the soliton/breather solutions. The study of nonlinear waves on curves and closed surfaces is important in its own right [18].

This paper is structured as follows. In the next section we study the NLD equation on conic sections such as a hyperbola, parabola and ellipse. In section 3 we show how the normalized charge density of a soliton is modified...
due to the curvature. In section 4 we consider Jacobi elliptic and other parameterizations of the various curves. Section 5 deals with space curves (i.e. with finite torsion) such as a helix, an elliptic helix and the helix of the hyperboloid of revolution. Finally in section 6 we discuss our findings and provide our main conclusions.

2. Nonlinear Dirac equation constrained to space curves in $2 + 1$ dimensions

First let us consider the solutions of the Nonlinear Dirac equation (NLD equation) in $2 + 1$ dimensions (often called the Gross-Neveu model [19]) when it is confined to a conic section which is a hyperbola, ellipse or parabola. The NLD equation in terms of the fermion field $\Psi$ including a mass $(m)$ term is given by:

$$-i(\gamma^a \partial_a + m)\Psi - g^2(\bar{\Psi}\Psi)\Psi = 0,$$

(2.1)

where $\gamma^a$ are Dirac matrices and $g^2$ is the dimensional coupling constant. It will be convenient in studying the transformation properties of this equation to introduce the quantity $\sigma$

$$\sigma \equiv -ig^2\bar{\Psi}\Psi,$$

(2.2)

so that we can rewrite the NLD equation in the suggestive form often used in the large-$N$ type expansions of the quantum version of this theory

$$-i[\gamma^a \partial_a + (m + \sigma)]\Psi = 0.$$

(2.3)

The fact that $\sigma$ transforms as a scalar is what will be important in what follows. Here we are using the general relativity convention for the metric $g_{\alpha\beta} = \text{diag}(-1,1,1)$, and the same convention on the $\gamma$ matrices as found in [20]. This is elaborated on in the appendix.

2.1. Hyperbola

First let us consider the hyperbola defined by $y^2 - x^2 = a^2$. With $u$ as an azimuthal angle, the parametrization

$$y = a \sec u, \quad x = a \tan u,$$

(2.4)

with $-\pi/2 < u < \pi/2$, fulfills this condition. We have that

$$ds^2 = dx^2 + dy^2 = [a^2 \sec^2(u)(\tan^2(u) + \sec^2(u))]du^2.$$

(2.5)

If we introduce a new coordinate $s$ and thus $ds = a \sec(u)\sqrt{\tan^2(u) + \sec^2(u)}du$, which is the arc length along the curve, we can solve the usual Dirac equation in terms of $s$ and then use the relationship:

$$s[a, u] = a(\sec(u)\sqrt{\tan^2(u) + \sec^2(u)} - 1)$$

(2.6)

to obtain the solutions as a function of $u$, $t$, The resulting $1 + 1$ dimensional NLD equation is

$$(\gamma^0 \partial_t + \gamma^1 \partial_s + \sigma + m)\Psi[s, t] = 0.$$

(2.7)

We see that because the spin connection is zero, one could have obtained this answer by realizing the metric in the following way:

$$(ds)^2 = -(dt)^2 + (dx)^2 + (dy)^2 = -(dt)^2 + a^2 \sec^2(u)(\tan^2(u) + \sec^2(u))du^2.$$

(2.8)

The $2 + 1$ dimensional metric can be transformed to a $1 + 1$ dimensional Minkowski metric by the transformation

$$ds = a \sec(u)\sqrt{\tan^2(u) + \sec^2(u)}du.$$

(2.9)

Then we would obtain the solutions of the NLD equation as a function of $s$, $t$ and then obtain the result in terms of $u$ using $s[u]$. The $1 + 1$ dimensional resulting NLDE is translationally invariant under $s \rightarrow s - s_0$, so there are solitary wave solutions centered at any value of the arc length $s_0$.

Note that if the transformation would have mixed space and time, such as in the light cone transformation performed in studying a scale invariant initial condition on the NLDE discussed in [21] this simple result would not have been possible. Then we need to resort to a more general formulation, namely that of vierbeins discussed in the appendix.

