Hamiltonian Analysis In New General Relativity

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It is known that one can formulate an action in teleparallel gravity which is equivalent to general relativity, up to a boundary term. In this geometry we have vanishing curvature, and non-vanishing torsion. The action is constructed by three different contractions of torsion with specific coefficients. By allowing these coefficients to be arbitrary we get the theory which is called “new general relativity”. In this note, the Lagrangian for new general relativity is written down in ADM-variables. In order to write down the Hamiltonian we need to invert the velocities to canonical variables. However, the inversion depends on the specific combination of constraints satisfied by the theory (which depends on the coefficients in the Lagrangian). It is found that one can combine these constraints in 9 different ways to obtain non-trivial theories, each with a different inversion formula.

Keywords: Teleparallel gravity; New general relativity; ADM-variables.

1. Conventions

Greek indices denote global coordinate indices running from 0 to 3, small Latin indices are spatial coordinate indices running from 1 to 3, whereas capital Latin indices denote Lorentz indices running from 0 to 3. We are always dealing with Lorentzian metrics. Sign convention for the Minkowski metric is \( \eta_{AB} = \text{diag}(-1,1,1,1) \).

2. Introduction

Gravity is conventionally described with the Levi-Civita connection which is induced by a pseudo-Riemannian metric. This means that the covariant derivative of the metric is zero, and the connection is torsion-free but has curvature. However, there are equivalent theories to general relativity¹. We will focus on teleparallel gravity where we have vanishing curvature, but non-vanishing torsion.

In particular we will perform the Hamiltonian analysis of “new general relativity” (NGR)². For discussions of certain issues with these theories see³⁴⁵. Previous work on the Hamiltonian analysis on teleparallel gravity theories have been performed in⁶⁷⁸. However, the full Hamiltonian analysis of NGR has not been performed. NGR is described by the following action:

\[
S_{\text{NGR}} = m_p^2 \int |\theta| \left( a_1 T_{\nu\rho}^\mu T_{\mu\nu}^{\rho} + a_2 T_{\nu\rho}^\mu T_{\rho\nu\mu} + a_3 T_{\rho\mu}^{\nu\rho} T_{\nu\mu} \right) \, d^4x,
\]

¹With NGR, we refer to the more general three-parameter teleparallel gravity in contrast to the special one-parameter teleparallel gravity theory which NGR originally referred to³.
where $m_{Pl}$ is the Planck mass, $T_{\mu \nu} = \Gamma_{\mu \nu} - \Gamma_{\nu \mu}$ is the torsion component with
\[\Gamma_{\mu \nu} = e^A_{\mu} \partial_\nu A^A + e^A_{\nu} (\Lambda^{-1})^A_\beta \partial_\mu A^B \theta^B_\nu,\]
with $\theta$ being the tetrad, $e$ its inverse and $\Lambda$ is a Lorentz matrix. Global spacetime indices are raised and lowered with $g_{\mu \nu} = \theta^A_\mu \theta^B_\nu \eta_{AB}$, while Lorentz indices are raised and lowered with $\eta_{AB}$. A theory equivalent to general relativity is obtained by setting $a_1 = \frac{1}{2}$, $a_2 = \frac{1}{2}$, and $a_3 = -1$.

Alternatively, the NGR action can be written down in the so-called axial, vector, and tensor decomposition. Then
\[
S_{NGR} = m_{Pl}^2 \int \sqrt{g} (c_1 T_{\text{ax}} + c_2 T_{\text{ten}} + c_3 T_{\text{vec}}),
\]
with $c_1 = -\frac{1}{18} c_1 + \frac{1}{2} c_2$, $c_2 = \frac{1}{2} c_1 + \frac{1}{2} c_2$, $c_3 = c_3 - \frac{1}{2} c_2$, and
\[
T_{\text{vec}} = T^\rho_{\mu \nu} T^\nu_{\rho \mu},
\]
\[
T_{\text{ax}} = -\frac{1}{18} (T_{\mu \nu} T^{\rho \mu \nu} - 2 T_{\rho \mu \nu} T^{\mu \rho \nu}),
\]
\[
T_{\text{ten}} = \frac{1}{2} (T_{\mu \nu} T^{\rho \mu \nu} + T_{\rho \mu \nu} T^{\mu \rho \nu}) - \frac{1}{2} T^\rho_{\mu \nu} T^\nu_{\rho \mu}.
\]

