Residuated operators and Dedekind-MacNeille completion

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Abstract The concept of operator residuation for bounded posets with unary operation was introduced by the first two authors. It turns out that in some cases when these operators are transformed into lattice terms and the poset $P$ is completed into a Dedekind-MacNeille completion $DM(P)$ then the complete lattice $DM(P)$ becomes a residuated lattice with respect to these transformed terms. It is shown that this holds in particular for Boolean posets and for relatively pseudocomplemented posets. More complicated situation is with orthomodular and pseudo-orthomodular posets. We show which operators $M$ (multiplication) and $R$ (residuation) yield operator left-residuation in a pseudo-orthomodular poset $P$ and if $DM(P)$ is an orthomodular lattice then the transformed lattice terms $\circ$ and $\to$ form a left residuation in $DM(P)$. However, it is a problem to determine when $DM(P)$ is an orthomodular lattice. We get some classes of pseudo-orthomodular posets for which their Dedekind-MacNeille completion is an orthomodular lattice and we introduce the so called strongly $D$-continuous pseudo-orthomodular posets. Finally we prove that, for a pseudo-orthomodular poset $P$, the Dedekind-MacNeille completion $DM(P)$ is an orthomodular lattice if and only if $P$ is strongly $D$-continuous.

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1 Introduction

Consider a bounded poset \( P = (P, \leq, ', 0, 1) \) with a unary operation \('\). For \( M \subseteq P \) denote by

\[ U(M) := \{ x \in P \mid y \leq x \text{ for all } y \in M \}, \]

the so-called upper cone of \( M \), and by

\[ L(M) := \{ x \in P \mid x \leq y \text{ for all } y \in M \}, \]

the so-called lower cone of \( M \). If \( M = \{a, b\} \) or \( M = \{a\} \), we will write simply \( U(a, b) \), \( L(a, b) \) or \( U(a) \), \( L(a) \), respectively.

The following concept was introduced in [2].

**Definition 1.** An operator left residuated poset is an ordered septuple \( P = (P, \leq, ', M, R, 0, 1) \) where \( (P, \leq, ', 0, 1) \) is a bounded poset with a unary operation and \( M \) and \( R \) are mappings from \( P^2 \) to \( 2^P \) satisfying the following conditions for all \( x, y, z \in P \):

\[ M(x, 1) \approx M(1, x) \approx L(x), \tag{1} \]
\[ M(x, y) \subseteq L(z) \text{ if and only if } L(x) \subseteq R(y, z), \tag{2} \]
\[ R(x, 0) \approx L(x'). \tag{3} \]

It is elementary to show that

\[ R(x, y) = P \text{ if and only if } x \leq y. \]

In what follows, we will work with posets \( P = (P, \leq, ', 0, 1) \) where \('\) is an antitone involution or a complementation. The precise definition is the following.

**Definition 2.** A poset with antitone involution is an ordered quintuple \( P = (P, \leq, ', 0, 1) \) such that \( (P, \leq, 0, 1) \) is a bounded poset and \('\) is a unary operation on \( P \) satisfying the following conditions for all \( x, y \in P \):

(i) \( x \leq y \) implies \( y' \leq x' \),
(ii) \( (x')' \approx x \).

A poset with complementation is a poset with antitone involution \( P = (P, \leq, ', 0, 1) \) satisfying the following LU-identities:

(iii) \( L(x, x') \approx \{0\} \) and \( U(x, x') \approx \{1\} \).

A subset \( S \subseteq P \) of a poset \( P \) with complementation such that \( s \leq t' \) for any pair \( s, t \in S, s \neq t \) is called orthogonal. \( P \) is said to have a finite rank if every orthogonal subset of \( P \) is finite.

A natural and interesting question is for which posets \( P = (P, \leq, ', 0, 1) \) the operators \( M \) and \( R \) can be constructed by means of the operators \( L \) and \( U \) similarly
as in (left) residuated lattices the operations $\odot$ and $\rightarrow$ can be expressed as term operations.

For reader’s convenience we recall that a lattice $L = (L, \lor, \land, 1)$ with the greatest element 1 is left residuated if there are two binary operations $\odot$ and $\rightarrow$ on $L$ such that for all $x, y, z \in L$ we have

\begin{align*}
x \odot 1 &\approx x \approx 1 \odot x, \quad (4) \\
x \odot y &\leq z \text{ if and only if } x \leq y \rightarrow z. \quad (5)
\end{align*}

In our treaty we do not ask that $\odot$ has to be associative, i.e., it need not be a t-norm. If $\odot$ is commutative then we simply say that $L$ is residuated.

It was shown by the first two authors in [3] that this is the case for Boolean algebras, orthomodular lattices and, as it is familiarly known, for relatively pseudo-complemented lattices.

For every poset $P = (P, \leq)$, its Dedekind-MacNeille completion $DM(P)$ is a complete lattice. In what follows, we say that an expression in operators $U$ and $L$ is $DM$-transformed if every expression $U(x,y)$ or $LU(x,y)$ is substituted by $x \lor y$ and every expression $L(x,y)$ is replaced by $x \land y$.

The aim of this paper is as follows. Having an operator left residuated poset $P = (P, \leq, ', M, R, 0, 1)$ we ask whether the operators $M$ and $R$ expressed in $U$ and $L$ can be $DM$-transformed such that the resulting expressions will be binary operations $\odot$ and $\rightarrow$ on $DM(P)$ satisfying (4) and (5) in the Dedekind-MacNeille completion $DM(P)$ of $P$.

