Ultrametric analogues of Ulam stability of groups

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Abstract

We study stability of metric approximations of countable groups with respect to groups endowed with ultrametrics, the main case study being a p-adic analogue of Ulam stability, where we take $\text{GL}_n(\mathbb{Z}_p)$ as approximating groups instead of $\text{U}(n)$. For finitely presented groups, the ultrametric nature implies equivalence of the pointwise and uniform stability problems, and the profinite one implies that the corresponding approximation property is equivalent to residual finiteness. Moreover, a group is uniformly stable if and only if its largest residually finite quotient is. We provide several examples of uniformly stable groups: these include finite groups, virtually free groups, some groups acting on rooted trees, and certain lamplighter and (Generalized) Baumslag–Solitar groups.

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1 Introduction

Let $\Gamma$ be a countable discrete group, and let $G$ be a family of groups $G$ equipped with bi-invariant metrics $d_G$. The question of stability of $\Gamma$ with respect to $G$ asks whether a map $\varphi : \Gamma \to G \in G$ that is a homomorphism up to a small error, is close to an actual homomorphism. This can be made rigorous in two different ways, that depend on whether one wants the errors and closeness to be pointwise or uniform. The notion of “error” is defined as follows:

**Definition 1.1.** Let $\varphi : \Gamma \to G \in G$ be a map. We define the *defect of $\varphi$ at $(g, h) \in \Gamma^2$* to be

$$\text{def}_{g,h}(\varphi) := d_G(\varphi(gh), \varphi(g)\varphi(h)).$$

The *defect of $\varphi$* is

$$\text{def}(\varphi) := \sup_{g,h \in \Gamma} \text{def}_{g,h}(\varphi).$$

A sequence $(\varphi_n : \Gamma \to G_n \in G)_{n \geq 1}$ is called a *pointwise asymptotic homomorphism* if $\text{def}_{g,h}(\varphi_n) \xrightarrow{n \to \infty} 0$ for all $(g, h) \in \Gamma^2$; and a *uniform asymptotic homomorphism* if $\text{def}(\varphi_n) \xrightarrow{n \to \infty} 0$.

Other commonly used terms are *almost-representation* [MM98] for the pointwise notion and *quasi-representation* [Sht98] or *$\varepsilon$-representation* [Kaz82] for the uniform notion. The notion of “closeness” is defined as follows:

**Definition 1.2.** Given two maps $\varphi, \psi : \Gamma \to G \in G$, we define their *distance at $g \in \Gamma$* to be

$$\text{dist}_g(\varphi, \psi) := d_G(\varphi(g), \psi(g));$$

and their *distance* to be

$$\text{dist}(\varphi, \psi) := \sup_{g \in \Gamma} \text{dist}_g(\varphi, \psi).$$

Two asymptotic homomorphisms $(\varphi_n, \psi_n : \Gamma \to G_n \in G)_{n \geq 1}$ are *pointwise asymptotically close* if $\text{dist}_g(\varphi_n, \psi_n) \xrightarrow{n \to \infty} 0$ for all $g \in \Gamma$; and *uniformly asymptotically close* if $\text{dist}(\varphi_n, \psi_n) \xrightarrow{n \to \infty} 0$.

This leads to the definition of two notions of stability, that we attribute to Ulam, after [Ula60]:

**Definition 1.3** (Ulam). The group $\Gamma$ is *pointwise $G$-stable* if any pointwise asymptotic homomorphism is pointwise asymptotically close to a sequence of homomorphisms. It is *uniformly $G$-stable* if any uniform asymptotic homomorphism is uniformly asymptotically close to a sequence of homomorphisms.

Early mentions of these problems can be found in works of von Neumann [vN29] and Turing [Tur38]. The problem of pointwise stability of $\mathbb{Z}^2$ with respect to certain families of matrices, for instance self-adjoint and with the operator norm, received a lot of attention during the second half of the twentieth century (see [Lin97] and the references therein). In [Ula60, Chapter 6], Ulam discusses more generally the question of stability of certain functional equations: because of this, the term *Ulam stability* was introduced in [BOT13] to refer to uniform stability with respect to unitary groups equipped with the distance induced by the operator norm (see below). Some of the most common families $G$ of approximating groups are the unitary groups $U(n)$ or the symmetric groups $S_n$.

$U(n)$ is typically considered with a metric induced by a norm defined on $M_n(\mathbb{C})$. The first example is that of the *operator norm*, where pointwise stability has striking topological and $K$-theoretic
interpretations [CGM90, Dad20], all amenable groups are known to be uniformly stable [Kaz82], and groups with non-trivial quasimorphisms are known not to be uniformly stable [BOT13]. Another example is that of the Frobenius norm \( \|A\|_{\text{Frob}} := \sqrt{|A_{ij}|^2} \), that is, the norm induced by the embedding of \( U(n) \) into \( \mathbb{C}^{n \times n} \): this has the advantage of allowing a cohomological criterion for pointwise stability [DCGLT20]. The third main example is given by the Hilbert–Schmidt norm \( \|A\|_{\text{HS}} := \frac{1}{\sqrt{n}}\|A\|_{\text{Frob}} \), which is the normalization of the Frobenius norm: this has the advantage of allowing a \( C^* \)-algebraic characterization of pointwise stability [HS18], as well as a simple algebraic characterization of uniformly stable groups among finitely generated residually finite ones [AD20].

On the other hand the groups \( S_n \) are studied with the normalized Hamming distance \( d_H(\sigma, \tau) := 1 - \frac{1}{n} |\text{Fix}(\sigma^{-1}\tau)| \). Pointwise stability of equations in permutation was initially considered by Glebsky and Rivera [GR09], then by Arzhantseva and Păunescu [AP15] who proved that this can be translated to a group property, as in Definition 1.3. Since then this pointwise stability problem has been under intense investigation, as well as some variants thereof: flexible [BL20, LLM19], quantitative [BM18], uniform, probabilistic [BC20], and connections to computer science [BML20].

The pointwise and uniform problems typically exhibit a very different behaviour. For example, consider the family \( \mathcal{G} = \{(U(n), \| \cdot \|_{op}) : n \geq 1\} \) of unitary groups equipped with the metric induced by the operator norm, and the two stability problems with respect to \( \mathcal{G} \). On the one hand, \( \mathbb{Z}^2 \) is not pointwise stable [Voi83], but it is uniformly stable, as are all amenable groups [Kaz82]. On the other hand, a non-abelian free group of finite rank is not uniformly stable [Rol09], but it is pointwise stable: if \( (\varphi_n)_{n \geq 1} \) is a pointwise asymptotic homomorphism, then letting \( \psi_n \) be the unique homomorphism that coincides with \( \varphi_n \) on a given free basis, \( (\psi_n)_{n \geq 1} \) is pointwise asymptotically close to \( (\varphi_n)_{n \geq 1} \).

In this paper, we study ultrametric versions of these problems, that is, we look at approximating families \( \mathcal{G} \) whose groups are ultrametric. The main example throughout the paper will be a \( p \)-adic analogue of Ulam stability: we choose \( \text{GL}_n(\mathbb{Z}_p) \) – which is maximal compact in \( \text{GL}_n(\mathbb{Q}_p) \) – as an analogue of \( U(n) \) – which is maximal compact in \( \text{GL}_n(\mathbb{C}) \). The natural norm on \( \mathbb{Q}_p \)-vector spaces, that is, the one that preserves the non-Archimedean nature, is the \( \ell^\infty \)-norm relative to the \( p \)-adic norm \( | \cdot |_p \) on \( \mathbb{Q}_p \). Keeping this and the case of \( U(n) \) in mind, there are three norms that one could choose to induce a distance on \( \text{GL}_n(\mathbb{Z}_p) \): the operator norm with respect to the \( \ell^\infty \)-norm on \( \mathbb{Q}_p^n \), the norm induced by the embedding into \( \mathbb{Q}_p^{n \times n} \) with the \( \ell^\infty \)-norm, and a normalized version of the latter. It turns out that all of these coincide (Lemma 3.15), and so

\[
\| \cdot \| : \text{Mat}_n(\mathbb{Q}_p) \to \mathbb{R}_{\geq 0} : A = (A_{ij})_{1 \leq i,j \leq n} \mapsto \max_{1 \leq i,j \leq n} |A_{ij}|_p
\]
is, in some sense, the canonical norm to consider. It induces a bi-invariant ultrametric \( d \), and moreover it reflects the profinite structure of \( \text{GL}_n(\mathbb{Z}_p) \): in fact \( \| A - I \| \leq p^{-k} \) if and only if \( A \equiv I \mod p^k \). We denote by \( \text{GL}(\mathbb{Z}_p) \) this family of metric groups, and will focus on this example of approximating family for the statements of the results, mentioning which properties we are using. Each result can be generalized to families of groups satisfying such properties, and the statements will be given in full generality in the paper.

To the author’s knowledge, the only previous mention of \( p \)-adic versions of stability is in Kazhdan’s work [Kaz82 Proposition 1], where it is shown that for every \( n \geq 1 \) the standard representative map \( \varphi : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}_p \) satisfies \( \text{def}(\varphi) = p^{-n} \) and \( \text{dist}(\varphi, \psi) = 1 \) for every homomorphism \( \psi \). This result does not however show that these groups are unstable with respect to the family \( \{ \mathbb{Z}_p \} \):
this is indeed never the case as we will see in Proposition 4.23. However, the fact that these bad estimates for stability arise when looking at maps from a finite $p$-group to a pro-$p$ group is not a coincidence, as appears from the results in Sections 6 and 7.

Using only the ultrametric inequality, we prove a relation between the pointwise and uniform stability problems, that as we have seen above does not hold in the Archimedean setting (Theorem 4.10):

**Theorem 1.4.** Let $\Gamma$ be finitely generated and pointwise $GL(\mathbb{Z}_p)$-stable. Then $\Gamma$ is uniformly $GL(\mathbb{Z}_p)$-stable. If moreover $\Gamma$ is finitely presented, then the converse holds.

The techniques developed for the proof of Theorem 4.10 also apply further: in Proposition 4.23 we show that if $\mathcal{G}$ is a finite family of compact ultrametric groups, then any finitely generated group is uniformly $\mathcal{G}$-stable. This also does not hold in the Archimedean setting: for instance if $\mathcal{G} = \{(U(1), \| \cdot \|_{op})\}$, then a non-abelian free group is not uniformly $\mathcal{G}$-stable [Rol09].

Using the fact that the metric reflects the profinite structure, we are able to reduce the uniform stability problem to residually finite groups (Theorem 4.14):

**Theorem 1.5.** Let $\Gamma$ be a group, and $R$ its largest residually finite quotient. Then $\Gamma$ is uniformly $GL(\mathbb{Z}_p)$-stable if and only if $R$ is. If $\Gamma$ is pointwise $GL(\mathbb{Z}_p)$-stable, then so is $R$.

This theorem implies in particular that a group without finite quotients is uniformly $GL(\mathbb{Z}_p)$-stable. An analogous result holds for pointwise stability, as we shall shortly see.

The techniques developed for the proof of Theorem 1.5 also apply further: in Propositions 4.22 and 4.25 we provide the complete solution to three other stability problems. The first one is with respect to the family $T(R)$ of invertible upper-triangular matrices over a commutative ring $R$; the second one is with respect to the family $Aut(X^*)$ of automorphism groups of regular rooted trees of increasing degrees; the third one is with respect to the family $Gal(K)$ of Galois groups associated to all Galois extensions of a field $K$ (which needs to admit only countably many finite Galois extensions for the groups to be metrizable: see Lemma 3.12). Under natural ultrametrics that reflect a projective structure, we prove that all finitely generated groups are uniformly stable in the last case, and all groups are uniformly stable in the first two cases.

Related to the stability problem is the corresponding approximation problem. We attribute the following definition to Gromov, after [Gro99]:

**Definition 1.6 (Gromov).** A sequence $(\phi_n : \Gamma \to G_n \in \mathcal{G})_{n \geq 1}$ is **asymptotically injective** if

$$\liminf_{n \to \infty} d_{G_n}(\phi_n(g), 1) > 0$$

for every $1 \neq g \in \Gamma$. A pointwise asymptotic homomorphism that is also asymptotically injective is called a $\mathcal{G}$-**approximation**: if one exists, $\Gamma$ is said to be $\mathcal{G}$-**approximable**.

This leads to the important notions of sofic groups, when $\mathcal{G} = \{(S_n, d_H) : n \geq 1\}$, introduced by Gromov [Gro99] and named by Weiss [Wei00]; and hyperlinear groups, when $\mathcal{G} = \{(U(n), \| \cdot \|_{HS}) : n \geq 1\}$, introduced by Radulescu [Rad00] in the context of the Connes embedding conjecture [Con76]. These classes of groups are very large, so large that no non-example is known to date. In contrast, the profinite nature of $GL_n(\mathbb{Z}_p)$ allows to characterize approximation in terms of other well-studied properties (Propositions 5.3 and 5.11):
Theorem 1.7. A countable group is \( \text{GL}(\mathbb{Z}_p) \)-approximable if and only if it is LEF (locally embeddable in the class of finite groups). In particular, a finitely presented group is \( \text{GL}(\mathbb{Z}_p) \)-approximable if and only if it is residually finite.

The class of LEF groups was formally introduced by Gordon and Vershik in [VG97], although it is already present in Malcev’s work [Mal40]: we refer the reader to Subsection 2.2 for the precise definitions. We are also able to characterize strong approximability, where the approximation is required to be a uniform asymptotic homomorphism, as being equivalent to residual finiteness, for arbitrary countable groups.

By an argument due to Arzhantseva and Păunescu [AP15] (see Lemma 2.2), a group that is both \( G \)-approximable and pointwise \( G \)-stable is fully residually-\( G \). Using this fact and Theorem 1.7 any group that is LEF but not residually finite is not pointwise \( \text{GL}(\mathbb{Z}_p) \)-stable (Corollary 5.12). This gives several examples of non-pointwise stable finitely generated groups (Examples 5.13 and 5.14), proving that both Theorem 1.4 and Theorem 1.5 are sharp. Indeed, there exist finitely generated groups that are uniformly stable but not pointwise stable; and there exist finitely generated groups that are not pointwise stable but whose largest residually finite quotient is. Moreover, the techniques developed for the proof of Theorem 1.7 allow to characterize approximability for a few other families (Corollaries 5.5 and 5.6), and to prove a pointwise version of Theorem 1.5 (Proposition 5.15), where the largest residually finite quotient is replaced by the largest LEF quotient. This last result is analogous to the fact that a group is pointwise stable in permutation if and only if its largest sofic quotient is.

Going back to stability, the strongest results that are proven in this paper concern ultrametric families with some restriction on the order of their finite quotients. Namely, using that the groups \( \text{GL}_n(\mathbb{Z}_p) \) are virtually pro-\( p \), we can prove stability results for fundamental groups of graphs of groups with some restriction on the orders of finite quotients. These include the following classes of examples (see Section 6):

**Theorem 1.8.** The following groups are uniformly \( \text{GL}(\mathbb{Z}_p) \)-stable:

1. Groups without finite virtual \( p \)-quotients.
2. Finitely generated virtually free groups without elements of order \( p \).
3. Baumslag–Solitar groups \( \text{BS}(m,n) \), whenever \( p \) divides exactly one of \( m \) and \( n \).
4. \( \mathbb{Z}[\frac{1}{mn}] \rtimes \mathbb{Z}_m \), for \( m, n \) as above, if moreover \( (m,n) = 1 \) and \( 1 \neq |m| \neq |n| \neq 1 \).
5. Wreath products \( G \wr \mathbb{Z} \), whenever \( G \) does not surject onto \( \mathbb{F}_p \).

Groups as in 1. include all periodic groups without elements of order \( p \) (Example 6.5), as well as groups of automorphisms of regular rooted trees of degree smaller than \( p \) with the congruence subgroup property (Example 6.6). Item 3., with the appropriate \( p \), applies to every non-Hopfian Baumslag–Solitar group and every residually finite Baumslag–Solitar group, with the exception of \( \mathbb{Z}^2 \) and the Klein bottle group [McS72]. The group from 4. is the largest residually finite quotient of \( \text{BS}(m,n) \) [Mol10], so it also provides an example of an infinitely presented pointwise stable group, by Item 3. and Theorem 1.5.
The proof of Theorem 1.8 relies on the Schur–Zassenhaus Theorem (Theorem 2.18), which states that any extension of finite groups with coprime orders splits, and that any two splittings are conjugate. The first part is used to prove Item 1, the second one is used to treat graphs of groups. All these results are quantitatively precise, in particular, the quantitative estimates involved with stability are optimal. Moreover, the statement about graphs of groups falls both in the framework of constraint stability [AP18] and of stability of epimorphisms [LL21], providing new examples of these notions.

These results only use that the groups GL($\mathbb{Z}_p$) are virtually pro-$p$, so in particular they also apply to the characteristic $p$ setting, where $\mathbb{Z}_p$ is replaced by $\mathbb{F}_q[[X]]$, for $q$ a power of $p$. But for the case of $\mathbb{Z}_p$ we can make these criteria more flexible: a cohomological argument implies an analogue of the Schur–Zassenhaus Theorem suitable to this setting (Lemma 2.20), that yields the following strengthening of Theorem 1.8 (see Section 7):

**Theorem 1.9.** The following groups are uniformly GL($\mathbb{Z}_p$)-stable:

1. Groups with a bound on the order of their finite virtual $p$-quotients.
2. Finitely generated virtually free groups.
3. Baumslag–Solitar groups $BS(m,n)$, whenever $\nu_p(m) \neq \nu_p(n)$.

Item 1. includes all groups with finite exponent, by Zelmanov’s solution of the restricted Burnside problem [Zel90, Zel91]. In Item 3. above, $\nu_p$ denotes the $p$-adic valuation: with the appropriate $p$ it applies to every non-residually finite Baumslag–Solitar group [Mes72], extending the remark following Theorem 1.8. Also these results are quantitatively precise, and the estimates involved are linear. Both stability results on Baumslag–Solitar groups are part of more general statements on Generalized Baumslag–Solitar groups (Corollaries 6.18 and 7.13), which give combinatorial and arithmetic conditions on the underlying weighted graphs that imply stability.

The methods used for the proof of Theorem 1.9 rely strongly on the fact that $\mathbb{Q}_p$ has characteristic 0, and in particular they cannot be used to determine whether finite $p$-groups are stable with respect to GL($\mathbb{F}_q[[X]]$), where $q$ is a power of $p$. Still, we are able to show that stability does hold for $\mathbb{Z}/2\mathbb{Z}$ in characteristic 2, with a quadratic estimate (Proposition 7.18). However our method relies on the solution of the similarity problem for representations of $\mathbb{Z}/2\mathbb{Z}$ over finite commutative local rings [BG78]. The analogous problem for all other $p$-groups in characteristic a power of $p$ is computationally wild [GP02].

The stability of virtually free groups can also be proven by another method, which is conceptually very different from the rest of the paper, and is reserved to Section 8. In [DCGLT20], the authors consider the family $\mathcal{G} := \{ (U(n), \| \cdot \|_{Frob}) : n \geq 1 \}$ and prove a cohomological criterion that ensures pointwise $\mathcal{G}$-stability of finitely presented groups. The key feature of the Frobenius norm that is exploited is its submultiplicativity, and indeed the same approach works for other submultiplicative norms on $U(n)$ [LO20]. The $C^\infty$-norm on GL$_n(\mathbb{Z}_p)$ also has this property (Lemma 3.15), which allows to carry over the arguments. But the non-Archimedean setting has a peculiarity of its own: the cocycles appearing in the proof are moreover bounded, so it is natural to state the result in terms of bounded cohomology instead. This is a rich theory over the reals (see e.g. [Mon01, Fri17]), whose study over non-Archimedean fields was recently initiated by the author [FF20]. The criterion is the following (Theorem 8.1, Corollary 8.10):
**Theorem 1.10.** Let $\Gamma$ be a finitely presented group such that $H_2^b(\Gamma, E) = 0$ for every Banach $\mathbb{Q}_p[\Gamma]$-module $E$ with a solid norm. Then $\Gamma$ is $GL(\mathbb{Z}_p)$-stable. In particular, this holds if $H^2(\Gamma, E) = 0$ for every such $E$.

A Banach norm $\| \cdot \|$ on $E$ is said to be **solid** if $\|E\| \subset |\mathbb{Q}_p|_p$: such spaces are isometrically classified \cite{PGS10} Theorem 2.5.4. The last statement implies that virtually free groups are $GL(\mathbb{Z}_p)$-stable. Similarly, the analogous statement in characteristic $p$ implies that virtually free groups without elements of order $p$ are $GL(\mathbb{F}_q[[X]])$-stable. We conjecture that these are the only examples that can be obtained via this theorem (Conjecture 8.11).

**Outline.** In Section 2 we review a few general facts about stability, approximation, residual properties and local embeddings. Moreover, we recall the interplay between lifting, splitting and cohomology, proving a useful technical lemma. In Section 3 we introduce the general framework of ultrametric families which will be the subject of our stability results, focusing on examples. In Section 4 we treat stability with respect to general ultrametric families, proving Theorems 1.4 and 1.5. In Section 5 we treat approximation and prove Theorem 1.7. In Section 6 we focus on families that are virtually pro-$\pi$ for some set $\pi$ of primes, and prove generalizations of Theorem 1.3. In Section 7 we focus on the case of $GL(\mathbb{Z}_p)$ (or more generally $GL(\mathfrak{o})$ where $\mathfrak{o}$ is the ring of integers of a finite extension of $\mathbb{Q}_p$), prove Theorem 1.9 and end by discussing the differences in the case of positive characteristic. Finally, in Section 8 we take a bounded-cohomological approach to stability, proving Theorem 1.10. Section 9 is dedicated to open questions and suggestions for further research.

**Remark.** The results of this paper are part of the author’s PhD project.

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2 Preliminaries

Notations and conventions. In the sequel, $p$ always denotes a prime. If $\pi$ is a set of primes, we denote by $\pi'$ its complement, in particular $p'$ is the set of primes other than $p$. For an integer $n$, the $p$-adic valuation of $n$ is denoted by $\nu_p(n)$. That is, $p^{\nu_p(n)}$ is the largest power of $p$ that divides $n$. The set of natural numbers $\mathbb{N}$ starts at 1. For simplicity, we use $x_n \to x$ instead of $x_n \xrightarrow{n \to \infty} x$ to denote convergence of sequences, whenever this does not lead to confusion.

$\Gamma$ denotes a countable discrete group. Given a set $S$ of letters, $F_S$ denotes the corresponding free group. We will always assume that $S \cap S^{-1} = \emptyset$. The trivial homomorphism onto any group will be denoted by $\text{id}$. If $R \subset F_S$ we denote by $\langle\langle R \rangle\rangle$ its normal closure, and $\langle S \mid R \rangle := F_S/\langle\langle R \rangle\rangle$. Once the presentation is fixed, we denote the projection map by $F_S \to \langle S \mid R \rangle: w \mapsto \overline{w}$. An extension $1 \to N \to \Gamma \to Q \to 1$ will be referred to as an extension of $N$ by $Q$.

2.1 Stability and approximation

Let $G$ be a family of groups equipped with arbitrary bi-invariant metrics. By bi-invariant we mean that if $(G,d_G) \in G$, and $g,h,k \in G$, we require $d_G(g,h) = d_G(kg,kh) = d_G(gk,hk)$. We denote by $G(\varepsilon)$ the closed ball of radius $\varepsilon > 0$ around the identity. The groups $G$ can thus be seen as topological groups.

Most of the paper is concerned with uniform stability. What follows is a n equivalent characterization which allows to make the statements more quantitative:

Lemma 2.1. The following are equivalent:

1. $\Gamma$ is uniformly $G$-stable.

2. For all $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $\varphi : \Gamma \to G \in G$ satisfies $\text{def}(\varphi) \leq \delta$, there exists a homomorphism $\psi : \Gamma \to G$ such that $\text{dist}(\varphi,\psi) \leq \varepsilon$.

If the function assigning to $\varepsilon$ the optimal $\delta$ is linear, polynomial, exponential... then we can attach the same adjective to the stability of $\Gamma$.

Proof. 2. $\Rightarrow$ 1. Suppose that 2. holds, and let $(\varphi_n : \Gamma \to G_n \in G)_{n \geq 1}$ be a uniform asymptotic homomorphism. For all $\varphi_n$ let $\psi_n : \Gamma \to G_n$ be a homomorphism that minimizes $\text{dist}(\varphi_n,\psi_n)$ up to $1/n$ (we cannot ask that a minimizing one exists in such a general situation). We need to show that $\text{dist}(\varphi_n,\psi_n) \to 0$, so let $\varepsilon > 0$ and let $\delta > 0$ be as in 2. for $\varepsilon/2$. Let $N$ be large enough so that $N \geq 2/\varepsilon$ and $\text{def}(\varphi_n) \leq \delta$ for all $n \geq N$. Then $\text{dist}(\varphi_n,\psi_n) \leq \varepsilon/2 + 1/n \leq \varepsilon$.

1. $\Rightarrow$ 2. Suppose that 2. does not hold. Then there exists $\varepsilon > 0$ with the following property: for all $n \geq 1$ there exists $\varphi_n : \Gamma \to G_n \in G$ such that $\text{def}(\varphi_n) \leq 1/n$ but for any homomorphism $\psi_n : \Gamma \to G_n$ we have $\text{dist}(\varphi_n,\psi_n) > \varepsilon$. The sequence $\varphi_n$ provides a counterexample to the uniform stability of $\Gamma$. 

\hfill $\square$
Rephrasing stability in terms of presentations (see Proposition 4.5) allows to give a quantitative characterization of pointwise stability of finitely presented groups, as in [AP15]. However it does not make sense to go into it here, since in the ultrametric setting uniform and pointwise stability coincide for finitely presented groups, as we will prove in Theorem 4.10.

The following simple observation, first made in [AP15, Theorem 4.3] (see also [GR09, Proposition 3]), is the key to connecting the notions of stability and approximation. Recall that given a group \( \Gamma \) and a family of groups \( \mathcal{G} \), we say that \( \Gamma \) is residually-\( \mathcal{G} \) if for all \( 1 \neq g \in \Gamma \) there exists a homomorphism \( \varphi : \Gamma \to G \in \mathcal{G} \) such that \( g \notin \ker(\varphi) \). It is fully residually-\( \mathcal{G} \) if for any finite subset \( K \subset \Gamma \) there exists a homomorphism \( \varphi : \Gamma \to G \in \mathcal{G} \) such that \( f|_K \) is injective. If the groups in \( \mathcal{G} \) are residually finite, and \( \Gamma \) is residually-\( \mathcal{G} \), then \( \Gamma \) is residually finite.

**Lemma 2.2** (Arzhantseva–Păunescu). Let \( \Gamma \) be a group that is both \( \mathcal{G} \)-approximable and pointwise \( \mathcal{G} \)-stable. Then \( \Gamma \) is fully residually-\( \mathcal{G} \).

This is mostly useful for counterexamples. For instance let \( \mathcal{G} \) be a family of residually finite groups. Then if \( \Gamma \) is not residually finite, it cannot be simultaneously approximable and pointwise stable: such classes include all families of finite groups, as well as the profinite families that will be defined in Section 3. Similarly, if \( \mathcal{G} \) is a family of locally residually finite groups, then the same holds under the additional hypothesis that \( \Gamma \) is finitely generated: such classes include all linear groups by a theorem of Malcev [Mal40]. This is the way the authors in [GR09] provide the first non-examples of pointwise \( (S_n, d_H) \)-stable equations. It is also the approach suggested in [AP15] and successfully realized in [DCGLT20], by which the authors provide the first example of non-approximable group, with respect to \( (U(n), \| \cdot \|_{\text{Frob}}) \).

Another useful equivalent characterization of pointwise stability, due to Arzhantseva and Păunescu [AP15, Theorem 4.2], is in terms of ultralimits and ultraproducts. We will only use it in Section 8 and apply it to finitely presented groups, so we state it in this setting which is the one from AP15. In the statement, \( \prod_{n \to \omega} G_n \) denotes the metric ultraproduct of the \( G_n \) with respect to the free ultrafilter \( \omega \). That is, \( \prod_{n \to \omega} G_n \) is the quotient of the direct product by the normal subgroup \( \{(g_n)_{n \geq 1} : d_{G_n}(g_n, 1_{G_n}) \to 0\} \).

**Lemma 2.3** (Arzhantseva–Păunescu). Let \( \Gamma = \langle S \mid R \rangle \) be finitely presented, and let \( \mathcal{G} \) be a family of groups equipped with bi-invariant metrics. The following are equivalent:

1. \( \Gamma \) is pointwise \( \mathcal{G} \)-stable.

2. For every free ultrafilter \( \omega \) on \( \mathbb{N} \) and any sequence \( (G_n)_{n \geq 1} \subset \mathcal{G} \), every homomorphism \( \Gamma \to \prod_{n \to \omega} G_n \) lifts to a homomorphism \( \Gamma \to \prod_{n \geq 1} G_n \).

### 2.2 Residual properties and local embeddings

See [CSC10, Chapters 2, 3, 7] for more detail.

Let \( \mathcal{C} \) be a class of groups, that for simplicity we assume to be closed under taking subgroups. If the class \( \mathcal{C} \) is closed under taking direct products, then a residually-\( \mathcal{C} \) group is automatically fully residually-\( \mathcal{C} \). The following result is standard, so we include it here for reference and omit the proof:
Lemma 2.4. Let \( \mathcal{C} \) be a class of groups closed under taking subgroups, \( \Gamma \) a group, \( K \) the intersection of all normal subgroups of \( \Gamma \) such that the quotient is in \( \mathcal{C} \), and \( R := \Gamma / K \). Then \( R \) is the largest residually-\( \mathcal{C} \) quotient of \( \Gamma \); that is, \( R \) is residually-\( \mathcal{C} \), and any homomorphism from \( \Gamma \) to a residually-\( \mathcal{C} \) group factors through \( R \).

The following will be common examples throughout this paper.

Example 2.5. Let \( \text{Sym}(\mathbb{Z}) \) be the group of permutations of the integers, let \( \text{Sym}_0(\mathbb{Z}) \) be the subgroup of permutations with finite support, and \( \text{Alt}_0(\mathbb{Z}) \) the subgroup of even permutations with finite support. Let \( T : \mathbb{Z} \to \mathbb{Z} : n \mapsto (n + 1) \) be the translation. We denote by \( G \) the group generated by \( \text{Sym}_0(\mathbb{Z}) \) and \( T \), and by \( G^+ \) the group generated by \( \text{Alt}_0(\mathbb{Z}) \) and \( T \), which has index 2 in \( G \). Then \( G \) splits as a semidirect product \( \text{Sym}_0(\mathbb{Z}) \rtimes \langle T \rangle \), and similarly \( G^+ \) splits as a semidirect product \( \text{Alt}_0(\mathbb{Z}) \rtimes \langle T \rangle \). Since \( \text{Alt}_0(\mathbb{Z}) \) has no non-trivial finite quotients, and \( \mathbb{Z} \) is residually finite, the largest residually finite quotient of \( G^+ \) is \( \mathbb{Z} \). Similarly the largest residually finite quotient of \( G \) is \( \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \).

Remark. The group \( G \) above was first considered by Malcev in [Mal40]. It is commonly referred to as Houghton’s second group, in reference to [Hou78], and denoted by \( H_2 \).

Example 2.6. Let \( G, H \) be groups, denote \( \Sigma_H G = \bigoplus_{h \in H} G_h \), where each \( G_h \) is an indexed copy of \( G \), and consider the wreath product \( G \wr H = \Sigma_H G \rtimes H \) where \( H \) acts on the direct sum by shifting the coordinates. Common examples in combinatorial group theory are the lamplighter groups, where \( H = \mathbb{Z} \) and \( G \) is finite (sometimes the name is used to refer to the specific case of \( G = \mathbb{Z}/2\mathbb{Z} \)).

Suppose that \( H \) is infinite. Then the projection of \( G \wr H \) onto its largest residually finite quotient factors through \( \text{Ab}(G) \wr H \), where \( \text{Ab}(G) \) is the abelianization of \( G \). This latter is residually finite if both \( \text{Ab}(G) \) and \( H \) are residually finite [Gri57, Theorem 3.2] (we will mostly be interested in the first statement).

In Section 5 we are concerned with local embeddings, which sit between approximability in the sense of Definition 4.6 and the corresponding residual property. The notion of local embedding was formally introduced in [VG97] although the idea goes back to Malcev [Mal40]. We refer the reader to [CSC10, Chapter 7] for details and proofs.

Definition 2.7 (Gordon–Vershik). Let \( \Gamma, C \) be groups, \( K \subseteq \Gamma \) a finite subset. A map \( f : \Gamma \to C \) is a \( K \)-local embedding if \( f|_K \) is injective and \( f(g)f(h) = f(gh) \) whenever \( g, h \in K \).

Let \( \mathcal{C} \) be a class of groups (closed under taking subgroups). The group \( \Gamma \) is locally embeddable into \( \mathcal{C} \) if for any finite subset \( K \subseteq \Gamma \) there exists a \( K \)-local embedding \( f : \Gamma \to C \in \mathcal{C} \). When \( \mathcal{C} \) is the class of finite groups, \( \Gamma \) is said to be LEF.

Here is an equivalent characterization of local embeddability (see e.g. [CSC10, Theorem 7.2.5]). In the statement, \( \prod_{n \to \omega} C_n \) denotes the set-theoretic ultraproduct of the \( C_n \) with respect to the free ultrafilter \( \omega \). That is, \( \prod_{n \to \omega} C_n \) is the quotient of the direct product by the normal subgroup \( \{(g_n)_{n \geq 1} : \{n : g_n = 1_{C_n}\} \in \omega \} \) (equivalently, the metric ultraproduct where the \( C_n \) are endowed with the discrete metric).

