I. INTRODUCTION

The Gottesmann-Knill theorem for continuous variables (CV) states that quantum computing based only on components described by one- or two-mode quadratic Hamiltonians, Gaussian states input, and measurements on canonical variables can be efficiently simulated by a classical computation. That means, although some of the algorithms are of fundamentally quantum nature, they do not provide any speedup over classical processes [1]. It has also been proved that construction of a CV universal quantum computer for transformations that are polynomial in those variables requires cubic or higher non-linear operations [2]. Therefore, investigation of the non-Gaussian transformations and states generation is crucial for a nontrivial quantum computation. More recently it has been noted that Kerr-like nonlinearities, in a variety of systems, enable high precision quantum metrology that would otherwise require entanglement to achieve it [3].

Over the last decade ion trap experiments have led the emerging technologies of coherent quantum control, especially in quantum information theory [4] and quantum computation [5]. Those applications require efficient creation and precise manipulation of both electronic and motional trapped ion state. Various theoretical proposals on how to produce nonclassical arbitrary states of ion motion have been discussed. In experiment, Fock number states [6], coherent states [7], vacuum squeezed states [8], and Schrödinger cat states [9] have been realized. In this latter case, the state is an entangled state of the vibrational and electronic degrees of freedom. On the contrary, in this paper we give a deterministic way to prepare the vibrational degree of freedom in a non-Gaussian state that is not entangled with the electronic states. According to our knowledge, neither non-Gaussian state (other than an entangled cat state) nor a superposition of more than two coherent states has been observed so far.

In the case of photons there is no practical method of non-Gaussian state generation so far. Efforts have been made to explore a class of the $\chi^{(3)}$-non-Gaussian states produced using a photon coherent state $|\alpha\rangle$ interacting with $\chi^{(3)}$ Kerr nonlinearity in an optical fiber

$$\Psi(\alpha, \tau) = e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{i\frac{n(n-1)}{2} \tau} |n\rangle. \tag{1}$$

This class of non-Gaussian states, parametrized by the unitless evolution parameter $\tau$ [10], is the most popular one. The state $|1\rangle$ is known also as the Kerr state. In general, this is a highly nonclassical state and after a certain time of evolution $\tau$ in the fiber its Wigner function would take negative values in the phase space $|1\rangle$. However, the nonlinearity in a fiber, or any other experimentally achievable Kerr medium, is too small, $\chi^{(3)} \simeq 10^{-22} m^2 W^{-2}$, to reach a highly nonlinear regime and thus produce the negativity in an experimentally reasonable time, before it is destroyed by dissipation [11]. Although microstructured fibers seem to be more promising with $\chi^{(3)} \simeq 10^{-16} m^2 W^{-2}$, their length does not exceed 1 m with current technology.

The most known examples of the Kerr state are the cat states $e^{-i\sigma/4} |\alpha\rangle + e^{i\sigma/4} |\alpha\rangle$ corresponding to $\tau = \pi$, which have been found useful for studies of quantum decoherence and quantum-classical boundary [12, 13].
larger cat states for which the two components $|i\alpha\rangle$ and $|-i\alpha\rangle$ are almost orthogonal ($\alpha > 1.5$) find their application in quantum information processing \cite{13} and quantum computation \cite{16,17}.

Recently, there has been introduced a probabilistic method of non-Gaussian state generation relying on a conditional photon subtraction from a squeezed vacuum state \cite{18,19,20,21}. Such a state is a good approximation for the cat state if the amplitude is small $\alpha < 1$. However, it is neither practical to subtract more than two photons in the experiment nor to produce a state with $\alpha > 1.6$ \cite{21}. Moreover, the state is not pure.

