Anomalous fermion number violation
and numerical simulations

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Abstract

After discussing the problem of lattice regularization of chiral gauge theories, a simple model for anomalous fermion number violation is formulated which can be numerically studied with present day technique. Exploratory results of numerical simulations of a two-dimensional U(1) Higgs model are presented.

1 Introduction

The anomalous baryon and lepton number violation is a consequence of the chirality of the electroweak gauge interactions [1]. This is a non-perturbative phenomenon and, therefore, a lattice formulation allowing for numerical simulations would be desirable. It is, however, well known that the non-perturbative formulation of chiral gauge theories is problematic (see, for instance, [2]).

In the first part of my talk some recent developments concerning lattice chiral fermions are reviewed and the possibility of existence of mirror fermions is discussed. Recent work on the limits of mirror fermion parameters is reviewed within the framework of a particular mixing scheme of fermion–mirror-fermion pairs. In the second part a simple model of anomalous fermion number violation is discussed which can be studied on the lattice [3]. In the last part a simplified toy model in two dimensions, with chiral U(1)-symmetry, is discussed and some recent numerical results concerning topological charge and Chern-Simons number are presented.

2 Chiral fermions on the lattice?

2.1 Recent developments

For an exact lattice formulation of the Standard Model on the lattice one has to solve the problem of chiral gauge theories. The recently proposed “domain wall fermion” formulation
can potentially lead to a solution by introducing an additional fifth dimension with four dimensional hypersurfaces representing the real world. A simpler, more economic, version of this idea is to throw away half of the five dimensional space, resulting in the “boundary fermion” formulation \[5\]. The difficulty in these formulations arises at switching on the gauge coupling. If in the fifth direction the gauge coupling is chosen to be strong, one has to fight against the dangerous “layered phase” \[7\], where subsequent four-dimensional hyperplanes do not interact and the theory becomes vector-like \[8\]. In case of a four-dimensional gauge field, identical in all hyperplanes, the decoupling of the chiral fermion from its mirror partner with opposite chirality seems unlikely, at least as long as exact gauge invariance is maintained.

A further obstacle for defining chiral gauge theories on the lattice is a recent extension of the Nielsen-Ninomiya no-go theorem including interactions and considering the continuum limit \[9\].

Avoiding the fermion doubling problem on random lattices has recently been also investigated, and does not seem to be promising, either \[10\].

The remaining possibilities for the lattice formulation of chiral gauge theories are not explicitly gauge invariant. In the modified Rome-approach capable to accommodate also fermion number violations \[11\] gauge fixing is essential. In case of the “reduced staggered fermions” \[12\] gauge symmetry is explicitly broken by lattice artifacts, but hopefully restored by the dynamics.

### 2.2 Mirror fermions

Since the non-perturbative formulation of the electroweak interactions is so difficult, a natural question is whether “chirality” is perhaps only a low-energy phenomenon, and at high energy the space-reflection symmetry is restored by the existence of opposite chirality “mirror fermions” \[13\]. If the presently known (almost complete) three fermion families were duplicated at the electroweak energy scale, in the range 100-1000 GeV, by three mirror fermion families with opposite chiralities and hence V+A couplings to the weak gauge vector bosons \[14\], then the whole fermion spectrum would be “vectorlike”. This would very much simplify the non-perturbative lattice formulation of the Standard Model \[15\].

The direct pair production of mirror fermions is not observed at LEP. This puts a lower limit on their masses of about 45 GeV. Heavier mirror fermions could be produced via their mixing to ordinary fermions. The present data imply some constraints on the mixing angles versus the masses which, however, strongly depend on the mixing scheme. (For an evaluation of the constraints implied by the high precision LEP data see the recent paper by Csáki and Csikor \[16\]).
2.3 Fermion–mirror-fermion mixing scheme

The strongest constraints on mixing angles between ordinary fermions and mirror fermions arise from the conservation of $c$, $\mu$, and $\tau$-lepton numbers and from the absence of flavour changing neutral hadronic currents. In a particular scheme these constraints can be implemented at tree level \[14\]. In this “monogamous mixing” scheme the structure of the mass matrix is such that there is a one-to-one correspondence between fermions and mirror fermions, due to the fact that the family structure of the mass matrix for mirror fermions is closely related to the one for ordinary fermions.

