COOLING FLOWS OF SELF-GRAVITATING, ROTATING, VISCOUS SYSTEMS

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Received 2001 July 31; accepted 2002 March 26

ABSTRACT

We obtain self-similar solutions that describe the dynamics of a self-gravitating, rotating, viscous systems. We use simplifying assumptions but explicitly include viscosity and the cooling due to the dissipation of energy. By assuming that the turbulent dissipation of energy is a power law of the density and the speed of sound and for a power-law dependence of viscosity on the density, pressure, and rotational velocity, we investigate turbulent cooling flows. It has been shown that for cylindrical and spherical cooling flows the similarity indices are the same, and they depend only on the exponents of the dissipation rate and the viscosity model. Depending on the values of the exponents, which the mechanisms of the dissipation and viscosity determine, we may have solutions with different general physical properties. The conservation of the total mass and the angular momentum of the system strongly depend on the mechanism of energy dissipation and the viscosity model.

Subject headings: hydrodynamics — instabilities — stars: formation — turbulence

1. INTRODUCTION

Detailed observations of the interstellar medium (ISM) or even the intergalactic medium have highlighted the need to provide a description that accounts for the turbulent pressure, thermal pressure, magnetic fields, rotation, and even stars themselves. A universal theory describing the complex structures of the ISM is far from complete and remains a challenge for the future. However, by dividing the ISM according to its properties, it is possible to present satisfactory theories for particular types of ISM. Gravitation, cooling, turbulence, and magnetic fields produce variations in the properties of the ISM such as the density and the temperature, which in turn define the structure of the ISM. Such structures include dense cores (e.g., Lee, Myers, & Tafalla 1999), filamentary clouds (e.g., Schneider & Elmegreen 1979; Harjunpaa et al. 1999), and even disks (e.g., Padgett et al. 1999). Impressive theoretical progress has been made on the properties and evolution of these structures during recent years under simplifying assumptions (e.g., Li 1998; Fiege & Pudritz 2000; Tsuribe 1999). Most of the theoretical models of interstellar clouds and clumps assume static or stationary configurations, in which there is an equilibrium between the self-gravity, centrifugal force, and some forms of internal energy in the cloud (e.g., Bertoldi & McKee 1992; Chieze 1987; Vázquez-Semadeni & Gazol 1995; Galli et al. 2001; Tomisaka, Ikeuchi, & Nakamura 1988; Shadmehri & Ghanbari 2001a).

Shu, Adams, & Lizano (1987) proposed a four stage scenario for the formation of an isolated low-mass star: (1) quasi-static formation and evolution of a molecular cloud core by ambipolar diffusion, (2) dynamic collapse of the core to a protostar and circumstellar disk, (3) breakout of a powerful bipolar outflow, and (4) clearing of the circumstellar envelope to reveal a pre–main-sequence star. Such theories of isolated star formation assert that gravitational collapse occurs onto a thermally supported core and motions are quasi-hydrostatic until very late times (e.g., Shu et al. 1987; Li 1998). Nevertheless, recent work has led to doubts as to whether their initial condition is a reasonable starting point. Rather than being quasi-static, the hierarchi-
& Phillips 1990). Recent numerical studies of turbulence have brought a new understanding of the physics of this complex phenomenon. For example, recently Padoan et al. (2001) suggested that because of the turbulent nature of supersonic motions in molecular clouds, the dense structures, such as filaments and clumps, are formed by shocks in a turbulent flow.

In this paper, we extend the work of SG to study turbulent cooling flows in self-gravitating, rotating, viscous clouds. We adopt an analytic approach and do not perform extensive numerical computations. Viscosity has an important role in the behavior of the cloud. Of course, molecular viscosity is totally negligible in the ISM compared to turbulent viscosity. This presents a problem as no comprehensive treatment of turbulent viscosity exists. For simplicity, we shall use the well-known $\alpha$-prescription (Shakura & Sunyaev 1973), which has been used for modeling accretion disks, and also a more general simplified model for viscosity. This study is a first step (at least, qualitatively) toward understanding the role of turbulent cooling flows in self-gravitating clouds. Clearly, turbulence is a complex phenomenon in the ISM and the results of this investigation are just approximations to the behavior of real turbulent cooling flows. In general, our treatment will be incomplete because we omit turbulent eddies with scales larger than the size of the cloud. Thus, we consider microturbulence, and this can only provide a first approximation to the real dynamics of ISM clouds. Nevertheless, our simple model provides useful information on the importance of turbulent cooling flows in star-forming regions.

