Impact of thermal radiation and non-uniform heat flux on MHD hybrid nanofluid along a stretching cylinder

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The current research investigates the thermal radiations and non-uniform heat flux impacts on magnetohydrodynamic hybrid nanofluid (CuO-Fe2O3/H2O) flow along a stretching cylinder, which is the main aim of this study. The velocity slip conditions have been invoked to investigate the slippage phenomenon on the flow. The impact of induced magnetic field with the assumption of low Reynolds number is imperceptible. Through the use of appropriate non-dimensional parameters and similarity transformations, the ruling PDE’s (partial differential equations) are reduced to set of ODE’s (ordinary differential equations), which are then numerically solved using Adams–Bashforth Predictor–Corrector method. Velocity and temperature fields with distinct physical parameters are investigated and explored graphically. The main observations about the hybrid nanofluid and non-uniform heat flux are analyzed graphically. A decrease in the velocity of the fluid is noted with addition of Hybrid nanofluid particles while temperature of the fluid increases by adding the CuO-Fe2O3 particles to the base fluid. Also, velocity of the fluid decreases when we incorporate the effects of magnetic field and slip. Raise in curvature parameter γ caused enhancement of velocity and temperature fields at a distance from the cylinder but displays opposite behavior nearby the surface of cylinder. The existence of heat generation and absorption for both mass dependent and time dependent parameters increases the temperature of the fluid.

Recently, many researchers attracted towards nano technology because of its vast applications in different fields. Nano fluids have different thermo-physical properties than their respective base fluids, which have poor ability to conduct heat. Nano fluid was firstly introduced by Choi1 in 1995 as a remedy to heat transfer enhancement. He found that nanofluid has greater thermal conductivity as compared to base fluid. Nanofluids are widely used in cancer therapeutics, nuclear reactor, refrigerators and also have many automobile and electronic applications. Buongiorno2 conducted a comprehensive study of convective transport in nanofluids. Khan and Pop3 investigated boundary layer flow of nanofluid across a stretched surface. They consider the thermophoresis effects and Brownian motion and solve the problem numerically. Nadeem and Lee4 analyze nanofluid flow via an exponentially stretched surface and use HAM to solve the problem analytically. Das et al.5 investigated the unsteady problem of nanofluid along stretching surface and used the shooting method in conjunction with the Runge–Kutta Fehlberg methodology to solve the problem numerically. The effects of radiation and varying fluid characteristics on unsteady boundary layer flow of a nanofluid are numerically discussed by Daba and Devaraj6. Awais et al.7 studied the polymeric material’s properties like nonlinear thermal radiation and solve the problem analytically by using HAM. Ali et al.8 used OHAM to solve the problem of three-dimensional Maxwell
nанофлюид посредством диффузии. Более того, на основе наложенной модели свободного потока, Vinita et al. 26 исследовали влияние на концентрацию и температуру нанофилюида. Устойчивость двухфазного перистальтического потока в окруженной среде была рассмотрена Qasim et al. 37. counted на динамическое моделирование с использованием модели перистальтического потока.

В последнее время, многие исследователи изучали теплообмен для турбулентного потока, который влияет на течение нанофилюида. В соответствии с тем, что Wang et al. 30 предложил модельный подход для описания нанофилюида. Использование нанофилюида позволяет улучшить теплопередачу. Poply et al. 39 также исследовали влияние нанофилюида на перистальтический поток в окружной среде.

