Uniform Linked Lists Contraction

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Abstract. We present a parallel algorithm (EREW PRAM algorithm) for linked lists contraction. We show that when we contract a linked list from size \( n \) to size \( n/c \) for a suitable constant \( c \) we can pack the linked list into an array of size \( n/d \) for a constant \( 1 < d \leq c \) in the time of 3 coloring the list. Thus for a set of linked lists with a total of \( n \) elements and the longest list has \( l \) elements our algorithm contracts them in \( O(n \log i/p + (\log^i n + \log i) \log \log l + \log l) \) time, for an arbitrary constructible integer \( i \), with \( p \) processors on the EREW PRAM, where \( \log^i n = \log n \) and \( \log^{i+1} n = \log \log^{(i+1)} n \) and \( \log n = \min\{i|\log^i n < 10\} \). When \( i \) is a constant we get time \( O(n/p + \log l) \) time. The previous best deterministic EREW PRAM algorithm has time \( O(n/p + \log n) \) and best CRCW PRAM algorithm has time \( O(n/p + \log n/\log \log n + \log l) \).

Keywords: Parallel algorithms, linked list, linked list contraction, uniform linked list contraction, EREW PRAM.

1 Introduction

Linked list contraction refers to the pairing of neighboring nodes on the linked list and replacing the two nodes with a newly created node in order to have the linked list contracted. In parallel computation we need to do this pairing-off for many pairs of nodes concurrently such that in one round the size of the linked list shrinks from \( n \) to \( n/c \) for a constant \( c > 1 \). Linked list contraction requires this process repeated such that the original linked list is finally contracted to a single node. Linked list contraction can be used to rank the nodes in the linked list such that nodes along the linked list receiving numbers 1, 2, 3, ..., \( n \).

Linked list contraction and/or ranking can then be used for many other computations for linked lists, trees, and graphs. Thus linked list contraction is regarded as a basic or fundamental computation in parallel computation.

The model we use is the EREW (Exclusive Read Exclusive Write) PRAM (Parallel Random Access Machine) \[8,9\]. PRAM is a machine where memory is shared among all processors. In one step on the EREW (CREW (Concurrent Read Exclusive Write), CRCW (Concurrent Read Concurrent Write)) PRAM multiple processors cannot (can, can) read the same memory cell and cannot (cannot, can) write the same memory cell. When concurrent write happens on the CRCW PRAM an arbitration rule needs to be used to determine what content is written into the memory cell. On the ARBITRARY CRCW PRAM an arbitrary processor will succeed in writing when concurrent write happens.
CRCW PRAM processors write into the same memory cell in a step have to write the same value and this value is written into the memory cell.

Early work on linked list contraction include the work of Wyllie [12] and Kruskal et al. [10]. Han achieved $O(n/p + \log n)$ time for linked list contraction of $n$ nodes on the CRCW PRAM with $p$ processors [4]. At the time it was difficult to achieve $O(n/p + \log n)$ time on the EREW PRAM because it was not known how to do uniform linked list contraction and dynamically balancing the load is difficult. In 1991 Anderson and Miller came up with a very clever accounting scheme to calculate the dynamic load and obtained an EREW PRAM algorithm [1] for linked list ranking (contraction) with time $O(n/p + \log n)$ (conference version of their paper was published in 1988). The existence of the log $n$ additive factor in the complexity of their algorithm comes from two causes. One is the pointer jumping [12] and the other reason is the dynamic load balancing scheme of their algorithm because their algorithm requires the contracted linked list be packed before the final pointer jumping stage [12]. We are unable to reduce the time complexity for pointer jumping and therefore in the case of contracting a single linked list of size $n$ we cannot improve on the $O(n/p + \log n)$ time complexity of Anderson and Miller’s algorithm. However, if we are contracting many linked lists stored in an array of size $n$ and the longest linked list has size $l$ then the only application of pointer jumping [12] will rank the linked lists in $O(n \log l/p + \log l)$ time. Therefore, the open question is that in the case of contracting or ranking multiple linked lists can we do better. The ideal case here is to achieve $O(n/p + \log l)$ time. Anderson and Miller’s algorithm cannot achieve this because their load balancing scheme cannot guarantee uniform contraction. Here we use uniform contraction to mean when linked lists are contracted they can be stored in a packed array using the same time for 3 coloring a linked list so that processors can be used efficiently to the contracted lists.

