Relation between the left and right cosets of an $\alpha$-normal subgroup

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Abstract. Let $G$ be a group and $\alpha$ be an automorphism of $G$. In 2016, Ganjali and Erfanian introduced the notion of a normal subgroup related to $\alpha$, called the $\alpha$-normal subgroup. It is basically known that if $N$ is an ordinary normal subgroup of $G$ then every right coset $Ng$ is actually the left coset $gN$. This fact allows us to define the product of two right cosets naturally, thus inducing the quotient group. This research investigates the relation between the left and right cosets of the relative normal subgroup. As we have done in the classic version, we then define the product of two right cosets in a natural way and continue with the construction of a, say, relative quotient group.

1. Introduction

One type of group that plays an important role in algebra is the normal subgroup. By the normal subgroup, we mean a subgroup $N$ of a group $G$ whose property that for every $n \in N$ and $g \in G$, $g^{-1}ng \in N$. This definition can be rewritten in several ways, for if we have a subgroup $N$ of $G$ then, the followings are equivalent [1]:
1. $N$ is normal in $G$.
2. For every $n \in N$ and $g \in G$, $g^{-1}Ng = N$.
3. For every $g \in G$, $gN = Ng$.

The property of the normal subgroup $N$ of $G$ allows us to define the product of two right cosets that induces the quotient group $G/N$ consists of all right cosets of $N$ in $G$.

Now suppose that $G$ be a group and $\alpha \in \text{Aut}(G)$. In [2], Ganjali and Erfanian introduced the notion of the normal subgroup that related to $\alpha$, called the $\alpha$-normal subgroup. In detail, a subgroup $N$ of $G$ is said to be an $\alpha$-normal subgroup if for every $n \in N$ and $g \in G$, $g^{-1}\alpha(n)g \in N$. By this definition, we can say that all normal subgroups are $1$-normal, where $1$ denotes the identity mapping of the underlying group. Thus, we may regard the concept of $\alpha$-normal subgroup as the generalization of the concept of a normal subgroup. Further results related to $\alpha$-normal subgroup can be found in [3] and [4].

In study of ordinary normal subgroup, we found that the equality of the left and right cosets was the key in defining the product of two right cosets that induces the structure of the quotient group. Hence, it is proper to investigate the relation between the left and right cosets for the $\alpha$ version. But, before we go,
let us consider the trivial cases: If $N = G$ then $N$ is $\alpha$-normal for any $\alpha \in \text{Aut}(G)$, and if $N = \{e\}$ then $N$ is $\alpha$-normal only for $\alpha = 1$. Thus, in this research, we assume that $N$ is a nontrivial subgroup of $G$.

2. Some basic properties
Before we go further, it is important to list some basic properties related to the concept of $\alpha$-normal subgroup. First, it is not difficult to see that a subgroup $N$ of $G$ is an $\alpha$-normal subgroup if and only if, for every $g \in G$, $g^{-1}N\alpha(g) = N$. So, we have one result similar to the classic version.

Next, notice that if $N$ is an $\alpha$-normal subgroup of $G$ then for every $g \in G$, $g^{-1}\alpha(g) = g^{-1}e\alpha(g) \in N$. Especially for every $n \in N$ we have $\alpha(n) \in N$. Thus, $\alpha(N) \subseteq N$.

Conversely, let $n \in N$. Since $\alpha$ is onto then $n = \alpha(m)$ for some $m \in G$. Now, from the previous result, we have $m^{-1}n = m^{-1}\alpha(m) \in N$. Thus, $m^{-1} \in N$ which means $m \in N$. So, $n = \alpha(m) \in \alpha(N)$ which shows that $N \subseteq \alpha(N)$. We conclude that $\alpha(N) = N$. We summarize these results in the following proposition.

**Proposition 2.1.** Let $G$ be a group, $\alpha \in \text{Aut}(G)$, and $N$ be an $\alpha$-normal subgroup of $G$. Then:
1. For every $g \in G$, $g^{-1}\alpha(g) \in N$, and
2. $\alpha(N) = N$.

