Attractor Networks on Complex Flag Manifolds

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Abstract
Robbin and Salamon showed in [10] that attractor-repellor networks and Lyapunov maps are equivalent concepts and illustrate this with the example of linear flows on projective spaces. In these examples the fixed points are linearly ordered with respect to the Smale order which makes the attractor-repellor network overly simple. In this paper we provide a class of examples in which the attractor-repellor network and its lattice structure can be explicitly determined even though the Smale order is not total. They are associated with special flows on complex flag manifolds. In the process we show that the Smale order on the set of fixed points can be identified with the well-known Bruhat order. This could also be derived from results of Kazhdan and Lusztig, but we give a new proof using the $\lambda$-Lemma of Palis. For the convenience of the reader we also introduce the flag manifolds via elementary dynamical systems using only a minimum of Lie theory.

1 Introduction

Given a dynamical system on a space $\mathcal{M}$, Smale considered the relation $x \leftarrow_{Sm} y$ defined by $W^-(y) \cap W^+(x) \neq \emptyset$, where $W^\pm(m)$ are the stable and unstable manifolds of $m \in \mathcal{M}$. This relation in general is not transitive, so one takes the transitive closure which is then called the Smale order and denoted by $\preceq_{Sm}$.

Let now $\mathcal{M}$ be a complex flag manifold. Viewing $\mathcal{M}$ as an adjoint orbit it is possible to construct gradient flows on $\mathcal{M}$ such that the Bruhat cells in $\mathcal{M}$ coincide with the unstable manifolds of this flow as Atiyah remarks in [1]. The set of Bruhat cells carries a natural order defined by $N_1 \preceq_{Br} N_2$ if $N_1 \subseteq \overline{N_2}$, where $N_2$ is the closure of $N_2$ in $\mathcal{M}$. This order is called the Bruhat order and it actually coincides with the combinatorial Bruhat order on the coset space of the Weyl group associated with $\mathcal{M}$ (see [2, Lemma 6.2.1]). This observation suggests a close relation between the Smale order and the Bruhat order. In fact, it turns out that the relation $\leftarrow_{Sm}$ in this case is a partial order which coincides with $\preceq_{Br}$ when the Weyl group coset space is identified with the set of...
fixed points. This allows us to describe the algebraic structure of the attractor network associated in [10] with the dynamical system on $\mathcal{M}$ in terms of the well studied order structures on Weyl groups.

The proofs we present are based on linear flows on projective space. The elementary techniques involved can also be used to provide quick proofs for most of the basic results on flag manifolds required in our context. We included them for the convenience of the reader not specializing in Lie theory.

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2 The Smale Order

Let $\mathcal{M}$ be a topological space and $\Phi: \mathbb{R} \times \mathcal{M} \to \mathcal{M}$ a continuous group action of $\mathbb{R}$ on $\mathcal{M}$. Then we call $\Phi$ a dynamical system with continuous time. We assume that $\mathcal{M}$ is compact and admits a metric $d$. For any invariant subset $A \subseteq \mathcal{M}$ the stable and unstable manifold of $A$ in $\mathcal{M}$ are defined by

$$W^\pm(A) := \{ x \in \mathcal{M} | \lim_{t \to \pm \infty} d(\Phi(t,x), A) = 0 \}.$$ 

Note that $W^\pm(A)$ are both invariant under the flow. One defines a relation $\leftarrow_{\text{Sm}}$ on the set of fixed points $\mathcal{M}_{\text{fix}}$ by

$$x \leftarrow_{\text{Sm}} y \iff W^-(y) \cap W^+(x) \neq \emptyset.$$ 

Then the transitive closure $\preceq_{\text{Sm}}$ of $\leftarrow_{\text{Sm}}$ is called the Smale order. Note that in general the relation $\preceq_{\text{Sm}}$ is not antisymmetric, i.e. it is not a partial order. In order to be able to derive a reasonable partial order from $\preceq_{\text{Sm}}$ one has to make some additional assumptions on the dynamical system.

Definition 2.1 We call a dynamical system $(\mathcal{M}, \Phi)$ admissible if $\mathcal{M}$ is compact, every flow line has a sink and a source, and, in addition, $\mathcal{M}_{\text{fix}}$ has only finitely many open closed subsets, i.e. $\mathcal{M}$ has only finitely many compact isolated subsets of fixed points.

Now suppose that $(\mathcal{M}, \Phi)$ is admissible. We denote the (finite) set of connected open closed subsets of $\mathcal{M}_{\text{fix}}$ by $\mathcal{M}_{\text{fix}}$. Then we can define a Smale order $\preceq_{\text{Sm}}$ also on $\mathcal{M}_{\text{fix}}$ as the transitive closure of the relation

$$p \preceq_{\text{Sm}} q \iff W^-(q) \cap W^+(p) \neq \emptyset$$

on $\mathcal{M}_{\text{fix}}$. 

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Remark 2.2 The set of subsets $F \subseteq \mathcal{M}_{\text{fix}}$ is in bijective correspondence with the set of open closed subsets $\mathcal{U} \subseteq \mathcal{M}_{\text{fix}}$ via

$$F \mapsto \mathcal{U}_F := \bigcup_{p \in F} p, \quad \mathcal{U} \mapsto F_\mathcal{U} := \{p \in \mathcal{M}_{\text{fix}} \mid p \subseteq \mathcal{U}\}.$$ 

A set $F \subseteq \mathcal{M}_{\text{fix}}$ is an upper set with respect to the Smale order if and only if $\mathcal{W}^+(\mathcal{U}_F) \cap \mathcal{M}_{\text{fix}} = \mathcal{U}_F$ which means that $\mathcal{U}_F$ is an upper set in $\mathcal{M}_{\text{fix}}$. Similarly, $F \subseteq \mathcal{M}_{\text{fix}}$ is a lower set with respect to the Smale order if and only if $\mathcal{U}_F$ is a lower set in $\mathcal{M}_{\text{fix}}$.

