Confirmation of the Secondary Constraint and Absence of Ghost in Massive Gravity and Bimetric Gravity

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Abstract: In massive gravity and in bimetric theories of gravity, two constraints are needed to eliminate the two phase-space degrees of freedom of the Boulware-Deser ghost. For recently proposed non-linear theories, a Hamiltonian constraint has been shown to exist and an associated secondary constraint was argued to arise as well. In this paper we explicitly demonstrate the existence of the secondary constraint. Thus the Boulware-Deser ghost is completely absent from these non-linear massive gravity theories and from the corresponding bimetric theories.

Keywords: massive gravity, bimetric gravity.
1. Introduction and summary

Generic theories of massive gravity and bimetric gravity contain a Boulware-Deser ghost mode [1] which renders these theories unstable. To avoid this inconsistency, the equations of motion of a healthy, ghost-free massive gravity or bimetric theory must contain two constraints which eliminate both the ghost field and the momentum canonically conjugate to it. In [2, 3, 4] it was established that one such constraint, the Hamiltonian constraint, exists in a special class of massive gravity and bimetric theories. In [2, 3, 4] an associated secondary constraint was argued to arise as well. Here we rigorously prove the existence of the secondary constraint and obtain an explicit expression for it.

The class of theories being considered are extensions of the massive gravity actions originally proposed in [5, 6] where they were formulated with a flat reference metric. In these works, the proposed actions were shown to be ghost-free in the decoupling limit and to possess a Hamiltonian constraint up to fourth order in perturbation theory. For complementary analyses, see [7, 8]. For other recent, related work, see [9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. For a recent review on massive gravity, see [19].

The full non-linear analysis of the Hamiltonian constraint was performed in [4, 5] and is based on a reformulation of these models presented in [20], where they are also extended to an arbitrary reference metric. In [4] a ghost-free bimetric theory based on these massive gravity theories was proposed in which the reference metric is promoted to a dynamical field. The existence of the Hamiltonian constraint for the bimetric theory was also established in [4]. For recent work related to the bimetric theory, see [21, 22, 23].
More specifically, the analyses of [2, 3, 4] showed that both the massive gravity and bimetric theories contain a constraint, say $C = 0$, that can be regarded as a generalization of the Hamiltonian constraint in General Relativity. A condition for consistency is the preservation of this constraint in time, $dC/dt = 0$. This condition will lead to a secondary constraint, provided the Poisson bracket of $C$ with itself vanishes on the constraint surface, $\{C(x), C(y)\} \approx 0$. In [2, 3, 4] this bracket was not computed, but was assumed to vanish considering that it vanishes in two very different limits of massive gravity: the zero mass limit, which is General Relativity, and the linearized limit, which is the linear massive Fierz-Pauli theory [24, 25]. It was then argued that a non-trivial secondary constraint should exist in the non-linear theory, as an extension of the secondary constraint in the Fierz-Pauli theory. This led to the conclusion that the Boulware-Deser ghost is eliminated by the two constraints in the non-linear theory.

However, it was recently argued in [26] that in massive gravity, the Poisson bracket $\{C(x), C(y)\}$ does not vanish at the non-linear level, hence the theory does not contain a secondary constraint. If true, this would imply that, even though the ghost field is eliminated by the Hamiltonian constraint, the momentum canonically conjugate to the ghost is not eliminated. The theory would thus have an odd dimensional phase space – an odd situation indeed. In this paper, we explicitly compute the Poisson bracket in massive gravity and show that $\{C(x), C(y)\} \approx 0$ indeed holds. Then we explicitly compute the secondary constraint $C^{(2)}$. We also argue that no other associated constraints exist. This result is then generalized from massive gravity to bimetric gravity. This conclusively proves the complete absence of the Boulware-Deser ghost both in non-linear massive gravity, as well as in non-linear bimetric gravity.

The apparent contradiction of this result with the work [26] can be understood in the following way. The constraint analysis of [26] is based on a straightforward counting of Lagrange multipliers and equations of motion. However, as the existence of the Hamiltonian constraint itself demonstrates, such a counting can be misleading. If the equations of motion are degenerate, then an equation which would otherwise determine a Lagrange multiplier instead becomes a constraint on the dynamical variables. The analysis of [26] is insensitive to this possibility, while here we show that this is indeed the case.

The paper is organized as follows: In section 2 we review non-linear massive gravity and its Hamiltonian constraint. Section 3 proves the existence of the secondary constraint and obtains an explicit expression for it. The absence of higher constraints is argued here as well. Section 4 extends the results to bimetric gravity.

2. Review of the Hamiltonian constraint in massive gravity

In this section we review the ADM formulation of massive gravity and the derivation of the Hamiltonian constraint based on [2, 3].

2.1 Massive gravity in ADM variables

We start by reviewing the derivation of the Hamiltonian constraint of the massive gravity theories proposed in [3, 4], as was carried out in [2, 3]. Here, we construct these theories
with respect to an arbitrary but non-dynamical reference metric \( f_{\mu \nu} \). In the last section of this paper we will discuss the generalization of these arguments to a theory with a dynamical \( f_{\mu \nu} \).

