We discuss some properties of higher derivative (HD) bulk gravity without Riemann tensor square term. Such a kind of gravity admits Schwarzschild Anti de Sitter (SAdS) black hole as exact solution. It is shown that induced brane geometry on such background is Friedmann-Robertson-Walker (FRW) radiation dominated Universe. We show that HD terms contributions appear in the Hawking temperature, entropy and Hubble parameter via the redefinition of 5-dimensional gravitational constant and AdS scale parameter. These HD terms do not destroy the AdS-dual description of radiation represented by strongly-coupled CFT. So-called Cardy-Verlinde formula which expresses cosmological entropy as square root from other parameters and entropies is also derived in $R^2$-gravity. This talk is based on works with Shin’ichi Nojiri and Sergei D. Odintsov.

1 Introduction

It has been realized recently that brane equations of motion are exactly FRW equations with radiation matter which plays the role of CFT in AdS/CFT correspondence. Furthermore, FRW equation can be rewritten in the form of so-called Cardy-Verlinde formula relating cosmological entropy with the one of CFT which exists on the boundary of AdS.

We will be interested in the further study of the CFT dominated Universe as the brane in the background of HD gravity which is known to possess AdS Black Hole (BH) solution. The interest in the HD bulk gravity is caused by the following: First of all, any effective stringy gravity includes HD terms of different order. Second, from the point of view of AdS/CFT correspondence the $R^2$-terms give next-to-leading terms in large $N$ expansion as it was directly checked in the calculation of holographic conformal anomaly from bulk $R^2$-gravity. Third, HD gravity may serve as quite good candidate for the construction of realistic brane-world cosmologies.

2 AdS Black holes in bulk $R^2$-gravity, Surface terms

Let us consider thermodynamics of AdS BH in bulk $R^2$-gravity. The calculation of thermodynamical quantities like mass and entropy will be necessary to relate them with the corresponding ones in brane FRW Universe. The general
action of $d+1$-dimensional $R^2$-gravity is given by
\[ I = \int d^{d+1}x \sqrt{-\hat{G}} \left\{ a\hat{R}^2 + b\hat{R}_{\mu\nu}\hat{R}^{\mu\nu} + \epsilon\hat{R}_{\mu\nu\xi\sigma}\hat{R}^{\mu\nu\xi\sigma} + \frac{1}{\kappa^2} \hat{R} - \Lambda \right\} \, . \quad (1) \]

When $\epsilon = 0$ Schwarzschild-anti de Sitter space is an exact solution:
\[ ds^2 = \hat{G}_{\mu\nu}dx^\mu dx^\nu = -e^{2\rho}dt^2 + e^{-2\rho}dr^2 + r^2 \sum_{i,j} g_{ij} dx^i dx^j \, , \]
\[ e^{2\rho} = \frac{1}{r^{d-2}} \left( -\mu + \frac{k_{r}^{d-2}}{d-2} + \frac{r^d}{l^2} \right) \, . \quad (2) \]

The curvatures have the form: $\hat{R} = -\frac{d(d+1)}{l^2} \, , \, \hat{R}_{\mu\nu} = -\frac{d}{r^2} \hat{G}_{\mu\nu} \, . \, \text{In (2), } \mu \text{ is the parameter corresponding to mass and the scale parameter } l \text{ is given by solving the equation of motion of } I \text{ with respect to } \hat{G}_{\mu\nu}:
\[ 0 = \frac{d^2(d+1)(d-3)a}{l^4} + \frac{d^2(d-3)b}{l^4} - \frac{d(d-1)}{\kappa^2 l^2} - \Lambda \, . \quad (3) \]

We also assume $g_{ij}$ expresses the Einstein manifold, defined by $r_{ij} = kg_{ij}$, where $r_{ij}$ is the Ricci tensor given by $g_{ij}$ and $k$ is a constant.

