AN ANALOGUE OF THE CARTAN DECOMPOSITION FOR \( p \)-ADIC REDUCTIVE SYMMETRIC SPACES

by

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Abstract. — Let \( k \) be a non Archimedean locally compact field of residue characteristic different from 2, let \( G \) be a connected reductive group defined over \( k \), let \( \sigma \) be an involutive \( k \)-automorphism of \( G \) and \( H \) an open \( k \)-subgroup of the fixed points group of \( \sigma \). We denote by \( G_k \) (resp. \( H_k \)) the group of \( k \)-points of \( G \) (resp. \( H \)). In this paper, we obtain an analogue of the Cartan decomposition for the reductive symmetric space \( H_k \backslash G_k \). More precisely, we obtain a decomposition of \( G_k \) as a union of \( H_k \)-cosets which is related to the \( H_k \)-conjugacy classes of maximal \( \sigma \)-anti-invariant \( k \)-split tori in \( G \). When \( G \) is \( k \)-split, we get a more precise result, involving the stabilizer of a special point of the Bruhat-Tits building of \( G \) over \( k \).

Résumé. — Soit \( k \) un corps localement compact non archimédien de caractéristique résiduelle impaire, soit \( G \) un groupe réductif connexe défini sur \( k \), soit \( \sigma \) un \( k \)-automorphisme involutif de \( G \) et \( H \) un \( k \)-sous-groupe ouvert du groupe des points de \( G \) fixes par \( \sigma \). On note \( G_k \) (resp. \( H_k \)) le groupe des \( k \)-points de \( G \) (resp. \( H \)). Dans cet article, nous obtenons un analogue de la décomposition de Cartan pour l’espace symétrique réductif \( H_k \backslash G_k \). Plus précisément, nous obtenons une décomposition de \( G_k \) sous la forme d’une réunion de classes modulo \( H_k \) reliée aux classes de \( H_k \)-conjugaision de tores \( k \)-déployés \( \sigma \)-anti-invariants maximaux de \( G \). Lorsque \( G \) est déployé sur \( k \), nous obtenons un résultat plus précis impliquant le stabilisateur d’un point spécial de l’immeuble de Bruhat-tits de \( G \) sur \( k \).

Introduction

Let \( k \) be a non Archimedean locally compact field of residue characteristic different from 2. Let \( G \) be a connected reductive group defined over \( k \), let \( \sigma \) be an involutive \( k \)-automorphism of \( G \) and \( H \) an open \( k \)-subgroup of the fixed
points group of \( \sigma \). We denote by \( G_k \) (resp. \( H_k \)) the group of \( k \)-points of \( G \) (resp. \( H \)). Harmonic analysis on the reductive symmetric space \( H_k \backslash G_k \) is the study of the action of \( G_k \) on the space of complex square integrable functions on \( H_k \backslash G_k \).

This study is related to the classification of \( H_k \)-distinguished representations of \( G_k \), that is representations having a nonzero space of \( H_k \)-invariant linear forms. The question of distinguishedness has been studied intensively for \( GL_n \) and related groups. See for instance \([1, 2, 12, 13, 14, 23, 24]\) for a (non-exhaustive) list of works on this question. Some other aspects of that problem, including the Plancherel formula, have been studied by Offen \([22]\) for spherical representations, in three particular cases related to \( GL_n \). Blanc and Delorme \([5]\) have studied parabolically induced representations for a general reductive symmetric space \( H_k \backslash G_k \). In this paper, we investigate the geometry of the space \( H_k \backslash G_k \).

Connected reductive groups over \( k \) can be considered as reductive symmetric spaces. Indeed, if \( G' \) is such a group, the map \( \sigma : (x, y) \mapsto (y, x) \) defines a \( k \)-involution of the connected reductive group \( G = G' \times G' \) whose fixed points group \( H \) is the diagonal image of \( G' \) in \( G \). Hence the reductive symmetric space \( H_k \backslash G_k \) naturally identifies with the group \( G_k' \). Moreover, if \( K' \) is a subgroup of \( G_k' \) and if we set \( K = K' \times K' \), then the \((H_k, K)\)-double cosets of \( G_k \) correspond to the \( K' \)-double cosets of \( G_k' \). In particular, if \( K' \) is the stabilizer in \( G_k' \) of a special point in the (reduced) Bruhat-Tits building of \( G' \) over \( k \), the decomposition of \( H_k \backslash G_k \) into \( K \)-orbits corresponds to the Cartan decomposition of \( G_k' \) relative to \( K' \) (see \([8\text{, Proposition 4.4.3}])\).

In this paper, we obtain an analogue of the Cartan decomposition for a general reductive symmetric space \( H_k \backslash G_k \). In \([15, 16, 17]\) A. and G. Helminck and Wang studied two kinds of objects which are related to our problem:

1. \( H_k \)-conjugacy classes of maximal \( \sigma \)-anti-invariant \( k \)-split tori of \( G \) (called maximal \((\sigma, k)\)-split tori in \([15]\), see also Definition 3.5);

2. \( H_k \)-conjugacy classes of the parabolic \( k \)-subgroups \( P \) of \( G \) which are opposite to \( \sigma(P) \) (called \( \sigma \)-split parabolic \( k \)-subgroups in \([17]\) and \( \sigma \)-parabolic \( k \)-subgroups in this paper, see Definition 3.7).
Let \( \{ A^j \mid j \in J \} \) be a set of representatives of the \( H_k \)-conjugacy classes of maximal \((\sigma, k)\)-split tori in \( G \). For each \( j \), we denote by \( W_{G_k}(A^j) \) (resp. \( W_{H_k}(A^j) \)) the quotient of the normalizer of \( A^j \) in \( G_k \) (resp. in \( H_k \)) by its centralizer. According to Helminck and Wang \([17]\), the set \( J \) is finite and, for \( j \in J \), the group \( W_{G_k}(A^j) \) is the Weyl group of a root system. Moreover, let \( A \) be a maximal \((\sigma, k)\)-split torus of \( G \), let \( S \) be a \( \sigma \)-stable maximal \( k \)-split torus of \( G \) containing \( A \) and \( P \) a minimal \( \sigma \)-parabolic \( k \)-subgroup of \( G \) containing \( S \). Then, according to \([16, \text{Theorem 3.6}]\), the finite union:

\[
\bigcup_{j \in J} W_{H_k}(A^j) \backslash W_{G_k}(A^j)
\]

classifies the open \((H_k, P_k)\)-double cosets of \( G_k \). For each \( j \in J \), we choose:

1. a set \( N_j \subset N_{G_k}(A^j) \) of representatives of \( W_{H_k}(A^j) \backslash W_{G_k}(A^j) \);
2. an element \( y_j \in G_k \) such that \( y_j A y_j^{-1} = A^j \);

and we denote by \( \mathcal{N} \) the set of all \( n y_j \) for \( j \in J \) and \( n \in N_j \). Note that \( \mathcal{N} \) is a set of representatives of \((0.1)\). Let \( \varpi \) be a uniformizer of \( k \), let \( \Lambda \) be the lattice formed by the images of \( \varpi \) by the various algebraic one-parameter subgroups of \( A \) and let \( \Lambda^- \) denote the subset of anti-dominant elements of \( \Lambda \) relative to \( P \). Then we can state our first main result (see Theorem 3.10):

**Theorem 0.1.** — There exists a compact subset \( \Omega \) of \( G_k \) such that:

\[
G_k = \bigcup_{n \in \mathcal{N}} H_k n \Lambda^- \Omega.
\]

In order to prove this result, we make a large use of the Bruhat-Tits theory \([8, 9]\). Let \( \mathcal{B} \) be the (reduced) Bruhat-Tits building of \( G \) over \( k \). It is endowed with an action of \( \sigma \). Then the proof of Theorem 0.1 is based on the following result (see Proposition 2.4):

**Proposition 0.2.** — \( \mathcal{B} \) is the union of its \( \sigma \)-stable apartments.

This result can be rephrased as follows. Let \( S \) be a \( \sigma \)-stable maximal \( k \)-split torus of \( G \), let \( N \) its normalizer in \( G \) and let \( \mathcal{G} \) be the set of all \( g \in G_k \) such
that $g^{-1}\sigma(g) \in N_k$. Then we have $G_k = \mathcal{O}K$, where $K$ is the stabilizer in $G_k$ of any point of the apartment corresponding to $S$ (see Proposition 3.4).

Let us mention that the question of the disjointness of the various components appearing in the decomposition of $G_k$ given by Theorem 0.1 has been investigated by Lagier [19].

When the group $G$ is $k$-split, we obtain a refinement of Theorem 0.1, which is based on the following refinement of Proposition 0.2 (see Proposition 4.4):

**Proposition 0.3.** — Let $x$ be a special point of $\mathcal{B}$. There is a $\sigma$-stable maximal $k$-split torus $S$ of $G$ such that the apartment corresponding to $S$ contains $x$, and such that the maximal $\sigma$-anti-invariant subtorus of $S$ is a maximal $(\sigma, k)$-split torus of $G$.

We thus obtain our second main result (see Theorem 4.8):

**Theorem 0.4.** — Let $K$ be the stabilizer in $G_k$ of a special point in $\mathcal{B}$. Then:
\[
G_k = \bigcup_{j \in J} H_k y_j S_k K.
\]

Note that Proposition 0.3 is no longer true for non-split groups, as proven in §5.3.

The paper is organized as follows. In §1 we recall the main properties of the Bruhat-Tits building attached to a connected reductive group defined over $k$. In §2 we study the set of all apartments containing a given $\sigma$-stable subset of the building, and we prove Proposition 0.2. In §3 we prove our first main result (Theorem 0.1). In §4 we are devoted to the case where $G$ is $k$-split. We prove Proposition 0.3 and Theorem 0.4. Finally, in §5 we study in more details the two following examples:

1. $G_k = \text{GL}_n(k)$ and $\sigma(g) = \text{transpose of } g^{-1}$.
2. $G_k = \text{GL}_n(k')$ with $k'$ quadratic over $k$ and $\text{id} \neq \sigma \in \text{Gal}(k'/k)$.

When $n = 2$ and $k'$ is totally ramified over $k$, Example (2) provides an example of a non-split group for which Proposition 0.3 is not satisfied.

After this work was finished, we learnt that Y. Benoist and H. Oh [4] proved a result equivalent to Theorem 0.1, with a weaker assumption on $k$ (they only
assume that its characteristic is not 2). They also use the Bruhat-Tits building, but in a different way.