2. GENERAL FORMULATION

We shall seek similarity solutions of the set of equations describing turbulent cooling flows in self-gravitating clouds. The simplest method of dealing with turbulence involves the assumption that the Navier-Stokes equation holds on each physical scale, with turbulence manifesting itself via renormalized viscosity and heat conduction coefficients that are generally not constant. We assume spherical (or cylindrical) symmetry and neglect heat conduction. For constructing the model, the continuity equation, the momentum equation in the radial direction, the equation of angular momentum conservation, the Poisson equation, and the energy equation can be rewritten explicitly in the following way:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho v_r) = 0,$$  (1)

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{1}{r} \frac{\partial p}{\partial r} + \frac{\partial}{\partial r} \left( \nu \frac{\partial v_r}{\partial r} \right) = \frac{v_r^2}{r},$$  (2)

$$\frac{\partial (rv_r)}{\partial t} + v_r \frac{\partial (rv_r)}{\partial r} = \frac{1}{r^{\lambda-1}} \frac{\partial}{\partial r} \left( \nu r^{\lambda+1} \frac{\partial \Psi}{\partial r} \right),$$  (3)

$$\frac{1}{r^{\lambda-1}} \frac{\partial}{\partial r} \left( r^{\lambda-1} \frac{\partial \Psi}{\partial r} \right) = 4\pi G \rho,$$  (4)

$$\frac{1}{\gamma - 1} \left( \frac{\partial p}{\partial t} + v_r \frac{\partial p}{\partial r} \right) + \frac{\gamma - 1}{\gamma - 1} \frac{\partial}{\partial r} \left( \rho v_r \right) + \Lambda(\rho, T) = 0,$$  (5)

where $\rho, v_r, v_r, p,$ and $\Psi$ denote the gas density, radial velocity, rotational velocity, pressure, and gravitational potential, respectively. Also, $\Omega = v_r/r$ is the angular velocity, and $\lambda$ determines the dimensionality of the model, with $\lambda$ being 2 in cylindrical geometry and 3 in spherical geometry.

To close this system of equations, the form of viscosity ($\nu$) and the viscosity cooling function ($\Lambda$) must be known. As was mentioned above, our understanding of turbulent viscosity is incomplete, and for this reason we adopt an empirical prescription. For example, the $\alpha$-prescription assumes that the molecular viscosity can be replaced by an isotropic turbulent viscosity, and so

$$\nu = \alpha \frac{p}{\rho^{3/2}},$$  (6)

where $\alpha$ is a constant (Shakura & Sunyaev 1973). It has been shown that $\alpha$ is not, in general, constant, but rather it depends on a number of factors (see, e.g., Brandenburg 1998). However, for simplicity, we will assume that the parameter $\alpha$ is constant. Also, we consider a more general viscosity model than the $\alpha$-model, viz.,

$$\nu = \alpha \sigma T^{\beta} \rho^{\eta},$$  (7)

where $\tau, \beta,$ and $\eta$ are three arbitrary exponents. A viscosity model similar to this form has been used by Begelman & Meier (1982). By imposing additional requirements like conservation of the total mass or angular momentum of the system, we shall find relations between these three exponents. It is clear that the $\alpha$-model corresponds to

$$\tau = 1, \quad \beta = -1, \quad \eta = -1,$$  (8)

and is one member of this family. Note that the constant $\alpha$ in equation (7) is, in general, different from the $\alpha$ in equation (6); they are the same only when the exponents have values given by equation (8). We use $\alpha$ as a free parameter to study the effect of viscosity.