Another approach is suggested by recent achievements in the the Polzik group \cite{22} based on their demonstration of teleportation of a quantum state from optical to matter degrees of freedom. If this process was reversed, so that a non-Gaussian state is either teleported or mapped from matter to light, optical Kerr states might be robustly generated and thus overcome obstacles which are met with in trying to generate optical non-Gaussian states directly. Of course this requires us to demonstrate a way to make non-Gaussian states in matter degrees of freedom. The proposals for transferring an arbitrary motional quantum state of an atom to a cavity field already exist \cite{23,24,25,26}. The effort to integrate ion traps and optical fields \cite{27} might offer a path based on the extraordinary level of coherent quantum control one has over the vibrational degrees of freedom for trapped ions.

In this paper we discuss two efficient and deterministic methods of $\chi^3$ non-Gaussian state generation for light using a single trapped ion. Both methods require ion motional state transfer to the cavity mode. The first method is based on the well-known protocol \cite{22} for an ion finite motional superposition state generation. It allows to produce the $\chi^3$ non-Gaussian states with arbitrary good approximation. We give criteria, based on Wigner function comparison and its measurement precision, which quantify the error resulting from the approximation. The second method is novel and it enables an exact non-Gaussian state generation using one laser pulse only. We point out that a Wigner function measurement of ion motional state can be performed using currently available technology and already existing experimental schemas. We also suggest a quantum metrology application, based on the work of Caves and co workers \cite{3}.

This paper is organized as follows. In section II we show that, applying a well-known protocol, one can produce an approximated $\chi^3$ non-Gaussian state of motion for a trapped ion. The method relies on using a series of laser pulses to couple electronic and vibrational degrees of freedom to effect the desired state preparation for the motional degree of freedom. We give the criteria for quantifying the extent to which the prepared state approximates the desired non-Gaussian state, and discuss the technical limitations of the method. In section III we present an alternative, and novel, method for generation of an exact $\chi^3$ non-Gaussian state. We also discuss the range of application of this method. We finish the paper with conclusions and a brief discussion of possible applications to quantum metrology.

II. SERIES OF LASER PULSES METHOD

An ion in a Paul trap \cite{28} may be prepared in an arbitrary state of the form

$$|\Psi\rangle = \delta|\Psi^e\rangle|e\rangle + |\Psi^g\rangle|g\rangle,$$

(2)

using the method proposed in \cite{27,29}, where

$$|\Psi^e\rangle = \sum_{n=0}^{M} w^e_{n}|n\rangle, \quad |\Psi^g\rangle = \sum_{n=0}^{M} w^g_{n}|n\rangle$$

(3)

are finite superpositions of ion motional states ($M < \infty$), $|n\rangle$ is a Fock state of a harmonic oscillator potential in the trap, $|g\rangle$ and $|e\rangle$ are the ion electronic ground and excited states respectively. Parameters $\delta$ and $\beta$ are complex numbers obeying the normalization constraint $|\delta|^2 + |\beta|^2 = 1$ and are set at the start of the experiment.

The method is based on applying a series of alternating laser pulses tuned, first to the carrier, and then the red sideband, transition of the trapped ion. It works both within and beyond the Lamb-Dicke regime. The ion is initially prepared in its electronic and vibronic (motional) ground state $|0, g\rangle$. Adjusting the Rabi frequencies and duration of each pulse properly, one could achieve $w^g_{n} = w^g_{n} = w_{n}$ equal to

$$w_{n} = \frac{1}{\sqrt{\sum_{k=0}^{M} |\alpha|^2 k!}} \alpha^n e^{i\frac{\pi}{2} n(n-1)}.$$  \hspace{1cm} (4)

These coefficients correspond to the coefficients of a quantum non-Gaussian state resulting from the unitary evolution of a self-Kerr interaction \cite{1} decomposed in the Fock basis, up to the normalization factor (which results from the fact that we cut off the infinite sum in \cite{1} and take into account only the first $M$ terms).

