EXTREMALS ON LIE GROUPS WITH ASYMMETRIC POLYHEDRAL FINSLER STRUCTURES

J. B. PRUDENCIO AND R. FUKUOKA

DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF MARINGÁ, 87020-900, MARINGÁ, PR, BRAZIL

Abstract. In this work we study extremals on Lie groups $G$ endowed with a left invariant polyhedral Finsler structure. We use the Pontryagin’s Maximal Principle (PMP) to find curves on the cotangent bundle of the group, such that its projections on $G$ are extremals. Let $\mathfrak{g}$ and $\mathfrak{g}^*$ be the Lie algebra of $G$ and its dual space respectively. We represent this problem as a control system $a'(t) = -\operatorname{ad}^* (u(t))(a(t))$ of Euler-Arnold type equation, where $u(t)$ is a measurable control in the unit sphere of $\mathfrak{g}$ and $a(t)$ is an absolutely continuous curve in $\mathfrak{g}^*$. A solution $(u(t), a(t))$ of this control system is a Pontryagin extremal and $a(t)$ is its vertical part. In this work we show that for a fixed vertical part of the Pontryagin extremal $a(t)$, the uniqueness of $u(t)$ such that $(u(t), a(t))$ is a Pontryagin extremal can be studied through an asymptotic curvature of $a(t)$.

1. Introduction

Let $M$ be a differentiable manifold, $T_x M$ the tangent space of $M$ at $x$ and $T^*_x M$ the cotangent space of $M$ at $x$. Let $TM = \{(x, y); x \in M, y \in T_x M\}$ and $T^*M = \{(x, \xi); x \in M, \xi \in T^*_x M\}$ be the tangent and cotangent bundles of $M$ respectively.

A $C^0$-Finsler structure on $M$ is a continuous function $F : TM \to \mathbb{R}$ such that $F(x, \cdot) : T_x M \to \mathbb{R}$ is an asymmetric norm for every $x \in M$ (see Definition 2.1). Polyhedral Finsler structures (or $p$-Finsler structures) are $C^0$-Finsler structures such that its restriction to tangent spaces are asymmetric norms which closed unit balls are polyhedra with the origin in its interior.

We use the term Finsler structures for the smooth case (see [7]). We can see, even in simple examples, that the behavior of geodesics observed in Riemannian and Finsler manifolds are not satisfied in $C^0$-Finsler manifolds. For instance, geodesics don’t need to be smooth and given $p$ in $M$ and $v \in T_p M$, we can have multiple geodesics $\gamma : (-\varepsilon, \varepsilon) \to M$ satisfying $\gamma(0) = p$ and $\gamma'(0) = v$ or else no geodesics at all (see [11], [13]).

Differential calculus is one of the main techniques to study Finsler manifolds, but it can not be applied directly on $C^0$-Finsler manifolds. We need other techniques as we explain in sequel.

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Metric geometry is one of these techniques. It can be used, for instance, to prove
the local existence of minimizing curves that connect two points (see [9, 18]). It can
also be used to calculate explicitly geodesics for some specific $C^0$-Finsler manifolds
(see [11]).

Another approach that can be useful for the study of $C^0$-Finsler manifolds $(M, F)$
is to approximate $F$ by a one-parameter family of Finsler structures $F_\varepsilon$. In [12],
the authors proved that we can approximate a $C^0$-Finsler structure $F$ by a one-
parameter family $F_\varepsilon$ of Finsler structures such that if $F$ is a Finsler structure,
then the various connections of the Finsler geometry and the flag curvature of $F_\varepsilon$
converge uniformly in compact subsets for the respective objects of $F$ when $\varepsilon$ tends
to zero. The technique used there was convolution with mollifiers.

We can assume that a $C^0$-Finsler structure has some kind of horizontal dif-
terentiability and we say that these structures are of Pontryagin type (See Item
3 of Definition 3.1). What is gained with this additional condition is that the
Pontryagin’s Maximum Principle (PMP) of the optimal control theory can be ap-
plied and the geodesic field of Finsler Geometry (and consequently, of Riemannian
Geometry) can be generalized: in [21], the authors define the extended geodesic
field $\mathcal{E}$ of a Pontryagin type $C^0$-Finsler manifold, which is a multivalued mapping
that associates each $(x, \xi) \in TM^*\setminus 0$ to a subset of $T_x^*(M^\ast \setminus 0)$. The integral
curves $(x(t), \xi(t))$ of the differential inclusion $(x'(t), \xi'(t)) \in \mathcal{E}(x(t), \xi(t))$ are the
Pontryagin extremals. The projection $x(t)$ of the Pontryagin extremals $(x(t), \xi(t))$
are extremals of $(M, F)$ and minimizing curves parametrized by arclength on the
domain of definition of the control vector fields are always extremals.

The use of the PMP for the study of extremals in $C^0$-Finsler manifolds was
available since its creation by Pontryagin and his students [20], because it deals
with variational problems on non-differentiable length structures. But in practice,
this type of approach took a long time to become popular. One of the first works
that proposes the use of PMP for a wide class of geometric problems is [2].

Now we present some works in the literature that are related to the subject of
this paper. This list is not exhaustive and some relevant but not directly related
subjects (such as sub-Riemannian geometry) were left out. For those interested in
this topic, see [1].

The PMP is well suited for Lie groups $G$ endowed with left invariant $C^0$-Finsler
structures because they are of Pontryagin type. Actually, a more general result
holds: Let $M$ be a differentiable manifold and $G$ a transitive group of transforma-
tions over $M$. Suppose that a $C^0$-Finsler structure $F$ on $M$ is $G$-invariant. Then
$F$ is of Pontryagin type (see [21]). In this work, we denote the Lie algebra of $G$ by $\mathfrak{g}$.

