Renormalisation group analysis of 4D spin models and self-avoiding walk

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Abstract

We give an overview of results on critical phenomena in 4 dimensions, obtained recently using a rigorous renormalisation group method. In particular, for the $n$-component $|\varphi|^4$ spin model in dimension 4, with small coupling constant, we prove that the susceptibility diverges with a logarithmic correction to the mean-field behaviour with exponent $(n + 2)/(n + 8)$. This result extends rigorously to $n = 0$, interpreted as a supersymmetric version of the model that represents exactly the continuous-time weakly self-avoiding walk. We also analyse the critical two-point function of the weakly self-avoiding walk, the specific heat and pressure of the $|\varphi|^4$ model, as well as scaling limits of the spin field close to the critical point.

1 Introduction and results

$|\varphi|^4$ model Our results apply to the $n$-component $|\varphi|^4$ model on the 4-dimensional integer lattice $\mathbb{Z}^d$ with $d = 4$. To define the model, we approximate $\mathbb{Z}^d$ by a discrete torus $\Lambda = \Lambda_N = \mathbb{Z}^d/L^N \mathbb{Z}^d$ of side length $L^N$ with $L$ fixed (large), and eventually $N \to \infty$. To define the model and set notation, for coupling constants $g > 0$, $\nu, z \in \mathbb{R}$, a subset $X \subseteq \Lambda$, and a field $\varphi : \Lambda \to \mathbb{R}^n$, set

$$V_{g,\nu,z}(\varphi, X) = \sum_{x \in X} \left( \frac{1}{2} z \varphi_x \cdot (-\Delta \varphi_x) + \frac{1}{2} \nu |\varphi_x|^2 + \frac{1}{4} g |\varphi_x|^4 \right).$$

(1)

The $|\varphi|^4$ model is then defined as the probability measure

$$\frac{1}{Z_{g,\nu,\Lambda}} e^{-V_{g,\nu,1}(\varphi, \Lambda)} \prod_{x \in \Lambda} d\varphi_x,$$

where $d\varphi_x$ is the Lebesgue measure on $\mathbb{R}^n$ and $Z_{g,\nu,\Lambda}$ is a normalisation constant (the partition function). Assuming (for now) existence of the limits, the two-point function and susceptibility are defined by

$$G_{g,\nu}(x) = \lim_{N \to \infty} \langle \varphi_0 \cdot \varphi_x \rangle_{g,\nu,\Lambda_N}, \quad \chi(g, \nu) = \sum_{x \in \mathbb{Z}^d} G_{g,\nu}(x).$$

(3)
where $\langle \cdot \rangle_{g,\nu,\Lambda}$ is the expectation of \cite{2}, and the pressure and (singular part of the) specific heat are $p(g, \nu) = \lim_{N \to \infty} \frac{1}{|\Lambda_N|} \log Z_{g,\nu,\Lambda_N}$ and $c_H(g, \nu) = \frac{\partial}{\partial \nu^2} p(g, \nu)$.

**Weakly self-avoiding walk** Let $X$ be a continuous-time simple random walk on $\mathbb{Z}^d$ and denote by $E_0$ the expectation for the process with $X(0) = 0 \in \mathbb{Z}^d$. The self-intersection local time up to time $T$ is the random variable

$$I(T) = \int_0^T \int_0^T 1_{X(t_1) = X(t_2)} dt_1 dt_2.$$  \hspace{1cm} \text{(4)}$$

For $g > 0$ and $\nu \in \mathbb{R}$, and $x \in \mathbb{Z}^d$, the continuous-time weakly self-avoiding walk two-point function and susceptibility are defined by the (possibly infinite) integrals

$$G_{g,\nu}(x) = \int_0^\infty E_0 \left( e^{-gI(T)} 1_{X(T) = x} \right) e^{-\nu T} dT, \quad \chi(g, \nu) = \sum_{x \in \mathbb{Z}^d} G_{g,\nu}(x).$$  \hspace{1cm} \text{(5)}$$

Overviews of results on (weakly) self-avoiding walks can be found in Refs. \cite{7,30}. The weakly self-avoiding walk is believed to be in the same universality class as the strictly self-avoiding walk. It is exactly related to a supersymmetric version of the $|\varphi|^4$ model with a complex bosonic and a complex fermionic field, and this is the starting point for our analysis \cite{16,31,32}. The fermionic components effectively count negatively, and we thus refer to weakly self-avoiding walk case as the case $n = 0$ of the $|\varphi|^4$ model (with 0 interpreted as $2 - 2$).