В работе настоящего исследования был проведен анализ нанофилюида, который использовался в качестве основного флюида. Hybrid nano-liquid was first extracted by Maxwell41 and is widely implemented in current flow investigator. Slip flow takes place where the flow pressure is very weak. Slip boundary condition was first extracted by Maxwell41 and is widely implemented in current flow investigator. Slip flows have various implementations in medical fields for example in the polishing of simulated heart valves and polymeric technology. Ibrahim and Shankar42 investigated MHD boundary layer flow and discussed heat transfer analysis while taking slip conditions into account. By addressing non-uniform heat flux effects, Das et al. 43 investigated slip phenomenon on MHD boundary layer flow of nanofluid over a vertical stretched sheet. The presence of thermal radiation, Haq et al. 44 examined the slip effects over stagnation point flow. The effects of temperature, velocity, and concentration slip on MHD nanofluid were discussed by Awais et al. 45. Raza et al. 46 discussed the effects of thermal radiation on Casson fluid stagnation point flow under slip conditions. Ali et al. 47 studied entropy generation on MHD peristaltic flow for two-phase nanofluid in the presence of slip effects. The slip effects on heat transfer study of nanofluid over a stretching cylinder are presented by Vinita and Poply48. Slip and heat effects on peristaltic flow of nanofluid with generalized complaint walls are shown by Awais et al. 49. On a non-linear stretching cylinder, Vinita et al. 50 investigate the velocity, temperature and concentration slip effects. Ali et al. 51 discuss the impact of Hall current and viscous dissipation on the slipage phenomenon in MHD peristaltic flow.

This work aims to determine the problem of hybrid nanofluid flow along a stretching cylinder. Hybrid nanofluids have been identified as prospective fluids and have gained considerable attraction from researchers, due to their wide range of applications in the medical and engineering sector. We have considered Copper Oxide (CuO) and Ferrous Oxide (Fe2O3) as nanometer size particles with water as a base fluid. Non-uniform heat flux and thermal radiation effects have been taken into account. The velocity slip condition will be invoked to study the slip effects on the flow. The mathematical modeling is carried out by using the continuity, momentum and energy equation. These governing equations are PDE's (partial differential equations), thus they are transformed into set of ODE's (ordinary differential equations) that may be solved numerically. Plots of different physical quantities were created to gain a better understanding of the subject under consideration.
Mathematical formulation

We have considered a two-dimensional, axisymmetric, unsteady, incompressible, boundary layer hybrid nano-fluid flow past a stretchable cylinder having radius $b$ in existence of uniform magnetic field. For this study, we have considered Copper Oxide (CuO) and Ferrous Oxide (Fe$_2$O$_3$) nanometer size particles with water as a base fluid. Cylindrical coordinate system has been considered with $(x, r)$-axes are taken along the cylinder's radial and axial direction respectively. A uniform magnetic field of strength $B_0$ is used in the radial direction. Magnetic field produced by induction is insignificant as compare to applied magnetic field with the supposition of small magnetic Reynolds number. The cylinder is being stretched in the axial direction, and the cylinder's stretching velocity is $U_w(x) = U_0 f(x)$, where $U_0$ is the associated velocity and $l$ is the characteristic length. The surface temperature $T_w(x)$ is believed to be higher than the ambient temperature $T_\infty$.

The vector form of the governing equations is:

\begin{align}
\nabla \cdot \mathbf{V} &= 0, \\
\rho_{\text{hf}} \frac{d\mathbf{V}}{dt} &= \mu_{\text{hf}} \nabla \cdot \mathbf{T} + \mathbf{J} \times \mathbf{B}, \\
(\rho \mathcal{C}_p)_{\text{hf}} \frac{dT}{dt} &= \kappa_{\text{hf}} \nabla^2 T - \nabla q_r + q'''.
\end{align}

where $\mathbf{V} = [v(r, x), 0, u(r, x)]$ is the velocity field in cylindrical coordinate system, $\frac{d}{dt}$ is the material time derivative, $\mathbf{J} \times \mathbf{B}$ is the Lorentz force vector calculated from ohm’s law, $\rho_{\text{hf}}$ is the hybrid nanofluid density, $\mu_{\text{hf}}$ is the hybrid nanofluid viscosity, $(\rho \mathcal{C}_p)_{\text{hf}}$ is the hybrid nanofluid heat capacity, $T$ is the fluids temperature and $\kappa_{\text{hf}}$ is the hybrid nanofluid thermal conductivity. $q_r$ is the nonlinear radiative heat flux\textsuperscript{32}, which may be derived using Rossland’s approximation and is given by:

\[ q_r = -\frac{4\sigma^*}{3k^*} \partial T^4 \partial r, \]

where $\sigma^*$ is the Stefan Boltzmann coefficient and $k^*$ is the mean absorption coefficient. Utilizing Taylor’s expansion of $T^4$ around $T_\infty$ which is the ambient temperature and ignoring higher order terms, we get,

\[ T^4 \approx 4T_\infty^3 T - 3T_\infty^4 \]

\[ q_r = -\frac{16\sigma^*}{3k^*} T_\infty^3 \partial T \partial r, \]

