Dynamics of a quantum phase transition with decoherence: the quantum Ising chain in a static spin environment

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We consider a linear quench from the paramagnetic to ferromagnetic phase in the quantum Ising chain interacting with a static spin environment. Both decoherence from the environment and non-adiabaticity of the evolution near a critical point excite the system from the final ferromagnetic ground state. For weak decoherence and relatively fast quenches the excitation energy, proportional to the number of kinks in the final state, decays like an inverse square root of a quench time, but slow transitions or strong decoherence make it decay in a much slower logarithmic way. We also find that fidelity between the final ferromagnetic ground state and a final state after a quench decays exponentially with a size of a chain, with a decay rate proportional to average density of excited kinks, and a proportionality factor evolving from 1.3 for weak decoherence and fast quenches to approximately 1 for slow transitions or strong decoherence. Simultaneously, correlations between kinks randomly distributed along the chain evolve from a near-crystalline anti-bunching to a Poissonian distribution of kinks in a number of isolated Anderson localization centers randomly scattered along the chain.

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I. INTRODUCTION

Phase transition is a fundamental change in the state of a system when one of its parameters passes through the critical point. In a second order phase transition, the fundamental change is continuous and the critical point is characterized by divergences in the correlation length and in the relaxation time. This critical slowing down implies that no matter how slowly a system is driven through the transition, its evolution cannot be adiabatic close to the critical point. As a result, ordering of the state after the transition, its evolution cannot be adiabatic near the critical point. In a second order phase transition, the critical point. As a result, ordering of the state after the transition, its evolution cannot be adiabatic close to the critical point. In a second order phase transition, the critical point. As a result, ordering of the state after the transition, its evolution cannot be adiabatic close to the critical point. In a second order phase transition, the critical point. As a result, ordering of the state after the transition, its evolution cannot be adiabatic close to the critical point.

This scenario was first described in the cosmological context by Kibble [1] who appealed to relativistic causality to set the size of the domains. The dynamical mechanism relevant for second order phase transitions was proposed by Zurek [2]. It is based on the universality of critical slowing down, and leads to a prediction that average size of the ordered domains scales with the transition time $\tau_Q$ as

$$\hat{\xi} \simeq \frac{Q}{\nu z}$$

where $\nu$ and $z$ are critical exponents. The Kibble-Zurek mechanism (KZM) for second order thermodynamic phase transitions was confirmed by numerical simulations of the time-dependent Ginzburg-Landau model [3] and successfully tested by experiments in liquid crystals [4], superfluid helium 3 [5], both high-$T_c$ [6] and low-$T_c$ [7] superconductors, and even in non-equilibrium systems [8]. With the exception of superfluid $^4$He – where the early detection of copious defect formation [9] was subsequently attributed to vorticity inadvertently introduced by stirring [10], and the situation remains unclear – experimental results are consistent with KZM, although more experimental work is clearly needed to allow for more stringent experimental tests of KZM. Quite recently, a new experiment was reported in Ref. [11] where they observe, for the first time, spontaneous appearance of vorticity during Bose-Einstein condensation driven by evaporative cooling confirming KZM predictions in Ref. [12].

The Kibble-Zurek mechanism is thus a universal theory of the dynamics of second order phase transitions whose applications range from the low temperature Bose-Einstein condensation (BEC) to the ultra high temperature transitions in the grand unified theories of high energy physics. However, the zero temperature quantum limit remained unexplored until very recently and quantum phase transitions are in many respects qualitatively different from transitions at finite temperature. Most importantly time evolution is unitary, so there is no damping, and there are no thermal fluctuations that initiate symmetry breaking in KZM. The recent progress on dynamical quantum phase transitions is mostly theoretical, see Refs. [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27] and, for an example of a disordered quantum system, Ref. [28], but there is already one significant exception: the experiment in Ref. [29] on a transition from paramagnetic to ferromagnetic phase in a dipolar BEC. Generic outcome of that experiment is a mosaic of finite-size ferromagnetic domains whose origin was attributed to the Kibble-Zurek mechanism. This explanation is further supported by theory in Ref. [30].

