LEGENDRE TRANSFORMATION FOR
REGULARIZABLE LAGRANGIANS IN FIELD THEORY

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ABSTRACT. Hamilton equations based not only upon the Poincaré–Cartan equivalent of a first-order Lagrangian, but rather upon its Lepagean equivalent are investigated. Lagrangians which are singular within the Hamilton–De Donder theory, but regularizable in this generalized sense are studied. Legendre transformation for regularizable Lagrangians is proposed, and Hamilton equations, equivalent with the Euler–Lagrange equations, are found. It is shown that all Lagrangians affine or quadratic in the first derivatives of the field variables are regularizable. The Dirac field and the electromagnetic field are discussed in detail.

KEYWORDS. Lagrangian, Poincaré–Cartan form, Lepagean form, Hamilton–De Donder equations, Hamilton \( p_2 \)-equations, regularity, regularizable Lagrangian, Legendre transformation, Dirac field, electromagnetic field

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1. INTRODUCTION

If \( \lambda \) is a Lagrangian defined on \( J^1Y \) (the first jet prolongation of a fibred manifold \( \pi: Y \to X \)), and \( \theta_\lambda \) is its Poincaré–Cartan form then the Euler–Lagrange equations are equations for local sections \( \gamma: X \to Y \) of \( \pi \) as follows: \( J^1\gamma^*i_\xi d\theta_\lambda = 0 \), for every vertical vector field \( \xi \) on \( J^1Y \). A geometric setting for the Hamilton theory on fibred manifolds goes back to Goldschmidt and Sternberg, who in their famous paper [7] proposed to consider Hamilton equations as an extension of the Euler–Lagrange equations to local
sections $\delta : X \to J^1 Y$, namely, $\delta^* i_\xi d\theta_\lambda = 0$. To become equivalent with the Euler–Lagrange equations, the Lagrangian has to satisfy the regularity condition

$$\det\left( \frac{\partial^2 L}{\partial y_i^\sigma \partial y_k^\tau} \right) \neq 0.$$

Goldschmidt–Sternberg’s approach, now adopted as standard (cf. [2], [5], [6], [8], [15-17] and many others), however seems not to be quite satisfactory. This turns out namely if generalizations to higher order are considered, or if concrete physical fields are studied: unfortunately, almost all of them are degenerate in the sense of the regularity condition (1.1). Parallely, Dedecker in 1977, and Krupka in 1983 considered another, from the mathematical point of view a more natural, extension of the Euler–Lagrange equations, based not upon a Poincaré–Cartan form as above, but rather upon a general Lepagean equivalent of a Lagrangian [3], [11]. As noticed already by Dedecker in [3], this approach opens a possibility to study “regularizations” of singular Lagrangians.

This paper develops the idea to understand Hamilton equations in the above mentioned generalized sense. However, we differ from Dedecker and Krupka in some points. First of all, we consider Lepagean equivalents which are at most 2-contact, i.e. of the form $\theta_\lambda +$ some auxiliary 2-contact term. Such Hamilton equations, first considered in [14] and called there Hamilton $p_2$-equations can be viewed as a “first correction” to the standard Hamilton equations. In [14], we studied relations with the Euler–Lagrange equations, and obtained appropriate regularity conditions, generalizing (1.1). The aim of this paper is to propose Legendre transformation for Hamilton $p_2$-equations, and to apply the results to concrete physically interesting first order Lagrangians, namely, Lagrangians affine or quadratic in the first derivatives of the field variables. Comparing our approach with Dedecker [3], one can see that our concept of regularity is stronger, and Legendre transformation is understood in a completely different way.

Contrary to the standard approach, where all affine and many quadratic Lagrangians are singular, we show that all these Lagrangians are regularizable, admit Legendre transformation, and provide Hamilton equations which are equivalent with the Euler–Lagrange equations (i.e., do not contain constraints). We also show that under certain additional conditions Hamilton $p_2$-equations of a Lagrangian coincide with the usual Hamilton equations of an appropriate equivalent Lagrangian. We study in detail the case of the Dirac field and the electromagnetic field, and find the corresponding “corrected” momenta and Hamiltonian which could be alternatively used for (unconstrained) quantization.

Finally, we note that results and techniques presented in this paper can be generalized to higher-order variational problems [13], [18].

2. Preliminaries

Throughout the paper all manifolds and mappings are smooth, and summation convention is used. We consider a fibred manifold $\pi : Y \to X$, $\dim X = n$, $\dim Y = m + n$, and its first (respectively, second) jet prolongation $\pi_1 : J^1 Y \to X$ (resp. $\pi_2 : J^2 Y \to X$). Natural fibred projections $J^k Y \to J^l Y$, where $0 \leq l < k \leq 2$, are denoted by $\pi_{k,l}$. A fibred chart on $Y$ (respectively, associated chart on $J^1 Y$) is denoted by $(V, \psi)$, $\psi = (x^i, y^\sigma)$ (respectively, $(V_1, \psi_1)$, where $V_1 = \pi_{1,0}^{-1}(V)$ and $\psi_1 = (x^i, y_1^\sigma, y_1^\sigma)$). We use the following
notations:

\[(2.1) \quad \omega_0 = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n, \quad \omega_i = i_{\partial/\partial x^i} \omega_0, \quad \omega_{ij} = i_{\partial/\partial x^j} \omega_i, \quad \cdots, \]

and

\[(2.2) \quad \omega^\sigma = dy^\sigma - y_j^\sigma dx^j. \]

It is important to note that \((dx^i, \omega^\sigma, dy^\sigma_j)\) is a basis of 1-forms on \(J^1Y\). A mapping \(\gamma : X \to Y\) defined on an open subset \(U \subset X\) is called a section of the fibred manifold \(\pi\) if the composite mapping \(\pi \circ \gamma\) is the identity mapping of \(U\). Quite analogously, a section of the fibred manifold \(\pi_1\) is defined. Notice that a section of \(\pi_1\) need not be of the form of prolongation of a section of \(\pi\). Accordingly, a section \(\delta\) of the fibred manifold \(\pi_1\) is called holonomic if \(\delta = J^1\gamma\) for a section \(\gamma\) of \(\pi\).

