CLIFFORD HOPF-EBRA AND BI-UNIVERSAL HOPF-EBRA

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Abstract. For a scalar product $\xi$ on co-vectors, the Clifford co-product $\Delta^\xi$ of multivectors is calculated from the dual Clifford algebra. With respect to this co-product $\Delta^\xi$, unit is not group-like and vectors are not primitive. For a scalar product $\eta$ on vectors the Clifford product $\wedge^\eta$ and the Clifford co-product $\Delta^\xi$ fits to the bi-ebra with respect to the family of the (pre)-braids. The Clifford bi-ebra is in a braided category iff $\xi = 0$ or $\eta = 0$.

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1. Multi-ary Bi-ebra

A multiplicative category is a category with a bifunctor of bin-ary operation, an anihilation $2 \to 1$, denoted by two initial leaves and one node. A co-multiplicative category possess a binary co-operation, a creation process $1 \to 2$. A pairing $2 \to 0$, a bin-ary anihilation $2 \to 1$, a bin-ary creation $1 \to 2$, and bin-ary scattering $2 \to 2$ are represented by the prime graph nodes in Diagram 1. All diagrams are directed and is recommendent to read them from the top to the bottom.

Diagram 1. A pairing, binary multiplication (anihilation), binary co-product i.e. creation and a scattering (braid)

1991 Mathematics Subject Classification. Primary 16W30; 17A42.
Key words and phrases. Clifford Hopf-ebra, Clifford bi-ebra, n-ary bigebra, braided bi-ebra.
The Author is a member of Sistema Nacional de Investigadores, México.
Submitted August 28, 1997, to Czechoslovak Journal of Physics. This paper is in final form and no version of it will be submitted for publication elsewhere.

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Throughout this paper \( k \) denotes a commutative ring. A pair of \( k \)-modules, say \( A \) and \( B \), with a pairing \( A \otimes B \to k \), is said to be a dual pair. Dual pair of al-gebras (co-gebras) extends to pair of co-gebras (al-gebras) and these structures, al-gebra and co-gebra, may close to bi-gebra for a family of pre-braids. We calculate these structures from the assumptions displayed on Diagrams 2-4.

Diagram 2. The product - co-product duality

The Diagram 2 imply that whenever product (co-product) is in variety then co-product (product) is in ‘the same’ co-variety. Diagram 2 extends to duality between \( n \)-ary multiplication and \( n \)-ary co-multiplication as shown for tern-ary operations, \( 1 \leftrightarrow 3 \), on Diagram 3.

Diagram 3. The tern-ary co-product - product duality

Diagram 4. The bin-ary bi-gebra with one (pre)-braid

A tern-ary bi-gebra is defined on Diagram 5 and this made clear the definition of the multi-ary bigebra.
2. Clifford Co-gebra and Antipode

Dirac in 1928 predicted the existence of an anti-matter, spin $\frac{1}{2}$ positrons, in terms of the Clifford algebra. To understand an anti-matter for any spin we need an action of the Clifford algebra $\mathcal{Cl}$ in a tensor product of $\mathcal{Cl}$-modules. The differential Dirac operator for mesons of zero and higher spins needs an action of the Clifford algebra on a tensor product of Clifford algebras and this action is illustrated on Diagram 6. This action depends on chosen co-product, the simplest one is known as the Duffin & Kemmer & Petiau co-product [Duffin 1938, Kemmer 1939, 1943]. The main problem is the classification of the co-products which fit the Clifford algebra into bi-gebra. The present paper is the introduction into this subject.

A category is said to be autonomous if $\forall M \in \text{obj}$, $\exists!$ a left dual $M^*$ and a right dual $^*M$ [Freyd & Yetter 1992]. An autonomous category is said to be pivotal if $M^* \simeq ^*M$.

