Study of fractional integral inequalities involving Mittag-Leffler functions via convexity

Zhihua Chen¹, Ghulam Farid², Maryam Saddiqa³*, Saleem Ullah⁴ and Naveed Latif⁵

Abstract

This paper studies fractional integral inequalities for fractional integral operators containing extended Mittag-Leffler (ML) functions. These inequalities provide upper bounds of left- and right-sided fractional integrals for \((\alpha, h - m)\)-convex functions. A generalized fractional Hadamard inequality is established. All the results hold for \(h\)-convex, \((h, m)\)-convex, \((\alpha, m)\)-convex, \((s, m)\)-convex, and associated functions.

Keywords: Convex function; \((\alpha, h - m)\)-convex function; Mittag-Leffler function; Fractional integral operators

1 Introduction

Convexity was introduced at the beginning of the twentieth century. Due to having many fascinating and important properties, a convex function plays a vital role in almost all areas of mathematical analysis, probability theory, optimization theory, graph theory, etc. It has been defined in different convenient ways, for example, graph of a convex function always lies below the chord joining any two points lying on its graph, the derivative of a differentiable convex function is increasing and vice versa, a convex function has line of support at each point of the interior of its domain, and many others. In the theory of inequalities it is frequently defined in the form of an inequality which can be interpreted very nicely in the plane. A function \(f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}\) satisfying the inequality \(f(ta + (1 - t)b) \leq tf(a) + (1 - t)f(b)\), where \(I\) is an interval, \(t \in [0, 1]\), and \(a, b \in I\), is called convex.

This analytic form of presentation of a convex function motivated the authors to define other types of convex functions for example \(m\)-convex, \(s\)-convex, \((s, m)\)-convex, \(h\)-convex, \((h, m)\)-convex, \((\alpha, m)\)-convex, exponentially convex, etc. In this age convex functions lead to the theory of convex analysis, theory of inequalities, a lot of research articles and books are dedicated to the literature which has been developed due to convex function, see [1, 3, 4, 20, 22, 25, 31].

The goal of this paper is to study the bounds of fractional integral operators involving Mittag-Leffler (ML) functions in their kernels by utilizing a generalized form of convex functions, namely \((\alpha, h - m)\)-convex functions which unify \(h\)-convex, \((h, m)\)-convex,
where \((\alpha, m)\)-convex and \((s, m)\)-convex functions. Therefore the results of this paper will simultaneously hold for all these kinds of convex functions.

In 2007, Varošanec introduced the \(h\)-convex function.

**Definition 1** ([30]) A function \(f : I \rightarrow \mathbb{R}\) is said to be \(h\)-convex if the following inequality holds:

\[
f(ta + (1 - t)b) \leq h(t)f(a) + h(1 - t)f(b),
\]

where \(h\) is a nonnegative function defined on \(I\), for all \(a, b \in I\), \(t \in [0, 1]\), and \(I\) and \(J\) are real intervals such that \((0, 1) \subset J\).

Özdemir introduced a generalization of \(h\)-convex function, namely \((h - m)\)-convex function.

**Definition 2** ([21]) Let \(J \subseteq \mathbb{R}\) be an interval containing \((0, 1)\), and let \(h : J \rightarrow \mathbb{R}\) be a nonnegative function. We say that \(f : [0, b] \rightarrow \mathbb{R}\) is an \((h - m)\)-convex function if \(f\) is nonnegative, and for all \(u, v \in [0, b], m \in [0, 1]\), and \(t \in (0, 1)\), one has

\[
f(tu + m(1 - t)v) \leq h(t)f(u) + mh(1 - t)f(v).
\]

Mihešan introduced the notion of \((\alpha, m)\)-convex function as follows.

**Definition 3** ([18]) A function \(f : [0, b] \subseteq \mathbb{R} \rightarrow \mathbb{R}\) is said to be \((\alpha, m)\)-convex function, where \((\alpha, m) \in [0, 1]^2\) and \(b > 0\), if for every \(u, v \in [0, b]\) and \(t \in [0, 1]\), we have

\[
f(tu + m(1 - t)v) \leq t^\alpha f(u) + m(1 - t^\alpha)f(v).
\]

**Definition 4** ([5]) A function \(f : [0, b] \rightarrow \mathbb{R}\) is said to be \((s, m)\)-convex function, where \((s, m) \in [0, 1]^2\) and \(b > 0\), if for every \(x, y \in [0, b]\) and \(t \in [0, 1]\), we have

\[
f(ta + m(1 - t)b) \leq tf(a) + m(1 - t)sf(b).
\]

Farid et al. unified the notions of \(h\)-convexity, \((\alpha, m)\)-convexity, \((h, m)\)-convexity, and \((s, m)\)-convexity in a single definition called \((\alpha, h - m)\)-convex function given as follows.

**Definition 5** ([15]) Let \(I \subseteq \mathbb{R}\) be an interval containing \((0, 1)\), and let \(h : I \rightarrow \mathbb{R}\) be a nonnegative function. We say that \(f : [0, b] \rightarrow \mathbb{R}\) is an \((\alpha, h - m)\)-convex function if \(f\) is nonnegative, and for all \(u, v \in [0, b], (\alpha, m) \in [0, 1]^2\), and \(t \in (0, 1)\), one has

\[
f(tu + m(1 - t)v) \leq h(t^\alpha)f(u) + mh(1 - t^\alpha)f(v).
\]

**Remark 1** By selecting a suitable function \(h\) and particular values of \(m\) and \(\alpha\), the above definition produces the functions as follows:

(i) By setting \(\alpha = 1\), \(h(t) = t^\alpha\), an \((s, m)\)-convex function can be obtained.

(ii) By setting \(h(t) = t\), an \((\alpha, m)\)-convex function can be obtained.

(iii) By setting \(\alpha = 1\), an \((h, m)\)-convex function can be obtained.
(iv) By setting $m = 1, \alpha = 1$, an $h$-convex function can be obtained.

Next we give the definition of Mittag-Leffler functions and associated definitions of fractional integral operators.