Recall that every \(q\)-form \(\eta\) on \(J^1Y\) admits a unique (canonical) decomposition into a sum of \(q\)-forms on \(J^2Y\) as follows:

\[(2.3) \quad \pi_{2,1}^* \eta = h(\eta) + \sum_{k=1}^{q} p_k(\eta), \]

where \(\pi_{2,1}\) is the canonical projection \(J^2Y \to J^1Y\), \(h(\eta)\) is a horizontal form, called the horizontal part of \(\eta\), and \(p_k(\eta), 1 \leq k \leq q\), is a \(k\)-contact form, called the \(k\)-contact part of \(\eta\) (see e.g. [9]). For our purposes it is sufficient to recall that in fibred coordinates a horizontal form on \(J^1Y\) is expressed by means of wedge products of the differentials \(dx^i\) only, with the components dependent upon the coordinates \((x^i, y^\sigma, y^\sigma_j)\). Similarly, a 1-contact (respectively, 2-contact) form contains only wedge products of the differentials \(dx^i\) with one (respectively, two) of the contact forms \(2.2)\).

By a first-order Lagrangian we shall mean a horizontal \(n\)-form \(\lambda\) on \(J^1Y\). This means that in every fibred chart,

\[(2.4) \quad \lambda = L \omega_0 \]

where \(L = L(x^i, y^\sigma, y^\sigma_j)\). A form \(\rho\) is called a Lepagean equivalent of a Lagrangian \(\lambda\) if (up to a projection) \(h(\rho) = \lambda\), and \(p_1(d\rho)\) is a \(\pi_{2,0}\)-horizontal form [9]. All (first order) Lepagean equivalents of a Lagrangian of order one take the form

\[(2.5) \quad \rho = \theta_\lambda + \nu, \]

where \(\theta_\lambda\) is the Poincaré–Cartan equivalent of \(\lambda\), i.e.

\[(2.6) \quad \theta_\lambda = L \omega_0 + \frac{\partial L}{\partial y^\sigma_j} \omega^\sigma \wedge \omega_j, \]

and \(\nu\) is an arbitrary at least 2-contact \(n\)-form, i.e. such that \(h(\nu) = p_1(\nu) = 0\).

If \(\rho\) is a Lepagean equivalent of \(\lambda\) then the \((n + 1)\)-form \(E_\lambda = p_1(d\rho)\) is called the Euler–Lagrange form of the Lagrangian \(\lambda\). Two Lagrangians \(\lambda_1\) and \(\lambda_2\) are called
equiv\textit{alent} if $E_{\lambda_1} = E_{\lambda_2}$. Recall that Lagrangians $\lambda_1$ and $\lambda_2$ are equivalent on an open set $U \subset J^1Y$ if and only if there exists an $(n - 1)$-form $\varphi$ such that $\lambda_2 = \lambda_1 + h(d\varphi)$ \[9\].

Besides the Poincaré–Cartan equivalent (2.6), the family (2.5) of Lepagean equivalents of a Lagrangian contains another distinguished Lepagean equivalent uniquely determined by the Lagrangian, namely,

\[
(2.7) \quad \rho_{\lambda} = L\omega_0 + \sum_{k=1}^{n} \left(\frac{1}{k!}\right)^2 \frac{\partial^k L}{\partial y_{j_1} \cdots \partial y_{j_k}} \omega_{j_1} \wedge \cdots \wedge \omega_{j_k} \wedge \omega_{j_1 \cdots j_k} [10] \quad (\text{cf. also} \ [1]).
\]

It is called the \textit{Krupka equivalent} of $\lambda$, and has the following important property: $d\rho_{\lambda} = 0$ \textit{if and only if} $E_{\lambda} = 0$; the latter condition, however, means that $\lambda = h(d\varphi)$ (a so called trivial Lagrangian).

With the help of Lepagean equivalents of a Lagrangian one obtains an intrinsic formulation of the \textit{Euler–Lagrange} and \textit{Hamilton equations} as follows (cf. \[9\], \[11\]). A section $\gamma$ of the fibred manifold $\pi$ is an \textit{extremal} of $\lambda$ if and only if

\[
(2.8) \quad J^{1\gamma} (i_{\pi\xi} d\rho) = 0
\]

for every $\pi$-vertical vector field $\xi$ on $Y$. A section $\delta$ of the fibred manifold $\pi_{1}$ is called a \textit{Hamilton extremal} of $\rho$ if

\[
(2.9) \quad \delta (i_{\pi\xi} d\rho) = 0,
\]

for every $\pi_{1}$-vertical vector field $\xi$ on $J^{1}Y$. The equations (2.8) and (2.9) are called the \textit{Euler–Lagrange} and the \textit{Hamilton equations}, respectively.

Notice that while the Euler–Lagrange equations (2.8) are uniquely determined by the Lagrangian, Hamilton equations (2.9) depend upon the choice of $\nu$. Consequently, one has many different “Hamilton theories” associated to a given variational problem.

Clearly, if $\gamma$ is an extremal then $J^{1\gamma}$ is a Hamilton extremal; conversely, however, a Hamilton extremal need not be holonomic, and thus a jet prolongation of some extremal. This suggests a definition of regularity as follows: A Lepagean form is called \textit{regular} if every its Hamilton extremal is holonomic \[12\].

Hamilton equations (2.9) where $\rho = \theta_{\lambda}$ (respectively, $\rho$ is \textit{at most} 2-contact) are called \textit{Hamilton–De Donder equations} \[4\], \[7\] (respectively, \textit{Hamilton} $p_2$-\textit{equations} \[14\]).

3. \textbf{HAMILTON} $p_2$-\textbf{EQUATIONS AND LEGENDRE TRANSFORMATION FOR FIRST-ORDER LAGRANGIANS}

In the sequel we consider Lepagean forms (2.5) where $\nu$ is \textit{2-contact}. Moreover, we suppose $\nu = p_2(\beta)$, where $\beta$ is defined on $Y$ and such that $p_i(\beta) = 0$ for all $i \geq 3$. Hence, in fibred coordinates

\[
(3.1) \quad \rho = L\omega_0 + \frac{\partial L}{\partial y_j^\sigma} \omega^\sigma \wedge \omega_j + g_{ij}^{ij} \omega^\sigma \wedge \omega^\nu \wedge \omega_{ij},
\]

where the functions $g_{ij}^{ij}$ do not depend upon the $y^j$’s and satisfy the conditions

\[
(3.2) \quad g_{ij}^{ij} = -g_{ji}^{ji}, \quad g_{ij}^{ij} = -g_{ij}^{ji}, \quad g_{ij}^{ij} = g_{ij}^{ij}.
\]

