The L(2,1)-Labeling of Planar Graphs with Neither 3-Cycles Nor Intersect 4-Cycles

Wenjuan Zhou, Lei Sun*

School of Mathematics and Statistics, Shandong Normal University, Jinan, China

*Corresponding author’s email: sunlei@sdnu.edu.cn

Abstract. Given a graph G, a k-L(2,1)-labelling of G is a function c: V(G)→{0,1,2,...,k} such that |φ(x)−φ(y)|≥2, if x is adjacent to y and |φ(x)−φ(y)|≥1, if x and y have a common neighbor. The least k denoted by λ_{2,1}(G) is the L(2, 1)-labelling number. In this article, we proved that: for every planar graph with neither 3-cycles nor intersect 4-cycles and Δ(G) ≥ 26, λ_{2,1}(G) ≤ Δ(G) + 12.

Keywords:-l(2,1)-labeling, cycle; intersect; maximum degree

1. Introduction

A k-L(2, 1)-labelling of G is a function c: V(G)→{0,1,2,...,k} such that |φ(x)−φ(y)|≥2, if x is adjacent to y and |φ(x)−φ(y)|≥1, if x and y have a common neighbor. The least k denoted by λ_{2,1}(G) is the L(2, 1)-labelling number. If every vertex x of G has its own set L(x) of admissible colors in which |L(x)|≥k, then we say that V(G) has a list L with size k. A graph G is said to be k-L(2,1)-choosable if G has an k-L(2,1)-labelling whenever |L(x)|≥k for every x ∈ V(G).

The L(2,1)-labelling of graphs was investigated by Griggs and Yeh [1] firstly and a series of conjectures are proposed. Then Chang and Kuo [2] improved their result in 1996. And Král et al. [3] and Conclaves [4] continued to improve the bound in 2005. Next, Van den Heuvel and McGuinness [5], Borodin et al. [6] and Molloy and Salavatipour [7] improved the result in turn. Zhu et al. [8] proved that every planar graph without 4,5-cycles, λ_{2,1}(G) ≤ Δ(G) + 13.

In this article, we shall prove the following Theorem 1.1.

Theorem 1.1. For every planar graph with neither 3-cycles nor intersect 4-cycles and Δ(G) ≥ 26, λ_{2,1}(G) ≤ Δ(G) + 12.
Figure 1. A graph $G_P$ with girth 4, $\Delta(G_P) = 2p$ and $\Delta_4(G_P) = 3p$.

We observe that the condition in Thm 1.1 cannot be deleted. In [6], Marthe Bonamy found that there are some planes with intersect 4-cycles satisfying that the difference between $\Delta_4(G)$ and $\Delta(G)$ can be arbitrarily large (see Figure 1). Obviously, $\Delta_4(G) \geq \Delta_4(G)$.

To prove this result, we suppose that it is false. In Section 3, we exhibit some properties of a minimal counterexample that contradicts Theorem 1.1. Depend on these properties, we can use Discharging to get a contradiction.

2. Notations
For a graph $G$, we use $\Delta(G)$, $\delta(G)$ and $\mathcal{F}(G)$, denote maximum degree, minimum degree and face set, respectively. For $v \in V(G)$, let $d_G(v)$ denote the degree of $v$ in $G$, simply $d(v)$. A vertex of degree $k$ (resp. at least $k$, at most $k$) will be called $k$-vertex (resp. $k+\text{-vertex}$, $k-\text{-vertex}$). Let us define faces similarly. Let $N_G(v)$ be the set of $v$'s neighbors. Let $v$'s $k$-neighbors be a vertex adjacent to $v$ with degree $k$. Let $n_k(v)$ denote the number of $v$'s $k$-neighbors.

For a $k$-vertex $v$, we denoted its neighbors in ascending order of degree by $v_1, v_2, \ldots, v_k$ and for a 2-vertex $v_i \in N_G(v)$, if $v_i'$ is a neighbor of $v_i$ except $v$, then we call it $v$'s weak adjacent vertex. Let $i(j)$ be a vertex with degree $i$ and with $j$ 2-neighbors.

A 3-vertex $v$ is a weak 3-vertex if $d(v) = 3$, $d(v) + d(v) \leq \Delta + 5$. A 3-vertex $v$ is a strong 3-vertex if $d(v) = 4$, $d(v) + d(v) \geq \Delta + 6$. A $v$ is a special 3-vertex if $d(v) = 4$, $d(v) + d(v) \leq \Delta + 5$. A 4-vertex $v$ is a weak 4-vertex if it is adjacent to at least one special 3-vertex.

We call two 4-cycles intersect when they have at least one vertex in common.

In the proof of the theorem1.1, we always get a partial $L(2, 1)$-labeling $\psi$ of subgraph $G'$ of $G$. Then we can extend $\psi$ to $L(2, 1)$-labeling $\varphi$ of $G$. Let $F(v) = \{ \varphi(u) | u \in N_1(v) \cup N_2(v) \}$ be a set of disable coloring, then $|F(v)|$ be the number of $v$'s disable coloring.

3. Structural Properties
To begin, we prove nine structural properties about minimal counterexample. Many of our arguments count the number of colors restricted from use on some uncolored element.

Lemma 3.1. [9] Let $P_k$ with a list assignment of $L$ is a path with $V(P_k) = \{v_1, v_2, \ldots, v_k\}$ and $E(P_k) = \{v_{[i]}, v_{[i+1]} | i = 1, 2, \ldots, k-1\}$. If $k = 2$ and $L$ satisfy the following condition that $|L(v_1)| = 2$, $|L(v_2)| = 2$ and $|L(v_1)| \neq |L(v_2)|$, the $P_2$ is $L(2, 1)$-choosable.

Property 1: A 2-vertex is not adjacent to 2-vertices.

Proof. Suppose that $xy \in E(G)$ and they are 2-vertices. By the minimality of $G$, $G-xy$ has a $(\Delta + 13)$-list-$L(2, 1)$-choosable $\psi$. We first erase the colors of $x$, $y$. Then $|F(x)| \leq \Delta + 3$, $|F(y)| \leq \Delta + 6$, so we can also recolor $x$ and $y$ successively. Thus $\psi$ is extended to a $(\Delta + 13)$-list-$L(2, 1)$-choosable of $G$, a contradiction.

