On the Complexity of Anonymous Communication Through Public Networks

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Abstract

Anonymous channels allow users to connect to websites or communicate with one another privately. Assume that either Alice or Allison is communicating with (a possibly corrupt) Bob. To protect the sender, we seek a protocol that provably guarantees that these two scenarios are indistinguishable to an adversary that can monitor the traffic on all channels of the network and control the internal operations in a constant fraction of the nodes.

Onion routing is the method of choice for achieving anonymous communication, used for example by Tor. In an onion routing protocol, messages travel through several intermediaries before arriving at their destinations; they are wrapped in layers of encryption (hence they are called “onions”). In this paper, we give the first rigorous characterization of the complexity of onion routing protocols for anonymous communication through public networks. We show that in order to provide anonymity, an onion routing scheme requires each participant to transmit, on average, a superlogarithmic number of onions. We match these negative results with a protocol in which every participant creates a polylogarithmic number of onions and participates in a polylogarithmic number of transmissions.

Keywords: Anonymity, privacy, onion routing.

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Anonymous channels allow users to connect to websites or communicate with one another privately. Without them, all of a user’s online activities may be traced or tracked. The question is: How can anonymous channels be implemented in a public network, such as the Internet? One requirement is that each user in the network must be able to send messages to any other user. Furthermore, an active adversary, one that monitors the network and controls some of its nodes, should not be able to tell who is communicating with whom.

An approach to anonymous channels that is widely used is onion routing [Cha81, DDM03, DMS04]. Suppose Alice wants to send a message to Bob through a communication network in which an active adversary controls a fraction of the nodes, possibly including Bob’s node, and monitors traffic on all channels of the network. Alice wishes to send her message without revealing to anyone that she is communicating with Bob. As a first step, Alice can protect the content of her message by encrypting the message so that only Bob can decrypt it. However, sending the encrypted message directly to Bob would allow the adversary to observe the flow of packets from Alice to Bob. Instead, Alice may designate several intermediate nodes, called mix-nodes, and send the ciphertext through them as follows. First, Alice encrypts her message $m$ under Bob’s public key. The resulting ciphertext $c$ is what she wishes Bob to receive. Now, suppose Alice randomly selects two intermediate nodes: Carol and David. Alice encrypts the instructions “send $c$ to Bob” under David’s public key, resulting in the ciphertext $c_D$. Next, she encrypts the instructions “send $c_D$ to David” under Carol’s public key, resulting in the ciphertext $c_C$. Alice’s message $m$ is wrapped in three layers of encryption, justifying the moniker onion. In our example, the onion is $c_C$, and each node is able to decrypt the appropriate layer until Bob receives the intended ciphertext $c$. Each mix-node, Carol or David, only knows the identities of adjacent nodes on the routing path, and the fact that the message Bob receives originated from Alice is concealed. Furthermore, since each encrypted onion that arrives at a node is computationally unrelated to its version that leaves that node, an adversary that monitors the traffic on the links cannot be certain about how to match onions entering a node to onions leaving it.

This simple protocol is at the heart of the most popular example of onion routing: Tor (The onion router) [DMS04]. Tor is a distributed network consisting of thousands of relay nodes and used by two million users per day to communicate anonymously. Tor boasts efficiency, fault tolerance (if an onion gets dropped somewhere, other onions still make it through), and scalability (the system continues to perform well even as more parties join). These practical benefits have so far outweighed concerns that onion routing has not been sufficiently analyzed and lacks provable security (in the setting where the adversary has full view of the network traffic). In fact, Tor is vulnerable to traffic correlation attacks [JWJ+13, SEF+17, WSJ+18], since an adversarial autonomous system (AS) is essentially an adversary with full view of the network traffic.

1. [https://metrics.torproject.org/userstats-relay-country.html](https://metrics.torproject.org/userstats-relay-country.html)
2. Tor’s security has been analyzed and even affirmed in weaker models where the adversary does not have full view of the network traffic, e.g., [BCKM12, BKM+13].
3. Johnson et al. [JWJ+13] showed that approximately 80% of users can be deanonymized by an adversarial relay node within six months, and almost all users can be deanonymized by AS within three months. This is detrimental when combined with the fact that one third of all Tor relay nodes are hosted by only six autonomous systems [SEF+17]. While the original attack by Øverlier and Syverson [OS06] assumed a static set up (without Internet churn), recent attacks (see Raptor attacks [SEF+17] and Tempest attacks [WSJ+18]) exploit the dynamic nature of the Internet to increase both the adversarial view as well as the adversary’s deanonymizing capabilities. Using such attacks, Sun et al. demonstrated that approximately 30% of Tor circuits can be deanonymized by an AS within three weeks [SEF+17].
Our goal. Suppose Alice is instructed to send a message to Betty instead of Bob and runs a network protocol. We want the adversary’s view in that scenario to be statistically indistinguishable\(^4\) from the one in which she sends her message to Bob. We refer to protocols that achieve this level of protection of Alice’s traffic as “anonymous protocols” (Definition 1). Up until now, it was not known whether practical, fault tolerant, and anonymous protocols for the active adversarial setting were even possible. The protocols constructed to date guarantee only differential privacy (a much weaker notion compared with anonymity) [vdHLZZ15, TGL+17, ALU18] or are not fault tolerant [TGL+17, KCDFT17] or are prohibitively inefficient (e.g., every participant sends an encrypted message to every other participant).

Our goal is to construct an onion routing protocol that is simultaneously anonymous, fault tolerant, and efficient as measured in the average number of onions handled (sent/received) by a party.

1.1 Our contributions

Following earlier examples of anonymity protocols [vdHLZZ15, TGL+17, KCDFT17, ALU18], our results are for the setting where everyone sends a message and is expecting to receive a message. We call this setting the “simple I/O setting.”

We present the first onion routing protocol that is simultaneously anonymous, fault tolerant, and efficient in the simple I/O setting. Using this construction, we prove the first nearly tight complexity bounds for anonymous onion routing protocols in the active adversary setting.

Suppose Alice sends a message to Bob through several intermediary nodes. A malicious adversary Eve, controlling the first intermediary, could target and drop Alice’s outgoing onion. In this case, the adversary might learn for whom Alice’s onion was intended, for example, by observing that Bob does not receive an onion but everyone else does. To subvert this attack, Alice may want to send many redundant onions (each carrying the same message to Bob) through many paths, hoping that many of them still start with honest nodes. In this way, it is more challenging for the adversary to target and drop Alice’s onions without also dropping many onions from other parties. However, even if Eve is unsuccessful in dropping all of Alice’s onions, she may be successful in dropping more of Alice’s onions than any other party’s. This may still be problematic; if Bob receives fewer onions than everyone else, Eve may deduce that he is Alice’s interlocutor. Thus to achieve privacy, an onion routing protocol must equalize the number of the remaining onions formed by different senders. In Section 5, we present the first fault tolerant onion routing protocol, \(\Pi_\triangle\) (pronounced “pi-tree” because of its binary tree structure), that achieves this.

In Section 6, we extend \(\Pi_\triangle\) to a more efficient onion routing protocol, \(\Pi_\infty\) (pronounced “pi-butterfly” because of its butterfly network structure). Using \(\Pi_\infty\), we prove our first bound:

**Theorem 1.** Let \(\lambda\) be the security parameter, and let \(N\) be the number of participants. For all constants \(0 \leq \kappa < 1\) and \(\gamma > 0\), there exists an onion routing protocol that, in the presence of the active adversary who corrupts up to \(\kappa\) fraction of the parties, is anonymous and fault tolerant requiring \(\log N \log^{2+\gamma} \lambda\) onions to be transmitted per user. (Increasing \(\gamma\) increases the rate at which the maximum statistical distance in the adversarial views between any two input settings shrinks. Increasing \(\kappa\) decreases this rate.)

We match our construction with a lower bound.

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\(^4\)Technically, since onion routing uses encryption, the adversary’s view cannot be statistically independent of the input, only computationally independent. However, as we will see, if we work in an idealized encryption model, such as Canetti’s \(\mathcal{F}_{\text{Enc}}\)-hybrid model [Can01], anonymity makes sense.
Informal Theorem 2. Let $\lambda$ be the security parameter, and let $N = \text{poly}(\lambda)$ be the number of participants. To be both anonymous and fault tolerant in the active adversary setting, there exists a sender who sends an expected $\omega(\log \lambda)$ onions.

1.2 Technical challenges

To be anonymous, an onion routing protocol must be secure in a setting where the adversary can view all packets traveling on all links. If all packets are fixed-length onions, the adversary’s view on the links is abstractly a sequence of sets of identifiers (e.g., numbers), where each set of identifiers represents the onions on a link and time step.

When an onion is received by an honest node, the adversary is not privy to how the onion evolves into another onion (when it is peeled). The adversary cannot link incoming onions at an honest node to outgoing onions. This mixing process creates uncertainty in the onions’ routing paths and is how onion routing can be used to anonymize network traffic. In contrast, the adversary knows how to map incoming onions to outgoing onions at corrupted nodes. No mixing occurs at corrupted nodes.

An active adversary can control the corrupted nodes to deviate from the protocol. It can essentially create, modify, or (possibly indefinitely) delay onions at corrupted nodes. A protocol can protect against onions created or modified by the adversary by using cryptographic tools, but protecting against dropped onions is more challenging. By changing the volume flow of onions, an active adversary may cause some information about the communication pattern to leak even in a protocol that is secure in the passive setting, where the corrupted nodes also follow the protocol.

For example, Ando et al. [ALU18] recently showed that a simple onion routing protocol can be anonymous from the passive adversary when the average load (per mix-node per round) and the number of rounds are both polylogarithmic in the security parameter. This protocol, however, is not secure when the adversary is active. With reasonable probability, the adversary can target Alice and drop her onion upfront and learn for whom Alice’s message was intended by observing who doesn’t receive a message. (This insight is the crux of our proof for our lower bound.)

Thus by mixing, we mean a much stronger notion than the one typically used to describe shuffling algorithms (e.g., [OGTU14, CV14, Czu15]). For an onion routing protocol to be secure against an active adversary, we require onions to be mixed at honest nodes enough times so that the adversary cannot map the onions at the end of the protocol run to their counterparts at the start of the run, given a view into all links and some of the nodes. This must hold even when some of the nodes deviate from the protocol. (We provide a formal definition of mixing in Section 3.3.)

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5 In the literature, it is often assumed that the adversary can access network traffic on all links, but that message packets are dropped only at corrupted nodes. To send a message packet “directly from Alice to Carol,” in actuality, the packet is routed through many physical nodes in the network; thus, it is often times unwise to assume that the message is not being logged somewhere on route. On the other hand, standard network protocols (i.e., TCP/IP) gives us some assurance that message packets are not dropped on the links.

6 For example, Czumaj [CV14, Czu15] showed that repeating the Thorp shuffle [Tho73] polylogarithmic (in the security parameter) times on a sequence of elements produces a distribution that is statistically close to the uniform distribution over the set of all possible permutations. In other words, the statistical distance (i.e., total variation distance) to the uniform distribution decays faster than any inverse polynomial in the security parameter. However, this result is for the unconditional distribution that does not depend on any adversarial observation (e.g., a view into all links and some nodes). Note that the butterfly topology is used in the Atom construction [KCDF17], but the authors rely on expensive verifiable shuffling steps to justify mixing using Czumaj’s result. Another example is Ohrimenko et al.’s Melbourne shuffle [OGTU14], an oblivious permutation algorithm which produces a distribution statistically close to the uniform distribution also in a polylogarithmic (in the security parameter) number of steps. However, in this case, mixing is defined with respect to a distribution conditioned on the adversary’s view into all the links but none of the nodes. Both the Thorp shuffle and the Melbourne shuffle are only proven to properly mix when each node behaves honestly and executes the algorithm correctly.
Challenge. Efficiently mix in the active adversary setting.

As illustrated by the attack above, an adversary can learn the recipient of a targeted sender by blocking all of her traffic. Preventing this kind of attack is easier when each party floods the network with redundant messages.

Challenge. Minimize the number of redundant onions used to guarantee that every honest party’s message is equally likely to be delivered.

1.3 Overview of our construction

Suppose that Alice creates many onions and sends them to Bob through different routing paths. The adversary may drop some of them. Thus, guaranteeing anonymity boils down to satisfying two conditions: (1) The adversary cannot drop all of Alice’s onions without also dropping so many other onions that each party will detect that an attack is underway. (2) In the end, Bob receives just as many onions as anyone else, even if the adversary targets Alice and drops as many of her onions as possible.

To satisfy (1), each party forms many onions. That way, some of Alice’s onions will be initially routed to honest intermediaries who won’t drop them. Still, the adversary can keep track of where these onions might end up and attempt to drop them as soon as she is given a chance. We cannot prevent an active adversary from dropping all the onions entrusted to an adversarial intermediary, but we can still give an all-or-nothing-or-random guarantee: Either all the messages are delivered, or all the messages get dropped, or messages are dropped at random. The probability that a message is delivered is a function only of how many total onions in the system the adversary has dropped and is independent of a message’s origin. This is accomplished by introducing “checkpoint dummy onions” whose purpose is to enable honest parties to check whether or not there is adversarial activity. The additional “message-bearing onions” are the ones that carry messages from senders to recipients.

To satisfy (2), we introduce a new technique of “merging onions.” Each party generates the routing paths of the message-bearing onions using a binary tree structure. Each path from a leaf node to the root node corresponds to a routing path of a message-bearing onion. If released simultaneously, these onions initially move independently from one another and then gradually come to coordinate their movements. When two onions become coordinated, the onions are merged into a single onion. (In actuality, merging is accomplished by simply dropping one of the two onions before the next round.) This way, even if the adversary drops a significant portion of Alice’s onions upfront, her recipient, Bob, receives (statistically) the same number of onions as everyone else. (See Section 5.)

1.4 Related work

The only prior negative result for onion routing is the Trilemma theorem due to Das et al. [DMMK18] presented at IEEE S&P 2018. This result differs from our lower bound, firstly, because it is for the passive adversary setting, whereas ours is for the active adversary setting, and, secondly, because the bound states that \(\omega(N)\) communication complexity is required, whereas our lower bound states that \(\omega(N \log \lambda)\) communication complexity is required, where \(\lambda\) is the security parameter.

Recently, Ando et al. [ALU18] gave an efficient onion routing protocol that satisfies anonymity against the passive adversary and one that satisfies differential privacy against the active adversary.

\footnote{For this reason, we assume that every party (in the so-called anonymity set) participates in the protocol and that the communication is synchronous.}
The work of van den Hooff et al. [vdHLZZ15] achieves differential privacy but assumes that all messages travel through the same set of dedicated servers and is, therefore, impractical. Other recently proposed systems, Stadium [TGL+17] and Atom [KCDF17], have good scalability (comparable to onion routing) but lack proof of security; they are also not robust. (A variant of Atom is robust but clearly not anonymous.)

Achieving anonymous channels using heavier cryptographic machinery has been considered also. One of the earliest examples is Chaum’s dining cryptographer’s protocol [Cha88]. Rackoff and Simon [RS93] use secure multi-party computation for providing security from active adversaries. Other cryptographic tools used in constructing anonymity protocols include oblivious RAM (ORAM) and private information retrieval (PIR) [CB95, CBM15]. Corrigian-Gibbs et al.’s Riposte solution makes use of a global bulletin board and has a latency of a couple of days [CBM15]. The aforementioned Stadium [TGL+17] is another solution for a public forum. Blaze et al. [BIK+09] provided an anonymity protocol in the wireless (rather than point-to-point) setting.