Note that (3.2) mean that only

\[
\binom{m}{2} \cdot \binom{n}{2} = \frac{1}{4} mn (m - 1)(n - 1)
\]

of the $mn \times mn$ functions $g_{ij}^{ij}$ are independent.
Theorem 1 [14]. Let $\lambda$ be a first-order Lagrangian, $\rho$ its Lepagean equivalent as above. Assume that the matrix

$$
\left( \frac{\partial^2 L}{\partial y^\sigma_i \partial y^\nu_j} - 4g^{ij}_{\sigma\nu} \right)
$$

with rows (respectively, columns) labelled by the pair $(\sigma, i)$ (respectively, $(\nu, j)$), is regular. Then $\rho$ is regular. Moreover, every Hamilton extremal $\delta$ of $\rho$ is of the form $\delta = J^1\gamma$, where $\gamma$ is an extremal of $\lambda$.

The proof is obtained by a direct calculation from (2.9), and can be found in [14].

In view of the above theorem we have the following concept:

Definition 1. Let $W \subset J^1Y$ be an open set. A Lagrangian $\lambda$ is called regularizable on $W$ if it has a regular Lepagean equivalent $\rho$ (3.1) defined on $W$. If $W = J^1Y$ we say that $\lambda$ is globally regularizable. We say that $\lambda$ is locally regularizable if it is regularizable in a neighbourhood of every point in $J^1Y$. The corresponding Lepagean equivalent $\rho$ is then called a (local) regularization of $\lambda$.

Note that for regularizable Lagrangians, the problem of solving the Euler–Lagrange equations is equivalent to the problem of solving (appropriate) Hamilton equations.

An important class of regularizable Lagrangians is characterized by the following proposition.

Proposition 1. Let $m \geq 2$. Then every Lagrangian $L$ such that

$$
L = a + b^i_j y^\sigma_j + c^{jk}_{\sigma\nu} y^\sigma_j y^\nu_k,
$$

where $a$, $b^i_j$ and $c^{jk}_{\sigma\nu}$ are functions of $(x^i, y^\rho)$, is locally regularizable.

In particular, every first-order Lagrangian affine (respectively, quadratic) in the $y^\rho_p$’s is locally regularizable.

Proof. By assumption, $\partial^2 L/\partial y^\sigma_i \partial y^\nu_j$ are functions defined on an open subset of the total space $Y$. Let $\det(\partial^2 L/\partial y^\sigma_i \partial y^\nu_j) = 0$ at a point $x \in Y$. Since $m > 2$, one can find functions $g^{ij}_{\sigma\nu}$, antisymmetric in $(\sigma\nu)$ and $(ij)$, defined in a neighbourhood of $x$ and such that at $x$ the condition (3.3) is satisfied. However, since the determinant is a continuous function, the corresponding matrix must be nondegenerate in a neighbourhood $U$ of $x$. With these $g$’s (independent of the $y^\rho_p$’s, as desired), the form $\rho = \theta_\lambda + g^{ij}_{\sigma\nu} \omega^\sigma \wedge \omega^\nu \wedge \omega_{ij}$ is a regularization of $\lambda$ on $W = \pi_{1,0}^{-1}(U)$. □

The following theorem provides a generalization of Legendre transformation to singular in the standard sense, but regularizable Lagrangians.

Theorem 2. Consider a Lepagean form $\rho$ given in a fibred chart $(V, \psi)$, $\psi = (x^i, y^\sigma)$ by (3.1) (3.2). Put

$$
p^i_\sigma = \frac{\partial L}{\partial y^\sigma_i} - 4g^{ij}_{\sigma\nu} y^\nu_j.
$$
Let \( x \in V_1 \subset J^1 Y \) be a point. If the matrix (3.3) is regular in a neighbourhood \( W \subset V_1 \) of \( x \), then \((x^i, y^\sigma, y^\nu_j) \rightarrow (x^i, y^\sigma, p^i_\sigma)\) is a coordinate transformation on \( W \).

**Proof.** The above theorem follows immediately from the fact that
\[
\frac{\partial p^i_\sigma}{\partial y^\nu_j} = \frac{\partial^2 L}{\partial y^\sigma_i \partial y^\nu_j} - 4g^{ij}_{\sigma \nu}.
\]

\[\square\]

Denote
\[(3.6) H = -L + \frac{\partial L}{\partial y^\sigma_i} y^\sigma_i - 2g^{ij}_{\sigma \nu} y^\sigma_i y^\nu_j = -L + p^i_\sigma y^\sigma_i + 2g^{ij}_{\sigma \nu} y^\sigma_i y^\nu_j.\]

Now the Lepagean form (3.1), (3.2) reads
\[(3.7) \rho = L \omega_0 + \frac{\partial L}{\partial y^\sigma_i} \omega^\sigma_0 \wedge y^\sigma_i + g^{ij}_{\sigma \nu} \omega^\sigma_i \wedge \omega^\nu_j + \omega_{ij} = \left( L - \frac{\partial L}{\partial y^\sigma_i} y^\sigma_i + 2g^{ij}_{\sigma \nu} y^\sigma_i y^\nu_j \right) \omega_0
\]
\[\quad + \left( \frac{\partial L}{\partial y^\sigma_i} - 4g^{ij}_{\sigma \nu} y^\nu_j \right) dy^\sigma \wedge \omega_i + g^{ij}_{\sigma \nu} dy^\sigma \wedge dy^\nu \wedge \omega_{ij}
\]
\[\quad = -H \omega_0 + p^i_\sigma dy^\sigma \wedge \omega_i + g^{ij}_{\sigma \nu} dy^\sigma \wedge dy^\nu \wedge \omega_{ij}.\]

In analogy with the standard terminology we shall call \( H \) the Hamiltonian and \( p^i_\sigma \) momenta of the Lepagean form \( \rho \) (3.1), (3.2), and the corresponding coordinate transformation *Legendre transformation*; accordingly, the coordinates \((x^i, y^\sigma, p^i_\sigma)\) will be referred to as Legendre coordinates of \( \rho \).

