DELIGNE GROUPOID REVISITED

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Abstract. We show that for a differential graded Lie algebra $\mathfrak{g}$ whose components vanish in degrees below $-1$ the nerve of the Deligne 2-groupoid is homotopy equivalent to the simplicial set of $\mathfrak{g}$-valued differential forms introduced by V. Hinich [Hinich, 1997].

1. Introduction

The principal result of the present note compares two spaces (simplicial sets) naturally associated with a nilpotent differential graded Lie algebra (DGLA) subject to certain restrictions. Our interest in this problem has its origins in formal deformation theory of associative algebras and, more generally, algebroid stacks ([Bressler, Gorokhovsky, Nest & Tsygan, 2007]). The results of the present note are used in [Bressler, Gorokhovsky, Nest & Tsygan, 2015] to deduce a quasi-classical description of the deformation theory of a gerbe from the formality theorem of M. Kontsevich ([Kontsevich, 2003]).

To a nilpotent DGLA $\mathfrak{h}$ which satisfies the additional condition

$$h^i = 0 \text{ for } i < -1$$

P. Deligne [Deligne, 1994] and, independently, E. Getzler [Getzler, 2009] associated a (strict) 2-groupoid which we denote $MC^2(\mathfrak{h})$ and refer to as the Deligne 2-groupoid.

Our principal result (Theorem 4.2) compares the simplicial nerve $\mathcal{N}MC^2(\mathfrak{h})$ of the 2-groupoid $MC^2(\mathfrak{h})$, $\mathfrak{h}$ a nilpotent DGLA satisfying (1), to another simplicial set, denoted $\Sigma(\mathfrak{h})$, introduced by V. Hinich [Hinich, 1997]:

1.1. Theorem. (Main theorem) Suppose that $\mathfrak{h}$ is a nilpotent DGLA such that $\mathfrak{h}^i = 0$ for $i < -1$. Then, the simplicial sets $\mathcal{N}MC^2(\mathfrak{h})$ and $\Sigma(\mathfrak{h})$ are weakly homotopy equivalent.

In the case when the nilpotent DGLA $\mathfrak{h}$ satisfies $\mathfrak{h}^i = 0$ for $i < 0$ and, consequently, $MC^2(\mathfrak{h})$ is an ordinary groupoid a homotopy equivalence between $\Sigma(\mathfrak{h})$ and the nerve of $MC^2(\mathfrak{h})$ was constructed by V. Hinich in [Hinich, 1997].

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Differential graded Lie algebras satisfying (1) arise in formal deformation theory of algebraic structures such as Lie algebras, commutative algebras, associative algebras to name a few. In what follows we shall concentrate on the latter example. Let \( k \) denote an algebraically closed field of characteristic zero. For an associative algebra \( A \) over \( k \) the shifted Hochschild cochain complex \( C^•(A)[1] \) has a canonical structure of a DGLA under the Gerstenhaber bracket; we denote this DGLA by \( g(A) \) for short. Suppose that \( m \) is a nilpotent commutative \( k \)-algebra (without unit). Then, \( g(A) \otimes_k m \) is a nilpotent DGLA which satisfies (1). Thus, the Deligne 2-groupoid \( \mathcal{MC}^2(g(A) \otimes_k m) \) is defined. For an Artin \( k \)-algebra \( R \) with maximal ideal \( m_R \) the 2-groupoid \( \mathcal{MC}^2(g(A) \otimes_k m_R) \) is naturally equivalent to the 2-groupoid of \( R \)-deformations of the algebra \( A \). In this sense the DGLA \( g(A) \) controls the formal deformation theory of \( A \).

The reason for considering the space \( \Sigma(h) \) is that it is defined not just for a DGLA (V. Hinich, [Hinich, 1997]), but, more generally, for any nilpotent \( L_\infty \) algebra (E. Getzler, [Getzler, 2009]). Homotopy invariance properties of the functor \( \Sigma \) (Proposition 3.9), the theory of J.W. Duskin ([Duskin, 2001/02]) and the theorem above yield the following result. If \( h \) is a DGLA satisfying (1), \( g \) is a \( L_\infty \) algebra \( L_\infty \)-quasi-isomorphic to \( h \) and \( m \) is a nilpotent commutative \( k \)-algebra, then \( \mathcal{N}MC^2(h \otimes_k m) \) is homotopy equivalent to \( \Sigma(g \otimes_k m) \). Thus, the 2-groupoid \( \mathcal{MC}^2(h \otimes_k m) \) can be reconstructed, up to equivalence, from the space \( \Sigma(g \otimes_k m) \). The situation envisaged above arises naturally. Any DGLA \( h \) is \( L_\infty \)-quasi-isomorphic to an \( L_\infty \) algebra with trivial univalent operation (the differential).

The paper is organized as follows. In Section 2 we review various constructions of nerves of 2-groupoids and their properties. In section 3 we recall the definitions of the functor \( \Sigma \) (3.4) and of the Deligne 2-groupoid (3.10) and prove basic properties thereof. The proof of the main theorem (Theorem 4.2) given in Section 4 proceeds by exhibiting canonical weak homotopy equivalences from \( \Sigma(h) \) and \( \mathcal{N}MC^2(h) \) to a third naturally defined simplicial set.

2. The homotopy type of a strict 2-groupoid

2.1. Nerves of simplicial groupoids.

2.1.1. Simplicial groupoids. In what follows a simplicial category is a category enriched over the category of simplicial sets. A small simplicial category consists of a set of objects and a simplicial set of morphisms for each pair of objects.

A simplicial category \( \mathcal{G} \) is a particular case of a simplicial object \( [p] \mapsto \mathcal{G}_p \) in \( \text{Cat} \) whose simplicial set of objects \( [p] \mapsto N_0\mathcal{G}_p \) is constant.

A simplicial category is a simplicial groupoid if it is a groupoid in each (simplicial) degree.

2.1.2. The naïve nerve. Suppose that \( \mathcal{G} \) is a simplicial category. Applying the nerve functor degree-wise we obtain the bi-simplicial set \( N\mathcal{G}: ([p], [q]) \mapsto N_q\mathcal{G}_p \) whose diagonal we denote by \( N\mathcal{G} \) and refer to as the naïve nerve of \( \mathcal{G} \).
2.1.3. The simplicial nerve. For a simplicial category \( G \) the simplicial nerve, also known as the homotopy coherent nerve, \( \mathfrak{N}G \) is represented by the cosimplicial object in \( [p] \mapsto \Delta^p \in \text{Cat}_\Delta \), i.e.

\[
\mathfrak{N}pG = \text{Hom}_{\text{Cat}_\Delta}(\Delta^p, G).
\]

Here, \( \Delta^p \) is the canonical free simplicial resolution of \( [p] \) which admits the following explicit description ([Cordier, 1982]).

