Coherent states with elliptical polarization

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Coherent states of the two dimensional harmonic oscillator are constructed as superpositions of energy and angular momentum eigenstates. It is shown that these states are Gaussian wave-packets moving along a classical trajectory, with a well defined elliptical polarization. They are coherent correlated states with respect to the usual cartesian position and momentum operators. A set of creation and annihilation operators is defined in polar coordinates, and it is shown that these same states are precisely coherent states with respect to such operators.

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1 Introduction

Coherent states were introduced by Glauber [1] in quantum optics and have a very wide range of application in quantum physics. They can be defined in several equivalent ways: as Gaussian wave packets evolving without spreading, and also as close analogues of classical states satisfying the minimum dispersion relation allowed by the Heisenberg uncertainty principle. One of the various generalizations of coherent states are the coherent correlated states introduced by Dodonov et al. [2], that satisfy the minimum Robertson-Schrödinger uncertainty relation.

A coherent state can be expressed as an infinite superposition of the number states of a harmonic oscillator. The generalization to two or more modes can be easily achieved by coupling several linear harmonic oscillators. This is the usual approach used in this kind of problems; for instance, Arickx et al. [3] have studied the evolution of Gaussian wave packets in two and three dimensions in cartesian coordinates.

However, polar coordinates are the natural coordinates for a two-dimensional oscillator with a single frequency, permitting to separate directly the angular momentum. Accordingly, it would be advantageous to have a definition of coherent states that takes into account the rotational symmetry of the problem. This is particularly relevant due the recent interest in the angular momentum of light, associated to electromagnetic fields with axial symmetry such as Bessel and Laguerre beams (see e.g. Padgett et al. [4]).

In this article, we propose a new kind of coherent states that are superpositions of states with well defined energy and angular momentum, and that can be directly related to axially symmetric fields. The analysis of these states is presented in Section 2, where it is also shown that they are coherent correlated states with respect to cartesian position and momentum, and correspond to two independent elliptical polarizations, rotating clockwise or counterclockwise. In Section 3, a set of creation and annihilation operators in polar coordinates are defined, and it is shown that the states are truly coherent with respect to these operators because they have minimum dispersion. Some numerical examples are presented.
2 Coherent correlated states

Consider a two dimensional harmonic oscillator with mass $M$ and frequency $\omega$. Choosing $M$, $\omega^{-1}$, and $(\hbar/m\omega)^{1/2}$ as units of mass, time and length respectively, the time dependent Schrödinger equation in cylindrical coordinates takes the form:

$$i\frac{\partial}{\partial t}\Psi(r, \phi, t) = \frac{1}{2}\left(-\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + r^2\right)\Psi(r, \phi, t).$$  \hspace{1cm} (2.1)

The eigenstates of energy and angular momentum are $|n, \pm l\rangle$ (we take $l \geq 0$). In coordinates representation, these states are given by the standard textbook solutions (see, e. g., [5]):

$$\langle r, \phi|n, \pm l\rangle = \left(\frac{n!}{\pi(n+l)!}\right)^{1/2} e^{\pm il\phi} e^{-r^2/2} r^l L_l^n(r^2),$$  \hspace{1cm} (2.2)

where $L_l^n(r^2)$ are Laguerre polynomials. The energy of these states is $E_{n,l} = 2n + l + 1$.

In analogy with the usual definition of coherent states, let us now define the following states:

$$|\alpha, \beta\rangle_\pm = N(\alpha, \beta) \sum_{n,l=0}^{\infty} \left(\frac{(n+l)!}{n!}\right)^{1/2} \frac{1}{l!} \alpha^n \beta^l |n, \pm l\rangle,$$  \hspace{1cm} (2.3)

where $N(\alpha, \beta)$ is a normalization constant. Clearly, the evolution of these states is given by

$$e^{-iH}|\alpha, \beta\rangle_\pm = |\alpha e^{-2it}, \beta e^{-it}\rangle_\pm.$$  \hspace{1cm} (2.4)

