A flat FRW model with dynamical $\Lambda$ as function of matter and geometry

Anirudh Pradhan$^1$, De Avik $^2$, Tee How Loo$^3$, Dinesh Chandra Maurya$^4$

$^1$Department of Mathematics, Institute of Applied Sciences & Humanities, GLA University, Mathura -281 406, Uttar Pradesh, India

$^2$Department of Mathematical and Actuarial Sciences, Universiti Tunku Abdul Rahman, Jalan Sungai Long, 43000 heras, Malaysia

$^3$Institute of Mathematical Sciences, University of Malaya, 50603 Kuala Lumpur, Malaysia

$^4$Department of Mathematics, Faculty of Engineering & Technology, IASE (Deemed to be University), Sardarshahar-331 403, Rajsthan, India

1Email:pradhan.anirudh@gmail.com

2Email:de.math@gmail.com

3Email:looth@um.edu.my

4Email:dcmaurya563@gmail.com

Abstract

We revisit the evolution of the scale factor in a flat FRW spacetime with a new generalized decay rule for the dynamic $\Lambda$-term under modified theories of gravity. It analyses certain cosmological parameters and examines their behaviours in this generalized setting which includes several decay laws in the literature. We have also obtained observational constraints on various model parameters and estimated the present values of cosmological parameters $\{\Omega_m, \Omega_\Lambda, q_0, t_0, \omega_0\}$ and have discussed with various observational results. Finite time past and future singularities in this model are also discussed.

Keywords FLRW universe, Modified gravity, Dynamical cosmological constant, Observational parameters

PACS No.: 98.80.Jk; 98.80.-k; 04.50.-h

1 Introduction

The idea of the cosmological constant $\Lambda$ was first introduced by Einstein in 1917 to obtain a static universe because of Einstein’s field equations in general relativity were showing a non-static universe i.e. the universe has either contracting or expanding nature and at that time observational studies supported an static universe due to presence limited observational data sets. Therefore, he revised his original field equations proposed in 1915 to remove this feature of size-changing. But the discovery of Hubble in 1929 suggested that our universe is in expanding nature which was already obtained by the Friedmann-Lamatire-Rabortson-Walker in 1922 in their solution of Einstein’s field equations using the same metric and this finding was once removed the newly added cosmological term $\Lambda$.

After the discovery of accelerating universe in 1998, the cosmological term came back in its place but at this time it did not prevent the expanding nature of universe, but in order to explain why it did not continues expanding nature with a constant rate. Although it should be possible for the known amount of matter present in the universe to exert adequate gravitational force to slow down the expansion, initiated with Big Bang. However, modern studies of standard candles showed that instead of decreasing, the rate at which it is expanding actually increases. The new generation of cosmologists proposes the existence of an
unknown, exotic, homogeneous energy density with high negative pressure referred to as dark energy (DE) to explain these observational findings. And the idea of cosmological term $\Lambda$ is the simplest contributing factor of this dark energy.

We still face the critical question of whether $\Lambda$ is always a fundamental constant or a mild dynamical variable constant after almost two decades. It turns out that the selection of $\Lambda$ as dark energy faces some serious issues in terms of (a) the “old” cosmological constant problem (or fine-tuning problem) and (b) the cosmic coincidence problem, if we consider it as a constant rather than a specific dynamical variable. Moreover, the overall fit to the cosmological observables SNIa+BAO+H(z)+LSS+BBN+CMB has also been shown to support the class of unique dynamic models of the so-called fundamental constant $\Lambda$ as well. Dynamic model of $\Lambda$ may ultimately be necessary not only as a new epitome to improve the EFE, but also phenomenologically to unwind a number of tensions between the concordance model and the observation that is indicating towards DE. There are innumerable $\Lambda(t)$ laws available in the literature although all of which might not be viable under theoretical and observational ground and they might not follow covariant action, but they are still interesting to study from phenomenological ground. For example, Carvalho et al. [11] and Waga [2] first considered the case $\Lambda \propto \left(\frac{a}{t}\right)^2$ as a generalization of Chen and Wu’s result advocating the possibility that the (effective) cosmological constant $\Lambda$ varies in time as $a^{-2}$. The case $\Lambda \propto \rho$ was first studied by Viswakarma [3] and the case $\Lambda \propto \frac{\dot{\rho}}{\rho}$ by Arbab [4], Ray et al. [5] and Viswakarma [6] showed that if we consider these three cases separately, the free parameters satisfy certain relations. Recently, Pan [7] investigated dynamical $\Lambda$ as a function of several form of polynomials of $H$ and $\dot{H}$, discussing the finite time future singularities for each case.