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1. The Bruhat-Tits building

Let \( k \) be a non Archimedean non discrete locally compact field, and let \( \omega \) be its normalized valuation. In this section, we recall the main properties of the Bruhat-Tits building attached to a connected reductive group defined over \( k \). The reader may refer to the original construction of Bruhat-Tits [8, 9] or to more concise presentations [20, 26, 29].

If \( G \) is a linear algebraic group defined over \( k \), the group of its \( k \)-points will be denoted by \( G_k \) or \( G(k) \), and its neutral component will be denoted by \( G^\circ \). If \( H \) is a subset of \( G \), then \( N_G(H) \) (resp. \( Z_G(H) \)) denotes the normalizer (resp. the centralizer) of \( H \) in \( G \).

If \( X \) is a subset of \( G \), then \( gX \) denotes the left conjugate of \( X \) by \( g \in G \).

1.1. Let \( G \) be a connected reductive group defined over \( k \), and let \( S \) be a maximal \( k \)-split torus of \( G \). We denote by \( X^\ast(S) = \text{Hom}(S, \text{GL}_1) \) (resp. by \( X_\ast(S) = \text{Hom}(\text{GL}_1, S) \)) the group of algebraic characters (resp. cocharacters) of \( S \). We define a map:

\[
X_\ast(S) \times X^\ast(S) \to \mathbb{Z}
\]

as follows. If \( \lambda \in X_\ast(S) \) and \( \chi \in X^\ast(S) \), then \( \chi \circ \lambda \) is an endomorphism of the multiplicative group \( \text{GL}_1 \), which corresponds to an endomorphism of the ring \( \mathbb{Z}[t, t^{-1}] \). It is of the form \( t \mapsto t^n \) for some \( n \in \mathbb{Z} \). This integer \( n \) is denoted by \( \langle \lambda, \chi \rangle \). The map (1.1) defines a perfect duality (see [6, §8.6]).
1.2. Let $N$ (resp. $Z$) denote the normalizer (resp. the centralizer) of $S$ in $G$. If we extend (1.1) by $\mathbb{R}$-linearity, there exists a unique group homomorphism:

$$\nu : Z_k \rightarrow X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$$

such that the condition:

$$\langle \nu(z), \chi \rangle = -\omega(\chi(z))$$

holds for any $z \in Z_k$ and any $k$-rational character $\chi \in X^*(Z)_k$ (see [29, §1.2]). According to [20, Proposition 1.2], the kernel of (1.2) is the maximal compact subgroup of $Z_k$. It will be denoted by $Z^1_k$.

Remark 1.1. — Note that the intersection $S_k \cap Z^1_k$ is equal to the maximal compact subgroup of $S_k$, which we denote by $S^1_k$. Indeed $S^1_k$ contains the compact subgroup $S_k \cap Z^1_k$ of $S_k$ and is contained in the maximal compact subgroup $Z^1_k$ of $Z_k$. According to [29, §1.2], the quotient $\Lambda_Z = Z_k/Z^1_k$ is a free abelian group of rank $\dim S$, and the image of $S_k$ in $\Lambda_Z$ has finite index.

1.3. Let $C$ denote the connected centre of $G$ and let $X_*(C)$ be the group of its algebraic cocharacters. It is a subgroup of the free abelian group $X_*(S)$. We denote by $\mathcal{A}$ the space:

$$V = (X_*(S) \otimes_{\mathbb{Z}} \mathbb{R})/(X_*(C) \otimes_{\mathbb{Z}} \mathbb{R})$$

considered as an affine space on itself and by $\text{Aff}(\mathcal{A})$ the group of its affine automorphisms. By making $V$ act on $\mathcal{A}$ by translations, we can think to $V$ as a subgroup of $\text{Aff}(\mathcal{A})$. It is the kernel of the natural group homomorphism $\text{Aff}(\mathcal{A}) \rightarrow \text{GL}(V)$ which associates to any affine automorphism its linear part.

1.4. The map (1.2) induces a homomorphism:

$$Z_k \rightarrow \text{Aff}(\mathcal{A})$$

which is still denoted by $\nu$. Its image is contained in $V$. An important property of this homomorphism is that it extends to a homomorphism $N_k \rightarrow \text{Aff}(\mathcal{A})$ (see [29, §1.2]). It does not extend in a unique way, but two homomorphisms
extending (1.4) to \(N_k\) are conjugated by a \textit{unique} element of \(\text{Aff}(\mathcal{A})\) (see [20, Proposition 1.8]).

1.5. The affine space \(\mathcal{A}\) endowed with an action of \(N_k\) defined by a group homomorphism \(\nu : N_k \to \text{Aff}(\mathcal{A})\) extending the homomorphism (1.4) is called the (reduced) \textit{apartment} attached to \(S\). It satisfies the conditions:

\begin{enumerate}
  \item[A1] \(\mathcal{A}\) is an affine space on \(V\);
  \item[A2] \(\nu\) is a group homomorphism \(N_k \to \text{Aff}(\mathcal{A})\) extending the canonical homomorphism \(Z_k \to V\).
\end{enumerate}

It has the following unicity property. If \((\mathcal{A}', \nu')\) satisfy A1 and A2, then there is a unique affine and \(N_k\)-equivariant isomorphism from \(\mathcal{A}'\) to \(\mathcal{A}\).

\textit{Remark 1.2.} — We obtain the \textit{non reduced} apartment \(\mathcal{A}_{nr}\) by replacing \(V\) by \(X^*(S) \otimes \mathbb{Z} \mathbb{R}\). This is the point of view of Tits [29]. The non reduced apartment is not as canonical as the reduced one: two homomorphisms extending the map \(\nu_{nr} : Z_k \to \text{Aff}(\mathcal{A}_{nr})\) to \(N_k\) are conjugated by an element of \(\text{Aff}(\mathcal{A}_{nr})\) which is not necessarily unique (see [20, §1] and also [29, §1.2]).

1.6. Let \(\Phi = \Phi(G, S)\) denote the set of roots of \(G\) relative to \(S\). It is a subset of \(X^*(S)\). Therefore, any root \(a \in \Phi\) can be seen as a linear form on \(X^*_s(S) \otimes \mathbb{R}\) which is trivial on the subspace \(X^*_s(C) \otimes \mathbb{R}\), hence as a linear form on \(V\) (see [20, §1]).

For \(a \in \Phi\), we denote by \(U_a\) the root subgroup associated to \(a\), which is a unipotent subgroup of \(G\) normalized by \(Z\) (see [6, Proposition 21.9]), and by \(s_a\) the reflection corresponding to \(a\), considered as an element of \(\text{GL}(V)\) — or, more precisely, of the quotient of \(\nu(N_k)\) by \(\nu(Z_k)\).

1.7. Let \(a \in \Phi\) and \(u \in U_a(k) - \{1\}\). The intersection:

\[U_{-a}(k)aU_{-a}(k) \cap N_k\]

(1.5)

consists of a single element, called \(m(u)\), whose image by \(\nu\) is an affine reflection whose linear part is \(s_a\) (see [7, §5]). The set \(\mathcal{H}_{a,u}\) of fixed points of \(\nu(m(u))\) is an affine hyperplane of \(\mathcal{A}\), which is called a \textit{wall} of \(\mathcal{A}\).
A chamber of $\mathcal{A}$ is a connected component of the complementary in $\mathcal{A}$ of the union of its walls. Note that a chamber is open in $\mathcal{A}$.

A point $x \in \mathcal{A}$ is said to be special if, for all root $a \in \Phi$, there is a root $b \in \Phi \cap \mathbb{R}_+ a$ and an element $u \in U_b(k) - \{1\}$ such that $x \in \mathcal{H}_{b,u}$ (see [21, §1.2.3] and also [29, §1.9]).

1.8. Let $\theta(a, u)$ denote the affine function $\mathcal{A} \to \mathbb{R}$ whose linear part is $a$ and whose vanishing hyperplane is the wall $\mathcal{H}_{a,u}$ of fixed points of $\nu(m(u))$. We fix a base point in $\mathcal{A}$, so that $\mathcal{A}$ can be identified with the vector space $V$. For $r \in \mathbb{R}$, we set:

$$U_a(k)_r = \{ u \in U_a(k) - \{1\} \mid \theta(a, u)(x) \geq a(x) + r \text{ for all } x \in \mathcal{A} \} \cup \{1\}. $$

Thus we obtain a filtration of $U_a(k)$ by subgroups. If we change the base point in $\mathcal{A}$, this filtration is only modified by a translation of the indexation.

1.9. Let $\Omega$ be a nonempty subset of $\mathcal{A}$. We set:

$$N_\Omega = \{ n \in N_k \mid \nu(n)(x) = x \text{ for all } x \in \Omega \},$$

and we denote by $U_\Omega$ the subgroup of $G_k$ generated by all the $U_a(k)_r$ such that the affine function $x \mapsto a(x) + r$ is non negative on $\Omega$. According to [20, §12], this subgroup is compact in $G_k$, and we have $nU_\Omega n^{-1} = U_{\nu(n)(\Omega)}$ for any $n \in N_k$. In particular $N_\Omega$ normalizes $U_\Omega$.

The subgroup $P_\Omega = N_\Omega U_\Omega$ is open in $G_k$ (see loc.cit., Corollary 12.12).

1.10. Let $\Phi = \Phi^- \cup \Phi^+$ be a decomposition of $\Phi$ into positive and negative roots. We denote by $U^+$ (resp. $U^-$) the subgroup of $G_k$ generated by the $U_a$ for $a \in \Phi^+$ (resp. $a \in \Phi^-$), and we set $U^+_\Omega = U_\Omega \cap U^+$ (resp. $U^-_\Omega = U_\Omega \cap U^-$).

Then the group $P_\Omega$ has the following Iwahori decomposition:

$$(1.6) \quad P_\Omega = U^-_\Omega U^+_\Omega N_\Omega$$

(see [20, Corollary 12.6] and also [8, §7.1.4]).
1.11. In [8, 9], Bruhat and Tits associate to the apartment \((\mathcal{A}, \nu)\) a \(G_k\)-set \(\mathcal{B} = \mathcal{B}(G, k)\) containing \(\mathcal{A}\), called the (reduced) building of \(G\) over \(k\) and satisfying the following conditions:

\(B1\) The set \(\mathcal{B}\) is the union of the \(g \cdot \mathcal{A}\) with \(g \in G_k\).

