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Electroosmotic flow of generalized Burgers’ fluid with Caputo–Fabrizio derivatives through a vertical annulus with heat transfer

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Abstract Applying the electric field to a fluid confined between capillary surfaces is the most recent technique used for studying the fluid movement. This technique is known as electroosmotic flow (EOF). The problem of the Caputo–Fabrizio time-fractional derivative of the electroosmotic generalized Burgers’ fluid through a vertical annulus with free convection heat transfer has been investigated. The annulus walls kept at constant values of a zeta potential. The dimensionless governing equations have been solved with the help of the Laplace and finite Hankel transforms. The inversion of Laplace for the transformed functions is calculated numerically. The essential features of the electroosmotic flow (EOF) and the related thermal characteristics are clearly illustrated by the variations in the velocity, the flow rate, the temperature and the Nusselt number. Moreover, comparisons with other non-Newtonian fluids have been discussed. It was found that the presence of the electric double layer (EDL) accelerates the fluid near the walls of the annulus, while in the core region, the reverse flow is noticed.

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1. Introduction

One of the fundamental electrokinetic phenomena is the electroosmotic flow (EOF), which is actuated by an applied external electric field in the region of a charged surface. Electroosmotic flow (EOF) in microchannels (microtubes) undergoes to the electric double layer (EDL), which is usually produced due to the interaction between the charged wall surface and ionized solution. The electroosmotic flow represents non-mechanical micropumps, where there are no movable parts. The non-mechanical micropumps transform the non-mechanical energy into kinetic momentum that drives the fluid contained within the microchannels (microtubes). The advantages of the electroosmotic flow are like efficient reconfigurability with electrical circuitry and simple design requirement brought it to the light of the microfluidic devices applications ([1,2]). The electroosmotic flow has a variety of applications in microfluidic devices and their applications in biology, medical science, analytical chemistry and micro-electro-mechanical systems ([3,4]). Due to the wide applications of the electroosmotic
flow, it has become one of the most appealing fields. Numerous numerical, theoretical and experimental problems of electroosmotic flows in nano/micro-shape have been discussed ([5–8]) and references therein. Due to the non-Newtonian fluids, have wide engineering applications of rheological problems in chemical, geophysics, cosmetic, biological science and petroleum industries compared with the Newtonian fluid. There have been growing number of researchers studied this class of fluids in the literature ([9–15] and the references therein). The generalized Burgers’ fluid can recover the complex rheological features more than other models ([16–20]). The topic of the electroosmotic flow of the non-Newtonian fluids has rapidly become one of the main topics of fundamental interest because of wide spectrum applications. The electroosmotic flow of a power-law fluid in parallel plate microchannels has been investigated analytically by Das and Chakraborty [21]. Further attempts have been made to expand the basic knowledge of these flows either numerically or analytically. A numerical solution of the Phan-Thien–Tanner equation for the non-trivial effects of viscous dissipation. There are many useful studies on the heat-transfer characteristics of such flow. Al-Mdallal and Mahfouz [38] discussed by Elnajjar et al. [37]. They found a unique solution over a shrinking permeable long tube with heat transfer was discussed by Elnejjar et al. [37]. They found a unique solution at a specific critical unsteadiness parameter and dual for both the fluid and heat transfer fields. Al-Mdallal and Mahfouz [38] discussed the problem of heat transfer of a cylinder performing circular motion in a uniform stream. Their findings showed that, the heat transfer rate increases appreciably in the high range of the relevant parameters.