Encryption schemes that are appropriate for onion routing are known [CL05, BGKM12]. Several papers attempted to define anonymity for communications protocols and to analyze Tor [SW06, EJS07, EJS12]. Backes et al. [BKM+13] were the first to consider a notion inspired by differential privacy [DMNS06] but, in analyzing Tor, they assume an adversary with only a partial view of the network. In contrast to our definition of anonymity, Backes et al.’s “sender anonymity” [BKM+13] is defined with respect to an indistinguishability game where the adversary chooses two challenger senders and a message (as opposed to specifying entire input vectors). There are also some studies on anonymity protocols, other than onion routing protocols, that were analyzed using information-theoretic measures [BFTS04, KB07, CPP08, DRS04, AAC+11].

2 Preliminaries

For a set \( S \), we denote the cardinality of \( S \) by \(|S|\), and \( s \leftarrow S \) is a sample from \( S \) chosen uniformly at random. For an algorithm \( A(x) \), \( y \leftarrow A(x) \) is the (possibly probabilistic) output \( y \) from running \( A \) on the input \( x \). In this paper, \( \log(x) \) is the logarithm of \( x \) base 2.

We say that a function \( f : \mathbb{N} \rightarrow \mathbb{R} \) is negligible in the parameter \( \lambda \), written \( f(\lambda) = \text{negl}(\lambda) \), if for a sufficiently large \( \lambda \), \( f(\lambda) \) decays faster than any inverse polynomial in \( \lambda \). When \( \lambda \) is the security parameter, an event \( E_\lambda \) is said to occur with (non-)negligible probability if the probability of \( E_\lambda \) can (not) be bounded above by a function negligible in \( \lambda \). An event occurs with overwhelming probability if its complement occurs with negligible probability. Two families \( \{D_{0,\lambda}\}_{\lambda \in \mathbb{N}} \) and \( \{D_{1,\lambda}\}_{\lambda \in \mathbb{N}} \) of probability distributions are said to be statistically close (also, statistically indistinguishable) if the total variation distance (a.k.a., statistical distance) \( \Delta(D_{0,\lambda}, D_{1,\lambda}) \) between \( D_{0,\lambda} \) and \( D_{1,\lambda} \) is negligible in \( \lambda \). When the security parameter is clear by context, we abbreviate this notion by \( D_0 \approx_{\lambda} D_1 \). We use the standard notion of a pseudorandom function [GoJ01, Ch. 3.6].

2.1 Onion encryption schemes

Following Camenisch and Lysyanskaya’s work [CL05] on cryptographic onions, an onion encryption scheme is a triple of algorithms: \( (\text{Gen}, \text{FormOnion}, \text{ProcOnion}) \). The algorithm \( \text{Gen} \) generates a public-key infrastructure for a set of parties. The algorithm \( \text{FormOnion} \) forms onions; and the algorithm \( \text{ProcOnion} \) processes onions.

Let \( [N] \) be a set of participants. For every \( i \in [N] \), let \((\text{pk}_i, \text{sk}_i) \leftarrow \text{Gen}(1^\lambda) \) be the key pair generated for party \( i \in [N] \), where \( \lambda \) is the security parameter.

Let \( M \) be the message space consisting of messages of the same fixed length, and let the nonce space \( \text{NonceSpace} \) consist of nonces of the same fixed length. \( \text{FormOnion} \) takes as input a message \( m \)
from \(\mathcal{M}\), an ordered list \((p_1, p_2, \ldots, p_{R+1})\) of parties from \([N]\), the public keys \((pk_{p_1}, pk_{p_2}, \ldots, pk_{p_{R+1}})\) associated with these parties, and a list \((s_1, s_2, \ldots, s_R)\) of (possibly empty) strings that are nonces from NonceSpace associated with layers of the onion. The party \(p_{R+1}\) is interpreted as the recipient of the message, and the list \((p_1, p_2, \ldots, p_{R+1})\) is the routing path of the message. The output of FormOnion is a sequence \((o_1, o_2, \ldots, o_{R+1})\) of onions. Such a sequence is referred to as an evolution, but every \(o_i\) in the sequence is an onion. Because it is convenient to think of an onion as a layered encryption object, where processing an onion \(o_{r+1}\), we sometimes refer to the process of revealing the next onion as decrypting the onion, or peeling the onion. For every \(r \in [R]\), only party \(p_r\) can peel onion \(o_r\) to reveal the next layer, \((p_{r+1}, o_{r+1}, s_{r+1}) \leftarrow \text{ProcOnion}(sk_{p_r}, o_r, p_r)\), which contains the peeled onion \(o_{r+1}\), the next destination \(p_{r+1}\), and the nonce \(s_{r+1}\). Only the recipient \(p_{R+1}\) can peel the innermost onion \(o_{R+1}\) to reveal the message, \(m \leftarrow \text{ProcOnion}(sk_{p_{R+1}}, o_{R+1}, p_{R+1})\).

In our constructions, \(\Pi_\Delta\) and \(\Pi_\infty\), a sender of a message \(m\) to a recipient \(j\) forms an onion by generating nonces and running the FormOnion algorithm on the message \(m\), a routing path \((p_1, p_2, \ldots, p_R, j)\), the public keys \((pk_{p_1}, pk_{p_2}, \ldots, pk_{p_R}, pk_j)\) associated with the parties on the routing path, and the generated nonces; the formed onion is the first onion \(o_1\) from the list of outputted onions. The sender (i.e., the party who formed the onion) sends \(o_1\) to the first party, \(p_1\), on the routing path, who processes \(o_1\) and sends the peeled onion \(o_2\) to the next destination, \(p_2\), and so on, until the last onion \(o_{R+1}\) is received by the recipient \(j\), who processes it to obtain the message \(m\).

Let \(o^0\) and \(o^1\) be any two onions that only (honest) party \(i\) can process, such that one of the onions was formed by an honest party using the message-recipient pair \((m^0, \text{recipient}^0)\), and the other was formed by an honest party using the pair \((m^1, \text{recipient}^1)\). Importantly, the adversary cannot tell which message-recipient pair produced which onion. See Camenisch and Lysyanskaya’s paper [CL05] for formal definitions.

For our result in Section 4 we use a definition for an onion routing scheme that captures a slightly broader class of algorithms than this standard notion; we allow a party to build upon any cryptographic onion it receives. (See Appendix A.)

3 Definitions

We consider a setting where \(N\) parties, labeled \(1, 2, \ldots, N\), participate in an onion routing (OR) protocol. We assume that the protocol progresses in global rounds and that an onion sent at round \(r\) is received instantaneously at the same round \(r\) to be processed in a subsequent round \(r' > r\). We assume that the number \(N\) of participants and every other quantity in the protocol is polynomially bounded in the security parameter \(\lambda\). For our analysis, effectively working in Canetti’s \(\mathcal{F}_\text{Enc}\)-hybrid model [Can01], we assume that each party has associated with it a public key generated using an ideal OR scheme; it is well understood how to relate this to security in the standard model.

In this paper, we analyze single executions (or runs) of OR protocols.

**Adversary model.** The results in this paper are for the setting where the adversary is active: Unless stated otherwise, we consider the active adversary who can observe the traffic on all links and, additionally, can non-adaptively corrupt and control a constant fraction of the parties. (By non-adaptively corrupt, we mean that the corruptions are made independently of any execution.) Once the adversary corrupts a party, she can observe the internal state and computations of the corrupted party and arbitrarily alter the behavior of the party.

We denote the set of all such active adversaries as \(\text{ActiveAdv}\). When a predicate is true for every adversary in \(\text{ActiveAdv}\), we simply say that it holds for the active adversary.
In general, when talking about multiple statistical quantities in a run of protocol states same setting (i.e., $V$ Definition 1 in forming onions, such an encryption scheme gives rise to onions that are information theoretically the length of the plaintext. This allows our analysis to rely solely on probability theory. When used are information theoretically unrelated to the plaintexts that they encrypt and reveal nothing but For our analysis, we assume an idealized version of an encryption scheme, in which the ciphertexts are statistically indistinguishable, i.e., $\approx$ for the corrupted parties, denoted $\sigma^0 \equiv_{\text{Bad}} \sigma^1$, if they belong to the same equivalence class imposed by $\text{Bad}$.

### 3.1 Anonymity

For our analysis, we assume an idealized version of an encryption scheme, in which the ciphertexts are information theoretically unrelated to the plaintexts that they encrypt and reveal nothing but the length of the plaintext. This allows our analysis to rely solely on probability theory. When used in forming onions, such an encryption scheme gives rise to onions that are information theoretically independent of their contents. The standard and natural notion of security under this model is,

**Definition 1 (Anonymity [ALU18 Def. 1]).** A communications protocol $\Pi(1^\lambda,$ $\text{pp, states, } \$, $\sigma)$ is anonymous from the adversary $A$ who corrupts up to a constant $0 \leq \kappa < 1$ fraction of the parties, for input vectors from the set $\Sigma$ if for any choice $\text{Bad}$ for corrupted parties such that $|\text{Bad}| \leq \kappa N$ and for all $\sigma^0, \sigma^1 \in \Sigma$ such that $\sigma^0 \equiv_{\text{Bad}} \sigma^1$, the adversary’s views $\mathcal{V}^{\Pi, A, \text{Bad}}(\sigma^0)$ and $\mathcal{V}^{\Pi, A, \text{Bad}}(\sigma^1)$ are statistically indistinguishable, i.e., $\mathcal{V}^{\Pi, A, \text{Bad}}(\sigma^0) \approx_s \mathcal{V}^{\Pi, A, \text{Bad}}(\sigma^1)$.

$\Pi$ is perfectly secure from $A$ if $\mathcal{V}^{\Pi, A, \text{Bad}}(\sigma^0) = \mathcal{V}^{\Pi, A, \text{Bad}}(\sigma^1)$ instead.

Note that in the case where the message content at corrupted parties essentially leaks the input vector, there is nothing that the channel can do to seal this leakage.
Simple I/O setting. A trusted third party can act as an ideal anonymous channel by mixing everyone’s messages and sending them out in random order. Unfortunately, not even a trusted third party can hide who is talking to whom if the adversary knows that Alice is sending a large file (e.g., a movie) to someone, and Bob isn’t receiving enough bits to suggest that he is the recipient of a large file. In this case, the adversary knows that Alice is not talking to Bob.

To efficiently statistically hide the communication pattern, let us consider the scenario where everyone sends exactly one message of the same length, and everyone receives exactly one message. Let SimpleIO be the set of all inputs of the form,

\[
\sigma = (\{(m_1, \pi(1))\}, \{(m_2, \pi(2))\}, \ldots, \{(m_N, \pi(N))\}),
\]

where \(m_1, m_2, \ldots, m_N \in M\) are messages from the message space \(M\), and \(\pi : [N] \mapsto [N]\) is a permutation function over the domain \([N]\). We refer to the setting where the input vector \(\sigma\) is constrained to SimpleIO as the simple I/O setting, and it is assumed that we are operating in this setting. For all definitions, when the input set \(\Sigma\) is left unspecified, it is understood that we mean \(\Sigma = \text{SimpleIO}\).

3.2 Onion cost per user

Our measure of efficiency of OR protocols is onion cost per user, which measures how many onions are transmitted by each user in the protocol. The onion cost is measured in unit onions, which is appropriate when the parties pass primarily onions to each other.

It is also an attractive measure of complexity because it is algorithm-independent: If we measured complexity in bits, it would change depending on which underlying encryption scheme was used. Since an onion contains as many layers as there are intermediaries, its bit complexity scales linearly with the number of intermediaries. (We assume that every message \(m\) can be contained in a single onion.) To translate our lower bound from onion complexity to bits, we will consider onions to be at least as long (in bits) as the message \(m\) being transmitted.

Let

- \(\text{out}^{\Pi, A}_i(\sigma) \overset{\text{def}}{=} \text{the number of honest onions that party } i \text{ transmits directly (to another party) in a protocol run of } \Pi \text{ interacting with adversary } A \text{ on input vector } \sigma\).

Definition 2 (Onion cost). The onion cost of the \(i\)-th party in an OR protocol \(\Pi(1^\lambda, pp, states, S, \sigma)\) interacting with an adversary \(A\) w.r.t. an input set \(\Sigma\) is \(\text{OC}^{\Pi, A}(\Sigma) \overset{\text{def}}{=} \mathbb{E}_{\sigma, S} \left[ \text{out}^{\Pi, A}_i(\sigma) \right]\), where \(N\) is the number of participants, and the expectation is taken over a uniformly random choice for an input vector \(\sigma \leftarrow \Sigma\) and the randomness \(S\) of the protocol.

3.3 Mixing

An onion formed by a corrupted party is distinguishable from honest onions. So when we say that an OR protocol mixes onions, we don’t mean to extend the guarantee to onions formed by corrupted parties. Instead, we mean that the protocol mixes some of the onions formed by the honest parties.

To avoid being pedantic, in this paper, an “onion” \(o\) can be either a type (e.g., \(o\) was produced by an honest party running \text{FormOnion}) or an instance (i.e., the specific sequence of bits).

We define mixing w.r.t. the game \(\text{MixingGame}(\Pi, A, \Sigma, \text{kickoff}, \text{freeze})\) below.

Mixing game. Let \(\Pi(1^\lambda, pp, states, S, \sigma)\) be an \(N\)-party OR protocol.

We define the following game \(\text{MixingGame}(\Pi, A, E_{\text{mix}}, \Sigma, \text{kickoff}, \text{freeze})\) between an adversary \(A\) and a challenger \(C\), parametrized by a set \(\Sigma\) of inputs to the protocol, a starting round \(\text{kickoff} \in [N]\), and an ending round \(\text{freeze} \in [N]\) such that \(\text{freeze} \geq \text{kickoff}\):
The game starts with the adversary $A$ choosing a vector $\sigma \in \Sigma$ of inputs to $\Pi$ as well as a set $\text{Bad} \subseteq [N]$ of parties to corrupt. The adversary $A$ sends both choices $\sigma$ and $\text{Bad}$ to the challenger $C$ and also chooses the keys for the corrupted parties in $\text{Bad}$. The challenger $C$ generates the keys for the remaining (honest) parties.

The adversary $A$ and the challenger $C$ interact in a run of protocol $\Pi$, with $C$ acting as the honest parties adhering to the protocol, and $A$ controlling the corrupted parties. At the end of the protocol run, the challenger $C$ identifies the set $O_{\text{mix}}$ of honest onions at round $\text{freeze}$ and sends $O_{\text{mix}}$ to the adversary $A$.

The adversary $A$ sends to the challenger $C$ two challenge onions $o^0$ and $o^1$ from $O_{\text{mix}}$. (For example, the onion $o^0 \in O_{\text{mix}}$ can be an onion layer that honest Apu formed to send a message $m^0$ to Bart through intermediaries Crazy Joe and Dr. Hibbert; and the onion $o^1 \in O_{\text{mix}}$ can be an onion layer that honest Andreas formed to send message $m^1$ to Emilia through intermediaries Benedetto, Cyrus, and Dario.)

The challenger $C$ chooses a uniformly random bit $b \leftarrow \{0,1\}$. (Recall that an \textit{evolution} is a sequence of onions produced when the challenger $C$ run the algorithm $\text{FormOnion}$ for forming onions; see Section 2 for details on onion encryption schemes.) If $o^b$ belongs unambiguously to an evolution $e = (e_1, e_2, \ldots, e_{R+1})$, and if there exists a unique onion $e_i$ that belongs to the same evolution $e$ and also to the set of honest onions at round $\text{kickoff}$, then the challenger $C$ selects $e_i$ as the challenge onion $o_{\text{ch}}$, i.e., $o_{\text{ch}} = e_i$. Otherwise, $C$ sets $o_{\text{ch}} = \bot$.

The challenger $C$ sends the challenge onion $o_{\text{ch}}$ to the adversary $A$, and $A$ outputs its guess $b'$ for the value of the chosen bit $b$ and \textit{wins} the game if $b' = b$.

