Splitting of an Extremal Reissner-Nordström Throat via Quantum Tunneling

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Abstract

The interior near the horizon of an extremal Reissner-Nordström black hole is taken as an initial state for quantum mechanical tunneling. An instanton is presented that connects this state with a final state describing the presence of several horizons. This is interpreted as a WKB description of fluctuations due to the throat splitting into several components.

“Fluctuat nec mergitur”
I. INTRODUCTION

The significance of geometries of non-Euclidean topology in General Relativity was first spelled out in Wheeler’s “geometrodynamics” [1]. Localized topological complications, such as Wheeler wormholes, are allowed even at the classical level. However, the Einstein dynamics of classical gravity coupled to classical matter fields provides no compelling reason that such non-Euclidean topologies should occur in nature, because the topology of spacelike Cauchy surfaces is conserved in the classical Einstein time development [2]. Wheeler has argued that in quantum geometry, wormholes can be created and destroyed, for example as part of the processes that constitute the vacuum fluctuations. According to this view the creation or destruction of a wormhole can be regarded as a quantum tunneling through a classically forbidden region.

Tunneling processes can be treated as propagation in imaginary (“Euclidean”) time [3], and this description is appropriate for quantum fluctuations of topology not least because the kinematical description of topological changes is simple in Riemannian manifolds. For example, for any pair of three-dimensional topologies one can find a Riemannian 4-manifold that interpolates between them. This is true for Lorentz manifolds only if one accepts closed timelike curves [4]. Another, practical reason for using the Euclidean description as part of a stationary phase (WKB) approximation is that we can have at least some confidence in it, lacking as we do today a complete theory of quantum gravity.

To implement this approximation one starts by looking for Riemannian solutions of the gravitational (and matter) field equations (“instantons”). An instanton can be considered as a tunneling history in the same way that a Lorentzian solution of the field equations represents an actual (classical) history of the fields. To make the development of this history more explicit one usually “slices” a space-time by spacelike surfaces, and one can similarly slice a 4-dimensional instanton by non-intersecting three-dimensional surfaces (some of which may be singular). The interpretation of the instanton as an approximate description of a wave functional in the 3-metric representation of quantum geometry is also analogous to that of classical (Lorentzian) solutions: In situations described by the latter, the wave functional $\Psi[3\mathcal{G}]$ is sharply peaked about Riemannian 3-geometries $\mathcal{G}$ that all fit (as spacelike surfaces) into one Lorentzian 4-geometry [5]; in the instanton case, $\Psi[\mathcal{G}]$ is sharply peaked about 3-geometries that all fit into one Riemannian 4-geometry [6]. The geometries that are connected by such a tunneling history are contained in the instanton 4-geometry as follows: the asymptotic region corresponds to the initial metastable state, and a maximally geodesic 3-surface, which has zero geometrical momentum as measured by the extrinsic curvature tensor, corresponds to the final state — the moment at which the decay product emerges. (Because the momenta vanish, the latter state is a suitable initial state for a Riemannian as well as for a Lorentzian development, and corresponds to the classical turning point or “bounce” in potential motion.) This interpretation has an

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1 In this paper we reserve the word Euclidean (without quotes) for $\mathbb{R}^n$ with the flat geometry whose $n = 2$ or $3$ version Euclid would recognize as the subject of his “Στοιχεία”. More general manifolds with a positive definite metric will be called Riemannian, as opposed to spacetimes with an indefinite metric, which will be called Lorentzian.
obvious generalization to the case when matter fields are also present.

A number of instanton solutions have been interpreted in this way, for example the production of Wheeler wormholes by a magnetic field [7]. However, to describe fluctuations of a wormhole that is already present one would expect an instanton related in some limit to a classical black hole background, for example the Schwarzschild-Kruskal solution. The very presence of a horizon, which characterizes black holes, usually prevents the existence of a suitable instanton, because in the analytic continuation to Euclidean time the manifold is regular at the horizon only if this time is a periodic coordinate [8]. Thus the manifold cannot have the desired asymptotic behavior in imaginary time. Equivalently, the problem is that a classical wormhole is not a stationary state, since its throat collapses, whereas the simple instanton picture of a classically forbidden decay is applicable only if the initial and/or final states are stationary. (The periodic instantons connected with black holes do have a physical interpretation, but it has to do with thermal phenomena such as Hawking radiation.)

