VIRTUAL POSETS, SHUFFLE ALGEBRAS AND ASSOCIATORS

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ABSTRACT. We provide a method to construct new associators out of Drinfel’d’s KZ associator. We obtain two analytic families of associators whose coefficients we can describe explicitly by a generalization of multiple zeta values. The two families contain two different paths that deform the Drinfel’d KZ associator into the trivial associator 1. We show that both paths are injective, that is, all of the associators parametrized by them are different. Our construction is based on the observation that one can recover multiple polylogarithms as generating functions of order polynomials of certain formally constructed posets.

1. INTRODUCTION

In [Dri90] Drinfel’d defined an associator, a group-like element $\phi(A, B)$ in the $\mathbb{C}$-algebra of formal power series in two non-commuting variables that satisfies certain conditions, called the pentagon and hexagon equations (see Section 2.1).

Associators appear in many fields of mathematics and physics. For instance, they allow to introduce a tensor product on the category of representations for a quasitriangular quasi-Hopf algebra $A$, and a morphism $a : (V_1 \otimes V_2) \otimes V_3 \simeq V_1 \otimes (V_2 \otimes V_3)$ where $V_1, V_2, V_3$ are representations of $A$, and $a$ satisfies the pentagon axiom of Mac Lane coherence theorem [Lan71]. Simply put, associators can be used to define an associative tensor product when constructing certain monoidal categories.

The set of associators is not empty; it contains the trivial associator $\phi(A, B) = 1$, the KZ associator $\phi_{KZ}$ which Drinfel’d constructed using the monodromy of the solutions to the Knizhnik–Zamolodchikov equation, and the anti-KZ associator $\phi_{aKZ}(A, B) = \phi_{KZ}(-A, -B)$.

An explicit expression for the Drinfel’d KZ associator was given by Le and Murakami [LM95] who described it as a power series in multiple zeta values with coefficients polynomial in $A, B$. Using the same techniques, Furusho [Fur03] found an equivalent formula for $\phi_{KZ}$ as a power series in $A, B$ with coefficients sums of multiple zeta values. Alekssev and Torossian constructed another associator [AT10], known as the AT associator, whose coefficients were also described by Furusho [Fur18]. A third associator was defined by Deligne and its coefficients were described by Brown in [Bro14]. An infinite number of associators interpolating $\phi_{KZ}$, $\phi_{AT}$ and $\phi_{aKZ}$ were constructed by Rossi and Willwacher.
Their coefficients are given in terms of integrals related to cocycles in Kontsevich’s graph complex.

In this paper we replace the coefficients of the Drinfel’d KZ associator, which are multiple zeta values, and prove that the resulting power series are new associators. The new coefficients are given by truncated multiple zeta values, for example,

\[
\zeta^m(k_1, k_2) = \sum_{m<n_1, n_1+m<n_2} \frac{1}{n_1^{k_1} n_2^{k_2}}, \quad m \in \mathbb{N}.
\]  

The main difficulty in this approach is that to define a function on multiple zeta values, we need to show that it does not depend on their representation as an iterated integral or as an iterated sum. To construct a well-defined function, we show that the coefficients of the Drinfel’d KZ associator are labeled by certain formally constructed posets, which we call virtual posets. This identification is based on [ANBDC21] where we constructed a power series representation of posets, generalizing Stanley’s order polynomials [Sta70].

For example the series associated to the poset consisting of two non-comparable points \(a, b\) is

\[
\frac{x}{(1-x)^2} + 2\frac{x^2}{(1-x)^3},
\]

while the series associated to two points with an order \(a < b\) is

\[
\frac{x^2}{(1-x)^3}.
\]

The introduction of virtual posets and the evaluation of their power series at \(x = 1\) enables us to encode multiple zeta values, both classical and of the form (1.1). From a sequence of maps of virtual posets we obtain then a sequence of maps of multiple zeta values. We prove that this maps preserve the structure of Shuffle algebras. This allows us to define a sequence of shuffle algebras over \(\mathbb{R}\). Each member of this sequence gives rise to a different associator by replacing the coefficients of the KZ associator accordingly.

The construction can be generalized in two ways: Firstly, using the iterated integral representation of (truncated) multiple zeta values we extend the former sequence to a disk of associators. It contains a path that interpolates between \(\phi_{KZ}\) and the trivial associator. Secondly, we can define additional associators by replacing the coefficients of members of the first family by (images of elements of) other shuffle algebras; we demonstrate this by describing two families of shuffle algebras together with an involution map that is the identity on multiple zeta values, but acts non-trivially on the other algebras.

The paper is organized as follows. In Section 2 we review the basic objects of our study, associators, iterated integrals and shuffle algebras.
Section 3 discusses the theory of order series of posets, based on the framework developed in [ANBDC21]. We introduce certain formally constructed virtual posets whose order series are polylogarithms and truncated versions of them (as in equation (1.1)). As a consequence we can express the coefficients of the Drinfel’d KZ associator $\phi_{KZ}$ by these virtual posets.

This will allow then to change these coefficients via certain maps of posets. We prove in Theorem 3.4 that these maps indeed give rise to morphisms of shuffle algebras.

In Section 4 we apply this to the study of associators. Our main result is Theorem 4.3 which explains how to construct a new associator out of $\phi_{KZ}$ given a shuffle algebra morphism from multiple zeta values. We describe its coefficients explicitly in Corollary 4.4. An immediate generalization of this construction gives rise to a disk of associators. We show that it contains a path from $\phi_{KZ}$ to 1 and that every point on this path describes a different associator (Theorem 4.5). In Section 4.3 we introduce an endomorphism of shuffle algebras to define a second path of associators from $\phi_{KZ}$ to 1 and prove that it is in fact different from the first one.

2. Main definitions

We recall the definition of associators (Section 2.1) and iterated integrals (Section 2.2). In Section 2.3 we abstract multiple zeta values as shuffle algebras and introduce some auxiliary results. Then we describe the coefficients of the Drinfel’d KZ associator.

2.1. The Drinfel’d KZ associator. Consider the ring $\mathbb{C} \langle \langle A, B \rangle \rangle$ of formal power series over $\mathbb{C}$ in non-commutative variables $A, B$. It is a Hopf algebra with coproduct

$$\Delta(A) = A \otimes 1 + 1 \otimes A, \quad \Delta(B) = B \otimes 1 + 1 \otimes B.$$ 

and antipode

$$S(A) = -A, \quad S(B) = -B.$$ 

An associator (or Drinfel’d series [Tom01]) is a power series $\phi(A, B) \in \mathbb{C} \langle \langle A, B \rangle \rangle$ whose leading term is 1, that is group-like, $\Delta \phi(A, B) = \phi(A, B) \otimes \phi(A, B)$, that satisfies the pentagon relation:

$$\phi(t_{12}, t_{23} + t_{24}) \phi(t_{12} + t_{23}, t_{34}) = \phi(t_{23}, t_{34}) \phi(t_{12} + t_{13}, t_{24} + t_{34}) \phi(t_{12}, t_{23}),$$

and, for some $\mu \in \mathbb{C}$, the series satisfies the hexagon relations:

$$e^{\mu t_{13} / 2 + t_{24}} = \phi(t_{13}, t_{12}) e^{\mu t_{14} / 2} \phi(t_{13}, t_{23})^{-1} e^{\mu t_{23} / 2} \phi(t_{12}, t_{23}),$$

$$e^{\mu t_{12} / 2 + t_{13}} = \phi(t_{23}, t_{13})^{-1} e^{\mu t_{14} / 2} \phi(t_{12}, t_{13})^{-1} e^{\mu t_{13} / 2} \phi(t_{12}, t_{23})^{-1}.$$

