Supercongruences for sums involving Domb numbers

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Abstract. We prove some supercongruence and divisibility results on sums involving Domb numbers, which confirm four conjectures of Z.-W. Sun and Z.-H. Sun. For instance, by using a transformation formula due to Chan and Zudilin, we show that for any prime \( p \geq 5 \),

\[
\sum_{k=0}^{p-1} \frac{3k + 1}{(-32)^k} \text{Domb}(k) \equiv (-1)^{\frac{p+1}{4}}p + p^3E_{p-3} \pmod{p^4},
\]

which is regarded as a \( p \)-adic analogue of the following interesting formula for \( 1/\pi \) due to Rogers:

\[
\sum_{k=0}^{\infty} \frac{3k + 1}{(-32)^k} \text{Domb}(k) = \frac{2}{\pi}.
\]

Here \( \text{Domb}(n) \) and \( E_n \) are the famous Domb numbers and Euler numbers.

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MR Subject Classifications: 11A07, 11Y55, 05A19, 33F10

1 Introduction

In 1960, Domb \[8\] first introduced the following sequence:

\[
\text{Domb}(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2n - 2k}{n - k},
\]

which are known as the famous Domb numbers. This sequence plays an important role in many research fields, including probability theory \[4\], special functions \[3\], Apéry-like differential equations \[1\], and combinatorics \[16\].

The Domb numbers are also connected to some interesting series for \( 1/\pi \). For instance, Chan, Chan and Liu \[5\] showed that

\[
\sum_{k=0}^{\infty} \frac{5k + 1}{64^k} \text{Domb}(k) = \frac{8}{\sqrt{3\pi}}.
\]
Another typical example is the following identity due to Rogers [17]:

\[
\sum_{k=0}^{\infty} \frac{3k+1}{(-32)^k} \text{Domb}(k) = \frac{2}{\pi}.
\]  \hspace{1cm} (1.1)

Let \( E_n \) denote the Euler numbers given by

\[
\frac{2}{e^x + e^{-x}} = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!}.
\]

The motivation of this paper is to prove the following interesting \( p \)-adic analogue of (1.1), which was originally conjectured by Z.-W. Sun [23, Conjecture 77 (ii)].

**Theorem 1.1** For any prime \( p \geq 5 \), we have

\[
\sum_{k=0}^{p-1} \frac{3k+1}{(-32)^k} \text{Domb}(k) \equiv (-1)^{\frac{n+1}{2}} p + p^3 E_{p-3} \pmod{p^4}.
\] \hspace{1cm} (1.2)

The proof of (1.2) heavily relies on the transformation formula due to Chan and Zudilin [7, Corollary 3.4]:

\[
\sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} \binom{2n-2k}{n-k} = \sum_{k=0}^{n} (-1)^k \binom{n+2k}{3k} \binom{2k}{k}^2 \binom{3k}{k} 16^{n-k}.
\] \hspace{1cm} (1.3)

The second purpose of this paper is to prove a related supercongruence conjectured by Z.-H. Sun [20, Conjecture 2.6] and two divisibility results on sums of Domb numbers conjectured by Z.-W. Sun [23, Conjecture 77 (i)].

**Theorem 1.2** For any prime \( p \geq 5 \), we have

\[
\sum_{k=0}^{p-1} \frac{3k+2}{(-2)^k} \text{Domb}(k) \equiv 2p(-1)^{\frac{n+1}{2}} + 6p^3 E_{p-3} \pmod{p^4}.
\] \hspace{1cm} (1.4)

We remark that Z.-W. Sun [22] conjectured the supercongruence (1.4) modulo \( p^3 \).