The set of objects of \( \Delta^p \) is \( \{0, 1, \ldots, p\} \). For \( 0 \leq i \leq j \leq p \) the simplicial set of morphisms is given by \( \text{Hom}_{\Delta^p}(i, j) = N\mathcal{P}(i, j) \). The category \( \mathcal{P}(i, j) \) is a sub-poset of \( 2^{\{0,\ldots,p\}} \) (with the induced partial ordering whereby viewed as a category) given by

\[
\mathcal{P}(i, j) = \{ I \subset \mathbb{Z} \mid (i, j) \in I \} \cup \{ k \in I \implies i \leq k \leq j \}.
\]

The composition in \( \Delta^p \) is induced by functors

\[
\mathcal{P}(i, j) \times \mathcal{P}(j, k) \to \mathcal{P}(i, k): (I, J) \mapsto I \cup J.
\]

In particular, \( \Delta^0 = [0] \) and \( \Delta^1 = [1] \)

We refer the reader to [Hinich, 2007] for applications to deformation theory and to [Lurie, 2009] for the connection with higher category theory. The simplicial nerve of a simplicial groupoid is a Kan complex which reduces to the usual nerve for ordinary groupoids.

Since \( \Delta^0 = [0] \) (respectively, \( \Delta^1 = [1] \)) it follows that \( \mathfrak{N}_0G \) (respectively, \( \mathfrak{N}_1G \)) is the set of objects (respectively, the set of morphisms) of \( G_0 \).

2.1.4. Comparison of nerves. We refer the reader to [Hinich, 2007] for the definition of the canonical map of simplicial sets \( \mathcal{N}G \to \mathfrak{N}G \). In what follows we will make use of the following result of loc. cit.

2.2. Theorem. ([Hinich, 2007], Corollary 2.6.3) For any simplicial groupoid \( G \) the canonical map \( \mathcal{N}G \to \mathfrak{N}G \) is a weak homotopy equivalence.

2.3. Strict 2-groupoids.

2.3.1. From strict 2-groupoids to simplicial groupoids. Suppose that \( G \) is a strict 2-groupoid, i.e. a groupoid enriched over the category of groupoids. Thus, for every \( g, g' \in G \), we have the groupoid \( \text{Hom}_G(g, g') \) and the composition is strictly associative.

The nerve functor \( [p] \mapsto N_p(\cdot) := \text{Hom}_{\text{Cat}_p}([p], \cdot) \) commutes with products. Let \( G_p \) denote the category with the same objects as \( G \) and with morphisms defined by \( \text{Hom}_{G_p}(g, g') = N_p \text{Hom}_G(g, g') \); the composition of morphisms is induced by the composition in \( G \). Note that the groupoid \( G_0 \) is obtained from \( G \) by forgetting the 2-morphisms.

The assignment \( [p] \mapsto G_p \) defines a simplicial object in groupoids with the constant simplicial set of objects, i.e. a simplicial groupoid which we denote by \( \tilde{G} \).
2.4. Lemma. The simplicial nerve $\mathfrak{N}\tilde{G}$ admits the following explicit description:

1. There is a canonical bijection between $\mathfrak{N}_0\tilde{G}$ and the set of objects of $G$.

2. For $n \geq 1$ there is a canonical bijection between $\mathfrak{N}_n\tilde{G}$ and the set of data of the form $((\mu_i)_{0 \leq i \leq n}, (g_{ij})_{0 \leq i \leq j \leq n}, (c_{ijk})_{0 \leq i < j < k \leq n})$, where $(\mu_i)$ is an $(n + 1)$-tuple of objects of $G$, $(g_{ij})$ is a collection of 1-morphisms $g_{ij}: \mu_j \to \mu_i$ and $(c_{ijk})$ is a collection of 2-morphisms $c_{ijk}: g_{ij}g_{jk} \to g_{ik}$ which satisfies

$$c_{ijl}c_{jkl} = c_{ikl}c_{ijk} \quad (2)$$

(in the set of 2-morphisms $g_{ij}g_{jk}g_{kl} \to g_{il}$).

For a morphism $f: [m] \to [n]$ in $\Delta$ the induced structure map $f^* : \mathfrak{N}_n\tilde{G} \to \mathfrak{N}_m\tilde{G}$ is given (under the above bijection) by $f^* ((\mu_i), (g_{ij}), (c_{ijk})) = ((\nu_i), (h_{ij}), (d_{ijk}))$, where $\nu_i = \mu_{f(i)}$, $h_{ij} = g_{f(i)f(j)}$, $d_{ijk} = c_{f(i)f(j)f(k)}$ (cf. [Duskin, 2001/02]).

Proof. An $n$-simplex of $\mathfrak{N}\tilde{G}$ is the following collection of data:

1. objects $\mu_0, \ldots, \mu_n$ of $G$;

2. morphisms of simplicial sets $N\mathcal{P}(i, j) \to N\text{Hom}_G(\mu_i, \mu_j)$ intertwining the maps induced on the nerves by composition functors $\mathcal{P}(i, j) \times \mathcal{P}(j, k) \to \mathcal{P}(i, k)$ and $\text{Hom}_G(\mu_i, \mu_j) \times \text{Hom}_G(\mu_j, \mu_k) \to \text{Hom}_G(\mu_i, \mu_k)$.

Since the nerve functor is fully faithful, the above data are equivalent to the following:

1. objects $\mu_0, \ldots, \mu_n$ of $G$;

2. for any $I \in N_0\mathcal{P}(i, j)$, a 1-morphism $g_I : \mu_j \to \mu_i$ in $G$;

3. for any morphism $J \to I$ in $\mathcal{P}(i, j)$, a 2-morphism $c_{IJ} : g_J \to g_I$, such that

$$c_{IJ}c_{JK} = c_{IK} \quad (3)$$

These data have to be compatible with the composition pairings $\mathcal{P}(i, j) \times \mathcal{P}(j, k) \to \mathcal{P}(i, k)$ and $\text{Hom}_G(\mu_i, \mu_j) \times \text{Hom}_G(\mu_j, \mu_k) \to \text{Hom}_G(\mu_i, \mu_k)$.

Let $g_{ij}: \mu_j \to \mu_i$ denote the morphism $g_{(i,j)}$. By compatibility with compositions, if $I = \{i, i_1, \ldots, i_k, j\}$ then $g_I = g_{ii_1} \cdots g_{ikj}$. Let $c_{ijk}$ denote the two-morphism $c_{\{i,j,k\};\{i,k\}} : g_{ik} \to g_{ij}g_{jk}$. Now, by virtue of (3) and of compatibility with compositions, $c_{ijk}$ satisfy the two-cocycle identity (3) and determine $c_{IJ}$ for any $I, J$. $\blacksquare$
In what follows, for a strict 2-groupoid \( \mathcal{G} \), we will denote by \( \mathcal{N}_2 \mathcal{G} \) (respectively \( \mathcal{N}_s \mathcal{G} \)) the naïve (respectively simplicial) nerve of the associated simplicial groupoid \( \mathcal{G} \).