Property 2: For a 3-vertex $v$:
(1) $v$ can not adjacent to two 2-vertices.
(2) If $n_2(v) = 1$, then $\sum_{i=2}^{k} d(v_i) \geq \Delta + 7$. 

2
(3) If \( v \) is a 3(0)-vertex, \( v \) can not adjacent to 3(1)-vertices; when \( v \) and \( v_1 \) are 3(0)-vertices, 
\[
\sum_{i=2}^{3} d(v_i) \geq \Delta + 6 \text{ or } d(u) + d(w) \geq \Delta + 6
\]
where \( u \) and \( w \) are \( v_1 \)'s neighbors except \( v \).

Proof. (2) Assume that \( d(v_1) = 2 \) and \( \sum_{i=2}^{3} d(v_i) \leq \Delta + 6 \). By the minimality of \( G \), \( G - vv_1 \) has a \((\Delta + 13)\)-list-L(2,1)-choosable. We first erase the colors of \( v \) and \( v_1 \). Then \( |F(v)| \leq \Delta + 11 \), \( |F(v_1)| \leq \Delta + 7 \), so we can also recolor \( v \) and \( v_1 \) successively. Therefore, \( G \) is \((\Delta + 13)\)-list-L(2,1)-choosable, a contradiction.

(3) According to (2) we can easily prove that 3(0)-vertex \( v \) can not adjacent to 3(1)-vertices. Assume that \( d(v_1) = 3 \), \( u \) and \( w \) are its neighbors except \( v \), 
\[
\sum_{i=2}^{3} d(v_i) \geq \Delta + 6 \text{ and } d(u) + d(w) \geq \Delta + 5
\]
where \( v_1 \)'s neighbors except \( v \).

(4) Suppose for all \( 1 \leq i \leq 4 \), \( d(v_i) = 3 \), \( d(v_{i1}) + d(v_{i2}) \geq \Delta + 5 \) where \( v_{i1} \) and \( v_{i2} \) are \( v_i \)'s neighbors except \( v \).

(5) Suppose for all \( 1 \leq i \leq 3 \), \( d(v_i) = 3 \), \( d(v_{i1}) + d(v_{i2}) \geq \Delta + 5 \) where \( v_{i1} \) and \( v_{i2} \) are \( v_i \)'s neighbors except \( v \) or \( d(v_i) \geq \Delta + 5 \).

Proof. (4) Suppose for all \( 1 \leq i \leq 4 \), \( d(v_i) = 3 \), \( d(v_{i1}) + d(v_{i2}) \leq \Delta + 4 \) where \( v_{i1} \) and \( v_{i2} \) are \( v_i \)'s neighbors except \( v \). By the minimality of \( G \), \( G - vv_1 \) has a \((\Delta + 13)\)-list-L(2,1)-choosable. We first erase the colors of \( v_1 \) and \( v \). \( |F(v_1)| \leq \Delta + 11 \), \( |F(v)| \geq 10 \), so we can recolor \( v_1 \) and \( v \) successively. Therefore, \( G \) is \((\Delta + 13)\)-list-L(2,1)-choosable, a contradiction.

(5) Suppose for all \( 1 \leq i \leq 3 \), \( d(v_i) = 3 \), \( d(v_{i1}) + d(v_{i2}) \leq \Delta - 6 \) where \( v_{i1} \) and \( v_{i2} \) are \( v_i \)'s neighbors except \( v \) and \( d(v_i) \leq \Delta - 6 \). By the minimality of \( G \), \( G - vv_1 \) has a \((\Delta + 13)\)-list-L(2,1)-choosable. We first erase the colors of \( v_1 \) and \( v \). \( |F(v_1)| \leq \Delta + 11 \), \( |F(v)| \leq \Delta + 11 \), so we can recolor \( v_1 \) and \( v \) successively. Therefore, \( G \) is \((\Delta + 13)\)-list-L(2,1)-choosable, a contradiction.

Property 3: For a 4-vertex \( v \):
(1) \( v \) can not adjacent to three 2-vertices.

(2) If \( n_2(v) = 2 \), then 
\[
\sum_{i=2}^{4} d(v_i) \geq \Delta + 6
\]

(3) If \( n_2(v) = 1 \), then 
\[
\sum_{i=2}^{4} d(v_i) \geq \Delta + 5
\]

(4) Suppose for all \( 1 \leq i \leq 4 \), \( d(v_i) = 3 \), \( d(v_{i1}) + d(v_{i2}) \geq \Delta + 5 \) where \( v_{i1} \) and \( v_{i2} \) are \( v_i \)'s neighbors except \( v \) or \( d(v_i) \geq \Delta + 5 \).

Proof. (4) Suppose for all \( 1 \leq i \leq 4 \), \( d(v_i) = 3 \), \( d(v_{i1}) + d(v_{i2}) \leq \Delta + 4 \) where \( v_{i1} \) and \( v_{i2} \) are \( v_i \)'s neighbors except \( v \). By the minimality of \( G \), \( G - vv_1 \) has a \((\Delta + 13)\)-list-L(2,1)-choosable. We first erase the colors of \( v_1 \) and \( v \). \( |F(v_1)| \leq \Delta + 11 \), \( |F(v)| \geq 20 \), so we can recolor \( v_1 \) and \( v \) successively. Therefore, \( G \) is \((\Delta + 13)\)-list-L(2,1)-choosable, a contradiction.

Property 4: For a 5-vertex \( v \):
(1) \( v \) can not adjacent to four 2-vertices.

(2) If \( n_2(v) = 3 \), then 
\[
\sum_{i=4}^{5} d(v_i) \geq \Delta + 5
\]

(3) If \( n_2(v) = 2 \), then 
\[
\sum_{i=3}^{5} d(v_i) \geq \Delta + 4
\]

Property 5: For a 6-vertex \( v \):
(1) \( v \) can not adjacent to five 2-vertices.

