Characteristic foliation on non-uniruled smooth divisors on hyperkähler manifolds

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Abstract

We prove that the characteristic foliation $F$ on a non-singular divisor $D$ in an irreducible projective hyperkähler manifold $X$ cannot be algebraic, unless the leaves of $F$ are rational curves or $X$ is a surface. More generally, we show that if $X$ is an arbitrary projective manifold carrying a holomorphic symplectic 2-form, and $D$ and $F$ are as above, then $F$ can be algebraic with non-rational leaves only when, up to a finite étale cover, $X$ is the product of a symplectic projective manifold $Y$ with a symplectic surface and $D$ is the pullback of a curve on this surface. When $D$ is of general type, the fact that $F$ cannot be algebraic unless $X$ is a surface was proved by Hwang and Viehweg. The main new ingredient for our results is the observation that the canonical class of the (orbifold) base of the family of leaves is zero. This implies, in particular, the isotriviality of the family of leaves of $F$. We show this, more generally, for regular algebraic foliations by curves defined by the vanishing of a holomorphic $(d-1)$-form on a complex projective manifold of dimension $d$.

1. Introduction

Let $X$ be a projective manifold equipped with a holomorphic symplectic form $\sigma$. Let $D$ be a smooth divisor on $X$. At each point of $D$, the restriction of $\sigma$ to $D$ has one-dimensional kernel. This gives a non-singular foliation $F$ on $D$, called the characteristic foliation. We say that $F$ is algebraic if all its leaves are compact complex curves.

If $D$ is uniruled, the characteristic foliation $F$ is always algebraic. Indeed, its leaves are the fibres of the rational quotient fibration on $D$ (see, for example, [3, Section 4]). On the other hand, J.-M. Hwang and E. Viehweg proved in [14] that $F$ cannot be algebraic when $D$ is of general type, except for the trivial case when $\dim(X) = 2$. The aim of this article is to classify the examples where $F$ is algebraic and $D$ is not uniruled, thus extending [14].

Our main results are as follows.

**Theorem 1.1.** Let $X$ be a projective manifold with a holomorphic symplectic form $\sigma$ and let $D$ be a smooth hypersurface in $X$. If $F$ as above is algebraic and the genus of its general leaf is $g > 0$, then the associated fibration is isotrivial and $K_D$ is nef and abundant, with $\nu(K_D) = \kappa(D) = 1$ when $g \geq 2$ and $\nu(K_D) = \kappa(D) = 0$ when $g = 1$.

Here $\nu$ denotes the numerical dimension and $\kappa$ the Kodaira dimension. In general, $\kappa(D)$ does not exceed $\nu(K_D)$, and $K_D$ is said to be abundant when the two dimensions coincide (by a result of Kawamata, this implies the semi-ampleness of $K_D$, so this notion is important in the minimal model program).

What we will actually prove is a slightly more general result. Consider a smooth projective variety $D$ of dimension $d$ carrying a nowhere vanishing holomorphic $(d-1)$-form $\omega$. Such a
form has one-dimensional kernel at each point and therefore defines a smooth rank-one foliation \( \mathcal{F} \). Alternatively, the foliations arising in this way are those defined by the subbundles of \( T_D \) isomorphic to the anticanonical bundle of \( D \). In this situation, we have the following theorem.

**Theorem 1.2.** If \( \mathcal{F} \) is algebraic, then the associated fibration \( f : D \to B \) is isotrivial without multiple fibres in codimension one and the canonical class \( K_B \) is trivial.

We refer to Section 2.1 for the definition and discussion of the fibration associated to a smooth algebraic foliation of rank one.

When \( D \) is a divisor in a holomorphic symplectic manifold \((X, \sigma)\) of dimension \( d + 1 = 2n \), one recovers the first part of Theorem 1.1 by taking the form \( \sigma \wedge (n-1) \) for \( \omega \), since the kernel of \( \sigma \) is then equal to that of \( \omega \); the assertions on the numerical and Kodaira dimension are deduced from Theorem 1.2 in a standard way.

The next two theorems are consequences of Theorem 1.1.

**Theorem 1.3.** Let \( X, D, \mathcal{F} \) be as in Theorem 1.1, and suppose moreover that \( X \) is irreducible (that is, simply connected and with \( h^{2,0}(X) = 1 \)). If \( \mathcal{F} \) is algebraic and \( D \) is not uniruled, then \( \dim(X) = 2 \).

By the Bogomolov decomposition theorem, up to a finite étale covering, any compact Kähler symplectic manifold is a product of a torus and several irreducible holomorphic symplectic manifolds. Since our assumptions on \( D \) and \( \mathcal{F} \) are preserved under finite étale coverings, Theorem 1.3 is valid for holomorphic symplectic manifolds with \( h^{2,0} = 1 \) and finite fundamental group. Moreover, we may consider only the case of such products in the sequel.

**Remark 1.4.** The smoothness assumption is essential, as one sees by considering the Hilbert square \( X \) of an elliptic K3 surface \( g : S \to \mathbb{P}^1 \): one has a fibration \( h : X \to \mathbb{P}^2 = \text{Sym}^2(\mathbb{P}^1) \). If \( C \subset \mathbb{P}^2 \) is the ramification conic of the natural 2-cyclic cover \( (\mathbb{P}^1)^2 \to \mathbb{P}^2 \), and \( L \subset \mathbb{P}^2 \) is a line tangent to \( C \), then the characteristic foliation on the singular divisor \( D := h^{-1}(L) \) is algebraic with \( g = 1 \). One obtains similar examples with \( g > 1 \) by considering the image of \( C \times S \) in the Hilbert square of \( S \), where \( S \) is an arbitrary K3 surface and \( C \subset S \) is a curve.

**Theorem 1.5.** Let \( X, D, \mathcal{F} \) be as in Theorem 1.1. Suppose that \( D \) is non-uniruled and \( \mathcal{F} \) is algebraic. Then, possibly after a finite étale covering, \( X = S \times Y \), where \( \dim(S) = 2 \), both \( S \) and \( Y \) are complex projective manifolds carrying holomorphic symplectic forms \( \sigma_S, \sigma_Y \), and \( D = C \times Y \), where \( C \subset S \) is a curve.

**Remark 1.6.** The surface \( S \) from Theorem 1.5 is, up to a finite cover, either K3 or abelian. In the first case, \( \sigma = p^*\sigma_S \oplus q^*\sigma_Y \) on \( TX \cong p^*TS \oplus q^*TY \) (where \( p, q \) denote the projections) by the Künneth formula.