The following theorem summarises the main results of Refs. \cite{1,6,16,20,33}. Here $A \sim B$ stands for $\lim A/B = 1$.

**Theorem 1.** Let $d = 4$, $n = 0, 1, 2, \ldots$, $p > 0$, and let $g > 0$ be small (depending on $n$ and $p$). For the $n$-component $|\varphi|^4$ model ($n \geq 1$), and for the weakly self-avoiding walk ($n = 0$), there exist critical values $\nu_c = \nu_c(g, n)$ such that, as $\varepsilon \downarrow 0$ respectively $|x| \to \infty$, the following hold (with constants $A, B, D > 0$ depending on $g, n$, and $C_p$ depending on $p$).

(i) (Ref. \cite{3}) For $n \geq 0$, the critical two-point function decays as

$$G_{g,\nu_c}(x) \sim B |x|^{-2}.$$  \hspace{1cm} \text{(6)}$$

(ii) (Refs. \cite{2,4}) For $n \geq 0$, the susceptibility obeys

$$\chi(g, \nu_c + \varepsilon) \sim A \varepsilon^{-1} (\log \varepsilon^{-1})^{(n+2)/(n+8)}.$$  \hspace{1cm} \text{(7)}$$

(iii) (Ref. \cite{8}) For $n \geq 0$, the correlation length of order $p > 0$ obeys

$$\frac{1}{\chi(g, \nu_c + \varepsilon)} \sum_x |x|^p G_{g,\nu_c+\varepsilon}(x) \sim C_p A^{-p/2} \varepsilon^{-p/2} (\log \varepsilon^{-1})^{p(n+2)/(2n+16)}.$$  \hspace{1cm} \text{(8)}$$

(iv) (Ref. \cite{3}) For $n \geq 1$, the specific heat obeys

$$c_H(g, \nu_c + \varepsilon) \sim D \begin{cases} (\log \varepsilon^{-1})^{(4-n)/(n+8)} & (n = 1, 2, 3) \\ \log \log \varepsilon^{-1} & (n = 4) \\ 1 & (n > 4) \end{cases}.$$  \hspace{1cm} \text{(9)}$$
(v) (Ref. [4] For $n \geq 1$, the spin field on the discrete torus of side length $L^N$ converges weakly to white noise if $\nu > \nu_c$, and to a massive Gaussian free field if $\nu \downarrow \nu_c$ as $N \to \infty$ appropriately.

(vi) (Ref. [35]) For $n \geq 0$, several multi-point functions have interesting $n$-dependent logarithmic corrections.

The limits defining the quantities on the left-hand sides are taken along the sequence $\Lambda_N$ with $L$ large enough, and the statement includes their existence in this case. For $n = 0,1,2$, independence of the sequence of most limits is known by other methods.

Item (iii) was obtained with Tomberg and Wallace, and (vi) with Tomberg. All results rely on a general renormalisation group method, outlined in the remainder of these proceedings. Several cases of the above results have been proved previously by different renormalisation group methods. In particular, (i) and a case of (vi) was proved for $n = 1$ in Refs. [23,24] (i) for $n = 1$ was independently proved in Ref. [22] versions of (ii), (iii) for $n = 1$ were obtained in Refs. [26,27], and (i) for a version of $n = 0$ in Ref. [28]. A hierarchical version of the 4-dimensional weakly self-avoiding walk was studied in Refs. [11,14,15,25] for complex $\nu$, which permits inversion of the Laplace transforms $G_{g,\nu}(x)$ in (5) and the analysis of the end-to-end distance. The above critical behaviour was first predicted over 40 years ago using non-rigorous methods; see in particular Refs. [10,29,34].

2 Method

The results of Theorem [11] are proved by a rigorous version of Wilson’s renormalisation group [35], developed in Refs. [1,5,6,13,16,20]. This method applies to bosonic fields (standard probability theory), fermionic fields (Grassmann fields), or both, and is compatible with supersymmetry. For brevity, we only discuss the (bosonic) $|\varphi|^4$ model.