Although several studies being done on these particular phenomenological models in the past, with more powerful observational analysis of the present days, most of these are detected to be suffering from certain theoretical and observational limitations. The $\Lambda \propto H^2$ and $\Lambda \propto \dot{H}$ cases are largely overruled now since the corresponding linear growth of cosmic perturbations is strongly disfavored by the observational data [8, 9]. If $\Lambda \propto \rho$ alone, one can trivially absorb it to the constant and hence there will be no dark energy scenario in the cosmological evolution. This is indeed very clear for a perfect fluid with equation of state $p = \omega \rho$. The introduction of $\Lambda \propto \rho$ effectively modifies the proportionality constant $\omega$ and the model becomes equivalent to one with merely a new fluid only. We tried to overcome this issue by introducing a new decay law which combines both matter and geometry into the dynamical cosmological term. Moreover, we can show that a motivation towards the present decay law of $\Lambda$ can be drawn from the f(R)-gravity or equivalently from Brans-Dicke scalar-tensor theory derived from a covariant action.

In cosmology, the fundamental variables are either constructed from a spacetime metric directly (geometrical), or depend upon properties of physical fields. Naturally the physical variables are model-dependent, whereas the geometrical variables are more universal. The most popular geometric variables are the Hubble constant $H$ and the deceleration parameter $q$ whose values depend on the first and second time derivatives of the scale factor, respectively. We use these two parameters extensively to resolve the mystery of the inflationary universe. However, a more sensitive discrimination of the expansion rate and hence dark energy can be performed by considering a third time derivative in the general form for the scale factor of the Universe

$$a(t) = a(t_0) + \dot{a}(t_0)(t-t_0) + \frac{\ddot{a}(t_0)}{2}(t-t_0)^2 + \frac{\dddot{a}(t_0)}{3}(t-t_0)^3 + \ldots$$

This led Sahni et al. [10] and Alam et al. [11] to introduce the statefinder parameters, a dimensionless pair $\{r,s\}$ from the scale factor and its third order time derivatives in concordance cosmology. In [12] this result is generalized to show that the hierarchy, $A_n = \frac{a^{(n)}}{2H^n}$, can take into account the deceleration parameter, the statefinder parameters, the snap parameter, the jerk parameter etc and further it can be expressed in terms of the deceleration parameter or the matter density parameter.

In observational cosmology, the beginning of the 21st century saw yet another significant milestone. BOOMERanG in the year 2000 collaboration (Balloon Measurements of Millimetric Extragalactic Radiation and Geophysics), a research on cosmic microwave history (CMB) a cosmos of flat spatial geometry ([13], [14]) was recorded using balloon-borne instruments. The MAXIMA (Millimeter Anisotropy Experiment Imaging Array) collaboration [15, 16], DASI (Degree Angular Scale Interferometer) [17], CBI (Cosmic Background
Imager) \cite{18} and WMAP (Wilkinson Microwave Anisotropy Probe) \cite{19} have also reported a similar result. These observations were an significant landmark in modern cosmology; assuming astronomy’s value of $\Omega M \sim 0.3$, the data directly pointed to a positive cosmological constant with a contribution of energy density of the order of $\Omega \Lambda \sim 0.7$. The observations from the supernova probes and the Hubble Space Telescope were ideally matched. The precise values of these parameters $\Omega M$ and $\Omega \Lambda$ as obtained by Sievers et al. \cite{18} and Spergel et al. \cite{20} are $[0.34 \pm 0.12, 0.67^{+0.10}_{-0.13}]$ and $[0.249^{+0.024}_{-0.031}, 0.719^{+0.021}_{-0.029}]$, respectively. More recently, the Lyman-α forest measurement of the baryon acoustic oscillations (BAO) by the Baryon Oscillation Spectroscopic Survey preferred an even smaller value of the matter density $\Omega_M$ than that obtained by cosmic microwave background data \cite{21}. We show that our model perfectly adapts to these observational data.

The present paper is organized in the following format: a brief review of literature is given in section-1, section-2 contains field equations with cosmological term $\Lambda$ and some specific cosmological solutions like scale factor $a(t)$, Hubble function $H(t)$, energy density parameters $\Omega_m$ and $\Omega_\Lambda$. An observational constraints on various model parameters using available observational data sets like $H(z)$, union 2.1 compilation of SNe Ia data sets and Joint Light Curve Analysis (JLA) etc. by applying $R^2$-test formula are obtained and discussed with various observational results in section-3. Some finite time singularities of the current model are discussed in the section-4 and in section-5 we have discussed some more cosmological parameters like statefinder with deceleration parameters. Finally conclusions are given in section-6.

## 2 Field equations with $\Lambda(t)$

We consider an action \cite{22}

$$S = \frac{1}{16\pi G} \int f(R)\sqrt{-g}d^4x + \int L_m\sqrt{-g}d^4x,$$

where $f(R)$ is an arbitrary function of the Ricci scalar $R$, $L_m$ is the matter Lagrangian density, and we define the stress-energy tensor of matter as $T_{\mu\nu} = -\frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}L_m.

