POISSON AND QUANTUM GEOMETRY OF ACYCLIC
CLUSTER ALGEBRAS

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Abstract. We prove that certain acyclic cluster algebras over the complex numbers are the coordinate rings of holomorphic symplectic manifolds. We also show that the corresponding quantum cluster algebras have no non-trivial prime ideals. This allows us to give evidence for a generalization of the conjectured variant of the orbit method for quantized coordinate rings and their classical limits.

CONTENTS

1. Introduction 1
2. Cluster Algebras 3
2.1. Cluster algebras 3
2.2. Upper cluster algebras 4
2.3. Poisson structures 4
2.4. Compatible Pairs and Their Mutation 5
2.5. Quantum Cluster Algebras 6
3. Intersections of Ideals with Clusters 8
4. Poisson ideals in acyclic cluster algebras 8
4.1. Symplectic Structure 9
5. Ideals in Acyclic Quantum Cluster Algebras 11
References 12

1. Introduction

In the present paper we investigate the Poisson geometry associated with cluster algebras over the complex numbers defined by acyclic quivers, and relate them to the ideal theory of the corresponding quantum cluster algebras. Our main motivation is the following conjectural analogue of Kirillov’s Orbit method for quantized coordinate rings which has been an open problem for roughly twenty years (see e.g. [3] or [15, Section 4.3] and [14] for the case of compact quantum groups). Let $G$ be a semisimple complex algebraic group and $\mathbb{C}[g]$ its coordinate ring while $\mathbb{C}_q[G]$ denote the corresponding quantized coordinate ring. It has been conjectured that there exists a homeomorphism between the space of primitive ideals in $\mathbb{R}_q[G]$ and the symplectic leaves of the standard Poisson structure on $G$. For an excellent introduction to this conjecture we refer the reader to Goodearl’s paper [9]. The conjecture appears extremely difficult to prove and it is only known to be true in the cases of $G = SL_2, SL_3$. 

1
The coordinate rings \( \mathbb{C}[G] \) are known to have an upper cluster algebra structure \( [1] \) while the quantized coordinate rings are conjectured to have a quantum (upper) cluster algebra structure \( [2, \text{Conjecture 10.10}] \). Indeed, it follows from recent results of Geiß, Leclerc and Schröer \( [7, \text{Section 12.4}] \) that \( \mathbb{C}[\text{SL}_n] \) has a quantum cluster algebra structure. Cluster algebras are nowadays very well-established, hence we do not recall any of the definitions here, and refer the reader to the literature, resp. our Section 2. Most importantly for our purposes, a cluster algebra over \( \mathbb{C} \) is defined by a combinatorial datum in a field of fractions \( \mathbb{C}(x_1, \ldots, x_n) \). We will denote this initial seed by \( (x, B) \) where \( x = (x_1, \ldots, x_n) \) and \( B \) is an integer \( m \times n \)-matrix with \( m \leq n \) such that its principal \( m \times m \) submatrix is skew-symmetrizable. The cluster variables \( x_{m+1}, \ldots, x_n \) are the frozen variables which we will call coefficients. A quantum cluster algebra is given by a quantum seed \( (x, B, \Lambda) \) where \( (B, \Lambda) \) where \( B \) is as above and \( \Lambda \) is a skew-symmetric \( n \times n \)-matrix such that \( (B, \Lambda) \) is a compatible pair (see Section 2 for details). The set \( x = (x_1, \ldots, x_n) \) now lives in the skew-field of fractions \( \mathbb{C}_\Lambda(x_1, \ldots, x_n) \) defined by \( \Lambda \). A compatible pair also defines a compatible Poisson structure in the sense of \( [8] \) on the cluster algebra given by \( (x, B) \). It is well-known that the conjectured quantum cluster algebra structures on the rings \( \mathbb{R}[G] \) and the standard Poisson structure on \( \mathbb{C}[G] \) arise from such a compatible pair. Therefore, we would like to suggest the following conjecture.

Conjecture 1.1. Let \( (B, \Lambda) \) be a compatible pair and let \( \mathfrak{A} \) and \( \mathfrak{A}_q \) be a cluster, resp. quantum cluster algebra defined by \( (x, B) \), resp. \( (x, B, \Lambda) \). Suppose further that \( \mathfrak{A} \) and \( \mathfrak{A}_q \) are Noetherian and that \( \mathfrak{A} \) is the coordinate ring of the affine variety \( X \). Then, there exists a homeomorphism between the space of primitive ideals of \( \mathfrak{A}_q \) and the symplectic leaves on \( X \) defined by \( \Lambda \).

In light of Conjecture 1.1 we may think of quantum affine space and quantum tori as cluster algebras where all cluster variables are frozen. In this case the corresponding homeomorphism is well known and easy to construct. The other extreme case are cluster algebras without coefficients and here the class that is usually easiest to study are the acyclic cluster algebras. For example, it is known that such a cluster algebra is always Noetherian and the coordinate ring of an affine variety (see \([1] \) and \([2] \) for the classical and quantum versions). It is our main objective to give evidence for Conjecture 1.1 by proving it in this very specific case. It is an immediate consequence of the following two main results.