The most general massive gravity actions can be written as \[ S = M_p^2 \int d^4x \sqrt{-\det g} \left[ R + 2m^2 \sum_{n=0}^{3} \beta_n e_n(\sqrt{g^{-1}f}) \right], \] where the square root of the matrix is defined such that \( \sqrt{g^{-1}f} \sqrt{g^{-1}f} = g^{\mu \lambda} f_{\lambda \nu} \). The \( e_k(\sqrt{g^{-1}f}) \) are elementary symmetric polynomials of the eigenvalues \( \lambda_n \) of the matrix \( \sqrt{g^{-1}f} \). For a \( 4 \times 4 \) matrix \( \mathcal{X} \) and using the notation \( [M] = \text{tr} M \), they can be written as,

\[
\begin{align*}
e_0(\mathcal{X}) &= 1, \\
e_1(\mathcal{X}) &= [\mathcal{X}], \\
e_2(\mathcal{X}) &= \frac{1}{2}([\mathcal{X}]^2 - [\mathcal{X}^2]), \\
e_3(\mathcal{X}) &= \frac{1}{6}([\mathcal{X}]^3 - 3[\mathcal{X}][\mathcal{X}^2] + 2[\mathcal{X}^3]), \\
e_4(\mathcal{X}) &= \frac{1}{24}([\mathcal{X}]^4 - 6[\mathcal{X}][\mathcal{X}^2] + 3[\mathcal{X}^2]^2 + 8[\mathcal{X}][\mathcal{X}^3] - 6[\mathcal{X}^4]), \\
e_k(\mathcal{X}) &= 0 \quad \text{for} \quad k > 4.
\end{align*}
\]

The \( \beta_n \) correspond to four free parameters, two of which are the graviton mass and the cosmological constant.

The physical content of these theories is most easily identified in the ADM formulation \[ [27] \] which is based on a 3 + 1 decomposition of the metric. Let \( N \) and \( N_i \) denote the lapse and shift functions of the metric \( g_{\mu \nu} \) while \( L \) and \( L_i \) denote the lapse and shift functions of the metric \( f_{\mu \nu} \), so that

\[
\begin{align*}
g_{\mu \nu}: \quad & N = (-g^{00})^{-1/2}, \quad N_i = g^{0i}, \quad \gamma_{ij} = g_{ij}, \\
f_{\mu \nu}: \quad & L = (-f^{00})^{-1/2}, \quad L_i = f^{0i}, \quad ^3f_{ij} = f_{ij}.
\end{align*}
\]

In massive gravity \( f_{\mu \nu} \) is non-dynamical. The lapse and shift variables \( N_\mu \) appear without time derivatives in \[ [2.1] \] and are thus non-dynamical as well. The components \( \gamma_{ij} \) are dynamical and, along with the their canonically conjugate momenta \( \pi_{ij} \), constitute 12 phase-space degrees of freedom, i.e., six potentially propagating modes. Of these, five correspond to the massive graviton while the sixth one, if not eliminated by the constraint equations, is the Boulware-Deser ghost.

In terms of these variables, the Lagrangian in \[ [2.1] \] becomes,

\[
\mathcal{L} = \pi_{ij} \partial_t \gamma_{ij} + N R^0 + R_i N^i + 2m^2 \sqrt{\det \gamma} N \sum_{n=0}^{3} \beta_n e_n(\sqrt{g^{-1}f}),
\]

where, as in General Relativity,

\[
R^0 = \sqrt{\det \gamma} \gamma^{ij} R + \frac{1}{\sqrt{\det \gamma}} \left( \frac{1}{2} \pi^i_j \pi^j_i - \pi^{ij} \pi_{ij} \right), \quad R^i = 2 \sqrt{\det \gamma} \gamma_{ij} \nabla_k \left( \frac{\pi^{jk}}{\sqrt{\det \gamma}} \right).
\]
The $N_\mu$ equations of motion derived from (2.5) contain a single constraint on the variables $\gamma_{ij}$ and $\pi^{ij}$. This is the Hamiltonian constraint which is necessary (though not sufficient) to remove the ghost. However, since the potential in (2.5) is highly nonlinear in the lapse $N$, the existence of a Hamiltonian constraint is far from apparent. To make this constraint manifest, we have to work with the appropriate set of variables. We define these in the following subsection.

2.2 The new shift-like variables

The existence of the Hamiltonian constraint is due to the fact that the four $N_\mu$ equations of motion in fact depend on only three functions $n^i$ (for $i = 1, 2, 3$) of the four variables $N_\mu$. Thus three combinations of these equations determine the $n^i$ and the remaining equation is the Hamiltonian constraint. This can be made explicit if we work with the new shift-like variables $n^i$ which are related to the ADM variables of (2.3) and (2.4) through (see [3] for details),

$$N^i - L^i = (L \delta^i_j + N D^i_j) n^j,$$

where $N^i = \gamma^{ij} N_j$ and $L^i = 3 f^{ij} L_j$. The matrix $D = D^i_j$ is determined by the condition,

$$\sqrt{x} D = \sqrt{(\gamma^{-1} - D nn^T D^T)^{3f}} ,$$

where,

$$x \equiv 1 - n^i (3 f^{ij}) n^j.$$

In (2.8) we have introduced the notation $n$ for the column vector $n^i$, and $n^T$ for its transpose. We will occasionally use this matrix notation when the meaning is clear. The matrix $D$ has the following important property

$$3 f^{ik} D^k_j = 3 f^{jk} D^k_i .$$

This identity will be used often to simplify our calculations. Although $D$ can be determined in terms of $n^i$, the explicit solution is not needed here.