In the second case, one still has \( \sigma = p^*\sigma_S \oplus q^*\sigma_Y \) when \( g > 1 \). Indeed, by the Künneth formula (and Bogomolov decomposition) one reduces to the case when \( Y \) is also an abelian variety, and the decomposition then follows by a straightforward linear-algebraic computation.

In contrast to these cases, when \( S \) is an abelian surface and \( g = 1 \), \( \sigma \) is not always a direct sum (see Example 4.1).

The main ideas of the proof of Theorem 1.2 are as follows. Suppose that \( D \) is not uniruled and that \( \mathcal{F} \) is algebraic. Then \( \mathcal{F} \) defines a holomorphic fibration \( f : D \to B \) such that its non-singular fibres are curves of genus \( g > 0 \), and the singular fibres are multiple curves with smooth reduction. The base has only quotient singularities by Reeb stability. We prove that the codimension of the locus of multiple fibres in \( D \) (and of its image in \( B \)) is at least two.
Therefore the form $\omega$ descends to $B$ outside of a codimension-two locus; this trivializes the canonical class of $B$. Moreover, an extension theorem due to Freitag implies that $B$ is not uniruled.

Miyaoka’s generic semi-positivity theorem now implies that the Iitaka dimension of the determinant of any subsheaf of the cotangent sheaf of $B$ is non-positive. On the other hand, Hwang and Viehweg construct such a subsheaf (coming from the Kodaira–Spencer map) with Iitaka dimension equal to the number of moduli of the fibres of $f$. Therefore the family $f$ must be isotrivial. This is the main new step of the arguments.

As an application, we deduce in Section 5 a certain case of the Lagrangian conjecture on a projective (and, more generally, compact Kähler) irreducible holomorphic symplectic manifold of dimension $2n$ from the abundance conjecture in dimension $2n - 1$. We therefore solve this case unconditionally for $n = 2$, since the abundance conjecture is known for 3-folds (by [17] in the projective case, and by [6] in the Kähler case). This was our initial motivation for this research. When the research was completed, Chenyang Xu informed us that for projective manifolds, this case of the Lagrangian conjecture follows from work by Demailly, Hacon and Paun [10]. Since no algebraic proof for [10] is known, our result gives a simple algebro-geometric alternative for projective hyperkähler manifolds. In the Kähler case, our application is new.

Section 2 is devoted to the proof of Theorems 1.1 and 1.2. Sections 3 and 4 prove Theorems 1.3 and 1.5, respectively. In Section 5, we treat our application to the Lagrangian conjecture.

2. Numerical invariants of the characteristic foliation

2.1. Smooth rank-one foliations

Let $D$ be a $d$-dimensional ($d \geq 2$) connected Kähler manifold carrying a non-singular holomorphic foliation $\mathcal{F}$ of rank one. The foliation $\mathcal{F}$ is called algebraic when all its leaves are compact complex curves. A non-singular algebraic foliation induces a proper holomorphic map $f : D \to \mathcal{C}(D)$ to a component $\mathcal{C}(D)$ of the cycle space of $D$. Indeed, the general leaves of $\mathcal{F}$ are smooth curves varying in a dominating family of cycles on $D$; by compactness of $\mathcal{C}(D)$, one has well-defined limit cycles which must be supported on the special leaves, and the multiplicity of such a cycle is uniquely determined by pairing with the Kähler class. Taking the normalization of the image if necessary, we obtain a proper holomorphic map $f : D \to B$ onto a $(d - 1)$-dimensional normal base $B$. It is well known that in such a situation, the holonomy groups of the leaves are finite (this amounts to the boundedness of the volume of the leaves which holds in the Kähler case; see, for example, [11]). Therefore by Reeb stability (see [23], or else [14] which develops the construction of [23] in the holomorphic case in some detail), locally in some saturated neighbourhood of each leaf $\mathcal{C}$ of holonomy group $G_C$, our foliation is the quotient of $T \times \hat{\mathcal{C}}$, where $T$ is a local transverse and $\hat{\mathcal{C}}$ is $G_C$-covering of $\mathcal{C}$, by the natural action of $G_C$.

In particular, $B$ has only quotient singularities and so is $\mathbb{Q}$-factorial, and $f$ is ‘quasi-smooth’, that is, the reduction of any of its fibres is a smooth projective curve.

Let $g$ denote the genus of a non-singular fibre of $f$. If $g = 0$, the holonomy groups are trivial and all fibres of $f$ are smooth reduced rational curves, $B$ is smooth, $f$ submersive. If $g > 0$, $f$ may have multiple fibres, of genus one when $g = 1$ and of genus greater than one (but possibly smaller than $g$) when $g > 1$.

If $g = 1$ and $B$ is compact, it is well known that $f$ must be isotrivial: indeed, the $j$-function then holomorphically maps $B$ to $\mathbb{C}$. In fact the holomorphicity of $j$ near the multiple fibres is easily checked: from Reeb stability we obtain the local boundedness of $j$, and then use the normality of $B$.

A pair $D, \mathcal{F}$ as above arises, for example, when $D$ is a smooth connected divisor in a $2n$-dimensional projective (or compact Kähler) manifold $X$ carrying a holomorphic symplectic 2-form $\sigma$. The foliation $\mathcal{F}$ is then given, at each $x \in D$, as the $\sigma$-orthogonal to $TD_x$ at $x$. In this
case $d = 2n - 1$. In general, $\mathcal{F}$ will not be algebraic. One particular case when $\mathcal{F}$ is algebraic is that of a uniruled $D$: the leaves of $\mathcal{F}$ are then precisely the fibres of the rational quotient fibration of $D$ (see, for instance, [3, Section 4]), so $g = 0$.

We will elucidate below the situation when $\mathcal{F}$ is algebraic and $g > 0$.

Note that in this example, the quotient bundle $\mathcal{T}_D/\mathcal{F}$ carries a symplectic form, so it has trivial determinant. Therefore the line bundle $\mathcal{F}$ is isomorphic to the anticanonical bundle of $D$ (by adjunction, this is $\mathcal{O}_D(-D)$).

The purpose of this section is to prove the following result, which is stated as Theorem 1.1 in the introduction.

