BOUNDDEDNESS AND ASYMPTOTIC STABILITY IN A QUASILINEAR TWO-SPECIES CHEMOTAXIS SYSTEM WITH NONLINEAR SIGNAL PRODUCTION

XU PAN AND LIANGCHEN WANG*

School of Science, Chongqing University of Posts and Telecommunications
Chongqing 400065, China

(Communicated by Enrico Valdinoci)

ABSTRACT. This paper deals with the following quasilinear two-species chemotaxis system
\[
\begin{align*}
\partial_t u_1 &= \nabla \cdot (D_1(u_1) \nabla u_1 - S_1(u_1) \nabla v) + f_1(u_1), & x \in \Omega, & t > 0, \\
\partial_t u_2 &= \nabla \cdot (D_2(u_2) \nabla u_2 - S_2(u_2) \nabla v) + f_2(u_2), & x \in \Omega, & t > 0, \\
\partial_t v &= \Delta v - v + g_1(u_1) + g_2(u_2), & x \in \Omega, & t > 0
\end{align*}
\]
under homogeneous Neumann boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^n$ ($n \geq 2$). The diffusivity and the density-dependent sensitivity are given by $D_i(s) \geq C_{d_i}(1 + s)^{-\alpha_i}$ and $S_i(s) \leq C_{s_i}s(1 + s)^{\beta_i-1}$ for all $s \geq 0$, respectively, where $C_{d_i}, C_{s_i} > 0$ and $\alpha_i, \beta_i \in \mathbb{R}$; the logistic source and the signal productions are given by $f_i(s) \leq r_is - \mu_is^{k_i}$ and $g_i(s) \leq s^{\gamma_i}$ for all $s \geq 0$ respectively, where $r_i \in \mathbb{R}, \mu_i, \gamma_i > 0$ and $k_i > 1$ ($i = 1, 2$). It is proved that this system possesses a global bounded smooth solution under some specific conditions with or without the logistic functions $f_i(s)$, which partially improves the results in [25]. Moreover, in case $r_i > 0$, if $\mu_i$ are sufficiently large, it is shown that the global bounded solution exponentially converges to \((\frac{r_i}{\mu_i})^{\frac{1}{\gamma_i-1}}, (\frac{r_i}{\mu_i})^{\frac{\gamma_i}{\gamma_i-1}}, (\frac{r_i}{\mu_i})^{\frac{\gamma_i}{\gamma_i-1}} + (\frac{r_i}{\mu_i})^{\frac{1}{\gamma_i-1}}\) as $t \to \infty$.

1. Introduction. The famous chemotaxis model, proposed by Keller and Segel [10], describes the aggregation of the Dictyostelium discoideum
\[
\begin{align*}
u_t &= \nabla \cdot (D(u) \nabla u - S(u) \nabla v) + f(u), & x \in \Omega, & t > 0, \\
\nu_t &= \Delta v - v + g(u), & x \in \Omega, & t > 0
\end{align*}
\]
where $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) is a bounded domain. If $f(u) = 0$: in case $g(u) = u$, the blow-up and global boundedness were obtained by Tao and Winkler in [23]; in case $0 < g(u) \leq u^\gamma$, the solution is bounded in $n \geq 2$ under the condition $\gamma \in (0, \frac{2}{n})$ [12]; if $\frac{S(u)}{D(u)} \leq C_e(1 + u)^{\eta}$ and $0 < \gamma \leq 1$, the global boundedness is obtained under the condition $\eta < \min \{1 + \frac{1}{n}, 1 + \frac{2}{n} - \gamma\}$ in [18]; Winkler [31] obtained the

2020 Mathematics Subject Classification. Primary: 92C17, 35K35; Secondary: 35A01, 35B35.

Key words and phrases. Boundedness, chemotaxis, nonlinear signal production, quasilinear, stabilization, two-species.

This work is supported by the Chongqing Research and Innovation Project of Graduate Students (No. CYS20271) and the Science and Technology Research Program of Chongqing Municipal Education Commission (No. KJQN202000618).

* Corresponding author.
radial blow-up solution of (1.1) under the condition $\gamma > \frac{2}{n}$ if the second equation is replaced by $0 = \Delta v - \frac{1}{|\Omega|} \int_{\Omega} u^2 + v^2$. Moreover, if $f(u) = \mu u(1 - u)$, in $n \leq 2$ [16], the blow-up can be prevented for arbitrarily small $\mu > 0$; in high-dimensional [32], the global boundedness of (1.1) was obtained whenever $\mu$ is sufficiently large. The global boundedness and the large time behavior of solution have also been studied in [2, 8, 28].

The following is the two-species and one-stimuli chemotaxis system, which describes the spatiotemporal evolution of two populations that reproduce and compete with Lotka-Volterra dynamics:

\[
\begin{align*}
\begin{cases}
    u_1 = \nabla \cdot (D_1(u)\nabla u - S_1(u)\nabla w) + \mu_1 u(1 - u - a_1 v), & x \in \Omega, \\ v_1 = \nabla \cdot (D_2(v)\nabla v - S_2(v)\nabla w) + \mu_2 v(1 - v - a_2 u), & x \in \Omega, \\ \tau v_1 = \Delta w - w + u + v, & x \in \Omega, 
\end{cases}
\end{align*}
\]

first of all, in the linear case: $D_1(u) = D_2(v) = 1$ and $S_1(u) = u$, $S_2(v) = v$. In case $\tau = 0$, the global existence in $n \geq 1$ and asymptotic stability in weakly competitive case of solution to (1.2) were studied by Black et al. [4]: Wang [27] further investigated the global boundedness in $n \geq 1$. In case $\tau = 1$, the global existence and asymptotic stability of solution to (1.2) was studied by Negreanu and Tello [15] if $a_1 = a_2 = 0$; recently, Bai and Winkler [1] obtained the global boundedness of solutions in $n \leq 2$ and the stabilization of solutions when $\mu_1$ and $\mu_2$ were large enough; the bounded and asymptotic results are optimized in [14]: the global existence and asymptotic stability of solution have also been investigated in [33] with high-dimensional. For (1.2) without logistic source in the quasilinear case, it has been investigated in [25], the diffusion functions $D_i(s)$ and the chemotactic sensitivity functions $S_i(s)$ satisfy $K_{0i}(s + 1)^{i-1} \leq D_i(s) \leq K_i(s + 1)^{i-1}$ and $\frac{\partial \tau}{\partial s} \leq K_i(s + 1)^{\alpha_i}$, $(i = 1, 2)$, the solutions are globally bounded under the conditions that $0 < \alpha_i < \frac{2}{n}$ for all $u_i > 1$; and the finite-time blow-up of solution was also obtained. For more results about two-species and one-stimuli chemotaxis system we refer the readers to [22, 26].

There are also many results about two-species and two-stimuli chemotaxis system

\[
\begin{align*}
\begin{cases}
    \partial_t u_1 = \Delta u_1 - \chi_1 \nabla \cdot (\nabla u_1 \nabla v_1) - \chi_2 \nabla \cdot (\nabla u_1 \nabla v_1) + f_1(u_1), & x \in \Omega, \\ \partial_t u_2 = \Delta u_2 - \chi_1 \nabla \cdot (\nabla u_2 \nabla v_1) - \chi_2 \nabla \cdot (\nabla u_2 \nabla v_2) + f_2(u_2), & x \in \Omega, \\ \partial_t v_1 = \Delta v_1 - \lambda_1 v_1 + \alpha_{11} u_1 + \alpha_{12} u_2, & x \in \Omega, \\ \partial_t v_2 = \Delta v_2 - \lambda_2 v_2 + \alpha_{21} u_1 + \alpha_{22} u_2, & x \in \Omega,
\end{cases}
\end{align*}
\]

in case $f_1(u_1) = f_2(u_2) = 0$. Espejo et al. [6] obtained the solution of system blow-up simultaneously in finite time under some specific parameter conditions. Moreover, in case $f_1(u_1) = \mu_1 u_1(1 - u_1 - a_1 u_2)$ and $f_2(u_2) = \mu_2 u_2(1 - u_2 - a_2 u_1)$, $\alpha_{11} = \alpha_{22} = \chi_1 = \chi_2 = a_1 = a_2 = 0$, the global boundedness solution has been studied by Black [3] for all $\mu_i > 0$ $(i = 1, 2)$ in $n = 2$, and when $\frac{\alpha_{11}}{\chi_1}$ and $\frac{\alpha_{22}}{\chi_2}$ were large enough, the large time behavior of the bounded solution was discussed. Wang et al. [30] obtained the the global bounded smooth solution in $n \leq 3$ if $\mu_i (i = 1, 2)$ were large enough, and the large time behavior was obtained under the condition that $\mu_1$ and $\mu_2$ were sufficiently large. Recently, the global boundedness of $n = 3$ was further investigated by Pan et al. [21] under the condition that

$$
\mu_1 > \max \left\{ \chi_1^2 + \frac{3(\sqrt{5} + \sqrt{2})}{\sqrt{\chi_1^2 + \chi_2^2}}, \frac{3(\sqrt{5} + \sqrt{2})}{2}\sqrt{\chi_1^2 + \chi_2^2} + 1 \right\}
$$
and
\[ \mu_2 > \max \left\{ \lambda_2^2 + \frac{3(\sqrt{\pi} + \sqrt{2})}{\sqrt{\lambda_1^2 + \lambda_2^2}}, \frac{3(\sqrt{\pi} + \sqrt{2})}{2\sqrt{\lambda_1^2 + \lambda_2^2} + 1} \right\}. \]

Moreover, Pan and Wang [17, 19] further investigated the nonlinear form of (1.3). For other results about (1.3), we refer to [20, 24, 29].

In this paper, we deal with the following quasilinear two-species Keller-Segel chemotaxis system

\[
\begin{aligned}
\partial_t u_1 &= \nabla \cdot (D_1(u_1) \nabla u_1 - S_1(u_1) \nabla v) + f_1(u_1), & x \in \Omega, & t > 0, \\
\partial_t u_2 &= \nabla \cdot (D_2(u_2) \nabla u_2 - S_2(u_2) \nabla v) + f_2(u_2), & x \in \Omega, & t > 0, \\
\partial_t v &= \Delta v - v + g_1(u_1) + g_2(u_2), & x \in \Omega, & t > 0, \\
\frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} = \frac{\partial v}{\partial n} = 0, & x \in \partial \Omega, & t > 0, \\
(u_1, u_2, v)(x, 0) &= (u_{10}(x), u_{20}(x), v_0(x)), & x \in \Omega
\end{aligned}
\]  

(1.4)

under homogeneous Neumann boundary conditions in a bounded domain \( \Omega \subset \mathbb{R}^n \) (\( n \geq 2 \)), and \( \partial/\partial n \) is the derivative of the normal with respect to \( \partial \Omega \); \( u_1 \) and \( u_2 \) represent the densities of different populations, \( v = v(x, t) \) denotes the concentrations of chemicals produced by populations. Throughout this paper, we assume that: the initial data is nonnegative and satisfies

\[ (u_{10}, u_{20}, v_0) \in C^0(\bar{\Omega}) \times C^0(\bar{\Omega}) \times C^1(\bar{\Omega}); \]

(1.5)

the diffusivity \( D_i(s) \in C^2([0, \infty)) \) and the density-dependent sensitivity \( S_i(s) \in C^2([0, \infty)) \) with \( S_i(0) = 0 \) satisfy

\[ D_i(s) \geq C_{d_i}(1 + s)^{-\alpha_i} \quad \text{and} \quad 0 \leq S_i(s) \leq C_{s_i}s(1 + s)^{\beta_i - 1} \quad \text{for all } s \geq 0 \]

(1.6)

with \( C_{d_i}, C_{s_i} > 0 \) and \( \alpha_i, \beta_i \in \mathbb{R} \); the signal productions \( g_i(s) \in C^1([0, \infty)) \) are nonnegative and satisfy

\[ g_i(s) \leq s^\gamma_i \quad \text{for all } s \geq 0 \quad \text{with } \gamma_i > 0; \]

(1.7)

the logistic sources \( f_i(s) \in C^0([0, \infty)) \) satisfy that

\[ f_i(s) \leq r_is - \mu_is^{k_i} \quad \text{for all } s \geq 0, \]

(1.8)

where \( r_i \in \mathbb{R}, \mu_i > 0 \) and \( k_i > 1, (i = 1, 2) \).

The purpose of this paper is to study the effects of the presence or absence of logistic sources on the boundedness and asymptotic stability of solutions, it is difficult to overcome the problems caused by the nontrivial effects of nonlinear diffusion, sensitivity, signal secretion and (without or with) logistic source. Yet, we can obtain the boundedness of solution without or with logistic source by establishing an appropriate a priori estimates under the some conditions; for the main part of this paper: the large time behavior, until now there have been no results on stabilization of quasilinear multi-population with one-chemical, fortunately, the corresponding results can be obtained by constructing some proper Lyapunov functionals under different parameters conditions. Now, we state our main results in this paper are stated as follows.

Theorem 1.1. Suppose that (1.5) – (1.7) hold and \( f_i \equiv 0 \ (i = 1, 2), \Omega \subset \mathbb{R}^n \) (\( n \geq 2 \)) is a smooth bounded domain. For \( i = 1, 2 \), if \( 0 < \gamma := \gamma_1 = \gamma_2 \leq 1 \) and

\[ \alpha_i + \beta_i < \min \left\{ 1 + \frac{1}{n}, 1 + \frac{2}{n} - \gamma \right\}, \]

then (1.4) possesses a classical solution \((u_1, u_2, v)\) which is globally bounded.
Remark 1. Theorem 1.1 improves previous results in [25]. For instance, in case $\gamma = 1$, the restriction in [25, Theorem 1.1] on the upper bound of the diffusion function was removed.

The globally bounded result of solution to (1.4) under logistic source are as follows.

**Theorem 1.2.** Suppose that (1.5) - (1.8) hold and $\Omega \subset R^n (n \geq 2)$. For $i = 1, 2$, there exists $\mu^* > 0$ such that if one of the following conditions

(i) $\gamma_i < k_i - \beta_i$;

(ii) $\gamma_1 < k_1 - \beta_1, \gamma_2 = k_2 - \beta_2, \mu_2 > \mu^*$;

(iii) $\gamma_1 = k_1 - \beta_1, \gamma_2 < k_2 - \beta_2, \mu_1 > \mu^*$;

(iv) $\gamma_i = k_i - \beta_i, \mu_i > \mu^*$

is satisfied, then (1.4) possesses a classical solution $(u_1, u_2, v)$ which is globally bounded.

Next, the exponential decay of solution to (1.4) is given.

**Theorem 1.3.** Under the hypothetical conditions of Theorem 1.2, assume that (1.6) holds, $g_i (s) = s^\gamma_i$ and $f_i (s) = r_i s - \mu_i s^{\beta_i} (i = 1, 2)$. There exists a large enough $\mu > 0$, if $r_i > 0$, $\gamma_i \geq \frac{1}{2}$ and $\mu_i > \mu$ such that for the initial data satisfying (1.5) with $u_{10}, u_{20} \not= 0$, then there exist $C, \lambda > 0$ such that the solution $(u_1, u_2, v)$ of (1.4) satisfies

$$
\|u_1 (\cdot, t) - u_{1*}\|_{L^\infty (\Omega)} + \|u_2 (\cdot, t) - u_{2*}\|_{L^\infty (\Omega)} + \|v (\cdot, t) - v_*\|_{L^\infty (\Omega)} < Ce^{-\lambda t}
$$

for all $t \geq 0$ with

$$
u_{1*} = \left( \frac{r_1}{\mu_1} \right) \frac{1}{1-s_1}, \quad u_{2*} = \left( \frac{r_2}{\mu_2} \right) \frac{1}{1-s_2}, \quad \text{and} \quad v_* = \left( \frac{r_1}{\mu_1} \right) \frac{1}{1-s_1} + \left( \frac{r_2}{\mu_2} \right) \frac{1}{1-s_2}.
$$

This paper is organized as below. Section 2 establishes the local-in-time existence of solution to (1.4) and gives some useful preliminaries. In Sections 3 and 4, we obtain the results of Theorems 1.1 and 1.2, respectively. For the main part of this article, Section 5, Theorem 1.3 will be obtained by constructing some appropriate Lyapunov functionals.

