The Muon \((g - 2)\) Spin Equations, the Magic \(\gamma\),
What’s small and what’s not.

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July 3, 2018

Abstract

We review the spin equations for the muon in the 1.45 T muon \((g - 2)\) storage ring, now relocated to Fermilab. Muons are stored in a uniform 1.45 T magnetic field, and vertical focusing is provided by four sets of electrostatic quadrupoles placed symmetrically around the storage ring. The storage ring is operated at a Lorentz factor centered on the “magic \(\gamma = 29.3\)”; the effect of the electric field on the muon spin precession cancels for muons at the magic momentum. We point out the relative sizes of the various terms in the spin equations, and show that for experiments that use the magic \(\gamma\) and electric quadrupole focusing to store the muon beam, any proposed effect that multiplies either the motional magnetic field \(\vec{\beta} \times \vec{E}\) or the muon pitching motion \(\vec{\beta} \cdot \vec{B}\) term, will be smaller by three or more orders of magnitude, relative to the spin precession due to the storage ring magnetic field. We use a recently proposed General Relativity correction [1] as an example, to demonstrate the smallness of any such contribution, and point out that their revised preprint [7] still contains a conceptual error, that significantly overestimates the magnitude of their proposed correction. We have prepared this document in the hope that future authors will find it useful, should they wish to propose corrections from some additional term added to the Thomas equation, Eq. 13, below. Our goal is to clarify how the experiment is done, and how the small corrections due to the presence of the radial electric field and the vertical pitching motion of the muons (betatron motion) in the storage ring are taken into account.

1 Introduction

In a recent preprint [1], Morishima et al., calculated a potential general relativity (GR) effect on the frequency of muon spin precession in a magnetic field. Several authors [2, 3, 4, 5, 6], have posted papers on arXiv, questioning various aspects of these calculations, and others have done so in private communications to us. More recently Morishima et al. updated their paper [7]. It is clear in both papers that these authors have misinterpreted the subtleties of the experimental technique, which if properly understood, would have prevented them from claiming such a large correction to the measured muon spin precession in their original paper [1], which is perpetuated in the update [7].

In this paper we examine the spin precession formulae, and calculate the magnitude of the electric field term for the Brookhaven National Laboratory (BNL) E821 experiment. To simplify this narrative, we have relegated details of the beam dynamics in the storage ring to appendices. The appendices include the derivation of corrections to the spin precession frequency from the electric field used to provide vertical focusing, along with the derivation
of the correction of the vertical pitching motion of the beam, to appendices. These two effects give rise to \( \simeq 0.5 \) ppm and \( \simeq 0.3 \) ppm corrections respectively.

First we briefly put the physics motivation of, and the results from the Brookhaven Muon \((g - 2)\) experiment in context, and then explain how the assumptions in Refs. [11, 7], or any other ppm or less effect, are not a relevant concern for the interpretation of the BNL E821 results, or from the ongoing muon \((g - 2)\) experiment, E989 at Fermilab.

A spin 1/2 lepton \((\ell = e, \mu, \tau)\) has an intrinsic magnetic moment due to its spin, given by the relationship

\[
\bar{\mu}_\ell = g_\ell \frac{Qe}{2m_\ell} \mathbf{s}, \quad g_\ell = 2(1 + a_\ell), \quad a_\ell = \frac{g_\ell - 2}{2},
\]

(1)

where \(Q = \pm 1, \; e > 0\) and \(m_\ell\) is the lepton mass. Dirac theory predicts that \(g = 2\), but experimentally, it is known to be greater than 2. This deviation from \(g = 2\), \(a_\ell\) in Eq. (1) is the magnetic anomaly, which arises from radiative corrections (quantum fluctuations).

In the Standard Model, \(a_\mu\) gets measurable radiative contributions from QED, the strong interaction, and from the electroweak interaction \([8, 9, 10, 11, 12, 13]\),

\[
a^{SM}_{\mu} = a_{QED} + a_{Had} + a_{Weak},
\]

(2)

which are shown diagrammatically in Fig. 1 along with the magnitudes of these contributions.

![Feynman graphs showing contributions to g for each of the Standard Model forces, ordered by size: (a) The Dirac interaction. (b) The lowest-order QED term \(\alpha/2\pi\), which dominates the value of the anomaly. (c) The hadronic vacuum polarization contribution. (d) The lowest-order electroweak contributions. (The one-loop Higgs contribution is negligible.) (e) Potential contribution from new BSM particles X and Y.](image)

The Standard Model value, \(a^{SM}_{\mu}\), has been theoretically calculated with an uncertainty of about \(\pm 0.3\) ppm \([10, 11, 12]\). The largest contribution comes from the mass-independent single-loop (Schwinger) diagram \([8]\,\text{shown in Fig. 1(b)}\). Should the experimental value \([14]\,

\[
a^{E821}_{\mu} = 116592091(63) \times 10^{-11}
\]

(3)

differ from the Standard Model value at a statistically significant level, it would reflect additional contributions from as yet undiscovered particles beyond those of the Standard Model \([15, 16]\). Using the hadronic contributions from Refs. [10, 11, 12], the difference between experiment and the Standard Model theory is positive, with a statistical significance between 3.5 and 4 standard deviations.