In [4,7] we showed that by resorting to the sublogarithmic CRCW array prefix sum algorithm [11] we can contract linked lists in $O(n/p + \log n / \log^{(3)} n + \log l)$ time. In [2] it is improved to $O(n/p + \log n / \log \log n + \log l)$ time due to the fact that array prefix sum can be done in $O(n/p + \log n / \log \log n)$ time [2].

Previously Han gave an $O(n \log i/p + \log^{(i)} n + \log i)$ time EREW PRAM algorithm [5] for finding a maximal independent set $i$ or a maximal matching for linked lists or a 3-coloring of the linked lists, where $i$ can be any constructible integer. Here we use this algorithm and we apply some new ideas of ours and reach deterministic time $O(n \log i/p + (\log^{(i)} n + \log i) \log \log l + \log l)$ for linked lists contraction on the EREW PRAM, where $n$ is the total number of nodes in all linked lists, $p$ is the number of processors used, $i$ is any constructible integer, $l$ is the length of the longest linked list.

In particular if $l = \Omega(\log^c n)$ for a constant $c$ we can reach $O(n/p + \log l)$ time.

Our uniform linked list contraction algorithm will pack the linked list of size $n/c$ for a constant $c$, after linked list 3 coloring and pairing-off nodes, into an array of size $n/d$ for a constant $1 < d \leq c$, using
the same time for linked list 3 coloring. Thus, if linked list 3 coloring can be done in constant time then the above linked lists contraction can be accomplished in $O(n/p + \log l)$ time for any values of $l$.

2 Preliminary

Linked list contraction is done by first finding a maximal independent set of the linked list nodes and then let nodes in the maximal independent set pair-off their neighboring node(s) with themselves (combine one’s neighboring node and itself into one node and thus the name of contraction). An independent set is a set of nodes such that there is no edge between any two nodes in the set. A maximal independent set is an independent set such that it cannot be enlarged and remains to be an independent set. If two nodes $v_1$ and $v_2$ in the maximal independent set intend to pair-off the same (non)maximal independent set node $w$ then an arbitrary one of $v_1$ or $v_2$ wins and succeeds in the pairing-off with $w$. A maximal independent set can be found after linked lists nodes are 3 colored. (Here 2 colored is OK also but 2 coloring is much slower than 3 coloring, use a maximum independent set is also OK but finding a maximum independent set is much slower than finding a maximal independent set).

Initially we may assume that linked lists are stored in a two dimensional array $A$ of 2 rows and $n/2$ columns. We show that, using the same time for 3 coloring a linked list, we can contract the linked list in $A$ and pack remaining nodes of the linked list to the bottom row of $A$.

As linked lists are contracted they remain in this array $A$ although many cells in $A$ are empty now. In order to design an efficient parallel algorithm we need to pack the remaining linked lists to a smaller size array. However, this packing operation is painful. We may assign 0’s to the remaining cells containing contracted linked lists nodes and 1’s to empty cells (where the node of the linked list is paired-off with other node of the list) and then sort array elements by these 0’s and 1’s in order to pack the remaining linked lists nodes to the front of the array. As is well known that this 0-1 sorting can be done by array prefix sum and it takes $O(n/p + \log n)$ time on the EREW PRAM (or $O(n/p + \log n/ \log \log n)$ time on the CRCW PRAM), where $p$ is the number of processors used. The additive factor $\log n$ on the EREW PRAM is not a pleasant factor and prevents the achievement of an optimal algorithm for linked list contraction on the EREW model.

