Generalized scale functions of standard processes with no positive jumps

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Abstract

As a generalization of scale functions of spectrally negative Lévy processes, we define scale functions of general standard processes with no positive jumps. For this purpose, we utilize excursion measures. Using our new scale functions we study Laplace transforms of hitting times, potential measures and duality.

1 Introduction

We first recall the basic facts of scale functions of spectrally negative Lévy processes. Let \((X, \mathbb{P}_x^X)\) with \(X = \{X_t : t \geq 0\}\) be a spectrally negative Lévy process with \(\mathbb{P}_x^X(X_0 = x) = 1\), i.e., a Lévy process which does not have positive jumps and monotone paths. Then there correspond to it the Laplace exponent \(\Psi\) and the \(q\)-scale function \(W(q)\) of \(X\) for all \(q \geq 0\). The Laplace exponent \(\Psi\) is a function from \([0, \infty)\) to \(\mathbb{R}\) defined by

\[
\Psi(\lambda) = \log \mathbb{E}^X_0 \left[ e^{\lambda X_1} \right], \quad \lambda \geq 0. \tag{1.1}
\]

The \(q\)-scale function \(W(q)\) is a function which is equal to 0 on \((-\infty, 0)\), is continuous on \([0, \infty)\), and satisfies

\[
\int_{0}^{\infty} e^{-\beta x} W(q)(x) dx = \frac{1}{\Psi(\beta) - q}, \quad \beta > \Phi(q), \tag{1.2}
\]

where \(\Phi(q) = \inf\{\lambda > 0 : \Psi(\lambda) > q\}\) (see, e.g., [5, Section 8] for the details). The scale function is useful since the Laplace transform of hitting times and the \(q\)-potential measure can be characterized as follows: for \(b < x < a\)

\[
\mathbb{E}^X_x \left[ e^{-qT_a^+} ; T_a^+ < T_b^- \right] = \frac{W(q)(x - b)}{W(q)(a - b)},
\]

\[
\mathbb{E}^X_x \left[ \int_{0}^{T_a^+ \wedge T_b^-} e^{-qt} f(X_t) dt \right] = \int_{b}^{a} f(y) \left( \frac{W(q)(x - b)}{W(q)(a - b)} W(q)(a - y) - W(q)(x - y) \right) dy,
\]

where \(T_a^+\) and \(T_b^-\) denote the first hitting times of \([a, \infty)\) and \((-\infty, b]\), respectively.

Kyprianou–Loeffen([6]) has introduced the refracted Lévy processes and Noba–Yano([7]) generalized their results to the generalized refracted Lévy processes, where the scale functions of the processes considered were defined and utilized. In particular, Noba–Yano([7]) proved that the scale functions satisfy

\[
W(q)(x) = \frac{1}{n_0^X} \left[ e^{-qT_a^+} ; T_a^+ < \infty \right], \quad q \geq 0, \quad x > 0, \tag{1.5}
\]
where $n^X_0$ is an excursion measure away from 0 subject to the normalization
\[ n^X_0[1 - e^{-qT_0}] = \frac{1}{\Phi'(q)}, \quad q > 0. \] 

In this paper, we define generalized scale functions of $\mathbb{R}$-valued standard processes by generalizing (1.5) and make identities which are analogies of (1.3) and (1.4) of generalized scale functions. In addition, we study a condition of generalized scale functions for two standard processes to be in duality.

The organization of this paper is as follows. In Section 2, we prepare some notation and recall preliminary facts about standard processes, local times and excursion measures. In Section 3, we give the definition of the generalized scale functions and obtain identities which are analogies of (1.3) and (1.4). In Section 4, we characterize duality of standard processes in terms of scale functions. In Appendix A, we discuss different definitions of scale functions of spectrally negative Lévy processes.

2 Preliminary

Let $\mathbb{D}$ denote the set of functions $\omega : [0, \infty) \to \mathbb{R} \cup \{\partial\}$ which are càdlàg and satisfy
\[ \omega(t) = \partial, \quad t \geq \zeta, \tag{2.1} \]
where $\partial$ is an isolated point and $\zeta = \inf\{t > 0 : \omega(t) = \partial\}$. Let $\mathcal{B}(\mathbb{D})$ denote the class of Borel sets of $\mathbb{D}$ equipped with the Skorokhod topology. For $\omega \in \mathbb{D}$, let
\[ T^-_x(\omega) := \inf\{t > 0 : \omega(t) \leq x\}, \tag{2.2} \]
\[ T^+_x(\omega) := \inf\{t > 0 : \omega(t) \geq x\}, \tag{2.3} \]
\[ T_x(\omega) := \inf\{t > 0 : \omega(t) = x\}. \tag{2.4} \]

We sometimes write $T^-_x, T^+_x, T_x$ simply for $T^-_X(x), T^+_X(x), T_X$, respectively, when we consider a process $(X, \mathbb{P}^X_x)$. Let $\mathbb{T}$ be an interval of $\mathbb{R}$ and set $a_0 = \sup \mathbb{T}$ and $b_0 = \inf \mathbb{T}$. We assume that the process $(X, \mathbb{P}^X_x)$ considered in this paper is a $\mathbb{T}$-valued standard process with no positive jumps with $\mathbb{P}^X_x(X_0 = x) = 1$, satisfying the following conditions:

(A1) $(x, y) \mapsto \mathbb{E}^X_x[e^{-T_y}] > 0$ is a $\mathcal{B}(\mathbb{T}) \times \mathcal{B}(\mathbb{T})$-measurable function.