3. Method

In order to go from the Lagrangian to the Hamiltonian analysis we need to identify the velocities, derive the conjugate momenta and express everything in canonical variables. We may decompose the torsion scalar in the ADM variables and $\Lambda$ is a Lorentz matrix. Global spacetime indices are raised and lowered with $g_{\mu \nu} = \theta^A_\mu \theta^B_\nu \eta_{AB}$, while Lorentz indices are raised and lowered with $\eta_{AB}$. A theory equivalent to general relativity is obtained by setting $a_1 = \frac{1}{2}$, $a_2 = \frac{1}{2}$, and $a_3 = -1$.

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3. Method

In order to go from the Lagrangian to the Hamiltonian analysis we need to identify the velocities, derive the conjugate momenta and express everything in canonical variables. We may decompose the torsion scalar in the ADM variables and the spatial components of the tetrad $\theta^A_i$:

\[
\theta^A_i = \theta^A_{1i} + \theta^A_{2i} T^B_{i} \theta^B_\alpha \eta_{\alpha \beta} + \theta^A_{3i} T^B_{i} \theta^B_\alpha \eta_{\alpha \beta} + \theta^A_{4i} T^{\alpha}_{i} \theta^B_\alpha \eta_{\alpha \beta}.
\]

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\theta^A_i = \theta^A_{1i} + \theta^A_{2i} T^B_{i} \theta^B_\alpha \eta_{\alpha \beta} + \theta^A_{3i} T^B_{i} \theta^B_\alpha \eta_{\alpha \beta} + \theta^A_{4i} T^{\alpha}_{i} \theta^B_\alpha \eta_{\alpha \beta}.
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\]
The velocities can now be inverted and expressed in canonical variables using

\[ S^A_i = \dot{\theta}^j M^i_{AB} j_B, \]  

(8)

with

\[ S^A_i = D_j (\alpha \xi^B + \beta^m \theta^B_m) M^i_{AB} j_B - T^B_{kl} [M^i_{AB} j^B - 2\alpha a h_{il} \xi^A \theta^B_k + 2\alpha a h_{il} \xi^A \theta^B_k] + \alpha \frac{\pi^i_A}{\sqrt{h}}, \]  

(9)

where \( D_i \) is the Levi-Civita covariant derivative with respect to the induced metric. However, \( M \) in equation (8) is singular for certain combinations of parameters of the theory and can hence only be inverted by the Moore-Penrose pseudo-inverse matrix. This is apparent if one decomposes the equation into irreducible representations of the rotation group, which generates the following constraints,

\[ 2a_1 + a_2 + a_3 =: V A = 0 \implies V C^i := S^i_A \xi^A = 0, \]  

(10)

\[ 2a_1 - a_2 =: A A = 0 \implies A_C^i j := S^k_A \theta^A(j h_{ij})k = 0, \]  

(11)

\[ 2a_1 + a_2 =: S A = 0 \implies S_C^i := S^k_A \theta^A (j h_{ij})k - \frac{1}{3} S^k_A \theta^A h_{ij} = 0, \]  

(12)

\[ 2a_1 + a_2 + 3a_3 =: T A = 0 \implies T := S^i_A \theta^A = 0. \]  

(13)

These are primary constraints, since these constrain both the tetrad field and their conjugate momenta, which also can be decomposed into irreducible parts. In the axial, vector, tensor decomposition we have that

\[ \frac{2}{9} c_1 + \frac{2}{3} c_2 = V A = 0, \]  

