BEAUTIFUL PAIRS

PABLO CUBIDES KOVACSICS, MARTIN HILS, AND JINHE YE

Dedicated to Bruno Poizat on the occasion of his 75th birthday

Abstract. We introduce an abstract framework to study certain classes of stably embedded pairs of models of a complete $\mathcal{L}$-theory $T$, called beautiful pairs, which comprises Poizat’s belles paires of stable structures and van den Dries-Lewenberg’s tame pairs of o-minimal structures. Using an amalgamation construction, we relate several properties of beautiful pairs with classical Fraïssé properties.

After characterizing beautiful pairs of various theories of ordered abelian groups and valued fields, including the theories of algebraically, $p$-adically and real closed valued fields, we show an Ax-Kochen-Ershov type result for beautiful pairs of henselian valued fields. As an application, we derive strict pro-definability of particular classes of definable types. When $T$ is one of the theories of valued fields mentioned above, the corresponding classes of types are related to classical geometric spaces and our main result specializes to their strict pro-definability. Most notably, we exhibit the strict pro-definability of a natural space of types associated to Huber’s analytifications. In this way, we also recover a result of Hrushovski-Loeser on the strict pro-definability of stably dominated types in algebraically closed valued fields, which corresponds to Berkovich’s analytifications.

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1. Introduction

In their paper [24], E. Hrushovski and F. Loeser presented a novel perspective on non-archimedean semi-algebraic geometry, with far-reaching implications for the topology of semi-algebraic subsets of Berkovich analytifications of algebraic varieties. Their approach introduces certain spaces of definable types as a model-theoretic counterpart of the analytification of an algebraic variety, which they call its stable completion. These spaces possess the remarkable property of being a strict pro-definable set, i.e., a projective limit of definable sets with surjective transition maps. This structural property endows the spaces with a definable structure that is in strong resemblance with familiar definable sets and enables Hrushovski and Loeser to leverage several model-theoretic tools, leading to the aforementioned applications.

Is there also a model-theoretic analogue of Huber’s adic spaces [25]? More specifically, is there a strict pro-definable structure on such spaces? In [13], the first and third authors provided a partial positive answer to this question by showing that the natural candidate for such a space has the structure of a pro-definable set. However, the crucial issue of strict pro-definability, i.e., the surjectivity of transition maps, proved to be significantly more intricate and remained unresolved. Thanks to the formalism presented in this paper, we can now provide a complete positive answer not only for Huber’s analytification of an algebraic variety but also for other notable geometric spaces such as the real analytification of semi-algebraic sets as defined in [26]. In other words, our work lays the foundations for a model-theoretic approach to non-archimedean geometry in various contexts.

Besides the geometric motivation, the formalism introduced here also holds significant model-theoretic importance. Rather than studying spaces of definable types directly, we instead investigate stably embedded pairs of models of a given theory $T$, following the approach outlined in [13]. The study of pairs of models of a complete theory is a well-established topic in model theory, dating back to early results of A. Robinson [38] on pairs of algebraically closed fields. B. Poizat [34] later introduced the notion of belles paires of models of a complete stable theory and extended Robinson’s results to this context. Y. Baisalov and Poizat [3] subsequently built on the work of D. Marker and C. Steinhorn [31] and A. Pillay [33] to develop similar ideas for stably embedded pairs of o-minimal structures, i.e., elementary pairs $(N, M)$ in which the trace in $M$ of every $N$-definable set is again $M$-definable. The formalism presented here unifies the stable and o-minimal settings into a more general context of stably embedded pairs of models of a complete theory $T$. In keeping with Poizat’s terminology, we refer to the pairs under consideration as beautiful pairs.

The following subsection provides a brief summary of the paper’s main results.

1.1. Summary of main results and new concepts. The main theorem of the present article is the following (see later Theorem 7.4.3, Theorem 7.4.6 and Theorem 7.4.8).

Main Theorem. The model-theoretic counterparts of the following spaces are strict pro-definable:

1. the analytification of an algebraic variety in the sense of Huber [25],
2. the Berkovich analytification of a (real) semi-algebraic set in the sense of [26],
3. the Zariski-Riemann space of an algebraic variety [49],
4. the $p$-adic spectrum of a $p$-adic algebraic variety in the sense of [41].

The proof of the Main Theorem uses properties of certain pairs of valued fields, which we call beautiful pairs. We established an Ax-Kochen-Ershov principle for such pairs in particular. However, the general machinery of beautiful pairs turned out to work in a completely abstract framework, encompassing not only valued fields but also other interesting structures and their
respective theories. As a result, this abstract formalism yields numerous significant findings about the corresponding first order structures. We will now describe our technical setup for beautiful pairs in complete generality.

For a complete first order $\mathcal{L}$-theory $T$, we distinguish a collection of well-behaved classes $\mathcal{K}$ of stably embedded pairs of $\mathcal{L}$-structures, called natural classes (Definition 2.2.8) and, for a given infinite cardinal $\lambda$, we define the notion of $\lambda$-$\mathcal{K}$-beautiful pairs as those pairs which are $\lambda$-rich (Definition 2.3.1). To have an intuition, the reader may think of natural classes and $\lambda$-$\mathcal{K}$-beautiful pairs in analogy to Fraïssé classes and Fraïssé limits, respectively. Poizat’s treatment of beautiful pairs of models of a stable theory transfers surprisingly smoothly to this more general context. However, some new phenomena arise from the fact that in an unstable theory not all types are definable. In a way, our framework generalizes the stable case in a direction which is orthogonal to the generalization to so-called “lovely pairs”, introduced by I. Ben-Yaacov, A. Pillay and E. Vassiliev in [5].

The following resumes some of the main results about beautiful pairs (see later Theorem 2.3.5, Corollary 2.3.6 and Theorem 2.3.12) in this general context.

**Theorem A.** Let $\mathcal{K}$ be a natural class. Then the following holds.

1. $\mathcal{K}$ has the amalgamation property if and only if $\lambda$-$\mathcal{K}$-beautiful pairs exist for all $\lambda \geq |T|^+$. Moreover, in this case, $\mathcal{K}$ has the extension property if and only if all beautiful pairs are elementary pairs of models of $T$.
2. $\mathcal{M} \equiv_{\infty, \lambda} \mathcal{N}$ for any two $\lambda$-$\mathcal{K}$-beautiful pairs $\mathcal{M}$ and $\mathcal{N}$, and therefore, all $\mathcal{K}$-beautiful pairs are elementarily equivalent.
3. If there is a $\lambda$-$\mathcal{K}$-beautiful pair which is $\lambda$-saturated for $\lambda \geq |T|^+$ (we call this property beauty transfer), then the common theory of $\mathcal{K}$-beautiful pairs $T_{\text{bp}}(\mathcal{K})$ admits quantifier elimination (in the language of beautiful pairs, see Definition 2.2.1), and the predicate $P$ is stably embedded and pure (i.e., there is no new induced structure).

Dually, one may represent natural classes in terms of global definable types (see Definition 2.4.1), and the following result is the desired link with strict pro-definability of spaces of definable types.

**Theorem B** (Later Theorem 2.4.8). Let $\mathcal{F}$ be a natural subclass of definable types and let $\mathcal{K}_{\mathcal{F}}$ be its associated natural class. Suppose $T_{\text{bp}}(\mathcal{K}_{\mathcal{F}})$ exists and has beauty transfer. Then $\mathcal{F}$ is strict pro-definable.

It is from Theorem B that the Main Theorem is derived. We also recover the strict pro-definability result of Hrushovski and Loeser [24] mentioned above.

Along the way, we characterize all completions of stably embedded pairs of models of $T$ when $T$ is one of the following theories: the theory of divisible ordered abelian groups, Presburger arithmetic and any completion of algebraically (resp. real and $p$-adically) closed valued fields. Moreover, we study beautiful pairs of certain henselian valued fields (called benign) by considering their $\mathcal{R}V$-theory, or even the theories of their residue field and value group. One of our main results consists in showing the following “Ax-Kochen-Ershov principle for beauty” (see also Theorem 8.1.2). For a benign valued field $(K,v)$ with value group $\Gamma$ and residue field $k$, we consider definitional $\mathcal{L}_{\mathcal{R}V}$-expansions with an additional set of sorts $\mathcal{A}$ for $k^*/(k^*)^n$ for all $n \geq 0$ and $\Gamma$ for the value group (with the natural quotients maps). Given natural classes $\mathcal{K}_{\mathcal{A}}$ and $\mathcal{K}_{\Gamma}$ of $\mathcal{S}E$-pairs (see Definition 2.2.4) in the theories of $k$ (in the sorts $\mathcal{A}$) and $\Gamma$, respectively, assume both $\mathcal{K}_{\mathcal{A}}$-beautiful pairs and $\mathcal{K}_{\Gamma}$-beautiful pairs exist, are elementary and satisfy beauty transfer.
Theorem C (Later Theorem 8.3.1). Let \( \mathcal{K} \) be the class of \( \mathcal{L}_{bp} \)-structures of valued fields \( \mathcal{M} \in \mathcal{K}_{\text{def}} \) induced by \( \mathcal{K}_{\text{cf}} \) and \( \mathcal{K}_{\Gamma} \). Then, \( \mathcal{K} \)-beautiful pairs exist. Moreover, \( T_{bp}(\mathcal{K}) \) has beauty transfer and is axiomatised by the following conditions on an \( \mathcal{L}_{bp} \)-structure \( \mathcal{M} = (M, P(M)) \):

- \( \mathcal{VF}(M) / \mathcal{VF}(P(M)) \) is us-defectless;
- \( P(M) \preceq M \models T \);
- \( (\mathcal{A} (M), \mathcal{A} (P(M))) \models T_{bp}(\mathcal{K}_{\text{cf}}) \) and \( (\mathcal{G}(M), \mathcal{G}(P(M))) \models T_{bp}(\mathcal{K}_{\Gamma}) \).

Part of the proof consists in showing a similar reduction result for short exact sequences in the framework recently investigated by M. Aschenbrenner, A. Chernikov, A. Gehret and M. Ziegler [1] (see Theorem 5.2.5).

Structure of the paper. The paper is organized as follows. Section 2 is divided in two main parts: in 2.1-2.3 we introduce the abstract framework of beautiful pairs and prove some of their main properties; in 2.4 we exhibit the relation with spaces of definable types and strict pro-definability. Section 3 is devoted to showing how our abstract framework is related to previous work, both for stable and \( \omega \)-minimal theories. In most of the remaining sections we study particular theories and their corresponding beautiful pairs. Section 4 focuses on ordered abelian groups. Beautiful pairs of short exact sequences are studied in Section 5. In Section 6 we gather various domination results in valued fields which are later used in Section 7 and Section 8 to study beautiful pairs of various theories of henselian valued fields and prove the Main Theorem.

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2. General theory of beautiful pairs

2.1. Preliminaries and notation. Let \( \mathcal{L} \) be a possibly multi-sorted language, \( T \) be a complete \( \mathcal{L} \)-theory and \( \mathcal{U} \) denote a universal domain (monster model) of \( T \). The sorts in \( \mathcal{L} \) are called the real sorts, while imaginary sorts are sorts in \( \mathcal{L}^{eq} \). Given a subset \( A \subseteq \mathcal{U} \) we let \( \langle A \rangle_{\mathcal{L}} \) denote the \( \mathcal{L} \)-substructure of \( \mathcal{U} \) generated by \( A \) and often omit the subscript \( \mathcal{L} \) when it is clear from the context.

Recall that given a subset \( A \subseteq M \models T \), a type \( p \in S_X(M) \) is \( A \)-definable (or definable over \( A \)) if for every \( \mathcal{L} \)-formula \( \varphi(x, y) \) there is an \( \mathcal{L}(A) \)-formula \( d_p \varphi(y) \) such that for every \( c \in M^y \)

\[
\varphi(x, c) \in p(x) \iff M \models d_p \varphi(c).
\]

The map \( \varphi(x, y) \mapsto d_p \varphi(y) \) is called a scheme of definition for \( p \), and the formula \( d_p \varphi(y) \) is called a \( \varphi \)-definition for \( p \). We say \( p \in S_X(M) \) is definable if it is \( M \)-definable. Given any set \( B \) containing \( M \), we use \( p|B \) to denote the type \( \{ \varphi(x, b) \mid b \in B^y \} \) such that \( \mathcal{U} \models d_p \varphi(b) \). We let \( S^\text{def}_X(A) \) denote the subset of \( S_X(\mathcal{U}) \) of \( A \)-definable types, and for an \( A \)-definable set \( X \), we let \( S^\text{def}_X(A) \) be the set of global \( A \)-definable types concentrating on \( X \). We refer the reader to [32, Section 1] for proofs and basic facts on definable types.

2.1.1. Definition. We say that \( T \) has uniform definability of types if for every \( \mathcal{L} \)-formula \( \varphi(x, y) \) there is an \( \mathcal{L} \)-formula \( \psi(y, z) \) such that for every model \( M \) of \( T \) and every definable type \( p \in S_X(M) \) there is a \( c \in M^z \) such that \( \psi(y, c) \) is a \( \varphi \)-definition for \( p \).

2.1.2. Remark. An alternative definition of uniform definability of types would be to require a uniform scheme as in the previous definition for definable \( \varphi \)-types instead of definable types in \( S_X(M) \). We will not consider this variant in the present article.
2.1.3. **Convention.** Throughout the remaining of the paper, unless otherwise stated, we assume that $T$ and $T^\text{eq}$ have quantifier elimination, and that $T$ has uniform definability of types. In addition, up to working in $\mathcal{L}^\text{eq}$, we will fix a uniform scheme of definition $\varphi(x,y) \mapsto d(\varphi)(y, z_\varphi)$ such that for every definable type $p \in S_\varphi(\mathcal{U})$ there is a unique $c = c(p, \varphi) \in \mathcal{U}^\text{eq}$ such that $d(\varphi)(y, c)$ is a $\varphi$-definition for $p$. The element $c$ is called a $\varphi$-canonical parameter for $p$.

2.2. **Stably embedded pairs.** We let $\mathcal{L}_P$ denote the language of pairs of $\mathcal{L}$-structures. Recall that an elementary pair of structures $M \preceq N \models T$ is stably embedded if for every finite tuple $a$ in $N$, $\text{tp}(a/M)$ is definable. We let $T_{\mathcal{SE}}$ be the $\mathcal{L}_P$-theory of stably embedded elementary pairs (note the class of such pairs is $\mathcal{L}_P$-elementary since $T$ has uniform definability of types).

2.2.1. **Definition.** We define the language of beautiful pairs $\mathcal{L}_{bp}$ (depending on $T$) as the following extension of $\mathcal{L}_P$. First, we add to $\mathcal{L}_P$ all $\mathcal{L}^\text{eq}$-sorts restricted to the predicate $P$. In addition, we add functions symbols $c_\varphi$ for each $\mathcal{L}^\text{eq}$-formula $\varphi(x,y)$ where $x$ is a tuple of variables in the real sorts and $y$ is a tuple of variables in $\mathcal{L}^\text{eq}$.

2.2.2. **Convention.** From now on, unless otherwise stated, every $\mathcal{L}_{bp}$-structure will satisfy the following axiom scheme, stating that the function $c_\varphi$ selects $\varphi$-canonical parameters:

$$(\forall x)(\forall y \in P)(\varphi(x,y) \leftrightarrow d(\varphi)(y, c_\varphi(x)))$$

where $\varphi(x,y)$ is an $\mathcal{L}^\text{eq}$-formula, $x$ is a tuple of real variables and $y$ is an arbitrary tuple of variables in $\mathcal{L}^\text{eq}$.

2.2.3. **Notation.** We denote $\mathcal{L}_{bp}$-structures by $\mathcal{A} = (A, P(\mathcal{A}))$. Note that this notation allows $A$ to be empty even when $P(\mathcal{A})$ is not.

2.2.4. **Definition.** An $\mathcal{L}_{bp}$-structure $\mathcal{A} = (A, P(\mathcal{A}))$ is called a stably embedded pair, in short, $se$-pair, if for any tuple of variables $x$ and $a \in A^x$, there is a model $N$ of $T$ containing $P(\mathcal{A})$ such that the set

$$\{\varphi(x,b) : \varphi(x,y) \text{ an } \mathcal{L}^\text{eq} \text{-formula with } x \text{ real, } b \in N^y, N \models d(\varphi)(b, c_\varphi(a))\}$$

is an element of $S_\varphi(N)$. We let $\mathcal{K}_{\text{def}}$ denote the class of all $se$-pairs.

2.2.5. **Remark.** The class $\mathcal{K}_{\text{def}}$ of all $se$-pairs is universally axiomatizable.

2.2.6. **Notation.**

- Given an $se$-pair $\mathcal{A} = (A, P(\mathcal{A}))$ and a small subset $C \subseteq \mathcal{U}^\text{eq}$ containing $P(\mathcal{A})$, we write $\text{tp}_{\mathcal{L}}(A/P(\mathcal{A}))|C$ for the unique type over $C$ given by the uniform scheme of definition encoded in the $\mathcal{L}_{bp}$-structure. Note that such a type is well-defined by the definition of $se$-pair.

- Given a substructure $A \subseteq \mathcal{U}^\text{eq}$, we associate to $A$ the $se$-pair $(A \cap \mathcal{U}, \text{dcl}^\text{eq}(A))$ which we denote by $A_{\text{triv}}$, the trivial $se$-pair associated to $A$.

- Given an $se$-pair $\mathcal{A}$, the $se$-definable closure of $\mathcal{A}$, denoted $se$-dcl($\mathcal{A}$), is the $se$-pair $\mathcal{B} = (B, P(\mathcal{B}))$ where $B$ is the real part of $\text{dcl}^\text{eq}(A \cup P(\mathcal{A}))$ and $P(\mathcal{B}) = \text{dcl}^\text{eq}(P(\mathcal{A}))$ (Note that the canonical parameter functions $c_\varphi$ extend uniquely to $se$-dcl($\mathcal{A}$), so $se$-dcl($\mathcal{A}$) is an $se$-pair).

Let $f : A \to B$ be an $\mathcal{L}_{bp}$-embedding between $se$-pairs. By the definition of $\mathcal{L}_{bp}$, we have that $f(c_\varphi(a)) = c_\varphi(f(a))$, which means that the uniform scheme of definition is preserved by $f$. In particular, $\text{tp}_{\mathcal{L}}(f(A)/P(\mathcal{B})) = \text{tp}(f(A)/f(P(\mathcal{A})))|P(\mathcal{B})$. 


The following lemma follows directly from the choice of the language $\mathcal{L}_{bp}$ and the definition of $se$-pairs.

2.2.7. **Lemma-definition** (Base extension). Let $A$ be an $se$-pair and $B$ be an $\mathcal{L}^{eq}$-structure such that $P(A) \subseteq B \subseteq \mathcal{U}^{eq}$. Then there is a unique amalgam $A_B$ of $se$-pairs

![Diagram]

such that $A_B = \langle g_1(A) \cup g_2(B_{triv}) \rangle_{\mathcal{L}_{bp}}$. We call the structure $A_B$ the base extension of $A$ to $B$. $\square$

2.2.8. **Definition.** Let $\mathcal{K}$ be a subclass of $\mathcal{K}_{def}$. We say $\mathcal{K}$ is a natural class if

(i) $\mathcal{K}$ is closed under isomorphism;
(ii) $A_{triv} \in \mathcal{K}$ for any small substructure $A \subseteq \mathcal{U}^{eq}$;
(iii) $\mathcal{K}$ is closed under $se$-dcl;
(iv) $\mathcal{A} \in \mathcal{K}$ if and only if every finitely generated substructure of $\mathcal{A}$ is in $\mathcal{K}$;
(v) $\mathcal{K}$ is closed under base extension.

For $\lambda \geq |T|^+$ we denote by $\mathcal{K}_{\lambda}$ the subclass of $\mathcal{K}$ of elements of cardinality less than $\lambda$.

Note that $\mathcal{K}_{def}$ is a natural class. In addition, by condition (iv) above, natural classes are closed under substructures and unions of chains.

2.3. **Beautiful pairs.**

2.3.1. **Definition** (Beautiful pairs). Let $\mathcal{K}$ be a natural class. For $\lambda \geq |T|^+$, an element $\mathcal{M} \in \mathcal{K}$ is called a $\lambda$-$\mathcal{K}$-beautiful pair if the following condition is satisfied:

(BP) whenever there are $\mathcal{L}_{bp}$-embeddings $f: \mathcal{A} \rightarrow \mathcal{M}$ and $g: \mathcal{A} \rightarrow \mathcal{B}$ with $\mathcal{A}, \mathcal{B} \in \mathcal{K}_{\lambda}$, there is an $\mathcal{L}_{bp}$-embedding $h: \mathcal{B} \rightarrow \mathcal{M}$ such that $f = h \circ g$.

We call an $\mathcal{L}_{bp}$-structure $\mathcal{M}$ a $\mathcal{K}$-beautiful pair if it is a $\lambda$-$\mathcal{K}$-beautiful pair for some $\lambda \geq |T|^+$, and we will omit the $\mathcal{K}$ when $\mathcal{K} = \mathcal{K}_{def}$.

2.3.2. **Remark.**

- If $\mathcal{M}$ is a $\lambda$-$\mathcal{K}$-beautiful pair of $T$, then $P(\mathcal{M})$ is a $\lambda$-saturated model of $T$. However, note that it might happen that $M$ is not a model of $T$. Although, in most cases of interest $\mathcal{K}$-beautiful pairs will satisfy that $P(\mathcal{M}) \preceq M \models T$. As we will see later (see Corollary 2.3.6), either all $\mathcal{K}$-beautiful pairs are elementary pairs or no $\mathcal{K}$-beautiful pair is an elementary pair.
- The notion of $\mathcal{K}$-beauty corresponds precisely to $\mathcal{K}$-richness in the terminology of usual (Fraïssé) amalgamation constructions.

The following is an equivalent way of defining beautiful pairs. The properties are slightly easier to verify.

2.3.3. **Lemma.** A structure $\mathcal{M} \in \mathcal{K}$ is $\lambda$-$\mathcal{K}$-beautiful if and only if the following holds:

(i) $P(\mathcal{M})$ is a $\lambda$-saturated model of $T$;
(ii) whenever there are embeddings \( f : A \to M \) and \( g : A \to B \) with \( A \in K_\lambda \), \( g(P(A)) = P(B) \) and \( B = (g(A) \cup \{ b \}) \) for some \( b \in B \), there is an embedding \( h : B \to M \) such that \( f = h \circ g \).