Proposition 2.8 (Gordon–Vershik). Let \( \Gamma \) be a countable group, \( \mathcal{C} \) a class of groups. Then \( \Gamma \) is locally embeddable into \( \mathcal{C} \) if and only if it embeds into \( \prod_{n \to \omega} C_n \) for some sequence \( (C_n)_{n \geq 1} \subseteq \mathcal{C} \).
The properties of being residually-$C$ and locally embeddable into $C$ are related by the following result (see e.g. [CSC10, Corollary 7.1.14, 7.1.21]):

**Proposition 2.9** (Gordon–Vershik). Let $C$ be a class of groups closed under taking subgroups.

1. Any fully residually-$C$ group is locally embeddable into $C$.
2. Any finitely presented group that is locally embeddable into $C$ is fully residually-$C$.

Importantly, Item 2 does not hold for general finitely generated groups.

**Example 2.10.** The group $G$ from Example 2.5 is finitely generated: by (12) and $T$. It is not residually finite: we computed its largest residually finite quotient in Example 2.5. However $G$ is LEF [VG97]. Similarly, $G^+$ is finitely generated (by (123) and $T$, or because it has finite index in $G$), not residually finite (again by Example 2.5), but it is LEF (since a subgroup of a LEF group is clearly LEF).

**Example 2.11.** Let $G, H$ be finitely generated LEF groups. Then their wreath product $G \wr H$ is finitely generated and LEF [VG97, Theorem 2.4 (ii)]. However, if $G$ is non-abelian, then $G \wr H$ is not residually finite by Example 2.6.

A natural framework in which to see this properties is that of *marked groups* introduced by Grigorchuk in [Gri84]. We use the point of view of normal subgroups, as in [CSC10] (see also [CG05]). Given a countable group $\Gamma$, the set of $\Gamma$-marked groups is the set of isomorphism classes of quotients of $\Gamma$, identified with the set of normal subgroups $\mathcal{N}(\Gamma)$ of $\Gamma$. The space of marked groups is this set endowed with the subspace topology $\mathcal{N}(\Gamma) \subset \mathcal{P}(\Gamma) \cong \{0,1\}^\Gamma$. This topology is totally disconnected, compact and, since $\Gamma$ is suppose to be countable, metrizable.

Given a class $C$ closed under taking subgroups, both the residual property and local embeddability admit characterizations in terms of the space of marked groups (see e.g. [CSC10, Proposition 3.4.3, Corollary 7.1.20]):

**Theorem 2.12** (Gordon–Vershik). Let $C$ be a class of groups closed under taking subgroups. Let $\Gamma = \langle S \mid R \rangle$ be a countable group, $N := \langle \langle R \rangle \rangle \leq F_S$. Then:

1. $\Gamma$ is fully residually-$C$ if and only if there exists a sequence $N_k \to \{1\} \in \mathcal{N}(\Gamma)$ such that $\Gamma/N_k \in C$ for all $k$.
2. $\Gamma$ is locally embeddable into $C$ if and only if there exists a sequence $N_k \to N \in \mathcal{N}(F_S)$ such that $\Gamma/N_k \in C$ for all $k$.

This point of view gives many more examples of finitely generated LEF groups that are not residually finite.

**Example 2.13.** Let $\Gamma = \langle S \mid R \rangle$, with $S$ finite and $R$ possibly infinite, be a presentation satisfying the $C'(1/6)$ small cancellation condition: we call $\Gamma$ a classical small cancellation group. Letting as usual $N = \langle \langle R \rangle \rangle \leq F_S$, there exists a sequence $N_k \to N \in \mathcal{N}(F_S)$, where the $N_k$ are normally generated by $k$ elements of $R$. Since the groups $F_S/N_k$ are defined by finite $C'(1/6)$ presentation, they are hyperbolic, and moreover they are residually finite. The last statement follows by combining the following three deep results: finitely presented $C'(1/6)$ groups are cubulable [Wis04], hyperbolic cubulable groups are virtually special-cubulable [Ago13], and special-cubulable groups
embed into RAAGs [HW08]. By Item 2. of Theorem 2.12 it follows that \( \Gamma \) is LEF. Let us point out that this is a special property of the \( C'(1/6) \) small cancellation condition: there exist finitely presented groups satisfying more relaxed small cancellation conditions that are not residually finite [Wis96].

On the other hand there are many examples of (infinitely presented) classical small cancellation groups that are not residually finite. A classical example is a group constructed by Pride in [Pri89]:

\[
\Gamma = \langle a, b \mid au_1, bv_1, au_2, bv_2, \ldots \rangle,
\]

where \( u_n, v_n \) are well-chosen words in \( a^n, b^n \) so that the presentation is \( C'(1/6) \). For any such choice, this group is infinite and has no proper finite-index subgroup, so it is in particular not residually finite. For more examples see [AO15, Section 2] and the references therein; the authors also explain how to construct continuum-many isomorphism classes of non-residually finite classical small cancellation groups.

### 2.3 Non-Archimedean fields

See [PGS10] for more detail.

Let \((K, | \cdot |)\) be a normed field. If the group \(|K^\times| \leq \mathbb{R}_{>0}^\times\) is discrete, it is either trivial, or of the form \( r^\mathbb{Z} \), for some \( 0 < r < 1 \), in the latter case we say that \( K \) is **discretely valued**. The norm is **non-Archimedean** if it satisfies the strong triangle inequality, namely \(|x + y| \leq \max\{|x|, |y|\}\).

Then the closed ball of radius 1 is a local ring, called the **ring of integers** and denoted by \( \mathfrak{o} \), whose maximal ideal is the open ball of radius 1, denoted by \( \mathfrak{p} \). The quotient \( \mathfrak{f} := \mathfrak{o}/\mathfrak{p} \) is called the **residue field** of \( K \), and its characteristic is called the **residual characteristic** of \( K \).

If the induced topology is locally compact and non-discrete then \( K \) is called a **local field**. Non-Archimedean local fields are precisely those that are discretely valued and have a finite residue field. It follows that \( \mathfrak{o} \) is compact, and that the maximal ideal \( \mathfrak{p} \) is principal, generated by an element \( \varpi \) such that \(|\varpi| = r\), using the notation above. Such an element is called a **uniformizer**. By Ostrowski’s Theorem a non-Archimedean local field \( K \) is either a finite extension of \( \mathbb{Q}_p \) (if it has characteristic 0) or of \( \mathbb{F}_q((X)) \) (if it has characteristic \( p \), where \( q \) is a power of \( p \)).

**Example 2.14.** We write elements of \( \mathbb{Q}_p \) in the usual series form \( x = \sum_{i \geq i_0} a_i p^i \), where \( a_i \in \{0, \ldots, p - 1\} \) and \( a_{i_0} \neq 0 \). Then \(|x|_p := p^{-i_0} \), so the norm takes values in \( p^\mathbb{Z} \): in the previous notation we have \( r = p^{-1} \). The ring of integers is \( \mathfrak{o} = \mathbb{Z}_p \) with uniformizer \( \varpi = p \) and maximal ideal \( \mathfrak{p} = p\mathbb{Z}_p \). The residue field is \( \mathfrak{f} \cong \mathbb{F}_p \).

**Example 2.15.** \( \mathbb{F}_q((X)) \) is the field of formal Laurent series, so it consists of elements of the form \( x = \sum_{i \geq i_0} a_i X^i \), where \( a_i \in \mathbb{F}_q \) and \( a_{i_0} \neq 0 \). Then \(|x|_q := q^{-i_0} \), so the norm takes values in \( q^\mathbb{Z} \): in the previous notation we have \( r = q^{-1} \). The ring of integers is \( \mathfrak{o} = \mathbb{F}_q[[X]] \) with uniformizer \( \varpi = X \) and maximal ideal \( \mathfrak{p} = X\mathbb{F}_q[[X]] \). The residue field is \( \mathfrak{f} \cong \mathbb{F}_q \).

We next review the basics of functional analysis over the local field \( K \), needed in Section 8. A normed \( K \)-vector space is a \( K \)-vector space \( E \) endowed with a norm \(| \cdot | : E \to \mathbb{R}_{\geq 0} \) that is positive-definite, \( K \)-multiplicative, and satisfies the strong triangle inequality. If the induced metric is complete, we say that \( E \) is a **\( K \)-Banach space**. The norm of \( E \) need not take the same set of values as that of \( K \). If this is the case, we say that the norm on \( E \) is **solid**. In our case \( K \) is a local
field, so it is in particular complete and discretely valued: such Banach spaces are isometrically classified [PGS10, Theorem 2.5.4].

Given two normed $\mathbb{K}$-vector space $E, F$, a linear map $T : E \to F$ is continuous if and only if it is bounded, that is if and only if $\|T\|_{op} = \inf\{C \geq 0 : \|Tx\|_F \leq C\|x\| \text{ for all } x \in E\} < \infty$. This applies in particular to the case when $E = \mathbb{K}$; then the space of bounded linear maps is called the continuous dual $E^*$, and is a normed $\mathbb{K}$-vector space endowed with the operator norm $\|T\|_{op}$. This is in general not equal to $\sup\{|Tx|_{\mathbb{K}} : \|x\| \leq 1\}$, but it is when the norm on $E$ is solid. Since $\mathbb{K}$ is complete, a dual space is always Banach.

**Definition 2.16.** A normed $\mathbb{K}[\Gamma]$-module is a normed $\mathbb{K}$-vector space $E$ with a linear isometric action of the group $\Gamma$. If $E$ is Banach, we say that $E$ is a Banach $\mathbb{K}[\Gamma]$-module. If $E = F^*$ is the dual of a $\mathbb{K}[\Gamma]$-module endowed with the operator norm and the dual action, we say that $E$ is a dual $\mathbb{K}[\Gamma]$-module.

The following basic theorem of functional analysis holds also in the non-Archimedean context [PGS10, Theorem 2.1.17]:

**Theorem 2.17** (Open Mapping Theorem). Let $T : E \to F$ be a bounded surjective linear map between Banach spaces. Then $T$ is open.

2.4 Splitting and lifting

See [Bro12, Chapter IV] and [Rot12, Chapter 7] for more detail. These results will be relevant starting from Section 6. All statements — apart from Lemma 2.20 — are standard, but we remind them here since we will need some specific constructions in the sequel.

Let $1 \to N \to E \to Q \to 1$ be a group extension. A splitting is a homomorphic section $\sigma : Q \to E$: if one exists we say that the extension splits, which is equivalent to $E \cong N \rtimes Q$, and the image of this section is a complement of $N$. Consider splittings $\sigma_1, \sigma_2 : Q \to E$ with complements $Q_1, Q_2$. The complements are conjugate if there exists $g \in E$ such that $gQ_1g^{-1} = Q_2$, which implies that there exists $g \in N$ such that $g\sigma_1g^{-1} = \sigma_2$. Because of this we may refer to the splittings themselves being conjugate, and the conjugating element may be assumed to lie in $N$.

Splitting problems are a special case of the more general lifting problems: given a group $\Gamma$, and a homomorphism $\Gamma \xrightarrow{\varphi} G/N$, can this be lifted to a homomorphism $\Gamma \xrightarrow{\psi} G$? Are all lifts conjugate?

A splitting problem is the special case in which $\Gamma = G/N$ and $\varphi$ is the identity. But any lifting problem can be reduced to a splitting problem. Consider the pullback $G \times_\varphi \Gamma := \{(x, g) \in G \times \Gamma : xN = \varphi(g) \in G/N\}$, denote by $pr_{1,2}$ the natural projections, and notice that there is a natural
embedding $j : N \rightarrow G \times _\varphi \Gamma : n \mapsto (n, 1)$. Then we have the following commutative diagram with exact rows:

$$
\begin{array}{ccccccc}
1 & \rightarrow & N & \xrightarrow{j} & G \times _\varphi \Gamma & \xrightarrow{pr_2} & \Gamma & \rightarrow & 1 \\
& & \phantom{=} & \downarrow{pr_1} & \phantom{=} & \downarrow{\varphi} & \phantom{=} & \\
1 & \rightarrow & N & \leftarrow & G & \rightarrow & G/N & \rightarrow & 1
\end{array}
$$

and the lift $\psi$ exists if and only if the exact sequence on the top row splits. Moreover, two lifts are conjugate if and only if the corresponding two splittings are conjugate, and by the discussion above the conjugating element may be chosen to lie in $N$. Lifting problems will appear more naturally in this paper, so we will state our results in those terms.

The following will be the fundamental tool in Section 6 [Rot12, Theorem 7.41]:

**Theorem 2.18 (Schur–Zassenhaus).** Let $\Gamma, N$ be finite groups such that the orders of $\Gamma$ and $N$ are coprime. Then, whenever $N \leq G$, any homomorphism $\Gamma \rightarrow G/N$ lifts to a homomorphism $\Gamma \rightarrow G$, and any two lifts are $N$-conjugate.

The conjugacy statement admits a simple proof in the case in which either $N$ or $\Gamma$ is solvable. This will be the setting for $\text{GL}(\mathbb{Q})$: in fact $N$ will even be a $p$-group. The general case follows from the Odd Order Theorem: if $N$ and $\Gamma$ have coprime order then at least one of them has odd order, and so is solvable. To the author’s knowledge, no simpler proof is known.

The Schur–Zassenhaus Theorem alone will be enough for Section 6. In Section 7 we will need a stronger lifting criterion, under additional hypotheses on $N$. Recall that given an extension $1 \rightarrow N \rightarrow E \rightarrow Q \rightarrow 1$ with $N$ abelian, the action of $E$ on $N$ by conjugacy induces an action of $Q$ on $N$, and so the corresponding cohomology groups $H^*(Q, N)$. To this extension one associates a second cohomology class $[E] \in H^2(Q, N)$. A cocycle representing it can be defined as follows: take a (set-theoretic) section $\tilde{\sigma} : Q \rightarrow E$, and define $c : Q \times Q \rightarrow N : (g, h) \mapsto c(g, h) = \tilde{\sigma}(gh)^{-1}\tilde{\sigma}(g)\tilde{\sigma}(h)$. If the extension splits, let us identify $E$ with $N \rtimes Q$ (which as a set is just the cartesian product $N \times Q$). Then each splitting $\sigma : Q \rightarrow N \rtimes Q$ defines a first cohomology class $[\sigma] \in H^1(Q, N)$. A cocycle representing it can be defined by $c : Q \rightarrow N : g \mapsto c(g)$ where $\sigma(g) = (c(g), g) \in N \rtimes Q$.

**Theorem 2.19 (Schreier).** The extension splits if and only if the associated cohomology class $[E] \in H^2(Q, N)$ vanishes. Two splittings $\sigma_1, \sigma_2$ are conjugate if and only if the associate cohomology classes $[\sigma_1], [\sigma_2] \in H^1(Q, N)$ coincide.

So cohomology vanishing gives rise to splitting and conjugacy results. The Schur–Zassenhaus Theorem is proven this way: one first reduces to the case where $N$ is an elementary abelian $p$-group, and then uses the fact that if $Q$ has order coprime to $p$, then $H^n(Q, N) = 0$ for $n \geq 1$. The following lemma strengthens this:

**Lemma 2.20.** Let $\Gamma, N$ be finite groups, with $N$ a $\mathbb{Z}/p^k\mathbb{Z}$-module, and $\nu_p(|\Gamma|) \leq l \leq k$. Let $H := p^lN$, which is characteristic in $N$. Then, whenever $N \leq G$, for any homomorphism $\Gamma \rightarrow G/H$,
the projection $\Gamma \to G/N$ lifts to a homomorphism $\psi : \Gamma \to G$.

Moreover, if they exist, any two lifts of $\varphi$ are $N$-conjugate.

Remark. Note that a lift of $\varphi$ is also a lift of $\varphi$, but the converse need not hold. The first part of the lemma only guarantees the existence of the latter, and the second one is a statement about the former.

Proof. The lift exists if and only if the extension $1 \to N \to G \times \varphi \Gamma \to \Gamma \to 1$ splits, which in turn is equivalent to the vanishing of the corresponding cohomology class in $H^2(\Gamma, N)$. A cocycle representing it is given by $(g, h) \mapsto \tilde{\sigma}(gh)^{-1} \tilde{\sigma}(g) \tilde{\sigma}(h)$, where $\tilde{\sigma} : \Gamma \to G \times \varphi \Gamma$ is any (set-theoretic) section. We can choose $\tilde{\sigma} : \Gamma \to G \times \varphi \Gamma \leq G \times \varphi \Gamma$, and then the corresponding cocycle will take values in $H$. Similarly, a lift of $\varphi$ defines a class in $H^1(\Gamma, N)$ such that the cocycle representing it takes values in $H$. Since $H = p^l N$, to show that these classes vanish it suffices to show that $p^l \cdot H^n(\Gamma, N) = 0$ for $n = 1, 2$. By [Bro12, Corollary III.10.2], we have $|\Gamma| \cdot H^n(\Gamma, N) = 0$ for all $n \geq 1$. Write $|\Gamma| = p^a m$ where $a \leq l$ and $(m, p) = 1$. The latter condition ensures that $m$ is a unit in $\mathbb{Z}/p^k\mathbb{Z}$, and so $m \cdot H^n(\Gamma, N) = H^n(\Gamma, N)$. Therefore $p^a \cdot H^n(\Gamma, N) = 0$ and so $p^l \cdot H^n(\Gamma, N) = 0$, which concludes the proof.
3 Ultrametric families

The subject of this paper is stability with respect to a family $\mathcal{G}$ all of whose groups are equipped with bi-invariant ultrametrics. Before moving to stability in the Section 4, here we prove some basic facts about such families, and present several examples.

3.1 Basic facts and terminology

**Definition 3.1.** We say that the metric group $(G, d)$ is **ultrametric** if the ultrametric inequality holds:

$$d(g, k) \leq \max\{d(g, h), d(h, k)\} \text{ for all } g, h, k \in G.$$  

We say that the family $\mathcal{G}$ is **ultrametric** if every $G \in \mathcal{G}$ is ultrametric. If moreover the groups in $\mathcal{G}$ are compact, the family $\mathcal{G}$ is called **profinite**.

The most important general property of ultrametric groups is contained in the following lemma.

**Lemma 3.2.** Let $(G, d)$ be a group equipped with a bi-invariant ultrametric. Then $G(\varepsilon)$ is a clopen normal subgroup of $G$.

**Proof.** Closed balls in ultrametric spaces are automatically open. If $g, h \in G(\varepsilon)$, then

$$d(gh^{-1}, 1) = d(g, h) \leq \max\{d(1, g), d(1, h)\} \leq \varepsilon.$$  

This shows that $G(\varepsilon)$ is a subgroup, and it is normal because $d$ is conjugacy-invariant. \qed

This allows to quotient out balls, leading to the following definition:

**Definition 3.3.** Let $(G, d)$ be an ultrametric group. The quotients $G/G(\varepsilon)$ are called **metric quotients** of $G$. Given an ultrametric family $\mathcal{G}$, we denote by $MQ(\mathcal{G})$ the family of metric quotients of groups in $\mathcal{G}$ and all subgroups thereof.

A metric quotient of $G$ is a discrete, since each $G(\varepsilon)$ is open, and comes equipped with a quotient metric. If $G$ is moreover compact, then all metric quotients are finite.

The possibility of quotienting out balls has very strong consequences for stability and approximation. These are based on the following lemma:

**Lemma 3.4.** Let $\Gamma$ be a group, $(G, d)$ an ultrametric group and $\varepsilon > 0$. Let $\varphi, \psi : \Gamma \rightarrow G$. Then:

1. $\text{def}(\varphi) \leq \varepsilon$ if and only if the map $\varphi(\varepsilon) : \Gamma \rightarrow G/G(\varepsilon)$ is a homomorphism.

2. If both $\text{def}(\varphi), \text{def}(\psi) \leq \varepsilon$, then $\text{dist}(\varphi, \psi) \leq \varepsilon$ if and only if the corresponding homomorphisms $\varphi(\varepsilon), \psi(\varepsilon)$ coincide.

**Proof.** 1. The map $\varphi(\varepsilon)$ is a homomorphism if and only if $\varphi(gh)G(\varepsilon) = \varphi(g)\varphi(h)G(\varepsilon)$ for all $g, h \in \Gamma$, which is equivalent to $\text{def}_{g, h}(\varphi) \leq \varepsilon$.

2. The homomorphisms coincide if and only if $\varphi(g)G(\varepsilon) = \psi(g)G(\varepsilon)$ for all $g \in \Gamma$, which is equivalent to $\text{dist}_{g}(\varphi, \psi) \leq \varepsilon$. \qed

**Definition 3.5.** With the notation of the previous lemma, we refer to the homomorphisms $\{\varphi(\varepsilon) : \varepsilon \geq \text{def}(\varphi)\}$ as the **homomorphisms induced by $\varphi$**.
3.2 Examples

A trivial example of bi-invariant metrics falls in the ultrametric framework.

Example 3.6. Given a discrete group \( G \), it is always possible to define a discrete metric on it by setting \( d(g, h) = 0 \) if and only if \( g = h \). This is a bi-invariant ultrametric. A family \( \mathcal{G} \) of discrete groups equipped with the discrete metric will be called a discrete family.

Probabilistic stability problems with respect to this metric are mostly used in property testing (see e.g. [Goï010] and [BC20]). In our deterministic setting, we will see that stability with respect to such families is less interesting (see Example 3.12).

Next, we present two constructions that allow to put natural ultrametrics onto groups, and we apply them to give examples of ultrametric families. Then we move on to the main example that will be treated in this paper, namely integral matrices over non-Archimedean fields. For the rest of this subsection, fix a strictly decreasing sequence \( \varepsilon = (\varepsilon_k)_{k \geq 0} \subset (0, 1] \) with \( \varepsilon_0 = 1 \) and \( \varepsilon_k \to 0 \).

3.2.1 Groups acting on filtered sets

Let \( \Omega \) be a set with a (possibly finite) filtration \((\Omega_k)_{k \geq 1}\); that is \( \Omega_k \subset \Omega_{k+1} \), and \( \Omega = \bigcup_{k \geq 0} \Omega_k \). Let \( G \) be a group acting on \( \Omega \) preserving each \( \Omega_k \). Define \( d(g, h) := \varepsilon_k \), where \( k \) is the maximal integer such that \( g|_{\Omega_k} = h|_{\Omega_k} \), and \( k = 0 \) if no such integer exists. This is a bi-invariant ultrametric: that it is a left-invariant ultrametric is clear, and right-invariance follows from the fact that \( G \) preserves each \( \Omega_k \).

Example 3.7. Let \( G = T_n(R) \) be the group of invertible upper-triangular \((n \times n)\) matrices over a commutative ring \( R \). Then we can set \( \Omega = R^n \) and \( \Omega_k = \text{span}\{e_1, \ldots, e_k\} \), so \( d_{\Omega}(g, h) \leq \varepsilon_k \) if and only if the first \( k \) columns of \( g \) and \( h \) are identical. We denote by \( T(R) \) the family \( \{ (T_n(R), d) : n \geq 1 \} \), or \( T(R)(\varepsilon) \) if we want to emphasize the choice of the sequence \( \varepsilon \). Similarly we can look at the subgroup \( UT_n(R) \) of upper-triangular matrices with ones on the diagonal, and obtain the family \( UT(R) \).

Given \( \varepsilon > 0 \), let \( k \geq 1 \) be the maximal integer such that \( \varepsilon_k \geq \varepsilon \). Then \( G(\varepsilon) = G(\varepsilon_k) \) is the subgroup consisting of upper-triangular matrices with a copy of \( I_k \) in the upper-left corner. It follows that the metric quotient \( G/G(\varepsilon) \) is isomorphic to \( T_k(R) \), or \( T_n(R) \) if \( k > n \). In particular, all metric quotients are solvable, and even nilpotent-by-abelian, so \( MQ(T(R)) \) is contained in the class of nilpotent-by-abelian groups. Similarly, metric quotients of \( UT_n(R) \) are isomorphic to \( UT_k(R) \) for some \( k \), in particular they are all nilpotent.

Example 3.8. See [Nek05] for more detail. Let \( X \) be a finite alphabet and \( \Omega := X^* \) the regular rooted tree of finite words on \( X \). We denote by \( \Omega_k \) the set of words of length at most \( k \). Then \( G = \text{Aut}(\Omega) \), the group of rooted tree automorphisms, can be equipped with this metric, so \( d_{\Omega}(g, h) \leq \varepsilon_k \) if and only if \( g \) and \( h \) act the same way on \( \Omega_k \), or equivalently they act the same way on the \( k \)-th level of the rooted tree. We denote by \( \text{Aut}(X^*_n) \) the family \( \{ (\text{Aut}(X^*_n), d) : n \geq 1 \} \), where \( X_n = \{1, \ldots, n\} \), or \( \text{Aut}(X^*_n)(\varepsilon) \) if we want to emphasize the choice of the sequence \( \varepsilon \).

Given \( \varepsilon > 0 \), let \( k \geq 1 \) be the maximal integer such that \( \varepsilon_k \geq \varepsilon \). Then \( G(\varepsilon) = G(\varepsilon_k) \) is the stabilizer of \( \Omega_k \), equivalently the stabilizer of the \( k \)-th level of the rooted tree. It follows that the metric quotient \( G/G(\varepsilon) \) is a \( k \)-fold iterated wreath product of the symmetric group \( \text{Sym}(X) \). In particular, all metric quotients are finite \( \pi \)-groups, where \( \pi \) is the set of primes \( p \leq |X| \). So \( MQ(\text{Aut}(X^*_n)) \) is the class of all finite groups (recall that we are also including subgroups of metric quotients in our definition of \( MQ \)).
3.2.2 Projective limits

Let \((A_k)_{k \geq 1}\) be an inverse system of discrete groups, indexed by the directed set \(\mathbb{N}\), and let \(G\) be the corresponding projective limit. Then we can define \(d(g, h) = \varepsilon_k\), where \(k \geq 1\) is the maximal integer such that \(g\) and \(h\) have the same image in \(A_k\). This is a bi-invariant ultrametric.

Let \(G_k\) be the kernel of the quotient map \(G \to A_k\). Then \(G_k = G(\varepsilon_k)\) and \(d(g, h) \leq \varepsilon_k\) if and only if \(gG_k = hG_k\). Given \(\varepsilon > 0\), let \(k \geq 1\) be the maximal integer such that \(\varepsilon_k \geq \varepsilon\). Then \(G(\varepsilon) = G(\varepsilon_k) = G_k\) and it follows that the metric quotient \(G/G(\varepsilon)\) is isomorphic to \(A_k\).

This construction applies to more general projective limits, where the defining system is countable but not necessarily indexed by \(\mathbb{N}\). More precisely, if \(G\) is the projective limit of \((A_i)_{i \in I}\), where \(I\) is a countable directed set, then we may choose a sequence \((i_k)_{k \geq 1}\) that is order-isomorphic to \(\mathbb{N}\) such that the corresponding system \((A_{i_k})_{k \geq 1}\) gives back \(G\).

Note that given an inverse system \((A_i)_{i \in I}\), the set \(I\) is countable if and only if \(G\) admits a countable neighbourhood basis of the identity, which for topological groups is equivalent to being first-countable. This shows that countability is a necessary and sufficient condition to put a metric on \(G\), since a metric space is always first-countable.

Example 3.9. Let \(G\) be a first-countable profinite group. Then there exists a strictly nested sequence \((G_k)_{k \geq 1}\) of finite-index normal subgroups that intersect trivially. The metric defined by \(d(g, h) = \varepsilon_k\), where \(k\) is the maximal integer such that \(gG_k = hG_k\), is a bi-invariant ultrametric.
The corresponding metric quotients are the finite groups \(G/G_k\). We denote this metric by \(d = d((G_k)_{k \geq 1}, \varepsilon)\).

Here are two specific examples of first-countable profinite groups metrized this way. The first we have already seen from another point of view.

Example 3.10. Using the same notation as in Example 3.8, let \(G = \text{Aut}(X^*)\) and \(G_k\) be the stabilizer of the \(k\)-th level. Then \((G_k)_{k \geq 1}\) is a strictly nested sequence of finite-index normal subgroups that intersect trivially. Letting \(d = d((G_k)_{k \geq 1}, \varepsilon)\) be the corresponding metric from Example 3.9, we obtain the same metric as in Example 3.8.

The next example falls into the more general construction, where the set \(I\) is countable but not naturally order-isomorphic to \(\mathbb{N}\). See [NSW13] for more information.

Example 3.11. Let \(K\) be a field, \(L/K\) a Galois extension and \(\text{Gal}(L/K)\) the corresponding Galois group: this is the projective limit of the Galois groups of all intermediate finite Galois extensions. Let \(G\) be the absolute Galois group, that is, the Galois group of the separable closure \(K^{\text{sep}}/K\) of \(K\): for any other Galois extension \(L/K\) the group \(\text{Gal}(L/K)\) is a quotient of \(G\) by a closed subgroup. Suppose that \(G\) is first-countable. Then we can choose a strictly nested sequence \(G_k\) of open finite-index normal subgroups, and project \(G_k\) onto \(\text{Gal}(L/K)\) for every other Galois extension: the resulting groups will have the same property. This defines a metric as in Example 3.9 on each \(\text{Gal}(L/K)\), with respect to the image of the sequence \((G_k)_{k \geq 1}\) and the sequence \(\varepsilon\).

We denote by \(\text{Gal}(K)\) the profinite family obtained this way, or \(\text{Gal}(K)((G_k)_{k \geq 1}, \varepsilon)\) if we want to emphasize the choices of the sequences \((G_k)_{k \geq 1}\) and \(\varepsilon\).

Here is a characterization of fields whose absolute Galois group is first-countable:

Lemma 3.12. Let \(L/K\) be a Galois extension of a field \(K\). Then the following are equivalent:

1. \(\text{Gal}(L/K)\) is first-countable;
2. \( L \) is a countable (increasing) union of intermediate finite Galois extensions;

3. There exist only countably many intermediate finite Galois extensions.

Proof. 1. \( \iff \) 2. A countable nested sequence of finite-index open normal subgroups corresponds to a countable increasing sequence of finite Galois extensions. If we choose the subgroups to form a basis, so to intersect trivially, the corresponding increasing union gives all of \( L \), and vice versa. Moreover the existence of a countable union implies the existence of an increasing one, since the compositum of finitely many finite Galois extensions is a finite Galois extension.

2. \( \iff \) 3. Write \( L = \bigcup_{i \geq 1} K_i \), where each \( K_i \) is a finite intermediate Galois extension. By the Primitive Element Theorem, each intermediate finite Galois extension is contained in some \( K_i \). By Galois Correspondence, there are only finitely many Galois subextensions of each \( K_i \). The other implication is clear. \( \square \)

The first easy example is the following:

**Example 3.13.** Let \( K \) be a countable field. Then \( K \) admits only countably many finite Galois extensions, since \( K^{sep} \) itself is countable. For instance the absolute Galois group of \( \mathbb{Q} \), or a finite field, is first-countable.

The second one is more involved. It relies on Krasner’s Lemma [NSW13, 8.1.6]: let \( K \) be a non-Archimedean complete normed field, \( \alpha \in K^{sep} \) with conjugates \( \alpha = \alpha_1, \ldots, \alpha_d \). If \( \beta \in K^{alg} \) satisfies \( |\beta - \alpha| < |\alpha - \alpha_i| \) for all \( i = 2, \ldots, d \), then \( K(\alpha) \subset K(\beta) \). This lemma is crucial in the proof that the field \( \mathbb{C}_p \) of \( p \)-adic complex numbers is algebraically closed [NSW13, Theorem 10.3.2], and in fact the following proof is very similar to that.

**Example 3.14.** Let \( K \) be a complete non-Archimedean field whose topology is separable. Then \( K \) admits only countably many finite Galois extensions. This applies to all non-Archimedean local fields.

Proof. By hypothesis there exists a countable dense subset \( D \subset K \), which we may assume is a field. We claim that \( K^{sep} = \bigcup_{\beta \in D^{sep}} K(\beta) \). Since \( D^{sep} \) is countable, this allows to conclude thanks to Lemma 3.12. So let \( \alpha \in K^{sep} \) and let \( f(X) \in K[X] \) be its minimal polynomial, whose roots \( \alpha = \alpha_1, \ldots, \alpha_d \) are the Galois conjugates of \( \alpha \), which are all distinct since \( \alpha \) is separable. We can choose a \( g(X) \in D[X] \) arbitrarily close to \( f(X) \), in fact we can choose \( g \) so that \( |g(\alpha_j)| = |g(\alpha_j) - f(\alpha_j)| \) is arbitrarily small, for all \( j \). Now write \( g(X) = \prod (X - \beta_i) \), where \( \beta_i \in K^{alg} \). This implies that for all \( j \) there exists \( i \) such that \( |\alpha_j - \beta_i| \) is small, say smaller than \( |\alpha_j - \alpha_k| \) for all \( j \neq k \). Since \( g \) has the same degree as \( f \), and the \( \alpha_j \) are all different, this association is a bijection. This implies that all \( \beta_i \) are distinct, so \( g \) is separable. Moreover, if \( \beta \) is the root close to \( \alpha \), we have \( K(\alpha) \subset K(\beta) \) by Krasner’s Lemma, and \( \beta \in D^{sep} \). \( \square \)

### 3.2.3 Integral matrices

Let \((K, |·|)\) be a non-Archimedean field with ring of integers \( o \), maximal ideal \( p \) and residue field \( \mathfrak{p} \) of characteristic \( p \geq 0 \). Then the matrix group GL(\( o \)) comes equipped with the \( \ell^\infty \)-norm induced by the inclusion into \( M_n(K) \):

\[
\|M\| := \max\{|M_{ij}| : 1 \leq i, j \leq n\}.
\]

This induces a distance by \( d(A, B) = \|A - B\| \), which we already mentioned in the introduction in the special case \( K = \mathbb{Q}_p \).
Lemma 3.15. Let $\| \cdot \|$ and $d$ be as above. Then

1. $d$ is an ultrametric.
2. $\| \cdot \|$ is submultiplicative.
3. $\| \cdot \|$ is equal to the operator norm, where $\mathbb{K}^n$ is also equipped with the $\ell^\infty$-norm.
4. If $A \in \text{GL}_n(\mathfrak{o})$, then $\|A\| = 1$.
5. If $A \in \text{GL}_n(\mathfrak{o})$ and $\|A - B\| < 1$, then $B \in \text{GL}_n(\mathfrak{o})$.
6. $\| \cdot \|$ is invariant under left or right multiplication by elements of $\text{GL}_n(\mathfrak{o})$.