(2) If \( n_2(v) = 4 \), then 
\[
\sum_{i=5}^{6} d(v_i) \geq \Delta + 4
\]

(3) If \( n_2(v) = 3 \), then 
\[
\sum_{i=4}^{6} d(v_i) \geq \Delta + 3
\]
(4) If \( n_2(v) = 2 \), then \( \sum_{i=3}^{6} d(v_i) \geq \Delta + 2 \).

Property 6: For a 7-vertex \( v \):
(1) \( v \) can not adjacent to six 2-vertices.
(2) If \( n_2(v) = 5 \), then \( \sum_{i=5}^{7} d(v_i) \geq \Delta + 3 \).
(3) If \( n_2(v) = 4 \), then \( \sum_{i=4}^{7} d(v_i) \geq \Delta + 2 \).
(4) If \( n_2(v) = 3 \), then \( \sum_{i=4}^{7} d(v_i) \geq \Delta + 1 \).
(5) If \( n_2(v) = 2 \), then \( \sum_{i=3}^{7} d(v_i) \geq \Delta \).
(6) If \( n_2(v) = 1 \), then \( \sum_{i=3}^{7} d(v_i) \geq \Delta - 1 \).

(7) In \( G \), every 7(0)-vertex is adjacent to at most six 3(1)-vertices, and when it is adjacent to six 3(1)-vertices, \( d(v_7) \geq -19 \).

Property 7: For a 8-vertex \( v \):
(1) If \( n_2(v) = 8 \), then \( d(v_i) = \Delta \) for all \( i \in \{1,2,...,8\} \).
(2) If \( n_2(v) = 7 \), then \( d(v_8) = \Delta \) or \( d(v_i) = \Delta \) for all \( i \in \{1,2,...,7\} \).
(3) If \( n_2(v) = 6 \), then \( \sum_{i=6}^{8} d(v_i) \geq \Delta + 2 \) or \( d(v_i) = \Delta \) for all \( i \in \{1,2,...,6\} \).
(4) If \( n_2(v) = 5 \), then \( \sum_{i=5}^{8} d(v_i) \geq \Delta + 1 \) or \( d(v_i) = \Delta \) for all \( i \in \{1,2,...,5\} \).
(5) If \( n_2(v) = 4 \), then \( \sum_{i=4}^{8} d(v_i) \geq \Delta \) or \( d(v_i) = \Delta \) for all \( i \in \{1,2,...,4\} \).
(6) If \( n_2(v) = 3 \), then \( \sum_{i=3}^{8} d(v_i) \geq \Delta - 1 \) or \( d(v_i) = \Delta \) for all \( i \in \{1,2,3\} \).

Proof. (2) Assume that there is an index \( i \in \{1,2,...,7\} \), \( d(v_i) \leq \Delta - 1 \). So we can suppose that \( d(v_i) \leq \Delta - 1 \) and \( d(v_8) \leq \Delta - 1 \). By the minimality of \( G \), \( G - vv_1 \) has a \( (\Delta + 13) \)-list-L(2,1)-choosable. We first erase the colors of \( v, v_2, v_3, v_4, v_5, v_6, v_7 \) and \( v_1 \). Then \( |F(v)| \leq \Delta + 8 \), \( |F(v_2)| \leq \Delta + 6 \), \( |F(v_3)| \leq \Delta + 7 \), \( |F(v_4)| \leq \Delta + 8 \), \( |F(v_5)| \leq \Delta + 9 \), \( |F(v_6)| \leq \Delta + 10 \), \( |F(v_7)| \leq \Delta + 11 \), \( |F(v_1)| \leq \Delta + 11 \), so we can recolor \( v, v_2, v_3, v_4 \) and \( v_1 \) successively. Therefore, \( G \) is \( (\Delta + 13) \)-list-L(2,1)-choosable, a contradiction.

Property 8: For a 9-vertex \( v \):
(1) If \( n_2(v) = 9 \), then \( d(v_i) = \Delta \) for all \( i \in \{1,2,...,9\} \).
(2) If \( n_2(v) = 8 \), then \( d(v_9) = \Delta \) or \( d(v_i) = \Delta \) for all \( i \in \{1,2,...,7\} \).
(3) If \( n_2(v) = 7 \), then \( \sum_{i=7}^{9} d(v_i) \geq \Delta + 1 \) or \( d(v_i) = \Delta \) for all \( i \in \{1,2,...,6\} \).
(4) If \( n_2(v) = 6 \), then \( \sum_{i=6}^{9} d(v_i) \geq \Delta \) or \( d(v_i) = \Delta \) for all \( i \in \{1,2,...,5\} \).

Property 9: For a 10-vertex \( v \):
(1) If \( n_2(v) = 10 \), then \( d(v_i) = \Delta \) for all \( i \in \{1,2,...,10\} \).
(2) If \(n_2(v) = 9\), then \(d(v_{10}) = \Delta\) or \(d(v_i) = \Delta\) for all \(i \in \{1,2,...,7\}\), \(d(v_{v_8}) \geq \Delta + 1\) and \(d(v_{v_9}) \geq \Delta + 2\).

4. Proof of Theorem 1.1

In this section, we give the proof of our main results by discharging method.

Proof. We prove Thm1.1 by contradiction. Let \(G\) be a plane with neither 3-cycles nor intersect 4-cycles and with fewest sum of the number of vertices and edges such that \(\chi^*_{2,1}(G) > \Delta(G) + 12\). That is to say, there exists a list \(L\) of size \(\Delta + 13\) of \(V(G)\), but \(G\) is not \(L(2, 1)\)-choosable. By the minimality of \(G\), \(G\) is connected.

According to Euler's formula and \(\sum_{v \in V} d(v) = 2|E|\), we get:

\[
\sum_{v \in V} \left(\frac{3}{2}d(v) - 5\right) + \sum_{f \in F} (d(f) - 5) = -10
\]

For all \(x \in V(G) \cup F(G)\), we define an initial weight function \(\omega\): if \(v \in V\), let \(\omega(v) = \frac{3}{2}d(v) - 5\); if \(f \in F\), let \(\omega(f) = d(f) - 5\). Then we get \(\sum_{x \in V(G) \cup F(G)} \omega(x) = -10\). If we obtain a new weight \(\omega'(x) \geq 0\) for all \(x \in V \cup F\) by transferring weights from one element to another, then we get a contradiction:

\[
0 \leq \sum_{x \in V \cup F} \omega'(x) = \sum_{x \in V \cup F} \omega(x) = -10.
\]

This contradiction shows the nonexistence of \(G\). Therefore, Theorem1.1 is true.