2. Preliminaries. In this section, to begin with, we state a basic result for local-in-time existence solution of (1.4), which are obtained by Banach contraction principle and parabolic regularity, the detailed proofs can be seen in [8, Theorem 3.1].

**Lemma 2.1.** Suppose that (1.5) - (1.8) hold and $\Omega \subset R^n (n \geq 1)$ is a smooth bounded domain, (1.4) possesses a non-negative solution $(u_1, u_2, v)$ from $C(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1} (\overline{\Omega} \times (0, T_{\max}))$ with $t \in (0, T_{\max})$, where $T_{\max} \in (0, \infty]$. Moreover,

$$
\lim_{t \uparrow T_{\max}} \{ \|u_1 (\cdot, t)\|_{L^\infty (\Omega)} + \|u_2 (\cdot, t)\|_{L^\infty (\Omega)} \} \rightarrow \infty \quad \text{if} \quad T_{\max} < \infty.
$$

We give the lemma for the variation of maximal Sobolev’s regularity [9, Lemma 2.1] in the following.

**Lemma 2.2.** Let $1 \leq n < m < \infty$, we consider the following system

$$
\begin{cases}
\frac{\partial f}{\partial t} = \Delta f - f + a + b, & (x, t) \in \Omega \times (0, T) \\
\frac{\partial f}{\partial \nu} = 0, & (x, t) \in \partial \Omega \times (0, T) \\
f(x, 0) = f_0 (x), & x \in \Omega.
\end{cases}
$$
For each nonnegative \( f_0 \in W^{2,m}(\Omega) \) and functions \( a, b \in L^m(0, T; L^m(\Omega)) \), there exists a unique solution

\[
f \in W^{1,m}((0, T); L^m(\Omega)) \cap L^m((0, T); W^{2,m}(\Omega)),
\]

moreover, if \( f(\cdot, t_0) \in W^{2,m}(\Omega) \) in \( t_0 \in [0, T) \) with \( \frac{\partial f(\cdot, t_0)}{\partial \nu} = 0 \) on \( \partial \Omega \), then there exists \( C(m) > 0 \) such that

\[
\int_{t_0}^T \int_{\Omega} e^{mt} |\Delta f|^m \leq C(m) \int_{t_0}^T \int_{\Omega} e^{mt} a^m + C(m) \int_{t_0}^T \int_{\Omega} e^{mt} b^m
\]

\[
+ C(m) e^{mt_0} \left( \|f(\cdot, t_0)\|_{L^{\infty}(\Omega)}^m + \|\Delta f(\cdot, t_0)\|_{L^m(\Omega)}^m \right).
\]

Next, we give some basic estimates of \( u_1, u_2, v \).

Lemma 2.3. Suppose that (1.5) – (1.7) hold and \( f_i = 0 \) (\( i = 1, 2 \), \( \Omega \subset R^n \) (\( n \geq 1 \)), then there hold that

\[
\|u_1(\cdot, t)\|_{L^1(\Omega)} = \|u_{10}\|_{L^1(\Omega)} \quad \text{and} \quad \|u_2(\cdot, t)\|_{L^1(\Omega)} = \|u_{20}\|_{L^1(\Omega)} \quad (2.1)
\]

for all \( t \in (0, T_{\text{max}}) \). Let \( 0 < \gamma := \gamma_1 = \gamma_2 \leq 1 \), if \( s \in [1, \frac{n}{(n\gamma-1\gamma)_{+}}) \), then there exists \( C > 0 \) such that

\[
\|v(\cdot, t)\|_{W^{1,\gamma}(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\text{max}}). \quad (2.2)
\]

Proof. Integrating the first and second equations of (1.4) over \( \Omega \), (2.1) can be directly obtained. (2.2) can be proved by the method of the Neumann semigroup in [11, Lemma 1].

3. Boundedness without logistic source. The purpose of this section is to obtain the uniformly bounded of solution to (1.4) without logistic source. The ideas come from [23, Lemma 3.3] and [28, Lemma 3.2]. We first give some basic estimates of \( u_1, u_2 \) and \( v \).

Lemma 3.1. Suppose that (1.5) – (1.7) hold and \( f_i \equiv 0 \) (\( i = 1, 2 \), let \( \Omega \subset R^n \) (\( n \geq 1 \)) and \( p_1 \geq 1 \), then we have

\[
\frac{d}{dt} \int_{\Omega} (1 + u_1)^{p_1} + \frac{2C_{d_1}p_1(p_1 - 1)}{(p_1 - \alpha_1)^2} \int_{\Omega} |\nabla(1 + u_1)^{\frac{p_1 - \alpha_1}{2}}|^2 \leq \frac{C_{s_1}^2}{2C_{d_1}} \int_{\Omega} (1 + u_1)^{p_1 + \alpha_1 + 2\beta_1 - 2} |\nabla v|^2 \quad (3.1)
\]

and

\[
\frac{d}{dt} \int_{\Omega} (1 + u_2)^{p_2} + \frac{2C_{d_2}p_2(p_2 - 1)}{(p_2 - \alpha_2)^2} \int_{\Omega} |\nabla(1 + u_2)^{\frac{p_2 - \alpha_2}{2}}|^2 \leq \frac{C_{s_2}^2}{2C_{d_2}} \int_{\Omega} (1 + u_2)^{p_2 + \alpha_2 + 2\beta_2 - 2} |\nabla v|^2 \quad (3.2)
\]

for all \( t \in (0, T_{\text{max}}) \). Let \( \gamma := \gamma_1 = \gamma_2 \) and \( q \geq 1 \), then for all \( t \in (0, T_{\text{max}}) \) there exists \( C_1 > 0 \) such that

\[
\frac{1}{q} \frac{d}{dt} \int_{\Omega} |\nabla v|^{2q} + \frac{q - 1}{q^2} \int_{\Omega} |\nabla|^{q} |\nabla v|^q \leq (4(q - 1) + n) \int_{\Omega} |\nabla v|^{2(q - 1)} (u_1^{2\gamma} + u_2^{2\gamma}) + (C_1 - 2) \int_{\Omega} |\nabla v|^{2q}. \quad (3.3)
\]
Proof. Testing the first equation of (1.4) by \( p_1(u_1 + 1)^{p_1 - 1} \) and integrating over \( \Omega \) by parts, according to Young’s inequality, then yields
\[
\frac{d}{dt} \int\Omega (1 + u_1)^{p_1} \leq -C_d p_1 (p_1 - 1) \int\Omega (1 + u_1)^{p_1 - \alpha_1 - 2} |\nabla u_1|^2 \\
+ C_s p_1 (p_1 - 1) \int\Omega (1 + u_1)^{p_1 + \beta_1 - 2} |\nabla u_1| |\nabla v| \\
\leq -\frac{C_d p_1 (p_1 - 1)}{2} \int\Omega (1 + u_1)^{p_1 - \alpha_1 - 2} |\nabla u_1|^2 \\
+ \frac{C_s^2 p_1 (p_1 - 1)}{2C_d} \int\Omega (1 + u_1)^{p_1 + \alpha_1 + 2\beta_1 - 2} |\nabla v|^2
\]
for all \( t \in (0, T_{\text{max}}) \), thus, we obtain (3.1), (3.2) can be obtained in the same way.

Using the third equation of (1.4), according to the point-wise identity \( \Delta |\nabla v|^2 = 2 |D^2 v|^2 + 2 \nabla v \cdot \nabla \Delta v \) and \( |\Delta v|^2 \leq n |D^2 v|^2 \), we have
\[
\frac{1}{q} \frac{d}{dt} \int\Omega |\nabla v|^{2q} + \frac{2}{n} \int\Omega |\nabla v|^{2(q - 1)} |\Delta v|^2 + 2 \int\Omega |\nabla v|^{2q} \\
\leq \int\Omega |\nabla v|^{2(q - 1)} \Delta |\nabla v|^2 + 2 \int\Omega |\nabla v|^{2(q - 1)} \nabla v \cdot \nabla (g_1(u_1) + g_2(u_2)) \\
= - (q - 1) \int\Omega |\nabla v|^{2(q - 2)} |\nabla |\nabla v| |\nabla v|^2 | + \int_{\partial \Omega} |\nabla v|^{2(q - 1)} |\nabla v|^2 | dS \\
- 2(q - 1) \int\Omega |\nabla v|^{2(q - 2)} g_1(u_1) |\nabla v|^2 | \cdot \nabla v - 2 \int\Omega |\nabla v|^{2(q - 1)} \Delta v \cdot g_1(u_1) \\
- 2(q - 1) \int\Omega |\nabla v|^{2(q - 2)} g_2(u_2) |\nabla v|^2 | \cdot \nabla v - 2 \int\Omega |\nabla v|^{2(q - 1)} \Delta v \cdot g_2(u_2) (3.4)
\]
for all \( t \in (0, T_{\text{max}}) \), applying the boundary integral without the convexity of domain [13, Lemma 4.2] and the trace inequality [7, Proposition 4.22, 4.24], which implies
\[
\int_{\partial \Omega} |\nabla v|^{2(q - 1)} |\nabla v|^2 | dS \leq 2\kappa_\Omega \int_{\Omega} |\nabla v|^{2q} dS \\
\leq \frac{q - 1}{q^2} \int_{\Omega} |\nabla v|^{2q} |^2 + C_1 \int_{\Omega} |\nabla v|^{2q} (3.5)
\]
with some \( C_1 > 0 \) and \( \kappa_\Omega > 0 \) is an upper bound for the curvatures of \( \partial \Omega \). Combining (3.4) and (3.5) with Young’s inequality, we obtain
\[
\frac{1}{q} \frac{d}{dt} \int\Omega |\nabla v|^{2q} + \frac{2}{n} \int\Omega |\nabla v|^{2(q - 1)} |\Delta v|^2 + 2 \int\Omega |\nabla v|^{2q} \\
\leq - \frac{q - 1}{2} \int\Omega |\nabla v|^{2(q - 2)} |\nabla |\nabla v| |\nabla v|^2 | + \frac{q - 1}{q^2} \int\Omega |\nabla v|^{2q} \\
+ \frac{2}{n} \int\Omega |\nabla v|^{2(q - 1)} |\Delta v|^2 + (4(q - 1) + n) \int\Omega |\nabla v|^{2(q - 1)} (g_1^2(u_1) + g_2^2(u_2)) \\
= - \frac{q - 1}{q^2} \int\Omega |\nabla v|^{2q} |^2 + C_1 \int_{\Omega} |\nabla v|^{2q} + \frac{2}{n} \int\Omega |\nabla v|^{2(q - 1)} |\Delta v|^2 \\
+ (4(q - 1) + n) \int\Omega |\nabla v|^{2(q - 1)} (g_1^2(u_1) + g_2^2(u_2)) (3.6)
\]
Thus, (3.3) is a direct result from (3.6) with (1.7) and \( \gamma := \gamma_1 = \gamma_2 \). \( \square \)
Combining (3.1) – (3.3) and taking appropriate parameters to establish the uniform boundedness of \( \|u_1\|_{L^{p_1}(\Omega)} \) and \( \|u_2\|_{L^{p_2}(\Omega)} \) with some \( p_1, p_2 > 1 \) in the following lemma.

**Lemma 3.2.** Suppose that (1.5) – (1.7) hold, \( f_i \equiv 0 \) and \( \Omega \subset R^n \) (\( n \geq 2 \)). If the conditions \( 0 < \gamma := \gamma_1 = \gamma_2 \leq 1 \) and 
\[
\alpha_i + \beta_i < \min \left\{ 1 + \frac{1}{n}, 1 + \frac{2}{n} - \gamma \right\}
\] (3.7) are satisfied \((i = 1, 2)\), then for all \( p_i \in [1, \infty) \) and \( q \in [1, \infty) \), there exists some \( C > 0 \) such that 
\[
\|u_1(\cdot, t)\|_{L^{p_1}(\Omega)} + \|u_2(\cdot, t)\|_{L^{p_2}(\Omega)} + \|\nabla v(\cdot, t)\|_{L^{\gamma}(\Omega)} \leq C\] (3.8)
for all \( t \in (0, T_{\text{max}}) \).

