Petri Net Model and Max-Plus Algebra on Queue in Clinic
UNS Medical Center

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Abstract. Patients come to the clinic UNS (Universitas Sebelas Maret) Medical Center to get a medical treatment. The clinic provides two health services include dental polyclinics and general polyclinics. Majority of patients do not know exactly how long the process of service from patient registration to getting medicine. Moving queues to each service is an event and people who queue up at each service are state. It is clear that the sum of all possible states is finite. This problem includes a Discrete Event System (DES). One of the uses of petri net is to model a DES. In this research, service flow was built in the clinic using Petri Net and service scheduling was modeled using Max-Plus Algebra. Furthermore, a coverability tree was built to determine the liveness and deadlock of the patient service queue and analyze the petri net flow. The selection of service start time can be chosen randomly to simulate the scheduling and length of service time.

1. Introduction
Health clinic is an institution that has a role to meet the needs of public health services. UNS Medical Center Clinic is a health clinic provided by Sebelas Maret University (UNS) through the health services of students, UNS community employees, and the community. Polyclinic provided include dental polyclinic and general polyclinic. The majority of patients do not know exactly how long the process of service from patient registration to getting medicines.

The queue flow system at the UNS Medical Center Clinic is a System Event Discrete (SED) because events change at discrete times. SED can be modeled using petri net. For an event to occur, several conditions must be met first. On Petri net, information about these events and situations is expressed in terms of transition and place, respectively. According to Cohen et. al. [4], the nature and behavior of the system can be analyzed using the max-plus approach. According to Subiono [8], algebra max plus pays attention to the aspect of system synchronization. According to Farlow [5], the use of simple max plus algebra is used on the railroad network. In 2018, Martha et al. [7] do the application of Petri net and max-plus algebra in a bank queue with two servers at each service. Therefore, there is a link between petri net, SED, and max-plus algebra in analyzing the system.

In this research, petri net service process flow was determined at the UNS Medical Center Clinic then a service scheduling model was built at the UNS Medical Center Clinic using Max Algebra Algebra. From the model obtained, it was then analyzed when scheduling patient services at the UNS Medical Center Clinic.

2. Preliminaries
In this preliminaries written theories that underlie the formulation of the problem, such as max-plus algebra, petri net, and max-plus algebra in timed petri net.
2.1 Max-plus algebra
In this section the concept of max-plus algebra is defined by Bacelli et. al. [2].

Notation. \( \mathbb{N} \) is the set of natural numbers, \( \mathbb{R} \) is the set of real numbers, \( \varepsilon = -\infty, e = 0 \), \( \mathbb{R}_{\max} = \mathbb{R} \cup \{ \varepsilon \} \).

Definition 2.1.1 Max-plus algebra is the set of \( \mathbb{R}_{\max} \) with the two operation \( \oplus \) and \( \otimes \) and is denoted by
\[
\mathbb{R}_{\max} = (\mathbb{R}_{\max}, \oplus, \otimes, \varepsilon, e),
\]
where the operation \( \oplus \) and \( \otimes \) define as
\[
x \oplus y = \max(x, y) \quad \text{and} \quad x \otimes y = x + y \quad (1)
\]
The following definition of semiring is given based on Königsberg [6].

Definition 2.1.2 A semiring is nonempty set \( \mathbb{R} \) endowed with two operations \( \oplus \mathbb{R}, \otimes \mathbb{R} \), and two elements \( \varepsilon \mathbb{R} \) and \( e \mathbb{R} \) such that
(i) \( \oplus \mathbb{R} \) is associative and commutative with zero element \( \varepsilon \mathbb{R} \).
(ii) \( \otimes \mathbb{R} \) is associative, distribute over \( \oplus \mathbb{R} \), and has unit element \( e \mathbb{R} \).
(iii) \( \varepsilon \mathbb{R} \) is absorbing for \( \otimes \mathbb{R} \) i.e., \( x \otimes \varepsilon = \varepsilon \) \( \otimes x = x \).

Such a semiring is denoted by \( \mathbb{R} = (\mathbb{R}, \oplus \mathbb{R}, \otimes \mathbb{R}, \varepsilon, e) \).

Theorem 2.1.1. The max-plus algebra \( \mathbb{R}_{\max} = (\mathbb{R}_{\max}, \oplus, \otimes, \varepsilon, e) \), has the algebraic structure of a commutative and idempotent semiring.