The advantage of the adversary $A$ is given as

$$\text{Adv}^{\Pi, A}(\Sigma, \text{kickoff}, \text{freeze}) = \left| \Pr[A \text{ wins MixingGame}(\Pi, A; \Sigma, \text{kickoff}, \text{freeze})] - \frac{1}{2} \right|.$$ 

We define the notion of \textit{mixing} (below) with respect to the mixing game.

**Definition 3** (Mixing). Let $\Pi(1^\lambda, \text{pp.states}, \$, \sigma)$ be an OR protocol, and let $\text{start}$ and $\text{end}$ denote the first and final rounds of $\Pi$. We say that $\Pi$ mixes onions from round $\text{kickoff} \in [\text{start}, \text{end}]$ to round $\text{freeze} \in [\text{kickoff}, \text{end}]$ \textit{for inputs from $\Sigma$} if for any adversary $A \in \text{ActiveAdv}$, $\text{Adv}^{\Pi, A}(\Sigma, \text{kickoff}, \text{freeze}) = \text{negl}(\lambda)$, $\Pi$ mixes onions by round partway if $\Pi$ mixes onions of class $E_{\text{mix}}$ from start to partway.

## 4 Polylog onion cost for a party is required

We show that to achieve anonymity from the active adversary, there must exists a participant who transmits a superlogarithmic (in the security parameter) number of onions (Theorem 2).

To prove this, we use Lemmas 1 and 2 below.

### 4.1 Lemmas

Let an onion be honest if it is formed by an honest party. Let an onion be addressed if it is delivered to a party $j$ who can peel (decrypt) the onion to reveal a message, and there exists a message $m$ such that the message-recipient pair $(m, j)$ is included in the input of the party who formed the onion.

Recall that $V^{\Pi, A}(\sigma)$ is the adversarial view. Let

- $V^{\Pi, A, \text{OAdd}}(\sigma) \overset{\text{def}}{=} $ the “filtered adversarial view” consisting only of the honest addressed onions (values, wires, and rounds) in the adversarial view $V^{\Pi, A}(\sigma)$. 

\[ \text{in}^{\Pi,A,\text{Addr}}(\sigma) \overset{\text{def}}{=} \text{the number of honest addressed onions transmitted directly to party } i \text{ in a protocol run of } \Pi \text{ interacting with adversary } A \text{ on input vector } \sigma. \]

We first show that if an OR protocol \( \Pi \) is anonymous from the active adversary, then the residual adversarial view depends on the adversary but not on the input to the protocol. Formally,

**Lemma 1.** If \( \Pi(1^\lambda, \text{pp}, \text{states}, \$; \sigma) \) is anonymous from the active adversary, then for every adversary \( A \in \text{ActiveAdv} \) for any choice \( \text{Bad} \) for corrupted parties, and for all \( \sigma^0, \sigma^1 \in \Sigma \) such that \( \sigma^0 \equiv \text{Bad} \sigma^1 \),

\[ \sqrt{\text{in}^{\Pi,A,\text{Addr}}(\sigma^0)} \approx_s \sqrt{\text{in}^{\Pi,A,\text{Addr}}(\sigma^1)}. \]

(The proof given in Appendix C.)

From Lemma 1 it suffices to show that an efficient protocol cannot always equalize the addressed onions. Our proof for this hinges on the following observation (Lemma 2, below): If an OR protocol \( \Pi \) has sufficiently low onion cost per user, then there exists a setting in which a party (Alice) is not an intermediary node on a routing path for a message to be received by another party (Bob). In Theorem 2, we will show that the active adversary can distinguish this setting from one in which Alice is instructed to send a message to Bob instead.

First, we introduce a few statistical quantities. For an honest party \( i \), let

- \( \text{hops}^{\Pi,A}_{i \rightarrow j} (\sigma) \overset{\text{def}}{=} \text{the number of onions created by party } i \text{ and received by party } j \text{ in a protocol run of } \Pi \text{ interacting with adversary } A \text{ on input vector } \sigma. \)
- \( \text{hops}^{\Pi,A}_{i \rightarrow j \rightarrow k} (\sigma) \overset{\text{def}}{=} \text{the number of onions created by party } i \text{ and received by party } j \text{ that will reach party } k \text{ (if allowed to continue to } k \text{) in a protocol run of } \Pi \text{ interacting with adversary } A \text{ on input vector } \sigma. \)

Let \( \Pi \) be an OR protocol, and let \( A \) be an adversary interacting with \( \Pi \) on input \( \sigma \in \text{SimpleIO} \). For honest participants \( i \) and \( j \), we say that “\( i \) does not affect \( j \)’s recipient” if

\[ \mathbb{E} \left[ \text{hops}^{\Pi,A}_{j \rightarrow i \rightarrow \text{recipient}(j)} (\sigma) \right] \leq \frac{1}{2}, \]

where \( \text{recipient}(j) \) is the recipient for party \( j \).

**Lemma 2.** If \( \Pi(1^\lambda, \text{pp}, \text{states}, \$; \sigma) \) is an \( N \)-party OR protocol with \( o(N) \) onion cost per user in the presence of the active adversary, then for every adversary \( A \in \text{ActiveAdv} \) and every party \( i \in [N] \), there exist an input vector \( \sigma^0 \in \text{SimpleIO} \) and party \( j \in [N] \) such that party \( i \) does not affect party \( j \)’s recipient, and \( \mathbb{E}_\$ \left[ \text{out}^{\Pi,A}_{i}(\sigma^0) \right] = O(1) \cdot \mathbb{E}_{\sigma,\$} \left[ \text{out}^{\Pi,A}_{i}(\sigma) \right]. \)

(The proof given in Appendix C.)

### 4.2 Main result

The naïve protocol in which every party sends an (encrypted) message to every other party is anonymous. Lemma 2 precludes this naïve protocol from being a possible solution when the communication cost per user is required to be sublinear in the number of participants. Still, a communications protocol can be both anonymous and arbitrarily efficient if it does not have to be functional. For example, a protocol in which nothing is ever transmitted achieves anonymity vacuously.

Our lower bound holds for protocols that are minimally functional (in the presence of an active adversary). We call this notion *robustness*:
Definition 4 (Robustness). An OR protocol $\Pi(1^\lambda, \text{pp}, \text{states}, \Sigma, \sigma)$ is robust w.r.t. an input set $\Sigma$ if for every adversary $A \in \text{ActiveAdv}$, for every input vector $\sigma \in \Sigma$, and for every (honest) party $i$, no onion formed by party $i$ is dropped by $A$ implies that, with overwhelming probability in the security parameter $\lambda$, party $i$’s messages are delivered to their respective recipients.

We are now ready to state and prove the theorem:

**Theorem 2.** Let $\Pi(1^\lambda, \text{pp}, \text{states}, \Sigma, \sigma)$ be an $N$-party OR protocol. If, in the presence of the active adversary, $\Pi$ is (i) anonymous and (ii) robust (Definition 4), then $\Pi$ must have $\omega(\log \lambda)$ onion cost per user.

**Proof.** For the sake of reaching a contradiction, let $\Pi$ be an OR protocol, which in the active adversary setting, is anonymous and robust while having only $O(\log \lambda)$ onion cost per user.

Let $A_{i,\kappa}$ be the adversary who corrupts a constant fraction $\kappa$ of the parties. Additionally, $A_{i,\kappa}$ targets party $i$ and drops every onion that $i$ transmits to a corrupted party. Otherwise, $A_{i,\kappa}$ follows the protocol.

Since $\Pi$ has sublinear onion cost per user, from Lemma 2 there exist an input vector $\sigma^0$ in the simple I/O setting and a party $j \neq i$, such that “$i$ does not affect $j$’s recipient,” i.e.,

$$\mathbb{E}\left[\text{hops}_{j \rightarrow i \rightarrow \text{recipient}}(\sigma^0)\right] \leq \frac{1}{2},$$

(1)

where **recipient** is the recipient of party $j$ in $\sigma^0$, and

$$\mathbb{E}\left[\text{out}_{i}^{\Pi, A_{i,\kappa}}(\sigma)\right] = O(\log \lambda).$$

(2)

Let $\sigma^1$ be the input vector that is the same as $\sigma^0$ except for the inputs to parties $i$ and $j$, which are swapped.

Let **isolated** denote the the event that $A_{i,\kappa}$ manages to drop every onion that $i$ transmits.

On input $\sigma^1$: Conditioned on **isolated**, **recipient** never receives his message, i.e.,

$$\Pr\left[\text{in}_{\text{recipient}}^{\Pi, A_{i,\kappa}, \text{OAbd}}(\sigma^1) | \text{isolated} = 0\right] = 1.$$

(3)

On input $\sigma^0$: Let **unaffected** denote the event that $\text{hops}_{j \rightarrow i \rightarrow \text{recipient}}(\sigma^0) = 0$. $\Pr[\text{unaffected}] \leq \frac{1}{2}$ from (1). From the robustness property, it follows that **recipient** receives his message with nonnegligible probability, i.e.,

$$\Pr\left[\text{in}_{\text{recipient}}^{\Pi, A_{i,\kappa}, \text{OAbd}}(\sigma^0) | \text{isolated} > 0\right] = \text{nonnegl}(\lambda).$$

(4)

Combining (3) and (4) and from Lemma 1 if **isolated** occurs with nonnegligible probability on input $\sigma_0$, then $\Pi$ cannot be anonymous from the active adversary. Thus to complete our proof, it suffices to prove that, on input $\sigma_0$, the probability of **isolated** is nonnegligible.

From (2), $\mathbb{E}\left[\text{out}_{i}^{\Pi, A_{i,\kappa}}(\sigma^0)\right] = O(\log \lambda)$. From Markov’s inequality, there exists a constant $\alpha > 0$, such that $\text{out}_{i}^{\Pi, A_{i,\kappa}}(\sigma^0) \leq \alpha \log \lambda$ with nonnegligible probability.

Let **droppable** denote the event that $\text{out}_{i}^{\Pi, A_{i,\kappa}}(\sigma^0) \leq \alpha \log \lambda$.

Let **isolated|droppable** denote the event **isolated** conditioned on **droppable**.

The probability of **isolated|droppable** is smallest when party $i$ chooses a different location for each of the (at most) $\alpha \log \lambda$ onions it transmits. This probability is bounded by the probability $p$ that
a random \((\alpha \log \lambda)\)-size sample from a set of \(N\) balls, \(\kappa N\) of them which are green, are all green. When \(\alpha \log \lambda \leq \sqrt{N}\),
\[
p = \left( \frac{\kappa N}{\alpha \log \lambda} \right) \cdot \left( \frac{\alpha \log \lambda}{N \alpha \log \lambda} \right) = \Theta \left( \frac{\kappa \log \lambda}{\alpha \log \lambda} \right) = \text{nonnegl}(\lambda).
\]
Thus, \(\Pr[\text{isolated} \land \text{droppable}] = \Pr[\text{droppable}] \cdot \Pr[\text{isolated} | \text{droppable}]\) is nonnegligible in \(\lambda\). It follows that \text{isolated} occurs with nonnegligible probability.

5 Our OR scheme \(\Pi_\triangle\) (pi-tree)

A "message-bearing onion" is an onion formed using an inputted message and a routing path ending in an inputted recipient.

An OR protocol that is anonymous from the active adversary must both mix and equalize honest message-bearing onions. Informally, an OR protocol mixes honest message-bearing onions by round partway if the adversary is unable to determine which of two honest message-bearing onions at partway belong to the same evolution as an onion from an earlier round. An OR protocol equalizes honest message-bearing onions after round partway if for every round \(r \geq \text{partway}\), the vector of numbers of message-bearing onions received by the \(N\) parties at \(r\) is statistically independent of the input to the protocol.

Recently, Ando, Lysyanskaya, and Upfal [ALU18] presented an OR protocol \(\Pi_a\) which is differentially private from the active adversary. Their construction relied on an OR technique that ensures that either (1) honest message-bearing onions are mixed (if the adversary doesn’t drop many onions), or (2) every honest party aborts the protocol (otherwise).

We briefly describe how the mixing technique works.

A "checkpoint onion" is an onion formed using the empty message "\(\bot\)" and essentially a random routing path. The checkpoint onions serve as a mechanism for letting the honest parties detect when too many onions have been dropped.

For every pair of honest parties \(i\) and \(j\) and round \(r\), with fixed frequency \(p\), the parties \(i\) and \(j\) each forms a checkpoint onion designed to reveal an expected checkpoint (nonce) value to the other party at round \(r\). Each party independently determines whether to form a "checkpoint onion" and, if so, which checkpoint value to embed by using a pseudorandom function keyed with the shared key \(sk_{i,j}\). If the number of expected nonces at every honest party and round is at least polylog (in the security parameter), the adversary is unable to drop too many onions without the honest parties noticing.

Here, we present a new technique for equalizing message-bearing onions using "mergeable onions." The routing paths for mergeable onions are structured (like paths from leaf nodes to root node in a binary tree graph) so that pairs of mergeable onions are designed to meet at the same location and round and merge; in actuality, merging two onions is accomplished by simply dropping one of the onions before the next round.

5.1 Overview

Our OR scheme \(\Pi_\triangle(1^\lambda, pp, states, s, \sigma)\) is designed to work in the simple I/O setting.

We use an onion encryption (OE) scheme \(OE = (\text{Gen}, \text{FormOnion}, \text{ProcOnion})\) as a primitive building block (see Section 2 for a description of OE schemes); and we assume that \(OE\) is ideal for the analysis. For every party \(i\), let \((pk_i, sk_i) \leftarrow \text{Gen}(1^\lambda)\) be party \(i\)'s key pair generated from running
the key generating algorithm Gen. For every pair \((i, j)\) of parties, let \(sk_{i,j}\) be the parties’ shared key known by only parties \(i\) and \(j\).  

**Onion-forming phase.** During the onion-forming phase, every honest party \(i\) creates three types of onions: (i) mergeable message-bearing onions, (ii) mergeable checkpoint onions, and (iii) regular (unmergeable) checkpoint onions.

**Execution phase.** All onions are created during the onion-forming phase and released simultaneously in the first round of the execution phase. After every round, each honest party processes the onions it received, tallies checkpoints and merges onions as needed, and then sends the remaining processed onions to their next destinations in random order in the next round.

Rounds of the execution are grouped into epochs of equal length, where each epoch lasts a polylog (in the security parameter) number of rounds. The last round of each epoch is a “diagnostic round,” in which each honest party independently runs a diagnostic to check whether the adversary dropped more than a predefined threshold number of onions. (Each party tallies the number of missing checkpoints which is strongly correlated with the total number of dropped onions, see Lemma 4 in Section 5.3.1 and Lemma 8 in Appendix E.) If a diagnostic test fails, the party stops participating in the protocol. All other rounds are “merging rounds,” in which every pair of mergeable onions received at honest parties are merged.

We now describe the protocol in more detail.

