Critical edge behavior in the modified Jacobi ensemble and Painlevé equations

Shuai-Xia Xu\(^1\) and Yu-Qiu Zhao\(^2\)

\(^1\) Institut Franco-Chinois de l’Energie Nucléaire, Sun Yat-sen University, GuangZhou 510275, People’s Republic of China
\(^2\) Department of Mathematics, Sun Yat-sen University, GuangZhou 510275, People’s Republic of China

E-mail: stszyq@mail.sysu.edu.cn

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Abstract

We study the Jacobi unitary ensemble perturbed by an algebraic singularity at \( t > 1 \). For fixed \( t \), this is the modified Jacobi ensemble studied by Kuijlaars et al. The main focus here, however, is the case when the algebraic singularity approaches the hard edge, namely \( t \to 1^+ \).

In the double scaling limit case when \( t - 1 \) is of the order of magnitude of \( 1/n^2 \), \( n \) being the size of the matrix, the eigenvalue correlation kernel is shown to have a new limiting kernel at the hard edge 1, described by the \( \psi \)-functions for a certain second-order nonlinear equation. The equation is related to the Painlevé III equation by a Möbius transformation. It also furnishes a generalization of the Painlevé V equation, and can be reduced to a particular Painlevé V equation via the Bäcklund transformations in special cases. The transitions of the limiting kernel to Bessel kernels are also investigated, with \( n^2(t - 1) \) being large or small.

In the present paper, the approach is based on the Deift–Zhou nonlinear steepest descent analysis for Riemann–Hilbert problems.

Keywords: Riemann–Hilbert approach, uniform asymptotic approximation, random matrix, modified Jacobi unitary ensemble, Painlevé III equation, Painlevé V equation

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1. Introduction and statement of results

Random matrices were introduced in nuclear physics by Wigner in the 1950s to describe the statistics of the energy level of quantum systems. In the 1960s, Dyson obtained the sine kernel limit of the correlations between eigenvalues in the Gaussian unitary ensemble. He then predicted that the same limit kernel should appear in general random matrix models. This is now known as the famous universality conjecture in random matrices theory.

A unitary ensemble is determined by the probability distributions (see [8, 24])

\[
\frac{1}{Z_n} e^{-\text{tr} V(M)} \, dM = \prod_{i=1}^{n} dM_{ii} \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} d\text{Re} M_{ij} d\text{Im} M_{ij} \quad (1.1)
\]

on the vector space of \( n \times n \) Hermitian matrices \( M = (M_{ij})_{n \times n} \), where \( V \) is a certain potential function, and \( Z_n = \int e^{-\text{tr} w(M)} \, dM \) is the normalization constant. For \( V(x) = x^2/2 \) on \( \mathbb{R} \), we have the classical Gaussian unitary ensemble. While the case \( V(x) = -\ln \left\{ (1-x)^\alpha (1+x)^\beta \right\} \) for \( x \in (-1, 1) \) gives the classical Jacobi unitary ensemble, where \( \alpha > -1 \) and \( \beta > -1 \). It is shown by Dyson [11] that the eigenvalues form a determinantal process with the correlation kernel

\[
K_n(x, y) = e^{-V(x)/2} e^{-V(y)/2} \sum_{k=0}^{n-1} p_k(x) p_k(y), \quad (1.2)
\]

where \( p_k(x) \) denotes the \( k \)th degree orthonormal polynomial with respect to the weight \( e^{-V(x)} \); see also [7, 8, 17, 22, 24]. Using the Christoffel–Darboux formula, (1.2) can be put into the following closed form

\[
K_n(x, y) = \gamma_k^2 e^{-V(x)/2} e^{-V(y)/2} \sum_{k=0}^{n-1} \frac{\pi_k(x) \pi_{k-1}(y) - \pi_{k-1}(x) \pi_k(y)}{x-y}, \quad (1.3)
\]

where \( \gamma_k \) is the leading coefficient of \( p_k(x) \), and \( \pi_k(x) \) is the monic polynomial such that \( p_k(x) = \gamma_k \pi_k(x) \). Thus, to justify the universality conjecture, or, more general, to study the limiting behavior of the kernel \( K_n \) as the size \( n \) tends to infinity, a major step is to obtain the asymptotics of the associated orthogonal polynomials.

There are several results worth mentioning. For the Jacobi unitary ensemble (JUE), we have the limiting mean eigenvalue density

\[
\lim_{n \to \infty} \frac{1}{n} K_n(x, x) = \frac{1}{\pi \sqrt{1-x^2}}, \quad x \in (-1, 1). \quad (1.4)
\]

Moreover, it is well known that in the bulk of the spectrum, the limiting behavior of \( K_n \) is given by the sine kernel

\[
S(x, y) := \frac{\sin \pi(x-y)}{x-y}, \quad (1.5)
\]

in the sense that the limit is independent of the precise reference point in the bulk. The bulk universality is rigorously proved for general unitary ensembles with real analytic potentials \( V \) in (1.1) by Deift et al [9] and for ensembles with continuous potentials by Lubinsky [18, 19].

However, at the hard edge of the spectrum, namely at \( \pm 1 \), the eigenvalue density (1.4) has a square-root singularity. The hard edge universality is investigated and described in this case by the Bessel kernel

\[
J_\alpha(x, y) := \frac{J_\alpha(\sqrt{x}) \sqrt{y} J'_\alpha(\sqrt{y}) - J_\alpha(\sqrt{y}) \sqrt{x} J'_\alpha(\sqrt{x})}{2(x-y)}; \quad (1.6)
\]
The hard edge universality is proved by Kuijlaars et al \cite{16, 17} in the modified JUE with potential
\[ V(x) = -\ln \left\{ (1 - x)^{\alpha} \left( 1 + x \right)^{\beta} \right\}, \quad x \in (-1, 1), \]
where \( \alpha > -1, \beta > -1 \) and \( h \) is positive and analytic on \([-1, 1]\). The reader is also referred to the work of Lubinsky \cite{20, 21} for a new approach to obtain the Bessel kernels near the hard edge \( x = 1 \) for unitary ensemble associated with the Jacobi weight perturbed by a continuous function \( h \) with \( h(1) > 0 \). The analysis and results in \cite{16} may be applied to other Szegö class weights of Jacobi type. However, in cases when the weights decay fast at the endpoints, other kernels, such as the Airy kernel, may be needed to describe the edge behavior; see \cite{31}.

It is worth pointing out that, in the double scaling limit, some kernels involving higher transcendental functions, such as the Painlevé functions, have appeared in certain critical situations. For example, when the eigenvalue density function vanishes at an interior point of the support, the Painlevé II kernel appears as an appropriate double scaling limit of the correlation kernel; see, e.g. \cite{2, 5}. Other limiting kernels involving Painlevé I transcendent and Painlevé III transcendent also appear in the sense of double scaling limits in critical situations, as where the eigenvalue density functions vanish to a higher order than square root \cite{6}, and the potential possesses a simple pole \cite{27}, respectively.

To see the appearance of the Painlevé type kernel in the double scaling limit, we mention a recent work on \( \alpha \)-generalized Airy kernel by Its, Kuijlaars and Östensson \cite{15}. For the Gaussian unitary ensemble GUE(\( n \)), described by the Gaussian measure
\[ \frac{1}{Z_{\text{GUE}(n)}} e^{-2\pi M F} dM, \]
it is well known that at the soft edge, namely, the edge of the support of the equilibrium measure, the scaling limit of the correlation kernel is the Airy kernel; see \cite{24, (24.2.1)}, see also \cite{8}. In \cite{15}, Its, Kuijlaars and Östensson have investigated the Gaussian unitary ensemble perturbed by an algebraic singularity at the soft edge, of the form
\[ \frac{1}{Z_n} \ln \left| \det (M - I) \right|^{2\alpha} e^{-2NtM F} dM. \]
The kernel \( K_n \), given in (1.2), is associated with the perturbed Gaussian weight
\[ |x - 1|^{2\alpha} e^{-2Nt x^2}, \quad x \in \mathbb{R}. \]
As \( N/n \to 1 \), the algebraic singularity in the perturbed term coalesces with the soft edge, which in this case is the edge of the support of the equilibrium measure, with respect to the external field \( 2 \frac{N}{n} x^2 \). Instead of the Airy kernel limit, a so-called \( \alpha \)-generalized Airy kernel is involved in this case. The generalized kernel is described in terms of a certain solution of a Painlevé XXXIV equation.

Similar Painlevé asymptotics can also be derived if the Gaussian weight is perturbed by a Heaviside step function,
\[ e^{-2Nt x^2} \begin{cases} 1, & x < 1; \\ \omega, & x > 1, \end{cases} \]
with \( \omega \) being a non-negative complex constant. The jump discontinuity in the perturbed term also approaches the soft edge as \( N/n \to 1 \); see Xu and Zhao \cite{28}.

In the present work, we consider the perturbed Jacobi unitary random matrix ensemble (pJUE)
\[ \frac{1}{Z_n} e^{-\text{tr} \ln w(M)} dM, \quad (1.7) \]
where the weight
\[
    w(x; t) = (1 - x^2)^\beta \left( t^2 - x^2 \right)^\alpha h(x), \quad x \in (-1, 1),
\]
with \( t \in (1, d], \ d > 1, \ \beta > -1, \ \alpha \in \mathbb{R} \) and \( h(z) \) is analytic in a domain containing \([-1, 1]\), such that \( h(x) > 0 \) for \( x \in [-1, 1] \).

We note that, if \( t \) keeps a positive distance from \([-1, 1]\), the ensemble (1.7) is reduced to the unitary ensemble of Jacobi type associated with the weight
\[
    (1 - x^2)^\beta h_1(x), \quad x \in (-1, 1),
\]
where \( h_1 \) is again analytic and positive on \([-1, 1]\), which furnishes a special case of the modified Jacobi weight considered in Kuijlaars et al [16, 17]. The scaling limit of the eigenvalue correlation kernel (1.2) at the edge \( x = 1 \) is the Bessel kernel \( J_\beta \) of order \( \beta \); see (1.6). While for \( t = 1 \), the same happens, the ensemble is again reduced to the modified Jacobi ensemble with weight
\[
    (1 - x^2)^\alpha h(x), \quad x \in (-1, 1),
\]
where a further restriction \( \alpha + \beta > -1 \) is brought in. Hence this time the scaling limit of the eigenvalue correlation kernel at \( x = 1 \) is the Bessel kernel \( J_{\alpha+\beta} \), of order \( \alpha + \beta \); see (1.6).

In the present paper, however, the main focus will be on the double scaling limit of the correlation kernel in the situation when the algebraic singularity approaches the hard edge, that is \( t \to 1 \), as \( n \to \infty \).

As mentioned earlier, the unitary ensemble of the modified Jacobi type has been studied in, e.g. Kuijlaars et al [16, 17]; see (1.5) and (1.6) for the limiting kernels. The results have been extended in a paper [23] by Martínez-Finkelshtein, McLaughlin and Saff, to a positive weight on the unit circle with Fisher–Hartwig singularities, of the form
\[
    w(z) \prod_{k=0}^{n} |z - e^{i\theta_k}|^{\alpha_k},
\]
where \( \theta_k \) are real, \( \alpha_k > -1 \), and \( w(z) \) is strictly positive and holomorphic on the unit circle. The asymptotics at the singular points can be expressed in terms of the Bessel functions of the first kind.

An example has been provided by Claeyts, Its and Krasovsky [3], with coalescing singularities of algebraic nature in the weight with jumps, of the form
\[
    (z - e^{\omega})^{\alpha+\beta} (z - e^{-\omega})^{\alpha-\beta} e^{-\pi i (\alpha+\beta) V(z)},
\]
where \( \alpha \pm \beta \neq -1, -2, \ldots, t > 0 \), and \( V(z) \) is a specific analytic function on the unit circle. The interesting part in [3] is the transition between the Szegő weight and Fisher–Hartwig weight, as \( t \to 0 \). A particular solution to a Painlevé V is used to describe the intermediate asymptotics. More recently, Claeyts and Krasovsky [4] have studied the Toeplitz determinants with merging algebraic singularities and jumps, with weight
\[
    e^{V(z)} z^{\beta_1+\beta_2} \prod_{j=1}^{2} |z - z_j|^{|\theta_j|} g_{z_j, \beta_j} z_j^{-\beta_j}, \quad z = e^{i\theta}, \ \theta \in [0, 2\pi),
\]
where \( z_1 = e^{i\nu}, \ z_2 = e^{i(2\pi-\rho)} \), and the step functions
\[
    g_{z_j, \beta_j} = \begin{cases} 
        e^{i\pi \beta_j} & 0 \leq \arg z < \arg z_j \\
        e^{-i\pi \beta_j} & \arg z_j \leq \arg z < 2\pi.
    \end{cases}
\]
Certain Painlevé V functions are also involved to describe the transition between the asymptotics of the Toeplitz determinants with different types of singularities.
By a change of variables \( x = \cos \theta \), the polynomials with respect to the weight (1.8) on the interval \([-1, 1]\) are converted to polynomials orthogonal on the unit circle with the weight
\[
w(\cos \theta) \sin \theta = 2^{-2\alpha-2\beta-1} e^{V(z)} |z|^2 - 1 |z - z_j|^\alpha, \quad z = e^{i\theta}, \quad \theta \in [0, 2\pi); \quad (1.9)
\]
see [26, theorem 11.5], where \( V(z) = \ln h(z^{1/2}) \) is analytic on the unit circle, \( z_1 = \psi(t), \ z_2 = 1/\psi(t), \ z_3 = -\psi(t), \ z_4 = -1/\psi(t), \) and \( \psi(t) = t + \sqrt{t^2 - 1} \) for \( t > 1 \). As \( t \to 1 \), the Fisher–Hartwig singularities \( z_j \to 1 \) for \( j = 1, 2 \) and \( z_j \to -1 \) for \( j = 3, 4 \). If the parameter \( \beta = -\frac{1}{2} \), there are no singularities in (1.9) at \( z = \pm 1 \) for \( t > 1 \), and the case was solved in [3].

In the particular case \( \beta = -\frac{1}{2} \), according to [3], the Painlevé V asymptotics is expected for the double scaling limit of the eigenvalue correlation kernel near the hard edge for the perturbed Jacobi weight (1.8), as \( t \to 1 \) and \( n \to \infty \). In a preceding paper [30], we showed this is true, and the Painlevé V kernel plays a role in describing the transition of the limiting kernel from \( J \) to \( J_{\alpha,\beta} \). However, for \( \beta \) other than half integers, there are extra Fisher–Hartwig singularities at \( z = \pm 1 \) in (1.9), and in general the Painlevé V asymptotics is no longer valid.

An alternative way to convert the orthogonality to the unit circle is to introduce a re-scaling in an earlier stage in the weight (1.8), so that
\[
w(tx) = t^{2\alpha} h(tx) (1 - r^2 x^2)^\beta (1 - x^2)^\alpha, \quad x \in (-1/t, 1/t), \quad t > 1.
\]

Then by the same change of variables \( x = \cos \theta \) and applying [26, theorem 11.5], we have the polynomials orthogonal on the unit circle with the weight
\[
\hat{V}(z) |z|^2 - 1 |z - e^{i\theta_j}|^\beta, \quad z = e^{i\theta}, \quad \theta \in (\theta_1, \theta_2) \cup (\theta_3, \theta_4), \quad (1.10)
\]
where \( \hat{V}(z) \) is analytic on the unit circle, \( \theta_1 = \arccos(1/t), \ \theta_2 = \pi - \arccos(1/t), \ \theta_3 = \pi + \arccos(1/t), \) and \( \theta_4 = 2\pi - \arccos(1/t) \). For \( t > 1 \), there are gaps on the unit circle. As \( t \to 1 \), the gaps around the singularities \( z = \pm 1 \) disappear, and the other Fisher–Hartwig singularities at the ends of the gaps merge to \( \pm 1 \). The transition asymptotics of the Toeplitz determinants in [3] and [4] is inspiring. It is of particular interest to study the asymptotics of the Hankel determinants with the weight (1.8), as has been addressed in a separate paper [32].

In the present paper, we will focus on the correlation kernel with respect to (1.8) in the general setting \( \beta > -1 \) and \( \alpha \in \mathbb{R} \). The main goal is to study the transition asymptotics of the eigenvalue correlation kernel for the perturbed Jacobi unitary ensemble (1.8), varying from the Bessel kernel \( J_{\alpha,\beta} \) to \( J_\beta \) as the parameter \( t \) varies from \( t = 1 \) to a fixed \( d > 1 \). It is interesting that a new limiting kernel is obtained, which involves a particular Painlevé III transcendent and, alternatively, a solution to a generalized Painlevé V equation. To obtain our main results, the nonlinear steepest descent method developed by Deift and Zhou is applied.

1.1. Modified Painlevé equation
To state our results, we briefly discuss several equations of Painlevé type.

**Proposition 1.** The function \( y(s) \) in the present paper satisfies the equation of the Painlevé type
\[
\frac{d^2 y}{ds^2} = \frac{2y}{y^3 - 1} \left( \frac{dy}{ds} \right)^2 + \frac{1}{s} \frac{dy}{ds} + \frac{y(y^2 + 1)}{4(y^2 - 1)} + \frac{y}{2s} = \Theta \frac{y}{s} + y \frac{y^2 + 1}{2s} = 0. \quad (1.11)
\]
where $\gamma$ and $\Theta$ are constants. The equation is converted to a generalized Painlevé V equation by putting $\omega = y^2$, so that

$$\frac{d^2\omega}{ds^2} - \left(\frac{1}{\omega - 1} + \frac{1}{2\omega}\right) \left(\frac{d\omega}{ds}\right)^2 + \frac{1}{s} \frac{d\omega}{ds} - \frac{(2\Theta - 1)\omega}{s} + \frac{\omega(\omega + 1)}{2(\omega - 1)} \pm \frac{\gamma\sqrt{\omega}}{s} = 0,$$

(1.12)

which is reduced to the classical Painlevé V equation for $\gamma = 0$. Alternatively, a change of unknown functions $v(s) = y(s) + 1 - \frac{1}{y(s)}$ transforms the equation (1.11) into the Painlevé III equation

$$\frac{d^2v}{ds^2} - \frac{1}{s} \left(\frac{\Theta - \gamma - \frac{1}{2}}{2} v^2 - \frac{\Theta + \gamma - \frac{1}{2}}{2}\right) - \frac{v^3}{16} + \frac{1}{16v} = 0.$$

(1.13)

Moreover, the equation (1.11) is the compatibility condition, namely $\Psi_1^{\lambda}(\lambda, s) = \Psi_1^s(\lambda, s)$, for the following Lax pair

$$\Psi_1^{\lambda}(\lambda, s) = \left(\frac{s\sigma_3}{2} + \frac{A(s)\lambda - 1}{2} + \frac{B(s)}{\lambda} + \frac{\gamma\sigma_1}{\lambda}\right) \Psi^{\lambda}(\lambda, s),$$

(1.14)

$$\Psi_1^s(\lambda, s) = \left(\frac{i\sigma_3}{2} + u(s)\sigma_1\right) \Psi^{\lambda}(\lambda, s),$$

(1.15)

where

$$A(s) = \sigma_1 B(s)\sigma_1, \quad \text{and} \quad B(s) = \left(\begin{array}{cc} b(s) + \frac{\Theta}{2} & -(b(s) + \Theta)y(s) \\
 b(s)/y(s) & -b(s) - \frac{\Theta}{2} \end{array}\right),$$

(1.16)

in which $\sigma_1$ and $\sigma_3$ are the Pauli matrices; see (2.7) below, and $y(s)$ is a specific solution of (1.11), while $b(s)$ and $u(s)$ are determined by the equations

$$s \frac{dy}{ds} = -sy^2 + \frac{b(y^2 - 1)^2}{y} + \Theta(y^2 - 1)y - \gamma(y^2 - 1)$$

(1.17)

and

$$u(s) = \frac{b(s)/y(s) - (b(s) + \Theta)y(s)}{s} + \frac{\gamma}{s}.$$

(1.18)

In section 2.4 below, we will also show that for integer $\gamma$, (1.12) can be reduced to the classic Painlevé V via Bäcklund transformations.

### 1.2. The $\Psi$-functions and the $\Psi$-kernel

We give a brief description of a pair of functions $\psi_1$ and $\psi_2$, upon which the limiting kernel will be constructed. The functions are determined via a model RH problem related to a special solution of (1.11); see (2.40)–(2.43). A detailed analysis will be carried out in section 2.2.

The model RH problem for $\Psi_0(\zeta, s)$ ($\Psi_0(\zeta)$, for short) is as follows.