Anderson and Miller [1] came up with a very clever way of overcoming this packing difficulty. They used a dynamic load balancing scheme to achieve the $O(n/p + \log n)$ time on the EREW PRAM for linked list contraction. However, because they do not pack linked list they cannot guarantee that the remaining linked list nodes after contraction are assigned evenly to all processors. This situation can be better explained by considering contracting multiple linked lists with the longest list has length $l$. These linked lists can be “contracted” by pointer jumping [12] in $\log l$ time with $n$ processors. If Anderson and Miller algorithm [1] is used to contract these linked lists it can be done in $O(n/p + \log n)$ time, but it cannot be done in $O(n/p + \log l)$ time. And this $O(n/p + \log l)$ time is what we are aiming for in this
paper. Basically, we show that we can pack the remaining contracted nodes of the linked lists in the same time for computing a 3 coloring, and this is what we call uniform linked list contraction.

The performance of our uniform linked list contraction algorithm is limited by the time for 3 coloring a linked list. If 3 coloring can be done in $T$ time, then we can pack the contracted linked lists into an array of size $n/d$ for a constant $d > 1$ in $T$ time.

3 The Algorithm

3.1 Localization

We use the localization technique. Localization basically says that we can separate linked lists into two or more (constant number of) parts such that we can view that there are no links between any two of these parts (each part is localized). Here we show how to store linked lists in two rows with each row containing $n/2$ nodes and then localize linked lists to each row (such that all links between two rows are deleted). In principle we can divide linked lists into any two parts and then localize them, i.e. we need not assume that any one part of the two parts are stored in a continuous block of memory.

Algorithm Localize

1. Store linked lists of $n$ nodes in a $2 \times n/2$ array $A$ such that each column has 2 nodes of the linked list. Call the 0th row as the upper level and 1st row as the lower level. We are going to localized the upper row as well as the lower row.

2. Our first goal is to let a sufficient long (at least 100 nodes but no more than 300 nodes) of a sublist to run at the same level (either upper level or lower level). In order to achieve this we do (a): first contract sublist runs at the lower level and if a sublist has less than 100 nodes running at the lower level before it connects nodes at the upper level then we contract all these nodes and eliminate them by having them contracted to the nodes at the upper level. If the sublist running at lower level has at least 100 nodes then we uncontract them and let this sublist run at the lower level. Then we do (b): contract sublist runs at upper level and if a sublist has less than 100 nodes running at the upper level before it connects nodes at the lower level then we contract all these nodes and eliminate them by having them contracted to the nodes at the lower level. If the sublist running at upper level has at least 100 nodes then we uncontract them and let this sublist run at the upper level but “delete” the links that connecting a node at lower level to an end node of this sublist at upper level (there are 2 such links for each sublist). In this way we now have many (sub)lists running either at the upper level or at the lower level.

The “deletion” of links connecting between upper level and lower level is for the purpose of viewing that the linked lists are localized. It does not mean that we will really cut any input linked list to multiple
lists. The localization is mainly to prevent linked list to alternatively going up and down between the upper level and the lower level, i.e. prevent sublist of the form \( a_1, b_2, a_3, b_4, \ldots \), where \( a_i \)'s are at the upper level and \( b_i \)'s are at the lower level.

We can repeatedly apply this localizing algorithm to localize linked lists to a constant number of parts. For example, we can localize linked lists to 4 rows.

### 3.2 Uniform Contraction

We first localize the input linked list to two rows of \( A \). Then we will 3 coloring the linked list on the bottom row. If a node is colored 2 instead of 0 or 1 then we will contract this node into one of its neighboring node. Thus now linked list on the bottom row is 0-1 colored. We then form 0-1 pairs. This is done by letting each node colored with 1 arbitrarily pairs with one of its neighboring node colored with 0. If two nodes colored with 1 both try to pair with the same 0 colored node then we let the 1 node with larger address wins the pair. Nodes colored with 0 or 1 did not form pairs will be contracted into the neighboring nodes that formed pairs. Thus now we have the linked list at the bottom row of \( A \) having nodes paired into 0-1 pairs.