Let $M$ be a differentiable manifold endowed with a completely non-holonomic
distribution $\mathcal{D}$ with constant rank $d$. A $C^0$-sub-Finsler structure $F$ is a correspon-
dence such that every $x \in M$ is associated to an asymmetric norm defined on $\mathcal{D}(x)$
in a continuous way, that is, if $X$ is a continuous horizontal vector field on $M$, then
$F(X) : M \to \mathbb{R}$ is continuous. In a connected Lie group endowed with a left
invariant distribution, the left invariant $C^0$-sub-Finsler structures can be studied
using the PMP.

In [13], the author classify all minimizing paths on the 2-dimensional non-abelian,
simply connected Lie group endowed with a left invariant symmetric $C^0$-Finsler
structure \( F \) (the restriction of \( F \) to each tangent space is a norm). She uses the PMP and solves explicitly the resulting equations.

In [4], the authors study left invariant \( C^0 \)-sub-Finsler structures of rank 2 in Cartan groups. They choose two left invariant vector fields \( X_1 \) and \( X_2 \) on \( G \) that generate the distribution and consider the maximum norm with respect to the frame given by the basis \( \{X_1, X_2\} \). They demonstrate that extremals of singular type are minimizers and extremals of type "bang-bang" are geodesics.

In [5], the authors study \( C^0 \)-sub-Finsler structures of rank 2 in unimodular Lie groups of dimension 3. The technique used here is convex trigonometry and they also use the control system defined by two left invariant vector fields that generate the distribution. They study several cases of \( C^0 \)-sub-Finsler structures, for example, with the unit ball of \( F \) being polyhedral, strictly convex, \( L^p \) norm, etc.

Let \( G \) be a 2-step Carnot group, that is, a Lie group such that the Lie algebra can be decomposed as \( g = V_1 \oplus V_2 \), with \([V_1, V_1] = V_2 \) and \([V_1, V_2] = 0\). Consider a left invariant \( C^0 \)-sub-Finsler structure \( F \) defined on the distribution corresponding to \( V_1 \). In [15], the author proves that if \( F \) is strictly convex (the restriction of \( F \) to each tangent space is strictly convex), then isometries from \( \mathbb{R} \) to \( G \) are affine applications, that is, they are compositions of group homomorphism with left translations.

Many recent works use PMP on Lie groups with left invariant \( C^0 \)-sub-Finsler structures of rank 2. Among them we can cite [5, 8, 17, 22]. In these works, the authors calculate extremals explicitly and study their properties.

Now we present this work.

In Section 2 we fix notations and present the prerequisites that are necessary for the development of this work.

Let \( G \) be a Lie group endowed with a left invariant \( C^0 \)-Finsler structure \( F \). We consider the control system given by the left invariant unit vector fields. The controls are measurable functions \( u(t) \) defined on the unit sphere \( S_{F_c} \) of \((g, F_c)\), where \( F_c \) is the restriction of \( F \) to \( g \). In Section 3 we obtain the Lie algebra equivalent of the differentiable inclusion \((x'(t), \xi'(t)) \in \mathcal{E}(x(t), \xi(t))\)), which is an Euler-Arnold type equation \( a'(t) = -\text{ad}^*\left((u(t))(a(t))\right) \), where \( a(t) \) is an absolutely continuous curve on the dual space \( g^* \) of \( g \) and the measurable control \( u(t) \) maximizes \( a(t) \) on \( S_{F_c} \) (for Euler-Arnold equation, see [6]). A solution \((u(t), a(t))\) of \( a'(t) = -\text{ad}^*\left((u(t))(a(t))\right) \) is called a Pontryagin extremal of \((G, F)\) and \( a(t) \) is its vertical part.

In Section 4 we introduce the limit flag curvature for Lie groups endowed with a left invariant \( p \)-Finsler structure. Let \( a \in g^* \setminus \{0\} \) be a functional and \( v_1 \) be a point that maximizes \( a \) in \( S_{F_c} \). Notice that if \( L \) is a face of \( S_{F_c} \) containing \( v_1 \), then \( L \subset \{v_1\} + \ker a \). We are interested in defining a curvature of a plane span\(\{v_1, v_2\}\) with respect to \( a \), where \( v_2 \) is chosen in \( \ker a \) for convenience. We consider a family of left invariant Riemannian metrics \( \{g_k\}_{k \in \mathbb{N}} \) on \( G \) such that the respective sequence of unit spheres \( S_{g_k} \subset (g, g_k|_g) \) centered at the origin contains \( v_1 \) and converges locally to \( \{v_1\} + \ker a \) in a neighborhood of \( v_1 \) when \( k \to \infty \). The sequence of sectional curvatures of span\(\{v_1, v_2\} \subset (g, g_k|_g) \) is a polynomial in \( k \) and we denote the coefficient of highest degree of this polynomial by \( K_B(a) \) (the notation \( B \) and technical details are explained in Section 4).

