Thermal Field Theory in Non-Equilibrium States

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Abstract

Conventional transport theory is not really applicable to non-equilibrium systems which exhibit strong quantum effects. We present two different approaches to overcome this problem. Firstly we point out how transport equations may be derived that incorporate a nontrivial spectral function as a typical quantum effect, and test this approach in a toy model of a strongly interacting degenerate plasma. Secondly we explore a path to include non-equilibrium effects into quantum field theory through momentum mixing transformations in Fock space. Although the two approaches are completely orthogonal, they lead to the same coherent conclusion.

Dedicated to the memory of Hiroomi Umezawa,
Friend and mentor,
Who had the vision to unify
Quantum field theory and statistical mechanics.
I. INTRODUCTION

In the past two decades the interest in non-equilibrium quantum systems has grown tremendously. It is therefore natural, that more and more examples have been found for which the applicability of traditional transport theory is doubtful, i.e., macroscopic Boltzmann-equations or Vlasov-Uehling-Uhlenbeck equations fail to describe the system adequately. Indeed some of these examples, like dynamical features of the early universe, collective effects in fusion plasmas, ultrashort phenomena in semi-conductors, nuclear matter in ultrarelativistic heavy-ion collisions and plasma drops composed of quarks and gluons have two things in common that require a new paradigm for their theoretical understanding.

The first of these common factors is the (space and time) density of interactions between the components of the system. Some of them are interacting weakly in the sense of having a small coupling constant, e.g., as between electrons and photons. However, even for weak coupling, a system of many particles experiences synergetic effects like collective participation in movement. As a consequence, the systems listed above have excitation spectra which differ significantly from those of rarefied gases of free particles. The latter however is a basic requirement for the applicability of traditional transport theory.

The second common factor is the importance of the non-equilibrium aspect on the level of the system components: Inhomogeneities occur on spatial and temporal scales that are comparable to the intrinsic scales of the physical problem. Particularly interesting examples are ultrarelativistic heavy-ion collisions, e.g. between lead nuclei $^{208}$Pb at $160\times208$ GeV laboratory energy. Here, time scales are fm/c and characteristic sizes are a few fm – not significantly different from the diameter of the nucleons which compose the initial nuclei. Such a significant difference however is another requirement for the description of the system in terms of macroscopic equations: Traditional transport theory is correct only to first order of gradients in density, temperature etc.

We may summarize the two common aspects of the systems listed above as 1. the breakdown of the quasi-particle picture, and 2. the non-separation of space-time scales. The necessary theoretical description therefore must account for these two aspects.

With the present paper we wish to contribute to this new paradigm from two sides: In section II we shall point out a way to incorporate nontrivial excitation spectra into transport theory (which already contains some non-equilibrium features), and in section III we will clean a path for the inclusion of non-equilibrium effects (gradients in external parameters) into quantum field theory. These two completely different approaches lead to the same consistent conclusion, summarized in section IV.

II. FROM STANDARD TO QUANTUM TRANSPORT THEORY

We perform this step by studying the reverse direction, i.e., starting from a Schwinger-Dyson equation for the full two-point function of a fermionic quantum field we derive transport equations. In contrast to the standard treatment however, we do not perform the usual quasi-particle approximation.

As has been pointed out by various authors, (see ref. [1] for an overview) the description of dynamical quantum phenomena in a statistical ensemble necessitates a formalism with a
doubled Hilbert space. For our purpose the relevant content of this formalism is, that its two-point Green functions are $2 \times 2$ matrix-valued. We prefer the technically simpler method of thermo field dynamics (TFD) [2], but will keep our derivation sufficiently general to perform similar calculations in the Schwinger-Keldysh, or Closed-Time Path (CTP) formalism [3].

Within this matrix formulation, we consider the Schwinger-Dyson equation for the full quark propagator $S = S_0 + S_0 \odot \Sigma \odot S$, where $S_0$ is the free and $S$ the full two-point Green function of the quark field, $\Sigma$ is the full self energy and the generalized product $\odot$ is a matrix product (thermal and spinor indices) and an integration (each of the matrices is a function of two space coordinates). Throughout this paper we use the convention to write space-time and momentum variables also as lower indices, e.g. $\Sigma_{xy} \equiv \Sigma(x, y)$.

In the CTP formulation as well as in the $\alpha = 1$ parameterization of TFD [4], the matrix elements of $S$, $S_0$ and $\Sigma$ obey

$$S_{11}^{(0)} + S_{22}^{(0)} = S_{12}^{(0)} + S_{21}^{(0)}; \quad \Sigma^{11} + \Sigma^{22} = -\Sigma^{12} - \Sigma^{21}. \quad (1)$$

Therefore the four components of the Schwinger-Dyson equation are not independent, the matrix equation can be simplified by a linear transformation which one may conveniently express as a matrix product [5]. It achieves a physical interpretation only in the TFD formalism, see ref. [4]. The transformation matrices $B$ are

$$B(n) = \begin{pmatrix} (1 - n) & -n \\ 1 & 1 \end{pmatrix}, \quad (2)$$

depending on one parameter only. For example, the third term in the Schwinger-Dyson equation becomes

$$B(n) \tau_3 S_0 \odot \Sigma \odot S (B(n))^{-1} = \begin{pmatrix} S_0^R \odot \Sigma^R \odot S^R \\ S_0^A \odot \Sigma^A \odot S^A \end{pmatrix} \cdot \text{something}. \quad (3)$$