All the above studies regarding the EOF in microchannels/microtubes are focused on integer-order derivatives of the mathematical model of fluids. The implementations of fractional-order may be useful in improving some of the models defined by a local derivative. Mathematical models with fractional derivatives are used in continuum mechanics and bioengineering problems to give new interprets for the non-linear problems. The concept of Caputo [39] is most commonly used for time-fractional derivative in continuum mechanics. To correct the serious disadvantages in the above concept, Caputo and Fabrizio [40] introduced a new definition of fractional-order
fractional derivative without singular kernel. The new model can be described as behaviors of electromagnetic systems, classical viscoelastic materials, thermal media, etc. Many investigators used the new definition of Caputo–Fabrizio in their studies. Abduhammed et al. [41,42] introduced analytical studies to investigate the electroosmotic flow of Maxwell (Oldroyd-B) fluids with Caputo–Fabrizio time-fractional derivatives through a circular tube. They concluded that the parameter of the fractional derivatives affected significantly on the fluid motion and heat transfer. Saqib et al. [43] presented a mathematical model for the prediction of dynamic response of the Jeffrey fluid over a vertical plate. The governing equation of the fluid are formulated with Caputo-Fabrizio fractional derivatives. They found that, fractional Jeffrey fluid moves faster than ordinary fluid for small values of time, but this effect is opposite in the case of larger values of time. Ali et al. [44] have studied the Caputo-Fabrizio and Atangana-Baleanu fractional models for generalized Jeffrey nanofluid in a vertical magnetic field. They solved their model by using Laplace transform technique and got the influence of the parameter of the fractional derivatives on the fluid motion. Recently, there have been new mathematical models applied the concept of Caputo–Fabrizio fractional derivative in fever, coronavirus and Pine Wilt diseases ([45–48]). The discrete version of the Caputo–Fabrizio derivative was discussed by Al-Refai and Abdeljawad ([50]). They have found an estimate of the Caputo fractional derivative of non-singular kernel at its extreme points. Based on the aforementioned literature, the current investigation aims to discuss the electroosmotic flow with thermal characteristics of generalized Burgers’ fluid with Caputo–Fabrizio derivatives through a vertical micro-annulus. The solution of the mathematical model will be achieved by means of Laplace transform and finite Hankel techniques. Due to the difficulty to get the inversion of Laplace of the transformed functions in the closed-form, we will utilize the numerical inversion algorithm established in [51].

2. Problem formulation and mathematical model

2.1. Problem description

Consider the unsteady electroosmotic flow of an electrolyte solution of generalized Burgers’ fluid through an annulus with radius \( R_1 < r < R_2 \). The walls of the annulus are keeping at the constant values of a zeta potential \( \psi_0 \). The fluid is subjected to the influence of an electric field \( E' = (0, -E, E) \) and a uniform magnetic field vertically upward with strength \( B' = (B_0, 0, 0) \). Fig. 1, delineates the detailed of the problem. For the present investigation, a few assumptions are taken in determining the mathematical model as follows:

1. The flow is axisymmetric and the velocity in the direction of the radial coordinate \( r' \) is zero.
2. The approximation of the Debye-Hückel is applied, which states that the electrical potential is physically small compared with the thermal energy of the charged species. Therefore, we have a linearized charge density \( \varrho_e = \frac{-2\lambda m r' \psi'}{K_T^e} \).

3. In the energy equation, the viscous dissipation ignored compared with the Joule heating.

2.2. The concept of Caputo–Fabrizio (CF) fractional derivative

The authors in [40] introduced a new concept of the derivative with fractional order without singular kernel as follows:

\[
\text{CFD}^\beta g(t) = \frac{R'(\beta)}{1 - \beta} \int_0^t g'(\tau) \exp\left[\frac{-\beta(t - \tau)}{1 - \beta}\right] d\tau, \tag{1}
\]

where \( \beta \in [0, 1), R'(\beta) \) represents a normalization function such that \( R'(0) = R'(1) = 1 \). If \( m \geq 1 \) and \( \beta \in [0, 1) \) the Caputo–Fabrizio \( \text{CFD}^\beta g(t) \) of order \( (\beta + m) \) is defined by:

\[
\text{CFD}^\beta (\text{CFD}^m g(t)) = \text{CFD}^{\beta + m} g(t). \tag{2}
\]

In fact the definition given in (2) is equivalent to that presented in ([52]) where the Lyapunov type inequalities were studied in the frame of fractional operators with non-singular exponential kernels. The Laplace transform of the Caputo–Fabrizio \( \text{CFD}^{\beta + m} g(t) \) is defined as follows [39].

\[
\mathcal{L}[\text{CFD}^{\beta + m} g(t)] = \left( \frac{1}{1 - \beta} \right) \mathcal{L}[g^{(m+1)}] \exp\left[\frac{-\beta t}{1 - \beta}\right] = s^{m+1} \mathcal{L}[g(t)] - s^m g(0) - s^{m-1} g'(0) - \ldots - g^{(m)}(0), \tag{3}
\]

For further details and properties see ([45]).