Let us denote doublet indices by $A = 1, 2$, colour indices by $c = 1, 2, 3$ in such a way that the leptons belong to the fourth value of colour $c = 4$, and family indices by $K = 1, 2, 3$. Diagonal entries in the mass matrix for “normal” fermions will be denoted by an index $\psi$, those for the mirror fermions by $\chi$ and the off-diagonal mixing masses between fermions and mirror fermions by $L$ or $R$ (depending on the chirality of the “normal” fermion). In general the elements of the mass matrix for three mirror pairs of fermion families are diagonal in isospin and colour, hence they have the form

$$\mu(\psi, \chi); A_2 c_2 K_2, A_1 c_1 K_1 = \delta A_2 A_1 \delta_{c_2 c_1} \delta (\psi)_{K_2 K_1}^{(A_1 c_1)};$$
$$\mu L; A_2 c_2 K_2, A_1 c_1 K_1 = \delta A_2 A_1 \delta_{c_2 c_1} \mu_L K_2 K_1; \quad \mu R; A_2 c_2 K_2, A_1 c_1 K_1 = \delta A_2 A_1 \delta_{c_2 c_1} \mu_R K_2 K_1. \quad (1)$$

The diagonalization of the mass matrix $M$ can be achieved for given indices $A$ and $c$ by two $6 \otimes 6$ unitary matrices $F_L^{(Ac)}$ and $F_R^{(Ac)}$ acting, respectively, on the L-handed and R-handed subspaces:

$$F_L^{(Ac)\dagger} (M^\dagger M)_L F_L^{(Ac)}, \quad F_R^{(Ac)\dagger} (M^\dagger M)_R F_R^{(Ac)}. \quad (2)$$

The main assumption of the “monogamous” mixing scheme is that in the family space $\mu_\psi, \mu_\chi, \mu_L, \mu_R$ are hermitian and simultaneously diagonalizable, that is

$$F_L^{(Ac)} = F_R^{(Ac)} = \begin{pmatrix} F^{(Ac)} & 0 \\ 0 & F^{(Ac)} \end{pmatrix}, \quad (3)$$

where the block matrix acts in $(\psi, \chi)$-space. The Kobayashi-Maskawa matrix of quarks is given by

$$C \equiv F^{(2c)\dagger} F^{(1c)}, \quad (4)$$

independently for $c = 1, 2, 3$. The corresponding matrix with $c = 4$ and $A = 1 \leftrightarrow 2$ describes the mixing of neutrinos, if the Dirac-mass of the neutrinos is nonzero. (Majorana masses of the neutrinos are not considered here, but in principle, they can also be introduced.)

An example for a mass matrix with “monogamous” mixing is the following:

$$\mu^{(Ac)}_{\psi; K_2 K_1} = \delta (\psi)_{K_2 K_1}^{(A_1 c_1)} + \delta_{K_2 K_1}^{(A_1 c_1)} \Delta^{(Ac)};$$
$$\mu^{(c)}_{\psi; K_2 K_1} = \delta_{K_2 K_1}^{(c)} \delta_L^{(c)}; \quad \mu^{(Ac)}_{R; K_2 K_1} = \delta (\psi)_{K_2 K_1}^{(A_1 c_1)} + \delta_{K_2 K_1}^{(A_1 c_1)} \delta_R^{(Ac)}; \quad (5)$$

