ON THE MAXIMAL $G$-COMPACTIFICATION OF PRODUCTS OF TWO $G$-SPACES

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Let $G$ be any Hausdorff topological group and let $\beta_G X$ denote the maximal $G$-compactification of a $G$-Tychonoff space $X$. We prove that if $X$ and $Y$ are two $G$-Tychonoff spaces such that the product $X \times Y$ is pseudocompact, then $\beta_G(X \times Y) = \beta_G X \times \beta_G Y$.

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1. Introduction

Let $G$ be any Hausdorff topological group and let $\beta_G X$ denote the maximal $G$-compactification of a $G$-Tychonoff space $X$ (i.e., a Tychonoff $G$-space possessing a $G$-compactification). Recall that a completely regular Hausdorff topological space is called pseudocompact if every continuous function $f : X \to \mathbb{R}$ is bounded.

In this paper, we prove that if $X$ and $Y$ are two $G$-Tychonoff spaces such that the product $X \times Y$ is pseudocompact, then $\beta_G(X \times Y) = \beta_G X \times \beta_G Y$ (see Theorem 2.2). This is a $G$-equivariant version of the well-known result of Glicksberg [16], which for $G$ a locally compact group was proved earlier by de Vries in [10]. Note that even in the case of a locally compact acting group $G$, our proof is shorter than that of [10, Theorem 4.1]. It follows from Proposition 2.7 that the equality $\beta_G(X \times Y) = \beta_G X \times \beta_G Y$ does not imply, in general, the pseudocompactness of $X \times Y$ even if $X$ and $Y$ both are infinite (cf. [16, Theorem 1]).

Theorem 2.10 says that if a pseudocompact group $G$ acts continuously on a pseudocompact space $X$, then $\beta_G X = \beta X$.

Let us introduce some terminology we will use in the paper.

Throughout the paper, all topological spaces are assumed to be Tychonoff (i.e., completely regular and Hausdorff). The letter “$G$” will always denote a Hausdorff (and hence, completely regular) topological group unless otherwise stated.

For the basic ideas and facts of the theory of $G$-spaces or topological transformation groups, we refer the reader to [5, 7, 11]. However, we recall below some more special notions and facts we need in the paper.
By a $G$-space we mean a Tychonoff space $X$ endowed with a continuous action $G \times X \to X$ of a topological group $G$. A continuous map of $G$-spaces $f : X \to Y$ is called a $G$-map or an equivariant map if $f(gx) = gf(x)$ for all $x \in X$ and $g \in G$.

If $X$ is a $G$-space and $S$ a subset of $X$, then $G(S)$ denotes the $G$-saturation of $S$, that is, $G(S) = \{ gs \mid g \in G, s \in S \}$. In particular, $G(x)$ denotes the $G$-orbit $\{ gx \mid g \in G \}$ of $x$. If $G(S) = S$, then $S$ is said to be an invariant set. The orbit space endowed with the quotient topology is denoted by $X/G$.

For a closed subgroup $H \subset G$, by $G/H$ we will denote the $G$-space of cosets $\{ gH \mid g \in G \}$ under the action induced by left translations.

On any product of $G$-spaces we always consider the diagonal action of $G$.

A $G$-compactification of a $G$-space $X$ is a pair $(b, bX)$, where $b : X \to bX$ is a $G$-homeomorphic embedding into a compact $G$-space $bX$ such that the image $b(X)$ is dense in $bX$. Usually $bX$ alone is a sufficient denotation. We will say that two $G$-compactifications $b_1 X$ and $b_2 X$ are equivalent if there exists a $G$-homeomorphism $f : b_1 X \to b_2 X$ such that $f(b_1(x)) = b_2(x)$ for all $x \in X$. Clearly, the equivalence of $G$-compactifications is an equivalence relation in the class of all $G$-compactifications of $X$. We will identify equivalent $G$-compactifications; any class of equivalent $G$-compactifications will be denoted by the same symbol $bX$, where $bX$ is any $G$-compactification from this equivalence class. An order relation in the family of all $G$-compactifications is defined as follows: $b_1 X \preceq b_2 X$ if there exists a $G$-map $f : b_2 X \to b_1 X$ such that $f b_2 = b_1$. It is easy to see that $b_1 X$ and $b_2 X$ are equivalent if and only if $b_1 X \preceq b_2 X$ and $b_2 X \preceq b_1 X$. We will write $b_1 X = b_2 X$ whenever $b_1 X$ and $b_2 X$ are equivalent $G$-compactifications. In a standard way, one can show that each nonempty family of $G$-compactifications of $X$ has a least upper bound with respect to the order $\preceq$. In particular, if a $G$-space $X$ has a $G$-compactification, then there exists a largest $G$-compactification $\beta G X$ with respect to the order $\preceq$; $\beta G X$ is called the maximal $G$-compactification of $X$.