Theorem 1.2. Let \( \mathfrak{A} \) be a cluster algebra with initial seed \( (x, B) \) defined by an acyclic quiver where \( B \) is invertible satisfying \([4.1]\), and suppose that it is the coordinate ring of an affine variety \( X \) and that \( (B, \Lambda) \) is a compatible pairs. Then \( X \) has the structure of a symplectic manifold, whose symplectic form is the corresponding Poisson bivector.

Theorem 1.3. Let \( \mathfrak{A}_q \) be a quantum cluster algebra with quantum seed \( (x, B, \Lambda) \) satisfying the assumptions of Theorem 1.2. Then \( \{0\} \) is the only proper two sided prime ideal in \( \mathfrak{A}_q \).

Our approach, is similar to that of \([16]\), however all the proofs are self-contained and much easier, as our set-up is less general. The main idea is to study the intersection of ideals with the polynomial ring generated by a given cluster—in this case the acyclic seed. We are able to derive rather strong conditions that Poisson prime ideals—resp. two-sided prime ideals in the quantum case—must satisfy and
are able to show that no non-trivial ideals satisfying them exist. A straightforward argument, then allows us to conclude that the variety \( X \) is a symplectic manifold. We should also remark that we do not know whether any acyclic cluster algebras exist that do not satisfy the assumptions made in Theorem 1.2.

The paper is organized as follows. We first briefly recall the definitions of cluster algebras and compatible Poisson structures, compatible pairs and quantum cluster algebras in Section 2. Thereafter, we continue with some technical key propositions (Section 3) and discuss in Section 4 the symplectic geometry of acyclic cluster algebras. The proof of Conjecture 1.1 in our specific case is completed in Section 5 by proving Theorem 1.3.

2. Cluster Algebras

2.1. Cluster algebras. In this section, we will review the definitions and some basic results on cluster algebras, or more precisely, on cluster algebras of geometric type over the field of complex numbers \( \mathbb{C} \). Denote by \( \mathcal{F} = \mathbb{C}(x_1, \ldots, x_n) \) the field of fractions in \( n \) indeterminates. Recall that a \( m \times m \)-integer matrix \( B' \) is called skew-symmetrizable if there exists a \( m \times m \)-diagonal matrix \( D \) with positive integer entries such that \( B' \cdot D \) is skew-symmetric. Now, let \( B \) be a \( m \times n \)-integer matrix such that its principal \( m \times m \)-submatrix is skew-symmetrizable. We call the tuple \((x_1, \ldots, x_n, B)\) the initial seed of the cluster algebra and \((x_1, \ldots, x_m)\) a cluster, while \( x = (x_1, \ldots, x_n) \) is called an extended cluster. The cluster variables \( x_{m+1}, \ldots, x_n \) are called coefficients. We will now construct more clusters, \((y_1, \ldots, y_n)\) and extended clusters \( y = (y_1, \ldots, y_n) \), which are transcendence bases of \( \mathcal{F} \), and the corresponding seeds \((y, B)\) in the following way.

Define for each real number \( r \) the numbers \( r^+ = \max(r, 0) \) and \( r^- = \min(r, 0) \). Given a skew-symmetrizable integer \( m \times n \)-matrix \( B \), we define for each \( 1 \leq i \leq m \) the exchange polynomial

\[
P_i = \prod_{k=1}^{n} x_{ik}^{r_k^+} + \prod_{k=1}^{n} x_{ik}^{-r_k^-}.
\]

We can now define the new cluster variable \( x'_i \in \mathcal{F} \) via the equation

\[
x_i x'_i = P_i.
\]

This allows us to refer to the matrix \( B \) as the exchange matrix of the cluster \((x_1, \ldots, x_n)\), and to the relations defined by Equation 2.2 for \( i = 1, \ldots, m \) as exchange relations.

We obtain that \((x_1, x_2, \ldots, \hat{x}_i, x'_i, x_{i+1}, \ldots, x_n)\) is a transcendence basis of \( \mathcal{F} \). We next construct the new exchange matrix \( B_i = B' = (b_{ij}') \), associated to the new (extended) cluster

\[x_i = (x_1, x_2, \ldots, \hat{x}_i, x'_i, x_{i+1}, \ldots, x_n)\]

via its coefficients \( b_{ij}' \) as follows:

- \( b_{ij}' = -b_{ij} \) if \( j \leq n \) and \( i = k \) or \( j = k \),
- \( b_{ij}' = b_{ij} + \frac{b_{ik}b_{kj} + b_{ij}b_{ik}}{2} \) if \( j \leq n \) and \( i \neq k \) and \( j \neq k \),
- \( b_{ij}' = b_{ij} \) otherwise.