Proof. Suppose \( M \) is \( \lambda \)-\( K \)-beautiful. Definition 2.3.1 clearly implies (ii), and condition (i) follows by Remark 2.3.2.

For the converse, let \( f : A \to M \) and \( g : A \to B \) with \( B \in K_\lambda \). Without loss of generality, suppose \( f \) is just inclusion.

Step 1: We may suppose \( g(P(A)) = P(B) \). Indeed, since \( P(M) \) is \( \lambda \)-saturated by (i), there is an \( L \)-embedding \( s : P(B) \to P(M) \) such that \( (s \circ g)|_{P(A)} = id|_{P(A)} \). Let \( A_{P(B)} \) be the base extension of \( A \) to \( P(B) \). By Lemma-definition 2.2.7, we may replace the base in our original amalgamation problem by \( A_{P(B)} \). Therefore, solving for \( A_{P(B)}, B \) and \( M \) gives also a solution for \( A, B \) and \( M \).

Step 2: Suppose by Step 1, that \( g(P(A)) = P(B) \). Let \( (b_\alpha)_{\alpha < \kappa} \) be an enumeration of \( B \setminus P(B) \). Inductively define \( \varepsilon \)-pairs \( B_\alpha \in K \)

1. \( B_0 := (A \cup \{ b_0 \}) \),
2. \( B_{\alpha+1} := (B_\alpha \cup \{ b_{\alpha+1} \}) \),
3. \( B_\alpha := \bigcup_{\beta < \alpha} B_\beta \) for \( \alpha \) limit.

Note that for each \( \alpha < \kappa \), since \( K \) is closed under substructures and unions of chains, \( B_\alpha \in K \). By (ii), there is an embedding \( h_0 : B_0 \to M \) such that \( h_0(P(B_0)) = P(A) \). By induction and (ii), for each \( \alpha < \kappa \) there is an embedding \( h_\alpha : B_\alpha \to M \) such that \( h_\alpha(P(B_\alpha)) = P(A) \) and moreover, for \( \alpha < \beta < \kappa \), \( h_\beta \) extends \( h_\alpha \).

From the usual Fraïssé theory, assuming \( K \) has the joint embedding property (JEP) and the amalgamation property (AP), one may construct \( K \)-beautiful pairs. Note that our assumptions on \( K \) entail that AP implies JEP. Indeed, it follows from the fact that substructures of \( U^{eq} \) satisfy JEP that any JEP problem in \( K \) can be converted to an AP problem in \( K \). Actually, the tensor product of definable types shows that \( K_{def} \) always satisfies JEP. However, in order to ensure that \( K \)-beautiful pairs are elementary pairs we need to impose on \( K \) the following additional property:

2.3.4. Definition. We say that \( K \) has the extension property (EP) if the following holds. Given a real sorted variable \( x \), a structure \( A \) from \( K \) and a consistent \( L(A) \)-formula \( \varphi(x) \), there is \( B \in K \), an embedding \( f : A \to B \) and \( b \in B \) satisfying \( \varphi(x) \).

2.3.5. Theorem. The following are equivalent:

1. \( K \) has the amalgamation property;
2. \( \lambda \)-\( K \)-beautiful pairs exist for all \( \lambda \geq |T|^+ \).

Moreover, assuming the above equivalent conditions hold, \( K \)-beautiful pairs are elementary pairs if and only if \( K \) has the extension property.

Proof. The equivalence between (1) and (2) is a standard Fraïssé construction by amalgamation. So suppose (1) and (2) hold.

For the left-to-right implication, let \( A \in K \) and \( \varphi(x) \) be a consistent \( L^{eq}(A) \)-formula. Take \( \lambda \) sufficiently big so that \( A \) embeds into a \( \lambda \)-\( K \)-beautiful pair \( M \). Taking \( M \) as \( B \) shows EP. The converse follows from the Tarski-Vaught test using EP.

We will later give examples of theories and natural classes for which EP fails (see later Section 2.4).
2.3.6. **Corollary.** Let \( \lambda \geq |T|^+ \), \( \mathcal{M}, \mathcal{N} \in \mathcal{K} \) and suppose that \( \mathcal{M} \) is \( \lambda \)-\( \mathcal{K} \)-beautiful. Then, the following are equivalent:

1. \( \mathcal{N} \) is \( \lambda \)-\( \mathcal{K} \)-beautiful;
2. the set of partial isomorphisms between substructures of \( \mathcal{M} \) and \( \mathcal{N} \) of size smaller than \( \lambda \) has the back-and-forth property.

In particular, \( \mathcal{M} \equiv_{\infty, \lambda} \mathcal{N} \) for any two \( \lambda \)-\( \mathcal{K} \)-beautiful pairs \( \mathcal{M} \) and \( \mathcal{N} \), and therefore, all \( \mathcal{K} \)-beautiful pairs are elementarily equivalent. \( \square \)

2.3.7. **Definition.** When \( \mathcal{K} \)-beautiful pairs exist, we will use \( T_{bp}(\mathcal{K}) \) to denote the common theory of \( \mathcal{K} \)-beautiful pairs. When \( \mathcal{K} = \mathcal{K}_{\text{def}} \), we will simply write \( T_{bp} \) instead of \( T_{bp}(\mathcal{K}) \).

Classical examples of \( T_{bp} \) will be gathered in Section 3.

2.3.8. **Remark.** The theory \( T_{bp} \) is an expansion by definitions of its reduct to \( L_P \).

The following standard fact mirrors the situation of Corollary 2.3.6.

2.3.9. **Fact.** Let \( T' \) be any complete theory, \( \lambda \geq |T'|^+ \), and \( \mathcal{M}, \mathcal{N} \) be two models of \( T' \) such that \( \mathcal{M} \) is \( \lambda \)-saturated. Then \( \mathcal{N} \) is \( \lambda \)-saturated if and only if \( \mathcal{M} \equiv_{\infty, \lambda} \mathcal{N} \). \( \square \)

2.3.10. **Lemma.** Suppose \( T_{bp}(\mathcal{K}) \) is consistent. The following are equivalent:

1. there is a cardinal \( \lambda \geq |T|^+ \) and \( \mathcal{M} \models T_{bp}(\mathcal{K}) \) which is \( \lambda \)-saturated and \( \lambda \)-\( \mathcal{K} \)-beautiful;
2. for every cardinal \( \lambda \geq |T|^+ \) and every model \( \mathcal{M} \models T_{bp}(\mathcal{K}) \), \( \mathcal{M} \) is \( \lambda \)-saturated if and only if \( \mathcal{M} \) is \( \lambda \)-\( \mathcal{K} \)-beautiful.

**Proof.** The implication \( (2) \Rightarrow (1) \) is trivial. For \( (1) \Rightarrow (2) \), let \( \mathcal{M}_0 \) be a \( \lambda \)-saturated \( \lambda \)-\( \mathcal{K} \)-beautiful pair and \( \lambda' \geq |T|^+ \) be a cardinal number. If \( \lambda' \leq \lambda \), \( (2) \) follows directly from Fact 2.3.9 and Corollary 2.3.6. So suppose \( \lambda < \lambda' \). Assume \( \mathcal{M} \) is a \( \lambda' \)-saturated model of \( T_{bp}(\mathcal{K}) \). Then it is also \( \lambda \)-saturated, and therefore, by Corollary 2.3.6, \( \lambda \)-\( \mathcal{K} \)-beautiful. By Lemma 2.3.3, it suffices to show that for a substructure \( \mathcal{A} \subseteq \mathcal{M} \) with \( \mathcal{A} \in \mathcal{K}_{\lambda'} \) and an embedding \( \mathcal{A} \to \mathcal{B} \) with \( \mathcal{B} = (\mathcal{A} \cup b) \) such that \( P(\mathcal{A}) = P(\mathcal{B}) \), there is an embedding \( \mathcal{B} \to \mathcal{M} \) which is the identity on \( \mathcal{A} \). Let \( (b_i)_{i \in I} \) be an enumeration of the set of finite tuples of \( \mathcal{B} \). For each \( i \in I \), let \( x_i \) be a tuple of variables associated to \( b_i \) and set \( x = (x_i)_{i \in I} \). Consider the partial \( L_{eq} \)-type

\[
\text{eq}(\lambda)(\mathcal{A}, \mathcal{B}) = \{(\forall y) (P(y) \rightarrow (\phi(x, y) \leftrightarrow d(\phi)(y, c\lambda(b_i)))) : \phi(x, y) \in L_{eq}\}.
\]

By \( \lambda' \)-saturation, if \( \Sigma \) is consistent, then it is realized in \( \mathcal{M} \). Moreover, any realization of \( \Sigma \) in \( \mathcal{M} \) gives us the desired embedding. Indeed, note that for any singleton \( b_i \in \mathcal{B} \), if \( \theta \) denotes the formula \( x_i = y \), then

\[
P(b_i) \leftrightarrow \models (\exists y) d(\theta)(y, c\lambda(b_i)),
\]

which shows that \( P \) is preserved by any such embedding. In addition, if \( e_i \in \mathcal{M} \) is the realization of the variables \( x_i \), the right-hand side of \( (\dagger) \) ensures, by the canonicity of \( c\lambda \), that \( c\lambda(b_i) = c\lambda(e_i) \).

Thus it suffices to show \( \Sigma(x) \) is consistent. For every \( i \in I \) and every finite subset \( A_0 \) of \( \mathcal{A} \), there is \( A_1 \) such that \( A_0 \subseteq A_1 \subseteq \mathcal{A}, A_1 \in \mathcal{K}_\lambda, P(A_1) = P(A_0) \) where \( b_i = (A_1 \cup b_i) \) and \( c\lambda(b_i) \in P(A_1) \) for every \( L \)-formula \( \varphi(x, y) \). Since \( \mathcal{M} \) is \( \lambda \)-\( \mathcal{K} \)-beautiful \( B_i \) embeds into \( \mathcal{M} \) over \( A_1 \), which shows that \( \Sigma(x) \) is finitely satisfiable in \( \mathcal{M} \).

For the converse, assume \( \mathcal{M} \) is a \( \lambda' \)-\( \mathcal{K} \)-beautiful pair. Let \( \mathcal{N} \) be a \( \lambda' \)-saturated elementary extension of \( \mathcal{M} \). By the the above implication, \( \mathcal{N} \) is \( \lambda' \)-\( \mathcal{K} \)-beautiful. Then \( \mathcal{M} \) is \( \lambda' \)-saturated by Corollary 2.3.6 and Fact 2.3.9. \( \square \)

2.3.11. **Definition.** Suppose \( T_{bp}(\mathcal{K}) \) is consistent. We say that \( T_{bp}(\mathcal{K}) \) has **beauty transfer** if one of the equivalent conditions in Lemma 2.3.10 holds.
2.3.12. **Theorem.** Suppose $T_{bp}(K)$ has beauty transfer. Then the following holds:

(1) $T_{bp}(K)$ is complete and admits quantifier elimination.
(2) The predicate $P$ is stably embedded in $T_{bp}(K)$ and pure (i.e., there is no new induced structure beyond $L$).

**Proof.** Part (1) follows from the fact that $|T|^+$-saturated models are $|T|^+$-$K$-beautiful by Lemma 2.3.10, and the system of partial isomorphisms between their substructures of size smaller than $|T|^+$ is thus a back-and-forth system by Corollary 2.3.6. Part (2) follows from (1). \qed

2.3.13. **Corollary.** Suppose $T_{bp}(K)$ has beauty transfer and $K$ has the extension property. Then if $T$ is NIP so is $T_{bp}(K)$.

**Proof.** By Theorem 2.3.12, the $L_P$-reduct of $T_{bp}(K)$ is bounded (i.e. every $L_P$-formula is equivalent to an $L_P$-formula where all quantifiers are over the predicate). We conclude by [8, Corollary 2.5]. \qed

2.3.14. **Remark.** By following carefully the proofs of [8, Theorem 2.4] and [8, Corollary 2.5], one could remove the assumption that $K$ has the extension property in Corollary 2.3.13.

2.4. **Natural classes and definable types.** In practice, we think of a natural class as induced by a given class of definable types. Let $\mathcal{F}$ denote a subclass of global definable types. For $A \subseteq U^{eq}$ and a tuple of real variables $x$ (possibly infinite), we let $\mathcal{F}_x(A)$ denote the subset of $\mathcal{F}$ of $A$-definable types in variables $x$. For $X$ an $A$-definable set in real sorts, we let $\mathcal{F}_X(A)$ be the subset of $\mathcal{F}$ of $A$-definable types concentrating on $X$.

2.4.1. **Definition.** A non-empty class $\mathcal{F}$ of global definable types is called *natural* if $\mathcal{F}$ satisfies the following properties for every $A \subseteq U^{eq}$:

- (Finitary) For any tuple $x$ of variables, $p \in \mathcal{F}_x(A)$ if and only if $p|_{x'} \in \mathcal{F}_{x'}(A)$ for any $x' \subseteq x$ and $x'$ finite;
- (Invariance) Given $\sigma \in \text{Aut}(U)$, $\sigma(\mathcal{F}(A)) = \mathcal{F}(\sigma(A))$.
- (Push forward) $\mathcal{F}(A)$ is closed under push forwards by $A$-definable functions.

2.4.2. **Definition.**

- Let $\mathcal{F}$ be a natural subclass of global definable types. We define the class of *se-pairs* $K_{\mathcal{F}}$ associated to $\mathcal{F}$ as the following class: for an se-pair $A$, $A \in K_{\mathcal{F}}$ if and only if the unique global extension of $tp_{L^{eq}}(A/P(A))$ given by the $c_{\varphi}$’s is in $\mathcal{F}$.
- Let $K$ be a natural class. Let $\mathcal{F}_K$ be the following class of global definable types. A type $p(x)$ is in $\mathcal{F}$ if there is $A \in K$ and $a \in A^x$ such that $p = tp(a/P(A))|U^{eq}$.

The following lemma follows easily from the definitions.

2.4.3. **Lemma.** If $\mathcal{F}$ is a natural subclass of global definable types, then $K_{\mathcal{F}}$ is a natural class. Conversely, if $K$ is a natural class, then $\mathcal{F}_K$ is a natural subclass of global definable types. In addition, the functions $\mathcal{F} \mapsto \mathcal{F}_K$ and $K \mapsto K_{\mathcal{F}}$ are inverses to each other.

2.4.4. **Examples.** The following are major examples that will be considered in the paper.

(1) Clearly $\mathcal{F} = S_{def}(U)$ is natural and the corresponding class is the class $K_{def}$ of all se-pairs.
(2) The subclass of definable types which are orthogonal to a given sort (or to a given invariant type) is natural.
(3) When $T$ is NIP, the subclass of generically stable types/stably dominated types is natural.
2.4.5. **Definition.** Given a natural class $\mathcal{F}$ of global definable types, for a finite tuple of real variables $x$ and a sufficiently saturated model $M$ of $T$, consider the map $\tau_{\mathcal{F}}$ sending a type $p \in \mathcal{F}(M)$ to the infinite tuple of its canonical parameters. More precisely,

$$\tau_{\mathcal{F}}: \mathcal{F}_x(M) \to \prod_{\varphi} M^{z_{\varphi}} \quad p \mapsto (c(p, \varphi))_{\varphi}$$

where $\varphi = \varphi(x, y)$ runs over all $\mathcal{L}^{eq}$-formulas and where $z_{\varphi}$ corresponds to the imaginary sort variable in $d(\varphi)(y, z_{\varphi})$ for $\varphi(x, y)$. We say $\mathcal{F}$ is pro-definable if the image of $\tau_{\mathcal{F}}$ is $*$-definable (in the sense of Shelah). Assuming $\mathcal{F}$ is pro-definable, we say $\mathcal{F}$ is strict pro-definable if the projection of $\mathcal{F}(M)$ onto any finite set of coordinates is a definable set. Equivalently, since one may encode finitely many formulas in one, this is equivalent to $\pi_\psi(\tau_{\mathcal{F}}(\mathcal{F}(M)))$ being definable for every formula $\psi$, where $\pi_\psi: \prod_{\varphi} M^{z_{\varphi}} \to M^{z_{\psi}}$ is the canonical projection.

2.4.6. **Remark.** By a result of M. Kamensky [28], the previous definition of pro-definability (resp. strict pro-definability) agrees with the standard one. We remit the reader to [13, Section 4] for details of the standard definition.

2.4.7. **Fact** ([13, Proposition 4.1]). Assuming uniform definability of types, $S_X^{eq}(U)$ is pro-definable for any $\mathcal{L}(U)$-definable set $X$.

The following theorem improves the above fact and extends it to natural classes of definable types.

2.4.8. **Theorem.** Let $\mathcal{F}$ be a natural subclass of definable types. Suppose $T_{bp}(K_{\mathcal{F}})$ exists and has beauty transfer. Then $\mathcal{F}$ is strict pro-definable.

**Proof.** Let $N$ be a $\lambda$-$K_{\mathcal{F}}$-beautiful pair with $\lambda$ big enough so that $P(N)$ is a monster model of $T$. Let $\psi(x, y)$ be an $\mathcal{L}^{eq}$-formula with $x$ real variables. Consider the $\mathcal{L}_{bp}$-definable subset $c_\psi(N)$ of $P(N)$. Since $N$ is $\lambda$-$K_{\mathcal{F}}$-beautiful, $\pi_\psi(\tau_{\mathcal{F}}(\mathcal{F}_x(P(N)))) = c_\psi(N)$. By Part (2) of Theorem 2.3.12, $c_\psi(N)$ is $\mathcal{L}^{eq}$-definable in $P(N)$.

2.4.9. **Definition.** Let $\mathcal{F}$ be a natural class of definable types. For a definable function $f: X \to Y$, we let $f^\mathcal{F}: \mathcal{F}_X(U) \to \mathcal{F}_Y(U)$ be the restriction of the pushforward $f_*$ to $\mathcal{F}_X(U)$. We say $\mathcal{F}$ has surjectivity transfer if whenever $f$ is surjective, so is $f^\mathcal{F}$. When $\mathcal{F} = S^{eq}$, we simply say that $T$ has surjectivity transfer.

2.4.10. **Remark.** If $T$ has uniform definability of types and $T^{eq}$ has surjectivity transfer, then $T^{eq}$ has uniform definability of types.

2.4.11. **Lemma.** Let $\mathcal{F}$ be a natural class of definable types and suppose $K_{\mathcal{F}}$ has the extension property. Then $\mathcal{F}$ has surjectivity transfer.

**Proof.** Let $f: X \to Y$ be a definable surjection and let $p(x) \in \mathcal{F}_Y(C)$, where $C$ is a small subset of $U$ over which the data are definable. Let $a$ be a realization of $p$ and $A$ be the $\mathcal{L}_{bp}$-structure given by $A = (Ca)_L$ and $P(A) = (C)_{L^{eq}}$, interpreting the canonical parameter functions $c_\varphi$ accordingly. By construction, $A \in K_{\mathcal{F}}$. Let $\varphi(x)$ be the formula $f(x) = a$. By the extension property, there is $B \subseteq K_{\mathcal{F}}$ such that $A \subseteq B$ together with $b \in B$ such that $\varphi(b)$ holds. Letting $q \in \mathcal{F}_X(P(B))$ be the corresponding global extension of $tp(b/P(B))$, we have that $f^\mathcal{F}(q) = p$.

Recall that $T$ has density of definable types if $S_X^{eq}(acl^{eq}(\mathcal{F}(X))) \neq \emptyset$ for every non-empty definable subset $X$ in the real sorts.
2.4.12. **Lemma.** Suppose $T$ has density of definable types. Then

1. the class $K_{\text{def}}$ has the extension property;
2. if $\mathcal{M} = (M, P(\mathcal{M}))$ is a beautiful pair for $T$, then $\mathcal{M}^* = (M^{\text{eq}}, P(M^{\text{eq}}))$ is a beautiful pair for $T^{\text{eq}}$.

**Proof.** For (1), let $\mathcal{A}$ be an $s\varepsilon$-pair and $\varphi(x)$ be a consistent $L^{\text{eq}}(\mathcal{A})$-formula with $x$ a real variable. Without loss of generality, we may assume $P(\mathcal{A}) \models T$. By density of definable types, there is $p \in S^{\text{eq}}(\text{acl}^{\text{eq}}(\mathcal{A}))$ where $X$ is the definable set associated to $\varphi$. In particular, $p$ is $\text{acl}^{\text{eq}}(\mathcal{A})$-definable. Letting $b$ be a realization of $p|\text{acl}^{\text{eq}}(\mathcal{A})$, we have that

$$\text{tp}(b, \text{acl}^{\text{eq}}(\mathcal{A})/P(\mathcal{A}))$$

is definable. We conclude by setting $\mathcal{B} = \langle Ab \rangle$.

For (2), let $\mathcal{A}$ be an $L^{\text{eq}}$-substructure of $\mathcal{M}^*$ and $A \to B$ be an $L^{\text{eq}}$-embedding such that $P(\mathcal{A}) = P(\mathcal{B})$. Without loss of generality, suppose $\text{acl}^{\text{eq}}(\mathcal{A}) = A$ and $P(\mathcal{A})$ is a model of $T^{\text{eq}}$. Take a small model $\mathcal{M}_0 \preceq_{L^{\text{eq}}} \mathcal{M}$ such that $A \subseteq \mathcal{M}_0$. Since density of definable types transfers to $T^{\text{eq}}$, there is a set of real elements $E$ such that $E^{\text{eq}}$ contains both $A$ and $P(\mathcal{A})$ with $\text{tp}_{L^{\text{eq}}}(E/A)$ admitting a global $A$-definable extension (it exists since $A$ is $\text{acl}^{\text{eq}}$-closed). Let $E' \models \text{tp}_{L^{eq}}(E/A)|\mathcal{M}_0$ and $\mathcal{A}'$ be $((E' \cup M_0)_{L^{\text{eq}}}, P(M_0))$. Note that $\mathcal{A}'$ is an $s\varepsilon$-pair for $T^{\text{eq}}$ since $\mathcal{M}_0$ is an $s\varepsilon$-pair containing $A$ and $\text{tp}(E' \cup M_0/P(M_0))$ is definable (by transitivity of definable types). We may $L^{\text{bp}}$-embed $\mathcal{A}'_{\mathcal{L}}$ in $\mathcal{M}$ over $(M_0)_{\mathcal{L}}$ by beauty of $\mathcal{M}$ and thus $L^{\text{bp}}$-embed $\mathcal{A}'$ in $\mathcal{M}^*$ over $\mathcal{M}_0$ since they are generated by real elements.