Here the variables $t_{ij}$ obey the condition of locality

$$[t_{ij}, t_{kl}] = 0 \quad \text{for distinct } i, j, k, l.$$
and the four term relations 4T

\[ [t_{ij}, t_{ik} + t_{jk}] = 0 \text{ for distinct } i, j, k. \] (2.5)

The Knizhnik–Zamolodchikov equation is

\[ G'(z) = \frac{1}{2\pi i} \left( \frac{A}{z} + \frac{B}{z - 1} \right) G(z), \] (2.6)

where \( G(z) \in \mathbb{C} \langle \langle A, B \rangle \rangle \) is supposed to be analytic with respect to the variable \( z \). According to [Dri90, Tom01, CDM12] there are two solutions of (2.6), denoted by \( G_{(\_)} \) and \( G_{(\_ \_)} \) that satisfy

- \( G_{(\_)}(z) = f(z)z^{A/2\pi \sqrt{-1}} \)
- \( G_{(\_ \_)}(z) = g(1 - z)(1 - z)^{B/2\pi \sqrt{-1}} \)

where \( f(z) \) and \( g(z) \) are analytic in a neighborhood of \( 0 \in \mathbb{C} \) with \( f(0) = g(0) = 1 \in \mathbb{C} \langle \langle A, B \rangle \rangle \).

Define the Drinfel’d KZ associator \( \varphi_{KZ} \) by

\[ G_{(\_ \_)}(z) = G_{(\_)}(z) \varphi_{KZ}. \]

By (2.6) it is independent of \( z \) and therefore an element of \( \mathbb{C} \langle \langle A, B \rangle \rangle \).

For simplicity we work with \( \phi_{KZ} \) defined by \( \phi_{KZ}(\frac{A}{2\pi}, \frac{B}{2\pi}) = \varphi_{KZ}(A, B) \).

**Proposition 2.1.** \( \phi_{KZ} \) is an associator.

**Proof.** See [Tom01, Proposition D.8]. \( \square \)

Note that the inverse of \( \phi_{KZ} \) satisfies

\[ \phi_{KZ}(A, B)^{-1} = \phi_{KZ}(B, A). \] (2.7)

2.2. **Iterated integrals.** Consider the number \( \zeta(3) = \sum_{n \geq 1} \frac{1}{n^3} \). It can be represented as an iterated integral,

\[ \zeta(3) = \int_0^1 \left( \int_0^{t_3} \left( \int_0^{t_2} \frac{dt_1}{1 - t_1} \right) \frac{dt_2}{t_2} \right) \frac{dt_3}{t_3} = \int_{t_1 \leq t_2 \leq t_3 \leq 1} \frac{dt_3}{t_3} \frac{dt_2}{t_2} \frac{dt_1}{t_1}. \]

One condition for integrals of this form to converge is that the first form that is integrated is \( \frac{dx}{1-x} \) and the last one is \( \frac{dx}{x} \) (cf. [Bro09, BB01]). A similar representation works for multiple zeta values:

\[ \zeta(k_1, \ldots, k_r) = \sum_{\substack{0 < n_1 < \ldots < n_r \leq \infty \ n_1^{k_1} n_2^{k_2} \cdots n_r^{k_r}}} \frac{1}{n_1^{k_1} n_2^{k_2} \cdots n_r^{k_r}} \]

\[ = \int_{0 \leq t_1, \ldots, t_r, k_r \leq 1} \frac{dt_{r,k_r}}{t_{r,k_r}} \frac{dt_{r,2}}{1 - t_{r,1}} \frac{dt_{r,1}}{t_{r,1}} \frac{dt_{1,k_1}}{t_{1,k_1}} \frac{dt_{1,2}}{1 - t_{1,1}}. \]

Note that in this paper we use the convention of [LM95, Tom01, Fur03, CDM12] on the indices of multiple zeta value and iterated sums, which differs from [LQ17] and [Eul75].
More general iterated integrals are defined as follows. Let $M$ be a smooth manifold and $\gamma : [0, 1] \to M$ a piece-wise smooth path on $M$. Given a family $\omega_i$ of smooth 1-forms on $M$ we define the iterated integral

$$ \int_{\gamma} \omega_n \cdots \omega_1 = \int_{0 \leq t_1 \leq \ldots \leq t_n \leq 1} f_n(t_n)dt_n \cdots f_1(t_1)dt_1 $$

where $f_i(t)dt = \gamma^* \omega_i$ denotes the pullback of $\omega_i$ along the path $\gamma$.

**Proposition 2.2.** [Bro09, Proposition 2.2] Iterated integrals satisfy the shuffle product formula

$$ \int_{\gamma} \omega_{r+s} \cdots \omega_{r+1} \int_{\gamma} \omega_r \cdots \omega_1 = \sum_{\sigma \in \Sigma(r,s)} \int_{\gamma} \omega_{\sigma(r+s)} \cdots \omega_{\sigma(1)}. $$

Here the group of $(r, s)$-shuffles $\Sigma(r, s)$ consists of all permutations of $\{1, \ldots, r + s\}$ that preserve the individual orders:

$$ \Sigma(r, s) = \{ \sigma \in \Sigma(r + s) : \sigma^{-1}(r) > \cdots > \sigma^{-1}(1) \text{ and } \sigma^{-1}(r + s) > \cdots > \sigma^{-1}(r + 1) \}. $$

2.3. **Shuffle algebras.** We now define the shuffle algebra of words on $\{A, B\}$ (compare with [LQ17] and [Fur10]). Let $R = \mathbb{Q}\langle A, B \rangle$ be the non-commutative graded $\mathbb{Q}$-polynomial ring generated by the variables $A$ and $B$ with $deg(A) = deg(B) = 1$. Let $h^0$ be the $\mathbb{Q}$-linear subspace $Q_{1_R} + ARB$ where $1_R$ is the empty word of the free group of words in the alphabet set $\{A, B\}$. Abusing notation we denote $1_R$ as a simple 1. We define the $\sqcup\sqcap$ product on $h^0$ by

- $1 \sqcup w = w \sqcup 1 = w$,
- $aw_1 \sqcup bw_2 = a(w_1 \sqcup bw_2) + b(aw_1 \sqcup w_2)$ for $a, b \in \{A, B\}$,

and by requiring $\sqcup\sqcap$ to be bilinear.

The algebra $(h^0, \sqcup\sqcap)$ is commutative. The existence of a shuffle product on multiple zeta values is equivalent to the existence of an algebra morphism

$$ \zeta : (h^0, \sqcup\sqcap) \longrightarrow (\mathbb{R}, \cdot), $$

$$ A^{q_n}B^{p_n} \cdots A^{q_1}B^{p_1} \longrightarrow \zeta(1, \ldots, 1, q_1 + 1, \ldots, 1, \ldots, 1, q_n + 1), $$

where $\cdot$ denotes the usual product on $\mathbb{R}$, see [LQ17, Corollary 2.2].

