Neutrino current in a gravitational plane wave collision background

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Abstract

The behaviour of a massless Dirac field on a general spacetime background representing two colliding gravitational plane waves is discussed in the Newman-Penrose formalism. The geometrical properties of the neutrino current are analysed and explicit results are given for the special Ferrari-Ibañez solution.

1 Introduction

Exact solutions of the Einstein equations representing colliding gravitational plane waves have been discussed extensively in the literature and the status of research on this topic is still best summarized in a monograph by Griffith[1] published in 1991.

The interest in such solutions is mainly due to the possibility they offer to understand better some nonlinear features of the gravitational interaction. In general, the spacetime geometry associated with two colliding gravitational plane waves is very rich: for example either a spacetime singularity[2] or a Killing-Cauchy horizon[3, 4, 5] can result from the nonlinear wave interaction. In general, such spacetimes contain four regions: a Minkowski region, representing the initially flat situation before the passage of the two oppositely directed plane waves, two Petrov type N regions, corresponding to the waves before the interaction, and an interaction region, generally of Petrov type I. Two commuting spacelike Killing vectors are always present, associated with the plane symmetry assumed for the two colliding waves.

Here we consider a massless Dirac field (neutrino) interacting with the colliding plane waves. Explicit results are given for the degenerate solutions found by Ferrari and Ibañez[4], in
which the interaction region of the colliding waves propagating along the common z direction is of Petrov type D, with a Killing-Cauchy horizon formed at a finite distance from the collision plane, after a finite time from the instant of collision. For this spacetime a careful analysis of timelike geodesics has only recently been performed[6].

Some special features of the neutrino current are discussed in detail, and previous work of Dorca and Verdaguer[7, 8] and Yurtsever[9] valid for a scalar field interacting with gravitational waves is extended here to spin 1/2 massless particles.

2 Colliding waves: the Dirac equation

The most general form of the spacetime metric representing two colliding waves with parallel polarization can be written as

\[ ds^2 = 2 g_{12} \, dx_1 \, dx_2 - g_{33} \, (dx_3)^2 - g_{44} \, (dx_4)^2 \, , \]

where \( g_{12}, g_{33}, g_{44} \) are real functions of the null coordinates \((x_1, x_2)\). The complete spacetime description of the collision must patch together two single-wave regions (the approaching waves) plus a portion of flat spacetime corresponding to the situation before the passage of the waves and the nontrivial interaction region where the metric depends on both \( x_1 \) and \( x_2 \). We distinguish the single-wave metrics by their dependence on a single null coordinate \((g_{ij} = g_{ij}(x_1) \) for the progressive wave and \( g_{ij} = g_{ij}(x_2) \) for the regressive one), while in flat spacetime these components are constants, \( g_{12} = 2, g_{33} = g_{44} = 1 \). Khan and Penrose[2] identified a standard procedure to extend continuously the metric over all the various regions by a proper use of Heaviside step functions. We will discuss the details of this extended spacetime in the case of the explicit solution of Ferrari-Ibáñez in the next section.

A wide variety of exact solutions can be found in the literature, the analysis of which is extremely simplified using the Newman-Penrose formalism[10]. Here the following NP null frame can be introduced

\[ l = \partial_{x_2} \, , \quad n = 1/g_{12} \, \partial_{x_1} \, , \quad m = \frac{1}{\sqrt{2}} \left[ \frac{1}{\sqrt{g_{33}}} \partial_{x_3} - i \frac{1}{\sqrt{g_{44}}} \partial_{x_4} \right] \, . \]

It is convenient to introduce the following two quantities

\[ \Xi = \sqrt{g_{12}/(2 \, g_{33})} \, , \quad \Phi = \sqrt{g_{12}/(2 \, g_{44})} \]

which play a role in a conformally rescaled form of the metric[11]

\[ ds^2 = \frac{g_{12}}{2} \left[ 4 \, dx_1 \, dx_2 - \Xi^{-2} \, (dx_3)^2 - \Phi^{-2} \, (dx_4)^2 \right] \, . \]

The Dirac equation for massless spin 1/2 particles then reduces to

\[ (D + \epsilon - \rho) \, F_1 + (\delta^* + \pi - \alpha) \, F_2 = 0 \, , \]
\[ (\delta + \beta - \tau) \, F_1 + (\Delta + \mu - \gamma) \, F_2 = 0 \, , \]

where the notation and conventions for the NP formalism follow those of Chandrasekhar[11]. A much simplified form of [13] results from rescaling the spin wave functions \( F_1 \) and \( F_2 \) as follows

\[ F_1 \rightarrow (g_{33} \, g_{44})^{-1/4}/\sqrt{g_{12}} \, H_1 = (-g)^{-1/4} \, H_1 \, , \]
\[ F_2 \rightarrow (g_{33} \, g_{44})^{-1/4} \, H_2 \, . \]
According to the Khan-Penrose procedure[2], we must first examine the Dirac equation for each of the four regions.

