BRYUNO FUNCTION AND THE STANDARD MAP

ALBERTO BERRETTI AND GUIDO GENTILE

Abstract. For the standard map the homotopically non-trivial invariant curves of rotation number \( \omega \) satisfying the Bryuno condition are shown to be analytic in the perturbative parameter \( \varepsilon \), provided \( |\varepsilon| \) is small enough. The radius of convergence \( \rho(\omega) \) of the Lindstedt series – sometimes called critical function of the standard map – is studied and the relation with the Bryuno function \( B(\omega) \) is derived: the quantity \( |\log \rho(\omega) + 2B(\omega)| \) is proved to be bounded uniformly in \( \omega \).

1. Introduction

We continue the study, started in [1], of the radius of convergence of the Lindstedt series for the standard map, for rotation numbers close to rational values. We consider real rotation numbers \( \omega \) satisfying the Bryuno condition (see below), and study how the corresponding radius of convergence depends on the Bryuno function \( B(\omega) \), introduced by Yoccoz in [2].

The standard map is a discrete time, one-dimensional dynamical system generated by the iteration of the area-preserving – symplectic – map of the cylinder into itself \( T_\varepsilon : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R} \), given by:

\[
T_\varepsilon : \begin{cases} 
  x' = x + y + \varepsilon \sin x, \\
  y' = y + \varepsilon \sin x.
\end{cases}
\]  

Given a real rotation number \( \omega \in [0, 1) \), we can look for (homotopically non-trivial) invariant curves described parametrically by:

\[
\begin{align*}
  x &= \alpha + u(\alpha, \varepsilon; \omega), \\
  y &= \alpha + u(\alpha, \varepsilon; \omega) - u(\alpha - 2\pi \omega, \varepsilon; \omega),
\end{align*}
\]  

such that the dynamics induced in the variable \( \alpha \) is given by rotations by \( \omega \):

\[
\alpha' = \alpha + 2\pi \omega.
\]  

For irrational rotation numbers \( \omega \), by imposing that the average of \( u \) over \( \alpha \) is 0, the (formal) conjugating function \( u \) is unique and odd in \( \alpha \), and has a formal expansion – known as Lindstedt series – of the form:

\[
u(\alpha, \varepsilon) = \sum_{\nu \in \mathbb{Z}} u_{\nu}(\varepsilon) e^{i\nu \alpha} = \sum_{k \geq 1} u^{(k)}(\alpha) \varepsilon^k = \sum_{k \geq 1} \sum_{\nu \in \mathbb{Z}} u_{\nu}^{(k)} e^{i\nu \alpha} \varepsilon^k,
\]  

the coefficients \( u_{\nu}^{(k)} \) can be expressed graphically in terms of sums over trees as explained shortly (see also [1] and references quoted therein). The radius of convergence of the series (1.4), called sometimes the critical function of the standard
map, is defined as:

$$\rho(\omega) = \inf_{\alpha \in \mathbb{T}} \left( \limsup_{k \to \infty} |u^{(k)}(\alpha)|^{1/k} \right)^{-1}. \quad (1.5)$$

Given \(\omega\), let \(\{p_n/q_n\}\) be the sequence of convergents defined by the standard continued fraction expansion of \(\omega\), and let:

$$B_1(\omega) = \sum_{n=0}^{\infty} \frac{\log q_{n+1}}{q_n}. \quad (1.6)$$

The irrational number \(\omega \in [0, 1)\) satisfies the Bryuno condition if \(B_1(\omega) < \infty\); we also say that in this case \(\omega\) is a Bryuno number. After Yoccoz [2], we define on the irrational numbers the Bryuno function \(B(\omega)\) by the functional equation:

$$\begin{cases} 
B(\omega) = -\log \omega + \omega B(\omega^{-1}) & \text{for } \omega \in (0, \frac{1}{2}) \text{ and irrational,} \\
B(\omega + 1) = B(-\omega) = B(\omega).
\end{cases} \quad (1.7)$$

It can be proved that such functional equation has a unique solution in \(L^p, p \geq 1\); moreover \(B(\omega)\) is related to the series \(B_1(\omega)\) by the inequality:

$$|B(\omega) - B_1(\omega)| < C_1, \quad (1.8)$$

for some constant \(C_1\). See [2] and [3] for the proofs of these statements.

We prove the following theorem.

**Theorem.** Consider the standard map (1.1) and let \(\omega\) be an irrational number, \(\omega \in [0, 1)\), satisfying the Bryuno condition. Then the radius of convergence (1.5) satisfies the bound:

$$|\log \rho(\omega) + 2B(\omega)| \leq C_0, \quad (1.9)$$

where \(C_0\) is a constant independent on \(\omega\).

An analogous result was proved by Davie [4] for the semistandard map (where the nonlinear term \(\sin x\) in (1.1) is replaced by \(e^{ix}\)); in the same paper it was also shown that the upper bound in (1.9) holds:

$$\log \rho(\omega) + 2B(\omega) < C_2, \quad (1.10)$$

for some constant \(C_2\). In ref. [5] it was proved, by “phase space renormalization” arguments, that \(\forall \eta > 0 \exists C_3\), depending on \(\eta\), such that:

$$\log \rho(\omega) + (2 + \eta)B(\omega) > C_3. \quad (1.11)$$

So our theorem improves the result of [5] (using also a different, direct technique, taken from [6] — and inspired to the works [7] and [8] —, in some sense more elementary than the one of [5]) and proves for the standard map the conjecture first stated in [9].

The paper is organized as follows. In sect. 2 we introduce the formalism and give the scheme of the proof of the theorem, elucidating the major difficulties, due to the accumulation of small divisors in the Lindstedt series, and showing that, in absence of such a phenomenon, the proof could be carried out by a detailed analysis of the single terms of the series. In sect. 3 and 4, we shall see how to handle the small divisors problem, by showing that there are cancellation mechanisms, operating to
all perturbative orders between different terms of the Lindstedt series, which assure its convergence. Finally sect. 5 and 6 deal with the proof of the main technical lemmata used in the proof of the theorem.

2. Formalism: trees, clusters and resonances

As in [1], we can express graphically the coefficients \( u^{(k)}_\nu \) in (1.4) in terms of trees. We shall only recall the definitions used in this paper and set up the notations, leaving the full details of the tree expansion for our problem to [1] and the references quoted therein.

A tree \( \vartheta \) consists of a family of lines arranged to connect a partially ordered set of points – nodes –, with the lower nodes to the right. All the lines have two nodes at their extremes, except the highest which has only one node, the last node \( u_0 \) of the tree; the other extreme \( r \) will be called the root of the tree and it will not be regarded as a node.

We denote by \( \preceq \) the partial ordering relation between nodes defined as follows: given two nodes \( u, v \), we say that \( u \preceq v \) if \( u \) is along the path of lines connecting \( v \) to the root \( r \) of the tree – they could coincide: we say that \( u \prec w \) if they do not. So our trees are “rooted trees”, following the terminology of [10].

We assign to each line \( \ell \) joining two nodes \( u, u' \) an “arrow” pointing from the highest to the lowest node according to the order relation just defined; if \( u' \prec u \), we say that the line \( \ell \) exists from \( u \) and enters \( u' \). We write \( u'_0 = r \) even if, strictly speaking, \( r \) is not considered a node. For each node \( u \) there is a unique exiting line, and \( m_u \geq 0 \) entering lines; as there is a one-to-one correspondence between lines and nodes, we can associate to each node \( u \) the line \( \ell_u \) exiting from it. The line \( \ell_{u_0} \) exiting the last node \( u_0 \) will be called the root line. Note that each line \( \ell \) can be considered the root line of the subtree consisting of the nodes satisfying \( v \preceq u \), and \( u' \) will be the root of such tree. The order \( k \) of the tree is defined as the number of its nodes.

To each node \( u \in \vartheta \) we associate a mode label \( \nu_u = \pm 1 \), and define the momentum flowing through the line \( \ell_u \) as:

\[
\nu_{\ell_u} = \sum_{w \in u} \nu_w, \quad \nu_w = \pm 1;
\]

(2.1)

note that no line can have zero momentum, as \( u_0^{(k)} = 0 \).

While in [1] we could get along considering only two “scales”, we need a full multiscale decomposition of the momenta associated to each line.

Given a rotation number \( \omega \in [0, 1) \setminus \mathbb{Q} \), let \( \{p_n/q_n\} \) be the sequence of convergents coming from the standard continued fraction expansion of \( \omega \). For \( x \in \mathbb{R} \), let:

\[
||x|| = \inf_{\nu \in \mathbb{Z}} |x - p|
\]

(2.2)

be the distance of \( x \) from the nearest integer. Let now:

\[
\gamma(\nu) = 2(\cos 2\pi \nu - 1);
\]

(2.3)
then we have the estimate:

$$|\gamma(\nu)| = 2|\cos 2\pi \omega \nu - 1| \geq \Gamma ||\omega \nu||^2,$$

(2.4)

for some constant $\Gamma$.

We introduce a $C^\infty$ partition of unity in the following way. Let $\chi(x)$ a $C^\infty$, non-increasing, compact-support function defined on $\mathbb{R}^+$, such that:

$$\chi(x) = \begin{cases} 1 & \text{for } x \leq 1, \\ 0 & \text{for } x \geq 2, \end{cases}$$

(2.5)

and define for each $n \in \mathbb{N}$:

$$\chi_n(x) = \chi(96q_n x) - \chi(96q_{n+1} x), \text{ for } n \geq 1.$$

(2.6)

Then for each line $\ell$ set:

$$g(\nu_{\ell}) = 1_{\nu_{\ell}}(\nu_{\ell}) = \sum_{n=0}^{\infty} \frac{\chi_n(\nu_{\ell})}{\gamma(\nu_{\ell})} = \sum_{n=0}^{\infty} g_n(\nu_{\ell}),$$

(2.7)

and call $g_n(\nu_{\ell})$ the propagator on scale $n$.

Given a tree $\vartheta$, we can associate to each line $\ell$ of $\vartheta$ a scale label $n_{\ell}$, using the multiscale decomposition (2.7) and singling out the summands with $n = n_{\ell}$. We shall call $n_{\ell}$ the scale label of the line $\ell$, and we shall say also that the line $\ell$ is on scale $n_{\ell}$.

**Remark 1.** Given a value $\nu_{\ell}$ there can be at most two possible – consecutive – values of $n$ such that the corresponding $\chi_n(\nu_{\ell})$ are not vanishing. This means that at most only two summands of the infinite series (2.7) really appear; nevertheless keeping all terms is more convenient, in order to have a label to characterize the “size” of the “propagators” $g(\nu_{\ell})$.

**Remark 2.** Note that if a line $\ell$ has momentum $\nu_{\ell}$ and scale $n_{\ell}$, then:

$$\frac{1}{96q_{n_{\ell}+1}} \leq ||\omega \nu_{\ell}|| \leq \frac{1}{48q_{n_{\ell}}},$$

(2.8)

provided that one has $\chi_{n_{\ell}}(\nu_{\ell}) \neq 0$.

A group $\mathcal{G}$ of transformations acts on the trees, generated by the permutations of all the subtrees emerging from each node with at least one entering line: $\mathcal{G}$ is therefore a cartesian product of copies of the symmetric groups of various orders. Two trees that can be transformed into each other by the action of the group $\mathcal{G}$ are considered identical.

Denote by $\mathcal{T}_{\nu,k}$ the set of trees, with nonvanishing value, of order $k$ and total momentum $\nu_{\ell_{u_0}} = \nu$, if $u_0$ is the last node of the tree. The number of elements in $\mathcal{T}_{\nu,k}$ is bounded by $2^k \cdot 2^k \cdot 2^{2k} = 2^{4k}$: the number of semitopological trees (see [1]) of order $k$ is bounded by $2^{2k}$, and there are two possible values for the mode label of each node and two possible values for the scale label of each line.

---

1 The number of semitopological trees can be bounded by the number of one-dimensional random walks with $2k - 1$ steps.
Then, as in [1] – to which we refer for more details and figures – one finds:

\[ v^{(k)}_\vartheta = \frac{1}{2\pi} \sum_{\vartheta \in T_{c,k}} \text{Val}(\vartheta), \quad \text{Val}(\vartheta) = -i \left[ \prod_{u \in \vartheta} \frac{\nu_u^{m_u+1}}{m_u!} \right] \left[ \prod_{\ell \in \vartheta} g_{n_{\ell}}(\nu_{\ell}) \right]; \quad (2.9) \]

the factors \( g_{n_{\ell}}(\nu_{\ell}) \) above are called propagators of small divisors on scale \( n_{\ell} \), and the quantity \( \text{Val}(\vartheta) \) will be called the value of the tree \( \vartheta \).

We define now the main combinatorial tools.

**Definition (Cluster).** Given a tree \( \vartheta \), a cluster \( T \) of \( \vartheta \) on scale \( n \) is a maximal connected set of lines of lines on scale \( \leq n \) with at least one line on scale \( n \). We shall say that such lines are internal to \( T \), and write \( \ell \in T \) for an internal line \( T \).

A node \( u \) is called internal to \( T \), and we write \( u \in T \), if at least one of its entering lines or exiting line is in \( T \). Each cluster has an arbitrary number \( m_T \geq 0 \) of entering lines but only one exiting line; we shall call external to \( T \) the lines entering or exiting \( T \) (which are all on scale \( > n \)). We shall denote with \( n_T \) the scale of the cluster \( T \), with \( n^o_T \) the minimum of the scales of the lines entering \( T \), with \( n^I_T \) the scale of the line exiting \( T \) and with \( k_T \) the number of nodes in \( T \).

Note that, despite the name, not all lines outside \( T \) are “external” to it: only those lines outside \( T \) which enter or exit \( T \) are external to it. On the contrary a line inside \( T \) is said to be “internal” to it. The use of such a terminology is inherited from Quantum Field Theory.

**Definition (Resonance).** Given a tree \( \vartheta \), a cluster \( V \) of \( \vartheta \) will be called a resonance with resonance-scale \( n = n^R_V \equiv \min\{n^I_V, n^o_V\} \), if:

1. the sum of the mode labels of its nodes is 0:
   \[ \nu_V = \sum_{u \in V} \nu_u = 0; \quad (2.10) \]
2. all the lines entering \( V \) are on the same scale except at most one, which can be on a higher scale;
3. \( n^I_V \leq n^o_V \) if \( m_V \geq 2 \), and \( |n^I_V - n^o_V| \leq 1 \) for \( m_V = 1 \);
4. \( k_T < q_n \);
5. \( m_V = 1 \) if \( q_{n+1} \leq 4q_n \);
6. if \( q_{n+1} > 4q_n \) and \( m_V \geq 2 \), denoting by \( k_0 \) the sum of the orders of the subtrees of order \( < q_{n+1}/4 \) entering \( V \), either
   - there is a only one subtree of order \( k_1 \geq q_{n+1}/4 \) entering \( V \) and \( k_1 + k_0 + k_T \geq q_{n+1}/4 \), \( k_0 < q_{n+1}/8 \), or
   - there is no such subtree and \( k_0 + k_T < q_{n+1}/4 \).

**Remark 3.** Note that for any resonance \( V \) one has \( n^R_V \geq n_V + 1 \), if \( n_V \) is the scale of the resonance \( V \) as a cluster. As in [11] we use the notation with a hyphen for the resonance-scale to avoid confusion between \( n^R_V \) and \( n_V \).

