Completeness for coherent states in a magnetic–solenoid field

V G Bagrov1,2, S P Gavrilov3,4, D M Gitman3 and K Górka3,5

1 Department of Physics, Tomsk State University, 634050 Tomsk, Russia
2 Tomsk Institute of High Current Electronics, SB RAS, 634034 Tomsk, Russia
3 Instituto de Física, Universidade de São Paulo, PO Box 66318, 05315-970 São Paulo, SP, Brazil
4 Department of General and Experimental Physics, Herzen State Pedagogical University of Russia, Moyka emb. 48, 191186 St Petersburg, Russia
5 H Niewodniczański Institute of Nuclear Physics, Polish Academy of Sciences, ul. Eljasza-Radzikowskiego 152, 31342 Kraków, Poland

E-mail: bagrov@phys.tsu.ru, gavrilovsergeyp@yahoo.com, gitman@fma.if.usp.br and kasia_gorska@o2.pl

Received 26 August 2011, in final form 11 December 2011
Published 30 May 2012
Online at stacks.iop.org/JPhysA/45/244008

Abstract
This paper completes our study of coherent states in the so-called magnetic–solenoid field (a collinear combination of a constant uniform magnetic field and Aharonov–Bohm solenoid field) presented in Bagrov et al (2010 J. Phys. A: Math. Theor. 43 354016, 2011 J. Phys. A: Math. Theor. 44 055301). Here, we succeeded in proving nontrivial completeness relations for non-relativistic and relativistic coherent states in such a field. In addition, we solve here the relevant Stieltjes moment problem and present a comparative analysis of our coherent states and the well-known, in the case of pure uniform magnetic field, Malkin–Man’ko coherent states.

This article is part of a special issue of Journal of Physics A: Mathematical and Theoretical devoted to ‘Coherent states: mathematical and physical aspects’.

PACS numbers: 03.65.Ge, 03.65.Pm

1. Introduction

A splitting of Landau levels in a superposition of the Aharonov–Bohm (AB) field and a parallel uniform magnetic field gives an example of the AB effect for bound states. In what follows, we call such a superposition the magnetic–solenoid field (MSF); more precisely, the MSF is a collinear combination of a constant uniform magnetic field of strength B and the AB field, i.e. the field of an infinitely long and infinitesimally thin solenoid with a finite constant magnetic flux \( \Phi \). Setting the \( z \)-axis along the AB solenoid, the MSF strength takes the form

\[
B_r = B + \Phi \delta(x) \delta(y) = B + \frac{\Phi}{\pi r} \delta(r), \quad B = \text{const}, \quad \Phi = \text{const}.
\]
We use the following electromagnetic potentials\(^6\) \(A^\mu\), assigned to MSF (1): \(A^0 = A^3 = 0\), and
\[
A_x = -y \left( \frac{\Phi}{2\pi r^2} + \frac{B}{2} \right), \quad A_y = x \left( \frac{\Phi}{2\pi r^2} + \frac{B}{2} \right),
\]
with \(x = r \cos \theta\) and \(y = r \sin \theta\). Henceforth, for our convenience, we will denote the flux \(\Phi\) as \(\Phi = \Phi_0(l_0 + \mu)\), where \(0 \leq \mu < 1\) and \(\Phi_0 = 2\pi \hbar e / e\) is the Dirac fundamental unit of magnetic flux.

Solutions of the Schrödinger equation with the MSF were first studied in [1]. Solutions of relativistic wave equations (Klein–Gordon and Dirac ones) with the MSF were obtained in [2] and then used in [3] to study the AB effect in cyclotron and synchrotron radiation. A profound study of these solutions and related problems can be found in [4–9] and [10]. It is important to stress that in contrast to the pure AB field case, where particles interact with the solenoid for a finite short time, moving in the MSF the particles interact with solenoid permanently. This opens more possibilities to study such an interaction and corresponds to a number of real physical situations.

Constructing coherent states (CS) for non-relativistic and relativistic particles in the MSF is a nontrivial problem, in particular, due to the non-quadratic structure of particle Hamiltonians in this case. For the first time, CS in the MSF were constructed both in non-relativistic and relativistic cases in [11, 12]. Such CS minimize uncertainty relations for some physical quantities (e.g. coordinates and momenta) at fixed time instants and means of particle coordinates, calculated with respect to the time-dependent CS, move along the corresponding classical trajectories. The CS are labelled by quantum numbers that have a direct classical analogue, namely by phase-space coordinates. The time-dependent CS maintain their form under the time evolution. However, some problems related to the constructed CS remain still open. In particular, the completeness relations for the CS were not presented. In this paper, we prove these relations for the non-relativistic and relativistic CS in the MSF. In addition, we solve the relevant Stieltjes moment problem and present a comparative analysis of the CS in the MSF and the well-known, in the case of the pure uniform magnetic field, Malkin–Man’ko CS [13].

2. Non-relativistic stationary states

Let us consider quantum behaviour of a non-relativistic spinless particle with the charge \(q = -e (e > 0)\) and the mass \(M\) in the MSF (see equation (1)) in the direction perpendicular to field \(B (B > 0)\), i.e. on the \(xy\)-plane. As shown in [11] such behaviour is described by two kinds of wavefunctions \(\Psi^{(j)}_{\mu,n_1}(t, \theta, r), j = 0, 1\), namely
\[
\Psi^{(j)}_{\mu,n_1}(t, \theta, r) = e^{-iE_{\mu}/\hbar} \Phi^{(j)}_{\mu,n_1}(\theta, \rho), \quad \rho = \frac{y r^2}{2}, \quad \gamma = \frac{eB}{\hbar c}, \quad j = 0, 1,
\]
which are the eigenfunctions of two operators commuting with each other: the Hamiltonian \(H_{\perp} = \frac{1}{2M}(\hat{P}_x^2 + \hat{P}_y^2)\) and the angular momentum \(\hat{L}_z = x \hat{p}_y - y \hat{p}_x\), where \(\hat{p}_k = \hat{p}_k + eA_k / c\), with \(\hat{p}_k = -i\hbar \partial_k\), \(k = x, y\), and the vector potential \(A\) is specified by equation (2). The eigenvalues of \(H_{\perp}\) and \(\hat{L}_z\) are given by \(E_{\mu} = \hbar eB/(Mc)(n_1 + 1/2)\) and \(h(l - l_0), l = 0, \pm 1, \ldots\), respectively. The existence of two kinds of states \(\Psi^{(j)}_{\mu,n_1}(t, \theta, r)\) is connected with the presence of the AB field \((\mu \neq 0)\) and, what follows from that, the breaking translation symmetry in the \(xy\)-plane.