**Proof.** Since the condition \( \alpha_i + \beta_i < \min \{1 + \frac{1}{n}, 1 + \frac{2}{n} - \gamma\} \), then we can select some \( s \in [1, \frac{n}{(n\gamma-1)^+}) \) such that 
\[
\gamma - \frac{1}{n} < \frac{1}{s} < 1 + \frac{1}{n} - \alpha_i - \beta_i.
\] (3.9)
For the choice of parameters for (3.10) – (3.12) below, we only give the results here, (the proof see [23, Lemma 2.1]). Let \( 1 < a_i < \min \left\{ \frac{n}{n-2}, \frac{s}{(s-2)^+} \right\} \) and \( b_i > \max \left\{ \frac{n}{2}, \frac{1}{2s} \right\} \), then we can select some \( p_i > \max \{1 + \frac{2\alpha_i}{2\beta_i}, \alpha_i + \frac{2\gamma}{2\beta_i} + 1\} \) and \( q > \max \{1 + \frac{2\alpha_i}{2\beta_i}, \frac{n-2}{(n\gamma-1)^+} \} \) such that for all \( p_i > p_i* \) and \( q > q* \), we have
\[
\frac{n-2}{n} \cdot \frac{p_i + \alpha_i + 2\beta_i - 2}{p_i - \alpha_i} < 1, \quad 1 - \frac{2\gamma}{s} < \frac{1}{p_i} < 1 - \frac{n-2}{n q}
\] (3.10) and 
\[
\frac{n-2}{n} \cdot \frac{p_i - \alpha_i}{p_i} < 1, \quad \frac{2\gamma}{s} < \frac{1}{b_i} < 1 - \frac{2\gamma}{b_i - 1}
\] (3.11) and 
\[
\frac{n-2}{n} \cdot \frac{p_i - \alpha_i}{p_i} < 1, \quad \frac{2\gamma}{s} < \frac{1}{b_i} < 1 - \frac{2\gamma}{b_i - 1}
\] (3.12)
Let \( p_i := p_2 + \alpha_1 - \alpha_2 \). Combining (3.1) – (3.3) to see that
\[
\frac{d}{dt} \int_{\Omega} \left( (1 + u_1)^{p_1} + (1 + u_2)^{p_2} + \frac{1}{q} |\nabla v|^{2q} \right) \leq C_2 \int_{\Omega} |\nabla v|^{2q} + \frac{q - 1}{q^2} \int_{\Omega} |\nabla |\nabla v|^{q}|^2
\]
\[
+ \frac{2C_{d_1}p_1(p_1 - 1)}{(p_1 - \alpha_1)^2} \int_{\Omega} |\nabla (1 + u_1)^{p_1 - \alpha_1}|^2 + 2C_{d_2}p_2(p_2 - 1) \int_{\Omega} |(1 + u_2)^{p_2 - \alpha_2}|^2 \leq C_2 \int_{\Omega} (1 + u_1)^{p_1 + \alpha_1 + 2\beta_1 - 2} |\nabla v|^2 + C_2 \int_{\Omega} (1 + u_2)^{p_2 + \alpha_2 + 2\beta_2 - 2} |\nabla v|^2
\]
\[
+ C_2 \int_{\Omega} (1 + u_1)^{2\gamma} |\nabla v|^{2(q-1)} + C_2 \int_{\Omega} (1 + u_2)^{2\gamma} |\nabla v|^{2(q-1)}
\] (3.13)
for all \( t \in (0, T_{\text{max}}) \) with \( C_2 := \max \left\{ \frac{C^2_{d_1}p_1(p_1 - 1)}{2C_{d_1}}, C_1 \right\} \), \( C_1 \geq 2, 4(q - 1) + n \). Since \( a_i, b_i > 1 \) \((i = 1, 2)\), let \( a_i' = \frac{a_i}{a_i - 1} > 1 \) and \( b_i' = \frac{b_i}{b_i - 1} > 1 \), using Hölder’s inequality to the terms on the right-hand side of inequality (3.13), then yields
\[
\int_{\Omega} (1 + u_1)^{p_1 + \alpha_1 + 2\beta_1 - 2} |\nabla v|^2 \leq \left( \int_{\Omega} (1 + u_1)^{(p_1 + \alpha_1 + 2\beta_1 - 2)a_1} \right)^{\frac{1}{a_1}} \left( \int_{\Omega} |\nabla v|^{2a_1} \right)^{\frac{1}{a_1}}, \quad (3.14)
\]
\[
\int_{\Omega} (1 + u_2)^{p_2 + \alpha_2 + 2\beta_2 - 2} |\nabla v|^2 \leq \left( \int_{\Omega} (1 + u_2)^{(p_2 + \alpha_2 + 2\beta_2 - 2)a_2} \right)^{\frac{1}{a_2}} \left( \int_{\Omega} |\nabla v|^{2a_2} \right)^{\frac{1}{a_2}},
\]
\[
\int_{\Omega} (1+u_1)^{2\gamma} |\nabla v|^{2(q-1)} \leq \left( \int_{\Omega} (1+u_1)^{2\gamma b_1} \right)^{\frac{1}{\gamma b_1}} \left( \int_{\Omega} |\nabla v|^{2(q-1)b'_1} \right)^{\frac{1}{b'_1}}
\]
and
\[
\int_{\Omega} (1+u_2)^{2\gamma} |\nabla v|^{2(q-1)} \leq \left( \int_{\Omega} (1+u_2)^{2\gamma b_2} \right)^{\frac{1}{\gamma b_2}} \left( \int_{\Omega} |\nabla v|^{2(q-1)b'_2} \right)^{\frac{1}{b'_2}}.
\]
By virtue of (2.1) and Gagliardo-Nirenberg inequality, there exist some \(C_3, C_4 > 0\) such that
\[
\left( \int_{\Omega} (1+u_1)^{p_1+\alpha_1+2\beta_1-2} \right)^{\frac{1}{p_1+\alpha_1+2\beta_1-2}} = \left\| (1+u_1)^{\frac{p_1+\alpha_1}{2}} \right\|_{L^{\frac{p_1+\alpha_1}{2}}(\Omega)}^{\frac{2(p_1+\alpha_1+2\beta_1-2)}{p_1+\alpha_1+2\beta_1-2}} \leq C_3 \left( \int_{\Omega} |\nabla (1+u_1)|^{\frac{p_1+\alpha_1}{2}} \right) + C_4,
\]
(3.15)
where \(\theta_1 = \frac{p_1+\alpha_1}{2} - \frac{2\alpha_1 + 2\beta_1 - 2}{\alpha + \beta - 2} \in (0, 1)\) is ensured by (3.10). According to (2.2) and G-N inequality again we derive
\[
\left( \int_{\Omega} |\nabla v|^{2\alpha} \right)^{\frac{1}{\alpha}} = \left\| |\nabla v|^{\frac{\alpha}{\alpha}} \right\|_{L^{\frac{\alpha}{\alpha}}(\Omega)}^{\frac{2\alpha}{\alpha}} \leq C_5 \left\| |\nabla v|^{\frac{\alpha}{\alpha}} \right\|_{L^{\frac{\alpha}{\alpha}}(\Omega)}^{\frac{2\alpha}{\alpha}} \leq C_6 \left( \int_{\Omega} |\nabla |\nabla v|^{\frac{\alpha}{\alpha}} \right)^{\frac{1}{\alpha}} + C_6
\]
(3.16)
with \(C_5, C_6 > 0\), where \(\delta_1 = \frac{\alpha + \beta - 2}{\alpha} \in (0, 1)\) is ensured by (3.11). Inserting (3.15) and (3.16) to (3.14) then yields
\[
\begin{align*}
C_2 & \int_{\Omega} (1+u_1)^{p_1+\alpha_1+2\beta_1-2} |\nabla v|^2 \\
\leq C_7 \left( \int_{\Omega} |\nabla (1+u_1)|^{\frac{p_1+\alpha_1+2\beta_1-2}{p_1+\alpha_1+2\beta_1-2}} \right)^{\frac{p_1+\alpha_1+2\beta_1-2}{p_1+\alpha_1+2\beta_1-2} \theta_1} \left( \int_{\Omega} |\nabla |\nabla v|^{\frac{\alpha}{\alpha}} \right)^{\frac{\delta_1}{\alpha}} + C_7
\end{align*}
\]
(3.17)
with some \(C_7 > 0\). Similarly, in view of (3.10), (3.11) and Lemma 2.3 again, there exist \(\theta_2 = \frac{p_2+\alpha_2+a+2\beta_2-2}{\alpha} \in (0, 1)\) and \(\delta_2 = \frac{\alpha + \beta - 2}{\alpha} \in (0, 1)\) such that
\[
\begin{align*}
C_2 & \int_{\Omega} (1+u_2)^{p_2+\alpha_2+2\beta_2-2} |\nabla v|^2 \\
\leq C_7 \left( \int_{\Omega} |\nabla (1+u_2)|^{\frac{p_2+\alpha_2+2\beta_2-2}{p_2+\alpha_2+2\beta_2-2}} \right)^{\frac{p_2+\alpha_2+2\beta_2-2}{p_2+\alpha_2+2\beta_2-2} \theta_2} \left( \int_{\Omega} |\nabla |\nabla v|^{\frac{\alpha}{\alpha}} \right)^{\frac{\delta_2}{\alpha}} + C_7
\end{align*}
\]
(3.18)
Similarly, by virtue of Lemma 2.3, the Gagliardo-Nirenberg inequality and (3.12) again we can obtain
\[
C_2 \int_\Omega (1 + u_1)^{2q} |\nabla v|^{2(q-1)} \\
\leq C_8 \left( \int_\Omega \left( \nabla (1 + u_1)^{\frac{p_1-\alpha_1}{2}} \right)^{2} \left( \int_\Omega |\nabla v|^q \right)^{\frac{(q-1)\delta_1}{q}} \right) + C_8 \tag{3.19}
\]
and
\[
C_2 \int_\Omega (1 + u_2)^{2q} |\nabla v|^{2(q-1)} \\
\leq C_8 \left( \int_\Omega \left( \nabla (1 + u_2)^{\frac{p_2-\alpha_2}{2}} \right)^{2} \left( \int_\Omega |\nabla v|^q \right)^{\frac{(q-1)\delta_2}{q}} \right) + C_8 \tag{3.20}
\]
with \( C_8 > 0 \) and
\[
\tilde{\theta}_i = \frac{p_i - \alpha_i}{2} - \frac{p_i - \alpha_i}{\gamma_i} (i = 1, 2) \in (0, 1) \quad \text{and} \quad \tilde{\delta}_i = \frac{\bar{q}}{s} + \frac{\bar{q}}{2(q-1)} - \frac{\bar{q}}{2} (i = 1, 2) \in (0, 1) \ (i = 1, 2).
\]
Thus, substituting (3.17) – (3.20) into (3.13), which implies
\[
\frac{d}{dt} \int_\Omega \left( (1 + u_1)^{p_1} + (1 + u_2)^{p_2} + \frac{1}{q} |\nabla v|^{2q} \right) - C_2 \int_\Omega |\nabla v|^{2q} + \frac{q - 1}{q^2} \int_\Omega |\nabla |\nabla v|^q \right|^{2} \\
+ \frac{2C_{d_1} p_1 (p_1 - 1)}{(p_1 - \alpha_1)^2} \int_\Omega |\nabla (1 + u_1)^{\frac{p_1-\alpha_1}{2}}|^2 \left( \int_\Omega |\nabla v|^q \right)^{\frac{4q}{q-1}} \tag{3.21}
\]
If \( \frac{p_i + \alpha_i + 2\beta_i - 2}{p_i - \alpha_i} \theta_i + \frac{\delta_i}{q} < 1 \) and \( \frac{2\gamma \bar{\theta}_i}{p_i - \alpha_i} + \frac{(q-1)\bar{\delta}_i}{q} < 1 \) are satisfied \((i = 1, 2)\), then there exists \( C_9 > 0 \) such that
\[
\frac{d}{dt} \int_\Omega \left( (1 + u_1)^{p_1} + (1 + u_2)^{p_2} + \frac{1}{q} |\nabla v|^{2q} \right) + C_{d_1} p_1 (p_1 - 1) \left( \int_\Omega |\nabla (1 + u_1)^{\frac{p_1-\alpha_1}{2}}|^2 \right) \\
+ \frac{2C_{d_2} p_2 (p_2 - 1)}{(p_2 - \alpha_2)^2} \int_\Omega |\nabla (1 + u_2)^{\frac{p_2-\alpha_2}{2}}|^2 + \frac{q - 1}{2q^2} \int_\Omega |\nabla |\nabla v|^q \right|^{2} \\
\leq C_2 \int_\Omega |\nabla v|^{2q} + C_9. \tag{3.22}
\]
Therefore, in order to satisfy the hypothesis in (3.21), let
\[
h_i(q) := \frac{p_i + \alpha_i + 2\beta_i - 2}{p_i - \alpha_i} \theta_i + \frac{\delta_i}{q} = \frac{p_i + \alpha_i + 2\beta_i - 2}{p_i - \alpha_i} - \frac{1}{2\alpha_i} + \frac{1}{\alpha_i} \left( \frac{1}{n - \frac{1}{2}} + \frac{p_i - \alpha_i}{2} \right) + \frac{1}{\alpha_i} \left( \frac{1}{n - \frac{1}{2}} + \frac{2}{s} \right). \]
and
\[ h_i(q(p_i)) < 1 \quad \text{and} \quad \tilde{h}_i(q(p_i)) < 1 \]

with \( q(p_i) := \frac{p_i - \alpha_i}{2} (i = 1, 2) \), where the assumption \( p_1 = p_2 + \alpha_1 - \alpha_2 \) ensures that \( q(p_1) = q(p_2) \). Since \( q(p_i) \to +\infty \) as \( p_i \to \infty \), there exists \( q \in [q_*, \infty) \) for all \( p_i \geq p_* \) and \( q \geq q_* \), fulfilling \( h_i(q) < 1 \) and \( \tilde{h}_i(q) < 1 \), therefore, the hypothesis in (3.21) are satisfied.

In order to make the inequality (3.22) satisfy the form of Gronwall’s inequality, applying Lemma 2.3 and the Gagliardo-Nirenberg inequality again, there exists \( C_{10} > 0 \) such that
\[
\int_{\Omega} (1 + u_1)^{p_1} = \left\| (1 + u_1)^{\frac{p_1 - \alpha_1}{2}} \right\|_{L^{\frac{p_1}{p_1 - \alpha_1}}(\Omega)}^{2p_1} \leq C_{10} \left( \int_{\Omega} \left| \nabla (1 + u_1)^{\frac{p_1 - \alpha_1}{2}} \right|^2 \right)^{\frac{p_1 \alpha_1}{p_1 - \alpha_1}} + C_{10} \quad (3.23)
\]
and
\[
\int_{\Omega} (1 + u_2)^{p_2} = \left\| (1 + u_2)^{\frac{p_2 - \alpha_2}{2}} \right\|_{L^{\frac{p_2}{p_2 - \alpha_2}}(\Omega)}^{2p_2} \leq C_{10} \left( \int_{\Omega} \left| \nabla (1 + u_2)^{\frac{p_2 - \alpha_2}{2}} \right|^2 \right)^{\frac{p_2 \alpha_2}{p_2 - \alpha_2}} + C_{10}, \quad (3.24)
\]
where \( \sigma_i = \frac{p_i - \alpha_i}{2 - \frac{\alpha_i}{2}} (i = 1, 2) \) are ensured because of \( p_i > 1 + \frac{\alpha_i}{2} \).

Similarly, there exist \( C_{11}, C_{12} > 0 \) such that
\[
\left( \frac{1}{q} + C_2 \right) \int_{\Omega} \nabla v|^2 q = \left( \frac{1}{q} + C_2 \right) \left\| \nabla v|^\frac{2}{q} \right\|_{L^2(\Omega)}^2 \leq C_{11} \left( \int_{\Omega} \left| \nabla \nabla v\right|^2 \right)^{\frac{\sigma_1 \sigma_1}{\sigma_1 - \sigma}} + C_{11} \leq q - 1 \int_{\Omega} \left| \nabla \nabla v\right|^2 + C_{12} \quad (3.25)
\]
where \( \sigma = \frac{\frac{1}{\sigma_1} - \frac{1}{p_2}}{\frac{1}{\sigma_1} + \frac{1}{p_2}} \in (0, 1) \) is ensured because of \( q > 1 + \frac{2}{p} \). Thus, inserting (3.23) – (3.25) into (3.22), there exist \( C_{13}, C_{14} > 0 \) such that
\[
\frac{d}{dt} \left( \int_{\Omega} (1 + u_1)^{p_1} + (1 + u_2)^{p_2} + \frac{1}{q} |\nabla v|^2 q \right) + C_{13} \left( \int_{\Omega} (1 + u_1)^{p_1} \right)^{\frac{p_1 - \alpha_1}{p_1 \sigma_1}} + C_{13} \left( \int_{\Omega} (1 + u_2)^{p_2} \right)^{\frac{p_2 \alpha_2}{p_2 - \alpha_2}} + \frac{1}{q} \int_{\Omega} |\nabla v|^2 q \leq C_{14}, \quad (3.26)
\]
in view of the ODI comparison argument with (3.26), then yields (3.8). □

Proof of Theorem 1.1. By virtue of [23, Lemmata 3.3 and A.1] with Lemma 3.2, the results of Theorem 1.1 are obatined. □
4. Boundedness with logistic source. The purpose of this section is to obtain the uniformly bounded of solution to (1.4) with logistic source. The idea comes from [28, Lemma 3.2].