A majority of theoretical work was devoted to the prototypical exactly solvable quantum Ising chain [13, 16, 19, 21, 26, 27, 30],

$$H_S = -\sum_{n=1}^{N} \left[ g \sigma_n^x + \sigma_n^z \sigma_{n+1}^z \right]$$  (2)
driven by a linear quench

\[ g(t) = -\frac{t}{\tau_Q} \]  

from \( g = \infty \) to \( g = 0 \) i.e. across the quantum phase transition from paramagnet to ferromagnet at \( g_c = 1 \). Since the system is gapless at \( g_c \), when \( \tau_Q \gg 1 \) the evolution becomes non-adiabatic at

\[ \hat{g} - g_c \approx \tau_Q^{-1/2}, \]

see Ref. [15]. The freezout at \( \hat{g} \) is the closer to \( g_c \) the slower is the transition. The Ising critical exponents \( \nu = 1 \) and \( z = 1 \) determine an average size of ferromagnetic domains at \( g = 0 \) proportional to the correlation length at \( \hat{g} \):

\[ \xi \approx \tau_Q^{1/2}, \]

or final density of kinks (domain walls) at \( g = 0 \)

\[ d \approx \xi^{-1} \approx \tau_Q^{-1/2} \]

which is proportional to excitation energy density. Other properties, like spin-spin correlation functions [19, 20] or entropy of entanglement between a block of consecutive spins and the rest of the chain [20] were analyzed in some detail and they all turned out to be determined by \( \xi \).

While the quench in the isolated system [22] seems to be well analyzed, relatively little is still known about dynamical transitions in open quantum systems subject to interaction with environment. Significant progress was made in Ref. [31] in a “global” case when interaction between the system \( S \) with the Hamiltonian [2] and its environment \( E \) is described by \( V = R(\sum_n \sigma_n^z) \), where \( R \) is a hermitian operator of the environment. This global model is solvable thanks to its translational invariance and its solutions indicate that decoherence is increasing density of excited kinks as compared to an isolated system. A local model, with the system [2] coupled to an Ohmic heat bath, was analyzed in Ref. [31] distinguishing between different regimes of parameters where defect production is dominated either by KZM or external heating. In this paper, we propose a quite realistic, but still solvable, model of local zero temperature decoherence from a static environment. Its solution predicts dramatic increase in the density \( d \) of excited kinks as compared to an isolated system with \( d \) decaying as only a logarithmic function of \( \tau_Q \).

Motivation for this study is twofold. It comes both from condensed matter physics, where it is virtually impossible to isolate a system from its environment, and adiabatic quantum computation, where a system is initially prepared in a simple ground state of a simple initial Hamiltonian \( H_0 \) and then it is driven adiabatically to a final Hamiltonian \( H_1 \) whose non-trivial ground state is the desired solution of a complex computational problem. The computation is complicated by a quantum critical point somewhere on the way from \( H_0 \) to \( H_1 \) which can make the adiabaticity problematics, but see Refs. [26, 27] for methods how to circumvent this problem. The Ising chain [2] is a toy model of adiabatic quantum computer with the final (trivial) ferromagnetic ground state at \( g = 0 \) playing the role of the desired “non-trivial” ground state. When the “computer” is isolated from environment, then Eq. (9) implies that the minimal “computation time” \( \tau_Q \) required to keep the evolution adiabatic or, equivalently, to make \( \xi \gg N \) is

\[ \tau_Q^{\text{isolated}} \approx N^2. \]

The “isolated” computation problem is polynomial in \( N \). In contrast, in our model of decoherence similar argument predicts

\[ \tau_Q^{\text{open}} \approx e^{\sqrt{N}} \]

which is non-polynomial in \( N \).

II. ISING CHAIN IN STATIC SPIN BATH

In this paper we couple the Ising chain [2] to an environment \( E \) of \( M \) spins through the interaction

\[ V = -\sum_{n=1}^{N} \sum_{m=1}^{M} \sigma_n^x \sigma_m^x \tau_m^x. \]

Here \( \tau_m \)'s are Pauli matrices of environmental spins. The spins are static, with \( H_E = 0 \), and the total Hamiltonian is just \( H = H_S + V \).

Initially at \( t \to -\infty \) the system is in the ground state \( |0_{g \to -\infty}\rangle \) of the pure ising chain [2] with all spins polarized along \( x \). This assumption is self-consistent in our open system because large initial energy gap of \( 2g \) makes the influence of the static environment so negligible that the initial states of \( S \) and \( E \) can be assumed uncorrelated: \( \rho_{S+E} = \rho_S \otimes \rho_E \) with \( \rho_S = |0_{g \to -\infty}\rangle \langle 0_{g \to -\infty}| \) and the environment is initially in a pure state

\[ \sum_{s_1,\ldots,s_M=-1,+1} c_{s_1,\ldots,s_M} |s_1\ldots s_M\rangle \langle s_1\ldots s_M|. \]

Here \( \tau_m|s_m\rangle = s_m|s_m\rangle \).