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\(^1\) With his famous calculation that obtained \(a = (\alpha/2\pi) = 0.00116\cdots\), Schwinger started an “industry”, which required Aoyama, Hayakawa, Kinoshita and Nio to calculate more than 12,672 diagrams to evaluate the tenth-order (five loop) contribution \([9]\).
2 Past Experiments and the Spin Equations

2.1 The Basics

When placed in a magnetic field at rest, the muon undergoes Larmor precession,

$$\vec{\omega}_{L\mu} = -g_\mu \frac{Qe}{2m} \vec{B},$$

(4)

where $e$ is the magnitude of the electron charge, $Q = \pm 1$ and $m$ is the muon mass. The very first measurements of the muon magnetic moment were done at rest [17, 18, 19]. The first two determined that $g$ was consistent with 2. The more precise third experiment demonstrated that $a_\mu$ was consistent with $\alpha/2\pi$, demonstrating that a muon behaved like an electron when placed in a magnetic field. The paper by Cassels et al. [18] pointed out that if the experiment were to be done with muons in flight, the difference between the spin precession frequency and the cyclotron frequency would depend directly on the radiative corrections, rather than on $g$. For this reason, all subsequent experiments were done with in-flight muons in a magnetic field, in order to measure $a_\mu$ directly.

In the simplest case, in the absence of an electric field and when the muon velocity is perpendicular to a uniform magnetic field, the rate at which the spin turns relative to the momentum is given by the difference between the spin rotation frequency $\omega_S$ and the cyclotron frequency $\omega_C$,

$$\vec{\omega}_{a_\mu} = \vec{\omega}_S - \vec{\omega}_C = -g_\mu \frac{Qe\vec{B}}{2m} - (1 - \gamma) \frac{Qe\vec{B}}{\gamma m} + \frac{Qe\vec{B}}{\gamma m} = -\left(\frac{g_\mu - 2}{2}\right) \frac{Qe}{m} \vec{B} = -a_\mu \frac{Qe}{m} \vec{B},$$

(5)

and is directly proportional to $a_\mu$.

2.1.1 Determination of $a_\mu$ from $\omega_{a_\mu}$

The experimental value of $a_\mu$ is derived from

$$\omega_{a_\mu} = \frac{e}{m_\mu} a_\mu B = \frac{2a_\mu \omega_{L\mu}}{g_\mu} = \frac{a_\mu \omega_{L\mu}}{1 + a_\mu},$$

(6)

with the appropriate small corrections such as the electric field and pitch corrections described in the appendices. This can be written as

$$a_\mu = \frac{\omega_{a_\mu}}{\omega_{L\mu} - \omega_{a_\mu}},$$

(7)

where we have used

$$a_\mu = \frac{g_\mu - 2}{2}, \quad \omega_{L\mu} = g_\mu \frac{e}{2m_\mu} B, \quad \hbar \omega_{L\mu} = 2|\mu_\mu| B.$$  

(8)

and $B = |B|$.

The $(g - 2)$ experiments measure two frequencies, $\omega_{a_\mu}$ and $\omega_{L\mu}$, the latter being the Larmor frequency of protons in nuclear magnetic resonance (NMR) probes used to monitor...
the magnetic field, which are calibrated to the Larmor frequency of a free proton.

\[ \omega_{Lp} = g_p \frac{e}{2m_p} B \quad \text{and} \quad \hbar \omega_{Lp} = 2\mu_p B. \]  

(9)

Dividing the numerator and denominator in Eq. 7 by \( \omega_{Lp} \), we get

\[ a_\mu = \frac{\omega_{a_\mu}}{\omega_{Lp}}/|\mu_\mu|/\mu_p - \frac{\omega_{a_\mu}}{\omega_{Lp}} = \frac{R}{\lambda - R}, \]  

(10)

where \( R = \omega_{a_\mu}/\omega_{Lp} \) was measured experimentally at BNL by E821 [14], and \( \lambda = \mu^+ / \mu_p = 3.183345142(71) \) [20], determined from muonium hyperfine structure splitting [21]. This equation was used to determine the value of \( a_\mu \) in Brookhaven E821, which required the assumption of CPT invariance to determine \( a_\mu^- \) [14].