Mittag-Leffler function $E_\xi(\cdot)$ for one parameter is defined as follows [19]:

$$
E_\xi(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\xi n + 1)},
$$

where $t, \xi \in \mathbb{C}, \Re(\xi) > 0$, and $\Gamma(\cdot)$ is the gamma function. It is a natural extension of exponential, hyperbolic, and trigonometric functions. This function and its extensions are useful in solving fractional integral/differential equations. It is also studied extensively in various fields of sciences; for details, see [2, 7, 10, 16, 17, 24, 26, 27].

Andrić et al. introduced the following extended Mittag-Leffler function.

**Definition 6** (6) Let $\mu, \xi, l, \gamma, c \in \mathbb{C}, \Re(\mu), \Re(\xi), \Re(l) > 0, \Re(c) > \Re(\gamma) > 0$ with $p \geq 0, \delta > 0$, and $0 < k \leq \delta + \Re(\mu)$. Then the extended generalized Mittag-Leffler function $E_{\mu,\xi,l}^{\gamma,k,c}(t; p)$ is defined by

$$
E_{\mu,\xi,l}^{\gamma,k,c}(t; p) = \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c - \gamma)} \frac{(c)_n}{\Gamma(\mu n + \alpha)} \frac{t^n}{(l)_n},
$$

where $\beta_p$ is defined by

$$
\beta_p(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}e^{-\frac{p}{t}} dt
$$

and $(c)_n = \frac{\Gamma(c+nk)}{\Gamma(c)}$.

A derivative formula of the extended Mittag-Leffler function is given in the following lemma.

**Lemma 1** ([6] If $m \in \mathbb{N}, \omega, \mu, \xi, l, \gamma, c \in \mathbb{C}, \Re(\mu), \Re(\xi), \Re(l) > 0, \Re(c) > \Re(\gamma) > 0$ with $p \geq 0, \delta > 0$, and $0 < k < \delta + \Re(\mu)$, then

$$
\left( \frac{d}{dt} \right)^m E_{\mu,\xi,l}^{\gamma,k,c}(t; p) = t^{x-m-1} E_{\mu,\xi-l}^{\gamma,k,c}(\omega t^n; p), \quad \Re(\xi) > m.
$$

**Remark 2** The extended Mittag-Leffler function (1.3) produces the related functions defined in [23, 24, 26–28], see [29, Remark 1.3].

Next, we give the definition of fractional integral operators containing the extended Mittag-Leffler function (1.3).

**Definition 7** (6) Let $\omega, \mu, \xi, l, \gamma, c \in \mathbb{C}, \Re(\mu), \Re(\xi), \Re(l) > 0, \Re(c) > \Re(\gamma) > 0$ with $p \geq 0, \delta > 0$, and $0 < k \leq \delta + \Re(\mu)$. Let $f \in L_1([a, b])$ and $x \in [a, b]$. Then the generalized fractional integral operators containing Mittag-Leffler function are defined by

$$
(c_{\mu,\xi,l,\omega,x}^{\gamma,k,c} f)(x; p) = \int_a^x (x-t)^{x-1} E_{\mu,\xi,l}^{\gamma,k,c}(\omega(x-t)^\mu; p)f(t) dt,
$$
\[
(e^{\gamma, b, k_c}_{\mu, \beta_d, j_0, b}, f)(x; p) = \int_{x}^{b} (t - x)^{\xi - 1} E^{\gamma, b, k_c}_{\mu, \beta_d, l, j}(\omega(t - x)^\mu; p) f(t) \, dt.
\] (1.6)

**Remark 3** Operators (1.5) and (1.6) produce in particular several kinds of known fractional integral operators, see [29, Remark 1.4].

The classical Riemann–Liouville fractional integral operator is defined as follows.

**Definition 8** ([28]) Let \( f \in L_1[a, b] \). Then Riemann–Liouville fractional integral operators of order \( \xi > 0 \) are defined by

\[
I_a^\xi f(x) = \frac{1}{\Gamma(\xi)} \int_{a}^{x} (x - t)^{\xi - 1} f(t) \, dt, \quad x > a,
\] (1.7)

\[
I_b^\xi f(x) = \frac{1}{\Gamma(\xi)} \int_{a}^{x} (t - x)^{\xi - 1} f(t) \, dt, \quad x < b.
\] (1.8)

It can be noted that \((e^{\gamma, b, k_c}_{\mu, \beta_d, l, a, \alpha}, f)(x; 0) = I_a^\alpha f(x)\) and \((e^{\gamma, b, k_c}_{\mu, \beta_d, l, b, \alpha}, f)(x; 0) = I_b^\alpha f(x)\). From fractional integral operators (1.5) and (1.6) we can write

\[
J_{\xi, a^+}(x; p) := (e^{\gamma, b, k_c}_{\mu, \beta_d, l, a, \alpha}, 1)(x; p) = (x - a)^{\xi} E^{\gamma, b, k_c}_{\mu, \beta_d, l, j, 1}(w(x - a)^\mu; p),
\] (1.10)

\[
J_{\xi, b^-}(x; p) := (e^{\gamma, b, k_c}_{\mu, \beta_d, l, b, \alpha}, 1)(x; p) = (b - x)^{\xi} E^{\gamma, b, k_c}_{\mu, \beta_d, l, j, 1}(w(b - x)^\mu; p).
\] (1.10)

In the upcoming section the extended Mittag–Leffler (ML) function (1.3) and the corresponding generalized fractional integral operators are used to evaluate the bounds of sum of left- and right-sided operators by using \((a, h - m)\)-convexity. Their particular cases are also discussed. Furthermore, the lower and upper bounds of sum of these operators are presented in the form of a Hadamard inequality for \((a, h - m)\)-convex functions. Also the presented results are connected with several already known results.

