Decay of solutions of diffusive Oldroyd-B system in $\mathbb{R}^2$

Joonhyun La *1

1Department of mathematics, Princeton University

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Abstract

We show that strong solutions of 2D diffusive Oldroyd-B systems in $\mathbb{R}^2$ decay at an algebraic rate, for a large class of initial data. The main ingredient for the proof is the following fact; an Oldroyd-B system is a macroscopic closure of a Fokker-Planck-Navier-Stokes system, and the free energy of this Fokker-Planck-Navier-Stokes system decays over time. In particular, $\|u\|_{L^\infty_tL^2_x}$ and $\|\nabla_xu\|_{L^2_tL^2_x}$ are uniformly bounded for all time.

1 Introduction

We are interested in the long-time dynamics of the following system, which is called a diffusive Oldroyd-B system. It reads the following:

\[
\begin{aligned}
\partial_t u(x,t) + u(x,t) \cdot \nabla_x u(x,t) &= \nu_1 \Delta_x u(x,t) - \nabla_x p(x,t) + \mu \nabla_x \cdot \tau(x,t), \\
\nabla \cdot u &= 0, \\
\partial_t \tau + u \cdot \nabla_x \tau &= (\nabla_x u) \tau + \tau (\nabla_x u)^T - 2k \tau + \rho (\nabla_x u + (\nabla_x u)^T) + \nu_2 \Delta_x \tau, \\
\partial_t \rho + u \cdot \nabla \rho &= \nu_2 \Delta \rho, \\
u(x,0) &= u_0(x), \rho(x,0) = \rho_0, \tau(x,0) = \tau_0,
\end{aligned}
\]

(1)

where $\nu_1, k, \mu > 0$, and $\nu_2 > 0$ are constants representing viscosity, inter-particle diffusivity, strength of the stress due to polymer, and center-of-mass diffusion effects, respectively. This system is widely used to model viscoelastic flows. We will consider the case of the spatial domain $\mathbb{R}^2$, although we will also briefly discuss the case of the spatial domain $\mathbb{T}^2$. In the case of $\mathbb{T}^2$, it is customary to put $\rho(x,t) \equiv 1$.

The system (1) is extensively studied in the literature. Barrett and Boyaval proved global existence of weak solution in [1]. In [5], Constantin and Klige proved global well-posedness of strong solution. Plan, Gupta, Vincenzi, and Gibbon calculated Lyapunov dimension of the system (1) under the assumption of the existence of a global attractor in [22]. This system is a formal macroscopic closure of Fokker-Planck-Navier-Stokes system, and a rigorous justification is provided recently by Barrett and S"uli in [2]. The author, in [16], extended the result in [2] to a larger class of data and provided a rigorous proof of the fact that the free energy of the system decreases over time.

The models with $\nu_2 = 0$ are called non-diffusive models, and they are also widely studied. Guillopé and Saut proved local existence, uniqueness of strong solution, and global existence of strong solution for small initial data, in the case of bounded domain, in [11] and in [12].

*joonhyun@math.princeton.edu
Fernández-Cara, Guillén, and Ortega extended the results of Guilloté and Saut to $L^p$ setting in [8], [9], and [10]. In addition, Hieber, Naito, and Shibata studied the system in the case of exterior domain in [13]. Chemin and Masmoudi studied the system in critical Besov spaces, and proved local well-posedness of the system and provided a Beale-Kato-Majda type ([3]) criterion in [4]. Other Beale-Kato-Majda type sufficient conditions were given by Kupferman, Mangoubi, and Titi in [15], and by Lei, Masmoudi, and Zhou in [18]. In addition, Lions and Masmoudi showed global existence of weak solution for corotational models in [21]. Hu and Lin proved in [11] global existence of weak solution for non-corotational models, given that the initial deformation gradient is close to the identity and the initial velocity is small. In [20], Lin, Liu, and Zhang developed an approach based on deformation tensor and Lagrangian particle dynamics. Lei and Zhou studied the system via incompressible limit in [19]. Moreover, in [7], Fang and Zi proved global well-posedness for initial data whose vertical velocity field can be large. Constantin and Sun proved global existence for small data and incompressible limit in [17]. Furthermore, Fang and Zi proved global well-posedness for initial data whose vertical velocity field can be large. Constantin and Sun proved global existence for small data with large gradients for Oldroyd-B, and considered regularization of Oldroyd-B model in [9].

The study of long-time behavior of the system will rely on the Fourier splitting technique. This technique is widely used to study long-time behavior of parabolic systems ([24], [25], [26], [23], [28]).

## 2 Oldroyd-B system, Fokker-Planck-Navier-Stokes system, and Free energy estimate

The system (1) can be obtained from the Fokker-Planck-Navier-Stokes system as a macroscopic closure of it: from the system

\[
\begin{align*}
\partial_t u(x,t) + u(x,t) \cdot \nabla_x u(x,t) &= -\nabla_x p(x,t) + \nu_1 \Delta_x u(x,t) + \mu \nabla_x \cdot \sigma(x,t), \\
\nabla \cdot u &= 0, \\
\partial_t f(x,m,t) + u \cdot \nabla_x f + (\nabla_x u)m \cdot \nabla_m f &= k \Delta_m f + k \nabla_m \cdot (mf) + \nu_2 \Delta_x f, \\
\rho(x,t) &= \int_{\mathbb{R}^2} f(x,m,t) dm, \\
\sigma(x,t) &= \int_{\mathbb{R}^2} m \otimes mf(x,m,t) dm, \\
u(x,0) &= u_0(x), f(x,m,0) = f_0(x,m),
\end{align*}
\]

(2)

by letting $\tau = \sigma - \rho \bar{a}$, we obtain (1). In [16], we proved the following:

**Theorem 1** ([16]). *Suppose that*

\[
\begin{align*}
&u_0 \in \mathbb{P}^{W^{2,2},} \\
f_0 \geq 0, \int_{\mathbb{R}_+^2 \times \mathbb{R}^2} f_0(x,m) dm dx = 1, \\
&M_{a,b}[f_0] \in W^{1,2} \text{ a } + b = 2p \leq 6, M_{a,b}[f_0] \in L^2 \text{ a } + b = 2q \leq 16, M_1[f_0] \in L^1, \\
&\int_{\mathbb{R}_+^2 \times \mathbb{R}^2} f_0 \log f_0 dm dx < \infty, \int_{\mathbb{R}^2} |\Lambda(x)|^2 M_{0,0}[f_0] dx < \infty, \Lambda(x) = \log(\max(|x|, 1)), \\
&\int M_{0,0}[f_0] \log (M_{0,0}[f_0]) dx < \infty
\end{align*}
\]

(3)
where
\[ M_{a,b}[f_0](x) = \int m^a_0 m^b_2 f_0(m,x)dm, \quad \tilde{M}_p[f_0](x) = \int |m|^p f_0(m,x)dm. \tag{4} \]

Then there is a unique global solution for the system (2), smooth enough for \( u \) and \( \sigma \), and it satisfies the following free energy estimate:
\[
\mu \int f(t) \log \frac{f(t)}{M_{0,0}[f(t)]} \omega dmdx + \|u(t)\|_{L^2}^2 + \nu_1 \int_0^t \|\nabla_x u(s)\|_{L^2}^2 ds \leq \|u_0\|_{L^2}^2 + \mu \int f_0 \log \frac{f_0}{M_{0,0}[f_0]} \omega dmdx, \tag{5}
\]

where \( f_E = \frac{e^{-\frac{|m|^2}{2}}}{\mathcal{Z}} \) is the equilibrium distribution.

Note that given \( \rho_0, \sigma_0 \) satisfying
\[
\rho_0 \in L^1 \cap W^{1,2}, \quad \int \rho_0 dx = 1, \quad \sigma_0 \in L^1 \cap W^{1,2} \quad \text{positive definite,}
\]
\[
\int \rho_0 \left( |A(x)|^2 + \log \rho_0 + \log |\sigma_0| + \frac{|\sigma_0|^2}{\rho_0^2} \right) dx < \infty \tag{6}
\]
\[
\left( \frac{1}{\rho_0} \right)^{p-1} \sum_{s \in S_p} \prod_{i=1}^p \sigma_{i,s(i)} \in W^{1,2}, \quad 0 \leq p \leq 3, \quad 0 \leq \rho \leq 8,
\]
the distribution \( f_0 \) given by
\[
f_0(x, m) = \frac{\rho_0(x)^2}{2\pi \sqrt{\det \sigma_0(x)}} \exp \left( -\frac{1}{2} m^T \rho_0(x) \sigma_0^{-1}(x) m \right) \tag{7}
\]
satisfies
\[
\int_{\mathbb{R}^2} f_0(x, m)dm = \rho_0(x),
\]
\[
\int_{\mathbb{R}^2} m \otimes m f_0(x, m)dm = \sigma_0(x) \tag{8}
\]
and \((u_0, f_0)\) satisfies the condition (3). By the uniqueness of the solution of diffusive Oldroyd-B system, we conclude that the for the solution \((u, f)\) of (2) with initial data \((u_0, f_0)\), \((u, \sigma[f] - \rho[f]I, \rho[f])\) is the solution of (1) with initial data \((u_0, \sigma[f_0] - \rho[f_0]I, \rho[f_0]) = (u_0, \sigma_0, \rho_0)\).

### 3 Decay of stress and Boundedness of velocity

In this section, we prove the following theorem.

**Theorem 2.** Suppose that \( u_0 \in PW^{2,2} \) and \( \rho_0, \sigma_0 \) satisfies (4) and \( \rho_0 \in L^\infty \). Then the system (1) has global unique solution whose initial data is \( u(0) = u_0, \rho(0) = \rho_0, \tau(0) = \sigma_0 - \rho_0 I \). Moreover,
\[
\|u\|_{L^\infty(0,\infty;W^{2,2})} + \|\tau\|_{L^2(0,\infty;W^{2,2})} + \|\rho\|_{L^\infty(0,\infty;W^{1,2})} + \|\tau(x,t)\|_{L^1} \leq C_* \frac{C_*}{(t+1)}, \quad \|\tau(t)\|_{L^2} + \|\rho(t)\|_{L^\infty} \leq C_* \frac{C_*}{(t+1)}, \tag{9}
\]
where \( C_* \) depends only on the initial data and coefficients.

Remark 1. In the periodic case \((x, t) \in \mathbb{T}^2 \times [0, \infty)\), the same argument in this section we obtain the corresponding result

\[
\|u\|_{L^\infty(0, \infty; W^{2, 2})} + \|\tau\|_{L^\infty(0, \infty; W^{1, 2})} + \|\tau\|_{L^\infty(0, \infty; W^{2, 2})} \leq C_*.
\] (10)

One difference between the system in \([5]\) and system \([1]\) is the presence of diffusion in the evolution of \(\rho\), and this allows us to use uniform decay result given in \([27]\). In this section, \(C\) denotes constants that do not depend on the size of initial data or parameters \(\nu_1, \nu_2, \mu,\) and \(k\). The constants depending on them are denoted by \(C_1, C_2\) and so on. First we briefly review the technique in \([24]\) and \([27]\) to show the decay of \(\rho\). From the fourth equation of \([1]\) we get

\[
\frac{d}{dt} \|\rho(t)\|_{L^2}^2 + 2\nu_2 \|\nabla_x \rho(t)\|_{L^2}^2 = 0.
\] (11)