We may assume by (1), that the real part of $B$ generates all of $\mathcal{B}$. Take $A''$ realizing $\text{tp}(A''/A)|B$ and set $\mathcal{B}' = ((A'' \cup B), P(B))$. By construction, $\mathcal{B}'$ is an $s\varepsilon$-pair extending $\mathcal{A}'$. Since $\mathcal{A}'$, $\mathcal{B}'$ are generated by real elements, a solution of the problem for $\mathcal{A}'_{\mathcal{L}}, \mathcal{B}'_{\mathcal{L}}$ and $\mathcal{M}$, induces the desired solution for $\mathcal{A}, \mathcal{B}$ and $\mathcal{M}^*$.

**2.4.13. Remark.** Given a natural class of definable types $\mathcal{F}$, Lemma 2.4.12 could also be shown for the class $K_{\mathcal{F}}$ assuming $T$ has density of $\mathcal{F}$-types, $\mathcal{F}$ is closed under $\text{acl}^{\text{eq}}$ and closed in towers.

**An example where surjectivity transfer fails: the leveled binary tree.** Consider the binary tree $2^{<\omega}$ and let $\ell: 2^{<\omega} \to \omega$ be the function sending an element $t \in 2^{<\omega}$ to its distance to the root $r \in 2^{<\omega}$, where $\ell(r) = 0$. We will study $(2^{<\omega}, \omega)$ as a two sorted structure in the following language.

2.4.14. **Definition.** We define the two sorted language of binary leveled meet trees $L_{\text{tree}}$ as follows. The *tree sort* $T$ contains a binary relation $<$, a binary function $\land$ and a unary function $\text{pred}_T$. The *value sort* $V$ contains a binary relation $<$ and a unary function $\text{pred}_V$. We also add a function symbol $\ell$: $T \to V$.

We interpret $M = (2^{<\omega}, \omega)$ as an $L_{\text{tree}}$-structure in the standard way, where $\land$ corresponds to the meet of two elements, $\text{pred}_T$ (resp. $\text{pred}_V$) to the predecessor function, extended by $\text{pred}_T(r) = r$ (resp. $\text{pred}_V(0) = 0$), and $\ell$ as the above distance function. Let $T$ be the $L_{\text{tree}}$-theory of $M = (2^{<\omega}, \omega)$. The following is folklore.

2.4.15. **Fact.** The theory $T$ has quantifier elimination.

2.4.16. **Corollary.** The theory $T$ does not have surjectivity transfer. In particular, $K_{\text{def}}$ does not have the extension property.
that the value sort is purely stably embedded. Moreover, note that every global definable type in the tree sort must be a realized type. In particular, the definable type at $+\infty$ in the value sort is not in the image under $\ell_\ast$ of any definable type, which shows that the $L_{\text{tree}}$-theory of $M = (2^{<\omega}, \mathbb{N})$ does not have surjectivity transfer. By Lemma 2.4.11, this theory also constitutes an example of a theory for which $K_{\text{def}}$ does not have the extension property.

It is easy to see that $T_{bp}$ is consistent and has beauty transfer. Indeed, models of $T_{bp}$ are of the form $M = (M, P(M))$, where the $L_{\text{tree}}$-reduct of $P(M) = (T(P(M)), V(P(M)))$ is a model of $T$, and for $M = (T(M), V(M))$ it holds that $T(M) = T(P(M))$ and $V(M)$ is a proper end-extension of $V(P(M))$.

We end our abstract study of beautiful pairs with two remarks on our global assumption that $T$ has uniform definability of types. 

2.4.17. Remark.
1. Assume that $K$ is a natural class of $se$-pairs. Our proofs in this section, in particular Theorems 2.3.12 and 2.4.8—the latter using the construction from Definition 2.4.5—work under the weaker assumption that merely the definable types in $\mathcal{F}_K$ are uniformly definable. In Section 4.3, we will exhibit examples of theories of regular ordered abelian groups not having uniform definability of types and for which yet certain classes of definable types do have uniform definability (see Proposition 4.3.5 and Theorem 4.3.6).
2. Even more generally, given an arbitrary complete theory $T$ with quantifier elimination, with no assumption on uniform definability of types, one may work with the class $\widetilde{K}_{\text{def}}$ of $L_P$-structures $\mathcal{A} = (A, P(A))$, such that $P(A) \models T$ and $tp_{L_P}(A/P(A))$ is definable, together with a notion of sub $se$-pair, where for $\mathcal{A}, \mathcal{B} \in \widetilde{K}_{\text{def}}$ we set $\mathcal{A} \sqsubseteq \mathcal{B}$ if and only if $\mathcal{A}$ is an $L_P$-substructure of $\mathcal{B}$ such that $tp_{L_P}(A/P(B)) = tp_{L_P}(A/P(A))|P(B)$. Let us call the corresponding notion of embedding an $se$-pairal embedding.

If $\widetilde{K}$ is a natural subclass of $\widetilde{K}_{\text{def}}$, then $\lambda$-$\widetilde{K}$-beautiful pairs may be defined using the notion of $se$-pairal embedding, and they exist precisely when $\widetilde{K}$ has the amalgamation property with respect to $se$-pairal embeddings. Any two beautiful pairs are then back-and-forth equivalent, and one may define beauty transfer as before. Assuming $\widetilde{K}$-beautiful pairs exist and $T_{bp}(\widetilde{K})$ has beauty transfer, it then follows that the definable types in $\mathcal{F}_{\widetilde{K}}$ are uniformly definable, so one may a posteriori expand the $L_P$-theory $T_{bp}(\widetilde{K})$ to an $L_{bp}$-theory and get back to the context of (1).

For ease of presentation, we chose to work under the global assumption that $T$ has uniform definability of types.

3. Relation with previous work and variants

In this section we show how our general context of beautiful pairs is related to two previous constructions, first in the context of stable theories and second, in the context of o-minimal theories. In the second context we also consider variants for o-minimal expansions of divisible ordered abelian groups without poles.

3.1. Beautiful pairs of stable theories. Suppose $T$ is stable. Recall that an elementary pair $(M, P(M))$ of models of $T$ is a belle paire in the sense of Poizat [34] if the following two conditions hold:

(i) $P(M)$ is $|T|^+$-saturated;
(ii) for every finite tuple $a$ of $M$, every type over $P(M) \cup \{a\}$ is realized in $M$. 

Proof. It follows from Fact 2.4.15 that the value sort is purely stably embedded. Moreover,
Recall that a theory $T$ has nfcp if no formula $\varphi(x,y)$ satisfies the following condition: for every integer $n$ there exists a subset $A$ of a model of $T$ and an inconsistent set $H \subseteq \{\varphi(x,a), \neg\varphi(x,a) : a \in A^y\}$ such that every subset of $H$ of size less than $n$ is consistent. The following is a reformulation of results of Poizat from [34]:

3.1.1. **Theorem** (Poizat). Let $T$ be stable. Then belles paires exist and any two belles paires are elementary equivalent. Moreover, the following are equivalent:

1. $T$ has nfcp;
2. there is a $|T|^+-saturated belle paire;
3. $S^\text{def}_X(U)$ is strict pro-definable for every definable set $X$.

Given $\lambda \geq |T|^+$, call an elementary pair $(M,P(M))$ a $\lambda$-belle paire if the following two conditions hold:

1. $P(M)$ is $\lambda$-saturated;
2. for every subset $A \subseteq M$ of cardinality $< \lambda$, every type over $P(M) \cup A$ is realized in $M$.

The following lemma follows essentially from the same argument as in the proof of Lemma 2.3.10.

3.1.2. **Lemma.** Given $\lambda \geq |T|^+$, let $M$ be a $\lambda$-saturated belle paire. Then $M$ is a $\lambda$-belle paire. □

By Remark 2.3.8, abusing of notation, given an $L_{bp}$-structure $M$, we also use $M$ for its $L_P$-reduct when no confusion arises. The following proposition is left as an exercise.

3.1.3. **Proposition.** Let $\lambda \geq |T|^+$. An $L_{bp}$-structure $M$ is a $\lambda$-beautiful pair if and only if it is a $\lambda$-belle paire. In particular, the $L_P$-reduct of $T_{bp}$ is the common theory of belles paires. □

As a corollary, by Lemma 2.3.10, $T_{bp}$ has beauty transfer if and only if any of the conditions (1)-(3) in Theorem 3.1.1 holds.

3.1.4. **Algebraically closed fields.** We finish this section with a brief description of beautiful pairs of algebraically closed fields. Although the results are classical, we present them here to provide an illustration of our framework.

Let $L$ denote the language of rings $L_{\text{ring}}$ and ACF denote a completion of the $L$-theory of algebraically closed fields. Let $\text{ACF}^2$ be the $L_{bp}$-theory of pairs of models of ACF and set $T_0 = \text{ACF}^2 \cup \{(\forall x)(P(x))\}$ and $T_1 = \text{ACF}^2 \cup \{(\exists x)(\neg P(x))\}$.

3.1.5. **Proposition.** The theories $T_0$ and $T_1$ axiomatise respectively the theories of beautiful pairs $\text{ACF}(K_{\text{triv}})$ and $\text{ACF}(K_{\text{def}})$. In particular, $T_0$ and $T_1$ are the only completions of pairs of models of ACF and they both have beauty transfer.

**Proof.** By Remark 2.3.8, a classical result of A. Robinson [39] shows that $T_0$ and $T_1$ are complete (and hence the only completions of $\text{ACF}^2$). Since ACF has nfcp, by Theorem 3.1.1, both $T_0$ and $T_1$ axiomatize respectively the theories of beautiful pairs $\text{ACF}(K_{\text{triv}})$ and $\text{ACF}(K_{\text{def}})$ and have beauty transfer. Note that in this precise example, quantifier elimination was already obtained by F. Delon in [14], who provided an explicit interpretation of the canonical parameter functions $c_\varphi$. □
3.2. **Beautiful pairs of \( \alpha \)-minimal theories.** Let \( T \) be a complete (dense) \( \alpha \)-minimal theory. By Marker-Steinhorn’s theorem [30], an elementary pair of models \( M \preceq N \) of \( T \) is stably embedded if and only if \( M \) is Dedekind complete in \( N \). In particular, the class of stably embedded elementary pairs of models of \( T \) is an elementary class (recall that we denote by \( T_{\mathrm{SE}} \) the corresponding \( \mathcal{L}_P \)-theory) and, by [13, Theorem 6.3], \( T \) has uniform definability of types. Recall that the non-realized definable types in one variable over a model \( M \) of \( T \) correspond to the type at infinity (resp. minus infinity) \( p_\infty \) (resp. \( p_{-\infty} \)) and to the types of elements infinitesimally close to an element of \( a \in M \) from the right (resp. left) \( p_{a^+} \) (resp. \( p_{a^-} \)).

In [33], A. Pillay provided an axiomatization of an \( \mathcal{L}_P \)-theory \( T^* \) of stably embedded pairs of models of \( T \) which, if consistent, is complete [33, Theorem 2.3]. It was later shown by B. Baisalov and B. Poizat that \( T^* \) is always consistent [3, Proposition, p.574]. They also provided the following simpler axiomatization (see the proof of [3, Proposition, p.574]), where a pair \( P(M) \preceq M \) is a model of \( T^* \) if

- \((M, P(M)) \models T_{\mathrm{SE}}\)
- Given \( \mathcal{L} \)-definable partial functions \( f(x, y), g(x, y) \) (possibly constant \( \pm \infty \)) and a tuple \( a \in M^\mathbb{N} \), if for all \( b_1, b_2 \in P(M)^\mathbb{N} \)
  \[ M \models -\infty \leq f(a, b_1) < g(a, b_2) \leq +\infty \]

  (whenever both sides are defined), then there is \( e \in M \) such that for all \( b \in P(M)^\mathbb{N} \)

\[ M \models f(a, b) < e < g(a, b). \]

As the following result shows, \( T^* \) axiomatises the theory of beautiful pairs of \( T \).

3.2.1. **Theorem.** The theory \( T_{\mathrm{bp}} \) exists and has beauty transfer. Moreover, it is axiomatized by \( T^* \).

**Proof.** Since \( T^* \) is consistent by [3, Proposition, p.574], Lemma 2.3.10 ensures it suffices to show that every \( |T|^+ \)-saturated model of \( T^* \) is a beautiful pair. Let \((M, P(M)) \) be such a model and \( M \) be its \( \mathcal{L}_{\mathrm{bp}} \)-expansion by definitions. We use Lemma 2.3.3. Condition (i) is clear. For condition (ii), let \( f: A \to M \) and \( g: A \to B \) with \( B \in K_{\mathrm{def}, |T|^+} \), \( g(P(A)) = P(B) \) and \( B = \langle A \cup \{b\} \rangle \). Without loss of generality, we may assume \( P(A) \) is a model of \( T \). Furthermore, we may assume that \( b \notin \text{def}^{\text{eq}}(A \cup P(M)) \). We are exactly in case (Iib) of the proof of [33, Theorem 2.3]. Proceeding word for word, we can find an \( \mathcal{L}_{\mathrm{bp}} \)-embedding \( g: B \to M \) over \( A \). \( \square \)

3.2.2. **Corollary.** The class \( K_{\mathrm{def}} \) has the amalgamation property and the extension property.

**Proof.** Immediate from Theorems 2.3.5 and 3.2.1. \( \square \)

3.2.3. **Bounded and convex pairs.** In this subsection we suppose that \( T \) expands the theory of divisible ordered abelian groups (DOAG).

3.2.4. **Definition.** Let \( M \preceq N \) be an elementary extension of models of \( T \) (or in general of \( \alpha \)-minimal structures).

- The pair \((N, M)\) is **bounded** (or that \( N \) is bounded by \( M \)) if for every \( b \in N \), there are \( c_1, c_2 \in M \) such that \( c_1 \leq b \leq c_2 \). For a small subset \( A \) of \( U \), a type \( p \in S^\text{def}_2(A) \) is **bounded** if for any small model \( M \) containing \( A \) and every realization \( a \models p|M \), there is an elementary extension \( M \preceq N \) with \( a \in N^2 \) and such that \((N, M)\) is a bounded pair.
• The pair \((N, M)\) is convex if \(M\) is a convex subset of \(N\), i.e., for every \(b \in N \setminus M\), either \(M < b\) or \(b < M\). We say a type \(p \in S^\text{def}_x(A)\) is at infinity if for for any small model \(M\) containing \(A\) and every realization \(a \models p|M\), there is an elementary extension \(M \leq N\) with \(a \in N^x\) and such that \((N, M)\) is a convex pair.

3.2.5. **Definition.** Let \(M\) be a model of \(T\). A pole in \(M\) is a definable continuous bijection between a bounded and an unbounded interval.

Observe that, by \(o\)-minimality, no model of \(T\) has a pole if and only if the types \(p_{0^+}\) and \(p_{+\infty}\) are orthogonal.

We let \(K_{\text{bdd}}\) be the class corresponding to \(K_\mathcal{F}\) for \(\mathcal{F}\) the collection of definable bounded types. We let \(K_\infty\) be the class corresponding to \(K_\mathcal{F}\) for \(\mathcal{F}\) the collection of definable types orthogonal to \(p_{0^+}\). Note that such classes are different from \(K_{\text{triv}}\) precisely when no model of \(T\) has a pole. Moreover, the classes of global bounded types and global types at infinity are natural classes of definable types, hence by Lemma 2.4.3, \(K_{\text{bdd}}\) and \(K_\infty\) are natural classes of \(s\mathcal{E}\)-pairs.

3.2.6. **Section assumption.** We assume from now that no model of \(T\) has a pole.

Consider the following \(L_{\text{bp}}\)-theories \(T^{*}_{\text{bdd}}\) and \(T^{*}_\infty\). The theory \(T^{*}_{\text{bdd}}\) consists of axioms expressing for a pair \((M, P(M))\) that

\[
\begin{align*}
(1)_{\text{bdd}} \quad & (M, P(M)) \models T_{\mathcal{E}} \text{ and } M \text{ is bounded by } P(M), \\
(2)_{\text{bdd}} \quad & \text{given } L\text{-definable partial functions } f(x, y), g(x, y) \text{ and a tuple } a \in M^x, \text{ if there are } \\
& c_1, c_2 \in P(M) \text{ such that for all } b_1, b_2 \in P(M)^y \\
& M \models c_1 < f(a, b_1) < g(a, b_2) < c_2 \\
& \text{(whenever both sides are defined), then there is } e \in M \text{ such that for all } b \in P(M)^y \\
& M \models f(a, b) < e < g(a, b).
\end{align*}
\]

The theory \(T^{*}_\infty\) consists of axioms expressing for a pair \((M, P(M))\)

\[
\begin{align*}
(1)_{\infty} \quad & (M, P(M)) \models T_{\mathcal{E}} \text{ and } (M, P(M)) \text{ is convex}, \\
(2)_{\infty} \quad & \text{given } L\text{-definable partial functions } f(x, y), g(x, y) \text{ (possibly constant } \pm \infty) \text{ and a tuple } \\
& a \in M^x, \text{ if for all } b_1, b_2 \in P(M)^y \\
& M \models P(M) < g(a, b_1) - f(a, b_2) \\
& \text{(whenever both sides are defined), then there is } e \in M \text{ such that for all } b \in P(M)^y \\
& M \models f(a, b) < e < g(a, b).
\end{align*}
\]

3.2.7. **Lemma.** The theories \(T^{*}_{\text{bdd}}\) and \(T^{*}_\infty\) are consistent.

**Proof.** We follow the same strategy of proof as in [3]. Let \(M_0\) be a model of \(T\) and let \(N\) be an \(|L(M_0)|^+\)-saturated elementary extension of \(M_0\).

We start with \(T^{*}_{\text{bdd}}\). Let \(M\) be maximal such that \(M_0 \preceq M \preceq N\), \(M_0 \preceq M\) is stably embedded and \(M\) is bounded by \(M_0\). Note that such a maximal \(M\) exists by Zorn’s lemma. Observe that for any \(a \in N \setminus M\), there is \(b \in \text{dcl}(Ma)\) such that \(\text{tp}(b/M_0)\) is either at infinity or not definable. Setting \((M, P(M)) = (M_0, M_0)\), we claim that \((M, P(M))\) is a model of \(T^{*}_{\text{bdd}}\). The axiom scheme \((1)_{\text{bdd}}\) is clear.

For \((2)_{\text{bdd}}\), let \(f(x, y), g(x, y)\) be \(L\)-definable partial functions and fix \(a \in M^x\). Let \(A\) (resp. \(B\)) be the image of \(f(a, y)\) (resp. the image of \(g(a, y)\)) under all \(b \in P(M)^y\) (whenever defined). Suppose there are \(c_1, c_2 \in P(M)\) such that \(c_1 < A < B < c_2\) and for a contradiction that there is no \(e \in M\) such that \(A < e < B\). Since the cardinalities of \(A\) and \(B\) are bounded
by the cardinality of $P(M)$, by saturation of $N$, we find $d \in N$ be such that $A < d < B$. Note that $\text{tp}(d/M)$ is definable as it is determined by the definable cut $A < B$, and it is also bounded (since there are no poles). So, setting $M' = \text{dcl}(Md)$ we contradict the maximality of $M$.

The proof for $T^*_\infty$ is similar. Let $M$ be maximal such that $M_0 \preceq M \preceq N$, $M_0 \preceq M$ is stably embedded and $(M, M_0)$ is convex. Again, such a maximal $M$ exists by Zorn’s lemma. It follows that for any $a \in N \setminus M$, there is $b \in \text{dcl}(Ma)$ such that $\text{tp}(b/M_0)$ is bounded and non-realized. Setting $(M, P(M)) = (M, M_0)$, we claim that $(M, P(M))$ is a model of $T^*_\infty$. The axiom scheme $(1)_{\infty}$ is clear.

For $(2)_{\infty}$, let $f(x, y), g(x, y)$ be $\mathcal{L}$-definable partial functions and fix $a \in M^2$. Let $A$ (resp. $B$) be the image of $f(a, y)$ (resp. the image of $g(a, y)$) under all $b \in P(M)^y$ (whenever defined). The cases where $B = \{+\infty\}$ or $A = \{-\infty\}$ are not excluded. Suppose that $P(M) < B - A$ and for a contradiction that there is no $e \in M$ such that $A < e < B$. Since the cardinalities of $A$ and $B$ are bounded by the cardinality of $P(M)$, by saturation of $N$, we find $d \in N$ such that $A < d < B$. The type $\text{tp}(d/M)$ is definable, hence either of the form $p_{a^+}, p_{a^-}$ for $a \in M$ or, $p_{-\infty}, p_{+\infty}$. The former two cannot happen: indeed $A$ (resp. $B$) has no largest finite element (resp. smallest finite element) since $P(M) < B - A$ and $\text{tp}(d/M)$ is determined by filling the cut $A < B$. Thus, $\text{tp}(d/M)$ is either $p_{-\infty}$ or $p_{+\infty}$. By the maximality of $M$, there is $e \in \text{dcl}(Md)$ such that $\text{tp}(e/M_0)$ is bounded and non-realized, but this contradicts that no model of $T$ has a pole.

3.2.8. **Theorem.** The theory $T_{\text{bdd}}$ axiomatisates $T(K_{\text{bdd}})$ and $T(K_{\text{bdd}})$ has beauty transfer. The theory $T^*_\infty$ axiomatisates $T(K_{\infty})$ and $T(K_{\infty})$ has beauty transfer.

*Proof.* Analogous to the proof of Theorem 3.2.1 using Lemma 3.2.7 respectively. \hfill \square

3.2.9. **Remark.** When $T$ extends RCF, A. H. Lewenberg and L. van den Dries proved in [44] that $T_{\text{bp}}$ is axiomatized by $T_{\mathcal{S}E} \cup \{(\exists x)(\neg P(x))\}$ (they call such pairs, tame pairs). In particular, as for ACF, there are only two completions of $T_{\mathcal{S}E}$ and (up to passing to an expansion by definitions) each one corresponds to either $T_{\text{bp}}(K_{\text{triv}})$ or $T_{\text{bp}}$. This suggests the natural question of whether this holds more generally for expansions of DOAG (see Question 9.3.5).

4. **Beautiful pairs of ordered abelian groups**

In this section we study beautiful pairs of complete theories of regular ordered abelian groups. In the cases of DOAG and Presburger arithmetic, we provide explicit axiomatizations and we characterize all completions of $T_{\mathcal{S}E}$.