3. Method
It is known that for a normal subgroup $N$ of $G$, every left coset $gN$ is actually the right coset $Ng$, for every $g \in G$. Starting from this fact, we then construct the group structure of $G/N$, called the quotient group. Thus, to obtain the analogous results, we start the research by investigating the relation between the left and the right cosets of an $\alpha$-normal subgroup.

Let $G$ be a group and $\alpha \in \text{Aut}(G)$. Suppose that $N$ is an $\alpha$-normal subgroup of $G$. Consider the left coset $gN$ and let $x \in gN$. Write $x = gn$ where $n \in N$. Now, since $N$ is $\alpha$-normal then $g\alpha(g^{-1}) \in N$ and hence $x = gn \in N(\alpha(g^{-1}))^{-1} = N\alpha(g)$. Thus, $gN \subseteq N\alpha(g)$.

Now consider the right coset $N\alpha(g)$ and let $y \in N\alpha(g)$. Write $y = n'\alpha(g)$ where $n' \in N$. Again, since $N$ is $\alpha$-normal then $g^{-1}n'\alpha(g) \in N$ and hence $y = n'\alpha(g) \in gN$. Thus, $N\alpha(g) \subseteq gN$ and hence we conclude that $gN = N\alpha(g)$.

Now suppose that $G$ is a group, $\alpha \in \text{Aut}(G)$, and $N$ is a subgroup of $G$ such that $gN = N\alpha(g)$ for every $g \in G$. Let $g \in G$ and $n \in N$. Since $gn \in gN = N\alpha(g)$, then $gn = n_0\alpha(g)$, for an $n_0 \in N$. Thus, $g\alpha(g^{-1}) = n_0\alpha(g)\alpha(g^{-1}) = n_0 \in N$. Since $g$ and $n$ are any, then we conclude that $N$ is $\alpha$-normal.

From the discussion above, we can conclude that if $G$ is a group, $\alpha \in \text{Aut}(G)$, and $N$ is a subgroup of $G$ then $N$ is $\alpha$-normal if and only if $gN = N\alpha(g)$ for every $g \in G$. Again, we obtained one result similar to the classic version.
4. Product of two right cosets

Let $G$ be a group, $\alpha \in Aut(G)$, and $N$ be an $\alpha$-normal subgroup of $G$. The best way to define the product of two right cosets of $N$ in $G$ is the natural way: if $Na$ and $Nb$ both are right cosets of $N$ in $G$, then $NaNb := \{xy \mid x \in Na, y \in Nb\}$.

Now let us view it further. If $x \in Na$ and $y \in Nb$ then there exist $n_1, n_2 \in N$ such that $x = n_1a$ and $y = n_2b$. Thus, $xy = (n_1a)(n_2b) = n_1(an_2)b$. Since $an_2 \in aN = N\alpha(a)$ then $an_2 = n_2\alpha(a)$, for an $n_2 \in N$. Hence, $xy = n_1(n_2\alpha(a))b = (n_1n_2)\alpha(a)b \in N\alpha(a)b$. Thus, $NaNb \subseteq N\alpha(a)b$.

Now let $z \in N\alpha(a)b$. Write $z = m_o\alpha(a)b$ where $m_o \in N$. Since $m_o\alpha(a) \in N\alpha(a) = aN$ then $m_o\alpha(a) = am_1$ for an $m_1 \in N$. Hence, $z = am_1b = eam_2b \in NaNb$. Thus, $N\alpha(a)b \subseteq NaNb$. From the previous result, we conclude that $NaNb = N\alpha(a)b$.

**Remark** Recall that we define the product of $Na$ and $Nb$ as the set $NaNb$ which is shown to be the same with the right coset $N\alpha(a)b$. Indeed, if $Na = Nc$ and $Nb = Nd$ then we must show that the right cosets $N\alpha(a)b$ and $N\alpha(c)d$ are the same. Note first that in this case we have that both $ac^{-1}$ and $bd^{-1}$ are in $N$, and our goal is to show that $\alpha(a)b(\alpha(c)d)^{-1} \in N$. Now, notice that

$$\alpha(a)b(\alpha(c)d)^{-1} = \alpha(a)bd^{-1}(\alpha(c))^{-1} \in \alpha(a)N\alpha(c)^{-1} = \alpha(a)c^{-1}N$$

from Section 3. Of course, if we can show that $\alpha(a)c^{-1} \in N$ then we are done. Now, since $ac^{-1} \in N$ then $a = nc$ for an $n \in N$. According to Proposition 1,

$$\alpha(a)c^{-1} = \alpha(nc)c^{-1} = \alpha(n)\alpha(c)c^{-1} = \alpha(n)(cc^{-1})^{-1} \in N$$

as desired.