Here, $\tau_3 = \text{diag}(1, -1)$, $\Sigma^{R,A}$ are the retarded and advanced full self energy function, and $S^{R,A}$ are the retarded and advanced full propagator (similarly for $S_0$)

$$\Sigma^R = \Sigma^{11} + \Sigma^{12}, \quad \Sigma^A = \Sigma^{11} + \Sigma^{21}$$
$$S^R = S^{11} - S^{12}, \quad S^A = S^{11} - S^{21}. \quad (4)$$

The diagonal elements of the transformed equation therefore are retarded and advanced Schwinger-Dyson equation. The off-diagonal element is a transport equation.

We now switch to the mixed (or Wigner) representation of functions depending on two space-time coordinates: $\tilde{\Sigma}_{XP} = \int d^4(x - y) \exp(i P_\mu (x - y)^\mu) \Sigma_{xy}$ with $X = (x + y)/2$, the $\tilde{}$-sign will be dropped henceforth. The Wigner transform of the convolution $\Sigma \odot G$ is a nontrivial step: Formally it may be expressed as a gradient expansion

$$\int d^4(x - y) \exp(i P_\mu (x - y)^\mu) \Sigma_{xz} \odot G_{zy} = \exp(-i \Diamond) \tilde{\Sigma}_{XP} \tilde{G}_{XP}. \quad (5)$$

$\Diamond$ is a 2nd order differential operator acting on both functions appearing behind it in the form of a Poisson bracket $\Diamond A_{XP} B_{XP} = \frac{1}{2} (\partial_X A_{XP} \partial_P B_{XP} - \partial_P A_{XP} \partial_X B_{XP})$. We will henceforth
use the infinite-order differential operator \( \exp(-i\Diamond) = \cos\Diamond - i\sin\Diamond \). Propagator and self energy are split into real Dirac matrix valued functions

\[
S_{XP}^{RA} = G_{XP} \mp i\pi A_{XP} \quad \Sigma_{XP}^{RA} = \text{Re}\Sigma_{XP} \mp i\pi \Gamma_{XP}.
\]

\( A_{XP} \) is the generalized spectral function of the quantum field.

Now consider the equations obtained by action of Dirac differential operators (= inverse free propagators) on the matrix-transformed Schwinger-Dyson equation \([6]\). The diagonal components are

\[
\text{Tr}[ (P^\mu \gamma_\mu - m) A_{XP} ] = \cos\Diamond \text{Tr}[ \text{Re}\Sigma_{XP} A_{XP} + \Gamma_{XP} G_{XP} ] \\
\text{Tr}[ (P^\mu \gamma_\mu - m) G_{XP} ] = \text{Tr}[1] + \cos\Diamond \text{Tr}[ \text{Re}\Sigma_{XP} G_{XP} - \pi^2 \Gamma_{XP} A_{XP} ].
\]

Two important facts about these equations have to be emphasized. First notice that these equations do not in general admit a \( \delta \)-function solution for \( A_{XP} \) even in zero order of \( \Diamond \). In fact, in contrast to other papers \([7]\) we find that there is not such thing as a mass shell constraint in quantum transport theory.

Secondly, the equations do not contain odd powers of the differential operator \( \Diamond \). This implies, that when truncating the Schwinger-Dyson equation to first order in \( \Diamond \) (the usual order for the approximations leading to kinetic equations), the spectral function \( A_{XP} \) may still be obtained as the solution of an algebraic equation.

**A. Transport equation**

The off-diagonal component of the transformed Schwinger-Dyson equation reads, after acting on it with the inverse free propagator \([4,6]\)

\[
\hat{S}_0^{-1} S_{xy}^K = \Sigma_{xz}^R \otimes S_{zy}^K - \Sigma_{xz}^A \otimes S_{zy}^A,
\]

with kinetic components \( S^K = (1-n) S^{12} + n S^{21} \) and \( \Sigma^K = (1-n) \Sigma^{12} + n \Sigma^{21} \). Inserting the real functions defined before, this leads to a differential equation, which contains all the features of our desired quantum transport theory \([4,6]\):

\[
\text{Tr}[ (\partial_X^\mu \gamma_\mu + 2\sin\Diamond \text{Re}\Sigma_{XP} + \cos\Diamond 2\pi \Gamma_{XP}) S_{XP}^K ] = \\
2i\text{Tr}[ i\sin\Diamond \Sigma_{XP}^K G_{XP} - \cos\Diamond \Sigma_{XP}^A i\pi A_{XP} ].
\]