\(^6\) We accept the following notation for 4- and 3-vectors: \(a = (a^0, a^i; i = 0, \ldots, 3) = (a^0 = a_0, a^1 = a_1, a^2 = a_2, a^3 = a_3), a_{i1} = -a_{i0}, a_{i2}, \ldots\) in particular, for the spacetime coordinates: \(x^0 = \sqrt{t^2 + r^2}, x^1 = x, x^2 = y, x^3 = z\), as well as cylindrical coordinates \((r, \theta)\), in the \(xy\)-plane, such that \(x = r \cos \theta, y = r \sin \theta\) and \(r^2 = x^2 + y^2\). Besides, \(dx = dt^0dx^0 + dx^1dx^1 + dx^2dx^2 + dx^3dx^3\) and the Minkowski tensor \(\eta_{\mu\nu} = \text{diag} (1, -1, -1, -1)\).
The presence of a non-zero flux $\Phi$ is also visible in two kinds of functions $\phi_{j,m_1}^{(1)}(\theta, \rho)$:

$$\phi_{j,1}^{(1)}(\theta, \rho) = N e^{i(l-j_0)\theta} I_{n_1}(\rho), \quad n_1 = m, \quad n_2 = m - l - \mu, \quad l < 0,$$

$$\phi_{j,2}^{(1)}(\theta, \rho) = N e^{i(l-j_0)\theta - i\pi} I_{n_2}(\rho), \quad n_1 = m + l + \mu, \quad n_2 = m, \quad l \geq 0,$$

which are the orthogonal sets on the $\alpha$-plane [11, 12]. $N = \sqrt{y/2\pi}$ is a normalization constant with respect to the inner product

$$(f, g) = \int f^*(\theta, \rho) g(\theta, \rho) \, d\theta \, d\rho$$

Here, $m = 0, 1, \ldots, I_{n,m}(\rho)$ is the Laguerre function [14] that is related to the associated Laguerre polynomials $L_{n}^{\alpha}(\rho)$ [14, 15] as follows:

$$L_{n}^{\alpha}(\rho) = \frac{m!}{\Gamma(1 + m + \alpha)} \int_0^\infty \frac{d\rho}{\rho^{m+\alpha}} e^{-\rho/\alpha} \rho^{\alpha} L_n^\alpha(\rho),$$

where $\alpha = 0$ or $r = 0$ when $l = 0$. It corresponds to the most natural self-adjoint extension of the differential symmetric operator $\hat{H}_L$ (all possible self-adjoint extensions of $\hat{H}_L$ were constructed in [9, 10]). Considering a regularized case of a finite-radius solenoid, one can demonstrate that the zero-radius limit yields such an extension, see [8].

The functions $\Psi_{j,m}^{(1)}(\theta, \rho)$ form a complete system in the corresponding Hilbert space, see for example [10] where a complete spectral analysis of all self-adjoint extensions $\hat{H}_L$ is presented in detail. However, it is useful to show explicitly that these functions satisfy the resolution on unity (completeness relation) on the $x'y'$-plane. For this purpose, we introduce the retarded $S_{\text{ret}}(x, x')$ Green function, which we define as follows:

$$S_{\text{ret}}(x, x') = \Theta(\Delta t) S(x, x'),$$

$$S(x, x') = i \sum_\alpha e^{-i\varepsilon_{\alpha} \Delta t / h} \phi_{j,1}^{(1)}(\theta, \rho) \phi_{j,2}^{(1)}(\theta', \rho'),$$

where $\Delta t = t - t'$ and $\alpha$ represents the integers $j$, $l$ and $m$, where $l$ and $m$ are determined in equation (4). $\Theta(z)$ is the Heaviside step function. Then, the unity resolution for states $\phi_{j,1}^{(1)}(\theta, \rho)$, being written with the help of $S_{\text{ret}}(x, x')$, has the form

$$-i S_{\text{ret}}(x, x')|_{\Delta t = 0} = \delta(x - x') \delta(y - y').$$

Note that $\delta(x - x') \delta(y - y') = \gamma \delta(\theta - \theta') \delta(\rho - \rho')$. It is convenient to introduce an auxiliary function $S_{j}^{(1)}(x, x')$ by using which $S(x, x')$ is represented as

$$S(x, x') = \sum_{j=0}^{l} \sum_{l=0}^{\infty} S_{j}^{(1)}(x, x'),$$

$$S_{j}^{(1)}(x, x') = i \sum_{m=0}^{\infty} e^{-i\varepsilon_{m} \Delta t / h} \phi_{j,1}^{(1)}(\theta, \rho) \phi_{j,2}^{(1)}(\theta', \rho'),$$

where $l < 0$ for $j = 0$ and $l \geq 0$ for $j = 1$.

Now, by employing the states (4) and formula (8.976.5) from [14], we represent $S_{j}^{(1)}(x, x')$ as

$$S_{j}^{(1)}(x, x') = \frac{\gamma}{2\pi} \exp \left[ i(l - l_0) \Delta \theta - i \frac{\hbar}{2M} (l + \mu) \Delta \theta - \frac{i}{2} (\rho + \rho') \cot(\hbar \gamma \Delta t / 2M) \right] \frac{\sin[\hbar \gamma \Delta t / 2M]}{\sin[\hbar \gamma \Delta t / 2M]} I_{j+1}^{(l+\mu)} \left( \frac{\sqrt{\rho \rho'}}{\sin[\hbar \gamma \Delta t / 2M]} \right),$$

$\gamma = \sqrt{y/2\pi}$.
where \( \Delta \theta = \theta - \theta' \) and \( I_{\nu}(z) \) is the modified Bessel function of the first kind. The upper sign in the index of \( I_{\nu}(z) \) is related to \( j = 0 \) and the lower is for \( j = 1 \). Representation (10) matches with the result obtained in [6]. Note that \( S_{\text{ret}}(x, x') \) is the integral kernel; it can be changed from convenience considerations by changing the integration path in the complex plane of \( \rho, \rho' \); e.g. for \( \rho = i \xi, \rho' = i \xi' \) with the real positive \( \xi, \xi' \), we have

\[
\int d\rho \, d\rho' S_{\text{ret}}(t, t', \theta, \theta', \rho, \rho') f(\rho) g(\rho') = i^2 \int d\xi \, d\xi' S_{\text{ret}}(t, t', \theta, \theta', i \xi, i \xi') f(i \xi) g(i \xi'),
\]

(11)

where \( f(\rho) \) and \( g(\rho') \) are the arbitrary integrable functions. Considering the limit \( \Delta t = 0^+ \) in equation (10), we can use the asymptotic formula (8.451.5) from [14] for the Bessel function.