We will redistribute the charge according to the ten discharging rules below, observe that each of the discharging rules preserves the sum of the charges. Hence, the sum of the charges after discharging is negative. By assuming that each of the nine structural properties above holds, we prove that after discharging each element has nonnegative charge, this is an obvious contradiction. Discharging rules can be listed below:

(R1) When \(d(f) = 4\). If \(f\) incident with 3-vertices, each 3-vertex sends \(\frac{1}{4}\) to \(f\); if \(f\) incident with 4\(^{+}\)-vertices, each 4\(^{+}\)-vertex sends \(\frac{1}{2}\) to \(f\).

(R2) Every 2-vertex get 1 from adjacent 3\(^{+}\)-vertices.

(R3) For a 3-vertex \(x\), \(xy \in E(G)\); if \(d(y) = 4\), \(x\) get \(\frac{1}{8}\) from \(y\); if \(d(y) = 5\), \(x\) get \(\frac{1}{4}\) from \(y\); if \(d(y) = 6\), \(x\) get \(\frac{1}{2}\) from \(y\); if \(7 \leq d(y) \leq 10\), \(x\) get \(\frac{3}{4}\) from \(y\); if \(11 \leq d(y) \leq \Delta - 3\), \(x\) get \(\frac{7}{8}\) from \(y\); if \(d(y) \geq \Delta - 3\), \(x\) get 1 from \(y\). In particular, each weak 3-vertex gets \(\frac{3}{4}\) from each adjacent strong 3-vertex.

(R4) For a 4-vertex \(x\), \(xy \in E(G)\); if \(d(y) = 5\), \(x\) get \(\frac{1}{4}\) from \(y\); if \(6 \leq d(y) \leq 10\), \(x\) get \(\frac{1}{2}\) from \(y\); if \(d(y) \geq 11\), \(x\) get 1 from \(y\). In particular, each weak 4-vertex gets \(\frac{1}{2}\) from each adjacent 3\(^{+}\)-vertex.

(R5) For a 5-vertex \(x\), \(xy \in E(G)\); if \(d(y) = 6\), \(x\) get \(\frac{1}{4}\) from \(y\); if \(7 \leq d(y) \leq \Delta - 2\), \(x\) get \(\frac{1}{2}\) from \(y\); if \(d(x) = \Delta - 1\), \(x\) get \(\frac{3}{4}\) from \(y\); if \(d(y) = \Delta\), \(x\) get 1 from \(y\).

(R6) For a 6-vertex \(x\), \(xy \in E(G)\); if \(7 \leq d(y) \leq \Delta - 2\), \(x\) get \(\frac{1}{2}\) from \(y\); if \(d(y) = \Delta - 1\), \(x\) get \(\frac{3}{4}\) from \(y\); if \(d(y) = \Delta\), \(x\) get 1 from \(y\).

(R7) For a 7-vertex \(x\), \(xy \in E(G)\); if \(8 \leq d(y) \leq 9\), \(x\) get \(\frac{1}{4}\) from \(y\); if \(10 \leq d(y) \leq \Delta - 1\), \(x\) get \(\frac{1}{2}\) from \(y\); if \(d(y) = \Delta\), \(x\) get \(\frac{3}{4}\) from \(y\).
(R8) For a 8-vertex \( x, xy \in E(G); \) if \( 9 \leq d(y) \leq \Delta - 1, \) \( x \) get \( \frac{1}{4} \) from \( y; \) if \( d(u) = \Delta, \) \( x \) get \( \frac{1}{2} \) from \( y. \)

Moreover, \( v \) gets \( \frac{3}{16} \) from each weak adjacent \( \Delta \)-vertex.

(R9) For a 9-vertex \( x, x \) get \( \frac{1}{9} \) from each weak adjacent \((\Delta - 1)^*\)-vertex.

(R10) For a 10-vertex \( x, x \) get \( \frac{1}{10} \) from each weak adjacent \((\Delta - 1)^*\)-vertex.

In the following, we will prove that \( \omega'(x) \geq 0 \) for all \( x \in V(G) \setminus F(G). \)

First we checking \( \omega'(x) \geq 0, \forall f \in F(G). \)

Case 1: \( d(f) = 4. \)

when \( f \) is incident with 2-vertices, but isn’t incident with 3-vertices, from property 1, and by R1, 
\( \omega'(f) \geq (4-5) + \frac{1}{2} \times 2 = 0. \) when \( f \) is incident with both 2-vertices and 3-vertices, from property 1 and 2, and by R1, 
\( \omega'(f) \geq (4-5) + \frac{1}{4} \times 2 + \frac{1}{2} = 0. \)

when \( f \) isn’t incident with 2-vertices, but is incident with 3-vertices, from property 2(3), and by R1, 
\( \omega'(f) \geq (4-5) + \frac{1}{4} \times 2 + \frac{1}{2} = \frac{3}{2} > 0. \)

Case 2: \( d(f) \geq 5. \) \( \omega'(f) = d(f) - 5 \geq 0. \)

Then we checking \( \omega'(v) \geq 0, \forall v \in V(G). \)

Case 1: \( d(v) = 2. \) From property 1, and by R2, 
\( \omega'(v) = (\frac{3}{2} \times 2 - 5) + 1 \times 2 = 0. \)

Case 2: \( d(v) = 3. \) From property 2 \( n_2(v) \leq 1. \)

Case 2.1: \( n_2(v) = 1. \) From property 2(2), 
\( (d(v_2); d(v_3)) \in \{(7, \Delta); (8, (\Delta - 1)^*); (9, (\Delta - 2)^*); (10, (\Delta - 3)^*); (11^+, 11^*)\}. \) By R3, \( v_2 \) and \( v_3 \) send at least \( \frac{7}{4} \) to \( v. \) By R1 and R2, 
\( \omega'(v) = (\frac{3}{2} \times 3 - 5) + 1 \times \frac{7}{4} = 0. \)

