CURVE OBSTRUCTION FOR AUTONOMOUS Diffeomorphisms ON SURFACES

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Abstract. Consider a Hamiltonian diffeomorphism $g$ on a surface. We describe several scenarios where a curve $L$ and its image $g(L)$ provide a simple evidence that $g$ is not autonomous.

1. Introduction and results

An autonomous Hamiltonian flow on a surface admits an invariant lamination $\mathcal{B}$ by the flow lines. Therefore, the time-$t$ map $g$ of such flow preserves $\mathcal{B}$. It follows that given a curve $L$, both $L$ and $g(L)$ have the same pattern of intersection with $\mathcal{B}$. However, not any pair of curves $L, g(L)$ admits existence of such a lamination $\mathcal{B}$. This is a potentially rich and unexplored source of combinatorial constraints that an autonomous map must satisfy. The present article is a little step in this direction, we describe an elementary obstruction in terms of combinatorics of intersections of $L$ with $g(L)$ that prevents $g$ from having a loop of fixed points. At the same time, there are many scenarios where topological or dynamical arguments ensure existence of such loops for every $g$ generated by an autonomous flow. In this case we obtain a tool which shows that if $g$ complies with the obstruction, it cannot be autonomous.

Fix a compact orientable symplectic surface $\Sigma$ (possibly with boundary) and a simple curve $L \subset \Sigma$. We ask that $L$ is either closed or connects boundary points of $\Sigma$, and does not touch $\partial \Sigma$ except at endpoints. We pick an orientation for $L$, it induces orientation also for $g(L)$. This article restricts to flows $\phi$ on $\Sigma$ that are tangent to $\partial \Sigma$ (e.g., flows supported in the interior of $\Sigma$). Such $\phi$ admit a time-$t$ map for all $t \in \mathbb{R}$ and preserve the connected components of $\partial \Sigma$ (as a set, but $\phi$ are allowed to rotate each boundary component). Let $c$ be a periodic trajectory of $\phi$ (that is, $c$ is homeomorphic to a circle). All $x \in c$ are periodic with the same period $T_c$. $\rho_x = 1/T_c$ is the rotation number of $x$. $c$ is a loop of fixed points of $\phi^1$ whenever $\rho_x \in \mathbb{N}$ for some (and hence all) $x \in c$.

We say that a family $\mathcal{M} \subset \text{Ham}(\Sigma)$ satisfies a fixed loop property detectable by $L$ (in short notation, $\mathcal{F}_L$ property) if every autonomous $g \in \mathcal{M}$ has a loop of fixed points which intersects $L$. In Section 3, we provide few scenarios where such property can be established by topological or dynamical considerations.

The main result is the following:

Theorem 1. Suppose $\mathcal{M} \subset \text{Ham}(\Sigma)$ has the $\mathcal{F}_L$ property, let $g \in \mathcal{M}$.

Assume $L \cap g(L)$ away from a neighborhood of $\partial \Sigma$ and there is a system of disjoint neighborhoods $\{U_i\}$ for transverse intersections in $L \cap g(L)$, such that in each neighborhood $(L \cup g(L)) \cap U_i$ is diffeomorphic to one of the configurations on Figure 1. Then $g$ is non-autonomous.

This result can be compared to [Kha1] which establishes a different kind of obstructions for a Hamiltonian $g$ to be autonomous. Similar to this article, the obstructions there can be described in terms of $L$ and its image $g(L)$. However, [Kha1] relies on more involved tools (various quasimorphisms) which reflect global dynamics of $g$, while here we use an elementary argument which is purely local near the intersection points.

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2. THE OBSTRUCTION

Let \( \phi \) be an autonomous Hamiltonian flow on \( \Sigma \). We call a closed trajectory \( c \) regular with respect to \( L \) if \( c \cap L \) and \( x \) is not a critical point for \( \rho \). This implies that near \( c \) one may find periodic trajectories \( c^- \) and \( c^+ \) with periods \( T_c^- < T_c < T_c^+ \).

