Multi-parameter estimates via operator-valued shifts

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Abstract

We prove the mixed-norm $L^p L^q$-boundedness of a general class of singular integral operators having a multi-parameter singularity and acting on vector-valued (UMD Banach lattice-valued) functions. Moreover, families of such operators with uniform assumptions are shown to be not only uniformly bounded but $\mathcal{R}$-bounded, a genuinely stronger property that is often needed in applications. Previous results of this nature only dealt with convolution-type or slightly more general paraproduct-free singular integrals. In contrast, our analysis specifically targets the array of different partial paraproducts that arise in the multi-parameter setting by interpreting them as paraproduct-valued one-parameter operators. This new point-of-view provides a conceptual simplification over the existing representation results for multi-parameter operators, which is a key to the proof of the boundedness of these operators.

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1. Introduction

Singular integral operators lie at the intersection of deep questions connecting harmonic analysis, functional analysis, partial differential equations and geometry. Generically, these operators take the form

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y) f(y)\,dy,$$

where different assumptions on the kernel $K$ lead to important classes of linear transformations arising across pure and applied analysis. These operators can be classified, among other things, according to the algebraic form of the kernel $K$, and the size of its singularity.

Convolution kernels of the special algebraic form $K(x,y) = k(x-y)$ are conveniently studied via the Fourier transform $f \mapsto \hat{f}$ due to formulae like

$$\hat{T}f(\xi) = \hat{k} \ast f(\xi) = \hat{k}(\xi)\hat{f}(\xi),$$

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which transforms the convolution operator into a pointwise multiplier in the frequency variable \( \xi \). In particular, the \( L^2 \)-boundedness of \( T \) is then equivalent to checking that \( \hat{k} \in L^\infty \); for the prototypical Hilbert transform with \( K(x,y) = 1/(x-y) \), we have \( \hat{k}(\xi) = -i\pi \mathrm{sgn}(\xi) \). The question of boundedness of \( T \) with a generic kernel \( K(x,y) \) is more delicate, and is often best answered by so-called \( T1 \) theorems, where the action of the operator \( T \) on the constant function 1 plays a critical role. The proofs of these \( T1 \) theorems (by now several in the literature) display a fundamental structural decomposition of singular integrals into their cancellative parts (bearing some resemblance with the classical convolution operators) and so-called paraproducts. Generically, a paraproduct refers to an expression obtained by expanding both factors of the usual pointwise product \( b \cdot f \) in terms of a suitable resolution of the identity, and dropping some of the terms in the resulting double expansion. The most relevant for us is the dyadic paraproduct

\[
 f \mapsto \Pi_b f = \sum_{Q,R \in \mathcal{D}} \{b,h_Q\} h_Q(f \ast h_R) h_R = \sum_{Q \in \mathcal{D}} \{b,h_Q\} (f)_Q h_Q,
\]

where \( h_Q \) denotes an \( L^2 \)-normalised cancellative Haar function on the dyadic cube \( Q \in \mathcal{D} \) and \( (f)_Q \) is the average of \( f \) over \( Q \); the deviation from the usual product is due to the restriction \( '\ell(Q) < \ell(R)' \) in the double sum.

While the original \( T1 \) theorem of David and Journé [4] was still based on Fourier analysis, it is dyadic analysis (in one form or another) that has proved most flexible for its various extensions to Banach spaces [6], non-homogenous measures [20] and sharp weighted inequalities [11], to name but a few. In particular, the dyadic representation theorem of [11] (extending an earlier special case from [23]) provides the further useful decomposition of the cancellative part of a singular integral into so-called dyadic shifts, which are suitable generalisations of dyadic martingale transforms, also known as Haar multipliers

\[
 f = \sum_{Q \in \mathcal{D}} \langle f,h_Q \rangle h_Q \mapsto \sum_{Q \in \mathcal{D}} \lambda_Q \langle f,h_Q \rangle h_Q,
\]

where \( (\lambda_Q) \) is a bounded multiplying sequence.

Classical one-parameter kernels are ‘singular’ (involve ‘division by zero’) exactly when \( x = y \). In contrast, the multi-parameter theory is concerned with kernels whose singularity is spread over the union of all hyperplanes of the form \( x_i = y_i \), where \( x,y \in \mathbb{R}^d \) are written as \( x = (x_i)_{i=1}^d \in \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_t} \) for a fixed partition \( d = d_1 + \cdots + d_t \). The bi-parameter case \( d = d_1 + d_2 = n + m \) is already representative of many of the challenges arising in this context. The prototype example is \( 1/[(x_1 - y_1)(x_2 - y_2)] \), the product of Hilbert kernels in both coordinate directions of \( \mathbb{R}^2 \), but general two-parameter kernels are neither assumed to be of the product nor of the convolution form. A bi-parameter \( T1 \) theorem, a criterion for \( L^2 \)-boundedness in this context, was first achieved by Journé [15], and recovered by one of us [18] through a bi-parameter dyadic representation theorem. The multi-parameter extension of this is by Ou [21].

An inherent complication of these bi-parameter and multi-parameter representations is the presence not only of ‘pure’ paraproducts and cancellative shifts, but also of their hybrid combinations that are completely new compared to the one-parameter case. Nevertheless, the bi-parameter representation has proved extremely useful, for example, in connection with bi-parameter commutators and weighted analysis, see Holmes–Petermichl–Wick [9] and Ou–Petermichl–Strouse [22], and sparse domination, see Barron–Pipher [2].

The aim of this paper is to recast the bi-parameter representation theorem of [18] into a more conceptual form by reinterpreting the various bi-parameter hybrids of shifts and paraproducts as one-parameter shifts with operator-valued (indeed, paraproduct-valued) coefficients, for example, in the case of the Haar multiplier (1.3) acting on functions \( f \) on \( \mathbb{R}^n \times \mathbb{R}^m \), we could have ‘partial pairings’ \( \langle f,h_Q \rangle = \int_{\mathbb{R}^m} f(x_1,x_2) h_Q(x_1) \, dx_1 \) with Haar functions on \( \mathbb{R}^n \), and each
\( \lambda_Q \) would be a dyadic paraproduct of the form (1.2) acting in the variable \( x_2 \in \mathbb{R}^m \). Such operator-valued dyadic shifts were recently studied in [8] under abstract operator-theoretic assumptions; for the present purposes, we need to verify the compatibility of these abstract conditions with the concrete situation of paraproducts. We also further develop the theory of operator-valued shifts.

While the broad underlying philosophy of viewing bi-parameter operators as ‘operator-valued one-parameter operators’ goes back (at least) to Journé [15], its combination with the modern dyadic representation theory is new, and its usefulness is justified by the following corollary for bi-parameter singular integrals, which generalises the \( T_1 \) type corollary of the representation theorem of [18] in three ways: we can handle UMD lattice-valued functions (instead of scalar-valued ones), we get the mixed-norm \( L^q L^p \) boundedness for all \( p, q \in (1, \infty) \) instead of just \( L^2 \) boundedness, and we get \( \mathcal{R} \)-boundedness results for families of bi-parameter singular integrals. (The relevant notions are recalled in Section 2.)

**Theorem 1.4.** Let \( E \) be a UMD space with property \((\alpha)\), and \( p, q \in (1, \infty) \). Let \( T \) be a bi-parameter singular integral satisfying \( T_1 \) type assumptions as in [18]. Suppose, in addition, that at least one of the following conditions holds:

1. \( E \) is a function lattice, or
2. \( T \) is paraproduct free.

Then we have

\[
\|T\|_{L^q(\mathbb{R}^n; L^p(\mathbb{R}^m; E)) \to L^q(\mathbb{R}^n; L^p(\mathbb{R}^m; E))} < \infty.
\]

Moreover, if \( T \) is a family of operators satisfying such assumptions uniformly, then this family is \( \mathcal{R} \)-bounded on \( L^q(\mathbb{R}^n; L^p(\mathbb{R}^m; E)) \).

We recall that the UMD (unconditional martingale differences) property is a well-known necessary and sufficient condition for the boundedness of various singular integrals already in the one-parameter case on \( L^p(\mathbb{R}^m; E) \) (see [12, Section 5.2.c and the Notes to Section 5.2]). It is also known that the additional ‘property \((\alpha)\)’ is necessary and sufficient for various bi-parameter extensions [13, Section 8.3.e].

By condition (2), we understand that all (both partial and full) paraproducts in the dyadic representation of \( T \) are required to vanish, which could also be stated in terms of (both partial and full) ‘\( T_1 = 0 \)’ type conditions. This is in particular the case for convolution (or Fourier multiplier) operators. All previous related results on the boundedness of bi-parameter singular integrals on Banach space-valued \( L^p \)-spaces were restricted to case (2) above: convolution (or Fourier multiplier) operators were treated by [10, 14], and the general paraproduct-free case recently by Di Plinio and Ou [5]. Our key contribution is the treatment of the various types of paraproducts arising in the general bi-parameter theory. We only achieved this under the formally stronger (although for main concrete examples practically the same) assumption (1) on the Banach space \( E \). Nevertheless, by inspection, our proof also covers the previously known case (2), since the assumption that \( E \) is a function lattice is only used in the estimation of the paraproducts.

The \( \mathcal{R} \)-boundedness in Theorem 1.4 is important not only as a final result, but also as a tool within the proof, as it is needed as an input to apply the abstract results on operator-valued dyadic shifts from [8]; that is, the \( \mathcal{R} \)-boundedness of families of one-parameter operators is used to obtain the boundedness (both with or without \( \mathcal{R} \)) of the two-parameter operators. This idea can be lifted to an induction on the number of parameters, and we develop this idea in some detail in the last section of the paper. However, for clarity, three parameters is the highest degree of parameters that we tackle explicitly.
2. Definitions and preliminaries

2.1. Vinogradov notation

We denote $A \lesssim B$ if $A \leq CB$ for some absolute constant $C$. The constant $C$ can at least depend on the dimensions of the appearing Euclidean spaces, on integration exponents and on Banach space constants (UMD and Pisier’s $(\alpha)$). We denote $A \sim B$ if $B \lesssim A \lesssim B$.

2.2. Dyadic notation

If $Q$ is a cube,

- $\ell(Q)$ is the side-length of $Q$;
- $\text{ch}(Q)$ denotes the dyadic children of $Q$.
- If $Q$ is in a dyadic grid, then $Q(k)$ denotes the unique dyadic cube $S$ in the same grid so that $Q \subset S$ and $\ell(S) = 2^k\ell(Q)$.
- If $D$ is a dyadic grid, then $D_i = \{Q \in D : \ell(Q) = 2^{-i}\}$.

In this paper we denote a dyadic grid in $\mathbb{R}^n$ by $D^n$. We are at most working in the tri-parameter setting, and then we will have three dyadic grids $D^n$, $D^m$, $D^k$ in $\mathbb{R}^n$, $\mathbb{R}^m$ and $\mathbb{R}^k$, respectively. Using the above notation $D^n_i$ denotes those $I \in D^n$ for which $\ell(I) = 2^{-i}$. The measure of a cube $I$ is simply denoted by $|I|$ no matter in what dimension we are in.

When $I \in D^n$ we denote by $h_I$ a cancellative $L^2$ normalized Haar function. This means the following. Writing $I = I_1 \times \cdots \times I_n$ we can define the Haar function $h_I^n$, $\eta = (\eta_1, \ldots, \eta_n) \in \{0,1\}^n$, by setting

$$h_I^n = h_{I_1}^{\eta_1} \otimes \cdots \otimes h_{I_n}^{\eta_n},$$

where $h_{I_i}^0 = |I_i|^{-1/2}1_{I_i}$ and $h_{I_i}^1 = |I_i|^{-1/2}(1_{I_i,l} - 1_{I_i,r})$ for every $i = 1, \ldots, n$. Here, $I_{i,l}$ and $I_{i,r}$ are the left and right halves of the interval $I_i$, respectively. If $\eta \neq 0$ the Haar function is cancellative: $\int h_{I}^\eta = 0$. We usually suppress the presence of $\eta$ and simply write $h_I$ for some $h_I^n$, $\eta \neq 0$.

For $I \in D^n$ and a locally integrable function $f : \mathbb{R}^n \to E$, where $E$ is a Banach space, we define the martingale difference

$$\Delta_I f = \sum_{I' \in \text{ch}(I)} [\langle f \rangle_{I'} - \langle f \rangle_I] 1_{I'}.$$ 

Here, $\langle f \rangle_I = \frac{1}{|I|} \int_I f$ (where the integral is the usual $E$-valued Bochner integral). Then $\Delta_I f = \sum_{\eta \neq 0} \langle f, h_I^\eta \rangle h_I^\eta$, or suppressing the $\eta$ summation, $\Delta_I f = \langle f, h_I \rangle h_I$. Here, $\langle f, h_I \rangle = \int f h_I$. In this paper the brackets $\langle \cdot, \cdot \rangle$ try to always refer to some kind of integral pairing, while $\langle \cdot, \cdot \rangle_E$ is used for the dual pairing of a Banach space $E$.

A martingale block is defined by

$$\Delta_K^n f = \sum_{I \in D^n} \Delta_I f, \quad K \in D^n.$$

2.3. Multi-parameter notation

We work either in the bi-parameter setting in the product space $\mathbb{R}^{n+m}$ or in the tri-parameter setting in the product space $\mathbb{R}^{n+m+k}$. In such a context $x$ (or $y$) is always a tuple, for example, if $x \in \mathbb{R}^{n+m+k}$, then $x = (x_1, x_2, x_3)$ with $x_1 \in \mathbb{R}^n$, $x_2 \in \mathbb{R}^m$ and $x_3 \in \mathbb{R}^k$. 

We often need to take integral pairings with respect to one or two of the variables only. For example, if \( f : \mathbb{R}^{n+m+k} \to E \), where \( E \) is a Banach space, then \( \langle f, h_I \rangle : \mathbb{R}^{n+k} \to E \) is defined by

\[
\langle f, h_I \rangle_1(x_2, x_3) = \int_{\mathbb{R}^n} f(y_1, x_2, x_3) h_I(y_1) \, dy_1,
\]

and \( \langle f, h_I \otimes h_J \rangle_{1,2} : \mathbb{R}^k \to E \) is defined by

\[
\langle f, h_I \otimes h_J \rangle_{1,2}(x_3) = \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} f(y_1, y_2, x_3) h_I(y_1) h_J(y_2) \, dy_1 \, dy_2.
\]

Moreover, an identification of the following kind is used all the time: a function \( f : \mathbb{R}^{n+m} \to E \) satisfying

\[
\left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^m} |f(x_1, x_2)|^p \, dx_2 \right)^{q/p} \, dx_1 \right)^{1/q} < \infty
\]

is identified with the function \( \phi_f \in L^q(\mathbb{R}^n; L^p(\mathbb{R}^m; E)) \), \( \phi_f(x_1) = f(x_1, \cdot) \).

We next define bi-parameter martingale differences. Let \( f : \mathbb{R}^n \times \mathbb{R}^m \to E \) be locally integrable. Let \( I \in \mathcal{D}^n \) and \( J \in \mathcal{D}^m \). We define the martingale difference

\[
\Delta_I^1 f : \mathbb{R}^{n+m} \to E, \Delta_I^1 f(x) := \Delta_I(f(\cdot, x_2))(x_1).
\]

(The reader should not confuse this with the martingale block notation of a one-parameter function from above). Define \( \Delta_J^2 f \) analogously. Then we set

\[
\Delta_{I \times J}^1 f : \mathbb{R}^{n+m} \to E, \Delta_{I \times J}^1 f(x) = \Delta_I^1(\Delta_J^2 f)(x) = \Delta_J^2(\Delta_I^1 f)(x).
\]

Note that \( \Delta_I^1 f = h_I \otimes \langle f, h_I \rangle_1 \), \( \Delta_J^2 f = \langle f, h_J \rangle_2 \otimes h_J \) and \( \Delta_{I \times J} f = \langle f, h_I \otimes h_J \rangle h_I \otimes h_J \) (suppressing the finite \( \eta \) summations).

Martingale blocks are defined in the natural way

\[
\Delta_{K \times V}^i f = \sum_{I : I^{(i)} = K} \sum_{J : J^{(i)} = V} \Delta_{I \times J}^i f = \Delta_K^i(\Delta_V^j f) = \Delta_V^j(\Delta_K^i f).
\]

2.4. Bounded mean oscillation spaces

We say that \( b \in L^1_{\text{loc}}(\mathbb{R}^n) \) belongs to the dyadic bounded mean oscillation (BMO) space \( \text{BMO}_{\mathcal{D}^n}(\mathbb{R}^n) = \text{BMO}_{\mathcal{D}^n} \) if

\[
\| b \|_{\text{BMO}_{\mathcal{D}^n}} := \sup_{I \in \mathcal{D}^n} \frac{1}{|I|} \int_I \left| b - \langle b \rangle_I \right| < \infty.
\]

The ordinary space \( \text{BMO}(\mathbb{R}^n) \) is defined by taking the supremum over all cubes.

**Bi-parameter product BMO.** Here, we define the (dyadic) bi-parameter product BMO space \( \text{BMO}^{\mathcal{D}^n, \mathcal{D}^m} = \text{BMO}^{\mathcal{D}^n, \mathcal{D}^m} \). For a sequence \( \lambda = (\lambda_{I,J}) \) we set

\[
\| \lambda \|_{\text{BMO}^{\mathcal{D}^n, \mathcal{D}^m}} := \sup_{\Omega} \left( \frac{1}{|\Omega|} \sum_{I \in \mathcal{D}^n, J \in \mathcal{D}^m} \left| \lambda_{I,J} \right|^2 \right)^{1/2},
\]

where the supremum is taken over those sets \( \Omega \subset \mathbb{R}^{n+m} \) such that \( |\Omega| < \infty \) and such that for every \( x \in \Omega \) there exists \( I \in \mathcal{D}^n, J \in \mathcal{D}^m \) so that \( x \in I \times J \in \Omega \).