The connection to the iterated integrals of the previous subsection is given by interpreting the letter $A$ as the 1-form $\frac{dx}{x}$ and $B$ as $\frac{dx}{1-x}$ (cf. [Bro09]). For example,

$$ A^2B \longrightarrow \int_{0 \leq t_0 \leq t_1 \leq t_2 \leq 1} \frac{dt_2 dt_1 dt_0}{t_2 t_1 1-t_0} = \sum_{n \geq 0} \frac{1}{n^3} = \zeta(3). $$

It follows that once we compute the structural constants of $(h^0, \sqcup\sqcap)$, we know how to describe the product of multiple zeta values as linear combinations of multiple zeta
values. For example, in [LQ17, Theorem 2.1] the structural constants $c_{\alpha_1, \ldots, \alpha_{n+m}} \in \mathbb{Q}$ of the following product

$$A^{k_n} B \cdots A^{k_1} B \sqcup A^{l_m} B \cdots A^{l_1} B$$

$$= \sum_{\sum_{i=1}^{n+m} \alpha_i = \sum_{i=1}^{k_1} k_i + \sum_{i=1}^{l_1} l_i, \alpha_i \geq 0, \alpha_{n+m} \geq 0} c_{\alpha_1, \ldots, \alpha_{n+m}} A^{\alpha_{n+m}} B \cdots A^{\alpha_1} B$$

were computed explicitly. Applying the map $\zeta$ gives

$$\zeta(k_1 + 1, \ldots, k_n + 1) \zeta(l_1 + 1, \ldots, l_m + 1)$$

$$= \sum_{\sum_{i=1}^{n+m} \alpha_i = \sum_{i=1}^{k_1} k_i + \sum_{i=1}^{l_1} l_i} c_{\alpha_1, \ldots, \alpha_{n+m}} \zeta(\alpha_1 + 1, \ldots, \alpha_{n+m} + 1).$$

Now assume there is a second shuffle morphism $\zeta': (\mathfrak{h}^0, \sqcup) \to (\mathbb{R}, \cdot)$.

**Lemma 2.3.** The morphism of vector spaces

$$\sigma: \mathbb{Q} \langle \zeta(w) | w \in \mathfrak{h}^0 \rangle \longrightarrow \mathbb{Q} \langle \zeta'(w) | w \in \mathfrak{h}^0 \rangle,$$

$$\zeta(w) \longmapsto \zeta'(w),$$

is a morphism of $\sqcup$-algebras.

**Proof.** Assume $w \sqcup w' = \sum_i d_i w_i$ with $d_i \in \mathbb{N}$. Then

$$\sigma(\zeta(w)) \sigma(\zeta(w')) = \zeta'(w) \zeta'(w')$$

$$= \sum_i d_i \zeta'(w_i)$$

$$= \sum_i d_i \sigma(\zeta(w_i))$$

$$= \sigma(\sum_i d_i \zeta(w_i))$$

$$= \sigma(\zeta(w) \zeta(w')).$$

□

We now recall the explicit description of the coefficients of the $KZ$ associator found in [Fur03].

A word is an element of $\mathcal{R}$ which is monic and monomial not including 1. For every word $w$ define $wt(w)$ to be the sum of the exponents of $A$ and $B$ in $w$, and $dp(w)$ to be the sum of exponents of $B$ in $w$. Consider the surjection $\mathcal{R} \to \mathcal{R}/(BR + RA)$ that kills all words which start with $B$ or end with $A$. There is a $\mathbb{Q}$-linear map $f: \mathcal{R} \to \mathcal{R}$, defined as the composition

$$f: \mathcal{R} \longrightarrow \mathcal{R}/(BR + RA) \simeq \mathbb{Q}1 + ARB \longrightarrow \mathcal{R}.$$
Proposition 2.4. [Fur03] Each coefficient of
\[ \phi_{KZ}(A, B) = 1 + \sum_{\text{words } w} I(w)w \]
can be expressed as follows.

- if \( w \in ARB \), then
  \[ I(w) = (-1)^{dp(w)} \zeta(w). \]

- if \( w = B^r V A^s \) with \( r, s \geq 0 \) and \( V \in ARB \), then
  \[ I(w) = (-1)^{dp(w)} \sum_{0 \leq a, b \leq s} (-1)^{a+b} \zeta(f(B^a \sqcup B^{r-a} V A^{s-b} \sqcup A^b)). \]

- if \( w = B^r A^s \) with \( r, s \geq 0 \), then
  \[ I(w) = (-1)^{dp(w)} \sum_{0 \leq a, r, b \leq s} (-1)^{a+b} \zeta(f(B^a \sqcup B^{r-a} A^{s-b} \sqcup A^b)). \]

The first coefficients are:
\[
\phi_{KZ}(A, B) = 1 - \zeta(2)[A, B] - \zeta(3)[A, [A, B]] + \zeta(1, 2)[[A, B], B] \\
+ \zeta(4)[A, [A, [A, B]]] - \zeta(1, 1, 2)[[[A, B], B], B] \\
+ \frac{\zeta(2)^2}{2} [A, B]^2 + \zeta(1, 3)[A, [[A, B], B]] + \ldots
\]
where we used commutators \([A, B] = AB - BA\) to assemble everything into a more compact form, and the identity \( \zeta(2, 2) + 2\zeta(1, 3) = \frac{\zeta(2)^2}{2} \).

We want to use Lemma 2.3 to construct new associators out of \( \phi_{KZ} \) by changing its coefficients. To define other shuffle morphisms \( \zeta' \) we will rely on a construction on partially ordered sets which we review in the next section.

3. Posets

In Section 3.1 we introduce order series of posets. In Section 3.2 we define virtual posets and show that their order series are given by polylogarithms. In Section 3.3 we define new \( \sqcup \)-algebras parametrized by virtual posets.

3.1. Order series. Consider the category \textbf{Poset} where any morphism \( f \) preserves the strict inequality \( x < y \Rightarrow f(x) < f(y) \) or the category \textbf{Poset}_+ where morphisms are only required to satisfy \( x < y \Rightarrow f(x) \leq f(y) \). We denote by \( \langle n \rangle \) the poset \( 1 < 2 < \cdots < n \).

In [ANBDC21] we associated to a finite poset \( X \in \textbf{Poset} \) a power series \( |X| \) whose \( n \)-th coefficient is the number of order preserving maps from \( X \) to \( \langle n \rangle \) (see also [Sta70]). In some cases, for example in the categories of \textit{Wixárika} and \textit{series-parallel posets}, (see
(3.1)

Consider the empty set ∅ as a poset. We view it as the initial object in the category of posets. For any poset X there is a unique map ∅ → X, in particular for X = ∅. This implies that

|∅| = \sum_{n \geq 0} x^n = \frac{1}{1 - x}.

In the following we need three operations on power series:

\begin{align*}
  f(x) \ast g(x) & := f(x)(1 - x)g(x), \\
  f(x) \ast^+ g(x) & := f(x)\frac{1 - x}{x}g(x), \\
  f(x) \sqcap g(x) & := f(x) \odot g(x),
\end{align*}

with ⊗ denoting the Hadamard product,

\[ \sum_{n \geq 1} a_n x^n \odot \sum_{n \geq 1} b_n x^n = \sum_{n \geq 1} a_n b_n x^n. \]

These operations are induced from operations on posets. The first two describe the concatenation operation that sends two posets X, Y to the poset W obtained by adding the relations x ≤ y for every maximal element x of X and every minimal element y of Y. It is shown in [ANBDC21] that |W| is given by |X| \ast |Y| in Poset and |X| \ast^+ |Y| in Poset+. Note the structural difference: For instance, there are fewer elements in Hom_{Poset}(⟨m⟩ \ast ⟨n⟩, ⟨s⟩) than in Hom_{Poset+}(⟨m⟩ \ast^+ ⟨n⟩, ⟨s⟩). This is the reason why they induce different multiplications on the level of power series. The third operation \sqcap is induced by disjoint union of elements in Poset. Furthermore, it can be shown that disjoint union with the one-element poset (henceforth called a point) is described by a differential operator,

(3.2)

\[ |X| \sqcap |1⟩ = x \frac{d}{dx} |X|. \]