where $\lambda^{(Ac)}_\chi, \Delta^{(Ac)}_\chi, \delta^{(c)}_L, \lambda^{(Ac)}_R, \delta^{(Ac)}_R$ do not depend on the family index.
The full diagonalization of the mass matrix on the \((\psi_L, \psi_R, \chi_L, \chi_R)\) basis of all three family pairs is achieved by the \(96 \otimes 96\) matrix

\[
\mathcal{O}^{(LR)}_{Nc'K',AcK} = \delta_{A'A} \delta_{c'c} F^{(Ac)}_{K'K}.
\]

\[\begin{pmatrix}
\cos \alpha_L^{(AcK)} & 0 & \sin \alpha_L^{(AcK)} & 0 \\
0 & \cos \alpha_R^{(AcK)} & 0 & \sin \alpha_R^{(AcK)} \\
-\sin \alpha_L^{(AcK)} & 0 & \cos \alpha_L^{(AcK)} & 0 \\
0 & -\sin \alpha_R^{(AcK)} & 0 & \cos \alpha_R^{(AcK)}
\end{pmatrix}. \quad (6)
\]

\(M^\dagger M\) is diagonalized by \(\mathcal{O}^{(LR)^\dagger} M^\dagger M \mathcal{O}^{(LR)}\), and \(MM^\dagger\) by \(\mathcal{O}^{(RL)^\dagger} MM^\dagger \mathcal{O}^{(RL)}\), where \(\mathcal{O}^{(RL)}\) is obtained from \(\mathcal{O}^{(LR)}\) by \(\alpha_L \leftrightarrow \alpha_R\).

In case of \(\mu_R = \mu_L\), which happens for instance in (5) if \(\lambda_R = 0\) and \(\delta_R = \delta_L\), the left-handed and right-handed mixing angles are the same:

\[
\alpha^{(AcK)} \equiv \alpha_L^{(AcK)} = \alpha_R^{(AcK)}. \quad (7)
\]

In Ref. [14] only this special case was considered. The importance of the left-right-asymmetric mixing was pointed out in Ref. [17], where the constraints arising from the measured values of anomalous magnetic moments were determined. It turned out that for the electron and muon the upper limit is

\[
|\alpha_L \alpha_R| \leq 0.0004, \quad (8)
\]

which is much stronger than the limits obtained from all other data [18]:

\[
\alpha^2_L, \alpha^2_R \leq 0.02. \quad (9)
\]

In case of the L-R asymmetric mixing the constraint (8) can be satisfied, for instance, if the right-handed mixing vanishes (or is very small): \(\alpha_R \simeq 0\).

The hypothetical mirror fermions can be discovered at the next generation of high energy colliders. At HERA the first family mirror fermions can be produced via mixing to ordinary fermions up to masses of about 200 GeV, if the mixing angles are close to their present upper limits [19, 20]. At SSC and LHC mirror lepton pair production can be observed up to masses of a few hundred GeV [21]. This has the advantage of being essentially independent of the small mixing. At a high energy \(e^+e^-\) collider, e. g. LEP-200 or NLC, mirror fermions can be pair produced and easily identified up to roughly half of the total energy, and also produced via mixing almost up to the total energy [22].

In a model with fermion–mirror-fermion pairs the anomaly in the baryon- and lepton-number current is zero. Therefore in such models there is no anomalous fermion number violation at the electroweak symmetry breaking scale. The baryon asymmetry of the Universe has to be produced at some higher energy scale, for instance, at the scale of grand unification [23].

\section{Fermion number anomaly on the lattice}

If mirror fermions do not exist, the lattice formulation of the Standard Model is problematic and the non-perturbative effects in the electroweak sector, like the anomalous fermion number...
violation, cannot be studied by numerical simulations. There is, however, an approximation of the electroweak sector of the standard model which can be studied with standard lattice techniques, namely the limit when the SU(3)colour \( \otimes U(1) \) hypercharge gauge couplings are neglected \([3]\).