### 5.2 Description

Our OR scheme \(\Pi_\Delta\) consists of an algorithm FormAllOnions and a protocol RouteOnions:

Each honest party runs FormAllOnions to form onions to be routed during the execution phase. FormAllOnions takes as input the security parameter \(1^\lambda\) (written in unary), the fraction \(\kappa \in [0,1)\) of participants that can be corrupted, the session id \(sid\), the participants’ public keys \(Keys\), the participants \([N]\), the message \(m\), and the recipient \(j\); and outputs onions. (See Figure 1)

| FormAllOnions\((1^\lambda, \kappa, sid, Keys, [N], m, j)\) |
|---|
| 1 : “NumMBOs = \(\lceil \log(2^{(1+\epsilon)} \lambda) \rceil\) |
| 2 : “h = \(\lfloor \log(NumMBOs) \rfloor + 1\) |
| 3 : “A = max \(\sqrt{N \log^{2+\epsilon} \lambda \log(2^{(1+\epsilon)} \lambda)}\) |
| 4 : “NumMCOs = \(2^{\lceil \log(A(\log(A+1))) \rceil}\) |
| 5 : “Nonces \leftarrow GenCkpts(1^\lambda, sid, NumMCOs, h, [N]) |
| 6 : “Onions = \emptyset |
| 7 : “OMBO \leftarrow FormMBOs(1^\lambda, NumMBOs, h, Keys, [N], m, j); add OMBO to Onions |
| 8 : “OMCO \leftarrow FormMCOs(1^\lambda, NumMCOs, h, Keys, [N], Nonces); add OMCO to Onions |
| 9 : “ORCO \leftarrow FormRCOs(1^\lambda, h, Keys, [N], Nonces); add ORCO to Onions |
| 10 : “return Onions |

Figure 1: Pseudocode for FormAllOnions. The parameter \(\epsilon\) is hardwired; when \(\epsilon\) is set to a higher value, the probability of insufficient mixing decays faster in the asymptotical sense.

These shared keys do not need to be set up in advance; it is known that they can be generated as needed from an existing PKI, e.g., using Diffie-Hellman.
FormAllOnions is implemented using the following subroutines. (See Figure 2)

(a) **GenPaths** generates routing paths (and sequences of nonces) for mergeable onions. It takes as input the security parameter $\lambda$, the number $\text{NumOnions}$ of mergeable onions, the number $h$ of epochs (which corresponds to the height of the binary tree graph), the participants $[N]$, (possibly) a set $\text{Nonces}$ of checkpoints, and a recipient $j$; and outputs routing paths.

(b) **GenCkpts** generates checkpoints. It takes as input the security parameter $\lambda$, the session id $\text{sid}$, the number $\text{NumMCOs}$ of mergeable checkpoint onions, the number $h$ of epochs, and the participants $[N]$; and outputs checkpoints.

(c) **FormMBOs** forms message-bearing onions. It takes as input the security parameter $\lambda$, the number $\text{NumMBOs}$ of message-bearing onions, the number $h$ of epochs, the participants’ public keys $\text{Keys}$, the participants $[N]$, the message $m$, and the recipient $j$; and outputs the mergeable message-bearing onions.

(d) **FormMCOs** forms mergeable checkpoint onions. It takes as input the security parameter $\lambda$, the number $\text{NumMCOs}$ of mergeable checkpoint onions, the number $h$ of epochs, the participants’ public keys $\text{Keys}$, the participants $[N]$, and the checkpoints $\text{Nonces}$ (generated by running **GenCkpts**); and outputs the mergeable checkpoint onions.

(e) **FormRCOs** forms regular checkpoint onions. It takes as input the security parameter $\lambda$, the number $h$ of epochs, the participants’ public keys $\text{Keys}$, the participants $[N]$, and the (remaining) checkpoints $\text{Nonces}$; and outputs the regular checkpoint onions.

Each honest party processes and routes onions using the algorithm **RouteOnions**, which runs **Batch** as a subroutine. (See Figure 3) **RouteOnions** takes as input the security parameter $\lambda$, the fraction $\kappa$ of participants that can be corrupted, the number $R \triangleq h \cdot \lceil \log^{1 + \epsilon} \lambda \rceil$ of intermediate rounds, and the onions $\text{Onions}$ formed in the onion-forming phase.

### 5.3 Security proof

**Proof sketch.** Within the first few rounds of the execution phase, the adversary can target Alice by disproportionately dropping more of her onions than any other party’s. However, we prove that either the adversary drops too many onions or else many of Alice’s onions survive the first epoch (Theorem 3).

If the protocol isn’t aborted at the first diagnostic round, then many of Alice’s onions were not dropped during the first epoch. However, there may be fewer of Alice’s onions remaining compared with those of another party, Allison.

As the protocol progresses past the first epoch, honest parties also reduce the number of onions in the system by following the protocol. If there are only a few of Alice’s message-bearing onions in the system, then each of Alice’s message-bearing onions is unlikely to be paired and so is likely to progress to the next epoch. If there are more of Allison’s message-bearing onions, each of Allison’s message-bearing onions is more likely to meet its mate and so is less likely to progress to the next epoch. In this way, the protocol equalizes the numbers of message-bearing onions. We show that the message-bearing onions are equalized after the midway point (halfway into the protocol run) if the protocol hasn’t been aborted by this point (Theorem 4).

Since the onions are mixed after a polylog number of rounds [ALU18], this proves that $\Pi_\triangle$ is anonymous from the active adversary.
Figure 2: Pseudocode for FormOnions’s subroutines. (a) In GenPaths: InitBinTree(NumOnions) initiates an empty binary tree with NumOnions number of leaves. G.nodes, G.leaves, G.root are G’s nodes, leaf nodes, and root node, respectively. CurNode.parent is the parent node of node CurNode.
(b) In GenCkpts: F is a pseudorandom function and \(f\) is a predicate function, such that evaluating \(f(F(\cdot, 0))\) outputs one with frequency \(2^\text{NumMCOs}/Nh\). (a), (c)-(d) The differences between generating message-bearing onions and mergeable checkpoint onions are boxed for emphasis.

5.3.1 Part 1: Analysis of the first epoch

For the analysis of \(\Pi_\Delta\), let us redefine what an honest onion is. When we refer to an honest onion, we will mean an onion which is either a message-bearing onion formed by an honest party or a checkpoint onion formed by an honest party for verification by another honest party. Onions other than these are distinguishable, and so we will not rely on their existence to prove security.
Figure 3: Pseudocode for RouteOnions and its subroutine Batch. The parameters $\epsilon$ and $\delta$ are hardwired; when set to higher values, the probability of error decays faster.

The adversary can add, modify, delay, or drop onions, but only dropping honest onions may help the adversary learn who is talking with whom. The onions formed by corrupted parties do not affect the honest onions (what happens to them), and so the adversary doesn’t learn anything from adding onions. Modified onions cannot be properly peeled, and so essentially they count as dropped onions. Finally, we can protect against delayed onions by simply embedding a round number in every layer of the onion and dropping onions that arrive too late.

We will first analyze just the first epoch of protocol $\Pi_\Delta$. We will show that if the adversary does not drop too many onions in the first epoch, then for every honest party, a constant fraction of their message-bearing onions will survive (not be dropped in) the first epoch.

Our proof essentially boils down to proving that the following undesirable events can happen with only negligible probability:

(a) For any honest party, the onions formed by the party do not travel together.
(b) The diagnostic test at the first diagnostic round fails.

Below, we show that events a and b can occur with only negligible probability. (The proofs for these lemmas are given in Appendix D.)

**Lemma 3.** Let $\lambda$ be the security parameter, and let $N' = (1 - \kappa)N \leq \text{poly}(\lambda)$ be the number of honest parties (or locations). Let $O$ be the set of honest onions at any round $1 \leq r \leq R$ of the protocol execution. If $\log^2 \lambda \leq |O| \leq (N' + 2)/2$, then, with overwhelming probability in $\lambda$, at least $|O|/\log \lambda$ participants receive at least one onion from $O$. 
Recall that $\text{NumMBOs}$ is the number of message-bearing onions formed by each honest party, and $h = \log \text{NumMBOs} + 1$ is the number of diagnostic rounds (or the number of epochs) in a full unaborted execution of $\Pi_\Delta$.

Lemma 4. In $\Pi_\Delta$: Let $N \leq \text{poly}(\lambda)$ be the number of participants. Suppose that the adversary drops at least $0 \leq \zeta \leq 1$ fraction of all honest onions, where $\zeta \cdot \frac{\text{NumMCOs}}{h} = \Omega(\log^{1+\epsilon} \lambda)$. If $F$ is a truly random function, then for all $0 < \delta \leq 1$, with overwhelming probability in $\lambda$, each honest party $k$ will notice at least $2(1-\delta)(1-\kappa)\zeta \frac{\text{NumMCOs}}{h}$ missing checkpoints at the first diagnostic round.

We will now prove Theorem 3.

Theorem 3. In $\Pi_\Delta$: Let $A \in \text{ActiveAdv}$ be an adversary that non-adaptively corrupts up to $0 \leq \kappa < 1$ fraction of the parties, and let $\sigma \in \text{SimpleIO}$ be any input vector in the simple I/O setting. Let $\text{FirstCheck}^A(\sigma, i)$ denote the number of message-bearing onions formed by party $i$ at the first diagnostic round in a run of $\Pi_\Delta$ interacting with $A$ on $\sigma$.

If an honest party doesn’t abort the protocol at the first diagnostic round, then for all $0 < \delta \leq 1$, with overwhelming probability in $\lambda$, for each honest party $i$,

$$\text{FirstCheck}^A(\sigma, i) \geq (1-\delta)(1-\kappa)^2 \text{NumMBOs},$$

i.e., the number of message-bearing onions formed by party $i$ at the previous round is at least $(1-\delta)(1-\kappa)^2 \text{NumMBOs}$.

Proof. Let $H$ denote the set of all honest onions, and consider only onions from this set.

Fix an input $\sigma$ in the simple I/O setting and an honest party $i$.

Let $A'$ be an adversary who non-adaptively corrupts up to $0 \leq \kappa < 1$ fraction of the parties and, at rounds 1 and 2, always drops an onion if it could have been formed by party $i$ and is droppable (delivered to a corrupted node).

Let $D(r)$ be the number of onions at round $r$ that could be one of party $i$’s message-bearing onions (from the adversary’s perspective). In the first round, party $i$ transmits $D(1)$ onions. For any arbitrarily small $0 < d \leq 1$, with overwhelming probability, $D(1) \geq 2(1-d)(1-\kappa)\text{NumMCOs}$, because party $i$ forms at least $2(1-d)(1-\kappa)\text{NumMCOs}$ dummy onions with a checkpoint for an honest party (Chernoff bounds; Corollary 4 in Appendix 4).

Let the span at round $r$, denoted $S(r)$, be the number of honest parties that, at round $r$, receive an onion that could be one of party $i$’s message-bearing onions (from the adversary’s perspective). From Corollary 4, at least $(1-d)(1-\kappa)$ of these $D(1)$ onions go to honest parties. So, from Lemma 4

$$S(1) \geq 2(1-d)^2(1-\kappa)^2 \frac{\text{NumMCOs}}{\log \lambda}.$$  \hspace{1cm} (5)

Each party in $S(1)$ receives at least $2(1-d)(1-\kappa)\text{NumMCOs}$ honest onions at round 1 (Chernoff bounds; Lemma 7 in Appendix 7). Combining this with (5), there are at least $4(1-d)^3(1-\kappa)^3 \frac{\text{NumMCOs}^2}{\log \lambda}$ honest onions that could have originated from party $i$ at round 2; that is, $D(2) \geq 4(1-d)^3(1-\kappa)^3 \frac{\text{NumMCOs}^2}{\log \lambda}$.

At least $(1-d)\kappa$ of these onions are routed to corrupted parties at round 2 (Chernoff bounds; Corollary 4); that is, the number of honest onions from $D(2)$ that go to corrupted parties is at least

$$4(1-d)^4(1-\kappa)^3 \kappa \frac{\text{NumMCOs}^2}{\log \lambda} \geq 4(1-d)^4(1-\kappa)^3 \kappa \frac{\lambda^2}{\log \lambda} (\log A + 1)^2$$

$$\geq 4(1-d)^4(1-\kappa)^3 \kappa (N \log^{1+\epsilon} \lambda)(\log A + 1)^2$$

$$\geq 4(1-d)^4(1-\kappa)^3 \kappa (N \log^{1+\epsilon} \lambda) h,$$
If $\rho$ In Lemma 6 (below), we show that the adversary can delay this point by at most a couple of rounds. Alice’s onions, thereby delaying the point at which the honest message-bearing onions are equalized. (and subsequent) epochs to attempt to further target Alice by disproportionately dropping more of to be Allison’s than Alice’s. The adversary may try to use this information during the second round: Suppose that the adversary drops more of Alice’s onions than Allison’s during the second round: Suppose that the adversary drops more of Alice’s onions than Allison’s during the

$$\zeta \geq \frac{4(1-d)^4(1-\kappa)^2\kappa(N \log^{1+\epsilon} \lambda)h}{3(1+d)(1-\kappa)\text{NumMCOs}N} = \left(\frac{4(1-d)^4(1-\kappa)^2\kappa}{3(1+d)}\right) \log^{1+\epsilon} \lambda \cdot \frac{h}{\text{NumMCOs}},$$

because there are less than $(1+d)(1-\kappa)\text{NumMCOs}$ honest onions in total (Chernoff bounds; Corollary 1). Let $1 - \delta = \frac{1-d^5}{1+d}$. From Lemma 1 each honest party $k$ will notice at least

$$ \frac{2\text{NumMBOs}}{h} \geq \left(\frac{4(1-d)^5(1-\kappa)^2\kappa}{3(1+d)}\right) \log^{1+\epsilon} \lambda \cdot \frac{h}{\text{NumMCOs}} \cdot \frac{2\text{NumMBOs}}{h} = \frac{4}{3}(1-\delta)(1-\kappa)^2 \kappa \log^{1+\epsilon} \lambda$$

missing checkpoints and will, therefore, abort the protocol. Thus, any adversary that drops at least as many onions as $A'$ will cause the honest parties to abort the protocol.

An adversary $A$ that drops at most as many onions as $A'$ can only do worse than $A'$; if $A$ can deviates from $A'$ either by dropping fewer onions or waiting to drop onions, then

$$\text{FirstCheck}^A(\sigma, i) \geq \text{FirstCheck}^{A'}(\sigma, i).$$

For any $0 < \delta \leq 1$, at least $(1-\delta)(1-\kappa)^2$ of party $i$’s message-bearing onions are randomly routed through only honest parties in rounds 1 and 2 (Chernoff bounds; Corollary 1); it follows that

$$\text{FirstCheck}^A(\sigma, i) \geq (1-\delta)(1-\kappa)^2 \text{NumMBOs}. \tag{7}$$

Combining (6) and (7), we obtain our desired result.

### 5.3.2 Part 2: Analysis of the rest of the execution

In the previous section, we proved that if the protocol is unaborted at the end of the first epoch, then for every sender, at least a constant fraction of her message-bearing onions remain undropped. Moreover, since there are $\Omega(N \log^{1+\epsilon} \lambda)$ honest onions per every round in the first epoch, the honest onions are mixed by partway through the first epoch $\text{ALUTS}$. (See Section 3.3 for a formal definition of mixing onions.) In other words, the adversary can no longer link an honest onion to its origin.

However, some unmixing occurs when mergeable onions rendezvous at the beginning of the second round: Suppose that the adversary drops more of Alice’s onions than Allison’s during the first epoch. Then a pair of mergeable onions at the beginning of the second epoch is more likely to be Allison’s than Alice’s. The adversary may try to use this information during the second (and subsequent) epochs to attempt to further target Alice by disproportionately dropping more of Alice’s onions, thereby delaying the point at which the honest message-bearing onions are equalized. In Lemma 6 (below), we show that the adversary can delay this point by at most a couple of rounds.

Let $h/2 \overset{\text{def}}{=} \lceil h/2 \rceil$; the $h/2$-th epoch formally marks the “midway point in the protocol run.” For every epoch $2 \leq t \leq h/2$, let a singleton at the $t$-th epoch be an onion at the first round of the epoch without a mate; it could be a regular checkpoint onion or a mergeable onion whose mate dropped off prematurely.

To prove Lemma 5 we rely on the diagnostic test to work; that is:

**Lemma 5.** In $\Pi_{\Delta}$: For every epoch $1 \leq t \leq h/2$, let $\beta_t$ be the fraction of the singletons that the adversary drops during the $t$-th epoch. Let $\rho_0 = 0$, and recursively define $\rho_t \overset{\text{def}}{=} \sum_{\tau=1}^{t} (1 - \rho_{\tau-1}) \beta_\tau$. If $\rho_t \geq \frac{1}{2t}$, then with overwhelming probability in $\lambda$, every honest party aborts by the $t$-th epoch.
Proof. The proof follows from combining Lemmas 8 and 9 in Appendix E.