3. Homotopy types associated with \( L_\infty \)-algebras

3.1. \( L_\infty \)-algebras. We follow the notation of [Getzler, 2009] and refer the reader to loc. cit. for details.

Recall that an \( L_\infty \)-algebra is a graded vector space \( g \) equipped with operations

\[
\bigwedge^k g \to g[2-k] : x_1 \wedge \ldots \wedge x_k \mapsto [x_1, \ldots, x_k]
\]

defined for \( k = 1, 2, \ldots \) which satisfy a sequence of Jacobi identities.

It follows from the Jacobi identities that the unary operation \( [.] : g \to g[1] \) is a differential, which we will denote by \( \delta \).

An \( L_\infty \)-algebra is abelian if all operations with valency two and higher (i.e. all operations except for \( \delta \)) vanish. In other words, an abelian \( L_\infty \)-algebra is a complex. An \( L_\infty \)-algebra structure with vanishing operations of valency three and higher reduces to a structure of a DGLA.

The lower central series of an \( L_\infty \)-algebra \( g \) is the canonical decreasing filtration \( F^i g \) with \( F^0 g = g \) for \( i \leq 1 \) and defined recursively for \( i \geq 1 \) by

\[
F^{i+1} g = \sum_{k=2}^{\infty} \sum_{i_1 + \ldots + i_k = i} [F^{i_1} g, \ldots, F^{i_k} g].
\]

An \( L_\infty \)-algebra is nilpotent if there exists an \( i \) such that \( F^i g = 0 \).

3.1.1. Maurer-Cartan elements. Suppose that \( g \) is a nilpotent \( L_\infty \)-algebra. For \( \mu \in g^1 \) let

\[
F(\mu) = \delta \mu + \sum_{k=2}^{\infty} \frac{1}{k!} [\mu^\wedge k].
\]

The element \( F(\mu) \) of \( g^2 \) is called the curvature of \( \mu \). For any \( \mu \in g^1 \) the curvature \( F(\mu) \) satisfies the Bianchi identity ([Getzler, 2009], Lemma 4.5)

\[
\delta F(\mu) + \sum_{k=1}^{\infty} \frac{1}{k!} [\mu^\wedge k, F(\mu)] = 0.
\]

An element \( \mu \in g^1 \) is called a Maurer-Cartan element (of \( g \)) if it satisfies the condition \( F(\mu) = 0 \). The set of Maurer-Cartan elements of \( g \) will be denoted \( \text{MC}(g) \):

\[
\text{MC}(g) := \{ \mu \in g^1 \mid F(\mu) = 0 \}.
\]

The set \( \text{MC}(g) \) is pointed by the distinguished element \( 0 \in g^1 \).

Suppose that \( a \) is an abelian \( L_\infty \)-algebra. Then,

\[
\text{MC}(a) = Z^1(a) := \ker(\delta : a^1 \to a^2),
\]

hence is equipped with a canonical structure of an abelian group.
3.1.2. **Central Extensions.** Suppose that $g$ is a $L_\infty$-algebra and $a$ is a subcomplex of $(g, \delta)$ such that $[a \wedge g^{\ast k}] = 0$ for all $k \geq 1$. In this case we will say that $a$ is central in $g$.

If $a$ is central in $g$, then there is a unique structure of an $L_\infty$-algebra on $g/a$ such that the projection $g \to g/a$ is a map of $L_\infty$-algebras. If $g$ is nilpotent, then so is $g/a$.

In what follows we assume that $g$ is a nilpotent $L_\infty$-algebra and $a$ is central in $g$.

3.2. **Lemma.**

1. The addition operation on $g^1$ restricts to a free action of the abelian group $MC(a)$ on the set $MC(g)$.

2. The map $MC(g) \to MC(g/a)$ is constant on the orbits of the action.

3. The induced map $MC(g)/MC(a) \to MC(g/a)$ is injective.

**Proof.** Suppose that $\alpha \in a^1$ and $\mu \in g^1$. Since $a$ is central in $g$, $[(\alpha + \mu)^{\ast k}] = [\mu^{\ast k}]$ for $k \geq 2$ and $\mathcal{F}(\alpha + \mu) = \delta \alpha + \mathcal{F}(\mu)$ (in the notation of (4)). Therefore, $MC(a) + MC(g) = MC(g)$. In other words, the addition operation in $g^1$ restricts to an action of the abelian group $MC(a)$ on the set $MC(g)$ which is obviously free. Since the map $MC(g) \to MC(g/a)$ is the restriction of the map $g \to g/a$ constant on the orbits of the action, i.e. factors through $MC(g)/MC(a)$, and the induced map $MC(g)/MC(a) \to MC(g/a)$ is injective. □

3.2.1. **The Obstruction Map.** The image of the map $MC(g) \to MC(g/a)$ may be described in terms of the obstruction map (6) which we construct presently.

If $\mu \in g^1$ and $\mu + a^1 \in MC(g/a)$, then $\mathcal{F}(\mu + a^1) = \mathcal{F}(\mu) + \delta a^1 \subset a^2$ and the Bianchi identity (5) reduces to $\delta \mathcal{F}(\mu + a^1) = 0$, i.e. the assignment $\mu + a^1 \mapsto \mathcal{F}(\mu + a^1)$ gives rise to a well-defined map

$$o_2: MC(g/a) \to H^2(a) \quad (6)$$

(notation borrowed from [Goldman, Millson, 1988], 2.6).

3.3. **Lemma.** The sequence of pointed sets

$$0 \to MC(g)/MC(a) \to MC(g/a) \xrightarrow{o_2} H^2(a) \quad (7)$$

is exact.