3 Linear Flows on Projective Space

Let $V$ be finite dimensional complex vector space with an inner product $\langle \cdot | \cdot \rangle$ and $\varphi \in \text{End}(V)$ a selfadjoint linear map. We denote the (real) spectrum of $\varphi$ by $\text{spec}(\varphi)$ and the eigenspace of $\varphi$ for $\lambda \in \text{spec}(\varphi)$ by $V_\lambda$. We consider the flow induced by $e^{\varphi}$ on the projective space $\mathbb{P}(V)$ the projective flow generated by $\varphi$ and denote it by $\Phi$.

The flow $\Phi$ of $\varphi$ on $\mathbb{P}(V)$ can be interpreted as a gradient flow. In fact, denote the natural projection $V \setminus \{0\} \to \mathbb{P}(V)$ by $\pi$ and equip $\mathbb{P}(V)$ with the Fubini-Study metric $\langle \cdot | \cdot \rangle_{FS}$, which turns $\mathbb{P}(V)$ into a Kähler manifold and which is characterised by the equation

$$\langle d\pi_v(u) | d\pi_v(w) \rangle_{FS} = \frac{\langle u | w \rangle}{\langle v | v \rangle}$$

at the point $[v]$, where $u$ and $w$ are orthogonal to $v$. Then we have

**Proposition 3.1** $[v] \mapsto -d\pi_v(\varphi v)$ is the gradient vector field of the height function

$$f([v]) = -\frac{1}{2} \frac{(\varphi v | v)}{\langle v | v \rangle},$$

where $(v | w) = \Re(v | w)$ and the Riemannian metric on $\mathbb{P}(V)$ is the real part of the Fubini-Study metric, i.e.,

$$\nabla f([v]) = -d\pi_v(\varphi v).$$

**Proof.** An elementary calculation shows that

$$d(f \circ \pi)_v(w) = -\frac{1}{(v | v)} \left( \varphi v - \frac{(\varphi v | v)}{(v | v)} v \right) | w).$$

On the other hand we have

$$d(f \circ \pi)_v(w) = \Re(\nabla f([v]) | d\pi_v(w))_{FS}.$$
Since $v$ is in the kernel of $d\pi_v$ the claim follows.

Note that Proposition 3.1 in particular shows that $f$ is strictly decreasing along non-constant flow lines. The fixed points of $e^{t\varphi}$ on $\mathbb{P}(V)$ are

$$F := \mathbb{P}(V)_{\text{fix}} = \{[v] \in \mathbb{P}(V) \mid (\exists \nu \in \text{spec}(\varphi)) v \in V_{\nu}\},$$

where $[v]$ is the line spanned by $v$. There is a natural embedding of $\mathbb{P}(V_\nu) \to \mathbb{P}(V)$ for each $\nu \in \text{spec}(\varphi)$. The height function takes the value $f([v]) = -\frac{1}{2}\nu$ on $\mathbb{P}(V_\nu)$. Identifying $\mathbb{P}(V_\nu)$ with its image under this embedding we obtain

$$F = \bigcup_{\nu \in \text{spec}(\varphi)} \mathbb{P}(V_\nu). \quad (1)$$

Given $x = [v] \in \mathbb{P}(V)$ we define

$$w_+(x) := \min\{r \in \mathbb{R} \mid v \in \bigoplus_{\nu \leq r} V_{\nu}\}, \quad w_-(x) := \max\{r \in \mathbb{R} \mid v \in \bigoplus_{r \leq \nu} V_{\nu}\}.$$

In particular, we have $w_\pm(x) = \nu$ for $x \in \mathbb{P}(V_\nu)$. In this case we simply write $w(x)$ for $w_+(x)$ and call it the $\varphi$-weight of $x$.

**Proposition 3.2** Each $e^{\mathbb{R}\varphi}$-orbit has limits for $t \to \pm\infty$. More precisely, for $x = [v]$ with $v = \sum_{\nu \in \text{spec}(\varphi)} v_{\nu}$ and $v_{\nu} \in V_{\nu}$ we have $\lim_{t \to \pm\infty} e^{\pm t\varphi} \cdot x = [v_{w_{\pm}(x)}]$.

We denote the resulting map $\mathbb{P}(V) \to \mathcal{F} \times \mathcal{F}$ by

$$\ell : \mathbb{P}(V) \to \mathcal{F} \times \mathcal{F}, \quad x \mapsto \lim_{t \to \pm\infty} (e^{-t\varphi} \cdot x, e^{t\varphi} \cdot x).$$

**Proof.** This is a straightforward calculation in homogeneous coordinates with respect to any basis which diagonalises $\varphi$.

**Remark 3.3** Proposition 3.2 together with (1) shows that $(\mathbb{P}(V), \Phi)$ is admissible and $\mathbb{P}(V)_{\text{fix}}$ with the Smale order is order anti-isomorphic to $\text{spec}(\varphi)$ with the natural order.

It is an elementary exercise to calculate the stable manifold and unstable manifolds:

**Proposition 3.4** Let $x = [v_\nu] \in \mathbb{P}(V_\nu) \subseteq \mathcal{F}$ for some $\nu \in \text{spec}(\varphi)$. Then

$$W^\pm(x) = \pi \left( v_\nu + \sum_{0 < \pm(\nu - \mu)} V_{\mu} \right).$$
In this section we review the construction and some basic properties of complex flag manifolds. The results presented are standard but the approach via dynamical systems is elementary and not so widely known. Note, however, that Duistermaat, Kolk, and Varadarajan in [6] provide plenty of information on this approach also in the more complicated case of real flag manifolds.