Given a loop of fixed points \( c \) of \( \phi \) such that \( c \cap L \neq \emptyset \), one can perturb \( \phi \) in \( C^1 \) topology into an autonomous Hamiltonian which has a regular loop of fixed points near \( c \): if \( c \cap L \) is not transverse, conjugate \( \phi \) with a \( C^\infty \)-small diffeomorphism \( h \) such that \( h(c) \cap L \neq \emptyset \) and \( h(c) \cap L \neq \emptyset \). \( h(c) \) is a loop of fixed points for the perturbed flow and has the same rotation number as \( c \) with respect to \( \phi \). If \( T_c \) is a critical value, one can accelerate (or slow down) the flow inside nearby trajectories on either side of \( c \) and destroy this condition.

Suppose that \( c \) is regular for \( g = \phi^1 \). Note, that since \( c \) is fixed pointwise by \( g \), \( c \cap L = c \cap g(L) \). That is, \( c \) is allowed to intersect \( L \cup g(L) \) only at \( L \cap g(L) \). Moreover, as \( g \) is orientation-preserving, \( L \) and \( g(L) \) induce the same coorientation at their transverse intersections with \( c \). In simple words, at an intersection point \( x \in c \cap L \cap g(L) \), both \( L \) and \( g(L) \) point to the same side of \( c \). If we pick a coordinate chart around \( p \in L \cap g(L) \) which sends \( L \) to the \( x \)-axis and \( g(L) \) to the \( y \)-axis (respecting orientations of curves), then \( c \) may cross the intersection either from the second quadrant to the fourth or vice versa. It cannot connect the first and the third quadrants as in this case \( L \) and \( g(L) \) have opposite coorientations.

We describe below a local obstruction to have a regular loop of fixed points \( c \) passing through \( x \in L \cap g(L) \). We provide two simple proofs for this phenomenon, one purely combinatorial and another having dynamical nature.

**Lemma 2.** Suppose there is a coordinate chart \( U \) which contains three intersection points of \( L \cap g(L) \) connected as it is shown in Figure 1. Then none of these intersection points lies on a regular loop of fixed points of \( g \).

Proof 1: Assume by contradiction that \( c \) is such a loop. In order to keep coorientation of \( L \) consistent with that of \( g(L) \), \( c \) has to intersect the curves in direction indicated by solid red lines in Figure 1. Moreover, if \( c \) crosses from one side of \( L \) to another and avoids \( L \cup g(L) \) except at their intersection points, it has to connect all three intersection points as described by the dotted line.
We look at the order of these three intersection points: the order along \( L \) does not match that along \( g(L) \) (even if \( L \) is a closed curve and we consider the circular order, it is still different). But if the points are fixed for \( g \), their order must be preserved. Indeed, let \( \gamma \) be the oriented arc of \( L \) which connects two intersection points \( x, y \in L \cap g(L) \). Then \( g(\gamma) \) is the oriented arc in \( g(L) \) which connects \( x \) to \( y \). A chain of three intersection points \( x, y, z \) connected by disjoint arcs \( \gamma_1, \gamma_2 \) is sent by \( g \) to the chain \( x, y, z \) connected by \( g(\gamma_1), g(\gamma_2) \) – the order of \( x, y, z \) along \( g(L) \) is the same as along \( L \). \( \square \)

Proof 2: Assume by contradiction that \( c \) is such a loop. As in Proof 1, coorientations constrain \( c \) to intersect the curves as indicated in the figure.