(a) $\Psi_0(\zeta)$ is analytic in $\zeta \in \mathbb{C} \setminus \bigcup_{j=1}^4 \Sigma_j$, the contours are depicted in figure 1;

(b) $\Psi_0(\zeta)$ satisfies the jump condition

$$\left(\begin{array}{c} e^{-\pi i\Theta\sigma_3} \\ \left(e^{-i(\Theta+y)+\frac{i}{4}}ight) 0 1 \\ 0 1 \end{array}\right), \quad \zeta \in \Sigma_1,$$

$$\left(\begin{array}{c} 1 \\ e^{i(-\Theta+y)+\frac{i}{4}} 0 1 \\ 0 1 \end{array}\right), \quad \zeta \in \Sigma_2,$$

$$\left(\begin{array}{c} \left(e^{-\pi i(-\Theta+y)+\frac{i}{4}}ight) 0 1 \\ 1 0 \end{array}\right), \quad \zeta \in \Sigma_3,$$

$$\left(\begin{array}{c} 0 1 \end{array}\right), \quad \zeta \in \Sigma_4;$$

(1.19)
The asymptotic behavior of $\Psi_0(\zeta)$ at infinity is
$$\Psi_0(\zeta) = \zeta^{\frac{1}{4}} \frac{I - i \sigma_1}{\sqrt{2}} \left( I + \frac{\sigma_3 + i u(s) \sigma_1}{\sqrt{2}} \right) + O \left( \frac{1}{\zeta} \right) e^{\frac{i}{2} \sigma_3 \zeta},$$  \hspace{1cm} (1.20)
for $\zeta \to \infty$, as $\arg \zeta \in (-\pi, \pi)$, where $s \in (0, \infty)$, $\sigma = (b + \frac{\Theta}{2}) s - (su)^{2}$; see (1.18) for $u(s)$.

The behavior of $\Psi_0(\zeta)$ at the origin is
$$\Psi_0(\zeta) = O \left( \frac{1}{\zeta} \right) \zeta^{\frac{1}{4} + \gamma} \left( \frac{O(1)}{O(1) \ln \zeta} \right),$$  \hspace{1cm} (1.21)
for $\zeta \in \Omega_4$, $\zeta \to 0$, the behavior in other sectors can be determined by (1.21) and the jump condition (1.19), and $c = 0$ for $\gamma - \frac{1}{2} \not\in \mathbb{N}$, $c = (-1)^{\gamma + \frac{1}{2}}$ for $\gamma - \frac{1}{2} \in \mathbb{N}$;

The behavior of $\Psi_0(\zeta)$ at $\zeta = \frac{1}{4}$ is
$$\Psi_0(\zeta) = \tilde{\Psi}^{(0)}(\zeta)(\zeta - 1/4)^{-\frac{1}{2} + \Theta}, \hspace{0.5cm} \zeta \to 1/4, \hspace{0.5cm} \arg(\zeta - 1/4) \in (-\pi, \pi),$$  \hspace{1cm} (1.22)
where $\tilde{\Psi}^{(0)}(\zeta)$ is analytic at $\zeta = \frac{1}{4}$.

Using a vanishing lemma argument; see lemma 1 in section 2.3 below, we have the following solvability result:

**Proposition 2.** Assuming $\gamma > -\frac{3}{2}$ and $\Theta \in \mathbb{R}$, for $\zeta \in (0, \infty)$, there exists a unique solution to the model RH problem for $\Psi_0(\zeta, s)$.

Now we put parameters $\gamma = \beta - \frac{1}{2} > -\frac{3}{2}$, $\Theta = -\alpha \in \mathbb{R}$ and the $\psi$-functions
$$\begin{pmatrix} \psi_1(x, s) \\ \psi_2(x, s) \end{pmatrix} = (\Psi_0)_a(x, s) \begin{pmatrix} e^{-\frac{\pi i (a+\beta)}{2}} \\ e^{\frac{\pi i (a+\beta)}{2}} \end{pmatrix}^T, \hspace{0.5cm} x < 0.$$  \hspace{1cm} (1.23)
Accordingly, we define the $\Psi$-kernel as
$$K_\psi(-u, -v; s) = \frac{\psi_1(-u, s) \psi_3(-v, s) - \psi_1(-v, s) \psi_3(-u, s)}{2\pi i (u - v)}$$  \hspace{1cm} (1.24)
for $u, v, s \in (0, \infty)$. 

**Figure 1.** Contours in the $\zeta$-plane of the RH problem for $\Psi_0$. 

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**Figure 1.** Contours in the $\zeta$-plane of the RH problem for $\Psi_0$.

Note that for real $\Theta$, the complex conjugate $\sigma_3(\zeta)$ also satisfies the model RH problem for $\Psi\theta$, then by proposition 2, $\sigma_3\Psi_0(\zeta)$ satisfies the model RH problem for $\Psi\theta$, then by proposition 2, and in view of the asymptotic conditions such as (1.20), we determine the coefficients of the Lax pair (1.16) and several quantities including $u(s)$ and $\sigma(s)$ mentioned above. These are real functions analytic for $s \in (0, \infty)$. Then from (1.18) and $\sigma = (b + \frac{\alpha}{2})s - (su)^2$, we see that the function $b(s)$ in (1.16) is real-valued and analytic in (0, $+\infty$), and $y(s)$, the special solution of (1.11), is real and meromorphic in (0, $+\infty$).

As a corollary of proposition 2, and taking into account the large-$s$ and small-$s$ asymptotics of the model RH problem provided in sections 5 and 6; see for example the large-$\zeta$ asymptotic behavior of $\Psi_0(\zeta, s)$ in (5.20) and (6.16), respectively for $s \to \infty$ and $s \to 0^+$, we have

**Proposition 3.** For $\gamma = \beta - \frac{1}{2} > -\frac{3}{2}, \Theta = -\alpha \in \mathbb{R}$, the functions $\sigma(s)$ and $u(s)$, appearing in (1.20), are real-valued, analytic in $(0, +\infty)$, with boundary behavior

$$\sigma(s) = \frac{-\alpha}{2} + O(1), \quad u(s) = O\left(\frac{1}{s}\right) \quad \text{as} \quad s \to \infty, \quad (1.25)$$

and

$$\sigma(s) = -\frac{\alpha^2}{2} + O\left(\frac{1}{s}\right), \quad u(s) = -\frac{1}{2s} \left(1 + O\left(\frac{1}{s}\right)\right) \quad \text{as} \quad s \to 0^+. \quad (1.26)$$

Also, $b(s)$ is analytic and $y(s)$ is meromorphic in $s \in (0, \infty)$, taking real values, with boundary behavior

$$b(s) = O\left(\frac{1}{s}\right) \quad \text{as} \quad s \to \infty \quad \text{and} \quad b(s) = \frac{-\alpha^2}{4} + O\left(\frac{1}{s}\right) \quad \text{as} \quad s \to 0^+, \quad (1.27)$$

and

$$y(s) = O(1) \quad \text{as} \quad s \to \infty \quad \text{and} \quad y(s) = 1 + O\left(\frac{1}{s}\right) \quad \text{as} \quad s \to 0^+. \quad (1.28)$$

In the small-$s$ behavior, the parameter $l = 2 \min\{1, 1 + \alpha + \beta\}$, and there is an additional restriction that $\alpha + \beta > -1$.

It is worth noting that by applying the invertible piecewise transformations (2.26) and (2.39), we obtain the matrix solution $\Psi(\lambda, s)$ of the Lax pair (1.14) and (1.15), of which (1.11) is the compatibility condition.

1.3. Main results

Now we are ready to present our main results, including a double scaling limit of the eigenvalue correlation kernel, in terms of a Painlevé type kernel, when parameter $s = 4n \ln\left(t + \sqrt{t^2 - 1}\right)$ is around a finite positive number, and the transition of the limiting kernel to the Bessel kernels, respectively as $s \to 0^+$ and $s \to +\infty$.

**Limiting kernel**

The first main result of the present paper is the $\Psi$-description of the limit of the weighted polynomial kernel (1.2), associated with the weight (1.8).

**Theorem 1.** Let $\alpha \in \mathbb{R}, \beta > -1,$ and $K_n(x, y)$ be the weighted polynomial kernel (1.2) associated with the weight (1.8). Then the following holds
(i) For $x \in (-1, 1)$, we have the limiting eigenvalue density

$$
\frac{1}{n} K_n(x, x) = \frac{1}{\pi \sqrt{1 - x^2}} + O\left(\frac{1}{n}\right), \quad \text{as } n \to \infty.
$$

The error term is uniform for $x$ in compact subsets of $(-1, 1)$.

(ii) For fixed $x \in (-1, 1)$, we have the following sine kernel limit uniformly for bounded real variables $u$ and $v$, as $n \to \infty$:

$$
\frac{\pi \sqrt{1 - x^2}}{n} K_n \left( x + \frac{\pi \sqrt{1 - x^2}}{n} u, x + \frac{\pi \sqrt{1 - x^2}}{n} v \right) = \frac{\sin[\pi (u - v)]}{\pi (u - v)} + O\left(\frac{1}{n}\right).
$$

(iii) At the edge of the spectrum, we have the double scaling limit as $n \to \infty$ and $t \to 1^+$ such that

$$
s = 4n \ln(t + \sqrt{t^2 - 1}) \to \tau, \quad \tau \in (0, \infty),
$$

$$
\frac{s^2}{8n^2} K_n \left( 1 - \frac{s^2 u}{8n^2}, 1 - \frac{s^2 v}{8n^2}; t \right) = K_\Psi(-u, -v; \tau) + O\left(n^{-2}\right) + O(s - \tau),
$$

uniformly for $u, v, \tau$ in compact subsets of $(0, \infty)$, where the $\Psi$-kernel $K_\Psi$ is defined in (1.24).

The formulas in (a) and (b) demonstrate the universality phenomenon. Whereas the edge behavior is of special interest since its limiting kernel involves a Painlevé type equation.

Generally, for $1 - r < x, y < 1$ with positive $r$ and $t \in (1, d]$, we have the uniform estimate of the correlation kernel

$$
K_n(x, y) = \frac{(-\psi_2(f_t(y)), \psi_1(f_t(y))) (I + O(x - y)) (\psi_1(f_t(x)), \psi_2(f_t(x)))^T}{2\pi i(x - y)},
$$

where the $\psi$-function is defined in (1.35) and the conformal mapping $f_t(x) = \frac{1}{2} \left( \ln \left( \frac{t + \sqrt{t^2 - 1}}{t - \sqrt{t^2 - 1}} \right) \right)^2$, $\varphi(t) = t + \sqrt{t^2 - 1}$; see (4.14). With the $\psi$-kernel as intermediate limiting kernel and in view of the uniform estimate, we proceed to the transition asymptotics between Bessel kernels.

**Transition to Bessel kernel as $s \to \infty$**

In theorem 1, we obtain the $\Psi$-kernel of Painlevé type in the double scaling limit of the correlation kernel near the hard edge. In the $\Psi$-kernel $K_\Psi(-u, -v; s)$, the parameter $s = 4n \ln(t + \sqrt{t^2 - 1}) \to \tau \in (0, \infty)$ describes the gap between the hard edge $x = 1$ and the singularity $x = t$ of the weight function (1.8). It is also of interest to consider the limit kernel as the parameter $s \to \infty$ or $s \to 0$, which reflects the separating and approaching of the hard edge $x = 1$ and the singularity at $x = t$; see the weight (1.8).

It is worth noting that the double scaling limit case corresponds to $t - 1 = O(1/n^2)$. As the distance between the hard edge $x = 1$ and the singularity $x = t$ becomes large in the sense that $t - 1 \gg 1/n^2$, one has $s = 4n \ln(t + \sqrt{t^2 - 1}) \to \infty$. It will be shown that the limit kernel is reduced to the Bessel kernel $J_{\beta}$, just as in the case when the hard edge 1 is separated from the fixed singularity $t > 1$, previously considered in Kuijlaars et al [16, 17].
Theorem 2. For \( \alpha \in \mathbb{R} \) and \( \beta > -1 \), we have the Bessel type approximation for large \( s \).
(a) The \( \Psi \)-kernel is approximated by the Bessel kernel as \( s \to \infty \)
\[
\frac{4}{s^2} K_{\Phi} \left( -\frac{4u}{s^2}, -\frac{4v}{s^2}; \frac{s}{s^2} \right) = \mathbb{J}_\beta(u, v) \left( 1 + O \left( \frac{1}{s} \right) \right),
\]
where the Bessel kernel \( \mathbb{J}_\beta(u, v) \) is given in (1.6), and the error term is uniform for \( u \) and \( v \) in compact subsets of \( (0, \infty) \).

(b) If the parameter \( t \in (1, \alpha] \) and \( n \to \infty \) such that
\[
s = 4n \ln(t + \sqrt{t^2 - 1}) \to \infty.
\]
Then we have the Bessel kernel limit for \( K_n(x, y) \):
\[
\frac{1}{2n^2} K_n \left( 1 - \frac{u}{2n^2}, 1 - \frac{v}{2n^2}; t \right) = \mathbb{J}_{\alpha+\beta}(u, v) + O \left( \frac{1}{n^2} \right) + O \left( \frac{1}{s} \right),
\]
where \( \mathbb{J}_{\alpha+\beta}(u, v) \) is given in (1.6) and the error terms are uniform in compact subsets of \( u, v \in (0, +\infty) \).

A proof of the theorem will be provided in section 5.

Transition to Bessel kernel as \( s \to 0 \)

As the distance becomes small in the sense that \( t - 1 \ll 1/n^2 \), then \( s = 4n \ln(t + \sqrt{t^2 - 1}) \to 0^+ \). We have the limiting kernel \( \mathbb{J}_{\alpha+\beta} \) in this case, just as the case when the singularity \( x = t \) coincides with the hard edge \( x = 1 \) in the perturbed weight (1.8), which again is the modified Jacobi weight investigated in Kuijlaars et al [16, 17].

Theorem 3. For \( \beta > -1 \) and \( \alpha + \beta > -1 \), we have the Bessel type approximation for small \( s \):
(a) The \( \Psi \)-kernel is approximated by the Bessel kernel as \( s \to 0 \)
\[
\frac{4}{s^2} K_{\Phi} \left( -\frac{4u}{s^2}, -\frac{4v}{s^2}; \frac{s}{s^2} \right) = \mathbb{J}_{\alpha+\beta}(u, v) \left( 1 + O \left( s^l \right) \right);
\]
see (1.6) for the Bessel kernel \( \mathbb{J}_{\alpha+\beta}(u, v) \), where the error term is uniform for \( u \) and \( v \) in compact subsets of \( (0, \infty) \), and \( l = 2 \min\{1, \alpha + \beta + 1\} \).

(b) If the parameter \( t \to 1 \) and \( n \to \infty \) such that
\[
s = 4n \ln(t + \sqrt{t^2 - 1}) \to 0,
\]
then we have the Bessel kernel limit for \( K_n(x, y) \):
\[
\frac{1}{2n^2} K_n \left( 1 - \frac{u}{2n^2}, 1 - \frac{v}{2n^2}; t \right) = \mathbb{J}_{\alpha+\beta}(u, v) + O \left( \frac{1}{n^2} \right) + O \left( s^l \right),
\]
where \( \mathbb{J}_{\alpha+\beta}(u, v) \) is given in (1.6), the error terms are uniform for compact subsets of \( u, v \in (0, +\infty) \), and \( l = 2 \min\{1, \alpha + \beta + 1\} \).

The rest of the paper is arranged as follows. In section 2, we start by studying the Lax pair for the generalized Painlevé V equation, and show that the compatibility of the Lax pair leads to the Painlevé III equation and the generalized Painlevé V equation. The RH problem for \( \Psi(\zeta, s) \) associated with the Lax pair is transformed to the model RH problem for \( \Psi_0(\zeta, s) \) with special monodromy data. The solvability for \( \Psi_0(\zeta, s) \) is then proved for \( s \in (0, \infty) \) by proving proposition 2. Specific Bäcklund transformations will also be established in this section, which implies that in the cases when \( \gamma \) are integers (or, equivalently, when the parameter \( \beta \) in (1.8) are
half-integers), the generalized Painlevé V can be reduced to the classic Painlevé V. In section 3, we carry out, in full detail, the Riemann–Hilbert analysis of the polynomials orthogonal with respect to the weight functions (1.8). Section 4 will be devoted to the proof of theorem 1, based on the asymptotic results obtained in the previous sections. In section 5, we investigate the transition from \( \Psi \)-kernel to the Bessel kernel \( J_\beta \) as \( s \to \infty \), and prove theorem 2. In the last section, section 6, we consider the transition of \( \Psi \)-kernel to the Bessel kernel \( J_{\alpha+\beta} \) as \( s \to 0^+ \), and prove theorem 3. Thus we complete the Bessel to Bessel transition as the parameter \( t \) in (1.8) varies from left to right in a finite interval \((1, d)\).

2. Equations of Painlevé type and a model Riemann–Hilbert problem

In the present section, we study a Lax pair system, of which the compatibility condition is a second-order nonlinear ordinary differential equation. The equation can be transformed to a generalized version of the Painlevé V. If a certain parameter \( \gamma = 0 \), the generalized Painlevé V is reduced to the classical Painlevé V. Also, the second-order equation can be transformed to the standard Painlevé III. Special cases are also investigated when \( \gamma \) are integers. Bäcklund transformations are determined, so that the generalized Painlevé V equation is turned into a special Painlevé V equation.

We also consider a model Riemann–Hilbert problem (RH problem) associated with the specific Lax pair. The solvability of the RH problem is justified. It is worth mentioning that the model problem will play a crucial role in the construction of a parametrix in later sections, and in the description of the edge behavior transition.

2.1. The Lax pair for the generalized Painlevé V

We consider the following Lax pair of first-order systems

\[
\begin{align*}
\Psi_\lambda &= L \Psi, \\
\Psi_s &= U \Psi,
\end{align*}
\]

where

\[
L(\lambda, s) = \frac{s \sigma_3}{2} \lambda - \frac{A(s)}{\lambda - \frac{1}{2}} + \frac{B(s)}{\lambda + \frac{1}{2}} + \frac{\gamma \sigma_1}{\lambda},
\]

\[
U(\lambda, s) = \frac{\lambda \sigma_3}{2} + u(s) \sigma_1,
\]

with coefficients

\[
A(s) = \sigma_1 B(s) \sigma_1, \quad B(s) = \begin{pmatrix} b(s) + \Theta \sigma_2 & -(b(s) + \Theta) y(s) \\ b(s)/y(s) & b(s) - \Theta \sigma_2 \end{pmatrix},
\]

and

\[
u(s) = \frac{b(s)/y(s) - (b(s) + \Theta) y(s)}{s} + \frac{\gamma}{s},
\]

where \( \gamma \) and \( \Theta \) are constants, and the Pauli matrices are defined as

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

For \( \gamma = 0 \) the systems (2.1) and (2.2) are reduced to the Lax pair for the Painlevé V equation of symmetry form, which differs from the ones in [13, 14] by a gauge transformation

\[
\Psi(\lambda, s) = e^{-\frac{i\pi}{4}} \Phi(\lambda + 1/2, s).
\]
It is easily seen that \([L, U] = LU - UL\) is a meromorphic function in \(\lambda\) with only possible singularities at \(\lambda = 0, \pm 1/2\) and \(\infty\), while straightforward calculation gives
\[
[L, U] = O(1) \text{ as } \lambda \to 0, \text{ and } [L, U] = O(1/\lambda) \text{ as } \lambda \to \infty.
\]
So, computing the singular parts of \([L, U]\) at \(\lambda = \pm 1/2\), we have
\[
[L, U] = \left[ A, \frac{1}{4} \sigma_3 + u\sigma_1 \right]_{\lambda = -\frac{1}{2}} + \left[ B, -\frac{1}{4} \sigma_3 + u\sigma_1 \right]_{\lambda = \frac{1}{2}},
\]
where
\[
[B, -\frac{1}{4} \sigma_3 + u\sigma_1] = -u \left\{ \frac{b}{y} + (b + \Theta)y \right\} \sigma_3 + iu(2b + \Theta)\sigma_2 - \frac{1}{2} \left( \begin{array}{cc} 0 & (b + \Theta)y \\ b/y & 0 \end{array} \right);
\]
and
\[
[A, \frac{1}{4} \sigma_3 + u\sigma_1] = \sigma_1 [B, -\frac{1}{4} \sigma_3 + u\sigma_1] \sigma_1.
\]
Thus, the compatibility condition \(L_y - U_x + [L, U] = 0\) is equivalent to
\[
\begin{aligned}
\frac{db}{ds} &= u(b/y + y(b + \Theta)) \\
\frac{dy}{ds} &= (b + \Theta)y + \frac{1}{2}(b/y) \\
\frac{d}{ds}((b + \Theta)y) &= u(2b + \Theta) - \frac{1}{2}(b + \Theta)y,
\end{aligned}
\tag{2.8}
\]
where \(u(s)\) is given in (2.6). It is readily verified that for \(2b + \Theta \neq 0\), (2.8) is in turn equivalent to the set of equations
\[
\begin{aligned}
\frac{db}{ds} &= \frac{b^2}{s^2} - (b + \Theta)^2y^2 + \gamma \left( \frac{b}{y} + y(b + \Theta) \right) \\
\frac{dy}{ds} &= -\frac{sy}{2} + \frac{b(y^2 - 1)^2}{y} + \Theta(y^2 - 1)y - \gamma(y^2 - 1).
\end{aligned}
\tag{2.9}
\]
From the second equation in (2.9) (see (1.17)), \(b(s)\) can be represented in terms of \(y(s)\) and \(y'(s)\). Substituting the representation into the first equation, we see that \(y(s)\) solves the following second-order nonlinear differential equation:
\[
\frac{d^2y}{ds^2} - \frac{2y}{y - 1} \left( \frac{dy}{ds} \right)^2 + \frac{1}{s} \frac{dy}{ds} + \frac{y(y^2 + 1)}{4(y^2 - 1)} + \frac{y}{2s} - \Theta \frac{y}{s} + \gamma \frac{y^2 + 1}{2s} = 0. \tag{2.10}
\]
Let \(\omega(s) = y^2(s)\), then we obtain the generalized Painlevé V equation
\[
\frac{d^2\omega}{ds^2} - \left( \frac{1}{\omega - 1} + \frac{1}{2\omega} \right) \left( \frac{d\omega}{ds} \right)^2 + \frac{1}{s} \frac{d\omega}{ds} + \frac{2(\Theta - 1)\omega}{s} + \frac{\omega(\omega + 1)}{2(\omega - 1)} \pm \gamma \frac{\sqrt{\omega}}{s} (\omega + 1) = 0. \tag{2.11}
\]
Note that for \(\gamma = 0\), the equation is reduced to a special Painlevé V equation; see [13, 29] and [30].

