Coupling Supersymmetric Nonlinear Sigma Models to Supergravity

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(Received April 1, 2010; Revised August 29, 2010)

It is known that supersymmetric nonlinear sigma models for the compact Kähler manifolds $G/H$ cannot be consistently coupled to supergravity, since the Kähler potentials are not invariant under $G$ transformation. We show that the supersymmetric nonlinear sigma models can be deformed such that the Kähler potential becomes exactly $G$-invariant if and only if one enlarges the manifolds by dropping all the $U(1)$'s in the unbroken subgroup $H$. Then, those nonlinear sigma models can be coupled to supergravity without losing the $G$ invariance.

Subject Index: 111, 116, 131, 146

§1. Introduction

One of the fundamental questions in particle physics is why nature chooses three families of quarks and leptons, but not more than that. Supersymmetric (SUSY) nonlinear sigma models may provide an answer to this question, since fermion partners of Nambu-Goldstone (NG) bosons may be identified with all quarks and leptons in the standard model. The number of families is determined by the geometry of a given coset-space $G/H$. In fact, a SUSY $E_7/SU(5) \times U(1)^3$ nonlinear sigma model is known to accommodate three families of quarks and leptons.² It is remarkable that, if one goes to larger nonlinear sigma models using $E_8$, one obtains a pair of extra family and anti-family in addition to the three families, and hence the net number of families remains three.

For general SUSY nonlinear sigma models, manifolds consisting of NG bosons should be complex manifolds. If $G/H$ itself is a Kähler manifold, the Kähler potential for the NG chiral multiplets is uniquely determined by the geometry of $G/H$. The Kähler potential $K$ transforms as $K(\phi, \phi^\dagger) \rightarrow K(\phi, \phi^\dagger) + F(\phi) + F^\dagger(\phi^\dagger)$ under global $G$ transformation, where $F$ is a holomorphic function of the NG superfields $\phi_i$. Since the Lagrangian is given by $\int d^2\theta d^2\bar{\theta} K(\phi, \phi^\dagger)$ in the rigid SUSY theory, it is invariant under global $G$ transformation.

However, the supergravity (SUGRA) Lagrangian is given by³,⁴

$$\left[ \Sigma \Sigma^\dagger e^{-K(\phi, \phi^\dagger)} \right]_D + \left[ \Sigma^3 W(\phi) \right]_F,$$

where $\Sigma$ is the chiral compensator of Weyl weight one in the superconformal tensor calculus, which leads to the so-called old minimal Poincaré supergravity, and
[V^{(w=2)}]_D and [\Phi^{(w=3)}]_F are the D-term and F-term superconformal invariant action formulae applicable to the general-type vector multiplet \( V^{(w=2)} \) of Weyl weight two and the chiral multiplet \( \Phi^{(w=3)} \) of Weyl weight three, respectively.\(^5\) The \( G \) invariance is maintained by a \( \Sigma \) transformation, \( \Sigma \to \Sigma e^F \), provided that the superpotential \( W \) vanishes.\(^6\) This is the case for a pure nonlinear sigma model system.

Actually, however, there are two problems here. First, even in the case of vanishing superpotential, the \( \Sigma \) transformation implies that the local dilatation and chiral \( U(1) \) transformations have to be performed on the system in order to maintain the canonical Einstein and Rarita-Schwinger form of supergravity. However, such local transformations suffer from anomaly so that the \( G \)-invariance is generally violated quantum-mechanically. Secondly, when describing the real world, we also have matter fields besides the nonlinear sigma-model fields that usually appear in the superpotential term. In reality, moreover, we need at least a constant term in \( W \) to get a (nearly) vanishing cosmological constant. Then, the \( G \)-invariance is explicitly broken even at a classical level and NG bosons get masses on the order of the gravitino mass \( m_3/2 \). These are the problems whenever a non-vanishing \( F(\phi) \) appears in the \( G \) transformation of the Kähler potential \( K(\phi, \phi^\dagger) \). This is, particularly, the case when \( G/H \) is a compact Kähler manifold.