Given a fixed point \( x \in c \) and a short oriented curve \( \gamma \) through \( x \) which is transverse to \( c \), we call \( x \) positive if the angle between \( \gamma \) and \( g(\gamma) \) at \( x \) is in counterclockwise direction. When \( c \) is a regular loop of fixed points, this angle cannot be zero. Note that positivity does not depend on the choice of \( \gamma \): pick \( v_\alpha, v_\gamma \in T_x \Sigma \) - non-zero vectors tangent to \( c, \gamma \), respectively, such that \( (v_\alpha, v_\gamma) \) is a basis with negative orientation. The linearization \( Dg_x : T_x \Sigma \to T_x \Sigma \) is represented in this basis by the matrix \( \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \). Indeed, \( c \) is fixed pointwise, hence \( v_\alpha \) is an eigenvector with the eigenvalue \( 1 \). \( g \) is Hamiltonian implies it is area-preserving hence the second eigenvalue must be \( 1 \) as well. \( c \) is regular means \( Dg_x \) cannot be the identity map, so \( \alpha \neq 0 \). The sign of \( x \) coincides with the sign of \( \alpha \) and \( \alpha \) depends continuously on \( \gamma'(x) \).

Note as well that the sign is constant at all points \( x \in c \). However, if \( c \) had intersected \( L \cap g(L) \) as indicated, it would have had alternating signs at the three intersection points - a contradiction. \( \square \)

Remark 3. Our obstruction is stable under \( C^1 \)-small perturbations of \( g \) (such perturbations do not change the combinatorics of \( L \cap g(L) \)). It follows that the obstruction prevents existence of general loops of fixed points for \( g \) (not only regular ones) that intersect \( L \) at \( x \): assume by contradiction that \( g \) is autonomous with a non-regular loop of fixed points \( c \). We perturb \( g \) into \( g' \) as it was described in the beginning of this section. \( g' \) has a regular loop of fixed points \( c' \) – but the obstruction persists also for \( g' \). A contradiction.

If all intersection points \( L \cap g(L) \) satisfy the obstruction, \( g \) does not have a loop of fixed points which intersects \( L \). This immediately implies Theorem 5.

In the next section we describe several examples of \((\Sigma, L)\) and families \( \mathcal{M} \subset \text{Ham}(\Sigma) \) where one may establish the \( \mathcal{F}_L \) property. In these scenarios our obstruction provides both an easy criterion to show non-autonomy of \( g \) and a method to perturb \( g(L) \) into \( L' \) so that any \( \{ h \in \mathcal{M} \mid h(L) = L' \} \) is not autonomous. Namely, go over all transverse intersection points \( L \cap g(L) \) and replace a small arc of \( g(L) \) around the intersection by a “snake” described in Figure 4 (pattern to the left if \( g(L) \) crosses \( L \) from the left to the right or pattern to the right for the opposite direction). This perturbation may be realized by a Hamiltonian \( h \) supported near the intersection point such that \( h \) has arbitrarily small \( C^0 \) and Hofer norms.
for all $x \in A$. That implies that the orbit $c$ orbits $r$ not closed, non-contractible loops in $A$.

Example. 3.1.1. Let $A = S^1 \times [0, 1]$ (we set $S^1 = \mathbb{R}/\mathbb{Z}$) equipped with the area form $\omega = d\theta \wedge ds$ so that $A$ has total area 1. For an autonomous flow $\phi$ on $A$ we may consider rotation numbers $r_A^\phi : A \to \mathbb{R}$ with respect to the $S^1$ coordinate of $A$. Looking at the Reeb graph of the Hamiltonian function which generates $\phi$, it is easy to show that $r_A(x)$ exists for all $x \in A$ and is continuous. Moreover, for periodic points $x$ with noncontractible orbits $r_A(x) = \pm \rho_x$ which was defined earlier (if the orbit of $x$ is either contractible or not closed, $r_A(x) = 0$).

In all examples below we set $L = \{0\} \times [0, 1]$. We describe families $\mathcal{M} \subset \text{Ham}(A)$ which guarantee existence of $x \in A$ with $r_A(x) \in \mathbb{Z} \setminus \{0\}$ for all autonomous $g \in \mathcal{M}$. That implies that the orbit $c$ of $x$ is a closed non-contractible loop with $\rho_x \in \mathbb{Z}$. As all non-contractible loops in $A$ intersect $L$, these $\mathcal{M}$ satisfy the fixed loop property detectable by $L$. (We have to exclude points with $r_A(x) = 0$ from our scope of interest as they may be non-periodic or belong to orbits which avoid $L$.)