(A2) $X$ has a reference measure $m$ on $\mathbb{T}$, i.e. for $q \geq 0$ and $x \in \mathbb{T}$, we have $R^{(q)}_X 1_{(-\infty, x]}(\cdot) \ll m(\cdot)$, where
\[ R^{(q)}_X f(x) := \mathbb{E}^X_x\left[\int_0^\infty e^{-qt} f(X_t) dt\right] \tag{2.5} \]
for non-negative measurable function $f$. Here and hereafter we use the notation $\int_{a}^{b} = \int_{[a,b] \cap \mathbb{T}}$. In particular, $\int_{b^-} = \int_{[b_0, b] \cap \mathbb{T}}$. 

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By [3, Theorem 18.4], there exist a family of processes \( \{L_{Z,x}^t\}_{x \in \mathbb{T}} \) with \( L_{Z,x}^t = \{L_{Z,x}^t\}_{t \geq 0} \) for \( x \in \mathbb{T} \) which we call local times such that the following conditions hold: for all \( q > 0 \), \( x \in \mathbb{T} \) and non-negative measurable function \( f \)

\[
\int_0^t f(X_s)ds = \int_\mathbb{T} f(y)L_{t}^{X,y}m(dy), \quad \text{a.s.} \tag{2.6}
\]

\[
R_X^{(q)}f(x) = \int_\mathbb{T} f(y)E_x^X\left[\int_0^\infty e^{-qt}dL_{t}^{X,y}\right]m(dy). \tag{2.7}
\]

We have the following two cases:

- **Case 1.** If \( x \in \mathbb{T} \) is regular for itself, this \( L_{Z,x}^t \) is the continuous local time at \( x \) ([1, p.216]). Note that \( L_{Z,x}^t \) has no ambiguity of multiple constant because of (2.6) or (2.7).

- **Case 2.** If \( x \in \mathbb{T} \) is irregular for itself, we have

\[
L_{Z,x}^t = l_x^Z \#\{0 \leq s < t : Z_s = x\}, \quad \text{a.s.} \tag{2.8}
\]

for some constant \( l_x^Z \in (0, \infty) \).

If \( x \) is regular for itself, the excursion measure \( n_x^X \) can be defined from the Poisson point process (see [4]). Let \( \eta^{X,x} \) denote the inverse local time of \( L^{X,x} \), i.e., \( \eta^{X,x}_t = \inf\{s > 0 : L^{X,x}_s > t\} \). Then, for all \( q > 0 \), we have

\[
- \log E_0^X\left[e^{-q\eta^{X,x}(1)}\right] = \delta_x^X q + n_x^X [1 - e^{-qT_x}] \tag{2.9}
\]

for a non-negative constant \( \delta_x^X \) called the stagnancy rate. We thus have

\[
E_x^X\left[\int_0^\infty e^{-qt}dL_{t}^{X,x}\right] = E_x^X\left[\int_0^\infty e^{-q\eta^{X,x}(s)}ds\right] = \frac{1}{\delta_x^X q + n_x^X [1 - e^{-qT_x}]}. \tag{2.10}
\]

If \( x \) is irregular for itself, we define \( n_x^X = \frac{1}{l_x^X}P_x^{X,x} \) where \( P_x^{X,x} \) denotes the law of \( X \) started from \( x \) and stopped at \( x \). Then we have

\[
E_x^X\left[\int_0^\infty e^{-qt}dL_{t}^{X,x}\right] = l_x^X \sum_{i=0}^{\infty} \left(\frac{E_x^X\left[e^{-qT_x}\right]}{E_x^X[1 - e^{-qT_x}]})^i = \frac{l_x^X}{l_x^X[1 - e^{-qT_x}] = \frac{1}{n_x^X [1 - e^{-qT_x}]}. \tag{2.11}
\]

**Remark 2.1.** Any point \( x \in \mathbb{T} \backslash \{a_0\} \) cannot be a holding point. In fact, assume \( x \) is. Then \( X \) leaves \( x \) by jumps (see, e.g., [10, Theorem 1 (vi)]). But \( X \) has no positive jumps, and thus \( X \) can not exceed \( x \), which contradicts (A1).

### 3 Generalized scale functions

In this section, we define generalized scale functions of standard processes with no positive jumps and study these fundamental properties.
Definition 3.1. For $q \geq 0$ and $x, y \in \mathbb{T}$, we define $q$-scale function of $X$ as

$$W^{(q)}_{X}(x, y) = \begin{cases} \frac{1}{n^X_y [e^{-qT^+_x}; T^+_x < \infty]}, & x \geq y, \\ 0, & x < y, \end{cases}$$

where $\frac{1}{\infty} = 0$.

Remark 3.2. All $x \in \mathbb{T}\setminus\{b_0\}$ is regular for $(x, \infty)$, i.e., $\mathbb{P}^X_x (T^+_x = 0) = 1$, thanks to the assumptions of no positive jumps and of (A1). When $x$ is irregular for itself, we have $W^{(q)}_{X}(x, x) = t^X_x$ by the definition of $n^X_x$.

