Weighted $L^2$ Holomorphic Functions on Ball-Fiber Bundles Over Compact Kähler Manifolds

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Abstract
Let $\tilde{M}$ be a complex manifold, $\Gamma$ be a torsion-free cocompact lattice of $\text{Aut}(\tilde{M})$ and $\rho: \Gamma \to SU(N, 1)$ be a representation. Suppose that there exists a $\rho$-equivariant totally geodesic isometric holomorphic embedding $\iota: \tilde{M} \to B^N$. Let $M := \tilde{M} / \Gamma$ and $\Sigma := B^N / \rho(\Gamma)$. In this paper, we investigate a relation between weighted $L^2$ holomorphic functions on the fiber bundle $\Omega := M \times_{\rho} B^N$ and the holomorphic sections of the pull-back bundle $\iota^*(S^n T^*_\Sigma)$ over $M$. In particular, $A^2_\alpha(\Omega)$ has infinite dimension for any $\alpha > -1$ and if $n < N$, then $A^2_{-1}(\Omega)$ also has the same property. As an application, if $\Gamma$ is a torsion-free cocompact lattice in $SU(n, 1)$, $n \geq 2$, and $\rho: \Gamma \to SU(N, 1)$ is a maximal representation, then for any $\alpha > -1$, $A^2_\alpha(B^n \times_{\rho} B^N)$ has infinite dimension. If $n < N$, then $A^2_{-1}(B^n \times_{\rho} B^N)$ also has the same property.

Keywords Compact submanifold in complex hyperbolic space forms · $L^2$ holomorphic functions · Holomorphic fiber bundles · $\bar{\partial}$-equations

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1 Introduction

For a complex manifold $X$, denote by $\text{Aut}(X)$ the set of holomorphic diffeomorphisms of $X$ onto itself and denote by $S^m T^*_X$ the $m$-th symmetric power of the holomorphic
and it satisfies the Milnor-Wood inequality.

Our primary result of this paper is

**Theorem 1.1** Let \( \widetilde{M} \) be a complex manifold, \( \Gamma \) be a torsion-free cocompact lattice of \( \text{Aut}(\widetilde{M}) \) and \( \rho: \Gamma \to SU(N, 1) \) be a representation. Suppose that there exists a \( \rho \)-equivariant totally geodesic isometric holomorphic embedding \( \iota: \widetilde{M} \to \mathbb{B}^N \). Let \( M := \widetilde{M} / \Gamma \) and \( \Sigma := \mathbb{B}^N / \rho(\Gamma) \). Let \( \Omega := M \times_{\rho} \mathbb{B}^N \) be a holomorphic \( \mathbb{B}^N \)-fiber bundle over \( M \) where any \( \gamma \in \Gamma \) acts on \( \mathbb{B}^N \) by \( (\xi, w) \mapsto (\gamma \xi, \rho(\gamma)w) \). Then there exists an injective linear map

\[
\Phi: \bigoplus_{m=0}^{\infty} H^0(M, i^*(S^m T^*_\Sigma)) \to \bigg\{ \bigwedge_{\alpha > -1} A_\alpha^2(\Omega) \subset \mathcal{O}(\Omega) \quad \text{if } n = N, \\
\bigwedge_{\alpha \geq -1} A_\alpha^2(\Omega) \subset \mathcal{O}(\Omega) \quad \text{if } n < N, \bigg\}
\]

which has a dense image in \( \mathcal{O}(\Omega) \) equipped with the compact open topology. In particular, \( \dim A_\alpha^2(\Omega) = \infty \) if \( \alpha > -1 \) and \( A_{-1}^2(\Omega) = \bigcap_{\alpha \geq -1} A_\alpha^2(\Omega) \) with \( \dim A_{-1}^2(\Omega) = \infty \) if \( n < N \).

We remark that under the condition of Theorem 1.1, \( \rho(\Gamma) \) acts on \( \mathbb{B}^N \) properly discontinuously and hence \( \Sigma \) is a complex manifold. Let \( \Gamma \subset SU(n, 1) \) be a cocompact lattice and \( \rho: \Gamma \to SU(N, 1) \) be a homomorphism. Denote by \( \omega_n \) and \( \omega_N \) the Kähler forms of the Bergman metrics of \( \mathbb{B}^n \) and \( \mathbb{B}^N \), respectively. Let \( f: \mathbb{B}^n \to \mathbb{B}^N \) be any smooth \( \rho \)-equivariant map and \( [\rho^* \omega_N] := [f^* \omega_N] \subset H^2_{dR}(\mathbb{B}^N / \Gamma) \) be the de Rham class of \( f^* \omega_N \) which only depends on \( \rho \). The Toledo invariant \( \tau(\rho) \) of \( \rho \) is defined by

\[
\tau(\rho) := \frac{1}{n!} \int_{\mathbb{B}^n / \Gamma} \rho^* \omega_N \wedge \omega_{\mathbb{B}^n / \Gamma}^{n-1}
\]

and it satisfies the Milnor-Wood inequality

\[
|\tau(\rho)| \leq \text{Vol}(\mathbb{B}^n / \Gamma) \tag{1.1}
\]

under suitable normalizations of the metrics. One says that \( \rho \) is a maximal representation if the equality holds in (1.1). In [5], Corlette showed that if \( \rho \) is a maximal representation with \( n \geq 2 \), then there exists a totally geodesic holomorphic \( \rho \)-equivariant embedding of \( \mathbb{B}^n \) into \( \mathbb{B}^N \). By Theorem 1.1, we have

**Corollary 1.2** Let \( \Gamma \subset SU(n, 1), n \geq 2 \), be a cocompact lattice and \( \rho: \Gamma \to SU(N, 1) \) be a maximal representation. Let \( \Omega := M \times_{\rho} \mathbb{B}^N \) be a holomorphic \( \mathbb{B}^N \)-fiber bundle over a complex hyperbolic space form \( M := \mathbb{B}^n / \Gamma \). Then for each \( \alpha > -1 \), the dimension of \( A_\alpha^2(\Omega) \) is infinite and \( \bigcap_{\alpha > -1} A_\alpha^2(\Omega) \) is dense in \( \mathcal{O}(\Omega) \) equipped with the compact open topology. Moreover if \( n < N \), then the dimension of \( A_\alpha^2(\Omega) \) is infinite for each \( \alpha \geq -1 \) and \( A_{-1}^2(\Omega) = \bigcap_{\alpha \geq -1} A_\alpha^2(\Omega) \) is dense in \( \mathcal{O}(\Omega) \) equipped with the compact open topology.
Theorem 1.1 is motivated by the following question.

**Question 1.3** Does any unit ball fiber bundle over a compact Kähler manifold admit nonconstant weighted $L^p$ ($1 \leq p \leq \infty$) holomorphic functions?

Remark that any $B^N$-fiber bundle over a compact Kähler manifold admits $C^\infty$ plurisubharmonic exhaustion function if either $n = 1$ ([7]) or $n \geq 2$ and $\rho$ is reductive ([10]).

Whether there exist holomorphic functions is a fundamental property of complex manifolds. If the complex manifold can be embedded holomorphically into a complex Euclidean space, then there exist a lot of holomorphic functions. A far deeper theorem is proved by Siu-Yau [11] and Greene-Wu [8] that a complete simply connected non-positively curved Kähler manifold of faster than quadratic sectional curvature decay is biholomorphic to $\mathbb{C}^n$ by constructing $n$ holomorphic functions. In [3], Bland presented two sufficient conditions given for a complete Kähler manifold of nonpositive sectional curvature to admit nonconstant bounded holomorphic functions.

On the other hand, if the manifold is compact, then there exist no nonconstant holomorphic functions. In [8], Greene-Wu showed that any complete simply connected Kähler manifold with nonpositive sectional curvature does not admit non-zero $L^p$ ($1 \leq p < \infty$) holomorphic functions. Moreover, if the curvature tensor has certain stronger conditions, then it does not have any nonconstant bounded holomorphic function neither.

Let $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk and $\Gamma \subset \text{Aut}(\Delta)$ be a cocompact torsion-free lattice. Then Ohsawa observed that there exists a holomorphic function induced from the Poincaré series given by

$$\sum_{\gamma \in \Gamma} (\gamma(z) - \gamma(w))^k$$

with any $k \geq 2$ on the disk bundle $\Delta \times \Delta / \Gamma$ over the Riemann surface $\Delta / \Gamma$ where $\Gamma$ acts on $\Delta \times \Delta$ by $\gamma(z, w) = (\gamma z, \gamma w)$. In [2], Adachi gave a concrete description of $\mathcal{O}(\Delta \times \Delta / \Gamma)$ and proved that there exist weighted $L^2$ holomorphic functions on $\Delta \times \Delta / \Gamma$ out of holomorphic sections of $K^\ell$ with $\ell \in \mathbb{N}$ where $K$ denotes the canonical line bundle over $\Delta / \Gamma$. In [9], the authors generalized his method to the $B^p$-fiber bundle $\mathbb{B}^n \times \mathbb{B}^n / \Gamma$ over a complex hyperbolic space form $\mathbb{B}^n / \Gamma$.

Now let $\widetilde{M}$ be a complex manifold and $\Gamma$ be a torsion-free cocompact lattice of $\text{Aut}(\widetilde{M})$ and $\rho : \Gamma \rightarrow \text{Aut}(\mathbb{B}^N)$ be a homomorphism. Suppose that there exists a $\rho$-equivariant holomorphic embedding $i : \widetilde{M} \rightarrow \mathbb{B}^N$, i.e., for any $\gamma \in \Gamma$, $i(\gamma \xi) = \rho(\gamma) i(\xi)$. Then the $\mathbb{B}^N$-fiber bundle $M \times_{\rho} \mathbb{B}^N : = \widetilde{M} \times \mathbb{B}^N / \Gamma$ over $M : = \widetilde{M} / \Gamma$ where any $\gamma \in \Gamma$ acts on $\widetilde{M} \times \mathbb{B}^N$ by $(\xi, \eta) \mapsto (\gamma \xi, \rho(\gamma) \eta)$ has a holomorphic function which is induced from the following Poincaré series

$$\sum_{\gamma \in \Gamma} \left(\rho(\gamma) j(i(\xi)) - \rho(\gamma) j(i(\eta))\right)^k$$

with $(\xi, \eta) \in \widetilde{M} \times \mathbb{B}^N$ and $k \geq N + 1$ (cf. Corollary 4.10 in [9]). In this paper, we generalize the method given in [2, 9] under the conditions when $i : \widetilde{M} \rightarrow \mathbb{B}^N$ is a
totally geodesic isometric embedding. It is worth to emphasize that \( \mathbb{B}^N / \rho(\Gamma) \) does not need to be compact.

Recently Deng-Fornæss [6] constructed a hyperconvex disk fiber bundle over a Hopf manifold which does not admit any nonconstant holomorphic function. Here hyperconvex means that there exists a bounded plurisubharmonic exhaustion function. Since \( \mathbb{B}^N \times \mathbb{B}^N / \Gamma \) is hyperconvex ([1, 10]), \( M \times_\rho \mathbb{B}^N \) is also hyperconvex for any \( N \).

2 Raising Operators for Submanifolds

2.1 Raising Operators

Let \( X \) be a Kähler manifold of dimension \( N \), \( g \) be its Kähler metric and \( G \in C^\infty(X, \Lambda^{1,1} T_X^* \) be its Kähler form. Let \( S^m T_X^* \) be the \( m \)-th symmetric power of holomorphic cotangent bundle \( T_X^* \) of \( X \). Let \( Y \) be a Kähler manifold of dimension \( n \) and \( \iota : Y \to X \) be a holomorphic map. We will denote by \( \Lambda^{p,q} T_Y^* \) the vector bundle of complex-valued \((p,q)\)-forms over \( X \).

For any \( p = 0, 1 \) and \( \tau : C^\infty(Y, i^* (S^m T_X^*)) \to C^\infty(Y, i^* (S^m T_X^*) \otimes i^* (\Lambda^{p,0} T_X^*) \otimes \Lambda^{0,q} T_Y^*) \), define a map

\[
\mathcal{R}_\tau^m : C^\infty(Y, i^* (S^m T_X^*)) \to C^\infty(Y, i^* (S^m T_X^*) \otimes \Lambda^{0,q} T_Y^*)
\]

by

\[
\mathcal{R}_\tau^m(u) = \sum \tau_{p,q}(u) a^p \otimes \overline{b}^q
\]

where \( \tau(u) = \sum_{|p|=p, |q|=q} \tau_{p,q}(u) \otimes a^p \otimes \overline{b}^q \) for \( u \in C^\infty(Y, i^* (S^m T_X^*)) \). Here \( a = (a_1, \ldots, a_N) \) is a local frame of \( i^*(T_X^*) \) and \( b = (b_1, \ldots, b_n) \) is a local frame of \( T_Y^* \).

