Functional Limit Theorems of moving averages of Hermite processes and an application to homogenization

Johann Gehringer
Imperial College London *

August 10, 2020

Abstract
We generalise the homogenisation theorem in [GL19, GL20a] for passive tracer in a fractional Gaussian field to fractional non-Gaussian fields. We also obtain the limit theorems of normalized functionals of Hermite-Volterra processes, extending the result in [DT18] to power series with fast decaying coefficients. We obtain either convergence to a Wiener process, in the short-range dependent case, or to a Hermite process, in the long-range dependent case. Furthermore, we prove convergence in the multivariate case with both, short and long-range dependent components and give an application to homogenization of fast/slow systems.

keywords: Central limit theorems, fractional noise, Hermite Ornstein-Uhlenbeck process, Hermite processes

MSC Subject classification: 60G22, 60F05, 60G10, 60G18

Contents
1 Introduction 2
2 Statement of the Theorems 3 4 5 6 8 9 11 15 16 20 25 25 28 30
2.1 Statement of Results 2
3 Notation 5
4 Preliminaries 5 6 8 9
4.1 Hermite processes . 5 6
5 Decomposition and convergence for building blocks 8 11 15 16
5.1 Decomposition . 8 9
5.2 Examining the scaling behaviour 11
5.3 Tightness in Hölder spaces 15
5.4 Short range dependent case 16
5.5 Long range dependent case 20
6 General Functions 25 28
6.1 Short range dependent case 25
6.2 Long range dependent case 28
7 Mixed multivariate case 30

*johann.gehringer18@imperial.ac.uk
1 Introduction

The motion of a passive tracer in a fractional turbulent noise field can be modelled by an equation of the form

\[ \dot{x}_t = F(x_t, y_t^\varepsilon), \]

where \( y_t^\varepsilon \) denotes the rescaled stationary fractional Ornstein-Uhlenbeck process. We will restrict to equations of the following form:

\[
\begin{cases}
\dot{x}^\varepsilon_t = \sum_{k=1}^{N} \alpha_k(\varepsilon) f_k(x^\varepsilon_t) G_k(y^\varepsilon_t), \\
x^\varepsilon_0 = x_0.
\end{cases}
\]

(1.1)

Let \( \mu \) be the standard Gaussian measure. A function \( G \in L^2(\mu) \) admits an expansion in Hermite polynomials \( G = \sum_{l=m}^{\infty} c_l H_l \). It is said to satisfy the fast chaos decay condition with parameter \( q \in \mathbb{N} \), if

\[
\sum_{l=0}^{\infty} |c_l| \sqrt{l!} (2q-1)^{\frac{l}{2}} < \infty.
\]

The lowest index \( l \) with \( c_l \neq 0 \) is called the Hermite rank of \( G \). If \( m \) is the Hermite rank we define \( H^*(m) = m(H-1) + 1 \).

A following homogenisation theorem is proved in \cite{GL19, GL20a}, see also \cite{GL19} for the preliminary version, for \( y_t^\varepsilon \) the normalised stationary fast fractional Ornstein-Uhlenbeck process with stationary measure \( \mu \).

**Theorem 1.1** \cite{GL19, GL20a} Let \( y_t \) be the fractional Ornstein-Uhlenbeck process with stationary measure \( \mu \). Let \( G_k \) be \( L^2 \cap L^p \) functions satisfying the fast chaos decay condition with parameter \( q \geq 4 \) for \( p_k \) sufficiently large. We order the functions \( \{G_k\} \) so that their Hermite ranks \( m_k \) satisfy \( H^*(m_k) < 0 \) for \( k \leq n \) and \( H^*(m_k) > \frac{1}{2} \) for \( k > n \) and some \( n \in \{0, \ldots, N\} \). Then, the solutions of

\[
\begin{cases}
\dot{x}^\varepsilon_t = \sum_{k=1}^{N} \alpha_k(\varepsilon) f_k(x^\varepsilon_t) G_k(y^\varepsilon_t), \\
x^\varepsilon_0 = x_0,
\end{cases}
\]

(1.2)

converges, as \( \varepsilon \to 0 \) to the solution of the following equation with the same initial data.

\[
dx_t = \sum_{k=1}^{n} f_k(x_t) \circ dX^k_t + \sum_{k=n+1}^{N} f_k(x_t) dX^k_t.
\]

(1.3)

where \( X^k_t \) is a Wiener process for \( k \leq n \) and otherwise a Gaussian or a non-Gaussian Hermite process and covariances between the processes are determined by the functions \( G_k \). In these equations, the symbol \( \circ \) denotes the Stratonovich integral and the other integrals are in the sense of Young integrals.

Here \( \alpha(\varepsilon, H^*(m)) \) are positive constants as follows, they depend on \( m, H \) and \( \varepsilon \) and tend to \( \infty \) as \( \varepsilon \to 0 \),

\[
\alpha(\varepsilon, H^*(m)) = \begin{cases}
\frac{1}{\sqrt{|\varepsilon|}}, & \text{if } H^*(m) < \frac{1}{2}, \\
\sqrt{|\ln(\varepsilon)|}, & \text{if } H^*(m) = \frac{1}{2}, \\
\varepsilon^{H^*(m)-1}, & \text{if } H^*(m) > \frac{1}{2}.
\end{cases}
\]

(1.4)
Note that the limit dynamics might be driven by Hermite processes, these are the non-Gaussian self-similar fractional fields sharing the same type of covariance functions as the fractional Brownian motions. It is therefore natural to consider passive tracers in Hermite noise fields. Our aim is to prove a similar type of theorem when the fractional environment for the passive tracer is given by the Ornstein-Uhlenbeck equation driven by the Hermite processes. Such processes falls within the category of the moving average process of Hermite processes. Therefore, Our main objective is to establish limit theorems for functionals of moving average process of Hermite processes.

2 Statement of the Theorems

Hermite processes, $Z_{t}^{H,m}$, are a two parameter family of self-similar stochastic processes with stationary increments. In case $m = 1$ they are given by fractional Brownian motions of Hurst-parameter $H > \frac{1}{2}$. They appear as renormalized limits of sums of Gaussian sequences, so called non-central limit theorems, see [BM07,MN09,MT13a,GL20b] and are used in models trying to capture long range dependence, see e.g. [SKME19]. Furthermore, one can define Wiener integrals with respect to them, see section 3 below or [MN19] for more details. Given a suitable regular kernel $x$, see Assumption 4, we may define a stationary process as follows,

$$y_{t} = \int_{-\infty}^{t} x(t-s)Z_{s}^{H,m}.$$

In case $m = 1$ and $x(s) = e^{-s}$ this gives the stationary fractional Ornstein-Uhlenbeck process. The problem we are now concerned with is, given a centred function $G \colon \mathbb{R} \to \mathbb{R}$, $(E[G(y_{0})] = 0)$, what can we say about the behaviour of

$$\varepsilon^{\alpha} \int_{0}^{T} G(y_{s})ds,$$

as $\varepsilon \to 0$ for a suitable scaling $\alpha$. Depending on the function $G$ we obtain different answers, either a Wiener or a Hermite process. In case $G = X^{2}$ it was shown in [DT18], that, after subtracting the average, the right scaling is given by $\alpha = 2H_{0} - 1$, where $H_{0} = 1 + \frac{H-1}{m}$, and the limiting process is given by a Rosenblatt process. Rosenblatt processes are Hermite processes, where $m = 2$. In this paper we are extending this result to functions $G(X) = \sum_{k=0}^{\infty} c_{k}X^{k}$, where $|c_{k}| \lesssim \frac{1}{k}$. In case $m = 1$ the stationary processes $y_{t}$ are Gaussian and for $G \in L^{2}(\mathbb{R},\mathcal{N}(0,1))$ one may expand $G(y_{s})$ into Hermite polynomials. This decomposition gives us a direct way to compute the projections of $G(y_{s})$ onto each Wiener chaos. In our setting there is no such straightforward way to compute these projections. Nevertheless, we may write $y_{t} = I_{m}(f_{t})$, for functions $f_{t}$ to be computed below and $I_{m}$ denotes a multiple Wiener stochastic integral. Now, in case $G$ is a polynomial we may, as in [DT18], apply the product formulae for multiple Wiener integrals, Lemma 4.1, iteratively to decompose $(y_{s})^{k}$ into its distinct chaos parts. Hence, for $G(X) = \sum_{k=0}^{\infty} c_{k}X^{k}$ we may decompose each monomial and then collect the contribution in each chaos. This is done in section 3.4.

In case $m = 1$ the Hermite rank of the function $G$ plays a key role. It is defined as the first degree of the first non zero coefficient in the Hermite expansion of $G$, hence a function with Hermite rank $m$ has the following form

$$G = \sum_{k=m}^{\infty} c_{k}H_{k},$$

where $H_{k}$ denote the Hermite polynomials. Usually one sees a different behaviour depending on the Hermite rank $m$ and the Hurst parameter of the underlying $f$Bm. In case $H^{*}(m) = (H-1)m + 1 > \frac{1}{2}$ one is in the long range dependent regime and obtains convergence to a Hermite process, whereas in the short range dependent setting $H^{*}(m) < \frac{1}{2}$ one converges to a Wiener process, which is a similar picture, where we use the concept of chaos rank instead of the Hermite rank to capture the order of the lowest chaos appearing.

After proving the functional limits theorems we apply the result to the homogenization of fast/slow systems. In order to obtain this result we apply the continuity theorem for young and rough differential equation. As continuous maps preserve weak convergence, applying the above mentioned theorems, enables us to conclude weak convergence of our ODE’s by proving weak convergence of the drivers in H"{o}lder/rough path topologies. This method has been applied recently, c.f. [KMT17,CFK19,GL19].
2.1 Statement of Results

Convention: if \( G_j(X) = \sum_{k=0}^{\infty} c_{j,k} X^k \), where \( j \in \{1, 2, \ldots, N\} \), are functions with chaos rank \( w_j \) with respect to \( y_t \), we order them in such a manner that \( H^*(w_j) < \frac{1}{2} \) for \( j \leq n \) and \( H^*(w_j) > \frac{1}{2} \) in case \( j > n \), for some \( n \in \{0, \ldots, N\} \). For \( T \in [0, 1] \), set

\[
\tilde{G}^{j} = \varepsilon^{H^*(w_j)} \int_0^T G^j(y_t) dt.
\]

The following is the main theorem of this article.

**Theorem A** Fix \( H \in (\frac{1}{2}, 1) \), \( m \in \mathbb{N} \). Let \( x \) be a kernel satisfying Assumption \( 2.1 \), and set

\[
y_t = \int_{-\infty}^t x(t-s) dZ^{H,m}.\]

For \( j \in \{1, \ldots, N\} \) let \( G_j(X) = \sum_{k=0}^{\infty} c_{j,k} X^k \) such that \( G_j \) has chaos rank \( w_j \) with respect to \( y_t \) and \( |c_{j,k}| \lesssim \frac{1}{T^{\frac{1}{2}}(j)} \). Assume further that \( H^*(w_j) \neq \frac{1}{2} \). Then the following statements hold.

- As \( \varepsilon \to 0 \), \( (\tilde{G}^{j})_{\varepsilon} \) converges weakly in \( C^\gamma([0, 1], \mathbb{R}^N) \).
- The limit is a vector valued process \( (W_T, Z_T) \), where the first part \( W_T \) is a \( n \)-dimensional Wiener process and \( Z_T \) a \( N-n \) dimensional Hermite process, for \( \gamma \in (0, \frac{1}{2}) \) in case \( n > 0 \) and \( \gamma \in (0, \min_{j \in \{1, \ldots, N\}} H^*(w_j)) \) otherwise. Furthermore,
  1. \( W_T \in \mathbb{R}^n \) and \( Z_T \in \mathbb{R}^{N-n} \) are independent.
  2. The Wiener part \( W_T \) of the process has the following covariance structure,
     \[
     \mathbf{E}[W_T^2 W_S^2] = 2(T \wedge S) \int_0^\infty \mathbf{E}[G^j(y_s)G^j(y_0)] ds
     \]
     \[
     = 2(T \wedge S) \sum_{k,k'=0}^{\infty} \sum_{d=0}^{\infty} \int_0^\infty \mathbf{E}[I_d(h_{s,k} h_{d,k'})] ds
     \]
  3. Each component of the Hermite part of the process \( Z_T \) has a representation by a Wiener process, which is the same for all components, that is independent of \( W_T \):
     \[
     Z_T = \left( \kappa_{n+1} Z_T^{H^*(w_{n+1}), w_{n+1}}, \ldots, \kappa_N Z_T^{H^*(w_N), w_N} \right),
     \]
     where \( \kappa_j = \lim_{\varepsilon \to 0} \| \tilde{G}^j \|_{L^2} \).

**Remark 2.1** For \( T, S \in [0, 1] \), \( Z \) has the following covariance structure:

\[
\mathbf{E}[Z_T^{H^*(w_j), w_j} Z_S^{H^*(w_l), w_l}] = \delta^{j,l} \kappa_j \kappa_l \int_{\mathbb{R}^{w_j}} d\xi \int_0^T \prod_{q=0}^{w_j} \prod_{q=1}^{H^*(w_j)-1} (\varepsilon^q - \xi_q)^{H^*(w_j)-1} q^{\frac{w_j}{2}} dr \int_0^S \prod_{q=1}^{w_l} (\varepsilon^q - \xi_q)^{H^*(w_l)-1} q^{\frac{w_l}{2}} dr'.
\]

**Remark 2.2** Given a function \( G \), it is not obvious how to construct a power series of a specified chaos rank. However, we can project \( G \) onto the first \( M \) Wiener chaoses and center it to obtain \( \bar{G}(y_t) = G(y_t) - P_{w-1}(G(y_t)) \). Then \( \bar{G}(y_t) \) behaves like a function with chaos rank \( w \).

Using the above result we obtain the following theorem.

**Theorem B** Let \( y_t \) and \( G \) be given as in Theorem A.

Fix \( f \in \mathcal{C}_b^\infty([0, 1], \mathbb{R}) \), \( x_0 \in \mathbb{R} \), and consider the equations

\[
dx_t^\varepsilon = \frac{\alpha(\varepsilon)}{\varepsilon} f(x_t^\varepsilon) G(y_t^\varepsilon) dt, \quad x_0^\varepsilon = x_0, \quad (2.1)
\]
where
\[
\alpha(\varepsilon) = \begin{cases} 
\frac{\sqrt{\varepsilon}}{\varepsilon H^*(w)} & \text{in case } H^*(w) < \frac{1}{2}, \\
\varepsilon H^*(w) & \text{in case } H^*(w) > \frac{1}{2}.
\end{cases}
\]

Then the following holds.

1. If \( H^*(w) > \frac{1}{2} \), \( x_t^\varepsilon \) converges weakly in \( C^\gamma([0,1],\mathbb{R}) \) to the solution to the Young differential equation \( d\tilde{x}_t = cf(\tilde{x}_t) \, dZ^H((w),w) \) with initial value \( \tilde{x}_0 \) for \( \gamma \in (0,H^*(w)) \) and

\[
c = \lim_{\varepsilon \to 0} \|\varepsilon H^*(w)\int_0^1 G(y_s)ds\|_{L^2(\Omega)}.
\]

2. If \( H^*(w) < \frac{1}{2} \), \( x_t^\varepsilon \) converges weakly in \( C^\gamma([0,1],\mathbb{R}) \) to the solution of the Stratonovich stochastic differential equation \( d\tilde{x}_t = cf(\tilde{x}_t) \circ dW_t \) with \( \tilde{x}_0 = x_0 \), where \( \gamma \in (0,\frac{1}{2}) \). \( W \) denotes a standard Wiener process and

\[
c = \lim_{\varepsilon \to 0} \|\varepsilon H^*(w)\int_0^1 G(y_s)ds\|_{L^2(\Omega)}.
\]

3 Notation

1. \((\Omega,\mathcal{F},\mathbb{P})\) denotes our underlying probability space
2. \(\lambda\) denotes the Lebesgue measure on the respective spaces or a parameter of the Hermite Ornstein-Uhlenbeck process
3. \(B\) denotes a two sided standard Brownian motion
4. \(\hat{B}\) denotes a Gaussian complex-valued random spectral measure to be defined in the section spectral measure
5. \(I_d\) denotes a \(d\) dimensional Wiener isometry given by iterated Wiener integrals
6. \(\hat{I}_d\) denotes a \(d\) dimensional iterated integral with respect to \(B\).
7. \(H\) denotes the self-similarity of our underlying Hermite process
8. \(m\) denotes the rank of our underlying Hermite process
9. \(H^* = 1 + \frac{H}{m}\)
10. \((H^*)^d = (H^* - 1)d + 1\)
11. \(f(x) \leq g(x)\) denotes less or equal up to a constant; there exists a constant \(M\) such that for all \(x\) \(f(x) \leq Mg(x)\)
12. \(f_+ = \min(0,f)\)
13. As we have to deal with many integrals we sometimes use the notation \(\int dx f(x)\) instead of \(\int f(x)dx\).
   Furthermore, instead of \(\int_{\mathbb{R}^k} ds_1 \ldots ds_k f(s_1,\ldots,s_k)\) we write \(\int_{\mathbb{R}^k} d^k s f(s_1,\ldots,s_k)\). Here \(d^k s\) denotes that we integrate over the \(s\) variables of which there are exactly \(k\). Sometimes the index is not \(1\) to \(k\), due to possible double indices, but, nevertheless, \(d^k s\) denotes that there are exactly \(k\) of them.
14. If a stationary process \(y_s\) belongs to the \(L^2\) space generated by a Wiener process and given a function \(G\) such that \(G(y_s) \in L^2(\Omega)\) we say that \(G\) has chaos rank \(w\) with respect to \(y_s\) if and only if all projections of \(G(y_s)\) onto the first \(w - 1\) Wiener chaoses are \(0\) and the projection of \(G(y_s)\) onto the \(w^{th}\) chaos is non-zero. Thus, \(G\) being centred with respect to the invariant distribution of \(y_s\) is equivalent to \(G\) having chaos rank bigger or equal 1. In case \(y_s\) is Gaussian this coincides with the Hermite rank of \(G\), see \[Taq75\].