**Corollary 1.** Let \((x^i, y^\sigma, p^i_\sigma)\) be the Legendre transformation associated with a Lepagean form \( \rho \) (3.1), (3.2). Then the matrix
\[(3.8) \begin{pmatrix} \frac{\partial^2 H}{\partial p^i_\sigma \partial p^j_\sigma} \end{pmatrix} \]
is regular and inverse to the matrix (3.3).

**Proof.** Explicit computations lead to
\[
\frac{\partial L}{\partial y^\sigma_i} = p^i_\sigma + 4g^{jk}_{\sigma \kappa} y^\kappa_j, \quad \frac{\partial L}{\partial y^\nu_j} = \frac{\partial L}{\partial y^\sigma_i} \frac{\partial y^\sigma_i}{\partial y^\nu_j} = \left( p^i_\sigma + 4g^{jk}_{\sigma \kappa} y^\kappa_j \right) \frac{\partial y^\nu_j}{\partial p^i_\sigma},
\]
\[
\frac{\partial H}{\partial p^i_\sigma} = -\frac{\partial L}{\partial p^i_\sigma} + y^\sigma_i + p^i_\sigma \frac{\partial y^\nu_j}{\partial p^i_\sigma} + 4g^{ik}_{\sigma \kappa} y^\kappa_i \frac{\partial y^\nu_j}{\partial p^i_\sigma} = y^\sigma_i, \quad \frac{\partial^2 H}{\partial p^i_\sigma \partial p^j_\sigma} = \frac{\partial y^\nu_j}{\partial p^i_\sigma},
\]
proving the assertion. \[\square\]

Expressing Hamilton \( p_2 \)-equations (2.9) in Legendre coordinates we get
\[
\begin{align*}
\frac{\partial H}{\partial y^\sigma} &= -\frac{\partial p^i_\sigma}{\partial x^i} + 4 \frac{\partial g^{ij}_{\sigma \kappa}}{\partial x^j} \frac{\partial y^\nu_j}{\partial x^i} + 2 \left( \frac{\partial g^{ij}_{\sigma \kappa}}{\partial y^\sigma} + \frac{\partial g^{ij}_{\sigma \kappa}}{\partial y^\nu} + \frac{\partial g^{ij}_{\sigma \kappa}}{\partial y^\kappa} \right) \frac{\partial y^\kappa}{\partial x^i} \frac{\partial y^\nu}{\partial x^j},
\frac{\partial H}{\partial p^i_\sigma} &= \frac{\partial y^\sigma}{\partial x^i},
\end{align*}
\]

or, equivalently,

\[
\frac{\partial H}{\partial y^\sigma} = - \frac{\partial p^i_\sigma}{\partial x^i} + 4 \frac{\partial g^{ij}_\sigma}{\partial x^j} \frac{\partial H}{\partial p^i_\nu} + 2 \left( \frac{\partial g^{ij}_k}{\partial y^\sigma} + \frac{\partial g^{ij}_\nu}{\partial y^\nu} + \frac{\partial g^{ij}_\sigma}{\partial y^\nu} \right) \frac{\partial H}{\partial p^i_k} \frac{\partial H}{\partial p^i_\nu},
\]

\[
\frac{\partial H}{\partial p^i_\sigma} = \frac{\partial y^\sigma}{\partial x^i}.
\]

**Corollary 2.** If the \(n\)-form

\[
\eta = g^{ij}_\sigma dy^\sigma \wedge dy^\nu \wedge \omega_{ij}
\]

is closed then the above Hamilton \(p_2\)-equations take the form

\[
\frac{\partial H}{\partial y^\sigma} = - \frac{\partial p^i_\sigma}{\partial x^i}, \quad \frac{\partial H}{\partial p^i_\sigma} = \frac{\partial y^\sigma}{\partial x^i}.
\]

**Remark 1.** Compared with Dedecker [3], we differ in both the definition of regularity and Legendre transformation. Dedecker’s regularity is weaker—Hamilton equations regular in his sense need not be equivalent with the Euler–Lagrange equations. Also, Legendre transformation is completely different: while Dedecker’s Legendre transformation is a map to a certain new space of higher dimension than that of the dynamical space, with unclear relations to regularity and to Lagrangian dynamics, Legendre transformation proposed above has similar properties as the Legendre transformation in classical mechanics (if the Lagrangian is regular in our sense, Legendre transformation becomes a local diffeomorphism of the space where the dynamics proceeds, providing a “canonical” form of the motion equations). Some further geometric properties of this Legendre transformation are clarified in [13].

### 4. Satellite Lagrangians

We shall investigate the meaning of the 2-contact term in the Lepagean equivalent \(\rho\) (3.1) of a Lagrangian. Keeping notations introduced so far, we start with the following interesting assertions:

**Lemma 1.** It holds (up to the projection \(\pi_{1,0}\))

\[
\eta = \rho_{h(\eta)}.
\]

**Proof.** Denote \(h(\eta) = l \omega_0\); then obviously,

\[
l = 2g^{ij}_\sigma y^\sigma_i y^\nu_j,
\]

and we obtain using (2.7) and (2.2)

\[
\rho_{h(\eta)} = l \omega_0 + \frac{\partial l}{\partial y^\sigma_i} \omega^\sigma \wedge \omega_i + \frac{1}{4} \frac{\partial^2 l}{\partial y^\sigma_i \partial y^\nu_j} \omega^\sigma \wedge \omega^\nu \wedge \omega_{ij}
\]

\[
= 2g^{ij}_\sigma y^\sigma_i y^\nu_j \omega_0 + 4g^{ij}_\sigma y^\sigma_j \omega^\sigma \wedge \omega_i + g^{ij}_\sigma \omega^\sigma \wedge \omega^\nu \wedge \omega_{ij}
\]

\[
= g^{ij}_\sigma dy^\sigma \wedge dy^\nu \wedge \omega_{ij} = \eta.
\]

\(\square\)
Lemma 2. Put
\[ \bar{\lambda} = \lambda - h(\eta) = h(\rho - \eta), \quad \text{i.e.} \quad \bar{L} = L - l. \]
Then
\[ \rho = \theta_{\bar{\lambda}} + \rho_{h(\eta)}, \quad \text{i.e.} \quad \theta_{\bar{\lambda}} = \rho - \eta. \]

Proof. Indeed, by (4.3) and (3.1), \( \rho_{h(\eta)} = \theta_{h(\eta)} + p_2(\rho) \), hence, by (4.4),
\[ \rho = \theta_{\lambda} + p_2(\rho) = \theta_{\lambda} - \theta_{h(\eta)} + \rho_{h(\eta)} = \theta_{\lambda} + \rho_{h(\eta)}. \]
\[ \square \]