Nibbelink and van Holten\(^7\) have proposed a method of making the Kähler potential \( G \)-invariant to circumvent the above problem.\(^*)\) They have added matter fields \( S_i \) and assigned a \( U(1) \) charge \( q_i \) to them such that the superpotential \( W(S) \) carries charge three. Then, after Kähler-Weyl transformation \( \Sigma^3 W(S) \to \Sigma^3 \), the Kähler potential becomes

\[
K_{NL}(\phi, \phi^\dagger) + \frac{1}{3} \ln |W(S)|^2 ,
\]

which become \( G \)-invariant if the matter fields are assumed to receive an additional transformation under \( G \) transformation:

\[
\delta_G^{\text{additional}} S_i = -q_i F(\phi) S_i \quad \Rightarrow \quad \delta_G W(S) = -3 F(\phi) W(S) ,
\]

where \( F(\phi) \) is the holomorphic shift of the nonlinear sigma model Kähler potential under \( G \) transformation: \( \delta_G K_{NL}(\phi, \phi^\dagger) = F(\phi) + F^\dagger(\phi^\dagger) \). The matter kinetic terms, \( S_i^\dagger e^{q_i K_{NL}(\phi, \phi^\dagger)} S_i \), or any of their functions, should also be added to the total Kähler potential \( K(\phi, \phi^\dagger, S, S^\dagger) \). If the logarithm of superpotential, \( \ln W(S) \), here is replaced with a single \( H \)-singlet matter \( \Phi \), then this proposal becomes equivalent to the recent proposal by Komargodski and Seiberg.\(^9\) One problem in this proposal, however, is that the origin of such a singlet field \( \Phi \) or the set of fields \( S_i \), which have very special transformation proportional to \( F(\phi) \), is not clear, although Komargodski and Seiberg identified \( \Phi \) with an extra field of the supergravity compensating multiplet appearing in a new extension of the minimal SUGRA formulation.

In this paper, therefore, we propose another way of solving the above problem. We show that if one drops all the \( U(1) \)'s in the unbroken subgroup \( H \) of the Kähler

\(*\) Essentially the same way out as theirs was proposed much earlier for the Fayet-Iliopoulos term problem by Ferrara et al.\(^8\)
manifold $G/H$, one can construct a SUSY nonlinear sigma model with an exactly $G$-invariant Kähler potential that can be safely coupled to SUGRA. Since the $U(1)$’s are broken, we have additional singlet NG multiplets. Those NG fields play the role of the above singlet $\Phi$ in solving the problem. We also show that the noninvariance under $G$ transformation has the same origin as the inconsistency problem of the Fayet-Iliopoulos terms in SUGRA.

§2. Invariant Kähler potential

Bando, Kuramoto, Maskawa and Uehara (BKMU)\textsuperscript{10} have presented three prescriptions, A-type to C-type, for constructing the invariant action for the supersymmetric system of the nonlinear realization of $G/H$. They have also shown that there is no A- or C-type invariant action and that only the B-type action is available in the Kählerian $G/H$ case. The B-type Kähler potential is generally noninvariant under nonlinearly realized $G/H$ transformation, but yields the so-called Kähler transformation $K(\phi, \phi^\dagger) \to K(\phi, \phi^\dagger) + F(\phi) + F^\dagger(\phi^\dagger)$. We determine which conditions are necessary and sufficient for the manifold $G/H$ in order for the holomorphic function $F(\phi)$ to always vanish for $G$ transformation. We discuss, in this paper, only the B-type action formulas, since phenomenologically interesting nonlinear sigma models are mostly the cases of the Kähler manifold $G/H$, like $E_7/SU(5) \times U(1)^3$, as discussed in the introduction. Moreover, this is also sufficient since the other A-type and C-type actions, if any, already give invariant Kähler potentials.

2.1. Example

Before going into the general consideration, it is helpful to recall an old explicit example that gives an exactly $G$-invariant Kähler potential. It is a supersymmetric nonlinear sigma model based on the manifold $U(4n_f + 2)/U(4n_f) \times SU(2)$, which the present authors and Uehara once discussed in Ref. 11), referred to as KUY henceforth, in the context of realizing weak $SU(2)$ gauge bosons as composite gauge fields of a hidden local symmetry.

