Pressure control system with nonlocal friction

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Abstract

We analyze the motion of a pressure control system described by a differential equation with nonlocal dissipative force. This system is composed by an oscillator, a membrane and a constant force. We consider the dissipative memory kernel consisting of two terms. One of them is described by the Dirac delta function which represents a local friction, whereas for the second one we consider two types: the exponential and power-law functions which represent nonlocal dissipative forces. For these cases, one can obtain exact solutions for the displacement and velocity. The long-time behaviors of these quantities are also investigated.

Key words: membrane, pressure control system, nonlocal friction
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1 Introduction

Vibration is an ubiquitous phenomenon in nature and one can perceive it in many macroscopic systems by the senses. In the microscopic scale this phenomenon is also present, for instance, the vibration of atoms inside of each molecule or vibration of molecules around their equilibrium positions in a solid body. In engineering the vibration is present in structural and flexible multibody systems. Usually these systems are accompanied by dissipation process. General speaking the dissipation process may be local or nonlocal. The nonlocal friction is related to memory effect which depends on the past motion of a system. This kind of friction has been used to study anomalous diffusion.

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processes which have been observed in various kinds of systems such as bacterial cytoplasm motion [1], conformational fluctuations within a single protein molecule [2,3] and fluorescence intermittency in single enzymes [4]. Further, it has been applied to structural damping systems, viscoelastic materials, seismic and vibration isolation (see [5,6,7,8,9,10,11] and references therein). In fact, nonlocal friction may be used to enhance the modelling of damping systems.

In this work we consider a system of vibration composed by an oscillator, a membrane and a constant force. This simple system can be used as a pressure control device such as shown in Fig. 1 [12]. We note that the pressure may be exerted by a fluid, some mechanical mechanism or a constant force such as the gravitational force of the earth. In this last case the motion of the system should be positioned, for convenience, in the same direction of the gravitational force. However, our purpose is to employ this system to investigating the dissipation process by the action of membrane in the system. For elastic membrane one can consider the system membrane-oscillator as an oscillator. If the system presents nonlocal dissipation due to membrane then the mass coupled to the oscillator will describe different motion in comparison with that system with local dissipation.

In order to model the above system we consider the following differential equation:

\[ m \frac{dv}{dt} + \int_0^t dt_1 \gamma(t-t_1) v(t_1) + kx = p\theta(t), \quad (1) \]

where \( m \) is the effective mass, \( k \) is the spring constant, \( v \) is the velocity, \( \gamma(t) \) is the dissipative memory kernel and \( \theta(t) \) is the Heaviside function. We note that the dissipative memory kernel \( \gamma(t) \) may be a very complicated function, but we will restrict it for some particular cases. We will consider that the friction force is described by a combination of local and nonlocal forces. The dissipative memory kernel for a local friction is represented by the Dirac \( \delta \) function, whereas for the nonlocal friction we consider the exponential and power-law functions. For this linear system, general expression for the solution can be obtained. We will present the explicit solutions for the relaxation function. We will also analyze the long-time behavior of the solutions.

2 Exact solutions and analysis of results

Before analyzing the details of the above system for some specific dissipative memory kernel, we present general expression for the solution of Eq. (1). Eq. (1) is linear then it can be solved by using the Laplace transform, with the
initial conditions \( x_0 = x(0) \) and \( v_0 = v(0) \). The application of the Laplace transform to Eq. (1) yields

\[
\overline{x}(s) = \frac{x_0 (ms + \overline{\gamma}) + mv_0 + p/s}{ms^2 + s\overline{\gamma} + k}, \tag{2}
\]

where \( \overline{\gamma} \) is the Laplace transform of the damping kernel \( \gamma(t) \). Now we rewrite Eq. (2) as follows:

\[
\overline{x}(s) = (x_0 (ms + \overline{\gamma}) + mv_0 + p/s) \overline{G}(s) \tag{3}
\]

where \( G(t) \) is the relaxation function and it is the Laplace inversion of

\[
\overline{G}(s) = \frac{1}{ms^2 + s\overline{\gamma} + k}. \tag{4}
\]