**Proof.** If $\mathcal{F}(\mu + a^1) \subset \delta a^1$, then there exists $\alpha \in a^1$ such that $\mathcal{F}(\mu + \alpha) = 0$, i.e. $\mu + a^1$ is in the image of $MC(g) \to MC(g/a)$. □

3.4. **The Functor $\Sigma$.** In what follows we denote by $\Omega_n$, $n = 0, 1, 2, \ldots$ the commutative differential graded algebra over $\mathbb{Q}$ with generators $t_0, \ldots, t_n$ of degree zero and $dt_0, \ldots, dt_n$ of degree one subject to the relations $t_0 + \cdots + t_n = 1$ and $dt_0 + \cdots + dt_n = 0$. The differential $d: \Omega_n \to \Omega_n[1]$ is defined by $t_i \mapsto dt_i$ and $dt_i \mapsto 0$. The assignment $[n] \mapsto \Omega_n$ extends in a natural way to a simplicial commutative differential graded algebra.
3.4.1. The simplicial set $\Sigma(g)$. For a nilpotent $L_\infty$-algebra $g$ and a non-negative integer $n$ let

$$\Sigma_n(g) = MC(g \otimes \Omega_n).$$

Equipped with structure maps induced by those of $\Omega_\bullet$, the assignment $n \mapsto \Sigma_n(g)$ defines a simplicial set denoted $\Sigma(g)$.

The simplicial set $\Sigma(g)$ was introduced by V. Hinich in [Hinich, 1997] for DGLA and used by E. Getzler in [Getzler, 2009] (where it is denoted $MC_\bullet(g)$) for general nilpotent $L_\infty$-algebras.

3.4.2. Abelian DGLA. If $a$ is an abelian $L_\infty$-algebra, then $\Sigma(a)$ is given by $\Sigma_n(a) = Z^1(\Omega_n \otimes a) = Z^0(\Omega_n \otimes a[1])$ and has a canonical structure of a simplicial abelian group. In particular, it is a Kan simplicial set.

Recall that the Dold-Kan correspondence associates to a complex of abelian groups $A$ a simplicial abelian group $K(A)$ defined by

$$K_n(A) = Z^0(C_\bullet([n];A)),$$

the group of cocycles of (total) degree zero in the complex of simplicial cochains on the $n$-simplex with coefficients in $A$.

The integration map $\int : \Omega_n \otimes a \to C_\bullet([n];a)$ induces a homotopy equivalence

$$\int : \Sigma(a) \to K(a[1]);$$

see [Getzler, 2009], Section 3. Thus, $\pi_i \Sigma(a) \cong H^{1-i}(a)$.

3.4.3. Central extensions. Suppose that $g$ is a nilpotent $L_\infty$-algebra and $a$ is a central subalgebra in $g$. Then, for $n = 0, 1, \ldots$, $\Omega_n \otimes a$ is central in $\Omega_n \otimes g$.

3.5. Lemma.

1. The addition operation on $(\Omega_n \otimes g)^1$ induces a principal action of the simplicial abelian group $\Sigma(a)$ on the simplicial set $\Sigma(g)$.

2. The map $\Sigma(g) \to \Sigma(g/a)$ factors through $\Sigma(g)/\Sigma(a)$.

3. The induced map $\Sigma(g)/\Sigma(a) \to \Sigma(g/a)$ is injective.

Proof. Follows from Lemma 3.2 and the naturality properties of the constructions in 3.1.2.

For $n = 0, 1, \ldots$ the map $([n] \to [0])^* : \mathbb{Q} \to \Omega_n$ is a quasi-isomorphism, with the quasi-inverse provided by the map induced by any morphism $[0] \to [n]$. Therefore, the map $a \to \Omega_n \otimes a$ is a quasi-isomorphism as well. The induced isomorphisms $H^2(a) \cong H^2(\Omega_n \otimes a)$ give rise to the isomorphism of the constant simplicial set $H^2(a)$ and $n \mapsto H^2(\Omega_n \otimes a)$.

The maps

$$o_{2,n} : \Sigma_n(g/a) = MC(\Omega_n \otimes g/a) \to H^2(\Omega_n \otimes a) \cong H^2(a)$$
assemble into the map of simplicial sets

\[ o_2: \Sigma(g/a) \to H^2(a). \]  

which factors as \( \Sigma(g/a) \to \pi_0\Sigma(g/a) \to H^2(a). \)

Let \( \Sigma(g/a)_0 = o_2^{-1}(0) \). Thus, by (7), \( \Sigma(g/a)_0 \) is a union of connected components of \( \Sigma(g/a) \) equal to the range of the map \( \Sigma(g)/\Sigma(a) \to \Sigma(g/a) \).

It follows that the map \( \Sigma(g) \to \Sigma(g/a)_0 \) is a principal fibration with group \( \Sigma(a) \), in particular, a Kan fibration ([May, 1967], Lemma 18.2).

3.6. Lemma. Suppose that \( g \) is a nilpotent \( L_\infty \)-algebra. Then, \( \Sigma(g) \) is a Kan simplicial set.

Proof. If \( g \) is an abelian \( L_\infty \)-algebra then \( \Sigma(g) \) is a simplicial group and therefore a Kan simplicial set.

Let \( F^*g \) denote the lower central series. Assume that \( Gr^*_ig \neq 0 \) if and only if \( 0 \leq i \leq n \); that is, \( g \) is nilpotent of length \( n \). By induction assume that \( \Sigma(h) \) is a Kan simplicial set for any nilpotent \( L_\infty \)-algebra \( h \) of length at most \( n - 1 \).

Since \( g \) is nilpotent of length \( n \), it follows that \( F^ng = Gr^n g \) is central in \( g \) and \( g/F^ng \) is nilpotent of length \( n - 1 \). Therefore, \( \Sigma(g/F^ng) \) is a Kan simplicial set and so is \( \Sigma(g/F^ng)_0 \). Since \( \Sigma(g) \to \Sigma(g/F^ng)_0 \) is a Kan fibration it follows that \( \Sigma(g) \) is a Kan simplicial set as well.

3.7. Lemma. Suppose that \( g \) is a nilpotent \( L_\infty \)-algebra such that \( g^q = 0 \) for \( q \leq -k \), \( k \) a positive integer. Then, for any connected component \( X \) of \( \Sigma(g) \), \( \pi_i(X) = 0 \) for \( i > k \).

Proof. Suppose that \( g \) is an abelian \( L_\infty \)-algebra. Then, \( \pi_i\Sigma(g) \cong H^{1-i}(g) \). For an \( L_\infty \)-algebra \( g \) which is not necessarily abelian the statement follows by induction on the nilpotency length, the abelian case establishing the base of the induction.

Let \( F^*g \) denote the lower central series. Assume that \( Gr^*_ig \neq 0 \) if and only if \( 0 \leq i \leq n \); that is, \( g \) is nilpotent of length \( n \). By induction assume that the conclusion holds for all nilpotent \( L_\infty \)-algebras of length at most \( n - 1 \).

Since \( g \) is nilpotent of length \( n \), it follows that \( F^ng = Gr^n g \) is central in \( g \) and \( g/F^ng \) is nilpotent of length \( n - 1 \). Let \( X \subseteq \Sigma(g) \) be a connected component of \( \Sigma(g) \) and let \( Y \subseteq \Sigma(g/F^ng) \) be the image of \( X \) under the map induced by the quotient map \( g \to g/F^ng \). Then, \( X \to Y \) is a principal fibration with group the connected component of the identity in \( \Sigma(F^ng) \). The desired vanishing of higher homotopy groups of \( X \) follows from the induction hypotheses using the long exact sequence of homotopy groups.