Note, that here even as well as odd powers of the operator \( \Diamond \) occur, and the solution in zero order \( \Diamond \) is not trivial. To see this more clearly, we define the generalized covariant distribution function \( N_{XP} \) through the equation

\[
(1 - N_{XP}) S_{XP}^{12} + N_{XP} S_{XP}^{21} = 0.
\]

It may be proven easily from the Kubo-Martin-Schwinger boundary condition \([8]\), that with this definition of \( N_{XP} \) one reaches the proper equilibrium limit \( N_{XP} \rightarrow n_F(E) \), where \( n_F(E) \) is the Fermi-Dirac equilibrium distribution function at temperature \( T \),

\[
n_F(E) = (e^{\beta(E-\mu)} + 1)^{-1}.
\]
With this definition, \( S_{XP}^{K} = 2\pi i \) \((N_{XP} - n)\) \( A_{XP} \), and therefore \( N_{XP} \) is the parameter which diagonalizes the full non-equilibrium matrix-valued propagator through the Bogoliubov matrix \( \mathcal{B} \) from (2) [4]:

\[
\mathcal{B}(N_{XP}) \tau_3 S_{XP} (\mathcal{B}(N_{XP}))^{-1} = \begin{pmatrix}
G_{XP} - i\pi A_{XP} \\
G_{XP} + i\pi A_{XP}
\end{pmatrix}.
\] (12)

Although one may use the generalized distribution function directly in the above eq. (9), we wish to compare this equation to traditional transport theory. Hence, one more step has to be performed, which is the consistent expansion to first order in the operator \( \star \).

As outlined before, we may then use a spectral function which is the solution of an algebraic equation. One may argue, that in non-equilibrium states a spectral representation of the propagator does not exist in general [2], but one may still exploit the fact that retarded and advanced propagator are by definition analytical functions of the energy parameter in the upper or lower complex energy half-plane:

\[
S_{R,A}^{xy}(E, p, X) = \text{Re} \, G_{XP} = \pm \frac{1}{2\pi \sin(\tau E)} \int_{-\infty}^{\infty} dE' A(E', p, X) \frac{1}{E - E' \pm i\epsilon},
\] (13)

which is nothing but the Wigner transform of \( S_{xy}^{R,A} = \mp 2\pi i \Theta(\pm(x_0 - y_0)) A_{xy} \).

Inserting this into the quantum transport equation the yields, correct to first order in \( \star \) (see refs. [3][4] for details):

\[
\text{Tr} \left[ A_{XP} \left\{ \left( P_\mu \gamma^\mu - m - \text{Re}\Sigma_{XP} \right), N_{XP} \right\} \right]
= \text{iTr} \left[ A_{XP} \left( N_{XP} \Sigma_{XP}^{21} - (N_{XP} - 1) \Sigma_{XP}^{12} \right) \right]
= \text{i} \int_0^\infty d\tau \int_{-\infty}^{\infty} \frac{dE}{2\pi} \sin(\tau E) \text{Tr} \left[ \left\{ A(X; P_0 + E, P), \right. \right.
\left. \left( N_{XP} \Sigma_{XP}^{21}(t + \tau/2, X; P) - (N_{XP} - 1) \Sigma_{XP}^{12}(t + \tau/2, X; P) \right) \right\} \right]_N.
\] (14)

In this equation, \( \{ \cdot, \cdot \} \) denotes the Poisson bracket, the index \( N \) means that the derivatives are not acting on \( N_{XP} \).

The integral over the history of our system constitutes a memory term of this generalized transport equation. The memory effect therefore can be attributed to the nonzero spectral width of the physical excitations in our system.

However, even when we send this spectral width parameter to zero in the above equation, a new effect remains. Using a simple quasi-particle spectral function

\[
A_{XP} = \left( P_\mu \gamma^\mu - m - \text{Re}\Sigma_{XP} \right)
\]
FIG. 1. Time dependent spectral width parameter $\gamma_t$. Parameters are $g=0.12$, $T_i = 1$ MeV, $T_f = 200$ MeV, $m = 10$ MeV.

Thin line: $\Gamma_t$ from eq. (17), thick line: $\gamma_t$ from eq. (19).

$$\delta \left( P_0 - E(t, x, P) \right) \left[ \frac{\partial ((P^\mu - \text{Re} \Sigma_{XP}^\mu)^2 - (M - \text{Re} \Sigma_{XP}^s)^2)}{\partial P_0} \right]^{-1},$$ 

where $E(t, x, P)$ is the (space-time and momentum dependent) generalized energy of the corresponding “particle”-like state, one obtains after integration over $P_0$

$$\left( \frac{\partial N(x, p, t)}{\partial t} + \frac{\partial E(t, x, P)}{\partial p} \frac{\partial N(t, x, p)}{\partial x} - \frac{\partial E(t, x, P)}{\partial x} \frac{\partial N(t, x, p)}{\partial p} \right)$$

$$= \text{St} \left[ N(t, x, p) \right] + \delta \text{St} \left[ N(t, x, p) \right].$$

The l.h.s. of this equation is the Vlasov (or streaming) part of the standard transport equation, it is free of dissipation.