### 2 Dedekind-MacNeille completion

In this section, we shall discuss several important classes of bounded posets $P$ with a unary operation which are operator residuated or operator left residuated and, moreover, the operator residuation from $P$ can be transformed into the residuation in $DM(P)$ by replacing $UL$-terms of $P$ into lattice terms of $DM(P)$.

We start with detailed definitions of these concepts.

It is well-known that every poset $(P, \leq)$ can be embedded into a complete lattice $L$. We frequently take the so-called Dedekind-MacNeille completion $DM(P, \leq)$ for this $L$.

Hence, let $P = (P, \leq)$ be a poset. Put $DM(P) := \{ B \subseteq P \mid LU(B) = B \}$. (We simply write $LU(B)$ instead of $L(U(B))$. Analogous simplifications are used in the sequel.) Then for $DM(P) = \{ L(B) \mid B \subseteq P \}$, $DM(P) := (DM(P), \subseteq)$ is a complete lattice and $x \mapsto L(x)$ is an embedding from $P$ to $DM(P)$ preserving all existing joins and meets, and an order isomorphism between posets $P$ and $(\{ L(x) \mid x \in P \}, \subseteq)$. We usually identify $P$ with $\{ L(x) \mid x \in P \}$.

For subsets $B$ and $C$ of a poset $(P, \leq)$ we will write $B \leq C$ if and only if $b \leq c$ for all $b \in B$ and $c \in C$. We write $b \leq C$ instead of $\{ b \} \leq C$ and $B \leq c$ instead of $B \leq \{ c \}$.
It is easy to see that if \( B, C \subseteq P \) such that \( B \leq C \) then \( \bigvee_{\text{DM}(P)} B = \bigwedge_{\text{DM}(P)} C \) if and only if \( \{ x \in P \mid x \leq C \} \subseteq \{ y \in P \mid B \leq y \} \).

By Schmidt [11] the Dedekind-MacNeille completion of a poset \( P \) is (up to isomorphism) any complete lattice \( L \) into which \( P \) can be supremum-densely and infimum-densely embedded (i.e., for every element \( x \in L \) there exist \( M, Q \subseteq P \) such that \( x = \bigvee \varphi(M) = \bigwedge \varphi(Q) \), where \( \varphi : P \to L \) is the embedding).

Let \( P \) be equipped with a binary operation \( * \). We introduce a new operation \( ** \) on \( \text{DM}(P) \) as follows:

\[
X ** Y := \bigcap_{a \in X, b \in U(Y)} L(a * b)
\]

for all \( X, Y \in \text{DM}(P) \).

Recall that a poset \( (P, \leq) \) is called relatively pseudocomplemented if for each \( a, b \in P \) there exists a greatest element \( c \) of \( P \) satisfying \( L(a, c) \subseteq L(b) \), see e.g. [4]. This element \( c \) is called the relative pseudocomplement of \( a \) with respect to \( b \) and it is denoted by \( a \ast b \). Every relative pseudocomplemented poset has a greatest element 1 since \( x \ast x = 1 \) for every \( x \in P \).

The following is known.

**Proposition 1.** [4, Theorem 3.1] Let \( P = (P, \leq) \) be a poset and \( * \) a binary operation on \( P \). Then the following are equivalent:

(i) \( P \) has the top element 1 and \( (P, *, 1) \) is a relatively pseudocomplemented poset;
(ii) \( (\text{DM}(P), *, P) \) is a relatively pseudocomplemented lattice satisfying the LU-identity

\[
L(x) * L(y) = L(x * y).
\]

Recall that a poset \( P \) is distributive if it satisfies one of the following equivalent identities:

\[
L(U(x,y),z) \approx LU(L(x,z),L(y,z)),
U_L(U(x,y),z) \approx UL(U(x,z),U(y,z)).
\]

A bounded poset \( P = (P, \leq, ', 0, 1) \) is called Boolean if it is a distributive poset and \( ' \) is the complementation.

**Example 1.** Fig. 1 shows two Boolean posets which are not Boolean algebras.

The following result was proved by Niederle [9].

**Proposition 2.** [9, Theorem 16] For every Boolean poset \( P = (P, \leq, ', 0, 1) \) its Dedekind-MacNeille completion \( \text{DM}(P) \) is a complete Boolean algebra.

Unfortunately, for other interesting classes of posets we do not have such a nice result. A poset with complementation \( P = (P, \leq, ', 0, 1) \) is called orthomodular if for all \( x, y \in P \) with \( x \leq y' \) there exists \( x \lor y \) and then \( P \) satisfies one of the following equivalent identities:
where $x \land y$ stands for $(x' \lor y')'$ (De Morgan laws).

It is known that for an orthomodular poset $P = (P, \leq, \lor, \land, 0, 1)$, its Dedekind-MacNeille completion $\text{DM}(P)$ need not be an orthomodular lattice.

Recall that a lattice with complementation $(L, \land, \lor, \lor', 0, 1)$ is orthomodular if and only if it satisfies the following identity [1, Theorem II.5.1]:

$$x \lor y \approx ((x \lor y) \land y') \lor y,$$

which in turn is equivalent to the following condition ([7, Chapter 1, 2. Theorem]):

if $x, y \in L$, $x \leq y$ and $x' \land y = 0$ then $x = y$.

The poset $P$ with complementation is called an orthocomplete poset if $\lor S$ exists in $P$ for every orthogonal subset $S \subseteq P$.