This method produces an approximated $\chi^3$ non-Gaussian state $|\Psi^{(M)}(\alpha, \tau)\rangle$ corresponding to the state reached via a Kerr interaction with an arbitrary value of evolution parameter $\tau$. Furthermore the effective value of $\tau$ can be much larger than can be achieved via unitary interaction under a realistic optical Kerr interaction were the bosonic degree of freedom an optical field mode. A proper choice of the cut-off value $M$ enables one to approximate the state \cite{1} arbitrarily closely. If we set $\delta = 0$ for simplicity, the ion state is given by

$$|\Psi^{(M)}(\alpha, \tau)\rangle = \frac{1}{\sqrt{\sum_{k=0}^{M} |\alpha|^2 k!}} \sum_{n=0}^{M} \frac{\alpha^n}{\sqrt{n!}} e^{i\frac{\pi}{2} n(n-1)} |n\rangle |g\rangle.$$  \hspace{1cm} (5)

The number of required pulses for preparing the state is equal to $2M$

$$|\Psi^{(M)}(\alpha, \tau)\rangle = R_{M} C_{M-1} \cdot \ldots \cdot R_{2} C_{0} |0, g\rangle,$$  \hspace{1cm} (6)
The hint for existence of a good non-Gaussian state approximation comes from a simple observation that only a finite amount of coefficients \( w_n \) contribute to the sum significantly and the exact number of them is strongly \( \alpha \) dependent. The coefficients, evaluated for exemplary values of the amplitude, \( \alpha = 2 \) and \( \alpha = 5 \), are depicted on Fig. 1.

For a given value of an amplitude \( \alpha \) the choice of the cut-off value \( M \) is based on three criteria. All criteria rely on comparing the Wigner function \( W^{(M)}(\tau = 2\pi, \gamma, \gamma^*) \) of an approximated state \( \Psi^{(M)} \) with the untruncated one \( W(\tau = 2\pi, \gamma, \gamma^*) \) evaluated for \( \Pi \). Please note that the comparison is made for \( \tau = 2\pi \). We chose this particular value of the evolution parameter because for this value the original Wigner function is given by a simple analytic formula, which can be computed directly.

The first criterion derives from an investigation of the approximated Wigner function isolines. The smaller the value of \( M \) the more the function deviates from an ideal Gaussian function for \( \tau = 2\pi \): the isolines are no longer circles. The isolines of interest can be chosen arbitrarily. We however chose \( 0.1, 0.3, 0.5 \) and calculate, for each isoline, the ratio between the most and the least distant point with respect to the point \((\alpha, 0)\) separately. For an ideal circle the ratio is always equal to 1.

In order to take into account the interference resulting from the approximate state, which takes the values around zero, we calculate the maximal and average ratio of the difference \( \frac{1}{2} |W(\tau = 2\pi, \gamma, \gamma^*) - W^{(M)}(\tau = 2\pi, \gamma, \gamma^*)| \) to the value of the ideal Wigner function, for all points \( \gamma \) in a phase space. Therefore, the second criterion gives the maximal and average relative error of the approximation respectively.

The third criterion reveals the percentage of points \( \gamma \) in a phase space for which \( W^{(M)}(\tau = 2\pi, \gamma, \gamma^*) = W(\tau = 2\pi, \gamma, \gamma^*) \) for a given precision. We assume a precision of order of either \( \pm 10^{-2} \) or \( \pm 10^{-3} \) to be good enough, since it relates to the accuracy of both the Wigner function computer visualisation using the density plots and the Wigner function reconstruction using quantum tomography [31].

The above criteria turn out to be more subtle than the well-known fidelity \( F \) which measures the overlap of the approximated and the original state in the phase space

\[
F = \left| \langle \Psi^{(M)}(\alpha, \tau) | \Psi(\alpha, \tau) \rangle \right|^2 = e^{-\alpha^2} \sum_{m=0}^{\infty} \frac{\alpha^{2m}}{m!}.
\]

Obviously, the fidelity approaches unity in the limit of \( M \to \infty \).

We would like to point out that having the ion already prepared in state \( \Psi^{(M)}(\alpha = 2, \tau = 2\pi, \gamma, \gamma^*) \) one could measure the Wigner function of its vibronic state using either the standard method of quantum tomography [32, 33] or the method of direct measurement developed in [34]. This would allow for detailed investigation of \( \chi^{(3)} \) non-Gaussian states, which has never been verified experimentally so far.