For dissipation, we could use the Mac Low (1999) result locally, which would give a dissipation time that would be always comparable to the local crossing time. In reality, the dissipation rate is not just a function of temperature. The $T$ varies all over because of shocks, and most of the dissipation is in shocks, not in a uniform thermal medium. So, if we have some characteristic scale, like the radius around a filament or the distance from the center of a sheet, we would take $\nu_{eq}/L$ as the dissipation rate for speed $v_{rms}$. Then $L = r$ is the distance from the axis of the filament, for example. Also, some authors have found power-law relations between $v_{rms}$ and $L$ by doing numerical simulations (e.g., Avila-Reese & Vázquez-Semadeni 2001). We note that by knowing the $v_{rms}$, it is possible to define the temperature $T$. Neglecting any viscous and external heating, however, it is possible to fit the cooling function (dissipation rate) by a power law,

$$\Lambda(\rho, T) = A \rho^2 T^\epsilon,$$  (9)

where $A$ and $\epsilon$ are constants. This general form enables us to study the behavior of the system by changing the values of $\epsilon$ and also $A$. In fact, the mechanism of turbulent dissipation determines the values of these parameters.

In certain optically thin systems, we can also use equation (9) to approximate the radiative cooling function (e.g., Spitzer 1978). Radiative cooling almost always involves a two-body process and, as such, also depends on the square of the density. The exponent of temperature $\epsilon$ depends on the
regime under consideration. In molecular clouds, cooling is dominated by dust or CO line emission, and the range of $\epsilon$ is from 1.5 to 3 (Goldsmith & Langer 1978). Also, for hot plasmas of cosmic composition, we have $\epsilon \ll 1$ for $T > 10^7$ K, when free-free emission dominates, and $\epsilon \approx -\frac{1}{2}$ for $10^5$ K $< T < 10^7$ K, when line cooling dominates (Gaetz & Salpeter 1983). This approach is useful for qualitative investigation of the effects of radiative cooling, but we must be aware of its limitations.

We are interested in quasi-hydrostatic flows in the clouds. For quasi-hydrostatic flows, equation (2) becomes

$$\frac{1}{\rho} \frac{\partial \rho}{\partial t} + \frac{\partial \Psi}{\partial r} = \frac{v_r^2}{r} .$$

(10)

To simplify the equations, we make the following substitutions:

$$\rho \rightarrow \check{\rho}, \quad p \rightarrow \check{p}, \quad \Psi \rightarrow \check{\Psi}, \quad T \rightarrow \check{T} , \quad v_r \rightarrow \check{v}_r, \quad r \rightarrow \check{r}, \quad t \rightarrow \check{t} ,$$

(11)

where

$$\check{\rho} = \rho_0, \quad \check{p} = p_0, \quad \check{T} = T_0 , \quad \check{\Psi} = \frac{\rho_0}{\rho_0} , \quad \check{r} = \left( \frac{p_0}{\pi G \rho_0^2} \right)^{1/2} ,$$

$$\check{t} = \frac{p_0}{(\gamma - 1) A_r \rho_0^2 T_0} , \quad \check{v}_r = \check{r}, \quad m = \frac{\check{v}_r^2 \rho_0}{\rho_0} .$$

(12)

Under these transformations, equations (1) and (10) do not change, but the other equations become

$$\frac{\partial (rv_r)}{\partial \check{t}} + v_r \frac{\partial (rv_r)}{\partial \check{r}} = \frac{\alpha}{m \lambda - 1} \frac{\partial}{\partial \check{r}} \left( \check{p} \frac{\partial \check{\Omega}}{\partial \check{r}} \right) ,$$

$$\frac{1}{r^{\lambda - 1}} \frac{\partial}{\partial \check{r}} \left( r^{\lambda - 1} \frac{\partial \Psi}{\partial \check{r}} \right) = \rho ,$$

$$\frac{\partial \check{p}}{\partial \check{t}} + v_r \frac{\partial \check{p}}{\partial \check{r}} + \gamma \frac{p}{r^{\lambda - 1}} \frac{\partial}{\partial \check{r}} (r^{\lambda - 1} v_r) + \check{r}^{\lambda - 1} p^\prime = 0 .$$

(15)

Notice the equations are not invariant under the transformation $t \rightarrow -t, \quad v_r \rightarrow -v_r,$ and $v_r \rightarrow -v_r,$ so the solutions cannot be time-reversible. Thus, these equations describe a nonlinear evolving system.