**Theorem 2.1.** Let $D$ be a smooth divisor in a projective holomorphic symplectic variety $(X,\sigma)$, and $\mathcal{F}$ the foliation on $D$ given by the kernel of $\sigma|_D$. If $D$ is non-uniruled and $\mathcal{F}$ is algebraic, then the corresponding fibration $f : D \to B$ is isotrivial, $K_D$ is nef and abundant, $\nu(K_D) = \kappa(D) = 1$ if $g \geq 2$, and $\nu(K_D) = \kappa(D) = 0$ if $g = 1$.

This will be a consequence of a more general isotriviality result stated as Theorem 1.2:

**Theorem 2.2.** Let $D$ be a complex projective manifold of dimension $d$ carrying a nowhere vanishing holomorphic $(d-1)$-form $\omega$. Let $\mathcal{F}$ be the foliation defined as the kernel of $\omega$. Suppose $\mathcal{F}$ is algebraic. Then the corresponding fibration $f : D \to B$ is isotrivial and submersive in codimension two, the canonical class of $B$ is trivial, and $B$ is not uniruled.

Our first idea is to introduce the orbifold base of a fibration and to show that the orbifold structure is actually trivial when the fibration is defined by a non-vanishing holomorphic $(d-1)$-form.

### 2.2. Orbifold base

Let $f : D \to B$ be a quasi-smooth fibration in curves, with $D$ smooth Kähler and $B$ $\mathbb{Q}$-factorial. We define (as in [4], but in a much less general situation than there) the orbifold base $(B, \Delta)$ for $f$ as follows: for each irreducible reduced Weil ($\mathbb{Q}$-Cartier) divisor $E \subset B$, set $E' = f^{-1}(E)$. This is an irreducible divisor, and $f^*(E) = m_f(E)E'$ for some positive integer $m_f(E)$. This integer is equal to 1 for all but finitely many $E$. Set $\Delta = \sum_{E \subset B} (1 - 1/m_f(E))E$. The divisor $\Delta$ thus carries the information about the multiple fibres of $f$ in codimension one, but the coefficients of $\Delta$ are ‘orbifold multiplicities’ varying between zero and one rather than the multiplicities of the fibres. Over a neighbourhood of a general point $b \in \Delta$ (that is, a point outside of $\text{Sing}(B)$ and $\text{Sing}(\Delta)$) the map $f$ is locally given by $(z_1, \ldots, z_{d-1}, w) \mapsto (z_1^m, \ldots z_{d-1}) = (u_1, \ldots, u_{d-1})$, where $m = m_f(E)$ for the component $E$ of $\Delta$ which contains $b$.

**Lemma 2.3.** Suppose that $f$ is given by the kernel of a non-vanishing holomorphic $(d-1)$-form $\omega$. Then $f$ has no multiple fibres in codimension one, that is, $\Delta = 0$, and $K_B$ is trivial. Moreover, $B$ is not uniruled.

**Proof.** The question concerning multiple fibres is local on $B$, and $B$ is smooth in codimension one. We can thus assume that $B$ is a polydisc in $\mathbb{C}^{d-1}$, with coordinates $(u_1, \ldots, u_{d-1}) = (u, u')$ ($u$ being the first coordinate and $u'$ the $(d-2)$-tuple of the rest), and that $f$ has multiple fibres of multiplicity $m > 1$ over the divisor $E$ defined by the equation $u = 0$. Since the form $\omega$ is $d$-closed, and its kernel is $\text{Ker}(df)^{\text{sat}}$, the saturation being taken in $TD$, it descends over $B - E$ to a holomorphic $(d-1)$-form $\alpha$ on $B - E$ such that $\omega = f^*(\alpha)$ on $f^{-1}(B - E)$ (see, for example, [26, Lemma 6], where the full argument is given for holomorphic symplectic forms; it immediately generalizes to our setting).
We will show that $\alpha$ extends holomorphically to $B$, and that $m = 1$. Write $\alpha = G(u, u')du \wedge du'$ (where $du'$ stands for the wedge product of $du_i$ for $i > 1$). We claim that $|G(u, u')| = e^g \cdot |u|^{-c}$, with $c = 1 - 1/m$, where $g$ is a real-valued bounded function, after possibly shrinking $B$ near $(0,0)$.

Indeed, let $B' \subset B$ be a smooth local multisecution of degree $m$ over $B$ meeting transversally the reduction of the fibre of $f$ over $(0,0) \in B$. We can choose the coordinates $(z_1, z_2, \ldots, z_{d-1}, w) = (z, z', w)$ on $D$ near the intersection point $(0,0)$ of $B'$ and the fibre $D_{(0,0)}$ of $f$ over $(0,0)$ in such a way that $f(z, z', w) = (z^m, z')$, and $B'$ is defined by the equation $w = 0$. Restricting $\omega$ to $B'$, we see that $f^*(\alpha) = G(z^m, z') \cdot m \cdot z^{m-1}dz \wedge dz' = \omega|_{B'} = h(z, z')dz \wedge dz'$, for some nowhere vanishing function $h(z, z') = H(z^m, z') = H(u, u')$, whenever $u = z^m \neq 0$.

Thus

$$|G(u, u')| = |G(z^m, z')| = \frac{|H(u, u')|}{m} \cdot \frac{1}{|u|} = e^{g(u, u')} \cdot \frac{1}{|u|}.$$

The following well-known fact now shows that $\alpha$ extends holomorphically to $B$, and hence $c$ must be zero and $m = 1$ as claimed.

Let $G(u, u')$ be a holomorphic function defined on $B - E$, where $B$ is a polydisc centred at $(0,0)$ in $\mathbb{C}^{m-1}$, and $E$ is the divisor defined by $u = 0$ in $B$. Assume that, for some $\varepsilon > 0$, $|G(u, u')| \leq C \cdot |u|^{-(1-\varepsilon)}$ for some positive constant $C$ independent of $u'$. Then $G(u, u')$ extends holomorphically across the divisor $u = 0$.

Indeed, fix $u'$. The Laurent expansion $G(u, u') = \sum_{k=-\infty}^{\infty} a_k(u') \cdot u^k$ of $G$ then has coefficients $a_k(u') = \frac{1}{2\pi i} \int_0^{2\pi} e^{-ikt}G(re^{it}, u') dt$ (cf. [8, p. 86, formula (2.1)]). The bound on $|G|$ implies that $|a_k(u')| \leq Cr^{-k-1+\varepsilon}$ for $0 < r \ll 1$. This implies that $a_k(u') = 0$ if $k < 0$, by letting $r \to 0^+$. Furthermore, the canonical divisor $K_B$ is trivial. Indeed the form $\omega$ descends to a non-vanishing holomorphic form on the smooth locus $B^{sm}$ (which is the complement of a codimension-two subset of $B$) and $K_B$ is by definition the pushforward of $K_{B^{sm}}$, trivialized by the section induced by $\omega$.