Then by Plancherel’s theorem we have

\[
\frac{d}{dt} \left(\int |\hat{\rho}|^2 \, dx\right) + \frac{2}{t+1} \int |\hat{\rho}|^2 \, dx \leq \frac{2}{t+1} \int A(t) |\hat{\rho}|^2 \, d\xi
\] (12)

where

\[
A(t) = \left\{ \xi : |\xi| \leq \left(\frac{2}{\nu_2(t+1)}\right)^{\frac{1}{2}} \right\}.
\] (13)

Noting that

\[
|\hat{\rho}(\xi, t)|^2 \leq \|\rho\|_{L^1}^2 = \|\rho_0\|_{L^1}^2
\] (14)

and multiplying \((t+1)^2\) to each side and integrating in time gives

\[
\|\rho(t)\|_{L^2}^2 \leq \frac{C}{(t+1)} \left(\|\rho_0\|_{L^1}^2 + \frac{1}{\nu_2} \|\rho_0\|_{L^1}^2\right).
\] (15)

For higher \(L^p\) norm, let \(p = 2^s, q = 2^{-1}\), with \(s \geq 2\). Multiplying \(\rho \rho^{-2}\) to the same equation and integrating, and applying integration by parts we get

\[
\frac{d}{dt} \int |\rho|^p \, dx = -\nu_2 p(p-1) \int |\nabla_x \rho|^2 \rho^{p-2} \, dx
\] (16)

and since

\[
|\nabla_x \rho|^2 \rho^{p-2} = |\rho^{q-1} \nabla_x \rho|^2 = \frac{1}{q^2} |\nabla_x (\rho^q)|^2
\] (17)

and \(\frac{p(p-1)}{q^2} \geq 1\) for \(s \geq 2\) we have

\[
\frac{d}{dt} \int |\rho|^q \, dx + \nu_2 \int |\nabla_x (\rho^q)|^2 \, dx \leq 0.
\] (18)

Applying the decay estimate for \(L^2\) repeatedly, and noting that by maximum principle \(\|\rho\|_{L^\infty} \leq \|\rho_0\|_{L^\infty}\) we have

\[
\|\rho(t)\|_{L^\infty} \leq \frac{C}{t+1} \max_{1 \leq r \leq \infty} (1 + \|\rho_0\|_{L^r}).
\] (19)

We establish estimates on \(u\) and \(\tau\). First we start with a simple lemma.

Lemma 1. Suppose that \(h : [0, \infty) \to [0, \infty)\) is a non-increasing function such that \(h(0) < \infty\) and \(\lim_{t \to \infty} \frac{h'(t)}{h(t)} = 0\). Then

\[
\int_0^t e^{-2k(t-s)} h(s) \, ds \leq Ch(t).
\] (20)
Proof. We only need to establish the bound for \( t \to \infty \), which is obvious by L'Hôpital. \( \square \)

Proof of Theorem 2. First, from the free energy estimate (5) we get

\[
\|u(t)\|_{L^2}^2 + 2\nu_1 \int_0^t \|\nabla_x u(s)\|_{L^2}^2 \, ds \leq C_1
\]

where \( C_1 \) depends only on initial data. Next, by multiplying \( \tau \) and integrating the third equation, we have

\[
\frac{d}{dt} \|\tau\|_{L^2}^2 + \nu_2 \|\nabla_x \tau\|_{L^2}^2 + 4k \|\tau\|_{L^2}^2 \leq \|\nabla_x u\|_{L^2} \|\tau\|_{L^2} + \|\nabla_x u\|_{L^2} \|\tau\|_{L^2} \|\rho\|_{L^\infty}.
\]

Then using (19) and Ladyzhenskaya's inequality

\[
\|\tau\|_{L^2}^2 \leq C \|\tau\|_{L^2} \|\nabla_x \tau\|_{L^2}
\]
valid in two dimension, we obtain

\[
\frac{d}{dt} \|\tau\|_{L^2}^2 + \nu_2 \|\nabla_x \tau\|_{L^2}^2 + (4k - \frac{C}{\nu_2} \|\nabla_x u\|_{L^2}^2) \|\tau\|_{L^2}^2 \leq \frac{C_2}{(1 + t)^2}
\]

where \( C_2 = C\nu_2 \max_{1 \leq t \leq \infty} (1 + \|\rho_0\|_{L^1})^2 \). Putting

\[
A(t) = 4k - \frac{C}{\nu_2} \|\nabla_x u(t)\|_{L^2}^2
\]

then we get

\[
\|\tau(t)\|_{L^2}^2 \leq \exp \left( - \int_0^t A(r) \, dr \right) \|\tau_0\|_{L^2}^2 + \int_0^t \exp \left( - \int_s^t A(r) \, dr \right) \frac{C_2}{(1 + s)^2} \, ds.
\]

but from (21) and Lemma 1 we have

\[
\|\tau(t)\|_{L^2}^2 \leq CC_2 \exp \left( \frac{CC_1}{\nu_2} \right) \frac{1}{(1 + t)^2} + \exp \left( \frac{CC_1}{\nu_2} - 4kt \right) \|\tau_0\|_{L^2}^2 \leq \frac{C_3}{(1 + t)^2},
\]

Also, estimates (24) and (27) give us

\[
\nu_2 \int_0^t \|\nabla_x \tau(s)\|_{L^2}^2 \, ds + 4k \int_0^t \|\tau(s)\|_{L^2}^2 \, ds \leq C_4 = \|\tau_0\|_{L^2}^2 + \int_0^t \frac{CC_3 \|\nabla_x u\|_{L^2}^2}{\nu_2 (1 + s)^2} \, ds + \int_0^t \frac{C_2 ds}{(1 + s)^2}.
\]