Proof. 1. This follows directly from the fact that the norm on $\mathfrak{o}$ family by $\text{GL}(n)$.

2. Let $A, B \in M_n(\mathbb{K})$. Then

$$\|AB\| = \max_{1 \leq i, j \leq n} |(AB)_{ij}| = \max_{1 \leq i, j \leq n} \left| \sum_{k=1}^{n} A_{ik} B_{kj} \right| \leq \max_{1 \leq i, j, k \leq n} |A_{ik} B_{kj}| \leq \max_{1 \leq i, j \leq n} |A_{ik}| \cdot \max_{1 \leq j, k \leq n} |B_{kj}| = \|A\| \cdot \|B\|.$$

3. The submultiplicativity also holds for matrix-vector multiplication, with the same proof: $\|A x\| \leq \|A\| \cdot \|x\|$ for all $x \in \mathbb{K}^n$, and so $\|A\|_{\text{op}} \leq \|A\|$. For the converse, suppose that $\|A_{ij}\|$ attains its maximum in the $i$-th column $A_i$. Then $\|A\|_{\text{op}} \geq \|A e_i\| = \|A_i\| = \|A\|$.

4. Suppose that $A \in \text{GL}_n(\mathfrak{o})$; this immediately implies $\|A\| \leq 1$. It also implies that $\det(A) \in \mathfrak{o}^\times$ so $|\det(A)| = 1$. Since $\det(A)$ is a polynomial on $A_{ij}$, this is not possible if $\|A\| < 1$. Note that the converse of 4. is not true: if $\|A\| = 1$, then $A$ is integral, but its determinant may lie in $\mathfrak{o} \setminus \mathfrak{o}^\times$, in which case $A^{-1}$ is not integral.

5. Clearly $B \in M_n(\mathfrak{o})$ so it suffices to show that $\det(B) \in \mathfrak{o}^\times$. This is because $A \equiv B \mod \mathfrak{p}$, so $\det(B \mod \mathfrak{p}) \neq 0 \in \mathfrak{k}$.

6. Let $A \in \text{GL}_n(\mathfrak{o})$ and $M \in M_n(\mathbb{K})$. Then, using 2. and 4.:

$$\|M\| = \|A^{-1} M\| \leq \|A^{-1}\| \cdot \|AM\| = \|AM\| \leq \|A\| \cdot \|M\| = \|M\|.$$ Therefore $\|AM\| = \|M\|$. Similarly $\|MA\| = \|M\|$. \hfill $\Box$

The family of groups $\text{GL}_n(\mathfrak{o})$ with the distance $d$ is thus an ultrametric family. We denote this family by $\text{GL}(\mathfrak{o})$. A special case is the family $\text{GL}(\mathbb{Z}_p)$ from the introduction.

In case $\mathbb{K}$ is a local field, $\text{GL}(\mathfrak{o})$ is a profinite family, and this norm can also be seen as a special case of Example 3.39. Since $\mathbb{K}$ is discretely valued we can choose as a sequence $\varepsilon_k := |\omega|^k$ (where $\omega$ is a uniformizer), and $\text{GL}_n(\mathfrak{o})_k$ will be the ball of radius $\varepsilon_k$ around the identity. This can be explicitly identified as the congruence subgroup:

$$\text{GL}_n(\mathfrak{o})_k := \{ I + \omega^k M : M \in M_n(\mathfrak{o}) \}.$$
where \( \overline{\omega} \) is a uniformizer. The metric quotients are the finite matrix groups \( \text{GL}_n(\mathfrak{o}/p^k) \). There is no restriction on the order of these groups: indeed any finite group embeds into \( \text{GL}_n(\mathfrak{o}/p) = \text{GL}_n(\mathfrak{t}) \) for \( n \) large enough. However there is a restriction on the order of the metric quotients of the principal congruence subgroup \( \text{GL}_n(\mathfrak{o})_1 \), which will be crucial in Sections 6 and 7.

**Lemma 3.16.** With the notation above, the principal congruence subgroup \( \text{GL}_n(\mathfrak{o})_1 \) is pro-p. More precisely, for all \( k \geq 1 \) there is an isomorphism:

\[
\text{GL}_n(\mathfrak{o})_k / \text{GL}_n(\mathfrak{o})_{2k} \rightarrow \left( \text{M}_n(\mathfrak{o}/p^k), + \right).
\]

**Proof.** Define the map \( \text{GL}_n(\mathfrak{o})_k \rightarrow \text{M}_n(\mathfrak{o}/p^k) : (I + \overline{\omega}^k M) \mapsto M \mod \overline{\omega}^k \). This is a homomorphism:

\[
(I + \overline{\omega}^k M)(I + \overline{\omega}^k N) = I + \overline{\omega}^k (M + N + \overline{\omega}^k M N),
\]

and the kernel is precisely \( \text{GL}_n(\mathfrak{o})_{2k} \). \( \square \)
4 Ultrametric stability

Let $\mathcal{G}$ be an ultrametric family. This section is concerned with stability with respect to $\mathcal{G}$, without additional assumptions, the main goals being Theorems 1.4 and 1.5. The main tool for the proof of Theorem 1.4 will be to rephrase stability in terms of presentations, following [AP15]: this is well-known in the general setting, but the ultrametric framework gives better quantitative estimates that we will use in the rest of the paper, so we go through the arguments in detail. The proof of Theorem 1.5 will be quite direct thanks to Lemma 3.4. We will end the section by giving complete solutions to uniform stability problems with respect to some of the families introduced in Section 3.

4.1 Lifting and inducing asymptotic homomorphisms

Let $\Gamma = \langle S | R \rangle$ be a presentation of $\Gamma$: for the moment we do not impose any finiteness condition. We define defects and distances for homomorphisms that are close to satisfying the relations: this is analogous to how one can look at homomorphisms on $\Gamma$ as homomorphisms that satisfy the relations. The following definitions are due to Arzhantseva–Păunescu in the finitely presented case [AP15], and to Becker–Lubotzky–Thom in the finitely generated case [BLT19].

Definition 4.1. Given a map $\hat{\varphi} : F_S \to G \in \mathcal{G}$ we define the defect of $\hat{\varphi}$ at $r \in \langle \langle R \rangle \rangle$ and the defect of $\hat{\varphi}$ to be

$$\text{def}_r(\hat{\varphi}) := d_G(\hat{\varphi}(r), 1_G); \quad \text{def}(\hat{\varphi}) := \sup_{r \in R} \text{def}_r(\hat{\varphi}).$$

Given two maps $\hat{\varphi}, \hat{\psi} : F_S \to G \in \mathcal{G}$ we define their distance at $w \in F_S$ and their distance to be

$$\text{dist}_w(\hat{\varphi}, \hat{\psi}) := d_G(\hat{\varphi}(s), \hat{\psi}(s)); \quad \text{dist}(\hat{\varphi}, \hat{\psi}) := \sup_{s \in S} \text{dist}_s(\hat{\varphi}, \hat{\psi}).$$

We will show in Lemma 4.3 the correspondence between these notions “at the level of $F_S$” and those “at the level of $\Gamma$” that we defined in the introduction. A good feature of these definitions is that they allow to give a unique quantity for the notions of defect (for finitely presented groups) and of distance (for finitely generated groups), even when dealing with pointwise asymptotic homomorphisms. For dealing with uniform almost-homomorphisms, one would instead have to define the defect with a supremum over all relations $r \in \langle \langle R \rangle \rangle$, and the distance with a supremum over all words $w \in F_S$. It turns out that this is not necessary under the ultrametric assumption:

Lemma 4.2. Let $\Gamma = \langle S | R \rangle$ and $\hat{\varphi}, \hat{\psi} : F_S \to G \in \mathcal{G}$ a homomorphism. Then:

1. For every $r \in \langle \langle R \rangle \rangle$ there exists a finite set $\{r_1, \ldots, r_k\} \subset R$ (independent of $\hat{\varphi}$) such that

$$\text{def}_r(\hat{\varphi}) \leq \max_i \text{def}_{r_i}(\hat{\varphi}).$$

In particular $\sup_{r \in \langle \langle R \rangle \rangle} \text{def}_r(\hat{\varphi}) = \sup_{r \in R} \text{def}_r(\hat{\varphi}) = \text{def}(\hat{\varphi}).$

2. For every $w \in F_S$ there exists a finite set $\{s_1, \ldots, s_k\} \subset S$ (independent of $\hat{\varphi}$ and $\hat{\psi}$) such that

$$\text{dist}_w(\hat{\varphi}, \hat{\psi}) \leq \max_i \text{dist}_{s_i}(\hat{\varphi}, \hat{\psi}).$$

In particular $\sup_{w \in F_S} \text{dist}_w(\hat{\varphi}, \hat{\psi}) = \sup_{s \in S} \text{dist}_s(\hat{\varphi}, \hat{\psi}) = \text{dist}(\hat{\varphi}, \hat{\psi}).$
Proof. 1. Let \( r \in \langle \langle R \rangle \rangle \), and write \( r = (w_1 r_1 w_1^{-1}) \cdots (w_k r_k w_k^{-1}) \) for \( w_i \in F_S \) and \( r_i \in R \). Then:

\[
\text{def}_r(\hat{\varphi}) = d_G(\hat{\varphi}(r), 1_G) = d_G(\hat{\varphi}(w_1 r_1 w_1^{-1}) \cdots \hat{\varphi}(w_k r_k w_k^{-1}), 1_G) \leq \\
\leq \max \{ d_G(\hat{\varphi}(w_2 r_2 w_2^{-1}) \cdots \hat{\varphi}(w_k r_k w_k^{-1}), 1_G), d_G(\hat{\varphi}(w_1 r_1 w_1^{-1}), 1_G) \} \leq \cdots \leq \\
\leq \max_i d_G(\hat{\varphi}(w_i r_i w_i^{-1}), 1_G) = \max_i d_G(\hat{\varphi}(r_i), 1_G) = \max_i \text{def}_r(\hat{\varphi}).
\]

2. Let \( w = s_1 \cdots s_k \in F_S \). Then, similarly:

\[
\text{dist}_w(\hat{\varphi}, \hat{\psi}) = d_G(\hat{\varphi}(w), \hat{\psi}(w)) = d_G(\hat{\varphi}(s_1) \cdots \hat{\varphi}(s_k) \hat{\psi}(s_k)^{-1} \cdots \hat{\psi}(s_1)^{-1}, 1_G) \leq \\
\leq \max_i d_G(\hat{\varphi}(s_i) \hat{\psi}(s_i)^{-1}, 1_G) = \max_i d_G(\hat{\varphi}(s_i), \hat{\psi}(s_i)) = \max_i \text{dist}_{s_i}(\hat{\varphi}, \hat{\psi}).
\]

\( \square \)

Let us now make the connection between such maps and those defined at the level of \( \Gamma \):

**Lemma 4.3.** Let \( \Gamma = \langle S \mid R \rangle \) and denote by \( F_S \to \Gamma : w \mapsto \overline{w} \) the projection.

1. Let \( \varphi : \Gamma \to G \in \mathcal{G} \) be a map. Let \( \hat{\varphi} : F_S \to G \) be the unique homomorphism coinciding with \( \varphi \) on \( S \). Then for any word \( w \in F_S \) there exists a finite set \( \{(g_1, h_1), \ldots, (g_k, h_k)\} \subset \Gamma^2 \) (independent of \( \varphi \)) such that

\[
d_G(\varphi(\overline{w}), \hat{\varphi}(w)) \leq \max_i \text{def}_{g_i, h_i}(\varphi).
\]

In particular, if \( r \in \langle \langle R \rangle \rangle \) is a relation, then \( \text{def}_r(\hat{\varphi}) \leq \max_i \text{def}_{g_i, h_i}(\varphi) \) and \( \text{def}(\hat{\varphi}) \leq \text{def}(\varphi) \).

2. Let \( \hat{\varphi} : F_S \to G \in \mathcal{G} \) be a homomorphism. Choose a (set-theoretic) section \( \sigma : \Gamma \to F_S \) and define \( \varphi := \hat{\varphi} \circ \sigma \). Then for every \( (g, h) \in \Gamma^2 \) there exists a finite set \( \{r_1, \ldots, r_k\} \subset R \) (independent of \( \hat{\varphi} \) but depending on \( \sigma \)) such that

\[
\text{def}_{g, h}(\varphi) \leq \max_i \text{def}_{r_i}(\hat{\varphi}).
\]

In particular \( \text{def}(\varphi) \leq \text{def}(\hat{\varphi}) \).

The proof that follows is unfortunately heavy in notation, mainly because we are making a rigorous distinction between elements of \( S \) and elements of \( S^{-1} \). All indices \( \varepsilon_i \) could be taken out if we could reduce to the case in which \( \varphi(s^{-1}) = \varphi(s)^{-1} \). However this assumption is equivalent to stability of the group \( \mathbb{Z}/2\mathbb{Z} \), which cannot be established in such great generality, as the next example shows.

**Example 4.4.** For each \( n \geq 1 \), let \( d_n \) be the metric on \( \mathbb{Z}_p \) defined by the non-Archimedean norm \(| \cdot |_p^n \), let \( G_n \) denote \( \mathbb{Z}_p \) equipped with this metric and \( \mathcal{G} \) the corresponding family, which is ultrametric. We claim that \( \mathbb{Z}/p\mathbb{Z} \) is not uniformly \( \mathcal{G} \)-stable (note that the pointwise and uniform notions coincide for finite groups). Indeed, let \( \varphi_n : \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}_p \) be the map sending \( k \mod p \) to \( k \), for \( 0 \leq k < p \). Since \( \mathbb{Z}_p \) is torsion-free, the only homomorphism \( \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}_p \) is the trivial one, which is at a distance 1 from \( \varphi_n \). On the other hand \( \text{def}(\varphi_n) = d_n(p, 0) = p^{-n} \to 0 \). Thus \( (\varphi_n : \mathbb{Z}/p\mathbb{Z} \to G_n)_{n \geq 1} \) is a uniform asymptotic homomorphism that is not uniformly asymptotically close to any homomorphism.

We proceed with the proof.
Proof of Lemma 4.2. 1. Fix a word \( w = s_1^{\varepsilon_1} \cdots s_k^{\varepsilon_k} \) written as a reduced product of elements of \( S \sqcup S^{-1} \); here \( s_i \in S \) and \( \varepsilon_i = \pm 1 \). Let
\[
\delta := \max_i d_G(\varphi(s_1^{\varepsilon_1} \cdots s_i^{\varepsilon_i}), \varphi(s_1^{\varepsilon_1} \cdots s_{i-1}^{\varepsilon_{i-1}})\varphi(s_i)^{\varepsilon_i}).
\]
Notice that this is bounded by finitely many terms of the form \( \text{def}_{g,h}(\varphi) \) for some \((g,h) \in \Gamma^2 \); here we are using that \( d_G(\varphi(s^{-1}), \varphi(s)^{-1}) \leq \max \{ \text{def}_{s,s^{-1}}(\varphi), \text{def}_{1,1}(\varphi) \} \). Now \( d_G(\varphi(s_1^{\varepsilon_1}), \varphi(s_1)^{\varepsilon_1}) \leq \delta \), and by induction
\[
d_G(\varphi(s_1^{\varepsilon_1} \cdots s_i^{\varepsilon_i}), \varphi(s_1^{\varepsilon_1} \cdots \varphi(s_i)^{\varepsilon_i}) \leq \max \{ d_G(\varphi(s_1^{\varepsilon_1} \cdots s_i^{\varepsilon_i}), \varphi(s_1^{\varepsilon_1} \cdots s_{i-1}^{\varepsilon_{i-1}})\varphi(s_i)^{\varepsilon_i}), d_G(\varphi(s_1^{\varepsilon_1} \cdots s_{i-1}^{\varepsilon_{i-1}}), \varphi(s_1)^{\varepsilon_1} \cdots \varphi(s_{i-1})^{\varepsilon_{i-1}}) \} \leq \delta.
\]
Therefore
\[
d_G(\varphi(\overline{w}), \varphi(w)) = d_G(\varphi(s_1^{\varepsilon_1} \cdots s_k^{\varepsilon_k}), \varphi(s_1)^{\varepsilon_1} \cdots \varphi(s_k)^{\varepsilon_k}) \leq \delta.
\]
The last statement follows by taking \( w \) to be a relation, so \( \varphi(\overline{w}) = \varphi(1_G) \) is close to \( 1_G \); more precisely \( d_G(\varphi(1), 1_G) = \text{def}_{1,1}(\varphi) \).

2. Fix \( g, h \in \Gamma \); and let \( r := \sigma(gh)(\sigma(g)\sigma(h))^{-1} \in \langle \langle R \rangle \rangle \). Then
\[
\text{def}_{g,h}(\varphi) = d_G(\varphi(gh), \varphi(g)\varphi(h)) = d_G(\hat{\varphi}(\sigma(gh)), \varphi(\sigma(g))\hat{\varphi}(\sigma(h))) = d_G(\hat{\varphi}(r), 1_G) = \text{def}_{r,1}(\hat{\varphi}).
\]
The result then follows from Item 1. of Lemma 4.2.

This implies the desired equivalent characterization of stability:

**Proposition 4.5.** Let \( \Gamma = \langle S \mid R \rangle \).

1. \( \Gamma \) is pointwise \( \mathcal{G} \)-stable if and only if for any sequence \( (\hat{\varphi}_n : F_S \to G_n \in \mathcal{G})_{n \geq 1} \) such that \( \text{def}_{r,1}(\hat{\varphi}_n) \to 0 \) for all \( r \in R \), there exists a sequence \( (\hat{\psi}_n : F_S \to G_n)_{n \geq 1} \) such that \( \text{def}(\hat{\psi}_n) = 0 \) and \( \text{dist}_s(\hat{\varphi}_n, \hat{\psi}_n) \to 0 \) for all \( s \in S \).

2. \( \Gamma \) is uniformly \( \mathcal{G} \)-stable if and only if for any sequence \( (\hat{\varphi}_n : F_S \to G_n \in \mathcal{G})_{n \geq 1} \) such that \( \text{def}(\hat{\varphi}_n) \to 0 \), there exists a sequence \( (\hat{\psi}_n : F_S \to G_n)_{n \geq 1} \) such that \( \text{def}(\hat{\psi}_n) = 0 \) and \( \text{dist}(\hat{\varphi}_n, \hat{\psi}_n) \to 0 \).

**Remark.** The condition \( \text{def}(\hat{\psi}) = 0 \) is equivalent to: \( \hat{\psi} \) descends to a homomorphism of \( \Gamma \).

**Proof.** Suppose that \( \Gamma \) is pointwise \( \mathcal{G} \)-stable and let \( \hat{\varphi}_n \) be a sequence as in the statement of 1. Item 2. of Lemma 4.2 gives a sequence \( (\varphi_n : \Gamma \to G_n)_{n \geq 1} \) such that \( \text{def}_{g,h}(\varphi_n) \to 0 \) for all \( (g,h) \in \Gamma^2 \). By pointwise stability the asymptotic homomorphism \( (\varphi_n)_{n \geq 1} \) is pointwise asymptotically close to a sequence of homomorphisms \( (\psi_n : \Gamma \to G_n)_{n \geq 1} \). This lifts to a sequence \( (\hat{\psi}_n : F_S \to G_n)_{n \geq 1} \) such that \( \text{def}(\hat{\psi}_n) = 0 \) and \( \text{dist}(\hat{\varphi}_n, \hat{\psi}_n) \to 0 \) for all \( s \in S \).

The converse is similar, using Item 1. of Lemma 4.2 instead, and the proof for the uniform case is the same.

So one can take the notions of stability to be defined by maps at the level of the free group that almost descend to \( \Gamma \). Then this proposition becomes less obvious than it looks, since it says that this notion does not depend on the choice of the presentation. This fact can also be proven directly by noticing, as in [AP15 Section 3], that this notion of stability of a presentation is not
affected by Tietze transformations

We can provide a further equivalent characterization of uniform stability by rephrasing the quantitative characterization from Lemma 2.1 in these terms:

**Corollary 4.6.** Let $\Gamma = \langle S \mid R \rangle$. The following are equivalent:

1. $\Gamma$ is uniformly $G$-stable.

2. For all $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $\hat{\varphi} : F_S \to G \in G$ satisfies $\text{def}(\hat{\varphi}) \leq \delta$, there exists $\hat{\psi} : F_S \to G$ that descends to $\Gamma$ and satisfies $\text{dist}(\varphi, \psi) \leq \varepsilon$.

**Proof.** The proof is the same as that of Lemma 2.1 using the characterization of uniform stability from Proposition 4.5.

One last thing we need to understand is when closeness on the generators implies closeness elsewhere. The following lemma showcases how useful moving up and down from $\Gamma$ to the corresponding free group can be.

**Lemma 4.7.** Let $\Gamma$ be generated by a set $S$, and consider two maps $\varphi, \psi : \Gamma \to G \in G$. Then for all $g \in \Gamma$ there exist finite sets $\{(x_1, y_1), \ldots, (x_k, y_k)\} \subset \Gamma^2$ and $\{s_1, \ldots, s_k\} \subset S$ (independent of $\varphi, \psi$) such that

$$\text{dist}_g(\varphi, \psi) \leq \max_i \{\text{def}_{x_i, y_i}(\varphi), \text{def}_{x_i, y_i}(\psi), \text{dist}_{s_i}(\varphi, \psi)\}.$$ 

In particular

$$\text{dist}(\varphi, \psi) \leq \max \{\text{def}(\varphi), \text{def}(\psi), \sup_{s \in S} \text{dist}_s(\varphi, \psi)\}.$$ 

**Proof.** Let $\hat{\varphi}, \hat{\psi} : F_S \to G$ be the homomorphisms obtained via Item 1 of Lemma 4.3. That statement allows us to work with $\hat{\varphi}, \hat{\psi}$ instead, up to finitely many defects of $\varphi$ and $\psi$. Then Item 2 of Lemma 4.2 implies that the distance at $g$ is bounded in terms of the distance at finitely many generators.

### 4.2 Finiteness conditions

Now we add finiteness conditions on the presentation. The following proposition gives general properties of asymptotic homomorphisms under such hypotheses:

**Proposition 4.8.** 1. Suppose that $\Gamma$ is generated by the finite set $S$, and that $(\varphi_n, \psi_n : \Gamma \to G_n \in G)_{n \geq 1}$ satisfy $\text{dist}_s(\varphi_n, \psi_n) \to 0$ for all $s \in S$. If $\varphi_n, \psi_n$ are pointwise (respectively, uniform) asymptotic homomorphisms, then they are pointwise (respectively, uniformly) asymptotically close.

2. Suppose that $\Gamma$ is finitely presented. Then any pointwise asymptotic homomorphism is pointwise asymptotically close to a uniform asymptotic homomorphism.

**Proof.** 1. This follows directly from Lemma 4.7.

2. Let $(\varphi_n)_{n \geq 1}$ be an asymptotic homomorphism. Using Item 1 of Lemma 4.3 we can lift $\varphi_n$ to $(\hat{\varphi}_n : F_S \to G_n)_{n \geq 1}$ such that $\text{def}_r(\hat{\varphi}_n)$ is bounded in terms of finitely many defects of $\varphi_n$ for any relator $r \in R$. Since $R$ is finite, the same holds for $\text{def}(\hat{\varphi}_n)$. Now $\hat{\varphi}_n$ induces a map on $\Gamma$ using...
Item 2. of Lemma 4.3 thus we obtain maps \( \psi_n : \Gamma \to G_n \) such that \( \text{def}(\psi_n) \leq \text{def}(\varphi_n) \). It follows that \( \psi_n \) is a uniform asymptotic homomorphism. It is pointwise asymptotically close to \( \varphi_n \) by the previous item.

Item 2. of Proposition 4.8 does not say that pointwise asymptotic homomorphisms are automatically uniform: this is false in general, as the next example shows.

**Example 4.9.** Consider the map \( \varphi_n : \mathbb{Z} \to \text{GL}_2(\mathbb{Z}_p) : k \mapsto \begin{cases} \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} & \text{if } |k| \leq n \\ I_2 & \text{otherwise.} \end{cases} \)

Then \( \text{def}(\varphi_n) = 1 \) for all \( n \), while \( \text{def}_{g,h}(\varphi_n) \to 0 \) for all \( (g,h) \in \mathbb{Z}^2 \). However note that, by construction, \( \varphi_n \) is pointwise (not uniformly) close to a homomorphism, in line with Item 2. of Proposition 4.8.

We can now prove Theorem 1.4:

**Theorem 4.10.** Let \( \Gamma \) be finitely generated and pointwise \( \mathcal{G} \)-stable. Then \( \Gamma \) is uniformly \( \mathcal{G} \)-stable. If moreover \( \Gamma \) is finitely presented, then the converse holds.

**Proof.** Let \( (\varphi_n : \Gamma \to G_n \in \mathcal{G})_{n \geq 1} \) be a uniform asymptotic homomorphism. Since \( \Gamma \) is pointwise \( \mathcal{G} \)-stable, this is pointwise asymptotically close to a sequence of homomorphisms \( (\psi_n : \Gamma \to G_n)_{n \geq 1} \). By Item 1. of Proposition 4.8 since \( \varphi_n \) and \( \psi_n \) are both uniform, they are uniformly asymptotically close.

Now suppose that \( \Gamma \) is finitely presented and uniformly \( \mathcal{G} \)-stable. Let \( (\varphi_n : \Gamma \to G_n \in \mathcal{G})_{n \geq 1} \) be a pointwise asymptotic homomorphism. By Item 2. of Proposition 4.8 this is pointwise asymptotically close to a uniform asymptotic homomorphism, which in turn is uniformly (thus pointwise) asymptotically close to a sequence of homomorphism, by uniform \( \mathcal{G} \)-stability.

We will see in Example 5.13 that there exist finitely generated groups that are uniformly but not pointwise \( \text{GL}(\mathfrak{o}) \)-stable. This theorem allows to unambiguously talk about \( \mathcal{G} \)-stability for finitely presented groups, since pointwise and uniform stability are equivalent.

**Example 4.11.** A free group of finite rank is \( \mathcal{G} \)-stable. This follows immediately by using the characterization of pointwise stability in Proposition 4.5. It also applies to free groups of countably infinite rank, proving both pointwise and uniform stability.

Although pointwise stability of free groups holds for any family \( \mathcal{G} \), as we remarked in the introduction, uniform stability is really special to the ultrametric setting. Indeed, free groups are not uniformly \( \mathcal{G} \)-stable, for \( \mathcal{G} = (\text{U}(n), \| \cdot \|) \), where \( \| \cdot \| \) is any \( \text{U}(n) \)-invariant norm on \( \text{M}_n(\mathbb{C}) \) [Rol09], or for \( \mathcal{G} = (S_n, d_H) \) [BC20].

**Example 4.12.** Let \( \mathcal{G} \) be a discrete family (Example 3.6). Then every group is uniformly \( \mathcal{G} \)-stable: if \( \varphi : \Gamma \to G \in \mathcal{G} \) satisfies \( \text{def}(\varphi) < 1 \), then \( \varphi \) is already a homomorphism. Theorem 4.10 implies that all finitely presented groups are pointwise \( \mathcal{G} \)-stable.

Pointwise \( \mathcal{G} \)-stability need not hold for arbitrary finitely generated groups: if \( \mathcal{G} \) is the discrete family of all finite groups and \( \Gamma \) is LEF but not residually finite, then \( \Gamma \) is not pointwise stable. This will be explained in more detail and generality in Section 5.
4.3 Homomorphisms onto metric quotients

We now move to the proof of Theorem 1.5. The main tool is given by Lemma 3.4 which relates asymptotic homomorphisms to $\mathcal{G}$ with true homomorphisms to $MQ(\mathcal{G})$, the family of metric quotients of $\mathcal{G}$ and subgroups thereof.

We start by proving a consequence of Lemma 3.4 which essentially gives one direction of Theorem 1.5. Let $G$ be an ultrametric group, and let $C$ be a class of groups that is closed under taking subgroups, and such that all metric quotients of $G$ are contained in $C$. By Lemma 2.4 there exists a largest residually-$C$ quotient of $\Gamma$, that we denote by $R$. Lemma 3.4 implies that maps onto $G$ with small defect almost factor through $R$.

Lemma 4.13. Let $\varphi : \Gamma \to G$ be such that $\text{def}(\varphi) \leq \varepsilon$. Then there exists a map $\overline{\varphi} : R \to G$ such that $\text{def}(\overline{\varphi}) \leq \varepsilon$ and $\text{dist}(\varphi, \overline{\varphi} \circ \pi_R) \leq \varepsilon$.

Proof. By Item 1. of Lemma 3.4 we can consider the homomorphisms $\varphi(\varepsilon) : \Gamma \to G/G(\varepsilon) \in C$ induced by $\varphi$. By the universal property of $R$ (Lemma 2.4) this map factors through a homomorphism $\phi : R \to G/G(\varepsilon)$. Take $\overline{\varphi}$ to be any lift of this homomorphism to a map $R \to G$. Then by construction $\phi : R \xrightarrow{\overline{\varphi}} G \to G/G(\varepsilon)$ is a homomorphism, so by Item 1. of Lemma 3.4 again $\text{def}(\overline{\varphi}) \leq \varepsilon$. Moreover, the induced homomorphisms $\varphi(\varepsilon), (\overline{\varphi}(\varepsilon) \circ \pi_R) : \Gamma \to G/G(\varepsilon)$ both coincide with $\phi \circ \pi_R$, so by Item 2. of Lemma 3.4 we have $\text{dist}(\varphi, \overline{\varphi} \circ \pi_R) \leq \varepsilon$. \hfill $\Box$

For the rest of this section, we let $R$ be the largest residually-$C$ quotient of $\Gamma$, where $C$ is a class of groups closed under taking subgroups and containing $MQ(\mathcal{G})$. That is, $R := \Gamma/K$ where $K$ is the intersection of all normal subgroups of $\Gamma$ with quotient in $C$ (see Lemma 2.4). For instance, if $\mathcal{G} = \text{GL}(\mathcal{O})$, then $MQ(\mathcal{G})$ is the class of all finite groups, and so $R$ can be taken to be the largest residually finite quotient of $\Gamma$.

Theorem 4.14. Let $\mathcal{G}, \Gamma, R$ be as above. Then $\Gamma$ is uniformly $\mathcal{G}$-stable if and only if $R$ is. If $\Gamma$ is pointwise $\mathcal{G}$-stable, then so is $R$.

Proof. Suppose that $R$ is uniformly $\mathcal{G}$-stable. We use the characterization from Lemma 2.4 for all $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $\text{def}(\varphi) \leq \delta$ there exists a homomorphism $\psi$ such that $\text{dist}(\varphi, \psi) \leq \varepsilon$. Fix $\varepsilon > 0$ and let $\delta > 0$ be given by uniform stability of $R$; we may assume that $\delta \leq \varepsilon$. Let $\varphi : \Gamma \to G \in \mathcal{G}$ be a map such that $\text{def}(\varphi) \leq \delta$. By Lemma 4.13 there exists a map $\overline{\varphi} : R \to G$ such that $\text{def}(\overline{\varphi}) \leq \delta$ and $\text{dist}(\varphi, \overline{\varphi} \circ \pi_R) \leq \delta$. By uniform stability of $R$ there exists a homomorphism $\overline{\psi} : R \to G$ such that $\text{dist}(\overline{\varphi}, \overline{\psi}) \leq \varepsilon$. Then $\overline{\psi} = \overline{\psi} \circ \pi_R : \Gamma \to G$ is a homomorphism and

$$\text{dist}(\varphi, \psi) \leq \max\{\text{dist}(\varphi, \overline{\varphi} \circ \pi_R), \text{dist}(\overline{\varphi} \circ \pi_R, \overline{\psi} \circ \pi_R)\} \leq \max\{\delta, \varepsilon\} \leq \varepsilon.$$ 

Suppose that $\Gamma$ is pointwise $\mathcal{G}$-stable, and let $(\varphi_n : R \to G_n \in \mathcal{G})_{n \geq 1}$ be a pointwise asymptotic homomorphism. Then $(\varphi_n \circ \pi_R : \Gamma \to G_n)_{n \geq 1}$ is also a pointwise asymptotic homomorphism: indeed $\text{def}_{g,h}(\varphi_n \circ \pi_R) = \text{def}_{\pi_R(g), \pi_R(h)}(\varphi_n)$ for all $(g, h) \in \Gamma^2$. By pointwise stability of $\Gamma$ there exists a sequence of homomorphisms $(\psi_n : \Gamma \to G_n)_{n \geq 1}$ that is pointwise asymptotically close to $(\varphi_n \circ \pi_R)_{n \geq 1}$. Since $G$ is residually-$MQ(\mathcal{G})$, we have that $\psi_n$ factors through $R$, and so there exists a homomorphism $\psi_n : R \to G$ such that $\psi_n \circ \pi_R = \psi_n$. Moreover, $\varphi_n$ and $\psi_n$ are pointwise asymptotically close: indeed $\text{dist}_{\pi_R(g)}(\varphi_n, \psi_n) = \text{dist} g(\varphi_n \circ \pi_R, \psi_n)$.