3.7.1. Homotopy invariance.

3.8. Lemma. Suppose that \( f: a \to b \) is a quasi-isomorphism of abelian \( L_\infty \)-algebras. Then, the induced map \( \Sigma(f): \Sigma(a) \to \Sigma(b) \) is a weak homotopy equivalence.
Proof. Note that $\Sigma(f)$ is a morphism of simplicial abelian groups. It is sufficient to show that the maps $\pi_n\Sigma(f): \pi_n\Sigma(a) \to \pi_n\Sigma(b)$ are isomorphisms for $n \geq 0$. To this end note that $\pi_n\Sigma(f)$ factors as the composition of isomorphisms

$$\pi_n\Sigma(a) \cong H^{1-n}(a) \xrightarrow{H^{1-n}(\Sigma(f))} H^{1-n}(\Sigma(b)) \cong \pi_n\Sigma(b).$$

3.9. Proposition. ([Getzler, 2009], Proposition 4.9) Suppose that $f: g \to h$ is a quasi-isomorphism of $L_\infty$-algebras and $R$ is an Artin algebra with maximal ideal $m_R$. Then, the map $\Sigma(f \otimes \text{Id}): \Sigma(g \otimes m_R) \to \Sigma(h \otimes m_R)$ is a weak homotopy equivalence.

Proof. We use induction on the nilpotency length of $m_R$, which is to say the largest integer $l$ such that $m^l_R \neq 0$.

If $m^2_R = 0$, then $f \otimes \text{Id}: g \otimes m_R \to h \otimes m_R$ is a quasi-isomorphism of abelian $L_\infty$-algebras and the claim follows from Lemma 3.8.

Suppose that $m^{l+1}_R = 0$. By the induction hypothesis

- the map $\Sigma(g \otimes m_R/m^l_R) \to \Sigma(h \otimes m_R/m^l_R)$ is a weak homotopy equivalence and
- the map $\pi_0\Sigma(g \otimes m_R/m^l_R) \to \pi_0\Sigma(h \otimes m_R/m^l_R)$ is a bijection.

The map $f \otimes \text{Id}|_{m^l_R}$ is a quasi-isomorphism of abelian $L_\infty$-algebras, therefore the map $H^2(g \otimes m^l_R) \to H^2(h \otimes m^l_R)$ is an isomorphism. The commutativity of

$$\begin{array}{ccc}
\pi_0\Sigma(g \otimes m_R/m^l_R) & \longrightarrow & \pi_0\Sigma(h \otimes m_R/m^l_R) \\
\downarrow & & \downarrow \\
H^2(g \otimes m^l_R) & \longrightarrow & H^2(h \otimes m^l_R)
\end{array}$$

implies that the map

$$\pi_0\Sigma(g \otimes m_R/m^l_R)_0 \to \pi_0\Sigma(h \otimes m_R/m^l_R)_0$$

is a bijection. Therefore, the map

$$\Sigma(g \otimes m_R/m^l_R)_0 \to \Sigma(h \otimes m_R/m^l_R)_0$$

is a weak homotopy equivalence. The map $\Sigma(f)$ restricts to a map of principal fibrations

$$\begin{array}{ccc}
\Sigma(g \otimes m_R) & \longrightarrow & \Sigma(h \otimes m_R) \\
\downarrow & & \downarrow \\
\Sigma(g \otimes m_R/m^l_R)_0 & \longrightarrow & \Sigma(h \otimes m_R/m^l_R)_0
\end{array}$$

relative to the map of simplicial groups $\Sigma(g \otimes m^l_R) \to \Sigma(h \otimes m^l_R)$. The latter is a weak homotopy equivalence by Lemma 3.8. Therefore, so is the map $\Sigma(g \otimes m_R) \to \Sigma(h \otimes m_R)$. □
3.10. Deligne groupoids.

3.10.1. Gauge transformations. Suppose that $\mathfrak{h}$ is a nilpotent DGLA. Then, $\mathfrak{h}^0$ is a nilpotent Lie algebra. The unipotent group $\exp \mathfrak{h}^0$ acts on the space $\mathfrak{h}^1$ by affine transformations. The action of $\exp X$, $X \in \mathfrak{h}^0$, on $\gamma \in \mathfrak{h}^1$ is given by the formula

$$\exp X \cdot \gamma = \gamma - \sum_{i=0}^{\infty} \frac{(\text{ad} X)^i}{(i + 1)!} (\delta X + [\gamma, X]). \quad (10)$$

The effect of the above action on the curvature $\mathcal{F}(\gamma) = \delta \gamma + \frac{1}{2} [\gamma, \gamma]$ is given by

$$\mathcal{F}((\exp X) \cdot \gamma) = \exp(\text{ad} X)(\mathcal{F}(\gamma)). \quad (11)$$

3.10.2. The functor $\mathcal{MC}^1$. Suppose that $\mathfrak{h}$ is a nilpotent DGLA. It follows from (11) that gauge transformations (10) preserve the subset of Maurer-Cartan elements $\mathcal{MC}(\mathfrak{h}) \subset \mathfrak{h}^1$.

We denote by $\mathcal{MC}^1(\mathfrak{h})$ the Deligne groupoid (denoted $\mathcal{C}(\mathfrak{h})$ in [Hinich, 1997]) defined as the groupoid associated with the action of the group $\exp \mathfrak{h}^0$ by gauge transformations on the set $\mathcal{MC}(\mathfrak{h})$.

Thus, $\mathcal{MC}^1(\mathfrak{h})$ is the category with the set of objects $\mathcal{MC}(\mathfrak{h})$. For $\gamma_1, \gamma_2 \in \mathcal{MC}(\mathfrak{h})$, $\text{Hom}_{\mathcal{MC}^1(\mathfrak{h})}(\gamma_1, \gamma_2)$ is the set of gauge transformations between $\gamma_1, \gamma_2$. The composition

$$\text{Hom}_{\mathcal{MC}^1(\mathfrak{h})}(\gamma_2, \gamma_3) \times \text{Hom}_{\mathcal{MC}^1(\mathfrak{h})}(\gamma_1, \gamma_2) \to \text{Hom}_{\mathcal{MC}^1(\mathfrak{h})}(\gamma_1, \gamma_3)$$

is given by the product in the group $\exp(\mathfrak{h}^0)$.