**2 Main results**

**Theorem 1** Let \( \varphi : [x_0, y_0] \to \mathbb{R} \) be a real-valued function. If \( \varphi \) is positive and \((a, h - m)\)-convex, \((a, m) \in [0, 1]^2, m \neq 0\), then for \( \xi, \eta \geq 1 \), the following fractional integral inequality for generalized fractional integral operators (1.5) and (1.6) holds:

\[
(e^{\gamma, b, k_c}_{\mu, \beta_d, j_0, \alpha}, \varphi)(x; p) + (e^{\gamma, b, k_c}_{\mu, \beta_d, j_0, \beta, \eta}, \varphi)(x; p)
\]

\[
\leq (x - x_0)J_{\xi, 1, \alpha^+}(x; p) \left( \varphi(x_0) \int_{0}^{1} h(z^\alpha) \, dz + m\varphi \left( \frac{x}{m} \right) \int_{0}^{1} h(1 - z^\alpha) \, dz \right)
\]

\[
+ (y_0 - x)J_{\eta, 1, \beta^+}(x; p) \left( \varphi(y_0) \int_{0}^{1} h(z^\alpha) \, dz + m\varphi \left( \frac{x}{m} \right) \int_{0}^{1} h(1 - z^\alpha) \, dz \right).
\] (2.1)

**Proof** Let \( x \in [x_0, y_0] \). Then, for \( t \in [x_0, x], \xi \geq 1 \), the following inequality holds:

\[
(x - t)^{\xi - 1} E^{\gamma, b, k_c}_{\mu, \beta_d, l, j}(\omega(x - t)^\mu; p) \leq (x - x_0)^{\xi - 1} E^{\gamma, b, k_c}_{\mu, \beta_d, l, j}(\omega(x - x_0)^\mu; p).
\] (2.2)

Using the definition of \((a, h - m)\)-convex function, we get

\[
\varphi(t) \leq h \left( \frac{x - t}{x - x_0} \right)^a \varphi(x_0) + mh \left( 1 - \left( \frac{x - t}{x - x_0} \right)^a \right) \varphi \left( \frac{x}{m} \right).
\] (2.3)
After multiplying (2.2) and (2.3), we integrate over \([x_0, x]\) to obtain the following inequality:

\[
\int_{x_0}^{x} (x - t)^{\xi-1} E_{\mu, \xi, \varphi}^{\eta, \lambda, \kappa, \epsilon} (\omega(x - t)^{\mu}; p) \varphi(t) \, dt \\
\leq (x - x_0)^{\xi-1} E_{\mu, \xi, \varphi}^{\eta, \lambda, \kappa, \epsilon} (\omega(x - x_0)^{\mu}; p) \left( \varphi(x_0) \int_{x_0}^{x} h \left( \frac{x - t}{x - x_0} \right)^{\alpha} \, dt \right) \\
+ m \varphi \left( \frac{x}{m} \right) \int_{x_0}^{x} h \left( 1 - \left( \frac{x - t}{x - x_0} \right)^{\alpha} \right) \, dt.
\]

Therefore the left fractional integral operator (1.5) satisfies the following upper bound:

\[
\left( E_{\mu, \xi, \varphi}^{\eta, \lambda, \kappa, \epsilon} \varphi \right)(x; p) \\
\leq (x - x_0) \int_{-1}^{x_0} (x; p) \left( \varphi(x_0) \int_{0}^{1} h(z^{\alpha}) \, dz + m \varphi \left( \frac{x}{m} \right) \int_{0}^{1} h(1 - z^{\alpha}) \, dz \right).
\] (2.4)

Similarly, for \(t \in (x, y_0]\) and \(\eta \geq 1\), the following inequality holds:

\[
(t - x)^{\eta-1} E_{\mu, \eta, \varphi}^{\xi, \lambda, \kappa, \epsilon} (\omega(t - x)^{\mu}; p) \leq (y_0 - x)^{\eta-1} E_{\mu, \eta, \varphi}^{\xi, \lambda, \kappa, \epsilon} (\omega(y_0 - x)^{\mu}; p),
\] (2.5)

again by using \((\omega, h - m)\)-convexity of \(\varphi\), we have

\[
\varphi(t) \leq h \left( \frac{t - x}{y_0 - x} \right)^{\alpha} \varphi(y_0) + m h \left( 1 - \left( \frac{t - x}{y_0 - x} \right)^{\alpha} \right) \varphi \left( \frac{x}{m} \right).
\] (2.6)

By multiplying (2.5) and (2.6), and integrating over \([x, y_0]\), we have

\[
\int_{x}^{y_0} (t - x)^{\eta-1} E_{\mu, \eta, \varphi}^{\xi, \lambda, \kappa, \epsilon} (\omega(t - x)^{\mu}; p) \varphi(t) \, dt \\
\leq (y_0 - x)^{\eta-1} E_{\mu, \eta, \varphi}^{\xi, \lambda, \kappa, \epsilon} (\omega(y_0 - x)^{\mu}; p) \left( \varphi(y_0) \int_{x}^{y_0} h \left( \frac{t - x}{y_0 - x} \right)^{\alpha} \, dt \right) \\
+ m \varphi \left( \frac{x}{m} \right) \int_{x}^{y_0} h \left( 1 - \left( \frac{t - x}{y_0 - x} \right)^{\alpha} \right) \, dt.
\]

Therefore the right fractional integral operator (1.6) satisfies the following upper bound:

\[
\left( E_{\mu, \eta, \varphi}^{\xi, \lambda, \kappa, \epsilon} \varphi \right)(x; p) \\
\leq (y_0 - x) \int_{y_0}^{1} (x; p) \left( \varphi(y_0) \int_{0}^{1} h(z^{\alpha}) \, dz + m \varphi \left( \frac{x}{m} \right) \int_{0}^{1} h(1 - z^{\alpha}) \, dz \right).
\] (2.7)

By adding (2.4) and (2.7), inequality (2.1) is obtained.

Some particular results are stated in the following corollaries.