The poset $P$ with complementation is called a pseudo-orthomodular poset if it satisfies one of the following equivalent conditions:

$$L(U(L(x,y),y'),y) \approx L(x,y),$$
$$U(L(U(x,y),y'),y) \approx U(x,y).$$

It is worth noticing that if the previous expressions are $\text{DM}$-transformed we obtain the orthomodular law which holds in orthomodular lattices. Unfortunately, if $P = (P, \leq, \lor, 0, 1)$ is a pseudo-orthomodular poset then its Dedekind-MacNeille completion $\text{DM}(P)$ need not be an orthomodular lattice.
Of course, every Boolean poset is pseudo-orthomodular and every orthomodular lattice is a pseudo-orthomodular poset.

We can state and prove the following result.

**Theorem 1.** Let \( P = (P, \leq, ', 0, 1) \) be a Boolean poset. Take \( M(x, y) = L(x, y) \) and \( R(x, y) = L(U(x', y)) \). Then

(i) \( P \) is operator residuated with respect to \( M \) and \( R \);

(ii) \( \text{DM}(P) \) is a complete Boolean algebra which is a residuated lattice with respect to the operations \( \odot \) and \( \to \) reached by the \( \text{DM} \)-transformation from \( M \) and \( R \), respectively, i.e., \( x \odot y = x \land y \) and \( x \to y = x' \lor y \).

**Proof.** (i) is proved in [2], the first part of (ii) is shown by Proposition 2, the \( \text{DM} \)-transformation is evident and the fact that \( \text{DM}(P) \) is a residuated lattice with respect to the operations \( \odot \) and \( \to \) is well-known. \( \square \)

Similar results can be stated for relatively pseudocomplemented posets.

Recall that a lattice \( L = (L, \lor, \land) \) is relatively pseudocomplemented if for each \( a, b \in L \) there exists the greatest element of the set \( \{ x \in L \mid a \land x \leq b \} \), the so-called relative pseudocomplement of \( a \) with respect to \( b \); it is denoted by \( a \ast b \). Evidently,

\[
a \land b \leq c \text{ if and only if } a \leq b \ast c.
\]

**Theorem 2.** Let \( (P, \leq, \ast, 0, 1) \) be a relatively pseudocomplemented poset. Take \( x' = x \ast 0 \), \( M(x, y) = L(x, y) \) and \( R(x, y) = L(x \ast y) \). Then

(i) \( P = (P, \leq, ', M, R, 0, 1) \) is operator residuated;

(ii) \( \text{DM}(P) \) is a complete relatively pseudocomplemented lattice which is a residuated lattice with respect to the operations \( \odot \) and \( \to \) reached by the \( \text{DM} \)-transformation from \( M \) and \( R \), respectively, i.e., \( x \odot y = x \land y \) and \( x \to y = x' \lor y \).

We need not get a proof because every of these assertions is familiarly known. Namely,

\[
M(x, y) \subseteq L(z) \iff L(x, y) \subseteq L(z) \iff L(x) \subseteq L(y \ast z) \iff L(x) \subseteq R(y, z).
\]

It was shown by the authors in [4] that the pseudocomplementation \( \ast \) in \( \text{DM}(P) \) for elements from \( P \) is the same as in \( P \).

### 3 Completion of pseudo-orthomodular posets

As mentioned above, the lattice \( \text{DM}(P) \) for a pseudo-orthomodular poset \( P = (P, \leq, ', 0, 1) \) need not be an orthomodular lattice. It was shown in [2] that for \( M(x, y) = L(U(x', y), y) \) and \( R(x, y) = L(U(L(x, y), x')) \), \( P \) becomes an operator left residuated poset. Unfortunately, making \( \text{DM} \)-transformation of \( M \) and \( R \), the Dedekind-MacNeille completion \( \text{DM}(P) \) need not be a left residuated lattice with
respect to \( x \odot y = (x \lor y') \land y \) and \( x \to y = (x \land y) \lor x' \) despite the fact that every orthomodular lattice is left residuated with respect to these operations.

The aim of this section is to show some cases of posets \( P \) for which DM(\( P \)) is an orthomodular lattice and when DM-transformation of \( M \) and \( R \) yields operations \( \odot \) and \( \to \) such that DM(\( P \)) is a left residuated lattice.

The horizontal sum of a family of bounded posets is obtained from their disjoint union by identifying the top elements and the bottom elements, respectively. Note that a horizontal sum of a family of bounded posets with antitone involution (complementation) is a bounded poset with antitone involution (complementation), respectively.

**Proposition 3.** Let \( P = (\mathcal{P}, \leq, 0, 1) \) be a bounded poset such that \( P \) is a horizontal sum of bounded posets \( P_\alpha = (\mathcal{P}_\alpha, \leq_\alpha, 0, 1), \alpha \in \Lambda \). Then DM(\( P \)) is order-isomorphic to a horizontal sum \( Q \) of complete lattices DM(\( P_\alpha \)), \( \alpha \in \Lambda \).

**Proof.** Clearly, a horizontal sum of complete lattices is a complete lattice. Moreover, \( P = (\mathcal{P}, \leq, 0, 1) \) is both join-dense and meet-dense in \( Q \) and we have an order embedding from \( P \) into \( Q \). It follows that DM(\( P \)) is order-isomorphic to \( Q \). \( \square \)

Using this, we can prove the following result.

**Proposition 4.** Let \( P = (\mathcal{P}, \leq', 0, 1) \) be a bounded poset such that \( P \) is a horizontal sum of pseudo-orthomodular posets \( P_\alpha = (\mathcal{P}_\alpha, \leq_\alpha', 0, 1), \alpha \in \Lambda \). Then \( P \) is a pseudo-orthomodular poset.

