Module forms for the $A_1$-tower

Martin Woitalla

June 21, 2017

In the 1960's Igusa determined the graded ring of Siegel modular forms of genus two. He used theta series to construct $\chi_5$, the cusp form of lowest weight for the group $\text{Sp}(2,\mathbb{Z})$. In 2010 Gritsenko found three towers of orthogonal type modular forms which are connected with certain series of root lattices. In this setting Siegel modular forms can be identified with the orthogonal group of signature $(2, 3)$ for the lattice $A_1$ and Igusa’s form $\chi_5$ appears as the roof of this tower. We use this interpretation to construct a framework for this tower which uses three different types of constructions for modular forms. It turns out that our method produces simple coordinates.

1 Introduction

Let $V$ be a real quadratic space of signature $(2, n)$ where $n \in \mathbb{N}_{\geq 3}$. The bilinear form of $V$ is denoted by $(\cdot, \cdot)$. The group of all isometries of $V$ is called the orthogonal group of $V$ and is given by

$$O(V) = \{g \in \text{GL}(V) \mid \forall v \in V : (gv, gv) = (v, v)\}.$$ 

We extend the bilinear form to $V \otimes \mathbb{C}$ by $\mathbb{C}$-linearity. We consider

$$D^\pm = \{[Z] \in \mathbb{P}(V \otimes \mathbb{C}) \mid (Z, Z) = 0, (Z, \overline{Z}) > 0\}$$

on which $O(V)$ acts as a linear group. The domain $D^\pm$ has two connected components. We choose one of them and denote it by $\mathcal{D}$. We define the subgroups

$$O(V)^+, \text{SO}(V)^+ = \{g \in O(V)^+ \mid \det(g) = 1\}$$

of index 2 and 4, respectively, which fix $\mathcal{D}$. The latter group is the connected component of the identity and is well-known to be a semisimple and noncompact Lie group.
Its maximal compact subgroup is given by \( K = \text{SO}(2) \times \text{SO}(n) \) and the Hermitian symmetric space \( \text{SO}(V)^+ / K \) is isomorphic to \( \mathcal{D} \). The affine cone is defined as

\[
\mathcal{D}^\bullet = \{ Z \in V \otimes \mathbb{C} \mid [Z] \in \mathcal{D} \}.
\]

Let \( L \subseteq V \) be a positive definite even lattice such that the dimension of \( L \otimes \mathbb{R} \) is \( n - 2 \) and let

\[
U, U_1 \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

be two integral hyperbolic planes. Denote by \( L(-1) \) the associated negative definite lattice. We consider the arithmetic subgroup

\[
\text{O}(L_2)^+ = \{ g \in \text{O}(V)^+ \mid g L_2 \subseteq L_2 \}
\]

where \( L_2 \cong U \perp U_1 \perp L(-1) \) is an even lattice in \( V \). For any subgroup \( \Gamma \leq \text{O}(L_2)^+ \) of finite index we consider the modular variety \( \Gamma \backslash \mathcal{D} \). This is a noncompact space. In \([4]\) and \([2]\) the Satake-Baily-Borel compactification of this space is considered. The boundary components of this compactification are usually called the cusps of \( \Gamma \backslash \mathcal{D} \). In \([2]\) the authors construct a general version of Siegel’s \( \Phi \)-operator to assign boundary values to automorphic forms with respect to \( \Gamma \). This is used in the following definition.

**Definition 1.1** Let \( \Gamma \) be a subgroup of \( \text{O}(L_2)^+ \). A modular form of weight \( k \in \mathbb{Z} \) and character \( \chi : \Gamma \to \mathbb{C}^\times \) with respect to \( \Gamma \) is a holomorphic function \( F : \mathcal{D}^\bullet \to \mathbb{C} \) such that

\[
F(tZ) = t^{-k}F(Z) \quad \text{for all } t \in \mathbb{C}^\times,
\]

\[
F(gZ) = \chi(g)F(Z) \quad \text{for all } g \in \Gamma.
\]

A modular form is called a cusp form if it vanishes at every cusp. The space of modular forms of weight \( k \) and character \( \chi \) for the group \( \Gamma \) will be denoted by \( \mathcal{M}_k(\Gamma, \chi) \). For the subspace of cusp forms we will write \( \mathcal{S}_k(\Gamma, \chi) \).

Let \( \Gamma \leq \text{O}(L_2)^+ \) be a subgroup of finite index and denote by \( \Gamma' = [\Gamma, \Gamma] \) the commutator subgroup of \( \Gamma \). We denote by

\[
\mathcal{A}(\Gamma') = \bigoplus_{k=0}^{\infty} \mathcal{M}_k(\Gamma', 1)
\]

the graded ring of modular forms. It is well-known that this ring is finitely generated. In the sequel the notation \( F_k \in \mathcal{A}(\Gamma') \) means that \( F \) is a homogeneous modular form of weight \( k \). We define the dual lattice of \( L \) as the \( \mathbb{Z} \)-module

\[
L^\vee := \{ x \in V \mid \forall l \in L : \langle x, l \rangle \in \mathbb{Z} \}.
\]

Since \( L \) is even we have \( L \subseteq L^\vee \). We define the discriminant group as the finite abelian group

\[
D(L) := L^\vee / L.
\]
The group $O(L^2)^+$ acts on the discriminant group $D(L^2)$. The kernel of this action is denoted by $\tilde{O}(L^2)^+$. This subgroup will be interesting for our further considerations. Another natural subgroup is the finite group $O(L)$ which consists of all automorphisms of the positive definite lattice $L$.

We put our focus to a special series of lattices. Denote by $(\cdot, \cdot)_m$ the standard scalar product on $\mathbb{R}^m$. If $\epsilon_1, \ldots, \epsilon_m$ denotes the standard basis of $\mathbb{R}^m$ we consider the following $\mathbb{Z}$-module of rank $m$

$$mA_1 = \langle \epsilon_1, \ldots, \epsilon_m \rangle_\mathbb{Z}.$$ 

If we equip $mA_1$ with the bilinear form $2(\cdot, \cdot)_m$ we obtain a series of (reducible) root lattices where $mA_1$ should be understood as an $m$-fold perpendicular sum of type $A_1$ root lattices. Due to some low-dimensional exceptional isogenies this series has connections to modular varieties of unitary and symplectic type.

**Case $m = 1$.** In this case $L_2(A_1)$ has signature $(2,3)$ and the group

$$\Gamma = O(L_2(A_1))^+ \cap SO(V)^+$$

is isomorphic to the projective symplectic group $PSp(2,\mathbb{Z})$, compare [16, Proposition 1.2]. The variety $\Gamma \backslash \mathcal{D}$ has been studied by Igusa in [20]. He showed that the graded ring of Siegel modular forms $\mathcal{A}(\Gamma')$ of genus two is generated by the Siegel Eisenstein series $E_4, E_6$ and cusp forms $\chi_5, \psi_{12}$ and $\chi_{30}$. For any modular form $F \in M_k(O(L_2(A_1))^+, \det^\kappa)$ where $\kappa, k \in \mathbb{N}_0$ the modularity conditions yield

$$(-1)^k F(Z) = F((-I_5Z)) = F(-Z) = (-1)^k F(Z).$$

Hence the determinant-character corresponds to the weight parity in the symplectic setting. According to Igusa’s result we have

$$\bigoplus_{k \in \mathbb{Z}} M_k(O(L_2(A_1))^+, 1) \cong \mathbb{C}[E_4, E_6, \chi_5^2, \psi_{12}]. \quad (1)$$

**Case $m = 2$.** We consider the Gaussian number field $K = \mathbb{Q}(\sqrt{-1})$ whose ring of integers equals $\mathfrak{o}_K = \mathbb{Z} + \mathbb{Z}\sqrt{-1}$. The special unitary group $SU(\mathfrak{o}_K) \subseteq SL(4, \mathfrak{o}_K)$ acts on the Hermitian half-plane of degree two. This can be used to show that $SO(L_2(2A_1))^+ / \{\pm I_6\}$ is isomorphic to $SU(\mathfrak{o}_K) / \{\pm I_4\}$, compare [28, Remark 3.3.4]. This case has been investigated by Freitag in [11] and later by Dern and Krieg in [9].

**Case $m = 4$.** Let $Q$ be the rational quaternion algebra of signature $(-1, -1)$. As a vector space over $\mathbb{Q}$ we have

$$Q = \mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}ij, \; i^2 = j^2 = -1.$$
A maximal order in \( \mathbb{Q} \) is given by \( \mathfrak{o} = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}\omega \) where \( \omega = \frac{1}{2}(1+i+j+ij) \). The order \( \mathfrak{o}_0 = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}ij \) is a sublattice of index 2 and is isomorphic to \( 4A_1 \). This lattice is also known as the ring of Lipschitz quaternions and \( \mathfrak{o} \) is the ring of Hurwitz quaternions. The corresponding modular group is \( \text{Sp}(2, \mathfrak{o}_0) \) and can be identified with a subgroup of \( \text{O}(L_2(4A_1))/\{\pm I_8\} \). The rings of quaternionic modular forms have been investigated by Freitag and Krieg in \( [12], [23] \) and \( [24] \).