Then, going back to the initial variables, we obtain

\[
S_{ij}(x, x')|_{\Delta t=0^+} = i \frac{\chi'}{2\pi} e^{(i-l_{ij})\Delta \theta} \delta(\rho - \rho').
\]

(12)

By using the representation

\[
\frac{1}{2\pi} \sum_{l=0}^{\infty} e^{il \Delta \theta} = \delta(\Delta \theta),
\]

(13)

we verify that relation (8) holds, such that the set of the functions \( \Psi_{n_1, n_2}^{(j)}(t, \theta, r) \) are complete. Note that the distribution \( S(x, x') \) is not defined for \( \Delta t = 0 \), that is why the time-dependent phase in equation (7) is important.

3. Non-relativistic CS

3.1. CS in the MSF

Following the idea of [11, 12], one has to introduce two kinds \((j = 0, 1)\) of instantaneous CS, which are the linear combinations of the states \( \phi_{n_1, n_2}^{(j)}(\theta, \rho) \) given by equations (4):

\[
\Phi_{z_1, z_2}^{(j)}(\theta, \rho) = \frac{1}{\sqrt{N_j(|z_1|^2, |z_2|^2)}} \sum_{l} \phi_{z_1, z_2}^{(j,l)}(\theta, \rho),
\]

\[
\Phi_{z_1, z_2}^{(j)}(\theta, \rho) = \sum_{m=0}^{\infty} \frac{z_1^m z_2^m}{\sqrt{m! (1 + n_1) (1 + n_2)}} \phi_{n_1, n_2}^{(j)}(\theta, \rho).
\]

(14)

The CS are labelled by the continuous complex parameters \( z_1 \) and \( z_2 \). Possible values of \( n_1 \) and \( n_2 \) depend on \( m, l \) and \( j \) according to equations (4). The normalization constants \( N_j(|z_1|^2, |z_2|^2) \) are calculated from the overlapping formula

\[
(\Phi_{z_1, z_2}^{(j)}, \Phi_{z_1', z_2'}^{(j')}) = \delta_{jj'} \frac{\mathcal{R}^{(j)}}{\sqrt{N_j(|z_1|^2, |z_2|^2)N_j(|z_1'|^2, |z_2'|^2)}},
\]

(15)

where

\[
\mathcal{R}^{(0)} = Q_{-\mu} \left( \sqrt{z_1 z_1'}, \sqrt{z_2 z_2'} \right), \quad \mathcal{R}^{(1)} = Q_{\mu} \left( \sqrt{z_1 z_2'}, \sqrt{z_2 z_1'} \right),
\]

\[
Q_{\mu}(u, v) = \sum_{l=0}^{\infty} \frac{u^l}{l!} I_{\mu+l}(2uv),
\]

(16)

for \( j = j' \) and \( z_k = z_k', k = 1, 2 \). Let us remark that \( \Phi_{z_1, z_2}^{(j)}(\theta, \rho) \) do not represent a kind of the Gazeau–Klauder coherent states\(^7\) (GKCS).

\(^7\) We recall that GKCS are constructed on the basis of a complete set of quantum states in a specific manner, see [16, 17].
3.2. Completeness relations

We are going to prove that CS (14) form a complete set on the xy-plane, that is, they allow a unity resolution with the measure \( d\nu_j(z_1, z_2) = W_j^\mu(|z_1|^2, |z_2|^2) d|z_1| d|z_2| \). This statement is equivalent to the relation

\[
\sum_{j=0,1} F^{(j)}(x, x') \bigg|_{\Delta t=0^+} = -i S^{\text{st}}(x, x') \bigg|_{\Delta t=0^+},
\]

(17)

where \( W_j^\mu(|z_1|^2, |z_2|^2) \) is the positive weight function and \( S^{\text{st}}(x, x') \) satisfies condition (8). We consider CS defined for almost equal times. We include the time-dependent phase \( e^{-i\hat{U}_j \Delta t/\hbar} \) into the definition of the distribution \( F^{(j)}(x, x') \) to provide the consistency of the limit \( \Delta t \to 0^+ \) in equation (17). To prove equation (17), we have to find the corresponding weight function \( W_j^\mu(|z_1|^2, |z_2|^2) \).

Let us check the relations

\[
F^{(j)}(x, x') \bigg|_{\Delta t=0^+} = -i \sum_n e^{-i \hat{U}_l \Delta t}/h \Phi_l^{(j)}(\theta, \rho) \Phi^{(j)}(\theta, \rho'),
\]

(18)

First, we consider the case \( j = 0 \), for which \( F^{(0)} \), after using the explicit form of \( \Phi_l^{(0)}(\theta, \rho) \) (see equations (14)), takes the form

\[
F^{(0)}(x, x') = \sum_{l, k} \sum_{m, n} e^{-i \hat{U}_l \Delta t/\hbar} \phi_{l, m, l, m}^{(0)}(\theta, \rho) \phi_{n, n, k}^{(0)}(\theta, \rho') G(m, n; l, k) / \sqrt{m! n!} (1 + m - l - \mu) \Gamma(1 - n - k - \mu).
\]

(19)

The auxiliary function \( G(m, n; l, k) \) is chosen as

\[
G(m, n; l, k) = \int d^2 z_1 d^2 z_2 W_0^\mu(|z_1|^2, |z_2|^2) N_0(|z_1|^2, |z_2|^2) c_i^{m} c_i^{n} c_j^{l} c_j^{k}
\]

\[
= \frac{1}{4\pi^2} \int_0^{2\pi} d\varphi_1 e^{i(m-n)\varphi_1} \int_0^{2\pi} d\varphi_2 e^{i(n-k-\mu)\varphi_2}
\]

\[
\times \int_0^\infty d|z_1|^2 d|z_2|^2 |z_1|^{m+n}|z_2|^{m+n-l-\mu} W_0^\mu(|z_1|^2, |z_2|^2)
\]

\[
= \delta_{m,n} \delta_{l,k} \int_0^\infty du d v u^{m-l-\mu} W_0^\mu(u, v),
\]