There is however one type of black hole that does not offer these problems, namely the “extremal Reissner-Nordström” black hole and its generalizations. Here the usual two Killing horizons of charged black holes coincide (degeneracy in the sense of Carter [9]). Two consequences are that the horizon is located at infinite distance on the spacelike surface \( t = \text{const} \), and that, in the analytic continuation, the Euclidean time does not have a finite period. Thus the existence of asymptotic regions of differing topology becomes possible.\(^2\)

The analogous picture in potential motion is given by a potential that has two degenerate minima at \( x_1, x_2 \). Here the motion in Euclidean time starts with an asymptotically slow roll-off from \( x_1 \), and ends with an asymptotically slow ascent to \( x_2 \) — there is no bounce back to \( x_1 \). In this case the instanton does not signal an instability where after a sufficiently long time the particle is found in the initial state with vanishingly small probability. Instead, the particle is found with some probability in either of the degenerate minima. The action of the instanton is related in the WKB approximation to the energy splitting between the two classically degenerate ground states, and therefore to the frequency with which the quantum system fluctuates between \( x_1 \) and \( x_2 \) [10].

The present paper presents an instanton with the corresponding properties in a space-time theory. It can be considered to have two asymptotic regions that can be foliated by 3-surfaces having topologies that differ between the two regions. Furthermore, in these asymptotic regions the geometry of the 3-surfaces agrees with the asymptotic interior of one respectively two (or several) extremal Reissner-Nordström wormholes.\(^3\) The instanton may therefore be considered a description of the topological fluctuations near the horizon of such extremal Reissner-Nordström wormholes.

Section II recalls the asymptotic form of the metric and fields near the extremal

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2 In the thermal interpretation the infinite period means that the temperature vanishes. Thus a scenario is possible in which all black holes that have magnetic charge and lose mass by Hawking radiation become magnetically charged extremal black holes. The fluctuations discussed here would then take over near this typical endpoint of the thermal evolution.

3 We use this term rather than “black hole” because we contemplate an interior free of any collapsing matter.
Reissner-Nordstrøm horizon (usually called the Bertotti-Robinson universe, and abbreviated here as BR) and its Euclidean version. Section III gives the instanton solution that connects one BR universe with several, and discusses the quantities that are conserved during the transition. In Section IV it is shown that the action of the instanton is finite, and remarks are made about the probable form of an instanton that would connect different numbers of wormholes not only in the extreme interior, but everywhere. Section V summarizes the conclusions.

II. LIMITING FORM OF EXTREMAL REISSNER-NORDSTRÖM GEOMETRY

The well-known Reissner-Nordstrøm metric for a magnetically charged black hole, in the case \( Q^2 = M^2 \), takes the form, in the usual static coordinates where \( r \geq M \) [9]

\[
\begin{align*}
    ds^2_{RN} &= (1 - M/r)^{-2} dr^2 + r^2 d\Omega^2 - (1 - M/r)^2 dt'^2 \\
    F_{RN} &= M \, d(\cos \theta) \wedge d\phi.
\end{align*}
\]

(1a)

(1b)

The geometry on a totally geodesic spacelike surface \( t' = \text{const.} \) is complete — on such a surface the horizon \( r = M \) is at infinite spatial distance from any point with \( r > M \). Thus this surface does not have the usual wormhole shape with two asymptotically flat regions; instead it is asymptotically flat at \( r \to \infty \), but becomes an infinitely long cylinder with a constant magnetic field along the cylinder’s generators in the limit \( r \to M \). In this limit it coincides with the BR solution [11] that describes this exact cylinder (infinite in both \( +z \) and \( -z \) directions),

\[
\begin{align*}
    ds^2_{BR} &= M^2 (dz^2 + d\Omega^2 - \cosh^2 z \, dr'^2) \\
    F_{BR} &= M \, d(\cos \theta) \wedge d\phi.
\end{align*}
\]