The calculation of thermodynamical quantities like free energy $F$, the entropy $S$ and the energy $E$ may be done with the help of the following method. After Wick-rotating the time variable by $t \rightarrow i\tau$, we assume $\tau$ has a period of $\frac{1}{T}$. The free energy $F$ can be obtained from the action $I$ in (1) where the classical solution is substituted because $F$ is related with $I$ as $F = -TI$. Substituting Eqs.(3) into (1) in the case of $d=4$ with $\epsilon=0$, we obtain free energy $F$. By using $F$, the entropy $S$ and energy $E$ are given by
\[ S = -\frac{dF}{dT_H} = \frac{dF}{dH} \frac{dH}{dT_H} = \frac{V_3 \pi r_H^3}{2} \left( \frac{8}{\kappa^2} - \frac{320a}{l^2} - \frac{64b}{l^2} \right) \, , \quad (4) \]
\[ E = F + TS = \frac{3V_3 \mu}{8} \left( \frac{8}{\kappa^2} - \frac{320a}{l^2} - \frac{64b}{l^2} \right) \, . \quad (5) \]

Here $V_3$ is the volume of 3-dimensional sphere, $r_H$ is the horizon radius given by solving the equation $e^{2\rho(r_H)} = 0$, and $T_H$ is the Hawking temperature given by $T_H = \frac{1}{2\pi} \frac{d}{r} |_{r=r_H} e^{2\rho}$. The above equations reproduce the standard Einstein theory results when $a = b = 0$. Note that one can consider the limit of $l \rightarrow 0$, where the background spacetime becomes flat Minkowski space.

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$^a$ For non-zero $\epsilon$ such S-AdS BH solution may be constructed perturbatively.

$^b$ For example, if $k > 0$ the boundary can be 4-dimensional de Sitter space (sphere when Wick-rotated), if $k < 0$, anti-de Sitter space or hyperboloid, or if $k = 0$, flat space. By properly normalizing the coordinates, one can choose $k = 2, 0$, or $-2$. 
Since the scalar curvature and Ricci tensor vanishes in the flat Minkowski, we cannot derive the thermodynamical quantities by evaluating the action $I$, which vanishes, if we start with the flat Minkowski background from the beginning. Then finite $l$ would give a kind of the regularization.

Before considering the dynamics of the brane, we review the problem of the variational principle in the Einstein gravity, whose action is given by $I_E = \frac{1}{\kappa^2} \int d^{d+1}x \sqrt{-G} R$. The scalar curvature contains the second order derivative of the metric tensor $\hat{G}_{\mu\nu}$ with respect to the coordinates. Therefore if there is a boundary, which we denote by $B$, under the variation $\delta \hat{G}_{\mu\nu}$, $\delta S_E$ contains, on the boundary, the derivative of $\delta \hat{G}_{\mu\nu}$ with respect to the coordinate perpendicular to the boundary, which makes the variational principle ill-defined. Therefore we need to add a surface term to the action, which is called the Gibbons-Hawking surface term $I_{GH} = 2 \eta \int_B d^{d+1}x \sqrt{-\hat{g}} \nabla_{\mu} n^{\mu}$, where $n^{\mu}$ is the unit vector perpendicular to the boundary and $\hat{g}_{mn}$ is the boundary metric induced from $\hat{G}_{\mu\nu}$. For the $R^2$-gravity, it is presumably also possible to assumes the surface terms in the following forms:

$$I_{b}^{(1)} = \frac{2}{\kappa^2} \int d^d x \sqrt{\hat{g}} \nabla_{\mu} n^{\mu}, \quad I_{b}^{(2)} = -\eta \int d^d x \sqrt{\hat{g}} .$$