By taking curl to the first equation of the system (1), we obtain

\[
\partial_t \omega + u \cdot \nabla_x \omega = \nu_1 \Delta_x \omega + \mu \nabla_x^2 \cdot \nabla_x \cdot \tau
\]

where \( \omega = \partial_1 u_2 - \partial_2 u_1 \) is the vorticity. Then we have

\[
\frac{d}{dt} \|\omega\|_{L^2}^2 + \nu_1 \|\nabla_x \omega\|_{L^2}^2 \leq \frac{\mu^2}{\nu_1} \|\nabla_x \tau\|_{L^2}^2,
\]

and by (28) we have

\[
\|\omega(t)\|_{L^2}^2 + \nu_1 \int_0^t \|\nabla_x \omega(s)\|_{L^2}^2 \, ds \leq \frac{\mu^2 C_4}{\nu_1 \nu_2} = C_5.
\]
To estimate $\|\nabla_x \tau\|_{L^2}^2$, we multiply $-\Delta_x \tau$ to the third equation of (1) and integrate: we have

$$\frac{1}{2} \frac{d}{dt} \|\nabla_x \tau\|_{L^2}^2 + 2k \|\nabla_x \tau\|_{L^2}^2 + \nu_2 \|\Delta_x \tau\|_{L^2}^2 = -\int (\nabla_x u) \cdot (\nabla_x \tau) \cdot (\nabla_x \tau) \, dx + \int (\nabla_x u) \tau (\nabla_x u)^T (-\Delta_x \tau) \, dx + \int \rho ((\nabla_x u) + (\nabla_x u)^T) (-\Delta_x \tau) \, dx. \tag{32}$$

The first term in the right hand side is bounded by, using Ladyzhenskaya’s inequality and Young’s inequality, we can bound the second term by

$$\left| \int (\nabla_x u) \cdot (\nabla_x \tau) \cdot (\nabla_x \tau) \, dx \right| \leq \frac{C}{\nu_2} \|\nabla_x u\|_{L^2}^2 \|\nabla_x \tau\|_{L^2}^2 + \frac{\nu_2}{8} \|\Delta_x \tau\|_{L^2}^2. \tag{33}$$

We apply the integration by parts, and then Ladyzhenskaya’s inequality to the second term to obtain

$$\int ((\nabla_x u) \tau + \tau (\nabla_x u)^T) (-\Delta_x \tau) \, dx \leq \frac{C}{\nu_2} \|\nabla_x u\|_{L^2}^2 \|\nabla_x \tau\|_{L^2}^2 + \frac{\nu_2}{8} \|\Delta_x \tau\|_{L^2}^2 \tag{34}$$

and by applying Young’s inequality we can bound the second term by

$$\frac{C}{\nu_2} \|\nabla_x u\|_{L^2}^2 \|\nabla_x \tau\|_{L^2}^2 + \frac{\nu_2}{4} \|\Delta_x \tau\|_{L^2}^2 + C \|\nabla_x \omega\|_{L^2} \|\tau\|_{L^2}^2 \|\nabla_x \tau\|_{L^2} \|\Delta_x \tau\|_{L^2}^2 \tag{35}$$

The last term is bounded by

$$\frac{C}{\nu_2} \|\rho\|_{L^\infty} \|\nabla_x u\|_{L^2}^2 + \frac{\nu_2}{8} \|\Delta_x \tau\|_{L^2}^2 \leq \frac{C_2 C_5}{\nu_2^2 (1 + t)^2} + \frac{\nu_2}{8} \|\Delta_x \tau\|_{L^2}^2. \tag{36}$$

To sum up, we have

$$\frac{d}{dt} \|\nabla_x \tau\|_{L^2}^2 + \left(4k - \frac{C}{\nu_2} \|\nabla_x u\|_{L^2}^2 - C \|\nabla_x \omega\|_{L^2}^2 \right) \|\nabla_x \tau\|_{L^2}^2 + \nu_2 \|\Delta_x \tau\|_{L^2}^2 \leq \frac{C_2 C_5}{\nu_2^2 (1 + t)^2} + \frac{1}{\nu_2} \|\tau\|_{L^2}^2. \tag{37}$$

Applying Grönwall with Lemma 11 we have

$$\|\nabla_x \tau(t)\|_{L^2}^2 \leq e^{\frac{CC_1}{\nu_1^2} + \frac{CC_3}{\nu_1}} \left( C \left( \frac{CC_5}{\nu_2^2} + \frac{C_3}{\nu_2} \right) \frac{1}{(1 + t)^2} + e^{-4k t} \|\nabla_x \tau_0\|_{L^2}^2 \right) \leq \frac{C_6}{(1 + t)^2} \tag{38}$$

and

$$4k \int_0^t \|\nabla_x \tau(s)\|_{L^2}^2 \, ds + \nu_2 \int_0^t \|\Delta_x \tau(s)\|_{L^2}^2 \, ds \leq \|\nabla_x \tau_0\|_{L^2}^2 + \left( C(C_1 + C_5)C_6 + \frac{CC_3}{\nu_1} + \frac{C C_2 C_5}{\nu_2^2} \right) = C_7. \tag{39}$$