3. SIMILARITY SOLUTIONS

3.1. Analysis

In following the nonlinear evolution of dynamically evolving systems, the technique of self-similar analysis is useful, as it allows a set of partial differential equations, such as those above, to be transformed into a set of ordinary differential equations. A similarity solution, although constituting only a limited part of the problem, is often useful in understanding the basic behavior of the system. In order to seek similarity solutions to the above equations, we introduce a similarity variable $\xi$ as

$$\xi = \frac{r}{(t_0 - t) \beta} ,$$

(16)

and assume that each physical quantity is given by the following form:

$$\rho(r, t) = (t_0 - t)^n \check{R} (\xi) ,$$

$$p(r, t) = (t_0 - t)^n \check{P} (\xi) ,$$

$$v_r(r, t) = (t_0 - t)^n \check{V} (\xi) ,$$

$$v_r(r, t) = (t_0 - t)^n \check{\Phi} (\xi) ,$$

$$\Psi(r, t) = (t_0 - t)^n \check{S} (\xi) ,$$

(17)

where the exponents $n, \check{v}_1, \check{v}_2, \check{v}_3, \check{v}_4$, and $\check{v}_5$ are constants that must be determined and $t < t_0$. By substituting the above equations into equations (1), (10), (13), (14), and (15), we obtain the general results:

$$\check{v}_1 = \frac{2n(1 - \epsilon) - 1}{\epsilon} , \quad \check{v}_2 = \frac{2n(2 - \epsilon) - 2}{\epsilon} ,$$

$$\check{v}_3 = h - 1 , \quad \check{v}_4 = \frac{2n - 1}{2\epsilon} , \quad \check{v}_5 = \frac{2n - 1}{\epsilon} .$$

(22)

and

$$n = \frac{\epsilon - [\beta + 2\tau + (\eta/2)]}{(\eta + 2\beta + 2\tau + 2\epsilon - (\eta + 2\beta + 4\tau)} .$$

(23)

In the specific case of the $\alpha$-model, we have $\tau = 1, \beta = \eta = -1$, which gives

$$n = \frac{2\epsilon - 1}{2(\epsilon - 1)} , \quad \check{v}_1 = -2 , \quad \check{v}_2 = \frac{3 - 2\epsilon}{\epsilon - 1} ,$$

$$\check{v}_3 = \frac{1}{2(\epsilon - 1)} , \quad \check{v}_4 = \frac{1}{2(\epsilon - 1)} , \quad \check{v}_5 = \frac{1}{\epsilon - 1} .$$

(24)

It is interesting that these similarity indices are independent of the dimensionality of the system, i.e., $\lambda$, but depend strongly on the exponents of the viscosity and the turbulent cooling function. Thus, in both cylindrical and spherical systems just the mechanisms of viscosity and dissipation of energy are important.

We can write the equations for the dependence of the physical quantities on the similarity variable as

$$-\check{v}_1 \check{R} + \check{n} \check{G} \frac{d \check{R}}{d \xi} + \frac{1}{\check{\xi}^{\lambda - 1}} \frac{1}{d \xi} (\check{\xi}^{\lambda - 1} \check{R} V) = 0 ,$$

$$\frac{1}{\check{R}} \frac{d \check{P}}{d \xi} + \frac{d \check{S}}{d \xi} = \frac{\check{\Phi}^2}{\check{\xi}} ,$$

$$-\check{v}_4 \check{\Phi} + \check{n} \check{G} \frac{d \check{\Phi}}{d \xi} + V \frac{d \check{\Phi}}{d \xi} (\check{\xi} \Phi) \right) ,$$

$$= \frac{1}{m \check{\xi}^{\lambda - 1}} \frac{d}{d \xi} \left[ \frac{1}{\check{\xi}^{\lambda - 1}} \check{R}^{\lambda - 1} \check{R} \check{\Phi} d \left( \frac{\check{\Phi}}{\check{\xi}} \right) \right] ,$$

$$-\check{v}_2 \check{P} + \check{n} \check{G} \frac{d \check{P}}{d \xi} + V \frac{d \check{P}}{d \xi} (\check{\xi}^{\lambda - 1} \check{V}) + R^{\lambda - 1} \check{P}^\prime = 0 ,$$

$$= \frac{1}{\check{\xi}^{\lambda - 1}} \frac{d}{d \xi} \left( \check{\xi}^{\lambda - 1} \check{S} \right) = \check{R} .$$