$q'''$ is the non-uniform heat flux and is defined as\textsuperscript{27}:

\[ q''' = \frac{\kappa_{\text{hf}} U_w}{x \nu_{\text{hf}}} \left[ X^*(T_w - T_\infty)f' + Y^*(T - T_\infty) \right] \]

Here, $X^*$ and $Y^*$ are space dependent and time dependent heat source and heat sink parameters. $X^* > 0$, $Y^* > 0$ symbolizes heat source and $X^* < 0$, $Y^* < 0$ symbolizes heat sink. Under the given assumptions, conservation laws of mass, momentum and energy Eqs. (1)–(3), in the presence of thermal radiation (4) and non-uniform heat generation/absorption (5) and using boundary layer approximations takes the following form\textsuperscript{36,37}:

\[ \frac{\partial}{\partial x} (ru) + \frac{\partial}{\partial r} (rv) = 0, \]
\[
\rho_{hf}\left(\frac{\partial u}{\partial x} + \nu \frac{\partial u}{\partial r}\right) = \mu_{hf}\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r}\right) - \sigma_{hf} B^2 u_x, \tag{7}
\]

\[
(\rho c_p)_{hf}\left(\frac{\partial T}{\partial x} + \nu \frac{\partial T}{\partial r}\right) = \kappa_{hf}\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r}\right) + \frac{16 \sigma^* T_s^3}{3k_s} \frac{\partial^2 T}{\partial r^2} + \frac{\kappa_{hf} U_w}{x \nu_{hf}} \left[X'(T_w - T_\infty) + Y'(T - T_\infty)\right]. \tag{8}
\]

The suitable boundary conditions affiliated with considered flow are:

\[
u = U_w + \mu_{hf} \frac{\delta u}{\partial r}, \eta = 0, T = T_w \text{ at } r = b, \quad \nu \to 0, T \to T_\infty \text{ as } r \to \infty. \tag{9}
\]

Let us specify dimensionless parameters as\[36,37]:

\[
\eta = \frac{r^2 - b^2}{2b} \left(\frac{U_w}{\nu_{hf}}\right)^{1/2}, \quad \psi = b \left(\frac{\nu_{hf}}{\nu_{hf}}\right)^{1/2} f(\eta), \quad \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}. \tag{10}
\]

Equation (6) is instinctively fulfilled with stream function \(\psi\) as

\[
u = \frac{1}{\eta} \frac{\partial \psi}{\partial r} \text{ and } \nu = -\frac{1}{\eta} \frac{\partial \psi}{\partial x}. \tag{11}
\]

Substituting Eq. (10) in Eqs. (7)–(9) we gain the resulting dimensionless form of equations:

\[
(1 + 2\nu_{hf}) f'''' + 2\nu_{hf} f''' - (1 - \phi_1)^{2.5}(1 - \phi_2) f'' + 2.5 \left(1 - \phi_2\right) \left(\phi_1 \left(\frac{\rho_{hf}}{\rho_f}\right) + (1 - \phi_1) \right) + \phi_2 \left(\frac{\rho_{hf}}{\rho_f}\right) \left(f''' - f''''\right) - (1 - \phi_1)^{2.5}(1 - \phi_2) f''
\]

\[
\left(\frac{\rho_{hf}}{\sigma_f}\right) \left[M_f' + \frac{\kappa_{hf} R_d}{\kappa_f} \right] (1 + 2\nu_{hf}) f'''' + 2\nu_{hf} f''' - \left(1 - \phi_1\right)^2 \left(1 - \phi_2\right) f'' + 2.5 \left(1 - \phi_2\right) \left(\phi_1 \left(\frac{\rho_{hf}}{\rho_f}\right) + (1 - \phi_1) \right) + \phi_2 \left(\frac{\rho_{hf}}{\rho_f}\right) \left(f''' - f''''\right) = 0. \tag{12}
\]

\[
f = 0, f' = 1 + \frac{\beta}{(1 - \phi_1)^2 (1 - \phi_2)^2} f'''' + \theta = 1 \text{ at } \eta = 0, \quad f' \to 0, \theta \to 0 \text{ as } \eta \to \infty. \tag{13}
\]