**Remark 4.** One would be tempted to give a simpler definition of resonance (for instance, by imposing only condition 1 to the cluster \( V \)). This temptation should be resisted, as it would make impossible to exploit the cancellations leading to the improvement of the bound discussed at the end of this section (in fact, no relation
would continue to subsist between momenta and scale labels and factorials would
arise from counting the summands generated by the renormalization procedure
described in sect. 4). On the other hand we shall see in sect. 5 that no problems
should arise if no resonances – exactly as they defined above – could appear.

In the following we shall need to introduce trees in which it can happen that a
line $\ell$ is on a scale $n_\ell$ and yet its momentum does not satisfy (2.8). The value of any
such tree $\vartheta$ is vanishing as $\chi_{n_\ell}(||\omega\nu_\ell||) = 0$; nevertheless it will be useful to write
$\text{Val}(\vartheta)$ as sum of two (possibly) nonvanishing terms: one of them will be used to
cancel terms arising from other tree values, so it will disappears, while the other one
is left and has to be bounded. This means that we shall have to deal with trees in
which there are lines $\ell$ with momentum $\nu_\ell$ and scale $n_\ell$ which do not satisfy (2.8).
What will be shown to hold is that for such lines a bound similar to (2.8), though
weaker, still holds; more precisely, a line $\ell$ with momentum $\nu_\ell$ will have only scales
$n_\ell$ such that

$$\frac{1}{768q_{n+1}} \leq ||\omega\nu_\ell|| \leq \frac{1}{8q_{n_\ell}},$$  \hspace{1cm} (2.11)$$

and, for fixed $\nu_\ell$, the number of possible scales to associate to $\ell$ is bounded by an
absolute constant.

As (2.11) is implied by (2.8), even for trees with nonvanishing value we shall use
that if a line is on scale $n_\ell$ then (2.11) holds.

Then, if $N_n(\vartheta)$, $n \in \mathbb{N}$, denotes the number of lines on scale $n$ in $\vartheta$, we have
trivially for a given tree $\vartheta$ the bound:

$$|\text{Val}(\vartheta)| \leq D_1 \prod_{n=0}^{\infty} (768q_{n+1})^{2N_n(\vartheta)},$$  \hspace{1cm} (2.12)$$

for some constant $D_1$ (actually $D_1 = 1/\Gamma$; see (2.4), (2.9) and (2.11)).

Given a tree $\vartheta$, let us denote with $N_n^R(\vartheta)$ the number of resonances with resonance-scale $n$ and by $P_n(\vartheta)$ the number of
resonances on scale $n$. Of course $N_0^R = 0$.

Remark 5. Note that the number $N_n^R(\vartheta)$ of resonances with resonance-scale $n$ can be counted by counting the number of lines exiting resonances with resonance-scale $n$; analogously $P_n(\vartheta)$ can be counted by counting the number of lines exiting resonances on scale $n$. Such counts will be performed in sect. 5.

The following simple lemmata contain all the arithmetic we shall need, and are
basically adapted from [4].

Lemma 1 (Davie’s lemma). Given $\nu \in \mathbb{Z}$ such that $||\omega\nu|| \leq 1/4q_n$, then

1. either $\nu = 0$ or $|\nu| \geq q_n$,
2. either $|\nu| \geq q_{n+1}/4$ or $\nu = sq_n$ for some integer $s$.

Lemma 2. If a tree $\vartheta$ has $k < q_n$ nodes, then $N_n(\vartheta) = 0$ and $P_{n-1}(\vartheta) = 0$.

Lemma 3. For any irrational number $\omega \in [0,1)$:

$$\sum_{n=0}^{\infty} \frac{\log q_n}{q_n} \leq D_2,$$  \hspace{1cm} (2.13)$$
for a constant $D_2$; here $q_n$ are the denominators of the convergents of $\omega$.

**Lemma 4.** Given a momentum $\nu$ such that

$$\frac{1}{768q_{n+1}} \leq ||\omega\nu|| \leq \frac{1}{8q_n},$$

(2.14)

then one can have $\chi_n'(||\omega\nu||) \neq 0$ only for $n'$ such that $n - 8 \leq n' \leq n + 8$.

**Proof of lemma 1.** If $\{q_n\}$ are the denominators of the convergents of $\omega$, then (see e.g. [12], Ch. 1, §3):

$$\frac{1}{2q_{n+1}} < ||\omega q_n|| < \frac{1}{q_{n+1}},$$

(2.15)

and:

$$\forall |\nu| < q_{n+1}, |\nu| \neq q_n: \quad ||\omega\nu|| > ||\omega q_n||.$$  

(2.16)

To prove 1 note that if $\nu = 0$ nothing has to be proved: so we assume $\nu \neq 0$. If $|\nu| < q_n$, by (2.16) and (2.15), $||\omega\nu|| \geq ||\omega q_{n-1}|| > 1/2q_n$, so that $||\omega\nu|| < 1/4q_n$ implies $|\nu| \geq q_n$, proving the first assertion of lemma 1.

To prove 2, again if $\nu = 0$ nothing has to be proved (and $s = 0$): so we assume $\nu \neq 0$, and proceed by reductio ad absurdum. If $0 < \nu < q_{n+1}/4$ and there does not exist any $s \in \mathbb{Z}$ such that $\nu = sq_n$, then one has $\nu = mq_n + r$, with $0 < r < q_n$ and $m < q_{n+1}/4q_n$; then, by (2.15), $||\omega mq_n|| \leq m||\omega q_n|| < m/q_{n+1} < 1/4q_n$, and, by (2.16), $||\omega r|| \geq ||\omega q_{n-1}|| > 1/2q_n$, as $r \neq 0$; so $||\omega\nu|| \geq ||\omega r|| - ||\omega mq_n|| > 1/4q_n$.

The case $0 > \nu > -q_{n+1}/4$ is identical as $|| \cdot ||$ is even.

**Proof of lemma 2.** If $k < q_n$, then for any $\ell \in \varnothing$ one has $|\nu\ell| \leq k < q_n$, so that, by (2.15) and (2.16), $||\omega\nu\ell|| \geq ||\omega q_{n-1}|| > 1/2q_n$, hence $n_\ell < n$ and so $N_n(\varnothing) = 0$.

If there are no lines on scale $n$, it is impossible to form a cluster on scale $n - 1$, a fortiori a resonance.

**Proof of lemma 3.** The denominators of the convergents $\{q_n\}$ of $\omega$ satisfy $q_0 = 1$, $q_1 \geq 1$ and $q_n \geq 2q_{n-2}$ for any $n \geq 2$. So we can write:

$$\sum_{n=0}^{\infty} \frac{\log q_n}{q_n} = \sum_{n=0}^{\infty} \frac{\log q_{2n}}{q_{2n}} + \sum_{n=0}^{\infty} \frac{\log q_{2n+1}}{q_{2n+1}},$$

(2.17)

using the fact that, for $x \geq e$, $x^{-1} \log x$ is decreasing, we obtain easily:

$$\sum_{n=0}^{\infty} \frac{\log q_n}{q_n} \leq 3 \max_{x \geq 1} \left\{ \frac{\log x}{x} \right\} + 2 \log 2 \sum_{k=2}^{\infty} \frac{k}{2^k} = 3(e^{-1} + \log 2) \equiv D_2,$$

(2.18)

which also gives an explicit value for the constant $D_2$.

**Proof of lemma 4.** Simply use that $q_{n+1} \geq q_n$ and $q_{n+2} \geq 2q_n$ for all $n \geq 0$, to deduce that $1/48q_{n+9} < 1/768q_{n+1}$ and $1/96q_{n-8} > 1/8q_n$.

The following “counting” lemma is the main result stated in this section, and it can be considered an adaption and extension of lemma 2.3 in [4]. We postpone its proof to sect. 5.
Lemma 5. Given a tree $\vartheta$, let $M_n(\vartheta) = N_n(\vartheta) + P_n(\vartheta)$. Then:

$$M_n(\vartheta) \leq \frac{k}{q_n} + \frac{8k}{q_{n+1}} + N_n^R(\vartheta), \quad (2.19)$$

where $k$ is the order of $\vartheta$.

Therefore we can rewrite the bound (2.12) on the tree value as:

$$|\text{Val}(\vartheta)| \leq D_1^k \prod_{n=0}^{\infty} (768q_{n+1})^{2(M_n(\vartheta) - P_n(\vartheta))} \leq D_1^k \prod_{n=0}^{\infty} (768q_{n+1})^{2(k/q_n + 8k/q_{n+1}) + N_n^R(\vartheta) - P_n(\vartheta)). \quad (2.20)$$

Note that at this point it would be very easy to prove the lower bound in (1.9) for the semistandard map and, by simple modifications of the same scheme, for Siegel problem, since in these cases no resonances appear. On the contrary, in the more difficult case of the standard map we lack, for the moment, a control on the number $N_n^R(\vartheta)$ of resonances in $\vartheta$ with resonance-scale $n$.

In sect. 3 and 4 we shall see how to improve the bound on the sum over the trees of fixed order and total momentum, in order to prove the theorem stated in sect. 1. We postpone to forthcoming sections the proofs, limiting ourselves here to a heuristic discussion in order to give an idea of the structure of the proof.

We perform a suitable resummation – described in sect. 3 and 4 – whose consequence is that, for each resonance $V$, it is as if one of the external lines on scale $n_V^R$ contributed $(768q_{n_V^R+1})^2$ instead of $(768q_{n_V^R+1})^2$. To obtain such a result, we shall perform on trees transformations which will lead to the introduction of new trees: so we extend $T_{\nu,k}$ to a larger set $T_{\nu,k}^*$. However we shall prove that the value of each single tree in $T_{\nu,k}^*$ still admits the bound (2.20) – even if, unlike the values of the trees in $T_{\nu,k}$, it fails to satisfy the same bound with 768 replaced with 96 – and the number of elements in $T_{\nu,k}^*$ is bounded by a constant to the power $k$ (i.e. no bad counting factors, like factorials, appear). Then we obtain, for the sum of the resummed trees, a bound of the form (2.20) with:

$$\prod_{n=0}^{\infty} (768q_{n+1})^{2N_n^R(\vartheta)}$$

replaced with:

$$D_3^k \prod_{n=0}^{\infty} (768q_{n+1})^{2P_n(\vartheta)},$$

for some constant $D_3$. By using that the number of trees in $T_{\nu,k}^*$ will be shown to be bounded by a constant to the power $k$, we obtain, for some constants $D_4, D_5$:

$$|u^{(k)}(\alpha)| \leq \sum_{|\nu| \leq k} \sum_{\vartheta \in T_{\nu,k}} |\text{Val}(\vartheta)| \leq \sum_{|\nu| \leq k} \sum_{\vartheta \in T_{\nu,k}} |\text{Val}(\vartheta)| \leq D_4^k \prod_{n=0}^{\infty} (768q_{n+1})^{2k/q_n + 16k/q_{n+1}} \leq D_5^k \exp \left[ 2k \sum_{n=0}^{\infty} \left( \frac{\log q_{n+1}}{q_n} + \frac{8 \log q_{n+1}}{q_{n+1}} \right) \right], \quad (2.21)$$
BRYUNO FUNCTION AND THE STANDARD MAP

which, by making use of lemma 3, gives:

$$\log \rho(\omega) + 2B_1(\omega) \geq -16D_2 - \log D_5.$$  \hspace{1cm} (2.22)

By making rigorous the above discussion in sect. 3 and 4, we shall complete the proof of the theorem, since the bound from above was already proved in [4].

3. Renormalization of resonances: set-up and the first step

Given a tree \( \vartheta \), let us consider maximal resonances, \textit{i.e.} resonances \textit{not} contained in any larger resonance; let us call them \textit{first generation resonances}. Inside the first generation resonances let us consider the “next maximal” resonances, \textit{i.e.} the resonances \textit{not} contained in any larger resonance except first generation resonances, and let us call them \textit{second generation resonances}. We can define in this way \textit{j-th generation resonances}, for \( j \geq 2 \), as resonances which are maximal within \((j-1)\)-th generation resonances.

Let \( V \) be the set of all resonances of a tree \( \vartheta \), and \( V_j \) the set of all resonances of \( j \)-th generation, with \( j = 1, \ldots, G \), for some integer \( G \), depending on \( \vartheta \).

Given a tree \( \vartheta \) and a resonance \( V \in V_j \) with \( m_V \) entering lines, define \( V_0 \) as the set of nodes and lines internal to \( V \) and outside any resonances contained in \( V \). Let \( L_V = \{ \ell_1, \ldots, \ell_{m_V} \} \) be the set of entering lines of \( V \); we define \( L^R_V \) as the subset of the lines in \( L_V \) which enter some resonances of higher generation contained inside \( V \) and \( L^0_V = L_V \setminus L^R_V \) as the subset of lines in \( L_V \) which enter nodes in \( V_0 \).

For any line \( \ell_m \in L^R_V \), let \( V(\ell_m) \) be the minimal resonance containing the node which the line \( \ell_m \) enters (\textit{i.e.} the highest generation resonance containing such a node) and \( V_0(\ell_m) \) the set of nodes and lines internal to \( V(\ell_m) \) and outside resonances contained in \( V(\ell_m) \). Define:

$$\tilde{V}(V) = \{ \tilde{V} \subset V : \tilde{V} = V(\ell_m) \text{ for some } \ell_m \in L^R_V \}. \hspace{1cm} (3.1)$$

Call \( m_{V_0} \) the number of lines in \( L^0_V \). The number of lines in \( L^R_V \) entering the same resonance \( \tilde{V} \in \tilde{V}(V) \) is not arbitrary: it is always 1, as it is shown by the following lemma.

\textbf{Lemma 6.} For \( j \geq 1 \), given a resonance \( W \in V_{j+1} \) contained inside a resonance \( V \in V_j \), only one among the entering lines \( W \) can also enter \( V \).

\textit{Proof.} By item 3 of the definition of resonance one has \( n^R_W \leq n_V \), otherwise \( V \) would be a cluster on scale \( < n^R_W \), so that all the lines external to \( W \) would be also external to \( V \) and \( V = W \), while we assumed that \( V \subset W \). Then if a line \( \ell \) enter both \( V \) and \( W \), one must have \( n_\ell > n^R_W \). But, by item 2 in the definition of resonance, all lines external to \( W \) have the same scale \( n^R_W \) except at most one. \( \Box \)

We define the \textit{resonance family} \( \mathcal{F}_V(\vartheta) \) of \( V \in \vartheta \) as the set of trees obtained from \( \vartheta \) by the action of a group of transformations \( P_V \) on \( \vartheta \), generated by the following operations:
1. Detach the line \( \ell_1 \), then if \( \ell_1 \in L^R_V \) reattach it to all nodes internal to \( V_0(\ell_1) \), while if \( \ell_1 \in L^U_V \) reattach it to all nodes in \( V_0 \); for each tree so obtained, do the same operations with the line \( \ell_2 \) and so forth for each line entering the resonance.