This algorithm is called matrix mutation. Note that \( B_i \) is again skew-symmetrizable (see e.g. [1]). The process of obtaining a new seed is called cluster mutation. The set of seeds obtained from a given seed \((x, B)\) is called the mutation equivalence class of \((x, B)\).
Definition 2.1. The cluster algebra $\mathfrak{A} \subset \mathcal{F}$ corresponding to an initial seed $(x_1, \ldots, x_n, B)$ is the subalgebra of $\mathcal{F}$ generated by the elements of all the clusters in the mutation equivalence class of $(x, B)$. We refer to the elements of the clusters as the cluster variables.

Remark 2.2. Notice that the coefficients, resp. frozen variables $x_{m+1}, \ldots, x_n$ will never be mutated. Of course, that explains their name.

We have the following fact, motivating the definition of cluster algebras in the study of total positivity phenomena and canonical bases.

Proposition 2.3. [6, Section 3] (Laurent phenomenon) Let $\mathfrak{A}$ be a cluster algebra with initial extended cluster $(x_1, \ldots, x_n)$. Any cluster variable $x$ can be expressed uniquely as a Laurent polynomial in the variables $x_1, \ldots, x_n$ with integer coefficients.

Moreover, it has been conjectured for all cluster algebras, and proven in many cases (see e.g. [12] and [4], [5]) that the coefficients of these polynomials are positive.

Definition 2.4. [1, Definition 1.10] Let $\mathfrak{A}$ be a cluster algebra and let $(x, B)$ be a seed. The lower bound $\mathcal{L}_B \subset \mathfrak{A}$ associated with $(x, B)$ is the algebra generated by the set $\{x_1, \ldots, x_n, y_1, \ldots, y_m\}$.

2.2. Upper cluster algebras. Berenstein, Fomin and Zelevinsky introduced the related concept of upper cluster algebras in [1].

Definition 2.5. Let $\mathfrak{A} \subset \mathcal{F}$ be a cluster algebra with initial cluster $(x_1, \ldots, x_n, B)$ and let, as above, $y_1, \ldots, y_m$ be the cluster variables obtained by mutation in the directions $1, \ldots, m$, respectively.

(a) The upper bound $U_{x, B}(\mathfrak{A})$ is defined as

$$U_{x, B}(\mathfrak{A}) = \bigcap_{j=1}^{m} \mathbb{C}[x_{j-1}^{\pm 1}, x_j, y_j, x_{j+1}^{\pm 1}, \ldots, x_m^{\pm 1}, x_{m+1}, \ldots, x_n]$$

(b) The upper cluster algebra $U(\mathfrak{A})$ is defined as

$$U(\mathfrak{A}) = \bigcap_{(x', B')} U_{x', B'}(\mathfrak{A})$$

where the intersection is over all seeds $(x', B')$ in the mutation equivalence class of $(x, B)$.

Observe that each cluster algebra is contained in its upper cluster algebra (see [1]).

2.3. Poisson structures. Cluster algebras are closely related to Poisson algebras. In this section we recall some of the related notions and results.

Definition 2.6. Let $k$ be a field of characteristic 0. A Poisson algebra is a pair $(A, \{\cdot, \cdot\})$ consisting of a commutative $k$-algebra $A$ and a bilinear map $\{\cdot, \cdot\} : A \otimes A \to A$, satisfying for all $a, b, c \in A$:

1. skew-symmetry: $\{a, b\} = -\{b, a\}$
and the upper cluster algebra \( U \)

Proposition 2.8. Following fact.

Proof. Denote as above by \( \Lambda \) the Poisson bracket on \( A \) by \( \Lambda \). Observe that the algebras \( \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \) are Poisson subalgebras of the Poisson algebra \( \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \) for each \( 1 \leq i \leq m \), as \( \{ x_i, y_1 \} \Lambda = \{ x_i, x_i^{-1} P_i \} \Lambda \in \mathbb{C}[x_1, \ldots, x_n] \). If \( A \) is a Poisson algebra and \( \{ B_i : i \in I \} \) is a family of Poisson subalgebras, then \( \bigcap_{i \in I} B_i \) is a Poisson algebra, as well. The assertion follows. \( \square \)

2.4. Compatible Pairs and Their Mutation. Section 2.4 is dedicated to compatible pairs and their mutation. Compatible pairs yield important examples of Poisson brackets which are compatible with a given cluster algebra structure, and as we shall see below, they are also integral in defining quantum cluster algebras. Note that our definition is slightly different from the original one in [2]. Let, as above, \( m \leq n \). Consider a pair consisting of a skew-symmetrizable \( m \times n \)-integer matrix \( B \) with rows labeled by the interval \( [1, m] = \{ 1, \ldots, m \} \) and columns labeled by \( [1, n] \) together with a skew-symmetrizable \( n \times n \)-integer matrix \( \Lambda \) with rows and columns labeled by \( [1, n] \).
Definition 2.9. Let $B$ and $\Lambda$ be as above. We say that the pair $(B, \Lambda)$ is compatible if the coefficients $d_{ij}$ of the $m \times n$-matrix $D = B \cdot \Lambda$ satisfy $d_{ij} = d_i \delta_{ij}$ for some positive integers $d_i$ ($i \in [1, m]$).