(20)

where \( z_k = |z_k|^2 \) \((k = 1, 2)\), \( u = |z_1|^2 \) and \( W_0^\mu(u, v) = \pi^2 W_0^\mu(u, v) \) which is an arbitrary positive function that provides equation (18). Taking \( W_0^\mu(u, v) = \exp(-u - v) \) and using the representation \( \Gamma(x) = \int_0^\infty x^{-1} e^{-x} dx \) of the gamma function, we obtain

\[
G(m, n; l, k) = \delta_{m,n} \delta_{l,k} \Gamma(1 + m) \Gamma(1 + m - l - \mu).
\]

(21)

This function being inserted in equation (19) gives a correct result for (18) with \( j = 0 \).

In the same manner, one can verify the case \( j = 1 \). Taking into account (8), we see that the validity of (17) is just the proof of the completeness of CS.

We point out that the choice of \( W_0^\mu(u, v) = \exp(-u - v) \) in equation (20) produces two Stieltjes moment problems \( \int_0^\infty dx x^k W(x) = \varrho(n) = \Gamma(1 + n) \), where \( x \) and \( n \) are, respectively, taken as \( u, v \) and \( m, \mu \). According to the Pakes criterion [18], the Stieltjes moment problem appearing here has a unique positive solution \( e^{-Ax} \), which leads to unambiguous, given the first time in the literature, weight function \( W_j^\mu(|z_1|^2, |z_2|^2) \) and at the same time to unambiguous positive measure \( d\nu_j(z_1, z_1) = W_j^\mu(|z_1|^2, |z_2|^2) d|z_1| d|z_2| \).
The weight functions $W_j^\mu(u, v)$ have the form
\[ W_j^\mu(u, v) = \pi^{-2} e^{-(u+v)} Q_{1-\mu}(\sqrt{u}, \sqrt{v}), \quad W_i^\mu(u, v) = \pi^{-2} e^{-(u+v)} Q_{\mu}(\sqrt{v}, \sqrt{u}). \] (22)
It turns out that $W_j^\mu(u, v)$ can be expressed via special functions only for $\mu = 0$ and $1/2$. The case of $\mu = 0$, which corresponds to the absence of the AB field, will be discussed in section 3.3. In the case $\mu = 1/2$, the weight functions are
\[ W_j^{1/2}(u, v) = \frac{1}{2\pi^2} [\text{erf}(\sqrt{u} + \sqrt{v}) - \text{erf}(\sqrt{u} - \sqrt{v})], \] (23)
where '+-' is for $j = 0$ and '++' for $j = 1$, erf$\,(z)$ is the ‘error function’ encountered in integrating the normal distribution [14].

3.3. Zero magnetic flux limit

Let us study the limit $\Phi = 0$ that corresponds to the pure magnetic field without the AB solenoid.

First of all, we consider such a limit for stationary states. All topological effects connected with the translation symmetry breaking vanish for $\mu = 0$ and, in particular, for $\Phi = 0$ ($l_0 = 0$). As a consequence, the shift of the Landau levels is absent for $\phi_{m,n}(\theta, \rho)$, and it is natural to consider a superposition of $j = 1$ and $j = 0$ states,
\[ \phi_{m,m-l}(\theta, \rho) + \phi_{m+1,m}(\theta, \rho) = \phi_{m,l}(\theta, \rho), \quad l = 0, \pm 1, \pm 2, \ldots \]
Next, we study the limit of $\Phi = 0$ in CS (14). Thus, we expect to obtain the Malkin–Man’ko CS [13]. To show this, we consider the following superposition of the CS:
\[ \Phi_{z_1,z_2}(\theta, \rho) = N_0^{1/2} \Phi_{z_1,z_2}(\theta, \rho) + N_1^{1/2} \Phi_{z_1,z_2}(\theta, \rho). \] (24)
At the beginning, we note that the probability distribution of $|\Phi_{z_1,z_2}(\theta, \rho)|^2$ calculated with respect to the inner product $(\Phi, \Phi)_L$ is equal to
\[ |\Phi_{z_1,z_2}(\theta, \rho)|^2 = N_0(|z_1|^2, |z_2|^2) + N_1(|z_1|^2, |z_2|^2) = e^{i|z_1|^2+i|z_2|^2}, \] (25)
where $N_j(|z_1|^2, |z_2|^2) = R_j^L$ at $z_k = z_k^*$ are given in equation (16) for $j = j'$. To derive equation (25), we employ formula (5.8.3.2) from [15] and the fact that $\phi_{j,j'}(\theta, \rho)$ are orthogonal for different $j$. The density $|\Phi_{z_1,z_2}(\theta, \rho)|^2$ is equal to the normalization constant of the Malkin–Man’ko CS, see equation (41) in [13]. Then, substituting $\Phi_{z_1,z_2}(\theta, \rho)$ into equation (25), we obtain
\[ \Phi_{z_1,z_2}(\theta, \rho) = \sum_{l,m} \frac{z_1^{m-l} z_2^{m+l} [\phi_{m,m-l}(\theta, \rho) + \phi_{m+1,m}(\theta, \rho)]}{\sqrt{m!(m + |l|)!}}, \]
\[ = \sum_{r_1,r_2=0}^{\infty} \frac{z_1^{r_1} z_2^{r_2}}{\sqrt{r_1! r_2!}} \phi_{r_1,r_2}(\theta, \rho). \] (26)
where $r_1 = m$ and $r_2 = m + |l|$. Comparing equation (26) with equation (41) from [13], we see that $\Phi_{z_1,z_2}(\theta, \rho)$ are the Malkin–Man’ko CS.