An interesting fact is that the equation (2.10) can be converted to a certain Painlevé III equation. Indeed, taking the following simple Möbius transformation of the unknown function
\[
v(s) = \frac{y(s) + 1}{y(s) - 1}, \tag{2.12}
\]
we obtain the Painlevé III equation
\[
\frac{d^2v}{ds^2} - \left( \frac{d}{s} \right)^2 + \frac{1}{s} \frac{dv}{ds} + \frac{1}{s} \left( \frac{\Theta - \gamma - 1}{2} \right) v^2 - \frac{\Theta + \gamma - \frac{1}{2}}{2} \right) - \frac{v^3}{16} + \frac{1}{16v} = 0; \tag{2.13}
\]
see [13, 14]. All the quantities involved in the Lax pair (2.1) and (2.2) can now be determined by the solution \( y \) to (2.10), or in turn by \( \omega \) in (2.11) and \( v \) in (2.13).

To complete the subsection, we derive an equation for the function \( u(s) \) given in (2.6).

Indeed, a combination of the last two equations in (2.8) yields

\[
\frac{d}{ds} \left( b - (b + \Theta)y \right) = \frac{1}{2} \left( b + (b + \Theta)y \right). \tag{2.14}
\]

Then, in view of (2.6), (2.8) and (2.14), we have

\[
\begin{align*}
\frac{d}{ds} (su) &= 2u \frac{d}{ds} (su) \\
\frac{d}{ds} (su) &= \frac{1}{2} \left( b + y(b + \Theta) \right) \\
\frac{d}{ds} (su) &= u(2b + \Theta) + \frac{1}{4} (su - \gamma).
\end{align*} \tag{2.15}
\]

Representing \( b(s) \) from the third equation of (2.15) and substituting it to the first equation, we find that

\[
2u(su)' = \frac{db}{ds} = \left( \frac{(su)''}{2u} - \frac{\Theta}{2} - \frac{s}{8} + \frac{\gamma}{8u} \right)',
\]

from which we obtain a third order nonlinear differential equation for \( u \)

\[
su'' + u'' \left( 3 - \frac{su'}{u} \right) - 2u^2 - 4su'u'^2 - 4u^3 - \frac{u}{4} - \frac{\gamma u'}{4u} = 0. \tag{2.16}
\]

For later use, we define an auxiliary function

\[
\sigma(s) = (b(s) + \Theta/2)s - (su)^2. \tag{2.17}
\]

Then it is readily seen from (2.15) that

\[
\sigma'(s) = b(s) + \frac{\Theta}{2}. \tag{2.18}
\]

### 2.2. A model Riemann–Hilbert problem

In the present subsection, we construct the two-by-two matrix-valued RH problem for \( \Psi(\lambda) = \Psi(\lambda, s) \) in (2.1) and (2.2). Note that we have introduced in the Lax pair an extra regular singularity at \( \lambda = 0 \), as compared with the Lax pair for the canonic Painlevé V; see [13, proposition 5.9] or [14]. Thus, for the RH problem for the Painlevé V, we have correspondingly an extra singularity at \( \lambda = 0 \); see (2.24) below. In the construction that follows, the symmetry relation \( \sigma_1 \Psi(-\lambda)\sigma_1 = \Psi(\lambda) \) will be used. The regions and contours are illustrated in figure 2.

(a) \( \Psi(\lambda) \) is analytic in \( \mathbb{C} \setminus \{ \Sigma_0 \cup \Sigma_1 \cup \Sigma_2 \cup \Sigma_2^- \} \);

(b) \( \Psi(\lambda) \) satisfies the jump condition

\[
\Psi_+(\lambda) = \Psi_-(\lambda)J, \quad \lambda \in \Sigma_0, \lambda \in \Sigma_1, \\
S_1 = \sigma_1 S_1 \sigma_1 = \begin{pmatrix} 1 & 0 \\ s_0 & 1 \end{pmatrix}, \quad \text{and} \quad S_2 = \sigma_1 S_2 \sigma_1 = \begin{pmatrix} 1 & 0 \\ s_0 & 1 \end{pmatrix},
\]

with

\[
J = E_{1/2}^{-1} e^{-\pi i \Theta/2} E_{1/2}, \quad S_1 = \begin{pmatrix} 1 & 0 \\ s_0 & 1 \end{pmatrix}, \quad \text{and} \quad S_2 = \sigma_1 S_1 \sigma_1 = \begin{pmatrix} 1 & 0 \\ s_0 & 1 \end{pmatrix},
\]

where \( s_0 \) is a complex constant, and the connection matrix \( E_{1/2} \) is constant, such that \( \det E_{1/2} = 1 \);
The asymptotic behavior of $\Psi(\lambda)$ at infinity is

$$
\Psi(\lambda) = \left( I + \frac{c_1(s)\sigma_3 + c_2(s)\sigma_2}{\lambda} + O\left(\frac{1}{\lambda^2}\right) \right) e^{\frac{1}{2}s\lambda\sigma_1}.
$$

(2.20)

Here use has been made of the fact that $\sigma_1\Psi(-\lambda)\sigma_1 = \Psi(\lambda)$. By substituting (2.20) into (2.1), and expanding both sides of (2.1) at infinity in powers of $1/\lambda$, we have

$$
c_1(s) = \sigma(s)/s \quad \text{and} \quad c_2(s) = -iu(s),
$$

where $u(s)$ and $\sigma(s)$ are introduced respectively in (2.6) and (2.17). In computing the coefficients, we have also used (2.3), (2.5), and the fact that by symmetry, the $O(1/\lambda^2)$ term in (2.20) has a leading behavior of the form $(\hat{c}_1(s)I + \hat{c}_2(s)\sigma_1)/\lambda^2$, with scalar functions $\hat{c}_1(s)$ and

$$
\hat{c}_2(s) = \frac{1}{s}\left( u + u\sigma - \frac{1}{2} \left( \frac{b}{y} + (b + \Theta)y \right) \right);
$$

(d) The behavior of $\Psi(\lambda)$ at $\pm \frac{1}{2}$ are respectively

$$
\Psi(\lambda) = \tilde{\Psi}_1(\lambda) (\lambda - 1/2)^{-\frac{1}{4}\Theta} E_{1/2} \quad \text{as} \quad \lambda \to 1/2,
$$

(2.21)

and

$$
\Psi(\lambda) = \tilde{\Psi}_{-1/2}(\lambda) (\lambda + 1/2)^{-\frac{1}{4}\Theta} E_{-1/2} \quad \text{as} \quad \lambda \to -1/2,
$$

(2.22)

where the connection matrices $E_{-1/2} = \sigma_1 E_{1/2}\sigma_1$, the functions $\tilde{\Psi}_{1,2}(\lambda)$ are analytic respectively at $\lambda = \pm \frac{1}{2}$, and the branch cut for $\lambda = 1/2$ is $\Sigma_1$, joined by the line segment $[0, 1/2]$, while the cut for $\lambda = -1/2$ is $[-1/2, 0] \cup \Sigma_2$;

(e) The behavior of $\Psi(\lambda)$ at $\lambda = 0$ can be described by

$$
\Psi_{0h}(\lambda) = \tilde{\Psi}_0(\lambda) \lambda^{\gamma_3} \begin{pmatrix} 1 & c \ln \lambda \\ 0 & 1 \end{pmatrix},
$$

(2.23)
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where \( c = 0 \) for \( \gamma = 0 \) \( \frac{1}{2} \), the branches are chosen such that \( \arg \gamma \in (-3\pi/4, 5\pi/4) \), and \( \tilde{\Psi}_0(\gamma) \) is analytic at \( \gamma = 0 \), with Maclaurin expansion

\[
\tilde{\Psi}_0(\gamma) = \frac{1}{\sqrt{2}} (I - i\sigma_2) \left[ 1 + \sum_{m=1}^{\infty} (a_{1m} + \alpha_{2m}) \gamma^{2m} + (b_{1m} + b_{2m}) \lambda^{2m-1} \right],
\]

with \( b_{11} = \frac{4(\lambda + \frac{1}{2}) + 2i(b + \theta_4)/\sqrt{2}}{4\gamma - 1} \) and \( b_{21} = \frac{8i(\lambda + \frac{1}{2}) + 2i(b + \theta_4)/\sqrt{2}}{4\gamma - 1} \).

The function \( \Psi(\lambda) \), behaving as (2.20) at infinity and fulfilling jump conditions (2.19) on \( \Sigma_2 \) and \( \Sigma_2^\nu \), is related to this function via a connection matrix \( E_0 \), det \( E_0 = 1 \), such that

\[
\Psi(\lambda) = \Psi_{01}(\lambda) E_0 \begin{cases} E_{1/2}^{1/2} e^{\pi i \theta_1} E_{1/2}, & \text{arg } \lambda \in (-3\pi/4, 0), \\ I, & \text{arg } \lambda \in (0, \pi/4), \\ S_1, & \text{arg } \lambda \in (\pi/4, \pi), \\ S_1 E_{-1/2}^{1/2} e^{-\pi i \theta_1} E_{-1/2}, & \text{arg } \lambda \in (\pi, 5\pi/4). \end{cases} \]  

(2.24)

Remark 1. The monodromy data \( \{S_1, S_2, E_0, E_{1/2}, E_{-1/2}\} \) are constrained by the cyclic condition,

\[
E_{1/2}^{-1} \frac{\pi i \theta_1}{E_{1/2} E_0^{-1} e^{-2\pi i \gamma \sigma_3}} \begin{pmatrix} 1 & -2c \pi i \\ 0 & 1 \end{pmatrix} E_0 (E_{-1/2} S_1)^{-1} e^{-\pi i \theta_1} (E_{-1/2} S_2) = S_1 S_2,
\]

(2.25)

where \( c = 0 \) for \( \gamma = 0 \) \( \frac{1}{2} \). Each of the matrices \( E_0, E_{1/2} \) and \( E_{-1/2} \) is determined by (2.25) up to a left-multiplicative diagonal matrix \( \text{diag}(d, d^{-1}) \); see [14], see also [13, p 69, p 108] for derivation of the cyclic condition, and [13, p 205] for a similar description of the behavior at the origin.

In view of the symmetry \( \sigma_1 \Psi(-\lambda) \sigma_1 = \Psi(\lambda) \), we expect a new RH problem on half of the \( \lambda \)-plane. To this end, we introduce a change of variables \( \lambda = \sqrt{\zeta} \), where \( \arg \zeta \in [-\pi, \pi] \), corresponding to \( \arg \lambda \in [-\pi/2, \pi/2] \). Now we take

\[
\hat{\Psi} (\zeta, s) = \zeta^{i\sigma_1} \frac{I + i\sigma_2}{\sqrt{2}} \Psi \left( \sqrt{\zeta}, s \right), \quad \text{arg } \zeta \in [-\pi, \pi]
\]

(2.26)

Then, \( \hat{\Psi} (\zeta, s) (\hat{\Psi} (\zeta), \text{short}) \) solves the following RH problem; see figure 3 for the contours, where \( \Sigma_1 \) and \( \Sigma_2 \) denote the images of the original ones, with the direction being adjusted.
(a) $\hat{\Psi}(\xi)$ is analytic in $\mathbb{C}\setminus \Sigma_j$, $j = 1, 2, 3$;
(b) $\hat{\Psi}(\xi)$ satisfies the jump condition
\[
\hat{\Psi}_+(\xi) = \hat{\Psi}_-(\xi) \begin{cases} E_{1/2}^{-1}e^{-\pi i \Theta \sigma_3}E_{1/2}, & \xi \in \Sigma_1, \\
S_1^{-1}, & \xi \in \Sigma_2, \\
\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, & \xi \in \Sigma_3; \end{cases}
\] (2.27)
(c) The asymptotic behavior of $\hat{\Psi}(\xi)$ at infinity is
\[
\hat{\Psi}(\xi) = \xi^{\frac{1}{4}} \left( I + \frac{i}{2} \frac{c_1(s) \sigma_3 + c_2(s) \sigma_2}{\sqrt{\gamma}} + O(1/\xi) \right) e^{\frac{1}{2} i \gamma \sigma_3},
\] (2.28)
where $\arg \xi \in (-\pi, \pi)$, $s c_1(s) = \sigma(s)$ and $c_2(s) = -i u(s)$; see (2.6) and (2.17) for definition of $u(s)$ and $\sigma(s)$;
(d) The behavior of $\hat{\Psi}(\xi)$ at $\xi = \frac{1}{4}$ is
\[
\hat{\Psi}(\xi) = \hat{\Psi}^{(1)}(\xi)(\xi - 1/4)^{-1/2} E_{1/2},
\] (2.29)
where $\hat{\Psi}^{(1)}(\xi)$ is analytic at $\xi = \frac{1}{4}$;
(e) The behavior of $\hat{\Psi}(\xi)$ at $\xi = 0$ is
\[
\hat{\Psi}(\xi) = \hat{\Psi}^{(0)}(\xi)(\xi + i\epsilon)^{\frac{1}{2} i \gamma \sigma_3} \begin{pmatrix} 1 & \frac{1}{2} \ln \xi \\ 0 & 1 \end{pmatrix},
\] (2.30)
as $\xi \to 0$, where $c = 0$ for $\gamma - \frac{1}{2} \notin \mathbb{N}$, and $\hat{\Psi}^{(0)}(\xi)$ is analytic at $\xi = 0$.

Now we rewrite the cyclic condition (2.25) as
\[
(JS_1 \sigma_1)^2 = E_0^{-1} e^{-2\pi i \gamma \sigma_1} \begin{pmatrix} 1 & -2c \pi i \\ 0 & 1 \end{pmatrix} E_0,
\] (2.31)
where $c = 0$ for $\gamma - \frac{1}{2} \notin \mathbb{N}$, and
\[
J = E_{1/2}^{-1} e^{-\pi i \Theta \sigma_3} E_{1/2} = \begin{pmatrix} e^{-\pi i \sigma_3} & 0 \\ 2i a \sin \pi \Theta & e^{\pi i \sigma_3} \end{pmatrix}
\] if $E_{1/2} := \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$.

We proceed to determine the connection matrices, assuming $E_{1/2}$ takes the specific form. Comparing the trace on both side of equation (2.31) gives
\[
soe^{\pi i \sigma_3} + 2ia \sin \pi \Theta = \pm 2i \sin(\pi \gamma).
\] (2.32)
Taking the minus sign in the above equation, we see that the factorization of
\[
JS_1 \sigma_1 = \begin{pmatrix} 0 & e^{-\pi i \sigma_3} \\ e^{\pi i \sigma_3} & -2i \sin(\pi \gamma) \end{pmatrix}
\] may take the following form: For $\gamma - \frac{1}{2} \notin \mathbb{N}$,
\[
JS_1 \sigma_1 = \begin{pmatrix} e^{-\pi i \sigma_3} & e^{-\pi i \sigma_3} \\ e^{-\pi i \sigma_3} & 0 \end{pmatrix} \begin{pmatrix} e^{-\pi i \sigma_3} & e^{-\pi i \sigma_3} \\ 0 & e^{-\pi i \sigma_3} \end{pmatrix}^{-1},
\]
and, for $\gamma - \frac{1}{2} \in \mathbb{N}$,
\[
JS_1 \sigma_1 = \begin{pmatrix} e^{-\pi i \sigma_3} & 0 \\ e^{-\pi i \sigma_3} & 1 \end{pmatrix} \begin{pmatrix} e^{-\pi i \sigma_3} & 1 \\ 0 & e^{-\pi i \sigma_3} \end{pmatrix} \begin{pmatrix} e^{-\pi i \sigma_3} & 0 \\ 1 & e^{-\pi i \sigma_3} \end{pmatrix}^{-1}.
\]
Therefore, comparing these with (2.31), we can determine
\[
E_{1/2} = \begin{pmatrix} 1 & 0 \\ -2i \sin(\pi \gamma)oe^{\pi i \sigma_3} & 1 \end{pmatrix}, \quad \Theta \notin \mathbb{Z}
\] (2.33)
and
\[
E_0 = \begin{cases} 
\left( \frac{e^{i\pi \gamma}}{2 \cos(\pi \gamma)} \right)^{1/2} \begin{pmatrix} -e^{-i\pi \gamma} & -e^{-i\pi \phi} \\ -e^{-i\pi \phi} & e^{-i\pi \phi} \end{pmatrix}, & \gamma - \frac{1}{2} \notin \mathbb{N}, \\
\begin{pmatrix} e^{i\pi \phi/2} & 0 \\ 0 & e^{-i\pi \phi/2} \end{pmatrix}, & \gamma - \frac{1}{2} \in \mathbb{N},
\end{cases}
\] (2.34)
each up to a left-multiplicative diagonal matrix \(d^\alpha\). Accordingly, we have
\[
c = \begin{cases} 
0, & \gamma - \frac{1}{2} \notin \mathbb{N}, \\
\frac{1}{2}(1)^{\gamma+\frac{1}{2}}, & \gamma - \frac{1}{2} \in \mathbb{N}.
\end{cases}
\] (2.35)
For later use, we choose the specific Stokes multiplier
\[
s_0 = -2i \sin(\pi(\gamma - \Theta)).
\] (2.36)
Substituting (2.36) in (2.33), we obtain
\[
E_{1/2} = \begin{pmatrix} 1 & 0 \\ -e^{i(\Theta - \gamma)} & 1 \end{pmatrix}.
\] (2.37)
In later sections, we also need to compute
\[
E_0E_{1/2}^{-1} = \begin{cases} 
\left( \frac{e^{i\pi \gamma}}{2 \cos(\pi \gamma)} \right)^{1/2} \begin{pmatrix} -2 \cos(\pi \gamma) & -2 \cos(\pi \phi) \\ 0 & e^{-i\pi \phi} \end{pmatrix}, & \gamma - \frac{1}{2} \notin \mathbb{N}, \\
\begin{pmatrix} e^{i\pi \phi/2} & 0 \\ 0 & e^{-i\pi \phi/2} \end{pmatrix}, & \gamma - \frac{1}{2} \in \mathbb{N}.
\end{cases}
\] (2.38)
Now we are in a position to state the model RH problem, to be applied later to the Riemann–Hilbert analysis, for the matrix function
\[
\Psi_0(\zeta, s) = \begin{cases} 
-e^{i\pi \phi} \overline{\Psi}(\zeta, s)E_{1/2}^{-1}e^{i\pi \phi}, & \zeta \in \Omega_1 \cup \Omega_4, \\
e^{-i\pi \phi} \overline{\Psi}(\zeta, s)e^{i\pi \phi}, & \zeta \in \Omega_2 \cup \Omega_3.
\end{cases}
\] (2.39)
The contours and regions are illustrated in figure 1.
(a) \(\Psi_0(\zeta, s)\) (\(\Psi_0(\zeta)\), for short) is analytic in \(\zeta \in \mathbb{C} \setminus \bigcup_{j=1}^{4} \Sigma_j\);
(b) \(\Psi_0(\zeta)\) satisfies the jump condition
\[
(\Psi_0)_+ (\zeta) = (\Psi_0)_- (\zeta) \begin{cases} 
e^{-i\pi \phi}, & \zeta \in \Sigma_1, \\
\begin{pmatrix} 1 \\ e^{i(-\phi+\frac{1}{2})} \\ 1 \end{pmatrix}, & \zeta \in \Sigma_2, \\
\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, & \zeta \in \Sigma_3, \\
\begin{pmatrix} 1 \\ e^{-i(-\phi+\frac{1}{2})} \\ 1 \end{pmatrix}, & \zeta \in \Sigma_4
\end{cases}
\] (2.40)
(c) The asymptotic behavior of \(\Psi_0(\zeta)\) at infinity is
\[
\Psi_0(\zeta) = \zeta^{\frac{1}{2}\eta_i} \frac{1 - i\sigma_1}{\sqrt{2}} \left( I + \frac{c_1(s)\sigma_3 + c_2(s)\sigma_1}{\sqrt{2}} + O \left( \frac{1}{\zeta} \right) \right) e^{i\frac{\pi}{2}\sigma_3} 
\] (2.41)
for \(\zeta \to \infty\), as \(\arg \zeta \in (-\pi, \pi)\), where \(s \in (0, \infty)\), \(sc_1(s) = \sigma(s)\) and \(c_2(s) = -i\sigma(s)\); see (2.6) and (2.17) for \(u(s)\) and \(\sigma(s)\), respectively;
(d) The behavior of \(\Psi_0(\zeta)\) at the origin is
\[
\Psi_0(\zeta) = O(1) \zeta^{(\frac{1}{2} + \frac{1}{2}\eta_i)} \begin{pmatrix} O(1) \\ O(1 + c \ln \zeta) \\ O(1) \end{pmatrix},
\] (2.42)
for \(\zeta \in \Omega_4, \zeta \to 0\), and the behavior in other sectors can be determined by (2.42) and the jump condition (2.40). Here use has also been made of (2.38), and \(c\) is given in (2.35).
(e) The behavior of \( \Psi_0(\zeta) \) at \( \zeta = \frac{1}{4} \) is
\[
\Psi_0(\zeta) = \tilde{\Psi}^{(0)}(\zeta)(\zeta - 1/4)^{-i\Theta_0}, \quad \zeta \to 1/4, \quad \arg(\zeta - 1/4) \in (-\pi, \pi),
\]
where \( \tilde{\Psi}^{(0)}(\zeta) \) is analytic at \( \zeta = \frac{1}{4} \).