(2a)

(2b)

(Whereas metric (2a) is geodesically complete, neither (1a) nor the Lorentzian counterpart of (5) below are geodesically complete, because timelike or null geodesics can reach the horizon at \( r = M \) resp. \( \rho = 0 \) in a finite affine parameter interval\(^4\); but the Riemannian versions, (3a) — a special case of (8a) — and (5) will turn out to be geodesically complete.) The single parameter \( M \) in (2) no longer has the interpretation as a mass, but it measures both the (transverse) size of the BR universe and the (constant) field strength. The total magnetic flux through the universe, \( \oint F \), is \( 4\pi M \). We will therefore call \( M \) the flux parameter.

Since these metrics are static they can easily be continued to “Euclidean” time by defining \( t' = it \) resp. \( \tau' = i\tau \), which merely changes the sign of the metrics’ last terms:

\[
\begin{align*}
    ds^2_{RN} &= (1 - M/r)^{-2} dr^2 + r^2 d\Omega^2 + (1 - M/r)^2 dt'^2 \\
    F_{RN} &= M \, d(\cos \theta) \wedge d\phi.
\end{align*}
\]

(3a)

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\(^4\) for a discussion of the analytic extension and of the relation between these metrics and (2a), which is complete, see Carter [9]
\[ ds^2_{BR} = M^2(dz^2 + d\Omega^2 + \cosh^2 z \, d\tau^2) \]  

\[ F_{RN} = F_{BR} = M \, d(\cos \theta) \wedge d\phi \]  

In metric (3b) we now introduce new coordinates [12]

\[ \rho = Me^{-\tau} \cosh z, \quad t = Me^{\tau} \tanh z \]  

and find

\[ ds^2_{BR} = (M/\rho)^2 d\rho^2 + M^2 d\Omega^2 + (\rho/M)^2 dt^2. \]  

This is identical to (3a) in the limit \( \rho = r - M \to 0 \).

Equation (3b) shows that the Riemannian BR geometry is the direct product of two spaces of constant curvature, the 2-sphere \( S^2(\theta, \phi) \) and the negative constant curvature hyperbolic space \( H^2(z, \tau) \). The coordinates are analogous in the sense that \( \theta - \frac{\pi}{2} \to iz, \phi \to i\tau \) transform the two-dimensional metrics into each other. The \( \rho, t \) coordinates, on the other hand, are a type of polar coordinates, with \( t = \text{const.} \) describing radial geodesics originating from a point at infinity.

III. AN EINSTEIN-MAXWELL INSTANTON

To find instantons that describe the change of one wormhole, or one BR universe, into several would a priori seem difficult because the instantons would be expected to have none of the continuous symmetries (except, in the case of the BR universe, the \( z \)-translation) that usually make solutions of the Einstein or Einstein-Maxwell equations possible. Remarkably, solutions are known that have no spatial symmetry, namely the conformastatic class [13]. The usual physical interpretation of this class is a set of arbitrarily placed extremal black holes that remains static because the gravitational attraction is balanced by the electric repulsion between the charges (which all have equal sign). But by appropriate choice of integration constants one can eliminate the asymptotically flat region and obtain a solution that is static and inhomogeneous in space, and that can be thought of as a BR-type universe containing a number of extremally charged black holes. In its Riemannian analytic extension one can identify the static direction with the BR \( z \)-direction, and consider the solution as inhomogeneous in imaginary time. This will then give use the desired instanton.

The Lorentzian conformastatic metric and Maxwell field have the form

\[ ds^2_{CS} = V^2 d\sigma^2 - V^{-2} dt^2, \quad d\sigma^2 = dX^2 + dY^2 + dZ^2 \]  

\[ *F = \xi \wedge dV \]  

where \( V \) satisfies

\[ \nabla^2 V = 0. \]
Here the Laplacian $\nabla^2$ is evaluated in the flat metric, $d\sigma^2$, and $\xi$ is the Killing form corresponding to time translation symmetry. To obtain a corresponding Riemannian solution we again replace $t'$ by $it$ and write the solution suitable for our purpose as

$$ds_{CS}^2 = V^2 d\sigma^2 + V^{-2} dt^2, \quad \ast F = \xi \wedge dV$$  \hspace{1cm} (8a)

$$V = \sum M_i/\rho_i, \quad \rho_i = \sqrt{(X - X_i)^2 + (Y - Y_i)^2 + (Z - Z_i)^2}$$  \hspace{1cm} (8b)

with $(X_i, Y_i, Z_i), \ i = 1\ldots n$ denoting $n$ different points in $\mathbb{R}^3$.

