Deformation of $\text{vect}(1)$-Modules of Symbols

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Abstract

We consider the action of the Lie algebra of polynomial vector fields, $\text{vect}(1)$, by the Lie derivative on the space of symbols $S_n^\delta = \bigoplus_{j=0}^n F_{\delta - j}$. We study deformations of this action. We exhibit explicit expressions of some 2-cocycles generating the second cohomology space $H^2_{\text{diff}}(\text{vect}(1), D_{\nu,\mu})$ where $D_{\nu,\mu}$ is the space of differential operators from $F_\nu$ to $F_\mu$. Necessary second-order integrability conditions of any infinitesimal deformations of $S_n^\delta$ are given. We describe completely the formal deformations for some spaces $S_n^\delta$ and we give concrete examples of non trivial deformations.

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1 Introduction

Let $\text{vect}(1)$ be the Lie algebra of polynomial vector fields on $\mathbb{R}$. Consider the 1-parameter deformation of the $\text{vect}(1)$-action on the space $\mathbb{R}[x]$ of polynomial functions on $\mathbb{R}$ defined by:

$$L^\lambda_X f = X f' + \lambda X' f,$$

where $X, f \in \mathbb{R}[x]$ and $X' := \frac{dX}{dx}$. Denote by $\mathcal{F}_\lambda$ the $\text{vect}(1)$-module structure on $\mathbb{R}[x]$ defined by this action for a fixed $\lambda$. Geometrically, $\mathcal{F}_\lambda$ is the space of polynomial weighted densities of weight $\lambda$ on $\mathbb{R}$:

$$\mathcal{F}_\lambda = \{ f dx^\lambda \mid f \in \mathbb{R}[x] \}.$$

The space $\mathcal{F}_\lambda$ coincides with the space of vector fields, functions and differential 1-forms for $\lambda = -1, 0$ and 1, respectively.

Denote by $D_{\nu,\mu} := \text{Hom}_{\text{diff}}(\mathcal{F}_\nu, \mathcal{F}_\mu)$ the $\text{vect}(1)$-module of linear differential operators with the $\text{vect}(1)$-action given by the formula

$$(1.1) \quad L_{\nu,\mu}^X (A) = L_\nu^X \circ A - A \circ L_\mu^X.$$

Each module $D_{\nu,\mu}$ has a natural filtration by the order of differential operators; the graded module $S_{\nu,\mu} := \text{gr} D_{\nu,\mu}$ is called the space of symbols. The quotient-module $D_{\nu,\mu}^k / D_{\nu,\mu}^{k-1}$ is isomorphic to the module of weighted densities $\mathcal{F}_{\mu-\nu-k}$, the isomorphism is provided by the principal symbol map $\sigma_{\text{pr}}$ defined by:

$$A = \sum_{i=0}^k a_i(x) \left( \frac{\partial}{\partial x} \right)^i \mapsto \sigma_{\text{pr}}(A) = a_k(x) (dx)^{\mu-\nu-k},$$

(see, e.g., [10]). As $\text{vect}(1)$-module, the space $S_{\nu,\mu}$ depends only on the difference $\delta = \mu - \nu$, so that $S_{\nu,\mu}$ can be written as $S_\delta$, and we have

$$S_\delta = \bigoplus_{k=0}^\infty \mathcal{F}_{\delta-k}$$

as $\text{vect}(1)$-modules. The space of symbols of order $\leq n$ is

$$S^n_\delta := \bigoplus_{j=0}^n \mathcal{F}_{\delta-j}.$$

The spaces $D_{\nu,\mu}$ and $S_\delta$ are not isomorphic as $\text{vect}(1)$-modules: $D_{\nu,\mu}$ is a deformation of $S_\delta$ in the sense of Richardson-Neijenhuis [13]. In the last two decades, deformations of various types of structures have assumed an ever increasing role in mathematics and physics. For each such deformation problem a goal is to determine if all related deformation obstructions vanish and many beautiful techniques been developed to determine when this is so. Deformations of Lie algebras with base and versal deformations were already considered by Fialowski in 1986 [6]. It was further developed, introducing a complete local algebra base (local means a commutative algebra which has a unique maximal ideal) by Fialowski in (1988) [7]. Also, in [7], the notion of miniversal (or formal versal) deformation was introduced in general, and
it was proved that under some cohomology restrictions, a versal deformation exists. Later Fialowski and Fuchs, using this framework, gave a construction for versal deformation [8].

We use the framework of Fialowski [7] (see also [1] and [2]) and consider (multi-parameter) deformations over complete local algebras. We construct the miniversal deformation of this action and define the complete local algebra related to this deformation.

According to Nijenhuis-Richardson [14], deformation theory of modules is closely related to the computation of cohomology. More precisely, given a Lie algebra \( \mathfrak{g} \) and a \( \mathfrak{g} \)-module \( V \), the infinitesimal deformations of the \( \mathfrak{g} \)-module structure on \( V \), i.e., deformations that are linear in the parameter of deformation, are related to \( H^1(\mathfrak{g}, \text{End}(V)) \).

Denote \( D := \mathcal{D}(\mathfrak{g}, \delta) \) the \( \text{vect}(1) \)-module of differential operators on \( S^n_{\mathfrak{g}} \). The infinitesimal deformations of the \( \text{vect}(1) \)-module \( S^n_{\mathfrak{g}} \) are classified by the space

\[
H^1_{\text{diff}}(\text{vect}(1), D) = \oplus_{\lambda, k} H^1_{\text{diff}}(\text{vect}(1), D_{\lambda, \lambda+k}),
\]

where \( H^i_{\text{diff}} \) denotes the differential cohomology; that is, only cochains given by differential operators are considered. Feigin and Fuchs computed \( H^1_{\text{diff}}(\text{vect}(1), D_{\lambda, \lambda'}) \), see [5]. They showed that non-zero cohomology \( H^1_{\text{diff}}(\text{vect}(1), D_{\lambda, \lambda'}) \) only appear for particular values of weights that we call resonant which satisfy \( \lambda' - \lambda \in \mathbb{N} \). Therefore, in formula (1.2), the summation \( \oplus_{\lambda, k} \) is over all \( \lambda \) and \( k \) satisfying \( 0 \leq \beta - \lambda - k < \beta - \lambda \leq \beta \).

In this paper we study the deformations of the structure of \( \text{vect}(1) \)-module on the space of symbols \( S^n_{\mathfrak{g}} \). We give the second-order integrability conditions which are sufficient in some cases. We will use the framework of Fialowski [6, 7] and Fialowski-Fuchs [5] (see also [1] and [2]) and consider (multi-parameter) deformations over complete local algebra base. For some examples, we will construct the miniversal deformation of this action and define the local algebra related to this deformation. The space \( H^1_{\text{diff}}(\text{vect}(1), D_{\lambda, \lambda+k}) \) was calculated in [5], and for space \( H^2_{\text{diff}}(\text{vect}(1), D_{\lambda, \lambda+k}) \) we can deduce the dimension from [5], see also [4]. We give explicit expressions of some 2-cocycles that span \( H^2(\text{vect}(1), D_{\lambda, \lambda+k}) \).

2 Cochains Spaces

Let \( \mathfrak{g} \) be a Lie algebra acting on a space \( V \). The space of \( n \)-cochains of \( \mathfrak{g} \) with values in \( V \) is the \( \mathfrak{g} \)-module

\[
C^n(\mathfrak{g}, V) := \text{Hom}(\wedge^n(\mathfrak{g}), V).
\]

The coboundary operator \( \partial^n : C^n(\mathfrak{g}, V) \to C^{n+1}(\mathfrak{g}, V) \) is a \( \mathfrak{g} \)-map satisfying \( \partial^n \circ \partial^{n-1} = 0 \). The kernel of \( \partial^n \), denoted \( Z^n(\mathfrak{g}, V) \), is the space of \( n \)-cocycles, among them, the elements in the range of \( \partial^n \) are called \( n \)-coboundaries. We denote \( B^n(\mathfrak{g}, V) \) the space of \( n \)-coboundaries.

By definition, the \( n^{th} \) cohomology space is the quotient space

\[
H^n(\mathfrak{g}, V) = Z^n(\mathfrak{g}, V)/B^n(\mathfrak{g}, V).
\]

We will only need the formula of \( \partial^n \) (which will be simply denoted \( \partial \)) in degrees 0, 1 and 2: for \( v \in C^0(\mathfrak{g}, V) = V \), \( \partial v(X) := Xv \), for \( b \in C^1(\mathfrak{g}, V) \),

\[
\partial b(X, Y) := Xb(Y) - Yb(X) - b([X,Y])
\]

and for \( \Omega \in C^2(\mathfrak{g}, V) \),

\[
\partial \Omega(X, Y, Z) := X\Omega(Y, Z) - \Omega([X,Y], Z) + \Omega([X,Z], Y)
\]
where \( \odot (X, Y, Z) \) denotes the summands obtained from the two written ones by the cyclic permutation of the symbols \( X, Y, Z \).