5. Results and discussion

After we succeed in defining the product of two right cosets of an $\alpha$-normal subgroup $N$ of $G$, it is proper to expect that this product defines the group structure on the set $G/N$ consists of all right cosets of $N$ in $G$.

Suppose that $Na$, $Nb$, and $Nc$ are right cosets of $N$ in $G$. Note that

$$(NaNb)Nc = N\alpha(a)bNc = N\alpha(\alpha(a)b)c = N\alpha^2(a)\alpha(b)c$$

while

$$Na(NbNc) = NaN\alpha(b)c = N\alpha(a)\alpha(b)c.$$  

As we expect the product to be associative, we need to show that $\alpha^2(a)\alpha(b)c(\alpha(a)\alpha(b)c)^{-1} \in N$. First, we have

$$\alpha^2(a)\alpha(b)c(\alpha(a)\alpha(b)c)^{-1} = \alpha(\alpha(a)a^{-1}).$$
Recall the Proposition 1. Since \( a\alpha(a^{-1}) \in N \) then its inverse \( \alpha(a)a^{-1} \) must be in \( N \) and thus, \( \alpha^2(a)\alpha(b)c(\alpha(a)\alpha(b)c)^{-1} \in \alpha(N) = N \) as desired. Hence, \((NaNb)Nc = Na(NbNc)\) which means such a product is associative.

Now notice that for every right coset \( Nz \) we have \( NNz = NeNz = N\alpha(e)z = Nz = Nz \) and \( NzN = NzNe = N\alpha(z)e = N\alpha(z) = Nz \). The last equality came from Proposition 1 that \( z\alpha(z^{-1}) \in N \).

We have shown that \( N \) acts as the identity element with respect to the product. Finally, for every \( u \in G \), define \( \bar{u} = \alpha^{-1}(u^{-1}) \). We see that \( Nu\alpha(u) = Nu\alpha(u)\bar{u} \). Now, since \( \alpha(u) = u^{-1} \) then \( (\bar{u}u)\alpha(u) = u^{-1}u^{-1} = u^{-1}\alpha(u) \in N \) by Proposition 1. Thus, \( u\bar{u} \in N \). Let us write \( u\bar{u} = n' \). Then \( \bar{u} = u^{-1}n' \) and hence

\[
\alpha(u)\bar{u} = \alpha(u)u^{-1}n' = \left(u\alpha(u^{-1})\right)^{-1}n' \in N
\]

by the basic property. Thus, \( NuN\bar{u} = N \). We see that for every right coset \( Nu \), the right coset \( N\bar{u} \) is the invers of \( Nu \).

We summarize the results we have obtained so far in the following theorems:

**Theorem 5.1.** Let \( G \) be a group, \( \alpha \in \text{Aut}(G) \), and \( N \) be a subgroup of \( G \). The followings are equivalent:
1. \( N \) is \( \alpha \)-normal in \( G \).
2. For every \( g \in G \), \( g^{-1}N\alpha(g) = N \).
3. For every \( g \in G \), \( gN = N\alpha(g) \).

**Theorem 5.2.** Let \( G \) be a group, \( \alpha \in \text{Aut}(G) \), and \( N \) be an \( \alpha \)-normal subgroup of \( G \). Then, for every \( a, b \in G \), \( NaNb = N\alpha(a)b \). Moreover, the set \( G/N \) consists of all right cosets of \( N \) in \( G \) is a group under this operation.

6. **Conclusion**

If \( G \) is a group, \( \alpha \in \text{Aut}(G) \), and \( N \) is an \( \alpha \)-normal subgroup of \( G \), then we can define a group structure to the set \( G/N \) consists of all right cosets of \( N \) in \( G \). The product of two elements in \( G/N \) is defined by \( NaNb = N\alpha(a)b \) for every \( Na, Nb \in G/N \).

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