We say that \( b \in L^1_{\text{loc}}(\mathbb{R}^{n+m}) \) belongs to the space \( \text{BMO}^{\mathcal{D}^n, \mathcal{D}^m} \) if

\[
\| b \|_{\text{BMO}^{\mathcal{D}^n, \mathcal{D}^m}} := \| (b, h_I \otimes h_J) \|_{\text{BMO}^{\mathcal{D}^n, \mathcal{D}^m}} < \infty.
\]
The (non-dyadic) product BMO space \( \text{BMO}_{\text{prod}}(\mathbb{R}^{n+m}) \) can be defined via the norm defined by the supremum of the above dyadic norms.

For two sequences \( \lambda = (\lambda_{I,J}) \), \( A = (A_{I,J}) \) we have the key estimate
\[
\sum_{I \in D^n, J \in D^m} |A_{I,J}| \|A_{I,J}\| \lesssim \|\lambda\|_{\text{BMO}_{\text{prod}}^{D^n,D^m}} \|S_{D^n,D^m}(A)\|_{L^1(\mathbb{R}^{n+m})},
\]
where
\[
S_{D^n,D^m}(A) := \left( \sum_{I \in D^n, J \in D^m} |A_{I,J}|^2 \frac{1_{I \times J}}{|I \times J|} \right)^{1/2}.
\]

For a simple proof see, for example, [19, Proposition 4.1]. This inequality is the key property of the product BMO for us. Of course, an analogous estimate holds in the one-parameter situation.

2.5. Paraproducts

Let \( E \) be a Banach space. A function \( b \in L^1_{\text{loc}}(\mathbb{R}^n) \) defines the dyadic paraproduct by the formula
\[
\pi_{D^n,b} f = \sum_{I \in D^n} (f)_l \Delta_I b, \quad f \in L^1_{\text{loc}}(\mathbb{R}^n, E).
\]

**Bi-parameter full paraproducts.** While our main object of study in this paper are the so-called partial paraproducts, we will also need to consider the so-called full bi-parameter paraproducts. For example, they appear in some of the partial tri-parameter paraproducts.

There are two types of full bi-parameter paraproducts: the standard ones and the mixed ones: for \( b \in L^1_{\text{loc}}(\mathbb{R}^{n+m}) \) we set
\[
\Pi_{D^n,D^m,b} f = \sum_{I \in D^n, J \in D^m} \langle b, h_I \otimes h_J \rangle (f)_{I \times J} h_I \otimes h_J;
\]
\[
\Pi_{D^n,D^m,b}^\text{mixed} f = \sum_{I \in D^n, J \in D^m} \langle b, h_I \otimes h_J \rangle \left( f, h_I \otimes \frac{1_J}{|J|} \cdot \frac{1_I}{|I|} \right) \otimes h_J, \quad f \in L^1_{\text{loc}}(\mathbb{R}^{n+m}, E).
\]

2.6. UMD and Pisier’s property (\( \alpha \))

A Banach space \( E \) is said to be a UMD space if
\[
\left\| \sum_{i=1}^N \epsilon_i d_i \right\|_{L^p(\Omega;E)} \lesssim \left\| \sum_{i=1}^N d_i \right\|_{L^p(\Omega;E)}
\]
for all \( E \)-valued \( L^p \)-martingale difference sequences \( (d_i)_{i=1}^N \) (defined on some probability space \( \Omega \)), and for all signs \( \epsilon_i \in \{-1,1\} \). The UMD property is independent of the choice of the exponent \( p \in (1,\infty) \). If \( E \) is UMD then so is \( E^* \) and \( L^p(\mathbb{R}^n;E) \).

A Banach space \( E \) has Pisier’s property (\( \alpha \)) if for all \( N \), all \( \alpha_{i,j} \) in the complex unit disc and all \( \epsilon_{i,j} \in E, 1 \leq i,j \leq N \), there holds
\[
\left( \mathbb{E} \epsilon^{E'} \sum_{1 \leq i,j \leq N} \epsilon_i \epsilon'_j \alpha_{i,j} \epsilon_{i,j} \right)^{2} \lesssim \left( \mathbb{E} \epsilon^{E'} \sum_{1 \leq i,j \leq N} \epsilon_i \epsilon'_j \epsilon_{i,j} \right)^{2} \frac{1}{2}.
\]
Here, \( (\epsilon_i) \) and \( (\epsilon'_i) \) are sequences of independent random signs. If \( E \) has Pisier’s property (\( \alpha \)) then so does \( L^p(\mathbb{R}^n;E) \).
The Kahane–Khintchine inequality says that
\[
\left( E \left| \sum_{i=1}^{N} \epsilon_i e_i \right|^q \right)^{\frac{1}{q}} \sim_q \left( E \left| \sum_{i=1}^{N} \epsilon_i e_i \right|^2 \right)^{\frac{1}{2}}
\]
for all \( 1 \leq q < \infty \) and Banach spaces \( E \). By a few applications of the Kahane–Khintchine inequality we see that we can use whatever exponent in the definition of property \( (\alpha) \).

2.7. \( \mathcal{R} \)-boundedness

If \( E \) and \( F \) are Banach spaces, we denote the space of bounded linear operators from \( E \) to \( F \) by \( \mathcal{L}(E, F) \). If \( E = F \) we simply write \( \mathcal{L}(E) \). A family of operators \( T \subset \mathcal{L}(E, F) \) is said to be \( \mathcal{R} \)-bounded if for all \( N, T_1, \ldots, T_N \in T \) and \( e_1, \ldots, e_N \in E \) we have
\[
\left( E \left| \sum_{i=1}^{N} \epsilon_i T_i e_i \right|^2 \right)^{\frac{1}{2}} \leq C \left( E \left| \sum_{i=1}^{N} \epsilon_i e_i \right|^2 \right)^{\frac{1}{2}}.
\]
The best constant \( C \) is denoted by \( \mathcal{R}(T) \). The Kahane–Khintchine inequality shows that one can replace in the definition the exponent 2 with any \( q \in [1, \infty) \).

2.8. Random sums and duality

For the definition of type and cotype of a Banach space the reader can, for example, consult Section 7 of the book [13]. However, for the following lemma a reader not familiar with the notion of type needs only to know that UMD spaces have non-trivial type (as all spaces of interest in this paper will be UMD) (see [13, Section 7.4.f] for a proof).

**Lemma 2.1.** Let \( E \) be a Banach space with non-trivial type and let \( F \subset E^* \) be a closed subspace of \( E^* \) which is norming for \( E \). Let \( p \in (1, \infty) \). Then for all finite sequences \( e_1, \ldots, e_N \in E \) we have
\[
\left( E \left| \sum_{i=1}^{N} \epsilon_i e_i \right|^p \right)^{\frac{1}{p}} \leq \sup \left\{ \left| \sum_{i=1}^{N} \{e_i, e_i^*\}_E \right| \right\},
\]
where the supremum is taken over all choices \( (e_i^*)_{i=1}^{N} \) in \( F \) such that
\[
\left( E \left| \sum_{i=1}^{N} \epsilon_i e_i^* \right|^{p'} \right)^{\frac{1}{p'}} \leq 1.
\]
The converse inequality trivially holds with a constant 1.

2.9. Operator-valued shifts

Here, we give the definition of ordinary (that is, one-parameter) operator-valued shifts as given by Hänninen–Hytönen [8]. Let \( E \) be a UMD space and \( i_1, i_2 \geq 0 \) be two indices. An operator-valued shift \( S_{D_n}^{i_1, i_2} \) in \( \mathbb{R}^n \) (defined using a fixed dyadic grid \( D^n \)) is an operator of the form
\[
S_{D_n}^{i_1, i_2} f = \sum_{K \in D^n} \Delta_{K}^{i_2} A_K \Delta_{K}^{i_1} f, \quad f \in L_{\text{loc}}^1(\mathbb{R}^n; E),
\]
where $A_K f : \mathbb{R}^n \to E$ is an averaging operator with an operator-valued kernel $a_K : \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{L}(E)$, that is,

$$A_K f(x) = \frac{1}{|K|} \int_K a_K(x, y) f(y) \, dy.$$ 

Provided that the family of kernels is $\mathcal{R}$-bounded, that is,

$$\mathcal{R}(\{a_K(x, y) \in \mathcal{L}(E) : K \in \mathcal{D}^n, x, y \in K\}) \subseteq C_a,$$

Hämmänen–Hytönen [8] proved that for all $1 < q < \infty$ we have

$$\|S^{i_1, i_2} f\|_{L^q(\mathbb{R}^n; E)} \lesssim (\max(i_1, i_2) + 1)C_a\|f\|_{L^q(\mathbb{R}^n; E)}.$$ 

The implicit constant depends on the UMD-constant of $E$ and on $q$. The result actually holds with $\min(i_1, i_2)$ in place of $\max(i_1, i_2)$. In this paper we need to prove the $\mathcal{R}$-boundedness of shifts (under the assumption that $E$ also has Pisier’s property $(a)$), and we do this with $\min$ in place of $\max$ (see Lemma 3.3).

**Operator-valued bi-parameter shifts.** Let $E$ be a UMD space satisfying the property $(a)$ of Pisier. An operator-valued bi-parameter dyadic shift in $\mathbb{R}^{n+m}$ with parameters $i_1$, $i_2$, $j_1$ and $j_2$ is an operator of the form

$$S^{i_1, i_2; j_1, j_2} f = \sum_{K \in \mathcal{D}^n, V \in \mathcal{D}^m} \Delta_{K \times V}^{i_2, j_2} A_{K, V} \Delta_{K \times V}^{i_1, j_1} f, \quad f \in L^1_{\text{loc}}(\mathbb{R}^{n+m}; E).$$

Here, each $A_{K, V}$ is an integral operator related to a kernel

$$a_{K, V} : \mathbb{R}^{n+m} \times \mathbb{R}^{n+m} \to \mathcal{L}(E)$$ 

by

$$A_{K, V} f(x) = \frac{1}{|K| |V|} \int_{K \times V} a_{K, V}(x, y) f(y) \, dy.$$ 

The family of kernels is assumed to be $\mathcal{R}$-bounded in the sense that

$$\mathcal{R}(\{a_{K, V}(x, y) : K \in \mathcal{D}^n, V \in \mathcal{D}^m, (x, y) \in K \times V\}) \subseteq C_a.$$ 

We will (among other things) prove in Section 3 that for all $p, q \in (1, \infty)$ we have

$$\|S^{i_1, i_2; j_1, j_2} f\|_{L^q(\mathbb{R}^n; L^p(\mathbb{R}^m; E))} \lesssim (\min(i_1, i_2) + 1)(\min(j_1, j_2) + 1)C_a\|f\|_{L^q(\mathbb{R}^n; L^p(\mathbb{R}^m; E))}.$$ 

2.10. **Function lattices**

An easy to read reference for this section is [17] (see also [16] and [1]). A normed space $E$ is a Banach function space (or a function lattice) if the following four conditions hold. Let $(\Omega, \mathcal{A}, \mu)$ be a $\sigma$-finite measure space.

1. Every $f \in E$ is a measurable function $f : \Omega \to \mathbb{R}$ (an equivalence class).
2. If $f : \Omega \to \mathbb{R}$ is measurable, $g \in E$ and $|f(\omega)| \leq |g(\omega)|$ for $\mu$-almost every $\omega \in \Omega$, then $f \in E$ and $|f|_E \leq |g|_E$.
3. There is an element $f \in E$ so that $f > 0$ (that is, $f(\omega) > 0$ for $\mu$-almost every $\omega \in \Omega$).
4. If $f_i, f$ are non-negative, $f_i \in E$, $f_i \leq f_{i+1}$, $f_i(\omega) \to f(\omega)$ for $\mu$-almost every $\omega \in \Omega$ and $\sup_i |f_i|_E < \infty$, we have $f \in E$ and $|f|_E \to |f|_E = \sup_i |f_i|_E$.

**Remark 2.2.** The definition of a function lattice seems to vary a little bit in the literature, and sometimes it is left quite vague what is the exact definition used. Here, we use the definition from [1] and [17].
Such a space is automatically a Banach space, and in fact a Banach lattice (for the definition and basic theory of Banach lattices see, for example, [16]). If \( g : \Omega \to \mathbb{R} \) is a measurable function such that \( fg \in L^1(\mu) \) for all \( f \in E \), we define
\[
e^*_g : E \to \mathbb{R}, \quad e^*_g(f) = \int_\Omega f(\omega)g(\omega) \, d\mu(\omega).
\]
In this case \( e^*_g \in E^* \). We define \( E' \subset E^* \) to consist of those element of \( E^* \) that have the form \( e^*_g \) for some \( g \) like above (and we freely identify \( e^*_g \) with \( g \)). The space \( E' \) — the Köthe dual of \( E \) — is a Banach function space, and a norming subspace of \( E^* \).

There is a condition called order continuity (the precise definition does not interest us here) of \( E \) which is equivalent with the fact that \( E' = E^* \). So, the dual of an order continuous Banach function space is a Banach function space (as \( E' \) is always a Banach function space). We will be working with UMD Banach function spaces — which we also call UMD function lattices. Such spaces are always reflexive, and reflexive Banach lattices are order continuous. So, in cases of interest to us \( E' = E^* \). In this case \( E \) automatically also satisfies the property \((\alpha)\) of Pisier.

This is because a Banach lattice satisfies Pisier’s property \((\alpha)\) if and only if it has finite cotype (see, for example, Theorem 7.5.20 in the book [13]). A UMD space certainly has finite cotype (see, for example, [13]).

This generality covers most of the naturally arising examples of UMD spaces satisfying the property \((\alpha)\) of Pisier. The only place where we really need that \( E \) is a function space (and not just a general UMD space satisfying Pisier’s property \((\alpha)\)) is Section 5, where we prove \( R \)-boundedness results for various families of paraproducts. That is to say, if one can generalise the results of Section 5, then this restriction can be lifted in other key results.

A key reason why we use function lattices instead of Banach lattices is because we want to use maximal function estimates by Bourgain [3] and Rubio de Francia [24]. Function lattices are certainly more convenient to use in other aspects too, but it could be the case that most other estimates could be performed in Banach lattices.

2.11. Estimates for martingales

We collect in this subsection a plethora of various estimates, many of them of standard nature. The main aim is Corollary 2.9. The results are stated in the bi-parameter case. For clarity we give some proofs.

**Lemma 2.3.** Let \( E \) be a UMD space. Then for all \( p, q \in (1, \infty) \) and fixed signs \( \epsilon_1, \epsilon_J \) we have
\[
\|f\|_{L^q(\mathbb{R}^n;L^p(\mathbb{R}^m;E))} \sim \left\| \sum_{I \in \mathcal{D}^n} \epsilon_I \Delta_I^1 f \right\|_{L^q(\mathbb{R}^n;L^p(\mathbb{R}^m;E))} \sim \left\| \sum_{J \in \mathcal{D}^m} \epsilon_J \Delta_J^2 f \right\|_{L^q(\mathbb{R}^n;L^p(\mathbb{R}^m;E))}.
\]

**Proof.** The first one is seen by using the fact that \( F := L^p(\mathbb{R}^m;E) \) is a UMD space and expanding the function \( f : \mathbb{R}^n \to F \). The second one is seen by expanding \( f(x_1,\cdot) : \mathbb{R}^m \to E \) for each fixed \( x_1 \in \mathbb{R}^n \). \( \square \)

**Lemma 2.4.** Let \( E \) be a UMD space satisfying Pisier’s property \((\alpha)\). Then for all \( p, q \in (1, \infty) \) and fixed signs \( \epsilon_{I,J} \) we have
\[
\|f\|_{L^q(\mathbb{R}^n;L^p(\mathbb{R}^m;E))} \sim \left\| \sum_{I \in \mathcal{D}^n} \epsilon_{I,J} \Delta_I \otimes J f \right\|_{L^q(\mathbb{R}^n;L^p(\mathbb{R}^m;E))}.
\]
Proof. Using Lemma 2.3 twice and taking expectations we get
\[
\left\| f \right\|_{L^q(\mathbb{R}^n; L^p(\mathbb{R}^m; E))} \sim \mathbb{E} \mathbb{E}' \left\| \sum_{I \in D^n} \sum_{J \in D^m} \epsilon_I \epsilon'_J \Delta_I \times Jf \right\|_{L^n(\mathbb{R}^n; L^n(\mathbb{R}^m; E))}. \tag{2.5}
\]
Using the fact that \( L^q(\mathbb{R}^n; L^p(\mathbb{R}^m; E)) \) satisfies Pisier’s property (a) we get
\[
\mathbb{E} \mathbb{E}' \left\| \sum_{I \in D^n} \sum_{J \in D^m} \epsilon_I \epsilon'_J \Delta_I \times Jf \right\|_{L^n(\mathbb{R}^n; L^n(\mathbb{R}^m; E))} \sim \mathbb{E} \mathbb{E}' \left\| \sum_{I \in D^n} \sum_{J \in D^m} \epsilon_I \epsilon'_I \epsilon_{J, I} \Delta_I \times Jf \right\|_{L^n(\mathbb{R}^n; L^n(\mathbb{R}^m; E))}.
\]
Applying the identity (2.5) to the function \( F = \sum_{I \in D^n} \sum_{J \in D^m} \epsilon_I \epsilon_I \Delta_I \times Jf \) we get the claim. 

\[\square\]

Lemma 2.6. Let \( E \) be a UMD space satisfying Pisier’s property (a). Then for all \( p, q \in (1, \infty) \) we have
\[
\mathbb{E} \left\| \sum_{j} \epsilon_j f_j \right\|_{L^q(\mathbb{R}^n; L^p(\mathbb{R}^m; E))} \sim \mathbb{E} \left\| \sum_{I \in D^n} \sum_{J \in D^m} \epsilon_I \epsilon'_{I,J} \Delta_I \times Jf_j \right\|_{L^n(\mathbb{R}^n; L^n(\mathbb{R}^m; E))}.
\]

Proof. We only prove the first estimate — the others are proved in the same way (just use Lemma 2.3 instead of Lemma 2.4). Using Lemma 2.4 to the function \( \sum_j \epsilon_j f_j \) and taking expectations we get that
\[
\mathbb{E} \left\| \sum_{j} \epsilon_j f_j \right\|_{L^q(\mathbb{R}^n; L^p(\mathbb{R}^m; E))} \sim \mathbb{E} \mathbb{E}' \left\| \sum_{I \in D^n} \sum_{J \in D^m} \epsilon_I \epsilon'_{I,J} \Delta_I \times Jf_j \right\|_{L^n(\mathbb{R}^n; L^n(\mathbb{R}^m; E))}.
\]
Using the fact that \( L^q(\mathbb{R}^n; L^p(\mathbb{R}^m; E)) \) satisfies Pisier’s property (a) we get
\[
\mathbb{E} \mathbb{E}' \left\| \sum_{I \in D^n} \sum_{J \in D^m} \epsilon_I \epsilon'_{I,J} \Delta_I \times Jf_j \right\|_{L^n(\mathbb{R}^n; L^n(\mathbb{R}^m; E))} \sim \mathbb{E} \mathbb{E}' \mathbb{E}'' \left\| \sum_{I \in D^n} \sum_{J \in D^m} \epsilon_I \epsilon'_{I,J} \epsilon''_{I,J} \Delta_I \times Jf_j \right\|_{L^n(\mathbb{R}^n; L^n(\mathbb{R}^m; E))}
\]
\[
= \mathbb{E}'' \left\| \sum_{I \in D^n} \sum_{J \in D^m} \epsilon''_{I,J} \Delta_I \times Jf_j \right\|_{L^n(\mathbb{R}^n; L^n(\mathbb{R}^m; E))}.
\]
\[\square\]
In the case that $E$ is a UMD function lattice (note that in this case $|e| \in E$ and $|e|^\alpha \in E$ are defined in the natural pointwise way for every $e \in E$) we prefer square function bounds. The previous estimates are translated to them using the following lemma. The important thing for us is that it holds with all UMD function lattices.