Remark 3.3. Let us characterize our scale functions of diffusion processes in terms of their characteristics. Let $m$ and $s$ be two $\mathbb{R}$-valued strictly increasing continuous functions on the interval $[0, \infty)$ satisfying $s(0) = 0$. Let $X$ be a $\frac{d}{dm}\frac{d}{ds}$-diffusion process with 0 being a reflecting boundary. Note that our $n^X_0$ coincides with the excursion measure defined in [11, Definition 2.1] up to scale transformation. Let $\psi^{(q)}$ denote the increasing eigenfunction $\frac{d}{dm}\frac{d}{ds}\psi^{(q)} = q\psi^{(q)}$ such that $\frac{d}{ds}\psi^{(q)}(0) = 1$. In other words, the $\psi^{(q)}$ is the unique solution of the integral equation

$$\psi^{(q)}(x) = s(x) + q \int_0^x (s(x) - s(y))\psi^{(q)}(y) dm(y), \quad x \in [0, \infty).$$

Then, by [11, Corollary 2.4], for $q > 0$ and $x \in (0, \infty)$, we have

$$\psi^{(q)}(x) = \frac{1}{n^X_0 [e^{-qT^+_x}; T^+_x < \infty]},$$

which shows that $W^{(q)}_{X}(x, 0) = \psi^{(q)}(x)$. In particular, we have $W^{(0)}_{X}(x, 0) = s(x)$.

We fix $b, a \in \mathbb{T}$ with $b < a$.

Theorem 3.4. For $q \geq 0$ and $x \in (b, a)$, we have

$$\mathbb{E}^X_x [e^{-qT^+_a}; T^+_a < T^+_b] = \frac{W^{(q)}_{X}(x, b)}{W^{(q)}_{X}(a, b)}.$$  \hspace{1cm} (3.4)

Proof. By the strong Markov property and since $X$ has no positive jumps, we have

$$n^X_b [e^{-qT^+_a}; T^+_a < \infty] = n^X_b [e^{-qT^+_a} \mathbb{E}_{T^+_a}^X [e^{-qT^+_x}; T^+_a < T^+_b]; T^+_x < \infty]$$

$$= n^X_b [e^{-qT^+_a}; T^+_x < \infty] \mathbb{E}^X_x [e^{-qT^+_x}; T^+_a < T^+_b]$$

and we obtain (3.4). \hfill $\square$

Lemma 3.5. For $q \geq 0$ and $x \in (b, a)$, we have

$$\mathbb{E}^X_x [e^{-qT^+_a}; T^+_a < T^+_b] = n^X_x [e^{-qT^+_a}; T^+_a < \infty] \mathbb{E}^X_x [\int_0^{T^+_a \wedge T^+_b} e^{-qL^X_t} dt].$$  \hspace{1cm} (3.7)
Proof. The proof is almost the same as that of [7, Lemma 6.1], but a slight difference lies in presence of stagnancy.

i) We assume that \( x \) is regular for itself.

Let \( p = \{ p(t) : t \in D(p) \} \) denote a Poisson point process with characteristic measure \( n^X_\cdot,\cdot \). We extend \( p(t) \) for all \( t \geq 0 \) by setting \( p(t) = \partial \) for \( t \notin D(p) \) where we abuse \( \partial \) for the path taking values in \( \partial \) for all time. For \( E \in \mathcal{B}(\mathbb{D}) \), we write \( \kappa_E = \inf\{ s \geq 0, p(s) \in E \} \).

We let \( A = \{ T^+_a < \infty \} \cup \{ T^-_b < \infty \} \cup \{ \zeta < \infty \} \). We denote by \( \epsilon^* = p(\kappa_A) \) the first excursion belonging to \( A \). Then we have

\[
\mathbb{E}^X_x \left[ e^{-qT^+_a} ; T^+_a < T^-_b \right] = \mathbb{E}^X_x \left[ e^{-qX^x(\kappa_A-)} e^{-qT^+_a(\epsilon^*)} ; T^+_a(\epsilon^*) < T^-_b(\epsilon^*) \right].
\] (3.8)

Since \( X \) has no positive jumps, we have \( T^+_a(\epsilon^*) = \infty \) and \( e^{-qT^+_a(\epsilon^*)} = 0 \) on \( \{ T^+_a(\epsilon^*) > T^-_b(\epsilon^*) \} \). So we have

\[
\mathbb{E}^X_x \left[ e^{-qX^x(\kappa_A-)} e^{-qT^+_a(\epsilon^*)} ; T^+_a(\epsilon^*) < T^-_b(\epsilon^*) \right] = \mathbb{E}^X_x \left[ e^{-qX^x(\kappa_A-)} e^{-qT^+_a(\epsilon^*)} \right]
\] (3.9)

By the renewal property of the Poisson point process, we have

\[
b\mathbb{E}^X_x \left[ e^{-qX^x(\kappa_A-)} e^{-qT^+_a(\epsilon^*)} \right] = \mathbb{E}^X_x \left[ e^{-qX^x(\kappa_A-)} \right] \frac{n^X_\cdot \left[ e^{-qT^+_a} ; A \right]}{n^X_\cdot \left[ A \right]}. \tag{3.10}
\]