**Corollary 1** If we set \(\xi = \eta\) in (2.1), then the following inequality is obtained:

\[
\left( E_{\mu, \xi, \varphi}^{\eta, \lambda, \kappa, \epsilon} \varphi \right)(x; p) + \left( E_{\mu, \xi, \varphi}^{\eta, \lambda, \kappa, \epsilon} \varphi \right)(x; p)
\] (2.8)
Remark

By setting Corollary 3 inequality is obtained

Along with the assumptions of Theorem Corollary 2

If defined in (1.5) that is,

Along with the assumptions of Theorem Theorem 2

(iii) If we set \( \alpha \)

(vi) If we set \( \alpha \)

(ii) If we set \( \alpha \)

(v) If we set \( \alpha \)

(i) If we set \( \alpha \)

Corollary 2 Along with assumptions of Theorem 1, if \( \psi \in L_\infty[x_0,y_0] \), then the following inequality is obtained:

\[
\left( e^{y,h,k,c}_{\mu,\xi,\lambda;\omega,\gamma,\delta,\varphi} \psi \right)(x;p) + e^{y,h,k,c}_{\mu,\xi,\lambda;\omega,\gamma,\delta,\varphi} \psi(x;p) \\
\leq \| \psi \|_\infty [ (x-x_0)I_{\xi-1,\lambda}^c(x;p) + (y_0-x)I_{\xi-1,\lambda}^c(x;p) ] \\
\times \left[ \int_0^1 h(z^o) \, dz + m \int_0^1 h(1-z^o) \, dz \right].
\]

Corollary 3 By setting \( \xi = \eta \) in (2.9), we get the following inequality:

\[
\left( e^{y,h,k,c}_{\mu,\xi,\lambda;\omega,\gamma,\delta,\varphi} \psi \right)(x;p) + e^{y,h,k,c}_{\mu,\xi,\lambda;\omega,\gamma,\delta,\varphi} \psi(x;p) \\
\leq \| \psi \|_\infty [ (x-x_0)I_{\xi-1,\lambda}^c(x;p) + (y_0-x)I_{\xi-1,\lambda}^c(x;p) ] \\
\times \left[ \int_0^1 h(z^o) \, dz + m \int_0^1 h(1-z^o) \, dz \right].
\]

Remark 4

(i) If we set \( h(t) = t \) in (2.1), then we obtain [12, Theorem 2.1].

(ii) If we set \( h(t) = t \) in (2.8), then we obtain [12, Corollary 2.1].

(iii) If we set \( \alpha = 1 \) in (2.1), then we obtain [9, Theorem 1].

(iv) If we set \( \alpha = m = 1 \) and \( h(t) = t \) in (2.1), then we obtain [9, Corollary 1].

(v) If we set \( \alpha = 1, \omega = p = 0 \) in (2.1), then we obtain [13, Theorem 1].

(vi) If we set \( \alpha = m = 1, \omega = p = 0, h(t) = t \) in (2.1), then we obtain [11, Theorem 1].

Theorem 2 Along with the assumptions of Theorem 1, if \( \psi \in L_\infty[x_0,y_0] \), then the operators defined in (1.5) and (1.6) are bounded and continuous.

Proof If \( \psi \in L_\infty[x_0,y_0] \), then from (2.4) we have

\[
\left| \left( e^{y,h,k,c}_{\mu,\xi,\lambda;\omega,\gamma,\delta,\varphi} \psi \right)(x;p) \right| \\
\leq \| \psi \|_\infty (x-x_0)I_{\xi-1,\lambda}^c(x;p) \int_0^1 \left( h(z^o) + mh(1-z^o) \right) \, dz \\
\leq \| \psi \|_\infty (y_0-x_0)I_{\xi-1,\lambda}^c(x;p) \int_0^1 \left( h(z^o) + mh(1-z^o) \right) \, dz,
\]

that is,

\[
\left| \left( e^{y,h,k,c}_{\mu,\xi,\lambda;\omega,\gamma,\delta,\varphi} \psi \right)(x;p) \right| \leq M \| \psi \|_\infty,
\]

(2.12)
Multiplication of \((2.2)\) with \((2.16)\) gives the following inequality:

\[
(x - t)^{\xi - 1} E_{\mu, \delta, I}^{\beta, \delta, \xi} \omega(x - t)^{\mu}; p) \psi'(t) \, dt 
\leq (x - x_0)^{\xi - 1} E_{\mu, \delta, I}^{\beta, \delta, \xi} \omega(x - x_0)^{\mu}; p) 
\times \left( h\left(\frac{x - t}{x - x_0}\right)^{\alpha} |\psi'(x_0)| + m h\left(1 - \left(\frac{x - t}{x - x_0}\right)^{\alpha}\right) |\psi'(x_0)| \right). 
\]

(2.17)

Now, integrating over \([x_0, x]\), we get

\[
\int_{x_0}^{x} (x - t)^{\xi - 1} E_{\mu, \delta, I}^{\beta, \delta, \xi} \omega(x - t)^{\mu}; p) \psi'(t) \, dt 
\leq (x - x_0)^{\xi - 1} E_{\mu, \delta, I}^{\beta, \delta, \xi} \omega(x - x_0)^{\mu}; p) \left( |\psi'(x_0)| \int_{x_0}^{x} h\left(\frac{x - t}{x - x_0}\right)^{\alpha} \right) dt 
\]

(2.16)
From (2.20) and (2.22), we get

\[
\int_0^x (x-t)^{\alpha-1} E_{\alpha,\beta}^{\gamma,k,k} (\omega(x-t)^{\alpha}; p) \psi'(t) \, dt.
\] (2.19)

Substituting \( x - t = r \), using the derivative property (1.4) of Mittag-Leffler function, we have

\[
\int_0^{x-t} r^{\alpha-1} E_{\alpha,\beta}^{\gamma,k,k} (\omega r^{\alpha}; p) \psi'(x-r) \, dr
\]
\[
= (x - x_0)^{\alpha-1} E_{\alpha,\beta}^{\gamma,k,k} (\omega(x - x_0)^{\alpha}; p) \psi(x_0) - \int_0^{x-x_0} r^{\alpha-2} E_{\alpha,\beta}^{\gamma,k,k} (\omega r^{\alpha}; p) \psi(x-r) \, dr.
\]