\(B2\) The subgroup \(N_k\) is the stabilizer of \(\mathcal{A}\) in \(G_k\), and \(n \cdot x = \nu(n)(x)\) for all \(x \in \mathcal{A}\) and \(n \in N_k\).

\(B3\) For all \(a \in \Phi\) and \(r \in \mathbb{R}\), the subgroup \(U_a(k)_r\) defined in §1.8 fixes the subset \(\{x \in \mathcal{A} \mid a(x) + r \geq 0\}\) pointwise.

The building has the following unicity property. If \(\mathcal{B}'\) is a \(G_k\)-set containing \(\mathcal{A}\) and satisfying \(B1, B2\) and \(B3\), then there is a unique \(G_k\)-equivariant bijection from \(\mathcal{B}'\) to \(\mathcal{B}\) (see [29, §2.1] and also [25, §1.9]).

1.12. The subsets of \(\mathcal{B}\) of the form \(g \cdot \mathcal{A}\) with \(g \in G_k\) are called apartments. According to \(B1\) the building is the union of its apartments.

For \(g \in G_k\), the apartment \(g \cdot \mathcal{A}\) can be naturally endowed with a structure of affine space and an action of \(^gN_k\) by affine isomorphisms. Upto unique isomorphism, it is the apartment attached to the maximal \(k\)-split torus \(^gS\) (see §1.5). This defines a unique \(G_k\)-equivariant map:

\[(1.7)\quad G \ni S' \mapsto \mathcal{A}(S') \subset \mathcal{B}\]

between maximal \(k\)-split tori of \(G\) and apartments of \(\mathcal{B}\), such that \(S\) maps to \(\mathcal{A}\).

Note that \(\mathcal{B}\) does not depend on the maximal \(k\)-split torus \(S\). Indeed, let \(S'\) be a maximal \(k\)-split torus of \(G\), let \((\mathcal{A}', \nu')\) be the apartment attached to \(S'\) and \(\mathcal{B}'\) the building of \(G\) over \(k\) relative to this apartment (see §1.11). If we identify \(\mathcal{A}'\) with the unique apartment of \(\mathcal{B}\) corresponding to \(S'\) via \((1.7)\), then \(\mathcal{B}' = \mathcal{B}\).

1.13. The building has the following important properties (see [8, §7.4] and also [20, §13]):

(1) Let \(\Omega\) be a nonempty subset of \(\mathcal{A}\). Then \(P_\Omega\) is the subgroup of \(G_k\) made of all elements fixing \(\Omega\) pointwise.
(2) Let \( g \in G_k \). There exists \( n \in N_k \) such that \( g \cdot x = n \cdot x \) for any element \( x \in \mathcal{A} \cap g^{-1} \cdot \mathcal{A} \).

In particular, Property (1) together with B2 imply that \( N_{\Omega} = N_k \cap P_\Omega \).

1.14. Let \( \sigma \) be a \( k \)-automorphism of \( G \). There is a unique bijective map from \( \mathcal{B} \) to itself, which we still denote by \( \sigma \), such that:

(i) the condition:
\[
\sigma(g \cdot x) = \sigma(g) \cdot \sigma(x)
\]
holds for any \( g \in G_k \) and \( x \in \mathcal{B} \);

(ii) the map \( \sigma \) permutes the apartments and, for any apartment \( \mathcal{A} \), the restriction of \( \sigma \) to \( \mathcal{A} \) is an affine isomorphism from \( \mathcal{A} \) onto its image.

This gives us an action of the group \( \text{Aut}_k(G) \) of \( k \)-automorphisms of \( G \) on the building (see [9, §4.2.12]).

1.15. Let \( V^1 \) denote the dual space of \( X^*(G) \otimes \mathbb{R} \). The extended building of \( G \) over \( k \) is the product \( \mathcal{B}^1 = \mathcal{B} \times V^1 \), where \( G_k \) acts on \( V^1 \) by:
\[
g \cdot \chi = -\omega(\chi(g)),
\]
for any \( g \in G_k \) and any \( k \)-rational character \( \chi \in X^*(G)_k \). The \( G_k \)-stabilizer of the reduced building \( \mathcal{B} \times \{0\} \), considered as a subset of the extended building \( \mathcal{B}^1 \), is denoted by \( G^1_k \). It is the subgroup of all \( g \in G_k \) such that \( \omega(\chi(g)) = 0 \) for any \( \chi \in X^*(G)_k \).

Remark 1.3. — Let \( D \) denote the maximal \( k \)-split torus of the connected centre \( C \) of \( G \). Then the quotient \( \Lambda_G = G_k/G^1_k \) is a free abelian group of rank \( \dim D \), and the image of \( D_k \) in \( \Lambda_G \) has finite index.

The action of \( \text{Aut}_k(G) \) on \( G \) induces an action of \( \text{Aut}_k(G) \) on \( V^1 \), hence on the extended building \( \mathcal{B}^1 \).

Remark 1.4. — Let \( \Gamma \) be a finite subgroup of \( \text{Aut}_k(G) \) whose order is prime to the residue characteristic of \( k \), and let \( H \) be the neutral component of the fixed points subgroup \( G^\Gamma \). Prasad and Yu [25] proved the existence of
a $H_k$-equivariant map $\iota: B^1(H, k) \to B^1(G, k)$ whose image is the set of $\Gamma$-invariant points. Moreover, such a map is *toral* in the sense of [20], which means that for any maximal $k$-split torus $T$ of $H$, there is a maximal $k$-split torus $S$ of $G$ containing $T$ such that $\iota$ maps $A_{nr}(H, T)$ to $A_{nr}(G, S)$ by an affine transformation (see [25, Theorem 1.9], and Remark 1.2 for the definition of the non reduced apartment $A_{nr}$).

2. Existence of $\sigma$-stable apartments

From now on, $k$ will be a non-Archimedean locally compact field of residue characteristic different from 2. Let $G$ be connected reductive group defined over $k$ and let $\sigma$ be a $k$-involution on $G$.

According to §1.14, the building $B$ of $G$ over $k$ is endowed with an action of $\sigma$. In this section, we prove that, for any $x \in B$, there exists a $\sigma$-stable apartment containing $x$. We keep using notations of Section 1.

2.1. Let $\Omega$ be a nonempty $\sigma$-stable subset of $B$ contained in some apartment, and let $\text{App}(\Omega)$ be the set of all apartments of $B$ containing $\Omega$. It is a nonempty set on which the group $P_\Omega$ acts transitively (see [20, Corollary 13.7]). Because $\Omega$ is $\sigma$-stable, both $P_\Omega$ and $\text{App}(\Omega)$ are $\sigma$-stable. Note that the $\sigma$-stable apartments containing $\Omega$ are exactly the $\sigma$-invariant points in $\text{App}(\Omega)$.

2.2. Let us fix an apartment $A \in \text{App}(\Omega)$ and an element $u \in P_\Omega$ such that $\sigma(A) = u \cdot A$. Let $N$ denote the normalizer in $G$ of the maximal $k$-split torus of $G$ corresponding to $A$. As $\sigma$ is involutive, we have:

\[(2.1) \quad \sigma(u)u \in P_\Omega \cap N_k = N_\Omega.\]

The map $\rho: g \mapsto g \cdot A$ induces a $P_\Omega$-equivariant bijection between the homogeneous spaces $P_\Omega/N_\Omega$ and $\text{App}(\Omega)$. The automorphism:

\[(2.2) \quad \theta: x \mapsto u^{-1}\sigma(x)u\]

of the group $G_k$ stabilises $P_\Omega$ and $N_\Omega$. Indeed $\sigma(N_k) = uN_ku^{-1}$, and:

\[\theta(N_\Omega) = u^{-1}\sigma(P_\Omega \cap N_k)u = P_\Omega \cap u^{-1}\sigma(N_k)u = N_\Omega.\]
Note that the condition (2.1) means that $\theta \circ \theta$ is conjugation by some element of $N_\Omega$. As $N_\Omega$ is $\theta$-stable, the map:

\begin{equation}
(\sigma, gN_\Omega) \mapsto u\theta(gN_\Omega), \quad g \in P_\Omega,
\end{equation}

defines an action of $\sigma$ on $P_\Omega/N_\Omega$, making $\rho$ into a $\sigma$-equivariant bijection. Note that this action differs from the natural action of $\sigma$ on $P_\Omega/N_\Omega$ (which obviously has fixed points).

2.3. Let $\Omega$ be a nonempty $\sigma$-stable subset of $B$ contained in some apartment.

**Proposition 2.1.** — Assume that $\Omega$ contains a point of a chamber of $B$. Then $\Omega$ is contained in some $\sigma$-stable apartment.

**Proof.** — First we describe $P_\Omega/N_\Omega$ as a projective limit of finite $\sigma$-sets. According to [11, §1.2], Example (f), the group $G_k$ is locally compact and totally disconnected. Therefore we can choose a decreasing filtration $(Q^i)_i \geq 0$ of the open subgroup $P_\Omega$ of $G_k$ satisfying the following properties:

(A) The intersection of the $Q^i$ is reduced to $\{1\}$.

(B) For any $i \geq 0$, the subgroup $Q^i$ is compact open and normal in $P_\Omega$.

For $i \geq 0$, let $P^i_\Omega$ denote the intersection of the subgroups $N_\Omega Q^i$ and $\theta(N_\Omega Q^i)$. The $P^i_\Omega$ form a decreasing filtration of $P_\Omega$, and we claim that such a filtration satisfies the following properties:

(1) The intersection of the $P^i_\Omega$ is reduced to $N_\Omega$.

(2) For any $i \geq 0$, the subgroup $P^i_\Omega$ is $\theta$-stable and of finite index in $P_\Omega$.

As $N_\Omega$ is $\theta$-stable, it is contained in the intersection of the $P^i_\Omega$. Let $g$ be in this intersection. For any $i \geq 0$, there exist $n_i \in N_\Omega$ and $q_i \in Q^i$ such that $g = n_i q_i$. Because of Property (A) above, $q_i$ converges to 1. Therefore $n_i$ converges to a limit contained in the closed subgroup $N_\Omega$, and this limit is $g$. This proves Property (1).