4.1. **Divisible ordered abelian groups.** We let $\mathcal{L}$ be the language of ordered abelian groups $\mathcal{L}_{\mathcal{O}G} = \{+, -, <, 0\}$ and $T$ be DOAG. We will show that $T_{\mathcal{S}E}$ has exactly 4 completions: $T_{\text{bp}}(K_{\text{triv}}), T_{\text{bp}}(K_{\text{bdd}}), T_{\text{bp}}(K_{\infty})$ and $T_{\text{bp}}$ (see Section 3.2.3 for the definition of $K_{\text{bdd}}$ and $K_{\infty}$). Consider the $\mathcal{L}_P$-sentences:

\begin{align*}
(E_{\infty}) \quad & (\exists x)(\forall y)(P(y) \rightarrow y < x); \\
(E_{0^+}) \quad & (\exists x)(\forall y)((P(y) \land 0 < y) \rightarrow 0 < x < y).
\end{align*}

4.1.1. **Theorem.** It holds that

1. $T_{\text{bp}}$ is axiomatized by $T_{\mathcal{S}E} \cup \{E_{0^+}, E_{\infty}\}$;
2. $T_{\text{bp}}(K_{\text{bdd}})$ is axiomatized by $T_{\mathcal{S}E} \cup \{E_{0^+}, \neg E_{\infty}\}$;
(3) $T_{hp}(K_{∞})$ is axiomatized by $T_{SE} \cup \{E_∞, -E_0^+\}$;
(4) $T_{hp}(K_{triv})$ is axiomatized by $T_{SE} \cup \{-E_∞, -E_0^+\}$.

Moreover, the above are all the completions of $T_{SE}$, and they all satisfy beauty transfer.

**Proof.** The first part of the last statement follows from the first one. Beauty transfer follows from Theorems 3.2.1 and 3.2.8.

Let us show the result for $T_{hp}$, the other cases being easily derived from the proof. Suppose $M$ is an $\mathbb{N}_1$-saturated model of $T_{SE} \cup \{E_0^+, E_∞\}$. By Lemma 2.3.3, it suffices to show condition $(ii)'$, so let $f: A \rightarrow M$ and $g: A \rightarrow B$ be $L_{bp}$-embeddings with $g(P(A)) = P(B)$, $B = (A \cup \{b\})$ and $A$ countable. After possibly increasing $P(A)$ and replacing $A$ with its divisible hull, we may suppose that $f$ and $g$ are the identity map and $P(A) \preceq_{\mathcal{L}} A \preceq_{\mathcal{L}} M$. We need to show that there is an $L_{bp}$-embedding $h: B \rightarrow M$ such that $h|_A = id$. There are two possibilities.

**Case 1:** Assume that up to interdefinability over $A$, there is some $b \models p_0^+|P(A)$ and $B$ is generated by $b$ over $A$. By saturation of $M$ and axiom $E_0^+$, there is $b' \in M$ realizing $p_0^+|P(M)$ with $tp_{\mathcal{L}}(b'/A) = tp_{\mathcal{L}}(b/A)$.

To show that the map extending $f$ and sending $b$ to $b'$ is an $L_{bp}$-embedding, it suffices to show the following:

$$b' \cup A \models p \models tp_{\mathcal{L}}(b \cup A/P(A))|P(M).$$

Let $x$ denote a variable intended for $b$ and $y$ variables for elements in $A$. Let $\varphi(x, y)$ be a $P(M)$-formula in $p$. By quantifier elimination, we may assume that $\varphi(x, y)$ is of the form $y + c \square x$ for some $c \in P(M)$ and $\square$ stands for $>$ or $<$ (note that $=$ cannot arise since $b \notin A$).

Without loss of generality, we may assume that the element $a \in A$ corresponding to $y$ is in the convex hull of $P(M)$. By possibly changing $c$, we may further assume $a \models p_0^+|P(A)$, $a = 0$ or $a \models p_0^-|P(A)$. In the case that $c = 0$, $\varphi(b', a)$ holds since $b$ and $b'$ realize the same cut over $A$. If $c \neq 0$, then $\varphi(0, a)$ holds, which implies that $\varphi(b', a)$ holds too.

**Case 2:** Assume that up to interdefinability over $A$, there is some $b \models p_{+\infty}^+|P(A)$ and $B$ is generated by $b$ over $A$ such that $b$ is not interdefinable with any $b' \models p_0^-|P(A)$ over $A$. By saturation of $M$, there is $b' \in M$ realizing $p_{+\infty}^+|P(M)$ and with $tp_{\mathcal{L}}(b'/A) = tp_{\mathcal{L}}(b/A)$. As in the previous case, we want to show that

$$p \models tp_{\mathcal{L}}(b \cup A/P(A))|P(M) = tp_{\mathcal{L}}(b' \cup A/P(M)).$$

Let $\varphi(x, y)$ be a $P(M)$-formula in which $y$ is intended for elements in $A$. We may assume that $\varphi(x, y)$ is of the form $y + c \square x$ where $\square$ stands for $=$, $>$ or $<$ and $c \in P(M)$. Since we are not in Case 1, equality never happens and the truth of the formula does not depend on $c \in P(M)$. Thus we may assume the formula is of the form $y > x$ or $y < x$, which is completely determined by $tp_{\mathcal{L}}(b'/A) = tp_{\mathcal{L}}(b'/A)$. \hfill \Box

### 4.2. Presburger arithmetic.

In this section we let $\mathcal{L}$ be the language of Presburger arithmetic $\mathcal{L}_{Pres} := \{+, -, \leq, 0, 1, (\equiv_n)_{n > 1}\}$ and $T = \text{PRES}$ be Presburger arithmetic. Recall that models of $T$ are also called $\mathbb{Z}$-groups.

**4.2.1. Fact ([9]).** Let $M$ be a $\mathbb{Z}$-group. Then, $p \in S_n(M)$ is definable if and only if for every realization $a \models p$, $M \subseteq M(a)$ is an end-extension. \hfill \Box

It follows that the class of stably embedded pairs of models of PRES is $\mathcal{L}_P$-elementary (see [13]). Note also that there is a unique way to turn a model of $T_{SE}$ into an $L_{bp}$-structure, hence we won’t make the distinction. We will show that, in this case, the theory $T_{SE}$ has two completions corresponding to $T_{hp}$ and $T_{hp}(K_{triv})$ respectively. Clearly the theory of trivial pairs is complete and has beauty transfer. In the non-trivial case, we have the following.
4.2.2. **Theorem.** The theory \( T_{SE} \cup \{ \exists x \neg P(x) \} \) is complete, axiomatizes \( T_{bp} \), and \( T_{bp} \) has beauty transfer.

**Proof.** We show that an \( \aleph_1 \)-saturated model \( M = (M, P(M)) \) of proper pairs of \( T_{SE} \) is a beautiful pair. The proof follows the same strategy as the proof of Theorem 4.1.1, and again, the result will follow using Corollary 2.3.6.

Let \( A \) be an \( L_{bp} \)-substructure of \( M \), and \( g: A \to B \) be an \( L_{bp} \)-embedding with \( P(A) = P(B) \). By taking definable closures, we may assume that \( P(A) \preceq A \) are models. Up to interdefinability, we may assume \( b \) is positive and \( b \) must realize some type at infinity over \( P(A) \). Moreover, note that \( b \) is not \( L \)-interdefinable over \( A \) with any \( b' \) with \( b' \) bounded by elements in \( P(A) \).

Using saturation of \( M \), we find an element \( b' \in M \) such that

\[ b' \models p_{+\infty}(P(M) \cup tp_{L}(b/A)). \]

It is now straightforward to check the following:

\[ p := tp_{L}(b \cup A/P(A))|P(M) = tp_{L}(b' \cup A/P(M)) \]

This will imply that the map \( b \mapsto b' \) induces an \( L_{bp}(A) \)-embedding of \( B \) into \( M \). To illustrate one case, suppose \( \varphi(x, y) \) is an \( L(P(M)) \)-formula where \( x \) is intended for \( b/b' \) and \( y \) is intended for elements in \( A \) and \( \varphi(x, y) \) is of the form \( nx + y + c \equiv_m 0 \) for \( n, m \in \mathbb{N} \) and \( c \in P(M) \). Let \( l \in \mathbb{N} \) be such that \( c \equiv_m l \). We see that \( 0 \equiv_m nx + y + c \leftrightarrow 0 \equiv_m nx + y + l \), and the truth of the right hand side is determined by \( tp_{L}(b/A) = tp_{L}(b'/A) \). \( \square \)

4.3. **Dense regular ordered abelian groups.** Recall that a (non-trivial) ordered abelian group is **discrete** if it has a minimum positive element, and otherwise it is called **dense**. We continue to work with \( L = L_{PRES} \). Any ordered abelian group \( M \) is viewed as an \( L \)-structure by interpreting \( +, 0, -, <, \equiv_n \) in the usual way, 1 as the minimum positive element if \( M \) is discrete and as 0 when \( M \) is dense.

Recall that an ordered abelian group \( M \) is called **regular** if one of the following equivalent conditions holds:

4.3.1. **Fact** ([40, 48, 10, 46]). For an ordered abelian group \( M \), the following are equivalent.

1. The theory of \( M \) has quantifier elimination in \( L \).
2. The only definable convex subgroups of \( M \) are \( \{0\} \) and \( M \).
3. The theory of \( M \) has an archimedean model, i.e. one embeddable in \( (\mathbb{R}, +, <) \) as an ordered abelian group.
4. For every \( n > 1 \), if the interval \([a, b]\) contains at least \( n \) elements, then it contains an element divisible by \( n \).
5. Any quotient of \( M \) by a nontrivial convex subgroup is divisible.

The following fact characterizes all complete \( L \)-theories of regular ordered abelian groups.

4.3.2. **Fact** ([40, 48]). Every discrete regular ordered abelian group is a model of \( PRES \), i.e., it is a \( \mathbb{Z} \)-group. If \( M, N \) are dense regular, then \( M \equiv N \) if and only if for every prime \( q \) the quotients \( M/qM \) and \( N/qN \) are either both infinite or have the same finite size.

The following lemma characterizes definable types in regular ordered abelian groups. We leave the proof to the reader, pointing out that the proof follows a similar argument given by G. Conant and S. Vojdani in the proof of Fact 4.2.1.

Given an ordered abelian group \( \Gamma \), we let \( \text{div}(\Gamma) \) be the divisible hull of \( \Gamma \) in an ambient divisible ordered abelian group.
4.3.3. **Lemma.** Let $T$ be a complete theory of regular ordered abelian groups, $\Gamma \preceq \Gamma' \models T$ and $a = (a_1, \ldots, a_n) \in \Gamma'$. Then $\text{tp}(a/\Gamma)$ is definable if and only if $\text{tp}(b/\Gamma)$ is definable for every element $b$ in $(\Gamma a)$. Moreover, for every element $e \in \Gamma'$, $\text{tp}(e/\Gamma)$ is definable if and only if $\text{tp}_{\text{DOAG}}(e/\text{div}(\Gamma))$ is definable. \hfill $\square$

4.3.4. **Proposition.** Let $\Gamma$ be an archimedean ordered abelian group. Then there is an archimedean elementary extension $\Gamma \preceq \Gamma'$ such that $\Gamma'$ is stably embedded in every elementary extension.

**Proof.** We may suppose $\Gamma$ is dense, as for discrete $\Gamma$ we have $\Gamma \cong \mathbb{Z}$ and we can thus take $\Gamma' = \Gamma$. Without loss of generality, $\Gamma \leq (\mathbb{R}, +, <)$. Let $(a_i)_{i \in I}$ be a $\mathbb{Q}$-basis of $\text{div}(\Gamma)$. Let $(b_j)_{j \in J}$ be such that $(a_i, b_j)_{i \in I, j \in J}$ is a $\mathbb{Q}$-basis of $\mathbb{R}$. Let $\Gamma'$ be $\Gamma' = \Gamma \oplus (\bigoplus_{j \in J} \mathbb{Q} b_j)$ equipped with the induced ordering by $\mathbb{R}$. The ordered abelian group $\Gamma'$ is archimedean and it is an elementary extension of $\Gamma$ by Facts 4.3.1 and 4.3.2. By construction, $\text{div}(\Gamma') = \mathbb{R}$, and therefore all types over $\Gamma'$ are definable by Lemma 4.3.3. \hfill $\square$

4.3.5. **Proposition.** Let $T$ be the theory of a regular ordered abelian group which is neither discrete nor divisible. Then, there is a proper stably embedded pair $(\Gamma, P(\Gamma))$ of models of $T$ which is not convex. In addition, there is an elementary extension $(\Gamma', P(\Gamma'))$ of $(\Gamma, P(\Gamma))$ which is not stably embedded. In particular, the class of stably embedded pairs of models of $T$ is not elementary and $T$ does not have uniform definability of types.

**Proof.** The last sentence follows directly from the first part. By Facts 4.3.1, 4.3.2 and Proposition 4.3.4, there is a stably embedded pair $(\Gamma, P(\Gamma))$ of models of $T$ that is not convex and such that $\Gamma$ is $|P(\Gamma)|^+$-saturated (take as $P(\Gamma)$ an archimedean model of $T$ which is stably embedded in every elementary extension).

In this proof, for every $a \in \text{div}(\Gamma)$, we write $p_{a^\pm}$ to denote the partial definable type in $T$ given by the corresponding definable quantifier-free DOAG-type.

Let $q$ be prime and $a \in P(\Gamma)$ so that $a$ is not $q$-divisible (which exists since $P(\Gamma)$ is not divisible). By saturation and Fact 4.3.1, there is $\varepsilon \in \Gamma$ such that $\varepsilon \models p_{a^+} P(\Gamma)$ and $a + \varepsilon$ is divisible. Now, consider any non-principal ultrapower (with index set $\mathbb{N}$) $(\Gamma^*, P(\Gamma^*))$ of $(\Gamma, P(\Gamma))$. We claim that $(\Gamma^*, P(\Gamma^*))$ is no longer stably embedded. Indeed, consider the element $a' \in \Gamma^*$ represented by the sequence $((a + \varepsilon)/q^i)_{i \in \mathbb{N}}$ in $\Gamma$. Let us show that $p = \text{tp}(a'/P(\Gamma^*))$ is not definable. By quantifier elimination in $L$, the only convex subsets definable in $P(\Gamma^*)$ are the ones with endpoints in the divisible hull of $P(\Gamma^*)$ (working in an ambient model of DOAG). If $p$ were definable, since $p$ is bounded, this would mean that there is $n \in \mathbb{N}$ and $b \in P(\Gamma^*)$ such that $p \vdash p_{(b/n)}|P(\Gamma^*)$ or $p \vdash p_{(b/n)}^{-}|P(\Gamma^*)$. Now if $b$ is represented by the sequence $(b_i)_{i \in \mathbb{N}}$, for almost every $i$ (with respect to the ultrafilter) $\text{tp}((a + \varepsilon)/q^i)/P(\Gamma) \vdash p_{(b_i/n)^\pm}$. On the other hand, $\text{tp}((a + \varepsilon)/q^i)/P(\Gamma) \vdash p_{(a/q^i)^\pm}$, so $a/q^i = b_i/n$ for almost all $i$, contradicting the fact that $a$ is not $q$-divisible. \hfill $\square$

Let $T$ now denote a complete theory of dense regular non-divisible ordered abelian groups. Due to the failure of uniform definability of types in $T$, in the following theorem, we need to work in the more general framework of $sc$-structures in the sense of Remark 2.4.17(1). We denote by $K$ the class of $L_{\text{op}}$-substructures of $sc$-pairs $A = (A, P(A))$ such that $P(A)$ is a model of $T$, $P(A)$ is convex in $A$ and for every prime $q$, the embedding $P(A)/qP(A)$ into $A/qA$ is a group isomorphism. It is not difficult to see that $K$ is a natural class. The proof of the following result is very similar to the one for Theorem 4.2.2. We leave the details to the reader.
4.3.6. **Theorem.** Let $T_{\infty}$ denote the theory of all proper convex elementary pairs $\mathcal{M}$ of models of $T$. Then $T_{\infty}$ is complete and axiomatizes $T_{bp}(K)$, and $T_{bp}(K)$ has beauty transfer.

Note that when $\mathcal{M}$ is a convex elementary pair of models of $T$, if follows from regularity that all cosets of $nM$ are represented in $P(M)$ for every $n > 0$.

Observe also that while $T$ does not have uniform definability of types by Proposition 4.3.5, by Remark 2.4.17, the class $\mathcal{F}$ of global definable types at infinity (i.e., those types which can be realized in convex elementary pairs) has uniform definability.

5. **BEAUTIFUL PAIRS OF PURE SHORT EXACT SEQUENCES**

We study in this section beautiful pairs of short exact sequences. Our motivation comes from valued fields, where we will apply the results of this section to the short exact sequence $1 \to K^n \to R V^x \to G \to 0$. Readers only interested in ACVF, RCVF, or $p$CF can skip this section and directly go to Section 6.

Let us first recall the setting of [1].

5.1. **The setting.** In this section, we consider a complete theory of an $\{A\}$-$\{C\}$-enrichment of a short exact sequence $M$ of abelian groups

$$0 \to A(M) \xrightarrow{\iota} B(M) \xrightarrow{\nu} C(M) \to 0$$

with $A(M)$ a pure subgroup of $B(M)$. It is well-known that $A$ and $C$ are stably embedded, orthogonal and pure sorts in $M$. Actually, as the sequence splits in any $\aleph_1$-saturated model, $\text{Th}(M)$ is a reduct of $\text{Th}(A(M) \times C(M))$, and these same properties hold in the product structure.

We need to consider various languages in what follows. Let $L_a = \{0, +, -\}$ be the language of abelian groups on sort $A$, and let $L_b$ and $L_c$ be similarly defined. Let $L_a^*$ be a relational enrichment of $L_a$ on the sort $A$, $L_c^*$ be a relational enrichment of $L_c$ on the sort $C$, and $T_a := \text{Th}_{L_a}(A(M))$, $T_c := \text{Th}_{L_c}(C(M))$, and let $T_a$ and $T_c$ be similarly defined (note that arbitrary enrichments of $L_a^*$ and $L_c^*$ may always be replaced by their Morleyzation).

Let $L_{abc}$ be the three sorted language given by $L_a \cup L_b \cup L_c \cup \{\iota, \nu\}$, and $L_{abc}^* = L_{abc} \cup L_a^* \cup L_c^*$. Let $T_{abc} = \text{Th}_{L_{abc}}(M)$ and $T_{abc} = \text{Th}_{L_{abc}}(M)$. In [1] the authors work in a definitional expansion $L_{abc} = L_{abc} \cup L_a \cup L_c \cup \{\iota, \nu\}$, and $L_{abc}^* = L_{abc} \cup L_a^* \cup L_c^*$. Let $T_{abc} = \text{Th}_{L_{abc}}(M)$ and $T_{abc} = \text{Th}_{L_{abc}}(M)$. In [1] the authors work in a definitional expansion $L_{abc} = L_{abc} \cup L_a \cup L_c \cup \{\iota, \nu\}$, and $L_{abc}^* = L_{abc} \cup L_a^* \cup L_c^*$. Let $T_{abc} = \text{Th}_{L_{abc}}(M)$ and $T_{abc} = \text{Th}_{L_{abc}}(M)$.

In what follows we denote by $\mathcal{F}$ the family of sorts $\{A/nA\}_{n \in \mathbb{N}}$. The proof of [1, Corollary 4.3] gives the following variant where we allow variables from all sorts from $\mathcal{F}$ and where we use that in our setting we only consider $\{A\}$-$\{C\}$-enrichments.

---

1See [37, Appendix A] for a formal definition of enrichment. For the sake of simplicity we assume no new sorts are added in such an enrichment.

2Note that in [1] the abelian group on the sorts $A/nA$ is not part of the language $L_{abc}$. 

---
5.1.1. **Fact.** Every $L_{abeq}^*$-formula $\phi(x_a, x_b, x_c)$, with $x_a, x_b, x_c$ variables from sorts $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$, respectively, is equivalent in $T_{abeq}^*$ to a boolean combination of formulas of the following forms:

(i) $\psi_a(x_a, \rho_n(t_1(x_b)), \ldots, \rho_n(t_l(x_b)))$ where $\psi_a(x_a, y_1, \ldots, y_l)$ is an $L_{eq}^*$-formula, each $t_i(x_b)$ is an $L_b$-term and $n, l_1, \ldots, l_l \in \mathbb{N}$,

(ii) $\psi_v(\nu(t_1(x_b)), \ldots, \nu(t_l(x_b)), x_c)$ where $\psi_v(y_1, \ldots, y_l, x_c)$ is an $L_c^*$-formula and each $t_i(x_b)$ is an $L_b$-term.

5.2. **Beautiful pairs.** Throughout this subsection we let $\mathcal{L}$ be $L_{abeq}^*$. Given $M$ and $N$ two superstructures of $K$, we sometimes write $K(M, N)$ instead of $\langle M, N \rangle$ to emphasize that $K$ is a common substructure of $M$ and $N$.

5.2.1. **Lemma.** Let $M_0 \preceq M \preceq \mathcal{U}$ be such that $\mathcal{U}$ is a sufficiently saturated and homogeneous model of $T_{abeq}^*$, and let $M_0 \subseteq N \subseteq \mathcal{U}$ be an $\mathcal{L}$-substructure of $\mathcal{U}$. Assume the following holds:

(1) $A/nA(N) \cap A/nA(M) = A/nA(M_0)$ for any $n \in \mathbb{N}$, and

(2) $C(N) \cap C(M) = C(M_0)$.

Let $\sigma$ and $\tau$ be two automorphisms of $\mathcal{U}$ over $M_0, \mathcal{A}(M)C(M)$. Then, there is an automorphism $\eta$ of $\mathcal{U}$ mapping $M_0 \langle M, N \rangle$ to $M_0 \langle \sigma(M), \tau(N) \rangle$ such that $\eta|_M = \sigma|_M$ and $\eta|_N = \tau|_N$.

Furthermore, $\mathcal{A}(M, N) = \mathcal{A}(M), \mathcal{A}(N)$ and $C(M, N) = C(M), C(N)$.

**Proof.** Without loss of generality, we may assume that $\tau$ is the identity and that $N$ is $\mathcal{B}$-generated. Indeed, the result for $(B(N))$ implies the result for $N$.

5.2.2. **Claim.** $B(N) \cap B(M) = B(M_0)$, equivalently $B(N) \oplus_{B(M_0)} B(M) \cong B(N) + B(M)$.