Let \( F_* \) be the dual norm of \( F_c \) on \( g^* \). \( F_* \) is also a polyhedral asymmetric norm. Due to the PMP, the vertical part of a Pontryagin extremal lies on a sphere \( S_{F_*}, [0, r] \subset g^* \). In Section 5 we study Lie groups endowed with a left invariant \( p \)-Finsler structure. We prove that the equation \( K_B(a) = 0 \ (\neq 0) \) depends only on
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and \( v_2 \) and we denote it by \( K(a, v_2) = 0 \) \(( \neq 0)\) (see Proposition 4.8 and Remark 4.9). If \( G \) is a three-dimensional Lie group, we prove that \( K(a, v_2) \) doesn’t depend on \( v_2 \) (see Proposition 5.7 and Remark 5.8). Theorems 5.5, 5.6, 5.10 and 5.11 are the main results of this work and they relate two properties of the vertical part \( a(t) \) of a Pontryagin extremal: \( a(t) \) admitting infinitely many controls \( \tilde{u}(t) \) such that \( (\tilde{u}(t), a(t)) \) is a Pontryagin extremal and the set \( I = \{ t \in I: \text{there exist } v_2 \in \ker a(t) \backslash \{0\} \text{ such that } K(a(t), v_2) = 0 \} \) having positive measure.

Finally Section 6 is devoted to final remarks and suggestions for future works.

In Finsler (and Riemannian) Geometry, geodesics and curvature are related by Jacobi fields. The main contribution of this work is to show that an asymptotic curvature can be defined on \( g^* \) and that it can be related with the behaviour of extremals in \( p \)-Finsler manifolds.

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2. Preliminaries

In this section we fix some notations and present some definitions that are used in this work. For asymmetric norms, compare with [10]. For convex polyhedral sets, see [14]. In Theorem 2.20 we prove the existence of \( p \)-Finsler structures on a smooth manifold. It is an immediate consequence of the theory presented here. Meanwhile, we also present some subjects of \( C^0 \)-Finsler geometry, convex geometry and duality between asymmetric norms that will be useful afterward.

Let \( V \) be a finite dimensional real vector space and \( V^* \) its dual space. We consider the usual topology on \( V \) and \( V^* \). If \( X \) is a subset of a topological space, its closure, interior and boundary will be denoted by \( \bar{X}, \text{int } X \) and \( \partial X \) respectively.

**Definition 2.1.** An asymmetric norm on \( V \) is a nonnegative function \( F : V \to \mathbb{R} \) such that

1. \( F(y) = 0 \) iff \( y = 0 \);
2. \( F(\alpha y) = \alpha F(y) \) for every \( \alpha > 0 \) and \( y \in V \) (positive homogeneity);
3. \( F(y + w) \leq F(y) + F(w) \) for every \( y, w \in V \).

The closed ball centered at \( y \in (V, F) \) with radius \( r \geq 0 \) is defined as

\[ B_F[y, r] = \{ w \in V; F(w - y) \leq r \} \]

and analogous notations hold for open balls \( B_F(y, r) \) and spheres \( S_F[y, r] \). The spheres centered at the origin with radius 1 will be simply denoted by \( S_F \).

In particular, a norm is an asymmetric norm.

The proof of the next proposition is relatively simple, so we omit it.

**Proposition 2.2.** Let \( F_1 \) and \( F_2 \) be asymmetric norms on a finite dimensional vector space \( V \) over \( \mathbb{R} \). Then there exist constants \( c, C > 0 \) such that

\[ c F_1(y) \leq F_2(y) \leq C F_1(y) \]

for every \( y \in V \). Moreover, if \( F \) is an asymmetric norm, then \( F \) is continuous.

**Definition 2.3.** A \( C^0 \)-Finsler structure \( F \) on a differentiable manifold \( M \) is a continuous function \( F : TM \to \mathbb{R} \) such that its restriction to each tangent space is an asymmetric norm. A differentiable manifold endowed with a \( C^0 \)-Finsler structure is a \( C^0 \)-Finsler manifold.
Now we present some fundamental theory of $C^0$-Finsler manifolds.

**Proposition 2.4.** Let $F_1$ and $F_2$ be two $C^0$-Finsler structures on $M$. Then every $p \in M$ admit a neighborhood $U$ and $c, C > 0$ such that

$$c.F_1(x, y) \leq F_2(x, y) \leq C.F_1(x, y)$$

for every $(x, y) \in TU$.

**Proof.** Fix a Riemannian metric $g$ on $M$ and let $SM$ be the sub-bundle of unit vectors of $TM$ with respect to $g$. If we choose a neighborhood $U$ of $p$ with compact closure in $M$ and set

$$C = \sup_{(x, y) \in SU} \frac{F_2(x, y)}{F_1(x, y)} \quad \text{and} \quad c = \inf_{(x, y) \in SU} \frac{F_2(x, y)}{F_1(x, y)},$$

then (1) holds. \qed

**Definition 2.5.** Let $\gamma : [a, b] \to (M, F)$ be a path which is continuously differentiable by parts. The length of $\gamma$ with respect to $F$ is defined by

$$\ell_F(\gamma) = \int_a^b F(\gamma(t), \gamma'(t)).dt.$$ 

We say that $\gamma$ is minimizing if $\ell_F(\gamma) \leq \ell_F(\eta)$ for every continuously differentiable by parts $\eta$ that connects $\gamma(a)$ and $\gamma(b)$. We say that $\gamma$ is a geodesic if $\gamma$ is locally minimizing, that is, if for every $t_0 \in [a, b]$, there exists a neighborhood $(t_0 - \varepsilon, t_0 + \varepsilon) \cap [a, b]$ such that $\gamma|_I$ is minimizing for every closed interval $I \subset (t_0 - \varepsilon, t_0 + \varepsilon) \cap [a, b]$.

**Remark 2.6.** If $\bar{\gamma}$ is the reverse curve of $\gamma$, then in general $\ell_F(\bar{\gamma}) = \ell_F(\gamma)$ doesn’t hold. Therefore $d_F : M \times M \to \mathbb{R}$ defined by

$$d_F(p, q) = \inf_{\gamma \in C^1_{p,q}} \ell_F(\gamma),$$

where $C^1_{p,q}$ is the family of paths that are continuously differentiable by parts and connects $p$ and $q$, isn’t a metric in general.