Now recall that $\theta \circ \theta$ is conjugation by some element of $N_\Omega$. This implies that $P^i_\Omega$ is $\theta$-stable. As $P^i_\Omega$ is open in $P_\Omega$ and contains $N_\Omega$, the quotient $P_\Omega/P^i_\Omega$ can be identified with the quotient of $U_\Omega$, which is compact, by some open subgroup. This gives the expected result.
Because of Property (2), the map:
\[(\sigma, gP^i_\Omega) \mapsto u\theta(gP^i_\Omega), \quad g \in P_\Omega,\]
defines an action of \(\sigma\) on the finite quotient \(P_\Omega/P^i_\Omega\). We get a projective system
\((P_\Omega/P^i_\Omega)_{i \geq 0}\) of finite \(\sigma\)-sets. Because \(P_\Omega\) is complete, and thanks to Property
(1), the natural \(\sigma\)-equivariant map from \(P_\Omega/N_\Omega\) to the projective limit of the
\(P_\Omega/P^i_\Omega\) is bijective.

**Lemma 2.2.** — Let \((\mathcal{X}^i)_{i \geq 0}\) be a projective system of finite \(\sigma\)-sets, and let
\(\mathcal{X}\) be its projective limit. Assume that, for each \(i \geq 0\), the cardinal of \(\mathcal{X}^i\) is
odd. Then \(\mathcal{X}\) has a \(\sigma\)-invariant point.

**Proof.** — Because each of the \(\mathcal{X}^i\) has an odd cardinal, each of them contains
a \(\sigma\)-invariant element. Suppose that we have constructed for some \(i \geq 1\) a \(\sigma\)-
invariant element \(x_i \in \mathcal{X}^i\). The fiber of \(x_i\) in \(\mathcal{X}^{i+1}\) is \(\sigma\)-stable and its cardinal
is the quotient of the cardinal of \(\mathcal{X}^{i+1}\) by the one of \(\mathcal{X}^i\). Therefore it is odd.
We deduce from this that there exists a \(\sigma\)-invariant element \(x_{i+1} \in \mathcal{X}^{i+1}\) whose
image in \(\mathcal{X}^i\) is \(x_i\). By induction, we get a \(\sigma\)-invariant element \(x \in \mathcal{X}\). \(\square\)

Let \(p\) denote the residue characteristic of \(k\). Recall that \(p\) is assumed to be
odd.

**Lemma 2.3.** — Let \(K\) be a normal subgroup of finite index in \(P_\Omega\) containing
\(N_\Omega\). Then the index of \(K\) in \(P_\Omega\) is a power of \(p\).

**Proof.** — Let \(S\) be the maximal \(k\)-split torus associated to \(\mathcal{A}\), let \(\Phi\) denote
the set of roots of \(G\) relative to \(S\) and let \(\Phi = \Phi^- \cup \Phi^+\) be a decomposition of
\(\Phi\) into positive and negative roots. According to §1.10, the group \(P_\Omega\) has the
following Iwahori decomposition:

\[(2.4) \quad P_\Omega = U^-_\Omega U^+_\Omega N_\Omega.\]

The fact that \(\Omega\) contains a point of a chamber of \(\mathcal{B}\) implies that the group \(N_\Omega\)
is reduced to \(\text{Ker}(\nu)\), hence normalizes the groups \(U^+_\Omega\) and \(U^-_\Omega\). The index of
K in $P_\Omega$ can be decomposed as follows:

$$\text{(2.5) } (P_\Omega : K) = (P_\Omega : U_\Omega^+ K) \cdot (U_\Omega^+ K : K).$$

In a first hand, the index $(U_\Omega^+ K : K) = (U_\Omega^+ : U_\Omega^+ \cap K)$ is a power of $p$, because $U_\Omega^+$ is a pro-$p$-group (i.e. a projective limit of finite discrete $p$-groups). In the other hand, the index $(P_\Omega : U_\Omega^+ K)$ is equal to $(U_\Omega^- : U_\Omega^- \cap U_\Omega^+ K)$, which is a power of $p$ because $U_\Omega^-$ is a pro-$p$-group. The result follows.

According to Lemma 2.3, the cardinal of each set $P_\Omega/P_\Omega^i$ with $i \geq 0$ is odd. Proposition 2.1 now follows from Lemma 2.2.

2.4. We now prove the main result of this section.

**Proposition 2.4.** — For any $x \in B$, there exists a $\sigma$-stable apartment containing $x$.

**Proof.** — Let $x$ be a point in $B$, and let $y$ be a point of a chamber of $B$ whose adherence contains $x$. The set $\Omega = \{y, \sigma(y)\}$ is a $\sigma$-stable subset of $B$ satisfying the conditions of Proposition 2.1. Hence we get a $\sigma$-stable apartment of $B$ containing $y$. Such an apartment contains the adherence of the chamber of $y$. In particular, it contains $x$. 

3. Decomposition of $H_k \backslash G_k$

Let $k$ be a non-Archimedean locally compact field of residue characteristic different from 2. Let $G$ be a connected reductive group defined over $k$, let $\sigma$ be an involutive $k$-automorphism of $G$ and let $H$ be an open $k$-subgroup of the fixed points group $G^\sigma$. Equivalently, $H$ is a $k$-subgroup of $G^\sigma$ containing the neutral component $(G^\sigma)^\circ$ (see [3]).

3.1. Let $S$ be a maximal $k$-split torus of $G$, and let $\mathcal{A}$ denote the corresponding apartment.

**Lemma 3.1.** — $\mathcal{A}$ is $\sigma$-stable if, and only if $S$ is $\sigma$-stable.
Proof. — This comes from the fact that the apartment corresponding to \( \sigma(S) \) is the image of \( \mathcal{A} \) by \( \sigma \).

3.2. We now assume that \( S \) is \( \sigma \)-stable. Let \( N \) (resp. \( Z \)) denote the normalizer (resp. the centralizer) of \( S \) in \( G \). Let \( \mathcal{O} = \mathcal{O}_S \) denote the set of all \( g \in G_k \) such that \( g^{-1}\sigma(g) \in N_k \).

**Proposition 3.2.** — \( \mathcal{O} \) is a finite union of \((H_k, Z_k)\)-double cosets.

Proof. — Let us fix a minimal parabolic \( k \)-subgroup \( P \) of \( G \) containing \( S \). According to [17, Proposition 6.8], the map \( g \mapsto H_k g P_k \) induces a bijection between the \((H_k, Z_k)\)-double cosets in \( \mathcal{O} \) and the \((H_k, P_k)\)-double cosets in \( G_k \). The result then follows from [17, Corollary 6.16].

We now give a direct proof of this result. We have an exact sequence:

\[
G_k^\sigma = H^0(G_k) \to H^0(G_k/N_k) \xrightarrow{\delta} H^1(N_k) \to H^1(G_k),
\]

where \( H^0 \) and \( H^1 \) denote respectively the set of \( \sigma \)-fixed points and the first set of nonabelian cohomology of \( \sigma \) (see [28, Chapter I, §5]). The transition map \( \delta \) induces an injective map from \( G_k^\sigma \) into \( H^1(N_k) \). Because \( Z_k \) (resp. \( H_k \)) is of finite index in \( N_k \) (resp. in \( G_k^\sigma \)), the finiteness of the number of \((G_k^\sigma, N_k)\)-double cosets of \( \mathcal{O} \) is equivalent to the finiteness of the number of \((H_k, Z_k)\)-double cosets of \( \mathcal{O} \). Therefore, it will be enough to prove that \( H^1(N_k) \) is finite.

Let \( M \) be a group with an action of \( \sigma \in \text{Aut}(M) \), and let \( M' \) be a \( \sigma \)-stable normal subgroup of \( M \). We can form the following exact sequence:

\[
H^1(M') \to H^1(M) \to H^1(M/M'),
\]

which proves that the finiteness of \( H^1(M') \) and \( H^1(M/M') \) implies the finiteness of \( H^1(M) \). Therefore we are reduced to proving that:

\[
H^1(N_k/Z_k), \quad H^1(Z_k/Z_k^1), \quad H^1(Z_k)
\]

are finite sets. Recall (see §1.2) that \( Z_k^1 \) denotes the maximal compact subgroup of \( Z_k \). Because \( N_k/Z_k \) is finite, the first case is immediate. Next, the
quotient $\Lambda = \mathbb{Z}_k/\mathbb{Z}_k^1$ is a finitely generated free abelian group. We have an exact sequence:

$$H^1(2\Lambda) \to H^1(\Lambda) \to H^1(\Lambda/2\Lambda).$$

Let $2m \in 2\Lambda$ be a cocycle, that is $2m + \sigma(2m) = 0$, and consider it as a cocycle in $\Lambda$. The identity $2m = m - \sigma(m)$ implies that the class $a(2m)$ is trivial in $H^1(\Lambda)$, hence that the map $a$ is null. Therefore $H^1(\Lambda)$ is embedded in $H^1(\Lambda/2\Lambda)$, which is finite because $\Lambda/2\Lambda$ is finite.

Now we treat the case of the compact subgroup $\mathbb{Z}_k^1$. Let $M$ be an open pro-$p$-subgroup of $\mathbb{Z}_k^1$. (Its existence is a topological property of $G_k$ asserted in [11, §1.2], Example (f).) The normalizer of $M$ in $\mathbb{Z}_k^1$ is open, hence of finite index, in $\mathbb{Z}_k^1$. We can therefore assume that $M$ is normal (if not, we replace it by the intersection of the finitely many $gM$ with $g \in \mathbb{Z}_k^1$). Moreover, we assume that $M$ is stable by $\sigma$ (if not, we replace it by $M \cap \sigma(M)$). Then $H^1(M)$ is trivial because $M$ is a pro-$p$-group and $p$ is odd, and $H^1(\mathbb{Z}_k^1/M)$ is finite because $M$ is of finite index in $\mathbb{Z}_k^1$. The finiteness of $H^1(\mathbb{Z}_k^1)$ follows. This ends our alternative proof of Lemma 3.2. \hfill \Box

3.3. Let $\mathcal{A}$ denote the $\sigma$-stable apartment corresponding to $S$.

**Lemma 3.3.** — We have $g \in \mathcal{O}$ if and only if $g \cdot \mathcal{A}$ is $\sigma$-stable.

**Proof.** — As $\mathcal{A}$ is $\sigma$-stable, the apartment $g \cdot \mathcal{A}$ is $\sigma$-stable if and only if $\sigma(g) \cdot \mathcal{A} = g \cdot \mathcal{A}$. This amounts to saying that $g^{-1}\sigma(g) \in \mathbb{N}_k$. \hfill \Box

For $x \in \mathcal{A}$, let $P_x$ denote the subgroup $P_\Omega$ (see §1.10) with $\Omega = \{x\}$.