Assuming that the Lagrangian density of matter $L_m$ depends only on the components of the metric tensor $g_{\mu\nu}$ and not on its components, we obtain derivatives,

$$T_{\mu\nu} = g_{\mu\nu}L_m - 2\frac{\partial L_m}{\partial g^{\mu\nu}}.$$

By varying the action $S$ of the gravitational field with respect to the metric tensor components $g^{\mu\nu}$ and using the least action principle we obtain the field equation

$$f_R(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_{\mu}\nabla_{\nu})f_R(R) = 8\pi G T_{\mu\nu},

(1)$$

where $\Box = \nabla^2 = \nabla_{\mu}\nabla^{\mu}$ represents the d’Alembertian operator and $f_R = \frac{\partial f(R)}{\partial R}$. EFE can be reawakened by putting $f(R) = R$. The contracted equation is given by

$$R f_R(R) - 2f(R) + 3\Box f_R(R) = 8\pi G T,$n

(2)

which transforms (1) to

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} + \frac{1}{6f_R(R)}[2kT + f(R) + R f_R(R)]g_{\mu\nu} = 8\pi G T_{\mu\nu}^{\text{eff}},

(3)$$

$$T_{\mu\nu}^{\text{eff}} = \frac{1}{f_R(R)}[T_{\mu\nu} + \frac{1}{8\pi G}\nabla_{\mu}\nabla_{\nu}f_R(R)].$$

The Friedmann-Robertson-Walker (FRW) metric is given by the line element

$$ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right],

(4)$$
where the curvature parameter \( k = -1, 0, 1 \) for open, flat and closed models of the universe, respectively and \( a(t) \) is the scale factor shaping the universe. Since current observations from CMB detectors such as BOOMERanG, MAXIMA, DASI, CBI and WMAP confirm a spatially flat universe, \( k = 0 \). Hence the Ricci scalar becomes \( R = 6\left(\frac{\ddot{a}}{a^2} + \frac{\dot{a}^2}{a^3}\right) \). Comparing the equation (3) with the Einstein’s field equations with cosmological constant

\[
R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi GT_{\mu\nu},
\]

we conclude that the dynamic cosmological term \( \Lambda(t) \) is possibly a function of \( \frac{\ddot{a}}{a}, \left(\frac{\dot{a}}{a}\right)^2 \) and \( \rho \) and we consider the simplest model as a linear combination

\[
\Lambda = \frac{l}{a}\frac{\ddot{a}}{a} + \lambda \left(\frac{\dot{a}}{a}\right)^2 + 4\pi G\eta \rho,
\]

where \( l, \lambda \) and \( \delta \) are arbitrary constants.

As usual, the Friedmann and Raychaudhuri equations for flat FRW line element are given by

\[
3\frac{\dddot{a}}{a} + 4\pi G(1 + 3w)\rho = \Lambda,
\]

\[
\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 - 4\pi G(1 - w)\rho = \Lambda,
\]

where \( w = \frac{p}{\rho} \).

From Eqs. (7) and (8) we obtain

\[
\left(\frac{\dot{a}}{a}\right)^2 - \frac{8\pi G\rho}{3} = \frac{\Lambda}{3}.
\]

Using Eq. (6), in terms of Hubble parameter \( H = \frac{\dot{a}}{a} \), Eq. (9) transforms into

\[
3H^2 - 8\pi G\rho = l\dot{H} + (l + \lambda)H^2 + 4\pi G\eta \rho.
\]

Therefore,

\[
4\pi G\rho = -\frac{l}{2 + \eta}\dot{H} + \frac{3 - l - \lambda}{2 + \eta}H^2
\]

and

\[
\Lambda = \frac{2l}{2 + \eta}\dot{H} + \frac{2l + 2\lambda + 3\eta}{2 + \eta}H^2.
\]

From Eq. (8) we obtain

\[
\frac{\ddot{H}}{H^2} = -(1 + w)(3 - l - \lambda)
\]

which gives us

\[
H = \frac{H_0[(2 + \eta) - l(1 + w)]}{H_0(1 + w)(3 - l - \lambda)(t - t_0) + (2 + \eta) - l(1 + w)}.
\]

Integrating again, we obtain

\[
\frac{a(t)}{a_0} = \left[1 + \frac{H_0(1 + w)(3 - l - \lambda)}{(2 + \eta) - l(1 + w)}(t - t_0)\right]^{\frac{(2 + \eta) - l(1 + w)}{(1 + w)(3 - l - \lambda)}}.
\]

Therefore, the vacuum energy density parameter \( \Omega_\Lambda = \frac{\Lambda}{3H^2} \) and the cosmic matter density parameter \( \Omega_m = \frac{8\pi G\rho}{3H^2} \) are calculated as
sets) respectively as of observational constraints (Union 2.1 compilation of SNe Ia data sets and Joint light curve analysis data sets of 

\[ 3\Omega_L = \frac{2l + 2\lambda + 3\eta - 3l(1 + w)}{(2 + \eta) - l(1 + w)} \]  

(16)

and

\[ 3\Omega_m = \frac{3 - l - \lambda}{(2 + \eta) - l(1 + w)} \],

(17)

respectively, which confirms \( \Omega_m + \Omega_L = 1 \).