4 Preliminaries

Given \(f \in L^2(\mathbb{R}^a,\lambda)\) we denote its \(a^{th}\) multiple Wiener-Itô integral by \(I_a(f) = \int_{\mathbb{R}^a} f(\xi_1,\ldots,\xi_a)dB_{\xi_1} \ldots dB_{\xi_a}\), where \(B\) is a two-sided Wiener process and the integral does not run over diagonals. For symmetric functions \(f\) we obtain \(I_a(f) = \frac{1}{a!} \int_{\mathbb{R}^a} f(\xi_1,\ldots,\xi_a)dB_{\xi_1} \ldots dB_{\xi_a}\). Furthermore, let \(\hat{f}\) denote \(f^{\prime}\)s symmetrization, then, \(I_0(f) = I_0(\hat{f})\). In particular, for \(f \in L^2(\mathbb{R}^a,\lambda)\) and \(g \in L^2(\mathbb{R}^b,\lambda)\),

\[
\mathcal{E}[I_a(f)I_b(g)] = \delta_{a,b} a! \langle \hat{f},\hat{g} \rangle_{L^2(\mathbb{R}^a,\lambda)}.
\]
For \( r \leq a \wedge b \) we denote the \( r \)th contraction between \( f \) and \( g \) by

\[
f \otimes_r g(\xi_1, \ldots, \xi_{a+b-2r}) = \int_{\mathbb{R}^r} f(\xi_1, \ldots, \xi_{a-r}, s_1, \ldots, s_r) g(\xi_{a-r+1}, \ldots, \xi_{a+b-2r}, s_1, \ldots, s_r) ds_1 \ldots ds_r.
\]

Moreover, we denote its symmetrization by \( f \otimes^* g \), see \[\text{Hermite-processes}\]. We conclude this section with an identity which we use plentifully in section \[\text{Hermite-processes}\].

Lemma 4.1 \textit{Given symmetric functions} \( f \in L^2(\mathbb{R}^a, \lambda) \) \textit{and} \( g \in L^2(\mathbb{R}^b, \lambda) \), \textit{then, the following relation holds}

\[
I_a(f)I_b(g) = \sum_{r=0}^{a\wedge b} \binom{a}{r} \binom{b}{r} I_{a+b-2r}(f \otimes_r g).
\]  \hspace{1cm} (4.1)  \hspace{1cm} \text{equation-p}

4.1 Hermite processes

Definition 4.2 \textit{Let} \( m \in \mathbb{N}, H \in \left( \frac{1}{2}, 1 \right) \) \textit{be given and recall,} \( H_0 = 1 + \frac{m}{2} \). \textit{The class of Hermite processes of rank} \( m \) \textit{is given by the following mean-zero processes,}

\[
Z^H_m(t) = K(H, m) \int_{\mathbb{R}^m} \prod_{j=1}^{m} (s - \xi_j)^{H_H - \frac{3}{2}} ds dB_{\xi_1} \ldots dB_{\xi_m},
\]  \hspace{1cm} (4.2)  \hspace{1cm} \text{definition}

where the integral over \( \mathbb{R}^m \) is to be understood as a multiple Wiener-Itô integrals, meaning no integration along the diagonals, and the constant \( K(H, m) \) is chosen so that their variances are \( 1 \) at \( t = 1 \). \textit{The number} \( H \) \textit{is also called Hurst parameter and} \( K(H, m)^2 = \frac{H(2H-1)}{B(H_0-\frac{3}{2}-2H_0)} \), where \( B \) denotes the beta function, \textit{c.f.} \[\text{Hermite-processes}\]. \textit{Hermite processes have stationary increments, are self-similar with exponent} \( H \),

\[
\lambda^H Z^H_m \sim Z^H_m,
\]

and their covariance is given by

\[
\mathbb{E}[Z^H_m(t)Z^H_m(s)] = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}).
\]  \hspace{1cm} (4.3)  \hspace{1cm} \text{definition-Hermite-processes}

They live in the \( m^{th} \) Wiener chaos and thus, by hypercontractivity, have finite moments of all orders. Using Kolmogorv’s theorem, one can show that the Hermite processes \( Z^H_m \) have sample paths of Hölder regularity up to \( H \). \textit{The rank} \( 1 \) \textit{Hermite processes} \( Z^H_1 \) \textit{are fractional Brownian motions for} \( H > \frac{1}{2} \), as above definition matches the Mandelbrot Van-Ness representation, see \[\text{Hermite-processes}\] and \[\text{Hermite-processes}\].

In \[\text{Hermite-processes}\] the following spectral representation for Hermite processes was obtained using a Gaussian random spectral measure \( B_{\xi} \), see section \[\text{Hermite-processes}\] for a brief summary,

\[
Z^H_m(t) = \hat{K}(H, m) \int_{\mathbb{R}^m} e^{it \sum_{j=1}^{m} \xi_j - \frac{1}{2} \sum_{j=1}^{m} |\xi_j|^{H_0-\frac{3}{2}}} dB_{\xi_1} \ldots dB_{\xi_m},
\]

where \( \hat{K}(H, m) \) is chosen such that \( \mathbb{E} \left[ \left( Z^H_m(t) \right)^2 \right] = 1 \).

4.1.1 Wiener integrals for Hermite processes

In \[\text{Hermite-processes}\] Wiener integrals with respect to Hermite processes were introduced. Via an isometry construction it was shown that for \( f \in \mathcal{H} \), where \( \mathcal{H} = \{ f : \mathbb{R} \to \mathbb{R} : \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(u)f(v)|u-v|^{2H-2} du dv < \infty \} \), the usual Wiener integral approach makes sense. In order to avoid integrability problems, we impose a bit more regularity
and restrict ourselves to the space $|H| = \{ f \in R^1 \to R^1 \mid f(0) = 0, \int f(u)f(v)|u - v|^{2H - 2} du dv < \infty \}$. Note that $L^1(R, \lambda) \cap L^2(R, \lambda) \subset |H|$, see [CKM03], [P100]. On $|H|$ the following relation holds,

$$\int_R f(s) dZ_s^{H,m} = K(H,m) \int_R \int_R f(s) \prod_{j=1}^m (s - \xi_j)^{H \cdot \frac{2}{2}} ds dB_{\xi_1} \ldots dB_{\xi_m}$$

and restrict ourselves to the space $L^1(R^m) \cap L^2(R^m)$. On $|H|$ the following relation holds,

$$\int_R f(s) dZ_s^{H,m} = K(H,m) \int_R f(s) \prod_{j=1}^m (s - \xi_j)^{H \cdot \frac{2}{2}} ds dB_{\xi_1} \ldots dB_{\xi_m}$$

where the integral over $R^m$ is to be understood as an iterated Wiener integral. By the above and using the identity

$$\int_R (u - y)^{a - 1} dy = \beta(a, 2a - 1)|u - v|^{2a - 1},$$

one obtains the following relation,

$$E \left[ \int_R f dZ_s^{H,m} \int_R g dZ_s^{H,m} \right] = H(2H - 1) \int_{R^2} f(u)g(v)|u - v|^{2H - 2} du dv.$$

Furthermore, we denote

$$||f||_H = H(2H - 1) \int_{R^2} |f(u)||f(v)||u - v|^{2H - 2} du dv$$

and

$$||f||_{|H|} = H(2H - 1) \int_{R^2} |f(u)||f(v)||u - v|^{2H - 2} du dv.$$

### 4.1.2 Hermite Ornstein-Uhlenbeck processes

Using this integration theory, the Hermite Ornstein-Uhlenbeck process was introduced in [Maejima-Ciprian]. It is the unique solution to the following SDE, where $\lambda > 0$,

$$y_t = y_0 - \lambda \int_0^t y_s ds + \sigma Z_t^{H,m},$$

which is given by

$$y_t = e^{-\lambda t}(y_0 + \sigma \int_0^t e^{\lambda s} dZ_s^{H,m}).$$

By choosing $y_0 \sim \sigma \int_0^\infty e^{\lambda s} dZ_s^{H,m}$ one obtains its stationary solution. Moreover, by the above formulae, solutions started with different initial conditions converge exponentially fast towards this stationary solution.

In [CMK03] a formulae for the covariance decay in case $m = 1$ was obtained. As the covariance does not change in $m$ and $e^{-r(t-s)}1_{s \leq t} \in L^1(R) \cap L^2(R)$ the same formulae also holds true for the Hermite Ornstein-Uhlenbeck processes and is given by

$$E[yy_{t+s}] = \frac{1}{2^2} \sigma^2 \sum_{n=1}^N \lambda^{-2n} \left( \prod_{j=0}^{2n-1} (2H - j) \right) s^{2H - 2n} + O(s^{2H - 2N - 2}),$$

for $s \to \infty$, see [Maejima-Ciprian]. In particular,

$$|E[yy_{t+s}]| \lesssim 1 \wedge s^{2H - 2}.$$
4.1.3 Volterra processes

For \( x \in \mathcal{H} \) we may also define

\[
y_t = \int_0^t x(t-s) dZ_s^{H,m}. \tag{4.12}
\]

and

\[
y_t = K(H,m) I_m \left( \int_0^t x(t-s) \prod_{j=1}^m (s - \xi_j)^{H_0 - \frac{2}{2}} ds \right). \tag{4.13}
\]

By setting \( x(s) = e^{-s} \) we obtain the Hermite Ornstein-Uhlenbeck process with initial value 0. Furthermore, setting

\[
y_t = \int_{-\infty}^t x(t-s) dZ_s^{H,m}. \tag{4.14}
\]

leads to a class of stationary processes, which in case \( x = e^{-s} \) equals the stationary Hermite Ornstein-Uhlenbeck process. In our analysis below we treat processes given as such Volterra integrals and, as in [DT18], impose the condition \( x \in L^1(\mathbb{R}) \cap |H| \)

Assumption 4.3 Assume \( x \in L^1(\mathbb{R}) \cap |H| \) and

\[
\int_{\mathbb{R}^2} |x(t-u)x(t'-v)||u-v|^{2H-2} dudv \lesssim 1 \land |t-t'|^{2H-2}.
\]

In particular, for \( y_t \) defined in equation (4.14) this leads to,

\[
|E[y_t y_{t'}]| \lesssim 1 \land |t-t'|^{2H-2}.
\]

Remark 4.4 In [NNZ16] amongst other things the case \( m = 1 \) in the short range dependent setting was treated. One of their assumption on the kernel \( x \) is the following integrability condition,

\[
\int_{\mathbb{R}} \left( \int_{[0,\infty]^2} x(u)x(v)|u-v-a|dudv \right)^m da < \infty,
\]

where \( m \) denotes the Hermite rank of the function \( G \). In case we are as well in the short range dependent regime, using the notion of the chaos rank of \( G \) instead of the Hermite rank, Assumption 4.3 implies this condition.

5 Decomposition and convergence for building blocks

In this section we first decompose each polynomial \((y_t)^k\) into its Wiener chaos components. To do so we apply the product formulae, Lemma 4.1, iteratively and collect all obtained "building blocks" belonging to the same Wiener chaos. This terms are then at the centre of our investigation as we can obtain the general case by sums of these objects. Next, we analyse the variance growth of these terms to obtain our scaling rate. Finally, we prove convergence in finite dimensional distributions for the rescaled integrals of our building blocks. Here we distinguish, as in the Gaussian setting, the short and long range dependent regime. In the first one our limit is
given by a Wiener process and we make use of the Fourth Moment Theorem to conclude our result. In the latter, the limits are again Hermite processes and, as in [DiT18], we argue via first rewriting everything as a multiple Wiener Itô integral with respect to a Gaußian complex-valued random spectral measure and then prove $L^2$ kernel convergence.

**Remark 5.1** Henceforth we suppress the constants $K(H,m)$ in our notation.

### 5.1 Decomposition

Given $y_t = I_m \left( \int_{-\infty}^{t} x(t-s) \prod_{j=1}^{m} (s-\xi_j)_+^{H_0-\frac{d}{2}} ds \right)$, we aim to calculate the contribution of $(y_t)^k$ to each distinct Wiener chaos. As this kernel appears again and again we set

$$f_t = \int_{-\infty}^{t} x(t-s) \prod_{j=1}^{m} (s-\xi_j)_+^{H_0-\frac{d}{2}} ds.$$

Now, by iteratively applying Lemma 4.1 we obtain,

\[
(y_t)^k = \left( I_m \left( \int_{-\infty}^{t} x(t-s) \prod_{j=1}^{m} (s-\xi_j)_+^{H_0-\frac{d}{2}} ds \right) \right)^k
= \left( I_m \left( \int_{-\infty}^{t} x(t-s) \prod_{j=1}^{m} (s-\xi_j)_+^{H_0-\frac{d}{2}} ds \right) \right)^{k-2} \left( \sum_{r_1=0}^{m} (r_1)! \left( \begin{array}{c} m \cr r_1 \end{array} \right) I_{2m-2r_1}(f_t \hat{\otimes} r_1 f_t) \right)
\]

\[
= \left( I_m \left( \int_{-\infty}^{t} x(t-s) \prod_{j=1}^{m} (s-\xi_j)_+^{H_0-\frac{d}{2}} ds \right) \right)^{k-3} \left( \sum_{r_1=0}^{m} (r_1)! \left( \begin{array}{c} m \cr r_1 \end{array} \right) \left( \prod_{r_2=0}^{2m-2r_1} \sum_{r_2} (r_2)! \left( \begin{array}{c} 2m-2r_1 \cr r_2 \end{array} \right) I_{3m-2r_1-2r_2}(f_t \hat{\otimes} r_1 f_t \hat{\otimes} r_2 f_t) \right) \right)
\]

\[
= \sum_{r_1, r_2, \ldots, r_k} C_1(r_1, \ldots, r_k, k, m) I_{km-2 \sum_{j=1}^{k} r_j} (f_t \hat{\otimes} r_1 f_t \hat{\otimes} r_2 \ldots \hat{\otimes} r_k f_t),
\]

where $r_1 \leq m, r_2 \leq (2m-2r_1) \land m, \ldots, r_k \leq (km-2 \sum_{j=1}^{k} r_j) \land m$ and

$$C_1(r_1, \ldots, r_k, k, m) = \prod_{j=1}^{k} (r_j)! \left( \begin{array}{c} m \cr r_j \end{array} \right) \left( jm - 2 \sum_{j=1}^{k-1} r_j \right).$$

denotes the arising constants. Henceforth, we denote the tuple $(r_1, \ldots, r_k)$ just by $r$ and set $\delta(k,r) = km - 2 \sum_{j=1}^{k} r_j$. The above calculation shows that we can decompose polynomials into their distinct Wiener chaos parts once we know how to compute the terms $f_t \hat{\otimes} r_1 f_t \hat{\otimes} r_2 \ldots \hat{\otimes} r_k f_t$.