For example, for an orthonormal frame \((e_1, \ldots, e_N)\), the Kähler form \( G = \sum e_\ell \otimes \overline{e_\ell} \) defines a map, which will be also denoted by \( G \),

\[
G : C^\infty(Y, i^* (S^m T_X^*)) \to C^\infty(Y, i^* (S^m T_X^*) \otimes \Lambda^{1,0} T_Y^* \otimes \Lambda^{0,1} T_Y^*)
\]

by \( u \mapsto \sum_{\ell=1}^N u \otimes e_\ell \circ \iota \otimes i^* \overline{e_\ell} \). We remark that we use the notation \( e_\ell \circ \iota \) or \( e_{|\iota(\zeta)} \), \( \zeta \in Y \), for the pull-back section and \( i^* e_\ell \) for the pull-back form. The corresponding raising operator \( \mathcal{R}_G \) is defined by

\[
\mathcal{R}_G : C^\infty(Y, i^* (S^m T_X^*)) \to C^\infty(Y, i^* (S^{m+1} T_X^*) \otimes \Lambda^{0,1} T_Y^*),
\]

\[
u = \sum_j u_j (e \circ \iota)^j \mapsto \sum_{\ell=1}^N \sum_j u_j (e \circ \iota)^j (e_\ell \circ \iota) \otimes i^* \overline{e_\ell}.
\]

(2.1)

Since this definition does not depend on the choice of orthonormal frame, it is well defined. By a similar way, for the Chern connection of \( i^* (S^m T_X^*) \) and its Chern curvature form, we can define the corresponding raising operators (cf. [9]).
2.2 Hodge Type Identities Over \( M \)

Let \( M \) be a compact Kähler manifold of dimension \( n \). Let \( \tilde{M} \) be its covering and \( \Gamma \) be a subgroup in \( \text{Aut}(\tilde{M}) \) so that \( M \) is biholomorphic to \( \tilde{M} / \Gamma \). Let \( \rho : \Gamma \rightarrow \text{Aut}(\mathbb{B}^N) \) be a homomorphism and \( i : \tilde{M} \rightarrow \mathbb{B}^N \) be a \( \rho \)-equivariant holomorphic embedding, i.e., for any \( \gamma \in \Gamma \) and \( \zeta \in \tilde{M} \), \( \rho(\gamma)(i(\zeta)) = i(\gamma(\zeta)) \). We will assume that \( \Sigma := \mathbb{B}^N / \rho(\Gamma) \) is a complex manifold and the map \( i \) induces a holomorphic embedding from \( M \) to \( \Sigma \). For simplicity, we also denote this map by \( i \).

Let

\[
g_{\mathbb{B}^N}(z) = \sum_{j,k=1}^N \frac{(1 - |z|^2)\delta_{kj} + z_j\overline{z}_k}{(1 - |z|^2)^2} \, dz_k \otimes d\overline{z}_j \quad (2.2)
\]

be the normalized Bergman metric of \( \mathbb{B}^N \). We assume that the normal bundle \( N = N_i := i^*T\Sigma / TM \) is holomorphically isomorphic to the orthogonal complement of \( TM \) in \( i^*T\Sigma \) with respect to the induced metric \( g \) from \( g_{\mathbb{B}^N} \). We emphasize that \( \Sigma \) does not need to be compact. Let \( h := i^*g \) be the pull-back metric of \( g \) on \( M \).

For any measurable section \( \phi \) of \( i^*(S^m T^*_{\Sigma}) \otimes \Lambda^{p,q} TM^* \), we define an \( L^2 \)-norm by

\[
\|\phi\|^2 = \int_M \langle \phi, \phi \rangle dV_M \quad (2.3)
\]

where \( \langle , \rangle \) and \( dV_M \) are induced by \( g \) and \( h = i^*g \) on \( M \). In particular, if we express any measurable section \( \phi \) of \( i^*(S^m T^*_{\Sigma}) \) with respect to a local orthonormal frame \( \{e_1, \ldots, e_N\} \) of \( T^*_{\Sigma} \) by \( \phi = \sum_I f_I(\zeta)e_I|_{i(\zeta)} \), we have

\[
\langle \phi, \phi \rangle = \sum_{|I|=m} \frac{1}{m!} |f_I|^2.
\]

The formal adjoint of \( \overline{\partial} \) on \( L^2(M, i^*(S^m T^*_{\Sigma}) \otimes \Lambda^{p,q} TM^*) \) with respect to the induced metric (2.3) will be denoted by \( \overline{\partial}^*_{(p,q),M} \). For simplicity, we will write \( \overline{\partial}^*_M \) instead of \( \overline{\partial}^*_{(p,q),M} \), if no confusion is likely to arise.

For the Kähler metric \( g \) on \( \Sigma \), let \( g^m \) denote the Hermitian metric on \( S^m T^*_{\Sigma} \) induced from \( g \) and let \( \Box^k_{m,M} \) be the complex Laplace operator

\[
\Box^k_{m,M} : C^\infty(M, i^*(S^m T^*_{\Sigma}) \otimes \Lambda^{0,k} TM^*) \rightarrow C^\infty(M, i^*(S^m T^*_{\Sigma}) \otimes \Lambda^{0,k} TM^*)
\]

given by

\[
\Box^k_{m,M} = \overline{\partial}_M \overline{\partial}^*_M + \overline{\partial}^*_M \overline{\partial}_M
\]

with respect to the metric \( g^m \) where \( k = 0, 1 \). We will omit \( k, m \) in the notation if there is no ambiguity. Especially we simply write \( \Box_M \) instead of \( \Box^0_M \). Let \( G^1 \) be the Green operator of \( \Box^1_M \).
Since the short exact sequence \( 0 \rightarrow TM \rightarrow i^*T_{\Sigma} \rightarrow N \rightarrow 0 \) holomorphically splits, we have

\[
\begin{align*}
\text{holomorphically} \\
\text{splits, we have}
\end{align*}
\]

\[
(2.4)
\]

\[ i^*(S^m T^*_\Sigma) \cong \bigoplus_{\ell=0}^m S^\ell T^*_M \otimes S^{m-\ell} N^*, \]

and as a consequence

\[
C^\infty(M, i^*(S^m T^*_\Sigma)) \cong \bigoplus_{\ell=0}^m C^\infty(M, S^\ell T^*_M \otimes S^{m-\ell} N*),
\]

\[
H^0(M, i^*(S^m T^*_\Sigma)) \cong \bigoplus_{\ell=0}^m H^0(M, S^\ell T^*_M \otimes S^{m-\ell} N*). \tag{2.4}
\]

**Lemma 2.1** Let \( m, \ell \) be non-negative integers with \( \ell \leq m \). The raising operator \( R_G \) is a linear injective map and for any \( u \in C^\infty(M, S^\ell T^*_M \otimes S^{m-\ell} N*) \),

\[
\|R_G(u)\|^2 = \frac{\ell + n}{m + 1} \|u\|^2 \tag{2.5}
\]

and

\[
\{\Box_M, R_G\}(u) := \Box_M R_G u - R_G \Box_M u = (m + \ell) R_G(u). \tag{2.6}
\]

**Proof** Let \( \{e_1, \ldots, e_N\} \) be a local orthonormal frame of \( T^*_\Sigma \) on a small open set \( U \) of \( \Sigma \) so that \( g = \sum_{\tau=1}^N e_\tau \otimes \bar{e}_\tau \) and \( h = \sum_{\tau=1}^n i^* e_\tau \otimes i^* \bar{e}_\tau \). Let \( u = \sum_{|I|=m} u_I e_I \) be a smooth section of \( i^*(S^m T^*_\Sigma) \). Since \( i^* e_\tau = 0 \) for any \( n + 1 \leq \tau \leq N \), we obtain

\[
R_G u = \sum_{\tau=1}^n \sum_{|I|=m} u_I e_\tau \otimes i^* e_\tau.
\]

If \( u \in C^\infty(M, S^\ell T^*_M \otimes S^{m-\ell} N*) \), then one has

\[
\langle R_G u, R_G u \rangle = \sum_{\tau, m=1}^n \langle u e_\tau \otimes i^* e_\tau, u e_m \otimes i^* e_m \rangle = \sum_{\tau=1}^n \langle u e_\tau, u e_\tau \rangle
\]

\[
= \sum_{|I|=m} \sum_{\tau=1}^n \frac{i_1! \cdots (i_\tau + 1)! \cdots i_N!}{(m + 1)!} |u_I|^2
\]

\[
= \sum_{|I|=m} \frac{\left(\sum_{j=1}^n i_j + n\right)}{m + 1} \frac{I!}{m!} |u_I|^2
\]

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\[ \sum_{|I|=m} \frac{\ell+n}{m+1} \frac{I!}{m!} |u_I|^2 = \frac{\ell+n}{m+1} \|u\|^2. \]

which implies (2.5).

To prove (2.6), let \( q \in M \) and \((\zeta_1, \ldots, \zeta_n)\) be a holomorphic normal coordinate system on a small open set \( q \in U \subset M \) such that \( U \cong \iota(U) \subset \Sigma \). Let \( p := \iota(q) \in \iota(U) \) and take a holomorphic normal coordinate system \((z_1, \ldots, z_N)\) at \( p \) such that for \( t_k := z_k \circ \iota, k = 1, \ldots, N \) we have

\[ \left. \frac{\partial u_m}{\partial \zeta_{\nu}} \right|_{q} = \begin{cases} \delta_{m\nu} (m = 1, \ldots, n), \\ 0 (m = n + 1, \ldots, N). \end{cases} \tag{2.7} \]

Let \( \{h_\gamma\} \) be a holomorphic normal frame of \( \iota^* (S^m T^*_\Sigma) \). For any smooth section \( u = \sum_\gamma u_\gamma h_\gamma \) of \( \iota^* (S^m T^*_\Sigma) \), we have

\[
\bar{\partial}_M^* R_G(u) = \sum_\gamma u_\gamma (g_{\alpha\beta} \circ \iota) h_\gamma (dz_\alpha \circ \iota) \otimes \iota^* dz_\beta
\]

By (2.7), we obtain

\[
\bar{\partial}_M^* R_G(u) = - \sum_{k=1}^n \sum_\gamma \frac{\partial u_\gamma}{\partial \zeta_k} h_\gamma (dz_k \circ \iota)
\]

at the point \( q \). Let \( \tilde{D}_m \) be the \((1, 0)\) part of the Chern connection of \( \iota^* (S^m T^*_\Sigma) \) induced from \((S^m T^*_\Sigma, g^m)\). Then

\[
\tilde{D}_m(u) = \tilde{D}_m \left( \sum_\gamma u_\gamma h_\gamma \right) = \sum_\gamma \sum_{k=1}^n \frac{\partial u_\gamma}{\partial \zeta_k} h_\gamma \otimes d\zeta_k + \sum_{k=1}^n \sum_{\gamma, \mu} u_\gamma \theta^\mu_{\gamma k} h_\mu \otimes d\zeta_k
\]
where $\theta_{\gamma}^{\mu}$ is the connection one form of $\tilde{D}_m$. Then it follows that

$$
\mathcal{R}_{\tilde{D}_m} \left( \sum_{\gamma} u_{\gamma} h_{\gamma} \right) = \sum_{k=1}^{n} \sum_{\gamma} \frac{\partial u_{\gamma}}{\partial \xi_k} h_{\gamma} (dz_k \circ \iota)
$$

$$
+ \sum_{k=1}^{n} \sum_{\gamma, \mu} u_{\gamma} \theta_{\gamma k}^{\mu} h_{\mu} (dz_k \circ \iota).
$$

Hence

$$
\tilde{\partial}_{M}^{*} \mathcal{R}_{G} u = - \mathcal{R}_{\tilde{D}_m} u
$$

at $q$.