Similarly, if $\Gamma$ is uniformly $\mathcal{G}$-stable, then so is $R$: the proof is the same. \hfill $\Box$
Example 4.15. Let $\Gamma$ be a group without non-trivial quotients in $MQ(\mathcal{G})$. Then $\Gamma$ is uniformly $\mathcal{G}$-stable. More precisely, if $\varphi : \Gamma \to G \in \mathcal{G}$ and $\text{def}(\varphi) \leq \varepsilon_k$, then $\text{dist}(\varphi, 1) \leq \varepsilon_k$. For instance, if $\Gamma$ is a simple group, then it can only be unstable if it belongs to $MQ(\mathcal{G})$.

If $\mathcal{G}$ is profinite, then $MQ(\mathcal{G})$ consists of finite groups, and so any infinite group without non-trivial finite quotients is uniformly $\mathcal{G}$-stable. Examples include Pride's group, which was already discussed in Example 2.13, as well as certain finitely presented groups such as Higman's group [Hig51] or Burger–Mozes groups [BM97]. Recently, more examples have been found in [CS18] among discrete subgroups of Isom$(\mathbb{H}^3)$.

If $MQ(\mathcal{G})$ consists of finite $\pi$-groups, for $\pi$ a set of primes, then this statement is a weaker version of Proposition 6.9.

Let us specialize to the case in which $\mathcal{G}$ is profinite, and so $MQ(\mathcal{G})$ consists of finite groups. The previous example implies that an infinite group without non-trivial finite quotients, such as Pride's example (Example 2.13), is uniformly $\mathcal{G}$-stable. However we can exploit Theorem 4.14 further than just the case in which $R = \{1\}$.

Example 4.16. The group $G^+$ from Example 2.5 has $\mathbb{Z}$ as largest residually finite quotient, which is uniformly $\mathcal{G}$-stable by Example 4.11. Therefore $G^+$ is uniformly $\mathcal{G}$-stable. Similarly the largest residually finite quotient of $G$ is $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, which as we will see in Corollaries 6.16 and 7.12 is uniformly GL($\mathfrak{o}$)-stable, whenever $\mathbb{K}$ has characteristic other than 2. Thus $G$ is uniformly GL($\mathfrak{o}$)-stable.

Example 4.17. Let $G, H$ be finitely generated, with $H$ infinite and residually finite. Then by Example 2.6 the largest residually finite quotient of $G \wr H$ is $\text{Ab}(G) \wr H$. In particular, if $G$ is perfect (that is, $\text{Ab}(G) = \{1\}$) and $H$ is uniformly $\mathcal{G}$-stable, then $G \wr H$ is uniformly $\mathcal{G}$-stable. For example the lamplighter group $G \wr \mathbb{Z}$ is uniformly $\mathcal{G}$-stable for every non-abelian finite simple group $G$, by Example 4.11. This will be strengthened for virtually pro-$\pi$ families in Corollary 6.17.

We will see in Example 5.13 that the groups from these examples are not pointwise GL($\mathfrak{o}$)-stable, where $\mathfrak{o}$ is the ring of integers of a non-Archimedean local field. This will prove that pointwise $\mathcal{G}$-stability of $R$ does not imply pointwise GL($\mathfrak{o}$)-stability of $\Gamma$, and also that a finitely generated uniformly $\mathcal{G}$-stable group need not be pointwise $\mathcal{G}$-stable.

Note that in the proof of Theorem 4.14 as well as that of Lemma 4.13 we only used that any homomorphism from $\Gamma$ to a residually-$MQ(\mathcal{G})$ group factors through $R$. The same holds for any intermediate quotient, and so we obtain the following generalization of Theorem 4.14:

Corollary 4.18. Let $K \leq \Gamma$ be a group that is contained in the kernel of the quotient $\Gamma \to R$. Then $\Gamma$ is uniformly stable if and only if $\Gamma/K$ is. If $\Gamma$ is pointwise $\mathcal{G}$-stable, then so is $\Gamma/K$.

Another interesting consequence of Theorem 4.14 is that the equivalence of Theorem 4.10 can also be extended to some infinitely presented residually-$MQ(\mathcal{G})$ groups.

Corollary 4.19. Let $\mathcal{G}$ be an ultrametric family, $\mathcal{C}$ a class of groups closed under taking subgroups and containing $MQ(\mathcal{G})$. Let $\Gamma$ be a (finitely generated, residually-$\mathcal{C}$) group, that can be expressed as the largest residually-$\mathcal{C}$ quotient of some finitely presented group. Then $\Gamma$ is pointwise stable if and only if it is uniformly stable.

Proof. By Theorem 4.10 we only need to show that uniform stability implies pointwise stability. Let $\hat{\Gamma}$ be the finitely presented group from the statement. If $\Gamma$ is uniformly stable, then $\hat{\Gamma}$ is uniformly stable by Theorem 4.14 being finitely presented it is also pointwise stable by Theorem 4.10 and so $\Gamma$ is pointwise stable again by Theorem 4.14.
In case $G$ is profinite and $C$ is the class of all finite groups, it is tempting to conjecture that all finitely generated residually finite groups have this property, and so the equivalence of Theorem 4.10 applies to all of them. This is not the case. Indeed, let $\Gamma$ be a finitely generated residually finite group. Suppose that we know:

1. Whenever $C$ is finitely presented and $C \to \Gamma$ is a surjective homomorphism, $C$ is large, that is, it virtually surjects onto $F_2$.

2. There exist finite groups onto which $\Gamma$ does not virtually surject.

Then $\Gamma$ cannot be the largest residually finite quotient of a finitely presented group. The paper [BGDLH13] contains many examples of such groups.

**Example 4.20.** Any finitely generated group that surjects onto $\mathbb{Z}$ with locally finite kernel has property 1. [BGDLH13, Post-Scriptum]. Thus the Lamplighter group $\mathbb{Z}/2\mathbb{Z}\wr \mathbb{Z}$ (which has property 2, since it is metabelian) cannot be the largest residually finite quotient of a finitely presented group.

**Example 4.21.** Here is a torsion-free example. The Basilica group, introduced in [GZ02], is finitely generated, residually finite, torsion-free, and every proper quotient is solvable, so it has property 2. It also has property 1. [BGDLH13, Section 2], so it cannot be the largest residually finite quotient of a finitely presented group.

Compare this with [CG05, Corollary 6.9]: every finitely generated residually free group is the largest residually free quotient of a finitely presented group.

### 4.4 Solution to some stability problems

We present here the complete solution to three uniform stability problems, with respect to families introduced in Section 3. The first two are with respect to the families $T(R)$ (Example 3.7) and $\text{Aut}(X^*_n)$ (Example 3.8) and admit a short and direct proof.

**Proposition 4.22.** Let $G$ be an ultrametric family with the following property: for every $G \in G$ and every $\varepsilon > 0$, the extension $1 \to G(\varepsilon) \to G \to G/G(\varepsilon) \to 1$ splits. Then all groups are uniformly $G$-stable. In particular, all groups are uniformly $T(R)$-stable and $\text{Aut}(X^*_n)$-stable.

**Proof.** We use the characterization from Lemma 2.1. By Lemma 3.4 it suffices to show that any homomorphism $\varphi : \Gamma \to G/G(\varepsilon)$ lifts to a homomorphism $\psi : \Gamma \to G$. Composing $\varphi$ with a section $G/G(\varepsilon) \to G$, we conclude.

Both $T(R)$ and $\text{Aut}(X^*_n)$ satisfy the hypothesis. The metric quotients of $T_n(R)$ are isomorphic to $T_{n-k}(R)$ for some $1 \leq k \leq n$, and the section is given by the inclusion in the upper-left corner. The metric quotients of $\text{Aut}(X^*_n)$ are isomorphic to the group of automorphisms of words of a given finite length, and the section is given by letting these elements act on the prefix of the appropriate length, and trivially on the other letters.

**Remark.** Note that the proof gives more that uniform stability: it implies moreover that the stability estimate is optimal. That is, if $\text{def}(\varphi) \leq \varepsilon$, then there exists a homomorphism $\psi$ with $\text{dist}(\varphi, \psi) \leq \varepsilon$. 

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In Example 5.13 we show that there exist (finitely generated) groups that are not pointwise Aut(X̂)-stable.

Our next goal is to give the complete solution for uniform stability of finitely generated groups with respect to the family Gal(K) (Example 3.11). We start by proving a stability result for finite families $G$. When studying uniform stability, even looking at families of a single group is interesting. For instance, when $G = \{(U(1), \|\cdot\|_{op})\}$, then non-abelian free groups are not uniformly $G$-stable [Rol09]. However, in the ultrametric setting, such a situation cannot occur:

**Proposition 4.23.** Let $G$ be a finite profinite family. Then any finitely generated group is uniformly $G$-stable.

This is a uniform version of [AC20, Proposition 6], in the ultrametric setting. For the proof we need the following equivalent characterization of uniform stability in terms of ultralimits, which is a uniform version of [AP13, Theorem 4.2] (see Lemma 2.3):

**Lemma 4.24.** Let $G$ be any family of groups equipped with bi-invariant ultrametrics and let $\Gamma = \langle S \mid R \rangle$ be a countable group. The following are equivalent:

1. $\Gamma$ is uniformly $G$-stable.

2. For every free ultrafilter $\omega \in \mathcal{P}(\mathbb{N})$ the following holds: for any sequence $(\hat{\varphi}_n : F_S \to G_n \in \mathcal{G})_{n \geq 1}$ such that $\lim_{n \to \omega} \varphi_n = 0$, there exists a sequence $(\hat{\psi}_n : F_S \to G_n)_{n \geq 1}$ of homomorphisms that descend to $\Gamma$ such that $\lim_{n \to \omega} \hat{\psi}_n = 0$.

**Proof.** We use the characterization of uniform $G$-stability from Corollary 4.6.

1. $\Rightarrow$ 2. Fix an ultrafilter $\omega$ and let $(\hat{\varphi}_n : F_S \to G_n \in \mathcal{G})_{n \geq 1}$ be such that $\lim_{n \to \omega} \varphi_n = 0$. For all $\hat{\varphi}_n$, let $\hat{\psi}_n : F_S \to G_n$ be a homomorphism that descends to $\Gamma$ and minimizes $\text{dist}(\hat{\varphi}_n, \hat{\psi}_n)$ up to $1/n$. We need to show that $\lim_{n \to \omega} \text{dist}(\hat{\varphi}_n, \hat{\psi}_n) = 0$, so let $\varepsilon > 0$ and let $\delta > 0$ be as in Corollary 4.6 for $\varepsilon/2 > 0$: this means that if $\lim_{n \to \omega} \varphi_n \leq \delta$ then $\lim_{n \to \omega} \text{dist}(\hat{\varphi}_n, \hat{\psi}_n) \leq \varepsilon/2 + 1/n$. Let $N \geq 2/\varepsilon$. Then if $\lim_{n \to \omega} \varphi_n \leq \delta$ and $n \geq N$ we have $\text{dist}(\hat{\varphi}_n, \hat{\psi}_n) \leq \varepsilon$. Therefore $\{n \geq 1 : \text{dist}(\hat{\varphi}_n, \hat{\psi}_n) \leq \varepsilon\} \supset \{n \geq N\} \cap \{n \geq 1 : \lim_{n \to \omega} \varphi_n \leq \delta\}$. The smaller set belongs to $\omega$ because $\lim_{n \to \omega} \varphi_n = 0$ and $\omega$ is free. Thus the larger set is also in $\omega$, and we conclude.

2. $\Rightarrow$ 1. Suppose that $\Gamma$ is not uniformly $G$-stable. By Corollary 4.6 there exists $\varepsilon > 0$ and a sequence $(\hat{\varphi}_n : F_S \to G_n \in \mathcal{G})_{n \geq 1}$ such that $\lim_{n \to \omega} \varphi_n \leq 1/n$ but for any sequence of $\hat{\psi}_n : F_S \to G_n$ descending to $\Gamma$ we have that $\lim_{n \to \omega} \text{dist}(\hat{\varphi}_n, \hat{\psi}_n) \geq \varepsilon$. This implies that for every free ultrafilter $\omega$ we have $\lim_{n \to \omega} \text{dist}(\hat{\varphi}_n, \hat{\psi}_n) \geq \varepsilon > 0$ for any sequence $\hat{\psi}_n$ of homomorphisms that descend to $\Gamma$. So 3. does not hold.

**Proof of Proposition 4.23.** Let $\Gamma = \langle S \mid R \rangle$ be a finitely generated group. Fix a free ultrafilter $\omega \in \mathcal{P}(\mathbb{N})$ and let $(\hat{\varphi}_n : F_S \to G_n \in \mathcal{G})_{n \geq 1}$ be a sequence such that $\lim_{n \to \omega} \varphi_n = 0$. Since $G$ is finite, up to restricting to a subset in the ultrafilter we may assume that $G_n = G$ is a fixed group for all $n \geq 1$. Since $G$ is a compact metric space, for all $s \in S$ the sequence $\hat{\varphi}_n(s)$ admits an ultralimit, which we denote by $\hat{\psi}(s) \in G$. Let $\hat{\psi} : F_S \to G$ be the corresponding homomorphism. Then $\hat{\psi}$ descends to $\Gamma$: indeed, for all $r \in R$ we have

$$d_G(\hat{\psi}(r), 1_G) = \lim_{n \to \omega} d_G(\hat{\varphi}_n(r), 1_G) = \lim_{n \to \omega} \text{def}(\hat{\varphi}_n) = 0.$$
Moreover, by definition \( \text{dist}\left(\hat{\varphi}_n, \hat{\psi}\right) \xrightarrow{n \to \omega} 0 \), and so, since \( S \) is finite, \( \text{dist}(\hat{\varphi}_n, \hat{\psi}) \xrightarrow{n \to \omega} 0 \). We conclude by Lemma 4.24.

We are now ready to solve uniform stability of finitely generated groups with respect to the family \( \text{Gal}(K) \).

**Proposition 4.25.** A group is uniformly \( \text{Gal}(K) \)-stable if and only if it is uniformly \( \{\text{Gal}(K^{\text{sep}}/K)\} \)-stable. In particular, all finitely generated groups are uniformly \( \text{Gal}(K) \)-stable.

**Proof.** We use the notation from Example 3.11: the absolute Galois group of \( K \) is denoted by \( G \), and the metric is constructed via the sequences \( (G_k)_{k \geq 1} \) and \( \sigma \). We denote by \( G_k^L \) the image of \( G_k \) in \( \text{Gal}(L/K) \).

We use the characterization of uniform stability from Lemma 2.1. Clearly if \( \Gamma \) is uniformly \( \text{Gal}(K) \)-stable, then it is uniformly \( \{G\} \)-stable. Now suppose that \( \Gamma \) is uniformly \( \{G\} \)-stable; fix \( \varepsilon > 0 \) and let \( \delta > 0 \) be as in Lemma 2.1. Since the defect only takes values in \( \{\varepsilon_k : k \geq 0\} \), we may assume that \( \delta = \varepsilon_k \) for some \( k \). Let \( \varphi : \Gamma \to \text{Gal}(L/K) \) be a map with \( \text{def}(\varphi) \leq \varepsilon_k \) and consider the induced homomorphism \( \varphi(\varepsilon_k) : \Gamma \to \text{Gal}(L/K)/G_k^L \). Now \( \text{Gal}(L/K)/G_k^L \cong G/G_k \), so we may lift this homomorphism to a map \( \hat{\varphi} : \Gamma \to G \) such that \( \text{def}(\hat{\varphi}) \leq \varepsilon_k \), by Lemma 3.4. By the choice of \( \delta = \varepsilon_k \), there exists a homomorphism \( \hat{\psi} : \Gamma \to G \) such that \( \text{dist}(\hat{\varphi}, \hat{\psi}) \leq \varepsilon_k \). Then \( \psi : \Gamma \xrightarrow{\hat{\psi}} G \to \text{Gal}(L/K) \) is a homomorphism, and \( \text{dist}(\varphi, \psi) \leq \varepsilon \).

We conclude that \( \Gamma \) is \( \text{Gal}(K) \)-stable. Since \( \{G\} \) is a finite profinite family, the last statement follows directly from Proposition 4.23.
5 Ultrametric approximation and pointwise stability

This section constitutes an interlude, in that we leave the question of stability to focus on the related approximation problem. The main goal is to prove Theorem 4.7 for the family \( \mathcal{G} \); combining this with Lemma 2.2 will produce several counterexamples to pointwise stability.

Recall from Definition 1.6 that \( \Gamma \) is \( \mathcal{G} \)-approximable if there exists a \( \mathcal{G} \)-approximation, namely an asymptotically injective pointwise asymptotic homomorphism \( (\varphi_n : \Gamma \to G_n \in \mathcal{G})_{n \geq 1} \). The following similar notion appears quite naturally in this context:

**Definition 5.1.** We say that a \( \mathcal{G} \)-approximation is strong if it is moreover a uniform asymptotic homomorphism. If \( \Gamma \) admits a strong \( \mathcal{G} \)-approximation, it is said to be strongly \( \mathcal{G} \)-approximable.

The results from Section 4 imply that this is not stronger for finitely presented groups:

**Lemma 5.2.** Let \( \mathcal{G} \) be an ultrametric family, \( \Gamma \) a finitely presented group. If \( \Gamma \) is \( \mathcal{G} \)-approximable, then it is strongly \( \mathcal{G} \)-approximable.

**Proof.** By Item 2 of Proposition 4.8, any pointwise asymptotic homomorphism of \( \Gamma \) is pointwise asymptotically close to a uniform one. Applying this to a \( \mathcal{G} \)-approximation gives a uniform asymptotic homomorphism that is still asymptotically injective: a strong \( \mathcal{G} \)-approximation. \( \square \)

5.1 From approximations to local embeddings

Well-studied approximation properties such as soficity or hyperlinearity are much weaker than residual finiteness, or local embeddability into finite groups. In this subsection we prove that \( \mathcal{G} \)-approximability, when \( \mathcal{G} \) is a profinite family, is stronger. This is essentially a reinterpretation of the interplay between local embeddability and convergence in the space of marked groups \([VG97]\) (see Theorem 2.12).

**Proposition 5.3.** Let \( \Gamma = \langle S \mid R \rangle \) be a countable group. If \( \Gamma \) is \( \mathcal{G} \)-approximable, then \( \Gamma \) is locally embeddable into MQ(\( \mathcal{G} \)). If \( \Gamma \) is strongly \( \mathcal{G} \)-approximable, then \( \Gamma \) is fully residually-MQ(\( \mathcal{G} \)). In particular, if \( \Gamma \) is \( \mathcal{G} \)-approximable and finitely presented, then \( \Gamma \) is fully residually MQ(\( \mathcal{G} \)).

**Remark.** The last statement follows from the general fact that finitely presented groups that are locally embeddable into MQ(\( \mathcal{G} \)) are also fully residually-MQ(\( \mathcal{G} \)) \([VG97]\) (see Proposition 2.9). However it also follows by combining the rest of the proposition with Lemma 5.2.

**Proof.** Let \((\varphi_n : \Gamma \to G_n)_{n \geq 1}\) be a pointwise asymptotic homomorphism, which we lift to \((\hat{\varphi}_n : F_S \to G_n)_{n \geq 1}\) using Lemma 4.3. Fix an enumeration of \( N = \langle \langle R \rangle \rangle \), denote by \( N(k) \) the first \( k \) elements, and fix a strictly decreasing sequence \( \varepsilon_k \to 0 \). Up to subsequence, we may assume that \( \text{def}_r(\hat{\varphi}_n) \leq \varepsilon_n \) for all \( r \in N(n) \), and we look at the induced homomorphism \( f_n : F_S \to G/G(\varepsilon_n) \). We get a sequence of \( F_S \)-marked groups with kernel \( N_n = \{ w \in F_S : d(\hat{\varphi}_n(w), 1) \leq \varepsilon_n \} \). Up to subsequence, this converges in \( \mathcal{N}(F_S) \) to:

\[
\{ w \in F_S : d(\hat{\varphi}_n(w), 1) \leq \varepsilon_n \text{ infinitely often} \} = \{ w \in \Gamma : d(\varphi_n(w), 1) \leq \varepsilon_n \text{ almost always} \}.
\]

Now \( N \) is contained in the left-hand side by choice of the subsequence. If moreover \( \varphi_n \) is asymptotically injective, then \( N \) contains the right-hand side, and so \( N_n \to N \in \mathcal{N}(F_S) \), which implies
that $\Gamma$ is locally embeddable into $MQ(\mathcal{G})$ by Item 2. of Theorem 2.12.

We can do the same for uniform asymptotic homomorphisms, by working over $\Gamma$-marked groups and taking $\varepsilon_n = \text{def}(\varphi_n)$. The corresponding sequence $N_n \in \mathcal{N}(\Gamma)$ converges to $\{1\}$, and we apply Item 1. of Theorem 2.12.

**Example 5.4.** A group is called *weakly hyperlinear* if it is approximable with respect to some family of compact metric groups $\mathbf{Gis}$ [AC20]. Similarly, a group is called *weakly sofic* if it is approximable with respect to some family of finite metric groups $\mathbf{GR08}$. Proposition 5.3 shows that, if we add the hypothesis that the approximating families are ultrametric, then all such groups are LEF.

According to the properties of the class $MQ(\mathcal{G})$, the conclusion of Proposition 5.3 can be strengthened. We look at two examples: the family $T(R)$ (Example 3.7) where $R$ is a finite ring, and the family $\text{Gal}(\mathbb{F})$ (Example 3.11), where $\mathbb{F}$ is a finite field.

**Corollary 5.5.** Let $R$ be a commutative ring, and $\Gamma$ a countable $T(R)$-approximable group. Then there exists a normal subgroup $\Gamma_0 \leq \Gamma$ such that $\Gamma/\Gamma_0$ embeds into $(R^\times)^{\mathbb{N}}$ and $\Gamma_0$ is locally embeddable into $UT(R)$, in particular it is locally embeddable into the class of nilpotent groups.

If moreover $R$ is finite and $\Gamma$ is finitely generated, then $\Gamma$ and $\Gamma_0$ have non-trivial abelian quotients, and $\Gamma/\Gamma_0$ embeds into $(R^\times)^n$ for some $n \geq 1$.

**Proof.** We will use the equivalent definition of local embeddability in terms of ultraproducts (Proposition 2.8) repeatedly throughout the proof.

By Proposition 5.3 and since all metric quotients of $T_n(R)$ are of the form $T_k(R)$ (Example 3.7), we know that $\Gamma$ is locally embeddable into $T(R)$. Then $\Gamma$ embeds into an ultraproduct $\prod_{n \to \omega} T_n(R)$.

This gives a homomorphism

$$\Gamma \to \prod_{n \to \omega} T_n(R) \to \text{Ab}\left(\prod_{n \to \omega} T_n(R)\right) \cong \prod_{n \to \omega} (R^\times)^n,$$

let $\Gamma_0$ be its kernel. Then $\Gamma_0$ embeds into $\prod_{n \to \omega} UT_n(R)$, and so it is locally embeddable into $UT(R)$.

Now $\Gamma/\Gamma_0$ embeds into $\prod_{n \to \omega} (R^\times)^n$, so it is locally embeddable into $\mathcal{C} := \{(R^\times)^n : n \geq 1\}$. Since $\Gamma/\Gamma_0$ is abelian, every finitely generated subgroup is finitely presented and locally embeddable into $\mathcal{C}$, so residually-$\mathcal{C}$ by Item 2. of Proposition 2.9, and so it embeds into $(R^\times)^{\mathbb{N}}$. Since $\Gamma$ is countable, $\Gamma/\Gamma_0$ embeds into $(R^\times)^{\mathbb{N}}$, too. If now $R^\times$ is finite, then $\mathbb{Z}$ cannot be residually-$\mathcal{C}$, since there is a bound on the order of cyclic subgroups of $(R^\times)^n$. So $\Gamma/\Gamma_0$ is a torsion group; if moreover $\text{Ab}(\Gamma)$ is finitely generated, then $\Gamma/\Gamma_0$ is finite, and being residually-$\mathcal{C}$ it embeds into $(R^\times)^n$ for some $n \geq 1$.

The statement about $\Gamma$ and $\Gamma_0$ having non-trivial abelian quotients is a consequence of [NST18], where it is proven that this holds for all finitely generated groups that are approximable in the class of finite solvable groups.

For the family $\text{Gal}(\mathbb{F})$, we can give a full characterization.

**Corollary 5.6.** Let $\mathbb{F}$ be a finite field, and $\Gamma$ a group. Then $\Gamma$ is $\text{Gal}(\mathbb{F})$-approximable if and only if it is abelian and all of its finite subgroups are cyclic. In particular, if $\Gamma$ is finitely generated, then it is $\text{Gal}(\mathbb{F})$-approximable if and only if it is of the form $\mathbb{Z}^r \times C$ for some finite cyclic group $C$.  

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Proof. The absolute Galois group \( \text{Gal}(\mathbb{F}^{\text{sep}}/\mathbb{F}) \) is isomorphic to the profinite completion \( \hat{\mathbb{Z}} \) of \( \mathbb{Z} \), which in turn is isomorphic to the direct product of \( \mathbb{Z}_p \), where \( p \) goes through all primes. It follows that any Galois extension of \( \mathbb{F} \) has pro-cyclic Galois group, so by Proposition 5.3, if \( \Gamma \) is \( \text{Gal}(\mathbb{F}) \)-approximable, then it locally embeddable in the class of finite cyclic groups. This implies two things: first, all finite subgroups of \( \Gamma \) are cyclic, since a finite group that is locally embeddable in a class automatically belongs to that class. Secondly, \( \Gamma \) is abelian; more generally, a group that is locally embeddable in the class of abelian groups is abelian: this follows directly from the ultraproduct characterization (Proposition 2.8).

We next show that all such groups are approximable. Since approximability is a local property it suffices to show that \( \Gamma = \mathbb{Z}^r \times C \) is approximable. We construct the approximations inside the absolute Galois group of \( \mathbb{F} \), which we identify with \( \hat{\mathbb{Z}} \cong \prod \mathbb{Z}_p \). This is metrized using a nested sequence \( G_k \) of finite-index open normal subgroups and a sequence of positive reals \( \varepsilon_k \to 0 \). Let us make another reduction: if we are able to approximate \( C = \mathbb{Z}/p^n\mathbb{Z} \) via maps that take values in \( \mathbb{Z}_p \leq \hat{\mathbb{Z}} \), then we are done. Indeed, we can write any finite cyclic group as a direct product of such groups, for a finite set \( \{p_1, \ldots, p_i\} \) of distinct primes, and take the direct product of these approximations into \( \prod_i \mathbb{Z}_{p_i} \leq \hat{\mathbb{Z}} \). Finally, we can embed \( \mathbb{Z}^r \) into a direct product of \( \mathbb{Z}_p \) for \( r \) distinct primes that we did not use yet.

So we are left to show that we can approximate \( C = \mathbb{Z}/p^n\mathbb{Z} \) for some \( n \geq 1 \) with an approximation taking values in \( \mathbb{Z}_p \). Given \( k \geq 1 \) let \( m \geq 1 \) be such that \( \mathbb{Z}_p \cap G_k = p^m\mathbb{Z}_p \); since the \( G_k \) get smaller, \( m \to \infty \) as \( k \to \infty \). Let \( k_n \) be the smallest integer such that the corresponding \( m \) is larger than \( n \), and let \( k_n \geq k_n \). Then we can embed \( C \) into \( \mathbb{Z}/p^m\mathbb{Z} \), compose with a section into \( \mathbb{Z}_p \), and finally include the latter in \( \hat{\mathbb{Z}} \). This gives a map \( \varphi : C \to \hat{\mathbb{Z}} \) that projects to an injective homomorphism into \( \hat{\mathbb{Z}}/G_k \). It follows that \( \text{def}(\varphi) \leq \varepsilon_k \), and moreover \( d(\varphi(x), 1) \geq \varepsilon_{k_n} \) for all \( x \neq 1 \).

Note that the proof actually shows that these groups are \( \{\text{Gal}(\mathbb{F}^{\text{sep}}/\mathbb{F})\} \)-approximable.

### 5.2 From local embeddings to approximations

The first easy examples of sofic and hyperlinear groups are residually finite, and more generally LEF groups. In this subsection we prove that these are also approximable with respect to certain ultrametric families. The precise statement will involve the following concept:

**Definition 5.7.** Let \( \mathcal{C} \) be a class of groups. We say that \( \mathcal{C} \) is \( \mathcal{G} \)-approximable if there exists \( \varepsilon > 0 \) such that for all \( C \in \mathcal{C} \) and for all \( \delta > 0 \) there exists a map \( \eta : C \to G \in \mathcal{G} \) such that \( \text{def}(\eta) < \delta \) and \( d_G(\eta(x), 1_G) > \varepsilon \) for all \( 1 \neq x \in C \).

So a class \( \mathcal{C} \) is \( \mathcal{G} \)-approximable if every group in \( \mathcal{C} \) is strongly \( \mathcal{G} \)-approximable, and moreover the injectivity gap can be taken uniformly for the entire class. For several profinite families \( \mathcal{G} \), the class of all finite groups is \( \mathcal{G} \)-approximable:

**Example 5.8.** The class of all finite groups is \( \text{GL}(\mathfrak{o}) \)-approximable, where \( \mathfrak{o} \) is the ring of integers of a non-Archimedean local field. Indeed any finite group \( C \) can be embedded into \( \text{GL}_n(\mathfrak{o}) \), for \( n \) large enough, using permutation matrices. If \( \eta \) denotes this embedding, then \( \text{def}(\eta) = 0 \) and \( d(\eta(x), I_n) = 1 \) for all \( 1 \neq x \in C \).

**Example 5.9.** The class of all finite groups is \( \text{Aut}(X_n^*) \)-approximable. Indeed, any finite group \( C \) can be embedded into \( S_n \), for \( n \) large enough, which in turn can be embedded into \( \text{Aut}(X_n^*) \) by
acting on the first letter of a word. If \( \eta \) denotes this embedding, then \( \text{def}(\eta) = 0 \) and \( d(\eta(x), \text{id}_{X_n}) = 1 \) for all \( 1 \neq x \in C \).

For other families, this definition is quite restrictive:

**Example 5.10.** Let \( K \) be a field and consider the family \( \text{Gal}(K) = \text{Gal}(K)((G_k)_{k \geq 1}, \varepsilon) \) of Galois groups of Galois extensions of \( K \): see Example 3.11 for the notation. We claim that an infinite class of finite groups cannot be \( \text{Gal}(K) \)-approximable.

Suppose that \( C \) is a \( \text{Gal}(K) \)-approximable family of finite groups, let \( \varepsilon > 0 \) be the uniform injectivity gap as in the definition and let \( k \geq 1 \) be such that \( \varepsilon \geq \varepsilon_k \). Then for all \( C \in C \) there exists \( G/N \in \text{Gal}(K) \) and a map \( \eta : C \to G/N \) such that \( \text{def}(\eta) \leq \varepsilon_k \) and \( d(\eta(x), 1) > \varepsilon \geq \varepsilon_k \) for all \( 1 \neq x \in C \). The first inequality implies that \( \eta \) induces a homomorphism \( C \to (G/N)/(G_kN/N) \cong G/G_kN \), and the second one implies that this homomorphism is injective. Therefore \( |C| \leq |G : G_kN| \leq |G : G_k| \). Since the inequality holds for any \( C \), this implies that \( C \) is finite.

By Proposition 5.3 a restriction of this kind is necessary. For instance if a finite group does not belong to \( MQ(G) \), then it cannot be \( G \)-approximable. But when the local embeddings take place in a \( G \)-approximable class, then we can prove a converse to Proposition 5.3.

**Proposition 5.11.** Let \( C \) be a \( G \)-approximable class of groups and \( \Gamma \) a countable group. If \( \Gamma \) is locally embeddable into \( C \), then \( \Gamma \) is \( G \)-approximable. If \( \Gamma \) is fully residually-\( C \), then \( \Gamma \) is strongly \( G \)-approximable.

**Proof.** Since \( \Gamma \) is countable, we can write it as an increasing union of finite sets \( (K_n)_{n \geq 1} \). If \( \Gamma \) is locally embeddable into \( C \), then for all \( n \) we can choose a \( K_n \)-local embedding \( f_n : \Gamma \to C_n \in C \). Since \( C \) is \( G \)-approximable, there exists \( \varepsilon > 0 \) such that: for all \( n \) there exists a map \( \eta_n : C_n \to G_n \in G \) with \( \text{def}(\eta_n) \leq 1/n \) and \( d_n(\eta_n(x), 1_{G_n}) > \varepsilon \) for all \( 1 \neq x \in C_n \). Then \( \varphi_n : \Gamma \xrightarrow{f_n} C_n \xrightarrow{\eta_n} G_n \) satisfies: \( \text{def}_{g,h}(\varphi_n) \leq 1/n \) for all \( (g, h)^2 \in K_n^2 \), and \( d_n(\varphi_n(g), 1_{G_n}) \geq \varepsilon \) for all \( 1 \neq g \in K_n \). It follows that \( \varphi_n \) is a \( G \)-approximation.

If \( \Gamma \) is fully residually-\( C \), then the \( K_n \)-local embeddings \( f_n : \Gamma \to C_n \) may be chosen to be homomorphisms that restrict to injective maps on \( K_n \). Then the resulting \( G \)-approximation is strong.

The requirement that the class \( C \) be \( G \)-approximable allows to deduce approximability from the existence of local embeddings of \( \Gamma \) into \( C \) without knowing what they look like. But this is not the only way to produce approximations. For instance, by Example 5.10 when \( G = \text{Gal}(K) \) we can only apply Proposition 5.11 to groups that are locally embeddable into some finite class \( C \) of finite groups, which are necessarily finite. But in Corollary 5.6 we saw that a \( \text{Gal}(\mathbb{F}) \)-approximable group can very well be infinite.