In \( [14] \) Gritsenko found three towers of reflective modular forms. In his construction Igusa’s modular form \( \chi_5 \) is the roof of the \( 4A_1 \)-tower. In the sequel we will develop a framework around Gritsenko’s tower without making use of the exceptional isogenies. We will use three different types of coordinates.

(i) The so called Eisenstein type modular forms constitute the first type. These forms are pullbacks of Gritsenko’s singular modular form for the even unimodular lattice of signature \( (2, 10) \). If we additionally take into account the heat operator for several variables considered in \( [30] \) we obtain non-cusp forms of weight 4 and 6. The common source function for all these forms is the classical Eisenstein series of weight 4 for the group \( \text{SL}(2, \mathbb{Z}) \).

(ii) The second family of modular forms arises as a natural extension of the \( 4A_1 \)-tower of reflective modular forms. These forms are called theta type modular forms and are investigated in \( [29] \). The source function of this tower is \( \Delta_{12} \), the first cusp form for the group \( \text{SL}(2, \mathbb{Z}) \).

(iii) The third family is of baby monster type (bm type) and arises as a quasi-pullback of Borcherds famous \( \Phi_{12} \)-function which is the denominator function of the fake monster Lie algebra, compare \( [3] \) and \( [18] \). In \( [13] \) an algorithm is presented to produce many reflective modular forms of baby monster type. We can again consider \( \Delta_{12} \) as the common source function of the bm type modular forms.

Besides the determinant-character the group \( \text{O}(L_2(mA_1))^+ \) admits two more finite characters. The discriminant group \( D(mA_1) \) is isomorphic to \( m \) copies of the cyclic group of order two. The quadratic form on \( L_2 \) induces the discriminant form on \( D(L_2) \) and we obtain the finite orthogonal group \( \text{O}(D(L_2)) \) as the image of the natural homomorphism

\[ \pi : \text{O}(L_2)^+ \rightarrow \text{O}(D(L_2)). \]

The kernel of this homomorphism is the stable orthogonal group \( \tilde{\text{O}}(L_2)^+ \). In our case \( \text{O}(D(L_2(mA_1))) \cong \text{O}(D(mA_1)) \) is isomorphic to the symmetric group on \( m \) letters \( S_m \) and \( \pi \) is surjective. This yields a binary character

\[ v_\pi : \text{O}(L_2(mA_1))^+ \rightarrow S_m \xrightarrow{\text{sgn}} \{ \pm 1 \}. \]

The construction of \( mA_1 \) implies that \( (x, y)_{mA_1} \in 2\mathbb{Z} \) for all \( x, y \in mA_1 \). In this case we can construct another binary character, see e.g. \( [22] \) Proposition 1.26 and \( [7] \).
Theorem 2.2:

\[ v_2 : O(L_2(mA_1))^+ \to \text{Sp}(2,\mathbb{F}_2) \to S_6 \xrightarrow{\text{sgn}} \{\pm 1\}. \]

The construction of \( v_2 \) implies

\[ \ker v_2 \cap SO(L_2(mA_1))^+ \leq SO(L_2(mA_1))^+ , \quad O(mA_1) \leq \ker v_2. \]

We set for abbreviation \( \Gamma_m := O(L_2(mA_1))^+ \) and \( \tilde{\Gamma}_m := \tilde{O}(L_2(mA_1))^+ \). In [28, Proposition 5.4.2] it is shown that \( \Gamma_m/\Gamma'_m \cong \langle \det, v_2, v_\pi \rangle \cong C_2^3 \) if \( m = 2, 3, 4 \) and \( \Gamma_1/\Gamma'_1 \cong \langle \det, v_2 \rangle \). The paper is organized in the following way.

In section 2 we introduce Jacobi forms of theta type. These forms are obtained by twisting powers of Jacobi's theta function of weight and index \( 1/2 \) with the weak Jacobi form of weight 0 and index 1 defined in [10] and a multiplication with suitable powers of Dedekind's eta function. Moreover Jacobi forms of Eisenstein type are introduced. The arithmetic lifting of these functions yields modular forms for the orthogonal group with trivial character. In section 3 we consider two refinements of theta type Jacobi forms which yield two more series of modular forms with respect to binary characters. The first one uses a variant of the arithmetic lifting for Jacobi forms of half-integral index given in [7]. The second series is obtained by considering a cusp form of weight 24 for the lattice \( D_4 \). We rewrite the coordinates of this function for the sublattice \( 4A_1 \) and obtain a series of length three by considering quasi-pullbacks. Finally the quasi-pullbacks of Borcherd's function \( \Phi_{12} \) produce another series of modular forms including Igusa's function \( \chi_{30} \). This enables us to state our main theorem.

**Theorem 1.2** Let \( m \in \{1, 2, 3, 4\} \). The graded ring of modular forms \( \mathcal{A}(\Gamma'_m) \) is generated by the \( m \)-th row of the following table:

| \( m \) | \( m \)-type | Eisenstein type | theta type |
|---|---|---|---|
| 1 | \( H_{30}^A \) | \( E_{4A}^1 \), \( E_{6}^1 \) | \( F_{12}^{A_1} \), \( \chi_{5}^{A_1} \) |
| 2 | \( H_{30}^A \) | \( E_{4A}^1 \), \( E_{6}^1 \) | \( F_{12}^{A_1} \), \( F_{10}^{2A_1} \), \( \chi_{4}^{2A_1} \), \( \Delta_{10}^{2A_1} \) |
| 3 | \( H_{30}^A \) | \( E_{4A}^1 \), \( E_{6}^1 \) | \( F_{12}^{A_1} \), \( F_{10}^{A_1} \), \( F_{8}^{3A_1} \), \( \chi_{3}^{3A_1} \), \( \Delta_{18}^{3A_1} \) |
| 4 | \( H_{30}^A \) | \( E_{4A}^1 \), \( E_{6}^1 \) | \( F_{12}^{A_1} \), \( F_{10}^{A_1} \), \( F_{8}^{4A_1} \), \( F_{6}^{4A_1} \), \( \chi_{2}^{4A_1} \), \( \Delta_{24}^{4A_1} \) |

where the index indicates the weight.
The pullback structure underlying the above table is explained in section 2 and 3. The generators in the cases $m = 1, 2, 3$ have been determined before by Igusa, Freitag, Dern, Krieg and Klöcker whereas generators in the case $m = 4$ have only been determined for the ring $A(\Gamma_4)$ in [24] best to the author’s knowledge. Finally in section 4 we give a number theoretical application. The construction of Eisenstein type modular forms allows us to express some numbers of lattice points lying on a sphere as special values of $L$-functions which appear in [6].

Acknowledgements The results of this paper are part of the author’s phd-thesis. The author would like to thank the supervisors Valery Gritsenko and Aloys Krieg for their guidance and support.

2 Jacobi forms

Let $L$ be a positive definite even lattice. The Jacobi group $\Gamma_J(L)$ is considered in [7] and is isomorphic to $\text{SL}(2, \mathbb{Z}) \ltimes H(L)$ where $H(L)$ denotes the integral Heisenberg group for the lattice $L$. We denote by $\mathbb{H}$ the upper half-plane in $\mathbb{C}$. Following [10] and [15] we define an action of the Jacobi group on the space of holomorphic functions defined on $\mathbb{H} \times (L \otimes \mathbb{C})$. By considering the generators of the Jacobi group we can use this action to introduce the notion of a Jacobi form.

Definition 2.1 Let $k, t \in \mathbb{N}_0$. A holomorphic function $\varphi : \mathbb{H} \times (L \otimes \mathbb{C}) \to \mathbb{C}$ is called a weak Jacobi form of weight $k$ and index $t$ with character $\chi$ if the following conditions are satisfied:

(i) For all $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$:

$$\varphi \left( \frac{a\tau + b}{c\tau + d}, \frac{\mathfrak{z}}{c\tau + d} \right) = \chi(A) (c\tau + d)^k e^{\pi it \frac{c(x,y)}{c\tau + d}} \varphi(\tau, \mathfrak{z}).$$

(ii) For all $x, y \in L$:

$$\varphi(\tau, \mathfrak{z} + x\tau + y) = \chi([x, y] : -(x, y)/2]) \cdot e^{-2\pi i t \frac{1}{4}(x,x)} \varphi(\tau, \mathfrak{z})$$

where $[x, y] : -(x, y)/2] \in H(L)$, compare [11] for the realization of $H(L)$.

(iii) The Fourier expansion of $\varphi$ has the shape

$$\varphi(\tau, \mathfrak{z}) = \sum_{n \in \mathbb{N}_0} \sum_{l \in \frac{1}{2}L} f(n, l) e^{2\pi i (n\tau + (l, \mathfrak{z}))}.$$
We call $\varphi$ a holomorphic Jacobi form if the Fourier expansion ranges over all $n, l$ such that $2nt - (l, l) \geq 0$ and $\varphi$ is called a Jacobi cusp form if it ranges over all $n, l$ satisfying $2nt - (l, l) > 0$.