3 Observational constraints

The Hubble parameter \( H \) is one of the key parameter in observational cosmology as well as in theoretical cosmology and it measures expansion rate of the universe. In our paper, we have obtained the Hubble function using Eqs. (14) & (15) as

\[ H(t) = H_0 \left( \frac{a_0}{a(t)} \right)^{(2 + \eta)/(2 + \eta - 3l(1 + w))} \]  

(18)

Several observational data sets are available in terms of redshift \( z \) and hence, to obtain constraints on various model parameters we have to convert the above relationship in terms of the redshift. Therefore, using the relation \( \frac{a_0}{a(t)} = 1 + z \) between scale factor \( a(t) \) and redshift \( z \), we obtain:

\[ H(z) = H_0 (1 + z)^{(2 + \eta)/(2 + \eta - 3l(1 + w))} \]  

(19)

We can also obtain the expression for luminosity distance \( D_L \) and apparent magnitude \( m(z) \) in the purpose of observational constraints (Union 2.1 compilation of SNe Ia data sets and Joint light curve analysis data sets) respectively as

\[ D_L = \frac{c}{H_0} \frac{2 + \eta - l(1 + \omega)}{2 + \eta + (\lambda - 3)(1 + \omega)} (1 + z)^{\frac{\lambda + 2\eta + (\lambda - 3)(1 + \omega)}{2 + \eta - l(1 + \omega)}} \]  

(20)

and

\[ m(z) = 16.08 + 5 \times \log_{10} \left( \frac{1}{0.026} \frac{2 + \eta - l(1 + \omega)}{2 + \eta + (\lambda - 3)(1 + \omega)} (1 + z)^{\frac{\lambda + 2\eta + (\lambda - 3)(1 + \omega)}{2 + \eta - l(1 + \omega)}} \right) \]  

(21)

Now, we consider the 29 Hubble data sets \( H(z) [23, 29] \), 580 data sets of apparent magnitude \( m(z) \) from union 2.1 compilation of SNe Ia data sets [30] and 51 data sets of \( m(z) \) from Joint Light Curve Analysis (JLA) data sets [31] and using the \( R^2 \)-test formula, we obtain the best fit values of various model parameters \( l, \lambda, \eta, \omega \) and \( H_0 \) for the best fit curve of Hubble function \( H(z) \) and apparent magnitude \( m(z) \). The case \( R^2 = 1 \) means the curve of the Hubble function is exact match with observational curve and using this technique of fitting we have obtained the constraints on model parameters for maximum \( R^2 \) values. The best fit values of \( l, \lambda, \eta, \omega \) and \( H_0 \) for the various data sets are mentioned in below Table-1 & 2 with their maximum \( R^2 \) value and root mean square error (RMSE).

Using the above estimated values of various parameters (as mentioned in Table-1 & 2) we have calculated the values of matter energy density parameter \( \Omega_m \), dark energy density parameter \( \Omega_L \), deceleration parameter \( q \), age of the present universe and statefinder parameters \( r, s \) for two types of universes one is dusty universe and second is the case of \( 0 < \omega < \frac{1}{3} \) which are mentioned in Table-3 & 4 given below.

Table-3 & 4 shows the present values of various cosmological parameters \{\( \Omega_m, \Omega_L, q_0, t_0, r, s \}\} and we see that for the deceleration in expansion of the universe dark energy parameter must be \( \Omega_L > 0.2493775520 \) and matter energy density parameter \( \Omega_m < 0.7506622480 \). In our model we have calculated the present value of deceleration parameter \( q_0 \) lies between \(-0.3115722925 \leq q_0 \leq 0.125933672 \). From these results we concludes that our present universe is in accelerating phase.
best fit curve of the Hubble function and the range of 1 data sets (H, Arbab’s [34] which was (3, previous results and also obtain the respective range of the parameters.

Table 1: Case-I dusty universe: The best fit values for the model parameters as well as the best fit values of the model parameters where the curvature parameter H are used in the best fit curve for the Hubble function H(z) and the apparent magnitude m(z) with different observational data sets (H(z), JLA, SNe Ia) with confidence level of constraints of 95%.

| Parameters(For \( w = 0 \)) | H(z)   | SNe Ia | JLA  |
|-------------------------------|---------|--------|------|
| \( l \)                      | 1.435   | 0.1707 | 0.2171 |
| \( \lambda \)                 | 0.8113  | 0.2855 | 0.7471 |
| \( \eta \)                    | 0.1044  | 1.119  | 0.8394 |
| \( H_0 \)                     | 59.22   | –      | –    |
| \( R^2 \)                     | 0.9115  | 0.9938 | 0.9214 |
| \( RMSE \)                    | 13.2672 | 0.2486 | 0.2782 |