Let $\theta_{\gamma}^{\mu} = \sum_{k} \theta_{\gamma k}^{\mu} \, d\xi_k$. Since the Chern curvature form $\Theta(i^* (S^m T^*_\Sigma))$ of $i^* (S^m T^*_\Sigma)$ satisfies $\Theta(i^* (S^m T^*_\Sigma)) = \tilde{\partial} \theta$ for $\theta := (\theta_{\gamma}^{\mu})$, it follows that

$$
\frac{\partial \theta_{\gamma k}^{\mu}}{\partial \xi_{\lambda}} = - R_{\gamma k \lambda}^{\mu}
$$

where $R_{\gamma k \lambda}^{\mu}$ is given by

$$
\Theta \left( i^* \left( S^m T^*_\Sigma \right) \right) = \sum_{\alpha, \beta, s, t=1}^{n} R_{\alpha s t}^{\beta} h_{s}^{\alpha} \otimes h_{t}^{\beta} d\xi_{s} \wedge d\bar{\xi}_{t}.
$$

Therefore by using (2.7), it follows that

$$
\tilde{\partial}_{M}^{*} \tilde{\partial}_{M} \mathcal{R}_{G} u = - \tilde{\partial}_{M} \mathcal{R}_{\tilde{D}_m} u
$$

$$
= - \sum_{k, \lambda=1}^{n} \sum_{\gamma} \frac{\partial^2 u_{\gamma}}{\partial \xi_k \partial \xi_{\lambda}} h_{\gamma} (dz_k \circ \iota) \otimes d\bar{\xi}_{\lambda}
$$

$$
+ \sum_{k, m=1}^{n} \sum_{\gamma, \mu} u_{\gamma} R_{\gamma k m}^{\mu} h_{\mu} (dz_k \circ \iota) \otimes d\bar{\xi}_{m} + O(|\xi|).
$$

Therefore by using (2.7), it follows that

$$
\tilde{\partial}_{M}^{*} \tilde{\partial}_{M} \mathcal{R}_{G} u
$$

$$
= \sum_{\alpha, \beta=1}^{N} \sum_{\gamma} \tilde{\partial}_{M}^{*} \tilde{\partial}_{M} \left( u_{\gamma} (g_{a \beta} \circ \iota) h_{\gamma} (dz_{\alpha} \circ \iota) \otimes i^* dz_{\beta} \right)
$$

$$
= \sum_{\alpha, \beta=1}^{N} \sum_{\gamma} \sum_{\lambda=1}^{n} \tilde{\partial}_{M}^{*} \tilde{\partial}_{M} \left( u_{\gamma} (g_{a \beta} \circ \iota) h_{\gamma} (dz_{\alpha} \circ \iota) \otimes \frac{\partial g_{a \beta}}{\partial \xi_{\lambda}} d\bar{\xi}_{\lambda} \right)
$$

$$
= \tilde{\partial}_{M}^{*} \left( \sum_{\alpha, \beta=1}^{N} \sum_{\gamma} \sum_{\lambda=1}^{n} \frac{\partial (u_{\gamma} (g_{a \beta} \circ \iota) \frac{\partial g_{a \beta}}{\partial \xi_{\lambda}})}{\partial \xi_{\lambda}} h_{\gamma} (dz_{\alpha} \circ \iota) \otimes d\bar{\xi}_{k} \wedge d\bar{\xi}_{\lambda} \right)
$$

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\[
\begin{align*}
&= - \sum_{\alpha, \beta = 1}^{N} \sum_{\lambda, k = 1}^{n} \sum_{\gamma} \left( \frac{\partial^2(u_Y(\gamma_{A\beta} \circ \iota))}{\partial \xi_k \partial \bar{\xi}_\lambda} - \frac{\partial^2(u_Y(\gamma_{A\beta} \circ \iota))}{\partial \xi_\beta \partial \bar{\xi}_k} \right) \alpha, \beta \cdot h_Y(dz_\alpha \circ \iota) + O(|\xi|) \\
&= - \sum_{\alpha, \beta = 1}^{N} \sum_{\lambda, k = 1}^{n} \sum_{\gamma} \left( \frac{\partial(u_Y(\gamma_{A\beta} \circ \iota))}{\partial \xi_k} - \frac{\partial(u_Y(\gamma_{A\beta} \circ \iota))}{\partial \xi_\lambda} \right) d\bar{\xi}_\lambda - \frac{\partial(u_Y(\gamma_{A\beta} \circ \iota))}{\partial \xi_\lambda} \right) d\bar{\xi}_k \right) h_Y(dz_\alpha \circ \iota) + O(|\xi|) \\
&= - \sum_{\alpha, \beta = 1}^{N} \sum_{\lambda, k = 1}^{n} \sum_{\gamma} \left( \frac{\partial(u_Y(\gamma_{A\beta} \circ \iota))}{\partial \xi_k} - \frac{\partial(u_Y(\gamma_{A\beta} \circ \iota))}{\partial \xi_\beta} \right) d\bar{\xi}_\lambda - \frac{\partial(u_Y(\gamma_{A\beta} \circ \iota))}{\partial \xi_\beta} \right) d\bar{\xi}_k \right) h_Y(dz_\alpha \circ \iota) + O(|\xi|). \\
\end{align*}
\]

Note that the last equality of (2.9) follows by

\[
\begin{align*}
&- \sum_{\alpha, \beta = 1}^{N} \sum_{\lambda, k = 1}^{n} \sum_{\gamma} \left( \frac{\partial(u_Y(\gamma_{A\beta} \circ \iota))}{\partial \xi_k} - \frac{\partial(u_Y(\gamma_{A\beta} \circ \iota))}{\partial \xi_\lambda} \right) d\bar{\xi}_\lambda - \frac{\partial(u_Y(\gamma_{A\beta} \circ \iota))}{\partial \xi_\lambda} \right) d\bar{\xi}_k \right) h_Y(dz_\alpha \circ \iota) + O(|\xi|) \\
&= - \sum_{\alpha, \beta = 1}^{N} \sum_{\lambda, k = 1}^{n} \sum_{\gamma} \left( \frac{\partial(u_Y(\gamma_{A\beta} \circ \iota))}{\partial \xi_k} - \frac{\partial(u_Y(\gamma_{A\beta} \circ \iota))}{\partial \xi_\lambda} \right) d\bar{\xi}_\lambda - \frac{\partial(u_Y(\gamma_{A\beta} \circ \iota))}{\partial \xi_\lambda} \right) d\bar{\xi}_k \right) h_Y(dz_\alpha \circ \iota) + O(|\xi|) \\
&= 0.
\end{align*}
\]

Moreover,

\[
\begin{align*}
\frac{\partial^2(u_Y(\gamma_{A\beta} \circ \iota))}{\partial \xi_k \partial \bar{\xi}_\lambda} - \frac{\partial^2(u_Y(\gamma_{A\beta} \circ \iota))}{\partial \xi_\beta \partial \bar{\xi}_k} &= \left( \frac{\partial^2(\gamma_{A\beta} \circ \iota)}{\partial \xi_k \partial \bar{\xi}_\lambda} - \frac{\partial^2(\gamma_{A\beta} \circ \iota)}{\partial \xi_\beta \partial \bar{\xi}_k} \right) u_Y \\
&+ (\gamma_{A\beta} \circ \iota) \left( \frac{\partial^2 u_Y}{\partial \xi_k \partial \bar{\xi}_\lambda} - \frac{\partial^2 u_Y}{\partial \xi_\beta \partial \bar{\xi}_k} \right) + O(|\xi|).
\end{align*}
\]

Since \( g \) is Kähler, it follows that

\[
\begin{align*}
\tilde{\partial}_M^* \tilde{\partial}_M R_{GU} &= - \sum_{\alpha = 1}^{N} \sum_{\beta, k = 1}^{n} \sum_{\gamma} (\gamma_{A\beta} \circ \iota) h_Y(dz_\alpha \circ \iota) \\
&\otimes \left( \frac{\partial^2 u_Y}{\partial \xi_k \partial \bar{\xi}_\lambda} - \frac{\partial^2 u_Y}{\partial \xi_\beta \partial \bar{\xi}_k} \right) + O(|\xi|).
\end{align*}
\]
Since
\[
\mathcal{R}_G \tilde{\partial}_M \partial M u = - \sum_{\alpha, \beta = 1}^{N} \sum_{\mu = 1}^{n} \sum_{\gamma} \frac{\partial^2 u_{\gamma \mu}}{\partial \zeta_{\mu} \partial \zeta_{\mu}} h_{\gamma} (g_{\alpha \beta} \circ \iota) (\partial z_{\alpha} \circ \iota) \otimes \tilde{r}^* dz_{\beta} + O(\vert \zeta \vert),
\]
by adding (2.8), (2.10), (2.11), and using (2.7) we have
\[
\begin{align*}
\{ \square, \mathcal{R}_G \} u &= \tilde{\partial}_M \partial M R_G u + \tilde{\partial}_M \partial M \mathcal{R}_G u - \mathcal{R}_G \tilde{\partial}_M \partial M u \\
&= \sum_{k, j = 1}^{n} \sum_{\gamma, \mu} u_{\gamma \mu} R_{\gamma k j}^\mu d \zeta_k h_\mu \otimes d \bar{\zeta}_j.
\end{align*}
\]
Moreover at \( q \), we have
\[
\sum_{k, m = 1}^{n} \sum_{\gamma, \mu} u_{\gamma \mu} R_{\gamma k m}^\mu d \zeta_k h_\mu \otimes d \bar{\zeta}_m = \sum_{\vert I \vert = m, j = 1}^{N} \sum_{i_j \in I} e_{i_j} \cdots e_{i_2} \cdots e_{i_1} \otimes \mathcal{R}_{\Theta^{(i^* T^*_\Sigma)}}(e_j)
\]
where \( \mathcal{R}_{\Theta^{(i^* T^*_\Sigma)}} \) is defined by
\[
\mathcal{R}_{\Theta^{(i^* T^*_\Sigma)}}(e_j) = \sum_{a = 1}^{N} \sum_{k, m = 1}^{n} e_a \otimes \Theta^{(T^*_\Sigma)}_{jkm} e_k \wedge \bar{e}_m.
\]
From
\[
\Theta^{(T^*_\Sigma)}_{jkm} = -(\delta_{ak} \delta_{jm} + \delta_{aj} \delta_{km}),
\]
one has
\[
\Theta^{(T^*_\Sigma)}_{jkm} = -\Theta^{(T^*_\Sigma)}^* = \delta_{jk} \delta_{am} + \delta_{ja} \delta_{mk}
\]
and hence
\[
\mathcal{R}_{\Theta^{(i^* T^*_\Sigma)}}(e_j) = \sum_{a = 1}^{N} \sum_{k, m = 1}^{n} e_a \otimes \Theta^{(T^*_\Sigma)}_{jkm} e_k \wedge \bar{e}_m
\]
\[
= \sum_{a = 1}^{n} e_a e_j \wedge e_a + e_j \otimes \sum_{r = 1}^{n} e_r \wedge \bar{e}_r
\]
where
\[
\epsilon_{\mu} = \begin{cases} 
1 & \text{if } \mu \in \{1, \cdots, n\}, \\
0 & \text{otherwise.}
\end{cases}
\]
\( \square \) Springer
Therefore, we have
\[
\sum_{|I|=m}^{N} \sum_{j=1}^{i_{j}} u_{I} e_{1}^{i_{1}} \cdots e_{j-1}^{i_{j-1}} \cdots e_{N}^{i_{N}} \cdot R_{G}(\iota^{*}T^{*}_{\Sigma}) (e_{j})
\]
\[
= \sum_{|I|=m}^{N} \sum_{j=1}^{N} i_{j} u_{I} e_{1}^{i_{1}} \cdots e_{j-1}^{i_{j-1}} \cdots e_{N}^{i_{N}} \cdot \left( \sum_{a=1}^{n} e_{a} \otimes e_{j} \wedge \bar{e}_{a} + e_{j} \otimes \sum_{r=1}^{n} e_{r} \wedge \bar{e}_{r} \right)
\]
\[
= \sum_{|I|=m}^{n} \sum_{j=1}^{n} \left( m + \sum_{j=1}^{n} i_{j} \right) u_{I} e_{1}^{i_{1}} \cdots e_{n}^{i_{n}} \cdots e_{N}^{i_{N}} \cdot e_{r} \otimes \bar{e}_{r}
\]
\[
= \sum_{|I|=m}^{n} \left( m + \sum_{j=1}^{n} i_{j} \right) R_{G}(u_{I} e^{I}).
\]

and it implies (2.6) if \( u \in C^\infty(M, S^{\ell}T^{*}_{M} \otimes S^{m-\ell}N^{*}) \).

\[\square\]

Remark 2.2

(1) If \( \iota \) is totally geodesic, since up to the composition with an automorphism of \( \mathbb{B}^{N} \), we have
\[
\iota(\tilde{M}) = \mathbb{B}^{N} \cap \{ (z_{1}, \ldots, z_{N}, 0, \cdots, 0) \in \mathbb{C}^{N} : z_{j} \in \mathbb{C}, \forall j \}
\]
and the normalized Bergman metric of \( \mathbb{B}^{N} \) is given by (2.2), it follows that the normal bundle \( \iota^{*}T_{\mathbb{B}^{N}}/T_{\tilde{M}} \) is holomorphically isomorphic to the orthogonal complement of \( T_{\tilde{M}} \) in \( \iota^{*}T_{\mathbb{B}^{N}} \). This implies that under the condition given in Theorem 1.1, the normal bundle \( \iota^{*}T_{\Sigma}/T_{M} \) is holomorphically isomorphic to the orthogonal complement of \( T_{M} \) in \( \iota^{*}T_{\Sigma} \) with respect to the induced metric \( g \) from the Bergman metric of \( \mathbb{B}^{N} \).