For simplicity, one considers \( x_0 = v_0 = 0 \), then Eq. (3) becomes

\[
\overline{x}(s) = \frac{p}{s} \overline{G}(s). \tag{5}
\]

Moreover, from Eq. (5) one can obtain \( \overline{v}(s) \) which is given by

\[
\overline{v}(s) = p\overline{G}(s). \tag{6}
\]

From this last equation we arrive at

\[
v(t) = pG(t) \tag{7}
\]

and

\[
x(t) = p \int_0^t dt_1 G(t_1). \tag{8}
\]

We see that the velocity is proportional to the relaxation function \( G(t) \). In order to obtain explicit solutions for \( x(t) \) and \( v(t) \) we should give the form of dissipative memory kernel \( \gamma(t) \). We consider three cases.

First case. We take the dissipation process as local friction. In this case the dissipative memory kernel is represented by the Dirac \( \delta \) function and the friction force is proportional to \( v(t) \). The function \( \gamma \) is given by \( \gamma(t) = \gamma_\delta \delta(t) \) and its Laplace transform is \( \overline{\gamma}(s) = \gamma_\delta \). Substituting it into Eq. (4) we obtain [12]

\[
x(t) = \frac{p}{k} \left\{ 1 - \left[ \cos(\omega_1 t) + \frac{\alpha}{\omega_1} \sin(\omega_1 t) \right] \exp(-\alpha t) \right\}, \quad \gamma_\delta^2 < 4mk, \tag{9}
\]
where \( E \) and \( \gamma \) are damped cases: Underdamped \( \gamma , \alpha \) where \( \lambda , \gamma \) and references therein). The Laplace transform of \( \delta \) model has been used to model linear elastic and viscoelastic systems (see \( x \) position which is given by \( \delta \)). Then we obtain from Eqs. (4), (7) and (8) the following results:

\[
v(t) = pG(t) = \frac{p}{m}\sin (\omega_1 t) \exp (-\alpha t), \quad \gamma_\delta^2 < 4mk, \quad (10)
\]

\[
x(t) = \frac{p}{k} \left\{ 1 - (1 + \alpha t) \exp (-\alpha t) \right\}, \quad \gamma_\delta^2 = 4mk, \quad (11)
\]

\[
v(t) = pG(t) = \frac{p}{m} \exp (-\alpha t), \quad \gamma_\delta^2 = 4mk, \quad (12)
\]

Second case. We take the combination of a local friction represented by the Dirac \( \delta \) function and exponential function for the frictional memory kernel which is given by \( \gamma (t) = \gamma_\delta \delta (t) + \gamma_\lambda e^{-\lambda t} \). We note that the exponential damping model has been used to model linear elastic and viscoelastic systems (see [11] and references therein). The Laplace transform of \( \gamma (t) \) is \( \mathcal{F}(s) = \gamma_\delta + \gamma_\lambda / (s + \lambda) \). Then we obtain from Eqs. (4), (7) and (8) the following results:

\[
x(t) = \frac{pt^2}{m} \sum_{n=0}^{\infty} \frac{(-k\lambda t^3)^n}{n!} \sum_{j=0}^{\infty} \frac{((-\gamma_\delta^2 + \gamma_\lambda + k)t^2)^j}{j!}
\]

\[
\times \left[ E_{1,3+j+2n}^{(n+j)} \left( -\left( \lambda + \frac{\gamma_\delta}{m} \right) t \right) + \lambda t E_{1,4+j+2n}^{(n+j)} \left( -\left( \lambda + \frac{\gamma_\delta}{m} \right) t \right) \right] \quad (15)
\]

and

\[
v(t) = \frac{pt}{m} \sum_{n=0}^{\infty} \frac{(-k\lambda t^3)^n}{n!} \sum_{j=0}^{\infty} \frac{((-\gamma_\delta^2 + \gamma_\lambda + k)t^2)^j}{j!}
\]