This means that $D = B \cdot \Lambda$ is a $m \times n$ matrix where the only non-zero entries are positive integers on the diagonal of the principal $m \times m$-submatrix.

The following fact is obvious.

Lemma 2.10. Let $(B, \Lambda)$ be a compatible pair. Then $B \cdot \Lambda$ has full rank.

Let $(B, \Lambda)$ be a compatible pair and let $k \in [1, m]$. We define for $\varepsilon \in \{+1, -1\}$ a $n \times n$ matrix $E_{k, \varepsilon}$ via

- $(E_{k, +1})_{ij} = \delta_{ij}$ if $j \neq k$,
- $(E_{k, +1})_{ij} = -1$ if $i = j = k$,
- $(E_{k, +1})_{ij} = \max(0, -\varepsilon b_{kj})$ if $i \neq j = k$.

Similarly, we define a $m \times m$ matrix $F_{k, \varepsilon}$ via

- $(F_{k, +1})_{ij} = \delta_{ij}$ if $i \neq k$,
- $(F_{k, +1})_{ij} = -1$ if $i = j = k$,
- $(F_{k, +1})_{ij} = \max(0, \varepsilon b_{kj})$ if $i = k \neq j$.

We define a new pair $(B_k, \Lambda_k)$ as

$$(2.4) \quad B_k = F^T_{k, +1} B E^T_{k, +1}, \quad \Lambda_k = E_{k, \varepsilon} \Lambda E^T_{k, \varepsilon},$$

where $X^T$ denotes the transpose of $X$. We chose this rather non-straightforward way of defining $E_{k, \varepsilon}$ and $F_{k, \varepsilon}$ in order to show how our definition relates to that of [2]. We will not need it in what follows. The motivation for the definition is the following fact.

Proposition 2.11. [2, Prop. 3.4] The pair $(B_k, \Lambda_k)$ is compatible. Moreover, $\Lambda_k$ is independent of the choice of the sign $\varepsilon$.

The following fact is clear.

Corollary 2.12. Let $\mathfrak{A}$ be a cluster algebra given by an initial seed $(\mathfrak{x}, B)$ where $B$ is a $m \times n$-matrix. If $(B, \Lambda)$ is a compatible pair, then $\Lambda$ defines a compatible Poisson bracket on $\mathfrak{A}$ and on $\mathcal{U}(\mathfrak{A})$.

Example 2.13. If $m = n$ (i.e. there are no coefficients/frozen variables) and $B$ has full rank, then $(B, \mu B^{-1})$ is a compatible pair for all $\mu \in \mathbb{Z}_{>0}$ such that $\mu B^{-1}$ is an integer matrix. It follows from [3, Theorem 1.4] that in this case all compatible Poisson brackets arise in this way.

Example 2.14. Recall that double Bruhat cells in complex semisimple connected and simply connected algebraic groups have a natural structure of an upper cluster algebra (see [1]). Berenstein and Zelevinsky showed that the standard Poisson structure is given by compatible pairs relative to this upper cluster algebra structure (see [2, Section 8]).

2.5. Quantum Cluster Algebras. In this section we recall the the definition of a quantum cluster algebra, introduced by Berenstein and Zelevinsky in [2]. We define, for each skew-symmetric $n \times n$-integer matrix $\Lambda$, the skew-polynomial ring $\mathbb{C}[x_1, \ldots, x_n]_{\Lambda}$ to be the $\mathbb{C}[t^{\pm 1}]$-algebra generated by $x_1, \ldots, x_n$ subject to the relations

$$x_i x_j = t^{\Lambda_{ij}} x_j x_i.$$
Analogously, the quantum torus $H^q_A = C^*_q[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is defined as the localization of $C^*_A[x_1, \ldots, x_n]$ at the monoid generated by $x_1, \ldots, x_n$, which is an Ore set. The quantum torus is clearly contained in the skew-field of fractions $\mathcal{F}_A$ of $C^*_A[x_1, \ldots, x_n]$, and the Laurent monomials define a lattice $L \subset H^q_A \subset \mathcal{F}_A$ isomorphic to $\mathbb{Z}^n$. Denote for each $e = (e_1, \ldots, e_n) \in \mathbb{Z}^n$ by $x^e$ the monomial $x_1^{e_1} \ldots x_n^{e_n}$.

We need the notion of a toric frame in order to define the quantum cluster algebra.