Fermilab E989 proposes to use the equivalent combination of constants [22] rather than \( \lambda \),

\[ a_\mu = \left( \frac{g_e}{2} \right) \left( \frac{m_\mu}{m_e} \right) \left( \frac{\mu_p}{\mu_e} \right) \left( \frac{\omega_{a_\mu}}{\omega_{Lp}} \right), \]  

(11)

to determine \( a_\mu \) from the two measured frequencies. The additional ratios appearing in Eq. 11 are well known from other experiments: \( g_e/2 = 1.00115965218073(28) \) (0.28 ppt) [23], \( m_\mu/m_e = 206.7682826(52) \) (22 ppb) [20, 21] and \( \mu_p/\mu_e = 658.2106866(20) \) (3 ppb) [20].

2.2 Vertical Focusing with an Electric Field

In a real experiment, the magnetic field that appears in Eq. 5 must be averaged over the muon ensemble, which consists of a range of momenta and positions in the storage ring. This averaging requires detailed information on the muon orbits, and the value of the magnetic field at each location \((x, y, z)\) in the muon orbits. Of three \((g - 2)\) experiments at CERN [25, 26, 27], the first two used magnetic gradients to contain the muons, the second being a storage ring experiment. However, to go beyond the precision of the second CERN experiment, it became necessary to find a different way to provide vertical focusing for the stored muon beam, since the presence of gradients in the magnetic field made it difficult to know the magnetic field averaged over the muon beam distribution to the parts per million (ppm) level.

To overcome this issue, the third CERN collaboration [28, 29] realized that one could employ a uniform magnetic dipole field and in addition an electrostatic quadrupole field for vertical focusing [2]. To a relativistic particle, an electric field is perceived to be a combination of electric and magnetic fields. The resulting motional magnetic field (MMF) through the \( \vec{\beta} \times \vec{E} \) term can cause the spin to precess. For the more general case where \( \vec{\beta} \cdot \vec{B} \neq 0 \) and \( \vec{E} \neq 0 \), the cyclotron rotation frequency becomes:

\[ \tilde{\omega}_C = -\frac{Qe}{m} \left( \frac{\vec{B}}{\gamma} - \frac{\gamma}{\gamma^2 - 1} \left( \frac{\vec{\beta} \times \vec{E}}{c} \right) \right), \]  

(12)

\footnote{This arrangement of magnetic and electric fields is essentially a Penning trap.}
and the spin rotation frequency becomes

\[ \vec{\omega}_S = -\frac{Qe}{m} \left[ \left( \frac{g}{2} - 1 + \frac{1}{\gamma} \right) \vec{B} - \left( \frac{g}{2} - 1 \right) \frac{\gamma}{\gamma + 1} (\vec{\beta} \cdot \vec{B}) \vec{\beta} - \left( \frac{g}{2} - \frac{\gamma}{\gamma + 1} \right) \left( \frac{\vec{\beta} \times \vec{E}}{c} \right) \right]. \]  

(13)

This equation was first discovered by L.H. Thomas in 1927 \cite{30}. We use the form given in Eq. 11.170 in Jackson’s text \cite{31}, which is equivalent to Thomas’ Eq. 4.121 in Ref. \cite{30}, but in modern notation.

Using \( a_\mu = (g_\mu - 2)/2 \), we find that the spin difference frequency is

\[ \vec{\omega}_{\text{diff}} = \vec{\omega}_S - \vec{\omega}_C \simeq \omega_{a_\mu} = -\frac{Qe}{m} \left[ a_\mu \vec{B} - a_\mu \left( \frac{\gamma}{\gamma + 1} \right) (\vec{\beta} \cdot \vec{B}) \vec{\beta} - \left( a_\mu - \frac{1}{\gamma^2 - 1} \right) \frac{\vec{\beta} \times \vec{E}}{c} \right]. \]  

(14)

With the presence of the electric field in the third term, and the fact that the velocity is not perpendicular to the magnetic field in the second term, small corrections must be accounted for in the determination of the muon anomaly, which are derived below in Appendices II and III.