\[
\times \left[ E_{1,2+j+2n}^{(n+j)} \left( -\left( \lambda + \frac{\gamma_\delta}{m} \right) t \right) + \lambda t E_{1,3+j+2n}^{(n+j)} \left( -\left( \lambda + \frac{\gamma_\delta}{m} \right) t \right) \right], \quad (16)
\]

where \( E_{\beta,\delta} (y) \) is the generalized Mittag-Leffler function [13,14] defined by

\[
E_{\beta,\delta} (y) = \sum_{n=0}^{\infty} \frac{y^n}{\Gamma (\beta n + \delta)}, \quad \beta > 0, \delta > 0, \quad (17)
\]

\[
E_{\beta,\delta}^{(k)} (y) = \frac{d^k}{dy^k} E_{\beta,\delta} (y) = \sum_{n=0}^{\infty} \frac{(n + k)!y^n}{n! \Gamma (\beta (n + k) + \delta)} \quad (18)
\]
and $\Gamma(z)$ is the Gamma function.

Note that relevant information of these quantities can be obtained by analyzing their asymptotic behaviors. Therefore we use the long-time limit of the generalized Mittag-Leffler function [14] given by

$$E_{\alpha,\beta}(z) \sim \frac{1}{z^{\beta-\alpha}}.$$  \hspace{1cm} (19)

From Eqs. (15), (16) and (19) yield

$$x(t) \sim \frac{p}{k} \left[ 1 - \frac{\gamma_\delta \lambda + \gamma_\lambda}{\gamma_\delta \lambda + \gamma_\lambda + k} \exp\left( -\frac{k\lambda t}{\gamma_\delta \lambda + \gamma_\lambda + k} \right) \right],$$  \hspace{1cm} (20)

and

$$v(t) \sim \frac{p\lambda (\gamma_\delta \lambda + \gamma_\lambda)}{(\gamma_\delta \lambda + \gamma_\lambda + k)^2} \exp\left( -\frac{k\lambda t}{\gamma_\delta \lambda + \gamma_\lambda + k} \right),$$  \hspace{1cm} (21)

where we consider the fact that $(\lambda + \gamma_\delta/m) t \gg 1$ and $(\lambda \gamma_\delta + \gamma_\lambda + k) \gtrsim m (\lambda + \gamma_\delta/m)^2$. We see that the displacement $x(t)$ and velocity $v(t)$ decay exponentially similar to those obtained from the system with an instantaneous friction. Moreover, we note that the constant force does not modify the exponential asymptotic form. From Eq. (20) we can obtain the final equilibrium position which is given by $x_f = p/k$. In Fig. 4 we verify the asymptotic result of $x(t)$ Eq. (15) by comparing with Eq. (20), whereas in Fig. 5 we verify the asymptotic result of $v(t)$ Eq. (16) by comparing with Eq. (21). The asymptotic results are very close to the exact ones.

**Third case.** We consider the combination of instantaneous friction represented by the Dirac $\delta$ function and long-time frictional memory kernel given by $\gamma(t) = \gamma_\delta \delta(t) + \gamma_\beta t^{-\beta}$, for $0 < \beta < 1$. Then, the Laplace transform of $\gamma(t)$ is $\mathcal{F} = \gamma_\delta + \gamma_\beta s^{\beta-1}$, where $\gamma_\beta = \gamma_\beta \Gamma(1-\beta)$. We note that the second term of $\gamma(t)$ describes a fractional derivative on Eq. (1) which corresponds to the Caputo fractional derivative [13]. The solutions for the displacement and velocity can be obtained from Eqs. (4), (7) and (8). The results are given by

$$x(t) = \frac{pt^2}{m} \sum_{n=0}^{\infty} \left( -\frac{\gamma_\delta}{m} t^2 - \beta \right)^n \sum_{j=0}^{\infty} \left( -\frac{\gamma_\delta}{m} t^2 - \beta \right)^j \frac{1}{(n+j)!} \frac{1}{(n+1+j)!} E_{1,3+n+(1-\beta)j}^{(n+j)} \left( -\frac{\gamma_\delta}{m} t \right),$$  \hspace{1cm} (22)