2.3. Mathematical model

Using the above hypotheses, the connection between \( \varrho_e \) and \( \psi' \), can be expressed by the Poisson equation:
where \( k' = z_{el} \left( \frac{\epsilon_{w}}{\epsilon_{e}} \right)^{1/2} \). The continuity, momentum and energy equations reduce to the following forms:

\[
\frac{\partial \rho w'}{\partial z} = 0, \\
\frac{\rho}{\rho_c} \frac{\partial \rho w'}{\partial z} - \sigma B_0 w' + \sigma B_0 E_0 = \kappa' \epsilon E_0 \psi' + \rho \varepsilon g(T - T_0) + \frac{\partial S_{w'}^{\phi}}{\partial z} + \frac{1}{r} \rho' S_{r'}^{\phi},
\]

\[
\frac{\rho c_p}{\partial T} = K \left( \frac{\partial^2 T}{\partial z^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right) + \sigma \left( E_0^2 + E_0^2 \right).
\]

It ought to be referenced here that feeble convective effect on ionic species transport has been applied in the Joule heating generation. The constitutive equation for generalized Burgers’ fluid, satisfies:

\[
\left( 1 + \frac{\partial}{\partial r} \right) w' + \frac{\partial^2}{\partial z^2} S_{w'}^{\phi} = \eta_0 \left( 1 + \frac{\partial}{\partial r} \right) \frac{\partial w'}{\partial r} + \frac{1}{r} \frac{\partial w'}{\partial z},
\]

Eliminating \( S_{w'}^{\phi} \) between Eqs. (6) and (8) leads to:

\[
\left( 1 + \frac{\partial}{\partial r} \right) \frac{\partial w'}{\partial r} + \frac{\partial^2}{\partial z^2} \left( \frac{\partial^2 w'}{\partial r^2} + \frac{1}{r} \frac{\partial w'}{\partial z} \right).
\]

The initial and boundary conditions are given by:

\[
w'(r,0) = T'(r,0) = \frac{\partial w'}{\partial r}(r,0) = \frac{\partial T'}{\partial r}(r,0) = 0, S_{w'}^{\phi}(r,0) = 0,
\]

\[
\frac{\partial S_{w'}^{\phi}}{\partial r}(r,0) = \frac{\partial^2 S_{w'}^{\phi}}{\partial r^2}(r,0) = 0 \psi'(R_1) = \psi_w, w'(R_1,t') = 0, T'(R_1,t') = T_1.
\]

The volumetric flow rate is defined by:

\[
Q(t') = 2 \pi \int_{r_1}^{r_2} r' w'(r',t') dr'.
\]

Introducing the following non-dimensional variables:

\[
w = \frac{w}{w_0}, \quad \psi = \frac{\psi}{\psi_0}, \quad r = \frac{r}{R_1}, \quad \Gamma = \frac{\eta_0}{\rho R_2^2}, \quad (t', \zeta_1, \zeta_2), \quad \psi'(R_1) = \psi_w, w'(R_1,t) = 0, T'(R_1,t') = T_1,
\]

\[
\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} - \kappa^2 \psi = 0,
\]

\[
Pr \frac{\partial \theta}{\partial r} = \frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \frac{\partial \theta}{\partial r} + S_\phi + S_e.
\]