Now, for \( x - r = t \) in the second term of the right-hand side of the above equation and then using (1.5), we get

\[
\int_0^{x-x_0} r^{\alpha-1} E_{\alpha,\beta}^{\gamma,k,k} (\omega r^{\alpha}; p) \psi'(x-r) \, dr
\]
\[
= (x - x_0)^{\alpha-1} E_{\alpha,\beta}^{\gamma,k,k} (\omega(x - x_0)^{\alpha}; p) \psi(x_0) - (\epsilon^{\gamma,k,k}_{\alpha,1,\mu_0} \omega)(x_0) \psi(x_0).
\]

Therefore (2.18) becomes

\[
I_{\alpha-1,\alpha}(x; p) \psi(x_0) - (\epsilon^{\gamma,k,k}_{\alpha,1,\mu_0} \omega)(x_0) \psi(x_0)
\]
\[
\leq (x - x_0) I_{\alpha-1,\alpha}(x; p) \left( |\psi'(x_0)| h(z^\alpha) \, dz + m \left| \frac{x}{m} \right| \int_0^1 h(1 - z^\alpha) \, dz \right).
\] (2.20)

Again from (2.15) we have

\[
\psi'(t) \geq - \left( h \left( \frac{x - t}{x - x_0} \right)^{\alpha} \right) \left| \psi'(x_0) \right| m \left( 1 - \left( \frac{x - t}{x - x_0} \right)^{\alpha} \right) \left| \psi'(x_0) \right|.
\] (2.21)

Similar as we did for (2.16), we can obtain

\[
(\epsilon^{\gamma,k,k}_{\alpha,1,\mu_0} \omega)(x; p) - I_{\alpha-1,\alpha}(x; p) \psi(x_0)
\]
\[
\leq (x - x_0) I_{\alpha-1,\alpha}(x; p) \left( |\psi'(x_0)| h(z^\alpha) \, dz + m \left| \frac{x}{m} \right| \int_0^1 h(1 - z^\alpha) \, dz \right).
\] (2.22)

From (2.20) and (2.22), we get

\[
|\epsilon^{\gamma,k,k}_{\alpha,1,\mu_0} \omega)(x; p) - I_{\alpha-1,\alpha}(x; p) \psi(x_0)|
\]
\[
\leq (x - x_0) I_{\alpha-1,\alpha}(x; p) \left( |\psi'(x_0)| h(z^\alpha) \, dz + m \left| \frac{x}{m} \right| \int_0^1 h(1 - z^\alpha) \, dz \right).
\] (2.23)
Now, for \( x \in [x_0, y_0] \) and \( t \in (x, y_0) \), again by using the \((\alpha, h - m)\)-convexity of \( |\varphi'| \), we have
\[
|\varphi'(t)| \leq h\left(\frac{t - x}{y_0 - x}\right)^\alpha |\varphi'(y_0)| + mh\left(1 - \left(\frac{t - x}{y_0 - x}\right)^\alpha\right)|\varphi'(x/m)|
\] (2.24)

Proceeding along similar lines as we did to get (2.23), we can obtain the following inequality:
\[
\left|\left(\epsilon_{\mu,h,k,c}^{\alpha,h,m,y_0} \varphi\right)(x;p) - I_{\eta-1,0,h}^{\alpha,h,m,y_0}(x;p)\varphi(y_0)\right|
\leq (y_0 - x)I_{\eta-1,0,h}^{\alpha,h,m,y_0}(x;p)\left(|\varphi'(y_0)| \int_0^1 h(z^\alpha) \, dz + m|\varphi'(x/m)| \int_0^1 h(1 - z^\alpha) \, dz\right)
\]
(2.25)

From inequalities (2.23) and (2.25), triangular inequality (2.14) can be obtained. \( \square \)

**Corollary 4** If we put \( \xi = \eta \) in (2.14), then the following inequality is obtained:
\[
\left|\left(\epsilon_{\mu,h,k,c}^{\alpha,h,m,y_0} \varphi\right)(x;p) + \left(\epsilon_{\mu,h,k,c}^{\alpha,h,m,y_0} \varphi\right)(x;p) - I_{\eta-1,0,h}^{\alpha,h,m,y_0}(x;p)\varphi(y_0)\right|
\leq (x - x_0)I_{\xi-1,0,h}^{\alpha,h,m,y_0}(x;p)\left(|\varphi'(x_0)| \int_0^1 h(z^\alpha) \, dz + m|\varphi'(x/m)| \int_0^1 h(1 - z^\alpha) \, dz\right)
\]
+ \( (y_0 - x)I_{\xi-1,0,h}^{\alpha,h,m,y_0}(x;p)\left(|\varphi'(y_0)| \int_0^1 h(z^\alpha) \, dz + m|\varphi'(x/m)| \int_0^1 h(1 - z^\alpha) \, dz\right)
\]
(2.26)

**Remark 5**
(i) If we take \( h(t) = t \) in (2.14), then we obtain [12, Theorem 2.2].
(ii) If we take \( h(t) = t^\beta \) in (2.26), then we obtain [12, Corollary 2.2].
(iii) If we take \( \alpha = 1 \) in (2.14), then we obtain [9, Theorem 2].
(iv) If we take \( \alpha = m = 1 \) and \( h(t) = t \) in (2.14), then we obtain [9, Corollary 2].
(v) If we take \( \alpha = 1, \omega = p = 0 \) in (2.14), then we obtain [13, Theorem 2].
(vi) If we take \( \alpha = m = 1, \omega = p = 0 \), and \( h(t) = t \) in (2.14), then we obtain [11, Theorem 2].

It is easy to prove the next lemma which will be helpful to establish estimations in the form of a Hadamard-type inequality.

