Infinite dimensional representations of finite dimensional algebras and amenability

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Abstract We present a novel quantitative approach to the representation theory of finite dimensional algebras motivated by the emerging theory of graph limits. We introduce the rank spectrum of a finite dimensional algebra $R$ over a countable field. The elements of the rank spectrum are representations of the algebra into von Neumann regular rank algebras, and two representations are considered to be equivalent if they induce the same Sylvester rank functions on $R$-matrices. Based on this approach, we can divide the finite dimensional algebras into three types: finite, amenable and non-amenable representation types. We prove that string algebras are of amenable representation type, but the wild Kronecker algebras are not. Here, the amenability of the rank algebras associated to the limit points in the rank spectrum plays a very important part. We also show that the limit points of finite dimensional representations of algebras of amenable representation type can always be viewed as representations of the algebra in the continuous ring invented by John von Neumann in the 1930’s. As an application in algorithm theory, we introduce and study the notion of parameter testing of modules over finite dimensional algebras, that is analogous to the testing of bounded degree graphs introduced by Goldreich and Ron. We shall see that for string algebras all the reasonable (stable) parameters are testable.

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1 Introduction

By Maschke’s theorem, the complex group algebra $\mathbb{C}G$ is semisimple if $G$ is a finite group. Hence all the infinite dimensional representations of $G$ can be decomposed into finite dimensional subrepresentations. The situation becomes much more interesting if the characteristic of the coefficient field divides the order of the group.

Example 1 Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ be the Klein group and $K$ be the field of two elements. Let $K(t)$ be the transcendent extension of $K$ by the element $t$. Then

$$\phi(a) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \phi(b) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

defines an indecomposable representation $\phi : G \to GL(2, K(t))$, where $G = \{1, a\} \times \{1, b\}$.

Infinite dimensional representations of finite dimensional algebras have been studied for decades (see e.g. [11,24]). The aim of our paper is to develop a theory for such representations guided by the convergence and limit theory of finite graphs (see the monograph of László Lovász [21]). Our philosophy is to view infinite dimensional representations of a given algebra (when it is possible) as a sort of limit of its finite dimensional representations.

The rank spectrum Let $K$ be a countable field and $R$ be a finite dimensional $K$-algebra. Let $R$-Mod denote the set of finitely generated (left) $R$-modules (up to isomorphism). If $M \in R$-Mod, then $\phi^M : R \to \text{End}_K(M) \cong \text{Mat}_{\dim(M) \times \dim(M)}(K)$ is the corresponding representation, where $\phi^M : r \mapsto rm$ and $\dim(M)$ is the $K$-dimension of the module $M$. The isomorphism $\text{End}_K(M) \cong \text{Mat}_{\dim(M) \times \dim(M)}(K)$ is, of course, depends on the choice of a $K$-basis for $M$. For any $k, l \geq 1$ the map $\phi^M$ extends to the homomorphism (using a $K$-basis of $M$ as above)

$$\phi^{M}_{k,l} : \text{Mat}_{k \times l}(R) \to \text{Mat}_{k \dim(M) \times l \dim(M)}(K).$$
If \( A \in \text{Mat}_{k \times l}(R) \), then let
\[
    rk_M(A) := \frac{\text{rank}(\phi^M_{k,l}(A))}{\dim(M)}.
\]

Then \( rk_M \) is a Sylvester rank function, that is
\begin{itemize}
    \item \( rk_M(I) = 1 \), where \( I \) is the unit element of \( R \) considered as a \( 1 \times 1 \)-matrix.
    \item \( rk_M \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = rk_M(A) + rk_M(B), \)
    \item \( rk_M \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \geq rk_M(A) + rk_M(B), \)
    \item \( rk_M(AB) \leq \min(rk_M(A), rk_M(B)) \),
\end{itemize}
provided that the \( R \)-matrices \( A, B, C \) have appropriate sizes. Observe that
\[
    rk_M = rk_{M \oplus M \oplus \cdots \oplus M}.
\]

We will show that if for two modules \( rk_M = rk_N \), then there exist \( k, l \geq 1 \) such that \( M^k \cong N^l \). We say that a sequence \( \{M_n\}_{n=1}^{\infty} \subseteq R\text{-Mod} \) is \textit{convergent} if for all pairs \( k, l \geq 1 \) and \( A \in \text{Mat}_{k \times l}(R) \), \( \lim_{n \to \infty} rk_{M_n}(A) \) exists. The notion of convergence is based on the Benjamini-Schramm convergence notion (see the Introduction of [15]). Since the set of all finite matrices over \( R \) is countable, from each sequence \( \{M_n\}_{n=1}^{\infty} \) one can pick a convergent subsequence. Let us consider the locally convex, topological vector space \( C = \mathbb{R}^{\text{Mat}(R)} \), where \( \text{Mat}(R) \) is the countable set of all finite matrices over \( R \). The set of all Sylvester rank functions on \( \text{Mat}(R) \), \( \text{Syl}(R) \) forms a convex, compact subset of \( C \). We will also consider the subset of Sylvester rank functions corresponding to finite dimensional representations
\[
    \text{Fin}(R) := \{rk_M \mid M \in R\text{-Mod}\}.
\]

Let \( K \) be a finite field as above. A \( K \)-algebra \( S \) is called a \textit{rank algebra} (see [17] for an introduction of rank functions on von Neumann regular rings) if
\begin{itemize}
    \item \( S \) is a von Neumann regular ring,
    \item \( \text{Mat}(S) \) is equipped with a Sylvester rank function \( \text{rank}_S \).
    \item \( \text{rank}_S(A) = 0 \implies A = 0. \)
\end{itemize}

The most important examples of rank algebras are skew fields, matrix rings over skew fields and more generally, semisimple Artinian rings. If we have a representation \( \rho : R \to S \) into a rank algebra we still have an associated Sylvester rank function
\[
    rk_{\rho}(A) := \text{rank}_S(\tilde{\rho}(A)),
\]
where \( \tilde{\rho} \) is the appropriate extension for matrices. We say that two such representations \( \rho_1 : R \to S_1 \) and \( \rho_2 : R \to S_2 \) are equivalent if \( rk_{\rho_1} = rk_{\rho_2} \). The \textit{rank spectrum} of \( R \), \( \text{Rank}(R) \) is the subset of Sylvester rank functions that are in the form \( rk_{\rho} \) for some representation \( \rho : R \to S \). So, an element of \( \text{Rank}(R) \) can be viewed as a
representation up to equivalence, where two representations are equivalent if they define the same rank function. We say that the infinite dimensional representation \( \rho : R \to S \) is the limit of the convergent sequence of finite dimensional representations \( \{ \rho_n : R \to \text{Mat}_{m_n \times m_n}(K) \}_{n=1}^{\infty} \) if for any \( A \in \text{Mat}(R) \),

\[
\lim_{n \to \infty} rk_{\rho_n}(A) = rk_{\rho}(A)
\]

that is, if \( rk_{\rho} \) is the limit point of the sequence \( \{ rk_{\rho_n} \}_{n=1}^{\infty} \) in \( \text{Syl}(R) \).

Example 2 Let \( \{ K(t_n) = E_n \}_{n=1}^{\infty} \) be a sequence of finite dimensional field extensions of \( K \) with generators \( \{ t_n \}_{n=1}^{\infty} \). Suppose that \( \dim_K(E_n) \to \infty \). Let \( G = \mathbb{Z}_2 \times \mathbb{Z}_2 \) as in Example 1 and define the representation \( \phi_n : KG \to \text{Mat}_{2 \times 2}(E_n) \) by

\[
\phi_n(a) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \phi_n(b) = \begin{pmatrix} 1 & t_n \\ 0 & 1 \end{pmatrix}.
\]

Then, the representations \( \{ \phi_n \}_{n=1}^{\infty} \) converge to the representation \( \phi : KG \to \text{Mat}_{2 \times 2}(K(t)) \) described in Example 1.

**Proposition 1.1** For any finite dimensional algebra \( R \) over a countable field, the set \( \text{Rank}(R) \) is a convex, compact subspace of \( \text{Syl}(R) \).

The following question is the somewhat analogous to the famous Connes Embedding Problem [7].

**Question 1** (see Theorem 3) Is the closure of \( \text{Fin}(R) \) in \( \text{Syl}(R) \) equal to \( \text{Rank}(R) \)?

Note that if \( rk \in \text{Syl}(R) \) is a Sylvester rank function such that the values of \( rk \) are in the set \( \mathbb{Z}/d \) for some \( d \in \mathbb{N} \), then by Schofield’s Theorem (Theorem 7.12 [25]), there exists a skew field \( D \) and a representation \( \rho : R \to \text{Mat}_{d \times d}(D) \) such that \( rk = rk_{\rho} \).

**Hyperfiniteness** Let \( R \) be a finite dimensional algebra as above and \( \{ M_n \}_{n=1}^{\infty} \subset R\text{-Mod} \) be a set of modules. We say that the family \( \{ M_n \}_{n=1}^{\infty} \) is hyperfinite, if for any \( \epsilon > 0 \), there exists some \( L_\epsilon > 0 \), such that for any \( n \geq 1 \), there exists a submodule \( P_n \subset M_n \), \( \dim_K(P_n) \geq (1 - \epsilon) \dim_K(M_n) \) and modules \( \{ N_i \}_{i=1}^{t_n} \subset R\text{-Mod} \), \( \dim_K(N_i) \leq L_\epsilon \) such that \( P_n \cong \bigoplus_{i=1}^{t_n} N_i \). An algebra \( R \) of infinite representation type (that is the set of indecomposable modules \( \text{Ind}(R) \) in \( R\text{-Mod} \) is infinite) is called of amenable representation type, if \( R\text{-Mod} \) itself is a hyperfinite family. In the course of the paper, we shall see (Proposition 10.1) that string algebras are of amenable representation type and the wild Kronecker algebras (Theorem 6) are not. These results suggest a relation between the tame/wild and the amenable/non-amenable dichotomies.

**Conjecture 1** Let \( K \) be a countable algebraically closed field and \( R \) be a finite dimensional algebra of infinite representation type over \( K \). Then \( R \) is of tame representation type if and only if \( R \) is of amenable representation type.

Let \( d_R \) be a metric on \( \text{Syl}(R) \) defining the compact topology. Two modules \( M, N \in R\text{-Mod} \) are \( \epsilon \)-close to each other if there exist submodules \( P \subset M, Q \subset N, P \cong Q \) such that \( \dim_K(P) \geq (1 - \epsilon) \dim_K(M), \dim_K(Q) \geq (1 - \epsilon) \dim_K(N) \). Motivated by graph theoretical results (Theorem 5. [14]), we make the following con.
Conjecture 2 Let $R$ be an finite dimensional algebra of amenable representation type over a field $K$. Then, for any $\epsilon > 0$ we have a $\delta > 0$ such that for two elements $M, N \subset R\text{-Mod}$, $1 - \delta \leq \frac{\dim_K(M)}{\dim_K(N)} \leq 1 + \delta$, $d_R(rk_M, rk_N) < \delta$, then $M$ and $N$ are $\epsilon$-close to each other.

The meaning of the conjecture is that if we have a finite dimensional algebra of amenable representation type, then the almost isomorphism of two $R$-modules of the same dimension can be decided by checking the ranks of finitely many matrices. This observation leads to the representational theoretical analogue of the Constant-Time Property Testing Algorithms developed by Goldreich and Ron for bounded degree graphs. (see Sect. 14). The main result of our paper is the following.

Theorem 1 Conjecture 2 holds if $R$ is a string algebra.

Amenability of infinite dimensional algebras Amenability and hyperfiniteness are intimately related notions in group theory. The notion of amenability for algebras were introduced by Misha Gromov [18] and plays an important part in the study of the rank spectrum.

Definition 1.1 An element $rk_\rho$ in the rank spectrum of $R$ is amenable, if there exists a hyperfinite sequence $\{M_n\}_{n=1}^\infty \subset R\text{-Mod}$ such that $\lim_{n \to \infty} rk_{M_n} = rk_\rho$.

The following conjecture is motivated by the theorem of Oded Schramm on hyperfinite graph sequences [26].

Conjecture 3 Let $K$ be a finite field and $R$ be a finite dimensional algebra over $K$. If $\rho : R \to S$ defines $rk_\rho$, where $S$ is an amenable rank algebra, then $rk_\rho$ is amenable.

We confirm this conjecture in the case, where $S = \text{Mat}_{l \times l}(D)$ and $D$ is an amenable skew field (Theorem 5). For the converse, we have the following con.

Conjecture 4 Let $K$ and $R$ be as above. If $rk_\rho$ is an amenable element of the rank spectrum, then there exists an amenable algebra $S$ and a homomorphisms $\rho' : R \to S$, such that $rk_\rho = rk_{\rho'}$.

The continuous algebra of John von Neumann It is well-known that any countable amenable group algebra has a trace-preserving embedding into the unique separable hyperfinite $II_1$-factor. In the world of ranks the role of the hyperfinite $II_1$-factor will be played by an algebra constructed by John von Neumann in the 1930’s [27]. The construction goes as follows. We consider the following sequence of diagonal embeddings.

$$K \to \text{Mat}_{2 \times 2}(K) \to \text{Mat}_{4 \times 4}(K) \to \text{Mat}_{8 \times 8}(K) \to \ldots,$$

where $K$ is an arbitrary field. Then all the embeddings are preserving the normalized ranks. Hence the direct limit $\lim_{\to} \text{Mat}_{2^k \times 2^k}(K)$ is a rank algebra. The addition, multiplication and the rank function extends to the metric completion $\mathcal{M}_K$ of the direct limit algebra. The resulting algebra $\mathcal{M}_K$ is a simple continuous rank algebra (see also [17]).
Theorem 2 Let \( K \) be a finite field and \( R \) be a finite dimensional algebra over \( K \). Let \( \mathcal{M} = \{M_n\}_{n=1}^{\infty} \subset R\text{-Mod} \) be a hyperfinite convergent sequence and \( rk_{\mathcal{M}} = \lim_{n \to \infty} rk_{M_n} \). Then there exists a representation \( \rho : R \to \mathcal{M}_K \) such that \( rk_{\rho} = rk_{\mathcal{M}} \).

The following conjecture is known by experts.

Conjecture 5 All the countable dimensional amenable rank algebras over the field \( K \) embed into \( \mathcal{M}_K \). Conversely, every countable dimensional rank subalgebra of \( \mathcal{M}_K \) is amenable.

Note that by Theorem 2, Conjecture 5 would imply Conjecture 4.

2 The continuous ultraproduct of rank algebras

First, let us prove Proposition 1.1. Let \( \{S_n\}_{n=1}^{\infty} \) be rank algebras and \( \{\rho_n : R \to S_n\}_{n=1}^{\infty} \) be a convergent sequence in the rank spectrum \( \text{Rank}(R) \), that is, for any matrix \( A \in \text{Mat}(R) \), \( \lim_{n \to \infty} rk_{\rho_n}(A) \) exists. Let \( \omega \) be a non-principal ultrafilter on the natural numbers and \( \lim_{\omega} \) be the associated ultralimit. We define the continuous ultraproduct \( \prod_{\omega} S_n \), a quotient of the classical ultraproduct, the following way. Let \( I \subset \prod_{n=1}^{\infty} S_n \) be the set of elements \( \{a_n\}_{n=1}^{\infty} \) such that \( \lim_{\omega} \text{rank}_{S_n}(a_n) = 0 \).

Clearly, \( I \) is a two-sided ideal in \( \prod_{n=1}^{\infty} S_n \). Then \( \Sigma = \prod_{\omega} S_n := \prod_{n=1}^{\infty} S_n/I \) is a von Neumann regular ring, since products of von Neumann regular rings and quotients of von Neumann regular rings are von Neumann regular rings as well. Now, if \( A \in \text{Mat}_k(\Sigma) = \{[A_n]_{n=1}^{\infty}\} \), then let \( \text{rank}_{\Sigma}(A) = \lim_{\omega} \text{rank}_{S_n}(A_n) \). Then \( \text{rank}_{\Sigma} \) is a Sylvester rank on \( \Sigma \) and \( \text{rank}_{\Sigma}(a) \neq 0 \) if \( a \in \Sigma \) and \( a \neq 0 \).

The representation \( \rho_{\Sigma} : R \to \Sigma \) is defined by \( \rho_{\Sigma}(r) := \{[\rho_n(r)]_{n=1}^{\infty}\} \). Then, \( rk_{\Sigma}(A) = \lim_{n \to \infty} rk_{\rho_n}(A) \) holds for any matrix \( A \in \text{Mat}(R) \). Hence, \( \text{Rank}(R) \) is a closed subset of \( \text{Syl}(R) \).

If the elements \( rk_1 \) resp. \( rk_2 \) are represented by homomorphisms \( \rho_1 : R \to S \) and \( \rho_2 : R \to T \), where \( S, T \) are rank algebras, then \( \lambda \cdot rk_1 + \mu \cdot rk_2, (\lambda, \mu \geq 0, \lambda + \mu = 1) \), is represented by \( \rho : R \to S \oplus T \), where

\[
\text{rank}_{S \oplus T} = \lambda \cdot \text{rank}_S + \mu \cdot \text{rank}_T.
\]

Hence, \( \text{Rank}(R) \) is a convex subset of \( \text{Syl}(R) \).

The following lemma is implicit in [15], nevertheless we include the proof for the sake of completeness.