Finally, the non-uniruledness of $B$ is a consequence of the extension theorem by Freitag [12, Satz 1, p. 99], according to which any holomorphic $p$-form on the regular part of a normal complex space with quotient singularities lifts to any desingularization as a holomorphic $p$-form. Since $h^0(B^{sm}, K_{B^{sm}}) = 1$, this implies that $h^0(B', K_B') = 1$ if $B'$ is any smooth model of $B$ (in particular, $B'$), and hence $B$ is not uniruled.

**Remark 2.4.** Another way to see the non-existence of multiple fibres in codimension one is by Reeb stability. Indeed in a neighbourhood of a multiple fibre $\omega$ must lift as a $G_C$-invariant form to the $G_C$-covering coming from Reeb stability, but this is impossible by the explicit local computation.

**Remark 2.5.** The map $f : D \to B$ given by a global non-vanishing $(d-1)$-form may have multiple fibres in codimension two: take, for instance, $D = (E \times E \times C)/G$, where $E$ is an elliptic curve, $C$ is a curve equipped with a fixed-point-free involution and $G$ a group of order two where the non-trivial element acts as $-Id$ on $E \times E$ and as that involution on $C$. Then the projection onto the quotient of $E \times E$ by $-Id$ has isolated multiple fibres, and is given by the kernel of a $2$-form which is the exterior product of $1$-forms on $E$.

**Remark 2.6.** If $B$ has quotient singularities and $K_B$ is torsion, $B$ can be uniruled. An example is the following Ueno surface. Let $A = E \times E$ be the product of two copies of the elliptic curve $E$ with complex multiplication by $i = \sqrt{-1}$. Let $S := A/\mathbb{Z}_4$, the generator acting
by $i$ simultaneously on both factors. Then $S$ is a rational surface with 16 quotient singularities, not all canonical. We have $2K_S = O_S$, but $S$ is uniruled.

2.3. Isotriviality of the fibration

As we have already remarked, the isotriviality of the family of curves $f : D \to B$ associated to $\mathcal{F}$ is clear when $g = 0$ or $g = 1$, so we assume in this section that $g \geq 2$. All varieties are assumed to be projective (or quasi-projective, when we work outside of a suitable codimension-two subset such as $\text{Sing}(B)$). Define the sheaf $\Omega_B^1$ as the direct image $j_*\Omega_{B}^{1m}$ where $j : B^{sm} \to B$ is the embedding of the smooth part of $B$ in $B$. The following theorem is a consequence of Miyaoka’s generic semi-positivity theorem (see [22, Corollary 8.6] or [24, Theorems 2.14 and 2.15, pp. 66–67] for a formulation adapted to our purposes).

**Theorem 2.7.** Let $B$ be a non-uniruled normal projective variety with quotient singularities such that $K_B$ is trivial. Let $L$ be a coherent rank-one subsheaf of $(\Omega^1_B)^{\otimes k}$ for some $k > 0$. Then $\deg_C(L_{|C}) \leq 0$ for a sufficiently general complete intersection curve $C$ cut out on $B$ by members of a linear system $|H|$, $l \gg 0$, where $H$ is an ample line bundle on $B$.

In particular, for any integer $m > 0$ one has $h^0(B^{sm}, L^{\otimes m}) \leq 1$ and so $\kappa(B^{sm}, det(\mathcal{F})) \leq 0$, for any coherent subsheaf $\mathcal{F} \subset \Omega^1_B$.

**Proof.** Since $B$ is not uniruled, one can apply Miyaoka’s generic semi-positivity theorem, which affirms that all quotients of $\Omega^1_B$ restricted to $C$ have non-negative degree. By general properties of slopes (see, for example, [9]) this is also true for quotients of $(\Omega^1_B)^{\otimes k}$. So the quotient $Q := ((\Omega^1_B)^{\otimes k}/L)$ restricted to $C$ has non-negative degree. The locally free sheaf $L_{|C}$ must thus have non-positive degree, since $\deg_C((\Omega^1_B)^{\otimes k}) = 0$.\qed

So the determinant of any subsheaf of $\Omega^1_B$ restricted to $B^{sm}$ has non-positive Kodaira dimension; this also remains true for finite coverings of $B$, étale over $B^{sm}$.

Following [14], we now construct a subsheaf of $\Omega^1_B$ (or more precisely of $\Omega^1_{B'}$, where $B'$ is such a covering) such that the Kodaira dimension of its determinant over $B^{sm}$ is equal to the variation of moduli of our family of curves; the argument is shorter here since we have remarked that $f$ is surjective in codimension one.

Indeed, it suffices to do so outside of a codimension-two algebraic subset in $B$, that is, over $B^0$ which is smooth and such that the restriction $f : D^0 \to B^0$ of $f : D \to B$ is a smooth family of curves. It is well known (see, for example, [14, Lemma 3.1]) that, after replacing $B^0$ by a finite étale covering, the family $f : D^0 \to B^0$ becomes the pullback of the universal family of curves with level $N$ structure $g : C^N_g \to M^N_g$ under a morphism $j : B^0 \to M^N_g$ for a suitable $N \gg 0$.

Since $D^0$ is now a smooth family of curves over a smooth base $B^0$, one can consider the ‘Kodaira–Spencer map’

$$f_* (\omega^\otimes_{D^0/B^0}) \to \Omega^1_{B^0}$$

obtained by dualizing the usual Kodaira–Spencer map from $T_{B^0}$ to $R^1f_*T_{D^0/B^0}$ associated to the family of curves $f : D^0 \to B^0$. Let $\mathcal{H} \subset \Omega^1_{B^0}$ be its image: it is a coherent subsheaf of $\Omega^1_{B^0}$.

Moreover, it is functorial in $B^0$, that is, its construction commutes with base change.

**Proposition 2.8** (cf. [14, Proposition 4.4]). Assume that $g \geq 2$. Then $\kappa(B^0, \det(\mathcal{H})) = \text{Var}(f) = \dim(\text{Im}(j))$.

