On the asymptotic behaviour of cosmic density-fluctuation power spectra of cold dark matter

Sara Konrad,1 ★ Yonadav Barry Ginat,2 † and Matthias Bartelmann1

1 Institute for Theoretical Physics, Heidelberg University, Germany
2 Faculty of Physics, Technion – Israel Institute of Technology, Haifa, 3200003, Israel

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ABSTRACT
We study the small-scale asymptotic behaviour of the cold dark matter density fluctuation power spectrum in the Zel’dovich approximation, without introducing an ultraviolet cut-off. Assuming an initially correlated Gaussian random field and spectral index $0 < n_s < 1$, we derive the small-scale asymptotic behaviour of the initial momentum-momentum correlations. This result is then used to derive the asymptotics of the power spectrum in the Zel’dovich approximation. Our main result is an asymptotic series, dominated by a $k^{-3}$ tail at large wave-numbers, containing higher-order terms that differ by integer powers of $k^{n_s-1}$ and logarithms of $k$. Furthermore, we show that dark matter power spectra with an ultraviolet cut-off develop an intermediate range of scales where the power spectrum is accurately described by the asymptotics of dark matter without a cut-off. These results reveal information about the mathematical structure that underlies the perturbative terms in kinetic field theory and thus the non-linear power spectrum. We also discuss the sensitivity of the small-scale asymptotics to the spectral index $n_s$.

Key words: cosmology: dark matter – cosmology: large-scale structure of Universe – cosmology: theory

1 INTRODUCTION

The Zel’dovich approximation (Zel’dovich 1970) is ubiquitous in the theory of structure formation (see White 2014 for a relatively recent review), as a key analytic approximation that encapsulates many of the physical effects pertinent to structure formation. One of its many advantages is that it allows for the derivation of a simple formula for the two-point correlation function (Schneider & Bartelmann 1995; Taylor & Hamilton 1996), and thus the power spectrum. While the Zel’dovich power spectrum accurately describes structures on linear scales, it is a non-linear power spectrum since it is a non-linear functional of the initial power spectrum.

In kinetic field theory (hereafter KFT), the Zel’dovich power spectrum plays an important role and is key to the full non-linear power spectrum (Bartelmann et al. 2017, 2019; Bartelmann et al. 2021; Konrad & Bartelmann 2022). It represents the leading order power spectrum of perturbation theory and additionally is the starting point of a mean-field theory within KFT, which is an alternative way to incorporate those particle interactions that are not already captured by the Zel’dovich approximation (Bartelmann et al. 2021). To derive the Zel’dovich power spectrum from KFT, three ingredients are necessary: (i) the specification of the initial conditions in classical phase-space, (ii) the assumption of straight particle trajectories1 that contain parts of the gravitational interactions and (iii) the expanding space-time as background (Bartelmann et al. 2016; Bartelmann et al. 2017).

We consider Gaussian initial conditions, implying that two-point correlation functions are sufficient to completely determine the initial probability distribution in phase space. We assume that initial momenta are correlated, while initial density-density and density-momentum correlations are neglected. This is reasonable on large scales and at late times, as they grow much more slowly than the initial momentum-momentum correlations. The initial velocity field can be considered to be curl-free, such that it can be expressed in terms of a gradient of a velocity potential $\psi$. Since the continuity equation has to hold initially, the initial density contrast has to satisfy Poisson’s equation

$$\delta^{(i)} = -\nabla^2 \psi.$$  

(All the constants that might appear are absorbed into $\psi$ here.) Consequently, the initial velocity potential power spectrum is related to the initial density fluctuation power spectrum via

$$P_\psi(k) = k^{-4} P_\delta^{(i)}(k).$$  

1 They are inertial if time is transformed to a specific non-linear function of itself.

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The initial momentum-momentum correlation matrix of two particles separated by $\mathbf{q}$ is then given by

$$
\hat{C}_{PP}(\mathbf{q}) = \int \frac{d^3k}{(2\pi)^3} \frac{(k' \otimes k')}{|k'|} p_p^{(i)}(k') e^{i\mathbf{k' \cdot q}}.
$$

(3)

The ansatz that the particle trajectories are straight is key to the Zel’dovich approximation. In KFT, this ansatz is expressed by splitting up the actual time dependent trajectory of a particle into the Zel’dovich part $\mathbf{q}^Z(t)$, that is linear in the initial phase space coordinates $\mathbf{q}^{(i)}$, $\mathbf{p}^{(i)}$, and an interaction part $\mathbf{q}^I(t)$ that contains all the remaining interactions that are not captured by the Zel’dovich approximation,

$$
\mathbf{q}(t) = \mathbf{q}^Z(t) + g_{qp}(t, t^{(i)}) \mathbf{p}^{(i)} + \mathbf{q}^I(t) = \mathbf{q}^Z(t) + \mathbf{q}^I(t),
$$

(4)

where $g_{qp}(t, t^{(i)})$ is the so-called propagator. Considering only $\mathbf{q}^Z(t)$ and identifying $g_{qp}(t, t^{(i)}) = D_+(t) - D_+(t^{(i)})$ with the linear growth factor $D_+$, yields the Zel’dovich approximation. In the following, we define the initial time $t^{(i)} = 0$ and then replace in our notation the propagator $g_{qp}(t, 0)$ by $t$. This not only simplifies the notation but also highlights that the propagator serves as a time coordinate. We emphasise that Eq. (4) is exact.

When these assumptions are combined, the unperturbed zeroth-order power spectrum of KFT is equivalent to the Zel’dovich power spectrum and is given by

$$
P(k, t) = e^{-Q} \int d^3q \left( e^{i \mathbf{k} \cdot \mathbf{q}} P_{\phi^2}(qk - 1) \right) e^{i \mathbf{q} \cdot \mathbf{q}},
$$

(5)

as shown in Bartelmann et al. (2016); Bartelmann et al. (2017). The exponent $Q$ is

$$
Q = \sigma_{1}^2 3 k^2 i^2,
$$

(6)

where $\sigma_1^2$ is one of the moments

$$
\sigma_n^2 = \frac{1}{2\pi^2} \int_0^\infty dk k^{2n+2} P^{(i)}_\phi(k),
$$

(7)

of the power spectrum $P^{(i)}_\phi$ of the initial velocity potential $\phi$. It is important for the present derivation of our results that $\sigma_2$ is finite, which is the case for $0 < n_s < 1$, but higher moments are allowed to be infinite.

We emphasise that nothing in this paper relies on features peculiar to KFT – indeed, all that we assume is that the power spectrum in the Zel’dovich approximation is given by (5), which can be reached from, e.g., Lagrangian perturbation theory, too (Taylor & Hamilton 1996).

The small-scale limit of the power spectrum is of interest to cosmologists for many reasons: it can be used to study the behaviour of dark matter inside haloes (Mao & Fry 2000), and thereby to gain some understanding on their universal density profile (see, e.g., Wang et al. 2020).