**Proposition 3.4.** — Let $x$ be in $\mathcal{A}$. Then we have $G_k = \mathcal{O}P_x$.

**Proof.** — For $g \in G_k$, we set $x' = g \cdot x$. According to Proposition 2.4, there is a $\sigma$-stable apartment $\mathcal{A}'$ containing $x'$. Let $g' \in \mathcal{O}$ be such that $\mathcal{A}' = g' \cdot \mathcal{A}$. According to Property (2) of §1.13, there exists $n \in \mathbb{N}_k$ such that we have $g'^{-1}g \cdot x = n \cdot x$. Hence we get $g \in \mathcal{O}N_kP_x$. As $\mathcal{O}N_k = \mathcal{O}$, we obtain the expected result. \hfill \Box
3.4. If $T$ is a $\sigma$-stable torus in $G$, we denote by $T^+$ (resp. $T^-$) the neutral component of $T \cap H$ (resp. of the subgroup \{t \in T \mid \sigma(t) = t^{-1}\}). Note that, as $T^+$ is open in the fixed points subgroup $T^\sigma$, we have $T^+ = (T^\sigma)^\circ$. The torus $T$ is the almost direct product (see [6, xi]) of $T^+$ and $T^-$, which means that $T$ is equal to the product $T^+T^-$ and the intersection $T^+ \cap T^-$ is finite.

**Definition 3.5 (Helminck-Wang [17], §4.4).** — A $\sigma$-stable torus $T$ of $G$ is said to be $(\sigma, k)$-split if it is $k$-split and if $T = T^-_k$.

Let us recall (see [17, Proposition 10.3]) that two arbitrary maximal $(\sigma, k)$-split tori of $G$ are $G_k$-conjugated.

3.5. Let $T$ be a $k$-split torus of $G$, and let $T^1_k$ denote its maximal compact subgroup. Let $\varpi$ be a uniformizer of $k$. The images of $\varpi$ by the various algebraic cocharacters of $T$ form a $\sigma$-stable lattice in $T_k$, which will be denoted by $\Lambda(T_k)$.

**Lemma 3.6.** — (i) $T_k$ is the direct product of $\Lambda(T_k)$ and $T^1_k$.

(ii) For any $g \in G_k$, we have $\Lambda(gT_k) = g\Lambda(T_k)$.

(iii) The subgroup generated by $\Lambda(T^+_k)$ and $\Lambda(T^-_k)$ has finite index in $\Lambda(T_k)$.

**Proof.** — Only (iii) is not immediate. First note that, as $k$ is a non Archimedean locally compact field of characteristic different from 2, the subgroup of squares of $k^\times$ is of finite index in $k^\times$. This implies that $T^2_k = \{t^2 \mid t \in T_k\}$ is of finite index in $T_k$.

For any $t \in T_k$, the element $t^2$ can be decomposed as the product of $t\sigma(t) \in T^+_k$ and $t\sigma(t)^{-1} \in T^-_k$. Indeed the image of $T$ by the map $t \mapsto t\sigma(t)$ is connected and contained in $T^\sigma$, thus in $T^+$. By a similar argument, the image of $T$ by $t \mapsto t\sigma(t)^{-1}$ is contained in $T^-$. Therefore $T^2_k$ is contained in $T^+_kT^-_k$, thus there is some finite subset $\mathcal{F}$ of $T_k$ such that $T_k = T^+_kT^-_k\mathcal{F}$. According to (i), this gives:

$$\Lambda(T_k)T^1_k = \Lambda(T^+_k)T^1_k\Lambda(T^-_k)T^1_k\mathcal{F} = \Lambda(T^+_k)\Lambda(T^-_k)T^1_k\mathcal{F}.$$
We obtain the expected result by computing the quotient of this equality by the subgroup $\Lambda(T_1^+)\Lambda(T_1^-)T_1^k$. \hfill $\blacksquare$

3.6. Let $\{A^j \mid j \in J\}$ be a set of representatives of the $H_k$-conjugacy classes of maximal $(\sigma, k)$-split tori in $G$. We denote by $W_{G_k}(A^j)$ (resp. $W_{H_k}(A^j)$) the quotient of the normalizer of $A^j$ in $G_k$ (resp. in $H_k$) by its centralizer. According to [17, Proposition 5.9], the group $W_{G_k}(A^j)$ is the Weyl group of a root system. In particular, it is a finite group. (If $\sigma$ is trivial on the isotropic factor of $G$ over $k$, then this group is trivial.)

**Definition 3.7.** A parabolic subgroup $P$ of $G$ is said to be $\sigma$-parabolic if it is opposite to $\sigma(P)$, that is if $P \cap \sigma(P)$ is a Levi subgroup of $P$ and $\sigma(P)$.

**Remark 3.8.** This differs from the terminology used in [17], where such parabolic subgroups are said to be $\sigma$-split.

3.7. Let $A$ be a maximal $(\sigma, k)$-split torus of $G$.

**Lemma 3.9.** There is a $\sigma$-stable maximal $k$-split torus of $G$ containing $A$.

**Proof.** Let $G'$ denote the neutral component of the centralizer of $A$ in $G$. It is a connected reductive $k$-group. Let $S$ be a $\sigma$-stable maximal $k$-split torus of $G'$, whose existence is asserted by Proposition 2.4 and Lemma 3.1 together. Such a torus $S$ is a $\sigma$-stable maximal $k$-split torus of $G$ containing $A$. \hfill $\blacksquare$

Let $S$ be a $\sigma$-stable maximal $k$-split torus of $G$ containing $A$ and $P$ a minimal $\sigma$-parabolic $k$-subgroup of $G$ containing $S$ (see [17, §4]). Let $\varpi$ be a uniformizer of $k$, set $\Lambda = \Lambda(A_k)$ and let $\Lambda^-$ denote the subset of anti-dominant elements of $\Lambda$ relative to $P$.

**Theorem 3.10.** For $j \in J$, let $\mathcal{N}_j \subset N_{G_k}(A^j)$ be a set of representatives of $W_{H_k}(A^j) \backslash W_{G_k}(A^j)$ and $y_j \in G_k$ such that $y_j A = A^j$. There exists a compact subset $\Omega$ of $G_k$ such that:

$$G_k = \bigcup_{j \in J} \bigcup_{n \in \mathcal{N}_j} H_k n y_j \Lambda^- \Omega.$$
Proof. — First let \( \{ u_i \mid i \in I \} \) be a set of representatives of \((H_k, Z_k)\)-double cosets in \( \mathcal{O} \). According to Lemma 3.2, such a set is finite. Let \( \mathcal{O} \) denote the apartment corresponding to \( S \), and let \( K \) be the stabilizer of \( x \) in \( G_k \). Then Proposition 3.4 can be rephrased as follows:

\[
G_k = \bigcup_{i \in I} H_k u_i Z_k K.
\]

Let \( K^1 \) denote the intersection \( K \cap G^1_k \) (see §1.15). It is the maximal compact subgroup of \( K \).

Lemma 3.11. — We have \( Z_k K = \Lambda(S_k) \mathcal{F} K^1 \) for some finite subset \( \mathcal{F} \subset G_k \).

Proof. — First note that \( Z_k \cap K^1 = Z_k^1 \). Indeed, any element of the group \( Z_k^1 \), which is the kernel of (1.2), acts trivially on \( x \). Therefore \( Z_k^1 \) is contained in \( K \), hence in its maximal compact subgroup \( K^1 \). Inversely, the compact group \( Z_k \cap K^1 \) is contained in \( Z_k \), hence in its maximal compact subgroup \( Z_k^1 \).

According to Remark 1.1, the group \( S_k Z_k^1 \) has finite index in \( Z_k \). Thus there exists a finite subset \( \mathcal{F}_1 \subset Z_k \) such that \( Z_k = \mathcal{F}_1 S_k (Z_k \cap K^1) \).

Let \( D \) denote the maximal \( k \)-split torus of the connected centre \( C \) of \( G \). According to Remark 1.3, the image of \( D_k \) in \( G_k / G^1_k \) has finite index, thus its image in \( K / K^1 \) too. This implies that \( D_k K^1 = \Lambda(D_k) K^1 \) has finite index in \( K \), thus that there exists a finite subset \( \mathcal{F}_2 \subset K \) such that \( K = \mathcal{F}_2 \Lambda(D_k) K^1 \).

Finally, we have:

\[
Z_k K = \mathcal{F}_1 S_k K = \mathcal{F}_1 \Lambda(S_k) K = \mathcal{F}_1 \Lambda(S_k) \mathcal{F}_2 \Lambda(D_k) K^1
\]

which gives the expected result with \( \mathcal{F} = \mathcal{F}_1 \mathcal{F}_2 \). \( \Box \)

For \( i \in I \), we set \( S^i = u_i S \). According to Lemmas 3.11 and 3.6(iii), there are finite subsets \( \mathcal{F} \subset G_k \) and \( \mathcal{V}_i \subset \Lambda(S^i_k) \), for \( i \in I \), such that:

\[
(3.2) \quad H_k u_i Z_k K = H_k \Lambda(S^i_k) \mathcal{V}_i u_i \mathcal{F} K^1.
\]
According to [16, Lemma 2.2], the \((\sigma, k)\)-split torus \(S^i\) is \(H_k\)-conjugated to a subtorus of \(A^j\) for some \(j \in J\). We can therefore assume that, for a suitable choice of the representative \(u_i\), the \((\sigma, k)\)-split torus \(S^i\) is contained in \(A^j\) for some \(j \in J\). For \(j \in J\), let \(\mathcal{W}_j\) be the union of the \(\mathcal{V}_{i} u_i \mathcal{F}\) such that \(A^j\) contains \(S^i\). Together with (3.1) and (3.2), this gives:

\begin{equation}
G_k = \bigcup_{j \in J} H_k \Lambda(A^j_k) \mathcal{W}_j K^1.
\end{equation}

For \(j \in J\), we fix a set \(\mathcal{M}_{H^j_k}\) of representatives of \(W_{H^j_k}(A^j)\) and we denote by \(\mathcal{N}_j\) the set \(\{hn | h \in \mathcal{M}_{H^j_k}, n \in \mathcal{N}_j\}\). It is a set of representatives of \(W_{G^j_k}(A^j)\). From (3.3) we have:

\begin{equation}
G_k = \bigcup_{j \in J} \bigcup_{n \in \mathcal{N}_j} H_k n \Lambda(A^j_k)^{-1} n^{-1} \mathcal{W}_j K^1,
\end{equation}

where \(\Lambda(A^j_k)^{-1}\) denotes the subset of anti-dominant elements of \(\Lambda(A^j_k)\) relative to the parabolic subgroup \(y_j P\). If we remark that \(\Lambda(A^j_k)^{-1} = y_j \Lambda^{-}\), and if we denote by \(\Omega\) the union of the \(y_j^{-1} n^{-1} \mathcal{W}_j K^1\) for \(j \in J\) and \(n \in \mathcal{N}_j\), then (3.4) becomes:

\begin{equation}
G_k = \bigcup_{j \in J} \bigcup_{n \in \mathcal{N}_j} H_k n y_j \Lambda^{-} \Omega.
\end{equation}

This gives us the expected result. \(\square\)

4. The split case

In this section, we keep using notations of Section 3. Moreover, we assume that the reductive group \(G\) is split over \(k\). Therefore, for any root \(a\) of \(G\) relative to some maximal \(k\)-split torus of \(G\), the root subgroup \(U_a\) is \(k\)-isomorphic to the additive group.

