ON THE C-VECTORS AND G-VECTORS OF THE MARKOV CLUSTER ALGEBRA

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Abstract. We describe the $c$-vectors and $g$-vectors of the Markov cluster algebra in terms of a special family of triples of rational numbers, which we call the Farey triples.

Introduction

In papers such as [3, 4, 11, 12, 15] (to mention just a few), the authors present (as examples and propositions) interesting properties of the cluster algebra arising from the quiver of Figure 1, as well as of the potentials, Jacobian algebras and cluster categories associated to it.

Figure 1. Double cyclic triangle

The cluster algebra arising from this quiver is frequently called the Markov cluster algebra. When this algebra has principal coefficients we denote it by $A_M$. In this paper, we describe the exchange matrices of $A_M$ and thus the set of its $c$-vectors, which can be considered as a generalization of a root system. In [13] and [14, Theorem 1.2], the authors show how the $c$-vectors of a cluster algebra associated to a quiver are related to its $g$-vectors. Thus, in our case, we can obtain the $g$-vectors. Section 1 is a reminder on cluster algebras with principal coefficients. In section 2 we introduce the Farey triples, which are triples of rational numbers satisfying a specific arithmetic condition. We define a mutation operation on Farey triples and show that the resulting exchange graph is a 3-regular tree $T_3$. In Section 3 we associate to each Farey triple $T$ an exchange matrix $M_T$ of $A_M$, and write the entries of $M_T$ in terms of the components of $T$.

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1. Preliminaries on cluster algebras

1.1. Cluster algebras with principal coefficients. In this section, we recall the construction of cluster algebras with principal coefficients from [10]. For an integer $x$, we use the notations

$$[x]_+ = \max(x, 0)$$

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and
\[ \text{sgn}(x) = \begin{cases} 
-1 & \text{if } x < 0 \\
0 & \text{if } x = 0 \\
1 & \text{if } x > 0 
\end{cases} \]

**Definition 1.1.1.** The tropical semifield on a finite family of variables \( u_j, j \in J \), is the abelian group (written multiplicatively) \( Trop(u_j : j \in J) \) freely generated by the \( u_j, j \in J \). It is endowed with an auxiliary addition \( \oplus \) defined by
\[ \prod_j u_j^{a_j} \oplus \prod_j u_j^{b_j} = \prod_j u_j^{\min(a_j, b_j)}. \]

**Remark 1.1.2.** From now on, we let \( n \) be a positive integer and \( \mathbb{P} \) be the tropical semifield on the indeterminates \( x_{n+1}, \ldots, x_{2n} \). Notice that \( \mathbb{QP} \), the group algebra on the abelian group \( \mathbb{P} \), is naturally identified with the algebra of Laurent polynomials with rational coefficients in the variables \( x_{n+1}, \ldots, x_{2n} \). We denote by \( \mathcal{F} \) the field of fractions of the ring of polynomials with coefficients in \( \mathbb{QP} \) in \( n \) indeterminates.

**Definition 1.1.3.** A seed in \( \mathcal{F} \) is a pair \((\hat{B}, \mathbf{x})\) formed by
- a \( 2n \times n \) integer matrix \( \hat{B} = (b_{ij}) \), whose principal part \( B \), i.e., the top \( n \times n \) submatrix, is antisymmetric (equivalently, associated to a quiver \( \Gamma \) with no oriented cycles of length \( \leq 2 \) via the formula \( b_{ij} = \text{number of arrows } i \rightarrow j \) in \( \Gamma \) \( - \text{number of arrows } j \rightarrow i \) in \( \Gamma \) \( ) \) for \( 1 \leq i, j \leq n \);
- a free generating set \( \mathbf{x} = \{x_1, \ldots, x_n\} \) of the field \( \mathcal{F} \).

The matrix \( \hat{B} \) is called the (extended) exchange matrix and the set \( \mathbf{x} \) the cluster of the seed \((\hat{B}, \mathbf{x})\).