The direct Wigner function measurement method is especially interesting. It relies on the fact that the Wigner function of a displaced vibronic state of an ion is related to the probability of finding the ion in the ground and the excited state. The probabilities are measured by detecting a fluorescence signal. Such a measurement has been done for a light in a cavity with accuracy of \( \pm 0.2 \) [35].

**Example:** \( \Psi^{(M)}(\alpha = 2, \tau = 2\pi) \) generation

Estimation of the minimal number of laser pulses for \( \Psi^{(M)}(\alpha = 2, \gamma, \gamma^*) \) generation, which approximates the original state \( \Psi(\alpha = 2, \gamma, \gamma^*) \) for any value of the evolution parameter \( \tau \) well, requires comparison of the approximated Wigner functions \( W^{(M)}(\tau = 2\pi, \gamma, \gamma^*) \) evaluated for a few different cut-off values \( M \), with the original Wigner function \( W(\tau = 2\pi, \gamma, \gamma^*) \). The selection of the best possible cut-off values is based on analysis of sig-
significant coefficients $w_n$. For $\alpha = 2$ only the coefficients ranging from $w_0$ to $w_{16}$ are greater than $10^{-3}$. The coefficients $w_{10}$ to $w_{13}$ are of order of $10^{-2}$. Therefore, we test $10 \leq M \leq 16$.

We next investigate the isolines. Table III shows the ratios of the most to the least distant point, with respect to the point of $(2, 0)$ for the isolines $(0.1, 0.3, 0.5)$ of the approximated Wigner functions $W^{(M)}(\tau = 2\pi, \gamma, \gamma^*)$. The minimal value of $M$ for which the ratios evaluated for $W^{(M)}(\tau = 2\pi, \gamma, \gamma^*)$ are equal to the ratios evaluated for the original Gauss function using the grid (discretized phase space) with a step equal to $\Delta \gamma = 0.04$ is equal to 14. It means that for the given grid and greater values of $M$, the numerical simulations of the Wigner function will not differ. Therefore, the value of $M = 14$ could be regarded as an appropriate cut-off value.

The series of 28 laser pulses is experimentally feasible [30]. However, decreasing the quality of approximation only a little, one can diminish the number of pulses to 20. The ratios evaluated for $M = 10$ ($M = 9$) and isolines 0.1, 0.3 and 0.5 differ from unity of 20% (33%), 10% (23%) and 8% (19%) respectively. This is the minimal cut-off value for which the average error equal to is below 1% (see table III). For $M = 9$ the error is equal to 1.67%. The maximal error is equal to 6.39% (11.30%). Within a given precision $10^{-2}$ for $M = 10$ ($M = 9$) there are around 76% (64%) points for which the numerically obtained Wigner function values are equal to the values of the ideal function. For the precision of $10^{-3}$ and $M = 10$ ($M = 9$) there are 36% (26%) of such points.

These results show that $M = 10$ is still a good approximation for the Wigner function analysis and measurement. The approximated functions for $M = 10$ and $M = 14$ with marked isolines are depicted on Fig. 2. We also include the plot for $M = 9$ for comparison.

The fidelities evaluated for $|\Psi^{(9)}(\alpha = 2, \tau)\rangle$, $|\Psi^{(10)}(\alpha = 2, \tau)\rangle$ and $|\Psi^{(14)}(\alpha = 2, \tau)\rangle$ are equal to $F = 0.9838$, $F = 0.9943$, and $F = 0.9999$ respectively.

