Bose–Einstein condensates in a homogeneous gravitational field

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The behavior of a Bose–Einstein condensate in a homogeneous gravitational field is analyzed. We consider two different trapping potentials. Firstly, the gas is inside a finite container. The effects of the finiteness of the height of the container in connection with the presence of a homogeneous gravitational field are mathematically analyzed and the resulting energy eigenvalues are deduced and used to obtain the corresponding partition function and the ensuing thermodynamical properties. Secondly, the trapping potential is an anisotropic harmonic oscillator and the effects of the gravitational field and of the zero–point energy on the condensation temperature are also considered. These results are employed in order to put forward an experiment which could test the so called Einstein Equivalence Principle.

I. INTRODUCTION

Gravity can be understood at the classical level as a purely geometric effect, i.e., the motion of a free classical particle moving in a curved manifold is given by the Weak Equivalence Principle (WEP) [1]. In other words, a particle moves along the geodesics of the corresponding manifold. The introduction of additional interactions is done resorting to the so–called Einstein Equivalence Principle [1], the famous “semicolon goes to coma rule”. This principle tells us that locally the laws of physics are the special–relativistic laws, we may rephrase this statement asserting that locally the gravitational field can be gauged away.

Additionally, we may state that the role of geometry is, classically, local. This phrase means that the dynamics of a free classical particle located at a certain point \( P \) of any Riemannian manifold is, according to General Relativity (GR), determined by the geometric properties of this manifold at \( P \) (the motion equations can be written in terms of the connection coefficients, which at \( P \) depend only on the values of the components of the metric and their derivatives evaluated at \( P \)), geometry at any other point plays no role in the determination of the motion when the particle is at \( P \). If we consider the geodesic deviation between two particles, then we would obtain information of the Riemann tensor, but once again only of the region where the motion of these classical particles takes place.

The experimental tests of GR contain a large number of proposals and none of them contains an experimental output which could be considered a counterexample to the theoretical predictions of GR [1, 2]. This last statement could be misleading since it could lead us to conclude that there are no conceptual problems in connection with GR. Nevertheless, we may assert that nowadays gravitational physics stands before a dead end. Indeed, on one hand the dominating belief states that gravity shall be described by a theory founded, among other concepts, upon quantum ideas [3–5]. On the other hand, there is, in this direction, no complete theory, neither mathematically nor physically, in spite of the advocates of these ideas which claim to be on the verge of deducing the final theory. These aforementioned facts have spurred a large number of theoretical attempts which can be considered in the phenomenological realm [6–9]. In other words, the quest for experimental results, either looking for new effects (for instance, the breakdown of Lorentz symmetry), or testing the experimental postulates of general relativity, has acquire a significative relevance. Recently the possibility of resorting to the Fermi telescope and use it for the detection quantum gravity effects stemming from light dispersion for high energy photons has been put forward. A deeper and ampler analysis of this proposal can be found in [10].

In this context the present work addresses the behavior of a Bose–Einstein condensate under the influence of a homogeneous gravitational field. The idea is to consider the possibility of testing the Einstein Equivalence Principle resorting to the temperature as the parameter to be monitored. Indeed, the usual Bose–Einstein experiments are carried out with the experimental device at rest with respect to the Earth’s surface [11]. We will deduce the changes in the condensation temperature as a consequence of the presence of a homogeneous gravitational field. This will be done for two different kind of trapping potential, namely, a container of volume \( V \) and an anisotropic three–dimensional harmonic oscillator. It will become clear that there is indeed a modification due to the presence of a non–vanishing...
value of \( g \), the acceleration of gravity. This result is no surprise since we know that the condensation temperature emerges at that point at which the chemical potential equals the energy of the ground state \[12\]. The introduction of a non–vanishing homogeneous gravitational field modifies the ground state of a particle either trapped by a container of volume \( V \) or by an anisotropic three–dimensional harmonic oscillator. An interesting point in this context is related to the fact that the change in the condensation temperature, here we mean if it grows or decreases with respect to the case in which \( g = 0 \), depends upon the trapping potential. Indeed, it will be shown that for a Bose gas trapped inside a container of volume \( V \) the condensation temperature, due to \( g \neq 0 \), is larger than for \( g = 0 \), whereas, for the harmonic oscillator case it is smaller.