Their Lagrangian is given by the arbitrary function of the $G$-invariant quantity

$$\det_{i,j} \left[ \Phi^i_{a} \Phi^\alpha_j \right]$$

for the $(4n_f + 2) \times 2$ matrix chiral fields of the form

$$\Phi^\alpha_i = \frac{2}{4n_f} \left( \begin{array}{c} e^\phi \delta^\alpha_i \\ \phi^\alpha_i \end{array} \right), \quad \begin{array}{c} (i = 1, 2) \\ (\alpha = 1, \cdots, 4n_f + 2) \\ (a = 3, \cdots, 4n_f + 2) \end{array} (2.2)$$

Here, $\phi$ is a single-component chiral field and $\phi^\alpha_a$ is $4n_f \times 2$-component one. This variable can be identified with the first 2 columns of the $(4n_f + 2) \times (4n_f + 2)$ BKMU’s basic matrix variable $\xi$ (explained later) in this case, and so it is nonlinearly transformed under $g \in G$ transformation as

$$\Phi' = g \Phi h^{-1}(g, \Phi), \quad h(g, \Phi) \in SU(2)^C. \quad (2.3)$$
Here, the $2 \times 2$ matrix $h(g, \Phi)$ is determined such that the first $2 \times 2$ component of the transformed $\Phi'$ also takes the diagonal form $\Phi'^\alpha_i = e^{\phi'} \delta^\alpha_i$.

The point is that the present manifold $U(4n_f + 2)/U(4n_f) \times SU(2)$ is not the Kähler manifold; it is different from the Kähler one $U(4n_f + 2)/U(4n_f) \times U(2)$ at the point that the unbroken $U(2)$ in the latter is further broken down to $SU(2)$. Therefore, the $2 \times 2$ matrix $h(g, \Phi)$ contains no $U(1)$ part so that
\[
\text{tr} \left[ \ln h(g, \Phi) \right] = 0 \quad \Rightarrow \quad \det[h(g, \Phi)] = 1. \tag{2.4}
\]
This is the reason why determinant (2.1) is invariant:
\[
\det \left[ \Phi'^\dagger \Phi' \right] = \det[h^{-1}(g, \Phi)] \cdot \det[\Phi^\dagger \Phi] \cdot \det[h^{-1}(g, \Phi)] = \det[\Phi^\dagger \Phi]. \tag{2.5}
\]
In the Kähler case, on the other hand, the first $2 \times 2$ component of the field $\Phi$ is constrained to be $\delta^\alpha_i$ and $h(g, \Phi)$ contains the $U(1)$ factor (which can be identified with $\exp(\phi - \phi')$ by inspection of the transformation law (2.3)).

2.2. General argument

This was totally generalized by Buchmüller and Ellwanger\textsuperscript{12}) following the general theory of supersymmetric nonlinear realization, which was developed by BKMU\textsuperscript{10}) and further elucidated by Itoh, Kugo, and Kunitomo (IKK)\textsuperscript{13}) for the Kählerian manifold case.

Any homogeneous Kähler manifold $G/H$ is characterized by the fact that $H$ is the centralizer of a torus $U(1)^n$;\textsuperscript{13}) that is, the unbroken subgroup $H$ has the form $H = H_{\text{S.S.}} \times U(1)^n$ with a semisimple $H_{\text{S.S.}}$ and consists of all the $G$ elements commutative with the torus. It has been known\textsuperscript{14), 15}) that spontaneous symmetry breaking in the supersymmetric theory does not lead to sigma models based on Kählerian $G/H$ but rather to those containing extra bosons (called quasi-NG bosons) in addition to the true NG bosons. The motivation of Buchmüller and Ellwanger was to construct nonlinear models that can be realized from the linear models by spontaneous symmetry breaking. Therefore, as the simplest possibility, they considered the sigma models based on the coset spaces $G/H$ with $H = H_{\text{S.S.}} \times U(1)^m$ $(m < n)$, where $n - m U(1)$ symmetries are further broken from the Kählerian case $H = H_{\text{S.S.}} \times U(1)^n$.

First, we have to recall the BKMU’s construction of the general Kähler potential in the Kählerian case $H = H_{\text{S.S.}} \times U(1)^n$ following IKK. Let us call the generators of the torus $U(1)^n$ ‘central charges’ $Q_i$ $(i = 1, \cdots, n)$. With respect to an arbitrarily chosen linear combination of these central charges, $Y \equiv \sum_i v_i Q_i$, the generators of $G^c/H^c$ are divided into positive $(X_I)$ and negative $(\bar{X}_I)$ generators carrying positive and negative $Y$-charges, respectively. Then, the generators $S_a$ of the semisimple subgroup $H_{\text{S.S.}}$, the central charges $Q_i$, and the positive generators $X_I$ span a complex subgroup $\hat{H}$, and the negative generators $\bar{X}_I$ correspond to $G^c/\hat{H}$:
\[
G^c = \{ ( X_I, S_a, Q_i ) \in \hat{H}, \quad \bar{X}_I \in G^c - \hat{H} \}. \tag{2.6}
\]