**Definition 2.7.** The open ball in $(M, F)$ with center $p$ and radius $r$ is defined by

$$B_F(p, r) = \{ q \in M ; d_F(p, q) < r \}.$$ 

The closed ball $B_F[p, r]$ and the sphere $S_F[p, r]$ are defined analogously.

**Remark 2.8.** If we choose a Riemannian metric $F_2 = g$ in (1) and consider that the family of open balls in a Riemannian manifold $(M, g)$ is a basis of the topology of the differentiable manifold $M$, then we can conclude that the family of open balls in $(M, F_1)$ is also a basis of the topology of $M$. In particular, the connected components of $M$ coincides with the connected components of $(M, F)$. In particular, $d_F(p, q) < \infty$ iff $p$ and $q$ lies in the same connected component of $M$.

Finally if $d_F$ is restricted to a connected component $M$ of $(M, F)$, then $d_F|_M$ is an asymmetric metric, that is, a mathematical object that satisfies every condition of a (finite) metric, except the symmetry.

**Definition 2.9.** Let $V$ be a finite dimensional vector space and $Z$ be a convex subset of $V$. The affine hull $\text{aff} Z$ of $Z$ is the smallest affine subset of $V$ that contains $Z$. The relative interior $\text{ri} Z$ of $Z$ is the interior of $Z$ considered as a subspace of $\text{aff} Z$. 
Now we return to the theory of polyhedral \( C^0 \)-Finsler structures. We begin with the theory on polyhedral asymmetric norms on finite dimensional real vector spaces \( V \). The definitions and results given here aren’t in the most general setting. We adapted the theory presented in [14] according to our necessities.

**Definition 2.10.** A non-empty subset of \( K \subset V \) is polyhedral if it is the intersection of a finite family of closed half spaces of \( V \).

**Definition 2.11.** A polyhedral asymmetric norm \( F : V \to \mathbb{R} \) on a vector space \( V \) is an asymmetric norm such that its closed unit ball \( B_F[0,1] \) is a polyhedral subset of \( V \).

**Definition 2.12.** A supporting hyperplane of a non-empty compact subset \( A \) of \( V \) is a hyperplane \( H \) of \( V \) such that \( A \) is contained in one of the closed half spaces determined by \( H \) and \( A \cap H \neq \emptyset \). A subset \( L \) of a polyhedral set \( K \subset V \) is a face of \( K \) if either \( L = \emptyset \), \( L = K \) or else if there exists a supporting hyperplane \( H \) of \( K \) such that \( L = K \cap H \). A 0-dimensional and a 1-dimensional face of \( K \) is also called a vertex and an edge of \( K \), respectively.

The polyhedral subsets we are interested in are closed balls \( B_F[0;r] \) in \( (V,F) \), which are bounded (compact) polyhedral subsets of \( V \) with the origin in its interior.

**Definition 2.13.** A maximal proper face of a polyhedral subset \( B \) of \( V \) is a facet of \( B \).

The next proposition gives a detailed information about the configuration of the faces of a compact polyhedral subset of \( V \) with non-empty interior (see Sections 2.6 and 3.1 of [14]). In order to follow Proposition 2.14 from this reference, some remarks are useful.

- Polytope is equivalent to bounded polyhedral subset;
- A subset of a polyhedral subset \( B \) of \( V \) is a poonem of \( B \) iff it is a face of \( B \);

**Proposition 2.14.** Let \( V \) be an \( n \)-dimensional real vector space and \( B \) be a bounded polyhedral subset of \( V \) with non-empty interior. Then

- There exists the smallest family of closed half spaces \( \{H_1^-, \ldots, H_m^-\} \) of \( V \) such that \( B = \cap_{i=1}^m H_i^- \). The affine hulls of the facets of \( B \) are \( H_i := \partial H_i^- \). In particular, the dimension of the facets is \( (n-1) \).
- Each facet of a facet of \( B \) is the intersection of two facets of \( B \);
- \( \partial B \) is the union of facets of \( K \);
- Every non-empty proper face of \( B \) is the the intersection of facets of \( B \).

Now we proceed studying polyhedral asymmetric norms.

**Remark 2.15.** The map \( \alpha \in V^* \setminus \{0\} \mapsto \alpha^{-1}(-\infty,1] \) is a bijection between \( V^* \setminus \{0\} \) and the family of closed half spaces that contains the origin in its interior.

**Proposition 2.16.** Let \( F \) be a polyhedral asymmetric norm. Suppose that \( B_F[0,1] = \bigcap_{i=1}^m H_i^- \), where \( H_i^- \) are closed half spaces. Then

\[
F = \max\{\alpha_1, \ldots, \alpha_m\},
\]

where \( \alpha_i \in V^* \setminus \{0\} \) and \( H_i^- = \alpha_i^{-1}(-\infty,1] \) for \( i \in \{1, \ldots, m\} \).
Definition 2.21. A differentiable manifold is called a polyhedral Finsler structure. By Proposition 2.20, every differentiable manifold admits a polyhedral Finsler structure.

Proposition 2.17. Let \( \{\alpha_1, \ldots, \alpha_m\} \subset V^* \setminus \{0\} \) be a family of functionals such that for every \( y \in V \setminus \{0\} \), there exist \( i \in \{1, \ldots, m\} \) satisfying \( \alpha_i(y) > 0 \). Then \( \max\{\alpha_1, \ldots, \alpha_m\} \) is a polyhedral asymmetric norm.