The parameter $\eta$ (brane tension) which is usually free parameter in brane-world cosmology is not free any more and can be determined by the condition that the leading divergence of bulk AdS should vanish when one substitutes the classical solution $(\hat{\zeta}, q)$ into the action $(1)$ with $\epsilon = 0$ and into $I_{b}^{(1)} + I_{b}^{(2)}$. Then we obtains $\eta$. For later convenience, we choose the metric:

$$ds^2 = dq^2 - e^{\zeta(q,\tau)} d\tau^2 + e^{\xi(q,\tau)} g_{ij} dx^i dx^j .$$

The variation of the actions leads two kinds of equations of motion. First equations of motion is derived from the condition that the coefficients in front of $\delta \zeta_{,q}$ and $\delta \xi_{,q}$ vanish.

$$\frac{1}{\kappa^2} = \frac{1}{\kappa^2} - \frac{2d(d+1)a}{l^2} - \frac{2db}{l^2} ,$$

And second equations of motion which is the original one is obtained by the condition that the coefficients in front of $\delta \zeta$ and $\delta \xi$ vanish: $(\zeta_{,q} + (d-1)\xi_{,q})|_{q=0} = (d-1)\xi_{,q}|_{q=0} = \kappa^2 \eta = \frac{2(d-1)}{l^2}$. We regard this equation is related to the dynamics of brane. Especially when $e^{\xi} = e^{\zeta} = l^2 e^{\frac{a}{l}}$, we obtains the relation: $A_{,q}|_{q=0} = \frac{A}{l}$. When $d = 4$, by using Eq.(8), the entropy (4) and the energy (5) are rewritten by $\kappa$ as follows:

$$S = \frac{4V_3 \pi r_H^3}{\kappa^2} , \quad E = \frac{3V_3 \mu}{\kappa^2} .$$
Therefore the corrections from the HD terms appear through the redefinition of gravitational coupling $\kappa$ to $\tilde{\kappa}$ through (3) and $l$ given by (3).

3 The FRW equations and Cardy-Verlinde formula

Let us rewrite the metric (2) of SAdS in a form of (7) with $e^\xi = e^{\zeta} = l^2 e^{2A}$. It is possible to change the coordinates from $(t, r)$ to $(q, \tau)$ as the metric on the brane takes FRW form:

$$ds^2_{\text{brane}} = -d\tilde{t}^2 + l^2 e^{2A} \sum_{i,j=1}^{d-1} g_{ij} dx^i dx^j,$$

where we choose $r = le^A$ and $\tilde{t}$ which is defined by $d\tilde{t} = le^A d\tau$. By using the relation between $(t, r)$ and $(q, \tau)$ to form FRW type metric, we have the square of the Hubble constant $H$ which is defined by $H = \frac{dA}{d\tilde{t}}$ and we can rewrite $H^2$ in the form of the FRW equation by using (2) and the relation $A_q |_{q=0} = \frac{1}{l}$:

$$H^2 = l^2 q^2 \frac{e^{-2A}}{l^2} = -\frac{k}{(d-2)r^2} + \frac{\kappa_d^2}{(d-1)(d-2)} \frac{\hat{E}}{V},$$

(10)

where $\hat{E} = \frac{(d-1)(d-2)\mu V_{d-1}}{\kappa_d^2}$ and $V = r^{d-1} V_{d-1}$; $V_{d-1}$ is the volume of the $(d - 1)$-dimensional sphere with a unit radius and $\kappa_d$ is the $d$-dimensional gravitational coupling, which is given by $\frac{\kappa_d^2}{2} = \frac{2\kappa^2}{d}$. By differentiating Eq.(10) with respect to $\tilde{t}$, since $H = \frac{\dot{a}}{a^2}$, we obtain the second FRW equation. From this equation, we can read the pressure of the matter on the brane as $p = \frac{(d-2)\mu}{r^2 \kappa_d^2}$. Thus we find the relation $0 = -\frac{\hat{E}}{V} + (d-1)p$, which tells that the trace of the energy-stress tensor coming from the matter on the brane vanishes. Therefore the matter on the brane can be regarded as the radiation, i.e., the massless fields. In other words, field theory on the brane should be conformal one as in case of Einstein brane. When $d = 4$, we find $\hat{E} = \frac{1}{2} E$. It should be noted that when $r$ is large, the metric (3) becomes the CFT metric which tells that the CFT time $\tilde{t}$ is equal to the AdS time $t$ times the factor $\frac{1}{l}$, that is $t_{\text{CFT}} = \frac{1}{l} t$. Therefore the relation between $E$ and $\hat{E}$ expresses that $\hat{E}$ is the energy in CFT. Assuming AdS/CFT correspondence, the HD terms in (3) correspond to the $1/N$ corrections in the large $N$ limit of some gauge theory, which could be a CFT on the brane. For $\kappa^2 a, \kappa^2 b \ll 1$, we can rewrite $\hat{E}$ using (3) as $\hat{E} \sim 2(d-1)(d-2) \kappa^2 \mu V_{d-1} \left( 1 + \frac{2(d+1)\kappa^2 a}{2d \kappa^2 b} + \frac{2db \kappa^2}{2d \kappa^2 b} \right)$. Then the parameters $a$ and $b$ could express the $1/N$ correction of the next-to-leading order of $1/N$ expansion.