Lemma 2.1 Let \( K \) be a countable field, \( R \) be a finite dimensional \( K \)-algebra and \( \rho : R \to S \) be a representation of \( R \) into a rank algebra. Then there exists a countable dimensional rank algebra \( S' \subset S \) that contains the image of \( \rho \). That is, all the elements of the rank spectrum can be represented by countable dimensional rank algebras.
Proof Let $X$ be a finite dimensional linear subspace of $S$ containing the unit. Denote by $P(X)$ the finite dimensional linear subspace of $S$ spanned by elements in the form 
\[ \{x_1 x_2 \mid x_1, x_2 \in X\} \].

Also, let $R(X)$ be an arbitrary finite set such that for any $x \in X$, there exists $y \in R(X)$ such that $xyx = x$. Let $X_1 = \text{Im} \rho(R)$ and $X_{n+1} = R(P(X_n))$. Then $\bigcup_{n=1}^{\infty} X_n$ is a countable dimensional von Neumann regular ring containing the image of $R$. \hfill \Box

3 The Ziegler spectrum

The goal of this section is to recall the notion of the Ziegler spectrum [28] (see also [23]) and show that finitely generated indecomposable modules are isolated points of the closure of $\text{Fin}(R)$.

Let $R$ be a finite dimensional algebra over a countable field $K$. A pp-formula $\phi$ of type $t$ is given by a matrix
\[ A = \{a_{ij}\}_{1 \leq i \leq m, 1 \leq j \leq n} \in \text{Mat}_{m \times n}(R). \]

Let $M$ be a module over $R$. We say that $v = \{v_1, v_2, \ldots, v_t\} \in M^t$, $n \geq t$ satisfies $\phi$ if there exists $\{y_{t+1}, y_{t+2}, \ldots, y_n\} \in M^{n-t}$ such that for any $1 \leq i \leq m$
\[ \sum_{j=1}^{t} a_{ij} v_j + \sum_{j=t+1}^{n} a_{ij} y_j = 0. \]

The elements $v$ satisfying $\phi$ form the pp-subspace $M(\phi) \subset M^t$. We say that two formulas (of the same type) are equivalent, $\phi \cong \psi$, if for any $M, M(\phi) = M(\psi)$. Also, $\phi \geq \psi$ if for any $M, M(\phi) \supset M(\psi)$. For any $t \geq 1$, the equivalence classes of pp-formulas of type $t$ form the modular lattice $\text{pp}^t(R)$. The Ziegler spectrum of $R, Zg(R)$, consists of the pure-injective indecomposable modules over $R$. The basic open sets of $Zg(R)$ are given by pp-pairs $\langle \phi, \psi \rangle$, where $\phi \geq \psi$.

\[ U_{\phi, \psi} = \{M \mid M(\phi) \supset M(\psi)\} \]

Then, the set of isolated points in $Zg(R)$ are exactly the finitely generated indecomposable modules and they form a dense subset in the quasi-compact (not necessarily Hausdorff) space $Zg(R)$ (Corollary 5.3.36. and Corollary 5.3.37. [23]). Let $\phi$ be a pp-formula of type $t$ and let $M$ be a finitely generated module. Set
\[ D_M([\phi]) := \frac{\dim_K(M(\phi))}{\dim_K(M)}. \]

Then, $D_M$ is a dimension function on $\text{pp}^t(R)$, that is
- $D_M(0) = 0$
\[ D_M(1) = 1 \]
\[ D_M(a \land b) + D_M(a \lor b) = D_M(a) + D_M(b) \]
\[ D_M(a) \leq D_M(b) \text{ if } a \leq b \]

(Note that we do not require that \( a \leq b \Rightarrow D_M(a) \leq D_M(b) \).) Now let \( \rho : R \to S \) be a representation (possibly infinite dimensional) into a rank algebra. Recall [17] that the finitely generated right submodules of \( S^k \) are projective, all exact sequences of such modules split and the rank function defines a dimension function \( \dim_S \) on finitely generated right submodules of finitely generated free \( S \)-modules. Let \( A : S^m \to S^n \) be a module endomorphism (we nonchalantly consider \( A \) as an \( m \times n \)-matrix in \( \text{Mat}(S) \)). Then

\[
\text{rank}_S(A) = \dim_S(\text{Im}(A)) = n - \dim_S(\text{Ker}(A)).
\]

Let us consider \( S \) as a left \( R \)-module.

**Lemma 3.1** For any \( \phi \in \text{pp}^i(R) \), \( S(\phi) \) is finitely generated and its dimension \( \dim_S(S(\phi)) \) can be computed by knowing only the element \( rk_{\rho} \) in the rank spectrum.

**Proof** Recall that \( S(\phi) \) is the set of elements \( \{s_1, s_2, \ldots, s_t\} \in S^t \) such that there exists \( \{y_{t+1}, y_{t+2}, \ldots, y_n\} \in S^{n-t} \) so that for any \( 1 \leq i \leq m \)

\[
\sum_{j=1}^{t} a_{ij}s_j + \sum_{j=t+1}^{n} a_{ij}y_j = 0,
\]

where the matrix \( A \) defines \( \phi \). Therefore, \( S(\phi) \) is a right \( S \)-module. Let \( Q \) be the set of elements \( \{z_1, z_2, \ldots, z_n\} \in S^n \) such that

\[
\sum_{j=1}^{n} a_{ij}z_j = 0,
\]

and \( z_i = 0 \), if \( 1 \leq i \leq t \). Then

\[
0 \to Q \to \text{Ker}(A) \to S(\phi) \to 0
\]

is an exact sequence, where \( Q \) is the kernel of a matrix \( B \in \text{Mat}(R) \subseteq \text{Mat}(S) \). That is, by Lemma 2.1 [17], \( Q \) and \( \text{Ker}(A) \) are both finitely generated right \( S \)-modules and \( S(\phi) \) is also finitely generated since it is a quotient of \( \text{Ker}(A) \). We have that

\[
D_S(\phi) = \dim_S(S(\phi)) = \dim_S(Q) - \dim_S(\text{Ker}(A)) = t + rk_{\rho}(B) - rk_{\rho}(A).
\]

\[ \square \]

**Corollary 3.1** If \( \{M_n\}_{n=1}^{\infty} \in R\text{-Mod} \) is a convergent sequence, then for any pp-formula \( \phi \), \( \{D_{M_n}(\phi)\}_{n=1}^{\infty} \) converges.
So, one can identify the elements of the rank spectrum with certain dimension functions on $pp'(R)$. Note that if $\{M_n\}_{n=1}^{\infty}$ is a convergent sequence of modules, $\langle \phi, \psi \rangle$ is a pp-pair, for any $n \geq 1$, $M_n(\phi) \neq M_n(\psi)$ but

$$\lim_{n \to \infty} (D_{M_n}(\phi) - D_{M_n}(\psi)) = 0.$$ 

then if $\rho : R \to S$ represents the limit of $\{M_n\}_{n=1}^{\infty}$ in the rank spectrum, we have $S(\phi) = S(\psi)$. This phenomenon signifies an important difference between the topology of the Ziegler-spectrum and the rank spectrum.

4 The geometry of the rank spectrum

The goal of this section is to understand the geometry of the closure of $\text{Fin}(R)$ (the ranks associated to finitely generated modules) in the rank spectrum. Let $M \in R\text{-Mod}$. By the Krull-Schmidt theorem, $M$ can be written in a unique way as $M \cong \bigoplus_{i=1}^{s} Q^n_i$, where the $Q_i$’s are indecomposable modules. The weight of an indecomposable module $Q_i$ in $M$ is defined as

$$w_{Q_i}(M) := \frac{n_i \dim_K(Q_i)}{\dim_K(M)}.$$

That is,

$$M \cong \bigoplus_{Q \in \text{Ind}(R)} Q^{w_{Q}(M) \dim_K(M)}_{\dim_K(Q)} \bigoplus_{i=1}^{s} Q^n_i,$$

and $\sum_{Q \in \text{Ind}(R)} w_{Q}(M) = 1$.

**Proposition 4.1** If $M, N \in R\text{-Mod}$, then $rk_M = rk_N$ if and only if $w_{Q}(M) = w_{Q}(N)$ for all $Q \in \text{Ind}(R)$. In particular, $rk_Q \neq rk_P$, whenever $P \neq Q \in \text{Ind}(R)$.

**Proof** First, suppose that $M \cong \bigoplus_{i=1}^{s} Q^n_i$.

**Lemma 4.1** $rk_M = \sum_{i=1}^{s} w_{Q_i}(M) rk_{Q_i}$

**Proof** Let $A \in \text{Mat}_{m \times n}(R)$ be a matrix. Then, using the definition in the Introduction,

$$\text{rank}(\phi^M_{m,n}(A)) = \sum_{i=1}^{s} w_{Q_i}(M) \frac{\dim_K(M)}{\dim_K(Q_i)} \text{rank}(\phi^{Q_i}_{m,n}(A)).$$

That is,

$$rk_M(A) = \frac{\text{rank}(\phi^M_{m,n}(A))}{\dim_K(M)} = \sum_{i=1}^{s} w_{Q_i}(M) rk_{Q_i}(A).$$

So, if for all $Q \in \text{Ind}(R)$, $w_{Q}(M) = w_{Q}(N)$ holds, then $rk_M = rk_N$. 

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Now, suppose that for some $Q, w_Q(M) \neq w_Q(N)$. Let $\langle \phi, \psi \rangle$ be a pp-pair, that isolates $Q$, that is

- $Q(\phi) \neq Q(\psi)$
- $P(\phi) = P(\psi)$ if $P \cong Q, P \in \text{Ind}(R)$.

By Lemma 1.2.3. [23], $M(\phi) = \bigoplus_{Q \in \text{Ind}(R)} Q^{w_Q(M) \dim K(M)}(\phi)$. Hence,

$$D_M(\phi) - D_M(\psi) = w_Q(M)(D_Q(\phi) - D_Q(\psi))$$

(1)

That is,

$$D_M(\phi) - D_M(\psi) = w_Q(M)(D_Q(\phi) - D_Q(\psi)) \neq D_N(\phi) - D_N(\psi).$$

Applying Lemma 3.1 for finite dimensional matrix algebras over $R$, we can observe that if $D_M(\phi) \neq D_N(\phi)$ for some $\phi$, then $rk_M \neq rk_N$. This finishes the proof of our proposition.

Note that (1), has the following corollary.

**Corollary 4.1** If $\{M_n\}_{n=1}^\infty \subset R\text{-}\text{Mod}$ is a convergent sequence, then for all $Q \in \text{Ind}(R)$ the sequence $\{w_Q(M_n)\}_{n=1}^\infty$ is convergent as well.

Now, let $Q_1, Q_2, \ldots$ be an enumeration of the indecomposable modules in $R\text{-}\text{Mod}$. Let $(a_n)_{n=1}^\infty$ be non-negative real numbers, such that $\sum_{n=1}^\infty a_n = 1$. Then $a = \sum_{n=1}^\infty a_n rk_{Q_n}$ is a well-defined element of $\text{Rank}(R)$. We denote these set of elements by $\text{Conv}(R)$.

**Proposition 4.2** Let $(a_i = \sum_{n=1}^\infty a_n i_n r_k Q_n)_{i=1}^\infty$ be a convergent sequence such that $\lim_{i \to \infty} a_i = b = \sum_{n=1}^\infty b_n r_k Q_n \in \text{Conv}(R)$. Then for any $n \geq 1$, $\lim_{i \to \infty} a_{n}^{i_k} = b_n$.

**Proof** Let suppose that for some $n \geq 1$ and subsequence $(i_k)_{k=1}^\infty$

$$\lim_{k \to \infty} a_{n}^{i_k} = c_n \neq b_n.$$ 

Let $\langle \phi, \psi \rangle$ be a pp-pair that isolates $Q_n$. Then,

$$\lim_{k \to \infty} (D_{\phi}^{i_k}(\phi) - D_{\phi}^{i_k}(\psi)) = c_n(D_{Q_n}(\phi) - D_{Q_n}(\psi))$$

$$\neq b_n(D_{Q_n}(\phi) - D_{Q_n}(\psi)) = D_{\phi}(\phi) - D_{\phi}(\psi),$$

leading to a contradiction.

**Corollary 4.2** The limit points of $\text{Ind}(R)$ in the rank spectrum form a non-empty closed set that is disjoint from $\text{Conv}(R)$.
5 Sofic algebras

Let $R$ be a finite dimensional algebra over the finite field $K$ as in the previous sections and $\rho : R \to S$ be a homomorphism to a rank algebra. In this section we will get a sufficient condition under which $rk_\rho$ is an element of the closure of $\text{Fin}(R)$ (see Question 1). Let $A$ be a countable dimensional algebra over the finite field $K$ with basis $\{1 = e_1, e_2, \ldots\}$. Following Arzhantseva and Paunescu [1], we call $A$ a sofic algebra

- if there exists a non-negative function $j : A \to \mathbb{R}$ such that $j(a) > 0$ if $a \neq 0$, and a sequence of real numbers $k_1 > k_2 > \ldots$ tending to zero,
- we have a sequence of integers $m_1 < m_2 < \ldots$ such that for any $n \geq 1$, there exists a unital, linear map $\phi_n : A \to \text{Mat}_{m_n \times m_n}(K)$ so that

$$\frac{\text{Rank}(\phi_n(ab) - \phi_n(a)\phi_n(b))}{m_n} < k_n,$$

whenever $a, b \in \text{Span}(e_1, e_2, \ldots, e_n)$,
- for any $a \in A$ there exists $N > 0$ such that $\frac{\text{Rank}(\phi_n(a))}{m_n} > \frac{j(a)}{2}$, if $n \geq N$.

We call such a sequence $\{\phi_n\}_{n=1}^\infty$ a sofic representation sequence of $A$. It is not hard to see (Proposition 5.1 [15]) that $A$ is sofic if and only if there exists an injective, unital homomorphism $\phi : A \to \text{Mat}_\omega^M$, where $\text{Mat}_\omega^M$ is the continuous ultraproduct of the matrix algebras $M := \{\text{Mat}_{m_n \times m_n}(K)\}_{n=1}^\infty$. Recall from Sect. 2 that the rank $rk_M$ on $\text{Mat}_\omega^M$ is given by $rk_M([a_n]) = \lim_\omega rk_{m_n}(a_n)$, where $rk_{m_n}$ is the normalized rank function on the matrix algebra $\text{Mat}_{m_n \times m_n}(K)$. By the proof of Proposition 1.1, it follows that any element of the closure of $\text{Fin}(R)$ is associated to a unital homomorphism $\rho : R \to \text{Mat}_\omega^M$. Now we prove the converse.

**Theorem 3** Let $A$ be a sofic algebra over our finite base field $K$ with a rank $rk_A$ derived from the injective homomorphism $\phi : A \to \text{Mat}_\omega^M$. Let $R$ be a finite dimensional algebra over $K$, let $\rho : R \to A$ be a homomorphism and let $rk_\rho$ be the corresponding element of the rank spectrum $\text{Rank}(R)$. Then $rk_\rho$ is in the closure of $\text{Fin}(R)$.

**Proof** By Proposition 5.1 [15], we have a sequence of unital maps $\{\phi_n : A \to \text{Mat}_{m_n \times m_n}(K)\}_{n=1}^\infty$ such that for any $\epsilon > 0$ and for any $a, b \in A$

$$\{n \mid rk_{m_n}(\phi_n(ab) - \phi_n(a)\phi_n(b)) < \epsilon\} \in \omega$$

and for any $k, l \geq 1$ and $A \in \text{Mat}_{k,l}(A)$

$$\left\{ n \mid \frac{\dim_K(\phi_n^{k,l}(A))}{m_n} - rk_\rho(A) < \epsilon \right\} \in \omega,$$
where $\phi_{n}^{k,l}$ is the extension of $\phi_{n}$ onto $\text{Mat}_{k,l}(A)$. Therefore, by taking a subsequence we can assure that for any $a, b \in A$

$$\lim_{n \to \infty} rk_{m_n}(\phi_{n}(ab) - \phi_{n}(a)\phi_{n}(b)) = 0 \quad (2)$$

and for any $k, l \geq 1$ and $A \in \text{Mat}_{k,l}(A)$

$$\lim_{n \to \infty} \frac{\dim_{K}(\phi_{n}^{k,l}(A))}{m_n} = rk_{\rho}(A). \quad (3)$$

If the sofic representation sequence $\{\phi_{n}\}_{n=1}^{\infty}$ satisfies (3), then we call it a **convergent sofic representation sequence**. It is enough to construct for all $n \geq 1$ a subspace $V_{n} \subset K^{m_n}$ such that

- $\lim_{n \to \infty} \frac{\dim_{K}(V_{n})}{m_n} = 1$,
- for any $a \in R$, $\phi_{n}'(a)(V_{n}) \subset V_{n}$,
- for any $a, b \in R$, $\phi_{n}'(a)\phi_{n}'(b)(v) = \phi_{n}'(ab)(v)$ for all $v \in V_{n}$,

where $\phi_{n}' = \phi_{n} \circ \rho$. Indeed, for such sequence of subspaces $\{V_{n}\}_{n=1}^{\infty}$ the maps $\phi_{n}'$ define $R$-module structures on $V_{n}$ and for any matrix $A \in \text{Mat}_{k,l}(A)$,

$$\lim_{n \to \infty} \left( \frac{\dim_{K}(\phi_{n}^{k,l}(A))}{m_n} - \frac{\dim_{K}(\psi_{n}^{k,l}(A))}{m_n} \right) = 0,$$

where $\psi_{n}^{k,l}$ is the restriction of $\phi_{n}^{k,l}$ onto $V_{n}^{l}$. By (2), for any pair $a, b \in R$,

$$\lim_{n \to \infty} \frac{\dim_{K}(\text{Ker}(\phi_{n}'(ab) - \phi_{n}'(a)\phi_{n}'(b)))}{m_n} = 0.$$

Hence, for $V_n = \bigcap_{a, b \in R} \text{Ker}(\phi_{n}'(ab) - \phi_{n}'(a)\phi_{n}'(b))$, $\lim_{n \to \infty} \frac{\dim_{K}(V_{n})}{m_n} = 1$. We need to show that if $v \in V_{n}$, then $\phi_{n}'(c)(v) \in V_{n}$, that is for any $a, b \in R$

$$\phi_{n}'(ab)\phi_{n}'(c)(v) = \phi_{n}'(a)\phi_{n}'(b)\phi_{n}'(c)(v). \quad (4)$$

Since $\phi_{n}'(ab)\phi_{n}'(c)(v) = \phi_{n}'(abc)(v)$ and $\phi_{n}'(a)\phi_{n}'(b)\phi_{n}'(c)(v) = \phi_{n}'(a)\phi_{n}'(bc)(v) = \phi_{n}'(abc)(v)$, (4) follows.