Case 2.2: \( n_2(v) = 0. \)

When \( v \) is adjacent to a 3-vertex \( v_1, \) if \( d(v_2) + d(v_3) \leq \Delta + 5, \) then \( v \) is a weak 3-vertex, and from property 2(3), \( d(u) + d(w) \geq \Delta + 6 \) where \( u \) and \( w \) are \( v_1 \)’s neighbors except \( v, \) so \( v_1 \) is a strong 3-vertex. At the same time \( v_2 \) and \( v_3 \) are 3*-vertices. Again by property 2(3), \( v_2 \) and \( v_3 \) are not weak 3-vertices. By R1 and R3, 
\( \omega'(v) = (\frac{3}{2} \times 3 - 5) + 3 \times \frac{1}{4} = 0. \)

Suppose \( d(v_2) + d(v_3) \geq 6 \) then \( v \) is a strong 3-vertex. So \((d(v_2), d(v_3)) \in \{(5^+, 10^*)\}. \) By R5, \( v_2 \) and \( v_3 \) send at least \( 1 \) to \( v. \) By R1, 
\( \omega'(v) = (\frac{3}{2} \times 3 - 5) + 1 \times \frac{1}{4} = \frac{1}{2} > 0. \)

When \( v \) is not adjacent to a 3-vertex, and by R1 and R3, 
\( \omega'(v) \geq (\frac{3}{2} \times 3 - 5) + 1 \times \frac{1}{4} = \frac{5}{2} > 0. \) In particular, if \( v \) is a special 3-vertex, then by R1, R3 and R4, 
\( \omega'(v) \geq (\frac{3}{2} \times 3 - 5) + 1 \times \frac{1}{4} + 1 \times \frac{1}{2} = \frac{1}{2} > 0. \)

Case 3: \( d(v) = 4. \) From property 3 \( n_2(v) \leq 2. \)

Case 3.1: \( n_2(v) = 2. \) From property 3(2), 
\( (d(v_3); d(v_4)) \in \{(6, \Delta); (7,(\Delta - 1)^*); (8, (\Delta - 2)^*); (9,(\Delta - 3)^*); (10, (\Delta - 4)^*); (11^+, (\Delta - 5)^*)\}. \) By R6, \( v_3 \) and \( v_4 \) send at least \( \frac{3}{2} \) to \( v. \) By R3 and R4, 
\( \omega'(v) \geq (\frac{3}{2} \times 4 - 5) - 1 \times 2 + \frac{3}{2} \times \frac{1}{2} = 0. \)
Case 3.2: \( n_2(v) = 1 \). From property 3(3), \( (d(v_2);d(v_3);d(v_4)) \in \{(3,3,(\Delta -1)^+));(3,4,(\Delta -2)^+));(3,5,(\Delta -3)^+));(3,6,(\Delta -4)^+));(3,7,(\Delta -5)^+));(3,8,(\Delta -6)^+));(3,9,(\Delta -7)^+));(3,10,(\Delta -8)^+));(3,11^+,11^+));(4^+,4^+,10^+)) \}. By R3 and R4, \( v_2, v_3 \) and \( v_4 \) send at least \( \frac{3}{4} \) to \( v \). By R1-R4, \( \omega'(v) > \left( \frac{3}{2} \times 4 - 5 \right) - 1 + \frac{3}{4} - \frac{1}{2} = 0 > 0. \)

Case 3.3: \( n_2(v) = 0 \).

Case 3.3.1: \( n_3(v) = 4 \). From property 3(4), there exists a vertex \( v_i \), \( d(v_i1) + d(v_i2) \geq \Delta + 5 \) for some \( i \in \{1,2,3,4\} \) where \( v_i1 \) and \( v_i2 \) are \( v_i \)’s neighbors except \( v \). So \( v_i \) is a special 3-vertex, by R1, R3 and R4, \( \omega'(v) = \left( \frac{3}{2} \times 4 - 5 \right) - 1 + \frac{3}{4} - \frac{1}{2} = 0 > 0. \)

Case 3.3.2: \( n_3(v) = 3 \).

If \( d(v_4) \geq \Delta - 5 \), then by R1, R3 and R4, \( \omega'(v) = 0. \) Suppose \( d(v_4) \leq \Delta - 6 \), then from property 3(5), there exists a vertex \( v_i \) such that \( d(v_i1) + d(v_i2) \geq \Delta + 5 \) for some \( i \in \{1,2,3\} \) where \( v_i1 \) and \( v_i2 \) are \( v_i \)’s neighbors except \( v \). So \( v_i \) is a special 3-vertex, by R1, R3 and R4, \( \omega'(v) = \left( \frac{3}{2} \times 4 - 5 \right) - 1 + \frac{3}{4} - \frac{1}{2} = 0. \)

Case 3.3.3: \( n_3(v) \leq 2 \).

By R1 and R3, \( \omega'(v) = 0. \)

Case 4: \( d(v) = 5 \). From property 4 \( n_2(v) \leq 3 \).

Case 4.1: \( n_2(v) = 3 \). From property 4(2), \( (d(v_4);d(v_5)) \in \{(5,\Delta);(6,(\Delta -1)^+));(7^+,7^+)) \}. By R5, \( v_4 \) and \( v_5 \) send at least 1 to \( v \). By R1 and R2, \( \omega'(v) = 0. \)

Case 4.2: \( n_2(v) = 2 \).

From property 4(3), \( (d(v_3);d(v_4);d(v_5)) \in \{(3,3,(\Delta -2)^+));(3,4,(\Delta -3)^+));(3,5,(\Delta -4)^+));(3,6^+,7^+));(3,7^+,7^+)) \}. By R4 and R5, \( v_3, v_4 \) and \( v_5 \) send at least \( \frac{1}{2} \) to \( v \). Then by R1 and R2, \( \omega'(v) = 0. \)

Case 4.3: \( n_2(v) \leq 1 \). Then by R1-R3, \( \omega'(v) = 0. \)

Case 5: \( d(v) = 6 \). From property 5 \( n_2(v) \leq 4 \).