**Definition 1.1.4.** Let \((\hat{B}, \mathbf{x})\) be a seed in \( \mathcal{F} \), and let \( 1 \leq k \leq n \) be an integer. The seed mutation \( \mu_k \) in the direction \( k \) transforms \((\hat{B}, \mathbf{x})\) into the seed \( \mu_k(\hat{B}, \mathbf{x}) = (\hat{B}', \mathbf{x}') \) defined as follows:
- The entries of \( \hat{B}' = (b'_{ij}) \) are given by
  \[ b'_{ij} = \begin{cases} 
  -b_{ij} & \text{if } i = k \text{ or } j = s; \\
  b_{ij} + \text{sgn}(b_{ik})[b_{ik}b_{kj}]_+ & \text{otherwise.} 
  \end{cases} \]
- The cluster \( \mathbf{x}' = \{x'_1, \ldots, x'_n\} \) is given by \( x'_j = x_j \) for \( j \neq k \), whereas \( x'_k \in \mathcal{F} \) is determined by the exchange relation
  \[ x'_k x_k = 2n \prod_{i=1}^{2n} x_i^{b_{ik}} + \prod_{i=1}^{n} x_i^{-b_{ik}}. \]

**Remark 1.1.5.** Mutation in a fixed direction is an involution, i.e., \( \mu_k \mu_k(\hat{B}, \mathbf{x}) = (\hat{B}, \mathbf{x}) \).

**Definition 1.1.6.** Let \( T_n \) be the \( n \)-regular tree, whose edges are labeled by the numbers \( 1, \ldots, n \) so that the \( n \) edges emanating from each vertex carry different labels. A cluster pattern is the assignment of a seed \((\hat{B}_t, \mathbf{x}_t)\) to each vertex \( t \) of \( T_n \) such that the seeds assigned to vertices \( t \) and \( t' \), linked by an edge with label \( k \), are obtained from each other by the seed mutation \( \mu_k \). Fix a vertex \( t_0 \) of the \( n \)-regular tree \( T_n \). Clearly, a cluster pattern is uniquely determined by the initial seed \((\hat{B}_{t_0}, \mathbf{x}_{t_0})\), which can be chosen arbitrarily. The e-vectors of a seed \((\hat{B}_t, \mathbf{x}_t)\) are elements of the set \( \{\mathbf{e}_{j:t} = (b'_{n+1,j}, \ldots, b'_{n+j}) \in \mathbb{Z}^n : 1 \leq j \leq n\} \), where \( \hat{B}_t = (b'_{ij}) \).
1.2. The $g$-vectors. In this section we recall the natural $\mathbb{Z}^n$-grading in a cluster algebra with principal coefficients. This leads us to the definition of the $g$-vectors.

**Definition 1.2.1.** Let $\tilde{A}(\tilde{B}, \mathbf{x})$ be the cluster algebra with principal coefficients at a vertex $t_0$, defined by the initial seed $(\tilde{B}, \mathbf{x}) = ((b_{ij}), \{x_1, \ldots, x_n\})$. The (natural) $\mathbb{Z}^n$-grading of the ring $\mathbb{Z}[x_1 \pm 1, \ldots, x_n^{\pm 1}; x_{n+1}, \ldots, x_{2n}]$ is given by

$$
\deg(x_j) = \begin{cases} 
  e_j & \text{if } 1 \leq j \leq n \\
  -b_j & \text{if } n + 1 \leq j \leq 2n 
\end{cases}
$$

where $e_1, \ldots, e_n$ are the standard basis (column) vector in $\mathbb{Z}^n$, and $b_j = \Sigma_i b_{ij} e_i$.

**Theorem 1.2.2.** ([10, Corollary 6.3]) Under the $\mathbb{Z}^n$-grading given by (1.4), the cluster algebra $\tilde{A}(\tilde{B}, \mathbf{x})$ is a $\mathbb{Z}^n$-graded subalgebra of $\mathbb{Z}[x_1 \pm 1, \ldots, x_n^{\pm 1}; x_{n+1}, \ldots, x_{2n}]$.