Let $\rho : R \to \text{Mat}_{n \times n}(D)$ be an infinite dimensional representation, where $D$ is a skew field over the base field $K$. Clearly, there exists a countable dimensional subskew field $E \subset D$ such that $\text{Im}(\rho) \subset \text{Mat}_{n \times n}(E)$. Since $\text{Mat}_{n \times n}(E)$ is sofic if and only if $E$ is sofic and $\text{Mat}_{n \times n}(E)$ has a unique rank, by Theorem 3, we have that if $E$ is sofic, then $rk_{\rho}$ is in the closure of $\text{Fin}(R)$. It is an open question, whether there are non-sofic skew fields or not. In [15] it was proved that all the amenable skew fields (we discuss them in Sect. 7) and the free skew field (we will prove that it is a non-amenable skew field in Sect. 7) are sofic. Hence for these skew fields and homomorphisms $\rho : R \to \text{Mat}_{n \times n}(E)$, $rk_{\rho}$ is always in the closure of $\text{Fin}(R)$.
6 Amenable elements in the rank spectrum

As we have seen in the previous section, one can construct convergent sequences of finite dimensional representations of a finite dimensional algebra $R$ by mapping $R$ into a sofic algebra $A$. Now we investigate that using this construction, how can we obtain amenable elements (see Definition 1.1) in the rank spectrum of $R$. So, let $A$ be a countable dimensional algebra over the finite field $K$, with basis $\{1 = e_1, e_2, \ldots \}$ and let $\{\phi_n : A \to \text{Mat}_{m_n \times m_n}(K)\}_{n=1}^{\infty}$ be unital linear maps forming a convergent, sofic representation system.

**Definition 6.1** We call the convergent, sofic representation system $\{\phi_n\}_{n=1}^{\infty}$ hyperfinite if for any $\epsilon > 0$, there exists $L_\epsilon > 0$ such that for any $n \geq 1$, we have independent $K$-linear subspaces $N_n^1, N_n^2, \ldots, N_n^{k(n)} \subset K^{m_n}$ satisfying the following three conditions:

- For any $1 \leq i \leq k(n)$, $\dim_K(N_n^i) \leq L_\epsilon$.
- $\sum_{i=1}^{k(n)} \dim_K(N_n^i) \geq 1 - \epsilon$.
- The subspaces $\left( \bigvee_{a \in \text{Span}\{e_1, e_2, \ldots, e_s\}} \phi_n(a)(N_n^i) \right)_{i=1}^{k(n)}$ are independent and for any $1 \leq i \leq k(n)$,
  \[
  \frac{\dim_K \left( \bigvee_{a \in \text{Span}\{e_1, e_2, \ldots, e_s\}} \phi_n(a)(N_n^i) \right)}{\dim_K(N_n^i)} \leq 1 + \epsilon, \quad \text{where } s \text{ is the integer part of } 1/\epsilon. 
  \]

So, as opposed to the case of finite dimensional algebras, the pieces $N_n^i$ are not exactly $A$-modules, only in an approximate sense. First, we prove a technical lemma that will make the proof of some of our results easier. It states that part of the third condition is not necessary to check in order to prove that a certain sequence is hyperfinite.

**Lemma 6.1** Let $\{\phi_n : A \to \text{Mat}_{m_n \times m_n}(K)\}_{n=1}^{\infty}$ be a convergent sofic representation sequence. Suppose that for any $\epsilon > 0$, there exists $L_\epsilon > 0$ such that for any $n \geq 1$, we have independent $K$-linear subspaces $Z_n^1, Z_n^2, \ldots, Z_n^{k(n)} \subset K^{m_n}$ satisfying the following three conditions:

- For any $1 \leq i \leq k(n)$, $\dim_K(Z_n^i) \leq L_\epsilon$.
- $\sum_{i=1}^{k(n)} \dim_K(Z_n^i) \geq 1 - \epsilon$.
- For any $1 \leq i \leq k(n)$,
  \[
  \frac{\dim_K \left( \bigvee_{a \in \text{Span}\{e_1, e_2, \ldots, e_s\}} \phi_n(a)(Z_n^i) \right)}{\dim_K(Z_n^i)} \leq 1 + \epsilon, \quad \text{where } s \text{ is the integer part of } 1/\epsilon. 
  \]

Then the family $\{\phi_n\}_{n=1}^{\infty}$ is hyperfinite.

**Proof** We start with a simple linear algebra lemma. \qed

**Lemma 6.2** Let $V \subset Z$ be finite dimensional $K$-spaces and $T, S \subset \text{End}_K(Z)$. Suppose that $\dim_K(T(V) + V) \leq (1 + \epsilon_T) \dim_K(V)$, $\dim_K(S(V) + V) \leq (1 + \epsilon_S) \dim_K(V)$, then

- There exists a linear subspace $W \subset V$ such that $T(W) \subset V$ and $\dim_K(W) \geq (1 - \epsilon_T) \dim_K(V)$.
- $\dim_K(TS(V) + V) \leq (1 + \epsilon_T + \epsilon_S) \dim_K(V)$.

\( \text{Springer} \)
Proof Let \( m_T : V \to \frac{T(V) + V}{V} \) be the natural quotient map. Since \( \dim_K(\text{Im}(m_T)) \leq \epsilon_T \dim_K(V) \), we have that

\[
\dim_K(\text{Ker}(m_T)) \geq (1 - \epsilon_T) \dim_K(V).
\]

On the other hand, \( T(\text{Ker}(m_T)) \subset V \). For the second part, let \( S(V) + V = Q \oplus V \) for some complementing space \( Q \), then \( TS(V) + V \subset T(Q) + T(V) + V \), hence

\[
\dim_K(TS(V) + V) \leq (1 + \epsilon_T + \epsilon_S) \dim_K(V)
\]

\( \square \)

**Lemma 6.3** For any \( n \geq 1 \) and \( 1 \leq i \leq k(n) \), there exists a subspace \( Q^i_n \subset Z^i_n \) so that

- \( \dim_K(Q^i_n) \geq (1 - \sqrt{\epsilon}) \dim_K(Z^i_n) \).
- For any \( a \in \text{Span}\{e_1, e_2, \ldots, e_{s(\epsilon)}\} \)

\[\phi_n(a)(Q^i_n) \subset Z^i_n,\]

where \( s(\epsilon) \) is given in such a way that \( |\text{Span}\{e_1, e_2, \ldots, e_{s(\epsilon)}\}| \leq \frac{1}{\sqrt{\epsilon}} \).

**Proof** For \( a \in \text{Span}\{e_1, e_2, \ldots, e_{s(\epsilon)}\} \), let \( W_a \subset Z^i_n \) (Lemma 6.2) such that

- \( \dim_K(W_a) \geq (1 - \epsilon) \dim_K(Z^i_n) \).
- \( \phi_n(a)(W_a) \subset Z^i_n \).

Let

\[Q^i_n = \bigcap_{a \in \text{Span}\{e_1, e_2, \ldots, e_{s(\epsilon)}\}} W_a.\]

Then,

\[
\dim_K(Q^i_n) \geq (1 - \sqrt{\epsilon}) \dim_K(Z^i_n)
\]

and for any \( a \in \text{Span}\{e_1, e_2, \ldots, e_{s(\epsilon)}\} \)

\[\phi_n(a)(Q^i_n) \subset Z^i_n \quad \square \]

The existence of the systems \( \{Q^i_n\}_{i=1}^{k(n)} \) clearly implies the statement in Lemma 6.1. \( \square \)

If \( \{\phi_n : A \to \text{Mat}_{m_n \times m_n}(K)\}_{n=1}^{\infty} \) is a hyperfinite sofic representation system, then the associated sofic representation system

\[\{\hat{\phi}_n : \text{Mat}_{l \times l}(A) \to \text{Mat}_{m_n \times m_n}(K)\}\]
is still hyperfinite. Indeed, the spaces $M^i_n := (N^i_n)^l \subset K^{mnl}$ will be the independent subspaces satisfying the approximate module condition. Now, let $\rho : R \rightarrow \text{Mat}_{l \times l}(A)$ be a unital homomorphism from a finite dimensional algebra $R$. By the comments in the proof of Theorem 3, we have a sequence of subspaces $V_n \subset K^{mnl}$ so that

- $\hat{\phi}_n \circ \rho(V_n) \subset V_n$ (that is $V_n$ is an $R$-module).
- $\lim_{n \to \infty} \frac{\dim_K(V_n)}{mnl} = 1$,

representing the element $rk_{\rho}$ in the rank spectrum.

**Proposition 6.1** The sequence of $R$-modules $\{V_n\}_{n=1}^\infty$ constructed above is hyperfinite, that is, $rk_{\rho}$ is an amenable element of the rank spectrum.

**Proof** The first step is to show that the sequence $\{V_n\}_{n=1}^\infty$ is hyperfinite in the weaker sense, described in Lemma 6.1. As opposed to the graph theoretical case, it is not a priori obvious. The reason is that although the subspaces $\{V_n\}_{n=1}^\infty$ are almost as big as the total spaces $K^{mnl}$ it is not true that they contain all (or even one single copy) of the pieces $M^i_n$. The following lemma resolves this problem. First, fix a basis $\{f_i\}_{i=1}^\infty$ for the algebra $\text{Mat}_{l \times l}(A)$. By taking a subsequence, we can suppose that the subspaces $\bigvee_{a \in \text{Span}\{f_1, f_2, \ldots, f_s\}} \hat{\phi}_n(a)(M^i_n)$ are independent, where $s$ is the integer part of $1/\epsilon$

and for any $1 \leq i \leq k(n)$,

$$\frac{\dim_K \left( \bigvee_{a \in \text{Span}\{f_1, f_2, \ldots, f_s\}} \hat{\phi}_n(a)(M^i_n) \right)}{\dim_K (M^i_n)} \leq 1 + \epsilon.$$

**Lemma 6.4** For any $\epsilon > 0$ and large enough $n \geq 1$, we have independent subspaces $Q^1_n, Q^2_n, \ldots, Q^{l(n)}_n$ in $V_n$ such that

- For any $1 \leq i \leq l(n)$, $\dim_K(Q^i_n) \leq L_\epsilon$.
- $\sum_{i=1}^{l(n)} \dim_K(Q^i_n) \geq 1 - 2\epsilon$.
- For any $1 \leq i \leq l(n)$, $\frac{\dim_K \left( \bigvee_{a \in \text{Span}\{f_1, f_2, \ldots, f_s\}} \hat{\phi}_n(a)(Q^i_n) \right)}{\dim_K (Q^i_n)} \leq 1 + \epsilon$, where $s$ is the integer part of $1/\epsilon$.

**Proof** We call the subspaces $M^i_n$ and $M^j_n$ equivalent if there exists a linear bijection $\alpha_{ij} : K^{mnl} \rightarrow K^{mnl}$ so that

- $\alpha_{ij}$ bijectively maps $M^i_n$ into $M^j_n$.
- For any $a \in \text{Span}\{f_1, f_2, \ldots, f_s\}$ and $v \in M^i_n$

$$\alpha_{ij}(\hat{\phi}_n(a)(v)) = \hat{\phi}_n(a)(\alpha_{ij}(v)).$$

Notice that by our finiteness conditions, there exists a constant $C \geq 0$ such that for any $n \geq 1$, the number of equivalence classes is at most $C$. Now, let $\mu = \{M^i_n, M^2_n, \ldots, M^{l(n)}_n\}$ be an equivalence class and let $\{\alpha_j : K^{mnl} \rightarrow K^{mnl}\}_{j=1}^{\mu(l)}$ be the
corresponding bijections mapping $M_n^{i_1}$ into $M_n^{i_j}$. For $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_{\mu(t)}\} \in K^{\mu(t)}$ define

$$M_n^{\hat{\lambda}} := \left( \sum_{j=1}^{\mu(t)} \lambda_j \alpha_j \right) (M_n^{i_1}).$$

Then

$$\dim_K \left( \bigvee_{a \in \text{Span}(f_1, f_2, \ldots, f_s)} \phi_n(a) (M_n^{\hat{\lambda}}) \right) \leq (1 + \epsilon) \dim_K (M_n^{\hat{\lambda}}).$$

Also, if $\hat{\lambda}_1, \hat{\lambda}_2, \ldots, \hat{\lambda}_q$ are independent vectors in $K^{\mu(t)}$, then the corresponding subspaces $M_n^{\hat{\lambda}_1}, M_n^{\hat{\lambda}_2}, \ldots, M_n^{\hat{\lambda}_q}$ are independent as well. We call an equivalence class $\mu$ large if

$$\mu(t) \dim_K (M_n^{\mu}) \geq \left( \frac{1 - \epsilon}{100C} \right) m_n l$$

holds, where $M_n^{\mu}$ is representing the class $\mu$. It is not hard to check that

$$\dim_K \left( \bigoplus_{\mu \text{ is large}} \bigoplus_{i=1}^{\mu(t)} M_n^{i, \mu} \right) \geq \left( 1 - \frac{3}{2} \epsilon \right) m_n l,$$

(5)

holds, where $\{M_n^{1, \mu}, M_n^{2, \mu}, \ldots, M_n^{\mu(t), \mu}\}$ are the subspaces in the class $\mu$. Now let $\mu_n$ be a large class for some $n \geq 1$ and $M_n^{i_1}$ be its representative. For $v \in M_n^{i_1}$, let $H_n^{\mu_n} \subset K^{\mu_n(t)}$ be the set of vectors $\hat{\lambda}$ such that $\sum_{j=1}^{\mu_n(t)} \lambda_j \alpha_j (v) \in V_n$. Since

$$\lim_{n \to \infty} \frac{\dim_K (V_n)}{m_n l} = 1,$$

we get

$$\lim_{n \to \infty} \frac{\dim_K (\cap_v H_n^{\mu_n})}{\mu_n(t)} = 1.$$

(6)

For each large class $\mu_n$, when $n$ is large enough we choose a basis $\hat{\lambda}_1, \hat{\lambda}_2, \ldots, \hat{\lambda}_p$ in $\cap_v H_n^{\mu_n}$ and set $Q_{n, \mu_n} := M_n^{\hat{\lambda}_i}$. Then, by (5) and (6) the subspaces $Q_{n, \mu_n}$ will satisfy the conditions of our proposition, provided that $n$ is large enough. This finishes the proof of our lemma.

So, we have a convergent sequence of representations

$$\{\psi_n : R \to \text{End}_K (V_n)\}_{n=1}^\infty$$

that are hyperfinite in the weaker sense, that is for any $\epsilon > 0$ we have $L_\epsilon > 0$ and for large enough $n$ independent subspaces $Q_n^1, Q_n^2, \ldots, Q_n^{(n)} \subset V_n$ such that

\[ \text{Springer} \]
For any $1 \leq i \leq l(n)$, $
abla^n_i \leq L\epsilon$. 

$\sum_{i=1}^{l(n)} \dim_K (\psi_n(r)(Q_n^n)) \geq (1 - 2\epsilon) \dim_K (V_n)$. 

For any $1 \leq i \leq l(n)$, 

$$\dim_K \left( \bigvee_{r \in R} \psi_n(r)(Q_n^n) \right) \leq (1 + \epsilon) \dim_K (\psi_n(r)(Q_n^n)).$$

Note that we used the fact that if $s$ is large then $\text{Span}\{f_1, f_2, \ldots, f_s\}$ contains $\rho(\text{R})$.

Our proposition immediately follows from the following lemma.

**Lemma 6.5** Let $\tau : R \rightarrow \text{End}_K (V)$ be a unital representation and $L \subset V$ be a subspace such that 

$$\dim_K \left( \bigvee_{r \in R} \tau(r)(L) \right) \leq (1 + \epsilon) \dim_K (L).$$

Then there exists a subspace $L' \subset L$, $\dim_K (L') \geq 1 - |R|\epsilon$ such that $L'$ is an $R$-module (that is for any $r \in R$, $\tau(r)L \subset L'$).