and

$$v(t) = \frac{pt}{m} \sum_{n=0}^{\infty} \left( -\frac{\gamma_\delta}{m} t^2 - \beta \right)^n \sum_{j=0}^{\infty} \left( -\frac{\gamma_\delta}{m} t^2 - \beta \right)^j \frac{1}{(n+j)!} \frac{1}{(n+1+j)!} E_{1,2+n+(1-\beta)j}^{(n+j)} \left( -\frac{\gamma_\delta}{m} t \right).$$  \hspace{1cm} (23)

For $\gamma_\delta = 0$ (nonlocal friction) we have
\[ x(t) = \frac{pt^2}{m} \sum_{n=0}^{\infty} \left( -\frac{k}{m} t^2 \right)^n \frac{E_{2,\beta,3+n}^{(n)}}{n!} \left( -\frac{\gamma_{\beta}^*}{m} t^{2-\beta} \right) \]

and

\[ v(t) = \frac{pt^2}{m} \sum_{n=0}^{\infty} \left( -\frac{k}{m} t^2 \right)^n \frac{E_{1,2+n}^{(n)}}{n!} \left( -\frac{\gamma_{\delta}^*}{m} t \right). \]

On the other hand, for \( \gamma_{\beta} = 0 \) (local friction) we have

\[ x(t) = \frac{pt^2}{m} \sum_{n=0}^{\infty} \left( -\frac{k}{m} t^2 \right)^n \frac{E_{1,3+n}^{(n)}}{n!} \left( -\frac{\gamma_{\delta}}{m} t \right) \]

and

\[ v(t) = \frac{pt^2}{m} \sum_{n=0}^{\infty} \left( -\frac{k}{m} t^2 \right)^n \frac{E_{1,2+n}^{(n)}}{n!} \left( -\frac{\gamma_{\delta}}{m} t \right). \]

We have numerically compared these last solutions (26) and (27) with (9)-(14) and they give similar results. This means that the solutions (26) and (27) contain the underdamped, critical damping and overdamped solutions.

Now we analyze the asymptotic behavior of the above quantities \( x(t) \) and \( v(t) \). By using the long-time limit of \( E_{\alpha,\beta}(y) \) (19) we obtain

\[ x(t) \sim \frac{p}{k} \left[ 1 - \frac{\gamma_{\delta}^* \sin(\pi \beta) \Gamma(\beta)}{k \pi t^\beta} \right], \]

\[ v(t) \sim \frac{p \gamma_{\delta}^* \sin(\pi \beta) \Gamma(1+\beta)}{k^2 \pi t^{1+\beta}}, \]

where we consider the fact that \( \gamma_{\delta} t/m \gg 1 \), \( \gamma_{\beta}^* m^{1-\beta} \gg \gamma_{\delta}^{(2-\beta)} \) and \( km^\beta \gg \gamma_{\beta}^* \gamma_{\delta}^\beta \). We can see that these asymptotic results decay as a power-law and they are dominated by the nonlocal dissipative force; The parameter \( \gamma_{\delta} \) does not appears in these leading terms. From Eq. (28) we can obtain the final equilibrium position which is given by \( x_f = p/k \). This means that the nonlocal dissipative force described by a long-time memory kernel may suppress the presence of a local friction in the long-time limit. Moreover, the expressions (28) and (29) do not depend on the mass \( m \), then the inertial term does not have significant influence on the long-time behavior of the system. In Fig. 6 we verify the asymptotic result of \( x(t) \) obtained by Eqs. (22) and (28), whereas in Fig. 7 we verify the asymptotic result of \( v(t) \) Eq. (23) by comparing with Eq. (29). The curves converge to the same behavior in the long-time limit. For small values of \( m \) the oscillator does not move across the final equilibrium position, whereas for large values of \( m \) the oscillator can move across the final
equilibrium position. In other words, the amplitude of oscillation increases with the increase of mass $m$. This result is due to the inertial term which has important influence for the initial movement of the system. We note that the solutions (22) and (23) decay slowly due to the power-law asymptotic behavior. These results are in contrast to those analyzed previously.