(25)

(26)

(27)

(28)

(29)

This is a system of nonlinear ordinary differential equations that has some critical points. Since the similarity indices are known, it is possible to study the general physical properties of the solutions. We see that the solution for each
physical quantity retains a similar form as the flow evolves, but the characteristic length scale of the flow increases proportionally to \((t_0 - t)^\alpha\), where equation (23) gives the value of \(n\). As we require that \(n\) be finite and nonzero, this relation shows that \(\epsilon\) may have any value except \(\epsilon = (\eta + 2\beta + 4\tau)/(\eta + 2\beta + 2\tau + 2)\) or \(\epsilon = \beta + 2\tau + (\eta/2)\). The sign of \(n\) determines whether the cloud expands or collapses; if \(n\) is positive, the radius of the cloud \(r_c\) decreases with time and the central density \(\rho_c\) increases with time. In the \(\alpha\)-model, \(n\) depends only on \(\epsilon\) and is negative for \(\epsilon\) between \(\frac{1}{3}\) and 1 and positive otherwise. Also, in this case, the density \(\rho\) at the center of the cloud increases in proportion to \((t_0 - t)^{-2}\), irrespective of the value of \(\epsilon\).

We now consider boundary conditions, such as conservation of the total mass, angular momentum, and constant external pressure. Conservation of mass is an important physical constraint and, as we shall show, fortuitously simplifies the equations and allows us to obtain some analytical results. The integrals representing the total mass \(M\) and angular momentum \(J\) are the following:

\[
M = \int \rho r^{k-1} dr, \\
J = \int \rho y r^{k+1} dr.
\]

Thus, the total mass \(M\) (nondimensional) is proportional to \((t_0 - t)^{k\lambda + \lambda\eta}\), and the total angular momentum \(J\) (nondimensional) is proportion to \((t_0 - t)^{k\lambda + \lambda\eta + (\lambda + 1)\eta}\). As these relations show, the conservation of \(M\) or \(J\) depends on the dimensionality of the system and the exponents of viscosity and cooling rate. First, we study the properties of the solutions for the \(\alpha\)-model. There is no value of \(\epsilon\) for which the mass of a cylindrical cloud can be conserved, and for spherical clouds the mass of the cloud is conserved only for \(\epsilon = -\frac{1}{3}\). The total angular momentum is conserved with \(\epsilon = (\lambda - 4)/(2(\lambda - 1))\). However, when the total mass of a spherical cloud is conserved \((\epsilon = -\frac{1}{3})\), the total angular momentum decreases: \(J \propto (t_0 - t)^{-1/3}\). Another interesting case is a cylindrical cloud with the general form of viscosity, in which the constraint of constant total mass gives

\[
\eta + 2\beta + 2\tau = 0.
\]

Note that this constraint involves only the indices of the viscosity model and not the index of the dissipation model, \(\epsilon\). Thus, this constraint is independent of the mechanism of dissipation. Under this constraint, equation (22) gives the similarity indices:

\[
\eta = -\frac{1}{3}, \quad \nu_1 = 1, \quad \nu_2 = 1, \\
\nu_3 = -\frac{1}{2}, \quad \nu_4 = 0, \quad \nu_5 = 0.
\]

Also, in this case, the total angular momentum decreases: \(J \propto (t_0 - t)^{-\lambda/2}\).

