Generalization Error Bounds Via Rényi-, $f$-Divergences and Maximal Leakage

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Abstract

In this work, the probability of an event under some joint distribution is bounded by measuring it with the product of the marginals instead (which is typically easier to analyze) together with a measure of the dependence between the two random variables. These results find applications in adaptive data analysis, where multiple dependencies are introduced and in learning theory, where they can be employed to bound the generalization error of a learning algorithm. Bounds are given in terms of $\alpha-$Divergence, Sibson’s Mutual Information and $f-$Divergence. A case of particular interest is the Maximal Leakage (or Sibson’s Mutual Information of order infinity), since this measure is robust to post-processing and composes adaptively. This bound can also be seen as a generalization of classical bounds, such as Hoeffding’s and McDiarmid’s inequalities, to the case of dependent random variables.

Index Terms

Sibson’s Mutual Information, Rényi-Divergence, f-Divergence, Maximal Leakage, Generalization Error, Adaptive Data Analysis.

I. INTRODUCTION

Let us consider two probability spaces $(\Omega, \mathcal{F}, P), (\Omega, \mathcal{F}, Q)$ and let $E \in \mathcal{F}$ be a measurable event. Given some divergence between the two distributions $\hat{D}(P, Q)$ (e.g., KL, Rényi’s $\alpha-$Divergence, etc.) our aim is to provide bounds of the following form:

$$P(E) \leq f(Q(E)) \cdot g(\hat{D}(P, Q)),$$

for some functions $f, g$. $E$ represents some “undesirable” event (e.g., large generalization error), whose measure under $Q$ is known and whose measure under $P$ we wish to bound. To that end, we use some notion of “distance” between $P$ and $Q$. Of particular interest is the case where $\Omega = \mathcal{X} \times \mathcal{Y}$, $P = P_{XY}$ (the joint distribution), and $Q = P_X P_Y$ (product of the marginals). This allows us to bound the likelihood of $E \subseteq \mathcal{X} \times \mathcal{Y}$ when two random variables...

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variables $X$ and $Y$ are dependent as a function of the likelihood of $E$ when $X$ and $Y$ are independent (a scenario typically much easier to analyze). Such a result can be applied in the analysis of the generalization error of learning algorithms, as well as in adaptive data analysis (with a proper choice of the dependence measure). Adaptive data analysis is a recent field that is gaining attention due to its connection with the “Reproducibility Crisis” [1], [2]. The idea is that, whenever you apply a sequence of analyses to some data (e.g., data-exploration procedures) and each analysis informs the subsequent ones, even though each of these algorithms is guaranteed to generalize well in isolation, this may no longer be true when they are composed together. The problem that arises with the composition is believed to be connected with the leakage of information from the data. The leakage happens because the output of each algorithm becomes an input to the subsequent ones. In order to be used in adaptive data analysis, a measure that provides such bounds needs to be robust to post-processing and to compose adaptively (meaning that we can bound the measure between input and output of the composition of the sequence of algorithms if each of them has bounded measure). Results of this form involving mutual information can be found in [3], [4], [5]. Via inequalities like in (1) we can provide bounds for adaptive mechanisms by treating them as non-adaptive and paying a “penalty” term (e.g., an information measure of statistical dependency) that measures how far is the mechanism from being non-adaptive.

With this aim, our main theorem provides a general bound in the form of (1) with $\mathcal{P} = \mathcal{P}_{XY}$ and $\mathcal{Q} = \mathcal{P}_X \mathcal{P}_Y$. As corollaries, we derive several families of interesting bounds:

- a family of bounds involving the Rényi’s divergence of order $\alpha$;
- a family of bounds involving Sibson’s Mutual Information of order $\alpha$;
- a bound involving Maximal Leakage [6];

Moreover, we derive a family of bounds using $f$-divergences, which provides a rich class of information measures. We focus in particular on the bounds involving Maximal Leakage, which is a secrecy metric that has appeared both in the computer security literature [7], and the information theory literature [8]. It quantifies the leakage of information from a random variable $X$ to another random variable $Y$, and is denoted by $\mathcal{L}(X \rightarrow Y)$. The basic insight is as follows: if a learning algorithm leaks little information about the training data, then it will generalize well. Moreover, similarly to differential privacy, maximal leakage behaves well under composition: we can bound the leakage of a sequence of algorithms if each of them has bounded leakage. It is also robust under post-processing. In addition, the expression to compute it is simply given by the following formula (for finite $X$ and $Y$):

\[
\mathcal{L}(X \rightarrow Y) = \log \sum_y \max_{x : P(x) > 0} P_{Y|X}(y|x),
\]

making it more amenable to analysis and relatively easy to compute, especially for algorithms whose randomness consists in adding independent noise to the outcomes. Despite the main focus being on a joint distribution and the corresponding product of the marginals, the proof techniques are more general and can be applied to any pair of joint distributions (under a mild condition of absolute continuity). Moreover, the Maximal Leakage result, as well as the bound using infinite-Rényi divergence, reduce to the classical concentration inequalities when independence holds (i.e., $\mathcal{P}_{XY} = \mathcal{P}_X \mathcal{P}_Y$)
A. Further related work

In addition to differentially private algorithms, Dwork et al. [1] show that algorithms whose output can be described concisely generalize well. They further introduce $\beta$-max information to unify the analysis of both classes of algorithms. Consequently, one can provide generalization guarantees for a sequence of algorithms that alternate between differential privacy and short description. In [2], the authors connect $\beta$-max information with the notion of approximate differential privacy, but show that there are no generalization guarantees for an arbitrary composition of algorithms that are approximate-DP and algorithms with short description length. With a more information-theoretic approach, bounds on the exploration bias and/or the generalization error are given in [5], [4], [9], [10], [11], [12], [13], using mutual information and other dependence-measures. Some results have also been found using Wasserstein distance [14], [15].

B. Notation

We will denote by calligraphic letters $\mathcal{P}$, $\mathcal{Q}$ probability measures and with capital letters $X$, $Y$, $Z$ random variables. Given two measures $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{P} \ll \mathcal{Q}$ denotes the concept of absolute continuity, i.e., for any measurable set $E$, $\mathcal{Q}(E) = 0 \Rightarrow \mathcal{P}(E) = 0$. Given two random variables $X, Y$ over the spaces $\mathcal{X}, \mathcal{Y}$ we will denote by $\mathcal{P}_{XY}$ a joint measure over the product space $\mathcal{X} \times \mathcal{Y}$, while with $\mathcal{P}_X \mathcal{P}_Y$ we will denote the product of the marginals, i.e., for any measurable set $E \subseteq \mathcal{X} \times \mathcal{Y}$, $\mathcal{P}_X \mathcal{P}_Y(E) = \int_{(x,y) \in E} d\mathcal{P}_X(x)d\mathcal{P}_Y(y)$.

Given a probability measure $\mathcal{P}$ and a random variable $X$ defined over the same space, we will denote with $\mathbb{E}_{\mathcal{P}}[X] = \int x d\mathcal{P}(x)$. \hspace{1cm} (3)

Furthermore, given a random variable $X$ we say that it is $\sigma^2$-sub-Gaussian if the following holds true for every $\lambda \in \mathbb{R}$:

$$\mathbb{E}[e^{\lambda X}] \leq e^{\frac{\lambda^2 \sigma^2}{2}} \hspace{1cm} (4)$$

C. Overview

In Section II we define the fundamental objects that will be used in this work:

- In Subsection II-A we consider Rényi’s-$\alpha$ Divergences, Sibson’s Mutual Information, Maximal Leakage and $f$–divergences;
- In Subsection II-B we provide an overview of the basic concepts in Learning Theory;

In Section III we prove our main results, categorized with respect to the information measure they consider. Some extension of our bounds to expected generalization error is also considered. In Section IV we consider the basic definitions of Adaptive Data Analysis and show how some of our results can be employed in the area. To conclude, in Section V we compare our results with recent results in the literature.
II. BACKGROUND AND DEFINITIONS

A. Information Measures

We will now briefly introduce the information measures that we will use to provide bounds. The idea is to try and capture the dependency between two random variables $X, Y$ through some information measure and employ it in order to provide bounds. We will consider $X$ to be the input of a learning algorithm $A$ and $Y = A(X)$ the corresponding (random) output. By controlling some measure of dependency, we will control how much the learning algorithm $A$ is over-fitting to the data.

1) Sibson’s $\alpha$–Mutual Information: Introduced by Rényi in an attempt to generalize the concept of Entropy and KL-Divergence, the $\alpha$-Divergence has then found many applications over the years in hypothesis testing, guessing and several other statistical inference problems [16]. Indeed, it has several useful operation interpretations (e.g., the number of bits by which a mixture of two codes can be compressed, the cut-off rate in block coding and hypothesis testing [17], [18] [19, p. 649]). It can be defined as follows [17]:

**Definition 1.** Let $(\Omega, \mathcal{F}, \mathcal{P}), (\Omega, \mathcal{F}, \mathcal{Q})$ be two probability spaces. Let $\alpha > 0$ be a positive real different from 1. Consider a measure $\mu$ such that $\mathcal{P} \ll \mu$ and $\mathcal{Q} \ll \mu$ (such a measure always exists, e.g. $\mu = (\mathcal{P} + \mathcal{Q})/2$) and denote with $p, q$ the densities of $\mathcal{P}, \mathcal{Q}$ with respect to $\mu$. The $\alpha$–Divergence of $\mathcal{P}$ from $\mathcal{Q}$ is defined as follows:

$$D_\alpha(\mathcal{P}\|\mathcal{Q}) = \frac{1}{\alpha - 1} \log \int p^\alpha q^{1-\alpha} d\mu.$$  

(5)

**Remark 1.** The definition is independent of the chosen measure $\mu$ whenever $\infty > \alpha > 0$ and $\alpha \neq 1$. It is indeed possible to show that $\int p^\alpha q^{1-\alpha} d\mu = \int \left(\frac{q}{p}\right)^{1-\alpha} d\mathcal{P}$, and that whenever $\mathcal{P} \ll \mathcal{Q}$ or $0 < \alpha < 1$ $\int p^\alpha q^{1-\alpha} d\mu = \int \left(\frac{p}{q}\right)^\alpha d\mathcal{Q}$, see [17].

It can be shown that if $\alpha > 1$ and $\mathcal{P} \ll \mathcal{Q}$ then $D_\alpha(\mathcal{P}\|\mathcal{Q}) = \infty$. The behavior of the measure for $\alpha \in \{0, 1, \infty\}$ can be defined by continuity. In particular, we have that $\lim_{\alpha \to 1} D_\alpha(\mathcal{P}\|\mathcal{Q}) = D(\mathcal{P}\|\mathcal{Q})$, i.e., the classical Kullback-Leibler divergence. For an extensive treatment of $\alpha$–Divergences and their properties we refer the reader to [17].

Starting from the concept of $\alpha$–Divergence, Sibson built a generalization of Mutual Information [20] that retains many interesting properties. The definition is the following [16]:

**Definition 2.** Let $X, Y$ be two random variables jointly distributed according to $\mathcal{P}_{XY}$. Let $\mathcal{P}_X$ be the corresponding marginal of $X$ (i.e., given a measurable set $A$, $\mathcal{P}_X(A) = \mathcal{P}_{XY}(A \times Y)$) and let $\mathcal{Q}_Y$ be any probability measure over $\mathcal{Y}$. Let $\alpha > 0$, the Sibson’s Mutual Information of order $\alpha$ between $X, Y$ is defined as:

$$I_\alpha(X; Y) = \min_{\mathcal{Q}_Y} D_\alpha(\mathcal{P}_{XY}\|\mathcal{P}_X\mathcal{Q}_Y).$$  

(6)

The following, alternative formulation is also useful [16]:

$$I_\alpha(X; Y) = \frac{\alpha}{\alpha - 1} \log \mathbb{E} \left[ \frac{\mathcal{P}_{Y|X}}{\mathcal{P}_Y} \right]$$  

(7)

$$= D_\alpha(\mathcal{P}_{XY}\|\mathcal{P}_X\mathcal{P}_Y) - D_\alpha(\mathcal{P}_{Y|X}\|\mathcal{P}_Y).$$  

(8)
where \( P_{Y_\alpha} \) is the measure minimizing \((6)\). In analogy with the limiting behavior of \( \alpha \)-Divergence we have that 
\[
\lim_{\alpha \to 1} I_\alpha(X; Y) = I(X; Y)
\]
while, when \( \alpha \to \infty \) we retrieve the following object:
\[
I_\infty(X; Y) = \log \mathbb{E}_{P_Y} \left[ \sup_{x: P_X(x) > 0} \frac{P_{XY}(x, Y)}{P_X(x)P_Y(Y)} \right].
\]
To conclude, let us list some of the properties of the measure:

**Proposition 1 ([16]).**

1) **Data Processing Inequality:** given \( \alpha > 0 \), \( I_\alpha(X; Z) \leq \min\{I_\alpha(X; Y), I_\alpha(Y; Z)\} \) if the Markov Chain \( X \rightarrow Y \rightarrow Z \) holds;

2) \( I_\alpha(X; Y) \geq 0 \) with equality iff \( X \) and \( Y \) are independent;

3) Let \( \alpha_1 \leq \alpha_2 \) then \( I_{\alpha_1}(X; Y) \leq I_{\alpha_2}(X; Y) \);

4) Let \( \alpha \in (0, 1) \cup (1, \infty) \), for a given \( P_X \), \( \frac{1}{\alpha - 1} \exp \left( \frac{\alpha - 1}{\alpha} I_\alpha(X; Y) \right) \) is convex in \( P_{Y|X} \);

5) \( I_\alpha(X; Y) \leq \min\{\log |X|, \log |Y|\} \);

For an extensive treatment of Sibson’s \( \alpha \)-MI we refer the reader to [16].