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Asymptotics of the Zeldovich Power Spectrum

2 SMALL-SCALE ASYMPOTICS OF THE INITIAL MOMENTUM CORRELATIONS

We start with the small-scale asymptotics of the initial momentum-momentum correlations of two particles that are separated by a vector $\mathbf{q}$. As shown in Bartelmann et al. (2017), the initial correlation matrix can be written as

$$
\hat{C}_{pp}(\mathbf{q}) = -\hat{q} \otimes \hat{q} a_2(q) - \frac{1}{3} a_1(q),
$$

(9)

with $\hat{q}$ being the unit vector pointing in the direction of $\mathbf{q}$ and

$$
a_1(q) \equiv \frac{\xi_\phi'(q)}{q} = -\frac{1}{2\pi^2} \int_0^\infty dk \frac{P^{(i)}_\delta (k)}{k} j_1(kq),
$$

(10)

$$
a_2(q) \equiv \xi_\phi''(q) = \frac{1}{2\pi^2} \int_0^\infty dk \frac{P^{(i)}_\delta (k)}{k} j_2(kq),
$$

(11)

where $\xi_\phi$ denotes the correlation function of the initial velocity potential and $j_n$ denotes the spherical Bessel function of order $n$. We take the initial power spectrum to be

$$
P^{(i)}_\delta(k) = \frac{8\pi^2 A_s}{25\Omega_m H_0^2} k_{eq}^{n_s} \left[ \frac{k}{k_{eq}} \right]^{n_s} T_D^2(\frac{k}{k_{eq}}) \equiv A \left[ \frac{k}{k_{eq}} \right]^{n_s} T_D^2(\frac{k}{k_{eq}}),
$$

(12)

where the wave number $k$ is defined below, and use a Dicus transfer function (Weinberg 2008, p. 307)

$$
T_D^2(\kappa) = \left( \frac{\ln\left( (\tilde{c}_1\kappa)^2 + 1 \right)}{(\tilde{c}_1\kappa)^2} \right)^2 \left( \frac{(\tilde{c}_2\kappa)^2 + (\tilde{c}_3\kappa)^4 + (\tilde{c}_4\kappa)^6 + 1}{(\tilde{c}_5\kappa)^2 + (\tilde{c}_6\kappa)^4 + (\tilde{c}_7\kappa)^6 + 1} \right),
$$

(13)

with $\tilde{c}_1 = 0.124$, $\tilde{c}_2 = 1.257$, $\tilde{c}_3 = 0.4452$, $\tilde{c}_4 = 0.2197$, $\tilde{c}_5 = 1.606$, $\tilde{c}_6 = 0.8568$, and $\tilde{c}_7 = 0.3927$.

We use the Planck 2016 cosmological parameters (Planck Collaboration et al. 2016), as given in Tab. 1. We stress that all results presented here are independent of the explicit choice of a transfer function for the initial power spectrum. The only important features are the asymptotics at small scales, i.e. at $k \to \infty$, that has to agree with the general form specified in (15) below; and that $0 < n_s < 1$.

To derive an asymptotic expansion, it is best to move from distances $q$ to a dimension-less variable $x$, defined by $x = qk_s$, where

$$
k_s = \frac{k_{eq}}{c_1},
$$

(14)

and switch from wave numbers $k$ to the dimension-less quantity $\kappa = k/k_s$. This effectively re-scales $\tilde{c}_1, \ldots, _\gamma \leftrightarrow c_1, \ldots, _\gamma \equiv \tilde{c}_1, \ldots, _\gamma$, which is what we shall do below. Likewise, we define $T_D(\kappa)$ in the same way as $T_D$, but with $c_1, \ldots, _\gamma$ instead of $\tilde{c}_1, \ldots, _\gamma$.

The initial power spectrum $P^{(i)}_\delta$ is taken to have an asymptotic expansion as $\kappa \equiv \frac{k}{k_s} \to \infty$ that reads

$$
P^{(i)}_\delta(\kappa) \sim \kappa^{n_s-4} \sum_{m=0}^\infty \kappa^{-m} \sum_{n=0}^2 c_{mn} \ln^n \kappa .
$$

(15)

Applying this to (13), one has

$$
c_{02} = \frac{4Ac_6}{c_7^4},
$$

(16)

and $c_{00} = c_{01} = 0$. We will not need the higher-order terms in this paper.

Applying the Mellin transform technique (cf. Appendix A) to the definitions of $a_{1,2}$, we find that, as $x = qk_s \to 0$,

$$
a_1(q) \sim -\frac{1}{6\pi^2} M \left[ P^{(i)}_\delta; 1 \right] + \frac{x^2}{60\pi^2} M \left[ P^{(i)}_\delta; 3 \right] - \frac{x^{3-n_s}}{2\pi^2} A \sum_{j=0}^2 \frac{2}{j} (-\ln x)^j M^{(2-j)}[j_1; n_s - 4] + o(x^{3-n_s}),
$$

(17)

$$
a_2(q) \sim \frac{x^2}{30\pi^2} M \left[ P^{(i)}_\delta; 3 \right] + \frac{x^{3-n_s}}{2\pi^2} A \sum_{j=0}^2 \frac{2}{j} (-\ln x)^j M^{(2-j)}[j_2; n_s - 3] + o(x^{3-n_s}),
$$

(18)

where $M^{(k)}[f; z]$ is the $k$-th derivative of the Mellin transform of the function $f$. $M[f; z]$, with respect to $z$. Note that the functions $a_{1,2}$ have units of area. Here, we have defined

$$
M \left[ P^{(i)}_\delta; z \right] \equiv A k_s \int_0^\infty k^{n_s+z-1} T_D^2(\kappa) d\kappa ,
$$

(19)
which also has units of area.

The coefficient \( \xi_1(x, n_s) \) of \( x^{3-n_s} \) in \( a_1 \) can be written as

\[
\xi_1(x, n_s) = -\frac{2}{\pi^2} \frac{c_s^4}{c_l^4} \left( M'[j_1; n_s - 4] - 2 \ln x M'[j_1; n_s - 4] + \ln^2 x M[j_1; n_s - 4] \right)
\]

\[
= : \xi_{10}(n_s) + \ln x \xi_{11}(n_s) + \ln^2 x \xi_{12}(n_s) ,
\]

and the coefficient \( \xi_2(x, n_s) \) of \( x^{3-n_s} \) in \( a_2 \) is likewise

\[
\xi_2(x, n_s) = \frac{2}{\pi^2} \frac{c_s^6}{c_l^4} \left( M''[j_2; n_s - 3] - 2 \ln x M'[j_2; n_s - 3] + \ln^2 x M[j_2; n_s - 3] \right)
\]

\[
= : \xi_{20}(n_s) + \ln x \xi_{21}(n_s) + \ln^2 x \xi_{22}(n_s) .
\]

Figure 1 shows that considering only the order \( q^2 \) asymptotics (green lines) is insufficient to describe the functions \( a_{1,2} \) (purple lines) accurately at cosmologically reasonable small scales. Since the next order, the \( q^{3-n_s} \) terms (dashed dark blue lines), are negative and \( |1 - n_s| \ll 1 \), they balance the order \( q^2 \) terms. The sum of both orders (light blue lines) describes the small scale asymptotics of \( a_{1,2} \) very well up to Mpc scales. Let us recall here, that our results (17) and (18) are only valid for \( n_s < 1 \). In the limit \( n_s \to 1 \), the analytical continuation of \( M \left[ p^{(i)}_\delta; z \right] \), i.e. the second moment of the initial velocity potential power spectrum, diverges (see Fig. 2). Thus, for \( n_s = 1 \), the resulting series contain terms with even powers of \( q \) and logarithms of higher order than 2 due to coinciding poles of \( M \left[ p^{(i)}_\delta; z \right] \) and \( M \left[ j_{1,2}; z \right] \). The coefficients may be computed by applying the second strategy of Case IV of the Mellin transform technique described in Appendix A.