The basic variable of the BKMU theory is the variable parametrizing the (right) coset space $G^c/\hat{H}$:
\[
\xi(\phi) \equiv \exp(\phi \cdot \bar{X}), \quad \phi \cdot \bar{X} \equiv \phi^I \bar{X}_I, \tag{2.7}
\]
where $\phi^I$ are the NG chiral superfields corresponding to complex broken generators $X_I$. The nonlinear transformation of the NG superfields, $\phi \rightarrow \phi'$, by the (real) group element $g \in G$ is defined as

$$g \xi(\phi) = \xi(\phi') \hat{h}(\phi, g), \quad \hat{h} \in \hat{H} \quad (2.8)$$

with

$$\hat{h}(\phi, g) = \exp(\alpha \cdot X) \exp(\beta \cdot S) \exp(i\gamma \cdot Q),$$

$$\alpha \cdot X = \alpha^I(\phi, g)X_I, \quad \beta \cdot S = \beta^a(\phi, g)S_a, \quad i\gamma \cdot Q = i\gamma^i(\phi, g)Q_i. \quad (2.9)$$

The most general Kähler potential for the Kählerian manifold $G/H$ takes the form

$$K(\phi, \bar{\phi}) = \sum_{i=1}^{n} c_i \ln \det_{\eta_i} \left( \xi^\dagger(\phi) \xi(\phi) \right), \quad (2.10)$$

with $n$ arbitrary coefficients $c_i$, where $\eta_i$ are projection matrices in the fundamental representation space$^1$ $V$ satisfying

$$\hat{H}\eta_i = \eta_i \hat{H}, \quad \eta_i^2 = \eta_i, \quad \eta_i^\dagger = \eta_i. \quad (2.11)$$

$\det_{\eta_i}$ denotes the determinant in the $\eta_i$ projected subspace $\eta_i V$. It was shown by IKK that the number of independent projection matrices indeed equals $n$, the number of central charges in $H$.

If we use the property (2.11) of the projection matrices $\eta_i$, we immediately obtain

$$\det_{\eta_i}(\xi^\dagger \xi') = \det_{\eta_i}(\hat{h}^{-1\dagger}) \det_{\eta_i}(\xi^\dagger \xi) \det_{\eta_i}(\hat{h}^{-1}). \quad (2.12)$$

This implies that the Kähler potential (2.10) is transformed under $G$ transformation as

$$K(\phi, \phi^\dagger) \xrightarrow{g \in G} K(\phi', \phi'^\dagger) = K(\phi, \phi^\dagger) + F(\phi) + F^\dagger(\phi^\dagger),$$

with $F(\phi) = \sum_i c_i F_i(\phi), \quad F_i(\phi) = \ln \det_{\eta_i}(\hat{h}^{-1}(\phi, g)). \quad (2.13)$

Since $F(\phi)$ and $F^\dagger(\phi^\dagger)$ are holomorphic and anti-holomorphic functions of chiral superfields, respectively, the integral $\int d^2\theta d^2\bar{\theta} K(\phi, \phi^\dagger)$ is $G$-invariant.

Up to here, the review of the BKMU theory is a la IKK in the Kählerian $G/H$ case.

Buchmüller-Ellwanger noted an important identity,

$$\det_{\eta_i} \hat{h}(\phi, g) = \exp[i \tr(\eta_i \gamma \cdot Q)], \quad (2.14)$$

which follows from the expression (2.9) and the fact that the generators $X_I$ and $S_a$ are traceless even in the projected subspace $\eta_i V$ owing to the property (2.11) of the

$^1$ For the B-type actions, it is sufficient to consider only the fundamental representation.\textsuperscript{10,13}

Therefore, all the generators and group elements are henceforth understood to be the representation matrices in the fundamental representation.
projection matrices $\eta_i$. Therefore, the noninvariance of the Kähler potential solely comes from the central charge (i.e., $U(1)$) terms in $H$:

$$\ln \det_{\eta_i}(\xi^\dagger \xi) = \ln \det_{\eta_i}(\xi^\dagger \xi) - i \text{tr} \left[ \eta_i (\gamma - \bar{\gamma}) \cdot Q \right].$$