2) **Maximal Leakage:** A particularly relevant dependence measure, strongly connected to Sibson’s Mutual Information is the maximal leakage. \( L(X \rightarrow Y) \) was introduced as a way of measuring the leakage of information from \( X \) to \( Y \), hence the following definition:

**Definition 3** (Def. 1 of [8]). Given a joint distribution \( P_{XY} \) on finite alphabets \( \mathcal{X} \) and \( \mathcal{Y} \), the maximal leakage from \( X \) to \( Y \) is defined as:
\[
L(X \rightarrow Y) = \sup_{U \sim X \rightarrow Y \sim U} \log \frac{\mathbb{P}(\{U = \hat{U}\})}{\max_{u \in U} \mathbb{P}_U(\{u\})},
\]
where \( U \) and \( \hat{U} \) take values in the same finite, but arbitrary, alphabet.

It is shown in [8] Theorem 1] that, for finite alphabets:
\[
L(X \rightarrow Y) = \log \sum_{y \in \mathcal{Y}} \max_{x \in \mathcal{X}: P_X(x) > 0} P_{Y|X}(y|x).
\]
If \( X \) and \( Y \) have a jointly continuous pdf \( f(x, y) \), we get [6 Corollary 4]:
\[
L(X \rightarrow Y) = \log \int_{\mathbb{R}^2} \sup_{x: f_X(x) > 0} f_{Y|X}(y|x) dy.
\]
One can show that \( L(X \rightarrow Y) = I_\infty(X; Y) \) i.e., Maximal Leakage corresponds to the Sibson’s Mutual Information of order infinity. This allows the measure to retain the properties listed in Proposition1 furthermore:

**Lemma 1 ([8]).** For any joint distribution \( P_{XY} \) on finite alphabets \( \mathcal{X} \) and \( \mathcal{Y} \), \( L(X \rightarrow Y) \geq I(X; Y) \).

Another relevant notion, important for its application to Adaptive Data Analysis, is Conditional Maximal Leakage:

**Definition 4** (Conditional Maximal Leakage [6]). Given a joint distribution \( P_{XYZ} \) on alphabets \( \mathcal{X}, \mathcal{Y}, \) and \( \mathcal{Z} \), define:
\[
L(X \rightarrow Y|Z) = \sup_{U: U \sim X \rightarrow Y \sim U} \log \frac{\mathbb{P}(\{U = \hat{U}(Y, Z)\})}{\mathbb{P}(\{U = U(Z)\})},
\]
where $U$ takes value in an arbitrary finite alphabet and we consider $\hat{U}, \bar{U}$ to be the optimal estimators of $U$ given $(Y,Z)$ and $Z$, respectively.

Again, it is shown in [6] that for discrete random variables $X,Y,Z$:

$$\mathcal{L}(X\rightarrow Y|Z) = \log \left( \max_{x:P_X(z)>0} \sum_y \max_{y:P_Y(x|z)>0} P_Y|XZ(y|xz) \right),$$

(13)

and

$$\mathcal{L}(X\rightarrow(Y,Z)) \leq \mathcal{L}(X\rightarrow Y) + \mathcal{L}(X\rightarrow Z|Y).$$

(14)

3) $f$–Mutual Information: Another generalization of the KL-Divergence can be obtained by considering a generic convex function $f : \mathbb{R} \rightarrow \mathbb{R}$, usually with the simple constraint that $f(1) = 0$. The constraint can be ignored as long as $f(1) < +\infty$ by simply considering a new mapping $g(x) = f(x) - f(1)$.

**Definition 5.** Let $(\Omega, \mathcal{F}, \mathcal{P}), (\Omega, \mathcal{F}, \mathcal{Q})$ be two probability spaces. Let $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a convex function such that $f(1) = 0$. Consider a measure $\mu$ such that $\mathcal{P} \ll \mu$ and $\mathcal{Q} \ll \mu$. Denoting with $p,q$ the densities of the measures with respect to $\mu$, the $f$–Divergence of $\mathcal{P}$ from $\mathcal{Q}$ is defined as follows:

$$D_f(\mathcal{P}\|\mathcal{Q}) = \int q f \left( \frac{p}{q} \right) d\mu.$$

(15)

Despite the fact that the definition uses $\mu$ and the densities with respect to this measure, it is possible to show that $f$–divergences are actually independent from the dominating measure [21]. Indeed, when absolute continuity between $\mathcal{P}, \mathcal{Q}$ holds, i.e. $\mathcal{P} \ll \mathcal{Q}$, an assumption we will often use, we retrieve the following [21]:

$$D_f(\mathcal{P}\|\mathcal{Q}) = \int f \left( \frac{d\mathcal{P}}{d\mathcal{Q}} \right) d\mathcal{Q}.$$  

(16)

Denoting with $\mathcal{F}_X$ the Sigma-field generated from the random variable $X$, (i.e., $\sigma(X)$), $f$-mutual information is defined as follows:

**Definition 6.** Let $X$ and $Y$ be two random variables jointly distributed according to $\mathcal{P}_{XY}$ over the a measurable space $(\mathcal{X} \times \mathcal{Y}, \mathcal{F}_{XY})$. Let $(\mathcal{X}, \mathcal{F}_X, \mathcal{P}_X), (\mathcal{Y}, \mathcal{F}_Y, \mathcal{P}_Y)$ be the corresponding probability spaces induced by the marginals. Let $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a convex function such that $f(1) = 0$. The $f$–Mutual Information between $X$ and $Y$ is defined as:

$$I_f(X;Y) = D_f(\mathcal{P}_{XY}\|\mathcal{P}_X\mathcal{P}_Y).$$

(17)

If $\mathcal{P}_{XY} \ll \mathcal{P}_X\mathcal{P}_Y$ we have that:

$$I_f(X;Y) = \int f \left( \frac{d\mathcal{P}_{XY}}{d\mathcal{P}_X\mathcal{P}_Y} \right) d\mathcal{P}_X\mathcal{P}_Y.$$  

(18)

It is possible to see that, if $f$ satisfies $f(1) = 0$ and it is strictly convex at 1, then $I_f(X;Y) = 0$ if and only if $X$ and $Y$ are independent [21]. This generalization includes the KL (by simply setting $f(t) = t \log(t)$) and allows to retrieve $\alpha$–Divergences through a one-to-one mapping. But it also includes many more divergences:

- Total Variation distance, with $f(t) = \frac{1}{2}|t - 1|;
- Hellinger distance, with $f(t) = (\sqrt{t} - 1)^2$;
Pearson $\chi^2$-divergence, with $f(t) = (t - 1)^2$.

Exploiting a bound involving $I_f(X;Y)$ for a broad enough set of functions $f$ allows to differently measure the dependence between $X$ and $Y$ and it may help us circumventing issues that commonly used measures, like Mutual Information, may suffer from. Consider for instance the following example [14]: let $S$ be a random vector, via Strong Data-Processing inequalities it is possible to show that, given the Markov Chain $S \rightarrow H \rightarrow Y$, where $\|H\| \leq k$ and $Y = H + N$ with $N$ Gaussian noise, the Total Variation distance between the joint and the product of the marginals of $S,Y$ is strictly less than 1, while $I(S;Y)$ may still be infinite. Furthermore, as presented in [22], different divergences between distributions can provide different convergence rates. It has been proved in [23] that it is possible to construct a random walk that converges in $2n \log n$ steps under KL, $n^2 \log n$ steps under the $\chi^2$-distance and $n \log n$ in total variation. This shows that even though several $f$-divergences may go to 0 with the number of steps (or samples, in the case of a generalization error bound), the rate of convergence obtainable can be quite different and this can possibly impact the sample complexity in the problems we will analyze in later sections.

**B. Learning Theory**

In this section we will provide some basic background knowledge on learning algorithms and concepts like generalization error. We are mainly interested in supervised learning, where the algorithm learns a classifier by looking at points in a proper space and the corresponding labels.

More formally, suppose we have an instance space $Z$ and a hypothesis space $\mathcal{H}$. The hypothesis space is a set of functions that, given a data point $s \in Z$ outputs the corresponding label $Y$. Suppose we are given a training data set $Z^n \ni S = \{z_1, \ldots, z_n\}$ made of $n$ points sampled in an i.i.d. fashion from some distribution $\mathcal{P}$. Given some $n \in \mathbb{N}$, a learning algorithm is a (possibly stochastic) mapping $A : Z^n \rightarrow \mathcal{H}$ that given as an input a finite sequence of points $S \in Z^n$ outputs some classifier $h = A(S) \in \mathcal{H}$. In the simplest setting we can think of $Z$ as a product between the space of data points and the space of labels i.e., $Z = \mathcal{X} \times \mathcal{Y}$ and suppose that $A$ is fed with $n$ pairs data-label $(x_i, y_i) \in Z$. In this work we will view $A$ as a family of conditional distributions $\mathcal{P}_{H|S}$ and provide a stochastic analysis of its generalization capabilities using the information measures presented so far. The goal is to generate a hypothesis $h : \mathcal{X} \rightarrow \mathcal{Y}$ that has good performance on both the training set and newly sampled points from $\mathcal{X}$. In order to ensure such property, the concept of generalization error is introduced.

**Definition 7.** Let $\mathcal{P}$ be some distribution over $Z$. Let $\ell : \mathcal{H} \times Z \rightarrow \mathbb{R}$ be a loss function. The error (or risk) of a prediction rule $h$ with respect to $\mathcal{P}$ is defined as

$$L_\mathcal{P}(h) = \mathbb{E}_{Z \sim \mathcal{P}}[\ell(h, Z)],$$

while, given a sample $S = (z_1, \ldots, z_n)$, the empirical error of $h$ with respect to $S$ is defined as

$$L_S(h) = \frac{1}{n} \sum_{i=1}^{n} \ell(h, z_i).$$

Moreover, given a learning algorithm $A : Z^n \rightarrow \mathcal{H}$, its generalization error with respect to $S$ is defined as:

$$\text{gen-err}_\mathcal{P}(A, S) = |L_\mathcal{P}(A(S)) - L_S(A(S))|.$$
The definition just stated considers general loss functions. An important instance for the case of supervised learning is the $0-1$ loss. Suppose again that $Z = \mathcal{X} \times \mathcal{Y}$ and that $\mathcal{H} = \{h| h : \mathcal{X} \rightarrow \mathcal{Y}\}$, given a couple $(x, y) \in Z$ and a hypothesis $h : \mathcal{X} \rightarrow \mathcal{Y}$ the loss is defined as follows:

$$\ell(h, (x, y)) = \mathbb{I}_{h(x) \neq y},$$

and the corresponding errors become:

$$L_P(h) = \mathbb{E}_{(x, y) \sim P}[\mathbb{I}_{h(x) \neq y}] = P(h(x) \neq y).$$

and

$$L_S(h) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}_{h(x_i) \neq y_i}.$$  \tag{24}

Another fundamental concept we will need is the sample complexity of a learning algorithm.

**Definition 8.** Fix $\epsilon, \delta \in (0, 1)$. Let $\mathcal{H}$ be a hypothesis class. The sample complexity of $\mathcal{H}$ with respect to $(\epsilon, \delta)$, denoted by $m_{\mathcal{H}}(\epsilon, \delta)$, is defined as the smallest $m \in \mathbb{N}$ for which there exists a learning algorithm $A$ such that, for every distribution $P$ over the domain $\mathcal{X}$

$$\mathbb{P}(\text{gen-err}_P(A, S) > \epsilon) \leq \delta.$$

If there is no such $m$ then $m_{\mathcal{H}}(\epsilon, \delta) = \infty$.

For more details we refer the reader to [24, Sections 2-3].

III. MAIN RESULTS

In this section we will present our main result. The bounds we provide will be categorized according to the information measure we are adopting. Notice that for the remainder of this paper $\log$ is always taken to the base $e$. Also, unless stated otherwise, we will always consider the following two probability spaces $(\mathcal{X} \times \mathcal{Y}, F, P_{XY}), (\mathcal{X} \times \mathcal{Y}, F, P_X P_Y)$ and assume that $P_{XY} \ll P_X P_Y$.

**Theorem 1.** Let $(\mathcal{X} \times \mathcal{Y}, F, P_{XY}), (\mathcal{X} \times \mathcal{Y}, F, P_X P_Y)$ be two probability spaces, and assume that $P_{XY} \ll P_X P_Y$.

Given $E \in F$ and $y \in \mathcal{Y}$, let $E_y = \{x : (x, y) \in E\}$, i.e. the “fibers” of $E$ with respect to $y$. Then,

$$P_{XY} (E) \leq \mathbb{E}_{P_Y} \mathbb{I}_{\gamma} \left[ P_X (E_{Y})^{\gamma'/\gamma} \right] \mathbb{E}_{P_Y} \mathbb{I}_{\alpha'} \left[ \left( \frac{dP_{XY}}{dP_{X} P_{Y}} \right)^{\alpha} \right],$$

where $\gamma, \alpha, \gamma', \alpha'$ are such that $1 = \frac{\alpha}{\gamma} + \frac{1}{\gamma'} = \frac{\alpha'}{\alpha} + \frac{1}{\gamma'}$.  \tag{26}
Proof. We have that:

\[ P_{XY}(E) = E_{P_{XY}}[1_E] \]  

(27)

\[ = E_{P_XP_Y} \left[ E_{P_X} \left[ \left( E_{P_Y} dP_{XY} \right) \right] \right] \]  

(28)

\[ = E_{P_Y} \left[ E_{P_X} \left[ \left( E_{P_Y} dP_{XY} \right) \right] \right] \]  

(29)

\[ \leq \left( E_{P_Y} \left[ E_{P_X} \left[ \left( E_{P_Y} dP_{XY} \right) \right] \right] \right)^{1/\gamma} E_{P_X}^{1/\alpha} \left( \left( dP_{XY} \right)^{\alpha} \right) \]  

(30)

where (a) and (b) follow from Holder’s inequality, given that \( \gamma, \alpha, \gamma', \alpha' \geq 1 \) and \( \frac{1}{\gamma} + \frac{1}{\alpha} = \frac{1}{\gamma'} + \frac{1}{\alpha'} = 1 \). \( \square \)

Remark 2. The proof above works for any couple of measures defined on the same measurable space. Although, we chose to state the theorem when the distributions considered are the joint \( P_{XY} \) and the corresponding product of the marginals \( P_XP_Y \) (i.e. informally, given a measurable set \( A, P_X(A) = P_{XY}(A \times Y) \) and similarly, \( P_Y(B) = P_{XY}(X \times B) \)). This helps us make a direct connection between what appears on the right-hand side of (26) and well-known information measures, later on.