Let \( V \) be a finite dimensional complex vector space and \( G \subseteq \text{GL}(V) \) be a closed connected subgroup. The Lie algebra of \( G \) is the set
\[
\mathfrak{g} := \{ X \in \text{End}_\mathbb{C}(V) \mid e^{RX} \subseteq G \}.
\]
Then \( \mathfrak{g} \) is a real vector subspace of \( \mathfrak{gl}(V) := \text{End}_\mathbb{C}(V) \) which is closed under the bilinear map
\[
\mathfrak{gl}(V) \times \mathfrak{gl}(V) \to \mathfrak{gl}(V), \quad (X, Y) \mapsto [X, Y] := XY - YX
\]
called the Lie bracket. The matrix exponential map \( \exp : \mathfrak{g} \to G, \ X \mapsto e^X \) is a local homeomorphism at 0 and can be used to define a \( G \)-invariant differential structure on \( G \) such that \( G \) is a closed real submanifold of the open subset \( \text{GL}(V) \) in \( \text{End}_\mathbb{C}(V) \). If \( \mathfrak{g} \) is a complex vector subspace of \( \mathfrak{gl}(V) \), then \( G \) is a complex submanifold of \( \text{GL}(V) \). The group generated by \( \exp(\mathfrak{g}) \) is all of \( G \).

Suppose now that \( V \) is equipped with an inner product \( (v, w) \mapsto \langle v \mid w \rangle \). Given \( \varphi \in \text{End}_\mathbb{C}(V) \) let \( \varphi^t \in \text{End}_\mathbb{C}(V) \) be the transpose of \( \varphi \) with respect to the inner product. The map
\[
\theta : \text{GL}(V) \to \text{GL}(V), \quad g \mapsto (g^{-1})^t
\]
is involutive and preserves the group multiplication. It is called the Cartan involution. Its derivative at the identity is the map
\[
\mathfrak{gl}(V) \to \mathfrak{gl}(V), \quad X \mapsto -X^t
\]
which preserves the Lie bracket. By abuse of notation it is also called Cartan involution and denoted by \( \theta \). If \( G \) is \( \theta \)-invariant, then it is easy to check that \( \mathfrak{g} \) is invariant under the (algebra) Cartan involution, so that we have a Cartan decomposition \( \mathfrak{g} = \mathfrak{t} + \mathfrak{p} \), where \( \mathfrak{t} \) and \( \mathfrak{p} \) are the \( \theta \)-eigenspaces for the eigenvalues 1 and \(-1\), respectively. The polar decomposition yields a global analogue, also called Cartan decomposition: \( G = KP \). Here \( K \) is the subgroup of \( \theta \)-fixed points in \( G \) and \( P = \exp(\mathfrak{p}) \).

A subspace of \( \mathfrak{gl}(V) \) is called abelian if the Lie bracket vanishes on it. Choose a maximal abelian subspace \( \mathfrak{a} \) in \( \mathfrak{p} \). Then \( \mathfrak{a} \) consists of commuting \( \mathbb{R} \)-split endomorphisms and is therefore simultaneously diagonalizable. Thus there exists a set \( \mathcal{P} \) of linear functionals on \( \mathfrak{a} \) such that
\[
V = \bigoplus_{\mu \in \mathcal{P}} V^\mu,
\]
where $V^\mu = \{ v \in V \mid (\forall X \in a) \ X v = \mu(X) v \}$ and $V^\mu \neq \{0\}$ for all $\mu \in \mathcal{P}$. The elements of $\mathcal{P}$ are called the weights of $G$ on $V$. Each element $X$ of $\mathfrak{g}$ defines a linear map $\text{ad}(X) : \mathfrak{g} \rightarrow \mathfrak{g}$ via $\text{ad}(X)Y = [X, Y]$, which is called the adjoint representation of $X$. Note that $\text{ad}(X)$ is self adjoint with respect to the inner product $(Y, Z) \mapsto \text{tr}(YZ^t)$ on $\mathfrak{g}$ whenever $X \in \mathfrak{p}$. Therefore these $\text{ad}(X)$ are $\mathbb{R}$-split. Since $a$ is abelian the maps $\text{ad}(X)$ with $X \in a$ commute. Thus, in analogy to the weight decomposition of $V$, we have a root decomposition

$$\mathfrak{g} = m \oplus a \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}^\alpha,$$

where $\mathfrak{g}^\alpha := \{ Y \in \mathfrak{g} \mid (\forall X \in a) \ \text{ad}(X)Y = \alpha(X) Y \}$, $\Delta := \{ \alpha \in a^* \mid \alpha \neq 0, \ \mathfrak{g}^\alpha \neq \{0\} \}$, and $m := \{ Y \in \mathfrak{k} \mid (\forall X \in a) \ [X, Y] = 0 \}$. The elements of $\Delta$ are called the restricted roots associated with the pair $(\mathfrak{g}, a)$. Note that $\theta(X) = -X$ for $X \in a$ implies that for $\alpha \in \Delta$ also $-\alpha$ is a restricted root and $\theta \mathfrak{g}^\alpha = \mathfrak{g}^{-\alpha}$. Choose an element $X_\alpha \in a$ such that $\alpha(X_\alpha) \neq 0$ for all $\alpha \in \Delta$ and set $\Delta^\pm := \{ \alpha \in \Delta \mid \pm \alpha(X_\alpha) > 0 \}$, the elements of $\Delta^+$ are called positive roots and $\Delta^- = -\Delta^+$ is the set of negative roots. The Lie bracket satisfies the Jacobi identity $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ from which one derives

$$\mathfrak{g}^\alpha(V^\mu) \subseteq V^{\mu + \alpha}, \quad [\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \subseteq \mathfrak{g}^{\alpha + \beta}.$$

Therefore $\mathfrak{n} := \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}^\alpha$ is a subalgebra of $\mathfrak{g}$ consisting of simultaneously triangularizable elements of $\text{End}_C(V)$ and we obtain the Bruhat decomposition

$$\mathfrak{g} = \theta \mathfrak{n} \oplus m \oplus a \oplus \mathfrak{n}.$$

On the other hand $Y + \theta(Y) \in \mathfrak{k}$ for $Y \in \theta \mathfrak{n}$ so that $\theta \mathfrak{n} \subseteq \mathfrak{k} + \mathfrak{n}$. This yields the Iwasawa decomposition

$$\mathfrak{g} = \mathfrak{k} + a + \mathfrak{n}$$

which is easily checked to be direct using the properties of $\theta$. The global version of the Iwasawa decomposition is $G = KAN$, where $A = \exp(a)$ and $N = \exp(n)$.