### 2.3 The Magic \( \gamma \) and a correction to the \( \vec{\beta} \times \vec{E} \) term

The discussion in Ref. \cite{1} is centered around the spin Eq. (14) above, which is used to describe the muon spin motion relative to the momentum vector in the Brookhaven muon \((g - 2)\) storage ring \cite{37, 38}. Under the approximation that the muon velocity is perpendicular to the magnetic field \((\vec{\beta} \cdot \vec{B} = 0)\), Eq. (14) reduces to

\[ \vec{\omega}_a = -\frac{Qe}{m} \left[ a_\mu \vec{B} - a_\mu \left( \frac{1}{\gamma^2 - 1} \right) \frac{\vec{\beta} \times \vec{E}}{c} \right]. \]  

(15)

If the value of the relativistic \( \gamma \)-factor is chosen to be \( \gamma_m = 29.304 \), \( p_m = 3.09 \text{ GeV}/c \), then

\[ \left( a_\mu - \frac{1}{\gamma^2 - 1} \right) = 0, \]  

(16)

and the \( \vec{\beta} \times \vec{E} \) term does not contribute to the muon spin precession. However, a well understood correction must be made to the measured spin frequency to account for the fact that not all muons are at the magic \( \gamma \) (see Appendix II). In the storage ring of the E821 and E989 experiments, there is an ensemble of muons with a range of momenta determined by the momentum acceptance of the storage ring, which is \( \pm 0.5\% \). Therefore, with the muons centered about the magic \( \gamma \), the maximum \( \gamma \) range of the stored muons is \( \gamma = \gamma_m \pm 0.5\% \). Due to the phase space acceptance of the storage ring, the distribution of momenta is approximately Gaussian with a width, \( \sigma \simeq 0.15\% \), as shown in Fig. 2.

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3Bargmann, Michel and Telegdi \cite{32} also studied this problem.

4Strictly speaking, the rate of change of the angle between the spin and the momentum vectors, \( |\vec{\omega}_{a_\mu}| \) = ‘precession frequency’, is equal to \( |\vec{\omega}_{\text{diff}}| \) only if \( \vec{\omega}_S \) and \( \vec{\omega}_C \) are parallel. For the E821 and E989 experiments, the angle between \( \vec{\omega}_S \) and \( \vec{\omega}_C \) is always small and the rate of oscillation of \( \vec{\beta} \) out of pure circular motion is fast compared to \( \omega_{a_\mu} \), allowing us in the following discussion the make the approximation that \( \vec{\omega}_{a_\mu} \simeq \vec{\omega}_{\text{diff}} \). More general calculations, where this approximation is not made, are found in References \cite{33, 34, 35, 36}. For the E821 and E989 experimental conditions, the results presented here are the same as those in these references.
Figure 2: The measured distribution of equilibrium radii in the \((g - 2)\) storage ring. This distribution was used to determine the radial electric field correction that is derived in Appendix II. (From Bennett et al. [14].)

Suppose that an effect \(\xi\), such as that proposed in Ref. [1], multiplies the \(\vec{\beta} \times \vec{E}\) term in Eq. 14. The question is: How would this term \(\xi\) change the spin difference precession frequency? To get an order of magnitude estimate we consider the term:

\[
\left[ \left( a_{\mu} - \frac{1}{\gamma^2 - 1} \right) \pm \xi \right] \frac{\vec{\beta} \times \vec{E}}{c} = \left( a_{\mu} - \frac{1}{\gamma^2 - 1} \right) \frac{\vec{\beta} \times \vec{E}}{c} \pm \xi \frac{\vec{\beta} \times \vec{E}}{c}
\]  

The reason that we separate these two effects is because the first term in Eq. 18 is used to calculate the radial electric field correction as we explain in Appendix II, and the effect of the \(\xi\) term must be calculated separately. So any physical effect that would change the muon spin difference frequency must be multiplied separately by the value of \((\vec{\beta} \times \vec{E})/c\) as indicated in Eq. 18. This is an important point, which Morishima et al. missed, as they worried about the magic \(\gamma\) cancellation for each muon in the beam.

We use the proposed correction of Morishima et al. [1], where \(\xi = -2.8 \times 10^{-9}\), as an example and we leave the the correctness of their General Relativity derivation for other authors to discuss.

2.4 Values of the E821 pitch and radial \(\vec{E}\) field corrections

Before we calculate the value of \((\vec{\beta} \times \vec{E})/c\) in Section 3, we present the corrections made to the E821 data for the vertical pitching of the beam (the \(\vec{\beta} \cdot \vec{B}\) term), and for the radial electric field (\(\vec{\beta} \times \vec{E}\) term). See Ref. [14] and the appendices below for more details. In each case, the correction requires that we increase the measured value of \(\omega_n\). For the 2001 data collection period [14], where data were collected at two different \(n\) values, the electric field corrections were \(C_E(\text{low} - n) = 0.47 \pm 0.054 \text{ ppm}\) and \(C_E(\text{high} - n) = 0.50 \pm 0.054 \text{ ppm}\). The

\footnote{This misunderstanding became clear in a discussion with two of the authors while BLR was visiting the University of Nagoya on February 16, 2018. Unfortunately, their revised preprint still contains this conceptual error.}

\footnote{Their proposed correction of \(2.8 \times 10^{-9}\) is roughly twice the electroweak contribution to \(a_{\mu}\) of \(1.54 \times 10^{-9}\) (1.31 ppm).}
pitch corrections were $C_P(\text{low} - n) = 0.27 \pm 0.036$ ppm and $C_P(\text{high} - n) = 0.32 \pm 0.036$ ppm. These corrections were the only corrections made to the muon spin precession frequency in E821, and their magnitude demonstrates that any additional effects that multiply $\vec{\beta} \cdot \vec{B}$ or $\vec{\beta} \times \vec{E}$ should be expected to be small.