2.1 Case 1: \( g_{ij} = g_{ij}(x_1, x_2) \)

The NP spin coefficients have a very simple and symmetric form in the general case

\[
\rho = -\frac{1}{4} \partial_{x_2} \log (g_{33} g_{44}) \quad , \quad \sigma = -\frac{1}{4} \partial_{x_2} \log \left(\frac{g_{33}}{g_{44}}\right) , \quad \\
\lambda = -\frac{1}{4g_{12}} \partial_{x_1} \log (g_{33} g_{44}) \quad , \quad \mu = -\frac{1}{4g_{12}} \partial_{x_1} \log (g_{33} g_{44}) , \quad \\
\epsilon = \frac{1}{2} \partial_{x_2} \log (g_{12}) \quad , \quad \alpha = \beta = \gamma = \kappa = \nu = \pi = \tau = 0 ,
\]

while the Weyl scalars are

\[
\Psi_0 = D \sigma - 2 \sigma (\rho + \epsilon) = \frac{1}{8} \left[ (\partial_{x_2} \log (g_{12} g_{33}))^2 - (\partial_{x_2} \log (g_{12} g_{44}))^2 \
- 2 \left( \frac{\partial_{x_2}^2 g_{33}}{g_{33}} - \frac{\partial_{x_2}^2 g_{44}}{g_{44}} \right) \right] , \quad \\
\Psi_1 = -\delta \epsilon = 0 , \quad \\
\Psi_2 = \frac{D \mu - \Delta \epsilon + 2(\mu \epsilon - \lambda \sigma)}{3} = \frac{1}{12g_{12}} \left( 2 \partial_{x_1} \log (g_{12}) \partial_{x_2} \log (g_{12}) \
- \frac{1}{2} \partial_{x_1} \log (g_{33} g_{44}) \partial_{x_2} \log (g_{33} g_{44}) \
- 2 \left( \frac{\partial_{x_1} \partial_{x_2} g_{12}}{g_{12}} + \frac{\partial_{x_1} \partial_{x_2} g_{33}}{g_{33}} + \frac{\partial_{x_1} \partial_{x_2} g_{44}}{g_{44}} \right) \right) , \quad \\
\Psi_3 = 0 , \quad \\
\Psi_4 = -\Delta \lambda - 2 \lambda \mu = \frac{1}{8g_{12}^2} \left[ (\partial_{x_1} \log (g_{12} g_{33}))^2 - (\partial_{x_1} \log (g_{12} g_{44}))^2 \
- 2 \left( \frac{\partial_{x_1}^2 g_{33}}{g_{33}} - \frac{\partial_{x_1}^2 g_{44}}{g_{44}} \right) \right] .
\]

(7)

Since the metric is independent of \((x_3, x_4)\) one can seek solutions (normal modes) of the form

\[
H_1 = e^{i(K_3 x_3 + K_4 x_4)} h_1(x_1, x_2) , \quad H_2 = e^{i(K_3 x_3 + K_4 x_4)} h_2(x_1, x_2) ,
\]

(9)

where \(K_3\) and \(K_4\) are real constants and \(h_1, h_2\) will depend on them. The Dirac equation then becomes

\[
\frac{\partial h_1(x_1, x_2)}{\partial x_2} = (K_4 \Phi - i K_3 \Xi) h_2(x_1, x_2) , \\
\frac{\partial h_2(x_1, x_2)}{\partial x_1} = (-K_4 \Phi - i K_3 \Xi) h_1(x_1, x_2) ,
\]

(10)

and the general solution can be given as superposition of modes

\[
\mathcal{H}_1 = \int dK_3 dK_4 H_1 , \quad \mathcal{H}_2 = \int dK_3 dK_4 H_2 .
\]

(11)
2.2 Case 2: \( g_{ij} = g_{ij}(x_1) \)

In this case, only \( \lambda \) and \( \mu \) remain nonzero among the spin coefficients. Since the metric is independent of \( (x_2, x_3, x_4) \) one can seek solutions of the form

\[
H_1 = e^{i(K_2 x_2 + K_3 x_3 + K_4 x_4)} h_1(x_1), \quad H_2 = e^{i(K_2 x_2 + K_3 x_3 + K_4 x_4)} h_2(x_1),
\]

where \( K_2, K_3 \) and \( K_4 \) are real constants and \( h_1 \) and \( h_2 \) will also depend on them. The Dirac equation then becomes

\[
h_1(x_1) = \frac{1}{K_2} \left( -K_3 \Xi - i K_4 \Phi \right) h_2(x_1),
\]

\[
h_2'(x_1) = \frac{i}{K_2} \left[ (K_3 \Xi)^2 + (K_4 \Phi)^2 \right] h_2(x_1),
\]

and the general solution can be given as superposition of these modes

\[
\mathcal{H}_1 = \int dK_2 dK_3 dK_4 H_1, \quad \mathcal{H}_2 = \int dK_2 dK_3 dK_4 H_2.
\]