We can also get an estimate for $\|\nabla_x \omega\|_{L^2}$; by multiplying $-\Delta_x \omega$ to the vorticity equation (29) and integrating, we have

$$\frac{1}{2} \frac{d}{dt} \|\nabla_x \omega(t)\|_{L^2}^2 + \frac{3}{4} \nu_1 \|\Delta_x \omega(t)\|_{L^2}^2 \leq \frac{C}{\nu_1} \|\Delta_x \tau\|_{L^2}^2 + \|\nabla_x \omega\|_{L^p}^2 \|\nabla_x u\|_{L^q} \tag{40}$$
where \( \frac{2}{p} + \frac{1}{q} = 1 \). Our choice of \( p \) and \( q \) are \( p = q = 3 \). By the Gagliardo-Nirenberg interpolation inequality
\[
\| \nabla_x \omega \|_{L_3} \leq C \| \Delta_x \omega \|_{L_2}^{\frac{2}{3}} \| \omega \|_{L_2}^{\frac{1}{3}}, \quad \| \nabla_x u \|_{L_3} \leq C \| \Delta_x u \|_{L_2}^{\frac{2}{3}} \| u \|_{L_2}^{\frac{1}{3}},
\]  
(41)
and Young’s inequality, we have
\[
\| \nabla_x \omega \|_{L_3}^2 \| \nabla_x u \|_{L_3} \leq C \| \Delta_x \omega \|_{L_2}^2 \| \nabla_x \omega \|_{L_2}^2 \| \omega \|_{L_2}^2 \| u \|_{L_2}^2 \leq \frac{\nu_1}{4} \| \Delta_x \omega \|_{L_2}^2 + \frac{C}{\nu_1} \| \omega \|_{L_2}^2 \| u \|_{L_2}^2 \| \nabla_x \omega \|_{L_2}^2.
\]  
(42)
To obtain
\[
\frac{d}{dt} \| \nabla_x \omega \|_{L_3}^2 + \nu_1 \| \Delta_x \omega \|_{L_2}^2 \leq \frac{C}{\nu_1} \| \Delta_x \tau \|_{L_2}^2 + \frac{C}{\nu_1^2} \| \omega \|_{L_2}^2 \| u \|_{L_2}^2 \| \nabla_x \omega \|_{L_2}^2.
\]  
(43)
Since \( \| u(t) \|_{L_2} \leq \sqrt{C}, \int_T^T \| \omega(t) \|_{L_2}^2 \, dt \leq \frac{C C_8}{\nu_1} \), we have
\[
\sup_{t \in \left[0, T\right]} \| \nabla_x \omega(t) \|_{L_3}^2 + \frac{\nu_1}{4} \int_0^T \| \Delta_x \omega(t) \|_{L_2}^2 \, dt \leq \exp \left( \frac{CC_7^3}{\nu_1^2} \right) \left( \| \nabla_x \omega(0) \|_{L_3}^2 + \frac{CC_7}{\nu_1 \nu_2} \right) = C_8.
\]  
(44)
Moreover, we have \( L^1 \) decay of \( \tau \). From
\[
\frac{d}{dt} \int \text{Tr} \tau = 2 \int \text{Tr} (\nabla_x u \tau) - 2k \int \text{Tr} \tau
\]  
(45)
we obtain
\[
\frac{d}{dt} \left( e^{2kt} \int \text{Tr} \tau \right) \leq e^{2kt} \| \nabla_x u \|_{L_2} \| \tau \|_{L_2} \leq e^{2kt} \frac{C \sqrt{C_5} C_3}{(1 + t)} = C_9 e^{2kt}
\]  
(46)
Therefore,
\[
\int \text{Tr} \tau(t) \, dt \leq e^{-2kt} \left( \frac{CC_9}{1 + t} \right) \ls \frac{CC_9}{1 + t}
\]  
(47)
Finally, by multiplying \(-\Delta_x \rho\) to the fourth equation of (1) and integrating, and using Ladyzhenskaya’s inequality we have
\[
\frac{d}{dt} \| \nabla_x \rho(t) \|_{L_2}^2 + \frac{\nu_2}{1} \| \Delta_x \rho(t) \|_{L_2}^2 \leq \frac{C}{\nu_2} \| \nabla_x u \|_{L_2}^2 \| \nabla_x \rho \|_{L_2}^2,
\]  
(48)
and by Grönwall
\[
\sup_{t \in \left[0, T\right]} \| \nabla_x \rho(t) \|_{L_2}^2 + \frac{\nu_2}{1} \int_0^T \| \Delta_x \rho(t) \|_{L_2}^2 \, dt \leq \exp \left( \frac{CC_1}{\nu_1 \nu_2} \right) \| \nabla_x \rho(0) \|_{L_2}^2 = C_{10}.
\]  
(49)
\[ \square \]
**Remark 2.** We will make an improvement for (24), (37) and (46). For (24) we used the bound
\[
\int \rho ((\nabla_x u) + (\nabla_x u)^T) \tau \, dx \leq C \| u \|_{L^\infty}^2 + \frac{C}{\nu_2} \| \nabla_x u \|_{L_2}^2 \| \tau \|_{L_2}^2.
\]  
(50)
We may instead use
\[
\int \rho ((\nabla_x u) + (\nabla_x u)^T) \tau \, dx \leq \frac{C}{k} \| \rho \|_{L^\infty}^2 \| \nabla_x u \|_{L_2}^2 + k \| \tau \|_{L_2}^2.
\]  
(51)
Then we have
\[ \frac{d}{dt} \| \tau \|_{L^2}^2 + \nu_2 \| \nabla x \tau \|_{L^2}^2 + \left( 3k - \frac{C}{\nu_2} \| \nabla u \|_{L^2}^2 \right) \| \tau \|_{L^2}^2 \leq \frac{C_2}{\kappa} \| \nabla u \|_{L^2}^2 \frac{1}{(1 + t)^2}. \] (52)

In the later section, we show that \( \| \nabla x u(t) \|_{L^2}^2 \leq \frac{C}{(t+1)} \) in time, so (52) gives faster decay of \( \| \tau \|_{L^2}^2, \| \nabla x \tau \|_{L^2}^2 \) by (57), and \( \| \tau \|_{L^1} \) by (60).

4 Decay of the strong solution

In this section, we prove the following theorem.