2. In a given tree, each node \( u \in V \) will have \( m_u \) entering lines, of which \( s_u \) are inside \( V \) and \( r_u = m_u - s_u \) are outside \( V \) (i.e. are entering lines of \( V \)); then we can apply to the set of lines entering \( u \) a transformation in the group obtained as the quotient of the group of permutations of the \( m_u \) lines entering \( u \) by the groups of permutations of the \( s_u \) internal entering lines and of permutations of the \( r_u \) entering lines outside \( V \); in this way for each node \( u \in V \) a number of trees equal to:
\[
\binom{m_u}{s_u} = \frac{m_u!}{s_u!r_u!}
\]
is obtained.

3. Flip simultaneously all the mode labels of the nodes internal to \( V \).

We shall call renormalization transformations (of type 1, 2, 3) the operations described above.

Remark 6. Note that in all such transformations the scales are not changed (by definition) and the set of resonance \( V \) remains the same (by construction). On the contrary the momenta flowing through the lines can change (because of the shift of the lines entering resonances) and in particular one can have for some lines \( \ell \), \( \chi_{n_{\ell}}(||\omega\nu_{\ell}||) = 0 \), if \( \nu_{\ell} \) is the modified momentum flowing through \( \ell \).

Remark 7. The definition of resonance families is aimed at grouping together the trees between which one will look for compensations, but in doing so one has to avoid overcountings. In fact, to each tree \( \vartheta \) we associate a value \( \text{Val}(\vartheta) \) according to (2.9); when applying the transformations of the group \( P_V \) on the tree \( \vartheta \), the same tree \( \vartheta' \) can be obtained, in general, in several ways; however, it has to be counted once. This means that \( P_V \), as a group, defines an equivalence class, and only inequivalent elements obtained through the transformations defining \( P_V \) have to be retained.

Let us call \( F_{V_1}(\vartheta) \) the family obtained by the composition of all transformations defining the resonance families \( F_{V_1}(\vartheta), V_1 \in V_1 \).

For any tree \( \vartheta_1 \in F_{V_1}(\vartheta) \), let \( V_2 \) be a resonance in \( V_2 \) and let us define the resonance family \( F_{V_2}(\vartheta_1) \) of \( V_2 \) in \( \vartheta_1 \) as the set of trees obtained from \( \vartheta_1 \) by the action of the group of transformations \( P_{V_2} \). The composition of all transformations defining the resonance families \( F_{V_2}(\vartheta_1) \), for all \( \vartheta_1 \in F_{V_1}(\vartheta) \) and all \( V_2 \in V_2 \), gives a family that we shall denote by \( F_{V_2}(\vartheta) \).

We continue by considering resonances of 3-rd generation, and so on until the \( G \)-th generation resonances are reached. At the end we shall have a family \( F(\vartheta) \) of trees obtained by the composition of all transformations of the groups \( P_V, V \in V \), defined recursively through the application of the renormalization transformations corresponding to resonances \( V \in V_j \) to all trees \( \vartheta' \) belonging to the family \( F_{V_{j-1}}(\vartheta) \).
Remark 8. Given a tree \( \vartheta \in \mathcal{T}_{\nu,k} \) and a family \( \mathcal{F}(\vartheta) \), when considering another tree \( \vartheta' \in \mathcal{F}(\vartheta) \) with nonvanishing value \( \text{Val}(\vartheta') \), the same family \( \mathcal{F}(\vartheta') = \mathcal{F}(\vartheta) \) is obtained (by construction). Note however that \( \mathcal{F}(\vartheta) \) can contain also trees with vanishing values, as they can have lines \( \ell \) such that \( \chi_{n_{\ell}}(\|\omega_{\nu}\|) = 0 \) (see remark 6).

Define also \( \mathcal{N}_{\mathcal{F}(\vartheta)} \) the number of trees in \( \mathcal{F}(\vartheta) \) whose value is not vanishing; of course \( \mathcal{N}_{\mathcal{F}(\vartheta)} \leq |\mathcal{F}(\vartheta)| \), if \( |\mathcal{F}(\vartheta)| \) is the number of elements in \( \mathcal{F}(\vartheta) \).

Write:

\[
\sum_{\vartheta \in \mathcal{T}_{\nu,k}} \text{Val}(\vartheta) = \sum_{\vartheta \in \mathcal{T}_{\nu,k}} \frac{1}{\mathcal{N}_{\mathcal{F}(\vartheta)}} \sum_{\vartheta' \in \mathcal{F}(\vartheta)} \text{Val}(\vartheta') = \sum_{\vartheta \in \mathcal{T}_{\nu,k}} \frac{1}{|\mathcal{F}(\vartheta)|} \sum_{\vartheta' \in \mathcal{F}(\vartheta)} \text{Val}(\vartheta'),
\]

where the factors \( \mathcal{N}_{\mathcal{F}(\vartheta)} \) and \( |\mathcal{F}(\vartheta)| \) have been introduced in order to avoid over-countings (see remark 8) and the last sum implicitly defines the set \( \mathcal{T}^*_{{\nu,k}} \); so \( \mathcal{T}^*_{{\nu,k}} \) is the set of inequivalent trees in \( \bigcup_{\vartheta \in \mathcal{T}_{\nu,k}} \mathcal{F}(\vartheta) \).

Consider a tree \( \vartheta \in \mathcal{T}^*_{{\nu,k}} \). Then \( \vartheta \in \mathcal{F}(\vartheta_0) \), for some tree \( \vartheta_0 \in \mathcal{T}_{\nu,k} \); however one has to bear in mind that \( \vartheta \), unlike \( \vartheta_0 \), could vanish.

Given a tree \( \vartheta \in \mathcal{T}^*_{{\nu,k}} \), if \( V \) is a first generation resonance, we define its resonance factor \( \mathcal{V}_V(\vartheta) \) as its contribution to the value of the tree \( \vartheta \), namely:

\[
\mathcal{V}_V(\vartheta) = \left[ \prod_{u \in V} \frac{\mu_{\nu_u}^{m_u} + 1}{m_u!} \right] \prod_{\ell \in V} g_{\nu_\ell}(\nu_\ell),
\]

which of course depends on the subset of \( \vartheta \) outside the resonance \( V \) only through the momenta of the entering lines of \( V \). Given a node \( u \in V \), let us denote with \( \mathcal{E}_u \) the set of lines entering \( V \) such that they end into nodes preceding \( u \).

For future notational convenience, we rewrite (3.3) as:

\[
\mathcal{V}_V(\vartheta) = U_V(\vartheta) L_V(\vartheta), \quad u_V(\vartheta) = \prod_{u \in V} \frac{\nu_u^{m_u} + 1}{m_u!}, \quad L_V(\vartheta) = \prod_{\ell \in V} g_{\nu_\ell}(\nu_\ell).
\]

In the following, we shall consider the quantities \( \omega \nu, \nu \in \mathbb{Z} \), modulo 1, and shall continue to use the symbol \( \omega \nu \) to denote the representative of the equivalence class within the interval \((-1/2,1/2] \).

For any node \( u \) contained in a resonance \( V \), we shall write:

\[
\nu_{\ell_u} = \nu_{\ell_u}^0 + \sum_{\ell' \in \mathcal{E}_u} \nu_{\ell'}, \quad \nu_{\ell_u}^0 = \sum_{w \in V \atop w \in \mathcal{E}_u} \nu_{\ell_u}^0,
\]

where the set \( \mathcal{E}_u \) was defined after (3.3).

We shall consider the resonance factor (3.3) as a function of the quantities \( \mu_1 = \omega \nu_{\ell_1}, \ldots, \mu_{m_V} = \omega \nu_{\ell_{m_V}} \), where \( \nu_{\ell_1}, \ldots, \nu_{\ell_{m_V}} \) are the momenta flowing through the lines \( \ell_1, \ldots, \ell_{m_V} \) entering \( V \). More precisely, we let:

\[
\mathcal{V}(\vartheta) \equiv \mathcal{V}_V(\vartheta; \omega \nu_{\ell_1}, \ldots, \omega \nu_{\ell_{m_V}}),
\]

and we write:

\[
\mathcal{V}_V(\vartheta; \omega \nu_{\ell_1}, \ldots, \omega \nu_{\ell_{m_V}}) = \mathcal{L} \mathcal{V}_V(\vartheta; \omega \nu_{\ell_1}, \ldots, \omega \nu_{\ell_{m_V}}) + \mathcal{R} \mathcal{V}_V(\vartheta; \omega \nu_{\ell_1}, \ldots, \omega \nu_{\ell_{m_V}}),
\]

where

\[
\mathcal{L} \mathcal{V}_V(\vartheta; \omega \nu_{\ell_1}, \ldots, \omega \nu_{\ell_{m_V}}) = \mathcal{V}_V(\vartheta; \omega \nu_{\ell_1}, \ldots, \omega \nu_{\ell_{m_V}}),
\]

and

\[
\mathcal{R} \mathcal{V}_V(\vartheta; \omega \nu_{\ell_1}, \ldots, \omega \nu_{\ell_{m_V}}) = \mathcal{V}_V(\vartheta; \omega \nu_{\ell_1}, \ldots, \omega \nu_{\ell_{m_V}}).
\]
where:
\[
\mathcal{L}V(\vartheta; \omega \nu_1, \ldots, \omega \nu_{m_V}) = \mathcal{V}_V(\vartheta; 0, \ldots, 0) + \sum_{m=1}^{m_V} \omega \nu_m \frac{\partial}{\partial \mu_m} \mathcal{V}_V(\vartheta; 0, \ldots, 0)
\]
(3.8)
is the localized part of the resonance factor, or localized resonance factor, while:
\[
\mathcal{R}V(\vartheta; \omega \nu_1, \ldots, \omega \nu_{m_V}) = \sum_{m,m'=1}^{m_V} \omega \nu_m \omega \nu_{m'} \cdot 
\int_0^1 dt \left(1 - t\right) \frac{\partial^2}{\partial \mu_m \partial \mu_{m'}} \mathcal{V}_V(\vartheta; t \omega \nu_1, \ldots, t \omega \nu_{m_V})
\]
(3.9)
is the renormalized part of the resonance factor, or renormalized resonance factor. In (3.7) \(L\) is called the localization operator and \(R = 1 - L\) is called the renormalization operator. Using the notations (3.4), we can write:
\[
\mathcal{L}V(\vartheta) = U_V(\vartheta) \mathcal{L}_V(\vartheta), \quad \mathcal{R}V(\vartheta) = U_V(\vartheta) \mathcal{R}_V(\vartheta),
\]
(3.10)
as only the factors in \(L_V(\vartheta)\) depend on the momenta flowing through the lines entering the resonance \(V\).

**Remark 9.** Note that in the localized part (3.8) the momentum \(\nu_\ell\) flowing through any line \(\ell\) internal to \(V\) is changed into \(\nu_0^\ell\) (see (3.5)).

Then we perform the renormalization transformations in \(P_V\) described above. By remark 9, for all trees obtained by applying the group \(P_V\) the contribution to the localized resonance factor arising from the \(L_V(\vartheta)\) term in (3.4) is the same, i.e.:
\[
\mathcal{L}_V(\vartheta) = \mathcal{L}_V(\vartheta'), \quad \forall \vartheta' \in \mathcal{F}_V(\vartheta),
\]
(3.11)
so that we can consider:
\[
\sum_{\vartheta' \in \mathcal{F}_V(\vartheta)} \mathcal{L}_V(\vartheta').
\]
(3.12)
The sum over all the trees in the resonance family \(\mathcal{F}_V(\vartheta)\) of the localized resonance factors produces zero, so that only the renormalized part has to be taken into account. The proof of this assertion is similar to the proof of the analogous statement in [1], and it is given in sect. 6 as a particular case of the proof of the more general statement in lemma 8 below.

Then only the second order terms have to be taken into account in (3.7). This leads to the following expression for the renormalized resonance factor:
\[
\mathcal{R}V(\vartheta) = U_V(\vartheta) \sum_{m,m'=1}^{m_V} \omega \nu_m \omega \nu_{m'} \cdot 
\left[ \sum_{\ell_1, \ell_2 \in V} \left( \frac{\partial}{\partial \mu_m} g_{n_{\ell_1}}(\nu_{\ell_1}) \right) \left( \frac{\partial}{\partial \mu_{m'}} g_{n_{\ell_2}}(\nu_{\ell_2}) \right) \left( \prod_{\ell \in V} g_{n_\ell}(\nu_\ell) \right) \right] + 
\left[ \sum_{\ell \in V} \left( \frac{\partial}{\partial \mu_m} g_{n_{\ell}}(\nu_{\ell}) \right) \left( \prod_{\ell \in V} g_{n_\ell}(\nu_\ell) \right) \right],
\]
(3.13)
from the very definition of the renormalized resonance factor (3.9), by noting that the two derivatives in (3.9) act either on two distinct propagators (the sum with \( \ell^2_V \neq \ell^2_V \) in (3.13)) or on the same propagator (the sum with only one line \( \ell_V \) in (3.13)).

Note that it can happen that \( \vartheta \in \mathcal{F}_V(\vartheta_0) \), for some tree \( \vartheta_0 \in \mathcal{T}_{\nu,k} \), i.e. for some tree \( \vartheta_0 \) with nonvanishing value, while \( V_V(\vartheta) = 0 \) (correspondingly there does not exist any tree in \( \mathcal{T}_{\nu,k} \) of that shape associated with the given choice of mode and scale labels). The tree \( \vartheta \) is obtained from \( \vartheta_0 \) through a transformation of \( \mathcal{P}_V \), so that there is a correspondence between the lines of \( \vartheta_0 \) and the lines of \( \vartheta \); we shall say that the lines are \emph{conjugate}. The tree \( \vartheta \) inherits the scale labels of the tree \( \vartheta_0 \), i.e. the lines in \( \vartheta \) have the same scales of the conjugate lines of \( \vartheta_0 \). So it can happen that in \( \vartheta_0 \) some line internal to \( V \) has a scale \( n_\ell \) and a momentum \( \tilde{\nu}_\ell \) such that \( \chi_n(\|\omega\tilde{\nu}_\ell\|) \neq 0 \), while the momentum \( \nu_\ell \) of the line \( \ell \) seen as a line of \( \vartheta \) (i.e. of the line of \( \vartheta \) conjugate to the line \( \ell \) of \( \vartheta_0 \)) is such that \( \chi_n(\|\omega\nu_\ell\|) = 0 \) (see remark 8).

This means that for such a line (2.8) does not hold. Nevertheless, as anticipated in remark 6, one finds that the momentum \( \nu_\ell \) can not change “too much” with respect to \( \tilde{\nu}_\ell \); more precisely:

\[
\frac{1}{768q_{n+1}} \leq \|\omega\nu_\ell\| \leq \frac{1}{24q_n},
\]

as we shall prove, using the following result.

**Lemma 7.** Given a tree \( \vartheta_0 \in \mathcal{T}_{\nu,k} \) and a resonance \( V \), let \( \vartheta \in \mathcal{T}^*_{\nu,k} \) be a tree obtained by the action of the group \( \mathcal{P}_V \), i.e. \( \vartheta \in \mathcal{F}_V(\vartheta_0) \). If \( \|\omega\nu_m\| \leq 1/8q_{nR} \) for any entering line \( \ell_m \) of \( V \), \( m = 1, \ldots, m_V \), then, for any line \( \ell \in V \), one has

\[
||\omega\nu_\ell|| - ||\omega\tilde{\nu}_\ell|| \leq \frac{1}{4q_{nR}}, \quad ||\omega\nu_\ell|| \geq \frac{1}{4q_{nR}}, \quad ||\omega\tilde{\nu}_\ell|| \geq \frac{1}{4q_{nR}},
\]

if \( \nu_\ell \) and \( \tilde{\nu}_\ell \) are the momenta flowing through \( \ell \) in \( \vartheta \) and \( \vartheta_0 \), respectively.