3 SMALL-SCALE ASYMPTOTICS OF THE ZEL’DOVICH POWER SPECTRUM

The small-scale asymptotics of the Zel’dovich power spectrum as \( k \to \infty \) are determined by the saddle points of the exponent of the integrand in (5). The saddle point, that dominates the \( k \to \infty \) asymptotics of the integral, is located at the origin \( q \to 0^+ \).

To prove this claim, let us now show that any potential saddle point of the form \( q = \text{ord} \left( k^{\beta} \right) \), where \( \beta > 0 \), gives an exponentially-decreasing contribution to \( P(k, \nu) \). As \( q \to \infty \), the Mellin-transform technique may be used to show that \( a_{1,2} \sim q^{1-n_s} \). (The explicit expression for the large-scale asymptotics of \( a_{1,2} \) is shown in Appendix B2.) This result implies that for any saddle point where \( k \to \infty \), we have \( \hat{C}_{pp}(q) = o(1) \), and therefore that the exponential,

\[
i^2 k^2 \left( \hat{C}_{pp} \right)_{ij} k^{ij} - Q ,
\]

with \( Q \sim k^2 \) from (6) is dominated by \( -Q \) in the large-\( k \) limit. That is, the contribution to \( P \) from such a saddle point is proportional to the exponentially-decaying factor

\[
\exp \left[ -k^2 \frac{\sigma_k^2}{3} + o \left( k^2 \right) \right] .
\]
In a similar fashion, when $\beta = 0$ and $q \neq 0$ (i.e. $q$ has a constant finite value), $\frac{t^2 k^i \hat{C}_{\mu \nu} k_j - Q}{k^2}$ is a negative, order unity quantity, and thus gives rise to an exponentially-decaying contribution. Consequently, if a saddle point exists with $\beta < 0$, i.e. at $q \to 0^+$, and if this saddle point yields a contribution that decays more slowly than exponentially, then it does dominate over all other saddle points with $\beta \geq 0$.

We are now in a position to evaluate the dominant contribution to the power spectrum, by inserting (17) and (18) into (9) and (5), and integrating over $q$. Explicitly, defining $\sigma^2 \equiv M\{P_{\delta}^{(3)}; 3\}$ (having dimensions of area; see (19)), we obtain, as $k \to \infty$,

$$
\mathcal{P}(k, t) \sim \int d^3 q \ e^{-t^2 (k - q)^i \left( x^2 \hat{a}_{ij} + x^3 - n_s \xi_i \right)} r^2 x^2 e^{-t^2 k^i (2 \mu^2 + 1) \frac{x^2}{2} + i k \cdot q} \exp \left( \frac{t^2 k^i x^3 - n_s (\xi_1 - \mu^2 \xi_2) \right),
$$

where we changed variables to the dimensionless radial coordinate $x = k_s q$. Recall that the functions $\xi_{1,2}$ both are quadratic polynomials in logarithms of $x$. Defining $\xi \equiv \xi_1 - \mu^2 \xi_2$ and using the Taylor expansion of the last exponential to generate an asymptotic series expression, we arrive at

$$
\mathcal{P}(k, t) \sim \frac{2 \pi}{(kk_s \sigma)^3} \sum_{m=0}^{\infty} \int_{-1}^{1} \mu \int_{0}^{\infty} dx x^2 e^{-t^2 k^2 (2 \mu^2 + 1) \frac{x^2}{2}} \cos (k \mu) \left( \frac{(2\mu^2 + 1) y^3}{m!} \right) e^{m} \left( \frac{x}{\sigma} \right)^m,
$$

as $k \to \infty$. Again, recall that $\frac{\xi}{\sigma^2}$ is a dimension-less quantity. The (dimension-less) integration variable $y$ is defined as $y = t x \sigma k$. So, as $k \to \infty$,

$$
\mathcal{P}(k, t) \sim \frac{2 \pi}{(tk_2 \sigma)^3} \sum_{m=0}^{\infty} \int_{-1}^{1} \mu \int_{0}^{\infty} dy y^2 \cos \left( \frac{y \mu}{t \sigma k_s} \right) \frac{e^{-(2\mu^2 + 1) y^3 / 3 \sigma^3 m(3-n_s)}}{m! (tk_2)^m (1-n_s)} \left( \frac{\xi}{\sigma^2} \right)^m + O \left( k^{-5} \right).
$$

One can summarise this expansion as

$$
\mathcal{P}(k, t) = \frac{2 \pi}{(k_2 \sigma)^3} \sum_{m=0}^{\infty} k_{1+n_s}^m \left( \frac{m(1-n_s)}{k_2 \sigma} \right)^m \mathcal{P}_m \left( t, \ln(k_2 / k_s) \right),
$$

with $\tau_2 \equiv t \sigma k_s$ and\footnote{Recall that $\tau_2$ is simply a monotonically increasing function of time.}

$$
\mathcal{P}_m \left( t, \ln(k_2 / k_s) \right) = \int_{-1}^{1} \mu \int_{0}^{\infty} dy y^2 \cos \left( \frac{y \mu}{\tau_2} \right) \frac{e^{-(2\mu^2 + 1) y^3 / 3 \sigma^3 m(3-n_s)}}{m!} \left( \frac{\xi}{\sigma^2} \right)^m.
$$

\textbf{Figure 2.} The moments of the initial velocity potential as a function of the spectral index $n_s$ for Planck parameters (Table 1). While the first moment $\sigma_1^2$ (left) varies only little with the spectral index, the second moment $\sigma_2^2$ and the analytic continuation of the third moment $\sigma_3^2$ diverge as $n_s \to 1$ (right). The Dicus transfer function (Weinberg 2008) is used.
We are now in a position to extract all of the $k$-dependence of the power spectrum. Like $\xi_{1,2}$, $\xi$ is a quadratic polynomial in $\ln k$, and therefore

$$P(k, t) \sim \frac{1}{(k \tau_2)^3} \sum_{m=0}^{\infty} \left( \frac{k_x}{k \tau_2} \right)^{m(1-n_s)} \sum_{n=0}^{2m} \mathcal{P}_{mn}(t) \ln^n \left( \frac{k \tau_2}{k_x} \right) + O \left( k^{-5} \right)$$

(29)

with

$$\mathcal{P}_{mn}(t) = \frac{2\pi}{m!(\sigma_z^2)^m} \int_0^1 d\mu \int_0^\infty dy \, e^{-\frac{1}{4}(2\mu^2+1)y^2} \, e^{i\mu \frac{2}{k \tau_2} y^{m(3-n_s)+2} \xi_{m,n}(\mu^2, \ln y)}$$

(30)

with $\xi_{m,0}(\mu^2, \ln y) \equiv \xi_{m,0}^o(\mu^2, \ln y)$ and for $n \neq 0$, the functions $\xi_{m,n}$ are given recursively by

$$\xi_{m,n}(\mu^2, \ln y) = \frac{1}{n k_0} \left[ (m-n+1)\xi_{1,1} \xi_{m,n-1} + (2m-n+2)\xi_{2,2} \xi_{m,n-2} \right]$$

(31)

with

$$\xi_{2,2}(\mu^2, \ln y) \equiv \xi_{12} - \mu^2 \xi_{22}$$

(32)

$$\xi_{1,1}(\mu^2, \ln y) \equiv -\left( \xi_{11} - \mu^2 \xi_{21} \right) - 2 \ln y \left( \xi_{12} - \mu^2 \xi_{22} \right)$$

(33)

$$\xi_{0,0}(\mu^2, \ln y) \equiv \ln^2 y \left( \xi_{12} - \mu^2 \xi_{22} \right) + \ln y \left( \xi_{11} - \mu^2 \xi_{21} \right) + \xi_{10} - \mu^2 \xi_{20}$$

(34)

with $\xi_{10, \xi_{11}, \xi_{12}, \xi_{20}, \xi_{21}, \xi_{22}$ given by Eqs. (20) and (21).