**Definition 1.2.3.** By iterating the exchange relation (1.3) we can express every cluster variable $x_{j:t}$ as a (unique) rational function in $x_1, \ldots, x_{2n}$ given by a subtraction-free rational expression; we denote this rational function by $X_{j:t}$. The $g$-vectors of a seed $(\tilde{B}_t, \mathbf{x}_t)$ are:

$$
g_{j:t} = \begin{pmatrix} 
  g_1 \\
  \vdots \\
  g_n 
\end{pmatrix} = \deg(X_{j:t}),
$$

we call a matrix of the form $(g_{1:t}, \ldots, g_{n:t})$ a $g$-matrix.

2. Farey triples

2.1. The Farey sum. Let $\mathbb{Q}_\infty = \mathbb{Q} \cup \{\infty\}$ be the totally ordered set of extended rational numbers. We let $\infty = \frac{a}{b}$ for all $a \neq 0$.

**Notation 2.1.1.** For $q \in \mathbb{Q}_\infty$, we denote by $d(q)$ and $r(q)$ the integers such that:

- $q = \frac{d(q)}{r(q)}$,
- $\gcd(d(q), r(q)) = 1$,
- $r(q) \geq 0$.

It is obvious that $d(q)$ and $r(q)$ are uniquely determined. In particular, $0 = \frac{0}{1}$ and $\infty = \frac{1}{0}$.

**Definition 2.1.2.** Given $q, q' \in \mathbb{Q}_\infty$ we define

$$
\Delta(q, q') := \left| \det \begin{pmatrix} 
  d(q) & d(q') \\
  r(q) & r(q') 
\end{pmatrix} \right|
$$

In case that $\Delta(q, q') = 1$, we call $q$ and $q'$ Farey neighbors.

**Definition 2.1.3.** Let $q, q' \in \mathbb{Q}_\infty$ be Farey neighbors. The Farey sum of $q$ and $q'$ is

$$
q \oplus q' := \frac{d(q) + d(q')}{r(q) + r(q')},
$$
and their Farey difference is

\[ q \odot q' := \frac{d(q) - d(q')}{r(q) - r(q')} \]

Note that both of these operations are commutative.

**Definition 2.1.4.** We call \((q_1, q_2, q_3) \in \mathbb{Q}_\infty^3\) a triple of neighbors provided that \(\Delta(q_i, q_j) = 1\) for \(1 \leq i < j \leq 3\). A Farey triple \([q_1, q_2, q_3]\) is a triple of neighbors considered up to permutation of its components. We use the indices \(\{f, s, t\} = \{1, 2, 3\}\) for a Farey triple to indicate that \(q_f < q_s < q_t\).

**Lemma 2.1.5.** Let \((q_1, q_2, q_3) \in \mathbb{Q}_\infty^3\) and \(q_1 < q_2 < q_3\). Then \((q_1, q_2, q_3)\) is a triple of neighbors if and only if \(q_2 = q_1 \oplus q_3\), i.e. the Farey triples are essentially of the form \([q_f, q_f \oplus q_t, q_t]\).

**Proof.** Fix \(q_1\) and \(q_3\) and consider the following system of equations in \(d(q_2)\) and \(r(q_2)\):

\[
\begin{align*}
d(q_3)r(q_2) - r(q_3)d(q_2) &= 1 \\
-d(q_1)r(q_2) + r(q_1)d(q_2) &= 1,
\end{align*}
\]

the determinant of the system is \(\Delta(q_1, q_3) = 1\), so it has a unique solution which is \(d(q_2) = d(q_1) + d(q_3)\) and \(r(q_2) = r(q_1) + r(q_3)\). \(\square\)

**Lemma and Definition 2.1.6.** For all \(q \in \mathbb{Q}\), there exist two unique Farey neighbors \(q', q'' \in \mathbb{Q}_\infty\) such that \(q = q' \oplus q''\). We call such expression the Farey decomposition of \(q\).