We denote \( D(p_A) = \{ s \in D(p) : p(s) \in A \} \) and \( p_A = p|_{D(p_A)} \) and extend \( p_A(t) \) for all \( t \geq 0 \) by setting \( p_A(t) = \partial \) for \( t \notin D(p_A) \). For \( q > 0 \), let \( e_q \) be an independent random variable which has the exponential distribution with intensity \( q \). We write \( \eta^X_{A^c}(s) = \eta^X_{A^c}(s) - \sum_{0 \leq s \leq T_x(p_A(u))} T_x(\partial) = 0. \) Since \( \eta^X_{A^c}(\kappa_A-) = \eta^X_{A^c}(\kappa) \) where \( \eta^X_{A^c} \) and \( \kappa_A \) are independent, we have

\[
\mathbb{E}^X_x \left[ e^{-qX^x(\kappa_A-)} \right] = \mathbb{E}^X_x \left[ \exp \left( -q\eta^X_{A^c}(e_{n^X_\cdot[A]}) \right) \right] \tag{3.11}
\]

\[
= n^X_\cdot [A] \mathbb{E}^X_x \left[ \int_0^\infty \exp \left( -ln^X_\cdot[A] \right) \exp \left( -q\eta^X_{A^c}(t) \right) dt \right] \tag{3.12}
\]

\[
= n^X_\cdot [A] \mathbb{E}^X_x \left[ \int_0^{e_{n^X_\cdot[A]}} \exp \left( -q\eta^X_{A^c}(t) \right) dt \right] \tag{3.13}
\]

\[
= n^X_\cdot [A] \mathbb{E}^X_x \left[ \int_0^{\kappa_A} \exp \left( -q\eta^X_{A^c}(t) \right) dt \right] \tag{3.14}
\]

\[
= n^X_\cdot [A] \mathbb{E}^X_x \left[ \int_{0^-}^{T^+_a \wedge T^-_b} e^{-qt} dL^X_t \right]. \tag{3.15}
\]

Therefore we obtain (3.7).

ii) We assume that \( z \) is irregular for itself.
Let $T^{(n)}_x$ denote the $n$-th hitting time to $x$ and let $T^{(0)}_x = 0$. Then we have
\[
\mathbb{E}_X \left[ \int_{0^-}^{T^{+}_a \wedge T^{+}_b} e^{-qt} dL^X_{t,x} \right] = t^{X}_x \sum_{i=0}^{\infty} \mathbb{E}_X \left[ e^{-qT^{(i)}_x}; T^{(i)}_x < T^{+}_a \wedge T^{+}_b \right] = t^{X}_x \sum_{i=0}^{\infty} \left( \mathbb{E}_X \left[ e^{-qT^{+}_x}; T^{+}_x < T^{+}_a \wedge T^{+}_b \right] \right)^i. \tag{3.16}
\]

On the other hand, we have
\[
\mathbb{E}_X \left[ e^{-qT^{+}_a}; T^{+}_a < T^{+}_b \right] = \sum_{i=0}^{\infty} \left( \mathbb{E}_X \left[ e^{-qT^{+}_x}; T^{+}_x < T^{+}_a \wedge T^{+}_b \right] \right)^i \mathbb{E}_X \left[ e^{-qT^{+}_a}; T^{+}_a < T^{+}_x \wedge T^{+}_b \right]. \tag{3.17}
\]

Therefore we obtain (3.7). \hfill \Box

By Theorem 3.4 and Lemma 3.5 for $q \geq 0$ and $x \in (b,a)$, we obtain
\[
\mathbb{E}_X \left[ \int_{0^-}^{T^{+}_a \wedge T^{+}_b} e^{-qt} dL^X_{t,x} \right] = \frac{W^{(q)}_X (x,b) W^{(q)}_X (a,x)}{W^{(q)}_X (a,b)}. \tag{3.18}
\]

For $q \geq 0$, $x \in (b,a)$ and non-negative measurable function $f$, we define
\[
\overline{P}^{(q;b,a)}_X f(x) := \mathbb{E}_X \left[ \int_0^{T^{+}_a \wedge T^{+}_b} e^{-qt} f(X_t) dt \right]. \tag{3.19}
\]

Then, for $q \geq 0$, we have
\[
\overline{P}^{(q;b,a)}_X f(x) = \int_{(b,a)} f(y) \mathbb{E}_X \left[ \int_0^{T^{+}_a \wedge T^{+}_b} e^{-qt} dL^X_{t,y} \right] m(dy). \tag{3.20}
\]

**Theorem 3.6.** For $q \geq 0$ and $x, y \in (b,a)$, we have
\[
\mathbb{E}_X \left[ \int_0^{T^{+}_a \wedge T^{+}_b} e^{-qt} dL^X_{t,x,y} \right] = \frac{W^{(q)}_X (x,b) W^{(q)}_X (a,y)}{W^{(q)}_X (a,b)} - W^{(q)}_X (x,y). \tag{3.21}
\]

**Proof.** i) We assume that $x = y$.

When $x$ is regular for itself, we have
\[
\mathbb{E}_X \left[ \int_0^{T^{+}_a \wedge T^{+}_b} e^{-qt} dL^X_{t,x} \right] = \mathbb{E}_X \left[ \int_0^{T^{+}_a \wedge T^{+}_b} e^{-qt} dL^X_{t,x} \right] \tag{3.22}
\]
and
\[
W^{(q)}_X (x, x) = 0. \tag{3.23}
\]
By (3.19), we obtain (3.22).