**Proof.** As \( V \) is a resonance, then for each line \( \ell \in V \) one has \( |\nu_\ell| \leq q_{nR} \) (see item 4 in the definition of resonance), so that

\[
||\omega\nu_\ell|| \geq ||\omega q_{nR}|| > \frac{1}{2q_{nR}},
\]

by (2.15) and (2.16). On the other hand

\[
||\omega\nu_\ell - \omega\nu_\ell|| \leq \sum_{m=1}^{m_V} ||\omega\nu_m||.
\]

if \( \nu_1, \ldots, \nu_{m_V} \) are the momenta flowing through the lines \( \ell_1, \ldots, \ell_{m_V} \) entering \( V \). By hypothesis

\[
||\omega\nu_m|| \leq \frac{1}{8q_{nR}}, \quad \forall m = 1, \ldots, m_V.
\]

If \( m_V \geq 2 \) then one must have \( q_{nR} > 4q_{nR} \) (see item 5 in the definition of resonance). In such a case if there is an entering line (say \( \ell_1 \)) which is the root line of a tree of order \( \geq q_{nR}+1/4 \), then all the other lines are the root lines of subtrees of orders \( k_2, \ldots, k_{m_V} \), such that \( 0 = k_2 + \ldots + k_{m_V} < q_{nR}+1/8 \) (see item 6a in the definition of resonance). Moreover, for each \( m = 2, \ldots, m_V \), \( k_m \geq q_{nR} \),
otherwise the line \( \ell_m \) would not be on scale \( \geq n^R_V \). By lemma 1, \( \nu_m = s_m q_n^R \) for all \( m = 2, \ldots, m_V \), with \( s_m \in \mathbb{Z} \), and
\[
|s_2| + \ldots + |s_{m_V}| \leq \frac{k_0}{q_n^R} \leq \frac{q_n^R + 1}{8q_n^R}, \tag{3.19}
\]
so that
\[
\sum_{m=1}^{m_V} ||\omega_{\nu_m}|| \leq \frac{1}{8q_n^R} + \sum_{m=2}^{m_V} |s_m| ||\omega q_n^R|| \leq \frac{1}{8q_n^R} + \frac{1}{8q_n^R} \leq \frac{1}{4q_n^R}, \tag{3.20}
\]
where use was made of (2.15). Therefore, when replacing \( \vartheta_0 \) with \( \vartheta \), (3.15) follows.

If there is no entering line of \( V \) which is the root line of a tree of order \( \geq q_n^R + 1/4 \) and the tree having as root line the exiting line of \( V \) is of order \( k < q_n^R + 1/4 \) (see item 6b in the definition of resonance), then
\[
\sum_{m=1}^{m_V} |s_m| q_n^R \leq k_1 + \ldots + k_{m_V} \equiv k - k_T < k \leq \frac{q_n^R + 1}{4}, \tag{3.21}
\]
so that
\[
\sum_{m=1}^{m_V} ||\omega_{\nu_m}|| \leq \sum_{m=1}^{m_V} |s_m| ||\omega q_n^R|| \leq \frac{q_n^R + 1}{4q_n^R} \cdot \frac{1}{q_n^R + 1} = \frac{1}{4q_n^R}, \tag{3.22}
\]
which implies again (3.15). If \( m_V = 1 \), then (3.15) follows immediately from (3.17) and (3.18).

We come back to the proof of (3.14). Note that inside \( V \) in \( \vartheta_0 \) (hence also in \( \vartheta \), see remark 6) only lines on scale \( n_T \) such that \( 1/48q_n > 1/4q_n^R \) are possible, by the second inequality in (3.15) and the definition of scale (see (2.8)).

As the entering lines of \( V \) satisfy (2.8), hence (2.11), lemma 7 applies. Then, given a line \( \ell \) internal to \( V \) on scale \( n_\ell \), one has
\[
||\omega_{\nu_\ell}|| \leq \frac{1}{48q_n} + \frac{1}{4q_n^R} \leq \frac{1}{48q_n} + \frac{1}{24q_n} \leq \frac{1}{24q_n}, \tag{3.23}
\]
Likewise, if \( 1/96q_{n+1} > 2/q_n^R \), one has
\[
||\omega_{\nu_\ell}|| \geq \frac{1}{96q_{n+1}} - \frac{1}{4q_n^R} \geq \frac{1}{96q_{n+1}} - \frac{1}{768q_{n+1}} \geq \frac{1}{96q_{n+1}} \left( 1 - \frac{1}{8} \right), \tag{3.24}
\]
while, if \( 1/96q_{n+1} < 2/q_n^R \), one has
\[
||\omega_{\nu_\ell}|| \geq \frac{1}{4q_n^R} \geq \frac{1}{768q_{n+1}}. \tag{3.25}
\]
by the third inequality in (3.15). Then (3.14) follows: so in particular the momentum \( \nu_\ell \) of the line \( \ell \in \vartheta \) still fulfills (2.11).

Note that (3.13) and (2.11) imply the following bound for the renormalized resonance factor of a first generation resonance:
\[
|RV_V(\vartheta)| \leq D_0D_k^{LV} \sum_{m,m'=1}^{m_V} ||\omega_{\nu_{\ell_m}}|| \cdot ||\omega_{\nu_{\ell_{m'}}}|| \cdot (768q_n^R + 1)^2 \left( \prod_{\ell \in V} (768q_{n+1})^2 \right), \tag{3.26}
\]
(for some constants $D_6$ and $D_7$), where the last product (times $\Gamma^{-k}$) represents a bound on the resonance factor (3.3). The proof of such an assertion again is as in [1] (see the proof of the Corollary in [1], §3), and follows immediately by noting that for any line $\ell \in V$ one has $n_\ell \geq n_V$. The only difference with respect to [1] is that now the derivatives can act also on the compact support functions: they were just missing in [1]; it is nevertheless straightforward to see that:

$$\left| \frac{\partial^p}{\partial^p \mu} \chi_n(||\omega_{n\ell}||) \right| \leq D_8 (768 q_{n+1}^2)^p,$$

(3.27)

with $p = 1, 2$, for some constant $D_8$, so that:

$$\left| \frac{\partial^p}{\partial^p \mu} g_n(\nu_\ell) \right| \leq D_9 (768 q_{n+1}^2)^{p+2},$$

(3.28)

with $p = 0, 1, 2$, for some constant $D_9$.

For any tree in $\mathcal{F}_V(\partial)$ the bound (2.11) holds, so that lemma 5 applies (see remark 15 in sect. 5).

Note that the two factors $||\omega_{n\ell_m}||, ||\omega_{n\ell_{m'}}||$ in (3.26) allow us to neglect the propagator corresponding to a line entering a resonance with resonance scale $n^R_V$, provided such a propagator is replaced by a factor $(768 q_{n+1}^2)^2$, where $n_V$ is the scale of the resonance as a cluster. Such a mechanism corresponds to the discussion leading to (2.21), as far as only the first generation resonances are considered.

In general a tree will contain more resonances, and the resonances can be contained into each other. Then the above discussion has to be extended to cover the more general case: which will be done in the next section.

4. Renormalization of resonances: the general step

We proceed following strictly the techniques of [6] and [13].

Consider a tree $\vartheta \in \mathcal{F}_{\nu,k}^*$ in (3.2). For each resonance $V$ of any generation, let us define a pair of derived lines $\ell_1^V, \ell_2^V$ internal to $V$ – possibly coinciding – with the following “compatibility” condition: if $V$ is inside some other resonance $W$, the set $\{\ell_1^V, \ell_2^V\}$ must contain those lines of $\{\ell_1^W, \ell_2^W\}$ which are inside $V$. Clearly there can be 0, 1 or 2 such lines, and correspondingly we shall say that the resonance $V$ is of type 2 if none of its derived lines is a derived line for one of the resonances containing it, of type 1 if just one of its two derived lines is a derived line for one of the resonances containing it, and of type 0 if both derived lines are derived lines for some resonances $W, W'$ – possibly coinciding – containing $V$; we shall use a label $z_V = 0, 1, 2$ to take note of the type of the resonance $V$. One associates also to each resonance $V$ a pair of entering lines $\ell_{m_1}^V, \ell_{m_1'}^V$ if $z_V = 2$ and a single line $\ell_m^V$ if $z_V = 1$, with $m, m' = 1, \ldots, m_V$. Moreover for each resonance we shall introduce
an interpolation parameter $t_V$ and a measure $\pi_{z_V}(t_V)\, dt_V$ such that:

$$\pi_z(t) = \begin{cases} (1 - t), & z = 2 \\ 1, & z = 1 \\ \delta(t - 1), & z = 0; \end{cases} \quad (4.1)$$

we shall denote with $t = \{t_V\}_{V\in\mathcal{V}}$ the set of all interpolation parameters.

The momentum flowing through a line $\ell_u$ internal to any resonance $V$ will be defined recursively as:

$$\nu_{\ell_u}(t) = \nu_{\ell_u}^0 + t_v \sum_{\ell \in \mathcal{E}_u} \nu_{\ell}(t), \quad \nu_{\ell_u}^0 = \sum_{\nu \in \mathcal{V}, \nu \not\in \mathcal{U}_{\ell_u}} \nu_u; \quad (4.2)$$

of course $\nu_{\ell_u}(t)$ will depend only on the interpolation parameters corresponding to the resonances containing the line $\ell_u$ (by construction).

For any resonance $V$ the resonance factor is defined as

$$\mathcal{V}_V(\vartheta) = U_V(\vartheta) \left[ \prod_{\ell \in V} g_{n_\ell}(\nu_{\ell}(t)) \right], \quad (4.3)$$

when $z_V = 2$, as

$$\mathcal{V}_V(\vartheta) = U_V(\vartheta) \left[ \left( \frac{\partial}{\partial \mu} g_{n_{\ell_1}}(\nu_{\ell_1}(t)) \right) \left( \prod_{\ell \in \mathcal{V}, \ell \not\in \ell_1} g_{n_\ell}(\nu_{\ell}(t)) \right) \right], \quad (4.4)$$

when $z_V = 1$ (and we have called $\ell_1^V$ the line in $\{\ell_1^V, \ell_2^V\}$ which belongs to the set $\{\ell_1^V, \ell_2^V\}$ for some resonance $W$ containing $V$), as

$$\mathcal{V}_V(\vartheta) = U_V(\vartheta) \left[ \left( \frac{\partial^2}{\partial \mu \partial \mu'} g_{n_{\ell_1}}(\nu_{\ell_1}(t)) \right) \left( \prod_{\ell \in \mathcal{V}, \ell \not\in \ell_1^V} g_{n_\ell}(\nu_{\ell}(t)) \right) \right], \quad (4.5)$$

when $z_V = 0$ and $\ell_1^V = \ell_2^V$, and as

$$\mathcal{V}_V(\vartheta) = U_V(\vartheta) \left[ \left( \frac{\partial}{\partial \mu} g_{n_{\ell_1}}(\nu_{\ell_1}(t)) \right) \left( \frac{\partial}{\partial \mu'} g_{n_{\ell_2}}(\nu_{\ell_2}(t)) \right) \right] \cdot \left( \prod_{\ell \in \mathcal{V}, \ell \not\in \ell_1^V, \ell_2^V} g_{n_\ell}(\nu_{\ell}(t)) \right), \quad (4.6)$$

when $z_V = 0$ and $\ell_1^V \neq \ell_2^V$.

In (4.4)-(4.6) one has $\mu = \omega_{\nu^W}$ and $\mu' = \omega_{\nu^{W'}}$, for some lines $\ell_{mW}$ and $\ell_{m'W'}$ (possibly coinciding) entering, respectively, some resonances $W$ and $W'$ (possibly coinciding) containing $V$.

We define the renormalization operator according to the type of the resonance; namely, if $z_V = 2$, then:

$$\mathcal{R} \mathcal{V}_V(\vartheta; \omega_{\nu_{\ell_1}}(t), \ldots, \omega_{\nu_{\ell_{mV}}}(t)) = \sum_{m,m'=1}^{m_V} \omega_{\nu_{\ell_m}}(t) \omega_{\nu_{\ell_{m'}}}(t) \cdot \int_0^1 dt_V \frac{\partial^2}{\partial \mu_{m} \partial \mu_{m'}} \mathcal{V}_V(\vartheta, t_V \omega_{\nu_{\ell_1}}(t), \ldots, t_V \omega_{\nu_{\ell_{mV}}}(t)); \quad (4.7)$$
if \( z_V = 1 \), then:

\[
\mathcal{R}\mathcal{V}_V(\vartheta; \omega \nu_t (t), \ldots, \omega \nu_{m_V} (t)) = \sum_{m=1}^{m_V} \omega \nu_{m} (t) \cdot
\int_0^1 dt \frac{\partial}{\partial \mu_m} \mathcal{V}_V(\vartheta, t \nu_t (t), \ldots, t \nu_{\nu m_V} (t));
\]

(4.8)

finally if \( z_V = 0 \), then:

\[
\mathcal{R}\mathcal{V}_V(\vartheta)(\vartheta; \omega \nu_t (t), \ldots, \omega \nu_{m_V} (t)) = \mathcal{V}_V(\vartheta)(\vartheta; \omega \nu_t (t), \ldots, \omega \nu_{\nu m_V} (t)).
\]

(4.9)

In all cases set \( L = 1 - \mathcal{R} \).

Remark 10. Note that \( z_V \) equals the order of the renormalization performed on the resonance \( V \).

Remark 11. If a resonance \( V \) has a resonance scale \( n^V_0 \), then there is a line \( \ell^0_0 \) on scale \( n^V_0 \) entering \( V \) such that \( \| \omega \nu_t \| \leq \| \omega \nu_{\nu 0} \| \) for each \( \ell \) entering \( V \). If there is ambiguity, \( \ell^0_0 \) can be chosen arbitrarily. For any resonance \( V \) one has a factor bounded by \( \| \omega \nu_{\nu 0} \|^z_v \), from (4.7), (4.8) and (4.9) and by the definition of \( \ell^0_0 \).

To each line \( \ell \) derived once one can associate the line \( \ell_{m}(\ell) \) corresponding to the quantity \( \mu_{m} = \omega \nu_{\nu m(\ell)} \) with respect to which the propagator \( \nu_{m_i} (\nu_t (t)) \) is derived. If the line \( \ell \) is derived twice one associates to it the two lines \( \ell_m(\ell) \) and \( \ell_{m'}(\ell) \) such that \( \mu_{m} = \omega \nu_{\nu m(\ell)} \) and \( \mu_{m'} = \omega \nu_{\nu m'(\ell)} \) are the quantities with respect to which the propagator \( \nu_{m_i} (\nu_t (t)) \) is derived.

Given a derived line \( \ell \), let \( V \) be the minimal resonance containing it. If the line \( \ell \) is derived once, then let \( W \) be the resonance for which \( \ell_{m}(\ell) \) is an entering line; if instead \( \ell \) is derived twice, let \( W, W' \subseteq W \) be the resonances for which the lines \( \ell_m(\ell), \ell_{m'}(\ell) \) respectively are entering lines.