From now on, we identify $V = (g, \nu, z) \in \mathbb{R}^3$ with the function $V_{g,\nu,z}$ defined in (1). Then for $m^2 > 0$ and $V_0 = (g_0, \nu_0, z_0)$ with $z_0 > -1$ and $g_0 > 0$, we define

$$Z_N(\varphi) = (\mathbb{E}_C \theta Z_0)(\varphi), \quad Z_0 = e^{-V_0(\varphi,\Lambda)}, \quad C = (-\Delta + m^2)^{-1} \quad (10)$$

where $\mathbb{E}_C \theta F$ denotes the convolution of $F$ with the Gaussian measure with covariance $C$. By a change of variables, the original model can be studied in terms of $Z_N$ with $g_0 = g(1+z_0)^2$ and $\nu_0 = (1+z_0)\nu - m^2$. It will be useful to carry out the analysis as a function of the four parameters $(m^2, g_0, \nu_0, z_0)$, and specialise later.

Progressive integration The starting point for the analysis of $Z_N$ is a positive definite finite-range decomposition [11,13] of the operator $(-\Delta + m^2)^{-1}$ ($m^2 > 0$) on $\Lambda_N$ as

$$(-\Delta_{\Lambda_N} + m^2)^{-1} = C_1 + \cdots + C_{N-1} + C_{N,N}, \quad (11)$$

satisfying $C_{j:x,y} = 0$ if $|x - y| > \frac{1}{2}L^j$ (finite range property), the estimates $|\nabla^a C_{j:x,y}| = O((1 + L^2(j-1)m^2)^{-s}L^{-(d-2+|a|)(j-1)})$ for any $s > 0$ and all $j < N$ (scaling estimates), and additional less significant properties. Moreover, similar estimates hold for $C_{N,N}$ for $m^2 \geq cL^{-2(N-1)}$, and we thus often write $C_N$ instead of $C_{N,N}$. Such a covariance decomposition enables a progressive evaluation [9] of $Z_N$ as the last element of

$$Z_{j+1} = \mathbb{E}_{C_{j+1}} \theta Z_j, \quad Z_0 = e^{-V_0(\Lambda)}. \quad (12)$$
The torus \( \Lambda_N \) is decomposed as the union over \( \mathcal{B}_j \) of disjoint blocks of side length \( L^j \) where \( \mathcal{B}_j \) is such that each block \( b \in \mathcal{B}_j \) is completely contained in a block \( B \in \mathcal{B}_{j+1} \). The set of polymers \( \mathcal{P}_j \) consists of unions of blocks in \( \mathcal{B}_j \). For any \( X \in \mathcal{P}_j \), we denote by \( \mathcal{B}_j(X) \) the blocks contained in \( X \). The finite range property asserts that the restrictions of a Gaussian field \( \zeta \) with covariance \( C_j \) to two polymers in \( \mathcal{P}_j \) that do not touch are independent.

**Renormalisation group**  
The renormalisation group map is a description of the *global* map \( Z_j \mapsto Z_{j+1} \) in terms of *local* coordinates \( I_j \) and \( K_j \), where \( I_j \) corresponds to the relevant and marginal directions in the Wilson renormalisation group [35], and \( K_j \) to the irrelevant directions. More concretely, there is an explicit function \( W_j \) such that the coordinate

\[
I_j(X, \varphi) = \prod_{B \in \mathcal{B}_j(X)} e^{-V_j(B, \varphi)}(1 + W_j(B, V_j, \varphi)), \quad (X \in \mathcal{P}_j)
\]

is completely determined by three *coupling constants* \( V_j = (g_j, \nu_j, z_j) \in \mathbb{R}^3 \), and \( I_j \) factors over \( j \)-blocks. The irrelevant coordinate \( K_j(X, \varphi) \) has the weaker factorisation property

\[
K_j(X \cup Y, \varphi) = K_j(X, \varphi)K_j(Y, \varphi) \quad \text{for } X, Y \in \mathcal{P}_j \text{ that do not touch.}
\]

Both \( I_j(X, \varphi) \) and \( K_j(X, \varphi) \) have the locality property that they only depend on \( \varphi \) in a neighbourhood of \( X \), as well as the normalisation \( I_j(\emptyset) = K_j(\emptyset) = 1 \). They can be multiplied by the *circle product* [21]

\[
(I_j \circ K_j)(X, \varphi) = \sum_{Y \in \mathcal{P}_j(X)} I_j(X \setminus Y, \varphi)K_j(Y, \varphi).
\]