2.3. Solvability of the Riemann–Hilbert problem

We turn to the solvability of the RH problem for \( \Psi_0(\zeta, s) \) for \( s \in (0, \infty) \). To this aim, we put (2.41) in the form
\[
\Psi_0(\zeta, s) = \left( I + (ic_1(s) - c_2(s))\sigma_x + O \left( \frac{1}{\zeta} \right) \right) \zeta^{\gamma/2} \frac{I - i\sigma_1 e^{i\sqrt{\sigma_1}}}{\sqrt{2}}, \quad \arg \zeta \in (-\pi, \pi),
\]
as \( \zeta \to \infty \), where \( \sigma_x = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \). Here use has been made of the fact that the leading \( O(1/\zeta) \) term in (2.41) is of the form \( (\tilde{c}_1(s)I + \tilde{c}_2(s)\sigma_2)/\zeta \), with scalar \( \tilde{c}_1 \) and \( \tilde{c}_2 \). A similar argument has previously been used in the derivation of (2.20).

We obtain the solvability of the RHP for \( \Psi_0(\zeta, s) \) by proving a vanishing lemma. Similar arguments can be found in, e.g. [13–15, 28]. In particular, a vanishing lemma for Painlevé \( V \) with special Stokes multipliers is given in [3, lemma 4.7], and a proof of the lemma, very similar to the one we will carry out, has been given in [30]. The main difference in the present case lies in an extra singularity at the origin, indicated by the parameter \( \gamma \).

**Lemma 1.** Assume that the homogeneous RH problem for \( \Psi_0^{(1)}(\zeta) = \Psi_0^{(1)}(\zeta, s) \) adapts the same jump (2.40) and the same boundary conditions (2.42)–(2.43) as \( \Psi_0(\zeta) \), with the behavior (2.44) at infinity being altered to
\[
\Psi_0^{(1)}(\zeta, s) = O \left( \frac{1}{\zeta} \right) \zeta^{\gamma/2} \frac{I - i\sigma_1 e^{i\sqrt{\sigma_1}}}{\sqrt{2}}, \quad \arg \zeta \in (-\pi, \pi), \quad \zeta \to \infty.
\]
If \( \gamma > -3/2 \), and the parameter \( s \in (0, \infty) \), then \( \Psi_0^{(1)}(\zeta, s) \equiv 0 \).

**Proof.** First, we remove the exponential factor at infinity and eliminate the jumps on \( \Sigma_2 \) and \( \Sigma_4 \) by defining
\[
\Psi_0^{(2)}(\zeta) = \begin{cases} 
\Psi_0^{(1)}(\zeta)e^{-\frac{i\sqrt{\sigma_1}}{2}}, & \text{for } \zeta \in \Omega_1 \cup \Omega_4, \\
\Psi_0^{(1)}(\zeta)e^{-\frac{i\sqrt{\sigma_1}}{2}} \begin{pmatrix} 1 & 0 \\ e^{\pi i(\theta + 1/2)} e^{-i\sqrt{\sigma_1}} & 1 \end{pmatrix}, & \text{for } \zeta \in \Omega_2, \\
\Psi_0^{(1)}(\zeta)e^{-\frac{i\sqrt{\sigma_1}}{2}} \begin{pmatrix} 1 & 0 \\ -e^{-\pi i(\theta + 1/2)} e^{-i\sqrt{\sigma_1}} & 1 \end{pmatrix}, & \text{for } \zeta \in \Omega_3;
\end{cases}
\]
See figure 1 for the regions \( \Omega_1 \cup \Omega_4 \), where \( \arg \zeta \in (-\pi, \pi) \). It is easily verified that \( \Psi_0^{(2)}(\zeta) \) solves the following RH problem:

(a) \( \Psi_0^{(2)}(\zeta) \) is analytic in \( \zeta \in \mathbb{C} \setminus \Sigma_1 \cup \Sigma_3 \) (see figure 1);
(b) \( \Psi_0^{(2)}(\zeta) \) satisfies the jump condition
\[
\left( \Psi_0^{(2)} \right)_+ (\zeta) = \left( \Psi_0^{(2)} \right)_- (\zeta) \begin{pmatrix} e^{-\pi i\Theta_1}, & 0 \\ e^{-\pi i\Theta_1} e^{-i\sqrt{\sigma_1}}, & e^{\pi i\Theta_1} e^{i\sqrt{\sigma_1}} \end{pmatrix}, \quad \zeta \in \Sigma_1,
\]
where \( \Theta_1 = \Theta - \gamma - 1/2 \), and \( \sqrt{\sigma_1} = i\sqrt{|\zeta|} \) for \( \zeta \in \Sigma_3 $;
(c) The asymptotic behavior of $\Psi_0^{(2)}(\zeta)$ at infinity is
\[ \Psi_0^{(2)}(\zeta) = O\left(\zeta^{-\frac{3}{4}}\right), \quad \zeta \to \infty; \tag{2.48} \]

(d) The behavior of $\Psi_0^{(2)}(\zeta)$ at the origin is
\[ \Psi_0^{(2)}(\zeta) = O(1) \zeta^{\frac{1}{4} + \frac{\gamma}{2} + \frac{i}{2} + c \ln \zeta), \quad \zeta \to 0; \tag{2.49} \]
where $c = 0$ for $\gamma - \frac{1}{2} \notin \mathbb{N};$

(e) The behavior of $\Psi_0^{(2)}(\zeta)$ at $\zeta = \frac{1}{4}$ is
\[ \Psi_0^{(2)}(\zeta) = O(1) (\zeta - 1/4)^{-\frac{1}{2}}. \tag{2.50} \]

We carry out yet another transformation to move the oscillating entries in the jump matrices to off-diagonal, as follows:
\[ \Psi_0^{(3)}(\zeta) = \begin{cases} 
\Psi_0^{(2)}(\zeta) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & \text{for } \operatorname{Im} \zeta > 0, \\
\Psi_0^{(2)}(\zeta), & \text{for } \operatorname{Im} \zeta < 0.
\end{cases} \tag{2.51} \]

Then $\Psi_0^{(3)}(\zeta)$ solves a RH problem with different jumps
\[ \left( \begin{array}{c}
\Psi_0^{(3)}(\zeta) \\
\Psi_0^{(3)}(\bar{\zeta})
\end{array} \right) = \left( \begin{array}{c}
\Psi_0^{(3)}(\zeta) \\
\Psi_0^{(3)}(\bar{\zeta})
\end{array} \right) J^{(3)}(\zeta), \tag{2.52} \]
where
\[ J^{(3)}(\zeta) = \begin{cases} 
1 & -e^{-\pi i (\theta - \gamma - 1/2)} e^{i \sqrt{\tau} \zeta} \\
e^{\pi i (\theta - \gamma)} e^{-i \sqrt{\tau} \zeta} & 0 \\
e^{\pi i (\theta - \gamma)} & 0 \\
0 & -1 \\
1 & 0
\end{cases}, \quad \zeta \in (-\infty, 0), \tag{2.53} \]
\[ J^{(3)}(\zeta) = \begin{cases} 
0 & 0 \\
0 & 0 \\
2 & 0 \\
0 & 0
\end{cases}, \quad \zeta \in (0, +\infty). \]

At infinity, $\Psi_0^{(3)}$ behaves the same as $\Psi_0^{(2)}$ does; see (2.48). Whereas the behavior at $\zeta = 0$ changes to
\[ \Psi_0^{(3)}(\zeta) = O(1) \zeta^{\frac{1}{4} + \frac{\gamma}{2} + \frac{i}{2} + c \ln \zeta), \quad \zeta \to 0; \tag{2.54} \]
as $\zeta \to 0$, where $c = 0$ for $\gamma - \frac{1}{2} \notin \mathbb{N}$, and the condition at $\zeta = 1/4$ now takes the form
\[ \Psi_0^{(3)}(\zeta) \sigma_2 = O(1) (\zeta - 1/4)^{-\frac{1}{2}} \begin{cases} 
\sigma_2, & \arg \zeta \in (0, \pi), \\
I, & \arg \zeta \in (-\pi, 0).
\end{cases} \tag{2.55} \]

It is readily seen that
\[ (J^{(3)}(\zeta))^* + J^{(3)}(\zeta) = \begin{cases} 
0 & 0 \\
0 & 0 \\
2 & 0 \\
0 & 0
\end{cases}, \quad \zeta \in (0, +\infty), \tag{2.56} \]
where $X^*$ denotes the Hermitian conjugate of a matrix $X$.

Next, we define an auxiliary matrix function
\[ H(\zeta) = \Psi_0^{(3)}(\zeta) \left( \Psi_0^{(3)}(\bar{\zeta}) \right)^* \quad \text{for } \zeta \notin \mathbb{R}. \tag{2.57} \]
Then $H(\zeta)$ is analytic in $\mathbb{C}\backslash \mathbb{R}$. Since $\Psi_0^{(3)}$ behaves the same as $\Psi_0^{(2)}$ at infinity, a combination of (2.57) and (2.48) yields

$$H(\zeta) = O(\zeta^{-3/2}) \quad \text{as} \quad \zeta \to \infty.$$  \hfill (2.58)

Similarly, combining (2.57) with (2.54) and (2.55) gives

$$H(\zeta) = O(1) \quad \text{as} \quad \zeta \to 1/4,$$  \hfill (2.59)

and

$$H(\zeta) = O \left( \zeta^{3+\gamma} \ln \zeta \right) \quad \text{as} \quad \zeta \to 0.$$  \hfill (2.60)

Here use has been made of the fact that $\tau^0 \sigma_2 \tau^0 = \sigma_2$ for non-vanishing scalar $\tau$, and that

$$\begin{pmatrix} O(1) & O(\ln \zeta) \\ 0 & O(1) \end{pmatrix} \sigma_2 \begin{pmatrix} O(1) & O(\ln \zeta) \\ 0 & O(1) \end{pmatrix}^* = \begin{pmatrix} O(\ln \zeta) & O(1) \\ O(1) & 0 \end{pmatrix}^*.$$

Thus, for $\gamma > -3/2$, applying Cauchy’s integral theorem, we have

$$\int_{\mathbb{R}} H_+ (\zeta) d\zeta = 0.$$  \hfill (2.61)

Now in view of (2.57), and adding to (2.61) its Hermitian conjugate, we have

$$2 \int_{-\infty}^0 \left( \Psi_0^{(3)}_+ (\zeta) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Psi_0^{(3)}_+ (\zeta) \right) d\zeta = 0.$$  \hfill (2.62)

A straightforward consequence is that the first column of $(\Psi_0^{(3)}_- (\zeta))$ vanishes for $\zeta \in (-\infty, 0)$. Furthermore, it follows from (2.53) that the second column of $(\Psi_0^{(3)}_+ (\zeta))$ also vanishes for $\zeta \in (-\infty, 0)$.

The jump $J^{(3)}_+ (\zeta)$ in (2.53) admits an analytic continuation in a neighborhood of $(-\infty, 0)$. Accordingly, using (2.52) we can extend $\Psi_0^{(3)}_+ (\zeta)$ from arg $\zeta \in (0, \pi)$ analytically to a larger sector arg $\zeta \in (0, 2\pi)$, such that $(\Psi_0^{(3)}_+ (\zeta))_{12} = (\Psi_0^{(3)}_+ (\zeta))_{22} = 0$ for arg $\zeta = \pi$. Hence we have

$$\begin{pmatrix} \Psi_0^{(3)}_+ (\zeta) \end{pmatrix}_{12} = \begin{pmatrix} \Psi_0^{(3)}_+ (\zeta) \end{pmatrix}_{22} = 0, \quad \text{Im} \, \zeta > 0.$$  \hfill (2.63)

Similarly, by analytically extending $\Psi_0^{(3)}_+ (\zeta)$ to arg $\zeta \in (-2\pi, 0)$, we have

$$\begin{pmatrix} \Psi_0^{(3)}_+ (\zeta) \end{pmatrix}_{11} = \begin{pmatrix} \Psi_0^{(3)}_+ (\zeta) \end{pmatrix}_{21} = 0, \quad \text{Im} \, \zeta < 0.$$  \hfill (2.64)

The reader is referred to [30] for a similar argument.

Now we proceed to examine the other entries of $\Psi_0^{(3)}_+ (\zeta)$ by appealing to Carlson’s theorem (see [25, p 236]). To this aim, for $k = 1, 2$, we define scalar functions

$$g_k (\zeta) = \begin{cases} \begin{pmatrix} \Psi_0^{(3)}_+ (\zeta) \end{pmatrix}_{k1}, & \text{for } 0 < \arg \zeta < \pi, \\ \begin{pmatrix} \Psi_0^{(3)}_+ (\zeta) \end{pmatrix}_{k2}, & \text{for } -\pi < \arg \zeta < 0. \end{cases}$$  \hfill (2.65)

From (2.53), (2.63) and (2.64), we see that each $g_k (\zeta)$ is analytic in $\mathbb{C}\backslash (-\infty, 1/4]$, and satisfies the jump conditions

$$(g_k)_+ (\zeta) = (g_k)_- (\zeta) e^{-\pi i (\gamma - \Theta + 1/2)} \sqrt{\zeta}, \quad \zeta \in (-\infty, 0),$$  \hfill (2.66)

where $\sqrt{\zeta_+} = i \sqrt{|\zeta|}$, and

$$(g_k)_+ (\zeta) = (g_k)_- (\zeta) e^{\pi i \Theta}, \quad \zeta \in (0, 1/4).$$  \hfill (2.67)
The sector of analyticity can be extended as follows:

\[
\hat{g}_k(\zeta) = \begin{cases} 
  g_k(e^{-2\pi i} \zeta) e^{-\pi i (\gamma - \Theta + 1/2)} e^{\sqrt{\zeta}}, & \text{for } \pi \leq \arg \zeta < 2\pi, \\
  g_k(e^{2\pi i} \zeta) e^{\pi i (\gamma - \Theta + 1/2)} e^{\sqrt{\zeta}}, & \text{for } -2\pi < \arg \zeta \leq -\pi.
\end{cases}
\]  

(2.68)

Thus, \( \hat{g}_k(\zeta) \) is now analytic in a cut-sector \(-2\pi < \arg \zeta < 2\pi\) and \( \zeta \notin [0, 1/4] \). It is worth noting that \( \hat{g}_k(\zeta) \) can be further extended analytically to \(-2\pi \leq \arg \zeta \leq 2\pi\) and \(|\zeta| > 1/4\), and that for \( s \in (0, \infty) \), the exponential term \( |e^{\sqrt{\zeta}}| \leq 1 \) for \( \pi \leq \arg \zeta < 2\pi \) and \(-2\pi < \arg \zeta \leq -\pi \).

If we put \( h_k(\zeta) = \hat{g}_k((\zeta + 1)^4) \) for \( \arg \zeta \in [-\pi/2, \pi/2] \),

(2.69)

then the above discussion implies that \( h_k(\zeta) \) is analytic in \( \Re \zeta > 0 \), continuous and bounded in \( \Re \zeta \geq 0 \), and satisfies the decay condition on the imaginary axis

\[
|h_k(\zeta)| = O \left( e^{-|\zeta|^2} \right), \quad \text{for } \Re \zeta = 0 \text{ as } |\zeta| \to \infty.
\]  

(2.70)

Hence, Carlson’s theorem applies, and we have \( h_k(\zeta) \equiv 0 \) for \( \Re \zeta > 0 \). Tracing back, we see that all entries of \( \Psi_0^\Theta(\zeta) \) vanish for \( \zeta \notin \mathbb{R} \); see (2.63)–(2.65). Therefore, \( \Psi_0^\Theta(\zeta) \) vanishes identically, which implies that \( \Psi_0^\Theta(\zeta) \) vanishes identically. This completes the proof of the vanishing lemma.

The solvability of the RH problem for \( \Psi_0 \) follows from the vanishing lemma. As briefly indicated in [13, p 104], the RH problem is equivalent to a Cauchy-type singular integral equations, the corresponding singular integral operator is a Fredholm operator of index zero. The vanishing lemma states that the null space is trivial, which implies that the singular integral equation (and thus \( \Psi_0 \)) is solvable as a result of the Fredholm alternative theorem. More details can be found in [15, proposition 2.4]; see also [8, 9, 13, 14] for standard methods connecting RH problems with integral equations.

Now we have the solvability result given in proposition 2, which states that for \( \gamma > -3/2, \Theta \in \mathbb{R}, \) and \( s \in (0, \infty) \), there exists a unique solution \( \Psi_0(\zeta, s) \) to the RH problem (2.40)–(2.43).

2.4. Bäcklund transformation

From (2.11), we see that the generalized Painlevé V equation is reduced to the classical Painlevé V equation as \( \gamma = 0 \). In this section, we study the Bäcklund transformation of the generalized Painlevé V equation. We will show that, by applying a certain Bäcklund transformation, (2.11) is turned into an equation of the same form, with only the parameter \( \gamma \) being replaced by \( \gamma' = -\gamma \pm 1 \). In particular, when \( \gamma \) is an integer, making use of such Bäcklund transformations \(|\gamma| \) times, the equation (2.11) can be transformed to a specific Painlevé V equation.

We seek a rational gauge transformation

\[
\tilde{\Psi}(\lambda, s) = F(\lambda, s) \Psi(\lambda, s),
\]  

(2.71)

which preserves the canonical asymptotic structure of \( \Psi \) at infinity and at the regular singularity \( \zeta = \pm 1/2 \); see (2.20)–(2.22), and shifts the formal monodromy exponent at the origin. In (2.71), we take

\[
F(\lambda, s) = I + \frac{F_1(s)}{\lambda};
\]  

(2.72)
compare, e.g. Fokas et al [13, (6.1.2)–(6.1.4)]. We assume that the form of the $\lambda$-equation (2.1) is preserved for $\Psi$, with $L$ being replaced by a certain
\[ \tilde{L}(\lambda, s) = \tilde{C}(s) + \tilde{A}(s) \frac{\lambda}{\lambda - \frac{1}{2}} + \tilde{B}(s) \frac{\lambda + \frac{1}{2}}{\lambda} + \tilde{\gamma} \sigma_1, \]
with a shifted $\tilde{\gamma}$. Similar discussion can be found in [13, chapter 6]. Substituting $\tilde{\Psi}$ into (2.1), we have
\[ F_i + FL = \tilde{L}F. \] (2.73)

The equation (2.73) splits into five equations
\[ \lambda^{-2}: \quad -F_1 + \gamma F_1 \sigma_1 = \tilde{\gamma} \sigma_1 F_1, \]
\[ 1 : \quad \frac{\gamma}{2} \sigma_1 = \tilde{C}, \]
\[ (\lambda - \frac{1}{2})^{-1}: \quad (I + 2 F_1) A = \tilde{A} (I + 2 F_1), \]
\[ (\lambda + \frac{1}{2})^{-1}: \quad (I - 2 F_1) B = \tilde{B} (I - 2 F_1), \]
\[ \lambda^{-1}: \quad \gamma \sigma_1 + \frac{1}{2} \tilde{A} \sigma_1 - 2 F_1 A + 2 F_1 B = \tilde{\gamma} \sigma_1 + \tilde{\gamma} \sigma_1 F_1 - 2 \tilde{\gamma} F_1 + 2 \tilde{B} F_1; \]
where $A$ and $B$ are the specified matrices given in (2.5).