In the first case, let \( W_i, i = 0, \ldots, p \) the resonances contained by \( W \) and containing \( V \), ordered naturally by inclusion:

\[
V = W_0 \subset W_1 \subset \cdots \subset W_p = W;
\]

(4.10)

We shall call the set \( W(\ell) = \{ W_0, \ldots, W_p \} \) the \textit{simple cloud} of \( \ell \).

In the second case, let \( W_i, i = 0, \ldots, p \), the resonances contained by \( W \) and containing \( V \), ordered naturally by inclusion:

\[
V = W_0 \subset W_1 \subset \cdots \subset W_p = W' \subset \cdots \subset W_p = W,
\]

(4.11)

with \( p' \leq p \). We shall say that \( W_-(\ell) = \{ W_0, \ldots, W_{p'} \} \) is the \textit{minor cloud} of \( \ell \) while \( W_+(\ell) = \{ W_0, \ldots, W_p \} \) is the \textit{major cloud} of \( V \).

When the renormalization of a resonance \( V \in \mathbf{V}_{j+1} \) is performed, a tree \( \vartheta^V_0 \in \mathcal{F}_V(\vartheta) \), with \( \vartheta \in \mathcal{T}_{\nu k} \), is replaced by the action of the group \( \mathcal{P}_V \) with a new tree \( \vartheta^V \). As this replacement is performed iteratively, one has the constraint that if \( V_1 \) and \( V_2 \) are two resonance such that \( V_1 \) is the minimal resonance containing \( V_2 \), then \( \vartheta^{V_1} = \vartheta^{V_2} \). At the end, the original tree \( \vartheta_0 \in \mathcal{T}_{\nu k} \) is replaced with a tree \( \vartheta \in \mathcal{T}_{\nu k} \). On each resonance \( V \in \mathbf{V} \) of \( \vartheta \) the renormalization operator \( \mathcal{R} \) acts: a tree whose resonance factors have been all renormalized will be called a \textit{renormalized} (or \textit{resummed}) tree.
As the replacement corresponding to each resonance settles a conjugation between lines of $\vartheta_0$ and those of $\vartheta^V$, in the end for each line of $\vartheta$ there will be a conjugate line of $\vartheta_0$.

Note that, as the transformations of the groups $\mathcal{P}_V$, $V \in \mathcal{V}$, do not modify the scales of $\vartheta_0$ (see remark 6), the scales of the lines of $\vartheta$ are the same as those of the conjugate lines of the tree $\vartheta_0$, so that, in order to apply lemma 5, we have only to verify that (2.11) is verified for the lines in $\vartheta$: this will be done below (after remark 12).

Now, we shall show that:

- the localized resonance factors can be neglected (in a sense that will appear clear shortly, see lemma 8 below),
- for any (renormalized) resonance we obtain a factor:
  \[
  (768q_{n+1})^2 ||\omega_{\ell^V_0}||^2,
  \]
  (4.12)

and

- the number of terms generated by the renormalization procedure is bounded by a constant to the power $k$,

so that the bound (2.20) can be replaced by a bound which leads to (2.21), as anticipated in sect. 2.

Note firstly that the localized part of the resonance factors can be dealt with as in sect. 3, when only first generation resonances were considered. More formally, we have the following result, which is proved in sect. 6.

**Lemma 8.** Given a tree $\vartheta$ and a resonance $V \in \vartheta$, the localized resonance factor $\mathcal{L}_{\mathcal{V}}(\vartheta)$ gives zero when the values of the trees belonging to the same resonance family $\mathcal{F}_V(\vartheta)$ are summed together.

Define the map $\Lambda$:

\[
\Lambda: \mathcal{V} \mapsto \Lambda \mathcal{V} = \{z_V, \ell^V_1, \ell^V_2, \{\ell^V_{m}, \ell^V_{m'}\}^*\}_{V \in \mathcal{V}},
\]

(4.13)

which associates to each resonance $V \in \mathcal{V}$ the derived lines $\ell^V_1, \ell^V_2$ and the lines in the set $\{\ell^V_{m}, \ell^V_{m'}\}^*$ defined as:

\[
\{\ell^V_{m}, \ell^V_{m'}\}^* = \begin{cases} 
\{\ell^V_{m}, \ell^V_{m'}\}, & \text{if } z_V = 2, \\
\ell^V_{m}, & \text{if } z_V = 1, \\
\emptyset, & \text{if } z_V = 0,
\end{cases}
\]

(4.14)

where $m, m' = 1, \ldots, m_V$ and $\ell^V_1, \ldots, \ell^V_{m_V}$ are the lines entering $V$.

Note that that the map $\Lambda$ gives a natural decomposition of the set $L$ of all lines of $\vartheta$ into $L = L_0 \cup L_1 \cup L_2$, where $L_j$ is the set of lines derived $j$ times.
Then, by using also lemma 6, one has
\[
\text{Val}(\nu) = \sum_{A' \subseteq V} \left( \prod_{V' \in V'} \int_0^1 \pi_{z_V}(t_V) \, dt_V \right) \left( \prod_{n \in \partial} \frac{g_{mn}^{m_n+1}}{m_n!} \right) \cdot \left( \prod_{\ell \in L_0} g_{n\ell}(\nu_\ell(t)) \right) \left( \prod_{\ell \in L_1} \omega_{\nu_\ell(t)} \frac{\partial}{\partial \mu_n} g_{n\ell}(\nu_\ell(t)) \right) \left( \prod_{\ell \in L_2} \omega_{\nu_{\ell'}(t)} \omega_{\nu_{\ell''}(t)} \frac{\partial^2}{\partial \mu_m \partial \mu_{m'}} g_{n\ell}(\nu_\ell(t)) \right).
\]

(4.15)

Remark 12. Note that no propagator is derived more than twice: this fact is essential for our proof since we have no control on the growth rate of the derivatives of the compact support functions (2.6).

After the renormalization procedure has been applied for all resonances, one check that the momenta of the lines in $\nu$ have changed, with respect to the original tree $\nu_0$ with nonvanishing value, in such a way that the bound (2.11) still hold.

Lemma 9. Consider a renormalized tree $\nu \in T_{\nu,k}$, obtained from $\nu \in T_{\nu,k}$ by the iterative replacements, described above, that take place each time a resonance appears. Then the lines of $\nu$ inherit the scales of the conjugate lines of $\nu_0$ and lemma 5 applies to $\nu$.

Proof. The first assertion follows by construction. The second one can be seen by induction on the generation of the resonances, by taking into account that for the first generation resonances the result has been already proved in sect. 3. So let us suppose that (2.14) holds for resonances of any generation $j'$, with $j' < j$. Consider a line $\ell$ contained inside a resonance $V \in \mathbf{V}_j$ and outside all resonances in $\mathbf{V}_{j+1}$ contained inside $V$: then there will be $j$ resonances $V = W_1 \subset \ldots \subset W_j$ containing $\ell$. Each renormalization produces a change on the momentum flowing through the line $\ell$, such that, if $\nu_\ell$ is the momentum flowing through the line $\ell$ in $\nu_0$ and $\nu_\ell$ is the momentum flowing through the conjugate line $\ell$ in $\nu$, then
\[
\frac{1}{90q_{n\ell}+1} - \sum_{i=1}^{j} \frac{1}{4q_{n\ell_i}} \leq \|\nu_\ell\| - \|\nu_\ell\| \leq \frac{1}{48q_{n\ell}} + \sum_{i=1}^{j} \frac{1}{4q_{n\ell_i}}.
\]

(4.16)

Call $\nu_0^V \in \mathcal{F}_V(\nu_0)$ the tree containing $V$ (which is not, in general, the originary tree $\nu_0$) and $\nu^V$ the tree in $\mathcal{F}_V(\nu_0^V)$ obtained by the action of the group $P_V$. As (2.11) is supposed to hold before renormalizing $V$, for all lines $\ell_m, m = 1, \ldots, m_V$, entering $V$ one has $\|\omega_{\nu_{\ell_m}}\| \leq 1/48q_{n\ell_m}$, so that, by reasoning as in sect. 3 to prove lemma 7, we can conclude that
\[
\|\omega_\nu\| - \|\nu_\ell\| \leq \frac{1}{4q_{n_\ell}}, \quad \|\omega_\nu\| \leq \frac{1}{4q_{n_\ell}}, \quad \|\nu_\ell\| \geq \frac{1}{4q_{n_\ell}},
\]

(4.17)

where $\nu_\ell$ is the momentum flowing through the line $\ell$ in $\nu^V$.

In order that $\ell$ be contained inside $V = W_1$, one must have $1/48q_{n\ell} \geq 1/4q_{n_\ell}$; moreover if $j_1 = \lfloor (j-1)/2 \rfloor$ and $j_2 = \lfloor j/2 \rfloor$ (here $\lfloor \cdot \rfloor$ denotes the integer part), one has
\[
q_{n\ell_{j_1}} \leq q_{n\ell_{j_2}} \leq \ldots \leq q_{n\ell_{j_1}j_1} \leq \frac{q_{n\ell_{j_2}}}{2j_1}, \quad q_{n\ell_{j_2}} \leq \frac{q_{n\ell_{j_1}}}{2j_2} \leq \ldots \leq q_{n\ell_{j_2}j_2},
\]

(4.18)
(simply use that \( q_{n+1} \geq q_n \) and \( q_{n+2} \geq 2q_n \) for any \( n \geq 0 \)). Then one can write
\[
||\omega_{\nu t}|| \leq \frac{1}{48q_{n_t}} + \frac{1}{4q_{n_{t}}^6} \left( \sum_{i=1}^{j_1} \frac{1}{2^i} + \sum_{i=1}^{j_2} \frac{1}{2^i} \right) \leq \frac{1}{48q_{n_t}} + \frac{1}{q_{n_{t}}^6};
\] (4.19)
this is bounded from above by \( 5/48q_{n_t} \). Likewise one finds
\[
||\omega_{\nu t}|| \geq \frac{1}{96q_{n_{t}+1}} - \frac{1}{q_{n_{t}}^6} \left( \sum_{i=1}^{j_1} \frac{1}{2^i} + \sum_{i=1}^{j_2} \frac{1}{2^i} \right) \geq \frac{1}{96q_{n_{t}+1}} - \frac{1}{q_{n_{t}}^6};
\] (4.20)
this is bounded from below by \( 1/192q_{n_{t}+1} \) if \( 1/96q_{n_{t}+1} > 2/q_{n_t}^6 \) and by \( 1/768q_{n_{t}+1} \) if \( 1/96q_{n_{t}+1} \leq 2/q_{n_t}^6 \).

Then (2.14) holds also for any line \( \ell \) contained inside \( V_0 \), if \( V \) is a resonance in \( V_j \). As any next renormalization is on resonances \( V \in V_j', \) with \( j' > j \), so that it does not shift the line \( \ell \), the momentum \( \nu_t \) changes no more, so that the inductive proof is complete. \( \Box \)

Then in (4.15) we can bound, for \( \ell \in L_1 \):
\[
\left| \omega_{\nu t}(\ell) \frac{\partial}{\partial \mu} g_{n_t}(\nu_t(t)) \right| \leq D_9||\omega_{\nu t}(\ell)|| (768q_{n_{t}+1})^3
\]
\[
\leq D_9||\omega_{\nu t}(\ell)|| (768q_{n_{t}+1})^3 \prod_{i=0}^{p-1} ||\omega_{\nu t_{W_i}}||
\]
\[
\leq D_9(768q_{n_{t}+1})^2 \left[ \prod_{i=0}^{p} ||\omega_{\nu t_{W_i}}|| \right] \left[ \prod_{i=0}^{p} (768q_{n_{W_i}+1}) \right],
\] (4.21)
where \( W(\ell) = \{W_0, \ldots, W_p\} \) is the simple cloud of \( \ell \), and, for \( \ell \in L_2 \):
\[
\left| \omega_{\nu t}(\ell) \omega_{\nu t}(\ell) \frac{\partial^2}{\partial \mu \partial \mu} g_{n_t}(\nu_t(t)) \right| \leq D_9||\omega_{\nu t}(\ell)|| ||\omega_{\nu t}(\ell)|| (768q_{n_{t}+1})^4
\]
\[
\leq D_9||\omega_{\nu t}(\ell)|| ||\omega_{\nu t}(\ell)|| (768q_{n_{t}+1})^4 \prod_{i=0}^{p-1} ||\omega_{\nu t_{W_i}}|| \prod_{j'=0}^{p'-1} ||\omega_{\nu t_{W_i}}||
\]
\[
\leq D_9(768q_{n_{t}+1})^2 \left[ \prod_{i=0}^{p} ||\omega_{\nu t_{W_i}}|| \right] \left[ \prod_{i=0}^{p} (768q_{n_{W_i}+1}) \right]
\]
\[
\left[ \prod_{j'=0}^{p'} ||\omega_{\nu t_{W_{i'}}}|| \right] \left[ \prod_{j'=0}^{p'} (768q_{n_{W_{i'}}+1}) \right],
\] (4.22)
where \( W_-(\ell) = \{W_0, \ldots, W_p\} \) is the minor cloud and \( W_+(\ell) = \{W_0, \ldots, W_p\} \) is the major cloud of \( \ell \).

Note that (4.21) and (4.22) give a factor
\[
||\omega_{\nu t_{W_i}}|| (768q_{n_{W_i}+1})
\] (4.23)
for each resonance \( W \) belonging to the (simple or minor or major) cloud of \( \ell \). As each resonance belongs to the cloud of some line internal to it and each resonance
contains two derived lines or one line derived twice (by definition of the renormalization procedure), then one concludes that a factor equal to the square of (4.23) is obtained for each resonance.

If we note that each underived propagator can be bounded again using (3.28) with \( p = 0 \), then we can summarize the bounds (4.21) ÷ (4.22) stating that, for each resummed tree \( \vartheta \), we have:

- for each resonance \( V \), a factor \( ||\omega_{\vartheta V}||^2 \) times a factor \( (768q_{nV+1})^2 \);
- for each line \( \ell \), a factor \( D_\vartheta(768q_{n\ell+1})^2 \) (as the factors \( (768q_{n\ell+1})^p \), \( p = 1, 2 \), appearing when the corresponding propagator is derived, are taken into account by the factors associated to the resonances, see the item above);

Then the statement concerning (4.12) is proved.

Once the single summand in (4.15) has been bounded, one is left with the problem of bounding the number of terms on which the sum is performed.

For each first generation resonance \( V \) at most \( m^2 V \) times \( k^2 V \) summands are generated by the renormalization procedure (see (3.13)). In general, for each (renormalized) resonance, we have to sum over the entering lines \( \{\ell^V_m, \ell^V_{m'}\}^* \) (corresponding to the quantities \( \mu_m \), \( m = 1, \ldots, mV \), in terms of which the renormalized resonance factor is considered a function) and over the internal lines \( \{\ell^V, \ell^V_\ell\} \) (corresponding to the factors on which the derivatives act). An estimates on the number of summands generated by the renormalization procedure can be obtained by using the counting lemma 6.

If \( V \in \text{V}_j, j \geq 1 \), let \( N^V_0 \) be the number of \((j + 1)\)-th generation resonances contained inside \( V \). Recall that \( V_0 \) is the set of lines internal to \( V \) which are outside any resonance contained in \( V \), and denote by \( k^V_0 \) the number of elements in \( V_0 \).