Now, let us consider the weight function $W_j^\mu(u, v)$ for $\mu = 0$. In the limit under consideration, we have
\[ W_0^\mu(u, v) + W_1^\mu(u, v) = \pi^{-2}, \] (27)
where formula (5.8.3.2) from [15] was used. Equation (27) is the weight function $W_0^\mu(u, v)$ for the Malkin–Man’ko CS.
4. Relativistic stationary states

Note that the relativistic spinless CS are reduced to the non-relativistic case. That is why in the relativistic case, only the CS of spinning particles are in a sense nontrivial. In spite of the fact that the algebra of the Dirac \( \gamma \)-matrices and the spin description in \( (2 + 1) \) and \( (3 + 1) \) dimensions are different, considering \( (3 + 1) \)-dimensional case, we can use technical results obtained for \( (2 + 1) \) dimensions. That is why in the beginning, we consider the spinning case in \( (2 + 1) \) dimensions.

The behaviour of an electron in the MSF in \( (2 + 1) \) dimensions is described by wavefunctions that obey the Dirac equation with such a field, see \([5]\). These wavefunctions for given ‘polarizations’ \( \xi = \pm 1 \) (related to one of two nonequivalent representations for \( \gamma \)-matrices) and particle/antiparticle energy \( cp_0 = \pm \varepsilon_{\pm} \) have the form

\[
\Psi = e^{-i(cpt_{0})/\hbar} \psi^{(1)}_{\pm}(x^1, x^2).
\]

(28)

In contrast to the \( (3 + 1) \)-dimensional case, particles and antiparticles in \( (2 + 1) \) dimensions have only one spin polarization state. Choosing \( \xi = +1 \), we deal with ‘spin-up’ particles, and choosing \( \xi = -1 \) with ‘spin-down’ particles. One can see that \( \psi^{(1)}_{\pm 1}(x^1, x^2) = \sigma^2 \psi^{(1)}_{\pm 0}(x^1, x^2) \), where \( \sigma^2 \) is a Pauli matrix. That is why we consider here only the case \( \xi = 1 \). The ‘spin-up’ particle \((+)\) and antiparticle \((-)\) states are denoted as \( \psi^{(1)}_{\pm 0} = \psi^{(1)}_{\pm \lambda} \).

The functions \( \psi^{(j)}_{\pm \alpha, \mu, \nu} \) are common eigenfunctions of the total angular momentum operator \( \hat{J} = -i\hbar \partial_0 + \hbar a^2 / 2 \) and the Hamiltonian \( \hat{H}^0 = c(\sigma \hat{P}_\perp + Mc\sigma^3) \). The eigenvalues are equal to \( \hbar(l + \frac{1}{2}) \) and \( \pm \varepsilon_{\pm} \), respectively. \( \hat{H}^0 \) represent a one-parameter family of self-adjoint Hamiltonians (self-adjoint extensions) that are determined by the corresponding boundary conditions. We consider only two special cases: \( \theta = \pi \) or \( \pm 1 \). They correspond to the most natural self-adjoint extensions \( \hat{H}^0 \). Considering a regularized case of a finite-radius solenoid, one can demonstrate that the zero-radius limit yields such extensions, see \([5]\). The functions \( \psi^{(j)}_{\pm \alpha, \mu, \nu} \) can be represented as \([12]\)

\[
\psi^{(1)}_{\pm \alpha, \mu, \nu} (\theta, \rho) = M_{j, \pm \alpha, \mu, \nu} [\sigma^3 \{ \pm \hat{P}_0(M) - a \hat{P}_\perp \} + Mc] u^{(1)}_{\alpha, \mu, \nu} (\theta, \rho),
\]

\[
\hat{P}_0(M) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-l^2 \tau^2} d\tau,
\]

\[
\hat{P}_\perp(M) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-l^2 \tau^2} d\tau,
\]

\[
u_0 = \frac{1}{\sqrt{\pi}}, \quad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

(29)

where \( \hat{P}_\perp = (\hat{P}_x, \hat{P}_y) \) and \( M_{j, \pm \alpha, \mu, \nu} \) are the normalization factors with respect to the inner product

\[
(\psi, \psi')_D = \gamma^{-1} \int_0^\infty d\rho \int_0^{2\pi} d\theta \psi^*_\alpha (\theta, \rho) \psi^*_\beta (\theta, \rho).
\]

(30)

For particles, the energy spectrum is \( \varepsilon_+ = [(Mc^2)^2 + (E^2_{\perp(+1)})]^{1/2} \); for antiparticles the energy spectrum is \( \varepsilon_- = [(Mc^2)^2 + (E^2_{\perp(-1)})]^{1/2} \). The energy \( E^2_{\perp(\sigma)} \) is given by \( E^2_{\perp(\sigma)} = 2\hbar e B [n_1 + (1 + \sigma) / 2] \). The functions \( \phi_{\pm \alpha, \mu, \nu}^{(j)}(\theta, \rho) \) have the form \([12]\)

\[
\phi_{\pm \alpha, \mu, \nu}^{(1)} (\theta, \rho) = e^{i(l-\mu)\theta} I_{n_2, \alpha, \rho} (\rho), \quad n_1 = m, \quad n_2 = m - l_\sigma - \mu, \quad l \leq -1 (1 - \sigma) / 2,
\]

\[
\phi_{\pm \alpha, \mu, \nu}^{(1)} (\theta, \rho) = e^{i(l-\mu)\theta} I_{m+\lambda, \alpha, \rho} (\rho), \quad n_1 = m + \nu = m, \quad n_2 = m, \quad \nu \geq 1 (1 + \sigma) / 2.
\]

(31)

where \( I_{\sigma} = l - (1 + \sigma) / 2 \) and \( I_{m+\sigma, \lambda} (\rho) \) are given by equation \((6)\). These functions form an orthogonal set on the semi-axis \( \rho > 0 \) with respect to the scalar product \( (f, g) \). The existence of two self-adjoint extensions is correlated with the irregular behaviour of the radial
functions \(I_{n,m}(\rho)\) at the origin when \(l = 0\) and either \(\sigma = -1\) for \(\vartheta = +1\) or \(\sigma = +1\) for \(\vartheta = -1\) \[5\]. Note that \(\tilde{\Pi}^2_0 u_{n_1, n_2; \pm 1}^{(j)} = \mathcal{E}^2_{\pm} u_{n_1, n_2; \pm 1}\), then \(\tilde{\Pi}^0_0 (M) u_{n_1, n_2; \pm 1}^{(j)} = \mathcal{E}_{\pm} u_{n_1, n_2; \pm 1}\). Thus, the spectrum of the operator \(\tilde{\Pi}^0_0 (M)\) is positive defined. We use the \(\tilde{\Pi}^0_0 (M)\) in representation (29) to simplify transition from these stationary states to CS in the following subsection.