By formally inverting this operation we will be able to construct more general power series which no longer come from real posets.
3.2. **Polylogarithms.** The classical polylogarithms are defined in terms of iterated sums as

\[ \text{Li}_{k_1,k_2,\ldots,k_r}(x) = \sum_{0 < n_1 < \cdots < n_r} \frac{x^{n_r}}{n_1^{k_1}n_2^{k_2}\cdots n_r^{k_r}}. \]

Note that

\[ \langle 1 \rangle = \sum_{n \geq 1} \frac{n}{1} x^n = \frac{x}{(1-x)^2} = \text{Li}_{-1}(x), \]

and, more generally, (3.2) implies

\[ \langle 1 \rangle \sqcup \cdots \sqcup \langle 1 \rangle = \text{Li}_{-k}(x). \]

Now we formally apply the following reciprocity theorem from [ANBDC21]. Let \( i \) denote the map \( x \mapsto x^{-1} \) and define a map \( \iota \) as follows. If \( f \) is the order series of a poset \( \mathcal{X} \), and \( n_f \) is the number of points in the poset, then

\[ \iota(f) := (-1)^{n_f+1} (f \circ i). \]

Given a poset \( \mathcal{X} \), denote by \( |\mathcal{X} \rangle \) its non-strict order series (the order series in \( \text{Poset}_+ \), i.e. with respect to weak order preserving maps).

**Proposition 3.1 ([ANBDC21]).** If \( X \neq \emptyset \) is a Wixárika poset, then

\[ \iota|\mathcal{X} \rangle = |\mathcal{X}_+ \rangle, \text{ and } \iota|\mathcal{X}_+ \rangle = |\mathcal{X} \rangle. \]

Define \( \emptyset \) as the virtual poset that satisfies

\[ |\emptyset| = \iota|\emptyset| = \frac{x}{1-x} = \text{Li}_0(x). \]

We think of \( \emptyset \) as a singleton that behaves as the empty set: Its element is unordered, so that there is exactly one (order preserving) map from \( \emptyset \) into every \( \langle n \rangle \), but none into the empty set. Another difference to \( \emptyset \) is that concatenation acts non-trivially, if we define \( \emptyset^* = \emptyset \ast \cdots \ast \emptyset \), then \( \emptyset^m \neq \emptyset^n \) for \( m \neq n \). We will use this fact below in Section 3.3.

Continuing to work formally, we can also define the left inverse of the disjoint union with a point, the operation of removing a point. At the level of power series it is defined by

\[ f(x)\langle 1 \rangle = \int_0^x f(y) \frac{dy}{y}. \]

If \( f(x)\langle 1 \rangle \) is defined, then the removal of a point satisfies

\[ (f(x)\langle 1 \rangle) \sqcup \langle 1 \rangle = f(x). \]

However, this disjoint union has no right inverse since \( \frac{x}{1-x} \sqcup \langle 1 \rangle = \frac{1}{1-x} \sqcup \langle 1 \rangle. \)
Moreover, this operation and its iterations are well-defined on $|\langle \rangle|$, 

$$
|\langle \rangle|(\langle \rangle)^k = \int_0^x \frac{dx_1}{x_1} \cdots \int_0^{x_3} \frac{dx_2}{x_2} \int_0^{x_2} \frac{dx_1}{1-x_1} = \sum_{n \geq 1} \frac{x^n}{n^k} = \text{Li}_k(x).
$$

We have thus encountered all the classical polylogarithms in their representation as iterated integrals. If $k > 1$, we find the zeta values by evaluating at $x = 1$, $\zeta(k) = \left|\langle \rangle|(\langle \rangle)^k\right|_{x=1}$.

Define a third binary operation on power series by

$$
f(x) \ast g(x) := \langle \rangle^+ \bigl( f(x) \ast g(x) \bigr).
$$

In terms of posets, $\mathcal{X} \ast \mathcal{Y}$ is the element of $\text{Poset}$ whose order series is given by $|\mathcal{X} \ast \mathcal{Y}| = |\mathcal{X}| \cdot |\mathcal{Y}|$. This means that $\ast$ acts as the usual product of functions on power series, in particular, $|\mathcal{X}| \ast |\emptyset| = |\mathcal{X}| \frac{1}{1-x}$. For a virtual poset $P$ define $P^{\ast n} = P \ast \cdots \ast P$, $n$-times.

When studying labelings of posets the emphasis lies on the coefficients of the power series $|\mathcal{X}|$ associated to a given poset, not on its particular values. However, with the formal operations introduced above we also need to consider evaluations. At the level of power series removing points requires integration which depends on the particular values of the power series (cf. Remark 3.3). Furthermore, we are interested in evaluations at $x = 1$, because this allows us to associate multiple zeta values to certain virtual posets.

**Proposition 3.2.** Let $r \in \mathbb{N}$ and $k_i \geq 0, k_r > 1$. Then

$$
\text{Li}_{k_1,k_2,\ldots,k_r}(x) = \left( \left( \left( \left( \left( |\langle \rangle| (\langle \rangle)^{k_1} \right) \ast |\langle \rangle| \right) (\langle \rangle)^{k_2} \right) \ast \cdots \right) \ast |\langle \rangle| \right) (\langle \rangle)^{k_r}.
$$

**Proof.** Note that

$$
\int_0^{x^k} \cdots \int_0^{x_{k-1}} \frac{dx_1}{x_1} \cdots \frac{dx_k}{x_k} = \frac{x^n}{m^k}.
$$

We prove the statement by induction on the depth $r$. The case $r = 1$ is clear. For $r = 2$ we calculate
\[(\left( |\psi\rangle \langle \psi| \right)^{k_1} \ast |\psi\rangle \langle \psi|)^{k_2} = \left( \frac{x}{1-x} \sum_{n \geq 1} \frac{x^n}{n^{k_1}} \right)^{k_2} \]
\[= \left( \sum_{m \geq 1} x^m \sum_{n \geq 1} \frac{x^n}{n^{k_1}} \right)^{k_2} \]
\[= \left( \sum_{m \geq 2} x^m \sum_{n \geq 1} \frac{1}{n^{k_1}} \right)^{k_2} \]
\[= \sum_{m \geq 2} m^{k_2} \sum_{n \geq 1} \frac{1}{n^{k_1}} \]
\[= \sum_{0 < n < m} \frac{x^m}{n^{k_1} m^{k_2}} \]
\[= \text{Li}_{k_1, k_2}(x). \]

Assuming that the formula holds for \( r - 1 \),
\[(\cdots \left( \left( |\psi\rangle \langle \psi| \right)^{k_1} \ast \cdots \right) \ast |\psi\rangle \langle \psi|)^{k_r} \]
\[= \left( \sum_{0 < n_1 < n_2 < \cdots < n_{r-1}} x^{n_{r-1}} \frac{1}{n_1^{k_1} n_2^{k_2} \cdots n_{r-1}^{k_{r-1}}} \right)^{k_r} \]
\[= \left( \sum_{n_1 = 1}^{\infty} x^n \left( \sum_{0 < n_1 < n_2 < \cdots < n_{r-1}} x^{n_{r-1}} \frac{1}{n_1^{k_1} \cdots n_{r-1}^{k_{r-1}}} \right) \right)^{k_r} \]
\[= \left( \sum_{n_r = r}^{\infty} x^{n_r} \sum_{n_{r-1} = r-1}^{n_r-1} \frac{1}{n_{r-1}^{k_{r-1}}} \cdots \sum_{n_1 = 1}^{n_r-1} \frac{1}{n_1^{k_1}} \right)^{k_r} \]
\[= \sum_{n_r = r}^{\infty} x^{n_r} \sum_{n_{r-1} = r-1}^{n_r-1} \frac{1}{n_{r-1}^{k_{r-1}}} \cdots \sum_{n_1 = 1}^{n_r-1} \frac{1}{n_1^{k_1}} \]
\[= \text{Li}_{k_1, k_2, \ldots, k_r}(x). \]