Consider $\beta \in B(N) \cap B(M)$. As $\nu(\beta) \in C(N) \cap C(M) = C(M_0)$, there is $\beta_0 \in B(M_0)$ such that $\nu(\beta) = \nu(\beta_0)$, i.e., $\beta - \beta_0 \in A(N) \cap A(M) = A(M_0)$, so finally $\beta \in B(M_0)$. This completes the proof of the claim.

5.2.3. **Claim.** For any $n \in \mathbb{N}$ we have $A/nA(M, N) = A/nA(M) + A/nA(N)$, where the latter is isomorphic to $A/nA(M) \oplus_{A/nA(M_0)} A/nA(N)$.

It is clear that $A/nA(M, N)$ is generated, as an abelian group, by $A/nA(M) \cup A/nA(N) \cup \rho_n(B(M) + B(N))$, and so it suffices to show that for any $b \in B(M)$ and $d \in B(N)$ we have $\rho_n(b + d) \in A/nA(M) + A/nA(N)$. We may assume that $\nu(b + d) \in nC(\mathcal{U})$. By assumption (2) and stability in $L_c$, $\text{tp}_{L_c}(\nu(b)/C(\mathcal{N}))$ is finitely satisfiable in $C(M_0)$. Then, there is $d_0 \in B(M_0)$ such that $\nu(b + d_0) = \nu(b) + \nu(d_0) \in nC(\mathcal{U})$. It follows that $\rho_n(b + d) = \rho_n((b + d_0) + (d - d_0)) = \rho_n(b + d_0) + \rho_n(d - d_0) \in A/nA(M) + A/nA(N)$. This completes the proof of the claim.

Since every element in $(B(M), B(N))$ is of the form $b_1 + b_2$ with $b_1 \in B(M)$ and $b_2 \in B(N)$, consider the map $\eta: (B(M), B(N)) \to \mathcal{U}$ given by $\eta(b_1 + b_2) = \sigma(b_1) + b_2$. Note that $\eta$ is well-defined by Claim 5.2.2 and the fact that $\tau$ is the identity on $M_0$. It is clear that $\eta$ defines a group homomorphism in all sorts. The proof of Claim 5.2.3 shows moreover that $\eta$ respects $\rho_n$ for every $n \geq 0$. By Fact 5.1.1, this shows that $\eta$ is an elementary map and hence, by homogeneity, it can be lifted to the desired automorphism.

The following is a domination result for short exact sequences. Part (1) slightly generalizes [43, Proposition 2.12].

5.2.4. **Lemma.** Let $M_0 \preceq \mathcal{U} \models T_{abeq}^*$ and let $M_0 \subseteq N \subseteq \mathcal{U}$, where $N$ is an $L_{abeq}^*$-substructure of $\mathcal{U}$. Then we have:
(1) \( tp_{\mathcal{L}}(N/M_0) \) is definable if and only if \( tp_{\mathcal{L}}(\mathcal{A}(N)/\mathcal{A}(M_0)) \) and \( tp_{\mathcal{L}}(\mathcal{C}(N)/\mathcal{C}(M_0)) \) are both definable;
(2) assuming that \( tp_{\mathcal{L}}(N/M_0) \) is definable and \( M_0 \lneq M \), we have
\[
\left( \begin{array}{c}
\forall \mathcal{A} \in \mathcal{L}
\mathcal{A} \rightarrow M
\end{array} \right) \cup \left( \begin{array}{c}
\forall \mathcal{A} \in \mathcal{L}
\mathcal{A} \rightarrow B
\end{array} \right) \impliedby \left( \begin{array}{c}
\mathcal{A} \rightarrow M
\mathcal{A} \rightarrow B
\end{array} \right) \impliedby \left( \begin{array}{c}
\forall \mathcal{A} \in \mathcal{L}
\mathcal{A} \rightarrow M
\mathcal{A} \rightarrow B
\end{array} \right).
\]

Proof. For (1), from left-to-right, the result follows from the fact that \( \mathcal{A}(M_0) \) is stably embedded in \( M_0 \) with induced structure given by \( \mathcal{L} \) and the fact that \( \mathcal{C}(M_0) \) is stably embedded in \( M_0 \) with induced structure given by \( \mathcal{L} \). For the converse, first note that \( T_{abcq} \) is stable, so \( tp_{\mathcal{L}}(N/M_0) \) is a definable type. Then, it follows from Fact 5.1.1 that \( tp_{\mathcal{L}}(N/M_0) \) is implied by \( tp_{\mathcal{L}}(N/M_0) \cup tp_{\mathcal{L}}(\mathcal{A}(N)/\mathcal{A}(M_0)) \cup tp_{\mathcal{L}}(\mathcal{C}(N)/\mathcal{C}(M_0)) \), which shows the result.

For part (2), by Lemma 5.2.1, any two realizations of the left-hand side of the condition in (2) are conjugates by an automorphism of \( \mathcal{U} \) fixing \( M \).

Suppose that \( \mathcal{K}_{abcq} \) is a natural class of \( s_{ec} \)-pairs of \( T_{abcq}^* \), satisfying the extension property, such that \( \mathcal{K}_{abcq} \)-beautiful pairs exist and \( T_{bp}(\mathcal{K}_{abcq}) \) has beauty transfer. Similarly, suppose that \( \mathcal{K}_{C} \) is a natural class of \( s_{ec} \)-pairs of \( T_{C}^* \), satisfying extension, such that \( \mathcal{K}_{C} \)-beautiful pairs exist and \( T_{bp}(\mathcal{K}_{C}) \) has beauty transfer. Let \( \mathcal{K} \) be the subclass of \( \mathcal{K}_{abcq} \) induced by \( \mathcal{K}_{abcq} \) and \( \mathcal{K}_{C} \), i.e., after base extending to a model of \( T_{abcq}^* \), the corresponding pairs lie in \( \mathcal{K}_{abcq} \) and \( \mathcal{K}_{C} \). It is not difficult to see \( \mathcal{K} \) is a natural class (for closure under base extension, one uses Lemma 5.2.1).

5.2.5. Theorem. Let \( \mathcal{K}_{abcq} \), \( \mathcal{K}_{C} \) and \( \mathcal{K} \) be as defined above. Then, \( \mathcal{K} \) satisfies the extension property and \( \mathcal{K} \)-beautiful pairs exist. Moreover, \( T_{bp}(\mathcal{K}) \) has beauty transfer and is axiomatized by the following conditions on an \( \mathcal{L}_{bp} \)-structure \( \mathcal{M} = (M, P(M)) \):

- \( P(M) \lneq M \models T_{abcq}^* \)
- \( (\mathcal{A}(M), \mathcal{A}(P(M))) \models T_{bp}(\mathcal{K}_{abcq}) \) and \( (\mathcal{C}(M), \mathcal{C}(P(M))) \models T_{bp}(\mathcal{K}_{C}) \).

Proof. Let \( T \) be the list of axioms given in the statement. Note that \( T \) is consistent, simply by taking the product of a model of \( T_{bp}(\mathcal{K}_{abcq}) \) and a model of \( T_{bp}(\mathcal{K}_{C}) \). So let \( \mathcal{M} \) be an \( |\mathcal{L}|^\ast \)-saturated model of \( T \). By Lemma 2.3.10, it suffices to show \( \mathcal{M} \) is an \( |\mathcal{L}|^\ast \)-\( \mathcal{K} \)-beautiful pair. So let \( \mathcal{A} \rightarrow \mathcal{M} \) and \( \mathcal{A} \rightarrow \mathcal{B} \) be \( \mathcal{L}_{bp} \)-embeddings with \( |\mathcal{B}| \leq |\mathcal{L}| \).

Step 0. Without loss of generality, we may suppose \( \mathcal{A} \subseteq \mathcal{M} \), and by Lemma-definition 2.2.7 (see also Step 1 of the proof of Lemma 2.3.3), we may assume \( P(\mathcal{A}) = P(\mathcal{B}) \). Moreover, we may also suppose \( P(\mathcal{A}) \) is a model of \( T_{abcq}^* \).

Step 1. Since \( \mathcal{B} \in \mathcal{K} \), we have \( \mathcal{A}(B) = (\mathcal{A}(B), \mathcal{A}(P(B))) \in \mathcal{K}_{abcq} \) and similarly \( \mathcal{C}(B) = (\mathcal{C}(B), \mathcal{C}(P(B))) \in \mathcal{K}_{abcq} \). By beauty transfer of \( T_{bp}(\mathcal{K}_{abcq}) \) and beauty transfer of \( T_{bp}(\mathcal{K}_{C}) \), \( \mathcal{A}(M) = (\mathcal{A}(M), \mathcal{A}(P(M))) \) is \( |\mathcal{L}|^\ast \)-\( \mathcal{K}_{abcq} \)-beautiful and \( \mathcal{C}(M) = (\mathcal{C}(M), \mathcal{C}(P(M))) \) is \( |\mathcal{L}|^\ast \)-\( \mathcal{K}_{C} \)-beautiful. In particular, there is an \( (\mathcal{L}_{abcq})_{bp} \)-embedding \( g : \mathcal{A}(B) \rightarrow \mathcal{A}(M) \) over \( \mathcal{A}(A) \), and similarly, there is an \( (\mathcal{L}_{C})_{bp} \)-embedding \( h : \mathcal{C}(B) \rightarrow \mathcal{C}(M) \) over \( \mathcal{A}(A) \). This yields that for any \( n \in \mathbb{N} \)

\[
\mathcal{A}/n\mathcal{A}(g(B)) \cap \mathcal{A}/n\mathcal{A}(P(M)) = \mathcal{A}/n\mathcal{A}(P(A)) \text{ and } \mathcal{C}(h(B)) \cap \mathcal{C}(P(M)) = \mathcal{C}(P(A)),
\]

which correspond to assumptions (1) and (2) in Lemma 5.2.1.

Step 2. By quantifier elimination (Fact 5.1.1) in \( \mathcal{L} \) and \( |\mathcal{L}|^\ast \)-saturation of \( \mathcal{M} \), there is an \( \mathcal{L} \)-embedding \( f : B \rightarrow M \) over \( A \) inducing the maps \( h \) and \( g \) from Step 1. Note that since
$P(B)$ is a model, the restriction of $f$ to $P(B) = P(A)$ induces an $\mathcal{L}_{bp}$-embedding. Moreover, we have that
\[
f(B) \models \text{tp}(B/P(A))
g(\mathfrak{A}(B)) \models \text{tp}(\mathfrak{A}(B)/\mathfrak{A}(P(A))) \mid \mathfrak{A}(P(M))
h(C(B)) \models \text{tp}(C(B)/C(P(A))) \mid C(P(M)).
\]
Part (2) of Lemma 5.2.4 yields that $f(B) \models \text{tp}(B/P(A)) \mid P(M)$, which shows that $f$ is an $\mathcal{L}_{bp}$-embedding (the predicate $P$ is preserved essentially by Claim 5.2.2). 

5.2.6. Remark (Application to ordered abelian groups). By Theorem 5.2.5 we obtain characterizations of beautiful pairs of products of regular ordered abelian groups with the lexicographic order. In particular, we obtain theories of beautiful pairs satisfying beauty transfer for $\text{Th}(\mathbb{Z}^n)$ and $\text{Th}(\mathbb{Z}^n \times \mathbb{Q})$ corresponding to combinations of the possibilities in Theorem 4.1.1 and Theorem 4.2.2. The results of the current section apply, as one may use induction on $n$ and the fact that in both $\mathbb{Z}^n$ and $\mathbb{Z}^n \times \mathbb{Q}$, all convex subgroups are definable.

6. Domination in valued fields

After setting up the notation and terminology on valued fields that will be used for the rest of the paper, we will prove some key domination results building on and generalizing [20, Proposition 12.11] and [15, Theorem 2.5] (see subsection 6.3). Some of these results have been recently obtained independently by M. Vicaria in [45]. Slight variants also appear in [22] by the second author and R. Mennuni. For the reader’s convenience we decided to keep all of our proofs.

6.1. Languages and theories of valued fields. For a valued field $(K, v)$, we let $\Gamma(K)$ denote its value group $v(K^\times)$, $\mathcal{O}(K)$ its valuation ring, $\mathcal{M}(K)$ its maximal ideal, $k(K)$ its residue field and $\text{res}: \mathcal{O}(K) \rightarrow k(K)$ the residue map. By default, unless otherwise stated, by a valued field extension $K \subseteq L$ we implicitly mean $(K \subseteq L, v)$ and always use $v$ for the underlying valuation. When working with multi-sorted structures, we will also use the notation $\mathbf{VF}(K)$ to denote the field $K$.

We will work in the following languages and theories of valued fields.

- The 3-sorted language of valued fields $\mathcal{L}_{\Gamma K}$ has sorts $\mathbf{VF}, k$, and $\Gamma_\infty$, where $\mathbf{VF}$ and $k$ are equipped with $\mathcal{L}_{\text{ring}}$, $\Gamma_\infty$ is equipped with $\mathcal{L}_{\text{og}}$ together with a new constant symbol for $\infty$, and we have additional functions symbols for the valuation $v$: $\mathbf{VF} \rightarrow \Gamma_\infty$ and the following residue map $\text{Res}: \mathbf{VF}^2 \rightarrow k$ interpreted in a valued field $(K, v)$ by
\[
\text{Res}(x, y) := \begin{cases} 
\text{res}(\frac{x}{y}) & \text{if } \infty \neq v(y) \leq v(x); \\
0 & \text{otherwise.}
\end{cases}
\]

We let $\Gamma$ be $\Gamma_\infty \setminus \{\infty\}$. For the remaining of the paper, unless otherwise stated, ACVF will denote the complete $\mathcal{L}_{\Gamma K}$-theory of an algebraically closed non-trivially valued field.

- The 3-sorted language of ordered valued fields $\mathcal{L}_{\text{ovf}}$ corresponds to $\mathcal{L}_{\Gamma K}$ together with additional binary predicates for the order in both $\mathbf{VF}$ and $k$. As usual, RCVF is the $\mathcal{L}_{\text{ovf}}$-theory of a real closed non-trivially valued field with convex valuation ring.

- The 3-sorted language of $p$-adically closed fields with $p$-ramification index $e$ and residue degree $f$ $\mathcal{L}_{\text{pct}}(e, f)$ corresponds to $\mathcal{L}_{\Gamma K}$ where on $\mathbf{VF}$ we replace $\mathcal{L}_{\text{ring}}$ by Macintyre language $\mathcal{L}_{\text{Mac}}$ (i.e., we add predicates $P_n$ for the $n^{\text{th}}$-powers) together $d = ef$ new constant symbols, and on $\Gamma_\infty$ we put Presburger’s language $\mathcal{L}_{\text{pres}}$ together with a new constant symbol for $\infty$. If $F$ is a finite extension of $\mathbb{Q}_p$, with $p$-ramification index $e$ and residue degree $f$, the new
constants are interpreted by elements in $O$ whose residues in $O/pO$ form an $\mathbb{F}_p$-basis. We let $p\text{CF}$ denote the complete $\mathcal{L}_{\text{pcf}}(e, f)$-theory of a finite extension of $\mathbb{Q}_p$ of $p$-ramification index $e$ and residue degree $f$.

- We will now consider $(K, v)$ in the leading term language $\mathcal{L}_{\text{RV}}$ of Flenner [17], consisting of a sort $\mathbf{VF}$ for the valued field (endowed with the ring language), a sort $\mathbf{RV}$ for the union of $\{0\}$ and the quotient group $K^\times/1 + \mathcal{M}(K)$ (endowed with the group language and a ternary relation $\oplus$) and the quotient map $rv: \mathbf{VF} \to \mathbf{RV}$ (extended by 0 $\mapsto$ 0). We let $\mathbf{RV}^\times$ denote $\mathbf{RV} \setminus \{0\}$. By an $\mathbf{RV}$-enrichment of $\mathcal{L}_{\text{RV}}$, we mean a language extending $\mathcal{L}_{\text{RV}}$ which may add new sorts but only adds functions, relations and constant symbols to $\mathbf{RV}$ and the new sorts. See [37, Appendix A] for a formal definition. Abusing of notation, given an $\mathbf{RV}$-enrichment $\mathcal{L}$ of $\mathcal{L}_{\text{RV}}$ and an $\mathcal{L}$-structure $M$, we continue to write $\mathbf{RV}(M)$ for the union of the $M$-points of $\mathbf{RV}$ together with the $M$-points of all new sorts from $\mathcal{L}$.

- We will denote by $\mathcal{L}_{\text{RV}, \Gamma}$ the language $\mathcal{L}_{\text{RV}}$ augmented by a sort $\Gamma_\infty$ (endowed with $\mathcal{L}_{\text{og}}$ together with a constant symbol for $\infty$) for the value group and the quotient map $v_{\text{rv}}: \mathbf{RV} \to \Gamma_\infty$ so that $v = v_{\text{rv}} \circ \text{rv}$.

- Recall that an angular component map is a multiplicative group homomorphism $ac: \mathbf{VF}^\times \to K^\times$ extended to $\mathbf{VF}$ by setting 0 $\mapsto$ 0 and such that $ac(a) = \text{res}(a)$ for any $a$ with $v(a) = 0$. The language $\mathcal{L}_{\text{Pas}}$ is the 3-sorted language with sorts $\mathbf{VF}$ (endowed with $\mathcal{L}_{\text{ring}}$, $\Gamma$ (endowed with $\mathcal{L}_{\text{og}}$ together with a constant symbol for $\infty$) and $K$ (endowed with $\mathcal{L}_{\text{ring}}$) and the maps $v: \mathbf{VF} \to \Gamma_\infty$ and $ac: \mathbf{VF} \to K$. Any $\aleph_1$-saturated valued field may be endowed with an ac map, and the $\mathcal{L}_{\text{Pas}}$-structure associated to a valued field $(K, v)$ is quantifier-free bi-interpretable with the $\mathcal{L}_{\text{RV}}$-structure associated to $(K, v)$ enriched by a splitting of the short exact sequence

$$1 \to K^\times \to \mathbf{RV}^\times \to \Gamma \to 0,$$

i.e., by a group homomorphism $s: \mathbf{RV}^\times \to K^\times$ such that $s \circ t = \text{id}_{K^\times}$. By a $\Gamma$-$K$-enrichment of $\mathcal{L}_{\text{Pas}}$ we mean a language extending $\mathcal{L}_{\text{Pas}}$ which may add functions, relations and constant symbols to $\Gamma_\infty$ and $K$, separately. Note that, contrarily to $\mathbf{RV}$-enrichments, here we do not allow new sorts.

6.1.1. **Definition.** We will call a valued field $(K, v)$ benign if its theory $T$ in $\mathcal{L}_{\text{RV}}$ admits elimination of $\mathbf{VF}$-quantifiers and if any model of $T$ admits a unique maximal immediate extension (up to isomorphism), which is an elementary extension.

6.1.2. **Fact.** The following valued fields are benign:

1. Henselian valued fields of equicharacteristic 0,
2. algebraically maximal Kaplansky valued fields of positive characteristic,
3. algebraically closed valued fields.

*Proof.* For elimination of $\mathbf{VF}$-quantifiers in cases (1), (2) and (3), respectively, see [17], [18] and [21]. The condition on maximal immediate extensions is well known in these cases. \square

6.1.3. **Remark.** In [43], Touchard defines benign valued fields as the valued fields belonging to the list given in Fact 6.1.2, although he works axiomatically, and the results he obtains work with the more general definition we are adopting. For this reason, we decided to stick to the same terminology.

By syntactic considerations (see [37, Proposition A.9]), elimination of $\mathbf{VF}$-quantifiers in benign valued fields yields the following.

6.1.4. **Fact.** Let $(K, v)$ be a benign valued field. Then the following holds.
6.2. Some lemmas about valuation independence. For $K \subseteq K'$, $L$ subfields of a common extension, we write $L \downarrow^\text{vd}_K K'$ if $L$ is linearly disjoint from $K'$ over $K$.

6.2.1. Fact. The following holds.

(1) Let $K \subseteq K'$, $L$ be fields all contained in an elementary extension $\mathcal{U}$ of $K$. If $\text{qftp}_{\mathcal{L}_{\text{ring}}}(L/K')$ is $K$-definable, then $L \downarrow^\text{id}_K K'$.

(2) For an arbitrary first order theory $T$, and $M \subseteq N, M'$ substructures all contained in an elementary extension $\mathcal{U} \models T$ of $M$, it holds that, if $\text{tp}(N/M')$ is $M$-definable, then $N \cap M' = M$. □

Given a valued field extension $K \subseteq L$ and a subset $A := \{a_1, \ldots, a_n\} \subseteq L$, we say that $A$ is $K$-valuation independent if $v(\sum_{i=1}^n c_i a_i) = \min_i (v(c_i a_i))$ for every $K$-linear combination $\sum_{i=1}^n c_i a_i$. The extension $L/K$ is called $vs$-defectless\(^3\) if every finitely generated $K$-vector subspace $V$ of $L$ admits a $K$-valuation basis, that is, a $K$-valuation independent set which spans $V$ over $K$. The following is a valuative analogue notion of linear disjointness.

6.2.2. Definition (Valuative disjointness). Let $K \subseteq K'$, $L$ be valued subfields of a common valued field. Suppose the extension $L/K$ is $vs$-defectless. We say $L$ is valuatively disjoint over $K$ from $K'$, denoted by $L \downarrow^\text{vd}_K K'$, if every $K$-valuation basis of a finite dimensional $K$-vector subspace in $L$ is also $K'$-valuation independent.

Observe that $L \downarrow^\text{vd}_K K'$ implies in particular that $L \downarrow^\text{id}_K K'$.

6.2.3. Definition. Let $K \subseteq L$ be a valued field extension and $B$ be a subset of $L$. We say $B$ is a normalized $K$-valuation independent set if it satisfies

(N1) for every $b, b' \in B$, if $v(b)$ and $v(b')$ lie in the same coset modulo $\Gamma(K)$, then $v(b) = v(b')$;

(N2) for every $b \in B$, the set $\{\text{res}(b'/b) \mid b' \in B$ and $v(b') = v(b)\}$ is $k(K)$-linearly independent;

(N3) if $b \in B$ and $v(b) \in \Gamma(K)$, then $v(b) = 0$;

(N4) if $b \in B$ and $\text{res}(b) \in k(K)$, then $\text{res}(b) = 1$.

Note that, as the name indicates, a normalized $K$-valuation independent set is $K$-valuation independent.