Table 2: Case-II 0 < \( \omega < \frac{1}{3} \): The best fit values for the model parameters \( l, \lambda, \eta \) and \( H_0 \) are used for the best fit curve for the Hubble function H(z) and the apparent magnitude m(z) for different observational data sets (H(z), JLA, SNe Ia) for confidence level of boundaries of 95%.

| Parameters | H(z)   | SNe Ia | JLA  |
|------------|--------|--------|------|
| \( \omega \) | 0.1256 | 0.1156 | 0.1218 |
| \( l \)    | 1.764  | 0.1881 | 0.1137 |
| \( \lambda \) | 0.7351 | 0.8632 | 0.9196 |
| \( \eta \) | 0.4916 | 0.7292 | 0.9696 |
| \( H_0 \) | 59.83  | –      | –    |
| \( R^2 \) | 0.9114 | 0.9938 | 0.9214 |
| \( RMSE \) | 13.5632| 0.2488 | 0.2812 |

Figures 1a & 1b represent the best fit curve of the Hubble function H(z) for the best fit values of the model parameters as well as the best fit values of the model parameters where the curvature parameter \( k = -1, 0, 1 \) for the open, flat and closed models displayed in Tables-1 & 2 for the three data sets as shown in the figures, respectively. Figures 2a & 2b depict the behaviour of cosmological constant \( \Lambda \) over redshift z and we see that it is an increasing function of \( \omega \) which is consistent with observational findings. Figures 3a & 3b show the age of the universe defined as \( (t_0 - t) = \int_0^z \frac{dz}{(1 + z)H(z)} \) for both cases of \( \omega \) and we have calculated the age of the present universe \( t_0 \) as 14.54821996 ≤ \( t_0 \) ≤ 20.62684904 Gyrs that is consistent with the observational results \( t_0 = 13.6 \) Gyrs.

From Table-3 & 4 we see that the value of total energy density parameter 0.9999999998 ≤ \( \Omega \) ≤ 1. These findings are consistent with the observational constraints obtained by MAXIMA-1 flight and COBE-DMR experiments [22], \( \Omega = 1.05 \pm 0.08 \) from CBI-DMR observations [18] and \( \Omega = 1.05 \pm 0.08 \) on the total energy density parameter \( \Omega = 1.00^{+0.25}_{-0.30} \). \( \Omega = 1.01 \pm 0.03 \) calculated from the first acoustic peak in the CMB fluctuation angular power spectrum [34].

Moreover, we can obtain ranges of \( l_0, \lambda_0 \) and \( \eta_0 \) (the values of \( l, \lambda, \eta \) in the present epoch) in the dust case \( (w = 0) \) from the above equations. The first flight of the MAXIMA combined with COBE-DMR resulted in \( 0.25 < \Omega_{m_0} < 0.50 \) and \( 0.45 < \Omega_{\Lambda_0} < 0.75 \). Observations of SNeIa combined with the total energy density constraints from CMB and combined gravitational lens and stellar dynamical analysis lead to \( \Omega_{m_0} \sim 0.3 \) and \( \Omega_{\Lambda_0} \sim 0.7 \). Sievers et al. and Sperger et al. obtained the values of \( [\Omega_{m_0}, \Omega_{\Lambda_0}] = [0.34 \pm 0.12, 0.67^{+0.10}_{-0.13}] \) and \( [0.249^{+0.024}_{-0.031}, 0.719^{+0.021}_{-0.029}] \) respectively [18][20]. Considering the varied observational dataset, many researchers prefer a range of 0.295 ≤ \( \Omega_{m_0} \) ≤ 0.365 [5].

For dust case, from [17] we obtain \( 3\Omega_m = 2\frac{3}{5} = \frac{3}{5} \). If we consider the case \( \lambda = 0 = \eta \), we obtain 3.7937 ≤ \( l_0 \) ≤ 4.2099. We note that using our model we are able to obtain the range of \( l \) smaller than that of Arbab’s [34] which was (3, 4.5) and Ray’s [5] which was [3.417, 4.674]. On a similar fashion we can find ranges of 1.905 ≤ \( l_0 \) ≤ 2.115 and 3.479 ≤ \( \eta_0 \) ≤ 4.780. So we see that using our model, we can generalize the previous results and also obtain the respective range of the parameters.
Figure 1: The plot of Hubble function $H(z)$ versus redshift $z$ for the best fit values of model parameters $l$, $\lambda$, $\eta$ and $H_0$ in both cases of $\omega$.

Figure 2: The plot of Cosmological constant $\Lambda(z)$ versus redshift $z$ for the best fit values of model parameters $l$, $\lambda$, $\eta$ and $H_0$ in both cases of $\omega$.