In particular, it follows that
\[\zeta(k_1, \ldots, k_r) = \left( \left( \left( \left( |\psi\rangle \langle \psi| \right)^{k_1} \ast |\psi\rangle \langle \psi| \right)^{k_2} \ast \cdots \ast |\psi\rangle \langle \psi| \right)^{k_r} \right) \mid_{x = 1}. \]

**Remark 3.3** (An interpretation of virtual posets in terms of quantum states). One may think of \( \bigotimes \) as the vacuum in the “universe” \textbf{Poset} (or \textbf{Poset}_+, or some subcategory thereof). The operation \( x \frac{d}{dx} \) adds a point – a poset particle – from which we can construct further posets (“states”) by iteration of disjoint union. The operator \( \int \frac{dx}{x} \) removes a point. Equivalently, we may think of this as disjoint union with a poset antiparticle. From this
point of view the two operators act like creation and annihilation operators on a quantum system, translating between different energy states,

\[ \ldots \otimes | -k \rangle \otimes \ldots \otimes | -1 \rangle \otimes | 0 \rangle \otimes | 1 \rangle \otimes \ldots \otimes | k \rangle \otimes \ldots, \]

where \(| \pm k \rangle\) denotes the disjoint union of \(k\) points/antipoints.

A crucial point here is the choice of integration constants. For instance, if we take in the step \(| 1 \rangle \rightarrow | 0 \rangle\) the solution

\[
|1\rangle \langle 1| = \int_c^x \frac{dx'}{x'(1-x')} = \frac{1}{1-x},
\]

we leave our universe. We get a different vacuum \(0\) and different anti-states, e.g.

\[
|1\langle 1| = \int_c^x \frac{dx'}{x'(1-x')} = \log x - \log(1-x) + C.
\]

In terms of physics, our states depend on a (arbitrary) choice of ground state energy.

3.3. **Truncated multiple zeta values.** Consider now a new virtual poset, \(\otimes^m\), the \(m\)-fold concatenation of \(\otimes\) with itself. A simple generalization of Proposition 3.2 shows that

\[
\begin{aligned}
|\otimes^m\rangle \langle 1|^{k_1} \otimes |\otimes^m\rangle \langle 1|^{k_2} \otimes \cdots \otimes |\otimes^m\rangle \langle 1|^{k_r} \\
= \sum_{m<n_1, n_1+m<n_2, \ldots, n_{r-1}+m<n_r} \frac{x^{n_r}}{n_1^{k_1} n_2^{k_2} \cdots n_r^{k_r}} \\
= \sum_{0<n_1<n_2<\ldots<n_r} \frac{x^{m+n_r}}{(m+n_1)^{k_1}(2m+n_2)^{k_2} \cdots (rm+n_r)^{k_r}}.
\end{aligned}
\]

The number \(m\) describes thus a truncation parameter for multiple zeta values.

For \(n \in \mathbb{N}\) define a map of vector spaces \(\zeta^n : \mathfrak{h}^0 \to \mathbb{R}\) by

\[
\zeta^n(A^{p_1} B^{p_2} \cdots A^{p_l}) =
(|\otimes^m\rangle \langle 1|)^{p_1} \langle 1|^{q_1} \otimes \cdots \otimes |\otimes^m\rangle \langle 1|^{q_{r-1}} \otimes \cdots \otimes |\otimes^m\rangle \langle 1|^{q_r} |x=1.
\]

**Theorem 3.4.** The map of vector spaces

\[
\sigma^n : \mathbb{Q} \langle \zeta(w) \mid w \in \mathfrak{h}^0 \rangle \to \mathbb{Q} \langle \zeta^n(w) \mid w \in \mathfrak{h}^0 \rangle,
\]

\[
\zeta(w) \mapsto \zeta^n(w),
\]

is a map of \(\otimes\)-algebras.

**Proof.** Explicitly, \(\sigma^n\) sends the number

\[
\int_{x_{r,k_r}, \ldots, x_{r,2}} \frac{dx_{r,1}}{1-x_{r,1}} \cdots \frac{dx_{1,k_1}}{1-x_{1,k_1}} \frac{dx_{1,2}}{1-x_{1,2}} \cdots \frac{dx_{1,1}}{1-x_{1,1}}
\]
to the number
\[
\int_{x_r,k_r} dx_{r,k_r} \cdots \int_{x_{r,2}}^{x_{r,1}} dx_{r,2} \cdot \frac{x_{r,1}^n}{1-x_{r,1}} \cdots \int_{x_{1,k_1}} dx_{1,k_1} \cdot \frac{x_{1,2}^n}{1-x_{1,2}} \cdots \int_{x_{k_1-1}} dx_{k_1-1} \cdot \frac{x_{1,1}^n}{1-x_{1,1}} dx_{1,1}.
\]

The image of each \( \sigma^n \) consists of (convergent) iterated sums with (convergent) integral representations where the integrand starts with \( \frac{dx}{x} \) and ends with \( \frac{x^n}{1-x} dx \).

We use Proposition 2.2 with \( \omega_i \in \{ \frac{x^n}{1-x} dx, \frac{dx}{x} \} \) and \( \omega_1 = \omega_{r+1} = \frac{x^n}{1-x} dx, \omega_r = \omega_{r+s} = \frac{dx}{x} \) to conclude that there is a shuffle product for the numbers \( \zeta^n(w) \). Lemma 2.3 implies that the map \( \sigma^n \) of \( \mathbb{Q} \)-vector spaces is a map of \( \mathbb{Q} \)-algebras. \( \square \)

4. Associators

Section 4.1 introduces a sequence of associators that starts at \( \phi_{KZ} \) and converges to 1. In Section 4.2 we extend this sequence to the domain \( \{ z \in \mathbb{C} \mid \Re(z) > 0 \} \). Finally, in Section 4.3 we show how an involution of shuffle algebras gives rise to a second disk of associators.

4.1. A sequence of associators. From Proposition 2.4 we know that each coefficient of \( \phi_{KZ} \) is a sum of multiple zeta values, and as shuffle algebras, these elements are parametrized by words in \( ARB \). Our goal is to define a new associator by changing the coefficients of \( \phi_{KZ} \), that is, by using a map on multiple zeta values.

Remark 4.1. We aim to define a function that depends on the representation of a number as an integral or iterated sum, but a number can have different representations, for example

\[
\sum_{0 < n < m} \frac{1}{n^2 m^2} = \sum_{0 < n < m < r < s} \frac{1}{n m^2 r s^2}
\]

which follows from the following duality relation [Zag94], [LM95, Proposition A.4],

\[
\zeta(k_1, \ldots, k_r) = \zeta(1, \ldots, 1, 2, \ldots, 2, \underbrace{1, \ldots, 1}_{k_r-1}, \underbrace{1, \ldots, 1}_{k_1-1}).
\]

We will formally choose one such representation using our virtual posets as labels.

In order to show that our construction actually produces an associator, we will need the following result.