2.2 Petri Net
Petri net is one of the tools to model discrete event systems. For an event to occur, several conditions must be met first. Information about these events and states is expressed in terms of transitions and places, respectively.

Definition 2.2.1 [3]. Petri net 4-tuple \( (P, T, A, w) \) with
\[
P : \text{ finite set of places, } P = \{p_1, p_2, \cdots, p_n\},
\]
\[
T : \text{ finite set of transitions, } T = \{t_1, t_2, \cdots, t_m\},
\]
\[
A : \text{ arc set, } A \subseteq (P \times T) \cup (T \times P), \text{ and}
\]
\[
w : \text{ weights function, } w : A \rightarrow \mathbb{N} \text{ finite.}
\]

Petri net can be described as directed graph. The point of the graph is place and transition. In petri net, graphs are allowed to use multiple arcs to connect two equivalent points by giving weight to each arc stating the number of arcs. This structure is known as multidigraph. The mechanism for firing petri net is to run a token across the network when the transition becomes enabled and this process changes the petrinet. All tokens in place input are reduced / taken as many as the weight of the arc connecting them.

Definition 2.2.2. A marking \( x \) in a petri net is a function
\[
x : P \rightarrow \mathbb{N} \quad x(p_i) \in \mathbb{N} \quad (2)
\]

Definition 2.2.3 Marked petri net is 5-tuple \( (P, T, A, w, x_0) \) where \( (P, T, A, w) \) is the petri net and \( x_0 \) is the initial marking.

The meaning of the matrix used in petri net refers to Subiono [8].

Definition 2.2.4. The backward (forward) incidence matrix representing petri net is a matrix of size \( n \times m \), with \( n \) the number of places and \( m \) the number of transitions, and the \( i \)-th and \( j \)-th elements are \( D^- \equiv w(p_i, t_j), D^+ \equiv w(t_j, p_i) \). From the backward and forward matrices, we get
\[
D = D^+ - D^-
\]
where
According to Subiono [8], the state of the petri net is said to be deadlock when the transition cannot be fired anymore. Deadlock can be caused by competition to get tokens. When all places don’t get tokens, a deadlock will occur.

**Definition 2.2.5.** A petri net with the initial state $x_0$ is called live if there is a path such that there is always a transition that can be fired for each state that can be reached from $x_0$.

Determining the location of the first token will affect which transitions can be fired so that it will also affect the location of the next token

$$X(k + 1) = X(k) + Du,$$

where

- $x(+1)$: the location of the token after a firing,
- $x(k)$: the location of the token before a firing,
- $D$: state incidence matrix $D = D^+ - D^-$,
- $u$: enable transition that will be fired.

A petri net flow is said to be not deadlock or live when there is an element in the matrix that is of one value. If all elements of the matrix are zero then the petri net path is said to be deadlocked.

It is given initial condition $x_0$, Cassandras [3] classifies the liveness into,

(i) dead or $L_0$-live, if the transition can never be firing sequence,
(ii) $L_1$-live, if transition can be fired at least once in some firing sequence,
(iii) $L_2$-live, for any positive integer $k$, transition can be fired at least $k$ times in some firing sequence, for $k > 1$,
(iv) $L_3$-live, if transition appears infinitely, often in some firing sequence, and
(v) live atau $L_4$-live, if transition is $L_1$-live for every possibility state from $x_0$.

According to Subiono [8], coverability tree is a technique used to complete several aspects of analysis in discrete event systems. If looping occurs, the petri net can be said to be not deadlocked. Based on Adzkiya [1], the concept of coverability has a relationship with $L_1$-live that is if there are certain firing lines so that the transition can be refined at least once. Only enabled transitions can be fired.

### 2.3 Max-Plus algebra model of timed petri net

This section explains the max-plus algebra model from petri net that refers to Subiono [8]. Timed petri net in queueing system is given in Figure 1.