3 Order of magnitude of the effect of $\xi$ on $\omega_{a_\mu}$

3.1 Correction from the Motional Magnetic field

An order of magnitude calculation, demonstrates that a $\xi = 2.8 \times 10^{-9}$ contribution to the motional magnetic field term ($\vec{\beta} \times \vec{E}$ term) makes a negligible contribution to the measurement of $a_\mu$ relative to the 0.5 ppm (BNL), and 0.14 ppm (projected Fermilab sensitivity) levels of precision.

The principal feature that must be recognized when calculating a new effect on the spin difference frequency, is that $\omega_{a_\mu}$ is dominated by the $a_\mu B$ term. Any quantity that multiplies the $\vec{\beta} \cdot \vec{B}$ or $\vec{\beta} \times \vec{E}$ term will be small, compared to the spin precession caused directly by the 1.45 T storage ring magnetic field,

$$- \frac{Qe}{m} a_\mu \vec{B}.$$  \hfill (19)

For the correction reported in Ref. [1], where $\xi = -2.8 \times 10^{-9}$, the shift in $\omega_{a_\mu}$ would be

$$- 2.8 \times 10^{-9} \left\langle \frac{\vec{\beta} \times \vec{E}}{c} \right\rangle_{\text{avg}},$$  \hfill (20)

where we have ignored for the moment the common factor $Qe/m$. The average refers to an average over the distribution of stored muons. This is the term that Refs. [1, 7] claim causes a sizable shift in the value of $a_\mu$. We now show in detail that their conclusion is incorrect, and in fact that this term is negligibly small compared to the dominant contribution to $\omega_{a_\mu}$, which comes from the term in Eq. (19).

The “magic momentum” is $p_m \approx 3.09$ GeV/c. The distribution of momenta in the storage ring is approximately Gaussian with $\sigma \approx 0.15\%$. With that range of momenta the $E$-field correction is about 0.5 ppm, which in E821 was known to $\approx 10\%$. We do not learn anything with any accuracy on $a_\mu$ by tuning for the magic momentum. Nearly all the information on $a_\mu$ is derived from the value of $\omega_{a_\mu}$, since the two are proportional to each other, due to the dominance of the $B$-field term on the spin precession frequency. The value of $\omega_{a_\mu}$ is completely insensitive to the magic $\gamma$ used in the $\vec{\beta} \times \vec{E}$, except at the fraction of a ppm level. Therefore we derive no information on $a_\mu$ from the $E$-field term because of the smallness of the $\vec{\beta} \times \vec{E}/c$ term. For example, tuning the momentum until the coefficient of the $E$-field term is zero will tell us nothing about the value of $a_\mu$ at the desired ppm sensitivity. In fact, because of the width of the momentum distribution, which is significantly larger than the ppm level, one has to treat the ensemble of muons as a whole, with the correction given in Appendix II.

To calculate the magnitude of the frequency change caused by the term $\xi$ in Eq. (19) has on the muon spin precession, we need to evaluate $\vec{\beta} \times \vec{E}$, averaged over the population of stored
muons. In the muon storage ring, the muon velocity (0.9994 c) is dominantly along the ring azimuth. We only need to consider the radial component of $\vec{E}$, since $\vec{\beta} \times \vec{E}$ is directed along the $B$-field and therefore is the only component that contributes to the precession frequency, $\omega_a \mu$, to first order.

The electric potential for a quadrupole electric field, with the poles situated along the radial and vertical ($B$-field) axes, is

$$V = -\frac{1}{2} \kappa (x^2 - y^2) \quad (21)$$

where $\kappa$ is the electric field gradient, $x = r - r_0$, $r$ is the radius of the muon trajectory perpendicular to the $B$-field, and $r_0 = 7.112$ m is the central radial position in the storage region. The radial field is given by

$$E_r \simeq E_x = -\frac{\partial V}{\partial x} = \kappa x. \quad (22)$$

The electric quadrupoles are split into four equal segments azimuthally around the storage ring, covering about 43% of the ring.