The Poisson–Boltzmann Eq. (13) can be solved with help of the boundary conditions Eq. (16) to get the electrical double layer potential distribution in the form:

\[
\psi = c_1 I_0(\kappa r) + c_2 K_0(\kappa r),
\]

where

\[
c_1 = \frac{K_0(k)}{I_0(\kappa k)} - \frac{K_0(ak)}{I_0(\kappa a k)},
\]

\[
c_2 = \frac{I_0(k)K_0(ak)}{I_0(\kappa a k)} - \frac{I_0(k)K_0(ak)}{I_0(\kappa k)},
\]

where \( I_0 \) and \( K_0 \) are the modified Bessel functions. Using the initial conditions, Eq. (16) and the Caputo–Fabrizio definition along with Laplace transform, Eqs. (18 and 19) will be:
The transformed boundary conditions are:
\[ \tilde{w}(a, s) = 0, \quad \tilde{\theta}(a, s) = 0 \]
\[ \tilde{w}(1, s) = 0, \quad \tilde{\theta}(1, t) = \frac{1}{t}. \]

To acquire the solution of Eq. (22), we will use the finite Hankel transform method with respect to \( r \), and is defined as follows:
\[ \tilde{w} = \int_0^1 r \tilde{w}(r, s) \phi(\mu r) dr, \]

\[ \tilde{w} = \frac{1}{2 \pi} \int_0^1 \frac{\tilde{w}(\mu r)}{\mu^2} \phi(\mu r) d \mu. \]

Fig. 2  Special cases.

Fig. 3  Variation of the velocity \( w \) (Panel (a)) and flow rate \( Q \) (Panel (b)) for different values of Debye–Hückel parameter \( \kappa \) at \( \alpha = 0.5, \beta = 0.4, \zeta_1 = 0.1, \zeta_2 = 0.1, \gamma_1 = 0.1, \gamma_2 = 0.1, H = 4, t = 0.5, Gr = 2, \Gamma = 0.5, S_\phi = 0.5, S_r = 0.5, \alpha = 0.1. \)
where \( \varphi(\beta_m, r) = Y_0(\alpha\beta_m)J_0(\beta_m r) - J_0(\alpha\beta_m)Y_0(\beta_m r) \) and \( \beta_m(m = 1, 2, \ldots) \) represents the positive root of \( \varphi(\beta_m, 1) = 0 \). The inverse of the finite Hankel transform is

\[
\hat{w}(r, s) = \frac{\pi^2}{2} \sum_{m=1}^{\infty} \frac{\beta_m^2 J_0'(\beta_m) \hat{w}(\beta_m, s)}{J_0'(\beta_m) - J_0'(\beta_m)}. \tag{26}
\]

Applying the finite Hankel transform to Eq. (22) and using the boundary conditions Eq. (24), we get:

\[
(s + H^2) \left[ 1 + \frac{s \xi_1}{s + 2(1 - s)} + \frac{s \mu}{s + 2(1 - s)} \right] \hat{w} - \beta_m^2 \left[ 1 + \frac{\xi_1}{s + \beta(1 - s)} + \frac{\mu}{s + 2\beta(1 - s)} \right] \hat{w} = \int_0^1 r \left[ Gr \left[ 1 + \frac{s \xi_1}{s + \beta(1 - s)} + \frac{s \chi_1}{s + 2\beta(1 - s)} \right] \psi \right] + \kappa^2 \psi \frac{s}{s} \frac{(\Gamma - HS)}{s} \varphi(\beta_m, r) dr. \tag{27}
\]

Fig. 4 Variation of the velocity \( w \) (Panel (a)) and flow rate \( Q \) (Panel (b)) for different values of the fractional-order parameter \( \alpha \) at \( \beta = 0.1, \xi_1 = 0.1, \xi_2 = 0.1, \chi_1 = 0.1, \chi_2 = 0.1, H = 0, \kappa = 8, Gr = 0, \Gamma = 0.5, \alpha = 0.1 \).