Assuming that $\det F(\lambda, s) \equiv 1$, or, equivalently, $tr F_1(s) \equiv 0$ and $det F_1(s) \equiv 0$, from the first equation we have non-vanishing $F_1$ if only $\gamma + \tilde{\gamma} = \pm 1$. More precisely, we can write
\[ F_1(s) = \kappa(s) (\sigma_1 \pm i \sigma_2) \quad \text{for} \quad \gamma + \tilde{\gamma} = \pm 1, \] (2.74)

where $\kappa(s)$ is a scalar function to be determined.

From the second to the fourth equation, we obtain
\[ \tilde{C} = \frac{s}{2} \sigma_3, \quad \tilde{A} = (I + 2 F_1) A (I - 2 F_1), \quad \text{and} \quad \tilde{B} = (I - 2 F_1) B (I + 2 F_1). \] (2.75)

Substituting these into the fifth equation, we see that
\[ \kappa(s) = \frac{\gamma - \tilde{\gamma}}{\pm 8 (b + \Theta) + \frac{4b}{\gamma} + 4(b + \Theta)y \pm s} \quad \text{for} \quad \gamma + \tilde{\gamma} = \pm 1. \] (2.76)

The rational gauge transformation (2.71) is thus determined.

The Bäcklund transformation can be deduced from (2.75). Indeed, assuming that $\tilde{B}(s)$ takes the form of $B(s)$ as in (2.5), for $\gamma + \tilde{\gamma} = 1$, we define a set of functions $\tilde{y}(s), \tilde{b}(s)$ and $\tilde{\Theta}(s)$ as
\[
\begin{align*}
\tilde{b} + \frac{\Theta}{2} &= (1 - 8 \kappa^2) \left( b + \frac{\Theta}{2} \right) - 2 \kappa (1 + 2 \kappa) \frac{b}{2} + 2 \kappa (1 - 2 \kappa) (b + \Theta)y, \\
- (\tilde{b} + \tilde{\Theta}) \tilde{y} &= 4 \kappa (1 - 2 \kappa) \left( b + \frac{\Theta}{2} \right) - 4 \kappa^2 \frac{b}{2} - (1 - 2 \kappa)^2 (b + \Theta)y, \\
\frac{\tilde{b} + \tilde{\Theta}}{\gamma} &= 4 \kappa (1 + 2 \kappa) \left( b + \frac{\Theta}{2} \right) + (1 + 2 \kappa) \frac{b}{2} + 4 \kappa^2 (b + \Theta)y.
\end{align*}
\] (2.77)

It is readily seen that $\det B(s) = \det \tilde{B}(s)$, which implies $\tilde{\Theta}^2 = \Theta^2$. Hence, $\tilde{\Theta}$ is independent of $s$, and we may put $\tilde{\Theta} = \Theta$. Straightforward verification then shows that
\[
\begin{align*}
\frac{d}{ds} \left( \tilde{b} + \tilde{\Theta}/2 \right) &= \tilde{u} \left( \tilde{b}/\tilde{y} + \tilde{y}(\tilde{b} + \Theta) \right), \\
\frac{d}{ds} \left( \tilde{b}/\tilde{y} \right) &= 2 \tilde{u} (\tilde{b} + \Theta/2) + \frac{1}{2} (\tilde{b}/\tilde{y}), \\
\frac{d}{ds} \left( (\tilde{b} + \Theta)\tilde{y} \right) &= 2 \tilde{u} (\tilde{b} + \Theta/2) - \frac{1}{2} (\tilde{b} + \Theta)\tilde{y},
\end{align*}
\] (2.78)

where
\[ \tilde{u}(s) = \frac{\tilde{b}(s) \tilde{y}(s) - (\tilde{b}(s) + \Theta)\tilde{y}(s)}{s} + \frac{\tilde{y}}{s}. \] (2.79)
bearing in mind that \( \gamma + \tilde{\gamma} = \pm 1 \). Comparing the equations (2.78) with (2.8), and the definition (2.79) with (2.6), we see a clear correspondence between the set of quantities such as \( \tilde{\gamma} \) and \( \gamma \), with \( \tilde{\gamma} \) corresponding to \( \gamma \). Hence, we obtain a differential system of the form (2.9), and, eventually, we see that \( \tilde{\gamma} \) solves the equation

\[
\frac{d^2 \gamma}{ds^2} - \frac{2y}{y^2 - 1} \left( \frac{dy}{ds} \right)^2 + \frac{1}{s} \frac{dy}{ds} + \frac{y(y^2 + 1)}{4(y^2 - 1)} + \frac{y}{2s} - \Theta \frac{y^2 + 1}{2s} = 0,
\]

(2.80)

which differs from (2.10) with only the constant \( \gamma \) being replaced with \( \tilde{\gamma} = -\gamma \pm 1 \). One more step further, we find that \( \tilde{\omega}(s) = \tilde{\gamma}^2(s) \) solves the generalized Painlevé V equation

\[
\frac{d^2 \omega}{ds^2} - \left( \frac{1}{w - 1} + \frac{1}{2\omega} \right) \left( \frac{d\omega}{ds} \right)^2 + \frac{1}{s} \frac{d\omega}{ds} = \frac{2(\Theta - 1)\omega}{s} + \frac{\omega(\omega + 1)}{2(w - 1)} \pm \tilde{\gamma} \sqrt{\omega} \frac{\omega}{s}(\omega + 1) = 0.
\]

(2.81)

Again, the equation differs from (2.11) in the parameter \( \tilde{\gamma} = -\gamma \pm 1 \).

We note that for \( |\gamma| \geq 1 \), we can always make \( |\tilde{\gamma}| = |\gamma| - 1 \) by choosing the proper sign in \( \pm \). Therefore, for integer \( \gamma \), applying the gauge transformation (2.71) (and correspondingly, the Bäcklund transformation (2.77)) \(|\gamma| \) times, the constant \( \tilde{\gamma} \) in (2.81) is turned into 0, and, as mentioned earlier the equation is thus reduced to a special Painlevé V equation; see [13] and [29, 30].

3. Nonlinear steepest descent analysis

We begin with a RH problem for \( Y \), associated with the orthogonal polynomials with respect to the specific weight \( w(x) \) given in (1.8). Such a remarkable connection between the orthogonal polynomials and RH problems is observed by Fokas, Its and Kitaev [12]. Then, we apply the nonlinear steepest descent analysis developed by Deift and Zhou et al [9, 10] to the RH problem for \( Y \); see also Bleher and Its [2]. The idea is to obtain, via a series of invertible transformations \( Y \rightarrow T \rightarrow S \rightarrow R \), eventually the RH problem for \( R \) with jumps in a sense close to the identity matrix. Tracing back, the uniform asymptotics of the orthogonal polynomials in the complex plane is obtained for large degree \( n \). A key step is the construction of a certain local parametrix in the neighborhood of the singular point \( t \) and the hard edge. Constructing the parametrix will be our main focus in this section.

3.1. Riemann–Hilbert problem for orthogonal polynomials

Initially, the Riemann–Hilbert problem for orthogonal polynomials is as follows (see [12]).

(Y1) \( Y(z) \) is analytic in \( \mathbb{C} \setminus [-1, 1] \);

(Y2) \( Y(z) \) satisfies the jump condition

\[
Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}, \quad x \in (-1, 1),
\]

(3.1)

where \( w(x) = (1 - x^2)^\beta (t^2 - x^2)^\alpha h(x) \) is the weight function defined in (1.8);

(Y3) The asymptotic behavior of \( Y(z) \) at infinity is

\[
Y(z) = (I + O(1/z)) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, \quad \text{as} \quad z \rightarrow \infty;
\]

(3.2)
The asymptotic behavior of $Y(z)$ at the endpoints $z = \pm 1$ are

$$Y(z) = \begin{cases} \frac{O(1)}{O((z \pm 1)^{\beta})}, & \text{for } -1 < \beta < 0, \\
\frac{O(1)}{O(\ln(z \pm 1))}, & \text{for } \beta = 0, \\
\frac{O(1)}{O(1)}, & \text{for } \beta > 0. \end{cases} \quad \text{(3.3)}$$

By virtue of the Sochacki–Plemelj formula and Liouville’s theorem, it is known that the above RH problem for $Y$ has a unique solution $Y(z) = Y(z; n)$,

$$Y(z) = \left( \frac{\pi_n(z)}{-2\pi i\gamma_n(z)} \right) \left( \pi_n(z) - \gamma_n(z) \int_{-1}^{1} \frac{\pi_n(x) w(x)}{x - z} \, dx \right), \quad \text{(3.4)}$$

where $\pi_n(z)$ is the monic polynomial, and $p_n(z) = \gamma_n \pi_n(z)$ is the orthonormal polynomial with respect to the weight $w(x) = w(x; t)$ in (1.8); see, e.g. [8] and [12].

3.2. The first transformation $Y \to T$

The first transformation $Y \to T$ is defined as

$$T(z) = 2^{n_{\sigma_3}} Y(z) \varphi(z)^{-n_{\sigma_3}}, \quad \text{(3.5)}$$

for $z \in \mathbb{C} \setminus [-1, 1]$, where $\varphi(z) = z + \sqrt{z^2 - 1}$ is a conformal map from $\mathbb{C} \setminus [-1, 1]$ onto the exterior of the unit circle, with the branches specified as $\arg(z \pm 1) \in (-\pi, \pi)$, such that $\varphi(z) \sim 2z$ as $z \to \infty$. The transformation (3.5) accomplishes a normalization of $Y(z)$ at infinity, and $T$ solves the RH problem:

(T1) $T(z)$ is analytic in $\mathbb{C} \setminus [-1, 1]$;

(T2) The jump condition is

$$T_+(x) = T_-(x) \begin{pmatrix} \varphi_+(x)^{-2n} & w(x) \\ 0 & \varphi_-(x)^{-2n} \end{pmatrix}, \quad x \in (-1, 1), \quad \text{(3.6)}$$

where $\varphi_\pm(x)$ are the boundary values of $\varphi(z)$, respectively from above $(-1, 1)$ and from below;

(T3) The asymptotic behavior of $T(z)$ at infinity

$$T(z) = I + O(1/z) \quad \text{as} \quad z \to \infty; \quad \text{(3.7)}$$

(T4) $T(z)$ behaves the same as $Y(z)$ at the end points $\pm 1$, as described in (3.3).

3.3. The second transformation $T \to S$

The Riemann–Hilbert problem for $T$ is oscillatory in the sense that the jump matrix in (3.6) has oscillating diagonal entries on the interval $(-1, 1)$. To remove the oscillation, we introduce the second transformation $T \to S$, based on a factorization of the oscillatory jump matrix

$$\begin{pmatrix} \varphi_+^{-2n} & w \\ 0 & \varphi_-^{-2n} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \varphi_-^{-2n} w^{-1} & 1 \end{pmatrix} \begin{pmatrix} 0 & w \\ -w^{-1} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varphi_+^{-2n} w^{-1} & 1 \end{pmatrix}, \quad \text{(3.8)}$$
where use has been made of the fact that $\varphi_+(x)\varphi_-(x) = 1$ for $x \in (-1, 1)$. Accordingly, we define a piecewise matrix-valued function

$$ S(z) = \begin{cases} 
T(z), & \text{for } z \text{ outside the lens-shaped region;} \\
T(z) \left( \begin{array}{cc}
\frac{1}{\varphi(z)^{-2n}w(z)^{-1}} & 0 \\
-\frac{\varphi(z)^{-2n}w(z)^{-1}}{1}
\end{array} \right), & \text{for } z \text{ in the upper lens region;} \\
T(z) \left( \begin{array}{cc}
\frac{1}{\varphi(z)^{-2n}w(z)^{-1}} & 0 \\
\frac{\varphi(z)^{-2n}w(z)^{-1}}{1}
\end{array} \right), & \text{for } z \text{ in the lower lens region,}
\end{cases} \quad (3.9) $$

where the regions are depicted in figure 4, and

$$ w(z) = (1 - z^2)^{\beta}(1 - z^2)^{\beta}h(z), \quad z \in \Omega\setminus\{(-\infty, -1] \cup [1, \infty)\} $$

denotes the analytic continuation of $w(x)$, with $\arg(1 \pm z) \in (-\pi, \pi)$ and $\arg(t \pm z) \in (-\pi, \pi)$, where $\Omega$ is the domain of analyticity of $h(z)$, such that $[-1, 1] \subset \Omega$.

Then $S$ solves the Riemann–Hilbert problem:

(S1) $S(z)$ is analytic in $\mathbb{C}\setminus\Sigma_S$, where $\Sigma_S$ are the deformed contours consisting of $(-1, 1)$ and the upper and lower lens boundaries, as illustrated in figure 4;

(S2) The jump condition is

$$ S_+(x) = S_-(x) \begin{bmatrix} w(x) & 0 \\
-w(x)^{-1} & 0 \\
\varphi(z)^{-2n}w(z)^{-1} & 1
\end{bmatrix}, \quad \text{for } x \in (-1, 1), \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\
1 & 0 \\
\varphi(z)^{-2n}w(z)^{-1} & 1
\end{bmatrix}, \quad \text{on the lens boundaries;} \quad (3.10) $$

(S3) The asymptotic behavior at infinity is

$$ S(z) = I + O(1/z), \quad \text{as } z \to \infty; \quad (3.11) $$

(S4) At the endpoints $\pm 1$, we have for $-1 < \beta < 0$

$$ S(z) = \begin{pmatrix} O(1) & O((z \pm 1)^{\beta}) \\
O(1) & O((z \pm 1)^{\beta})
\end{pmatrix}, \quad \text{as } z \to \pm 1, \quad (3.12) $$

while for $\beta = 0$,

$$ S(z) = O(\ln(z \pm 1)) \quad \text{as } z \to \pm 1, \quad (3.13) $$

and for $\beta > 0$,

$$ S(z) = \begin{pmatrix} O((z \pm 1)^{-\beta}) & O(1) \\
O((z \pm 1)^{-\beta}) & O(1) \\
O(1) & O(1)
\end{pmatrix}, \quad \text{as } z \to \pm 1, \quad \text{inside of the lens,} \quad (3.14) $$

as $z \to \pm 1, \quad \text{outside of the lens.} \quad (3.14) $$
### 3.4. Global parametrix

From (3.10), we see that the jump matrix for $S$ on the lens-shaped boundary is of the form $J_2(z) = I$, plus an exponentially small term. The only jump of significance is attached to $(-1, 1)$. We are now in a position to solve the following limiting Riemann–Hilbert problem for $N_t(z)$,

(N1) $N_t(z)$ is analytic in $\mathbb{C}\setminus[-1, 1]$;

(N2) The jump condition is

$$
(N_t)_+ (x) = (N_t)_- (x) \begin{pmatrix} 0 & w(x) \\ -w(x)^{-1} & 0 \end{pmatrix} \quad \text{for} \quad x \in (-1, 1);
$$

(N3) The asymptotic behavior at infinity is

$$
N_t(z) = I + O(1/z), \quad \text{as} \quad z \to \infty. \tag{3.16}
$$

Since $i\sigma_2 = M_1^{-1}(-i\sigma_3)M_1$, where $M_1 = (I + i\sigma_1)/\sqrt{2}$; see (2.7) for the definition of the Pauli matrices, a solution to the above RH problem can be constructed explicitly as (see [17])

$$
N_t(z) = D_t(\infty)^{\sigma_2} M_1^{-1} a(z)^{\sigma_3} M_1 D_t(z)^{-\sigma_1}, \tag{3.17}
$$

where $a(z) = (\frac{-1}{z+1})^{1/4}$ for $z \in \mathbb{C}\setminus[-1, 1]$, the branches are chosen such that $a(x)$ is positive for $x > 1$ and $a(x)/a(-x) = i$ for $x \in (-1, 1)$, and the Szegő function associated with $w(x)$ takes the form

$$
D_t(z) = \left(\frac{z^2 - 1}{\psi(z)^2}\right)^{\beta/2} \exp \left(\frac{\sqrt{z^2 - 1}}{2\pi} \int_{-1}^{1} \frac{\ln \{t^2 - x^2\} \alpha h(x)}{\sqrt{1 - x^2}} \frac{dx}{z - x} \right), \quad z \in \mathbb{C}\setminus[-1, 1], \tag{3.18}
$$

which is a non-zero analytic function on $\mathbb{C}\setminus[-1, 1]$ such that $(D_t)_+ (x) (D_t)_- (x) = w(x)$ for $x \in (-1, 1)$. In (3.18) the principal branches are taken, namely, $\arg(z \pm 1) \in (-\pi, \pi)$, and $\psi(z)$ for $z \in \mathbb{C}\setminus[-1, 1]$ is defined in (3.5). It is readily seen that the limit at infinity is

$$
D_t(\infty) = 2^{\beta/2} \exp \left(\frac{1}{2\pi} \int_{-1}^{1} \frac{\ln \{t^2 - x^2\} \alpha h(x)}{\sqrt{1 - x^2}} \frac{dx}{1 - x^2} \right).
$$

For each $t > 1$, the jump for $SN_t^{-1}$ is close to the unit matrix in the open curves $\Sigma_t \setminus \{\pm 1\}$, yet this is not true at the endpoints. $SN_t^{-1}$ is not even bounded near $\pm 1$. Thus local parametrices have to be constructed in neighborhoods of these endpoints.

### 3.5. Local parametrix $P^{(1)}(z)$

In the present subsection, we focus on the construction of the parametrix at the right endpoint $z = 1$, or, more precisely, in the neighborhood $U(1, r) = \{z : |z - 1| < r\}$, $r$ being fixed and sufficiently small. The parametrix $P^{(1)}(z)$ should solve the following RH problem:

(a) $P^{(1)}(z)$ is analytic in $U(1, r) \setminus \Sigma_t$, where $\Sigma_t$ are the deformed contours depicted in figure 4;

(b) On $\Sigma_t \cap U(1, r)$, $P^{(1)}(z)$ satisfies the same jump conditions as $S(z)$ does, see (3.10);

(c) $P^{(1)}(z)$ fulfills the following matching condition on $\partial U(1, r)$:

$$
P^{(1)}(z)N_t^{-1}(z) = I + O(n^{-1}), \tag{3.19}
$$

(d) The asymptotic behavior of $P^{(1)}(z)$ at the endpoint $z = 1$ is as described in (3.12)–(3.14).
To construct $P^{(1)}(z)$, we transform the RH problem for $P^{(1)}$ to a new RH problem for $\hat{P}^{(1)}$, with constant jump matrices, as

$$\hat{P}^{(1)}(z) = P^{(1)}(z)\psi(z)^{\gamma\sigma}W(z)^{\frac{1}{\sigma}}, \quad (3.20)$$

in which

$$W(z) = (z^2 - 1)^{\beta}(z^2 - t^2)^{\delta}h(z), \quad z \in \Omega \setminus (-\infty, t],$$

such that arg$(z \pm 1) \in (-\pi, \pi)$ and arg$(z \pm t) \in (-\pi, \pi)$, where $\Omega$ is the domain of analyticity of $h(z)$ such that $[-1, 1] \subset \Omega$. We note that $W(z)$ is related to, but different from, the function $w(z)$ introduced in (3.9). Then $\hat{P}^{(1)}$ solves the following RH problem:

(a) $\hat{P}^{(1)}(z)$ is analytic in $U(1, r) \setminus \Sigma_S$ (see figure 4);

(b) $\hat{P}^{(1)}(z)$ possesses the following constant jumps

$$\hat{P}^{(1)}_+(z) = \hat{P}^{(1)}_-(z) \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{ if } z \in (1 - r, 1), \\ e^{(\alpha + \beta)\pi i} & 0, & \text{ on the upper lens boundary}, \\ e^{-\alpha\pi i n_1} & 1, & \text{ on the lower lens boundary}, \end{cases} \quad (3.21)$$

(c) The behavior of $\hat{P}^{(1)}(z)$ at $z = t$ is

$$\hat{P}^{(1)}(z) = O(1) \left( z - t \right)^{\hat{\gamma} n_1}, \quad \text{ as } z \to t; \quad (3.22)$$

(d) The behavior of $\hat{P}^{(1)}(z)$ at $z = 1$ is, for $-1 < \beta < 0$,

$$\hat{P}^{(1)}(z) = O \left( (z - 1)^{\beta/2} \right), \quad \text{ as } z \to 1, \quad (3.23)$$

while for $\beta = 0$,

$$\hat{P}^{(1)}(z) = O(\ln(z - 1)) \quad \text{ as } z \to 1, \quad (3.24)$$

and for $\beta > 0$,

$$\hat{P}^{(1)}(z) = \begin{cases} O \left( (z - 1)^{-\beta/2} \right) & \text{ as } z \to 1, \text{ inside of the lens}, \\ O(1) \left( z - 1 \right)^{\frac{\hat{\gamma} n_1}{\sigma_1}} & \text{ as } z \to 1, \text{ outside of the lens}. \end{cases} \quad (3.25)$$

The RH problem for $\hat{P}^{(1)}$ shares exactly the same jumps as those for $\Psi_0$ in (2.40), with the parameter $\Theta = -\alpha, \gamma = \beta - \frac{1}{2}$. The behavior of $\hat{P}^{(1)}$ at $z = t$ in (3.22) is the same as that of $\Psi_0$ at $\xi = \frac{1}{2}$ in (2.43). We proceed to construct $\hat{P}^{(1)}(z)$ out of $\Psi_0(\xi, s)$, bearing in mind the matching condition (3.19).