We can use (2.7) to eliminate the shifts $N^i$ in favor of $n^i$. The resulting form of the Lagrangian (2.5) is then linear in $N^i$. It can be compactly written as,

$$\mathcal{L} = \pi^{ij} \partial_t \gamma_{ij} - \mathcal{H}_0 + \mathcal{C} .$$

Here $\mathcal{H}_0$ and $\mathcal{C}$ stand for

$$\mathcal{H}_0 = -(L n^i + L^i) R_i - 2m^2 L \sqrt{\det \gamma} U,$$

$$\mathcal{C} = R^0 + R_i D^i_j n^j + 2m^2 \sqrt{\det \gamma} V .$$

The functions $U$ and $V$ appearing above are given by the following expressions,

$$U \equiv \beta_1 \sqrt{x} + \beta_2 \left[ \sqrt{x^2} D^i_i + n^i f^{ij} D^j_k n^k \right] + \beta_3 \left[ \sqrt{x} (D^i_i n^i f^{ij} D^j_k n^k - D^i_k n^i n^j f^{ij} D^j_i n^l) + \frac{1}{2} \sqrt{x^3} (D^i_i D^j_j - D^i_j D^j_i) \right] ,$$

$$V \equiv \beta_0 + \beta_1 \sqrt{x} D^i_i + \frac{1}{2} \beta_2 \sqrt{x^2} \left[ D^i_i D^j_j - D^i_j D^j_i \right] + \frac{1}{6} \beta_3 \sqrt{x^3} \left[ D^i_i D^j_j D^k_k - 3 D^i_i D^j_j D^k_k + 2 D^i_j D^j_i D^k_k \right] .$$
The massive gravity Lagrangian is now given entirely in terms of $\gamma_{ij}, \pi^{ij}, N$ and $n^i$. It also involves the ADM parameters of $f_{\mu\nu}$ as non-dynamical fields.

### 2.3 The Hamiltonian constraint

In the Lagrangian (2.11), the four variables $N$ and $n^i$ are non-propagating while the six modes described by ($\gamma_{ij}, \pi^{ij}$) potentially propagate and include the ghost field and its conjugate momentum. The Hamiltonian constraint is identified after imposing the $n^k$ equations of motion. These were derived in [2, 3]. Here we outline the derivation. Varying $L$ in (2.11) with respect to $n^k$ gives the equations of motion,

$$
\frac{\partial L}{\partial n^k} = -\frac{\partial H_0}{\partial n^k} + N \frac{\partial C}{\partial n^k} = 0. \tag{2.16}
$$

To determine the variation of the two terms in (2.16), it is helpful to use the following relations, derived from the expression (2.8),

$$
\frac{\partial}{\partial n^k} \text{tr}(\sqrt{x} D) = \frac{1}{\sqrt{x}} n^T J_f \frac{\partial (Dn)}{\partial n^k},
$$

$$
\frac{\partial}{\partial n^k} \text{tr}(\sqrt{x} D)^2 = -2 n^T J_f D \frac{\partial (Dn)}{\partial n^k},
$$

$$
\frac{\partial}{\partial n^k} \text{tr}(\sqrt{x} D)^3 = -3 \sqrt{x} n^T J_f ^2 \frac{\partial (Dn)}{\partial n^k}. \tag{2.17}
$$

One then obtains the useful expressions,

$$
\frac{\partial H_0}{\partial n^k} = -LC_k , \quad \frac{\partial C}{\partial n^k} = C_i \frac{\partial (D^i_j n^j)}{\partial n^k}. \tag{2.18}
$$

Note that both these variations are proportional to $C_i$, given by,

$$
C_i = R_i - 2m^2 \sqrt{\det \gamma} \frac{f_{ij} f^{ij}}{\sqrt{x}} \left[ \beta_1 \delta_i^j + \beta_2 \sqrt{x} \left( \delta_i^j D^m_m - D_i^j \right) 
\right. 

\left. + \beta_3 \sqrt{x} \left( \frac{1}{2} \delta_i^j (D^m_m D^m_n - D^m_n D^m_m) + D_i^m D^m_i - D_i^j D^m_m \right) \right]. \tag{2.19}
$$

Thus the variation of $L$ becomes,

$$
\frac{\partial L}{\partial n^k} = C_i \left[ L \delta_i^j + N \frac{\partial (D^i_j n^j)}{\partial n^k} \right] = 0. \tag{2.20}
$$

The matrix within the square brackets is the Jacobian of the transformation (2.7) and is invertible. Hence the $n^i$ equations of motion are

$$
C_i (\gamma, \pi, n) = 0. \tag{2.21}
$$

These are independent of the lapse $N$ and can in principle be solved to determine $n^i$ in terms of $\gamma_{ij}$ and $\pi^{ij}$. So far, the explicit solution is known only for the minimal massive action corresponding to $\beta_2 = \beta_3 = 0$. But, as shown below, the equations (2.21) alone are
enough to demonstrate the absence of the ghost. Note that (2.21) implies the vanishing of the individual variations in (2.18).

