Gorenstein dimensions in trivial ring extensions

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Abstract. In this paper, we show that the Gorenstein global dimension of trivial ring extensions is often infinite. Also we study the transfer of Gorenstein properties between a ring and its trivial ring extensions. We conclude with an example showing that, in general, the transfer of the notion of Gorenstein projective module does not carry up to pullback constructions.

Key Words. (Gorenstein) projective dimension, (Gorenstein) injective dimension, (Gorenstein) flat dimension, trivial ring extension, global dimension, weak global dimension, quasi-Frobenius ring, perfect ring

1 Introduction

Throughout this work, all rings are commutative with identity element and all modules are unital. Let $R$ be a ring and $M$ an $R$-module. We use $\text{pd}_R(M)$, $\text{id}_R(M)$ and $\text{fd}_R(M)$ to denote the usual projective, injective and flat dimensions of M, respectively. It is convenient to use “local” to refer to (not necessarily Noetherian) rings with a unique maximal ideal.

In 1967-69, Auslander and Bridger [1,2] introduced the concept of G-dimension for finitely generated modules over Noetherian rings. Several decades later, Enochs, Jenda and Torrecillas [10,11,12] extended this notion by introducing three homological dimensions called Gorenstein projective, injective, and flat dimensions, which...
have all been studied extensively by their founders and also by Avramov, Christensen, Foxby, Frankild, Holm, Martsinkovsky, and Xu among others [3, 8, 9, 14, 16, 22]. For a ring $R$, the Gorenstein projective, injective and flat dimension of an $R$-module $M$ denoted $\text{Gpd}_R(M)$, $\text{Gid}_R(M)$ and $\text{Gfd}_R(M)$, respectively, is defined in terms of resolutions of Gorenstein projective, injective and flat modules, respectively (see [16]). The Gorenstein projective dimension is a refinement of projective dimension to the effect that $\text{Gpd}_R(M) \leq \text{pd}_R(M)$ and equality holds when $\text{pd}_R(M)$ is finite.

Recently, in [5], the authors introduced three classes of modules called strongly Gorenstein projective, injective and flat modules. These modules allowed for nice characterizations of Gorenstein projective and injective modules [5, Theorem 2.7], similar to the characterization of projective modules via the free modules. In [6], the authors started the study of Gorenstein homological dimensions of a ring $R$; namely, the Gorenstein global dimension of $R$, denoted $G-\text{gldim}(R)$, and the Gorenstein weak (global) dimension of $R$, denoted $G-\text{wdim}(R)$, and defined as follows: $G-\text{gldim}(R) = \sup\{\text{Gpd}_R(M) \mid M \text{ an } R\text{-module}\}$ [6, Theorem 3.2] and $G-\text{wdim}(R) = \sup\{\text{Gfd}_R(M) \mid M \text{ an } R\text{-module}\}$. They proved that, for any ring $R$, $G-\text{wdim}(R) \leq G-\text{gldim}(R)$ [6, Theorems 4.2] and that the Gorenstein weak and global dimensions are refinements of the classical ones, i.e., $G-\text{gldim}(R) \leq \text{gldim}(R)$ and $G-\text{wdim}(R) \leq \text{w.gl.dim}(R)$ with equality holding if the weak global dimension of $R$ is finite [6, Propositions 3.11 and 4.5].

This paper studies the Gorenstein dimensions in trivial ring extensions. Let $A$ be a ring and $E$ an $A$-module. The trivial ring extension of $A$ by $E$ is the ring $R := A \ltimes E$ whose underlying group is $A \times E$ with multiplication given by $(a, e)(a', e') = (aa', ae' + a'e)$ [17, 18]. Specifically, we investigate the possible transfer of Gorenstein properties between a ring $A$ and its trivial ring extensions. Section 2 deals with the descent and ascent of the (strongly) Gorenstein properties between $A$-modules and $R$-modules, where $R$ is a trivial ring extension of $A$ (Theorem 2.1, Corollary 2.3 and Proposition 2.4). The last part of this section is dedicated to the Gorenstein global dimension (Theorem 2.5). In Section 3, we compute $G-\text{gldim}(A \ltimes E)$ when $(A, m)$ is a local ring with $mE = 0$ (Theorem 3.1) as well as $G-\text{gldim}(D \ltimes E)$ when $D$ is an integral domain and $E$ is an qf($D$)-vector space (Theorem 3.5). The last theorem gives rise to an example showing that, in general, the notion of Gorenstein projective module does not carry up to pullback constructions (Example 3.10).
2 Transfer of Gorenstein properties to trivial ring extensions