![Figure 1. Timed Petri Net](image)

In Figure 1, $a(k)$ denotes time when the the $k$--th patient come, $v_{a,k}$ denotes time length of the $k$-th patient arrive, $s(k)$ states time for the $k$th services begin, $d(k)$ states the time for the $k$-th patient leave the services, and $v_{d,k}$ denotes time length for the $k$-th patient leave the services, state $Q$ is the queue and $B$ is busy, it is obtained.
\[
\begin{align*}
a(k) &= v_{a,k} + a(k-1) \\
s(k) &= \max\{a(k), d(k-1)\} \\
d(k) &= v_{d,k} + s(k) \\
&= v_{d,k} + \max\{a(k), d(k-1)\} \\
&= \max\{v_{d,k} + v_{a,k}, a(k-1), v_{d,k} + d(k-1)\}
\end{align*}
\]

or it can be written as
\[
\begin{align*}
\left(\frac{a(k)}{d(k)}\right) &= \left(\frac{v_{a,k}}{v_{d,k}}\right) \bigotimes \left(\frac{c}{v_{d,k}}\right) \bigotimes \left(\frac{a(k-1)}{d(k-1)}\right)
\end{align*}
\]

where \( c \) is chosen so
\[
c = v_{a,k} \bigotimes a(k-1) \bigotimes (c \bigotimes d(k-1)) = v_{a,k} \bigotimes a(k-1) \text{ for } k = 1, 2, \ldots \text{ and initial condition } a(0) = d(0) = 0.
\]

3. Main results

3.1 Flow of the clinic queue

In general, the flow of services in each clinic is almost the same. However, it depends on the privacy of each clinic and what services are provided there. This study uses two main services in each clinic, a polyclinic service and a pharmacy service. The flow is given by the flow chart illustrated in Figure 2..

![Figure 2. Flow of patient services at clinic UNS Medical Center](image)

Patients come to the queue number machine to retrieve the service queue number. Then, they sit in the waiting chairs to be called the registration number. After that, customers will enter the registration queue depending on their number and wait until their number is called, then presented at the counter. They will be served at counters 1 or 2 or 3, then their complaints will be asked. If their goal is to go to a general
practitioner they immediately enter the queue of general practitioners. That also applies to dentists. After getting services, patients will get prescription drugs from doctors and will take them to the pharmacy. The patient puts a prescription in the locker then waits to be called to get medicine from the pharmacist. After the service is finished, they can choose whether they still need other services or not

3.2 Petri net model
From the flow of patient services at the UNS medical Center clinic in Figure 2, the petri net flow is then made as shown in Figure 3

![Petri net flow](image)

By defining transitions and places in the service flow as follows

- $P_0$: Patient takes a queue number
- $P_1$: Process is waiting for the registration officer to be served
- $P_2$: Patient processes are served in Counter 1
- $P_3$: Patient processes are served in Counter 2
- $P_4$: Patient processes are served in Counter 3
- $P_5$: Patients waiting to be served by a dental doctor
- $P_6$: Patients waiting to be served by general doctor
- $P_7$: Patient processes are served in the dental polyclinic
- $P_8$: Patient processes are served in the general polyclinic
- $P_9$: Patient is waiting to be served by a pharmacist
- $P_{10}$: Patient processes are is served at the pharmacy
- $P_{11}$: Patients double check whether they still need service
- $T_0$: Patient are coming
- $T_1$: Patient enters the registration officer queue
- $T_2$: Patients are served in Counter 1
- $T_3$: Patients are served in Counter 2
- $T_4$: Patients are served in Counter 3
- $T_5$: Patients enter the queue of dental and general doctor
- $T_6$: Patients are served in the dental polyclinic
- $T_7$: Patients are served at the general polyclinic
- $T_8$: Patient enters the pharmacist queue
- $T_9$: Patients are served at the pharmacy
- $T_{10}$: Patient has finished serving
- $T_{11}$: Patient leaves the clinic
- $T_{12}$: Patients need service again

3.3 Petri Net Flow Analysis
From the petri net flow of patient services at UNS medical Center clinics such as Figure 3, a forward matrix, backward matrix and $X(p)$ matrix can be formed. The forward $D^+$ and backward matrix is $12 \times 14$. With equation (4) ($D = D^+ - D^-$) an incidence combination matrix is obtained as follows
The petri net flow in Figure 3 has selected the transition to be refined which can only be done once. To find out that petri net is not deadlock it will be analyzed through the coverability tree. For example, the transition $T_1$ that will be fired so that $u$ is obtained is $u = [0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T$. With matrix $X(k)$ at the initial state $k = 0$

$$X(k = 0) = [1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T,$$

which means there is one token in $P_1$. Furthermore, by combining the matrix $X(p)$, matrix $u$ and matrix $D$ into equation (5), and done repeatedly by choosing the transitions $T_2, T_5, T_6, T_8, T_9, T_{10},$ dan $T_{12}$ that will be fired so we get the coverability tree presented as follows