Lemma 6. In $\Pi_\Delta$: For every party $i$, let $MBO^{(i)}_t$ be the number of message-bearing onions formed by party $i$’s at the beginning of the $t$-th epoch. Suppose that the adversary $A \in \text{ActiveAdv}$ drops only singleton onions from the second epoch to the midway point.

If an honest party does not abort the protocol before the midway point, then with overwhelming probability in $\lambda$, for every honest party $i$,

$$MBO^{(i)}_{h/2} = \frac{\text{NumMBOs}}{2^{h/2-1}}.$$  

Proof. Fix a party $i$, and consider only party $i$’s message-bearing onions. Let $t$ be any epoch between 2 and $h/2$. Let $\zeta_t = \left\lfloor \frac{|V_t|}{2^t} \right\rfloor$ be the ratio between the actual number $|V_t|$ of party $i$’s onions at the $t$-th merging round and the maximum number $|U_t| = \frac{\text{NumMBOs}}{2^t} = \Omega\left(\log^{1+\epsilon} \lambda\right)$.

Fix $0 \leq \chi \leq \frac{1}{2}$, and let $0 \leq \rho \leq \chi$ be any fraction between zero and $\chi$.

We analyze what happens when the adversary $A$ drops $\rho$ fraction of the singletons at round $t$ and another $\frac{\chi - \rho}{\chi}$ fraction of the singletons at round $t+1$.

At the $t$-th epoch, there are an expected (approx.) $\zeta_t^2|U_t|$ paired onions and an expected (approx.) $\zeta_t(1-\zeta_t)|U_t|$ singletons (Lemma 10 and Corollary 2). If the adversary $A$ drops $\rho$ fraction of the singletons at round $t$, then for any small constant $\epsilon \geq \frac{|V_{h/2}|}{|V_{h/2}-1|} - 1$,

$$\zeta_{t+1} \geq \frac{|U_t|}{|U_{t+1}|} \left( (1-\epsilon)\frac{\zeta_t^2}{2} + (1-\epsilon)(1-\rho)\zeta_t(1-\xi_t) \right)$$

$$= 2(1-\epsilon) \left( \frac{\zeta_t^2}{2} + (1-\rho)\zeta_t(1-\xi_t) \right)$$

$$= (1-\epsilon)\zeta_t^2 + 2(1-\epsilon)\zeta_t - 2(1-\epsilon)\zeta_t^2 - 2(1-\epsilon)\rho\zeta_t + 2(1-\epsilon)\rho^2 \zeta_t^2$$

$$\overset{\text{def}}{=} \xi_{t+1}$$

with overwhelming probability (Corollary 2 in Appendix E).

At the $(t+1)$-st epoch, there are an expected (approx.) $\zeta_{t+1}^2|U_{t+1}|$ paired onions and an expected (approx.) $\zeta_{t+1}(1-\zeta_{t+1})|U_{t+1}|$ singletons (Lemma 10 and Corollary 2). So if the adversary $A$ drops $\beta = \frac{\chi - \rho}{\chi}$ fraction of the singletons, then

$$\zeta_{t+2} \geq \frac{|U_{t+1}|}{|U_{t+2}|} \left( (1-\epsilon)\frac{\zeta_{t+1}^2}{2} + (1-\epsilon)(1-\beta)\zeta_{t+1}(1-\xi_{t+1}) \right)$$

$$= 2(1-\epsilon) \left( \frac{\zeta_{t+1}^2}{2} + (1-\beta)\zeta_{t+1}(1-\xi_{t+1}) \right)$$

$$= 2(1-\epsilon)(1-\beta)\zeta_{t+1} + 2(1-\epsilon) \left( \beta - \frac{1}{2} \right) \xi_{t+1}^2$$

$$\overset{\text{def}}{=} \xi_{t+2}$$

with overwhelming probability.

Taking a derivative of $\xi_{t+2}$ with respect to $\rho$, we get

$$\frac{d^2\xi_{t+2}}{d\rho} = \left( \frac{d\xi_{t+2}}{d\xi_{t+1}} \right) \left( \frac{d\xi_{t+1}}{d\rho} \right)$$

$$= (2(1-\epsilon)(1-\beta + (2\beta - 1)\xi_{t+1})) \left( 2(1-\epsilon) \left( \zeta_t^2 - \zeta_t \right) \right)$$

$$= 4(1-\epsilon)^2 (\zeta_t^2 - \zeta_t) (1 - \beta + (2\beta - 1)\xi_{t+1})$$

$$\leq 0,$
since \( \frac{ds_{0+1}}{dp} = 2(1-\epsilon) \left( 1 - \beta + (2\beta - 1)\xi_{t+1} \right) \) from (8). This last inequality follows because: \((1-\epsilon)^2 \geq 0, \) since \( \epsilon \geq 0; \) \( (\zeta_2^2 - \zeta_t) \leq 0, \) since \( \zeta_t \leq 1; \) and \( \xi_{t+1} \leq \frac{1-\beta}{1-\rho}, \) since \( \beta = \frac{1-\rho}{1-\rho} \leq \frac{1}{2}. \)

For every \( 1 \leq t \leq h/2, \) let \( \beta_t \) be the fraction of singleton onions that the adversary drops in the \( t-\)th epoch, and let \( \beta = (\beta_1, \beta_2, \ldots, \beta_{h/2}) \) be the adversary’s strategy. Let \( \rho_0 = 0, \) and for every \( 1 \leq t \leq h/2, \) let \( \rho_t = \sum_{\tau=1}^{t} (1 - \beta_{\tau-1}) \beta_{\tau}. \)

If \( \rho_{h/2} \geq \frac{1}{2}, \) from Lemma 5 every honest party aborts the protocol by the \( h/2-\)th epoch with overwhelming probability.

If \( \rho_{h/2} < \frac{1}{2}, \) then the adversary cannot do better than dropping half of the singleton onions upfront (in the second epoch) so that \( \zeta_3 \geq (1-\epsilon)\zeta_2. \) Assume for the sake of reaching a contradiction that there exists some strategy where \( \beta_\tau > 0 \) for some \( \tau > 2 \) that performs strictly better. Let \( t \) be the largest of these \( \tau \)’s. Then from (9), the strategy \( \beta', \) where \( \beta_{t-1} = \beta_{t-1} + (1 - \rho_t) \beta_{t+1}, \beta'_t = 0, \) and \( \beta'_\tau = \beta_\tau \) for all other \( \tau \)’s, performs just as well.

From Theorem 3, \( \zeta_2 \geq (1-\delta)(1-\kappa)^2. \) So if the adversary drops half of the singleton onions upfront, then \( \zeta_3 \geq (1-\epsilon)(1-\delta)(1-\kappa)^2. \) If the adversary does not drop any other onions, then using a known concentration bound \[HS05\] for the hypergeometric distribution, the protocol succeeds in equalizing the numbers of remaining message-bearing onions, i.e., with overwhelming probability, for all party \( i, \) \( \zeta_{h/2} = 1. \)

We are finally ready to prove that protocol \( \Pi_\triangle \) is anonymous.

**Theorem 4.** Protocol \( \Pi_\triangle \) is anonymous from the active adversary.

**Proof.** We prove this by cases.

**Case 1.** In the first case, the adversary \( \mathcal{A} \) drops too many onions before the midway point, and all honest parties abort the protocol run by this point. With overwhelming probability, no honest onion reaches its final destination.

**Case 2.** In the second case, the adversary \( \mathcal{A} \) doesn’t drop enough onions to cause the honest parties to abort by the midway point.

Each epoch lasts for a polylog number of rounds. Thus for each epoch \( 2 \leq t \leq h/2, \) there exists a round \( \text{partway}_t \) such that the honest onions mix from the start of the epoch to round \( \text{partway}_t. \) [ALU18]. The implication of this is that the adversary’s choice on how to drop honest onions can depend on only the numbers of honest singleton onions and paired (merging mergeable) onions for the epochs.

For every epoch \( t, \) let \( \text{Singletons}_t \) denote the set of singleton onions, and let \( \text{Pairs}_t \) denote the set of pairs of mated onions. For any onion in \( \text{Singletons}_t, \) the adversary may have an opportunity to choose to deviate from the protocol and drop it, or to let it progress to the next epoch. Likewise, for every pair of onions in \( \text{Pairs}_t, \) the adversary may have an opportunity to drop the pair or to let one of the onions progress to the next epoch. (If the adversary chooses to keep both onions, then with overwhelming probability, one of the two will be dropped by an honest party before the next epoch.)

We now assume that at every epoch \( t, \) the adversary may adaptively choose a fraction \( \alpha_t \) of the onions in \( \text{Singletons}_t, \) to drop as well as a fraction \( \beta_t \) of pairs of onions in \( \text{Pairs}_t, \) to drop. In reality, the adversary cannot control what happens to the onions at honest parties, but we will prove that \( \Pi_\triangle \) is secure even when the adversary can choose which onions are dropped at the beginning of each epoch. Thus, the adversary’s strategy (actions that affect the numbers of message-bearing onions at every round) is captured fully by two sequences: \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{h/2}) \) and \( \beta = (\beta_1, \beta_2, \ldots, \beta_{h/2}). \)

**Case 2a.** Suppose that the adversary chooses to drop only singleton onions. In other words, \( \alpha = \vec{0} \) is the zero vector. Then, \( \Pi_\triangle \) is anonymous from Lemma 6.
Case 2b. The adversary also drops merging pairs. In other words, there exists an epoch $t$ such that $\alpha_t > 0$. This is suboptimal: For every epoch, the adversary cannot drop more than half of the paired onions. Otherwise, with overwhelming probability, every honest party aborts the protocol (Lemma 8 and Lemma 10). Thus, the adversary just wastes her budget for dropping onions, while helping to equalize the message-bearing onions quicker.

Since the honest onions are mixed by the midway point [ALU18], this concludes our proof. □

Rather than evaluating $\Pi_\Delta$ up to the midway point, we could have evaluated $\Pi_\Delta$ up to the $\left\lceil \frac{h}{\epsilon_1} \right\rceil$-th epoch for any constant fraction $0 < \frac{1}{\epsilon_1} \leq 1$ instead. Thus, while we explicitly showed that there exists a statistically private OR protocol requiring $\log^{2(1+\epsilon)} \lambda$ message-bearing onions per user, this result extends to $\log^{\epsilon_1(1+\epsilon_2)} \lambda$ messages-bearing onions per user for any $\epsilon_1 > 1$ and $\epsilon_2 > 0$.

6 Polylog onion cost per party is sufficient

In Section 5, we presented the first non-trivial OR protocol, $\Pi_\Delta$, that is both robust and anonymous from the active adversary. Unfortunately, $\Pi_\Delta$ is impractical requiring $\omega(\sqrt{N} \cdot \text{polylog}(\lambda))$ onion cost per user, where $N$ is the number of participants, and $\lambda$ is the security parameter.

In this section, we present an extension of $\Pi_\Delta$, called $\Pi_{\text{pi-butterfly}}$, that decreases the onion cost per user to $\text{polylog}(\lambda)$ in exchange for $\text{polylog}(\lambda)$ more rounds.

Let us first understand the reason for $\Pi_\Delta$’s inefficiency: The adversary can succeed in breaking the protocol’s security if she can isolate an honest party by eliminating all the traffic coming out of the honest party without causing the protocol to be aborted. In $\Pi_\Delta$, each party sends many (more than $\Theta(\sqrt{N} \cdot \text{polylog}(\lambda))$) onions to ensure that his/her message-bearing onions spread out rapidly (in a couple of rounds) across the network. In this way, the adversary cannot drop all of a targeted party’s message-bearing onions without also dropping many other honest dummy onions and thus enabling all honest parties to detect that an attack is underway.

By mixing onions in smaller batches, it becomes easier for the honest parties to detect when an attack is underway. This translates into requiring fewer onions. In $\Pi_{\text{pi-butterfly}}$, we do this by initially restricting the communication to (virtual) butterfly network connections.

6.1 Description

Our OR scheme $\Pi_{\text{pi-butterfly}}(\lambda, \text{pp, states, } \$, \sigma)$ is designed to work in the simple I/O setting.

We use an onion encryption (OE) scheme $\mathcal{OE} = (\text{Gen, FormOnion, ProcOnion})$ as a primitive building block (see Section 2 for a description of OE schemes); and we assume that $\mathcal{OE}$ is ideal for the analysis. For every party $i$, let $(pk_i, sk_i) \leftarrow \text{Gen}(\lambda)$ be party $i$’s key pair generated from running the key generating algorithm $\text{Gen}$. For every pair $(i, j)$ of parties, let $sk_{i,j}$ be the parties’ shared key known by only parties $i$ and $j$.

Our OR scheme $\Pi_{\text{pi-butterfly}}$ consists of an algorithm $\text{FormAllOnions}$ (for forming onions during the onion-forming phase) and a protocol $\text{RouteOnions}$ (for routing onions during the execution phase) and is essentially the same as our original OR scheme $\Pi_\Delta$ with a mixing subphase replacing the first epoch the execution phase. We refer to the rest of the execution phase as the merging subphase.

**Onion-forming phase.** Let $\epsilon_1 > 1$ and $\epsilon_2 > 0$ be some small and system-specified parameters. By running $\text{FormAllOnions}$, each honest party forms $\lceil \log^{\epsilon_1(1+\epsilon_2)} \lambda \rceil$ message-bearing onions and (expected) $2\lceil \log^{\epsilon_1(1+\epsilon_2)} \lambda \rceil$ checkpoint onions, and the routing path of each onion is extended.
to include a subpath for routing during the mixing subphase. In Figure 4, the modifications to FormAllOnions’s original subroutines are boxed for emphasis.

| GenPaths($1^\lambda$, NumOnions, $h$, $h_{\text{sec}}$, $[N]$, Nonces, $j$) |
|---|
| 1: $G \leftarrow \text{InitBinTree}(\text{NumOnions})$ |
| 2: for $\ell \in [2\text{NumOnions} - 1]$ |
| 3: $G$.nodes[$\ell$].parties $\leftarrow [N]^{\lceil \log 1+\epsilon \lambda \rceil}$ |
| 4: $G$.nodes[$\ell$].nonces $\leftarrow \text{NonceSpace}[\log 1+\epsilon \lambda]$ |
| 5: Paths = $\emptyset$ |
| 6: for $\ell \in [\text{NumOnions}]$ |
| 7: $p = ()$ |
| 8: $s = ()$ |
| 9: CurNode = $G$.leaves[$\ell$].parent |
| 10: for count $\in [h - 1]$ |
| 11: $p$.append(CurNode.parties) |
| 12: $s$.append(CurNode.nonces) |
| 13: CurNode = CurNode.parent |
| 14: if Nonces $\neq \emptyset$ |
| 15: nonce = ⊥ |
| 16: $(t, k, c) \leftarrow \text{Nonces (w/o replacement)}$ |
| 17: if $t > h_{\text{sec}} \log 1+\epsilon \lambda$ |
| 18: $p[t \cdot \log 1+\epsilon \lambda] = k$ |
| 19: $s[t \cdot \log 1+\epsilon \lambda] = (\text{ckpt}, c)$ |
| 20: else |
| 21: nonce = $(t, k, c)$ |
| 22: $(p', s') \leftarrow \text{GenBlfyPath}(1^\lambda, \text{Servers}, \text{nonce})$ |
| 23: $p'$.append($p$) |
| 24: $s'$.append($s$) |
| 25: $p'$.append($j$) |
| 26: add $(p', s')$ to Paths |
| 27: return Paths |

| FormRCOs($1^\lambda$, $h$, $h_{\text{sec}}$, Keys, $[N]$, Nonces) |
|---|
| 1: $R = (h + h_{\text{sec}} - 1) \cdot \lceil \log 1+\epsilon \lambda \rceil$ |
| 2: Onions = $\emptyset$ |
| 3: while Nonces $\neq \emptyset$ |
| 4: $p \leftarrow (p_1, p_2, \ldots, p_{R+1}) \leftarrow [N]^{R+1}$ |
| 5: $s \leftarrow (s_1, s_2, \ldots, s_R) \leftarrow \text{NonceSpace}^R$ |
| 6: nonce $\leftarrow \bot$ |
| 7: $(t, k, c) \leftarrow \text{Nonces (w/o replacement)}$ |
| 8: if $t > h_{\text{sec}} \lceil \log 1+\epsilon \lambda \rceil$ |
| 9: $p[t \cdot \log 1+\epsilon \lambda] = k$ |
| 10: $s[t \cdot \log 1+\epsilon \lambda] = (\text{ckpt}, c)$ |
| 11: else |
| 12: nonce $= (t, k, c)$ |
| 13: $(p', s') \leftarrow \text{GenBlfyPath}(1^\lambda, \text{Servers}, \text{nonce})$ |
| 14: $p'$.append($p$) |
| 15: $s'$.append($s$) |
| 16: $o \leftarrow \text{FormOnion}(\bot, p', \text{Keys}, s')$ |
| 17: add $o$ to Onions |
| 18: return Onions |

Figure 4: Pseudocode for GenPaths and FormRCOs.