Proof. It is straightforward that Items (1) and (2) of Definition 2.1 holds for \( \max\{\alpha_1, \ldots, \alpha_m\} \). For the triangular inequality, if \( y, w \in V \), then there exist \( j \in \{1, \ldots, m\} \) such that
\[
\max_{i=1,\ldots,m} \alpha_i(y + w) = \alpha_j(y + w)
\]
and
\[
\max_{i=1,\ldots,m} \alpha_i(y + w) = \alpha_j(y) + \alpha_j(w) \leq \max_{i=1,\ldots,m} \alpha_i(y) + \max_{i=1,\ldots,m} \alpha_i(w),
\]
what settles the proposition.

Lemma 2.18. Let \( F_1 \) and \( F_2 \) be polyhedral asymmetric norms and \( a_1, a_2 > 0 \). Then \( a_1F_1 + a_2F_2 \), is a polyhedral asymmetric norm.

Proof. It is enough to prove that if \( F_1 \) and \( F_2 \) are polyhedral asymmetric norms and \( a > 0 \), then \( aF_1 + aF_2 \) and \( aF_1 \) are polyhedral asymmetric norms. The latter statement is trivial. For the proof of the former statement, set \( F_1 = \max_{i=1,\ldots,k} \alpha_i \) and \( F_2 = \max_{j=1,\ldots,m} \beta_j \), where \( \{\alpha_i\}_{i=1,\ldots,k} \) and \( \{\beta_j\}_{j=1,\ldots,m} \) are families of linear functionals satisfying the conditions of Proposition 2.17. Notice that
\[
F_1 + F_2 = \max_{i=1,\ldots,k} \{\alpha_i\} + \max_{j=1,\ldots,m} \{\beta_j\} = \max_{i=1,\ldots,k} \{\alpha_i + \beta_j\}
\]
and \( F_1 + F_2 \) is a polyhedral asymmetric norm due to Proposition 2.17.

Definition 2.19. Let \( M \) be a differentiable manifold. A polyhedral Finsler structure (or shortly, p-Finsler structure) \( F \) on \( M \) is a \( C^0 \)-Finsler structure such that \( F_x := F(x, \cdot) : T_xM \to \mathbb{R} \) is a polyhedral asymmetric norm for every \( x \in M \).

Proposition 2.20. Every differentiable manifold \( M \) admits a p-Finsler structure.

Proof. Let \( \{(U_i, x)\}_{i \in \mathbb{N}} \) be a locally finite cover by coordinate neighborhoods on \( M \). Let \( \{\eta_i : M \to \mathbb{R}\}_{i \in \mathbb{N}} \) be a smooth partition of unity subordinated to \( \{(U_i, x)\}_{i \in \mathbb{N}} \). Consider these coordinate neighborhoods endowed with constant p-Finsler structures \( F_i(x, y) = F_i(y) \) and set \( F = \sum_{i \in \mathbb{N}} \eta_i F_i \). Then this sum is locally finite and \( F \) is a p-Finsler structure on \( M \) due to Lemma 2.18.

Definition 2.21. A differentiable manifold \( M \) endowed with a p-Finsler structure is called a p-Finsler manifold.
We end this section with some results about an asymmetric norm $F$ and its dual asymmetric norm $F_*$. The basic reference here is Section 3.4 of [14]. It is necessary to remark that in this work, the author study finite dimensional vector space $V$ endowed with an inner product $\langle \cdot, \cdot \rangle$, and several types of duality are between objects of $(V, \langle \cdot, \cdot \rangle)$. But the inner product induces an isomorphism $v \mapsto \langle v, \cdot \rangle$ between $V$ and $V^*$, and this isomorphism induces the corresponding duality between objects of $V$ and $V^*$ that will be used in this work.

**Remark 2.22.** Let $F_* : V^* \to \mathbb{R}$ be the dual asymmetric norm

$$F_*(a) = \max_{v \in B_F[0,1]} a(v)$$

of $F$. Then the unit ball $B_F[0,1] \subset V^*$ is the polar set of $B_F[0,1]$. Therefore $B_F[0,1]$ is a polyhedral subset of $V^*$ and consequently $F_*$ is a polyhedral asymmetric norm (see Exercise 5 (viii), Section 3.4 of [14]). The inclusion reversing correspondence $\Psi$ between the faces of $S_F$ and $S_{F_*}$ is given as follows: If $L$ is a $k$-dimensional face of $S_F$, then $L^* = \Psi(L) = \{a \in S_{F_*} ; a(v) = 1 \text{ for every } v \in L\}$ is a $(n-k-1)$-dimensional face of $S_{F_*}$. Similarly $\Psi^{-1}$ is given by $L = \Psi^{-1}(L^*) = \{v \in S_F ; a(v) = 1 \text{ for every } a \in L^*\}$ (see [14] Section 3.4).

**Remark 2.23.** Given $a \in \text{ri} L^*$, we claim that $a$ is maximized in $S_F$ by $L = \Psi^{-1}(L^*)$. In fact, notice that $a(v) = 1$ for every $v \in L$ due to $L = \{v \in S_F ; a(v) = 1 \text{ for every } a \in L^*\}$. But we can’t have $a(v) = 1$ for $v \notin L$ otherwise $a$ would be maximized by a face properly containing $L$, which would imply that $a$ lies in a face of $S_F$, properly contained in $L^*$, contradicting $a \in \text{ri} L^*$.

### 3. The Extended Geodesic Field

In [21] the authors define the Pontryagin type $C^0$-Finsler structures which are structures that satisfy the minimum requirements of Pontryagin Maximum Principle.