Constant external pressure is another interesting case in which the cloud is surrounded by an ambient gas: \(p(r \to \infty) = \text{const}\). Of course, the total mass need not be conserved, as inflow or outflow of the gas is allowed. In the \(\alpha\)-model this corresponds to \(\epsilon = 3/2\), regardless of dimensionality, and the total mass and the angular momentum of the system decreases: \(M \propto (t_0 - t)^{2(\lambda - 1)}\) and \(J \propto (t_0 - t)^{(\lambda + 1)}\).

From the above, it is clear that the solutions show different behaviors depending on the values of the exponents of the dissipation of energy and the viscosity (that is, on the dominating mechanisms of dissipation and viscosity). Also, these results are independent of the magnitude of the turbulent viscosity.

### 3.2. Numerical Solutions

Once \(\epsilon, \eta, \beta, \) and \(\tau\) are selected, we can solve the set of ordinary differential equations (25)–(29). The full range of possibilities becomes enormous if we regard all the exponents of the viscosity model and the cooling rate as free parameters. For this reason, we shall investigate only the case of constant total mass. First, we study cylindrical cooling flows with constant total mass, in which the constraint given by equation (32) must be satisfied. If we require a finite density and zero velocity at the cloud center, equation (25) gives

\[
V = -n\xi,
\]

where \(n = \frac{1}{3}\). Substituting this equation into equation (28) gives an algebraic relation between the similarity functions of the pressure and the density:

\[
P = P_0 R^\eta,
\]

where \(P_0 = (\gamma - 1)^{1/(\epsilon - 1)}\) and \(q = (\epsilon - 2)/(\epsilon - 1)\). It is interesting that this relation for turbulent and rotating cooling flows is the same as the one derived by SG for radiative cooling flows. Note that there is no critical value for \(\gamma\), because for all values of \(\gamma > 1\), the relation between the density and the pressure is well defined. Also, \(\epsilon\) may have any values except \(\epsilon = 1, 2\). Going back to equations (27) and (29) and using this result, we have

\[
\frac{d}{d\xi} \left[ \epsilon^3 - \eta R^{\gamma + \beta + 1} \Phi \frac{d\Phi}{d\xi} \right] + \frac{m}{2\alpha P_0} \xi^2 \Phi R = 0,
\]

\[
\frac{d}{d\xi} \left( \Phi^2 - qP_0 \xi R^{\varepsilon - 2} \frac{dR}{d\xi} \right) - \xi R = 0.
\]

We have to solve these equations subject to the boundary conditions:

\[
R(\xi = 0) = 1, \quad \frac{dR}{d\xi} |_{\xi=0} = 0,
\]

\[
\Phi(\xi = 0) = 0, \quad \frac{d\Phi}{d\xi} |_{\xi=0} = 0.
\]

Of course, there is another constraint: conservation of mass. Since we are not interested in the exact value of the mass of the system, we do not parameterize the solutions by the total mass. In fact, we integrate the equations by the Runge-Kutta method to seek solutions satisfying the boundary conditions, and then we can calculate the mass of the system. However, it could be simply checked that if \(q = (4 - \tau)/(\tau - 2)\), then the above equations are invariant under a homology transformation. Indeed, if \(R(\xi)\) and \(\Phi(\xi)\) are a set of solutions of the equations, then \(A^{(2q-3)/2(q-2)} R(A\xi)\) and \(A^{(2q-3)/2(q-2)} \Phi(A\xi)\) are also a set of solutions, where \(A\) is an arbitrary constant. Using this transformation, it is possible to find \(A\) so that the solutions are satisfying the boundary conditions and the constraint of constant mass. Thus, if \(\tau\) and \(q\) are satisfying the relation \(q = (4 - \tau)/(\tau - 2)\), we can parameterize the solutions by the line \(\Phi\) of the cylindrical cloud.
In Figure 1, we represent the distributions of the density $R(\xi)$ in the self-similar space for $\gamma = 5/3$, $m = 4.0$, $\alpha = 0.007$, $\tau = \frac{1}{5}$, $\eta = -1$, and $\beta = 0$. Shown are the normalized density $R(\xi)$ for different values of $\epsilon$: $\epsilon = -4$ (solid curve), $-3$ (dotted curve), $4$ (dashed curve), and $3$ (long-dashed curve). All the solutions tend to zero as $\xi \to \infty$.