**Definition 2.15.** A toric frame in $\mathfrak{F}$ is a mapping $M : \mathbb{Z}^n \to \mathfrak{F} - \{0\}$ of the form 

$$M(c) = \phi(X^{\eta(c)}) ,$$

where $\phi$ is a $\mathbb{Q}(\frac{1}{2})$-algebra automorphism of $\mathfrak{F}$ and $\eta : \mathbb{Z}^n \to L$ an isomorphism of lattices.

Since a toric frame $M$ is determined uniquely by the images of the standard basis vectors $\phi(X^{\eta(e_1)}), \ldots, \phi(X^{\eta(e_n)})$ of $\mathbb{Z}^n$, we can associate to each toric frame a skew commutative $n \times n$-integer matrix $\Lambda_M$. We can now define the quantized version of a seed.

**Definition 2.16.** [2 Definition 4.5] A quantum seed is a pair $(M, B)$ where

- $M$ is a toric frame in $\mathfrak{F}$.
- $B$ is a $n \times m$-integer matrix with rows labeled by $[1, m]$ and columns labeled by $[1, n]$.
- The pair $(B, \Lambda_M)$ is compatible.

Now we define the seed mutation in direction of an exchangeable index $k \in [1, m]$. For each $\varepsilon \in \{1, -1\}$ we define a mapping $M_k : \mathbb{Z}^n \to \mathfrak{F}$ via

$$M_k(c) = \sum_{p=0}^{c_k} \binom{c_k}{p} q^{p k^2} M(E_\varepsilon c + \varepsilon p b_k) , \quad M_k(-c) = M_k(c)^{-1} ,$$

where we use the well-known $q$-binomial coefficients (see e.g. [2 Equation 4.11]), and the matrix $E_{k,\varepsilon}$ defined in Section 2.3. Define $B_k$ to be obtained from $B$ by the standard matrix mutation in direction $k$, as in Section 2.1. One obtains the following fact.

**Proposition 2.17.** [2 Prop. 4.7] (a) The map $M_k$ is a toric frame, independent of the choice of sign $\varepsilon$.

(b) The pair $(B_k, \Lambda_{M_k})$ is a quantum seed.

Now, given an initial quantum seed $(B, \Lambda_M)$ denote, in a slight abuse of notation, by $X_1 = M(e_1), \ldots, X_r = M(e_r)$, which we refer to as the cluster variables associated to the quantum seed $(M, B)$. Here our nomenclature differs slightly from [2], since there one considers the coefficients not to be cluster variables. We now define the seed mutation

$$X'_k = M(-e_k + \sum_{b_{ik} > 0} b_{ik} e_i) + M(-e_k - \sum_{b_{ik} < 0} b_{ik} e_i) .$$

We obtain that $X'_k = M_k(e_k)$ (see [2 Prop. 4.9]). We say that two quantum seeds $(M, B)$ and $(M', B')$ are mutation-equivalent if they can be obtained from one another by a sequence of mutations. Since mutations are involutive (see [2 Prop 4.10]), the quantum seeds in $\mathfrak{F}$ can be grouped in equivalence classes, defined by the relation of mutation equivalence. The quantum cluster algebra generated
by a seed \((M, B) \subset \mathcal{G}\) is the \(\mathbb{C}[t^{\pm 1}]\)-subalgebra generated by the cluster variables associated to the seeds in an equivalence class.

**Remark 2.18.** There are definitions of quantum lower bounds, upper bounds and quantum upper cluster algebras (see \([2] \text{ Sections 5 and 7}\)), analogous to the classical case.

3. Intersections of Ideals with Clusters

In the present chapter we consider the intersection between Poisson ideals in a cluster algebra and individual clusters. Moreover, we prove quantum analogues of the propositions whenever available.

**Proposition 3.1.** Let \(x\) be a cluster, \(\operatorname{rank}(\Lambda) = n\) and \(I\) be a non-zero Poisson ideal. Then the ideal \(I\) contains a monomial in \(x^m \in \mathbb{C}[x]\).

**Proof.** Notice first that \(I_x = I \cap \mathbb{C}[x_1, \ldots, x_n] \neq 0\). Indeed, let \(0 \neq f \in I\). We can express \(f\) as a Laurent polynomial in the variables \(x_1, \ldots, x_n\); i.e., \(f = x_1^{-c_1} \cdots x_n^{-c_n} g\) where \(c_1, \ldots, c_n \in \mathbb{Z}_{\geq 0}\) and \(0 \neq g \in \mathbb{C}[x_1, \ldots, x_n]\). Clearly, \(g = x_1^{c_1} \cdots x_n^{c_n} f \in I_x\).

We complete the proof by contradiction. Let \(f = \sum_{w \in \mathbb{Z}^n} c_w x^w \in I_x\). We assume that \(f\) has the smallest number of nonzero summands such that no monomial term \(c_w x^w\) with \(c_w \neq 0\) is contained in \(I\). It must therefore have at least two monomial terms.