Let $M$ be $k$-module and $\eta$ and $\xi$ be scalar products,

$$\eta \in \text{lin}(M, M^*) \simeq M^* \otimes M \simeq \text{lin}(M \otimes 2, k),$$

$$\xi \in \text{lin}(M^*, M) \simeq M \otimes 2 \simeq \text{lin}(M^* \otimes 2, k).$$

A pair of the mutually dual Clifford $k$-algebras is paired by determinant (scalar product independent),

$$\mathcal{Cl}(M, \eta) \simeq \{M^\wedge, \wedge^\eta\}, \quad \mathcal{Cl}(M^*, \xi) \simeq \{M^{*\wedge}, \wedge^\xi\}, \quad (2.1)$$

$$M^{*\wedge} \otimes M^\wedge \xrightarrow{\det} k. \quad (2.2)$$

A $\xi$-dependent co-multiplication $\Delta^\xi : M^\wedge \longrightarrow M^\wedge \otimes M^\wedge$ is calculated from the product - co-product duality of Diagram 2.
Diagram 6. A co-product dependent action of $\mathcal{C}_\ell$ on tensor product of $\mathcal{C}_\ell$-modules

In the sequel $\{e_\mu \in M\}$ denotes a basis and $\{\varepsilon_\mu \in M^* \equiv \text{lin}(M, k)\}$ is a dual basis, $\varepsilon_\mu e_\nu = \delta^\mu_\nu$. For $1 \in k < M^\wedge$ and $v, w \in M$,

$$\Delta^\xi 1 = 1 \otimes 1 + \sum \xi(\varepsilon^\mu \otimes \varepsilon^\nu)(e_\mu \otimes e_\nu)$$

$$- \sum \xi(\varepsilon^{\mu_1} \wedge \varepsilon^{\mu_2}, \varepsilon^{\nu_1} \wedge \varepsilon^{\nu_2})(e_{\nu_1} \wedge e_{\nu_2}) \otimes (e_{\mu_1} \wedge e_{\mu_2})$$

$$- \sum \xi(\text{tri-co-vectors} \otimes 2 \text{trivectors} \otimes 2 + \ldots) \left[ \frac{1}{2} \text{grade} \right] \ldots$$

$$\Delta^\xi v = 1 \otimes v + v \otimes 1 + \sum \xi(\varepsilon^\mu \otimes \varepsilon^\nu)(e_\mu \otimes (v \wedge e_\nu) - (v \wedge e_\nu) \otimes e_\mu) + \ldots$$

$$\Delta^\xi (v \wedge w) = 1 \otimes (v \wedge w) + (v \wedge w) \otimes 1 - v \otimes w + w \otimes v$$

$$+ \sum \xi(\varepsilon^\mu \otimes \varepsilon^\nu)[(w \wedge e_\mu) \otimes (e_\nu \wedge v) - (v \wedge e_\nu) \otimes (e_\mu \wedge w)] + \ldots$$

If $\xi = 0$ then $1 \in \mathcal{C}_\ell$ is group-like, vectors are $(1, 1)$-primitive and $\Delta^\xi = 0$ is the Duffin & Kemmer & Petiau co-product [Duffin 1938, Kemmer 1939, 1943].

The above co-product $\Delta^\xi$ (as well as (3.1) late on) is co-unital

(2.3) $\varepsilon \in \text{lin}(M^\wedge, k)$, $k \ni \varepsilon w = \begin{cases} 0 & \text{if grade } w \neq 0, \\ w & \text{if grade } w = 0. \end{cases}$

However co-unit (2.3) is not an algebra map iff $\eta \neq 0$ and unit $u \in \text{lin}(k, M^\wedge)$ is not cogebrat map iff $\xi \neq 0$,

$$\eta \neq 0 \iff \varepsilon \not\in \text{alg}(\wedge^n, k),$$

$$\xi \neq 0 \iff u \not\in \text{cog}(k, \Delta^\xi).$$

Conjecture 2.1. A condition $\xi \circ \eta \neq \text{id}_M$ is a necessary and sufficient condition that exists an antipode $S \in \text{End}(M^\wedge)$ (see example below).