Let us now explain a little bit deeper the whole idea. According to the so–called Einstein Equivalence Principle (EEP) in a freely falling frame, for sufficiently short experimental times, the result of any experiment coincides with the outcome when \( g = 0 \), i.e., locally gravity can be gauged away \[1\]. The idea is to perform the experiment in a freely falling frame and detect the condensation temperature. If EEP is valid, then this temperature shall be equal to the case in which \( g = 0 \). Afterwards, knowing the result of the same experiment when carried out on the Earth’s surface we may compare the corresponding experimental outputs can contrasted them against the theoretical predictions arising from EEP. In this sense we would have a test of EEP resorting to the concept of temperature. Maxwell–Boltzmann (MB) statistics predicts the appearance of condensation only if the temperature of the system vanishes. Indeed, for a gas, obeying MB, comprising \( N \) non–interacting particles and discarding also internal degrees of freedom, the internal energy \( U \) satisfies the condition \( U = N < K > = 3N\kappa T/2 \), according to the theorem of equipartition of energy \[13\], here \( < K > \) denotes the average kinetical energy per particle, and \( \kappa \) is Boltzmann constant. This last expression tells us that \( < K > = 0 \) if and only if \( T = 0 \). The lowest energy appears only when \( T = 0 \). On the other hand, the case of Bose–Einstein statistics is quite different. Indeed, this situation involves a certain temperature, denoted condensation temperature \( T_c \), at which a large number of particles are in the corresponding ground state, usually the behavior has the form \( N_0/N \sim (1 - T/T_c)\alpha \), where \( N_0 \) is the number of particles in the ground state and \( \alpha \) is a positive real number which depends on the trapping potential, the number of dimensions, etc \[11\]. We may rephrase the previous argumentations stating that classical behavior (here this statement means MB) predicts condensation only if \( T = 0 \), whereas, quantum properties, Bose–Einstein statistics, entail a macroscopical number of particles in the ground state even if \( T > 0 \). In this sense we could discuss the possible interpretation of the present proposal as a quantum test for EEP since classical statistics requires inexorably \( T = 0 \).

II. CONDENSATION IN A HOMOGENEOUS GRAVITATIONAL FIELD

A. Gas in a container

As a first example let us consider the case of a finite container immersed in a homogeneous gravitational field, for the sake of simplicity we will assume that the gravitational field is along the \( z \)–axis and that the container has cylindrical symmetry with its axis parallel to \( z \). The problem of a quantum particle in a gravitational field is usually solve assuming that the particle vanishes in the limit \( z \to \infty \), i.e., the particle is to be found in the interval \( z \in [0, \infty] \). Under these conditions the solution reduces to one of the so–called Airy functions \[14\]. Our situation is quite different, indeed, the potential along the \( z \)–direction is comprised by two terms, namely,

\[
V(z) = \begin{cases} 
0, & \text{para } 0 < z < L \\
\infty, & \text{elsewhere,}
\end{cases}
\]

whose eigenvalues and eigenfunctions read:

\[
E_n^{(0)} = \frac{\hbar^2 \pi^2}{2mL^2} n^2, \quad (1)
\]

\[
\psi(z) = \sqrt{\frac{2}{L}} \sin \left( \frac{n\pi z}{L} \right). \quad (2)
\]
Clearly, we must add an extra term containing the gravitational interaction

\[ V_g = mgz. \]  

(3)

This term will be treated as a perturbation for the Hamiltonian given by

\[ H_0 = \frac{p_z^2}{2m} + V(z). \]  

(4)

In other words, our complete Hamiltonian is

\[ H = H_0 + \lambda V_g. \]  

(5)

A lengthy calculation, up to second order in \( \lambda \), renders the following eigenenergies

\[ \varepsilon_n = \frac{\hbar^2 \pi^2}{2mL^2} n^2 + \lambda \frac{mgL}{2} \bigg( 4m^3g^2L^4 \left( \frac{\zeta(5)}{n} - \frac{\pi^4}{48n^2} \right) \bigg). \]  

(6)

Of course, \( n \) is an integer equal or larger than 1. This last expression takes into account only the contribution to the energy as a consequence of the Hamiltonian along the \( z \) axis. It is obvious that the contributions of the corresponding Hamiltonians along \( x \) and \( y \) have to be included. These contributions are simply the eigenenergies of a particle inside a two–dimensional box whose form is a square (whose sides have a length equal to \( a \)), namely,

\[ \varepsilon_{(n_x,n_y)} = \frac{\hbar^2 \pi^2}{2ma^2} \left( n_x^2 + n_y^2 \right). \]  

(7)

The energies are provided by the sum of these last two expressions.