The commutator algebra of a Lie algebra $\mathfrak{g}$ is the space $\mathfrak{g}'$ spanned by all $[X, Y]$ with $X, Y \in \mathfrak{g}$. It is easy to check that it is actually a subalgebra of $\mathfrak{g}$ under the Lie bracket. One can show that the Lie algebra of the commutator subgroup $G'$ of $G$ is $\mathfrak{g}'$. The Lie algebra $\mathfrak{g}$ is called solvable if the sequence of successive commutator algebras

$$\mathfrak{g} \supseteq \mathfrak{g}' \supseteq (\mathfrak{g}')' \supseteq \ldots$$

reaches zero.

Suppose now that $\mathfrak{g}$ is a complex subspace of $\mathfrak{gl}(V)$. Then $\mathfrak{k} = i\mathfrak{p}$ and hence $a_C = a + i a$ is maximal abelian in $\mathfrak{g}$. As a consequence $\mathfrak{b} := m + a + n$ is a complex solvable subalgebra of $\mathfrak{g}$ called the Borel subalgebra. The (closed) subgroup of $G$ generated by $\exp \mathfrak{b}$ is called Borel subgroup and denoted by $B$. The group $\exp(a_C)$ is denoted by $A_C$. The abelian algebra $a_C$ is its own normaliser in $\mathfrak{g}$, i.e., $[X, a_C] \subseteq a_C$ implies $X \in a_C$. Such algebras are called Cartan subalgebras
and one can show that all Cartan subalgebras of $\mathfrak{g}$ are conjugate under $G$. Using Jordan canonical forms, linear flows on $\mathbb{P}(V)$ can be used to give an elementary proof of the following version of Borel’s Fixed Point Theorem:

**Theorem 4.1** Let $S \subseteq \text{GL}(V)$ be a closed connected subgroup with complex solvable Lie algebra and $C \subseteq \mathbb{P}(V)$ a closed $S$–invariant subset. Then $C$ contains an $S$–fixed point.

Let $\mathcal{M} := G \cdot x \subseteq \mathbb{P}(V)$ be a closed $G$–orbit. Then the above theorem applied to the group $B$ shows that there exists a $B$–fixed point $x_c \in \mathcal{M}$. In other words, $B$ is contained in the stabiliser $G_{x_c}$ of $x_c$.

Subgroups of $G$ containing $B$ are called standard parabolic subgroups of $G$. Conjugates of standard parabolic subgroups are called parabolic subgroups. Thus all the stabilisers $G_y$ of points $y \in \mathcal{M}$ are parabolic subgroups. Conversely, suppose that $B \subseteq G_y$ for some $y \in \mathbb{P}(V)$. The Iwasawa decomposition shows that $G/G_y \cong K/K_y$ is compact so that $G \cdot y$ is closed in $\mathbb{P}(V)$. One can even show that all parabolic subgroups can be obtained in this way. A complex flag manifold of $G$ is a homogeneous space of the form $G/P$ with $P$ parabolic. Thus our discussion shows that the complex flag manifolds are precisely the closed $G$–orbits in $\mathbb{P}(V)$.

We close this section with a technical lemma which is the tool we need to characterise the fixed point set of our flows on complex flag manifolds. Let $N_K(\mathfrak{a}) = \{k \in K \mid k\mathfrak{a}k^{-1} = \mathfrak{a}\}$ and $Z_K(\mathfrak{a}) = \{k \in K \mid (\forall X \in \mathfrak{a}) \ kXk^{-1} = X\}$ be the normaliser and the centraliser of $\mathfrak{a}$ in $K$, respectively.

**Lemma 4.2** (i) $Z_K(\mathfrak{a})$ is a normal subgroup of $N_K(\mathfrak{a})$ and the quotient group $W := N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ is finite.

(ii) Suppose that $x, y \in \mathcal{M}$ are both fixed under $A_C$. Then there exists a $k \in N_K(\mathfrak{a})$ such that $k \cdot x = y$.

**Proof.** (i) The first part is obvious. To prove the second part note first that the annihilator $\Delta^\perp$ of $\Delta$ in $\mathfrak{a}$ is central in $\mathfrak{g}$. The group $N_K(\mathfrak{a})$ permutes the elements of $\Delta$ (dual action). If an element $k \in N_K(\mathfrak{a})$ fixes all $\alpha \in \Delta$, then $k$ acts trivially on $\mathfrak{a}/\Delta^\perp$ and by the above remark also on $\mathfrak{a}$.