### 3.1 Model for fermion number violation

A simple prototype model is the standard SU(2)\(_L\) Higgs model coupled to an even number \(2N_f\) of fermion doublets. In the standard model we have \(N_f = 6\) (for simplicity, we consider Dirac-neutrinos, but the massless neutral right-handed neutrinos decouple \([24]\)). One can take, for simplicity, \(N_f = 1\) but the extension to \(N_f > 1\) is trivial. The lattice action depends on the matrix scalar field \(\varphi_x = \phi_{0x} + i\phi_{sx}\tau_s\) (with four real fields \(\phi_{S=0,...,3}\)) and the fermion doublet fields \(\psi(1,2)_x\):

\[
S = S_{\text{scalar}} + S_{\text{fermion}}. \tag{10}
\]

The standard scalar-gauge Higgs-model action is

\[
S_{\text{scalar}} = -\frac{1}{2} \sum_x \left( \frac{\mu_0}{2} [\psi_{1x}^T \epsilon C \psi_{1x} - (\psi_{1x}^T \epsilon C \bar{\psi}_{2x})] + \frac{\lambda}{16} \left[ \psi_{1x}^T \psi_{1x} + \psi_{1x} \psi_{1x}^T \right] \right)
\]

\[
+ \frac{1}{2} \sum_x \left[ (\psi_{1x}^T \epsilon C \bar{\psi}_{1x} + (\psi_{1x} \psi_{1x}^T) \bar{\psi}_{1x} + (\psi_{1x}^T \epsilon C \bar{\psi}_{1x}) \bar{\psi}_{1x}) \right]
\]

\[
+ \frac{1}{2} \sum_x \left[ (\psi_{2x}^T \epsilon C \psi_{1x} - (\psi_{2x} \epsilon C \bar{\psi}_{2x}) - (\psi_{2x}^T \epsilon C \bar{\psi}_{2x})) \right]
\]

\[
+ \left( \psi_{2x}^T \epsilon C \bar{\psi}_{1x} + (\psi_{1x} \psi_{1x}^T) \bar{\psi}_{2x} + (\psi_{1x} \psi_{1x}^T) \bar{\psi}_{2x}) \right)
\]

\[
+ \left( \psi_{1x}^T \epsilon C \bar{\psi}_{1x} + (\psi_{1x} \psi_{1x}^T) \bar{\psi}_{2x} + (\psi_{1x} \psi_{1x}^T) \bar{\psi}_{2x}) \right)
\]

\[
+ \left( \psi_{1x}^T \epsilon C \bar{\psi}_{1x} + (\psi_{1x} \psi_{1x}^T) \bar{\psi}_{2x} + (\psi_{1x} \psi_{1x}^T) \bar{\psi}_{2x}) \right)
\]

The fermionic part contains the chiral gauge fields (with \(U_{x\mu} \in SU(2)\)) and \(P_{L,R} = (1 \pm \gamma_5)/2\)

\[
U_{(L,R)x\mu} = P_{(L,R)} U_{x\mu} + P_{(R,L)}, \tag{12}
\]

and is given by

\[
S_{\text{fermion}} = -\frac{1}{2} \sum_x \left( \frac{\mu_0}{2} [\psi_{1x}^T \epsilon C \psi_{1x} - (\psi_{1x}^T \epsilon C \bar{\psi}_{2x})] + \frac{\lambda}{16} \left[ \psi_{1x}^T \psi_{1x} + \psi_{1x} \psi_{1x}^T \right] \right)
\]

\[
+ \frac{1}{2} \sum_x \left[ (\psi_{1x} \psi_{1x}^T) \bar{\psi}_{1x} + (\psi_{1x} \psi_{1x}^T) \bar{\psi}_{1x} + (\psi_{1x} \psi_{1x}^T) \bar{\psi}_{1x}) \right]
\]

\[
+ \frac{1}{2} \sum_x \left[ (\psi_{2x} \epsilon C \psi_{1x} - (\psi_{2x} \epsilon C \bar{\psi}_{2x}) - (\psi_{2x} \epsilon C \bar{\psi}_{2x})) \right]
\]