Assume, as above that \(c_w, c_w' \neq 0\) and denote by \(v\) the difference \(v = w - w'\). Since \(\Lambda\) has full rank, there exists \(i \in [1, n]\) such that \(\{x_i, x^v\} \neq 0\). Therefore, \(\{x_i, x^w\} = c x i x^w \neq d x i x^{w'} \{x_i, x^{w'}\}\) for some \(c, d \in \mathbb{C}\). Note that \(\{x_i, f\} = \sum_{w \in \mathbb{Z}^n} c_w \lambda_w x^w x_i\) for certain \(\lambda_w \in \mathbb{Z}\). Clearly, \(c x i f - \{x_i, f\} \in I\) and

\[
 c x i f - \{x_i, f\} = (c - c) c w x^w x_i + (c - d) c_w x^{w'} x_i + \ldots .
\]

Hence, \(c x i f - \{x_i, f\} \neq 0\) and it has fewer monomial summands than \(f\) which contradicts our assumption. Therefore, \(I\) contains a monomial. The proposition is proved.

The following fact is an obvious corollary of Proposition 3.1.

**Proposition 3.2.** Let \(\mathfrak{A}\) be the cluster algebra defined by a Poisson seed \((x, B, \Lambda)\) with \(\operatorname{rank}(\Lambda) = n\). If \(I \subset \mathfrak{A}\) is a non-zero Poisson prime ideal, then \(I\) contains a cluster variable \(x_i\).

We also have the following quantum version of Proposition 3.1 which is proved analogously to the classical case.

**Proposition 3.3.** Let \((x, B, \Lambda)\) be a quantum seed, \(\operatorname{rank}(\Lambda) = n\) and \(I\) a non-zero two-sided ideal. Then the ideal \(I\) contains a monomial in \(x^m \in \mathbb{C}_\Lambda[x]\).

**Remark 3.4.** We do not have a quantum version of Proposition 3.2 because we do not know whether prime ideals in quantum cluster algebras are completely prime.

4. Poisson ideals in acyclic cluster algebras

In this section we recall results from our previous paper \([16]\). As the proofs are rather short, we shall include them for convenience. Recall e.g. from \([1]\) that acyclic cluster algebras associated with an acyclic quiver and with trivial coefficients...
correspond, up to a reordering of the variables of the acyclic seed, to cluster algebras
defined by a seed \((x, B)\) where \(B\) is a skew-symmetric \(n \times n\)-matrix with \(b_{ij} > 0\) if
\(i < j\).

Berenstein, Fomin and Zelevinsky proved in \[1\] that such a cluster algebra \(\mathfrak{A}\) is
equal to both its lower and upper bounds. Thus, it is Noetherian and, if \(B\) has
full rank, a Poisson algebra with the Poisson brackets given by compatible pairs
\((B, \Lambda)\) with \(\Lambda = \mu B^{-1}\) for certain \(\mu \in \mathbb{Z}\) (see Example 2.13). In order for
\(B\) to have full rank we have to assume that \(n = 2k\) is even. Let \(P_i = m_i^+ + m_i^-\)
where \(m_i^+\) and \(m_i^-\) denote the monomial terms in the exchange polynomial. Then
\(\{y_i, x_i\} = \mu_1 m_i^+ + \mu_2 m_i^-\) for some \(\mu_1, \mu_2 \in \mathbb{Z}\). We, additionally, want to require
that \(\mu_1 \neq \mu_2\). To assure this, we assume that
\[
(4.1) \sum_{j=1}^{n} (b^{-1})_{ij} (\max(b_{ij}, 0) + \min(b_{ij}, 0)) \neq 0
\]
for all \(i \in [1, n]\). We have the following result.

**Theorem 4.1.** \[10\] Let \(\mathfrak{A}\) be an acyclic cluster algebra over \(\mathbb{C}\) with trivial coef-
ficients of even rank \(n = 2k\), given by a seed \((x_1, \ldots, x_n, B)\) where \(B\) is a skew-
symmetric \(n \times n\)-integer matrix satisfying \(b_{ij} > 0\) if \(i < j\) and suppose that \(B\) and
\(B^{-1}\) satisfy Equation 4.1 for each \(i \in [1, n]\). Then, the Poisson cluster algebra
defined by a compatible pair \((B, \Lambda)\) where \(\Lambda = \mu B^{-1}\) with \(0 \neq \mu \in \mathbb{Z}\) contains no
non-trivial Poisson prime ideals.

**Proof.** Suppose that there exists a non-trivial Poisson prime ideal \(I\). Then, \(I \cap x\)
is nonempty by Proposition 3.1, hence \(I \cap x = \{x_{i_1}, \ldots, x_{i_j}\}\) for some \(1 \leq i_1 \leq
i_2 \leq \ldots \leq i_j \leq 2k\). Additionally, observe that \(P_i = m_i^+ + m_i^-\) has to be contained
in \(I\), as well as
\[
\{y_{i_1}, x_{i_1}\} = \mu_1 m_{i_1}^+ + \mu_2 m_{i_1}^-.
\]
By our assumption, we have \(\mu_1 \neq \mu_2\), and therefore \(m_{i_1}^- \in I\). Hence, \(x_h \in I\) for
some \(h \in [1, i_1 - 1]\) or \(1 \in I\). But \(b_{i,h} < 0\) implies that \(h < i\) and we obtain the
desired contradiction. The theorem is proved.