(14)

\[ 3 \frac{2}{9} c_2 = A A = 0, \]  

(15)

\[ \frac{3}{2} c_2 = S A = 0, \]  

(16)

\[ 3c_3 = T A = 0. \]  

(17)

In this language the primary constraints get some further geometrical meaning. Equations (14) and (15) together imposes the teleparallel equivalent to general relativity and impose invariance of the Lagrangian under pure tetrad local Lorentz transformations. This is, however, not more apparent from the axial, vector, tensor decomposition we made. What is more interesting are the constraints imposed by equations (16) and (17). In this decomposition of the torsion scalar they exactly correspond to putting \( T_{ten} \) and \( T_{vec} \) to zero respectively.

4. Results

Different combinations of (10)-(13) yield 9 non-trivial classes of theories.
Any other solutions would be trivial \((c_1 = c_2 = c_3 = 0)\). Excluding these trivial solutions we can normalize our parameters to

\[
\tilde{c}_i = \frac{c_i}{\sqrt{c_1^2 + c_2^2 + c_3^2}},
\]

for \(i = 1, 2, 3\), which means that we can make a 2-dimensional plot to visualize these theories in the normalized parameter-space. This can be nicely visualized in polar coordinates \((\theta, \phi)\) on the unit sphere with

\[
\tilde{c}_1 = \cos \theta, \quad \tilde{c}_2 = \sin \theta \cos \phi, \quad \tilde{c}_3 = \sin \theta \sin \phi.
\]

Every pair of antipodal points on the sphere corresponds to a ray in the 3-dimensional parameter space, whose elements describe the same theory. Hence, it suffices to display only the upper half sphere \(\tilde{c}_1 \geq 0\), which is done in figure 1. However, note that points on the equator \(\tilde{c}_1 = 0\) still appear twice, and both copies should be identified with each other. This applies in particular to the two purple points in figure 1 both describing the class of theories defined by pure vector torsion \(\tilde{c}_1 = \tilde{c}_2 = 0\). The Hamiltonian is found to always appear with four Lagrange multipliers (linearity in lapse and shifts) with,

\[
H = \alpha \mathcal{H} \left( \theta, M^{-1} \right) + \beta^k \mathcal{H}_k \left( \theta, M^{-1} \right) + D_i \left[ \left( \alpha \xi^A + \beta^j \theta^A \right) \pi_A^i \right],
\]

in the unconstrained case.\(^7\)

5. Discussion

One can distinguish 9 different classes of NGR theories by the presence or absence of primary constraints appearing in their Hamiltonian formulation. What remains to be determined is how many secondary constraints are induced by demanding closure of the constraint algebra. Some considerations in this direction have been studied in\(^6,18\), however, our work invites for further investigation. The theories satisfying \(A_I \neq 0, \forall I \in \{V, A, S, T\}\) can be parameterized by two free parameters (and a global rescaling of the Lagrangian, fixing the value of the Planck mass, which does not affect the presence or absence of primary constraints). Models which exhibit one primary constraint \(A_I = 0\) have one free parameter left, while for those with
more primary constraints all parameters are fixed. The free parameters might affect the vanishing, or non-vanishing of certain Poisson brackets, which therefore have to be calculated in order to obtain the number of degrees of freedom.

The number of degrees of freedom can be compared with polarization modes in gravitational waves\cite{20}. Furthermore, it can be compared with the linear level in order to find out if the theories are strongly coupled. One may extend this analysis to $f(T_{ax}, T_{ten}, T_{vec})$\cite{19} or include parity violating terms.