Fig. 5 Variation of the velocity \( w \) (Panel (a)) and flow rate \( Q \) (Panel (b)) for different values of the fractional-order parameter \( \beta \) at \( \alpha = 0.5, \xi_1 = 0.5, \xi_2 = 0.5, \chi_1 = 0.3, \chi_2 = 0.4, H = 0.5, \kappa = 5, Gr = 0, \Gamma = 0.5, \alpha = 0.1 \).
Simplifying Eq. (27) and applying the inverse of the finite Hankel transform, we obtain:

\[
\tilde{w}(r,s) = \frac{\pi^2}{2} \sum_{m=1}^{\infty} \frac{\beta_m^2 J_0(\beta_m r)}{\left(\tilde{J}_0^{(0)}(\beta_m r) - \tilde{J}_0^{(1)}(\beta_m r)\right)} \left\{s + \frac{s \zeta_1}{s + 2 \zeta_1} + \frac{s \zeta_2}{s + 2 \zeta_2}\right\} \times \int_0^1 r \left\{G_1 \left[1 + \frac{s \zeta_1}{s + 2 \zeta_1} + \frac{s \zeta_2}{s + 2 \zeta_2}\right] + \frac{s \zeta_2}{s + 2 \zeta_2} \varphi(\beta_m r) dr \right\}
\]

The solution of the heat Eq. (23) with the help of the boundary conditions Eq. (24), we get:

\[
\frac{\partial}{\partial r}(r, s) = c_1 I_0(\sqrt{Ar}) + c_2 K_0(\sqrt{Ar}) - \frac{\beta}{\sqrt{r}},
\]

where

\[
c_1 = -\frac{\delta_0(\sqrt{\zeta}) - \delta_0(\zeta) + \delta_0(\sqrt{\zeta})}{\delta_0(\sqrt{\zeta}) - \delta_0(\zeta) + \delta_0(\sqrt{\zeta})}
\]

\[
c_2 = -\frac{\delta_0(\sqrt{\zeta}) - \delta_0(\zeta) + \delta_0(\sqrt{\zeta})}{\delta_0(\sqrt{\zeta}) - \delta_0(\zeta) + \delta_0(\sqrt{\zeta})}
\]

Fig. 6 Variation of the velocity \( w \) (Panel (a)) and flow rate \( Q \) (Panel (b)) for different values of \( \zeta_1, \zeta_2 \) at \( \alpha = 0.5, \beta = 0.5, \chi_1 = 0.1, \chi_2 = 0.3, H = 0.5, \kappa = 5, Gr = 0, \Gamma = 0, \alpha = 0.1 \).

Fig. 7 Variation of the velocity \( w \) (Panel (a)) and flow rate \( Q \) (Panel (b)) for different values of \( \chi_1, \chi_2 \) at \( \alpha = 0.5, \beta = 0.5, \zeta_1 = 0.5, \zeta_2 = 0.5, H = 2, \kappa = 5, Gr = 0, \Gamma = 0, \alpha = 0.1 \).
The instantaneous volumetric flow rate will be:

\[ Q_s(s) = 2\pi \int_{a}^{b} r \bar{v}(r, s) dr. \] (30)

On account of the difficulty to obtain the inverse of Laplace of the transformed functions \( \bar{w}, \bar{\theta} \) and \( \bar{Q} \) in the closed-form, we will utilize the numerical inversion algorithm established in [51] (see Appendix A).

To validate the solutions by the numerical inversion algorithm method, some limiting and special cases will be additionally discussed in the next section.

4. Graphical results and discussion

Throughout the previous section, the electroosmotic velocity, flow rate, and temperature were obtained semi analytically. The flow is driven by an amalgamation of the exerted electrical, magnetic forces, the applied pressure gradient, convection, and the electroosmotic forces. In the current section, we will discuss the influence of the relevant parameters on the velocity, the flow rate, and the heat transfer. Thus it is important for us to address the permissible ranges of these related parameters. We take the values of the relevant parameters as follows

Fig. 8 Variation of the velocity \( w \) (Panel (a)) and flow rate \( Q \) (Panel (b)) versus \( H \) at \( \chi_1 = 0.3, \chi_2 = 0.4, z = 0.5, \beta = 0.5, \zeta_1 = 0.4, \zeta_2 = 0.4, Gr = 0, \kappa = 5, \Gamma = 0, t = 2. \)

Fig. 9 Variation of the velocity \( w \) (Panel (a)) and flow rate \( Q \) (Panel (b)) for different values of \( \Gamma \) at \( z = 0.5, \beta = 0.4, \zeta_1 = 0.1, \zeta_2 = 0.1, \chi_1 = 0.1, \chi_2 = 0.1, H = 4, Gr = 0, \kappa = 2, a = 0.1. \)
a = 0 (tube) up to 0.2, H = 0 (absence of the magnetic field) up to 4, the order of parameter $S$ can be achieved, $O(S) \sim 1 - 100$ and $Gr = 0$ (absence of the free convection force) up to 6.