**Proof.** Consider the diophantine equation \(d(q)y - r(q)x = 1\). Let \(x = a\), \(y = b\) be a solution of the equation with \(0 \leq b < r(q)\). Put \(q' = \frac{a}{b}\) and \(q'' = q \oplus q'\), then \(q = q' \oplus q''\). To see the uniqueness of \(q'\) and \(q''\), suppose that \(q = p \oplus p'\) is another decomposition. Without loss of generality we can assume that \(p, q' < q < p'q''\). Since \(\Delta(q, p) = \Delta(q, q') = 1\) and \(p, q' < q\), it follows that \(d(q)(r(q') - r(p)) = r(q)(d(q') - d(p))\). In particular \(q' \neq p\) if and only if \(r(q') \neq r(p)\). Suppose that \(q' \neq p\). Then we have

\[
\frac{d(q)}{r(q)} = \frac{d(q') - d(p)}{r(q') - r(p)},
\]

but \(-r(q) < r(q') - r(p) < r(q)\), a contradiction. Hence \(q' = p\). We do the same for \(q''\) and \(p'\). \(\square\)

**Definition 2.1.7.** Let \([q_f, q_s, q_t]\) be a Farey triple (with \(q_f < q_s < q_t\) see \(\ref{2.1.4}\)) and \(k \in \{f, s, t\}\). The mutation \(\mu_k\) in direction \(k\) of \([q_f, q_s, q_t]\) is the Farey triple \(\mu_k[q_f, q_s, q_t]\) defined as follows:

\[
\mu_k[q_f, q_s, q_t] = \begin{cases} 
[q_s, q_s \oplus q_t, q_t] & \text{if } k = f \\
[q_f, q_f \oplus q_t, q_t] & \text{if } k = s \\
[q_f, q_f \oplus q_s, q_s] & \text{if } k = t
\end{cases}
\]

In view of Lemma \(\ref{2.1.6}\) we have that the mutation is an involution i.e. we have \(\mu_k \mu_k[q_f, q_s, q_t] = [q_f, q_s, q_t]\) for each \(k\).

**Definition 2.1.8.** The exchange graph of the Farey triples is the graph whose vertices are the Farey triples, and whose edges connect triples related by a single mutation.

**Proposition 2.1.9.** The exchange graph of the Farey triples is a 3-regular tree \(T_3\).

**Proof.** First we show that the graph is connected. It is sufficient to show that any Farey triple is connected with the triple \([\frac{1}{T}, \frac{0}{T}, \frac{1}{T}]\). Call this triple the initial triple and all others non-initial triples. For any Farey triple \([q_f, q_s, q_t]\), define its complexity as \(c[q_f, q_s, q_t] = ...\)
Some isomorphisms of graph contain an always changes under mutation, the graph does not contains loops. Now, notice that if the remains to show that the graph does not contain cycles. Since the complexity of a triple increases only in directions 0 and 3.

In this section, we use the Farey triples in order to give an explicit description of the exchange matrices of the Markov cluster algebra (with principal coefficients) \( A_M \). An important property of the quiver is that the mutation at any vertex only

\[ |d(q_s)| + r(q_s) \]

Since \( c[q_f, q_s, q_t] = 1 \) implies that \( q_f, q_s, q_t = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \), it will be sufficient to show that for any non-initial triple, there is a (unique) direction \( k \in \{ f, s, t \} \) such that \( c(\mu_k[\chi_{f,s}^k, 0, 1, 0]) < c[q_f, q_s, q_t] \). If \( c[q_f, q_s, q_t] = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \) with \( z \geq 1 \), then take \( k = f \).

In any other case we have either \( q_i \geq 0 \) or \( q_i \leq 0 \) for all \( i \), so \( k = s \). It only remains to show that the graph does not contain cycles. Since the complexity of a triple always changes under mutation, the graph does not contains loops. Now, notice that if the graph contain an \( n \)-cycles with \( n \geq 2 \), then we can find a vertex of the cycle for which the complexity decreases in two different directions, a contradiction.