**Lemma 2.7.** Let $E$ be a Banach function space with finite cotype. For all $q \in (1, \infty)$, we have

$$\| \left( \sum_j |f_j|^2 \right)^{1/2} \|_{L^q(\mathbb{R}^n;E)} \sim \left\| \sum_j \varepsilon_j f_j \right\|_{L^q(\mathbb{R}^n;E)}.$$

**Proof.** Applying Kahane–Khintchine inequality multiple times we see that

$$\left\| \sum_j \varepsilon_j f_j \right\|_{L^q(\mathbb{R}^n;E)} \sim \left( \left\| \sum_j \varepsilon_j f_j \right\|_E \right)^{1/2}.$$ 

So, things boil down to the equivalence

$$\left( \left\| \sum_j \varepsilon_j f_j \right\|_E \right)^{1/2} \sim \left( \sum_j \left\| \varepsilon_j f_j \right\|_E \right)^{1/2}$$

for all $e_j \in E$. But (2.8) holds in all Banach lattices with finite cotype – this is a theorem of Khintchine–Maurey. For a proof see [13, Theorem 7.2.13].

**Corollary 2.9.** Let $E$ be a UMD function lattice. For all $p, q \in (1, \infty)$ we have

$$\left\| \left( \sum_j |f_j|^2 \right)^{1/2} \right\|_{L^q(\mathbb{R}^n;L^p(\mathbb{R}^m;E))} \sim \left\| \left( \sum_j \sum_{I \in D^n} |\Delta_I f_j|^2 \right)^{1/2} \right\|_{L^q(\mathbb{R}^n;L^p(\mathbb{R}^m;E))}$$

$$\sim \left\| \left( \sum_j \sum_{I \in D^n} |\Delta_I f_j|^2 \right)^{1/2} \right\|_{L^q(\mathbb{R}^n;L^p(\mathbb{R}^m;E))}$$

2.12. **Estimates for maximal functions**

We introduce the lattice maximal function theory of Bourgain [3] and Rubio de Francia [24]. Let $E$ be a Banach function space. For each (simple) locally integrable function $f: \mathbb{R}^n \to E$ we define $M_{D^n,E}f: \mathbb{R}^n \to E$ by setting

$$M_{D^n,E}f(x, \omega) = \sup_{I \in D^n} \frac{1}{|I|} \int_I |f(y, \omega)| dy.$$
The following definitions are in line with our usual notational conventions. If \( f : \mathbb{R}^{n+m} \to E \) we define \( M^p_{D^n,E} f : \mathbb{R}^{n+m} \to E \) by setting \( M^p_{D^n,E} f(x_1, x_2) = M^p_{D^n,E}(f(\cdot, x_2))(x_1) \). The operator \( M^p_{D^n,E} \) is defined similarly.

The bi-parameter strong maximal function \( \mathcal{M}_{D^n,D^m,E} \) is defined for \( f : \mathbb{R}^{n+m} \to E \) by setting

\[
\mathcal{M}_{D^n,D^m,E} f(x_1, x_2, \omega) = \sup_{I \in D^n, J \in D^m} \frac{1}{|I||J|} \int_{I \times J} |f(y_1, y_2, \omega)| \, dy_1 \, dy_2.
\]

If \( E \) is the scalar field we simply write \( M_{D^n,E} \) etc.

The following proposition is due to Bourgain \([3]\) and Rubio de Francia \([24]\) in the case of the torus. A weighted version of this in \( \mathbb{R}^n \) is proved in \([7]\). See also \([1]\) and \([17]\). Again, the point made in Remark 2.2 applies here.

**Proposition 2.10.** Let \( E \) be a UMD function lattice. Then for all \( q \in (1, \infty) \) we have

\[
\|M^q_{D^n,E} f\|_{L^q(\mathbb{R}^n; E)} \lesssim \|f\|_{L^q(\mathbb{R}^n; E)}, \quad f \in L^q(\mathbb{R}^n; E).
\]

**Remark 2.11.** If \( E \) is a UMD function lattice, then so is \( E^* = E' \). Therefore, we also have for each \( q \in (1, \infty) \) that

\[
\|M^q_{D^n,E^*} f\|_{L^q(\mathbb{R}^n; E^*)} \lesssim \|f\|_{L^q(\mathbb{R}^n; E^*)}, \quad g \in L^q(\mathbb{R}^n; E^*).
\]

Proposition 2.10 extends, for instance, to the case where the function lattice \( E \) is replaced with the function lattice \( L^p(\mathbb{R}^m; \ell^r(E)) \), where \( p, r \in (1, \infty) \). Let \( q \in (1, \infty) \). A function \( f \in L^q(\mathbb{R}^n; L^p(\mathbb{R}^m; \ell^r(E))) \) can be viewed as \( f = \{f_j\}_{j \in \mathbb{Z}} \), where each \( f_j \) is an \( E \)-valued function on \( \mathbb{R}^{n+m} \), and

\[
\|f\|_{L^q(\mathbb{R}^n; L^p(\mathbb{R}^m; \ell^r(E)))} = \left\| \left( \sum_j |f_j|^r \right)^{1/r} \right\|_{L^q(\mathbb{R}^n; L^p(\mathbb{R}^m; \ell^r(E)))}.
\]

The maximal function \( M^q_{D^n,L^p(\mathbb{R}^m; \ell^r(E))} \) acting on \( f \) gives

\[
M^q_{D^n,L^p(\mathbb{R}^m; \ell^r(E))} f = \{M^q_{D^n,E} f_j\}_j.
\]

Thus, the extension of Proposition 2.10 gives the following corollary.

**Corollary 2.12.** Let \( E \) be a UMD function lattice. For all \( p, q, r \in (1, \infty) \) we have

\[
\left\| \left( \sum_j |M^q_{D^n,E} f_j|^r \right)^{1/r} \right\|_{L^q(\mathbb{R}^n; L^p(\mathbb{R}^m; \ell^r(E)))} \lesssim \left\| \left( \sum_j |f_j|^r \right)^{1/r} \right\|_{L^q(\mathbb{R}^n; L^p(\mathbb{R}^m; \ell^r(E)))}.
\]

Corollary 2.12 in turn directly leads to Corollary 2.13:

**Corollary 2.13.** Let \( E \) be a UMD function lattice. For all \( p, q, r \in (1, \infty) \) we have

\[
\left\| \left( \sum_j |M^q_{D^n,D^m,E} f_j|^r \right)^{1/r} \right\|_{L^q(\mathbb{R}^n; L^p(\mathbb{R}^m; \ell^r(E)))} \lesssim \left\| \left( \sum_j |f_j|^r \right)^{1/r} \right\|_{L^q(\mathbb{R}^n; L^p(\mathbb{R}^m; \ell^r(E)))}.
\]
Proof. We first apply the inequality $M_{D^n, D^m, E} f_j \lesssim M_{D^n, E}^2 M_{D^m, E} f_j$. In the inner integral over $\mathbb{R}^m$ we use Proposition 2.10 with $E$ replaced by $\ell'(E)$ to obtain

$$\left( \sum_j |M_{D^n, D^m, E} f_j|^r \right)^{1/r} \lesssim \left( \sum_j |M_{D^n, E} f_j|^r \right)^{1/r}.$$ 

Corollary 2.12 then readily gives the result.

\[ Q.E.D. \]

3. Various estimates for operator-valued shifts

3.1. $\mathcal{R}$-boundedness results for operator-valued shifts

For various reasons we need the following lemma on the $\mathcal{R}$-boundedness of one-parameter dyadic shifts. This actually also reproves the boundedness of operator-valued dyadic shifts from Hänninen–Hytönen [8] with a bit better dependence on the pair of indices $(j_1, j_2)$. The proof is similar, though. However, Lemma 3.10 offers a more interesting variant, which will also be of key importance to us.

Let us first introduce some definitions related to the so-called decoupling estimate. Let $V \in D^m$. We denote by $Y_V$ the measure space $(V, \text{Leb}_m(V), \nu_V)$. Here, $\text{Leb}_m(V)$ is the collection of Lebesgue measurable subset of $V$ and $\nu_V = dx [V/V]$, where $dx[V]$ is the $m$-dimensional Lebesgue measure restricted to $V$. With these we define the product probability space

$$(Y^m, \mathcal{A}_m, \nu_m) := \prod_{V \in D^m} Y_V.$$ 

If $y \in Y^m$ and $V \in D^m$, we denote by $y^2$ the coordinate related to $Y_V$.

Suppose $i \in \{0, 1, \ldots \}$ and $j \in \{0, \ldots, i\}$. Let $D^m_{i,j}$ be the collection

$$D^m_{i,j} := \{V \in D^m : \ell(V) = 2^{k(i+1)+j} \text{ for some } k \in \mathbb{Z}\}.$$ 

We have for all $p \in (1, \infty)$ and $f \in L^p(\mathbb{R}^m; E)$, where $E$ is a UMD space, that

$$\int_{\mathbb{R}^m} \left( \sum_{V \in D^m_{i,j}} \Delta^p_{i,j} f \right) \frac{dx}{E} \sim_p E \int_{Y^m} \int_{\mathbb{R}^m} \left( \sum_{V \in D^m_{i,j}} 1_V(x) \Delta^p_{i,j} f(y_V) \right) \frac{dx}{E} d\nu_m(y),$$

where the implicit constant is independent of $i$ and $j$. This is a special case of [8, Theorem 3.1]. See also [8] for the details and the history of this estimate. The reason why the collections $D^m_{i,j}$ are used is to guarantee that $\Delta^p_{i,j} f$ is constant on $V'$ if $V, V' \in D^m_{i,j}$ and $V' \subseteq V$. This is a technical detail needed to apply the underlying abstract decoupling principle behind (3.2).

Lemma 3.3. Let $E$ be a UMD space with Pisier’s property $(\alpha)$. Fix two parameters $j_1, j_2 \geq 0$. Suppose $\{S^{j_1, j_2}_{D^m, k}\}_{k \in K}$ is a family of operator-valued dyadic shifts. For every $k \in K$ let $\{a_{V,k} \}_{V \in D^m}$ be the family of kernels related to the shift $S^{j_1, j_2}_{D^m, k}$.

Assume that there exists a constant $C_\alpha$ so that

$$\mathcal{R}(\{a_{V,k}(x, y) \in \mathcal{L}(E) : k \in K, V \in D^m, x, y \in V\}) \leq C_\alpha.$$ 

Then, for every $q \in (1, \infty)$,

$$\mathcal{R} \left( \{S^{j_1, j_2}_{D^m, k} \in \mathcal{L}(L^q(\mathbb{R}^m; E)) : k \in K\} \right) \lesssim (\min(j_1, j_2) + 1) C_\alpha.$$
Proof. Fix some \( q \in (1,\infty) \). Let \( I \subset K \) be a finite subset, and suppose that for every \( k \in I \) we have a function \( f_k \in L^q(\mathbb{R}^m; E) \). The UMD property of \( E \) implies that

\[
\mathbb{E} \left\| \sum_{k \in I} \varepsilon_k S_{D^m}^{j_1,j_2} f_k \right\|_{L^q(\mathbb{R}^m; E)} = \mathbb{E} \left\| \sum_{V \in D^m} \sum_{k \in I} \varepsilon_k A_{V,k} \Delta_V^{j_1} f_k \right\|_{L^q(\mathbb{R}^m; E)} \sim \mathbb{E} \left\| \sum_{V \in D^m} \sum_{k \in I} \varepsilon_k A_{V,k} \Delta_V^{j_1} f_k \right\|_{L^q(\mathbb{R}^m; E)} \leq \mathbb{E} \left( \sum_{V \in D^m} \sum_{k \in I} \sum_{\nu} \varepsilon_k \varepsilon'_k 1_{V} a_{V,k}(x,y) \Delta_V^{j_1} f_k(y_V) \right)^q_{L^q(\mathbb{R}^m; E)} 1/q.
\]

(3.5)

where we applied Stein’s inequality in the second to last step, and the Kahane–Khintchine inequality in the last. The UMD-valued version of Stein’s inequality is by Bourgain, for a proof see, for example, Theorem 4.2.23 in the book [12].

Let \( V \in D^m \) and \( k \in I \). Applying the probability space \( Y^m \), which was introduced in the beginning of this section, we can write

\[
A_{V,k} \Delta_V^{j_1} f_k(x) = \frac{1_V(x)}{|V|} \int_V a_{V,k}(x,y) \Delta_V^{j_1} f_k(y) \, dy
\]

\[
= \int_{Y_V} 1_V(x) a_{V,k}(x,y_V) \Delta_V^{j_1} f_k(y_V) \, d\nu_V(y_V)
\]

\[
= \int_{Y^m} 1_V(x) a_{V,k}(x,y_V) \Delta_V^{j_1} f_k(y_V) \, d\nu_m(y).
\]

Applying this in the right-hand side (RHS) of (3.5), and using Hölder’s inequality related to the appearing \( Y^m \) integral, we see that the RHS of (3.5) is bounded by

\[
\left( \mathbb{E} \left( \int_{Y^m} \left| \sum_{k \in I} \sum_{V \in D^m} \varepsilon_k \varepsilon'_k 1_V a_{V,k}(x,y_V) \Delta_V^{j_1} f_k(y_V) \right|_{L^q(\mathbb{R}^m; E)}^q \, d\nu_m(y) \, dx \right)^{1/q} \right).
\]

Fix for the moment two points \( x \in \mathbb{R}^m \) and \( y \in Y^m \). The \( \mathcal{R} \)-boundedness assumption (3.4) together with Pisier’s property (α) show that

\[
\mathbb{E} \left( \sum_{k \in I} \sum_{V \in D^m} \varepsilon_k \varepsilon'_k 1_V a_{V,k}(x,y_V) \Delta_V^{j_1} f_k(y_V) \right)^q_{L^q(\mathbb{R}^m; E)} \leq \mathbb{E} \left( \sum_{k \in I} \sum_{V \in D^m} \varepsilon''_k 1_V a_{V,k}(x,y_V) \Delta_V^{j_1} f_k(y_V) \right)^q_{L^q(\mathbb{R}^m; E)} \leq C^\alpha \mathbb{E} \left( \sum_{k \in I} \sum_{V \in D^m} \varepsilon''_k 1_V a_{V,k}(x,y_V) \Delta_V^{j_1} f_k(y_V) \right)^q_{L^q(\mathbb{R}^m; E)} .
\]
We have shown that

$$\mathbb{E} \left\| \sum_{k \in \mathcal{I}} \epsilon_k S_{D^m,k}^{j_1,j_2} f_k \right\|_{L^q(\mathbb{R}^m; E)} \lesssim C_a \left( \mathbb{E}^\prime \int_{\mathbb{R}^m} \int_{\mathcal{Y}^m} \left\| \sum_{k \in \mathcal{I}} \sum_{V \in \mathcal{D}^m} \epsilon_k \epsilon_V \Delta_V^{j_1} f_k (y_V) \right\|^q_{E} \, d\nu_m(y) \, dx \right)^{1/q} \quad (3.6)$$

$$\lesssim C_a \left( \mathbb{E}^\prime \int_{\mathbb{R}^m} \int_{\mathcal{Y}^m} \left\| \sum_{k \in \mathcal{I}} \sum_{V \in \mathcal{D}^m} \epsilon_k \epsilon_V \Delta_V^{j_1} f_k (y_V) \right\|^q_{E} \, d\nu_m(y) \, dx \right)^{1/q}$$

where in the last step we again applied the property (α). Let $k \in \{0, \ldots, j_1\}$ and let $\mathcal{D}_{j_1,k}^m \subset \mathcal{D}^m$ be the related collection as defined in (3.1). The decoupling estimate (3.2) gives that

$$\mathbb{E}^\prime \int_{\mathbb{R}^m} \int_{\mathcal{Y}^m} \left\| \sum_{k \in \mathcal{I}} \sum_{V \in \mathcal{D}_{j_1,k}^m} \epsilon_k \epsilon_V \Delta_V^{j_1} f_k (y_V) \right\|^q_{E} \, d\nu_m(y) \, dx = \mathbb{E}^\prime \int_{\mathbb{R}^m} \int_{\mathcal{Y}^m} \sum_{V \in \mathcal{D}_{j_1,k}^m} \epsilon_V \Delta_V^{j_1} \left( \sum_{k \in \mathcal{I}} \epsilon_k f_k (y_V) \right) \, d\nu_m(y) \, dx \quad (3.7)$$

$$\sim \int_{\mathbb{R}^m} \left\| \sum_{V \in \mathcal{D}_{j_1,k}^m} \Delta_V^{j_1} \left( \sum_{k \in \mathcal{I}} \epsilon_k f_k \right) \right\|^q_{E} \, dx \lesssim \int_{\mathbb{R}^m} \left\| \sum_{k \in \mathcal{I}} \epsilon_k f_k \right\|^q_{E} \, dx.$$