Remark 3. It is clear from the proof that one can similarly bound \( E[g(X, Y)] \) for any positive function \( g(X, Y) \) that is \( P_{XY} \)-integrable. But the shape of the bound becomes more complex as one in general does not have that \( g(X, Y)^\gamma = g(X, Y) \) for every \( \gamma \geq 1 \).

A. \( \alpha \)-Divergences and Sibson’s Mutual Information

Based on the choices of \( \alpha, \alpha' \) one has different bounds. Two are of particular interest to us and rely on different choices of \( \alpha' \). Choosing \( \alpha' = \alpha \) and thus \( \gamma' = \gamma \) in Theorem 1, we retrieve:

Corollary 1. Let \( E \in \mathcal{F} \) we have that:

\[ P_{XY}(E) \leq (P_XP_Y(E))^{1/\gamma} \exp \left( \frac{\alpha - 1}{\alpha} D_{\alpha}(P_{XY} \mid \mid P_XP_Y) \right). \]  

(33)

Proof 1 of Corollary 1 Choosing \( \alpha' = \alpha \) and \( \gamma' = \gamma \) in Theorem 1 one gets:

\[ P_{XY}(E) \leq E_{P_Y}^{1/\gamma} [P_X(E_Y)] E_{P_X}^{1/\alpha} \left[ E_{P_X} \left[ \left( dP_{XY} \right)^{\alpha} \right] \right] \]  

(34)

\[ = (P_XP_Y(E))^{1/\gamma} \exp \left( \frac{\alpha - 1}{\alpha} D_{\alpha}(P_{XY} \mid \mid P_XP_Y) \right). \]  

(35)

\( \square \)
Proof 2 of Corollary 7. Let us denote with \( p = P_{XY}(E), q = P_X P_Y(E), \bar{p} = 1 - p, \bar{q} = 1 - q \)

\[
D_\alpha (P_{XY} \| P_X P_Y) \geq D_\alpha (Ber(p) \| Ber(q)) \tag{36}
\]

\[
= \frac{1}{\alpha - 1} \log \left( p^\alpha q^{1-\alpha} + \bar{p}^\alpha \bar{q}^{1-\alpha} \right) \tag{37}
\]

\[
\geq \frac{1}{\alpha - 1} \log p^\alpha q^{1-\alpha}, \tag{38}
\]

where \( \mathbf{[c]} \) follows from the Data-Processing inequality for \( \alpha \)-Divergences. Re-arranging the terms one gets:

\[
p^\alpha q^{1-\alpha} \leq \exp \left( (\alpha - 1) D_\alpha (P_{XY} \| P_X P_Y) \right) \iff \tag{39}
\]

\[
p^\alpha \leq \exp \left( (\alpha - 1) D_\alpha (P_{XY} \| P_X P_Y) \right) q^{\alpha - 1} \iff \tag{40}
\]

\[
p \leq \exp \left( \frac{(\alpha - 1)}{\alpha} D_\alpha (P_{XY} \| P_X P_Y) \right) q^{\frac{\alpha - 1}{\alpha}}. \tag{41} \]

\[ \square \]

Alternatively, choosing \( \alpha' \to 1 \), which implies \( \gamma' \to +\infty \) we retrieve:

**Corollary 2.** Let \( E \in F \) we have that:

\[
P_{XY}(E) \leq \left( \operatorname{ess sup}_{P_y} P_X(E_y) \right)^{1/\gamma} \mathbb{E}_{P_Y} \left[ \mathbb{E}_{P_X}^{1/\alpha} \left[ \frac{dP_{XY}}{dP_y dP_X} \right]^\alpha \right] \tag{42}
\]

\[
= \left( \operatorname{ess sup}_{P_y} P_X(E_y) \right)^{1/\gamma} \exp \left( \frac{\alpha - 1}{\alpha} I_\alpha(X;Y) \right). \tag{43}
\]

where \( I_\alpha(X;Y) \) is the Sibson’s Mutual Information of order \( \alpha \) \[16\].

**Remark 4.** An in-depth study of \( \alpha \)-Mutual Information appears in \[16\], where a slightly different notation is used. For reference, we can restate Eq. (42) in the notation of \[16\] to obtain:

\[
P_{XY}(E) \leq \left( \operatorname{ess sup}_{P_y} P_X(E_y) \right)^{1/\gamma} \mathbb{E}_{P_Y} \left[ \mathbb{E}_{P_X}^{1/\alpha} \left[ \frac{dP_{Y|X}}{dP_Y} \right]^\alpha \right] \tag{44}
\]

Moreover, for a fixed \( \alpha \) due to the property that Holder’s conjugates need to satisfy, we have that \( \frac{1}{\gamma} = \frac{\alpha - 1}{\alpha} \) and the bound in (43) can also be rewritten as:

\[
P_{XY}(E) \leq \exp \left( \frac{\alpha - 1}{\alpha} \left( I_\alpha(X;Y) + \log \left( \operatorname{ess sup}_{P_y} P_X(E_y) \right) \right) \right). \tag{45}
\]

An interesting property of Sibson’s \( \alpha \)-Mutual Information is that the information measure is non-decreasing with respect to \( \alpha \) \[16\]. Considering the right hand side of (26) we have that, for \( \alpha_1 \leq \alpha_2 \):

\[
\frac{\alpha_1 - 1}{\alpha_1} I_{\alpha_1}(X;Y) \leq \frac{\alpha_2 - 1}{\alpha_2} I_{\alpha_2}(X;Y), \tag{46}
\]

thus, choosing a smaller \( \alpha \) yields a better dependence on \( I_\alpha(X;Y) \) in the bound, but given that \( \frac{1}{\gamma} = \frac{\alpha - 1}{\alpha} \) we also have that \( \frac{1}{\gamma_1} \leq \frac{1}{\gamma_2} \) and being \( \operatorname{ess sup}_{P_y} P_X(E_y) \leq 1 \) it implies that

\[
\left( \operatorname{ess sup}_{P_y} P_X(E_y) \right)^{\frac{1}{\gamma_1}} \geq \left( \operatorname{ess sup}_{P_y} P_X(E_y) \right)^{\frac{1}{\gamma_2}}. \tag{47}
\]
with a worse dependence on \( \left( \text{ess sup}_{P_y} P(E_y) \right)^{1/\gamma} \) on the bound. This leads to a trade-off between the two quantities.

In the bounds of interest \( P_X(E_y) \) is typically exponentially decaying with the number of samples and this trade-off can be explicitly seen in the sample complexity of a learning algorithm:

**Corollary 3.** Let \( \mathcal{X} \times \mathcal{Y} \) be the sample space and \( \mathcal{H} \) be the set of hypotheses. Let \( \mathcal{A} : \mathcal{X}^n \times \mathcal{Y}^n \to \mathcal{H} \) be a learning algorithm that, given a sequence \( S \) of \( n \) points, returns a hypothesis \( h \in \mathcal{H} \). Suppose \( S \) is sampled i.i.d according to some distribution \( P \) over \( \mathcal{X} \times \mathcal{Y} \), i.e., \( S \sim P^n \). Let \( \ell \) be the 0–1 loss function as defined in (42).

Given \( \eta \in (0, 1) \), let \( E = \{(S, h) : |L_P(h) - L_S(h)| > \eta\} \). Fix \( \alpha \geq 1 \) then,

\[
\mathbb{P}(E) \leq \exp \left( \frac{\alpha - 1}{\alpha} \left( I_\alpha(S; \mathcal{A}(S)) + \log 2 - 2n\eta^2 \right) \right). \tag{48}
\]

**Proof.** Fix \( \eta \in (0, 1) \) and \( \alpha \geq 1 \). Let \( \frac{1}{\gamma} = \frac{\alpha - 1}{\alpha} \) and let us denote with \( E_h \) the fiber of \( E \) over \( h \) for some \( h \in \mathcal{H} \), i.e. \( E_h = \{(S, h) : |L_P(h) - L_S(h)| > \eta\} \). Consider \( S, \hat{S} \in \{\mathcal{X} \times \mathcal{Y}\}^n \), where \( S = ((x_1, y_1), \ldots, (x_n, y_n)) \) and \( \hat{S} = ((\hat{x}_1, \hat{y}_1), \ldots, (\hat{x}_n, \hat{y}_n)) \). If \( S, \hat{S} \) differ only in one position \( j \), i.e. \( (x_i, y_i) = (\hat{x}_i, \hat{y}_i) \forall i \in [n] \setminus \{j\} \) and \((x_j, y_j) \neq (\hat{x}_j, \hat{y}_j)\) we have that for every \( h \in \mathcal{H} \),

\[
|L_S(h) - L_{\hat{S}}(h)| = \frac{1}{n} \left| \sum_{i=1}^n 1_{h(x_i) \neq y_i} - \sum_{i=1}^n 1_{h(\hat{x}_i) \neq \hat{y}_i} \right| \tag{49}
\]

\[
= \frac{1}{n} \left| (1_{h(x_j) \neq y_j} - 1_{h(\hat{x}_j) \neq \hat{y}_j}) \right| \leq \frac{1}{n}. \tag{50}
\]

By McDiarmid’s inequality and Inequality (50) we have that for every hypothesis \( h \in \mathcal{H} \),

\[
P_S(E_h) \leq 2 \cdot \exp(-2n\eta^2). \tag{51}
\]

Then it follows from Corollary 2 and Inequality (51) that:

\[
\mathbb{P}(E) \leq \exp \left( \frac{\alpha - 1}{\alpha} I_\alpha(S; \mathcal{A}(S)) \right) \cdot (2 \exp(-2n\eta^2))^{1/\gamma} \tag{52}
\]

\[
= \exp \left( \frac{\alpha - 1}{\alpha} \left( I_\alpha(S; \mathcal{A}(S)) + \log 2 - 2n\eta^2 \right) \right). \tag{53}
\]

**Corollary 4.** Under the same assumptions of Corollary 3 fix \( \alpha \geq 1 \). In order to ensure a confidence of \( \delta \in (0, 1) \), i.e. \( \mathbb{P}(E) \leq \delta \), it is sufficient to have \( n \) samples where

\[
m \geq I_\alpha(S; \mathcal{A}(S)) + \log 2 + \gamma \log \left( \frac{1}{\delta} \right). \tag{54}
\]

**Proof.** From Corollary 2 we have that

\[
\mathbb{P}(E) \leq \exp \left( \frac{\alpha - 1}{\alpha} \left( I_\alpha(S; \mathcal{A}(S)) + \log 2 - 2n\eta^2 \right) \right). \tag{55}
\]

Fix \( \delta \in (0, 1) \), our aim is to have that:

\[
\exp \left( \frac{\alpha - 1}{\alpha} \left( I_\alpha(S; \mathcal{A}(S)) + \log 2 - 2n\eta^2 \right) \right) \leq \delta,
\]

solving the inequality with respect to \( n \) gives us Equation (54).
Smaller $\alpha$ means that $I_{\alpha}(S; A(S))$ will be smaller, but it will imply a larger value for $\gamma = \frac{\alpha}{\alpha-1}$ and thus a worse dependency on $\log(1/\delta)$ in the sample complexity. Let $Z$ be the sample space and $H$ be the set of hypotheses. An immediate generalization of Corollary 3 follows by considering loss functions such that $\ell(h, Z)$ is $\sigma^2$-sub-Gaussian for every $h \in H$ and some $\sigma > 0$.

**Corollary 5.** Let $A : Z^n \to H$ be a learning algorithm that, given a sequence $S$ of $n$ points, returns a hypothesis $h \in H$. Suppose $S$ is sampled i.i.d according to some distribution $P$ over $Z$. Let $\ell : H \times Z \to \mathbb{R}$ be a loss function s.t. $\ell(h, Z)$ is $\sigma^2$-sub-Gaussian random variable for every $h \in H$. Given $\eta \in (0, 1)$, let $E = \{(S, h) : |L_{P}(h) - L_{S}(h)| > \eta\}$. Fix $\alpha \geq 1$ Then,

$$P(E) \leq \exp \left( \frac{1}{\gamma} \left( I_{\alpha}(S; A(S)) + \log 2 - n \frac{\eta^2}{2\sigma^2} \right) \right).$$

*(56)*

**Proof.** Fix $\eta \in (0, 1)$. Let us denote with $E_h$ the fiber of $E$ over $h$ for some $h \in H$, i.e. $E_h = \{S : |L_{P}(h) - L_{S}(h)| > \eta\}$. By assumption we have that $\ell(h, Z)$ is $\sigma^2$-sub-Gaussian for every $h \in H$. We can thus use Hoeffding’s inequality for every hypothesis $h \in H$,

$$P_S(E_h) \leq 2 \cdot \exp \left( -n \frac{\eta^2}{2\sigma^2} \right).$$

*(57)*

Then it follows from Corollary 3 and Ineq. *(57)* that:

$$P(E) \leq \exp \left( \frac{\alpha - 1}{\alpha} I_{\alpha}(S; A(S)) \right) \cdot \left( 2 \exp \left( -n \frac{\eta^2}{2\sigma^2} \right) \right)^{1/\alpha} \leq \exp \left( \frac{\alpha - 1}{\alpha} \left( I_{\alpha}(S; A(S)) + \log 2 - n \frac{\eta^2}{2\sigma^2} \right) \right).$$

*(58)*

$$= \exp \left( \frac{\alpha - 1}{\alpha} \left( I_{\alpha}(S; A(S)) + \log 2 - n \frac{\eta^2}{2\sigma^2} \right) \right).$$

*(59)*



### B. $f$-Divergences

A similar approach yields bounds involving $f$-Divergences and $f$-Mutual Information.