Equation (29) is the main result of this paper: it shows the full dependence of the Zel’’dovich power spectrum on $k$, for large values of $k/k_x$, up to corrections of order $k^{-5}$ (for the Dicus transfer function, which are in principle calculable using the Mellin transform technique. Here, as opposed to earlier works (Konrad & Bartelmann 2021; Chen & Pietroni 2020), higher moments of the initial power spectrum need not be finite. There, a UV cutoff of the initial power spectrum had to be introduced. In its presence, a full asymptotic expansion for $k \rightarrow \infty$ is given by (8). Without a UV cutoff in the initial power spectrum, the asymptotics of the evolved Zel’’dovich power spectrum is instead given by (29) up to $O \left( k^{-5} \right)$. With a UV cutoff, (29) reduces to the single non-vanishing term with $m = 0$, expressing the leading-order $k^{-3}$ asymptotics of the Zel’’dovich power spectrum. We emphasize that terms with $m > 0$ are not only negligible, but strictly absent in presence of a UV cutoff since they only appear due to the poles of the Mellin transform of the initial power spectrum. Our result (29) also shows why terms beyond the order $k^{-3}$ are necessary to describe the Zel’’dovich power spectrum on cosmologically relevant scales: the weak increase of logarithms in the numerator almost precisely cancels the weak increase of $k^{m(1-n_s)}$ (for realistic $n_s$) in the denominator. The relevance of high orders can be seen in Fig. 4, which shows the asymptotic series up to a maximum value of $m \leq M$, for various $M$. This maximum value $M$ is chosen such that contributions way beyond order $k^{-5}$ are considered, indicating that the missing $O(k^3)$-terms in (29) are less relevant; see also Appendix C. We will discuss this in more detail for redshift $z = 0$ in the next section.
Figure 4. The dimension-less cold dark matter power spectrum in the Zel’dovich approximation (purple line) at today’s redshift \( z = 0 \), compared with the analytical prediction of (35), up to high orders \( M \). The Dicus transfer function (Weinberg 2008) and the parameters in table 1 are used.

4 LATE TIME EVOLUTION

As \( t \) grows, \( \tau_2 \) becomes quite large. The argument of the cosine in (28) tends to 0 at late times, thus

\[
P_m \to \int_{-1}^{1} d\mu \int_{0}^{\infty} dy \, y^2 e^{-(\mu^2+1)y^2/30} \frac{y^{m(3-n_s)}}{m!} \left[ \frac{\xi}{\sigma^2} \right]^m,
\]

which is time-independent and simpler to evaluate numerically. More precisely, in the late-time limit \( \tau_2 \to \infty \), \( P_{mn} \) from (30) can be expanded into the series

\[
P_{mn}(t) \to \frac{2\pi}{m!(\sigma_2^2)^m} \int_{-1}^{1} d\mu \int_{0}^{\infty} dy \, e^{-\frac{\pi y^2}{\tau_2^2}} \frac{1}{(2\mu)} \sum_{l=0}^{\infty} \frac{(-1)^l}{(2l)!} \left( \frac{Y}{\tau_2} \right)^{2l} y^{m(3-n_s)+2} \xi_{m,n} \left( \mu^2, \ln y \right).
\]

For the parameters in Table 1, \( \tau_2 \approx 259 \) at redshift \( z = 0 \), so one can safely use the late-time expression. The results are shown in Fig. 4. One can see excellent agreement with the asymptotic expansion at scales \( k \geq 10^3 \) Mpc\(^{-1} \) when going to high orders in the asymptotics: for low numbers of terms (\( M = 15 \), green line), the asymptotics underestimate the actual Zel’dovich power spectrum by more than an order of magnitude. Summing more and more orders improves the result significantly. Including terms up to order \( M = 1000 \) leads to an agreement of the asymptotics with the Zel’dovich power spectrum at small scales. We note that the computation of the double integral in (35) is more involved for large \( m \), and we explain the method we used in Appendix D. Furthermore, let us note that the order \( k^{-5} \) term can be neglected although we go to very high orders in \( M \), which we discuss in Appendix C.

5 IMPLICATIONS FOR DARK MATTER

If dark matter is made of massive fermions, then the initial power spectrum does not decay like \( k^{n_s-4} \log^2 k \). It may rather have a cutoff, determined by the particle mass and the particle production mechanism, which is related to the type(s) of interaction the particle undergoes (Green et al. 2004). The ultraviolet cut-off \( k_D \) is also typically related to a temperature of dark matter. In this section, we discuss two aspects of the Zel’dovich power spectrum that are related to the physically relevant scenario of dark matter with a finite temperature. Konrad & Bartelmann (2021) have shown that the Zel’dovich power spectrum for large \( k \) can be expanded in a series with terms of order \( k^{-3-2m} \), when
the initial power spectrum has a cut-off and all moments of the initial velocity potential power spectrum exist. The first two coefficients in the series \(8\) are given by

\[
\begin{align*}
\mathcal{P}^{(0)}(t) &= 3(4\pi)^{3/2} \left(\frac{5}{2\pi^2}\right)^{3/2} e^{-\frac{5}{2\pi^2}}, \\
\mathcal{P}^{(1)}(t) &= \frac{(4\pi)^{3/2} \sigma_2^5}{28} \left(\frac{5}{2\pi^2}\right)^{5/2} e^{-\frac{5}{2\pi^2}} \left[ 123 - 132 \left(\frac{5}{2\pi^2}\right) + 20 \left(\frac{5}{2\pi^2}\right)^2 \right],
\end{align*}
\]

where \(\mathcal{P}(k, t) \sim \mathcal{P}^{(0)}(t)k^{-3} + \mathcal{P}^{(1)}(t)k^{-5}\) as \(k \to \infty\). While the first term describes the asymptotic \(k^{-3}\) tail very well, the intersection with the second term, of order \(k^{-5}\), engenders a time-dependent scale \(k_{\text{asymp}} = \sqrt{\frac{\mathcal{P}^{(0)}(t)}{\mathcal{P}^{(1)}(t)}}\), below which the asymptotics fail to reproduce the Zel’’dovich power spectrum. Thus, an analytical description of the latter between this scale and the linear regime was missing.