Statistical Mechanics teaches us that the knowledge of the associated eigenenergies allows us to calculate the corresponding partition function \[13\]. In our case we will resort to the Grand–Canonical partition function, the one tells us that the number of particles in thermodynamical equilibrium is given by

\[ N = \sum_n \langle n \rangle = \sum_{\varepsilon} \frac{1}{z - 1 e^{\beta \varepsilon} - 1}. \]  

(8)

Here \( \beta = \frac{1}{kT} \) and \( z \) is the so–called fugacity, defined as follows: \( z = e^{\mu/kT} \), with \( \mu \) the chemical potential. This last expression can be cast in the following form

\[ N = \frac{z}{1 - z} + \sum_{\varepsilon_A=1}^{\infty} \sum_{\varepsilon_w=1}^{\infty} \frac{1}{z - 1 e^{\beta (\varepsilon_A + \varepsilon_w)} - 1}. \]  

(9)

In this last expression it has been introduced the fact that the energy can be divided into two contributions, namely; (i) one stemming from the motion on the plane perpendicular to the gravitational field, this term has been denoted by \( \varepsilon_A \); (ii) the energy coming from the motion along the \( z \)–axis, \( \varepsilon_w \).

Particles in the ground state are given by

\[ N_0 = \frac{z}{1 - z}. \]  

(10)

The number of particles in the excited states is given by \( N - N_0 \), which for our particular case takes the form

\[ N - N_0 = \frac{V}{\Lambda} g_{3/2}(z e^{-\frac{mgL}{\hbar kT}}). \]  

(11)

Here \( g_v(z) \) are called Bose–Einstein functions and \( \Lambda = \frac{(2\pi m kT)^{\frac{1}{2}}}{\hbar} \) is de Broglie thermal wavelength \[13\].
We may at this point understand that the presence of a gravitational field does indeed impinge upon the condensation temperature, among other thermodynamical parameters. To fathom this last statement, let us recall that condensation appears (when it is possible) at that temperature in which the chemical potential equals the ground energy. For our particular case this last assertion entails

$$\mu(T = T_c) = \frac{mgL}{2}. \quad (12)$$

This last expression can be contemplated as an implicit definition of the condensation temperature, $T_c$. Indeed, for this case we obtain

$$(T_c^o)^{3/2} = T_c^{3/2} \left[ 1 - \frac{1}{\zeta(3/2)} \sqrt{\frac{\pi mgL}{2\kappa T_c}} \right]. \quad (13)$$

This last expression can be approximately written as follows

$$T_c \approx T_c^o \left[ 1 + \frac{2}{3} \frac{1}{\zeta(3/2)} \sqrt{\frac{\pi mgL}{2\kappa T_c^o}} \right]. \quad (14)$$

Here we have introduced the following definition which is the condensation temperature without gravitational field.

$$T_c^o = \frac{\hbar^2}{2\pi mk} \left( \frac{N}{V \zeta(3/2)} \right)^{2/3}. \quad (15)$$

Our last results show clearly that there is an increase in the condensation temperature provoked by the presence of a non–vanishing homogeneous gravitational field. Since the achievement of very low temperatures is an experimental difficulty this fact seems to be an advantage. Nevertheless, there are two points that have to be underlined in connection with this first example. Firstly, current technology achieves condensation resorting to trapping potential which are not our simple box. Secondly, this feature, increase of $T_c$ due to the presence of a non–vanishing $g$, is not a general behavior, as will be shown below.