**Proof** By Lemma 6.2, for any $r \in R$, there exists a subspace $L_r \subset L$ such that 

$$\dim_K (L_r) \geq (1 - \epsilon) \dim_K (L)$$

and $\tau(r)L_r \subset L$. Then $\dim_K (\cap_{r \in R} L_r) \geq (1 - |R|\epsilon) \dim_K (L)$ and for any $r \in R$, $\tau(r)(\cap_{\tau \in R} L_r) \subset L$. Hence the subspace $L' = \bigvee_{r \in R} (\tau(r)\cap_{r \in R} L_r)$ satisfies the condition of our lemma. \qed

**Proposition 6.2** Let $K[X]$ be the polynomial algebra over an arbitrary field $K$. Then the family of all the finite dimensional representations $K[X]-\text{Mod}$ is hyperfinite.

**Corollary 6.1** If $K$ is a finite field, $R$ is finite dimensional algebra over $K$ and $\phi : R \rightarrow \text{Mat}_{l \times l}(K[X])$ is a homomorphism, then $\phi^*(K[X]-\text{Mod})$ is a hyperfinite family in $R$-Mod.

**Proof** (of Proposition 6.2) It is enough to show that the class of indecomposable finite dimensional $K[X]$-modules is hyperfinite. By Jordan’s theorem indecomposable elements in $K[X]-\text{Mod}$ are in the form of $K[X]/f^n K[X]$, where $n \geq 1$ and $f \in K[X]$ is an irreducible, monic polynomial. The module structure of $M_f^n = K[X]/f^n K[X]$ is given the following way,

- $M_f^n = \text{Span}\{1, x, x^2, \ldots, x^{\deg(f) n - 1}\}$
- $X \cdot t^i = t^{i+1}$, if $1 \leq i \leq \deg(f) n - 2$
- $X \cdot t^{\deg(f) n - 1} = t^{\deg(f) n - f^n(t)}.$

Let $\epsilon > 0$ and $k \geq 1/\epsilon$. We will show that for any module $M_f^n$, we have independent subspaces $N_{f^n}^1, N_{f^n}^2, \ldots, N_{f^n}^k \subset M_f^n$ such that
• For any $1 \leq i \leq k(f^n)$, $\dim_K(N_{f^n}^i) \leq 2k^2$
• For any $1 \leq i \leq k(f^n)$,

$$\frac{\dim_K(X(N_{f^n}^i) + N_{f^n}^i)}{\dim_K(N_{f^n}^i)} \leq 1 + \epsilon$$

• $\sum_{i=1}^{k(f^n)} \dim_K(N_{f^n}^i) \geq (1 - \epsilon) \dim_K(M_{f^n})$.

If $\deg(f)n \leq 2k^2$, let $k(f^n) = 1$ and $N_{f^n}^1 = M_{f^n}$. If $\deg(f)n > 2k^2$, let $C$ be the integer such that

$$Ck - 1 < \deg(f)n \leq (C + 1)k - 1$$

and for $1 \leq i \leq C + 1$ let

$$N_{f^n}^i := \text{Span}\{t^{(i-1)k}, t^{(i-1)k} + 1, \ldots, t^{ik-1}\}.$$

Then $\dim_K(X(N_{f^n}^i) + N_{f^n}^i) = \dim_K(N_{f^n}^i) + 1$ and all the three inequalities above are satisfied.

### 7 Amenable and non-amenable skew fields

In this section we recall some basic definitions and results about amenable and non-amenable skew fields. We will confirm Conjecture 3 for amenable algebras $S = \text{Mat}_{l \times l}(D)$, when $D$ is an amenable skew field. We also prove the non-amenable-ness of the free skew field $D_K^r$. This result is hopefully interesting on its own right, in any case, it will be important for us in the proof of Theorem 6.

Let $\mathcal{A}$ be a countable dimensional algebra over the base field $K$. We say that the algebra is (left) amenable [12,18] if there exists a sequence of finite dimensional nonzero $K$-linear subspaces $\{W_n\}_{n=1}^{\infty} \subset \mathcal{A}$ such that for each $a \in \mathcal{A}$

$$\lim_{n \to \infty} \frac{\dim_K(W_n + aW_n)}{\dim_K(W_n)} = 1.$$ 

Let us recall some basic facts on amenable algebras.

• If $\mathcal{A}$ is an amenable domain, then it has the Ore property and its classical skew field of quotients is amenable as well (Proposition 2.1, [12]).
• If $\mathcal{A} = K \Gamma$ is a group algebra, then $\mathcal{A}$ is amenable if and only if $\Gamma$ is an amenable group [2,12].
• If $E \subset D$ are skew fields over the base field $K$, and $E$ is non-amenable, then so is $D$ (Theorem 1. [12]).
• If $D$ and $E$ are amenable skew fields and $D \otimes_K E$ is a domain, then $D \otimes_K E$ is amenable (Proposition 2.4 [12]).
Now let $K F_r$ be the free algebra $K \langle x_1, x_2, \ldots, x_t \rangle$ on $r$ indeterminates. Let $K F_r$-Mod be the set of all finite dimensional modules over $K F_r$. So, any element $M \in K F_r$-Mod defines a Sylvester rank function on $K F_r$ and the convergence of such representations is well-defined (see [15]) exactly the same way as for finite dimensional algebras. Note that a convergent sequence of representations can always be viewed as a sofic approximation of the algebra $K F_r/I$, where $I$ is the ideal of elements $a \in K F_r$ for which $\lim_{n \to \infty} rk_{M_n}(a) = 0$.

Let $M = \{ M_n \}_{n=1}^{\infty} \subset K F_r$-Mod be a convergent sequence of modules and $\phi_M : K F_r \to \text{Mat}_\omega$ be the ultraproduct representation as in the previous sections (we suppose that $K$ is a finite field). Recall that if $R \subset S$ are rings, then $R$ is division-closed in $S$ if for any element $a \in R$ for which $a^{-1} \in S$ exists, $a^{-1} \in R$. The division closure of a subring $L$ in a ring $S$ is the smallest division-closed ring $D(L) \subset S$ containing $L$. By Theorem 2. [15], the division closure of the image of $K F_r$ in $\text{Mat}_\omega$ is a skew field if and only if the rank function $rk_\omega$ is integer-valued.

Furthermore, the skew field $D$ is an amenable if and only if the convergent sequence $M$ is hyperfinite (Theorem 3 [15]). According to Proposition 7.1 [15] the free skew field $D^*_r$ on $r$ generators over the base field $K$ is sofic (for the free skew field see e.g. [8].) That is, there exists a convergent sequence $\{ M_n \}_{n=1}^{\infty} \subset K F_r$-Mod , for which the division closure of $K F_r$ in $\text{Mat}_\omega$ is the free skew field. In [12] using operator algebraic methods we proved that the free field over the complex numbers is non-amenable. Now we prove the following theorem.

**Theorem 4** The free skew field on $r > 1$ generators over an arbitrary base field $K$ is non-amenable.

**Proof** First, observe that for any base field $K$, the free algebra $K F_r$ is isomorphic to its opposite algebra $K F_r^{op}$ (using the homomorphism that maps $x_{n_1}x_{n_2}\ldots x_{n_l}$ to $x_{n_l}x_{n_{l-1}}\ldots x_{n_1}$), hence the free skew field $D = D^*_K$ is isomorphic to its opposite algebra $D^{op}$. Recall that $D \otimes_K D$ is a domain (Theorem 3.1, [10]). Hence, it is enough to prove that there exists an embedding of $D \otimes D^{op}$-modules

\[
\pi : (D \otimes D^{op})^2 \to D \otimes D^{op}. \tag{7}
\]

Indeed, if $D$ is amenable, then by the properties listed above $D \otimes D^{op}$ is an amenable domain and therefore $D \otimes D^{op}$ has the Ore property. Thus two non-trivial left ideals of $D \otimes D^{op}$ have non-zero intersection, in contradiction with the existence of the embedding (7). In order to construct the embedding, we follow the lines of Lemma 4.21 in [5] (I am indebted to Andrey Lazarev for calling my attention to this paper). First, consider the two-terms free resolution of $K F_r$ as a bimodule over itself

\[
0 \to K F_r \otimes_K [x_1, x_2] \otimes_K K F_r \to K F_r \otimes_K K F_r \to K F_r \to 0
\]
as in [5]. Here $[x_1, x_2]$ denotes the 2-dimensional vector space spanned by $x_1, x_2$. If we tensor the resolution above by $D$ on the right, we obtain an exact sequence of $K F_r - D$-bimodules

\[
0 \to K F_r \otimes_K [x_1, x_2] \otimes_K D \to K F_r \otimes_K D \to D \to 0,
\]
since \( \text{Tor}^1_{K^r} (K F_r, D) = 0 \) by the definition of the functor \( \text{Tor} \). Now let us tensor this exact sequence by \( D \) on the left. Notice that \( \text{Tor}^1_{K^r} (D, D) = 0 \) ([25] Theorems 4.7, 4.8). Hence, we obtain an embedding of \( D - D \)-bimodules

\[
i : D \otimes_K [x_1, x_2] \otimes_K D \to D \otimes D.
\]

That is we have the embedding of left \( D \otimes D^{op} \)-modules

\[
j : (D \otimes D^{op})^2 \to D \otimes D^{op}
\]

we sought for. This finishes the proof of our theorem.

Theorem 5 Let \( \rho : R \to \text{Mat}_{l \times l}(D) \) be an infinite dimensional representation of a finite dimensional algebra \( R \) over the finite field \( K \), where \( D \) is a countable dimensional amenable skew field. Then the associated element \( r k_\rho \in \text{Rank}(R) \) is amenable.

Proof Let \( \{d_1, d_2, \ldots, d_r\} \) be the set of all entries in the matrices \( \{\rho(s)\}_{s \in R} \). Let \( \pi : K F_r \to D \) be the unital homomorphism mapping \( x_i \) to \( d_i \). Let \( E \subset D \) be the sub skew field generated by \( \{d_1, d_2, \ldots, d_r\} \) (that is the division closure of \( \text{Im} (\pi) \)). From now on, we can suppose that \( \rho \) maps \( R \) into \( \text{Mat}_{l \times l}(E) \) and \( \pi \) maps \( K F_r \) into \( E \). Let \( S := K F_r / \text{Ker} (\pi) \), and \( \theta : K F_r \to S \) and \( \hat{\pi} : S \to E \) be the natural quotient maps, so \( \pi = \hat{\pi} \circ \theta \). Then, the homomorphism \( \zeta : R \to \text{Mat}_{l \times l}(S) \) can be defined in a unique way to satisfy \( \hat{\pi} \circ \zeta = \rho \) (note that slightly abusing the notation, \( \hat{\pi} \) is viewed here as a homomorphism from \( \text{Mat}_{l \times l}(S) \) to \( \text{Mat}_{l \times l}(E) \)). Finally, pick a lifting \( \mu : S \to K F_r \). that is \( \mu \) is an injective, unital, linear map such that \( \theta \circ \mu = \text{Id}_S \). Then, \( \pi \circ \mu = \hat{\pi} \).

Since \( E \) is amenable, it is sofic. Hence, we have a sofic representation sequence \( \{\phi_n : E \to \text{Mat}_{m_n \times m_n}(K)\}_{n=1}^\infty \). Therefore, we have the sofic representation sequence

\[
\{\phi_n \circ \hat{\pi} : S \to \text{Mat}_{m_n \times m_n}(K)\}_{n=1}^\infty
\]

that factors through the sofic representation system

\[
\{\phi_n \circ \pi : K F_r \to \text{Mat}_{m_n \times m_n}(K)\}_{n=1}^\infty.
\]

By Proposition 11.1 [15], \( \{\phi_n \circ \pi\}_{n=1}^\infty \) is hyperfinite hence \( \{\phi_n \circ \hat{\pi}\}_{n=1}^\infty \) is a hyperfinite sofic representation. Therefore by Proposition 6.1, \( rk_\zeta = rk_\rho \) is an amenable element of the rank spectrum.

\[\square\]

8 The wild Kronecker algebras are of non-amenable representation types

First recall the notion of Kronecker quiver algebras and the wildness phenomenon (see e.g. [3]). Let \( r \geq 3 \) and \( Q_r \) be the finite dimensional algebra with basis \( p_1, p_2, \{e_i\}_{i=1}^r \), where

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\[ p_1^2 = p_1, \quad p_2^2 = p_2, \quad p_1 + p_2 = 1, \quad p_1 p_2 = p_2 p_1 = 0. \]

For any \( 1 \leq i, j \leq r \), \( e_i p_1 = e_i \), \( p_1 e_i = 0 \), \( e_i p_2 = 0 \), \( p_2 e_i = e_i \), \( e_i e_j = 0 \).

The algebra \( Q_r \) defined the way above is the Kronecker quiver algebra of index \( r \). The wildness of \( Q_r \) means that the classification of finitely generated indecomposable modules over \( Q_r \) is as complicated as the classification of finitely generated indecomposable modules over noncommutative free algebras. To make it precise, there exists a representation \( \pi_r : Q_r \to \text{Mat}_{2 \times 2}(K F_{r-1}) \) (where \( K \) is a finite field) such that

- The associated functor \( \pi^*_r : K F_{r-1}\text{-Mod} \to Q_r\text{-Mod} \) is injective.
- If \( M \in K F_{r-1}\text{-Mod} \) is indecomposable, then \( \pi^*_r(M) \) is indecomposable as well.

The following representation satisfies the conditions above: \( \pi_r(p_1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \pi_r(p_2) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \pi_r(e_1) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \pi_r(e_j) = \begin{bmatrix} 0 & 0 \\ x_{j-1} & 0 \end{bmatrix} \) if \( j > 1 \).

**Proposition 8.1** Let \( \pi_r : Q_r \to \text{Mat}_{2 \times 2}(K F_{r-1}) \) be the representation as above. Then the family of modules \( \{\pi^*_r(M_n)\}_{n=1}^\infty \subset Q_r\text{-Mod} \) is hyperfinite if and only if \( \{M_n\}_{n=1}^\infty \subset K F_{r-1}\text{-Mod} \) is hyperfinite.

**Proof** The “if” part follows from Proposition 6.1. For the converse, let us suppose that \( \{\pi^*_r(M_n)\}_{n=1}^\infty \) is a hyperfinite family. That is, for any \( \epsilon > 0 \), we have \( L_\epsilon > 0 \) and for any \( n \geq 1 \) we have independent \( Q_r\text{-submodules} N_1^1, N_2^2, \ldots, N_k^{k(n)} \subset \pi^*_r(M_n) \) such that

- \( \dim_K(N_i^j) \leq L_\epsilon \)
- \( \sum_{i=1}^{k(n)} \dim_K(N_i^j) \geq (1 - \epsilon)2 \dim_K(M_n) \).

Note that as a vector space \( \pi^*_r(M_n) \) is isomorphic to \( M_n \oplus M_n \) and \( p_i \) acts on \( \pi^*_r(M_n) \) as the projection onto the \( i \)-th coordinate.

Let us consider the subspaces \( p_1 N_1^i \subset M_n \oplus 0 \), \( p_2 N_2^i \subset 0 \oplus M_n \). We use the notation \([p_1 N_1^i]\) resp. \([p_2 N_2^i]\) for the subspaces in \( M_n \) that are the projections of the spaces above onto the first resp. second component of \( M_n \oplus M_n \). Then, for any \( n \geq 1, 1 \leq i \leq k(n) \)

- \([p_1 N_1^i]\) \( \subset \) \([p_2 N_2^i]\)
- \([x_j p_1 N_1^i]\) \( \subset \) \([p_2 N_2^i]\) if \( 1 \leq j \leq r - 1 \).

Also, the subspaces \( \{[p_1 N_1^i]\}_{i=1}^{k(n)} \) are independent. Since

\[
\sum_{i=1}^{k(n)} (\dim_K[p_1 N_1^i] + \dim_K[p_2 N_2^i]) \geq (1 - \epsilon)2 \dim_K(M_n),
\]

we have the inequality

\[
\sum_{i=1}^{k(n)} \dim_K[p_1 N_1^i] \geq (1 - 2\epsilon) \dim_K(M_n).
\]
Lemma 8.1 Let $C > 0, \{a_i\}_{i=1}^{k(n)}, \{b_i\}_{i=1}^{k(n)}$, be positive real numbers satisfying the following inequalities.

- $\sum_{i=1}^{k(n)} a_i \geq (1 - 2\epsilon)C$
- $\sum_{i=1}^{k(n)} b_i \leq C$
- $a_i \leq b_i$.

Let $S = \{i \mid \frac{b_i}{a_i} < 1 + \sqrt{\epsilon}\}$. Then $\sum_{i \notin S} a_i \leq 2\sqrt{\epsilon}C$.