The renormalization procedure, for each renormalized resonance, generates a single or double sum over the entering lines whose momenta appear in the quantities \( \omega_{\mu_1}(t), \ldots, \omega_{\mu_{mV}}(t) \), in terms of which the resonance factor is expanded: the sum is single if the localization is to first order and double if the localization is to second order (see (4.7) and (4.8)).

Then we find, using lemma 6, that in the renormalization procedure each sum over the entering lines of a first generation resonance \( V \) is on \( m^2 V \) terms, each sum over the entering lines of all second order resonances \( V' \subset V \) is on \( k^V_0 + N^V_0 \) terms, each sum over the entering lines of all third generation resonances \( V'' \subset V' \subset V \) is on \( k^V_0 + N^V_0 \), and so on; in general, each sum over the entering lines of all the resonances \( V'' \in \text{V}_{j+1} \) contained inside a resonance \( V \in \text{V}_j \) is bounded by \( k^V_0 + N^V_0 \).

Once all generations of resonances have been considered, the overall number of summands generated by the renormalization procedure – by taking also into account the sum over the derived lines and using remark 12 – is bounded by:

\[
\left[ \prod_{V \in \text{V}_1} k^2_V \right] \left[ \prod_{V \in \text{V}_1} m^2_V \right] \left[ \prod_{V \in \text{V}_0} (k^V_0 + N^V_0)^2 \right] \leq e^{6k},
\]

(4.24)
where $k$ is the order of the tree $\vartheta$. In fact, just use $x \leq e^x$ and the obvious inequalities:

$$\sum_{V \in V_1} k_V \leq k,$$

$$\sum_{V \in V_1} m_V + \sum_{V \in V} k_{V_0} \leq k,$$

$$\sum_{V \in V} N_V \leq k. \quad (4.25)$$

Then the statement after (4.12) is proved and the constant $D_3 = e^6$.

Finally one has to count the number of trees. The bound given in sect. 2 is no more valid, as a line $\ell \in \vartheta$ can have more than two scale labels. However lemma 4 proves that to each line at most $D_{10} = 17$ scale labels can be associated, so that the number of trees in $T_{\nu,k}$ is bounded by $2^k D_{10}^k$. Then the bound (2.21) follows, with $D_4 = 2^3 D_3 D_9 D_{10}$: this concludes the proof of the theorem.

### 5. Proof of Lemma 5

We shall prove inductively on the order $k$ the following bounds:

$$M_n(\vartheta) = 0, \quad \text{if } k < q_n, \quad (5.1a)$$

$$M_n(\vartheta) \leq \frac{2k}{q_n} - 1 + N_n^R(\vartheta), \quad \text{if } k \geq q_n, \quad (5.1b)$$

for any $n \geq 0$, and:

$$M_n(\vartheta) = 0, \quad \text{if } k < q_n, \quad (5.2a)$$

$$M_n(\vartheta) \leq \frac{k}{q_n} + N_n^R(\vartheta), \quad \text{if } q_n \leq k < \frac{q_{n+1}}{4}, \quad (5.2b)$$

$$M_n(\vartheta) \leq \frac{k}{q_n} + \frac{8k}{q_{n+1}} - 1 + N_n^R(\vartheta), \quad \text{if } k \geq \frac{q_{n+1}}{4}, \quad (5.2c)$$

for $q_{n+1} > 4q_n$, where $k$ is the order of the tree $\vartheta$.

Note that (5.1a) and (5.2a) are simply a consequence of lemma 2 of sect. 2, so we have to prove only (5.1b), (5.2b) and (5.2c).

**Remark 13.** If we were only interested in proving the analyticity of the invariant curves for rotation numbers satisfying the Bryuno condition, then equations (5.1) would be sufficient – as it would be easy to check by proceeding along the lines of sect. 3 and 4. However, in order to find the optimal dependence of the radius of convergence $\rho(\omega)$ on $\omega$, which is the main focus of this paper, the more refined bounds (5.2) are necessary.

**Remark 14.** The proof of (5.1) is easier, as it is obvious since it is a weaker result. After dealing with (5.2), the proof of (5.1) could be left as an exercise: we shall prove it explicitly for completeness, and as it could be read as an introduction to the more involved proof of (5.2).
We shall prove first (5.2) (case \( q_{n+1} > 4q_n \)) in cases [1] \( \div [3] \) below, then (5.1) in items [4] \( \div [6] \) below. We proceed by induction, and assuming that (5.1), (5.2) hold for any \( k' < k \) we shall show that they hold for \( k \) also; their validity for \( k = 1 \) being trivial, lemma 5 is proved. Recall also remark 5 in sect. 2 about the way of counting the resonances on scale \( n \) and the resonances with resonance-scale \( n \).

- So consider first \( q_{n+1} > 4q_n \).

[1] If the root line \( \ell \) of \( \vartheta \) has scale \( \neq n \) and it is not the exiting line of a resonance on scale \( n \), let us denote with \( \ell_1, \ldots, \ell_m \) the lines entering the last node \( u_0 \) of \( \vartheta \) and \( \vartheta_1, \ldots, \vartheta_m \) the subtrees of \( \vartheta \) whose root lines are those lines. By construction \( M_n(\vartheta) = M_n(\vartheta_1) + \cdots + M_n(\vartheta_m) \) and \( N_n^R(\vartheta) = N_n^R(\vartheta_1) + \cdots + N_n^R(\vartheta_m) \): the bounds (5.2) follow inductively by noting that for \( k \geq q_{n+1}/4 \) one has \( 8k/q_{n+1} - 1 \geq 1 \).

[2] If the root line \( \ell \) of \( \vartheta \) has scale \( n \), then we can reason as follows. Let us denote with \( \ell_1, \ldots, \ell_m \) the lines on scale \( \geq n \) which are the nearest to the root line of \( \vartheta \),\(^{2}\) and let \( \vartheta_1, \ldots, \vartheta_m \) be the subtrees with root lines \( \ell_1, \ldots, \ell_m \). If \( m = 0 \) then (5.2) follow immediately from lemma 2 of sect. 2; so let us suppose that \( m \geq 1 \). Then the lines \( \ell_1, \ldots, \ell_m \) are the entering lines of a cluster \( T \) (which can degenerate to a single point) having the root line of \( \vartheta \) as the exiting line. As \( \ell \) cannot be the exiting line of a resonance on scale \( n \), one has:

\[
M_n(\vartheta) = 1 + M_n(\vartheta_1) + \cdots + M_n(\vartheta_m). \tag{5.3}
\]

In general \( \tilde{m} \) subtrees among the \( m \) considered have orders \( \geq q_{n+1}/4 \), with \( 0 \leq \tilde{m} \leq m \), while the remaining \( m_0 = m - \tilde{m} \) have orders \( < q_{n+1}/4 \). Let us numerate the subtrees so that the first \( \tilde{m} \) have orders \( \geq q_{n+1}/4 \).

Let us distinguish the cases \( k < q_{n+1}/4 \) and \( k \geq q_{n+1}/4 \).

[2.1] If \( k < q_{n+1}/4 \), then \( \tilde{m} = 0 \) and each line entering \( T \), by lemma 1 of sect. 2, has a momentum which is a multiple of \( q_n \) and, by lemma 2, has a scale label \( n \). Therefore the momentum flowing through the root line is \( \nu = \nu_T + s_0q_n \), for some \( s_0 \in \mathbb{Z} \), with:

\[
\nu_T \equiv \sum_{u \in T} \nu_u. \tag{5.4}
\]

Moreover also the root line of \( \vartheta \) has scale \( n \), by assumption, and momentum \( \nu = sq_n \) for some \( s \in \mathbb{Z} \), by lemma 1, so that \( \nu_T = (s - s_0)q_n = s'q_n \), for some integer \( s' \).

[2.1.1] If \( s' \neq 0 \), then \( kT \geq |\nu_T| \geq q_n \), giving:

\[
M_n(\vartheta) \leq 1 + \frac{k_1 + \cdots + k_m}{q_n} + N_n^R(\vartheta_1) + \cdots + N_n^R(\vartheta_m) \leq 1 + \frac{k - kT}{q_n} + N_n^R(\vartheta) \leq \frac{k}{q_n} + N_n^R(\vartheta), \tag{5.5}
\]

as \( N_n^R(\vartheta) = N_n^R(\vartheta_1) + \cdots + N_n^R(\vartheta_m) \), and (5.2b) follows.

[2.1.2] If \( s' = 0 \) and \( kT \geq q_n \), one can reason as in case [2.1.1].

\(^{2}\)That is, such that no other line along the paths connecting the lines \( \ell_1, \ldots, \ell_m \) to the root line is on scale \( \geq n \).
[2.1.3] If \( s' = 0 \) and \( k_T < q_n \), then \( T \) is a resonance with resonance-scale \( n \), and:

\[
M_{n}(\vartheta) \leq 1 + \frac{k_1 + \cdots + k_m}{q_n} + N_n^R(\vartheta_1) + \cdots + N_n^R(\vartheta_m) \leq 1 + \frac{k}{q_n} + N_n^R(\vartheta_1) + \cdots + N_n^R(\vartheta_m) \leq \frac{k}{q_n} + N_n^R(\vartheta),
\]

(5.6)
as \( N_n^R(\vartheta) = 1 + N_n^R(\vartheta_1) + \cdots + N_n^R(\vartheta_m) \), and again (5.2b) follows.

[2.2] If \( k \geq q_{n+1}/4 \), assume again inductively the bounds (5.2). From (5.3) we have:

\[
M_{n}(\vartheta) \leq 1 + \sum_{j=1}^{\tilde{m}} \left( \frac{k_j}{q_n} + \frac{8k_j}{q_{n+1}} - 1 \right) + \sum_{j=\tilde{m}+1}^{m} \frac{k_j}{q_n} + \sum_{j=1}^{m} N_n^R(\vartheta_j),
\]

(5.7)
where \( k_j \) is the order of the subtree \( \vartheta_j \), \( j = 1, \ldots, m \).

[2.2.1] If \( \tilde{m} \geq 2 \), then (5.2c) follows immediately.

[2.2.2] If \( \tilde{m} = 0 \), then (5.7) gives:

\[
M_{n}(\vartheta) \leq 1 + \frac{k_1 + \cdots + k_m}{q_n} + \sum_{j=1}^{m} N_n^R(\vartheta_j) \leq 1 + \frac{k}{q_n} + \sum_{j=1}^{m} N_n^R(\vartheta_j) \leq \frac{8k}{q_{n+1}} - 1 + \frac{k}{q_n} + N_n^R(\vartheta),
\]

(5.8)
as we are considering \( k \) such that \( 1 \leq 8k/q_{n+1} - 1 \) and \( N_n^R(\vartheta_1) + \cdots + N_n^R(\vartheta_m) = N_n^R(\vartheta) \).

[2.2.3] If \( \tilde{m} = 1 \), then (5.7) gives:

\[
M_{n}(\vartheta) \leq 1 + \left( \frac{k_1}{q_n} + \frac{8k_1}{q_{n+1}} - 1 \right) + \sum_{j=2}^{m} \frac{k_j}{q_n} + \sum_{j=1}^{m} N_n^R(\vartheta_j) = \frac{k_1}{q_n} + \frac{8k_1}{q_{n+1}} + \frac{k_0}{q_n} + \sum_{j=1}^{m} N_n^R(\vartheta_j),
\]

(5.9)
where \( k_0 = k_2 + \cdots + k_m \).

[2.2.3.1] If in such case \( k_0 \geq q_{n+1}/8 \), then we can bound in (5.9):

\[
\frac{k_1}{q_n} + \frac{8k_1}{q_{n+1}} + \frac{k_0}{q_n} \leq \frac{k_1 + k_0}{q_n} + \frac{8(k_1 + k_0)}{q_{n+1}} - \frac{8k_0}{q_{n+1}} \leq \frac{k}{q_n} + \frac{8k}{q_{n+1}} - 1,
\]

(5.10)
and \( N_n^R(\vartheta_1) + \cdots + N_n^R(\vartheta_m) = N_n^R(\vartheta) \), so that (5.2c) follows.

[2.2.3.2] If \( k_0 < q_{n+1}/8 \), then, denoting with \( \nu \) and \( \nu_1 \) the momenta flowing through the root line \( \ell \) of \( \vartheta \) and the root line \( \ell_1 \) of \( \vartheta_1 \) respectively, one has:

\[
||\omega(\nu - \nu_1)|| \leq ||\omega\nu|| + ||\omega\nu_1|| \leq \frac{1}{4q_n},
\]

(5.11)
as both \( \ell \) and \( \ell_1 \) are on scale \( n \) (see remark 2 in sect. 2 and use (2.14)). Then either \( |\nu - \nu_1| \geq q_{n+1}/4 \) or \( \nu - \nu_1 = \delta q_n, \delta \in \mathbb{Z} \), by lemma 1 of sect. 2.

[2.2.3.2.1] If \( |\nu - \nu_1| \geq q_{n+1}/4 \), noting that \( \nu = \nu_1 + \nu_T + \nu_0 \), where \( \nu_0 = s_0 q_n \) (with \( s_0 \in \mathbb{Z} \) and \( |\nu_0| \leq k_0 < q_{n+1}/8 \)) is the sum of the momenta flowing through
the root lines of the $m_0$ subtrees entering $T$ with orders $< q_{n+1}/4$ and $\nu_T$ is defined by (5.4), one has:

$$k_T \geq |\nu_T| \geq |\nu - \nu_1| - |\nu_0| \geq \frac{q_{n+1}}{8}, \quad (5.12)$$

so that in (5.9) one can bound:

$$\frac{k_1}{q_n} + \frac{8k_1}{q_{n+1}} + \frac{k_0}{q_n} \leq \frac{k - k_T}{q_n} + \frac{8(k - k_0 - k_T)}{q_{n+1}} \leq \frac{k}{q_n} + \frac{8(k - k_T)}{q_{n+1}} \leq \frac{k}{q_n} + \frac{8k}{q_{n+1}} - 1, \quad (5.13)$$

and $N_n^R(\vartheta_1) + \cdots + N_n^R(\vartheta_m) = N_n^R(\vartheta)$, so that (5.2c) follows again.

2.2.3.2.2] If $\nu - \nu_1 = \bar{s}q_n$, $\bar{s} \in \mathbb{Z}$, then:

$$\nu_T = \nu - \nu_1 - \nu_0 = (\bar{s} - s_0) = s\bar{q}_n, \quad (5.14)$$

where $s \in \mathbb{Z}$.

2.2.3.2.2.1] If $s \neq 0$, then $k_T \geq q_n$, so that in (5.3) one has:

$$\frac{k_1}{q_n} + \frac{8k_1}{q_{n+1}} + \frac{k_0}{q_n} \leq \frac{k - k_T}{q_n} - \frac{8k}{q_{n+1}} \leq \frac{k}{q_n} - \frac{1}{q_n} + \frac{8k}{q_{n+1}}, \quad (5.15)$$

and $N_n^R(\vartheta_1) + \cdots + N_n^R(\vartheta_m) = N_n^R(\vartheta)$, so implying (5.2c).