Before continuing further with the investigation of the relevant outcomes, the validity of the present results in some limiting case is established by comparing the values of the electroosmotic velocity for the tube. Fig. 2(a) shows that our results are similar to those obtained by [53] for the generalized Maxwell fluids at $\xi_2 = \chi_1 = \chi_2 = 0$, $Gr = 0$, $\Gamma = 0$, $S = 0$, $a = 0$ in a tube. Moreover, Fig. 2(b) shows that our results are similar to those obtained by [36] for electroosmotic Newtonian fluid in a tube.

4.1. Velocity and flow rate

In this subsection the velocity and flow rate of the electroosmotic flow of generalized Burgers’ fluid through a vertical annulus, were discussed by means of a fractional model based on the Caputo–Fabrizio time-fractional derivative.

Fig. 3 illustrates the effect of the electrokinetic width $\kappa$ on the velocity $w$ (Panel (a)) and flow rate $Q$ (Panel (b)). It is clear that for high values of $\kappa$ (presence of a thin EDL near the walls of the annulus) the velocity accelerates near the walls of the annulus, while in the core region, the reverse flow is noticed. It is also noticed that the flow rate increases with time and goes to steady-state after a short time. Moreover, the variation of the flow rate is noticeable when there is a change in $\kappa$. The effect of the fractional-order $\alpha$ on the velocity $w$ and the flow rate $Q$ delineated in Fig. 4. From Fig. 4(a) it is noticed that the velocity is affected by increasing the fractional-order $\alpha$ and the fluid oscillates after the beginning but for a long time, the velocity oscillation is dampened. The same behavior of the flow rate is noticed as seen from Fig. 4(b). Fig. 5 illustrates the influence of the fractional-order $\beta$ on the velocity $w$ and the flow rate $Q$. It is clear that both of the velocity and the flow rate go faster to the steady-state case by elevating the fractional-order $\beta$. The figure also shows that the period required for the flow to increase with increasing the relaxation time parameter. The figure also shows that for higher values of the retardation time parameter, more time is required to achieve steady-state flow. Physically, the elevation of the retardation time boosts the viscoelastic effect of the fluid, so it needs additional time.

Fig. 10 Variation of the velocity $w$ for different values of $Gr$ at $\Gamma = 0.5$, $\alpha = 0.4$, $\xi_1 = 0.1$, $\xi_2 = 0.1$, $\chi_1 = 0.1$, $\chi_2 = 0.1$, $H = 4$, $\kappa = 2$, $a = 0.1$.

Fig. 11 Variation of the temperature $\theta$ for different values of the fractional-order parameter $\alpha$ at $S_\phi = 2$, $S_r = 2$, $Pr = 6.2$, $a = 0.1$. 
for the stress response to deformation, which outcomes in a reduction in its unsteady flow velocity. The effects of the material parameters $v_1$ and $v_2$ on the velocity $w$ and the flow rate $Q$ are shown in Figs. 6 and 7. It is clear that the velocity and the flow rate of the generalized Burgers’ fluid is smaller than Burgers’ fluid ($v_2 = 0$) and Oldroyd-B fluid ($v_1 = v_2 = 0$), for a short period of time. After this period the behavior of the electroosmotic flow of the generalized Burgers’ fluid and its special cases is similar. Fig. 8 shows the variation of $w$ and $Q$ with $H$ for various values of $a$ at fixed values of the pertinent parameters. The figure shows that the higher velocity (flow rate) achieves at small values of the Hartmann number $H$. This behavior is due to the Lorentz force which comprises of two parts; the aiding force and the inhibiting force. For the small values of $H$, the aiding magnetic force dominates on the impeding force. But for higher values of $H$, the impeding force prevail. The figure also shows that the velocity and the flow rate through the annulus is higher than the tube. Fig. 9 displays the velocity and flow rate profiles for various values of the pressure gradient $C$. When $C = 0$ represents pure electroosmotic flow, and the cases $C > 0$ and $C < 0$ are correspond to flows with adverse and favorable pressure gradients, respectively. Fig. 10 depicts the variation of the Grashof number $Gr$ on the velocity for to different profiles. It is evident that the free convection force ($Gr > 0$) can boost the electroosmotic flow after a short time from start-up flow. Furthermore, this reinforcement appears in the core region.