(2) For a compact manifold \( M \) and a holomorphic embedding, not necessarily totally geodesic, \( \iota : M \to \Sigma := \mathbb{B}^{N}/\Gamma \), let \( \{ e_{1}, \cdots, e_{N} \} \) be a local orthonormal frame of \( T^{*}_{\Sigma} \) so that \( g = \sum_{\ell=1}^{N} e_{\ell} \otimes \bar{e}_{\ell} \) and \( h := \iota^{*}g = \sum_{\ell=1}^{n} i^{*}e_{\ell} \otimes i^{*}\bar{e}_{\ell} \). For any smooth section
\[
u = \sum_{i_{1}+\cdots+i_{N}=m} u_{i_{1} \cdots i_{N}} e_{1}^{i_{1}} \cdots e_{n}^{i_{n}} \cdots e_{N}^{i_{N}}
\]
of \( \iota^{*}(S^{m}T^{*}_{\Sigma}) \), we have
\[
\| R_{G}(u) \|^{2} = \sum_{|I|=m}^{i_{1} + \cdots + i_{n} + n} \frac{m+1}{m} \| u_{I} e^{I} \|^{2}
\]
and
\[
\{ \Box_{M}, R_{G} \}(u) := \Box_{M}R_{G}u - R_{G} \Box_{M}u = \sum_{|I|=m}^{(m+1)\cdots+i_{n}} (m+1) \cdots + i_{n}) R_{G}(u_{I} e^{I}).
\]
For each positive integer $m$, non-negative integer $\ell$ with $\ell \leq m$ and $k = 0, 1$, let $\Box_{m,M}^{k,\ell}$ denote the complex Laplace operator on

$$S^\ell T^*_M \otimes S^{m-\ell} N^* \otimes \Lambda^0, k T^*_M \subset i^* (S^m T^*_\Sigma) \otimes \Lambda^0, k T^*_M$$

over $M$.

**Corollary 2.3** Let $\ker^\perp (\Box_{m,M}^{0,\ell} - \lambda I)$ be the orthogonal complement of $\ker (\Box_{m,M}^{0,\ell} - \lambda I)$ in $L^2(M, S^\ell T^*_M \otimes S^{m-\ell} N^*)$. Then one has

1. $\mathcal{R}_G (\ker (\Box_{m,M}^{0,\ell} - \lambda I)) \subset \ker (\Box_{m+1,M}^{1,\ell} - (\lambda + m + \ell) I)$,
2. $\mathcal{R}_G (\ker^\perp (\Box_{m,M}^{0,\ell} - \lambda I)) \subset \ker^\perp (\Box_{m+1,M}^{1,\ell} - (\lambda + m + \ell) I)$.

**Proof** (1) is a consequence of the equation (2.6). In view of [4, Corollary 3.16], $\ker^\perp (\Box_{m,M}^{0,\ell} - \lambda I)$ is the direct sum of $\ker (\Box_{m,M}^{0,\ell} - \lambda I)$ and eigenspaces of $\Box_{m,M}^{0,\ell}$ whose eigenvalues are different from $\lambda$. Therefore, (2) follows by Lemma 2.1 and self-adjointness of $\Box_{m+1,M}^{1,\ell}$.  

3 Construction of Holomorphic Functions on $M \times \rho \mathbb{B}^N$

**3.1 Preliminaries**

Let $z$ be a fixed point in the unit ball $\mathbb{B}^N$. For one-dimensional vector space $[z]$ spanned by $z$, we define an orthogonal projection $P_z$ from $\mathbb{C}^N$ onto $[z]$. Another orthogonal projection $Q_z$ is defined by $P_z + Q_z = Id z$. Consider an automorphism $T_z$ of $\mathbb{B}^N$ given by

$$T_z(w) = \frac{z - P_z(w) - s_z Q_z(w)}{1 - w \cdot \bar{z}}$$

where $s_z = \sqrt{1 - |z|^2}$ with $|z|^2 = z \cdot \bar{z}$. We remark that $T_z$ is an involution, i.e., $T_z \circ T_z = Id_{\mathbb{B}^N}$.

Let $A = (A_{jk}) := dT_z(z)$ and let

$$e_j := \sum_{k=1}^N A_{jk} dz_k. \quad (3.1)$$

Then $\{e_j\}_{j=1}^N$ is an orthonormal frame of $T^*_\mathbb{B}^N$ with respect to the Bergman metric on $\mathbb{B}^N$ (see [9]). Let $\{X_j\}_{j=1}^N$ be the dual frame of $\{e_j\}_{j=1}^N$ on $T^*_\mathbb{B}^N$, i.e.,

$$X_j = \sum_{k=1}^N A_{kj} \frac{\partial}{\partial z_k}$$
where \((A^{kj})_{j,k=1}^{N}\) is the inverse matrix of \((A_{jk})_{j,k=1}^{N}\).

Let \(\tilde{e}_1, \ldots, \tilde{e}_n\) be a local orthonormal frame on \(T^*_M\). Then there exist locally defined smooth functions \(b_{kl}\) such that

\[
i^*e_k = \sum_{l=1}^{n} b_{kl} \tilde{e}_l.
\]

Let \(Y_1, \ldots, Y_n\) be the local dual frame of \(\tilde{e}_1, \ldots, \tilde{e}_n\) on \(T^*_M\). Then there exist locally defined smooth functions \(y_{lj}\) on \(M\) and \(a_{lk}\) on \(i(M)\) such that

\[
Y_j = \sum_{l=1}^{n} y_{lj}(\xi) \frac{\partial}{\partial \xi_l} \tag{3.2}
\]

and

\[
i_*Y_k = \sum_{l=1}^{N} (a_{lk} \circ i)(\xi) X_l. \tag{3.3}
\]

**Lemma 3.1** \(b_{km} = a_{km} \circ i\)

**Proof** Since one has

\[
i^*e_k(Y_m) = \sum_{l} b_{kl} \tilde{e}_l(Y_m) = b_{km}
\]

and

\[
i^*e_k(Y_m) = e_k(i_*Y_m) = e_k \left( \sum a_{lm} \circ i X_l \right) = a_{km} \circ i,
\]

we obtain the lemma. \(\square\)

**Lemma 3.2** For each \(\mu = 1, \ldots, N\),

\[
\sum_{l=1}^{N} (a_{lk} \circ i)(\xi) A^{\mu l} = \sum_{l=1}^{n} y_{lk}(\xi) \frac{\partial i_{\mu}}{\partial \xi_l}.
\]

**Proof** Since we have

\[
i_*Y_k = i_* \left( \sum_{l=1}^{n} y_{lk} \frac{\partial}{\partial \xi_l} \right) = \sum_{l=1}^{N} \sum_{\mu=1}^{N} y_{lk}(\xi) \frac{\partial i_{\mu}}{\partial \xi_l} \frac{\partial}{\partial z_{\mu}} \bigg|_{z = i(\xi)}
\]

and

\[
\sum_{l=1}^{N} (a_{lk} \circ i)(\xi) X_l = \sum_{l=1}^{N} \sum_{\mu=1}^{N} (a_{lk} \circ i)(\xi) A^{\mu l} \frac{\partial}{\partial z_{\mu}} \bigg|_{z = i(\xi)},
\]

by (3.3) the proof is completed. \(\square\)
3.2 Definition of Formal Series

First, we note that $T^*_\Sigma$ is Griffiths positive. Since $i$ is an embedding, $i^*T^*_\Sigma$ is also Griffiths positive and so it is ample. Since $i^*(S^mT^*_\Sigma) \cong S^m(i^*T^*_\Sigma)$, we know that $\bigoplus_{m=0}^{\infty} H^0(M, i^*(S^mT^*_\Sigma)) \cong \bigoplus_{m=0}^{\infty} H^0(M, S^m(i^*T^*_\Sigma))$ is infinite dimensional.

By the decomposition (2.4) any symmetric differential $\psi \in H^0(M, i^*(S^mT^*_\Sigma))$ is of the form

$$\psi = \sum_{\ell=0}^{m} \psi^\ell_m$$

where $\psi^\ell_m \in H^0(M, S^\ell T^*_M \otimes S^{m-\ell}N^*)$. Fix $\psi = \sum_{\ell=0}^{m_0} \psi^\ell_{m_0} \in H^0(M, i^*(S^{m_0}T^*_\Sigma))$. For each $\ell = 0, \ldots, m_0$, we define a sequence of vector bundles $\{F^\ell_k\}_{k=0}^{\infty}$ by

$$F^\ell_k = \begin{cases} i^*(S^k T^*_\Sigma) & \text{if } k < m_0, \\ S^{\ell+k-m_0} T^*_M \otimes S^{m_0-\ell}N^* & \text{if } k \geq m_0. \end{cases}$$

and consider the sequence

$$\{\varphi^\ell_k\}_{k=0}^{\infty} \in \bigoplus_{k=0}^{\infty} C^\infty(M, F^\ell_k)$$

such that

$$\varphi^\ell_k = \begin{cases} 0 & \text{if } k < m_0, \\ \psi^\ell_{m_0} & \text{if } k = m_0, \\ \text{the minimal solution of} \\ \bar{\partial}_M \varphi^\ell_k = -(k - 1) \mathcal{R}_G \varphi^\ell_{k-1} & \text{if } k > m_0. \end{cases} \quad (3.4)$$

The minimal solution of the equation

$$\bar{\partial}_M \varphi^\ell_k = -(k - 1) \mathcal{R}_G \varphi^\ell_{k-1} \quad (3.5)$$

exists by the following lemma for each $k$.

**Lemma 3.3** For any symmetric differential $\psi = \sum_{\ell=0}^{m_0} \psi^\ell_{m_0} \in H^0(M, i^*(S^{m_0}T^*_\Sigma))$ and each $\ell = 0, \ldots, m_0$, the sequence $\{\varphi^\ell_k\}_{k=0}^{\infty}$ given by (3.4) is well defined and it satisfies

$$\|\varphi^\ell_{m_0+m}\|^2 = \left( \prod_{j=1}^{m} \left( \frac{(\ell + j) + (n - 1)}{m_0 + j} \right) \right) \left( \frac{(m_0 + \ell - 1)! \cdot ((m_0 + m - 1)!^2 \cdot 1)}{(m_0 - 1)!^2 \cdot (m_0 + \ell + m - 1)! \cdot m!} \right) \|\psi^\ell_{m_0}\|^2$$
for any \( m \geq 1 \). Moreover for any \( m \geq 0 \), \( \varphi^\ell_{m_0+m} \) satisfies
\[
\Box_{m_0+m,M}^0 (\varphi^\ell_{m_0+m}) = (m^2 + (m_0 + \ell - 1)m)\varphi^\ell_{m_0+m}.
\]

**Proof** We will use induction with respect to the index \( k \). If \( k \leq m_0 \), then (3.4) holds trivially. Suppose that there is the minimal solution of (3.5) for any \( p \leq m_0 + m - 1 \). First we will show that \( R_G(\varphi^\ell_{m_0+m-1}) \) is \( \bar{\delta}_M \)-closed. Take a point \( q \in M \) and small open set \( q \in U \subset M \) such that \( U \cong \iota(U) \subset \Sigma \). Let \((z_1, \cdots, z_N)\) be a local coordinate system at \( p := \iota(q) \in \iota(U) \) such that
\[
\iota(U) = \{(z_1, \cdots, z_N) : z_{n+1} = \cdots = z_N = 0 \} \quad \text{near} \quad p = (0, \cdots, 0).
\]

Then we obtain
\[
\text{If} \quad k \leq m_0 + m - 1, \quad \text{then} \quad (3.4) \text{ holds.}
\]

As a result, for each \( j \),
\[
\varphi^\ell_{m_0+m-1} = \sum_{|L| = \ell + m - 1, |J| = m_0 - \ell} \varphi^\ell_{LJ} dz^L \otimes dz^J.
\]

Then we obtain
\[
\bar{\delta}_M \varphi^\ell_{m_0+m-1} = \sum_{j=1}^n \sum_{|L| = \ell + m - 1, |J| = m_0 - \ell} Y_{\ell} \varphi^\ell_{LJ} dz^L \otimes dz^J \otimes \bar{e}_j
\]
and by (3.5)
\[
\bar{\delta}_M \varphi^\ell_{m_0+m-1} = -(m_0 + m - 2) \sum_{\mu=1}^n \varphi^\ell_{m_0+m-2} (e_\mu \circ \iota) \otimes i^* e_\mu
\]
\[
= -(m_0 + m - 2) \sum_{\mu,j=1}^n \varphi^\ell_{m_0+m-2} (e_\mu \circ \iota) B_{\mu j} \otimes \bar{e}_j.
\]

As a result, for each \( j = 1, \ldots, n \),
\[
\sum_{|L| = \ell + m - 1, |J| = m_0 - \ell} Y_{\ell} \varphi^\ell_{LJ} dz^L \otimes dz^J = -(m_0 + m - 2) \sum_{\mu=1}^n \varphi^\ell_{m_0+m-2} B_{\mu j} (e_\mu \circ \iota).
\]