**Remark and Definition 2.2** (a) The action can be extended for $k, t \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ and $\chi|_{\text{SL}(2,\mathbb{Z})}$ being a multiplier system for $\text{SL}(2,\mathbb{Z})$. Here we have to replace $\text{SL}(2,\mathbb{Z})$ by the metaplectic cover $\text{Mp}(2,\mathbb{Z})$, see e.g. [5]. In this more general situation we use the notation $J^{(\text{cusp})}_{k,L,t}(\chi) \subseteq J_{k,L,t}(\chi) \subseteq J^{(\text{weak})}_{k,L,t}(\chi)$ for the corresponding spaces of Jacobi forms. If $\chi = 1$ we write $J^{(\ast)}_{k,L,t}$ for each of these spaces.

(b) The notion of a Jacobi form is compatible with Definition 1.1. To see this we note that we have an affine model for the homogeneous domain $\mathcal{D}$ given by

$$\mathcal{H}(L_2) = \left\{ (\omega, \mathfrak{J}, \tau) \in \mathbb{C} \times (L \otimes \mathbb{C}) \times \mathbb{C} \mid \omega_i, \tau_i > 0, 2\omega_i \tau_i - (\mathfrak{J}_i, \mathfrak{J}_i) > 0 \right\}$$

where we have used the abbreviations

$$\omega_i := \text{Im}(\omega) \quad , \quad \tau_i := \text{Im}(\tau) \quad , \quad \mathfrak{J}_i := \text{Im}(\mathfrak{J}) .$$

Let $\varphi \in J_{k,L,t}(\chi)$ where we assume $k \in \mathbb{Z}$. We define a holomorphic function on $\mathcal{H}(L_2)$ by

$$\tilde{\varphi}(\tau, \mathfrak{J}) = \varphi(\tau, \mathfrak{J}) e^{2\pi i t \omega} .$$

Since $\mathcal{D}$ and $\mathcal{H}(L_2)$ are biholomorphically equivalent we can interpret $\tilde{\varphi}$ as an element in $\mathcal{M}_k(\Gamma^J(L), \chi)$.

The following two examples are the basic ingredients to define theta type Jacobi forms.

**Example 2.3** (a) Dedekind’s eta function is a Jacobi form of weight $1/2$ and index $0$ for every positive definite even lattice $L$, thus $\eta \in J^{(\text{cusp})}_{1/2,L,0}(\nu_\eta)$ where $\nu_\eta$ is a multiplier system for $\text{SL}(2,\mathbb{Z})$.

(b) The Jacobi theta series of characteristic $(\frac{1}{2}, \frac{1}{2})$ is given as

$$\vartheta(\tau, z) = \sum_{n \in \mathbb{Z}} \left( -\frac{4}{n} \right) q^{\frac{n^2}{16}} \tau^{\frac{n}{2}} = \sum_{n \in \mathbb{Z}, n \equiv 1 \mod 2} (-1)^{\frac{n-1}{2}} \exp \left( \frac{\pi in^2 \tau}{4} + \piinz \right)$$

where $q = e^{2\pi i \tau}, \tau \in \mathbb{H}$ and $r = e^{2\pi iz}, z \in \mathbb{C}$. This function was originally discovered by Carl Gustav Jacob Jacobi. In [19] the authors reinterpreted this.
function as a modular form of half-integral weight and index. Jacob's triple
identity yields
\[ \vartheta(\tau, z) = -q^{1/8}r^{-1/2} \prod_{n \geq 1} (1 - q^{n-1}r)(1 - q^n r^{-1})(1 - q^n). \]

The function has the properties
\[ \vartheta(\tau, -z) = -\vartheta(\tau, z), \]
\[ \vartheta(\tau, z + x\tau + y) = (-1)^{x+y} \exp(-\pi i (x^2 \tau + 2xz)) \vartheta(\tau, z) \]
for all \( x, y \in \mathbb{Z} \) and the set of zeroes of \( \vartheta \) equals
\[ \{ x\tau + y \mid x, y \in \mathbb{Z} \}. \]

In the sequel let \( M \) be a positive definite even lattice and \( L \leq M \) be a sublattice of \( M \). We define
\[ L_M := \{ m \in M \mid \forall l \in L : (l, m) = 0 \} \]
and note that this is again a positive definite sublattice in \( M \). The direct sum
\[ L \oplus L_M \leq M \]
is a sublattice of finite index. The next Lemma can be found in [7], Proposition 3.1.

Lemma 2.4 Let \( L \leq M \) be a sublattice such that \( \text{rank } L < \text{rank } M \) and let \( \varphi \in J_{k,M,t}(\chi) \) be a Jacobi form of weight \( k \) and index \( t \) for the character \( \chi \). Consider the decomposition \( \mathfrak{z}_M = \mathfrak{z}_L \oplus \mathfrak{z}_L^\perp \in M \otimes \mathbb{C} = (L \oplus L_M) \otimes \mathbb{C} \). We define the pullback of \( \varphi \) to \( L \) as the function \( \varphi \downharpoonright_L \) on \( \mathbb{H} \times (L \otimes \mathbb{C}) \)
\[ \varphi \downharpoonright_L (\tau, \mathfrak{z}_L) := \varphi(\tau, \mathfrak{z}_L \oplus 0). \]

Then \( \varphi \downharpoonright_L \in J_{k,L,t}(\chi|_{\Gamma^L(L)}) \) and the pullback maps cusp forms to cusp forms.

Definition 2.5 Let \( L \leq M \) be a sublattice of \( M \) such that \( \text{rank } L < \text{rank } M \) and \( \varphi \in J_{k,M,t}^{(\text{weak})} \) be a Jacobi form of weight \( k \) and index \( t \). Let \( \psi \in J_{k,L,t}^{(\text{weak})} \). We say that \( \psi \) is a pullback of \( \varphi \) if there exists some \( \alpha \in \mathbb{C}^\times \) such that \( \psi = \alpha \cdot \varphi \downharpoonright_L \). In this case we use the notation \( \varphi \rightarrow \psi \). We set \( \varphi \downharpoonright_L := \varphi \) if \( \text{rank } L = \text{rank } M \).

We define
\[ \vartheta_L : \mathbb{H} \times (L \otimes \mathbb{C}) \rightarrow \mathbb{C}, \quad \vartheta_L(\tau, \mathfrak{z}) := \prod_{j=1}^m \vartheta(\tau, (\mathfrak{z}, \varepsilon_j)). \]
This leads us to the notion of theta type Jacobi forms.
Definition 2.6 Let \( L \subseteq \mathbb{R}^m \) be a positive definite even lattice and \( \varphi \in J_{k,L,1} \). We say that \( \varphi \) is of theta type if there exists a sublattice \( L' \subseteq L \), \( \alpha \in \mathbb{C}^\times \) and integers \( a, b \in \mathbb{Z}_{\geq 0} \) such that

\[
\varphi \downharpoonright_{L'} (\tau, z') = \alpha \cdot \eta(\tau)^a \vartheta_{L}(\tau, z')^b.
\]

Note that \( \Delta_{12}(\tau) = \eta(\tau)^{24} \) is of theta type because \( J_{k,L,0} \) is isomorphic to the space of weight \( k \) modular forms for the group \( SL(2,\mathbb{Z}) \).

For \( z \in (mA_1) \otimes \mathbb{C} \) we write

\[
z = \sum_{j=1}^{m} z_j e_j =: (z_1, \ldots, z_m), \quad z_j \in \mathbb{C}.
\]

(2)

For the next Proposition we note that

\[
\vartheta_{4A_1}(\tau, z) = \vartheta(\tau, z_1)\vartheta(\tau, z_2)\vartheta(\tau, z_3)\vartheta(\tau, z) \in J_{2,4A_1;1/2}(\chi_2)
\]

has already been constructed in [14] where \( \chi_2 \) is a binary character. In the next Proposition the square of this function is denoted by \( \psi_{4,4A_1} \). In [29, Proposition 3.7] the author constructs a tower of theta type Jacobi forms for the lattice \( 4A_1 \).

Proposition 2.7 There exists the following diagram of theta type Jacobi forms for the \( A_1 \)-tower

\[
\begin{array}{cccccc}
\Delta_{12} & \frac{\partial^2}{\partial \tau^2} |_{z_1=0} & \varphi_{12,1A_1} & \psi_{10,1A_1} & \frac{\partial^2}{\partial z_2^2} |_{z_2=0} & \varphi_{12,2A_1} & \varphi_{10,2A_1} & \psi_{8,2A_1} & \frac{\partial^2}{\partial z_3^2} |_{z_3=0} & \varphi_{12,3A_1} & \varphi_{10,3A_1} & \varphi_{8,3A_1} & \psi_{6,3A_1} & \frac{\partial^2}{\partial z_4^2} |_{z_4=0} & \varphi_{12,4A_1} & \varphi_{10,4A_1} & \varphi_{8,4A_1} & \psi_{6,4A_1} & \psi_{4,4A_1}
\end{array}
\]

where \( \psi_{k,mA_1}, \varphi_{k,mA_1} \in J_{k,mA_1;1} \). Except for the last line all forms are cusp forms.