For $a = 1$ we get the curve shown in figure 1.

2.2. NLD equation constrained to an ellipse

For the other conical sections the formalism is the same, just the parametric representation is changed. Since the spin connection is zero for these transformations, one can obtain the transformation to the $1 + 1$ dimensional Dirac equation directly from the metric. For the ellipse centered at the coordinate origin

$$x = a \cos \theta, \quad y = b \sin \theta,$$

(2.10)
so that

$$ds^2 = -dt^2 + dx^2 + dy^2 = -dt^2 + (a^2 \sin^2 \theta + b^2 \cos^2 \theta)\, d\theta^2.$$  \hspace{1cm} (2.11)

We see that we can reduce the metric to a 1 + 1 dimensional Minkowski form by the transformation:

$$ds = \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} \, d\theta.$$  \hspace{1cm} (2.12)

Explicitly

$$s[\theta] = bE\left(\theta \mid 1 - \frac{a^2}{b^2}\right).$$  \hspace{1cm} (2.13)

where $E(x, m)$ is the incomplete elliptic integral of the second kind with modulus $m$ \cite{22}. When $a = 1$, $b = 2$ we get the transformation shown in figure 2. For a circle ($a = b$) the above equation reduces to $s(\theta) = a\theta$.

2.3. NLD equation constrained to a parabola

For the parabola $x^2 = 4ay$, the parametric equations are

$$x = 2a\eta, \quad y = a\eta^2.$$  \hspace{1cm} (2.14)

Then

$$ds^2 = -dt^2 + dx^2 + dy^2 = -dt^2 + 4a^2(1 + \eta^2)\, d\eta^2.$$  \hspace{1cm} (2.15)
Here changing variables to

$$ds = 2a \int \sqrt{1 + \eta^2} \, d\eta$$

(2.16)

brings the metric back into the Minkowski form. Explicitly

$$s[\eta] = a(\eta \sqrt{\eta^2 + 1 + \sinh^{-1}(\eta)}).$$

(2.17)

When $a = 1$ we obtain the curve shown in figure 3.

3. Normalized charge density

What we have shown for the NLD equation in $2 + 1$ dimensions constrained to the three conic sections, that we can reduce the $2 + 1$ dimensional NLDE to the flat space $1 + 1$ dimensional NLDE if we use the arc length variable. The static solutions of the NLDE in $1 + 1$ dimensions can be written in the form:

$$\psi(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = \begin{pmatrix} R(x) \cos \theta \\ i R(x) \sin \theta \end{pmatrix},$$

(3.1)

where

$$u^2 = R^2 \cos^2 \theta = \frac{2}{g^2} \frac{(m^2 - \omega^2)(m + \omega) \cosh^2 \beta x}{(m + \omega \cosh 2 \beta x)^2},$$

$$v^2 = R^2 \sin^2 \theta = \frac{2}{g^2} \frac{(m^2 - \omega^2)(m - \omega) \sinh^2 \beta x}{(m + \omega \cosh 2 \beta x)^2},$$

(3.2)

as discussed in [23] and more recently for arbitrary nonlinearity in [4]. As a function of $\omega/m$ the shape of the solitary wave changes from single humped to double humped as one lowers $\omega/m$. As a function of the arc length $s$ the density of the solitary wave is given by $\rho(s) = R^2(s)$

$$\rho(s) = \Psi^* \Psi = \frac{2 \beta^2}{g^2 (m + \omega)} \frac{1 + \alpha^2 \tanh^2 \beta s}{(1 - \alpha^2 \tanh^2 \beta s)^2} \sech^2 \beta s.$$

(3.3)

Here (with frequency $\omega$)

$$\beta = \sqrt{m^2 - \omega^2}, \quad \alpha^2 = \frac{m - \omega}{m + \omega},$$

(3.4)

and the total charge is

$$Q = \int ds \Psi^* \Psi = \frac{2 \beta}{g^2 \omega}.$$

(3.5)