Since we have
\[
\bar{\delta}_M R_G(\varphi^\ell_{m_0+m-1}) = \bar{\delta}_M \left( \sum_{\mu=1}^n \varphi^\ell_{m_0+m-1} (e_\mu \circ \iota) \otimes i^* e_\mu \right)
\]
\[
= \sum_{\mu,j,L,J} \left( Y_{\ell} \varphi^\ell_{LJ} dz^L (e_\mu \circ \iota) \otimes dz^J \otimes \bar{e}_j \otimes i^* e_\mu \right)
\]
+ \sum_{\tau, j, \mu, s} \varphi_{m_0 + m - 1}^\ell \, dz_\tau \otimes \tilde{\partial}_M \left( (A_{\mu \tau} \circ t)(A_{\mu j} \circ t) \frac{\partial t_j}{\partial \zeta_s} \right) \wedge d\zeta_s,

\sum_{\mu, j | L|=\ell + m - 1 |J|=m_0 - \ell} \bar{Y}_j \varphi_{LJ}^\ell \, dz^L (e_\mu \circ t) \otimes dz^J \otimes \bar{e}_j \wedge i^* e_\mu

= -(m_0 + m - 2) \sum_{\mu, j, \eta} \bar{b}_{\eta j} \varphi_{m_0 + m - 2}^\ell (e_\mu \circ t)(e_\eta \circ t) \otimes \bar{e}_j \wedge i^* e_\mu

= -(m_0 + m - 2) \sum_{\mu, j} \varphi_{m_0 + m - 2}^\ell (e_\mu \circ t)(e_\eta \circ t) i^* e_\eta \wedge i^* e_\mu = 0

by (3.6), and

\begin{align*}
\sum_{j, \mu, s} \tilde{\partial}_M \left( (A_{\mu \tau} \circ t)(A_{\mu j} \circ t) \frac{\partial t_j}{\partial \zeta_s} \right) \wedge d\zeta_s

= \sum_{j, \mu, s} \tilde{\partial}_M \left( (A_{\mu \tau} \circ t)(A_{\mu j} \circ t) \right) \wedge \frac{\partial t_j}{\partial \zeta_s} d\zeta_s

+ \sum_{j, \mu, \eta, s} (A_{\mu \tau} \circ t)(A_{\mu j} \circ t) \frac{\partial^2 t_j}{\partial \zeta_\eta \partial \zeta_s} d\zeta_\eta \wedge d\zeta_s

= \sum_{j, \mu, \eta, \sigma, s} \left( \frac{\partial A_{\mu \tau}}{\partial \zeta_\eta} (A_{\mu j} \circ t) + (A_{\mu \tau} \circ t) \frac{\partial A_{\mu j}}{\partial \zeta_\eta} \right) \frac{\partial t_\eta}{\partial \zeta_\sigma} d\zeta_\sigma \wedge \frac{\partial t_j}{\partial \zeta_s} d\zeta_s

= \sum_{j, \mu, \eta} \left( \frac{\partial A_{\mu \tau}}{\partial \zeta_\eta} (A_{\mu j} \circ t) + (A_{\mu \tau} \circ t) \frac{\partial A_{\mu j}}{\partial \zeta_\eta} \right) d\zeta_\eta \wedge d\zeta_j = 0,
\end{align*}

(3.7)

one has $\tilde{\partial}_M R_G(\varphi_{m_0 + m - 1}^\ell) = 0$. Here the last equality in (3.7) holds by the same argument given in the proof of Lemma 4.12 in [9].

Now we claim that $\varphi_{m_0 + k}^\ell$, $k \leq m - 1$ is an eigenfunction of $\Box^0_{m_0 + k, M}$. Denote $E_{m_0, k}$ be its eigenvalue. Since $\varphi_{m_0}^\ell \in H^0(M, S^\ell T^*_M \otimes S^{m_0 - \ell} N^*)$, one has $E_{m_0, 0}^\ell = 0$. Assume that $\varphi_{m_0 + k}^\ell$ is an eigenvector of $\Box^1_{m_0 + k, M}$ for some $k \geq 0$. By (2.6) and self-adjointness of $\Box^1_{m_0 + k + 1, M}$, we know

$$R_G(\varphi_{m_0 + k}^\ell) \perp \ker \Box^1_{m_0 + k + 1, M}.$$  

(3.8)

Moreover, by Corollary 2.3 and (3.8), we obtain

$$R_G(\varphi_{m_0 + k}^\ell) = \Box^1_{m_0 + k + 1, M} G^1 R_G(\varphi_{m_0 + k}^\ell) = G^1 R_G((E_{m_0, k}^\ell + (\ell + k) + (m_0 + k))\varphi_{m_0 + k}^\ell)$$

and by properties of the Green operator $G^1$, it follows that

$$\Box^0_{m_0 + k + 1, M} (\varphi_{m_0 + k + 1}^\ell) = (E_{m_0, k}^\ell + (\ell + k) + (m_0 + k))\varphi_{m_0 + k + 1}^\ell.$$
The eigenvalue of $\varphi_{m_0+k+1}^\ell$ for $\Box^0_{m_0+k+1,M}$ is

$$E_{m_0,k+1}^\ell = E_{m_0,k}^\ell + (\ell + k) + (m_0 + k).$$

Hence

$$E_{m_0,k}^\ell = (\ell + \ell + 1 + \cdots + (\ell + k - 1)) + (m_0 + (m_0 + 1) + \cdots + (m_0 + k - 1))$$
$$= \frac{k(2m_0 + k - 1)}{2} + \frac{k(\ell + k - 1)}{2}. \tag{3.9}$$

Now we will show that (3.5) has a solution when $k = m$. By the Hodge decomposition, the solvability of (3.5) follows by (3.8). By (2.5), (3.5), and Corollary 2.3, we have

$$\|\varphi_{m_0+m}^\ell\|^2 = (m_0 + m - 1)^2 \langle (\bar{\partial} G^1 R G \varphi_{m_0+m-1}^\ell, \bar{\partial} G^1 R G \varphi_{m_0+m-1}^\ell) \rangle$$
$$= \frac{(m_0 + m - 1)^2}{E_{m_0,m-1}^\ell + (\ell + m - 1) + (m_0 + m - 1)} \|R G \varphi_{m_0+m-1}^\ell\|^2$$
$$= \frac{(\ell + m - 1) + n}{(m_0 + m - 1) + 1} \frac{(m_0 + m - 1)^2}{E_{m_0,m}^\ell} \|R G \varphi_{m_0+m-1}^\ell\|^2.$$

Therefore,

$$\|\varphi_{m_0+m}^\ell\|^2 = \left( \prod_{j=1}^m \left( \frac{\ell + j - 1 + n}{m_0 + j - 1 + 1} \right) \right) \frac{2(m_0 + m - j)^2}{(m - j + 1)((2m_0 + m - j) + (2\ell + m - j))} \|\varphi_{m_0}^\ell\|^2$$
$$= \left( \prod_{j=1}^m \left( \frac{\ell + j + (n - 1)}{m_0 + j} \right) \right) \frac{(m_0 + \ell - 1)!((m_0 + m - 1)!)^2}{(m_0 - 1)!^2 m!(m_0 + \ell + m - 1)!} \|\varphi_{m_0}^\ell\|^2.$$

For non-negative integer $k$ define $\varphi_k \in C^\infty(M, i^*(S^k T^*_\Sigma)) \cong \bigoplus_{\mu=0}^k C^\infty(M, S^\mu T^*_M \otimes S^{k-\mu} N^*)$ and $\varphi \in \bigoplus_{k=0}^\infty C^\infty(M, i^*(S^k T^*_\Sigma))$ by

$$\varphi_k := 0 \quad \text{for } k < m_0, \quad \varphi_k := \sum_{\ell=0}^{m_0} \varphi_{m_0+(k-m_0)}^\ell \quad \text{for } k \geq m_0,$$

and

$$\varphi := \sum_k \varphi_k \tag{3.10}$$

where $\varphi_{m_0+(k-m_0)}^\ell \in C^\infty(M, S^{\ell+(k-m_0)} T^*_M \otimes S^{m_0-k} N^*)$. Using the frame $e = (e_1, \ldots, e_N)$ given in (3.1), we write

$$\varphi_k(\xi) = \sum_{|I|=k} f_I(i(\xi)) e_{I,1}(\xi) \tag{3.11}$$

for $\xi \in M$. 

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Lemma 3.4 In the above setting, the following identity holds:

\[ \| \varphi_k \|^2 = \sum_{\ell=0}^{m_0} \| \varphi_{m_0+(k-m_0)}^\ell \|^2 \] for \( k \geq m_0 \).

**Proof** Let \( \{ U_\alpha \} \) be a finite open cover of \( i(M) \) in \( \Sigma \) satisfying that \( T^*_\Sigma \) on \( U_\alpha \) has a local orthonormal frame \( \{ e_1, \ldots, e_N \} \) such that \( g = \sum_{\mu=1}^N e_\mu \otimes \bar{e}_\mu \) and \( h = \sum_{\mu=1}^n t^* e_\mu \otimes t^* \bar{e}_\mu \). Write \( L = (i_1, \ldots, i_n) \), \( J = (i_{n+1}, \ldots, i_N) \) and \( e^L = e_1^{i_1} \cdots e_n^{i_n} \), \( e^J = e^{i_{n+1}} \cdots e^{i_N} \) accordingly. Then \( \{ e^L \otimes e^J \}_{|L|=|\ell+(k-m_0), |J|=m_0-\ell} \) becomes a local orthonormal frame of \( S^{\ell+(k-m_0)} T^*_M \otimes S^{m_0-\ell} N^* \) on \( i^{-1}(U_\alpha) \), and locally

\[ \varphi_k^\ell (\zeta) = \sum_{|L|=|\ell+(k-m_0), |J|=m_0-\ell} f_{LJ}(i(\zeta)) e^L \big|_{i(\zeta)} \otimes e^J \big|_{i(\zeta)} \]

by (3.10), where \( f_{LJ} \) is a smooth function on \( U_\alpha \). Since \( \iota \) is an embedding, \( \{ \iota^{-1}(U_\alpha) \} \) becomes a finite open cover of \( M \). Let \( \{ \chi_\alpha \} \) be a partition of unity subordinate to \( \{ \iota^{-1}(U_\alpha) \} \). Then

\[
\| \varphi_k \|^2 = \int_M \langle \varphi_k, \varphi_k \rangle dV_M
\]

\[
= \sum_{\alpha} \sum_{\ell=0}^{m_0} \sum_{|L|=|\ell+(k-m_0), |J|=m_0-\ell} \int_{\iota^{-1}(U_\alpha)} \chi_\alpha |(f_{LJ} \circ \iota)(\zeta)|^2 \langle e^L, e^L \rangle \langle e^J, e^J \rangle dV_M
\]\n
\[
= \sum_{\alpha} \sum_{\ell=0}^{m_0} \sum_{|L|=|\ell+(k-m_0), |J|=m_0-\ell} \frac{L! J!}{k!} \int_{\iota^{-1}(U_\alpha)} \chi_\alpha |(f_{LJ} \circ \iota)(\zeta)|^2 dV_M.
\]

Since

\[
\| \varphi_k^\ell \|^2 = \sum_{\alpha} \sum_{|L|=|\ell+(k-m_0), |J|=m_0-\ell} \frac{L! J!}{k!} \int_{\iota^{-1}(U_\alpha)} \chi_\alpha |(f_{LJ} \circ \iota)(\zeta)|^2 dV_M,
\]

the proof is completed. \( \square \)

By using (3.11), we define a formal sum \( f \) on \( \tilde{M} \times \mathbb{B}^N \) by

\[
f(\zeta, w) := \sum_{|J|=0}^{\infty} f_J(i(\zeta))(T_i(\zeta) w)^J.
\]

(3.12)

In view of Lemma 3.5 below, we may consider \( f \) as a function on \( \Omega := M \times_\rho \mathbb{B}^N \).