\(^3\)This is the same as “separated” in W. Baur’s and F. Delon’s terminology.
6.2.4. **Lemma** ([12, Lemma 2.24]). Let $K \subseteq L$ be a valued field extension and $B = \{b_1, \ldots, b_n\}$ a $K$-valuation independent set. Then there is a set $\{c_i \in K^\times : 1 \leq i \leq n\}$ such that $B' := \{c_ib_i : 1 \leq i \leq n\}$ is a normalized $K$-valuation independent set. □

The following fact follows directly from the definition of valuation independence.

6.2.5. **Fact.** Let $K \subseteq L$ be valued fields. Suppose $\{c_1, \ldots, c_n\} \subseteq L$ is $K$-valuation independent. Then for $a_i \in K$

$$rv\left(\sum_{i=1}^{n} c_ia_i\right) = \bigoplus_{i=1}^{n} rv(c_ira_i).$$

6.2.6. **Lemma.** Let $K \subseteq K'$, $L$ be valued subfields of a common valued field. Suppose that $L/K$ is $vs$-defectless, $\Gamma(L) \cap \Gamma(K') = \Gamma(K)$ and $k(L) \downarrow^{id}_{k(K)} k(K')$. Then, $L \downarrow^{vl} K'$. Moreover, $\Gamma(LK') = \Gamma(L) + \Gamma(K')$, $k(LK')$ is the field compositum of $k(L)$ and $k(K')$ and $LK'/K'$ is $vs$-defectless. In particular, $RV(LK')$ is generated by $RV(L)$ and $RV(K')$.

**Proof.** Let $B = \{b_1, \ldots, b_n\}$ be a $K$-valuation basis of a $K$-vector subspace of $L$. By Lemma 6.2.4, we may suppose $B$ is a normalized $K$-independent set. Suppose for a contradiction, $B$ is not $K'$-valuation independent. Hence, there are $a_1, \ldots, a_n \in K'$ such that

$$v\left(\sum_{i=1}^{n} b_i a_i\right) > \gamma := \min\{v(b_i a_i) : 1 \leq i \leq n\}.$$  

Let $J := \{i : v(b_i a_i) = \gamma\}$. In particular, $J$ is non-empty and $v(\sum_{i \in J} b_i a_i) > \gamma$. For $i, j \in J$

$$v(b_i/b_j) = v(a_i/a_j)$$

and since $\Gamma(L) \cap \Gamma(K') = \Gamma(K)$, all $v(b_i)$ lie in the same coset modulo $\Gamma(K)$. Therefore, by (N1), $v(b_i) = v(b_j)$ for all $i, j \in J$. Fixing some $i_0 \in J$, by (N2), the elements in $E := \{\text{res}(b_i/b_{i_0}) : i \in J\}$ are $k(K')$-linearly independent. Since $k(L)$ is linearly disjoint from $k(K')$ over $k(K)$, they are also $k(K')$-linearly independent. But by [16, Lemma 3.2.2], this implies that $\{b_i : i \in J\}$ is $K'$-valuation independent, a contradiction.

The moreover part follows for $\Gamma$ and $k$ as in the proof of [20, Proposition 12.11]. This implies the result for $RV$ since $RV$ sits in the exact sequence $1 \rightarrow k^\times \rightarrow RV^\times \rightarrow \Gamma \rightarrow 0$. □

For the definition of valued vector spaces and the following fact see [2, Definition 3.2.23 and Corollary 3.2.26].

6.2.7. **Fact.** Let $(K, v)$ be a maximally complete valued field. Then, every finite dimensional valued $K$-vector space $(V, w)$ admits a $K$-valuation basis.

6.2.8. **Lemma.** Let $K$ be valued field having an elementary extension which is maximal. Let $K \subseteq L \subseteq M$ be valued field extensions with $K \preceq M$. If $qftp_{\Gamma_L K}(L/K)$ is definable, then the extension $L/K$ is $vs$-defectless.

**Proof.** Let $V = \text{Span}_K(b_1, \ldots, b_n)$ where $b_1, \ldots, b_n \in L$ are $K$-linearly independent. The pair $(V, v)$ is a valued $K$-vector space.

Let $\preceq$ be the preorder on $K^n$ given by

$$a \preceq a' \iff v\left(\sum_{i=1}^{n} a_ib_i\right) \leq v\left(\sum_{i=1}^{n} a'_ib_i\right)$$

and let $\sim$ be the associated equivalence relation on $K^n$. Since $qftp_{\Gamma_L k}(L/K)$ is definable, $\preceq$ and $\sim$ are $K$-definable, and thus the quotient $\Gamma_V := K^n/\sim$ together with the total ordering $\leq$ on $\Gamma_V$ induced by $\preceq$ is $K$-interpretable.
Letting $w: K^n \rightarrow \Gamma_V$ be the function sending $a$ to $a/\sim$, it is clear that the valued $K$-vector space $(V, v)$ is isomorphic to the $K$-interpretable valued $K$-vector space $(K^n, w)$. Let $K'$ be a maximally complete elementary extension of $K$ and $((K')^n, w')$ be the corresponding definable vector space. By Fact 6.2.7, $((K')^n, w')$ admits a $K'$-valuation basis. But this is a definable property, so $(K^n, w)$ has also a $K$-valuation basis, and hence so does $(V, v)$.

**6.3. Domination.** Let $\mathcal{L}$ be either $\mathcal{L}_{\Gamma_k}, \mathcal{L}_{ovf}$ or $\mathcal{L}_{pct}(e, f)$ and $T$ be either a completion of ACVF, RCVF or $p$CF, each in their corresponding language. Let $K \subseteq K'$ and $K \subseteq L$ be $\mathcal{L}$-structures all being $\mathcal{L}$-substructures of some sufficiently large model $\mathcal{U}$ of $T$. We let $\mathbf{VF}(K)[\mathbf{VF}(L), \mathbf{VF}(K')]$ denote the ring generated by $\mathbf{VF}(L)$ and $\mathbf{VF}(K')$ over $\mathbf{VF}(K)$ and $\mathbf{VF}(K)[\mathbf{VF}(L), \mathbf{VF}(K')]$ denote the field compositum of $\mathbf{VF}(L)$ and $\mathbf{VF}(K')$ over $\mathbf{VF}(K)$. We let $K(L, K')$ denote the $\mathcal{L}$-structure generated by $L$ and $K'$ over $K$.

**6.3.1. Proposition.** Let $K \subseteq F$ and $K \subseteq L$ be $\mathcal{L}_{\Gamma_k}$-structures all being $\mathcal{L}_{\Gamma_k}$-substructures of some sufficiently saturated and homogeneous algebraically closed valued field $\mathcal{U}$. Suppose that

- $K$ is $\mathbf{VF}$-generated and $\mathbf{VF}(K), \mathbf{VF}(F), \mathbf{VF}(L)$ are fields,
- $\mathbf{VF}(L)/\mathbf{VF}(K)$ is vs-defectless,
- $v(\mathbf{VF}(L)) \cap v(\mathbf{VF}(F)) = v(\mathbf{VF}(K))$ and
- $\text{res}(O(L)) \cup \text{res}(O(K)) = \text{res}(O(F))$.

Let $\sigma$ and $\tau$ be two automorphisms of $\mathcal{U}$ over $K\Gamma(L)k(L)$ with $\sigma(L) = L'$ and $\tau(F) = F'$. Then there is an automorphism $\rho$ of $\mathcal{U}$ mapping $N = K\langle L, F \rangle$ to $N' = K\langle L', F' \rangle$ and such that $\rho|_L = \sigma|_L$ and $\rho|_F = \tau|_F$. Furthermore, $\mathbf{VF}(N)$ (resp. $k(N))$ is the ring generated by $\mathbf{VF}(L)$ and $\mathbf{VF}(F)$ (resp. by $k(L)$ and $k(F)$), $\Gamma(N) = \Gamma(L) + \Gamma(F)$, and $\mathbf{VF}(L) \downarrow \mathbf{VF}(F)$.

**Proof.** Without loss of generality we may assume $\tau$ is the identity. Furthermore, we may suppose that $F$ and $L$ are also $\mathbf{VF}$-generated. Indeed, let $\bar{F} \subseteq F$ and $\bar{L} \subseteq L$ be their respective $\mathbf{VF}$-generated parts, $\bar{L}' = \sigma(\bar{L})$, $\bar{F}' = \tau(\bar{F})$ and $\bar{\rho}$ be the automorphism sending $\bar{N} = K\langle \bar{L}, \bar{F} \rangle$ to $\bar{N}' = K\langle \bar{L}', \bar{F}' \rangle$. In addition, $\mathbf{VF}(\bar{N})$ is the field compositum of $\mathbf{VF}(\bar{L})$ and $\mathbf{VF}(\bar{F})$ over $\mathbf{VF}(K)$. What remains in $N \setminus \bar{N}$ only appears in the sorts $\Gamma$ and $k$. Then, the union of the maps $\bar{\rho}|_{\bar{N}}$ and $\sigma|_{\bar{N}}\bar{N}$ defines an $\mathcal{L}_{\Gamma_k}$-isomorphism between $N$ and $N'$. By homogeneity it extends to an automorphism $\rho$ on $\mathcal{U}$ which, by construction, satisfies $\rho|_L = \sigma|_L$ and $\rho|_F = \tau|_F$. The last statement of the proposition follows from the assumptions, together with the corresponding statement for $\bar{N}$.

Assuming that all structures are $\mathbf{VF}$-generated, we may apply Lemma 6.2.6. The rest of the proof follows word for word the proof of [20, Proposition 12.11]. The fact that $\mathbf{VF}(L) \downarrow \mathbf{VF}(F)$ ensures that the map $\rho$ defined on $\mathbf{VF}(K)[\mathbf{VF}(L), \mathbf{VF}(F)]$ by

$$\rho \left( \sum_{i=1}^{n} l_i f_i \right) := \sum_{i=1}^{n} \sigma(l_i) f_i$$

is well-defined. Note that since $N = (\mathbf{VF}(K)(\mathbf{VF}(L), \mathbf{VF}(F)))$, it suffices to show that $\rho: \mathbf{VF}(N) \rightarrow \mathbf{VF}(N')$ is a valued field isomorphism. This is also proved in [20, Proposition 12.11], and follows from the fact that $\mathbf{VF}(L) \downarrow \mathbf{VF}(F)$.

The following result is new in the case of $p$CF.

**6.3.2. Proposition.** Let $\mathcal{L}$ be either $\mathcal{L}_{ovf}$ or $\mathcal{L}_{pct}(e, f)$ and $T$ be either RCVF or $p$CF, each in their corresponding language.
Let $K \subseteq F$ and $K \subseteq L$ be $\mathcal{L}$-structures all being $\mathcal{L}$-substructures of some sufficiently saturated and homogeneous model $\mathcal{U}$ of $T$. Suppose that

- $K$ is $\text{VF}$-generated and $\text{VF}(K), \text{VF}(F), \text{VF}(L)$ are fields,
- $\text{VF}(L)/\text{VF}(K)$ is vs-defectless,
- $v(\text{VF}(L)) \cap v(\text{VF}(F)) = v(\text{VF}(K))$ and
- $\text{res}(\mathcal{O}(L)) \downarrow_{\text{res}(\mathcal{O}(K))} \text{res}(\mathcal{O}(F))$.

Let $\sigma$ and $\tau$ be two automorphisms of $\mathcal{U}$ over $K\Gamma(L)k(L)$ with $\sigma(L) = L'$ and $\tau(F) = F'$. Then there is an automorphism $\rho$ of $\mathcal{U}$ mapping $N = K\langle L, F \rangle$ to $N' = K\langle L', F' \rangle$ and such that $\rho|_L = \sigma|_L$ and $\rho|_F = \tau|_F$. Furthermore, $\text{VF}(N)$ (resp. $k(N)$) is the ring generated by $\text{VF}(L)$ and $\text{VF}(F)$ (resp. by $k(L)$ and $k(F)$), $\Gamma(N) = \Gamma(L) + \Gamma(F)$, and $\text{VF}(L) \downarrow_{\text{VF}(K)} \text{VF}(F)$.

**Proof.** As in the proof of Proposition 6.3.1, we may suppose that: $\tau$ is the identity, $F$ and $L$ are $\text{VF}$-generated. Moreover, it holds that $\text{VF}(L) \downarrow_{\text{VF}(K)} \text{VF}(F)$ and that the map $\rho$ on $\text{VF}(K)(\text{VF}(L), \text{VF}(F))$ defined by $\rho(\sum_{i=1}^n l_i f_i) = \sum_{i=1}^n \sigma(l_i) f_i$, is an $\mathcal{L}_{\Gamma_k}$-isomorphism sending $N$ to $N'$. It remains to show that $\rho$ is also an $\mathcal{L}_{\text{ovf}}$-isomorphism when $T$ is RCVF and an $\mathcal{L}_{\text{pcf}(e,f)}$-isomorphism when $T$ is $p\text{CF}$. For RCVF, this is contained in the proof of [15, Theorem 2.5]. Let us see it for $p\text{CF}$.

It suffices to show that $\rho$ preserves the predicates $P_n$. Suppose $x = \sum_{i=1}^m l_i f_i$ with $l_i \in \text{VF}(L)$ and $f_i \in \text{VF}(F)$. Without loss of generality, we may assume that $\{l_i\}_{1 \leq i \leq n}$ are $\text{VF}(K)$-valuation independent and so $\text{VF}(F)$-valuation independent by Lemma 6.2.6. Let us first show the result for $m = 1$, so $x = l_1 f_1$. Let $d = ef$ and $u_1, \ldots, u_d \in \mathcal{O}(\mathcal{U})$ be the interpretations of the $d = ef$ constant symbols from $\mathcal{L}_{\text{pcf}(e,f)}$ (which, by definition, represent an $\mathcal{F}_p$-basis of $\mathcal{O}(\mathcal{U})/p\mathcal{O}(\mathcal{U})$). Since the completion of $\mathbb{Q}(u_1, \ldots, u_d)$ is a model of $T$ and $\mathbb{Q}(u_1, \ldots, u_d) \subseteq K$, $K$ contains representatives of $\mathcal{U}^\times/P_n(\mathcal{U})$ as $P_n(\mathcal{U})$ is an open subgroup. In particular, by possibly dividing by an element of $K$, we may suppose that $l_1$ is already an $n^{\text{th}}$-power. Therefore, $l_1 f_1$ is an $n^{\text{th}}$-power if and only if $f_1$ is an $n^{\text{th}}$-power, and therefore $\rho(l_1 f_1) = \sigma(l_1) f_1$ is an $n^{\text{th}}$-power if and only if $f_1$ is an $n^{\text{th}}$-power. For $m > 1$, since the residue field is finite and $\{l_i\}_{1 \leq i \leq m}$ is $\text{VF}(F)$-valuation independent, we have that $v(l_1), \ldots, v(l_n)$ all lie in different $\Gamma(F)$-cosets, and we may thus assume that $v(l_1 f_1) < v(l_i f_i)$ for all $i > 1$. In particular, $x = l_1 f_1(1 + \epsilon)$ for some $\epsilon > \mathbb{Z}$. Note that $P_n(1 + \epsilon)$ holds for all $n \geq 1$, since $P_n(\mathcal{U})$ contains a ball centered at 1 with radius in $\mathbb{Z}$. Then, $x$ is an $n^{\text{th}}$-power if and only if $l_1 f_1$ is an $n^{\text{th}}$-power, and we are reduced to the case $n = 1$. \hfill \Box

Part (1) of the following lemma generalizes results in [11] and [13].

**6.3.3. Proposition.** Let $\mathcal{L}$ be either $\mathcal{L}_{\Gamma_k}$, $\mathcal{L}_{\text{ovf}}$ or $\mathcal{L}_{\text{pcf}(e,f)}$ and $T$ be either a completion of ACVF, RCVF or $p\text{CF}$, each in their corresponding language.

Let $K \subseteq L \subseteq \mathcal{U}$ be $\mathcal{L}$-structures with $K \preccurlyeq \mathcal{U} \models T$ and $\text{VF}(L)$ a field (but $L$ not necessarily $\text{VF}$-generated). Then

1. $\text{tp}(L/K)$ is definable if and only if $\text{VF}(L)/\text{VF}(K)$ is $\text{vs}$-defectless and $\text{tp}(k(L)/k(K))$ and $\text{tp}(\Gamma(L)/\Gamma(K))$ are both definable;
2. if $\text{tp}(L/K)$ is definable and $K \subseteq K' \subseteq \mathcal{U}$, then

\[\text{tp}(L/K) \cup \text{tp}(k(L)/k(K)) \ | k(K') \cup \text{tp}(\Gamma(L)/\Gamma(K)) \ | \text{tp}(L/K) | K'.\]

**Proof.** One direction of part (1) follows by Lemma 6.2.8, and stable embeddedness of the value group and residue field. The converse direction follows the same argument as in [13, Theorem 5.9] (where the assumption that $L$ is a model plays no essential role).
For part (2), let \( f: L \to \mathcal{U} \) be an \( \mathcal{L} \)-embedding such that \( f(L) \) realizes the left-hand side of (\ref{eq}). Possibly passing to the fields of fractions, we may suppose \( k(L) \) is a field. Let \( h: \Gamma(L) \to \Gamma(\mathcal{U}) \) and \( g: k(L) \to k(\mathcal{U}) \) be the corresponding induced embeddings. By Fact 6.2.8, \( k(g(L)) \downarrow_{k(L)} k(K') \) and \( \Gamma(h(L)) \cap \Gamma(K') = \Gamma(K) \). By Part (1), the extension \( \text{VF}(L)/\text{VF}(K) \) is vs-defectless. We are thus in the situation of Proposition 6.3.1 or Proposition 6.3.2, respectively, which yields the result.

We now present further domination results in benign valued fields.

6.3.4. Proposition. Let \((K,v)\) be benign. Work in the language \( \mathcal{L}_{\text{RV},\Gamma} \), possibly \( \text{RV} \)-enriched, and let \( K \subseteq L \subseteq K' \), with \( K \preceq K' \) and \( \text{VF}(L) \) a field, where \( L \) is a (not necessarily \( \text{VF} \)-generated) substructure of \( K' \). Then the following holds:

(1) \( \text{tp}(L/K) \) is definable if and only if \( \text{VF}(L)/\text{VF}(K) \) is vs-defectless and \( \text{tp}(\text{RV}(L) \cup \Gamma(L))/\text{RV}(K) \) is definable (in the theory of \( \text{RV}(K) \));

(2) assuming that \( \text{tp}(L/K) \) is definable and letting \( K \subseteq N \subseteq \mathcal{U} \), with \( K \preceq \mathcal{U} \) and \( N \) a substructure, it holds that

\[
\text{tp}(L/K) \cup \text{tp}(\text{RV}(L) \cup \Gamma(L)/\text{RV}(K)) \mid \text{RV}(N) \cup \Gamma(N) \vdash \text{tp}(L/K) \mid N.
\]

Proof. To prove (1), we argue as in the proof of \cite[Theorem 2.17]{43}, where this is shown in the special case when \( L \) is an elementary extension of \( K \).

Suppose \( \text{tp}(L/K) \) is definable. By Lemma 6.2.8, the extension \( \text{VF}(L)/\text{VF}(K) \) is vs-defectless. That the type \( \text{tp}(\text{RV}(L) \cup \Gamma(L)/\text{RV}(K)) \) is definable follows from the assumption and the fact that \( \text{RV}(K) \) is purely stably embedded (Fact 6.1.4).

For the converse, let \( \varphi(x,y) \) be an \( \mathcal{L} \)-formula, with \( y \) a tuple from sort \( \text{VF} \) and \( a \in L^{[x]} \), with \( a_0 \) the subtuple given by elements from sort \( \text{VF} \).

By \( \text{VF} \)-quantifier elimination (in the theory of \( K \)), we may suppose the formula \( \varphi(a,y) \) is of the form

\[
\psi(\text{rv}(P_1(y),\ldots,\text{rv}(P_m(y))),b)
\]

where \( P_i(y) \) are polynomials with coefficients in \( K(a_0) \) and \( b \) is a tuple of elements in \( \text{RV}(L) \cup \Gamma(L) \), and where \( \psi \) is a formula in the (enrichment of) the language \( \mathcal{L}_{\text{RV},\Gamma} \). Let \( c = (c_1,\ldots,c_n) \) be a valuation basis of the \( K \)-vector space spanned by the coefficients of the polynomials \( P_i(y),\ldots,P_m(y) \). Then, each polynomial \( P_i \) can be expressed as a sum of the form \( \sum_{j=1}^n c_j Q_{ij}(y) \) where \( Q_{ij} \in K[y] \). By Fact 6.2.5, this sum factors through \( \text{rv} \), namely

\[
\text{rv}(P_i(y)) = \bigoplus_{j=1}^n \text{rv}(c_j)\text{rv}(Q_{ij}(y))
\]

for any \( y \) from \( K \), and therefore the subset of \( K \) defined by the formula \( \varphi(a,y) \) may also be defined by a formula of the form

\[
\theta(\text{rv}(Q_{ij}(y)),\text{rv}(c_1),\ldots,\text{rv}(c_n),b).
\]

Introduce new \( \text{RV} \)-variables \( z = (z_{ij})_{ij} \). Since the type \( \text{tp}(\text{RV}(L) \cup \Gamma(L)/\text{RV}(K)) \) is definable, we may find an \( \text{RV}(K) \)-formula \( \theta'(z) \) such that \( \theta'(\text{RV}(K)) = \theta(\text{RV}(K),\text{rv}(c_1),\ldots,\text{rv}(c_n),b) \).

It follows that \( \theta'((\text{rv}(Q_{ij}(y)))_{ij}) \) defines the same set in \( K \) as \( \varphi(a,y) \).

To prove (2), we proceed as in Part (2) of Proposition 6.3.3. Indeed we may suppose that \( k(L) \) is a field. Moreover \( \text{tp}(\text{RV}(L) \cup \Gamma(L)/\text{RV}(K)) \mid \text{RV}(N) \cup \Gamma(N) \) yields \( k(L) \downarrow_{k(L)} k(N) \) as well as \( \Gamma(L) \cap \Gamma(N) = \Gamma(K) \). This shows, by Proposition 6.3.1, that the type \( \text{qftp}_{\mathcal{L}_{\Gamma}(L/K)}|N \) is determined by \( \text{tp}(L/K) \cup \text{tp}(\text{RV}(L) \cup \Gamma(L)/\text{RV}(K)) \mid \text{RV}(N) \cup \Gamma(N) \) which completes the result by \( \text{VF} \)-quantifier elimination (Fact 6.1.4). 

\[\square\]
The following is an immediate consequence of Proposition 6.3.4, taking into account part (3) of Fact 6.1.4 and that RV is canonically isomorphic to $k \times \Gamma_\infty$, if an angular component map is added to the language.