Case 5.1: \( n_2(v) = 4 \).

From property 5(2), \( (d(v_5);d(v_6)) \in \{(5,\Delta);(6,(\Delta -1)^+));(6^+,7^+)) \}. By R4-R6, \( v_5 \) and \( v_6 \) send at least \( \frac{1}{2} \) to \( v \). Then by R1 and R2, \( \omega'(v) = 0. \)

Case 5.2: \( n_2(v) = 3 \).

From property 5(3), \( (d(v_4);d(v_5);d(v_6)) \in \{(3,3,(\Delta -3)^+));(3,4,(\Delta -4)^+));(3,5,(\Delta -5)^+));(3,6^+,7^+));(4,4,(\Delta -5)^+));(4,5^+,7^+));(5^+,5^+,7^+)) \}. By R3-R6, \( v_4 \) and \( v_5 \) send at least \( \frac{1}{2} \) to \( v \). Then by R1 and R2, \( \omega'(v) = 0. \)

Case 5.3: \( n_2(v) = 2 \).

From property 5(4), \( (d(v_3);d(v_4);d(v_5);d(v_6)) \in \{(3,3,(\Delta -7)^+));(3,4,(\Delta -8)^+));(3,5,(\Delta -9)^+));(3,6^+,7^+));(3,7^+,7^+));(4,4^+,4^+,7^+));(4^+,4^+,7^+)) \}. By R3-R6, \( v_3, v_4, v_5 \) and \( v_6 \) send at least \( \frac{1}{2} \) to \( v \). Then by R1 and R2, \( \omega'(v) = 0. \)

Case 5.4: \( n_2(v) \leq 1 \). Then by R1-R5, \( \omega'(v) = 0. \)

Case 6: \( d(v) = 7 \). From property 6 \( n_2(v) \leq 5 \).

Case 6.1: \( n_2(v) = 5 \).
From property 6(2), \( d(v_5) + d(v_6) \geq \Delta + 4 \). So \( d(v_5); d(v_6) \in \{(3, \Delta); (4, (\Delta - 1)^+); (5, (\Delta - 2)^+); (6, (\Delta - 3)^+); (7, 7^+)\} \). By R3-R7, \( v_5 \) and \( v_7 \) send at least 0 to \( v \). Then by R1 and R2, \( \omega(v) \geq \left(\frac{3}{2} \times 7 - 5\right) - 1 \times 5 - \frac{1}{2} = 0 \).

Case 6.2: \( n_2(v) = 4 \).

From property 6(3), \( (d(v_5); d(v_6); d(v_7)) \in \{(3, 3, (\Delta - 4)^+)); (3, 4, (\Delta - 5)^+); (3, 5, (\Delta - 6)^+); (3, 6, (\Delta - 7)^+); (3, 7+, 7+); (4+, 4+, 7+)\} \). By R3-R7, \( v_5, v_6 \) and \( v_7 \) send at least -1 to \( v \). Then by R1 and R2, \( \omega(v) = \left(\frac{3}{2} \times 7 - 5\right) - 1 \times 3 - \frac{3}{4} \times 3 - \frac{1}{2} - \frac{1}{2} = \frac{1}{4} > 0 \).

Case 6.3: \( n_2(v) = 3 \).

Case 6.3.1: \( n_3(v) = 3 \). By R1-R3 and R7, \( \omega(v) = \left(\frac{3}{2} \times 7 - 5\right) - 1 \times 3 - \frac{3}{4} \times 3 = \frac{1}{4} > 0 \).

Case 6.3.2: \( n_3(v) = 2 \). If \( d(v_5) \geq 4, d(v_6) \geq (\Delta - 9), \) then \( v \) transfer most weight; by R1-R6, \( \omega(v) = \left(\frac{3}{2} \times 7 - 5\right) - 1 \times 3 - \frac{3}{4} \times 2 - \frac{1}{2} - \frac{1}{2} = \frac{1}{4} > 0 \).

Case 6.3.3: \( n_3(v) = 1 \). If \( d(v_5) \geq 4, d(v_7) \geq (\Delta - 13), \) then \( v \) transfer most weight; by R1-R6, \( \omega(v) = \left(\frac{3}{2} \times 7 - 5\right) - 1 \times 3 - \frac{3}{4} \times 2 + \frac{1}{2} - \frac{1}{2} = \frac{1}{2} > 0 \).

Case 6.4: \( n_2(v) = 2 \).

Case 6.4.1: \( n_3(v) = 4 \). By R1-R3, \( \omega(v) = \left(\frac{3}{2} \times 7 - 5\right) - 1 \times 3 = \frac{1}{4} > 0 \).

Case 6.4.2: \( n_3(v) = 3 \). If \( d(v_5) \geq 4, d(v_7) \geq (\Delta - 19), \) then \( v \) transfer most weight; by R1-R6, \( \omega(v) = \left(\frac{3}{2} \times 7 - 5\right) - 1 \times 3 - \frac{3}{4} \times 2 + \frac{1}{2} - \frac{1}{2} = \frac{1}{2} > 0 \).

Case 6.5: \( n_2(v) = 0 \).

Case 6.5.1: \( n_3(v) = 5 \). By R1-R3, \( \omega(v) = \left(\frac{3}{2} \times 7 - 5\right) - 1 \times 3 - \frac{3}{4} \times 3 + \frac{1}{2} - \frac{1}{2} = \frac{1}{2} > 0 \).

Case 6.5.2: \( n_3(v) = 4 \). By R1-R3, \( \omega(v) = \left(\frac{3}{2} \times 7 - 5\right) - 1 \times 3 - \frac{3}{4} \times 2 + \frac{1}{2} = \frac{1}{2} > 0 \).

Case 6.6: \( n_2(v) = 1 \).