2.2. Some isomorphisms of \( T_3 \setminus \{ 0, 1, 1 \} \). From now on, we write \( \mu_0, \mu_1 \) and \( \mu_{\infty} \) to denote the mutations in direction of the fractions of the form \( \frac{even}{odd} \), \( odd \) and \( odd \), respectively. We label the edges of \( T_3 \) with the same indices, and denote by \( T_3 \) the connected component of \( T_3 \setminus \{ 0, 1, 1 \} \) having the triple \( \mu_1[0, 1, 1] \) as a vertex.

Remark 2.2.1. From now on, even if the Farey triples are considered up to permutation of their components, when we write a Farey triple, we assume that it is written in the order \( \{ even, odd, odd, even \} \).

Definition 2.2.2. We define a graph isomorphism \( \phi : T_3 \setminus \{ 0, 1, 1 \} \rightarrow T_3 \) as follows:

- The triple \( \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \) is mapped to the triple \( \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \mu_{\infty}[0, 1, 1] \).
- The edges with label \( -1, 0, \infty \) are mapped to the edges with label \( -1, 0, \infty \), resp.

Analogously, we define a graph isomorphism \( \psi : T_3 \setminus \{ 0, 1, 1 \} \rightarrow T_3 \) as follows:

- The triple \( \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \) is mapped to the triple \( \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \mu_0[0, 1, 1] \).
- The edges with label \( -1, 0, \infty \) are mapped to the edges with label \( 0, \infty, -1, \), resp.

Proposition 2.2.3. The isomorphisms of \( \phi, \psi \) can be expressed in terms of the following formulas:

\[ \phi[q_0, q_{-1}, q_{\infty}] = \begin{bmatrix} -r(q_0) \\ r(q_0) + d(q_\infty) \\ -r(q_{-1}) \\ r(q_{-1}) + d(q_\infty) \end{bmatrix}, \]

\[ \psi[q_0, q_{-1}, q_{\infty}] = \begin{bmatrix} -r(q_0) + d(q_{-1}) \\ r(q_{-1}) + d(q_\infty) -r(q_0) + d(q_\infty) \\ r(q_{-1}) + d(q_\infty) -r(q_{-1}) + d(q_\infty) \end{bmatrix}. \]

Proof. We prove it only for \( \phi \) using induction on \( c[q_0, q_{-1}, q_{\infty}] \). We see that the formula in Definition 2.2.2 for \( \phi[0, 1, 1] \) satisfies the statement, so we take it as base for the induction. Without loss of generality, we assume that \( q_0 < q_{-1} < q_{\infty} \). Thus, the complexity of the triple increases only in directions \( 0 \) and \( \infty \). We consider only direction \( 0 \), direction \( \infty \) is similar.

By induction we obtain:

\[ \phi(\mu_0[q_0, q_{-1}, q_{\infty}]) = \mu_{-1} \begin{bmatrix} -r(q_{\infty}) \\ r(q_{\infty}) + d(q_{-1}) \\ -r(q_{-1}) \\ r(q_{-1}) + d(q_{-1}) \end{bmatrix}, \]

Since \( q_{-1} + q_{\infty} = \frac{d(q_{-1}) + d(q_{\infty})}{r(q_{-1}) + r(q_{\infty})} \), it is clear that the result follows for \( \phi \).
reverses the orientation of the arrows.