Figure 3: The plot of Cosmic time $(t_0 - t)H_0$ versus redshift $z$ for the best fit values of model parameters $l$, $\lambda$, $\eta$ and $H_0$ in both cases of $\omega$. 
The deceleration parameter $\ddot{a}/aH^2 = -\frac{H}{H} - 1$ in cosmology is a dimensionless measure of the cosmic acceleration of the expansion of space. The statefinder parameters $(r, s)$ is defined originally by

$$r = \frac{\ddot{a}}{aH^3}, \quad s = \frac{r - 1}{3(q - 1/2)}.$$  

We may link the $(r, s)$ parameters to the Hubble parameters $H$ and the deceleration parameter $q$ for our present model as

$$r = 2 \left( \frac{\dot{H}}{H^2} \right)^2 + 3 \frac{\dot{H}}{H^2} + 1 = q(2q + 1)$$  

$$s = -\frac{2\dot{H}}{3H^2} = \frac{2}{3}(q + 1).$$  

4 Finite time singularities in the current model

We investigate equation [15] for possible finite time past or future cosmic singularities present in the current model. We consider distinct cases depending on the sign of $(2 + \eta) - l(1 + w)$.

**Case 1:** $\frac{2 + \eta}{l} > 1 + w$.

If $(1 + w)(3 - l - \lambda) > 0$, then there is a finite past singularity at $t_p = t_0 - \frac{2 + \eta - l(1 + w)}{H_0(1 + w)(3 - l - \lambda)} < t_0$ whereas if $(1 + w)(3 - l - \lambda) < 0$, then the scale factor diverges at some finite future time $t_f = t_0 - \frac{2 + \eta - l(1 + w)}{H_0(1 + w)(3 - l - \lambda)} > t_0$.

**Case 2:** $\frac{2 + \eta}{l} < 1 + w$.

If $(1 + w)(3 - l - \lambda) < 0$, then there is a finite past singularity at $t_p = t_0 - \frac{2 + \eta - l(1 + w)}{H_0(1 + w)(3 - l - \lambda)} < t_0$ whereas if $(1 + w)(3 - l - \lambda) > 0$, then the scale factor diverges at some finite future time $t_f = t_0 - \frac{2 + \eta - l(1 + w)}{H_0(1 + w)(3 - l - \lambda)} > t_0$.

From this discussion it is clear that we can control the free parameters $l, \lambda, \eta$ to avoid any future singularity in the system.

5 Some cosmological parameters

The deceleration parameter $q = -\frac{\ddot{a}}{aH^2} = -\frac{H}{H} - 1$ in cosmology is a dimensionless measure of the cosmic acceleration of the expansion of space. The statefinder parameters $(r, s)$ is defined originally by

$$r = \frac{\ddot{a}}{aH^3}, \quad s = \frac{r - 1}{3(q - 1/2)}.$$  

We may link the $(r, s)$ parameters to the Hubble parameters $H$ and the deceleration parameter $q$ for our present model as

$$r = 2 \left( \frac{\dot{H}}{H^2} \right)^2 + 3 \frac{\dot{H}}{H^2} + 1 = q(2q + 1)$$  

$$s = -\frac{2\dot{H}}{3H^2} = \frac{2}{3}(q + 1).$$  

Table 3: Case-I dusty universe: The present values of various parameters obtained for different data sets.

| Parameters (for $\omega = 0$) | $H(z)$ | SNe Ia | JLA |
|-----------------------------|--------|--------|-----|
| $\Omega_m$                 | 0.7560224480 | 0.5752015285 | 0.5175609198 |
| $\Omega_{\Lambda}$         | 0.2493775520 | 0.4247984713 | 0.4824390803 |
| $q$                        | 0.125933672 | -0.1371977072 | -0.2236586203 |
| $t_0$                      | 14.56595745 | 18.74417427 | 20.62684904 |
| $r$                        | 0.1576522515 | -0.0995512854 | -0.1236122635 |
| $s$                        | 0.7560224480 | 0.5752015286 | 0.5175609198 |

Table 4: Case-II of $0 < \omega < \frac{1}{3}$: The present values of various parameters obtained for different data sets.
We also obtain a relation between \( r \) and \( s \) as
\[
    r = \frac{9}{2}s(s - 1) + 1. \tag{24}
\]
In general, the statefinder hierarchy \( A_n = \frac{a(n)}{aHn} \) for our present model can be calculated as
\[
    A_{2k+1} = q(2q + 1)(3q + 2) \cdots (2kq + 2k - 1),
\]
\[
    A_{2k} = -q(2q + 1)(3q + 2) \cdots ((2k - 1)q + 2k - 2). \tag{25}
\]
In this regard, it is to be mentioned that these handy relations between the various cosmological parameters as obtained in this section are valid for any time independent \( w \) and with any of the decay laws considered in [1, 2, 3, 4] or any of their linear combinations.