We recall that there exists a unique (up to isomorphism) positive definite even lattice \( E_8 \) in dimension 8 which is unimodular. Following [15] we can attach a Jacobi theta series to \( E_8 \):

\[
\Theta_{E_8}(\tau, z) = \sum_{l \in E_8} \exp(\pi i (l,l) \tau + 2\pi i (l,z)).
\]
Then $\Theta_{E_8} \in J_{A_1, E_8}$ is a singular Jacobi form for $E_8$. We fix a chain of embeddings

$$A_1 \hookrightarrow 2A_1 \hookrightarrow 3A_1 \hookrightarrow 4A_1 \hookrightarrow E_8.$$ 

We will investigate the pullbacks

$$\epsilon_{m, A_1} = \Theta_{E_8} \downarrow_{mA_1} \in J_{m, A_1}.$$ 

**Proposition 2.8** Let $\sigma \in O(A_1)$ for $m \in \{1, 2, 3, 4\}$. Then $\sigma$ can be extended to $O(E_8)$ and for any sublattice $L \leq E_8$ where $L \cong mA_1$ there exists some $g \in O(E_8)$ such that $gL = mA_1$. Moreover $\epsilon_{m, A_1}$ is invariant under the transformation induced by $\sigma$ such that for all $\tau, \beta$ one has

$$\epsilon_{m, A_1}(\tau, \sigma \beta) = \epsilon_{m, A_1}(\tau, \beta).$$

**Proof**

We denote by $K_m := (mA_1)_{E_8}$ the orthogonal complement of $mA_1$ in $E_8$. For the proof of the statement we will have to investigate the discriminant form

$$q : D(L) \to \mathbb{Q}/2\mathbb{Z}, \quad x + L \mapsto (x, x) + 2\mathbb{Z}.$$ 

In the following list one can find a root system which is isomorphic to $K_m$

| $m$ | $1$ | $2$ | $3$ | $4$ |
|-----|-----|-----|-----|-----|
| $K_m$ | $E_7$ | $D_6$ | $A_1 \oplus D_4$ | $4A_1$ |

From [26 Proposition 1.6.1] we know that $\sigma$ can be extended to $O(E_8)$ if the natural homomorphism $O(K_m) \to O(D(K_m))$ is surjective. (3)

According to [21 Chapter 4, Section 8.2] we have $O(D(E_7)) = \{id\}$ because $E_7^\vee = E_7 \cup (v + E_7)$ where $q(v + E_7) = \frac{3}{2} + 2\mathbb{Z}$ which grants the surjectivity in this case. The lattice $D_m, m \geq 3$ can be realized as the $\mathbb{Z}$-module with basis

$$\varepsilon_2 + \varepsilon_1, \varepsilon_2 - \varepsilon_1, \varepsilon_3 - \varepsilon_2, \ldots, \varepsilon_m - \varepsilon_{m-1}.$$ 

Moreover this lattice can be described as the following subset in $\mathbb{Z}^m$:

$$D_m = \{x \in \mathbb{Z}^m \mid x_1 + \cdots + x_m = 0 \text{ mod } 2\}$$

We define $w := \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_m) \in D_m^\vee$. The values of the discriminant form for the representatives of $D(D_m)$ are given as follows

$$\begin{array}{c|ccccc}
 l & 0 & \varepsilon_1 & w & \varepsilon_1 + w \\
 q(l) & 0 + 2\mathbb{Z} & 1 + 2\mathbb{Z} & \frac{m}{2} + 2\mathbb{Z} & \frac{m}{2} + 2\mathbb{Z}
\end{array}$$

10
If \( m \neq 4 \mod 8 \) one has \( \text{O}(D(D_m)) \cong \mathbb{Z}/2 \) and this group is generated by the permutation of the classes represented by \( w \) and \( \varepsilon_1 + w \). This element is induced by \( \sigma_{\varepsilon_1} \in \text{O}(D(D_m)) \), the reflection at the hyperplane perpendicular to \( \varepsilon_1 \). Moreover \( \text{O}(D(D_4)) \cong S_3 \). In this case the group is generated by the permutation of the classes \( w \) and \( \varepsilon_1 + w \) and the permutation of \( \varepsilon_1 \) and \( \varepsilon_1 + w \). The latter element is induced by the reflection \( \sigma_w \in \text{O}(D_4) \). Finally we note that the natural homomorphism

\[
\text{O}(mA_1) \rightarrow \text{O}(D(mA_1))
\]
is surjective. Summarizing these considerations we see that the assumption (3) is satisfied for each \( m = 1, \ldots, 4 \). This proves the first part and the invariance property of the pullbacks as a direct consequence. \( \square \)

In the next step we construct a differential operator. This operator is well-known and a treatment can be found in [30] for the general case or in [10] for classical Jacobi forms. The heat operator is given as

\[
H = 4\pi i \det(S) \frac{\partial}{\partial \tau} - \det(S) S^{-1} \left[ \frac{\partial}{\partial \tau} \right].
\]

We recall the definition of the quasi-modular Eisenstein series of weight 2

\[
G_2(\tau) = -\frac{1}{24} + \sum_{n \geq 1} \sigma_1(n) e^{2\pi in\tau}, \quad \sigma_k(n) = \sum_{d|n} d^k
\]

which transforms under \( \text{SL}(2, \mathbb{Z}) \) as

\[
G_2 \left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^2 G_2(\tau) - \frac{c(c\tau + d)}{4\pi i}.
\]

and denote by \( G_2 \bullet \) the operator which multiplies a function by \( G_2 \). By virtue of the transformation property of \( G_2 \) we obtain a quasi-modular operator. We fix the notation

\[
\tau^l := \exp(2\pi i(l, \delta)) \quad , \quad l \in L^\vee, \delta \in L \otimes \mathbb{C}.
\]

**Lemma 2.9** For every \( k \in \mathbb{N} \) there is a quasi-modular differential operator \( H_k : J_{k,L,1} \rightarrow J_{k+2,L,1} \) defined by the formula

\[
H_k = H + (4\pi i)^2 \det(S) \left( k - \frac{m}{2} \right) G_2 \bullet
\]

where \( m = \text{rank}(L) \). The operator \( H \) acts on \( q^n \tau^l \), \( n \in \mathbb{N}, l \in L^\vee \) by multiplication with \( (2\pi i)^2 \det(S) (2n - (l, l)) \).
The first part can be deduced from the considerations in [30] and the second part is a direct verification.

Using this operator we define

\[ \epsilon_{6,mA_1} = H_4(\epsilon_{4,mA_1}) \in J_{6,mA_1,1} \, , m \in \{1, 2, 3, 4\} \, . \]

These functions inherit the invariance under coordinate permutations from \( \epsilon_{4,mA_1} \).

**Corollary 2.10** Let \( m \in \{1, 2, 3, 4\} \). For all \( \sigma \in \text{O}(mA_1) \) we have

\[ \epsilon_{6,mA_1}(\tau, \sigma \cdot z) = \epsilon_{6,mA_1}(\tau, z) \, . \]

In particular \( \epsilon_{6,mA_1} \) is invariant with respect to the action of \( \text{O}(D(mA_1)) \cong S_m \).

The next Lemma is about cusp forms and follows immediately from Lemma 2.9

**Lemma 2.11** Let \( \varphi \in J_{k,L,1} \). Then \( \varphi \) is a cusp form if and only if \( H_k(\varphi) \) is a cusp form.

Let \( L \leq M \) be a primitive sublattice. We extend the pullback notation of Definition 2.6 to modular forms in the canonical way. In [15] the arithmetic lifting of Jacobi forms has been defined. We denote this lifting operator by \( \text{A-Lift}() \) and define

\[ \mathcal{E}_m^{mA_1} := \text{A-Lift}(\epsilon_{k,mA_1}) \in \mathcal{M}_k(\tilde{\Gamma}_m, 1) \, . \]

Moreover the operator \( H_k \) is extended to the Maaß space by the following convention:

\[ H_k(\text{A-Lift}(\varphi)) := \text{A-Lift}(H_k(\varphi)) \, , \, \varphi \in J_{k,L,1} \, . \]

The following Theorem describes all modular forms obtained by the previous considerations.