Lemma 3.5 \( f(\zeta, w) \) is \( \Gamma \)-invariant, i.e., \( f(\gamma \zeta, \rho(\gamma) w) = f(\zeta, w) \) for all \( \gamma \in \Gamma \), \( \zeta \in \tilde{M} \) and \( w \in \mathbb{B}^N \).
Proof} Fix $\gamma \in \Gamma$. There exists a unitary matrix $U_\zeta$ depending only on $\zeta$ satisfying
\[
T_{\rho(\gamma)(i(\zeta))}\rho(\gamma)w = U_\zeta T_i(\zeta)w. \tag{3.13}
\]
Since $\varphi \in \bigoplus_{k=0}^\infty H^0(M, i^*(S^k T_{\Sigma}^*)) \cong \bigoplus_{k=0}^\infty H^0(\tilde{M}, S^k T_{\Sigma}^*|\tilde{M})$ and $i(M) \cong \tilde{M}/\rho(\Gamma)$, we have $\rho(\gamma)^*\varphi_k = \varphi_k$. Note that
\[
\rho(\gamma)^*e_j = \sum_k A_{jk} \circ \rho(\gamma)d(\rho \circ \gamma)_k = \sum_{k,m,l} A_{jk} \circ \rho(\gamma)\frac{\partial(\rho \circ \gamma)_k}{\partial \zeta_l} A^{lm} e_m
\]
where $(A^{lm})$ denotes the inverse matrix of $A$, i.e., for $e = (e_1, \ldots, e_N)$
\[
\rho(\gamma)^*e |_{i(\zeta)} = \left( A \circ \rho(\gamma)d\rho(\gamma)A^{-1} \right)^* e |_{i(\zeta)} = U^*_\zeta e |_{i(\zeta)}.
\]
This implies
\[
\sum_{|I|=0}^\infty f_I(i(\zeta))e_I = \sum_{|I|=0}^\infty f_I(\rho(\gamma)i(\zeta))\rho(\gamma)^*(e_I)
\]
\[
= \sum_{|I|=0}^\infty f_I(\rho(\gamma)i(\zeta))U^*_\zeta(e_I) \tag{3.14}
\]
where $\rho(\gamma)^*$ and $U^*_\zeta$ are understood as the pull-back of symmetric differential forms. Hence by (3.13) and (3.14), we have
\[
f(\gamma \zeta, \rho(\gamma)w) = \sum_{|I|=0}^\infty f_I(i(\gamma \zeta))(T_{\rho(\gamma) i(\zeta)}\rho(\gamma)w)^I
\]
\[
= \sum_{|I|=0}^\infty f_I(\rho(\gamma)i(\zeta))(T_{\rho(\gamma) i(\zeta)}\rho(\gamma)w)^I
\]
\[
= \sum_{|I|=0}^\infty f_I(\rho(\gamma)i(\zeta))(U^*_\zeta T_i(\zeta)w)^I
\]
\[
= \sum_{|I|=0}^\infty f_I(i(\zeta))(T_i(\zeta)w)^I = f(\zeta, w).
\]

\[\square\]

3.3 $L^2$ Convergence of Formal Series

Let $\Omega := M \times_{\rho} \mathbb{B}^N$ and $K : \mathbb{B}^N \times \mathbb{B}^N \to \mathbb{C}$ be the (normalized) Bergman kernel on $\mathbb{B}^N$, i.e.,
\[
K(z, w) = \frac{1}{(1 - z \cdot \overline{w})^{N+1}}
\]
where \(z \cdot \overline{w} = \sum_{i=1}^{N} z_i \overline{w_i}\). We define a Kähler form \(\omega\) on \(\Omega\) by

\[
\omega|_{\{\zeta, w\}} = \tilde{H} + \frac{\sqrt{-1}}{N+1} \partial \overline{\partial} \log K(w, w)
\]

with the Kähler form \(\tilde{H}\) for \((\tilde{M}, i^* g_{B^N})\), where \(i^* g_{B^N}\) is the pull-back metric on \(\tilde{M}\) of the normalized Bergman metric \(g_{B^N}\) on \(B^N\). One can check that \(\omega\) is an \((1, 1)\) form on \(M \times \rho B^N\). We define the volume form on \(\Omega\) by

\[
dV_{\omega} = \sqrt{-1} N K(w, w) \tilde{H}^n \wedge dw \wedge \overline{dw},
\]

(3.15)

where \(d\omega := dw_1 \wedge \cdots \wedge dw_N\). Now for measurable sections \(f_1, f_2\) on \(\Lambda^{p,q} T^*_\Omega\) and \(\alpha > -1\), we set

\[
\langle\langle f_1, f_2 \rangle\rangle_\alpha := c_\alpha \int_{\Omega} \langle f_1, f_2 \rangle_\omega \delta^{\alpha+N+1} dV_{\omega}
\]

where \(c_\alpha = \frac{\Gamma(N+\alpha+1)}{\Gamma(\alpha+1) N!}\) and \(\delta = 1 - |T_i(\zeta)|^2\).

**Lemma 3.6** If \(f_1, f_2\) are measurable sections on \(\Lambda^{p,q} T^*_\Omega\), then

\[
\langle\langle f_1, f_2 \rangle\rangle_\alpha = c_\alpha \int_{\Omega} \langle f_1, f_2 \rangle_\omega \delta^{\alpha} \left| \frac{K(w, t(\zeta))^2}{K(t(\zeta), t(\zeta))} \right| \tilde{H}^n \wedge dw \wedge \overline{dw}.
\]

**Proof** By (3.15) and

\[
1 - |T_z w|^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - z \cdot \overline{w}|^2} = \left( \frac{K(z, z) K(w, w)}{|K(z, w)|^2} \right)^{-\frac{1}{N+1}},
\]

the lemma follows. \(\square\)

For \(\alpha > -1\), we define a weighted \(L^2\)-space by setting

\[
L^2_{(p,q),\alpha}(\Omega) := \{ f : f \text{ is a measurable section on } \Lambda^{p,q} T^*_\Omega, \|f\|^2_\alpha := \langle\langle f, f \rangle\rangle_\alpha < \infty \}
\]

and a weighted Bergman space by \(A^2_{\alpha}(\Omega) := L^2_{(0,0),\alpha}(\Omega) \cap O(\Omega)\). In this setting, we extend \(\tilde{\partial}\)-operator on \(\Omega\) as the maximal extension of \(\tilde{\partial}\) on \(\Omega\) which acts on smooth \((p, q)\) forms on \(\Omega\).

**Lemma 3.7** For any partial sum

\[
F_{m_0+m}(\zeta, w) = \sum_{|I|=0}^{m_0+m} f_I(t(\zeta))(T_i(\zeta)w)^I
\]

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of \( f \) in (3.12), the following identity holds:

\[
\|F_{m_0+m}\|_\alpha^2 = \frac{2^N \pi^N}{N!} \sum_{|I|=0}^{m_0+m} \|\varphi_I\|_\alpha^2 \frac{|I|! \Gamma(N + |I| + \alpha + 1)}{\Gamma(N + |I| + \alpha + 1)}.
\] (3.16)

\textbf{Proof} Let \( \hat{M} \) denote the fundamental domain of \( M \) in \( \tilde{M} \) and \( \tilde{\Omega} \) denote the corresponding domain of \( \Omega \). Note that \( \tilde{\Omega} = \hat{M} \times {\mathbb{B}}^N \subset \Omega = M \times {\mathbb{B}}^N \). By Lemma 3.6, \( \|F_{m_0+m}\|_\alpha^2 \) is equal to

\[
c_\alpha 2^N \int_{\tilde{\Omega}} \left| \sum_{|I|=0}^{m_0+m} f_I(t(\zeta))(T_{t(\zeta)}w)^I \right|^2 (1 - |T_{t(\zeta)}w|^2)^\alpha \tilde{H}^n \frac{|K(w, t(\zeta))|^2}{K(t(\zeta), t(\zeta))} d\lambda_w
\] (3.17)

where \( \lambda_w = \left( \sqrt{-1} \right)^N dw_1 \wedge dw_1 \wedge \cdots \wedge dw_N \wedge d\overline{w}_N \) denotes the Lebesgue measure of \( \mathbb{B}^N \).

Since \( t = T_{t(\zeta)}w, J_{\mathbb{R}} T_{t(\zeta)}(0) = (1 - |t(\zeta)|^2)^{N+1}, d\lambda_w = |J_{\mathbb{C}} T_{t(\zeta)}|2d\lambda_t, \)

\[
K(t(\zeta), w) = K(T_{t(\zeta)}0, T_{t(\zeta)}t) = \frac{K(0, t)}{J_{\mathbb{C}} T_{t(\zeta)}(0) J_{\mathbb{C}} T_{t(\zeta)}(t)} = \frac{1}{J_{\mathbb{C}} T_{t(\zeta)}(0) J_{\mathbb{C}} T_{t(\zeta)}(t)},
\]

and

\[
K(t(\zeta), t(\zeta)) = K(T_{t(\zeta)}0, T_{t(\zeta)}0) = \frac{K(0, 0)}{|J_{\mathbb{C}} T_{t(\zeta)}(0)|^2} = \frac{1}{|J_{\mathbb{C}} T_{t(\zeta)}(0)|^2},
\]

by (3.17), we obtain

\[
\|F_{m_0+m}\|_\alpha^2 = 2^N c_\alpha \int_{\tilde{M}} \tilde{H}^n \int_{\mathbb{B}^N} \left| \sum_{|I|=0}^{m_0+m} f_I(t(\zeta))(T_{t(\zeta)}w)^I \right|^2 (1 - |t|^2)^\alpha d\lambda_t
\] (3.18)

The second equality in (3.18) can be induced by the orthogonality of polynomials with respect to the inner product \( \int_{\mathbb{B}^n} f \overline{g}(1 - |t|^2)^\alpha d\lambda_t \) ([13]).

Since

\[
\|\varphi_\ell\|_\alpha^2 = \sum_{|I|=\ell} \frac{I!}{\ell!} \int_{\tilde{M}} |f_I \circ t|^2 \tilde{H}^n,
\]
by (3.18) one has
\[ \| F_{m_0+m} \|^2_{L^2_\alpha} = \frac{2^N \pi N^{m_0+m}}{N!} \sum_{|I|=0} \| \varphi_{|I|} \|^2 |I|! \Gamma(N + \alpha + 1) \frac{\Gamma(N + |I| + \alpha + 1)}{\Gamma(N + |I| + \alpha + 1)}. \]

Lemma 3.8 For any \( \alpha > -1 \), the formal sum \( f \) converges in \( L^2_\alpha(\Omega) \). Moreover, if \( n \neq N \), then \( f \) converges in \( L^2_{-1}(\Omega) \).

Proof By (3.16) and Lemma 3.4, the partial sum
\[ F_{m_0+m}(\zeta, w) = \sum_{|I|=0}^{m_0+m} f_I(\zeta) (T_{\ell_I}(\zeta)) w^I \]
satisfies
\[ \| F_{m_0+m} \|^2_{L^2_\alpha} = \frac{2^N \pi N}{N!} \sum_{k=0}^{m} \| \varphi_{m_0+k} \|^2 \frac{(m_0 + k)! \Gamma(N + \alpha + 1)}{\Gamma(N + m_0 + k + \alpha + 1)} \]
\[ = \frac{2^N \pi N}{N!} \sum_{k=0}^{m} \left( \sum_{\ell=0}^{m_0} \| \varphi_{m_0+k}^{\ell} \|^2 \right) \frac{(m_0 + k)! \Gamma(N + \alpha + 1)}{\Gamma(N + m_0 + k + \alpha + 1)} \]
\[ = \frac{2^N \pi N \Gamma(N + \alpha + 1) (m_0 + 1)}{\Gamma(m_0 + N + \alpha + 1)} \sum_{\ell=0}^{m_0} \sum_{k=0}^{m} a_{k}^{\ell}, \]
where
\[ a_{k}^{\ell} := \frac{(m_0 + 1)_k}{(N + m_0 + \alpha + 1)_k} \frac{m_0(m_0)_k}{(m_0 + \ell)_k} \frac{1}{k!} \left( \prod_{j=1}^{k} \frac{((\ell + j) + (n - 1))}{(m_0 + j)} \right) \| \varphi_{m_0+k}^{\ell} \|^2 \]
\[ = \frac{(m_0 + 1)_k}{(N + m_0 + \alpha + 1)_k} \frac{m_0(m_0)_k}{(m_0 + \ell)_k} \frac{1}{k!} \left( \prod_{j=1}^{k} \frac{((\ell + j) + (n - 1))}{(m_0 + j)} \right) \| \varphi_{m_0+k}^{\ell} \|^2 \]
\[ = \frac{(m_0 + 1)_k}{(N + m_0 + \alpha + 1)_k} \frac{m_0(m_0)_k}{(m_0 + \ell)_k} \frac{1}{k!} \left( \prod_{j=1}^{k} \frac{((\ell + j) + (n - 1))}{(m_0 + j)} \right) \| \varphi_{m_0+k}^{\ell} \|^2 \]

and \( (m_0)_k := m_0(m_0 + 1) \cdots (m_0 + k - 1) \). Note that for each fixed \( \ell = 0, \cdots, m, \)
\[ k \left( \frac{a_{k}^{\ell}}{a_{k+1}^{\ell}} - 1 \right) = k \left( \frac{(k+1)(m_0 + \ell + 1)(k + N + m_0 + \alpha + 1)}{(k + m_0)^2(k + n + \ell)} - 1 \right) \]
\[ = k \left( \frac{(\ell - m_0 + 1)k - m_0^2}{(k + m_0)^2} \right) + (N + m_0 + \alpha + 1 - (n + \ell)) \frac{k(k+1)(m_0 + \ell + 1)}{(k + m_0)^2(k + n + \ell)} \]
\[ \to (\ell - m_0 + 1) + (N + m_0 + \alpha + 1 - n - \ell) = 1 + (N - n + \alpha + 1) \]
as \( k \to \infty \). Hence the series \( \sum_{k=0}^{\infty} a_{k}^{\ell} \) converges when \( \alpha > n - (N+1) \) by the Raabe’s test. Since \( F_{m_0+m} \) is the partial sum of \( f \), the lemma is now proved. □
Remark 3.9 Set $\alpha < -1$. For any formal sum $f$ given by (3.12) define $L^2$-norm for any partial sum of $f$ by (3.16). Then Lemma 3.8 tells us that $f$ converges in $L^2_\alpha(\Omega)$ if $\alpha > n - (N + 1)$ (cf. [12, Chapter 12]).