**Definition 3.1.** A $C^0$-Finsler manifold $(M, F)$ is of Pontryagin type at $p \in M$ if there exist a neighborhood $U$ of $p$, a coordinate system $\phi = (x^1, \cdots, x^n) : U \to \mathbb{R}^n$, with the respective natural coordinate system $d\phi = (x^1, \cdots, x^n, y^1, \cdots, y^n) : TU \to \mathbb{R}^{2n}$ on the tangent bundle, and a family of $C^1$ unit vector fields

$$\{x \mapsto X_u(x) = (y^1(x^1, \cdots, x^n, u), \cdots, y^n(x^1, \cdots, x^n, u)); u \in S^{n-1}\}$$

on $U$ parametrized by $u \in S^{n-1}$ such that

1. $u \mapsto X_u(x)$ is a homeomorphism from $S^{n-1} \subset \mathbb{R}^n$ onto the unit sphere of $(T_xM, F(x, \cdot))$ for every $x \in U$;
2. $(x, u) \mapsto (y^1(x^1, \cdots, x^n, u), \cdots, y^n(x^1, \cdots, x^n, u))$ is continuous;
3. $(x, u) \mapsto \left(\frac{\partial y^1}{\partial x^i}(x^1, \cdots, x^n, u), \cdots, \frac{\partial y^n}{\partial x^i}(x^1, \cdots, x^n, u)\right)$ is continuous for every $i \in \{1, \cdots, n\}$.

We say that $F$ is of Pontryagin type on $M$ if it is of Pontryagin type for every $p \in M$.

**Remark 3.2.** This definition doesn’t depend on the choice of the coordinate system (see [21] Remark 3.2).
Denote $X_u(x) = f^i(x,u)\frac{\partial}{\partial x^i} = (f^1(x,u),\cdots,f^n(x,u))$ (From now on the Einstein summation convention is in place). The extended geodesic field is the rule $\mathcal{E}$ that associates each $(x,\xi) \in T^*U\backslash\{0\}$ to the set

$$\mathcal{E}(x,\xi) = \left\{ f^i(x,u)\frac{\partial}{\partial x^i} - \xi_j \frac{\partial f^j}{\partial x^i}(x,u) \frac{\partial}{\partial \xi_i}; \; u \in \mathcal{C}(x,\xi) \right\},$$

where $\mathcal{C}(x,\xi) = \{ u \in S^{n-1}; \; \xi(X_u(x)) = \max_{v \in S^{n-1}} \xi(X_v(x)) \}$. An absolutely continuous curve $\gamma : [a,b] \to T^*M$ such that $\gamma'(t) \in \mathcal{E}(\gamma(t))$ for almost every $t \in [a,b]$ is called a Pontryagin extremal of $(M,F)$.

As a consequence of Pontryagin Maximum Principle, if $x(t)$ is a length minimizer on $U$, then there exist a Pontryagin extremal $(x(t),\xi(t))$ of $(U,F)$ and

$$\mathcal{M}(x(t),\xi(t)) = \max_{u \in S^{n-1}} \xi(t)(X_u(x(t))) = F_u(x(t),\xi(t))$$

is constant.

Let $G$ be a Lie group with identity element $e$. Consider the left translation $L_x : G \to G$, $L_x(z) = x \cdot z$. Let $F : TG \to \mathbb{R}$ be a left invariant $C^0$-Finsler structure $F$, that is,

$$F(x,z,(dL_x)_z(y)) = F(z,y)$$

for every $x, z \in G$ and $y \in T_zG$. In [3], the authors prove that if $\varphi : \tilde{G} \times M \to M$ is a transitive differentiable action of a Lie group $\tilde{G}$ on a differentiable manifold $M$ and $F$ is a $\tilde{G}$-invariant $C^0$-Finsler structure on $M$, then $F$ is of Pontryagin type. In particular, Lie groups endowed with left invariant $C^0$-Finsler structures are of Pontryagin type.

From now on, $G$ is a Lie group endowed with a left invariant $C^0$-Finsler structure $F$. We denote its Lie algebra and its dual by $\mathfrak{g}$ and $\mathfrak{g}^*$ respectively. The aim of this section is to represent the extended geodesic field of $(G,F)$ on $\mathfrak{g}^*$. Here we replace the control set $S^{n-1}$ by the unit sphere $S_{F_e} \subset \mathfrak{g}$ because it is more convenient than the Euclidean sphere $S^{n-1}$.

In [3], the authors consider a Hamiltonian $h$ defined on $T^*G$ and they calculate the Hamiltonian system associated to $h$. They also consider the case where $h$ is left invariant and they calculate the reduction of the Hamiltonian system to $\mathfrak{g}^*$. These calculations can be adapted for our case where a control is also considered. But we will give the complete proof of the representation of the extended geodesic field of $(G,F)$ on $\mathfrak{g}^*$ because it is more helpful for the reader than to point out the necessary adaptations which are necessary to prove this result in a format that we need.

The left invariant vector fields $x \mapsto X_u(x) = d(L_x)_e(u)$ define a family of unit vector fields on $G$ parametrized by $u$ satisfying all conditions of the definition of Pontryagin type. Choose a basis $\{e_1,\cdots,e_n\}$ of $\mathfrak{g}$ and let $(x^1,\cdots,x^n)$ be a coordinate system in a neighborhood $U$ of $e$ such that $\frac{\partial}{\partial x^i}(e) = e_i$, $i \in \{1,\cdots,n\}$. For every $i \in \{1,\cdots,n\}$, let $X_i$ be the left invariant vector field such that $X_i(e) = e_i$. We can write

$$X_i(x) = d(L_x)_e(e_i) = b_i^j(x) \frac{\partial}{\partial x^j} \in TU,$$

where $b_i^j \in C^\infty(U)$. Since $X_u$ is left invariant,

$$f^i(x,u)\frac{\partial}{\partial x^i} = X_u(x) = d(L_x)_e(u) = d(L_x)_e(u^i e_i) = u^i b_i^j(x) \frac{\partial}{\partial x^j},$$

implying that $f^i(x,u) = u^i b_i^j(x)$.
Therefore, in a Lie group, the extended geodesic field \( E \) turns out to be

\[
E(x, \xi) = \left\{ u^i b^j_i(x) \frac{\partial}{\partial x^i} - \xi_j u^k \frac{\partial b^j_i}{\partial x^k}(x) \frac{\partial}{\partial \xi_i}; \quad u \in C(x, \xi) \right\}.
\] (5)

Now let us prove that \( E \) can be represented on \( g^* \). In what follows, \( \text{ad}^* \) is the infinitesimal coadjoint representation on \( g^* \) (see [24]).