The non-dimensional quantities existing in the Eqs. (11)–(13) are magnetic term \(M\), Prandtl number \(Pr\), curvature \(\gamma\), radiation \(Rd\) and slip parameter \(\beta\) defined as:

\[
M = \frac{\sigma_{hf} B^2 l}{\rho_f U_0}, \quad \beta = \frac{\delta \mu_{hf}}{\rho_f U_0} \left(\frac{U_0}{\nu_{hf}}\right)^{1/2}, \quad Rd = \frac{4 \sigma^* T_s^3}{k_s \nu_{hf}}, \quad \gamma = \frac{1}{U_0 b^2}, \quad Pr = \frac{\mu_{hf} (\rho c_p)_{hf}}{\rho_f \nu_{hf}}. \tag{14}
\]

The skin friction coefficient and the Nusselt number are two important physical quantities, which are defined as follows:

\[
C_f = \frac{\mu_{hf}}{\sigma_f \rho_f U_0} \left(\frac{\partial u}{\partial r}\right)_{r=b}, \quad Nu_x = -\frac{\kappa_{hf} x}{k_f (T_w - T_\infty)} \left(\frac{\partial T}{\partial r}\right)_{r=b}. \tag{15}
\]

Using the similarity variables Eq. (10) and stream function definitions into Eq. (15), we derive the following non-dimensional form of skin friction coefficient and Nusselt number:

\[
Re_x^{1/2} C_f = \frac{\mu_{hf}}{\mu_f} f''(0), \quad Re_x^{-1/2} Nu_x = -\frac{\kappa_{hf}}{k_f} \theta'(0), \tag{16}
\]

where \(Re_x = U_w x / \nu_{hf}\) is the local Reynolds number.

**Thermophysical properties**

Table 1 displays the experimental relationships of hybrid nanofluid based on different thermal properties while Table 2 indicates the computational values for thermo-physical characteristics of nanoparticles and fluid yield to find the computational values of hybrid nanofluid properties.
Adams–Bashforth technique we established first order system of equations along with boundary conditions (13) quantity. Secondly, we use Adams–Moulton method as corrector step which refines the initial approximation. In first we use Adams–Bashforth method as prediction step which calculates a rough approximation of the desired rather than discarding it and thus refer to several previous points and derivative values. It works in two steps, Multistep methods try to enhance the efficiency preserving and utilizing the information from prior phases the numerical approach Adams–Bashforth Predictor–Corrector method, which is a linear multistep method. System of non-dimensional equations subject to the given boundary conditions. For this purpose we have utilized the numerical strategy which is utilized for determining the solution of our physical properties H2O Fe2O3 (ϕ1) CuO (ϕ2).

### Expressions for Hybrid nanofluid

| Properties | Expressions for Hybrid nanofluid |
|------------|---------------------------------|
| Density | \( \frac{\rho_{nf}}{\rho} = (1 - \phi_2) \left[ \phi_1 \left( \frac{\rho_1}{\rho} \right) + (1 - \phi_1) \right] + \phi_2 \left( \frac{\rho_2}{\rho} \right) \) |
| Viscosity | \( \frac{\mu_{nf}}{\mu} = \frac{1}{(1 - \phi_2)^{1.5}} \left[ \phi_1 \left( \frac{\mu_1}{\mu} \right)^{1.5} + (1 - \phi_1) \right] + \phi_2 \left( \frac{\mu_2}{\mu} \right)^{1.5} \) |
| Heat capacity | \( \frac{(v_P)_{nf}}{(v_P)_{P}} = (1 - \phi_2) \left[ \phi_1 \left( \frac{(v_P)_{1}}{(v_P)_{P}} \right) + (1 - \phi_1) \right] + \phi_2 \left( \frac{(v_P)_{2}}{(v_P)_{P}} \right) \) |
| Thermal conductivity | \( \frac{\kappa_{nf}}{\kappa} = \frac{\kappa_1 + (n - 1)\kappa_2 - (n - 1)\phi_2 (\kappa_2 - \kappa_1)}{\kappa_1 + (n - 1)\kappa_2 - (n - 1)\phi_2 (\kappa_2 - \kappa_1)} \) |
| Electric conductivity | \( \frac{\sigma_{nf}}{\sigma} = \frac{\sigma_1 + 2\sigma_2 - 2\phi_2 (\sigma_2 - \sigma_1)}{\sigma_1 + 2\sigma_2 - 2\phi_2 (\sigma_2 - \sigma_1)} \) |