\[
+ \left( \psi_{2x} \epsilon C \bar{\psi}_{1x} + (\psi_{1x} \psi_{1x}^T) \bar{\psi}_{2x} + (\psi_{1x} \psi_{1x}^T) \bar{\psi}_{2x}) \right)
\]

\[
+ \left( \psi_{1x} \psi_{1x}^T) \bar{\psi}_{2x} + (\psi_{1x} \psi_{1x}^T) \bar{\psi}_{2x}) \right)
\]

\[
+ \left( \psi_{1x} \psi_{1x}^T) \bar{\psi}_{2x} + (\psi_{1x} \psi_{1x}^T) \bar{\psi}_{2x}) \right)
\]

Here \(\epsilon = i\tau_2\) acts in isospin space, and \(C\) is the fermion charge conjugation matrix. The Yukawa-couplings \(G_{1,2}\) can, in general, be arbitrary diagonal matrices in isospin space. In case of degenerate doublets \(G_{1,2}\) are proportional to the unit matrix.

Instead of the off-diagonal Majorana mass \(\mu_0\) and Majorana-like Wilson term (proportional to \(r\)), it is technically more convenient to consider a Dirac-like form with \(\psi \equiv \psi_1\) and the mirror fermion field \(\chi\) defined by

\[
\chi_x = -e^{-1} C \psi^T \psi_{2x}, \quad \bar{\chi}_x = \psi^T \psi_{2x} C. \tag{14}
\]
Figure 1: The value of the lattice sum $I_C$ on $100^4$ lattice as a function of the bare mass $\mu_0$. The upper curve is for periodic, the lower one for antiperiodic boundary conditions.

Figure 2: The same as fig. 1 on $200^4$ lattice.
In terms of $\psi$ and $\chi$ one obtains the mirror fermion action for chiral gauge fields \cite{15}, which is well suited for studying the physically relevant phase with broken symmetry.

In the symmetric (i.e. confinement) phase there is a natural alternative choice in terms of the reshuffled combinations

$$\psi_{Cx} \equiv \psi_{Lx} + \chi_{Rx}, \quad \psi_{Nx} \equiv \chi_{Lx} + \psi_{Rx}.$$  \hspace{1cm} (15)

On this basis the vectorlike nature of the model becomes explicit ($\gamma_5$’s appear only in the Yukawa-couplings). The SU(2) gauge field couples only to the “charged field” $\psi_C$, and the neutral doublet $\psi_N$ has only its Yukawa-coupling.

The fermion number is the difference of the number of fermions ($\psi$-field) and mirror fermions ($\chi$-field). The gauge invariant fermion number current can be defined as

$$J_{x\mu} \equiv \frac{1}{2} \left[ (\bar{\psi}_{x+\hat{\mu}} \gamma_\mu U_{Lx\mu} \psi_x) + (\bar{\psi}_x \gamma_\mu U_{Lx\mu}^{\dagger} \psi_{x+\hat{\mu}}) - (\overline{\chi}_{x+\hat{\mu}} \gamma_\mu U_{Rx\mu} \chi_x) - (\overline{\chi}_x \gamma_\mu U_{Rx\mu}^{\dagger} \chi_{x+\hat{\mu}}) \right].$$  \hspace{1cm} (16)

3.2 Volume dependence

The anomalous Ward-Takahashi identity can be derived, as usual, on a weak and smooth background gauge field with field strength

$$F_{\mu\nu}^a(x) = \partial_\mu A_{\nu}^a(x) - \partial_\nu A_{\mu}^a(x) + g \epsilon_{stu} A_{\mu}^t(x) A_{\nu}^u(x).$$  \hspace{1cm} (17)