If $\xi = 0$ then antipode exists and $S | M^\wedge 2 = 0$. 
3. Co-gebra (Co-field) of Co-complex Numbers and Antipode

In the sequel if \( \alpha \in M^* \) then \( \alpha^2 \in \kappa \) stands for \( \xi(\alpha \otimes \alpha) \) and if \( v \in M \) then \( v^2 \in \kappa \) stands for \( \eta(v \otimes v) \).

In sections 3-4 \( \dim_R M = 1 \). Let \( i \in M \). If \( \eta(i \otimes i) = -1 \) then \( \mathbb{C} \simeq \mathcal{C}(M, \eta) \).

If \( \xi \otimes \eta = \text{id}_M \) and \( \alpha v = 1 \in \kappa \), then \( \alpha^2 v^2 \equiv \xi(\alpha \otimes \alpha)\eta(v \otimes v) = 1 \).

For \( j \in \mathbb{C}^* \equiv \text{lin}_R(\mathbb{C}, \mathbb{R}) \),

\[
\wedge(1 \otimes 1) = 1, \quad \wedge(i \otimes i) = \wedge_0 i = \eta(i \otimes i) \in \mathbb{R},
\]

\[
\wedge(i \otimes 1) = \wedge_0 i = i, \quad \wedge(1 \otimes 1) = \wedge_0 1 = i,
\]

\[
\Delta_i = 1 \otimes 1 + \xi(j \otimes j)i \otimes i,
\]

\[
(\wedge, \Delta_i) = 1 \otimes i + i \otimes 1.
\]

For \( z \in \mathbb{C} \) and \( i^2 j^2 = 1 \), \( \Delta z = 1 \otimes z + \frac{1}{z} i \otimes iz \). From now on

\[
a \equiv i^2 j^2 = \eta(i \otimes i)\xi(j \otimes j) \in \mathbb{R}.
\]

**Proposition 3.1.** With respect to co-unit \((\wedge, \Delta_i)\), an antipode \( S \in \text{lin}(\mathbb{C}, \mathbb{C}) \) exists iff \( \xi \otimes \eta \neq \text{id}_M \), i.e. iff \( a \neq 1 \), and then

\[
S 1 = \frac{1}{1 - a}, \quad S i = -\frac{i}{1 - a}.
\]

4. Hopf-gebra and Bi-gebra of Complex Numbers

Diagram 4 is a relation among three tensors, a product \( \wedge \), co-product \( \Delta \) and a bin-ary scattering \( \sigma \). The purpose of this section is to determine from Diagram 4 the set of all possible scatterings \( \sigma \in \text{End}(\mathbb{C}^{\otimes 2}) \) for \( (\eta, \xi) \)-dependent product and coproduct \((\wedge, \Delta_i)\) on two-dimensional \( \mathbb{R} \)-space span by \( \{1, i\} \in \mathbb{C} \). Then the set \( \{\wedge, \Delta_i, \sigma\} \) is a bin-ary bi-gebra.

If \( i^2 j^2 \neq 1 \) then exists the unique scattering \( \sigma \in \text{End}_\mathbb{R}(\mathbb{C}^{\otimes 2}) \),

\[
\sigma(1 \otimes 1) = \left(1 - \frac{a^2}{1 - a}\right) 1 \otimes 1 - \frac{j^2}{1 - a} i \otimes i,
\]

\[
\sigma(i \otimes i) = -\frac{1}{1 - a} (i \otimes i + i^2 \cdot 1 \otimes 1),
\]

\[
\sigma(1 \otimes i) = \frac{1}{1 - a} (i \otimes 1 + a \cdot 1 \otimes i),
\]

\[
(\wedge, \Delta_i, \sigma) = 1 \otimes i + a \cdot i \otimes 1).
\]

The minimum polynomial of \((\wedge, \Delta_i, \sigma)\) is of the fourth order

\[
b = \frac{1 + i^2 j^2}{1 - i^2 j^2}, \quad (\sigma + \text{id}) \circ (\sigma - b \cdot \text{id}) \circ (\sigma^2 + ab \cdot \sigma - b \cdot \text{id}) = 0.
\]