**Proof.** The sheaf $f_* (\omega^\otimes_{D^0/B^0})$ is the pullback by $j$ of $g_* (\omega^\otimes_{C^N_g/M^N_g})$, and the latter is ample by [14, Proposition 4.3]. We conclude by [14, Lemma 4.2].\qed
Corollary 2.9. The fibration \( f : D \to B \) is isotrivial.

Indeed, by Theorem 2.7 we know that the Kodaira dimension of the determinant of any subsheaf of \( \Omega_{D_{\text{sm}}}^1 \) is non-positive, and so \( \kappa(B^m, \det(H)) = \text{Var}(f) = 0 \). This finishes the proof of Theorem 2.2.

2.4. A more general conjectural isotriviality statement

Corollary 2.9 is a special case of the following more general conjectural statement, which slightly generalizes [27].

Conjecture 2.10. Let \( f : X \to B \) be a proper, connected, quasi-smooth纤维 of quasi-projective varieties, where \( X \) is smooth and \( B \) is normal. Assume that the (reduced) fibres of \( f \) have semi-ample canonical class, and that the orbifold base \( (B, \Delta) \) of \( f \) is special in the following sense (cf. [5]): for any \( p > 0 \) and any coherent rank-one subsheaf \( L \subset (f^*(\Omega_B^p))^{\text{sat}} \), where the saturation takes place in \( \Omega_X^p \), one has \( \kappa(X, L) < p \). Then \( f \) is isotrivial.

We would like to remark that the special case of this conjecture when \( f : X \to B \) is a family of curves and the orbifold canonical bundle of the base is trivial can be proved by an argument similar to the one just given, using the orbifold generic semi-positivity of [7] (instead of [22]), and the argument of [14].

2.5. Consequences of isotriviality

Our goal now is to get the information on \( K_D \) once the isotriviality is established. All arguments work in the compact Kähler case. Let us first remark that the relative canonical divisor \( K_D/B \) is well defined as a \( \mathbb{Q} \)-Cartier divisor, and \( K_D \equiv K_D/B \) since \( K_B \) is trivial.

We first make a normalized base change to remove all multiple fibres.

Lemma 2.11. Let \( D \) be a compact connected Kähler manifold with a smooth rank-one foliation \( \mathcal{F} \) with compact leaves of genus \( g \geq 1 \). Let \( f : D \to B \) be the associated proper fibration. Consider the normalized base change \( f'_D : (D \times_B D) \to D \). Then \( f'_D \) is smooth.

Proof. By definition of a foliation, a neighbourhood of \( x \in D \) is isomorphic to \( U' \times F \), where \( F \) is a small open subset of the leaf through \( x \) and \( U' \) is a local transverse to the foliation. Moreover, by Reeb stability, a small neighbourhood \( U \) of \( b \in B \) is \( U'/G \) where \( G \) is the holonomy group, and \( D'_U = (D \times_B U') \) is smooth over \( U' \) and étale over \( D_U = f^{-1}(U) \). Hence \( (D \times_B D)' \), which locally in a neighbourhood of \( x \) is naturally isomorphic to \( (D \times_U (U' \times F))' = D'_U \times F \), is smooth over \( D \); indeed the projection to \( D \) is, locally, the composition of the smooth projection to \( D'_U \) with the natural étale projection from \( D'_U \) to \( D_U \).

Denote by \( f' : D' \to B' \) our new smooth family (so that \( B' = D \) and \( D' = (D \times_B D)' \)) and by \( s : D' \to D \) the natural projection. Note that since the normalization procedure only concerns the codimension-two locus, we have \( K_{D'/B'} \equiv s^*K_{D/B} \).

It is well known that a smooth isotrivial family of curves of genus \( g \), after a suitable finite base change, becomes a product when \( g \geq 2 \), and a principal fibre bundle when \( g = 1 \). More precisely, we have the following lemma.

In [27], the conjecture is established when \( B \) is smooth and \( \Delta = 0 \).

That is, the reduction of every fibre is smooth.
Lemma 2.12. There exists a finite proper map $h' : B'' \to B'$ such that after base-changing $f'$ by $h'$, we get $f'' : D'' \to B''$ and $s'' : D'' \to D'$ with the following properties: $D'' \cong F \times B''$ over $B''$ when $g \geq 2$, and $f'' : D'' \to B''$ is a principal fibre bundle if $g = 1$. Moreover, $K_{D''/B''}$ is nef, $\kappa(D'', K_{D''/B''}) = \nu(D'', K_{D''/B''}) = 1$ if $g \geq 2$, and $\kappa(D'', K_{D''/B''}) = \nu(D'', K_{D''/B''}) = 0$ if $g = 1$.

Here $\nu$ denotes the numerical dimension.

Proof. The smooth isotrivial family $f'$ is a locally trivial bundle with structure group $\text{Aut}(F)$, where $F$ is a fibre. If $g \geq 2$, this is a finite group, so that the bundle trivializes after a finite covering $h' : B'' \to B'$. If $g = 1$, we get the principal bundle structure after a finite covering corresponding to the quotient of $\text{Aut}(F)$ by the translation subgroup. The second claim is obvious when $g \geq 2$. When $g = 1$, we remark that $K_{D''/B''}$ is dual to $f''^*(R^1 f''_*(\mathcal{O}_{D''}))$, and the latter is trivial since translations on an elliptic curve operate trivially on cohomology.

Corollary 2.13. Let $f : D \to B$ be as above. Then $K_D$ is nef, $\kappa(D) = \nu(D, K_D) = 1$ if $g \geq 2$, and $\kappa(D) = \nu(D, K_D) = 0$ if $g = 1$.

Proof. Since

$$NK_{D''/B''} \equiv Ns^*(K_{D''/B'}) \equiv Ns^*(s^*(K_{D/B})) \equiv N(h \circ h')^*(K_D),$$

this follows from the preceding lemma, by the preservation of nefness, numerical dimension and Kodaira–Moishezon dimension under inverse images (by [28, Theorem 5.13, p. 61]).

This finishes the proof of Theorem 1.1 in the projective case. Remark that when $g = 1$, this argument also proves the Kähler case, since the isotriviality, for which the projectivity assumption was needed, is then automatic. This will be used in the proof of Corollary 5.3.

In the next section, we shall give a proof of Theorem 1.3.

3. Divisors on irreducible hyperkähler manifolds

We suppose now that $X$ is a projective irreducible holomorphic symplectic manifold of dimension $2n \geq 4$, $D \subset X$ is a smooth non-uniruled divisor on $X$ and the fibres of $f : D \to B$ are curves of non-zero genus tangent to the kernel of the restriction of the holomorphic symplectic form $\sigma$ to $D$. Recall that on the second cohomology of $X$ there is a non-degenerate bilinear form $q$, the Beauville–Bogomolov form.