In figure 4 we plot $\rho/Q[\omega]$ versus $s$ for two values, $\omega = 0.9$ which is single humped and $\omega = 1/4$ which is double humped.
3.1. Hyperbola

We now wish to discuss the change in shape of the solitary wave as we go from the arc variable $s$ to the parametric representation of the curve variable $u$ when the two dimensional motion is confined to a hyperbola. The arc length in terms of $u$ is given by

$$s[a, u] = a(\sec(u)\sqrt{\tan^2(u) + \sec^2(u)} - 1).$$  \hspace{1cm} (3.6)

In what follows we will always choose the mass $m = 1$, and take $g = 1$. Later on we will use the symbol $m$ to describe the modulus of Jacobi elliptic functions. This is not to be confused with the Dirac mass $m$ which we set to one. In figure 5 we plot $\rho/Q[\omega]$ versus $u$ for the same two values $\omega = 0.9$ which is single humped and $\omega = 1/4$ which is double humped. We see that the solitary wave shape is quite different (i.e. contracted, see the horizontal axis) when the solitary wave is plotted versus $u$. Thus the charge density is ‘distorted’ in response to the local curvature. This effect could be possibly observed in Q1D curved Bose condensates.

3.2. Ellipse

Choosing for the parameters of the ellipse $a = 1$, $b = 2$, we have that the arc length is given by

$$s[\theta] = 2E\left(\theta \left| \frac{3}{4} \right. \right).$$  \hspace{1cm} (3.7)

The curve for the normalized charge densities for the curve constrained to the ellipse with $b = 2$, $a = 1$ is shown in figure 6 and is contracted. Again, this particular distortion of the charge density could be possibly observed in Q1D curved Bose condensates.
3.3. Parabola
Choosing for the parameters of the parabola $a = 1$, we have that the arc length is given by

$$s[\eta] = \eta \sqrt{\eta^2 + 1} + \sinh^{-1}(\eta).$$

(3.8)

The curve for the normalized charge densities for the curve constrained to a parabola is shown in figure 7. It is even more contracted. Comments similar to the distortion of charge density in the case of a hyperbola and an ellipse apply in the present case as well.

4. Other parameterizations of the conic sections

Of course there are other parameterizations of the conic sections which give different expressions for the arc length in terms of these parameters. For example, we can parametrize the ellipse, using doubly periodic functions—namely the Jacobi elliptic functions and we can parametrize the hyperbola and the parabola using hyperbolic functions.

4.1. Jacobi elliptic parametrization of the ellipse
In the simplest case we can let

$$x = a \text{cn}(\theta, m), \quad y = b \text{sn}(\theta, m),$$

(4.1)
when the modulus \( m \to 0 \) the Jacobi elliptic functions \( \text{cn}(x, k) \) and \( \text{sn}(x, k) \) become the circular functions \( \cos(u) \) and \( \sin(u) \), respectively. The spatial part of the metric is

\[
dx^2 + dy^2 = dn(\theta|m)^2(a^2 \text{ sn}(\theta|m)^2 + b^2 \text{ cn}(\theta|m)^2)(d\theta)^2,
\]

so that the change of variables is:

\[
ds = dn(\theta|m)\sqrt{a^2 \text{ sn}(\theta|m)^2 + b^2 \text{ cn}(\theta|m)^2} \, d\theta.
\]

Integrating we get:

\[
s = bE\left(\text{am}(\theta|m)\right)\left|1 - \frac{a^2}{b^2}\right| \theta,
\]

which simplifies to our previous result when \( b = 2a \). Here \( \text{am}(\theta, m) \) is the Jacobi amplitude function. The Jacobi ellipse is shown, with \( a = 3 \), \( b = 5 \), and \( m = 1/3 \) in figure 8. For the densities parametrized as a function of \( \theta \) of the Jacobi elliptic function we find the results shown in figure 9.