The new subroutine GenBlfyPath (Figure 5) generates the subpath for the mixing subphase. It takes as input the security parameter $1^\lambda$, the servers Servers (a subset of the participants), and a nonce nonce; and outputs a subpath.

Without loss of generality, let the set Servers of servers be the set $[N]$ of all participants, and let the number $N$ of participants be a power of two.

Similar to $\Pi_\Delta$, the execution phase progresses in epochs, each epoch lasting $\lceil \log 1+\epsilon \lambda \rceil$ rounds. During the mixing subphase, the onions are routed through an iterated butterfly topology $H$ with $\lceil \log 1+\epsilon \lambda \rceil$ iterations and $h_{\text{sec}} = \lceil \log N \rceil$ stages per iteration, and such that the switching nodes at every stage are the $N$ servers. When the servers are indexed base two, each epoch in the mixing
GenBflyPath(1^\lambda, Servers, nonce)

1: if nonce = ⊥ // If message-bearing onion
2: epoch = 1
3: party ← Servers
4: value ← NonceSpace
5: else
6: (epoch, party, value) = nonce
7: \(p_{epoch,1}, p_{epoch,2}, \ldots, p_{epoch,\left\lceil \log^{1+\lambda} \right\rceil} \) ← Subnet(Servers, epoch, party)
8: \(s_{epoch,1}, s_{epoch,2}, \ldots, s_{epoch,\left\lceil \log^{1+\lambda} \right\rceil} \) ← NonceSpace
9: \(p_{epoch,\log^{1+\lambda}} = party\)
10: \(s_{epoch,\log^{1+\lambda}} = (\text{chpt}, \text{value})\)
11: for \(t\) from epoch + 1 to \(h_{\text{opt}} \left\lceil \log^{1+\lambda} \right\rceil / \) Random walk forwards
12: party = \(p_{t-1,\log^{1+\lambda}}\)
13: \(p_{t,1}, p_{t,2}, \ldots, p_{t,\left\lceil \log^{1+\lambda} \right\rceil} \) ← Subnet(Servers, \(t, \) party)
14: \(s_{t,1}, s_{t,2}, \ldots, s_{t,\left\lceil \log^{1+\lambda} \right\rceil} \) ← NonceSpace
15: for \(t\) from 1 to epoch − 1 // Random walk backwards
16: party = \(p_{t+1,1}\)
17: \(p_{t,1}, p_{t,2}, \ldots, p_{t,\left\lceil \log^{1+\lambda} \right\rceil} \) ← Subnet(Servers, \(t, \) party)
18: \(s_{t,1}, s_{t,2}, \ldots, s_{t,\left\lceil \log^{1+\lambda} \right\rceil} \) ← NonceSpace
19: return \((p, s)\)

Figure 5: Pseudocode for the subroutine GenBflyPath. The function Subnet(Servers, \(t, \) party) outputs the set of parties which belong to the subgroup containing party party at epoch \(t\).

subphase corresponds to a bit position of the server label. At every epoch \(t\), every switch \(S\) is connected to another switch \(S'\) with the label that is the same as that of \(S\) in all but the \(t\)-th bit. We call such a pair \((S, S')\) of switches a subnet and denote it \(\text{Subnet}(\text{Servers}, t, S)\) or alternatively, \(\text{Subnet}(\text{Servers}, t, S')\).

In the subroutine GenBflyPath, an onion’s routing subpath is generated by a random walk through \(H\). The random walk provides the locations for the onion at the starting rounds of the epochs in the mixing subphase. Each epoch corresponds to a subnet in \(H\); and for every round \(r\) of an epoch, the onion’s destination at round \(r\) is chosen independently and uniformly at random from the subnet corresponding to the epoch. See Figure 5.

**Execution phase.** During the merging subphase, honest parties process and route onions essentially as specified in the unmodified Batch code for \(\Pi_\Delta\). At every round of the mixing subphase, every honest party peels and processes the onions received in the round. If the round is a diagnostic round (the last round of an epoch), and more than half of the checkpoints that were expected for the round were not received, then the party aborts the protocol. Otherwise, the peeled onions are sent out to their next destinations in random order.

6.2 Security proof sketch

We now briefly explain why \(\Pi_{\text{opt}}\) works.

The purpose of the mixing subphase is to provide the same safety guarantees that the first epoch
of $\Pi_\Delta$ provides (only using much fewer honest onions). These guarantees are two-fold: Firstly, for every honest party, a constant fraction of his/her message-bearing onions survive the mixing subphase (if the adversary does not drop too many honest onions during the mixing subphase). Secondly, the honest onions are mixed by partway through the mixing subphase.

Thus, to prove that $\Pi_\Delta$ is anonymous from the active adversary, it suffices to prove that the protocol provides these two safety guarantees:

Let a subnet be called *okay* if it contains at least one honest party. (A subnet is not okay if both parties in the subnet are corrupted.)

**Guarantee 1.** Fix an honest party $i$.

Party $i$ forms a polylogarithmic number of honest onions (message-bearing onions and honest checkpoint onions). A constant fraction of these honest onions can be received by a corrupted party in the first round of the mixing subphase. However, from Chernoff bounds (Corollary 1 in Appendix B), with overwhelming probability, a constant fraction of these honest onions are first received by an honest server (in an okay subnet).

Let $O_{ok}$ be the set of honest onions received first received by an honest server. The adversary’s best strategy for dropping all the onions in $O_{ok}$ is to do so upfront in the first epoch. Let $\text{Subnets}_{ok}$ be the okay subnets that hold an onion in $O_{ok}$, and let $\mathcal{A}$ be the adversary that, for each subnet in $\text{Subnets}_{ok}$, drops half of the onions held by the subnet during the first epoch. Even if $\mathcal{A}$ doesn’t cause the honest parties to abort the protocol, from Chernoff bounds (Corollary 1 in Appendix B), with overwhelming probability, at least a constant fraction of the onions in $O_{ok}$ survive the mixing subphase.

**Guarantee 2.** We want to prove that there exists some round partway before the end of mixing subphase, such that all honest onions are mixed by round partway. Every honest onion at the midway point (halfway through the mixing subphase) is routed through the virtual butterfly graph a polylogarithmic number of times. From Chernoff bounds (Corollary 1 in Appendix B), with overwhelming probability, for every stage $1 \leq t \leq h_\Delta$ of the butterfly graph, there exists an iteration for which the onion mixed at an okay subnet. Since every stage corresponds to a bit of the server label, with overwhelming probability, the final location of the onion is statistically close to the uniform distribution over the set of all servers.

### 7 Conclusion

We presented the first complexity bounds for onion routing protocols. We showed that in order for an OR protocol to be both robust and anonymous from the active adversary, there exists a party who transmits at least a polylog (in the security parameter) number of onions. We also present a construction $\Pi_\Delta$ nearly matching this bound; it is robust and anonymous, and on average, every party sends a polylog (in the security parameter) number of onions. Using essentially the same proof used to prove our lower bound, it is possible to show that $\Pi_\Delta$ forms the fewest number of onions for simultaneously achieving robustness and anonymity among protocols in which the onions are formed in an initial onion-forming phase.

In this paper, we considered the adverse effects of dropping honest onions. Clearly, our lower bound holds for adversaries who deviate from the protocol in other ways. But what about our upper bound? Could the adversary succeed in learning the input if she were to try other tactics?

First, we note that the adversary does not learn anything by forming additional onions, by modifying existing ones, or by delaying onions (by an arbitrary number of rounds). Second, the adversary cannot mount an "eclipse attack" to skew an honest party’s perception in a meaningful way. The adversary can only persuade an honest party to abort the protocol prematurely when it
is safe to continue, but she cannot convince an honest party to continue when it is unsafe to do so. Third, rerouting onions (e.g., “BGP attacks”) doesn’t help either since we already assume that the adversary can view the traffic on all links, and a corrupted party is unable to peel onions meant for honest parties.

Of course, no protocol (including ours) can provide anonymity with respect to the original set of parties if the adversary can drop all traffic originating from an honest party. (In our analysis, we make the simplifying assumption that this cannot happen; onions are maliciously dropped only at corrupted nodes.) Nonetheless, our protocols are resilient to “denial-of-service” (DoS) attacks; anonymity with respect to a constant fraction (e.g., one-half) of the honest parties can be guaranteed even if the adversary can effectively drop onions at honest parties by mounting a successful DoS attack on them.

Future work. We rely on a few simplifying assumptions to prove anonymity. Following previous work in the area, we assume to be operating in the simple I/O setting. Instructing each party to send one message is easy, but guaranteeing that only one party is instructed to send a message to each party is challenging. Known efficient solutions to this problem are only differentially private (e.g., Vuvuzela’s dialing protocol [vdHLZZ15]); that is, it is not known whether statistically secure solutions exist. Extending our results for the general input set may require essentially solving this problem.

In this paper, we modeled the network as a synchronous and stable complete graph. A natural direction for future work is to extend the results of this paper for asynchronous systems and or systems with high churn.

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A Onion cryptosystem

We begin a formal definition of onion cryptosystems and conforming onion routing by recalling the convenient programming convention of a data type. In most programming languages, variables are classified by type. Since we are only worried about the programs that are run by honest participants, we may assume that the code of an onion cryptosystem and an onion routing protocol can be written in a programming language that recognizes the onion data types defined below:

Definition 5. An onion-ready programming language is a programming language that recognizes the following six data types: SkType (variables of this type are secret keys for an onion cryptosystem),PkType (for public keys for an onion cryptosystem), NodeNameType (for names of mix-nodes in an onion routing scheme), OnionType (for the onion ciphertexts), MessageType (for messages), OtherType (for all other data, including more complex data structures with sub-fields of the above five types).

Definition 6 (Onion cryptosystem). Let (KeyGen, Wrap, Peel) be algorithms, written in an onion-ready programming language. If they satisfy the following three conditions, then we call them an onion cryptosystem:

Input/Output Let 1^k be the security parameter. Let \( M(1^k) \) be the message space (and any value drawn from it is of type MessageType), \( N(1^k) \) be the space of names for the mix nodes (so any value drawn from it is of type NodeNameType), and \( D(1^k) \) be some finite space such that \( M(1^k) \subseteq D(1^k) \) (so that values drawn from it may be of type MessageType or of type OtherType).

- KeyGen takes 1^k as input and outputs (PkType pk, SkType sk).
- Wrap takes three inputs: a vector of public keys \( pk = (PkType pk_0, \ldots, PkType pk_\ell) \), a vector of names \( name = (NodeNameType name_1, \ldots, NodeNameType name_\ell) \), and a piece of data \( d \in D(1^k) \) (so \( d \) can be of type MessageType or of type OtherType). It outputs a vector of onions \( onion = (onion_0, \ldots, onion_\ell) \). \( \ell \geq 0 \) is a parameter; Wrap need
not be well-defined for all $\ell$ but it needs to be well-defined for $\ell = 0$. Input data $d$ may be a data structure of type OtherType; in that case, it may contain at most one sub-structure of type OnionType, and no sub-structures of type MessageType. Input data $d$ may also be of type MessageType, in which case it contains a unique message $m$.

- **Peel** takes as input a secret key $(SkType sk)$ and an onion $(OnionType onion)$ and outputs either $(NodeNameType name, OnionType onion')$ (i.e. an onion whose intended recipient is the mix node whose name is name), or message $(MessageType message)$ (i.e. the intended recipient is the holder of sk so it recovered the message $m$) or (OtherTyped) (i.e. the intended recipient is the holder of sk and it recovered the data $d$), or Wrap fails.

Correctness $(KeyGen, Wrap, Peel)$ are correct over $(M, N, D)$ if for all $k$, for all $(pk, sk) = (pk_0, sk_0), \ldots, (pk_\ell, sk_\ell) \in \text{KeyGen}(1^k)^{\ell+1}$, for all $(name_0, \ldots, name_\ell) \in N^\ell$, for all $d \in D$, for all $(onion_0, \ldots, onion_\ell) \in \text{Wrap}(pk, name, d)$,

- Peel$(sk_\ell, onion_\ell) = (name_{i+1}, onion_{i+1})$ for all $i \in \{0, \ldots, \ell - 1\}$, and
- Peel$(sk_\ell, onion_\ell) = d$.

**Uniqueness of onions** This property ensures that if an onion was created by an honest party, it is straightforward to identify, by checking all honest parties’ computation histories, which honest party created it and on what call to Wrap.

Formally, we require that the probability that a polynomial-time adversary $A$ wins in the following experiment is a negligible function of the security parameter: (1) key pairs $(pk_i, sk_i)$ are sampled using $\text{KeyGen}(1^k)$ and given to $A$ until $A$ directs this process to stop; (2) $A$ selects two sequences of the public keys, $pk = (pk_{a_0}, \ldots, pk_{a_\ell})$ and $pk' = (pk'_{b_0}, \ldots, pk'_{b_\ell})$, two sequences of names $name = (name_1, \ldots, name_\ell)$ and $name' = (name'_1, \ldots, name'_\ell)$ (each name is in $N$) and two data inputs $d, d' \in D$; note that we do not require that these values be distinct; (3) two vectors of onions are sampled as follows: onion = $(onion_0, \ldots, onion_\ell) \leftarrow \text{Wrap}(pk, name, d)$, onion' = $(onion'_0, \ldots, onion'_\ell) \leftarrow \text{Wrap}(pk', name', d')$; (4) $A$ wins if for some $i$ and $j$, onion$_i$ = onion'$_j$.

Note that if the uniqueness requirement is not satisfied, then the onion cryptosystem is either incorrect or insecure. If a collision can be produced for inputs where $(pk, name, i) \neq (pk', name', j)$ or $d \neq d'$, then it is easy to see that the cryptosystem does not satisfy correctness. If a collision cannot be produced for such diverging inputs, but can be produced for inputs where $(pk, name) = (pk', name')$ and $d = d'$, then the onion cryptosystem does not satisfy semantic security: to distinguish if an onion $onion$ was formed on input $d_0 = d$ for which a collision occurs non-negligibly or on input $d_1 \neq d$, a distinguisher samples onion' = Wrap$(pk, name, d)$; if onion'$_i$ = onion$_i$ it outputs 0 else it outputs 1.