Below we present the time of pulses duration and their phase required to generate an exemplary compass state $|\Psi^{(10)}(\alpha = 2, \tau = \pi/2)\rangle$, which is a superposition of four coherent states

| $\varphi_R^C$ | $t_R^C$ | $\varphi_C^R$ | $t_C^R$ |
|-------------|---------|-------------|---------|
| $R_{10}$    | $\pi$   | 0.99 ms     | $C_0$   | 2.89 $\mu$s |
| $R_9$       | 0.47    | 0.39 ms     | $C_8$   | 0.83 1.16 $\mu$s |
| $R_8$       | 7.23    | 0.35 ms     | $C_7$   | 1.33 1.30 $\mu$s |
| $R_7$       | 4.26    | 0.44 ms     | $C_6$   | 2.41 2.21 $\mu$s |
| $R_6$       | 5.00    | 0.47 ms     | $C_5$   | 0.05 1.60 $\mu$s |
| $R_5$       | 1.82    | 0.55 ms     | $C_4$   | 0.86 2.44 $\mu$s |
| $R_4$       | 2.21    | 0.55 ms     | $C_3$   | 2.97 1.92 $\mu$s |
| $R_3$       | 1.30    | 0.75 ms     | $C_2$   | 3.93 2.84 $\mu$s |
| $R_2$       | 0.95    | 0.81 ms     | $C_1$   | 0.09 2.84 $\mu$s |
| $R_1$       | 3.23    | 1.37 ms     | $C_0$   | 4.19 1.04 $\mu$s |

These quantities were evaluated for the following Rabi frequencies $\Omega_C = 1$ MHz, $\Omega_R = 100$ kHz corresponding to $2 \pi / 84$ ms.
FIG. 3: The Wigner function for an approximated \(\chi^{(3)}\) non-Gaussian state: \(|\Psi^{(9)}(\alpha = 2, \tau = \frac{\pi}{2})\rangle\) – the top figure, \(|\Psi^{(10)}(\alpha = 2, \tau = \frac{\pi}{2})\rangle\) – the middle figure, \(|\Psi(\alpha = 2, \tau = 2\pi)\rangle\) – the bottom figure.

to carrier transition and red sideband respectively, and Lamb-Dicke parameter \(\eta = 0.02\). The total time of state build-up is equal to \(t_c \simeq 6.7\) ms.

The Wigner function of the approximated compass state for \(M = 10\) is presented on Fig. 3. We also present the Wigner functions for \(M = 9\) and the original one for \(|\Psi(\alpha = 2, \tau = \frac{\pi}{2})\rangle\) for comparison.

**Technical limitations**

At the end of this section we estimate the maximum value of the amplitude \(\alpha\) for which the method works. The maximum value seems to be \(\alpha \simeq 2.3\).

This limitation does not result from the decoherence of the ion state. Both the time of single pulse duration and the total duration of state build-up remain within the coherence of the ion for the vibronic state 190 ms. The coherence time for the electronic state is equal to \(1.4\) ms.

The number of laser pulses is limited due to the finite binding energy in the trap. The cut-off value \(M\) corresponds to the maximal ion excitation \(|M\rangle\). The \(|M = 17\rangle\) is the upper limit for trapping the ion so far [36].

**III. ONE LASER PULSE METHOD**

The one laser pulse method of \(\chi^{(3)}\) non-Gaussian state generation requires one carrier resonance pulse applied to the ion cooled down to a Lamb-Dicke regime. The vibronic state of the ion is initially prepared in a coherent state \(|\alpha\rangle\). The advantage of this method over the first one is that it allows for generation of the original Kerr state \(|\Psi^{(1)}\rangle\) without any approximations. However, not all values of the evolution parameter are accessible: the pulse duration is limited due to the ion decoherence.

The interaction between an ion and a laser pulse of Rabi frequency \(\Omega\) is governed by the following Hamiltonian

\[
H = \frac{\hbar \Omega}{2} \left\{ \sigma^+ e^{i\eta(ae^{-i\nu} + a^\dagger e^{i\nu})} + h.c. \right\},
\]

(8)

where \(\sigma^+\) is an electronic state rising operator, \(a\) is a vibronic state annihilation operator, \(\nu\) is a trapping frequency, and \(\eta\) is a Lamb-Dicke parameter. Using the expansion of the exponent function into a Taylor series

\[
e^{i\eta(ae^{-i\nu} + a^\dagger e^{i\nu})} = \sum_{k=0}^{L} \frac{(i\eta)^k}{k!} (ae^{-i\nu} + a^\dagger e^{i\nu})^k,
\]