Throughout this section, we adopt the following notation: $A$ is a ring, $E$ an $A$-module and $R = A \ltimes E$, the trivial ring extension of $A$ by $E$. We study the transfer of (strongly) Gorenstein projective and injective notions between $A$ and $R$. We start this section with the following theorem which handles the transfer of strongly Gorenstein properties between $A$-modules and $R$-modules.

Theorem 2.1 Let $M$ be an $A$-module. Then:

1. (a) Suppose that $\text{pd}_A(E) < \infty$. If $M$ is a strongly Gorenstein projective $A$-module, then $M \otimes_A R$ is a strongly Gorenstein projective $R$-module.

(b) Conversely, suppose that $E$ is a flat $A$-module. If $M \otimes_A R$ is a strongly Gorenstein projective $R$-module, then $M$ is a strongly Gorenstein projective $A$-module.

2. Suppose that $\text{Ext}^p_A(R, M) = 0$ for all $p \geq 1$ and $\text{fd}_A(R) < \infty$. If $M$ is a strongly Gorenstein injective $A$-module, then $\text{Hom}_A(R, M)$ is a strongly Gorenstein injective $R$-module.

Proof. (1) (a) Suppose that $M$ is a strongly Gorenstein projective $A$-module. Then there is an exact sequence of $A$-modules:

$$0 \to M \to P \to M \to 0 \quad (\ast)$$

where $P$ is projective [5, Proposition 2.9]. It is known that $R = A \oplus_A E$ and since $\text{pd}_A(E) < \infty$ we have $\text{pd}_A(R) < \infty$ and from the exact sequence ($\ast$), $\text{Tor}_i^A(M, R) = 0, \forall i \geq 1$. Then the sequence $0 \to M \otimes_A R \to P \otimes_A R \to M \otimes_A R \to 0$ is exact. Note that $P \otimes_A R$ is a projective $R$-module. On the other hand, for any $R$-module projective $Q$, $\text{pd}_A(Q) < \infty$ [7, Exercise 5, page 360]. Then, since $M$ is strongly Gorenstein projective, $\text{Ext}_R(M \otimes_A R, Q) = \text{Ext}_A(M, Q) = 0$ [7, page 118]. Therefore $M \otimes_A R$ is a strongly Gorenstein projective $R$-module [5, Proposition 2.9].

(b) If $E$ is a flat $A$-module, then $R = A \ltimes E$ is a faithfully flat $A$-module. Suppose that $M \otimes_A R$ is strongly Gorenstein projective; combining [5, Remark 2.8] and [5, Proposition 2.9], there is an exact sequence of $R$-modules:

$$0 \to M \otimes_A R \to F \to M \otimes_A R \to 0 \quad (***)$$

where $F = R^{(J)}$ is a free $R$-module. Then the sequence (***$)$ is equivalent to the exact sequence:

$$0 \to M \otimes_A R \to A^{(J)} \otimes_A R \to M \otimes_A R \to 0.$$
Since $R$ is a faithfully flat $A$-module, the sequence of $A$-module $0 \to M \to A^{(J)} \to M \to 0$ is exact. On the other hand, let $P$ be a projective $A$-module. Then $P \otimes_A R$ is a projective $R$-module and $\text{Ext}_A^k(M, P \otimes_A R) = \text{Ext}_R^k(M \otimes_A R, P \otimes_A R) = 0$, since $\text{Tor}_A^i(M, R) = 0$ and by [7, Proposition 4.1.3, page 118]. But $0 = \text{Ext}_A^k(M, P \otimes_A R) \cong \text{Ext}_A^k(M, P) \otimes_A \text{Ext}_A^k(M, P \otimes_A E)$, then $\text{Ext}_A^k(M, P) = 0$. Therefore $M$ is a strongly Gorenstein projective $A$-module.