3.4 Holomorphicity of Formal Series

Lemma 3.10 The formal sum $f$ given by (3.12) is holomorphic.

Proof Note that since $f$ is holomorphic in $w$, we only need to show that $f$ is holomorphic in $\zeta$. Let

$$F_m(\zeta, w) := \sum_{|I|=0}^m \hat{f}_I(\iota(\zeta))(T_{\iota(\zeta)}w)^I$$

be the finite sum of $f$ and let $\Gamma^j_{i\mu} := \sum_{k,s} A^k j \frac{\partial A_{ls}}{\partial \bar{z}_k} A^{s\mu}$. Since

$$\frac{\partial F_m(\zeta, w)}{\partial \xi_j} = \frac{\partial \tilde{F}_m(\zeta, T_{\iota(\zeta)}w)}{\partial \xi_j} + \sum_{k,v} \frac{\partial \tilde{F}_m(\zeta, T_{\iota(\zeta)}w)}{\partial t_k} \frac{\partial (T_{\iota(\zeta)}w)_k}{\partial \bar{z}_v} \bigg|_{z=\iota(\zeta)} \frac{\partial \bar{t}_v}{\partial \xi_j}$$

with $\tilde{F}_m(\zeta, t) := \sum_{|I|=0}^m \hat{f}_I(\iota(\zeta))t^I$, we obtain

$$\bar{\phi}_s F_m = \sum_{|I|=0}^m \bar{\phi}_s \tilde{F}_m(\zeta, T_{\iota(\zeta)}w) + \sum_{|I|=0}^m \bar{\phi}_s \tilde{F}_m(\zeta, T_{\iota(\zeta)}w) \sum_{k,v} \frac{\partial \tilde{F}_m(\zeta, T_{\iota(\zeta)}w)}{\partial t_k} \frac{\partial (T_{\iota(\zeta)}w)_k}{\partial \bar{z}_v} \bigg|_{z=\iota(\zeta)} \frac{\partial \bar{t}_v}{\partial \xi_j}$$

Here, the second equality holds by Lemma 3.2 and the third equality holds by the equation (4.8) and Lemma 4.8 in [9]. If we express $\phi_s = \sum_{|I|=s} \hat{f}_I(\iota(\zeta))e^I|_{\iota(\zeta)}$, then we have

$$\tilde{\phi}_s = \sum_{|I|=s} \sum_{\mu=1}^n \left( \bar{\phi}_s \tilde{F}_m(\zeta, T_{\iota(\zeta)}w) + \sum_{\tau=1}^N \left( \sum_{k} \iota_k (f_I \circ \iota)(\Gamma^k_{\tau}) + |I|(f_I \circ \iota)(T_{\iota(\zeta)}w)_\tau \right) \right) e^I|_{\iota(\zeta)} \otimes \bar{e}_\mu.$$  (3.20)
On the other hand, one has

\[
\bar{\partial}_s \varphi = -(s - 1) \mathcal{R}_G(\varphi_{s-1})
\]

\[
= -(s - 1) \sum_{\mu=1}^{N} \sum_{\tau=1}^{n} (a_{\tau \mu} \circ \iota) \sum_{|J|=s-1} (f_J \circ \iota)(e^J e_\tau)|_{t(\zeta)} \otimes \tilde{e}_\mu
\]

(3.21)

by (3.4), (3.10), and the definition of \( \mathcal{R}_G \) with Lemma 3.1. Hence by comparing (3.20) and (3.21), one obtains

\[
- (s - 1) \sum_{|J|=s-1} \sum_{\tau=1}^{n} (a_{\tau \mu} \circ \iota)(f_J \circ \iota)t^J t_\tau = \sum_{|J|=s} \left( \overline{Y}_\mu(f_J \circ \iota) \right.
\]

\[
+ \sum_{\tau=1}^{n} (a_{\tau \mu} \circ \iota) \left( \sum_{k} i_k (f_J \circ \iota) \Gamma_k^r \right. + \sum_{k} (i_k + 1) \sum_{q \neq k} \left( f_{i_1 \ldots i_k + 1 \ldots i_q - 1 \ldots i_N} \circ \iota \Gamma_k^r \right) \left. \right) t^J t_\tau.
\]

Therefore, we obtain

\[
\overline{Y}_\mu F_m = m \sum_{|I|=m} \sum_{\tau=1}^{n} (a_{\tau \mu} \circ \iota)(f_I \circ \iota)(T_I(\zeta)w)^J (T_I(\zeta)w)_\tau.
\]

If \( f_1 \) and \( f_2 \) are monomials in \( \iota \) with \( f_1 \neq c f_2 \) for any \( c \in \mathbb{R} \), we have \( \int_{\mathbb{B}^n} f_1 f_2 (1 - |t|^2)^{\alpha} d\lambda_t = 0 \). Hence, one obtains

\[
\|
\bar{\partial}_s F_m \|^2 \leq m^2 \sum_{\tau=1}^{n} \left\| \sum_{|I|=m} \sum_{\tau=1}^{n} (a_{\tau \mu} \circ \iota)(f_I \circ \iota)(\zeta)(T_I(\zeta)w)^J (T_I(\zeta)w)_\tau \right\|^2
\]

\[
\lesssim m^2 \sum_{|I|=m} \left( \sum_{\tau=1}^{n} |t|^J t_\tau|^2 (1 - |t|^2)^{\alpha} d\lambda_t \right)
\]

\[
\lesssim m^2 \sum_{|I|=m} \left\| \varphi_I \right\|^2 \frac{m! \Gamma(N + 2) \pi^N (i_1 + \cdots + i_n + n)}{N! \Gamma(N + m + 3)}
\]

\[
\lesssim m^2 \sum_{|I|=m} \left\| \varphi_I \right\|^2 \frac{m! \Gamma(N + 2) \pi^N (m + n)}{N! \Gamma(N + m + 3)}
\]

\[
\lesssim \sum_{\ell=0}^{m_0} \left( \frac{1}{(N/m + 1 + 2/m) \left( \frac{N}{m} + 1 + \frac{1}{m} \right) (m + N)!} \right) \left\| \varphi^\ell_m \right\|^2
\]

for \( m \geq m_0 \) by using Lemma 3.4.
Note that
\[
\frac{1}{(\frac{N}{m} + 1 + \frac{2}{m}) (\frac{N}{m} + 1 + \frac{1}{m})} \frac{m!(m+n)}{(m+N)!} = O(m^{-(N-1)}).
\]

Moreover,
\[
\frac{(m_0 + \ell - 1)![(m_0 + (m-m_0) - 1)]^2}{(m_0 - 1)!^2(m-m_0)!(m_0 + \ell + (m-m_0) - 1)!} = O(m^{(n-1)-(m_0-\ell)})
\]
and
\[
\prod_{j=1}^{m-m_0} \frac{(\ell+j)+(n-1)}{m_0+j} = \frac{m_0!(\ell + (m - m_0) + (n - 1))!}{(\ell + (n-1))(m_0 + (m-m_0))!} = O(m^{(n-1)-(m_0-\ell)})
\]
by Lemma 3.3 and Stirling’s formula. Hence \(\|\tilde{\partial}F_m\|_1^2 = O(m^{n-N-1}) \to 0\) as \(m \to \infty\). Therefore by the distribution theory, we conclude that \(f\) is holomorphic. \(\Box\)

Let \(f\) be a holomorphic function on \(\Omega = M \times_\rho B^N\). Using \(i\), we may regard \(M \times_\rho B^N\) as a quotient of \(i(\tilde{M}) \times B^N\) under the diagonal action of \(\rho(\Gamma)\) and it becomes a complex submanifold of \(B^N \times B^N \rho(\Gamma)\) which is a quotient of \(B^N \times B^N\) under the same action. So we may identify \(f \in \mathcal{O}(\Omega)\) with \(f \in \mathcal{O}(i(\tilde{M}) \times B^N)\) which satisfies \(\tilde{f}(i(\xi), w) = f((\rho \circ \gamma)i(\xi)), (\rho \circ \gamma)(w))\) for any \(\gamma \in \Gamma\).

Let \((z, w) \in i(\tilde{M}) \times B^N \subseteq B^N \times B^N\). Since \(\tilde{f}(z, t) := f(z, T_z t) = f(z, w)\) is holomorphic for \(t = Tzw\), we may express \(\tilde{f}\) by
\[
\tilde{f}(z, t) = \sum_{|I| = 0}^\infty f_I(z) t^I, \quad \text{where} \quad f_I(z) = \frac{1}{I!} \frac{\partial^{|I|}}{\partial t^{|I|}} \tilde{f}(z, 0) \in C^\infty(i(\tilde{M}))
\]
Hence
\[
f(i(\xi), w) = \sum_{|I| = 0}^\infty f_I(i(\xi))(T_{i(\xi)}w)^I
\]
on \(\tilde{M} \times B^N\). We associate \(\sum_{|I| = 0}^\infty f_I(i(\xi))(T_{i(\xi)}w)^I\) to a set of sections \(\{\varphi_m\}\) with \(\varphi_m \in C^\infty(M, i^*(S^m T^*_\Sigma))\) which is defined by
\[
\varphi_m := \sum_{|I| = m} f_I(i(\xi)) e^I|_{i(\xi)}
\]
where \(e^I = e_1^{i_1} \cdots e_N^{i_N}\) and \(i_1 + \cdots + i_N = m\). We call \(\{\varphi_m\}\) the associated differential of \(f\) on \(M\). Note that by a similar argument of Lemma 3.7, we obtain
\[
\|f\|^2 = \frac{2^{N} \pi^N}{N!} \sum_{|I| = 0}^\infty \|\varphi_I\|^2 \frac{|I|!\Gamma(N + \alpha + 1)}{\Gamma(N + |I| + \alpha + 1)}
\]
(3.22)
The Hardy space $A^2_{-1}(\Omega)$ is defined by

$$A^2_{-1}(\Omega) := \{ f \in \mathcal{O}(\Omega) : \| f \|_{-1}^2 < \infty \}$$

where the norm $\| f \|_{-1}^2$ is given by

$$\| f \|_{-1}^2 := \frac{2^N \pi^N}{N!} \sum_{|I|=0}^{\infty} \| \varphi_{|I|} \|_2^2 \frac{|I|! \Gamma(N)}{\Gamma(N + |I|)}.$$  \hspace{1cm} (3.23)

with the associated differential $\{ \varphi_{|I|} \}$ of $f$.

**Lemma 3.11** If $N > n$, then for any $\alpha > -1$, $A^2_{-1}(\Omega) \subset A^2_\alpha(\Omega)$.

**Proof** For any $|I| \geq 1$ the inequality

$$\frac{|I|! \Gamma(N + \alpha + 1)}{\Gamma(N + |I| + \alpha + 1)} < \frac{|I|! \Gamma(N)}{\Gamma(N + m)}$$

is equivalent to

$$\frac{(\alpha + 1)_N}{(\alpha + 1)_{N+|I|}} < \frac{(N - 1)!}{(N + |I| - 1)!}$$  \hspace{1cm} (3.24)

and (3.24) holds whenever $\alpha > -1$. Therefore, the lemma follows from (3.22), (3.23) and the comparison test. \hfill \Box

Now we define a linear map

$$\Phi : \bigoplus_{m=0}^{\infty} H^0(M, \iota^*(S^m T_{\Sigma}^*)) \to \mathcal{O}(\Omega),$$

For a constant function $\psi \in H^0(M, \iota^*(S^0 T_{\Sigma}^*))$, identifying $S^0 T_{\Sigma}^*$ with the trivial line bundle $\Sigma \times \mathbb{C}$, we associate $\psi$ to the constant function $\Phi(\psi)$ of the same constant value. For a non-zero $\psi \in H^0(M, \iota^*(S^m T_{\Sigma}^*)) \cong \bigoplus_{\ell=0}^{m} H^0(M, \iota^*(S^\ell T_{\Sigma}^* \otimes S^{m-\ell} N^*))$, we consider sequences $\{ \varphi_0^\ell \}, \ldots, \{ \varphi_m^\ell \}$ for $\psi$ described in (3.4) and Lemma 3.3 and define $\Phi(\psi)$ by the formal sum $f$ given by (3.10), (3.11), and (3.12). Then by (3.22) and Lemma 3.8, the image of $\Phi$ is contained in $A^2_\alpha(\Omega)$ for any $\alpha > -1$. If $n \neq N$, then $\Phi(\psi)$ belongs to $A^2_{-1}(\Omega)$.