### 2.1 The First Cohomology Space

The first cohomology space \( H^1_{\text{diff}}(\text{vect}(1), D_{\lambda, \lambda+k}) \) was calculated by Feigin and Fuks in [5]. The result is as follows

**Theorem 2.1.** The space \( H^1_{\text{diff}}(\text{vect}(1), D_{\lambda, \lambda+k}) \) has the following structure:

\[
H^1_{\text{diff}}(\text{vect}(1), D_{\lambda, \lambda+k}) \simeq \begin{cases} \mathbb{R} & \text{if } k = 0, 2, 3, 4 \text{ for all } \lambda, \\ \mathbb{R}^2 & \text{if } \lambda = 0 \text{ and } k = 1, \\ \mathbb{R} & \text{if } \lambda = 0 \text{ or } \lambda = -4 \text{ and } k = 5, \\ \mathbb{R} & \text{if } \lambda = -\frac{5+\sqrt{19}}{2} \text{ and } k = 6, \\ 0 & \text{otherwise}. \end{cases}
\] (2.3)

These cohomology spaces are spanned by the cohomology classes of the 1-cocycles, \( C_{\lambda, \lambda+k} : \text{vect}(1) \to D_{\lambda, \lambda+k}, \) that are collected in the following table. We write, for \( X \frac{df}{dx} \in \text{vect}(1) \) and \( fdx^\lambda \in F_\lambda, \)

\[
C_{\lambda, \lambda+k}(X \frac{df}{dx})(fdx^\lambda) = C_{\lambda, \lambda+k}(X)(fxdx^\lambda).
\]

**Table 1. Cocycles that span \( H^1_{\text{diff}}(\text{vect}(1), D_{\lambda, \mu}) \)**

| \( C_{\lambda, \lambda}(X)(f) \) | \( C_{0,0}(X)(f) = X^f \) |
| --- | --- |
| \( C_{0,1}(X)(f) = X^ff \) |
| \( C_{\lambda, \lambda+2}(X)(f) = X^{(5)}f + 2X^ff \) |
| \( C_{\lambda, \lambda+3}(X)(f) = X^{(3)}f + X^ff \) |
| \( C_{\lambda, \lambda+4}(X)(f) = -\lambda X^{(5)}f + X^{(4)}f - 6X^{(3)}f - 4X^ff \) |
| \( C_{0,5}(X)(f) = 2X^{(5)}f - 5X^{(4)}f + 10X^{(3)}f + 5X^ff \) |
| \( C_{-4,1}(X)(f) = 12X^{(6)}f + 22X^{(5)}f + 5X^{(4)}f - 10X^{(3)}f - 5X^ff \) |
| \( C_{a_1, a_1+6}(X)(f) = \alpha_1X^{(7)}f - \beta_1X^{(6)}f' - \gamma_1X^{(5)}f'' - 5X^{(4)}f^{(3)} + 5X^{(3)}f^{(4)} + 2X^ff^{(3)} \) |

where

\[
a_1 = -\frac{5+\sqrt{19}}{2}, \quad a_1 = -\frac{22+5\sqrt{19}}{4}, \quad \beta_1 = \frac{31+7\sqrt{19}}{2}, \quad \gamma_1 = \frac{25+7\sqrt{19}}{2},
\]

\[
a_2 = -\frac{5-\sqrt{19}}{2}, \quad a_2 = -\frac{22-5\sqrt{19}}{4}, \quad \beta_2 = \frac{31-7\sqrt{19}}{2}, \quad \gamma_2 = \frac{25-7\sqrt{19}}{2}.
\]

The maps \( C_{\lambda, \lambda+j}(X) \) are naturally extended to \( S^\mu_\delta = \bigoplus_{j=0}^\mu F_{\delta-j}. \)

### 2.2 The Second Cohomology Space

Let \( g \) a Lie algebra and \( V \) a \( g \)-module, the *cup-product* defined, for arbitrary linear maps \( C_1, C_2 : g \to \text{End}(V) \), is defined by:

\[
[C_1, C_2] : g \otimes g \to \text{End}(V)
\]

\[
[C_1, C_2](x, y) = [C_1(x), C_2(y)] + [C_2(x), C_1(y)].
\] (2.4)
Therefore, it is easy to check that for any two 1-cocycles \( C_1 \) and \( C_2 \in Z^1(\mathfrak{g}, \text{End}(V)) \), the bilinear map \([C_1, C_2]\) is a 2-cocycle. Moreover, if one of the cocycles \( C_1 \) or \( C_2 \) is a 1-coboundary, then \([C_1, C_2]\) is a 2-coboundary. Therefore, we naturally deduce that the operation (2.4) defines a bilinear map:

\[
H^1(\mathfrak{g}, \text{End}(V)) \otimes H^1(\mathfrak{g}, \text{End}(V)) \rightarrow H^2(\mathfrak{g}, \text{End}(V)).
\] (5.5)

Thus, we can deduce the expressions of some 2-cocycles by computing the cup-products of 1-cocycles. That is especially important if we know the dimension of \( H^2(\mathfrak{g}, \text{End}(V)) \).

The second cohomology space \( H^2_{\text{diff}}(\text{vect}(1), D_{\lambda, \lambda+k}) \) of \( \text{vect}(1) \) can be deduced from the work of Feigin-Fuks [5] (see also [4]). The result is as follows

**Theorem 2.2.** The space \( H^2_{\text{diff}}(\text{vect}(1), D_{\lambda, \lambda+k}) \) has the following structure:

\[
H^2_{\text{diff}}(\text{vect}(1), D_{\lambda, \lambda+k}) \cong \begin{cases} 
\mathbb{R} & \text{if } \begin{cases} k = 1, \lambda = 0, \\
k = 2, 3, 4, 7, 8, 9, 10, 11 \text{ for all } \lambda, \\
k = 12, 13, 14 \text{ but } \lambda \text{ is either } \frac{1-k}{2} \text{ or } \frac{1-k}{2} \pm \frac{\sqrt{12k-23}}{2}, \end{cases} \\
\mathbb{R}^2 & \text{if } k = 5, \lambda = 0, -4, \text{ or } k = 6, \lambda = a_1, a_2, \\
0 & \text{otherwise.}
\end{cases}
\]

In the sequel, we consider some 2-cocycles \( \Omega_{\lambda, \lambda+k} : \text{vect}(1) \times \text{vect}(1) \rightarrow D_{\lambda, \lambda+k} \). For \( X \frac{d}{dx}, Y \frac{d}{dx} \in \text{vect}(1) \) and \( f dx^{\lambda} \in \mathcal{F}_\lambda \), we write

\[
\Omega_{\lambda, \lambda+k}(X \frac{d}{dx}, Y \frac{d}{dx})(fdx^{\lambda}) = \Omega_{\lambda, \lambda+k}(X, Y)(f)(dx)^{\lambda+k}.
\]

We need the following two lemmas:

**Lemma 2.3.** Let \( b_{\lambda, \lambda+2} \in C^1(\text{vect}(1), D_{\lambda, \lambda+2}) \) defined as follows: for \( X \frac{d}{dx} \in \text{vect}(1) \) and \( f dx^{\lambda} \in \mathcal{F}_\lambda \)

\[
b_{\lambda, \lambda+2}(X)(f) = \sum_{j=0}^{3} \alpha_j X^{(3-j)} f^{(j)}
\] (6.6)

where the coefficients \( \alpha_j \) are constant. Then the map \( \partial b_{\lambda, \lambda+2} : \text{vect}(1) \times \text{vect}(1) \rightarrow D_{\lambda, \lambda+2} \) is given by

\[
\partial b_{\lambda, \lambda+2}(X, Y)(f) = -\lambda \alpha_3 XY^{(4)} f - \lambda \alpha_2 X^3 Y^{(3)} f - \alpha_3 (3\lambda + 1) XY^{(3)} f' - \alpha_2 (2\lambda + 1) X' Y^{''} f' \\
- \alpha_3 (3\lambda + 3) XY^{'''} f'' - \alpha_3 XY' f^{(3)} - (X \leftrightarrow Y).
\] (7.7)

Proof. Straightforward computation.

\( \square \)

**Lemma 2.4.** Any \( \partial b_{0,5} \in C^2(\text{vect}(1), D_{0,5}) \) has the general following form: for \( X \frac{d}{dx}, Y \frac{d}{dx} \in \text{vect}(1) \) and \( f \in \mathcal{F}_0 \),

\[
\partial b_{0,5}(X, Y)(f) = \alpha_0 (9X''Y(5) + 5X^3 Y^{(4)} f - \alpha_6 X Y^{(6)} f' - \alpha_5 X^2 Y^{(5)} f' + (\alpha_2 + 5\alpha_1 - \alpha_4)X'' Y^{(4)} f' - 6\alpha_2 X Y^{(5)} f'' - 5\alpha_5 X' Y^{(4)} f'' + (2\alpha_2 + 3\alpha_3 - 4\alpha_4)X'' Y^{(3)} f'' - 15\alpha_6 X Y^{(4)} f^{(3)} - 10\alpha_5 X' Y^{(3)} f^{(3)} \\
- 20\alpha_6 X Y^{(3)} f^{(4)} - 10\alpha_5 X' Y'' f^{(4)} - 15\alpha_6 X Y^{''} f^{(5)} + \alpha_5 X Y' f^{(6)}) - (X \leftrightarrow Y).
\] (8.8)
where the coefficients $\alpha_j$ are constant and the map $b_{0,5}: \text{Vect}_{\text{Poly}}(1) \rightarrow D_{0,5}$ is given by

$$b_{0,5}(X)(f) = \sum_{j=0}^{6} \alpha_j X^{(6-j)} f^{(j)}.$$  (2.9)

Proof. Straightforward computation. \qed

**Proposition 2.5.** The cohomology spaces $H^2_{\text{diff}}(\text{vect}(1), D_{\lambda, \lambda+k})$, for $k = 1, 2, 3, 4$, are spanned by the cohomology classes of the nontrivial 2-cocycles $\Omega_{\lambda, \lambda+k}$ defined by

\[
\begin{align*}
\Omega_{0,1}(X,Y)(f) &= (X'Y'' - X''Y')f, \\
\Omega_{\lambda, \lambda+2}(X,Y)(f) &= (X^{(3)}Y' - X'Y^{(3)})f + 2(X''Y' - X'Y'')f', \\
\Omega_{\lambda, \lambda+3}(X,Y)(f) &= (X''Y^{(3)} - X^{(3)}Y'')f + (X^{(3)}Y' - X'Y^{(3)})f', \\
\Omega_{\lambda, \lambda+4}(X,Y)(f) &= -\lambda X'Y^{(5)}f + X'Y'(4f' - 6X'Y''f'' - 4X'Y''f'') - (X \leftrightarrow Y)
\end{align*}
\]

Proof. The map $\Omega_{0,1}$ is the cup-product of the 1-cocyles $C_{0,1}$ and $C_{1,1}$. So, the map $\Omega_{0,1}$ is a 2-cocycle, therefore, we will need only to prove that it is nontrivial since the space $H^2_{\text{diff}}(\text{vect}(1), D_{0,1})$ is one dimensional. Let $b_{0,1} \in C^1(\text{vect}(1), D_{0,1})$ defined by

$$b_{0,1}(X, f) = \sum_{j=0}^{2} \alpha_j X^{(2-j)} f^{(j)}$$

Thus,

$$\partial b_{0,1}(X, Y)f = \alpha_2(X''Y - XY'')f' + \alpha_2(X'Y - XY')f''$$

and therefore, it is clear that

$$\Omega_{0,1}(X,Y)f \neq \partial b_{0,1}(X,Y)f, \quad \text{for all } \alpha_0, \alpha_1, \alpha_2 \in \mathbb{R}.$$

The map $\Omega_{\lambda, \lambda+2}$ is the cup-product $[C_{\lambda, \lambda+3}, C_{\lambda, \lambda}]$. By Lemma 2.3 it is easy to check that $\Omega_{\lambda, \lambda+2}$ is a nontrivial 2-cocycle.