We denote by $B^+$ (resp. $B^-$) the matrix associated with the left hand (resp. right hand) quiver of 3.1. Then the initial matrix of $\mathcal{A}_M$ is:

$$
\tilde{B}_0 = \begin{pmatrix}
0 & -1 & \infty \\
\downarrow & \downarrow & \downarrow \\
0 & -2 & 2 \\
2 & 0 & -2 \\
-2 & 2 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
$$

We consider the following assignment defined recursively. To the initial Farey triple $[\frac{0}{1}, \frac{-1}{1}, \frac{1}{0}]$, assign the initial matrix $\tilde{B}_0$. To any Farey triple $\mu_{a_n} \cdots \mu_{a_1} [\frac{0}{1}, \frac{-1}{1}, \frac{1}{0}]$ with $a_i \in \{0, -1, \infty\}$, assign the extended exchange matrix $\mu_{a_n} \cdots \mu_{a_1} (\tilde{B}_0)$. Note that this assignment is a surjection onto the set of the exchange matrices of $\mathcal{A}_M$. We use the following notation to denote the assignment $[q_0, q_{-1}, q_{\infty}] \rightarrow M$ (see Remark 2.2.1):

$$
M = \begin{pmatrix}
q_0 & q_{-1} & q_{\infty} \\
B^\pm & a & d & g \\
b & e & h \\
c & f & i
\end{pmatrix}
$$

Thus we may speak of mutating both, a triple and the associated matrix. Our goal is to describe all the extended exchange matrices using the Farey triples. First we describe the matrices obtained by alternating mutations in direction $-1$ and $0$.

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \xrightarrow{\mu_{-1}}
\begin{pmatrix}
1 & 0 & 0 \\
2 & -1 & 0 \\
0 & 0 & 1
\end{pmatrix}
$$

$$
\begin{pmatrix}
3 & -2 & 0 \\
4 & -3 & 0 \\
0 & 0 & 1
\end{pmatrix} \xrightarrow{\mu_{-1}}
\begin{pmatrix}
-1 & 2 & 0 \\
-2 & 3 & 0 \\
0 & 0 & 1
\end{pmatrix}
$$

$$
\begin{pmatrix}
-3 & 4 & 0 \\
-4 & 5 & 0 \\
0 & 0 & 1
\end{pmatrix} \xrightarrow{\mu_{-1}} \cdots
$$

These matrices correspond to triples of the form $[\frac{q}{1}, \frac{a\pm1}{1}, \frac{1}{b}]$. The next result describes all the matrices associated to triples of that form.
Proposition 3.1.1. The matrices obtained by alternating the mutations $\mu_{-1}$ and $\mu_0$ (beginning with $\mu_{-1}$) are of the following form:

\[
\begin{pmatrix}
\frac{a}{T} & \frac{a-1}{T} & 1 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
B^+
\end{pmatrix}
\quad \text{or} \quad
\begin{pmatrix}
\frac{a}{T} & \frac{a+1}{T} & 1 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
B^-
\end{pmatrix}
\]

Proof. We proceed by induction on $a$. Take as base for the induction the calculus made in (3.2). Suppose that \([\frac{a}{T}, \frac{a-1}{T}, 1, 0] \rightarrow M\) satisfies the statement, mutating both in direction $-1$ we obtain:

\[
M = \begin{pmatrix}
-a+1 & a & 0 \\
-a & a+1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
B^+
\end{pmatrix}
\]

\[
\begin{pmatrix}
\frac{a+1}{T} & -a & 0 \\
a+2 & -(a+1) & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
B^-
\end{pmatrix}
\]

In this case we are done. The second case is obtained in the same way. \(\square\)

Notice that the matrices just described are associated to a triple lying in $T_{-1}^3$. The next theorem describes the rest of the matrices associated to the triples in $T_{-1}^3$. All other exchange matrices will be obtained from those described in Proposition 3.1.1 and Theorem 3.1.2 below.