(2.15)

Or, more explicitly, the holomorphic shift $F_i(\phi)$ of the Kähler potential $\ln \det_{\eta_i}(\xi^\dagger \xi)$ under $G$ transformation is given by

$$F_i(\phi) = -i \sum_j A_{ij} \gamma^j, \quad A_{ij} \equiv \text{tr} \left[ \eta_i Q_j \right].$$

(2.16)

Now, we turn to the cases of $G/\tilde{H}$ with $\tilde{H} = H_{S,S} \times U(1)^m$ ($m < n$), that is, the cases where $n - m$ $U(1)$ symmetries are further broken from the Kählerian case $H = H_{S,S} \times U(1)^n$. The $n$ central charges $Q_i$ split into the $n - m$ broken charges $Q_k$ ($k = 1, \ldots, n - m$) and $m$ unbroken ones $\tilde{Q}_\ell$ ($\ell = 1, \ldots, m$):

$$\{ Q_i (i = 1, \ldots, n) \} = \{ Q_k (k = 1, \ldots, n-m), \tilde{Q}_\ell (\ell = 1, \ldots, m) \}.$$

(2.17)

For distinction from the previous Kähler case, the complex unbroken subgroup is now denoted as $\tilde{H}$. Then the splitting of the generators of $G^c$ now becomes

$$G^c = \{ (X_I, S_a, \tilde{Q}_\ell) \in \tilde{H}, \quad (X_I, Q_k) \in G^C - \tilde{H} \}.$$

(2.18)

We should note that the same projection matrices $\eta_i$ as before still gives the complete set of projection matrices satisfying the desired property (2.11) even when $\tilde{H}$ is replaced with $\tilde{H}$.

Denoting the NG superfields corresponding to the additional broken $U(1)$ charges $Q_k$ as $\Phi^k$, we can take the following form for the BKMU variables parametrizing the coset $G^c/\tilde{H}$ in this case:

$$\zeta(\phi, \Phi) = \exp(\phi^I X_I) \exp(i \Phi^k Q_k)$$

$$= \xi(\phi) \exp(i \Phi \cdot \bar{Q}),$$

(2.19)

where $\xi(\phi)$ is the same part as the previous BKMU variable in (2.7) in the Kählerian case $G/H$. The nonlinear $G$ transformation is defined as is usual as

$$g \zeta(\phi, \Phi) = \zeta(\phi', \Phi') \tilde{h}(\phi, \Phi, g),$$

$$\tilde{h}(\phi, \Phi, g) = \exp(\bar{\alpha} \cdot X) \exp(\bar{\beta} \cdot S) \exp(i \bar{\gamma} \cdot \bar{Q}) \in \tilde{H}.$$

(2.20)

Then, the same argument as above gives the transformation law

$$\ln \det_{\eta_i}(\zeta^\dagger \zeta') = \ln \det_{\eta_i}(\zeta^\dagger \zeta) - i \text{tr} \left[ \eta_i \sum_{\ell=1}^{m} (\bar{\gamma} - \bar{\gamma}) \bar{Q}_\ell \right],$$

(2.21)

since the noninvariance of $\ln \det_{\eta_i}(\zeta^\dagger \zeta)$ again solely comes from the central charge ($U(1)$) terms contained in $\tilde{H}$. However, the relation (2.19) between the variables $\zeta$ and $\xi$ tells us that

$$\ln \det_{\eta_i}(\zeta^\dagger \zeta) = \ln \det_{\eta_i}(\xi^\dagger \xi) + i \text{tr} \left[ \eta_i (\Phi - \bar{\Phi}) \cdot \bar{Q} \right].$$