This combined with (3.6) shows that

$$\mathbb{E} \left\| \sum_{k \in \mathcal{I}} \epsilon_k S_{D^m,k}^{j_1,j_2} f_k \right\|_{L^q(\mathbb{R}^m; E)} \lesssim (j_1 + 1) C_a \left( \mathbb{E} \left\| \sum_{k \in \mathcal{I}} \epsilon_k f_k \right\|^q_{L^q(\mathbb{R}^m; E)} \right)^{1/q}$$

$$\sim (j_1 + 1) C_a \mathbb{E} \left\| \sum_{k \in \mathcal{I}} \epsilon_k f_k \right\|_{L^q(\mathbb{R}^m; E)}.$$

So far we have proved the claim with the constant $(j_1 + 1)$. The constant $(\min(j_1, j_2) + 1)$ is achieved via duality. Consider some shift $S_{D^m,k}^{j_1,j_2}$. Its adjoint $(S_{D^m,k}^{j_1,j_2})^*$ is the operator acting on functions $g \in L^q(\mathbb{R}^m; E^*)$ by

$$(S_{D^m,k}^{j_1,j_2})^* g = \sum_{V \in \mathcal{D}^m} \Delta_V^{j_1} A_{V,k} \Delta_V^{j_2} g,$$

where each $A_{V,k}$ is an integral operator

$$A_{V,k} \varphi(y) = \frac{1}{V(y)} \int_V a_{V,k}(x,y) \varphi(x) \, dx, \quad \varphi \in L^1_{\text{loc}}(\mathbb{R}^m; E^*).$$

Since $E$ is a UMD space, we know that if $\mathcal{T} \subset \mathcal{L}(E)$ is an $\mathcal{R}$-bounded operator family, then the family $\mathcal{T}^* := \{T^* \in \mathcal{L}(E^*) : T \in \mathcal{T}\}$ is also $\mathcal{R}$-bounded and $\mathcal{R}(\mathcal{T}^*) \lesssim \mathcal{R}(\mathcal{T})$. This can be seen using Lemma 2.1. Thus,

$$\mathcal{R}(\{a_{V,k}(x,y)^* \in \mathcal{L}(E^*) : k \in \mathcal{K}, V \in \mathcal{D}^m, x, y \in V\}) \lesssim C_a. \quad (3.8)$$

Also, because $E$ is a UMD space, Pisier’s property (α) of $E$ implies that also $E^*$ has the property (α) with comparable constants. See Proposition 7.5.15 in the book [13].
Hence, we see that \( \{(S_{D_{\text{sym}}, k}^{j_2})^*\}_{k \in K} \) is a family of dyadic shifts with parameters \((j_2, j_1)\), and the related family of kernels satisfies the \( R \)-boundedness condition (3.8). The above proof shows that
\[
\mathcal{R}\left( \left\{ (S_{D_{\text{sym}}, k}^{j_2})^* \in \mathcal{L}(L^q(R^m; E^*)) : k \in K \right\} \right) \lesssim (j_2 + 1)C_a.
\]
Using Lemma 2.1 again we have
\[
\mathcal{R}\left( \left\{ S_{D_{\text{sym}}, k}^{j_2} \in \mathcal{L}(L^q(R^m; E)) : k \in K \right\} \right) \lesssim (j_2 + 1)C_a.
\]
This concludes the proof. \( \square \)

Next, we investigate shifts \( S_{D_{\text{sym}}, k}^{j_2} \) related to families of kernels \( a_K : R^n \times R^n \to L^p(R^m; E) \), where \( E \) is a UMD space with the property \((\alpha)\) of Pisier and \( p \in (1, \infty) \) is fixed. This time we are interested in estimates of the form
\[
\|S_{D_{\text{sym}}, k}^{j_2} f\|_{L^p(R^m; L^q(R^n; E))} \lesssim \|f\|_{L^p(R^m; L^q(R^n; E))}
\]
for a given \( q \in (1, \infty) \). Note that if we had the norm of \( L^q(R^n; L^p(R^m; E)) \) instead, we could apply Lemma 3.3.

Let \( Y^n \) be the probability space related to decoupling in \( R^n \), and suppose \( T \in \mathcal{L}(L^p(R^m; E)) \).

If \( f : R^n \times R^n \times Y^n \to E \) is a finite sum
\[
f(x_1, x_2, y) = \sum_i 1_{A_i}(x_2)1_{B_i}(x_1, y)e_i,
\]
where \( A_i \subset R^m, B_i \subset R^n \times Y^n \) are sets of finite measure and \( e_i \in E \), then we define
\[
Tf(x_1, x_2, y) := \sum_i T(1_{A_i}e_i)(x_2)1_{B_i}(x_1, y)e_i.
\]
The function \( Tf \) is well defined, that is, independent of the representation of \( f \). We say that \( T \) can be extended to an operator in \( \mathcal{L}(L^p(R^m; L^q(R^n \times Y^n; E))) \) if there exists \( \tilde{T} \in \mathcal{L}(L^p(R^m; L^q(R^n \times Y^n; E))) \) so that \( \tilde{T}f = Tf \) for all \( f \) of the form (3.9). This extension, if it exists, is unique since functions as in (3.9) are dense in \( L^p(R^m; L^q(R^n \times Y^n; E)) \).

**Lemma 3.10.** Suppose \( E \) is a UMD space with Pisier’s property \((\alpha)\). Let \( p, q \in (1, \infty) \) and \( i_1, i_2 \in \{0, 1, \ldots\} \) be fixed. Assume that \( \{a_{K,k}\}_{K \in D^n, k \in K} \) is a family of kernels
\[
a_{K,k} : R^n \times R^n \to \mathcal{L}(L^p(R^m; E)),
\]
so that each \( a_{K,k}(x_1, y_1) \in \mathcal{L}(L^p(R^m; E)) \) can be extended to an operator in
\[
\mathcal{L}(L^p(R^m; L^q(R^n \times Y^n; E))).
\]
In addition, it is assumed that the kernels are of the form
\[
a_{K,k}(x_1, y_1) = \sum_{l \in J_{K,k}} a_{K,k,l} 1_{S_{K,k,l}}(x_1, y_1),
\]
where \( (S_{K,k,l})_{l \in J_{K,k}} \) is a finite partition of \( K \times K \) and \( a_{K,k,l} \in \mathcal{L}(L^p(R^m; E)) \) Suppose that there exists a constant \( C_a \) so that
\[
\mathcal{R}(\{a_{K,k}(x_1, y_1) \in \mathcal{L}(L^p(R^m; L^q(R^n \times Y^n; E))) : k \in K, K \in D^n, x_1, y_1 \in K\}) \leq C_a.
\]
For every \( k \in K \), let \( S_{D_{\text{sym}}, k}^{i_2} \) be the operator-valued dyadic shift related to the family \( \{a_{K,k}\}_{K \in D^n} \). Then, every \( S_{D_{\text{sym}}, k}^{i_2} \) can be extended to an operator in \( \mathcal{L}(L^p(R^m; L^q(R^n; E))) \), and
\[
\mathcal{R}(\{S_{D_{\text{sym}}, k}^{i_2} \in \mathcal{L}(L^p(R^m; L^q(R^n; E))) : k \in K\}) \lesssim C_a(min(i_1, i_2) + 1).
\]
Remark 3.12. The assumptions are stronger than in Lemma 3.3 in the sense that they imply that
\[ R(\{a_{K,k}(x_1, y_1) \in \mathcal{L}(L^p(\mathbb{R}^m; E)): k \in \mathcal{K}, K \in D, x_1, y_1 \in K \}) \leq C_a. \]
Therefore, we have by Lemma 3.3 that for all \( a \) \leq (1, \infty) \) there holds
\[ R(\{S_{D_n,k}^{i_1,i_2} \in \mathcal{L}(L^p(\mathbb{R}; L^p(\mathbb{R}^m; E))): k \in \mathcal{K} \}) \leq C_a(\min(i_1, i_2) + 1). \]
The assumption (3.11) is satisfied in all the applications of this lemma below.

Proof of Lemma 3.10. Let \( \{f_k\}_{k \in \mathcal{I}} \), where \( \mathcal{I} \subset \mathcal{K} \) is finite, be a sequence of functions \( f_k: \mathbb{R}^n \times \mathbb{R}^m \to E \) of the form
\[ f_k(x_1, x_2) = \sum_i 1_{A_{k,i}}(x_1)1_{B_{k,i}}(x_2)e_{k,i}, \tag{3.13} \]
where the sum is finite, \( A_{k,i} \subset \mathbb{R}^n \) and \( B_{k,i} \subset \mathbb{R}^m \) are sets of finite measure, and \( e_{k,i} \in E \). By the Remark 3.12, \( S_{D_n,k}^{i_1,i_2} \) is well defined for every \( k \). We will show that
\[ E \left\| \sum_{k \in \mathcal{I}} \varepsilon_{k} S_{D_n,k}^{i_1,i_2} f_k \right\|_{L^p(\mathbb{R}^m; L^q(\mathbb{R}; E))} \lesssim C_a(i_1 + 1)E \left\| \sum_{k \in \mathcal{I}} \varepsilon_{k} f_k \right\|_{L^p(\mathbb{R}^n; L^q(\mathbb{R}^m; E))}, \]
which proves Lemma 3.10 (the minimum can be attained using duality as before).

Below we view the functions \( f_k \) as functions in \( L^q(\mathbb{R}^m; L^p(\mathbb{R}^m; E)) \), so that the martingale differences \( \Delta_{K,k} f_k \) have the usual meaning as \( L^p(\mathbb{R}^m; E) \)-valued functions. Begin by estimating (operate in \( L^q(\mathbb{R}^m; E) \) with a fixed \( x_2 \in \mathbb{R}^m \)) to introduce random signs and to get rid of the martingales, and use Kahane–Khintchine):
\[ E \left\| \sum_{k \in \mathcal{I}} \varepsilon_{k} S_{D_n,k}^{i_1,i_2} f_k \right\|_{L^p(\mathbb{R}^m; L^q(\mathbb{R}; E))} = E \left\| \sum_{K \in D_n} \Delta_{K,k}^{i_1} f_k \right\|_{L^p(\mathbb{R}^m; L^q(\mathbb{R}; E))} \lesssim \sum_{K \in D_n} E E \left\| \sum_{k \in \mathcal{I}} \varepsilon_{k} A_{K,k} \Delta_{K,k}^{i_1} f_k \right\|_{L^p(\mathbb{R}^m; L^q(\mathbb{R}; E))} \quad \tag{3.14} \]
\[ \sim E \left\| \sum_{K \in D_n} \sum_{k \in \mathcal{I}} \varepsilon_{K,k} A_{K,k} \Delta_{K,k}^{i_1} f_k \right\|_{L^p(\mathbb{R}^m; L^q(\mathbb{R}; E))}. \]
The last step applied the property (\( \alpha \)) of \( L^p(\mathbb{R}^m; L^q(\mathbb{R}; E)) \).

As in Lemma 3.3, we write
\[ A_{K,k} \Delta_{K,k}^{i_1} f_k(x_1) = \frac{1_K(x_1)}{|K|} \int_K a_{K,k}(x_1, y_1) \Delta_{K,k}^{i_1} f_k(y_1) \, dy_1 \]
\[ = \int_{\mathbb{R}^n} 1_K(x_1) a_{K,k}(x_1, y_K) \Delta_{K,k}^{i_1} f_k(y_K) \, d\nu_n(y). \]
The interpretation here is that \( \Delta_{K,k}^{i_1} f_k(y_K) \in L^p(\mathbb{R}^m; E) \), to which \( a_{K,k}(x_1, y_K) \in \mathcal{L}(L^p(\mathbb{R}^m; E)) \) hits giving \( a_{K,k}(x_1, y_K) \Delta_{K,k}^{i_1} f_k(y_K) \in L^p(\mathbb{R}^m; E) \). This can further be evaluated at \( x_2 \in \mathbb{R}^m \) to get an element of \( E \). This will simply be written as \( a_{K,k}(x_1, y_K) \Delta_{K,k}^{i_1} f_k(y_K)(x_2) \in E \).

We can assume that the kernels \( A_{K,k} \) are supported in \( K \times K \), and so we can stop writing the indicator \( 1_K(x_1) \). Thus, the RHS of (3.14) is dominated by
\[ E \left\| \sum_{K \in D_n} \sum_{k \in \mathcal{I}} \varepsilon_{K,k} a_{K,k}(x_1, y_K) \Delta_{K,k}^{i_1} f_k(y_K)(x_2) \right\|_{L^p(dx_2; L^q(dx_1 \times \nu_n(y); E))}. \tag{3.15} \]
To proceed, we aim to apply the fact that the kernels \( a_{K,k} \) are of the form (3.11). Kahane–Khintchine inequality implies that (3.15) is equivalent with
\[
\left\| \int_{\mathbb{R}^n} \int_{Y^n} \mathbb{E} \left[ \sum_{K \in \mathcal{D}^n} \sum_{k \in \mathcal{I}} \epsilon_{K,k} a_{K,k}(x_1, y_K) \Delta_{K}^{i_k} f_k(y_K)(x_2) \right]^q \, d\nu_n(y) \, dx_1 \right\|_{L^p(dx_2)}^{1/q}.
\]

Fix \( x_1 \in \mathbb{R}^n \) and \( y \in Y^n \). Let
\[
\{ \epsilon_{K,k,l} : K \in \mathcal{D}^n, k \in \mathcal{K}, l \in \mathcal{J}_{K,k} \}
\]
be another independent sequence of random signs. By the identical distribution of
\[
\{ \epsilon_{K,k} : K \in \mathcal{D}^n, k \in \mathcal{K}, x_1 \in \mathcal{K} \}
\]
and
\[
\{ \epsilon_{K,k,l} : K \in \mathcal{D}^n, k \in \mathcal{K}, l \in \mathcal{J}_{K,k} \}
\]
we have
\[
\mathbb{E} \left[ \sum_{K \in \mathcal{D}^n} \sum_{k \in \mathcal{I}} \epsilon_{K,k} a_{K,k}(x_1, y_K) \Delta_{K}^{i_k} f_k(y_K)(x_2) \right]^q
\]
\[
= \mathbb{E} \left[ \sum_{k \in \mathcal{I}} \sum_{l \in \mathcal{J}_{K,k}} \epsilon_{K,k} S_{K,k,l}(x_1, y_K) a_{K,k,l} \Delta_{K}^{i_k} f_k(y_K)(x_2) \right]^q.
\] (3.16)

Using (3.16) and applying the Kahane–Khintchine inequality again we have shown that (3.15) is equivalent with
\[
\mathbb{E} \left[ \sum_{k \in \mathcal{I}} \sum_{l \in \mathcal{J}_{K,k}} \epsilon_{K,k,l} S_{K,k,l}(x_1, y_K) a_{K,k,l} \Delta_{K}^{i_k} f_k(y_K)(x_2) \right],
\] (3.17)
where the norm \( \| \cdot \| \) refers to \( \| \cdot \|_{L^p(dx_2; L^q(dx_1 \times \nu_n(y); E))} \). Define for every \((K, k, l)\) the function
\[
1_{S_{K,k,l}}(x_1, y_K) \Delta_{K}^{i_k} f_k(y_K)(x_2) =: F_{K,k,l}(x_1, x_2, y).
\]

Using the fact that \( f \) is of the form (3.13) we see that
\[
1_{S_{K,k,l}}(x_1, y_K) a_{K,k,l} \Delta_{K}^{i_k} f_k(y_K)(x_2) = a_{K,k,l} F_{K,k,l}(x_1, x_2, y),
\]
where in the RHS we interpreted \( a_{K,k,l} \) as the extended operator in \( \mathcal{L}(L^p(\mathbb{R}^n); L^q(\mathbb{R}^n \times Y^n; E)) \). Now, the assumed \( \mathcal{R} \)-boundedness gives that
\[
(3.17) = \mathbb{E} \left[ \sum_{k \in \mathcal{I}} \sum_{l \in \mathcal{J}_{K,k}} \epsilon_{K,k,l} a_{K,k,l} F_{K,k,l}(x_1, x_2, y) \right]
\]
\[
\lesssim C_n \mathbb{E} \left[ \sum_{k \in \mathcal{I}} \sum_{l \in \mathcal{J}_{K,k}} \epsilon_{K,k,l} 1_{S_{K,k,l}}(x_1, y_K) \Delta_{K}^{i_k} f_k(y_K)(x_2) \right]
\]
\[
\sim C_n \mathbb{E} \left[ \sum_{k \in \mathcal{I}} \sum_{l \in \mathcal{J}_{K,k}} \epsilon_{K,k,l} 1_{K_1}(x_1) \Delta_{K}^{i_k} f_k(y_K)(x_2) \right],
\]
where in the last step we converted back to the random signs \( \epsilon_{K,k} \) as in (3.16).
Finally, using the property \((\alpha)\) of \(L^p(dx_2; L^q(dx_1 \times \nu_n(y); E))\) and then decoupling similarly as in (3.7) it is seen that

\[
\mathbb{E} \left\| \sum_{k \in I} \sum_{K \in \mathcal{D}^n} \epsilon_{K,k} 1_K(x_1) \Delta^i_K f_k(y_K)(x_2) \right\|_{L^p(dx_2; L^q(dx_1 \times \nu_n(y); E))} \\
\sim \mathbb{E} \mathbb{E}^\prime \left\| \sum_{K \in \mathcal{D}^n} \epsilon_K 1_K(x_1) \Delta^i_K \left( \sum_{k \in I} \epsilon_k f_k \right)(y_K)(x_2) \right\|_{L^p(dx_2; L^q(dx_1 \times \nu_n(y); E))} \\
\lesssim (i_1 + 1) \mathbb{E} \left\| \sum_{k \in I} \epsilon_k f_k \right\|_{L^p(R^m; L^q(R^n; E))}
\]

This concludes the proof. \(\square\)

3.2. Bi-parameter operator-valued shifts

We turn to show that operator-valued bi-parameter shifts are bounded (even \(\mathcal{R}\)-bounded as a family).