Therefore \( \sigma \) is invertible iff \( i^2 j^2 \neq \pm 1 \). We conjecture that \( \sigma \in \text{End}(M^{\otimes 2}) \) is a (pre)-braid operator i.e. \( \sigma \) is a solution of the Artin braid equation (this
indeed is the case if $i^2j^2 = 0$), 

\begin{equation}
(\sigma \otimes \text{id}_M) \circ (\text{id}_M \otimes \sigma) \circ (\sigma \otimes \text{id}_M) = (\text{id}_M \otimes \sigma) \circ (\sigma \otimes \text{id}_M) \circ (\text{id}_M \otimes \sigma),
\end{equation}

Proposition 4.1. The Clifford bi-gebra $\{\mathbb{R}^2, \wedge, \Delta^\xi, \sigma\}$, (3.1)-(4.1), is $\sigma$-braided iff $\eta = 0$ or $\xi = 0$.

The formulas (3.1)-(3.2)-(4.1) describe two-parameter $\{i^2, j^2\}$-family of Hopf-gebras (and this include the field of complex numbers) for which neither unit nor co-unit are respecting co-product and product respectively.

If $\xi \circ \eta = \text{id}_C$ i.e. if $i^2j^2 = 1$, then exists 12-parameters family of mappings $\sigma \in \text{End}_k(\mathbb{C}^{\otimes 2})$ which fit to bi-gebra. Among other this include the following solution for $p + q + r = 0 \in \mathbb{R}$,

\begin{align*}
\sigma(1 \otimes 1) &= 1 \otimes 1, \\
\sigma(1 \otimes i) &= i \otimes 1 + p \cdot 1 \otimes i, \\
\sigma(i \otimes 1) &= 1 \otimes i + q \cdot i \otimes 1, \\
\sigma(i \otimes i) &= r \cdot i \otimes i - i^2 \cdot 1 \otimes 1.
\end{align*}

5. Bi-universal Hopf-gebra

| Some notation | Description |
|---------------|-------------|
| $\mathbb{k}$  | is a commutative ring |
| $\mathbb{k}$-mod | a category of $\mathbb{k}$-modules (of $\mathbb{k}$-alphabets) |
| $\mathbb{k}$-alg | a category of associative unital $\mathbb{k}$-algebras |
| $T : \mathbb{k}$-mod $\to$ $\mathbb{k}$-alg | the tensor algebra functor |
| $F : \mathbb{k}$-alg $\to$ $\mathbb{k}$-mod | the forgetful functor; |
| $\otimes$ | bifunctor of tensor product: $\mathbb{k}$-mod $\times$ $\mathbb{k}$-mod $\to$ $\mathbb{k}$-mod. |
| $\otimes$ | means $\otimes_k$ if not otherwise stated; |
| lin $\equiv$ lin$_k$, End $\equiv$ End$_k$ | are both sided $\mathbb{k}$-linear bifunctors; |
| $M \in \mathbb{k}$-mod | is a $\mathbb{k}$-module (a $\mathbb{k}$-alphabet); |
| $M^{\otimes} = FTM$ | a $\mathbb{Z}$-graded $\mathbb{k}$-vocabulary in a $\mathbb{k}$-alphabet $M$; |
| $M^* \equiv$ lin$(M, \mathbb{k})$ | a dual $\mathbb{k}$-module of co-vectors. |

A bi-associative (i.e. an associative and co-associative) and bi-unital (i.e. unital and co-unital) Hopf-gebra in a braided monoidal category (≡ a braided Hopf-gebra or a ‘braided group’) has been introduced by Majid in series of papers in years.
1991-1993. In [Oziewicz et al. 1995] we generalized a braided Hopf-algebra to pre-braided Hopf-algebra when a braid needs not to be invertible. This generalization was motivated by the following problem: does exist pre-braid for which exists a pre-braided bi-universal (i.e. universal and co-universal) Hopf-algebra? This is illustrated by Diagram 4 in the case of the fixed universal product and of the co-universal co-product. We showed that pre-braided bi-universal bi-associative and bi-unital Hopf-algebra exists for zero pre-braid only [Oziewicz et al. 1995].