After evolution for time \( \Delta t \) reduced density matrix of the system \( \rho_S = \text{Tr}_E \rho_{S+E} \) becomes

\[ \rho_S(\Delta t) = \sum_{\vec{s}} |c_{\vec{s}}|^2 U(\Delta t, \vec{s}) \langle 0_\infty | \langle 0_\infty | U(\Delta t, \vec{s})^\dagger \]

\[ \equiv \frac{1}{\sqrt{N}} \sum_{\vec{s}} U(\Delta t, \vec{s}) \langle 0_\infty | \langle 0_\infty | U(\Delta t, \vec{s})^\dagger. \]

Here \( \vec{s} = (s_1, \ldots, s_M) \),

\[ U(\Delta t, \vec{s}) = T \exp \left[ -i \int_0^{\Delta t} dt' H(t', \vec{s}) \right] \]

where \( H(t', \vec{s}) = \sum_{\vec{s},\vec{m}} \sigma_{\vec{s},\vec{m}} H_{\vec{s},\vec{m}}(t') \).
and
\[ H(t, \vec{s}) = -\sum_{n=1}^{N} \{ [g(t) + \Gamma_n] \sigma_n^x + \sigma_n^z \sigma_{n+1}^z \} \] (13)

with random magnetic fields of the static environment
\[ \Gamma_n = \sum_{m=1}^{M} V_{nm} s_m. \] (14)

The overline in Eq. (11) is an average over \( \vec{s} \) with probability distribution \( |c_s|^2 \), but it can also be interpreted as an average over random “disorder” field \( \Gamma_n \). \( \rho_S(\Delta t) \) is an average over states \( U(\Delta t, \vec{s})|0_\infty \rangle \) obtained in quenches with different disordered Hamiltonians (13). In this way, our original problem of a quench in the open pure Ising model (2) is mapped to an average over quenches in the isolated random Ising model (2).

In the following, rather than struggle with the problem in its full generality, we assume that each spin of the environment couples to only one spin of the system or, in other words, each spin of the system has its own local environment. Consequently, \( \Gamma_m \) and \( \Gamma_n \) are statistically independent when \( m \neq n \). We also assume that each \( \Gamma_n \) has the same Gaussian probability distribution
\[ f(\Gamma) = \frac{e^{-\Gamma^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}, \] (15)

where \( \sigma \) is strength of disorder/decoherence.

The Hamiltonian (13) belongs to the universality class of the well known random Ising chain (22, 33). It has a continuous quantum critical point at \( g_c \) when
\[ \ln |g_c + \Gamma| = \int_{-\infty}^{\infty} dt \ f(\Gamma) \ln |g_c + \Gamma| = 0. \] (16)

\( g_c \) depends on \( \sigma \) as shown in Fig. 1A. There is no critical point for \( \sigma > 1.887 \) when the disorder is too strong. No matter how weak \( \sigma \) is, renormalization group transformations drive the model (13) towards an infinite disorder fixed point with different critical exponents than in the pure Ising chain: the random chain has \( \nu = 2 \) instead of \( \nu = 1 \) and \( z \to \infty \) instead of \( z = 1 \) (33). A straightforward application of the standard KZM formula (11) gives \( \xi \approx 1 \) i.e. a domain size that does not depend on the quench time \( \tau_Q \). A more careful argument in Ref. (28), going back to the basics of KZM, predicts a logarithmic dependence
\[ \xi \approx \frac{\ln^2 |\alpha \tau_Q|}{\ln^2 [\ln(\alpha \tau_Q)]}, \] (17)

with a non-universal \( \alpha \approx 1 \). This equation leads to the estimate in Eq. (3). Since Eq. (17) is based on the universality class alone, it is valid for any model of this class when \( \tau_Q \) is long enough for the quench to become non-adiabatic close enough to \( g_c \) to be affected by disorder. The estimate (17) was confirmed by numerics in the model of Ref. (28), where it was also found that for weak disorder and relatively fast \( \tau_Q \) one recovers Eqs. (16) as in the pure model Eq. (2).

In the present effective model (13), it is simple to estimate how slow a quench needs to be for Eqs. (16) and, more importantly, Eq. (7) to be not valid. Assuming that influence of \( \Gamma_n \) is negligible, evolution becomes non-adiabatic at a field \( \tilde{g} \) in Eq. (4). This assumption is not self-consistent when the remaining distance from \( \tilde{g} \) to \( g_c \), \( \tilde{g} - g_c \approx \tau_Q^{-1/2} \), is less than the strength \( \sigma \) of disorder field \( \Gamma_n \), or equivalently
\[ \tau_Q \sigma^2 \gg 1. \] (18)

Thus, no matter how weak the decoherence is, its influence is not negligible when the transition is slow enough: \( \tau_Q \gg \sigma^{-2} \). In consequence, there is a maximal number of qubits
\[ N \ll \frac{1}{\sigma}, \] (19)

which can be simulated with polynomial efficiency, compare Eq. (7) and (18).

In the next Section, we review static properties of the random Ising model (13) many of which, we believe, are described in this form for the first time.