**Proof.** If \( x \in \{0, 1\} \) or \( y \in \{0, 1\} \) then clearly \( L(U(L(x,y'), y'), y) \approx L(x,y) \). Assume that \( x, y \in \mathcal{P} \setminus \{0, 1\} \). Suppose first that \( x \in \mathcal{P}_\alpha \setminus \{0, 1\} \) and \( y \in \mathcal{P}_\beta \setminus \{0, 1\} \), \( \alpha, \beta \in \Lambda, \alpha \neq \beta \). It follows that \( L(x,y) = \{0\} \). Hence \( U(L(x,y'), y') = U(y') \) and \( L(U(y'), y) = \{0\} \), i.e., we have again \( L(U(L(x,y'), y'), y) \approx L(x,y) \). To the end, assume that \( x, y \in \mathcal{P}_\alpha \setminus \{0, 1\} \). We have \( L(x,y) = L_{\mathcal{P}_\alpha}(x,y), U(L(x,y), y') = U_{\mathcal{P}_\alpha}(L_{\mathcal{P}_\alpha}(x,y), y') \) and \( L(U(L(x,y), y'), y) = L_{\mathcal{P}_\alpha}(U_{\mathcal{P}_\alpha}(L_{\mathcal{P}_\alpha}(x,y), y'), y) \). This yields \( L(U(L(x,y'), y'), y) \approx L(x,y) \) since \( P_\alpha \) is a pseudo-orthomodular poset. \( \square \)

**Theorem 3.** Let \( P = (\mathcal{P}, \leq', 0, 1) \) be a bounded poset such that \( P \) is a horizontal sum of pseudo-orthomodular posets \( P_\alpha = (\mathcal{P}_\alpha, \leq_\alpha', 0, 1), \alpha \in \Lambda \), and any DM(\( P_\alpha \)) is a complete orthomodular lattice. Then DM(\( P \)) is a complete orthomodular lattice.

**Proof.** From Proposition 3 we know that DM(\( P \)) is order-isomorphic to a horizontal sum \( Q \) of complete lattices DM(\( P_\alpha \)). It is evident that the isomorphism preserves the antitone involution as well. Since any DM(\( P_\alpha \)) is a complete orthomodular lattice we have that DM(\( P \)) is orthomodular. \( \square \)

We obtain the following corollary of Theorem 3 and Proposition 2.

**Corollary 1.** Let \( P = (\mathcal{P}, \leq', 0, 1) \) be a bounded poset such that \( P \) is a horizontal sum of Boolean posets \( P_\alpha = (\mathcal{P}_\alpha, \leq_\alpha', 0, 1), \alpha \in \Lambda \), and \( M(x, y) = L(U(x,y'), y) \) and \( R(x, y) = LU(L(x,y), x') \). Then DM(\( P \)) is a complete orthomodular lattice. Moreover, DM(\( P \)) is a left residuated lattice with respect to \( \odot \) and \( \to \) reached by the DM-transformation from \( M \) and \( R \), respectively.
The proof of the last assertion in Corollary 1 follows from the fact that every orthomodular lattice is a left residuated lattice with respect to $x \odot y = (x \lor y') \land y$ and $x \to y = (x \land y) \lor x'$, see [3] for details.

Hence, horizontal sums of non-trivial Boolean posets form a class of pseudo-orthomodular posets which can be extended to an orthomodular lattice and the residuation of the latter can be reached by the DM-transformation.

**Example 2.** Consider the horizontal sum $P$ of the Boolean poset $P_1$ where $P_1 = \{0, a, b, c, d, e, e', b', d', 1\}$ and an four-element Boolean algebra $P_2$ where $P_2 = \{0, f, f', 1\}$ and whose Hasse diagram is depicted in Fig. 2:

![Hasse diagram](image)

According to Proposition 4 and Corollary 1, $P$ is a pseudo-orthomodular poset and $\text{DM}(P)$ is a nonmodular orthomodular lattice.

We can solve our problem also from the opposite direction. Namely, we can assume that $\text{DM}(P)$ is really an orthomodular lattice and ask what is $P$. The answer is as follows.

**Theorem 4.** Let $P = (P, \leq', 0, 1)$ be a complemented poset such that $\text{DM}(P)$ is an orthomodular lattice. Then $P$ is pseudo-orthomodular.

**Proof.** Let $\text{DM}(P)$ be an orthomodular lattice and let $x, y \in P$. We compute:

$$L(U(x, y)) = x \lor_{\text{DM}(P)} y = ((x \lor_{\text{DM}(P)} y') \land_{\text{DM}(P)} y') \lor_{\text{DM}(P)} y = LU(L(U(x, y), y'), y).$$

It follows that $U(L(U(x, y), y'), y) = U(x, y)$, i.e., $P$ is pseudo-orthomodular.

Let us note that the result of Theorem 4 justifies the concept of a pseudo-orthomodular poset. With respect to the completion into an orthomodular lattice it is more appropriate than the concept of an orthomodular poset. It will be emphasized also by Corollary 2 and Theorem 7 below.
In what follows we will show that for finite orthomodular posets $P$ such that $P$ is not a lattice their Dedekind-MacNeille completions $\text{DM}(P)$ are not orthomodular.

We will need the following definitions and theorem from Kalmbach [7] reformulated as in Svozil and Tkadlec [12].