For the E821 experiment, for positive muon storage, typically $V \simeq -25,000$ V at the horizontal electrodes located at $(r_0 - 0.05)$ m and $(r_0 + 0.05)$ m. Similarly the vertical electrodes have $V = +25,000$ V at $y = 0.05$ m and $y = -0.05$ m. Thus the electric field gradient is

$$\kappa = \frac{2 \times 25000 \text{ V}}{(0.05)^2 \text{ m}^2} = 2 \times 10^7 \text{ V/m}^2. \quad (23)$$

The typical radial electric field as a function of $x$ is therefore

$$E_r = E_x = -\frac{\partial V}{\partial x} = \kappa x = 2 \times 10^7 \times x \text{ V/m}. \quad (24)$$

The average value of $\vec{\beta} \times \vec{E}$ is simply $\beta = 0.9994$ times the average value of $E_r$. The average radial position of the stored muon beam in E821 is about 7.115 m, or 0.003 m larger than the central ring radius. The average value of $E_r$ is reduced by a factor of 0.43 to account for the partial coverage of electric quadrupoles inside the ring, thus the average value

$$(E_r)_{avg} = 0.43 \times 0.003 \times 2 \times 10^7 = 2.6 \times 10^4 \text{ V/m}. \quad (25)$$

The relative contribution of $\xi = -2.8 \times 10^{-9}$ to the precession frequency, $\Delta \omega_{a\mu}/\omega_{a\mu}$, which is very nearly $\Delta a_{\mu}/a_{\mu}$, is thus given by dividing the $\xi$ contribution to the $E$-field term, by the much larger $B$-field term, Eq. [19]

$$\frac{\Delta a_{\mu}}{a_{\mu}} = \frac{\xi \times \beta \times (E_r)_{avg}}{cBa_{\mu}} \quad (26)$$

To calculate a numerical value, we use $B = 1.45$ T, $\beta = 0.9994$, $a_{\mu} = 1.1659 \times 10^{-3}$. Using these values and $\xi = 2.8 \times 10^{-9}$ one gets:

$$\frac{\Delta a_{\mu}}{a_{\mu}} = 1.4 \times 10^{-10}, \quad (27)$$
or about 0.14 ppb.

We have shown that the proposed GR contribution of Eq. 40 of Ref. [1], and in section 3.4 of Ref. [7] when applied correctly to the storage-ring experiments with electric quadrupole focusing, is completely negligible when compared to the experimental uncertainty of 0.54 ppm of BNL E821, or to the uncertainty of 0.140 ppm expected at Fermilab. In order to have an observable effect on the determination of $a_{\mu}$, one must have $\xi \geq 1 \times 10^{-6}$. Thus the conclusions in section 3 of Ref. [1] and section 3.4 of Ref. [7] are incorrect.

3.2 Correction from the $\vec{\beta} \cdot \vec{B}$ term

We now briefly consider the vertical pitching motion, which introduces a term proportional to $\vec{\beta} \cdot \vec{B}$,

$$\vec{\omega}_{a_{\mu}} \simeq \vec{\omega}_{diff} = -\frac{Qe}{m} \left[ a_{\mu} \vec{B} - a_{\mu} \left( \frac{\gamma}{\gamma + 1} \right) (\vec{\beta} \cdot \vec{B}) \vec{\beta} \right]$$

(28)

into the spin difference equation. As mentioned above this term caused a small change of $\simeq 0.3$ ppm correction to the precession frequency, which is derived in Appendix III. The situation is similar to the motional magnetic field correction, viz. an additional term would also multiply a small number that is on the order of 0.3 ppm, and if it is the same order of magnitude of $\xi$ discussed above would have a negligible effect on the spin precession.

4 Summary

We have outlined the approximate expressions for spin precession used to derive $a_{\mu}$ from experimental data in BNL E821. We have included discussion of corrections associated with the focusing electric field and the betatron motion of muons in the storage ring. We have shown that any small effect $\xi$, perhaps from general relativity or from some other source, that multiplies the $(\vec{\beta} \times \vec{E})$ term in the spin equations will not contribute to the muon spin rotation at a measurable level, unless it is greater than $10^{-6}$. For larger values, the exact details would need to be understood, in order to calculate the effect on the $(g - 2)$ measurement. Thus the $\xi = 2.8 \times 10^{-9}$ contribution proposed by Morishima, et al. [1, 7] does not contribute at a measurable level to the muon $(g - 2)$ experiments, since when calculated correctly, its effect on the measurement is at the tenths of a part per billion level, three orders of magnitude smaller than the expected experimental precision of the Fermilab E989 final result. Therefore, the conclusions in §3 of Ref. [1], and conclusions in §4.3, including Eq. 77 of Ref. [7] are incorrect. The authors have misinterpreted the means by which the $(g - 2)$ experiments have determined $a_{\mu}$, and have consequently greatly overestimated the size of the correction based on their proposed GR term.