Lemma 4.1. Suppose that (1.5) – (1.8) hold and \( \Omega \subset \mathbb{R}^n \) (\( n \geq 2 \)) is a smooth bounded domain. For \( p_i \in (1, \infty) \), \( (i = 1, 2) \), there exists \( \mu^* > 0 \) such that if one of the following conditions

(i) \( \gamma_i < k_i - \beta_i \);  
(ii) \( \gamma_1 < k_1 - \beta_1, \gamma_2 = k_2 - \beta_2, \mu_2 > \mu^* \);  
(iii) \( \gamma_1 = k_1 - \beta_1, \gamma_2 < k_2 - \beta_2, \mu_1 > \mu^* \);  
(iv) \( \gamma_i = k_i - \beta_i, \mu_i > \mu^* \)

is satisfied, then there exists \( C > 0 \) such that

\[
\| u_1 (\cdot, t) \|_{L^{p_1}(\Omega)} + \| u_2 (\cdot, t) \|_{L^{p_2}(\Omega)} \leq C \quad \text{for all} \ t \in (0, T_{\max}) .
\]  

(4.1)

Proof. Let \( p_i > \max \{1, n(k_i - \beta_i) + 1 - k_i\} \) and \( p_1, p_2 \) further satisfy \( p_1 = \frac{k_1 - \beta_1}{k_2 - \beta_2} (p_2 + k_2 - 1) + 1 - k_1, t_0 := \min \{1, \frac{1}{2} T_{\max}\} \). Testing the first equation of (1.4) by \( (1 + u)^{p_1 - 1} \) and integrating over \( \Omega \) by parts, then yields

\[
\frac{1}{p_1} \frac{d}{dt} \int_\Omega (1 + u_1)^{p_1} = - (p_1 - 1) \int_\Omega (1 + u_1)^{p_1 - 2} |\nabla u_1|^2 + \int_\Omega (1 + u_1)^{p_1 - 1} f_1(u_1) \\
+ (p_1 - 1) \int_\Omega (1 + u_1)^{p_1 - 2} S_1(u) \nabla u_1 \cdot \nabla v \quad \text{for all} \ t \in (t_0, T_{\max}) .
\]  

(4.2)

In view of Young’s inequality and (1.8), then there exists \( C_1 > 0 \) such that

\[
\int_\Omega (1 + u_1)^{p_1 - 1} f_1(u_1) \leq r_1 \int_\Omega (1 + u_1)^{p_1 - 1} u_1 - \mu_1 \int_\Omega (1 + u_1)^{p_1 - 1} u_1^{k_1} \leq r_1 \int_\Omega (1 + u_1)^{p_1 - 1} u_1 + \mu_1 \int_\Omega (1 + u_1)^{p_1 - 1} \\
= \frac{\mu_1}{2^{k_1 - 1} - 1} \int_\Omega (1 + u_1)^{p_1 + k_1 - 1} \leq 2r_1 + \mu_1 \int_\Omega (1 + u_1)^{p_1} - \frac{\mu_1}{2^{k_1 - 1} - 1} \int_\Omega (1 + u_1)^{p_1 + k_1 - 1} + C_1 ,
\]  

(4.3)

where we used the elementary inequality \( (1 + u)^{k_1} \leq 2^{k_1 - 1} (u^{k_1} + 1) \). Let

\[
\varphi_j (u) := (p_1 - 1) \int_0^u (1 + \sigma)^{p_1 - 2} S_1(\sigma) d\sigma \quad \text{for all} \ j \geq 0 ,
\]

the assumptions \( p_1 > n(k_1 - \beta_1) + 1 - k_1 \) and \( k_1 > \beta_1 \) guarantee that \( p_1 + \beta_1 - 1 > 0 \), in virtue of (1.6), we obtain

\[
0 \leq \varphi_1 (u_1) \leq \frac{C_{\varphi_1} (p_1 - 1)}{p_1 + \beta_1 - 1} (1 + u_1)^{p_1 + \beta_1 - 1} \quad \text{for all} \ u_1 \geq 0 ,
\]

thus, for the last term of (4.2), there exists \( C_2 > 0 \) such that

\[
(p_1 - 1) \int_\Omega (1 + u_1)^{p_1 - 2} S_1(u_1) \nabla u_1 \cdot \nabla v \\
= \int_\Omega \nabla \varphi_1 (u_1) \cdot \nabla v \leq \frac{C_{\varphi_1} (p_1 - 1)}{p_1 + \beta_1 - 1} \int_\Omega (1 + u_1)^{p_1 + \beta_1 - 1} |\Delta v|
\]
Let 

$$m_i := \frac{p_i + k_i - 1}{k_i - \beta_i}, \quad (i = 1, 2), \tag{4.5}$$

$m_i > n$ is satisfied because of $p_1 > n(k_1 - \beta_1) + 1 - k_1$. Therefore, a combination of (4.2)–(4.5), according to Young’s inequality with $k_1 > 1$, there exists $C_3 > 0$ such that

$$\frac{1}{p_1} \int_\Omega (1 + u_1)^{p_1} \leq - \frac{\mu_1}{2k_1} \int_\Omega (1 + u_1)^{p_1 + k_1 - 1} + C_2 \int_\Omega |\Delta v|^{m_1} + \left(\frac{m_1}{p_1} + 2\gamma_1\right) \int_\Omega (1 + u_1)^{p_1 - 1} \tag{4.6}$$

(4.6) in conjunction with the constant variation formula yields that

$$\frac{1}{p_1} \int_\Omega (1 + u_1)^{p_1} \leq - \frac{\mu_1}{2k_1} \int_\Omega (1 + u_1)^{p_1 + k_1 - 1} + C_2 \int_\Omega |\Delta v|^{m_1} dx + C_3,$$

with $C_4 = \frac{C_3}{m_1} + \frac{1}{p_1} \int_\Omega (1 + u_1(x, t_0))^{p_1} dx$, in virtue of Lemma 2.2 and (1.7), this results in

$$\frac{1}{p_1} \int_\Omega (1 + u_1)^{p_1} \leq - \frac{\mu_1}{2k_1} \int_\Omega \int_{t_0}^t e^{-m_i(t-\tau)}(1 + u_1)^{p_1 + k_1 - 1} dx d\tau + C_5 \int_{t_0}^t \int_\Omega e^{-m_i(t-\tau)} \|v\|_{L^s(\Omega)}^{m_1} + \|\Delta v\|_{L^s(\Omega)}^{m_1} \, dx d\tau + C_4.$$
where \( C_5 > 0 \), \( C_6 = C_5 (||v(\cdot, t_0)||^{m_1}_{L^{m_1}(\Omega)} + ||\Delta v(\cdot, t_0)||^{m_1}_{L^{m_1}(\Omega)}) + C_4 \). Similarly, we also have
\[
\frac{1}{p_2} \int_\Omega (1 + u_2)^{p_2} \arint_{t_0}^t \int_\Omega e^{-m_2(t-\tau)} (1 + u_2)^{p_2 + k_2 - 1} dx d\tau + C_8 \int_{t_0}^t \int_\Omega e^{-m_2(t-\tau)} g_2^{m_2} (u_2) dx d\tau \\
+ C_8 e^{-m_2(t-t_0)} (||v(\cdot, t_0)||^{m_2}_{L^{m_2}(\Omega)} + ||\Delta v(\cdot, t_0)||^{m_2}_{L^{m_2}(\Omega)}) + C_7 \\
+ C_8 \int_{t_0}^t \int_\Omega e^{-m_2(t-\tau)} g_2^{m_2} (u_2) dx d\tau \\
\leq - \frac{\mu_2}{2k_2} \int_{t_0}^t \int_\Omega e^{-m_2(t-\tau)} (1 + u_2)^{p_2 + k_2 - 1} dx d\tau + C_8 \int_{t_0}^t \int_\Omega e^{-m_2(t-\tau)} u_2^{m_2\gamma_2} dx d\tau \\
+ C_8 \int_{t_0}^t \int_\Omega e^{-m_2(t-\tau)} u_2^{m_2\gamma_2} dx d\tau + C_9,
\]
where \( C_8 > 0 \), \( C_9 = C_8 (||v(\cdot, t_0)||^{m_2}_{L^{m_2}(\Omega)} + ||\Delta v(\cdot, t_0)||^{m_2}_{L^{m_2}(\Omega)}) + C_7 \). According to (4.8) and (4.9), we have
\[
\frac{1}{p_1} \int_{t_0}^t \int_\Omega (1 + u_1)^{p_1} + \frac{1}{p_2} \int_{t_0}^t \int_\Omega (1 + u_2)^{p_2} + \frac{\mu_1}{2k_1} \int_{t_0}^t \int_\Omega e^{-m_1(t-\tau)} (1 + u_1)^{p_1 + k_1 - 1} dx d\tau \\
+ \frac{\mu_2}{2k_2} \int_{t_0}^t \int_\Omega e^{-m_2(t-\tau)} (1 + u_2)^{p_2 + k_2 - 1} dx d\tau - C_6 - C_9
\]
\[
\leq C_5 \int_{t_0}^t \int_\Omega e^{-m_1(t-\tau)} (1 + u_1)^{m_1\gamma_1} dx d\tau + C_5 \int_{t_0}^t \int_\Omega e^{-m_1(t-\tau)} (1 + u_2)^{m_1\gamma_2} dx d\tau \\
+ C_8 \int_{t_0}^t \int_\Omega e^{-m_2(t-\tau)} (1 + u_1)^{m_2\gamma_1} dx d\tau + C_8 \int_{t_0}^t \int_\Omega e^{-m_2(t-\tau)} (1 + u_2)^{m_2\gamma_2} dx d\tau.
\]
Thanks to (4.5) and \( p_1 = \frac{k_1 - \gamma_1}{k_2 - \gamma_2} (p_2 + k_2 - 1) + 1 - k_1 \) we have \( m_1 = m_2 \). We derive the uniformly bounded of solution to (1.4) by dividing it into four cases in the following.

**Case (i)** \( \gamma_i < k_i - \beta_i \) (i = 1, 2).

The conditions (4.5) and \( \gamma_i < k_i - \beta_i \) guarantee that \( m_1\gamma_i < p_i + k_i - 1 \), according to Young’s inequality and \( m_1 = m_2 \), there exists some \( C_{10} > 0 \) satisfies
\[
(C_5 + C_8) \int_{t_0}^t \int_\Omega e^{-m_1(t-\tau)} (1 + u_1)^{m_1\gamma_1} dx d\tau \\
\leq \frac{\mu_1}{2k_1} \int_{t_0}^t \int_\Omega e^{-m_1(t-\tau)} (1 + u_1)^{p_1 + k_1 - 1} dx d\tau + C_{10}.
\]

Therefore, inserting (4.11) into (4.10) results in
\[
\frac{1}{p_1} \int_{t_0}^t \int_\Omega (1 + u_1)^{p_1} + \frac{1}{p_2} \int_{t_0}^t \int_\Omega (1 + u_2)^{p_2} \leq C_{11}
\]
for all \( t \in (t_0, T_{\max}) \) with some \( C_{11} > 0 \).

**Case (ii)** \( \gamma_1 < k_1 - \beta_1, \gamma_2 = k_2 - \beta_2 \).

Similar to Case (i), the conditions (4.5) with \( \gamma_1 < k_1 - \beta_1 \) and \( \gamma_2 = k_2 - \beta_2 \) guarantee that \( m_1\gamma_1 < p_1 + k_1 - 1 \) and \( m_2\gamma_2 = p_2 + k_2 - 1 \). Therefore, for some \( \mu_2 \geq \mu_2^*: = 2^{k_1} (C_5 + C_8) > 0 \), inserting (4.11) with \( i = 2 \) into (4.10) results in (4.12).

**Case (iii)** \( \gamma_1 = k_1 - \beta_1, \gamma_2 < k_2 - \beta_2 \).


Case (iii) is similar to Case (ii), so we omit it here.

Case (iv) \( \gamma_i = k_i - \beta_i \) (i = 1, 2).

Case (iv) can be handled in the same way.

Therefore, (4.1) is obtained, this proof is finished.

**Proof of Theorem 1.2.** By virtue of [23, Lemmata 3.3 and A.1] with Lemma 4.1, the results of Theorem 1.2 are obtained.

5. Stabilization. The purpose of this main section is to derive the asymptotic behavior of solutions to (1.4) with \( r_i > 0 \) (i = 1, 2). The ideas come from [1, Section 3.1] (also [5]). To achieve this goal, we first give the energy functional

\[
F(t) := \int_\Omega b_1(u_1) + \int_\Omega b_2(u_2) + (a_1 + a_2) \int_\Omega (v - \xi_1^2 - \xi_2^2)^2 > 0
\]

(5.1)

for all \( t > 0 \), where \( \xi_i := (\frac{r_i}{\mu_i})^{\frac{1}{r_i - 1}} \) and

\[
a_1 := \frac{C_2^2 C_{u_1}}{4 C_d_1} \cdot \frac{r_1}{\mu_1} > 0 \quad \text{and} \quad a_2 := \frac{C_2^2 C_{u_2}}{4 C_d_2} \cdot \frac{r_2}{\mu_2} > 0
\]

(5.2)

and

\[
b_i(s) := s - \xi_i - \xi_i \ln (\xi_i^{-1} s) \geq 0 \quad \text{for all} \quad s > 0 \quad (i = 1, 2),
\]

(5.3)

where \( C_{u_1} := (\|u_1\|_{L^\infty} + 1)^{\alpha_1 + 2\beta_1 - 2} > 0 \) and \( C_{u_2} := (\|u_2\|_{L^\infty} + 1)^{\alpha_2 + 2\beta_2 - 2} > 0 \). In virtue of the Taylor expansion, there exists \( \theta(s) \in (0, 1) \) such that

\[
\lim_{s \to \xi_i} \frac{b_i(s)}{(s - \xi_i)^2} = \lim_{s \to \xi_i} \frac{(s - \xi_i)^2}{2(\theta s + (1 - \theta) \xi_i)^2} = \frac{1}{2\xi_i^2}.
\]

(5.4)

The standard parabolic regularity theory lemma (see [30, Lemma 5.1]) are important for proving Theorem 1.3.

**Lemma 5.1.** Suppose that \((u_1, u_2, v)\) is a global bounded classical solution of (1.4) and \( \Omega \subset \mathbb{R}^n (n \geq 1) \) is a bounded smooth domain, then there exist \( \theta \in (0, 1) \) and \( c > 0 \) such that

\[
\|u_1\|_{C^{2+\theta,1+\frac{\theta}{2}}(\bar{\Omega} \times [t, t+1])} + \|u_2\|_{C^{2+\theta,1+\frac{\theta}{2}}(\bar{\Omega} \times [t, t+1])} + \|v\|_{C^{2+\theta,1+\frac{\theta}{2}}(\bar{\Omega} \times [t, t+1])} \leq c
\]

for all \( t > 1 \).

Through the above preparation, we can design different Lyapunov functionals in the case of \( k_i \geq 2 \) and \( k_i < 2 \) (i = 1, 2) that meet the requirements.