**Definition 3.** A diagram is a pair $(V,E)$, where $V \neq \emptyset$ is a set of atoms (drawn as points) and $E \subseteq \exp V \setminus \{\emptyset\}$ is a set of blocks (drawn as line segments connecting corresponding points). A loop of order $n \geq 2$ ($n$ being a natural number) in a diagram $(V,E)$ is a sequence $(e_1, \ldots, e_n) \in E^n$ of mutually different blocks such that there are mutually distinct atoms $v_1, \ldots, v_n$ with $v_i \in e_i \cap e_{i+1}$ ($i = 1, \ldots, n$, $e_{n+1} = e_1$).

In particular, we precise it as follows (see e.g. [7]).

**Definition 4.** A Greechie diagram is a diagram satisfying the following conditions:

1. Every atom belongs to at least one block.
2. If there are at least two atoms then every block is at least 2-element.
3. Every block which intersects with another block is at least 3-element.
4. Every pair of different blocks intersects in at most one atom.
5. There is no loop of order 3.

Recall that a block in an orthomodular poset is a maximal Boolean subalgebra of it. An element $a$ of a poset $P$ with least element 0 is an atom if $0 < a$ and there is no $x \in P$ such that $0 < x < a$. A poset $P$ with a least element 0 is

1. atomic if every element $b > 0$ has an atom $a$ below it,
2. atomistic if every element is a join of atoms of $P$.

**Theorem 5.** [7] Loop Lemma] For every Greechie diagram with only finite blocks there is exactly one (up to an isomorphism) orthomodular poset such that there are one-to-one correspondences between atoms and atoms and between blocks and blocks which preserve incidence relations. The poset is a lattice if and only if the Greechie diagram has no loops of order 3.

We use the notion Greechie logic for an orthomodular poset that can be represented by a Greechie diagram with only finite edges. Recall that every element of a Greechie logic is a supremum of a finite orthogonal set of atoms and suprema (infima) of elements from a block of the Greechie logic coincide with their suprema (infima) in the whole Greechie logic, respectively.

Using this, we can construct the promised example.

**Example 3.** Let $P = (P, \leq, 0, 1)$ be the finite Greechie logic given by the Greechie diagram in Fig. 3 (see also [7] Exercise 3, page 259).

The Greechie logic $P$ has 4 blocks $B_0$, $B_1$, $B_2$ and $B_3$. The maximal respective orthogonal sets of atoms of $P$ are $\{x,y,z\}$, $\{z,t,s\}$, $\{s,u,v\}$ and $\{v,w,x\}$. Denote the set of all atoms of $P$ by $A$.

We have that $y \notin L(s',x') \cap A = \{v,z\}$. It follows that $U(L(s',x'),v) = U(L(s',x') \cap A,v) = \{1\}$. Hence $L(U(L(s',x'),v),x') = L(\{1\},x') = L(x')$. We conclude that $P$ is not pseudo-orthomodular, i.e., by Theorem $\text{DM}(P)$ is not orthomodular.
Motivated by the above example we will prove the following.

**Theorem 6.** Let $\mathbf{P} = (P, \leq, 0, 1)$ be an orthocomplete atomic orthomodular poset. The following conditions are equivalent:

(i) $\mathbf{P}$ is pseudo-orthomodular.

(ii) $\mathbf{P}$ is a complete orthomodular lattice.

(iii) $\text{DM}(\mathbf{P})$ is orthomodular.

**Proof.** (ii) $\implies$ (iii) is evident and (iii) $\implies$ (i) follows by Theorem 4.

(i) $\implies$ (ii): Let $\mathbf{P}$ be a pseudo-orthomodular poset and denote the set of all atoms of $\mathbf{P}$ by $A$. Since $\mathbf{P}$ is an orthocomplete atomic orthomodular poset it is atomistic (namely, any element $x$ of $\mathbf{P}$ is a join of a maximal orthogonal set of atoms lying under $x$). Let us show that $\mathbf{P}$ is a lattice. Assume that $v, z \in P, v, z \notin \{0, 1\}$ (the case when $v \in \{0, 1\}$ or $z \in \{0, 1\}$ is trivial) and let us prove that $v \lor z$ exists.

Suppose first that there is a maximal orthogonal set of atoms $A_1 \subseteq A$ such that $v = \bigvee A_v, z = \bigvee A_z$ and $A_v \cup A_z \subseteq A_1$. We show that $v \lor z = \bigvee (A_v \cup A_z)$. Since $\mathbf{P}$ is orthocomplete $\bigvee (A_v \cup A_z)$ exists and $v, z \leq \bigvee (A_v \cup A_z)$. Let $c \in P, v, z \leq c$. Then $A_v \leq c$ and $A_z \leq c$. We conclude that $A_v \cup A_z \leq c$ and again by orthocompleteness of $\mathbf{P}$ we have $\bigvee (A_v \cup A_z) \leq c$.

Now assume that there is no maximal orthogonal set of atoms $A_1 \subseteq A$ such that $v = \bigvee A_v, z = \bigvee A_z$ and $A_v \cup A_z \subseteq A_1$.

If $v \lor z$ exists then we are finished. Assume that $v \lor z$ does not exist. From the fact that $\mathbf{P}$ is an orthomodular poset we have $z \not\leq v'$ (equivalently, $v \not\leq z'$).

Since $v \lor z$ does not exist $v' \land z'$ does not exist as well. Hence there are two different orthogonal sets of atoms $A_\alpha$ and $A_\beta$ such that $A_\alpha$ and $A_\beta$ are maximal elements from $\{C \subseteq A \mid C \leq \{v', z'\}, C \text{ orthogonal}\}$, $A_\alpha \not\subseteq A_\beta$ and $A_\beta \not\subseteq A_\alpha$. Put...
\( s = \sqrt{A_\alpha} \) and \( x = \sqrt{A_\beta} \). Then \( s \not\leq x \), \( x \not\leq s \), \( v \leq s' \), \( v \leq x' \), \( z \leq s' \) and \( z \leq x' \). Moreover, \( x \not\leq s' \) (equivalently, \( s \not\leq x' \)).