The proof of Proposition 5.11 implies something (a priori) slightly stronger than approximation: namely, such groups are \( G \)-approximable with a uniform injectivity gap:

\[
\inf_{g \in \Gamma} \left( \liminf_{n \to \infty} d_n(\varphi_n(g), 1_{G_n}) \right) > 0.
\]

This is taken to be the definition of \( G \)-approximation in some of the literature (see e.g. [AC20]). This ambiguity is due to the fact that in several approximation problems the two notions coincide: for instance, for sofic groups, this follows from a well-known amplification trick due to Elek and
Putting together Proposition 5.3, Examples 5.8 and 5.9, and Proposition 5.11, we obtain that a group is LEF if and only if it is $\text{GL}(\mathcal{O})$-approximable, or $\text{Aut}(X^*)$-approximable. In other words, the notions of approximability with respect to these two families coincides with that of approximability with respect to the family of all finite groups equipped with the discrete metric (see Example 3.6). However the quantitative versions of approximability are potentially distinct: for instance to embed a finite group into $\text{GL}_n(\mathcal{O})$ one does not always need the degree $n$ to be equal to the order. The quantitative study of approximation properties was initiated in [AC20], and the case of local embeddings into finite groups was recently studied in more detail in [Bra21].

5.3 Counterexamples to pointwise stability

We now apply the results in this section to give counterexamples to pointwise stability. These all stem from the following corollary, which applies to both $\mathcal{G} = \text{GL}(\mathcal{O})$ and $\mathcal{G} = \text{Aut}(X^*)$.

**Corollary 5.12.** Let $\mathcal{G}$ be a profinite family, such that the class finite groups is $\mathcal{G}$-approximable. Then if $\Gamma$ is LEF but not residually finite, it is not pointwise $\mathcal{G}$-stable.

**Proof.** This is just a combination of Lemma 2.2 and Propositions 5.3 and 5.11.

It is worth noticing that there is no hypothesis of finite generation in this statement, in contrast to the analogous statement for families of unitary groups. This is because the groups $\text{GL}_n(\mathcal{O})$ are not only locally residually finite, as guaranteed for all linear groups by a theorem of Malcev [Mal40], but they are themselves residually finite, being profinite.

In the examples below, $\text{GL}(\mathcal{O})$ may be replaced with any class satisfying the hypotheses of the corollary, for instance $\text{Aut}(X^*)$, or the discrete family of all finite groups (Example 3.6).

**Example 5.13.** Let $\Gamma$ be a classical small cancellation group that is not residually finite, for instance Pride’s group (Example 2.13). Then $\Gamma$ is not pointwise $\text{GL}(\mathcal{O})$-stable. So even the simplest example of a finitely generated uniformly stable group — a group without non-trivial finite quotients (see Example 4.15) — need not be pointwise $\text{GL}(\mathcal{O})$-stable.

**Example 5.14.** Let $G = \text{Sym}_0(\mathbb{Z}) \rtimes \mathbb{Z}$ and $G^+ = \text{Alt}_0(\mathbb{Z}) \rtimes \mathbb{Z}$ be the groups from Example 2.5. These groups are finitely generated, and their largest residually finite quotient is $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, respectively $\mathbb{Z}$, so they are not residually finite. But by Example 2.10 these groups are LEF, therefore they are not pointwise $\text{GL}(\mathcal{O})$-stable by Corollary 5.12. On the other hand, by Example 4.16 the group $G^+$ is uniformly $\text{GL}(\mathcal{O})$-stable (and when $\mathbb{K}$ does not have characteristic 2, so is $G$, by Example 6.10 and Proposition 7.4). The same result holds for lamplighter groups by Examples 2.6, 2.11 and 4.17 for instance if $G$ is a non-abelian finite simple group, then $G \rtimes \mathbb{Z}$ is finitely generated, LEF, but not residually finite, so it is not pointwise $\text{GL}(\mathcal{O})$-stable, even though it is uniformly $\text{GL}(\mathcal{O})$-stable.

These examples show that two previous results are sharp. First, the last statement of Theorem 4.10 does not hold for general finitely generated groups: there exists a finitely generated group that is uniformly but not pointwise $\text{GL}(\mathcal{O})$-stable. Secondly, the converse of the last statement of Theorem 4.14 fails in general: there exists a (finitely generated) group that is not pointwise $\text{GL}(\mathcal{O})$-stable, and whose largest residually finite quotient is pointwise $\text{GL}(\mathcal{O})$-stable.
5.4 A pointwise version of Theorem 4.14

Let $\mathcal{C}$ be a class of groups closed under taking subgroups such that $MQ(\mathcal{G}) \subset \mathcal{C}$, and let $R$ be the largest residually-$\mathcal{C}$ quotient of $\Gamma$, that we assume to be countable as usual. We further assume that $\mathcal{C}$ is closed under taking directed products, from which it follows that residually-$\mathcal{C}$ groups are fully residually-$\mathcal{C}$, and so locally embeddable into $\mathcal{C}$ by Proposition 2.9. Recall from Theorem 4.14 that $\Gamma$ is uniformly stable if and only if $R$ is. The methods from this section allow to prove a pointwise version of this theorem, where again the residual property is replaced by local embeddability. To this end let $L$ be the largest quotient of $\Gamma$ that is locally embeddable into $\mathcal{C}$. As in Lemma 2.3, this is the quotient $\Gamma/K$, where $K$ is the intersection of all kernels of morphisms of $\Gamma$ into a group that is locally embeddable into $\mathcal{C}$. By construction $L$ has the factoring property, and the fact that $\mathcal{C}$ is closed under taking direct products implies that $L$ is locally embeddable into $\mathcal{C}$, since this condition can be verified on finite subsets.

The following proposition is analogous to the fact that a group is pointwise stable in permutation if and only if its largest sofic quotient is.

**Proposition 5.15.** Let $\mathcal{G}, \Gamma, L$ be as above. Then $\Gamma$ is pointwise $\mathcal{G}$-stable if and only if $L$ is. If $\Gamma$ is uniformly $\mathcal{G}$-stable, then so is $L$.

*Proof.* If $\Gamma$ is (pointwise or uniformly) $\mathcal{G}$-stable, then so is $L$, by the same proof as Theorem 4.14.

Suppose that $L$ is pointwise $\mathcal{G}$-stable. By Lemma 2.3, this means that any homomorphism of $L$ to a metric ultraproduct of $G_n \in \mathcal{G}$ lifts to the direct product. To prove the same for $\Gamma$ it suffices to show that any homomorphism of $\Gamma$ to a metric ultraproduct of $G_n$ factors through $L$. Now the image of such a homomorphism is $\mathcal{G}$-approximable, so locally embeddable into $MQ(\mathcal{G})$ by Proposition 5.3, in particular locally embeddable into $\mathcal{C}$.

**Example 5.16.** A finitely presented non-residually-$\mathcal{C}$ group is not locally embeddable into $\mathcal{C}$, by Proposition 2.9. Take such a group, and embed it into a finitely generated simple group $\Gamma$ [Hal74, Gor74]. Then $\Gamma$ contains a subgroup that is not locally embeddable into $\mathcal{C}$, so it has the same property. Being simple, the corresponding group $L$ from Proposition 5.15 is trivial, and so $\Gamma$ is pointwise $\mathcal{G}$-stable. For instance, if the family $\mathcal{G}$ is profinite, we can start with any finitely presented non-residually finite group, and obtain a finitely generated non-LEF simple group that is pointwise $\mathcal{G}$-stable. Note that Hall’s construction [Hal74] always outputs an infinitely presented group. So these are our first examples of finitely generated infinitely presented pointwise stable groups. We will give other explicit examples in Corollary 6.23.
6 Virtually pro-\(\pi\) stability

We now specialize our study of stability to profinite families \(G\) whose metric quotients have restricted orders. This will allow us to provide various examples of uniformly stable groups with respect to such families, some of which are listed in Theorem 1.8.

The basic idea of this approach can be traced back to \[Pom73\], where the author uses the conjugacy part of the Hall Theorem on solvable groups to prove a kind of stability result for the conjugacy relation, under some coprimality assumption on the order of the elements. Here we will go much further, and this is made possible by the use of the more general Schur–Zassenhaus Theorem (see Theorem 2.18).

**Definition 6.1.** Let \(G\) be a profinite family. Given a class \(C\) of finite groups, we say that \(G\) is virtually pro-\(C\) if there exists some \(\varepsilon > 0\) such that \(G(\varepsilon)\) is pro-\(C\) for all \(G \in G\). This section we focus on the class \(C\) of \(\pi\)-groups, where \(\pi\) is a fixed set of primes: we say that \(G\) is virtually pro-\(\pi\).

The condition that \(G(\varepsilon)\) be pro-\(C\) is equivalent to all metric quotients of \(G(\varepsilon)\) being in \(C\). Notice that we are asking that the \(\varepsilon > 0\) be uniform for the whole family.

**Example 6.2.** Let \(\mathfrak{o}\) be the ring of integers of a non-Archimedean local field whose residue field has characteristic \(p\). Then \(GL(\mathfrak{o})\) is virtually pro-\(p\): indeed by Lemma 3.16 the principal congruence subgroups \(GL_n(\mathfrak{o})_1\) are pro-\(p\), and we can take \(\varepsilon = p^{-1}\).

The key to this approach is the interpretation of stability in terms of the following lifting property. A map \(\varphi : \Gamma \to G\) with \(\text{def}(\varphi) \leq \delta\) induces a homomorphism \(\varphi(\delta) : \Gamma \to G/G(\delta)\). Then a homomorphism \(\psi : \Gamma \to G\) satisfies \(\text{dist}(\varphi, \psi) \leq \varepsilon\), where \(\delta \leq \varepsilon\), if and only if it is a lift of the induced homomorphism \(\varphi(\varepsilon) : \Gamma \to G/G(\varepsilon)\). This is just a rephrasing of Lemma 3.4.

6.1 \(\pi\)-free groups

In this subsection we use the lifting part of the Schur–Zassenhaus Theorem to prove stability with respect to virtually pro-\(\pi\) families of groups whose finite quotients have restricted orders.

**Definition 6.3.** A group \(\Gamma\) is \(\pi\)-free if all of its finite quotients are \(\pi^\prime\)-groups; that is, if all of its finite quotients have order divisible only by primes not in \(\pi\). Equivalently, a group is \(\pi\)-free if it has no finite virtual \(p\)-quotients, for any \(p \in \pi\).

Clearly a group is \(\pi\)-free if and only if it is \(p\)-free for all \(p \in \pi\). This terminology is inspired from the terminology in \[Sch75, FF20\] introduced by Schikhof: indeed a group is \(p\)-free according to Definition 6.3 if and only if its profinite completion is \(p\)-free according to Schikhof’s definition. This class of groups clearly contains groups without finite quotients, whose uniform stability was already established in Example 4.15. But there are more examples, including residually finite ones.

**Example 6.4.** Finite \(\pi^\prime\)-groups are \(\pi\)-free.

**Example 6.5.** More generally, let \(\Gamma\) be a periodic group without elements of order \(p\) for all \(p \in \pi\). Then \(\Gamma\) is \(\pi\)-free. Indeed, the order of any element in a finite quotient of \(\Gamma\) must divide the order of any preimage thereof. For example Grigorochuk’s first group is a finitely generated periodic residually finite 2-group, so it is \(2^\prime\)-free.
Example 6.6. Let $X$ be a finite alphabet, and let $\text{Aut}(X^*)$ be the group of rooted tree automorphisms (see Example 3.8). Following [BSZ12], we say that a subgroup $\Gamma \leq \text{Aut}(X^*)$ – which is necessarily residually finite – has the congruence subgroup property if every finite-index subgroup of $\Gamma$ contains some level stabilizer $\text{Aut}(X^*)_k$.

Let $\pi$ be the set of primes $p \leq |X|$. Then $\text{Aut}(X^*)$ is pro-$\pi$, and so any subgroup $\Gamma \leq \text{Aut}(X^*)$ with the congruence subgroup property is $\pi'$-free. This gives many examples of $\pi$-free groups, among which are many branch groups [BSZ12], and in particular all Grigorchuk–Gupta–Sidki groups with non-constant defining vector [Per07, FAGUA17]. See [GUA19, TUA20] for more examples.

Example 6.7. Building on the previous example, there exist finitely generated residually finite torsion-free groups that are $\pi'$-free [FAGUA17].

We now prove stability of such groups. We first prove a quantitative lemma, and deduce the stability statement.

Lemma 6.8. Let $\mathcal{G}$ be a virtually pro-$\pi$ family, and let $\varepsilon_0$ be such that $G(\varepsilon_0)$ is pro-$\pi$ for all $G \in \mathcal{G}$. Let $\varphi : \Gamma \to G \in \mathcal{G}$ be such that $\text{def}(\varphi) \leq \varepsilon \leq \varepsilon_0$, and suppose that the image $C$ of $\Gamma$ in $G/G(\varepsilon)$ is a $\pi'$-group. Then there exists a homomorphism $\psi : \Gamma \to G$ such that $\text{dist}(\varphi, \psi) \leq \varepsilon$. Moreover, $\psi(\Gamma) \leq G$ is a finite group isomorphic to $C$, in particular it is a $\pi'$-group.

Proof. Consider the induced homomorphism $\varphi(\varepsilon) : \Gamma \to G/G(\varepsilon)$ given by Lemma 3.4. By hypothesis $C = \varphi(\varepsilon)(\Gamma)$ is a $\pi'$-group. Given $\delta < \varepsilon$, we have the following lifting problem

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{\varphi(\varepsilon)} & C \\
& & \downarrow \ \\
& G/G(\varepsilon) & \cong (G/G(\delta))/(G(\varepsilon)/G(\delta))
\end{array}
\]

Since $G(\varepsilon)/G(\delta)$ is a finite $\pi$-group, by the Schur–Zassenhaus Theorem there exists a lift: a homomorphism $\varphi(\delta) : \Gamma \to G/G(\delta)$ such that the projection onto $G/G(\varepsilon)$ gives back $\varphi(\varepsilon)$. Moreover $\varphi(\delta)(\Gamma) \cong C$, since it is a quotient of $C$ and it surjects onto it. Repeating this process by induction on a sequence $\varepsilon \geq \delta_1 \to 0$ gives a sequence of homomorphisms $\varphi(\delta_i) : \Gamma \to G/G(\delta_i)$ that are all compatible with the projections, and such that all images are groups isomorphic to $C$. Since $G$ is the projective limit of the groups $G/G(\delta_i)$, this induces a homomorphism $\psi : \Gamma \to G$ such that $\psi(\varepsilon) : \Gamma \to G/G(\varepsilon)$ coincides with $\varphi(\varepsilon)$ and $\psi(\Gamma)$ is a finite group isomorphic to $C$. Therefore $\text{dist}(\varphi, \psi) \leq \varepsilon$ by Lemma 3.4.

Proposition 6.9. Let $\mathcal{G}$ be a virtually pro-$\pi$ family and $\Gamma$ a $\pi$-free group. Then $\Gamma$ is uniformly $\mathcal{G}$-stable.

Proof. Since $\Gamma$ is $\pi$-free, all finite quotients of it are $\pi'$-groups. Therefore given a map $\varphi : \Gamma \to G$ with small enough defect, the previous lemma applies and $\varphi$ is close to a homomorphism. We conclude by Lemma 2.1.

Lemma 6.8 shows that the estimate for stability is the optimal one: there exists $\varepsilon_0 > 0$ such that if $\varphi : \Gamma \to G$ satisfies $\text{def}(\varphi) \leq \varepsilon \leq \varepsilon_0$, then there exists a homomorphism $\psi : \Gamma \to G$ such that $\text{dist}(\varphi, \psi) \leq \varepsilon$. 

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Example 6.10. Let $\Gamma$ be a $p$-free group, $\mathfrak{o}$ the ring of integers of a non-Archimedean local field of residual characteristic $p$. Then $\Gamma$ is uniformly $\text{GL}(\mathfrak{o})$-stable. Quantitatively, for any map $\varphi : \Gamma \to \text{GL}_n(\mathfrak{o})$ such that $\text{def}(\varphi) \leq \varepsilon \leq |\mathfrak{o}|$ (where $|\mathfrak{o}|$ is a uniformizer), there exists a homomorphism $\psi : \Gamma \to \text{GL}_n(\mathfrak{o})$ such that $\text{dist}(\varphi, \psi) \leq \varepsilon$.

Note that, even for finite groups, the hypothesis of $\pi$-freeness is necessary. Indeed, we saw in Example 4.4 that there exists a family of pro-$2$ groups with respect to which $\mathbb{Z}/2\mathbb{Z}$ is unstable. On the other hand, we will see that $\mathbb{Z}/2\mathbb{Z}$ is $\text{GL}(\mathfrak{o})$-stable also when $\mathbb{K}$ has residual characteristic $2$ (Example 7.3 and Proposition 7.18).

The following example is a hint at the relation with bounded cohomology that will be explored in Section 8.

Example 6.11. Let $\mathbb{K}$ have characteristic $p$, and let $\Gamma$ be a normed $\mathbb{K}$-amenable group [FF20] Definition 1.1]. Then $\Gamma$ is $\text{GL}(\mathfrak{o})$-stable: indeed such groups are characterized as being locally finite (thus periodic) and without elements of order $p$ [FF20] Theorem 6.2].

6.2 Graphs of groups

In this subsection we exploit the conjugacy part of the Schur–Zassenhaus Theorem to prove stability with respect to virtually pro-$\pi$ families of several fundamental groups of graphs of groups. This part of the Schur–Zassenhaus Theorem depends on the Odd Order Theorem, but this can be avoided if either the kernel or the quotient of the extension to which the theorem is being applied is solvable. As in the proof of Proposition 6.9 the extensions to which we apply the Schur–Zassenhaus Theorem are with a $\pi$-kernel and a $\pi'$-quotient, so if $\pi = \{p\}$ then the kernel is solvable. Similarly if $\pi = \{p, q\}$ then the kernel is solvable by Burnside’s Theorem. The same kind of statements can be given for the quotient. It may also be possible that we know for other reasons that $G$ is virtually prosolvable: for instance this is the case for $\text{Gal}(F)$ when $F$ is a finite field (see Corollary 6.9). For the general case, however, we need the full power of the Schur–Zassenhaus Theorem, and so the general statements in this subsection depend on the Odd Order Theorem.

Let us fix the definitions and notation (see [Ser77] for more detail). Let $X = (V, E)$ be a connected graph with vertex set $V$ and edge set $E$, maps $\pm : E \to V : e^{\pm}$ giving the source and target of an edge, and a fix-point free involution $e \mapsto \bar{e}$ reversing the orientation of each edge. A graph of groups is composed by the following data: a graph $X = (V, E)$, groups $\Gamma_v$ for all $v \in V$ and $\Gamma_e$ for all $e \in E$ such that $\Gamma_e = \Gamma_{\bar{e}}$, and injective morphisms $\iota^+_e : \Gamma_e \to \Gamma_{e^+}$. By abuse of notation we use $X$ to denote both the graph of groups and the underlying abstract graph.

Let $T$ be a spanning tree of $X$. The fundamental group of this graph of groups is the group generated by all vertex groups, together with an element $t_e$ for each $e \in E$, with the additional relations: $t_{\bar{e}} = t_e^{-1}$, $t_e \iota_e^-(x) t_e^{-1} = \iota_e^+(x)$ for all $x \in \Gamma_e$ (a generating set of $\Gamma_e$ suffices), and $t_e = 1$ if $e \in T$. The isomorphism type of the fundamental group is independent of the choice of $T$. A presentation of the fundamental group is thus given by

$$\langle \{S_v : v \in V\} \cup \{t_e : e \in E\} | \{R_v : v \in V\} \cup \{R_e : e \in E\}\rangle,$$

where $\langle S_v | R_v \rangle$ is a presentation of $\Gamma_v$, and $R_e$ are the relations describing the identification $t_{\bar{e}} = t_e^{-1}$, the effect of conjugacy by $t_e$, and the relation $t_e = 1$ if $e \in T$. 

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As in the previous subsection, we first prove a quantitative lemma, and then deduce two stability statements. Since the fundamental group is defined in terms of a presentation, the most natural approach is by working in terms of it, which is possible by Proposition 4.5 and Corollary 4.6.

**Lemma 6.12.** Let \( G \) be a virtually pro-\( \pi \) family, and let \( \varepsilon_0 \) be such that \( G(\varepsilon_0) \) is pro-\( \pi \) for all \( G \in G \). Let \( X \) be a connected graph of groups with vertex groups \( \Gamma_v \), edge groups \( \Gamma_e \) and edge inclusions \( i_v^e : \Gamma_e \to \Gamma_e^\pm \). Let \( \Gamma \) be the fundamental group of \( X \), with the standard presentation \( \langle S \mid R \rangle \) as above.

Let \( \hat{\varphi} : F_S \to G \in G \) be a map with \( \text{def}(\hat{\varphi}) \leq \varepsilon \leq \varepsilon_0 \). Suppose further that for all \( v \in V \) the restriction of \( \hat{\varphi} \) to \( F_{S_v} \) descends to a homomorphism \( \varphi_v : \Gamma_v \to G \) such that, if \( e^\pm = v \), the image of \( \varphi_v(i_v^e(\Gamma_e)) \) in \( G/G(\delta) \) is a \( \pi^t \)-group, for all \( \delta \leq \varepsilon \).

Then there exists a homomorphism \( \hat{\psi} : F_S \to G \) such that \( \text{dist}(\hat{\varphi}, \hat{\psi}) \leq \varepsilon \) and \( \hat{\psi} \) descends to a homomorphism of \( \Gamma \).

Before proceeding with the proof, let us comment on how this lemma is of interest independently of our applications (namely Propositions 6.13 and 6.14 below and their corollaries in the next subsections). Indeed, it shows that such fundamentals groups of graphs of groups are examples of two notions related to stability, introduced recently in the literature and of which few examples are known so far.

First, Lemma 6.12 is a statement about *constraint stability*, a notion introduced by Arzhantseva and Păunescu in [AP18]. Given a group \( \Gamma \) and a subgroup \( \Lambda \leq \Gamma \), let us say that \( \Gamma \) is *constraint stable* with respect to \( \Lambda \), if for any asymptotic homomorphism \( (\varphi_n : \Gamma \to G_n)_{n \geq 1} \) such that its restriction to \( \Lambda \) is close to a sequence of homomorphisms \( (\psi_n : \Lambda \to G_n)_{n \geq 1} \), we can extend \( (\psi_n)_{n \geq 1} \) to a homomorphism of \( \Gamma \) that is close to \( (\varphi_n)_{n \geq 1} \). As usual, this can be formalized to a pointwise notion and a uniform one. Similarly we can talk of \( \Gamma \) being stable with respect to a set of subgroups. Then Lemma 6.12 is a statement about constraint stability of \( \Gamma \) with respect to the set of vertex subgroups.

Secondly, Lemma 6.12 is a statement about *stability of an epimorphism*, a notion introduced by Lazarovich and Levit in [LL21]. We say that an epimorphism \( \Gamma \to \Gamma \) is *stable* if any asymptotic homomorphism of \( \Gamma \) that almost descends to \( \Gamma \) (where “almost descends” is meant as in Proposition 4.5) is close to a sequence of homomorphisms of \( \Gamma \) that descend to \( \Gamma \). Again, this leads to a pointwise and a uniform notion. Then Lemma 6.12 is a statement about stability of the epimorphism of \( ((*, \Gamma_v) * (*, t_e)) \) onto \( \Gamma \). Interestingly, this is precisely the setting in [LL21], where the authors prove stability of the same epimorphism in the case of virtually free groups, to deduce that all virtually free groups are stable in permutation.

We proceed with the proof.

**Proof of Lemma 6.12.** By hypothesis \( \hat{\varphi} \) already satisfies all relations \( R_v \). Our goal is to modify \( \hat{\varphi} \) step by step so that it keeps this property, changes by at most \( \varepsilon \), and it also satisfies all relations \( R_e \). For the rest of this proof, we denote the reduction map \( G \to G/G(\delta) \) by \( (\cdot \mod \delta) \). So if \( A \leq G \), its image in \( G/G(\delta) \) is denoted by \( A \mod \delta \).

First of all, we can choose a set \( E_+ \) of positively oriented edges, and replace \( \hat{\varphi}(t_e) \) by \( \hat{\varphi}(t_e)^{-1} \) for all \( e \notin E_+ \). Since \( \text{def}(\hat{\varphi}) \leq \varepsilon \), these two elements are at a distance at most \( \varepsilon \), and so this substitution does not affect the other relations being satisfied in \( G/G(\varepsilon) \). Similarly we can replace...
\( \hat{\varphi}(t_e) \) by 1 for all \( e \in T \). This leaves us with the conjugacy relations.

We start with the edges in \( T \): we will modify \( \hat{\varphi} \) at the vertex groups so that it still restricts to a homomorphism and moreover it satisfies the amalgamations. Fix a vertex \( v_0 \) in \( X \), with neighbours \( v_1, \ldots, v_r \) and edges \( e_1, \ldots, e_r \in T \), where \( v_0 = e^-_0 \) and \( v_i = e^+_i \) for \( i = 1, \ldots, r \). Let \( A^-_i \) := \( \varphi_{v_0}(e^-_i(\Gamma_{e_i})) \leq \varphi_{v_0}(\Gamma_{v_0}) \) and \( A^+_i \) := \( \varphi_{v_i}(e^+_i(\Gamma_{e_i})) \leq \varphi_{v_i}(\Gamma_{v_i}) \). By hypothesis, for all \( \delta \leq \varepsilon \) the reduction modulo \( \delta \) of both \( A^-_i \) and \( A^+_i \) is a \( \pi^- \)-group. This implies in particular that for all \( \delta \leq \varepsilon \) the projection map \( A^\pm \mod \delta \rightarrow A^\pm \mod \varepsilon \) is an isomorphism: indeed the kernel is contained in \( G(\varepsilon)/G(\delta) \) which is a finite \( \pi^- \)-group since \( G(\varepsilon) \leq G(\varepsilon_0) \) is pro-\( \pi^- \).

Now def(\( \hat{\varphi} \)) \( \leq \varepsilon \), and the copies of \( \Gamma_{e_i} \) in \( \Gamma_{v_0} \) and \( \Gamma_{v_i} \) are amalgamated in any quotient of \( \Gamma \). Thus by Lemma 3.4 the reduction modulo \( \varepsilon \) of the homomorphisms \( \varphi_{v_0} \circ t^-_{e_i} \), \( \varphi_{v_i} \circ t^+_{e_i} : \Gamma_{e_i} \rightarrow A^\pm \) is the same, denoted \( f(\varepsilon) \). Therefore the reductions modulo \( \varepsilon \) of the same maps are two (a priori distinct) solutions to the following lifting problem:

\[
\begin{array}{ccc}
\Gamma_{e_i} & \longrightarrow & f(\varepsilon)(\Gamma_{e_i}) \\
\downarrow & & \downarrow \\
G/G(\varepsilon) & \cong & (G/G(\delta))/(G(\varepsilon)/G(\delta))
\end{array}
\]

Since \( G(\varepsilon)/G(\delta) \) is a finite \( \pi^- \)-group, the Schur–Zassenhaus Theorem implies that these two lifts are \( G(\varepsilon)/G(\delta) \)-conjugate. Let \( t \in G(\varepsilon) \) be a lift of this conjugating element. We can then replace \( \varphi_{v_i} \) by \( x \mapsto t\varphi_{v_i}(x)t^{-1} \). This is still a homomorphism of \( \Gamma_{v_i} \), and since \( t \in G(\varepsilon) \) it remains \( \varepsilon \)-close to \( \varphi_{v_i} \), but now the groups \( A^\pm \) coincide modulo \( \delta \). We can repeat this process inductively on a sequence \( \varepsilon \geq \delta_k \rightarrow 0 \): at each step we modify \( \varphi_{v_i} \) by conjugating it by an element of \( G(\delta_k) \), so that the groups \( A^\pm \) are amalgamated modulo \( \delta_{k+1} \). This sequence of conjugating elements converges to an element \( t \in G(\varepsilon) \), and conjugating by it we have modified \( \varphi_{v_i} \) so that it is still \( \varepsilon \)-close to \( \varphi \), but it moreover satisfies the amalgamation \( A^-_i = A^+_i \).

We can do this for all \( i \), and so we get the desired relation for each edge \( e_i \). These modifications are compatible, since they only affect \( \varphi_{v_i} \) for \( i \geq 1 \) and not for \( i = 0 \). We can now apply the same procedure to all neighbours of \( v_i \) connected by edges of \( T \) other than \( e_0 \), and take care of those edges without affecting the behaviour of \( \varphi_{v_i} : i = 0, \ldots, r \). Since \( T \) is a tree we can keep on doing this until we have covered all edges of \( T \). We have thus obtained \( \hat{\varphi} : F_S \rightarrow G \) with the same properties as before, but now it also satisfies all relations \( \{ R_e : e \in T \} \).

We next move to edges not in \( T \). For such an edge \( e \), let \( A^\pm := \varphi(e^\pm(e)) \leq G \) as before, so that \( A^\pm \mod \varepsilon \) is a \( \pi^- \)-group. Since \( \hat{\varphi} \mod \varepsilon \) descends to a homomorphism of \( \Gamma \), we know that \( A^+ \) and \( A^- \) are conjugate modulo \( \varepsilon \) by \( \hat{\varphi}(t_e) \). So \( A^+ \) and \( \hat{\varphi}(t_e)A^- \hat{\varphi}(t_e)^{-1} \) satisfy the same hypotheses as in the previous step. By the same argument, there exists \( t \in G(\varepsilon) \) such that \( t\hat{\varphi}(t_e)A^- \hat{\varphi}(t_e)^{-1}t^{-1} = A^+ \). We can thus replace \( \hat{\varphi}(t_e) \) by \( t\hat{\varphi}(t_e) \), which is congruent to \( \hat{\varphi}(t_e) \) modulo \( \varepsilon \). This takes care of all such relations. Note that we have only modified the images of the edge elements, so this does not affect the definition of \( \hat{\varphi} \) at the vertex groups, or at the other edge groups.

We are left with a homomorphism \( \hat{\psi} : F_S \rightarrow G \) such that \( \text{dist}(\hat{\varphi}|_{F_S}, \hat{\psi}|_{F_S}) \leq \varepsilon \) for every vertex \( v \), and \( d(\hat{\varphi}(t_e), \hat{\psi}(t_e)) \leq \varepsilon \) for every edge \( e \), and moreover \( \hat{\psi} \) satisfies all of the defining relations of \( \Gamma \). Thus \( \psi \) is \( \varepsilon \)-close to \( \hat{\varphi} \) and it descends to a homomorphism \( \Gamma \rightarrow G \).
Here is our first stability result for graphs of groups:

**Proposition 6.13.** Let $G$ be a virtually pro-$\pi$ family and $\Gamma$ the fundamental group of a graph of groups such that all vertex groups are uniformly $G$-stable with a uniform estimate, and such that for every edge $e$ adjacent to a vertex $v$, the image of $\Gamma_e$ in any finite quotient of $\Gamma_v$ is a $\pi'$-group. Then $\Gamma$ is uniformly $G$-stable.

**Remark.** By “uniformly $G$-stable with a uniform estimate” we mean that the $\delta = \delta(\varepsilon)$ from Lemma 2.1 may be chosen uniformly for all $\Gamma_v$. This is automatically satisfied if the graph is finite.

**Proof.** We use the characterization of uniform stability from Corollary 4.6. Let $\langle S \mid R \rangle$ be the standard presentation of $\Gamma$. Fix $0 < \varepsilon \leq \varepsilon_0$. Let $\hat{\varphi}: F_S \to G \in G$ be a homomorphism with $\text{def}(\hat{\varphi}) \leq \delta$, where $\delta = \delta(\varepsilon)$ is given by uniform $G$-stability of the $\Gamma_v$. This allows to modify $\hat{\varphi}$ by at most $\varepsilon$ on the vertex generators $S_v$ so that $\hat{\varphi}|_{F_{S_v}}$ descends to a homomorphism of $\Gamma_v$. Now $\hat{\varphi}$ satisfies the hypotheses of Lemma 6.12, and so there exists $\hat{\psi}: F_S \to G$ such that $\text{dist}(\hat{\varphi}, \hat{\psi}) \leq \varepsilon$ and $\hat{\psi}$ descends to a homomorphism of $\Gamma$.

Lemma 6.12 shows that the estimate for stability is at least as good as the uniform estimate for the vertex groups. For instance if the graph is finite, this proposition gives as an estimate of stability the one of the vertex group with the least efficient estimate. This slightly different result combines Lemma 6.12 with the last part of Lemma 6.8.

**Proposition 6.14.** Let $G$ be a virtually pro-$\pi$ family and $\Gamma$ the fundamental group of a graph of groups such that for every vertex $v$ the image of $\Gamma_v$ in any finite quotient of $\Gamma$ is a $\pi'$-group. Then $\Gamma$ is uniformly $G$-stable.

**Proof.** Let $\langle S \mid R \rangle$ be the standard presentation of $\Gamma$. Fix $0 < \varepsilon \leq \varepsilon_0$ and let $\hat{\varphi}: F_S \to G \in G$ be a homomorphism with $\text{def}(\hat{\varphi}) \leq \varepsilon$. By Lemma 3.4, this induces a homomorphism $\Gamma \to G/G(\varepsilon)$, whose restriction to $\Gamma_v$ is a $\pi'$-group. Now Lemma 6.8 allows to modify $\hat{\varphi}$ by at most $\varepsilon$ on the vertex generators $S_v$ so that $\hat{\varphi}|_{F_{S_v}}$ descends to a homomorphism $\Gamma_v \to G$ whose image is a finite $\pi'$-group. Then we apply Lemma 6.12 and conclude as in Proposition 6.13.

Here the proof shows that the estimate for stability is optimal.

### 6.3 First corollaries

We now apply Propositions 6.13 and 6.14 to obtain examples of uniformly $G$-stable groups.

The following is a direct consequence of Proposition 6.13.

**Corollary 6.15.** Let $G$ be a virtually pro-$\pi$ family. The following groups are uniformly $G$-stable:

1. Fundamental groups of connected graphs of groups, with $\pi$-free vertex groups.

2. Fundamental groups of finite, connected graphs of groups, with uniformly $G$-stable vertex groups and $\pi$-free edge groups.