A continuous real-valued function $f : X \to \mathbb{R}$ on a $G$-space $X$ is said to be $G$-uniform if for any $\varepsilon > 0$, there exists a neighborhood $U$ of the identity element in $G$ such that $| f(gx) - f(x) | < \varepsilon$ for all $x \in X, g \in U$.

A $G$-space $X$ is said to be $G$-Tychonoff if for any closed set $A \subset X$ and any point $x \in X \setminus A$, there exists a $G$-uniform function $f : X \to [0,1]$ such that $f(x) = 0$ and $A \subset f^{-1}(1)$.

It is evident that each continuous function on a compact $G$-space is $G$-uniform, and hence every compact $G$-space is $G$-Tychonoff. Since an invariant subspace of a $G$-Tychonoff space is again $G$-Tychonoff, we see that if a $G$-space has a $G$-compactification, then it is $G$-Tychonoff. The converse is also true (see, e.g., [1, 2]). Thus, a $G$-space is $G$-Tychonoff if and only if it admits a $G$-compactification, and in particular, a maximal $G$-compactification. In [8, 9], it was proved that if $G$ is a locally compact group, then every Tychonoff $G$-space is $G$-Tychonoff. The local compactness of $G$ is essential here (see [18]).

Given a space $Z$, we will denote by $C(Z, \mathbb{R})$ the space of all continuous real-valued functions $f : Z \to \mathbb{R}$ equipped with the compact-open topology (see, e.g., [13, Chapter 12, Section 1]). A subset $K \subset C(Z, \mathbb{R})$ is called equicontinuous at a point $z_0 \in Z$ if for any $\varepsilon > 0$, there exists a neighborhood $O$ of $z_0 \in Z$ such that $| f(z) - f(z_0) | < \varepsilon$ for all $z \in O$.
and $f \in K$. If $K$ is equicontinuous at each point $z_0 \in Z$, then we will say that it is an equicontinuous set.

If additionally $Z$ is a $G$-space for a group $G$, then one can define the following (in general not continuous) action of $G$ on $C(Z, \mathbb{R})$:

$$(g\psi)(z) = \psi(g^{-1}z), \quad \psi \in C(Z, \mathbb{R}), \ z \in Z, \ g \in G. \quad (1.1)$$

If $G$ is locally compact, then this action is continuous, otherwise it may be discontinuous (see, e.g., [7, Chapter I, Section 2.1]). However, the following result is true.

**Lemma 1.1.** Let $Z$ be a $G$-space and $K$ an invariant equicontinuous subset of $C(Z, \mathbb{R})$. Then the closure $\overline{K}$ is also an invariant set and the restriction of the action (1.1) to $G \times \overline{K}$ is continuous.

**Proof.** For every $g \in G$, define the map $g_* : C(Z, \mathbb{R}) \to C(Z, \mathbb{R})$ by setting $g_*(\psi) = g\psi$, where $g\psi$ is defined as in (1.1). First we show that $g_*$ is a continuous map.

Indeed, let $C$ be a compact set in $Z$, $U$ an open set in $\mathbb{R}$, and $M(C, U) = \{ \psi \in C(Z, \mathbb{R}) \mid \psi(C) \subset U \}$. Since all the sets of the form $M(C, U)$ constitute a subbase of the compact-open topology of $C(Z, \mathbb{R})$ and $g_*^{-1}(M(C, U)) = M(g^{-1}C, U)$, we infer that $g_*$ is continuous.