Therefore, we now investigate contractions of $f_t$ with itself a bit more. In [DiT18] the following was shown,

\[
\begin{align*}
f_t \otimes r_1 f_t(\xi_1, \ldots, \xi_{m-r_1}, \xi_2, \ldots, \xi_{2m-r_1}) &= \int_{[-\infty, t]^2} ds_1 ds_2 x(t-s_1) x(t-s_2) \prod_{i=1}^{m-r_1} (s_1 - \xi_{1,i})_+^{H_0-\frac{d}{2}} (s_2 - \xi_{2,i})_+^{H_0-\frac{d}{2}} \int_{\mathbb{R}^m} d^{r_1} z \prod_{j=1}^{r_1} (s_1 - z_j)_+^{H_0-\frac{d}{2}} (s_2 - z_j)_+^{H_0-\frac{d}{2}} \left( \int_{\mathbb{R}^{r_1}} dz (s_1 - z)_+^{H_0-\frac{d}{2}} (s_2 - z)_+^{H_0-\frac{d}{2}} \right)^{r_1} \\
&= C_2(H_0)^{r_1} \int_{[-\infty, t]^2} ds_1 ds_2 x(t-s_1) x(t-s_2) \prod_{i=1}^{m-r_1} (s_1 - \xi_{1,i})_+^{H_0-\frac{d}{2}} (s_2 - \xi_{2,i})_+^{H_0-\frac{d}{2}} |s_1 - s_2|^{r_1(2H_0-2)},
\end{align*}
\]
where \( C_2(H_0) = B(H_0 - \frac{1}{2}, 2 - 2H_0) \) and \( B \) denotes the beta function, see also Equation \( 10 \). Hence,

\[
 f_t \bar{\otimes} r_1 f_t = \frac{C_2(H_0)^{r_1}}{(2m-2r_1)!} \sum_{\psi_1 \in S_{2m-2r_1}} \int_{[\infty,t]^2} ds_1 ds_2 x(t-s_1)x(t-s_2)
\]

\[
\prod_{l=1}^{m-r_1} (s_1 - \xi_{\psi_l(1,l)})^{H_0 - \frac{2}{3}} \prod_{l=1}^{m-r_2} (s_2 - \xi_{\psi_l(2,l)})^{H_0 - \frac{2}{3}} |s_1-s_2|^{|r_2|2(2H_0-2)},
\]

where \( S_{2m-2r_1} \) denotes the symmetric group of order \( 2m-2r_1 \) and we implicitly make the identifications \( (1, l) = l \) and \( (2, l) = m - r_1 + l \). From now on we freely use such implicit identifications of indices to lighten the notation.

When computing the next contraction, \( f_t \bar{\otimes} r_1 f_t \otimes r_2 f_t \), we face the problem that for different choices of \( \psi_1 \in S_{2m-2r_1} \) we eventually integrate different terms. This is due to the fact that contractions use the "last" \( r_2 \) variables, which is well defined for the symmetric function \( f_t \bar{\otimes} r_1 f_t \), however, in each of the summands the notion of the "last" variables depends on the permutation. Nevertheless, we know that exactly \( r_2 \xi' \)'s are consumed in the next round and we denote by \( r_{2,1}(\psi_1) \) and \( r_{2,2}(\psi_1) \) the amount which contracts with the \( s_1 \) and the \( s_2 \) terms respectively.

Performing the same calculation as above, we obtain,

\[
 f_t \bar{\otimes} r_1 f_t \otimes r_2 f_t = \frac{C_2(H_0)^{r_1} C_2(H_0)^{r_2}}{(2m-2r_1)! (3m-2r_1-2r_2)!} \sum_{\psi_1 \in S_{2m-2r_1}} \sum_{\psi_2 \in S_{3m-2r_1-2r_2}} \int_{[\infty,t]^3} ds_1 ds_2 ds_3 x(t-s_1)x(t-s_2)x(t-s_3)
\]

\[
\prod_{l=1}^{m-r_1-r_{2,1}(\psi_1)} (s_1 - \xi_{\psi_l(1,l)})^{H_0 - \frac{2}{3}} \prod_{l=1}^{m-r_1-r_{2,2}(\psi)} (s_2 - \xi_{\psi_l(2,l)})^{H_0 - \frac{2}{3}} \prod_{l=1}^{m-r_2} (s_3 - \xi_{\psi_l(3,l)})^{(2H_0-2)} |s_1-s_2|^{|r_1|2(2H_0-2)} |s_1-s_3|^{|r_{2,1}(\psi_1)|2(2H_0-2)},
\]

together with the algebraic constraint \( r_{2,1}(\psi_1) + r_{2,2}(\psi_1) = r_2 \). After forming the symmetrization of this expression one ends up with,

\[
 f_t \bar{\otimes} r_1 f_t \otimes r_2 f_t = \frac{C_2(H_0)^{r_1}}{(2m-2r_1)!} \frac{C_2(H_0)^{r_2}}{(3m-2r_1-2r_2)!} \sum_{\psi_1 \in S_{2m-2r_1}} \sum_{\psi_2 \in S_{3m-2r_1-2r_2}} 
\int_{[\infty,t]^3} ds_1 ds_2 ds_3 x(t-s_1)x(t-s_2)x(t-s_3)
\]

\[
\prod_{l=1}^{m-r_1-r_{2,1}(\psi_1)} (s_1 - \xi_{\psi_l(1,l)})^{H_0 - \frac{2}{3}} \prod_{l=1}^{m-r_1-r_{2,2}(\psi)} (s_2 - \xi_{\psi_l(2,l)})^{H_0 - \frac{2}{3}} \prod_{l=1}^{m-r_2} (s_3 - \xi_{\psi_l(3,l)})^{(2H_0-2)} |s_1-s_2|^{|r_1|2(2H_0-2)} |s_1-s_3|^{|r_{2,1}(\psi_1)|2(2H_0-2)} |s_2-s_3|^{|r_{2,2}(\psi_1)|2(2H_0-2)}.
\]

Thus, each time we contract once more, we integrate over one more variable resulting in the gain of an additional kernel, giving rise to the term \( x(t-s_1)x(t-s_2)x(t-s_3) \), which does not further interact with anything else. However, the terms \( \prod (s_j - \xi_{\psi_l(j,l)}) \) lead to an entanglement of the old variables with the new one, in case the contraction number is greater than 0. Although this entangling depends on the choice of the permutation, the obtained structure is the same.

If we perform \( k \) contractions of \( f_t \) with itself keeping track of all the constants \( C_2(H_0)^{r_1} \), renormalization factors \( \frac{1}{(2m-2r_1)!} \), sums \( \sum_{\psi_1 \in S_{2m-2r_1}} \), as well as the range of the products is notionally quite intense. Hence, we assume from now on that we performed \( k \) contractions with contraction numbers \( r = (r_1, r_2, \ldots, r_k) \), giving rise to one object that represents all possible choices, \( \psi \in \bar{S}_r = S_{2m-2r_1} \times S_{3m-2r_2-2r_2} \times \cdots \times S_{km-2} \sum_{j=1}^{\infty} r_j \) and denote the arising constants by \( C^k(\psi, r, H_0) \). Furthermore, we introduce constants \( M^k_j(\psi) \) and \( \beta^k_j(\psi) \) to keep track of the range of the products as well as the exponent of \( |s_j - s_0| \) respectively. The expression we then end up with looks more structured and we summarize the conclusions of the above discussion in the following Lemma.
Lemma 5.2 Given a kernel $x$ such that $f_t = \int_{-\infty}^{t} x(t-s) \prod_{j=1}^{m}(s-\xi_j)^{H_0-\frac{d}{2}} ds \in L^2(\mathbb{R}, \lambda)$ and contraction numbers $r = (r_1, \ldots, r_k)$, where $r_1 \leq m$, $r_2 \leq (2m - 2r_1) \land m$, $\ldots$, $r_k \leq \left((km - 2 \sum_{j=1}^{k} r_j\right) \land m$, then, the following identity holds,

$$f_t \otimes r_1 f_t \otimes r_2 f_t \ldots \otimes r_k f_t$$

(5.6)

$$= \sum_{\psi \in S_r} C_3(\psi, r, H_0) \int_{-\infty}^{t} d^k s \prod_{j=1}^{k} x(t-s_j) \prod_{l=1}^{M_j(\psi)} (s_j - \xi_{\psi}(\xi, l)^{H_0-\frac{d}{2}} \sum_{q=1}^{j-1} |s_j - s_q|^{\beta_{j,q}(\psi)(2H_0-2)},$$

(5.7)

$$\sum_{q=1}^{j-1} \beta_{j,q}(\psi) = r_j, \quad \sum_{j=1}^{k} M_j(\psi) = km - 2 \sum_{j=1}^{k} r_j = \delta(k,r), \quad \text{independently of the choice of } \psi \in S_r.$$

Hence, to decompose $(y_t)^k$ into its distinct Wiener chaos parts it is only left to collect all terms $f_t \otimes r_1 f_t \otimes r_2 f_t \ldots \otimes r_k f_t$ which give the same value for $\delta(k,r) = km - 2 \sum_{j=1}^{k} r_j$. To simplify our notation we set $g_{t,c}^{r} = f_t \otimes r_1 f_t \otimes r_2 f_t \ldots \otimes r_k f_t$ and $h_t^{d,k} = \sum_{(r, \delta(k,r) = d)} C_4(\psi, r, k, m) g_{t,c}^{r}$. Now, the terms $h_t^{d,k}$ equal the projection of $(y_t)^k$ onto the $d$th Wiener chaos, thus, by Equation (5.8)

$$\left(\begin{array}{c}
(y_t)^k = \sum_{d=0}^{km} I_d \sum_{(r, \delta(k,r) = d)} C_1(r, k, m) g_{t,c}^{r} \\
= \sum_{d=0}^{km} I_d(h_t^{d,k})
\end{array}\right)$$

5.2 Examining the scaling behaviour

In this section we bound the variance growth of terms of the form $\int_{0}^{T} h_t^{d,k} dt$ to obtain our scaling rate. Typically, one expects that this growth decreases with $d$ and after some critical value stays at the Wiener scaling as long as no cancellation occurs, see also Remark 4.3 and [10q/9] [11t35] GL19. This change of behaviour appears due to the algebraic decay of the correlations proved in this section. Terms in the $d$th chaos admit a decay of the form $1 \land s^{d(2H_0-2)}$. Thus, $\int_{0}^{T} 1 \land s^{d(2H_0-2)} ds$ either converges as $\varepsilon \to 0$, if $d$ is large enough, or diverges at rate $\varepsilon^{-d(2H_0-2)+1}$ leading to the change of behaviour.

Lemma 5.3 Given $h_t^{d,k}$ as defined above for a kernel $x$ satisfying Assumption 5.3 then, the following estimate holds,

$$\left|\mathbb{E} \left[ I_d(h_t^{d,k}) I_d'(h_t^{d',k'}) \right] \right| \leq C_4(k, k', m, d) \left(1 \land t^{(2H_0-2)d} \right),$$

where $C_4(k, k', m, d) = \sqrt{(km)(k’m)(k+k')^m(k+k')^m} \land m$ and $\mathcal{L} = C_2(H_0) + 3 + \|x\|_H + K(H, m)$.

Proof. In case $d \neq d'$ the expectation is 0 by orthogonality of different Wiener chaoses. Thus, henceforward we assume $d = d'$. Due to the decomposition,

$$\mathbb{E} \left[ I_d(h_t^{d,k}) I_d'(h_t^{d',k'}) \right] = \sum_{(r, \delta(k,r) = d)} C_1(r, k, m) \sum_{(r', \delta(k',r') = d)} C_1(r', k', m) \mathbb{E} \left[ I_d(g_{t,c}^{r}) I_d'(g_{t,c}^{r'}) \right]$$

11
and the finiteness of both sums, we may restrict ourselves to the analysis of the behaviour of $E \left[ I_d \left( g^{k,r}_t \right) I_d \left( g^{k',r'}_{t'} \right) \right]$ as long as our bounds only depend on $d$, $k$, $k'$, $m$ and $H_0$. By the Wiener-Ito isometry $E \left[ I_d \left( g^{k,r}_t \right) I_d \left( g^{k',r'}_{t'} \right) \right] = d! \langle g^{k,r}_t, g^{k',r'}_{t'} \rangle_{L^2(\mathbf{R}^d)}$ which is equivalent to computing the $d^{th}$ contraction between $g^{k,r}_t$ and $g^{k',r'}_{t'}$. As both terms arise from iterated contractions of $f$ we may argue similarly as above to find a good expression for their $L^2$ norm. We treat the behaviour of multiplicate constants separately, thus, we suppress them in the following calculations. We have,

$$E \left[ I_d \left( g^{k,r}_t \right) I_d \left( g^{k',r'}_{t'} \right) \right] = \int_{\mathbf{R}^d} \int_{S_d} \int_{[-\infty,t]^k} \int_{[-\infty,t']^{k'}} d^k s \sum_{j=1}^{\infty} x(t - s_j) \prod_{j=1}^{k} \left( s_j - \xi_{j,t} \right) H_{0} - 2 \prod_{q=1}^{j-1} |s_j - s_q|^{\beta_{j,q}(\psi)(2H_0 - 2)} \prod_{q'=1}^{j-1} \left| s_{j'} - s_{q'} \right|^{\beta_{j',q'}(\psi')(2H_0 - 2)}.$$

When we computed $g^{k,r}_t = \otimes_{r_1} f_t \otimes_{r_2} f_t \otimes_{r_3} f_t \ldots \otimes_{r_k} f_t$ we saw that, depending on the choice of permutations $\psi$, two variables $s_j$ and $s_q$ either interact with each other or not leading to the terms $|s_j - s_q|^{\beta_{j,q}(2H_0 - 2)}$, importantly, their exponents need to satisfy certain algebraic constraints. We may view each of the $k$ variables $s_j, j = 1, \ldots, k$ as the node of a graph with $m$ "degrees of freedoms". Entangling two variables, $s_j$ and $s_q$, can be seen as taking away this freedoms and connecting them with $\beta_{j,q}$ edges. Once a node has $m$ connections it can no longer interact with any other variables in following contractions.

However, in case $\delta(k, r) = d$ there are in total $d$ freedoms left, no matter which permutation we consider. When computing the variance of this objects we viewed this as having two, possibly distinct, graphs, with $k$ and $k'$ nodes respectively such that each graph has $d$ "degrees of freedom" left. What is now left to do, is to literally connect the dots to obtain the formulae below by the same calculations as in section 5. This graph analogy will be used over and over and provides a good picture of what is going on. The exponents $\gamma_{j',q'}$ now depend on both permutations and denote how many edges are connecting the node representing $s_j$ to the one representing $s_{j'}$. We again obtain constants of the form $C_k(2H_0)$ which we, together with the constants previously obtained, suppress in the following computations. We end up with,

$$E \left[ I_d \left( g^{k,r}_t \right) I_d \left( g^{k',r'}_{t'} \right) \right] \leq \sum_{\psi \in S_d, \psi' \in S_d} \int_{[-\infty,t]^k} \int_{[-\infty,t']^{k'}} d^k \!s' \!\int_{[-\infty,t]^k} \int_{[-\infty,t']^{k'}} d^k \!s \! \prod_{j=1}^{k} \prod_{j'=1}^{k'} |x(t - s_j) x(t' - s_{j'})|$$

$$\left| s_j - s_{j'} \right|^{\gamma_{j',q'}(\psi, \psi')(2H_0 - 2)} \prod_{q=1}^{j-1} |s_j - s_q|^{\beta_{j,q}(\psi)(2H_0 - 2)} \prod_{q'=1}^{j'-1} \left| s_{j'} - s_{q'} \right|^{\beta_{j',q'}(\psi')(2H_0 - 2)},$$

and, by the triangle inequality,

$$\left| \sum_{\psi \in S_d, \psi' \in S_d} \int_{[-\infty,t]^k} \int_{[-\infty,t']^{k'}} d^k \!s' \!\int_{[-\infty,t]^k} \int_{[-\infty,t']^{k'}} d^k \!s \! \prod_{j=1}^{k} \prod_{j'=1}^{k'} |x(t - s_j) x(t' - s_{j'})|$$

$$\left| s_j - s_{j'} \right|^{\gamma_{j',q'}(\psi, \psi')(2H_0 - 2)} \prod_{q=1}^{j-1} |s_j - s_q|^{\beta_{j,q}(\psi)(2H_0 - 2)} \prod_{q'=1}^{j'-1} \left| s_{j'} - s_{q'} \right|^{\beta_{j',q'}(\psi')(2H_0 - 2)},$$

From now on we also suppress dependencies on the permutations $\psi$ and $\psi'$. Furthermore, we focus on one generic summand and rely on arguments independent of the specific choice of permutations. Using the above introduced graph analogy we observe that $\sum_{j} \gamma_{j,\psi} + \sum_{j} \beta_{j,\psi} = m$ as each node has exactly $m$ connections either within its own graph, the $\beta$'s, or with the other one, the $\gamma$'s, independently of the choice of permutations. Looking at the
integral with respect to \( s_k \) and pulling the kernels \( x(t'-s_{j'}) \) and \( x(t-s_j) \), with the right exponents according to the calculation below, into the integral, we may apply the generalized Hölder inequality, as \( \sum_{j'} \frac{x_{k,j'}}{m} + \sum_j \frac{x_{\beta_k,j}}{m} = 1 \) and obtain,

\[
\int_{-\infty}^{t} ds_k \prod_{j'=1}^{k'} |x(t-s_k)^{\frac{\gamma_{k,j'}}{m}} x(t'-s_{j'})^{\frac{\gamma_{k,j'}}{m}}||s_k-s_{j'},|^{\gamma_{k,j'}(2H_0-2)} \prod_{q=1}^{k-1} |x(t-s_q)^{\frac{\beta_{k,j'}}{m}} x(t-s_q)|^{\gamma_{k,j'}(2H_0-2)} \leq \prod_{j'=1}^{k'} \left( \int_{-\infty}^{t} ds_k |x(t-s_k)x(t'-s_{j'})||s_k-s_{j'}|^{m(2H_0-2)} \right)^{\frac{\gamma_{k,j'}}{m}} .
\]

Philosophically, Hölder's inequality enables us to factorize the integral with respect to \( s_k \) by trading the terms \( |x(t'-s_{j'})\frac{\gamma_{k,j'}}{m}| \) for \( \left( \int_{-\infty}^{t} ds_k |x(t-s_k)x(t'-s_{j'})||s_k-s_{j'}|^{m(2H_0-2)} \right)^{\frac{\gamma_{k,j'}}{m}} \), the terms \( |x(t-s_q)\frac{\beta_{k,j'}}{m}| \) for \( \left( \int_{-\infty}^{t} ds_k |x(t-s_k)x(t-s_q)|^{m(2H_0-2)} \right)^{\frac{\beta_{k,j'}}{m}} \), and getting rid of the terms \( |s_k-s_{j'}|^{\gamma_{k,j'}(2H_0-2)} \) as well as \( |s_k-s_j|^{\beta_{k,j}(2H_0-2)} \). Note that the exponents stay the same, meaning we trade a term with exponent \( \frac{\gamma_{k,j'}}{m} \) for one which again has exponent \( \frac{\gamma_{k,j'}}{m} \) and similarly for the \( \beta \) terms. Hence, looking at the \( s_{k'} \) term we find the same situation as for the \( s_k \) terms, at least exponent wise and we may repeat the procedure all over again.