When $x$ is irregular for itself, we have
\[
E^X \left[ \int_0^{T_b^+ \wedge T_a^+} e^{-qt} dL^X_t \right] = E^X \left[ \int_0^{T_b^+ \wedge T_a^+} e^{-qt} dL_t \right] - l^X_x. \tag{3.25}
\]

By (3.19) and Remark 3.2, we obtain (3.22).

\(\text{ii)}\) We assume that $x < y$.

By the strong Markov property, we have
\[
E^X \left[ \int_0^{T_b^+ \wedge T_a^+} e^{-qt} dL^X_t \right] = E^X \left[ e^{-qT_y}; T_y < T_b^+ \right] E^X \left[ \int_0^{T_b^+ \wedge T_a^+} e^{-qt} dL_t \right]. \tag{3.26}
\]

So by Theorem 3.4 and (3.19), we have
\[
(3.26) = \frac{W^{(q)}_X(x, b) W^{(q)}_X(y, b) W^{(q)}_Y(a, y)}{W^{(q)}_X(y, b) W^{(q)}_Y(a, b)} \tag{3.27}
\]
and we obtain (3.22).

\(\text{iii)}\) We assume that $x > y$.

By the strong Markov property, we have
\[
E^X \left[ \int_0^{T_b^+ \wedge T_a^+} e^{-qt} dL^X_t \right] = E^X \left[ e^{-qT_y}; T_y < T_b^+ \wedge T_a^+ \right] E^X \left[ \int_0^{T_b^+ \wedge T_a^+} e^{-qt} dL_t \right]. \tag{3.28}
\]

Since
\[
E^X \left[ e^{-qT_y}; T_y < T_a^+ < T_b^+ \right] = E^X \left[ e^{-qT_y}; T_y < T_b^+ \wedge T_a^+ \right] E^X \left[ e^{-qT_y}; T_a^+ < T_b^+ \right], \tag{3.29}
\]
we have
\[
(3.29) = \frac{E^X \left[ e^{-qT_y}; T_y < T_a^+ < T_b^+ \right]}{E^X \left[ e^{-qT_y}; T_a^+ < T_b^+ \right]}
\]
\[
= \frac{E^X \left[ e^{-qT_y}; T_a^+ < T_b^+ \right] - E^X \left[ e^{-qT_y}; T_a^+ < T_y \right]}{E^X \left[ e^{-qT_y}; T_a^+ < T_b^+ \right]}
\]
\[
= \frac{E^X \left[ e^{-qT_y}; T_a^+ < T_b^+ \right]}{E^X \left[ e^{-qT_y}; T_a^+ < T_b^+ \right]} \tag{3.30}
\]
\[
= \frac{E^X \left[ e^{-qT_y}; T_a^+ < T_b^+ \right]}{E^X \left[ e^{-qT_y}; T_a^+ < T_b^+ \right]} \tag{3.31}
\]

By Theorem 3.4 and (3.19), we have
\[
(3.31) = \frac{W^{(q)}_X(a, b) W^{(q)}_Y(x, b)}{W^{(q)}_X(y, b) W^{(q)}_Y(a, b)} \tag{3.32}
\]
and we obtain (3.22).
Corollary 3.7. For $x, y \in (b_0, a_0)$, we define

$$Z^q_X(x, y) = \begin{cases} 1 + q \int_{(y,x)} W^q_X(x, z)m(dz), & x > y, \\ 1, & x \leq y. \end{cases} \tag{3.33}$$

Then we have

$$E^X_x\left[e^{-q_{T_b^-}}; T_b^- < T_a^+\right] = Z^q_X(x, b) - \frac{W^q_X(x, b)}{W^q_X(a, b)} Z^q_X(a, b). \tag{3.34}$$

Proof. We have

$$E^X_x\left[e^{-q_{T_b^-}}; T_b^- < T_a^+\right] = E^X_x\left[e^{-q(T_b^- \wedge T_a^+)}; T_b^- \wedge T_a^+ < \infty\right] - E^X_x\left[e^{-q T_a^+}; T_a^+ < T_b^-\right]. \tag{3.35}$$

By Theorem 3.6, we have

$$E^X_x\left[e^{-q(T_b^- \wedge T_a^+)}; T_b^- \wedge T_a^+ < \infty\right] = 1 - q E^X_x\left[\int_0^{T_b^- \wedge T_a^+} e^{-qt} dt\right] \tag{3.36}$$

$$= 1 - q \int_{(b,a)} \left(\frac{W^q_X(x, b)}{W^q_X(a, b)} W^q_X(a, y) - W^q_X(x, y)\right) m(dy). \tag{3.37}$$

By (3.35), (3.37) and Theorem 3.4, we have

$$\tag{3.35} = 1 + q \int_{(b,a)} W^q_X(x, y)m(dy) - \frac{W^q_X(x, b)}{W^q_X(a, b)} \left(1 + q \int_{(b,a)} W^q_X(a, y)m(dy)\right), \tag{3.38}$$

and therefore we obtain (3.34). \qed

4 Relation between duality and scale functions

In this section, we characterize duality of standard processes in terms of our scale functions.

Let $X$ be a $T$-valued standard process with no positive jumps satisfying (A1) and (A2). Let ($\hat{X}, \hat{P}^\hat{X}_x$) with $\hat{X} = \{\hat{X}_t : t \geq 0\}$ be a $T$-valued standard process with no negative jumps satisfying the following conditions:

(B1) $(x, y) \mapsto E^X_x[\hat{e}^{-T_y}] > 0$ is a $B(T) \times B(T)$-measurable function.