Note that by (4.5) and (3.7),
\[ \theta_{\lambda} = -H\omega_0 + p^i_\sigma dy^\sigma \wedge \omega_i. \]

Let us denote by \( \tilde{p}^i_\sigma(L) \) and \( \tilde{H}(L) \) the De Donder momenta and Hamiltonian associated with a Lagrangian \( L \). Recall that
\[ \tilde{p}^i_\sigma(L) = \frac{\partial L}{\partial y^i_\sigma}, \quad \tilde{H}(L) = -L + \tilde{p}^i_\sigma(L) y^\sigma_i \]
[4], [7]. Note that for our momenta and Hamiltonian (3.5) and (3.6) we obtain
\[ p^i_\sigma = \tilde{p}^i_\sigma(L) - \tilde{p}^i_\sigma(l), \quad H = \tilde{H}(L) - \tilde{H}(l) = \tilde{H}(L) - l. \]
Consequently, we have the following lemma.

Lemma 3. It holds
\[ H = \tilde{H}(\bar{L}), \quad p^i_\sigma = \tilde{p}^i_\sigma(\bar{L}). \]
Moreover, the regularity condition (3.3) is equivalent with the “standard” regularity condition (1.1) for \( \bar{L} \), i.e., with
\[ \det \left( \frac{\partial^2 \bar{L}}{\partial y^i_\sigma \partial y^k_\eta} \right) \neq 0. \]

In view of the above results we shall call the Lagrangian \( h(\eta) \) a satellite of \( \lambda \), and the Lagrangian \( \bar{\lambda} \) a dedonderization of \( \lambda \).

Now, from Lemma 1 and 2 we immediately obtain the following important result:

Proposition 2. If the form \( \eta \) is closed then the Lagrangian \( \bar{\lambda} \) is equivalent with \( \lambda \), and
\[ d\rho = d\theta_{\bar{\lambda}}. \]

In Section 3 we introduced regularization as a procedure to find for a Lagrangian appropriate Hamilton \( p_2 \)-equations (i.e., a certain “correction” to the Hamilton DeDonder equations of \( L \)) which are equivalent with the Euler–Lagrange equations, hence represent a suitable (unconstrained) alternative for solving the extremal problem. Now, taking into account all the above properties of satellite Lagrangians we conclude that regularization can be understood also in a different way as a procedure to find to a given Lagrangian an appropriate satellite in such a way that the Hamilton–De Donder equations of the new Lagrangian would be equivalent with the Euler–Lagrange equations:

Theorem 3. Let \( \lambda \) be a regularizable Lagrangian. Then for every its (local) regularization \( \rho \) such that \( d\eta = 0 \), the Lagrangian \( \bar{\lambda} = \lambda - h(\eta) \) is equivalent with \( \lambda \), satisfies the “standard” regularity condition (4.10), and the Hamilton \( p_2 \)-equations of \( \lambda \) based upon \( \rho \) coincide with the Hamilton–De Donder equations of the Lagrangian \( \bar{\lambda} \).
5. Examples of Legendre Transformations for First-Order Lagrangians

The above results can be directly applied to concrete Lagrangians. Let us consider two important cases: Lagrangians affine in the first derivatives of the field variables (in particular, the Dirac field), and the electromagnetic field.

5.1. Affine Lagrangians. Recall that by Proposition 1, if the fibre dimension $m$ is at least 2, all Lagrangians affine in the first derivatives are locally regularizable and admit Legendre transformation introduced in Section 3. Assume

$$L = L_0 + L^i_\sigma y^\sigma_i$$

where $L_0$ and $L^i_\sigma$ ($1 \leq i \leq n$, $1 \leq \sigma \leq m$) are functions of $(x^i, y^\nu)$. Note that the De Donder momenta and Hamiltonian of (5.1) take the form $\tilde{p}^i_\sigma = L^i_\sigma$, $\tilde{H} = -L_0$, i.e., they are defined on an open subset of the total space $Y$, and the corresponding Hamilton equations must be treated as constrained, within the range of the Dirac theory of constrained systems (cf. eg. [6]). On the other hand, we get by (3.5) and (3.6),

$$p^i_\sigma = L^i_\sigma - 4g^{ij}_{\sigma\nu}y^\nu_j, \quad H = -L_0 - 2g^{ij}_{\sigma\nu}y^\sigma_i y^\nu_j,$$

where $(g^{ij}_{\sigma\nu})$ is a regular $(mn \times mn)$-matrix. We can see that the domain of definition of the functions (5.2) is a polynomial of degree 2 in momenta. By Theorem 3, we should choose the $g^{ij}_{\sigma\nu}$ in such a way that the form $\eta$ (3.11) be closed. Then the corresponding satellite Lagrangian is trivial, and Hamilton equations take the form (3.12).

Let us discuss in more detail the case $m = 2$, and $n = 2$ (respectively, $n = 4$).

(i) $n = \text{dim} X = 2$. The conditions (3.2) on the $g^{ij}_{\sigma\nu}$’s mean that only one of these functions is independent, say, $g^{12}_{12}$. Denote $u = 4g^{12}_{12}$, and assume $u \neq 0$. The condition $d\eta = 0$, i.e., $du \wedge dy^1 \wedge dy^2 = 0$ means that $u = u(y^1, y^2)$.