**Proposition 3.18.** Let \(E\) be a UMD space satisfying the property \((\alpha)\) of Pisier, and let \(i_1, i_2, j_1, j_2 \geq 0\) be fixed parameters. Suppose \(\{S_{D_n, D_m, k}^{i_1, j_2, j_1, j_2}\}_{k \in \mathcal{K}}\) is a family of operator-valued bi-parameter dyadic shifts as in Section 2.9. For every \(k \in \mathcal{K}\) let

\[
\{a_{K,V,k}: K \in \mathcal{D}^n, V \in \mathcal{D}^m\}
\]

be the family of kernels related to the shift \(S_{D_n, D_m, k}^{i_1, j_2, j_1, j_2}\). Assume that there exists a constant \(C_a\) so that

\[
\mathcal{R}(\{a_{K,V,k}(x,y): k \in \mathcal{K}, K \in \mathcal{D}^n, V \in \mathcal{D}^m, (x,y) \in K \times V\}) \leq C_a.
\]

Then for all \(p, q \in (1, \infty)\) we have

\[
\mathcal{R}(\{S_{D_n, D_m, k}^{i_1, j_2, j_1, j_2} \in \mathcal{L}(L^2(R^n; L^p(R^m; E))): k \in \mathcal{K}\}) \lesssim (\text{min}(i_1, i_2) + 1)(\text{min}(j_1, j_2) + 1)C_a.
\]

**Proof.** For each fixed \(k \in \mathcal{K}, K \in \mathcal{D}^n\) and \(x_1, y_1 \in K\) we define the one-parameter operator-valued dyadic shift in \(R^m\) by the formula

\[
S_{D_m,K,x_1,y_1}^{j_2,j_1}\varphi := \sum_{V \in \mathcal{D}^m} \Delta^j_V A_{V,k}^{K,x_1,y_1} \Delta^i_V \varphi, \quad \varphi \in L^1_{loc}(R^m; E),
\]

where

\[
A_{V,k}^{K,x_1,y_1}(x_2) := \frac{1_V(x_2)}{|V|} \int_V a_{K,V,k}(x_1, x_2, y_1, y_2) \varphi(y_2) \, dy_2.
\]

The assumptions and Lemma 3.3 show that for all \(p \in (1, \infty)\) we have

\[
\mathcal{R}(\{S_{D_m,K,x_1,y_1}^{j_2,j_1} \in \mathcal{L}(L^p(R^m; E))): k \in \mathcal{K}, K \in \mathcal{D}^n, x_1, y_1 \in K\}) \lesssim (\text{min}(j_1, j_2) + 1)C_a.
\]

(3.19)

Next, fix \(p \in (1, \infty)\), and for each \(k \in \mathcal{K}\) define the one-parameter operator-valued dyadic shift in \(R^n\) by the formula

\[
S_{D_n,k}^{i_1,j_2}\psi := \sum_{K \in \mathcal{D}^n} \Delta^j_K A_{K,k} \Delta^i_K \psi, \quad \psi \in L^1_{loc}(R^n; L^p(R^m; E)),
\]
where
\[ A_{K,k} \psi(x_1) = \frac{1_K(x_1)}{|K|} \int_K a_{K,k}(x_1,y_1) \psi(y_1) \, dy_1 \]
and
\[ a_{K,k}(x_1,y_1) = S^{j_1,j_2}_{D_n,k,K,x_1,y_1}. \]
Recall that \( L^p(\mathbb{R}^m; E) \) is a UMD space with the property (a) of Pisier. Using (3.19) and Lemma 3.3 we see that for all \( \eta \in (1, \infty) \) there holds
\[ \mathcal{R}(\{ S^{j_1,j_2}_{D_n,k} \in \mathcal{L}(L^q(\mathbb{R}^n; L^p(\mathbb{R}^m; E))): k \in K \} \leq (\min(i_1,i_2) + 1)(\min(j_1,j_2) + 1)C_\eta. \]
To conclude the proof, we only need to check that
\[ S^{j_1,j_2}_{D_n,k} f = S^{j_1,j_2,j_1,j_2}_{D_n,D_n,k} f. \]
For this identity we consider \( k \in K \) fixed, and suppress it from the notation. A straightforward, however tedious, way is to expand both sides using Haar functions. Clearly, \( S^{j_1,j_2,j_1,j_2}_{D_n,D_n} f \) equals
\[
\sum_{K \in D_n, V \in D_m} \frac{1}{|K||V|} \sum_{I_1, I_2 \in D_n} \sum_{J_1, J_2 \in D_m} \left( \int \int I_1 \times I_1 \int \int J_2 \times J_2 (h_{I_1} \otimes h_{J_1})(y)(h_{I_2} \otimes h_{J_2})(z) \right.
\]
\[ \times a_{K,V}(z,y) \langle f, h_{I_1} \otimes h_{J_1} \rangle \, dz \, dy \right) h_{I_2} \otimes h_{J_2}. \]
We now check that also \( S^{j_1,j_2}_{D_n} f \) equals this. Since \( \Delta_{j_1}^i \mathcal{R}(f) = \sum_{I_1 \in D_n} h_{I_1}(y_1) \langle f, h_{I_1} \rangle_1, \)
we have
\[ A_K \Delta_{j_1}^i \mathcal{R}(f)(z_1) = \frac{1_K(z_1)}{|K|} \sum_{I_1 \in D_n} \int_{I_1} h_{I_1}(y_1) S^{j_1,j_2}_{D_n,K,z_1,y_1} \langle f, h_{I_1} \rangle_1 \, dy_1 \]
and
\[ \Delta_{j_2}^i A_K \Delta_{j_1}^i \mathcal{R}(f)(x_1) = \frac{1}{|K|} \sum_{I_1, I_2 \in D_n, I_1^{(i_1)} = K} h_{I_1}(x_1) \int \int h_{I_1}(y_1) h_{I_2}(z_1) \]
\[ \times S^{j_1,j_2}_{D_n,K,z_1,y_1} \langle f, h_{I_1} \rangle_1 \, dz_1 \, dy_1 \right). \]
Since
\[ \Delta_{j_2}^i \mathcal{R}(f, h_{I_1})(y_2) = \sum_{J_1 \in D_m} \langle f, h_{I_1} \otimes h_{J_1} \rangle h_{J_1}(y_2), \]
we have
\[ A^{K,z_1,y_1}_V \Delta_{j_2}^i \mathcal{R}(f, h_{I_1})(z_2) = \frac{1_V(y_2)}{|V|} \sum_{J_1 \in D_m} \int_{J_1} h_{J_1}(y_2) \]
\[ \times a_{K,V}(z_1, z_2, y_1, y_2) \langle f, h_{I_1} \otimes h_{J_1} \rangle \, dy_2 \]
and

\[
\Delta_{V}^{j_{2}}A_{V}^{i_{1},y_{1}}\Delta_{V}^{j_{1}}(f, h_{I_{1}})_{1}(x_{2}) = \frac{1}{|V|} \sum_{j_{1}, j_{2} \in \mathbb{D}^{m}} \left( \int_{I_{1}} \int_{I_{2}} h_{j_{1}}(y_{2}) h_{j_{2}}(z_{2}) \right).
\]

Combining, we readily see that \(S_{D^{n},D^{n}}^{i_{1},i_{2}} f\) has the same Haar expansion as \(S_{D^{n},D^{n}}^{i_{1},i_{2}} f\), and the proof is complete. \(\Box\)

4. Model operators

Fix a UMD space \(F\). Suppose that for every \(K, I_{1}, I_{2} \in \mathbb{D}^{n}\) we are given an operator \(B_{K,I_{1},I_{2}} \in \mathcal{L}(F)\). Fix two indices \(i_{1}, i_{2} \geq 0\), and define the model operator

\[
P_{D^{n},b}^{i_{1},i_{2}} f(x) = \sum_{K \in \mathbb{D}^{n}} \sum_{I_{1}, I_{2} \in \mathbb{D}^{n}} h_{I_{2}}(x) B_{K,I_{1},I_{2}} ([f, h_{I_{1}}]),
\]

where \(x \in \mathbb{R}^{n}\) and \(f: \mathbb{R}^{n} \to F\) is locally integrable. The next proposition considers a family of these operators \(P_{D^{n},b}^{i_{1},i_{2}}\), where \(b \in \mathcal{B}\) and \(\mathcal{B}\) is some index set.

**Proposition 4.1.** Let \(F\) be a UMD space with the property (\(\alpha\)) of Pisier. Suppose that

\[
\mathcal{R} \left( \left\{ \frac{|K|}{|I_{1}|^{1/2} |I_{2}|^{1/2}} B_{K,I_{1},I_{2},b} \in \mathcal{L}(F): K, I_{1}, I_{2} \in \mathbb{D}^{n}, b \in \mathcal{B} \right\} \right) \leq C_{0}.
\]

Let \(P_{D^{n},b}^{i_{1},i_{2}}\) be a model operator associated with the operators \(B_{K,I_{1},I_{2},b}\). Then for all \(q \in (1, \infty)\) we have

\[
\mathcal{R} (\{ P_{D^{n},b}^{i_{1},i_{2}} \in \mathcal{L}(L^{q}(\mathbb{R}^{n}; F)): b \in \mathcal{B} \}) \lesssim (\min(i_{1}, i_{2}) + 1) C_{0}.
\]

**Proof.** This is essentially just an estimate for operator-valued shifts in a form which is a priori slightly different. To see the simple connection define the operator-valued kernels

\[
a_{K,b}^{i_{1},i_{2}} : \mathbb{R}^{n} \times \mathbb{R}^{n} \to \mathcal{L}(F)
\]

by setting

\[
a_{K,b}^{i_{1},i_{2}}(x, y) = |K| \sum_{I_{1}, I_{2} \in \mathbb{D}^{n}} h_{I_{1}}(y) h_{I_{2}}(x) B_{K,I_{1},I_{2},b}.
\]

We then define the averaging operator \(A_{K,b}^{i_{1},i_{2}}\), mapping locally integrable functions \(f : \mathbb{R}^{n} \to F\) to \(A_{K,b}^{i_{1},i_{2}} f : \mathbb{R}^{n} \to F\), by the formula

\[
A_{K,b}^{i_{1},i_{2}} f(x) = \frac{1}{|K|} \int_{\mathbb{R}^{n}} a_{K,b}^{i_{1},i_{2}}(x, y) f(y) \, dy.
\]

Define the operator-valued shift

\[
S_{D^{n},b}^{i_{1},i_{2}} f := \sum_{K \in \mathbb{D}^{n}} A_{K,b}^{i_{1},i_{2}} f = \sum_{K \in \mathbb{D}^{n}} \Delta_{K}^{i_{2}} A_{K,b}^{i_{1},i_{2}} \Delta_{K}^{i_{1}} f.
\]

By Lemma 3.3 we have for all \(q \in (1, \infty)\) that

\[
\mathcal{R} (\{ S_{D^{n},b}^{i_{1},i_{2}} \in \mathcal{L}(L^{q}(\mathbb{R}^{n}; F)): b \in \mathcal{B} \}) \lesssim (\min(i_{1}, i_{2}) + 1) R.
\]
where
\[ R := \mathcal{R}(\{ a_{K,b}^{i_1,i_2}(x,y) \in \mathcal{L}(F) : b \in B, K \in \mathcal{D}^n, x, y \in K \}). \]

Consider a fixed tuple \((b, K, x, y)\) so that \(a_{K,b}^{i_1,i_2}(x,y) \neq 0\). Then there are unique \(I_1, I_2 \in \mathcal{D}^n\) (depending on \((K, x, y)\)) so that \(I_1^{(i_1)} = I_2^{(i_2)} = K\), \(y \in I_1\) and \(x \in I_2\). Now, we have
\[ a_{K,b}^{i_1,i_2}(x,y) = |K| h_{I_1}(y) h_{I_2}(x) B_{K,I_1,I_2,b} = \pm \frac{|K|}{|I_1|^{1/2} |I_2|^{1/2}} B_{K,I_1,I_2,b}. \]

Thus, we have \(R \leq C_0\). It remains only to note that \(S_{D^n,b}^{i_1,i_2} f = P_{D^n,b}^{i_1,i_2} f\), which follows from the fact that
\[ A_{K,b}^{i_1,i_2} f(x) = \sum_{I_1,I_2 \in \mathcal{D}^n} h_{I_2}(x) B_{K,I_1,I_2,b} \left( \int_I f(y) h_{I_1}(y) \, dy \right). \]

Let us now consider the special case of model operators, where \(F = L^p(\mathbb{R}^m; E)\) for some fixed \(p \in (1, \infty)\) and UMD space \(E\). We formulate a condition for verifying the boundedness of \(P_{D^n,b}^{i_1,i_2}\) from \(L^p(\mathbb{R}^m; L^q(\mathbb{R}^n; E))\) to \(L^q(\mathbb{R}^n; L^p(\mathbb{R}^m; E))\). Note that the previous proposition only allows to conclude that under certain conditions \(P_{D^n,b}^{i_1,i_2}\) is bounded from \(L^q(\mathbb{R}^n; L^p(\mathbb{R}^m; E)) \to L^q(\mathbb{R}^m; L^p(\mathbb{R}^m; E))\) for all \(q \in (1, \infty)\). The condition will now depend both on \(p\) and \(q\). These assumptions are stronger than above (that is, they also imply the conclusion of Proposition 4.1) (see Remark 3.12).

When we talk about extensions, we always mean tensor extensions as in Section 3.

**Proposition 4.2.** Let \(E\) be a UMD space with the property \((\alpha)\) of Pisier, \(p, q \in (1, \infty)\) and \(F = L^p(\mathbb{R}^m; E)\). Suppose we are given operators \(B_{K,I_1,I_2,b} \in \mathcal{L}(F)\) that can be extended to operators in \(\mathcal{L}(L^p(\mathbb{R}^m; L^q(\mathbb{R}^n \times Y^n; E)))\), and that
\[ \mathcal{R} \left( \left\{ \frac{|K|}{|I_1|^{1/2} |I_2|^{1/2}} B_{K,I_1,I_2,b} \in \mathcal{L}(L^p(\mathbb{R}^m; L^q(\mathbb{R}^n \times Y^n; E))) : K, I_1, I_2 \in \mathcal{D}^n, b \in B \right\} \right) \leq C_0. \]
Let \(P_{D^n,b}^{i_1,i_2}\) be a model operator associated with the operators \(B_{K,I_1,I_2,b}\). Then we have
\[ \mathcal{R}(\{ P_{D^n,b}^{i_1,i_2} \in \mathcal{L}(L^p(\mathbb{R}^m; L^q(\mathbb{R}^n; E))) : b \in B \}) \lesssim (\min(i_1, i_2) + 1) C_0. \]

**Proof.** As in the proof of Proposition 4.1 we have that \(P_{D^n,b}^{i_1,i_2} = S_{D^n,b}^{i_1,i_2}\) for a certain operator-valued shift. Using Lemma 3.10 we get the claim exactly as before. \(\square\)

For the purposes of tri-parameter theory let us still go one step further. So, suppose now that \(F = L^p(\mathbb{R}^m; L^r(\mathbb{R}^k; E))\). Note that if we now consider boundedness in \(L^q(\mathbb{R}^n; F)\) we can use Proposition 4.1. On the other hand, if we consider boundedness in \(L^p(\mathbb{R}^m; L^{r_1}(\mathbb{R}^n; L^q(\mathbb{R}^k; E)))\) we need to use Proposition 4.2. The case \(L^p(\mathbb{R}^m; L^{r_1}(\mathbb{R}^k; L^q(\mathbb{R}^n; E)))\) requires a new proposition.

**Proposition 4.3.** Let \(E\) be a UMD space with the property \((\alpha)\) of Pisier, \(p, q, r \in (1, \infty)\) and \(F = L^p(\mathbb{R}^m; L^r(\mathbb{R}^k; E))\). Suppose we are given operators \(B_{K,I_1,I_2,b} \in \mathcal{L}(F)\) that can be extended to operators in \(\mathcal{L}(L^p(\mathbb{R}^m; L^{r_1}(\mathbb{R}^k; L^q(\mathbb{R}^n \times Y^n; E))))\), and that
\[ \mathcal{R} \left( \left\{ \frac{|K|}{|I_1|^{1/2} |I_2|^{1/2}} B_{K,I_1,I_2,b} \in \mathcal{L}(L^p(\mathbb{R}^m; L^{r_1}(\mathbb{R}^k; L^q(\mathbb{R}^n \times Y^n; E)))) : K, I_1, I_2 \in \mathcal{D}^n, b \in B \right\} \right) \leq C_0. \]
Let $P_{D^n,b}^{i_1,i_2}$ be a model operator associated with the operators $B_{K,I_1,I_2,b}$. Then we have
\[ \mathcal{R}(\{ P_{D^n,b}^{i_1,i_2} \in \mathcal{L}(L^p(\mathbb{R}^m; L^r(\mathbb{R}^k; L^q(\mathbb{R}^n; E))) : b \in B \}) \lesssim (\min(i_1,i_2) + 1)C_0. \]

**Proof.** As in the proof of Proposition 4.1 we have that $P_{D^n,b}^{i_1,i_2} = S_{D^n,b}^{i_1,i_2}$ for a certain operator-valued shift. Using an obvious variant of Lemma 3.10 (the proof is essentially the same) we get the claim. \( \square \)

5. $\mathcal{R}$-boundedness results for paraproducts

5.1. $\mathcal{R}$-boundedness of one-parameter paraproducts

We begin by giving a nice and elementary argument showing the $\mathcal{R}$-boundedness of paraproducts $\pi_{D^m,b_i} \in \mathcal{L}(L^p(\mathbb{R}^m))$ when $\| b_i \|_{\text{BMO}} \leq 1$. This proof may be of independent interest. However, it is not needed as, using other techniques, we prove a more general result right after.

In this more general result we study $\mathcal{R}$-boundedness in $L^p(\mathbb{R}^m; E)$ for a UMD function lattice $E$. Recall that UMD-valued paraproducts are bounded in $L^p$. Therefore, it seems reasonable to suspect that Proposition 5.4 is true in the generality that $E$ is a UMD space satisfying Pisier’s property $(\alpha)$. However, showing that would certainly require different methods.