**Theorem 3.** Suppose that \( u_0 \in PW^{2,2} \) and \( \rho_0, \sigma_0 \) satisfies (6) and \( \rho_0 \in L^\infty \). Furthermore, suppose that \( u_0 \in L^1 \). Then there is a constant \( C_{**} \) depending on initial data and parameter such that
\[ \| u(t) \|_{W^{2,2}} \leq \frac{C_{**}}{(t+1)^{\frac{1}{2}}}, \quad \| \tau(t) \|_{W^{1,2} \cap L^1} \leq \frac{C_{**}}{(t+1)^{\frac{1}{2}}}. \] (53)

**Proof.** We first follow the methods presented in [23]. Let
\[ S(t) = \left\{ \xi : |\xi| \leq \left( \frac{2}{\nu_1(t + 1)} \right)^{\frac{1}{2}} \right\}. \] (54)

We apply the estimate for the decay of \( \| \omega(t) \|_{L^2}^2 \). From the vortex equation we have
\[ \frac{d}{dt} \| \omega(t) \|_{L^2}^2 \leq -\nu_1 \| \nabla \omega(t) \|_{L^2}^2 + \frac{C \mu^2}{\nu_1} \| \nabla x \tau(t) \|_{L^2}^2, \] (55)
and by Plancherel's theorem
\[ \frac{d}{dt} \| \omega(t) \|_{L^2}^2 \leq -\nu_1 \int_{\mathbb{R}^2} |\xi|^2 |\hat{\omega}(t)|^2 d\xi + \frac{CC_*}{\nu_1(t + 1)^2} \leq -\nu_1 \int_{S(t)^c} |\xi|^2 |\hat{\omega}(t)|^2 d\xi + \frac{CC_*}{\nu_1(t + 1)^2} \]
\[ \leq -\frac{2}{t + 1} \int_{S(t)^c} |\hat{\omega}(t)|^2 d\xi + \frac{CC_*}{\nu_1(t + 1)^2} \leq -\frac{2}{t + 1} \| \omega(t) \|_{L^2}^2 + \frac{2}{t + 1} \int_{S(t)} |\hat{\omega}(t)|^2 d\xi + \frac{CC_*}{\nu_1(t + 1)^2}. \] (56)

To summarize, we have
\[ \frac{d}{dt} \| \omega(t) \|_{L^2}^2 \leq -\frac{2}{t + 1} \| \omega(t) \|_{L^2}^2 + \frac{2}{t + 1} \int_{S(t)} |\hat{\omega}(t)|^2 d\xi + \frac{CC_{**}}{(t+1)^2}. \] (57)

We need a pointwise estimate of \( \hat{\omega}(\xi, t) \). For that purpose we investigate \( \hat{u}(\xi, t) \). The phase space version of the velocity equation of (11) reads
\[ \partial_t \hat{u} + \nu_1 |\xi|^2 \hat{u} = -\xi \cdot \left( I - \frac{\xi \otimes \xi}{|\xi|^2} \right) \mathcal{F}(u \otimes u - \mu \tau), \] (58)
and from \( \| \mathcal{F}(u \otimes u)(t) \|_{L^\infty} \leq \| u(t) \|_{L^2}^2 \leq C_* \) and \( \| \mathcal{F}(\tau(t)) \|_{L^1} \leq \| \tau(t) \|_{L^1} \leq \frac{C_*}{(t + 1)^{\frac{1}{2}}} \), we have
\[ \hat{u}(\xi, t) = e^{-\nu_1 |\xi|^2 t} \hat{u}(\xi, 0) - \int_0^t e^{-\nu_1 |\xi|^2 (t-s)} \xi \cdot \left( I - \frac{\xi \otimes \xi}{|\xi|^2} \right) \mathcal{F}(u \otimes u - \mu \tau)(s) ds, \]
\[ |\hat{u}(\xi, t)| \leq C_* \left( \frac{1}{|\xi| + 1} \right). \] (59)
given that \( u_0 \in L^1 \). Therefore, since \( |\dot{\omega}(\xi, t)| \leq |\xi| |\dot{\omega}(\xi, t)| \leq C_*(|\xi| + 1) \). Therefore, 
\[
\int_{S(t)} |\dot{\omega}(t)|^2 d\xi \leq \frac{C_*}{(t+1)^{\frac{3}{2}}},
\]
and by multiplying \((t + 1)^2\) to \(67\) we obtain
\[
\frac{d}{dt} (t + 1)^2 \|\omega(t)\|_{L^2}^2 \leq CC^*,
\]
and therefore
\[
\|\omega(t)\|_{L^2}^2 \leq \frac{C^*}{(t + 1)^{\frac{3}{2}}}.
\]
In addition, as we mentioned in Remark 2, this gives a better decay rate for \( \tau \):
\[
\|\tau(t)\|_{L^2} \leq \frac{C^*}{(t + 1)^{\frac{3}{2}}} \|\nabla_x \tau(t)\|_{L^2} \leq \frac{C^*}{(t + 1)^{\frac{3}{2}}} \|\tau(t)\|_{L^1} \leq \frac{C^*}{(t + 1)^{\frac{3}{2}}}.
\]
Finally, from \(43\), using \( \|\nabla_x \omega\|_{L^2}^2 \leq C_* \) and previous estimates we have
\[
\frac{C}{\nu_1} \|\omega(t)\|_{L^2}^2 \|u(t)\|_{L^2} \|\nabla_x \omega(t)\|_{L^2}^2 \leq \frac{CC^*}{(t + 1)^{\frac{3}{2}}}
\]
and adding multiples of \(67\) to \(43\) to cancel \( \frac{C}{\nu_1} \|\Delta_x \tau\|_{L^2}^2 \) in the right side of \(43\), we obtain
\[
\frac{d}{dt} \left( \|\nabla_x \omega(t)\|_{L^2}^2 + \frac{C}{\nu_1 \nu_2} \|\nabla_x \tau(t)\|_{L^2}^2 \right) + \frac{C}{\nu_1 \nu_2} \left( 4k - \frac{C}{\nu_2} \|\nabla_x u(t)\|_{L^2}^2 - C \|\nabla_x \omega(t)\|_{L^2}^2 \right) \|\nabla_x \tau(t)\|_{L^2}^2 
\]
\[-\frac{C}{\nu_1} \|\Delta_x \tau(t)\|_{L^2}^2 + \nu_1 \|\Delta_x \omega(t)\|_{L^2}^2 \leq \frac{C^*}{(1 + t)^{\frac{3}{2}}} + \frac{C^* C_*}{(1 + t)^{\frac{3}{2}}}.
\]
Then applying the same Fourier splitting technique, together with the pointwise bound
\[
|\mathcal{F}(\nabla_x \tau)(\xi, t)| \leq |\xi| \|\tau(t)\|_{L^1} \leq \frac{C_* |\xi|}{(1 + t)}, \quad |\mathcal{F}(\nabla_x \omega)(\xi, t)| \leq C_* |\xi| (1 + |\xi|)
\]
we obtain
\[
\frac{d}{dt} \left( (t + 1)^2 \left( \|\nabla_x \omega(t)\|_{L^2}^2 + \frac{C}{\nu_1 \nu_2} \|\nabla_x \tau(t)\|_{L^2}^2 \right) \right) \leq C^* \left( \frac{1}{(1 + t)} + (t + 1)^{\frac{1}{2}} \right),
\]
and from this we conclude that
\[
\|\nabla_x \omega(t)\|_{L^2}^2 \leq \frac{C^*}{(t + 1)^{\frac{3}{2}}}.
\]
We put \(67\) into \(63\), and then we repeat the last estimate to obtain
\[
\|\nabla_x \omega(t)\|_{L^2}^2 \leq \frac{C^*}{(t + 1)^{\frac{3}{2}}},
\]
Now we are ready to prove decay of \(\|u(t)\|_{L^2}\). We follow the argument in [28].