For a \( \Lambda \)CDM model the statefinder pair \((r, s)\) have the value \((1, 0)\). In our present model we obtain this if and only if \( \dot{H}/H^2 = 0 \) which means the scale factor \( a(t) \) increases exponentially with time. It also implies that \( q = -1 \), which is true if \( \alpha + \beta = 3 \), but this condition also provide \( \Lambda \) is truly a constant.

**Dust Case:**

Below we obtain some important results for the particular case \( w = 0 \).

First and foremost, we have
\[
    q = \frac{3 - l - \lambda}{2 + \eta - l} - 1. \tag{26}
\]
Hence for an accelerating universe \( (q < 0) \) we obtain a relation \( \lambda + \eta > 1 \).

We notice that for dust case, the standard model formula \( \Omega_m = 2q \) is now replaced by \( \Omega_m = \frac{2}{3}q + \frac{2}{3} \), same as the result of Arbab [4]. However, both models give \( q = 1/2 \) for a critical density \( \rho_c = \frac{3H^2}{8\pi G} \). We also obtain \( s = \Omega_m \).

Furthermore, using equations (13) and (16) we obtain that the deceleration parameter \( q \) and the vacuum density parameter \( \Omega_\Lambda \) are connected by the relation
\[
    3\Omega_\Lambda = 1 - 2q + 2w \frac{3 - l - \lambda}{(2 + \eta) - l(1 + w)}, \tag{27}
\]
and so for dust case, an accelerating universe \((q < 0)\) requires \( \Omega_\Lambda > 1/3 \), which perfectly fits the modern observational data as the present accepted value of \( \Omega_\Lambda \) is 0.7, much larger than 1/3. Hence our model fits an accelerating universe.

### 6 Conclusion

We have considered a spatially homogeneous and isotropic spacetime in the presence of a dynamic cosmological term \( \Lambda(t) \) satisfying \( \Lambda = l(\dot{a}/a)^2 + \lambda \ddot{a}/a + 4\pi G\rho \) motivated by modification of the Einstein-Hilbert action. This generalizes several results of the previous models in flat FRW spacetime and also enables us to find the ranges for \( l, \lambda, \eta \) from the observational data apart from the interrelation we obtain between these parameters for accelerating universe \((q < 0)\). We explore the recently defined statefinder hierarchy for this generalized model of ours and prove some effective results for any time independent \( w \). Furthermore, we show that our model fits in perfectly with the modern observational datasets.

We have found that the scale factor \( a(t) \) vanishes at \( t = t_0 - \frac{(2+\eta)-(1+w)}{H_0(1+w)(3-l-X)} \) while for this value of time the Hubble function \( H \) and the cosmological term \( \Lambda \) get infinitely large values. We have found that with increasing time or decreasing redshift the scale factor increases but the \( H \) and \( \Lambda \) decreases to a finite small values in late time universe. These behaviours of cosmological parameters reveal that our universe model
starts with a finite time big-bang singularity stage and goes on expanding till late time. We have also noticed that the total energy density parameter \( \Omega \leq 1 \) which indicates towards the closed and flat geometry of the universe and it was compatible with several observational results. We have also found a constraint on dark energy density parameter \( \Omega_\Lambda > \frac{1}{4} \) to provide an acceleration in expansion of the universe which is compatible with the result \( \Omega_\Lambda > \frac{1}{3} \). We have also calculated the age of the present universe in the range \( 14 < t_0 < 21 \) Gys.

**Acknowledgments**

De Avik and Tee How Loo are supported by the grant FRGS/1/2019/STG06/UM/02/6. A. Pradhan would like to express his appreciation to the Inter-University Centre for Astronomy & Astrophysics (IUCAA), India for supporting under the visiting associateship.

**References**

[1] J. C. Carvalho, J. A. S. Lima and I. Waga, Cosmological consequences of a time-dependent \( \Lambda \) term, Phys. Rev. D 46 2404 (1992).

[2] I. Waga, Decaying vacuum flat cosmological models: Expressions for some observable quantities and their properties, Astrophys. J. 414 436 (1993).

[3] R. G. Vishwakarma, A Machian model of dark energy, Class. Quantum Grav. 17 3833 (2000).

[4] A.I. Arbab, Cosmic acceleration with a positive cosmological constant, Class. Quantum Grav. 93 (2003).

[5] S. Ray, U. Mukhopadhyay and S. B. Dutta, Dark energy models with a time-dependent gravitational constant, Gravit. & Cosmol. 142 (2007).

[6] R. G. Vishwakarma, Consequences on variable \( \Lambda \)-models from distant type Ia supernovae and compact radio sources, Class. Quantum Grav. 18 1159 (2001).

[7] S. Pan, Exact solutions, finite time singularities and non-singular universe models from a variety of \( \Lambda(t) \) cosmologies, Mod. Phys. Lett. A 33 1850003 (2018).

[8] S. Basilakos, M. Plionis and J. Sola, Hubble expansion and structure formation in time varying vacuum models, Phys. Rev. D 80 083511 (2009).