(2.22)
This relation and the comparison of the transformation laws (2.15) and (2.22) gives
\(-i \text{tr} [\eta_i (\gamma - \bar{\gamma}) \cdot Q] + i \text{tr} [\eta_i (\Phi' - \bar{\Phi}') \cdot Q] = +i \text{tr} [\eta_i (\Phi - \bar{\Phi}) \cdot Q] - i \text{tr} [\eta_i (\bar{\gamma} - \gamma) \cdot \bar{Q}].\)
\[(2.23)\]
Noting that this holds for \(\forall \eta_i\) and \(\{Q_i\} = \{Q_k, \bar{Q}_\ell\}\), we find from this that
\(-\gamma^{i=k}(\phi, g) + \Phi'^{k} = \Phi^{k}, \quad (k = 1, \ldots, n-m)\)
\(\gamma^{i=\ell}(\phi, g) = \bar{\gamma}^{\ell}(\phi, g).\)
\[(2.24)\]
The first equations imply that the noninvariance of \(\ln \det_{\eta_i}(\xi^\dagger \xi)\) proportional to the central charge \(Q_i=k = Q_k\) is canceled by the new NG chiral superfield \(\Phi^{k}\) if the \(U(1)\) charge is spontaneously broken. Therefore, if all the central charges \(Q_i\) contained in \(H\) in the Kählerian manifold \(G/H\) are spontaneously broken, that is, when \(\tilde{H} = H_{S.S.}\) (i.e., \(m = 0\)), then all the determinant factors
\[\det_{\eta_i}(\xi^\dagger \xi) \quad (i = 1, \ldots, n)\]
become \(G\)-invariant. Therefore, their arbitrary functions are \(G\)-invariant. Our conclusion is that we need to break all the \(U(1)\) factors in \(H\) in order to have \(G\)-invariant Kähler potentials \textit{generically}.

However, if we are contented with \textit{special} Kähler potentials, we need not break all the \(U(1)\) factors in \(H\). Indeed, even when \(m \neq 0\), there are \(n-m\) NG chiral multiplets \(\Phi^{k}\) and so \(n-m\) exactly \(G\)-invariant linear combinations can be constructed from the \(n \ln \det_{\eta_i}(\xi^\dagger \xi)\). Their arbitrary functions give exactly \(G\)-invariant Kähler potentials. We should, however, note that such a Kähler potential is absolutely nongeneric since it excludes some combinations of \(\ln \det_{\eta_i}(\xi^\dagger \xi)\), and hence may not even contain the kinetic term of some of the NG superfields.

The fact that one can construct some \(G\)-invariants even if not all the \(U(1)\)'s are broken was first pointed out by Buchmüller and Ellwanger in a generic context. However, a concrete example of such a phenomenon has already been given prior to their report; it was nothing but the KUY’s sigma model based on \(G/H = U(4n_f+2)/U(4n_f) \times SU(2)\), which we explained in the preceding subsection. There, \(H\) still contains a \(U(1)\); nevertheless, we had a \(G\)-invariant Kähler potential. To understand the situation, we note that actually two projection operators exist there, corresponding to the \(H\)-irreducible decomposition of \(4n_f+2\)-dimensional vector space into a 2-dimensional one and a \(4n_f\)-dimensional one:
\[\eta_1 = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \quad \eta_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1_{4n_f} \end{pmatrix}.\]
\[(2.26)\]
Actually, \(\eta_2\) is the unit operator in the total vector space so that it is not a genuine projection operator, but it is an allowed “projection” operator giving a nontrivial Kähler potential in this case since \(G = U(4n_f+2) = SU(4n_f+2) \times U(1)\) itself contains a \(U(1)\). KUY’s Kähler potential contains only the determinant factor (2.1), which is identified with \(\det_{\eta_1}(\xi^\dagger \xi)\) using the \(\eta_1\) projection operator alone, then the \(U(1)\) factor group element contained in \(H = U(4n_f) = SU(4n_f) \times U(1)\) does not contribute to \(\text{tr} [\eta_1 \ln (\hat{h})]\) since the \(U(1)\) is acting only in the \(4n_f\)-dimensional subspace \(1-\eta_1\)\(V.\)
Finally, in this section, we should note the following. The linear combination of Eq. (2.22) gives the following equation for the generic Kähler potential $K(\phi, \phi^\dagger)$ in Eq. (2.10) for the Kählerian manifold $G/H$:

$$\sum_{i=1}^{n} c_i \ln \det_{\eta_i} (\zeta \zeta^\dagger) = K(\phi, \phi^\dagger) + \Phi + \Phi^\dagger, \quad (2.27)$$

where

$$\Phi \equiv -i \sum_{i,j=1}^{n} c_i A_{ij} \Phi_j, \quad A_{ij} = \text{tr} [\eta_i Q_j]. \quad (2.28)$$