**Theorem 1.3** Let \( n \) be a positive integer. Then

\[
\frac{1}{n} \sum_{k=0}^{n-1} (2k+1) \text{Domb}(k) 8^{n-1-k} \quad \text{and} \quad \frac{1}{n} \sum_{k=0}^{n-1} (2k+1) \text{Domb}(k) (-8)^{n-1-k}
\]

are all positive integers.
The sums of cubes of binomial coefficients:

\[ f_n = \sum_{k=0}^{n} \binom{n}{k}^3 \]

are known as Franel numbers [9]. The proofs of Theorems 1.2 and 1.3 respectively make use of the identity due to Z.-H. Sun [19, Lemma 3.1]:

\[
\sum_{k=0}^{n} \binom{n}{k}^2 \binom{2n}{k} \binom{2n-2k}{n-k} = \sum_{k=0}^{[n/2]} \binom{2k}{k}^2 \binom{3k}{k} \binom{n+k}{3k} 4^{n-2k}, \tag{1.5}
\]

and the other identity due to Chan, Tanigawa, Yang and Zudilin [6, (2.27)]:

\[
\sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2n-2k}{n-k} = (-1)^n \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-8)^{n-k} f_k. \tag{1.6}
\]

In the past few years, supercongruences for sums of Domb numbers have been widely discussed by many researchers (see, for example, [14, 15, 19, 20, 22, 25]).

The rest of the paper is organized as follows. Section 2 lays down some preparatory results on combinatorial identities involving harmonic numbers and related congruences. We prove Theorems 1.1–1.3 in Sections 3–5, respectively.

2 Preliminary results

Let

\[ H^{(r)}_n = \sum_{j=1}^{n} \frac{1}{j^r} \]

denote the nth generalized harmonic number of order \( r \) with the convention that \( H_n = H^{(1)}_n \). The Fermat quotient of an integer \( a \) with respect to an odd prime \( p \) is given by \( q_p(a) = (a^{p-1} - 1)/p \).

**Lemma 2.1** For any non-negative integer \( n \), we have

\[
\sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{n+i}{i} (H_{2i} - H_i) = (-1)^{n+1} \sum_{i=1}^{n} \frac{(-1)^i}{i}, \tag{2.1}
\]

\[
\sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{n+i}{i} \left( (H_{2i} - H_i)^2 - H_{2i}^{(2)} - H_i^{(2)} \right) = 2(-1)^n \left( \sum_{i=1}^{n} \frac{(-1)^i}{i^2} + \sum_{i=1}^{n} \frac{(-1)^i}{i} H_i \right). \tag{2.2}
\]
Proof. The identities (2.1) and (2.2) are discovered and proved by the symbolic summation package Sigma developed by Schneider [18]. One can also refer to [12, 13] for the same approach to finding and proving identities of this type. □

Lemma 2.2 (See [21, Lemma 2.4] and [2, Lemma 2.9].) For any prime \( p \geq 5 \), we have

\[
\sum_{i=1}^{\frac{p-1}{2}} \frac{(-1)^i}{i^2} \equiv (-1)^{\frac{p-1}{2}} 2E_{p-3} \pmod{p},
\]

(2.3)

\[
\sum_{i=1}^{\frac{p-1}{2}} \frac{(-1)^i}{i} H_i \equiv \frac{1}{2} q_p(2)^2 + (-1)^{\frac{p-1}{2}} E_{p-3} \pmod{p}.
\]

(2.4)

Lemma 2.3 For any prime \( p \geq 5 \), we have

\[
\sum_{i=1}^{\frac{p-1}{2}} \frac{(-1)^i}{i} \equiv -q_p(2) + \frac{1}{2} pq_p(2)^2 - p(-1)^{\frac{p-1}{2}} E_{p-3} \pmod{p^2}.
\]

(2.5)

Proof. We begin with the following congruence [11, (43)]:

\[
\sum_{i=1}^{\lfloor p/4 \rfloor} \frac{1}{p - 4i} \equiv \frac{3}{4} q_p(2) - \frac{3}{8} pq_p(2)^2 \pmod{p^2}.
\]

(2.6)

Since for \( 1 \leq i \leq \lfloor p/4 \rfloor \),

\[
\frac{1}{p - 4i} \equiv -\frac{1}{4i} - \frac{p}{(4i)^2} \pmod{p^2},
\]

we have

\[
\sum_{i=1}^{\lfloor p/4 \rfloor} \frac{1}{p - 4i} \equiv -\frac{1}{4} H_{\lfloor p/4 \rfloor} - \frac{p}{16} H^{(2)}_{\lfloor p/4 \rfloor} \pmod{p^2}.
\]

(2.7)