As above, we consider Lepagean equivalents of the Lagrangian (5.1) in the form

$$\rho = \left( L_0 + L^i_\sigma y^\sigma_i \right) dx^1 \land dx^2 + L^i_\sigma \omega^\sigma \land \omega_i + u \omega^1 \land \omega^2$$

(5.3)

(where summation runs through $1 \leq i, \sigma \leq 2$). The regularity condition (3.3) reads

$$\det \begin{pmatrix} 0 & 0 & 0 & -u \\ 0 & 0 & u & 0 \\ -u & u & 0 & 0 \end{pmatrix} \neq 0,$$

(5.4)

and is clearly satisfied. Momenta become

$$p^1_1 = L^1_1 - uy^2_2, \quad p^1_2 = L^1_2 + uy^1_2, \quad p^2_1 = L^2_1 + uy^2_1, \quad p^2_2 = L^2_2 - uy^1_1.$$

Since the inverse to the Legendre transformation takes the form

$$y^1_1 = \frac{1}{u} \left( L^2_2 - p^2_2 \right), \quad y^1_2 = \frac{1}{u} \left( L^1_1 - p^1_1 \right), \quad y^2_1 = -\frac{1}{u} \left( L^2_1 - p^2_1 \right), \quad y^2_2 = -\frac{1}{u} \left( L^1_2 - p^1_2 \right),$$

(5.5)
The matrix (3.3) takes the form
\[(5.11) \quad \bar{\epsilon} \quad \text{where} \quad u \quad \text{However, for every nonzero function} \quad \text{and it is apparently degenerate in the sense of the standard regularity condition (1.1).} \]

Hamilton \(p_2\)-equations in the Legendre coordinates take the form
\[(5.7) \quad \frac{\partial H}{\partial y^1} = -\frac{\partial p_1}{\partial x^1}, \quad \frac{\partial H}{\partial y^2} = -\frac{\partial p_1}{\partial x^2}, \quad \frac{\partial H}{\partial p_1} = \frac{\partial y_1}{\partial x^1}, \quad \frac{\partial H}{\partial p_2} = \frac{\partial y_1}{\partial x^2}, \quad \frac{\partial H}{\partial y^2} = \frac{\partial y_2}{\partial x^1}, \quad \frac{\partial H}{\partial p_2} = \frac{\partial y_2}{\partial x^2}. \]

Note that for every fixed \( u \neq 0 \) we have obtained Hamilton equations equivalent with the Euler-Lagrange equations.

As an illustration, let us consider the Dirac field. In this case we have \( X = R^2, Y = R^2 \times R^2 \), i.e. \( J^1 Y = R^2 \times R^2 \times R^4 \), with the global coordinates denoted by \((x^\mu, \psi, \bar{\psi}, \partial_\mu \psi, \partial_\mu \bar{\psi})\), \( \mu = 1, 2 \), Lagrangian \( L \) takes the form
\[(5.9) \quad L = \frac{1}{2} (\bar{\psi} \gamma^\mu \partial_\mu \psi + \partial_\mu \bar{\psi} \gamma^\mu \bar{\psi}) - \bar{\psi} m \psi, \]
and it is apparently degenerate in the sense of the standard regularity condition (1.1). However, for every nonzero function \( u(\psi, \bar{\psi}) \), the form \( \rho (5.3) \) is a global regularization of \( L \). Computing the corresponding satellite Lagrangian for (5.9) we get
\[(5.10) \quad l = -u \epsilon^{\mu \nu} \partial_\mu \bar{\psi} \partial_\nu \psi, \]
where \( \epsilon^{\mu \nu} \) is the Levi-Civita symbol. Now the Lagrangian
\[(5.11) \quad \bar{L} = \frac{1}{2} (\bar{\psi} \gamma^\mu \partial_\mu \psi + \partial_\mu \bar{\psi} \gamma^\mu \bar{\psi}) - \bar{\psi} m \psi + u \epsilon^{\mu \nu} \partial_\mu \bar{\psi} \partial_\nu \psi \]
is a dedonderization of the Dirac field Lagrangian, which is regular in the standard sense. Accordingly, Hamilton–De Donder equations of (5.11) are equivalent with the Dirac field equations, and in this sense, the (standard) De Donder Hamiltonian and momenta of (5.11) (which, however, are precisely the functions \( H \) and \( p \')s given by (5.7) and (5.5)), should represent possibly a better physical alternative for (unconstrained) quantization of the Dirac field. Explicitly, the “new” Hamiltonian reads by (4.8)
\[(5.12) \quad H = u \epsilon^{\mu \nu} \partial_\mu \bar{\psi} \partial_\nu \psi + \bar{\psi} m \psi. \]

Note that in the above formulas the most simple admissible choice is \( u \neq 0 \) a constant function.

(ii) \( n = \operatorname{dim} X = 4 \). In this case we get 6 independent functions \( g_{ij}^{ij} \). Denote
\[(5.13) \quad u_1 = 4 g_{12}^{12}, \quad u_2 = 4 g_{12}^{13}, \quad u_3 = 4 g_{12}^{14}, \quad u_4 = 4 g_{12}^{23}, \quad u_5 = 4 g_{12}^{24}, \quad u_6 = 4 g_{12}^{34}. \]
The matrix (3.3) takes the form
\[(5.14) \quad \begin{pmatrix} 0 & -M \\ M & 0 \end{pmatrix}, \quad \text{where} \quad M = \begin{pmatrix} 0 & u_1 & u_2 & u_3 \\ -u_1 & 0 & u_4 & u_5 \\ -u_2 & -u_4 & 0 & u_6 \\ -u_3 & -u_5 & -u_6 & 0 \end{pmatrix}. \]
We can see that for any choice of functions $u_k(x^i, y^\sigma)$, $1 \leq k \leq 6$, such that $\det M \neq 0$ we obtain a regular Hamilton $p_2$-theory, based upon the Lepagean form

$$
\rho = \left( L_0 + L^\alpha y_\alpha^\sigma \right) dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 + L^\alpha_\alpha \omega^\sigma \wedge \omega_i
$$

(5.15)

$$
+ u_1 \omega^1 \wedge \omega_1 + u_2 \omega^1 \wedge \omega_{12} + u_2 \omega^1 \wedge \omega_{13} + u_3 \omega^1 \wedge \omega_{1} + u_4 \omega^1 \wedge \omega_{14} + u_5 \omega^1 \wedge \omega_{23} + u_6 \omega^1 \wedge \omega_{24} + u_6 \omega^1 \wedge \omega_{2} \wedge \omega_{34}.
$$

Expressing the corresponding satellite Lagrangian for the Dirac field explicitly, we easily obtain

$$
l = -\sum_{\mu, \nu} u_{(\mu, \nu)} \epsilon^\mu_\nu \partial_\mu \bar{\psi} \partial_\nu \psi
$$

(5.16)

where $u_{(\mu, \nu)} = u_{(\nu, \mu)}$ and the notation $u_1 = u_{(1,2)}$, $u_2 = u_{(1,3)}$, $u_3 = u_{(1,4)}$, $u_4 = u_{(2,3)}$, $u_5 = u_{(2,4)}$, $u_6 = u_{(3,4)}$ is used. “Corrected” momenta can now be obtained by a short routine calculation from (3.5), and the Hamiltonian takes by (4.8) the form $H = l + \bar{\psi}m\psi$.