Therefore, any correct and at least semantically secure cryptosystem would satisfy our uniqueness requirement.

**Definition 7** (Conforming onion routing scheme). An $N$-party protocol $P$ using an onion cryptosystem $(KeyGen, Wrap, Peel)$ is a conforming onion routing scheme if the program executed by an honest participant $P$ in $P$ satisfies the following rules:

- Variables of type PkType and SkType can only be created by a call to KeyGen or given as input or sent to $P$. Variables of type NodeNameType can only be sampled from $N$ or given as input or sent to $P$. Variables of type MessageType must be given as input or sent to $P$. Variables of type OnionType may only be created by a call to Wrap or sent to $P$. The creation of variables of type OtherType is not restricted.
• Once created, variables of type $Pkm$, $SkType$, $Message$, $NodeName$, $OnionType$ can be sent to other protocol participants, copied, deleted, and incorporated into more complex data structures, but they must be treated atomically: that is they cannot be broken up into individual bits and transformed in any way by any algorithm other than $Wrap$ and $Peel$, and the way that they are given to $Wrap$ and $Peel$ must conform to the input/output type specification restrictions described in Definition 6. Variables of type $OtherType$ are not subject to the atomicity restriction; but if a data structure of type $OtherType$ contains a sub-structure of a restricted type, the restriction applies to the sub-structure. (For example, an array of onions would be of type $OtherType$, but each individual onion in this array is subject to the atomicity restriction.)

B Useful Chernoff bounds

We make use of the following bounds, which allow us to make the arguments that certain favorable events occur with overwhelming probability as opposed to merely occurring in expectation.

Lemma 7. [ALU18 Lemma 16a] Let $λ$ be a (security) parameter. Let $N$ be polynomially bounded in $λ$; that is, for sufficiently large $λ$, $N \leq \text{poly}(λ)$. Let $ε > 0$ be any positive constant. If $N$ balls are thrown independently and uniformly at random into $n = O(N/\log^{1+ε} λ)$ bins, then for any $δ \in (0, 1]$, the number of balls thrown into any bin is at least $(1 - δ)N/n$ and at most $(1 + δ)N/n$ with overwhelming probability in $λ$.

Sketch. For any fixed bin, the number of balls that falls into the bin is sharply concentrated at the expected value (Chernoff for Poisson trials). Since the number of bins is polynomially bounded, the probability that any bin contains too few or too many balls is also negligibly small (union bound).

Corollary 1. [ALU18 Lemma 16b] Let $λ$ be a (security) parameter. Let $N$ be polynomially bounded in $λ$. Let $ε > 0$ be any positive constant. If $N$ balls are thrown independently and uniformly at random into $n = O(N/\log^{1+ε} λ)$ bins, then for any $δ \in (0, 1]$, the number of balls thrown into any set of $k$ bins is at least $(1 - δ)kN/n$ and at most $(1 + δ)kN/n$ with overwhelming probability in $λ$.

Sketch. Assume for the sake of reaching a contradiction that with non-negligible probability, there is a set of $k$ bins with either less than $(1 - δ)kN/n$ balls or more than $(1 + δ)kN/n$ balls. This implies that with non-negligible probability, there is a bin with either $(1 - δ)N/n$ balls or more than $(1 + δ)N/n$ balls (pigeonhole principle), which contradicts Lemma 7.

C Supplementary proofs for Section 4

Proof. [of Lemma 1] Let $Π$ be an anonymous OR protocol.

For the sake of reaching a contradiction, assume that there exist an adversary $A_{S} \in \text{ActiveAdv}$ (using fixed randomness $ζ_{c}$ for choosing which parties to corrupt) and input vectors $σ^{0}, σ^{1} \in \text{SimpleIO}$ that differ only on the honest parties' inputs and outputs, such that $V^{Π,A_{S}\cdot\text{Add} (σ^{0})} ≠ s V^{Π,A_{S}\cdot\text{Add} (σ^{1})}$. That is, there exists a party $\text{recipient}$ such that $in_{\text{recipient}}^{Π,A_{S}\cdot\text{Add} (σ^{0})} ≠ s in_{\text{recipient}}^{Π,A_{S}\cdot\text{Add} (σ^{1})}$.

Let $Bad_{S}$ denote the set of parties corrupted by the adversary $A_{S}$.

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If \( \text{recipient} \in \text{Bad}_s \), then \( \text{in}^{\Pi_A, O_{\text{Bad}}} \) is part of the filtered adversarial view \( \mathcal{V}^{\Pi_A, O_{\text{Bad}}} (\sigma^0) \) on input \( \sigma^0 \), and \( \text{in}^{\Pi_A, O_{\text{Bad}}} (\sigma^1) \) is part of the filtered adversarial view \( \mathcal{V}^{\Pi_A, O_{\text{Bad}}} (\sigma^1) \) on input \( \sigma^1 \). In this case, \( \mathcal{V}^{\Pi_A, O_{\text{Bad}}} (\sigma^0) \neq \emptyset \), \( \mathcal{V}^{\Pi_A, O_{\text{Bad}}} (\sigma^1) \).

If \( \text{recipient} \notin \text{Bad}_s \), then there exists another adversary \( A'_s \) such that \( A'_s \) can distinguish \( \mathcal{V}^{\Pi_A, O_{\text{Bad}}} (\sigma^0) \) from \( \mathcal{V}^{\Pi_A, O_{\text{Bad}}} (\sigma^1) \). The adversary \( A'_s \) corrupts every party in \( \text{Bad}_s \) and additionally corrupts a party chosen uniformly at random from the set \( [N] \setminus \text{Bad}_s \) of remaining parties. Otherwise, \( A'_s \) behaves just like the original adversary \( A_s \). (The additionally corrupted party acts as though she were honest.)

**Proof.** Let \( A \in \text{ActiveAdv} \) be any active adversary, and let \( i \) be any party.

From Markov’s inequality, \( \Pr[\sigma \mid \text{out}^{\Pi_A} (\sigma) \geq 2E_{\sigma,S} \left[ \text{out}^{\Pi_A} (\sigma) \right]] \leq \frac{1}{2} \). Since \( \Pi \) has \( o(N) \) onion cost per user in the presence of the adversary \( A \), there exist an input vector \( \sigma^0 \in \text{SimpleIO} \) such that \( E_S \left[ \text{out}^{\Pi_A} (\sigma^0) \right] < 2E_{\sigma,S} \left[ \text{out}^{\Pi_A} (\sigma) \right] < \frac{N}{2} \) (for sufficiently large \( N \)).

Since \( E_S \left[ \text{out}^{\Pi_A} (\sigma^0) \right] < \frac{N}{2} \), there are at most \( N - 1 \) distinct party \( j' \neq i' \), such that

\[
E \left[ \text{hops}^{\Pi_A}_{j' \rightarrow \text{recipient}(j)} (\sigma^0) \right] \geq \frac{1}{2}.
\]

(If this weren’t the case, then the expected number of onions that party \( i \) transmits would be at least \( \frac{N}{2} \).

**D Supplementary proofs for Section 5.3.1**

**Proof.** Let \( D = |\mathcal{O}| \) denote the size of \( \mathcal{O} \).

We recast this problem as a balls-and-bins problems, where the onions in \( \mathcal{O} \) are the balls, and the participants (or locations) are the bins. To prove the lemma, we show that when \( D = |\mathcal{O}| \) balls are thrown independently and uniformly at random into the \( N' \) bins, with overwhelming probability in \( \lambda \), the number of non-empty bins is at least \( \frac{D}{\log \lambda} \).

Let \( b_1, \ldots, b_{N'} \) denote the \( N' \) bins. For each \( i \in [N'] \), let \( X_i \) denote the indicator random variable that is one if the \( i \)-th bin \( b_i \) is empty (and zero, otherwise). The probability that \( b_i \) remains empty is given as \( \Pr[X_i = 1] = \mathbb{E}[X_i] = (1 - \frac{1}{N'})^D \). The total number \( X = \sum_{i=1}^{N'} X_i \) of empty bins is the summation of all the \( X_i \)'s. By the linearity of expectation, \( \mathbb{E}[X] = \sum_{i=1}^{N'} \mathbb{E}[X_i] = N' \left( 1 - \frac{1}{N'} \right)^D \).

Let \( W \) be the total number of non-empty bins; i.e., \( W = N' - X \). Again by linearity of expectation,

\[
\mathbb{E}[W] = N' - \mathbb{E}[X] \\
= N' - N' \left( 1 - \frac{1}{N'} \right)^D \\
> N' - N' \left( 1 - \frac{D}{N'} + \frac{D(D-1)}{N'^2} \right) \\
= D \left( 1 - \frac{D-1}{N'} \right) \\
\geq \frac{D}{2},
\]

(10)
where (10) holds since \((1 - \frac{1}{N'})^D\) is strictly less than the first three terms of its Laurent series, and (11) holds since \(D \leq \frac{N'+2}{2}\) by the hypothesis.

For every \(i \in [D]\), let \(Y_i\) be the location of the \(i\)-th ball, and let \(Z_i = \mathbb{E}[X|Y_1, \ldots, Y_i]\). The sequence \(Z_1, \ldots, Z_D\) is a Doob martingale by construction, satisfying the Lipschitz condition with constant bound one, i.e., \(|Z_i - Z_{i-1}| \leq 1\). Thus, we may apply the Azuma-Hoeffding inequality as follows: For \(\delta \geq \log \lambda - 2\log \lambda\),

\[
\Pr[X - \mathbb{E}[X] \geq \delta \mathbb{E}[W]] \leq \exp \left(-\frac{\delta^2 \mathbb{E}[W]^2}{2 \sum_{i=1}^D 1}\right) \leq \exp \left(-\frac{\delta^2 D/2^2}{2D}\right) = \exp \left(-\frac{\delta^2 D}{8}\right) \leq \exp \left(-\frac{(\log \lambda - 2)^2 \log^2 \lambda}{8 \log^2 \lambda}\right) \leq \exp \left(-\frac{(\log \lambda - 2)^2}{8}\right) = \text{negl}(\lambda),
\]

where (12) follows directly from (11), and (13) holds since \(\delta \geq \log \lambda - 2\log \lambda\), and \(D \geq \log^2 \lambda\) from the hypothesis.

In other words, with overwhelming probability in \(\lambda\), \(X - \mathbb{E}[X] \leq \delta \mathbb{E}[W]\). Thus, it follows that, with overwhelming probability in \(\lambda\),

\[
W \geq (1 - \delta) \mathbb{E}[W] = \left(1 - \frac{\log \lambda - 2}{\log \lambda}\right) \mathbb{E}[W] = \frac{2}{\log \lambda} \mathbb{E}[W] \geq \frac{D}{\log \lambda},
\]

where (14) follows directly from (11).

**Proof.** [of Lemma 4] We recast this problem as a balls-and-bins problem with red balls, white balls, and green balls. The different categories of balls correspond to different categories of onions (explained below). The bins correspond to the \(N\) parties.

Recall that \(\kappa\) is the fraction of the parties that are corrupted. The number of honest parties is \((1 - \kappa)N\). Each honest party forms \(\text{NumMBOs} = \Omega\left(\log^{2(1+\epsilon)} \lambda\right)\) message-bearing onions. Thus, there are \(X = (1 - \kappa)\text{NumMBOs}N = \Omega\left(N \log^{2(1+\epsilon)} \lambda\right)\) honest message-bearing onions. (These correspond to the red onions/balls.)

We now consider the honest dummy onions.

For every triple \((i, j, u)\) consisting of a diagnostic round \(u\) and honest parties \(i\) and \(j\), let \(Y_{i \rightarrow i}^u\) be one if party \(i\) forms a dummy onion to be verified by party \(j\) at the \(u\)-th diagnostic round (and zero, otherwise).
Fix an honest party $k$ and a round $v$. Let $Z$ denote the set of all dummy onions for verification by party $k$ at the $v$-th diagnostic round. (These correspond to the green onions/balls. All other dummy onions correspond to the white onions/balls.) Let $Z = |Z|$.

If $F$ is a truly random function, the onions in $Z$ are i.i.d. Bernoulli random variables, each having probability $q = \frac{2\text{NumMCOs}}{N}$ of success. It follows that $Z \sim \text{Binomial}((1 - \kappa)N, q)$.

Using Chernoff bounds for Poisson trials, for any $0 < \delta_1 \leq 1$: \[
\Pr[|Z - \mathbb{E}[Z]| > \delta_1 \mathbb{E}[Z]] \leq 2 \exp\left(-\Omega\left(\log^{2(1+\epsilon)} \lambda\right)\right) = \text{negl}(\lambda). \tag{15}
\]

Thus, with overwhelming probability in $\lambda$, $Z$ falls between $(1 - \delta_1)\mathbb{E}[Z] = 2(1 - \delta_1)(1 - \kappa)\frac{\text{NumMCOs}}{h}$ and $(1 + \delta_1)\mathbb{E}[Z] = 2(1 + \delta_1)(1 - \kappa)\frac{\text{NumMCOs}}{h}$.

Consider the honest onions at the $v$-th diagnostic round: Let the message-bearing onions be the red onions/balls, let the dummy onions with checkpoints for verification by party $k$ be the green onions/balls, and let all other dummy onions be the white onions/balls.

Let $\mathcal{H}$ denote the set of all honest onions, i.e., the set of all red, white, and green onions/balls.

Since the adversary cannot distinguish between any two honest onions, the cumulative set $\mathcal{E} \subseteq \mathcal{H}$ of honest onions eliminated (or dropped) by the adversary by round $v$ is a random subset of the set $\mathcal{H}$ of all honest onions.

Let $\zeta = \frac{|\mathcal{E}|}{|\mathcal{H}|}$ denote the fraction of onions in $\mathcal{H}$ dropped by the adversary by round $v$. Using a known concentration bound for the hypergeometric distribution, when the expected number $\zeta Z$ of green balls in the random sample $\mathcal{E}$ is at least polylogarithmic in $\lambda$, i.e., $\zeta Z = \Omega(\log^{1+\epsilon} \lambda)$, the actual number $W$ of green balls in $\mathcal{E}$ is close to the expected value $\zeta Z$, i.e., for any $0 < \delta_2 \leq 1$, $(1 - \delta_2)\zeta Z \leq W \leq (1 + \delta_2)\zeta Z$. Combining this with (15) above, with overwhelming probability in $\lambda$, the number of green balls in the random sample $\mathcal{E}$ falls between $2(1 - \delta_1)(1 - \delta_2)(1 - \kappa)\zeta h \frac{\text{NumMCOs}}{h}$ and $2(1 - \delta_1)(1 - \delta_2)(1 - \kappa)\zeta h \frac{\text{NumMCOs}}{h}$.

By choosing an appropriate $\delta$ such that $1 - \delta \leq (1 - \delta_1)(1 - \delta_2)$ and $1 + \delta \geq (1 + \delta_1)(1 + \delta_2)$, we obtain our desired bounds. \hfill \square

### E Supplementary proofs for Section 5.3.2

Lemma 8 says that dropping a fraction of the dummy onions is reflected in the number of missing checkpoints at every honest party.

**Lemma 8.** For all $0 < d \leq 1$, with overwhelming probability:

1. The total number of regular checkpoint onions formed by honest parties is at least $(1 - d)(1 - \kappa)\text{NumMCOs}N$.

2. If $\text{RCO}_t$ fraction of all regular checkpoint onions are dropped by the $t$-th epoch, then every honest party notices $\left(1 - \frac{d}{1 + d}\right)^2 (1 - \frac{1}{2^{1+\epsilon}} + \text{RCO}_t) h \frac{\text{NumMCOs}}{h}$ missing checkpoints at the $t$-th diagnostic round.