2.3 Case 3: \( g_{ij} = g_{ij}(x_2) \)

Here the non-zero spin coefficients are \( \rho, \sigma \) and \( \epsilon \). Since the metric is independent of \( (x_1, x_3, x_4) \) one can seek solutions of the form

\[
H_1 = e^{i(K_1 x_1 + K_3 x_3 + K_4 x_4)} h_1(x_2), \quad H_2 = e^{i(K_1 x_1 + K_3 x_3 + K_4 x_4)} h_2(x_2),
\]

where \( K_1, K_3 \) and \( K_4 \) are real constants and as in the previous cases \( h_1 \) and \( h_2 \) will also depend on them. The Dirac equation then becomes:

\[
h_1'(x_2) = \frac{i}{K_1} \left[ (K_3 \Xi)^2 + (K_4 \Phi)^2 \right] h_1(x_2),
\]

\[
h_2(x_2) = \frac{1}{K_1} \left( -K_3 \Xi + i K_4 \Phi \right) h_1(x_2),
\]

and the general solution can be given as superposition of these modes:

\[
\mathcal{H}_1 = \int dK_1 dK_3 dK_4 H_1, \quad \mathcal{H}_2 = \int dK_1 dK_3 dK_4 H_2.
\]

2.4 Case 4: \( g_{ij} = \) (positive) constants

For a metric with constant components the spin coefficients are obviously all zero. We assume without any loss of generality: \( g_{12} = 2 \) and \( g_{33} = g_{44} = 1 \). Since the metric is constant one can seek solutions of the form

\[
H_1 = e^{i(K_1 x_1 + K_2 x_2 + K_3 x_3 + K_4 x_4)} h_1, \quad H_2 = e^{i(K_1 x_1 + K_2 x_2 + K_3 x_3 + K_4 x_4)} h_2,
\]

with \( h_1 \) and \( h_2 \) also constants. The Dirac equation reduces to

\[
K_2 h_1 + (K_3 + i K_4) h_2 = 0,
\]

\[
(K_3 - i K_4) h_1 + K_1 h_2 = 0,
\]
which implies (by setting to zero the determinant of the coefficient matrix of the linear system)

\[ K_1 K_2 = K_3^2 + K_4^2. \]  

(20)

In the next section we will specialize these cases to the horizon-forming Ferrari-Ibañez metric and discuss their solutions.

3 The horizon-forming Ferrari-Ibañez metric

In 1987 Ferrari and Ibañez\[4, 5\] found a type D solution of the Einstein equations that can be interpreted as describing the collision of two linearly polarized gravitational plane waves propagating along a common direction \( z \) in opposite senses and developing a non-singular Killing-Cauchy horizon when they collide. Using the standard coordinates \((t, z, x, y)\), with an appropriate choice of the amplitude parameters associated with the strength of the waves, the metric takes the form:

\[ ds^2 = (1 + \sin t)^2 (dt^2 - dz^2) - \frac{1 - \sin t}{1 + \sin t} dx^2 - \cos^2 z (1 + \sin t)^2 dy^2. \]  

(21)

The interaction region where this form of the metric is valid (designated as “Region I”, as in most of the literature\[7\]) is represented in the \((t, z)\) diagram by a triangle whose vertex (representing the initial event of collision) can be identified with the origin of the coordinate system; the horizon is mapped onto the base of the shaded triangle in Fig. 1.

In order to describe the larger spacetime of which this is only one region, one must introduce the two null coordinates

\[ u = (t - z)/2, \quad v = (t + z)/2 \iff t = u + v, \quad z = v - u, \]  

(22)

in terms of which the metric (21) takes the form

\[ ds^2 = 4 [1 + \sin(u + v)]^2 du dv - \frac{1 - \sin(u + v)}{1 + \sin(u + v)} dx^2 - \cos^2(u - v)[1 + \sin(u + v)]^2 dy^2. \]  

(23)

Following Khan-Penrose\[2\] one can easily extend the formula for the metric from the interaction region to the remaining parts of the spacetime representing the single wave zones and the flat spacetime zone before the waves arrive. The interaction region corresponds to the triangular region in the \((u, v)\) plane bounded by the lines \( u = 0, \quad v = 0 \) and \( u + v = \pi/2 \). One need only make the following substitutions in (23):

\[ u \to u H(u) \quad v \to v H(v) \]

which give rise to the four regions

\[ u \geq 0, \quad v \geq 0, \quad u + v < \pi/2 \quad \text{Region I} \quad \text{Interaction region} \]
\[ 0 \leq u < \pi/2, \quad v < 0 \quad \text{Region II} \quad \text{Single u-wave region} \]
\[ u < 0, \quad 0 \leq v < \pi/2 \quad \text{Region III} \quad \text{Single v-wave region} \]
\[ u < 0, \quad v < 0 \quad \text{Region IV} \quad \text{Flat space} \]

(24)

shown in fig. 1. In this way the extended metric in general is \( C^0 \) (but not \( C^1 \)) along the null boundaries \( u = 0 \) and \( v = 0 \). It is worth noting that certain calculations are more easily done
in one or the other of these two sets of coordinates, so we will switch back and forth between them as needed.