Now choose $\varphi \in \overline{K}$ and $h \in G$ arbitrary. One needs to show that $h\varphi \in \overline{K}$. Let $V$ be a neighborhood of $h\varphi$. Since the above-defined map $h_*$ is continuous, the set $h_*^{-1}(V) = h^{-1}V$ is a neighborhood of $\varphi$. Consequently, $h^{-1}(V) \cap K \neq \emptyset$, which is equivalent to $V \cap hK \neq \emptyset$. But $hK = K$ because $K$ is invariant. Hence, $V \cap K \neq \emptyset$, as required. Thus, the proof that the closure $\overline{K}$ is an invariant subset is complete.

Next we observe that the closure of an equicontinuous set is again equicontinuous [17, Chapter 7, Theorem 14]; so $\overline{K}$ is an equicontinuous invariant subset of $C(Z, \mathbb{R})$.

Now the continuity of the restriction of the action (1.1) to $G \times \overline{K}$ follows easily from the continuity of the evaluation map $\omega : \overline{K} \times Z \to \mathbb{R}$ defined by $\omega(\psi, z) = \psi(z), \ \psi \in \overline{K}, \ z \in Z$ (see, e.g., [17, Chapter 7, Theorem 15]). We refer the reader to [2, Lemma 2] for more details. \[\square\]

We will need this lemma in the proof of Theorem 2.2.

In what follows, we will need also the following two characterizations of the maximal $G$-compactification $\beta_G X$ established in [8] (see also [4]).

**Proposition 1.2.** Let $G$ be a group and $X$ a $G$-Tychonoff space. Then the following hold.

1. Each $G$-map $f : X \to B$ to a compact $G$-space has a unique $G$-extension $F : \beta_G X \to B$.
2. Let $bX$ be a $G$-compactification of $X$ such that every $G$-map $f : X \to B$ to a compact $G$-space has a $G$-extension $F : bX \to B$. Then $bX$ is equivalent to $\beta_G X$.

**Proposition 1.3.** Let $G$ be a group and $X$ a $G$-Tychonoff space. Then the following hold.

1. Each bounded $G$-uniform function $f : X \to \mathbb{R}$ possesses a unique continuous extension $F : \beta_G X \to \mathbb{R}$.
2. If $bX$ is a $G$-compactification such that each bounded $G$-uniform function $f : X \to \mathbb{R}$ admits a continuous extension $F : bX \to \mathbb{R}$, then $bX$ is equivalent to $\beta_G X$. 
2. Main results

**Lemma 2.1.** Let $G$ be any group, $X$ a $G$-space, and $A$ a dense $G$-subset of $X$. Assume that $f : X \to \mathbb{R}$ is a continuous map such that the restriction $f|_A : A \to \mathbb{R}$ is $G$-uniform. Then $f$ is $G$-uniform as well.

**Proof.** Define the map $f' : X \to C(G, \mathbb{R})$ by setting $f'(x)(g) = f(gx)$, $x \in X$, $g \in G$. The continuity of $f'$ follows from the fact that the compact-open topology is proper (see [14, Theorem 3.4.1]).

It is easy to see that the $G$-uniformness of $f$ is just equivalent to the equicontinuity of the image $f'(X)$ in $C(G, \mathbb{R})$. Since the restriction $f|_A$ is $G$-uniform, we infer that the set $f'(A)$ is equicontinuous. But closure of an equicontinuous set is again equicontinuous [17, Chapter 7, Theorem 14]; so $f'(A)$ is equicontinuous. By continuity of $f'$, $f'(X) \subset f'(A)$, yielding that $f'(X)$ is also equicontinuous. Hence, $f$ is $G$-uniform. \[\square\]

**Theorem 2.2.** Let $G$ be any group and let $X$ and $Y$ be $G$-Tychonoff spaces such that $X \times Y$ is pseudocompact. Then $\beta_G(X \times Y) = \beta_GX \times \beta_GY$.