2.2.3.2.2.2] If $s = 0$ (i.e. $\nu_T = 0$) and $k_T \geq q_n$, one can proceed as in case 2.2.3.2.2.1.

2.2.3.2.2.3] If $s = 0$ and $k_T < q_n$, then $T$ is a resonance with resonance-scale $n$,\(^3\)

so that $N_n^R(\vartheta) = 1 + N_n^R(\vartheta_1) + \cdots + N_n^R(\vartheta_m)$, hence (5.9) gives:

$$M_n(\vartheta) \leq \frac{k}{q_n} + \frac{8k}{q_{n+1}} - 1 + \sum_{j=1}^{m} N_n^R(\vartheta_j) \leq \frac{k}{q_n} + \frac{8k}{q_{n+1}} - 1 + N_n^R(\vartheta), \quad (5.16)$$

and (5.2c) follows.

[3] If the root line $\ell$ of $\vartheta$ is on scale $> n$ and it is the exiting line of a resonance $V_n$ on scale $n$, let us denote with $\ell_1, \ldots, \ell_m$ the lines on scale $\geq n$ which are the nearest to the root line of $\vartheta$, and let $\vartheta_1, \ldots, \vartheta_m$ be the subtrees with root lines $\ell_1, \ldots, \ell_m$; some of these lines – at least one – are lines on scale $n$ inside $V_n$.\(^4\) Let $T$ be the cluster which the lines $\ell_1, \ldots, \ell_m$ enter; of course $T \subset V_n$ and $T$ can degenerate into a single point. As in case [2], let $\tilde{m}$ be the number of subtrees among the $m$ considered which have orders $\geq q_{n+1}/4$, and again let us numerate the subtrees in such a way that the ones with orders $\geq q_{n+1}/4$ are the first $\tilde{m}$.

Note that $k \geq q_{n+1}$ (otherwise $\ell$ could not be on scale $> n$) and

$$M_n(\vartheta) = 1 + M_n(\vartheta_1) + \cdots + M_n(\vartheta_m), \quad (5.17)$$

as the root line $\ell$ contributes one unit to $P_n(\vartheta)$ and does not contribute to $N_n(\vartheta)$.

Note also that if $T$ is a resonance then its resonance scale is $n$.

\(^3\)If $m_0 = 0$, then $n \equiv \nu = \nu_1$, so that $n_{\ell} \leq n_{\ell_1} \leq n_{\ell} + 1$, by construction and by remark 2.

\(^4\)Otherwise $V_n$ would not contain any line on scale $n$, so that it would not be a resonance on scale $n$ as we are supposing.
[3.1] If \( T \) is not a resonance, then:

\[
N^R_n(\vartheta) = N^R_n(\vartheta_1) + \cdots + N^R_n(\vartheta_m).
\]  

(5.18)

By induction (5.2) and (5.17) imply:

\[
M_n(\vartheta) \leq 1 + \sum_{j=1}^{m} \left( \frac{k_j}{q_{n+1}} + \frac{8k_j}{q_{n+1}} - 1 \right) + \frac{m}{q_n} \sum_{j=1}^{m} N^R_n(\vartheta_j),
\]

(5.19)

where \( k_j \) are the orders of the subtrees \( \vartheta_j, j = 1, \ldots, m \).

[3.1.1] If \( \bar{m} = 2 \), then (5.2c) follows immediately.

[3.1.2] The case \( \bar{m} = 0 \) is impossible because \( T \) is contained inside a resonance \( V_n \) on scale \( n \), so that at least one of the subtrees entering \( T \) must have order \( \geq q_{n+1}/4 \) – otherwise no line on scale \( > n \) could enter \( V_n \), see lemma 2.

[3.1.3] If \( \bar{m} = 1 \) let \( k_0 = k_2 + \cdots + k_m \); then the case \( k_0 \geq q_{n+1}/8 \) can be dealt with as in case [2.2.3.1]; if \( k_0 < q_{n+1}/8 \), we deduce from lemma 1 that either \( |\nu - \nu_1| \geq q_{n+1}/4 \) or \( \nu - \nu_1 = s q_n \), using the same notations of case [2.2.3.2].

The first case can be discussed as in case [2.2.3.2.1], while in the second case we find, as in case [2.2.3.2.2], that \( \nu_T = \nu - \nu_1 - \nu_0 = sq_n \), with either \( s \neq 0 \) or \( s = 0 \) and \( k_T \geq q_n \) (otherwise \( T \) would be a resonance), so that the conclusions in cases [2.2.3.2.2.1] and [2.2.3.2.2.2] can be inherited in the present case and (5.2c) follows again.

[3.2] If \( T \) is a resonance, then its resonance-scale is \( n \) (and all its entering lines are on scale \( n \); see item 1 in the definition of resonance), so that:

\[
N^R_n(\vartheta) = 1 + N^R_n(\vartheta_1) + \cdots + N^R_n(\vartheta_m).
\]  

(5.20)

The discussion goes on as in case [3.1] above, with the only difference that now, when \( \bar{m} = 1 \) (and \( k_T < q_n, k_0 < q_{n+1}/8 \)), the case \( \nu_T = 0 \) (i.e. \( \nu_T = sq_n \), with \( s = 0 \)) is the only possible since \( T \) is a resonance. In such a case:

\[
M_n(\vartheta) \leq 1 + \frac{k_1}{q_n} + \frac{8k_1}{q_{n+1}} - 1 + \frac{k_0}{q_n} + \sum_{j=1}^{m} N^R_n(\vartheta_j) \leq \frac{k}{q_n} + \frac{8k}{q_{n+1}} - 1 + N^R_n(\vartheta),
\]

(5.21)

and (5.2c) follows once more.

* Now we prove (5.1).

[4] If the root line \( \ell \) of \( \vartheta \) as scale \( \neq n \) and it is not the entering line of a resonance on scale \( n \), let us denote with \( \ell_1, \ldots, \ell_m \) the lines entering the last node \( u_0 \) of \( \vartheta \). By construction \( M_n(\vartheta) = M_n(\vartheta_1) + \cdots + M_n(\vartheta_m) \) and \( N^R_n(\vartheta) = N^R_n(\vartheta_1) + \cdots + N^R_n(\vartheta_m) \) so that the bound (5.1) follows immediately by induction.

[5] If the root line \( \ell \) of \( \vartheta \) has scale \( n \), using the same notations as in case [2], denote with \( \ell_1, \ldots, \ell_m \) the lines on scale \( \geq n \) which are nearest to the root line of \( \vartheta \), and let \( \vartheta_1, \ldots, \vartheta_m \) be the subtrees with these lines as root lines. Then such lines are the entering lines of a cluster \( T \) (which can degenerate into a single point) having the root line of \( \vartheta \) as the exiting line. We have:

\[
M_n(\vartheta) = 1 + M_n(\vartheta_1) + \cdots + M_n(\vartheta_m).
\]  

(5.22)
Assuming again inductively the bounds (5.1), from (5.22) we have:

\[ M_n(\vartheta) \leq 1 + \sum_{j=1}^{m} \left( \frac{2k_j}{q_n} - 1 \right) + \sum_{j=1}^{m} N^R_n(\vartheta_j), \]  

(5.23)

where \( k_j \) is the order of the subtree \( \vartheta_j, j = 1, \ldots, m \).

[5.1] If \( m \geq 2 \), then (5.1b) follows immediately.

[5.2] If \( m = 0 \), then \( M_n(\vartheta) = 1 \). As \( \ell \) is on scale \( n \), the order \( k \) of \( \vartheta \) has to be \( k \geq q_n \), so that:

\[ M_n(\vartheta) = 1 \leq \frac{2k}{q_n} - 1, \quad N^R_n(\vartheta) = 0, \]  

(5.24)

and (5.1b) follows again.

[5.3] If \( m = 1 \), then (5.23) gives:

\[ M_n(\vartheta) \leq 1 + \left( \frac{2k_1}{q_n} - 1 \right) + N^R_n(\vartheta_1) = \frac{2k_1}{q_n} + N^R_n(\vartheta_1). \]  

(5.25)

Denoting with \( \nu \) and \( \nu_1 \) the momenta flowing, respectively, through the root line \( \ell \) of \( \vartheta \) and through the root line \( \ell_1 \) of \( \vartheta_1 \), we have:

\[ ||\omega(\nu - \nu_1)|| \leq ||\omega\nu|| + ||\omega\nu_1|| \leq \frac{1}{4q_n}, \]  

(5.26)

as both \( \ell \) and \( \ell_1 \) are on scale \( \geq n \) (see remark 2 in page 4 and use (2.14)). Then, as \( \nu_T = \nu - \nu_1 \), either \( |\nu_T| \geq q_n \) or \( \nu_T = 0 \).

[5.3.1] If \( |\nu_T| \geq q_n \), then \( k_T \geq |\nu_T| \geq q_n \) and \( N^R_n(\vartheta_1) + \cdots + N^R_n(\vartheta_m) = N^R_n(\vartheta) \) (since \( T \) is not a resonance), so that (5.25) gives:

\[ M_n(\vartheta) \leq \frac{2k}{q_n} - \frac{2k_T}{q_n} + N^R_n(\vartheta_1) \leq \frac{2k}{q_n} - 1 + N^R_n(\vartheta_1), \]  

(5.27)

and (5.1b) follows.

[5.3.2] If \( \nu_T = 0 \) and \( k_T \geq q_n \), one can reason as in case [5.3.1].

[5.3.3] If \( \nu_T = 0 \) and \( k_T < q_n \), then \( \nu_1 = \nu \) and either \( n_\ell_1 = n \) or \( n_\ell_1 = n + 1 \) (see remark 1 in page 4); then \( T \) is a resonance with resonance scale \( n \), so that:

\[ 1 + N^R_n(\vartheta_1) + \cdots + N^R_n(\vartheta_m) = N^R_n(\vartheta), \]  

(5.28)

and (5.1) follows again.

[6] If the root line \( \ell \) of \( \vartheta \) is on scale \( > n \) and it is the exiting line of a resonance \( V_n \), as in case [3] above, denote with \( \ell_1, \ldots, \ell_m \) the lines on scale \( \geq n \) which are nearest to the root line of \( \vartheta \), and let \( \vartheta_1, \ldots, \vartheta_m \) be the subtree of \( \vartheta \) of which these lines are root lines. Some of these lines – at least one – are lines on scale \( n \) inside \( V_n \). Let \( T \) be the cluster which the lines \( \ell_1, \ldots, \ell_m \) enter; of course \( T \subset V_n \), and \( T \) can degenerate into a single point.

Note that as in case [3]:

\[ M_n(\vartheta) = 1 + M_n(\vartheta_1) + \cdots + M_n(\vartheta_m), \]  

(5.29)
as the root line $\ell$ contributes one unit to $P_n(\vartheta)$ and does not contribute to $N_n(\vartheta)$, and that if $T$ is a resonance then its resonance scale is $n$.

[6.1] If $T$ is not a resonance, then:

$$N^R_n(\vartheta) = N^R_n(\vartheta_1) + \cdots + N^R_n(\vartheta_m).$$

By induction, (5.1) and (5.29) imply:

$$M_n(\vartheta) \leq 1 + \sum_{j=1}^{m} \left( \frac{2k_j}{q_n} - 1 \right) + \sum_{j=1}^{m} N^R_n(\vartheta_j),$$

where $k_j$ are the orders of the subtrees $\vartheta_j$, $j = 1, \ldots, m$.

[6.1.1] If $m = 2$, then (5.1b) follows immediately.

[6.1.2] The case $m = 0$ is impossible (see case [3.1.2]).

[6.1.3] If $m = 1$ in (5.31), we have $\nu_T = \nu - \nu_1$, so that $|\nu_T| \geq q_n$ (as $\nu_T \neq 0$, otherwise $T$ would be a resonance). Then we can go on along the lines of case [5.3.1] in order to obtain (5.1b).

[6.2] If $T$ is a resonance, then its resonance scale is $n$, so that:

$$N^R_n(\vartheta) = 1 + N^R_n(\vartheta_1),$$

and the discussion goes on as in case [6.1], with the only difference that now, for $m = 1$, the case $\nu_T = 0$ is the only possible as $T$ is supposed to be a resonance. In such a case:

$$M_n(\vartheta) \leq 1 + \left( \frac{2k}{q_n} - 1 \right) + N^R_n(\vartheta_1) \leq \frac{2k}{q_n} - 1 + N^R_n(\vartheta),$$

implying again (5.1b).

Finally, to deduce (2.19) from (5.1) and (5.2), simply note that, for $q_{n+1} \leq 4q_n$, we have $2k/q_n \leq 8k/q_{n+1}$; then lemma 5 follows.

**Remark** 15. Note that the correspondence between momenta and scale labels has been used only through the inequality (2.11). As we have seen in sect. 4 the renormalization procedure can shift the “original” momenta flowing through the lines of a bounded quantity which does not alter such an inequality. This allow us to apply lemma 4 also to the renormalized trees, as it was repeatedly claimed in the previous sections.

6. **Proof of lemma 8**

As far as only the localized resonance factor is involved, the momenta flowing through the lines entering any resonance are set to zero, so that it does not matter if such momenta are interpolated or not (i.e. if they are of the form $\nu$ or $\nu(t)$). In particular, the case of first generation resonances (discussed in sect. 3) is included in lemma 8.
A basic property of the trees belonging to the resonance family \( \mathcal{F}_V(\vartheta) \) is that the difference between their values is only in the resonance factor: for any tree \( \vartheta' \in \mathcal{F}_V(\vartheta) \), we can write:

\[
\text{Val}(\vartheta') = A(\vartheta) \mathcal{V}_V(\vartheta'),
\]

for some factor \( A(\vartheta) \) which is the same for all \( \vartheta' \in \mathcal{F}_V(\vartheta) \). This simply follows from the fact that the transformations in \( \mathcal{P}_V \) do not touch the part of the tree \( \vartheta \) which is outside the resonance \( V \). Therefore a cancellation between localized resonance factors yields a cancellation between tree values (in which the resonance factor has been localized of course).