**Lemma 5.2.** In case \( r_i > 0 \), assume that (1.6) – (1.8) and (5.1) – (5.3) hold, if \( k_i \geq 2 \) (i = 1, 2), then the global bounded smooth solution of (1.4) satisfies

\[
\frac{d}{dt} F(t) + 2(a_1 + a_2) \int_\Omega |\nabla v|^2 + (a_1 + a_2) \int_\Omega (v - \xi_1^2 - \xi_2^2)^2 \\
+ \frac{r_1}{\xi_1} \int_\Omega (u_1 - \xi_1)^2 + \frac{r_2}{\xi_2} \int_\Omega (u_2 - \xi_2)^2 \\
\leq \frac{C_2^2 \xi_1}{2 C_d_1} \int_\Omega (u_1 + 1)^{\alpha_1 + 2\beta_1 - 2} |\nabla v|^2 + \frac{C_2^2 \xi_2}{2 C_d_2} \int_\Omega (u_2 + 1)^{\alpha_2 + 2\beta_2 - 2} |\nabla v|^2 \\
+ 2(a_1 + a_2) \int_\Omega H_1(u_1)(u_1 - \xi_1)^2 + 2(a_1 + a_2) \int_\Omega H_2(u_2)(u_2 - \xi_2)^2
\]

(5.5)
for all \( t > 0 \) with
\[
H_1(u_1) := \begin{cases} 
4^{1-\gamma_1} \xi_1^{2\gamma_1-2} & \text{if } \gamma_1 < 1, \\
\gamma_1^2(u_1 + \xi_1)^{2\gamma_1-2} & \text{if } \gamma_1 \geq 1
\end{cases}
\]  \( (5.6) \)

and
\[
H_2(u_2) := \begin{cases} 
4^{1-\gamma_2} \xi_2^{2\gamma_2-2} & \text{if } \gamma_2 < 1, \\
\gamma_2^2(u_2 + \xi_2)^{2\gamma_2-2} & \text{if } \gamma_2 \geq 1.
\end{cases}
\]  \( (5.7) \)

Proof. Testing the first equation of (1.4) by \( (1 - \frac{\xi_1}{u_1}) \) and integrating over \( \Omega \) by parts, then yields
\[
\frac{d}{dt} \int_\Omega b_1(u_1) = \int_\Omega \frac{u_1 - \xi_1}{u_1} (\nabla \cdot (D_1(u_1) \nabla u_1) - \nabla \cdot (S_1(u_1) \nabla v) + r_1 u_1 - \mu_1 u_1^{k_1})
\]
\[
= -\xi_1 \int_\Omega \frac{D_1(u_1)}{u_1^2} |\nabla u_1|^2 + \xi_1 \int_\Omega \frac{S_1(u_1)}{u_1^2} |\nabla v|^2 
- \mu_1 \int_\Omega (u_1 - \xi_1) (u_1^{k_1-1} - \frac{r_1}{\mu_1}) 
\]
\[
\geq \frac{\xi_1}{2} \int_\Omega |\nabla u_1|^2 + \frac{\xi_1}{2} \int_\Omega |\nabla v|^2 
- \mu_1 \int_\Omega (u_1 - \xi_1) (u_1^{k_1-1} - \frac{r_1}{\mu_1}) \geq \xi_1^{k_1-2} (u_1 - \xi_1)^2, \tag{5.8}
\]

for all \( t > 0 \), using Young’s inequality we have
\[
\int_\Omega \frac{S_1(u_1)}{u_1^2} |\nabla u_1|^2 \leq \frac{1}{2} \int_\Omega \frac{D_1(u_1)}{u_1^2} |\nabla u_1|^2 + \frac{1}{2} \int_\Omega S_1^2(u_1) |\nabla v|^2 
\leq \frac{1}{2} \int_\Omega \frac{D_1(u_1)}{u_1^2} |\nabla u_1|^2 + \frac{C_1^2}{2C_{d_1}} \int_\Omega (u_1 + 1)^{\alpha_1 + 2\beta_1 - 2} |\nabla v|^2. \tag{5.9}
\]

For the last terms of (5.8), in view of the conditions \( k_i \geq 2 \ (i = 1, 2) \), we derive
\[
(u_1 - \xi_1) (u_1^{k_1-1} - \frac{r_1}{\mu_1}) = (u_1 - \xi_1) (u_1^{k_1-1} - \xi_1^{k_1-1}) \geq \xi_1^{k_1-2} (u_1 - \xi_1)^2, \tag{5.10}
\]

therefore, in conjunction with (5.8) – (5.10) we have
\[
\frac{d}{dt} \int_\Omega b_1(u_1) + \frac{r_1}{\xi_1} \int_\Omega (u_1 - \xi_1)^2 \leq \frac{C_1^2}{2C_{d_1}} \int_\Omega (u_1 + 1)^{\alpha_1 + 2\beta_1 - 2} |\nabla v|^2. \tag{5.11}
\]

for all \( t > 0 \), by similar treatment to the second equation of (1.4), combining with (5.11) results in
\[
\frac{d}{dt} \int_\Omega (b_1(u_1) + b_2(u_2)) + \frac{r_1}{\xi_1} \int_\Omega (u_1 - \xi_1)^2 + \frac{r_2}{\xi_2} \int_\Omega (u_2 - \xi_2)^2 \leq \frac{C_2^2}{2C_{d_2}} \int_\Omega (u_1 + 1)^{\alpha_2 + 2\beta_2 - 2} |\nabla v|^2. \tag{5.12}
\]

for all \( t > 0 \). Testing the third equation of (1.4) by \( v - \xi_1^2 - \xi_2^2 \) and integrating over \( \Omega \) by parts then yields
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega (v - \xi_1^2 - \xi_2^2)^2 + \int_\Omega |\nabla v|^2 \leq - \int_\Omega (v - \xi_1^2 - \xi_2^2)^2 + \int_\Omega (u_1^2 + u_2^2 - \xi_1^2 - \xi_2^2) (v - \xi_1^2 - \xi_2^2) 
\leq - \frac{1}{2} \int_\Omega (v - \xi_1^2 - \xi_2^2)^2 + \frac{1}{2} \int_\Omega (u_1^2 + u_2^2 - \xi_1^2 - \xi_2^2)^2 \leq - \frac{1}{2} \int_\Omega (v - \xi_1^2 - \xi_2^2)^2 + \int_\Omega (u_1^2 - \xi_1^2)^2 + \int_\Omega (u_2^2 - \xi_2^2)^2 \tag{5.13}
\]
for all $t > 0$, where we used the elementary inequality $(u_i^2 - \xi_i^1 + u_j^2 - \xi_j^2)^2 \leq 2(u_i^2 - \xi_i^1)^2 + 2(u_j^2 - \xi_j^2)^2$. We can deal with it under different ranges of $\gamma_i$ ($i = 1, 2$) for the last two terms of (5.13) in the following.

In the case of $\gamma_i \in (0, 1)$, if $u_1(x, t) \leq \frac{\xi_1}{2}$ for $(x, t) \in \Omega \times (0, \infty)$, we derive

$$|u_i^1 - \xi_i^1| \leq |u_1 - \xi_1| = |u_1 - \xi_1| |\gamma_i - 1| |u_1 - \xi_1| \leq 2^{1 - \gamma_i}|\xi_i^1 - 1||u_1 - \xi_1|;$$  \hspace{1cm} (5.14)

if $u_1(x, t) > \frac{\xi_1}{2}$ for $(x, t) \in \Omega \times (0, \infty)$, let $h_i(s) := s^{\gamma_i}$ on $s \in (\frac{\xi_1}{2}, \infty)$, in view of the mean value theorem and $h_i'(s) = \gamma_i s^{\gamma_i - 1}$ is a monotone decreasing function on $(\frac{\xi_1}{2}, \infty)$, then we can obtain

$$|u_i^1 - \xi_i^1| = h_1(u_1 - h_1(\xi_1)) \leq h_1'(u_1 - \theta_1 u_1 + \theta_1 \xi_1)|u_1 - \xi_1|$$

$$\leq \gamma_i^{1 - \gamma_i}|\xi_i^1 - 1||u_1 - \xi_1|$$  \hspace{1cm} (5.15)

with some $\theta_1 \in (0, 1)$, where we also used $u_1 - \theta_1 u_1 + \theta_1 \xi_1 > \frac{\xi_1}{2}$ if $u_1 \geq \frac{\xi_1}{2}$.

In the case of $\gamma_i \geq 1$, in virtue of the Mean value theorem and the monotone increasing function $h_i(s) = \gamma_i s^{\gamma_i - 1}$, then we can obtain

$$|u_i^1 - \xi_i^1| = |h_1(u_1 - h_1(\xi_1))| \leq h_1'(u_1 - \theta_1 u_1 + \theta_1 \xi_1)|u_1 - \xi_1|$$

$$\leq \gamma_i(u_1 + \xi_1)\gamma_i^{-1}||u_1 - \xi_1|.$$  \hspace{1cm} (5.16)

Therefore, a combination of (5.14) – (5.16) we get

$$(u_i^1 - \xi_i^1)^2 \leq H_1(u_1)(u_1 - \xi_1)^2;$$  \hspace{1cm} (5.17)

where $H_1(u_1)$ is defined in (5.6). Similarly, according to $H_2(u_2)$ in (5.7) we also have

$$(u_i^2 - \xi_i^2)^2 \leq H_2(u_2)(u_2 - \xi_2)^2.$$  \hspace{1cm} (5.18)

Substituting (5.17) and (5.18) into (5.13) results in

$$\frac{d}{dt} F(t) + 2(a_1 + a_2) \int_\Omega |
abla v|^2 + (a_1 + a_2) \int_\Omega (v - \xi_1^1 - \xi_2^2)^2$$

$$+ r_1 \left( u_1^{k_1 - 1} - \xi_1^{k_1 - 1} \right)^2 + r_2 \left( u_2^{k_2 - 1} - \xi_2^{k_2 - 1} \right)^2$$

$$\leq \frac{C_2^2}{2C_{d_1}} \int_\Omega (u_1 + 1)^{a_1 + 2\beta_1 - 2} |\nabla v|^2 + \frac{C_2^2}{2C_{d_2}} \int_\Omega (u_2 + 1)^{a_2 + 2\beta_2 - 2} |\nabla v|^2$$

$$+ 2(a_1 + a_2) \int_\Omega H_1(u_1)(u_1^{k_1 - 1} - \xi_1^{k_1 - 1})^2 + 2(a_1 + a_2) \int_\Omega H_2(u_2)(u_2^{k_2 - 1} - \xi_2^{k_2 - 1})^2$$

$$\text{for all } t > 0$$

Lemma 5.3. Suppose that (1.6) – (1.8) and (5.1) – (5.3) hold, $r_i > 0$, if $k_i < 2$ ($i = 1, 2$), then the global bounded smooth solution of (1.4) satisfies

$$\frac{d}{dt} F(t) + 2(a_1 + a_2) \int_\Omega |
abla v|^2 + (a_1 + a_2) \int_\Omega (v - \xi_1^1 - \xi_2^2)^2$$

$$+ r_1 \left( u_1^{k_1 - 1} - \xi_1^{k_1 - 1} \right)^2 + r_2 \left( u_2^{k_2 - 1} - \xi_2^{k_2 - 1} \right)^2$$

$$\leq \frac{C_2^2}{2C_{d_1}} \int_\Omega (u_1 + 1)^{a_1 + 2\beta_1 - 2} |\nabla v|^2 + \frac{C_2^2}{2C_{d_2}} \int_\Omega (u_2 + 1)^{a_2 + 2\beta_2 - 2} |\nabla v|^2$$

$$+ 2(a_1 + a_2) \int_\Omega H_1(u_1)(u_1^{k_1 - 1} - \xi_1^{k_1 - 1})^2 + 2(a_1 + a_2) \int_\Omega H_2(u_2)(u_2^{k_2 - 1} - \xi_2^{k_2 - 1})^2$$

$$\text{for all } t > 0$$

with

$$\tilde{H}_1(u_1) := \begin{cases} 
\left( 1 - \frac{1}{2(\gamma_1 - 1)} \right)^{\frac{2}{\gamma_1 - 2}} k_1 - 1 \gamma_1^{2k_1 - 2k_1 + 2} u_1^{k_1 - 1} - \xi_1^{k_1 - 1} & \text{if } \gamma_1 < k_1 - 1, \\
\left( \frac{\gamma_1^2}{(k_1 - 1)^2} \right)^{\frac{2}{\gamma_1 - 2}} k_1 - 1 \frac{\gamma_1^{2k_1 - 2k_1 + 2}}{(k_1 - 1)^2} u_1^{k_1 - 1} - \xi_1^{k_1 - 1} & \text{if } \gamma_1 \geq k_1 - 1.
\end{cases}$$  \hspace{1cm} (5.21)
and
\[ H_2(u_2) := \begin{cases} (1 - \frac{1}{2^{2^{k_2}-1}})^{\frac{2^{2^{k_2}-1}}{2^{k_2}-2}} & \text{if } \gamma_2 < k_2 - 1, \\
\frac{\gamma_2^2}{(k_2-1)^2} (u_2^{k_2-1} + \xi_2^{k_2-1})^{\frac{2^{2^{k_2}-1}}{2^{k_2}-2}} & \text{if } \gamma_2 \geq k_2 - 1. \end{cases} \] (5.22)

**Proof.** In view of (5.8) - (5.9) we get
\[ \frac{d}{dt} \int_{\Omega} b_1(u_1) \leq \frac{C_2}{2C_1} \int_{\Omega} (u_2 + 1)^{\alpha_1+2\beta_1-2}|\nabla v|^2 - \mu_1 \int_{\Omega} (u_1 - \xi_1)(u_1^{k_1-1} - \frac{r_1}{\mu_1}) \] (5.23)
for all \( t > 0 \). Thanks to the conditions \( k_i < 2 \) (\( i = 1, 2 \)), we derive
\[ (u_1 - \xi_1)(u_1^{k_1-1} - \frac{r_1}{\mu_1}) = (u_1 - \xi_1)(u_1^{k_1-1} - \xi_1^{k_1-1}) \geq \xi_1^{2-k_1}(u_1^{k_1-1} - \xi_1^{k_1-1})^2, \] (5.24)
therefore, in conjunction with (5.23) and (5.24) we obtain
\[ \frac{d}{dt} \int_{\Omega} b_1(u_1) \leq \frac{C_2}{2C_1} \int_{\Omega} (u_1 + 1)^{\alpha_1+2\beta_1-2}|\nabla v|^2 - r_1 \xi_1^{3-2k_1} \int_{\Omega} (u_1^{k_1-1} - \xi_1^{k_1-1})^2, \] (5.25)
by similar treatment to the second equation of (1.4), combining with (5.25) results in
\[ \frac{d}{dt} \int_{\Omega} (b_1(u_1) + b_2(u_2)) \leq \frac{C_2}{2C_1} \int_{\Omega} (u_1 + 1)^{\alpha_1+2\beta_1-2}|\nabla v|^2 + \frac{C_2}{2C_2} \int_{\Omega} (u_2 + 1)^{\alpha_2+2\beta_2-2}|\nabla v|^2 \]
\[ - r_1 \xi_1^{3-2k_1} \int_{\Omega} (u_1^{k_1-1} - \xi_1^{k_1-1})^2 - r_2 \xi_2^{3-2k_2} \int_{\Omega} (u_2^{k_2-1} - \xi_2^{k_2-1})^2. \] (5.26)
From (5.13) we have
\[ \frac{d}{dt} \int_{\Omega} (v - \xi_1^{\gamma_1} - \xi_2^{\gamma_2})^2 + 2 \int_{\Omega} |\nabla v|^2 + \int_{\Omega} (v - \xi_1^{\gamma_1} - \xi_2^{\gamma_2})^2 \]
\[ \leq 2 \int_{\Omega} (u_1^{\gamma_1} - \xi_1^{\gamma_1})^2 + 2 \int_{\Omega} (u_2^{\gamma_2} - \xi_2^{\gamma_2})^2. \] (5.27)
For the last two terms of (5.27), the treatment is similar to Lemma 5.2, we deal with it under different ranges of \( \gamma_i \) (\( i = 1, 2 \)) for the last two terms of (5.13) in the following.