When looking at the graph picture this procedure is a bit like cutting with a mince knife. We start at the top of the first graph and "cut" \( s_k \) out of the picture, then, we move to the other side and cut \( s_{k'} \). As when cutting herbs we now again move to the other side with our knife and tackle \( s_{k-1} \) followed by \( s_{k-1} \) and so on until we have cut all edges and, hopefully, what remains is less entangled and easier to deal with. Another useful visualisation is that as soon as the knife hits a node with \( m \) edges it splits into \( m \) nodes each has one edge. This alone does not help as at first the other end of the edge is possibly still entangled, but as soon as the mince knife hits the other node we obtain a simple integral, corresponding to that edge. In the end we obtain a product over all these integrals.

Picking up our mince knife, if we look at the terms which include \( s_{k'} \), again after pulling in other kernels with the right exponents, we end up with,

\[
\int_{-\infty}^{t'} ds_{k'} \prod_{j=1}^{k'-1} |x(t'-s_{k'})^{1-\frac{\gamma_{k',j}}{m}} x(t-s_j)^{\frac{\gamma_{k',j}}{m}}||s_{k'}-s_j|^{\gamma_{k',j}(2H_0-2)} \prod_{q=1}^{k'-1} |x(t'-s_{k'})^{\frac{\beta_{k',j'}}{m}} x(t'-s_{k'})|^{\gamma_{k',j'}(2H_0-2)} x(t'-s_{q'})\frac{\beta_{k',j'}}{m} ||s_{k'}-s_{q'}|^{\beta_{k',j'}(2H_0-2)} \left( \int_{-\infty}^{t} ds_k |x(t-s_k)x(t'-s_{k'})||s_k-s_{k'}|^{m(2H_0-2)} \right)^{\frac{\gamma_{k,k'}}{m}} .
\]

Note that we have already used up some part of the \( x(t'-s_{k'}) \) kernel leading to the \( 1-\frac{\gamma_{k',j}}{m} \) exponent. As indicated above the sum of the exponents has the same structure, hence we obtain,

\[
\int_{-\infty}^{t'} ds_{k'} \prod_{j=1}^{k'-1} |x(t'-s_{k'})^{1-\frac{\gamma_{k',j}}{m}} x(t-s_j)^{\frac{\gamma_{k',j}}{m}}||s_{k'}-s_j|^{\gamma_{k',j}(2H_0-2)} \prod_{q=1}^{k'-1} |x(t'-s_{k'})^{\frac{\beta_{k',j'}}{m}} x(t'-s_{k'})|^{\gamma_{k',j'}(2H_0-2)} \left( \int_{-\infty}^{t} ds_k |x(t-s_k)x(t'-s_{k'})||s_k-s_{k'}|^{m(2H_0-2)} \right)^{\frac{\gamma_{k,k'}}{m}} \leq \prod_{j'=1}^{k-k} \left( \int_{-\infty}^{t} ds_{k'} |x(t'-s_{k'})x(t-s_j)||s_j-s_{k'}|^{m(2H_0-2)} \right)^{\frac{\gamma_{k,k'}}{m}} \prod_{q'=1}^{k'-1} \left( \int_{-\infty}^{t} ds_{k'} |x(t'-s_{k'})x(t'-s_{q'})||s_{k'}-s_{q'}|^{m(2H_0-2)} \right)^{\frac{\beta_{k',j'}}{m}} .
\]
The term
\[ \int_{-\infty}^{t'} ds_k \int_{-\infty}^t ds_k |x(t-s_k)x(t'-s_{k'})||s_k-s_{k'}|^{m(2H_0-2)} \]
now obeys the decay imposed in Assumption \textit{H} and we obtained it to the exponent \( \gamma_{k,k'} \). Cutting of \( s_{k-1} \) we obtain, in particular, the term,
\[ \left( \int_{-\infty}^t ds_k ds_{k-1} |x(t-s_k)x(t-s_{k-1})||s_k-s_{k-1}|^{m(2H_0-2)} \right)^{\beta_{k,k-1}(2H_0-2)}. \]
This term behaves like a power of \( \|x\|_{\mathfrak{H}} \), thus, the term only contributes a constant and we do not gain any decay from it. Summarizing, we end up with the following picture, edges within a graph do not lead to any decay, however, the ones between the graphs do with a rate given by powers of
\[ \int_{-\infty}^{t'} ds_k \int_{-\infty}^t ds_k |x(t-s_k)x(t'-s_{k'})||s_k-s_{k'}|^{m(2H_0-2)}. \]
As the \( \gamma \)'s represent the edges between the graphs and there are exactly \( d \) of them, \( \sum_{j=1}^k \sum_{j'=1}^{k'} \gamma_{j,j'} = d \), we obtain,
\[ \left| \mathbf{E} \left[ I_{d} \left( g_r^{k,r'} \right) I_{d} \left( g_r^{k',r'} \right) \right] \right| \leq \prod_{j=1}^k \prod_{j'=1}^{k'} \left( \int_{-\infty}^{t'} dv \int_{-\infty}^t du |x(t-u)x(t'-v)||u-v|^{m(2H_0-2)} \right)^{\gamma_{j,j'}} = \left( \int_{-\infty}^{t'} dv \int_{-\infty}^t du |x(t-u)x(t'-v)||u-v|^{m(2H_0-2)} \right)^{\gamma_{j,j'}} \leq 1 \wedge |t-t'|^{m(2H_0-2)} \leq 1 \wedge |t-t'|^{(2H_0-2)d}, \]
by Assumption \textit{H}.

It is left to prove the bound on the proportionality constant. We picked up constants corresponding to powers of \( \|x\|_{\mathfrak{H}} \) from the edges between each graph. As there are at most \( (k+k')m \) factors we can bound them by \( (1+\|x\|_{\mathfrak{H}})^{(k+k')m} \). Next, we bound the amount of possible contractions \( r \) such that \( \delta(k,r) = d \), the case \( \delta(k',r') \) can be bounded analogously. As each single contraction can at most have rank \( m \) we can control this quantity by \( m^k \). The constants \( C_3(\psi, r, H_0) \) can be bounded by \( (C_2(H_0) + 2)^{(k+k')} \). Set \( \mathcal{I} = C_2(H_0) + 3 + \|x\|_{\mathfrak{H}} + K(H, m) \) to combine these constants and add in the normalizations \( K(H, m) \). Next, the constants \( C_1(r, k, m) \), by simply bounding \( r_j \) by \( m! \) and each binomial coefficient by \( k^m \), can be bounded by \( (m!)^k m^m \). Finally, we deal with the constant \( d! \) obtained from using the Wiener-Itô isometry, cf. Equation (\ref{eq:wiener_ito_isometry}). As \( \delta \leq (k+k')m \) \( d \) may bound \( d! \) by \( \sqrt{(km)!/(k'm)!} \) and add the multiplicative constant \( \frac{(k+k')m}{d^2} \), in order to have a summable decay in \( d \), to conclude the proof.

\section*{Lemma 5.4}
Given \( h_t^{d,k} \) as above, then, the following holds,
\[ \frac{1}{C_4(k,k',m,d)} \int_0^Z \int_0^Z \mathbf{E} \left[ I_{d} \left( h_t^{d,k} \right) I_{d} \left( h_t^{d',k'} \right) \right] \cdot dt \cdot dt' \begin{cases} T, & \text{if } H^*(d) < \frac{1}{2}, \\ T \ln(\varepsilon), & \text{if } H^*(d) = \frac{1}{2}, \\ (\frac{1}{\varepsilon})^{2H_0-2d+2}, & \text{if } H^*(d) > \frac{1}{2}. \end{cases} \]
Lemma 5.6

\[ \frac{1}{C_4(k,k',m,d)} \int_0^T \int_0^T \left[ \mathbb{E} \left[ I_d(h_t^{d,k}) I_d(h_{t'}^{d,k'}) \right] \right] dt dt' \lesssim \int_{[0,T]^2} 1 \wedge (2H_0 - 2d) dt dt' \lesssim \frac{T}{\varepsilon} \int_0^T 1 \wedge |u|(2H_0 - 2d) du. \]

Depending on the exponent \( (2H_0 - 2d) \), the above integral is either finite or diverges with rate \( \varepsilon^{(2H_0 - 2d) + 1} \), resulting in the proclaimed rate. Overall we obtain,

\[ \frac{1}{C_4(k,k',m,d)} \mathbb{E} \left[ \left( \int_0^T I_d(h_t) dt \right)^2 \right] \lesssim \begin{cases} \frac{T}{\varepsilon}, & \text{if } (2H_0 - 2d) < -1, \\ \frac{T}{\varepsilon} \ln(\varepsilon), & \text{if } (2H_0 - 2d) = -1, \\ \left( \frac{T}{\varepsilon} \right)^{(2H_0 - 2d) + 2}, & \text{if } (2H_0 - 2d) < -1, \end{cases} \quad (5.10) \]

concluding the proof.

**Convention 5.5** We call building blocks \( y_t^{k,r} \) short range dependent if \( H^*(\delta(k,r)) < \frac{1}{2} \) and long range dependent in case \( H^*(\delta(k,r)) > \frac{1}{2} \). For a function \( G \) we say it is short range dependent if all its building blocks are short range dependent and long range dependent in case it contains at least one long range dependent building block.

### 5.3 Tightness in Hölder spaces

In this section we apply Lemma 5.7 as well as a basic hypercontractivity estimate to obtain tightness in Hölder spaces via a Kolmogorov type argument. In case the function \( G \) is a finite polynomial the \( L^2 \) bound obtained above directly gives us an \( L^p \) bound via hypercontractivity. In order to treat the infinite series case, we rely on the decay rate of the coefficients.

**Lemma 5.6** Given \( G(\mathbb{X}) = \sum_{k=0}^{\infty} c_k \mathbb{X}^k \) such that \( G \) has chaos rank \( w \geq 1 \) with respect to \( y_t \), \( |c_k| \lesssim \frac{1}{k^w} \) and \( H^*(w) \in (-\infty, 1) \setminus \{ \frac{1}{2} \} \), then, for every \( p > 2 \),

\[ \| e^{H^*(w)\frac{1}{2}} \int_0^T G(y_t) ds \|_{L^p(\Omega)} \lesssim |T - s| H^*(w) \frac{1}{2}, \]

**Proof.** Firstly, by stationarity of \( y_t \) we may restrict our analysis to the case \( S = 0 \). Secondly, we want to remark at this point that by assumption the chaos rank of \( G \) is greater than 1, hence, its average with respect to the distribution of \( y_t \) is 0. Thirdly, we recall the following hypercontractivity estimate, for a random variable \( X \) belonging to the \( d^{th} \) Wiener chaos the following holds true for all \( p > 2 \), c.f. [Nualart],

\[ \| X \|_{L^p(\Omega)} \leq (p - 1)^{\frac{1}{2}} \| X \|_{L^2(\Omega)}. \]

Using this bound and the triangle inequality, we compute,

\[ \| e^{H^*(w)\frac{1}{2}} \int_0^T G(y_t) ds \|_{L^p(\Omega)} = \| \sum_{d=w}^{\infty} \sum_{k=0}^{\infty} c_k e^{H^*(w)\frac{1}{2}} \int_0^T I_d(h_t^{d,k}) dt \|_{L^p(\Omega)} \]

\[ \leq \sum_{d=w}^{\infty} \| \sum_{k=0}^{\infty} c_k e^{H^*(w)\frac{1}{2}} \int_0^T I_d(h_t^{d,k}) dt \|_{L^p(\Omega)} \]

\[ \lesssim \sum_{d=w}^{\infty} (p - 1)^{\frac{1}{2}} \| \sum_{k=0}^{\infty} c_k e^{H^*(w)\frac{1}{2}} \int_0^T I_d(h_t^{d,k}) dt \|_{L^2(\Omega)}. \]

15
Now, we observe that the polynomial \((y_t)^k\) has contributions only in the chaoses up to order \(km\), hence, using Lemma \ref{lemma:kolmogorov-bound},

\[
\sum_{d,w} (p-1)^{\frac{\bar{d}}{2}} \left\| \sum_{k=0}^{\infty} c_k \varepsilon H^{\ast}(w)^{\frac{d}{2}} \int_0^{\epsilon} I_d(h_t^{d,k}) dt \right\|_{L^2(\Omega)}
\]

\[
= \sum_{d,w} (p-1)^{\frac{\bar{d}}{2}} \left\| \sum_{k=0}^{\infty} c_k c_{k'} \varepsilon^{2H^{\ast}(w)} H^{d,k'} \int_0^{\epsilon} \int_0^{\epsilon} \mathbb{E} \left[ I_d(h_t^{d,k}) I_d(h_t^{d,k'}) \right] dt dt' \right\|_{L^2(\Omega)}
\]

\[
\leq \sum_{d,w} (p-1)^{\frac{\bar{d}}{2}} \left\| \sum_{k=0}^{\infty} c_k c_{k'} \varepsilon^{2H^{\ast}(w)} H^{d,k'} \int_0^{\epsilon} \int_0^{\epsilon} 1 \wedge |t - t'|^{(2H_0 - 2) d} dt dt' \right\|_{L^2(\Omega)}
\]

\[
\leq \sum_{d,w} (p-1)^{\frac{\bar{d}}{2}} \left\| \sum_{k=0}^{\infty} c_k c_{k'} \varepsilon^{2H^{\ast}(w)} H^{d,k'} \int_0^{\epsilon} \int_0^{\epsilon} 1 \wedge |t - t'|^{(2H_0 - 2) d} dt dt' \right\|_{L^2(\Omega)}
\]

\[
\lesssim \sum_{d,w} (p-1)^{\frac{\bar{d}}{2}} \left\| \sum_{k=0}^{\infty} c_k c_{k'} \varepsilon^{H^{\ast}(w)} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} c_k c_{k'} \varepsilon^{H^{\ast}(w)} H^{d,k'} \int_0^{\epsilon} G_j(y_t) dt \right\|_{L^2(\Omega)}
\]

\[
\lesssim |T|^{H^{\ast}(w)} \sum_{k=0}^{\infty} (k+k') \sum_{d,w} (p-1)^{\frac{(k+k')}{2}} c_k c_{k'} C_4(k,k',m,d) |T|^{H^{\ast}(w)} \frac{1}{2}
\]

as \(\sum_{d,w} (p-1)^{\frac{\bar{d}}{2}} \left\| \sum_{k=0}^{\infty} c_k \varepsilon H^{\ast}(w)^{\frac{d}{2}} \int_0^{\epsilon} I_d(h_t^{d,k}) dt \right\|_{L^2(\Omega)} < \infty\), due to the decay condition imposed on the coefficients \(c_k\).

\(\square\)

**Proposition 5.7** Fix \(H \in (\frac{1}{2}, 1)\). Let, for each \(j = 1, \ldots, N\), a function of the form \(G_j(X) = \sum_{k=0}^{\infty} c_{j,k} X^k\) such that \(G_j\) has chaos rank \(w_j\) with respect to \(y_t\) and \(|c_{j,k}| \lesssim \frac{1}{\gamma^k}\) be given. Assume further that for each \(j\), \(w_j \in (-\infty, 1) \setminus \{\frac{1}{2}\}\). For \(T \in [0, 1]\) set

\[
\tilde{G}_T^{\gamma,\varepsilon} = \varepsilon^{H^{\ast}(w_j)} \int_0^{\epsilon} G_j(y_t) dt.
\]

Then,

\[
\left( \tilde{G}_T^{\gamma,\varepsilon}, \ldots, \tilde{G}_T^{\gamma,\varepsilon} \right)
\]

is tight in \(C^\gamma([0, 1], \mathbb{R}^N)\) for \(\gamma \in (0, \frac{1}{2})\) in case there is at least on component such that \(H^\ast(w_j) < \frac{1}{2}\) and \(\gamma \in (0, \min_{j=1,\ldots,N} H^\ast(w_j))\) otherwise.

**Proof.** By Lemma \ref{lemma:kolmogorov-bound} and Kolmogorov’s Theorem each component is tight in \(C^\gamma\) for \(\gamma \in (0, \frac{1}{2} \lor H^\ast(w_j))\). Therefore, taking the minimum over these values, we may conclude the proof.