Now, varying the Lagrangian with respect to $N$ gives,

$$C(\gamma, \pi, n) = 0.$$  \hfill (2.22)

On eliminating the $n^i$ using the solutions$^1$ of (2.21), this equation becomes a constraint on the 12 dynamical variables $\gamma_{ij}$ and $\pi^{ij}$, reducing the number of independent degrees of freedom to 11. This is the Hamiltonian constraint. The degree of freedom it eliminates is the ghost field. One more constraint is necessary to eliminate the momentum canonically conjugate to the ghost field. This is obtained in the next section.

3. The secondary constraint

In this section we show that the Hamiltonian constraint gives rise to a secondary constraint and we obtain its explicit expression. This new constraint eliminates the momentum canonically conjugate to the ghost. Thus there remain 10 degrees of freedom out of an original 12.

3.1 Existence of a secondary constraint

In order to be consistent, the Hamiltonian constraint $C = 0$ must be preserved under the time evolution of $\gamma_{ij}$ and $\pi^{ij}$, that is, $dC/dt = 0$. This requirement can be implemented in the Hamiltonian formulation. From the Lagrangian (2.11), the Hamiltonian is given by,

$$H = \int d^3x \left( H_0 - NC \right).$$  \hfill (3.1)

In this expression the $n^i$ are determined by their equations of motion (2.21), while $N$ is a Lagrange multiplier. The only dynamical variables are $\gamma_{ij}$ and $\pi^{ij}$.

In the Hamiltonian formulation, the consistency condition on the time evolution of the constraint becomes,

$$\frac{d}{dt} C(x) = \{ C(x), H \} = 0,$$  \hfill (3.2)

where the Poisson bracket is defined as,

$$\{ C(x), H \} = \int d^3z \left( \frac{\delta C(x)}{\delta \gamma_{mn}(z)} \frac{\delta H}{\delta \pi^{mn}(z)} - \frac{\delta C(x)}{\delta \pi^{mn}(z)} \frac{\delta H}{\delta \gamma_{mn}(z)} \right).$$  \hfill (3.3)

The $\delta$ denote total variations, including dependence on $\gamma_{ij}$ and $\pi^{ij}$ through the $n^k$. Using (3.1), the consistency condition (3.2) becomes,

$$\{ C(x), H \} = \int d^3y \left( C(x), H_0(y) \right) - \int d^3y N(y) \{ C(x), C(y) \} = 0.$$  \hfill (3.4)

It is necessary for this condition to hold only on the constraint surface, i.e., when $C = 0$. When imposing the above consistency condition, two possibilities could arise:

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$^1$That this is always possible follows from the fact that when the $n^i$ equations of motion are satisfied we have $\partial C/\partial n^i = 0$. Hence on the space of solutions, $C$ depends only on $\gamma_{ij}$ and $\pi^{ij}$.  

---
1. $\{C(x), C(y)\} \neq 0$, where the symbol $\approx$ is used for equalities that hold on the constraint surface, i.e., when $C = 0$. In this case (3.4) becomes an equation for $N$ and does not act as a constraint on $\gamma_{ij}$ and $\pi^{ij}$. This leads to a theory with an odd dimensional phase space, or equivalently, with 5.5 propagating modes. Such a possibility was raised in [20] and was argued to be the case in massive gravity.

2. $\{C(x), C(y)\} \approx 0$. In this case, the condition (3.4) is independent of $N$ and thus acts as a secondary constraint on $\gamma_{ij}$ and $\pi^{ij}$,

$$C_{(2)}(x) = \{C(x), H_0\} = 0.$$ (3.5)

where, $H_0 = \int d^3 y \mathcal{H}_0(y)$. The secondary constraint eliminates the momentum canonically conjugate to the ghost field and leads to a theory with 5 propagating modes. The Boulware-Deser ghost mode is thus entirely absent. This is the scenario advocated in [2, 3, 4].

In this section we will explicitly show that massive gravity indeed corresponds to the second case given above and is, therefore, entirely free of the Boulware-Deser ghost. We will show in section four that this argument also holds for the theories of bimetric gravity based on these massive gravity actions [4].

To prove the existence of the secondary constraint, we now show that,

$$\{C(x), C(y)\} \approx 0.$$ (3.6)

To evaluate the Poisson bracket, note that $C$ depends on $\gamma_{ij}$ and $\pi^{ij}$ explicitly as well as implicitly, through the $n^i$. However, from (2.18) it follows that when the $n^i$ equations of motion are satisfied one has $\delta C_{\partial n^i} = 0$. Thus the most general variation with respect to the canonical variables is given by,

$$\delta C = \frac{\delta C}{\delta \gamma_{mn}} \delta \gamma_{mn} + \frac{\delta C}{\delta \pi^{mn}} \delta \pi^{mn},$$ (3.7)

where the derivatives are now evaluated at fixed $n^i$. This means that the Poisson bracket (3.6) can be evaluated by replacing the total variations $\delta$ by the partial ones evaluated at fixed $n^i$. From now on we only consider these types of variations.