In Figure 1, we represent the distributions of the density $R(\xi)$ in the self-similar space for $\gamma = 5/3$, $m = 4.0$, $\alpha = 0.007$, $\tau = \frac{1}{5}$, $\eta = -1$, and $\beta = 0$, and different values of $\epsilon$: $\epsilon = 3, 4, -3, -4$. They are obtained by solving the equations with the fourth-order Runge-Kutta method. This figure shows that the general behaviors of the similarity density function $R(\xi)$ for different values of $\epsilon$ are almost the same. In fact, all the solutions tend to zero as $\xi \to \infty$. It means that we can define the radius of the cloud by the condition $R(\xi_c) \leq 0$, where $r_c = \xi_c(t_0 - t)^{-3/2}$ (see eq. [33]).

Figures 2 and 3 show the behavior of the similarity rotational velocity function $\Phi(\xi)$ for the same parameters as Figure 1. From Figure 1 and 2 we see that the whole of the cloud is rotating for $\epsilon = 3$ and 4. But Figure 3 shows that only part of the cloud is rotating for $\epsilon = -3$ and $-4$. Also, each profile shows a maximum value for the rotational velocity.

For the $\alpha$-model viscosity, we showed that only for $\epsilon = -\frac{1}{2}$ the spherically cooling flows describe a system with constant total mass, and this value of $\epsilon$ corresponds to free-free emission for cosmic compositions. Figure 4 shows the density profile of spherically symmetric flows for different values of $\alpha$. As the figure shows, the profile of $R(\xi)$ hardly depends on the value of $\alpha$. It means that the distribution of the mass in spherically symmetric cooling flows is independent of the magnitude of viscosity. But as Figure 5 shows, the distribution of angular momentum in such systems strongly depends on the magnitude of viscosity. In Figure 5, we can see the normalized rotational velocity $\Phi(\xi)$ for different values of $\alpha$, and it shows that as the magnitude of viscosity increases, the maximum value of the profile of $\Phi(\xi)$ increases.

4. DISCUSSION AND CONCLUSIONS

We have explored in this paper a new set of similarity solutions of the equations relevant to cooling flows of self-gravitating, rotating, viscous systems. Analogously to the case with no rotation and viscosity (SG; Meerson et al. 1996), we were able to obtain similarity solutions that describe cooling flows in rotating and viscous systems. However, we did not investigate all of the solutions of the
system. In fact, we restrict ourselves to the quasi-hydrostatic flows with constant total mass under a wide range of viscosity and cooling models. When the central density is high and the velocities low, we can apply our solutions that are regular at the center. Even if we solve the equations without the quasi-steady assumption by including the time-dependent terms in the momentum equation, we shall find the same similarity indices as equation (22).

It may seem that physical constraints on our solutions, such as mass or angular momentum conservation, limit the acceptable range of similarity indices. For example, the \( \alpha \)-viscosity law is incompatible with mass conservation in our models. But it does not mean that the self-similarity assumption is not a good assumption. As for the viscosity model, we mentioned that since within the turbulent viscosity assumption the main difficulty is to point out a plausible source of turbulence, empirical viscosity prescriptions (e.g., \( \alpha \)-model) have been broadly used. Although the \( \alpha \)-model enables a reasonable global description of stationary, thin and non-self-gravitating disks (Frank, King, & Raine 1992), we must note that the \( \alpha \)-model is only an empirical prescription. Few improvements of the \( \alpha \)-prescription have been made (Narayan 1992; Narayan, Loeb, & Kumar 1994; Godon 1995), and it has been shown that, in general, \( \alpha \) is not constant (Brandenburg 1998). Thus, just according to considerations about viscosity models, we cannot investigate the validity of our solutions. Due to these facts, we introduced a general form for the viscosity law, i.e., equation (7). However, one should demonstrate that the acceptable viscosity laws have reasonable dependencies on physical quantities. But such a study is out of the scope of this paper.