Besides, by direct computation, as before, we show that the cup-products $[C_{\lambda, \lambda+3}, \Omega_{\lambda, \lambda}]$ and $[C_{\lambda+4, \lambda+4}, \Omega_{\lambda, \lambda+4}]$ are nontrivial 2-cocycles. So, the spaces $H^2_{\text{diff}}(\text{vect}(1), D_{\lambda, \lambda+3})$ and $H^2_{\text{diff}}(\text{vect}(1), D_{\lambda, \lambda+4})$ can be spanned respectively by the cohomology classes of the nontrivial 2-cocycles $\Omega_{\lambda, \lambda+3}$ and $\Omega_{\lambda, \lambda+4}$ defined by

$$\Omega_{\lambda, \lambda+3} = [C_{\lambda, \lambda+3}, C_{\lambda, \lambda}] \quad \text{and} \quad \Omega_{\lambda, \lambda+4} = [C_{\lambda+4, \lambda+4}, C_{\lambda, \lambda+4}].$$

Now, we consider the cohomology spaces $H^2_{\text{diff}}(\text{vect}(1), D_{\lambda, \lambda+k})$ for $k = 5, 6$. These spaces are generically trivial, but, for $k = 5$ and $\lambda = -4$, 0 or $k = 6$ and $\lambda = a_1, a_2$ (where $a_1 = -\frac{5+\sqrt{13}}{2}$ and $a_2 = -\frac{5-\sqrt{13}}{2}$), they are two dimensional. In the following proposition we exhibit a basis for each of them.
Proposition 2.6. The cohomology spaces \( H^2_{\text{d}iff}(\text{vect}(1), \mathcal{D}_{0,5}) \), \( H^2_{\text{d}iff}(\text{vect}(1), \mathcal{D}_{-4,1}) \) and \( H^2_{\text{d}iff}(\text{vect}(1), \mathcal{D}_{a_i,a_i+6}) \), \((i=1, 2)\), are respectively spanned by the cohomology classes of the nontrivial following 2-cocycles:

\[
\begin{align*}
\Omega_{0,5}(X,Y)(f) &= (X^{(5)}Y'' + X^{(4)}Y^{(3)})f + 4X^{(4)}Y''f' + 3X^{(3)}Y''f'' - (X \leftrightarrow Y), \\
\tilde{\Omega}_{0,5}(X,Y)(f) &= 2X^{(5)}Y'f' + 5X^{(4)}Y^{(4)}f'' + 10X^{(3)}Y'f^{(3)} + 5X^{(3)}Y'f^{(4)} - (X \leftrightarrow Y), \\
\Omega_{-4,1}(X,Y)(f) &= 2X^{(4)}Y''f' + 3X^{(3)}Y''f'' - (X \leftrightarrow Y), \\
\tilde{\Omega}_{-4,1}(X,Y)(f) &= 12X^{(6)}Y'f' + 22X^{(5)}Y''f'' + 5X^{(4)}Y''f'' + 10X^{(3)}Y'f^{(3)} + 5X^{(3)}Y'f^{(4)} \\
&- (X \leftrightarrow Y), \\
\Omega_{a_i,a_i+6}(X,Y)(f) &= (X^{(5)}Y'' + X^{(4)}Y^{(3)})f' + 3X^{(4)}Y''f'' + 2X^{(3)}Y''f^{(3)} - (X \leftrightarrow Y) \\
\tilde{\Omega}_{a_i,a_i+6}(X,Y)(f) &= \alpha_iX^{(7)}Y'f + \beta_iX^{(6)}Y^{(6)}f' + \gamma_iX^{(5)}Y^{(5)}f'' + 5X^{(4)}Y^{(4)}f^{(3)} + 5X^{(3)}Y'f^{(4)} \\
&+ 2X^{(3)}Y'f^{(5)} - (X \leftrightarrow Y), \quad i = 1, 2.
\end{align*}
\]

Proof. The 2-cocycles \( \Omega_{0,5} \) and \( \tilde{\Omega}_{0,5} \) are defined as follows:

\[
\begin{align*}
\Omega_{0,5} &= [C_{2,5}, C_{0,2}], \quad \text{and} \quad \tilde{\Omega}_{0,5} = [C_{5,5}, C_{0,5}].
\end{align*}
\]

By Lemma 2.4 it is easy to show that these 2-cocyles are nontrivial. Indeed, for instance, computing the term in \( f \) in both the expressions of \( \Omega_{0,5} \) and of \( \partial_0 \) given in (2.8), we see obviously that \( \Omega_{0,5} \) cannot be a coboundary.

Similarly, we show that the 2-cocycles \( \Omega_{-4,1} = [C_{-1,1}, C_{-4,-1}] \), \( \tilde{\Omega}_{-4,1} = C_{1,1}, C_{-4,1} \), \( \Omega_{a_i,a_i+6} = [C_{a_i+3,a_i+6}, C_{a_i,a_i+3}] \) and \( \tilde{\Omega}_{a_i,a_i+6} = [C_{a_i+6,a_i+6}, C_{a_i,a_i+6}] \) are nontrivial. \( \square \)

Now, we give basis for the spaces \( H^2_{\text{d}iff}(\text{vect}(1), \mathcal{D}_{\lambda,\lambda+7}) \) when \( \lambda \notin \{0, -6\} \), and for the spaces \( H^2_{\text{d}iff}(\text{vect}(1), \mathcal{D}_{\lambda,\lambda+8}) \) when \( 2\lambda \neq -7 \pm \sqrt{39} \).

Proposition 2.7. The spaces \( H^2_{\text{d}iff}(\text{vect}(1), \mathcal{D}_{\lambda,\lambda+7}) \) where \( \lambda \neq 0, -6 \) and the spaces \( H^2_{\text{d}iff}(\text{vect}(1), \mathcal{D}_{\lambda,\lambda+8}) \) where \( 2\lambda \neq -7 \pm \sqrt{39} \) are respectively spanned by the cohomology classes of the nontrivial following 2-cocycles:

\[
\begin{align*}
\Omega_{\lambda,\lambda+7} &= [C_{\lambda+3,\lambda+7}, C_{\lambda,\lambda+3}], \\
\Omega_{\lambda,\lambda+8} &= [C_{\lambda+4,\lambda+8}, C_{\lambda,\lambda+4}].
\end{align*}
\]

Proof. By direct computation, we show that the cup-product \( [C_{\lambda+3,\lambda+7}, C_{\lambda,\lambda+3}] \) is a nontrivial 2-cocycle if and only if \( \lambda \neq 0, -6 \). Similarly, the cup-product \( [C_{\lambda+4,\lambda+8}, C_{\lambda,\lambda+4}] \) is a nontrivial 2-cocycle if and only if \( 2\lambda \neq -7 \pm \sqrt{39} \). \( \square \)

For \( k \geq 9 \), there are only few cases where we can exhibit 2-cocycles by computation of cup-products of 1-cocycles. Theses 2-cocycles generating the corresponding cohomology spaces are collected in the following proposition.

Proposition 2.8. The cohomology spaces \( H^2_{\text{d}iff}(\text{vect}(1), \mathcal{D}_{a_i,a_i+9}) \), \( H^2_{\text{d}iff}(\text{vect}(1), \mathcal{D}_{a_i,a_i-3,a_i+6}) \), \( H^2_{\text{d}iff}(\text{vect}(1), \mathcal{D}_{-8,1}) \), \( H^2_{\text{d}iff}(\text{vect}(1), \mathcal{D}_{0,9}) \), \( H^2_{\text{d}iff}(\text{vect}(1), \mathcal{D}_{-5,4,5}) \), \( H^2_{\text{d}iff}(\text{vect}(1), \mathcal{D}_{a_i,a_i+10}) \), \( H^2_{\text{d}iff}(\text{vect}(1), \mathcal{D}_{a_i-4,a_i+6}) \) are respectively spanned by the cohomology classes of the nontrivial following 2-cocycles:

\[
\begin{align*}
\Omega_{a_i,a_i+9} &= [C_{a_i+6,a_i+9}, C_{a_i,a_i+6}], \\
\Omega_{-8,1} &= [C_{-4,1}, C_{-8,4}], \\
\Omega_{-4,5} &= [C_{1,5}, C_{-4,1}], \\
\Omega_{a_i-4,a_i+6} &= [C_{a_i,a_i+6}, C_{a_i-4,a_i+6}].
\end{align*}
\]

Here we omit the explicit expressions of these last 2-cocycles as they are too long. But, as before, by direct computation, we show that they are nontrivial.
3 The General Framework

In this section we define deformations of Lie algebra homomorphisms and introduce the notion of miniversal deformations over complete local algebras. Deformation theory of Lie algebra homomorphisms was first considered with only one-parameter of deformation \[8, 14, 17\]. Recently, deformations of Lie algebras with multi-parameters were intensively studied (see, e.g., \[1, 2, 15, 16\]). Here we give an outline of this theory.