(2) If $M$ is a strongly Gorenstein injective $A$-module, there exists an exact sequence of $A$-modules:

$$0 \to M \to I \to M \to 0$$

where $I$ is an injective $A$-module. Since $\text{Ext}_A(R, M) = 0$, the sequence

$$0 \to \text{Hom}_A(R, M) \to \text{Hom}_A(R, I) \to \text{Hom}_A(R, M) \to 0$$

is exact. Note that $\text{Hom}_A(R, I)$ is an injective $R$-module. On the other hand, for any injective $R$-module $J$, we have $\text{id}_A(J) < \infty$ (since $\text{fd}_A(R) < \infty$ and by [7, Exercise 5, page 360]) and $\text{Ext}_R^i(J, \text{Hom}_A(R, M)) \cong \text{Ext}_A^i(J, M) = 0$ [7, Proposition 4.1.4, page 118]. Therefore $\text{Hom}_A(R, M)$ is a strongly Gorenstein injective $R$-module.

**Remark 2.2** The statements (1)(a) and (b) in Theorem 2.1 hold for any homomorphism from $A$ to $R$ of finite projective dimension in (a) and faithfully flat in (b), respectively. But here we restrain our study to trivial ring extensions.

**Corollary 2.3** Let $M$ be an $A$-module. Then:

1. Suppose that $\text{pd}_A(E) < \infty$. If $M$ is a Gorenstein projective $A$-module, then $M \otimes_A R$ is a Gorenstein projective $R$-module.

2. Suppose that $\text{Ext}_A^p(R, M) = 0$ for all $p \geq 1$ and $\text{fd}_A(R) < \infty$. If $M$ is a Gorenstein injective $A$-module, then $\text{Hom}_A(R, M)$ is a Gorenstein injective $R$-module.

Next we compare the Gorenstein projective (resp., injective) dimension of an $A$-module $M$ and the Gorenstein projective (resp., injective) dimension of $M \otimes_A R$ (resp., $\text{Hom}_A(R, M)$) as an $R$-module.

**Proposition 2.4** Let $M$ be an $A$-module. Then:

1. Suppose that $\text{Tor}_A^k(M, R) = 0$, $\forall \ k \geq 1$. Then:

   $$\text{Gpd}_A(M) \leq \text{Gpd}_R(M \otimes_A R).$$
2. Suppose that $\text{Ext}^k_A(R, M) = 0, \forall k \geq 1$. Then:

$$\text{Gid}_A(M) \leq \text{Gid}_R(\text{Hom}_A(R, M)).$$

Proof.

(1) By hypothesis $\text{Tor}^k_A(M, R) = 0$ for all $k \geq 1$. So, by [7, Proposition 4.1.3, page 118], for any $A$-module $P$ and all $n \geq 1$ we have

$$\text{Ext}^k_A(M, P \otimes_A R) \cong \text{Ext}^k_R(M \otimes_A R, P \otimes_A R).$$

Suppose that $\text{Gpd}_R(M \otimes_A R) \leq d$ for some integer $d \geq 0$. Let $P$ be a projective $A$-module. Then by [16, Theorem 2.20], $0 = \text{Ext}^d_R(M \otimes_A R, P \otimes_A R) \cong \text{Ext}^{d+1}_A(M, P \otimes_A R)$. But $0 = \text{Ext}^{d+1}_A(M, P \otimes_A R) \cong \text{Ext}^{d+1}_A(M, P) \oplus \text{Ext}^{d+1}_A(M, P \otimes_A E)$. So $\text{Ext}^{d+1}_A(M, P) = 0$ for any projective $A$-module $P$. Therefore $\text{Gpd}_A(M) \leq d$.

(2) The proof is essentially dual to (1). Here we use [7, Proposition 4.1.4, page 118] instead of [7, Proposition 4.1.3, page 118].

The following theorem gives a relation between $\text{G-\text{gldim}}(A)$ and $\text{G-\text{gldim}}(R)$.

**Theorem 2.5** Suppose that $\text{G-\text{gldim}}(A)$ is finite and $\text{fd}_A(E) = r$, for some integer $r \geq 0$. Then:

$$\text{G-\text{gldim}}(A) \leq \text{G-\text{gldim}}(R) + r.$$ 

Proof.