### 4.2. A different hyperbolic function parametrization of the hyperbola

Let us consider the following parametrization of the hyperbola in terms of hyperbolic functions

\[
x = a \sinh(\theta), \quad y = a \cosh(\theta).
\]

For this alternative parametrization of the hyperbola we find

\[
ds^2 = a^2 \cosh(2\theta) d\theta^2,
\]

and

\[
s = -iaE(i|2).
\]

In terms of this parameterization we get the densities shown in figure 10 when we choose \( a = 2 \).

### 4.3. Hyperbolic function parametrization of a parabola

Instead of the parametrization given above of the parabola we could have used a parametrization in terms of hyperbolic functions:
so that again $x^2 = 4 ay$. Note that for small $\eta$ this parametrization goes over to the parametrization used earlier. The spatial part of the metric then becomes

$$dx^2 + dy^2 = 4 a^2 \cosh^4(\eta) \, d\eta^2.$$  

(4.9)

Thus the appropriate change of variables to a $1 + 1$ dimensional metric is captured in

$$ds = 2 a \cosh^2(\eta) \, d\eta.$$  

(4.10)

In this way, we obtain

$$s(\eta) = a[\eta + \sinh(2\eta)/2].$$  

(4.11)

One obtains for the normalized densities in the two cases studied before the results shown in figure 11.

5. Solitary wave solutions for the $3 + 1$ dimensional NLDE confined to a space curve

5.1. Helix

The helix is a space curve with parametric equations:

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = c \theta.$$  

(5.1)

Here $r$ is the radius of the helix and $2\pi c$ gives the vertical separation of the helix’s hoops, i.e. pitch. The curvature of the helix $\kappa$ is given by
which is constant and the torsion $\tau$ is given by

$$\tau = \frac{c}{r^2 + c^2}.$$  

(5.3)

which is also constant. For this problem we find the metric is given by

$$ds^2 = -dr^2 + dx^2 + dy^2 + dz^2 = -dr^2 + (r^2 + c^2)d\theta^2.$$  

(5.4)

Therefore, for this problem, the simple transformation to the $1 + 1$ Minkowski metric is again in terms of the arc length

$$ds = \sqrt{r^2 + c^2}d\theta.$$  

(5.5)

So again in the $1 + 1$ dimensional space the transformation is given by

$$ds = \sqrt{-g}d\theta.$$  

(5.6)

5.2. Elliptic helix

The elliptic helix, i.e. a helix with an elliptic cross-section on projection, is defined by

$$x = a \cos \theta, \quad y = b \sin \theta, \quad z = k \int \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}d\theta.$$  

(5.7)

Note that $dz^2 = k^2(dx^2 + dy^2)$ so that the 3D arc length is proportional to the 2D arc length. The curvature is variable in this case. Performing the integral we get

$$z = aE\left(\theta \left| 1 - \frac{b^2}{a^2}\right.\right).$$  

(5.8)

The metric is given by

$$ds^2 = -dr^2 + dx^2 + dy^2 + dz^2 = -dr^2 + \left(1 + k^2(a^2 \cos^2 \theta + b^2 \sin^2 \theta)\right)d\theta^2,$$  

(5.9)

so that again one can transform this to a Minkowski metric with the transformation utilizing the arc length

$$ds = \sqrt{-g}d\theta = \sqrt{1 + k^2(a^2 \cos^2 \theta + b^2 \sin^2 \theta)}d\theta.$$  

(5.10)

Thus, the explicit transformation is (for $a > b > 0$)

$$s = \sqrt{1 + k^2}aE\left(\theta \left| 1 - \frac{b^2}{a^2}\right.\right).$$  

(5.11)
5.3. Jacobi elliptic helix

The Jacobi elliptic helix can be parametrized as follows:

\[ x = a \cn(\theta | m), \quad y = b \sn(\theta | m), \quad z = k \int d\theta \, \dn(\theta | m) \sqrt{(a^2 \sn(\theta | m)^2 + b^2 \cn(\theta | m)^2)}. \]  

(5.12)

Performing the integral we have

\[ z = kbE \left( \am(\theta | m) \left| 1 - \frac{a^2}{b^2} \right. \right). \]  