**Lemma 2** Let \( \varphi : [x_0, my_0] \to \mathbb{R} \) be an \((\alpha, h - m)\)-convex function. If \( \varphi\left(x_0 + my_0 - x\right) = \varphi(x) \) and \((\alpha, m) \in [0,1]^2, m \neq 0 \), then the following inequality holds:
\[
\varphi\left(x_0 + my_0/2\right) \leq \varphi(x)\left(h\left(\frac{1}{2^\alpha}\right) + mh\left(1 - \frac{1}{2^\alpha}\right)\right).
\] (2.27)

**Proof** Since \( \varphi \) is an \((\alpha, h - m)\)-convex function, for \( t \in [0, 1] \), we have
\[
\varphi\left(x_0 + my_0/2\right) \leq h\left(\frac{1}{2^\alpha}\right)\varphi((1-t)x_0 + my_0) + mh\left(1 - \frac{1}{2^\alpha}\right)\varphi\left(ta + (1-t)y_0\right).
\] (2.28)
Let \( x = x_0(1 - t) + my_0 \). Then we have \( x_0 + my_0 - x = ta + m(1 - t)y_0 \).

\[
\varphi \left( \frac{x_0 + my_0}{2} \right) \leq h \left( \frac{1}{2^a} \right) \varphi(x) + mh \left( 1 - \frac{1}{2^a} \right) \varphi \left( \frac{x_0 + my_0 - x}{m} \right). \tag{2.29}
\]

Hence, by using \( \varphi \left( \frac{x_0 + my_0}{m} \right) = \varphi(x) \), inequality (2.27) can be obtained. \( \square \)

**Theorem 4** Let \( \varphi : [x_0, y_0] \rightarrow \mathbb{R}, 0 \leq x_0 < my_0, \) be a real-valued function. If \( \varphi \) is positive \((\alpha, h - m)\)-convex, \((\alpha, m) \in [0, 1]^2, m \neq 0, \) and \( \varphi \left( \frac{x_0 + my_0}{m} \right) = \varphi(x) \), then for \( \xi, \eta > 0, \) the following fractional integral inequality holds:

\[
\begin{align*}
\frac{1}{h^\frac{1}{2^a}} & + mh \left( 1 - \frac{1}{2^a} \right) \varphi \left( \frac{x_0 + my_0}{2} \right) \left[ J_{\eta,1+y_0} (x_0;p) + J_{\xi,1+y_0} (y_0;p) \right] \\
& \leq (\epsilon_{\gamma,\delta, k} \varphi)_\mu (x_0;p) + (\epsilon_{\eta,\xi, k} \varphi) (y_0;p) \\
& \leq \left[ J_{\eta,1+y_0} (x_0;p) + J_{\xi,1+y_0} (y_0;p) \right] (y_0 - x_0)^2 \\
& \times \left( \varphi(y_0) \int_0^1 h(z^\alpha) \, dz + mf \left( \frac{x_0}{m} \right) \int_0^1 h(1 - z^\alpha) \, dz \right). \tag{2.30}
\end{align*}
\]

**Proof** For \( x \in [x_0, y_0], \eta > 0, \) we have

\[
(x - x_0)^\eta \frac{E_{\mu,\eta} (\omega(x - x_0)^\mu; p)}{E_{\mu,\eta} (\omega(x - x_0)^\mu; p)} \leq (y_0 - x_0)^\eta \frac{E_{\mu,\eta} (\omega(y_0 - x_0)^\mu; p)}{E_{\mu,\eta} (\omega(y_0 - x_0)^\mu; p)}. \tag{2.31}
\]

Since the function \( \varphi \) is \((\alpha, h - m)\)-convex, for \( x \in [x_0, y_0], \) we have

\[
\varphi(x) \leq h \left( \frac{x - x_0}{y_0 - x_0} \right)^\alpha \varphi(y_0) + mf \left( \frac{x_0}{m} \right) h \left( 1 - \left( \frac{x - x_0}{y_0 - x_0} \right)^\alpha \right). \tag{2.32}
\]

Multiplying (2.31) and (2.32) and then integrating over \([x_0, y_0],\) we get

\[
\int_{x_0}^{y_0} (x - x_0)^\eta \frac{E_{\mu,\eta} (\omega(x - x_0)^\mu; p)}{E_{\mu,\eta} (\omega(x - x_0)^\mu; p)} \varphi(x) \, dx \\
\leq (y_0 - x_0)^\eta \frac{E_{\mu,\eta} (\omega(y_0 - x_0)^\mu; p)}{E_{\mu,\eta} (\omega(y_0 - x_0)^\mu; p)} \left( \varphi(y_0) \int_{x_0}^{y_0} h \left( \frac{x - x_0}{y_0 - x_0} \right)^\alpha \, dx \right) \\
+ mf \left( \frac{x_0}{m} \right) \int_{x_0}^{y_0} h \left( 1 - \left( \frac{x - x_0}{y_0 - x_0} \right)^\alpha \right) \, dx,
\]

from which we can get the following inequality:

\[
(\epsilon_{\gamma,\delta, k} \varphi)_\mu (x_0;p) \\
\leq (y_0 - x_0)^2 J_{\eta,1+y_0} (x_0;p) \left( \varphi(y_0) \int_0^1 h(z^\alpha) \, dz + mf \left( \frac{x_0}{m} \right) \int_0^1 h(1 - z^\alpha) \, dz \right). \tag{2.33}
\]

On the other hand, for \( x \in [x_0, y_0], \xi > 0, \) we have

\[
(y_0 - x)^\xi \frac{E_{\mu,\xi} (\omega(y_0 - x)^\mu; p)}{E_{\mu,\xi} (\omega(y_0 - x)^\mu; p)} \leq (y_0 - x)^\xi \frac{E_{\mu,\xi} (\omega(y_0 - x)^\mu; p)}{E_{\mu,\xi} (\omega(y_0 - x)^\mu; p)}. \tag{2.34}
\]
Multiplying (2.32) and (2.34), and then integrating over \([x_0, y_0]\), we get

\[
\int_{x_0}^{y_0} (y_0 - x)^\alpha E_{\nu, \beta, \kappa}^{\gamma} x \mu (\omega(y_0 - x)^\mu; p) \varphi(x) \, dx
\]

\[
\leq (y_0 - x_0)^\alpha E_{\nu, \beta, \kappa}^{\gamma} x \mu (\omega(y_0 - x_0)^\mu; p) \left( \varphi(y_0) \int_{x_0}^{y_0} h \left( \frac{x - x_0}{y_0 - x_0} \right)^\alpha \, dx \right)
\]

\[
+ mf \left( \frac{x_0}{m} \right) \int_{x_0}^{y_0} h \left( 1 - \left( \frac{x - x_0}{y_0 - x_0} \right)^\alpha \right) \, dx,
\]

from which we can get the following inequality:

\[
\left( E_{\mu, \nu, \kappa}^{\gamma, \delta, \kappa} \varphi \right)(y_0; p)
\]

\[
\leq (y_0 - x_0)^2 f_{\xi - 1, \omega} (y_0; p) \left( \varphi(y_0) \int_{0}^{1} h(x^\alpha) \, dx + mf \left( \frac{x_0}{m} \right) \int_{0}^{1} h(1 - x^\alpha) \, dx \right). \tag{2.35}
\]