In Fig. 5 we show the dimensionless Zel’’dovich power spectra of strictly cold dark matter without an initial ultraviolet cut-off, i.e. when dark matter is treated as a non-relativistic Vlasov fluid at all scales (purple line), at redshift \(z = 0\), together with those of three types of warm dark matter, where the initial (linear) power spectrum is exponentially cut off at a scale \(k_D = \{10^2, 10^4, 10^6\} \text{ Mpc}^{-1}\) by a multiplicative Gaussian \(\exp(-k^2/k_D^2)\) (yellow, blue and green line, respectively). The latter corresponds to a power spectrum of WIMP dark matter with particle mass of \(\sim 100 \text{ GeV}\) (Green et al. 2004). For small and intermediate wave numbers, the Zel’’dovich power spectra align with the spectrum of the strictly cold dark matter. At large wave numbers, the spectra become flat, reaching the \(k^{-3}\) asymptotics (coloured dashed lines). The wave number where the spectra with finite initial temperature deviate from the strictly cold dark matter spectrum is in all shown cases indicated very well by the intersection of the \(k^{-3}\) (coloured dashed lines) and the \(k^{-5}\) (dotted dashed lines) asymptotics. In case of the WIMP dark matter, the wave number regime just below validity of the \(k^{-3}\) asymptotics, i.e. below \(10^4 \text{ Mpc}^{-1}\), is now very well described by the asymptotics of the strictly cold dark matter (thick dashed dark blue line) that we derived in this paper up to order \(M = 1000\). This is remarkable, since the latter contains information about the asymptotic tail of the initial power spectrum, that is actually cut-off for WIMP dark matter, but ultimately originates from the fact that \(\sigma_2^2/k_D^4 \gg M]\) [\(\mathcal{P}^{(1)}(t)\)], where the former is computed with a cut-off, and the latter – without one.

It may be surprising, prima facie that the asymptotic tails of the Zel’’dovich power spectra with initial ultraviolet cut-off have a higher amplitude at redshift \(z = 0\) the stronger the ultraviolet cut-off is, i.e. the smaller \(k_D\). The reason for this is that the time evolution of the asymptotic amplitude \(\mathcal{P}^{(0)}(t), (37)\), depends on the product \(\sigma_2^2/\kappa^2\). This implies that smaller values of \(\sigma_2\) lead to a slower evolution of the asymptotic tail. Since at redshift \(z = 0\), all of these amplitudes are decreasing in time due to re-expansion, the spectra with larger \(k_D\), i.e. larger \(\sigma_2^2\) already progressed further in re-expansion. The time evolution of the amplitude is discussed in detail in Konrad & Bartelmann (2021).

6 SUMMARY AND DISCUSSION

In this paper we derived an asymptotic expansion for the cold dark matter power spectrum in the Zel’’dovich approximation, at large wave numbers \(k\), by using the Mellin transform technique; the result, Eq. (29), shows that in addition to the expected \(k^{-3}\), \(k^{-5}\), etc. terms, there are terms that go like negative integer powers \(k^{1-n_s}\), accompanied by positive powers of \(\log k\). These are a sine qua non in any practical use of asymptotic expansions of the Zel’’dovich power spectrum, since only by going to high order, can a reasonable accuracy be achieved (see Fig. 4).

This differs from the case where the power spectrum contains an ultraviolet cut-off, \(k_D\) (Konrad & Bartelmann 2021). In this case, there exists a cut-off and time dependent wave number \(k_{\text{asymp}}\) above which the Zel’’dovich power spectrum aligns with the \(k^{-3}\) asymptotics. However if \(k_D \gg k_s\), there is an intermediate regime, where \(k_s \ll k \ll k_{\text{asymp}}\). There, physically, the cut-off should have no effect, and for these values of \(k\), we indeed see in Fig. 5 that \(\mathcal{P}(k, t)\) follows (29), rather than the asymptotic expansion of Konrad & Bartelmann (2021); Chen & Pietroni (2020). This already means that for such \(k\) one should not use the expansion with a cut-off, but rather treat the cut-off as non-existent. The applicability of an expansion without a cut-off, well below the cut-off in asymptotic expansions is known to exist elsewhere in cosmology, e.g. in the distribution of the astrophysical gravitational-wave background (Ginat et al. 2020). We therefore have shown that (29) is valid even for realistic models of dark matter, which do include some ultraviolet cut-off, as long as one considers scales below \(k_{\text{asymp}}\). We therefore conclude that for Zel’’dovich power spectra of dark matter with an initial ultraviolet cut-off, the \(k \to \infty\) asymptotics are correctly described by the result of Konrad & Bartelmann (2021), (8), rather than (29). This is because the \(m > 0\) terms in (29) appear only due to the poles of the Mellin transform of the initial power spectrum when no initial ultraviolet cut-off is imposed. For some intermediate regime of finite values of \(k\), however, at wave numbers below which the series (8) fails to reproduce the warm dark matter Zel’’dovich power spectra, warm dark matter and strictly cold dark matter spectra align and the warm dark matter spectra can be described by the asymptotics of the cold dark matter spectra (29) derived in this work. We suspect that the mathematical reason for this is that there might exist an expansion for the warm dark matter spectra in the \(k \to \infty\) limit for which the \(m > 0\) terms do not vanish strictly, but are suppressed by an \(O(k^{-\infty})\) function (i.e. one that falls off more steeply than any power law), with the suppression controlled by the cut-off scale \(k_D\). Since such terms would be non-perturbative, they would not appear in the asymptotic expansion (8). In between those two regimes, i.e. at wave numbers of about \(k_{\text{asymp}}\), the warm dark matter power spectrum follows some interpolation of the two asymptotic expressions. It will be subject to future work to investigate the connection between those two expansions and to formally derive the correct interpolation.
Asymptotics of the Zeldovich Power Spectrum

While one has to include many terms in the series (29) for it to be accurate, knowing the functional form of the \( k \)-dependence of the power spectrum is of theoretical significance in itself. Evaluating the Zel’dovich power spectrum can be achieved by solving the integral (5) numerically. This expansion will help to find out more about the mathematical structure of the Zel’dovich power spectrum in general; we anticipate that once this series is known, in the future one would be able to re-sum it to arrive at more accurate results without needing to go to very high orders. The Zel’dovich power spectrum resembles the zeroth order power spectrum of KFT and is a fundamental building-block of perturbative terms; the result of this paper will therefore guide future work on KFT perturbation theory and thereby on analytical expressions for the non-linear power spectrum of dark matter, and put them on mathematically solid ground. Here, we have approximated particle trajectories by straight lines. Due to this Zel’dovich approximation, virialization is not expected to occur. However, virialization is an emergent effect in KFT and thus need not be introduced explicitly. Incorporating gravitational interactions beyond the Zel’dovich approximation will lead to more persistent structures on small scales, since gravity prevents the structures formed in the Zel’dovich approximation from re-expanding. Nonetheless, we have convincing reasons to believe that the shape of the asymptotics will not be changed even by interactions: Taking them into account in a mean-field approximation, we could show that the multiplicative interaction term becomes scale-independent for large \( k \) (Bartelmann et al. 2021; Konrad & Bartelmann 2022). Then, the amplitude of the power spectrum will increase, but the shape will remain unchanged. The analytical result based on the mean-field approximation recovers fully numerical results for non-linear power spectra to an accuracy of a few percent up to wave numbers \( k \lesssim 10^2 \, h \, \text{Mpc}^{-1} \). In future work we will extend the present calculation beyond the Zel’dovich approximation, and endeavour to find an asymptotic series of the form (29) for the full cold dark-matter power spectrum.

We did not consider baryonic effects either here, but dark matter only as a back-bone for cosmic structure formation. This is of particular cosmological interest because the question as to which small-scale structures in (warm and cold) dark-matter only universes form is not completely solved yet, e.g. in the notorious “core-cusp” problem (Genina et al. 2018). While baryonic effects are certain to change the small-scale structures built from dark matter, it is yet unclear which small-scale structures actually arise in dark-matter only universes, as numerical simulations are limited in resolution. We are convinced that this is an important question to solve, prior to incorporating any baryonic effects.