Since $\det_{\eta_i} (\zeta \zeta^\dagger)$ are $G$-invariant when all the $U(1)$’s in $H$ are spontaneously broken, the RHS combination $K(\phi, \phi^\dagger) + \Phi + \Phi^\dagger$ is also $G$-invariant. This implies that the combination (2.28) of central charge $NG$ superfields $\Phi_j$ plays a similar role to the special massless matter field $\Phi$ of Komargodski and Seiberg,\textsuperscript{9) which cancels the holomorphic shift $F(\phi)$ of the Kähler potential $K(\phi, \phi^\dagger)$. If we consider only this combination (2.27) of the Kähler potential with the coefficients $c_i$ fixed, then the single matter field $\Phi$ of Eq. (2.28) is enough to cancel the holomorphic shift $F(\phi)$. However, if one considers the most general cases, we need $n$ central charge $NG$ superfields $\Phi_j$. Moreover, our fields $\Phi^i$ have a clear basis for existence as NG superfields corresponding to the spontaneous breaking of the $U(1)$ charges $Q_j$ so that they obey the low-energy theorem and their Lagrangian is naturally determined by the symmetry principle.

§3. Common origin with the Fayet-Iliopoulos term problem

Komargodski and Seiberg\textsuperscript{9) have already noted the similarity between the difficulties in coupling the system to supergravity in two cases of the nonlinear sigma model and Fayet-Iliopoulos term. However, these two problems are not merely similar but actually are of the same origin. This is because all the nonlinear $G/H$ sigma-model Lagrangians can be cast into the form in which the hidden local symmetry $H_{\text{local}}$ is made manifest by introducing auxiliary gauge field variables. If $H$ contains $U(1)$ factor groups, the rewritten Lagrangian with a manifest hidden-local symmetry contains the Fayet-Iliopoulos term for the hidden-local $U(1)$ vector superfield.

In the simplest case of the Grassmannian Kähler manifold $U(n + m)/U(n) \times U(m)$, this fact has been known for a long time. Zumino\textsuperscript{16) has written the Kähler potential for this coset space in the form

$$K(\phi, \phi^\dagger) = \text{tr} \ln (1_n + \phi^\dagger \phi) = \ln \det (1_n + \phi^\dagger \phi), \quad (3.1)$$

where $\phi$ is a chiral superfield valued $m \times n$ matrix, which is related to the BKMU coset variable $\xi \in G^C/\hat{H}$ by

$$\xi = n \begin{pmatrix} 1_n & m \\ \phi & 1_n \end{pmatrix}, \quad (3.2)$$
and Zumino’s action is identical to BKMU’s

\[ K(\phi, \phi^\dagger) = \ln \det \eta_1 (\xi^\dagger \xi) = \ln \det \left[ \begin{pmatrix} 1_n & \phi^\dagger \\ \phi & 1_n \end{pmatrix} \right], \tag{3.3} \]

with the unique projection operator \( \eta_1 \) in this case:

\[ \eta_1 = \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.4} \]

Aoyama\(^{17}\) has shown for the first time that this Lagrangian is equivalently rewritten into the following form possessing \( U(n)_{\text{local}} \) symmetry:

\[ \mathcal{L} = \int d^2 \theta d^2 \bar{\theta} K(\Phi, \Phi^\dagger, V), \quad K(\Phi, \Phi^\dagger, V) = \text{tr} (\Phi^\dagger \Phi e^V) - g \text{tr} V, \tag{3.5} \]

where \( \Phi \) is an \((n+m) \times n\) matrix chiral superfield and \( V \) is an \( n \times n \) \( U(n) \) gauge superfield. This Lagrangian is manifestly invariant under global \( U(n+m) \) and local \( U(n) \) transformations:

\[ \Phi \rightarrow g \Phi h^{-1}(x, \theta, \bar{\theta}), \quad g \in U(n+m), \quad h(x, \theta, \bar{\theta}) \in U(n)_{\text{local}}. \tag{3.6} \]

Since the \( U(n) \) gauge superfield \( V \) is an auxiliary field with no kinetic term, the \( V \) equation of motion can be solved:

\[ \frac{\delta \mathcal{L}}{\delta V} = 0 \quad \Rightarrow \quad g e^{-V} = \Phi^\dagger \Phi \quad \Rightarrow \quad -V = \ln (\Phi^\dagger \Phi). \tag{3.7} \]

Substituting this back into Eq. (3.5), we find that the Kähler potential becomes

\[ K = g \text{tr} \ln (\Phi^\dagger \Phi) = g \ln \det (\Phi^\dagger \Phi). \tag{3.8} \]

This still possesses the \( U(n)_{\text{local}} \) gauge symmetry with a chiral superfield parameter, so that we can take

\[ \Phi = \frac{n}{m} \begin{pmatrix} 1_n \\ \phi \end{pmatrix}, \tag{3.9} \]

as a gauge fixing condition. Then it reduces to the original Zumino’s form of the Kähler potential.