Theorem 4.2 ([Fur10, BND12]). Let \( \phi = 1 + c_2(AB - BA) + \cdots \) be a group-like element of \( \mathbb{C} \langle \langle A, B \rangle \rangle \). Suppose that \( \phi \) satisfies the pentagon equation of Drinfel’d (2.1). Then there is an element \( \mu = \pm (24c_2)^\frac{1}{2} \) such that the pair \( (\mu, \phi) \) satisfies the hexagon equations (2.2), (2.3).
The Drinfel’d KZ associator satisfies the pentagon equation
\[
\phi_{KZ}(t_{12}, t_{23} + t_{24})\phi_{KZ}(t_{12} + t_{23}, t_{34}) - \phi_{KZ}(t_{23}, t_{34})\phi_{KZ}(t_{12} + t_{13}, t_{24} + t_{34})\phi_{KZ}(t_{12}, t_{23}) = 0.
\]
We can study this equation degree by degree. For example, the degree two part is
\[
\zeta(2)(t_{12}(t_{23} + t_{24}) - (t_{23} + t_{24})t_{12}) + \zeta(2)((t_{12} + t_{23})t_{34} - t_{34}(t_{12} + t_{23}))
- \zeta(2)(t_{23}t_{34} - t_{34}t_{23}) - \zeta(2)((t_{12} + t_{13})(t_{24} + t_{34}) - (t_{24} + t_{34})(t_{12} + t_{13})) - \zeta(2)(t_{12}t_{23} - t_{23}t_{12}) = 0.
\]
For higher degrees \( n \) the coefficients in these equations will be sums of products of multiple zeta values.

**Theorem 4.3.** Given a shuffle algebra \( \zeta': \mathfrak{h}^0 \to \mathbb{R} \) with a map of shuffle algebras
\[
\sigma': \mathbb{Q}\langle\zeta(w) | w \in \mathfrak{h}^0\rangle \longrightarrow \mathbb{Q}\langle\zeta'(w) | w \in \mathfrak{h}^0\rangle,
\]
\( \zeta(w) \mapsto \zeta'(w) \)
we can change the coefficients of the Drinfel’d KZ associator to obtain a new associator.

**Proof.** For every \( n \in \mathbb{N} \) we consider
\[
\sum_{dp(w_s) + dp(w_l) = n} I(w_s)I(w_l)w_s(t_{12}, t_{23} + t_{24})w_l(t_{12} + t_{23}, t_{34}),
\]
the terms of degree \( n \) in the first part of the pentagon equation (left hand side) where the sum is over pairs of words \( w_s, w_l \) with \( dp(w_s) + dp(w_l) = n \). We also consider
\[
\sum_{dp(w_s) + dp(w_m) + dp(w_l) = n} I(w_s)I(w_m)I(w_l)w_s(t_{23}, t_{34})w_m(t_{12} + t_{13}, t_{24} + t_{34})w_l(t_{12}, t_{23}),
\]
the terms of degree \( n \) in the second part of the pentagon equation (right hand side). Here the \( t_{ij} \) satisfy the locality and the 4T relations, and we work on \( \mathfrak{A}_4 \), the degree completion of the algebra of cord diagrams with four strands (see [BN98]). Since the multiple zeta values satisfy the degree \( n \) pentagon equation on \( \mathfrak{A}_4 \), the isomorphic shuffle algebra will satisfy it too:
\[
\sum \sigma^n(I(w_s))\sigma^n(I(w_l))w_s(t_{12}, t_{23}, t_{24})w_l(t_{12}, t_{23}, t_{34})
- \sum \sigma^n(I(w_s))\sigma^n(I(w_m))\sigma^n(I(w_l))w_s(t_{23}, t_{34})w_m(t_{12}, t_{13}, t_{24}, t_{34})w_l(t_{12}, t_{23})
= \sigma^n\left(\sum I(w_s)I(w_l)w_s(t_{12}, t_{23}, t_{24})w_l(t_{12}, t_{23}, t_{34})
- \sum I(w_s)I(w_m)I(w_l)w_s(t_{23}, t_{34})w_m(t_{12}, t_{13}, t_{24}, t_{34})w_l(t_{12}, t_{23})\right)
= 0.
\]
It remains to show that \( \sigma^n(\phi_{KZ}) \) is group-like. Then the hexagon equations will follow from Theorem 4.2.
To show that $\Delta(\sigma^n(\phi_{KZ})) = \sigma^n(\phi_{KZ}) \otimes \sigma^n(\phi_{KZ})$, note that this is equivalent to a sequence of identities

$$
\left( \sum I(w_s) \right) w_1(A, B) \otimes w_2(A, B) - I(w_1)I(w_2)w_1(A, B) \otimes w_2(A, B) = 0
$$

for all possible pairs of finite words $w_1(A, B), w_2(A, B)$, and $\Delta(w_s) = w_1(A, B) \otimes w_2(A, B) + \cdots$. Since the coefficients of the Drinfel’d KZ associator satisfy these identities, so do their images under the map $\sigma^n$. □

**Corollary 4.4.** Let $\sigma^m$ be the map from Theorem 3.4. For every $m \in \mathbb{N}$, consider $\phi_{KZ, m} = \sigma^m(\phi_{KZ})$, the power series in which we replace every multiple zeta coefficient of $\phi_{KZ}$

$$
\sum_{0<n_1<n_2<\cdots<n_r} \frac{1}{n_1^{k_1}n_2^{k_2} \cdots n_r^{k_r}}
$$

by its truncated version

$$
\sum_{m<n_1,n_1+m<n_2,\ldots,n_r-1+m<n_r} \frac{1}{n_1^{k_1}n_2^{k_2} \cdots n_r^{k_r}}.
$$

Then we have a sequence of different associators $\{\phi_{KZ, m}\}_{m \in \mathbb{N}}$, from $\phi_{KZ} = \phi_{KZ, 0}$ to $1 = \phi_{KZ, \infty}$ with explicit coefficients.

**Proof.** According to Lemma 3.4 the function $\sigma^m$ is well defined and satisfies the hypothesis of the previous theorem. Therefore, $\phi_{KZ, m}$ is an associator for every $m \in \mathbb{N}$.

The sequence $\{a_\ell\}_{\ell \in \mathbb{N}}$ with

$$
a_\ell = \sum_{\ell<n_1,n_1+\ell<n_2,\ldots,n_r-1+\ell<n_r} \frac{1}{n_1^{k_1}n_2^{k_2} \cdots n_r^{k_r}}
$$

is strictly decreasing. This implies that the coefficients of consecutive associators, and hence the associators themselves, are different. It is clear that $\lim_{m \to \infty} \sigma^m \zeta(k_1, \cdots, k_r) = 0$ and thus $\lim_{m \to \infty} \phi_{KZ, m} = 1$. □

### 4.2. A disk of associators.

We can actually strengthen the above statement.

The sequence $\phi_{KZ, m}$ is a discrete subset of an open disk in the set of associators with $\phi_{KZ}$ and 1 lying on its boundary. To see this, note that replacing $m$ by $z \in \mathbb{C}$ in the integral expression for $\sigma^m$ (given in the proof of Theorem 3.4) defines a function $\sigma^z$ that is generally of the form

$$
\sigma^z \zeta(k) = \int_{\Delta} \left( \prod_{\ell} x_{\ell} \right)^z f(x) dx
$$

where $\Delta = \{0 \leq x_1 \leq \cdots \leq x_{|k|} \leq 1\}$ and $f$ is a product of terms $\frac{1}{x_{\ell}}$ and $\frac{1}{1-x_{\ell}}$. Since the integral converges absolutely for $\text{Re}(z) > 0$, we can differentiate under the integral to conclude that it defines an analytic function on $\{\text{Re}(z) > 0\}$. 
One shows, completely analogous to the integral expression for multiple zeta values, that its value at $z$ is given by
\[
\sum_{0 < n_1 < n_2 < \ldots < n_r} \frac{1}{(z + n_1)^{k_1}(2z + n_2)^{k_2} \cdots (rz + n_r)^{k_r}}.
\]
It follows that the limit $z \to 0$ exists and that it is $\zeta(k_1, \ldots, k_r)$.