We can apply the results of the previous section using the coordinates \((u, v, x, y)\) and the extended metric (23) – (24).

3.1 Region I: wave interaction

The metric

\[
\begin{align*}
g_{12} &= 2 \left[ 1 + \sin(u + v) \right]^2, \\
g_{33} &= \frac{1 - \sin(u + v)}{1 + \sin(u + v)}, \\
g_{44} &= \cos^2(v - u) \left[ 1 + \sin(u + v) \right]^2. \\
\end{align*}
\] (25)

is of Petrov type D, the NP frame is not principal (i.e. not aligned along the two repeated principal null directions of the spacetime) and the Dirac equation (10) becomes

\[
\begin{align*}
\frac{\partial h_1(u, v)}{\partial v} &= \left( \frac{K_y}{\cos(v - u)} - i K_x \frac{1 + \sin(u + v)}{\cos(u + v)} \right) h_2(u, v), \\
\frac{\partial h_2(u, v)}{\partial u} &= \left( - \frac{K_y}{\cos(v - u)} - i K_x \frac{1 + \sin(u + v)}{\cos(u + v)} \right) h_1(u, v),
\end{align*}
\] (26)

where a convenient notational change has been made with respect to the previous section, denoting \(K_{1,2,3,4}\) as \(K_{u,v,x,y}\).

To exploit the symmetries of (26) we again transform the Dirac variables

\[
\begin{align*}
h_1(u, v) &= s_1(u, v) + s_2(u, v), \\
h_2(u, v) &= s_1(u, v) - s_2(u, v),
\end{align*}
\] (27)
and then we switch back to the spacetime coordinates \((t, z, x, y)\), introducing the abbreviated notation
\[
\alpha(t) = \frac{(1 + \sin t)^2}{\cos t}, \quad \beta(z) = \frac{1}{\cos z}.
\]

Equations (26) then become
\[
\begin{align*}
\frac{\partial s_1(t, z)}{\partial t} + i K_x \alpha(t) s_1(t, z) - K_y \beta(z) s_2(t, z) &= 0, \\
\frac{\partial s_2(t, z)}{\partial z} + i K_x \alpha(t) s_2(t, z) + K_y \beta(z) s_1(t, z) &= 0.
\end{align*}
\]

These equations can be separated by assuming \(s_1(t, z) = T_1(t) Z_1(z)\) and \(s_2(t, z) = T_2(t) Z_2(z)\). In fact dividing the first by \(T_2(t) Z_1(z)\) and the second by \(T_1(t) Z_2(z)\), we obtain
\[
\begin{align*}
\frac{T_1'(t)}{T_2(t)} + \frac{Z_2'(z)}{Z_1(z)} + i K_x \alpha(t) \frac{T_1(t)}{T_2(t)} + K_y \beta(z) \frac{Z_2(z)}{Z_1(z)} &= 0, \\
\frac{T_2'(t)}{T_1(t)} - \frac{Z_1'(z)}{Z_2(z)} - i K_x \alpha(t) \frac{T_2(t)}{T_1(t)} - K_y \beta(z) \frac{Z_1(z)}{Z_2(z)} &= 0,
\end{align*}
\]
and hence
\[
\begin{align*}
T_1'(t) + i K_x \alpha(t) T_1(t) &= -K T_2(t), \\
T_2'(t) - i K_x \alpha(t) T_2(t) &= -J T_1(t), \\
Z_1'(z) - K_y \beta(z) Z_1(z) &= J Z_2(z), \\
Z_2'(z) + K_y \beta(z) Z_2(z) &= K Z_1(z),
\end{align*}
\]
where \(J\) and \(K\) are arbitrary constants. By differentiating these equations the functions \(T, Z\) are seen to satisfy a system of four completely separated second-order differential equations of the following form:
\[
\frac{df(x)}{dx^2} - [a^2 A(x)^2 + a A'(x) + J K] f(x) = 0,
\]
with the respective sets of values for \(f(x), a, A:\)
\[
\begin{align*}
f(x) &\Rightarrow \left\{ \begin{array}{c}
 T_1 \\
 T_2 \\
 Z_1 \\
 Z_2 \end{array} \right\}, \\
a &\Rightarrow \left\{ \begin{array}{c}
 -i K_x \\
 i K_x \\
 K_y \\
 -K_y \end{array} \right\}, \\
A &\Rightarrow \left\{ \begin{array}{c}
 \alpha(t) \\
 \alpha(t) \\
 \beta(z) \\
 \beta(z) \end{array} \right\}.
\end{align*}
\]
With the restriction \(J K = -\left(\frac{\pi}{2}\right)^2, n \in \mathbb{N}\), the \(31\) for \(Z_1, Z_2\) can be integrated analytically. In fact with the variable change \(z = \pi/2 - \theta\) and the rescaling \(Z_{1,2}(\theta) = \sqrt{\sin \theta} \Theta_{1,2}(\theta)\) the resulting equation for \(\Theta_{1,2}(\theta)\) becomes
\[
\Theta''(\theta) + \cot \theta \Theta'(\theta) + \left[\frac{2 m s (m^2 - s^2)}{\sin^2 \theta} + l (l + 1)\right] \Theta(\theta) = 0,
\]
where \(J K = -[l (l + 1) + s^2]\) and \(s = \mp 1/2\) must hold in order to have regular solutions, with \(\Theta_1(\theta)\) corresponding to \(s = -1/2, m = K_y\) (or \(s = 1/2, m = -K_y\)) and \(\Theta_2(\theta)\) corresponding
to \( s = 1/2, m = K_y \) (or \( s = -1/2, m = -K_y \)). A class of solutions of (33) is represented by the spin-weighted spherical harmonics \( sY^\pm_{l}(\theta, 0) \) (first introduced by Newman and Penrose\cite{10} as a natural extension of the scalar harmonics to the Hilbert space of the square-integrable spin-s functions defined on the 2-dimensional unit sphere). An explicit definition of these functions, together with a list of their main properties and features is given in the Appendix. The corresponding solutions of Eqs. (30–32) can then be denoted by