(ii) Using the global Iwasawa decomposition again, we see that $K$ acts transitively on $\mathcal{M}$, so we find a $k_1 \in K$ such that $y = k_1 \cdot x$. Therefore the stabilisers of $x$ and $y$ are conjugate under $k_1$, i.e. $G_y = G_{k_1 \cdot x} = k_1 G_x k_1^{-1}$. This shows that $A_C$ and $k_1^{-1} A_1 k_1$ are contained in $G_x$. Therefore $A_C$ and $k_1 A_C k_1^{-1}$ are contained in the Lie algebra $\mathfrak{g}_y$ of $G_y$ and hence automatically Cartan subalgebras of $\mathfrak{g}_y$. Now we use the fact that all Cartan subalgebras of $\mathfrak{g}_y$ (which is complex) are conjugate under $G_y$. This means we can find an $h \in G_y$ such that $A_C = h k_1 A_C (h k_1)^{-1}$. Then $h k_1$ is contained in the $\theta$–invariant set $N_{C}(A_C)$, which according to the global Cartan decomposition, is equal to $N_K(\mathfrak{a})A$. Therefore we can write $h k_1 = k a_1$ with $a_1 \in A$ and obtain

$$k a_1^{-1} h k_1 a_1 (h k_1)^{-1} = h k_1 a (h k_1)^{-1} = a$$
The group \( W = N_K(a)/Z_K(a) \) is called the Weyl group of \( G \). One can show that \( Z_K(a) \) is connected (being the centraliser in \( K \) of the maximal abelian subalgebra \( i\mathfrak{a} \) in \( \mathfrak{k} \), cf. [7, p. 287]). Therefore it is equal to \( \exp(i\mathfrak{a}) \) and thus contained in \( B \). Thus we can define a subgroup \( W_M \) of \( W \) via

\[
W_M := \{kZ_K(a) \mid k \cdot x_c = x_c\}.
\]

5 Special Flows on Complex Flag Manifolds

In this section we introduce the kind of flow we want to study on complex flag manifolds.

**Definition 5.1** Suppose that \( X \in \mathfrak{a} \) has the following properties:

(i) \( \alpha(X) < 0 \) for all \( \alpha \in \Delta^+ \).

(ii) \( \mu(X) \neq \nu(X) \) for all \( \mu, \nu \in \mathcal{P} \) with \( \mu \neq \nu \).

Then we call the linear flow \( \Phi_X \) of \( X \) on \( \mathbb{P}(V) \) special.

Suppose that the flow of \( X \in \mathfrak{a} \) is special. Then the fixed points of \( \Phi_X(\mathbb{R}) = e^{\mathbb{R}X} \) in \( \mathbb{P}(V) \) are precisely the points \( [v] \) with \( v \in V^\mu \) for some \( \mu \in \mathcal{P} \). In fact, \( [v] \) is a fixed point for \( e^{\mathbb{R}X} \) if and only if \( \mathbb{C}v \) is invariant under \( X \) which in turn is the same as saying that \( v \) is an eigenvector of \( X \). But our hypothesis on \( X \) implies that the eigenvalues of \( X \) are the \( \mu(X) \) with \( \mu \in \mathcal{P} \) and that \( \mu(X) \) determines \( \mu \). Thus any eigenvector of \( X \) belongs to some \( V^\mu \) with \( \mu \in \mathcal{P} \). As a consequence we see that an \( e^{\mathbb{R}X} \)-fixed point in \( \mathbb{P}(V) \) is automatically an \( A_{\mathbb{C}} \)-fixed point, where \( A_{\mathbb{C}} = \exp(a_{\mathbb{C}}) \). We restrict the special flow \( e^{\mathbb{R}X} \) to \( \mathcal{M} = G \cdot x_c \subseteq \mathbb{P}(V) \) and consider the set

\[
\mathcal{F}_\mathcal{M} = \mathcal{F} \cap \mathcal{M} = \{x \in \mathcal{M} \mid A_{\mathbb{C}} \cdot x = x\}
\]

of \( A_{\mathbb{C}} \)-fixed points in \( \mathcal{M} \) as well as the corresponding stable and unstable manifolds \( W^\pm_{\mathcal{M}}(x) = W^\pm(x) \cap \mathcal{M} \) in \( \mathcal{M} \).

**Theorem 5.2** (i) The Weyl group \( W \) acts transitively on \( \mathcal{F}_\mathcal{M} \). The stabiliser of \( x_c \) in \( W \) is \( W_M \). Therefore \( \mathcal{F}_\mathcal{M} \) is parametrised by \( W/W_M \) and \((\mathcal{M}, \Phi)\) is admissible with \( \mathcal{M}_{\text{fix}} = \mathcal{M}_\text{fix} = \mathcal{F}_\mathcal{M} \).

(ii) \( \mathcal{M} = \bigcup_{x \in \mathcal{F}_\mathcal{M}} W^+(x) = \bigcup_{x \in \mathcal{F}_\mathcal{M}} W^-(x) \).

(iii) \( W^+_{\mathcal{M}}(x) = N \cdot x \) and \( W^-_{\mathcal{M}}(x) = (\theta N) \cdot x \) for all \( x \in \mathcal{F}_\mathcal{M} \).
Proof. (i) The first part follows from Lemma 4.2. The second part is an immediate consequence of the definition of $W_M$.

(ii) Since all eigenvalues of $X$ are real, the limit of $e^{tX} \cdot y$ for $t \to \pm \infty$ exists and is contained in $\mathcal{F}_M$ by Proposition 3.2.

(iii) Recall that $N = \exp(\sum_{\alpha \in \Delta} g^\alpha)$. We calculate

$$\exp(tX) \exp(\sum Y^\alpha) \cdot x = \exp(tX) \exp(\sum Y^\alpha) \exp(-tX) \cdot x$$

$$= \exp(\sum e^{\sigma tX} Y^\alpha) \cdot x$$

$$= \exp(\sum e^{t\alpha(X)} Y^\alpha) \cdot x$$

and note that this expression converges to $x$ for $t \to \infty$ by our hypothesis on $X$. As a result we obtain $N \cdot x \subseteq W_M^+(x)$. Analogously we find $\theta N \cdot x \subseteq W_M^-(x)$.