Proof We have

$$C \geq \sum_{i=1}^{k(n)} b_i \geq \sum_{i \notin S} (1 + \sqrt{\epsilon})a_i + \sum_{i \in S} a_i \geq (1 - 2\epsilon)C + \left(\sum_{i \notin S} a_i\right)\sqrt{\epsilon}.$$ 

Therefore $\sum_{i \neq S} a_i \leq 2\sqrt{\epsilon}C$. \qed

Let us apply the lemma above, for $C = \dim K(M_n), a_i = [p_1 N^n_i], b_i = [p_2 N^n_i]$. We get that

- $\sum_{i \notin S} \dim_K ([p_1 N^n_i]) \leq 2\sqrt{\epsilon} \dim_K (M_n)$.
- If $i \in S$, then $\dim_K \left([p_1 N^n_i] \oplus \bigoplus_{j=1}^{r-1} x_j [p_1 N^n_i]\right) \leq (1 + \sqrt{\epsilon}) \dim_K ([p_1 N^n_i])$.
- $\dim_K ([p_1 N^n_i]) \leq L_\epsilon$.

Hence by definition, $\{M_n\}_{n=1}^\infty$ is a hyperfinite family of $KF_{r-1}$-modules. \qed

Theorem 6 The wild Kronecker quiver algebras are of non-amenable representation type.

Proof Let $r \geq 2$ and $\mathcal{M} = \{M_n\}_{n=1}^\infty \subset KF_r$-Mod be a convergent sequence of $KF_r$-modules such that for all $KF_r$-matrices $A$

$$\lim_{n \to \infty} rk_{M_n}(A) \in \mathbb{Z}.$$ 

Let $\phi_\omega : KF_r \to \text{Mat}_\omega^\mathcal{M}$ be the limit representation of the sequence $\{M_n\}_{n=1}^\infty$. Then by Proposition 8.1 [15], the division closure of $\phi_\omega(KF_r)$ in the ring $\text{Mat}_\omega^\mathcal{M}$ is a skew field. We also know that

- The division closure above is an amenable skew field if and only if $\{M_n\}_{n=1}^\infty$ is hyperfinite (Proposition 11.1 and 12.1 [15]).
- There exists a convergent sequence of modules $\{M_n\}_{n=1}^\infty$ such that the division closure is the free skew field on $r$-generators with base field $K$ (Proposition 7.1 [15]).

Hence by Proposition 8.1 and Theorem 4, there exist non-hyperfinite families of modules for the algebra $Q_r, r \geq 3$. \qed
9 Hyperfinite families and the continuous algebra of von Neumann

The goal of this section is to prove Theorem 2. In fact, we will prove a stronger interpolation theorem. Let $K$ be a finite field, $R$ be a finite dimensional algebra over $K$ and let $\mathcal{M} = \{M_n\}_{n=1}^{\infty} \subset R\text{-}\text{Mod}$ be a hyperfinite convergent sequence of modules and $\omega$ be a non-principal ultrafilter.

**Theorem 7** Let $\rho_\omega : R \to \text{Mat}_\omega^\mathcal{M}$ be the ultraproduct representation corresponding to the sequence of modules above. Then there exists a subalgebra $\text{Im}(\rho_\omega) \subset S \subset \text{Mat}_\omega^\mathcal{M}$ such that $S \cong \mathcal{M}_K$ (where $\mathcal{M}_K$ is the algebra defined in the Introduction).

According to the theorem, when we pick a countable dimensional subalgebra from the ultraproduct, we can always land inside a copy of $\mathcal{M}_K$.

**Proof** The proof will be given in a series of lemmas and propositions.

**Lemma 9.1** Let $a_1, a_2, \ldots, a_r$ be positive integers, $\epsilon > 0$ and $U \in \omega$. Let us have an integer $m_i$ for each $i \in U$ (we suppose that $m_i$ tends to infinity as $i$ tends to infinity). Suppose that for any $i \in U$ and $1 \leq j \leq r$ we have a non-negative integer $c_{ij}$ such that

$$(1 - \epsilon)m_i \leq \sum_{j=1}^{r} c_{ij} a_j \leq m_i.$$

Then we have a subset $V \subset U$, $V \in \omega$, and non-negative integers $\{p_j\}_{j=1}^{r}, \{l_k\}_{k=1}^{\infty}$ such that for any $i \in V$

$$(1 - 2\epsilon)m_i \leq l_i \left(\sum_{j=1}^{r} p_j a_j\right) \leq m_i.$$

**Proof** For large enough $i \in U$, let

$$\frac{\epsilon m_i}{2r(\max_{1 \leq j \leq r} a_j)} < l_i < \frac{\epsilon m_i}{r(\max_{1 \leq j \leq r} a_j)}$$

be an integer. Then, for any $1 \leq j \leq r$, we have $c_{ij} = l_i d_{ij} + g_{ij}$, where $0 \leq g_{ij} < l_i$ and

$$0 \leq d_{ij} \leq \frac{2r \max_{1 \leq j \leq r} a_j}{\epsilon}.$$ 

Since

$$\sum_{j=1}^{r} l_i a_j \leq \frac{\epsilon m_i}{r(\max_{1 \leq j \leq r} a_j)} \sum_{j=1}^{r} a_j \leq \epsilon m_i,$$
we have that $\sum_{j=1}^{r} g_{ij}a_j \leq \epsilon m_i$ that is

$$(1 - 2\epsilon)m_i \leq l_i \sum_{j=1}^{r} d_{ij}a_j \leq m_i.$$  

Since the set $\{d_{ij}\}_{i \in U, 1 \leq j \leq r}$ is bounded, there are non-negative integers $p_1, p_2, \ldots, p_r$ such that

$$V = \{i \mid d_{ij} = p_j, \text{ for any } 1 \leq j \leq r\} \in \omega.$$  

Therefore if $i \in V$, we have

$$(1 - 2\epsilon)m_i \leq l_i \left(\sum_{j=1}^{r} p_ja_j\right) \leq m_i.$$  

\[\square\]

**Definition 9.1** The module $A \in R\text{-Mod} \epsilon$-tiles the module $B \in R\text{-Mod}$, if there exists $k \geq 1$ and $A^k \cong C \subset B$ such that $\dim_K(C) \geq (1 - \epsilon) \dim_K(B)$.

**Lemma 9.2** Let $U \in \omega$. Suppose that $\{M_i\}_{i \in U}$ is a hyperfinite family. Then for any $\epsilon$, there exists $A \in R\text{-Mod}$ and $V \subset U$, $V \in \omega$ such that for each $i \in V$ the module $A \epsilon$-tiles $M_i$.

**Proof** By hyperfiniteness, we have modules $A_1, A_2, \ldots, A_r \in R\text{-Mod}$ such that for each $n \geq 1$ there exist constants $\{c_{nj}\}_{1 \leq j \leq r}$ so that

$$\bigoplus_{j=1}^{r} A_{n}^{c_{nj}} \cong B_n \subset M_n$$

satisfying the inequality

$$\left(1 - \frac{\epsilon}{2}\right) \dim_K(M_n) \leq \dim_K(B_n).$$

By Lemma 9.1, there exist constants $\{p_j\}_{j=1}^{r}$, a module $A = \bigoplus_{j=1}^{r} A_j^{p_j}$ and $V \subset U$, $V \in \omega$ so that $A \epsilon$-tiles $M_i$ if $i \in V$. \[\square\]

**Lemma 9.3** Suppose that the module $A \in R\text{-Mod} \epsilon$-tiles all elements of the sequence $\{B_n\}_{n=1}^{\infty} \subset R\text{-Mod}$, where $\lim_{n \to \infty} \dim_K(B_n) = \infty$. Then for any $l \geq 1$, $A^l 2\epsilon$-tiles $B_n$, if $n$ is large enough.

(the proof is straightforward)

**Lemma 9.4** Let $0 < \epsilon < 1$ and $A^k \cong C \subset B$ be finitely generated $R$-modules such that

$$(1 - \epsilon) \dim_K(B) \leq \dim_K(C).$$

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Let $\delta \leq \frac{\epsilon}{\dim_K(A)}$ and $B' \subset B$ is a submodule such that

$$(1 - \delta)\dim_K(B) \leq \dim_K(B').$$

Then, there exists $k' \geq 0$ and $A^{k'} \cong C' \subset B'$ such that

$$(1 - 2\epsilon)\dim_K(B') \leq \dim_K(C').$$

**Proof** Let $b = \dim_K(B)$, that is,

$$(1 - \epsilon)b \leq k \dim_K(A) \leq b.$$  

Let us write $A^k$ into the form of $A \otimes K^k$ and for each $a \in A$ define the vector space

$$V_a := \{v \mid a \otimes v \in B'\}.$$ 

Since

$$\dim_K(a \otimes K^k) + \dim_K(B') = \dim_K((a \otimes K^k) \cap B') + \dim_K((a \otimes K^k) \lor B')$$

we have that

$$k + (1 - \delta)b \leq k + \dim_K(B') \leq \dim_K(V_a) + b.$$ 

That is, $k - \delta b \leq \dim_K(V_a)$. Hence, $k - \dim_K(A)\delta b \leq \dim_K(\cap_{a \in A} V_a)$. Observe that

$$C' = A \otimes_K (\cap_{a \in A} V_a) \subset B'$$

and

$$(1 - 2\epsilon)b \leq k \dim_K(A) - \delta b\dim_K^2(A) \leq \dim_K(C').$$

This finishes the proof of our lemma. \hfill \Box

Now we put together the previous lemmas to have a single technical proposition.

**Proposition 9.1** Let $\{M_n\}_{n=1}^{\infty} \subset R\text{-Mod}$ be a convergent, hyperfinite sequence. Then there exist modules $\{N_i\}_{i=1}^{\infty} \subset R\text{-Mod}$, subsets $N_i \supseteq V_1 \supseteq V_2 \supseteq \ldots, V_i \in \omega$ and a monotonically decreasing sequence of real numbers $\{\alpha_n\}_{n=1}^{\infty}, \alpha_n \to 0$ such that

- For any $i$, $N_i \alpha_i$-tiles $M_j$, whenever $j \in V_i$.
- For any $i$, $N_i \alpha_i$-tiles $N_{i+1}$.
- For any $i$, $\dim_K(N_{i+1})$ is divisible by $\dim_K(N_i)$.
- For any $k, l \geq 1$ and matrix $A \in \text{Mat}_{k \times l}(R)$.

$$\lim_{n \to \infty} rk_{M_n}(A) = \lim_{i \to \infty} rk_{N_i}(A).$$

\(\text{Springer}\)
Proposition 9.1. Set

\[ \top \]

Proof We proceed by induction. Our inductional hypothesis goes as follows. Suppose that \( N_1, N_2, \ldots N_l \) have already been constructed together with the sets \( V_1 \supseteq V_2 \supseteq \cdots \supseteq V_l \) and \( \alpha_1 > \alpha_2 > \cdots > \alpha_l, \alpha_i < \frac{1}{7} \), satisfying the following conditions.

- For any \( 1 \leq i \leq l-1 \), \( N_i \alpha_i \)-tiles \( M_j \) if \( j \in V_i \).
- For any \( 1 \leq i \leq l-1 \), \( N_i \alpha_i \)-tiles \( N_{i+1} \).
- \( N_i \alpha_i /2 \)-tiles \( M_j \) if \( j \in V_i \).
- For any \( 1 \leq i \leq l-1 \), \( \dim_K(N_{i+1}) \) is divisible by \( \dim_K(N_i) \).

By Lemma 9.2, we have a set \( V_{l+1} \subseteq V_l \), \( V_{l+1} \in \omega \) and \( N_{l+1} \in R\text{-Mod} \) such that the module \( N_{l+1} /2 \)-tiles \( M_j \) if \( j \in V_{l+1} \), where \( \frac{\alpha_{l+1}}{2 \dim_K(N_{l+1})} \). Then by Lemma 9.4, there exists some \( m \geq 1 \) such that \( N_i \alpha_i \)-tiles \( (N_{l+1})^m \). Hence, \( N_i \alpha_i \)-tiles \( (N_{l+1})^m \dim_K(N_i) \) as well. Let \( N_{l+1} = (N_{l+1})^m \dim_K(N_i) \). By Lemma 9.3, there exists \( n_j > 0 \) such that \( N_{l+1} \alpha_i \)-tiles \( M_j \) provided that \( j \in V_{l+1} \) and \( j > n_j \). So, let

\[ V_{l+1} = V_{l+1} \cap [n \mid n > n_j] \]

Clearly, \( V_{l+1} \in \omega \). Then, we satisfy the inductional hypothesis with \( V_1 \supseteq V_2 \supseteq \cdots \supseteq V_{l+1} \) and the modules \( N_1, N_2, \ldots, N_{l+1} \). In order to finish the proof of our proposition, we need to show that for any \( k, l \geq 1 \) and matrix \( A \in \text{Mat}_k \times l(R) \)

\[ \lim_{n \to \infty} r_k M_n(A) = \lim_{i \to \infty} r_k N_i(A). \]

Therefore, it is enough to prove the following approximation lemma.

**Lemma 9.5** Let \( N \subset M \) be finitely generated \( R \)-modules such that \( \dim_K(N) \geq \dim_K(M)(1 - \epsilon) \) for some \( 0 < \epsilon < 1 \). Let \( k, l \geq 1 \) and \( A \in \text{Mat}_k \times l(R) \). Then

\[ |r_k M(A) - r_k N(A)| \leq 2\epsilon l. \]

**Proof** By definition,

\[ r_k N(A) = \frac{\text{rank}_N(A)}{\dim_K(N)}, \quad r_k M(A) = \frac{\text{rank}_M(A)}{\dim_K(M)}, \]

where \( A \) is viewed as a linear map from \( M^l \) to \( M^k \) (and from \( N^l \) to \( N^k \)) and \( \text{rank}_M(A) := \dim_K(\text{Im} \ A) \). Then,

\[ |r_k M(A) - r_k N(A)| = \left| \frac{\text{rank}_M(A)}{\dim_K(M)} - \frac{\text{rank}_N(A)}{\dim_K(N)} \right| \left| \frac{\text{rank}_M(A)}{\dim_K(M)} \right| + \left| \frac{\text{rank}_N(A)}{\dim_K(N)} \right| \leq 2\epsilon l \]

Hence our lemma and the proposition follows. \( \square \)

Now we turn to the proof of Theorem 7. We will use the spaces and constants of Proposition 9.1. Set \( n_i = \dim_K(N_i), m_j = \dim_K(M_j) \). We need three maps for the
proof of our theorem, \( \rho_\omega : R \to \prod_\omega \text{End}_K(M_j) = \text{Mat}_\omega^\mathcal{M} \) is already constructed in Sect. 2.

The map \( \rho : R \to \mathcal{M}_K \):

Let \( V, W \) be finite dimensional \( K \)-spaces such that \( \dim_K(W) = l \dim_K(V) \) and let \( W = W_1 \oplus W_2 \oplus \cdots \oplus W_l \) be a decomposition together with isomorphisms \( s_j : V \to W_j, 1 \leq j \leq l \). Then, we have the corresponding diagonal homomorphism \( D : \text{End}_K(V) \to \text{End}_K(W) \) defined by

\[
D(A)(w_1 \oplus w_2 \oplus \cdots \oplus w_l) = \left( s_1(A(s_1^{-1}(w_1))) \oplus s_2(A(s_2^{-1}(w_2))) \oplus \cdots \oplus s_l(A(s_l^{-1}(w_l))) \right).
\]

Let \( N_{i+1} = N_i^{l_i} \oplus V_{i,1} \oplus \cdots \oplus V_{i,b_i} \), where the first component is the submodule defined in Proposition 9.1 and \( \{V_{i,j}\}_{j=1}^{b_i} \) are arbitrary subspaces satisfying \( \dim_K(V_{i,j}) = n_i \). Note that the subspaces \( V_{i,j} \) can be constructed by the divisibility condition and that \( l_in_i/n_{i+1} \geq 1 - \alpha_i \). Let \( \tau_i : \text{End}_K(N_i) \to \text{End}_K(N_{i+1}) \) be the corresponding diagonal map. So, we have a sequence of injective maps

\[
\text{End}_K(N_1) \xrightarrow{\tau_1} \text{End}_K(N_2) \xrightarrow{\tau_2} \cdots
\]

Hence we have a direct limit of matrix algebras

\[
T = \lim_{\to} \text{End}_K(N_i).
\]

Let \( \overline{T} \) be the closure of the algebra with respect to the unique rank metric, where \( d(A, B) = \text{Rank}(A - B) \). Then by [19], \( \overline{T} \cong \mathcal{M}_K \). Note that this result is originally due to von Neumann. Let \( \rho_i : R \to \text{End}_K(N_i) \) be the map given by the \( R \)-module structure. Observe that

\[
\text{rk}_{N_{i+1}}(\rho_{i+1}(r) - \tau_i \circ \rho_i(r)) \leq \alpha_i.
\]

Hence, the sequence \( \{\rho_i(r)\}_{i=1}^\infty \) is Cauchy in \( \overline{T} \). Thus, \( \lim_{i \to \infty} \rho_i(r) = \rho(r) \) defines a homomorphism \( \rho : R \to \overline{T} \cong \mathcal{M}_K \).

The map \( \Phi : \mathcal{M}_K \to \prod_\omega \text{End}_K(M_j) \):

Let \( j \in V_i \) and \( M_j \cong N_i^{l_i,j} \oplus Z_{i,j} \) a decomposition into linear subspaces, where the first component \( N_i^{l_i,j} \) is the submodule obtained by the \( \alpha_i \)-tiling of \( M_j \) by the module \( N_i \) and \( Z_{i,j} \) is an arbitrary complementing subspace. So, we have

\[
\dim_K(Z_{i,j}) \leq \alpha_i \dim_K(M_j).
\]

Using the decomposition above, we define a non-unital injective homomorphism \( \Phi_{i,j} : \text{End}_K(N_i) \to \text{End}_K(M_j) \) by

\[
\Phi_{i,j}(A) = A \oplus A \oplus \cdots \oplus A \oplus 0,
\]
where there are \( t_{i,j} \) copies of \( A \) in \( \Phi_{i,j}(A) \). For \( j \in \mathbb{N} \), let

- \( q(j) = j \) if \( j \in V_j \).
- \( q(j) = \max\{i \mid j \in V_i\} \), if \( j \notin V_j \).