### III. RANDOM ISING CHAIN

Here we assume for convenience that \( N \) is even and following Refs. (28, 33) make the Jordan-Wigner transformation \( \sigma^x_n = 1 - 2c_n^\dagger c_n \) and \( \sigma^z_n = -(c_n + c_n^\dagger) \prod_{m<n}(1 - 2c_m^\dagger c_m) \) introducing spinless fermionic operators \( c_n \). The Hamiltonian (13) becomes \( H = P^+ H^+ P^+ + P^- H^- P^- \) where \( P^\pm = \frac{1}{2} \left[ 1 \pm \prod_{n=1}^{N}(1 - 2c_n^\dagger c_n) \right] \) are projectors on subspaces with even (+) and odd (−) numbers of c-quasiparticles and
\[ H^\pm = \sum_{n=1}^{N} \left( g_n c_n^\dagger c_n - c_n^\dagger c_{n+1} - c_{n+1}^\dagger c_n - \frac{g_n}{2} \right) + \text{h.c.} \] (20)

are quadratic Hamiltonians. Here
\[ g_n = g + \Gamma_n \] (21)

for short. The \( c_n \)'s in \( H^- \) satisfy periodic boundary conditions \( c_{N+1} = c_1 \), but the \( c_n \)'s in \( H^+ \) must obey \( c_{N+1} = -c_1 \), what we call “antiperiodic” boundary conditions.

The parity of the number of c-quasiparticles is a good quantum number and the ground state has even parity for any value of \( g \). Assuming that the quench begins in the ground state we can confine to the subspace of even parity. In this subspace the quadratic \( H^+ \) is diagonalized by a Bogoliubov transformation \( c_n =
\[ \sum_{m=1}^{N} (u_{nm} \gamma_m + v_{nm} \gamma_m^\dagger) \]. The index \( m \) numerates (Bogoliubov) eigenmodes of the stationary Bogoliubov-de Gennes equations

\[ \omega_m \frac{du_{nm}^\pm}{dt} = 2g_n u_{nm}^\pm - 2u_{n-1,m}^\pm, \tag{22} \]

with \( \omega_m > 0 \). Here we define \( u_{nm}^\pm = u_{nm} + v_{nm} \) and assume the anti-periodic boundary conditions: \( u_{N+1,m}^\pm = -u_{1,m}^\pm \) and \( u_{N,m}^\pm = -u_{N-1,m}^\pm \). The eigenstates \( (u_{nm}, v_{nm}) \), normalized so that \( \sum_n (|u_{nm}|^2 + |v_{nm}|^2) = 1 \), define quasiparticle operators \( \gamma_m = u_{nm} c_n + v_{nm} c_n^\dagger \). After the Bogoliubov transformation the Hamiltonian becomes \( H^+ = \sum_{m=1}^{N} \omega_m (\gamma_m^\dagger \gamma_m - 1/2) \) which is a simple-looking sum of quasiparticles. However, thanks to the projection \( P^+ H^+ P^+ \) only states with even numbers of quasiparticles belong to the spectrum of \( H \).

When \( g = \sigma = 0 \) the final system Hamiltonian \( \mathbf{12} \) has \( N \) degenerate quasiparticles with \( \omega = 2 \). We choose an orthonormal basis

\[ u_{nm}^{(0)} = \frac{1}{2} (\delta_{n+1,m} - \delta_{n,m}), \tag{23} \]
\[ v_{nm}^{(0)} = \frac{1}{2} (\delta_{n+1,m} + \delta_{n,m}), \tag{24} \]

antiperiodic in \( n \). These eigenmodes define Bogoliubov quasiparticles \( \gamma_m^{(0)} \) which are simply kinks localized at bonds between sites \( m \) and \( m+1 \). The kinks are related to quasiparticles \( (u_{nm}, v_{nm}) \) at finite \((g, \Gamma)\) by a Bogoliubov transformation

\[ \gamma_a = U_{ba}^{\ast} \gamma_b^{(0)} + V_{ba}^{\ast} \gamma_b^{(0)\dagger}, \tag{25} \]

where

\[ U_{ba} = \sum_n (u_{nb}^{(0)} u_{na} + \gamma_{nb}^{(0)} v_{na}), \tag{26} \]
\[ V_{ba} = \sum_n (v_{nb}^{(0)} u_{na} + u_{nb}^{(0)} v_{na}) \tag{27} \]

leading, for example, to a simple expression for average kink density in the ground state \( |0_{g,\Gamma}\rangle \) of the Hamiltonian \( \mathbf{13} \)

\[ d(g, \sigma) = \frac{1}{N} \langle 0_{g,\Gamma}| \sum_{m=1}^{N} \gamma_m^{(0)\dagger} \gamma_m^{(0)} |0_{g,\Gamma}\rangle = \text{Tr} V^\dagger V \frac{N}{N}, \tag{28} \]

shown in Fig. \( \mathbf{13} \). It is finite at any \( g > 0 \), both in the ferromagnetic \((g < g_c)\) and paramagnetic phase \((g > g_c)\), but, as we will see, kink-kink correlations are qualitatively different in the two phases.