The orthogonality condition determines $N(\alpha, \beta)$, that is:

$$\pm \langle \alpha, \beta|\alpha, \beta\rangle_\pm = |N(\alpha, \beta)|^2 \sum_{n,l=0}^{\infty} \frac{(n+l)!}{n!(l!)^2} |\alpha|^{2n} |\beta|^{2l} = 1.$$  \hspace{1cm} (2.5)

Using the formula

$$\sum_{n=0}^{\infty} \frac{(n+l)!}{n!} x^n = \frac{l!}{(1-x)^{l+1}},$$  \hspace{1cm} (2.6)

it follows that

$$|N(\alpha, \beta)| = \sqrt{1 - |\alpha|^2} \exp\left\{-\frac{|\beta|^2}{2(1 - |\alpha|^2)}\right\},$$  \hspace{1cm} (2.7)
with the condition $|\alpha| < 1$.

It can also be seen that

$$
\pm \langle \alpha', \beta' | \alpha, \beta \rangle \pm = \frac{\sqrt{1 - |\alpha|^2}}{1 - \alpha \alpha'} \sqrt{1 - |\alpha'|^2} \times \exp \left\{ - \frac{|\beta|^2}{2(1 - |\alpha|^2)} - \frac{|\beta'|^2}{2(1 - |\alpha'|^2)} + \frac{\beta \beta'}{1 - \alpha \alpha'} \right\},
$$

(2.8)

and

$$
\mp \langle \alpha', \beta' | \alpha, \beta \rangle \pm = \frac{\sqrt{1 - |\alpha|^2}}{1 - \alpha \alpha'} \sqrt{1 - |\alpha'|^2} \times \exp \left\{ - \frac{|\beta|^2}{2(1 - |\alpha|^2)} - \frac{|\beta'|^2}{2(1 - |\alpha'|^2)} \right\}.
$$

(2.9)

Let us now look for the explicit form of the states $|\alpha, \beta \rangle \pm$ in coordinates representation. Since

$$
\langle r, \phi | \alpha, \beta \rangle \pm = \frac{1}{\pi^{1/2}} \frac{N(\alpha, \beta)}{(1 - |\alpha|^2)^{1/2}} \sum_{n,l=0}^{\infty} \frac{1}{n!} \alpha^n \beta^l e^{i\phi} e^{-r^2/2} r^l L_n^l(r^2),
$$

(2.10)

and using the generating function formula:

$$
\sum_{n=0}^{\infty} a^n L_n^l(x) = (1-a)^{-l-1} e^{x a/(a-1)},
$$

(2.11)

it follows that

$$
\langle r, \phi | \alpha, \beta \rangle \pm = \frac{N_{\pm}(\alpha, \beta)}{\pi^{1/2}(1 - |\alpha|^2)} e^{-(1+\alpha)r^2/2(1-\alpha)} \sum_{l=0}^{\infty} \frac{1}{l!} \left( \frac{\beta e^{i\phi}}{1-\alpha} r \right)^l.
$$

(2.12)

Accordingly:

$$
\Psi_{\alpha \beta}(r, \phi) \equiv \langle r, \phi | \alpha, \beta \rangle \pm
$$

$$
= \frac{1}{\pi^{1/2}} \frac{1 + \alpha}{1 - \alpha} \exp \left\{ - \frac{1 + \alpha}{1 - \alpha} \frac{r^2}{2} + \frac{\beta e^{i\phi}}{1 - \alpha} r - \frac{|\beta|^2}{2(1 - |\alpha|^2)} \right\}.
$$

(2.13)

Now, writing the exponent in this last equation in the form:

$$
- \frac{1 + \alpha}{1 - \alpha} \frac{r^2}{2} + \frac{\beta e^{i\phi}}{1 - \alpha} r = -\frac{1}{2}(x - q)(U^{-1} + iV)(x - q) + i \mathbf{p} \cdot \mathbf{x},
$$