In the case of \( \gamma_1 \in (0, k_1 - 1) \), if \( u_1(x, t) \leq \frac{\xi_1}{2} \) on \( (x, t) \in \Omega \times (0, \infty) \), we derive
\[ |u_1^{\gamma_1} - \xi_1^{\gamma_1}| = \left| \left( u_1^{k_1-1} \right)^{\frac{\gamma_1-1}{k_1-1}} - \left( \xi_1^{k_1-1} \right)^{\frac{\gamma_1-1}{k_1-1}} \right| \leq \left| u_1^{k_1-1} - \xi_1^{k_1-1} \right| \]
\[ = u_1^{k_1-1} - \xi_1^{k_1-1} \left| \frac{\gamma_1-1}{k_1-1} \right| u_1^{k_1-1} - \xi_1^{k_1-1} \]
\[ \leq \left( 1 - \frac{1}{2^{k_1-1}} \right)^{\frac{\gamma_1-1}{k_1-1}} \xi_1^{1-k_1+1} \left| u_1^{k_1-1} - \xi_1^{k_1-1} \right|, \] (5.28)
if \( u_1(x, t) > \frac{\xi_1}{2} \) on \( (x, t) \in \Omega \times (0, \infty) \), let \( \hat{h}_1(s) := s^{\frac{\gamma_1-1}{k_1-1}} \) on \( s \in \left( \frac{\xi_1}{2}, \infty \right) \), in view of the mean value theorem again and \( \hat{h}_1(s) = \frac{\gamma_1}{k_1-1} s^{\frac{\gamma_1-1}{k_1-1}} \) is a monotone decreasing function on \( \left( \frac{\xi_1}{2}, \infty \right) \), then we can obtain
\[ |u_1^{\gamma_1} - \xi_1^{\gamma_1}| = \left| \hat{h}_1 \left( u_1^{k_1-1} \right) - \hat{h}_1 \left( \xi_1^{k_1-1} \right) \right| \]
\[
\lesssim \tilde{h}_1' \left( u_1^{k_1-1} - \theta_2 u_1^{k_1-1} + \theta_2 \xi_1^{k_1-1} \right) \left| u_1^{k_1-1} - \xi_1^{k_1-1} \right|
\]
\[
\leq \frac{\gamma_1}{k_1 - 1} \left| u_1^{k_1-1} - \xi_1^{k_1-1} \right|
\]
with some \( \theta_2 \in (0, 1) \), where we also used \( u_1^{k_1-1} - \theta_2 u_1^{k_1-1} + \theta_2 \xi_1^{k_1-1} > \frac{\xi_1^{k_1-1}}{2} \) if \( u_1 > \frac{\xi_1}{2} \).

In the case of \( \gamma_1 \geq k_1 - 1 \), in virtue of the Mean value theorem and the monotone increasing function \( \tilde{h}_1(s) = \frac{\gamma_1}{k_1 - 1} s^{\gamma_1 - k_1 + 1} \), then we can obtain
\[
\left| u_1^{\gamma_1} - \xi_1^{\gamma_1} \right| = \left| \tilde{h}_1 \left( u_1^{k_1-1} \right) - \tilde{h}_1 \left( \xi_1^{k_1-1} \right) \right|
\]
\[
\leq \frac{\gamma_1}{k_1 - 1} \left( u_1^{k_1-1} - \theta_2 u_1^{k_1-1} + \theta_2 \xi_1^{k_1-1} \right) \left| u_1^{k_1-1} - \xi_1^{k_1-1} \right|
\]
\[
\leq \frac{\gamma_1}{k_1 - 1} \left( u_1^{k_1-1} + \xi_1^{k_1-1} \right)^{\gamma_1 - k_1 + 1} \left| u_1^{k_1-1} - \xi_1^{k_1-1} \right|. \quad (5.30)
\]

Therefore, a combination of (5.28) – (5.30) we get
\[
\left( u_1^{\gamma_1} - \xi_1^{\gamma_1} \right)^2 \leq \tilde{H}_1(u_1) \left( u_1^{k_1-1} - \xi_1^{k_1-1} \right)^2, \quad (5.31)
\]
where \( \tilde{H}_1(u_1) \) is defined in (5.21). Similarly, according to \( \tilde{H}_2(u_2) \) in (5.22) we also have
\[
\left( u_2^{\gamma_2} - \xi_2^{\gamma_2} \right)^2 \leq \tilde{H}_2(u_2) \left( u_2^{k_2-1} - \xi_2^{k_2-1} \right)^2. \quad (5.32)
\]

Substituting (5.31) and (5.32) into (5.27) results in
\[
\frac{d}{dt} \int_\Omega (v - \xi_1^{\gamma_1} - \xi_2^{\gamma_2})^2 + 2 \int_\Omega \left| \nabla v \right|^2 + \int_\Omega (v - \xi_1^{\gamma_1} - \xi_2^{\gamma_2})^2
\]
\[
\leq 2 \int_\Omega \tilde{H}_1(u_1) \left( u_1^{k_1-1} - \xi_1^{k_1-1} \right)^2 + 2 \int_\Omega \tilde{H}_2(u_2) \left( u_2^{k_2-1} - \xi_2^{k_2-1} \right)^2 \quad (5.33)
\]
for all \( t > 0 \). Therefore, (5.20) is the result of combination of (5.26) and (5.33). \( \square \)

Proof of Theorem 1.3. For \( (i = 1, 2) \), let \( \gamma_i \geq \frac{1}{2} \) and \( \xi_i := \left( \frac{r_i}{\mu_i} \right)^{-\frac{1}{\gamma_i - 1}} \), there exist
\( C_{u_1} := (\|u_1\|_{L^\infty} + 1)^{\alpha_1 + 2\beta_i - 2} > 0 \) and \( C_{u_2} := (\|u_2\|_{L^\infty} + 1)^{\alpha_2 + 2\beta_i - 2} > 0 \) such that
\[
\frac{C_{u_1}^{\gamma_1}}{2C_{d_1}} \int_\Omega (u_1 + 1)^{\alpha_1 + 2\beta_i - 2} |\nabla v|^2 + \frac{C_{u_2}^{\gamma_2}}{2C_{d_2}} \int_\Omega (u_2 + 1)^{\alpha_2 + 2\beta_i - 2} |\nabla v|^2
\]
\[
\leq \left( \frac{C_{u_1}^{\gamma_1} r_1}{2C_{d_1} \mu_1} \right)^{\frac{1}{\gamma_i - 1}} + \left( \frac{C_{u_2}^{\gamma_2} r_2}{2C_{d_2} \mu_2} \right)^{\frac{1}{\gamma_i - 1}} \int_\Omega |\nabla v|^2 \quad (5.34)
\]
for all \( t > 0 \). We derive the large time behavior of global bounded solution by dividing the ranges of \( k_i \) and \( \gamma_i \) \( (i = 1, 2) \) for eight cases in the following.

Case 1. \( k_i \geq 2, 1 \leq \gamma_i < 1 \) \( (i = 1, 2) \).

According to (5.6), (5.7) and \( \xi_i := \left( \frac{r_i}{\mu_i} \right)^{-\frac{1}{\gamma_i - 1}} \) then yields
\[
2(a_1 + a_2) \int_\Omega H_1(u_1)(u_1 - \xi_1)^2 + 2(a_1 + a_2) \int_\Omega H_2(u_2)(u_2 - \xi_2)^2 \quad (5.35)
\]
\[
= 2(a_1 + a_2) 4^{1-\gamma_1} \left( \frac{r_1}{\mu_1} \right)^{\frac{2-\gamma_1}{\gamma_1 - 1}} \int_\Omega (u_1 - \xi_1)^2 + 2(a_1 + a_2) 4^{1-\gamma_2} \left( \frac{r_2}{\mu_2} \right)^{\frac{2-\gamma_2}{\gamma_2 - 1}} \int_\Omega (u_2 - \xi_2)^2
\]
with \( a_i \) are defined in (5.2). Combining (5.5), (5.34) and (5.35) we have
\[
\frac{d}{dt} F(t) + (a_1 + a_2) \int_\Omega (v - \xi_1^{\gamma_1} - \xi_2^{\gamma_2})^2
\]
In view of (5.3) and (5.4), we infer that

\[ \mu \] for all terms of (5.36), as follows by the definitions of \( a \), and 1

\[ \int_\Omega |\nabla v|^2 \]

\[ + \left( 2(a_1 + a_2)4^{1-\gamma_1} \left( \frac{r_1}{\mu_1} \right)^{\frac{\gamma_1-2}{\gamma_1-1}} - \left( \frac{r_1}{\mu_1} \right)^{\frac{\gamma_1-2}{\gamma_1-1}} \right) \int_\Omega (u_1 - \xi_1)^2 \]

\[ + \left( 2(a_1 + a_2)4^{1-\gamma_2} \left( \frac{r_2}{\mu_2} \right)^{\frac{\gamma_2-2}{\gamma_2-1}} - \left( \frac{r_2}{\mu_2} \right)^{\frac{\gamma_2-2}{\gamma_2-1}} \right) \int_\Omega (u_2 - \xi_2)^2 \]  

(5.36)

for all \( t > 0 \), and \( \frac{\gamma_1}{\mu_1} \). Similarly, we also have

\[ 2(a_1 + a_2)4^{1-\gamma_1} \left( \frac{r_1}{\mu_1} \right)^{\frac{\gamma_1-2}{\gamma_1-1}} \]

\[ \mu \]

\[ \rightarrow \infty \] thanks to \( \gamma_1 \geq \frac{1}{2} \), therefore, there exists a large enough \( \mu_{31} > 0 \) such that

\[ r_1 \left( \frac{r_1}{\mu_1} \right)^{-\frac{1}{\gamma_1-1}} > 2(a_1 + a_2)4^{1-\gamma_1} \left( \frac{r_1}{\mu_1} \right)^{\frac{\gamma_1-2}{\gamma_1-1}} + 1 \]  

(5.38)

with \( \mu_i > \mu_{31} \). Similarly, we also have

\[ d \]

\[ \frac{d}{dt} F(t) + \int_\Omega (u_1 - \xi_1)^2 + \int_\Omega (u_2 - \xi_2)^2 + (a_1 + a_2) \int_\Omega (v - \xi_1^2 - \xi_2^2)^2 \leq 0 \]  

(5.40)

with \( \mu_i > \mu_{31} \) and \( \gamma_i \geq \frac{1}{2} \) \( i = 1, 2 \). From (5.40) we obtain

\[ \int_1^\infty \int_\Omega (u_1 - \xi_1)^2 < \infty \]  

and \( \int_1^\infty \int_\Omega (u_2 - \xi_2)^2 < \infty \).

Therefore, in virtue of [1, Lemma 3.1] and Lemma 5.1 then yields

\[ \|u_1(\cdot, t) - \xi_1\|_{L^2(\Omega)} \rightarrow 0 \]  

and \( \|u_2(\cdot, t) - \xi_2\|_{L^2(\Omega)} \rightarrow 0 \) as \( t \rightarrow \infty \),

this combined with Gagliardo-Nirenberg inequality we obtain

\[ u_1(\cdot, t) \rightarrow \xi_1 \]  

and \( u_2(\cdot, t) \rightarrow \xi_2 \) in \( L^\infty(\Omega) \) as \( t \rightarrow \infty \).

In view of (5.3) and (5.4), we infer that

\[ \frac{1}{4\xi_1}(u_1(\cdot, t) - \xi_1)^2 \leq b_1(u_1(\cdot, t)) \leq \frac{1}{\xi_1}(u_1(\cdot, t) - \xi_1)^2 \]  

(5.41)

and

\[ \frac{1}{4\xi_2}(u_2(\cdot, t) - \xi_2)^2 \leq b_2(u_2(\cdot, t)) \leq \frac{1}{\xi_2}(u_2(\cdot, t) - \xi_2)^2 \]  

(5.42)

for all \( t \geq t_0 > 0 \), in conjunction with (5.41), (5.42) and (5.40), there exists \( c_2 > 0 \) such that \( \frac{d}{dt} F(t) + c_2 F(t) \leq 0 \) for all \( t > t_0 > 0 \), therefore, this combined with the
Gronwall inequality and there exists $c_3, l > 0$ such that $F(t) \leq c_3 e^{-lt}$ for all $t > 0$. In virtue of (5.41) and (5.42) again results in
\[
\int_{\Omega} \left( u_1(\cdot, t) - \left( \frac{r_1}{\mu_1} \right)^{\frac{1}{1-\gamma}} \right)^2 + \int_{\Omega} \left( u_2(\cdot, t) - \left( \frac{r_2}{\mu_2} \right)^{\frac{1}{2 + \gamma}} \right)^2 \\
\int_{\Omega} \left( v(\cdot, t) - \left( \frac{r_1}{\mu_1} \right)^{\frac{1}{1-\gamma}} - \left( \frac{r_2}{\mu_2} \right)^{\frac{1}{2 + \gamma}} \right)^2 \leq c_4 F(t) \leq c_5 e^{-lt} \tag{5.43}
\]
for all $t > t_0 > 0$ with some $c_4, c_5 > 0$, therefore, this combined with Lemma 5.1 and the Gagliardo-Nirenberg inequality we obtain
\[
\|u_1(\cdot, t) - \left( \frac{r_1}{\mu_1} \right)^{\frac{1}{1-\gamma}}\|_{L^\infty(\Omega)} + \|u_2(\cdot, t) - \left( \frac{r_2}{\mu_2} \right)^{\frac{1}{2 + \gamma}}\|_{L^\infty(\Omega)} \\
\leq c_6 \|u_1(\cdot, t)\|_{W^{1, \infty}(\Omega)} \|u_1(\cdot, t) - \left( \frac{r_1}{\mu_1} \right)^{\frac{1}{1-\gamma}}\|_{L^2(\Omega)} \\
+ c_6 \|u_2(\cdot, t)\|_{W^{1, \infty}(\Omega)} \|u_2(\cdot, t) - \left( \frac{r_2}{\mu_2} \right)^{\frac{1}{2 + \gamma}}\|_{L^2(\Omega)} \\
\leq c_7 e^{-\frac{1}{2} t}
\]
for all $t > t_0 > 0$ with some $c_6, c_7 > 0$. Similarly, we also have
\[
\|v(\cdot, t) - \left( \frac{r_1}{\mu_1} \right)^{\frac{1}{1-\gamma}} - \left( \frac{r_2}{\mu_2} \right)^{\frac{1}{2 + \gamma}}\|_{L^\infty(\Omega)} \leq c_8 e^{-\frac{1}{2} t}
\]
for all $t > t_0 > 0$ with $c_8 > 0$.