These correlations can be indirectly probed by average fidelity \( F \) between the final ground state \( |0\rangle \) of \( H_S \) at \( g = 0 \) (state with no kinks \( \gamma^{(0)} \)) and ground states \( |g, \Gamma_n\rangle \) at finite \( g \) or \( \sigma \):

\[ |0_{g,\Gamma}\rangle = \mathcal{N} e^{\frac{i}{2} \sum_{a,b=1}^{N} Z_{ab} \gamma_{(0)\dagger}^{(0)} \gamma_{b}^{(0)\dagger}} |0\rangle. \tag{29} \]

Here \( Z = V^\dagger (U^\ast)^{-1} \) and \( \mathcal{N} \) is a normalization factor. The average fidelity is

\[ F = \frac{\langle 0_{g,\Gamma}| \sum_{m=1}^{N} \gamma_m^{(0)\dagger} \gamma_m^{(0)} |0_{g,\Gamma}\rangle}{\text{Tr} V^\dagger V}, \tag{30} \]

Here \( \hat{Z} = \frac{1}{N} \sum_{a,b} Z_{ab} \gamma_{a}^{(0)\dagger} \gamma_{b}^{(0)\dagger} \) for short.

We found that the fidelity is exponential in \( N \)

\[ F(g, \sigma, N) \sim (1 - c d)^N \tag{31} \]

when \( F \ll 1 \). Here \( d(g, \sigma) \) is the density of kinks in Eq. \( \mathbf{28} \) and Fig. \( \mathbf{13} \). The coefficient \( c(g, \sigma) \) is shown in Fig. \( \mathbf{13} \). For weak disorder, when \( \sigma \ll 1 \), we have \( c \approx \frac{1}{\sigma} \) when \( g < g_c \), with \( c \) increasing towards \( 1 \) when \( g \gg g_c \) or \( \sigma \gg 1 \). These two limits can be explained as follows.

When the magnetic fields \( g_0 = \Gamma_n + g \) in the effective Hamiltonian \( \mathbf{13} \) are strong, because either \( g \) or \( \sigma \) or both are strong, then in any ground state \( |0_{g,\Gamma}\rangle \) all spins are polarized along \( \sigma^2 \). Fidelity to the \( \sigma^2 \)-ferromagnet is \( F = 1/2^N \) and density of kinks is \( d = 1/2 \), hence \( F = (1 - c d)^N \) with \( c = 1 \). This \( c \) is consistent with the data shown in Fig. \( \mathbf{13} \) when \( g \) or \( \sigma \) are strong.
In the opposite limit of weak magnetic fields $|g_n| = |g + \Gamma_n| \ll 1$, the ground states are

$$|0_{g,r}\rangle \approx \prod_{n=1}^{N} \left| \frac{1}{\sqrt{1 + \frac{q_d^2}{4}}} \right| ^{n_n}.$$  \hspace{1cm} (32)

Their fidelity to the $\sigma^z$-ferromagnet is

$$F = \prod_{n=1}^{N} \frac{1}{1 + \frac{q_d^2}{4}} \approx \left(1 - \frac{1}{2^d}\right)^N.$$  \hspace{1cm} (33)

when $F \ll 1$. Here $d = \frac{1}{2} |\Gamma_n| \ll 1$ is small density of kinks and $c = 1/2$. This $c$ is consistent with the data in Fig. [1C] when $\sigma \ll 1$ and $g < g_c \approx 1$.

It is interesting to interpret the widely different values of $\frac{1}{2} \leq c < 1$ in terms of a simple Poissonian model where each of $N$ bonds is either excited (with probability $d_{exc}$) or not excited (with probability $1 - d_{exc}$) independently of other bonds. Here $d_{exc}$ is average density of excitations. The fidelity is a probability that none of the $N$ independent bonds is excited

$$F = (1 - d_{exc})^N.$$  \hspace{1cm} (34)

Comparing Eqs. (31,34) we obtain density of independent excitations

$$d_{exc} = c \, d.$$  \hspace{1cm} (35)

We can conclude that $c = d_{exc}/d$ measures correlations between kinks: $c < 1$ means bunching and an eventual $c > 1$ would mean anti-bunching of kinks randomly distributed along the spin chain.

This simple interpretation of $c$ follows from the fact that any ground state $|0_{g,r}\rangle$ is a BCS state of kinks $\gamma^{(0)}$, see Eq. (29) where $Z_{ab}$ is a wave-function for a (Cooper) pair of kinks. Depending on $g$ and $\sigma$, this BCS state can be either a condensate of tightly bound Cooper pairs of kinks (with $c = 1/2$), or just a weakly coupled BCS state with pairing (anti-)correlations manifesting as the (anti-)bunching of kinks.