Moreover, as is visible from Fig. 2, the coefficients \( P_m(t, \ln k) \) (or \( P_{mn}(t) \)) are extremely sensitive to the spectral index \( n_s \). We believe that the primordial \( n_s \) may leave an imprint on cosmic structure formation at early times. Since the cooling time of the baryonic matter is finite, the formation of dense baryonic structures is expected to set in later than small-scale structures form by stream-crossing in cold dark matter.

Figure 5. The cold dark matter power spectrum in the Zel’dovich approximation without small scale smoothing (purple line), compared with power spectra that are initially smoothed on small scales. For larger smoothing wave numbers, i.e. smaller smoothing scales, the initially smoothed power spectra align with the unsmoothed spectrum up to higher wave numbers \( k \). This implies that the intermediate regime of initially smoothed Zel’dovich power spectra is well approximated by the asymptotics of the exact cold dark matter power spectrum. The Dicus transfer function (Weinberg 2008) and the parameters in Table 1 are used.

\[ k_D = 10^2 \, \text{Mpc}^{-1} \]
\[ k_D = 10^4 \, \text{Mpc}^{-1} \]
WIMP CDM, \( k_D = 10^6 \, \text{Mpc}^{-1} \)
strictly CDM
CDM asymptotics

\[ k^3 P(k) \]
\[ k \, [\text{Mpc}^{-1}] \]

\( k \) values are shown for different asymptotic regimes.

\[ 10^{-3} \]
\[ 10^{-2} \]
\[ 10^{-1} \]
\[ 10^0 \]
\[ 10^1 \]
\[ 10^2 \]
\[ 10^3 \]
\[ 10^4 \]
\[ 10^5 \]
\[ 10^6 \]
\[ 10^7 \]
These small-scale, cold dark-matter structures will in turn influence the mass and time scales of baryonic structure formation. These different time scales might themselves be reflected in observables related to reionization or the recombination epoch. It is thus quite possible that the primordial $n_s$ may be related to observables of cosmic structure formation. It will be the subject of future work to investigate these time scales and predict observables, with initial conditions for our theory set during the matter-dominated era. In future surveys, if they could be inferred experimentally, they could furnish an additional probe of this cosmological parameter.

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DATA AVAILABILITY

The data underlying this article will be shared on reasonable request to the corresponding authors.

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APPENDIX A: MELLIN TRANSFORM TECHNIQUE

In this section, we briefly sketch the Mellin transform technique as described in Bleistein & Handelsman (1986), in-so-far as it is used in this paper. This technique is a powerful tool to derive the asymptotics of functions that are described by integral transforms. The Mellin transform of a function $f(k)$ is defined as

$$\mathcal{M} [f; z] = \int_0^\infty dk \ k^{z-1} f(k).$$  \hspace{1cm} (A1)
The H-transform of a function $f$ with respect to a function $h$ is defined as

$$H[f; \lambda] \equiv \int_0^\infty dk \ f(k)h(\lambda k)$$

(A2)

where $c \in \mathbb{R}$ lies in the strip of analyticity of $M[h; z]$ and $M[f; 1 - z]$. Upon defining

$$G(z) \equiv M[h; z]M[f; 1 - z]$$

(A4)

and using the residue theorem, the H-transform has the asymptotic expansion as $\lambda \to \infty$

$$H[f; \lambda] = - \sum_{c < \text{Re} \ z < R} \text{Res} \left\{ \lambda^{-z}G(z) \right\} + \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} G(z)dz$$

(A5)

where $R$ is chosen such that $\lambda^{-z}G(z)$ has no poles for $\text{Re} \ z = R$. The latter integral is $O(\lambda^{-R})$ as $\lambda \to \infty$, where $G$ is analytically continued as much as possible, preferably to a meromorphic function of $z$.

To calculate the asymptotics as $\lambda \to \infty$ explicitly, we assume that, as $t \to \infty$,

$$h(t) \sim e^{-\lambda t} \sum_{m=0}^{\infty} t^{-r_m} \sum_{n=0}^{N(m)} c_{mn} \ln^n t$$

(A6)

where $d \geq 0$, $\nu > 0$, $\text{Re} \ r_m \to \infty$, $0 \leq N(m)$ is finite. We also assume that as $t \to 0$,

$$f(t) \sim e^{-\alpha t} \sum_{m=0}^{\infty} r_m^{a_m} \sum_{n=0}^{\bar{N}(m)} p_{mn} \ln^n t$$

(A7)

where $q \geq 0$, $\mu > 0$, $\text{Re} \ a_m \to \infty$, and $0 \leq \bar{N}(m)$ is finite. The relevant theorem, quoted here from Bleistein & Handelsman (1986) for completeness, considers four cases:

- **Case I:** $d \neq 0 \neq q$:
  $$H[f; \lambda] = o(\lambda^{-R}) \ \forall R > 0$$

(A8)

- **Case II:** $d \neq 0$, $q = 0$:
  $$H[f; \lambda] \sim \sum_{m=0}^{\bar{N}(m)} \sum_{n=0}^{N(m)} c_{mn} \ln^n t \lambda^{-1-\alpha_m} M(\lambda^{n-j}h; z)_{j=1+a_m}$$

(A9)

- **Case III:** $d = 0$, $q \neq 0$:
  $$H[f; \lambda] \sim \sum_{m=0}^{\infty} \sum_{n=0}^{N(m)} c_{mn} \ln^n t \lambda^{n-j} M(\lambda^{n-j}f; z)_{j=1-r_m}$$

(A10)

- **Case IV:** $d = 0$, $q = 0$:
  - $r_m \neq a_n + 1$ for all pairs $n, m$, then the resulting asymptotics is simply given by the sum of Cases II and III.
  - When $r_m = a_n + 1$ for one or several pairs $n, m$, then one has to consider the residue
    $$\text{Res}_{r_m} \left\{ \lambda^{-z}M[h; z]M[f; 1 - z] \right\},$$

(A11)

which follow from the equation below for the residue. Note that the multiplicity (order) of the pole is increased, which produces logarithmic terms even if there weren't any in $f$ or $h$ to begin with.

In all the above cases, $M$ denotes the Mellin transform, or its analytical continuation. $M^{(n-j)}$ denotes the $(n-j)$-th derivative of the Mellin transform, or its analytical continuation.

**APPENDIX B: ASYMPTOTICS OF THE INITIAL MOMENTUM CORRELATIONS**

### B1 Small Scale Asymptotics

We assume that the initial power spectrum $p^{(i)}(k)$ has an asymptotic expansion as $k \to \infty$ that is given by

$$p^{(i)}(k) = A \left( \frac{k}{k_s} \right)^{n_s} T^2 \left( \frac{k}{k_s} \right) \sim \left( \frac{k}{k_s} \right)^{n_s - 1} \sum_{m=0}^{\infty} c_{mn} \ln^m \left( \frac{k}{k_s} \right)$$

(B1)

Note that with $\kappa = k/k_s$ and $k_s$ as defined in (14) the argument is actually dimension-less.
The functions $a_1$ and $a_2$ that characterise the initial momentum correlations are given by

$$a_1(q) := \frac{\xi'(q)}{q} = -\frac{1}{2\pi^2} \int_0^\infty dk \frac{P^{i1}_\delta(k) j_1(kq)}{kq},$$

$$a_2(q) := \xi''(q) - \frac{\xi'(q)}{q} = \frac{1}{2\pi^2} \int_0^\infty dk \frac{P^{i2}_\delta(k) j_2(kq)}{kq}.$$  \hspace{1cm} (B2)

let us derive the asymptotics of these functions for small scales, i.e. when $q \to 0$, by applying the Mellin transform technique.