(5.13)

The space curve is shown in figure 12, and

\[ dx^2 + dy^2 + dz^2 = (1 + k^2)(\dn(\theta | m))^2 \left( a^2 \sn(\theta | m)^2 + b^2 \cn(\theta | m)^2 \right). \]  

(5.14)

So we see that we can transform this to a 1D Dirac equation by letting

\[ ds = \sqrt{1 + k^2} \, \dn(\theta | m) \sqrt{(a^2 \sn(\theta | m)^2 + b^2 \cn(\theta | m)^2)} \, d\theta. \]  

(5.15)

Integrating, we get the arc variable

\[ s = \sqrt{1 + k^2} \, bE \left( \am(\theta | m) \left| 1 - \frac{a^2}{b^2} \right. \right). \]  

(5.16)

5.4. Helix of the hyperboloid of revolution

The hyperboloid of revolution is defined by

\[ \frac{x^2 + y^2}{a^2} - \frac{z^2}{b^2} = 1. \]  

(5.17)
The helix of the one-sheeted hyperboloid of revolution can be parametrized as follows:

\[ x = a \cosh \eta \cos f(\eta), \quad y = a \cosh \eta \sin f(\eta), \quad z = b \sinh \eta, \]

where

\[ \frac{df}{d\eta} = \sqrt{\left(\frac{b}{a \tan \alpha}\right)^2 - \tanh^2 \eta}. \]

The curve \( f[\eta] \) depends on whether \( \frac{b}{a \tan \alpha} \) is less than one or greater than or equal to one. In the latter case \( f[\eta] \) is a straight line. In our plot, figure 13, we consider the cases \( \beta = 1/2, 1, 10 \) to exemplify the possibilities. Explicitly,

\[
\begin{align*}
 f_{1/2}(\eta) &= \cosh(\eta)\left(\frac{\sqrt{6 - 2 \cosh(2\eta) \sinh^{-1}(\sinh(\eta))} - 2 \cosh(2\eta) - 3 \tanh^{-1}\left(\frac{2 \sinh(\eta)}{\sqrt{\cosh(2\eta) - 3}}\right)}{\cosh(2\eta) - 3}\right), \\
 f_1(\eta) &= \cosh(\eta)\left(4 - 2 \tanh^2(\eta)\right)\left(\sinh^{-1}\left(\frac{\sinh(\eta)}{\sqrt{3}}\right) + \tan^{-1}\left(\frac{\sqrt{3} \sinh(\eta)}{\sqrt{\cosh(2\eta) + 3}}\right)\right), \\
 f_{10}(\eta) &= \cosh(\eta)\left(20 - 2 \tanh^2(\eta)\right)\left(3 \sinh^{-1}\left(\frac{3 \sinh(\eta)}{\sqrt{10}}\right) + \tan^{-1}\left(\frac{\sqrt{3} \sinh(\eta)}{9 \cosh(2\eta) + 11}\right)\right). \\
\end{align*}
\]

From equation (5.18) one can show that the differential arc length is

\[ ds = b \csc(\alpha) \cosh(\eta). \]

Since

\[ dz = b \cosh(\eta), \]

therefore the parameter \( \alpha \) has the meaning

\[ \frac{dz}{ds} = \sin \alpha \]

and the arc length is given by

\[ s = b \csc(\alpha) \sinh(\eta). \]

The helices for \( \beta = 1/2, 1, 10 \) are shown in figure 14.
6. Discussion and conclusions

The modification of the soliton (and breather) profiles in the presence of specific curves is known for the NLS equation [16, 17]. However, it is worthwhile asking the question how this interplay of nonlinearity and geometry (i.e. curvature) is modified in the relativistic case of the NLD equation. This is what we have tried to elucidate here, particularly noting that one of the NLD solitons is double humped. We find that the shape and ‘distortion’ of the soliton depends in interesting ways on the planar or space curve it resides. The specific distortion may thus be amenable to actual study in some experimental situations such as curved Bose condensates. Interestingly, the corresponding comprehensive study for the nonrelativistic counterpart, the NLS equation, is still lacking for a variety of curves which we hope to undertake in the near future.