Let \( A \in T \) such that \( A \notin \text{End}_K(N_i) \) but \( A \notin \tau_{i-1}(\text{End}_K(N_{i-1})) \). Then, let

- \( \Psi_j(A) = 0 \) if \( j \notin V_i \).
- \( \Psi_j(A) = \Phi_{q(j),j}(\tau_{q(j)}^{-1} \circ \tau_{q(j)}^{-2} \circ \cdots \circ \tau_i(A)) \) if \( j \in V_i \), that is \( q(j) \geq i \).

Finally, let

- \( \Psi_j(A) = 0 \) if \( j \notin V_i \).
- \( \Psi_j(A) = \Phi_{q(j)}(\tau_{q(j)}^{-1} \circ \cdots \circ \tau_i(A)) \) if \( j \in V_i \), that is \( q(j) \geq i \).

Let \( A \in \text{End}_K(N_s) \), \( A \notin \tau_{s-1}(\text{End}_K(N_{s-1})) \) and \( B \in \text{End}_K(N_t) \), \( B \notin \tau_{t-1}(\text{End}_K(N_{t-1})) \). Then, by Proposition 9.1,

\[
\{ j \in \mathbb{N} \mid \Psi_j(AB) = \Psi_j(A)\Psi_j(B) \} \in \omega
\]

and

\[
\{ j \in \mathbb{N} \mid \Psi_j(A + B) = \Psi_j(A) + \Psi_j(B) \} \in \omega.
\]

Since the direct limit ring has a unique rank, the only thing remains to be proved is that \( \Phi' \) is unital. By (8),

\[
\text{rk}_{M_j}(\Psi_j(1) - \text{Id}_{\text{End}_K(M_j)}) \leq \alpha_{q(j)}.
\]

Since for any \( i \geq 1 \),

\[
\{ j \in \mathbb{N} \mid q(j) \geq i \} \in \omega
\]

we have that

\[
\lim_{\omega}(\text{rk}_{M_j}(\Phi_j(1) - \text{Id}_{\text{End}_K(M_j)})) = 0.
\]

Thus, \( \Phi' \) is indeed unital. \( \square \)

Now let \( \Phi : \mathcal{M}_K \to \prod_{\omega} \text{End}_K(M_j) \) be the closure of \( \Phi' \) that is

\[
\Phi \left( \lim_{n \to \infty} x_n \right) = \lim_{n \to \infty} \Phi'(x_n).
\]

Since \( \Phi' \) is rank preserving, \( \Phi \) is a well-defined homomorphism. There, we have three homomorphisms.
\begin{itemize}
  \item $\rho : R \to \mathcal{M}_K$
  \item $\rho_\omega : R \to \prod_\omega \text{End}_K(M_j)$.
  \item $\Phi : \mathcal{M}_K \to \prod_\omega \text{End}_K(M_j)$.
\end{itemize}

The following lemma implies that $\text{Im}(\Phi)$ contains $\text{Im}(\rho_\omega)$, hence it completes the proof of Theorem 7.

**Lemma 9.7** $\Phi \circ \rho = \rho_\omega$.

**Proof** Let $\kappa_j : R \to \text{End}_K(M_j)$ be the homomorphism given by the module structure. By (8), if $j \in V_i$ and $r \in R$,

$$rk_{M_j}(\Phi_i,j(\rho_i(r)) - \kappa_j(r)) \leq \alpha_i.$$ 

Therefore, for any $i \geq 1$

$$rk_\omega(\Phi \circ \rho_i(r) - \rho_\omega(r)) \leq \alpha_i.$$ 

Since $\lim_{i \to \infty} \rho_i(r) = \rho(r)$ and $\Phi$ is a rank preserving map, $\Phi \circ \rho(r) = \rho_\omega(r)$. \hfill $\Box$

### 10 String algebras are of amenable representation types

The significance of string algebras is due to the fact that the system of f.g. indecomposable modules over such algebras can explicitly be described (see e.g. [23] and the references therein). First, let us recall the notion of a string algebra. Let $Q$ be a finite quiver with vertex set $Q_0$ and arrow set $Q_1$. Let $KQ$ be the associated path algebra and $I \triangleleft KQ$ be an ideal, generated by monomials in the path algebra, satisfying the following four conditions.

1. For any vertex $a \in Q_0$, there are at most two arrows with source $a$, and at most two arrows with target $a$.
2. For any arrow $\alpha \in Q_1$, there exists at most one $\beta \in Q_1$ such that $s(\beta) = t(\alpha)$ and $\beta \alpha \notin I$.
3. For any arrow $\alpha \in Q_1$, there exists at most one $\beta \in Q_1$ such that $t(\beta) = s(\alpha)$ and $\alpha \beta \notin I$.
4. There exists $q \geq 1$ such that any $Q$-path of length at least $q$ is inside $I$.

**Example 3** $KQ = K\langle x, y \rangle$, $I = \langle x^n, y^n, xy, yx \rangle$ is a string algebra. The path algebra of the 2-Kronecker quiver is itself a string algebra.

Let $R = KQ/I$ be a string algebra. First we describe the so-called string modules over $R$. For every arrow $\alpha \in Q_1$, $\alpha^{-1}$ will denote its formal inverse such that $s(\alpha) = t(\alpha^{-1})$ and $t(\alpha) = s(\alpha^{-1})$. We denote by $Q_1^{-1}$ the set of all inverse arrows. The elements of $Q_1 \cup Q_1'$ are called letters. A **string** of length $n \geq 1$ is a sequence $C_1C_2 \ldots C_n$ of letters such that

1. $t(C_{i+1}) = s(C_i)$ if $1 \leq i \leq n - 1$.
2. $C_i \neq C_{i+1}^{-1}$ if $1 \leq i \leq n - 1$. 

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3. No substring $C_jC_{j+1}\ldots C_k$ or its inverse $C_k^{-1}\ldots C_{j+1}^{-1}C_j^{-1}$ lies in the ideal $I$.

Additionally, any vertex $a \in Q_0$ is considered to be a string of length 0. Note that for any string $C_1C_2\ldots C_n$, $(C_1C_2\ldots C_n)^{-1} = C_n^{-1}C_{n-1}^{-1}\ldots C_1^{-1}$ is also a string. The set of strings is denoted by $W$. For each string $C_1C_2\ldots C_n = S \in W$ we can associate an $R$-module $M(S)$ the following way. A $K$-basis for $M(S)$ is $z_0, z_1, \ldots, z_n$. For a vertex $a \in Q_0$ and $e_a$ (path of length zero starting and ending at $a$):

$$e_a z_i = \begin{cases} 
  z_i & \text{if } i > 0 \text{ and } s(C_i) = a \\
  z_0 & \text{if } i = 0 \text{ and } t(C_1) = a \\
  0 & \text{otherwise}
\end{cases}$$

For an arrow $\alpha \in Q_1$ we define

$$\alpha z_i = \begin{cases} 
  z_{i-1} & \text{if } C_{i-1} = \alpha \\
  z_{i+1} & \text{if } C_i = \alpha^{-1} \\
  0 & \text{otherwise}
\end{cases}$$

Observe that $M(S) \cong M(S^{-1})$. On the other hand, if $S_1 \neq S_2$, $M(S_1) \not\cong M(S_2)$ whenever $S_1 \neq S_2^{-1}$.

Now we describe the band modules over the string algebra $R$. A string $S = S_{(0)} = C_1C_2\ldots C_n$ is called **cyclic** if all of its powers $S^k = C_1C_2\ldots C_nC_1C_2\ldots C_n\ldots C_1C_2\ldots C_n$ are also strings as well. A **band** is a primitive cyclic string $S$, that is, $S \neq T^l$ for any string $T$ and $l \geq 2$. Let $S = C_1C_2\ldots C_n$ be a band and $V$ be a finite dimensional vector space over $K$. Fix an indecomposable transformation $\phi$ on $V$ by identifying $V$ with $K[x]/f(x)^n$, where $f$ is an irreducible monic polynomial and $\phi$ is the multiplication by $x$. The band module $M(S, \phi)$ is defined the following way. $M(S, \phi) = \bigoplus_{i=1}^n V_i$, where $V_i \cong V$, that is for each $v \in V$ and $1 \leq i \leq n$ we fix $v_i \in V_i$. For any $\alpha \in Q_1$ and $i \neq 1, 2$:

$$\alpha v_i = \begin{cases} 
  v_{i-1} & \text{if } C_i = \alpha \\
  v_{i+1} & \text{if } C_{i+1} = \alpha^{-1} \\
  0 & \text{otherwise}
\end{cases}$$

Also,

$$\alpha v_1 = \begin{cases} 
  v_n & \text{if } C_1 = \alpha \\
  (\phi^{-1}(v))_2 & \text{if } C_2 = \alpha^{-1} \\
  0 & \text{otherwise}
\end{cases}$$

$$\alpha v_2 = \begin{cases} 
  (\phi(v))_1 & \text{if } C_2 = \alpha \\
  v_3 & \text{if } C_3 = \alpha^{-1} \\
  0 & \text{otherwise}
\end{cases}$$
By the Butler-Ringel classification of indecomposable modules [6], a band module and a string module are never isomorphic and any finitely generated indecomposable module over a string algebra is either a string module or a band module.

**Proposition 10.1** Any string algebra $R$ is of hyperfinite type.

The proof will be given in two lemmas.

**Lemma 10.1** The family of string modules over a string algebra $R$ is hyperfinite.

**Proof** By 6.1, it is enough to show that there exists some $m_\epsilon > 0$ such that if $n \geq m_\epsilon$ and $w = C_1C_2\ldots C_n$ is a string, then there exist independent subspaces $W_0, W_1, \ldots, W_t \subset M(w)$ such that

\[
\dim_K \left( W_i \oplus \bigoplus_{\alpha \in Q_1} \alpha W_i \right) \leq (1 + \epsilon) \dim_K(W_i)
\]

(we only need to check the elements $\alpha \in Q_1$ not the whole path algebra by Lemma 6.2)

\[
\dim_K \left( \bigoplus_{i=0}^t W_i \right) \geq (1 - \epsilon) \dim_K(M(w)).
\]

Let $m$ be an integer such that $\frac{m-2}{m} < 1 + \epsilon$ and let $m_\epsilon \geq \frac{m}{\epsilon}$. Finally, let $t \geq 1$ such that $tm \leq n < (t + 1)m$.

Let $W_i$ be the subspace spanned by the vectors \(\{z_{im}, z_{im+1}, \ldots, z_{(i+1)m-1}\}\). Then,

\[
\dim_K \left( W_i \oplus \bigoplus_{\alpha \in Q_1} \alpha W_i \right) \leq m + 2 \leq (1 + \epsilon) \dim_K(W_i).
\]

Also,

\[
\dim_K(M(w)) - \dim_K(\bigoplus_{i=0}^t W_i) \leq m \leq \epsilon \dim_K(M(w)).
\]

\[\square\]

**Lemma 10.2** The family of band modules over a string algebra $R$ is hyperfinite.

**Proof** The proof will be very similar to the previous one. Fix $\epsilon > 0$. It is enough to prove that there exists $L_\epsilon > 0$, such that for any band module $M(S, \phi)$ there exist independent subspaces \(\{V_{\rho}\}_{\rho \in G}\) such that

- For all $\rho \in G$, $\dim_K(V_{\rho}) \leq L_\epsilon$.
- $\dim_K(\sum_{\alpha \in Q_1} \alpha V_{\rho} + V_{\rho}) \leq (1 + \epsilon) \dim_K(V_{\rho})$
- $\sum_{\rho \in G} \dim_K(V_{\rho}) \leq (1 - \epsilon) \dim_K(M(S, \phi))$
Let \( m \in \mathbb{N} \) such that \( \frac{m+2}{m} \leq 1 + \epsilon \) and \( m_\epsilon > \frac{2m}{\epsilon} \). Let \( S = C_1 C_2 \ldots C_n \) be a band, \( \phi \) and \( f \) be as above and \( d = \deg(f) \). First, suppose that \( n > m_\epsilon \). Consider the basis \( \{ 1, x, x^2, \ldots, x^{d-1} \} \) for \( V \) and let \( t \in \mathbb{N} \) such that \( tm + 2 \leq n < (t+1)m + 1 \). Then, for any pair \( 0 \leq i \leq t - 1 \) and \( 0 \leq j \leq d - 1 \) let \( W^i_j \) be the subspace of \( M(S, \phi) \) spanned by the set \( \{ x^j_{i+m+2}, x^j_{i+m+3}, \ldots, x^j_{(i+1)m+1} \} \). Then, it is easy to see that for any pair \( i, j \)

\[
\dim_K \left( \sum_{\alpha \in Q_1} \alpha W^i_j + W^i_j \right) \leq (1 + \epsilon) \dim_K (W^i_j).
\]

and

\[
\sum_{0 \leq i \leq t - 1} \sum_{0 \leq j \leq d - 1} \dim_K (W^i_j) \leq (1 - \epsilon) \dim_K M(S, \phi).
\]

Now, suppose that for the length of the cyclic string, we have \( n \leq m_\epsilon \) and for the dimension of \( V \) we have \( d > m_\epsilon \). Let \( t \) be as above, and for \( 0 \leq q \leq t - 1 \) let \( Z_q \in M(S, \phi) \) be the subspace spanned by the set

\[
\{ \cup_{j=1}^n \cup_{j=qm}^{(q+1)m-1} x^j_i \}.
\]

By the definition of the band module structure,

\[
\dim_K \left( \sum_{\alpha \in Q_1} \alpha Z_q + Z_q \right) \leq (1 + \epsilon) \dim_K (Z_q).
\]

and

\[
\sum_{q=0}^{t-1} \dim_K Z_q \geq (1 - \epsilon) \dim_K M(S, \phi).
\]

Also, for any \( q \), \( \dim_K (Z_q) \leq mm_\epsilon \). Observe that there are only finitely many band modules \( M(S, \phi) \) for which \( n, d \leq m_\epsilon \). Let \( M \) be the maximal \( K \)-dimension of these modules. So, in order to finish the proof of the lemma we need to set \( L_\epsilon := \max(mm_\epsilon, M) \).

11 Benjamini–Schramm convergence implies the convergence of string modules

For the next three sections we fix a string algebra \( R \) over the finite field \( K \). The sole purpose of the next two sections is to establish a relation between the convergence of string modules and the Benjamini-Schramm convergence of the associated edge-colored graphs. This will be the preparation for the proof of Theorem 1. Let \( M(S) \) be
the string module corresponding to the string $S = (C_1C_2\ldots C_n)$. We can associate a
graph $G_S$ to $S$ in a very natural way.

- The vertex set $V(G_S)$ has $n$ elements $\{x_0, x_1, \ldots, x_n\}$.
- For any $1 \leq i \leq n$, we have a directed edge between $x_{i-1}$ and $x_i$. If $C_i = \alpha \in Q_1$, 
  then the edge is directed from $x_{i-1}$ to $x_i$ and colored by $\alpha$. If $C_i = \alpha^{-1} \in Q_1$, then 
  the edge is directed from $x_i$ to $x_{i-1}$ and colored by $\alpha$. That is, all the edge-colors 
  are from $Q_1$.

Note that the graphs $G_S$ and $G_T$ are isomorphic as edge-colored, directed graphs if 
$S = T$ or $S = T^{-1}$. By definition, if $x$ is a vertex of $G_S$ and $e, f$ are edges both 
pointing into resp. pointing out of $x$, then the edge-colors of $e$ and $f$ are different.

**Definition 11.1** A directed $Q_1$-edge colored graph is called a string graph if all of its 
components are in the form of $G_S$ for some string $S$. So, for each finitely generated 
$R$-module $M$ that is a sum of string modules, we have a unique string graph $G_M$ 
and two $R$-modules are isomorphic if and only if the corresponding string graphs are 
isomorphic.

Now we recall the Benjamini–Schramm graph convergence (see [4] and [14]). A rooted 
string graph is a connected string graph with a distinguished vertex $x$ (the root). The 
radius of a rooted string graph is the shortest path distance between a vertex of $H$ 
and the root. The finite set of rooted string graphs of radius less or equal than $r$ 
are isomorphic.

- In a very natural way.

**Definition 11.2** A sequence of string graphs $\{G_n\}_{n=1}^\infty$ is convergent (in the sense 
of Benjamini and Schramm) if for any $r \geq 1$ and $H \in U^r$, \( \lim_{n \to \infty} p(H, G_n) \) exists.

Note that for any $H, G$ and $k \geq 1$, $p(H, G) = p(H, G^k)$, where $G^k$ is the disjoint 
union of $k$ copies of $G$ and it is easy to see that $p(H, G_1) = p(H, G_2)$ for all $H$ if and 
only if $G_1^n \cong G_2^n$ for some $m, n \geq 1$. So, by Proposition 4.1, we have the following 
lemma.