6.3.5. **Corollary.** Let $(K,v)$ be benign. Work in the language $L_{FS}$, possibly $\Gamma$-$k$-enriched, and let $K \subseteq L \subseteq K'$, with $K \preceq K'$, where $L$ is a (not necessarily VF-generated) substructure of $K'$. Then the following holds:

1. $tp(L/K)$ is definable if and only if $VF(L)/VF(K)$ is vs-defectless and both $tp(k(L)/k(K))$ and $tp(\Gamma(L)/\Gamma(K))$ are definable;
2. assuming that $tp(L/K)$ is definable and letting $K \subseteq N \subseteq \mathcal{U}$, with $K \preceq \mathcal{U}$ and $N$ a substructure, it holds that

$$tp(L/K) \cup tp(k(L)/k(K)) \cup k(N) \cup tp(\Gamma(L)/\Gamma(K)) \cup \Gamma(N) \vdash tp(L/K) \mid N.$$

$\Box$

7. **Beautiful pairs of valued fields**

In this section we characterize all completions of stably embedded pairs of ACVF, RCVF and $p$CF modulo the completions of stably embedded pairs of the corresponding value group and residue field. An Ax-Kochen type result will be given in Section 8 for beautiful pairs of benign henselian valued fields (see Theorem 8.3.1).

7.1. **Beautiful pairs of ACVF in the three sorted language.** Throughout Section 7.1, we let $\mathcal{L}$ denote the three sorted language of valued fields $\mathcal{L}_{\Gamma k}$. Recall that ACVF denotes a completion of the $\mathcal{L}_{\Gamma k}$-theory of algebraically closed non-trivially valued fields. Note that ACVF has uniform definability of types by [13, Theorem 6.3].

Recall that there are only two completions of the theory of stably embedded pairs of ACF and both correspond to theories of beautiful pairs with beauty transfer (see Proposition 3.1.5). Similarly, by Theorem 4.1.1, the 4 completions of the theory of stably embedded pairs of DOAG also correspond to theories of beautiful pairs with beauty transfer.

7.1.1. **Definition.** Let $T_k$ and $T_\Gamma$ be completions of the theory of stably embedded pairs of ACF and DOAG, respectively. Let $\mathcal{K}_k$ (resp. $\mathcal{K}_\Gamma$) be the natural classes such that $T_k$ (resp. $T_\Gamma$) is the theory of $\mathcal{K}_k$-beautiful pairs (resp. $\mathcal{K}_\Gamma$-beautiful pairs).

- We let $ACVF(T_k, T_\Gamma)$ be the $L_{bp}$-theory of stably embedded pairs of algebraically closed valued fields, for which the corresponding pair of residue fields is a model of $T_k$, and the corresponding pair of value groups is a model of $T_\Gamma$.
- We define $\mathcal{K}$ to be the class of $se$-pairs $\mathcal{A} \in \mathcal{K}_{def}$ such that $(k(\mathcal{A}'), k(P(\mathcal{A}'))) \in \mathcal{K}_k$ and $(\Gamma(\mathcal{A}'), \Gamma(P(\mathcal{A}'))) \in \mathcal{K}_\Gamma$ where $\mathcal{A}'$ is the base extension $\mathcal{A}' = \mathcal{A}_B$ with $B$ some model of ACVF containing $P(\mathcal{A})$. The class $\mathcal{K}$ is said to be induced by $\mathcal{K}_k$ and $\mathcal{K}_\Gamma$.

7.1.2. **Remark.** The class $\mathcal{K}$ is closed under base extension. Indeed, for $\mathcal{A} \in \mathcal{K}$, taking into account the definition of $\mathcal{K}$, we may assume that $P(\mathcal{A}) \models ACVF$. Now, given $P(\mathcal{A}) \subseteq B \subseteq \mathcal{U}_{eq}$, we may further suppose $B \models ACVF$. Setting $\mathcal{A}' = \mathcal{A}_B$, it holds that $(k(\mathcal{A}'), k(P(\mathcal{A}'))) \in \mathcal{K}_k$ and $(\Gamma(\mathcal{A}'), \Gamma(P(\mathcal{A}'))) \in \mathcal{K}_\Gamma$ (note no extra base extension is needed). This follows by Lemma 6.2.6, as $A/P(\mathcal{A})$ is vs-defectless by Lemma 6.2.8. This shows $\mathcal{A}' \in \mathcal{K}$.

7.1.3. **Lemma.** Given $\mathcal{K}_k$, $\mathcal{K}_\Gamma$ and $\mathcal{K}$ as above, $\mathcal{K}$ is a natural class.

**Proof.** Properties (i)-(iv) in Definition 2.2.8 are straightforward. Property (v) (closure under base extension) follows from Remark 7.1.2. $\Box$

7.1.4. **Remark.** By Part (1) of Proposition 6.3.3, the theory $ACVF(T_k, T_\Gamma)$ is axiomatized by axioms stating for a pair $\mathcal{M} = (M, P(\mathcal{M}))$
• \(VF(M)/VF(P(M))\) is vs-defectless;
• \((k(M), k(P(M))) \models T_k\) and \((\Gamma(M), \Gamma(P(M))) \models T_\Gamma\).

It follows easily that ACVF\((T_k, T_\Gamma)\) is consistent.

7.1.5. **Theorem.** Let \(K_k, K_\Gamma, K, T_k\) and \(T_\Gamma\) be as in Definition 7.1.1. Then, \(K\)-beautiful pairs exist. Moreover, ACVF\(_{bp}(K)\) has beauty transfer and is axiomatized by ACVF\((T_k, T_\Gamma)\).

**Proof.** Let \(M\) be an \(\aleph_1\)-saturated model of ACVF\((T_k, T_\Gamma)\). By Lemma 2.3.10, it suffices to show that \(M\) is an \(\aleph_1\)-\(K\)-beautiful pair. So let \(A \to M\) and \(A \to B\) be \(L_{bp}\)-embeddings with \(B\) countable.

**Step 0.** Without loss of generality, we may suppose \(A \subseteq M\), and by Lemma-definition 2.2.7 (see also Step 1 of the proof of Lemma 2.3.3), we may assume \(P(A) = P(B)\). Moreover, we may also suppose \(P(A)\) is a model of ACVF, and the valued field sorts of \(A\) and \(B\) are fields.

**Step 1.** Since \(B \in K\), we have \((k(B), k(P(B))) \in K_k\) and \(\Gamma(B) = (\Gamma(B), \Gamma(P(B))) \in K_\Gamma\). By beauty transfer of \(T_k\) and beauty transfer of \(T_\Gamma\), \((k(M), k(P(M))) \models K_k\)-beautiful and \(\Gamma(M) = (\Gamma(M), \Gamma(P(M))) \models K_\Gamma\)-beautiful. In particular, there is an \((L_{ring})_{bp}\)-embedding \(\gamma: \Gamma(B) \to \Gamma(M)\) and similarly, \(L_{log}\)-embedding \(\delta: \Gamma(B) \to \Gamma(M)\) over \(\Gamma(A)\). Fact 6.2.1 yields that \(k(g(B)) \downarrow_{k(P(A))} k(P(M))\) and \(\Gamma(h(B)) \cap \Gamma(P(M)) = \Gamma(P(A))\).

**Step 2.** By quantifier elimination in \(L\) and \(\aleph_1\)-saturation of \(M\), there is an \(L\)-embedding \(f: B \to M\) over \(A\) inducing the maps \(h\) and \(g\) from Step 1. Note that since \(P(A) = P(B)\), the restriction of \(f\) to \(P(B)\) is trivially an \(L_{bp}\)-embedding. Moreover, we have that

\[
\begin{align*}
f(B) &\models tp(B/P(A)) \\
g(k(B)) &\models tp(k(B)/k(P(A))) \mid k(P(M)) \\
h(\Gamma(B)) &\models tp(\Gamma(B)/\Gamma(P(A))) \mid \Gamma(P(M)).
\end{align*}
\]

Part (2) of Proposition 6.3.3 yields that \(f(B) \models tp(B/P(A)) \mid P(M)\), which shows that \(f\) induces an \(L_{bp}\)-embedding. (Note that it is clear that \(f\) respects the predicate \(P\) and is thus an \(L_p\)-embedding.)

Combining Proposition 3.1.5 and Theorems 4.1.1 and 7.1.5 we obtain the following result.

7.1.6. **Corollary.** There are exactly 8 possible completions of ACVF\(_{SE}\) (see Section 7.4 for a description), each corresponding to a theory of beautiful pairs, satisfying beauty transfer and of the form ACVF\((T_k, T_\Gamma)\).

7.2. **Beautiful pairs of RCF and \(p\)CF in the three sorted language.** Throughout this subsection we let \(L\) denote either the 3-sorted language of ordered valued fields \(L_{ovf}\) or the 3-sorted language \(L_{pcf}(e, f)\). Our results for RCF and \(p\)CF use the same pattern as those in the case of ACVF presented in the previous section.

We first consider the case of RCF. Recall that there are only two completions of the theory of stably embedded pairs of RCF and both correspond to theories of beautiful pairs (see Remark 3.2.9). Let \(T_k\) and \(T_\Gamma\) be completions of the theory of stably embedded pairs of RCF and DOAG, respectively. Let \(K_k\) (resp. \(K_\Gamma\)) be the natural classes such that \(T_k\) (resp. \(T_\Gamma\)) is the theory of \(K_k\)-beautiful pairs (resp. \(K_\Gamma\)-beautiful pairs). Note that \(T_k\) and \(T_\Gamma\) have beauty transfer (by Theorem 3.2.1 and Theorem 4.1.1, respectively)
Let $\mathcal{K}$ be the class of $se$-pairs $\mathcal{M} \in \mathcal{K}_{def}$ induced by $\mathcal{K}_k$ and $\mathcal{K}_\Gamma$.\textsuperscript{4} The argument given in Lemma 7.1.3 shows that $\mathcal{K}$ is a natural class. We let $RCVF(T_k, T_\Gamma)$ be the $\mathcal{L}_{bp}$-theory of stably embedded pairs of real closed valued fields, for which the corresponding pair of residue fields is a model of $T_k$, and the corresponding pair of value groups is a model of $T_\Gamma$. Moreover, by Part (1) of Proposition 6.3.3, the theory $RCVF(T_k, T_\Gamma)$ is axiomatizable and it is easy to see it is consistent.

7.2.1. **Theorem.** Let $\mathcal{K}_k, \mathcal{K}_\Gamma, \mathcal{K}, T_k$ and $T_\Gamma$ be as above. Then, $\mathcal{K}$-beautiful pairs exist. Moreover, $RCVF_{bp}(\mathcal{K})$ has beauty transfer and is axiomatized by $RCVF(T_k, T_\Gamma)$.

*Proof.*** The proof follows exactly the same strategy as that of Theorem 7.1.5, using the RCVF part of Proposition 6.3.3. □

7.2.2. **Corollary.** There are exactly 8 possible completions of $RCVF_{SE}$, each corresponding to a theory of beautiful pairs, satisfying beauty transfer and of the form $RCVF(T_k, T_\Gamma)$. □

In the case of $p$-adically closed fields the situation is much simpler. Combining Proposition 6.3.3 and Theorem 4.2.2, and following the same strategy of the proof of Theorem 7.1.5, we have the following. Recall that $PRES$ denotes the theory of Presburger arithmetic.

7.2.3. **Theorem.** The theory $pCF_{bp}$ is consistent and has beauty transfer. It is axiomatized by the theory of $us$-defectless elementary pairs of models of $pCF$ for which the corresponding pair of value groups is a model of $(PRES)_{bp}$. The theories $pCF_{bp}$ and $pCF_{bp}(K_{triv})$ are the only 2 completions of $pCF_{SE}$.

By Corollary 2.3.13, we have the following.

7.2.4. **Corollary.** Any completion of the theory of stably embedded pairs of $ACVF$, $RCVF$ and $pCF$ is NIP. □

7.3. **Beautiful pairs of $ACVF$ in the geometric language.** In this subsection, we consider the theory $ACVF$ in the geometric sorts $G$ of Haskell-Hrushovski-Macpherson [19, 20]. The corresponding language is denoted by $\mathcal{L}^G$. By the main result of [19], $ACVF$ eliminates imaginaries in $\mathcal{L}^G$.

7.3.1. **Theorem.** Let $T_\Gamma$ be a completion of the theory of stably embedded pairs of DOAG and let $T_k = ACF_{bp}$. Let $\mathcal{K}_\Gamma$ and $\mathcal{K}_k$ be their corresponding natural classes. Let $\mathcal{K}$ be the class of $se$-pairs (as $\mathcal{L}_{bp}^G$-structures) $\mathcal{M} \in \mathcal{K}_{def}$ induced by $\mathcal{K}_\Gamma$ and $\mathcal{K}_k$ (as in Definition 7.1.1). Then, $\mathcal{K}$-beautiful pairs exist and are elementary pairs. Moreover, $ACVF_{bp}(\mathcal{K})$ has beauty transfer and is axiomatized by $ACVF(T_k, T_\Gamma)$.

*Proof.*** We will reduce the statement to Theorem 7.1.5. Let $\mathcal{M}$ be an $\aleph_1$-saturated model of $ACVF(ACF_{bp}, T_\Gamma)$ in the language $\mathcal{L}^G_{bp}$. By Lemma 2.3.10, it suffices to show $\mathcal{M}$ is an $\aleph_1$-$K$-beautiful pair. So let $A \rightarrow M$ and $A \rightarrow B$ be $\mathcal{L}^G_{bp}$-embeddings with $B \in K$ countable.

As in the proof of Theorem 7.1.5, we may suppose $A \subseteq M$, $A \subseteq B$, $P(A) = P(B)$ is a model of $ACVF$, and the valued field sorts of $A$ and $B$ are fields.

We let $S_n$ denote the sort of $n$-dimensional lattices and $S$ the union of all $S_n$ for all $n > 0$. For any lattice $s \in S(B)$ the type $p_s$ of a generic $O$-basis of $s$ is a generically stable global $s$-definable type (see [27, Paragraph after Definition 3.6]). For $s \in S(B)$, choose $b_s \models p_s \mid B$, such that the tuple $\langle b_s \rangle_{s \in S(B)}$ is independent over $B$. Then $B_1 := \langle B \cup \{b_s \mid s \in S(B)\} \rangle, P(A) \in K, \text{ as } \Gamma(B_1) = \Gamma(B)$. Setting $A_1 := \langle A \cup \{b_s \mid s \in S(A)\}, P(A) \rangle$, we get $A \subseteq A_1, B \subseteq B_1$.

\textsuperscript{4}This is defined as in the case of $ACVF$, see Definition 7.1.1.
By $n_1$-saturation of $\mathcal{M}$ and since $k(M) \supseteq P(k(M))$, we find an $AP(\mathcal{M})$-independent family $(c_s)_{s \in S(A)}$ in $M$ with $c_s \models p_s \mid AP(\mathcal{M})$ for all $s \in S(A)$. It follows that $b_s \mapsto c_s$ defines an $L^\text{bp}_\text{bp}(\mathcal{A})$-embedding of $\mathcal{A}_1$ into $\mathcal{M}$.

We now iterate the preceding procedure $\omega$ times and obtain increasing unions $(\mathcal{A}_n)_{n \geq 1}$ and $(B_n)_{n \geq 1}$. Set $A' := \bigcup_{n \geq 1} A_n$ and $B' := \bigcup_{n \geq 1} B_n$.

By construction, every lattice $s \in S_n(A')$ has a basis in $\text{VF}(A')$, similarly for $B'$. Thus $s/\text{ms}$ is $\text{VF}(A')$-definably isomorphic to $k^n$ for all $s \in S_n(A')$, and it follows that $dcl^B(A') = dcl^B(\text{VF}(A') \cup k(A'))$. Similarly, $dcl^B(B') = dcl^B(\text{VF}(B') \cup k(B'))$.

This shows in particular that $A'$ and $B'$ are generated by their restriction to the sorts in $L_k$. The $(L_k)_{\text{bp}}$-embedding given by Theorem 7.1.5 between (the $(L_k)_{\text{bp}}$-reducts of) $B'$ and $\mathcal{M}$ over $A'$ induces thus an $L^\text{bp}_\text{bp}$-embedding.

7.3.2. **Remark.** It seems plausible that an analogue of Theorem 7.3.1 holds for RCVF in the geometric language using prime resolutions as introduced in [15, Section 4].

7.4. **Strict pro-definability of spaces of definable types.** In this section, we apply our results to show that various spaces of definable types in ACVF, RCVF and $p$CF are strict pro-definable. Using the dual description of natural classes of $s\varepsilon$-pairs in terms of natural classes of definable types given in Section 2.4, each theory of beautiful pairs $T_{\text{bp}}(K)$ (where $T$ is one of the above theories) induces a corresponding class of definable types $\mathcal{F}$ for which $K = \mathcal{K}_{\mathcal{F}}$. By Theorem 2.4.8, the class $\mathcal{F}$ is strict pro-definable. We now describe the classes of types they correspond to. It is worth to point out that it was already shown in [13] that $S^{\text{def}}(\mathcal{U})$ is pro-definable for ACVF, RCVF and $p$CF.

7.4.1. **Remark.** In the following list of spaces of types in ACVF (resp. later in RCVF), when we say that a certain class of types $X$ is the model-theoretic analogue of a certain geometric space $Y$, we mean that they are related via the restriction functor in [24, Chapter 14]. In particular, when $K$ is a maximally complete model with value group $\mathbb{R}$ (and residue field $\mathbb{R}$ in the case of RCVF), we have that the restriction induces a canonical bijection between $X(K)$ and $Y(K)$.

7.4.2. **Spaces of definable types in ACVF:** For $T_k$ and $T_\Gamma$ as in Definition 7.1.1 let $\mathcal{F}$ be the corresponding natural class of definable types such that $\text{ACVF}(T_k, T_\Gamma)$ axiomatizes $\text{ACVF}_{\text{bp}}(\mathcal{K}_{\mathcal{F}})$. We now describe the 8 corresponding possible classes $\mathcal{F}$. Suppose first that $T_k$ corresponds to the theory of proper pairs of algebraically closed fields $\text{ACF}_{\text{bp}}$ and let $V$ be an algebraic variety over a small valued field $K$.

1. When $T_\Gamma$ is the theory of trivial pairs of DOAG, $\mathcal{F}$ corresponds to the class of stably dominated (equivalently generically stable) types. The set $\mathcal{F}_V(K)$ of $K$-definable types in $\mathcal{F}$ concentrating on $V$ is denoted $\widehat{V}(K)$, and can be viewed as a model theoretic analogue of the Berkovich analytification $V^\text{an}$ of $V$. It was already shown to be strict pro-definable in [24].

2. When $T_\Gamma$ corresponds to bounded pairs of DOAG, $\mathcal{F}$ corresponds to the class of bounded definable types in ACVF, meaning the types $p \in S^{\text{def}}(\mathcal{U})$ that have a realization in a model whose value group is bounded by $\Gamma(\mathcal{U})$. The set $\mathcal{F}_{V}(K)$ is denoted by $\widehat{\mathcal{V}}(K)$ and studied in a subsequent work. It can be seen as the model theoretic analogue of the adic space, in the sense of Huber [25], associated to $V$.

3. When $T_\Gamma$ is the theory of proper pairs of DOAG such that $P(\mathcal{M})$ is convex in $M$, $\mathcal{F}$ corresponds to those definable types that are orthogonal to $p_{0^+}$. We are unaware of a geometric interpretation of such type spaces.
(4) When $T_Γ$ is the theory that corresponds to all definable types in DOAG, $ℱ$ corresponds to the class of all definable types. The set $ℱ_V(K)$ can be seen as a model theoretic analogue of the Zariski-Riemann space associated to $V$.

Suppose now that $T_k$ corresponds to the theory of trivial elementary pairs of ACF.

(5) When $T_Γ$ is the theory of trivial pairs of DOAG, $ℱ$ corresponds to the class of realized types.

(6) When $T_Γ$ corresponds to the class of bounded types, the class $ℱ$ corresponds to those definable types which are orthogonal to the residue field sort and $p_{0^+}$. We are unaware of a geometric interpretation of such type spaces.

(7) When $T_Γ$ is the theory of proper pairs of DOAG such that $P(M)$ is convex in $M$, as for ACVF, we are unaware of a geometric interpretation of such spaces.

(8) When $T_Γ$ is the theory that corresponds to the class of all definable types in DOAG, $ℱ$ corresponds to the class of definable types orthogonal to the residue field sort. Again, we are unaware of a geometric interpretation of such type spaces.

7.4.3. Theorem. Let $X$ be an interpretable set in ACVF. Then, for $ℱ$ as in classes (1)-(5) above, $ℱ_X$ is strict pro-definable. When $X$ is a definable set in the 3-sorted language of ACVF, for $ℱ$ as in (6)-(8) above, $ℱ_X$ is strict pro-definable.

Proof. When $X$ is definable, the result follows for (1)-(8) by Theorems 7.1.5 and 2.4.8. When $X$ is only assumed to be interpretable, the result follows for (1)-(5) by Theorems 7.3.1 and 2.4.8. □

7.4.4. Remark. ACVF has surjectivity transfer in $L^G$ [24, Lemma 4.2.6]. This gives an alternative proof of the strict pro-definability of $S^\text{def}(U)$ when $X$ is interpretable. In the remaining cases, using the same method, one would have to show surjectivity transfer for the corresponding classes of types. Since these conditions have not been checked, invoking Theorem 7.3.1 is necessary when $X$ is interpretable.

7.4.5. Spaces of definable types in RCVF: For $T_k$ and $T_Γ$ as in Definition 7.1.1 let $ℱ$ be the corresponding natural class of definable types such that $RCVF(T_k, T_Γ)$ axiomatizes $RCVF_{bp}(K, ℵ)$. We now describe the 8 corresponding possible classes $ℱ$. Suppose first that $T_k$ corresponds to the theory of proper pairs of real closed fields $RCF_{bp}$ and let $V$ be a semi-algebraic set in $VF$.

(1) When $T_Γ$ is the theory of trivial pairs of DOAG, $ℱ$ corresponds to the class of definable types which are orthogonal to $Γ$. The set $ℱ_V(K)$, also denoted by $\tilde{V}(K)$, can be viewed as the model theoretic analogue of the Berkovich analytification of semi-algebraic sets as defined in [26].