Let’s commence with the function $a_1$ for the large parameter $x^1 - k_s^{-1} q^{-1}$. We start by transforming the integral expression for $a_1$ by using $\kappa = k/k_s$

$$a_1(q) = -\frac{1}{2\pi^2} \int_0^\infty dk \frac{P^{i1}_\delta(k) j_1(k\kappa)}{k\kappa} = -\frac{1}{2\pi^2} x^1 k_s \int_0^\infty dk \frac{P^{i1}_\delta(k_s\kappa^{-1}) j_1(k)}{k} = -\frac{1}{2\pi^2} x^1 k_s \int_0^\infty dk \frac{\Lambda(k\kappa) T_D^2(k\kappa) j_1(k)}{k}. \hspace{1cm} (B4)$$

Since the spherical Bessel function $j_1$ has a convergent Taylor series at $\kappa = 0$,

$$j_1(k) = \sum_{m=0}^\infty \frac{(-)^m}{m!} \left(\frac{k}{2}\right)^{2m},$$

and $T_D^2$ has a power law tail with logarithms (B1), the resulting asymptotics is given by Case IV of the appendix A. Since we assume $0 < n_s < 1$, the exponents of the series of $P^{i1}_\delta$ in (B1) are not integers, which implies that the asymptotic series is a sum of Cases II and III:

$$a_1(q) = -\frac{1}{2\pi^2} x^1 k_s \sum_{m=0}^\infty \frac{(-)^m}{(2m+3)(2m+1)!} M \left[P^{i1}_\delta:1+2m\right] - \frac{1}{2\pi^2} x^1 n_s - n_s \sum_{m=0}^\infty x^m \sum_{n=0}^\infty \frac{\zeta_{mn}}{A} \sum_{j=0}^n \frac{n}{j} (-\ln x)^j M^{(n-j)} [j_1(z)]_{s=n_s - 3}.$$  \hspace{1cm} (B6)

where we have defined

$$M \left[P^{i1}_\delta:z\right] \equiv A k_s \int_0^\infty k^{n_s + z} T_D^2(k) d\kappa.$$  \hspace{1cm} (B7)

Note that for $z = 2m - 1$, these terms are scaled versions of the analytical continuation of the moments of the initial velocity power spectrum (7),

$$M \left[P^{i1}_\delta:2m - 1\right] = 2\pi^2 \frac{\sigma^2_m}{k_s^{2m-2}}.$$  \hspace{1cm} (B8)

Similarly, we get for $a_2$ as $q \to 0$

$$a_2(q) = \frac{1}{2\pi^2} \int_0^\infty dk \frac{P^{i2}_\delta(k) j_2(kq)}{kq} = \frac{1}{2\pi^2} \int_0^\infty dk \frac{P^{i2}_\delta(k_s\kappa^{-1}) j_2(k)}{k} = \frac{1}{2\pi^2} \int_0^\infty dk \frac{\Lambda(k\kappa) T_D^2(k\kappa) j_2(k)}{k}.$$  \hspace{1cm} (B9)

$$- \frac{1}{2\pi^2} \sum_{m=0}^\infty x^{2m+2} \frac{(-)^m}{(2m+3)(2m+1)!} M \left[P^{i2}_\delta:3+2m\right] + \frac{1}{2\pi^2} x^{n_s - n_s} \sum_{m=0}^\infty x^m \sum_{n=0}^\infty \frac{\zeta_{mn}}{A} \sum_{j=0}^n \frac{n}{j} (-\ln x)^j M^{(n-j)} [j_2(z)]_{s=n_s - 3}. \hspace{1cm} (B10)$$

B2 Large Scale Asymptotics

We assume that the initial power spectrum $P^{i1}_\delta$ has an asymptotic expansion as $k \to 0$ that is given by

$$p^{i1}_\delta \sim k^{n_s} \sum_{m=0}^\infty p_m k^m,$$  \hspace{1cm} (B11)

with $\kappa = k/k_s$ and coefficients $p_m$ that have units of volume. We consider again the integral expressions of $a_1$ and $a_2$ in terms of the dimensional integration variable $x$ and $x = k_s q$,

$$a_1(q) = -\frac{1}{2\pi^2} k_s \int_0^\infty dk \frac{P^{i1}_\delta(k_s\kappa) j_1(k\kappa)}{k\kappa},$$

$$a_2(q) = \frac{1}{2\pi^2} k_s \int_0^\infty dk \frac{P^{i2}_\delta(k_s\kappa) j_2(k\kappa)}{k\kappa}. \hspace{1cm} (B12)$$

Since the spherical Bessel functions $j_n(x)$ are oscillating for $x \to \infty$, in order to compute the asymptotics of $a_1$ and $a_2$ for $q \to \infty$, Case II of the Mellin transform technique applies for both integrals. Thus, as $x = k_s q \to \infty$,

$$a_1(q) = \frac{1}{2\pi^2} \sum_{m=0}^\infty x^{-n_s-m} k_s p_m M [j_1: n_s + m] = \frac{x^{-n_s}}{2\pi^2} k_s p_0 \sqrt{\pi} \frac{\Gamma \left(\frac{n_s+1}{2}\right)}{\Gamma \left(\frac{4-n_s}{2}\right)} + O \left(q^{-2-n_s}\right), \hspace{1cm} (B14)$$

$$a_2(q) = \frac{1}{2\pi^2} \sum_{m=0}^\infty x^{-n_s-m} k_s p_m M [j_2: 1 + n_s + m] = \frac{x^{-n_s}}{2\pi^2} k_s p_0 \sqrt{\pi} \frac{\Gamma \left(\frac{n_s+1}{2}\right)}{\Gamma \left(\frac{4-n_s}{2}\right)} + O \left(q^{-2-n_s}\right). \hspace{1cm} (B15)$$

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B3 Computing the analytic continuation of the Mellin transform

In order to actually evaluate the coefficients appearing in the small scale asymptotics of \(a_1\) and \(a_2\), we need to compute the analytical continuation of the moments of the initial power spectrum \(M \left[ P_\delta^{(i)} ; 2m-1 \right] \) into the right-hand half-plane numerically. We assumed that \(P_\delta^{(i)}\) has an asymptotic expansion as \(k \to \infty\) that is given by

\[
P_\delta^{(i)} \sim \sum_{m=0}^{\infty} \tilde{k}^{-r_m} \frac{2}{\pi} c_{mn} \ln^n k .
\]  

The analytic continuation of its Mellin transform is calculated by first defining the function

\[
h_p(k) := \frac{\sum_{\text{Re } r_m < p} \kappa^{-r_m} \sum_{n=0}^{N(m)} c_{mn} \ln^n k}{0} , 
\]

for \(k \geq 1\), else.