4.2. Heat characteristics

In this subsection, we will discuss the influences of the fractional-order parameter ($\alpha$) and the Joule heating parameters ($S_0$ and $S_z$) on the heat transfer in Figs. 11 and 12 and on the Nusselt number in Figs. 13 and 14. It is clear from Fig. 11, that the heat transfer enhances by elevating the fractional-order parameter. Moreover, at high values of the fractional-order parameter, the heat transfer goes fast to the steady-state. Both of the Joule heating parameters ($S_0$ and $S_z$) dissipate the heat transfer as shown through Fig. 12.

Fig. 13 displays the variation of the Nusselt number $Nu$ for different values of $\alpha$ at $S_0 = 2, S_z = 2, Pr = 6.2, \alpha = 0.1$.

Fig. 12 Variation of the temperature $\theta$ for different values $S_0$& $S_z$ at $z = 0.6, Pr = 6.2, \alpha = 0.1$. 

Fig. 13 Variation of the Nusselt number $Nu$ for different values of $\alpha$ at $S_0 = 2, S_z = 2, Pr = 6.2, \alpha = 0.1$. 

Fig. 10 Variation of the temperature $\theta$ for different values $S_0$& $S_z$ at $z = 0.6, Pr = 6.2, \alpha = 0.1$. 
5. Concluding remarks

A mathematical model has been developed to discuss the Caputo–Fabrizio time-fractional derivative of the electroosmotic generalized Burgers’ fluid flow through a vertical annulus. Moreover, the heat transfer and the Nusselt number have been discussed also. The dimensionless governing equations have been solved with the help of the Laplace and finite Hankel transforms. The inversion of Laplace for the transformed functions calculated numerically. The main results obtained, are as follows:

- The velocity accelerates near the walls of the annulus due to the presence of a thin EDL, while in the core region, the reverse flow is noticed.
- The velocity (flow rate) affected by increasing the fractional-order and the fluid oscillates after the beginning but for a long time, the velocity oscillation is dampened.
- The required period to reach the steady-state flow increases with increasing the relaxation time parameter.
- For higher values of the retardation time parameter, more time is required to achieve steady-state flow.
- The free convection force can boost the electroosmotic flow after a short time from start-up flow. Furthermore, this reinforcement appears in the core region.
- At high values of the fractional-order parameter, the heat transfer goes fast to the steady-state.
- The Nusselt number rises by elevating the fractional-order parameter near the inner tube until we reach a critical point in the annulus the behavior is reversed.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. To find the inverse of Laplace transform functions \( \Psi(s) \) numerically. The inverse can be approximated as follows [51]:

\[
\Psi(t) \simeq \frac{\exp(\sigma t)}{T} \left\{ \frac{\Psi(\sigma)}{2} + \sum_{m=1}^{\infty} \text{Re} \left[ \Psi(\sigma) + \frac{m\pi i}{T} \cos\left(\frac{m\pi i}{T}\right) \right] \right. \\
\left. - \text{Im} \left[ \Psi(\sigma) + \frac{m\pi i}{T} \sin\left(\frac{m\pi i}{T}\right) \right] \right\},
\]

(A-1)

where \( \sigma = \sigma_0 - \frac{m\pi i}{2T} \) and \( 2T > t_{\max} \). The relative errors is to be no grater than \( E', \) And \( \sigma_0 \) should be chosen as a number slightly larger than \( \text{Max} \{\text{Re}(q)\} \), where \( q \) is a pole of \( \Psi(s) \).

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