(2.14)
we can compare our results with those of Arick et al [3]. Remembering that 
\( \alpha(t) = \alpha e^{-2it} \) and \( \beta(t) = \beta e^{-it} \) (taking \( \alpha \) and \( \beta \) real in the rest of this section without loss of generality), it follows that the position \( q \) and momentum \( p \) of the wave-packet center are:

\[
q = \left\{ \frac{\beta \cos t}{1 + \alpha}, \frac{-\beta \sin t}{1 - \alpha} \right\},
\]
\[
(2.15)
\]

\[
p = \left\{ \frac{\beta \sin t}{1 + \alpha} \left(1 + \frac{2\alpha \cos^2 t}{1 - \alpha e^{-2it}}\right), \frac{\beta \cos t}{1 - \alpha} \left(1 - \frac{2\alpha \sin^2 t}{1 - \alpha e^{-2it}}\right) \right\}.
\]
\[
(2.16)
\]

Similarly, we find:

\[
U = \frac{|1 - \alpha e^{2it}|^2}{1 - \alpha^2} 1
\]
\[
(2.17)
\]

and

\[
V = \frac{2\alpha \sin 2t}{|1 - \alpha e^{-2it}|^2} 1;
\]
\[
(2.18)
\]

these matrices determine the probability density,

\[
\rho(x) = \exp\{-(x - q)U^{-1}(x - q)\}
\]

and the current density

\[
j(x) = \rho(x) [p + V(x - q)]
\]

(see Ref. [3]).

The above relations describe a Gaussian wave-packet with a periodically varying variance, and a center moving along the classical trajectory of a harmonic oscillator. This trajectory is an ellipse of eccentricity

\[
\frac{2\sqrt{\alpha}}{1 + \alpha},
\]

and with semi-major and semi-minor axis determined by the prameter \( \beta \) :

\[
\frac{\beta}{1 + \alpha}.
\]

Following the formulas of Ref. [3], we can readily calculate the dispersions:

\[
\langle \Delta x^2 \rangle = \langle \Delta y^2 \rangle = \frac{1}{2} \frac{|1 - \alpha e^{-2it}|^2}{1 - \alpha^2},
\]
\[
(2.19)
\]
which show that the width of the Gaussian wave-packet oscillates sinusoidally between the values $\frac{1}{2}(1-\alpha)/(1+\alpha)$ and $\frac{1}{2}(1+\alpha)/(1-\alpha)$. Similarly,

$$\langle \Delta p_x^2 \rangle = \langle \Delta p_y^2 \rangle = \frac{1 - |\alpha|^2}{1 - \alpha^2} \left( 1 - e^{-2\alpha|t|^2} \right),$$

and

$$\langle \Delta x \Delta p_x \rangle = \langle \Delta y \Delta p_y \rangle = \frac{\alpha \sin 2t}{1 - \alpha^2}.$$  

From these formulas, we notice that:

$$\langle \Delta x^2 \rangle \langle \Delta p_x^2 \rangle - \langle \Delta x \Delta p_x \rangle^2 = \langle \Delta y^2 \rangle \langle \Delta p_y^2 \rangle - \langle \Delta y \Delta p_y \rangle^2 = \frac{1}{4},$$

which is precisely the definition of a coherent correlated state [2].

### 3 Polar operators and coherent states

In order to compare with the standard results of the two-dimensional harmonic oscillator, we consider the usual creation and annihilation operators in Cartesian coordinates:

$$a_i = \frac{1}{\sqrt{2}} \left( x_i + \frac{\partial}{\partial x_i} \right)$$

$$a_i^\dagger = \frac{1}{\sqrt{2}} \left( x_i - \frac{\partial}{\partial x_i} \right),$$

with $(x_1, x_2) = (x, y) = (r \cos \phi, r \sin \phi)$. They satisfy the commutation relations:

$$[a_i, a_j^\dagger] = \delta_{ij}.$$

Define now the operators

$$A = \frac{1}{\sqrt{2}} (a_1 + ia_2)$$

$$B = \frac{1}{\sqrt{2}} (a_1 - ia_2).$$

Their explicit forms in polar coordinates are:

$$A = \frac{1}{2} e^{i\phi} \left( r + \frac{\partial}{\partial r} + \frac{i}{r} \frac{\partial}{\partial \phi} \right)$$
\[ B = \frac{1}{2} e^{-i\phi} \left( r + \frac{\partial}{\partial r} - \frac{i}{r} \frac{\partial}{\partial \phi} \right) \]
\[ A^\dagger = \frac{1}{2} e^{-i\phi} \left( r - \frac{\partial}{\partial r} + \frac{i}{r} \frac{\partial}{\partial \phi} \right) \]
\[ B^\dagger = \frac{1}{2} e^{i\phi} \left( r - \frac{\partial}{\partial r} - \frac{i}{r} \frac{\partial}{\partial \phi} \right), \quad (3.26) \]

and they satisfy the commutation relations:
\[ [A, B] = 0 = [A, B^\dagger] \]
\[ [A, A^\dagger] = 1 = [B, B^\dagger]. \quad (3.27) \]

If we now express the wave functions in coordinate representations as in Eq. (2.13), then it is easy to see that:
\[ e^{i\phi} \left( \frac{\partial}{\partial r} + \frac{i}{r} \frac{\partial}{\partial \phi} \right) \Psi_{\alpha\beta}(r, \phi) + \alpha\beta = -1 + \beta \Psi_{\alpha\beta}(r, \phi) \quad (3.28) \]
and
\[ e^{-i\phi} \left( \frac{\partial}{\partial r} - \frac{i}{r} \frac{\partial}{\partial \phi} \right) \Psi_{\alpha\beta}^+(r, \phi) = \left[ -1 + \frac{\alpha}{\alpha - 1} e^{-i\phi} \alpha \Psi_{\alpha\beta}(r, \phi) \right] + \frac{2\beta}{\alpha - 1} \Psi_{\alpha\beta}^+(r, \phi). \quad (3.29) \]

These last two relations can be written in a coordinate independent form in terms of the operators defined in Eqs. (3.26). Thus, for the righthanded polarized states:
\[ (A + \alpha B^\dagger)\alpha\beta = 0 \quad (3.30) \]
\[ (\alpha A^\dagger + B)\alpha\beta = \beta \alpha\beta \quad (3.31) \]

From these expression it is easy to find the expectation values of the \( A \) and \( B \) operators in the righthanded states:
\[ +\langle A \rangle_+ = -\frac{\alpha^*}{1 - |\alpha|^2} \]
\[ +\langle B \rangle_+ = \frac{\beta}{1 - |\alpha|^2}. \quad (3.32) \]

In a similar way, it can be seen that for the lefthanded polarized states:
\[ (\alpha A^\dagger + B)\alpha\beta = 0 \quad (3.33) \]
\[ (A + \alpha B^\dagger)\alpha\beta = \beta \alpha\beta \quad (3.34) \]
and the expectations values in these states are:

\[
-\langle A \rangle_- = \frac{\beta}{1 - |\alpha|^2} \\
-\langle B \rangle_- = -\frac{\alpha\beta^*}{1 - |\alpha|^2}.
\]

Notice that the eigenvalue equations for the \( |\alpha\beta\rangle_- \) are the same for the \( |\alpha\beta\rangle_+ \) states if the interchange \( A \leftrightarrow B \) is made.

As for the expectation values of quadratic operators, some lengthy but straightforward calculations show that:

\[
+\langle A\dagger A \rangle_+ = \frac{|\alpha|^2|\beta|^2}{(1 - |\alpha|^2)^2},
\]

\[
+\langle B\dagger B \rangle_+ = \frac{|\beta|^2}{(1 - |\alpha|^2)^2},
\]

\[
+\langle A\dagger B \rangle_+ = -\frac{\alpha^*\beta^2}{(1 - |\alpha|^2)^2},
\]

\[
+\langle A^2 \rangle_+ = \frac{\alpha^2\beta^*\beta^2}{(1 - |\alpha|^2)^2},
\]

\[
+\langle B^2 \rangle_+ = \frac{\beta^2}{(1 - |\alpha|^2)^2},
\]

\[
+\langle AB \rangle_+ = -\frac{\alpha|\beta|^2}{(1 - |\alpha|^2)^2}.
\]

The equivalent expressions for the expectation values in the left-handed state can be obtained directly by the interchange \( A \leftrightarrow B \) combined with \( +\langle \ldots \rangle_+ \leftrightarrow -\langle \ldots \rangle_- \).