Let $M$ be an $A$-module and let

$$P_r \xrightarrow{f_r} P_{r-1} \xrightarrow{f_{r-1}} \ldots \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0 \quad (i)$$

be an exact sequence of $A$-modules where each $P_i$ is projective. Since $R = A \oplus A E$ as an $A$-modules, $\text{fd}_A(E) = \text{fd}_A(R) = r$. Then for all $k \geq 1$ we have

$$\text{Tor}^k_A(Im f_r, R) \cong \text{Tor}^{k+r}_A(M, R) = 0 \quad (ii)$$

If $\text{G-\text{gldim}}(R) \leq n$, then $\text{Gpd}_R(Im f_r \otimes_A R) \leq n$, and by Proposition 2.4 we have $\text{Gpd}_A(Im f_r) \leq n$. From (i), we have $0 = \text{Ext}^{r+n+1}_A(Im f_r, P) \cong \text{Ext}^{r+n+1}_A(M, P)$, for every projective $A$-module $P$. Therefore $\text{Gpd}_A(M) \leq n + r$ and so $\text{G-\text{gldim}}(A) \leq n + r$ [6, Lemma 3.3].
3 Gorenstein global dimension of some trivial ring extensions

In this section, we study the Gorenstein global dimension of particular trivial ring extensions. We start by investigating the Gorenstein global dimension of $R = A \ltimes E$, where $(A, m)$ is a local ring with maximal ideal $m$ and $E$ is an $A$-module such that $mE = 0$. Recall that a Noetherian ring $R$ is quasi-Frobenius if $\id_R(R) = 0$ and a ring $R$ is perfect if all flat $R$-modules are projective [21].

Next we announce the first main result of this section.

**Theorem 3.1** Let $(A, m)$ be a local ring with maximal ideal $m$ and $E$ an $A$-module such that $mE = 0$. Let $R = A \ltimes E$. Then:

1. If $A$ is a Noetherian ring which is not a field and $E$ is a finitely generated $A$-module (i.e., $R$ is Noetherian), then $G\text{-gldim}(R) = \infty$.

2. If $A$ is a perfect ring, then $G\text{-gldim}(R)$ is either $\infty$ or $0$. Moreover, in the case $G\text{-gldim}(R) = 0$, necessarily $A = K$ is a field and $E$ is a $K$-vector space with $\dim_K E = 1$ (i.e., $R = K \ltimes K$).

To prove this thm, we need the following Lemmas.

**Lemma 3.2 ([6, Lemma 3.4])** Let $R$ be a ring with $G\text{-gldim}(R) < \infty$ and let $n \in \mathbb{N}$. Then the following statements are equivalent:

1. $G\text{-gldim}(R) \leq n$;

2. $\text{pd}_R(I) \leq n$, for all injective $R$-modules $I$.

The next Lemma gives a characterization of quasi-Frobenius rings.

**Lemma 3.3 ([20, Theorem 1.50])** For a ring $R$, the following statements are equivalent:

1. $R$ is quasi-Frobenius;

2. $R$ is Noetherian and $\Ann_R(\Ann_R(I)) = I$ for any ideal $I$ of $R$, where $\Ann_R(I)$ denotes the annihilator of $I$ in $R$. 


Recall that the finitistic Gorenstein projective dimension of a ring $R$, denoted by $FPD(R)$, is defined in [16] as follows:

$$FPD(R) := \{ \text{Gpd}_R(M) \mid M \text{ R-module and } \text{Gpd}_R(M) < \infty \}.$$ 

**Proof.** [Proof of Theorem 3.1] (1) Suppose that $G\text{-gldim}(R) = n < \infty$ for some positive integer $n$. If $n \geq 1$, let $I$ be an injective $R$-module. By [6, Lemma 3.4], $pd_R(I) \leq n$. Then there is an exact sequence of $R$-modules

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow I \longrightarrow 0$$

with $P_i$ projective and hence free ($R$ is local). Since $A$ is local and $mE = 0$, every finitely generated ideal of $R$ has a nonzero annihilator. From [15, Corollary 3.3.18], $\text{coker}(P_n \longrightarrow P_{n-1})$ is flat. Then $f\text{d}_R(I) \leq (n - 1)$. Therefore from [6, Theorem 4.11] and [16, Theorem 3.14] we obtain

$$G\text{-wdim}(R) \leq n - 1 = G\text{-gldim}(R) - 1 \quad (\star)$$

On the other hand, $R$ is Noetherian by ([13, Theorem 25.1]), and from [6, Corollary 2.3] we get

$$G\text{-wdim}(R) = G\text{-gldim}(R). \quad (\star\star)$$

So from $(\star)$ and $(\star\star)$ we conclude that $G\text{-gldim}(R) = \infty$.