Now we can compute $\{C(x), C(y)\}$ using the expression (2.13) for $C$. To simplify, note that $R^0$ and $V$ depend, respectively, only on $\pi$ and $\gamma$ and not on their derivatives. Thus the term $\{R^0(x), \sqrt{\det \gamma} V(y)\}$ is proportional to the Dirac delta function $\delta^3(x - y)$. The net contribution of this term and its conjugate, $-\{R^0(y), \sqrt{\det \gamma} V(x)\}$, to the bracket $\{C(x), C(y)\}$ is therefore zero, due to the antisymmetry of the bracket. Taking this into account gives,

$$\{C(x), C(y)\} = \{R^0(x), R^0(y)\} + \{R_i(x), R_j(y)\} D^i_k n^k(x) D^j_l n^l(y) + \{R^0(x), R_i(y)\} D^i_k n^k(y) - \{R^0(y), R_i(x)\} D^i_k n^k(x) + S^{mn}(x) \frac{\delta R_i(y)}{\delta \pi^{mn}(x)} D^i_k n^k(y) - S^{mn}(y) \frac{\delta R_i(x)}{\delta \pi^{mn}(y)} D^i_k n^k(x).$$ (3.8)
Here, the quantity $S^{mn}$ stands for

$$S^{mn}(x) = R_j(x) \frac{\delta(D^j, n^r)}{\delta \gamma_{mn}}(x) + 2m^2 \frac{\delta(\sqrt{\det \gamma} V)}{\delta \gamma_{mn}}(x). \quad (3.9)$$

The expression (3.8) involves the standard Poisson brackets of General Relativity (see, for example, [28]),

$$\{ R^0(x), R^0(y) \} = - \left[ R^i(x) \frac{\partial}{\partial x^i} \delta^3(x - y) - R^i(y) \frac{\partial}{\partial y^i} \delta^3(x - y) \right],$$

$$\{ R^0(x), R_i(y) \} = - R^0(y) \frac{\partial}{\partial y^i} \delta^3(x - y), \quad (3.10)$$

$$\{ R_i(x), R_j(y) \} = - \left[ R_j(x) \frac{\partial}{\partial x^i} \delta^3(x - y) - R_i(y) \frac{\partial}{\partial y^j} \delta^3(x - y) \right].$$

These terms as well as the remaining terms in (3.8) involve derivatives of the delta function.

In order to manipulate these unambiguously, we introduce localized smoothing functions $f(x)$ and $g(y)$, and define

$$F \equiv \int d^3x f(x) C(x), \quad G \equiv \int d^3y g(y) C(y). \quad (3.11)$$

Then, the Poisson bracket $\{ F, G \}$ is given by

$$\{ F, G \} = \int d^3x \int d^3y f(x) g(y) \{ C(x), C(y) \}. \quad (3.12)$$

We will first compute this bracket and then from it extract $\{ C(x), C(y) \}$.

To determine the last two terms in (3.8), we note that, on using the expression (2.6) for $R_j$, for any vector field $v^j = \gamma^{jk} v_k$ we have,

$$\int d^3x \frac{\delta R_j(x)}{\delta \gamma_{mn}(y)} v^i(x) = - [\nabla_m v_n(y) + \nabla_n v_m(y)]. \quad (3.13)$$

Using (3.10) and (3.13) in expression (3.8), and then carrying out one of the integrals in (3.12), one gets,

$$\{ F, G \} = - \int d^3x \left[ f R^i \partial_i g + f \partial_i (g R^0 D^j, n^k) + f R_i D^j, n^k \partial_j (g D^j, n^l) + 2f S^{mn} \nabla_m (g \gamma_{nj} D^j, n^k) - (f \leftrightarrow g) \right], \quad (3.14)$$

where everything under the integral is now a function of $x$. Again, due to the antisymmetry of the right-hand-side under the interchange of $f$ and $g$, the only terms that do not cancel are the ones where a derivative directly acts on $f$ or $g$. These can be written as

$$\{ F, G \} = - \int d^3x \left( f \partial_i g - g \partial_i f \right) P^i, \quad (3.15)$$

where,

$$P^i = (R^0 + R_j D^j, n^k) D^i, n^l + R^i + 2S^{il} \gamma_{lj} D^j, n^k. \quad (3.16)$$
To extract the Poisson bracket \( \{ \mathcal{C}(x), \mathcal{C}(y) \} \) from (3.15), we write \( g(x) = \int d^3y \, g(y) \, \delta^3(x-y) \) and perform an analogous rewriting of \( f \). Then, comparing with (2.12) gives,

\[
\{ \mathcal{C}(x), \mathcal{C}(y) \} = - \left[ P^i(x) \frac{\partial}{\partial x^i} \delta^3(x-y) - P^i(y) \frac{\partial}{\partial y^i} \delta^3(x-y) \right].
\] (3.17)