(B2) $\hat{X}$ has a reference measure $m$ on $T$. 

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For $q \geq 0$ and non-negative measurable function $f$, we denote
\[ R^{(q)}_X f(x) = \mathbb{E}_x^\hat{X} \left[ \int_0^\infty e^{-qt} f(\hat{X}_t)dt \right]. \] (4.1)

We define local times $\{L^{\hat{X}}_{x,t}\}_{x \in \mathbb{T}}$, excursion measures $\{n^\hat{X}_x\}_{x \in \mathbb{T}}$ and scale functions $\{W_{-\hat{X}}^{(q)}\}_{q \geq 0}$ of $\hat{X}$ by the same way as $X$’s in Section 3.

**Definition 4.1** (See, e.g., [2]). Let $m$ be a $\sigma$-finite Radon measure on $\mathbb{T}$. We say that $X$ and $\hat{X}$ are in duality (relative to $m$) if for $q > 0$, non-negative measurable functions $f$ and $g$,
\[ \int_{\mathbb{T}} f(x)R^{(q)}_X g(x)m(dx) = \int_{\mathbb{T}} R^{(q)}_X f(x)g(x)m(dx). \] (4.2)

**Theorem 4.2** (See, e.g., [2] or [9]). Suppose $X$ and $\hat{X}$ be in duality relative to $m$. Then, for each $q > 0$, there exists a function $r^{(q)}_X : \mathbb{T} \times \mathbb{T} \to [0, \infty)$ such that

(i) $r^{(q)}_X$ is $\mathcal{B}(\mathbb{T}) \times \mathcal{B}(\mathbb{T})$-measurable.

(ii) $x \mapsto r^{(q)}_X(x,y)$ is $q$-excessive and finely continuous for each $y \in \mathbb{T}$.

(iii) $y \mapsto r^{(q)}_X(x,y)$ is $q$-coexcessive and cofinely continuous for each $x \in \mathbb{T}$.

(iv) For all non-negative function $f$,
\[ R^{(q)}_X f(x) = \int_{\mathbb{T}} f(y)r^{(q)}_X(x,y)m(dy), \quad R^{(q)}_X f(y) = \int_{\mathbb{T}} f(x)r^{(q)}_X(x,y)m(dx). \] (4.3)

**Remark 4.3.** We suppose that $X$ and $\hat{X}$ are in duality. Then if a function $f$ is finely continuous at $x \in \mathbb{T}$, then $f$ is right continuous at $x$. If $f$ is cofinely continuous at $x \in \mathbb{T}$, then $f$ is left continuous at $x$.

**Remark 4.4.** We suppose that $X$ and $\hat{X}$ are in duality relative to $m$. Then, by the proof of Lemma 4.7 and Theorem 3.6 for $x, y \in (b_0, a_0)$, the function $x \mapsto W^{(q)}(x,y)$ is finely continuous and $y \mapsto W^{(q)}(x,y)$ is cofinely continuous.

By [9] Proposition of Section V.1], if $X$ and $\hat{X}$ are in duality relative to $m$, there exist local times $\{L^X_{x,t}\}_{x \in \mathbb{T}}$ of $X$ and $\{L^{\hat{X}}_{x,t}\}_{x \in \mathbb{T}}$ of $\hat{X}$ satisfying
\[ \mathbb{E}_x^X \left[ \int_0^\infty e^{-qt} dL^X_{i,t} \right] = r^{(q)}_X(x,y), \quad \mathbb{E}_y^{\hat{X}} \left[ \int_0^\infty e^{-qt} d\hat{L}^x_{i,t} \right] = r^{(q)}_X(x,y). \] (4.4)

for all $q > 0$. When $X$ and $\hat{X}$ are in duality, we always use the normalization of the local times above. In other cases, we use the normalization of the local times in Section 3.
Theorem 4.5. If $X$ and $\hat{X}$ are in duality relative to $m$, then we have

$$W_X^{(q)}(x,y) = W_{-\hat{X}}^{(q)}(-y,-x), \quad x,y \in (b_0,a_0).$$

(4.5)

If $\mathbb{T}$ is open, then the converse is also true.

Remark 4.6. If $X$ is a spectrally negative Lévy process, we have that $\hat{X} = -X$ is a dual process relative to $m$ when $m$ is the Lebesgue measure. By stationary independent increments and the definition of generalized scale functions, we have

$$W_X^{(q)}(x,y) = W^{(q)}(x-y) = W_{-\hat{X}}^{(q)}(-y,-x),$$

(4.6)

where $W^{(q)}$ is the right continuous function defined by (1.2). We thus obtain (4.5).

Lemma 4.7. Suppose $X$ and $\hat{X}$ be in duality relative to $m$. Then, for all $b < a \in \mathbb{T}$ and $x,y \in (b,a)$, we have

$$E_x\left[ \int_0^{T_b \wedge T_a^+} e^{-qt} dL_t^{X,y} \right] = E_{\hat{X}}\left[ \int_0^{T_b \wedge T_a^+} e^{-qt} dL_t^{\hat{X},x} \right].$$

(4.7)

Proof. Let $X^{(b,a)}$ and $\hat{X}^{(b,a)}$ denote the $X$ and $\hat{X}$ killed on exiting $(b,a)$, respectively. We denote by $R^{(q)}_{X^{(b,a)}}$ and $R^{(q)}_{\hat{X}^{(b,a)}}$ the $q$-resolvent operators of $X^{(b,a)}$ and $\hat{X}^{(b,a)}$, respectively. For each $q > 0$, there exists a function $r^{(q)}_{X^{(b,a)}} : (b,a) \times (b,a) \rightarrow [0,\infty)$ such that

(i) $r^{(q)}_{X^{(b,a)}}$ is $\mathcal{B}(b,a) \times \mathcal{B}(b,a)$-measurable.