Furthermore, before the service scheduling model is built, first a petri net is provided for patient service using time. The flow of customer service petri net that is connected with time is the same as Figure 4. But in each transition a sequence of services is given $(k)$ so that each transition previously denoted $T_0$ becomes $T_0(k)$. Also at each transition is given the length of service time $V_{T_0,k}$. Laying the length of service time is adjusted to each transition. By taking the groove in the dental polyclinic service, the following data are obtained

$$T_0(k) = V_{T_0,k} + T_0(k - 1)$$

$$T_1(k) = \max\{T_0(k), T_{12}(k - 1)\}$$

$$T_2(k) = T_1(k) + V_{T_1,k}$$

$$T_5(k) = T_2(k) + V_{T_2,k}$$

$$T_6(k) = T_5(k) + V_{T_5,k}$$

$$T_8(k) = T_6(k) + V_{T_6,k}$$

$$T_9(k) = T_8(k) + V_{T_8,k}$$

$$T_{10}(k) = T_9(k) + V_{T_9,k}$$

$$T_{12}(k) = T_{10}(k) + V_{T_{10},k}$$
Furthermore, combining all equations (8) - (15) is obtained,
\[ T_{12}(k) = \max \{V_{T_1,k} + V_{T_2,k} + V_{T_5,k} + V_{T_6,k} + V_{T_9,k} + V_{T_{10},k} + T_0(k - 1), (V_{T_{12},k} + V_{T_2,k} + V_{T_5,k} + V_{T_6,k} + V_{T_8,k} + V_{T_9,k} + V_{T_{10},k} + T_0(k - 1)) \} \]  
(16)

From equations (8) and (16) a matrix can be formed from the max-plus algebraic model of the petri net service system which is time related
\[
\begin{bmatrix} T_{12}(k) \\ T_0(k) \end{bmatrix} = \begin{bmatrix} W & Y \\ V_{T_0,k} & C \end{bmatrix} \otimes \begin{bmatrix} T_{12}(k - 1) \\ T_0(k - 1) \end{bmatrix}
\]  
(17)

where
\[ W = V_{T_2,k} + V_{T_5,k} + V_{T_6,k} + V_{T_8,k} + V_{T_9,k} + V_{T_{10},k} \]
\[ Y = V_{T_2,k} + V_{T_5,k} + V_{T_6,k} + V_{T_8,k} + V_{T_9,k} + V_{T_{10},k} \]

and \( C \) is chosen so that
\[ (V_{T_0,k} \otimes T_{12}(k - 1)) \oplus (C \otimes T_0(k - 1)) = V_{T_0,k} \otimes T_{12}(k - 1), \]
with the initial state \( T_0(0) = T_{12}(0) = 0 \).

If it is assumed the time taken by the patient to take the queue number \( V_{T_0,k} \) is 1 minute and each other service requires a waiting time of about 5 minutes. So the time needed for the patient to leave the service system \( V_{T_{12},k} \) is estimated at 30 minutes then obtained
\[ W = 30, Y = 25, \]
(18)

So equation (17) becomes
\[
\begin{bmatrix} T_{12}(k) \\ T_0(k) \end{bmatrix} = \begin{bmatrix} 30 & 25 \\ 1 & C \end{bmatrix} \otimes \begin{bmatrix} T_{12}(k - 1) \\ T_0(k - 1) \end{bmatrix}
\]  
(19)

and \( C \) is chosen so that
\[ (V_{T_0,k} \otimes T_{12}(k - 1)) \oplus (C \otimes T_0(k - 1)) = V_{T_0,k} \otimes T_{12}(k - 1), \]
with the initial state \( T_0(0) = T_{12}(0) = 0 \), matrix in equation (19) as a matrix for scheduling patients in the flow of dental clinic services at the UNS Medical Center clinic.

4. Conclusions

The petri net flow from the queue at the UNS Medical Center clinic is depicted in Figure 3 and the scheduling of patients in the dental clinic service flow is presented in equation (19). Petri net flow in a queueing system at UNS Medical Center clinic does not experience a deadlock because it forms a loop. Simulation of scheduling the arrival of patients going for treatment at a dental polyclinic service at the UNS Medical Center clinic, for example the first patient at the UNS Medical Center clinic takes a queue number \( T_0(0) \) at 08.00 the total service process ends at 31 minutes later at 08.31.

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