**Proposition 5.1.** Suppose that $\| b_i \|_{\text{BMO}} \leq 1$, $i \in \mathcal{I}$, and $p \in (1, \infty)$. Then
\[ \mathcal{R}(\{ \pi_{D^m,b_i} \in \mathcal{L}(L^p(\mathbb{R}^m)) : i \in \mathcal{I} \}) \lesssim 1. \]

**Proof.** This proof has the benefit that we can work with the $L^1$ definition of BMO directly, and we even don’t need to use the John–Nirenberg inequality.

We start with simple stopping time preliminaries. Consider first a single function $b$ for which $\| b \|_{\text{BMO}} \leq 1$, and a fixed $J_0 \in \mathcal{D}^m$. We set $\mathcal{F}_b(J_0) = \{ J_0 \}$, and let $\mathcal{F}_b^0(J_0)$ consist of the maximal $J \in \mathcal{D}^m$, $J \subset J_0$, for which $|\langle b \rangle_J - \langle b \rangle_{J_0}| > 4$. Note that for all $J \in \mathcal{F}_b^0(J_0)$ we have
\[ 4 < |\langle b \rangle_J - \langle b \rangle_{J_0}| \leq \frac{1}{|J|} \int_J |b - \langle b \rangle_{J_0}|, \]
so that
\[ \sum_{J \in \mathcal{F}_b^0(J_0)} \Delta_J b \lesssim 1, \quad J \in \mathcal{F}_b(J_0). \]

Another stopping time we use is the standard principal cubes of a function $f \in L^1_{\text{loc}}(\mathbb{R}^m)$. This means that $\mathcal{S}_f^0(J_0) = \{ J_0 \}$, and we let $\mathcal{S}_f^1(J_0)$ consist of the maximal $J \in \mathcal{D}^m$, $J \subset J_0$, for
which \(|f|\rangle_J > 4\langle f|\rangle_{J_0}\). This time it is perhaps even more trivial that

\[
\bigg| \bigcup_{J \in S^1_f(J_0)} J \bigg| \leq \frac{|J_0|}{4}.
\]

Iterating this we get the sparse family of stopping cubes defined by \(S_f(J_0)\).

Our final stopping time is established by combining these two in the following sense. Let \(F_b,J(J_0) = \{J_0\}\) and let \(F^1_b,J(J_0)\) be the maximal cubes of \(F^1_b(J_0) \cup S^1_f(J_0)\). The final sparse collection, established by iterating this, is denoted by \(F_{b,J}(J_0) = \bigcup_{j=1}^{\infty} F^{(j)}_{b,J}(J_0)\).

After these preliminaries we give the actual proof. We need to show that given a finite \(J \subset I\) and \(f_j \in L^p(\mathbb{R}^m)\) we have

\[
\left\| \left( \sum_{j \in J} |\pi_{b_j} f_j|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^m)} \lesssim \left\| \left( \sum_{j \in J} |f_j|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^m)},
\]

where we abbreviated \(\pi_{b_j} = \pi_{D^m,b_j}\).

Using calculations as in Section 2.11 it is clear that the following one-parameter analog of the results of that section holds:

\[
\left\| \left( \sum_{j \in J} |f_j|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^m)} \sim \left\| \left( \sum_{j \in J} \sum_{Q \in D^m} |\Delta_Q f_j|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^m)}.
\]

This is also stated in [25, Lemma 2.1]. Using this we have

\[
\left\| \left( \sum_{j \in J} |\pi_{b_j} f_j|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^m)} \sim \left\| \left( \sum_{j \in J} \sum_{Q \in D^m} |\langle f_j \rangle_Q \Delta_Q b_j|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^m)}.
\]

Therefore, it is enough to fix an arbitrary \(J_0 \in D^m\) and prove the estimate

\[
\left\| \left( \sum_{j \in J} \sum_{Q \subset J_0} \langle f_j \rangle_Q \Delta_Q b_j \right)^{1/2} \right\|_{L^p(\mathbb{R}^m)} \lesssim \left\| \sum_{j \in J} |f_j|^2 \right\|_{L^p(\mathbb{R}^m)}.
\]

For every \(j \in J\) we set \(F_j := F_{b_j,f_j}(J_0)\). Then, we estimate

\[
\left\| \left( \sum_{j \in J} \sum_{Q \in D^m} \langle f_j \rangle_Q \Delta_Q b_j \right)^{1/2} \right\|_{L^p(\mathbb{R}^m)} \lesssim \left\| \sum_{j \in J} \sum_{Q \in D^m} \langle f_j \rangle_Q \sum_{\pi_{F_j} Q = J} \Delta_Q b_j \right\|_{L^p(\mathbb{R}^m)}^{1/2}
\]

\[
\sim \left\| \sum_{j \in J} \sum_{Q \in D^m} \langle f_j \rangle_Q \sum_{\pi_{F_j} Q = J} \Delta_Q b_j \right\|_{L^p(\mathbb{R}^m)}^{2/2}.
\]

(5.3)
The last step applied (5.2) again. Since for every \( j \in J \) and \( J \in F_j \) we have

\[
| \sum_{Q \in D^m_{\pi_{F_j}Q=J}} \Delta_Q b_j | \lesssim 1_j,
\]

the RHS of (5.3) is further bounded by

\[
\left\| \left( \sum_{j \in J} \sum_{J \in F_j} \langle |f_j| \rangle J \right)^{1/2} \right\|_{L^p(\mathbb{R}^m)} \lesssim \left\| \left( \sum_{j \in J} |f_j|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^m)}.
\]

The last step used a version of the Carleson embedding theorem stated at least in \([25, \text{Lemma 2.2}]\). \( \square \)

With a different proof we can manage the generality that \( E \) is a UMD function lattice. The method is very similar to those that we use with bi-parameter paraproducts below.

**Proposition 5.4.** Suppose that \( \|b_i\|_{\text{BMO}_{D^m}} \leq 1 \), \( i \in I \), \( E \) is a UMD function lattice and \( p \in (1, \infty) \). Then

\[
\mathcal{R}(\{\pi_{D^m, b_i} \in \mathcal{L}(L^p(\mathbb{R}^m; E)) : i \in I\}) \lesssim 1.
\]

**Proof.** We denote \( \pi_{b_i} = \pi_{D^m, b_i} \). For all finite subsets \( J \subset I \) and \( f_j \in L^p(\mathbb{R}^m; E) \), \( g_j \in L^{p'}(\mathbb{R}^m; E^*) \) (recall that \( E^* = E' \)) we will show that

\[
\left| \sum_{j \in J} \int_{\mathbb{R}^m} \{\pi_{b_i} f_j(x), g_j(x)\} E \, dx \right| \lesssim \left( \sum_{j \in J} |f_j|^2 \right)^{1/2} \left( \sum_{j \in J} |g_j|^2 \right)^{1/2}.
\]

This is enough as can be seen by using Lemmas 2.1 and 2.7. As \( g_j(x) \in E^* = E' \) the pairings \( \{\cdot, \cdot\}_E \) are just integrals (over some space \( \Omega \) appearing in the definition of function lattices, see Section 2.10). This is convenient for checking the validity of some of the manipulations below.

Recalling that

\[
\pi_{b_i} f_j(x) = \sum_{J \in D^m} \langle b_i, h_J \rangle \langle f_j \rangle J h_J(x), \quad x \in \mathbb{R}^m,
\]

we get

\[
\left| \sum_{j \in J} \int_{\mathbb{R}^m} \{\pi_{b_i} f_j(x), g_j(x)\} E \, dx \right| = \sum_{j \in J} \sum_{J \in D^m} \langle b_i, h_J \rangle \{\langle f_j \rangle J, \langle g_j \rangle h_J \}_E \leq \sum_{j \in J} \sum_{J \in D^m} |\langle b_i, h_J \rangle| |\{\langle f_j \rangle J, \langle g_j \rangle h_J \}_E| \lesssim \sum_{j \in J} \int_{\mathbb{R}^m} \left( \sum_{J \in D^m} |\{\langle f_j \rangle J, \langle g_j \rangle h_J \}_E|^2 1_{J(y)} |J| \right)^{1/2} dy,
\]
where for each \( j \) we used \( \|b_j\|_{\text{BMO}_{D^m}} \leq 1 \) via the following inequality

\[
\sum_{j \in D^m} |\langle b_j, h_j \rangle| a_j \lesssim \int_{\mathbb{R}^m} \left( \sum_{j \in D^m} |a_j|^{2 \frac{1}{|J|}} \right)^{1/2} dy.
\]

Here, \((a_j)_{j \in D^m}\) can be an arbitrary sequence of scalars (see Section 2.4).

For a fixed \( y \in \mathbb{R}^m \) we have (by using \( \ell^2 \) duality) that

\[
\left( \sum_{j \in D^m} |\langle f_j, \langle g_j, h_j \rangle \rangle_E| \right)^{1/2} \lesssim \left\{ M_{D^m,E} f_j(y), \left( \sum_{j \in D^m} |\langle g_j, h_j \rangle|^{2 \frac{1}{|J|}} \right)^{1/2} \right\}_E.
\]

Therefore, we have

\[
\left| \sum_{j \in J} \int_{\mathbb{R}^m} \left\{ \pi_{b_j} f_j(x), g_j(x) \right\}_E dx \right| \lesssim \int_{\mathbb{R}^m} \sum_{j \in J} \left\{ M_{D^m,E} f_j(y), \left( \sum_{j \in D^m} |\langle g_j, h_j \rangle|^{2 \frac{1}{|J|}} \right)^{1/2} \right\}_E dy
\]

\[
\lesssim \int_{\mathbb{R}^m} \left\{ \left( \sum_{j \in J} |M_{D^m,E} f_j(y)|^{2} \right)^{1/2}, \left( \sum_{j \in J} \sum_{j \in D^m} |\langle g_j, h_j \rangle|^{2 \frac{1}{|J|}} \right)^{1/2} \right\}_E dy
\]

\[
\lesssim \left\| \left( \sum_{j \in J} |M_{D^m,E} f_j|^{2} \right)^{1/2} \right\|_{L^p(\mathbb{R}^m; E^e)} \left\| \left( \sum_{j \in J} \sum_{j \in D^m} |\langle g_j, h_j \rangle|^{2 \frac{1}{|J|}} \right)^{1/2} \right\|_{L^p(\mathbb{R}^m; E^e)}
\]

That

\[
\left\| \left( \sum_{j \in J} |M_{D^m,E} f_j|^{2} \right)^{1/2} \right\|_{L^p(\mathbb{R}^m; E)} \lesssim \left\| \left( \sum_{j \in J} |f_j|^{2} \right)^{1/2} \right\|_{L^p(\mathbb{R}^m; E)}
\]

holds follows from Proposition 2.10 (see also the discussion below the proposition). The next estimate follows from the one-parameter version of Corollary 2.9:

\[
\left\| \left( \sum_{j \in J} \sum_{j \in D^m} |\langle g_j, h_j \rangle|^{2 \frac{1}{|J|}} \right)^{1/2} \right\|_{L^{p'}(\mathbb{R}^m; E^e)} \lesssim \left\| \left( \sum_{j \in J} |g_j|^{2} \right)^{1/2} \right\|_{L^{p'}(\mathbb{R}^m; E^e)} \quad \Box
\]

5.2. \( R \)-boundedness of bi-parameter paraproducts

**Proposition 5.5.** Suppose that \( \|b_i\|_{\text{BMO}_{D^m,D^k}} \leq 1, i \in \mathcal{I}, E \) is a UMD function lattice and \( p, r \in (1, \infty) \). Then

\[
\mathcal{R}(\{\Pi_{D^m,D^k} b_i \in L^p(\mathbb{R}^m; L^r(\mathbb{R}^k; E)) : i \in \mathcal{I}\}) \lesssim 1.
\]
Proof. We denote \( \Pi_{b_j} := \Pi_{D^m, D^k, b_j} \). We will show that given a finite \( \mathcal{J} \subset \mathcal{I} \), \( f_j \in L^p(\mathbb{R}^m; L^r(\mathbb{R}^k; E)) \) and \( g_j \in L^{p'}(\mathbb{R}^m; L^{r'}(\mathbb{R}^k; E^*)) \) we have
\[
\left| \sum_{j \in \mathcal{J}} \iint_{\mathbb{R}^{m+k}} \{ \Pi_{b_j}, f_j(x_1, x_2), g_j(x_1, x_2) \} \right|_E dx_1 \, dx_2 \\
\lesssim \left( \sum_{j \in \mathcal{J}} |f_j|^2 \right)^{1/2}_{L^p(\mathbb{R}^m; L^r(\mathbb{R}^k; E))] \left( \sum_{j \in \mathcal{J}} |g_j|^2 \right)^{1/2}_{L^{p'}(\mathbb{R}^m; L^{r'}(\mathbb{R}^k; E^*))}.
\]
That this is enough is justified like in Proposition 5.4.

Recall that
\[
\Pi_{b_j} f_j(x_1, x_2) = \sum_{V \in D^m} \sum_{U \in D^k} \lambda_{j, V, U} (f_j)_{V \times U} h_V(x_1) h_U(x_2),
\]
where the scalars \( \lambda_{j, V, U} \) satisfy for all \( j \) and all scalars \( A_{V, U} \) that
\[
\sum_{V \in D^m} \sum_{U \in D^k} |\lambda_{j, V, U}| |A_{V, U}| \lesssim \iint_{\mathbb{R}^{m+k}} \left( \sum_{V \in D^m} \sum_{U \in D^k} |A_{V, U}|^2 \frac{1_V \otimes 1_U}{|V||U|} \right)^{1/2}.
\]
This gives
\[
\left| \sum_{j \in \mathcal{J}} \iint_{\mathbb{R}^{m+k}} \{ \Pi_{b_j}, f_j, g_j \} \right|_E \leq \sum_{j \in \mathcal{J}} \sum_{V \in D^m} \sum_{U \in D^k} |\lambda_{j, V, U}| \left\{ (f_j)_{V \times U}, (g_j, h_V \otimes h_U) \right\}_E \\
\lesssim \sum_{j \in \mathcal{J}} \iint_{\mathbb{R}^{m+k}} \left( \sum_{V \in D^m} \sum_{U \in D^k} \left\{ M_{D^m, D^k, E} f_j, (g_j, h_V \otimes h_U) \right\}_E^2 \frac{1_V \otimes 1_U}{|V||U|} \right)^{1/2},
\]
which is further bounded by
\[
\left| \sum_{j \in \mathcal{J}} \iint_{\mathbb{R}^{m+k}} \left\{ M_{D^m, D^k, E} f_j, \left( \sum_{V \in D^m} \sum_{U \in D^k} |(g_j, h_V \otimes h_U)|^2 \frac{1_V \otimes 1_U}{|V||U|} \right)^{1/2} \right\}_E \\
\lesssim \left\| \sum_{j \in \mathcal{J}} \left| M_{D^m, D^k, E} f_j \right|^2 \right\|_{L^p(\mathbb{R}^m; L^r(\mathbb{R}^k; E))} \\
\times \left\| \sum_{j \in \mathcal{J}} \sum_{V \in D^m} \sum_{U \in D^k} |(g_j, h_V \otimes h_U)|^2 \frac{1_V \otimes 1_U}{|V||U|} \right\|^{1/2}_{L^{p'}(\mathbb{R}^m; L^{r'}(\mathbb{R}^k; E^*))}.
\]
The proof is finished by applying Corollaries 2.9 and 2.13. \( \square \)
Proposition 5.6. Suppose $\|b_i\|_{\text{BMO}_D^{m,k}} \leq 1$, $i \in I$, $E$ is a UMD function lattice and $p, r \in (1, \infty)$. Then

$$\mathcal{R}(\{\Pi_{b_i}^{\text{mixed}} \in L(L^p(\mathbb{R}^m; L^r(\mathbb{R}^k; E)) : i \in I\}) \lesssim 1.$$  

Proof. We denote $\Pi = \Pi_{b_i}^{\text{mixed}}$. We will show that given a finite $J \subset I$, $f_j \in L^p(\mathbb{R}^m; L^r(\mathbb{R}^k; E))$ and $g_j \in L^p(\mathbb{R}^m; L^r(\mathbb{R}^k; E^*))$ we have

$$\left| \sum_{j \in J} \int_{\mathbb{R}^{m+k}} \{ \Pi_{b_j}^{\text{mixed}} f_j(x_1, x_2), g_j(x_1, x_2) \}_E \, dx_1 \, dx_2 \right| \lesssim \left( \sum_{j \in J} |f_j|^2 \right)^{1/2} \left( \sum_{j \in J} |g_j|^2 \right)^{1/2}.$$  

That this is enough is justified like in Proposition 5.4.