3.10.3. The functor $\mathcal{MC}^2$. For $\mathfrak{h}$ as above satisfying the additional vanishing condition $\mathfrak{h}^i = 0$ for $i < -1$ we denote by $\mathcal{MC}^2(\mathfrak{h})$ the Deligne 2-groupoid as defined by P. Deligne [Deligne, 1994] and independently by E. Getzler, [Getzler, 2009]. Below we review the construction of Deligne 2-groupoid of a nilpotent DGLA following [Getzler, 2009, Getzler, 2002] and references therein.

The objects and the 1-morphisms of $\mathcal{MC}^2(\mathfrak{h})$ are those of $\mathcal{MC}(\mathfrak{h})$. That is, for $\gamma_1, \gamma_2 \in \mathcal{MC}(\mathfrak{h})$ the set $\text{Hom}_{\mathcal{MC}^2(\mathfrak{h})}(\gamma_1, \gamma_2)$ is the set of objects of the groupoid $\text{Hom}_{\mathcal{MC}^2(\mathfrak{h})}(\gamma_1, \gamma_2)$. The morphisms in $\text{Hom}_{\mathcal{MC}^2(\mathfrak{h})}(\gamma_1, \gamma_2)$ (i.e. the 2-morphisms of $\mathcal{MC}^2(\mathfrak{h})$) are defined as follows.

For $\gamma \in \mathcal{MC}(\mathfrak{h})$ let $[\cdot, \cdot]_\gamma$ denote the Lie bracket on $\mathfrak{h}^{-1}$ defined by

$$[a, b]_\gamma = [a, \delta b + [\gamma, b]]. \quad (12)$$

Equipped with this bracket, $\mathfrak{h}^{-1}$ becomes a nilpotent Lie algebra. We denote by $\exp_\gamma \mathfrak{h}^{-1}$ the corresponding unipotent group, and by

$$\exp_\gamma : \mathfrak{h}^{-1} \to \exp_\gamma \mathfrak{h}^{-1}$$
the corresponding exponential map. If $\gamma_1, \gamma_2$ are two Maurer-Cartan elements, then the group $\exp_{\gamma_2} h^{-1}$ acts on $\text{Hom}_{\text{MC}^1(h)}(\gamma_1, \gamma_2)$. For $\exp_{\gamma_2} t \in \exp_{\gamma_2} h^{-1}$ and $\text{Hom}_{\text{MC}^1(h)}(\gamma_1, \gamma_2)$ the action is given by

$$(\exp_{\gamma_2} t) \cdot (\exp X) = \exp(\delta t + [\gamma_2, t]) \exp X \in \exp h^0.$$ 

By definition, $\text{Hom}_{\text{MC}^2(h)}(\gamma_1, \gamma_2)$ is the groupoid associated with the above action.

The horizontal composition in $\text{MC}^2(h)$, i.e. the map of groupoids

$$\otimes : \text{Hom}_{\text{MC}^2(h)}(\exp X_{23}, \exp Y_{23}) \times \text{Hom}_{\text{MC}^2(h)}(\exp X_{12}, \exp Y_{12}) \rightarrow \text{Hom}_{\text{MC}^2(h)}(\exp X_{23} \exp X_{12}, \exp X_{23} \exp Y_{12}),$$

where $\gamma_i \in \text{MC}(h)$, $\exp X_{ij}, \exp Y_{ij}$, $1 \leq i, j \leq 3$ is defined by

$$\exp_{\gamma_3} t_{23} \otimes \exp_{\gamma_2} t_{12} = \exp_{\gamma_3} t_{23} \exp_{\gamma_3}(\exp(\text{ad} X_{23})(t_{12})), $$

where $\exp_{\gamma_j} t_{ij} \in \text{Hom}_{\text{MC}^2(h)}(\exp X_{ij}, \exp Y_{ij})$.

3.11. REMARK. There is a canonical map of 2-groupoids $\text{MC}^1(h) \rightarrow \text{MC}^2(h)$ which induces a bijection $\pi_0(\text{MC}^1(h)) \rightarrow \pi_0(\text{MC}^2(h))$ on sets of isomorphism classes of objects.

3.12. PROPERTIES OF $\mathcal{N}\text{MC}^2$.

3.12.1. ABELIAN DGLA.

3.13. LEMMA. Suppose that $\mathfrak{a}$ is an abelian DGLA satisfying $\mathfrak{a}^i = 0$ for $i < -1$. Then, the simplicial sets $\mathcal{N}\text{MC}^2(\mathfrak{a})$ and $K(\mathfrak{a}[1])$ are isomorphic naturally in $\mathfrak{a}$.

PROOF. The claim is an immediate consequence of the definitions and the explicit description of the nerve of $\text{MC}^2(\mathfrak{a})$ given in Lemma 2.4. ■

Combining Lemma 3.13 with the integration map (8) we obtain the map of simplicial abelian groups

$$\int : \Sigma(\mathfrak{a}) \rightarrow \mathcal{N}\text{MC}^2(\mathfrak{a})$$

(13)

which is a weak homotopy equivalence.

3.13.1. CENTRAL EXTENSIONS. Suppose that $\mathfrak{g}$ is a nilpotent DGLA satisfying $\mathfrak{g}^i = 0$ for $i < -1$ and $\mathfrak{a}$ is a central subalgebra in $\mathfrak{g}$. Note that $\text{MC}^2$ commutes with products, $\mathcal{N}$ commutes with products and the addition map $+: \mathfrak{a} \times \mathfrak{g} \rightarrow \mathfrak{g}$ is a morphism of DGLAs. Thus, we obtain an action of the simplicial abelian group $\mathcal{N}\text{MC}^2(\mathfrak{a})$ on the simplicial set $\mathcal{N}\text{MC}^2(\mathfrak{g})$

$$\mathcal{N}\text{MC}^2(+): \mathcal{N}\text{MC}^2(\mathfrak{a}) \times \mathcal{N}\text{MC}^2(\mathfrak{g}) \rightarrow \mathcal{N}\text{MC}^2(\mathfrak{g}).$$

Note that the group structure on $\mathcal{N}\text{MC}^2(\mathfrak{a})$ is obtained from the case $\mathfrak{a} = \mathfrak{g}$. Clearly, the action is free and the map $\mathcal{N}\text{MC}^2(\mathfrak{g}) \rightarrow \mathcal{N}\text{MC}^2(\mathfrak{g}/\mathfrak{a})$ factors through $\mathcal{N}\text{MC}^2(\mathfrak{g})/\mathcal{N}\text{MC}^2(\mathfrak{a})$.