The expression for \( P^i \) can now be simplified. This is the key step in our proof of the existence of the secondary constraint. First, in the expression for \( S^{mn} \) (3.9), we eliminate \( R_j \) in favor of \( \mathcal{C}_j \) using (2.19). We then compute \( \partial V / \partial \gamma_{mn} \) using the following identities that are derived from the relation (2.5),

\[
\begin{align*}
\frac{\partial}{\partial \gamma_{mn}} \text{tr}(\sqrt{x} D) &= -\frac{1}{\sqrt{x}} \left( n^T 3 f \frac{\partial (Dn)}{\partial \gamma_{mn}} - \frac{3}{2} f \frac{D^{-1}}{\partial \gamma_{mn}} \right), \\
\frac{\partial}{\partial \gamma_{mn}} \text{tr}(\sqrt{x} D)^2 &= -2 \left( n^T 3 f D \frac{\partial (Dn)}{\partial \gamma_{mn}} - \frac{3}{2} f D \frac{D^{-1}}{\partial \gamma_{mn}} \right), \\
\frac{\partial}{\partial \gamma_{mn}} \text{tr}(\sqrt{x} D)^3 &= -3 \sqrt{x} \left( n^T 3 f D^2 \frac{\partial (Dn)}{\partial \gamma_{mn}} - \frac{3}{2} f D \frac{D^{-1}}{\partial \gamma_{mn}} \right).
\end{align*}
\] (3.18)

This gives,

\[
S^{mn}(x) = C_i \frac{\partial (D^i n^j)}{\partial \gamma_{mn}} + m^2 \sqrt{\det \gamma} (V \gamma^{mn} - \bar{V}^{mn}),
\] (3.19)

where,

\[
\bar{V}^{mn} \equiv \gamma^{mj} \left[ \beta_1 \frac{1}{\sqrt{x}} 3 f_{ik} (D^{-1})^k_j + \beta_2 3 f_{ik} (D^{-1})^k_j D^l_i - 3 f_{ij} \\
+ \beta_3 \sqrt{x} \left( 3 f_{ik} D^k_j - 3 f_{ij} D^k_k + \frac{1}{2} 3 f_{ik} (D^{-1})^k_j (D^l_i D^h_j - D^l_i D^h_l) \right) \right] \gamma^n.
\] (3.20)

Since \( C_i \) is zero by the \( n^i \) equations of motion, this is simply

\[
S^{mn}(x) = m^2 \sqrt{\det \gamma} (V \gamma^{mn} - \bar{V}^{mn}).
\] (3.21)

Thus \( P^i \) (3.10) becomes,

\[
P^i = \left( R^0 + R_j D^j_i n^k + 2 m^2 \sqrt{\det \gamma} V \right) D^ijn^l + R^i - 2 m^2 \sqrt{\det \gamma} \bar{V}^{ijl} D^j_k n^k.
\] (3.22)

The expression within the parentheses is simply \( C \) (2.13). Also, from the expression for \( \bar{V}^{mn} \) (3.20) and on using (2.10), it is easy to verify that the remaining terms give \( C_i \) (2.19). Thus, (3.22) is in fact, \( P^i = C D^i n^l + C_i \gamma^{li} \). Again, since \( C_i \) is zero on the \( n^i \) equations of motion, this is equivalent to

\[
P^i = C D^i n^l.
\] (3.23)

Thus the Poisson bracket (3.17) finally becomes

\[
\{ \mathcal{C}(x), \mathcal{C}(y) \} = - \left[ C(x) D^i j n^l(x) \frac{\partial}{\partial x^i} \delta^3(x-y) - C(y) D^i j n^l(y) \frac{\partial}{\partial y^i} \delta^3(x-y) \right].
\] (3.24)
As \( C(x) = 0 \) on the constraint surface, it is clear that,

\[
\{ C(x), C(y) \} \approx 0.
\]  
(3.25)

This corresponds to case 2 discussed following equation (3.4) and proves the existence of a secondary constraint.

### 3.2 Evaluation of the secondary constraint

The consistency condition \( dC/dt = 0 \) now acts as a secondary constraint given by (3.3),

\[
C^{(2)}(x) = \int d^3y \{ C(x), H_0(y) \} = 0.
\]  
(3.26)

To compute the Poisson bracket, note again equation (2.18),

\[
\frac{\partial H_0}{\partial n^k} = -L C_k, \quad \frac{\partial C}{\partial n^k} = C_i \frac{\partial (D^i_{jk} n^j)}{\partial n^k}.
\]  
(3.27)

Given that \( C_i \) vanishes on the \( n^i \) equations of motion, we can therefore consider variations of both \( C \) and \( H_0 \) at fixed \( n^i \). Then, using the expressions for \( H_0 \) and \( C \) in (2.12) and (2.13), we obtain,

\[
\left\{ C(x), H_0(y) \right\} = \left\{ -\{ R^0(x), R_i(y) \} (Ln^i + L^i)(y) - (D^i_{jk} n^k)(x) \{ R_i(x), R_j(y) \} (Ln^i + L^i)(y) \right.
\]

\[
+ m^2 L \frac{\delta R^0(x)}{\delta \pi^{mn}(y)} \sqrt{\det \gamma} U^{mn}(y) + m^2 L (D^i_{jk} n^k)(x) \frac{\delta R_i(x)}{\delta \pi^{mn}(y)} \sqrt{\det \gamma} U^{mn}(y)
\]

\[
- S^{mn}(x) \frac{\delta R_i(y)}{\delta \pi^{mn}(x)} (Ln^i + L^i)(y)\right\}.
\]  
(3.28)

where we have introduced the notation,

\[
U^{mn} \equiv \frac{2}{\sqrt{\det \gamma}} \frac{\delta \left( \sqrt{\det \gamma} U \right)}{\delta \gamma_{mn}}.
\]  
(3.29)

\( S^{mn} \) is given by (3.3) and again reduces to the expression (3.21).