\[
T_1(t) = T^K\_l^{s}(u + v),
T_2(t) = T^{-K}\_l^{s}(u + v)
\]

\[
Z_1(z) = \sqrt{\cos(v - u)}\ Y^K\_l^{s}(\pi/2 + u, 0)
\]

\[
Z_2(z) = \sqrt{\cos(v - u)}\ Y^{-K}\_l^{s}(\pi/2 + u, 0)
\]

(34)

where indices 1, 2 correspond to the signs \( \pm \) respectively. The general solution will then be a superposition of such modes

\[
h_1(u, v) = \sqrt{\cos(v - u)}\ \sum_{l=|K_y|}^{\infty} C_l\ T^K\_l^{s}(u + v)\ Y^K\_l^{s}(\pi/2 + u, 0) +
\]

\[
T^{-K}\_l^{s}(u + v)\ Y^{-K}\_l^{s}(\pi/2 + u, 0)
\]

(35)

with an analogous expression for \( h_2 \), where but the functions \( T^K\_l^{s}(u + v) \) can only be given numerically.

### 3.1.1 Teukolsky Master Equation

We notice here that as this portion of spacetime is Petrov type D actually one can write, following notation and conventions of Teukolski\cite{12}, a Master Equation for any spin \( s = 0, \pm 1/2, \pm 1, \pm 3/2, \pm 2 \) perturbations of the form

\[
\psi(t, z, x, y) = e^{iK_y x} e^{iK_y y} T(t) Z(z)
\]

(36)

of this background. To do this it is convenient to adopt a (Kinnersley) principal null tetrad, different from the one introduced above for the Dirac field, aligned with the two repeated principal null directions of the spacetime

\[
l = \frac{1}{2 \cos t} \left[ \partial_t + \frac{(1 + \sin t)^2}{\cos t} \partial_x \right],
\]

\[
n = \frac{\cos t}{(1 + \sin t)^2} \partial_t - \partial_x,
\]

\[
m = \frac{1}{\sqrt{2}(1 + \sin t)} \left[ \partial_x + \frac{i}{\cos z} \partial_y \right],
\]

(37)

so that one gets two decoupled equations, one for \( Z(z) \)

\[
\frac{d^2 Z(z)}{dz^2} - \tan z \frac{dZ(z)}{dz} - V_z(z) Z(z) = 0,
\]

(38)

where

\[
V_z(z) = \frac{K_y^2 - 2K_y s \sin z + s^2}{\cos^2 z} + \Omega,
\]

(39)
with $\Omega$ a separation constant, and one for $T(t)$

$$\frac{d^2T(t)}{dt^2} - (1 + 2s) \tan t \frac{dT(t)}{dt} - V(t)T(t) = 0 ,$$

(40)

where

$$V(t) = \Omega + s(s + 1) - K_x^2 \frac{(1 + \sin t)^4}{\cos^2 t} - 2iK_x s \left( \sin t + 2 \frac{1 + \sin t}{\cos^2 t} \right) .$$

(41)

Again, by introducing the new variable $\theta = \pi/2 + z$ Eq. (38) takes the same form of Eq. (33), and with $\Omega = -l(l+1)$, with $l, m = K_y$ integers or half-integers and $s = 0, \pm 1/2, \pm 1, \pm 3/2, \pm 2$, its solutions are the spin-weighted spherical harmonics, with spin-weight $s$, generalizing the results already given for the Dirac field.

To accomplish the task of extending this description all over the spacetime it is obviously necessary to match the equations for the various spin fields also in the other two regions of type N and in the Minkowski portion. This can be done for spin $s = 1$ in complete analogy to what we have done for the $s = 1/2$ case. The gravitational case $s = 2$ is to be more carefully handled because of the nonvanishing radiative contribution of the background gravitational field in the two N regions. However this analysis goes beyond the aims of the present paper and it will be studied subsequently.