**Lemma 11.1** Let $M, N$ be finite direct sums of string modules. They represent the 
same element in the rank spectrum if and only if $p(H, G_M) = p(H, G_N)$ holds for 
all $r \geq 1$ and $H \in U^r$.

Now we can state our main proposition in the section.

**Proposition 11.1** Let $\{M_n\}_{n=1}^\infty \subset R$-Mod be a sequence of modules such that for any 
n $\geq 1$, $M_n$ is the sum of string modules. Suppose that the graphs $\{G_{M_n}\}_{n=1}^\infty$ converge 
in the sense of Benjamini and Schramm. Then $\{M_n\}_{n=1}^\infty$ is a convergent sequence of 
$R$-modules.

**Proof** The proof will be given in a series of lemmas. First we need the definition of 
$\epsilon$-isomorphism for string graphs.
Definition 11.3 The string graphs $G_1$ and $G_2$ are $\epsilon$-isomorphic, if they contain subgraphs $J_1 \subset G_1$ and $J_2 \subset G_2$ such that $J_1$ and $J_2$ are isomorphic string graphs and $|V(J_1)| \geq (1 - \epsilon)|V(G_1)|$, $|V(J_2)| \geq (1 - \epsilon)|V(G_2)|$.

Lemma 11.2 For any $\epsilon > 0$, there exists $\delta > 0$ and $n \geq 1$ such that if the string graphs $G_1$ and $G_2$ has the same number of vertices and $|p(H, G_1) - p(H, G_2)| < \delta$ for any $H \in U^r$, $1 \leq r \leq n$, then $G_1$ and $G_2$ are $\epsilon$-isomorphic.

Proof (of Lemma 11.2) Our lemma holds for simple planar graphs with a vertex degree bound by the main result of [22] (see also Theorem 5. in [15]). Our strategy is to reduce our lemma to the Newman–Sohler result, by encoding our $Q_1$-edge colored, directed graph with a simple planar graph. Let $G$ be a string graph. The encoding simple graph $\hat{G}$ is constructed as follows. The vertex set of $\hat{G}$ consists of the vertex set of $G$ plus two vertices $a_{xy}, b_{xy}$ per each edge of $G$. If $(x, y)$ is an edge of $G$, then $(x, a_{xy}), (a_{xy}, b_x, y), (b_{xy}, y)$ will be edges of $\hat{G}$. That is, we substitute the original edge with a path of length three. For each element of $\alpha \in Q_1$, we choose a planar graph (the edge-color coding graph) $T_\alpha$ with a distinguished vertex $t_\alpha$, such that each $T_\alpha$ have the same number of vertices and all the vertex degrees of $T_\alpha$ are at least three. For each edge $(x, y)$ of $G$ we stick a copy of $T_\alpha$ to $a_{xy}$ by identifying $a_{xy}$ and the distinguished vertex $t_\alpha$ if the edge is directed towards $x$ and its color is $\alpha$. On the other hand, we stick a copy of $T_\alpha$ to $b_{xy}$ if the edge is directed towards $y$ and its color is $\alpha$. Clearly,

- We can reconstruct $G$ from $\hat{G}$.
- If $(G_n)_{n=1}^{\infty}$ is convergent then $(\hat{G}_n)_{n=1}^{\infty}$ is a convergent sequence of simple planar graphs.

So, by the Newman–Sohler Theorem, for any $\epsilon' > 0$, there exists $n \geq 1$ and $\delta > 0$ such that if $|p(H, G_1) - p(H, G_2)| < \delta$ holds for any $H \in U^r$, $1 \leq r \leq n$ and $G_1$ and $G_2$ have the same amount of vertices, than $\hat{G}_1$ and $\hat{G}_2$ are $\epsilon'$-isomorphic. Also, it is easy to see that for any $\epsilon > 0$ there exists $\epsilon' > 0$ such that if $\hat{G}_1$ and $\hat{G}_2$ are $\epsilon'$-isomorphic, than $G_1$ and $G_2$ are $\epsilon$-isomorphic. Hence our lemma follows.

Lemma 11.3 For any $\epsilon > 0$, there exists $n \geq 1$ such that if for the string graphs $G_1$ and $G_2$

$$1 - \delta \leq \frac{|V(G_1)|}{|V(G_2)|} \leq 1 + \delta$$

and $|p(H, G_1) - p(H, G_2)| < \delta$ for any $H \in U^r$, $1 \leq r \leq n$, then $G_1$ and $G_2$ are $\epsilon$-isomorphic.

Proof First, pick a $\delta$ and $n$ to the constant $\epsilon/2$ as in Lemma 11.2. Let $0 < \delta' \leq \min(\delta/2, \epsilon/10)$ be a constant such that for any $H \in U^r$, $1 \leq r \leq n$,

$$|p(H, G_2) - p(H, G_3)| < \frac{\delta}{2}$$

provided that $G_2 \subset G_3$ and $|V(G_3)| \leq (1 + \delta')|V(G_2)|$. The existence of such $\delta'$ can easily be seen. Now we show that $\delta'$ and $n$ satisfy the condition of our lemma. We can
suppose that $|V(G_2)| \leq |V(G_1)|$. By adding strings of one single vertex, we can get $G_2 \subset G_3$ such that $|V(G_3)| = |V(G_1)|$. Then $|V(G_3)| \leq (1 + \delta')|V(G_2)|$. Hence for any $H \in U^r$, $1 \leq r \leq n$, 

$$|p(H, G_1) - p(H, G_3)| < \delta.$$ 

Therefore, by Lemma 11.2, $G_1$ and $G_2$ are $\epsilon/2$-isomorphic. Since $\delta' < \epsilon / 10$, $G_2$ and $G_1$ are $\epsilon$-isomorphic. □

**Lemma 11.4** For any $\epsilon > 0$, there exists $\delta > 0$ such that if $G_M$ and $G_N$ are $\delta$-isomorphic string graphs, then the modules $M$ and $N$ are $\epsilon$-isomorphic.

**Proof** We need some notations. If $x \in G_M$, then let $\delta_x$ be the corresponding element in $M$. If $x \xrightarrow{\alpha} y$ is an edge, then $\alpha \delta_x = \delta_y$. Also, if there is no edge pointing out of $x$ colored by $\alpha$, then $\alpha \delta_x = 0$. By the definition of the string algebra, there exists $q > 0$ such that any path of the quiver $Q$ of length at least $q$ is in $I$, where $R = KQ/I$.

Therefore, if the $q$-neighborhood of a vertex $x \in G_M$ is in a subgraph $L$, then $R \delta_x \subseteq \text{Span}\{\delta_y \mid y \in V(L)\}$. (9)

Suppose that $L_1 \subset G_M$, $L_2 \subset G_N$ are isomorphic string graphs such that $|V(L_1)| \geq (1 - \delta)|V(G_M)|$, $|V(L_2)| \geq (1 - \delta)|V(G_N)|$. We call $x \in V(L_1)$ an inside point if its $q$-neighborhood is contained in $L_1$. Obviously, if $\delta$ is small enough, then $|I_1| \geq (1 - \epsilon)|V(G_M)|$, $|I_2| \geq (1 - \epsilon)|V(G_N)|$, where $I_1$ resp. $I_2$ are the sets of inside points in $L_1$ resp. in $L_2$. Since $L_1$ and $L_2$ are isomorphic, the modules generated by $\{\delta_x \mid x \in I_1\}$ and by $\{\delta_x \mid x \in I_2\}$ are also isomorphic. Thus our lemma follows. □

Therefore, from the previous lemmas we have the following corollary.

**Corollary 11.1** For any $\epsilon > 0$, there exists $\delta > 0$ and $n \geq 1$ such that if $M$ and $N$ are sums of string modules and for any $H \in U^r$, $1 \leq r \leq n$, 

$$|p(H, G_M) - p(H, G_N)| < \delta$$

and also, $1 - \delta \leq \frac{\dim_K(M)}{\dim_K(N)} \leq 1 + \delta$, then $M$ and $N$ are $\epsilon$-isomorphic.

Now, by Lemma 9.5, our proposition follows. □

12 Convergence of string modules implies Benjamini–Schramm convergence

The goal of this section is to prove the following converse of Proposition 11.1.

**Proposition 12.1** Let $\{M_n\}_{n=1}^{\infty} \subset R\text{-Mod}$ be a convergent sequence of modules such that each module $M_n$ is a sum of string modules. Then $\{G_{M_n}\}_{n=1}^{\infty}$ is convergent in the sense of Benjamini and Schramm.
Lemma 12.2 Let \( G \) be a string graph and \( S = C_1 C_2 \ldots C_k \) be a string. Let \( L \cong G_S \) be a subgraph in \( G \). We have \( V(L) = \{ l_0, l_1, \ldots, l_k \} \), where

- The edge \( (l_{i-1}, l_i) \) is directed towards \( l_{i-1} \) and colored by \( C_i \), if \( C_i \in Q_1 \).
- The edge \( (l_{i-1}, l_i) \) is directed towards \( l_i \) and colored by \( C_i^{-1} \), if \( C_i \in Q_1^{-1} \).

We call \( l_k \) the right resp. \( l_0 \) the left endvertex of \( L \). By the definition of the strings, if \( L_1, L_2 \) are subgraphs in \( G \) isomorphic to \( G_S \) and their right resp. left endvertices coincide, then \( L_1 = L_2 \). For the string \( S \) and the string graph \( G \), let \( R(S, G) \) be the set of vertices \( x \) in \( G \) such that \( x \) is the right vertex of a substring \( L \) of \( G \) isomorphic to \( G_S \). Let

\[
\frac{\dim_{V}(\alpha)}{|\alpha|} = \frac{|R(S, G)|}{|V(G)|}.
\]

We say that \( \{G_n\}_{n=1}^{\infty} \) is \textbf{stringconvergent} if for any \( S \), \( \lim_{n \to \infty} r(S, G_n) \) exists. The following combinatorial lemma is straightforward to prove.

Lemma 12.1 If \( \{G_n\}_{n=1}^{\infty} \) is stringconvergent, then it is convergent in the sense of Benjamini and Schramm.

Let \( N \in R\text{-Mod} \) be a sum of string modules and \( G_N \) be its string graph. That is, for any \( x \in V(G_N) \), we have a base element \( \delta_x \in N \) and if \( x \xrightarrow{\alpha} y \) is an edge of \( G_N \), then \( \alpha(\delta_x) = \delta_y \).

Lemma 12.2 (The String Counting Lemma) For any string \( S = C_1 C_2 \ldots C_k \), there exists a pp-pair \( \langle \phi_S, \psi_S \rangle \) such that

\[
\dim_K(N(\phi_S)) - \dim_K(N(\psi_S)) = R(S, G_N)
\]

holds for any module \( N \) that is a sum of string modules.

Proof For \( 1 \leq i \leq k \) let \( E_i \) be the equation

- \( C_i n_i - n_{i-1} = 0 \), if \( C_i \in Q_1 \).
- \( C_i n_{i-1} - n_i = 0 \), if \( C_i \in Q_1^{-1} \).

Also, let \( E_0 \) be the equation \( n_o = 0 \).

Let

\[
N(\phi_S) = \{ n_k \in N \mid \exists n_0, n_1, \ldots, n_{k-1} \subset N \text{ such that } E_1, E_2, \ldots, E_k \text{ holds} \}.
\]
\[
N(\psi_S) = \{ n_k \in N \mid \exists n_0, n_1, \ldots, n_{k-1} \subset N \text{ such that } E_0, E_1, \ldots, E_k \text{ holds} \}.
\]

Then clearly, \( \phi_S \geq \psi_S \). Let \( \pi : N(\phi_S) \to \text{Span}\{\delta_x : x \in R(S, G_N)\} \) be the natural restriction map.

Lemma 12.3 The map \( \pi \) is surjective and \( \text{Ker}(\pi) = N(\psi_S) \).

Proof Let \( x \in R(S, G_N) \) and \( L = \{x_0, x_1, \ldots, x_k \} \), \( x_k = x \) be the subgraph of \( G_N \) isomorphic to \( G_S \). Then for \( n_i = \delta_{x_i}, 0 \leq i \leq k \) the equations \( E_1, E_2, \ldots, E_k \) hold. Hence \( \pi \) is surjective. If \( z \in N(\phi_S) \) and the \( \delta_x \)-coordinate of \( z \) is \( \lambda \), then the \( \delta_{x_0} \)-coordinate of \( z \) is \( \lambda \) as well. Hence, if \( \pi(z) = 0 \), then \( z \in N(\psi_S) \). \( \square \)
So we have

\[ r(S, M_n) = \frac{\dim_K (M_n(\phi_S))}{\dim_K (M_n)} - \frac{\dim_K (M_n(\psi_S))}{\dim_K (M_n)}. \]

By Corollary 3.1, the sequence \( \{r(S, M_n)\}_{n=1}^{\infty} \) converges whenever the modules \( \{M_n\}_{n=1}^{\infty} \) converge. Hence our proposition follows from Lemma 12.1. \( \square \)

13 The proof of Theorem 1

Let \( d_R \) be any metric on \( \text{Syl}(R) \) defining the compact topology. Then by compactness, for any \( \epsilon > 0 \) there exists \( \delta > 0 \) and matrices \( A_1, A_2, \ldots, A_n \in \text{Mat}(R) \) so that for all \( 1 \leq i \leq n \), \(|rk_1(A_i) - rk_2(A_i)| < \delta\), then \( d_R(rk_1, rk_2) < \epsilon \). Conversely, for any set of matrices \( A_1, A_2, \ldots, A_n \in \text{Mat}(R) \) and \( \delta > 0 \), there exists \( \epsilon > 0 \) so that if \( d_R(rk_1, rk_2) < \epsilon \), then for all \( 1 \leq i \leq n \), \(|rk_1(A_i) - rk_2(A_i)| < \delta\). In particular, by Lemma 9.5, for any \( \epsilon > 0 \) there exists \( \delta > 0 \) such that if \( M, N \) are \( \delta \)-isomorphic, then \( d_R(rk_M, rk_N) < \epsilon \). First, we prove a weak version of Theorem 1.

**Proposition 13.1** For any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that if \( M, N \in R\text{-Mod} \) are sums of string modules, \( 1 - \delta \leq \frac{\dim_K (M)}{\dim_K (N)} \leq 1 + \delta \) and \( d_R(rk_M, rk_N) < \delta \), then \( M \) and \( N \) are \( \epsilon \)-isomorphic.

**Proof** We proceed by contradiction. Suppose that \( \{M_n\}_{n=1}^{\infty}, \{N_n\}_{n=1}^{\infty} \) are sums of string modules such that \( \lim_{n \to \infty} \frac{\dim_K (M_n)}{\dim_K (N_n)} = 1 \) and none of the pairs \( M_n, N_n \) are \( \epsilon \)-isomorphic. We can also assume that \( \{M_n\}_{n=1}^{\infty}, \{N_n\}_{n=1}^{\infty} \) are convergent sequences. By Proposition 12.1, \( \{G_{M_n}\}_{n=1}^{\infty} \) and \( \{G_{N_n}\}_{n=1}^{\infty} \) are convergent in the sense of Benjamini and Schramm. Hence, by Lemma 11.3, for any \( \delta > 0 \) there exists \( n_\delta \) such that for any \( n \geq n_\delta \), \( G_{M_n} \) and \( G_{N_n} \) are \( \delta \)-isomorphic. So, by Lemma 11.4, \( M_n \) and \( N_n \) are always \( \epsilon \)-isomorphic, if \( n \) is large enough, leading to a contradiction. \( \square \)

Now suppose that Theorem 1 does not hold. Again, we have two convergent sequence of modules \( \{M_n\}_{n=1}^{\infty} \) and \( \{N_n\}_{n=1}^{\infty} \) such that

- \( \lim_{n \to \infty} \frac{\dim_K (M_n)}{\dim_K (N_n)} = 1 \).
- \( \lim_{n \to \infty} rk_{M_n} = \lim_{n \to \infty} rk_{N_n} \).
- For each \( n \geq 1 \), \( M_n, N_n \) are not \( \epsilon \)-isomorphic.

In the proof of Lemma 10.2 we observed that for any \( \kappa > 0 \), there exists \( T_\kappa \) \( \geq 0 \) such that if for a band module \( B \), \( \dim_K (B) \geq T_\kappa \) holds, then there is a module \( L_B \) such that

- \( L_B \) is a sum of string modules.
- \( B \) and \( L_B \) are \( \kappa \)-isomorphic.

Now, we fix some constants. Let \( \kappa_1 > 0 \) such that if \( L_1, L_2 \) are sums of string modules and \( d_R(L_1, L_2) < \kappa_1 \), then \( L_1 \) and \( L_2 \) are \( \epsilon/100 \)-isomorphic. Let \( \kappa_2 > 0 \) be a constant such that \( \kappa_2 < \epsilon/100 \) and if \( M, N \in R\text{-Mod} \) are \( \kappa_2 \)-isomorphic, then \( d_R(M, N) \leq \kappa_1/3 \). For each \( n \geq 1 \), we consider the decompositions,

\[ M_n = M_n^1 \oplus M_n^2, \quad N_n = N_n^1 \oplus N_n^2, \]
where \( M_n^1, N_n^1 \) are sums of band modules of dimension less than \( T_{k_2} \) and \( M_n^2, N_n^2 \) are sums of string modules and band modules of dimension greater or equal than \( T_{k_2} \).