The theorem has the following corollary which was also independently proved by
Muller very recently \[11\], though in more generality.

**Corollary 4.2.** Let \(\mathfrak{A}\) be as in Theorem 4.1. Then, the variety \(X\) defined by
\(\mathfrak{A} = \mathbb{C}[X]\) is smooth.

**Proof.** The singular subset is contained in a Poisson ideal of co-dimension greater
or equal to one by a result of Polishchuk \[13\]. It is well known that the Poisson
ideal must be contained in a proper Poisson prime ideal (see also \[10\]). The assertion
follows.

**Remark 4.3.** The assumption that the cluster algebra has even rank is very im-
portant. Indeed, Muller has recently shown that the variety corresponding to the
cluster algebra of type \(A_3\) has a singularity \[11\ Section 6.2\).

4.1. Symplectic Structure.
4.1.1. Symplectic geometry of Poisson varieties. In this section we will recall some well-known properties of the symplectic structure on Poisson varieties. Our discussion follows along the lines of [3, Part III.5]. If \( A \) is a Poisson algebra over a field \( k \), then each \( a \in A \) defines a derivation \( X_a \) on \( A \) via

\[
X_a(b) = \{a, b\}.
\]

This derivation is called the Hamiltonian vectorfield of \( a \) on \( A \).

Now suppose that \( A \) is the coordinate ring of an affine complex variety \( Y \). We will associate to the Poisson bracket the Poisson bivector \( u \in \Lambda^2 T(Y) \) where \( T(Y) \) denotes the tangent bundle of \( Y \). Let \( p \in Y \) be a point and \( m_p \subset A \) the corresponding maximal ideal. Let \( \alpha, \beta \in m_p/m_p^2 \) be elements of the cotangent space and let \( f, g \) be lifts of \( \alpha \) and \( \beta \), respectively. We define \( u_p \in \Lambda^2 T_p(Y) \)

\[
u_p(\alpha, \beta) = \{f, g\}(p).
\]

Note that \( u_p \) is a well-defined skew-symmetric form. Indeed, if \( I \subset A \) is an ideal and \( b \in I^2 \), then for all \( a \in A \)

\[\{a, b\} \in I\]

by the Leibniz rule. The form \( u_p \) may be degenerate, indeed if it is non-degenerate at every point \( p \in Y \), then we call \( u_p \) symplectic and, moreover, if \( Y \) is connected, then \( Y \) is smooth and a (holomorphic) symplectic manifold. Define

\[N(p) = \{\alpha \in T^*_p Y : u_p(\alpha, \cdot) = 0\},\]

and \( H(p) \subset T_p Y \) its orthogonal complement. The space \( H(P) \) is the tangent space of the linear span of the Hamiltonian vectorfields at \( p \). Recall that by the Theorem of Frobenius, a Poisson variety \( Y \) decomposes as a disjoint union of symplectic leaves, maximal symplectic submanifolds. The tangent space of the symplectic leaf at the point \( p \) is \( H(p) \).

4.1.2. The Main Theorem. Corollary 4.2 implies that \( X \) is smooth, hence it has the structure of a complex manifold. We have the following result.

**Theorem 4.4.** Let \( \mathfrak{A} \) be an acyclic cluster algebra as defined above, \( X \) an affine variety such that \( \mathfrak{A} = \mathbb{C}[X] \) and let \( \Lambda \) define a compatible Poisson bracket. Then, \( X \) is a holomorphic symplectic manifold.

**Proof.** First, let \( p \in X \) be a generic point, by which we mean that \( x_i(p) \neq 0 \) for all \( i = 1, \ldots, n \). Set \( x_i(p) = p_i \). It is easy to see that the Hamiltonian vectorfield \( X_{x_i} \) at \( p \) evaluates in the local coordinates \( (x_1, \ldots, x_n) \) as

\[X_{x_i}(p) = (\lambda_{i1}p_1, \ldots, \lambda_{ii}p_i),\]

where \( \Lambda = (\lambda_{ij})_{i,j=1}^n \).