If we take this gauge condition in the Lagrangian (3.5), global \( G \) transformation induces the local gauge transformation \( h(\phi, g) \in U(n)_{\text{local}} \) in order to keep the form (3.9). Then it is actually the Fayet-Iliopoulos term \(-g \text{tr} V\) that yields the holomorphic term shift \( F(\phi) = \text{tr} \ln h(\phi, g) \) of the Kähler potential. This clearly shows the equivalence of the two problems of the nonlinear sigma model and Fayet-Iliopoulos term.

This rewriting of the Lagrangian into the form in which the hidden local symmetry \( \hat{\mathcal{H}} \) is manifest was given for more general Kähler manifold cases in Ref. 18). Therefore, we can generally see the common root of the two problems.
§4. Discussion and conclusions

The SUSY nonlinear sigma model for $E_7/SU(5) \times U(1)^3$ is very interesting, since it accommodates just three families of quark and lepton chiral multiplets as NG multiplets. However, this nonlinear sigma model has two independent problems. One is that it suffers from the so-called nonlinear sigma model anomalies.\(^{19}\) That is, the fermion path-integral is ill-defined on the Kähler manifold. The other problem is that the model cannot be coupled to SUGRA unless the superpotential vanishes, $W = 0$.\(^{6}\)

It was pointed out\(^{20}\) that the former problem can be solved if one introduces one extra matter multiplet transforming as $5^*$ of the unbroken $SU(5)$. In this paper, we have proposed to eliminate all $U(1)$ subgroups from the unbroken subgroup $H$ in order to couple the nonlinear sigma models to SUGRA. We have also clarified that we can make the Kähler potentials for nonlinear sigma models completely invariant under the global symmetry $G$ so that the nonlinear sigma models can be coupled to SUGRA without any explicit breaking of the $G$ symmetry.

Now we propose an $E_7/SU(5)$ nonlinear sigma model coupled to SUGRA, which consists of three $(5^* + 10 + 1 + 1) + 5$ as NG multiplets and one extra matter $5^*$. We must introduce gauge interactions of the $SU(3) \times SU(2) \times U(1)$ subgroup of the unbroken $SU(5)$ and also Yukawa couplings for quark and lepton chiral multiplets to make the model realistic. They are explicit breaking terms of the total $G$ symmetry. We do not know yet the origin of the breaking terms, but if they are the only sources of the explicit breaking we may have interesting predictions testable at LHC.\(^{21}\) Namely, all squarks and sleptons are massless at the tree level even after the SUSY is spontaneously broken. This is because the $G$ invariance is kept unbroken in the limit of all gauge and Yukawa interactions switched off, and the masslessness of the NG bosons is guaranteed as long as the SUSY breaking sector never breaks the $G$ symmetry. Moreover, the introduction of gauge and Yukawa interactions do not contribute to the NG boson’s SUSY-breaking masses at the tree level. Therefore, it is reasonable to assume that all squarks and sleptons are massless at the cutoff scale, say the Planck or GUT scale. Then, the squarks and sleptons receive their masses from higher-order corrections from gauge and Yukawa interactions, which are calculable. It is extremely interesting to test this hypothesis at LHC.

Acknowledgements

The authors thank K.-I. Izawa and K. Yonekura for useful discussions. T. Kugo is grateful to the hospitality of staff of IPMU, where this work was performed. TK is supported by a Grant-in-Aid for the Global COE Program “The Next Generation of Physics, Spun from Universality and Emergence” from the Ministry of Education, Culture, Sports, Science and Technology (MEXT) of Japan. TK is also partially supported by a Grant-in-Aid for Scientific Research (B) (No. 20340053) from the Japan Society for the Promotion of Science. This work was supported by the World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan.
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