**Theorem 2.** Let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex function such that $\phi(1) = 0$, and assume $\phi$ is non-decreasing on $[0, +\infty)$. Suppose also that $\phi$ is such that for every $y \in \mathbb{R}^+$ the set $\{t \geq 0 : \phi(t) > y\}$ is non-empty, i.e. the generalized inverse, defined as $\phi^{-1}(y) = \inf\{t \geq 0 : \phi(t) > y\}$, exists. Let $\phi^*(t) = \sup_{\lambda \geq 0} \lambda t - \phi(\lambda)$ be the Fenchel-Legendre dual of $\phi(t)$ [25, Section 2.2]. Given an event $E \in \mathcal{F}$, we have that:

$$P_{XY}(E) \leq P_X P_Y(E) \cdot \phi^{-1} \left( \frac{I_{\phi}(X; Y) + (1 - P_X P_Y(E))\phi^*(0)}{P_X P_Y(E)} \right).$$

*(60)*
\textbf{Proof 1 of Theorem} \(^2\) \(\forall \lambda > 0:\)

\[ \mathcal{P}_{XY}(E) = \mathbb{E}_{\mathcal{P}_{XY}}[\mathbb{I}_E] \]

(61)

\[ = \mathbb{E}_{\mathcal{P}_X \mathcal{P}_Y} \left[ \mathbb{I}_E \frac{d\mathcal{P}_{XY}}{d\mathcal{P}_X \mathcal{P}_Y} \right] \]

(62)

\[ \leq \frac{1}{\lambda} \mathbb{E}_{\mathcal{P}_X \mathcal{P}_Y} \left[ \phi^*(\lambda \mathbb{I}_E) + \phi \left( \frac{d\mathcal{P}_{XY}}{d\mathcal{P}_X \mathcal{P}_Y} \right) \right] \]

(63)

\[ \leq \frac{I_\phi(X;Y) + \phi^*(\lambda)\mathbb{E}_{\mathcal{P}_X \mathcal{P}_Y}[\mathbb{I}_E] + \phi^*(0)\mathbb{E}_{\mathcal{P}_X \mathcal{P}_Y}[1 - \mathbb{I}_E]}{\lambda} \]

(64)

\[ = \frac{I_\phi(X;Y) + \phi^*(\lambda)\mathcal{P}_X \mathcal{P}_Y(E) + \phi^*(0)(1 - \mathcal{P}_X \mathcal{P}_Y(E))}{\lambda} \]

(65)

(66)

where (d) follows from Young’s inequality and where \(\phi^*\) is the Legendre-Fenchel dual of \(\phi\), (e) follows from our definition of \(\phi\)-Mutual Information and (f) follows as, being \(\mathbb{I}_E \in [0,1]\) and we can write:

\[ \phi^*(\lambda \mathbb{I}_E) = \phi^*(\lambda(\mathbb{I}_E + (1 - \mathbb{I}_E)0))) \]

(67)

\[ \leq \mathbb{I}_E \phi^*(\lambda) + (1 - \mathbb{I}_E)\phi^*(0). \]

(68)

To get the best bound over \(\mathcal{P}_{XY}\) we can minimize (66) over all \(\lambda > 0:\)

\[ \mathcal{P}_{XY}(E) \leq \inf_{\lambda > 0} \frac{I_\phi(X;Y) + \phi^*(\lambda)\mathcal{P}_X \mathcal{P}_Y(E) + (1 - \mathcal{P}_X \mathcal{P}_Y(E))\phi^*(0)}{\lambda} \]

(69)

\[ = \mathcal{P}_X \mathcal{P}_Y(E) \cdot \inf_{\lambda > 0} \frac{I_\phi(X;Y) + (1 - \mathcal{P}_X \mathcal{P}_Y(E))\phi^*(0)}{\mathcal{P}_X \mathcal{P}_Y(E)} + \phi^*(\lambda) \]

(70)

\[ \equiv \mathcal{P}_X \mathcal{P}_Y(E) \cdot \phi^{-1} \left( \frac{I_\phi(X;Y) + (1 - \mathcal{P}_X \mathcal{P}_Y(E))\phi^*(0)}{\mathcal{P}_X \mathcal{P}_Y(E)} \right), \]

(71)

with (g) following from [25] Lemma 2.4. In order to use [25] Lemma 2.4 the convex function needs to respect a set of properties. Using the notation of [25], the result is obtained by making the following substitution \(\psi = \phi^*, \psi^* = \phi\).

The properties that the function has to respect in the premise of the Lemma \((\phi^*(0) = \phi^*(0)) = 0\) have the purpose, analyzing the proof, to ensure that \(\phi\) is non-negative, convex and non-decreasing. Since \(\phi\) is convex by assumption, we have that \((\phi^*)^* = \phi\) and thus \((\phi^*)^*\) is convex and non-decreasing by assumption. As for the non-negativity, it is required in order to make sure that for a given \(\lambda > 0\), we have that \(\phi(t) \geq \lambda t - \phi^*(\lambda)\) is unbounded and the set \(\{t \geq 0 : \phi(t) > y\}\) is non-empty for every \(y \geq 0\). Thus, the non-negativity of \(\phi\) is a stronger assumption enforced in order to have a well defined generalized inverse \(\phi^{-1}(y) = \inf\{t \geq 0 : \phi(t) > y\}\), and can be omitted when this is always non-empty.

Another proof can be constructed using the variational representation of \(\phi\)-divergences, for a convex function \(\phi\).

\textbf{Proof 2 of Theorem} \(^2\) Let \(\phi : \mathbb{R} \to \mathbb{R}\) be a convex function respecting all the assumption of Theorem \(^2\). We have that, given two measures \(\mathcal{P}, \mathcal{Q}\) \(^26\):

\[ D_\phi(\mathcal{P} \parallel \mathcal{Q}) \geq \mathbb{E}_\mathcal{P}[f] - \mathbb{E}_\mathcal{Q}[\phi^*(f)], \]

(72)
for every $\mathcal{P}$–measurable function $f$. Let $\mathcal{P} = \mathcal{P}_{XY}, Q = \mathcal{P}_X \mathcal{P}_Y$ and $f(x, y) = \lambda 1_{(x, y) \in E}$ for $\lambda > 0$. By Inequality (72) we have that:

$$I_\phi(X; Y) = D_\phi(\mathcal{P}_{XY} \parallel \mathcal{P}_X \mathcal{P}_Y)$$

$$\geq \mathbb{E}_{\mathcal{P}_{XY}}[1_E] - \mathbb{E}_{\mathcal{P}_X \mathcal{P}_Y} [\phi^*(1_E)]$$

$$= \lambda \mathcal{P}_{XY}(E) - \mathbb{E}_{\mathcal{P}_X \mathcal{P}_Y} [\phi^*(1_E)].$$

We thus have that:

$$\mathcal{P}_{XY}(E) \leq \frac{I_\phi(X; Y) + \mathbb{E}_{\mathcal{P}_X \mathcal{P}_Y} [\phi^*(1_E)]}{\lambda}.$$  \hspace{1cm} (76)

The conclusion can be drawn by following the same steps of the other proof starting from Inequality (64).

**Remark 5.** A simpler form of the Equation (60) can be found when $-\infty < \phi^*(0) \leq 0$. Indeed, it is possible to start from Eq. (67) and further upper-bound Ineq. (68) to obtain the following:

$$\phi^*(\lambda 1_E) \leq \frac{\phi^*(\lambda)}{\mathbb{E}\phi \left( \frac{\lambda 1_E}{1_E} \right)}.$$ \hspace{1cm} (77)

Proof 3 of Theorem 2. Let us denote with $p = \mathcal{P}_{XY}(E), q = \mathcal{P}_X \mathcal{P}_Y(E), \bar{p} = 1 - \mathcal{P}_{XY}(E), \bar{q} = 1 - \mathcal{P}_X \mathcal{P}_Y(E)$.

$$I_\phi(X; Y) = D_\phi(\mathcal{P}_{XY} \parallel \mathcal{P}_X \mathcal{P}_Y) \geq D_\phi(Ber(p) \parallel Ber(q))$$

$$= q\phi \left( \frac{p}{q} \right) + \bar{q}\phi \left( \frac{\bar{p}}{\bar{q}} \right)$$

$$\geq q\phi \left( \frac{p}{q} \right) + \bar{q} \left( \bar{p} \bar{q} - \phi^*(y) \right)$$

$$\equiv q\phi \left( \frac{p}{q} \right) - \bar{q}\phi^*(0).$$ \hspace{1cm} (82)

where (h) follows from the Data-Processing Inequality for $f$–divergences, (i) follows from Young’s inequality, which applies for every $y \geq 0$ and (j) follows from choosing $y = 0$. Re-arranging the terms one gets:

$$I_\phi(X; Y) + \bar{q}\phi^*(0) \geq qf \left( \frac{p}{q} \right) \iff$$

$$\frac{I_\phi(X; Y) + \bar{q}\phi^*(0)}{q} \geq f \left( \frac{p}{q} \right) \iff$$

$$q\phi^{-1} \left( \frac{I_\phi(X; Y) + \bar{q}\phi^*(0)}{q} \right) \geq p.$$ \hspace{1cm} (85)

We can now see that we can obtain Corollary 1 as a Corollary to Theorem 2. We restate it here for ease of reference.

**Corollary 6.** Let $E \in \mathcal{F}$ we have that:

$$\mathcal{P}_{XY}(E) \leq (\mathcal{P}_X \mathcal{P}_Y(E))^{1/\gamma} \exp \left( \frac{\alpha - 1}{\alpha} D_\alpha(\mathcal{P}_{XY} \parallel \mathcal{P}_X \mathcal{P}_Y) \right).$$ \hspace{1cm} (86)
Proof. Fix $\alpha > 1$ and consider the following convex function:

$$\phi_\alpha(t) = \frac{t^\alpha - 1}{\alpha - 1},$$

i.e. the Hellinger Divergence. The restriction of $\phi_\alpha(t)$ to $[0, +\infty)$ is increasing and thus invertible. Since we will consider only ratios between measures, the restriction is sufficient and Theorem 2 is applicable. It follows that:

$$\phi_\alpha^{-1}(t) = ((\alpha - 1) t + 1)^{1/\alpha},$$

and that:

$$\phi_\alpha^*(t) = t \left( \frac{(\alpha - 1) t}{\alpha} \right)^{1/\alpha - 1} \frac{((\alpha - 1) t + 1)^{\alpha/\alpha - 1}}{\alpha - 1} + \frac{1}{\alpha - 1},$$

from which we can deduce that:

$$\phi_\alpha^*(0) = \frac{1}{\alpha - 1}.$$

We also have that for a given $\alpha > 0$ and two measures $P, Q$:

$$D_\alpha(P \| Q) = \frac{1}{\alpha - 1} \log(1 + (\alpha - 1) D_{f_\alpha}(P \| Q)),$$

then, with $\phi = f_\alpha$ and computing the right-hand side of Ineq. (91) we retrieve:

$$\phi^{-1} \left( \frac{I_\phi(X; Y) + (1 - P_X P_Y(E)) / (\alpha - 1)}{P_X P_Y(E)} \right) = \phi^{-1} \left( \frac{D_\alpha(P_{XY} \| P_X P_Y) + (1 - P_X P_Y(E)) / (\alpha - 1)}{P_X P_Y(E)} \right)$$

$$= \left( \frac{(\alpha - 1) D_{f_\alpha}(P_{XY} \| P_X P_Y) + (1 - P_X P_Y(E))}{P_X P_Y(E)} + 1 \right)^{1/\alpha}$$

$$= \left( \frac{(\alpha - 1) D_{f_\alpha}(P_{XY} \| P_X P_Y) + 1}{P_X P_Y(E)} \right)^{1/\alpha} = \exp \left( \frac{\alpha - 1}{\alpha} D_\alpha(P_{XY} \| P_X P_Y) \right) / P_X P_Y(E)^{1/\alpha},$$

where [k] follows from [91]. To conclude, substitute [96] in (60):

$$P_{XY}(E) \leq P_X P_Y(E) \exp \left( \frac{\alpha - 1}{\alpha} D_\alpha(P_{XY} \| P_X P_Y) \right).$$

since $\frac{\alpha - 1}{\alpha} = \frac{1}{\gamma}$ is the Holder’s conjugate of $\frac{1}{\alpha}$ we recover Corollary 11.