The main results of this section are Proposition 4.4 and Theorem 4.8.

4.1. Let \(S\) be a \(\sigma\)-stable maximal \(k\)-split torus of \(G\), let \(\mathcal{A}\) be the apartment corresponding to \(S\) and \(\Phi\) the set of roots of \(G\) relative to \(S\).
Let \( x \in \mathcal{A} \) be a special point, and let \( U_x \) denote the subgroup \( U_\Omega \) (see \( \S 1.10 \)) with \( \Omega = \{x\} \). Let \( a \in \Phi \) be a \( \sigma \)-invariant root, which means that \( a \circ \sigma = a \).

**Lemma 4.1.** — Assume that \( U_{-a}(k) \) is contained in \( \{g \in G_k \mid \sigma(g) = g^{-1}\} \). Then there are \( n \in \mathbb{N}_k \) and \( c \in U_x \) such that \( n = c^{-1}\sigma(c) \) and \( \nu(n) \) is the affine reflection of \( \mathcal{A} \) which let \( x \) invariant and whose linear part is \( s_a \).

**Proof.** — We fix a base point in the apartment \( \mathcal{A} \), so that it can be identified with the vector space \( V \). For any \( b \in \Phi \), this defines a filtration of the group \( U_b(k) \) (see \( \S 1.8 \)). For \( u \in U_b(k) - \{1\} \), we denote by \( \varphi_b(u) \) the greatest real number \( r \in \mathbb{R} \) such that \( u \in U_b(k)^r \).

Let us choose \( w \in U_{-a}(k) - \{1\} \) such that \( x \) is contained in the wall \( \mathcal{H}_{-a,w} \). Thus \( \nu(m(w)) \) is the affine reflection of \( \mathcal{A} \) which fixes \( x \) and whose linear part is \( s_a \), and we can set:

\[
 n = m(w) \in \mathbb{N}_k.
\]

Moreover \( \theta(-a, w) \), which is the unique affine function from \( \mathcal{A} \) to \( \mathbb{R} \) whose linear part is \(-a\) and whose vanishing hyperplane is \( \mathcal{H}_{-a,w} \), vanishes on \( x \). Therefore it is equal to the map:

\[
 y \mapsto -a(y) + a(x),
\]

which implies that \( \varphi_{-a}(w) = a(x) \). According to B3 (see \( \S 1.11 \)), it follows that \( w \) fixes \( x \).

The subgroup \( U_{-a}(k) \) is isomorphic to the additive group of \( k \). Thus, for any \( r \in \mathbb{R} \), the subgroup \( U_{-a}(k)_r \) corresponds through this isomorphism to a nontrivial \( \mathfrak{o} \)-submodule of \( k \), where \( \mathfrak{o} \) denotes the ring of integers of \( k \) (see [20, Proposition 7.7]). Therefore there is a unique element \( v \in U_{-a}(k) \) such that \( w = v^2 \) and \( \varphi_{-a}(v) = \varphi_{-a}(w) \). Thus \( v \in U_x \).

The map \( U_a(k) \times U_a(k) \rightarrow G_k \) defined by \( (u, u') \mapsto uu' \) is injective and the intersection given by (1.5) consists of a single element, namely \( n \). If we choose \( u, u' \in U_a(k) \) such that \( uu' = n \), then the element:

\[
(4.1) \quad \sigma(u')^{-1}w\sigma(u)^{-1} = \sigma(n)^{-1}
\]
is contained in the intersection (1.5). Hence $\sigma(n)^{-1}$ is equal to $n$, and the unicity property implies that $u' = \sigma(u)^{-1}$. Moreover, according to [20, Lemma 7.4(ii)], the real numbers $\varphi_a(u)$ and $\varphi_a(\sigma(u))$ are both equal to $-\varphi_{-a}(w)$. This implies that $u$ and $\sigma(u)$ are contained in $U_x$. Since $v$ is $\sigma$-anti-invariant and $w = v^2$, we get the expected result with $c = (uv)^{-1}$.

\begin{remark}
Note that $\sigma(c) \in U_x$. Indeed we have $\sigma(v) = v^{-1} \in U_x$ and $\sigma(u) \in U_x$. Hence $n = c^{-1} \sigma(c) \in N_k \cap U_\Omega$, which is contained in $N_\Omega$ with $\Omega = \{x, \sigma(x)\}$.
\end{remark}

4.2. Let $\mathcal{D}G$ denote the derived subgroup of $G$, and recall that $C$ denotes the connected centre of $G$. This latter subgroup is a $k$-split torus of $G$.

\begin{lemma}
Let $T$ be a $k$-split torus of $G$.

(i) There is a $k$-subtorus $T'$ of $C$ such that the groups $T \cdot \mathcal{D}G$ and $T' \cdot \mathcal{D}G$ are equal.

(ii) If $T$ is $(\sigma, k)$-split, then any $T'$ satisfying (i) is $(\sigma, k)$-split.

(iii) Assume that $\mathcal{D}G$ is contained in $H$ and $T$ is $(\sigma, k)$-split. Then any $T'$ satisfying (i) is $(\sigma, k)$-split and has the same dimension as $T$.
\end{lemma}

\begin{proof}
We set $\tilde{G} = G/\mathcal{D}G$ and, for any $k$-subgroup $K$ of $G$, we denote by $\tilde{K}$ the image of $K$ in $\tilde{G}$. According to [6, Proposition 14.2], the group $G$ is the almost direct product of $C$ and $\mathcal{D}G$, which means that $G$ is equal to the product $C \cdot \mathcal{D}G$ and that the intersection $C \cap \mathcal{D}G$ is finite. This implies that $\tilde{C} = \tilde{G}$. Let $f$ denote the $k$-rational map $C \to \tilde{C}$. It is surjective with finite kernel. Hence $\tilde{G}$ is a $k$-split torus, and we denote by $\tilde{\sigma}$ the involutive $k$-automorphism of $\tilde{G}$ induced by $\sigma$. We now prove the lemma in three steps.

(i) According to [6, Proposition 8.2(c)], the neutral component of the inverse image $f^{-1}(\tilde{T})$ is a $k$-split subtorus of $C$ which we denote by $T'$. It has finite index in $f^{-1}(\tilde{T})$. The image $f(T')$ is then a subtorus of finite index in the connected group $\tilde{T}$, so that $\tilde{T}' = \tilde{T}$.

(ii) Now assume that $T$ is $(\sigma, k)$-split, and let $T'$ satisfy (i). Let us consider the map $t \mapsto t\sigma(t)$ from $T'$ to itself. As $\tilde{T}' = \tilde{T}$ is a $(\tilde{\sigma}, k)$-split torus, the image
of this map is a connected \( k \)-subgroup contained in the kernel of \( f \), which is finite.

(iii) Assume that \( \mathcal{D}G \) is contained in \( H \) and \( T \) is \((\sigma, k)\)-split. Then the map \( T \rightarrow \tilde{T} \) has finite kernel, which implies that \( T \) and \( \tilde{T} \) have the same dimension. Now let \( T' \) satisfy (i). According to (ii), such a torus is \((\sigma, k)\)-split, and it has the same dimension as \( \tilde{T}' = \tilde{T} \).

This ends the proof of Lemma 4.3.

\[ \square \]

4.3. Let \( \mathcal{B} \) denote the building of \( G \) over \( k \).

**Proposition 4.4.** — Let \( x \) be a special point of \( \mathcal{B} \). There exists a \( \sigma \)-stable maximal \( k \)-split torus \( S \) of \( G \) such that the apartment corresponding to \( S \) contains \( x \) and such that \( S^- \) is a maximal \((\sigma, k)\)-split torus of \( G \).

**Remark 4.5.** — In \( \S 5.3 \) we give an example of a non split \( k \)-group \( G \) such that Proposition 4.4 does not hold.

**Proof.** — Let \( \mathcal{A} \) be a \( \sigma \)-stable apartment containing \( x \) (see Proposition 2.4) and let \( S \) be the corresponding maximal \( k \)-split torus of \( G \). Assume that \( \mathcal{A} \) has been chosen such that the dimension of the \((\sigma, k)\)-split torus \( S^- \) is maximal. If it is a maximal \((\sigma, k)\)-split torus of \( G \), then Proposition 4.4 is proved. Assume that this is not the case, and let \( A \) be a maximal \((\sigma, k)\)-split torus of \( G \) containing \( S^- \). The dimension of \( A \) is greater than \( \dim S^- \) (if not, the containment \( S^- \subset A \) would imply that \( S^- = A \)). If we get a contradiction, the proposition will be proved.

Let \( G' \) be the neutral component of the centralizer of \( S^- \) in \( G \). It is a \( k \)-split connected reductive subgroup of \( G \) containing \( S \) and \( A \), which is naturally endowed with a nontrivial action of \( \sigma \). Let \( C' \) denote the connected center of \( G' \).

**Lemma 4.6.** — There is \( a \in \Phi(G', S) \) such that the corresponding root subgroup \( U'_a \) is not contained in \( H \), and such a root is \( \sigma \)-invariant.

**Proof.** — Assume that \( U'_a \subset H \) for each root \( a \in \Phi(G', S) \). Thus the derived subgroup \( \mathcal{D}G' \), which is generated by the \( U'_a \) for \( a \in \Phi(G', S) \), is contained
in \( H \) (see [18, Theorem 27.5(e)]). According to Lemma 4.3(iii), there exists a \((\sigma, k)\)-subtorus \( A' \) of \( C' \) such that \( A \cdot D G' = A' \cdot D G' \) and \( \dim(A) = \dim(A') \).