The next corollary relies on Dunwoody’s characterization of groups of cohomological dimension (denoted $cd$) at most 1 [Dun79]:

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Corollary 6.16. Let $G$ be a virtually pro-$\pi$ family. If $cd_{\mathbb{F}_p}(\Gamma) \leq 1$ for all $p \in \pi$, then $\Gamma$ is uniformly $G$-stable. In particular finitely generated virtually free groups without elements of order $p$, for all $p \in \pi$, are uniformly $G$-stable.

Proof. By [Dun79], a group has $\mathbb{F}_p$-cohomological dimension at most 1 if and only if it is the fundamental group of a connected graph of groups whose vertex groups are finite and $p$-free. Even if the underlying graph is infinite, the estimate for uniform $G$-stability of the vertex groups is uniform (in fact, optimal) by Lemma 6.8, and so we can apply Proposition 6.13. The statement about virtually free groups is the finitely generated case of [Dun79], but it also follows more directly from Stallings’s Theorem on groups with infinitely many ends, without going through cohomological dimension (see [Sta68] for the torsion-free case and [Sta72] 5.A.9 for the general case).

Recall from Example 4.17 that if $G$ is perfect, then $G \wr \mathbb{Z}$ is uniformly $G$-stable for any profinite family $G$. The following corollary of Proposition 6.14 strengthens this:

Corollary 6.17. Let $G$ be a virtually pro-$\pi$ family. If $G$ does not surject onto $\mathbb{F}_p$, for any $p \in \pi$, then $G \wr \mathbb{Z}$ is uniformly $G$-stable.

Proof. We use the notation from Example 2.6. Note that $G \wr \mathbb{Z}$ is the fundamental group of a loop with vertex group and edge group $G$, where the edge inclusions are $G \cong \to G_0$ and $G \cong \to G_1$. So to apply Proposition 6.14 it suffices to show that in any finite quotient of $G \wr \mathbb{Z}$ the image of $\sum Z_G$ has order coprime to $p$. By Example 2.6, this image must be abelian, and a finite abelian group of order divisible by $p$ surjects onto $\mathbb{F}_p$. This is ruled out by the hypothesis.

Proposition 6.14 can also be applied to Generalized Baumslag–Solitar groups, which will be the subject of the next subsection.

Note that – except for Item 2. of Corollary 6.15 which depends on the stability estimates of the vertex groups – all other examples have an optimal estimate for stability: see the discussions after the proofs of Propositions 6.9, 6.13 and 6.14.

6.4 GBS groups

We now apply Proposition 6.14 to many Generalized Baumslag–Solitar (from now on: GBS) groups. We refer the reader to [Rob11] for more details on GBS groups.

Recall that the Baumslag–Solitar group $BS(m,n)$ is defined by the presentation $\langle s, t \mid ts^m t^{-1} = s^n \rangle$, so it is the fundamental group of a loop with vertex group and edge group $\mathbb{Z}$, where the edge inclusions are $\mathbb{Z} \to \to \mathbb{Z}$, $1 \mapsto n, m$. More generally, a GBS group is the fundamental group of a finite connected graph of groups $X = (V, E)$ all of whose vertex and edge groups are infinite cyclic. The information on the edge inclusions can be summarized in to weight functions $w_+ : E \to \mathbb{Z} \setminus \{0\}$, that is $\iota^+_e : \Gamma_e \cong \to \mathbb{Z} \to \Gamma_e^+ \cong \to \mathbb{Z} : 1 \mapsto w_+(e)$. Note that it suffices to know $w_+$, or to know $w_+$ on a set of positively oriented edges, in order to recover all the information; indeed $\iota^{-}_e = \iota^+_\tau$. We denote the graph of groups associated to a GBS group by $(X, w)$, where $X$ is the underlying graph and $w = (w_-, w_+)$ are the weight functions.

It will be convenient to extend the weight functions from oriented edges to oriented paths. So given an oriented path $P : v_1 \xrightarrow{e_1} v_2 \to \cdots \to v_k \xrightarrow{e_k} v_{k+1}$, we denote by $w_+(P) := \prod_i w_+(e_i)$. 

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If \( p \) is a prime, we have \( \nu_p(w_\pm(P)) = \sum_i \nu_p(w_\pm(e_i)) \). In particular \( \nu_p(w_\pm(P)) = 0 \) if and only if \( \nu_p(w_\pm(e_i)) = 0 \) for all \( i = 1, \ldots, k \).

The following corollary to Proposition 6.14 gives a combinatorial criterion that ensures that a GBS group is \( \mathcal{G} \)-stable (since such groups are finitely presented, by Theorem 1.10 we need not specify whether the stability is pointwise or uniform).

**Corollary 6.18.** Let \( \mathcal{G} \) be a virtually pro-\( \pi \) family, and let \( \Gamma \) be a GBS group corresponding to the weighted graph \((X, w)\). Suppose that for all \( p \in \pi \) there exists a set of oriented cycles \( C \) satisfying \( \nu_p(w_-(C)) = 0 < \nu_p(w_+(C)) \), and that for every vertex \( y \) there exists a vertex \( x \) belonging to one of these cycles, and a path \( x \xrightarrow{P} y \) with \( \nu_p(w_+(P)) = 0 \). Then \( \Gamma \) is \( \mathcal{G} \)-stable.

Note that the condition only requires that such cycles and paths exist for any given \( p \in \pi \): we are allowed to choose different ones for each prime in \( \pi \). We will prove that such groups satisfy the conditions of Proposition 6.14. So we deduce not only stability, but also that the estimate is optimal. The simplest example is that of Baumslag–Solitar groups, which features in Theorem 1.8. Here there is only one vertex so the condition on the existence of special paths is not needed, and the special cycle is given by the loop.

**Corollary 6.19.** Let \( \mathcal{G} \) be a virtually pro-\( \pi \) family and suppose that each \( p \in \pi \) divides exactly one of \( m, n \). Then \( \text{BS}(m, n) \) is \( \mathcal{G} \)-stable.

Here is a more complex example of graph \((X, w)\) that satisfies the conditions of Corollary 6.18 where \( \pi = \{p\} \) is a single prime. We draw a set of positively oriented edges \( e \), labeled by the weights \((w_-(e), w_+(e))\). Each weight labeled 0 may be replaced by any integer coprime to \( p \), each weight labeled 1 by any non-zero multiple of \( p \), and * by any non-zero integer.

\[
\begin{array}{c}
(0,*) & u & \xrightarrow{(0,*)} & v & (0,1) & \xrightarrow{(*,*)} & x & (*,0) & \xrightarrow{(*,*)} & y & (1,0) & \xrightarrow{(*,*)} & z
\end{array}
\]

If \( C : u \rightarrow v \rightarrow x \rightarrow u \), then \( \nu_p(w_-(C)) = \nu_p(0) + \nu_p(0) + \nu_p(0) = 0 \), while \( \nu_p(w_+(C)) \geq \nu_p(1) > 0 \), so it satisfies the hypothesis. Similarly, the loop at \( z \) satisfies the hypothesis: \( \nu_p(0) = 0 < \nu_p(0) \).

The only vertex left to check is \( y \), and for this we use the path \( P : z \rightarrow y \), that satisfies \( \nu_p(w_+(P)) = \nu_p(0) = 0 \).

This example also clarifies that although the condition is stated in notation-heavy terms, it is quite easy to check, and there is no need to precisely compute \( \nu_p(w_\pm(C, P)) \). For instance \( \nu_p(w_-(C)) = 0 < \nu_p(w_+(C)) \) simply means that the negative weights along \( C \) are all coprime to \( p \), and that there is at least one positive weight that is divisible by \( p \). Similarly \( \nu_p(w_+(P)) = 0 \) simply means that the positive weights along \( P \) are all coprime to \( p \).

**Proof of Corollary 6.18.** The proof will be split in a sequence of technical lemmas. Fix a GBS group \( \Gamma \) with graph \((X, w)\) and \( p \in \pi \). Given a vertex \( x \) we denote by \( s_x \) the corresponding generator. By Proposition 6.14 we need to show that for every vertex \( x \) the image of \( s_x \) in any finite quotient of \( \Gamma \) has order coprime to \( p \). For the sake of brevity, let us say that such a vertex \( x \) is \( p \)-free. The first lemma shows that the condition on the existence of special paths reduces the question to the vertices belonging to special cycles:
Lemma 6.20. Suppose that $x$ is $p$-free, and let $x \xrightarrow{P} y$ be an oriented path with $\nu_p(w_+(P)) = 0$. Then $y$ is also $p$-free.

Proof. By induction on the length of the path, it suffices to show this for paths of length 1. So suppose that $x \xrightarrow{e} y$ and let $(w_-(e), w_+(e)) = (m, n)$. By definition of the fundamental group, $s^m_x$ is conjugate to $s^n_y$ in $\Gamma$, and by hypothesis $\nu_p(n) = 0$. Let $f : \Gamma \to K$ be a finite quotient of $\Gamma$, let $o_x$ be the order of $f(s_x)$ and $o_y$ the order of $s_y$. Conjugacy implies that $f(s_x)^m$ and $f(s_y)^n$ have the same order, that is $o_x/(o_x, m) = o_y/(o_y, n)$. Since $x$ is $p$-free, $\nu_p(o_x) = 0$, so $\nu_p(o_y) = \nu_p(o, y) \leq \nu_p(n) = 0$. Since $K$ was arbitrary, we conclude that $y$ is $p$-free. \hfill \Box

So we only need to show that if $C$ is a cycle such that $\nu_p(w_-(C)) = 0 < \nu_p(w_+(C))$, then every vertex of $C$ is $p$-free. Here is a sufficient condition for a vertex to be $p$-free:

Lemma 6.21. Let $x$ be a vertex such that $s^m_x$ is conjugate to $s^n_y$. If $\nu_p(m) = 0 < \nu_p(n)$, then $x$ is $p$-free.

Proof. Let $f : \Gamma \to K$ be a finite quotient of $\Gamma$, and let $o$ be the order of $s_x$. As in the previous lemma, the conjugacy implies that $o/(o, m) = o/(o, n)$ and so $(o, m) = (o, n)$. Since $p$ divides $n$ but not $m$, this is only possible if $p$ does not divide $o$. \hfill \Box

Note that this lemma alone is enough to conclude the proof in the case in which all cycles are loops, in particular it concludes the proof in the case of the Baumslag–Solitar group (which does not even need Lemma 6.20). For more general cycles, we use instead the next lemma:

Lemma 6.22. Let $x \xrightarrow{P} y$ be an oriented path. Then $s^w_+(P)$ is conjugate to $s^w_+(P)$.

Proof. We prove the statement by induction on the length of $P$. It is clear if $P$ has length 1. Now suppose that the statement is true for $x \xrightarrow{P} y$ and let us prove it for $x \xrightarrow{P} y \xrightarrow{z}$ for some edge $e$. By induction hypothesis $s^w_+(P)$ is conjugate to $s^w_+(P)$, and so $(s^w_+(P))_{w_-(e)}$ is conjugate to $(s^w_+(P))_{w_-(e)}$. Since by definition $s^w_+(e)$ is conjugate to $s^w_+(e)$, this implies that $(s^w_+(e))_{w_+(P)}$ is conjugate to $(s^w_+(e))_{w_+(P)}$. Thus $s^w_+(P)_{w_-(e)}$ is conjugate to $s^w_+(P)_{w_+(e)}$. \hfill \Box

In the case in which $P = C$ is a cycle such that $\nu_p(w_-(C)) = 0 < \nu_p(w_+(C))$, Lemma 6.22 implies that every vertex in $C$ satisfies a relation as in Lemma 6.21 and so is $p$-free. Lemma 6.20 allowed to reduce to looking at the vertices in cycles, hence this concludes the proof of Corollary 6.18. \hfill \Box

Corollary 6.19 has a further consequence. We know from Theorem 4.10 that for finitely presented groups the notions of pointwise and uniform stability coincide. For finitely generated infinitely presented groups, we have mostly seen examples of uniform stability, and non-examples of pointwise stability (Example 5.13), with one family of examples of pointwise stability (Example 5.16). But we know from Theorem 4.14 that the largest residually finite quotient of a pointwise stable group is pointwise stable. We will apply this to the largest residually finite quotient of the non-residually finite Baumslag–Solitar groups, which were identified by Moldavanskii in [Mol10]. Thus we obtain:

Corollary 6.23. Let $G$ be a virtually pro-$\pi$ family and suppose that each $p \in \pi$ divides exactly one of $m, n$. Let $d := (m, n)$ be the greatest common divisor of $m, n$, and suppose that $|m|, |n|$ are distinct from each other and from 1. Then the group

$$\Gamma = \langle a, b_i : i \in \mathbb{Z} \mid [b_i^d, b_j] = 1, b_i^m = b_i^n, ab_i^{-1} = b_i^{-1} : i \in \mathbb{Z} \rangle$$

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is finitely generated, infinitely presented, and pointwise $G$-stable.

Writing $m = du$, $n = dv$, this group fits into an extension

$$1 \to \mathbb{Z} \left[ \frac{1}{uv} \right] \to \Gamma \to (\mathbb{Z}/d\mathbb{Z} \ast \mathbb{Z}) \to 1,$$

and in case $d = 1$ it is isomorphic to $\mathbb{Z} \left[ \frac{1}{mn} \right] \rtimes \frac{m}{n} \mathbb{Z}$.

Proof. By Corollary 6.19 the group $BS(m, n)$ is $G$-stable. The condition on $|m|, |n|$ is equivalent to $BS(m, n)$ being not residually finite [Mes72], and the group above is its largest residually finite quotient [Mol10, Equation (1)], which is pointwise stable by Theorem 4.14. It is infinitely presented by [Mol10, Theorem 2].

The description of the group is in [Mol10, Proposition 3], except the author uses the presentation $C = \langle e_k : k > 0 \mid e_k = e_{k+1}^{uv} \rangle$ for the kernel of the extension [Mol10, Proposition 4]. This is isomorphic to $\mathbb{Z}[1/uw]$ under the isomorphism $\varphi : C \to \mathbb{Z}[1/uw] : e_k \mapsto (uv)^{-k}$, with inverse $\varphi^{-1} : \mathbb{Z}[1/uw] \to C : a(uv)^{-k} \mapsto e_k^a$. \hfill $\square$

Example 6.24. The group $\mathbb{Z} \left[ \frac{1}{6} \right] \rtimes \frac{2}{3} \mathbb{Z}$ is finitely generated, infinitely presented, residually finite and pointwise $GL(\mathfrak{o})$-stable, whenever $K$ has residual characteristic 2 or 3.
7 GL(\(o\))-stability

In this section we focus on the family GL(\(o\)), where \(o\) is the ring of integers of a non-Archimedean local field of characteristic 0. By Ostrowski’s Theorem, \(K\) is a finite extension of \(\mathbb{Q}_p\), where \(p\) is the residual characteristic of \(K\). The stability results will be similar to the ones in Section 6 but more flexible: this will be achieved by applying Lemma 2.20 (instead of the Schur–Zassenhaus Theorem) to ensure existence or conjugacy in the lifting problems that occur. This will allow to expand the class of examples that we presented in Section 6, some of which are included in Theorem 1.9.

We fix the following notation for the rest of the section (see Subsection 2.3): \(K\) is a finite extension of \(\mathbb{Q}_p\), with a norm \(|\cdot|\) that we may assume restricts to the \(p\)-adic norm of \(\mathbb{Q}_p\). Let \(o\) be the ring of integers, \(\omega\) a uniformizer, so the maximal ideal of \(o\) is \(p = \omega o\), and \(|\omega| = r < 1\) so that \(|K^\times| = rZ\). Since \(|p| = |p|_p = p^{-1}\), there exists \(a \geq 1\) such that \(r^a = p^{-1} = |p|\), that is \(p \in p^a\). Whenever \(n\) does not vary, we denote \(G := \text{GL}_n(o)\) and the congruence subgroups by \(G_k := \text{GL}_n(o)_k\).

The last Subsection 7.4 switches to the case in which \(K\) has characteristic \(p\): we will prove stability of \(\mathbb{Z}/2\mathbb{Z}\) in this case (Proposition 7.18) and discuss why the method does not work for other finite \(p\)-groups.

7.1 Virtually \(p\)-free groups

In this subsection we use the lifting part of Lemma 2.20 to prove stability of groups that are only required to be virtually \(p\)-free. This covers in particular all finite groups. Here is a characterization:

**Lemma 7.1.** Let \(\Gamma\) be a group and \(p\) be a prime. Then \(\Gamma\) is virtually \(p\)-free if and only if

\[
\sup\{\nu_p(|C|) : C \text{ is a finite quotient of } \Gamma\} < \infty.
\]

**Proof.** Suppose that \(\Gamma\) is virtually \(p\)-free, and let \(H\) be a \(p\)-free finite-index subgroup. A finite-index subgroup of a \(p\)-free group is \(p\)-free, so we may assume that \(H\) is normal. We claim that the supremum is achieved at \(\Gamma/H\). Indeed, let \(K\) be any other finite-index normal subgroup of \(\Gamma\). Then

\[
\nu_p(|\Gamma/K|) \leq \nu_p(|\Gamma/(K \cap H)|) = \nu_p(|\Gamma/H|) \cdot \nu_p(|H/(K \cap H)|) = \nu_p(|\Gamma/H|),
\]

where the last equality uses that \(H\) is \(p\)-free.

Conversely, suppose that the supremum is achieved at \(\Gamma/H\), where \(H\) is a finite-index normal subgroup of \(\Gamma\). Then \(H\) is \(p\)-free. Indeed, if \(K\) is a finite-index normal subgroup of \(H\), let \(N \leq K\) be a finite-index normal subgroup of \(\Gamma\); then

\[
\nu_p(|H/K|) \leq \nu_p(|H/N|) = \nu_p(|\Gamma/N|)/\nu_p(|\Gamma/H|) = 1,
\]

where the last equality uses that \(\nu_p(|\Gamma/N|) \geq \nu_p(|\Gamma/H|)\), and the latter is maximal. \(\square\)

Therefore a group is virtually \(p\)-free if and only if there is a bound on the order of its finite virtual \(p\)-quotients.

**Example 7.2.** Let \(\Gamma\) be a locally finite group with a bound on the order of its finite \(p\)-subgroups. Say \(\Gamma\) has no subgroup of order \(p^k\) (and so no subgroup of order \(p^l\) for \(l \geq k\)). Then it cannot admit a group of order \(p^k\) as a virtual quotient. Lemma 7.1 implies that \(\Gamma\) is virtually \(p\)-free.
We now prove the analogues of Lemma 6.8 and Proposition 6.9. The proofs are essentially the same, but they use Lemma 2.20 instead of the Schur–Zassenhaus Theorem.

**Lemma 7.3.** Let \( \varphi : \Gamma \to G \in \text{GL}(\mathfrak{o}) \) be such that \( \text{def}(\varphi) \leq r^{ak} = p^{-k} \) for some \( k \geq 1 \), and suppose that the image of \( \varphi \) in \( G/G_{ak} \) is a group \( C \) with \( \nu_p(|C|) \leq l < k/2 \). Then there exists a homomorphism \( \psi : \Gamma \to G \) such that \( \text{dist}(\varphi, \psi) \leq r^{a(k-l)} = p^l \cdot p^{-k} \). Moreover, \( \psi(\Gamma) \leq G \) is a finite group of isomorphic to a quotient of \( C \).

**Proof.** Let \( \varphi_k = \varphi(r^{ak}) : \Gamma \to G/G_{ak} \) denote the induced homomorphism; we also get a homomorphism \( \varphi_{k-l} : \Gamma \to G/G_{a(k-l)} \). So we have the following lifting problem:

\[
\begin{array}{ccc}
G/G_{2a(k-l)} & \xrightarrow{\varphi_{k-l}} & G/G_{ak} \\
\varphi_k & \downarrow & \downarrow \\
G/G_{a(k-l)} & \xrightarrow{} & G/G_{ak}
\end{array}
\]

Now \( G/G_{ak} \cong (G/G_{2a(k-l)})/(G_{ak}/G_{2a(k-l)}) \) and \( G/G_{a(k-l)} \cong (G/G_{2a(k-l)})/(G_{a(k-l)}/G_{2a(k-l)}) \). Moreover by Lemma 3.16 we have an isomorphism

\[
G_{a(k-l)}/G_{2a(k-l)} = \text{GL}_n(\mathfrak{o})_{a(k-l)}/\text{GL}_n(\mathfrak{o})_{2a(k-l)} \to M_n(\mathfrak{o}/p^{a(k-l)}).
\]

So \( G_{a(k-l)}/G_{2a(k-l)} \) is a \( \mathbb{Z}/p^{(k-l)}\mathbb{Z} \)-module (because \( \mathfrak{o}/p^{a(k-l)} \) is) and the image under multiplication by \( p^l \) is \( G_{ak}/G_{2a(k-l)} \). Since \( \nu_p(|C|) \leq l \) we are in the situation of Lemma 2.20, which means that the lift exists. Lifting this in turn to a map \( \psi : \Gamma \to G \), we have \( \text{def}(\psi) \leq r^{2a(k-l)} \) and \( \text{dist}(\varphi, \psi) \leq r^{a(k-l)} \).

The hypothesis \( k > 2l \) implies that the defect of \( \psi \) is strictly smaller than that of \( \varphi \), so we can apply the above procedure to \( \psi \). This leads to a sequence \( \psi_i : \Gamma \to G \) such that \( \text{def}(\psi_i) \to 0 \) and \( r^{a(k-l)} \geq \text{dist}(\psi_i, \psi_{i-1}) \to 0 \). The latter condition implies that \( \psi_i \) is Cauchy with respect to the uniform norm, so it converges to a homomorphism \( \psi \). The inequality \( \text{dist}(\varphi, \psi) \leq r^{a(k-l)} \) holds because it does for all \( \psi_i \), by the ultrametric inequality. \( \square \)

**Proposition 7.4.** Let \( \Gamma \) be virtually \( p \)-free. Then \( \Gamma \) is uniformly \( \text{GL}(\mathfrak{o}) \)-stable.

**Proof.** Let \( l \) be the supremum from Lemma 7.1 for any finite quotient \( C \) of \( \Gamma \), it holds \( \nu_p(|C|) \leq l \). Therefore given a map \( \varphi : \Gamma \to \text{GL}_n(\mathfrak{o}) \) with small enough defect, the previous lemma applies and \( \varphi \) is close to a homomorphism. We conclude by Lemma 2.1. \( \square \)

Lemma 7.3 shows that the estimate for stability is the linear: if \( \varphi : \Gamma \to G \) satisfies \( \text{def}(\varphi) < r^{a2l} = p^{-2l} \) (so the \( k \) in the lemma is indeed larger than \( 2l \)), then there exists a homomorphism \( \psi : \Gamma \to G \) such that \( \text{dist}(\varphi, \psi) \leq p^l \text{def}(\varphi) \). Note how this is reminiscent of a well-known generalization of Hensel’s Lemma: if \( f \in \mathbb{Z}_p[X] \) and \( a \in \mathbb{Z}_p \) satisfy \( p^n = |f(a)|_p \), \( p^h = |f'(a)|_p \) and \( r > 2h \) (that is, \( |f(a)|_p < |f'(a)|_p^2 \)), then there exists a unique root \( a' \) of \( a \) such that \( a' \equiv a \mod p^{r-h} \).

**Example 7.5.** All finite groups are \( \text{GL}(\mathfrak{o}) \)-stable. More precisely, let \( \Gamma \) be a finite group, and \( \nu_p(\Gamma) = l \). Then for every \( \varphi : \Gamma \to \text{GL}_n(\mathfrak{o}) \) such that \( \text{def}(\varphi) \leq p^{-2l} \), there exists a homomorphism \( \psi : \Gamma \to \text{GL}_n(\mathfrak{o}) \) such that \( \text{dist}(\varphi, \psi) \leq p^l \text{def}(\varphi) \). Compare this with Example 4.4 and Kaz82 Proposition 1.
Example 7.6. A finitely generated group of finite exponent is uniformly $GL(o)$-stable. Indeed, by Zelmanov’s solution of the restricted Burnside problem [Zel90, Zel91], any such group has only finitely many finite quotients. In particular, free Burnside groups of finite rank are uniformly $GL(o)$-stable.

We saw in Example 6.11 that normed $K$-amenable groups are $GL(o)$-stable when $K$ has characteristic $p$. The following example completes the picture:

Example 7.7. Let $K$ have characteristic 0, and let $\Gamma$ be a normed $K$-amenable group [FF20, Definition 1.1]. Then $\Gamma$ is $GL(o)$-stable: indeed such groups are characterized as being locally finite and with a bound on the order of their finite $p$-subgroups [FF20, Theorem 6.2].

7.2 Graphs of groups

We now use the conjugacy part of Lemma 2.20 to strengthen the results on stability of graphs of groups from Subsection 6.2 from which we borrow the notation. As before, we start with the analogue of Lemma 6.12 and then prove the analogues of Propositions 6.13 and 6.14. Also here, the lemma gives examples of constraint stability [AP18] and stability of an epimorphism [LL21]: see the discussion after the statement of Lemma 6.12.

Lemma 7.8. Let $X$ be a connected graph of groups with vertex groups $\Gamma_v$, edge groups $\Gamma_e$ and edge inclusions $i^\pm_e : \Gamma_e \to \Gamma_{e\pm}$. Let $\Gamma$ be the fundamental group of $X$, with the standard presentation $\langle S \mid R \rangle = \langle S_v, t_e \mid R_v, R_e \rangle$.

Let $\hat{\phi} : F_S \to G \in GL(o)$ be a map with $\text{def}(\hat{\phi}) \leq r^a_k = p^{-k}$ for some $k \geq 1$. Suppose further that for all $v \in V$ the restriction of $\hat{\phi}$ to $F_S_v$ descends to a homomorphism $\varphi_v : \Gamma_v \to G$ such that, for all $m \geq k$, if $e^\pm = v$, the image $C$ of $\varphi_v(i^\pm_e(\Gamma_e))$ in $G/G_{am}$ satisfies $\nu_p(|C|) \leq l < k/2$.

Then there exists a homomorphism $\hat{\psi} : F_S \to G$ such that $\text{dist}(\hat{\phi}, \hat{\psi}) \leq r^{a(k-l)} = p^l \cdot p^{-k}$ and $\hat{\psi}$ descends to a homomorphism of $\Gamma$.

Proof. The proof is essentially the same as that of Lemma 6.12 using Lemma 2.20 as we did in the proof of Lemma 7.3.

We start by setting $\hat{\phi}(t_e) = 1$ for all $e \in T$. Next we modify $\hat{\phi}$ at the vertex groups so that it satisfies the conjugacy relations given by edges in $T$. Using the same induction argument it suffices to treat the case $(v_0 \xrightarrow{e_i} v_1) \in T$: we need to find $t \in G_{a(k-l)}$ that conjugates the image of $\Gamma_{e_i}$ in $\varphi(\Gamma_{v_0})$ to that of $\varphi(\Gamma_{v_1})$. Considering the following lifting problem:

\[ \begin{array}{ccc}
\Gamma_{e_i} & \xrightarrow{f_k(\Gamma_{e_i})} & G/G_{ak} \\
& & \downarrow \\
& & G/G_{a(k-l)} \\
\end{array} \]

and using Lemma 2.20 to prove that any two lifts of the horizontal arrow are $G_{a(k-l)}$-conjugate, we obtain an element $t \in G_{a(k-l)}$ that conjugates the two images modulo $r^{2a(k-l)}$. Reiterating this process yields a sequence that converges to the desired conjugating element.

Finally we modify $\hat{\phi}$ at the generators of edges not in $T$, using the same argument.
Proposition 7.9. Let $\Gamma$ the fundamental group of a graph of groups such that all vertex groups are uniformly $\text{GL}(\mathbf{g})$-stable with a uniform estimate, and such that there exists $l \geq 1$ such that for every edge $e$ adjacent to a vertex $v$, the image of $\Gamma_e$ in any finite quotient of $\Gamma_v$ has no subgroup of order $p^l$. Then $\Gamma$ is uniformly $\text{GL}(\mathbf{g})$-stable.

Remark. By the proof of Lemma 7.1, the second condition is really asking for the image of $\Gamma_e$ inside $\Gamma_v$ to be “virtually $p$-free relative to $\Gamma_v$”: that is, there exists a finite-index subgroup $\Gamma'_e \leq \Gamma_e$ such that the image of $\Gamma'_e$ in any finite quotient of $\Gamma_v$ is a $p'$-group. We state it by first fixing $l$ because we need this condition to be uniform on the vertices. If however the graph is finite, then the uniformity is automatically satisfied.

Proof. As in the proof of Proposition 6.13, we start with $\hat{\phi} : F_S \rightarrow \text{GL}_n(\mathbf{g})$ of small defect, use stability of the vertex groups to modify it at the vertex generators, then apply Lemma 7.8 to obtain a homomorphism $\hat{\psi} : F_S \rightarrow \text{GL}_n(\mathbf{g})$ close to $\hat{\phi}$ that descends to a homomorphism of $\Gamma$. 

Lemma 7.8 shows that the estimate for stability is linear in terms of the uniform estimate for the vertex groups.

Proposition 7.10. Let $\Gamma$ be the fundamental group of a graph of groups such that there exists $l \geq 1$ such that for every vertex $v$ the image of $\Gamma_v$ in any finite quotient of $\Gamma$ has no subgroup of order $p^l$. Then $\Gamma$ is uniformly $\mathcal{G}$-stable.

Proof. As in the proof of Proposition 6.14, we start with $\hat{\phi} : F_S \rightarrow \text{GL}_n(\mathbf{g})$ of small defect, apply Lemma 7.3 to modify it at the vertex groups so that it descends to homomorphisms with finite image on each vertex group, and finally apply Lemma 7.8 to obtain a homomorphism $\psi : F_S \rightarrow \text{GL}_n(\mathbf{g})$ close to $\hat{\phi}$ that descends to $\Gamma$.

Here the proof shows that the estimate for stability is linear. More precisely, if $l$ is as in the statement of the proposition, and $\phi : \Gamma \rightarrow \text{GL}_n(\mathbf{g})$ satisfies $\text{def}(\phi) \leq r^{2d} = p^{-2l}$, then there exists a homomorphism $\psi : \Gamma \rightarrow \text{GL}_n(\mathbf{g})$ such that $\text{dist}(\phi, \psi) \leq p^l \text{def}(\phi)$.

7.3 Corollaries

We now apply Propositions 7.9 and 7.10 to obtain some examples of uniformly $\text{GL}(\mathbf{g})$-stable groups, which strengthen those in Subsections 6.3 and 6.4.

Corollary 7.11. The following groups are uniformly $\text{GL}(\mathbf{g})$-stable:

1. Fundamental groups of finite, connected graphs of groups, with virtually $p$-free vertex groups.

2. Fundamental groups of finite, connected graphs of groups, with uniformly $\text{GL}(\mathbf{g})$-stable vertex groups and virtually $p$-free edge groups.

Proof. This is a direct consequence of Propositions 7.9 and 7.10. We restrict to finite graphs of groups in order to have the integer $l$ in the statements be uniform.

A special case of Item 2. is when edge groups are finite. Such fundamental groups are precisely the finitely presented groups with infinitely many ends, by Stallning’s Theorem [Sta68, Sta72].

Corollary 7.12. Finitely generated virtually free groups are $\text{GL}(\mathbf{g})$-stable.
Proof. This follows again from Proposition 7.9 and [Dun79] or [Sta68, Sta72]: such groups are fundamental groups of finite connected graphs of groups, with finite vertex groups. The statement does not specify the type of stability because such groups are finitely presented. □

As for GBS groups, we have a criterion for stability much simpler than that of Corollary 6.18

Corollary 7.13. Let $\Gamma$ be a GBS group corresponding to the graph $(X, w)$. Suppose that there exist a cycle $C$ in $X$ satisfying $\nu_p(w_-(C)) \neq \nu_p(w_+(C))$. Then $\Gamma$ is uniformly $\text{GL}(\mathfrak{o})$-stable.

Proof. The proof is similar to Corollary 6.18. Given a vertex $x$ denote by $s_x$ the corresponding generator. By Proposition 7.10 we need to show that for every vertex $x$ the image of $s_x$ in any finite quotient of $\Gamma$ has order divisible by at most a uniformly bounded power of $p$. For the sake of brevity, let us say that such a vertex is virtually $p$-free.

We only need to show this for a single vertex: we claim that if some $x$ is virtually $p$-free, then all vertices are. Since the graph $X$ is finite and connected, it suffices to show that if $x \xrightarrow{e} y$ is an edge in $e$ and $x$ is virtually $p$-free, then so is $y$. Let $(w_-(e), w_+(e)) = (m, n)$ be the weights of $e$. Then $s_x^m$ is conjugate to $s_y^n$, so as in Lemma 6.20 if $o_x, o_y$ are the orders of $s_x, s_y$ in a finite quotient of $\Gamma$, we have $o_x/(o_x, m) = o_y/(o_y, n)$. Thus

$$\nu_p(o_y) = \nu_p(o_x) - \nu_p(o_x, m) + \nu_p(o_y, n) \leq \nu_p(o_x) + \nu_p(n)$$

is uniformly bounded, and $y$ is virtually $p$-free.