Another interesting application of Theorem 2 is for $\phi(t) = (t - 1)^2$. This function allows us to retrieve the Pearson’s $\chi^2$–divergence between two distributions. We will denote, through a slight abuse of notation, with $\chi^2(X; Y) = \chi^2(P_{XY} \| P_X P_Y)$. The bound we retrieve is the following:

**Corollary 7.** Let $\phi(t) = t^2 - 1$, we have that $I_\phi(X; Y) = \chi^2(X; Y)$. Let $E \subseteq X \times Y$ we have that:

$$P_{XY}(E) \leq \sqrt{\chi^2(X; Y) + 1} P_X P_Y(E)$$

$$\leq \sqrt{\exp \left( L(X \rightarrow Y) \right)} P_X P_Y(E).$$
Proof. We have that $\phi^*(t) = t^2/4 + 1$ and thus $\phi^*(0) = 1$. We also have that $\phi^{-1}(t) = \sqrt{t + 1}$. Applying Theorem 2 we have that:

$$P_{XY}(E) \leq P_X P_Y(E) \sqrt{\frac{\chi^2(X;Y) + 1 - P_X P_Y(E)}{P_X P_Y(E)}} + 1 \quad (100)$$

$$= P_X P_Y(E) \sqrt{\frac{\chi^2(X;Y) + 1}{P_X P_Y(E)}} \quad (101)$$

$$= \sqrt{(\chi^2(X;Y) + 1)} P_X P_Y(E). \quad (102)$$

Step (l) (in the statement of the corollary) then follows from $\chi^2(X;Y) \leq \exp(L(X \rightarrow Y)) - 1$, see \[10\]. Another proof for this result can be obtained from Corollary 1 with $\alpha = 2$ and $\gamma = 2$ and using the following equality:

$$D_2(P\|Q) = \log(1 + \chi^2(P\|Q)). \quad (103)$$

In the same fashion of Corollary 3 where we derived a generalization error and a sample complexity bound for Sibson’s Mutual Information, one can do the same for the inequalities obtained via $f$-divergences. In particular, considering Corollary 7 where the measure $\chi^2(X;Y)$ is involved, one retrieves the following sample complexity result:

**Corollary 8.** Let $X \times Y$ be the sample space and $H$ be the set of hypotheses. Let $A : X^n \times Y^n \rightarrow H$ be a learning algorithm that, given a sequence $S$ of $n$ points, returns a hypothesis $h \in H$. Suppose $S$ is sampled i.i.d according to some distribution $P$ over $X \times Y$, i.e., $S \sim P^n$. Let $\ell$ be the $0-1$ loss function. Given $\eta \in (0,1)$, let $E = \{(S,h) : |L_P(h) - L_S(h)| > \eta\}$. In order to ensure a confidence of $\delta \in (0,1)$, i.e., $P(E) \leq \delta$, it is sufficient to have $m$ samples where:

$$m \geq \frac{\log(\chi^2(X;Y) + 1) + 2\log\left(\frac{1}{\delta}\right)}{2\eta^2}. \quad (104)$$

C. Maximal Leakage

An interesting special case of Corollary 2 is to let $\alpha \rightarrow \infty$. In this scenario, in the right-hand side of Eq. 43 we obtain Maximal Leakage \[8\]. Maximal Leakage has gained growing interest in the last few years and enjoys a series of properties that are of particular interest to us and we will soon analyze. The result will be thus stated independently with an alternative proof. Considering the other extreme, i.e., $\alpha \rightarrow 1$ we retrieve a trivial bound. Indeed, letting $\alpha \rightarrow 1$ in any of our results leads to a bound of 1 on $P_{XY}(E)$. This means that our approach does not provide bounds that exploit either the Kullback-Leibler divergence or the mutual information. Nonetheless, we will provide some comparison with this result and some analogous result obtained for mutual information (although, through a different approach \[3, 4\]) in Section V-A.

**Theorem 3.** Let $E \in F$ we have that:

$$P_{XY}(E) \leq \left(\text{ess sup}_{P_y} P_X(E_y)\right) \exp(L(X \rightarrow Y)) \quad (105)$$
Proof 1 of Theorem 3. The proof follows directly from Corollary 2 and by seeing that when $\alpha \to \infty$ then $\gamma \to 1$ and $\mathcal{L}(X\to Y) = I_\infty(X;Y)$ \cite{8}.

An alternative proof is the following:

Proof 2 of Theorem 3. Let $P, Q$ be two measure over the same $\sigma$-field we have that the $\alpha$-divergence of order infinity $D_\infty(P\|Q) = \log(\ess sup_{\mathcal{Q}} \frac{dP}{dQ})$ \cite{17}. Suppose that the measurable space $(X \times \mathcal{Y}, \mathcal{F})$ has the regular conditional probability property \cite{27} and let us denote the conditional measure of $P_{XY}$ with respect to the random variables $X, Y$ as $P_{X|Y=y}$ and $P_{Y|X=x}$. We have that, for every $y \in \mathcal{Y}$:

$$
P_{X|Y=y}(E_y) \leq \ess sup_{P_y} P_X(E_y) \cdot \exp \left(D_\infty \left(P_{X|Y=y}\|P_X\right)\right) \tag{106}
$$

$$
= \ess sup_{P_y} P_X(E_y) \cdot \exp \left(D_\infty \left(P_{Y|X=x}\|P_Y\right)\right) \tag{107}
$$

And thus:

$$
P_{XY}(E) = \mathbb{E}_{P_Y} \left[P_{X|Y=y}(E_y)\right] \tag{108}
$$

$$
\leq \ess sup_{P_y} P_X(E_y) \mathbb{E}_{P_Y} \left[\exp \left(D_\infty \left(P_{Y|X=x}\|P_Y\right)\right)\right] \tag{109}
$$

$$
= \ess sup_{P_y} P_X(E_y) \cdot \exp \left(\mathcal{L}(X\to Y)\right), \tag{110}
$$

as by \cite{6} Thm. 7:

$$
\mathbb{E}_{P_Y} \left[\exp \left(D_\infty \left(P_{Y|X=x}\|P_Y\right)\right)\right] = \mathbb{E}_{P_Y} \left[\ess sup \frac{dP_{X|Y=y}}{dP_Y}\right] \tag{111}
$$

Some reasons why this result has more relevance (than the other $\alpha$–Divergences bounds) are:

- Maximal Leakage is more amenable to analysis due to its semi-closed form (e.g., it is possible to easily compute the maximal leakage of noise-addition mechanisms);
- A conditional version of Maximal Leakage allows us to provide adaptive composition results;
- The absence of the power $\frac{1}{\gamma}$ on the right-hand side of (43) allows us to provide a generalization of the classical concentration of measure results in adaptive scenarios.

Next, we illustrate Theorem 3 by giving two examples where Ineq. (105) is met with equality, both when $X$ is independent from $Y$ and when they are strongly dependent.

Example 1 (independent case). Suppose that $E$ is such that $P_X(E_y) = \alpha$ for all $y \in \mathcal{Y}$. In that case we have that, if $X$ and $Y$ are independent:

$$
\alpha = \mathbb{E}_{P_Y} \left[P_X(E_y)\right] = P_{XY}(E) \leq \alpha. \tag{112}
$$

Example 2 (strongly dependent case). Consider the example presented in \cite{3}. Suppose $X = Y \sim U([n])$ then we have that $\mathcal{L}(X\to Y) = \log n$ and if $E = \{(x, y) \in [n] \times [n]|x = y\}$ then,

$$
1 = P_{XY}(E) \leq \frac{1}{n} \cdot n = 1. \tag{113}
$$
Thus, when the worst-case behavior (i.e., \( \max_y \mathcal{P}_X(E_y) \)) matches with the average-case behavior (i.e., \( \mathbb{E}_{\mathcal{P}_Y}[\mathcal{P}_X(E_y)] = \mathcal{P}_X \mathcal{P}_Y(E) \)), like with McDiarmid’s or Hoeffding’s inequality, our bound represents a generalization of the classical concentration of measure inequalities, in adaptive settings.

We will now explore how this result can be applied in providing bounds on the generalization error of learning algorithms.

**Corollary 9.** Under the same assumptions of Corollary 2

\[ \mathbb{P}(E) \leq 2 \cdot \exp(\mathcal{L}(S \rightarrow \mathcal{A}(S)) - 2n\eta^2). \]  

**Proof.** Fix \( \eta \in (0,1) \) and \( \alpha \geq 1 \). Let \( \frac{1}{\gamma} = \frac{a-1}{a} \). Let us denote with \( E_h \) the fiber of \( E \) over \( h \) for some \( h \in \mathcal{H} \), i.e. \( E_h = \{ S : |L_P(h) - L_S(h)| > \eta \} \). By McDiarmid’s inequality and the same reasoning used in the proof of Corollary 3 (i.e., Ineq. (50)) we have that for every hypothesis \( h \in \mathcal{H} \)

\[ \mathcal{P}_S(E_h) \leq 2 \cdot \exp(-2n\eta^2). \]  

(115)

Then it follows from Theorem 3 and Ineq. (115) that:

\[ \mathbb{P}(E) \leq \exp(\mathcal{L}(S \rightarrow \mathcal{A}(S))) \cdot 2 \exp(-2n\eta^2). \]  

(116)

\[ \square \]

Whenever \( \mathcal{A} \) is independent from the samples \( S \) we have that \( \exp(\mathcal{L}(S \rightarrow \mathcal{A}(S))) = 1 \) and we immediately fall back to the non-adaptive scenario: \( \mathbb{P}(E) \leq 2 \cdot \exp(-2n\eta^2) \) i.e., McDiarmid’s inequality with sensitivity \( 1/n \). A more general form to the generalization-error bound can be given for loss functions such that \( \ell(h,z) \) is \( \sigma^2 \)-sub-Gaussian for every \( h \in \mathcal{H} \).

**Corollary 10.** Let \( Z \) be the sample space and \( \mathcal{H} \) be the set of hypotheses. Let \( \mathcal{A} : Z^n \rightarrow \mathcal{H} \) be a learning algorithm that, given a sequence \( S \) of \( n \) points, returns a hypothesis \( h \in \mathcal{H} \). Suppose \( S \) is sampled i.i.d according to some distribution \( \mathcal{P} \) over \( Z \). Let \( l : \mathcal{H} \times Z \rightarrow \mathbb{R} \) be a loss function s.t. \( l(h,z) \) is \( \sigma^2 \)-sub-Gaussian random variable for every \( h \in \mathcal{H} \). Given \( \eta \in (0,1) \), let \( E = \{(S,h) : |L_P(h) - L_S(h)| > \eta \} \). Then,

\[ \mathbb{P}(E) \leq 2 \cdot \exp \left( \mathcal{L}(S \rightarrow \mathcal{A}(S)) - n\frac{\eta^2}{2\sigma^2} \right). \]  

(117)

**Proof.** Fix \( \eta \in (0,1) \). Let us denote with \( E_h \) the fiber of \( E \) over \( h \) for some \( h \in \mathcal{H} \), i.e. \( E_h = \{ S : |L_P(h) - L_S(h)| > \eta \} \). By assumption we have that \( \ell(h,z) \) is \( \sigma^2 \)-sub-Gaussian for every \( h \). We can thus use Hoeffding’s inequality for every hypothesis \( h \in \mathcal{H} \), and retrieve that for every \( h \in \mathcal{H} \)

\[ \mathcal{P}_S(E_h) \leq 2 \cdot \exp \left( -n\frac{\eta^2}{2\sigma^2} \right). \]  

(118)

Then it follows from Theorem 3 and Ineq. (118) that:

\[ \mathbb{P}(E) \leq \exp(\mathcal{L}(S \rightarrow \mathcal{A}(S))) \cdot 2 \exp \left( -n\frac{\eta^2}{2\sigma^2} \right). \]  

(119)

\[ \square \]
D. Bounds on Expected Generalization Error

Given the bounds proposed in Corollaries 3, 5 and 9, one may ask how these reflect in results on the expected value of the generalization-error. To give a meaningful bound one needs to make some assumptions on the probability of our event $E$, in particular we will assume this probability to be exponentially decreasing with the number of samples $n$ (as it often happens in the literature [28], [25]). The following result is inspired by [24, p. 419] with a different (slightly improved, for our purposes) proof.

**Lemma 2.** Let $X$ be a random variable and let $\hat{x} \in \mathbb{R}$. Suppose that there exist $a \geq 0$ and $b \geq c$ such that for every $\eta > 0$ $\mathcal{P}_X(\|X - \hat{x}\| \geq \eta) \leq 2b \exp \left(-\frac{\eta^2}{a^2}\right)$ then $\mathbb{E}[X - \hat{x}] \leq a \left(2 \sqrt{\log b} + \frac{1}{\sqrt{\log b}}\right)$.

**Proof.** Since $|X - \hat{x}|$ is a positive random variable we have that

\[
\mathbb{E}[|X - \hat{x}|] = \int_0^{+\infty} \mathcal{P}_X(|X - \hat{x}| \geq \eta) d\eta. \tag{120}
\]

Since for small values of $\eta$ the exponential bound may be exceedingly loose, instead of trivially upper-bounding (120) we do the following:

\[
\mathbb{E}[|X - \hat{x}|] = \int_0^{+\infty} \mathcal{P}_X(|X - \hat{x}| \geq \eta) d\eta \leq \int_0^{+\infty} \min\left(1, 2b \exp \left(-\frac{\eta^2}{a^2}\right)\right) d\eta \leq \int_0^{\sqrt{a^2 \log 2b}} d\eta + \int_{\sqrt{a^2 \log 2b}}^{+\infty} 2b \exp\left(-\frac{\eta^2}{a^2}\right) d\eta \leq \sqrt{a^2 \log 2b} + \frac{a^2}{\sqrt{a^2 \log 2b}} \int_0^{+\infty} \frac{2b\eta}{a^2} \exp\left(-\frac{\eta^2}{a^2}\right) d\eta = a \left(\sqrt{\log 2b} + \frac{1}{\sqrt{\log 2b}}\right). \tag{125}
\]

**Theorem 4.** Let $A : \mathcal{Z}^n \rightarrow \mathcal{H}$ be a learning algorithm and let $I_\alpha(S; A(S))$ be the dependence measure chosen. Suppose that the loss function $\ell : \mathcal{Z} \times \mathcal{H} \rightarrow \mathbb{R}$ is such that $\forall h, \mathcal{P}_{S \sim \mathcal{D}^n}(\|L_S(h) - \mathbb{E}[L(h)]\| > \eta) \leq 2 \exp\left(-\frac{\eta^2}{2\sigma^2 n}\right)$ for some $\sigma > 0$ (e.g. $\ell(h, Z) - \mathbb{E}[\ell(h, Z)]$ is $\sigma^2$-sub-Gaussian for each $h$), then:

\[
\mathbb{E}[\|L_S(H) - \mathbb{E}[L(H)]\|] \leq \sqrt{\frac{2\sigma^2}{n}} \left(\sqrt{\frac{\log 2 + I_\alpha(S; A(S))}{\gamma}} + \frac{1}{2\sqrt{\log 2 + I_\alpha(S; A(S))}}\right). \tag{126}
\]

**Proof.** The proof is a simple application of Lemma 2 and Corollary 5 with $a = \sqrt{\gamma \sigma}/\sqrt{n}$ and $b = \exp\left(\frac{I_\alpha(A(S); S) + \log 2}{2}\right)$ with some fixes due to the presence of $2^{\frac{1}{2}}$ as multiplicative factor in (122) instead of 2.