Theorem 3.1.2. Let $M$ be the exchange matrix associated to a triple $[q_0, q_{-1}, q_{\infty}] \neq [\frac{a}{T}, \frac{a+1}{T}, 1, 0]$ lying in $T_{-1}^3$. Consider the following six cases:

(i) $q_0 < q_{\infty} < q_{-1}$  
(ii) $q_{-1} < q_{\infty} < q_0$  
(iii) $q_0 < q_{-1} < q_{\infty}$  
(iv) $q_{\infty} < q_{-1} < q_0$  
(v) $q_{-1} < q_0 < q_{\infty}$  
(vi) $q_{\infty} < q_0 < q_{-1}$

Let $q_0 = \frac{a}{T}$, $q_{-1} = \frac{a-1}{T}$ and $q_{\infty} = \frac{1}{T}$. Then, in each case, the matrix $M$ is of the form:

(i) \[
\begin{pmatrix}
\frac{a+1}{T} & -c+1 & c-a-1 \\
\frac{a+b+1}{T} & -c-d+1 & c+d-a-b-1 \\
b+1 & -d+1 & d-b-1
\end{pmatrix}
\]

(ii) \[
\begin{pmatrix}
-a+1 & c+1 & a-c-1 \\
-a-b+1 & c+d+1 & a+b-c-d-1 \\
b+1 & d+1 & b-d-1
\end{pmatrix}
\]

(iii) \[
\begin{pmatrix}
a+1 & e-a-1 & -e+1 \\
\frac{a+b+1}{T} & e+f-a-b-1 & -e-f+1 \\
b+1 & f-b-1 & -f+1
\end{pmatrix}
\]

(iv) \[
\begin{pmatrix}
-a+1 & a-e-1 & e+1 \\
-a-b+1 & a+b-e-f-1 & e+f+1 \\
b+1 & b-f-1 & f+1
\end{pmatrix}
\]
In cases (i), (iv) and (v), the principal part of $M$ is $B^+$, in the other cases it is $B^-$. 

Proof. First we show that the matrices obtained by mutating those of Proposition 3.1.1 in direction $\infty$ satisfy the statement:

$\begin{pmatrix}
e - c - 1 & c + 1 & - e + 1 \\
e + f - c - d - 1 & c + d + 1 & - e - f + 1 \\
f - d - 1 & d + 1 & - f + 1
e - 1 & e + 1 \\
e + d - e - f - 1 & - c - d + 1 & e + f + 1 \\
d - f - 1 & - d + 1 & f + 1
\end{pmatrix}$

For the matrices obtained by the assignment $\mu_\infty$, we do the same. Suppose that $[q_{-1}, q_0, q_{\infty}] = \mu_\infty[a, a + 1, 1/2]$ for some $a > 0$, we proceed by induction on $c[q_{-1}, q_0, q_{\infty}]$. Take as base for the induction is the first part of the proof. Now, notice that the cases (for the triples) are related by mutation in the directions described in the following diagram (we only consider directions which increase the complexity):

Thus, we will be done if we prove that the matrices of the statement satisfies the same diagram. Once again, we only prove one case and in one direction, it will be obvious that all cases are similar. Consider case (i), and mutate in direction 0:
\[
\begin{bmatrix}
\frac{a}{b} & \frac{c}{a} & \frac{a+c}{b+d} \\
\frac{a+1}{b+1} & \frac{-c+1}{-c-d+1} & \frac{c-a-1}{c+d-a-b-1} \\
\frac{a+b+1}{b+1} & \frac{-c-d+1}{-d+1} & \frac{c+d-a-b-1}{d-b-1}
\end{bmatrix}
\]

We have described all the matrices associated to a triple in \(T_{3}^{-1}\). In order to describe the rest, we consider the action of the alternating group \(A_{3} = \{\text{id}, (123), (132)\}\) on \(\text{Mat}(3 \times 3, \mathbb{Z})\) by cyclic permutation of its “diagonals”, i.e., if \(B = (b_{ij})\) then \(\sigma \cdot B = (b_{\sigma(i)\sigma(j)})\).

**Proposition 3.1.3.** Let \(M\) be the matrix associated to a triple \([q_{0}, q_{-1}, q_{\infty}] \in T_{3}^{-1}\). Then \(\phi[q_{0}, q_{-1}, q_{\infty}] \rightarrow (123) \cdot M\) and \(\psi[q_{0}, q_{-1}, q_{\infty}] \rightarrow (132) \cdot M\).