By item 1 in the definition of resonance and by definition of \( V_0 \), one has

\[
\sum_{u \in V_0} \nu_u = 0; \tag{6.2}
\]

moreover, given an entering line \( \ell_m \) of \( V \), if \( \ell_m \in L^R_V \) and \( \tilde{V}_0 = V_0(\ell_m) \), then

\[
\sum_{u \in \tilde{V}_0} \nu_u \equiv \sum_{u \in V_0(\ell_m)} \nu_u = 0. \tag{6.3}
\]

In general we can write, for any tree \( \vartheta' \in \mathcal{F}_V(\vartheta) \),

\[
\mathcal{L} \mathcal{V}_V(\vartheta') = \mathcal{B}(\vartheta') \mathcal{L} \mathcal{V}_0(\vartheta') \prod_{\ell \in L^R_V} \mathcal{L} \mathcal{V}_{V(\ell)}(\vartheta'), \tag{6.4}
\]

where \( \mathcal{V}_0(\vartheta') \) and \( \mathcal{V}_{V(\ell)}(\vartheta') \) are defined as the resonance factor \( \mathcal{V}_V(\vartheta') \), but with the product ranging only over nodes and lines internal to \( V_0 \) and \( V(\ell) \), respectively, while \( \mathcal{L} \mathcal{V}_0(\vartheta') \) and \( \mathcal{L} \mathcal{V}_{V(\ell)}(\vartheta') \) are obtained from \( \mathcal{V}_0(\vartheta') \) and \( \mathcal{V}_{V(\ell)}(\vartheta') \), respectively, by replacing \( \nu_\ell \) with \( \nu^0_\ell \) in \( V \), for all lines \( \ell \in V \). In (6.4) \( \mathcal{B}(\vartheta') \) takes into account all other factors (if there are any), always evaluated with \( \nu_\ell \) replaced with \( \nu^0_\ell \), \( \ell \in V \). Note that, as \( A(\vartheta) \) in (6.1), also \( \mathcal{B}(\vartheta') \) is the same for all \( \vartheta' \in \mathcal{F}_V(\vartheta) \), so that one can set \( \mathcal{B}(\vartheta') = \mathcal{B}(\vartheta) \) and write:

\[
\text{Val}(\vartheta') = A(\vartheta) \mathcal{V}_V(\vartheta'), \quad \mathcal{L} \mathcal{V}_V(\vartheta') = \mathcal{B}(\vartheta) \mathcal{L} \mathcal{V}_0(\vartheta') \prod_{\ell \in L^R_V} \mathcal{L} \mathcal{V}_{V(\ell)}(\vartheta'). \tag{6.5}
\]

[1] If \( z_V = 1 \) the localized resonance factor is given by the resonance factor computed for \( \mu_1 = \cdots = \mu_m = 0 \).

Summing the localized resonance factors corresponding to the trees belonging to \( \mathcal{F}_V(\vartheta) \), we can group them into subfamilies of inequivalent trees whose contributions are different as for each node \( u \in V \) there is a factor:

\[
\frac{1}{m_u!} \left( \frac{m_u}{s_u! r_u!} \right) \tag{6.6}
\]

as all terms which are obtained by permutations are summed together (this gives the binomial coefficient in the left hand side of the above equation), times a factor:

\[
\nu_u^{m_u+1} = \nu_u^{(s_u+1)+r_u}, \tag{6.7}
\]

times a propagator \( g_{n_{\ell_u}}(\nu^0_{\ell_u}) \) (the last factor is missing if corresponding to the line exiting \( V \); see definitions (4.3)÷(4.6)).
Then for \(\mu_1 = \cdots = \mu_m = 0\) we can write:

\[
\sum_{\vartheta' \in \mathcal{F}_V(\vartheta')} \mathcal{L} \mathcal{V}_V(\vartheta') = \sum_{\vartheta' \in \mathcal{F}_V(\vartheta')} \left[ \prod_{\upsilon \in V} \frac{\nu_u^{s_u+1}}{s_u!} \right] \left[ \prod_{\ell \in V} g_{n_{x}(\nu_{\ell}^{b})} \right].
\]

\[
\cdot \left( \prod_{u \in V_0} \frac{\nu_u^{s_u+1}}{r_u!} \right) \left( \prod_{\ell \in L^0_\nu u \in V_0(\ell)} \frac{\nu_{\ell}^{s_\ell}}{r_{\ell}!} \right) = \left[ \prod_{u \in V} \frac{\nu_u^{s_u+1}}{s_u!} \right] \left[ \prod_{\ell \in V} g_{n_{x}(\nu_{\ell}^{b})} \right].
\]

\[
= \left[ \prod_{u \in V \setminus V_0} \frac{\nu_u^{s_u+1}}{r_u!} \right] \left( \prod_{\ell \in L^0_\nu u \in V_0(\ell)} \frac{\nu_{\ell}^{s_\ell}}{r_{\ell}!} \right),
\]  

where we have used the fact that for \(\mu_1 = \cdots = \mu_m = 0\) the factors in square brackets have the same value for all \(\vartheta' \in \mathcal{F}_V(\vartheta)\) (see (3.11) and take into account what observed at the beginning of this section). The last sum in (6.8) can be rewritten as:

\[
\sum_{\vartheta' \in \mathcal{F}_V(\vartheta')} \left( \prod_{u \in V_0} \frac{\nu_u^{s_u}}{r_u!} \right) \left( \prod_{\ell \in L^0_\nu u \in V_0(\ell)} \frac{\nu_{\ell}^{s_{\ell}}}{r_{\ell}!} \right) = \left( \sum_{u \in V_0} \nu_u \right)^{m_{V_0}} \left( \prod_{V \in V(V)} \nu_V \right) \left( \sum_{u \in V_0} \nu_u \right),
\]

which is zero by definition of resonance (see (6.2) and (6.3) above).

[2] If \(z_V = 2\) the localized resonance factor, with respect to the previous case, contains also the first order terms (again computed in \(\mu_1 = \cdots = \mu_m = 0\)).

The zero-th order contribution can be discussed as for the case \(z_V = 1\), and the same result holds. Also the second order contribution vanishes, after summing over the trees \(\vartheta' \in \mathcal{F}_V(\vartheta)\). To prove this we shall consider separately the cases \(m_V = 2\) and \(m_V = 1\).

In the first case, when the derivative \((\partial/\partial \mu_m)\mathcal{V}_V(\vartheta; 0, \ldots, 0)\) is considered, let us compare all the trees \(\vartheta'\) in the subfamily of \(\mathcal{F}_V(\vartheta)\) in which the line \(\ell_m\) is kept fixed (call \(\bar{u}\) the node which such a line enters), while all other lines are shifted (i.e. detached and reattached to all nodes inside the resonance). The difference with respect to the previous case, discussed above, is that the line with momentum \(\nu_{\ell_m}\) can be chosen in \(r_{\bar{u}}\) ways among the \(r_{\bar{u}}\) lines entering the node \(\bar{u} \in V\) and outside \(V\). This means that we can write:

\[
\frac{\nu_u^{m_u+1}}{m_u!} \left( \frac{s_u}{s_u!} r_u! \right) = \nu_u^{(s_u+1)+r_u} \frac{s_u! r_u!}{s_u! r_u!}
\]

for all nodes \(u \neq \bar{u}\), and:

\[
\frac{\nu_{\bar{u}}^{m_u}}{m_{\bar{u}}!} \left( \frac{s_{\bar{u}}}{s_{\bar{u}}!} r_{\bar{u}} \right) = \nu_{\bar{u}}^{(s_{\bar{u}}+1)+(r_{\bar{u}}-1)} \frac{s_{\bar{u}}! (r_{\bar{u}}-1)!}{s_{\bar{u}}! (r_{\bar{u}}-1)!}.
\]
for \( \bar{u} \). Then we have an expression analogous to (6.8), with the only difference that the labels \( \{r_u\} \) have to be replaced with labels \( \{r'_u\} \), defined as:

\[
r'_u = r_u - \delta_{u\bar{u}}, \quad \forall u \text{ either in } V_0 \text{ or in } \bigcup_{\tilde{V} \in V(V)} \tilde{V},
\]

such that

\[
\sum_{u \in V_0} r'_u + \sum_{\tilde{V} \in V(V)} \sum_{u \in \tilde{V}_0} r'_u = m_V - 1;
\]

so the last sum in the second line of (6.8) has to be replaced by:

\[
\sum_{\vartheta' \in F(V)} \left( \prod_{u \in V_0} \frac{\nu'_{r_u}}{r_u} \right) \prod_{\ell \in L} \prod_{u \in V_0} \frac{\nu'_{r_u}}{r_u} \nu_{r_u} = \frac{1}{m_{V_0}^*} \left( \sum_{u \in V_0} \nu_{r_u} \right) \prod_{\tilde{V} \in V(V)} \left( \sum_{u \in \tilde{V}} \nu_{r_u} \right) \zeta^*(\tilde{V}),
\]

where

\[
m_{V_0}^* = \begin{cases} m_{V_0}, & \text{if } \bar{u} \notin V_0, \\ m_{V_0} - 1, & \text{if } \bar{u} \in V_0, \end{cases} \quad \zeta^*(\tilde{V}) = \begin{cases} 1, & \text{if } \bar{u} \notin \tilde{V}_0, \\ 0, & \text{if } \bar{u} \in \tilde{V}_0, \end{cases}
\]

so that we have again vanishing contributions (as \( m_V \geq 2 \)).

On the contrary, if \( m_V = 1 \), the above reasoning does not apply, as there is only one entering line. Anyway the function \((\partial/\partial \mu_1)V(\vartheta;0)\) is an odd function, as all the propagators are even in their arguments, so that the derived one\(^5\) becomes odd, and the numerator contains an even number of \( \nu_{r_u} \)'s. Then by reversing the signs of the labels \( \nu_{r_u} \), \( u \in V \), the numerator will not change, while the overall sign of the denominator will change, so that the sum over the first order contributions of the localized resonance factors of the two tree values being considered vanishes.\(^6\)

[3] Finally if \( z_V = 0 \) the localization operator \( L \) gives zero when acting on the resonance factors, so that nothing has to be proved.

---

\(^5\) If \( z_V = 2 \), then there is only one derived propagator, arising from the renormalization of the resonance \( V \) itself.

\(^6\) Note that the renormalization transformations of type 3 are explicitly used in order to implement the cancellation mechanism only in the case of a resonance \( V \) with \( z_V = 2 \) and \( m_V = 1 \). In general not all the transformations are used for all resonances: in particular, when \( z_V = 0 \), we consider separately all terms generated by the action of the group \( \mathcal{P}_V \), as there is no need of additional renormalizations.
7. Conclusions

Our theorem can be related to the result and the methods of [1]. There we proved that, for \( \omega \in \mathbb{C} \), if \( \omega \) tends to a rational number \( p/q \) through a path in the complex plane non-tangential to the real axis, then the radius of convergence satisfies:

\[
\left| \log \rho(\omega) + \frac{2}{q} \log |\omega - \frac{p}{q}| \right| < C_4
\]

for some constant \( C_4 \).

If instead we consider a sequence of real, irrational numbers tending to a rational value \( p/q \), the situation is quite more complex. In fact, the limit and its very existence may depend on the arithmetic properties of the numbers of the sequence we consider, and on their uniformity in \( k \); namely:

1. The sequence \( \{\omega_k\} \) can tend to \( p/q \) but, though all the \( \omega_k \) are irrational, some of them are not Bryuno numbers so that for those \( B(\omega_k) = +\infty \) and \( \rho(\omega_k) = 0 \).

2. The sequence \( \{\omega_k\} \) can tend to \( p/q \) through Bryuno numbers, or even Diophantine numbers, but they are not uniformly such in \( k \) so that \( B(\omega_k) \) diverges faster than \( \log(|\omega_k - p/q|^{1/q}) \) (and so \( \rho(\omega_k) \) tends to zero faster than \( |\omega_k - p/q|^{2/q} \)). An example can be the sequence of Diophantine (actually even "noble") numbers:

\[
\omega_k = \frac{1}{k + \frac{1}{2k^2 + \gamma}},
\]

where \( \gamma \) denotes the “golden mean”:

\[
\gamma = \frac{1}{1 + \frac{1}{1 + \cdots}} = \frac{\sqrt{5} - 1}{2};
\]

a simple calculation using the recursion relation (1.7) shows that indeed \( B(\omega_k) = O(k) \) while \( \omega_k = O(1/k) \), so that, by taking into account also logarithmic corrections in \( B(\omega_k) \), \( \rho(\omega_k) = O(\omega_k^2 e^{-2/\omega_k}) \), that is much faster than \( \omega_k^2 \).

3. Finally, the sequence \( \{\omega_k\} \) can tend to \( p/q \) through a sequence of Bryuno numbers satisfying uniform estimates in \( k \), so that an estimate like (7.1) holds (note that decays slower than \( |\omega_k - p/q|^{2/q} \) are not possible); an example can be given by the sequence:

\[
\omega_k = \frac{1}{k + \gamma},
\]

where again \( \gamma \) is the golden mean (7.3).

Notice that in the numerical calculations of [14] only real sequences of type 3 were considered, and that sequences of type 2 are practically inaccessible from the numerical point of view.

Finally, one may ask how much these results can be extended to more complicated, and realistic, symplectic maps and continuous time Hamiltonian systems. We
believe that while some additional complications may arise, the really hard problem
(i.e., how to handle resonances) is already present in the standard map and it was
solved by carefully using the trees formalism and the multiscale decomposition of
the propagators. More general maps and Hamiltonian systems, though, as already
pointed out in [1], may have different, more complicated interpolation properties
for the radius of convergence of their Lindstedt series: this is an area where still
much work has to be done.

REFERENCES

[1] Berretti, A., and Gentile, G., Scaling Properties for the Radius of Convergence of a Lindstedt
Series: the Standard Map, to appear in Journal de Mathématiques Pures et Appliquées.
[2] Yoccoz, J. C., Théorème de Siegel, nombres de Brjuno and polinômes quadratiques,
Astérisque 231 (1995), 3–88
[3] Marmi, S., Moussa, P., and Yoccoz, J. C., The Brjuno Functions and their Regularity Prop-
terties, Communications in Mathematical Physics 186 (1997), 265–293
[4] Davie, A. M., The Critical Function for the Semistandard Map, Nonlinearity 7 (1994), 219–
229
[5] Davie, A. M., Renormalization for Analytic Area-Preserving Maps, unpublished
[6] Gentile, G. and Mastropietro, V., Methods for the Analysis of the Lindstedt Series for KAM
tori and Renormalizability in Classical Mechanics. A Review with Some Applications, Re-
views in Mathematical Physics 8, No. 3 (1996), 393–444
[7] Eliasson, L. H., Absolutely Convergent Series Expansions for Quasi-periodic Motions,
University of Stockholm preprint (1988), and Math. Phys. Elect. J. 2, No. 4 (1996),
http://mpej.unige.ch/mpej/Vol/2/4.ps
[8] Gallavotti, G., Twistless KAM tori, Communications in Mathematical Physics 164 (1994),
145–156
[9] Marmi, S., and Stark, J., On the Standard Map Critical Function, Nonlinearity 5 (1992),
743–761
[10] Harary, P., and Palmer, E., Graphical Enumeration, Academic Press, (New York, 1973)
[11] Gallavotti, G., Gentile, G., Majorant series convergence for twistless KAM tori, Ergodic
Theory and Dynamical Systems 15 (1995), 857–869
[12] Schmidt, W. M., Diophantine Approximation, Lecture Notes in Mathematics 785, Springer
(Berlin, 1980)
[13] Bonetto, F., Gallavotti, G., Gentile, G., and Mastropietro, V., Lindstedt Series, Ultraviolet
Divergences and Moser’s Theorem, Annali della Scuola Normale Superiore di Pisa Cl. Sci.
(4), 26, No. 3 (1998), 545–593
[14] Berretti, A., and Marmi, S., Scaling near Resonances and Complex Rotation Numbers for the
Standard Map, Nonlinearity 7 (1994), 603–621

ALBERTO BERRETTI, DIPARTIMENTO DI MATEMATICA, II UNIVERSITÀ DI ROMA (TOR VERGATA),
VIA DELLA RICERCA SCIENTIFICA, 00133 ROMA, ITALY AND ISTITUTO NAZIONALE DI FISICA NU-
CLEASE, SEZ. TOR VERGATA
E-mail address: berretti@roma2.infn.it

GUIDO GENTILE, DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI ROMA 3, LARGO S. LEONARDO
MURIALDO 1, 00146 ROMA, ITALY
E-mail address: gentile@matrm3.mat.uniroma3.it