We now do the same for the linked list at the top row of \( A \) and form 0-1 pairs for nodes in the linked list on the top row of \( A \).

Now comes the key step. If two nodes at the bottom row \( A[i, 0] \) and \( A[j, 0] \) form a 0-1 pair we want to enforce that \( A[i, 0] \) and \( A[j, 0] \) at the top row are colored with the same color. We call this the uniformity step. Note that, in general, \( A[i, 0] \) and \( A[j, 0] \) do not form 0-1 pair, for if \( A[i, 0] \) and \( A[j, 0] \) form a 0-1 pair then we contract \( A[i, 0] \) and \( A[j, 0] \) into one node and place it at \( A[i, 0] \), and we contract \( A[i, 0] \) and \( A[j, 0] \) into one node and place it at \( A[i, 0] \).

To achieve the uniformity step, let us look at case here: \( A[i, j_0] \) and \( A[i, j_1] \) form a 0-1 pair and \( A[j_0, 0] \) and \( A[j_1, 0] \) are of different color. Say \( A[j_0, 0] \) is colored with 0 and \( A[j_1, 0] \) is colored with 1. Now look at the colors of \( A[j_0, 1] \), \( A[j_1, 1] \), \( A[j_0, 3] \) and \( A[j_1, 3] \). If \( A[j_0, 1] \) is colored with 0 then let \( A[j_0, 1] \) be the pair of \( A[j_0, 3] \) that is colored with 1. Let \( A[j_0, 1] \) and \( A[j_1, 1] \) form a 0-1 pair. If \( A[j_0, 3] \) is colored with 0 then we swap the position of \( A[j_0, 3] \) and \( A[j_1, 3] \) and thus accomplishing the uniformity step. If \( A[j_0, 3] \) is colored with 1 then we contract \( A[j_0, 3] \) and \( A[j_1, 3] \) into one node and place it at \( A[j_0, 3] \) and we move \( A[j_0, 3] \) to \( A[j_1, 3] \) and thus accomplish the uniformity step.

If \( A[j_0, 1] \) is colored with 1 then we form a chain (directed linked list) of \( A[j_0, 1], A[j_1, 1], A[j_2, 1], A[j_3, 1], \ldots \), where \( A[j_0, 1], A[j_2, 1], A[j_4, 1], \ldots \) are colored with 0 and \( A[j_1, 1], A[j_3, 1], A[j_5, 1], \ldots \) are colored with 1. A \( A[j_0, 2] \) and \( A[j_2, 2] \) form a 0-1 pair. We group \( A[j_0, 2] \) and \( A[j_2, 2] \) into one node because \( A[j_0, 2] \) and \( A[j_2, 2] \) is a pair. Thus, we get a directed linked list \( L \) of nodes \( a_0, a_1, a_2, \ldots \), where \( a_0 \) represents \( A[j_0, 0] \) and \( A[j_0, 1] \), \( a_1 \) represents \( A[j_0, 2] \) and \( A[j_0, 3] \), \( \ldots \). We 3 color the nodes of \( L \). If \( A[j_0, 2] \) and \( A[j_2, 2] \) is colored 0, \( A[j_0, 2] \) and
\( A[0, j_{2i+3}] \) is colored with 1, \( A[0, j_{2i+4}] \) and \( A[0, j_{2i+1+5}] \) is colored with 2, then we swap \( A[0, j_{2i+1}] \) and \( A[0, j_{2i+2}] \), contract \( A[0, j_{2i+3}] \) and \( A[0, j_{2i+4}] \) into one node and place it at \( A[0, j_{2i+4}] \), move \( A[0, j_{2i+5}] \) to \( A[0, j_{2i+3}] \). Thus accomplishing the uniformity step. If \( A[0, j_{2i}] \) and \( A[0, j_{2i+1}] \) is colored 0, \( A[0, j_{2i+2}] \) and \( A[0, j_{2i+3}] \) is colored with 1, \( A[0, j_{2i+4}] \) and \( A[0, j_{2i+1+5}] \) is colored with 0, then we swap \( A[0, j_{2i+1}] \) and \( A[0, j_{2i+2}] \) and thus accomplishing the uniformity step.