**Proof.** Since the matrices \(B^{\pm}\) and the \(3 \times 3\) identity matrix are invariant under the action of \((123)\) and \((132)\), and the functions \(\phi\) and \(\psi\) preserve the mutation in the sense of 2.2.2, it is sufficient to verify the statement for the matrices associated to \(\phi[\frac{a}{b}, \frac{a-1}{a}, \frac{1}{b}]\) and \(\psi[\frac{a}{b}, \frac{a+1}{a}, \frac{1}{b}]\), but this follows at once. \(\square\)

**Observation 3.1.4.** Each c-vector of \(A_{M}\) has either only non negative entries or only non positive entries. The corresponding fact is known for each antisymmetric cluster algebra of geometric type, cf. section 1 of [14].

In the same way as we did with the exchange matrices, we assign to each Farey Triple a g-matrix.

**Theorem 3.1.5.** Let \(M\) be the g-matrix associated to a triple \([q_{0}, q_{-1}, q_{\infty}] \in T_{3}^{-1}\). Consider the following three cases:

(i) \([q_{0}, q_{-1}, q_{\infty}] = [\frac{a}{b}, \frac{a-1}{a}, \frac{1}{b}]\)

(ii) \([q_{0}, q_{-1}, q_{\infty}] = [\frac{a}{b}, \frac{a+1}{a}, \frac{1}{b}]\)

(iii) \(q_{0} = \frac{a}{b}, \ q_{-1} = \frac{a+1}{a}, \ q_{\infty} = \frac{c}{d} \neq \frac{1}{b}\)

Then, in each case, the matrix \(M\) is of the form:

(i) \(\begin{bmatrix}
    a+1 & a & 0 \\
    -a & -a+1 & 0 \\
    0 & 0 & 1
\end{bmatrix}\)

(ii) \(\begin{bmatrix}
    a+1 & a+2 & 0 \\
    -a & -(a+1) & 0 \\
    0 & 0 & 1
\end{bmatrix}\)
Proof. In view of [14, Theorem 1.2], we can calculate $M$ by inverting and transposing the complementary part of the exchange matrices of Proposition 3.1.1 and Theorem 3.1.2. We can easily see that for the cases (i) and (ii) we are done. Now assume that $[q_0, q_{-1}, q_\infty]$ is as in case (iii). Notice that the matrices of Theorem 3.1.2 can be expressed in terms of a single matrix, namely

$$
\lambda \begin{pmatrix}
e - c + cf - de & a - e + bf - af & c - a + ad - bc \\
e + f - c - d + cf - de & a + b - e - f + bf - af & c + d - a - b + ad - bc \\
f - d + cf - de & b - f + bf - af & d - b + ad - bc
\end{pmatrix},
$$

where $\lambda = (cf - dc + bc - af + ad - bc)^{-1}$. An straightforward calculation shows that the inverse of such matrix is

$$
\begin{pmatrix}
a + 1 & b - a - 1 & 1 - b \\
c + 1 & d - c - 1 & 1 - d \\
e + 1 & f - e - 1 & 1 - f
\end{pmatrix}.
$$

□

Proposition 3.1.6. Let $M$ be the $g$-matrix associated to a triple $[q_0, q_{-1}, q_\infty] \in T_3^{-1}$. Then $\phi[q_0, q_{-1}, q_\infty] \rightarrow (123) \cdot M$ and $\psi[q_0, q_{-1}, q_\infty] \rightarrow (132) \cdot M$.

Proof. The operations of transposing and inverting a matrix commute with the action $A_3$. □

We notice that the $g$-vectors of $A_M$ lie in the plane $x + y + z = 1$. The first vectors obtained by iterated mutation of the initial seed are represented in the Figure 3.1. Another interpretation of this figure is given in [6].

Figure 2. The $g$-vectors of $A_M$
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