Case 6.6.1: \( n_3(v) = 6 \). If \( d(v_7) + d(v_8) \geq (\Delta - 2), \) then \( (d(v_7); d(v_8)) \in \{(3, (\Delta - 1)^+); (4, (\Delta - 2)^+); (5, (\Delta - 3)^+); (6, (\Delta - 4)^+); (7, (\Delta - 5)^+); (8^+, 8^+)\} \). By R3-R7, \( v_5, v_6 \) and \( v_7 \) send at least -1 to \( v \). Then by R1 and R2, \( \omega(v) = \left(\frac{3}{2} \times 8 - 5\right) - 1 \times 6 - \frac{1}{2} - \frac{1}{2} = 0 \).
=0. Suppose \(d(v_7) + d(v_8) \leq \Delta + 1\), from property 7(3) \(d(v_i) = \Delta\ (1 \leq i \leq 6)\). By R1-R8, \(\omega'(v) \geq (\frac{3}{2} \times 8 - 5) - 1 \times 6 - \frac{3}{4} \times 2 - \frac{3}{16} \times 6 = \frac{1}{8} > 0\).

Case 7.4: \(n_2(v) = 5\).

If \(d(v_6) + d(v_7) + d(v_8) \geq \Delta + 1\), so \((d(v_6); d(v_7); d(v_8)) \in \{(3, 3, (\Delta - 5)^+)\}; (3, 4^+, 8^+); (4^+, 4^+, 8^+)\). R3-R8, \(v_6, v_7\) and \(v_8\) send at least \(-\frac{5}{4}\) to \(v\). Then by R1 and R2, \(\omega'(v) \geq (\frac{3}{2} \times 8 - 5) - 1 \times 5 - \frac{5}{4} - \frac{1}{2} = \frac{1}{4} > 0\).

Suppose \(d(v_6) + d(v_7) + d(v_8) \leq \Delta\), from property 7(4) \(d(v_i) = \Delta\ (1 \leq i \leq 6)\). By R1-R8, \(\omega'(v) \geq (\frac{3}{2} \times 8 - 5) - 1 \times 5 - \frac{3}{4} - \frac{3}{16} \times 5 = \frac{3}{16} > 0\).

Case 7.5: \(n_2(v) = 4\).

If \(\sum_{i=5}^{8} d(v_i) \geq \Delta\), \(v\) transfer most weight when \(n_3(v) = 3\). By R1-R3, \(\omega'(v) \geq (\frac{3}{2} \times 8 - 5) - 1 \times 4 - \frac{3}{4} \times 4 - \frac{1}{2} - \frac{3}{16} \times 4 = \frac{1}{4} > 0\).

By R1-R8, \(\omega'(v) \geq (\frac{3}{2} \times 8 - 5) - 1 \times 4 - \frac{3}{4} \times 4 - \frac{1}{2} = \frac{1}{2} > 0\).

Suppose \(\sum_{i=5}^{8} d(v_i) \leq \Delta - 1\), from property 7(5) \(d(v_i) = \Delta\ (1 \leq i \leq 4)\).

By R1-R8, \(\omega'(v) \geq (\frac{3}{2} \times 8 - 5) - 1 \times 3 - \frac{3}{4} \times 3 - \frac{1}{2} = \frac{1}{2} > 0\).

Case 7.6: \(n_2(v) = 3\).

If \(\sum_{i=4}^{8} d(v_i) \geq \Delta - 1\). If \(n_3(v) = 4\) and \(d(v_8) \geq \Delta - 13\), then \(v\) transfer most weight; by R1-R3, \(\omega'(v) \geq (\frac{3}{2} \times 8 - 5) - 1 \times 2 - \frac{3}{4} \times 6 - \frac{1}{2} = 0\).

Suppose \(\sum_{i=4}^{8} d(v_i) \leq \Delta - 2\), from property 7(6) \(d(v_i) = \Delta\ (1 \leq i \leq 3)\). By R1-R8, \(\omega'(v) \geq (\frac{3}{2} \times 8 - 5) - 1 \times 3 - \frac{3}{4} \times 5 - \frac{1}{2} + \frac{3}{16} \times 3 = \frac{5}{16} > 0\).

Case 7.7: \(n_2(v) \leq 2\). Then by R1-R8, \(\omega'(v) \geq (\frac{3}{2} \times 9 - 5) - 1 \times 2 - \frac{3}{4} \times 6 - \frac{1}{2} = 0\).

Case 8: \(d(v) = 9\). From property 8 \(n_2(v) \leq 9\).

Case 8.1: \(n_2(v) = 9\). From property 8(1) \(d(v_i) = \Delta\ (1 \leq i \leq 9)\). By R1, R2 and R9, \(\omega'(v) \geq (\frac{3}{2} \times 9 - 5) - 1 \times 9 - \frac{1}{2} + \frac{3}{9} \times 9 = 0\).

Case 8.2: \(n_2(v) = 8\).

If \(d(v_9) = \Delta\); by R1 and R2, \(\omega'(v) = (\frac{3}{2} \times 9 - 5) - 1 \times 8 - \frac{1}{2} = 0\). Suppose \(d(v_9) \leq \Delta - 1\). From property 8(2) \(d(v_i) = \Delta\ (1 \leq i \leq 8)\), \(v\) is adjacent to nine weak \((\Delta - 1)^+\)-vertices, or eight weak \(\Delta\)-vertices. By R1-R9, \(\omega'(v) \geq (\frac{3}{2} \times 9 - 5) - 1 \times 8 \times \frac{3}{4} \times \frac{1}{2} + \frac{1}{9} \times 8 = \frac{5}{36} > 0\).

Case 8.3: \(n_2(v) = 7\).

If \(d(v_8) + d(v_9) \geq \Delta + 1\), then \((d(v_8); d(v_9)) \in \{(3, (\Delta - 2)^+); (4, (\Delta - 3)^+); (5, (\Delta - 4)^+); (6, (\Delta - 5)^+); (7, (\Delta - 6)^+); (8, (\Delta - 7)^+); (9^+, 9^+)\}\). R3-R8, \(v_8\) and \(v_9\) send at least \(-\frac{3}{4}\) to \(v\). Then by R1 and R2, \(\omega'(v) \geq (\frac{3}{2} \times 9 - 5) - 1 \times 7 - \frac{3}{4} - \frac{1}{2} = \frac{1}{4} > 0\). Suppose \(d(v_8) + d(v_9) \leq \Delta\), from property 8(3) \(d(v_i) = \Delta\ (1 \leq i \leq 7)\), \(v\) is adjacent to
eight weak $(\Delta - 1)^+$-vertices, or seven weak $\Delta$-vertices. By R1-R9, $\omega'(v) \geq (\frac{3}{2} \times 9 - 5) \times 1 \times 7 - \frac{3}{4} \times 2 - \frac{1}{2} + \frac{1}{9} \times 7 = \frac{17}{36} > 0$.