Now if $G\text{-gldim}(R) = 0$, then $R$ is quasi-Frobenius. First we claim that $A$ is a quasi-Frobenius ring. Since $A$ is Noetherian and by Lemma 3.3 we must prove only that $\text{Ann}_A(\text{Ann}_A(I)) = I$ for any ideal $I$ of $A$. Let $I$ be an ideal of $A$. Since $R$ is quasi-Frobenius it is easy to see that $\text{Ann}_R(\text{Ann}_R(I \times E)) = \text{Ann}_A(\text{Ann}_A(I)) \times E = I \times E$. Hence $I = \text{Ann}_A(\text{Ann}_A(I))$ and $A$ is quasi-Frobenius; thus $G\text{-gldim}(A) = 0$. On the other hand, since $R$ is quasi-Frobenius, $R$ is self-injective. Then $\text{Ext}_R^i(A, R) = 0$ for any integer $i \geq 1$ and so $\text{id}_A(m \oplus A) = \text{id}_R(R) = 0$ by [13, Lemma 4.35]. Hence, $m \oplus A$ is a projective $A$-module by Lemma 3.2 in particular $E$ is a projective $A$-module and so $E$ is free since $A$ is local. Contradiction since $mE = 0$ and $m \neq 0$. Therefore, we conclude that $G\text{-gldim}(R) = \infty$.

(2) First, suppose that $G\text{-gldim}(R) < \infty$. Note that since $A$ is perfect, $R$ is perfect too by [13, Proposition 1.15]. Combining [4, cor 7.12] and [16, Theorem 2.28] we conclude that $FGPD(R) = FPD(R) = 0$ and so $G\text{-gldim}(R) = FGPD(R) = 0$. Then from Lemma 3.2 and [20, Theorem 7.56] $R$ is quasi-Frobenius. In particular $R$ is Noetherian and by (1) $A = K$ is a field. Now we claim that $\text{dim}_K E = 1$. Assume that $\text{dim}_K E \geq 2$ and let $E' \subseteq E$ be a proper submodule of $E$. Obviously $0 \times E \subseteq \text{Ann}_R(\text{Ann}_R(0 \times E')) \neq 0 \times E'$, this is a contradiction since $R$ is quasi-Frobenius and by Lemma 3.3. Therefore $\text{dim}_K E = 1$ and $E \cong K$. Then $R = K \times K$. 


Example 3.4 Let $K$ be a field, $X_1, X_2, ..., X_n$ indeterminates over $K$, $A = K[[X_1, ..., X_n]]$, the power series ring in $n$ variables over $K$, and $R := A \times K$. Then, $G\text{-gldim}(R) = \infty$.

Next we announce the second main thm of this section.

Theorem 3.5 Let $D$ be an integral domain which is not a field, $K$ its quotient field, $E$ a $K$-vector space, and $R := D \times E$. Then $G\text{-gldim}(R) = \infty$.

To prove this thm we need the following Lemmas.

Lemma 3.6 ([6, Remarks 3.10]) For a ring $R$, if $G\text{-gldim}(R)$ is finite, then

$$G\text{-gldim}(R) = \sup \{ \text{Gpd}_R(R/I) \mid I \text{ ideal of } R \} = \sup \{ \text{Gpd}_R(M) \mid M \text{ finitely generated } R\text{-module} \}.$$ 

Lemma 3.7 ([20, Corollary 1.38]) Let $A$ be a ring. If $A$ is self-injective (i.e., $\text{id}_A(A) = 0$), then $\text{Ann}_A(\text{Ann}_A(I)) = I$ for any finitely generated ideal $I$ of $A$.