3.13.2. THE OBSTRUCTION MAP.
3.14. Lemma. The obstruction map (6) factors as

\[ \text{MC}(\mathfrak{g}/\mathfrak{a}) \to \pi_0 \text{MC}^2(\mathfrak{g}/\mathfrak{a}) \to H^2(\mathfrak{a}) \]

Proof. Suppose \( \mu + a^1 \in \text{MC}(\mathfrak{g}/\mathfrak{a}) \). It follows from the formula (10) that

\[ \exp(X + a^0) \cdot (\mu + a^1) = (\exp X) \cdot \mu + a^1. \]

The formula (11) implies that

\[ F(\exp(X + a^0) \cdot (\mu + a^1)) = (\exp a^0 X) + \delta a^1 = \exp(ad X)(F(\mu) + \delta a^1). \]

Since \( F(\mu) + \delta a^1 \subset a^2 \), it follows that \( \exp(ad X)(F(\mu) + \delta a^1) = F(\mu) + \delta a^1 \) or, equivalently,

\[ o_2(\exp(X + a^0) \cdot (\mu + a^1)) = o_2(\mu + a^1). \]

Recall (Lemma 2.4) that an \( n \)-simplex of \( \mathfrak{H} \text{MC}^2(\mathfrak{g}/\mathfrak{a}) \), i.e. an element of \( \mathfrak{H}n \text{MC}^2(\mathfrak{g}/\mathfrak{a}) \) includes, among other things, a collection of \( n + 1 \) gauge-equivalent Maurer-Cartan elements of \( \mathfrak{g}/\mathfrak{a} \). By Lemma 3.14 all of these Maurer-Cartan elements give rise to the same element of \( H^2(\mathfrak{a}) \) under the map (6). Therefore, the assignment of this common value to an element of \( \mathfrak{H}n \text{MC}^2(\mathfrak{g}/\mathfrak{a}) \) give rise to a well-defined map

\[ o_{2,n} : \mathfrak{H}n \text{MC}^2(\mathfrak{g}/\mathfrak{a}) \to H^2(\mathfrak{a}) \]

for each \( n = 0, 1, 2, \ldots \) such that the sequence of pointed sets

\[ 0 \to \mathfrak{H}n \text{MC}^2(\mathfrak{g})/\mathfrak{H}n \text{MC}^2(\mathfrak{a}) \to \mathfrak{H}n \text{MC}^2(\mathfrak{g}/\mathfrak{a}) \xrightarrow{o_{2,n}} H^2(\mathfrak{a}) \]

is exact. The maps (14) assemble into a map of simplicial sets

\[ o_2 : \mathfrak{H} \text{MC}^2(\mathfrak{g}/\mathfrak{a}) \xrightarrow{o_2} H^2(\mathfrak{a}), \]

where \( H^2(\mathfrak{a}) \) is constant. Let \( \mathfrak{H} \text{MC}^2(\mathfrak{g}/\mathfrak{a})_0 = o_2^{-1}(0) \). The simplicial subset \( \mathfrak{H} \text{MC}^2(\mathfrak{g}/\mathfrak{a})_0 \) is a union of connected components of \( \mathfrak{H} \text{MC}^2(\mathfrak{g}/\mathfrak{a}) \) equal to the range of the map \( \mathfrak{H} \text{MC}^2(\mathfrak{g})/\mathfrak{H} \text{MC}^2(\mathfrak{a}) \to \mathfrak{H} \text{MC}^2(\mathfrak{g}/\mathfrak{a}) \).

It follows that \( \mathfrak{H} \text{MC}^2(\mathfrak{g}) \to \mathfrak{H} \text{MC}^2(\mathfrak{g}/\mathfrak{a})_0 \) is a principal fibration with the group \( \mathfrak{H} \text{MC}^2(\mathfrak{a}) \).

4. \( \mathfrak{H} \text{MC}^2 \) vs. \( \Sigma \)

In this section we show that for a DGLA \( \mathfrak{h} \) satisfying \( \mathfrak{h}^i = 0 \) for \( i < -1 \) the simplicial sets \( \mathfrak{H} \text{MC}^2(\mathfrak{h}) \) and \( \Sigma(\mathfrak{h}) \) are isomorphic in the homotopy category of simplicial sets.
4.1. The main theorem. Let $\Sigma^2_2(h) = MC^2(\Omega_n \otimes h)$, where the latter is the simplicial groupoid associated with the strict 2-groupoid $MC^2(\Omega_n \otimes h)$ (see 2.3.1). Let $\Sigma^2_2(h) \colon [n] \mapsto \Sigma^2_2(h)$ denote the corresponding simplicial object in simplicial groupoids. Note that $\Sigma(h)$ is the simplicial set of objects of $\Sigma^2_2(h)$, hence there is a canonical map

$$\Sigma(h) \to \mathcal{N}\Sigma^2_2(h). \quad (15)$$

The map $Q \to \Omega_\bullet$ of simplicial DGA induces the map of simplicial objects in simplicial groupoids

$$MC^2(h) \to \Sigma^2_2(h). \quad (16)$$

Consider the diagram

$$\Sigma(h) \quad \xrightarrow{(15)} \quad \mathcal{N}\Sigma^2_2(h) \xrightarrow{\mathcal{N}(16)} \mathcal{N}MC^2(h). \quad (17)$$

4.2. Theorem. Suppose that $h$ is a nilpotent DGLA satisfying $h^i = 0$ for $i < -1$. Then, the morphisms (15) and $\mathcal{N}(16)$ are weak homotopy equivalences so that the diagram (17) represents an isomorphism $\Sigma(h) \cong \mathcal{N}MC^2(h)$ in the homotopy category of simplicial sets.

The rest of Section 4 is devoted to a proof of Theorem 4.2 which borrows techniques from the proof of Proposition 3.2.1 of [Hinich, 2004].

4.3. The map (15) is a weak homotopy equivalence. Let $\Sigma^1_2(h)$ denote the simplicial object in groupoids defined by $\Sigma^1_2(nh) = MC^1(\Omega_n \otimes h)$. Note that $\Sigma(h)$ is the simplicial set of objects of $\Sigma^1_2(h)$ and hence there is a canonical map

$$\Sigma(h) \to \mathcal{N}\Sigma^1_2(h); \quad (18)$$

by Remark 3.11 there is a canonical map of simplicial objects in simplicial groupoids

$$\Sigma^1_2(h) \to \Sigma^2_2(h). \quad (19)$$

The map (15) is equal to the composition

$$\Sigma(h) \xrightarrow{(18)} \mathcal{N}\Sigma^1_2(h) \xrightarrow{\mathcal{N}(19)} \mathcal{N}\Sigma^2_2(h) \to \mathcal{N}\Sigma^2_2(h),$$

where the last map is the weak homotopy equivalence of Theorem 2.2.