**Case 2.** $k_i \geq 2, 0 < \frac{1}{2} \leq \gamma_1 < 1, \gamma_2 \geq 1$ (i = 1, 2).

Similar to Case 1, according to (5.6), (5.7) and $\xi = (\frac{r_1}{\mu_1})^{\frac{1}{1-\gamma}}$, then yields
\[
2(a_1 + a_2) \int_{\Omega} H_1(u_1)(u_1 - \xi_1)^2 + 2(a_1 + a_2) \int_{\Omega} H_2(u_2)(u_2 - \xi_2)^2 \\
= 2(a_1 + a_2)4^{1-\gamma_1} \left( \frac{r_1}{\mu_1} \right)^{\frac{2\gamma_1 - 2}{1-\gamma_1}} \int_{\Omega} (u_1 - \xi_1)^2 + 2(a_1 + a_2)\gamma_2^2 \int_{\Omega} (u_2 + \xi_2)^{2\gamma_2 - 2}(u_2 - \xi_2)^2 \\
\leq 2(a_1 + a_2)4^{1-\gamma_1} \left( \frac{r_1}{\mu_1} \right)^{\frac{2\gamma_1 - 2}{1-\gamma_1}} \int_{\Omega} (u_1 - \xi_1)^2 \\
+ 2(a_1 + a_2)\gamma_2^2 \left( C_{u_4} + \left( \frac{r_2}{\mu_2} \right)^{\frac{1}{2 + \gamma}} \right)^{2\gamma_2 - 2} \int_{\Omega} (u_2 - \xi_2)^2 \tag{5.44}
\]
for all $t > 0$ with some $C_{u_4} = C_{u_4}(\|u_2\|_{L^\infty(\Omega)}) > 0$ and $a_i$ are defined in (5.2). Combining (5.2), (5.5), (5.34) and (5.44) we have
\[
\frac{d}{dt} F(t) + (a_1 + a_2) \int_{\Omega} (v - \xi_1^{\gamma_1} - \xi_2^{\gamma_2})^2 \\
\leq 2(a_1 + a_2)4^{1-\gamma_1} \left( \frac{r_1}{\mu_1} \right)^{\frac{2\gamma_1 - 2}{1-\gamma_1}} - r_1 \left( \frac{r_1}{\mu_1} \right)^{\frac{1}{1-\gamma_1}} \int_{\Omega} (u_1 - \xi_1)^2 \\
+ 2(a_1 + a_2)\gamma_2^2 \left( C_{u_4} + \left( \frac{r_2}{\mu_2} \right)^{\frac{1}{2 + \gamma}} \right)^{2\gamma_2 - 2} - r_2 \left( \frac{r_2}{\mu_2} \right)^{\frac{1}{2 + \gamma}} \int_{\Omega} (u_2 - \xi_2)^2 \tag{5.45}
\]
for all $t > 0$. Thus, similar to (5.37), there exists a large enough $\mu_{32} > 0$ such that
\[
    r_1 \left( \frac{r_1}{\mu_1} \right)^{-\frac{1}{\gamma_1-1}} > 2(a_1 + a_2) 4^{1-\gamma_1} \left( \frac{r_1}{\mu_1} \right)^{\frac{2\gamma_1-2}{\gamma_1-1}} + 1
\]  
(5.46)
with $\mu_i > \mu_{32}$ and $\gamma_1 \geq \frac{1}{2}$. Similarly, we also have
\[
    r_2 \left( \frac{r_2}{\mu_2} \right)^{-\frac{1}{\gamma_2-1}} > 2(a_1 + a_2) \gamma_2^2 \left( C_{u_4} + \left( \frac{r_2}{\mu_2} \right)^{\frac{1}{\gamma_2-1}} \right)^{2\gamma_2-2} + 1
\]  
(5.47)
with $\mu_i > \mu_{32}$ and $\gamma_2 \geq 1$. Thus, substituting (5.46) and (5.47) into (5.45) yields
\[
    \frac{d}{dt} F(t) + \int_{\Omega} (u_1 - \xi_1)^2 + \int_{\Omega} (u_2 - \xi_2)^2 + (a_1 + a_2) \int_{\Omega} (v - \xi_1^{\gamma_1} - \xi_2^{\gamma_2})^2 \leq 0
\]
with $\mu_i > \mu_{32}$ and $\gamma_i \geq \frac{1}{2}$ and $\gamma_2 \geq 1$. The rest is handled similarly to Case 1, so we omit it here.

**Case 3.** $k_1 \geq 2$, $\gamma_1 \geq 1$, $\frac{1}{2} \leq \gamma_2 < 1$ $(i = 1, 2)$.
Case 3 and Case 2 are treated exactly the same, so we omit it.

**Case 4.** $k_1 \geq 2$, $\gamma_i \geq 1$ $(i = 1, 2)$.

According to (5.6), (5.7) and $\xi_i = \left( \frac{r_i}{\mu_i} \right)^{\frac{1}{\gamma_i-1}}$ then yields
\[
    2(a_1 + a_2) \int_{\Omega} H_1(u_1)(u_1 - \xi_1)^2 + 2(a_1 + a_2) \int_{\Omega} H_2(u_2)(u_2 - \xi_2)^2
\]
\[
    = (a_1 + a_2)^2 \gamma_1^2 \int_{\Omega} (u_1 + \xi_1)^{2\gamma_1-2}(u_1 - \xi_1)^2 + 2(a_1 + a_2) \gamma_2^2 \int_{\Omega} (u_2 + \xi_2)^{2\gamma_2-2}(u_2 - \xi_2)^2
\]
\[
    \leq 2(a_1 + a_2)^2 \gamma_1^2 \left( C_{u_3} + \left( \frac{r_1}{\mu_1} \right)^{\frac{1}{\gamma_1-1}} \right) \int_{\Omega} (u_1 - \xi_1)^2
\]
\[
    + 2(a_1 + a_2)^2 \gamma_2^2 \left( C_{u_4} + \left( \frac{r_2}{\mu_2} \right)^{\frac{1}{\gamma_2-1}} \right) \int_{\Omega} (u_2 - \xi_2)^2
\]  
(5.48)
for all $t > 0$ with some $C_{u_3} = C_{u_3}(\|u_1\|_{L^\infty}, \Omega) > 0$, $C_{u_4} = C_{u_4}(\|u_2\|_{L^\infty}, \Omega) > 0$ and $a_i$ are defined in (5.2). Combining (5.2), (5.5), (5.34) and (5.48) we have
\[
    \frac{d}{dt} F(t) + (a_1 + a_2) \int_{\Omega} (v - \xi_1^{\gamma_1} - \xi_2^{\gamma_2})^2
\]
\[
    \leq \left( 2(a_1 + a_2)^2 \gamma_1^2 \left( C_{u_3} + \left( \frac{r_1}{\mu_1} \right)^{\frac{1}{\gamma_1-1}} \right) \int_{\Omega} (u_1 - \xi_1)^2 - r_1 \left( \frac{r_1}{\mu_1} \right)^{\frac{1}{\gamma_1-1}} \right) \int_{\Omega} (u_1 - \xi_1)^2
\]
\[
    + \left( 2(a_1 + a_2)^2 \gamma_2^2 \left( C_{u_4} + \left( \frac{r_2}{\mu_2} \right)^{\frac{1}{\gamma_2-1}} \right) \int_{\Omega} (u_2 - \xi_2)^2 - r_2 \left( \frac{r_2}{\mu_2} \right)^{\frac{1}{\gamma_2-1}} \right) \int_{\Omega} (u_2 - \xi_2)^2
\]  
(5.49)
for all $t > 0$. Thus, similar to (5.37), there exists a large enough $\mu_{34} > 0$ such that
\[
    r_1 \left( \frac{r_1}{\mu_1} \right)^{-\frac{1}{\gamma_1-1}} > 2(a_1 + a_2) \gamma_1^2 \left( C_{u_3} + \left( \frac{r_1}{\mu_1} \right)^{\frac{1}{\gamma_1-1}} \right)^{2\gamma_1-2} + 1
\]  
(5.50)
and
\[
    r_2 \left( \frac{r_2}{\mu_2} \right)^{-\frac{1}{\gamma_2-1}} > 2(a_1 + a_2) \gamma_2^2 \left( C_{u_4} + \left( \frac{r_2}{\mu_2} \right)^{\frac{1}{\gamma_2-1}} \right)^{2\gamma_2-2} + 1
\]  
(5.51)
with $\mu_i > \mu_{34}$ and $\gamma_i \geq 1$ $(i = 1, 2)$. Thus, substituting (5.50) and (5.51) into (5.49) yields

$$\frac{d}{dt} F(t) + \int_{\Omega} (u_1 - \xi_1)^2 + \int_{\Omega} (u_2 - \xi_2)^2 + (a_1 + a_2) \int_{\Omega} (v - \xi_1^{\gamma_1} - \xi_2^{\gamma_2})^2 \leq 0$$

with $\mu_i > \mu_{34}$ and $\gamma_i \geq 1$. The rest is handled similarly to Case 1, so we omit it here. 

**Case 5.** $k_i < 2, \frac{1}{2} \leq \gamma_i < k_i - 1$ $(i = 1, 2)$.

According to (5.21) and (5.22) and $\xi_i = \left(\frac{r_i}{\mu_i}\right)^{\frac{1}{k_i-1}}$, then yields

$$2(a_1 + a_2) \int_{\Omega} \hat{H}_1(u_1) \left(u_1^{k_1-1} - \xi_1^{k_1-1}\right)^2 + 2(a_1 + a_2) \int_{\Omega} \hat{H}_2(u_2) \left(u_2^{k_2-1} - \xi_2^{k_2-1}\right)^2$$

$$= 2(a_1 + a_2) \left(1 - \frac{1}{2k_1-1}\right) \int_{\Omega} \left(4^{k_1-1 - \gamma_1} \xi_1^{2\gamma_1 - 2k_1 + 2} - \int_{\Omega} \left(u_1^{k_1-1} - \xi_1^{k_1-1}\right)^2\right)$$

$$+ 2(a_1 + a_2) \left(1 - \frac{1}{2k_2-1}\right) \int_{\Omega} \left(4^{k_2-1 - \gamma_2} \xi_2^{2\gamma_2 - 2k_2 + 2} - \int_{\Omega} \left(u_2^{k_2-1} - \xi_2^{k_2-1}\right)^2\right)$$

for all $t > 0$ and $a_i$ are defined in (5.2). Combining (5.2), (5.20), (5.34) and (5.52) we have

$$\frac{d}{dt} F(t) + (a_1 + a_2) \int_{\Omega} (v - \xi_1^{\gamma_1} - \xi_2^{\gamma_2})^2 \leq \left(2(a_1 + a_2) \left(1 - \frac{1}{2k_1-1}\right) \int_{\Omega} \left(4^{k_1-1 - \gamma_1} \xi_1^{2\gamma_1 - 2k_1 + 2} - \int_{\Omega} \left(u_1^{k_1-1} - \xi_1^{k_1-1}\right)^2\right)$$

$$+ (2(a_1 + a_2) \left(1 - \frac{1}{2k_2-1}\right) \int_{\Omega} \left(4^{k_2-1 - \gamma_2} \xi_2^{2\gamma_2 - 2k_2 + 2} - \int_{\Omega} \left(u_2^{k_2-1} - \xi_2^{k_2-1}\right)^2\right)$$

for all $t > 0$. Thus, similar to (5.37) we derive

$$2(a_1 + a_2) \left(1 - \frac{1}{2r_i-1}\right) \frac{2^{\gamma_i} - 2}{r_i} \frac{4^{k_i-1 - \gamma_i} \xi_i^{2\gamma_i - 2k_i + 2}}{r_i}$$

$$= \left(\frac{C_{s_1}}{2C_{d_1}} \cdot \frac{r_1}{\mu_1} \right)^{\frac{1}{k_1-1}} + \left(\frac{C_{s_2}}{2C_{d_2}} \cdot \frac{r_2}{\mu_2} \right)^{\frac{1}{k_2-1}} \cdot \frac{1}{r_1} \left(1 - \frac{1}{2r_i-1}\right) \frac{2^{\gamma_i} - 2}{r_i} \frac{4^{k_i-1 - \gamma_i}}{r_i}$$

as $\mu_i \to \infty$ with $\gamma_i \geq \frac{1}{2}$, thus, there exists a large enough $\mu_{35} > 0$ such that

$$r_1 \xi_1^{3-2k_1} > 2(a_1 + a_2) \left(1 - \frac{1}{2k_1-1}\right) \frac{2^{\gamma_1} - 2}{r_1} \frac{4^{k_1-1 - \gamma_1} \xi_1^{2\gamma_1 - 2k_1 + 2}}{r_1} + 1$$

with $\mu_i > \mu_{35}$. Similarly, we also have

$$r_2 \xi_2^{3-2k_2} > 2(a_1 + a_2) \left(1 - \frac{1}{2k_2-1}\right) \frac{2^{\gamma_2} - 2}{r_1} \frac{4^{k_2-1 - \gamma_2} \xi_2^{2\gamma_2 - 2k_2 + 2}}{r_1} + 1$$

with $\mu_i > \mu_{35}$ and $\gamma_2 \geq \frac{1}{2}$. In order to get the desire result, we also need to deal with

$$(k_1 - 1)^2 \int_{\Omega} (u_1 - \xi_1)^2 \leq \int_{\Omega} \left(u_1^{k_1-1} - \xi_1^{k_1-1}\right)^2$$

(5.56)
for all \(u_1 \leq 2\xi_1\) and
\[
(k_2 - 1)^2(2\xi_2)^{2k_2 - 4} \int_\Omega (u_2 - \xi_2)^2 \leq \int_\Omega (u_2^{k_2 - 1} - \xi_2^{k_2 - 1})^2 \tag{5.57}
\]
for all \(u_2 \leq 2\xi_2\). Thus, substituting (5.54) – (5.57) into (5.53) yields
\[
\frac{d}{dt} F(t) + (k_1 - 1)^2(2\xi_1)^{2k_1 - 4} \int_\Omega (u_1 - \xi_1)^2 + (k_1 - 1)^2(2\xi_2)^{2k_2 - 4} \int_\Omega (u_2 - \xi_2)^2 \\
+ (a_1 + a_2) \int_\Omega (v - \xi_1^{\gamma_1} - \xi_2^{\gamma_2})^2 \leq 0
\]
with \(\mu_i > \mu_{36}\) and \(\gamma_i \geq \frac{1}{2}\). The rest is handled similarly to Case 1, so we omit it here.