The first piece on the r.h.s. is the standard “collision integral” of the kinetic equation, while $\delta \text{St}$ is a time-local correction which now contains the three-dimensional Poisson bracket of the $\Sigma^{12}, \Sigma^{21}$ self energies with the function $N(t, x, p)$ [9]. We will consider this equation in more detail in section III of this paper.

Before solving the generalized transport equation for a simple example, we would like to point out that with our derivation we have indeed taken into account the two topics mentioned in the introductory statements. The breakdown of the quasi-particle approximation is accounted for by the nontrivial spectral function, and the non-separation of time-scales is accounted for by the Poisson bracket on the r.h.s. of eq. (14).

B. Solution of the generalized transport equation

In the following, we will concentrate on the memory effects hidden in eq. (14). To isolate them from the the gradients in the generalized collision integral $\delta \text{St} \left[ N(t, x, p) \right]$ of
eq. (16), we consider a very simple model system, tailored to mimic the thermalization of a plasma composed of quarks and gluons. We cannot possibly explain all the relevant physics in the present short paper, rather refer to the literature on such systems [10]. In view of the previous derivations, we are interested in a calculation of relaxation time scales for such a plasma, and make some simple approximations:

1. We assume, that a gas of bosons (gluons) is instantaneously heated to a very high temperature. In this gas then eventually quark-antiquark pairs start to pop up, until at the very end a thermal equilibrium in the sense of a degenerate plasma is reached.

2. We assume, that the self energy function for the quarks is dominated by gluonic contributions, and that it does not depend on the energy of the quarks nor on the space coordinates.

3. The gluon background is dominated by external conditions, i.e., we neglect the back-reaction of quarks on the gluon distribution.

4. We neglect the influence of anti-quarks in the spectral function. This restriction is removed in an extended version of this application, ref. [11].

We summarize these assumptions in the following ansatz for the imaginary part of the self energy function and for the spectral function of quarks:

\[ \Gamma_{XP} \equiv \Gamma_t = \gamma^0 g T(t) = \gamma^0 g (T_i \Theta(-t) + T_f \Theta(t)) , \]  

(17)

\[ \mathcal{A}(t, E, p) = \frac{\gamma^0}{\pi} \frac{\gamma_t}{(E - \omega_t)^2 + \gamma_t^2} . \]  

(18)
Hence, we approximate the quark spectral function by two time-dependent parameters \( \omega_t \) and \( \gamma_t \), which we may interpret as effective energy and effective spectral width. Arguments for the validity of this approach are given in ref. [11].

With the above spectral function the coupled system (7) reduces to a single nonlinear equation for \( \gamma_t \), plus the condition \( \omega_t^2 = \omega_0^2 = p^2 + m^2 \). This latter condition is more complicated, when the anti-particle piece of the spectral function is taken into account [11]. The energy parameter is chosen as \( E = \omega_0 \), which yields instead of eq. (7) as the Schwinger-Dyson equation for the retarded (or advanced) two-point function of the quarks:

\[
\gamma_t = gT_i + g(T_f - T_i) \Theta(t) \left(1 - e^{-2\gamma_t t}\right)
\]  (19)

In Fig. 1, the solution of this equations is plotted in comparison to the time dependent imaginary part of the self energy function from eq. (17). It is obvious, that the solution of the nonlinear equation (19) approaches the imaginary part of the self energy function with a characteristic delay time. In ref. [11] it is discussed how this delay time is calculated from the system parameters.

We now consider three different levels of transport theory for this model, the corresponding generalized distribution functions relabeled to \( N_t, N_t^B \) and \( N_t^G \). First of all, due to the simplicity of our ansatz we obtain as the full quantum transport equation (9):

\[
\frac{d}{dt}N_t = -2 \gamma_t \left(N_t - n_F(m, T(t)) \right)
\]  (20)

with \( T(t) \) as defined in eq. (17). This equation looks surprisingly similar to a kinetic equation in relaxation time approach. However, this similarity is superficial: The kinetic equation, or Boltzmann equation, derived for our simple model system reads

\[
\frac{d}{dt}N_t^B = -2 \Gamma_t \left(N_t^B - n_F(m, T(t)) \right).
\]  (21)

Finally, the generalized transport equation (14) is, correct up to first order in the gradients:

\[
\frac{d}{dt}N_t^G = -2 \Gamma_t \left(N_t^G - n_F(m, T(t)) \right) + 4t \Theta(t) \left(g(T_f - T_i)\right)^2 \exp(-2\gamma_t t) \left(N_t^G - n_F(m, T(t)) \right) \left( N_t^G - n_F(m, T(t)) \right).
\]  (22)

In Fig.2 we show the numerical solution for \( N_t^G \), and compare it to the Boltzmann solution \( N_t^B \) as well as the full quantum transport solution \( N_t \). This comparison of the three methods shows, that the full quantum transport equation results in a much slower equilibration process than the Boltzmann equation. This result is in agreement with other attempts to solve the quantum relaxation problem [12,13]: The quantum system exhibits a memory, it behaves in an essentially non-Markovian way.