**Lemma 3.12** Let $f$ be a holomorphic function on $\Omega$. Then the associated differential $\{ \varphi_m \}$ of $f$ on $M$ satisfies

$$\bar{\partial}_M \varphi_{m} = -(m - 1) R_G \varphi_{m-1}.$$
Proof Take a point \( q \in \tilde{M} \) and small open set \( q \in U \subset \tilde{M} \) such that \( U \cong i(U) \subset i(\tilde{M}) \). Consider a local coordinate system \((z_1, \ldots, z_N)\) at \( p := i(q) \in i(U) \) such that

\[ i(U) = \{(z_1, \ldots, z_N) : z_{n+1} = \cdots = z_N = 0\} \quad \text{near} \quad p = (0, \ldots, 0).\]

Since the holomorphicity of \( f \) on \( \Omega \) implies

\[ 0 = \frac{\partial}{\partial \xi_j} f(i(\xi), w) = \sum_{\mu=1}^n \frac{\partial f}{\partial \bar{z}_\mu} \frac{\partial \bar{t}_\mu}{\partial \xi_j} + \sum_{k=1}^N \sum_{\mu=1}^n \frac{\partial f}{\partial t_k} \frac{\partial (T_z w)_k}{\partial \bar{z}_\mu} \bigg|_{z=i(\xi)} \frac{\partial \bar{t}_\mu}{\partial \xi_j} \text{ for } j = 1, \ldots, n. \tag{3.25} \]

Therefore

\[ 0 = \frac{1}{I!} \frac{\partial^{|I|}}{\partial t^I} \bigg|_{t=0} \left( \sum_{\mu=1}^n \frac{\partial f}{\partial \bar{z}_\mu} \frac{\partial \bar{t}_\mu}{\partial \xi_j} + \sum_{k=1}^N \sum_{\mu=1}^n \frac{\partial f}{\partial t_k} \frac{\partial (T_z w)_k}{\partial \bar{z}_\mu} \bigg|_{z=i(\xi)} \frac{\partial \bar{t}_\mu}{\partial \xi_j} \right). \]

By a similar computation to Proposition 4.9 in [9] with Lemma 3.2,

\[ 0 = Y_q(f_{i_1 \cdots i_N} \circ i) + \sum_{l=1}^N (a_{i_l q} \circ i) \left( \sum_{k=1}^N \binom{N}{k} \Gamma_{k}^{l_k} + \sum_{\tau \neq k} (i_k + 1) (f_{i_1 \cdots i_k+1 \cdots i_{n-l-1} \cdots i_N} \circ i) \Gamma_{\tau}^{l_k} \right) + (|I|-1) (f_{i_1 \cdots i_{n-l-1} \cdots i_N} \circ i). \tag{3.26} \]

If we express \( \varphi_l = \sum_{|I|=l} f_l(i(\xi)) e^I |_{i(\xi)} \), we have

\[ \bar{\partial} \varphi_{i_1 \cdots i_N} = \sum_{q=1}^n \left( Y_q(f_{i_1 \cdots i_N} \circ i) + \sum_{k,l=1}^N (a_{i_l q} \circ i) i_k (f_{i_1 \cdots i_N} \circ i) \Gamma_{k}^{l_k} \right) e_1^{i_1} \cdots e_N^{i_N} \otimes \bar{e}_q \]

\[ + \sum_{q=1}^n \left( \sum_{k,l=1}^N a_{i_l q} i_k (f_{i_1 \cdots i_N} \circ i) \Gamma_{k}^{l_k} \right) e_1^{i_1} \cdots e_{l_k-1}^{i_{l_k}} \cdots e_{l_{\tau}+1}^{i_{l_{\tau}}} \cdots e_N^{i_N} \otimes \bar{e}_q \]

Therefore, the lemma follows by (3.26). \( \square \)
Remark 3.13  By Lemma 3.12, for any \( f \in \mathcal{O}(\Omega) \) which vanishes up to \( m \)-th order with nonvanishing \((m+1)\)-th order on \( D := \{ ((\zeta, t(\zeta)) \} \in \Omega : \zeta \in \hat{M} \} \), there exists a non-zero holomorphic section \( \varphi_{m+1} \) of \( t^*(S^{m+1}T^*_{\Sigma}) \) associated to \( f \).

Proposition 3.14  The linear map

\[
\Phi : \bigoplus_{m=0}^{\infty} H^0(M, t^*(S^mT^*_{\Sigma})) \to \mathcal{O}(\Omega)
\]

has a dense image in \( \mathcal{O}(\Omega) \) equipped with the compact open topology.

Let

\[
\Omega_\epsilon := \{ ((\zeta, w)) \in \Omega : |T_{t(\zeta)}w| < \epsilon \}
\]

with \( 0 < \epsilon < 1 \). These \( \Omega_\epsilon \) exhausts \( \Omega \). Define

\[
L^2(\Omega_\epsilon) := \{ f : f \text{ is measurable function on } \Omega_\epsilon \text{ such that } \| f \|^2_{0,\epsilon} := \langle \langle f, f \rangle \rangle^2_{0,\epsilon} < \infty \},
\]

where

\[
\langle \langle f, g \rangle \rangle^2_{0,\epsilon} := \int_{\Omega_\epsilon} f \overline{g} \delta^{N+1}dV_\omega.
\]

The Bergman space \( A^2(\Omega_\epsilon) \) is given by \( L^2(\Omega_\epsilon) \cap \mathcal{O}(\Omega_\epsilon) \).

Proof  Since the proof of the proposition is similar to those in [2] and [9], we will only give a sketch of it. By the Cauchy estimate, it suffices to show that the image of \( \Phi \) is dense in \( A^2(\Omega_\epsilon) \) for any \( 0 < \epsilon < 1 \). For a contradiction, suppose that there exists a non-zero holomorphic function \( f \in A^2(\Omega_\epsilon) \) which is orthogonal to the image of \( \Phi \) in \( A^2(\Omega_\epsilon) \). Then, for the associated differential \( \{ \varphi_k \} \) of \( f \) on \( M \), there exists \( m_0 \in \mathbb{N} \) such that \( \varphi_k = 0 \) for any \( k < m_0 \), but \( \varphi_{m_0} \neq 0 \). Since \( \varphi_{m_0} \neq 0 \), there exists an \( \ell \), \( 0 \leq \ell \leq m_0 \) such that \( \varphi_{m_0}^\ell \neq 0 \) and \( H^0(M, S^{\ell\ell}T^*_M \otimes S^{m_0-\ell}N^*_\Sigma) \).

Now we define orthogonal projections:

\[
\Pi_{m_0+m, E_{m_0+m}}^{\ell, \ell} : L^2(M, S^{\ell\ell}T^*_M \otimes S^{m_0-\ell}N^*_\Sigma \otimes \Lambda^{0,\ell}T^*_M) \to \ker(\square_{m_0+m,M}^{\ell,\ell} - E_{m_0+m}^{\ell,\ell})
\]

for \( i = 0, 1 \) where \( E_{m_0,m} \) is given by (3.9). Let \( \{ \tilde{\varphi}_k^{\ell} \} \) be the sequence satisfying (3.4) with respect to the symmetric differential \( \varphi_{m_0}^{\ell} \). Since \( \langle \varphi_{m_0+m}, \tilde{\varphi}_{m_0+m}^{\ell} \rangle = \langle \varphi_{m_0}^{\ell}, \tilde{\varphi}_{m_0+m}^{\ell} \rangle \) holds for each \( m \) and \( \ell \), if we prove that the sequence \( \{ \tilde{\varphi}_k^{\ell} \} \) equals to \( \{ \Pi_{m_0+m, E_{m_0+m}}^{\ell,\ell} \varphi_{m_0+m} \}_{m=0}^{\infty} \), then it gives a contradiction. For this, we will use induction.

Suppose that this claim is true for any \( m \leq k - 1 \). Since \( f - \Phi(f) \) is also holomorphic, by Lemma 3.12 we have

\[
\overline{\partial}_M (\varphi_{m_0+k}^{\ell} - \tilde{\varphi}_{m_0+k}^{\ell}) = -(m_0 + k - 1) \nabla G (\varphi_{m_0+k-1}^{\ell} - \tilde{\varphi}_{m_0+k-1}^{\ell}).
\]
If we prove
\[
\Pi^{1,\ell}_{m_0+k,E_{m_0,k}} \partial (\varphi^{\ell}_{m_0+k} - \tilde{\varphi}^{\ell}_{m_0+k}) = \partial (\Pi^{0,\ell}_{m_0+k,E_{m_0,k}} (\varphi^{\ell}_{m_0+k} - \tilde{\varphi}^{\ell}_{m_0+k}))
\]
(3.27)
and
\[
\Pi^{1,\ell}_{m_0+k,E_{m_0,k}} \mathcal{R}_G (\varphi^{\ell}_{m_0+k-1} - \tilde{\varphi}^{\ell}_{m_0+k-1}) = \mathcal{R}_G (\Pi^{0,\ell}_{m_0+k-1,E_{m_0,k-1}} (\varphi^{\ell}_{m_0+k-1} - \tilde{\varphi}^{\ell}_{m_0+k-1})),
\]
(3.28)
then by \( \ker \partial \perp \ker (\Pi^{0,\ell}_{m_0+k,E_{m_0,k}} I) \), it follows that \( \tilde{\varphi}^{\ell}_{m_0+k} = \Pi^{0,\ell}_{m_0+k,E_{m_0,k}} (\varphi^{\ell}_{m_0+k}) \) and therefore the claim is proved. Since (3.27) follows by a straightforward computation and (3.28) follows by Corollary 2.3 and the assumption, the proof is completed.

\[\square\]

**Proof of Theorem 1.1** To show that \( \Phi \) is injective, since \( \Phi(\mathcal{H}^0(M, S^{\ell} T^*_M \otimes S^{m-\ell} N^*)) \) are orthogonal to each other if \( m \) or \( \ell \) is different by Lemma 3.3, we only need to consider when \( \psi_1, \psi_2 \) belong to \( \mathcal{H}^0(M, i^*(S^m T^*_\Sigma)) \) such that \( \psi_1 \neq \psi_2 \). However, in this case, \( \Phi(\psi_1) \) and \( \Phi(\psi_2) \) are different by the construction (3.12). By Proposition 3.14 and Lemma 3.10, the proof is completed.

\[\square\]

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**Declarations**

**Conflict of interest** The authors declare that they have no conflict of interest.

**References**

1. Adachi, M.: On a hyperconvex manifold without non-constant bounded holomorphic functions. In: Byun, J., Cho, H. (eds.) Geometric Complex Analysis. Springer Proceedings in Mathematics & Statistics, vol. 246, pp. 1–10. Springer, Singapore (2018)
2. Adachi, M.: On weighted Bergman spaces of a domain with Levi-flat boundary. Trans. Am. Math. Soc. 374(10), 7499–7524 (2021)
3. Bland, J.S.: On the existence of bounded holomorphic functions on complete Kähler manifolds. Invent. Math. 81(3), 555–566 (1985)
4. Bertin, J., Demailly, J.-P ., Illusie, L., Peters, C.: Introduction to Hodge theory. Translated from the 1996 French original by James Lewis and Peters. SMF/AMS Texts and Monographs, vol. 8, p. 232. American Mathematical Society, Providence, RI; Société Mathématique de France, Paris (2002)
5. Corlette, K.: Flat G-bundles with canonical metrics. J. Differ. Geom. 28(3), 361–382 (1988)
6. Deng, F.: Fornæss, John Erik Flat bundles over some compact complex manifolds. J. Geom. Anal. 30(4), 3484–3497 (2020)
7. Diederich, K.: Ohsawa, Takeo Harmonic mappings and disc bundles over compact Kähler manifolds. Publ. Res. Inst. Math. Sci. 21(4), 819–833 (1985)
8. Greene, R.E., Wu, H.: Function Theory on Manifolds Which Possess a Pole. Lecture Notes in Mathematics, vol. 699. Springer, Berlin (1979)
9. Lee, S., Seo, A.: Symmetric differentials and Jets extension of $L^2$ holomorphic functions, to appear in Indiana University Mathematics Journal
10. Seo, A.: Weakly 1-completeness of holomorphic fiber bundles over compact Kähler manifolds. J. Lond. Math. Soc. (2) **106**(3), 2305–2341 (2022)
11. Siu, Y.T., Yau, S.T.: Complete Kähler manifolds with nonpositive curvature of faster than quadratic decay. Ann. Math. (2) **105**(2), 225–264 (1977)
12. Zhao, R., Zhu, K.: Theory of Bergman spaces in the unit ball of $\mathbb{C}^n$. Mém. Soc. Math. Fr. (N.S.) **115**(2008), 103 (2009)
13. Zhu, K.: Spaces of Holomorphic Functions in the Unit Ball. Graduate Texts in Mathematics, vol. 226, p. 271. Springer, New York (2005)

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