\(\square\)

### 5.4 Short range dependent case

In this section we establish the convergence of building blocks in finite dimensional distributions in the short range dependent setting. Thus, we deal with terms of the form \(\sqrt{\varepsilon} \int_0^{\epsilon} g_{k,r}^{s} dt\) such that \(\delta(k, r) = d\) and \(H^\ast(d) < \frac{1}{2}\). Our main tool to prove this convergence is the fourth-moment theorem, which is stated below.

**Theorem 5.8** [Fourth Moment Theorem] Let \(2 \leq L, 1 \leq d_1 \leq \cdots \leq d_L\) be fixed integers and \(f_j^{\varepsilon} \in L^2(\mathbb{R}^{d_j})\) for \(1 \leq j \leq L\) be given. Then, under the condition that \(\lim_{\varepsilon \to 0} \mathbb{E} \left[ I_{d_j}(f_1^{\varepsilon}) I_{d_l}(f_1^{\varepsilon}) \right] = \Lambda_{j,l} \) exists for \(1 \leq j, l \leq L\), the following are equivalent:

---

16
1. For $1 \leq j \leq L$ and $p = 1, \ldots, d_j$, 
\[ \lim_{\varepsilon \to 0} \| f^{j, \varepsilon} \otimes_p f^{j, \varepsilon} \|_{L^2([\mathbb{R}^{d_j}, -2p])} = 0. \]

2. The vector $(I_{d_1}(f^1, \varepsilon), \ldots, I_{d_L}(f^L, \varepsilon))$ converges in distribution to a $L$ dimensional Gaussian vector with mean zero and covariance matrix $\Lambda$.

**Remark 5.9** Up to know we labelled our contraction numbers by $r = (r_1, \ldots, r_k)$. In case we need to denote several of such vectors we make the convention that upper indices denote different vectors and subscripts the position within the vector.

Next, we prove that the conditions necessary to apply the fourth moment theorem are indeed satisfied in our regime. In the short range dependent case we set, for $T \in [0, 1]$,
\[ g_T^{k, r, \varepsilon} = \sqrt{\varepsilon} \int_0^T g_t^{k, r} dt. \]

**Lemma 5.10** For each $k, k', r, r'$ such that $\delta(k, r) = d, \delta(k', r') = d'$, $H^+(d) < \frac{1}{2}$, and $g_T^{k, r, \varepsilon}, g_T^{k', r', \varepsilon}$ for $T, T' \in [0, 1]$ defined as above, the following holds,
\[ \lim_{\varepsilon \to 0} \mathbb{E}[I_d(g_T^{k, r, \varepsilon}) I_d(g_{T'}^{k', r', \varepsilon})] = \delta_{d, d'} 2(T \wedge T') \int_0^\infty \mathbb{E}[I_d(g_{t'}^{k', r'}) I_d(g_{t}^{k, r})] du. \]

**Proof.** Firstly, if $d \neq d'$ the expression is 0 by orthogonality of distinct Wiener chaoses. Hence, we assume $d = d'$ and w.l.o.g $T \leq T'$ from now on. In this case
\[ \mathbb{E}[I_d(g_T^{k, r, \varepsilon}) I_d(g_{T'}^{k', r', \varepsilon})] = \varepsilon \int_0^\frac{T}{\sqrt{\varepsilon}} \int_0^\frac{T'}{\sqrt{\varepsilon}} \mathbb{E}[I_d(g_{t'}^{k', r'}) I_d(g_t^{k, r})] dt' dt + \varepsilon \int_0^\frac{T}{\sqrt{\varepsilon}} \int_0^\frac{T'}{\sqrt{\varepsilon}} \mathbb{E}[I_d(g_{t'}^{k', r'}) I_d(g_{t}^{k, r})] dt' dt \]

By Lemma 5.3 and the change of variables $u = t - t'$, we obtain for the second term,
\[ \varepsilon \int_0^\frac{T}{\sqrt{\varepsilon}} \int_0^\frac{T'}{\sqrt{\varepsilon}} \mathbb{E}[I_d(g_{t'}^{k', r'}) I_d(g_t^{k, r})] dt' dt \lesssim \varepsilon \int_0^\frac{T}{\sqrt{\varepsilon}} \int_0^\frac{T'}{\sqrt{\varepsilon}} 1 \wedge |t - t'|^{2H_0-2} dt' dt \]
\[ \lesssim \int_0^\frac{T}{\sqrt{\varepsilon}} \int_0^\frac{T'}{\sqrt{\varepsilon}} 1 \wedge u^{2H_0-2} du. \]

By assumption $H^+(d) < \frac{1}{2}$, leading to $2H_0 - 2 < -1$, hence, the above term converges to 0 as $\varepsilon \to 0$ in case $T' > 0$. If $T = 0$, however, the whole expression equals 0 independent of $\varepsilon > 0$. Now we deal with the first term.

As $\mathbb{E}[I_d(g_{t'}^{k', r'}) I_d(g_{t}^{k, r})] = \mathbb{E}[I_d(g_{t'}^{k', r'}) I_d(g_{0}^{k, r'})]$ we obtain, again via the change of variables $u = t - t'$,
\[ \varepsilon \int_0^\frac{T}{\sqrt{\varepsilon}} \int_0^\frac{T'}{\sqrt{\varepsilon}} \mathbb{E}[I_d(g_{t'}^{k', r'}) I_d(g_{t}^{k, r})] dt' dt = \varepsilon \int_{T/\sqrt{\varepsilon}}^{T/\sqrt{\varepsilon}} \int_{T'/\sqrt{\varepsilon}}^{T'/\sqrt{\varepsilon}} \mathbb{E}[I_d(g_{t'}^{k', r'}) I_d(g_{0}^{k, r'})] dt' dt \]
\[ = 2T \int_0^\frac{T}{\sqrt{\varepsilon}} \frac{T - u}{T} \mathbb{E}[I_d(g_{u}^{k, r}) I_d(g_{0}^{k, r'})] du \]
\[ \to 2T \int_0^\infty \mathbb{E}[I_d(g_{u}^{k, r}) I_d(g_{0}^{k, r'})] du, \]
when $\varepsilon \to 0$ by dominated convergence.
Lemma 5.11 For each $k, r$ such that $\delta(k, r) = d$, $T \in [0, 1]$, $H^*(d) < \frac{1}{2}$, $p \leq d - 1$, and $g^{k,r,e}_T$ as above,

$$\lim_{\varepsilon \to 0} \|g^{k,r,e}_T \otimes_p g^{k,r,e}_T\|_{L^2(\mathbb{R}^{d-2p}, \lambda)} = 0.$$  

**Proof.** We compute,

$$\|g^{k,r,e}_T \otimes_p g^{k,r,e}_T\|_{L^2(\mathbb{R}^{d-2p})}^2 = \varepsilon^2 \int_{\mathbb{R}^d} d^{d-p} \xi_1 \int_{\mathbb{R}^d} d^{d-p} \xi_2 \left| \varepsilon \int_0^T dt_0 \int_0^T dt_1 \int_0^T dt_2 \int_0^T dt_3 \right|^2$$

The integrals of the form

$$\int_{\mathbb{R}^d} d^{d-p} \xi_1 \int_{\mathbb{R}^d} d^{d-p} \xi_2 \int_0^T dt_0 \int_0^T dt_1 \int_0^T dt_2 \int_0^T dt_3$$

are just further contractions of $f_j$ with itself and can be computed as in section 5.1. We obtain, up to the constants $C_2(H_0)$,

$$\int_{\mathbb{R}^d} d^{d-p} \xi_1 \int_{\mathbb{R}^d} d^{d-p} \xi_2 \int_0^T dt_0 \int_0^T dt_1 \int_0^T dt_2 \int_0^T dt_3$$

and similarly for

$$\int_{\mathbb{R}^d} d^{d-p} \xi_1 \int_{\mathbb{R}^d} d^{d-p} \xi_2 \int_0^T dt_0 \int_0^T dt_1 \int_0^T dt_2 \int_0^T dt_3$$

Computing the integrals with respect to $\xi_1$ and $\xi_2$ we again score constants $C_2(H_0)$ which we can neglect and end up with,

$$\sum_{\psi_1, \psi_2 \in \mathcal{S}_r} \int_{[0, \varepsilon]} d^{4-t} \int_{-\infty}^{t} d^{4k} s[x(t_j - s_{j,l})] \prod_{j=1}^{4} \prod_{l=1}^{k} |s_j - s_{j,l}|^{\frac{H_0-\frac{1}{2}}{2}}$$

where we make the convention $t = t_1, t' = t_2, \tau = t_3, \tau' = t_4$ and

$$A_{j',l'}^{j,l}(\psi_j, \psi_{j'}) = \begin{cases} \beta_{j,l}(\psi_j) & \text{if } j = j', \\ \gamma_{l,l'}(\psi_j, \psi_{j'}) & \text{if } j \neq j'. \end{cases}$$

Now, we have to consider four graphs. Nevertheless, as in the discussions above we focus on one particular choice of permutations and rely on arguments which only depend on $k, r, m$ and $H_0$, thus, we suppress dependencies on the $\psi$’s from now on. Hence, the term we need to deal with is given by,

$$\varepsilon^2 \int_{[0, \varepsilon]} d^{4-t} \int_{-\infty}^{t} d^{4k} s[x(t_j - s_{j,l})] \prod_{j=1}^{4} \prod_{l=1,t'=1}^{k} |s_j - s_{j,l,t'}|^{A_{j',l'}^{j,l}(\psi_j, \psi_{j'})}$$

where $A_{j',l'}^{j,l}$ depends on the four permutations we choose when picking graphs and denotes how many edges run between the $l^{th}$ node in the $j^{th}$ graph to the $l^{th}$ node in the $j^{th}$ one. Keeping the graph view, the following illustration shows the entanglements between the four objects, where $B_{j,j'} = \sum_{l=1,t'=1}^{k} A_{j',l'}^{j,l}(\psi_j, \psi_{j'})$ is the number of all edges going from graph $j$ to graph $j'$. 

18
As previously we have some algebraic constraints, namely, $B_{1.3} + B_{1.4} = d - p$ and $B_{2.3} + B_{2.4} = d - p$. By the same arguments as in section 5.2 we can iteratively apply the generalized Hölder inequality and bound the above integrand by

$$\prod_{j \geq j'} \left( \int_{-\infty}^{t_j} \int_{-\infty}^{t_j'} |x(t_j - s)x(t_j' - r)||s - r|^{m(2H_0 - 2)} dr ds \right)^{B_{j, j'}}.$$ 

Terms for which $j = j'$ give rise to constants as they represent connections within a graph. For the other terms we obtain the same decay as in Lemma 5.3.

Hence, we obtain, for $\varrho(s) = 1 \land |s|^{(2H_0 - 2)m}$,

$$\int_{[0, \tau]^4} d^4 t \prod_{j=1}^4 \int_{[-\infty, t_j]} ds x(t_j - s_j) \prod_{j'=1}^4 \prod_{l=1, l'=1}^k |s_{j,l} - s_{j',l'}|^{(2H_0 - 2)}.$$

As $B_{1.3} + B_{1.4} = d - p$ we obtain $\varrho(t_1 - t_3)^{B_{1.3}} \varrho(t_1 - t_4)^{B_{1.4}} \leq \varrho(t_1 - t_3)^{d-p} + \varrho(t_1 - t_4)^{d-p}$ and similarly $\varrho(t_2 - t_3)^{B_{2.3}} \varrho(t_2 - t_4)^{B_{2.4}} \leq \varrho(t_2 - t_3)^{d-p} + \varrho(t_2 - t_4)^{d-p}$. Thus,

$$\int_{[0, \tau]^4} d^4 t \varrho(t_1 - t_2)^p \varrho(t_3 - t_4)^p \varrho(t_1 - t_3)^{B_{1.3}} \varrho(t_1 - t_4)^{B_{1.4}} \varrho(t_2 - t_3)^{B_{2.3}} \varrho(t_2 - t_4)^{B_{2.4}} \leq \int_{[0, \tau]^4} d^4 t \varrho(t_1 - t_2)^p \varrho(t_3 - t_4)^p \varrho(t_2 - t_3)^{d-p} + \varrho(t_2 - t_4)^{d-p}. \varrho(t_1 - t_3)^{d-p} + \varrho(t_1 - t_4)^{d-p}.$$

We just treat the term $\varrho(t_1 - t_2)^p \varrho(t_3 - t_4)^p \varrho(t_2 - t_3)^{d-p} \varrho(t_1 - t_3)^{d-p}$ as the others are analogous. Applying the
Proposition 5.12

Proof. Now, the first integral is finite as by assumption \( H^*(d) < \frac{1}{2} \). Furthermore, \( \int_0 \varrho((t_s)^d)p_{dt} \leq \varepsilon^{-(2H_0-2)(d-1)} \), cf. the proof of Lemma 5.5. Thus, the expression is of order \( \varepsilon^{-(2H_0-2)d-3} \) and as \( H^*(d) < \frac{1}{2} \) this expression is of order \( o(\varepsilon^2) \), which concludes the proof.

Now we have all tools necessary to conclude the proclaimed convergence in finite dimensional distributions for our building blocks.

**Proposition 5.12** Given the process \( \left( I_{d_1}(g_{k_1, r^1, \varepsilon}), \ldots, I_{d_L}(g_{k_L, r^L, \varepsilon}) \right) \), for which \( d_j = \delta(k_j, r^j) \) such that for each \( 1 \leq j \leq L \), \( H^*(d_j) < \frac{1}{2} \), then,

\[
\left( I_{d_1}(g_{k_1, r^1, \varepsilon}), \ldots, I_{d_L}(g_{k_L, r^L, \varepsilon}) \right) \rightarrow (W_1, \ldots, W_L),
\]

in finite dimensional distributions, where \((W_1, \ldots, W_L)\) is a multidimensional Wiener process with covariance matrix

\[
\mathbb{E}[W_j(1)W_l(1)] = \Lambda_{j,l} = \lim_{\varepsilon \to 0} \mathbb{E}
\left[
I_{d_j}(g_{1, k_j, r^j, \varepsilon})I_{d_l}(g_{1, k_l, r^l, \varepsilon})
\right] = \delta_{d_j, d_l} 2 \int_0^{\infty} \mathbb{E}
\left[
I_{d_j}(g_{1, k_j, r^j})I_{d_l}(g_{0, k_l, r^l})
\right] du,
\]

**Proof.** By the Fourth Moment Theorem \( \text{f.m.t.} \) and an application of the Cramer Wold Theorem \( \text{c.w.t.} \), the claim follows by Lemma 5.9 and Lemma 5.10.

**5.5 Long range dependent case**

In this section we establish the convergence of building blocks in finite dimensional distributions in the long range dependent setting. Thus, we deal with terms of the form \( \varepsilon H^*(d) \int_0^2 g_{k, r} dt \) such that \( \delta(k, r) = d \) and \( H^*(d) > \frac{1}{2} \).

Our main tool to prove this convergence is \( L^2 \) kernel convergence.

**5.5.1 Spectral representation**

In this section we use a connection between iterated Wiener integrals and a Gaussian complex-valued random spectral measure \( \hat{B} \) such that for all Borel sets the following holds, \( \mathbb{E}\left[\hat{B}(A)\right] = 0 \), \( \mathbb{E}\left[\hat{B}(A_1)\hat{B}(A_2)\right] = \lambda(A_1 \cap A_2) \), \( \hat{B}(A) = \overline{B(-A)} \) and for disjoint Borel sets \( \hat{B}(\bigcup_{j=1}^n A_j) = \sum_{j=1}^n \hat{B}(A_j) \). The random variables \( \text{Re}\left(\hat{B}(A)\right) \) and \( \text{Im}\left(\hat{B}(A)\right) \) are independent Gaussians with zero mean and variance \( \frac{\lambda(A)}{2} \). One can now define multiple Wiener
Lemma 5.14 This gives rise to an isometric mapping \( \hat{I}_d : \mathcal{CH}_m \to L^2(\Omega) \) via,
\[
\hat{I}_d(f) = d! \int_{\mathbb{R}^d} d^d\xi \hat{f}(\xi_1, \ldots, \xi_d),
\]
see [DT18, Maj44, Dob79]. The next lemma gives us a way to relate \( I_d \) and \( \hat{I}_d \).

**Lemma 5.13** ([Taq79] Lemma 6.1) Let \( h \in L^2(\mathbb{R}^d, \lambda) \) be a real valued symmetric function and denote its Fourier transform by \( \hat{h} \), then,
\[
I_d(h) = \int_{\mathbb{R}^d} d^d\xi \ h(\xi_1, \xi_2, \ldots, \xi_d) = \int_{\mathbb{R}^d} d^d\xi \ \hat{h}(\xi_1, \xi_2, \ldots, \xi_d) = \hat{I}_d(h),
\]
where \( \hat{B} \) is a Gaussian complex valued random spectral measure given as above and the second equality is to be understood in law.

This relation also holds in the multivariate setting.