3.1 Infinitesimal deformations

Let \( \rho_0 : g \to \text{End}(V) \) be an action of a Lie algebra \( g \) on a vector space \( V \). When studying deformations of the \( g \)-action \( \rho_0 \), one usually starts with infinitesimal deformations:

\[
\rho = \rho_0 + tC,
\]

where \( C : g \to \text{End}(V) \) is a linear map and \( t \) is a formal parameter. The homomorphism condition

\[
[\rho(x), \rho(y)] = \rho([x, y]),
\]

where \( x, y \in g \), is satisfied in order 1 in \( t \) if and only if \( C \) is a 1-cocycle. Moreover, two infinitesimal deformations \( \rho_1 = \rho_0 + tC_1 \) and \( \rho_2 = \rho_0 + tC_2 \), are equivalents if and only if \( C_1 - C_2 \) is a coboundary:

\[
(C_1 - C_2)(x) = [\rho_0(x), A] := \partial A(x),
\]

where \( A \in \text{End}(V) \) and \( \partial \) stands for differential of cochains on \( g \) with values in \( \text{End}(V) \). So, the space \( H^1(g, \text{End}(V)) \) determines and classifies the infinitesimal deformations up to equivalence. (see, e.g., \[9, 14\]). If \( H^1(g, \text{End}(V)) \) is multi-dimensional, it is natural to consider multi-parameter deformations. More precisely, if \( \dim H^1(g, \text{End}(V)) = m \), then choose 1-cocycles \( C_1, \ldots, C_m \) representing a basis of \( H^1(g, \text{End}(V)) \) and consider the infinitesimal deformation

\[
\rho = \rho_0 + \sum_{i=1}^{m} t_i C_i,
\]

with independent parameters \( t_1, \ldots, t_m \).

In our study, an infinitesimal deformation of the \( \text{vect}(1) \)-action on \( S^p_{\delta} \) is of the form

\[
\mathcal{L}_X = L_X + \mathcal{L}_X^{(1)},
\]

where \( L_X \) is the Lie derivative of \( S^p_{\delta} \) along the vector field \( X \frac{d}{dx} \) defined by (1.1), and

\[
\mathcal{L}_X^{(1)} = \sum_{\lambda} \sum_{j=0}^{6} t_{\lambda, \lambda+j} C_{\lambda, \lambda+j}(X) + \tilde{t}_{0,1} \tilde{C}_{0,1}(X),
\]

and where \( t_{\lambda, \lambda+j} \) and \( \tilde{t}_{0,1} \) are independent parameters, \( \delta - \lambda \in \mathbb{N}, \delta - n \leq \lambda, \lambda + j \leq \delta \) and the 1-cocycles \( C_{\lambda, \lambda+j} \) and \( \tilde{C}_{0,1} \) are defined in Table 1. We mention here that the term \( \tilde{t}_{0,1} \tilde{C}_{0,1}(X) \) don’t appear in the expression of \( \mathcal{L}_X^{(1)} \) if \( \delta - n \notin \mathbb{Z}_- \) or \( \delta \notin \mathbb{N}^* \).
3.2 Integrability conditions

Consider the problem of integrability of infinitesimal deformations. Starting with the infinitesimal deformation \( (3.10) \), we look for a formal series

\[
\rho = \rho_0 + \sum_{i=1}^{m} t_i C_i + \sum_{i,j} t_i t_j \rho_{ij}^{(2)} + \cdots ,
\]

(3.13)

where the highest-order terms \( \rho_{ij}^{(2)}, \rho_{ijk}^{(3)} , \ldots \) are linear maps from \( g \) to \( \text{End}(V) \) such that

\[
\rho : g \to \text{End}(V) \otimes \mathbb{C}[t_1, \ldots, t_m]
\]

(3.14)

satisfies the homomorphism condition in any order in \( t_1, \ldots, t_m \).

However, quite often the above problem has no solution. Following \[6, 7\] and \[2\], we will impose extra algebraic relations on the parameters \( t_1, \ldots, t_m \). Let \( R \) be an ideal in \( \mathbb{C}[t_1, \ldots, t_m] \) generated by some set of relations, the quotient

\[
A = \mathbb{C}[t_1, \ldots, t_m]/R
\]

(3.15)

is a local algebra with unity, and one can speak about deformations with base \( A \), see \[6, 7\] for details. The map \( (3.14) \) sends \( g \) to \( \text{End}(V) \otimes A \).

**Example 3.1.** Consider the ideal \( R \) generated by all the quadratic monomials \( t_i t_j \). In this case

\[
A = \mathbb{C} \otimes \mathbb{C}^m
\]

(3.16)

and any deformation is of the form \( (3.10) \). In this case any infinitesimal deformation becomes a deformation with the base \( A \) since \( t_i t_j = 0 \) in \( A \), for all \( i, j = 1, \ldots, m \).

Given an infinitesimal deformation \( (3.10) \), one can always consider it as a deformation with base \( (3.16) \). Our aim is to find \( A \) which is big as possible, or, equivalently, we look for relations on \( t_1, \ldots, t_m \) which are necessary and sufficient for integrability (cf.\[1\], \[2\]).

3.3 Equivalence and the miniversal deformation

The notion of equivalence of deformations over commutative associative algebras has been considered in \[8\].

**Definition 3.1.** Two deformations, \( \rho \) and \( \rho' \) with the same base \( A \) are called equivalent if there exists an inner automorphism \( \Psi \) of the associative algebra \( \text{End}(V) \otimes A \) such that

\[
\Psi \circ \rho = \rho' \text{ and } \Psi(I) = I,
\]

where \( I \) is the unity of the algebra \( \text{End}(V) \otimes A \).

The following notion of miniversal deformation is fundamental. It assigns to a \( g \)-module \( V \) a canonical commutative associative algebra \( A \) and a canonical deformation with base \( A \).

**Definition 3.2.** A deformation \( \rho \) with base \( A \) is called miniversal, if

(i) for any other deformation, \( \rho' \) with base (local) \( A' \), there exists a homomorphism \( \psi : A' \to A \) satisfying \( \psi(1) = 1 \), such that

\[
\rho = (I \otimes \psi) \circ \rho'.
\]
(ii) in the notations of (i), if \( A \) is infinitesimal then \( \psi \) is unique.

If \( \rho \) satisfies only the condition (i), then it is called versal.

The miniversal deformation corresponds to the smallest ideal \( \mathcal{R} \). We refer to [8] for a construction of miniversal deformations of Lie algebras and to [2] for miniversal deformations of \( g \)-modules.

## 4 Second-order Integrability Conditions

Assume that the infinitesimal deformation (3.11) can be integrated to a formal deformation

\[
\mathcal{L}_X = L_X + \mathcal{L}_X^{(1)} + \mathcal{L}_X^{(2)} + \mathcal{L}_X^{(3)} + \cdots
\]

(4.17)

where \( \mathcal{L}_X^{(1)} \) is given by (3.12) and \( \mathcal{L}_X^{(2)} \) is a quadratic polynomial in the parameters \( t_{\lambda,\lambda+k} \) with coefficients in \( D_\delta \). We compute the conditions for the second-order terms \( \mathcal{L}^{(2)} \). Consider the quadratic terms of the homomorphism condition

\[
[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}.
\]

(4.18)

The homomorphism condition (4.18) gives for the second-order terms the following (Maurer-Cartan) equation

\[
\partial \mathcal{L}^{(2)} = -\frac{1}{2} [\mathcal{L}^{(1)}, \mathcal{L}^{(1)}],
\]

(4.19)

so that the right hand side of (4.19) is automatically a 2-cocycle. In our case, we obtain:

\[
\partial \mathcal{L}^{(2)} = -\frac{1}{2} \left[ \sum_{\lambda} \sum_{j=0}^6 t_{\lambda,\lambda+j} C_{\lambda,\lambda+j} + \tilde{t}_{0,1} \tilde{C}_{0,1}, \sum_{\lambda} \sum_{j=0}^6 t_{\lambda,\lambda+j} C_{\lambda,\lambda+j} + \tilde{t}_{0,1} \tilde{C}_{0,1} \right].
\]

(4.20)

Let us consider the 2-cocycles: \( B_{\lambda,\lambda+k} \in Z^2_{\text{diff}}(\text{vect}(1), D_{\lambda,\lambda+k}) \), for \( k = 0, \ldots, 10 \), defined by:

\[
B_{\lambda,\lambda+k} = \sum_{j=2}^k t_{\lambda+j,\lambda+k} t_{\lambda,\lambda+j} [C_{\lambda+j,\lambda+k}, C_{\lambda,\lambda+j}].
\]

(We consider also \( \tilde{C}_{0,1} \) in the expression of \( B_{\lambda,\lambda+k}(X,Y) \) when it is possible: \( \lambda = 0 \) and \( j = 1 \) or \( \lambda + j = 0 \) and \( \lambda + k = 1 \)). Necessary conditions for the integrability of the infinitesimal deformation (3.12) are that any 2-cocycle \( B_{\lambda,\lambda+k}, k = 0, \ldots, 10 \), must be a coboundary:

\[
B_{\lambda,\lambda+k} = \partial b_{\lambda,\lambda+k},
\]

(4.21)

where \( b_{\lambda,\lambda+k} \in C^1(\text{vect}(1), D_{\lambda,\lambda+k}) \). We easily see that \( B_{\lambda,\lambda} = 0 \), so, there are no integrability conditions for \( k = 0 \). In the following, we study, successively, the second-order integrability conditions for \( k = 1, \ldots, 10 \).
**Proposition 4.1.** For $k = 1, 2, 3$, we have the following second-order integrability conditions of the infinitesimal deformation (3.11):

\begin{align*}
t_{0,1}(t_{0,0} - t_{1,1}) - t_{1,1}\tilde{t}_{0,1} &= 0, \quad (k = 1, \lambda = 0) \\
t_{\lambda,\lambda+2}(t_{\lambda,\lambda} - t_{\lambda+2,\lambda+2}) &= 0, \quad (k = 2) \\
t_{\lambda,\lambda+3}(t_{\lambda,\lambda} - t_{\lambda+3,\lambda+3}) &= 0, \quad (k = 3).
\end{align*}