By [11] page 359, we have

\[
H^{(2)}_{\lfloor p/4 \rfloor} \equiv (-1)^{\frac{p-1}{2}} 4E_{p-3} \pmod{p}.
\]

(2.8)

Combining (2.6)–(2.8), we arrive at

\[
H_{\lfloor p/4 \rfloor} \equiv -3q_p(2) + \frac{3}{4} pq_p(2)^2 - p(-1)^{\frac{p-1}{2}} E_{p-3} \pmod{p^2}.
\]

(2.9)

Furthermore, we have

\[
\sum_{i=1}^{\frac{p-1}{2}} \frac{(-1)^i}{i} = H_{\lfloor p/4 \rfloor} - H_{(p-1)/2},
\]

(2.10)

and the following result (see [11] (45)):

\[
H_{(p-1)/2} \equiv -2q_p(2) + pq_p(2)^2 \pmod{p^2}.
\]

(2.11)

Finally, substituting (2.9) and (2.11) into (2.10), we complete the proof of (2.5). □
3 Proof of Theorem 1.1

By (1.3), we have

\[
\sum_{k=0}^{p-1} 3k + 1 \Domb(k) = \sum_{k=0}^{p-1} 3k + 1 (-32)^k \sum_{i=0}^{k} (-1)^i \binom{k + 2i}{3i} \binom{2i}{i} \binom{3i}{i} 16^{k-i}
\]

\[
= \sum_{i=0}^{p-1} \frac{1}{(-16)^i} \binom{2i}{i}^2 \binom{3i}{i} \sum_{k=i}^{p-1} 3k + 1 \binom{k + 2i}{3i}.
\] (3.1)

It can be easily proved by induction on \( n \) that

\[
\sum_{k=i}^{n-1} 3k + 1 \binom{k + 2i}{3i} = (n - i) \binom{n + 2i}{3i} (-2)^{1-n}.
\] (3.2)

It follows from (3.1) and (3.2) that

\[
\sum_{k=0}^{p-1} 3k + 1 (-32)^k \Domb(k) = \sum_{i=0}^{p-1} \frac{2^{1-p(p-i)}}{(-16)^i} \binom{2i}{i}^2 \binom{3i}{i} \binom{p + 2i}{3i}.
\] (3.3)

Now we split the sum on the right-hand side of (3.3) into two pieces:

\[
S_1 = \sum_{i=0}^{(p-1)/2} (\cdot) \quad \text{and} \quad S_2 = \sum_{i=(p+1)/2}^{p-1} (\cdot).
\]

For \( 0 \leq j \leq (p-1)/2 \), we have

\[
(-1)^{j(p - j)} \binom{3i}{i} \binom{p + 2i}{3i} = \frac{p(-1)^j(p + 2i) \cdots (p + 1)(p - 1) \cdots (p - i)}{i!(2i)!}
\]

\[
= \frac{p(-1)^j(p + 2i) \cdots (p + i + 1)(p^2 - 1) \cdots (p^2 - i^2)}{i!(2i)!}
\]

\[
\equiv \frac{p!/(p + 2i) \cdots (p + i + 1)}{(2i)!} \left( 1 - p^2 H^{(2)}_i \right)
\]

\[
\equiv \frac{p!/(p + 2i) \cdots (p + i + 1)}{(2i)!} - p^3 H^{(2)}_i \pmod{p^4}.
\]

Furthermore, we have

\[
\frac{p!/(p + 2i) \cdots (p + i + 1)}{(2i)!}
\]