In the continuum limit of the background field the result for $2N_f$ fermion doublets is

$$\langle \partial_\mu J_{\mu}(x) \rangle_f = \lim_{a \to 0} \langle \Delta_{b\mu} J_{x\mu} \rangle_f a^{-4} = N_f g^2 \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu}^a(x) F_{\rho\sigma}^a(x) \mathcal{I}(r, \mu_0).$$  \hspace{1cm} (18)

Here the lattice integral $\mathcal{I}$ is given by

$$\mathcal{I}(r, \mu_0) \equiv \frac{1}{(2\pi)^4} \int_{-\pi}^{\pi} \frac{\mu_k \cos k_1 \cos k_2 \cos k_3 \cos k_4}{(k^2 + \mu_k^2)^3} \left[ r \sum_{\alpha=1}^{4} \bar{k}_\alpha / \cos k_\alpha - \mu_k \right] d^4k,$$  \hspace{1cm} (19)

and the notations are

$$\mu_k = \mu_0 + \frac{r}{2} \bar{k}_2, \quad \bar{k}_\mu = \sin k_\mu, \quad \hat{k}_\mu = 2 \sin \frac{k_\mu}{2}.$$  \hspace{1cm} (20)

The integral $\mathcal{I}$ is the same as the one occurring in the chiral anomaly, and one can prove (see e.g. \cite{23, 26})

$$\mathcal{I}(r, 0) = \frac{1}{32\pi^2} \quad \text{(independently from } r).$$  \hspace{1cm} (21)

\cite{18} and \cite{21} show that the correct continuum anomaly is reproduced at vanishing bare (Majorana-) fermion mass $\mu_0 = 0$. It is, however, interesting to investigate the $\mu_0$ dependence of the lattice integral in \cite{19}. The numerical evaluation of the corresponding lattice sum $\mathcal{I}_L$ on finite ($L^4$) lattices shows that $\mathcal{I} = \lim_{L \to \infty} \mathcal{I}_L$ is very small, probably $\mathcal{I}(\mu_0, r) = 0$ for every positive $\mu_0$. (See fig. 1 and 2.) The deviation from zero for small $\mu_0$ depends on the boundary
condition and, as it can be seen, is a function of $L\mu_0$. The value of $L\mu_0$ comes close to zero near $L\mu_0 \simeq 10$. This behaviour implies that at every positive $\mu_0$ the non-conservation of the current (16) resulting from the topological charge of the background gauge field exactly cancels the non-conservation due to the finite bare mass.

In case of a quantized gauge field, on a fluctuating background, the fermion bilinears and the topological density are renormalized and mixed with each other. The form of the anomaly equation becomes a matter of convention (see, for instance, [27]).

### 4 Numerical simulations in a 2d U(1) model

Before doing numerical simulations in the model for anomalous fermion number violation discussed in the previous section, it is useful to study a corresponding U(1) toy model in two dimensions, which has often been studied in this context (see e. g. [28]).

The lattice action depending on the compact U(1) gauge field $U_{x\mu} = \exp(iA_\mu(x))$, ($\mu = 1, 2$) and, for simplicity, fixed length Higgs scalar field $\phi(x)$, $|\phi(x)| = 1$ can be written as

$$S = \beta \sum_x \sum_{\mu=1,\nu=2} [1 - \cos(F_{\mu\nu}(x))] - 2\kappa \sum_x \sum_{\mu=1}^2 \phi^*(x + \hat{\mu})U_{x\mu}\phi(x),$$

where the lattice field strength is defined for $\mu = 1, \nu = 2$ as

$$F_{\mu\nu}(x) = A_\nu(x + \hat{\mu}) - A_\nu(x) - A_\mu(x + \hat{\nu}) + A_\mu(x).$$

Real angular variables $-\pi < \theta_{x\mu} \leq \pi$ on the links can be introduced by

$$U_{x\mu} \equiv \exp(i\theta_{x\mu}) \quad \theta_{x\mu} = A_\mu(x) - 2\pi \cdot NINT(A_\mu(x)/2\pi).$$

Fermions in this two dimensional model are introduced in the mirror fermion basis $(\psi, \chi)$, according to (14), but will not be explicitly considered here.