**Lemma 5.14** ([BT13b] Lemma A.2) Let, for each \( j = 1, \ldots, N \), a symmetric function \( h^j \in L^2(\mathbb{R}^{d_j}, \lambda) \) be given and denote its Fourier transform by \( \hat{h}^j \), then, the following equivalence holds in law,
\[
(I_d(h^1), \ldots, I_d(h^N)) = (\hat{I}_d(h^1), \ldots, \hat{I}_d(h^N)),
\]
where \( I_d \) and \( \hat{I}_d \) denote the isometries defined in sections 4.7 and 4.8, respectively.

**Lemma 5.15** Given \( g_{t}^{k,r} = f_t \hat{\otimes}_r f_t \hat{\otimes}_r f_t \ldots \hat{\otimes}_r f_t \) such that \( \delta(k, r) = d \), which is a symmetric function in \( L^2(\mathbb{R}^d, \lambda) \), then,
\[
g_{t}^{k,r}(\xi_1, \ldots, \xi_d) = \left( \frac{\Gamma(H_0 - \frac{1}{2})}{\sqrt{2\pi}} \right)^d \sum_{\psi \in S_r} C_3(\psi, r, H_0) \left( \varphi_{t}^{k,r,\psi} \left( \sum_{l=1}^{M_3(\psi)} \xi_{1,l}, \ldots, \sum_{l=1}^{M_3(\psi)} \xi_{k,l} \right) \prod_{j=1}^{k} \prod_{l=1}^{M_3(\psi)} \xi_{j,l}^{2-H_0} C(\xi_{j,l}) \right),
\]
where \( \varphi_{t}^{k,r,\psi}(s) = 1_{s \leq t} \prod_{j=1}^{k} \prod_{q=1}^{j-1} |s_j - s_q|^{\beta_{j,q}(\psi)(2H_0-2)}. \) Furthermore, the following equivalence holds in law,
\[
\int_{\mathbb{R}^d} d^d\xi \ g_{t}^{k,r} = \int_{\mathbb{R}^d} d^d\xi \ \hat{g}_{t}^{k,r} \left( \frac{\Gamma(H_0 - \frac{1}{2})}{\sqrt{2\pi}} \right)^d \sum_{\psi \in S_r} C_3(\psi, r, H_0) \left( \varphi_{t}^{k,r,\psi} \left( \sum_{l=1}^{M_3(\psi)} \xi_{1,l}, \ldots, \sum_{l=1}^{M_3(\psi)} \xi_{k,l} \right) \prod_{j=1}^{k} \prod_{l=1}^{M_3(\psi)} \xi_{j,l}^{2-H_0} \right).
\]

**Proof.** By section 4.11
\[
g_{t}^{k,r} = \sum_{\psi \in S_r} C_3(\psi, r, H_0) \int_{(-\infty,t]^r} d^k s \ \prod_{j=1}^{k} \int_{(-\infty,t]^r} d^k s \ \prod_{j=1}^{k} (s_j - \xi_{\psi(j,l)})^{H_0-\frac{1}{2}} \prod_{j=1}^{k} |s_j - s_q|^{\beta_{j,q}(\psi)(2H_0-2)}.
\]
As Fourier transforming commutes with symmetrizing we again restrict our computation to one \( \psi \) as our computations are independent of the choice for \( \psi \). Thus, we suppress \( \psi \) in our notation as well the constants \( C_3, \)
Following Taqqu [Taq92], care is needed as \( \xi^{H_0-\frac{2}{3}} \) does not belong to \( L^1 \) nor \( L^2 \). Instead we first consider

\[
g_{s,t}(\xi) = g_{s,t}(\xi)1_{[-K,K]}(\xi).
\]

Substituting \( u_{j,t} = s_j - \xi_j,t \) leads us to,

\[
g^{k,r,K}_{s,t} = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} d^d \xi \ e^{i \sum_{j=1}^d \xi_j t (j)} \int_{[-\infty,t]^{K}} d^k s \ \prod_{j=1}^k x(t - s_j) \ \prod_{l=1}^M \ (s_j - \xi_j,t)^{H_0 - \frac{2}{3}} 1_{[0,t]}(s_j) \ \prod_{q=1}^j |s_j - s_q|^{\beta_{j,q}(2H_0-2)}
\]

\[
= \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} d^d u \ \int_{[-\infty,t]^{K}} d^k s \ e^{i \sum_{j=1}^d \xi_j u_{j,t} (j)} \ \prod_{j=1}^k x(t - s_j) \ \prod_{l=1}^M \ (u_{j,t})^{H_0 - \frac{2}{3}} 1_{[s_j - K,s_j + K]}(u_{j,t}) \ \prod_{q=1}^j |s_j - s_q|^{\beta_{j,q}(2H_0-2)}
\]

\[
= \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} d^d u \ \int_{[-\infty,t]^{K}} d^k s \ e^{i \sum_{j=1}^d \xi_j u_{j,t} (j)} \ \prod_{j=1}^k \ \prod_{l=1}^M (u_{j,t})^{H_0 - \frac{2}{3}} 1_{s_j - K,s_j + K}(u_{j,t}) e^{\sum_{j=1}^d \xi_j s_j x(t - s_j) \ \prod_{q=1}^j |s_j - s_q|^{\beta_{j,q}(2H_0-2)}},
\]

Although we cannot separate the \( u \) and \( s \) integrals as \( e^{-i \sum_{j=1}^d \xi_j u_{j,t} (j)} \ \prod_{j=1}^k \ \prod_{l=1}^M (u_{j,t})^{H_0 - \frac{2}{3}} 1_{s_j - K,s_j + K}(u_{j,t}) \) depends on \( s \), as \( K \to \infty \) this dependency vanishes, thus, we go ahead and analyse its behaviour as \( K \to \infty \). The \( u_{j,t} \)’s terms split so we can restrict ourselves to

\[
Q_{\xi_{j,t}}(a,b) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{iu_{j,t} \xi_j t} du_{j,t},
\]

for \( 0 \leq a \leq b < \infty \). The following estimate was shown in [Taq92].

\[
\sup_{0 \leq a \leq b < \infty} |Q_{\xi_{j,t}}(a,b)| \leq \frac{1}{\sqrt{2\pi}} \left( \frac{1}{H_0 - \frac{2}{3}} + \frac{2}{|\xi_{j,t}|} \right).
\]

Thus,

\[
|g_{s,t}^{k,r,K}(\xi)| \leq \int_{[-\infty,t]^{K}} d^k s \ \prod_{j=1}^k x(t - s_j) \ \prod_{q=1}^j |s_j - s_q|^{\beta_{j,q}(2H_0-2)} \ \prod_{j,l} B_{\xi_{j,t}}(\max(0,s_j - K), \max(0,s_j + K))
\]

\[
\leq \int_{[-\infty,t]^{K}} d^k s \ |\varphi_{\xi_{j,t}}^{k,r}(s)| \ \prod_{j,l} \frac{1}{\sqrt{2\pi}} \left( \frac{1}{H_0 - \frac{2}{3}} + \frac{2}{|\xi_{j,t}|} \right).
\]

By arguments similar to the ones in Lemma 3.1 one can show \( \varphi_{\xi_{j,t}}^{k,r} \in L^r(\mathbb{R}^d, \lambda) \). We compute,

\[
\|\varphi_{\xi_{j,t}}^{k,r}\|_{L^1(\mathbb{R}^d, \lambda)} = \int_{[-\infty,t]^{K}} d^k s \ \prod_{j=1}^k |x(t - s_j)||s_k - s_j|^{\beta_{k,j}(2H_0-2)}.
\]

Looking at the integral with respect to \( s_k \), and again pulling the kernels \( x(t - s_j) \), with the right exponents, in the integral, we obtain, setting \( B_k = m - \sum_{j=1}^{k} \beta_{k,j} \),

\[
\int_{-\infty}^t ds_k |x(t - s_k)| \frac{\beta_{k,j}}{m} \ \prod_{j=1}^k |x(t - s_j)|^{-\frac{\beta_{k,j}}{m}} |s_k - s_j|^{\beta_{k,j}(2H_0-2)}
\]

\[
\leq \left( \int_{-\infty}^t ds_k |x(t - s_k)| \frac{\beta_{k,j}}{m} \right) \prod_{j=1}^k \left( \int_{-\infty}^t ds_k |x(t - s_k)| x(t - s_j) |s_k - s_j|^{(2H_0-2)} \right)^{\frac{\beta_{k,j}}{m}}
\]

\[
\leq \|x\|_{L^1(\mathbb{R}^d, \lambda)} \prod_{j=1}^k \left( \int_{-\infty}^t ds_k |x(t - s_k)| x(t - s_j) |s_k - s_j|^{(2H_0-2)} \right)^{\frac{\beta_{k,j}}{m}}.
\]
\[ \varphi_t^{k,r} \leq \left\| \sum_{j=1}^{k} \frac{B_j}{d} \prod_{j,q=1}^{k} \left( \int_{-\infty}^{t} ds \int_{-\infty}^{t} dr |x(t-s)x(t-r)||s-r|^{(2H_0-2)} \right)^{\frac{\beta+\kappa}{2}} \right\|_{L^2(\mathbb{R}^k,\lambda)} \]

Thus, \( g_t^{k,r,K} \) is finite and uniformly bounded with respect to \( K \) as soon as all \( \hat{\xi}_{j,l} \) are different from 0. By the substitution \( u \to u||\hat{\xi}_{j,l}| \), we obtain

\[ g_t^{k,r} = \lim_{K \to \infty} g_t^{k,r,K} \]

\[ = \int_{[-\infty,t]^k} d^{k}se \sum_{j=1}^{k} \xi_{j,l} s_j \sum_{j=1}^{k} \int_{-\infty}^{t} ds \int_{-\infty}^{t} dr \left| s_j - s_q \right|^{\beta_j q |(2H_0-2)|} \prod_{j,q=1}^{k} \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{1}{2}u^2} du \]

For the identity \( \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{1}{2}u^2} du = \Gamma(H_0 - \frac{1}{2}) \) we refer to the appendix of \( [\text{Diu-Tran}] \). Furthermore, \( C(\xi) = e^{i\xi^{\frac{1}{2}}(H_0 - \frac{1}{2})} \) for \( \xi > 0 \), thus, \( C(-\xi) = C(\xi) \) and \( |C(\xi)| = 1 \) for \( \xi \neq 0 \). Hence, see \( [\text{Dobrnjak}],[\text{Dobrnjak}] \), \( C(\hat{\xi}_{j,l})dW_{\hat{\xi}_{j,l}} \sim dW_{\hat{\xi}_{j,l}} \). This concludes the proof.

So, what did we gain from dealing with \( \hat{g}_t^{k,r} \) instead of \( g_t^{k,r} \)? In the kernel representation for \( g_t^{k,r} \) we see that the \( s \) variables are in a way convoluted with the \( \xi \) ones. As we saw in the graph picture above, when computing \( L^2 \) norms we could entangle them with our mince knife, however, for obtaining kernel convergence this method is too harsh. Nevertheless, as Fourier transforms change convolutions to multiplications this in a sense entangles the variables between the graphs from the ones within each graph.

5.5.2 Kernel Convergence

In the long range dependent case we set, for \( T \in [0,1] \),

\[ g_t^{k,r} = e^{H^*(d)} \int_{0}^{T} g_t^{k,r} dt, \]

where \( d = \delta(k, r) \).

Proposition 5.16 Given the process \( \left( I_{d_1}(g_t^{k_1,r_1}), \ldots, I_{d_L}(g_t^{k_L,r_L}) \right) \), for which \( d_j = \delta(k_j, r_j) \) such that for each \( 1 \leq j \leq L \), \( H^*(d_j) > \frac{1}{2} \) and \( T \in [0,1] \), then,

\[ \left( I_{d_1}(g_T^{k_1,r_1}), \ldots, I_{d_L}(g_T^{k_L,r_L}) \right) \rightarrow \left( \kappa_1 Z_T^{H^*(d_1), d_1}, \ldots, \kappa_L Z_T^{H^*(d_L), d_L} \right), \]

where \( \kappa_j = \lim_{t \to 0} \| I_{d_t}(g_t^{k_j, r_j}) \|_{L^2} \) and \( \left( Z_T^{H^*(d_1), d_1}, \ldots, Z_T^{H^*(d_L), d_L} \right) \) is a multidimensional Hermite process such that each component is defined via the same Wiener process, in the sense of finite dimensional distributions.
Proof. By Lemma 13, the Fourier transform of a building block is given by

$\hat{g}_{t,r}^{k,r} = \left(\frac{\Gamma(H_0 - \frac{1}{2})}{\sqrt{2\pi}}\right)^d \sum_{\psi \in S_t} C_0(\psi, r, H_0) \hat{f}_{t,r}^{k,r,\psi} \left(\sum_{l=1}^{M_1(\psi)} \hat{\xi}_{j,l} + \cdots + \sum_{l=1}^{M_k(\psi)} \hat{\xi}_{j,k,l}\right) \prod_{j=1}^{k} \prod_{l=1}^{M_j(\psi)} \hat{\xi}_{j,l}^{H_0} C(\hat{\xi}_{j,l}),$

where the constants $C(\hat{\xi}_{j,l})$ can be absorbed into the measures. We perform the following computations for one building block, however, the changes of variables can be done simultaneously for all components and hence preserve equivalence in law. As usual we drop constants and restrict ourselves to one choice of $\psi$ as our analysis is independent of this choice. Denoting by $\hat{g}_{t,r}^{k,r,\psi}$ the Fourier transform of $\hat{g}_{t,r}^{k,r,\psi}$, we compute,

$\hat{I}_d(\hat{g}_{t,r}^{k,r,\psi}) = C^{H_0}(d) \int_0^T dt \int_{\mathbb{R}^d} d\hat{B}_\xi \prod_{j=1}^{d} \left(\hat{\xi}_{j,l}^{\frac{1}{2} - H_0}\right) \left(\sum_{l=1}^{M_1(\psi)} \hat{\xi}_{j,l} + \cdots + \sum_{l=1}^{M_k(\psi)} \hat{\xi}_{j,k,l}\right) \prod_{j=1}^{k} \prod_{l=1}^{M_j(\psi)} \hat{\xi}_{j,l}^{H_0} C(\hat{\xi}_{j,l}),$  

In the following we use the self-similarity of $\hat{B}$, namely, $\hat{B}(\hat{\xi}) = \hat{\xi} s \hat{B}(\hat{\xi})$, cf. [30]. Applying the changes of variables $t \rightarrow \epsilon t$, $\hat{\xi}_{j,l} \rightarrow \frac{1}{\epsilon} \hat{\xi}_{j,l}$ and $s_j \rightarrow \epsilon - s_j$ we obtain,

$\hat{I}_d(\hat{g}_{t,r}^{k,r,\psi}) = \epsilon^{H_0}(d-1) \int_0^T dt \int_{\mathbb{R}^d} d\hat{B}_\xi \prod_{j=1}^{d} \left(\hat{\xi}_{j,l}^{\frac{1}{2} - H_0}\right) \left(\sum_{l=1}^{M_1(\psi)} \hat{\xi}_{j,l} + \cdots + \sum_{l=1}^{M_k(\psi)} \hat{\xi}_{j,k,l}\right) \prod_{j=1}^{k} \prod_{l=1}^{M_j(\psi)} \hat{\xi}_{j,l}^{H_0} C(\hat{\xi}_{j,l}),$  

Now, fix $T \in [0, 1]$. By the $L^2$ isometry property obtained in section 15.1 we can also work with the kernel in order to prove $L^2(\Omega)$ convergence. Hence, by dominated convergence we obtain the pointwise result

$\int_0^T dt \int_{\mathbb{R}^d} d\hat{\xi} \prod_{j=1}^{d} \left(\hat{\xi}_{j,l}^{\frac{1}{2} - H_0}\right) \left(\sum_{l=1}^{M_1(\psi)} \hat{\xi}_{j,l} + \cdots + \sum_{l=1}^{M_k(\psi)} \hat{\xi}_{j,k,l}\right) \prod_{j=1}^{k} \prod_{l=1}^{M_j(\psi)} \hat{\xi}_{j,l}^{H_0} C(\hat{\xi}_{j,l}) = \int_{\mathbb{R}^d} d\xi \prod_{j=1}^{m} \left(\xi_{j,l}^{\frac{1}{2} - H_0}\right) \left(\sum_{l=1}^{M_1(\psi)} \xi_{j,l} + \cdots + \sum_{l=1}^{M_k(\psi)} \xi_{j,k,l}\right) \prod_{j=1}^{k} \prod_{l=1}^{M_j(\psi)} \xi_{j,l}^{H_0} C(\xi_{j,l}),$

which is up to the constant $\int_{\mathbb{R}^k} d\xi \prod_{j=1}^{k} x(s_j) \prod_{q=1}^{k} |s_j - s_q|^\beta_{\psi,\sigma}(2H_0-2)$ the spectral representation of $Z^{\psi}(d,d).$
As mentioned above, the changes of variables $t \to \varepsilon t$, $\tilde{\xi}_{j,t} \to \varepsilon \tilde{\xi}_{j,t}$ and $s_j \to \frac{t}{\varepsilon} - s_j$ can be performed simultaneously and we obtain the following equivalence in law,

$$
\left( I_{d_1}(g_{T_{1,1}^{(r_1)^r}}), \ldots, I_{d_L}(g_{T_{Q_L,1}^{(r_1)^r}}) \right) = \left( I_{d_1}(g_{T_{1,1}^{(r_1)^r}}), \ldots, I_{d_L}(g_{T_{Q_L,1}^{(r_1)^r}}) \right).
$$

By the above each component of the later converges in $L^2(\Omega)$ to the spectral representation of the proclaimed limit. As $L^2(\Omega)$ convergence in each component implies $L^2(\Omega)$ convergence of the whole vector this concludes the proof.  