Since, \( \Lambda \) is non-degenerate, we obtain that the Hamiltonian vectorfields span the tangent space \( T_p X \) at \( p \). It remains to consider the case when \( p_i = 0 \) for some \( i \in [1, n] \). Suppose that \( p \in X \) such that \( p_i = 0 \) and \( p_j \neq 0 \) for all \( j < i \). We have to show that the symplectic leaf containing \( p \) is not contained in the hyper-surface \( x_i = 0 \). We may assume, employing induction, that if \( p_1, \ldots, p_i \neq 0 \) then the symplectic leaf at \( p \) has full rank. We now claim that \( \{x_i, y_j\}(p) \neq 0 \). Indeed, suppose that \( \{x_i, y_j\}(p) = (\mu_1m^+\mu_2m^-)(p) = 0 \). Since \( p_i = 0 \) implies that \( (m^+ + m^-)(p) = 0 \), we would conclude as in the proof of Theorem 4.1 that \( m^+(p) = m^-(p) = 0 \), but that is a contradiction to our assumption that \( p_j \neq 0 \) for all \( j < i \). Denote by \( u \in \Lambda^2 T(Y) \) the Poisson bivector. We obtain that \( u_p(\frac{\delta}{\delta x_i}, \cdot) \neq 0 \), hence the symplectic leaf...
containing \( p \) is not tangent to the hypersurface \( x_i = 0 \) at \( p \). It must contain a point in an analytic neighborhood of \( p \) at which \( x_i(p) \neq 0 \) and \( x_j(p) \neq 0 \) for all \( j < i \) by our assumption. We obtain the desired contradiction and, hence, every symplectic leaf has dimension \( n \). But the manifold \( X \) is connected and, therefore, cannot be a union of disjoint open submanifolds of equal dimension, hence \( X \) contains only one symplectic leaf and the theorem is proved.

\[ \square \]

**Remark 4.5.** This result can be easily generalized to acyclic cluster algebras with invertible coefficients (using Remark 2.7). This would imply that in the set-up of locally acyclic cluster algebras it should be easy to show that the spectrum of a locally acyclic cluster algebra is a holomorphic symplectic manifold.

5. Ideals in Acyclic Quantum Cluster Algebras

**Theorem 5.1.** Let \( \mathfrak{A}_q \) be a quantum cluster algebra with quantum seed \( (\mathbf{x}, B, \Lambda) \) satisfying Equation (4.1). Then, \( \mathfrak{A}_q \) does not contain any non-trivial proper prime ideals.

**Proof.** Let \( \mathcal{I} \) be a prime ideal in \( \mathfrak{A}_q \). We obtain from Proposition 3.3 that \( \mathcal{I} \) contains a monomial \( x^v \) with \( v \in \mathbb{Z}_+^n \). It is easy to observe that we can choose \( v \) to be minimal with respect to the lexicographic order on \( \mathbb{Z}_+^n \). Recall that the lexicographic ordering defines \( u < w \) for \( u, w \in \mathbb{Z}_+^n \) if and only if there exists \( i \in [1, n] \) such that \( u_i, w_i \) and \( u_j = w_j \) if \( j > i \). Recall that this defines a total ordering on \( \mathbb{Z}_+^n \). There exists some \( i \in [1, n] \) such that \( v_i \neq 0 \) and \( v_k = 0 \) for all \( k > i \), and we write \( x^v = x^v x_i^n \) with \( v' = v - v_i e_i \).

Recall that an ideal \( \mathcal{I} \) in a non-commutative ring \( R \) is prime if \( arb \in \mathcal{I} \) for all \( r \in R \) implies that \( a \in \mathcal{I} \) or \( b \in \mathcal{I} \). Now, since \( \mathfrak{A}_q \) is acyclic, we know from [2, Theorem 7.3 and 7.5] that it is isomorphic to its lower bound. Hence, employing the notation of Definition 2.4 each element \( a \in \mathfrak{A}_q \) can be written as

\[
a = \sum_{p=1}^r c_p x^{w_p} y_i^{h_p} y^{w'} = \sum_{p=1}^r c_p q^{\lambda_p} x^{w_p} y_i^{h_p} x^v y^{w'} \in \mathcal{I},
\]

for all \( a \in \mathfrak{A}_q \) and certain \( \lambda_p \in \mathbb{Z} \). Hence, since \( \mathcal{I} \) is a prime ideal, \( x^v' \in \mathcal{I} \) or \( x_i^n \in \mathcal{I} \). But \( v' < v \) and \( v_i e_i \leq v \), and therefore, \( x_i^n = x^v \in \mathcal{I} \). But we have assumed in Equation (4.1) that \( x_i y_i \) and \( y_i x_i \) are not linearly dependent, hence \( x_i^n y_i \) and \( y_i x_i^n \) are not linearly dependent. As in the proof of Theorem 4.1, we now argue that \( x_i^{n_i-1} m_i^{+} \in \mathcal{I} \), where \( P_i = m_i^{+} + m_i^{-} \) (see Section 4). But this is a contradiction to our minimality assumption. The theorem is proved.

\[ \square \]

Theorem 5.1 and Theorem 4.4 have the following immediate corollary.

**Corollary 5.2.** Let \( (\mathbf{x}, B, \Lambda) \) be an acyclic Poisson or quantum seed, as defined above. Then the space of primitive ideals in the quantum cluster algebra (one point corresponding to the 0 ideal) and the space of symplectic leaves (also just one point) are homeomorphic.
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