Whenever $(X, Y, Z)$ approaches a particular, say the $j$th, of the $n$ points, the $j$th term in the sum dominates, so that $V$ approaches $M_j/\rho_j$. Thus the metric takes on the BR form of Eq. (5) in the limit $\rho_j \to 0$. Similarly, in the limit $(X, Y, Z) \to \infty$ we have $V \to M_\infty/\rho$, where $\rho^2 = X^2 + Y^2 + Z^2$ and $M_\infty = \sum M_i$. In this case the metric also takes on the BR form (5), but in the limit $\rho \to \infty$. Thus the geometry and field described by Eq. (8a) interpolate between the $n+1$ BR universes corresponding to these asymptotic regions. The flux parameter of each of the $n$ universes in the limit $\rho_j \to 0$ is $M_j$, and that of the $n+1$st universe, in the limit $\rho \to \infty$, is $\sum M_i$, as expected from flux conservation.

The Maxwell tensor $F$ of this solution is not given directly by Eq. (8a), but by way of its dual, $\ast F$. In the Lorentzian spacetime, replacing $F$ by $\ast F$ (i.e., interchanging $E$ and $B$) yields another Maxwell-Einstein solution with the same geometry. But only the solution with a magnetic field produces a real instanton, because the electric field becomes imaginary under the replacement of $t'$ by $it$. (This is clear from the transformation of the components $F^{0i}$, or from the fact that the Killing form $\xi$ becomes imaginary under this replacement). The magnetic field, on the other hand, remains real because the $\epsilon$-tensor, which defines the dual, also picks up some imaginary components in the analytic continuation. Alternatively, and more simply, we can regard $\xi$ as the real Killing vector of the Riemannian manifold and $\ast$ as its normal dual operation. Any electric fields “induced” in the Riemannian development will then be real\(^5\), and satisfy the Riemannian Maxwell equations (whose main difference from the Lorentzian Maxwell equations is that there is not the negative sign usually associated with Lenz’ law).

As usual with magnetic charges, the field $F$ of Eq. (8a) does not come from a globally defined vector potential, but its dual can be so derived (the usual restrictions on existence of potentials in wormhole spacetimes [14] do not apply to these extremal configurations),

$$\ast F = dC, \quad C = V \xi.$$  \hspace{1cm} (9)

Finally we note that our instanton is geodesically complete with finite Maxwell field everywhere, because the coordinates of Eqs. (8) are good everywhere except in the limits $\rho_j \to 0$ and $\rho \to \infty$, where the geometry is asymptotically BR, that is $S^2 \times H^2$, and the Maxwell field is finite.

\(^5\) The situation is analogous to particle motion in real vs. imaginary time. The particle has a real momentum in the classically allowed region, and a real momentum in the classically forbidden region after the potential has been turned “upside down”; but the two motions can connect only at the turning point, where the momentum vanishes.
IV. THE ACTION

The instanton of the last section has infinite extent in the $t$-direction, and is homogeneous in that direction. (Locally this corresponds to an axis of the asymptotic BR universes, that is, a $z$-direction in the metrics of Section II.) The instanton can have a finite action only if the contribution from any range of $t$ is independent of that range — that is, the contribution must have the form of a suitable surface integral. It should also depend only on the number and values of the flux parameters $M_j$, and not on the location of the points $(X_j, Y_j, Z_j)$. This is so because our instanton is an analytic continuation of an Einstein-Maxwell spacetime that is static in the region considered ("static" here includes vanishing of the electromagnetic momentum variable, i.e., of the electric field). The action of this spacetime therefore consists of a potential energy term only. But for extremal black holes, the gravitational attraction is balanced by electromagnetic repulsion. Therefore this potential, and hence the action, should not change with the location of the holes.