Recently the FRW equation in $d$-dimension can be regarded as a $d$-
dimensional analogue of the Cardy formula of 2-dimensional CFT:

\[ \tilde{S} = 2\pi \sqrt{\frac{c}{6} \left( L_0 - \frac{k}{d - 2} \frac{c^2}{24} \right)} . \quad (11) \]

In the present case, 4-dimensional entropy \( S \) is obtained by identifying
\( \frac{2\pi}{\kappa} \tilde{E} r \Rightarrow 2\pi L_0 \), \( \frac{2\pi H V}{\kappa^2} \Rightarrow \frac{c}{24} \), \( \frac{8\pi H V}{\kappa^4} \Rightarrow \tilde{S} \). Since the FRW-like equation
\[ (10) \] has the form \( (11) \). Then one can evaluate holographic entropy \( \tilde{S} \) when the brane crosses the horizon \( r = r_H \). When \( r = r_H \), Eq.\( (10) \) tells that \( H = \pm \frac{1}{l} \) where the plus sign corresponds to the expanding brane universe and the minus sign to the contracting one. Taking the expanding case, we find \( \tilde{S} = 4\pi r_H^3 V_3 / \kappa^2 \). Thus we realize that the entropy \( \tilde{S} \) is identical with \( S \) in \( (4) \), which is nothing but the black hole entropy. We now stress again that compared with the Einstein gravity case, the corrections from the HD terms always appear through the redefinition of gravitational coupling \( \kappa \) to \( \tilde{\kappa} \) via \( (8) \) and when the length scale \( l \) is given by \( (3) \). From the viewpoint of AdS/CFT correspondence, the HD terms correspond to the \( 1/N \) corrections in the large \( N \) limit of some gauge theory, which could be a CFT on the brane. Then \( \tilde{E} = \frac{1}{\kappa} E \) and \( \tilde{S} = S \) would tell that AdS/CFT correspondence could be valid in the next-to-leading order of the \( 1/N \).

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References

1. S. Nojiri, S.D. Odintsov and S. Ogushi, [hep-th/0108172, hep-th/0105117].
2. I. Savonije and E. Verlinde, Phys. Lett. B507 (2001) 305.
3. J. Maldacena, Adv. Theor. Math. Phys. 2(1998) 231.
4. E. Verlinde, [hep-th/0008140].
5. O. Aharony, A. Fayyazuddin and J. Maldacena, JHEP 9807(1998)013.
6. S. Nojiri and S.D. Odintsov, Int.J.Mod.Phys. Lett. A15(2000)413;
   Further references are contained therein.
7. S. Nojiri, S.D. Odintsov and S. Ogushi, Prog.Theor.Phys. 105(2001) 869;
   Further references are contained therein.
8. S. Nojiri and S.D. Odintsov, Phys. Lett. B493 (2000)153.
9. G. Gibbons and S.W. Hawking, Phys. Rev.D15 (1977)2752.