[9] A. Gomez-Valent, J. Sola and S. Basilakos, Dynamical vacuum energy in the expanding Universe confronted with observations: a dedicated study JCAP 01 004 (2015), arXiv:1409.7048 [astro-ph.CO].

[10] V. Sahni, T. D. Saini, A. A. Starobinsky and U. Alam, Statefinder-a new geometrical diagnostic of dark energy, JETP Lett. 77 201 (2003).

[11] U. Alam, V. Sahni, T. D. Saini and A. A. Starobinsky, Exploring the expanding universe and dark energy using the Statefinder diagnostic, Mon. Not. R. Astron. Soc. 344 1057 (2003).

[12] M. Arabsalmani, V. Sahni, The Statefinder hierarchy: An extended null diagnostic for concordance cosmology, Phys. Rev. D 83 043501 (2011), arXiv:1101.3436 [astro-ph.CO].

[13] P. de Bernardis, P. A. R. Ade, N. Vittorio, A flat Universe from high-resolution maps of the cosmic microwave background radiation, Nature 404 955 (2000).

[14] P. de Bernardis et al., Multiple peaks in the angular power spectrum of the cosmic microwave background: Significance and consequences for cosmology, Astrophys. J. 564 559 (2002).

[15] S. Hanany et al., MAXIMA-1: A measurement of the cosmic microwave background anisotropy on angular scales of 10′-5°, Astrophys. J. 545 L5 (2000).
[16] A. T. Lee et al., A High spatial resolution analysis of the MAXIMA-1 cosmic microwave background anisotropy data, Astrophys. J. 561 L1 (2001).

[17] N. W. Halverson et al., Degree angular scale interferometer first results: a measurement of the cosmic microwave background angular power spectrum, Astrophys. J. 568 38 (2002).

[18] J. L. Sievers et al., Cosmological parameters from Cosmic Background Imager observations and comparisons with BOOMERANG, DASI, and MAXIMA, Astrophys. J. 591 599 (2003).

[19] C. L. Bennett et al., First-Year Wilkinson Microwave Anisotropy Probe (WMAP) observations: Preliminary maps and basic results, Astrophys. J. Suppl. 148 1 (2003).

[20] D. N. Spergel et al., Wilkinson Microwave Anisotropy Probe (WMAP) Three Year Results: Implications for Cosmology, Astrophys. J. Suppl. 170 377 (2007), astro-ph/0603449.

[21] T. Delubac, et al. Baryon acoustic oscillations in the Ly-α forest of BOSS DR11 quasars, Astron. Astrophys. 574 A59 (2015).

[22] T. Harko, F. S. N. Lobo, f(R, L_m) gravity, Eur. Phys. J. C 70 (2010).

[23] A. Stephen, Appleby and E. V. Linder, Probing dark energy anisotropy, Phys. Rev. D 87 023532 (2013).

[24] D. Adak, et al., Reconstructing the equation of state and density parameter for dark energy from combined analysis of recent SNe Ia, OHD and BAO data, arXiv:1102.4722 [astro-ph.CO] (2011).

[25] L. Amendola and S. Tsujikawa, Dark energy: theory and observations, Cambridge University Press, (2010).

[26] R. Aurich and F. Steiner, Dark energy in a hyperbolic universe, Mon. Not. Roy. Astron. Soc. 334, 735 (2002).

[27] M. Arabsalmani, V. Sahni and T. D. Saini, Reconstructing the properties of dark energy using standard sirens, Phys. Rev. D 87, 083001 (2013).

[28] S. W. Allen, et al., Improved constraints on dark energy from Chandra X-ray observations of the largest relaxed galaxy clusters, Mon. Not. Roy. Astron. Soc. 383, 879 (2008); arXiv:0706.0033 [astro-ph].

[29] L. Anderson, et al., The clustering of galaxies in the SDSS-III Baryon Oscillation Spectroscopic Survey: baryon acoustic oscillations in the Data Release 9 spectroscopic galaxy sample, Mon. Not. Roy. Astron. Soc. 427, 3435 (2012).

[30] N. Suzuki, et al., The Hubble Space Telescope cluster supernova survey. V. Improving the dark-energy constraints above z > 1 and building an early-type-hosted supernova sample, Astrophys. J. 85 746 (2012).

[31] M. Betoule, et al., Improved cosmological constraints from a joint analysis of the SDSS-II and SNLS supernova samples, arXiv:1401.4064v2 [astro-ph.CO].

[32] A. Balbi et al., Constraints on Cosmological Parameters from MAXIMA-1, Astrophys. J. Lett. 545 L1 (2001).

[33] R. Rebolo, Cosmological parameters, Nucl. Phys. B (Proc. Suppl.) 114 3 (2003).

[34] A.I. Arbab, The equivalence between different dark (matter) energy scenarios, Astrophys. Space. Sci. 291 141 (2004).