In particular, for the physical scenario studied here, the time to reach 1-1/e \( \approx 86\% \) of the equilibrium quark occupation number is almost doubled (14.7 fm/c as compared to 8.2 fm/c in the Boltzmann case). We furthermore find, that with the generalized transport
equation one does at least partially describe the memory effects in a quantum system (the characteristic time now is 11.4 fm/c).

Thus, although we have only used a toy model, it might turn out that quantum effects (= memory as described in this contribution) substantially hinder the thermalization of a strongly interacting over long time scales. A more thorough discussion of this physical result is carried out in ref. [11].

III. FROM QUANTUM FIELD THEORY TO INHOMOGENEOUS TFD

In the following we describe a system of charged bosons in a non-equilibrium state. For simplicity we first consider a space-inhomogeneous situation, but neglect the spectral width of the bosons. This view therefore is orthogonal to the previous section, where we ended in considering a time-dependent system with translational invariance. While this gave us a window to isolate the memory effects of quantum transport theory, we will isolate some non-equilibrium quantum features in the present picture.

As we have pointed out, the description of non-equilibrium systems requires to double the Hilbert space. The reason is, that the occupation number, i.e., the interpretation of a state as “particle” or “hole” (and hence its temporal boundary condition) may change from point to point in order to ensure causality.

The proper formulation is achieved by expressing each physical (causal) excitation operator as the superposition of an operator evolving forward in time and a backward evolving operator. Such a superposition is known in the Liouville space, we refer to the literature for an introduction [14,15]. Obviously, there is also a corresponding anti-causal (orthogonal) linear combination. Denoting the causal single-particle creation and annihilation operators by \( a_{kl} \), \( a_{kl}^\dagger \) (the two different charges are distinguished by a lower index \( l = \pm \)) and the anti-causal operators with \( \tilde{\ } \)-signs, one may conveniently express this superposition as a matrix in three-momentum space [14,17]:

\[
\begin{pmatrix}
  a_{kl}(t) \\
  a_{kl}^\dagger(t)
\end{pmatrix}
= \int d^3 q \left( \mathcal{B}_l^{-1}(t, q, k) \right)^* \frac{\xi_{ql}}{\xi_{ql}^*} e^{-iE_{ql}t}
\]

\[
\begin{pmatrix}
  a_{kl}(t) \\
  a_{kl}^\dagger(t)
\end{pmatrix}^T
= \int d^3 q \begin{pmatrix}
  \xi_{ql}^\# \\
  -\xi_{ql}
\end{pmatrix}^T \mathcal{B}_l(t, q, k) e^{iE_{ql}t}.
\]

\( k \) is the three-momentum of the modes, therefore in this notation \( a_{kl}^\dagger(t) \) creates a negatively charged physical excitation with momentum \( k \), while \( a_{kl}(t) \) annihilates a positive charge.

The operator \( \xi_{ql}^\# \) creates a mode with momentum \( q \) and energy \( E_q \equiv E(q) \), which propagates forward in time. \( \xi_{ql} \) annihilates a state with the same momentum and energy, but propagating backwards in time. These two operators therefore may be combined in a Bogoliubov transformation, with a \( 2 \times 2 \) matrix (compare to the fermion case, eq. (12)):

\[
\mathcal{B}_l(t, q, k) = \begin{pmatrix}
  (\delta^3(q - k) + N_l(t, q, k)) & -N_l(t, q, k) \\
  -\delta^3(q - k) & \delta^3(q - k)
\end{pmatrix}.
\]

For simplicity, let us choose a local equilibrium state, i.e., \( N(t, q, k) \) is the Fourier transform of a space-local Bose-Einstein distribution function.
\[ N_l(t, q, k) = \frac{1}{(2\pi)^3} \int d^3 z \, e^{-i(q-k)z} n_l(t, z, (q+k)/2) \]
\[ n_l(t, z, p) = \left[ e^{\beta(t,z)(E_p - \mu_l(t,z))} - 1 \right]^{-1}. \] (25)

With these creation and annihilation operators we construct two mutually commuting complex scalar fields \( \phi_x, \tilde{\phi}_x \); see ref. 1,4,13 for details:

\[ \phi_x = \int \frac{d^3 k}{\sqrt{(2\pi)^3}} \left( a_{k-}(t) e^{-i k x} + a_{k+}(t) e^{i k x} \right) \]
\[ \tilde{\phi}_x = \int \frac{d^3 k}{\sqrt{(2\pi)^3}} \left( a_{k-}^\dagger(t) e^{i k x} + \tilde{a}_{k+}(t) e^{-i k x} \right). \] (26)

Each of these is a representation of the canonical commutation relations, and they may be combined in a statistical doublet \( \Phi_x = (\phi_x, \tilde{\phi}_x)^T \).

Before proceeding we would like to emphasize that the above heuristic formulation has led us to the creation and annihilation operators known from thermo field dynamics (TFD, 2). However, the construction of two commuting representations in non-equilibrium field theory is also achieved in other formulations 1, and their existence has been noticed independently by several authors 14,15.