Adding (2.33) and (2.35), we get

\[
\left( E_{\mu, \nu, \kappa}^{\gamma, \delta, \kappa} \varphi \right)(x_0; p) + \left( E_{\mu, \nu, \kappa}^{\gamma, \delta, \kappa} \varphi \right)(y_0; p)
\]

\[
\leq \left[ f_{\omega - 1, \gamma} (x_0; p) + f_{\omega - 1, \gamma} (y_0; p) \right] (y_0 - x_0)^2
\]

\[
\times \left( \varphi(y_0) \int_{0}^{1} h(x^\alpha) \, dx + mf \left( \frac{x_0}{m} \right) \int_{0}^{1} h(1 - x^\alpha) \, dx \right) \tag{2.36}
\]

Multiplying (2.27) with \((x - x_0)^\alpha E_{\nu, \beta, \kappa}^{\gamma} x \mu (\omega(x - x_0)^\mu; p)\) and integrating over \([x_0, y_0]\), we get

\[
\varphi \left( \frac{x_0 + my_0}{2} \right) \int_{x_0}^{y_0} (x - x_0)^\alpha E_{\nu, \beta, \kappa}^{\gamma} x \mu (\omega(x - x_0)^\mu; p) \, dx
\]

\[
\leq \left( h \left( \frac{1}{2^\alpha} \right) + mh \left( 1 - \frac{1}{2^\alpha} \right) \right) \int_{x_0}^{y_0} (x - x_0)^\alpha E_{\mu, \nu, \kappa}^{\gamma} x \mu (\omega(x - x_0)^\mu; p) \varphi(x) \, dx \tag{2.37}
\]

By using (1.6) and (1.9), we get

\[
\varphi \left( \frac{x_0 + my_0}{2} \right) f_{\omega - 1, \gamma} (x_0; p)
\]

\[
\leq \left( h \left( \frac{1}{2^\alpha} \right) + mh \left( 1 - \frac{1}{2^\alpha} \right) \right) \left( E_{\mu, \nu, \kappa}^{\gamma, \delta, \kappa} \varphi \right)(x_0; p). \tag{2.38}
\]

Multiplying (2.27) with \((y_0 - x)^\alpha E_{\nu, \beta, \kappa}^{\gamma} x \mu (\omega(y_0 - x)^\mu; p)\) and integrating over \([x_0, y_0]\), using (1.5) and (1.9), we get

\[
\varphi \left( \frac{x_0 + my_0}{2} \right) f_{\omega - 1, \gamma} (y_0; p)
\]

\[
\leq \left( h \left( \frac{1}{2^\alpha} \right) + mh \left( 1 - \frac{1}{2^\alpha} \right) \right) \left( E_{\mu, \nu, \kappa}^{\gamma, \delta, \kappa} \varphi \right)(y_0; p). \tag{2.39}
\]

Adding (2.38) and (2.39), we get

\[
\frac{1}{h(\frac{1}{2^\alpha}) + mh(1 - \frac{1}{2^\alpha})} \left( \varphi \left( \frac{x_0 + my_0}{2} \right) f_{\omega - 1, \gamma} (x_0; p) + f_{\omega - 1, \gamma} (y_0; p) \right)
\]
\[
\left( e^{\gamma,k,c,\frac{x_0}{\mu_{\xi}+1,\omega_{\xi}}\varphi}(x_0; p) + (\gamma,k,c,\frac{x_0}{\mu_{\xi}+1,\omega_{\xi}}\varphi)(y_0; p) \right). \tag{2.40}
\]

Now, by combining (2.36) and (2.40), inequality (2.30) is established.

\begin{corollary}
If we put \( \xi = \eta \) in (2.30), then the following inequality is obtained:
\[
\frac{1}{h(t)} + mh(1-\frac{t}{2^\mu}) \left( J_{\xi-1,\gamma}(x_0; p) + J_{\xi-1,\gamma}(y_0; p) \right)
\leq \left( e^{\gamma,k,c,\frac{x_0}{\mu_{\xi}+1,\omega_{\xi}}\varphi}(x_0; p) + (\gamma,k,c,\frac{x_0}{\mu_{\xi}+1,\omega_{\xi}}\varphi)(y_0; p) \right)
\leq \left[ J_{\xi-1,\gamma}(x_0; p) + J_{\xi-1,\gamma}(y_0; p) \right] \left( y_0 - x_0 \right)^2
\times \left( \varphi(y_0) \int_0^1 h(z^\mu) \, dz + mh \left( \frac{x_0}{m} \right) \int_0^1 h(1-z^\mu) \, dz \right). \tag{2.41}
\end{corollary}

\begin{remark}
(i) If we take \( h(t) = t \) in (2.30), then we get [12, Theorem 2.3].
(ii) If we take \( h(t) = t \) in (2.41), then we get [12, Corollary 2.3].
(iii) If we take \( \alpha = 1 \) in (2.30), then we get [9, Theorem 3].
(iv) If we take \( \alpha = m = 1 \) and \( h(t) = t \) in (2.30), then we get [9, Corollary 3].
(v) If we take \( \alpha = 1, \omega = p = 0 \) in (2.30), then we get [13, Theorem 3].
(vi) If we take \( \alpha = m = 1, \omega = p = 0 \), and \( h(t) = t \) in (2.30), then we obtain [11, Theorem 3].
\end{remark}

### 3 Concluding remarks

In this work, we have studied fractional integral inequalities for a generalized convexity called \((\alpha, h - m)\)-convexity. The presented results provide bounds of fractional integral operators involving an extended Mittag-Leffler function and a new fractional Hadamard inequality for \((\alpha, h - m)\)-convex functions. The results proved in [8, 9, 11–14] are direct consequences of the theorems of this paper.