**Lemma 13.1** The sequences \( \{M_n^1\}_{n=1}^\infty, \{M_n^2\}_{n=1}^\infty, \{N_n^1\}_{n=1}^\infty \) and \( \{N_n^2\}_{n=1}^\infty \) are all convergent. Also, the limits \( \lim_{n \to \infty} \frac{\dim_K(M_n^1)}{\dim_K(M_n^2)} \), \( \lim_{n \to \infty} \frac{\dim_K(M_n^2)}{\dim_K(M_n^1)} \), \( \lim_{n \to \infty} \frac{\dim_K(N_n^1)}{\dim_K(N_n^2)} \) and \( \lim_{n \to \infty} \frac{\dim_K(N_n^2)}{\dim_K(N_n^1)} \) exist.

Let \( B_1, B_2, \ldots, B_3 \) be the set of band modules in \( R\text{-Mod} \) that have dimension less than \( T_{k_2} \). By Corollary 4.1, for any \( 1 \leq i \leq s \), the limit \( \lim_{n \to \infty} w_{B_i}(M_n) \) exists. By the definition of the rank (see also Lemma 4.1) if \( P_1, P_2, \ldots, P_l \in R\text{-Mod} \), then for any \( A \in \text{Mat}(R) \),

\[
 r^k_{\oplus_{i=1}^l P_i}(A) = \sum_{i=1}^l \frac{\dim_K(P_i)}{\dim_K(\oplus_{i=1}^l P_i)} r^k_{P_i}(A). \tag{10}
\]

Recall that

\[
 M_n^1 = \oplus_{i=1}^s \frac{w_{B_i}(M_n)}{\dim_K(B_i)} B_i. \tag{11}
\]

Therefore, \( \frac{\dim_K(M_n^1)}{\dim_K(M_n^2)} = \sum_{i=1}^s \frac{w_{B_i}(M_n)}{\dim_K(B_i)} \). Then by (11), \( \{M_n^1\}_{n=1}^\infty \) and \( \{N_n^1\}_{n=1}^\infty \) are convergent, hence by (10), \( \{M_n^2\}_{n=1}^\infty \) and \( \{N_n^2\}_{n=1}^\infty \) are convergent as well. \( \square \)

Also, for large \( n, B_i/\dim_K(B_i) \) and \( B_i/\dim_K(B_i) \) are \( \epsilon \)-isomorphic, hence for large \( n, M_n^1 \) and \( N_n^1 \) are \( \epsilon \)-isomorphic. By our assumptions, we have modules \( L_n \subset M_n^2, O_n \subset M_n^2 \) that are sums of string modules such that for any \( n \geq 1, L_n \) and \( M_n^2 \) resp. \( O_n \) and \( N_n^2 \) are \( \kappa_2 \)-isomorphic. So, by the definition of \( \kappa_2 \), we have that \( d_R(L_n, M_n^2) \leq \kappa_1/3, d_R(O_n, N_n^2) \leq \kappa_1/3 \). By the previous lemma, if \( n \) is large enough, then \( d_R(M_n^2, N_n^2) \leq \kappa_1/3 \). Therefore if \( n \) is large enough, then \( d_R(L_n, O_n) \leq \kappa_1 \). Thus by the definition of \( \kappa_1, L_n \) and \( O_n \) are \( \epsilon /100 \)-isomorphic. Since \( \kappa_2 < \epsilon /100 \), we can see that \( M_n^2 \) and \( N_n^2 \) are \( \epsilon \)-isomorphic. Since for large \( n, M_n^1, N_n^1 \) are \( \epsilon \)-isomorphic, we get that for large \( n, M_n^2 \) and \( N_n^2 \) are \( \epsilon \)-isomorphic as well, leading to a contradiction. Hence our theorem follows.

### 14 Parameter testing for modules

A module parameter \( p \) is a bounded real function on \( R\text{-Mod} \), where \( R \) is a finite dimensional algebra over a finite field \( K \). So, if \( M \cong N \) then \( p(M) = p(N) \) (see [13] for parameters of bounded degree graphs).

**Examples:**

- Let \( G(M) \) be the smallest generating system of the module \( M \). Then \( g(M) := \frac{|G(M)|}{\dim_K(M)} \) is a module parameter. This parameter is analogous to the covering number of finite graphs.

- Let \( I(M) \) be the largest system \( \{m_1, m_2, \ldots, m_{I(M)}\} \) of elements in the module \( M \) so that \( \sum_{i=1}^{I(M)} r_i m_i = 0 \) implies that \( r_i = 0 \) for any \( 1 \leq i \leq |I(M)| \). Then \( i(M) := \frac{|I(M)|}{\dim_K(M)} \) is a module parameter, analogous to the independence number of a finite graph for a module.
Let $Q \in R$-Mod. Then the weight function $w_Q(M)$ (defined in Section 4) is a module parameter.

Let $Q \in R$-Mod. Then $L_Q(M) := \frac{\dim_K (\operatorname{Hom}_K(Q,M))}{\dim_K(M)}$ resp. $R_Q(M) := \frac{\dim_K (\operatorname{Hom}_K(M,Q))}{\dim_K(M)}$ left and right homomorphism numbers are module parameters analogous to the left and right homomorphism numbers of finite graphs.

**Definition 14.1** The module parameter $p : R$-Mod $\rightarrow \mathbb{R}$ is stable if it satisfies the following two conditions:

1. For any $\epsilon > 0$ there exists a $\delta > 0$ such that if $N \subseteq M$ and $\dim_K(N) \geq (1 - \delta) \dim_K(M)$ then $|p(M) - p(N)| < \epsilon$.
2. For any $M \in R$-Mod the limit $\lim_{k \rightarrow \infty} p(M^k)$ exists.

By Lemma 9.4, for any matrix $A \in \text{Mat}(R)$, the matrix parameter $p_A(M) = rk_M(A)$ is stable.

**Proposition 14.1** The parameters $g, i, w_Q, L_Q$ and the parameter $R_Q$ provided that $Q$ is an injective module are stable parameters.

**Proof** Clearly, $w_Q(M^k) = w_Q(M), L_Q(M^k) = L_Q(M), R_Q(M^k) = R_Q(M)$. Also, $|G(M \oplus N)| \leq |G(M)| + |G(N)|$ and $|I(M \oplus N)| \leq |I(M)| + |I(N)|$. So, by Fekete’s Theorem on Subadditive and Superadditive Functions

$$
\lim_{k \rightarrow \infty} \frac{|G(M^k)|}{k \dim_K(M)} \quad \text{and} \quad \lim_{k \rightarrow \infty} \frac{|I(M^k)|}{k \dim_K(M)}
$$

exist. Note that similar observations were made by Cohn on projective module parameters in [9]. It remains to show that the first condition of stability holds for our parameters.

$g : R$-Mod $\rightarrow \mathbb{R}$. Let $W \subseteq M$ be a $K$-linear subspace such that $N \oplus W = M$. Clearly, if $\{t_1, t_2, \ldots, t_s\}$ is a basis of the space $W$ and $\{n_1, n_2, \ldots, n_q\}$ is a generating system for $N$, then $\{t_1, t_2, \ldots, t_s, n_1, n_2, \ldots, n_q\}$ is a generating system for $M$. Therefore

$$
G(M) \leq G(N) + \delta \dim_K(M)
$$

(12)

if $\dim_K(N) \geq (1 - \delta) \dim_K(M)$. Now let $\{m_1, m_2, \ldots, m_l\}$ be a generating system for $M$ and $\{w_i\}_{i=1}^l \subseteq W$ be elements such that $m_i + w_i \in N$, for any $1 \leq i \leq l$. Let $n \in N$ and $n = \sum_{i=1}^l r_i m_i$. Then

$$
n = \sum_{i=1}^l r_i (m_i + w_i) - \sum_{i=1}^l r_i w_i
$$

(13)

hence, $\sum_{i=1}^l r_i w_i \in N$. Let $[W]$ be the $R$-module generated by $W$, so $\dim_K([W]) \leq \dim_K(R) \dim_K(W)$. Let $\{u_1, u_2, \ldots, u_z\}$ be a $K$-basis for $[W] \cap N$. Then by (13),
{u_1, u_2, \ldots, u_s} \cup \{m_1 + w_1, m_2 + w_2, \ldots, m_l + w_l\} is a generating system for N. That is,
\[ G(N) \leq G(M) + \delta \dim_K(M) \dim_K(R). \] (14)

Now, (12) and (14) imply that the first condition of stability holds for the parameter \( i : R\text{-Mod} \to \mathbb{R} \). Clearly, \( I(N) \subset I(M) \). Suppose that we have a sequence \( \{N_k \subset M_k\}_{k=1}^{\infty} \) such that
\[ \dim_K(M_k) \to \infty. \]
\[ \lim_{k \to \infty} \frac{\dim_K(N_k)}{\dim_K(M_k)} = 1, \] such that \( |i(N_k) - i(M_k)| \geq \epsilon \).

We can suppose that \( I(M_k) \geq \epsilon \dim_K(M_k) \). Let \( Z_1, Z_2, \ldots, Z_t \) be the finite set of isomorphism classes of principal \( R \)-modules. For each \( k \), we have \( S_k \subset M_k, S_k \cong \bigoplus_{i=1}^{t} Z_i^{r_i} \) such that \( \sum_{i=1}^{t} q_i = I(M_k) \). By Lemma 9.4, we have \( r_i \leq q_i \) and \( T_k \subset N_k, T_k = \sum_{i=1}^{t} Z_i^{r_i}, \sum_{i=1}^{t} r_i \leq I(N_k) \) such that
\[ \lim_{k \to \infty} \frac{\dim_K(Z_i)(q_i - r_i)}{\dim_K(M)} = 0. \]

Therefore, \( \lim_{k \to \infty} |i(N_k) - i(M_k)| = 0 \) in contradiction with our assumption. This implies that the first condition of stability holds for the parameter \( i \).

\( w_Q : R\text{-Mod} \to \mathbb{R} \). Again, suppose that we have a sequence \( \{N_k \subset M_k\}_{k=1}^{\infty} \) such that
\[ \dim_K(M_k) \to \infty. \]
\[ \lim_{k \to \infty} \frac{\dim_K(N_k)}{\dim_K(M_k)} = 1 \] and for any \( k \geq 1 \)
\[ |w_Q(M_k) - w_Q(N_k)| \geq \epsilon. \] (15)

Taking a subsequence, we can assume that \( \{M_k\}_{k=1}^{\infty} \) is convergent. That is, by Lemma 9.5, for any matrix \( A \in \text{Mat}(R) \),
\[ \lim_{k \to \infty} |r_{kM_k}(A) - r_{kN_k}(A)| = 0. \]

By (1),
\[ w_Q(M_k) = \frac{D_{M_k}(\phi) - D_{M_k}(\psi)}{D_{Q}(\phi) - D_{Q}(\psi)} \]
\[ w_Q(N_k) = \frac{D_{N_k}(\phi) - D_{N_k}(\psi)}{D_{Q}(\phi) - D_{Q}(\psi)} \]
that is \( \lim_{k \to \infty} |w_Q(M_k) - w_Q(N_k)| = 0 \) in contradiction with (15). So, we established the first stability condition for the parameter \( w_Q \).

\( R_Q : R\text{-Mod} \to \mathbb{R} \). Since \( Q \) is injective, any homomorphism \( \phi : N \to Q \) extends to a homomorphism \( \phi' : M \to Q \). Therefore,
\[ \dim_K(\text{Hom}(N, Q)) \leq \dim_K(\text{Hom}(M, Q)). \] (16)
Let \( \pi : \text{Hom}(M, Q) \to \text{Hom}(N, Q) \) be the restriction map. Clearly,

\[
\dim_K(\ker(\pi)) \leq (\dim_K(M) - \dim_K(N)) \dim_K(Q).
\]

that is,

\[
\dim_K(\text{Hom}(M, Q)) \leq \dim_K(\text{Hom}(N, Q)) + (\dim_K(M) - \dim_K(N)) \dim_K(Q).
\] (17)

Now, (16) and (17) imply the first stability condition for the parameter \( R_Q \), provided that \( Q \) is injective.

\( L_Q : R\text{-Mod} \to \mathbb{R} \). Let us consider the following equations on \( M^Q \).

- For any pair \( q_1, q_2 \in Q \)
  
  \[
m_{q_1} + m_{q_2} - m_{q_1+q_2} = 0.
  \]

- For any pair \( r \in R, q \in Q \)
  
  \[
m_{rq} - rm_q = 0.
  \]

The solution set of the equations above is isomorphic with \( \text{Hom}(Q, M) \). So, the first stability condition for the parameter \( L_Q \) immediately follows from the fact that all matrix parameters \( p_A(M) = rk_M(A) \) are stable.

The theory of constant-time graph algorithms was developed in the last decade (see e.g. [16]). Let us briefly recall the main idea. Say, we want to estimate the value of a certain graph parameter \( p \) for an immensely large graph \( G \) of small vertex degrees. For certain parameters (such as the matching number) we can do the estimate in constant-time (that is, independently of the size of the graph). A parameter \( p \) is called testable for the class of graphs \( G \), if for any \( \epsilon > 0 \) there exists \( \delta > 0 \) and a family of test-graphs \( F_1, F_2, \ldots, F_t \) such that if we learn all the \( F_i \)-subgraph densities for \( G \in G \) up to an error of \( \delta \), we can compute \( p(G) \) up to an error of \( \epsilon \). The point is that using a fixed amount of random samplings of \( G \), we can estimate all the subgraph densities above up to an error of \( \delta \) with very high probability, no matter how large our graph \( G \) is. We have an analogous definition for modules.

**Definition 14.2** A module parameter \( p : R\text{-Mod} \to \mathbb{R} \) is testable if for any \( \epsilon > 0 \), there exists \( \delta > 0 \), test-matrices \( A_1, A_2, \ldots, A_t \) and \( n \geq 1 \), such that if for a modules \( M \in R\text{-Mod}, \dim_K(M) \geq n \), and we know all the \( rk_M(A_i) \)'s up to an error of \( \delta \), then we can compute \( p(M) \) up to an error of \( \epsilon \).

The following theorem is motivated by the results of Newman-Sohler [22] and Hassidim–Kelner–Nguyen–Onak [20].

**Theorem 8** If Conjecture 2 holds for an algebra \( R \) (e.g. \( R \) is a string algebra), then every stable parameter \( p : R\text{-Mod} \to \mathbb{R} \) is testable.
Proof Let $p$ be a stable parameter for $R$. By our assumptions, we have matrices $A_1, A_2, \ldots, A_t$ and $\kappa > 0$ such that $|rk_M(A_i) - rk_N(A_i)| \leq \kappa$ holds for any $1 \leq i \leq t$ and

$$1 - \kappa < \frac{\dim_K(M)}{\dim_K(N)} \leq 1 + \kappa$$

then $|p(M) - p(N)| \leq \epsilon/3$. We can pick $\lambda > 0$ in such a way that

- $\lambda \leq \kappa$
- if a module $Q$ $\lambda$-tiles $M$ (see Section 9), then $|rk_Q(A_i) - rk_M(A_i)| \leq \kappa/2$ holds for any $1 \leq i \leq t$.

In the proof of Lemma 9.2 we saw that there exists a finite set of modules $\{Q_1, Q_2, \ldots, Q_s\}$ such that for any $M \in R$-Mod at least one of the $Q_j$’s $\lambda$-tiles $M$. Pick $n$ so large that if $1 \leq j \leq s$ and $\dim_K(Q_j) \geq n/2$, then $\lim_{k \to \infty} p(Q^k) - p(Q_j) \leq \epsilon/3$. We also assume that $\dim_K(Q_j) \leq \kappa n$. Let $p_i = p(Q_j^n)$, then for each $1 \leq j \leq s$

$$| \lim_{k \to \infty} p(Q^k) - p_i | \leq \epsilon/3.$$  \(18\)

Our algorithm goes as follows. We set $\delta = \kappa/2$. If we learn all the values $rk_M(A_i)$’s up to an error of $\delta$, we can find at least one $1 \leq j \leq s$ such that

$$|rk_{Q_j}(A_i) - rk_M(A_i)| \leq \kappa$$  \(19\)

holds for all $1 \leq i \leq t$. These comparisons are all the computations we do after learning the estimated values $rk_M(A_i)$. Hence, the computation time does not depend on the dimension of $M$. Notice however, the role of $n \geq 1$. In the graph parameter case, all the interesting parameters are additive with respect to disjoint union. Here we only have the stability property.

Note that we might find more than one $Q_j$’s for which (19) holds and we cannot be sure that the $Q_j$ we choose $\lambda$-tiles $M$. Nevertheless by our assumption, for some $c > 0$

$$1 - \kappa \leq \frac{\dim_K(M)}{\dim_K(Q_j^n)} \leq 1 + \kappa,$$

hence by the choice of $\kappa$, $|p(M) - p(Q_j^n)| \leq \epsilon/3$, whenever $\dim_K(M) \geq n$. Therefore by (18), $|p_j - p(M)| \leq \epsilon$. So, $p_j$ estimates $p(M)$ up to an error of $\epsilon$. $\square$

Remark One can easily see that the parameters $L_Q$ and $w_Q$ are testable for any finite dimensional algebra $R$. Indeed, they can be computed from the rank of some matrices.

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