The brackets in the first line of (3.28) are the brackets of General Relativity given in (3.10). Under the integral of (3.26), the ordinary derivatives appearing in (3.10) can be consistently converted to covariant derivatives. In addition, using (2.6) and (3.13), we find

\[
\frac{\delta R^0(x)}{\delta \pi^{mn}(y)} = \frac{1}{\sqrt{\det \gamma}} \left( \gamma_{mn}(x) \pi^k(x) - 2 \pi_{mn}(x) \right) \delta^3(x - y),
\]  
(3.30)

\[
\frac{\delta R_i(x)}{\delta \pi^{mn}(y)} = - (\gamma_{im}(x) \nabla_y^n + \gamma_{in}(x) \nabla_y^m) \delta^3(x - y).
\]  
(3.31)

To evaluate the secondary constraint we perform the integration of (3.26). The derivatives of \( \delta^3(x - y) \) that appear in the above expressions can be manipulated under the integral.
After performing various integrations by parts to move the derivatives from the delta functions and after dropping the corresponding boundary terms, we determine the secondary constraint (3.26) to be,

$$C^{(2)} \approx m^2 L \left( \gamma_{mn} \pi^k_k - 2 \pi_{mn} \right) U^{mn} + 2m^2 L \sqrt{\det \gamma} (\nabla_m U^{mn}) \gamma_{mi} D^i_k n^k + \left( R_j D^i_k n^k - 2m^2 \sqrt{\det \gamma} \gamma_{jk} \bar{V}^{ki} \right) \nabla_i (L n^i + L^i) + \sqrt{\det \gamma} \left[ \nabla_i \left( \frac{R^0}{\sqrt{\det \gamma}} \right) + \nabla_i \left( \frac{R_i}{\sqrt{\det \gamma}} \right) D^j_k n^k \right] (L n^i + L^i).$$

Here we have dropped a term $C \nabla_i (L n^i + L^i)$ as this vanishes on the constraint surface.

That the entire secondary constraint does not somehow vanish on the constraint surface can be seen by considering the form of the first term in (3.32) which appears nowhere in either $C$ or the equation of motion $C_i = 0$.

Now the consistency condition $C^{(2)} = 0$ can be used to eliminate the component of $\pi^{ij}$ that is canonically conjugate to the ghost component of $\gamma_{ij}$, which was itself eliminated by the primary constraint $C = 0$. This leaves only five propagating modes, establishing the absence of the Boulware-Deser ghost in massive gravity.

### 3.3 Absence of a tertiary constraint

It is important to verify that no further constraints are generated by the secondary constraint and thus no additional degrees of freedom are eliminated\(^2\). In other words, the consistency condition

$$\frac{d}{dt} C^{(2)}(x) = \{ C^{(2)}(x), H \} = 0,$$

should determine $N$ as a function of the remaining variables, rather than act as a constraint on the dynamical variables. For this to be the case, we must have both

$$\{ C^{(2)}(x), H(0)(y) \} \neq 0 \quad \text{and} \quad \{ C^{(2)}(x), C(y) \} \neq 0.$$

For both conditions, it is sufficient to consider the nonlinear theory at lowest order in $\gamma$ and $\pi$. The theories considered here reproduce the Fierz-Pauli Hamiltonian at lowest order, by construction. Both constraints $C$ and $C^{(2)}$ reduce to the Fierz-Pauli primary and secondary constraints at lowest order. Since the above two conditions are satisfied by the Fierz-Pauli theory, they will also be satisfied by the full nonlinear theory. Thus the above consistency condition (3.33) becomes an equation for the lapse $N$, rather than a tertiary constraint.

### 4. Extension to bimetric gravity

It is straightforward to extend the above analysis to the case of bimetric gravity in which the reference metric $f_{\mu\nu}$ becomes dynamical. The most general bimetric theory is given by

---

\(^2\)There are instances in which we expect massive gravity to have fewer than five propagating modes. For instance, massive gravity in the background of de Sitter spacetime is known to have only four propagating modes when the Higuchi bound is saturated [29, 30]. Such a scenario is describable by the framework presented here. However, we expect that in these cases, the additional mode is removed by an additional gauge symmetry and not by tertiary and quaternary constraints.
We have introduced two different Planck masses for the $f$ and $g$ sectors of the theory and defined an effective mass,

$$M_{\text{eff}}^2 = \left( \frac{1}{M_g^2} + \frac{1}{M_f^2} \right)^{-1}. \quad (4.2)$$