**A. Effective interaction and diffusion**

In this subsection we will rely more on results obtained in the TFD formalism than in the other parts of this paper. However, we believe that the approach described in the following is also valid and useful for other methods used in thermal field theory.

First of all we note, that due to the introduction of the momentum mixing terms the equation of motion for the physical creation/annihilation operators is

\[ \frac{i}{\hbar} \frac{\partial}{\partial t} \left( \begin{pmatrix} a_{kl}(t) \\ \tilde{a}_{kl}(t) \end{pmatrix} \right) = \left[ \begin{pmatrix} a_{kl}(t) \\ \tilde{a}_{kl}(t) \end{pmatrix}, \hat{H} + \hat{Q} \right], \] (27)

with a “bare Hamiltonian”

\[ \hat{H} = \sum_{l=\pm} \int d^3 k \, E_k \left( \begin{pmatrix} a_{kl}^\dagger(t) \\ -\tilde{a}_{kl}(t) \end{pmatrix} \right)^T \left( \begin{pmatrix} a_{kl}(t) \\ \tilde{a}_{kl}(t) \end{pmatrix} \right) = \sum_{l=\pm} \int d^3 k \, E_k \left( a_{kl}^\dagger(t) a_{kl}(t) - \tilde{a}_{kl}^\dagger(t) \tilde{a}_{kl}(t) \right) \] (28)

and a second part that vanishes for homogeneous systems:

\[ \hat{Q} = \sum_{l=\pm} \int d^3 k \, d^3 q \left( \begin{pmatrix} a_{ql}^\dagger(t) \\ -\tilde{a}_{ql}(t) \end{pmatrix} \right)^T \left( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right) \left( -i \partial_t + E_k - E_q \right) N_l(t, k, q) \left( \begin{pmatrix} a_{ql}(t) \\ \tilde{a}_{ql}(t) \end{pmatrix} \right). \] (29)

Consequently, by introducing the above linear combinations for the physical creation and annihilation operators, the physical fields are no longer free. The time evolution acquires a term mixing causal and anti-causal field.
Let us emphasize at this point, that one should think of $\hat{H} + \hat{Q}$ as the Liouville operator of our quantum system, which is the generator of the time evolution for the density matrix. Its spectral properties then are immediately obvious \([13,14,18,19]\), e.g., it is not bounded from below.

The mixing part $\hat{Q}$ of the full Liouvillean vanishes when $N_l(t, k, q)$ is time independent and proportional to $\delta(k - q)$, which according to \((13)\) is the case when $n_l(t, z, p)$ does not depend on the space-time coordinates $(t, z)$. Consequently $\hat{Q}$ couples the systems to gradients in the function $n_l(t, z, p)$.

To see this more explicitly, we henceforth introduce a “trivial” dispersion relation $E_q = \sqrt{q^2 + m^2}$ and obtain

$$ (E_k - E_q) N_l(t, k, q) = -i \frac{Q}{2E_Q(2\pi)^3} \int d^3 z e^{-i(q-k)z} \nabla_z n_l(t, z, Q) + \mathcal{O}(\nabla_z^2), \quad (30) $$

where $Q = (q + k)/2$. One may argue, that this is an ad-hoc introduction of a coupling into the system dynamics. However, this formulation has a deeper foundation: Consider for a moment an equilibrium system, because then the Liouville operator is identical to eq. \((28)\) and invariant under the thermal Bogoliubov transformation as $\left(\hat{a}_k^\dagger \hat{a}_k - \hat{\tilde{a}}_k^\dagger \hat{\tilde{a}}_k\right) = \left(\xi_k^\dagger \xi_k - \tilde{\xi}_k^\dagger \tilde{\xi}_k\right)$. This invariance constitutes of a continuous symplectic symmetry of the Liouville space – and such a global symmetry can be made local: We search for those operators $\xi$, which diagonalize the Liouville operator locally in space and time. Such a requirement connects the parameter $n_l(t, z, p)$ of the Bogoliubov transformation with the system dynamics.

Therefore, this process of gauging the symplectic symmetry is entirely equivalent to the space-time local Bogoliubov transformation diagonalizing the single-particle propagator that was carried out in sect. II.

On the other hand, it is well known that gauging of a symmetry automatically introduces the coupling of the field $\phi_x$ to gradients in an external scalar field. Here, this external scalar field is the distribution function “field” $n_l(t, z, p)$, it is classical and does not possess any dynamical feature. To summarize these arguments: The new gradient terms in the time evolution occur, because we are implementing causal boundary conditions locally in space and time.