We define a conformal mapping in a $z$-neighborhood $U(1, r)$ of $z = 1$ and $z = t$ as follows

$$f_t(z) = \frac{(\ln \psi(z))^2}{\rho_t} = \frac{2(z - 1)}{\rho_t} (1 + O(z - 1)), \quad z \in U(1, r) \quad (3.26)$$

with $f_t(1) = 0$ and $f_t(t) = \frac{1}{2}$, where $\rho_t = 4(\ln \psi(t))^2 = 8(t - 1) + O \left( (t - 1)^2 \right)$ as $t \to 1$. Making use of the conformal mapping, we seek a $\hat{P}^{(1)}$ of the form

$$\hat{P}^{(1)}(z) = E(z)\Psi_0 \left( f_t(z), 2n \sqrt{\rho} \right), \quad z \in U(1, r),$$

accordingly,

$$P^{(1)}(z) = E(z)\Psi_0 \left( f_t(z), 2n \sqrt{\rho} \right) \psi(z)^{-\gamma\sigma}W(z)^{-\frac{1}{\sigma_1}}, \quad (3.27)$$
where \( \Psi_0(\zeta) = \Psi_0(\zeta, s) \) is the solution to the RH problem (2.40)–(2.43), and \( E(z) \) is an analytic matrix-valued function in the neighborhood \( U(1, r) \), to be determined by the matching condition (3.19).

First, we introduce

\[
E(z) = N_t(z)W(z)\hat{\zeta}^{\sigma_1} \{ G(f_t(z)) \}^{-1},
\]

(3.29)

where \( G(\zeta) \) is a specific matrix function defined as

\[
G(\zeta) = \zeta^{i\sigma_1} \frac{I - i\alpha \sigma_1}{\sqrt{2}} \exp \left\{ \int_0^1 \frac{1}{\sqrt{\tau - \zeta}} \frac{i}{\sqrt{\tau}} \frac{d\tau}{\tau} \right\}, \quad \zeta \in \mathbb{C}\setminus(\infty, 1/4],
\]

(3.30)

with \( \text{arg} \zeta \in (-\pi, \pi) \), and satisfying the jump conditions

\[
G_+(x) = G_-(x)(i\sigma_2) \quad \text{for} \quad x \in (-\infty, 0), \quad \text{and} \quad G_+(x) = G_-(x)e^{i4\alpha\sigma_1} \quad \text{for} \quad x \in (0, 1/4).
\]

(3.31)

In view of (3.15) and (3.31), it is readily verified from (3.29) that

\[
E_+(x) = E_-(x) \quad \text{for} \quad x \in U(1, r) \cap \mathbb{R}.
\]

Next, we show that \( E(z) \) is also analytic at \( z = 1 \) and \( z = t \). Indeed, it follows from (3.17) and (3.18) that

\[
N_t(z)W(z)\hat{\zeta}^{\sigma_1} = O \left( (z - 1)^{-i\frac{\alpha}{\sqrt{2}}} \right) \quad \text{as} \quad z \to 1, \quad \text{and} \quad N_t(z)W(z)\hat{\zeta}^{\sigma_1} = O(1)(z - t)^{\frac{i\alpha}{\sqrt{2}}} \quad \text{as} \quad z \to t.
\]

Also, from (3.26) and (3.30) we see that \( G(f_t(z)) = ((z - 1)^{1/4}) \quad \text{as} \quad z \to 1 \), and the integral in (3.30) implies that \( G(f_t(z)) = O(1)(z - t)^{\frac{i\alpha}{2}} \quad \text{as} \quad z \to t \). Substituting these estimates into (3.29) gives

\[
E(z) = O \left( (z - 1)^{-1/2} \right) \quad \text{as} \quad z \to 1,
\]

and

\[
E(z) = O(1) \quad \text{as} \quad z \to t,
\]

which means that \( E(z) \) has at most isolated weak singularities at \( z = 1, t \), and hence the singularity is removable. Thus \( E(z) \) is analytic in the neighborhood \( U(1, r) \).

Next, we note that, from section 5.1 below, \( G(\zeta) \) solves a limiting RH problem with jumps (3.31), such that \( \Psi_0(\zeta, s) \) is approximated by \( G(\zeta) \) for \( s \) large, and \( \zeta \) being kept away from the origin; see (5.19) below. Hence, we have

\[
\Psi_0 \left( f_t(z), 2n\sqrt{\rho_t} \right) \varphi(z)^{-i\alpha \sigma_1} = G(f_t(z)) \left( I + O \left( \frac{1}{n\sqrt{\rho_t}} \right) \right),
\]

(3.32)

uniformly for \( z \in \partial U(1, r) \) as \( n\sqrt{\rho_t} \to \infty \). For \( z \in \partial U(1, r) \), we have

\[
W^{\sigma_1}(z), \quad N_t^{-1}(z), \quad a^{\sigma_1}(z), \quad D_t^{\rho}(z) = O(1); \quad \text{see section 3.4 for the definitions of these quantities. Thus, by combining (3.28) with (3.32) and (3.33), we see that}
\]

\[
P^{(1)}N_t^{-1} = N_tW^{\sigma_1} \left( I + O \left( \frac{1}{n\sqrt{\rho_t}} \right) \right) W^{-1}N_t^{-1} = I + O \left( \frac{1}{n\sqrt{\rho_t}} \right),
\]

(3.34)

for \( |z - 1| = r \), where \( t \) is taken so that \( n\sqrt{\rho_t} \to \infty \). Thus, the matching condition (3.19) is fulfilled with obvious modification if \( n\sqrt{\rho_t} \) is unbounded.
Figure 5. The remaining contours $\Sigma_R$: contours of the RH problem for $R(z)$.

If $n^{1/2} \approx 2^{n/2} n^{1/2} / \sqrt{1-r} \in (0, \delta)$ bounded, then the conformal mapping (3.26) satisfies

$$ \frac{1}{f_t(z)} = O\left(\frac{1}{n^2}\right) $$

for $|z - 1| = r$. Then, for $\xi = f_t(z) \gg 1/4$, the matrix function $G(\xi)$ in (3.30) is approximated as

$$ G(f_t(z)) = \left( f_t(z) \right)^{1/4} n^{1/4} \left( I + O\left(\frac{1}{n}\right) \right). \quad (3.36) $$

Thus, combining (3.36) with the expansion (2.41) of $\Psi_0$ at infinity and (6.15), we have

$$ P^{(1)}(z) = N_t W^{1/2} \left( I + O\left(\frac{1}{n}\right) \right) W^{-1/2} N_t^{-1} = I + O\left(\frac{1}{n}\right) \quad (3.37) $$

for $|z - 1| = r$. Thus, the matching condition (3.19) is also fulfilled for bounded $n^{1/2}$.

We have completed the construction of the local parametrix $P(\pm 1)(z)$ at the right edge $z = 1$, in which a generalized fifth Painlevé transcendent is involved. Similarly, we can state and construct the parametrix $P(\mp 1)(z)$ at the left edge $z = -1$.

3.6. The final transformation $S \to R$

Now we bring in the final transformation by defining

$$ R(z) = \begin{cases} 
S(z) N_t^{-1}(z), & z \in \mathbb{C} \setminus \{ U(-1, r) \cup U(1, r) \cup \Sigma_S \}; \\
S(z) (P^{(1)})^{-1}(z), & z \in U(-1, r) \setminus \Sigma_{P^{(1)}};
\end{cases} \quad (3.38) $$

comparing figure 5 for the regions involved, where $U(\pm 1, r)$ are the disks of radius $r$, centered respectively at $\pm 1$. So defined, the matrix-valued function $R(z)$ satisfies a Riemann–Hilbert problem on the remaining contours $\Sigma_R$ illustrated in figure 5, as follows:

(R1) $R(z)$ is analytic in $\mathbb{C} \setminus \Sigma_R$ (see figure 5);
(R2) $R(z)$ satisfies the jump conditions

$$ R_+(z) = R_-(z) J_R(z), \quad z \in \Sigma_R, $$

where

$$ J_R(z) = \begin{cases} 
N_t(z) (P^{(1)})^{-1}(z), & z \in \partial U(-1, r), \\
N_t(z) (P^{(1)})^{-1}(z), & z \in \partial U(1, r), \\
N_t(z) J_S(z) N_t^{-1}(z), & \text{otherwise},
\end{cases} $$

where $J_S(z)$ is the jump for $S$, given in (3.10);
(R3) $R(z)$ demonstrates the following behavior at infinity:

$$ R(z) = I + O\left(\frac{1}{z}\right), \quad \text{as } z \to \infty. $$

(3.40)
We note that $R(z)$ has removable singularities at $z = \pm 1$. Indeed, since $S(z)$ and $P^{(1)}(z)$ share the same jump within $U(1, r)$, we see that $z = 1$ is at most an isolated singularity for $R(z)$. Again, $S(z)$ and $P^{(1)}(z)$ satisfy the same behavior (3.12)–(3.14), hence we have

$$R(z) = O((z - 1)^{\beta}) \text{ as } z \to 1, \quad -1 < \beta < 0 \text{ and } R(z) = O\left((\ln(z))^2\right) \text{ as } z \to 1, \quad \beta = 0.$$  

While for the case $\beta > 0$, we have

$$R(z) = \begin{cases} O\left((z - 1)^{-\beta}\right), & \text{as } z \to 1 \text{ from inside of the lens}, \\ O(1), & \text{as } z \to 1 \text{ from outside of the lens}, \end{cases}$$

Thus, in each case, $R(z)$ has removable singularity at $z = 1$. A similar argument applies to $z = -1$.

It follows from the matching condition (3.19) of the local parametrices, the definition of $\varphi$, and the definition of $N_1$ in (3.15) that

$$J_R(z) = \begin{cases} I + O(n^{-1}), & z \in \partial U(\pm 1, r), \\ I + O(e^{-cn}), & z \in \Sigma_1 \setminus \partial U(\pm 1, r), \end{cases}$$  

(3.41)

where $c$ is a positive constant, and the error term is uniform for $z$ on the corresponding contours. Hence, we have

$$\|J_R(z) - I\|_{L^2(\Sigma_1 \setminus \Sigma_2)} = O(n^{-1}).$$  

(3.42)

Then, applying the now standard procedure of norm estimation of Cauchy operator and using the technique of deformation of contours (see [8, 10]), it follows from (3.42) that

$$R(z) = I + O(n^{-1}),$$  

(3.43)

uniformly for $z$ in the whole complex plane.

This completes the nonlinear steepest descent analysis. In the next section, we will show that the orthogonal polynomial kernel (the Christoffel–Darboux kernel) can be represented in terms of the solution to the RH problem for $Y$, formulated at the very beginning of this section. The large-$n$ asymptotic behavior of the kernel can then be obtained.

### 4. Proof of theorem 1

The orthonormal polynomials $p_n(z) = \gamma_n \pi_n(z)$ satisfy the three-term recurrence relation

$$B_n p_{n+1}(z) + (A_n - z) p_n(z) + B_{n-1} p_{n-1}(z) = 0, \quad n = 0, 1, \cdots,$$  

(4.1)

where $B_{-1} = 0$, and $B_n = \gamma_n / \gamma_{n+1}$ for $n = 0, 1, 2, \cdots$. From the three-term recurrence relation, it is readily seen that the following Christoffel–Darboux formula holds (see, e.g. [26]):

$$\sum_{k=0}^{n-1} p_k(x) p_k(y) = \gamma_n^2 \frac{\pi_n(x) \pi_{n-1}(y) - \pi_n(y) \pi_{n-1}(x)}{x - y}. $$  

(4.2)

Hence, in terms of the matrix-valued function $Y(z)$ defined in (3.4), the kernel $K_n(x, y)$ in (1.2) can be written as

$$K_n(x, y) = \sqrt{w(x)w(y)} \left\{Y^{-1}_+(y) Y_n(x)\right\}_{21}, \quad x, y \in (-1, 1).$$  

(4.3)
4.1. Proof of theorem 1 (ii): the sine kernel limit

Assume that \( x, y \in I_\delta = [-1 + \delta, 1 - \delta] \), with \( \delta > 0 \) fixed, such that \( 0 < r < \delta \), where \( r \) is the radius of \( U(\pm 1, r) \). Substituting the transformations (3.5) and (3.9)–(4.3), we have

\[
K_n(x, y) = \frac{\sqrt{w(x)w(y)}}{2\pi i(x - y)} \left\{ \left( \begin{array}{cc} 1 & 0 \\ \frac{1}{w(y)} & 1 \end{array} \right) \varphi_{n\delta y}^{-1}(y)S_{n\delta}^{-1}(y)S_n(x)\varphi_{n\delta y}^{-1}(x) \left( \begin{array}{cc} 1 & 0 \\ \frac{1}{w(x)} & 1 \end{array} \right) \right\},
\]

(4.4)

On the other hand, in view of the jump condition (3.39) and the uniform estimate (3.42), we have

\[
R(z) = I + \frac{1}{2\pi i} \int_{\Sigma_R} \frac{R_-(\zeta)(J_R(\zeta) - I) d\zeta}{\zeta - z}, \quad z \notin \Sigma_R,
\]

from which we conclude that

\[
R(x) = I + O\left(\frac{1}{n}\right), \quad \frac{dR}{dz}\bigg|_{z=x} = O\left(\frac{1}{n}\right),
\]

as long as \( I_\delta \) keeps a constant distance from \( \Sigma_R \). Hence, we see that

\[
R^{-1}(y)R(x) = I + O\left((x - y)/n\right),
\]

uniformly for \( x, y \in I_\delta \). Observing that both \( (N_t)_+ \) and \( \frac{d}{dx} (N_t)_+ \) are uniformly of \( O(1) \) on \( I_\delta \), and accordingly \( (N_t)_+^{-1}(y) (N_t)_+(x) = I + O(x - y) \), uniformly again for \( x, y \in I_\delta \), and combining these with (3.38), we have

\[
S_{n\delta}^{-1}(y)S_n(x) = (N_t)_+^{-1}(y)R^{-1}(y)R(x) (N_t)_+(x) = I + O(x - y).
\]

(4.5)

Substituting (4.5) in (4.4) then yields

\[
K_n(x, y) = \frac{1}{2\pi i(x - y)} \left[ \frac{w(y)}{w(x)} \left( \varphi_n(y) \right)^n - \frac{w(x)}{w(y)} \left( \varphi_n(x) \right)^n \right] + O(1),
\]

(4.6)

uniformly for \( x, y \in I_\delta \). In deriving (4.6), use has been made of the fact that \( \varphi_n(x) = e^{i\arccos x} \), so that \( |\varphi_n(x)| = 1 \). Noting that \( \sqrt{w(x)/w(y)} \), \( \sqrt{w(y)/w(x)} = 1 + O(x - y) \), from (4.6) we further obtain

\[
K_n(x, y) = \frac{\sin[n(\arccos y - \arccos x)]}{\pi(x - y)} + O(1).
\]

(4.7)

It then readily follows that

\[
\frac{\pi \sqrt{1 - x^2}}{n} K_n \left( x + \frac{\pi \sqrt{1 - x^2}}{n} u, x + \frac{\pi \sqrt{1 - x^2}}{n} v \right) = \frac{\sin[\pi(u - v)]}{\pi(u - v)} + O\left(\frac{1}{n}\right)
\]

(4.8)

by expanding \( \arccos(x + t) = \arccos x - t/\sqrt{1 - x^2} + \cdots \) for fixed \( x \in (-1, 1) \) and small \( t \). The large-\( n \) limit (4.8) holds uniformly for bounded real \( u \) and \( v \). Thus, we complete the proof of theorem 1 (ii): the sine kernel limit. We see that the universality property is preserved in the bulk of the spectrum; see [8] and [17].

4.2. Proof of theorem 1 (i): the limiting eigenvalue density

The \( O(1) \) term in (4.7) is uniform with respect to all \( x, y \in [-1 + \delta, 1 - \delta] \) for positive \( \delta \). Hence, we can take the limit \( y \to x \). As a result, we have

\[
K_n(x, x) = \frac{n}{\pi \sqrt{1 - x^2}} + O(1),
\]

(4.9)

for \( x \in (-1, 1) \) fixed and \( n \to \infty \). This proves theorem 1 (i), as stated in (1.29).
4.3. Proof of theorem 1 (iii): the Painlevé kernel limit

Now we turn to the neighborhood \( U(1, r) = \{ z : |z - 1| < r \} \), in which the parametrix \( P^{(1)}(z) \) is constructed. A combination of (3.5), (3.9), (3.28) and (3.38) gives
\[
Y_n(x) = 2^{-n}\sigma_1 R(x) E(x) ([\Psi_0]_{s} \cdot (f_2(x), s)) e^{-\frac{\pi}{2} (\omega + \beta s)} (\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} w(x)^{-\frac{1}{n}}, 1 - r < x < 1,
\]
where \( s = 2n\sqrt{p_1} = 4n \ln \varphi(t) \); see (3.26). Substituting it into (4.3), we have
\[
K_n(x, y) = \frac{(-\psi_2(f_1(y)), \psi_1(f_1(y))) E^{-1}(y) R(1)(y) R(x) E(x) (\psi_1(f_1(x)), \psi_2(f_1(x)))^T}{2\pi i(x - y)}.
\]
(4.10)
where
\[
\begin{pmatrix} \psi_1(\zeta) \\ \psi_2(\zeta) \end{pmatrix} = \left( \begin{pmatrix} \psi_1(\zeta, s) \\ \psi_2(\zeta, s) \end{pmatrix} = (\Psi_0)_+ (\zeta, s) \begin{pmatrix} e^{-\frac{\pi}{2} (\omega + \beta s)} \\ e^{\frac{\pi}{2} (\omega + \beta s)} \end{pmatrix} \right)^T
\]\nfor \( \zeta < 0 \) and \( s = 2n\sqrt{p_1} \).

Now specifying
\[
x = 1 - \frac{p_1}{2} u, \quad y = 1 - \frac{p_2}{2} v \quad \text{with} \quad u, v \in D,
\]\nwith \( s^2D \) being a compact subset of \((0, +\infty)\), where again \( s = 2n\sqrt{p_1} = 4n \ln \varphi(t) \approx 4\sqrt{2} n \ln \ln \varphi(t) \approx 1 \). Then it follows from (3.26) and (4.12) that
\[
f_1(x) = -u \left( 1 + O \left( n^{-2} \right) \right), \quad f_2(y) = -v \left( 1 + O \left( n^{-2} \right) \right),
\]\nwhere the \( O \left( n^{-2} \right) \) terms are uniform respectively for \( u, v \in D \). Similarly, the analyticity of \( R(z) \) in \( U(1, r) \) implies that
\[
R^{-1}(y) R(x) = I + O(x - y) = I + O \left( n^{-2} \right),
\]\nagain with uniform error terms. Hence, substituting all these into (4.10) yields
\[
K_n(x, y) = \frac{(-\psi_2(f_1(y)), \psi_1(f_1(y))) (1 + O(x - y)) (\psi_1(f_1(x)), \psi_2(f_1(x)))^T}{2\pi i(x - y)}.
\]\n(4.14)
the error term is actually uniform for \( t \in (1, d] \) and for \( 1 - r < x, y < 1 \), with \( d > 1 \) and \( r > 0 \) being constants.

Now we consider the double scaling limit when \( n^2(t - 1) \) approaches a positive number as \( n \to \infty \) and \( t \to 1^+ \). In such a case, we can regard \( s \) as a positive constant. The formula (4.13) implies that
\[
\psi_2(f_1(x), s) = \psi_2(-u, s) \left( 1 + O \left( n^{-2} \right) \right) \quad \text{and} \quad \psi_2(f_1(y), s) = \psi_2(-v, s) \left( 1 + O \left( n^{-2} \right) \right)
\]\n(4.15)
for \( k = 1, 2 \), where the error terms are uniform for \( u, v \) in compact subsets of \((0, \infty)\).