To show the converse, note first that the tangent space $T_xM$ is given by

$$T_xM = \{X \cdot x | X \in \theta n + n\}$$

since $A_C$ fixes $x$. Therefore the above calculation shows that locally (close to $x$) $N \cdot x$ agrees with $W_M^+(x)$ and $\theta N \cdot x$ agrees with $W_M^-(x)$. Since $\exp(tX)$ normalises $N$ and $\theta N$ this proves the claim. □

As a corollary we obtain a generalization of the global version of the Bruhat decomposition which explains why we call the $N$-orbits in $M$ Bruhat cells:

**Corollary 5.3** Let $m_1, \ldots, m_l \in N_K(a)$ be such that

$$m_1Z_K(a), \ldots, m_lZ_K(a) \in W$$

is a system of representatives for $W/W_M$. Then we have a decomposition

$$G = \bigcup_{j=1}^l Nm_jB.$$

The following lemma shows that special flows have desirable properties from the point of view of dynamical systems. It will allow us to apply the $\lambda$-Lemma of Palis (see [9, §2.7]), which will eventually show the transitivity of the Smale relation.

**Lemma 5.4** The vector field $X$ on $M$ is Morse-Smale. In fact, we even have:

(i) $-Xv$ is the gradient vector field of the function

$$f([v]) = -\frac{1}{2} \frac{(Xv \mid v)}{(v \mid v)},$$

where $(v \mid w) = \Re\langle v \mid w \rangle$ and the Riemannian metric on $\mathbb{P}(V)$ is the Fubini-Study metric.
(ii) $W_M(x)$ and $W_M^+(y)$ intersect transversally for all $x, y \in F_M$.

(iii) All critical points of $X$ are hyperbolic.

Proof. (i) follows immediately from Proposition\textsuperscript{3.1}

(ii) All $N$– and $\theta N$–orbits of elements in $F_M$ are $A_C$–invariant since $A_C$ normalises $N$ and $\Theta N$. Therefore we have

$$T_z M = (n + \theta n) \cdot z = T_z (N \cdot z) + T_z (\theta N \cdot z) = T_z (W_M^+ y) + T_z (W_M^-)$$

for each point $z \in W_M^-(x) \cap W_M^+(y)$.

(iii) This follows from the fact that $X$ is semisimple with real eigenvalues. ■

6 The Bruhat Order and its Relation to the Smale Order

Consider the relation

$$x \preceq_{Br} y \iff N \cdot x \subseteq N \cdot y$$

on the set $F_M$ of $A_C$–fixed points in $M$. It clearly is reflexive and transitive. We will see below that it actually also is antisymmetric, i.e., a partial order. This order is called the Bruhat order on $F_M$. Recall from Theorem\textsuperscript{5.2} that $F_M$ is parametrised by $W/W_M$. A result of Kostant (see [12, Lemma 1.1.2.15]) shows that there is a canonical choice of representatives $W_M^W$ for $W/W_M$ in $W$ and using the generalised Bruhat decomposition of $G$ (see [4, Chap. 4, §2(5), Prop. 2]) one can show that the induced order on $W^M$ actually is the Bruhat order as defined by generators and relations for $W$ viewed as a Coxeter group (see [8, §5.10] and also [2, Lemma 6.2.1]).

The Smale relation and the Smale order in this context are given by

$$x \leftarrow_{Sm} y \iff (\theta N) \cdot y \cap N \cdot x \neq \emptyset,$$

$$x \preceq_{Sm} y \iff x \leftarrow_{Sm} \cdots \leftarrow_{Sm} y.$$

In view of Lemma\textsuperscript{5.4} the $\lambda$-Lemma of Palis now shows that given $x \in F_M$ and two submanifolds $S_\pm$ of $M$ with $S_\mp$ intersecting $W_M^\pm(x)$ transversally in $x_\mp$ one can find a $t > 0$ such that $e^{tX} \cdot S_- \cap S_+ \neq \emptyset$. This is the key observation in the proof of our main tool for the determination of the attractor networks of complex flag manifolds:

**Theorem 6.1**

(i) The relations $\preceq_{Br}, \leftarrow_{Sm}$, and $\preceq_{Sm}$ on $F_M$ agree.

(ii) $N \cdot x = \bigcup_{y \preceq_{Br} x} N \cdot y$.

(iii) $\theta (N) \cdot x = \bigcup_{y \preceq_{Sm} x} \theta (N) \cdot y$.  

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(iv) \( \preceq \) is a partial order.

Proof. (i) Suppose that \( a^{-} \leftarrow_{\text{Sm}} x \leftarrow_{\text{Sm}} a^{+} \) and set \( S_{\pm} = W_{\mathcal{M}}^{\pm}(a^{\mp}) \). Then the \( \lambda \)-Lemma implies \( a^{-} \leftarrow_{\text{Sm}} a^{+} \). This argument proves that \( \leftarrow_{\text{Sm}} \) is already transitive and hence agrees with \( \preceq_{\text{Sm}} \).

Next we show that \( W^{+} \leftarrow_{\text{Sm}} W^{+} \leftarrow_{\text{Sm}} \). The inclusion \( " \subseteq " \) follows since the right hand side is an attractor for \( e^{-tX} \) and as such closed. For the converse we again use the \( \lambda \)-Lemma: Let \( x \leftarrow_{\text{Sm}} y \) and \( z \in W_{\mathcal{M}}^{+}(y) \). Choose a small disc \( S_{-} \) through \( z \) intersecting \( W_{\mathcal{M}}^{+}(y) \) transversally. Further we set \( S_{+} = W_{\mathcal{M}}^{+}(x) \). Then \( x \leftarrow_{\text{Sm}} y \) implies that \( S_{+} \) intersects \( W_{\mathcal{M}}^{+}(y) \) (transversally), so that we find a \( t > 0 \) with \( e^{tX} S_{-} \cap S_{+} \neq \emptyset \). But then also \( S_{-} \cap S_{+} = S_{-} \cap e^{tX} S_{+} \neq \emptyset \). Since \( S_{-} \) was arbitrarily small this proves the inclusion \( " \supseteq " \).