Let us now turn to the generalized transport equation used in the previous section. Rewritten for bosons, with an effective energy parameter $E$ that is independent of $x$ it reads

$$ \left(\frac{\partial n_l(t, x, k)}{\partial t} + \frac{\partial E(k)}{\partial k} \frac{\partial n_l(t, x, k)}{\partial x}\right) = \text{St} \left[ n_l(t, x, k) \right] + \delta \text{St} \left[ n_l(t, x, k) \right]. \quad (31) $$

According to our derivation in section II we know that $\delta \text{St}$ contains a three-dimensional Poisson bracket of $n$ and some interaction probability. Obviously this implies that

$$ \int \frac{d^3 k}{(2\pi)^3} \int d^3 x \delta \text{St} \left[ n_l(t, x, k) \right] = 0. \quad (32) $$

and therefore we may write the momentum integral as the three-dimensional divergence of a current:

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\[
\int \frac{d^3 k}{(2\pi)^3} \delta \text{St} \left[ n_l(t, x, k) \right] = -\nabla_x J_l^\delta(t, x) \tag{33}
\]

with \( \nabla_x \equiv \partial/\partial x \). We use the fact, that the momentum integral over the “standard” collision term \( \text{St} \left[ n(t, x, k) \right] \) is zero \([20]\), and therefore obtain for the integrated transport equation

\[
\partial_t n_l(t, x) + \nabla_x \left( j_l(t, x) + J_l^\delta(t, x) \right) = 0 . \tag{34}
\]

This is the expression for current conservation, with

\[
\dot{j}_l(t, x) = \int \frac{d^3 k}{(2\pi)^3} \frac{k}{E_k} n_l(t, x, k) \tag{35}
\]
as the convective part of this current. Eq. (34) implies, that in our boson gas a current \( J_l^\delta(t, x) \) arises even for zero “convection” \( j_l(t, x) \): An inhomogeneous temperature distribution gives rise to diffusion. For small gradients of the temperature distribution the diffusion current \( J_l^\delta(t, x) \) will be proportional to \( \nabla_x n_l(t, x) \). The momentum integrated transport equation then is nothing but Fick’s law.

More detailed derivations within a non-relativistic field model, as well as exemplaric calculations of particle flow and energy flow may be found in ref. [21].

**B. Calculation of the Diffusion Coefficient**

Having outlined the diffusion problem in the previous subsection, we need to generalize the momentum mixing to excitations with continuous mass spectrum. This amounts to an approximation of the fully interacting quantum fields by generalized free fields \([19]\) as already discussed in section II, and will certainly complicate matters very much.

On the other hand we have outlined above, that only the spatial gradient terms enter into the diffusion problem. Hence, for the following we may neglect the time dependence of the mixing parameter \( n \). Instead of eq. (23) we therefore write

\[
\left( \begin{array}{c} a_{kl}(t) \\ a_{kl}^\dagger(t) \end{array} \right) = \int_0^\infty dE \int d^3 q \, A_l^{1/2}(E, k) \left( B_l^{-1}(E, q, k) \right)^* \left( \begin{array}{c} \xi_{Eql} \\ \bar{\xi}_{Eql} \end{array} \right) e^{-iEt}
\]
\[
\left( \begin{array}{c} a_{kl}(t) \\ -a_{kl}^\dagger(t) \end{array} \right)^T = \int_0^\infty dE \int d^3 q \, A_l^{1/2}(E, k) \left( \begin{array}{c} \xi_{Eql} \\ -\bar{\xi}_{Eql} \end{array} \right)^T B_l(E, q, k) e^{iEt} , \tag{36}
\]

where the \( 2 \times 2 \) Bogoliubov matrices \( B \) have the same form as already given in eq. (24), but \( N_l \) is replaced by

\[
N_l(E, q, k) = \frac{1}{(2\pi)^3} \int d^3 z \, e^{-i(q-k)z} n_l(E, z)
\]

\[
n_l(E, z) = \left[ e^{\beta(z)(E-\mu(z))} - 1 \right]^{-1} . \tag{37}
\]

The \( \xi \)-operators have commutation relations
\[ \left[ \xi_{Ekl}, \xi_{E'k'l'}^\# \right] = \delta_{ll'} \delta(E - E') \delta^3(k - k') . \]  

(38)

Similar relations hold for the \( \tilde{\xi} \) operators, all other commutators vanish, see [19].

The weight functions \( A_l(E, k) \) are positive and have support only for positive energies, their normalization is

\[ \int_0^\infty dE E A_l(E, k) = \frac{1}{2} , \quad \int_0^\infty dE A_l(E, k) = Z_{kl} . \]  

(39)

The principles of this expansion have been discussed in ref. [19], its generalization to non-equilibrium states was introduced in ref. [4]. For equilibrium states the combination \( A_B(E, k) = A_+ (E, k) \Theta(E) - A_-(E, -k) \Theta(-E) \) is the spectral function of the boson field \( \phi_x \) and the limit of free particles with mass \( m \) is recovered when \( A_B(E, k) \rightarrow \) sign\( (E) \delta(E^2 - k^2 - m^2) = \) sign\( (E) \delta(E^2 - \omega_k^2) \). As we have seen in the previous section, the concept of a local spectral function may be applied to non-equilibrium states up to first order in the gradient operator \( \sigma \).

Using these relations we are then able to express the current response of the quantum field to the gradients in the distribution function by using the Heisenberg equation of motion for time-dependent operators, eq. (27), to obtain

\[ J^{\delta}(t, x) = i \int_{t_0}^t d\tau \langle \left[ \hat{j}(t - \tau, x), \hat{Q} \right] \rangle \]  

(40)

where \( \hat{j}(x) = i(\phi^\dagger_x \nabla_x \phi_x - \phi_x \nabla_x \phi^\dagger_x) \) is the current operator.