References

1. J. Beltrán Jiménez, L. Heisenberg and T. S. Koivisto, “The Geometrical Trinity of Gravity,” [arXiv:1903.06830 [hep-th]].
2. M. Krssak, R. J. Van Den Hoogen, J. G. Pereira, C. G. Boehmer and A. A. Coley, “Teleparallel Theories of Gravity: Illuminating a Fully Invariant Approach,” [arXiv:1810.12932 [gr-qc]].
3. K. Hayashi and T. Shirafuji, “New General Relativity,” Phys. Rev. D 19 (1979) 3524 Addendum: [Phys. Rev. D 24 (1982) 3312].
4. J. M. Nester, “Is there really a problem with the teleparallel theory?,” Class. Quant. Grav. 5. (1988) no. 7, 10.1088
5. W. Kopczynski, “Problems with the metric-teleparallel theories of gravitation”, J. Phys. A: Math. Gen. 15. (1982) 493
6. W. H. Cheng, D. C. Chern and J. M. Nester, “Canonical Analysis of the One
Parameter Teleparallel Theory,” Phys. Rev. D 38 (1988) 2656.
7. D. Blixt, M. Hohmann and C. Pfeifer, “Hamiltonian and primary constraints of
new general relativity,” Phys. Rev. D 99 (2019) no.8, 084025 [arXiv:1811.11137
[gr-qc]].
8. D. Blixt, M. Hohmann and C. Pfeifer, “On the gauge fixing in the Hamiltonian
analysis of general teleparallel theories,” arXiv:1905.01048 [gr-qc].
9. M. Blagojevic and I. A. Nikolic, “Hamiltonian structure of the teleparallel for-
mulation of GR,” Phys. Rev. D 62 (2000) 024021 [hep-th/0002022].
10. J. W. Maluf and J. F. da Rocha-Neto, “Hamiltonian formulation of general
relativity in the teleparallel geometry,” Phys. Rev. D 64 (2001) 084014.
11. J. W. Maluf, “Hamiltonian formulation of the teleparallel description of general
relativity,” J. Math. Phys. 35 (1994) 335.
12. R. Ferraro and M. J. Guzmán, “Hamiltonian formulation of teleparallel grav-
ity,” Phys. Rev. D 94 (2016) no.10, 104045 [arXiv:1609.06766 [gr-qc]].
13. M. Li, R. X. Miao and Y. G. Miao, “Degrees of freedom of $f(T)$ gravity,” JHEP
1107 (2011) 108 [arXiv:1105.5934 [hep-th]].
14. R. Ferraro and M. J. Guzmán, “Hamiltonian formalism for $f(T)$ gravity,” Phys.
Rev. D 97 (2018) no.10, 104028 [arXiv:1802.02130 [gr-qc]].
15. Y. C. Ong and J. M. Nester, “Counting Components in the Lagrange Multi-
plier Formulation of Teleparallel Theories,” Eur. Phys. J. C 78 (2018) no.7, 568
[arXiv:1709.00068 [gr-qc]].
16. A. Okolów, “ADM-like Hamiltonian formulation of gravity in the teleparallel
geometry: derivation of constraint algebra,” Gen. Rel. Grav. 46 (2014) 1636
[arXiv:1309.4685 [gr-qc]].
17. A. Okolów, “ADM-like Hamiltonian formulation of gravity in the teleparallel
geometry,” Gen. Rel. Grav. 45 (2013) 2569 [arXiv:1111.5498 [gr-qc]].
18. A. Okolow and J. Swiezewski, “Hamiltonian formulation of a simple theory of
the teleparallel geometry,” Class. Quant. Grav. 29 (2012) 045008
[arXiv:1111.5490 [math-ph]].
19. S. Bahamonde, C. G. Böhmer and M. Krčskák, “New classes of modified telepar-
allel gravity models,” Phys. Lett. B 775 (2017) 37 [arXiv:1706.04920 [gr-qc]].
20. M. Hohmann, M. Krčskák, C. Pfeifer and U. Ualikhanova, “Propagation of grav-
itational waves in teleparallel gravity theories,” Phys. Rev. D 98 (2018) no.12,
124004 [arXiv:1807.04580 [gr-qc]].