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### Author details

1 Institute of Computing Science and Technology, Guangzhou University, Guangzhou, 510006, China. 2 Department of Mathematics, COMSATS University Islamabad, Attock Campus, Attock, Pakistan. 3 Department of Mathematics, Air University, Islamabad, Pakistan. 4 General Studies Department Jubail Industrial College, Jubail Industrial City, Jubail 31961, Kingdom of Saudi Arabia.

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References

1. Agarwal, P.: Some inequalities involving Hadamard-type k-fractional integral operators. Math. Methods Appl. Sci. 11, 3882–3891 (2017)
2. Agarwal, P., Al-Mdallal, Q., Cho, Y.J.: Fractional differential equations for the generalized Mittag-Leffler function. Adv. Differ. Equ. 2018, 58 (2018)
3. Agarwal, P., Dragomir, S.S., Jlelli, M., Samet, B.: Fractional Differential Equations for the Generalized Mittag-Leffler Function. Advances in Mathematical Inequalities and Applications. Springer, Berlin (2018)
4. Agarwal, P., Jlelli, M., Tomar, M.: Certain Hermite–Hadamard type inequalities via generalized k-fractional integrals. J. Inequa. Appl. 2017, 58 (2017)
5. Anastassiou, G.A.: Generalized fractional Hermite–Hadamard inequalities involving m-convexity and (s,m)-convexity. Ser. Math. Inform. 28, 107–126 (2013)
6. Andrić, M., Farid, G., Pečarić, J.: A further extension of Mittag-Leffler function. Fract. Calc. Appl. Anal. 21, 1377–1395 (2018)
7. Arshad, M., Choi, J., Mubeen, S., et al.: A new extension of Mittag-Leffler function. Commun. Korean Math. Soc. 33, 549–560 (2018)
8. Chen, L., Farid, G., Butt, S.I., Akbar, S.B.: Boundedness of fractional integral operators containing Mittag-Leffler functions. Turkish J. Ineq. 7, 1931–1939 (2016)
9. Farid, G.: Some Riemann–Liouville fractional integral inequalities for convex functions. J. Anal. 27, 1095–1102 (2019)
10. Farid, G.: Bounds of fractional integral operators containing Mittag-Leffler function. UPB Sci. Bull. 81, 133–142 (2019)
11. Farid, G.: Bounds of Riemann–Liouville fractional integral operators. Comput. Methods Differ. Eq. (to appear)
12. Farid, G., Akbar, S.B., Rehman, S.U., Pečarić, J.: Boundedness of fractional integral operators containing Mittag-Leffler functions via (s,m)-convexity. AIMS Math. 5, 966–978 (2020)
13. Farid, G., Rehman, A.U., Ain, Q.U.: k-fractional integral inequalities of Hadamard type for (h – m)-convex functions. Comput. Methods Differ. Equ. 8, 119–140 (2020)
14. Gorenflo, R., Kilbas, A.A., Mainardi, F., Rogosin, S.V.: Mittag-Leffler Function, Related Topics and Applications. Springer, Berlin (2014)
15. Haubold, H.J., Mathai, A.M., Saxena, R.K.: Mittag-Leffler functions and their applications. J. Appl. Math. 2011, Article ID 298628 (2011)
16. Miculescu, V.: A Generalization of the Convexity. Seminar on Functional Equations, Approx. Convex. Cluj-Napoca, Romania (1993)
17. Mittag-Leffler, G.: Sur la nouvelle fonction Eα(x). C. R. Acad. Sci. Paris 137, 554–558 (1903)
18. Niculescu, C.P., Persson, L.E.: Convex Functions and Their Applications: A Contemporary Approach. Springer, Berlin (2006)
19. Ozdemir, M.E., Akdemri, A.O., Set, E.: On (h – m)-convexity and Hadamard-type inequalities. J. Math. Mech. 8, 51–58 (2016)
20. Pečarić, J., Proschan, F., Tong, Y.L.: Convex Functions, Partial Orderings, and Statistical Applications. Academic Press, New York (1992)
21. Prabhakar, T.R.: A singular integral equation with a generalized Mittag-Leffler function in the kernel. Yokohama Math. J. 19, 7–15 (1971)
22. Rahman, G., Baleanu, D., Qarashi, M.A., et al.: The extended Mittag-Leffler function via fractional calculus. J. Nonlinear Sci. Appl. 10, 4244–4253 (2017)
23. Roberts, A.W., Varberg, D.E.: Convex Functions. Academic Press, New York (1973)
24. Salim, T.O., Faraj, A.W.: A generalization of Mittag-Leffler function and integral operator associated with integral calculus. J. Fract. Calc. Appl. 3, 1–13 (2012)
25. Shukla, A.K., Prajapati, J.C.: On a generalization of Mittag-Leffler function and its properties. J. Math. Anal. Appl. 336, 797–811 (2007)
26. Srivastava, H.M., Tomovski, Z.: Fractional calculus with an integral operator containing generalized Mittag-Leffler function in the kernel. Appl. Math. Comput. 211, 198–210 (2009)
27. Ullah, S., Farid, G., Khan, K.A., et al.: Generalized fractional inequalities for quasi-convex functions. Adv. Differ. Equ. 2019, 15 (2019)
28. Varošanec, S.: On h-convexity. J. Math. Anal. Appl. 326, 303–311 (2007)
29. Wang, G., Agarwal, P., Chand, M.: Certain Grüss type inequalities involving the generalized fractional integral operator. J. Inequa. Appl. 2014, 147 (2014)