(ii) $x \mapsto r^{(q)}_{X^{(b,a)}}(x,y)$ is $q$-excessive and finely continuous for each $y \in (b,a)$.

(iii) $y \mapsto r^{(q)}_{X^{(b,a)}}(x,y)$ is $q$-coexcessive and cofinely continuous for each $x \in (b,a)$.

(iv) For all non-negative function $f$,

$$R^{(q)}_{X^{(b,a)}} f(x) = \int_{(b,a)} f(y) r^{(q)}_{X^{(b,a)}}(x,y) m(dy), \quad R^{(q)}_{\hat{X}^{(b,a)}} f(y) = \int_{(b,a)} f(x) r^{(q)}_{\hat{X}^{(b,a)}}(x,y) m(dx).$$

(4.8)

By definition, we have

$$R^{(q)}_{X^{(b,a)}} f(y) = E_x^{X^{(b,a)}} f(y) = \int_{(b,a)} f(y) E_x^{X^{(b,a)}} \left[ \int_0^{T_b \wedge T_a^+} e^{-qt} dL_t^{X,y} \right] m(dy).$$

(4.9)

So, for all $x \in (b,a)$, we have $r^{(q)}_{X^{(b,a)}}(x,\cdot) = E_x^{X^{(b,a)}} \left[ \int_0^{T_b \wedge T_a^+} e^{-qt} dL_t^{X,y} \right]$, m.a.e. Since

$$E_x^{X^{(b,a)}} \left[ \int_0^{T_b \wedge T_a^+} e^{-qt} dL_t^{X,y} \right] = E_x^{\infty} e^{-qt} dL_t^{X,y} - E_x^{X^{(b,a)}} e^{-q(T_b \wedge T_a^+)} E_x^{X^{(b,a)}} \left[ \int_0^{\infty} e^{-qt} dL_t^{X,y} \right],$$

(4.10)
and the dominated convergence theorem, the function \( y \mapsto \mathbb{E}_x^X \left[ \int_0^{T^-_X + T^+_X} e^{-qt} dL_t^X \right] \) is a cofinely continuous function. By cofine continuity, for all \( x, y \in (b, a) \), we have

\[
\frac{r_{X,(b,a)}^{(q)}(x,y)}{r_{X,(a,b)}^{(q)}(a,b)} = \frac{W_{-X}^{(q)}(-y,-a)}{W_{-X}^{(q)}(-b,-a)} \frac{W_{-X}^{(q)}(-b,-x)}{W_{-X}^{(q)}(-b,-a)}. \tag{4.11}
\]

By the same way, we have

\[
\frac{r_{X,(b,a)}^{(q)}(x,y)}{r_{X,(a,b)}^{(q)}(a,b)} = \mathbb{P}_y^X \left[ \int_0^{T^-_X + T^+_X} e^{-qt} dL_t^{\hat{X},x} \right]. \tag{4.12}
\]

By (4.11) and (4.12), we obtain (4.7). \( \square \)

**Proof of Theorem 4.5.** Let us assume that \( X \) and \( \hat{X} \) are in duality relative to \( m \). First, we fix \( b, y, a \in \mathbb{T} \) with \( b < y < a \). By Lemma 4.7 and Theorem 3.6, for all \( q \geq 0 \) and \( x \in (b, y) \), we have

\[
\frac{W_X^{(q)}(x,b)}{W_X^{(q)}(a,b)} W_{X}^{(q)}(a,y) = \frac{W_X^{(q)}(-y,-a)}{W_{-X}^{(q)}(-b,-a)} \frac{W_{-X}^{(q)}(-b,-x)}{W_{-X}^{(q)}(-b,-a)}. \tag{4.13}
\]

Here there exists a function \( \gamma_1 : [0, \infty) \times \mathbb{T} \to (0, \infty) \) satisfying

\[
W_X^{(q)}(x,b) = \gamma_1(q,b) W_X^{(q)}(-b,-x) \quad x \in (b, a_0). \tag{4.14}
\]

Second, we fix \( b, x, a \in \mathbb{T} \) with \( b < x < a \). For \( q \geq 0 \) and \( y \in (x, a) \), we have (4.13). Thus there exists a function \( \gamma_2 : [0, \infty) \times \mathbb{T} \to (0, \infty) \)

\[
W_X^{(q)}(a,y) = \gamma_2(q,a) W_{-X}^{(q)}(-y,-a) \quad y \in (b_0, a). \tag{4.15}
\]

By (4.14) and (4.15), for \( q \geq 0 \) and \( a, b \in (b_0, a_0) \), we have \( \gamma_1(q,b) = \gamma_2(q,a) \), so \( \gamma_1 \) and \( \gamma_2 \) depend on only \( q \geq 0 \). By (4.13), \( \gamma_1 = \gamma_2 = 1 \). Thus, for \( y, x \in (b_0, a_0) \) with \( y < x \), we have \( W_X^{(q)}(x,y) = W_X^{(q)}(-y,-x) \). By the fine continuity of \( W_X^{(q)} \) and the cofine continuity \( W_{-X}^{(q)} \), for \( x \in (b_0, a_0) \), we have \( W_X^{(q)}(x,x) = W_X^{(q)}(-x,-x) \).