We are left to show that a vertex $x$ lying on a cycle $C$ satisfying $\nu_p(w_-(C)) \neq \nu_p(w_+(C))$ is virtually $p$-free. Let $m := w_-(C), n := w_+(C)$. By Lemma 6.22 we know that $s_x^m$ is conjugate to $s_y^n$. Now let $o$ be the order of $s_x$ in a finite quotient of $\Gamma$. As in Lemma 6.21 we have $(o, m) = (o, n)$, and so

$$\min\{\nu_p(o), \nu_p(n)\} = \nu_p(o, n) = \nu_p(o, m) = \min\{\nu_p(o), \nu_p(m)\}.$$

Since $\nu_p(m) \neq \nu_p(n)$, this is only possible if $\nu_p(o) \leq \min\{\nu_p(m), \nu_p(n)\}$, which gives a uniform bound on $\nu_p(o)$ and concludes the proof. □

We similarly obtain corollaries about the special case of Baumslag–Solitar groups and their largest residually finite quotient:

Corollary 7.14. Suppose that $\nu_p(m) \neq \nu_p(n)$. Then BS$(m, n)$ is uniformly $\text{GL}(\mathfrak{o})$-stable.

Corollary 7.15. Suppose moreover that $|m|, |n|$ are distinct from each other and from 1 and let $d := (m, n)$ be the greatest common divisor of $m, n$. Then the group

$$\Gamma = \langle a, b_i : i \in \mathbb{Z} | [b_i^d, b_j] = 1, b_i^m = b_{i+1}^n, ab_i a^{-1} = b_{i+1} : i \in \mathbb{Z} \rangle$$

is finitely generated, infinitely presented, and pointwise $\text{GL}(\mathfrak{o})$-stable.

Note that the special case $d = 1$, which admits the nicer description $\mathbb{Z} \left[ \frac{1}{mn} \right] \times \mathbb{Z}$, is not more general than Corollary 6.23 if $\nu_p(m) \neq \nu_p(n)$ and $(m, n) = 1$, then $p$ must divide exactly one of $m, n$.

Also in these examples – except for Item 2. of Corollary 7.11 which depends on the stability estimates of the vertex groups – we obtain a linear estimate for stability.
7.4 Positive characteristic

Let \( K \) have characteristic \( p > 0 \) for the rest of this subsection. The proofs in this section until now all relied on the cohomological Lemma 2.20. This cannot have as strong of an analogue in characteristic \( p \). For instance if \( M \) is an \( \mathbb{F}_p \)-module, seen as a trivial \( \mathbb{F}_p[\mathbb{Z}/p\mathbb{Z}] \)-module, then \( H^n(\mathbb{Z}/p\mathbb{Z}, M) \cong M \) for all \( n \geq 1 \). Therefore, analogues of the stability results we proved so far need a different approach.

In Sections 6 and 7 many results reduced to statements about finite groups. So we restrict our attention to those, and ask:

**Question 7.16.** Are all finite groups \( \text{GL}(\mathfrak{o}) \)-stable?

We already know that finite groups without elements of order \( p \) are \( \text{GL}(\mathfrak{o}) \)-stable, by Proposition 6.9. So the next natural question is whether finite \( p \)-groups are \( \text{GL}(\mathfrak{o}) \)-stable. Using a Theorem of Gaschütz [Rot12, Theorem 7.43] and an argument similar to the one of Lemma 7.3, one can show that if \( \Gamma \) is a finite group such that all \( p \)-Sylow subgroups are \( \text{GL}(\mathfrak{o}) \)-stable with a subquadratic estimate, then \( \Gamma \) is stable. The hypothesis of the subquadratic estimate is needed in order to have the kernel \( G_k/G_{2k} \) of the corresponding lifting problem be abelian; an example due to Zassenhaus implies that this hypothesis is necessary in Gaschütz’s Theorem [Hig54, Postscriptum].

However, stability with a subquadratic estimate for finite \( p \)-groups is too strong of a requirement to have useful applications. Indeed, the next example shows that the estimate for \( \mathbb{Z}/p^k\mathbb{Z} \) is, at best, polynomial of degree \( p^k \).

**Example 7.17.** Let \( A \in M_n(\mathfrak{o}) \) be such that \( \|A\| \leq \varepsilon \). Then, using that \( K \) has characteristic \( p \):

\[
\|(I + A)^{p^k} - I\| = \|A^{p^k}\| \leq \|A\|^{p^k} \leq \varepsilon^{p^k}.
\]

On the other hand, if we chose \( A \) so that \( \|A^{p^k}\| = \varepsilon^{p^k} \), then \( (I + A) \) is \( \varepsilon \)-far from any matrix \( (I + A') \) satisfying \( (I + A')^{p^k} = I \).

Notice that for all groups whose stability was proven so far, the estimates were always linear. Thus Example 7.17 shows that Question 7.16 is quite different from the stability problems we encountered until now.

Here is a special case that admits a positive answer:

**Proposition 7.18.** Let \( K \) have characteristic 2. Then \( \mathbb{Z}/2\mathbb{Z} \) is \( \text{GL}(\mathfrak{o}) \)-stable.

The proof will show that the estimate is quadratic: if \( \|A^2 - I\| \leq \varepsilon^2 \), then there exists \( A' \) such that \( (A')^2 = I \) and \( \|A - A'\| \leq \varepsilon \). This estimate is sharp by Example 7.17.

Combining Proposition 7.18 with Proposition 6.13, we obtain more examples of \( \text{GL}(\mathfrak{o}) \)-stable groups that are not covered by the results in Section 6.

**Example 7.19.** The infinite dihedral group \( D_\infty \cong \mathbb{Z}/2\mathbb{Z} \ast \mathbb{Z}/2\mathbb{Z} \) is \( \text{GL}(\mathfrak{o}) \)-stable; more generally, free Coxeter groups are \( \text{GL}(\mathfrak{o}) \)-stable. The modular group \( \text{PSL}_2(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \ast \mathbb{Z}/3\mathbb{Z} \) is \( \text{GL}(\mathfrak{o}) \)-stable.

The proof of Proposition 7.18 relies on the following theorem [BG78, Theorem 3.1], which classifies similarity classes of involutory matrices over the quotient rings \( \mathfrak{o}/p^k \):
Theorem 7.20 (Brawley–Gamble). Let $R$ be a finite commutative local ring of characteristic a power of 2 and maximal ideal $m$. Let $A \in \text{M}_n(R)$ be such that $A^2 = I$. Then there exists $P \in \text{GL}_n(R), 0 \leq l \leq n/2$ and $B \in I_{n-2l} + M_{n-2l}(m)$ such that
\[
PAP^{-1} = \begin{pmatrix} 0 & I_l \\ I_l & 0 \\ & B \end{pmatrix}.
\]

This has the following implication in our setting:

Corollary 7.21. Let $A \in \text{GL}_n(o)$ be such that $\|A^2 - I\| \leq r^k < 1$. Then there exists $P \in \text{GL}_n(o)$ and $B \in I_{n-2l} + M_{n-2l}(p)$ such that
\[
PAP^{-1} \equiv \begin{pmatrix} 0 & I_l \\ I_l & 0 \\ & B \end{pmatrix} \mod p^k.
\]

Proof. Apply Theorem 7.20 to the reduction of $A$ modulo $p^k$, which is involutory over $o/p^k$; then lift the matrices $P$ and $B$ to elements of $\text{M}_n(o)$. Since $k \geq 1$, the lift to $\text{GL}_n(o)$ of any matrix in $\text{GL}_n(o/p^k)$ is invertible.

We proceed with the proof. This will use repeatedly the identity $(I + M)^2 = I + M^2$, which holds since $K$ has characteristic 2.

Proof of Proposition 7.18. Let $\varepsilon > 0$. We will show that if $A \in \text{M}_n(o)$ is such that $\|A^2 - I\| \leq \varepsilon^2$, then there exists $A' \in \text{M}_n(o)$ such that $\|A - A'\| \leq \varepsilon$. Since $\|\cdot\|$ takes values in $\mathbb{R}^+$, we may assume that $\varepsilon =: r^a$, and since the statement is trivial when $\varepsilon \geq 1$, we may assume $a \geq 1$. The proof is by induction on $n$. For $n = 1$, we compute $\varepsilon^2 \geq |a^2 - 1| = |(a - 1)^2| = |a - 1|^2$, and so $|a - 1| \leq \varepsilon$. So let $n > 1$ and suppose by induction that the statement holds up to $(n - 1)$ for all $\varepsilon > 0$.

Since $\|A^2 - I\| \leq r^{2a}$, by Corollary 7.21 there exist $P \in \text{GL}_n(o), 0 \leq l \leq n/2$ and $B \in I_{n-2l} + M_{n-2l}(p)$ such that
\[
PAP^{-1} \equiv \begin{pmatrix} 0 & I_l \\ I_l & 0 \\ & B \end{pmatrix} \mod p^{2a}.
\]

Now $\|B^2 - I_{n-2l}\| \leq \max\{r^{2a}, \|PA^2P^{-1} - I_n\|\} \leq \varepsilon^2$. If $l \geq 1$, it follows by induction that there exists $B' \in \text{M}_{n-2l}(o)$ such that $(B')^2 = I_{n-2l}$ and $\|B - B'\| \leq \varepsilon$. Then
\[
A' := P^{-1} \begin{pmatrix} 0 & I_l \\ I_l & 0 \\ & B' \end{pmatrix} P
\]
is the desired matrix. So in case $l \geq 1$, the statement is true for all $\varepsilon > 0$ as well.

Assume finally that $l = 0$, and so $A = I + o^kM$ for some $M \in \text{M}_n(o)$, where we take $k \geq 1$ to be maximal. If $r^k \leq \varepsilon$ we may take $A' = I$ and conclude. Otherwise:
\[
\varepsilon^2 \geq \|A^2 - I\| = \|[I + o^{2k}M^2 - I]\| = r^{2k}\|M^2\|;
\]
so \((I + M)^2 - I\) = \(\|M^2\| \leq (r^{-k}\varepsilon)^2 < 1\). The choice of \(k\) implies that \((I + M)\) is not congruent to the identity modulo \(p\), so \((I + M)\) falls in the previous case. Therefore there exists \(M' \in M_n(\mathfrak{o})\) such that \((I + M')^2 = I\) and \(\|(I + M) - (I + M')\| \leq r^{-k}\varepsilon\). Then \(A' := I + \omega^k M'\) satisfies \((A')^2 = I\) and

\[
\|A - A'\| = \|\omega^k (M - M')\| = r^k \|(I + M) - (I + M')\| \leq r^k \cdot r^{-k}\varepsilon = \varepsilon.
\]

The fundamental tool for the proof of Proposition 7.18 was Theorem 7.20, which provides simple representatives for each conjugacy class of representations \(\mathbb{Z}/2\mathbb{Z} \to \text{GL}_n(\mathfrak{o}/p^k)\). A similar result for other finite \(p\)-groups could similarly be used to prove stability. However the group \(\mathbb{Z}/2\mathbb{Z}\) is really special from this point of view. Indeed, it is shown in [GP02] that for any other finite \(p\)-group \(\Gamma\) and any commutative local ring \(R\) of characteristic a power of \(p\), the analogous problem for representations \(\Gamma \to \text{GL}_n(R)\) is computationally wild. More precisely, this problem contains the problem of simultaneous similarity classes of pairs of matrices over the residue field \(R/m\), which is a long-standing open problem in its general form (see e.g. [BS03]). Therefore, although the few known special cases could be used to prove further stability results, the general solution to Question 7.16 needs a different approach.
8 Stability via bounded cohomology

The goal of this section is to prove the following bounded cohomological criterion for uniform \( \text{GL}(\mathfrak{o}) \)-stability, where \( \mathfrak{o} \) is the ring of integers of a non-Archimedean local field \( \mathbb{K} \):

**Theorem 8.1.** Let \( \Gamma \) be finitely presented and suppose that \( H^2_b(\Gamma, E) = 0 \) for every Banach \( \mathbb{K}[[\Gamma]] \)-module \( E \) with a solid norm. Then \( \Gamma \) is \( \text{GL}(\mathfrak{o}) \)-stable.

Note that by Theorem 4.10 we do not need to specify whether the stability is pointwise or uniform. The approach follows that in [DCGLT20], and the reason we restrict to finitely presented groups is the same: the ultraproduct techniques work best for pointwise stability, but the quantitative approach needs a quantity that controls all local defects, and this is only possible for finitely presented groups. The ultrametric inequality gives bounded cocycles, and so the bounded cohomological approach is the more natural one to take in this case. By [FF20, Corollary 8.7] (see Proposition 8.2), cohomology vanishing is, a priori, stronger than bounded cohomology vanishing, and this will imply the cohomological analogue of Theorem 8.1 (Corollary 8.10).

The boundedness of the cocycles is a consequence of the fact that pointwise asymptotic homomorphisms are asymptotically close to uniform ones for finitely presented groups, by Item 2. of Proposition 4.8. So the reader may suspect that a bounded cohomological criterion for uniform stability should also hold in the Archimedean setting. However, in that case the situation is more delicate and requires the introduction of a different cohomology theory, called asymptotic cohomology [GMLR]. The advantage of our setting is that, while cocycles are bounded thanks to the uniform nature of the problem, we can still use the same ultraproduct techniques that apply to the pointwise setting of [DCGLT20].

We will use Theorem 8.1 to deduce the stability results for virtually free groups we obtained in Sections 6 and 7. However we are not able to produce examples other than these, and we conjecture that in fact this method cannot produce other examples (Conjecture 8.11). We end by discussing how a stronger criterion could potentially produce more examples, and justify why this seems to be a hard problem.

For the rest of this section, we fix a non-Archimedean local field \( \mathbb{K} \) with ring of integers \( \mathfrak{o} \), uniformizer \( \varpi \), maximal ideal \( \mathfrak{p} = \varpi \mathfrak{o} \), residue field \( \mathfrak{k} \), and value group \( \mathbb{Z} = |\mathbb{K}^\times| \), where \( r = |\varpi| \in (0, 1) \).

### 8.1 Bounded cohomology

We review here the basics of bounded cohomology of discrete groups that are needed for the rest of the section. For more information, see [Mon01, Fri17] for bounded cohomology over the reals, and [FF20] for bounded cohomology over non-Archimedean fields. All of the material presented here is also contained in [FF20].

We will work with the bar resolution throughout, since it is the easiest one to treat lifting problems with. Let \( \Gamma \) be a group, \( E \) a \( \mathbb{K}[[\Gamma]] \)-module, without any specified norm. Let

\[ C^n(\Gamma, E) := \{ f : \Gamma^n \to E \}, \]
which is a $\mathbb{K}$-vector space with pointwise addition and scalar multiplication. Define the coboundary map $\delta^n : C^n(\Gamma, E) \to C^{n+1}(\Gamma, E)$ by the formula:

$$\delta^n(f)(g_1, \ldots, g_{n+1}) := g_1 \cdot f(g_2, \ldots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i f(g_i, g_1, g_{i+1}, \ldots, g_{n+1}) + (-1)^{n+1} f(g_1, \ldots, g_n).$$

This defines a cochain complex of $\mathbb{K}$-vector spaces $(C^*(\Gamma, E), \delta^*)$: we denote the cocycles by $Z^*(\Gamma, E)$, the coboundaries by $B^*(\Gamma, E)$, and the cohomology by $H^*(\Gamma, E)$.

Now suppose that $E$ is a normed $\mathbb{K}[\Gamma]$-module. Let

$$C^n_0(\Gamma, E) := \{ f : \Gamma \to E : \|f\|_{\infty} < \infty \} \subset C^n(\Gamma, E),$$

which is a normed $\mathbb{K}$-vector space with the supremum norm $\| \cdot \|_{\infty}$, and even a Banach space if $E$ is also Banach. With the same coboundary map, we obtain the cochain complex $(C^n_0(\Gamma, E), \delta^*)$, the bounded cocycles $Z^*_b(\Gamma, E)$, the bounded coboundaries $B^*_b(\Gamma, E)$, and the bounded cohomology $H^*_b(\Gamma, E)$.

The inclusion $C^n_0(\Gamma, E) \to C^n(\Gamma, E)$ is a chain map, that induces the comparison map

$$c^n : H^*_b(\Gamma, E) \to H^n(\Gamma, E).$$

The kernel of this map, called exact bounded cohomology, is denoted by $\text{EH}^*_b(\Gamma, E)$. In the real case it is very rich and interesting, even in the simple case where $n = 2$ and $E$ is the trivial $\Gamma$-module $\mathbb{R}$, leading to the theory of quasimorphisms. However, in the non-Archimedean case, the exact bounded cohomology in degree 2 is trivial for finitely generated groups [FF20, Corollary 8.7]:

**Proposition 8.2.** Let $\Gamma$ be a finitely generated group, $E$ a normed $\mathbb{K}[\Gamma]$-module. Then the comparison map $c^2 : H^2_b(\Gamma, E) \to H^2(\Gamma, E)$ is injective.

Theorem 8.1 applies to groups such that $H^2_b(\Gamma, E) = 0$ for every Banach $\mathbb{K}[\Gamma]$-module $E$ with a solid norm. The next lemma shows that this vanishing is in some sense uniform.

**Lemma 8.3.** Let $n \geq 1$ and suppose that $H^2_b(\Gamma, E) = 0$ for any Banach $\mathbb{K}[\Gamma]$-module $E$ with a solid norm. Then there exists $C \geq 1$ such that for any such $E$ and any $z \in Z^n_b(\Gamma, E)$ there exists a primitive $b \in C^{n-1}_b(\Gamma, E)$ such that $\|b\|_{\infty} \leq C\|z\|_{\infty}$.

**Proof.** We start by showing that for every Banach $\mathbb{K}[\Gamma]$-module $E$ with a solid norm we can choose a constant $C = C(E)$ that works for $Z^n_b(\Gamma, E)$ (this is the same proof as in the real case [MM85]). The map $\delta^{n-1} : C^{n-1}_b(\Gamma, E) \to C^n_b(\Gamma, E)$ is a bounded linear map between Banach spaces. Since $H^2_b(\Gamma, E) = 0$, it is surjective onto $Z^n_b(\Gamma, E)$, which is closed in $C^n_b(\Gamma, E)$ and thus Banach. It follows from the Open Mapping Theorem (Theorem 2.17) that $\delta^{n-1}$ is open and so the induced isomorphism $\delta^{n-1} : C^{n-1}_b(\Gamma, E)/Z^{n-1}_b(\Gamma, E) \to Z^n_b(\Gamma, E)$ is bi-Lipschitz, where the left-hand side is endowed with the quotient norm. The Lipschitz constants give the desired result.

Next, we need to show that the constant $C(E)$ can be chosen independently of $E$. First, note that it is only necessary to uniformly bound $C(E_i)$, where $\{E_i\}_{i \in I}$ is the set of Banach $\mathbb{K}[\Gamma]$-modules with a solid norm of cardinality at most $2^{\aleph_0}$. Indeed, any $z \in Z^n_b(\Gamma, E)$ takes values in
some $E_i$, more precisely the smallest $\Gamma$-invariant closed subspace of $E$ containing the countable set $z(\Gamma^n)$. Now take their Banach direct sum, which is the completion of the direct sum $E = \bigoplus_{i \in I} E_i$ with respect to the norm $\|(x_i)_{i \in I}\| := \max_{i \in I} \|x_i\|_{E_i}$ (this is allowed because we have taken a set of modules, and not the proper class of all modules). This is a Banach $\mathbb{K}[\Gamma]$-module, the norm is still solid, and we have $C(\tilde{E}) \geq C(E_i)$ for all $i \in I$, which gives the desired uniform bound.

8.2 Reducing the defect

Lemma 2.3 rephrases stability in terms of a lifting property, which is what allows to use cohomology to prove stability. However the kernel in this lifting problem is not tractable with cohomology: it is not even abelian. The goal of this subsection is to show that a weaker quantitative statement (intuitively: every asymptotic homomorphism is close to one with smaller defect) can be related to a simpler lifting problem, where the kernel is more approachable and will be analyzed in the next subsection.

Let us fix some terminology and notation concerning ultrafilters. Fix a free ultrafilter $\omega$. We say that an event $E_n$ holds for most $n$ if it holds for a set of $n$ inside $\omega$. Accordingly we denote $\varepsilon_n \not\in \omega$, $\delta_n \leq \omega \varepsilon_n$, and so on. Given a sequence $\varepsilon_n \not\in \omega$, we write $\delta_n = O(\varepsilon_n)$ if there exists $C \geq 1$ such that $\delta_n \leq C \varepsilon_n$. The minimal such $C$ can be characterized as $\lim_{n \to \omega} \delta_n / \varepsilon_n$: this limit makes sense since $\varepsilon_n \not\in \omega$. If this limit is 0, we write $\delta_n = o(\varepsilon_n)$.

Let $G_n := \text{GL}_{k_n}(\mathfrak{o})$, and fix a sequence $\{\hat{\varphi}_n : F_S \to G_n\}_{n \geq 1}$ with $\varepsilon_n := \text{def}(\hat{\varphi}_n) \xrightarrow{n \to \omega} 0$. Since the metric on $G_n$ takes values on $\mathbb{R}^+$, the same holds for the sequence $\varepsilon_n$. Moreover we may assume that $\varepsilon_n \notin \omega$, since otherwise $\text{def}(\hat{\varphi}_n)$ is already close to a homomorphism. The asymptotic homomorphism $(\hat{\varphi}_n)_{n \geq 1}$ induces a homomorphism onto the ultraproduct (see Lemma 2.3). Here we will use a modified ultraproduct that takes into account the sequence $\varepsilon_n$ as well. In analogy with notation which will shortly be introduced, we denote $G(0) := \prod_{n \geq 1} G_n$.

For a sequence $\delta_n \xrightarrow{n \to \omega} 0$, denote

$$N(\delta_n) := \{(A_n)_{n \geq 1} \in G(0) : \|A_n - I_{k_n}\| \leq \omega \delta_n\},$$

and similarly $N(O(\varepsilon_n))$ and $N(o(\varepsilon_n))$.

Lemma 8.4. $N(\delta_n), N(O(\varepsilon_n))$ and $N(o(\varepsilon_n))$ are normal subgroups of $G(0)$.

The fact that $N(\delta_n)$ is a subgroup relies strongly on the ultrametric inequality, and will allow to streamline a few arguments, and to make them more quantitatively precise, which allows to use bounded cohomology instead of cohomology. By contrast, in the Archimedean case one needs to work with $N(O(\varepsilon_n))$.

Proof. We prove the statement for $N(\delta_n)$ (which is the only specific to the ultrametric case), the rest is similar. Suppose that $(A_n)_{n \geq 1}, (B_n)_{n \geq 1} \in N(\delta_n)$. Then $\|A_n B_n - I_{k_n}\| \leq \max\{\|A_n B_n - B_n\|, \|B_n - I_{k_n}\|\} \leq \omega \delta_n$, where we have used that $\| \cdot \|$ is right-invariant. If now $(C_n)_{n \geq 1} \in G(0)$, then $\|C_n A_n C_n^{-1} - I_{k_n}\| = \|A_n - I_{k_n}\| \leq \omega \delta_n$.

We will denote $G(\delta_n) := G(0)/N(\delta_n)$. Given a constant $C \geq 1$, we have $\varepsilon_n \leq \omega C \varepsilon_n$, so $N(o(\varepsilon_n)) \leq N(\varepsilon_n) \leq N(C \varepsilon_n)$, and so there are quotient maps $G(o(\varepsilon_n)) \to G(\varepsilon_n) \to G(C \varepsilon_n)$. The asymptotic homomorphism $(\hat{\varphi}_n)_{n \geq 1}$ induces homomorphisms $\varphi(\varepsilon_n) : \Gamma \to G(\varepsilon_n)$, as well as
homomorphisms \( \varphi(C\varepsilon_n) : \Gamma \to G(C\varepsilon_n) \). This yields to the following lifting problem similar to the one from Lemma 2.20.

\[
\begin{array}{c}
\Gamma \\
\downarrow \varphi(C\varepsilon_n) \\
G(C\varepsilon_n) \\
\end{array} \xrightarrow{\psi} \begin{array}{c}
\Gamma \\
\downarrow \varphi(C\varepsilon_n) \\
G(C\varepsilon_n) \\
\end{array} \xrightarrow{G(\alpha(\varepsilon_n))} \begin{array}{c}
G(\varepsilon_n) \\
\end{array}
\]

Lemma 8.5. The existence of a solution \( \psi \) to the above lifting problem is equivalent to the existence of a sequence \( (\hat{\psi}_n) : F_\varepsilon \to G_n \) such that \( \text{def}(\hat{\psi}_n) = \alpha(\varepsilon_n) \) and \( \text{dist}(\hat{\phi}_n, \hat{\psi}_n) \leq \omega C\varepsilon_n \).

Proof. The proof is the same as that of [AP15, Theorem 4.2] (see Lemma 2.3).

We denote by \( E_C := N(C\varepsilon_n) / N(\alpha(\varepsilon_n)) = \ker \{ G(\alpha(\varepsilon_n)) \to G(C\varepsilon_n) \} \). As in Subsection 2.4, this lifting problem reduces to a splitting problem

\[ 1 \to E_C \to \left( G(\alpha(\varepsilon_n)) \times_{\varphi(C\varepsilon_n)} \Gamma \right) \to \Gamma \to 1 \]

and in turn, if \( E_C \) is abelian, to a cohomology vanishing problem with coefficients in \( E_C \). Moreover, as in Lemma 2.20, the particular form of this lifting problem implies that the relevant cocycle takes values in \( N_1 = N(\varepsilon_n) / N(\alpha(\varepsilon_n)) = \ker \{ G(\alpha(\varepsilon_n)) \to G(\varepsilon_n) \} \).

8.3 Additional structures on the kernel

In this subsection we show not only that \( E_C \) is abelian, which allows to apply cohomology to the above lifting problem, but moreover that it is the closed \( C \)-ball of a Banach \( \mathbb{K}[\Gamma] \)-module. This Banach \( \mathbb{K}[\Gamma] \)-module will be \( E := N(\alpha(\varepsilon_n)) / N(\alpha(\varepsilon_n)) \). We can characterize \( N(\alpha(\varepsilon_n)) \) as

\[
N(\alpha(\varepsilon_n)) := \bigcup_{C \geq 1} N(C\varepsilon_n) = \{(A_n)_{n \geq 1} \in G(0) : \lim_{\varepsilon_n \to} \frac{\|A_n - I_k\|}{\varepsilon_n} < \infty \}.
\]

Here we are using that \( \varepsilon_n \neq 0 \), and that a bounded sequence always admits an ultralimit. We will denote by \( A = (A_n)_{n \geq 1} \) elements of \( G(0) \), and if \( A \in N(\alpha(\varepsilon_n)) \) we will denote by \( [A] \) its image in \( E \). We also denote \( I := (I_{k_n})_{n \geq 1} \).

We start by showing that \( E \) is a \( \mathbb{K} \)-vector space. It already has a well-defined product structure, but to underline that it is abelian we will denote it by \( [A] + [B] := [AB] \). The scalar multiplication will be given by a stretch fixing the identity, namely \( \lambda [A] = [\lambda A + (1 - \lambda)I] = [I + \lambda(A - I)] \).

Lemma 8.6. With these operations, \( E \) is a \( \mathbb{K} \)-vector space.

Proof. We already know that \( (E, +) \) is a group. It is moreover abelian: for this we need to show that given \( A, B \in N(\alpha(\varepsilon_n)) \) we have \( AB^{-1}A^{-1}B \in N(\alpha(\varepsilon_n)) \). This follows from the submultiplicativity of the norm \( \| \cdot \| \) (see Lemma 3.15):

\[
\|A_nB_nA_n^{-1}B_n^{-1} - I_{k_n}\| = \|A_nB_n - B_nA_n\| = \|(A_n - I_{k_n})(B_n - I_{k_n}) - (B_n - I_{k_n})(A_n - I_{k_n})\| \leq \|A_n - I_{k_n}\| \|B_n - I_{k_n}\| = O_\omega(\varepsilon_n^2) = \alpha(\varepsilon_n).
\]
The scalar multiplication is well-defined: indeed
\[ \lambda A_n + (1 - \lambda) I_{k_n} = I_{k_n} + \lambda(A_n - I_{k_n}), \]
is close to the identity, thus invertible, when \( \|A_n - I_{k_n}\| \) is small enough. It is also easy to see that it is still in \( N(O_\omega(\varepsilon_n)) \), and that \( (\lambda \mu)[A] = \lambda(\mu[A]) \). Finally we prove bilinearity of the scalar multiplication. First, given \( \lambda, \mu \in K, [A] \in N(O_\omega(\varepsilon_n)) \), we have:
\[
\lambda \cdot [A] + \mu \cdot [A] = [I + \lambda(A - I)] + [I + \mu(A - I)] = \\
[I + (\lambda + \mu)(A - I) + \lambda \mu(A - I)^2] = (\lambda + \mu)[A],
\]
where we used that \( \|(A_n - I_n)^2\| = o_\omega(\varepsilon_n) \). Similarly, given \( \lambda \in K, [A], [B] \in N(O_\omega(\varepsilon_n)) \), we have:
\[
\lambda \cdot [A] + \lambda \cdot [B] = [\lambda(A - I) + I] + [\lambda(B - I) + I] = \\
[\lambda^2(A - I)(B - I) + \lambda(A - I) + \lambda(B - I) + I] = [\lambda(A + B - 2I) + I] = \\
[-\lambda(A - I)(B - I) + \lambda(AB - I) + I] = [\lambda(AB - I) + I] = \lambda \cdot [AB]
\]
\[ \square \]

Now since \( E \) is abelian, the splitting problem after Lemma 8.5 gives an action of \( \Gamma \) on \( E \) by conjugacy. Concretely, this action is defined on the free group by \( w \) -conjugation by elements of \( GL_k \), and taking the ultralimit shows that \( (\hat{\varphi}(w)A_n \hat{\varphi}(w)^{-1})_{n \geq 1} \), and it descends to \( \Gamma \) since \( \text{def}(\hat{\varphi}) \leq_\omega \varepsilon_n \).

Next we introduce a norm on \( E \), namely \( \| [A] \|_E := \lim_{n \to \omega} \frac{\|A_n - I_{k_n}\|}{\varepsilon_n} \).

**Lemma 8.7.** \( \| \cdot \|_E \) is a solid Banach norm on \( E \), and the action of \( \Gamma \) is isometric.

**Proof.** \( \| \cdot \|_E \) is well-defined on \( N(O_\omega(\varepsilon_n)) \) and it is zero precisely on \( N(o_\omega(\varepsilon_n)) \), by very definition of these two spaces. Now
\[
\frac{\|A_n B_n - I_{k_n}\|}{\varepsilon_n} \leq \max\{\|A_n B_n - B_n\|, \|B_n - I_{k_n}\|\} = \max\left\{\frac{\|A_n - I_{k_n}\|}{\varepsilon_n}, \frac{\|B_n - I_{k_n}\|}{\varepsilon_n}\right\},
\]
and taking the ultralimit shows that \( \|AB\|_E \leq \max\{\|A\|_E, \|B\|_E\} \). This implies at once that \( \| \cdot \|_E \) is well-defined on \( E \), and that it satisfies the strong triangle inequality. That it is \( K \)-linear is clear. It takes values in \( r^K \) because the maximum norm does, so \( \| \cdot \|_E \) is a solid norm on \( E \). Since \( \Gamma \) acts by conjugation by elements of \( GL_k(\omega) \), and the maximum norm is bi-invariant, the action is isometric.

We are left to that \( E \) is Banach. So let \( \{[A^k] = [(A_n^k)_{n \geq 1}]\}_{k \geq 1} \) be a Cauchy sequence. Explicitly, this means that
\[
\| [A^k] - [A^l] \|_E = \lim_{n \to \omega} \frac{\|A^k_n (A^l_n)^{-1} - I_{k_n}\|}{\varepsilon_n} = \lim_{n \to \omega} \frac{\|A^k_n - A^l_n\|}{\varepsilon_n} \xrightarrow{k,l \to \infty} 0.
\]
By Tychonoff's Theorem the product space \( G(0) \) is compact, so up to subsequence we may assume that a sequence of representatives \( A^k \) converges in this topology to some \( A \in G(0) \). This means pointwise convergence, that is: \( \|A^k_n - A_n\| \xrightarrow{k \to \infty} 0 \) for all \( n \), although the convergence is not necessarily uniform in \( n \). We need to show that \( A \in N(O_\omega(\varepsilon_n)) \) and that \( \|[A^k] - [A]\|_E \xrightarrow{k \to \infty} 0 \).
Let $\delta > 0$ be fixed. We will show that there exists some $K = K(\delta) \geq 1$ such that for all $k \geq K$ and for most $n$ we have: $\frac{\|A^k_n - A_n\|}{\varepsilon_n} < \delta$. An application of the triangle inequality then shows that $A \in N(O_{\omega}(\varepsilon_n))$, and letting $\delta \to 0$ we also have $\omega$-convergence. We choose $K$ to be such that $\|A^k_l - [A^l]\|_E < \delta$ for all $k, l \geq K$. By definition of the ultralimit it follows that

$$X := \left\{ n \geq 1 : \varepsilon_n \neq 0, \frac{\|A^k_n - A_n\|}{\varepsilon_n} < \delta \quad \forall k, l \geq K \right\} \in \omega.$$

For all $n \in X$, let $K_n$ be such that $\|A^l_n - A_n\| < \delta \varepsilon_n$ for all $l \geq K_n$ (we can do this because $n \in X$ and so $\varepsilon_n \neq 0$). Then for all $k \geq K$, given $n \in X$ and choosing $l \geq \max\{K_n, K\}$ we have:

$$\frac{\|A^k_n - A_n\|}{\varepsilon_n} \leq \max \left\{ \frac{\|A^k_n - A_n\|}{\varepsilon_n}, \frac{\|A^l_n - A_n\|}{\varepsilon_n} \right\} < \max \left\{ \delta, \frac{\delta \varepsilon_n}{\varepsilon_n} \right\} = \delta.$$

This concludes the proof. \hfill \Box

**Lemma 8.8.** $E_C$ is the closed $C$-ball in $E$ with respect to the norm $\| \cdot \|_E$.

We will not go into detail here, but this Banach $K[\Gamma]$-module $E$ is isometrically $\Gamma$-isomorphic to one which admits a nicer description, namely the matrix ultraproduct $\prod_{n \to \omega} M_{k_n}(K)$ with the maximum norm. This is the quotient of the subspace of bounded sequences in the direct product by the subspace of sequences $(M_n)$ such that $\|M_n\| \to \infty$, and can be endowed with a natural norm $\|M\| = \lim_{n \to \omega} \|M_n\|$ and a $\Gamma$-action by conjugacy via $(\hat{\psi}_n)_{n \geq 1}$. Then the isometric $\Gamma$-isomorphism is given by

$$\prod_{n \to \omega} M_{k_n}(K) \to E : [M = (M_n)_{n \geq 1}] \mapsto [(I_n + \varepsilon_n^{-1} M_n)].$$

**8.4 Proof of Theorem 8.1**

We are ready to prove Theorem 8.1. For the rest of this subsection, assume that $\Gamma$ satisfies the cohomology vanishing criterion. Lemma 8.3 has the following consequence for our problem:

**Lemma 8.9.** There exists $C \geq 1$ such that the following holds. For every sequence $(\hat{\varphi}_n : F_S \to G_n)_{n \geq 1}$ such that $\text{def}(\hat{\varphi}_n) \to \infty$ there exists a sequence $(\hat{\psi}_n : F_S \to G_n)_{n \geq 1}$ such that $\text{def}(\hat{\psi}_n) = o_\omega(\text{def}(\hat{\varphi}_n))$ and $\text{dist}(\hat{\varphi}_n, \hat{\psi}_n) \leq \omega C \varepsilon_n$.