The subgroup generated by \( C' \) and \( S \) is a \( k \)-torus of \( G' \). As \( G' \) is \( k \)-split, \( S \) is a maximal torus of \( G' \), hence it contains \( C' \). Therefore \( S^- \) contains \( A' \) which has the same dimension as \( A \), and this dimension is greater than \( \dim S^- \). This gives us a contradiction.

Now let \( a \) be a root in \( \Phi(G', S) \) such that \( U'_a \) is not contained in \( H \). The root \( a \) and its conjugate \( a \circ \sigma \) coincide on \( S^+ \) and are both trivial on \( S^- \). As \( S \) is the almost direct product of \( S^+ \) and \( S^- \) (see §3.4), they are equal. Therefore \( a \) is \( \sigma \)-invariant. This ends the proof of Lemma 4.6. \( \square \)

Let \( a \in \Phi(G', S) \) as in Lemma 4.6. If we think to \( a \) as a root in \( \Phi(G, S) \), the root subgroup \( U_a \) is \( \sigma \)-stable and is not contained in \( H \). Moreover, we have the following result.

**Lemma 4.7.** — \( U_a(k) \) is contained in \( \{ g \in G_k \mid \sigma(g) = g^{-1} \} \).

**Proof.** — As \( G \) is \( k \)-split, \( U_a \) is \( k \)-isomorphic to the additive group. Thus the action of \( \sigma \) on \( U_a(k) \) corresponds to an involutive automorphism of the \( k \)-algebra \( k[t] \). It has the form \( t \mapsto \lambda t \) for some \( \lambda \in k^\times \) with \( \lambda^2 = 1 \). As \( U_a \) is not contained in \( H \), we have \( \lambda = -1 \). This gives the expected result. \( \square \)

According to Lemma 4.1, there are \( n \in N_k \) and \( c \in U_x \) such that \( n = c^{-1} \sigma(c) \) and \( \nu(n) \) is the affine reflection of \( \mathcal{A} \) which let \( x \) invariant and whose linear part is \( s_a \). For any \( t \in S \), note that we have:

\[
\sigma(ctc^{-1}) = c n \sigma(t)n^{-1}c^{-1} = cs_a(\sigma(t))c^{-1}.
\]

Let \( \mathcal{A}' \) denote the apartment \( c \cdot \mathcal{A} \) and let \( S' = 'S \) be the corresponding maximal \( k \)-split torus of \( G \). Then \( \mathcal{A}' \) contains \( x \) and is \( \sigma \)-stable. Moreover, as the root \( a \) is trivial on \( S^- \) and \( s_a \) fixes the kernel of \( a \) pointwise, the conjugate \( c(S^-) \) is a \((\sigma, k)\)-split subtorus of \( S' \). Thus \( S'^- \) has dimension not smaller than \( \dim S^- \).
Now let $S_a$ denote the maximal $k$-split torus in the set of all $t \in S$ such that $s_a(t) = t^{-1}$. As $a$ is $\sigma$-invariant, such a torus is $\sigma$-stable. Moreover, it is one dimensional and its intersection with $\text{Ker}(a)$ is finite. Therefore the conjugate $cS_a$ is a nontrivial $(\sigma, k)$-split subtorus of $S'$ which is not contained in $c(S^-)$. Thus the dimension of $S'^-$, which contains $c(S_aS^-)$, is greater than $\dim S^-$, which contradicts the maximality property of $\mathcal{A}$. This ends the proof of Proposition 4.4.

4.4. Let $A$ be a maximal $(\sigma, k)$-split torus of $G$ and $S$ a $\sigma$-stable maximal $k$-split torus of $G$ containing $A$. Let $\{A_j \mid j \in J\}$ be a set of representatives of the $H_k$-conjugacy classes of maximal $(\sigma, k)$-split tori in $G$. Let $x$ be a special point of the building and let $K$ be its stabilizer in $G_k$.

**Theorem 4.8.** — For $j \in J$, let $y_j \in G_k$ such that $y_j A = A_j$. We have:

$$G_k = \bigcup_{j \in J} H_k y_j S_k K.$$ 

**Proof.** — We fix $g \in G_k$. According to Proposition 4.4, there is a $\sigma$-stable maximal $k$-split torus $S'$ of $G$ such that the apartment corresponding to it contains $g \cdot x$ and such that $S'^-$ is a maximal $(\sigma, k)$-split torus of $G$. Let $j \in J$ be such that $S'^- = H_k$-conjugate to $A_j$. According to [16, Lemma 2.2], there is $h \in H_k$ such that $S' = h y_j S$. Hence $g \cdot x$ is contained in $h y_j \cdot \mathcal{A}$. According to Property (2) of §1.13, there exists $n \in N_k$ such that $g \cdot x = h y_j n \cdot x$.

Therefore $G_k$ is the union of the $H_k y_j N_k K$ for $j \in J$. As $x$ is special, we have $N_k K = S_k K$ and we get the expected result.

5. Examples

Let $k$ be a non-Archimedean locally compact field of residue characteristic different from 2. Let $\mathcal{o}$ be its ring of integers and $\mathfrak{p}$ its maximal ideal.

5.1. Here we consider the connected reductive $k$-group $G = \text{GL}_n$, endowed with the $k$-involution $\sigma : g \mapsto t^g g^{-1}$, where $^t g$ denotes the transpose of $g \in G$. 

We set $K = \text{GL}_n(\mathfrak{o})$ and $H = \text{G}^\sigma$, which is an orthogonal group, and we denote by $S$ the diagonal torus of $G$.

We start with the following lemma.

**Lemma 5.1.** — Let $V$ be a finite dimensional $k$-vector space and $B$ a symmetric bilinear form on $V$. Then any free $\mathfrak{o}$-submodule of finite rank of $V$ has a basis which is orthogonal relative to $B$.

**Proof.** — Let $\Lambda$ be a free $\mathfrak{o}$-submodule of finite rank of $V$. The proof goes by induction on the rank of $\Lambda$. If $B$ is null, then the result is trivial. If not, we denote by $B_\Lambda$ the restriction of $B$ to $\Lambda \times \Lambda$. Its image is of the form $p^m$ for some integer $m \in \mathbb{Z}$. If $\varpi$ is a uniformizer of $k$, then the form $B^0_\Lambda = \varpi^{-m}B_\Lambda$ has image $\mathfrak{o}$ on $\Lambda \times \Lambda$. Therefore, it defines a non trivial bilinear form:

$$\bar{B}^0_\Lambda : \Lambda/p\Lambda \times \Lambda/p\Lambda \to \mathfrak{o}/p.$$ 

Let $e \in \Lambda$ be a vector whose reduction mod. $p$ is not isotropic relative to $\bar{B}^0_\Lambda$, which means that $B^0_\Lambda(e,e)$ is a unit of $\mathfrak{o}$. Then $\Lambda$ is the direct sum of $\mathfrak{o}e$ and $\Lambda \cap ke^\perp$, where $ke^\perp$ denotes the orthogonal of $ke$ in $V$. Indeed, it follows from the decomposition:

$$x = \frac{B(e,x)}{B(e,e)}e + \left( x - \frac{B(e,x)}{B(e,e)}e \right)$$

for any $x \in \Lambda$. As $\Lambda \cap ke^\perp$ is a free $\mathfrak{o}$-submodule of finite rank of $V$ whose rank is smaller than the rank of $\Lambda$, we conclude by induction.

We introduce the set $\mathcal{E}$ of all $g \in G_k$ such that $^tg \in S_k$ (compare §3.2). We have the following decomposition of $G_k$, which is more precise than the one given by Proposition 3.4.

**Proposition 5.2.** — We have $G_k = \mathcal{E}K$.

**Proof.** — We make $G_k$ act on the quotient $G_k/K$, which can be identified to the set of all $\mathfrak{o}$-lattices (that is, cocompact free $\mathfrak{o}$-submodules) of the $k$-vector space $V = k^n$. Let $B$ denote the symmetric bilinear form on $V$ making the canonical basis of $V$ into an orthonormal basis. According to Lemma 5.1, for
any \( g \in G_k \), the \( \mathfrak{o} \)-lattice \( \Lambda \) corresponding to the class \( gK \) has a basis which is orthogonal relative to \( B \). This means that there exists \( u \in K \) such that the element \( g' = gu^{-1} \in gK \) maps the canonical basis of \( V \) to an orthogonal basis of \( \Lambda \). Therefore we have \( g' \in \mathcal{E} \), thus \( g \in \mathcal{E}K \).

We now investigate the maximal \((\sigma, k)\)-split tori of \( G \). Note that \( S \) is a maximal \((\sigma, k)\)-split torus of \( G \).

**Proposition 5.3.** — The map \( g \mapsto gS \) induces a bijection between \( (H_k, N_k) \)-double cosets of \( \mathcal{E} \) and \( H_k \)-conjugacy classes of maximal \((\sigma, k)\)-split tori of \( G \).

**Proof.** — One immediately checks that this map is well defined and injective. For \( g \in G_k \), the conjugate \( gS \) is a maximal \((\sigma, k)\)-split torus of \( G \) if and only if \( g^{-1}\sigma(g) \in S_k \), which amounts to saying that \( g \in \mathcal{E} \) and proves surjectivity.

Let \( \mathcal{D} \) denote the set of all equivalence classes of non degenerate quadratic forms on \( k^n \). For \( a = \text{diag}(a_1, \ldots, a_n) \in S_k \) we denote by \( Q_a \) the diagonal quadratic form \( a_1X_1^2 + \ldots + a_nX_n^2 \). Note that the map \( a \mapsto Q_a \) induces a surjective map from \( S_k \) to \( \mathcal{D} \).

**Proposition 5.4.** — (i) The map \( g \mapsto t^gg \) induces an injection \( \iota \) from the set of \( (H_k, N_k) \)-double cosets of \( \mathcal{E} \) to \( H^1(N_k) \).

(ii) For \( a \in S_k \), the class of \( a \) in \( H^1(N_k) \) is in the image of \( \iota \) if and only if \( Q_a \sim X_1^2 + \ldots + X_n^2 \).

**Proof.** — We have an exact sequence:

\[
H_k \rightarrow H^0(G_k/N_k) \rightarrow H^1(N_k) \rightarrow H^1(G_k),
\]

where the map from \( H^0(G_k/N_k) \) to \( H^1(N_k) \) is induced by \( g \mapsto t^gg \). As the set of \( (H_k, N_k) \)-double cosets of \( \mathcal{E} \) is a subset of \( H_k \backslash H^0(G_k/N_k) \), we get (i). To get (ii), it is enough to remark that \( H^1(G_k) \) canonically identifies with \( \mathcal{D} \).