**Proof.** According to Proposition 1.3, it suffices to prove that every bounded $G$-uniform function $f : X \times Y \to \mathbb{R}$ has a continuous extension $F : \beta_GX \times \beta_GY \to \mathbb{R}$.

The idea is first to extend $f$ to a bounded $G$-uniform function $\varphi : \beta_GX \times Y \to \mathbb{R}$, and then to extend in a similar way $\varphi$ to obtain the desired extension $F$. In the nonequivariant case, this is due to Todd [21].

Define the map $f' : X \to C(G \times Y, \mathbb{R})$ by setting

$$f'(x)(g, y) = f(gx, gy) \quad \forall x \in X, (g, y) \in G \times Y. \quad (2.1)$$

Continuity of $f'$ follows from the fact that the compact-open topology is proper (see [13, Theorem 3.1]).

**Claim 2.3.** The image $f'(X)$ is an equicontinuous set in $C(G \times Y, \mathbb{R})$.

**Proof of the claim.** Let $\varepsilon > 0$ and $(g_0, y_0) \in G \times Y$. We have to show that there exist neighborhoods $U$ of $g_0$ and $V$ of $y_0$ such that

$$|f'(x)(g, y) - f'(x)(g_0, y_0)| < \varepsilon \quad \forall x \in X, g \in U, y \in V. \quad (2.2)$$

Since $f$ is a $G$-uniform function, one can choose a neighborhood $U$ of the unity in $G$ such that

$$|f(tx, ty) - f(x, y)| < \frac{\varepsilon}{3} \quad \forall (x, y) \in X \times Y, t \in U. \quad (2.3)$$

Then

$$|f'(x)(g, y) - f'(x)(g_0, y_0)| = |f(gx, gy) - f(g_0x, g_0y_0)| \leq |f(gx, gy) - f(gx, g_0y_0)| + |f(gx, g_0y_0) - f(gx, gy)| \quad (2.4)$$

$$+ |f(gx, gy) - f(g_0x, g_0y_0)|.$$
It follows from (2.3) that for all \( x \in X \) and \( g \in U_{g_0} \), we have
\[
\left| f(gx,gy_0) - f(g_0x,g_0y_0) \right| < \frac{\varepsilon}{3},
\] (2.5)

It is known that the formula
\[
\varphi(y) = \sup_{x \in X} \left| f(x,y) - f(x,g_0y_0) \right|, \quad y \in Y,
\] (2.6)
defines a continuous function \( \varphi : Y \rightarrow \mathbb{R} \) (see [15, Lemma 1.3]).

Since \( \varphi(g_0y_0) = 0 \), we conclude that there is a neighborhood \( V \) of \( g_0y_0 \) in \( Y \) such that \( \varphi(v) < \varepsilon/3 \) for all \( v \in V \). Hence, one has
\[
\left| f(x,v) - f(x,g_0y_0) \right| < \frac{\varepsilon}{3} \quad \forall v \in V, \ x \in X.
\] (2.7)

By continuity of the action on \( Y \), there exist neighborhoods \( O \) and \( W \) of \( g_0 \) and \( y_0 \), respectively, such that \( OW \subset V \) and \( O \subset U_{g_0} \). Consequently, if \( g \in O \) and \( y \in W \), then \( gy \in V \) and \( gy_0 \in V \). Hence, (2.7) yields for all \( x \in X \)
\[
\left| f(gx,gy) - f(gx,g_0y_0) \right| < \frac{\varepsilon}{3}, \quad \left| f(gx,gy) - f(gx,g_0y_0) \right| < \frac{\varepsilon}{3}.
\] (2.8)

Now, (2.4), (2.5), and (2.8) imply for all \( g \in U_{g_0} \) and \( y \in W \) that
\[
\left| f'(x)(g,y) - f'(x)(g_0,y_0) \right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\] (2.9)
as required. Thus, \( f'(X) \) is indeed an equicontinuous set, and the proof of the claim is complete. \( \square \)