**Definition 1.** For given \( u_0 \in L^2(\mathbb{R}^2) \) and \( f \in L^1(0, +\infty; L^2(\mathbb{R}^2)) \), we say that \((u_0, f)\) belong to the set \(D_\alpha^{(n)}\) with \(\alpha \geq 0\) if there is a constant \( C \) such that
\[
\|v(t)\|_{L^2}^2 + (1 + t)^2 \|f(t)\|_{L^2}^2 \leq C(1 + t)^{-\alpha}
\]
where \(v(t)\) is the solution of the heat equation
\[
\partial_t v - \nu_1 \Delta_x v = f, \quad v(0) = u_0.
\]
Theorem 4 ([28]). Let $u$ be a strong solution of Navier-Stokes system

$$
\begin{align*}
\partial_t u + u \cdot \nabla_x u &= -\nabla_x p + \nu \Delta_x u + f, \\
\nabla_x u &= 0, \quad u(0) = u_0, \\
\end{align*}
$$

(71)

$u \in C_w([0, +\infty), L^2(\mathbb{R}^2)) \cap L^1_{loc}(0, +\infty; W^{1,2}(\mathbb{R}^2))$.

Let $u_0 \in L^2(\mathbb{R}^2)$, $f \in L^1(0, +\infty; L^2(\mathbb{R}^2))$. Then we have the following.

1. $\|u(t)\|_{L^2} \to 0$ for $t \to \infty$.

2. If $(u_0, f) \in D_1^{(2)}$, then $\|u(t)\|_{L^2}^2 \leq C(1 + t)^{-1}$.

3. Furthermore, the solution $u$ is asymptotically equivalent to the heat system (70) in the sense that

$$
\|u(t) - v(t)\|_{L^2}^2 \leq C (\log(t + e))^2 (t + 1)^{-2}.
$$

(72)

The proof of Theorem 4 is based on the Fourier splitting technique. The proof of Theorem 3 is completed by the following Lemma.

Lemma 2. Suppose that $u_0, \tau_0, \rho_0$ satisfy the initial conditions in Theorem 3. Then $(u_0, \nabla_x \cdot \tau)$ belong to the set $D_1^{(2)}$, where $(u, \tau, \rho)$ is the solution of (4) with initial data $(u_0, \tau_0, \rho_0)$.

proof of the Lemma. Since $\|\nabla_x \tau(t)\|_{L^2} \leq \frac{C_{\star}}{(t+1)^{2}}, \nabla_x \tau \in L^1(0, +\infty; L^2(\mathbb{R}^2)^2)$. Furthermore, by the same Fourier splitting technique,

$$
\frac{d}{dt} \|v(t)\|_{L^2}^2 \leq -\frac{2}{(t+1)} \|v(t)\|_{L^2}^2 + \frac{2}{(t+1)} \int_{\mathcal{S}(t)} |\hat{v}(t)|^2 d\xi + \frac{C_{\star\star}}{(t+1)^{3}}.
$$

(73)

However, in the Fourier space

$$
\begin{align*}
\partial_t \hat{v}(t) &= -\nu_1 |\xi|^2 \hat{v}(t) - i \xi \cdot \hat{\tau}, \\
\hat{v}(t) &= e^{-\nu_1 |\xi|^2 t_0} - i \int_0^t \xi \cdot e^{-\nu_1 |\xi|^2 (t-s)} \hat{\tau}(s) ds,
\end{align*}
$$

(74)

and since $|\hat{\tau}(s, \xi)| \leq \|\tau(s)\|_{L^1} \leq \frac{C_{\star}}{(s+1)^{2}}$,

$$
|\hat{v}(t, \xi)| \leq \|u_0\|_{L^1} + C_{\star\star} |\xi|,
$$

(75)

then we have $\|v(t)\|_{L^2}^2 \leq \frac{C_{\star\star}}{(t+1)}$, as desired.