3. If at any epoch $2 \leq t \leq h$, the fraction of all merging checkpoint onions dropped by the $t$-th epoch is $1 - \frac{1}{2^{1+\epsilon}} + \text{MCO}_t$, then every honest party notices $\left(1 - \frac{d}{1 + d}\right)^2 (1 - \frac{1}{2^{1+\epsilon}} + \text{MCO}_t) h \frac{\text{NumMCOs}}{h}$ missing checkpoints at the $t$-th diagnostic round.

**Proof.** Recall that $\kappa$ is the fraction of the parties that are corrupted. The number of honest parties is $(1 - \kappa)N$. Each honest party forms $\text{NumMCOs} = \log^{1+\epsilon} \lambda$ message-bearing onions. Thus, there are $X = (1 - \kappa)\text{NumMCOs}N$ honest message-bearing onions.
We now consider the honest dummy onions.

For every triple \((i, j, \tau)\) consisting of a diagnostic round \(\tau\) and honest Parties \(i\) and \(j\), let \(Y_{i \rightarrow i}^\tau\) be one if Party \(i\) forms a dummy onion to be verified by Party \(j\) at the \(\tau\)-th diagnostic round (and zero, otherwise).

Fix a Party \(k\) and a round \(t\). Let \(Z\) denote the set of all dummy onions for verification by Party \(k\) at the \(t\)-th diagnostic round. Let \(Z = |Z|\).

Let \(\mathcal{Y}_1\) be the set of all (for all \(i\)’s and all \(\tau\)’s) dummy onions formed by Party \(i\) for Party \(i\) at round \(\tau\), excluding the (possible) onion formed by Party \(k\) for verification by Party \(k\) at round \(t\). Let \(Y_1' = |\mathcal{Y}_1|\).

Let \(\mathcal{Y}_2\) be the set of all (for all \(i\)’s, all \(j\)’s, and all \(\tau\)’s) dummy onions formed by Party \(i\) for Party \(j > i\), excluding any onion for verification by Party \(k\) at round \(t\) as well as any onion formed by Party \(k\) for verification at round \(t\). Let \(Y_2' = |\mathcal{Y}_2|\).

Since \(Y_{i \rightarrow j}^\tau = 1 \iff Y_{j \rightarrow i}^\tau = 1\) (i.e., Party \(i\) creates an onion to be verified by Party \(j\) at the \(\tau\)-th epoch iff Party \(j\) creates a symmetric onion to be verified by Party \(i\) at the \(\tau\)-th epoch), it follows that the total (over all \(i\)’s, all \(j\)’s, and all \(\tau\)’s) number of dummy onions formed by Party \(i\) for Party \(j \neq i\) is \(2 \left( Y_2' + \sum_{i \neq k} Y_{i \rightarrow k}^\tau \right)\).

Let \(Y_1 = Y_1' + Y_{k \rightarrow k}^\tau\), and let \(Y_2 = Y_2' + \sum_{i \neq k} Y_{i \rightarrow k}^\tau\). The total number \(Y\) of dummy onions is given by

\[
Y = (Y_1' + Y_{k \rightarrow k}^\tau) + 2 \left( Y_2' + \sum_{i \neq k} Y_{i \rightarrow k}^\tau \right) = Y_1 + 2Y_2.
\]

If \(F\) is a truly random function, the onions in \(Z \cup \mathcal{Y}_1 \cup \mathcal{Y}_2\) are i.i.d. Bernoulli random variables, each having probability \(q = \frac{2^{\text{NumMCOs}}}{N_h}\) of success. It follows that

\[
Z \leftarrow \text{Binomial}((1 - \kappa)N, q), \\
Y_1 \leftarrow \text{Binomial}((1 - \kappa)Nh, q) , \\
Y_2 \leftarrow \text{Binomial}\left(\left(\frac{(1 - \kappa)N}{2}\right)h, q\right).
\]

Using Chernoff bounds for Poisson trials, for any \(0 < d \leq 1\):

\[
\Pr[|Z - \mathbb{E}[Z]| > d \mathbb{E}[Z]] \leq 2 \exp\left(-\Omega\left(\log^{2(1+\epsilon)} \lambda\right)\right) = \text{negl}(\lambda) \tag{16}
\]

\[
\Pr[|Y_1 - \mathbb{E}[Y_1]| > d \mathbb{E}[Y_1]] \leq 2 \exp\left(-\Omega\left(\log^{2(1+\epsilon)} \lambda\right)\right) = \text{negl}(\lambda) \tag{17}
\]

\[
\Pr[|Y_2 - \mathbb{E}[Y_2]| > d \mathbb{E}[Y_2]] \leq 2 \exp\left(-\Omega\left(\log^{2(1+\epsilon)} \lambda\right)\right) = \text{negl}(\lambda). \tag{18}
\]

Thus, with overwhelming probability in \(\lambda\), \(Z\) falls between \((1 - d) \mathbb{E}[Z] = (1 - d)(1 - \kappa)\frac{2^{\text{NumMCOs}}}{N_h}\) and \((1 + d)(1 - \kappa) \mathbb{E}[Z] = (1 + d)\frac{2^{\text{NumMCOs}}}{N_h}\), \(Y_1\) falls between \((1 - d) \mathbb{E}[Y_1]\) and \((1 + d) \mathbb{E}[Y_1]\), and \(Y_2\) falls between \((1 - d) \mathbb{E}[Y_2]\) and \((1 + d) \mathbb{E}[Y_2]\).
It follows that,
\[ Y = (Y_1 + Y_{k ightarrow k}^v) + 2(Y_2 + Z) \]
\[ \leq (1 + d)(E[Y_1] + 2E[Y_2]) \]
\[ = (1 + d) \left( (1 - \kappa)Nh + 2 \left( \frac{(1 - \kappa)N((1 - \kappa)N - 1)}{2}h \right) \right)q \]
\[ = (1 + d)(1 - \kappa)^2N^2h \left( \frac{2\text{NumMCOs}N}{Nh} \right) \]
\[ = (1 + d) \cdot 2(1 - \kappa)^2\text{NumMCOs}N, \]
where (19) follows (16)-(18), and (20) holds because \( E = (1 - \kappa)Nhq \) and \( E[Y_2] = \left( \frac{(1 - \kappa)N((1 - \kappa)N - 1)}{2}h \right)q \). (Following a similar argument, we have \( Y \geq (1 - d) \cdot 2(1 - \kappa)^2\text{NumMCOs}N \).)

Let \( Q \overset{\text{def}}{=} Y - (1 - \kappa)^2\text{NumMCOs}N \) denote the number of (honest) regular checkpoint onions\(^9\) and let \( \text{DO}_t \) be the fraction of all dummy onions that are dropped by the \( t \)-th epoch, including (honest) merging checkpoint onions.

\[ \text{DO}_t \geq \left( 1 - \frac{1}{2^{t-1}} \right) \frac{(1 - \kappa)^2\text{NumMCOs}N}{Y} + Q \cdot \text{RCO}_t \]
\[ \geq \left( 1 - \frac{1}{2^{t-1}} \right) \frac{(1 - \kappa)^2\text{NumMCOs}N}{(1 + d) \cdot 2(1 - \kappa)^2\text{NumMCOs}N} \]
\[ + \left( \frac{(1 - d) \cdot 2(1 - \kappa)^2\text{NumMCOs}N - (1 - \kappa)^2\text{NumMCOs}N}{(1 + d) \cdot 2(1 - \kappa)^2\text{NumMCOs}N} \right) \text{RCO}_t \]
\[ \geq \left( 1 - \frac{2d}{1 + d} \right) \left( 1 - \frac{1}{2^{t-1}} + \text{RCO}_t \right) \left( \frac{1}{2} \right). \]

In other words, if \( \text{RCO}_t \) fraction of the regular checkpoint onions are dropped, then with overwhelming probability, at least \( \frac{1 - 2d}{1 + d} \) fraction of all dummy onions are dropped.

We recast this problem as a balls-and-bins problem. Consider the honest onions/balls at the \( t \)-th diagnostic round: Let the dummy onions with checkpoints for verification by Party \( k \) be the green onions/balls, and let all other dummy onions be the white onions/balls. Let \( \mathcal{H} \) denote the set of all honest dummy onions, i.e., the set of all green and white onions/balls.

Since the adversary cannot distinguish between any two onions in \( \mathcal{H} \), the cumulative set \( \mathcal{E} \subseteq \mathcal{H} \) of honest onions eliminated (or dropped) by the adversary by round \( t \) is a random subset of the set \( \mathcal{H} \) of all honest onions.

Using a known concentration bound [HS05] for the hypergeometric distribution, when the expected number \( \text{DO}_t Z = \Omega \left( \log^{1 + \epsilon} \lambda \right) \) of green balls in the random sample \( \mathcal{E} \) is at least polylog in \( \lambda \), the actual number \( W \) of green balls in \( \mathcal{E} \) is close to the expected value \( \text{DO}_t Z \), i.e., \( (1 - d)\text{DO}_t Z \leq W \leq (1 + d)\text{DO}_t Z \). Combining this with (16) and (21) above, with overwhelming probability in \( \lambda \), the number of green balls in the random sample \( \mathcal{E} \) is at least \( \frac{(1 - d)(1 - 2d)}{1 + d} \left( 1 - \frac{1}{2^{t - 1}} + \text{RCO}_t \right) \frac{\text{NumMCOs}}{h} \).

This proves Condition 2 of the theorem statement. Condition 3 follows using a similar argument.

\[ \square \]

At every epoch, the adversary observes a set of paired onions and a set of singleton onions. We first relate the cumulative fraction of regular checkpoint onions that are dropped by the \( t \)-th epoch to the fractions of all singletons dropped in the epochs (i.e., \( \beta_1, \beta_2, \ldots, \beta_t \)).

\[ ^9 \text{Recall that there are exactly} (1 - \kappa)^2\text{NumMCOs}N \text{ merging checkpoint onions.} \]
Lemma 9. In $\Pi_\Delta$: For every epoch $1 \leq t \leq h/2$, let $\beta_t$ be the fraction of the singletons that the adversary drops during the $t$-th epoch. Let $\rho_0 = 0$, and recursively define $\rho_t = \sum_{\tau=1}^{t}(1 - \rho_{t-1})\beta_{\tau}$. Let $\text{sdo}_t$ be the fraction of regular checkpoint onions (remaining at the $t$-th epoch) that are dropped during the $t$-th epoch. Let $0 \leq \text{RCO}_t \leq 1$ be the cumulative number of regular checkpoint onions that are dropped by the $t$-th epoch, divided by the total number of regular checkpoint onions at the first round (before anything is dropped).

For all $0 < d \leq 1$, with overwhelming probability in $\lambda$, $(1 - d)\rho_t \leq \text{RCO}_t \leq (1 + d)\rho_t$.

Proof. We prove this by induction; we show that for all $0 < d \leq 1$ and for all $1 \leq t \leq h/2$,

$$(1 - d)\rho_t \leq \text{sdo}_t \leq (1 + d)\rho_t.$$ 

Base case: This follows from a known concentration bound \cite{HS05} for the hypergeometric distribution.

Inductive step: Assume that $(1 - d)\rho_{t-1} \leq \text{sdo}_{t-1} \leq (1 + d)\rho_{t-1}$. We obtain the lower bound for the $t$-th epoch as follows:

$$\text{RCO}_t = \text{sdo}_{t-1} + (1 - \text{sdo}_{t-1})\text{sdo}_t \geq (1 - d)\rho_{t-1} + (1 - (1 + d)\rho_{t-1})\text{sdo}_t \geq (1 - d)\rho_{t-1} + (1 - d)(1 - (1 + d)\rho_{t-1})\beta_t \geq (1 - d)\rho_{t-1} + (1 - d)\beta_t - (1 - d)\rho_{t-1}\beta_t = (1 - d)\rho_t,$$

where (22) follows from the inductive hypothesis, (23) follows from a known concentration bound \cite{HS05} for the hypergeometric distribution, and (24) follows because $1 - d^2 \geq 1 - d$.

We obtain the upper bound using a symmetric argument. \hfill $\square$

Lemma 10 and Corollary 2 (below) are the concentration bounds for the numbers of paired onions and the numbers of singleton onions, respectively.

Lemma 10. Let $U$ be a set of $2u$ balls paired into $u = \Omega\left(\log^{1+\epsilon} \lambda\right)$ distinct pairs of balls, and let $V$ be a random subset of $U$, such that $\zeta \overset{\text{def}}{=} \frac{|V|}{|U|}$ is a constant factor. For any constant $\frac{2v - \zeta}{2v - 1} - 1 < \epsilon \leq 1$, with overwhelming probability in $\lambda$, the number $W$ of paired balls in $V$ is at least $(1 - \epsilon)\zeta|V|$ and at most $(1 + \epsilon)\zeta|V|$.

Proof. Let $v \overset{\text{def}}{=} \frac{|V|}{2}$. For every $i \in [u]$, let $\omega_i$ be one if both onions that comprise the $i$-th pair in $U$ are in $V$, and zero otherwise.

$$\mathbb{E}[\omega_i] = \Pr[\omega_i = 1] = \frac{\binom{2u-2}{2v-2}}{\binom{2u}{2v}} = \frac{(2u - 2)!}{(2v - 2)!(2u - 2v)!} \cdot \frac{(2v)! (2u - 2v)!}{(2u)!} = \frac{2v(2v - 1)}{2u(2u - 1)},$$

since there are $\binom{2u-2}{2v-2}$ ways to choose $2v - 2$ balls from $2u - 2$ balls; and likewise, there are $\binom{2u}{2v}$ ways of choosing $2v$ balls from $2u$ balls.

Let $\omega$ denote the number of pairs in $V$ From the linearity of expectation,

$$\mathbb{E}[\omega] = \sum_{i=1}^{u} \mathbb{E}[\omega_i] = u \cdot \frac{2v(2v - 1)}{2u(2u - 1)} = \frac{2v - 1}{2u - 1} \cdot v.$$
Recall that \( \zeta \overset{\text{def}}{=} \frac{u}{v} = \frac{|V|}{|U|} \). It follows that
\[
E[W] = (\frac{2v - 1}{2u - 1}) |V| = \left( \frac{2v - 1}{2v - \zeta} \right) \left( \frac{\zeta(2u - 1)}{2u - 1} \right) |V| = \left( \frac{2v - 1}{2v - \zeta} \right) \zeta |V|.
\]

For each \( i \in [2v] \), let \( Y_i \) be the \( i \)-th chosen ball in \( V \), and let
\[
Z_i = E[W|Y_1, Y_2, \ldots, Y_i].
\]

Then, \( Z_0, Z_1, \ldots, Z_{2v} \) is a Doob martingale by construction satisfying the Lipschitz condition with bound \( 1 \). Thus, from the Azuma-Hoeffding inequality,
\[
\Pr[|W - E[W]| \geq \delta E[W]] \leq 2 \exp \left( -\frac{\delta^2 E[W]^2}{\sum_{i=1}^{2v} 1} \right) = 2 \exp \left( -\frac{\delta^2 (2v - 1)^2}{(2u - 1)^2} \cdot 2v \right) = 2 \exp (\Theta(1) \cdot 2v) = \text{negl}(\lambda).
\]

This completes our proof. \( \square \)

**Corollary 2.** Let \( U \) be a set of \( 2u \) balls paired into \( u = \Omega(\log^{1+\epsilon} \lambda) \) distinct pairs of balls, and let \( V \) be a random subset of \( U \), such that \( \zeta \overset{\text{def}}{=} \frac{|V|}{|U|} \) is a constant factor. For any constant \( \frac{2v - \zeta}{2u - 1} - 1 < \epsilon \leq 1 \), with overwhelming probability in \( \lambda \), the number \( X \) of unpaired balls in \( V \) is at least \((1 - \epsilon)(1 - \zeta)|V|\) and at most \((1 + \epsilon)(1 - \zeta)|V|\).