Estimating as in the previous proposition we get

$$\left| \sum_{j \in J} \int_{\mathbb{R}^{m+k}} \{ \Pi_{b_j}^{\text{mixed}} f_j, g_j \}_E \right| \lesssim \sum_{j \in J} \int_{\mathbb{R}^{m+k}} \left( \sum_{V \in D^m \atop U \in D^k} \left| \left\{ \left( f_j, h_V \otimes \frac{1_U}{|U|} \right), \left( g_j, \frac{1_V}{|V|} \otimes h_U \right) \right\}_E \right| \frac{1}{|V||U|} \right)^{1/2} \lesssim \sum_{j \in J} \int_{\mathbb{R}^{m+k}} \left( \sum_{V \in D^m \atop U \in D^k} \left\{ M_{D^k, E}^{m}(f_j, h_V)_1, M_{D^m, E^*}(g_j, h_U)_2 \right\}_E \frac{1}{|V||U|} \right)^{1/2}.$$  

Note that

$$\left( \sum_{V \in D^m \atop U \in D^k} \left\{ M_{D^k, E}^{m}(f_j, h_V)_1, M_{D^m, E^*}(g_j, h_U)_2 \right\}_E \frac{1}{|V||U|} \right)^{1/2} \lesssim \left( \sum_{V \in D^m} \left( \frac{1}{|V|^2} M_{D}^{k}(f_j, h_V)_1 \right)^{1/2} \left( \sum_{U \in D^k} \left[ M_{D^m, E^*}(g_j, h_U)_2 \right]^2 \frac{1}{|U|} \right)^{1/2} \right)^{1/2} \lesssim \left( \sum_{V \in D^m} \left( \frac{1}{|V|} \left[ M_{D^k, E}^{m}(f_j, h_V)_1 \right]^2 \right)^{1/2} \left( \sum_{U \in D^k} \left[ M_{D^m, E^*}(g_j, h_U)_2 \right]^2 \frac{1}{|U|} \right)^{1/2} \right)^{1/2}.$$
so that

\[
\left| \sum_{j \in J} \int \int_{\mathbb{R}^{m+k}} \Pi_{b_j}^{\text{mixed}} f_j(x_1, x_2), g_j(x_1, x_2) \right|_{E} dx_1 dx_2 \\
\lesssim \left\| \left( \sum_{j \in J} \sum_{V \in D^m} \frac{1}{|V|} \otimes [M_{D^k, E} \langle f_j, h_V \rangle_1]^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^m; L^r(\mathbb{R}^k; E))} \\
\times \left\| \left( \sum_{j \in J} \sum_{U \in D^k} [M_{D^m, E^*} \langle g_j, h_U \rangle_2^2 \otimes \frac{1}{|U|}] \right)^{1/2} \right\|_{L^p(\mathbb{R}^m; L^r(\mathbb{R}^k; E^*))}.
\]

There holds

\[
\left\| \left( \sum_{j \in J} \sum_{V \in D^m} \frac{1}{|V|} \otimes [M_{D^k, E} \langle f_j, h_V \rangle_1]^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^m; L^r(\mathbb{R}^k; E))} \\
\lesssim \left\| \left( \sum_{j \in J} \sum_{V \in D^m} \frac{1}{|V|} \otimes (f_j, h_V)^2_1 \right)^{1/2} \right\|_{L^p(\mathbb{R}^m; L^r(\mathbb{R}^k; E))} \\
\lesssim \left\| \left( \sum_{j \in J} \frac{|f_j|}{|I_j|} \right)^{1/2} \right\|_{L^p(\mathbb{R}^m; L^r(\mathbb{R}^k; E))}.
\]

In the first step we used Proposition 2.10 (see again also the discussion after the proposition) in the inner integral over \( \mathbb{R}^k \). The second step was an application of Corollary 2.9. The term related to the sequence \( \{g_j\} \) is handled with a corresponding argument, except that the first step is now an application of Corollary 2.12. \( \square \)

**Remark 5.7.** The estimate in \( L^r(\mathbb{R}^k; L^p(\mathbb{R}^m; E)) \) follows with the same proof.

### 6. Bi-parameter partial paraproducts

A bi-parameter partial paraproduct is an operator of the form

\[
P_{D^m, D^m}^{I_1, I_2} f = \sum_{K \in D^n} \sum_{I_1, I_2 \in D^n} h_{I_2} \otimes \pi_{D^m, b_{K, I_1, I_2}}(\langle f, h_{I_1} \rangle_1), \quad f \in L^1_{\text{loc}}(\mathbb{R}^{n+m}; E),
\]

where \( E \) is a UMD space with the property (\( \alpha \)) of Pisier and \( \pi_{D^m, b_{K, I_1, I_2}} \) is a dyadic (one-parameter) paraproduct for some function

\[
b_{K, I_1, I_2} : \mathbb{R}^m \to \mathbb{R}
\]

satisfying

\[
\|b_{K, I_1, I_2}\|_{\text{BMO}_{D^m}} \leq \frac{|I_1|^{1/2}|I_2|^{1/2}}{|K|}.
\]
With $E = \mathbb{R}$ (or $\mathbb{C}$) this is the exact form in which these operators appear in the bi-parameter representation theorem [18]. Of course, such operators appear also in the form that contains the dual paraproducts $\pi_{D^m, b_{K, i_1, i_2}}^*$, and in the form that the paraproduct component is in $\mathbb{R}^n$.

**Theorem 6.1.** Let $E$ be a UMD function lattice, $p, q \in (1, \infty)$ and $i_1, i_2 \geq 0$. Suppose that for each $k \in K$ we are given a partial paraproduct

$$
P_{D^n, D^m, k}^{i_1, i_2} f = \sum_{K \in D^n} \sum_{I_1, I_2 \in D^n} h_{I_2} \otimes \pi_{D^m, b_{K, i_1, i_2, k}}(\langle f, h_{I_1} \rangle_1), \quad f \in L^1_{\text{loc}}(\mathbb{R}^{n+m}; E),$$

where

$$||b_{K, i_1, i_2, k}||_{\text{BMO}_{D^m}} \leq \frac{|I_1|^{1/2}|I_2|^{1/2}}{|K|}.$$ 

Then we have

$$\mathcal{R}(\{P_{D^n, D^m, k}^{i_1, i_2} \in \mathcal{L}(L^q(\mathbb{R}^n; L^p(\mathbb{R}^m; E))): k \in K\}) \lesssim \min(i_1, i_2) + 1$$

and

$$\mathcal{R}(\{P_{D^n, D^m, k}^{i_1, i_2} \in \mathcal{L}(L^p(\mathbb{R}^n; L^q(\mathbb{R}^m, E))): k \in K\}) \lesssim \min(i_1, i_2) + 1.$$

**Proof.** Fix $p \in (1, \infty)$. We can see the partial paraproduct $P_{D^n, D^m, k}^{i_1, i_2}$, $k \in K$, as a model operator when it acts on locally integrable functions $f: \mathbb{R}^n \to F$, where $F = L^p(\mathbb{R}^m; E)$. Proposition 4.1 says that for all $q \in (1, \infty)$ we have

$$\mathcal{R}(\{P_{D^n, D^m, k}^{i_1, i_2} \in \mathcal{L}(L^q(\mathbb{R}^n; L^p(\mathbb{R}^m, E))): k \in K\}) \lesssim (\min(i_1, i_2) + 1)R_p(E),$$

where

$$R_p(E) := \mathcal{R}\left(\left\{ \frac{|K|}{|I_1|^{1/2}|I_2|^{1/2}} \pi_{D^m, b_{K, i_1, i_2, k}} \in \mathcal{L}(L^p(\mathbb{R}^m; E)): k \in K, K, I_1, I_2 \in D^n \right\}\right).$$

Proposition 5.4 says that $R_p(E) \lesssim 1$, as $E$ is a UMD function lattice.

On the other hand, Proposition 4.2 says that for all $p, q \in (1, \infty)$ we have

$$\mathcal{R}(\{P_{D^n, D^m, k}^{i_1, i_2} \in \mathcal{L}(L^p(\mathbb{R}^m; L^q(\mathbb{R}^n; E))): k \in K\}) \lesssim (\min(i_1, i_2) + 1)R_{p,q}(E),$$

where

$$R_{p,q}(E) = \mathcal{R}\left(\left\{ \frac{|K|}{|I_1|^{1/2}|I_2|^{1/2}} \pi_{D^m, b_{K, i_1, i_2, k}} \in \mathcal{L}(L^p(\mathbb{R}^m; L^q(\mathbb{R}^n \times Y^n; E))): k \in K, K, I_1, I_2 \in D^n \right\}\right).$$

Since $E$ is a UMD function lattice, we can apply Proposition 5.4 with $L^q(\mathbb{R}^n \times Y^n; E)$. This gives that $R_{p,q}(E) \lesssim 1$. 

**Remark 6.2.** We wrote the proof like we did to illustrate the following point. Note that even if we were interested only in the case $E = \mathbb{R}$, the second part of the proof would require applying Proposition 5.4 to the UMD function lattice $L^q(\mathbb{R}^n \times Y^n)$. That is, we would need the $\mathcal{R}$-boundedness of ordinary paraproducts in some vector-valued setting anyway! Of course, in the $p = q$ case we could just use the easier proof, that is, the first part of the proof.
7. Application to bi-parameter singular integrals

The definition of a bi-parameter singular integral $T$ is somewhat lengthy. We only give a brief idea here, for the full details see [18].

The definition involves first of all the structural assumption that $T$ should have a full kernel representation, that is, $\langle T f, g \rangle$ can be written as an integral operator with a kernel $K: (\mathbb{R}^{n+m} \times \mathbb{R}^{n+m}) \setminus \{(x,y) \in \mathbb{R}^{n+m} \times \mathbb{R}^{n+m} : x_1 = y_1 \text{ or } x_2 = y_2 \} \to \mathbb{R}$, when $f = f_1 \otimes f_2$ and $g = g_1 \otimes g_2$ are of tensor product form with $\text{spt } f_1 \cap \text{spt } g_1 = \emptyset$ and $\text{spt } f_2 \cap \text{spt } g_2 = \emptyset$. The kernel $K$ needs to satisfy various estimates: the size estimate, Hölder estimates, and mixed Hölder and size estimates. Some Calderón–Zygmund structure on $\mathbb{R}^n$ and $\mathbb{R}^m$ is also demanded separately. This entails having regular enough kernel representations when only $\text{spt } f_1 \cap \text{spt } g_1 = \emptyset$ or $\text{spt } f_2 \cap \text{spt } g_2 = \emptyset$.

Next, we demand various boundedness and cancellation assumptions: the weak boundedness assumption, some diagonal BMO conditions, and finally that $T_1, T^{*}_1, T_2(1)$ and $T^{*}_2(1)$ belong to $\text{BMO}_{\text{prod}}(\mathbb{R}^{n+m})$. Here, $T_1$ is the first partial adjoint of $T$, that is, $\langle T_1(f_1 \otimes f_2), g_1 \otimes g_2 \rangle = \langle (Tf_1 \otimes f_2), f_1 \otimes g_2 \rangle$.

With such assumptions it was shown in [18] that a specific dyadic representation of bi-parameter singular integrals holds. For all bounded and compactly supported $f, g: \mathbb{R}^{n+m} \to \mathbb{R}$, we have

$$\langle Tf, g \rangle = C_T \mathbb{E}_{u^n} \mathbb{E}_{w^m} \sum_{(i_1,i_2) \in \mathbb{Z}_2^n} \sum_{(j_1,j_2) \in \mathbb{Z}_2^m} \alpha_{i_1,i_2,j_1,j_2} \sum_u \langle D^{n,i,j}_{D^n,D^m,u} f, g \rangle,$$

where $\alpha_{i_1,i_2,j_1,j_2} = 2^{-\max(i_1 + i_2)/2} 2^{-\max(j_1 + j_2)/2} (\delta$ appears on the Hölder estimates of the kernels) and $u$ runs over some finitely many integers. Here, $S^{i_1,i_2,j_1,j_2}_{D^n,D^m,u}$ is a bi-parameter shift (such as we have defined in this paper even in the operator-valued setting) with the scalar-valued kernels being uniformly bounded by $1$. In the other remaining cases $S^{i_1,i_2,j_1,j_2}_{D^n,D^m,u}$ can either be a parameter shift, some partial paraproduct with the paraproduct component in $\mathbb{R}^n$ or $\mathbb{R}^m$ (BMO bounds normalised like $(|I_1|^{1/2}/|I_2|^{1/2})/|K|$ or $(|J_1|^{1/2}|J_2|^{1/2})/|V|$), or a full bi-parameter paraproduct, either of standard type or of mixed type, associated with a product BMO function of norm at most $1$. We have one full standard paraproduct, one adjoint of such, one mixed paraproduct and an adjoint of such, and these appear in the case $(i_1,i_2,j_1,j_2) = (0,0,0,0)$. Moreover, the average is taken over all random dyadic grids $D^n = D^n(w^n)$ and $D^m = D^m(w^m)$.

If $E$ is a UMD space with the property $(\alpha)$ of Pisier, we can apply this to simple functions $f = \sum_{a=1}^A f_a e_a$ and $g = \sum_{b=1}^B g_b e^*_b$, where $f_a: \mathbb{R}^{n+m} \to \mathbb{R}$ and $g_b: \mathbb{R}^{n+m} \to \mathbb{R}$ are bounded and compactly supported, $e_a \in E$ and $e^*_b \in E^*$. This gives that

$$\int_{\mathbb{R}^{n+m}} \{T f(x), g(x)\}_E \, dx = C_T \mathbb{E}_{u^n} \mathbb{E}_{w^m} \sum_{(i_1,i_2) \in \mathbb{Z}_2^n} \sum_{(j_1,j_2) \in \mathbb{Z}_2^m} \sum_u \int_{\mathbb{R}^{n+m}} \{S^{i_1,i_2,j_1,j_2}_{D^n,D^m,u} f(x), g(x)\}_E \, dx.$$

Here, the interpretation is clear: $T f(x_1, x_2) = \sum_{a=1}^A T f_a(x_1, x_2) e_a$ (and the same with the shifts).

**Theorem 7.1.** Suppose $I$ is an index set, and that for each $i \in I$ we are given a bi-parameter SIO as in [18]. Suppose that the Hölder exponents of the kernels of these operators are uniformly bounded from below, and that all the other constants in the assumptions are
uniformly bounded from above. Let $E$ be a UMD function lattice, and $p, q \in (1, \infty)$. Then we have

$$\mathcal{R}(\{T_i \in \mathcal{L}(L^q(\mathbb{R}^n; L^p(\mathbb{R}^m; E))): i \in I\}) \lesssim 1.$$  

**Proof.** Fix a finite $K \subset I$ and simple functions $f_k: \mathbb{R}^{n+m} \to E$, $k \in K$. We need to prove that

$$\left( \mathbb{E} \left\| \sum_{k \in K} \epsilon_k T_k f_k \right\|_{L^q(\mathbb{R}^n; L^p(\mathbb{R}^m; E))}^2 \right)^{1/2} \lesssim \left( \mathbb{E} \left\| \sum_{k \in K} \epsilon_k f_k \right\|_{L^q(\mathbb{R}^n; L^p(\mathbb{R}^m; E))}^2 \right)^{1/2}.$$  

Lemma 2.1 says that the RHS is dominated by

$$\sup_{K \subset I} \left| \sum_{k \in K} \int_{\mathbb{R}^{n+m}} \{T_k f_k(x), g_k(x)\}_E dx \right|,$$

where the supremum is taken over those simple $g_k: \mathbb{R}^{n+m} \to E$ for which

$$\left( \mathbb{E} \left\| \sum_{k \in K} \epsilon_k g_k \right\|_{L^{q'}(\mathbb{R}^n; L^{p'}(\mathbb{R}^m; E^*))}^2 \right)^{1/2} \lesssim 1.$$  

With any fixed such sequence $(g_k)_{k \in K}$ we write

$$\left| \sum_{k \in K} \int_{\mathbb{R}^{n+m}} \{T_k f_k(x), g_k(x)\}_E dx \right| \leq C \mathbb{E}_{w_n} \mathbb{E}_{w_m} \sum_{\alpha_{i_1, i_2, j_1, j_2} \in \mathbb{Z}^2_+} \sum_{\alpha_{i_1, i_2, j_1, j_2} \in \mathbb{Z}^2_+} \alpha_{i_1, i_2, j_1, j_2} \sum_{\alpha_{i_1, i_2, j_1, j_2} \in \mathbb{Z}^2_+} \int_{\mathbb{R}^{n+m}} \left\{ S^{i_1, i_2, j_1, j_2}_{D^{i_1, i_2, j_1, j_2}, D^{i_2, j_1, j_2}, k, u} f_k(x), g_k(x) \right\}_E dx \right| \leq C \mathbb{E}_{w_n} \mathbb{E}_{w_m} \alpha_{i_1, i_2, j_1, j_2} \sum_{\alpha_{i_1, i_2, j_1, j_2} \in \mathbb{Z}^2_+} \sum_{\alpha_{i_1, i_2, j_1, j_2} \in \mathbb{Z}^2_+} \int_{\mathbb{R}^{n+m}} \left\{ S^{i_1, i_2, j_1, j_2}_{D^{i_1, i_2, j_1, j_2}, D^{i_2, j_1, j_2}, k, u} f_k(x), g_k(x) \right\}_E dx \right|.

We have

$$\left| \sum_{k \in K} \int_{\mathbb{R}^{n+m}} \left\{ S^{i_1, i_2, j_1, j_2}_{D^{i_1, i_2, j_1, j_2}, D^{i_2, j_1, j_2}, k, u} f_k(x), g_k(x) \right\}_E dx \right| \leq \mathbb{E} \left| \int_{\mathbb{R}^{n+m}} \left\{ \sum_{k \in K} \epsilon_k S^{i_1, i_2, j_1, j_2}_{D^{i_1, i_2, j_1, j_2}, D^{i_2, j_1, j_2}, k, u} f_k(x), \sum_{k' \in K} \epsilon_k' g_{k'}(x) \right\}_E dx \right| \leq \left( \mathbb{E} \left\| \sum_{k \in K} \epsilon_k S^{i_1, i_2, j_1, j_2}_{D^{i_1, i_2, j_1, j_2}, D^{i_2, j_1, j_2}, k, u} f_k \right\|^2_{L^q(\mathbb{R}^n; L^p(\mathbb{R}^m; E))} \right)^{1/2}.$$  

If for the fixed $i_1, i_2, j_1, j_2$ and $u$ the appearing operators $S^{i_1, i_2, j_1, j_2}_{D^{i_1, i_2, j_1, j_2}, D^{i_2, j_1, j_2}, k, u}$, $k \in K$, are bi-parameter shifts, Proposition 3.18 (a special case where the kernels are scalar-valued and pointwise
uniformly bounded by 1) gives
\[
\left( E \left\| \sum_{k \in K} \epsilon_k S^{i_1,i_2,j_1,j_2}_{D^m,D^m,k,u} f_k \right\|^2_{L^q(\mathbb{R}^n;L^p(\mathbb{R}^m;E))} \right)^{1/2} 
\leq (\min(i_1,i_2)+1)(\min(j_1,j_2)+1) \left( E \left\| \sum_{k \in K} \epsilon_k f_k \right\|^2_{L^q(\mathbb{R}^n;L^p(\mathbb{R}^m;E))} \right)^{1/2}.
\]

If they are partial paraproducts, then Theorem 6.1 gives the same bound. Note that even if we
have here arbitrarily chosen the symmetry \(L^q(\mathbb{R}^n;L^p(\mathbb{R}^m;E))\) (and not \(L^p(\mathbb{R}^m;L^q(\mathbb{R}^k;E))\)), we 
ever also need the second inequality of Theorem 6.1 here (to handle the case \(S^{0,0,0,0}_{D^m,D^m,k,u}\) where the paraproduct component is \(\mathbb{R}^n\)). Finally, Propositions 5.5 and 5.6 give the same bound
in the case that the operators \(S^{0,0,0,0}_{D^m,D^m,k,u}\), \(k \in K\), are full paraproducts of the same type. The
proof is complete.