Table 1. | Empirical relations for thermophysical characteristics of hybrid nanofluid.

### Physical properties

| Physical properties | H2O | Fe2O3 (ϕ1) | CuO (ϕ2) |
|---------------------|-----|------------|-----------|
| \( \rho \) (kg m\(^{-3}\)) | 997.1 | 3970 | 6500 |
| \( \sigma_2 \) (J kg\(^{-1}\) K\(^{-1}\)) | 4179 | 765 | 531.8 |
| \( \kappa \) (W m\(^{-1}\) K\(^{-1}\)) | 0.613 | 40 | 0.85 |
| \( \sigma \) (Ω m\(^{-1}\)) | 0.05 | \( 35 \times 10^6 \) | \( 59.6 \times 10^6 \) |

Table 2. | Computational values for thermophysical characteristics of base fluid and nanoparticles.

### Numerical procedure

In this portion we have presented the numerical strategy which is utilized for determining the solution of our system of non-dimensional equations subject to the given boundary conditions. For this purpose we have utilized the numerical approach Adams–Bashforth Predictor–Corrector method, which is a linear multistep method. Multistep methods try to enhance the efficiency preserving and utilizing the information from prior phases rather than discarding it and thus refer to several previous points and derivative values. It works in two steps, first we use Adams–Bashforth method as prediction step which calculates a rough approximation of the desired quantity. Secondly, we use Adams–Moulton method as corrector step which refines the initial approximation. In Adams–Bashforth technique we established first order system of equations along with boundary conditions (13) in terms of \( f(\eta) \) and \( \theta(\eta) \) from dimensionless Eqs. (11)–(12). First order system for \( f(\eta) \) is framed as:

\[
\begin{align*}
    f_1 &= f' \ , f_2 = f'_1 \ , f_3 = f'_2 \\
    f_5 &= f'_2 = -\frac{1}{(1 + 2\gamma \eta)} \left[ A_1 A_2 (f_1^2 - f_2) + A_1 A_3 M f_1 + 2\gamma f_2 \right]
\end{align*}
\]

and temperature equation \( \theta(\eta) \) as:

\[
\begin{align*}
    \theta_1 &= \theta, \ \theta_2 = \theta' \\
    \theta_2 &= \theta'_1 = -\frac{1}{(A_5 + Rd)(1 + 2\gamma \eta)} \left[ \left( 2A_5 + Rd \right) \gamma + A_4 Pr f' \right] \theta_1 + A_1 A_2 a_5 \left( X^* f' + Y^* \theta \right)
\end{align*}
\]

The suitable boundary conditions for \( f(\eta) \) and \( \theta(\eta) \) are

\[
\begin{align*}
    f(0) &= 0, \ f_1(0) = 1 + \frac{B}{A_1} f_2(0), \\
    f_1(1) &= 0, \ \theta(0) = 1, \ \theta(1) = 0.
\end{align*}
\]

where
\[ A_1 = (1 - \phi_1)^{2.5} (1 - \phi_2)^{2.5}, \]
\[ A_2 = \left[ (1 - \phi_2) \left( \phi_1 \left( \frac{\rho_1}{\rho_f} \right) + (1 - \phi_1) \right) + \phi_2 \left( \frac{\rho_2}{\rho_f} \right) \right], \]
\[ A_3 = \frac{\sigma_{\text{nf}}}{\sigma_f}, \]
\[ A_4 = \left[ (1 - \phi_2) \left( \phi_1 \left( \frac{\rho c_p}{\rho c_p} \right) + (1 - \phi_1) \right) + \phi_2 \left( \frac{\rho c_p}{\rho c_p} \right) \right], \]
\[ A_5 = \frac{k_{\text{nf}}}{k_f}. \]