The "Euclidean" action, $\hat{I}$, for the Einstein-Maxwell fields is given by [15]

$$
-16\pi \hat{I} = \int (R - F_{\mu\nu}F^{\mu\nu})\sqrt{g} \, d^4x + 2 \oint K\sqrt{3g} \, d^3x + c. \tag{10}
$$

Here the surface integral over $K$ (the trace of the surface’s second fundamental form) is needed to convert the Einstein-Hilbert action, $\int R\sqrt{g} \, d^4x$, into the first-order form appropriate for quantum gravity. The term $c$ is included to annul the action of the initial state. (If the initial state is flat space one usually puts $c = -2 \oint K^0\sqrt{3g} \, d^3x$, where $K^0$ is the value of $K$ when the boundary is embedded in flat space.) The analogous normalization in the case of tunneling through a potential fixes the energy of the classical initial state at zero, because only then is the exponential fall-off of the WKB wavefunction given correctly by $\exp(-\hat{I}/\hbar)$.

For an Einstein-Maxwell solution the Einstein-Hilbert action $\int R\sqrt{g}d^4x$ vanishes, since the Maxwell stress-energy is traceless. Similarly, if $*F$ is derived from a potential, Eq. (9), then the Maxwell part of the action can be converted into a volume integral that vanishes when evaluated on a solution, plus a surface integral:

$$
\frac{1}{2} \int F_{\mu\nu}F^{\mu\nu}\sqrt{g} \, d^4x = \int *F \wedge F = \int dC \wedge F = \oint C \wedge F - \int C \wedge dF
$$

Since $dF = 0$ by Maxwell’s equations, the action becomes the pure boundary integral

$$
-8\pi \hat{I} = \oint (K + C_\mu *F^{\mu\nu}n_\nu)\sqrt{3g} \, d^3x + c. \tag{11}
$$

For the boundary three-surfaces we choose the cylinders with "mantles" $\rho = P$, $t \in (-T, T)$ and $\rho_j = P_j$, $t \in (-T, T)$, and "top" and "bottom" surfaces $\rho \leq P$, $\rho_j \geq P_j$, $t = T$ resp. $-T$, all in the limit $P \to \infty$, $P_j \to 0$, $T \to \infty$. The top and bottom are totally geodesic and normal to $C$ in the gauge of Eq. (9). Hence both terms in the integrand of Eq. (11) vanish there.
On the mantle surfaces, $\rho = P$ resp. $\rho_j = P_j$, we find, in the limit $P \to \infty$ resp. $P_j \to 0$,

$$K = -1/M_j + O(P_j^2) \quad \text{resp.} \quad 1/M_\infty + O(P^{-2}) \quad (12a)$$

and

$$C_\mu * F^{\mu
u} n_\nu = 1/M_j + O(P_j^2) \quad \text{resp.} \quad -1/M_\infty + O(P^{-2}). \quad (12b)$$

Since $K \neq 0$, neither of these limits represents totally geodesic surfaces as would be appropriate for a bounce solution. But this instanton does not involve a bounce: both “initial” and “final” states are reached only asymptotically in “imaginary time.” Hence it is appropriate to evaluate the action in both regions in the way one evaluates it in the one asymptotically Euclidean region of bounce-type instantons. That is, we take the limits and choose $c$ in such a way that we obtain zero when the action is evaluated for the initial, single BR universe of Eq. (5), for which $M = M_\infty = M_1$ and all other $M_j = 0$. We can achieve this by taking the limits $P \to \infty$, $P_j \to 0$ before the limit $T \to \infty$, and setting $c = 0$. Because the terms of Eqs. (12a) and (12b) then cancel for any set of $M_j$’s, the contribution to the action from the mantle surfaces always vanishes as well.