Assume first that \( s \lor v = 1 \). Then \( s = v' \). We also have \( s \lor z \lor (s' \land z') = 1 \). It follows that \( z \lor (s' \land z') = v \), i.e., \( z \leq v \), a contradiction with \( z \not\leq v \). Hence \( s \lor v \neq 1 \), i.e., \( 0 < u = s' \land v' < 1 \). Clearly, \( u \not\leq z' \). Otherwise we would have \( v' = u \lor s \leq z' \), i.e., \( z \leq v \), a contradiction. By the same arguments we obtain that \( u \not\leq x' \). Similarly by symmetry \( 0 < w = x' \land v' < 1 \), \( w \not\leq z' \) and \( w \not\leq s' \). Then \( 0 < t = s' \land z' < 1 \), \( t \not\leq v' \) and \( t \not\leq x' \) and \( 0 < y = x' \land z' < 1 \), \( y \not\leq v' \) and \( y \not\leq s' \). Hence we obtain the same picture as in Fig. 3 (although the elements need not be atoms).

Let \( c \in L(u,t) \) be an atom. Then \( c \leq u \land s' \land v' \leq s', c \leq v' \), \( c \leq z' \) and \( c \not\leq s \). Hence \( A_\alpha \cup \{ c \} \in \{ C \subseteq A \mid C \leq \{ v', z' \}, C \text{ orthogonal} \} \), a contradiction with the maximality of \( A_\alpha \). We have that \( u \land t = 0 \). Similarly, \( w \land y = 0 \).

We assert that \( P \) is not pseudo-orthomodular. The reason is: \( v,z \in L(x', s') \) and \( u \not\in L(x', s') \). Let \( q \) be any upper bound of the set \( \{ v, z, s \} \). It follows that \( q \geq v \land s = u' \) and \( q \geq z \lor s = t' \). Hence \( q' \in L(u,t) \) \( = \{ 0 \} \), i.e., \( q = 1 \). Then it is easy to see that \( U(L(x', s'), s) = \{ 1 \} \). This shows that \( u \in L(U(L(x', s'), s'), s') = L(\{ 1 \}, s') = L(s') \), contradicting our assumptions.

Therefore every two elements of \( P \) have a join and \( P \) is a complete orthomodular lattice.

As our final result on orthomodular posets we show that even for a finite orthomodular poset \( P \) its Dedekind-MacNeille completion \( DM(P) \) is not orthomodular. This disqualifies these posets for operator left residuation.

**Corollary 2.** Let \( P = (P, \leq', 0, 1) \) be a finite orthomodular poset which is not a lattice. Then its Dedekind-MacNeille completion \( DM(P) \) is not orthomodular.

**Proof.** Assume that \( DM(P) \) is orthomodular. From Theorem 4 we have that \( P \) is pseudo-orthomodular. From Theorem 6 we obtain that \( P \) is a lattice, a contradiction. 

**Corollary 3.** Any non-lattice Greechie logic does not possess an orthomodular Dedekind-MacNeille completion.

**Proposition 5.** Let \( P = (P, \leq', 0, 1) \) be an atomic pseudo-orthomodular poset. Then any element of \( P \) is a join of an orthogonal set of atoms lying under it and \( P \) is an atomistic poset.

**Proof.** Assume that \( x \in P \) and let \( A_x \) be a maximal orthogonal set of atoms under \( x \). Clearly, \( x \in U(A_x) \). Let \( y \in U(A_x) \). We have to show that \( x \leq y \). Evidently, \( A_x \subseteq L(x, y) \). We conclude that \( U(A_x, x') = \{ 1 \} \). Namely, let \( q \in U(A_x, x') \), \( q \neq 1 \). Then there is an atom \( a \in P \) such that \( a' \in U(A_x, x') \), \( a' \geq q \). Consequently, \( a \leq x \) and \( a \leq b' \) for all \( b \in A_x \), a contradiction with the maximality of \( A_x \).

We conclude that \( U(L(x, y), x') = \{ 1 \} \), hence \( L(x, y) = L(U(L(x, y), x'), x) = L(\{ 1 \}, x) = L(x) \), i.e., \( x \leq y \).
Remark 1. Recall that Finch \[5\] Proposition (3.2)] has shown, for a complemented poset \(P\), that its Dedekind-MacNeille completion \(\text{DM}(P)\) is orthomodular if and only if for any non-empty subset \(X\) of \(P\) and any maximal orthogonal subset \(S\) of \(LU(X)\) one has \(LU(S) = LU(X)\).

In Corollary 2 we proved that no finite non-lattice orthomodular poset has an orthomodular Dedekind-MacNeille completion. This is the reason why we have to modify the definition of orthomodularity in posets to obtain a more favorable result. It turns out that our concept of a pseudo-orthomodular poset can serve for this reason. Hence, we prove the following.

**Theorem 7.** Let \(P = (P, \leq, 0, 1)\) be an atomic pseudo-orthomodular poset with finite rank. Then \(\text{DM}(P)\) is orthomodular.

**Proof.** By Remark 1 it is enough to check that for any non-empty subset \(X\) of \(P\) and any maximal orthogonal subset \(S\) of \(LU(X)\) one has \(LU(S) = LU(X)\).