Thus, in view of the fact that \( \frac{\partial}{\partial s} \) is analytic in \( s \), a combination of (4.14) and (4.15) gives
\[
\frac{s^2}{8n^2} K_n \left( 1 - \frac{s^2 u}{8n^2}, 1 - \frac{s^2 v}{8n^2} \right) = K_\psi(-u, -v; s) + O \left( \frac{1}{n^2} \right).
\]\n(4.16)
for large \( n \), where
\[
K_\psi(-u, -v; s) = \frac{\psi_1(-u, s) \psi_2(-v, s) - \psi_1(-v, s) \psi_2(-u, s)}{2\pi i(u - v)}
\]
is the Painlevé type kernel, and the error term \( O \left( n^{-2} \right) \) is uniform for \( u, v \) in compact subsets of \((0, \infty)\), and \( t - 1 = O \left( 1/n^2 \right) \), thus completing the proof of theorem 1.
5. Transition to the Bessel kernel $J_\beta$ as $s \to \infty$

When $t > 1$ fixed, the weight in (1.8) can be written as $w(x) = (1 - x^2)^\beta h_1(x)$, where $h_1(z) = (t^2 - z^2)^\beta h(z)$ is an analytic function for $z \in \Omega \setminus \{(-\infty, -t] \cup [t, \infty)\}$, $\Omega$ being a neighborhood of $[-1, 1]$. This is a special case investigated in [16, 17]. The local behavior at $x = 1$ is described via the kernel $J_\beta$ given in (1.6).

In the $\Psi$-kernel $K_\Psi(u, v; s)$, we use the parameter $s = 4n \ln(t + \sqrt{t^2 - 1})$ to describe the location of $t$. As $t$ varies to $d > 1$ fixed, the parameter $s \to \infty$ as $n \to \infty$. In the present section, we begin with an asymptotic study of the model RH problem for $\Psi_0(\zeta, s)$ with specified parameters $\Theta = -\alpha$ and $\gamma = \beta - \frac{1}{2}$, and as $s \to \infty$. Then, we apply the results to obtain a transition of the limit kernel from $K_\Psi(u, v; s)$ to the classical Bessel kernel $J_\beta$, as $s \to \infty$. As a by-product, we obtain the asymptotics for the nonlinear equation $b(s)$, $u(s)$, and $y(s)$.

5.1. Nonlinear steepest descent analysis of the RH problem for $\Psi_0(\zeta, s)$ as $s \to \infty$

Taking the normalization of $\Psi_0(\zeta, s)$ at infinity as $U(\zeta, s) = \Psi_0(\zeta, s)e^{-\frac{\pi i \alpha \sigma_3}{2}}$, arg $\zeta \in (-\pi, \pi)$, (5.1) where $\Psi_0(\zeta, s)$ solves the model RH problem (2.40–2.43), we see that $U(\zeta, s)$ satisfies the following RH problem:

(a) $U(\zeta)$ is analytic in $\mathbb{C} \setminus \bigcup_{j=1}^4 \Sigma_j$ (see figure 1); 
(b) $U(\zeta)$ satisfies the jump conditions,

\[
U_+(\zeta) = U_-(\zeta) \begin{cases} 
\begin{bmatrix} 1 & 0 \\
e^{-i\sqrt{\pi(1+\beta)}} & 1 \\
0 & 1 \\
-1 & 0 \\
0 & 1 \\
e^{-i\sqrt{\pi(1+\beta)} } & 1 \\
\end{bmatrix}, & \text{for } \zeta \in \Sigma_1, \\
\begin{bmatrix} 0 & 0 \\
1 & 0 \\
0 & 1 \\
\end{bmatrix}, & \text{for } \zeta \in \Sigma_2, \\
\begin{bmatrix} 0 & 0 \\
1 & 0 \\
\end{bmatrix}, & \text{for } \zeta \in \Sigma_3, \\
\begin{bmatrix} 0 & 0 \\
1 & 0 \\
\end{bmatrix}, & \text{for } \zeta \in \Sigma_4; 
\end{cases}
\]

(c) The asymptotic behavior of $U(\zeta)$ at infinity is

\[
U(\zeta) = \zeta^{\frac{i \alpha}{2}} \frac{1 - i \sigma_1}{\sqrt{2}} \left( I + O \left( \frac{1}{\sqrt{\zeta}} \right) \right), \quad \arg \zeta \in (-\pi, \pi);
\]

(d) The behavior of $U(\zeta)$ at the origin is

\[
U(\zeta) = O(1) \zeta^{\frac{i \beta \sigma_3}{2}} \begin{pmatrix} O(1) & O(1 + c \ln \zeta) \\
0 & O(1) \end{pmatrix},
\]

for $\zeta \in \Omega_4$, $\zeta \to 0$, and the behavior in other sectors can be determined by the jump condition (5.2). Here $c$ is given in (2.35) such that $c = 0$ for $\beta \notin \mathbb{N}$;

(e) The behavior of $U(\zeta)$ at $\zeta = \frac{1}{4}$ is

\[
U(\zeta) = \tilde{\Psi}(0)(\zeta - 1/4)^{-\frac{i \beta \sigma_3}{2}} e^{-\frac{\pi i \alpha \sigma_3}{2}};
\]

where $\tilde{\Psi}(0)(\zeta)$ is analytic at $\frac{1}{4}$.
We observe that the jumps along $\Sigma_2$ and $\Sigma_4$ in (5.2) differ from the identical matrix by exponentially small errors, as $s \to +\infty$ and $\zeta$ being kept away from the origin. Hence, we may consider the following limiting RH problem for $G$:

- $G(\zeta)$ is analytic in $\mathbb{C} \setminus \{ \Sigma_1 \cup \Sigma_3 \}$ (see figure 1);
- $G(\zeta)$ satisfies the jump conditions
  \[
  G^+(\zeta) = G^-(\zeta) \begin{pmatrix} e^{\pi i \sigma_3}, & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{for} \quad \zeta \in \Sigma_1 = (0, 1/4),
  \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad \text{for} \quad \zeta \in \Sigma_3 = (-\infty, 0); \tag{5.6}
  \]
- The asymptotic behavior of $G(\zeta)$ at infinity is
  \[
  G(\zeta) = \zeta^{1/4} I - \frac{i \sigma_1}{\sqrt{2}} \left( I + O\left(\frac{1}{\sqrt{\zeta}}\right) \right), \tag{5.7}
  \]

For this RH problem with simple jump curve, a solution can be constructed as

\[
G(\zeta) = \zeta^{1/4} I - \frac{i \sigma_1}{\sqrt{2}} \exp\left\{ \frac{\sqrt{2}}{2} \int_0^{1/4} \frac{1}{\sqrt{\tau - \zeta}} \frac{d\tau}{\sqrt{\tau - \zeta}} \right\} \sigma_3 \quad \text{for} \quad \zeta \in \mathbb{C} \setminus (-\infty, 1/4], \tag{5.8}
\]

where branches are chosen such that $\text{arg} \zeta \in (-\pi, \pi)$.

At the origin, $G(\zeta)$ is no longer a good approximation of $U(\zeta)$; see the jumps (5.2) and (5.6). Hence, in the disk $|\zeta| < 1/4$, we consider a local parametrix $P^{(0)}(\zeta)$, which obeys the same jump conditions (5.2) and the same behavior (5.4) at the origin as $U(\zeta)$, and fulfills the following matching condition at the boundary of the disk:

\[
P^{(0)}(\zeta) \sim G(\zeta) \quad \text{as} \quad |\zeta| = 1/4. \tag{5.9}
\]

We seek a solution, involving a re-scaling of the variable, of the form

\[
P^{(0)}(\zeta) = E_1(\zeta) \Phi \left( \frac{1}{16} \zeta^2 \right) \begin{pmatrix} e^{\pi i \sigma_3}, & \arg \zeta \in (0, \pi), \\ e^{-\pi i \sigma_3}, & \arg \zeta \in (-\pi, 0) \end{pmatrix}, \tag{5.10}
\]

with analytic $E_1$ in the disk $|\zeta| < 1/4$. Here $\Phi(\zeta)$ solves the following model RH problem:

- $\Phi(\zeta)$ is analytic in $\mathbb{C} \setminus \bigcup_{j=2}^4 \Sigma_j$ (see figure 6);
(b) \( \Phi(\zeta) \) satisfies the jump condition with parameter

\[
\Phi_+ (\zeta) = \Phi_- (\zeta) \begin{cases} 
\left( \begin{array}{cc} 1 & 0 \\ \e^{\beta \pi i} & 1 \end{array} \right), & \text{for } \zeta \in \Sigma_2, \\
\left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), & \text{for } \zeta \in \Sigma_3, \\
\left( \begin{array}{cc} 1 & 0 \\ \e^{-\beta \pi i} & 1 \end{array} \right), & \text{for } \zeta \in \Sigma_4;
\end{cases}
\]

(c) The asymptotic behavior of \( \Phi(\zeta) \) at infinity is

\[
\Phi(\zeta) = (4\pi^2 \zeta)^{-\frac{1}{4} \sigma_1} I_{\frac{1}{2} \sigma_1} \left( \frac{1}{\sqrt{\zeta}} \right) e^{2 \sqrt{\zeta} \sigma_1},
\]

for \( \arg \zeta \in (-\pi, \pi) \), \( \zeta \to \infty \).

A solution to the RH problem for \( \Phi(\zeta) \) can be constructed in terms of the modified Bessel functions as

\[
\Phi(\zeta) = \begin{cases} 
\left( \begin{array}{cc} I_{\beta}(2 \sqrt{\zeta}) & \frac{1}{2} K_{\beta}(2 \sqrt{\zeta}) \\
2\pi i \sqrt{\zeta} I'_{\beta}(2 \sqrt{\zeta}) & -2 \sqrt{\zeta} K'_{\beta}(2 \sqrt{\zeta}) \end{array} \right), & \text{for } \zeta \in I, \\
\left( \begin{array}{cc} I_{\beta}(2 \sqrt{\zeta}) & \frac{1}{2} K_{\beta}(2 \sqrt{\zeta}) \\
2\pi i \sqrt{\zeta} I'_{\beta}(2 \sqrt{\zeta}) & -2 \sqrt{\zeta} K'_{\beta}(2 \sqrt{\zeta}) \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\
-\e^{\beta \pi i} & 1 \end{array} \right), & \text{for } \zeta \in II, \\
\left( \begin{array}{cc} I_{\beta}(2 \sqrt{\zeta}) & \frac{1}{2} K_{\beta}(2 \sqrt{\zeta}) \\
2\pi i \sqrt{\zeta} I'_{\beta}(2 \sqrt{\zeta}) & -2 \sqrt{\zeta} K'_{\beta}(2 \sqrt{\zeta}) \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\
\e^{-\beta \pi i} & 1 \end{array} \right), & \text{for } \zeta \in III,
\end{cases}
\]

where \( \arg \zeta \in (-\pi, \pi) \); see figure 6 for the regions, and see [16] for such a construction. To verify the jump condition, one may use the analytic continuation formulas in [1, (9.6.30) and (9.6.31)]. The asymptotic at infinity (5.12) can be obtained by expanding the Bessel functions asymptotically in sectors; see [1, (9.6.31) and section 9.7], see also [16].

Taking into consideration the matching condition (5.9), and the asymptotic approximation of \( \Phi(\zeta) \) at infinity (5.12), we chose

\[
E_1(\zeta) = G(\zeta) \begin{cases} 
\e^{-\frac{i}{2} \pi \sigma_1} I_{\frac{1}{2} \sigma_1} \left( \frac{\pi^2 s^2 \zeta}{t^2} \right)^{\frac{1}{4} \sigma_1}, & \text{for } \zeta \in (0, \pi), \\
\e^{\frac{i}{2} \pi \sigma_1} I_{\frac{1}{2} \sigma_1} \left( \frac{\pi^2 s^2 \zeta}{t^2} \right)^{\frac{1}{4} \sigma_1}, & \text{for } \zeta \in (-\pi, 0),
\end{cases}
\]

which is analytic in the disk \(|\zeta| < 1/4\). Indeed, it is readily verified that \( E_1(\zeta) \) has no jump on \( \Sigma_1 \cup \Sigma_3 \), and hence has at most an isolated singularity at the origin. Furthermore, from the fact that \( \int_{\frac{1}{2}}^{\infty} \frac{1}{\sqrt{\zeta} t - \zeta} = \int_{\frac{1}{2}}^{\infty} \frac{1}{\sqrt{\zeta} t + \zeta} = \frac{\pi}{2} + O(1) \) for \( \zeta \to 0 \) and \( \arg \zeta \in (0, 2\pi) \), we see that the possible singularity of \( E_1(\zeta) \) at \( \zeta = 0 \) is weak, and hence is removable.

We note that \( p(0)(\zeta) \) given in (5.10) behaves the same as \( U(\zeta) \) in (5.4) at the origin. Indeed, for \( \zeta \in I \) \(|\arg \zeta| < 2\pi/3\), from (5.10) and (5.13) we have \( p(0)(\zeta) = O(1) \zeta^{\frac{\beta}{2} \sigma_3} \) as \( \zeta \to 0 \). Combining it with (5.4), we see that \( U(\zeta)(P(0))^{-1}(\zeta) \) has isolated singularity at \( \zeta = 0 \), such that \( U(\zeta)(P(0))^{-1}(\zeta) = O \left( \zeta^{-1/2} \ln \zeta \right) \) for \( \zeta \to 0 \) and \( \zeta \in I \), so far as \( \beta > -1 \). The same order estimates can be obtained in other sectors. Therefore, the singularity at the origin is weak, and is hence removable.

We are now in a position to introduce

\[
R_1(\zeta) = \begin{cases} 
U(\zeta)(P(0))^{-1}(\zeta), & |\zeta| < 1/4, \\
U(\zeta)G^{-1}(\zeta), & |\zeta| > 1/4.
\end{cases}
\]

So defined, \( R_1(\zeta) \) is a piecewise analytic function in \( \mathbb{C} \setminus \Sigma_{R_1} \), where the remaining contour \( \Sigma_{R_1} \) consists of the circle \(|\zeta| = 1/4\) oriented clockwise, and portions of \( \Sigma_2 \) and \( \Sigma_4 \), such that
We prove theorem 2 by applying the nonlinear steepest descendent analysis of 5.2. Proof of theorem 2

\[ J_{R_i}(\xi) = \begin{cases} 1 + O(s^{-1}), & |\xi| = 1/4, \\ 1 + O(e^{-c_s}), & \xi \in \Sigma_j \cap \Sigma_{R_i}, \ j = 2, 4, \end{cases} \]  

(5.16)

where \( c \) is a positive constant. Analysis similar to those in section 3.6 leads to

\[ R_4(\xi) = I + O(1/s), \]

(5.17)

with uniform error term in \( C \setminus \Sigma_{R_i} \).

For later use, we need the following sharper estimate for \( R_1(\xi) - I \) for large \( \xi \), as can be derived from the jump estimate (5.16) and the Cauchy type resolvent operator of \( R_1(\xi) \):

\[ R_1(\xi) = I + O\left(\frac{1}{s^2}\right), \quad \xi \to \infty \text{ and } s \to +\infty. \]

(5.18)

5.2. Proof of theorem 2

We prove theorem 2 by applying the nonlinear steepest descendent analysis of \( \Psi_0(\xi, s) \), as \( s \to \infty \).

From (5.1), (5.15) and (5.17), we get

\[ \Psi_0(\xi, s) e^{i\alpha \sigma_3} = \left( I + O\left(\frac{1}{s^2}\right) \right) G(\xi), \]

(5.19)

uniformly for \( |\xi| \geq r \) as \( s \to +\infty \).

Expanding the integral representation for \( G(\xi) \) in (5.8), we have

\[ \Psi_0(\xi, s) e^{-i\alpha \sigma_3} = \left( I + O\left(\frac{1}{s^2}\right) \right) \xi^{1/4} \frac{I - i\sigma_3}{\sqrt{2}} \left( I - \frac{\alpha \sigma_3}{2\sqrt{\xi}} + O\left(\frac{1}{\xi}\right) \right) \]

as \( \xi \to \infty \).

Thus, from (2.41) and (5.20), we obtain

\[ c_1(s) = \sigma(s)/s = -\alpha/2 + O(1/s) \quad \text{and} \quad c_2(s) = -i\alpha(s) = O(1/s) \quad \text{as} \quad s \to \infty. \]

(5.21)

Accordingly, we have

\[ u(s) = O(1/s) \quad \text{and} \quad \sigma(s) = -\alpha/2 + O(1) \quad \text{as} \quad s \to \infty. \]

(5.22)

Substituting (2.6) and (2.17) into (5.22), and recalling that \( \Theta = -\alpha \) and \( \gamma = \beta - \frac{1}{2} \) in the present case, we get

\[ b(s) = O(1/s) \quad \text{and} \quad y(s) = O(1) \quad \text{as} \quad s \to \infty, \]

(5.23)

so long as \( \alpha \neq 0 \).

A combination of (5.1), (5.10) and (5.15) gives

\[ \Psi_0(\xi) = R_4(\xi) E_1(\xi) \Phi \left( -\frac{s^2 \xi}{16} \right) e^{i\pi \gamma} \quad \text{for} \quad \arg \xi \in (0, \pi) \quad \text{with} \quad |\xi| < \frac{1}{4}, \]

(5.24)

where \( E_1 \) is analytic in the disk and \( \Phi \) is defined in (5.13). Now, further specifying \( \xi \in II; \) see figure 6, taking into account the definitions in (5.13) and (4.11), we have

\[ \begin{pmatrix} \psi_1(\xi, s) \\ \psi_2(\xi, s) \end{pmatrix} = R_1(\xi) E_1(\xi) \begin{pmatrix} i \beta \left( \frac{1}{2} s \sqrt{\xi} \right) \\ \frac{1}{2} s \sqrt{\xi} J_\beta \left( \frac{1}{2} s \sqrt{\xi} \right) \end{pmatrix} \begin{pmatrix} e^{-\frac{1}{2} z_0} \\ 0 \end{pmatrix} \]

Recalling that \( e^{-\frac{1}{2} z_0} J_\beta(z) = J_\beta(ze^{-\frac{1}{2} i}) \) for \( z \in (0, \pi/2] \) (corresponding to \( \arg \xi \in (0, \pi) \)); see [1, (9.6.3) and (9.6.30)], we obtain

\[ \begin{pmatrix} \psi_1(\xi, s) \\ \psi_2(\xi, s) \end{pmatrix} = R_1(\xi) E_1(\xi) \begin{pmatrix} J_\beta \left( \frac{1}{2} s \sqrt{\xi} e^{-\frac{1}{2} i} \right) \\ \frac{1}{2} s \sqrt{\xi} J_\beta \left( \frac{1}{2} s \sqrt{\xi} e^{-\frac{1}{2} i} \right) \end{pmatrix}. \]

(5.25)
Thus, substituting (5.25) into (4.16), by a similar argument as that in section 4.3, we obtain the following reduction of the $K_\Psi$ kernel, as $s \to +\infty$

$$\frac{4}{s^2}K_{\Psi} \left( -\frac{4u}{s^2}, -\frac{4v}{s^2} \right) = \psi_1 \left( -\frac{4u}{s^2}, s \right) \psi_2 \left( -\frac{4u}{s^2}, s \right) - \psi_1 \left( -\frac{4v}{s^2}, s \right) \psi_2 \left( -\frac{4v}{s^2}, s \right)$$

$$= J_\rho(u, v) \left( I + O \left( \frac{1}{s} \right) \right),$$

where $J_\rho(u, v)$ is defined in (1.6), with $u$ and $v$ being in compact subsets of $(0, +\infty)$. In deriving the second equality, use has been made of the fact that $\det \{ R_1(\zeta)(E_1(\zeta)) \} = 1$, and that $X^T \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right) X = \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right)$ as det $X = 1$. Here, as before, $X^T$ stands for the transpose of a matrix $X$.

It follows from (4.16) and (5.26) that

$$\frac{1}{2n^2}K_n \left( 1 - \frac{u}{2n^2}, 1 - \frac{v}{2n^2} \right) = \frac{4}{s^2}K_{\Psi} \left( -\frac{4u}{s^2}, -\frac{4v}{s^2} \right) \left( 1 + O \left( \frac{1}{n^2} \right) \right)$$

$$= J_\rho(u, v) \left( 1 + O \left( \frac{1}{n^2} \right) \right),$$

where $J_\rho(u, v)$ is given in (1.6), and the error terms are uniform in compact subsets of $u, v \in (0, +\infty)$ (that is $u/s^2, v/s^2 \in D$, see (4.12)). Thus completes the proof of theorem 2.