As has been pointed out in [4,13], the \( i \)-th vector component of the \( l \)-charged currents generated by the inhomogeneity of the system is

\[ J_{l}^{\delta(i)}(t, x) = 2\pi \int \frac{d^3Q}{(2\pi)^3} \frac{Q^{(i)}}{2Z_Q^2} \int dE (A_l(E, Q))^2 \times \int dE' E' \left\{ \frac{\partial n_{l'}(E', Q, x)}{\partial x^{(i)}} \frac{\partial A_l(E', Q)}{\partial Q^{(j)}} \right\} . \]  

(41)

In principle, this equation may be used to calculate the diffusion constant for interacting boson fields. The result of such a calculation made for pions coupled to nuclear matter has led to results which are bigger than the semi-classical transport coefficients [22,20] by factors of 10–100 [13,4]. Since the transport coefficients are proportional to the relaxation time of the system, this corresponds to the result obtained in section II: Quantum effects lead to a slowdown of the relaxation process.

In order to establish our approach more safely, we now insert a boson spectral function with a constant spectral width. However, in contrast to the fermionic case discussed in section II, we may then no longer neglect the anti-particles. Hence we use

\[ A_B(E, k) = \frac{1}{\pi} \frac{2E\gamma_B}{(E^2 - E_k^2 - \gamma_B^2)^2 + 4E^2\gamma_B^2} . \]  

(42)
Similar to the fermionic spectral function (18), one obtains this form by summing over four simple poles in the complex energy plane, each with the same distance from the real $E$-axis and with the same residue. A short calculation then gives the approximate results

$$\frac{1}{2} \frac{d}{dE} (A(E, Q))^2 \approx \frac{1}{2\pi\gamma_B} + \frac{2}{\pi^2 E_Q} + O(\gamma_B)$$

and

$$\int dE' E' \frac{\partial n_l(E', Q, x)}{\partial x^{(j)}} \frac{\partial A_l(E', Q)}{\partial Q^{(j)}} = \frac{Q^{(j)}}{|Q|} \frac{\partial}{\partial |Q|} \frac{\partial}{\partial x^{(j)}} n_l(E_Q, x),$$

such that an integration by parts gives, up to first order in the gradients of the distribution function,

$$J^j_i(x) = -\left( \frac{1}{2\gamma_B} + O(1) \right) \nabla_x n_l(t, x) = -D \nabla_x n_l(t, x).$$

The diffusion coefficient $D$ therefore diverges when $\gamma_B \to 0$. In traditional calculations of transport coefficients this diffusion coefficient is obtained as $D = \tau \langle v^2 \rangle / 3$, i.e., as the product of relaxation time and square average of the particle velocity [20]. As we have shown before, the relaxation time of a non-equilibrium quantum system is infinite if the particles do not have a spectral width – and therefore our result has the correct free-particle limit.

For more detailed considerations however, we would have to define a model for an interacting system, which e.g. yields the spectral width parameter as function of the system characteristics (temperature, chemical potential etc.). This would overstretch the goal set with the present paper.

**IV. CONCLUSION**

With the present paper we have demonstrated, how one may apply modern techniques of “thermal” field theory to non-equilibrium states. This has become important because of the quantum character of non-equilibrium systems studied today: A definite breakdown of the quasi-particle approximation and the emergence of space-time inhomogeneities on scales comparable to those of system components are the reasons for the non-applicability of “standard” transport theory.

We have started with the derivation of a generalized transport equation, which accounts for the breakdown of the quasi-particle approximation. In Fig. 2 we have compared the solution of this equation with the full Schwinger-Dyson equation as well as with standard transport theory. We find, that the nontrivial spectral function one encounters when going beyond the quasi-particle picture leads to strong memory effects in transport phenomena. In particular, we studied a system which exhibits a characteristic delay time. The memory effects are partially taken into account by the generalized transport equation – but it turns out, that gradients in the external parameters are equally important to describe the physical systems listed in the introduction to this work.

In the second major part of this work we therefore pointed out, how one may incorporate gradient effects into “thermal” field theory. This is achieved by introducing a mixing of operators in momentum space. Firstly we have shown that such an approach indeed
leads to relaxation currents in inhomogeneous systems. Secondly we have used the mixing transformation to achieve a linear response result for transport coefficients.

This transport coefficient calculation was not pursued numerically. However, we may rely on previous numerical estimates of transport coefficients achieved with this method \[4\]. Those numerical estimates give a result very similar to the one obtained in the quantum transport model of section II: Quantum systems are relaxing \emph{slower} than anticipated from standard transport theory.

Hence, the two approaches we have demonstrated in the present paper lead to a coherent result, which is also in agreement with more traditional calculations from statistical mechanics. We therefore believe that a clear path exists towards an improved treatment of quantum systems in non-equilibrium states, and we express our gratitude to our teacher Hiroomi Umezawa for leading us on this path.
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