**Proof.** Let $C$ be as in Lemma 8.3. By Lemma 8.3 and the discussion thereafter, the existence of such a sequence is equivalent to the vanishing of a cohomology class in $H^2_0(\Gamma, E_C)$, which admits a representative cocycle taking values in $E_1$. By Lemma 8.3 cocycles taking values in $E_1$ are cocycles taking values in $E$ whose norm is at most 1, and so the existence of a primitive taking values in $E_C$ is guaranteed by Lemma 8.3. \hfill \Box

Now fix a sequence $\varepsilon_n \to \infty$ and define $\text{Hom}_{\varepsilon_n}(\Gamma, \text{GL}(\mathfrak{o})) := \{ (\hat{\varphi}_n : F_S \to G_n)_{n \geq 1} : \text{def}(\hat{\varphi}_n) \leq \omega \varepsilon_n \}$. We need to show that for any $(\hat{\varphi}_n)_{n \geq 1} \in \text{Hom}_{\varepsilon_n}(\Gamma, \text{GL}(\mathfrak{o}))$ there exists a sequence $(\pi_n : F_S \to G_n)_{n \geq 1}$ that descends to $\Gamma$ and such that $\text{dist}(\hat{\varphi}_n, \pi_n) \to 0$. Define

$$\text{Hdist}(\hat{\varphi}) := \inf \{ \text{dist}(\hat{\varphi}, \pi) : \pi \text{ descends to } \Gamma \}.$$
Therefore we need to show that for any \((\hat{\varphi}_n)_{n \geq 1} \in \text{Hom}_{\varepsilon_n}(\Gamma, \text{GL}(\mathfrak{o}))\) we have \(\text{Hdist}(\hat{\varphi}_n) \xrightarrow{n \to \omega} 0\).

We equip the space of functions \(\{f : S \to G_n\}\) with the product topology, making it into a compact space such that the functions \(\text{def}\) and \(\text{Hdist}\) are continuous. Let \(X_n\) be the subspace of maps such that \(\text{def}(\hat{\varphi}_n) \leq \varepsilon_n\), which is closed, thus compact. The map

\[
\theta : X_n \rightarrow \mathbb{R}_{\geq 0} : \hat{\varphi}_n \mapsto \text{Hdist}(\hat{\varphi}_n) - 2C \text{def}(\hat{\varphi}_n)
\]

is continuous, and so there exists an asymptotic homomorphism \((\hat{\varphi}^M_n)_{n \geq 1} \in \text{Hom}_{\varepsilon_n}(\Gamma, \text{GL}(\mathfrak{o}))\) maximizing \(\theta\) for each \(n\). We claim that \(\theta(\hat{\varphi}^M_n) = \omega 0\).

Let \((\hat{\psi}_n)_{n \geq 1}\) be given by Lemma \ref{8.9} so that \(\text{def}(\hat{\psi}_n) = o_\omega(\text{def}(\hat{\varphi}^M_n))\) and \(\text{dist}(\hat{\varphi}^M_n, \hat{\psi}_n) \leq \omega C \text{def}(\hat{\varphi}^M_n)\). The first condition implies that \(\text{def}(\hat{\psi}_n) \leq \frac{1}{4} \text{def}(\hat{\varphi}^M_n) \leq \omega \varepsilon_n\). Then we have

\[
\text{Hdist}(\hat{\varphi}^M_n) \leq \omega \text{Hdist}(\hat{\psi}_n) + \text{dist}(\hat{\varphi}^M_n, \hat{\psi}_n) \leq \omega \text{Hdist}(\hat{\psi}_n) + C \text{def}(\hat{\varphi}^M_n).
\]

By maximality:

\[
\text{Hdist}(\hat{\psi}_n) - 2C \text{def}(\hat{\psi}_n) \leq \omega \text{Hdist}(\hat{\varphi}^M_n) - 2C \text{def}(\hat{\varphi}^M_n) \leq \omega \text{Hdist}(\hat{\psi}_n) - C \text{def}(\hat{\varphi}^M_n),
\]

whence

\[
\text{def}(\hat{\varphi}^M_n) \leq \omega 2 \text{def}(\hat{\psi}_n) \leq \omega \frac{1}{2} \text{def}(\hat{\varphi}^M_n).
\]

It follows that \(\text{def}(\hat{\varphi}^M_n) = \omega 0\), so \(\hat{\varphi}^M_n\) is a homomorphism for most \(n\), and \(\text{Hdist}(\hat{\varphi}^M_n) = \omega 0\) too. In particular \(\theta(\hat{\varphi}^M_n) = \omega 0\), which proves the claim.

Finally, since \(\hat{\varphi}^M_n\) maximizes \(\theta\) for all \(n\), for all \((\hat{\varphi}_n)_{n \geq 1} \in \text{Hom}_{\varepsilon_n}(\Gamma, \text{GL}(\mathfrak{o}))\) we have

\[
\lim_{n \to \omega} \text{Hdist}(\hat{\varphi}_n) = \lim_{n \to \omega} \theta(\hat{\varphi}_n) \leq \lim_{n \to \omega} \theta(\hat{\varphi}^M_n) = 0,
\]

which concludes the proof of Theorem \ref{8.1}.

\section{Applying the criterion}

By analogy with the notion of cohomological dimension, let us say that \(\Gamma\) has \(\mathbb{K}\)-bounded cohomological dimension at most 1, denoted \(\text{bcd}_{\mathbb{K}}(\Gamma) \leq 1\) if it satisfies the condition on Theorem \ref{8.1} that is \(H^2_{\mathbb{K}}(\Gamma, E) = 0\) for every Banach \(\mathbb{K}[\Gamma]\)-module \(E\) with a solid norm. Proposition \ref{8.2} and Theorem \ref{8.1} together imply:

**Corollary 8.10.** Let \(\Gamma\) be finitely presented and suppose that \(H^2(\Gamma, E) = 0\) for every Banach \(\mathbb{K}[\Gamma]\)-module \(E\) with a solid norm. Then \(\Gamma\) is \(\text{GL}(\mathfrak{o})\)-stable.

In particular, if \(\Gamma\) is finitely presented and has \(\mathbb{K}\)-cohomological dimension at most 1, then it is stable. Dunwoody characterized such groups in [\text{Dun79}]: if \(\mathbb{K}\) has characteristic 0, then these are precisely the finitely generated virtually free groups, and if \(\mathbb{K}\) has characteristic \(p\), these are precisely the finitely generated virtually free groups without elements of order \(p\). These examples are already contained in Corollary \ref{7.12} and \ref{6.16} respectively.

Corollary \ref{8.10} is an analogue of the main theorem in [\text{DCGLT20}]: a finitely presented group is \((U(n), \| \cdot \|_{Frob})\)-stable if \(H^2(\Gamma, E) = 0\) for any unitary representation \(E\). The so-called Garland
method, initially introduced in [Gar73] and since then vastly generalized to include even general Banach coefficients [Opp20], allows to give many examples of groups satisfying this condition. Our criterion asks for vanishing over Banach spaces, which at first glance may seem too restrictive compared to the Archimedean setting. However, on the one hand there is no analogue of Hilbert spaces over non-Archimedean fields [PGS10, 2.4], and on the other hand the hypothesis of the norm being solid has strong implications: such spaces are isometrically classified [PGS10, Theorem 2.5.4]. But even then it seems hard to prove a cohomology vanishing criterion by adapting Garland’s method, since the distinction between positive and negative real eigenvalues plays a fundamental role.

Theorem 8.1 is a priori stronger than Corollary 8.10: it asks for vanishing of $H^2_b(\Gamma, E)$ which is a subspace of $H^2(\Gamma, E)$. The hypothesis of $\Gamma$ being finitely presented may play an important role here. Indeed, the comparison map $c^2 : H^2_b(\Gamma, \mathbb{K}) \to H^2(\Gamma, \mathbb{K})$ is an isomorphism when $\Gamma$ is finitely presented, and $\mathbb{K}$ is seen as the trivial $\mathbb{K}[\Gamma]$-module [FF20, Corollary 8.13], although we do not know whether this holds with non-trivial coefficients.

We conjecture that the only finitely presented groups satisfying $bcd_{\mathbb{K}}(\Gamma) \leq 1$ are virtually free. More generally:

**Conjecture 8.11.** Let $\Gamma$ be an arbitrary group such that $bcd_{\mathbb{K}}(\Gamma) \leq 1$, and let $p$ be the characteristic of the residue field of $\mathbb{K}$. Then the following holds:

1. If $\mathbb{K}$ has characteristic $p$, then $cd_{\mathbb{K}}(\Gamma) \leq 1$, and so $\Gamma$ is the fundamental group of a graph of groups whose vertex groups are finite without elements of order $p$.

2. If $\mathbb{K}$ has characteristic 0, then there exists $k \geq 1$ such that $\Gamma$ is the fundamental group of a graph of groups whose vertex groups are finite with $p$-subgroups of order at most $p^k$.

Let us give some motivation behind this conjecture. Let $E$ be a Banach $\mathbb{K}[\Gamma]$-module. Since $\Gamma$ acts isometrically on it, it also acts on the reduction $\overline{E}_k$ for all $k \geq 1$, that is, the quotient of the closed ball $B_1$ of radius 1 by the closed ball $B_{r^k}$ of radius $r^k$. Note that $\overline{E}_k$ is an $(\mathfrak{g}/\mathfrak{p}^k)[\Gamma]$-module, in particular $\overline{E}_1$ is a $\mathfrak{g}[\Gamma]$-module. The reduction plays an important role in the classification of Banach spaces with a solid norm [PGS10, Theorem 2.5.4]: if $X$ is a $\mathfrak{g}$-basis of $\overline{E}_1$, then $E$ is isometrically isomorphic to

$$c_0(X) = \{ f : X \to \mathbb{K} : \# \{ x \in X : |f(x)| < \varepsilon \} < \infty \text{ for all } \varepsilon > 0 \},$$

with the supremum norm.

Suppose that $bcd_{\mathbb{K}}(\Gamma) \leq 1$. By using a dimension-shifting argument, one would prove that $H^2_b(\Gamma, E) = 0$ for all Banach $\mathbb{K}[\Gamma]$-modules with a solid norm, too. This would need a functorial approach to bounded cohomology over non-Archimedean fields, as in the real case [Mon01], but this has yet to be developed (see [FF20, Section 7] for a discussion). Assuming this step, by Lemma 8.3 there exists a constant $C \geq 1$ such that for $n = 2, 3$ and any $z \in Z^n_b(\Gamma, E)$ there exists $b \in C^{n-1}_b(\Gamma, E)$ such that $\|b\|_\infty \leq C\|z\|_\infty$.

**Claim 8.12.** Suppose that, in the above setting, we have $C = 1$. Then $H^2(\Gamma, \overline{E}_1) = 0$.

**Proof.** Let $z \in C^2(\Gamma, \overline{E}_1)$. We lift this to an element $t \in C^2(\Gamma, B_1) \leq C^2_b(\Gamma, E)$, whose coboundary takes values in $B_1$; in other words $\|t\|_\infty \leq 1$ and $\|\delta^2t\|_\infty \leq r$. Now $\delta^2t \in Z^3_b(\Gamma, E)$, so there exists $c \in C^2_b(\Gamma, E)$ such that $\delta^2c = \delta^2t$ and $\|c\|_\infty \leq \|\delta^2t\|_\infty \leq r$. We can thus consider $\hat{z} := (t - c) \in$
Let $Z^2_b(\Gamma, B_1)$ whose image in $Z^2(\Gamma, \overline{E}_1)$ is $z$. Then there exists $\hat{b} \in C^1_b(\Gamma, E)$ such that $\delta^1 \hat{b} = \hat{z}$ and $\|\hat{b}\|_\infty \leq \|z\|_\infty \leq 1$. Denoting by $b$ the image of $\hat{b}$ in $C^1(\Gamma, \overline{E}_1)$, we conclude that $z = \delta^1 b$ is a coboundary. \hfill \qed

This conclusion is quite strong: for instance by taking $E = c_0(\Gamma)$ we have $\overline{E}_1 \approx \ell[\Gamma]$, and so $H^2(\Gamma, \ell[\Gamma]) = 0$. This implies that either $\Gamma$ has $\ell$-cohomological dimension at most 1 (and so it satisfies the conjecture by Dunwoody’s characterization), or it has infinite $\ell$-cohomological dimension. Conversely:

**Claim 8.13.** Suppose that $H^2(\Gamma, \overline{E}_1) = 0$. Then $H^2_b(\Gamma, E) = 0$; more precisely for all $z \in Z^2_b(\Gamma, E)$ there exists a primitive $b \in C^1_b(\Gamma, E)$ such that $\|b\|_\infty \leq \|z\|_\infty$.

**Proof.** Let $z \in Z^2_b(\Gamma, E)$. Then the projection $\overline{z} \in Z^2(\Gamma, \overline{E}_1)$ of the normalization $z/\|z\|_\infty$ admits a primitive $\overline{b} \in C^1(\Gamma, \overline{E}_1)$. Lifting $\overline{b}$ and rescaling it to an element $b_1 \in C^1_b(\Gamma, E)$ with $\|b_1\|_\infty \leq \|\overline{z}\|_\infty$ we have $\|z - \delta^1 b_1\|_\infty \leq r \cdot \|\overline{z}\|_\infty$. We then apply the same procedure to $(z - \delta^1 b_1)$, and so on inductively. This yields a sequence $b_i$ such that $\|b_i\|_\infty \to 0$ and $\|z - (\delta^1 \sum_{j<i} b_j)\|_\infty \to 0$. Since $C^1_b(\Gamma, E)$ is Banach, $b := \sum b_i$ exists, it satisfies $\|b\|_\infty \leq \|z\|_\infty$ and $\delta^1 b = z$. \hfill \qed

The formulation of the conjecture for the more general case in which $C \geq 1$ is done by analogy with the way Schikhof’s notion of $K$-amenability [Sch75] was generalized to the author’s notion of normed $K$-amenability [FF20, Definition 1.1]. Intuitively, Schikhof’s notion of $K$-amenability (where $K$ is local, or more generally spherically complete) is a “norm 1” notion, similar to the condition $C = 1$ above, and the notion only depends on $p$, not on $K$ or its characteristic. When making this notion more flexible by allowing “bounded norms”, similar to allowing $C \geq 1$ above, we obtained the notion of normed $K$-amenability, which stays the same for characteristic $p$, and in characteristic 0 it replaces the absence of elements of order $p$ by a bound on the order of finite $p$-subgroups.

Looking at the real setting, one may hope that Theorem 8.1 could be strengthened by only asking for vanishing with dual $K[\Gamma]$-modules. For instance, in the real setting all amenable groups have vanishing bounded cohomology with dual coefficients [Fri17, Chapter 3], and in degree 2 this even applies to high-rank lattices [BM99]. In our setting, there is a significant obstacle, namely that no infinite-dimensional $K$-Banach space is reflexive [PGS10, Corollary 7.4.20], so proving that a Banach $K[\Gamma]$-module is dual would require an explicit construction of a pre-dual. This seems hard considering that the spaces appearing in our setting are quite complicated: they are matrix ultraproducts with the $\Gamma$-action induced by an asymptotic homomorphism. Using the classification of Banach spaces, and assuming that the degree $k_n \to \infty$ (else Proposition 4.23 applies), we are able to show that all spaces appearing in the proof are isometric to $\ell^\infty(N)$, which is the dual of $c_0(N)$. So there is a chance that these spaces are dual $K[\Gamma]$-modules, but to show that the action is dual one would probably have to construct an explicit pre-dual. We formulate this as an open question:

**Question 8.14.** Does stability still hold if in Theorem 8.1 bounded cohomology vanishing is assumed only for dual modules?

Even if such a strengthening were possible, all vanishing results over local fields with dual modules that are known so far [FF20, Theorem 7.4, Corollary 7.13] apply to groups whose stability has already been proved in Sections 3 and 7. So such a strengthening would be of interest only if one were able to prove more general vanishing results in degree 2, possibly by adapting the work of Burger and Monod [BM99] to lattices in $K$-analytic groups.
9 Further remarks and open questions

In this section we survey some open questions on ultrametric stability, give a few partial answers, and propose directions for further research. The first two subsections contain open questions about \( \text{GL}(\mathfrak{o}) \)-stability of certain groups, with special attention to \( \mathbb{Z}^2 \). Lastly Subsection 9.3 proposes other ultrametric families whose study may be of interest. We refer the reader to Subsections 7.4 and 8.5 for further open questions, about stability of finite groups in positive characteristic and bounded cohomology vanishing, respectively.

Throughout this section, \( \mathfrak{o} \) is the ring of integers of a non-Archimedean local field \( K \) of residual characteristic \( p \), uniformizer \( \mathfrak{p} \) and value group \( |K^\times| = r^\mathbb{Z} \).

9.1 Finding non-examples

Most of this paper has been concerned with giving positive results on stability. The negative results have been few and far apart: in Example 4.4 we constructed a pro-\( p \) family \( G_p \) such that \( \mathbb{Z}/p\mathbb{Z} \) is not \( G_p \)-stable, and in Corollary 5.12 we showed that a non-residually finite LEF group is not pointwise \( \text{GL}(\mathfrak{o}) \)-stable. In particular, the following questions remain open:

**Question 9.1.** Does there exist a finitely generated group that is not uniformly \( \text{GL}(\mathfrak{o}) \)-stable?

**Question 9.2.** Does there exist a finitely generated residually finite group that is not pointwise \( \text{GL}(\mathfrak{o}) \)-stable?

**Question 9.3.** Does there exist a finitely presented group that is not \( \text{GL}(\mathfrak{o}) \)-stable?

It would be very surprising if some of these questions had a negative answer. Good candidates seem to be free abelian groups, surface groups, free nilpotent groups and free solvable groups. The case of \( \mathbb{Z}^2 \) is discussed in detail in Subsection 9.2, the other ones are briefly mentioned after Example 9.12. Other potential non-examples are given by graphs of groups as in Subsection 6.2, where the coprimality conditions on finite quotients are not satisfied. For instance a good candidate for the uniform part of Question 9.2 is the lamplighter group \( \mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z} \): we believe this group not to be uniformly \( \text{GL}(\mathfrak{o}) \)-stable when \( p = 2 \). This would also give a positive solution to the following question:

**Question 9.4.** Does there exist a group that is (pointwise, uniformly) \( \text{GL}(\mathfrak{o}) \)-stable in some, but not all, (residual) characteristics?

Analogously, an interesting family of examples to look at is that of GBS-tree products, that is, GBS groups whose underlying graph is a tree, to which the criteria from Corollaries 6.18 and 7.13 cannot apply. In particular:

**Question 9.5.** Which torus knot groups \( K_{m,n} \) are \( \text{GL}(\mathfrak{o}) \)-stable? Does some condition on \( \nu_p(m) \) and \( \nu_p(n) \) imply stability, or instability?

All our examples of finitely generated groups that are uniformly but not pointwise stable use Corollary 5.12, so they cannot be residually finite. Therefore we ask a more precise version of Question 9.2:

**Question 9.6.** Does there exist a finitely generated residually finite group that is uniformly but not pointwise \( \text{GL}(\mathfrak{o}) \)-stable?
Recall from Corollary 4.19 that if Γ is a finitely generated residually finite group that can be expressed as the largest residually finite quotient of a finitely presented group, then the uniform and pointwise GL(σ)-stability of Γ are equivalent. We saw in the discussion after Corollary 4.19 that not all finitely generated groups satisfy this. The examples given there, all coming from [BGDLH13], could provide a positive answer to Question 9.6.

Proposition 5.15 seems to suggest that by working on the space of marked groups one could adapt the results from Sections 6 and 7 to the pointwise setting.

**Question 9.7.** Do the results from Sections 6 and 7 have a pointwise counterpart?

However, this presents more technical subtleties, since one would have to prove lifting results for local homomorphisms to the metric quotients: we are not aware of such results, or of a connection to cohomology analogous to the classical one (see Subsection 2.4).

Before moving on to specific examples in the next subsections, let us comment on why a general method for producing non-examples of stability does not work for GL(σ). This is commonly known as the projection trick. For instance when G is a family of unitary groups, one starts with \((n + 1)\)-dimensional irreducible unitary representations of the finitely generated group Γ, and restricts to the top \((n \times n)\) corner to obtain an asymptotic homomorphism. Assuming this is close to a homomorphism, one arrives at a contradiction with the irreducibility of the initial representation. The same idea works for permutations, by starting with a transitive action of Γ on \(\{1, \ldots, n\}\). To our knowledge this method was first used in [BL20] to prove that if Γ is infinite, sofic and has property (T), then it is not pointwise stable in permutations. It also appears in [BC20] and [AD20], where it is shown that uniform stability, in permutations and with respect to unitary groups with the Hilbert–Schmidt norm respectively, are very restrictive properties.

It is key in the arguments that by looking at \(n\) out of \((n + 1)\) entries (or \(n^2\) out of \((n + 1)^2\)) one does not lose much in terms of normalized metrics. This cannot be the case for the \(\ell^\infty\)-norm on GL(σ), where a big difference in a single entry is detected as a big difference overall.

This projection trick is precisely the motivation behind introducing notions of flexible stability [BL20], that have proven fruitful in some contexts [LLM19]. The discussion above shows that it seems hard to define an analogous notion of flexible GL(σ)-stability. Given the rigidity of this context, it is even possible that naïve definitions of flexible GL(σ)-stability are equivalent to ordinary GL(σ)-stability.

**Question 9.8.** Is there a meaningful notion of flexible GL(σ)-stability? Is it different from ordinary GL(σ)-stability?

### 9.2 (In)stability of \(\mathbb{Z}^2\)

The following is the main open question we would like to draw attention to:

**Question 9.9.** Is \(\mathbb{Z}^2\) GL(σ)-stable?

An answer for free abelian groups of arbitrary finite rank would be ideal, but we stick to \(\mathbb{Z}^2\) for this discussion. An algebraic-geometric approach seems to be the most appropriate: indeed in [Zor17, Theorem D, Example 10.1] the author proves a result – including an explicit example – that suggests that \(\mathbb{Z}^2\) is not constraint GL(σ)-stable with respect to a direct factor (see [AP18] for
the definition of constraint stability). The notions of stability and constraint stability can be quite distinct, for instance \( Z^2 \) is stable in permutation \([AP13]\) but not constraint stable with respect to a direct factor \([AP18]\). Still, Zordan’s result is the closest instance to a result on \( \text{GL}(\mathfrak{a}) \)-instability of \( Z^2 \) that we were able to find in the literature.

Zordan’s result is in terms of Lie algebras, not of Lie groups. This is equivalent to our setting: more precisely, instead of working with \( \text{GL}_n(\mathfrak{a}) \), one could work with \( M_n(\mathfrak{a}) \) and prove stability of the ring-theoretic commutator.

**Lemma 9.10.** The following are equivalent:

1. \( Z^2 \) is \( \text{GL}(\mathfrak{a}) \)-stable.

2. For all \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \( A, B \in M_n(\mathfrak{a}) \) satisfy \( \|AB - BA\| < \delta \), then there exist \( A', B' \in M_n(\mathfrak{a}) \) such that \( A'B' = B'A' \) and \( \|A - A'\|, \|B - B'\| < \varepsilon \).

**Proof.** We use the characterization from Corollary [46]. Since \( \|ABA^{-1}B^{-1} - I\| = \|AB - BA\| \) by \( \text{GL}(\mathfrak{a}) \)-invariance of \( \|\cdot\| \) (Lemma [3.15]), instability of \( Z^2 \) implies that 2. does not hold. Conversely, given matrices \( A, B \) contradicting 2, for a given \( \varepsilon > 0 \), the matrices \( (I + \omega A), (I + \omega B) \) contradict the characterization of stability from Corollary [46] for a rescaled \( \varepsilon \). Indeed, they are invertible by Lemma [3.15] and

\[
\|(I + \omega A)(I + \omega B) - (I + \omega B)(I + \omega A)\| = r^2 \|AB - BA\|.
\]

\( \square \)

Since the norm \( \|\cdot\| \) on \( M_n(\mathbb{K}) \) coincides with the operator norm (Lemma [3.15]), it is tempting to try and adapt Voiculescu’s counterexample \([Voi83]\) to prove instability of \( Z^2 \). Voiculescu’s matrices proving that \( Z^2 \) is not pointwise stable with respect to \( \{(U(n), \|\cdot\|_{\text{op}}) : n \geq 1\} \) are the permutation matrix \( P \) corresponding to the cycle \( (1 \cdots n) \) and the diagonal matrix \( D := \text{diag}(1, \omega_n, \omega_n^2, \ldots, \omega_n^{n-1}) \), where \( \omega_n = e^{2\pi i/n}. \) It is easy to adapt this example to produce asymptotic homomorphisms from \( Z^2 \) to \( \text{GL}(\mathfrak{a}) \), but they are all close to homomorphisms. This is true even for an arbitrary matrix \( D \), as long as \( P \) is monomial (that is, it has a unique non-zero entry in each row and column):

**Lemma 9.11.** Let \( P \in \text{GL}_n(\mathfrak{a}) \) be a monomial matrix, and let \( D \in M_n(\mathbb{K}) \). Then there exists \( D' \in M_n(\mathbb{K}) \) such that \( PD' = D'P \) and \( \|D - D'\| \leq \|PD - DP\| \).

**Proof.** Let \( \varepsilon := \|PD - DP\| = \|PD(P^{-1} - D)\| \). Let \( \sigma \in S_n \) be the permutation such that \( P_{ij} \neq 0 \) precisely when \( j = \sigma(i) \), and set \( \lambda_i := P_{\sigma(i)} \), which is in \( \mathfrak{a}^{\times} \) since \( P \in \text{GL}_n(\mathfrak{a}) \). Then \( (PD(P^{-1} - D))_{ij} = \lambda_i \lambda_j^{-1} D_{\sigma(i)\sigma(j)} \), so \( \|D_{\sigma(i)\sigma(j)} - \lambda_i \lambda_j D_{ij}\| \leq \varepsilon \) for each \( (i, j) \in \{1, \ldots, n\}^2 \). By induction it follows that

\[
\left| D_{\sigma^k(i)\sigma^k(j)} - \left( \prod_{l=0}^{k-1} \lambda_{\sigma^l(i)} \lambda_{\sigma^l(j)} \right) \cdot D_{ij} \right| \leq \varepsilon
\]

for every \( k \geq 0 \).

Now choose representatives for each orbit of the diagonal action of \( \sigma \) on \( \{1, \ldots, n\}^2 \), and for each representative \((i, j)\) and each \( k \geq 0 \) smaller than the size of the corresponding orbit, set

\[
D'_{\sigma^k(i)\sigma^k(j)} := \left( \prod_{l=0}^{k-1} \lambda_{\sigma^l(i)} \lambda_{\sigma^l(j)} \right) \cdot D_{ij}.
\]

Then \( \|D - D'\| \leq \varepsilon \) and \( PD' = D'P \). \( \square \)
In particular if $D \in \text{GL}_n(o)$ and $\|PD - DP\| < 1$, then $\|D - D'\| < 1$ and so $D' \in \text{GL}_n(o)$ as well, by Lemma 3.15. This shows that such matrices produce asymptotic homomorphisms that are close to homomorphisms, even with an optimal estimate. Moreover there is no need to modify $P$, so this also does not even work as a counterexample to constraint $\text{GL}(o)$-stability of $\mathbb{Z}^2$.

Let us end by noticing that the stability estimate of $\mathbb{Z}^2$ is, at best, quadratic. So this example really is different from the ones treated in this paper, where the stability estimates were always linear with the exception of Subsection 7.4.

**Example 9.12.** Let $A, B \in M_n(o)$ be such that $\|A\|, \|B\| \leq \varepsilon < 1$, and consider the map $\varphi : F_2 \rightarrow \text{GL}_n(o)$ sending the generators to $(I + A)$ and $(I + B)$, which are invertible by Lemma 3.15. This homomorphism almost descends to $\mathbb{Z}^2$ with a defect of $\varepsilon^2$:

$$\|(I + A)(I + B) - (I + B)(I + A)\| = \|AB - BA\| \leq \|A\| \cdot \|B\| \leq \varepsilon^2.$$ 

On the other hand, if $A$ and $B$ are chosen so that $\|AB - BA\| = \varepsilon^2$, then this homomorphism is $\varepsilon$-far from any homomorphism that descends to $\mathbb{Z}^2$.

With the same idea one can show that free abelian groups and surface groups have at best quadratic estimates. For free nilpotent groups, applying the above argument inductively on the length of the lower central series, one can show that the estimate is at best polynomial, with the degree increasing together with the length. Similarly for free solvable groups with the length of the derived series.

### 9.3 Other ultrametric families

Most of this paper was concerned with $\text{GL}(o)$-stability, where $o$ is the ring of integers of a non-Archimedean local field. The groups $\text{GL}_n(o)$ are compact because $o$ is, and compactness played an important role in our arguments, especially to have finiteness of the metric quotients. The general picture could be more complicated:

**Question 9.13.** Study $\text{GL}(o)$-stability, where $o$ is the ring of integers of a (not necessarily local) non-Archimedean field with residual characteristic $p > 0$. How does it compare to the case of local fields? Does completeness play a role? Does spherical completeness?

It is likely that some results from Section 6 and 7 carry over, assuming at least completeness. In this case the residue field $\mathfrak{k}$ is not finite, but at least it has characteristic $p$, which makes it possible to recover some arguments. In case $\mathbb{K}$ has residual characteristic 0, the analogy with local fields breaks down, so this is likely to need a separate study:

**Question 9.14.** Study $\text{GL}(o)$-stability, where $o$ is the ring of integers of a non-Archimedean field with residual characteristic 0. Does completeness play a role? Does spherical completeness?

Another direction in which to generalize $\text{GL}(o)$ while retaining compactness is to look at other compact $\mathbb{K}$-analytic groups equipped with suitable bi-invariant ultrametrics. For instance, using a result of Segal [Seg99], some of the stability results on graphs of groups from Section 7 could be generalized to any family of compact $p$-adic analytic groups equipped with a suitable metric.

**Question 9.15.** Study $G$-stability, for other families $G$ of compact $\mathbb{K}$-analytic groups equipped with suitable bi-invariant ultrametrics.
In the introduction we mentioned that the $\ell^\infty$-norm on $M_n(K)$ has the special feature of being at once an ultrametric analogue of the operator norm, of the Frobenius norm, and of the Hilbert–Schmidt norm on $U(n)$. There is a fourth norm on matrix groups that one could consider, namely the normalized rank, leading to the rank metric: the corresponding approximable groups are called linear sofic and are studied in [AP17]. The rank metric can also be defined on non-Archimedean fields, however it is not an ultrametric, so it does not fall in the framework of this paper. Therefore we ask:

**Question 9.16.** Does the rank metric admit an ultrametric analogue?

For the family $\text{Gal}(K)$ we proved in Proposition 4.25 that every finitely generated group is uniformly stable. This essentially followed from the fact that every group in the family is a quotient of the absolute Galois group. Another interesting family of Galois groups that does not have this feature is $\mathcal{G} := \{(\text{Gal}(\mathbb{Q}^{\text{sep}}/\mathbb{Q}_p), d_p) : p \text{ prime}\}$, where $d_p$ is a bi-invariant ultrametric obtained as in Example 3.9 with respect to a fixed sequence $\varepsilon$.

**Question 9.17.** Study stability with respect to the family $\mathcal{G}$ above.

Our first trivial example of ultrametric family was a family of discrete groups $\mathcal{G}$ equipped with discrete metrics. We saw that stability is less interesting in this setting (Example 4.12) but as we mentioned in the discussion after Example 3.6 probabilistic versions of stability with respect to such families appear often in property testing [Gol10]. Therefore it would be interesting to develop a general framework of probabilistic ultrametric stability, analogously to what is done for the family $\{(S_n, d_H) : n \geq 1\}$ in [BC20], which could produce new results in property testing.

**Question 9.18.** Study probabilistic analogues of ultrametric stability.

Finally, it would be interesting to produce and study more example of ultrametric families of finite groups. Beyond discrete families, the only case we treated is the family $T(R)$, where $R$ is a finite commutative ring: we proved in Corollary 5.5 a result concerning approximation with respect to such families, and stability was treated in Proposition 4.22 without the finiteness hypothesis. While we studied in detail an ultrametric analogue of $U(n)$, it would be interesting to find an ultrametric analogue of $(S_n, d_H)$ and compare the corresponding stability problems.

**Question 9.19.** Produce new examples of ultrametric families of finite groups, and study the corresponding stability problems.
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