Now we continue with the proof of Theorem 2.2. Consider \( G \times Y \) as a \( G \)-space endowed with the action \( h \ast (g,y) = (gh^{-1},hy) \). Then the induced action (1.1) becomes the following action:
\[
(h\psi)(g,y) = \psi(gh,h^{-1}y) \quad \forall \psi \in C(G \times Y, \mathbb{R}), \ g,h \in G, \ y \in Y.
\] (2.10)

We claim that \( f' \) is algebraically equivariant, that is, \( hf'(x) = f'(hx) \) for all \( x \in X \) and \( h \in G \). Indeed, if \( (g,y) \in G \times Y \), then we have
\[
(hf'(x))(g,y) = f'(x)(gh,h^{-1}y) = f(ghx,gy) = f'(hx)(g,y) = (hf'(x))(g,y),
\] (2.11)
which means that \( hf'(x) = f'(hx) \).

Consequently, \( f'(X) \) is an invariant subset of \( C(G \times Y, \mathbb{R}) \). By Lemma 1.1 and the above claim, the closure \( T = \overline{f'(X)} \) also is an invariant subset of \( C(G \times Y, \mathbb{R}) \), and the restriction of the action (2.10) to \( G \times T \) is continuous.

Further, since \( f'(X) \) is a bounded subset of \( C(G \times Y, \mathbb{R}) \), it follows from the Arzelà-Ascoli theorem [13, Theorem 6.4] that \( T \) is compact.

Thus, \( T \) is a compact \( G \)-space. Next, since \( f' : X \rightarrow T \) is a \( G \)-map, by Proposition 1.2, \( f' \) extends to a \( G \)-map \( F' : \beta_{G}X \rightarrow T \subset C(G \times Y, \mathbb{R}) \).
Define the map $\phi : \beta G X \times Y \to \mathbb{R}$ by the formula $\phi(z,y) = F'(z)(e, y)$, where $(z, y) \in \beta G X \times Y$ and $e$ is the unity of $G$. Clearly, $\phi$ is bounded.

Since the evaluation map $\omega : T \times (G \times Y) \to \mathbb{R}$ defined by $\omega(\psi, t) = \psi(t)$, $\psi \in T$, $t \in G \times Y$, is continuous (see, e.g., [17, Chapter 7, Theorem 15]), we infer that $\phi$ is also continuous.

If $(x, y) \in X \times Y$, then $\phi(x, y) = F'(x)(e, y) = f'(x)(e, y) = f(x, y)$, showing that $\phi$ extends $f$. Since $f$ is $G$-uniform, it follows from Lemma 2.1 that $\phi$ is $G$-uniform.

Since the product of a pseudocompact space and a compact space is pseudocompact (see, e.g., [14, Corollary 3.10.27]), $\beta G X \times Y$ is a pseudocompact $G$-space. Consequently, by the same way, one can prove that the bounded $G$-uniform function $\phi : \beta G X \times Y \to \mathbb{R}$ extends to a continuous function $F : \beta G X \times \beta G Y \to \mathbb{R}$, which is the desired extension of $f$. This completes the proof. \qed

**Remark 2.4.** For $G$ a locally compact group, Theorem 2.2 was proved earlier by de Vries in [10] in a different way. If $G$, as a topological space, is a $k$-space (i.e., a quotient image of a locally compact space) and $X$ is a pseudocompact $G$-space, then $\beta G X = \beta X$ (see [10, Lemma 5.5]). Hence, Theorem 2.2 follows in this case directly from the classical result of Glicksberg [16] (this is just [10, Corollary 5.7]).

In the following lemma, we just list two known important cases when the product of two pseudocompact spaces is pseudocompact.

**Lemma 2.5.** The product $X \times Y$ of two spaces is pseudocompact, if at least one of the following conditions is fulfilled:

1. $X$ is a pseudocompact $k$-space and $Y$ is a pseudocompact space;
2. $X$ is a pseudocompact topological group and $Y$ is a pseudocompact space.