For a \( k \)-module \( M \), \( M^\otimes \) denotes \( \mathbb{Z} \)-graded \( k \)-module (not an algebra) i.e. a totality of all finite sentences in \( M \). By definition functors \( T \) and \( F \) are adjoint [Kan 1958]: bifunctors \( \text{lin}_{\mathbb{k}}(\cdot, F \cdot) \) and \( \text{alg}_k(T \cdot, \cdot) \) are naturally equivalent. This means that a natural set bijection holds,

\[
\forall M \times A \in \mathbb{k}\text{-mod} \times \mathbb{k}\text{-alg}, \quad \text{lin}_k(M, FA) \ni \ell \mapsto \ell^A \in \text{alg}_k(TM, A), \quad \ell^A | M \equiv \ell.
\]

(5.1)

\[
\ell^m \equiv \ell^A = m^\otimes \circ \ell^\otimes \in \text{alg}(TM, A),
\]

\[
\ell^\Delta \equiv \ell^C = \ell^\otimes \circ \Delta^\otimes \in \text{cog}(C, ShM),
\]

\[\begin{array}{ccc}
M & \xrightarrow{\text{injection}} & M^\otimes \\
\downarrow \ell & & \downarrow \ell^\otimes \\
A^\otimes & \xrightarrow{\ell^\otimes} & M^\otimes \\
\downarrow m^\otimes & & \downarrow \ell^A \\
A & \xrightarrow{\ell^A} & M
\end{array}\]

\[\begin{array}{ccc}
M & \xrightarrow{\text{projection}} & M^\otimes \\
\downarrow \ell & & \downarrow \ell^\otimes \\
C^\otimes & \xrightarrow{\ell^\otimes} & M^\otimes \\
\downarrow \Delta^\otimes & & \downarrow \ell^C \\
C & \xrightarrow{\ell^C} & M
\end{array}\]

Diagram 7. The universal and co-universal lifts

An example of realization of \( M \)-universal tensor \( k \)-algebra is \( TM \simeq \{M^\otimes, \otimes\} \)

\( i.e. \) a \( \mathbb{Z} \)-graded \( k \)-module \( M^\otimes \) of all finite words in an alphabet \( M \) with a concatenation \( \otimes \) as a multiplication of grade \( \otimes = 0 \).

An example of realization of \( M \)-co-universal co-algebra is \( ShM \simeq \{M^\otimes, \text{sh}\} \), \( \text{sh} \) is short for ‘shuffle’ [Sweedler 1969], \( T(M^*) \simeq (ShM)^* \).

A deformation of bi-algebra is said to be preferred if either product or co-product is not deformed [Gerstenhaber & Schack 1992, Bonneau et al. 1994]. One can consider two pre-braid dependent bi-associative preferred deformations of 0-braided bi-universal Hopf-algebra: an universal Hopf-algebra which is not co-universal (an universal product is not deformed) and co-universal Hopf-algebra which is not universal (a co-universal shuffle co-product is not deformed). These families generalize for arbitrary pre-braid the Sweedler construction for the switch [Sweedler 1969, chapter XII].

There exists an unique pre-braid dependent homomorphism (extending an identity mapping on generating space) of universal Hopf-algebra into co-universal Hopf-algebra [Oziewicz et al. 1995, Różański 1996]. This Hopf-algebra homomorphism
1. is a pre-braid dependent deformation of an identity,
2. commutes with an antipod and
3. for invertible braid coincide with the braid-dependent ‘(anti)symmetrizer’ introduced by Woronowicz in 1989.

The image of this Hopf-gebra homomorphism is said to be an exterior Hopf-gebra. An exterior Hopf-gebra is co-universal and pre-braided.

An open problem is to find necessary and sufficient conditions (on braid, scalar product, Lie algebra, etc.) that exists pre-braided filtered algebra as tensor-dependent quantizations and (the Chevalley) deformations of exterior Hopf-gebra.

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