(9)
in the rotating wave approximation and in an interaction picture the Hamiltonian is approximated by the first five terms of the expansion

\[
H_{\text{int}}^{\text{rwa}} = \frac{\hbar \Omega}{2} \left\{ \frac{1}{2} - \frac{\eta^2}{2} + \frac{\eta^4}{8} + \left( -\frac{\eta^2}{2} + \frac{\eta^4}{2} \right) a^\dagger a + \frac{\eta^4}{4} a^2 a^2 \right\} (\sigma^+ + \sigma^-).
\]

(10)
The terms in the first line in the above formula govern the free ion evolution. The last term in (10) we associate with a self-Kerr interaction Hamiltonian, up to the electronic state operators. Therefore, the unitary evolution operator resulting from the nonlinear part of the Hamiltonian is given by

\[
U(t) = e^{-i\frac{\Omega a^\dagger a}{8} t} a^2 (\sigma^+ + \sigma^-).
\]

(11)
It allows for reading out the effective value of the evolution parameter

$$\tau_{\text{eff}} = \frac{\Omega^2 t}{4}.$$  \hspace{1cm} (12)

In the further discussion we neglect the electronic state evolution since it may be made to take no part in the dynamics. To do this one first needs to prepare the electronic state in an eigenstate of $\sigma_x$ using a $\pi/2$ pulse tuned to the carrier transition, but the pulse must not excite the vibrational degree of freedom in any way. This can be done by making the pulse propagate orthogonal to the vibrational axis.

Please note that by way of contrast to the method described in the previous section this method is independent of the initial amplitude $\alpha$. The laser pulse duration required for the Kerr state generation is the same for all values of the amplitude. In other words, the time required for generation e.g. $|\Psi(\alpha = 2, \tau)\rangle$ and $|\Psi(\alpha = 5, \tau)\rangle$ is the same. However, accessibility of the evolution parameter $\tau_{\text{eff}}$ is limited instead. The greater Rabi frequency $\Omega$ and Lamb-Dicke parameter value are the shorter pulse duration $t$ must be to obtain $\tau_{\text{eff}}$. On the other hand, the ion has to remain within the Lamb-Dicke regime in order to hold the expansion (10) true.

Figure 3 shows the time of the pulse duration assuming its Rabi frequency $\Omega = 10$ MHz and the Lamb-Dicke parameters $\eta = 0.1$, $\eta = 0.3$, $\eta = 0.02$, and $\eta = 0.03$. For the first two Lamb-Dicke parameters it is less than 800 $\mu$s, which is within the coherence time of vibronic ion state (190 ms), to obtain $\tau_{\text{eff}} = \pi$ and therefore produce the cat state. For the two remaining $\eta$ values the duration of the pulse is of order of tenth of second.

The duration of the pulse required for $|\Psi(\alpha, \tau = 0.04)\rangle$ generation is equal to $t = 0.16$ ns for $\eta = 0.1$ and $t = 1.98$ $\mu$s for $\eta = 0.3$ ($t = 0.1$ s for $\eta = 0.02$ and $t = 0.2$ $\mu$s for $\eta = 0.03$). For $\tau = 0.04$ the first negativities in the Wigner function become visible [11].

This method allows for obtaining also the other coherent state superpositions easily. For example, for $\tau_{\text{eff}} = \frac{\pi}{4}$ one achieves superposition of six states and if $\tau_{\text{eff}} = \frac{\pi}{2}$ one achieves superposition of four states (the compact state [37]). For $\eta = 0.3$ and $|\Psi(\alpha, \tau = \pi/3)\rangle$ the pulse duration is equal to $t = 51.71$ $\mu$s, and for $|\Psi(\alpha, \tau = \pi/2)\rangle$ it is equal to $t = 77.57$ $\mu$s. These values are experimentally feasible.