The summation in the above action now runs from $n = 0$ to 4. The contribution of the new $n = 4$ term is

$$\sqrt{-\text{det} g} \beta_4 e_4(\sqrt{g^{-1}f}) = \beta_4 \sqrt{-\text{det} f}. \quad (4.3)$$

Let us rewrite this action in terms of ADM variables, using the same shift-like variables $n^i$ introduced in (2.7), in place of the shift $N^i$. The Lagrangian becomes,

$$\mathcal{L} = M_g^2 \pi^{ij} \partial_t \gamma_{ij} + M_f^2 p^{ij} \partial_t \gamma_{ij} - \mathcal{H}_0 + NC. \quad (4.4)$$

Here $\mathcal{H}_0$ and $C$ stand for

$$\mathcal{H}_0 = -L^i \left( M_g^2 R^{(s)}_i + M_f^2 R^{(f)}_i \right) - L \left( M_f^2 R^{(f)}_i + M_g^2 n^i R^{(g)}_i + 2m^2 M_{\text{eff}}^2 \sqrt{\text{det} \gamma} U' \right), \quad (4.5)$$

$$C = M_g^2 R^{(s)} + M_f^2 R^{(g)} + 2m^2 M_{\text{eff}}^2 \sqrt{\text{det} \gamma} V. \quad (4.6)$$

In order to account for the new term, $\beta_4 \sqrt{-\text{det} f}$, we have introduced

$$\sqrt{\text{det} \gamma} U' \equiv \sqrt{\text{det} \gamma} U + \beta_4 \sqrt{\text{det} f}, \quad (4.7)$$

where $U$ and $V$ in the above expressions are defined as before in (2.14) and (2.15) respectively.

The $p^{ij}$ are the momentum canonically conjugate to the dynamical variables $\gamma_{ij}$. The lapse $L$ and shift $L^i$ of $f_{\mu\nu}$ remain non-dynamical in the bimetric theory. Thus the $\gamma_{ij}$, $\pi^{ij}$, $\gamma_{ij}$ and $p^{ij}$ represent 24 potentially propagating phase-space degrees of freedom. We argue now that the primary constraint $\mathcal{C}$ given above generates a secondary constraint, as in the massive gravity case, eliminating two phase-space degrees of freedom. In addition, we will argue for the existence of eight additional constraints, related to the restored general coordinate invariance of the bimetric theory, that remove an additional eight phase-space degrees of freedom. As a result, the bimetric theory has 14 phase-space degrees of freedom, or seven propagating modes, consistent with the seven modes one expects for a massive spin-2 field and a massless spin-2 field.

The bimetric constraint $\mathcal{C}$ in (4.6) is identical to that defined above for massive gravity (2.13) (up to irrelevant overall factors), but now depends on the dynamical variables $\gamma_{ij}$, both explicitly and through the shift-like variables $n^i$. However, $\mathcal{C}$ is independent of $p^{ij}$,
the momentum canonically to \(3f_{ij}\). To see this, note that the \(ni\) equations of motion, determined from the Lagrangian (4.4), are

\[
M^2 g_i^{(g)} - 2m^2 M^2_{\text{eff}} \sqrt{\text{det} \gamma} \frac{n^i f_{ij}}{\sqrt{\gamma}} \left[ \beta_1 \delta^i_j + \beta_2 \sqrt{\gamma} (\delta^i_m D^j_m - D^j_i) + \beta_3 \sqrt{\gamma}^2 \left( \frac{1}{2} \delta^i_j (D^m_m D^n_n - D^m_m D^n_n) + D^i_m D^m_i - D^j_i D^m_m \right) \right] = 0. \quad (4.8)
\]

These equations can be used to fix the \(ni\) in terms of the dynamical variables. In addition to being independent of \(N, L\) and \(L_i\), these equations are also independent of \(p^{ij}\). Thus \(n^i = n^i(\gamma_{ij}, 3f_{ij}, \pi^{ij})\). As a result, the constraint \(C\) is independent of \(p^{ij}\) as well.

So, when calculating the Poisson bracket \(\{C(x), C(y)\}\) for bimetric gravity, one can in fact consistently treat the metric \(f_{\mu\nu}\) as non-dynamical, since the bracket evaluated with respect to \(\{f_{ij}, p^{ij}\}\) is zero. Thus the proof of existence of the secondary constraint in massive gravity holds for bimetric gravity as well. The primary constraint \(C\) commutes with itself on the constraint surface. Thus the consistency condition \(dC/dt = 0\) acts as a secondary constraint \(C^{(2)}\) on the dynamical variables, \(\gamma_{ij}, \pi^{ij}, 3f_{ij}\) and \(p^{ij}\).

In addition to these two constraints, we note that the Lagrangian (4.4) is manifestly linear in the lapse \(L\) and shift \(L^i\). Thus the variation of the action with respect to these variables produces four additional constraints on the dynamical variables. These four constraints, along with the four general coordinate invariances of the bimetric theory and the primary and secondary constraints \(C\) and \(C^{(2)}\), reduce the initial 24 potentially dynamical phase-space variables to 14, consistent with the seven modes of a massive spin-2 field coupled to a massless spin-2 field.

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