To summarize, we have studied the behavior of the nonlinear Dirac equation [4, 23] on planar and space curves. We have shown here how the arc length variable is the relevant choice to study the nonlinear Dirac equation on planar and space curves. We studied different parameterizations of various curves including those involving Jacobi elliptic functions [22] and studied the charge density in the presence of a soliton. We found the change in soliton shape in terms of narrowing or broadening of the soliton profile in the curved region. These results illustrate an insightful interplay between solitons of the nonlinear Dirac equation and the curvature (and torsion) of a variety of curves. Our results are relevant to curved Q1D Bose condensates assuming they can be realized experimentally. It would be instructive to (numerically) study collisions of NLD solitons on various curves to explore how curvature (and torsion) affect their interaction and collision dynamics including bounce windows.

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Appendix: The vierbein formalism applied to a hyperbola

The vierbein introduced by Hermann Weyl [15] is also called a tetrad or a frame field (in general relativity). We shall use the metric convention (−++++) for the 2+1 space-time which is commonly used in the curved-space
literature. In what follows, we use Greek indices for the curvilinear coordinates \( \eta \) or \( \theta \), and Latin indices for the Minkowski coordinates \( x, y \) and \( t \). To obtain the fermion evolution equations in the new coordinate system it is simplest to use a coordinate covariant action such as that used in field theory in curved spaces, even though here the actual curvature is zero. For the constrained system, the 2 + 1 dimensional Minkowski line element
\[
(ds)^2 = -(dt)^2 + (dx)^2 + (dy)^2 = -(dt)^2 + a^2 \cosh 2\eta (dt)^2 \equiv -(dt)^2 + (ds)^2,
\] (A.1)
so that the original Minkowski metric
\[
g_{\alpha \beta} = \text{diag}\{ -1, 1, 1 \}
\] (A.2)
reduces to an effective curved space 1 + 1 dimensional metric tensor:
\[
g_{\mu \nu} = \text{diag}\{ -1, a^2 \cosh 2\eta \}
\] (A.3)
with inverse
\[
g^{\mu \nu} = \text{diag}\{ -1, \frac{1}{a^2 \cosh 2\eta} \}.
\] (A.4)

Vierbeins transform the curved 1 + 1 dimensional space to a locally Minkowski 1 + 1 dimensional space via \( g_{\mu \nu} = V_\mu^a V_\nu^b \eta_{ab} \), where \( \eta_{ab} = (-1, 1) \) is the flat Minkowski metric. A convenient choice for the vierbein is
\[
V_\mu^a = \text{diag}\{ 1, a\sqrt{\cosh 2\eta} \},
\] (A.5)
so that
\[
V_\mu^a = \text{diag}\left\{ 1, \frac{1}{a\sqrt{\cosh 2\eta}} \right\}.
\] (A.6)

The determinant of the metric tensor is given by
\[
\det V = \sqrt{-g} = a\sqrt{\cosh 2\eta}.
\] (A.7)
The action for our model in general curvilinear coordinates is
\[
S[\Psi, \sigma] = \int d^3x \det V \left\{ -\frac{i}{2} \bar{\Psi} \gamma^\mu \nabla_\mu \Psi + \frac{i}{2} \left( \nabla_\mu \bar{\Psi} \gamma^\mu \Psi - i(m\bar{\Psi}\Psi) - \frac{g^2}{2}(\bar{\Psi}\Psi)^2 \right) \right\}.
\] (A.8)
The coordinate dependent gamma matrices \( \tilde{\gamma}^\mu \) are obtained from the usual Dirac gamma matrices \( \gamma^a \) via
\[
\tilde{\gamma}^\mu = \gamma^a V_\mu^a (x).
\] (A.9)
The coordinate independent Dirac matrices \( \gamma^a \) satisfy the usual gamma matrix algebra:
\[
\{ \gamma^a, \gamma^b \} = 2\eta^{ab}.
\] (A.10)