Recall that $\nu(K_D)$ denotes the numerical dimension of $K_D$ as a divisor on the variety $D$, whereas $\nu(D)$ denotes the numerical dimension of $D$ considered as a divisor on $X$.

By Corollary 2.13, $\nu(K_D) \leq 1 < \dim(X)/2$. On the other hand, we have the following well-known lemma (see, for instance, [19, Lemma 1], keeping in mind that by the Fujiki formula $D^{2n}$ is proportional to $q(D, D)^n$ with non-zero coefficient, and that the numerical dimension $\nu(D)$ of a nef divisor $D$ is the maximal number $k$ such that the cycle $D^k$ is numerically non-trivial).

Lemma 3.1. Let $D$ be a non-zero nef divisor on an irreducible hyperkähler manifold $X$. Then either $\nu(D) = \dim(D)$ (if $q(D, D) > 0$), or $\nu(D) = \dim(X)/2$ (if $q(D, D) = 0$).

Note that $\nu(D) = \nu(K_D) + 1$, since $K_D = D|_D$. Therefore $\nu(D) \leq 2$ and the only possibility is $\dim(X) = 4$, $\nu(D) = 2$, $\nu(K_D) = \kappa(D) = 1$, $g \geq 2$. This case can be excluded as follows: since $\nu(D) = \nu(K_D)$, $D$ is a good minimal model and the Iitaka fibration $\phi : D \to C$ is a regular
map. Its fibres $S$ are proportional to $D^2$ as cycles on $X$, and therefore are lagrangian. Indeed, it follows from the definition of the Beauville–Bogomolov form $\sigma$ on $X$ that
\[
\int_S \sigma^2 = q(D, D) = 0,
\]
and this implies that the restriction of $\sigma$ to $S$ is zero. So the leaves of the characteristic foliation must be contained in the fibres of $\phi$, giving the fibration of $S$ in curves of genus at least 2. But this is impossible on $S$, since $S$ is a minimal surface of Kodaira dimension zero.

This proves Theorem 1.3.

4. Divisors on general projective symplectic manifolds

The purpose of this section is to prove Theorem 1.5.

Recall the setting: $(X, \sigma)$ is a holomorphic symplectic projective variety, $D \subset X$ is a smooth hypersurface such that its characteristic foliation $\mathcal{F}$ is algebraic and the genus $g$ of the leaves is strictly positive. We wish to prove that up to a finite étale covering, $X$ is a product with a surface and $D$ is the inverse image of a curve under projection to this surface.

By the Bogomolov decomposition theorem, we may assume that $X$ is the product of a torus $T$ and several irreducible hyperkähler manifolds $H_j$ with $q(H_j) = 0$ (here $q$ denotes the irregularity $h^{1,0}$) and $h^{2,0}(H_j) = 1$.

We distinguish two cases:

Case 1: $X$ is not a torus. We proceed by induction on the number of non-torus factors in the Bogomolov decomposition of $X$.

Since $X$ is not a torus, there is an irreducible hyperkähler factor $H$ in the Bogomolov decomposition. If $X = H$, we are done. Otherwise, write $X = H \times Y$, where $Y$ is the product of the remaining factors. By the Künneth formula, we have $\sigma_X = \sigma_H + \sigma_Y$ on $TX \cong TH \oplus TY$, since $q(H) = 0$. For $y \in Y$ general, let $D_y = D \cap (H \times \{y\})$. If this is empty, then $D = H \times D_Y$ for some divisor $D_Y$ of $Y$, which is smooth with algebraic characteristic foliation. Indeed, at any point of $D$ the $\sigma_X$-orthogonal to $TD$ is contained in the $\sigma_Y$-orthogonal to $TD \subset TD$, whereas $TH^\perp = TY$ since $\sigma_X$ is a direct sum. We conclude by induction in this case.

Therefore, we may suppose that $D$ dominates $Y$. For $y \in Y$ generic, $D_y$ is a smooth non-uniruled divisor on $H \times y$. At any point $(h, y) \in D$ such that $D_y \neq H \times y$ is smooth at $h$, we have $TD_y = TD \cap TH$. Moreover, at such a point $TH \not\subseteq TD$ and thus, taking the $\sigma$-orthogonals, $\mathcal{F} \not\subseteq TY$. We get $(TD_y)^\perp = TD^\perp \oplus TH^\perp = \mathcal{F} \oplus TY$.

Since $\sigma$ is a direct sum, the $\sigma_H$-orthogonal of $TD_y$ in $TH$ is the projection of $\mathcal{F}$ to $TH$.

In other words, the characteristic foliation $\mathcal{F}_{D_y}$ of $D_y$ inside $H$ is the projection on $TH$ of the characteristic foliation $\mathcal{F} \subset TX$ along $D_y$. The leaves of $\mathcal{F}_{D_y}$ are thus the étale $p_H$-projections of the leaves of $\mathcal{F}$ along $D_y$, and so $\mathcal{F}_{D_y}$ is algebraic, with non-uniruled leaves. From Theorem 1.3, we deduce that $H$ is a K3 surface, and the divisors $D_y$ are curves of genus $g > 0$ for $y \in Y$ generic.

When $D_y$ is singular at $h$, one has $TH \subset TD$ at $(h, y)$, and therefore at such points $\mathcal{F} \subset TY$.

Fix any $h \in H$ and let $C_y$ denote the leaf of the characteristic foliation of $D$ through $(h, y)$. By isotriviality, all the curves $C_y$ are isomorphic to each other. When $y$ varies in the fibre of $D$ over $h$, we thus have a positive-dimensional family of non-constant maps $p_H : C_y \to H$ parameterized by a compact (but possibly not connected) variety $D^h$, and all images pass through the point $h \in H$. After a base change $\alpha : Z \to D^h$ (not necessarily finite, but with $Z$ still compact) of the family of the leaves, we have a map $p : C_y \times Z \to H$ mapping a section $c \times Z$ to a point. By the rigidity lemma, all images $p_H(C_y)$ coincide when $y$ varies in a connected component of $Z$; therefore there are finitely many curves $C_y$ through any $h \in H$. By the same reason, such a curve (that is, the projection of a leaf of $\mathcal{F}$ to $H$) does not intersect its small deformations in the family of the projections of leaves. The family of such curves is thus
at most a one-parameter family, and there are only finitely many of them through any given point of $H$.