3 Conclusion

In this work we have investigated the motion of a particle governed by the classical Newtonian equation (1) under the influence of the combination of local and nonlocal dissipative forces, linear external force given by $U(x) = -kx$ and a constant load force $p$. This system can be used to model a pressure control device. In particular, we have employed the exponential and power-law functions for the dissipative memory kernel. Exact and asymptotic solutions for the relaxation function $G(t)$, $x(t)$ and $v(t)$ have been obtained. The asymptotic results have permitted us to obtain the final equilibrium position $x_f = p/k$. In the case of exponential memory kernel the asymptotic results of $x(t)$ and $v(t)$ decay exponentially similar to the system described by a local friction. For the power-law memory kernel we have shown the asymptotic results of $x(t)$ and $v(t)$ which decay as a power-law and the leading terms are independent of the parameter of local friction $\gamma_\delta$ and mass $m$; This means that the long-time memory friction may suppress the presence of an instantaneous friction in the long-time limit. We note that the system designed in Fig. 1 can be made experimentally. Then we hope the model described by Eq. (1) may be used to investigate the dynamics of membranes. If necessary other kinds of function for dissipative memory kernel can also be used and the solutions can be obtained by Eq. (4).

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Figure Captions

Fig. 1 - The elements of a pressure control system.

Fig. 2 - Behavior of $x(t)$ for different values of the mass $m$, in arbitrary units. The parameters $k$, $p$ and $\gamma_\delta$ have the following values: $k = 0.25$, $p = 0.1$ and $\gamma_\delta = 0.3$. The dotted line is obtained by Eq (11) with $m = 0.09$.

Fig. 3 - Behavior of $v(t)$ for different values of the mass $m$, in arbitrary units. The parameters $k$, $p$ and $\gamma_\delta$ have the following values: $k = 0.25$, $p = 0.1$ and $\gamma_\delta = 0.3$. The dotted line is obtained by Eq (12) with $m = 0.09$.

Fig. 4 - Behavior of $x(t)$ for different values of the mass $m$, in arbitrary units. The parameters $\lambda$, $k$, $p$, $\gamma_\delta$ and $\gamma_\lambda$ have the following values: $\lambda = 0.1$, $k = 0.25$, $p = 0.1$, $\gamma_\delta = 0.3$ and $\gamma_\lambda = 0.13$. The dotted line is obtained by Eq (20).

Fig. 5 - Behavior of $v(t)$ for different values of the mass $m$, in arbitrary units. The parameters $\lambda$, $k$, $p$, $\gamma_\delta$ and $\gamma_\lambda$ have the following values: $\lambda = 0.1$, $k = 0.25$, $p = 0.1$, $\gamma_\delta = 0.3$ and $\gamma_\lambda = 0.13$. The dotted line is obtained by Eq (21).

Fig. 6 - Behavior of $x(t)$ for different values of the mass $m$, in arbitrary units. The parameters $\beta$, $k$, $p$, $\gamma_\delta$ and $\gamma_\beta$ have the following values: $\beta = 0.5$, $k = 0.25$, $p = 0.1$, $\gamma_\delta = 0.3$ and $\gamma_\beta = 0.13$. The dotted line is obtained by Eq (28).

Fig. 7 - Behavior of $v(t)$ for different values of the mass $m$, in arbitrary units. The parameters $\beta$, $k$, $p$, $\gamma_\delta$ and $\gamma_\beta$ have the following values: $\beta = 0.5$, $k = 0.25$, $p = 0.1$, $\gamma_\delta = 0.3$ and $\gamma_\beta = 0.13$. The dotted line is obtained by Eq (29).