\[
\equiv p \left( 1 + p(H_{2i} - H_i) + \frac{p^2}{2} \left( (H_{2i} - H_i)^2 - H^{(2)}_{2i} + H^{(2)}_i \right) \right) \pmod{p^4}.
\]
It follows that
\[
(-1)^i (p - i) \binom{3i}{i} \binom{p + 2i}{3i} \equiv p + p^2 (H_{2i} - H_i) + \frac{p^3}{2} \left( (H_{2i} - H_i)^2 - H_{2i}^{(2)} - H_i^{(2)} \right) \pmod{p^4},
\]
and so
\[
S_1 \equiv 2^{1-p} p \sum_{i=0}^{(p-1)/2} \frac{1}{16^i} \binom{2i}{i}^2 \times \left( 1 + p (H_{2i} - H_i) + \frac{p^2}{2} \left( (H_{2i} - H_i)^2 - H_{2i}^{(2)} - H_i^{(2)} \right) \right) \pmod{p^4}. \tag{3.4}
\]
Note that for \(0 \leq i \leq \frac{p-1}{2}\),
\[
(-1)^i \binom{(p - 1)/2}{i} \binom{(p - 1)/2 + i}{i} = \frac{\left( \left( \frac{1}{2} \right)^2 - \left( \frac{p}{2} \right)^2 \right) \left( \left( \frac{3}{2} \right)^2 - \left( \frac{p}{2} \right)^2 \right) \cdots \left( \left( \frac{2i-1}{2} \right)^2 - \left( \frac{p}{2} \right)^2 \right)}{i!^2} \equiv \frac{1}{16^i} \binom{2i}{i}^2 \pmod{p^2}. \tag{3.5}
\]
Letting \(n = \frac{p-1}{2}\) in (2.1) and (2.2) and using (3.5), we obtain
\[
\sum_{i=0}^{(p-1)/2} \frac{1}{16^i} \binom{2i}{i}^2 (H_{2i} - H_i) \equiv (-1)^{\frac{p+1}{2}} \sum_{i=1}^{(p-1)/2} \frac{(-1)^i}{i} \pmod{p^2}, \tag{3.6}
\]
\[
\sum_{i=0}^{(p-1)/2} \frac{1}{16^i} \binom{2i}{i}^2 \left( (H_{2i} - H_i)^2 - H_{2i}^{(2)} - H_i^{(2)} \right) \equiv 2(-1)^{\frac{p+1}{2}} \left( \sum_{i=1}^{(p-1)/2} \frac{(-1)^i}{i^2} + \sum_{i=1}^{(p-1)/2} \frac{(-1)^i}{i} H_i \right) \pmod{p^2}. \tag{3.7}
\]
Substituting (2.3) - (2.5) into the right-hand sides of (3.6) and (3.7) gives
\[
\sum_{i=0}^{(p-1)/2} \frac{1}{16^i} \binom{2i}{i}^2 (H_{2i} - H_i) \equiv \left( -q_p(2) + \frac{1}{2} p q_p(2)^2 \right) + p E_{p-3} \pmod{p^2} \tag{3.8}
\]
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and
\[
\sum_{i=0}^{(p-1)/2} \frac{1}{16^i} \binom{2i}{i}^2 \left( (H_{2i} - H_i)^2 - H_{2i}^{(2)} - H_i^{(2)} \right) \\
\equiv (-1)^{\frac{p-1}{2}} q_p(2)^2 + 6E_{p-3} \pmod{p}. \quad (3.9)
\]

Moreover, by [21, (1.7)] we have
\[
\sum_{i=0}^{(p-1)/2} \frac{1}{16^i} \binom{2i}{i}^2 \equiv (-1)^{\frac{p-1}{2}} + p^2E_{p-3} \pmod{p^3}. \quad (3.10)
\]

Substituting (3.8)–(3.10) into (3.4) and using the Fermat's little theorem, we arrive at
\[
S_1 \equiv (-1)^{\frac{p-1}{2}} p + 5p^3E_{p-3} \pmod{p^4}. \quad (3.11)
\]

Next, we evaluate \( S_2 \) modulo \( p^4 \). For \((p+1)/2 \leq i \leq p-1\), we have \( (\binom{2i}{i})^2 \equiv 0 \pmod{p^2} \), and
\[
(-1)^i 2^{1-p} (p-i) \binom{3i}{i} \binom{p + 2i}{3i} = (-1)^i 2^{1-p} p(p + 2i) \cdots (p + 1)(p-1) \cdots (p-i) \\
\quad \quad \quad \equiv \frac{p(p + 1) \cdots (p + 2i)}{(2i)!} \pmod{p^2} \\
\quad \quad \quad \equiv \frac{p(p + 1)(p + 2) \cdots 2p \cdots (p + 2i)}{1 \cdot 2 \cdots p \cdots 2i} \\
\quad \quad \quad \equiv 2p \pmod{p^2},
\]
where we have utilized the Fermat's little theorem in the second step. Thus,
\[
S_2 \equiv 2p \sum_{i=(p+1)/2}^{p-1} \frac{1}{16^i} \binom{2i}{i}^2 \pmod{p^4}.
\]