**Proof.** For the first statement, see, for example, [14, Theorem 3.10.26]. The second one is proved in [20, Corollary 2.14]. \qed

**Corollary 2.6.** Let $G$ be any group, $H$ a closed subgroup of $G$ such that $G/H$ is compact, and let $X$ be a pseudocompact $G$-Tychonoff space. Then $\beta G(G/H \times X) = G/H \times \beta G X$.

The following simple result shows that the converse of Theorem 2.2 is not true even if $X$ and $Y$ both are infinite (cf. [16, Theorem 1]).

**Proposition 2.7.** Let $G$ be any group, $H$ a closed subgroup of $G$ such that $G/H$ is compact, and let $X$ be a Tychonoff space endowed with the trivial action of $G$. Then $\beta G(G/H \times X) = G/H \times \beta X$.

**Proof.** Evidently, $G/H \times \beta X$ is a $G$-compactification of $G/H \times X$. Hence, according to Proposition 1.3, it suffices to prove that every bounded $G$-uniform function $f : G/H \times X \to \mathbb{R}$ has a continuous extension $F : G/H \times \beta X \to \mathbb{R}$.

Define a function $f' : X \to C(G/H, \mathbb{R})$ by $f'(x)(t) = f(t, x)$, where $(t, x) \in G/H \times X$. Then $f'$ is continuous, and it follows from the $G$-uniformness of $f$ that the image $f'(X)$ is an equicontinuous set in $C(G/H, \mathbb{R})$. Besides, the set $f'(X)(t_0) = \{ f'(x)(t_0) \mid x \in X \}$ is bounded for all $t_0 \in G/H$. Consequently, by the Arzela-Ascoli theorem [13, Theorem 6.4], $f'(X)$ has a compact closure $\overline{f'(X)}$ in $C(G/H, \mathbb{R})$. Hence, $f'$ has a continuous extension
$F': \beta X \to f'(X) \subset C(G/H, \mathbb{R})$. Define $F: G/H \times \beta X \to \mathbb{R}$ by $F(t,z) = f'(z)(t)$. The compactness of $G/H$ insures that $F$ is continuous (see, e.g., [14, Theorem 3.4.3]). It remains only to observe that $F$ extends $f$. \hfill \square

Recall that a $G$-space $X$ is called free if for every $x \in X$, the equality $gx = x$ implies that $g = e$, the unity of $G$.

Below, we will need the following well-known result.

**Lemma 2.8.** Let $G$ be a compact group and $X$ a free $G$-space. Then $(G \times X)/G$ is $G$-homeomorphic to $X$, where $G$ acts on the orbit space $(G \times X)/G$ according to the rule $h \ast G(g,x) = G(gh^{-1},x)$.

**Proof.** The desired $G$-homeomorphism $f: (G \times X)/G \to X$ is defined as follows:

$$f(G(g,x)) = g^{-1}x \quad \forall (g,x) \in G \times X,$$

where $G(g,x)$ stands for the $G$-orbit of the pair $(g,x)$.

It is easy to verify that $f$ is continuous and bijective. The closedness of $f$ follows from that of the map $G \times X \to X$, $(g,x) \mapsto g^{-1}x$ (see [5, Chapter I, Theorem 1.2]). \hfill \square

If the action of $G$ on $X$ is not trivial, then Proposition 2.7 is no longer true. Namely, we have the following proposition.

**Proposition 2.9.** Let $G$ be an infinite, compact, metrizable group and $X$ a finite-dimensional, paracompact, noncompact, free $G$-space. Then $\beta G(G \times X) \neq G \times \beta GX$.

**Proof.** Suppose the contrary, that $\beta G(G \times X) = G \times \beta GX$. Passing to the orbit spaces, we have

$$\frac{G \times \beta GX}{G} = \frac{\beta G(G \times X)}{G}. \quad (2.13)$$

Using the formula $(\beta GZ)/G = \beta (Z/G)$ (see [4, Corollary 4.10]), we get

$$\frac{\beta G(G \times X)}{G} = \beta \left( \frac{G \times X}{G} \right). \quad (2.14)$$

Hence,

$$\frac{G \times \beta GX}{G} = \beta \left( \frac{G \times X}{G} \right). \quad (2.15)$$

It is known that a finite-dimensional, paracompact, free $G$-space has a free $G$-compactification and in this case $\beta G X$ is also a free $G$-space (see [3, Proposition 3.7]). Consequently, by virtue of Lemma 2.8, one has that $(G \times X)/G = X$ and $(G \times \beta GX)/G = \beta GX$. In sum, we get $\beta X = \beta G X$, which implies that each bounded continuous function $f: X \to \mathbb{R}$ is $G$-uniform. However, this is not true.