The BCS state is particularly simple at weak magnetic fields $g_n = g + \Gamma_n$ when, crudely speaking, the ground state $|0_{g,r}\rangle$ in Eq. (32) is approximately the ferromagnet $|\uparrow\uparrow\uparrow\uparrow\ldots\uparrow\rangle$ but with occasional spins reversed to $|\downarrow\rangle$ by weak magnetic fields $g_n \sigma^n_z$. Each reversed spin is a tightly bound Cooper pair of two kinks sitting on nearest neighbor bonds with a relative distance of $r = 1$ lattice sites between the two kinks. No wonder that $d_{exc} = d/2$ in this case.

As we could see in Fig. [1C], the picture with tightly bound Cooper pairs is accurate for weak disorder in the ferromagnetic phase, but with $g$ increasing above $g_c$, the Cooper pairs begin to dissociate into free kinks and antikinks and $c$ increases well above $\frac{1}{2}$. This dissociation can be clearly seen in Fig. [1D] where we show a correlation function $C_r$

$$C_r \sim \sum_m |Z_{m+r,m}|^2$$  \hspace{1cm} (36)

between two kinks making a Cooper pair. Here $Z_{a,b}$ is the unnormalized wavefunction of a Cooper pair of two kinks in the BCS state (29) and $r$ is a relative distance between the two kinks. The $C_r$ shown in Fig. [1D] is localized when $g \ll g_c$ (tight Cooper pairs) and delocalized when $g > g_c$ (dissociated pairs).

A similar $C_r$ is shown in Fig. [1E] for a finite $\sigma = 0.8$ in the ferromagnetic phase at $g = 0$. In addition, the same figure shows reduced probability distribution

$$P_n \sim \sum_m |Z_{m,n}|^2$$  \hspace{1cm} (37)

for a single kink. Unlike in the pure case, this $P_n$ is not uniform and shows Anderson localization.

In the ferromagnetic phase below $g_c$ all kinks are bound into tight Cooper pairs. The small density of Cooper pairs, each of them a tightly bound pair of kink and antikink, does not destroy long range ferromagnetic order in $\sigma^z$. With $g$ increasing above $g_c$ the transition to the paramagnetic phase takes place when the Cooper pairs begin to dissociate into free kinks and antikinks which scramble long-range ferromagnetic correlations.

In the context of adiabatic quantum computation there are two generally accepted measures of how far the final state $\rho_S(0)$ from the desired final ground state $|0\rangle$ is. One is energy of excitation of the final state above the energy of the desired final ground state, and the other is fidelity between the final state and the final ground state. Both quantities are tractable in our model, where the excitation energy is proportional to the number of kinks. We study them in the following two Sections.

IV. DENSITY OF KINKS AFTER A QUENCH

In our simulations of a quench the system is initially prepared in its ground state at a large initial value of $g \gg 1$ i.e. in a Bogoliubov vacuum state for quasiparticles at an initial $g \gg 1$. As the magnetic field is being turned off to zero across $g_c$, the state of the system $|\psi(t)\rangle$ is getting excited from its adiabatic ground state. However, in a similar way as in Refs. [14,28], we use the Heisenberg picture where the state remains a vacuum for quasiparticle operators

$$\tilde{\gamma}_m = u^*_{nm}(t)c_n + v^*_{nm}(t)c^\dagger_n$$  \hspace{1cm} (38)

with the Bogoliubov modes $u_{nm}(t)$ and $v_{nm}(t)$ solving time-dependent Bogoliubov-de Gennes equations

$$\frac{du^\pm_{n,m}}{dt} = 2g_n(t)uc^\mp_{n,m} - 2u^\mp_{n-1,m}.$$

Initially each mode is a positive frequency eigenmode of the stationary Eqs. (22). Equations (38) were integrated by a split-step method for each realization of $\Gamma_n$. Their solutions $u^\pm_{nm}(0)$ at the final $g = 0$ determine final states $|\psi(0)\rangle$ whose average over disorder gives final density matrix $\rho_s(0) = \langle \psi(0)| \psi(0) \rangle$. For each realization of $\Gamma_n$, a
final state $|\psi(0)\rangle$ is a vacuum for $\tilde{\gamma}_m$’s which are related to the kinks $\gamma_0$ by the transformation Eq. (25). Final densities of kinks follow from Eq. (28).

For the system Hamiltonian (2) at the final $g = 0$ the final excitation energy is simply twice the final number of kinks excited in the desired ferromagnetic ground state. In Figure 2 we show density of kinks $d$ as a function of $g(t)$. For the early large $g$ the density follows the density $d(g,\sigma)$ in a ground state $|0_{g,\sigma}\rangle$, which is also shown in Fig. 1 B, but as the quench is approaching the critical point $g_c \approx 1$ the evolution becomes non-adiabatic and the density $d$ is excited above its ground state level $d(g,\sigma)$. The slower the quench, the closer to the critical point the non-adiabatic stage begins and the quench is probing the critical point more closely.