We claim that in the limit $z \to \infty$ the expression vanishes. We can restrict to a sequence of reals, because
\[
\left| \sum_{0 < n_1 < \ldots < n_r} \frac{1}{(z + n_1)^{k_1}(r z + n_r)^{k_r}} \right| \leq \sum_{0 < n_1 < \ldots < n_r} \frac{1}{|z + n_1|^{k_1} \cdots |r z + n_r|^{k_r}} \leq \sum_{0 < n_1 < \ldots < n_r} \frac{1}{|\text{Re}(z) + n_1|^{k_1} \cdots |r \text{Re}(z) + n_r|^{k_r}}.
\]
But for a sequence $(x_m)_{m \in \mathbb{N}} \subset \mathbb{R}$ with $\lim_{m \to \infty} x_m = \infty$ we have the following estimate
\[
\sum_{0 < n_1 < \ldots < n_r} \frac{1}{|x_m + n_1|^{k_1} \cdots |r x_m + n_r|^{k_r}} \leq \sum_{0 < n_1 < \ldots < n_r} \frac{1}{|x_m + n_1|^{k_1} \cdots |r |x_m| + n_r|^{k_r}}
\]
where $\lfloor \cdot \rfloor$ denotes the floor function $\mathbb{R} \to \mathbb{N}$. Now, since $\lfloor x_m \rfloor$ is a monotonically increasing sequence of natural numbers, we can use the alternative representation (3.4) of truncated multiple zeta values to deduce that (4.2) converges to zero for $m \to \infty$.

A similar line of arguments shows that the map $x \mapsto \sigma^x(\zeta(w))$ is a strictly decreasing map on $[0, \infty)$. From Corollary 4.4 we know this is true for $x \in \mathbb{N}$. Computing its derivative gives
\[
\frac{\partial}{\partial x} \sum_{0 < n_1 < \ldots < n_r} \frac{1}{(x + n_1)^{k_1}(2x + n_2)^{k_2} \cdots (rx + n_r)^{k_r}} = \sum_{j=1}^r \sum_{0 < n_1 < \ldots < n_r} \frac{-jk_j}{(x + n_1)^{k_1}(2x + n_2)^{k_2} \cdots (jx + n_i)^{k_i+1} \cdots (rx + n_r)^{k_r}}
\]
which is negative for every $x \in [0, \infty)$.

Putting everything together, we have proven our main theorem.

**Theorem 4.5.** The open disk $\{\text{Re}(z) > 0\}$ parametrizes a family of associators $\phi_{KZ, z}$.

We can approach two points on its boundary via the limits $z \to 0$ and $z \to \infty$, giving $\phi_{KZ}$ and 1, respectively.

In particular, the line $[0, \infty]$ parametrizes a path of associators, interpolating between the Drinfel’d KZ and the trivial associator, and every point on this line gives a different associator.
Remark 4.6. Note that we can repeat the whole construction outlined in this section (and in the following) for the anti-KZ associator $\phi_{aKZ}(A, B) = \phi_{KZ}(-A, -B)$.

4.3. A second family of shuffle algebras. We construct another shuffle algebra via an endomorphism of shuffle algebras that fixes the algebra of multiple zeta values.

There is a difference between the formal shuffle algebra $\mathfrak{h}^0$ and its image $\zeta$ in $\mathbb{R}$, the multiple zeta values satisfy identities such as (4.1). Define a map $\kappa$ on $\mathfrak{h}^0$ by

$$\kappa(A^{\sigma_1}B^{\sigma_2} \cdots A^{\sigma_n}) = (A^{\sigma_1}B^{\sigma_2} \cdots A^{\sigma_n}B^{\sigma_1}).$$

Then (4.1) is equivalent to

$$\zeta(A^{\sigma_1}B^{\sigma_2} \cdots A^{\sigma_n}) = \zeta(\kappa(A^{\sigma_1}B^{\sigma_2} \cdots A^{\sigma_n})).$$

see [LM95, Lemma 3.1.4].

We will see below that this identity does not hold for truncated multiple zeta values. We can thus define a non-trivial map $\tilde{\zeta}^n : (\mathfrak{h}^0, \shuffle) \to \mathbb{R}$ by

$$\tilde{\zeta}^n(A^{\sigma_1}B^{\sigma_2} \cdots A^{\sigma_n}) = \zeta^n(\kappa(A^{\sigma_1}B^{\sigma_2} \cdots A^{\sigma_n})).$$

Unraveling the definitions, we obtain for example

$$A^{k_1-1}B A^{k_2-1}B \cdots A^{k_{n-1}-1}B \cdots \sigma^n(1, \ldots, 1, 2, \ldots, 1, \ldots, 1, 2)$$

$$= \sum_{m_1 + \ldots + m_n = m_1 + 1} 1 \frac{m_{r, 1} \cdots m_{r, k_r - 1}m_{r, k_r}^2 \cdots m_{1, 1} \cdots m_{1, k_1 - 1}m_{1, k_1}^2}{m_{1, k_1}}$$

$$= \sum_{0 < r, 1 \ldots < m_1, k_1} 1 \frac{(n + m_{r, 1}) \cdots ((k_r - 1)n + m_{r, k_r - 1})(k_r n + m_{r, k_r})^2 \cdots (k_1 n + m_{1, k_1})^2}{k_1 n + m_{1, k_1}}.$$

Let us take a detailed look at the word $A^2 B$. We have $\kappa(A^2 B) = AB^2$ so that for $n = 0$

$$\tilde{\zeta}(A^2 B) = \zeta(AB^2) = \sum_{0 < m < k} \frac{1}{mk^2} \quad \text{and} \quad \tilde{\zeta}^1(A^2 B) = \sum_{1 < m} \frac{1}{m^2}.$$

For $n = 1$ we get

$$\tilde{\zeta}^1(A^2 B) = \sum_{1 < m, m+1 < k} \frac{1}{mk^2} \quad \text{and} \quad \tilde{\zeta}(A^2 B) = \sum_{1 < m} \frac{1}{m^3}.$$

To show that these two expression do not coincide, consider the difference

$$\tilde{\zeta}(A^2 B) - \tilde{\zeta}^1(A^2 B) = \sum_{1 < m} \frac{1}{m^2} + \sum_{1 < m} \frac{1}{m(m + 1)^2}.$$

From

$$\sum_{0 < m < k} \frac{1}{mk^2} = \sum_{m=1}^{\infty} \frac{1}{m(m + 1)^2} = \sum_{m=1}^{\infty} \frac{1}{m^2} - \frac{1}{m + 1} = \frac{1}{17} - \frac{1}{(m + 1)^2} = 2 - \zeta(2)$$
we conclude that $\tilde{\zeta}(A^2B) - \zeta^1(A^2B) = \zeta(2) - 1 + 2 - \zeta(2) - \frac{1}{4} = \frac{3}{4}$. Finally, from $\zeta(A^2B) - \zeta^1(A^2B) = 1$ it follows that indeed

$$\zeta^1(A^2B) > \zeta^1(A^2B).$$

We can generalize this statement.