#### 4.1 Topological charge

The topological charge of U(1) lattice gauge field configurations can be defined as a sum over the contributions of plaquettes. The basic assumption is the existence of a piecewise continuous interpolation of the gauge field [29, 30]. The gauge invariant topological charge on the torus corresponding to periodic boundary conditions is obtained either from the “transition functions” [29] or from the “sections” [31] of this interpolated gauge field.

Introducing the plaquette angle $-\pi < \theta_{x\mu\nu} \leq \pi$ by

$$\theta_{x\mu\nu} \equiv \theta_{x\mu} + \theta_{x+\hat{\mu},\nu} - \theta_{x\nu} - \theta_{x+\hat{\nu},\mu} - 2\pi n_{x\mu\nu},$$

one can show that the topological charge $Q$ is given by

$$Q = \sum_x n_{x12}.$$
4.2 Gauge field kinks

It follows from the definition in (25) that the integers \( n_{x12} \) can have the values 0, ±1, ±2. The plaquettes with \( n_{x12} \neq 0 \) can be imagined to carry “Dirac strings” or “gauge field kinks” \( [32] \). These local contributions to the topological charge play an important rôle in the dynamics of lattice U(1) gauge fields and has been intensively studied in the literature. For recent references in higher dimensions see, for instance, \([33, 34]\).

In the scalar-gauge model defined by the lattice action (22), where the pure gauge part corresponds to the Wilson action with compact U(1) variables, in a typical configuration there are a lot of gauge field kinks. As an example, a typical configuration at the large \( \beta, \kappa \) values \( \beta = 8.0, \kappa = 1.0 \) is shown in fig. 3. As one can see, a considerable part of the plaquettes has a kink (with positive \( n_{12} \)) or antikink (negative \( n_{12} \)).

It is important to keep in mind that the individual terms in the sum over plaquettes (26) are not gauge invariant, in contrast to the total topological charge \( Q \). Namely, performing large gauge transformations on the two ends of a link can create or annihilate a kink-antikink pair on the two plaquettes which both contain the link.

In order to decrease the number of gauge kinks one can fix the gauge in some way. One possibility is to define the “minimal gauge” by minimizing the sum of squares (or absolute value) of the link angles \( [35] \):

\[
\sum_x \sum_{\mu=1}^2 \theta_{x\mu}^2 .
\]  
(27)

This removes some part of the kinks in fig. 3, but still a lot of them remains.

The rôle of the gauge kinks can be important for quantities sensitive to the topological charge, or to the smoothness of the gauge field. Therefore it is instructive to study the model with some modified actions suppressing kinks. One possibility is to suppress the kinks by introducing a “chemical potential” for them \( [36] \), which can in fact also completely remove them from the space of allowed configurations.

Another modification of the gauge field action is suggested by the distribution of link angles in the minimal gauge. For small gauge couplings \( g^2 \equiv \beta^{-1} \ll 1 \), where most of the link variables are concentrated near \( \theta_{x\mu} = 0 \), there are, namely, also some secondary maxima at \( \theta_{x\mu} = \pm \pi/2 \). This is shown on the example of a configuration at \( \beta = 8.0, \kappa = 0.8 \) in fig. 4. The secondary peaks are due to the fact that the Wilson U(1) gauge action is periodic for a shift of the plaquette angles by \( 4 \cdot \pi/2 = 2\pi \). Therefore, one way to push in the \( \beta \to \infty \) limit all link angles to zero is to introduce a modified gauge field action like

\[
S_4 = 16\beta_4 \sum_x \sum_{\mu=1,\nu=2} \left\{ 1 - \cos[(\theta_{x\mu} + \theta_{x+\hat{\mu},\nu} - \theta_{x\nu} - \theta_{x+\hat{\nu},\mu})/4] \right\} .
\]  
(28)

The factor 16 in front of \( \beta_4 \) is introduced in order to have in the continuum limit of the action the same normalization as for \( \beta \).