Now we see that \( y \preceq_{\text{Sm}} x \) is equivalent to \( N \cdot y = W_{\mathcal{M}}^{+}(y) \subseteq W_{\mathcal{M}}^{+}(x) = N \cdot x \) which in turn just means \( y \preceq_{\text{Br}} x \). This proves (i) and (ii).

To prove (iii) we observe that the time reversed flow interchanges the roles of stable and unstable manifolds and flips the Smale order around. Therefore we only have to apply Theorem 5.2.

(iv) Suppose that \( x \preceq_{\text{Br}} y \preceq_{\text{Br}} x \). Then \( x \leftarrow_{\text{Sm}} y \leftarrow_{\text{Sm}} x \) which implies \( x = y \) since we have a gradient flow.

We note that this theorem could also be derived from results of Kazhdan and Lusztig (see [5]).

## 7 Attractor-Repellor Pairs

In this section we determine the attractor-repellor pairs for special flows on complex flag manifolds and show how one can identify the attractor network with the lattice of upper sets in \( \mathcal{F}_{\mathcal{M}} \). For details concerning attractor networks we refer to [10].

We begin by reviewing some basic material concerning attractors and repellors for general continuous time dynamical systems. An attractor for \( \Phi \) in \( \mathcal{M} \) is a compact invariant set \( \mathcal{A} \subseteq \mathcal{M} \) which admits a neighborhood \( U \) such that

\[
\Phi(t, U) \subseteq \text{int } U \quad \forall t > 0
\]

and

\[
\mathcal{A} = \bigcap_{t \geq 0} \Phi(t, U),
\]

where \( \text{int } U \) is the interior of \( U \). A repellor for \( (\mathcal{M}, \Phi) \) is an attractor for the time-reversed dynamical system. We denote the set of all attractors for \( \Phi \) by
\( \mathcal{A}(\mathcal{M}, \Phi) \) and the set of all repellors for \( \Phi \) by \( \mathcal{A}(\mathcal{M}, \Phi^{-1}) \). Then \( \mathcal{A}(\mathcal{M}, \Phi) \) is a lattice with respect to the set theoretic meet and join operations. For any subset \( \mathcal{A} \) of \( \mathcal{M} \) consider the set
\[
\mathcal{A}_{\text{fix}} := \{ x \in \mathcal{A} \mid (\forall t \in \mathbb{R})\Phi(t, x) = x \}
\]
of fixed points in \( \mathcal{A} \).

**Proposition 7.1** Assume that every flow line in \( \mathcal{M} \) has a sink and a source. In other words:
\[
\lim_{t \to \pm \infty} \Phi(t, x) \quad \forall x \in \mathcal{M}.
\]
Then for any closed invariant subset \( \mathcal{A} \subseteq \mathcal{M} \) we have
(i) \( W^\pm(\mathcal{A}) = \bigcup_{x \in \mathcal{A}_{\text{fix}}} W^\pm(x) \).
(ii) \( W^\pm(\mathcal{A})_{\text{fix}} = \mathcal{A}_{\text{fix}} \).

If \( \mathcal{A} \) is an attractor, then \( \mathcal{A}^* = \mathcal{M} \setminus W^+(\mathcal{A}) \) is a repellor, called the dual repellor of \( \mathcal{A} \). Then \((\mathcal{A}, \mathcal{A}^*)\) is called an attractor-repellor pair. From Proposition 7.1 one derives

**Proposition 7.2** Let \((\mathcal{A}, \mathcal{A}^*)\) be an attractor-repellor pair. Then we have
(i) \( \mathcal{A}^*_{\text{fix}} = \mathcal{M}_{\text{fix}} \setminus \mathcal{A}_{\text{fix}} \).
(ii) \( \mathcal{A}^* = \bigcup_{x \in \mathcal{A}^*_{\text{fix}}} W^+(x) \) is closed.
(iii) \( \mathcal{A} = \bigcup_{x \in \mathcal{A}_{\text{fix}}} W^-(x) \) is closed.
(iv) \( \mathcal{M} = \mathcal{A} \cup (W^+(\mathcal{A}) \cap W^-(\mathcal{A}^*)) \cup \mathcal{A}^* \) is a disjoint union.

Recall from [10] that \( \mathcal{A}(\mathcal{M}, \Phi) \) is a lattice with respect to the usual set operations and that an attractor network is a finite sublattice of \( \mathcal{A}(\mathcal{M}, \Phi) \). Before we can formulate and prove the general theorem that leads the way to a description of attractor networks for special flows in complex flag manifolds we have to introduce some more order theoretic concepts.

To each partially ordered set \( P \) one assigns the dual distributive lattice
\[
P^* = \text{Hom}_{\text{order}}(P, \{0, 1\})
\]
of order homomorphisms, where the two point set \( \{0, 1\} \) is a lattice with respect to operations max and min. A different way to view \( P^* \) is to identify an element \( \alpha \in P^* \) with \( \alpha^{-1}(1) \). This set is an upper set with respect to the ordering. This means that for any \( p \in \alpha^{-1}(1) \) and \( p \leq q \in P \) we have \( q \in \alpha^{-1}(1) \). Conversely each upper set \( U \subseteq P \) gives rise to an order homomorphism \( \chi_U : P \to \{0, 1\} \), where \( \chi_U \) is the characteristic function of \( U \). In this way \( P^* \) gets identified with the lattice of upper subsets of \( P \) under the usual set theoretic join (union) and meet (intersection).
Theorem 7.3 Suppose that the dynamical system \((M, \Phi)\) is admissible with isolated fixed points and \(\preceq_{\text{Sm}}\) is a partial order. Then all the attractor-repellor pairs \((A, A^*)\) are of the form \((A, A^*) = (A_u, A^*_u)\) with

\[
A_u = \bigcup_{x \in U} W^-(x), \quad A^*_u = \bigcup_{x \in L} W^+(x),
\]

where \(U\) is a compact isolated upper subset of \(M_{\text{fix}}\) with respect to the Smale order and \(L = M_{\text{fix}} \setminus U\). The map

\[
M^*_{\text{fix}} \mapsto A(M, \Phi), \quad U \mapsto A_U
\]

is a lattice isomorphism.