The final density of kinks at $g = 0$ is shown in Fig. 2A. For large $\tau_Q$, the density tends to saturate at $d(g = 0,\sigma)$ shown in Fig. 1 B i.e. the density of kinks in the ground state of the random Hamiltonian (13). In Fig. 3B we show a difference $\delta d = d - d(g = 0,\sigma)$ which can be attributed to the non-adiabaticity of the transition described by KZM. If $\delta d = \sigma Q^{\infty} \log_{10} \alpha + w(\log_{10} \tau_Q)$, but this is not the case when $\tau_Q \gg \sigma^{-2}$ as in Eq. (18). At best we can think of a local slope $w(\tau_Q)$ which can be estimated by fitting to pairs of nearest neighbor data points. In the legend we give ranges of local slopes $w$ obtained for different $\sigma$’s (with error bars on their last digits). For weak $\sigma$ and small $\tau_Q$ the slope $w$ is close to the $-1/2$ characteristic for the pure model (2): fast quenches, when $\tau_Q \ll \sigma^{-2}$, become non-adiabatic far enough from the critical point not to see any effect of weak disorder. At stronger $\sigma$ or longer $\tau_Q$ the local slopes are less steep and for a fixed $\sigma$ they become less steep with increasing $\tau_Q$. For example, at the strongest $\sigma = 0.8$ the local slope falls to a mere $|w| = 0.04$ for the longest $\tau_Q$. These observations are consistent with the predicted logarithmic dependence of the dynamical correlation length $\xi$ in Eq. (17).

V. FIDELITY AND CORRELATIONS AFTER A QUENCH

In the context of adiabatic quantum computation, it is important to know how close the final state $\rho_S(0)$ to the desired ground state of the final Hamiltonian $H_S$ is. The closeness is measured by fidelity

$$F = \langle 0|\rho_S(0)|0 \rangle = \langle 0|\psi(0)\rangle\langle\psi(0)|0 \rangle$$

(40)
given by Eq. (39).

Without decoherence, or in a pure Ising chain with $\sigma = 0$, the fidelity can be obtained analytically from the exact solution in Refs. (16, 20). $F$ is a probability that not a single pair of $\gamma(0)$-quasiparticles with opposite quasimomenta $(k, -k)$ is excited after a quench, $F = \prod_{k>0}(1 - p_k)$. Here $p_k \approx \exp(-2\pi\tau_Q k^2)$ when $\tau_Q \gg 1$. When $Nd \gg 1$ we obtain

$$\ln F \approx -\frac{Nd}{2} \int_0^\infty ds \ln \left[1 - e^{-s^2}\right] \approx -1.3Nd.$$ (41)
Here \( d = \int_{-\pi}^{\pi} \frac{4k}{2\pi} p_k = 1/2\pi \sqrt{2\tau_Q} \). \( F \) is exponential in \( N \) as in the static case Eq. (31). Given that \( d \ll 1 \) for \( \tau_Q \gg 1 \) we obtain that \( c \approx 1.3 > 1 \) implying anti-bunching of kinks.

The anti-bunching can also be seen in the state \(|\psi(0)\rangle\) after a quench which is a BCS state of kinks in Eq. (29). Figure 4A shows a probability distribution \( P_n \) for a kink in Eq. (37) and the correlation function \( C_r \) between two kinks in a Cooper pair in Eq. (30) after a quench with \( \sigma = 0 \) and \( \tau_Q = 16 \). We have \( C_0 = 0 \) because the kinks are fermions, there is a broad maximum in \( C_r \) in the range \(|r| = 20 \ldots 40 \), and a flat distribution for \(|r| > 40 \) when the Cooper pair is dissociated. If \( C_r \) were flat everywhere (except \( r = 0 \)), then \( c = 1 \), but the broad maximum means that even when the kinks get close to each other they prefer to keep a safe distance in the range \( 20 \ldots 40 \). This short range repulsion is consistent with the anti-bunching observed in Eq. (41) where \( c = 1.3 > 1 \).

The same anti-bunching can also be seen in the ferromagnetic correlation function after a quench at \( g = 0 \)

\[
\langle \sigma_i^z \sigma_{i+R}^z \rangle \approx \exp(-1.55R_d) \cos(2.95R_d - \varphi)
\]

(42)

accurate when \( 1 \ll R \ll \sqrt{Q} \log \tau_Q \), see Ref. [20]. The oscillatory cosine factor means that kinks tend to order into a crystal lattice. This very imperfect crystalline order implies the anti-bunching seen in \( c = 1.3 > 1 \). We can conclude that in the state after a quench with \( \sigma = 0 \) there is a similar connection between \( c, C_r \), and ferromagnetic correlations as in the static case.