(4.22)

**Proof.** Obviously, for $k = 1$ and for $\lambda \neq 0$, we have $B_{\lambda,\lambda+1} = 0$, since, $C_{\lambda,\lambda+1} = 0$, therefore, there are no conditions in this case. For $\lambda = 0$, we have

$$B_{0,1} = t_{0,1}t_{0,0}[C_{0,1}, C_{0,0}] + t_{1,1}t_{0,1}[C_{1,1}, C_{0,1}] + t_{1,1}\tilde{t}_{0,1}[[C_{0,1}, C_{0,0}]].$$

By a straightforward computation, we show that

$$[C_{1,1}, C_{0,1}] = -[C_{0,1}, C_{0,0}] = [[C_{0,1}, C_{0,0}]] = \Omega_{0,1}.$$

Therefore

$$B_{0,1} = (t_{1,1}t_{0,1} + t_{1,1}\tilde{t}_{0,1} - t_{0,1}t_{0,0})\Omega_{0,1}.$$ 

According to Proposition [2.5], $\Omega_{0,1}$ is a nontrivial 2-cocycle, so, the first integrability condition: $t_{1,1}t_{0,1} + t_{1,1}\tilde{t}_{0,1} - t_{0,1}t_{0,0} = 0$, holds

For $k = 2$, we have

$$B_{\lambda,\lambda+2} = t_{\lambda,\lambda+2}t_{\lambda,\lambda}[C_{\lambda,\lambda+2}, C_{\lambda,\lambda}] + t_{\lambda,\lambda+2}t_{\lambda+2,\lambda+2}[C_{\lambda+2,\lambda+2}, C_{\lambda,\lambda+2}].$$

But, it is easy to show that $[C_{\lambda,\lambda+2}, C_{\lambda,\lambda}] = -[C_{\lambda+2,\lambda+2}, C_{\lambda,\lambda+2}] = \Omega_{\lambda,\lambda+2}$. Thus, we get the integrability condition: $t_{\lambda,\lambda+2}(t_{\lambda,\lambda} - t_{\lambda+2,\lambda+2}) = 0$, since, by Proposition [2.5], $\Omega_{\lambda,\lambda+2}$ is a nontrivial 2-cocycle.

Similarly, for $k = 3$, we obtain $t_{\lambda,\lambda+3}(t_{\lambda,\lambda} - t_{\lambda+3,\lambda+3}) = 0.$

\[\boxed{\text{(4.23)}}\]

**Proposition 4.2.** For $k = 4, 5, 6$, we have the following second-order integrability conditions of the infinitesimal deformation (3.11), where in the first line $\lambda \notin \{0, -3\}$:

\begin{align*}
t_{\lambda,\lambda+4}(t_{\lambda,\lambda} - t_{\lambda+4,\lambda+4}) &= 0, \\
t_{-3,1}(t_{-3,1} - t_{-3,1}) + \frac{1}{10}t_{-3,0}\tilde{t}_{0,1} &= 0, \\
t_{0,4}(t_{0,0} - t_{4,4}) = 0, \\
t_{-3,0}t_{0,5} - 12t_{0,1}t_{1,5} - 12t_{0,1}t_{1,5} + t_{0,2}t_{2,5} &= 0, \\
t_{0,0}t_{0,5} + \frac{2}{5}t_{0,1}t_{1,5} - t_{0,5}t_{5,5} &= 0, \\
t_{-4,1}t_{4,1} + \frac{2}{5}t_{-4,0}\tilde{t}_{0,1} - t_{-4,1}t_{-4,1} &= 0, \\
t_{a_1,a_2,a_3,a_4,a_5} - R_{a_1,a_2,a_3,a_4,a_5} + S_{a_1,a_2,a_3,a_4,a_5} + T_{a_1,a_2,a_3,a_4,a_5} = 0.
\end{align*}

(4.23a)

where $R_{i}$, $S_{i}$ and $T_{i}$ are some constants which we leave out their explicit expressions as they are so complicated.

**Proof.** 1) First, we show that $[C_{\lambda+2,\lambda+4}, C_{\lambda,\lambda+2}] = \partial b_{\lambda,\lambda+k}$ where

$$b_{\lambda,\lambda+4}(X)f = \frac{2}{5}(\lambda - 1)X^{(5)}f - 2X^{(4)}f'. $$

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For $\lambda \notin \{0, -3\}$, the 2-cocycle $B_{\lambda, \lambda+4}$ is defined by

$$B_{\lambda, \lambda+4} = t_{\lambda, \lambda+4}t_{\lambda+4}\big[ C_{\lambda, \lambda+4}, C_{\lambda, \lambda} \big] + t_{\lambda, \lambda+4}t_{\lambda+4, \lambda+4}[C_{\lambda+4, \lambda+4}, C_{\lambda, \lambda+4}]$$

$$+ t_{\lambda, \lambda+2}t_{\lambda+2, \lambda+4}[C_{\lambda+2, \lambda+4}, C_{\lambda, \lambda+2}].$$

We check that $\big[ C_{\lambda, \lambda+4}, C_{\lambda, \lambda} \big] = -\big[ C_{\lambda+4, \lambda+4}, C_{\lambda, \lambda+4} \big] = \Omega_{\lambda, \lambda+4}$. Thus, we get the following condition:

$$t_{\lambda, \lambda+4}(t_{\lambda, \lambda} - t_{\lambda+4, \lambda+4}) = 0.$$

For $\lambda = 0$, the cup-products $[C_{1,4}, C_{0,1}]$ and $[C_{1,4}, \tilde{C}_{0,1}]$ appear in the expression of $B_{0,4}$. But, we check that $[C_{1,4}, \tilde{C}_{0,1}] = \frac{1}{10}\Omega_{0,4}$ and $[C_{1,4}, \tilde{C}_{0,1}] = \partial \tilde{b}_{0,4}$ where

$$\tilde{b}_{0,4}(X)f = -\frac{1}{6}X(5)f + X''f(3).$$

For $\lambda = -3$, the cup-products $[C_{0,1}, C_{-3,0}]$ and $[\tilde{C}_{0,1}, C_{-3,0}]$ appear in the expression of $B_{-3,1}$. We check that $[\tilde{C}_{0,1}, C_{-3,0}] = -\frac{1}{10}\Omega_{-3,1}$ and $[C_{1,4}, \tilde{C}_{0,1}] = \partial \tilde{b}_{-3,1}$ where

$$\tilde{b}_{-3,1}(X)f = -\frac{3}{10}X(5)f - \frac{1}{2}X''f'.$$

Thus, for $\lambda \in \{0, -3\}$, we obtain the following integrability conditions:

$$t_{-3,1}(t_{-3, -3} - t_{1, 1}) - \frac{1}{10}t_{-3, 0}t_{0, 1} = t_{0, 4}(t_{0, 0} - t_{4, 4}) + \frac{1}{10}t_{0, 1}t_{1, 4} = 0.$$

2) Now, if $\lambda \notin \{0, -4\}$ the 2-cocycle $B_{\lambda, \lambda+5}$ is defined by

$$B_{\lambda, \lambda+5} = t_{\lambda, \lambda+2}t_{\lambda+2, \lambda+5}[C_{\lambda+2, \lambda+5}, C_{\lambda, \lambda+2}] + t_{\lambda, \lambda+3}t_{\lambda+3, \lambda+5}[C_{\lambda+3, \lambda+5}, C_{\lambda, \lambda+3}].$$

But, by a direct computation, we show that

$$[C_{\lambda+2, \lambda+5}, C_{\lambda, \lambda+2}] = \partial b_{\lambda, \lambda+5} \quad \text{and} \quad [C_{\lambda+3, \lambda+5}, C_{\lambda, \lambda+3}] = \partial \tilde{b}_{\lambda, \lambda+5}$$

where, for $\lambda \neq -2$,

$$b_{\lambda, \lambda+5}(X)(f) = \frac{-1}{\lambda^2 + 6\lambda + 8} \left( \frac{7}{30}(\lambda + 4)X(6)f + \frac{1}{10}(10\lambda^2 + 39\lambda - 4)X(5)f' 
+ \frac{1}{12}(4\lambda^2 + 17\lambda + 4)X(4)f'' + \frac{1}{33}(3\lambda^2 + 11\lambda - 4)X(3)f(3) \right),$$

$$\tilde{b}_{\lambda, \lambda+5}(X)(f) = \frac{-1}{\lambda^2 + 6\lambda + 8} \left( -\frac{7}{30}X(6)f + \frac{21}{16}X(5)f' + (\lambda + \frac{11}{2})X(4)f'' + (\lambda + \frac{13}{3})X(3)f(3) \right)$$

and

$$b_{-2,3}(X)(f) = -\frac{1}{12}X(5)f' - \frac{1}{6}X(4)f'' - \frac{3}{2}X(3)f(3); \quad \tilde{b}_{-2,3}(X)(f) = -\frac{7}{12}X(5)f' - \frac{7}{6}X(4)f'' - \frac{7}{2}X(3)f(3).$$

For $\lambda = 0$, recall that the cohomology space $H^3_{\text{diff}}(\text{vect}(1), D_{0,5})$ is spanned by the cohomology classes of the 2-cocycles $\Omega_{0,5} = [C_{2,5}, C_{0,2}]$ and $\tilde{\Omega}_{0,5} = [C_{5,5}, C_{0,5}]$ (see Proposition 2.6). Moreover, in this case, we have

$$B_{0,5} = t_{0,1}t_{1,5}[C_{1,5}, C_{0,1}] + \tilde{t}_{0,1}t_{1,5}[C_{1,5}, \tilde{C}_{0,1}] + t_{0,0}t_{0,5}[C_{0,5}, C_{0,0}].$$
But, by direct computation, we check that

\[ [C_{1,5}, C_{0,0}] = -12 \Omega_{0,5} + \partial b_{0,5}, \]
\[ [C_{1,5}, \tilde{C}_{0,1}] = -12 \Omega_{0,5} + \frac{2}{5} \Omega_{0,5} + \partial \tilde{b}_{0,5}, \]
\[ [C_{0,5}, C_{0,0}] = 30 \Omega_{0,5} - \tilde{\Omega}_{0,5} + \partial \tilde{b}_{0,5}, \]

where

\[ b_{0,5}(X)(f) = -X(6)f - 55 X(4)f'' - 20 X' f(5), \]
\[ \tilde{b}_{0,5}(X)(f) = -X(6)f - 45 X(4)f'' - 15 X'' f(4) + \frac{1}{5} X' f(5), \]
\[ \tilde{b}_{0,5}(X)(f) = 3 X(6)f + 135 X(4)f'' + 45 X'' f(4). \]