Note some pairs at the top row have be contracted into one node but this node is the only node at the top row for the pairs at the same column and bottom row. That is, if \( A[0, j_{2i+1}] \) and \( A[0, j_{2i+2}] \) are contracted into one node and placed at \( A[0, j_{2k}] \) then \( A[0, j_{2k+1}] \) is vacant. Thus, we can form 0-1 pairs for these contracted nodes and need not worry about the uniformity requirement.

Now we enforce the uniformity requirement for the bottom row. That is for each 0-1 pair at the top row \( A[0, i] \) and \( A[0, j] \) we require that \( A[1, i] \) and \( A[1, j] \) be colored with the same color. The approach is the same as we did for the uniformity of the top row.

After we enforced uniformity step, the colors of \( 2 \ast \text{color}(A[0, j]) + \text{color}(A[1, j]) \) along the linked list form periodic pattern of 0, 1, 3, 2 and thus it provides an orientation of the linked list \( A[*], j, A[*], j + 1, A[*], j + 2, ... \), with links running alternatively at the top and bottom rows. This linked list orientation then allows us to contract the nodes at \( A[0, j_{2i+1}] \) and \( A[0, j_{2i+2}] \) into one node and place it at \( A[1, j_{2i+2}] \) and contract nodes at \( A[1, j_{2i}] \) and \( A[1, j_{2i+1}] \) into one node and place it at \( A[1, j_{2i+1}] \).

**Main Theorem:** Linked lists can be uniformly contracted. \( \square \)

Because we can 3 color linked lists in \( O(n \log i/p + \log^{(i)} n + \log i) \) time and therefore if the longest linked list has length \( l = \Omega(\log^{(c)} n) \) for a constant \( c \) then we can 3 color linked lists in \( O(n/p + \log^{(c)} n) \) time. We can place input linked lists on a two dimensional array of \( \log^{(c+1)} n \) rows and \( n/\log^{(c+1)} n \) columns. If we use \( p = n/\log^{(c)} n \) processors then each row can be 3 colored in \( O(n/(p \log^{(c+1)} n)) \) time and be contracted into the rows below. Thus, we eliminate one row in \( O(n/(p \log^{(c+1)} n)) \) time. When the last row remains we can use pointer jumping to finish the linked lists contraction. Thus, it takes \( O(n/p + \log l) \) time to do the linked lists contraction.

**Corollary 1:** If multiple linked lists are placed into an array of size \( n \) and the longest linked list has length \( l = \Omega(\log^{(c)} n) \) then linked lists contraction can be done in \( O(n/p + \log l) \) time using \( p \) processors on the EREW PRAM. \( \square \)

In general because we compute 3 coloring in \( O(n \log i/p + \log^{(i)} n + \log i) \) time with \( p \) processors, thus if we use \( n/\log l \) processors then after \( O(\log l \log i + (\log^{(i)} n + \log i) \log \log l) \) steps the size of the array containing all unfinished (contraction) linked lists has size \( n/\log l \) and we can use pointer jumping to
finish the linked lists contraction in $O(\log l)$ time. Thus we have

**Corollary 2:** If multiple linked lists are placed into an array of size $n$ then linked lists contraction can be done in $O(n \log i/p + (\log^i n + \log i) \log \log l + \log l)$ time using $p$ processors on the EREW PRAM.

4 Conclusion

We presented a deterministic algorithm for uniform linked list contraction. Because it is uniform and thus it brings some advantages such as for linked lists contraction.

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