In comparison with the usual formulas they differ by additional terms—(De Donder) momenta and Hamiltonian of the satellite (5.16) of the Lagrangian (5.9).

Note that on the fibred manifold $R^4 \times R^2 \to R^4$ the most simple global regularization of the Dirac Lagrangian is obtained for $u_i$, $1 \leq i \leq 6$, constant functions; by (5.13) some of them may even equal zero (a simple choice is e.g. $u_3, u_4 \neq 0, u_1 = u_2 = u_5 = u_6 = 0$).

5.2. The electromagnetic field. For the electromagnetic field Lagrangian

$$
L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} (y_\nu^\sigma y_\sigma^\nu - g^{\sigma\nu} g_{\mu\rho} y_\sigma^\rho y_\nu^\mu)
$$

(5.17)

(where $F_{\mu\nu} = A_{\mu, \nu} - A_{\nu, \mu}$, $(g_{\sigma\nu})$ denotes the Lorentz metric, $g_{\sigma\nu} = 0$ for $\sigma \neq \nu$, $-g_{11} = g_{22} = g_{33} = g_{44} = 1$, and $y_\sigma = g^{\sigma\nu} A_\nu$), the standard regularity condition (1.1) gives that $L$ is degenerate. For example, in the $n = 2$ case, the De Donder momenta and Hamiltonian read

$$
\vec{p}_1^1 = \vec{p}_2^2 = 0, \quad \vec{p}_1^2 = \vec{p}_2^2 = y_1^2 + y_2^2, \quad \vec{H} = -\frac{1}{2} (y_1^2 + y_2^2)^2 + \vec{p}_2 y_1 + \vec{p}_1 y_2.
$$

(5.18)

Hence, momenta are not independent, and the corresponding Hamilton equations must be treated as constrained. However, as we proved in Sec. 3, Lagrangian (5.17) is regularizable (and admits many global regularizations). Let us choose one of them and compute the corresponding Hamiltonian and momenta. Put

$$
g_{\sigma\nu}^{\alpha\beta} = \frac{\partial^2 L}{\partial y_\alpha^\sigma \partial y_\beta^\nu} - \frac{\partial^2 L}{\partial y_\beta^\sigma \partial y_\alpha^\nu}
$$

(5.19)

(apparently, these $g$’s do not depend upon the $y_i^\sigma$’s, and satisfy (3.2), as desired). The Lepagean equivalent (3.1) now reads,

$$
\rho = L \omega_0 + \frac{\partial L}{\partial y_\alpha^\sigma} \omega^\sigma \wedge \omega_\alpha + \left( \frac{\partial^2 L}{\partial y_\alpha^\sigma \partial y_\beta^\nu} - \frac{\partial^2 L}{\partial y_\beta^\sigma \partial y_\alpha^\nu} \right) \omega^\sigma \wedge \omega^\nu \wedge \omega_{\alpha\beta},
$$

(5.20)
and the regularity condition (3.3) leads to checking regularity of the following matrix

\[
\begin{pmatrix}
4 \frac{\partial^2 L}{\partial y_\alpha^2 \partial y_\beta^2} - 3 \frac{\partial^2 L}{\partial y_\alpha^2 \partial y_\beta^2}
\end{pmatrix}.
\]

(i) Let \( X = R^2 \). Then we have \( Y = R^2 \times R^2 \), i.e. \( m = 2 \), and the Lagrangian (5.17) reads

\[
L = \frac{1}{2} (y_1^2 + y_1^2).
\]

For the matrix (5.21) we obtain

\[
\begin{pmatrix}
0 & 0 & 0 & 4 \\
0 & 1 & -3 & 0 \\
0 & -3 & 1 & 0 \\
4 & 0 & 0 & 0
\end{pmatrix},
\]

i.e., it is regular. Consequently, the related Hamilton \( p_2 \)-equations are equivalent with the Maxwell equations. For the momenta we easily obtain

\[
p_1^1 = 4y_2^1, \quad p_1^2 = -3y_2^1 + y_1^2, \quad p_2^2 = 4y_1^1, \quad p_1^1 = y_1^2 - 3y_2^1.
\]

The inverse transformation to the Legendre transformation exists and takes the form

\[
y_1^1 = \frac{1}{4}p_2^2, \quad y_2^1 = -\frac{3}{8}p_2^1 - \frac{1}{8}p_1^1, \quad y_2^2 = \frac{1}{4}p_1^1, \quad y_1^2 = -\frac{1}{8}p_2^1 - \frac{3}{8}p_1^1.
\]

The Hamiltonian in the Legendre coordinates reads

\[
H = \frac{1}{4} p_1^1 p_2^2 - \frac{3}{8} p_2^1 p_1^1 - \frac{1}{16} (p_2^1)^2 - \frac{1}{16} (p_1^1)^2.
\]

(ii) Let \( X = R^4 \). We have \( m = 4 \), and the Lagrangian (5.17) takes the form

\[
L = \frac{1}{2} (y_2^1 + y_1^1)^2 + \frac{1}{2} (y_3^1 + y_1^3)^2 + \frac{1}{2} (y_4^1 + y_4^1)^2 - \frac{1}{2} (y_3^1 - y_2^1)^2. \tag{5.24}
\]

The matrix (5.21) becomes

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 4 \\
0 & 1 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -3 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -3 & 0 \\
0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & -3 & 0 & 0 & -1 & 0 \\
4 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]
and one can easily check that it is regular. For the momenta we get

\[
\begin{align*}
p_1^1 &= 4y_2^1 + 4y_3^1 + 4y_4^1, \\
p_1^2 &= y_2^1 - 3y_1^1, \\
p_1^3 &= y_3^1 - 3y_1^1, \\
p_1^4 &= y_4^1 - 3y_1^1, \\
p_2^1 &= 4y_1^1 + 4y_3^1 + 4y_4^1, \\
p_2^2 &= y_1^1 - 3y_2^1, \\
p_2^3 &= -y_3^1 - 3y_2^1, \\
p_2^4 &= -y_4^1 - 3y_2^1, \\
p_3^1 &= 4y_1^1 + 4y_2^1 + 4y_4^1, \\
p_3^2 &= y_1^3 - 3y_3^1, \\
p_3^3 &= -y_2^3 - 3y_3^1, \\
p_3^4 &= -y_4^3 - 3y_3^1, \\
p_4^1 &= 4y_1^1 + 4y_2^1 + 4y_3^1, \\
p_4^2 &= y_1^3 - 3y_4^1, \\
p_4^3 &= -y_2^3 - 3y_4^1, \\
p_4^4 &= -y_3^3 - 3y_4^1,
\end{align*}
\]