6 General Functions

In the previous section we established joint convergence in finite dimensional distributions for our building blocks. As $G(y_s)$ may consist of infinitely many such objects we first prove so called reduction theorems in both the short and long range dependent setup, cf. [14, 15, 16, 31, 38]. In the Gaussian setup it suffices to look at finitely many Hermite polynomials in the SRD regime and only at the lowest rank one in the LRD case. In our case each term $(y_s)^k$ could give us a contribution in each chaos up to order $km$, hence, we do not just make a cut-off in the chaos rank, but also in the ranks of the polynomials. However similar to the Gaussian case it suffices to consider only the lowest chaos rank contributions in the long range dependent case. Furthermore, we use the decay assumption imposed on the coefficients $c_k$ to ensure that our estimates from sections 5.2 and 5.3 can be carried over.

6.1 Short range dependent case

**Definition 6.1** Given $G(X) = \sum_{k=0}^{\infty} c_k X^k$ such that $|c_k| \leq \frac{1}{k}$ we denote by $G_M(y_t)$ the projection of $\sum_{k=0}^{M} c_k(y_t)^k$ onto the first $M$ Wiener chaoses minus the higher polynomial contributions in the $0^{th}$ chaos. Thus, $G_M(y_t) = \sum_{k=0}^{M} c_k \sum_{d=M}^{\infty} I_d(h_t^{1,k}) + \sum_{k=M+1}^{\infty} c_k I_0(h_t^{0,k}).$

**Remark 6.2** The term $\sum_{k=M+1}^{\infty} I_0(h_t^{0,k})$ ensures that $E[G_M(y_s)] = 0$, hence, also $E[G(y_s) - G_M(y_s)] = 0$, for centred functions $G$.

**Lemma 6.3** Given $G(X) = \sum_{k=0}^{\infty} c_k X^k$ such that $G$ has chaos rank $w \geq 1$ with respect to $y_t$, $|c_k| \leq \frac{1}{k}$ and $H^*(w) \in (-\infty, \frac{1}{2})$, then,

$$
\lim_{M \to \infty} \limsup_{\varepsilon \to 0} E \left[ \left( \sqrt{\varepsilon} \int_{0}^{\varepsilon} G(y_t) - G_M(y_t) \, dt \right)^2 \right] = 0.
$$

25
Proof.

\[
E \left[ \left( \sqrt{\varepsilon} \int_0^T G(y_{t}) - G_M(y_t) dt \right)^2 \right]
\]

\[
= \varepsilon \int_0^T \int_0^T dt'dt' E \left[ (G(y_{t}) - G_M(y)) (G(y_{t'}) - G_M(y_{t'}) ) \right]
\]

\[
= \varepsilon \int_0^T \int_0^T dt'dt' E \left[ \left( \sum_{k=0}^{\infty} \sum_{d=M+1}^{\infty} c_k I_d (h^d_{t, k}) + \sum_{k=M+1}^{\infty} \sum_{d=1}^{\infty} c_k I_d (h^d_{t, k}) \right) \right]
\]

\[
\left( \sum_{k'=0}^{\infty} \sum_{d'=M+1}^{\infty} c_{k'} I_{d'} (h^{d'}_{t', k'}) + \sum_{k'=M+1}^{\infty} \sum_{d'=0}^{\infty} c_{k'} I_{d'} (h^{d'}_{t', k'}) \right)
\]

\[
= \sum_{d=M+1}^{\infty} \sum_{k,k'=0}^{\infty} c_k c_{k'} \varepsilon \int_0^T \int_0^T dt'dt' E \left[ I_d (h^d_{t, k}) I_d (h^{d'}_{t', k'}) \right] + \sum_{d=1}^{\infty} \sum_{k,k'=M+1}^{\infty} c_k c_{k'} \varepsilon \int_0^T \int_0^T dt'dt' E \left[ I_d (h^d_{t, k}) I_d (h^{d'}_{t', k'}) \right]
\]

\[
\leq \sum_{d=M+1}^{\infty} \sum_{k,k'=0}^{\infty} c_k c_{k'} C_4 (k, k', m, d) + \sum_{d=1}^{\infty} \sum_{k,k'=M+1}^{\infty} c_k c_{k'} C_4 (k, k', m, d).
\]

The first sum represents the part belonging to chaos of rank bigger than \( M \) and the second one the parts in a low order chaos from high order polynomials. Furthermore, \( C_4 (k, k', m, d) = \frac{\sqrt{(km)(k'm)(m+k')(k'+m)}}{d!} \) satisfies,

\[
\sum_{d=M+1}^{\infty} \sum_{k,k'=0}^{\infty} c_k c_{k'} C_4 (k, k', m, d) + \sum_{d=1}^{\infty} \sum_{k,k'=M+1}^{\infty} c_k c_{k'} C_4 (k, k', m, d)
\]

\[
\leq \sum_{d=M+1}^{\infty} \frac{1}{d^2} + \sum_{k,k'=M+1}^{\infty} \frac{(m!(k+k')!(m+k')m^2m+k'g(k+k')m)}{k!k!} \rightarrow 0 \text{ as } M \rightarrow \infty,
\]

proving the claim. \( \square \)

Lemma 6.4 Let, for each \( j = 1, \ldots, N \), a function of the form \( G^j (X) = \sum_{k=0}^{\infty} c_{j,k} X^k \) such that \( |c_{j,k}| \leq \frac{1}{k!} \) be given. If for every \( M \in \mathbb{N} \),

\[
\left( \sqrt{\varepsilon} \int_0^T G^1_M (y_{t}) dt, \ldots, \sqrt{\varepsilon} \int_0^T G^N_M (y_{t}) dt \right),
\]

converges in finite dimensional distribution to a Wiener process \( W^1_{M,T} = (W^1_{M,T} , \ldots, W^N_{M,T}) \) with covariance structure \( \Lambda_{M,1}^{j,1} = \Lambda_{M}^{j,1} \), then,

\[
\left( \sqrt{\varepsilon} \int_0^T G^1 (y_{t}) dt, \ldots, \sqrt{\varepsilon} \int_0^T G^N (y_{t}) dt \right)
\]

converges to a Wiener process \( W_T \) with covariance structure \( \Lambda_{M}^{j,1} \).

Proof. By Lemma 9.2, the condition on the \( L^2 (\mu) \) norm imposed in Theorem 9.2 is satisfied by the processes \( \left( \sqrt{\varepsilon} \int_0^T G^1_M (y_{t}) dt, \ldots, \sqrt{\varepsilon} \int_0^T G^N_M (y_{t}) dt \right) \) and \( \left( \sqrt{\varepsilon} \int_0^T G^1 (y_{t}) dt, \ldots, \sqrt{\varepsilon} \int_0^T G^N (y_{t}) dt \right) \). Thus, an application of Theorem 9.2 and the Cramer-Wold Theorem 9.5 concludes the proof. \( \square \)
Proposition 6.5  Given a collection of functions $G^j, j = 1, \ldots, N$, where $G^j = \sum_{k=0}^\infty c_{j,k}X^k$ such that $|c_{j,k}| \lesssim k!$, $G^j_M(y_s)$ as above and set $\tilde{G}^j_M, T = \sqrt{\varepsilon} \int_0^T G^j_M(y_s)ds$. Then, for every $M \in \mathbb{N}$ and finite collection of times $T_{i,j} \in [0,1]$ the vector $(\tilde{G}^j_M, T_{i,j})$, where $0 \leq i \leq Q$ and $1 \leq j \leq N$, converges jointly to a multivariate normal distribution $(W^j_M, T_{i,j})$ with covariance structure

$$\mathbb{E}[W^j_M, T_{i,j} W^j_M, T_{i',j'}] = \lim_{\varepsilon \to 0} \mathbb{E}[\tilde{G}^j_M, T_{i,j} \tilde{G}^j_M, T_{i',j'}] = 2(T_{i,j} \wedge T_{i',j'}) \sum_{k=0}^M \sum_{k'=0}^M \int_0^\infty \mathbb{E}[I_d(h_{0,k}^d)I_d(h_{0,k'}^d)]ds.$$  

Proof. As we now deal with finitely many terms $h_{d,k}$, there are infinitely many terms in the $0^{th}$ chaos, however by assumption they sum up to 0, we can collect all buildings blocks $g^{k_{0,r'}}$ such that $k_{0} \leq M$ and $0 < \delta(k_{0}, r') \leq M$. By Proposition 6.5, the vector $(\tilde{G}^j_M, T_{i,j})$ converges jointly to a multivariate normal distribution $(W^j_M, T_{i,j})$ with covariance as summation is a continuous operation, and

$$\tilde{G}^j_M, T_{i,j} = \sum_{k=0}^M c_k \sum_{d=0}^M \sum_{\delta(k,r)=d} I_d(\tilde{g}_{i,j}^{k,r}) - \mathbb{E}\left[\sum_{k=0}^M c_k \sum_{d=0}^M \sum_{\delta(k,r)=d} I_d(\tilde{g}_{i,j}^{k,r})\right]$$

$(\tilde{G}^j_M, T_{i,j})$ also converges to a multivariate Gaussian with the proclaimed covariances. □

Proposition 6.6  Fix $H \in \left(\frac{1}{2},1\right)$, $m \in \mathbb{N}$, a kernel $x$ satisfying assumptions (2,3) and set $y_t = \int_{-\infty}^t (1 - s) xZ_s$. Let, for each $j = 1, \ldots, N$, a function of the form $G^j(X) = \sum_{k=0}^\infty c_{j,k}X^k$ such that $G^j$ has chaos rank $w^j$ with respect to $y_t$ and $|c_{j,k}| \lesssim \frac{1}{k!}$ be given. Assume further that for each $j$, $H^j(w^j) < \frac{1}{2}$. For $T \in [0,1]$ set,

$$\tilde{G}^j_T = \sqrt{\varepsilon} \int_0^T G^j(y_t)dt.$$  

Then, the vector

$$(\tilde{G}^1_T, \ldots, \tilde{G}^N_T),$$

converges as $\varepsilon \to 0$ weakly in $C^\gamma([0,1], \mathbb{R}^N)$, for $\gamma \in (0, \frac{1}{2})$ to a multivariate Wiener process

$$W_T = (W^1_T, \ldots, W^N_T)$$

with covariance structure, for $T,S \in [0,1],$

$$\mathbb{E}[W^j_T W^j_S] = 2(T \wedge S) \int_0^\infty \mathbb{E}[G^j(y_t)G^j(y_0)]ds.$$  

Proof. Combining Lemma 6.3, Proposition 6.5 and the computation

$$\lim_{M \to \infty} \mathbb{E}[W^j_M, T W^j_M, S] = \lim_{M \to \infty} 2(T \wedge S) \sum_{k=0}^M \sum_{k'=0}^M \int_0^\infty \mathbb{E}[I_d(h_{0,k}^d)I_d(h_{0,k'}^d)]ds$$

$$= 2(T \wedge S) \int_0^\infty \mathbb{E}[G^j(y_t)G^j(y_0)]ds$$

$$= \mathbb{E}[W^j_T W^j_S]$$

gives us convergence in finite dimensional distributions. Furthermore, by Proposition 6.5 we have for each $j$ and $p > 2$,

$$||\tilde{G}^j_T - \tilde{G}^j_S||_{L^p(\Omega)} \lesssim \sqrt{T - S}.$$

Thus, by an application of Kolmogorov’s Theorem each $\tilde{G}^j_T$ is tight in $C^\gamma([0,1], \mathbb{R}^N)$ for $\gamma \in (0, \frac{1}{2})$. As tightness in $C^\gamma([0,1], \mathbb{R}^N)$ is equivalent to tightness in each component this concludes the proof. □
6.2 Long range dependent case

In the long range dependent case the reduction is in fact easier as we only need to consider the lowest order chaos. We denote by \( G_{w,M}(y_s) \) the projections of the first \( M \) polynomials of \( G(y_s) \), thus \( G_{w,M}(y_s) = \sum_{k=0}^{M} c_k I_w(h^{w,k}). \)

**Lemma 6.7** Given \( G(X) = \sum_{k=0}^{\infty} c_k X^k \) such that \( |c_k| \lesssim \frac{1}{k!} \), \( G \) has chaos rank \( w \geq 1 \) with respect to \( y_s \), where \( H^*(w) > \frac{1}{2} \), and \( T \in [0,1] \), then,

\[
\lim_{M \to \infty} \limsup_{\varepsilon \to 0} \| \varepsilon^{H^*(w)} \left( \int_0^T G(y_t) - G_{w,M}(y_t) dt \right) \|_{L^2(\Omega)} = 0.
\]

**Proof.** Using the estimates obtained in Lemma 6.2 we compute, similar to Lemma 6.4:

\[
E \left[ \varepsilon^{H^*(w)} \left( \int_0^T G(y_t) - G_{w,M}(y_t) dt \right) \right]^2 = \varepsilon^{H^*(w)} \int_{[0,T]^2} E \left[ (G(y_t) - G_{w,M}(y_t))(G(y_{t'}) - G_{w,M}(y_{t'})) \right] dt dt'
\]

\[
= \varepsilon^{H^*(w)} \int_{[0,T]^2} \left( \sum_{k=0}^{\infty} \sum_{d=0}^{\infty} c_k I_d(h^{d,k}_w) + \sum_{k=M+1}^{\infty} c_k I_w(h^{w,k}_w) \right) \left( \sum_{k'=0}^{\infty} \sum_{d'=0}^{\infty} c_{k'} I_{d'}(h^{d',k'}_{w'}) \right) \int_0^T C_4(k,k',m,d) + \sum_{k,M+1}^{\infty} c_k c_{k'} C_4(k,k',m,w).
\]

Furthermore, \( C_4(k,k',m,d) = \frac{(m!)^{k+k'}((k+k')m)!}{k!k!(k+m+k')!} \) satisfies,

\[
\sum_{d=0}^{\infty} \sum_{k'=0}^{\infty} c_k c_{k'} C_4(k,k',m,d) + \sum_{k,M+1}^{\infty} c_k c_{k'} C_4(k,k',m,w)
\]

\[
\lesssim \sum_{d=0}^{\infty} \frac{1}{d^2} + \sum_{k,M+1}^{\infty} \frac{(m!)^{k+k'}((k+k')m)!}{k!k!(k+m+k')!} \frac{C_4(k,k',m,w)}{k!k!}
\]

hence, \( \varepsilon^{H^*(w)} \left( \sum_{d=0}^{\infty} \sum_{k'=0}^{\infty} c_k c_{k'} C_4(k,k',m,d) \right) \to 0 \) as \( \varepsilon \to 0 \) and \( \sum_{k,M+1}^{\infty} c_k c_{k'} C_4(k,k',m,w) \to 0 \) as \( M \to \infty \), proving the claim. \( \square \)

**Proposition 6.8** Given a collection of functions \( G_j, j \in \{1, \ldots, N\} \), where \( G_j = \sum_{k=0}^{\infty} c_{j,k} X^k \) such that \( |c_{j,k}| \lesssim k! \) with chaos rank \( w_j \), \( G_j^{w_j,M}(y_s) \) as above and set \( \tilde{G}^{w_j,M}_{j,t}(r) = \varepsilon^{H^*(w_j)} \int_0^t G_j^{w_j,M}(y_s) ds \). Then, for every \( M \in \mathbb{N} \) and finite collection of times \( t_{i,j} \in [0,1] \) the vector \( (\tilde{G}^{w_j,M}_{j,t_{i,j}})_{1 \leq i \leq Q, 1 \leq j \leq N} \), converges jointly to the marginals of a multivariate Hermite process \( (Z_{w,M,T_{i,j}}^{H^*(w_j)}) \). In particular,

\[
\left( \tilde{G}^{1}_{w,M,T}, \ldots, \tilde{G}^{N}_{w,M,T} \right).
\]
converges in the sense of finite dimensional distributions to a multivariate Hermite process

\[
\left( Z_{M,T}^{H^{*}(w_1),w_1}, \ldots, Z_{M,T}^{H^{*}(w_N),w_N} \right),
\]

where each component is defined via the same Wiener process.