Similarly, for \( \lambda = -4 \), the cohomology space \( H^2_{\text{diff}}(\text{vect}(1), D_{-4,1}) \) is spanned by the cohomology classes of \( \Omega_{-4,1} = [C_{-1,1}, C_{-4,-1}] \) and \( \tilde{\Omega}_{-4,1} = C_{1,1}, C_{-4,1} \), and we check that

\[ [\tilde{C}_{0,1}, C_{-4,0}] = -12 \Omega_{-4,1} + \frac{2}{5} \tilde{\Omega}_{-4,1} + \partial b_{-4,1}, \]
\[ [C_{0,1}, C_{-4,0}] = -12 \Omega_{-4,1} + \partial b_{-4,1}, \]
\[ [C_{-4,1}, C_{-4,-4}] = 30 \Omega_{-4,1} - \tilde{\Omega}_{-4,1} + \partial \tilde{b}_{-4,1} \]

where

\[ \tilde{b}_{-4,1}(X)(f) = -\frac{17}{13} X(5)f' - \frac{30}{13}(X(4)f'' + X'' f(4)) - \frac{30}{13} X(3) f(3) + \frac{1}{5} X' f(5), \]
\[ b_{-4,1}(X)(f) = -\frac{17}{13} X(5)f' - \frac{30}{13}(X(4)f'' + X'' f(4)) - \frac{30}{13} X(3) f(3), \]
\[ \tilde{b}_{-4,1}(X)(f) = \frac{36}{13} X(5)f' + \frac{435}{32}(X(4)f'' + X'' f(4)) + \frac{345}{20} X(3) f(3). \]

Thus, we obtain the integrability conditions corresponding to the case \( k = 5 \).

3) If \( \lambda \notin \{a_1, a_2\} \) then \( B_{\lambda, \lambda+6} \) is coboundary. More precisely, \( B_{\lambda, \lambda+6} \) is defined by

\[ B_{\lambda, \lambda+6} = t_{\lambda, \lambda+2} t_{\lambda+2, \lambda+6} [C_{\lambda+2, \lambda+6}, C_{\lambda+2}] + t_{\lambda, \lambda+3} t_{\lambda+3, \lambda+6} [C_{\lambda+3, \lambda+6}, C_{\lambda+3}] + t_{\lambda, \lambda+4} t_{\lambda+4, \lambda+6} [C_{\lambda+4, \lambda+6}, C_{\lambda+4}], \]

but, we show that

\[ [C_{\lambda+2, \lambda+6}, C_{\lambda+2}] = \partial b_{\lambda, \lambda+6}, \]
\[ [C_{\lambda+3, \lambda+6}, C_{\lambda+3}] = \partial \tilde{b}_{\lambda, \lambda+6} \text{ and } [C_{\lambda+4, \lambda+6}, C_{\lambda+4}] = \partial \tilde{b}_{\lambda, \lambda+6} \]

where

\[ b_{\lambda, \lambda+6}(X)(f) = \frac{1}{11} \lambda(2\lambda^2 + 10\lambda + 3) \left( (-12 - 9\lambda + 97\lambda^2 + 90\lambda^3 + 24\lambda^4) X(6)f \\
+ (-72 - 404\lambda - 41\lambda^2 + 127\lambda^3 + 60\lambda^4) X(5)f'' \\
+ (-180 - 163\lambda + 83\lambda^2 + 160\lambda^3 + 80\lambda^4) X(4)f(3) \\
+ (-240 - 922\lambda - 13\lambda^2 + 155\lambda^3 + 60\lambda^4) X(3)f(4) \\
+ (-180 - 56\lambda - 15\lambda^2 + 90\lambda^3 + 24\lambda^4) X'' f(5) \right), \]
\[ \tilde{b}_{\lambda, \lambda+6}(X)(f) = \frac{1}{11} \lambda(2\lambda^2 + 10\lambda + 3) \left( (-12\lambda^2 - \lambda) X(6)f - 2(44\lambda^2 + 10\lambda) X(5)f'' \\
+ (-68\lambda^2 - 32\lambda) X(4)f(3) + (12\lambda^2 + 4\lambda) X'' f(5) \right), \]
\[ \tilde{b}_{\lambda, \lambda+6}(X)(f) = \frac{1}{11} \lambda(2\lambda^2 + 10\lambda + 3) \left( (-9\lambda^2 - 118\lambda^3) X(6)f \\
- (16\lambda + 183\lambda^2 + 118\lambda^3) X(5)f'' - (5\lambda + 195\lambda^2 + 580\lambda^3) X(4)f(3) \\
- (58\lambda + 323\lambda^2 + 435\lambda^3) X(3)f(4) - (103\lambda + 377\lambda^2 + 174\lambda^3) X'' f(5) \right). \]
For \( \lambda = a_i \), the cohomology space \( H^2_{\text{diff}}(\text{vect}(1), D_{a_i,a_i+6}) \) is spanned by the cohomology classes of the 2-cocycles \( \Omega_{a_i,a_i+6} = [C_{a_i+3,a_i+6}, C_{a_i,a_i+3}] \) and \( \tilde{\Omega}_{a_i,a_i+6} = [C_{a_i+6,a_i+6}, C_{a_i,a_i+6}] \). Moreover, in this case, we have

\[
B_{a_i,a_i+6} = t_{a_i,a_i+6}[C_{a_i+3,a_i+6}, C_{a_i,a_i}] + t_{a_i,a_i+2}t_{a_i+2,a_i+6}[C_{a_i+6,a_i+2}, C_{a_i,a_i+2}]
+ t_{a_i,a_i+3}t_{a_i+3,a_i+6}[C_{a_i+3,a_i+6}, C_{a_i,a_i+3}] + t_{a_i,a_i+4}t_{a_i+4,a_i+6}[C_{a_i+4,a_i+6}, C_{a_i,a_i+4}].
\]

But, by direct computation, we show that

\[
\begin{align*}
[C_{a_i,a_i+6}, C_{a_i,a_i}] &= -R_i \Omega_{a_i,a_i+6} - \tilde{\Omega}_{a_i,a_i+6} + \partial b_{a_i,a_i+6}, \\
[C_{a_i+2,a_i+6}, C_{a_i,a_i+2}] &= S_i \Omega_{a_i,a_i+6} + \partial \tilde{b}_{a_i,a_i+6}, \\
[C_{a_i+4,a_i+6}, C_{a_i,a_i+4}] &= -T_i \Omega_{a_i,a_i+6} + \partial \tilde{b}_{a_i,a_i+6}
\end{align*}
\]

where the maps \( b_{a_i,a_i+6}, \tilde{b}_{a_i,a_i+6} \) and \( \tilde{\tau}_{a_i,a_i+6} \) are all proportional to the map \( b \) defined by

\[
b(X)(f) = X^{(5)} f''.
\]

We omit here the explicit expressions of the scalar factors because they are too complicated. Thus, we obtain the integrability conditions corresponding to the case \( k = 6 \).

**Proposition 4.3.** For \( k = 7, 8, 9, 10 \), we have the following second-order integrability conditions of the infinitesimal deformation \((x, \Omega)\), where in the first line \( \lambda \notin \{-9, -4\} \) and in the fourth \( \lambda \notin \{-7, 0, a_i - 2, -4, -\frac{7+\sqrt{39}}{2}\} \):

\[
\begin{align*}
(2\lambda + 13)t_{\lambda,3,\lambda+3} &+ (1 - 2\lambda)t_{\lambda,6,\lambda+4} = 0, \\
45 t_{-2,0,t_0+5} - 36 t_{-2,1,t_1+5} &- 2t_{-2,2,t_2,5} = 0, \\
20 t_{-4,1,3} + 36 t_{-4,0,t_0+3} &+ 45 t_{-4,1,1} = 0, \\
t_{\lambda,\lambda+4}t_{\lambda,4,\lambda+8} &+ 60 t_{-7,3,t_3+1} + t_{-7,4,t_4+1} = 0, \\
-60 t_{0,4,t_4+8} &+ t_{0,5,5,8} = 0, \\
4 t_{-4,0,t_4} - t_{-4,1,1,4} &+ \eta_i t_{a_i-2,a_i,t_{a_i,a_i}+6} + \theta_i t_{a_i-2,a_i+2,t_{a_i+2,a_i}+6} = 0, \\
\mu_1 t_{a_i,a_i+6}t_{a_i+6,a_i+8} &+ \nu_1 t_{a_i,a_i+4}t_{a_i+4,a_i+8} = 0, \\
t_{a_i-3,a_i,t_{a_i,a_i}+6} &+ t_{-3,4,t_4-1} = 0, \\
t_{-3,4,t_4-1} &+ t_{-4,0,t_0+5} - t_{-4,1,1,5} = 0, \\
t_{0,5,5,9} &+ t_{a_i-4,a_i,t_{a_i,a_i}+10} = 0, \\
t_{a_i-4,a_i+1} &+ t_{a_i,a_i+6} = 0,
\end{align*}
\]

where

\[
\begin{align*}
\eta_i &= 7(76437 + 53739\sqrt{19}), & \theta_i &= 64(1160123 + 30689\sqrt{19}), \\
\mu_1 &= 8947638 + 205273\sqrt{19}, & \nu_1 &= 96(474174 + 108783\sqrt{19})
\end{align*}
\]