With respect to the self-adjoint operators \(\tilde{\mathcal{H}}^\vartheta\), it is known that for any \(\vartheta\) the functions \(\psi_{\pm 1, n_2}^{(j)}(\theta, \rho)\) form a complete system on the \(xy\)-plane, see for example [10] where a complete spectral analysis of all self-adjoint extensions \(\tilde{\mathcal{H}}^\vartheta\) is presented in detail.

To prove this directly and to find an explicit form of the unity resolution, we, similar to the non-relativistic case, introduce the retarded Green function \(S^\text{ret}(x, x')\) of the Dirac equation

\[
S^\text{ret}(x, x') = \Theta(\Delta t)|S^-(x, x') + S^+(x, x')| = \Theta(\Delta t)|S'(x, x') - S^x(x, x')|,
\]

\[
S'(x, x') = \Theta(\Delta t)S^-(x, x') - \Theta(-\Delta t)S^+(x, x'),
\]

\[
S^x(x, x') = i \sum_{j, m, l} e^{\pm iE_\sigma |\Delta t|} \psi_{\pm 1, n_2}^{(j)}(\theta, \rho) \psi_{\pm 1, n_2}^{\dagger}(\theta', \rho') \sigma^3,
\]

where summation in equation (32) is over all possible quantum numbers \(j, m\) and \(l\) as specified in equation (31). The functions \(S'(x, x')\) and \(S^x(x, x')\) are the causal and anticausal Green functions, respectively. The resolution of unity is satisfied if the following relation holds:

\[
-i\sigma^3S^\text{ret}(x, x')|_{\Delta t=0} = \delta(x - x')\delta(y - y') 1,
\]

where \(1\) is an \(2 \times 2\) identity matrix. We are going to prove that equation (33) takes place in our case. To this end, we represent \(S'(x, x')\) and \(S^x(x, x')\) in the form of the Fock–Schwinger proper time integral [6]:

\[
S'(x, x') = [\sigma^3(\tilde{\rho}_0 - \sigma \tilde{\mathcal{P}}_\perp) + M\mathcal{C}] \Delta'(x, x'),
\]

\[
S^x(x, x') = [\sigma^3(\tilde{\rho}_0 - \sigma \tilde{\mathcal{P}}_\perp) + M\mathcal{C}] \Delta(x, x'),
\]

\[
\Delta'(x, x') = \int_0^\infty ds f(x, x', s), \quad \Delta(x, x') = \int_{-\infty}^\infty ds f(x, x', s),
\]

where \(\tilde{\rho}_0 = \frac{\hbar}{2\sigma}\) and the kernel \(f(x, x', s)\) is given by

\[
f(x, x', s) = \sum_{\sigma=\pm 1} \sum_{l=\pm \infty} f_{\sigma, l}(x, x', s), \quad f_{\sigma, l}(x, x', s) = A_{\sigma, l}(s) B_{\sigma, l}(s) \Xi_{\sigma},
\]

\[
A_{\sigma, l}(s) = \frac{\mathcal{Y}}{2\pi^3/2^3/4\sin(y/s)} \exp\left\{ -i (c\Delta t)^2 / 4s + i / 2 (\rho + \rho') \cot(\gamma s) \right\} \times \exp\left\{ i \pi r \right\} - i (M\mathcal{C}^{-1})^2 s + i(I_{\sigma} - l_0) \Delta \theta - i(I_{\sigma} + \sigma + \mu) \gamma s \right\},
\]

\[
B_{\sigma, l}(s) = I_{|\rho_{\sigma, l}|}(z) \quad \text{if} \quad \vartheta \neq 1, \quad B_{\sigma, 0}(s) = I_{|\rho_{\sigma, l}|}(z) \quad \text{if} \quad \vartheta = +1,
\]

\[
B_{\sigma, 0}(s) = I_{|\rho_{\sigma, l}|}(z) \quad \text{if} \quad \vartheta = -1, \quad \vartheta = e^{-iz} \sqrt{\rho \rho'} / \sin(y/s), \quad \Xi_{\pm 1} = \frac{1}{2}(1 \pm \sigma^3).
\]

The integration path over \(s\) is deformed so that it goes slightly below the singular points \(s_k = k\pi / \gamma\) and \(-s_k, k = 1, 2, \ldots\). Negative values for \(s\) are defined as \(s = |s| e^{-iz}\). The kernel \(f(x, x', s)\) satisfies the following differential equation:

\[
i \frac{d}{ds} f(x, x', s) = \hbar^{-2} [(M\mathcal{C})^2 - [\sigma^3(\tilde{\rho}_0 - \sigma \tilde{\mathcal{P}}_\perp)]^2] f(x, x', s).
\]

Remembering that \(S^\text{ret}(x, x')\) is the integral kernel of an integral over the variables \(\rho, \rho'\), we can carry out the transformation used above in the non-relativistic case and change the integral path
on the complex plane of $\rho$, $\rho'$ to a form where $\rho = i\xi$, $\rho' = i\xi'$ with the real positive $\xi$, $\xi'$. With respect to irregular behaviour of the quantities $B_{\sigma,0}(s)$ as $|\rho\rho'| \to 0$, we restrict the range of $|\rho\rho'|$ to $0 < \delta < |\rho\rho'| < \infty$, with arbitrary $\delta \ll 1$. Then, we take the limit $s \to 0^+$. Under such a condition we use the asymptotic expansion of the Bessel function as $s \to 0^+$ and thereafter, return to the original variable $\rho$, $\rho'$ on the real semi-axis. Thus, we find that

$$\lim_{s \to 0^+} f_{\sigma,\ell}(x',s) = \frac{ie^{i\xi'}}{2\pi} e^{i(\xi'-\xi)\Delta t} \delta(c\Delta t) \delta(\rho - \rho') \Xi_{\sigma}$$

for both self-adjoint extensions $\ell = \pm 1$. Finally, using representation (13), we obtain

$$\lim_{s \to 0^+} f(x',s) = i\hbar \delta(c\Delta t) \delta(x - x') \delta(y - y') I.$$  \hspace{1cm} (37)