For \( j = 0 \) one then has \( Z_j(\varphi) = e^{-u_j|\Lambda|(I_j \circ K_j)(\Lambda, \varphi)} \), with \( u_0 = 0 \), \( W_0 = 0 \), and \( K_0(X, \varphi) = 1_{X=\emptyset} \). The *renormalisation group map* is a lifting of the map \( Z_j \mapsto Z_{j+1} \) to a map \((u_j, I_j, K_j) \mapsto (u_{j+1}, I_{j+1}, K_{j+1})\), with \( u_j \in \mathbb{R} \), such that

\[
e^{-u_j|\Lambda|} \mathbb{E}_{C_{j+1}} \theta(I_j \circ K_j)(\Lambda, \varphi) = e^{-u_{j+1}|\Lambda|}(I_{j+1} \circ K_{j+1})(\Lambda, \varphi).
\]

**Flow of coupling constants**  
In Ref. [5] the map \( V_j \mapsto V_{j+1} \) is defined to second order by perturbation theory. In Refs. [19, 20] the non-perturbative correction and the complete map \((V_j, K_j) \mapsto (V_{j+1}, K_{j+1})\) are defined, as well suitable function spaces of \( K_j \) and estimates that show that \( K_j \) is contractive in these spaces.

In particular, the evolution of \( V_j \) and thus \( I_j \) is determined by a flow of coupling constants, which similarly as in Wilson’s non-rigorous analysis, are given by

\[
g_{j+1} = g_j - \beta_j g_j^2 + r_{g,j} \tag{17}
\]

\[
\mu_{j+1} = L^2 \mu_j \left(1 - \frac{n + 2}{n + 8} \beta_j g_j \right) + (\cdots) + r_{\mu,j}. \tag{18}
\]

Here \( \mu_j = L^{2j} \nu_j \), the \( (\cdots) \) denote other explicit terms which are at most quadratic in \( V \), and the \( r \) are non-perturbative remainders that depend on \( K_j \) and are third order in \( V \). The explicit flow
of $z_j$ is also important, but conceptually less significant, and we mostly ignore it in this exposition. The coefficients $\beta_j$ are given by

$$\beta_j = \sum_{x \in \mathbb{Z}^d} \left( w_{j+1}(x)^2 - w_j(x)^2 \right), \quad w_j(x) = \sum_{k=1}^j C_k(x).$$

To study the approach of the critical point rather than only the critical point itself, the $\beta_j$ here depend on $m^2 > 0$ through the covariances $C_k$. They have asymptotic behaviour $\lim_{j \to \infty} \beta_j \sim (n+8)(\log L)/(16\pi^2)$ as $j \to \infty$, and obey $\lim_{N \to \infty} \sum_j \beta_j \to (n+8)B_{m^2}$ where $B_{m^2} = \sum_{x \in \mathbb{Z}^d} [(-\Delta_x + m^2)^{-1} - 1]_2^2 \sim (n+8) \log m^2/(16\pi^2)$ is the bubble diagram of the free Green function. The logarithmic divergence of $B_{m^2}$ is ultimately responsible for the criticality of $d = 4$ and the logarithmic corrections in Theorem 1.

The control of $K_j$ is at the heart of the issues to obtain a mathematically rigorous result. The analysis in Refs. [19,20] exploits the finite range property of the covariances $C_k$ to avoid the need for cluster expansions. An example of this approach in a simpler context can be found in Ref. [12].

The (non-hyperbolic) dynamical system $(V_j, K_j) \mapsto (V_{j+1}, K_{j+1})$ is analysed in Refs. [4,6]. For $(m^2, g_0) \in (0, \delta)^2$ with $\delta > 0$ small, initial conditions $(\nu_0, \nu_0) = (\nu_0^c(m^2, g_0), \nu_0^c(m^2, g_0))$ are determined such that $V_j$ remains bounded and $K_j \to 0$, as $j \to \infty$. Along this renormalisation group trajectory the observables discussed in Theorem 1 are studied. This will be exemplified in the case of the susceptibility. The susceptibility is also fundamental to relate $(\nu_0^c, z_0^c)$ to the critical points $\nu_c(g)$ of the original models, using the change of variables mentioned below [10] and implicit function theory.