**Theorem 3.3.** Let \((G, F)\) be a \(C^0\)-Finsler Lie group with Lie algebra \( g \) and its dual \( g^* \). Let \( \tilde{E} \) be the rule that associates each element of \( g^* \setminus \{0\} \) to the set

\[
\tilde{E}(a) = \{- \text{ad}^*(u)(a); \quad u \in C(a)\},
\] (6)

where \( C(a) := \{ u \in S_{F}; \quad a(u) = \max_{v \in S_{F}} a(v)\} \). Then finding Pontryagin extremals of \((G, F)\) is equivalent to find \((u(t), a(t))\) in \( S_{F} \times g^* \setminus \{0\} \) such that \( a(t) \) is a solution of the differential inclusion

\[
a'(t) \in \tilde{E}(a(t)),
\] (7)

and \( u(t) \in C(a(t)) \) is a measurable control.

**Proof.** First of all we prove that finding Pontryagin extremals \((x(t), \xi(t))\) of \((G, F)\) is equivalent to find \((u(t), a(t))\) in \( S_{F} \times g^* \setminus \{0\} \) such that \( u(t) \) is a measurable control, \( a(t) \) is an absolutely continuous function, \( a'(t) = -\text{ad}^*(u(t))(a(t)) \) and \( u(t) \in C(a(t)) \).

Let \((x(t), \xi(t))\) a Pontryagin extremal of \((G, F)\) and \( u(t) \in C(x(t), \xi(t)) \) the corresponding admissible control. Then,

\[
\begin{cases}
\frac{dx^i}{dt} = u^j(t) b^j_i(x(t)) \\
\frac{d\xi}{dt} = -\xi_j(t) u^k(t) \frac{\partial b^j_i}{\partial x^k}(x(t))
\end{cases}
\] (8)

Using the pullback of \( L_{x(t)} \) at \( e \), we define

\[
a(t) = d(L^*_{x(t)})_e(\xi(t)) = \xi(t)(d(L_{x(t)})_e) \in g^*.
\] (9)

Notice that given \( g \in g \), we have

\[
d(L_x)_e(g) = d(L_x)_e(y^i e_i) = y^i d(L_x)_e(e_i) = y^i b^j_i(x) \frac{\partial}{\partial x^j}.
\]

Then,

\[
a(t)(y) = d(L^*_{x(t)})_e(\xi(t))(y) = \xi(t)(d(L_{x(t)})_e(y)) = \xi_k(t) d x^k \left( y^j b^j_i(x(t)) \frac{\partial}{\partial x^j} \right) = \xi_j(t) b^j_i(x(t)) y^j,
\]

and writing \( a(t) = a_i(t) e^i \), where \( \{e^1, \ldots, e^n\} \) is the dual basis of \( \{e_1, \ldots, e_n\} \), we have

\[
a_i(t) = \xi_j(t) b^j_i(x(t)).
\] (10)
Calculating the derivative of the last equality and using the expressions of $\xi'_i$ and $(x^j)'$ from (8) we get

$$a'_i(t) = \xi'_i(t)b_k^i(x(t)) + \xi_j(t)(x^j)'(t)\frac{\partial b_k^j}{\partial x^j}(x(t))$$

$$= -\xi_j(t)u^k(t)\frac{\partial b_k^j}{\partial x^j}(x(t))b'_j^i(x(t)) + \xi_j(t)u^k(t)b_k^j(x(t))\frac{\partial b_k^j}{\partial x^j}(x(t))$$

$$= \xi(t)\left[u^k(t)b_k^j(x(t))\frac{\partial}{\partial x^j}b'_j^i(x(t))\frac{\partial}{\partial x^j}\right]$$

$$= \xi(t)[X_{u(t)}(x(t)), d(L_{x(t)})e(e_i)]$$

$$= \xi(t)\circ d(L_{x(t)})e[u(t), e_i]$$

$$= a(t)(\text{ad}(u(t))(e_i))$$

$$= -\text{ad}^*(u(t))(a(t))(e_i).$$

Therefore

$$a'(t) = -\text{ad}^*(u(t))(a(t)), \quad (12)$$

where $u(t) \in C(a(t)) := \{u \in S_{F_e}; a(t)(u) = \max_{v \in S_{F_e}} a(t)(v)\}$ is an admissible control and

$$\mathcal{M}(a(t)) = \max_{u \in S_{F_e}} a(t)(u) \quad (13)$$

is constant. Therefore a Pontryagin extremal $(x(t), \xi(t))$ induces a function $(u(t), a(t))$ which is essentially $(x'(t), \xi(t))$ represented in $g \times g^*$ and satisfies (12).