**Theorem 2.12** We have the following diagram of modular forms for the \( A_1 \)-tower

\[ \begin{array}{cccccc}
\mathcal{E}_4^{A_1} & \xrightarrow{H_4} & \mathcal{E}_6^{A_1} & \\
\uparrow & & \uparrow & \\
\mathcal{E}_4^{2A_1} & \xrightarrow{H_4} & \mathcal{E}_6^{2A_1} & \\
\uparrow & & \uparrow & \\
\mathcal{E}_4^{3A_1} & \xrightarrow{H_4} & \mathcal{E}_6^{3A_1} & \\
\uparrow & & \uparrow & \\
\mathcal{E}_4^{4A_1} & \xrightarrow{H_4} & \mathcal{E}_6^{4A_1} & \\
\end{array} \]

\[ \begin{array}{cccccc}
F_{12}^{A_1} & \xrightarrow{H_{12}} & F_{10}^{A_1} & \\
\uparrow & & \uparrow & \\
F_{12}^{2A_1} & \xrightarrow{H_{12}} & F_{10}^{2A_1} & G_{8}^{2A_1} & \\
\uparrow & & \uparrow & \\
F_{12}^{3A_1} & \xrightarrow{H_{12}} & F_{10}^{3A_1} & G_{6}^{3A_1} & \\
\uparrow & & \uparrow & \\
F_{12}^{4A_1} & \xrightarrow{H_{12}} & F_{10}^{4A_1} & G_{4}^{4A_1} & \\
\end{array} \]

12
where \( F_k^{mA_1}, G_k^{mA_1} \in M_k(\Gamma_m, 1) \) for each form appearing in the diagram. The forms inside the rectangle are cusp forms.

**Proof**

We define the \( F_k^{mA_1}, G_k^{mA_1} \) as the arithmetic liftings of the functions in Proposition 2.7 with the same arrangement for the weights and the lattices. This yields functions which belong to \( M_k(\tilde{\Gamma}_m, 1) \). The arithmetic lifting maps cusp forms to cusp forms if the lattice \( mA_1 \) is a maximal even lattice. This is precisely the case for \( m = 1, 2, 3 \). Hence the statements on cusp forms follow from Proposition 2.7 and the preceding construction of \( \epsilon_k^{mA_1} \). Note that the operator \( A\text{-Lift} \) commutes with pullbacks as one immediately extracts from its definition. The Jacobi forms appearing in Proposition 2.8 and Corollary 2.10 are invariant with respect to the permutations of \( z_1, \ldots, z_m \). However, the theta type forms listed in Proposition 2.7 are not except for the functions \( \psi_{12-2m, mA_1} \) on the diagonal. Hence we can apply the operator

\[
J_{k, mA_1; 1} \mapsto J_{k, mA_1; 1}, \quad \varphi \mapsto \frac{1}{m!} \sum_{\sigma \in S_m} \sigma \cdot \varphi
\]

where \( (\sigma \cdot \varphi)(\tau, z) := \varphi(\tau, \sigma \cdot z) \) for any \( \sigma \in S_m \). After application of (5) all the functions appearing in Proposition 2.7 are symmetric. Hence the maximal modular group of these liftings is \( \Gamma_m \). \( \square \)

### 3 Rings of Modular Forms

For any \( r \in L_2 \otimes \mathbb{Q} \) satisfying \( (r, r) < 0 \) we define the rational quadratic divisor as

\[
D_r = \{ [Z] \in D \mid (Z, r) = 0 \}.
\]

For any \( m \in \mathbb{N} \) we fix the notation

\[
D^m := D(L_2(mA_1)).
\]

In particular consider \( \varepsilon_m \in mA_1(-1) \subseteq L_2(mA_1) \). Then one has

\[
D_{\varepsilon_m}^m \cong D^{m-1}.
\]

In [14, Theorem 5.1] it was proved that this divisor is attached to \( G_k^{mA_1}_{12-2m} \) for \( m = 1, \ldots, 4 \).

**Theorem 3.1** Let \( m \in \{1, 2, 3, 4\} \). The divisor of the modular form \( G_k^{mA_1}_{12-2m} \) consists of the \( \Gamma_m \)-orbit of \( D_{\varepsilon_m}^m \). The vanishing order is two on each irreducible component of \( \text{div}(G_k^{mA_1}_{12-2m}) \). Moreover there exists a modular form

\[
\chi_{6-m} \in M_{6-m}(\Gamma_m, \det v_2)
\]

whose square equals \( G_k^{mA_1}_{12-2m} \). If \( m \neq 4 \), then \( \chi_{6-m} \) is a cusp form.
This yields the structure of the graded ring for the $A_1$-tower with trivial character.

**Theorem 3.2** Let $m \in \{1, 2, 3, 4\}$. The graded ring $A(\Gamma_m)$ is a polynomial ring in the $m + 3$ functions which are given by the $m$-th row of the diagram in Theorem 2.12.

**Proof**

Since $-I_{m+4} \in \Gamma_m$ for all $m$ there are no modular forms of odd weight in $A(\Gamma_m)$. For the proof we consider the following reduction process:

(i) The starting point is the case $m = 1$. Our construction yields the classical result of Igusa

$$A(\Gamma_1) = \mathbb{C}[\mathcal{E}_{4}^{A_1}, \mathcal{E}_{6}^{A_1}, G_{10}^{A_1}, F_{12}^{A_1}]$$

as we have already investigated in (1).

(ii) We have an embedding $\Gamma_m \hookrightarrow \Gamma_{m+1}$ for each $m \in \mathbb{N}$. Hence the restriction map

$$M_k(\Gamma_{m+1}, 1) \to M_k(\Gamma_m, 1), F \mapsto F|_{D^m}$$

is well-defined for any even $k \in \mathbb{N}_0$. This map extends to a homomorphism of the graded algebras

$$\text{Res}_{m+1}^m : A(\Gamma_{m+1}) \to A(\Gamma_m).$$

We shall show that this map is surjective for $m = 1, 2, 3$.

(iii) Let $k \in \mathbb{N}_0$ and consider $F \in M_k(\Gamma_m, 1)$ with the property

$$F|_{D^{m-1}} \equiv 0.$$

Let $m \geq 2$ and define

$$M := \text{diag}(1, 1, K, 1, 1) \text{ where } K := \text{diag}(1, \ldots, -1) \in \text{GL}(m, \mathbb{Z}).$$

Since $M$ belongs to $\Gamma_m$ the Taylor expansion of $F$ around 0 with respect to $z_m$ shows that $F$ vanishes of order at least two on $D^{m-1}$. According to Theorem 5.1 we can divide $F$ by $G_{12, -2m}^{A_1}$ and obtain a holomorphic modular form in $M_{k-12, 2m}(\Gamma_m, 1)$ by Koecher’s principle for automorphic forms, compare [1, p. 209]. Note that we have used the identification (6), here.

Now starting from (1) the statement follows by induction on the weight using (i)-(iii) for the surjectivity of $\text{Res}_{m+1}^m$ for $m = 1, 2, 3$ is extracted from Theorem 2.12.

In the following we construct three modular form with respect to the character $v_\pi$. Following [29] we consider the following three theta type Jacobi forms with respect to the coordinates introduced in (2):

$$
\begin{align*}
\vartheta^{(1)}_{A_1}(\tau, \mathfrak{z}_4, A_1) &= \vartheta(\tau, z_1 - z_2)\vartheta(\tau, z_1 + z_2)\vartheta(\tau, z_3 - z_4)\vartheta(\tau, z_3 + z_4), \\
\vartheta^{(2)}_{A_1}(\tau, \mathfrak{z}_4, A_1) &= \vartheta(\tau, z_3 - z_2)\vartheta(\tau, z_3 + z_2)\vartheta(\tau, z_1 - z_4)\vartheta(\tau, z_1 + z_4), \\
\vartheta^{(3)}_{A_1}(\tau, \mathfrak{z}_4, A_1) &= \vartheta(\tau, z_1 - z_3)\vartheta(\tau, z_1 + z_3)\vartheta(\tau, z_2 - z_4)\vartheta(\tau, z_2 + z_4).
\end{align*}
$$
such that \( \vartheta^{(j)}_{4A_1} \in J_{2,4A_1;1}(v_2^{12}) \) for \( j = 1, 2, 3 \). By multiplying each of the three functions by \( \eta(\tau)^{12} \) and considering the arithmetic lifting of these functions we obtain three modular forms \( \Delta^{(j)}_{8,4A_1}, j = 1, 2, 3 \) which belong to \( \mathcal{M}_8(\tilde{\Gamma}_4, 1) \).

**Proposition 3.3** There is a cusp form

\[
\Delta^{4A_1}_{24} \in S_{24}(\Gamma_4, v_\pi)
\]

satisfying

\[
\Delta^{4A_1}_{24} = \Delta^{(1)}_{8,4A_1} \Delta^{(2)}_{8,4A_1} \Delta^{(3)}_{8,4A_1}.
\]

The divisor equals the \( \Gamma_4 \)-orbit of \( D^{4A_1}_{\varepsilon_1+\varepsilon_4} \).

**Proof**

This function coincides with the cusp form of weight 24 for the lattice \( D_4 \) which was constructed in [29, Theorem 4.4]. Since \( 4A_1 \) is a sublattice of \( D_4 \) we obtain a function with the modular behaviour stated above where the character \( v_\pi \) appears due to the definition of \( \vartheta^{(j)}_{4A_1} \) above. For a maximal even lattice the arithmetic lifting of a Jacobi form is a cusp form if the Fourier expansion ranges over all parameters with positive hyperbolic norm, compare [15, Theorem 3.1]. This characterization can be extended to all lattices with the property that every isotropic subgroup of \( D(L_2) \) is cyclic, see [17, Theorem 4.2] for a proof. Since this is the case for the lattice \( L_2(4A_1) \) the function \( \Delta^{4A_1}_{24} \) is a cusp form. □

The Proposition yields two more cusp forms for the tower.