Proof. Given an attractor \(A\) and an attracting neighbourhood \(U \subseteq M\) as described in (2) and (3) we see that \(L := U \cap M_{\text{fix}} = A^*_u\). This implies that \(L\) is an open and closed lower set in \(M_{\text{fix}}\). We set \(U = M_{\text{fix}} \setminus L\). Then Proposition 7.2 proves the equalities \(A_u = A\) and \(A^*_u = A^*\). Conversely, since there are only finitely many open and closed subsets in \(M_{\text{fix}}\) we can apply [10, Thm. 1.3] to see that each attractor has to be of this form. The last claim is now clear. \(\blacksquare\)

Theorem 7.3 says that for the partially ordered (with respect to the Smale order) set \(M_{\text{fix}}\) of isolated fixed points of \(\Phi(t, x)\) the dual lattice is the lattice of attractors of the flow. The correspondence is given by

\[
A_\alpha = \bigcup_{\alpha(p) = 0} W^-(p)
\]

for \(\alpha \in M^*_{\text{fix}}\). For each attractor the dual repeller is given by

\[
A^*_\alpha = \bigcup_{\alpha(p) = 1} W^+(p).
\]

8 The Attractor Network for Special Flows on Complex Flag Manifolds

Let \(V\) be finite dimensional complex vector space with an inner product \(\langle \cdot \mid \cdot \rangle\) and \(\varphi \in \text{End}(V)\) a selfadjoint linear map. For each eigenvalue \(\nu \in \text{spec}(\varphi)\) of \(\varphi\) we set \(V_+^{(\nu)} := \sum_{\nu \leq \mu} V_\mu\) and \(V_-^{(\nu)} := \sum_{\nu < \mu} V_\mu\). Then \(V = V_+^{(\nu)} \oplus V_-^{(\nu)}\).

Proposition 8.1 The attractor-repellor pairs for \((\mathbb{P}(V), \Phi)\) are given by

\[
A_\nu := \mathbb{P} \left( V_+^{(\nu)} \right) \quad \text{and} \quad A^*_\nu := \mathbb{P} \left( V_-^{(\nu)} \right)
\]

for \(\nu \in \text{spec}(\varphi)\).
Proof. This follows from (1), Remark 3.3 and Theorem 7.3.

We return to the particular case of a complex flag manifold \( M = G \cdot x_c \subseteq \mathbb{P}(V) \) and a special flow \( \Phi_X \) on \( M \) considered in § 5. Theorem 7.3 shows that the hypotheses for Theorem 7.3 are satisfied and that the Smale order agrees with the Bruhat order.

Let \( A \) be an attractor in \( M \) for the flow \( \Phi_X \) and \( A^* \) its dual repellor. Then, by Proposition 7.1, we have

\[
A = W^-(A) = \bigcup_{x \in A^\text{fix}} W^-(x) = \bigcup_{x \in A^\text{fix}} (\theta N) \cdot x
\]

and

\[
A^* = W^+(A^*) = \bigcup_{x \in A^{\ast \text{fix}}} W^+(x) = \bigcup_{x \in A^{\ast \text{fix}}} N \cdot x.
\]

Since repellors are closed, the Theorem 6.1 shows that \( A^{\ast \text{fix}} \subseteq M^{\text{fix}} \) is a lower set with respect to the Bruhat order. This means that if \( x \in A^{\ast \text{fix}} \), then all \( y \in M^{\text{fix}} \) with \( y \preceq_{\text{Br}} x \) also belong to \( A^{\ast \text{fix}} \). Similarly \( A^{\text{fix}} \subseteq M^{\text{fix}} \) is an upper set with respect to the Bruhat order. In fact, it turns out that all lower and upper sets can be obtained in this way.

**Theorem 8.2** Let \( L(F_M) \) be the lattice of lower sets in \( F_M \) and \( U(F_M) \) be the lattice of upper sets in \( F_M \). Then the maps

\[
U(F_M) \rightarrow A(M, \Phi), \quad U \mapsto A_U := \bigcup_{x \in U} (\theta N) \cdot x
\]

and

\[
L(F_M) \rightarrow A(M, \Phi^{-1}), \quad L \mapsto R_L := \bigcup_{x \in L} N \cdot x
\]

are lattice isomorphisms and \( R_L \) is the dual repellor of \( A_{F_M \setminus L} \).

Proof. In view of Theorem 7.3 only the last claim remains to be proved. To do that, let \( L = F_M \setminus U \). Then from Proposition 7.2 we see

\[
M = \bigcup_{(x,x') \in F_M \times F_M} N \cdot x \cap (\theta N) \cdot x',
\]

\[
A_U = \bigcup_{(x,x') \in F_M \times U} N \cdot x \cap (\theta N) \cdot x',
\]

\[
R_L = \bigcup_{(x,x') \in L \times F_M} N \cdot x \cap (\theta N) \cdot x',
\]

\[
W^u(A_U) \cap W^s(R_L) = \bigcup_{(x,x') \in U \times L} N \cdot x \cap (\theta N) \cdot x'.
\]
so that we have a disjoint union

\[ \mathcal{M} = \mathcal{A}_U \cup (W^s(\mathcal{A}_U) \cap W^u(\mathcal{R}_L)) \cup \mathcal{R}_L. \]

Now [10] Prop. 1.4 shows that \( \mathcal{A}_U \) is an attractor and \( \mathcal{R}_L = \mathcal{A}_U^* \) is its dual repellor.

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