3. Examples

3.1. The annulus. Let $A = S^1 \times [0, 1]$ (we set $S^1 = \mathbb{R}/\mathbb{Z}$) equipped with the area form $\omega = d\theta \wedge ds$ so that $A$ has total area 1. For an autonomous flow $\phi$ on $A$ we may consider rotation numbers $r_A^\phi : A \to \mathbb{R}$ with respect to the $S^1$ coordinate of $A$. Looking at the Reeb graph of the Hamiltonian function which generates $\phi$, it is easy to show that $r_A(x)$ exists for all $x \in A$ and is continuous. Moreover, for periodic points $x$ with noncontractible orbits $r_A(x) = \pm \rho_x$ which was defined earlier (if the orbit of $x$ is either contractible or not closed, $r_A(x) = 0$).

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3.1.1. Example. given $g \in \text{Ham}(A)$ let $\tilde{g} \in \text{Ham}(\mathbb{R} \times [0, 1])$ be its lift to the universal cover of $A$.

$$\mathcal{M} = \{g \in \text{Ham}(A) \mid \pi_\mathbb{R}(\tilde{g}(0, 1)) - \pi_\mathbb{R}(\tilde{g}(0, 0)) > 4\}.$$  

Note that while the projection $\pi_\mathbb{R}(\tilde{g}(0, 1))$ depends on the lift, the difference $\pi_\mathbb{R}(\tilde{g}(0, 1)) - \pi_\mathbb{R}(\tilde{g}(0, 0))$ is determined by $g$. Moreover, it may be described in the following way: let $g(L)$ be a lift of $g(L)$ to $\mathbb{R} \times [0, 1]$.

$$\pi_\mathbb{R}(\tilde{g}(0, 1)) - \pi_\mathbb{R}(\tilde{g}(0, 0)) = \pi_\mathbb{R}(\tilde{\tilde{x}})$$

where $\tilde{x}, \tilde{y}$ are the endpoints of $\tilde{g}(L)$. That is, the family $\mathcal{M}$ is determined by the curve $g(L)$.

It is well-known that for an autonomous flow $\varphi : S^1 \to S^1$ the rotation number $\rho^\varphi$ satisfies $\varphi^1(x) - x - 1 < \rho^\varphi < \varphi^1(x) - x + 1$ where $\varphi : \mathbb{R} \to \mathbb{R}$ is the lift of $\varphi$ to the universal cover such that $\varphi^0 = 1$ and $x \in \mathbb{R}$ is arbitrary. Suppose $g \in \mathcal{M}$ is the time-1 map of an autonomous Hamiltonian flow $\phi$. $\phi$ restricts to a flow on the boundary components so

$$r_A^\phi(0, 1) - r_A^\phi(0, 0) > (\pi_\mathbb{R}(\tilde{\tilde{x}}) - 1) - (\pi_\mathbb{R}(\tilde{\tilde{x}}) + 1) > 2.$$
By continuity $r_A^\phi$ has at least two different integer values in its image, so at least one value in $\mathbb{Z} \setminus \{0\}$. It follows that $\mathcal{M}$ satisfies the $F_L$ property.

**Remark 5.** The rotation number $r_A^\phi$ may depend on the isotopy $\phi$ rather than on its time-1 map $g$, different choices of isotopy may give rise to rotation numbers shifted by an integer. We have shown above that there exists an orbit with non-zero integer rotation number for every choice of autonomous isotopy that generates $g$.