Acquired differential system for \( f(\eta) \) and \( \theta(\eta) \) are generally represented as follows:

\[ \begin{aligned}
\frac{df}{d\eta} &= q(\eta, f), \quad f(\eta_0) = f_0 \\
\frac{d\theta}{d\eta} &= q(\eta, \theta), \quad \theta(\eta_0) = \theta_0.
\end{aligned} \]

The general expression for two step Adams–Bashforth approach for \( f(\eta) \) and \( \theta(\eta) \) are given respectively as follows

\[ \begin{aligned}
f_{k+1} &= f_k + \frac{h}{2} \left[ 3q(\eta_k, f_k) - q(\eta_{k-1}, f_{k-1}) \right] \\
\theta_{k+1} &= \theta_k + \frac{h}{2} \left[ 3q(\eta_k, \theta_k) - q(\eta_{k-1}, \theta_{k-1}) \right],
\end{aligned} \]

where \( h \) is a step size parameter. The flow chart of the scheme is given below in Fig. 2:

**Graphical analysis**

In this portion, we have presented the impact of distinct quantities on the flow profiles. Equations (11)–(13) have been determined by utilizing numerical technique. Physical changes in velocity and temperature fields against distinct parameters are drafted in Figs. 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14. These distinct parameters are radiation, magnetic, slip and heat generation/absorption parameters along with Prandtl number \( \text{Pr} \). The consequences of magnetic effects on velocity field have been expressed in Fig. 3. It is shown that velocity field decays with increase in magnetic, slip and heat generation/absorption parameters along with Prandtl number as shown in Fig. 4. Due to influence of slip, fluid velocity nearby the stretchable surface is no more equivalent to velocity of the fluid. Increment in \( \text{Pr} \) increases the rate of heat transfer from cylinder to the surface, thus temperature of the fluid drops near the surface and elevates distance from the cylinder surface. Physically, the Lorentz force act as a decelerating agent which reduces the fluid speed and momentum boundary layer thickness. Consequently higher values of \( \text{Pr} \) strengthens away from the surface. Figure 10 demonstrates the consequence of \( \eta \) on temperature field. Conduction is further dominated close to surface, so near the wall thickness of thermal boundary layer and temperature decreases and increases away from the cylinder surface. The reason for this behavior is that, increasing velocity field decays with increase in \( \eta \) from cylinder to the surface, thus temperature of the fluid drops near the surface and strengthens away from the surface. Figure 10 demonstrates the consequence of \( \eta \) on temperature field. It decays considering substantial values of \( \text{Pr} \), because \( \text{Pr} \) is fraction of mass diffusivity to thermal diffusivity therefore, increase in \( \text{Pr} \) slows down the heat diffusion rate which causes both temperature field and boundary layer thickness decays.
Figure 2. Flow chart of numerical scheme.

Figure 3. Magnetic effects on velocity field.
Figures 11, 12, 13, 14 show the outcomes of space dependent parameter \(X^*\) and time dependent parameter \(Y^*\) for internal heat generation (positive values) and internal heat absorption (negative values) on temperature profile. The existence of heat generation (space and time dependent heat greater than zero) raises the temperature of the fluid by adding more heat to the system and decreases the thickness of thermal boundary layer. Also, for heat absorption (space and time dependent heat less than zero) take in heat from thermal boundary layer leading to drop the temperature profile.
Tables 3 and 4 shows the numerical values of the skin friction coefficient and heat transfer rate for different values of the involved physical parameters. It is noted that skin friction coefficient increases by increasing the Hartmann number, while slip parameter decreases the shear stress on the surface. Also, the skin friction coefficient is inversely related to curvature of the stretching cylinder. Radiation and Prandlt number increases the rate of mass transfer on the surface. In Table 5 nomenclature is given.
Conclusion
In this paper, we have presented the investigations of MHD hybrid nanofluid over a stretching cylinder. We have incorporated the effects of thermal radiation and non-uniform heat flux with velocity slip condition. The mathematical modeling is carried out by using the continuity, momentum and energy equation. A set of suitable transformations and non-dimensional variables have been utilized to transform the governing partial differential
Figure 13. $X^* < 0$ on temperature field.