6. Reduction to Bessel kernel $J_{\alpha+\beta}$ as $s \to 0$

When $t = 1$ fixed, the weight in (1.8) can be written as $w(x) = (1 - x^2)^{\alpha+\beta}h(x)$. The local behavior at $x = 1$ is described via the kernel $J_{\alpha+\beta}$ given in (1.6) for $\alpha + \beta > -1$; see [17].

In the $\Psi$-kernel $K_{\Psi}(-u, -v; s)$ in (4.16), the parameter $s = 4n \ln(t + \sqrt{t^2 - 1}) \to 0$ as $t$ varies to 1°. Similar to the derivation in section 5, we study in the present section the asymptotics of the model RH problem for $\Psi_0(\zeta, s)$, with the parameters $\Theta = -\alpha, \gamma = \beta - \frac{1}{2}$ and $\alpha + \beta > -1$, as $s \to 0$. Then we apply the asymptotic results to reduce $K_{\Psi}(-u, -v; s)$ to the classical Bessel kernel $J_{\alpha+\beta}$. And we also obtain the asymptotics for the solution $b(s)$, $u(s)$ and $y(s)$ to the nonlinear equations given in section 1.2, as $s \to 0$°.

6.1. Nonlinear steepest descent analysis of the RH problem for $\Psi_0(\zeta, s)$ as $s \to 0$

$\Psi_0(\zeta, s)$ solves the model RH problem formulated in (2.40)–(2.43). Accordingly, $\Psi_0(\zeta/s^2, s)$ solves a re-scaled version of the RH problem. As $s \to 0$, the jump contour $\Sigma_1$ for $\Psi_0(\zeta/s^2, s)$ becomes the shrinking line segment $(0, s^2/4)$. Ignoring the constant jump on $\Sigma_1$, the RH problem is then reduced to the the Bessel model RH problem $\Phi$ formulated in (5.11) and (5.12), with the parameter $\beta$ being replaced by $\alpha + \beta$. Thus $\Psi_0(\zeta/s^2, s)$ is approximated by $\Phi$. However, the approximation is not true for $\zeta \in (0, s)$. So we need a local parametrix near the origin. A similar argument can be found in [30].

First we recall the well-known formulas for the modified Bessel functions

$$I_{\alpha+\beta}(z) = z^{\alpha+\beta} \sum_{n=0}^{\infty} \frac{z^{2n}}{n!(n + \alpha + \beta + 1)}.$$ $K_{\alpha+\beta}(z) = \frac{\pi}{2} \frac{I_{-\alpha-\beta}(z) - I_{\alpha+\beta}(z)}{\sin((\alpha + \beta)\pi)},$$ (6.1)

where arg $z \in (-\pi, \pi)$, and $\alpha + \beta \not\in \mathbb{Z}$; see [1, 9.6.2 and 9.6.10]. Applying these formulas, we can rewrite the function $\Phi$ in (5.13), using $\alpha + \beta$ instead of $\beta$, as

$$\Phi(\zeta) = E_2(\zeta) \frac{\psi^{i\alpha+\beta}(\zeta)}{\psi^{i\alpha+\beta}(0)} \left( \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right) \left( \begin{array}{c} J_1, \zeta \in I \\ J_{11}, \zeta \in II \\ J_{111}, \zeta \in III \end{array} \right),$$ (6.2)
See figure 6 for the regions, where
\[ J_I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J_{II} = \begin{pmatrix} 1 & 0 \\ e^{\pi i (\alpha + \beta)} & 1 \end{pmatrix}, \quad \text{and} \quad J_{III} = \begin{pmatrix} 1 & 0 \\ e^{-\pi i (\alpha + \beta)} & 1 \end{pmatrix}, \]
and \( E_2(\zeta) \) is a matrix-valued entire function, explicitly given as
\[
E_2(\zeta) = \begin{pmatrix} \zeta^{-(\alpha + \beta)/2} I_{\alpha + \beta}(2\sqrt{\zeta}) & \frac{1}{2\pi i} \frac{\zeta^{(\alpha + \beta)/2} I_{-(\alpha + \beta)}(2\sqrt{\zeta})}{\sin((\alpha + \beta)\pi)} \\ 2\pi i \zeta^{(1-\alpha-\beta)/2} I'_{\alpha + \beta}(2\sqrt{\zeta}) & 2\pi i \zeta^{(1+\alpha+\beta)/2} I'_{-(\alpha + \beta)}(2\sqrt{\zeta}) \end{pmatrix}. \tag{6.3}
\]

A straightforward comparison shows that \( \Psi_0(\zeta / s^2, s) \) and \( \left( \frac{\zeta}{s} \right)^{\frac{1}{2}\sigma_1} (-i\sigma_1) \Phi(\zeta / 16) \) share the same jumps and the same behavior at infinity, as long as \(|\zeta| > s^2/4\); see (2.40)–(2.41) and (5.11)–(5.12), in which \( \beta \) being replaced with \( \alpha + \beta \). For \(|\zeta| < s^2/4\), \( \Phi(\zeta) \) fails to approximate \( \Psi_0(\zeta / s^2, s) \) due to the appearance of the extra contour \( \Sigma_1 \) for \( \Psi_0 \). Then it is natural to consider a local parametrix, say, \( M(\zeta) \), in a small neighborhood \( U_\epsilon : |\zeta| < \epsilon, 0 < \epsilon < 1 \). For small \( s \), we see that the re-scaled \( \Sigma_1 \) lies in \( U_\epsilon \).

We state the RH problem for \( M(\zeta) \) as follows:

(a) \( M(\zeta) \) is analytic in \( U_\epsilon \setminus \cup_{j=1}^3 \Sigma_j \), where \( \Sigma_j \) are re-scaled version of those depicted in figure 1, such that \( \Sigma_1 = (0, s^2/4) \);

(b) \( M(\zeta) \) satisfies the same constant jump conditions (2.40), with \( \Psi_0 \) on \( U_\epsilon \cap \Sigma_j, j = 1, 2, 3, 4 \), specifying \( \Theta = -\alpha \) and \( \gamma = \beta - \frac{1}{2} \);

(c) The matching condition on the boundary \( \partial U_\epsilon \), as the parameter \( s \to 0 \), is
\[
M(\zeta) = (I + O(s^4))\Phi(\zeta / 16), \quad |\zeta| = \epsilon, \tag{6.4}
\]
where \( l = 2 \min\{1, \alpha + \beta + 1\} \), and \( \alpha + \beta > -1 \).

We seek a solution of the form
\[
M(\zeta) = \tilde{E}_2(\zeta) \begin{pmatrix} m(\zeta / s^2) & 0 \\ 0 & 1 \end{pmatrix} \left( \frac{\zeta}{s} \right)^{\sigma_1} \left( 1 - \frac{s^2}{4} \right)^{\sigma_3} \begin{pmatrix} \gamma_1 & 0 \\ 0 & 1 \end{pmatrix} \left( \frac{1}{2\pi i \sin((\alpha + \beta)\pi)} \right) \begin{pmatrix} J_I & 0 \\ 0 & J_{II} \end{pmatrix}. \tag{6.5}
\]
for \( \arg \zeta \in (-\pi, \pi) \) and \( \arg(\zeta - \zeta_0) \in (-\pi, \pi) \), where the constant matrices \( J_I - J_{III} \) are given in (6.2), the sectors \( I-II \) are illustrated in figure 6, and \( \tilde{E}_2(\zeta) = E_2(\zeta / 16) 4^{-(\alpha + \beta)\sigma_3} \) is entire, in which \( E_2(\zeta) \) is explicitly defined in (6.3).

Assuming that \( m(\zeta) \) is an analytic scalar function in \( \mathbb{C} \setminus [0, \frac{1}{4}] \), it is easily shown that the jump conditions (2.40) on \( \Sigma_2 - \Sigma_4 \) are satisfied automatically by \( M(\zeta) \). The remaining jump condition for \( M(\zeta) \) on the re-scaled contour \( \Sigma_1 = (0, s^2/4) \) is equivalent to the jump for \( m(\zeta) \) in (6.6) below. Hence, it suffices to solve the scalar RH problem:

(a) \( m(\zeta) \) is analytic in \( \mathbb{C} \setminus [0, \frac{1}{4}] \);

(b) \( m(\zeta) \) satisfies the jump condition
\[
m_+ (\zeta) - m_- (\zeta) = -\frac{\sin(\alpha \pi)}{\sin(\alpha + \beta) \pi} \left( \frac{1}{2} - \frac{\gamma}{\alpha + \beta} \right) \left( \frac{1}{4} - \zeta \right)^{\alpha} \] for \( \zeta \in (0, 1/4) \); \tag{6.6}

(c) The behavior of \( m(\zeta) \) at infinity is
\[
m(\zeta) = O(1/\zeta). \tag{6.7}
\]

The RH problem can be solved by using the Sokhotski–Plemelj formula. We have
\[
m(\zeta) = -\frac{\sin(\alpha \pi)}{2\pi i \sin(\alpha + \beta) \pi} \int_{\tau}^\zeta \frac{\epsilon^\beta (1 - \tau)^{\alpha}}{\tau - \zeta} d\tau, \quad \zeta \in \mathbb{C} \setminus [0, 1/4]. \tag{6.8}
\]
In view of (6.5) and (6.8), we see that the matching condition (6.4) is fulfilled.
straightforward manner, that the isolated singularities at \( \zeta = \zeta_m(\zeta) \) are removable. Indeed, by the same argument as in the non-integer case, we construct a local parametrix \( M(\zeta) \) which is an analytic matrix function and \( M(\zeta) = I + O(s^2 \ln s) \Phi(\zeta/16) \), where \( \Phi(\zeta) \) is an analytic matrix function and \( \zeta \in \mathbb{C} \setminus [0, 1/4] \). The relations of Bessel functions in (6.1) should be modified, and a logarithmic singularity may appear in the off-diagonal entry in (6.2). Instead of (6.2), we have

\[
\Phi(\zeta) = \tilde{E}_2(\zeta)\zeta^{\frac{i\alpha_1+\beta}{2}} \begin{pmatrix} 1 & \frac{-1}{\pi i} \\ 0 & 1 \end{pmatrix} \left( \ln \zeta - \frac{\pi}{2} \right),
\]

where \( \tilde{E}_2(\zeta) \) is an analytic matrix function and \( \zeta \in \mathbb{C} \). Then, by the same argument as in the non-integer case, we construct a local parametrix \( M(\zeta) \) in the form of (6.5), with \( m(\zeta) \) defined as

\[
m(\zeta) = \frac{\sin(\alpha \pi)(-1)^{\alpha+\beta}s^2(\alpha+\beta)}{2\pi i} \int_0^1 \frac{\tau^\beta \left( \frac{1}{4} - \tau \right)^\alpha \ln(s^2 \tau) d\tau}{\tau - \zeta}, \quad \zeta \in \mathbb{C} \setminus [0, 1/4].
\]

And, the matching condition (6.4) is now slightly modified as

\[
M(\zeta) = (I + O(s^2 \ln s))\Phi(\zeta/16), \quad |\zeta| = \epsilon, \quad \text{for } \alpha + \beta = 0,
\]

and

\[
M(\zeta) = (I + O(s^2))\Phi(\zeta/16), \quad |\zeta| = \epsilon,
\]

for \( \alpha + \beta \) being a positive integer, where use is made of the condition that \( \alpha + \beta > -1 \).

Now we proceed to consider

\[
R_2(\zeta) = \begin{cases} i\sigma_1 \left( \frac{s^2}{4} \right)^{\frac{-1}{2}} \psi_0 \left( \frac{\zeta^2}{s^2}, s \right) M^{-1}(\zeta), & |\zeta| < \epsilon, \\
\frac{\sqrt{\pi}}{2} i \sigma_1 \left( \frac{1}{2} \right)^{\frac{-1}{2}} \psi_0 \left( \frac{1}{2}, s \right) \Phi^{-1}(\frac{\zeta}{16}), & |\zeta| > \epsilon.
\end{cases}
\]

The matrix function \( R_2 \) is analytic in \( |\zeta| \neq \epsilon \). Indeed, we need only to verify, in a straightforward manner, that the isolated singularities at \( \zeta = 0, s^2/4 \) are removable. For example, a combination of (2.43) (with \( \Theta = -\alpha \), (6.5) and (6.11) gives

\[
R_2(\zeta) = O(1) \begin{pmatrix} O(1) & 0 \\ 0 & O(1) \end{pmatrix} O(1) \quad \text{as } \zeta \rightarrow s^2/4.
\]

Thus \( \zeta = s^2/4 \) is a weak singularity, and hence is removable. A similar argument applies to \( \zeta = 0 \). Here use has been made of the fact that the scalar function defined in (6.8) has the boundary behavior

\[
m(\zeta) = O(1) + O \left( \zeta^\beta \right), \quad \zeta \rightarrow 0 \quad \text{and} \quad m(\zeta) = s^{2(\alpha+\beta)} \left[ O(1) - \frac{\zeta^\beta (\zeta - 1/4)^\alpha}{2i \sin(\alpha+\beta \pi)} \right], \quad \zeta \rightarrow 1/4,
\]
for $\zeta \notin [0, 1/4]$, where $\arg \zeta \in (-\pi, \pi)$ and $\arg(\zeta - 1/4) \in (-\pi, \pi)$ in the approximation at $\zeta = 1/4$. Also, it follows from the matching condition that the jump

$$J_R(\zeta) = I + O(s^l), \quad |\zeta| = \epsilon,$$

(6.12)

where $l = 2 \min\{1, \alpha + \beta + 1\}$, with $\alpha + \beta > -1$. So, by an argument similar to section 3.6, we have

$$R_2(\zeta) = I + O(s^l), \quad s \to 0^+,$$

(6.13)

where the error term $O(s)$ is uniform in $\zeta$. A sharper estimate is available for large $\zeta$, namely,

$$R_2(\zeta) = I + O\left(\frac{s^l}{\zeta}\right), \quad s \to 0^+ \text{ and } \zeta \to \infty,$$

(6.14)

where $l = 2 \min\{1, \alpha + \beta + 1\}$, and $\alpha + \beta > -1$.

6.2. Proof of theorem 3

We apply the asymptotic formulas to obtain the asymptotic properties of several functions introduced in section 2.1, as $s \to 0^+$. To begin with, we see from (5.12), (6.11) and (6.14) that

$$\Psi_0 \left(\frac{\zeta}{s^4}, s\right) e^{\frac{2\pi}{\sigma_1} s} = s^{-\frac{i}{\sigma_1}} \left(I + O\left(\frac{s^l}{\zeta}\right)\right) \frac{I - i\sigma_1}{\sqrt{2}} \left(I + O\left(\frac{1}{\sqrt{\zeta}}\right)\right)$$

(6.15)

for $\zeta \to \infty$ and $s \to 0^+$, where $l = 2 \min\{1, \alpha + \beta + 1\}$, and $\alpha + \beta > -1$. Refinement is available by using (6.19) below and expanding $\Phi(\zeta)$ in (5.13) for large $\zeta$, with $\beta$ being replaced by $\alpha + \beta$. As a result, we have

$$\Psi_0 \left(\frac{\zeta}{s^2}, s\right) e^{\frac{2\pi}{\sigma_1} s} = s^{-\frac{i}{\sigma_1}} \left(I + O\left(\frac{s^l}{\zeta}\right)\right) \frac{I - i\sigma_1}{\sqrt{2}} \left(I + \frac{C_{R, 1}}{\sqrt{\zeta}} + \frac{C_{R, 2}}{\zeta} + O\left(\frac{1}{\zeta^{3/2}}\right)\right)$$

(6.16)

as $\zeta \to \infty$ and $s \to 0$, where the first two coefficients of the large-$\zeta$ expansion for $\Phi(\zeta)$ are

$$C_{R, 1} = -\frac{i}{2} \sigma_1 - \left\{\frac{(\alpha + \beta)^2 + 1}{4}\right\} \sigma_3 \quad \text{and} \quad C_{R, 2} = \frac{4(\alpha + \beta)^2 - 1}{8} \left\{\left(\alpha + \beta\right)^2 + \frac{3}{4}\right\} I + 3\sigma_2.$$

Thus, comparing (2.41) with (6.16), we have the behavior for $\sigma(s)$, $u(s)$ and $\hat{c}_2(s)$, such that

$$\sigma = -\left((\alpha + \beta)^2 + \frac{1}{4}\right) + O\left(s^l\right), \quad u = -\frac{1}{2s} + O\left(s^{l-1}\right) \quad \text{and} \quad \hat{c}_2 = \frac{3(4(\alpha + \beta)^2 - 1)}{8s^2} + O\left(s^{l-2}\right)$$

(6.17)

as $s \to 0^+$, where $\sigma, u$ and $\hat{c}_2$ appear in the coefficient of the asymptotic behavior at infinity of $\Psi$ and $\Psi_0$; see (2.20) and (2.41). Then, a combination of (2.6), (2.17) and (6.17), with $\Theta = -\alpha$ and $\gamma = \beta - \frac{i}{2}$, yields

$$\begin{cases} b(s) = -\frac{(\alpha + \beta)^2}{2} + \frac{1}{2} + O\left(s^{l-1}\right), \\ y(s) = 1 + O\left(s^l\right) \end{cases}$$

(6.18)

as $s \to 0^+$, where $l = 2 \min\{1, \alpha + \beta + 1\}$ and use has been made of the fact that

$$y = \frac{\gamma + 2u + 2u\sigma - us - 2\hat{c}_2 s}{2(b - \alpha)}.$$
Now we are in a position to prove theorem 3. For $|\zeta| > \epsilon$, it follows directly from (6.11) that
\[
\Psi_0 \left( \frac{\zeta}{s^2}, s \right) = -i \left( \frac{\pi s}{2} \right)^{-\frac{1}{4} \sigma_1} \sigma_1 R_2(\zeta) \Phi \left( \frac{\zeta}{16} \right). \tag{6.19}
\]
Thus, for $\zeta \in II$ and $|\zeta| > \epsilon$, a combination of (4.11), (6.19) and (5.13), again with $\alpha + \beta$ taking the place of $\beta$, gives
\[
\left( \begin{array}{c}
\psi_1 \left( \frac{\zeta}{s^2}, s \right) \\
\psi_2 \left( \frac{\zeta}{s^2}, s \right)
\end{array} \right) = \left( \frac{\pi s}{2} \right)^{-\frac{1}{4} \sigma_1} (-i \sigma_1) R_2(\zeta) e^{\frac{\alpha + \beta}{\pi i \sigma_1}} \left( \begin{array}{c}
I_{\alpha + \beta} \left( \frac{\sqrt{\pi s}}{2} \right) \\
\frac{\pi i \sigma_1}{2} J_{\alpha + \beta} \left( \frac{\sqrt{\pi s}}{2} \right)
\end{array} \right).
\tag{6.20}
\]
Here use has been made of the fact that $e^{-\frac{1}{4} \nu s} I_{\nu}(z) = J_{\nu}(ze^{-\frac{1}{4} \nu i})$ for $\arg z \in (0, \pi/2]$.

Thus, by a similar argument leading to (4.16), or to (5.26), we get from (6.20) the approximation of $K_\nu$ by the Bessel kernel as follows,
\[
\frac{\psi_1(-\frac{4u}{\pi}, s) \psi_2(-\frac{4u}{\pi}, s) - \psi_1(-\frac{4v}{\pi}, s) \psi_2(-\frac{4v}{\pi}, s)}{2\pi i(u-v)} = \mathbb{I}_{\alpha + \beta}(u, v) \left( 1 + O \left( s^4 \right) \right), \tag{6.21}
\]
where $l = 2 \min\{1, \alpha + \beta + 1\}$, $\alpha + \beta > -1$, and the Bessel kernel is defined in (1.6), and the error term is uniform in compact subsets of $u, v \in (0, \infty)$.

Thus, by (4.16) and (6.21), we obtain
\[
\frac{1}{2n^2} K_n \left( 1 - \frac{u}{2n^2}, 1 - \frac{v}{2n^2} \right) = \frac{\psi_1(-\frac{4u}{\pi}, s) \psi_2(-\frac{4u}{\pi}, s) - \psi_1(-\frac{4v}{\pi}, s) \psi_2(-\frac{4v}{\pi}, s)}{2\pi i(u-v)} \times \left( 1 + O \left( \frac{1}{n^2} \right) \right)
\]
\[
= \mathbb{I}_{\alpha + \beta}(u, v) \left( 1 + O(s^4) + O(1/n^2) \right), \tag{6.22}
\]
where $l = 2 \min\{1, \alpha + \beta + 1\}$, $\alpha + \beta > -1$, $\mathbb{I}_{\alpha + \beta}(u, v)$ is given in (1.6), and the error terms are uniform in compact subsets of $u, v \in (0, +\infty)$. And we complete the proof of theorem 3.

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