We are thus left with two cases: either all leaves of $\mathcal{F}$ project to the same curve on $H$, so that $p_H(D) = C \subset H$ is a curve and we are finished; or $p_H(D) = H$. In this last case, $H$ is covered by a one-parameter family of curves $C_t$, which we may suppose irreducible, such that $C_t$ does not intersect its small deformations and there are finitely many curves $C_t$ through a given point.

Note also that these $C_t$ have to coincide with the connected components of the divisors $D_y$ and therefore the generic $C_t$ is smooth. By the adjunction formula, it is an elliptic curve and $H$ is fibred in curves $C_t$.

We claim that every $C_t$ is non-singular. Indeed, suppose that some $C_t$ is singular at $h \in H$. It has to be a connected component of a $D_y$ for some $(h, y)$ on a leaf of $\mathcal{F}$ projecting to $C_t$. As we have remarked above, the singularity of $D_y$ at $h$ means that $TH \subset TD$ and therefore $\mathcal{F} \subset TY$ along a connected component of $p_H^{-1}(h)$. But such a component is of strictly positive dimension and therefore would contain a leaf of $\mathcal{F}$. So there are at least two leaves of $\mathcal{F}$ through $(h, y)$, one projecting to $C_t$ and another to a point, which is absurd.

Since $H$ is a K3 surface, it does not admit an elliptic fibration without singular fibres for topological reasons (non-vanishing of the Euler number). This is the contradiction excluding $p_H(D) = H$, and thus establishing Theorem 1.5 when $X$ is not a torus.

Case 2: $X = T$ is a torus. We shall use Ueno’s structure theorem for subvarieties of tori [28, Theorem 10.9].

If $g > 1$, then $\kappa(D) = 1$. By Ueno’s theorem there is a subtorus $K$ of codimension 2 such that $D := p^{-1}(C)$ is the inverse image of a smooth curve $C \subset S$ under the quotient map $p : T \to S := T/K$, $C \subset S$ being a curve of genus $g' > 1$ on the abelian surface $S$. The $\sigma$-orthogonal space to $K$ gives canonically a two-dimensional linear foliation $\mathcal{F}_T$ on $T$, such that the intersections of its leaves with $D$ are the leaves of $\mathcal{F}$, hence smooth compact curves which project in an étale way by $p$ onto $C$ (by construction, we indeed have $TD = p^*(TC)$, and so by taking the $\sigma$-orthogonals, we have that $\mathcal{F}$ is transverse to $K$ at any point of $D$).

Let us show that the leaves of $\mathcal{F}_T$ are compact. Take a leaf $C$ of $\mathcal{F}$ through a point $x \in T$. It is contained in the leaf $L$ of $\mathcal{F}_T$ through $x$. Choose a group structure on $T$ in such a way that $x = 0$. The translate of $C$ by any point $a \in C$ is still contained in the leaf $L$ since $L$ is linear; on the other hand, it is not equal to $C$ for $a$ outside of a finite set, since $g(C) > 1$. Since $L$ is two-dimensional and contains a family of compact curves parameterized by a compact base, $L$ must itself be compact.

Therefore, the leaves of $\mathcal{F}_T$ are translates of an abelian surface $S'$. It suffices now to take a finite étale base change from $S$ to $S'$ to get the desired form $T' = K \times S'$, $D' = K \times C$, $\sigma$ direct sum of symplectic forms on $S'$ and $K$.

If $g = 1$, then $\kappa(D) = 0$, and $D$ is a subtorus of codimension one with an elliptic fibration. There thus exist an elliptic curve $C \subset T$ and a quotient $\pi : T \to R = T/C$ such that $D = \pi^{-1}(V)$, where $V$ is a codimension-one subtorus of the torus $R$. Project $\rho : R \to R/V$, and consider the composition $p : T \to S := R/V$. Then $S$ is an abelian surface, and $C' := p(C)$ is an elliptic curve on it. Moreover, $D = p^{-1}(C)$. Let $K$ be the kernel of $p$: this is a subtorus of $T$ of codimension two. By Poincaré reducibility, there exists an abelian surface $S' \subset T$ such that $S' \cap K$ is finite. After a finite étale cover, $T' = S' \times K$, and $D' = C \times K$ is of the claimed form. $\square$

Remark 4.1. In this last case, $\sigma_T$ is in general not the direct sum of symplectic forms on $S'$ and $K$. Take, for example, $T = S \times A$, $D = E \times A$, for $S$, $A$, $E \subset S$ abelian varieties of dimensions $2$, $n - 2$, 1 respectively, with linear coordinates $(x, y)$ on $S$, $(z_1, \ldots, z_{n-2})$ on $A$, and $E$ given by $x = 0$. Take $\sigma_S := dx \wedge dy$, $\sigma_A$ arbitrary on $A$, and $\sigma = \sigma_S + \sigma_A + dx \wedge dz$, for any non-zero linear form $z$ on $TA$. 
5. **Application to the Lagrangian conjecture**

Our aim is Corollary 5.3 below. First we prove the following proposition.

**Proposition 5.1.** Let $D \subset X$ be a smooth hypersurface in a connected compact Kähler manifold $X$ of dimension $2n$, carrying a holomorphic symplectic 2-form $\sigma$. Denote by $F$ the characteristic foliation on $D$ defined by $\sigma$. Assume that $D$ admits a holomorphic fibration $\psi : D \rightarrow S$ onto an $(n-1)$-dimensional connected complex manifold $S$, such that its general fibre is a lagrangian subvariety of $X$ with nef canonical divisor and of zero Kodaira dimension. Then:

(i) the foliation $F$ is $\psi$-vertical (that is, tangent to the fibres of $\psi$);

(ii) either the smooth fibres of $\psi$ are tori, and then $\psi$ is the restriction to $D$ of a holomorphic lagrangian fibration $\psi'$ on some open neighborhood of $D$ in $X$; or their irregularity $q(F)$ is equal to $n-1$. In this case, the Albanese map $a_F : F \rightarrow \text{Alb}(F)$ is surjective and connected, and its fibres are elliptic curves which are the leaves of $F$. Moreover, $F$ has a finite étale covering which is a torus.

**Proof.** The first claim is obvious, since, at any generic $x \in D$, the $\sigma$-orthogonal to $TD_x$ is included in the $\sigma$-orthogonal to $TF_x$ (where $F$ denotes the fibre of $\psi$ through $x$), which is equal to itself since $F$ is lagrangian.