Proof. of Theorem 3.5. First we claim that $\frac{R}{0 \times E}$ is not a Gorenstein projective $R$-module. For this, let $0 \neq a \in D$ a non-invertible element, then $R(0, a) = 0 \times Da$ is an ideal of $R$. Clearly, $0 \times Da \subseteq 0 \times E \subseteq \text{Ann}_R(\text{Ann}_R(0 \times Da))$, and by Lemma 3.7, $\text{id}_R(R) \neq 0 = \text{id}_D(E)$, then $\text{Ext}^i_\text{R}(\frac{R}{0 \times E}, R) \cong \text{Ext}^i_\text{R}(D, R) \neq 0$, for some $i \geq 1$ [13 Proposition 4.35]. So $\frac{R}{0 \times E}$ is not a Gorenstein projective $R$-module [16 Proposition 2.3]. Now we claim that $0 \times E$ is not a Gorenstein projective $R$-module. Deny. $0 \times E$ is a Gorenstein projective $R$-module. Then there is an exact sequence of $R$-modules

$$0 \rightarrow 0 \times E \rightarrow F \rightarrow G \rightarrow 0 \quad (1)$$

where $F \cong R^I$ is a free $R$-module and $G$ is Gorenstein projective by [16 Proposition 2.4]. Consider the pushout diagram

$$
\begin{array}{ccccccc}
0 & \rightarrow & 0 \times E & \rightarrow & R & \rightarrow & \frac{R}{0 \times E} & \rightarrow & 0 \\
| & | & \downarrow & | & \downarrow & | & \downarrow \\
0 & \rightarrow & 0 \times E & \rightarrow & R^I & \rightarrow & C' & \rightarrow & 0 \\
\end{array}
$$

where $R^I = R''$. 

$$
\begin{array}{ccccccc}
R'' & \rightarrow & \frac{R}{0 \times E} & \rightarrow & 0 \\
\downarrow & | & \downarrow & | & \downarrow \\
0 & \rightarrow & 0
\end{array}
$$
Combining the exact sequence (1) and the short exact sequence in the pushout $0 \to 0 \times E \to R^I \to C' \to 0$, yields $C' \cong G$ is Gorenstein projective. Then from the short exact sequence $0 \to \frac{R}{0 \times E} \to C' \to R^I \to 0$, we get $\frac{R}{0 \times E}$ is Gorenstein projective [16 Theorem 2.5]. But this contradicts the fact that $\frac{R}{0 \times E}$ is not Gorenstein projective in the first part of the proof. Then $0 \times E$ is not Gorenstein projective. On the other hand, from the short exact sequence $0 \to (0 \times E)^J \to R^J \to 0 \times E \to 0$ we obtain $\text{Gpd}_R(0 \times E) = \infty$ [16 Proposition 2.18]. Therefore $G{-\text{gldim}}(R) = \infty$.

Note that the condition “$D$ is not a field” in Theorem 3.5 is necessary. For, the next corollary shows that for any field $K$, $G{-\text{gldim}}(K \times K) = 0$. However [19 Lemma 2.2] asserts that $\text{gldim}(K \times K) = \infty$.

**Corollary 3.8** Let $K$ be a field. Then:

1. $G{-\text{gldim}}(K \times K) = 0$.
2. $G{-\text{gldim}}(K \times K^n) = \infty$, for any $n \geq 2$.

**Example 3.9** Let $R := \mathbb{Z} \times \mathbb{Q}$, where $\mathbb{Z}$ is the ring of integers and $\mathbb{Q}$ the field of rational numbers. Then $G{-\text{gldim}}(R) = \infty$.

Next we exhibit an example showing that, in general, the transfer of the notion of Gorenstein projective module does not carry up to pullback constructions.

**Example 3.10** Let $(D, m)$ be a discrete valuation domain and $K = \text{qf}(D)$. Consider the following pullback

$$
\begin{array}{ccc}
R = D \times K & \longrightarrow & T = K \times K \\
D \cong \frac{R}{0 \times K} & \longrightarrow & K
\end{array}
$$

Let $0 \neq a \in m$ and $I = 0 \times Da$. Consider the following short exact sequence of $R$-modules

$$
0 \longrightarrow 0 \times K \longrightarrow R \overset{u}{\longrightarrow} 0 \times Da \longrightarrow 0
$$

where $u(b, e) = (b, e)(0, a) = (0, ba)$. Similar arguments used in the proof of Theorem 3.5 yield $0 \times Da$ is not Gorenstein projective, $I \otimes_R T \cong 0 \times K$ is a Gorenstein projective ideal of $T$, and $I \otimes_R \frac{R}{0 \times K} \cong \frac{R}{0 \times K}$ is a free $\frac{R}{0 \times K}$-module, then Gorenstein projective.
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