**Remark 6.** Notice that, even though we provide a concrete example (Theorem 4) that uses $\sigma^2$ sub-Gaussianity the assumption is not strictly necessary. Lemma 2 only requires that the probability of $X$ diverging from $\hat{x}$ decays exponentially fast. But this can be true also for other classes of random variables, like sub-Weibull ones with an opportune choice of parameters 29. Moreover, unlike 3, 5, Corollary 1 and 2 are more general and do not require any assumption about the convergence rate.
An interesting application of Theorem 4 can be found, as before, by considering $L(S \to A(S))$ and the 0−1 loss (hence, 1/4-sub-Gaussian). Specifically, we get the following corollary:

**Corollary 11.** Let $A : Z^n \to \mathcal{H}$. Consider the 0−1 loss, then $\forall h \in \mathcal{P}_S \sim \mathcal{D}^n(\{L_S(h)−E[L(h)]\} > \eta) \leq 2\exp(-2\eta^2n)$, and:

$$\mathbb{E}[|L_S(h)−E[L(S)|] \leq \frac{1}{\sqrt{2n}} \left( \sqrt{\log 2 + L(S \to A(S))} + \frac{1}{\sqrt{\log 2 + L(S \to A(S))}} \right).$$ (127)

E. Examples

A simple way of keeping the Maximal Leakage of an algorithm $A(X)$ bounded (and thus ensure generalization) is to add noise (e.g., $\hat{Y} = A(X) + N$ with $A$ a real-valued function). The proofs for this section can be found in Appendix B.

**Lemma 3 (Laplacian Noise).** Let $g : \mathcal{X}^n \to \mathbb{R}$ be a function such that $g(x) \in [a,c], a < c \forall x \in \mathcal{X}^n$. The mechanism $M(x) = g(x) + N$ where $N \sim \text{Lap}(b)$ is such that:

$$L(X \to M(X)) = \log \left(1 + \frac{(c-a)}{b}\right).$$ (128)

Similar results can be obtained analyzing different types of noise.

**Lemma 4 (Gaussian Noise).** Let $g : \mathcal{X}^n \to \mathbb{R}$ be a function such that $\forall x \in \mathcal{X}^n g(x) \in [a,c], a < c$. The mechanism $M(x) = g(x) + N$ where $N \sim \mathcal{N}(0,\sigma^2)$ is such that:

$$L(X \to M(X)) = \log \left(1 + \frac{(c-a)}{\sqrt{2\pi\sigma^2}}\right).$$ (129)

**Lemma 5 (Exponential Noise).** Let $f : \mathcal{X}^n \to \mathbb{R}$ be a function such that $\forall x \in \mathcal{X}^n f(x) \in [a,c], c > 0$. The mechanism $M(x) = f(x) + N$ where $N \sim \text{Exp}(\lambda)$ (i.e. $\mathbb{E}[N] = 1/\lambda = b$) is such that:

$$L(X \to M(X)) = \log \left(1 + \frac{(c-a)}{b}\right).$$ (130)

The addition of carefully calibrated noise to control maximal leakage can be used in practice to obtain generalization guarantees of learning algorithms. As an exact analogy to [4, Corollary 4] we can state the following Corollary, involving a noisy version of the Empirical Risk Minimization (ERM) algorithm.

**Corollary 12.** Let us consider the following algorithm:

$$A(S) = \arg \min_{h \in H} (L_S(h) + N_h),$$ (131)

where $N_h$ is exponential noise drawn independently from the input, added to the empirical risk of each hypothesis on a given data-set $S$. Let $|H| = k$ and denote with $N_i$ the noise added to the hypothesis $h_i$, we have that for every $\eta \in (0,1)$:

$$\mathbb{P}(\text{gen−err}(A) \geq \eta) \leq 2\exp\left(\sum_{i=1}^{k} \log \left(1 + \frac{1}{b_i}\right) - 2n\eta^2\right).$$ (132)
Choosing $b_i = i^{1/3} / n^{1/3}$ we retrieve:
\[
\mathbb{P}(\text{gen} - \text{err}(A) \geq \eta) \leq 2 \exp \left( -n(2\eta^2 - 11/n^{2/3}) \right).
\] (133)

Furthermore:
\[
\mathbb{E}[\text{gen} - \text{err}(A)] = O \left( \sqrt{\log n} \right) \quad (134)
\]

**Remark 7.** This example shows how simply the maximal leakage bound can be used, in contrast with the mutual information one. Indeed, following the proof of [4, Corollary 4], the mutual information of the same mechanism analyzed here is hard to compute directly and the quantity $I(S; H)$ is, in the end, upper-bounded using maximal leakage:
\[
I(S; H) \leq \sum_{i=1}^{k} \log \left( 1 + \frac{L_{\mu}(h_i)}{b_i} \right)
\] (135)
\[
\leq \sum_{i=1}^{k} \log \left( 1 + \frac{1}{b_i} \right) = \mathcal{L}(S \to H).
\] (136)

IV. **Adaptive Data Analysis**

Other than providing a nice generalization of the classical bounds for adaptive scenarios, maximal leakage can also be employed in adaptive data analysis. The model of adaptive composition we will be considering is identical to the setting in [30], [1], [2] and defined as follows:

**Definition 9.** Let $\mathcal{X}$ be a set. Let $S$ be a random variable over $\mathcal{X}^n$. Let $(A_1, \ldots, A_m)$ be a sequence of algorithms such that $\forall i : 1 \leq i \leq m$ \(A_i : \mathcal{X}^n \times Y_1 \times \ldots \times Y_{i-1} \to Y_i\). Denote with $Y_1 = A_1(S), Y_2 = A_2(S, Y_1), \ldots, Y_m = A_m(S, Y_1, \ldots, Y_{m-1}).$ The adaptive composition of $(A_1, \ldots, A_m)$ is an algorithm that takes as an input $S$ and sequentially executes the algorithms $(A_1, \ldots, A_m)$ as described by the sequence $(Y_i, 1 \leq i \leq m)$.

This level of generality allows us to formalize the behavior of a data analyst who, after viewing the previous outcomes of the analysis performed, decides what to do next. A potential analyst would execute a sequence of algorithms that are known to have a certain property (e.g. generalize well) when used without adaptivity. The question we would like to address is the following: is this property also maintained by the adaptive composition of the sequence? The answer is not trivial as, for every $i$, the outcome of $A_i$ depends both on $S$ and on the previous outputs, that depend on the data themselves. However, when this property is guaranteed by some measure that composes adaptively itself (like differential privacy or, as we will show soon, maximal leakage) then it can be preserved. Indeed, being robust to post-processing, Maximal Leakage allows us to retain the generalization guarantees it provides, regardless of how one may manipulate the outcome of the algorithm:

**Lemma 6** (Robustness to post-processing). Let $\mathcal{X}$ be the sample space and let $X$ be distributed over $\mathcal{X}$. Let $\mathcal{Y}$ and $\mathcal{Y}'$ be output spaces, and consider $A : \mathcal{X} \to \mathcal{Y}$ and $B : \mathcal{Y} \to \mathcal{Y}'$. Then, $\mathcal{L}(X \to B(A(X))) \leq \mathcal{L}(X \to A(X))$.

The proof is a direct application of the data processing inequality for maximal leakage. The useful implication of this result is as follows: in terms of maximal leakage, any generalization guarantees provided by $A$ cannot be
invalidated by further processing the output of $A$. Regarding adaptive composition of two algorithms, we retrieve the following:

**Lemma 7** (Adaptive Composition of Maximal Leakage). Let $A : X \rightarrow \mathcal{Y}$ be an algorithm such that $\mathcal{L}(X \rightarrow A(X)) \leq k_1$. Let $B : X \times \mathcal{Y} \rightarrow Z$ be an algorithm such that for all $y \in \mathcal{Y}$, $\mathcal{L}(X \rightarrow B(X, y)) \leq k_2$. Then $\mathcal{L}(X \rightarrow (A(X), B(X, A(X)))) \leq k_1 + k_2$.

The proof of this lemma relies crucially on the fact that maximal leakage depends on the marginal $P_X$ only through its support and can be found in Appendix A, along with the other proofs for this section.

In order to generalize the result to the adaptive composition of $n$ algorithms, we need to lift the property stated in the inequality (14) to more than two random variables.

**Lemma 8.** Let $n \geq 1$ and $X, A_1, \ldots, A_n$ be random variables.

$$\mathcal{L}(X \rightarrow (A_1, \ldots, A_n)) \leq \mathcal{L}(X \rightarrow A_1) + \mathcal{L}(X \rightarrow A_2|A_1) + \ldots + \mathcal{L}(X \rightarrow A_n|(A_1, \ldots, A_{n-1})).$$

(137)

The proof can be found in Appendix A. An immediate application of Lemma 8 leads us to the following result.

**Lemma 9.** Consider a sequence of $k \geq 1$ algorithms: $(A_1, \ldots, A_k)$ where for each $1 \leq i \leq k$, $A_i : X \times \mathcal{Y}_1 \times \ldots \times \mathcal{Y}_{i-1} \rightarrow \mathcal{Y}_i$. Suppose that for all $1 \leq i \leq k$ and for all $(y_1, \ldots, y_{k-1}) \in \mathcal{Y}_1 \times \ldots \times \mathcal{Y}_{i-1}$, $\mathcal{L}(X \rightarrow A_i(X, y_1, \ldots, y_{i-1})) \leq j_i$. Then, denoting by $A_1, \ldots, A_n$ the (random) outputs of the algorithms:

$$\mathcal{L}(X \rightarrow (A_1, \ldots, A_k)) = \mathcal{L}(X \rightarrow A_k) \leq \sum_{i=1}^{n} j_i.$$  

(138)

The conclusion to be drawn is straightforward: given a collection of algorithms that have bounded leakage (and thus good generalizations capabilities) even if the outcome of one of them is used to inform a subsequent analysis (hence, creating multiple dependencies on the data) the generalization guarantees of the composition can still be maintained.

Another interesting application of Corollary 3 in adaptive scenarios may be the following (same setting of [2]): consider the problem of bounding the probability of making a false discovery, when the statistic to apply is selected with some data dependent algorithm $T$. In this context, the classical guarantees that allow to upper-bound this probability by the significance value no longer hold. Measuring the information leaked from the data through $T$ with the maximal leakage we retrieve the following:

**Corollary 13.** Let $A : X^n \rightarrow T$ be a data dependent algorithm for selecting a test statistic $t \in T$. Let $X$ be a random dataset over $X^n$. Suppose that $\sigma \in [0, 1]$ is the significance level chosen to control the false discovery probability. Denote with $E$ the event that $A$ selects a statistic such that the null hypothesis is true but its $p$-value is at most $\sigma$. Then,

$$\mathbb{P}(E) \leq \exp(\mathcal{L}(X \rightarrow A(X))) \cdot \sigma.$$  

(139)

If the analyst wishes to achieve a bound of $\delta$ on the probability of making a false discovery in adaptive settings, the significance level $\sigma$ to be used should be no higher than $\delta / \exp(\mathcal{L}(X \rightarrow A(X)))$. Once again, if $A$ is independent
from $X$, we recover the bound of $\sigma$.

V. COMPARISON WITH OTHER BOUNDS

A. Maximal Leakage and Mutual Information

One interesting result in the field, that connects the generalization error with Mutual Information, under the same assumptions of Corollary 9, is the following (Theorem 8 of [3]):

$$P(E) \leq \frac{I(S; A(S)) + \log 2}{2n\eta^2 - \log 2}.$$

(140)

Let us compare this result with Corollary 9 in terms of sample complexity. From Corollary 9, it follows that using a sample size of

$$m \geq \left( \mathcal{L}(S \rightarrow A(S)) + \log(2/\delta) \right),$$

yields a learner for $\mathcal{H}$ with accuracy $\eta$ and confidence $\delta$ and this, in turn, implies that

$$m_{\mathcal{H}}(\eta, \delta) = O \left( \frac{\mathcal{L}(S \rightarrow A(S)) + \log(1/\delta)}{\eta^2} \right).$$

(142)

Using the same reasoning with inequality (140), we get :

$$m \geq \left( \frac{I(S; A(S)) + 1 + \delta^2}{2\eta^2 \delta} \right),$$

(143)

and thus,

$$m_{\mathcal{H}}(\eta, \delta) = O \left( \frac{I(A(S); S) \cdot 1}{\eta^2} \cdot \frac{1}{\delta} \right).$$

(144)

Since $\mathcal{L}(X \rightarrow Y) \geq I(X; Y)$ [8], in the regime where the two measures behave similarly, the reduction in the sample complexity is exponential in $\delta$. Moreover, as shown in [3], if we consider the case where $X = [d]$ and $\mathcal{H} = \{0, 1\}^X$, we have that the VC-dimension of $\mathcal{H}$ is $d$ and, being $\mathcal{L}(S \rightarrow A(S)) \leq \log(|\mathcal{H}|) \leq d$, our bound recovers exactly the VC-dimension bound [24], which is always sharp. Another source of comparison can be found in Example 1 and 2. Considering the same two scenarios, when $X$ is independent from $Y$, with the mutual information bound we retrieve:

$$P_{XY}(E) \leq \frac{1}{-\log(\max Y P_{X}(E_y))} = \frac{1}{-\log(\alpha)},$$

(145)

which is much weaker than the bound $P_{XY}(E) \leq \alpha$ that can be obtained directly from Ineq. (105). When $X = Y \sim \mathcal{U}([n])$ and $\alpha = 1/n$ we have that Ineq. (143) recovers:

$$1 = P_{XY}(E) \leq 1 + \frac{1}{\log n},$$

(146)

that is asymptotically tight, while with Ineq. (105) we recover:

$$1 = P_{XY}(E) \leq \frac{1}{n} \cdot n = 1,$$

(147)

and thus, our bound is matched with an exact equality.
B. Maximal Leakage and Differential Privacy

In this section we will compare our results with the generalization guarantees provided by differential privacy. The definition of $(\epsilon, \delta)$-DP is the following:

**Definition 10.** Let $A : \mathcal{X}^n \to \mathcal{Y}$ be a randomized algorithm. $A$ is $(\epsilon, \delta)$-differentially private if for every $S \subseteq \mathcal{Y}$ and every $x, y \in \mathcal{X}^n$ that differ only in one position:

$$
\Pr(A(x) \in S) \leq e^\epsilon \Pr(A(y) \in S) + \delta.
$$

(148)

A relationship with Maximal Leakage can be established:

**Lemma 10.** Let $A : \mathcal{X}^n \to \mathcal{Y}$ be an $\epsilon$-Differentially Private randomized algorithm, then

$$
\mathcal{L}(X \to A(X)) \leq \epsilon \cdot n.
$$

(149)

**Proof.** Let $Y = A(X)$ and assume, for simplicity, that $Y$ is a discrete random variable (the proof for continuous $Y$ follows very similar arguments). Fix some $\hat{x} \in \mathcal{X}^n$, $\forall x \in \mathcal{X}^n$ we have that $x$ and $\hat{x}$ differ in at most $n$ positions and, iteratively applying the definition of Differential Privacy, we have that $\Pr(Y = y | X = x) \leq e^{\epsilon \cdot n} \Pr(Y = y | X = \hat{x})$.