Figure 14. $Y^* < 0$ on temperature field.

Table 3. Numerical values of skin friction coefficient $-\frac{1}{\chi_1} f''(0)$.
equations into set of non-dimensional ordinary differential equations, which are then solved numerically. Plots of several physical quantities have been prepared to get the right insight of the considered problem. It is observed that, for realistic values of the controlled parameters, we observe that the velocity of the fluid decreases as the Hartmann number, the slip parameter, and nanoparticles volume fraction increases; it increases away from the surface with increasing the curvature parameter. The temperature of the fluid increases as radiation parameter, nanoparticles volume fraction, and non-uniform heat source/sink parameters increases. For increasing curvature parameter, it decreases near the surface and increases away from the surface. The Prandtl number also decreases the temperature of the fluid. We noted that hybrid nanomaterial work efficiently in processes involving high temperatures. These includes solar energy, refrigeration systems, air conditioning applications, heat exchanger, coolants in machining and auto motives, transformer cooling, nuclear system etc.

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| $M$ | $\beta$ | $\gamma$ | $Rd$ | Pr | $\phi_1, \phi_2$ | $-A_5\theta'(0)$ |
|-----|---------|---------|------|----|-----------------|-----------------|
| 4   | 0.4     | 0.2     | 0    | 6.2| 0.2             | 2.00336         |
|     |         |         | 0.3  |  | 2.83816         |
|     |         |         | 0.6  |  | 4.24755         |
|     |         |         | 0.9  |  | 7.5965          |
| 4   | 0.4     | 0.2     | 0.6  | 4  | 0.2             | 4.02301         |
|     |         |         | 5    |  | 4.11232         |
|     |         |         | 6    |  | 4.22268         |
|     |         |         | 7    |  | 4.3574          |
| 4   | 0.4     | 0.2     | 0.6  | 6.2| 0               | -5.49465        |
|     |         |         | 0.05 |  | -5.49465        |
|     |         |         | 0.1  |  | 2.3497          |
|     |         |         | 0.15 |  | 1.75107         |

Table 4. Numerical values of Nusselt number $-A_5\theta'(0)$.

| Nomenclature | l, v $x$ and $r$ components of velocity (m s$^{-1}$) |Greek symbol| Nanoparticles volume fraction $\phi_1, \phi_2$ | Fluid density (kg m$^{-3}$) $\rho$ | Stream function $\Psi$ | Dynamic viscosity (kg m$^{-1}$ s$^{-1}$) $\mu$ | Kinematic viscosity (m$^2$ s$^{-1}$) $\nu$ | Specific heat (J kg$^{-1}$ K) $c_p$ | Thermal conductivity (W m$^{-1}$ K$^{-1}$) $\kappa$ |
|--------------|---------------------------------|------------|---------------------------------|---------------------------------|---------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| $u, v$       | $T$ $\theta$ $T_w$ $B_0$ $R_d$ $U_w$ $\sigma^*$ $X^*$ | $\alpha$ $\tau$ $\nu$ $\phi$ $\chi$ $\beta$ $\gamma$ $\rho$ $\mu$ $\nu$ $c_p$ $\kappa$ |

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**Author contributions**

A.A. and T.C. wrote the main manuscript text and M.A. prepared all the figures. Z.S. and P.K. contributed in the numerical computations and editing the manuscript in the revision. P.T. worked on the revised manuscript and edit it grammatically. All authors finalized the manuscript after its internal evaluation.

**Competing interests**

The authors declare no competing interests.

**Additional information**

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