Recall the following supercongruence [21 (1.9)]:
\[
\sum_{i=(p+1)/2}^{p-1} \frac{1}{16^i} \binom{2i}{i}^2 \equiv -2p^2E_{p-3} \pmod{p^3}.
\]

It follows that
\[
S_2 \equiv -4p^3E_{p-3} \pmod{p^4}. \quad (3.12)
\]

Then the proof of (1.2) follows from (3.3), (3.11) and (3.12).
4 Proof of Theorem 1.2

By (1.5), we have
\[
\sum_{k=0}^{p-1} 3k + 2 \frac{Domb(k)}{(-2)^k} = \sum_{k=0}^{p-1} 3k + 2 \sum_{i=0}^{[k/2]} \left( \frac{2i}{i} \right)^2 \left( \frac{3i}{i} \right) \left( \frac{k + i}{3i} \right) 4^{k-2i}
\]
\[
= \sum_{i=0}^{(p-1)/2} \frac{1}{16^i} \left( \frac{2i}{i} \right)^2 \left( \frac{3i}{i} \right) \sum_{k=2i}^{p-1} (-2)^k (3k + 2) \left( \frac{k + i}{3i} \right) . \tag{4.1}
\]

Recall the following identity [10] (2.4):
\[
\sum_{k=2i}^{n-1} (-2)^k (3k + 2) \left( \frac{k + i}{3i} \right) = (-1)^{n-1} (n - 2i) \left( \frac{n + i}{3i} \right) 2^n , \tag{4.2}
\]
which can be easily proved by induction on \( n \). It follows from (4.1) and (4.2) that
\[
\sum_{k=0}^{p-1} 3k + 2 \frac{Domb(k)}{(-2)^k} = \sum_{i=0}^{(p-1)/2} \frac{2^p (p - 2i)}{16^i} \left( \frac{2i}{i} \right)^2 \left( \frac{3i}{i} \right) \left( \frac{p + i}{3i} \right) \tag{4.3}
\]

For \( 0 \leq i \leq (p - 1)/2 \), we have
\[
(p - 2i) \left( \frac{3i}{i} \right) \left( \frac{p + i}{3i} \right) = \frac{p(p + i) \cdots (p + 1)(p - 1) \cdots (p - 2i)}{i!(2i)!}
\]
\[
= \frac{p(p^2 - 1)(p^2 - 2^2) \cdots (p^2 - i^2)(p - i - 1) \cdots (p - 2i)}{i!(2i)!}
\]
\[
\equiv \frac{p(-1)^i i! (p - i - 1) \cdots (p - 2i)}{(2i)!} \left( 1 - p^2 H_i^{(2)} \right)
\]
\[
\equiv \frac{p(-1)^i i! (p - i - 1) \cdots (p - 2i)}{(2i)!} - p^3 H_i^{(2)} \pmod{p^4} .
\]

Furthermore, we have
\[
\frac{(-1)^i i! (p - i - 1) \cdots (p - 2i)}{(2i)!}
\]
\[
\equiv 1 - p(H_{2i} - H_i) + \frac{p^2}{2} \left( (H_{2i} - H_i)^2 - H_{2i}^{(2)} + H_i^{(2)} \right) \pmod{p^3} .
\]

Thus,
\[
(p - 2i) \left( \frac{3i}{i} \right) \left( \frac{p + i}{3i} \right)
\]
\[
\equiv p - p^2 (H_{2i} - H_i) + \frac{p^3}{2} \left( (H_{2i} - H_i)^2 - H_{2i}^{(2)} - H_i^{(2)} \right) \pmod{p^4} . \tag{4.4}
\]
Combining (4.3) and (4.4) gives
\[
\sum_{k=0}^{p-1} 3k + 2 \left( -\frac{2}{p} \right)^k \text{Domb}(k) \equiv 2^p p \sum_{i=0}^{(p-1)/2} \frac{1}{16^i} \binom{2i}{i}^2 \\
\times \left( 1 - p(H_{2i} - H_i) + \frac{p^2}{2} \left( (H_{2i} - H_i)^2 - H_{2i}^{(2)} - H_i^{(2)} \right) \right) \pmod{p^4}. \tag{4.5}
\]