In the same manner, we can obtain

$$\lim_{s \to 0^+} f(x',s) = -i\hbar \delta(c\Delta t) \delta(x - x') \delta(y - y') I.$$  \hspace{1cm} (38)

Taking into account that the kernel $f(x',s)$ has no singularity in the lower part of the complex plane of $s$, the integral $\int_{\mathrm{ret}} \Delta^{\mathrm{ret}}(x,x') = \Theta(\Delta t)[\Delta'(x,x') - \Delta^0(x,x')]$ is represented as

$$\int_{\mathrm{ret}} \Delta^{\mathrm{ret}}(x,x') = \Theta(\Delta t) \int_{\Gamma} ds f(x',s),$$  \hspace{1cm} (39)

where $\Gamma$ is a clockwise circle, which connects the points $s = +0$ and $s = 0 \cdot e^{-i\pi}$, and passes in the lower part of the complex plane of $s$. If conditions (37) and (38) hold, the function

$$S^{\mathrm{ret}}(x',s) = [\sigma^y(\hat{p}_0 - \sigma \hat{P}_+) + Mc] \Delta^{\mathrm{ret}}(x,x')$$  \hspace{1cm} (40)

satisfies equation (33) and is indeed the retarded Green function of the corresponding Dirac equation [19].

5. Relativistic CS

Generally speaking, in the relativistic case, the Dirac Hamiltonian is not quadratic in the momenta. Due to this fact, the time evolution of instantaneous CS on the $xy$-plane (see details in [12]) is not trivial. However, because the time evolution of these states is unitary, it is enough to show that the set of such initial CS is complete.

For instance, CS for the massive spinning (spin up) particle in the MSF on the $xy$-plane and in $(2 + 1)$ dimensions are

$$\Psi^{(j)}_{\pm,z_1,z_2}(\theta, \rho) = [\sigma_1^y \pm \hat{P}_0(M) - \sigma \hat{P}_+ + Mc] u^{(j)}_{\pm,z_1,z_2,\pm 1}(\theta, \rho),$$

$$u^{(j)}_{\pm,z_1,z_2,\pm 1}(\theta, \rho) = \Phi^{(j)}_{\pm,z_1,z_2,\pm 1}(\theta, \rho) v_{\sigma},$$  \hspace{1cm} (41)

where $\Phi^{(j)}_{\pm,z_1,z_2,\pm 1}(\theta, \rho)$ are defined in a similar way as the CS of the non-relativistic electron, see equations (14). Taking equation (29) into account, one can see that the CS (41) can be written as

$$\Psi^{(j)}_{\pm,z_1,z_2}(\theta, \rho) = \frac{1}{\sqrt{M_{j,\pm}(|z_1|^2,|z_2|^2)}} \sum_{l} \Psi^{(j),l}_{\pm,z_1,z_2}(\theta, \rho),$$

$$\Psi^{(j),l}_{\pm,z_1,z_2}(\theta, \rho) = \sum_{m=0}^{\infty} \frac{n_1 n_2}{\sqrt{\Gamma(1 + n_1)\Gamma(1 + n_2)}} \psi^{(j)}_{\pm,n_1,n_2}(\theta, \rho),$$  \hspace{1cm} (42)

where $n_1$, $n_2$, $j$ and $l$ change according to equation (31). The normalization constants $M_{j,\pm}(|z_1|^2,|z_2|^2)$ can be calculated from the overlapping formula

$$\langle \Psi^{(j)}_{\pm,z_1,z_2}, \Psi^{(j)}_{\pm',z_1',z_2'} \rangle_{D} = \frac{2Mc}{\sqrt{M_{j,\pm}(|z_1|^2,|z_2|^2)M_{j',\pm}(|z_1'|^2,|z_2'|^2)}}$$

$$\times \langle \Phi^{(j)}_{\pm,z_1,z_2,\pm 1} [\pm \hat{P}_0(M) + Mc] \Phi^{(j),l}_{\pm,z_1',z_2',\pm 1} \rangle_{\perp},$$  \hspace{1cm} (43)

where the inner products $\langle \cdot, \cdot \rangle_D$ and $\langle \cdot, \cdot \rangle_\perp$ are defined by (30) and (5), respectively.
Representation (42) is like (14), so that the unity resolution in CS can be done in a form similar to equation (17). Taking into account the structure of the retarded Green function $S^{ret}(x, x')$ given by (32), we find

$$
\sum_{j=0,1} F^{(j)}(x, x') |_{\Delta t=0^+} = -i\sigma^j S^{ret}(x, x') |_{\Delta t=0^+}, \quad k = 1, 2,
$$

$$
F^{(j)}(x, x') = \int d^2 z_1 \, dz_2 \, W_{\mu}^j(|z_1|^2, |z_2|^2) \sum_{\zeta=\pm} e^{-i(\epsilon_{\zeta}(M) \Delta t) / \hbar} \Phi^{(j)}_{\zeta, z_1, z_2}(\theta, \rho) \Phi^{(j)\dagger}_{\zeta, z_1, z_2}(\theta', \rho'),
$$

where the weight function $W_{\mu}^j(|z_1|^2, |z_2|^2)$ is the same as in (17). As in the non-relativistic case, the consistent limit as $\Delta t \to 0^+$ can be considered due to the inclusion of an appropriate time-dependent phase factor in the definition of $F^{(j)}(x, x')$ in (44). In the relativistic case under consideration, we have two such different factors, one for particle states and the other for antiparticle states. The proof of the resolution identity (44) is quite similar to the one for $\Phi^{(j)}_{\zeta, z_1, z_2}(\theta, \rho)$. Using the same weight function $W_{\mu}^j(|z_1|^2, |z_2|^2)$, we obtain a similar result for the massless fermions.