Assume that \(X \subseteq P, X \neq \emptyset\) and \(S \subseteq LU(X)\), \(S\) maximal orthogonal. Since \(P\) has finite rank, \(S\) is finite: let \(S = \{s_1, \ldots, s_k\}\). Put \(d_S = \bigvee_{\text{DM}(P)} S\). Then \(d_S \leq LU(X)\). If \(d_S = LU(X)\) we are done. Suppose that \(d_S < LU(X)\). From Proposition 5 we know that \(P\) is atomistic. Since any element of \(\text{DM}(P)\) is a join of elements of \(P\) also \(\text{DM}(P)\) is atomistic with the same set of atoms. We conclude that there is an atom \(a \in P\) such that \(a \not\leq d_S\) and \(a \leq LU(X)\).

We put

\[
l_S = \max\{j \in \{2, \ldots, k\} \mid a \in P \text{ is an atom}, a \not\leq d_S, a \leq LU(X), a \leq s'_1, \ldots, a \leq s'_{j-1}, a \not\leq s'_j\}.
\]

Note that \(l_S\) is correctly defined since by maximality of \(S\) there is no atom \(a\) such that \(a \leq LU(X)\) and \(a \leq l'_1, \ldots, a \leq l'_k\). Let \(a \in P\) be an atom of \(P\) such that \(a \not\leq d_S\), \(a \leq LU(X)\), \(a \leq s'_1, \ldots, a \leq s'_{k-1}, a \not\leq s'_k\).

We have \(LU(a, s_k) = LU(L(U(a, s_k), s'_k), s_k) \leq L(s'_1, \ldots, s'_{k-1})\) since \(P\) is pseudo-orthomodular and both \(a\) and \(s_k\) are in \(L(s'_1, \ldots, s'_{k-1})\). Moreover, \(LU(a, s_k) \leq LU(X)\) and \(LU(a, s_k) \not\leq d_S\) since \(a, s_k \leq LU(X)\) and \(a \not\leq d_S\). We conclude that there is an atom \(b\) of \(P\) such that \(b \in L(U(a, s_k), s'_k), b \not\leq d_S, b \leq LU(X), b \leq s'_1, \ldots, b \leq s'_{k-1}, b \not\leq s'_k\), a contradiction with the maximality of \(l_S\). Hence \(d_S = LU(X)\) and \(\text{DM}(P)\) is orthomodular. \(\Box\)

Getting together the previous results we can formulate the following corollary which is a full analogy for finite pseudo-orthomodular posets to the results on Boolean or relatively pseudo-complemented posets as stated in Theorem 1 or Theorem 2, respectively. Hence, we conclude

**Corollary 4.** Let \(P = (P, \leq, 0, 1)\) be a finite pseudo-orthomodular poset, \(M(x, y) = L(U(x, y'), y)\) and \(R(x, y) = LU(L(x, y), x')\). Then \(\text{DM}(P)\) is a complete orthomodular lattice. Moreover, \(\text{DM}(P)\) is a left residuated lattice with respect to \(\odot\) and \(\rightarrow\) reached by the \(\text{DM}\)-transformation from \(M\) and \(R\), respectively.
The next definition and theorem are suggested by a similar result of Niederle for Boolean posets ([9, Theorem 17]).

**Definition 5.** Let $P = (P, \leq, ', 0, 1)$ be a complemented poset. A subset $X$ of $P$ is **complement-closed and doubly dense in $P$** if the following conditions are satisfied:

1. $(\forall a \in P)(a = \bigvee_{P}(L(a) \cap X) = \bigwedge_{P}(U(a) \cap X)$,
2. $x \in X \implies x' \in X$,
3. $0, 1 \in X$.

**Remark 2.** Recall that any complement-closed and doubly dense subset $X$ in $P$ is a complemented poset with induced order and complementation. Moreover, if $P = (P, \leq, ', 0, 1)$ is a complemented poset then $P$ is a complement-closed and doubly dense subset in its Dedekind-MacNeille completion $DM(P)$. This can be shown by the same arguments as in ([9, Theorem 16]) or can be directly deduced from ([8, Theorem 2.5]) so we omit it.

**Theorem 8.** Embedding theorem for finite pseudo-orthomodular posets.

**Finite pseudo-orthomodular posets are precisely complement-closed and doubly dense subsets of finite orthomodular lattices.**

**Proof.** We have just proved in Corollary 4 that every finite pseudo-orthomodular posets has a finite orthomodular Dedekind-MacNeille completion. Hence it is a complemented closed and doubly dense subset of a finite orthomodular lattice. Conversely, let $P = (P, \leq, ', 0, 1)$ be a complemented closed and doubly dense subset of a finite orthomodular lattice $(L, \land, \lor, ', 0, 1)$. Then $P$ is a complete complemented poset. Let us show that $P$ is pseudo-orthomodular. Let $x, y \in P$. We can proceed similarly as in Theorem 1.1. Let $a \in P$. We have:

$$a \in U(x, y) \iff x, y \leq a \iff x \lor_{L} y \leq a \iff ((x \lor_{L} y) \land_{L} y') \lor_{L} y \leq a$$

$$\implies ((x \lor_{L} y) \land_{L} y') \leq a \land y \leq a$$

$$\implies (\forall z \in P)(z \leq x \lor_{L} y \land z \leq y') \implies z \leq a \land a \in U(y)$$

$$\implies (\forall z \in P)(z \leq U(x, y) \land z \in L(y')) \implies z \leq a \land a \in U(y)$$

$$\implies (\forall z \in P)(z \in L(U(x, y), y')) \implies z \leq a \land a \in U(y)$$

$$\implies a \in U(L(U(x, y), y'), y) \land a \in U(y)$$

$$\implies a \in U(L(U(x, y), y'), y).$$

We conclude that $U(L(U(x, y), y'), y) = U(x, y)$, i.e., $P$ is pseudo-orthomodular. □

Motivated by a paper [10] we introduce the following definition.