4.4. Lemma. ([Hinich, 2004], Proposition 3.2.1) The map (18) is a weak homotopy equivalence.

Proof. Let $G_n(h) := \exp((\Omega_n \otimes h)^0)$. Then, $G(h) \colon [n] \mapsto G_n(h)$ is a simplicial group acting on $\Sigma(h)$, and $\Sigma(h)$ is the associated groupoid. Therefore,

$$N_q\Sigma(h) = \Sigma(h) \times G(h)^{\times q}$$

and the map

$$\Sigma(h) \to N_q\Sigma(h)$$

is a weak homotopy equivalence because $G(h)$ is contractible. ■
4.5. PROPOSITION. The map \( N((19)) \) is a weak homotopy equivalence.

PROOF. Let \( \Gamma^1(h) \) (respectively, \( \Gamma^2(h) \)) denote the full subcategory of \( \Sigma^1(h) \) (respectively, of \( \Sigma^2(h) \)) whose set of objects is \( MC(h) \) (a constant simplicial set). There is a commutative diagram

\[
\begin{array}{ccc}
\Gamma^1(h) & \longrightarrow & \Gamma^2(h) \\
\downarrow & & \downarrow \\
\Sigma^1(h) & \longrightarrow & \Sigma^2(h)
\end{array}
\]

The vertical arrows induce weak homotopy equivalences on respective nerves since, for each \( n \), the functors \( \Gamma^1(h)_n \to \Sigma^1(h)_n = MC^1(\Omega_n \otimes h) \) and \( \Gamma^2(h)_n \to \Sigma^2(h)_n = MC^2(\Omega_n \otimes h) \) are equivalences by [Hinich, 2001], Proposition 8.2.5.

The map \( \Gamma^1(h) \to \Gamma^2(h) \) induces a bijection between sets of isomorphism classes of objects. For \( \mu \in MC(h) \), \( \text{Hom}_{\Gamma^2(h)}(\mu, \mu) \) is naturally identified with the nerve of the groupoid associated to the action of the simplicial group \( H(h, \mu): [n] \to \exp((\Omega_n \otimes h)_\mu) \) on the simplicial set \( \text{Hom}_{\Gamma^1(h)}(\mu, \mu) \). Since the group \( H(h, \mu) \) is contractible (it is isomorphic as a simplicial set to \( [n] \to \Omega^0_n \otimes h^{-1} \)) the induced map \( \text{Hom}_{\Gamma^1(h)}(\mu, \mu) \to \text{Hom}_{\Gamma^2(h)}(\mu, \mu) \) is an equivalence.

4.6. THE MAP \( N((16)) : \mathfrak{M} MC^2(h) \to \mathfrak{M} \Sigma^2(h) \) IS A WEAK HOMOTOPY EQUIVALENCE. It suffices to show that the map

\[ \mathfrak{M} MC^2(h) \to \mathfrak{M} MC^2(\Omega_n \otimes h) \]

is a weak homotopy equivalence for all \( n \). This follows from Proposition 4.7.

4.7. PROPOSITION. Suppose that \( h \) is a nilpotent DGLA concentrated in degrees greater than or equal to \( -1 \). The functor

\[
MC^2(h) \to MC^2(\Omega_n \otimes h)
\]

is an equivalence.

PROOF. The induced map \( \pi_0((20)) \) is a bijection by Remark 3.11 and (the proof of) [Hinich, 1997], Lemma 2.2.1. The result now follows from Lemma 4.8 below.

4.8. LEMMA. Suppose \( \mu \in MC(h) \). The functor

\[
\text{Hom}_{MC^2(h)}(\mu, \mu) \to \text{Hom}_{MC^2(\Omega_n \otimes h)}(\mu, \mu)
\]

is an equivalence.
Proof. According to the description given in 3.10.3, for any nilpotent DGLA \((\mathfrak{g}, \delta)\) with \(\mathfrak{g}^i = 0\) for \(i < -1\) and \(\mu \in \text{MC}(\mathfrak{g})\) the groupoid \(\text{Hom}_{\text{MC}(\mathfrak{g})}(\mu, \mu)\) is isomorphic to the groupoid associated with the action of the group \(\exp\mu \mathfrak{g}^{-1}\) on the set \(\exp(\ker(\delta^{-1})) \subset \exp(\mathfrak{g}^0)\) where \(\delta_{\mu} = \delta + [\mu, \cdot]\).

Note that, for any \(X \in \ker(\delta_{\mu}^{-1})\), the automorphism group \(\text{Aut}(\exp(X))\) is isomorphic to (the additive group) \(\ker(\delta_{\mu}^{-1})\).

The map
\[
([n] \to [0])^* \otimes \text{Id}: (\mathfrak{h}, \delta) \to (\Omega_n \otimes \mathfrak{h}, d + \delta)
\]  
(22)
is a quasi-isomorphism of DGLA with the quasi-inverse given by the evaluation map \(\text{ev}_0 := ([0] \to [n])^* \otimes \text{Id}: \Omega_n \otimes \mathfrak{h} \to \mathfrak{h}\) (for any choice of a morphism \([0] \to [n]\)) which is a morphism of DGLA as well. The same maps are mutually quasi-inverse quasi-isomorphisms of DGLA \((\mathfrak{h}, \delta_{\mu}) \leftrightarrow (\Omega_n \otimes \mathfrak{h}, d + \delta_{\mu})\).

Since (22) is a quasi-isomorphism and both DGLA are concentrated in degrees greater than or equal to \(-1\), the induced map \(\ker(\delta_{\mu}^{-1}) \to \ker((d + \delta_{\mu})^{-1})\) is an isomorphism, hence so are the maps of automorphism groups.

Since the map (21) admits a left inverse (namely, \(\text{ev}_0\)) it remains to show that the induced map on sets of isomorphism classes is surjective. Note that, since \(\text{ev}_0\) is a surjective quasi-isomorphism, the map \(d + \delta_{\mu}: \ker(\text{ev}_0)^{-1} \to \ker(\text{ev}_0)^0 \cap \ker((d + \delta_{\mu})^0)\) is an isomorphism.

Consider \(X \in (\Omega_n \otimes \mathfrak{g})^0\). Then, \(X = \text{ev}_0(X) + Y\) with \(Y \in \ker(\text{ev}_0)\), and \((d + \delta_{\mu})X = 0\) if and only if \(\delta_{\mu} \text{ev}_0(X) = 0\) and \((d + \delta_{\mu})Y = 0\).

Suppose \(X \in \ker((d + \delta_{\mu})^0)\). Then, \(\exp(X) = \exp(\text{ev}_0(X)) \cdot \exp(Z)\) where \(Z \in \ker(\text{ev}_0)^0 \cap \ker((d + \delta_{\mu})^0)\), and, therefore, \(Z = (d + \delta_{\mu})U\) for a uniquely determined \(U\).

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