To complete our consideration, we study the $(3 + 1)$-dimensional case. The domains of the $(3 + 1)$-Dirac Hamiltonian in the MSF are trivial extensions of the corresponding domains mentioned in the $(2 + 1)$ case, that is why we use for the self-adjoint $(3 + 1)$-Dirac Hamiltonian the same notation $\hat{H}^0$. Of course, in this case $\hat{H}^0 = c \rho^j (\sum_{k=1,2,3} \gamma^j \hat{P}_k + Mc)$, $\hat{P}_i = i\hbar \partial_i$ and $\gamma^0, \gamma^k$ are the $4 \times 4$ Dirac gamma matrices. In particular, we consider the CS for the spinning particle in $(3 + 1)$ dimensions, which can be constructed by using the set of orthogonal stationary states defined in equation (A.39) from [12]. The latter can be reduced to the $(2 + 1)$ dimensions as follows:

$$
\Psi^{(j)}_{\pm, s, p_1, n_1, n_2}(x) = \exp \left[ -i / \hbar (c \tilde{\Pi}_0(M) + p_3 z) \right] \Psi^{(j)}_{\pm, s, p_1, n_1, n_2}(x_\perp), \quad s = \pm 1,
$$

$$
\Psi^{(j)}_{\pm, s, p_1, n_1, n_2}(x_\perp) = M_{p_1} \begin{cases} 
M^{-1}(p^3/c + s\tilde{M}) + 1] \tilde{\Psi}^{(j)}_{\pm, n_1, n_2}(\theta, \rho), \\
M^{-1}(p^3/c + s\tilde{M}) - 1] \sigma^3 \tilde{\Psi}^{(j)}_{\mp, n_1, n_2}(\theta, \rho) 
\end{cases}.
$$

Here, $\tilde{M} = \sqrt{\tilde{M}^2 + (p_3/c)^2}$, $p_3$ is the $z$-component of the momentum, $\tilde{\Pi}_0(M) = \tilde{\Pi}_0(M)|_{M=\tilde{M}}$, where $\tilde{\Pi}_0(M)$ is defined in (29), $M_{p_1}$ is an additional normalization factor, $s$ are eigenvalues of the spin operator $\hat{S}_z$,

$$
\tilde{S}_z = \frac{1}{2} (\hat{H}^0 \Sigma_z + \Sigma_z \hat{H}^0) / \tilde{M}^2,
$$

and $\tilde{\Psi}^{(j)}_{\pm, n_1, n_2}(\theta, \rho) = \tilde{\Psi}^{(j)}_{\pm, n_1, n_2}(\theta, \rho)|_{M=\tilde{M}}$, where $\tilde{\Psi}^{(j)}_{\pm, n_1, n_2}(\theta, \rho)$ is defined in (29).

Using the matrix structure (45) we can find instantaneous CS in $(3 + 1)$ dimensions in the similar form

$$
\Psi^{(j)}_{\pm, s, p_1, z_1, z_2}(\theta, \rho) = M_{p_1} \begin{cases} 
[M^{-1}(p^3/c + s\tilde{M}) + 1] \tilde{\Psi}^{(j)}_{\pm, z_1, z_2}(\theta, \rho), \\
[M^{-1}(p^3/c + s\tilde{M}) - 1] \sigma^3 \tilde{\Psi}^{(j)}_{\mp, z_1, z_2}(\theta, \rho) 
\end{cases}.
$$

Here, the two component column $\tilde{\Psi}^{(j)}_{\pm, z_1, z_2}(\theta, \rho)$ is defined as

$$
\tilde{\Psi}^{(j)}_{\pm, z_1, z_2}(\theta, \rho) = \Psi^{(j)}_{\pm, z_1, z_2}(\theta, \rho)|_{M=\tilde{M}},
$$

where $\Psi^{(j)}_{\pm, z_1, z_2}(\theta, \rho)$ are CS in $(2 + 1)$ dimensions given by (42). Thus, we see that for given $p_3$ and $s$, the unity resolution in CS for the $(3 + 1)$-dimensional case is reduced to $(2 + 1)$ dimensions considered above. Note that representations (45) and (46) are convenient for the non-relativistic limit.
Another type of instantaneous CS in $(3 + 1)$ dimensions that allows one to construct relativistic time-dependent CS was obtained in the work \[12\], see equation \((40)\) there. They have another matrix structure. In the same work, it was demonstrated that instantaneous CS in $(3 + 1)$ dimensions are reduced to ones in $(2 + 1)$ dimensions. That is why the unity resolution in terms of CS in $(3 + 1)$ dimensions is also reduced to the $(2 + 1)$-dimensional case considered above. This allows one to prove that the relativistic time-dependent CS given by equation \((89)\) from \[12\] form a complete system on the light cone hyper-surface $ct - z = \text{const}$.

Acknowledgments

KG acknowledges support from Fundação de Amparo á Pesquisa do Estado de S˜ao Paulo (FAPESP, Brazil) under program no 2010/15698-5; VGB thanks FAPESP (Brazil) and Russian Science and Innovations Federal Agency under contract no 02.740.11.0238 and Russia President grant SS-3400.2010.2 for support; SPG acknowledges support of the program Bolsista CAPES/Brazil and thanks the University of S˜ao Paulo for hospitality. DMG acknowledges the permanent support of FAPESP and CNPq.

References

[1] Lewis R R 1983 Phys. Rev. A \textbf{28} 1228
[2] Bagrov V G, Gitman D M and Tlyachev V B 2001 J. Math. Phys. \textbf{42} 1933
[3] Bagrov V G, Gitman D M, Levin A D and Tlyachev V B 2001 Mod. Phys. Lett. A \textbf{16} 1171
[4] Falomir H and Pisani P A G 2001 J. Phys. A: Math. Gen. \textbf{34} 4143
[5] Gavrilov S P, Gitman D M and Smirnov A A 2003 Eur. Phys. J. C \textbf{32} s119
[6] Gavrilov S P, Gitman D M and Smirnov A A 2005 Phys. Rev. A \textbf{67} 024103
[7] Gavrilov S P, Gitman D M and Smirnov A A 2004 J. Math. Phys. \textbf{45} 1873
[8] Gavrilov S P, Gitman D M, Smirnov A A and Voronov B L 2004 Dirac fermions in a magnetic–solenoid field

Focus on Mathematical Physics Research ed C V Benton (New York: Nova Science Publishers) p 131
[9] (arXiv:hep-th/0308093v2)
[10] (arXiv:hep-th/0308093v2)
[11] (arXiv:hep-th/0308093v2)
[12] (arXiv:hep-th/0308093v2)
[13] (arXiv:hep-th/0308093v2)
[14] (arXiv:hep-th/0308093v2)
[15] (arXiv:hep-th/0308093v2)
[16] (arXiv:hep-th/0308093v2)
[17] (arXiv:hep-th/0308093v2)
[18] (arXiv:hep-th/0308093v2)
[19] (arXiv:hep-th/0308093v2)