Let us assume that \( \mathbb{T} \) is open and that (4.15) is satisfied. Then, for \( b < a \in \mathbb{T} \) and \( x, y \in (b, a) \), we have

\[
\frac{W_X^{(q)}(x,b)}{W_X^{(q)}(a,b)} W_X^{(q)}(a,y) - W_X^{(q)}(x,y) = \frac{W_X^{(q)}(-y,-a)}{W_{-X}^{(q)}(-b,-a)} W_{-X}^{(q)}(-b,-x) - W_{-X}^{(q)}(-y,-x). \tag{4.16}
\]

By Theorem 3.6, the first term and the second term of (4.16) are potential densities of \( X \) and \( \hat{X} \) killed on exiting \((b, a)\), respectively. We therefore conclude the duality of killed processes, which yields that of the original processes. \( \square \)
The case of spectrally negative Lévy processes

When $X$ is a spectrally negative Lévy process, the usual definition of the scale function $W(q)(x)$ is based on the Laplace transform

$$
\int_0^\infty e^{-qt} W(q)(x) dx = \frac{1}{\Psi(\beta) - q}, \quad \beta > \Phi(q).
$$

It satisfies

$$
W(q)(x) = e^{\Phi(q)x} r(q)(0+) - r(q)(-x)
$$

where $r(q)$ is the right-continuous potential density of $X$ with respect to the Lebesgue measure (see [8]) and, as we have mentioned it in Section 1, it also satisfies

$$
W(q)(x) = \frac{1}{m_X \left[ e^{-qT_x^+} ; T_x^+ < \infty \right]}.
$$

Pistorius([8]) provided a potential theoretic viewpoint for the scale functions in the sense that he started from (A.1) and proved (1.2). We now provide another viewpoint.

Let $X$ be a spectrally negative Lévy process and $m$ are the Lebesgue measure. By Section 3 and 4, we can define local times, excursion measures and scale functions. Since $X$ has stationary independent increment property, for $q \geq 0$ and $x, y \in \mathbb{R}$, we have

$$
E^X_0 \left[ \int_0^\infty e^{-qt} dL_t^{X,y-x} \right] = E^X_0 \left[ \int_0^\infty e^{-qt} dL_t^{X,y} \right].
$$

So there exists a left continuous function $r(q) : \mathbb{R} \to [0, \infty)$ and a right continuous function $W(q) : \mathbb{R} \to [0, \infty)$ satisfying

$$
r(q)(y-x) = E^X_x \left[ \int_0^\infty e^{-qt} dL_t^{X,y} \right], \quad x, y \in \mathbb{R},
$$

$$
W(q)(x-y) = W(q)(x, y), \quad x, y \in \mathbb{R},
$$

for all $q \geq 0$. Then $r(q)$ is a càdlàg function. In fact, we have

$$
\lim_{h \downarrow 0} r(q)(x+h) = \lim_{h \downarrow 0} E^X_{-h} \left[ \int_0^\infty e^{-qt} dL_t^{X,x} \right]
$$

$$
= \lim_{h \downarrow 0} E^X_{-h} \left[ e^{-qT_0^+} ; T_0^+ \leq T_x \right] E^X_0 \left[ \int_0^\infty e^{-qt} dL_t^{X,x} \right]
$$

$$
+ \lim_{h \downarrow 0} E^X_{-h} \left[ e^{-qT_x} ; T_x < T_0^+ \right] E^X_x \left[ \int_0^\infty e^{-qt} dL_t^{X,x} \right]
$$

$$
= E^X_0 \left[ \int_0^\infty e^{-qt} dL_t^{X,x} \right]
$$

where (A.7) uses

$$
E^X_0 \left[ e^{-qT_x^+} ; T_x^+ < \infty \right] = e^{-\Phi(q)x}, \quad x > 0,
$$

and $\lim_{x \downarrow 0} E^X_0 \left[ e^{-qT_x^+} ; T_x^+ < \infty \right] = \lim_{x \downarrow 0} e^{-\Phi(q)x} = 1$ (see, e.g., [5, Theorem 3.2]).
Theorem A.1. For all $q \geq 0$, we have (1.2).

Proof. By the equation obtained from (3.7) when $b$ limits to infinity, for $x > 0$, we have

$$W(q)(x) = \frac{1}{n_0^X[e^{-qT_x^+}; T_x^+ < \infty]}$$

(A.9)

By (A.8), for $\beta > \Phi(q)$, we have

$$\int_0^\infty e^{-\beta x} W(q)(x) dx = \int_0^\infty \left( e^{-\beta \Phi(q)} x \right) (0+) - e^{-\beta x} r(q)(-x) \right) dx$$

(A.13)

By analytic extension, we have (A.20) for $\beta > \Phi(q)$. By (A.14) and (A.20), we obtain (1.2).

Remark A.2. The latter part of the proof of Theorem A.1 is almost the same as a part of the proof of [8, Theorem 1(i)–(iii)].
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