(\( \eta_2, \theta_2, \mu_2, \nu_2 \) are the conjugates respectively of \( \eta_1, \theta_1, \mu_1, \nu_1 \)).
Proof. 1) If \( \lambda \notin \{0, -2, -4, -6\} \), we have

\[
B_{\lambda, \lambda+7} = t_{\lambda+3, \lambda+7} + t_{\lambda+4, \lambda+7}[C_{\lambda+3, \lambda+7}, C_{\lambda, \lambda+3}] + t_{\lambda+4, \lambda+8}[C_{\lambda+4, \lambda+7}, C_{\lambda, \lambda+4}],
\]

but we show that

\[
[C_{\lambda+4, \lambda+7}, C_{\lambda, \lambda+4}] = \frac{1-2\lambda}{2\lambda+13} \Omega_{\lambda, \lambda+7} + \partial b_{\lambda, \lambda+7}.
\]

So, we obtain the second-order integrability conditions for \( k = 7 \) and for generic \( \lambda \). Besides, we study, as before, singular values of \( \lambda \) and then we obtain the corresponding second-order integrability conditions. More precisely, the map \( B_{\lambda, \lambda+7} \) has the following form:

\[
B_{\lambda, \lambda+7} = \omega_{\lambda, \lambda+7}^1(t) \Omega_{\lambda, \lambda+7} + \omega_{\lambda, \lambda+7}^2(t) \partial b_{\lambda, \lambda+7},
\]

where

\[
\omega_{\lambda, \lambda+7}^1(t) = \begin{cases} 
(2\lambda + 13) t_{\lambda+3, \lambda+7} + (1 - 2\lambda) t_{\lambda+4, \lambda+7} & \text{if } \lambda \notin \{0, -2, -4, -6\} \\
-\frac{9}{4} t_{-2,0} t_{0,5} + \frac{9}{4} t_{-1,1} t_{1,5} + t_{-2,2} t_{2,5} & \text{if } \lambda = -2, \\
\frac{5}{4} t_{-4,1} t_{1,3} & \text{if } \lambda = -4, \\
0 & \text{if } \lambda = 0, -6.
\end{cases}
\]

Hereafter we omit the expressions of the maps \( b_{\lambda, \lambda+k} \) and \( \omega_{\lambda, \lambda+k}^2 \) as they are too long.

2) Now, for \( k = 8 \) and \( 2\lambda \neq -7 \pm \sqrt{39} \), the spaces \( H^2(\text{vect}(1), D_{\lambda, \lambda+8}) \) are spanned by the cohomology classes of the 2-cocycle \( \Omega_{\lambda, \lambda+8} = [C_{\lambda+4, \lambda+8}, C_{\lambda, \lambda+4}] \) and generically we have

\[
B_{\lambda, \lambda+8} = t_{\lambda+4, \lambda+8}[C_{\lambda+4, \lambda+8}, C_{\lambda, \lambda+4}].
\]

But, for singular values of \( \lambda \), other cup-products appear in the expression of \( B_{\lambda, \lambda+8} \). More precisely, we show that

\[
B_{\lambda, \lambda+8} = \omega_{\lambda, \lambda+8}^1(t) \Omega_{\lambda, \lambda+8} + \omega_{\lambda, \lambda+8}^2(t) \partial b_{\lambda, \lambda+8},
\]

where

\[
\omega_{\lambda, \lambda+8}^1(t) = \begin{cases} 
\frac{t_{\lambda+4, \lambda+8} t_{\lambda, \lambda+4}}{t_{-3,1} t_{-7,3} + \frac{1}{60} t_{-4,1} t_{-7,4}} & \text{if } \lambda \neq 0, -7, -4, -\frac{7 + \sqrt{39}}{2}, a_i, a_i - 2 \\
t_{4,7} t_{0,4} - \frac{1}{50} t_{5,8} & \text{if } \lambda = -7, \\
t_{-4,0} t_{0,4} - \frac{1}{4} t_{-4,1} t_{1,4} & \text{if } \lambda = 0, \\
\frac{\mu_i}{a_i} t_{a_i+6, a_i+8 t_{a_i, a_i+6} + t_{a_i+4, a_i+8 t_{a_i, a_i+4}} & \text{if } \lambda = a_i, \\
\frac{\mu_i}{a_i} t_{a_i+6 t_{a_i-2 a_i} + t_{a_i+2 a_i+6 t_{a_i-2 a_i+2}} & \text{if } \lambda = a_i - 2, \\
0 & \text{if } \lambda = -7 + \sqrt{39}. \end{cases}
\]

2) For \( k = 9 \), the maps \( B_{\lambda, \lambda+9} \) exist only for some singular values of \( \lambda \). More precisely, we show that, for \( \lambda \neq -4 \),

\[
B_{\lambda, \lambda+9} = \omega_{\lambda, \lambda+9}^1(t) \Omega_{\lambda, \lambda+9},
\]

15
where

\[
\omega_{\lambda,\lambda+9}(t) = \begin{cases} 
0 & \text{if } \lambda \neq a_i, a_i - 4, \\
t_{-8,-4}t_{-4,1} & \text{if } \lambda = -8, \\
t_{0,5}t_{5,9} & \text{if } \lambda = 0, \\
t_{a_i-3,a_i}t_{a_i,a_i+6} & \text{if } \lambda = a_i - 3, \\
t_{a_i,a_i+6}t_{a_i+6,a_i+9} & \text{if } \lambda = a_i
\end{cases}
\]

and

\[
B_{-4,5} = (t_{-4,0}t_{0,5} - t_{-4,1}t_{1,5})\Omega_{-4,5} + t_{-4,1}t_{1,5}b_{-4,5}.
\]

3) Finally, we show that

\[
B_{\lambda,\lambda+10} = \omega_{\lambda,\lambda+10}(t)\Omega_{\lambda,\lambda+10},
\]

where

\[
\omega_{\lambda,\lambda+10}(t) = \begin{cases} 
0 & \text{if } \lambda \neq a_i, a_i - 4, \\
t_{a_i,a_i+6}t_{a_i+6,a_i+10} & \text{if } \lambda = a_i, \\
t_{a_i-4,a_i}t_{a_i,a_i+6} & \text{if } \lambda = a_i - 4.
\end{cases}
\]

Now, in the following theorem, we recapitulate the second-order integrability conditions for the infinitesimal deformation (3.12). More precisely, we give the necessary conditions to have the second term \( \mathcal{L}^{(2)} \) of (3.12). We give all conditions of second order, but, any space \( \mathcal{S}_0^n \) is concerned only by relations between monomials \( t_{\lambda,\lambda+j}t_{\lambda+j,\lambda+k} \), where \( \delta - n \leq \lambda, \lambda + k \leq \delta \) and \( 0 \leq j \leq k \leq 10 \).

Our main result in this paper is the following

**Theorem 4.4.** The conditions [4.22], [4.23] and [4.24] are necessary and sufficient for second-order integrability of the infinitesimal deformation (3.12).

Proof. Of course, these conditions are necessary as it was shown in Propositions 4.1, 4.2 and 4.3. Now, under these conditions, the second term \( \mathcal{L}^{(2)} \) of the \( \mathfrak{sl}(2) \)-trivial infinitesimal deformation (3.12) is a solution of the Maurer-Cartan equation (4.20). This solution is defined up to a 1-coboundary and it has been shown in [3.2] that different choices of solutions of the Maurer-Cartan equation correspond to equivalent deformations. Thus, we can always choose

\[
\mathcal{L}^{(2)} = \frac{1}{2} \sum_{\lambda} t_{\lambda,\lambda+2}t_{\lambda+2,\lambda+4}b_{\lambda,\lambda+4} + \frac{1}{2}t_{0,1}t_{1,4}\tilde{b}_{0,4} + \frac{1}{2}t_{-3,0}t_{0,1}\tilde{b}_{-3,1} + \frac{1}{2} \sum_{\lambda} t_{\lambda,\lambda+2\lambda+2,\lambda+5}b_{\lambda,\lambda+5} + \frac{1}{2} \sum_{\lambda} t_{\lambda,\lambda+3\lambda+3,\lambda+5}\tilde{b}_{\lambda,\lambda+5}
\]

\[
+ \frac{1}{2} \sum_{\lambda=0,4} t_{\lambda,\lambda+5}\tilde{b}_{\lambda,\lambda+5} + \frac{1}{2} \sum_{\lambda} t_{\lambda,\lambda+2\lambda+2,\lambda+6}b_{\lambda,\lambda+6} + \frac{1}{2} \sum_{\lambda} t_{\lambda,\lambda+3\lambda+3,\lambda+6}\tilde{b}_{\lambda,\lambda+6} + \frac{1}{2} \sum_{\lambda} t_{\lambda,\lambda+4\lambda+4,\lambda+6}\tilde{b}_{\lambda,\lambda+6} + \frac{1}{2} \sum_{\lambda} \omega_{\lambda,\lambda+7}(t)b_{\lambda,\lambda+7} + \frac{1}{2} \sum_{\lambda} \omega_{\lambda,\lambda+8}(t)b_{\lambda,\lambda+8} + t_{-4,1}t_{1,5}b_{-4,5}.
\]

Of course, any \( t_{\lambda,\lambda+k} \) appear in the expressions of \( \mathcal{L}^{(1)} \) or \( \mathcal{L}^{(2)} \) if and only if \( \delta - \lambda \) and \( k \) are integers satisfying \( \delta - n \leq \lambda, \lambda + k \leq \delta \). Theorem 4.4 is proved. \( \square \)
Remark 4.5. There are no second-order conditions for integrability in the following cases:

i) \( k = 0 \), for all \( \lambda \),

ii) \( k = 1 \) and \( \lambda \neq 0 \),

iii) \( k = 5 \) and \( \lambda \neq 0, -4 \),

iv) \( k = 6 \) and \( \lambda \neq a_i \),

v) \( k = 7 \) and \( \lambda = 0, -6 \),

vi) \( k = 8 \) and \( \lambda = \frac{-7 \pm \sqrt{37}}{2} \),

vii) \( k = 9 \) and \( \lambda \neq a_i, 0, -4, -8, a_i - 3 \),

viii) \( k = 10 \) and \( \lambda \neq a_i, a_i - 4 \).