There is, however, a nonzero contribution to the action from the cylinder’s two-dimensional “edges” at $\rho = P$, $\rho_j = P_j$, $t = \pm T$. There the curvature $K$ has a $\delta$-function behavior, because the boundary surface makes a sharp $90^\circ$ turn. The contribution to the integral of Eq (11) from the $j^{th}$ pair of these edges is $2(\frac{\pi}{2}) A_j$, where $A_j$ is the area of the edge $P_j \to 0$, that is $4\pi M_j^2$ as given by Eq. (8). From the sum of these contributions we must again subtract the value of a term of this type for the single BR universe. Thus the final result for the “Euclidean” action is

$$\hat{I} = \frac{1}{8} (A_\infty - \sum A_j) = \frac{\pi}{2} [(\sum M_j)^2 - \sum M_j^2]. \quad (13)$$

(Of course this does not take into account any contribution to the action due to possible modifications of (10) by topological invariants.) This finite value of the action makes the solution (8) a proper instanton.

It is curious that $\hat{I}$ is half the difference in the entropies of the black holes whose asymptotic interiors are approximated by these BR universes, so that the square of the WKB wavefunction agrees with the probability $\exp(\Delta S)$ associated by statistical mechanics with a fluctuation in which the entropy $S$ deviates by $\Delta S$ from its equilibrium value.\footnote{For pointing this out I thank T. Jacobson and F. Dowker, who also suggested the interpretation given to Eq. (13) in the discussion, below.}

The instanton and the corresponding WKB wavefunction describe the splitting of one BR universe into two or several. An analogous instanton would be expected to describe the splitting of an extreme Reissner-Nordström wormhole into two or several, agreeing with the BR instanton in the extreme, cylindrical interiors. The action of this instanton can have a finite value, since the above shows that there is only a finite contribution from the infinitely long cylinder. (In fact, it is likely that the action in the wormhole case is similar to (13): although the corresponding instanton is not known, to evaluate (11) one only needs the geometry near its boundary surfaces, which one can estimate.)
V. DISCUSSION

The numerical value of the instanton action allows one to calculate such quantities as the probability of wormhole (or BR universe) splitting, or the relative probabilities of different numbers of wormholes (or BR universes) being present in the fluctuating state characterized by a given mass (or flux) parameter. Eq. (13) suggests that the process of one extremal wormhole breaking into many is suppressed by \( \exp(-2\hat{I}/\hbar) \), that is, according to how badly the second law of black hole thermodynamics is broken. Thus the present calculation supports the idea that the topology-changing process is important on the Planck scale, for example when \( M_\infty \) is of the order of a Planck mass.

What other information can we read out of the instanton? To understand how a Lorentzian spacetime describes the development of geometry and fields from an initial to a final state it is appropriate to “slice up” the spacetime by non-intersecting spacelike surfaces. The same procedure is appropriate for an instanton, except that the spacelike requirement is no restriction, and that the surfaces typically cannot all be regular if topology change is involved. The resulting “imaginary time history” shows how the fluctuation comes about in the sense that it provides a sequence of geometries and fields that are most likely to be present during the fluctuation.

For our instanton, a particularly simple slicing is by surfaces \( V = \text{const.} \). If we suppress the trivial axial \((t-)\) direction, the corresponding 2-surfaces are easily visualized as embedded in three-dimensional space: except for the conformal factor \( V^2 \) they are simply the equipotentials of \( n \) charges \( M_j \). Thus, to visualize, say, the fluctuation history of one parent BR universe into two equal daughters, think of the equipotentials of two equal charges. At large distances we have spheres, corresponding to the parent universe. By applying the correction prescribed by the conformal factor \( V^2 \) in Eqs. (8) we find that their true radius is \( 2M_1 \). As we move inward, the spheres distort and pinch off, forming two surfaces that in the limit \( \rho_1 \to 0 \) or \( \rho_2 \to 0 \) again become spherical — the daughter universes, each of true radius \( M_1 \). This particular slicing is distinguished by the vanishing of the electric field everywhere. The magnetic field, being in the \( t \)-direction perpendicular to the 2-surfaces, could be visualized as a density of conserved points that avoid the pinch-off region, where \( dV = 0 \) in Eq. (6b).

The fields and geometries of this slicing may also be useful for suggesting minisuperspaces in which such topology-changing processes can be further investigated.

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