\section{5 Conclusion}

We proved that strong solutions of 2D diffusive Oldroyd-B systems in $\mathbb{R}^2$ decay over time at an algebraic rate, given that the initial data satisfy mild restrictions originating from the kinetic considerations.

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References

[1] J. W. Barrett and S. Boyaval. Existence and approximation of a (regularized) Oldroyd-B model. *Math. Models Methods Appl. Sci.*, 21(9):1783–1837, 2011.

[2] J. W. Barrett and E. Süli. Existence of global weak solutions to the kinetic Hookean dumbbell model for incompressible dilute polymeric fluids. *Nonlinear Anal. Real World Appl.*, 39:362–395, 2018.

[3] J. T. Beale, T. Kato, and A. Majda. Remarks on the breakdown of smooth solutions for the 3-D Euler equations. *Comm. Math. Phys.*, 94(1):61–66, 1984.

[4] J.-Y. Chemin and N. Masmoudi. About lifespan of regular solutions of equations related to viscoelastic fluids. *SIAM J. Math. Anal.*, 33(1):84–112, 2001.

[5] P. Constantin and M. Kliegl. Note on global regularity for two-dimensional Oldroyd-B fluids with diffusive stress. *Arch. Ration. Mech. Anal.*, 206(3):725–740, 2012.

[6] P. Constantin and W. Sun. Remarks on Oldroyd-B and related complex fluid models. *Commun. Math. Sci.*, 10(1):33–73, 2012.

[7] D. Fang and R. Zi. Global solutions to the Oldroyd-B model with a class of large initial data. *SIAM J. Math. Anal.*, 48(2):1054–1084, 2016.

[8] E. Fernández-Cara, F. Guillén, and R. R. Ortega. Some theoretical results for viscoplastic and dilatant fluids with variable density. *Nonlinear Anal.*, 28(6):1079–1100, 1997.

[9] E. Fernández-Cara, F. Guillén, and R. R. Ortega. Some theoretical results concerning non-Newtonian fluids of the Oldroyd kind. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 26(1):1–29, 1998.

[10] E. Fernández-Cara, F. Guillén, and R. R. Ortega. Mathematical modeling and analysis of viscoelastic fluids of the Oldroyd kind. In *Handbook of numerical analysis, Vol. VIII*, Handb. Numer. Anal., VIII, pages 543–661. North-Holland, Amsterdam, 2002.

[11] C. Guillopé and J.-C. Saut. Existence results for the flow of viscoelastic fluids with a differential constitutive law. *Nonlinear Anal.*, 15(9):849–869, 1990.

[12] C. Guillopé and J.-C. Saut. Global existence and one-dimensional nonlinear stability of shearing motions of viscoelastic fluids of Oldroyd type. *RAIRO Modél. Math. Anal. Numér.*, 24(3):369–401, 1990.

[13] M. Hieber, Y. Naito, and Y. Shibata. Global existence results for Oldroyd-B fluids in exterior domains. *J. Differential Equations*, 252(3):2617–2629, 2012.

[14] X. Hu and F. Lin. Global solutions of two-dimensional incompressible viscoelastic flows with discontinuous initial data. *Comm. Pure Appl. Math.*, 69(2):372–404, 2016.

[15] R. Kupferman, C. Mangoubi, and E. S. Titi. A Beale-Kato-Majda breakdown criterion for an Oldroyd-B fluid in the creeping flow regime. *Commun. Math. Sci.*, 6(1):235–256, 2008.

[16] J. La. On diffusive 2d fokker-planck-navier-stokes systems. *to be appeared*, 2018.

[17] Z. Lei, C. Liu, and Y. Zhou. Global solutions for incompressible viscoelastic fluids. *Arch. Ration. Mech. Anal.*, 188(3):371–398, 2008.

[18] Z. Lei, N. Masmoudi, and Y. Zhou. Remarks on the blowup criteria for Oldroyd models. *J. Differential Equations*, 248(2):328–341, 2010.

[19] Z. Lei and Y. Zhou. Global existence of classical solutions for the two-dimensional Oldroyd model via the incompressible limit. *SIAM J. Math. Anal.*, 37(3):797–814, 2005.
[20] F.-H. Lin, C. Liu, and P. Zhang. On hydrodynamics of viscoelastic fluids. *Comm. Pure Appl. Math.*, 58(11):1437–1471, 2005.

[21] P. L. Lions and N. Masmoudi. Global solutions for some Oldroyd models of non-Newtonian flows. *Chinese Ann. Math. Ser. B*, 21(2):131–146, 2000.

[22] E. L. C. V. M. Plan, A. Gupta, D. Vincenzi, and J. D. Gibbon. Lyapunov dimension of elastic turbulence. *J. Fluid Mech.*, 822:R4, 12, 2017.

[23] M. Schonbek. The Fourier splitting method. In *Advances in geometric analysis and continuum mechanics (Stanford, CA, 1993)*, pages 269–274. Int. Press, Cambridge, MA, 1995.

[24] M. E. Schonbek. Decay of solutions to parabolic conservation laws. *Comm. Partial Differential Equations*, 5(5):449–473, 1980.

[25] M. E. Schonbek. $L^2$ decay for weak solutions of the Navier-Stokes equations. *Arch. Rational Mech. Anal.*, 88(3):209–222, 1985.

[26] M. E. Schonbek. Large time behaviour of solutions to the Navier-Stokes equations. *Comm. Partial Differential Equations*, 11(7):733–763, 1986.

[27] M. E. Schonbek. Uniform decay rates for parabolic conservation laws. *Nonlinear Anal.*, 10(9):943–956, 1986.

[28] M. Wiegner. Decay results for weak solutions of the Navier-Stokes equations on $\mathbb{R}^n$. *J. London Math. Soc. (2)*, 35(2):303–313, 1987.