The one laser pulse method of $\chi^{(3)}$ non-Gaussian state generation is limited due to introduced cut-off in the expansion [39]. It is valid only if we take the expansion for a small parameter. To estimate the small parameter we approximate the quantum operators by the classical amplitudes $a \simeq \alpha$

$$e^{i\eta(ac - i\epsilon + a^\dagger c^\dagger)} = \sum_{k=0}^{L} \frac{(2i\eta|\alpha|)^k}{k!} \left( \frac{ac - i\epsilon}{2|\alpha|} \right)^k$$ \hspace{1cm} (13)

and read out the small parameter in the expansion to be $2\eta\alpha$. The above series is decreasing if $\eta\alpha < \frac{1}{2}$.

This constraint is enough to ensure that the contribution from the higher order terms is negligible. For example, for $\alpha = 5$ and $\eta = 0.1$, obeying $\alpha = \frac{1}{2\eta}$ ($\eta = 0.09$ and $\alpha < \frac{1}{2\eta}$) the coefficients $(2i\eta|\alpha|)^k/k!$ are equal to: 0.5 (0.405) for $k = 2$, 0.041 (0.027) for $k = 4$, 0.0013 (0.0007) for $k = 6$. For coefficients obeying $\alpha < \frac{1}{2\eta}$ the sixth order coefficient is of two orders of magnitude smaller than the forth order coefficient.

### IV. WEAK FORCE DETECTION

We now show how the cat-state weak force detection protocol proposed in Munro et al [38] can be implemented in an ion trap given the ability to engineer an approximate cat state as we have described. It must be admitted that there is no compelling case to use the vibrational motion of a trapped ion as a weak force detector. However, the method we propose here would be a nice demonstration of how non-Gaussian states can beat the standard quantum limit for weak force detection.

Suppose one was able to prepare the vibrational degree of freedom in the non-Gaussian state

$$|\psi_i\rangle = \frac{1}{\sqrt{2}} \left( e^{i\pi/4} |\alpha\rangle + e^{-i\pi/4} | -\alpha\rangle \right)$$ \hspace{1cm} (14)

with the amplitude $\alpha$ real. When a weak force acts it can be described by the action of the unitary displacement operator $D(ie) = \exp(ica^\dagger - ic\alpha)$ acting on the initial state $|\psi_i\rangle$ to give the output state $|\psi_o\rangle = D(ie)|\psi_i\rangle$. Using the result that

$$D(ie)|\alpha\rangle = e^{d\mu'(i\epsilon)}|\alpha + i\epsilon\rangle$$ \hspace{1cm} (15)

we find that for $\epsilon \ll \alpha$, and $\alpha$ real that

$$|\psi_o\rangle = \cos(\pi/4 + \alpha\epsilon)|+\rangle + i \sin(\pi/4 + \alpha\epsilon)|-\rangle$$ \hspace{1cm} (16)

where the even and odd parity states are given by

$$|\pm\rangle = \frac{1}{\sqrt{2}} (|\alpha\rangle \pm |-\alpha\rangle)$$ \hspace{1cm} (17)

In other words, the weak force is well approximated by a rotation in the two dimensional parity subspace. As in [38] it then follows that the minimum detectable force is then given by

$$\epsilon_{\text{min}} = \frac{1}{2\alpha}$$ \hspace{1cm} (18)

which beats the standard quantum limit by a factor of $(\alpha)^{-1}$. Ion trap provide a simple way to reach this lower bound. Suppose that after the weak force has acted, the total vibrational and electronic state is given by Eq. [10]. The first step is to make a $\pi/2$ rotation of the electronic state, followed by the conditional rotation

$$R = e^{-i\pi a^\dagger a\sigma_z}$$ \hspace{1cm} (19)
which can easily be done \[39\]. Finally another \(\pi/2\) rotation gives the state
\[ |\psi_\alpha\rangle = \cos(\pi/4 + \alpha |+\rangle|g\rangle + i \sin(\pi/4 + \alpha |\rangle|e\rangle. \] (20)
The electronic state can now be readout, and the probability to find the ion in, say, the excited state is
\[ P_+ = \frac{1}{2} (1 - \sin(2\alpha)). \] (21)
Sampling this distribution gives the estimation for the force with a minimum detectable force that is inversely proportional to \(\alpha\). For example, the minimum force required to shift the interference distribution by one fringe is \(\epsilon_{\text{min}} = \pi/(4\alpha)\).