**Case 6.** \(k_i < 2, \frac{1}{2} \leq \gamma_1 < k_1 - 1, \gamma_2 \geq \max\{k_2 - 1, \frac{1}{2}\} \ (i = 1, 2)\).

According to (5.21) and (5.22) and \(\xi_i = \left(\frac{\tau_i}{\mu_i}\right)^{\frac{1}{k_i - 1}}\) then yields
\[
2(a_1 + a_2) \int_\Omega \tilde{H}_1(u_1) \left(u_1^{k_1 - 1} - \xi_1^{k_1 - 1}\right)^2 + 2(a_1 + a_2) \int_\Omega \tilde{H}_2(u_2) \left(u_2^{k_2 - 1} - \xi_2^{k_2 - 1}\right)^2 \\
= 2(a_1 + a_2) \left(1 - \frac{1}{2^{k_1 - 1}}\right)^{2, 1 - 2} 4^{k_1 - 1 - \gamma_1} \xi_1^{2\gamma_1 - 2k_1 + 2} \int_\Omega (u_1^{k_1 - 1} - \xi_1^{k_1 - 1})^2 \\
+ 2(a_1 + a_2) \frac{\gamma_2^2}{(k_2 - 1)^2} \int_\Omega (u_2^{k_2 - 1} + \xi_2^{k_2 - 1})^{2, 2 - 2} 4^{k_2 - 1 - \gamma_2} \xi_2^{2\gamma_2 - 2k_2 + 2} (u_2^{k_2 - 1} - \xi_2^{k_2 - 1})^2 \\
\leq 2(a_1 + a_2) \left(1 - \frac{1}{2^{k_1 - 1}}\right)^{2, 1 - 2} 4^{k_1 - 1 - \gamma_1} \xi_1^{2\gamma_1 - 2k_1 + 2} \int_\Omega (u_1^{k_1 - 1} - \xi_1^{k_1 - 1})^2 \\
+ \left(\frac{2(a_1 + a_2) \gamma_2^2 C_{u_2}}{(k_2 - 1)^2} \left(\frac{\tau_2}{\mu_2}\right)^{2(\gamma_2 - (k_2 - 1))} - r_2 \xi_2^{3 - 2k_2}\right) \int_\Omega (u_2^{k_1 - 1} - \xi_1^{k_1 - 1})^2 \tag{5.58}
\]
for all \(t > 0\) with some \(C_{u_2} = C_{u_2}(k_2, \gamma_2, \mu_2, r_2, \|u_2\|_{L^\infty, \Omega}) > 0\) and \(a_i \ (i = 1, 2)\) are defined in (5.2), where we also used the elementary inequality \((u_2^{k_2 - 1} + \xi_2^{k_2 - 1})^{2, 2 - 2} 4^{k_2 - 1 - \gamma_2} \xi_2^{2\gamma_2 - 2k_2 + 2} \leq 4^{k_2 - 1} \xi_2^{3 - 2k_2}\). Combining (5.2), (5.20), (5.34) and (5.58) we have
\[
\frac{d}{dt} F(t) + (a_1 + a_2) \int_\Omega (v - \xi_1^{\gamma_1} - \xi_2^{\gamma_2})^2 \tag{5.59}
\]
\[
\leq 2(a_1 + a_2) \left(1 - \frac{1}{2^{k_1 - 1}}\right)^{2, 1 - 2} 4^{k_1 - 1 - \gamma_1} \xi_1^{2\gamma_1 - 2k_1 + 2} - r_1 \xi_1^{3 - 2k_1} \int_\Omega (u_1^{k_1 - 1} - \xi_1^{k_1 - 1})^2 \\
+ \left(\frac{2(a_1 + a_2) \gamma_2^2 C_{u_2}}{(k_2 - 1)^2} \left(\frac{\tau_2}{\mu_2}\right)^{2(\gamma_2 - (k_2 - 1))} - r_2 \xi_2^{3 - 2k_2}\right) \int_\Omega (u_2^{k_1 - 1} - \xi_1^{k_1 - 1})^2
\]
for all \(t > 0\). Thus, similar to (5.54), there exists a large enough \(\mu_{36} > 0\) such that
\[
r_1 \xi_1^{3 - 2k_1} > 2(a_1 + a_2) \left(1 - \frac{1}{2^{k_1 - 1}}\right)^{2, 1 - 2} 4^{k_1 - 1 - \gamma_1} \xi_1^{2\gamma_1 - 2k_1 + 2} + 1 \tag{5.60}
\]
with \(\mu_i > \mu_{36}\) and \(\gamma_1 \geq \frac{1}{2}\). Similarly, we also have
\[
r_2 \xi_2^{3 - 2k_2} > 2(a_1 + a_2) \gamma_2^2 C_{u_2} \left(\frac{\tau_2}{\mu_2}\right)^{2(\gamma_2 - (k_2 - 1))} + 1 \tag{5.61}
\]
with $\mu_i > \mu_{36}$ and $\gamma_2 \geq \frac{1}{2}$. Thus, substituting (5.60), (5.61), (5.56) and (5.57) into (5.59) yields
\[
\frac{d}{dt} F(t) + (k_1 - 1)^2(2\xi_1)^{2k_1 - 4} \int_\Omega (u_1 - \xi_1)^2 + (k_2 - 1)^2(2\xi_2)^{2k_2 - 4} \int_\Omega (u_2 - \xi_2)^2 \\
+ (a_1 + a_2) \int_\Omega (v - \xi_1^2 - \xi_2^2)^2 \leq 0
\]
with $\mu_i > \mu_{36}$ and $\gamma_i \geq \frac{1}{2}$. The rest is handled similarly to Case 1, so we omit it here.

**Case 7.** $k_1 < 2, \gamma_1 \geq \max\{k_2 - 1, \frac{1}{2}\}, \frac{1}{2} \leq \gamma_2 < k_1 - 1$ ($i = 1, 2$).

Case 7 and Case 6 are treated exactly the same, so we omit it.

**Case 8.** $k_1 < 2, \gamma_1 \geq \max\{k_2 - 1, \frac{1}{2}\}, \gamma_2 \geq \max\{k_1 - 1, \frac{1}{2}\}$ ($i = 1, 2$).

According to (5.21) and (5.22) and $\xi_i = \left(\frac{r_i}{\mu_i}\right) \frac{1}{k_i}$, similar to (5.58) then yields
\[
2(a_1 + a_2) \int_\Omega H_1(u_1) (u_1^{k_1 - 1} - \xi_1^{k_1 - 1})^2 + 2(a_1 + a_2) \int_\Omega H_2(u_2) (u_2^{k_2 - 1} - \xi_2^{k_2 - 1})^2 \\
\leq 2(a_1 + a_2) \frac{\gamma_1^2}{(k_1 - 1)^2} \int_\Omega (u_1^{k_1 - 1} + \xi_1^{k_1 - 1}) \frac{2\gamma_1}{k_1 - 1} - 2 (u_1^{k_1 - 1} - \xi_1^{k_1 - 1})^2 \\
+ 2(a_1 + a_2) \frac{\gamma_2^2}{(k_2 - 1)^2} \int_\Omega (u_2^{k_2 - 1} + \xi_2^{k_2 - 1}) \frac{2\gamma_2}{k_2 - 1} - 2 (u_2^{k_2 - 1} - \xi_2^{k_2 - 1})^2 \\
\leq 2(a_1 + a_2) \gamma_1^2 C_{u_1} \cdot \left(\frac{r_1}{\mu_1}\right) \frac{2\gamma_1}{k_1 - 1} \int_\Omega (u_1^{k_1 - 1} - \xi_1^{k_1 - 1})^2 \\
+ 2(a_1 + a_2) \gamma_2^2 C_{u_2} \cdot \left(\frac{r_2}{\mu_2}\right) \frac{2\gamma_2}{k_2 - 1} \int_\Omega (u_2^{k_2 - 1} - \xi_2^{k_2 - 1})^2
\]
for all $t > 0$ with some $C_{u_1} = C_{u_1}(k_1, \gamma_1, \mu_1, r_1, \|u_1\|_{L^\infty, \Omega}) > 0$ and $C_{u_2} = C_{u_2}(k_2, \gamma_2, \mu_2, r_2, \|u_2\|_{L^\infty, \Omega}) > 0$, $a_1$ ($i = 1, 2$) are defined in (5.2). Combining (5.2), (5.20), (5.34) and (5.62) we have
\[
\frac{d}{dt} F(t) + (a_1 + a_2) \int_\Omega (v - \xi_1^2 - \xi_2^2)^2 \\
\leq \left(2(a_1 + a_2) \gamma_1^2 C_{u_1} \cdot \left(\frac{r_1}{\mu_1}\right) \frac{2\gamma_1}{k_1 - 1} - r_1 \xi_1^{3 - 2k_1}\right) \int_\Omega (u_1^{k_1 - 1} - \xi_1^{k_1 - 1})^2 \\
+ \left(2(a_1 + a_2) \gamma_2^2 C_{u_2} \cdot \left(\frac{r_2}{\mu_2}\right) \frac{2\gamma_2}{k_2 - 1} - r_2 \xi_2^{3 - 2k_2}\right) \int_\Omega (u_2^{k_2 - 1} - \xi_1^{k_2 - 1})^2
\]
for all $t > 0$. Thus, similar to (5.61), there exists a large enough $\mu_{38} > 0$ such that
\[
r_1 \xi_1^{3 - 2k_1} > \frac{2(a_1 + a_2) \gamma_1^2 C_{u_1}}{(k_1 - 1)^2} \cdot \left(\frac{r_1}{\mu_1}\right) \frac{2\gamma_1}{k_1 - 1} + 1
\]
and
\[
r_2 \xi_2^{3 - 2k_2} > \frac{2(a_1 + a_2) \gamma_2^2 C_{u_2}}{(k_2 - 1)^2} \cdot \left(\frac{r_2}{\mu_2}\right) \frac{2\gamma_2}{k_2 - 1} + 1
\]
with \( \mu_i > \mu_{38} \) and \( \gamma_i \geq \frac{1}{2} \) \( (i = 1, 2) \). Thus, substituting (5.64), (5.65), (5.56) and (5.57) into (5.63) yields

\[
\frac{d}{dt} F(t) + (k_1 - 1)^2 (2 \xi_1)^{2k_1 - 4} \int_{\Omega} (u_1 - \xi_1)^2 + (k_2 - 1)^2 (2 \xi_2)^{2k_2 - 4} \int_{\Omega} (u_2 - \xi_2)^2 \\
+ (a_1 + a_2) \int_{\Omega} (v - \xi_1^\gamma_1 - \xi_2^\gamma_2)^2 \leq 0
\]

with \( \mu_i > \mu_{38} \) and \( \gamma_i \geq \frac{1}{2} \). The rest is handled similarly to Case 1, so we omit it here.

Therefore, the proof of Theorem 1.3 is completed.

Acknowledgments. The authors are very grateful to the anonymous reviewers for their carefully reading and valuable suggestions which greatly improved this work.

REFERENCES

[1] X. Bai and M. Winkler, Equilibration in a fully parabolic two-species chemotaxis system with competitive kinetics, Indiana Univ. Math. J., 65 (2016), 553–583.

[2] N. Bellomo, A. Bellouquid, Y. Tao and M. Winkler, Toward a mathematical theory of Keller-Segel models of pattern formation in biological tissues, Math. Models Methods Appl. Sci., 25 (2015), 1663–1763.

[3] T. Black, Global existence and asymptotic stability in a competitive two-species chemotaxis system with two signals, Discrete Contin. Dyn. Syst. Ser. B., 22 (2017), 1253–1272.

[4] T. Black, J. Lankeit and M. Mizukami, On the weakly competitive case in a two-species chemotaxis model, IMA J. Appl. Math., 81 (2016), 860–876.

[5] M. Ding, W. Wang, S. Zhou and S. Zheng, Asymptotic stability in a fully parabolic quasilinear chemotaxis model with general logistic source and signal production, J. Diff. Equations., 268 (2020), 6729–6777.

[6] E. Espejo, K. Vilches and C. Conca, A simultaneous blow-up problem arising in tumor modeling, J. Math. Biol., 79 (2019), 1357–1399.

[7] D. D. Haroske, H. Triebel, Distributions, Sobolev Spaces, Elliptic Equations, European Mathematical Society, Zurich, 2008.

[8] D. Horstmann and M. Winkler, Boundedness vs. blow-up in a chemotaxis system, J. Differ. Equ., 215 (2005), 52–107.

[9] S. Ishida, K. Seki and T. Yokota, Boundedness in quasilinear Keller-Segel systems of parabolic-parabolic type on non-convex bounded domains, J. Differ. Equ., 256 (2014), 2993–3010.

[10] E. F. Keller and L. A. Segel, Initiation of slime mold aggregation viewed as an instability, J. Theor. Biol., 26 (1970), 399–415.

[11] R. Kowalczyk and Z. Szymańska, On the global existence of solutions to an aggregation model, J. Math. Anal. Appl., 343 (2008), 379–398.

[12] D. Liu and Y. Tao, Boundedness in a chemotaxis system with nonlinear signal production, Appl. Math. J. Chin. Univ. Ser. B., 31 (2016), 379–388.

[13] N. Mizoguchi and P. Souplet, Nondegeneracy of blow-up points for the parabolic Keller-Segel system, Ann. Inst. H. Poincaré Anal. Non Linéaire., 31 (2014), 851–875.

[14] M. Mizukami, Improvement of conditions for asymptotic stability in a two-species chemotaxis-competition model with signal-dependent sensitivity, Discrete Contin. Dyn. Syst. Ser. S., 13 (2020), 269–278.

[15] M. Negreanu and J. I. Tello, Asymptotic stability of a two species chemotaxis system with non-diffusive chemotacticant, J. Differ. Equ., 258 (2015), 1592–1617.

[16] K. Osaki and A. Yagi, Global existence for a chemotaxis-growth system in \( R^2 \), Adv. Math. Sci. Appl., 12 (2002), 587–606.

[17] X. Pan and L. Wang, Boundedness in a two-species chemotaxis system with nonlinear sensitivity and signal secretion, J. Math. Anal. Appl., (2021), 125078.

[18] X. Pan and L. Wang, Improvement of conditions for boundedness in a fully parabolic chemotaxis system with nonlinear signal production, C. R. Math., 359 (2021), 161–168.

[19] X. Pan and L. Wang, On a quasilinear fully parabolic two-species chemotaxis system with two chemicals, Discrete Contin. Dyn. Syst. Ser. B., (2021).
[20] X. Pan, L. Wang and J. Zhang, Boundedness in a three-dimensional two-species and two-stimuli chemotaxis system with chemical signalling loop, Math. Method. Appl. Sci., 43 (2020), 9529–9542.

[21] X. Pan, L. Wang, J Zhang and J Wang, Boundedness in a three-dimensional two-species chemotaxis system with two chemicals, Z. Angew. Math. Phys., 71 (2020), 15pp.

[22] C. Stinner, J.I. Tello and M. Winkler, Competitive exclusion in a two-species chemotaxis model, J. Math. Biol., 68 (2014), 1607–1626.

[23] Y. Tao and M. Winkler, Boundedness in a quasilinear parabolic-parabolic Keller-Segel system with subcritical sensitivity, J. Differ. Equ., 252 (2012), 692–715.

[24] Y. Tao and M. Winkler, Boundedness vs. blow-up in a two-species chemotaxis system with two chemicals, Discrete Contin. Dyn. Syst. Ser. B., 20 (2015), 3165–3183.

[25] M. Tian and S. Zheng, Global boundedness versus finite-time blow-up of solutions to a quasilinear fully parabolic Keller-Segel system of two species, Commun. Pure Appl. Anal., 15 (2016), 243–260.

[26] J. I. Tello and M. Winkler, Stabilization in a two-species chemotaxis system with a logistic source, Nonlinearity, 25 (2012), 1413–1425.

[27] L. Wang, Improvement of conditions for boundedness in a two-species chemotaxis competition system of parabolic-parabolic-elliptic type, J. Math. Anal. Appl., 481 (2020), 123705.

[28] L. Wang, Y. Li and C. Mu, Boundedness in a parabolic-parabolic quasilinear chemotaxis system with logistic source, Discrete Contin. Dyn. Syst., 34 (2014), 789–802.

[29] L. Wang and C. Mu, A new result for boundedness and stabilization in a two-species chemotaxis system with two chemicals, Discrete Contin. Dyn. Syst. Ser. B., 25 (2020), 4585–4601.

[30] L. Wang, J. Zhang, C. Mu and X. Hu, Boundedness and stabilization in a two-species chemotaxis system with two chemicals, Discrete Contin. Dyn. Syst. Ser. B., 25 (2020), 191–221.

[31] M. Winkler, A critical blow-up exponent in a chemotaxis system with nonlinear signal production, Nonlinearity, 31 (2018), 2031–2056.

[32] T. Xiang, How strong a logistic damping can prevent blow-up for the minimal Keller-Segel chemotaxis system? J. Math. Anal. Appl., 459 (2018), 1172–1200.

[33] L. Xie and Y. Wang, On a fully parabolic chemotaxis system with Lotka-Volterra competitive kinetics, J. Math. Anal. Appl., 471 (2019), 584–598.

Received October 2020; revised March 2021.

E-mail address: panx_math@163.com
E-mail address: wanglc@cqupt.edu.cn