Indeed, since $X$ is paracompact and noncompact, it is not countably compact [14, Theorem 3.10.3]. Hence, there exists a locally finite, disjoint, countable family $\{U_1,U_2,\ldots\}$ of open subsets of $X$. Since $G$ is infinite, one can choose a countable base $\{O_1,O_2,\ldots\}$ of neighborhoods of the unity in $G$. For each $n \geq 1$, choose a point $x_n \in U_n$ arbitrary. Then,
by continuity of the $G$-action at $x_n \in X$, there exists an element $g_n \in O_n$ such that $g_n$ is different from the unity of $G$ and $g_n x_n \in U_n$, $n = 1, 2, \ldots$. Since $X$ is a free $G$-space, we see that $g_n x_n \neq x_n$, $n \geq 1$.

Now, let $f_n : X \to [0,1]$ be a continuous function such that $f_n(x_n) = 1$, $f_n(g_n x_n) = 0$ and $f_n(X \setminus U_n) = \{0\}$. Define $f(x) = \sum_{n=0}^\infty f_n(x)$, $x \in X$. Since $\{U_1, U_2, \ldots\}$ is disjoint and locally finite, $f$ is a well-defined, continuous, bounded function $X \to \mathbb{R}$. Hence, it should be also $G$-uniform, which yields a neighborhood $Q$ of the unity in $G$ such that $|f(gx) - f(x)| < 1/2$ for all $x \in X$ and $g \in Q$. We choose $n \geq 1$ so large that $O_n \subset Q$. This implies that $g_n \in Q$, and hence $1 = |f(g_n x_n) - f(x_n)| < 1/2$, a contradiction. \hfill $\square$

In general, if the acting group $G$ is not discrete, an action $G \times X \to X$ cannot be extended (continuously) to an action $G \times \beta X \to \beta X$; the natural rotation-action of the circle group on the plane $\mathbb{R}^2$ provides a counterexample (see [19, Section 1.5]). However, the following result holds true.

**Theorem 2.10.** Let $G$ be a pseudocompact group and $X$ a pseudocompact $G$-space. Then $X$ is $G$-Tychonoff and $\beta_G X = \beta X$.

**Proof.** The action $\alpha : G \times X \to X$ uniquely extends to a continuous map $\varphi : \beta(G \times X) \to \beta X$. By Lemma 2.5(2), the product $G \times X$ is pseudocompact, and hence, according to Glicksberg’s theorem [16], $\beta(G \times X) = \beta G \times \beta X$. Thus, $\varphi$ can be treated as a continuous map of $\beta G \times \beta X$ in $\beta X$ which extends $\alpha$. But remember that $\beta G$ is a topological group containing $G$ as a dense subgroup (see, e.g., [6, Theorem 4.1(f)]).

Further, the fact that $\alpha$ satisfies the two algebraic conditions of action implies easily that the map $\varphi : \beta G \times \beta X \to \beta X$ satisfies these conditions as well. Thus, $\varphi$ is an action, and hence $\beta X$ is a $\beta G$-space. In particular, $\beta X$ is a $G$-space. Consequently, $\beta X$ is a $G$-compactification of $X$, and hence $X$ is a $G$-Tychonoff space. It is also clear that $\beta X$ is the maximal $G$-compactification of $X$, that is, $\beta_G X = \beta X$, as required. \hfill $\square$

**Remark 2.11.** It is worth to mention that there exists a pseudocompact group whose underlying topological space is not a $k$-space (see, e.g., [12, 20]).

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