Case 8.4: $n_2(v) = 6$. If $d(v_7) + d(v_8) + d(v_9) \geq \Delta$, so $(d(v_7); d(v_8); d(v_9)) \in \{(3, 3, (\Delta - 6)^+));(3, 3, (\Delta - 7)^+);(4^+, 4^+, (\Delta - 8)^+)\}.$

By R3-R8, $v_6, v_7$ and $v_8$ send at least $\frac{3}{2} \times 2$ to $v$. Then by R1 and R2, $\omega'(v) \geq (\frac{3}{2} \times 9 - 5) - \frac{3}{4} \times 6 - \frac{1}{2} = \frac{9}{2} > 0.$

Suppose $d(v_7) + d(v_8) + d(v_9) \leq \Delta - 1$, from property 8(4) $d(v_i^\prime) = \Delta \leq \Delta$, so $v$ is adjacent to seven weak $(\Delta - 1)^+$-vertices, or seven weak $\Delta$-vertices. By R1-R9, $\omega'(v) \geq (\frac{3}{2} \times 9 - 5) - \frac{3}{4} \times 6 + \frac{1}{2} + \frac{1}{9} \times 7 = \frac{5}{12} > 0$.

Case 8.5: $n_2(v) \leq 5$. Then by R1-R8, $\omega'(v) \geq (\frac{3}{2} \times 9 - 5) - \frac{3}{4} \times 6 - \frac{1}{2} = 0$.

Case 9: $d(v) = 10$. From property 9 $n_2(v) \leq 10$.

Case 9.1: $n_2(v) = 10$. From property 9(1) $d(v_i^\prime) = \Delta \leq \Delta$. By R1, R2 and R10, $\omega'(v) \geq (\frac{3}{2} \times 9 - 5) - \frac{3}{4} \times 6 - \frac{1}{2} = 0$.

Case 9.2: $n_2(v) = 9$. If $d(v_1)$ $\Delta = \Delta , by R3, R4, R8 and R9, \omega'(v) = (\frac{3}{2} \times 9 - 5) - \frac{3}{4} \times 6 - \frac{1}{2} > 0.$ Suppose $d(v_{10}) \leq \Delta - 1$, $v$ is adjacent to nine weak $(\Delta - 2)^+$-vertices, or eight weak $(\Delta - 1)^+$-vertices, or seven weak $\Delta$-vertices. By R1, R2 and R10, $\omega'(v) \geq (\frac{3}{2} \times 9 - 5) - \frac{3}{4} \times 6 - \frac{1}{2} = 0$.

Case 9.3: $n_2(v) \leq 8$. Then by R1-R8, $\omega'(v) \geq (\frac{3}{2} \times 10 - 5) - \frac{3}{4} \times 6 - \frac{1}{2} = 0$.

Case 10: $11 \leq d(v) \leq (\Delta - 3)$.

By R2-R10, $v$ send at most 1 to adjacent $8^-$-vertices. Then by R1, $\omega'(v) = (\frac{3}{2} \times 9 - 5) - d(v) - \frac{1}{2} \geq 0$.

Case 11: $d(v) = (\Delta - 2) = 24$.

By R2-R10, $v$ send at most 1 to adjacent $10^-$-vertices and $\frac{1}{10}$ to each weak $10$-vertex. Then by R1, $\omega'(v) = (\frac{3}{2} \times 9 - 5) - d(v) - \frac{1}{10} \geq 0$.

Case 12: $d(v) = (\Delta - 1) = 25$.

By R2-R10, $v$ send at most 1 to adjacent $10^-$-vertices and $\frac{1}{10}$ to each weak $10$-vertex. Then by R1, $\omega'(v) = (\frac{3}{2} \times 9 - 5) - d(v) - \frac{1}{10} \geq 0$.

By R2-R10, $v$ send at most 1 to adjacent $8^-$-vertices. Then by R1, $\omega'(v) = (\frac{3}{2} \times 9 - 5) - d(v) - \frac{1}{10} \geq 0$.

This completes the proof of Thm1.1.

Acknowledgments
This work is supported by the National Natural Science Foundation of China (Grant No. 12071265) and the Natural Science Foundation of Shandong Province (Grant No. ZR2019MA032).

References
[1] J. R. Griggs, and R. K. Yeh 1992 Labeling graphs with a condition at distance 2 SIAM J. Discrete Math 5 pp 586-595
[2] G. J. Chang, and D. Guo 1996 The L(2, 1)-labelling problem on graphs SIAM J. Discrete Math. 9 pp 309-316
[3] D. Král et al. 2003 A theorem about the channel assignment problem SIAM J. Discrete Math 16 pp 426-437
[4] D. Concalves 2005 On the L(p,1)-labelling of graphs, Discrete Math Theoret. Comput. Sci. AE pp 81-86
[5] J. Van Den Heuvel, and S. McGuinness 2003 Coloring of the square of a planar graph J. Graph Theory 42 pp 110-124
[6] O. V. Borodin, H. J. Broersma, A. Glebov, and J. Van Den Heuvel 2002 Stars and bunches in planar graphs Part: General planar graphs and colorings, CD AM, Research report
[7] M. Molloy, and M. R. Salavatipour 2005 A bound on the chromatic number of the square of a planar graph J. Combinatorial Theory. B. 94 pp 189-213
[8] H. Y. Zhu, X. Z. Lu, C. Q. Wang, and M. Chen 2012 Labeling planar graphs without 4,5-cycles with a condition on distance two SIAM J. Discrete Math. 26 pp 52-64