5 Acknowledgments

We wish to thank Gianguido Dall’Agata, Alexander Keshavarzi and Massimo Passera for useful comments on this manuscript, and our g-2 collaborators for helpful conversations. This work was supported in part by the U.S. Department of Energy, Office of High Energy Physics.
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Appendix 1: Beam dynamics in the \((g-2)\) storage ring.

In E821 at BNL, and in the upcoming experiment at Fermilab, muons are stored in a magnetic storage ring \([37]\) with a uniform magnetic field and electrostatic quadrupoles \([38]\), shown in Fig. 3 to provide vertical focusing. With the weak focusing electric quadrupoles, the \((g-2)\) storage ring acts like a weak-focusing betatron. \([22]\). The muons execute harmonic motion in both the vertical and horizontal directions as they go around the storage ring.

An important quantity of the storage ring is the field index \(n\) which is given by:

\[
n = \frac{\kappa R_0}{vB_0} \tag{29}
\]

where \(\kappa\) is the quadrupole gradient strength. The frequencies of the radial \((x)\) and vertical \((y)\) betatron motions are determined by the field index:

\[
f_x = f_C \sqrt{1 - n} \simeq 0.929 f_C \quad \text{and} \quad f_y = f_C \sqrt{n} \simeq 0.37 f_C, \tag{30}
\]

where \(f_C = 6.7\) MHz is the cyclotron frequency. A significant amount of the E821 data were taken with \(n = 0.137\), where the potentials on the quadrupole plates were \(\pm 25\) kV. The muon spin precession frequency is \(f_a = 229.073\) kHz.

The field index also determines the angular acceptance of the ring. The maximum horizontal and vertical angles of the muon momentum are given by

\[
\theta_x^{\text{max}} = \frac{x_{\text{max}} \sqrt{1 - n}}{R_0}, \quad \text{and} \quad \theta_y^{\text{max}} = \frac{y_{\text{max}} \sqrt{n}}{R_0}, \tag{31}
\]

where \(x_{\text{max}}, y_{\text{max}} = 45\) mm is the radius of the storage aperture, and \(R_0 = 7.112\) m. For a betatron amplitude \(A_x\) or \(A_y\) less than 45 mm, the maximum angle is reduced, as can be seen from the above equations.
Appendix II: The radial electric field correction

We consider the $\vec{\beta} \times \vec{E}$ term first. For muons not at the magic radius (magic $\gamma$) we have the following correction.

$$\omega'_a = \omega_a \left[ 1 - \beta \frac{E_r}{cB_y} \left( 1 - \frac{1}{a_{\mu} \beta^2 \gamma^2} \right) \right], \quad (32)$$

where $\omega_a = -a_{\mu} \frac{Qe}{m} B$. Using $p = \beta \gamma m = (p_m + \Delta p)$. Note that the quantity $(E_r/c)B_y$ is the ratio of the motional magnetic field to the 1.45 T storage ring field.

After some algebra one finds

$$\frac{\omega'_a - \omega_a}{\omega_a} = -2 \frac{\beta E_r}{cB_y} \frac{\Delta p}{p_m}, \quad (33)$$

Now

$$\frac{\Delta p}{p_m} = (1 - n) \frac{\Delta R}{R_0} = (1 - n) \frac{x_e}{R_0}, \quad (34)$$

where $x_e$ is the muon’s equilibrium radius of curvature relative to the central orbit. The mean electric quadrupole field experienced by the muon is

$$\langle E_r \rangle = \kappa x = \frac{n \beta c B_y}{R_0} x_e, \quad (35)$$

where $x_e$ is the equilibrium radius of the muon. We obtain

$$\frac{\Delta \omega}{\omega} = -2n(1 - n) \beta^2 \frac{x_e}{R_0^2}, \quad (36)$$

so the effect of muons not at the magic momentum is to lower the observed frequency. For a quadrupole focusing field plus a uniform magnetic field, the time average of $x$ is just $x_e$, the equilibrium radius, so the electric field correction is given by

$$C_E = \frac{\Delta \omega}{\omega} = -2n(1 - n) \beta^2 \frac{\langle x_e^2 \rangle}{R_0^2}, \quad (37)$$

where $\langle x_e^2 \rangle$ is determined from experimental data (see Ref. [14])
Appendix III: The pitch correction

The “pitch” correction from the $\vec{\beta} \cdot \vec{B} \neq 0$ term also contributes a small correction to the measured precession frequency. In this derivation, we follow J. Paley \[39\]. More general derivations can be found in Refs. \[33, 35, 36\].