Finally, substituting (3.8)–(3.10) into (4.5) and using the Fermat’s little theorem, we obtain
\[
\sum_{k=0}^{p-1} 3k + 2 \left( -\frac{2}{p} \right)^k \text{Domb}(k) \equiv 2p (-1)^{n-1} \left( (2^{p-1} - 1)^3 + 1 \right) + 6p^3 E_{p-3} \\
\equiv 2p (-1)^{n-1} + 6p^3 E_{p-3} \pmod{p^4},
\]
as desired.

5 Proof of Theorem 1.3

By (1.6), we have
\[
\sum_{k=0}^{n-1} (2k + 1) \text{Domb}(k) 8^{n-1-k} = \sum_{k=0}^{n-1} (2k + 1) 8^{n-1-i} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \binom{k+i}{i} f_i \\
= \sum_{i=0}^{n-1} (-1)^i 8^{n-1-i} f_i \sum_{k=i}^{n-1} (2k + 1) \binom{k}{i} \binom{k+i}{i}.
\]

Note that
\[
\sum_{k=i}^{n-1} (2k + 1) \binom{k}{i} \binom{k+i}{i} = \frac{n(n-i)}{i+1} \binom{2i}{i} \binom{n+i}{2i},
\]
which can be easily proved by induction on \( n \). Thus,
\[
\frac{1}{n} \sum_{k=0}^{n-1} (2k + 1) \text{Domb}(k) 8^{n-1-k} = \sum_{i=0}^{n-1} (-1)^i 8^{n-1-i} \frac{n(n-i)}{i+1} \binom{2i}{i} \binom{n+i}{2i} f_i. \tag{5.1}
\]

Since the Catalan numbers \( C_i = \frac{(2i)!}{(i+1)} \) are always integral, we conclude that the left-hand side of (5.1) is always a positive integer.

In a similar way, by using (1.6) and the following identity:
\[
\sum_{k=i}^{n-1} (-1)^k (2k + 1) \binom{k}{i} \binom{k+i}{i} = (-1)^{n-1} n \binom{n-1}{i} \binom{n+i}{i},
\]

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we obtain

$$\frac{1}{n} \sum_{k=0}^{n-1} (2k + 1) \text{Domb}(k) (-8)^{n-1-k} = \sum_{i=0}^{n-1} (-1)^i 8^{n-1-i} \binom{n-1}{i} \binom{n+i}{i} f_i.$$  \hspace{1cm} (5.2)

It is easy to see that the left-hand side of (5.2) is always an integer.

Next, we show that the left-hand side of (5.2) is positive. From [24, Proposition 2.8], we conclude that the sequence \( \{ \text{Domb}(k+1)/\text{Domb}(k) \}_{k \geq 0} \) is strictly increasing. For \( k \geq 2 \), we have

$$\frac{\text{Domb}(k+1)}{\text{Domb}(k)} \geq \frac{\text{Domb}(3)}{\text{Domb}(2)} = \frac{64}{7} > 8,$$

and so the sequence \( \{ \text{Domb}(k)/8^k \}_{k \geq 2} \) is strictly increasing. Let

$$a_k = \frac{(2k + 1) \text{Domb}(k)}{8^k}.$$

We immediately conclude that the sequence \( \{ a_k \}_{k \geq 0} \) is strictly increasing (the cases \( k = 0, 1 \) can be easily verified by hand). Thus,

$$a_{n-1} - a_{n-2} + a_{n-3} - \cdots + (-1)^{n-1} a_0 > 0,$$

and so

$$\sum_{k=0}^{n-1} (2k + 1) \text{Domb}(k) (-8)^{n-1-k} = 8^{n-1} (a_{n-1} - a_{n-2} + a_{n-3} - \cdots + (-1)^{n-1} a_0) > 0.$$

This proves the positivity for the left-hand side of (5.2).

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