**B. Gas in a harmonic trap**

As mention before, in the experimental realm the condensation process does not resort to a gas within a container, the trapping potential has a more sophisticated structure. Indeed, there are several kind of traps, for instance magneto–optical traps (MOT), Optical traps (OT), etc. The mathematical description of the available magnetic traps, at least for alkali atoms, is that the corresponding confining potential can be approximated by a three–dimensional harmonic oscillator

$$U(x, y, z) = \frac{m}{2} \left( w_1^2 x^2 + w_2^2 y^2 + w_3^2 z^2 \right). \quad (16)$$

We now consider the presence of a homogeneous gravitational field along the $z$–axis, hence, the complete potential becomes

$$U(x, y, z) = \frac{m}{2} \left( w_1^2 x^2 + w_2^2 y^2 + w_3^2 \left( z + \frac{g}{w_3^2} \right)^2 \right) - \frac{1}{2} \frac{mg^2}{w_3^2}. \quad (17)$$

The energy eigenvalues are given by

$$\varepsilon = \hbar w_1 \left( n_x + \frac{1}{2} \right) + \hbar w_2 \left( n_y + \frac{1}{2} \right) + \hbar w_3 \left( n_z + \frac{1}{2} \right) - \frac{1}{2} \frac{mg^2}{w_3^2}, \quad n_x, n_y, n_z \in \mathbb{N}. \quad (18)$$
The density of states is provided by

$$\Omega(\varepsilon) = \frac{(\varepsilon + \frac{mg}{2w_3} - \frac{\hbar}{2}(w_1 + w_2 + w_3))^2}{2\hbar^3w_1w_2w_3}. \quad (19)$$

This last expression allows us to calculate the average number of particles \(N\)

$$N = \int_0^\infty \frac{\Omega(\varepsilon)}{z^{-\frac{1}{2}e^\beta\varepsilon - 1}} d\varepsilon + \frac{1}{z^{-\frac{1}{2}e^\beta\varepsilon_0 - 1}}. \quad (20)$$

Here the energy of the ground state reads

$$\varepsilon_0 = \frac{\hbar}{2}(w_1 + w_2 + w_3) - \frac{1}{2}\frac{mg^2}{w_3}. \quad (21)$$

It is readily seen that the presence of a non–vanishing homogeneous gravitational field modifies the ground state energy, and therefore, this field entails a change in the condensation temperature. Indeed, it can be shown that a modification in the ground state energy due to the presence of an interaction, as the case here, does imply a new condensation temperature provided by [11]

$$\Delta T_c = -\frac{\zeta(2)}{3\zeta(3)} \frac{\Delta \mu}{\kappa}. \quad (22)$$

For our particular case the change in the chemical potential, \(\Delta \mu = -(0.456)\frac{mg^2}{2\omega_3} \), namely, the condensation temperature under the presence of a homogeneous gravitational field reads

$$T_c^{(g)} = T_c^{(0)} - (0.456)\frac{mg^2}{2\kappa \omega_3}. \quad (23)$$

In this last expression \(T_c^{(g)}\) is the condensation temperature if there is a non–vanishing gravitational field, whereas \(T_c^{(0)}\) denotes the condensation temperature without gravitational field. If \(g = 0\) we recover the usual condensation temperature.

C. Conclusions

In the present work the effects upon the condensation temperature of a homogeneous gravitational field have been calculated. This has been done for two particular trapping potential, namely, a container of volume \(V\) and a three–dimensional harmonic oscillator, the latter is a case which mimics some of the current technological possibilities in the experimental realm.

It has been shown that for the case of a trapping potential modelled by a container of volume \(V\) the presence of a homogeneous gravitational field entails an increase of the condensation temperature. For a three–dimensional harmonic oscillator the presence of this simple gravitational field has precisely the opposite effect, i.e., the condensation temperature suffers a decrease. This last situation entails that, according to EEP, the temperature of a freely falling condensate, trapped by a harmonic oscillator, should be higher than the corresponding temperature if the condensate lies at rest with respect to the surface of the Earth.

In order to have a quantitative idea of the modifications upon the condensation temperature due to the presence of a non–vanishing homogeneous gravitational field, for the case of a gas trapped by harmonic oscillators, we now show a table for three elements, namely, Rubidium, Sodium, and Lithium. The frequency \(\omega_z\) is equal to \(10^3\,\text{s}^{-1}\), and the condensation temperatures (under the presence of a gravitational field, i.e., \(T_c^{(g)}\) ) read \(50 \times 10^{-9}\,\text{K}, 2 \times 10^{-6}\,\text{K},\) and \(300 \times 10^{-9}\,\text{K},\) for Rubidium, Sodium, and Lithium, respectively, [15–17].
These last results display clearly the fact that, experimentally, the best option seems to be Lithium since it has the largest percentage of change, whereas Sodium is the worst one. The percentages are all smaller than one percent and oscillate between 0.31 and 0.62.

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