For quenches with \( \sigma > 0 \) we also find an exponential tail

\[
F(\tau_Q, \sigma) \sim (1 - cd)^N
\]

(43)

when \( F \ll 1 \), compare panels B and D in Fig. 4. Here \( d(\tau_Q, \sigma) \) is an asymptotic value of average kink density.
obtained for a sufficiently large lattice size $N$, such that $Nd(\tau_Q, \sigma) \gg 1$, see panels A and C in the same Fig. 4. In contrast, when $N \ll 1/d(\tau_Q, \sigma)$, then a finite gap at $g = g_c$ results in an adiabatic transition and a final density of kinks which is less than the asymptotic $d(\tau_Q, \sigma)$ for large $N$.

Fits to the exponential tails of fidelity in Figs. 4B and D give correlation coefficients $\sigma$ for different $\tau_Q$ and $\sigma$ which are shown in Fig. 5. For comparison, we also show in the same figure $c$ at $\sigma = 0$ which saturates at $c = 1.3$ when $\tau_Q \gg 1$. The results suggest that for a strong $\sigma$ or large $\tau_Q$, when even a weak $\sigma$ has strong effect, the correlation coefficient decays towards $c \approx 1$ suggesting random Poissonian trains of kinks and exponential ferromagnetic correlation functions.

This picture is corroborated by typical correlation functions $C_r$ and probability distributions $P_n$ shown in panels B, C, D, and E of Fig. 6. Here it is clear, especially for the larger $\tau_Q = 1024$ (green) or the stronger $\sigma = 0.8$ (panels C and E), that a kink gets localized in isolated Anderson localization centers. A close inspection of corresponding $C_r$’s reveals that $C_r$ is localized at $r$’s equal to distances between the localization centers in $P_n$. There is only one exception from this rule: $C_r$ is negligible when $r \approx 0$ or $N$ because, apparently, a (Cooper) pair of fermionic kinks avoids being trapped in the same localization center. It seems that each kink in a Cooper pair chooses its localization center at random, independently of the other kink, except for avoiding the same localization center. A quantitative proof of this simple picture is provided in Figs. 7B and C where we plot a convolution

$$PP_r = \sum_{n=1}^{N} P_n P_{n+r} \quad (44)$$

If the two kinks in a Cooper pair were independent, then this convolution would reconstruct the corresponding $C_r$’s in Figs. 7B and C. Comparing panels B and C in the two figures we see that it does, with the expected exceptions when $r \approx 0, N$ and in the case of a fast quench ($\tau_Q = 16$) at weak $\sigma = 0.1$ (black in Fig. 7B). We can conclude that, with some idealization, for long $\tau_Q$ or strong $\sigma$, final kinks are distributed as if each kink were choosing at random one of the isolated Anderson localization centers randomly distributed along the chain. This approximate Poissonian model is consistent with the observed $c \approx 1$.

When $\tau_Q \sigma^2 \gg 1$, then the final states at $g = 0$ are qualitatively different from the ground state of the random Hamiltonian \( \delta d = d - d(0, \sigma) \) whose origin can be attributed to non-adiabaticity, is small as compared to $d(0, \sigma)$.

These striking properties of the final states after a quench are inherited from the ground state of the random Hamiltonian \( g \leq g_c \) just above $g_c$. Its properties are relevant here because, as we could see in Fig. 4 in a slow quench a state $|\psi(t)\rangle$ follows an adiabatic ground state of the random Hamiltonian until a $\hat{g}$ just above $g_c$ when, in accordance with KZM, the evolution becomes non-adiabatic. In Fig. 8 we show some properties of the ground state just above $g_c$ where $P_n$ is fragmented into isolated localization centers, and $C_r$ is identical with $PP_r$ everywhere except $r = 0$. Apparently, the following evolution from $\hat{g}$ to $g = 0$ does not change this qualitative picture.

**Conclusion**

A static spin environment increases non-adiabaticity of the transition in a dramatic way; density of quasiparticles (kinks) decays no longer as a power of the transition time but in a much slower logarithmic way. This means, in the context of adiabatic quantum computation, that coupling to a static environment may transform a polynomial computational problem into a non-polynomial one.
Fidelity between a final state after a quench and the desired final ground state decays exponentially with a chain size. The rate of this decay is equal to the density of kinks times a correlation coefficient equal, for fast transitions and weak decoherence, to 1, and for slow transitions or strong decoherence, close to 1. Corresponding kink-kink correlations are, respectively, anti-bunching in a near-crystalline ordering and a simple Poissonian distribution of kinks in isolated Anderson localization centers randomly distributed along a chain.

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