**Proof.** As we now deal with finitely many terms \(h_{i,k}\) we can view all buildings blocks \(g_{k,r}^{q}\) such that \(k \leq M\) and \(\delta(k_q,r^q) = w\). By Proposition 5.7 the vector \((\bar{g}_{T,i,j})\) converges jointly to a multivariate Hermite distribution \((Z_{w,M,T,i,j}^{H^{*}})\), where each component is defined via the same Wiener process. As summation is a continuous operation, and

\[
\bar{G}_{w,M,T,i,j}^{j} = \sum_{k=0}^{M} \sum_{r \geq 0} I_d(\bar{g}_{T,i,j}^{k,r}),
\]

also \((G_{M,T,i,j}^{j})\) converges to a multivariate Hermite distribution with the proclaimed covariances. \(\square\)

**Proposition 6.9** Fix \(H \in \left(\frac{1}{2}, 1\right), m \in \mathbb{N}\), a kernel \(x\) satisfying assumptions in Lemma 5.17 and set \(y_t = \int_{-\infty}^{t} x(t-s)dz_s\). Let, for each \(j \in \{1, \ldots, N\}\), a function of the form \(G_j^j(X) = \sum_{k=0}^{\infty} c_{j,k} X^k\) such that \(G_j^j\) has chaos rank \(w_j\) with respect to \(y_t\) and \(|c_{j,k}| \leq \frac{1}{n^k}\) be given. Assume further that for each \(j\), \(H^*(w_j) > \frac{1}{2}\). For \(T \in [0,1]\) set

\[
G_{T}^{j,\varepsilon} = \sqrt{\varepsilon} \int_{0}^{\varepsilon} G^j(y_t)dt.
\]

Then, the vector,

\[
(G_{T}^{1,\varepsilon}, \ldots, G_{T}^{N,\varepsilon}),
\]

converges as \(\varepsilon \to 0\) weakly in \(C^\gamma([0,1], \mathbb{R}^N)\), for \(\gamma \in (0, \min_{j=1, \ldots, N} H^*(w_j))\) to a multivariate Hermite process

\[
(k_1 Z_{T}^{H^*(w_1),w_1}, \ldots, k_N Z_{T}^{H^*(w_N),w_N}),
\]

where \(k_j = \lim_{\varepsilon \to 0} \|G_{T}^{j,\varepsilon}\|_{L^2(\Omega)}\).

**Proof.** Setting \(E_{w,M} = (G_{T}^{1,\varepsilon}, \ldots, G_{T}^{N,\varepsilon}) - (\bar{G}_{w,M,T}^{1,\varepsilon}, \ldots, \bar{G}_{w,M,T}^{N,\varepsilon})\), we obtain

\[
(G_{T}^{1,\varepsilon}, \ldots, G_{T}^{N,\varepsilon}) = (\bar{G}_{w,M,T}^{1,\varepsilon}, \ldots, \bar{G}_{w,M,T}^{N,\varepsilon}) + E_{w,M},
\]

where by Lemma 6.7

\[
\lim_{M \to \infty} \limsup_{\varepsilon \to 0} \|E_{w,M}\|_{L^2} = 0.
\]

Moreover, by Proposition 6.8 \((G_{w,M,T}^{1,\varepsilon}, \ldots, G_{w,M,T}^{N,\varepsilon})\) converges in finite dimensional distributions to

\[
(k_{w_1,M,1} Z_{w,M,T}^{H^*(w_1),w_1}, \ldots, k_{w_N,M,N} Z_{w,M,T}^{H^*(w_N),w_N}).
\]

As \(k_{w_j,M,j} \to k_j\) as \(M \to \infty\) and all our Hermite processes are defined via Wiener integrals over the same Wiener process we may apply Theorem 6.6 to conclude the first part of the proof.

Concerning weak convergence in Hölder spaces, by Proposition 6.7 we have for each \(j\) and \(p > 2\),

\[
\|G_{T}^{j,\varepsilon} - G_{S}^{j,\varepsilon}\|_{L^p(\Omega)} \lesssim |T - S|^{H^*(w_j)}.
\]

Thus, by an application of Kolmogorv’s Theorem each \(G_{T}^{j,\varepsilon}\) is tight in \(C^\gamma([0,1], \mathbb{R}^N)\) for \(\gamma \in (0, \min_{j=1, \ldots, N} H^*(w_j))\). As tightness in \(C^\gamma([0,1], \mathbb{R}^N)\) is equivalent to tightness in each component this concludes the proof. \(\square\)
7 Mixed multivariate case

In this section we put together the Proposition\textsuperscript{6.4, 6.7} and \textsection\textsuperscript{6.10} into Theorem\textsuperscript{6.8}. To do so we rely on an asymptotic independence argument proven in\textsection\textsuperscript{7.12} and \textsection\textsuperscript{7.16}. Hence, we deal with finitely many terms given as iterated Wiener integrals. Furthermore for $j \leq n$ the functional $G_{j,M}^r$ is constructed by objects of the form $\tilde{g}_T^{k,r}$ for which $\delta(k,r) = 1$ such that $H^*(d) < \frac{1}{2}$. In particular they converge to a Wiener process, thus, given two such terms, by the Fourth Moment Theorem\textsuperscript{5.8}

$$\|\tilde{g}_T^{k_1,r_1} \otimes \tilde{g}_S^{k_2,r_2}\|_{L^2(\mathbb{R}^{d_1+d_2-2r_1,\lambda})} \to 0, \quad r = 1, \ldots, \min(k_1-1, k_2-1).$$

Applying Cauchy-Schwarz we obtain for $r = 1, \ldots, \min(k_1-1, k_2-1)$,

$$\|\tilde{g}_T^{k_1,r_1} \otimes \tilde{g}_S^{k_2,r_2}\|_{L^2(\mathbb{R}^{d_1+d_2-2r_1,\lambda})} \leq \|\tilde{g}_T^{k_1,r_1}\|_{L^2(\mathbb{R}^{d_1-r_1,\lambda})} \|\tilde{g}_S^{k_2,r_2}\|_{L^2(\mathbb{R}^{d_2-r_2,\lambda})} \to 0,$$

for all $S, T \in [0, 1]$. Now, an application of Proposition\textsuperscript{9.1} and the Cramer Wold Theorem\textsuperscript{7.3} gives us the convergence in finite dimensional distributions of the vector

$$(\tilde{G}_1^r, \ldots, \tilde{G}_N^r).$$

Therefore, the moments bounds obtained in Lemma\textsuperscript{5.16} and an application of Kolmogorv’s Theorem proof convergence in the proclaimed Hölder spaces.

To show that the Wiener process defining the Hermite processes is independent of limiting one in case $j \leq n$, note that in case the chaos rank of one component equals 1 the limit is a fractional Brownian motion. It was shown in\textsection\textsuperscript{5.6} and an application of Kolmogorv’s Theorem that the filtration between this fBM and the Wiener process defining it are identical, thus, as the fBM is independent of the limit for $j \leq n$ so is the defining Wiener process. The proclaimed covariances were proved in the sections above, hence, this concludes the proof.

8 Application to the homogenization of slow/fast systems

In this section we give an application of Theorem\textsuperscript{8.8} to a homogenization problem using Young/rough path integration theory.

In the following proof require the following theorem from rough path theory for details and the corresponding version for Young differential equations we refer to\textsection\textsuperscript{6.13} and \textsection\textsuperscript{8.10}. We denote the space of rough paths of regularity $\gamma$ by $\mathcal{C}^\gamma$.

**Theorem 8.1** Let $Y_0 \in \mathbb{R}$, $\gamma \in (\frac{1}{3}, \frac{1}{2})$, $f \in \mathcal{C}^\gamma([0, 1], \mathbb{R})$, and $X \in \mathcal{C}^\gamma([0, T], \mathbb{R})$. Then, the differential equation

$$Y_t = Y_0 + \int_0^t f(Y_s)\,dX_s$$

has a unique solution which belongs to $\mathcal{C}^\gamma([0, 1], \mathbb{R})$. Furthermore, the solution map $\Phi_f : \mathbb{R} \times \mathcal{C}^\gamma([0, 1], \mathbb{R}) \to \mathcal{C}^\gamma([0, 1], \mathbb{R})$, where the first component is the initial condition and the second one the driver $X$, is continuous.

For more details concerning rough path theory we refer to\textsection\textsuperscript{8.13} and \textsection\textsuperscript{8.14}.

**Proof of Theorem 8.1**

**Proof.** Set $X_t^\varepsilon = \alpha(\varepsilon) \int_0^\frac{t}{\varepsilon} G(y_s)\,ds$. Thus, we may rewrite Equation\textsuperscript{8.1} as

$$dx_t^\varepsilon = f(x_t^\varepsilon)\,dX_t^\varepsilon, \quad x_0^\varepsilon = x_0.$$

30
Therefore, in case 1 the claim follows from Proposition 9.1 and the continuity of solutions to Young differential equation, the equivalent of Theorem 3.2 in the Young setting, as by assumption $H^*(w) > \frac{1}{2}$.

In case 2 we need to lift $X^\varepsilon$ to a rough path. However, as we restrict ourselves to 1 dimensions the rough path lift $X^\varepsilon_{s,t}$ is just given by $\frac{1}{2}(X^\varepsilon_{s,t})^2$ by symmetry. Although the function $x^2$ is not bounded, due to a truncation argument and our integrability assumptions one can show that $\frac{1}{2}(X^\varepsilon_{s,t})^2 \to \frac{1}{2}(X_{s,t})^2$ in finite dimensional distributions. Again by symmetry the moment bounds from Lemma 5.16 carry over to $X^\varepsilon_{s,t}$ and we obtain convergence of $X^\varepsilon = (X^\varepsilon_{s,t}, X^\varepsilon_{s,t})$ to $(W_{s,t}, W_{s,t})$, where $W$ denotes a standard Wiener process and $W$ its Stratonovich lift, in $\mathcal{C}^\gamma([0, 1], \mathbb{R})$ for $\gamma \in (\frac{1}{3}, \frac{1}{2})$. Therefore, we may conclude the proof with an application of Theorem 9.1.

\[ \Box \]

9 Appendix

9.1 Asymptotic Independence

For the proof of Theorem 9.2 we need the following Proposition which slightly modifies results from \[ \text{NNP16, which can be found in } \text{GL20b}. \]

\[ \text{Proposition 9.1 Let } q_1 \leq q_2, \cdots \leq q_n \leq p_1 \leq p_2, \cdots \leq p_m. \text{ Let } f_i^r \in L^2(\mathbb{R}^n), \ g_i^r \in L^2(\mathbb{R}^n), \ F^\varepsilon = (I_{p_1}(f_1^r), \ldots, I_{p_m}(f_m^r)) \text{ and } G^\varepsilon = (I_{q_1}(g_1^r), \ldots, I_{q_m}(g_m^r)). \text{ Suppose that for every } i, \text{ and any } 1 \leq r \leq q_i: \]

\[ \|f_i^r \otimes g_i^r\| \to 0. \]

Then $F^\varepsilon \to U$ and $G^\varepsilon \to V$ weakly imply that $(F^\varepsilon, G^\varepsilon) \to (U, V)$ jointly, where $U$ and $V$ are taken to be independent random variables.

9.2 Reduction

\[ \text{Theorem 9.2 (Theorem 3.2 [Bil99]) Given random variables } X^\varepsilon_{M} \text{ and } X^\varepsilon \text{ such that } X^\varepsilon_{M} \text{ converges weakly to } X^\varepsilon \text{ as } \varepsilon \to 0, \ X^\varepsilon \to X \text{ as } M \to \infty, \text{ and } \]

\[ \lim_{M \to \infty} \limsup_{\varepsilon \to 0} \|X^\varepsilon_{M} - X^\varepsilon\|_{L^2(\Omega)} = 0, \]

then, $X^\varepsilon \to X$ weakly.

\[ \text{Theorem 9.3 (Cramer-Wold) Given random variables } (X^1, x, \ldots, X^N, x) \text{ and } (X^1, \ldots, X^N). \text{ Then, } (X^1, x, \ldots, X^N) \to (X^1, \ldots, X^N) \text{ in law if and only if for every } (t_1, \ldots, t_N) \in \mathbb{R}^N, \]

\[ \sum_{j=1}^N t_j X^j \sim \sum_{j=1}^N t_j X^j, \]

in law.

References

[BH02] Samir Ben Hariz. Limit theorems for the non-linear functional of stationary Gaussian processes. J. Multivariate Anal., 80(2):191–216, 2002.

[Bil99] Patrick Billingsley. Convergence of probability measures. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons, Inc., New York, second edition, 1999. A Wiley-Interscience Publication.

[BT13a] Shuyang Bai and Murad S. Taqqu. Multivariate limit theorems in the context of long-range dependence. J. Time Series Anal., 34(6):717–743, 2013.
Shuyang Bai and Murad S. Taqqu. Multivariate limit theorems in the context of long-range dependence. *J. Time Series Anal.*, 34(6):717–743, 2013.

Ilya Chevyrev, Peter K. Friz, Alexey Korepanov, Ian Melbourne, and Huilin Zhang. Multiscale systems, homogenization, and rough paths. In *Probability and Analysis in Interacting Physical Systems*, 2019.

Patrick Cheridito, Hideyuki Kawaguchi, and Makoto Maejima. Fractional Ornstein-Uhlenbeck processes. *Electron. J. Probab.*, 8:no. 3, 14, 2003.

R. L. Dobrushin and P. Major. Non-central limit theorems for nonlinear functionals of Gaussian fields. *Z. Wahrsch. Verw. Gebiete*, 50(1):27–52, 1979.

R. L. Dobrushin. Gaussian and their subordinated self-similar random generalized fields. *Ann. Probab.*, 7(1):1–28, 1979.

T. T. Du Tran. Non-central limit theorems for quadratic functionals of Hermite-driven long memory moving average processes. *Stoch. Dyn.*, 18(4):1850028, 18, 2018.

Peter K. Friz and Martin Hairer. *A course on rough paths*. Universitext. Springer, Cham, 2014. With an introduction to regularity structures.

Peter K. Friz and Nicolas B. Victoir. *Multidimensional stochastic processes as rough paths*, volume 120 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010. Theory and applications.

J. Gehringer and Xue-Mei Li. Tagged particles in fractional noise field. In Arxiv, where the title is “Homogenization in fractional random field, 2019.

J. Gehringer and Xue-Mei Li. Diffusive and rough homogenisation in fractional noise field. This is an improved version of part II of [arXiv:1911.12600](https://arxiv.org/abs/1911.12600) 2020.

J. Gehringer and Xue-Mei Li. Functional limit theorem for fractional OU. This is an improved version of [arXiv:1911.12600](https://arxiv.org/abs/1911.12600) 2020.

Martin Hairer. Ergodicity of stochastic differential equations driven by fractional Brownian motion. *Ann. Probab.*, 33(2):703–758, 2005.

David Kelly and Ian Melbourne. Deterministic homogenization for fast-slow systems with chaotic noise. *J. Funct. Anal.*, 272(10):4063–4102, 2017.

Terry J. Lyons, Michael Caruana, and Thierry Lévy. *Differential equations driven by rough paths*, volume 1908 of *Lecture Notes in Mathematics*. Springer, Berlin, 2007. Lectures from the 34th Summer School on Probability Theory held in Saint-Flour, July 6–24, 2004, With an introduction concerning the Summer School by Jean Picard.

Terry Lyons. Differential equations driven by rough signals. I. An extension of an inequality of L. C. Young. *Math. Res. Lett.*, 1(4):451–464, 1994.

Péter Major. *Multiple Wiener-Itô integrals*, volume 849 of *Lecture Notes in Mathematics*. Springer, Cham, second edition, 2014. With applications to limit theorems.

Makoto Maejima and Ciprian A. Tudor. Wiener integrals with respect to the Hermite process and a non-central limit theorem. *Stoch. Anal. Appl.*, 25(5):1043–1056, 2007.

Ivan Nourdin, David Nualart, and Giovanni Peccati. Strong asymptotic independence on Wiener chaos. *Proc. Amer. Math. Soc.*, 144(2):875–886, 2016.
[NNZ16] Ivan Nourdin, David Nualart, and Rola Zintout. Multivariate central limit theorems for averages of fractional Volterra processes and applications to parameter estimation. *Stat. Inference Stoch. Process.*, 19(2):219–234, 2016.

[NR14] Ivan Nourdin and J. Rosinski. Asymptotic independence of multiple wiener-ito integrals and the resulting limit laws. *The Annals of Probability*, 42(2):497–526, 2014.

[Nua06] David Nualart. *The Malliavin calculus and related topics*. Probability and its Applications (New York). Springer-Verlag, Berlin, second edition, 2006.

[PT00] Vladas Pipiras and Murad S. Taqqu. Integration questions related to fractional Brownian motion. *Probab. Theory Related Fields*, 118(2):251–291, 2000.

[SRMF19] Stoyan V. Stoyanov, Svetlozar T. Rachev, Stefan Mitnik, and Frank J. Fabozzi. Pricing derivatives in Hermite markets. *Int. J. Theor. Appl. Finance*, 22(6):1950031, 27, 2019.

[Taq75] Murad S. Taqqu. Weak convergence to fractional Brownian motion and to the Rosenblatt process. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 31:287–302, 1975.

[Taq79] Murad S. Taqqu. Convergence of integrated processes of arbitrary Hermite rank. *Z. Wahrsch. Verw. Gebiete*, 50(1):53–83, 1979.