Thus:

$$
\mathcal{L}(X \to Y) = \log \sum_{y \in \mathcal{Y}} \max_{x \in \mathcal{X}^n} \Pr(Y = y | X = x)
$$

(150)

$$
\leq \log \sum_{y \in \mathcal{Y}} e^{\epsilon \cdot n} \Pr(Y = y | X = \hat{x})
$$

(151)

$$
= n \cdot \epsilon
$$

(152)

This suggests an immediate application of Corollary 9. Indeed, suppose $A$ is an $\epsilon$-DP algorithm, then:

$$
\exp(\mathcal{L}(X \to Y) - 2n\eta^2) \leq \exp(\epsilon n - 2n\eta^2)
$$

(153)

$$
= \exp(-n(2\eta^2 - \epsilon)).
$$

(154)

In order for the bound to be decreasing with $n$, we need $2\eta^2 - \epsilon > 0$ leading us to $\epsilon < 2 \cdot \eta^2$, where $\eta$ represents the accuracy of the generalization error and $\epsilon$ the privacy parameter. Thus, for fixed $\eta$, as long as the privacy parameter is smaller than $2 \cdot \eta^2$, we have guaranteed generalization capabilities for $A$ with an exponentially decreasing bound.

For $\epsilon \leq \eta/2$, it is shown in [30, Theorem 9] that $\Pr(E) \leq 1/4 \exp \left(-n\eta^2/12\right)$. It is easy to check that, for large enough $n$, our bound is tighter if $\epsilon \leq 23/12\eta^2$.

It is possible to see that enforcing differential privacy on some algorithm $A$ induces generalization guarantees similar to those stated in Corollary 3: suppose $A$ is an $\epsilon$-DP, with

$$
\epsilon \leq \sqrt{\frac{\log(1/\beta)}{2n}},
$$

(155)

and let $\max_y \mathcal{P}_X(E_y) \leq \beta$ then [30, Theorem 11]:

$$
\Pr(E) \leq 3\sqrt{\beta}.
$$

(156)
The results we are providing are qualitatively different: we do not require the imposition of some (possibly very strong) privacy criteria on the algorithm but rather propose a way of estimating how the probabilities we are interested in change, by measuring the level of dependence through Maximal Leakage. Moreover, given an \( \epsilon \)-Differential Private algorithm the bound obtained via Ineq. (149) can be tighter for certain regimes of \( \epsilon \). Indeed, let:

\[
\epsilon < \frac{\log (3/\sqrt{\beta})}{n} \leq \sqrt{\frac{\log(1/\beta)}{2n}}, \tag{157}
\]

using (156) we get a fixed bound of \( 3\sqrt{\beta} \), while with Corollary 5 and Lemma 10 we obtain that:

\[
\exp(\mathcal{L}(X \rightarrow Y)) \cdot \beta < \exp (\log (3/\sqrt{\beta})) \cdot \beta = 3\sqrt{\beta}. \tag{158}
\]

Hence, whenever the privacy parameter is lower than \( 1/n \log (3/\sqrt{\beta}) \) we are able to provide a better bound. Notice that Lemma 10 can be quite loose: using Lemma 3 it is possible to see that for classical mechanisms that imply \( \epsilon \)-DP, Maximal Leakage can be much lower that \( \epsilon \cdot n \). Indeed, using the result proven in Lemma 3, we can find such an example:

**Corollary 14.** Let \( g : \mathcal{X}^n \rightarrow \mathbb{R} \) be a function of sensitivity \( 1/n \) and let \( N \sim \text{Lap}(1/\epsilon n) \) then the mechanism \( \mathcal{M}(x) = g(x) + N \) is \( \epsilon \)-DP. Without loss of generality we have that \( |g(x)| \leq 1 \) (e.g. 0-1 loss) and thus:

\[
\mathcal{L}(X \rightarrow \mathcal{M}(X)) = \log(1 + \epsilon \cdot n) < \epsilon \cdot n. \tag{159}
\]

More importantly, the family of algorithms with bounded Maximal Leakage is not restricted to the differentially private ones. It is easy to see, for instance, that whenever there is a deterministic mapping and \( \epsilon \)-Differential Privacy is enforced on it, \( \epsilon \geq +\infty \). Trying to relax it to \( (\epsilon, \delta) \)–Differential Privacy does not help either, as one would need \( \delta \geq 1 \) rendering it practically useless. On the other hand, if the algorithm has a bounded range the Maximal Leakage from input to output is always bounded, since \( \mathcal{L}(X \rightarrow Y) \leq \min\{\log |\mathcal{X}|, \log |\mathcal{Y}|\} \). This simple observations allows us to immediately retrieve another result [1, Theorem 9]: \( \mathbb{P}(E) \leq |\mathcal{Y}| \cdot \beta \), where \( \beta \) is such that \( \mathbb{P}(E_{y}) \leq \beta \) for every \( y \). Indeed, given a a random variable \( Y \) with bounded support, \( \mathcal{L}(X \rightarrow Y) \leq \log |\mathcal{Y}| \) and from Corollary 3 we have that:

\[
\mathbb{P}(E) \leq \max_{y} \mathbb{P}(E_{y}) \exp (\mathcal{L}(X \rightarrow Y)) \leq \beta \cdot |\mathcal{Y}|. \tag{160}
\]

This shows how Corollary 3 is more general than both Theorems 6 and 9 of [1].

To conclude the comparison let us now state Corollary 9 with a general sensitivity \( c \):

\[
\mathbb{P}(E) \leq 2 \cdot \exp \left( \mathcal{L}(X \rightarrow Y) - \frac{2\eta^2}{c^2 n} \right). \tag{161}
\]

By contrast, [1, Cor. 7] states that whenever an algorithm \( A : \mathcal{X}^n \Rightarrow \mathcal{Y} \) outputs a function \( f \) of sensitivity \( c \) and is \( \eta/(cn) \)–DP then, denoting with \( S \) a random variable distributed over \( \mathcal{X}^n \) and with

\[
E = \{(S, f) : f(S) - \mathbb{E}(f) \geq \eta\},
\]

we have that:

\[
\mathbb{P}(E) \leq 3 \exp(-\eta^2/(c^2 n)). \tag{163}
\]

It is easy to see that we have a tighter bound whenever the accuracy \( \eta > n \cdot c \).
C. Maximal Leakage and Max Information

Another tool used in the line of work started by Dwork et al.\cite{1, 2} is the concept of max-information. The definition is the following:

**Definition 11.**\cite{1, Def. 10} Let $X, Y$ be two random variables jointly distributed according to $\mathcal{P}_{XY}$ and with marginals $\mathcal{P}_X, \mathcal{P}_Y$. The max-information between $X$ and $Y$, is defined as follows:

$$I_\infty(X; Y) = \log \sup_{(x,y) \in X \times Y} \frac{\mathcal{P}_{XY}\{(x,y)\}}{\mathcal{P}_X\{(x)\}\mathcal{P}_Y\{(y)\}},$$

while, the $\beta$–approximate max-information is defined as:

$$I_\infty^\beta(X; Y) = \log \sup_{\mathcal{O} \subseteq X \times Y, \mathcal{P}_{XY}(\mathcal{O}) > \beta} \frac{\mathcal{P}_{XY}(\mathcal{O}) - \beta}{\mathcal{P}_X\mathcal{P}_Y(\mathcal{O})}.$$  

One of the main reasons that led to the definition of approximate max-information is related to the generalization guarantees it provides, now recalled for convenience.

**Lemma 11.**\cite{1, Thm. 13} Let $X$ be a random dataset in $\mathcal{X}^n$ and let $A : \mathcal{X}^n \to \mathcal{Y}$ be such that for some $\beta \geq 0$, $I_\infty^\beta(X, A(X)) = k$. Let $Y = A(X)$ then, for any event $E \subseteq \mathcal{X}^n \times \mathcal{Y}$:

$$\mathcal{P}_{XY}(E) \leq e^k \mathcal{P}_X\mathcal{P}_Y(E) + \beta.$$  

The result looks quite similar to Corollary 9 but the two measures, Max-Information and Maximal Leakage, although related, can be quite different. In this section we will analyze the connections and differences between the two measures underlining the corresponding implications.

**Lemma 12.** Let $A : \mathcal{X}^n \to \mathcal{Y}$ be a randomized algorithm such that $I_\infty(X; A(X)) \leq k$. Then, $\mathcal{L}(X \to A(X)) \leq k$.

*Proof.* Denote with $Y = A(X)$. Having a bound of $k$ on the Max-Information of $A$ means that for all $x \in \mathcal{X}^n$, and $y \in \mathcal{Y}$, $\mathbb{P}(Y = y|X = x) \leq e^k \cdot \mathbb{P}(Y = y)$; and this implies that $\mathcal{L}(X \to Y) \leq k$. \qed

With respect to $\beta$-approximate max-information instead, we can state the following.

**Lemma 13.** Let $A : \mathcal{X}^n \to \mathcal{Y}$ be a randomized algorithm. Let $X$ be a random variable distributed over $\mathcal{X}^n$ and let $Y = A(X)$. Suppose $X, Y$ are discrete random variables and denote with $\mathcal{P}_{XY}$ the joint distribution and with $\mathcal{P}_X, \mathcal{P}_Y$ the corresponding marginals. For any $\beta \in (0, 1)$

$$I_\infty^\beta(X; A(X)) \leq \mathcal{L}(X \to A(X)) + \log \left(\frac{1}{\beta}\right).$$

*Proof.* Fix any $\beta > 0$. Using \cite{1, Lemma 18} we have that if

$$\mathcal{P}_{XY}\left(\left\{(x,y) \in \mathcal{X} \times \mathcal{Y}\mid \frac{\mathcal{P}_{XY}\{(x,y)\}}{\mathcal{P}_X\{(x)\}\mathcal{P}_Y\{(y)\}} \geq e^k\right\}\right) \leq \beta,$$

then

$$I_\infty^\beta(X; Y) \leq k.$$
Denote with $Y = \mathcal{A}(X)$. Notice that $\mathcal{L}(X \rightarrow Y) = \log \mathbb{E}_{p_Y} \left[ \sup_{x \in \mathcal{X}} \frac{p_{XY}(\{x,y\})}{p_X(\{x\}) p_Y(\{y\})} \right]$.

We have that

$$\mathcal{P}_{XY} \left( \left\{ (x, y) \in \mathcal{X} \times \mathcal{Y} \mid \frac{\mathcal{P}_{XY}(\{x, y\})}{\mathcal{P}_X(\{x\}) \mathcal{P}_Y(\{y\})} \geq \frac{e^{\mathcal{L}(X \rightarrow Y)}}{\beta} \right\} \right) \leq \frac{\mathbb{E}_{p_{XY}} \left[ \frac{p_{XY}(\{x, y\})}{p_X(\{x\}) p_Y(\{y\})} \right] \cdot \beta}{e^{\mathcal{L}(X \rightarrow Y)}} \leq \frac{\mathbb{E}_{p_{XY}} \left[ \sup_{x \in \mathcal{X}} \frac{p_{XY}(\{x, y\})}{p_X(\{x\}) p_Y(\{y\})} \right] \cdot \beta}{e^{\mathcal{L}(X \rightarrow Y)}} = \beta.$$ (170)

Hence, $I_\beta^\mathcal{X}(X; \mathcal{A}(X)) \leq \log \left( \frac{e^{\mathcal{L}(X \rightarrow Y)}}{\beta} \right) = \mathcal{L}(X \rightarrow Y) + \log \left( \frac{1}{\beta} \right).$ (171)

The role played by $\beta$ can lead to undesirable behaviors of $\beta$-approximate max-information. The following example, indeed, shows how $\beta$-approximate max-information can be unbounded while, in the discrete case, the Maximal Leakage between two random variables is always bounded by the logarithm of the smallest cardinality.

Example 3. Let us fix a $\beta \in (0, 1)$. Suppose $X \sim \text{Ber}(2/3)$. We have that $\mathcal{L}(X \rightarrow X) = \log |\text{supp}(X)| = \log 2$. For the $\beta$-approximate max-information we have: $I_\beta^\mathcal{X}(X; X) \geq \log((2/3) / (\beta^2)) = \log(1/\beta)$. It can thus be arbitrarily large.