**Corollary 3.4** There are cusp forms

\[
\Delta^{2A_1}_{10} \in S_{10}(\Gamma_2, v_\pi) \quad \text{and} \quad \Delta^{3A_1}_{18} \in S_{18}(\Gamma_3, v_\pi)
\]

whose divisor equals the \( \Gamma_m \)-orbit of

\[
D^{m}_{\varepsilon_1+\varepsilon_m}, \quad m = 2 \text{ or } 3, \text{ respectively.}
\]

**Proof**

We consider the cusp form \( \Delta^{4A_1}_{24} \) in Proposition 3.3. The divisor is the sum of the six \( \Gamma_m \)-orbits which are represented by \( D^1 + D^2 \) where

\[
D^1 := D^{4}_{\varepsilon_1+\varepsilon_2} + D^{4}_{\varepsilon_1+\varepsilon_3} + D^{4}_{\varepsilon_2+\varepsilon_3},
\]

\[
D^2 := D^{4}_{\varepsilon_1+\varepsilon_4} + D^{4}_{\varepsilon_2+\varepsilon_4} + D^{4}_{\varepsilon_3+\varepsilon_4}.
\]

The group \( O(D(4A_1)) \cong S_4 \) acts 2-fold transitive on the set

\[
\{\varepsilon_\mu | 1 \leq \mu \leq 4\}
\]
by relabelling the indices. The group $S_4$ contains $S_3$ as the subgroup fixing $\varepsilon_4$. The sets $D_{\varepsilon_k+\varepsilon_4}, D_{\varepsilon_k-\varepsilon_4}$ where $k = 1, 2, 3$ belong to the same $\tilde{\Gamma}_4$-orbit but constitute different $\Gamma_3$-orbits. Hence the restriction of $\Delta_{24}^{A_1}$ to $D^3$ has the divisor
\[ D^3_j + 2D^3_2 \]
with respect to the action of $\tilde{\Gamma}_3$ where
\[ D^3_j := D^j_4 \cap D^3, \quad j = 1, 2. \]
Moreover the $\Gamma_3$-orbit of the restriction is represented by
\[ D_{\varepsilon_1+\varepsilon_3}^3 + 2D_{\varepsilon_3}^3. \]
According to Theorem 3.1 we can define
\[ \Delta_1^{3A_1} := \frac{\Delta_{24}^{3A_1}}{G_6^{3A_1}} \bigg|_{D^3} \]
and obtain a modular form with the desired properties by Koecher’s principle. Now the same construction is done with $\Delta_1^{3A_1}$ instead and one defines
\[ \Delta_1^{2A_1} := \frac{\Delta_{18}^{2A_1}}{G_8^{2A_1}} \bigg|_{D^3} \]
which has the correct divisor. An analysis of the Fourier expansion of $\Delta_1^{2A_1}, \Delta_1^{3A_1}$ yields that both functions are cusp forms. □

In the following we consider another type of modular forms. Let $II_{2,26}$ be the unique (up to isomorphism) even unimodular lattice of signature $(2, 26)$. We define
\[ R_{-2,26} := \{ r \in II_{2,26} \mid (r, r) = -2 \}. \]
The following statement is due to Borcherds and can be found in [3, Theorem 10.1 and Example 2].

**Theorem 3.5 (Borcherds)** There is a holomorphic modular form $\Phi_{12}$ with the properties
\[ \Phi_{12} \in M_{12}(O(II_{2,26})^+, \det) \quad \text{div}(\Phi_{12}) = \bigcup_{r \in R_{-2,26}^{II_{2,26}}} D_r(II_{2,26}), \]
where the vanishing order is exactly one on each irreducible component.
In [3, Example 2] Borcherds computes the Fourier expansion of $\Phi_{12}$. It turns out that $\Phi_{12}$ reflects the Weyl denominator formula for the fake monster Lie algebra.

Let $\mathcal{N}$ be the Niemeier lattice with root system $24A_1$, see [23]. Since $II_{2,26} \cong U \perp U_1 \perp \mathcal{N}$

where

$$U = \langle e, f \rangle, U_1 = \langle e_1, f_1 \rangle$$

are two integral hyperbolic planes we can consider the natural embedding

$L_2(mA_1) \hookrightarrow U \perp U_1 \perp \mathcal{N}$, $m \in \{1, 2, 3, 4\}$

(8)

which is induced by $mA_1 \hookrightarrow 24A_1$. Let $K_m$ be the orthogonal complement of $L_2(mA_1)$ in $II_{2,26}$. Each vector $r \in II_{2,26}$ has a unique decomposition

$r = \alpha(r) + \beta(r)$, $\alpha(r) \in L_2(mA_1)^\vee$, $\beta(r) \in K_m^\vee$.

We set

$R_{-2}(K_m) = \{ r \in II_{2,26} | (r, r) = -2, r \perp L_2(mA_1) \}$

which is contained in the negative definite lattice $K_m$ and hence finite. Consequently we define $N(K_m) = \frac{1}{2}R_{-2}(K_m) \in \mathbb{N}$. The next statement is a special case of [18, Theorem 8.2 and Corollary 8.12] and describes the construction of a quasi-pullback from Borcherds function $\Phi_{12}$.

**Theorem 3.6** Consider a primitive embedding $L_2 \hookrightarrow II_{2,26}$ and denote by $K$ the orthogonal complement of $L_2$ in $II_{2,26}$.

$$\Phi_{L_2}^{(QP)}(Z) = \frac{\Phi_{12}(Z)}{\prod_{r \in R_{-2}(K)/\{\pm 1\}} (Z, r)} \bigg|_{D(L_2)}$$

where in the product one fixes a set of representatives for $R_{-2}(K)/\{\pm 1\}$. Then $\Phi_{L_2}^{(QP)}$ belongs to $M_{12+N(K)}(\tilde{O}(L_2)^\dagger, \det)$ and vanishes exactly on all rational quadratic divisors

$D_{\alpha(r)} = \{ [Z] \in D(L_2) | (Z, \alpha(r)) = 0 \}$

where $r$ runs through the set $R_{-2}^{II_{2,26}}$ and $(\alpha(r), \alpha(r)) < 0$. If $N(K) > 0$ we say that $\Phi_{L_2}^{(QP)}$ is a quasi-pullback of $\Phi_{12}$. In this case $\Phi_{L_2}^{(QP)}$ is a cusp form.

The choice of the embedding [3] yields another four modular forms with respect to a character.
**Theorem 3.7** Let \( m \in \{1, 2, 3, 4\} \). There exists a cusp form

\[
F_{36-m}^{mA_1} \in S_{36-m}(\Gamma_m, \det v_n^\kappa), \quad \kappa \in \{0, 1\}
\]

whose divisor is represented by the sum of the two different \( \Gamma_m \)-orbits

\[
D_{e_1-f_1}^m + D_{\varepsilon_m}^m.
\]

**Proof**

The results can be extracted from [13]. However, we give a sketch of the proof, here, using Theorem 3.6. The strategy of the proof is to construct modular forms as quasi-pullbacks of \( \Phi_{12} \) with respect to the embedding (8). We denote these forms by \( F_{mA_1}^{k_m} \) for \( m = 1, 2, 3, 4 \). Since the number of \(-2\)-roots for the lattice \( mA_1 \) is exactly \( 2m \) we have \( N(K_m) = 24 - m \) and the weight \( k_m \) of \( F_{mA_1}^{k_m} \) is exactly \( 12 + N(K_m) = 36 - m \).

The divisor of \( F_{36-m}^{mA_1} \) is determined by all vectors \( \alpha \in L_{2}(mA_1)^{\vee} \), \( (\alpha, \alpha) < 0 \) such that there exists a \( \beta \in K_2^v \) satisfying \( \alpha + \beta \in R_{-2}^{26} \). The choice of our embedding already implies \( (\alpha, \alpha) = -2 \) and \( \beta(r) = 0 \). There are \( m + 1 \) different orbits of \(-2\)-roots in \( L_2(mA_1) \) with respect to \( \tilde{\Gamma}_m \) represented by \( e_1 - f_1, \varepsilon_1, \ldots, \varepsilon_m \).

The divisor is represented by the sum of these orbits. In [26, Corollary 15.2] the author gave a criterion to decide whether an element \( g \in L_2(mA_1) \) extends to \( O(H_{2,26})^{+} \). In our case this is always possible, compare [13, p. 122]. Hence the maximal modular group is indeed larger and the divisor with respect to the larger group is represented by the two orbits stated above. \( \square \)

We define the four functions

\[
H_{30}^{mA_1} := \frac{F_{36-m}^{mA_1}}{\lambda_46-m} \quad \text{where} \quad m \in \{1, 2, 3, 4\}
\]

whose divisor is represented by the \( \Gamma_m \)-orbit of \( D_{e_1-f_1}^m \). The next Lemma is useful in order to determine the graded ring \( A(\Gamma_0) \).