8. Tri-parameter partial paraproducts

We conclude the paper by studying boundedness properties of tri-parameter partial paraproducts. They come in essentially two different forms, here called type 1 and type 2. The proved estimates are a key step in proving that tri-parameter singular integrals are
proved estimates are a key step in proving that tri-parameter singular integrals are
L\(\infty\) bounded for all \(p,q,r\in(1,\infty)\) and UMD function lattices \(E\), that is, for proving the analog
of Theorem 7.1 for tri-parameter singular integrals. In addition to these partial paraproducts one would also need to consider the full tri-parameter paraproducts of different flavours and tri-parameter shifts. The shifts would be straightforward to handle, and we believe that the full paraproducts should not pose problems either.

8.1. Tri-parameter partial paraproducts of type 1

A tri-parameter partial paraproduct of type 1 is an operator of the form
\[
P^{i_1,i_2}_{D^m,D^m,D^k} f = \sum_{K \in \mathbb{D}^m} \sum_{l_1,l_2 \in \mathbb{D}^n \atop l_1^{(i_1)}=l_2^{(i_2)}=K} h_{l_2} \otimes B_{D^m,D^k,K,l_1,l_2}(\langle f,h_{l_1} \rangle_1),
\]
where \(f \in L^1_{\text{loc}}(\mathbb{R}^{n+m+k};E)\), \(E\) is a UMD space satisfying Pisier’s property \((\alpha)\), and
\(B_{D^m,D^k,K,l_1,l_2}\) is a standard full bi-parameter paraproduct \(\Pi^{\text{mixed}}_{D^m,D^k,b_{K,l_1,l_2}}\) for all \(K,l_1,l_2\), or a mixed full bi-parameter paraproduct \(\Pi^{\text{mixed}}_{D^m,D^k,b_{K,l_1,l_2}}\) for all \(K,l_1,l_2\). The functions
\[
b_{K,l_1,l_2} : \mathbb{R}^{m+k} \to \mathbb{R}
\]
satisfy
\[
\|b_{K,l_1,l_2}\|_{\text{BMO}^{\text{mixed}}_{D^m,D^k}} \leq \frac{|l_1|^{1/2}|l_2|^{1/2}}{|K|}.
\]

**Proposition 8.1.** Let \(E\) be a UMD function lattice, \(p,q,r \in (1,\infty)\) and \(i_1,i_2 \geq 0\). Suppose
that for each \(s \in \mathcal{S}\) we are given a tri-parameter partial paraproduct of type 1
\[
P^{i_1,i_2}_{D^m,D^m,D^k,s} f = \sum_{K \in \mathbb{D}^m} \sum_{l_1,l_2 \in \mathbb{D}^n \atop l_1^{(i_1)}=l_2^{(i_2)}=K} h_{l_2} \otimes B_{D^m,D^k,K,l_1,l_2,s}(\langle f,h_{l_1} \rangle_1),
\]
where either \( B_{D_m^1,D_m^k,K,I_1,I_2,s} = \Pi_{D_m^1,D_m^k,K,I_1,I_2,s} \) for all \( s \in S \) and \( K, I_1, I_2 \in \mathcal{D}^n \) or \( B_{D_m^1,D_m^k,K,I_1,I_2,s} = \Pi_{D_m^1,D_m^k,K,I_1,I_2,s}^{\text{mixed}} \) for all \( s \in S \) and \( K, I_1, I_2 \in \mathcal{D}^n \). Moreover, assume that
\[
\|b_{K,I_1,I_2,s}\|_{\text{BMO}_{D_m^1,D_m^k}} \leq \frac{|I_1|^{1/2}|I_2|^{1/2}}{|K|}.
\]
Then we have
\[
R(\{P_{D_m^1,D_m^k,K,I_1,I_2,s}^{i_1,i_2} \in L(L^q(\mathbb{R}^n; L^p(\mathbb{R}^m; L^r(\mathbb{R}^k; E))) : s \in S\}) \lesssim \min(i_1, i_2) + 1,
\]
and the same is true with all the other permutations of \( L^q, L^p \) and \( L^r \).

**Proof.** We view \( P_{D_m^1,D_m^k,K,I_1,I_2,s}^{i_1,i_2} \) as a model operator acting on locally integrable functions \( f: \mathbb{R}^n \to F \), where \( F = L^p(\mathbb{R}^m; L^r(\mathbb{R}^k; E)) \). Proposition 4.1 says that
\[
R(\{P_{D_m^1,D_m^k,K,I_1,I_2,s}^{i_1,i_2} \in L(L^q(\mathbb{R}^n; L^p(\mathbb{R}^m; L^r(\mathbb{R}^k; E))) : s \in S\}) \lesssim (\min(i_1, i_2) + 1)R_{p,r}(E),
\]
where \( R_{p,r}(E) \) is the constant
\[
R\left( \left\{ \left| \frac{|K|}{|I_1|^{1/2}|I_2|^{1/2}} \right| B_{D_m^1,D_m^k,K,I_1,I_2,s} \in L(L^p(\mathbb{R}^m; L^r(\mathbb{R}^k; E))) : s \in S, K, I_1, I_2 \in \mathcal{D}^n \right\} \right).
\]
Propositions 5.5 and 5.6 say that \( R_{p,r}(E) \approx 1 \).

Proposition 4.2 says that
\[
R(\{P_{D_m^1,D_m^k,K,I_1,I_2,s}^{i_1,i_2} \in L(L^p(\mathbb{R}^m; L^q(\mathbb{R}^n; L^r(\mathbb{R}^k; E)))) : s \in S\}) \lesssim (\min(i_1, i_2) + 1)R_{p,q,r}^1(E),
\]
where \( R_{p,q,r}^1(E) \) is the constant
\[
R\left( \left\{ \left| \frac{|K|}{|I_1|^{1/2}|I_2|^{1/2}} \right| B_{D_m^1,D_m^k,K,I_1,I_2,s} \in L(L^p(\mathbb{R}^m; L^q(\mathbb{R}^n \times Y^n; L^r(\mathbb{R}^k; E)))) : s \in S, K, I_1, I_2 \in \mathcal{D}^n \right\} \right).
\]
To bound this constant we need a bit strange versions of Propositions 5.5 and 5.6. So, we need to check the \( R \)-boundedness of the extensions of these full bi-parameter parafunctionals in a space of the form \( L^p(\mathbb{R}^n; L^q(X; L^r(\mathbb{R}^k; E))) \), where \( X \) is some measure space. Let \( f_j = \sum_{a \in A_j} 1_{X_j,a} F_{j,a} \), where \( F_{j,a}: \mathbb{R}^{m+k} \to E \) and \( X_{j,a} \subset X \), and \( g_j = \sum_{c \in C_j} 1_{X_{j,c}} G_{j,c} \), where \( G_{j,c}: \mathbb{R}^{m+k} \to E^* \) and \( \hat{X}_{j,c} \subset X \). Following the proofs of the said propositions one ends up with the need to show that
\[
\left\| \left( \sum_j \left[ M_{D_m^1,D_m^k,E}^{1,3} f_j \right]^2 \right]^{1/2} \right\|_{L^p(\mathbb{R}^m; L^q(X; L^r(\mathbb{R}^k; E)))}
\]
and
\[
\left\| \left( \sum_j \sum_{V \in D^m} \left\langle g_j, h_V \otimes h_U \right\rangle_{1,3} \right)^2 \left( \left\| V \right\| \left\| U \right\| \right)^{1/2} \right\|_{L^p(\mathbb{R}^m; L^{q'}(X; L^{r'}(\mathbb{R}^k; E^*)))}
\]
are bounded by
\[ \left\| \left( \sum_j |f_j|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^m;L^q(X;L^r(\mathbb{R}^k;E)))} \] and
\[ \left\| \left( \sum_j |g_j|^2 \right)^{1/2} \right\|_{L^{p'}(\mathbb{R}^m;L^{q'}(X;L^{r'}(\mathbb{R}^k;E^*))))}, \]
respectively, and the same with
\[ \left\| \left( \sum_j \sum_{V \in D^m} \left| M_{D^k,E}^j \left( \frac{1}{|V|^{1/2}} \otimes \langle f_j, h_V \rangle_1 \right) \right|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^m;L^q(X;L^r(\mathbb{R}^k;E)))} \]
and
\[ \left\| \left( \sum_j \sum_{U \in D^k} \left| M_{D^m,E^*}^j \left( \langle g_j, h_U \rangle_3 \otimes \frac{1}{|U|^{1/2}} \right) \right|^2 \right)^{1/2} \right\|_{L^{p'}(\mathbb{R}^m;L^{q'}(X;L^{r'}(\mathbb{R}^k;E^*))}} . \]
But these are all bounded essentially in the same way as in the original propositions. Therefore, we have \( R^1_{p,q,r}(E) \leq 1. \)
Proposition 4.3 says that
\[ \mathcal{R}(\{ P_{D^m,D^m,D^k}^{i_1,i_2,j_1,j_2}, s \in L^p(\mathbb{R}^m;L^q(\mathbb{R}^k;L^r(\mathbb{R}^n;E))) : s \in \mathcal{S} \}) \lesssim (\min(i_1,i_2)+1) R^2_{p,q,r}(E), \]
where \( R^2_{p,q,r}(E) \) is the constant
\[ \mathcal{R}\left\{ \frac{|K|}{|I_1|^{1/2}|I_2|^{1/2}} B_{D^m,D^k,K,I_1,I_2,s} \in L^p(\mathbb{R}^m;L^q(\mathbb{R}^k;L^p(\mathbb{R}^n \times Y^n;E))) : \right. \]
satisfying
\[ \left. s \in \mathcal{S}, K, I_1, I_2 \in D^m \right\}. \]
That \( R^2_{p,q,r}(E) \leq 1 \) follows from Propositions 5.5 and 5.6 applied with \( E \) replaced by \( L^q(\mathbb{R}^n \times Y^n;E) \).

We can do all of the above but just starting with \( F = L^r(\mathbb{R}^k;L^p(\mathbb{R}^n;E)) \), so all of the symmetries follow. \( \square \)

### 8.2. Tri-parameter partial paraproducts of type 2

A tri-parameter partial paraproduct of type 2 is an operator of the form
\[ P_{D^m,D^m,D^k}^{i_1,i_2,j_1,j_2} f \]
\[ = \sum_{K \in D^m} \sum_{V \in D^m} \sum_{I_1,J_2 \in D^m} \sum_{J_1,J_2 \in D^m} h_{I_2} \otimes h_{J_2} \otimes \pi_{D^k,b_{K,V,I_1,j_2,J_1,J_2}} \langle (f, h_{I_1} \otimes h_{J_1}) \rangle_{I_2,J_2}, \]
where \( f \in L^1_{\text{loc}}(\mathbb{R}^{n+m+k};E) \), \( E \) is a UMD space satisfying Pisier’s property \((\alpha)\), and \( \pi_{D^k,b_{K,V,I_1,j_2,J_1,J_2}} \) is a dyadic paraproduct for some function
\[ b_{K,V,I_1,J_2,J_1,J_2} : \mathbb{R}^k \to \mathbb{R} \]
satisfying
\[ \|b_{K,V,I_1,J_2,J_1,J_2}\|_{\text{BMO}_{D^k}} \leq \frac{|I_1|^{1/2}|J_2|^{1/2} |J_1|^{1/2}|J_2|^{1/2}}{|K|}. \]
Proposition 8.2. Let $E$ be a UMD function lattice, $p, q, r \in (1, \infty)$ and $i_1, i_2, j_1, j_2 \geq 0$. Suppose that for each $s \in S$ we are given a tri-parameter partial paraproduct of type 2

$$P_{D^n, D^m, D^k, s}^{i_1, i_2, j_1, j_2} f = \sum_{K \in D^n} \sum_{I_1, I_2 \in D^m} \sum_{J_1, J_2 \in D^m} h_{I_2} \otimes h_{J_2} \otimes \pi_{D^k, b_{K,V,I_1, i_2, J_1, j_2, s}}((f, h_{I_1} \otimes h_{J_1}), 1, 2),$$

where

$$\|b_{K,V,I_1, i_2, J_1, j_2, s}\|_{\text{BMO}_{D^k}} \lesssim \frac{|I_1|^{1/2}|I_2|^{1/2}|J_1|^{1/2}|J_2|^{1/2}}{|K||V|}.$$

Then we have

$$R\left\{ P_{D^n, D^m, D^k, s}^{i_1, i_2, j_1, j_2} \in \mathcal{L}(L^q(\mathbb{R}^n; L^p(\mathbb{R}^m; L^r(\mathbb{R}^k; E))); s \in S) \right\} \lesssim (\min(i_1, i_2) + 1)(\min(j_1, j_2) + 1),$$

and the same is true with all the other permutations of $L^q$, $L^p$ and $L^r$.

Proof. Define for each $K, I_1, I_2 \in D^n$ and $s \in S$ the bi-parameter partial paraproduct

$$P_{D^m, D^k, K, I_1, I_2, s}^{i_1, i_2, j_1, j_2} g := \sum_{V \in D^m} \sum_{J_1, J_2 \in D^m} h_{J_2} \otimes \pi_{D^k, b_{K,V,I_1, i_2, J_1, j_2, s}}((g, h_{J_1}), 1),$$

where $g \in L^1_{\text{loc}}(\mathbb{R}^{m+k}; E)$. We have by Theorem 6.1 that

$$R\left\{ \frac{|K|}{|I_1|^{1/2}|I_2|^{1/2}} P_{D^m, D^k, K, I_1, I_2, s}^{i_1, i_2, j_1, j_2} \in \mathcal{L}(L^p(\mathbb{R}^m; L^r(\mathbb{R}^k; E))); s \in S, K, I_1, I_2 \in D^n \right\} \lesssim \min(j_1, j_2) + 1$$

and

$$R\left\{ \frac{|K|}{|I_1|^{1/2}|I_2|^{1/2}} P_{D^m, D^k, K, I_1, I_2, s}^{i_1, i_2, j_1, j_2} \in \mathcal{L}(L^r(\mathbb{R}^k; L^p(\mathbb{R}^m; E))); s \in S, K, I_1, I_2 \in D^n \right\} \lesssim \min(j_1, j_2) + 1.$$

Our original operators

$$P_{D^n, D^m, D^k, s}^{i_1, i_2, j_1, j_2} f = \sum_{K \in D^n} \sum_{I_1, I_2 \in D^m} \sum_{J_1, J_2 \in D^m} h_{I_2} \otimes h_{J_2} \otimes \pi_{D^k, b_{K,V,I_1, i_2, J_1, j_2, s}}((f, h_{I_1}), 1),$$

where $f \in L^1_{\text{loc}}(\mathbb{R}^{n+m+k}; E)$, are model operators of the above form. It follows from Proposition 4.1 that

$$R\left\{ P_{D^n, D^m, D^k, s}^{i_1, i_2, j_1, j_2} \in \mathcal{L}(L^q(\mathbb{R}^n; L^p(\mathbb{R}^m; L^r(\mathbb{R}^k; E))); s \in S) \right\} \lesssim (\min(i_1, i_2) + 1)(\min(j_1, j_2) + 1)$$

and

$$R\left\{ P_{D^n, D^m, D^k, s}^{i_1, i_2, j_1, j_2} \in \mathcal{L}(L^r(\mathbb{R}^k; L^p(\mathbb{R}^m; L^r(\mathbb{R}^k; E))); s \in S) \right\} \lesssim (\min(i_1, i_2) + 1)(\min(j_1, j_2) + 1).$$
Proposition 4.2 gives
\[
\mathcal{R}(\{P^{i_1,i_2,j_1,j_2}_{D^m,D^k,S} \in \mathcal{L}(L^r(\mathbb{R}^k; L^q(\mathbb{R}^n; L^p(\mathbb{R}^m; E))): s \in S\})
\]
\[
\lesssim (\min(i_1, i_2) + 1) \mathcal{R} \left( \left\{ \frac{|K|}{|I_1|^{1/2}|I_2|^{1/2} |J_1|^{1/2}|J_2|^{1/2}} \prod_{D^k,S} P^{j_1,j_2}_{D^m,D^k,K,I_1,I_2,s} \right. \right.
\]
\[
\left. \in \mathcal{L}(L^r(\mathbb{R}^k; L^q(\mathbb{R}^n \times Y^n; L^p(\mathbb{R}^m; E))): s \in S, K, I_1, I_2 \in \mathcal{D}^n \right\}.
\]
Viewing \(\pi_{D^k,b,K,V,I_1,i_2,i_1,j_2,s}\) as a bounded operator in \(F = L^r(\mathbb{R}^k; L^q(\mathbb{R}^n \times Y^n; E))\) and \(P^{j_1,j_2}_{D^m,D^k,K,I_1,I_2,s}\) as a model operator in \(L^m\) composed of the paraproduct operators \(\pi_{D^k,b,K,V,I_1,i_2,i_1,j_2,s}\), we see using Proposition 4.3 that the RHS is further dominated by \((\min(i_1, i_2) + 1)(\min(j_1, j_2) + 1)\) multiplied with
\[
\mathcal{R} \left( \left\{ \frac{|K|}{|I_1|^{1/2}|I_2|^{1/2} |J_1|^{1/2}|J_2|^{1/2}} \pi_{D^k,b,K,V,I_1,i_2,i_1,j_2,s} \right. \right.
\]
\[
\left. \in \mathcal{L}(L^r(\mathbb{R}^k; L^q(\mathbb{R}^n \times Y^n; L^p(\mathbb{R}^m \times Y^m; E))): s \in S, K, I_1, I_2 \in \mathcal{D}^n, V, J_1, J_2 \in \mathcal{D}^m \right\}.
\]
This constant is bounded by Proposition 5.4.

We can run the argument by decomposing in the symmetric way so that the partial paraproducts are formed in \(\mathbb{R}^{n+k}\). This gives that we can change \(\mathbb{R}^n\) and \(\mathbb{R}^m\) above, which yields all the symmetries. \(\square\)

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