Since the deformations of our lagrangian fibres $F$ cover $D$, we have $q(F) = h^0(F, \Omega^1_F) = h^0(F, N_{F/X}) \geq \dim(D) - \dim(F) = n - 1$. In the opposite direction, $q(F) \leq n$, since $\kappa(F) = 0$, and the Albanese map $a_F : F \rightarrow \text{Alb}(F)$ is surjective with connected fibres by [16].

If $q(F) = n$, $a_F : F \rightarrow \text{Alb}(F)$ is bimeromorphic. Since the canonical divisor of $F$ is nef, this map is an isomorphism, by Lemma 5.2 below. We recall, for example from [25] (see also [18, 30]) that the deformations of lagrangian submanifolds are unobstructed, and conclude that $F$ deforms in an $n$-dimensional family and this gives a fibration of a neighbourhood of $F$ in $X$ (indeed, the normal bundle to $F$ in $X$ is trivial since it is isomorphic to the cotangent bundle by the lagrangian condition).

Otherwise, $q(F) = n-1$ and the fibres of the Albanese map $a_F$ are one-dimensional. In fact these are elliptic curves by $C_{n,n-1}$ (see [29]), and this also implies that $F$ has a finite étale covering which is a torus.

Finally, the leaves of $F$ inside $F$ are tangent to the fibres of $a_F$. Indeed, since $q = n-1$ and $F$ moves inside an $(n-1)$-dimensional smooth and unobstructed family of deformations (the fibres of $\psi$), all deformations of $F$ stay inside $D$, and the natural evaluation map $ev : H^0(F, N_{F/X}) \otimes O_F \rightarrow TX|_F$ must take its values in $T_{D/F}$.

Assume the leaves of $F$ are not the fibres of $a_F$. We can then choose a 1-form $u$ on $\text{Alb}(F)$ such that $v = a_F^*(u)$ does not vanish on $F$ at the generic point $z$ of $F$. The vanishing hyperplane of $v_z$ in $TF_z$ is, however, $\sigma$-dual to a vector $t_z \in TX_z$, unique and non-zero modulo $TF_z$, which corresponds to the 1-form $v_z$ under the isomorphism $(N_F)_z \cong (\Omega^1_F)_z$ induced by $\sigma$ on the lagrangian $F$. Since $v$ does not vanish on $F$, by assumption, $t_z \notin (TD)_z$, which contradicts the fact that all first-order infinitesimal deformations of $F$ are contained in $D$. \hfill $\Box$

**Lemma 5.2.** Let $a : V \rightarrow W$ be a holomorphic proper bimeromorphic map between compact complex connected manifolds. Assume that $K_{V/W}$ is nef (that is, has non-negative degree on any compact complex curve contained in the exceptional set of $a$). Then $a$ is an isomorphism.

**Proof.** This follows (by contradiction) from three facts. First of all, the lemma is obvious if $a$ is a composition of smooth blow-ups. Next, by Hironaka’s flattening one can dominate $a$ by a composition $a' : V' \rightarrow W$ of such blow-ups. Finally, by the smoothness assumptions, $K_{V'/V}$ is a positive linear combination of the irreducible components of the exceptional divisors of
shows that the lagrangian conjecture is true for $q\psi_*$ is semi-ample. Assume that $X$ is an irreducible curve in a fibre of the last blow-up of $a'$, we get the conclusion (we may indeed assume that none of the blow-ups of $a'$ factorizes through $a$, so that $a''(C) \neq 0$).

**Corollary 5.3.** Assume that $X$ is an irreducible hyperkähler manifold of dimension $2n$ (not necessarily projective), and $D \subset X$ a smooth reduced and irreducible divisor. Assume that $K_D$ is semi-ample. Then $\mathcal{O}_X(D)$ is semi-ample.

**Proof.** If the Beauville–Bogomolov square $q(D, D)$ is positive, then $D$ is big, $X$ is projective and the statement follows from the Kawamata base point freeness theorem. So the interesting case is when $D$ is Beauville–Bogomolov isotropic. We have $K_D = \mathcal{O}_X(D)|_D$. If $K_D$ is semi-ample, its Kodaira dimension is equal to $\nu(K_D) = n - 1$ (Lemma 3.1) and the Iitaka fibration $\psi$ is regular. The relative dimension of $\psi$ is equal to $n$. In fact $q(D, D) = 0$ implies that $\psi$ is lagrangian in the same way as in [20] (using that $K_D = \mathcal{O}_D(D)$ and that a suitable positive multiple $m.F$ of the fibre $F$ is $\psi^*(H^{n-1})$ for some very ample line bundle $H$ on $S$). By Proposition 5.1, we have two possibilities: either $F$ is a torus, and then the fibration $\psi$ extends near $D$, since $F$ must deform in an $n$-dimensional family; or $F$ is of Albanese dimension $n - 1$ and the characteristic foliation on $D$ is algebraic.

In the first case we conclude by [13, 15] and [21]. In the second case, we notice that since $F$ has numerically trivial canonical bundle, the fibres of the characteristic foliation, which by Proposition 5.1 are tangent to $F$, must be elliptic curves by the adjunction formula. Therefore the characteristic foliation is isotrivial, and Corollary 2.13 and the proof of Theorem 1.3 together imply that this is impossible unless in the case $n = 1$, which is well known.

Recall that the lagrangian conjecture affirms that a non-zero nef Beauville–Bogomolov isotropic divisor is semi-ample (and thus there is a lagrangian fibration associated to some multiple of such a divisor). Corollary 5.3 shows that the lagrangian conjecture is true for an effective smooth divisor on a holomorphic symplectic manifold of dimension $2n$, if the abundance conjecture holds in dimension $2n - 1$. Since the abundance conjecture is known in dimension three, we have the following result.

**Corollary 5.4.** Let $X$ be an irreducible hyperkähler manifold of dimension four, and $D$ a nef divisor on $X$. Assume that $D$ is effective and smooth. Then $\mathcal{O}_X(D)$ is semi-ample.

Note that if $\dim(X) = 4$, we can use [1] instead of [13] and [15], and [2] instead of [21], so that the proof becomes more elementary in this case.

**Acknowledgements.** We are grateful to Jorge Pereira who suggested looking at $(d - 1)$-forms rather than just at 2-forms, and to Michael McQuillan who asked a question which led us to a simplification of the original argument.

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