### 3.1.3. Example

In this example we restrict to $\text{Ham}^\prime(A) \subset \text{Ham}(A)$ generated by Hamiltonian functions $H_t : A \to \mathbb{R}$ which are locally constant near $\partial A$ (that is, the Hamiltonian flow is stationary near the boundary, but the flux through $L$ might be not zero).

$$\mathcal{M} = \{ g \in \text{Ham}^\prime(A) \mid \text{flux}_L(g) > 1 \}.$$  

The flux may be computed by

$$\text{flux}_L(g) = \int_{\Sigma} H_t(1,0)dt - \int_{\Sigma} H_t(0,0)dt$$

where $H_t$ is a Hamiltonian function which generates $g$. Alternatively, $\text{flux}_L(g)$ is equal to the area bounded between $L$ and $\tilde{g}(L)$ where $\tilde{L}$ is a lift of $L$ to the universal cover of $A$ and $\tilde{g}$ is the lift of $g$ which restricts to the identity near the boundary. In this setting $r_A(x)$ is well defined for all autonomous $g$ (does not depend on the isotopy $\phi \subset \text{Ham}^\prime(A)$). Moreover, for autonomous $g$

$$\text{flux}_L(g) = \int_A r_A(x)\omega > 1$$

(this identity is easy to verify using action-angle coordinates near periodic orbits of $\phi$). Therefore there exists $x \in A$ with $r_A(x) \geq 1$. At the same time $r_A = 0$ near $\partial A$. Continuity implies that $r_A$ has value 1 in its image and the $F_L$ property follows.

### 3.1.3. Example

We restrict to $\text{Ham}_c(A) \subset \text{Ham}(A)$ generated by Hamiltonian functions $H_t : A \to \mathbb{R}$ which are compactly supported in the interior of $A$. (that is, the Hamiltonian flow is stationary near the boundary, and the flux through $L$ is zero).

Let $F_t : \Sigma \to \mathbb{R}, t \in [0,1]$ be a time-dependent smooth function with compact support in the interior of $\Sigma$. We define $\overline{\text{Cal}}(F_t) = \int_0^1 (\int_{\Sigma} F_t \omega)dt$. If the symplectic form $\omega$ is exact (this is the case for an annulus or a disk), $\text{Cal}$ descends to a homomorphism $\text{Cal}_c : \text{Ham}(\Sigma) \to \mathbb{R}$ which is called the **Calabi homomorphism**.

We also need the **Calabi quasimorphism** by Entov-Polterovich (see [EP]). The authors construct a homogeneous quasimorphism $\text{Cal}_{S^2} : \text{Ham}(S^2) \to \mathbb{R}$. It has many wonderful properties related to the group structure and geometry of $\text{Ham}(S^2)$, here we will state just the following one: for an autonomous Hamiltonian $f : S^2 \to S^2$,

$$\text{Cal}_{S^2}(f) = \int_{S^2} F\omega - \text{Area}(S^2) \cdot F(X),$$

where $F$ is a generating function for $f$ and $X$ is a certain level set of $F$ (the median, consider [EP] for details).

We embed $A$ into a sphere $S^2_{a,b}$ of area $1 + a + b$ by gluing a disk of area $a$ to $S^1 \times \{0\}$ and a disk of area $b$ to $S^1 \times \{1\}$. Denote this embedding by $i_{a,b} : A \to S^2_{a,b}$. A Hamiltonian function $F : A \to \mathbb{R}$ supported in the interior of $A$ extends by zeros to a function $S^2 \to \mathbb{R}$. This allows us to pull $\text{Cal}_{S^2}$ back to a quasimorphism $i_{a,b}^* \text{Cal}_{S^2} : \text{Ham}_c(A) \to \mathbb{R}$.

Let

$$r_{a,b} = \frac{1}{1 + a + b} \cdot \left( \text{Cal}_A - i_{a,b}^* \text{Cal}_{S^2} \right)$$

be the normalized difference between the Calabi homomorphism on $A$ and the pullback of the Calabi quasimorphism. Using methods from [EP], one shows that $r_{a,b}$ vanishes on Hamiltonians supported in a small disk hence is $C^1$-continuous.
Fix a parameter $h \in [0, 1]$. Let $G : A \to \mathbb{R}$ be a Hamiltonian function, $g$ its time-1 map and set $a = 1, b = 1 + 2h$. For a generic $G$, $r_{1,1+2h}(g) = G(X_h)$ where $X_h$ is the level set of $G$ which is sent to a median of $i_{1,1+2h,*}(G)$ on $S^2$. One may show that $X_h$ is not contractible in $A$ and $X_h$ together with $S^1 \times \{0\}$ bound a subannulus $A^h_g$ of area at most $h$. The flux of $g$ in $A^h_g$ is precisely $G(X_h) = r_{1,1+2h}(g)$. From here we continue as in the previous example.