### 3.2 Region II: single $u$-wave

The metric

$$g_{12} = 2 (1 + \sin u)^2 ,$$

$$g_{33} = \frac{1 - \sin u}{1 + \sin u} ,$$

$$g_{44} = \cos^2(u) (1 + \sin u)^2 ,$$

(42)

is of Petrov type N, the NP frame (2) is principal and the Dirac equation (13) gives

$$h_1(u) = -\frac{K_x (1 + \sin u)^2 - i K_y}{K_x \cos u} h_2(u) ,$$

$$h_2'(u) = i \frac{K_y^2 + K_x^2 (1 + \sin u)^4}{K_y \cos^2 u} h_2(u) .$$

(43)

The ordinary differential equation for $h_2(u)$ can be solved exactly

$$h_2(u) = -\sqrt{2} N \frac{K_y}{K_x + i K_y} e^{\frac{i}{K_y} [K_x^2 A(u) + K_y^2 \tan u]} ,$$

(44)

where $N$ is a normalization constant and the function $A(x)$ is defined by

$$A(x) = \frac{(\sin x + 8) (\cos^2 x + 2)}{2 \cos x} + 7 \tan x - \frac{15}{2} x - 12 .$$

(45)
3.3 Region III: single \( \nu \)-wave

The metric

\[
\begin{align*}
g_{12} &= 2 (1 + \sin \nu)^2, \\
g_{33} &= \frac{1 - \sin \nu}{1 + \sin \nu}, \\
g_{44} &= \cos^2(\nu) (1 + \sin \nu)^2
\end{align*}
\]

is again of Petrov type N, the NP frame is principal and the Dirac equation is:

\[
\begin{align*}
h'_1(\nu) &= i \frac{K_x^2 (1 + \sin \nu)^2 + K_y^2}{K_u \cos^2 \nu} h_1(\nu), \\
h_2(\nu) &= - \frac{K_x (1 + \sin \nu)^2 - i K_y}{K_u \cos \nu} h_1(\nu).
\end{align*}
\]

The solution for \( h_1(\nu) \) is

\[
h_1(\nu) = \sqrt{2} N e^{\frac{1}{K_u} [K_x^2 A(\nu) + K_y^2 \tan \nu]},
\]

where \( N \) is a normalization constant and \( A(\nu) \) is defined in (45). Here one has the same problems matching the solution to region I as for region II.

3.4 Region IV: flat spacetime

The metric

\[
g_{12} = 2, \quad g_{33} = g_{44} = 1.
\]

is the Minkowski metric. The relation between the K-constants that comes from the Dirac equation is

\[
K_u K_v = K_x^2 + K_y^2
\]

(which corresponds to energy-momentum conservation for the test particle[6]). A straightforward solution for the amplitudes of the wave functions is:

\[
\begin{align*}
h_1 &= \sqrt{2} N, \\
h_2 &= - \sqrt{2} N \frac{K_v}{K_x + i K_y} = - \sqrt{2} N \frac{K_x - i K_y}{K_u}.
\end{align*}
\]

This list of the various cases concludes the analysis of the Dirac equation. We have seen that an explicit solution can be given completely in the two type N regions and of course in the flat Minkowski region. In the type D region the Dirac equation can be separated in its \( (t, z) \) dependence, but only the separated \( z \) equation can be solved exactly in terms of spin-weighted spherical harmonics.

3.5 Tracking the Dirac field

It is now clear how to obtain the formal solution of the Dirac equation in the four spacetime regions. However, since the general solution in region I can only be given numerically, the
problem of (continuously) matching at the boundaries $I - II$ and $I - III$ can only be approached numerically. To be more explicit, consider the example of a transition from region I to region II for $h_1$. Approaching the $u$ axis ($v \to 0$) one has

$$
\lim_{v \to 0} h_1(u, v) = \sqrt{\cos u} \sum_{l=|K_y|}^{\infty} C_l [T^K_{l-1/2}(u) Y^K_{l}(\pi/2 + u, 0) + \bar{T}^K_{l-1/2}(u) Y^K_{l}(\pi/2 + u, 0)],
$$

and this limit must coincide with the solution $h_1(u)$ of Eq. (43) in region II, so that it can be seen as a series representation for $h_1(u)$ in terms of spin-weighted spherical harmonics. But the coefficients of this expansion are not constant and thus we can no longer use the orthonormality properties of the harmonics to extract the coefficients $C_l$ analytically.

Special cases can be studied numerically. For example, once the $(K_u, K_v, K_x, K_y)$ parameters are assigned in region IV, the solution is continuously propagated into regions II and III without any arbitrariness, reaching the boundaries of region I.

At each crossing point a single mode solution of region I can be numerically constructed and made continuous with the cited single wave solutions of region II and III. Even if they could be superimposed, these single mode solutions can account for a qualitative description of the behaviour of the Dirac field as scattered by the gravitational waves.

In the next section the neutrino current will be analyzed separately in the four regions of the spacetime following this procedure.

4 Neutrino current

In order to study the properties of the solution and its matching across the boundary of the various regions of the spacetime, we analyze the associated neutrino current.