In the approximation that all muons are at the magic $\gamma$, we set $a_\mu - 1/(\gamma^2 - 1) = 0$ in Equation \[14\] and obtain

$$\bar{\omega}_{a_\mu} \simeq \bar{\omega}_{diff} = -\frac{Qe}{m_\mu} \left[ a_\mu \bar{B} - a_\mu \left( \frac{\gamma}{\gamma + 1} \right) (\vec{\beta} \cdot \vec{B}) \frac{\gamma}{\gamma + 1} \right]. \quad \text{(38)}$$

The geometry is shown in Fig. 4. The pitch angle $\psi = \psi_0 \cos \omega_{b_y} t$ oscillates with the vertical betatron frequency

$$\omega_{b_y} = 2\pi \sqrt{n f_C} \simeq 2\pi \times 2.5 \text{ MHz} \quad \text{(39)}$$

where $n$ is the field index from Appendix I, and $f_C$ is the cyclotron frequency. Thus $\omega_{b_y}$ is significantly larger than the spin precession frequency $\omega_{a_\mu} \simeq 2\pi \times 0.23 \text{ MHz}$, so we are justified in averaging out the pitching motion.

We adopt the (rotating) coordinate system shown in Figure 4 where $\vec{\beta}$ lies in the $yz$-plane, $z$ being the direction of propagation, and $y$ being vertical in the storage ring. The $x$ and $z$ axes rotate with the angular frequency

$$\omega = \frac{Qe}{m_\mu \gamma} B_y. \quad \text{(40)}$$

The frequency consists of two components, $\omega_\perp$ and $\omega_\parallel$,

$$\omega_\perp = \omega_{a_\mu} = \omega_y \cos \psi - \omega_z \sin \psi \quad \text{(41)}$$

Assuming $\vec{B} = \hat{y} B_y$, $\vec{\beta} = \hat{z} \beta_z + \hat{y} \beta_y = \hat{z} \beta \cos \psi + \hat{y} \beta \sin \psi$,

$$\bar{\omega}'_{a_\mu} = -\frac{Qe}{m_\mu} \left[ a_\mu \hat{y} B_y - a_\mu \left( \frac{\gamma}{\gamma + 1} \right) \beta_y B_y (\hat{z} \beta_z + \hat{y} \beta_y) \right]. \quad \text{(42)}$$
we get
\[ \omega_a = -\frac{Qe}{m}a_B \beta_y \left[ 1 - \left( \frac{\gamma}{\gamma + 1} \right) \beta_y^2 \right] = -\frac{Qe}{m}a_B \beta_y \left[ 1 - \left( \frac{\gamma}{\gamma + 1} \right) \beta_y^2 \right] \] \hspace{1cm} (43)

Using
\[ \frac{\beta_y}{\beta} = \sin \psi \simeq \psi, \quad \text{and} \quad \frac{\gamma \beta^2}{\gamma + 1} = \frac{\gamma - 1}{\gamma}, \] \hspace{1cm} (44)
we get
\[ \omega_a = -\frac{Qe}{m}a_B \left[ 1 - \left( \frac{\gamma - 1}{\gamma} \right) \sqrt{\psi}^2 \right] \] \hspace{1cm} (45)

For \( \omega_{az} \) we have
\[ \omega_{az} = \frac{Qe}{m}a_B \left( \frac{\gamma}{\gamma + 1} \right) \beta_y \beta_z = \omega_{az} = \frac{Qe}{m}a_B \left( \frac{\gamma}{\gamma + 1} \right) \beta^2 \frac{\beta_y^2 \beta_z}{\beta_y^2 \beta_z}. \] \hspace{1cm} (46)

Using \( \frac{\beta_y}{\beta_z} = \tan \psi \simeq \psi \)
\[ \omega_{az} = -\frac{Qe}{m}a_B \left( \frac{\gamma - 1}{\gamma} \right) \sqrt{\psi}. \] \hspace{1cm} (47)

Substituting Eq. 45 and Eq. 47 into Eq. 41 gives
\[ \omega_a = \omega_{az} \simeq -\frac{Qe}{m}a_B \left( 1 - \frac{\psi^2}{2} \right) = -\frac{Qe}{m}a_B \left( 1 - \frac{\psi^2}{2} \right) \] \hspace{1cm} (48)

Taking the time average of the oscillating term we get
\[ -\frac{Qe}{m}a_B \left( 1 - \frac{\psi^2}{4} \right) \] \hspace{1cm} (49)

So the pitch correction is
\[ C_p = -\frac{\langle \psi^2 \rangle}{4} = -\frac{n \langle y^2 \rangle}{2 R_0^2}, \] \hspace{1cm} (50)
where we have used Equation 31 \( \langle \psi^2 \rangle = n \langle y^2 \rangle / R_0^2 \), and \( \langle y^2 \rangle = 0.5 \langle y_0^2 \rangle \). In E821, the quantity \( \langle y^2 \rangle \) was both determined experimentally and from simulations.