5 Examples

The second-order conditions given in Theorem (4.4) are not, in general, sufficient, but they are in some cases. In this section we give examples of symbol spaces \( S^\delta_n \) for which the corresponding second-order integrability conditions are also sufficient and then we describe completely the formal deformations of these spaces. Finally, we consider an example for which the second-order integrability conditions are not sufficient, but we exhibit the higher-order integrability conditions and then we describe also, in this case, the formal deformations.

Example 5.1. Consider \( S^2_3 \). The infinitesimal deformation of the vect(1)-action on \( S^2_3 \) is of the form \( L_X + \mathcal{L}_X^{(1)} \), where \( L_X \) is the Lie derivative of \( S^2_3 \) along the vector field \( X \frac{d}{dx} \) defined by (1.1), and

\[
\mathcal{L}_X^{(1)} = \sum_{\lambda=1}^{3} \sum_{j=0}^{3-\lambda} t_{\lambda,\lambda+j} C_{\lambda,\lambda+j}(X) = (t_{1,1} C_{1,1} + t_{1,3} C_{1,3} + t_{2,2} C_{2,2} + t_{3,3} C_{3,3})(X). \tag{5.27}
\]

We have the unique equation:

\[
t_{1,3}(t_{1,1} - t_{3,3}) = 0 \tag{5.28}
\]

as necessary integrability condition of this infinitesimal deformation. The following proposition shows that this condition is also sufficient.

Proposition 5.1. There are two deformations of the vect(1)-action on \( S^2_3 \) with three independent parameters given by:

\[
\mathcal{L}_X = L_X + t_1 (C_{1,1}(X) + C_{3,3}(X)) + t_2 C_{2,2}(X) + t_3 C_{1,3}(X), \tag{5.29}
\]
or

\[
\mathcal{L}_X = L_X + t_1 C_{1,1}(X) + t_2 C_{2,2}(X) + t_3 C_{3,3}(X). \tag{5.30}
\]
Proof. We consider the infinitesimal deformation \( (5.27) \) and then we consider solutions of \( (5.28) \). The first solution is: \( t_{1,1} = t_{3,3} \), we put \( t_1 = t_{1,1} = t_{3,3} \), \( t_2 = t_{2,2} \) and \( t_3 = t_{1,3} \) and then we obtain \( (5.29) \). The second solution is: \( t_{1,3} = 0 \), we put \( t_1 = t_{1,1} \), \( t_2 = t_{2,2} \) and \( t_3 = t_{3,3} \) and then we obtain the second deformation. The solution \( L^{(2)} \) of \( (1.19) \) can be chosen identically zero. Choosing the highest-order terms \( L^{(m)} \) with \( m \geq 3 \), also identically zero, one obviously obtains a deformation (which is of order 1 in \( t \)). \( \square \)

**Example 5.2.** Now consider \( S_3^3 \). In this case we have

\[
L_X^{(1)} = t_{1,1}C_{1,1}(X) + t_{1,2}C_{1,3}(X) + t_{1,4}C_{1,4}(X) + t_{2,2}C_{2,2}(X) \\
+ t_{2,4}C_{2,4}(X) + t_{3,3}C_{3,3}(X) + t_{4,4}C_{4,4}(X). \tag{5.31}
\]

**Proposition 5.2.** The space \( S_3^3 \) admits eight different formal deformations with four independent parameters. They are all equivalent to infinitesimal ones.

Proof. The following equations are the necessary integrability conditions of the infinitesimal deformation \( L_X + L_X^{(1)} \) (they are also sufficient):

\[
t_{1,3}(t_{1,1} - t_{3,3}) = 0, \\
t_{2,4}(t_{2,2} - t_{4,4}) = 0, \\
t_{1,4}(t_{1,1} - t_{4,4}) = 0. \tag{5.32}
\]

There are eight solutions for these equations, so \( S_3^3 \) admits eight different deformations with four independent parameters. Like in the first example all these deformations are equivalent to infinitesimal ones. \( \square \)

**Example 5.3.** Consider \( S_3^3 \).

\[
L_X^{(1)} = t_{0,0}C_{0,0}(X) + t_{0,1}C_{0,1}(X) + \tilde{t}_{0,1}C_{0,1}(X) + t_{0,2}C_{0,2}(X) \\
+ t_{0,3}C_{0,3}(X) + t_{1,1}C_{1,1}(X) + t_{2,2}C_{2,2}(X) + t_{3,3}C_{3,3}(X). \tag{5.33}
\]

The integrability conditions are

\[
t_{0,1}(t_{0,0} - t_{1,1}) = t_{1,1}\tilde{t}_{0,1}, \\
t_{0,2}(t_{0,0} - t_{2,2}) = 0, \\
t_{0,3}(t_{0,0} - t_{3,3}) = 0. \tag{5.34}
\]

In this case also these conditions are sufficient and any formal deformation of \( S_3^3 \) is equivalent to infinitesimal one satisfying \( (5.34) \).

One can construct a great number of examples of deformations of \( S_3^3 \) with 4 (or less) independent parameters, but the deformation

\[
L_X = L_X + L_X^{(1)}
\]

is the miniversal one of \( S_3^3 \) with base \( \mathcal{A} = \mathbb{C}[t]/\mathcal{R} \), where \( \mathbb{C}[t] = \mathbb{C}[t_{0,0}, t_{0,1}, \ldots] \) and \( \mathcal{R} \) is the ideal generated by

\[
t_{0,1}(t_{0,0} - t_{1,1}) - t_{1,1}\tilde{t}_{0,1}, \quad t_{0,2}(t_{0,0} - t_{2,2}) \quad \text{and} \quad t_{0,3}(t_{0,0} - t_{3,3}).
\]
Example 5.4. Consider $\mathcal{S}_5^4$. In this case we have

$$L^{(1)}_X = t_{1,1}C_{1,1}(X) + t_{1,3}C_{1,3}(X) + t_{1,4}C_{1,4}(X) + t_{1,5}C_{1,5}(X) + t_{2,2}C_{2,2}(X) + t_{2,4}C_{2,4}(X) + t_{2,5}C_{2,5}(X) + t_{3,3}C_{3,3}(X) + t_{3,5}C_{3,5}(X) + t_{4,4}C_{4,4}(X) + t_{5,5}C_{5,5}(X).$$  \hspace{1cm} (5.35)

Proposition 5.3. Any formal deformation of $\mathcal{S}_5^4$ is equivalent to a polynomial one with degree $\leq 3$.

Proof. The second-order integrability conditions in this case are

$$t_{1,3}(t_{1,1} - t_{3,3}) = 0, \\
t_{2,4}(t_{2,2} - t_{4,4}) = 0, \\
t_{3,5}(t_{3,3} - t_{5,5}) = 0, \\
t_{1,4}(t_{1,1} - t_{4,4}) = 0, \\
t_{2,5}(t_{2,2} - t_{5,5}) = 0, \\
t_{1,5}(t_{1,1} - t_{5,5}) = 0. \hspace{1cm} (5.36)$$

Under these conditions the second-order term $L^{(2)}_X : \mathcal{F}_1 \rightarrow \mathcal{F}_5$ is defined by

$$L^{(2)}_X f = \frac{1}{2} t_{1,3} t_{3,5} b_{1,5}(X) f = -t_{1,3} t_{3,5} X^{(4)} f'.$$

The third-order term $L^{(3)}_X : \mathcal{F}_1 \rightarrow \mathcal{F}_5$ is a solution of the third-order Maurer-Cartan equation:

$$\partial L^{(3)} = -\frac{1}{2} \sum_{i+j=3} [L^{(i)}, L^{(j)}]. \hspace{1cm} (5.37)$$

We compute the right side of the equation (5.37), so this equation becomes

$$\partial L^{(3)}(X,Y)f = \frac{1}{2} t_{1,1} t_{1,3} t_{3,5} X^{(4)} Y'' f + \frac{1}{2} (t_{1,1} - t_{5,5}) t_{1,3} t_{3,5} X^{(4)} Y' f' - (X \leftrightarrow Y).$$

Under the following third-order integrability condition:

$$(t_{1,1} - t_{5,5}) t_{1,3} t_{3,5} = 0, \hspace{1cm} (5.38)$$

the equation (5.37) has a solution:

$$L^{(3)}_X f = \frac{1}{5} t_{1,1} t_{1,3} t_{3,5} X^{(5)} f.$$

Now, we compute the fourth-order term $L^{(4)}_X : \mathcal{F}_1 \rightarrow \mathcal{F}_5$. It is a solution of:

$$\partial L^{(4)} = -\frac{1}{2} \sum_{i+j=4} [L^{(i)}, L^{(j)}] = -\frac{1}{2} t_{1,1} [C_{1,1}, L^{(3)}] - \frac{1}{2} t_{5,5} [L^{(3)}, C_{5,5}]. \hspace{1cm} (5.39)$$

It is easy to see that, under the conditions (5.36) and (5.38), the right hand side of (5.39) is identically zero. Thus, the solution $L^{(4)}$ of (5.39) can be chosen identically zero. Choosing the highest-order terms $L^{(m)}$ with $m \geq 5$, also identically zero, one obviously obtains a deformation (which is of order 3 in $t$).
Now, by studying the equations (5.36) and (5.38), we can see that, up to equivalence, the Lie derivative on $S_5^4$ admits a formal deformation with seven independent parameters, this deformation corresponds to the solution $t_{i,i} = t_{j,j}$ of the equations (5.36) and (5.38). A great number of non-trivial deformations with $k$ independent parameters can be constructed if $k < 7$, each deformation corresponds to a solution to equations (5.36) and (5.38). All these deformations are polynomial of order equal or less than 3 in $t$.

**Remark 5.5.** In the previous four examples we obtain the same results if we substitute $S_{\lambda+n}^m$ for $S_n^m$ where $\lambda \in \mathbb{R}_+^*$.  

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