Then, by defining

\[
P_\delta^{(i)}(k) := A r^{N} \hat{T}_D(k) = P_\delta^{(i)}(k) ,
\]

we get for the following expression

\[
M \left[ P_\delta^{(i)} ; \zeta \right] = k_s \int_0^{\infty} dk \kappa^{-1} \hat{P}_\delta^{(i)}(k) = k_s \int_0^{1} dk \kappa^{-1} \hat{P}_\delta^{(i)}(k) + k_s \int_1^{\infty} dk \kappa^{-1} \hat{P}_\delta^{(i)}(k)
\]

\[
= k_s \int_0^{1} dk \kappa^{-1} P_\delta^{(i)}(k) + k_s \int_1^{\infty} dk \kappa^{-1} \left[ P_\delta^{(i)}(k) - h_p(k) \right] + k_s \int_1^{\infty} dk \kappa^{-1} h_p(k) ,
\]

where the first two integrals can be calculated numerically and the last one is given by its analytical continuation

\[
\int_1^{\infty} dk \kappa^{-1} h_p(k) = \sum_{\text{Re } r_m < p} \sum_{n=0}^{N(m)} \left( -1 \right)^{n+1} c_{mn} n ! .
\]

APPENDIX C: NEGLECTING OTHER ORDERS IN THE ASYMPTOTICS

While for dark matter with an initial ultraviolet cut-off the \(k^{-5}\) asymptotics typically dominate the \(k^{-3}\) asymptotics on small scales (see figure 5), this is not the case for strictly cold dark matter. To compute the asymptotics for the Zel’dovich power spectrum of strictly cold dark matter, we therefore exclusively added terms from the sum (25), i.e. we did not add the order \(k^{-5}\) term, although \(M \approx 1000\) implies that we went up to order \(-M(1-n_s) = -33.3\). Nevertheless, as can be seen in figure C1: at today’s redshift \(z = 0\) this term is negligible at small scales, because the logarithms in the numerator almost cancel the powers of \(k\) in the denominator.

APPENDIX D: LARGE M

The integral in (35) is a double integral over \(y\) and \(\mu\). Simply evaluating it numerically in Mathematica works only up to \(m \leq 50\). For larger \(m\), one might presume naively that a simple Laplace-type approximation would be appropriate, but this is true only for the \(y\) integral.

Let us start, therefore, with the \(y\) integral, and write

\[
\mathcal{P}_m = \frac{1}{m!} \int_0^{\infty} dy \int_{-1}^{1} d\mu \ \frac{e^{m(1-n_s) \ln y - (2\mu^2 + 1) y^2/30}}{\sqrt{\xi}} \left[ \frac{\xi}{\sigma^2} \right]^m .
\]

This is a moveable saddle problem, and therefore we need to re-scale \(y\) appropriately before proceeding. The exponent

\[
m(3-n_s) \ln y - \frac{2\mu^2 + 1}{30} y^2
\]

is stationary when

\[
y^2 = m \frac{15(3-n_s)}{2\mu^2 + 1} .
\]

i.e. the stationary point is \(\sqrt{m}\). So, let us change variables to \(y = u \sqrt{m}\), and define

\[
u_0(\mu) = \sqrt{\frac{15(3-n_s)}{1 + 2\mu^2}} .
\]
\( \mathcal{P}_m \) then becomes

\[
\mathcal{P}_m \sim \frac{1}{m!} \int_{-1}^{1} d\mu \int_{0}^{\infty} du \frac{m^{3m(3-n_s)/2} \sigma^2}{2n_0} \left[ \frac{\xi(y = \sqrt{m}u_0(\mu)/k\tau_2)}{\sigma^2} \right]^m \exp \left[ \frac{m(3-n_s)}{2} \left( \ln(u_0^2) - 1 \right) \right] \exp \left[ -m \frac{1 + 2\mu^2}{15} (u - u_0)^2 + \ldots \right]
\]  

\( \text{(D5)} \)

\[
\frac{m^{3m(3-n_s)/2}}{m!} e^{-m(3-n_s)/2} \int_{-1}^{1} d\mu \frac{u_0^{2m(3-n_s)}}{\sqrt{1+2\mu^2}} \frac{\xi(y = \sqrt{m}u_0(\mu)/k\tau_2)}{\sigma^2} \right] \int_{-\infty}^{\infty} du \exp \left[ -m(1+2\mu^2) \frac{(u - u_0)^2}{\sigma^2} \right]
\]  

\( \text{(D6)} \)

\[
\mathcal{P}_m \sim \frac{m^{3m(3-n_s)/2}}{m!} e^{-m(3-n_s)/2} \sqrt{15\pi} \int_{-1}^{1} dx \frac{u_0^2(x) e^{m\varphi(x)}}{\sqrt{1+2x^2}}.
\]  

\( \text{(D7)} \)

where, in moving to the penultimate line from the one above it, we used Laplace’s method for \( m \gg 1 \).

Equation (D7) can be used to evaluate \( \mathcal{P}_m \) numerically with Mathematica up to \( m \leq 90 \) – after that, the power of \( u_0 \) and \( \xi \) in the integral becomes too large. Upon changing \( \mu \) to \( x \), Eq. (D7) may be written as

\[
\mathcal{P}_m \sim \frac{m^{3m(3-n_s)/2}}{m!} e^{-m(3-n_s)/2} \sqrt{15\pi} \int_{-1}^{1} dx \frac{u_0^2(x) e^{m\varphi(x)}}{\sqrt{1+2x^2}}.
\]  

\( \text{(D8)} \)

where

\[
\varphi(x) \equiv (3-n_s) \ln u_0(x) + \ln \left[ \frac{\xi(y = \sqrt{m}u_0(x)/k\tau_2, x)}{\sigma^2} \right].
\]  

\( \text{(D9)} \)

Laplace’s method works only if \( \varphi \) has variations of order one about it maximum \( x = 0 \). However, here, \( \varphi \) is almost constant, for the parameters in table 1 and, e.g. \( k \gg 10 \). For \( 80 \leq m \leq 1000, m\varphi \) is still almost constant for \( x \in [-1, 1] \) (i.e. \( m|\varphi''(0)|x^2 \ll 1 \)), and therefore a Laplace-type

Figure C1. The cold dark matter power spectrum in the Zel’dovich approximation at \( z = 0 \) (solid purple line), compared with the analytical prediction of the asymptotics (35), up to high orders. On the scales shown here, the \( k^3 \) asymptotics (dashed purple line) underestimate the dimension-less power spectrum by approximately an order of magnitude and the \( \mathcal{O}(k^{-5}) \) term is completely negligible. The Dicus transfer function and the parameters in table 1 are used.
approximation would be inaccurate. Instead, we simply approximate \( \varphi(x) \approx \varphi(0) + \varphi''(0)x^2/2 \), and write

\[
\mathcal{P}_m \approx \frac{1 + m(3-n_s)}{2} e^{-m(3-n_s)/2} m! \sqrt{\frac{15\pi}{1 + 2x^2}} \int_{-1}^{1} dx \frac{u_0^2(x)}{\sqrt{1 + 2x^2}} \left( 1 + m \varphi''(0) \frac{x^2}{2} \right)
\]

(D10)

\[
\mathcal{P}_m \approx \frac{m^{1 + m(3-n_s)/2} e^{-m(3-n_s)/2} m! \sqrt{15\pi}}{m!} [15(3-n_s)] \left( \frac{2}{\sqrt{3}} + \frac{m \varphi''(0)}{2} \left( \frac{\text{arcsinh}(\sqrt{2})}{\sqrt{2}} - \frac{1}{\sqrt{3}} \right) \right)
\]

(D11)

For figure 4, we use (D7) for \( m < 85 \), and for \( m > 85 \) we use (D11). Finally, let us remark that for \( n_s < 1 \), Stirling’s approximation for \( m! \) implies that the series (27) diverges very slowly, for fixed \( k \), but is of course still asymptotic as \( k \to \infty \).

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