**Susceptibility**

We sketch the proof of (17). For a test function $h : \Lambda \to \mathbb{R}$, set $\Sigma_N(h) = \mathbb{E}_C(Z_0(\varphi)e^{(h, Ch)})$. Then, by completion of the square,

$$\frac{\Sigma_N(h)}{\Sigma_N(0)} = e^{-\frac{1}{2}(h, Ch) Z_N(Ch)} \frac{I_N(\Lambda, Ch) + K_N(\Lambda, Ch)}{I_N(\Lambda, 0) + K_N(\Lambda, 0)},$$

using that $I_N \circ K_N(\Lambda) = I_N(\Lambda) + K_N(\Lambda)$ since there is only one $N$-block on $\Lambda_N$. Assuming that $(g, \nu)$ and $(m^2, g_0, \nu_0, z_0)$ are related as below [10], the susceptibility is obtained (up to a factor $(1 + z_0^2)$ by differentiating twice with respect to a constant test function $h$. In particular, if $(m^2, g_0, \nu_0, z_0) = (m^2, g_0, \nu_0^c, z_0^c)$ is critical according to the dynamical system analysis, then $K_N \to 0$ in a suitable norm, and using $C1 = m^{-2}1$ for constant test function $1_x = 1$ as well as the explicit form of $I_N$, we obtain the identity

$$\chi(g, \nu) = (1 + z_0) \lim_{N \to \infty} \left( \frac{1}{m^2} - \nu_0 \right) \left( \frac{1}{m^4} D^2 W(\Lambda; 0, 1, 1) + D^2 K(\Lambda; 0, 1, 1) \right) \frac{1}{m^2}.$$

In particular, $\nu \downarrow \nu_c(g)$ corresponds to $m^2 \downarrow 0$ under the critical choice of the four coupling constants, and the singular behaviour of $\chi$ at $\nu = \nu_c(g)$ is encoded in the relationship between $m^2$ and $(g, \nu)$. To understand $\chi$, we derive an equation for $\frac{\partial}{\partial g} \chi = (1 + z_0) \frac{\partial}{\partial \nu_0} \chi$. The derivative can be taken inside the limit in (21), and is taken with $m^2, g_0, z_0$ fixed. Then the $\nu_0$-derivative of $1/m^2$...
vanishes and the main contribution to $\frac{\partial}{\partial \nu} \chi$ is given by $-\nu_N/m^4$ (with the contribution due to $K_N$ again subleading), where the prime denotes the derivative with respect to $\nu_0$. By differentiating (17)–(18), along the critical trajectory, for which coupling constants are controlled, it can be shown that

$$\nu_j' \sim (1 + O(g)) \left( \frac{g_j}{g_0} \right)^{(n+2)/(n+8)}.$$  \hfill(22)

The coupling constant $g_j$ tends to an $m^2$-dependent limit $g_\infty$. As $m^2 \downarrow 0$,

$$g_\infty \sim \frac{1}{(n + 8) B m^2} \sim \frac{16\pi^2}{(n + 8) \log m^2}.$$  \hfill(23)

This leads to

$$\frac{\partial \chi}{\partial \nu}(g, \nu) = \frac{(1 + z_0)^2}{m^4} \lim_{N \to \infty} \left( -\nu_N' + \frac{\partial}{\partial \nu_0} \frac{1}{|\Lambda|} \left( D^2 W(\Lambda; 0; 1, 1) + D^2 K(\Lambda; 0; 1, 1) \right) \right)$$

$$\sim c \frac{(\log m^2)^{n+2}/(n+8)}{m^4}.$$  \hfill(24)

From (21) and (24) we obtain $\frac{\partial}{\partial \nu} \chi \sim c (\log \chi)^{(n+2)/(n+8)} \chi^2$ as $m^2 \downarrow 0$, and the claim

$$\chi(g, \nu_c(g) + \varepsilon) \sim A \varepsilon^{-1} (\log \varepsilon^{-1})^{(n+2)/(n+8)} (\varepsilon \downarrow 0),$$  \hfill(25)

follows.

**Other observables** The analysis of the specific heat follows a similar strategy as that for the susceptibility. The pointwise analysis of the two-point and multi-point functions require the analysis of an additional flow of *observable coupling constants*, which depends on the bulk flow (17)–(18), but not vice-versa. In particular, it is also shown that $G_{\nu_c}(x) \sim (1 + z_0)(-\Delta)^{-1}_{0x}$ as $|x| \to \infty$. Together with (21) this allows to characterise $m^2$ as the *renormalised mass* and $1 + z_0$ as the *field strength renormalisation*. The scaling limit result is obtained by analysing (20) with general smooth test functions $h$.

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