From the Newman-Penrose formalism, the expression for the neutrino current null vector is the following:

$$
J = 2 \left[ |F_1|^2 + |F_2|^2 + m (F_1 F_2^*) + \bar{m} (F_1^* F_2) \right].
$$

In terms of the formalism developed so far the neutrino current density can be expressed in the form

$$
j = \sqrt{-g} J = j^u \partial_u + j^v \partial_v + j^x \partial_x + j^y \partial_y,
$$

that can easily be specialized to the various regions by identifying the correct dependence of $h_1, h_2, \Xi, \Phi$. The current conservation law in the wave interaction region results in the simple condition

$$
\partial_u j^u + \partial_v j^v = 0,
$$

which trivially becomes $\partial_u j^u = 0$ in region II and $\partial_v j^v = 0$ in region III. Introducing the (closed) “projected current 1–form”

$$
I = j^u dv - j^v du, \quad dI = 0
$$

allows the use of Stokes’ theorem over a bounded portion $\Omega$ of spacetime inside region I: $\int_{\partial \Omega} I = 0$. In order to get information on the interaction of the neutrino current with the
colliding gravitational waves we examine the behaviour of the neutrino current vector limiting ourselves to the \((u, v)\) plane, drawing the corresponding field lines

\[
\frac{du}{dv} = \frac{J^u}{J^v} = \frac{j^u}{j^v} = \frac{|h_2|^2}{|h_1|^2}.
\]  

(57)

Along such lines the 1-form \(J\) vanishes identically by definition.

This equation can be exactly integrated in the single wave regions, giving

\[
v = \frac{K_y^2}{K_v^2} \tan v + \frac{K_x^2}{K_u^2} \hat{A}(v) + C_1
\]

(58)
in region II and

\[
u = \frac{K_y^2}{K_u^2} \tan v + \frac{K_x^2}{K_v^2} \hat{A}(u) + C_2
\]

(59)
in region III, where \(C_1\) and \(C_2\) are integration constants and

\[
\hat{A}(x) = A(x) + 12.
\]

(60)

Moreover, one can show by explicit calculation that the following remarkable property holds (in the single wave regions only): the (null) neutrino current rescaled by the wavefront volume element

\[
J_r^\alpha = \frac{1}{\sqrt{g_{33} g_{44}}} J^\alpha
\]

(61)
is geodesic, i.e. satisfies a speciality property as discussed by Wainwright\[13\].

The lines of the neutrino current are plotted in Figs. 2-4 for selected values of the (global) parameters \(K_x\) and \(K_y\) for a particle entering the single \(u\)-wave region from the flat spacetime regime, with different values of energy (i.e. for different values of \(K_v\)). In regions II and III the integral curves of the current density are null geodesics.

In all the plots, the corresponding null geodesics are superimposed on the current lines. These are the same in the single wave region but evolve differently in the interaction region. Several different mode current lines are plotted, entering the collision wave region, i.e. a sort of mode-by-mode analytical extension of the single wave region current is attempted, since the general solution for \(h_1(u, v)\) and \(h_2(u, v)\) is not available here. These current lines are also superimposed on the geodesics. In any case, this is enough to study the general behaviour of the process: in the interaction region (considering the single modes for various \(l\) as elementary solutions of the Dirac equation) the single mode neutrino current is no longer geodesic, i.e. the helicity of the Dirac particle couples to the spacetime curvature; for generic values of \(K_x\) and \(K_y\), this current shows a typical oscillatory behaviour that is more and more damped as one approaches the horizon; for \(K_x = 0\) the behaviour is rather different in the sense that the oscillations are not as damped as in all the other cases, including the companion one \(K_y = 0\).

It is worth noting that once again the \(\partial_x\) direction has special properties which we have studied in terms of Papapetrou fields and Killing directions in a previous paper\[6\].

5 Concluding remarks

The Dirac equation in the background spacetime of two colliding gravitational waves with parallel polarization is studied in detail in the Newman-Penrose formalism. Explicit results
have been considered for the so called Ferrari-Ibañez degenerate solution. In this case the equation is separated in all the four spacetime regions, but the integration is complete in the single wave and flat spacetime regions only. In the collision region instead, only a partial analytic form of the solution can be given, involving the spin-weighted spherical harmonics. Continuously matching the solution across the regions’ boundaries is briefly discussed too. The characteristics of the associated neutrino current are analysed, showing that it is geodesic in the single wave regions, and the results are graphically summarized (see figs. 2-6).