**Lemma 3.8** Let \( F \in M_k(\Gamma_m, \lambda) \) for some finite character \( \lambda : \Gamma_m \to \mathbb{C}^* \) and \( m = 1, 2, 3, 4 \).

(i) If \( \lambda = \det v_a^b v_x^b \) where \( a, b \in \{0, 1\} \) then \( D_{e_1}^m \leq \text{div}(F) \).

(ii) If \( \lambda = v_a \psi_x^b \) where \( a \in \{0, 1\} \) then \( D_{e_1-\varepsilon_m}^m \leq \text{div}(F) \).

(iii) If \( \lambda = v_2 v_a^b \) where \( a \in \{0, 1\} \) then \( D_{e_1-f_1}^m \leq \text{div}(F) \).

18
Proof
(i) The reflection $\sigma_{\varepsilon_m}$ satisfies $\lambda(\sigma_{\varepsilon_m}) = -1$. Hence its fixed locus $\mathcal{D}^m_{\varepsilon_m}$ is contained in $\text{div}(F)$.

(ii) We have $\lambda(\sigma_{\varepsilon_j - \varepsilon_l}) = -1$ where $j, l$ are distinct numbers in $\{1, \ldots, m\}$ and this reflection induces the permutation $z_j \mapsto z_k, z_k \mapsto z_j$ with respect to our standard basis $\mathcal{O}$. This shows that the fixed locus of this reflection is part of $\text{div}(F)$.

(iii) We consider the reflection $\sigma_1 = \sigma_{\varepsilon_1 - f_1}$. From [22, Section 1.6.3] and the formula for $p$ in the proof of [22, Corollary 1.23] we deduce that $v_2(\sigma_1) = -1$. As $\mathcal{D}^m_{\varepsilon_1 - f_1}$ is exactly the fixed locus of this reflection we are done.

□

Now we are able to prove the main theorem by extending the diagram given in Theorem 2.12.

Proof of Theorem 1.2
Let $F \in \mathcal{A}(\Gamma_m' \cap \Gamma_0)$. Without loss of generality we can assume that $F$ is homogeneous of weight $k$. If the character of $F$ is trivial the assertion follows from Theorem 3.2. In the general situation we can use Lemma 3.8 to divide $F$ by one of the forms $H_{30}, \chi_{mA_1}$ or $\Delta_{mA_1}$ in the case $m \neq 1$. By virtue of Koecher’s principle this process yields a modular form with trivial character and we are back in the first case.

In the case $m = 2, 3, 4$ it is easy to show that the only relations among the generators are induced by the representations of $(\Delta_{mA_1})^2$ and $(H_{30})^2$ as polynomials in the functions given in Theorem 3.2. In the case $m = 1$ our methods yield Igusa’s description.

4 Eisenstein series
Let $L$ be a positive definite even lattice. In [6] the authors investigated vector valued Eisenstein series. We denote the vector valued Eisenstein series of weight $k$ with respect to the simplest cusp by $\vec{e}_{L,k}$, compare [5, p. 23]. We denote by $(\varepsilon_{\gamma})_{\gamma \in \mathcal{D}(L)}$ the standard basis for the group algebra $\mathbb{C}[\mathcal{L}/L]$. For $k \geq 5/2, k \in \mathbb{Z}/2$ these functions are vector valued modular forms with respect to the dual of the Weil representation for $L$ with Fourier expansion

$$\vec{e}_{L,k}(\tau) = \sum_{\gamma \in \mathcal{D}(L)} \sum_{n \geq 0} c(n, \gamma, L, k) \exp(2\pi i n \tau) \varepsilon_{\gamma},$$

where $\tau \in \mathbb{H}$.

In [6] Theorem 4.6] an explicit formula for $c(n, \gamma, L, k)$ is given. There is a well-known correspondence between vector valued modular forms and Jacobi forms for the lattice $L$, compare [27, Proposition 1.6]. Let $m \in \mathbb{N}, m \leq 8$. We define

$$s_m(n, l) := \sharp \{ x \in ((mA_1)_{E_8})^\vee | l + x \in E_8, (l + x, l + x) = 2n \} \in \mathbb{Z}.$$
Proposition 4.1  

(a) Let $$\epsilon$$ and $$\delta$$ be given as follows:

$$\epsilon = \frac{1}{2} \sum_{n \in \mathbb{Z}_{\geq 0}} \sum_{l \in m A_1^\vee} s_m(n, l)q^n r^l.$$  

Since the lattice $$E_8$$ is a maximal even lattice there are no nontrivial isotropic subgroups of $$D(E_8)$$. Hence we can rewrite this quantity in the simplified form

$$s_m(n, l) = \sharp \{x \in ((m A_1) E_8) \setminus \{(l + x, l + x) = 2n\} \mid n \in \mathbb{N}, l \in (mA_1)^\vee\}. \quad (9)$$

We describe the numerical values of the Eisenstein-like Jacobi forms if $$J_{k, mA_1; 1} = \{0\}$$ and $$m \leq 3$$.

Proposition 4.1 (a) Let $$m = 1, 2, 3$$. For any $$n \in \mathbb{Z}_{\geq 0}, l \in (mA_1)^\vee$$ we have the identity

$$2s_m(n, l) = c(n - (l, l)/2, l \bmod L, mA_1, 4 - m/2).$$

The first Fourier coefficients of $$\epsilon_m$$ are given as follows:

$$\epsilon_{4, A_1}(\tau, \zeta) = 1 + (r^{-1} + 56r^{-1/2} + 126 + 56r^{1/2} + r)q$$

$$+ (126r^{-1} + 576r^{-1/2} + 756 + 576r^{1/2} + 126r^2)q^2$$

$$+ (56r^{-3/2} + 756r^{-1} + 1512r^{-1/2} + 2072 + 1512r^{1/2} + 756r + 56r^{3/2})q^3$$

$$+ (\ldots)q^4,$$

$$\epsilon_{4, 2A_1}(\tau, \zeta) = 1 + (60 + 32r(\pm 1/2, 0) + 32r(0, \pm 1/2) + 12r(\pm 1/2, \pm 1/2) + r(\pm 1, 0) + r(0, \pm 1))q$$

$$+ (252 + 192r(\pm 1/2, 0) + 192r(0, \pm 1/2) + 160r(\pm 1/2, \pm 1/2)$$

$$+ 60r(\pm 1, 0) + 60r(0, \pm 1) + 32r(\pm 1, \pm 1/2) + 32r(\pm 1/2, \pm 1) + r(\pm 1, \pm 1))q^2$$

$$+ (\ldots)q^3,$$

$$\epsilon_{4, 3A_1}(\tau, \zeta) = 1 + (26 + 16r(\pm 1/2, 0) + 16r(0, \pm 1/2, 0) + 16r(0, 0, \pm 1/2) + 8r(\pm 1/2, \pm 1/2, 0)$$

$$+ 8r(\pm 1/2, 0, \pm 1/2) + 8r(0, \pm 1/2, \pm 1/2) + 2r(\pm 1/2, \pm 1/2, \pm 1/2)$$

$$+ r(\pm 1, 0) + r(0, \pm 1, 0) + r(0, 0, \pm 1))q$$

$$+ (\ldots)q^2.$$

(b) Let $$m = 1, 2$$ and define the numbers $$r_m(n, l)$$ by

$$\epsilon_{6, mA_1}(\tau, \zeta) = \sum_{n \in \mathbb{Z}_{\geq 0}} \sum_{l \in m A_1^\vee} r_m(n, l)q^n r^l.$$
For any \( n \in \mathbb{Z}_{\geq 0}, l \in (mA_1)^\vee \) we have
\[
\frac{3 \cdot 2^{1-m}}{16 - 2m} \cdot r_m(n, l) \in \mathbb{Z}\pi^2.
\]

**Proof**

We consider the subspace of \( J_{k,mA_1,1} \) consisting of all functions which belong to the kernel of the symmetrization operator \([5]\). Due to Theorem 5.2 this space is one-dimensional if \((m,k)\) is contained in the set
\[
\Omega = \{(1,4), (2,4), (3,4), (1,6), (2,6)\}.
\]

In each case the space is generated by \( \epsilon_{k,mA_1} \). Any zero-dimensional cusp of the modular variety \( \Gamma_m \backslash \mathcal{D}^m \) is represented by a primitive isotropic vector in \( L_2(mA_1) \). Two zero-dimensional cusps are equivalent if they belong to the same \( \Gamma_m \)-orbit. In the cases where \( m < 4 \) all zero-dimensional cusps are equivalent to the simplest cusp. In this case the codimension of the subspace \( J_{k,mA_1;1}^{(\text{cusp})} \subseteq J_{k,mA_1;1} \) is one. In [5] the author defined Eisenstein series for every zero-dimensional cusp represented by a vector \( c \in L^\vee \) such that \((c,c) \in 2\mathbb{Z}\). The corresponding Jacobi forms are denoted by \( e_{L,c}^k \). The span of these functions, as \( c \) runs through all zero-dimensional cusps, constitutes the complementary space of \( J_{k,mA_1;1}^{(\text{cusp})} \). We call the members of this space Jacobi-Eisenstein series. They can be viewed as the natural generalization of the classical Jacobi-Eisenstein series investigated in [10]. For any permutation \( \sigma \in S_m \) the formula for the Fourier coefficients of \( e_{L,c}^k \) yields
\[
\sigma e_{L,c}^k = e_{L,\sigma c}^k.
\]

Since \( \sigma c \) is equivalent to \( c \) this identity shows that \( e_{L,c}^k \) belongs to the kernel of the operator \([5]\) for any \( c \) satisfying \((c,c) \in 2\mathbb{Z}\). Choosing \( c = 0 \) we infer that for any \((m,k) \in \Omega\) there exists some \( \lambda \in \mathbb{C}^* \) such that
\[
\epsilon_{k,mA_1} = \lambda e_{k,0}^{mA_1}.
\]

Using [6, Theorem 4.6] we can express the Fourier coefficients of \( \epsilon_{k,mA_1} \) as special values of \( L \)-functions.