From the action equation (A.8) we obtain the Heisenberg field equation for the fermions,
\[
(\tilde{\gamma}^\mu \nabla_\mu + \sigma + m)\Psi = 0,
\] (A.11)
where it is to be understood:
\[
\sigma = -ig^2 \bar{\Psi}\Psi.
\] (A.12)
This can be generalized easily to the case where the nonlinear term in the NLDE is of the form [4]
\[
-g^2(\bar{\Psi}\Psi)^k \bar{\Psi},
\] (A.13)
with \( k \) denoting arbitrary nonlinearity. This is done by letting
\[
\sigma = -ig^2 (\bar{\Psi}\Psi)^k
\] (A.14)
in the equation of motion. This again is space and time dependent mass, but a scalar as we will show below.

Here \( \tilde{\gamma}^\mu = \gamma^a V_\mu^a, \nabla_\mu = \partial_\mu + \Gamma_\mu \), and \( \Gamma_\mu \) is the spin connection given by [24, 25]
\[
\Gamma_\mu = \frac{1}{2} \Sigma^{ab} V_\nu (\partial_\mu V_\nu + \Gamma_\nu V_\mu V_\nu), \quad \Sigma^{ab} = \frac{1}{4} [\gamma^a, \gamma^b].
\] (A.15)
We have
\[
\tilde{\gamma}^0 = \gamma^0; \quad \tilde{\gamma}^1 = \frac{1}{a\sqrt{\cosh 2\eta}} \gamma^1.
\] (A.16)

We will choose for convenience the following representation for the matrices \( \gamma^0 \) and \( \gamma^1 \) in the transformed local Minkowski 1 + 1 dimensional Dirac equation.
\[ i\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]  
\[ \gamma^i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

We have that the scalar field \( \sigma \) is unchanged in form
\[ \sigma = -ig^2(\Psi^0\gamma^0 \Psi) = -ig^2(\Psi^0 \gamma^0 \Psi), \]  
if we used equation (A.14) to define \( \sigma \), it would again be unchanged in form.

The Christoffel symbols have the usual definition,
\[ \Gamma_{\mu\nu}^\sigma = \frac{1}{2}g^{\sigma\rho}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}). \]  
Since the only non-zero derivative is
\[ \partial_\mu = a^2 \sinh 2\eta, \]
we find the only nonzero Christoffel symbol is
\[ \Gamma_{22} = a^2 \sinh 2\eta \]
so that
\[ \Gamma_{22}^2 = \tanh 2\eta. \]

We find the spin connection is zero for the following reason. If we consider
\[ F_{\mu\nu}^\sigma = \partial_\mu V_\nu^\sigma + \Gamma_\mu^\sigma V_\nu^\lambda, \]
only \( F_{22}^2 \) could have a non-zero contribution but we find
\[ \partial_\nu V_2^2 = -\frac{\tanh 2\eta}{a\sqrt{\cosh 2\eta}}; \quad \Gamma_{22}^2 V_2^2 = +\frac{\tanh 2\eta}{a\sqrt{\cosh 2\eta}}, \]
so that \( F_{22}^2 = 0 \) which implies that the spin connection is zero.

From equation (A.11), and the fact that the spin connection is zero we find that the equation of motion for \( \Psi \) is
\[ (\gamma^\mu \partial_\mu + \sigma + m)\Psi = 0. \]

Writing this out more explicitly, we obtain
\[ \left( \gamma^0 \partial_t + \frac{1}{a\sqrt{\cosh 2\eta}} \gamma^i \partial_i + \sigma + m \right) \Psi[\eta, t] = 0. \]

We recognize that
\[ \frac{1}{a\sqrt{\cosh 2\eta}} \gamma^i \partial_i = \gamma^i \partial_i, \]
so that the equation reduces to a flat space \( 1 + 1 \) dimensional Dirac equation in terms of the arc length \( s \).

ORCID iDs
Fred Cooper @ https://orcid.org/0000-0002-2594-4169
Avadh Saxena @ https://orcid.org/0000-0002-3374-3236

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