$$\mathcal{M}_h = \{ g \in \text{Ham}_c(A) \mid r_{1,1+2h}(g) > h \}.$$ 

As before, for generic autonomous $g$

$$\text{flux}_{A^h_g}(g) = \int_{A^h_g} r_A(x) \omega > h$$

hence there exists $x \in A$ with $r_A(x) = 1$. $r_{1,1+2h}$ is $C^0$-continuous, so applying a perturbation we may drop the requirement that $g$ is generic. The $F_L$ property follows.

**Remark 6.** Using tools described in [Kha1], one may show that the condition $r_{1,1+2h}(g) > h$ can be determined (up to a bounded defect) by the pair $L, g(L)$.

### 3.2. The disk

Let $D$ be the unit disk in $\mathbb{R}^2$ equipped with the area form $\omega = \frac{1}{\pi} dx \wedge dy$ so that $D$ has total area 1. $L$ is the diameter $[-1, 1] \times \{0\}$. We use a similar construction to Example 3.1.3.

Embed $D$ into a sphere $S^2_a$ of area $1 + a$ by gluing a disk of area $a$ to $\partial D$. Denote this embedding by $i_a : D \to S^2_a$. Let

$$r_a = \frac{1}{1 + a} \cdot (\text{Cal}_D - i_a^* \text{Cal}_{S^2})$$

be the normalized difference between the Calabi homomorphism on $D$ and the pullback of the Calabi quasimorphism of $S^2_a$.

Let $G : D \to \mathbb{R}$ be a Hamiltonian function compactly supported in the interior of $D$, $g$ its time-1 map and set $a = 1 - 2h$ for a fixed parameter $h \in (0, 0.5)$. For a generic $G$, $r_{1-2h}(g) = G(X_h)$ where $X_h$ is the level set of $G$ which is sent to the median of $i_{1-2h}(G)$ on $S^2$. One shows that if $G(X_h) \neq 0$, $X_h$ together with $\partial D$ bound a subannulus $A^h_g$ of area at most $h$. The flux of $g$ in $A^h_g$ is precisely $G(X_h) = r_{1-2h}(g)$. From here we continue as before.

$$\mathcal{M}_h = \{ g \in \text{Ham}_c(D) \mid r_{1-2h}(g) > h \}.$$ 

Once again, for generic autonomous $g$

$$\text{flux}_{A^h_g}(g) = \int_{A^h_g} r_{A^h_g}(x) \omega > h$$

hence there exists $x \in A$ with $r_{A^h_g}(x) = 1$. The orbit $c$ of $x$ is a non-contractible loop in $A^h_g$ hence $c$ encircles $D \setminus A^h_g$. Therefore the region bounded by $c$ is a topological disk of area at least $1 - h > 1/2$. By area considerations, $c$ must intersect $L$. $r_{1-2h}$ is $C^0$-continuous, so applying a perturbation we may drop the requirement that $g$ is generic. The $F_L$ property follows.

**Remark 7.** Using the argument from [Kha2], one shows that the condition $r_{1-2h}(g) > h$ can be determined (up to a bounded defect) by the pair $L, g(L)$.

### 3.3. Final remarks

**Remark 8.** In close analogues of the examples above the family $\mathcal{M}$ can be described in terms of the curve $g(L)$ rather than the map $g$ itself. That is, we have constructed an obstruction that depends only on the pair of curves $L$ and $g(L)$. 
Remark 9. In all examples except for the first one, we may refine the obstruction. Instead of blocking all possible intersections of loops of fixed points $c$ with $L$ it is enough to obstruct only those intersection points $x \in L \cap g(L)$ that have Maslov index 2, as these are the intersection points that arise when we have rotation number $r_A(x) = 1$.

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