(a) A comparison of the Fourier coefficients yields \( \lambda = \frac{1}{2} \) if \( k = 4 \) and we obtain the numerical values of \( \epsilon_{4,mA_1} \) by evaluating the formula for the Fourier coefficients of \( e_{k,0}^{mA_1} \).

(b) The Fourier coefficient of the constant term of \( \epsilon_{6,A_1} \) equals
\[
-(2\pi i)^2 \det(A_1) \frac{14}{24} = \frac{14}{3}\pi^2.
\]
The Fourier coefficients of \( \frac{1}{2} \epsilon_{6,0} \) are well-known to be integral, see \([10, \text{Theorem 1.2.1}]\) and \([8]\). Now the statement in the case \( m = 1 \) follows by a comparison of the constant term. In the case \( m = 2 \) the constant term of \( \epsilon_{6,2A_1} \) is given by
\[-(2\pi)^2 \det(2A_1) \frac{12}{24} = 8\pi^2.\]

From Lemma \([23]\) we obtain the Fourier expansion of \( \epsilon_{6,2A_1}(\tau, \delta) \) as
\[
(8\pi^2) \cdot 24 G_2(\tau) \cdot \epsilon_{4,2A_1}(\tau, \delta) - \sum_{\substack{n \in \mathbb{Z}_{\geq 0} \\ell \in (2A_1)^\vee \\quad 2n - (l,l) \geq 0}} \alpha(n,l) q^n r^j
\]
where
\[
\alpha(n,l) := (8\pi^2) \cdot (4n - 2(l,l)) s_m(n,l).
\]

Since \((4n - 2(l,l)) \in \mathbb{Z}\) for all \( n \in \mathbb{Z}_{\geq 0}, l \in (2A_1)^\vee \) we obtain the assertion as a consequence of formula \([11]\) and \([9]\). \(\square\)

The last Proposition justifies the notion Eisenstein type. In the cases considered there the space complementary to the space of cusp forms is always one-dimensional. If \( m = 4 \) there is no Jacobi-Eisenstein series to be considered since \( 4 - m/2 = 2 < 5/2 \). In this case \( \epsilon_{4,4A_1} \) is the correct replacement for the missing Eisenstein-series. Moreover there are two inequivalent zero-dimensional cusps. Here the complementary space is two-dimensional. If \( k = 4 \) the first generator of this space is given by the Eisenstein type modular form \( \mathcal{E}^A_4 \) and the second generator coincides with the square of \( \chi_{2A_1}^4 \) which is the main function of the \( A_1 \)-tower. By virtue of the identities
\[
\sharp \{ l \in E_7 \mid (l,l) = 2n \} = c(n,0,A_1,7/2)/2
\]
\[
\sharp \{ l \in D_6 \mid (l,l) = 2n \} = c(n,0,2A_1,3/2)
\]
\[
\sharp \{ l \in A_1 \oplus D_4 \mid (l,l) = 2n \} = c(n,0,3A_1,5/2)/2
\]
Proposition \([11]\) yields a new description of these representation numbers of quadratic forms as special values of \( L \)-functions.

References

[1] W.L. Baily. *Introductory lectures on automorphic forms*, volume 1. Princeton University Press and Iwanami, Shoten, 1973.

[2] W.L. Baily and A. Borel. Compactification of arithmetic quotients of bounded symmetric domains. *Ann. of Math.*, 84(2):442–528, 1966.
3] R.E. Borcherds. Automorphic forms on $O_{s+2,2}(R)$ and infinite products. *Invent. Math.*, 120:161–213, 1995.

[4] A. Borel and L. Ji. *Compactifications of symmetric and locally symmetric spaces*. Mathematics: Theory & Applications. Birkhäuser, Boston, 2006.

[5] J. H. Bruinier. *Borcherds Products on O(2, l) and Chern classes of Heegner Divisors*, volume 1. Springer-Verlag, Berlin, Heidelberg, 2002.

[6] J. H. Bruinier and M. Kuss. Eisenstein series attached to lattices and modular forms on orthogonal groups. *Manuscr. Math.*, 106:443–459, 2001.

[7] F. Clery and V.A. Gritsenko. Modular forms of orthogonal type and Jacobi theta-series. *Abh. Math. Semin. Univ. Hambg.*, 83:187–217, 2013.

[8] H. Cohen. Sums Involving the Values at Negative Integers of L-Functions of Quadratic Characters. *Ann. of Math.*, 217:271–285, 1975.

[9] T. Dern and A. Krieg. Graded Rings of Hermitian Modular Forms of Degree 2. *Manuscr. Math.*, 110:251–272, 2003.

[10] M. Eichler and D. Zagier. *The Theory of Jacobi Forms*. Birkhäuser, Boston, Basel, Stuttgart, 1985.

[11] E. Freitag. Modulformen zweiten Grades zum rationalen und Gauß’schen Zahlkörper. *Sitzungsberichte der Heidelberger Akad.d.Wiss.*, 1.Abh., 1967.

[12] E. Freitag and C.F. Hermann. Some modular varieties of low dimension. *Advances in Mathematics*, 152:203–287, 2000.

[13] B. Grandpierre. Produits automorphes, classification des reseaux et theorie du codage. PhD thesis, Lille 2009.

[14] V.A. Gritsenko. Reflective modular forms in Algebraic geometry. *ArXiv: 1005.3753 [math.AG]*. 28 pp.

[15] V.A. Gritsenko. Modular forms and moduli spaces of abelian and K3 surfaces. *St. Petersburg Math. J.*, 6(6):1179–1208, 1995.

[16] V.A. Gritsenko and K. Hulek. Minimal Siegel Modular Threefolds. *Math. Proc. Cambridge Philos. Soc.*, 123:461–485, 1998.

[17] V.A. Gritsenko, K. Hulek, and G.K. Sankaran. The Kodeira dimension of the moduli of K3 surfaces. *Invent. Math.*, 169:519–567, 2007.

[18] V.A. Gritsenko, K. Hulek, and G.K. Sankaran. *Moduli of K3 Surfaces and Irreducible Symplectic Manifolds*, volume 24 of *Adv. Lect. Math. (ALM)*. Int. Press, Somerville, MA, 2013.
[19] V.A. Gritsenko and V. Nikulin. Automorphic forms and Lorentzian Kac-Moody algebras. II. *International J. Math.*, 9(1):201–275, 1998.

[20] J.I. Igusa. On Siegel Modular Forms of Genus Two. *Amer. Journal of Math.*, 84(1):175–200, 1962.

[21] J.H. Conway and N.J.A. Sloane. *Sphere Packings, Lattices and Groups*, volume 21. Springer-Verlag, New York, Berlin, Heidelberg, London, Paris, Tokyo, 1988.

[22] I. Klöcker. Modular Forms for the Orthogonal Group O(2, 5). PhD thesis, Aachen 2005.

[23] A. Krieg. The graded ring of quaternionic modular forms of degree 2. *Math Z.*, 251:929–942, 2005.

[24] A. Krieg. Another Quaternionic Maass Space. *Number Theory. Ramanujan Math. Society, Lect. Notes Series*, 15:43–50, 2011.

[25] H.V. Niemeier. Definite quadratische Formen der Dimension 24 und Diskriminante 1. *J. Number Theory*, 5:142–178, 1973.

[26] V. Nikulin. Integral symmetric bilinear forms and some of their applications. *Math. USSR Izvestija*, 14(1):103–167, 1980.

[27] T. Shintani. On construction of holomorphic cusp forms of half integral weight. *Nagoya Math. J.*, 58:83–126, 1975.

[28] M. Woitalla. A framework for some distinguished series of orthogonal type modular forms. PhD thesis, Aachen 2016.

[29] M. Woitalla. Theta type Jacobi forms. *ArXiv: 1705.04526 [math.AG]*. To appear in *Acta Arith*.

[30] Y. Choie and H. Kim. Differential operators and Jacobi forms of several variables. *J. Number Theory*, 82:40–63.