**Lemma 4.7.** For all $n \geq 1$, we have $\tilde{\zeta}^n(A^2B) \neq \zeta^n(A^2B)$.

**Proof.** Fix $n \geq 1$. We claim

$$\sum_{n,m,m+k} \frac{1}{mk^2} - \sum_{n,m} \frac{1}{m^3} > 0.$$

The function $f_n(x) = \frac{1}{(x+2n)^2}$ is monotone decreasing on $[0, \infty)$. Given $N \in \mathbb{N}$, we apply the Maclaurin–Cauchy formula to obtain

$$\sum_{k=N}^{\infty} \frac{1}{(k+2n)^2} \geq \int_{N}^{\infty} \frac{1}{(x+2n)^2} dx = \frac{1}{N + 2n}.$$

Using this, together with the alternative representation of truncated multiple zeta values (3.4), we estimate

$$\sum_{n,m,m+k} \frac{1}{mk^2} - \sum_{n,m} \frac{1}{m^3} = \sum_{0 < m < k} \frac{1}{(m+n)(k+2n)^2} - \sum_{0 < m} \frac{1}{(m+n)^3}$$

$$= \sum_{0 < m} \frac{1}{m+n} \left( \left( \sum_{m < k} \frac{1}{(k+2n)^2} \right) - \frac{1}{(m+n)^2} \right)$$

$$\geq \sum_{0 < m} \frac{1}{m+n} \left( \frac{1}{m+1+2n} - \frac{1}{(m+n)^2} \right)$$

$$= \sum_{0 < m} \left( \frac{m(m-1+2n)+n(n-2)-1}{(m+n)^3(m+1+2n)} \right) > 0.$$

\[\square\]

Let $\tilde{\phi}_{KZ,n}$ be the power series in which we replace every coefficient $\zeta$ of $\phi_{KZ}$ by $\tilde{\zeta}^n$. Then all arguments of the previous section apply verbatim showing that $\tilde{\phi}_{KZ,n}$ is also an associator. This means we have found another sequence of associators $(\tilde{\phi}_{KZ,n})_{n \in \mathbb{N}}$ satisfying $\tilde{\phi}_{KZ,0} = \phi_{KZ}$ and $\lim_{n \to \infty} \tilde{\phi}_{KZ,n} = 1$. Moreover, the terms in the sequences of coefficients are strictly decreasing, so that, again, they are all different.

We can once more promote $n$ to a complex variable $z$ as described at the end of the previous section. This means we have another family of associators $\tilde{\phi}_{KZ,z}$, parametrized by the disk $\{\text{Re}(z) > 0\}$.

It is natural to ask whether the members of this family are all different from the former one. We can answer this in terms of the two paths $\phi_{KZ,s}$ and $\tilde{\phi}_{KZ,s}$, $s \geq 0$. 

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Proposition 4.8. The two paths of associators $\phi_{KZ,s}$ and $\tilde{\phi}_{KZ,s}$, $s \in [0, \infty) \subset \mathbb{R}$, are different.

Proof. We aim to show that one path is not a reparametrization of the other. For this it suffices to study the terms of degree three in the derivative of both paths at the origin.

We first calculate
\[
\frac{\partial}{\partial s} \tilde{\zeta}^s(A^2B) = \frac{\partial}{\partial s} \sum_{0 < m_1 < m_2} \frac{1}{(s + m_1)(2s + m_2)^2} - \frac{4}{(s + m_1)(2s + m_2)^3}.
\]
With $\zeta(1,3) + \zeta(2,2) = \zeta(4)$ and $\zeta(2,2) = 3/4\zeta(4)$ [A19] we evaluate this at $s = 0$:
\[
\frac{\partial}{\partial s} \tilde{\zeta}^s(A^2B)|_{s=0} = -\zeta(2,2) - 4\zeta(1,3) = -\zeta(2,2) - 4(\zeta(4) - \zeta(2,2)) = -4\zeta(4) + 3\zeta(2,2) = -4\zeta(4) + 9/4\zeta(4) = -5/4\zeta(4).
\]
On the other hand,
\[
\frac{\partial}{\partial s} \zeta^s(A^2B) = \frac{\partial}{\partial s} \sum_{0 < m_1} \frac{1}{(s + m_1)^3} = \sum_{0 < m_1} \frac{-3}{(s + m_1)^4}
\]
which evaluates at $s = 0$ to $-3\zeta(4)$.

Since $\zeta^s(AB^2) = \zeta^s(A^2B)$ and $\tilde{\zeta}^s(A^2B) = \zeta^s(AB^2)$ this determines all the degree three terms of the derivatives at the origin.

For the first path we get
\[
-\frac{\partial}{\partial s} \zeta^s(A^2B)|_{s=0} \cdot [A, [A, B]] + \frac{\partial}{\partial s} \tilde{\zeta}^s(AB^2)|_{s=0} \cdot [[A, B], B]
= 3\zeta(4) \cdot [A, [A, B]] - \frac{5}{4} \zeta(4) \cdot [[A, B], B],
\]
while for the second one we find
\[
-\frac{\partial}{\partial s} \zeta^s(AB^2)|_{s=0} \cdot [A, [A, B]] + \frac{\partial}{\partial s} \tilde{\zeta}^s(A^2B)|_{s=0} \cdot [[A, B], B]
= \frac{5}{4} \zeta(4) \cdot [A, [A, B]] - 3\zeta(4) \cdot [[A, B], B].
\]

Now assume that there is a smooth reparametrization $\rho : [0, 1] \to [0, 1]$ so that $\zeta^{\rho(s)} = \tilde{\zeta}^s$ (or $\zeta^s = \tilde{\zeta}^{\rho(s)}$). Then
\[
\frac{\partial}{\partial s} \zeta^{\rho(s)}|_{s=0} = \frac{\partial}{\partial s} \tilde{\zeta}^s|_{s=0} = \frac{\partial \zeta^{\rho(0)}}{\partial s} \frac{\partial \rho(0)}{\partial s}.
\]
But there is no $\lambda \in \mathbb{C}$ that satisfies $\lambda(3\zeta(4)) = \frac{5}{4}\zeta(4)$ and $\lambda(\frac{5}{4}\zeta(4)) = -3\zeta(4)$ simultaneously. Therefore there is no reparametrization of one path into the other. \[\square\]
It is natural to ask whether the above presented methods allow to describe the behaviour of the paths close to the trivial associator 1.

For each coefficient of $\phi_{KZ,s}$ and $\bar{\phi}_{KZ,s}$ replace $s$ by $t^{-1}$, compute the derivative with respect to $t$, and evaluate the limit $t \to 0$. Each derivative is a series of fractions as above, multiplied by an overall factor $-t^{-2}$. A simple power counting argument shows that the denominator of every fraction is at least linear in $t^{-1}$. For example,

$$\frac{\partial}{\partial t} s^{-1}(A^2 B) = \sum_{0 < m_1 < m_2} \frac{1}{t^2 (t^{-1} + m_1)^2 (2t^{-1} + m_2)^2} + \sum_{0 < m_1 < m_2} \frac{4}{t^2 (t^{-1} + m_1)(2t^{-1} + m_2)^3}$$

$$= \sum_{0 < m_1 < m_2} \frac{1}{(1 + tm_1)^2 (2t^{-1} + m_2)^2} + \sum_{0 < m_1 < m_2} \frac{4}{(t + t^2 m_1)(2t^{-1} + m_2)^3}.$$

This means that every coefficient of the derivative vanishes when $s \to \infty$, so that $\phi_{KZ,s}$ and $\bar{\phi}_{KZ,s}$ together with their first derivative become equal as $s \to \infty$. However, note that $\frac{\partial^2}{\partial t^2} s^{-1}(AB)|_{t=0}$ diverges. A different strategy is needed to compare the two paths around 1.

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