(2) When $T_Γ$ corresponds to bounded pairs of DOAG, $ℱ$ corresponds to the class of bounded types in RCVF (see point (2) in Section 7.4.2). The set $ℱ_V(K)$ could be thought of as (a model theoretic analogue of) the adic space associated to $V$.

(3) When $T_Γ$ is the theory of proper pairs of DOAG such that $P(M)$ is convex in $M$, as for ACVF, we are unaware of a geometric interpretation of such spaces.

(4) When $T_Γ$ is the theory that corresponds to all definable types in DOAG, $RCVF(T_k, T_Γ)$ corresponds to the class of all definable types. The set $ℱ_V(K)$ is a model theoretic analogue of the Zariski-Riemann space above a given ordering on the real spectrum of $V$. 
7.4.6. Theorem. Let $X$ be an $L_{oef}$-definable set in $RCVF$. Then, for $\mathcal{F}$ as in (1)-(8) above, $\mathcal{F}_X$ is strict pro-definable.

7.4.7. Remark. As noted in [13], adapting the proof of [24, Lemma 4.2.6 (1)], $RCVF$ has surjectivity transfer. A careful look of the proof shows that $RCVF$ has density of definable types and moreover, density of definable bounded types. By Theorem 7.2.1 and Lemma 2.4.12 (and Remark 2.4.13), this shows that whenever $X$ is interpretable, both $S_{X}^\text{def}(U)$ and the set of bounded definable types concentrating on $X$ are strict pro-definable.

Similar methods give us the following:

7.4.8. Theorem. Let $X$ be definable set in $pCF$ in $L_{pcf}$, then $S_{X}^\text{def}$ is strict pro-definable.

In this case, $S_{V}^\text{def}$ is a model-theoretic analogue of the $p$-adic spectrum of a variety $V$ in the sense of [41] (see also [4]).

8. An Ax-Kochen-Ershov principle for beauty

8.1. Pairs of henselian valued fields in RV-enrichments. Let $\mathcal{L}$ be an $RV$-enrichment of $\mathcal{L}_{RV}$ in which we allow a set of auxiliary sorts from $RV^{eq}$ which always contains $\Gamma$. Recall that $RV$ denotes the $RV$-sort together with the auxiliary sorts. Let $\mathcal{L}^-$ be the restriction of $\mathcal{L}$ to the sorts in $RV$. For an $\mathcal{L}$-structure $A$, let $A|\mathcal{L}^-$ be its reduct to $RV$.

Let $T$ be the $\mathcal{L}$-theory of a benign valued field $(K,v)$. Let $K_{RV}$ be a natural class of $se$-pairs in the $\mathcal{L}^-$-theory of $K|\mathcal{L}^-$ (in sorts $RV$). Assume that $K_{RV}$-beautiful pairs exist and that $T_{bp}(K_{RV})$ has beauty transfer. Moreover, for ease of presentation, we will assume in addition that $K_{RV}$ has the extension property (recall that this implies that $K_{RV}$-beautiful pairs are elementary pairs by Theorem 2.3.5).

Let $K$ be the class of $L_{bp}$-structures $M \in K_{def}$ induced by $K_{RV}$ analogously as in Definition 7.1.1.

8.1.1. Lemma. $K$ is a natural class.

Proof. We argue as in the proof of Lemma 7.1.3. Properties (i)-(iv) are clear. As for Property (v) (closure under base extension), let us first assume that $RV$ is unenriched. In this case, the result follows exactly as in Remark 7.1.2. If $RV$ is enriched, let $M \in K$ with $P(M) \models T$ and $P(M) \preceq B$. For $M' := M_{B}$, since we work with an $RV$-enrichment, it still holds that $RV(M') = RV(M)_{RV(B)}$. Since $K_{RV}$ is natural, $M' \in K$.

8.1.2. Theorem. Let $K$ be the class of $L_{bp}$-structures $M \in K_{def}$ induced by $K_{RV}$. Then, $K$ has the extension property and $K$-beautiful pairs exist. Moreover, $T_{bp}(K)$ has beauty transfer and is axiomatised by the following conditions on $M = (M, P(M))$

- $VF(M)/VF(P(M))$ is vs-defectless;
- $(M, P(M))$ is an elementary pair of models of $T$;
- $(M|\mathcal{L}^-, P(M)|\mathcal{L}^-) \models T_{bp}(K_{RV})$.

Proof. The proof follows exactly the same strategy as that of Theorem 7.1.5, using Proposition 6.3.4 instead of Proposition 6.3.3.
8.2. Pairs of henselian valued fields with angular component. Let $T$ be the theory of a benign valued field $(K, v)$, where we work in a $\Gamma$-k-enrichment of $\mathcal{L}_{\text{Pas}}$. Let $\mathcal{K}_k$ and $\mathcal{K}_\Gamma$ be natural classes of $\infty$-pairs in the theory of the residue field $k$ and of the value group $\Gamma$ of $(K, v)$, respectively. As before, for ease of presentation, we will assume both classes $\mathcal{K}_k$ and $\mathcal{K}_\Gamma$ have the extension property and the amalgamation property, so that $\mathcal{K}_k$-beautiful pairs and $\mathcal{K}_\Gamma$-beautiful pairs exist. Assume moreover that $T_{\text{bp}}(\mathcal{K}_k)$ and $T_{\text{bp}}(\mathcal{K}_\Gamma)$ have beauty transfer. Let $\mathcal{K}$ be the class of $\infty$-pairs $\mathcal{M} \in \mathcal{K}_{\text{def}}$ induced by $\mathcal{K}_k$ and $\mathcal{K}_\Gamma$. Recall that adding an angular component map is bi-interpretable to adding a splitting of the short exact sequence $1 \to k^\times \to RV^\times \to \Gamma \to 0$, and thus we obtain a uniquely determined associated $RV$-enrichment, in which moreover $\Gamma_\infty$ and $k$ are orthogonal (as this holds in any $\Gamma$-k-enrichment of the $\mathcal{L}_{\text{Pas}}$-theory of $(K, v)$) and in which we have a definable product decomposition $RV = \Gamma_\infty \times k$. In particular, $\mathcal{K}_k$ and $\mathcal{K}_\Gamma$ induce a natural class $\mathcal{K}_{RV}$, which in turn induces (going back to the original enrichment of $\mathcal{L}_{\text{Pas}}$) the same class $\mathcal{K}$. As a corollary of Theorem 8.1.2 we thus obtain:

8.2.1. Corollary. Let $\mathcal{K}$ be the class of $\mathcal{L}_{\text{bp}}$-structures $\mathcal{M} \in \mathcal{K}_{\text{def}}$ induced by $\mathcal{K}_k$ and $\mathcal{K}_\Gamma$. Then, $\mathcal{K}$ has the extension property and $\mathcal{K}$-beautiful pairs exist. Moreover, $T_{\text{bp}}(\mathcal{K})$ has beauty transfer and is axiomatised by the following conditions on $\mathcal{M} = (M, P(\mathcal{M}))$:

- $\mathcal{VF}(\mathcal{M})/\mathcal{VF}(P(\mathcal{M}))$ is vs-defectless;
- $(M, P(\mathcal{M}))$ is an elementary pair of models of $T$;
- $(\kappa(M), \kappa(P(\mathcal{M}))) \models T_{\text{bp}}(\mathcal{K}_k)$ and $(\Gamma(M), \Gamma(P(\mathcal{M}))) \models T_{\text{bp}}(\mathcal{K}_\Gamma)$.

8.3. Ax-Kochen-Ershov principle. Let $T$ be the theory of the benign valued field $(K, v)$. Consider the (definitional) $\mathcal{L}_{\text{RV}}$-expansion where we add sorts $\mathcal{A}$ for $k^*/(k^*)^n$ for all $n \geq 0$ and $\Gamma$ (with the natural quotient maps). We are in the setting of Section 8.1.

Let $\mathcal{K}_\mathcal{A}$ and $\mathcal{K}_\Gamma$ be natural classes of $\infty$-pairs in the theory of the residue field $k$ (in the sorts $\mathcal{A}$) and of the value group $\Gamma$ of $(K, v)$, respectively. Assume both $\mathcal{K}_\mathcal{A}$ and $\mathcal{K}_\Gamma$ have extension and amalgamation. By 2.3.5, amalgamation is equivalent to the fact that $\mathcal{K}_\mathcal{A}$-beautiful pairs and $\mathcal{K}_\Gamma$-beautiful pairs exist. Assume in addition that $T_{\text{bp}}(\mathcal{K}_\mathcal{A})$ and $T_{\text{bp}}(\mathcal{K}_\Gamma)$ have beauty transfer.

8.3.1. Theorem. Let $\mathcal{K}$ be the class of $\mathcal{L}_{\text{bp}}$-structures $\mathcal{M} \in \mathcal{K}_{\text{def}}$ induced by $\mathcal{K}_\mathcal{A}$ and $\mathcal{K}_\Gamma$. Then, $\mathcal{K}$-beautiful pairs exist. Moreover, $T_{\text{bp}}(\mathcal{K})$ has beauty transfer and is axiomatised by the following conditions on an $\mathcal{L}_{\text{bp}}$-structure $\mathcal{M} = (M, P(\mathcal{M}))$:

- $\mathcal{VF}(\mathcal{M})/\mathcal{VF}(P(\mathcal{M}))$ is vs-defectless;
- $P(\mathcal{M}) \preceq M \models T$;
- $(\mathcal{A}(M), \mathcal{A}(P(\mathcal{M}))) \models T_{\text{bp}}(\mathcal{K}_\mathcal{A})$ and $(\Gamma(M), \Gamma(P(\mathcal{M}))) \models T_{\text{bp}}(\mathcal{K}_\Gamma)$.

Proof. This follows directly from Theorems 8.1.2 and 5.2.5. □

8.3.2. Examples.

1. All completions of stably embedded elementary pairs of models of $\text{Th}(\mathbb{R}((t)))$ or $\text{Th}(\mathbb{C}((t)))$ correspond to theories of beautiful pairs for appropriate choices of natural classes in the residue field and value group. Note that the same result holds for stably embedded elementary pairs of $\text{Th}(\mathbb{Q}_p((t)))$ with respect to the $t$-adic valuation. Moreover, since $\mathbb{Q}_p((t))$ with the $t$-adic valuation is bi-interpretable with $\mathbb{Q}_p((t))$ with the total valuation, we get the mixed-characteristic result for $\mathbb{Q}_p((t))$ using the above machinery.

2. Note that both Theorem 7.1.5 and Theorem 7.2.1 can be recovered from Theorem 8.3.1. In the latter case, the only non-trivial quotients $k^*/(k^*)^n$ in $\mathcal{A}$ appear for even $n$ when they are of size 2, allowing to define the ordering on the valued field without $\mathcal{VF}$-quantifiers.
(3) The results can also be applied to any algebraically maximal Kaplansky field in positive characteristic. A particular example of interest is $K = \mathbb{F}_p^\text{alg}(\Gamma)$, with $\Gamma = 1/p^\infty\mathbb{Z}$ the subgroup of $\mathbb{Q}$ consisting of the elements that can be expressed as $m/p^n$. The beautiful pairs of models of $\text{Th}(K)$ are axiomatized by the corresponding beautiful pairs of the residue fields and value groups. The residue field pairs are characterized by Proposition 3.1.5. For the value group, due to the fact that stably embedded elementary pairs are not elementary (see Proposition 4.3.5), the situation is slightly more complicated since we need to restrict to convex elementary pairs in the sense of Theorem 4.3.6, where the beautiful pairs still satisfy beauty transfer. Together with the beautiful pairs of $\text{ACF}_p$ these give certain theories of beautiful pairs of $\text{Th}(K)$ satisfying beauty transfer by Theorem 8.3.1.

9. Concluding remarks and open questions

9.1. Variants in mixed characteristic. We sketch in this section an analogue Ax-Kochen-Ershov type result in mixed characteristic. This applies in particular to the field of Witt vectors $W(\mathbb{F}_p^\text{alg})$. The strategy being completely analogous to the one given for benign valued fields, we will only point to the main changes without giving any proof.

We work in a slight variant of the language $\mathcal{L}_{RV}$, with sorts $\mathcal{RV}_n$ for $n \geq 0$ as defined in [17]. Abusing notation, we let $\mathcal{RV}$ denote the union of all sorts $\mathcal{RV}_n$ (including all new sorts when working in an $\mathcal{RV}$-enrichment) and $\mathcal{L}_{RV}$ the corresponding language. See [22, Section 7] for more details about this context.

Let $\mathcal{L}$ be an $\mathcal{RV}$-enrichment of $\mathcal{L}_{RV}$ and $T$ be the $\mathcal{L}$-theory of a mixed characteristic finitely ramified henselian valued field $K$ with perfect residue field. Recall $T$ eliminates $\mathbf{VF}$-quantifiers by [17]. Suppose $\mathcal{L}$ contains a sort for $\Gamma$. Let $\mathcal{K}_{RV}$ be a natural class of $\mathcal{SE}$-pairs in the $\mathcal{L}^-$-theory of $K|\mathcal{L}^-$ ($\mathcal{L}^-$ as in Section 8). Assume that $\mathcal{K}_{RV}$-beautiful pairs exist, $\mathcal{K}_{RV}$ has the extension property and $T_{bp}(\mathcal{K}_{RV})$ has beauty transfer. Let $\mathcal{K}$ be the class of $\mathcal{L}_{bp}$-structures $\mathcal{M} \in \mathcal{K}_{def}$ induced by $\mathcal{K}_{RV}$. As in Lemma 8.1.1, $\mathcal{K}$ is a natural class.

9.1.1. Theorem. The analogue statements of Proposition 6.3.4 and Theorem 8.1.2 hold for $T$.

Proof. This follows from an easy adaptation of the corresponding proofs. \hfill \Box

The reduction from $\mathcal{L}_{RV}$ to $\mathcal{L}_{\Gamma k}$ is a little more subtle as it needs a variant of the results in Section 5 for short exact sequences. Indeed, one needs to consider more generally inverse systems as explained in [22, Remark 7.6]. For $T = \text{Th}(W(\mathbb{F}_p^\text{alg}))$ the situation is somewhat easier, as the sequence of kernels $W_n[\mathbb{F}_p^\text{alg}]^\times$ for $n \geq 0$ of the maps $\mathcal{RV}_n \rightarrow \Gamma$ is bi-interpretable with the residue field, and hence no quotients are needed since the residue field eliminates imaginaries. In this special case, one obtains the following result.

9.1.2. Corollary. Let $T_k = T_{bp}(\mathcal{K}_k)$ and $T_\Gamma = T_{bp}(\mathcal{K}_\Gamma)$ be completions of $(\text{ACF}_p)_{\mathcal{SE}}$ and of $\text{PRES}_{\mathcal{SE}}$, respectively, where $\mathcal{K}_k$ and $\mathcal{K}_\Gamma$ are the corresponding natural classes of $\mathcal{SE}$-pairs. Let $\mathcal{K}$ be the class of $\mathcal{L}_{bp}$-structures $\mathcal{M} \in \mathcal{K}_{def}$ induced by $\mathcal{K}_k$ and $\mathcal{K}_\Gamma$. Then, $\mathcal{K}$ is a natural class satisfying the extension property and $\mathcal{K}$-beautiful pairs exist. Moreover, $T_{bp}(\mathcal{K})$ has beauty transfer and is axiomatized by the following conditions on $\mathcal{M} = (M, P(\mathcal{M}))$:

- $\text{VF}(M)/\text{VF}(P(\mathcal{M}))$ is $\text{vs}$-defectless;
- $(M, P(\mathcal{M}))$ is an elementary pair of models of $T$;
- $(k(M), k(P(\mathcal{M}))) \models T_k$ and $(\Gamma(M), \Gamma(P(\mathcal{M}))) \models T_\Gamma$.
In particular, there are exactly 4 completions of the theory of stably embedded elementary pairs of models of \( T = \text{Th}(W(\mathbb{F}_p^\text{alg})) \), all corresponding to theories of beautiful pairs satisfying beauty transfer.

9.2. Valued differential fields. We now briefly discuss a similar context of valued differential fields in which an Ax-Kochen-Ershov principle for beauty applies. Let VDF be the theory of existentially closed valued differential fields \((K, v, \partial)\) of residue characteristic 0 satisfying \( v(\partial(x)) \geq v(x) \) for all \( x \). This theory had first been studied by Scanlon [42], and it was further investigated by Rideau-Kikuchi [36]. The theory VDF is NIP. In VDF the value group is stably embedded and a pure model of DOAG, and the residue field is stably embedded and a pure model of DCF\(_0\), with the derivation induced by \( \partial \). As DCF\(_0\) is stable with nfcp, \((\text{DCF}_0)_{\text{bp}}\) exists and has beauty transfer by Theorem 3.1.1 and Proposition 3.1.3.

9.2.1. Remark. Using the methods from [22, Section 8] and following the proof strategy for Theorem 7.1.5 one may show an analogue of Theorem 7.1.5 for VDF. Indeed, let \( \mathcal{K}_k \) and \( \mathcal{K}_\Gamma \) be natural classes of SE-pairs in DCF\(_0\) and DOAG, respectively. For ease of presentation, we will assume both classes \( \mathcal{K}_k \) and \( \mathcal{K}_\Gamma \) have the extension property and the amalgamation property. Assume moreover that \( T_{\text{bp}}(\mathcal{K}_k) \) and \( T_{\text{bp}}(\mathcal{K}_\Gamma) \) have beauty transfer. Let \( \mathcal{K} \) be the class of SE-pairs \( M \in \mathcal{K}_{\text{def}} \) induced by \( \mathcal{K}_k \) and \( \mathcal{K}_\Gamma \). Then, in the 3-sorted language with sorts VF, RV and \( \Gamma \), beautiful pairs exist. Moreover, VDF\(_{\text{bp}}(\mathcal{K})\) has beauty transfer and is axiomatized by the following conditions on \( M = (M, P(M)) \):

- VF\((M)/VF(P(M))\) is vs-defectless;
- \((M, P(M))\) is an elementary pair of models of VDF;
- \((k(M), k(P(M)))\) \models (DCF\(_0\))\(_{\text{bp}}(\mathcal{K}_k)\) and \((\Gamma(M), \Gamma(P(M)))\) \models \text{DOAG}\(_{\text{bp}}(\mathcal{K}_\Gamma)\).

9.2.2. Remark. Let \( X \) be a definable set in a model of VDF, in sorts VF, RV and \( \Gamma \). Then the class of definable types concentrating on \( X \) in the classes appearing in Remark 9.2.1 are strict pro-definable. In particular, this holds for the space \( \hat{X} \) of all stably dominated types concentrating on \( X \), and for the space \( S_X^{\text{def}} \) of all definable types.

Note that, for \( \hat{X} \), strict pro-definability has already been shown in work by the second author, Kamensky and Rideau-Kikuchi [21].

9.3. Some open questions. We gather in this section some open questions.

9.3.1. Question. Is there a theory \( T \) (with uniform definability of types) such that \( \mathcal{K}_{\text{def}} \) does not have the amalgamation property?

9.3.2. Remark. Since the first version of this manuscript has been made public, R. Mennuni and the second author have found a theory \( T \) in which definable types do not amalgamate ([23]). Although, in this example the definable types are not uniformly definable, so this only provides a partial answer to Question 9.3.1.

When \( T \) is stable, by Theorem 3.1.1, both strict pro-definability and beauty transfer are equivalent to nfcp. This suggests the following question.

9.3.3. Question. Is there a theory \( T \) such that \( T_{\text{bp}} \) is consistent, \( S^{\text{def}}(U) \) (resp. a natural subclass \( \mathcal{F} \) of definable types) is strict pro-definable, but \( T_{\text{bp}} \) (resp. \( T_{\text{bp}}(\mathcal{K}_{\mathcal{F}}) \)) does not have beauty transfer?

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5This theory is sometimes denoted by VDF\(_{\text{cc}}\).
Corollary 2.3.13 shows that, assuming $T_{bp}(K)$ has beauty transfer, NIP is preserved from $T$ to $T_{bp}(K)$. It seems plausible that, using the methods from [7], the same holds NTP$_2$. We leave it as a question.

9.3.4. **Question.** Suppose $T_{bp}(K)$ has beauty transfer. Does it hold that if $T$ has NTP$_2$ so does $T_{bp}(K)$? If not, does this hold if $K$ has the extension property?

Recall that Theorem 4.1.1 shows that DOAG$_{SE}$ has only four completions. Moreover, RCF$_{SE}$ has only two completions by Remark 3.2.9. This suggests the following.

9.3.5. **Question.** Suppose $T_{bp}(K)$ has beauty transfer. Is every completion of $T_{bp}(K)$ one of the following: $T_{bp}(K_{triv})$, $T_{bp}(K_{bdd})$, $T_{bp}(K_{\infty})$ or $T_{bp}$?

Let $p$ be a prime and $e \in \mathbb{N}$. Let SCVF$_{p,e}$ be the theory of separably closed non-trivially valued fields of characteristic $p$ and imperfection degree $e$. In [21, Corollary 7.7] it is shown that SCVF$_{p,e}$ is metastable and that for any interpretable set $X$, the space $\hat{X}$ of stably dominated types concentrating on $X$ is strict pro-definable. We expect that the following question does have a positive answer, which would yield a generalization of this result.

9.3.6. **Question.** Does $(SCVF_{p,e})_{bp}$ exist and have beauty transfer? Similarly, denoting by $K$ the natural class of se-pairs corresponding to the class of stably dominated types, does $(SCVF_{p,e})_{bp}(K)$ exist and have beauty transfer?

The proof of Theorem 7.3.1 relies on generic resolutions, and thus the pair of residue fields needs to be non-trivial. This leads to the following question.

9.3.7. **Question.** Does the analogous statement of Theorem 7.3.1 also hold in the cases when the pair of residue fields is assumed to be trivial?

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Departamento de Matemáticas, Universidad de los Andes, Carrera 1 # 18A - 12, 111711, Bogotá, Colombia

Email address: p.cubideskovacsics@uniandes.edu.co

Institut für Mathematische Logik und Grundlagenforschung, Westfälische Wilhelms-Universität Münster, Einsteinstr. 62, D-48149 Münster, Germany

Email address: hils@uni-muenster.de

Mathematical Institute, University of Oxford, Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Road, Oxford, United Kingdom, OX2 6GG

Email address: jinhe.ye@maths.ox.ac.uk