The lattice action in (28) is not exactly gauge invariant because large gauge transformations of the links can cause a jump by \( 2\pi \) in the link angles, due to the limitation in (24). This kind
Figure 3: The plaquette contributions to the topological charge $q \equiv n_{12}$ on $64^2$ lattice at $\beta = 8.0$, $\kappa = 1.0$. 
of gauge non-invariance is, however, natural since we want to suppress exactly such large gauge transformations creating kink-antikink pairs. $S_4$ remains gauge invariant if both link angles and gauge transformation angles are kept small, for instance, less than $\pi/3$ in absolute value. Therefore, the breaking of gauge invariance does not influence weak and smooth fields relevant in perturbation theory.

The modified gauge action $S_4$ suppressing kinks is by no means unique. One could, for instance, also use the “non-compact” formulation with the gauge field action

$$S_{NC} = \frac{\beta}{2} \sum_x \sum_{\mu=1,\nu=2} F_{\mu\nu}(x)F_{\mu\nu}(x).$$

We want, however, to stay close to the physically interesting model with SU(2) gauge field in four dimensions, where usually the compact formulation is used. The generalization of the gauge action $S_4$ to SU(2) is possible and stays closer to the Wilson action than the non-compact formulation.

The effect of reducing the density of gauge kinks by the modified gauge action $S_4$ is dramatic. This can be seen on fig. 5.
Figure 5: The same as fig. 3 for the modified gauge action $S_4$ in minimal gauge at $\beta_4 = 2.0$, $\kappa = 0.8$. 
4.3 Topological susceptibility and C-S numbers

An interesting physical quantity is the topological susceptibility

$$\chi_Q \equiv \langle Q^2 \rangle - \langle Q \rangle^2 .$$

(30)

This is strongly influenced by the density of gauge kinks, as is shown by figs. 6 and 7. Although the equality of bare couplings $\beta$ and $\beta_4$ does not mean that, for instance, the correlation lengths are the same, it is clear that $\chi_Q$ gets substantially reduced by the suppression of gauge kinks. Even if $\beta$ is increased to $\beta = 8$, $\chi_Q$ is only reduced by roughly a factor of 10, not by a factor more than 1000 as between fig. 6 and 7.

Another important topological feature is the distribution of Chern-Simons number. This can be defined on a $L_1 \cdot L_2$ lattice in the temporal gauge with $U_{x_1,x_2,2} = 1$, $x_2 = 0, 1, \ldots, L_2 - 2$ by the sum of link angles

$$N_{CS}(x_2) \equiv \frac{1}{2\pi} \sum_{x_1=0}^{L_1-1} \theta_{x_1,x_2,1} .$$

(31)

As an example, the distribution is shown in fig. 8 and 9.

The time dependence (here $x_2$ dependence) of $N_{CS}(x_2)$ can be characterized by the expectation value of $[N_{CS}(x_2) - N_{CS}(x_2 - 1)]^2$, which is illustrated by fig. 10.

In summary, it seems that the density of gauge field kinks has a strong influence on the topological features in the two dimensional U(1) Higgs model. In order to clarify the physical significance of this, detailed studies has been started recently [37].
**topological susceptibility at beta_4=2.**

Figure 7: The same as fig. 6 with the modified action.

**C-S numbers at beta=8. kappa=.8**

Figure 8: The distribution of Chern-Simons number $N_{CS}$ on $64^2$ lattice at $\beta = 8.0$, $\kappa = 0.8$. 
Figure 9: The same as fig. 8 in larger bins.

Figure 10: The expectation value of the squared time difference of the Chern-Simons number on $64^2$ lattice at $\beta = 2.0$ as a function of $\kappa$. 
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