**Definition 6.** Let $P = (P, \leq, ', 0, 1)$ be a complemented poset. Then $P$ is called **strongly D-continuous** if and only if for all $B, C \subseteq P$ with $B \leq C$ the following condition is satisfied:

**\((SDC) \bigwedge_{P}\{g \in P \mid g \in C \text{ or } g' \in B\} = 0\) if and only if every lower bound of$C$ is under every upper bound of $B$.**
Remark 3. Recall that the implication:

If \( B, C \subseteq P \) for a complemented poset \( P \) are such that \( B \leq C \) then \( \{ a \in P \mid a \leq C \} \leq \{ d \in P \mid B \leq d \} \) implies that \( \bigwedge_P \{ g \in P \mid g \in C \text{ or } g' \in B \} = 0 \)

from the condition (SDC) is valid in any complemented poset \( P \) since it follows from the fact that the Dedekind-MacNeille completion of a complemented poset is always complemented (see \([8, \text{Theorem 2.3.}, \text{Theorem 2.4.}]\)). This fact was explained and used for Boolean posets in \([6]\).

In the following, we establish a characterization of complemented posets with orthomodular Dedekind-MacNeille completion.

**Theorem 9.** Let \( P = (P, \leq, ', 0, 1) \) be a complemented poset. \( P \) has an orthomodular Dedekind-MacNeille completion if and only if \( P \) is a strongly \( D \)-continuous pseudo-orthomodular poset.

**Proof.** (1) Since \( \text{DM}(P) \) is a complemented lattice it is enough to check the following condition:

if \( X, Y \in \text{DM}(P), X \subseteq Y \) and \( X' \wedge Y = 0 \) then \( X = Y \).

We put \( B = X \) and \( C = U(Y) \). Then \( B \leq C \) and \( \bigwedge_P \{ d \in P \mid d \in C \text{ or } d' \in B \} = 0 \).

We conclude from (SDC) that \( X = \bigvee_{\text{DM}(P)} B = \bigwedge_{\text{DM}(P)} C = Y \). Hence \( \text{DM}(P) \) is orthomodular.

(2) Let \( (\text{DM}(P), \wedge, \vee, ', 0, 1) \) be the orthomodular Dedekind-MacNeille completion of \( P \). It is enough to verify the following implication:

if \( B, C \subseteq P \) are such that \( B \leq C \) then \( \bigwedge_P \{ g \in P \mid g \in C \text{ or } g' \in B \} = 0 \) implies that \( \{ a \in P \mid a \leq C \} \leq \{ d \in P \mid B \leq d \} \)

from the condition (SDC). Let \( X = \bigvee_{\text{DM}(P)} B \) and \( Y = \bigwedge_{\text{DM}(P)} C \). Then \( X \subseteq Y \) and \( X' \wedge Y = 0 \). Since \( \text{DM}(P) \) is orthomodular we obtain that \( X = Y \). We conclude that \( \{ a \in P \mid a \leq C \} \leq \{ d \in P \mid B \leq d \} \), i.e., \( P \) is strongly \( D \)-continuous. \( \Box \)

**Corollary 5.** Every complemented strongly \( D \)-continuous poset is pseudo-orthomodular. Every finite pseudo-orthomodular poset is strongly \( D \)-continuous.

Similarly as for finite pseudo-orthomodular posets we have the following theorem.

**Theorem 10.** Embedding theorem for strongly \( D \)-continuous pseudo-orthomodular posets. Strongly \( D \)-continuous pseudo-orthomodular posets are precisely complement-closed and doubly dense subsets of complete orthomodular lattices.

**Proof.** From Theorem 9 we know that every strongly \( D \)-continuous pseudo-orthomodular poset is a complement-closed and doubly dense subset in its orthomodular Dedekind-MacNeille completion. Conversely, let \( P = (P, \leq, ', 0, 1) \) be a complement-closed and doubly dense subset of a complete orthomodular lattice \( (L, \wedge, \vee, ', 0, 1) \).

Then \( P \) is a complemented poset. Let us show that \( P \) is strongly \( D \)-continuous. As in Theorem 9 it is enough to verify the following implication:
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if \( B, C \subseteq P \) are such that \( B \leq C \) then \( \bigwedge_P \{ g \in P \mid g \in C \text{ or } g' \in B \} = 0 \) implies that \( \{ a \in P \mid a \leq C \} \leq \{ d \in P \mid B \leq d \} \)

from the condition (SDC). Let \( X = \bigvee_L B \) and \( Y = \bigwedge_L C \). Then \( X \leq Y \) and \( X' \wedge Y = 0 \) (since \( u \in P, u \leq X' \wedge Y \) implies \( u \leq g \) for all \( g \in P \) such that \( g \in C \) or \( g' \in B \), i.e., \( u = 0 \)). Since \( L \) is orthomodular we obtain that \( X = Y \). Now, let \( a \in P, a \leq C \) and \( d \in P, B \leq d \). Then \( a \leq Y = X \leq d \), i.e., \( P \) is strongly \( D \)-continuous and from Corollary we have that is also pseudo-orthomodular.

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