Recently, Elmegreen (1999) investigated the role of wave-driven turbulence in interstellar clouds. In his study, a dense region forms because of a convergence and high pressure from external magnetic waves and the thermal cooling that follows at the compressed interface. However, turbulent dissipation at the interface is not as important as thermal cooling for this density enhancement (Elmegreen 1999). Thus, if in our model we consider the dissipation rate simply as a thermal cooling function (not as a turbulent dissipation rate), it will probably be possible to apply the results of this study for the compressed interfaces in Elmegreen’s model.

In fact, some cosmological structures, diffuse H I mediums, and molecular clouds, all show some cooling properties. Thus, our solutions might have applications to the formation of these structures at early stages. In this regard, this work is an extension of the same cooling flow problem considered in SG and Meerson et al. (1996) in order to study the effects of viscosity and rotation on cooling flows. For filamentary clouds, SG showed that if the mass of the system is conserved, then the density at the center increases \( \rho_c \propto (t_0 - t)^{-1} \) and the radius of the filament decreases \( r_c \propto (t_0 - t)^{-1/2} \), irrespective of \( \epsilon \), i.e., the mechanism of cooling. We showed that if we consider the rotation and viscosity, conservation of the total mass of the system depends on the mechanism of cooling and the viscosity model. For example, for the \( \alpha \)-model, all the cooling mechanisms lead to a system with nonconstant mass. But for the other models of viscosity, which the exponents satisfy into equation (32), the total mass of the system is conserved and we have \( \rho_c \propto (t_0 - t)^{-2} \) and \( r_c \propto (t_0 - t)^{1/2} \). Also, for spherical cooling flows the value of \( \epsilon = -\frac{1}{2} \) corresponds to constant mass. In this case, we show that \( \rho_c \propto (t_0 - t)^{-2} \) and \( r_c \propto (t_0 - t)^{1/2} \). However, Meerson et al. (1996) showed that for nonviscous and nonrotating spherical cooling flows, the behaviors of \( r_c \) and \( \rho_c \) depend on \( \epsilon \) and the mass of the system is conserved: \( r_c \propto (t_0 - t)^{1/(\gamma + 2)} \) and \( \rho_c \propto (t_0 - t)^{-3/(2(\gamma + 3))} \).

A dispersion relation has been derived by Elmegreen (1989) for gravitational instabilities in a medium with cloud collisional cooling. The cooling function due to the cloud-cloud collisions with an isotropic, Maxwellian distribution of cloud velocities is a power law of the density and the temperature (Elmegreen 1987), similar to the form of our cooling rate in this paper. Thus, in a cloudy medium without much star formation activity (i.e., without heating), the energy dissipation is given by equation (9). Elmegreen showed that the regions with such conditions will clump into cloud complexes on a variety of scales. Since, on the largest scales, the rotation and shear of the galaxy become important, our results describe the general properties of structure formation in such regions.

This simple model nevertheless has some limitations. We have not included explicit heating terms in our energy equation, in order to minimize the number of parameters. However, we can consider additional terms in the governing equations. For example, one could invoke the flow inertia in the momentum equation or heat conduction in the energy equation. Although the similarity indices would be selected in these cases, we can consider only limited values of the exponents of the cooling function and the viscosity model. Also, the assumption of similarity solutions could be dropped, although we should not forget the complex nature of the governing partial differential equations. Also, we assumed a power law for the form of the cooling rate and viscosity, and the results depend strongly on the form of these functions. However, as we mentioned in SG, it seems that we can use piecewise solutions. This means that for each part of the evolution of the solutions, we can find the suitable exponents and then join one set of solutions to another one. Thus, it will be possible to consider a much wider range of cooling functions and viscosity models with the same analytical-type of solution.

This work was completed while one of the authors (M. S.) was visiting the Instituto de Astronomia, Universidad...
Nacional Autónoma de México, and he acknowledges gratefully the hospitality of Professor Enrique Vázquez-Semadeni during his visit. We also thank Alan Watson and Bruce Elmegreen for carefully reading early versions of the manuscript and providing us with detailed comments. M. S. acknowledges a Research Fellowship from Ferdowsi University of Mashhad.

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