Appendix. Spin-weighted spherical harmonics

1. Analytic, implementable definition

\[ sY_l^m(\theta, \phi) = e^{im\phi} \sqrt{\frac{2l+1}{4\pi} \frac{(l+m)!}{(l-s)!}} \left( \sin \frac{\theta}{2} \right)^{2l-s} \times \sum_{r=0}^{l} (-1)^{l+m-r}\left( \begin{array}{c} l-s \\ r-s \end{array} \right) \left( \begin{array}{c} l+s \\ r-m \end{array} \right) \left( \cot \frac{\theta}{2} \right)^{2r-m-s} \] (62)

2. Compatibility with spherical harmonics

\[ \phi Y_l^m(\theta, \phi) = Y_l^m(\theta, \phi) \] (63)

3. Conjugation relation

\[ sY_l^{m*}(\theta, \phi) = (-1)^{m+s} -s Y_l^{-m}(\theta, \phi) \] (64)

4. Orthonormality relation

\[ \int d\Omega \ [sY_l^{m*}(\theta, \phi)] [sY_l^{m'}(\theta, \phi)] = \delta_{l,l'} \delta_{m,m'} \] (65)

5. Completeness relation

\[ \sum_{l,m} [sY_l^{m*}(\theta, \phi)] [sY_l^{m'}(\theta', \phi')] = \delta(\phi - \phi') \delta(\theta - \theta') \] (66)

6. Parity relation

\[ sY_l^m(\theta, \phi) = (-1)^l Y_l^m(\theta, \phi) \] (67)

7. Clebsch-Gordan relation

\[ \langle s_1 Y_{l_1}^{m_1} \rangle (s_2 Y_{l_2}^{m_2}) = \sqrt{(2l_1+1)(2l_2+1)} \sum_{l,m,s} \langle l_1,l_2;m_1,m_2|l_1,l_2;l,m \rangle \times \langle l_1,l_2; -s_1,-s_2|l_1,l_2;l,-s \rangle \sqrt{\frac{4\pi}{2l+1}} \] (68)
8. Spin raising and lowering operators

\[
s \partial^+ s Y^m_l = - \sin^s \theta \left[ \partial_\theta (\sin^{-s} \theta s Y^m_l) + \frac{i}{\sin \theta} \partial_\phi (\sin^{-s} \theta s Y^m_l) \right] \\
= \sqrt{l-s} \sqrt{l+s+1} (s+1 Y^m_{l+1}),
\]

\[
s \partial^- s Y^m_l = - \sin^{-s} \theta \left[ \partial_\theta (\sin^s \theta s Y^m_l) - \frac{i}{\sin \theta} \partial_\phi (\sin^s \theta s Y^m_l) \right] \\
= \sqrt{l+s} \sqrt{l-s+1} (s-1 Y^m_{l-1}).
\]

Harmonics with \( s = 1/2 \) and different \( l \) and \( m \) are listed in the following table.

| (l, m) | \( \frac{1}{2} Y^m_l \) |
|-------|------------------|
| (1/2, -1/2) | \(-\frac{i}{\sqrt{2\pi}} e^{-i/2} \cos(\theta/2)\) |
| (1/2, 1/2) | \(\frac{i}{\sqrt{2\pi}} e^{i/2} \sin(\theta/2)\) |
| (3/2, -3/2) | \(-\sqrt{\frac{3}{\pi}} i e^{-3i/2} \cos^2(\theta/2) \sin(\theta/2)\) |
| (3/2, -1/2) | \(-\frac{i}{2\sqrt{\pi}} e^{-i/2} \cos(\theta/2) (3 \cos \theta - 1)\) |
| (3/2, 1/2) | \(\frac{i}{2\sqrt{\pi}} e^{i/2} \sin(\theta/2) (3 \cos \theta + 1)\) |
| (3/2, 3/2) | \(-\sqrt{\frac{3}{\pi}} i e^{3i/2} \cos(\theta/2) \sin^2(\theta/2)\) |

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Figure 2: Several lines of the current field are plotted here, corresponding to the case $K_x = 1$, $K_y = 1/2$, $0.2 < |K_v| < 2.2$, $l = 1/2$, where the particle is conventionally set to enter the single-wave zone at the point $(-\pi/2, 0)$. For $v < 0$ the lines follow geodesic paths in this $(u, v)$ slice, while in region I they decouple (current lines are dashed, geodesic lines are solid). The current clearly oscillates around $u - v = \text{const.}$ axes, damping its oscillations as the horizon (the oblique line at the right) is approached.

Figure 3: Figure 2 repeated for $l = 3/2$, showing the influence of this parameter: it just slightly changes the amplitude of the oscillations, while having almost no effect on the horizon-crossing value of the field lines.
Figure 4: Figure 3 repeated with instead $K_y = 3/2$. Neither the current lines nor the geodesics change qualitatively.

Figure 5: Figure 2 repeated for $K_x = 0, K_y = 1/2, 0.1 < |K_y| < 1.1, l = 1/2$. This choice of $K_x$ changes the behaviour of the current field in region I: the lines of force do not oscillate around directions perpendicular to the horizon (i.e. around fixed values of $z$) anymore and they are strongly different from geodesic trajectories.
Figure 6: This figure shows how a multiplet of current field lines forms in the transition between region II and region I. Here $K_x = 1$, $K_y = 1/2$, $|K_v| = 1$ and $l$ goes from $1/2$ to $9/2$: the oscillations are damped for higher values of $l$.

Figure 7: Here $|h_1|$ and $|h_2|$ (the widerly and finerly dotted lines, respectively) are plotted, with $K_x = 1$, $K_y = 1/2$, $|K_v| = 1/8$ and $l = 1/2$, and compared to the current (solid line).