V. CONCLUSIONS

In the paper we have presented two experimentally feasible methods of \(\chi^{(3)}\) non-Gaussian states generation for light using a trapped ion. The first method is based on a well-known protocol (a series of laser pulses) and allows for generation of an arbitrarily well approximated non-Gaussian state. The approximation is quantified by three criteria based on Wigner function analysis. This method is only limited by technical parameters of the trap: the binding energy.

The second method enables an exact non-Gaussian state generation using one laser pulse. It is limited by the decoherence time of the motional ion state. However, adjusting the laser pulse and trap parameters properly one is able to produce a cat state.

Based on the proposed protocols \[23, 24, 25\] we believe that non-Gaussian state transfer from ion motion to a light beam is possible in a foreseeable future. It will enable application of a nontrivial quantum computing protocols based for example on coherent states. This is also a step towards exploring so far unknown branch of quantum optics such as non-Gaussian states.

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| Isoline | M    | 9  | 10  | 11  | 12  | 13  | 14  | 15  | 16  |
|---------|------|----|-----|-----|-----|-----|-----|-----|-----|
| 0.1     | 1.33 (33\%) | 1.20 (20\%) | 1.10 (10\%) | 1.04 (–) | 1.01 (–) | 1.00 (–) | 1.00 (–) | 1.00 (–) |
| 0.3     | 1.23 (23\%) | 1.10 (10\%) | 1.04 (–) | 1.01 (–) | 1.00 (–) | 1.00 (–) | 1.01 (–) | 1.01 (–) |
| 0.5     | 1.19 (19\%) | 1.08 (8\%) | 1.01 (–) | 1.04 (–) | 1.04 (–) | 1.04 (–) | 1.04 (–) | 1.04 (–) |

TABLE I: The ratios of the most to the least distant point, with respect to the point \((2, 0)\), for given isolines 0.1, 0.3, 0.5, of the approximated Wigner function \(W^{(M)}(\tau = 2\pi, \gamma, \gamma^*)\) for \(9 \leq M \leq 16\).

| Accuracy | M    | 9  | 10  | 11  | 12  | 13  | 14  | 15  | 16  |
|----------|------|----|-----|-----|-----|-----|-----|-----|-----|
| \(10^{-4}\) | 64\% | 76\% | 90\% | 99\% | 100\% | 100\% | 100\% | 100\% |
| \(10^{-3}\) | 26\% | 36\% | 44\% | 53\% | 65\% | 80\% | 97\% | 100\% |

TABLE II: The number of points \(\gamma\) in the phase space for which \(W^{(M)}(\tau = 2\pi, \gamma, \gamma^*) = W(\tau = 2\pi, \gamma, \gamma^*)\) at a given accuracy for \(9 \leq M \leq 16\).

| Error     | M    | 9  | 10  | 11  | 12  | 13  | 14  | 15  | 16  |
|-----------|------|----|-----|-----|-----|-----|-----|-----|-----|
| Average   | 1.67\% | 0.98\% | 0.55\% | 0.30\% | 0.15\% | 0.08\% | 0.04\% | 0.02\% |
| Maximal   | 11.30\% | 6.39\% | 3.48\% | 1.84\% | 0.94\% | 0.47\% | 0.22\% | 0.11\% |

TABLE III: The average and the maximal error estimated for \(\alpha = 2\) and for \(9 \leq M \leq 16\).
FIG. 4: The time of pulse duration required for Kerr state $|\Psi(\alpha, \tau)\rangle$ generation for Lamb-Dicke parameter $\eta = 0.1$ – the top figure, $\eta = 0.3$ – the top middle figure, $\eta = 0.02$ – the bottom middle figure, $\eta = 0.03$ – the bottom figure. The red horizontal line denotes time duration of 1 s.