Defining the Hermitian operators \( Q_{A,B} \) and \( P_{A,B} \) as \( A, B = \frac{1}{2}(Q_{A,B} + iP_{A,B}) \), it follows from the above formulas that

\[
\pm \langle (\Delta Q_{A,B})^2 \rangle_\pm = \pm \langle (\Delta P_{A,B})^2 \rangle_\pm = 1.
\]

Therefore, the states under consideration are indeed coherent states, that is, they have minimum dispersion with respect to the \( A \) and \( B \) operators.

Finally, we recall that the operators \( a_i \) and \( a_i^\dagger \) also generate an \( U(2) \) algebra (see, e. g., [5]). In terms of the standard operators \( a_i \) and \( a_i^\dagger \) and the
new operators \(A\) and \(B\), the components of an angular momentum operator \(J\) can be defined as:

\[
J_x + iJ_y \equiv \frac{1}{2}[(a_x^\dagger + ia_y^\dagger)(a_x + i a_y)]
\]

\[
= B^\dagger A = \frac{1}{4} e^{2i\phi} \left[ r^2 + \frac{1}{r} \frac{\partial}{\partial r} - \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + 2i \left( \frac{1}{r} \frac{\partial}{\partial \phi} - \frac{1}{r} \frac{\partial^2}{\partial \phi \partial r} \right) \right], \quad (3.43)
\]

\[
J_z \equiv \frac{i}{2}(a_y^\dagger a_x - a_x^\dagger a_y)
\]

\[
= -\frac{1}{2} (AA^\dagger - BB^\dagger) = \frac{i}{2} \frac{\partial}{\partial \phi}, \quad (3.44)
\]

and the number operator is

\[
N = a_x^\dagger a_x + a_y^\dagger a_y = A^\dagger A + B^\dagger B, \quad (3.45)
\]

which is simply related to the Hamiltonian appearing in Eq. (2.1): \(H = N + 1\).

### 4 Results and conclusions

We have performed a numerical calculations in order to illustrate the evolution of the states described in this paper. In Fig. 1, we present a state corresponding to left handed polarization with particular values of the parameters \(\alpha\) and \(\beta\). As expected, the Gaussian wave packet rotates along an ellipse and changes its dispersion periodically; the form of the wave-packet is recovered after each cycle, in accordance with the results obtained by Arickx et al. [3]. A cross section of the \(|\Psi_{\alpha\beta}(r, \phi)|^2\) is shown in the figure.

An interesting feature of the states we have defined in this article is that they are coherent correlated states with respect to the cartesian position and momentum operators. However, a new set of position and momentum operators can be defined in polar coordinates, and these same states turn out to be linear combinations of the eigenstates of these operators, with the peculiarity of being truly coherent with respect to these operators, having the minimum dispersion allowed by the Heisenberg uncertainty principle.

In conclusion, the use of polar coordinates permits to see some features that are not evident in cartesian coordinates. The relation between classical and quantum motions is made explicit including the rotational degrees of freedom that correspond to different polarizations.
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Figure caption

Fig. 1- An horizontal cross section of the square modulus $|\Psi_{\alpha\beta}^{\pm}(r, \phi)|^2$ with parameters $\alpha = 0.4$ and $\beta = 5$, in two-dimensional coordinates space. The cut is at $|\Psi_{\alpha\beta}^{\pm}(r, \phi)|^2 = 0.05$. The classical trajectory, an ellipse with eccentricity $2\sqrt{\alpha}/(1+\alpha)$, is added to illustrate the corresponding classical motion.