**Remark 5.5.** — Recall (see [27, IV §2.3]) that for \( a, b \in S_k \), the nondegenerate quadratic forms \( Q_a, Q_b \) are equivalent if, and only if they have the same discriminant and the same Hasse invariant.
Proposition 5.6. — Let \( \{ u_j \mid j \in J \} \subset S_k \) form a set of representatives of \( \text{Im}(\iota) \) in \( H^1(N_k) \). For \( j \in J \), we choose \( y_j \in \mathcal{E} \) such that \( y_j y_j^t = a_j \). Then:

\[
G_k = \bigcup_{j \in J} H_k y_j S_k K.
\]

Proof. — Propositions 5.2 and 5.3 imply that \( G_k \) is the union of the components \( H_k y_j N_k K \) for \( j \in J \). As \( N_k K = S_k K \) we get the expected result.

Example 5.7. — In the case where \( n = 2 \), we give an explicit description of \( \text{Im}(\iota) \). Let \( \varpi \) denote a uniformizer of \( \mathfrak{o} \) and \( \xi \in \mathfrak{o}^\times \) a non square unit of \( \mathfrak{o} \), so that \( \{ 1, \xi, \varpi, \xi \varpi \} \) is a set of representatives of \( k^\times \) modulo \( k^{\times 2} \). The set of elements of \( k^\times \) which are represented by the quadratic form \( Q_1 = X^2 + Y^2 \) depends on the image of \( p \) in \( \mathbb{Z}/4\mathbb{Z} \). If \( p \equiv 1 \) mod \( 4 \), all elements of \( k^\times \) are represented by \( Q_1 \). If \( p \equiv 3 \) mod \( 4 \), an element of \( k^\times \) is represented by \( Q_1 \) if and only if its normalized valuation if even. We set:

\[
J = \begin{cases} 
\{1, \xi, \varpi, \xi \varpi\} & \text{if } p \equiv 1 \text{ mod } 4, \\
\{1, \xi\} & \text{if } p \equiv 3 \text{ mod } 4. 
\end{cases}
\]

For each \( j \in J \), set \( a_j = \text{diag}(j, j) \). Then the elements \( a_j \) form a set of representatives of \( \text{Im}(\iota) \) in \( H^1(N_k) \).

5.2. In this paragraph we consider the connected reductive \( k \)-group \( G = \text{Res}_{k'/k} \text{GL}_n \), where \( k' \) is a quadratic extension of \( k \), endowed with the involutive \( k \)-automorphism \( \sigma \) of \( G \) induced by the nontrivial element of \( \text{Gal}(k'/k) \).

We set \( H = G^\sigma \), so that we have \( G_k = \text{GL}_n(k') \) and \( H_k = \text{GL}_n(k) \). We denote by \( S \) the diagonal torus of \( G \) and by \( K \) the maximal compact subgroup \( \text{GL}_n(o') \) of \( G_k \), where \( o' \) denotes the ring of integers of \( k' \). Note that \( S \) is \( \sigma \)-invariant, that is \( S = S^+ \).

As usual, \( N \) (resp. \( Z \)) denotes the normalizer (resp. the centralizer) of \( S \) in \( G \). Let \( \mathfrak{S}_n \) denote the group of permutation matrices in \( G_k \), so that \( N_k \) is the semidirect product of \( \mathfrak{S}_n \) by \( Z_k \). Note that \( S_k \) (resp. \( Z_k \)) is the subgroup of all diagonal matrices of \( G_k \) with entries in \( k \) (resp. in \( k' \)).
Lemma 5.8. — $H^1(N_k)$ can be identified with the set of conjugacy classes of elements of $S_n$ of order 1 or 2.

Proof. — According to Hilbert’s Theorem 90, the group $H^1(Z_k)$ is trivial. Therefore we have an exact sequence:

\[(5.2) \ 1 \rightarrow H^1(N_k) \rightarrow H^1(N_k/Z_k).\]

As $\sigma$ acts trivially on $N_k/Z_k \cong S_n$, the set $H^1(N_k/Z_k)$ can be identified to the set of $S_n$-conjugacy classes of $\text{Hom}(Z/2Z, S_n)$, that is, to the set of conjugacy classes of elements of $S_n$ of order $\leq 2$. This proves that $H^1(N_k)$ can be naturally embedded in the set of conjugacy classes of elements of $S_n$ of order $\leq 2$.

Now two elements $w, w' \in S_n$ define the same class in $H^1(N_k)$ if and only if they are conjugate in $S_n$, thus if and only if $wZ_k$ and $w'Z_k$ define the same class in $H^1(N_k/Z_k)$. Therefore (5.2) is a bijection. \[\square\]

Proposition 5.9. — (i) The number of $H_k$-conjugacy classes of $\sigma$-stable maximal $k$-split tori in $G_k$ is $[n/2] + 1$.

(ii) There is a unique $H_k$-conjugacy class of maximal $(\sigma, k)$-split tori in $G_k$.

Proof. — (i) Let $\mathcal{O}$ denote the set of all $g \in G_k$ such that $g^{-1}\sigma(g) \in N_k$. Then the map $g \mapsto gS$ defines an injective map from the set of $(H_k, N_k)$-double cosets of $\mathcal{O}$ to $H^1(N_k)$. Therefore we are reduced to proving that this map is surjective, and (i) will follow from Lemma 5.8

For $n = 2$, let $\tau$ denote the nontrivial element of $S_2$ and choose an element $a \in k'$ which is not in $k$. Then the element:

\[(5.3) \quad u = \begin{pmatrix} a & \sigma(a) \\ 1 & 1 \end{pmatrix} \in \text{GL}_2(k')\]

satisfies the relation $u^{-1}\sigma(u) = \tau$. For an arbitrary integer $n \geq 2$, let $w \in S_n$ have order $\leq 2$. Then there is an integer $0 \leq i \leq [n/2]$ such that $w$ is conjugate to the element:

$$\tau_i = \text{diag}(\tau, \ldots, \tau, 1, \ldots, 1) \in \text{GL}_n(k'),$$
where $\tau \in \text{GL}_2(k')$ appears $i$ times and $1 \in \text{GL}_1(k')$ appears $n - 2i$ times. Thus the matrice:

\begin{equation}
(5.4) \quad u_i = \text{diag}(u, \ldots, u, 1, \ldots, 1) \in \text{GL}_n(k')
\end{equation}

satisfies the relation $u_i^{-1}\sigma(u_i) = \tau_i$. Therefore any cocycle in $N_k$ is $G_k$-cohomologous to the neutral element $1 \in G_k$, which proves (i).

(ii) For any $0 \leq i \leq \lfloor n/2 \rfloor$, the dimension of the $(\sigma, k)$-split torus $(u^iS)^-$ is equal to $i$. According to (i), the map:

\[ H_k g N_k \mapsto \text{class of } g^{-1}\sigma(g) \text{ in } H^1(N_k) \]

is a bijection from the set of $(H_k, N_k)$-double cosets of $\mathcal{O}$ to $H^1(N_k)$, and the elements of this latter set are the classes of the $\tau_i$ for $0 \leq i \leq \lfloor n/2 \rfloor$. This gives us the expected result.

This ends the proof of Proposition 5.9.

\begin{proof}
According to the proof of Proposition 5.9, the set $\mathcal{O}$ is the union of the double cosets $H_ku_iN_k$ with $0 \leq i \leq \lfloor n/2 \rfloor$. The result then follows from Proposition 3.4 and from the fact that $N_kK = Z_kK$.
\end{proof}

\section*{5.3.}
Here we give an example (due to Bertrand Lemaire) of a non-split $k$-group such that Proposition 4.4 does not hold. We set $G = \text{Res}_{k'/k} \text{GL}_2$, where $k'$ is now a \textit{totally ramified} quadratic extension of $k$. The $k$-involution $\sigma$ is still induced by the nontrivial element of $\text{Gal}(k'/k)$ and we set $H = \text{GL}_2$. Let $\mathcal{B}'$ (resp. $\mathcal{B}$) denote the building of $G$ (resp. $H$) over $k$.

In [10], Bruhat and Tits give a description of the faces of $\mathcal{B}$ in terms of hereditary $\mathfrak{o}$-orders of $M_2(k)$. More precisely, there is a bijective correspondence:

\begin{equation}
(5.5) \quad F \mapsto \mathcal{M}_F
\end{equation}
between the faces of $B$ and the hereditary $\mathfrak{o}$-orders of $M_2(k)$, such that the stabilizer of $F$ in $GL_2(k)$ in the normalizer of $\mathcal{M}_F$ in $GL_2(k)$. For $x \in B$, we will denote by $\mathcal{M}_x$ the hereditary order corresponding to the face of $B$ which contains $x$. Of course, we have a similar correspondence between faces of $B'$ and hereditary $\mathfrak{o}'$-orders of $M_2(k')$. Moreover, as $k'$ is tamely ramified over $k$, there is a bijective correspondence $j$ from the set $B'^{\sigma}$ of $\sigma$-invariant points of $B'$ to $B$ such that, for any $x \in B'^{\sigma}$, we have:

$$\mathcal{M}_j(x) = \mathcal{M}_x \cap M_2(k).$$

Let $q$ denote the cardinal of the residue field of $k$. As $k'$ is totally ramified over $k$, any vertex of $B$ (resp. $B'$) has exactly $q+1$ neighbours in $B$ (resp. in $B'$). Let $x$ be a $\sigma$-invariant point of $B'$. Recall that, according to Proposition 2.4, it is contained in a $\sigma$-stable apartment.

1. If $j(x)$ is in a chamber of $B$, then $x$ has $q+1$ neighbours in $B'$ but only two $\sigma$-invariant ones. Thus $x$ has non-$\sigma$-invariant neighbours.

2. If $j(x)$ is a vertex of $B$, then $x$ has $q+1$ neighbours in $B'$ as in $B$. Thus any neighbour of $x$ in $B'$ is $\sigma$-invariant, which implies that any $\sigma$-stable apartment containing $x$ is $\sigma$-invariant. For instance, this is the case of the vertex $x$ corresponding to the $\mathfrak{o}'$-order $M_2(\mathfrak{o}')$, because its image $j(x)$ corresponds to the maximal $\mathfrak{o}$-order $M_2(\mathfrak{o}) \cap M_2(k) = M_2(\mathfrak{o})$. Such a special point does not satisfy Proposition 4.4.

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