(5.25)

and the Hamiltonian in the Legendre coordinates takes the form

\[
H = -\frac{1}{16} \left( (p_1^1)^2 + (p_2^1)^2 + (p_3^1)^2 + (p_4^1)^2 - p_1^1 p_2^1 - p_1^1 p_3^1 - p_1^1 p_4^1 - p_2^1 p_3^1 - p_2^1 p_4^1 - p_3^1 p_4^1 \right)
- \frac{1}{16} \left( (p_1^2)^2 + (p_2^2)^2 + (p_3^2)^2 + (p_4^2)^2 + (p_4^1)^2 \right)
- \left( (p_1^3)^2 - (p_2^3)^2 - (p_3^3)^2 - (p_2^4)^2 - (p_3^4)^2 \right)
- \frac{3}{8} \left( p_1^2 p_1^2 + p_3^1 p_3^1 + p_4^1 p_4^1 + p_3^2 p_2^2 + p_2^3 p_4^2 + p_3^3 p_3^3 \right).
\]

(5.26)

Let us compute the corresponding satellite Lagrangian for (5.17). Since

\[
g_{\alpha \beta} = \delta_\alpha^\gamma \delta_\beta^\delta - \delta_\gamma^\beta \delta_\delta^\alpha,
\]

we obtain

\[
l = 2(A_\mu^\alpha A_\alpha^\mu - A_\mu^\alpha A_\alpha^\nu) = 2(\text{Tr}(A^2) - (\text{Tr} A')^2),
\]

(5.28)

where \(A'\) denotes the matrix \((\partial_\alpha A^\beta)\). Now, the dedonderization Lagrangian and the “new” Hamiltonian for the electromagnetic field read

\[
\tilde{L} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + 2(A_\mu^\alpha A_\alpha^\nu - A_\mu^\alpha A_\nu^\mu), \\
H = \tilde{H} + 2(A_\mu^\alpha A_\alpha^\nu - A_\nu^\mu A_\mu^\alpha).
\]

(5.29)

Remark 2. Note that the Lepagean equivalent \(\rho_\lambda (2.7)\) is not an appropriate regularization of the electromagnetic field. Indeed, in this case the matrix (3.3) takes the form

\[
\left( \frac{\partial^2 L}{\partial y_\alpha^\mu \partial y_\alpha^\nu} + \frac{\partial^2 L}{\partial y_\mu^\nu \partial y_\beta} \right),
\]

and it is singular for the Lagrangian (5.17).

References

1. D. E. Betounes, Extension of the classical Cartan form, Phys. Rev. D 29 (1984), 599–606.
2. F. Cantrijn, J. A. Ibort and M. de León, Hamiltonian structures on multisymplectic manifolds, Rend. Sem. Mat. Univ. Pol. Torino 54 (1996) 225–236.
3. P. Dedecker, On the generalization of symplectic geometry to multiple integrals in the calculus of variations, in: Lecture Notes in Math. 570 (Springer, Berlin, 1977) 395–456.
4. Th. De Donder, Théorie Invariantive du Calcul des Variations, Gauthier–Villars, Paris, 1930.
5. G. Giachetta, L. Mangiarotti and G. Sardanashvily, New Lagrangian and Hamiltonian Methods in Field Theory, World Scientific, Singapore, 1997.
6. G. Giachetta, L. Mangiarotti and G. Sardanashvily, Covariant Hamilton equations for field theory, J. Phys. A: Math. Gen. 32 (1999) 6629–6642.
7. H. Goldschmidt and S. Sternberg, The Hamilton–Cartan formalism in the calculus of variations, Ann. Inst. Fourier, Grenoble 23 (1973), 203–267.
8. M. J. Gotay, A multisymplectic framework for classical field theory and the calculus of variations, I. Covariant Hamiltonian formalism, in: Mechanics, Analysis and Geometry: 200 Years After Lagrange, M. Francaviglia and D. D. Holm, eds. (North Holland, Amsterdam, 1990) 203–235.
9. D. Krupka, A geometric theory of ordinary first order variational problems in fibred manifolds. I. Critical sections, II. Invariance, J. Math. Anal. Appl. 49 (1975), 180–206; 469–476.
10. D. Krupka, A map associated to the Lepagean forms of the calculus of variations in fibred manifolds, Czechoslovak Math. J. 27 (1977), 114–118.
11. D. Krupka, On the higher order Hamilton theory in fibred spaces, in: Geometrical Methods in Physics, Proc. Conf. Diff. Geom. Appl., Nové Město na Moravě, 1983, D. Krupka, ed. (J.E. Purkyně University, Brno, Czechoslovakia, 1984) 167–183.
12. D. Krupka and O. Štěpánková, On the Hamilton form in second order calculus of variations, in: Geometry and Physics, Proc. Int. Meeting, Florence, Italy, 1982, M. Modugno, ed. (Pitagora Ed., Bologna, 1983) 85–101.
13. O. Krupková, Hamiltonian field theory revisited: A geometric approach to regularity, Preprint GA 11/2000 (Silesian University, Opava, 2000) 19pp; submitted to Proc. Colloq. Diff. Geom., Debrecen, July 2000.
14. O. Krupková and D. Smetanová, On regularization of variational problems in first-order field theory, Rend. Circ. Mat. Palermo Suppl., to appear.
15. M. de León and P.R. Rodrigues, Generalized Classical Mechanics and Field Theory, North-Holland, Amsterdam, 1985.
16. J.E. Marsden and S. Shkoller, Multisymplectic geometry, covariant Hamiltonians, and water waves, Math. Proc. Camb. Phil. Soc. 125 (1999) 553–575.
17. D.J. Saunders, The regularity of variational problems, Contemporary Math., 132 (1992) 573–593.
18. D. Smetanová, On Hamilton $p_2$-equations in second order field theory, Proc. Colloq. Diff. Geom., Debrecen, Hungary, July 2000, submitted.