Another interesting characteristic of max-information is that, differently from differential privacy, it can be bounded even if we have deterministic algorithms; this observation is implied by the connection with what in the literature is known as “description length” of an algorithm, and synthesized in the following result [1]: Let $\mathcal{A} : \mathcal{X}^n \rightarrow \mathcal{Y}$ be a randomized algorithm, for every $\beta > 0$,

$$I_\beta^\mathcal{X}(\mathcal{A}, n) \leq \log \left( \frac{\mathcal{Y}}{\beta} \right).$$ (173)

In contrast, with Maximal Leakage we have

$$\mathcal{L}(X \rightarrow \mathcal{A}(X)) \leq \log(|\mathcal{Y}|).$$ (174)

Clearly, being $0 < \beta$ typically very small in the key applications, the corresponding multiplicative factors in the bounds are $(|\mathcal{Y}| / \beta)$ and $|\mathcal{Y}|$, and the difference between the two bounds can be substantial. It is also worth noticing that (173) can be seen as a consequence of Lemma [13] and (174). The difference between the two measures is not uniquely restricted to deterministic mechanisms. The following is a simple example of a randomized mapping where Maximal Leakage is smaller than $\beta$-approximate-max-information, for small $\beta$.

Example 4. Consider $X \sim \text{Ber}(1/2)$ and a random variable $Y$ with support $\mathcal{Y} = \{0, 1, e\}$. Consider also the following randomized mapping: $\mathbb{P}(Y = e \mid X = x) = \alpha$ and $\mathbb{P}(Y = x \mid X = x) = 1 - \alpha$. That is, $Y$ can be interpreted as passing $X$ through a binary erasure channel with erasure probability $\alpha$. In this case, the Maximal Leakage is $\mathcal{L}(X \rightarrow Y) = \log(2 - \alpha)$ [3]; while, for $\beta$-Approximate max-information one finds (after a series of computations) that: $I_\beta^\mathcal{X}(X; Y) = \log(2 \cdot \max\{(1 - \alpha - \beta)/(1 - \alpha), (1 - \beta)/(1 + \alpha)\})$; It is easy to see how for a fixed $\alpha$ and for $\beta$ going to 0, Approximate Max-Information approaches $\log 2$ while Maximal Leakage is strictly smaller.
VI. CONCLUSION

Our aim was to bound the probability of an event $E$ under the joint distribution $\mathcal{P}_{X,Y}$ via information measures and the probability of the same event under the product of the marginals $\mathcal{P}_X \mathcal{P}_Y$. We provided a family of bounds characterized by four parameters $\alpha, \gamma, \alpha', \gamma' \geq 1$, constrained by the following equality $\frac{1}{\alpha} + \frac{1}{\gamma} = \frac{1}{\alpha'} + \frac{1}{\gamma'}$ (i.e., Holder’s conjugates). We explicit and analyze the following choices of parameters:

- with $\alpha' = \alpha$ and $\gamma' = \gamma$ we retrieve a family of bounds involving the Rényi’s divergence of order $\alpha$;
- with $\alpha' \to 1$ and consequently, $\gamma' \to \infty$ we retrieve a family of bounds involving Sibson’s Mutual Information of order $\alpha$;
- with $\alpha' \to 1$, $\gamma' \to \infty$, $\alpha \to \infty$ and $\gamma \to 1$, we retrieve a bound involving Maximal Leakage;

We also provided a family of bounds involving $f-$divergences where $f$ is an invertible convex function. We focused in particular on Maximal Leakage, since its semi-closed form and the dependence on $\mathcal{P}_X$ only through the support make it more amenable to analysis. Moreover, we show that the measure is robust under post-processing and composes adaptively. Although the robustness to post-processing is true for any information measure satisfying the data-processing inequality, the lack of a definition of conditional Sibson’s MI or $f-$mutual information, does not allows us, for the moment, to fully address the issue and verify whether or not such measures compose adaptively. Another interesting property of Maximal Leakage, instead, is that the bound it provides, represents a possible generalization of the classical inequalities in adaptive mechanisms. The comparison with the other approaches showed how this measure is less strict than Differential Privacy and yet still provides strong generalization guarantees. We also showed how, in regimes where Mutual Information and Maximal Leakage behave similarly, the leakage bound provides an exponential improvement in the sample complexity. Some bounds on expected generalization error were also provided but, probably as an artifact of the analysis, they are generally worse (for finite samples $n$) than the ones that use Mutual Information [5], [3], [12].

APPENDIX A

PROPERTIES OF MAXIMAL LEAKAGE

Lemma (Adaptive Composition of Maximal Leakage). Let $A : \mathcal{X} \to \mathcal{Y}$ be an algorithm such that $\mathcal{L}(X \to A(X)) \leq k_1$. Let $B : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ be an algorithm such that for all $y \in \mathcal{Y}$, $\mathcal{L}(X \to B(X,y)) \leq k_2$. Then $\mathcal{L}(X \to (A(X),B(X,A(X)))) \leq k_1 + k_2$.

The proof of this lemma relies crucially on the fact that maximal leakage depends on the marginal $\mathcal{P}_X$ only through its support.

Proof. Let us denote with $R_X$ the support of a random variable $X$. If we consider the second constraint in our
assumption and denoting with $Z_y = B(X, y)$, we get:

$$\forall y \in Y \mathcal{L}(X \to Z_y) \leq k_2 \iff (175)$$

$$\forall y \in Y \sum_{z_y \in R_{X|y}} \max_{x \in R_X} \mathbb{P}(z_y|x) \leq \exp(k_2) \iff (176)$$

$$\forall y \in Y \sum_{z_y \in R_{Z|Y=y}} \max_{x \in R_X} \mathbb{P}(z|x, y) \leq \exp(k_2). (177)$$

The last step holds, since every $y$ generates a family of conditional distributions $\mathbb{P}(z_y|x)$ through $B$ and this probability is just $\mathbb{P}(z|x, y)$, with $z = B(x, y)$. Using this observation in the conditional leakage of (14):

$$\mathcal{L}(X \to Z|Y) = \log \max_{y \in R_Y} \sum_{z \in R_{Z|Y=y}} \max_{x \in R_X} \mathbb{P}(z|x, y)$$

$$\leq \log \max_{y \in R_Y} \sum_{z \in R_{Z|Y=y}} \max_{x \in R_X} \mathbb{P}(z|x, y)$$

$$\leq \log \max_{y \in R_Y} \exp(k_2)$$

$$= k_2, (181)$$

leading us to the desired bound.

**Lemma.** Let $n \geq 1$ and $X, A_1, \ldots, A_n$ be random variables.

$$\mathcal{L}(X \to (A_1, \ldots, A_n)) \leq \mathcal{L}(X \to A_1) + \mathcal{L}(X \to A_2|A_1) + \cdots + \mathcal{L}(X \to A_n|(A_1, \ldots, A_{n-1})). (182)$$

**Proof.**

$$\mathcal{L}(X \to (A_1, \ldots, A_n)) = \mathcal{L}(X \to A^n)$$

$$= \mathcal{L}(X \to (A^{n-1}, A_n)), (184)$$

then the result follows from recursively applying the same argument to $\mathcal{L}(X \to A^{n-1})$.  

**APPENDIX B**

**EXAMPLES**

**Lemma** (Laplacian Noise). Let $g: \mathcal{X}^n \to \mathbb{R}$ be a function such that $g(x) \in [a, c], a < c \forall x \in \mathcal{X}^n$. The mechanism $\mathcal{M}(x) = g(x) + N$ where $N \sim \text{Lap}(b)$ is such that:

$$\mathcal{L}(X \to \mathcal{M}(X)) = \log \left( 1 + \frac{(c-a)}{b} \right) (185)$$
Proof. Let $Y = g(X) + N$, starting from Eq. (11),
\[
\exp(\mathcal{L}(X \rightarrow Y)) = \int_{\mathbb{R}} \sup_{x: f_X(x) > 0} f_{Y \mid X}(y \mid x) dy
\]
\[= \int_{\mathbb{R}} \sup_{x: f_X(x) > 0} f_N(y - g(x)) dy \quad (186)
\]
\[= \frac{1}{2b} \left( \int_{-\infty}^{+\infty} \sup_{x: f_X(x) > 0} \exp\left(-\frac{|y - g(x)|}{b}\right) dy \right) \quad (187)
\]
\[= \frac{1}{2b} \left( \int_{-\infty}^{a} \exp\left(-\frac{|y - a|}{b}\right) dy + \int_{a}^{c} dy \right) \quad (188)
\]
\[= \frac{1}{2b} \left( \int_{-\infty}^{0} \exp\left(-\frac{|z|}{b}\right) dz + (c - a) \right) \quad (189)
\]
\[+ \frac{1}{2b} \left( \int_{c}^{+\infty} \exp\left(-\frac{|w|}{b}\right) dw \right) \quad (190)
\]
\[= \frac{1}{2b} \left( (c - a) + 2 \int_{0}^{+\infty} \exp\left(-\frac{w}{b}\right) dw \right) \quad (191)
\]
\[= \frac{1}{2b} \left( (c - a) + 2b \right) = \left( 1 + \frac{(c - a)}{2b} \right) \quad (192)
\]

Corollary. Suppose that the hypothesis class is s.t. $\mathcal{H}$ is countable and is indexed in such a way that a hypothesis with a lower index is preferred over one with a higher index. Also suppose $\ell \in [0, 1]$. Let $S \in \mathbb{Z}^n$ be a sequence of samples, Noisy ERM is an algorithm $A : \mathbb{Z}^n \rightarrow \mathcal{H}$ defined in the following way:
\[
A(S) = \arg \min_{h \in \mathcal{H}} (L_S(h) + N_h),
\]
(193)
with $N_h$ exponential noise independently added to the empirical risk of each hypothesis on $S$. Let $|\mathcal{H}| = k$ and denote with $N_i$ the noise added to the hypothesis $h_i$. Let $\mathbb{E}[N_i] = b_i$ for every $i$, we have that for every $\eta \in (0, 1)$:
\[
\mathbb{P}(\text{gen}-\text{err}(A) \geq \eta) \leq 2 \exp\left( - \sum_{i=1}^{k} \log \left( 1 + \frac{1}{b_i} \right) - 2n\eta^2 \right). \quad (194)
\]
Choosing $b_i = i^{1.1/n^{1/3}}$ we retrieve:
\[
\mathbb{P}(\text{gen}-\text{err}(A) \geq \eta) \leq 2 \exp\left( -n(2\eta^2 - 11/n^{2/3}) \right) \quad (195)
\]

Proof. Suppose the hypothesis space is countable i.e. $|\mathcal{H}| = k$, suppose also that $\mathbb{E}[N_i] = b_i$ [4] (with $N_i$ being the noise added to the $i$-th hypothesis), we have that, by the data-processing inequality:
\[
\mathcal{L}(S \rightarrow H) \leq \mathcal{L}(L_S(h_i)_{i \in [k]} \rightarrow (L_S(h_i) + N_i)_{i \in [k]}). \quad (196)
\]

Proof. Suppose the hypothesis space is countable i.e. $|\mathcal{H}| = k$, suppose also that $\mathbb{E}[N_i] = b_i$ [4] (with $N_i$ being the noise added to the $i$-th hypothesis), we have that, by the data-processing inequality:
\[
\mathcal{L}(S \rightarrow H) \leq \mathcal{L}(L_S(h_i)_{i \in [k]} \rightarrow (L_S(h_i) + N_i)_{i \in [k]}). \quad (197)
\]
Also, denoting with $X_i = L_S(h_i)$ and with $Y_i = X_i + N_i$:

\[
\exp(\mathcal{L}((X_1, \ldots, X_k) \to (Y_1, \ldots, Y_k))) = \int \cdots \int_{-\infty}^{+\infty} \max_{x^n} f(y^n|x^n) dy^n
\]

(199)

\[
= \int \cdots \int_{-\infty}^{+\infty} \max_{x^n} \left( \prod_{i=1}^{k} f_N(y_i - x_i) \right) dy^n
\]

(200)

\[
= \int \cdots \int_{-\infty}^{+\infty} \max_{x^n} \left( \prod_{i=1}^{k} \frac{1}{b_i} e^{-\frac{(y_i - x_i)}{b_i}} \right) dy^n
\]

(201)

\[
= \prod_{i=1}^{k} \int_{-\infty}^{+\infty} \max_{x_i} \left( \frac{1}{b_i} e^{-\frac{(y_i - x_i)}{b_i}} \right) dy
\]

(202)

\[
= \prod_{i=1}^{k} \left( 1 + \frac{1}{b_i} \right).
\]

(203)

Equation (203), along with Corollary 9, implies that:

\[
P(\text{gen} - \text{err}(A) \geq \eta) \leq 2 \exp\left( \mathcal{L}(S\to H) - 2n\eta^2 \right)
\]

(204)

\[
= \exp\left( \sum_{i=1}^{k} \log \left( 1 + \frac{1}{b_i} \right) - 2n\eta^2 \right).
\]

(205)

Now, suppose that $b_i = i^{1.1}/n^{1/3}$,

\[
\mathcal{L}(S\to H) \leq \sum_{i=1}^{k} \log(1 + n^{1/3}/i^{1.1})
\]

(206)

\[
\leq n^{1/3} \sum_{i=1}^{+\infty} \frac{1}{i^{1.1}}
\]

(207)

\[
\leq (n^{1/3}) \cdot 11.
\]

(208)

We have that

\[
P(E) \leq 2 \exp(-n(2\eta^2 - 11/n^{2/3})).
\]

(209)

Thus, for every $\eta \in (0, 1)$, with $n$ large enough the bound approaches 0 exponentially fast. Moreover, using Corollary 11, we get:

\[
E[\text{gen} - \text{err}(A)] \leq \frac{1}{\sqrt{2n}} \left( \sqrt{\log(22n^{1/3})} + \frac{1}{\sqrt{\log(22n^{1/3})}} \right)
\]

(210)

\[
\square
\]

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