Reciprocally, fix a point $x_0 \in G$ and a solution $(u(t), a(t))$ of (12) such that $a(t)$ is absolutely continuous and $u(t) \in C(a(t))$ is bounded and measurable. Then, by the Carathéodory existence and uniqueness theorem (see [16]), there is a unique absolutely continuous curve $x(t)$ which is solution of

$$\dot{x} = d(L_{x(t)})e(u(t)) = X_{u(t)}(x) \quad (14)$$

and $x(0) = x_0$. The proof that this solution extends to the whole interval of definition $I$ of $(u(t), a(t))$ can be done adapting the proof of the classical Hopf-Rinow theorem of Riemannian geometry. Fix a left-invariant Riemannian metric $g$ on $G$. Then $(G, g)$ is a complete Riemannian manifold. Notice that $e^\ell_F(x) \leq \ell_F(x) \leq C e^\ell_g(x)$ due to Proposition 1 and the left invariance of $F$ and $g$. If $(t_i)$ is a Cauchy sequence in $I$, then $(x(t_i))$ is a Cauchy sequence in $(G, g)$ because $x(t)$ is parametrized by arclength in $(G, F)$. Therefore any solution $x(t)$ of (14) defined on an open interval $(a, b)$, with $b \in I$, can be extended continuously to $b$. If $b$ is an interior point of $I$, then the solution of (14) can be extended beyond $b$. Consequently $x(t)$ can be extended to $I$ and

$$\frac{dx^i}{dt} = u^j(t)b_j^i(x(t))$$

holds in $I$.

Define $\xi(t) = d(L_{x(t)}^{-1})x(t)(a(t)) \in T_{x(t)}^*G$, which is equivalent to (11) and implies (10). Since the product of two absolutely continuous functions defined on a closed interval is absolutely continuous and $a(t)$ is absolutely continuous, so is $\xi$ due to (10), the smoothness of $b_j^i$ and the fact that the matrix $(b_j^i(x))$ is everywhere
invertible. Moreover, from (10) and (12), we get (11) and

\[-\xi_j(t)u^k(t) \frac{\partial b_i^j}{\partial x^k}(x(t))b_i^l(x(t)) + \xi_j(t)(b_i^l(x(t)))' = a_i'(t) = (\xi_j(t)b_i^j(x(t)))'.\]

Therefore

\[
[(\xi_i(t))' + \xi_j(t)u^k(t) \frac{\partial b_i^j}{\partial x^k}(x(t))]b_i^l(x(t)) = 0
\]

and since the matrix \((b_i^j(x))\) is invertible, we have

\[
\frac{d\xi_i(t)}{dt} = -\xi_j(t)f^k(e, u(t))\frac{\partial b_i^j}{\partial x^k}(x(t))
\]

for every \(l \in \{1, \cdots, n\}\). Thus \((x(t), \xi(t))\) is solution of (8) for \(u(t) \in C(a(t))\).

Finally notice that finding \((u(t), a(t))\) in \(S_F \times g^* \setminus \{0\}\) such that \(a(t)\) is a solution of (6) and \(u(t) \in C(a(t))\) is a measurable control is equivalent to find \((u(t), a(t))\) satisfying (12), what settles the proof. \(\square\)

**Definition 3.4.** A solution \((u(t), a(t))\) of \(a'(t) = -ad^*(u(t))(a(t))\), where \(u(t) \in C(a(t))\) is a measurable function, is also called a Pontryagin extremal of \((G, F)\) and \(a(t)\) is called the vertical part of a Pontryagin extremal.

**Remark 3.5.** Notice that the solution \(a(t)\) of the differential inclusion \(a'(t) \in \tilde{E}(a(t))\) isn’t enough to determine a Pontryagin extremal. We also need a control \(u(t)\) such that \((u(t), a(t))\) is a Pontryagin extremal.

4. The Limit Flag Curvature

In this section, we introduce the asymptotic expansion of the flag curvature for \(p\)-Finsler Lie groups with respect to \(a \in g^* \setminus \{0\}\). This definition is given through approximation by sectional curvatures of a family of Riemannian metrics.

**Remark 4.1.** The term “flag curvature” comes from Finsler geometry (see [7]). It is a generalization of sectional curvature of Riemannian geometry. Let us explain with more details.

In Riemannian geometry, the Riemannian metric \(g\) is determined by its components \(g_{ij} = g(\partial/\partial x_i, \partial/\partial x_j)\) with respect to a coordinate system \(\{x^1, \cdots, x^n\}\) on \(M\). It is well known that the sectional curvatures are determined in terms of \(g_{ij}\).

Let \(\pi : TM \to M\) be the natural projection. In Finsler geometry, the fundamental tensor \(\tilde{g}\) is defined on the pulled back tangent bundle \(\pi^*TM\). In the natural coordinate system \((x, y) = (x^1, \cdots, x^n, y^1, \cdots, y^n)\) of the slit tangent bundle \(TM \setminus 0 := \{(x, y) \in TM, y \neq 0\}\), \(\tilde{g}\) is given by

\[
\tilde{g}_{ij}(x, y)dx^i \otimes dx^j = \frac{1}{2} \frac{\partial F^2}{\partial y^i \partial y^j} dx^i \otimes dx^j \quad (\text{See [7]})
\]

Notice that in Finsler geometry, \(\tilde{g}_{ij}\) also depends on \(y\). The key idea here is that \(\tilde{g}_{ij}(x, y)dx^i \otimes dx^j\) in \((x, y) \in TM \setminus 0\) can be seen as the best Riemannian approximation of \(F^2|_{T_yM}\) in a neighborhood of \(y \in T_xM\).

The flag curvature of a Finsler manifold is completely determined by its fundamental tensor, and it depends not only on the two-dimensional subspace \(\sigma\) of \(T_xM\), but also on the non-zero vector \(y \in \sigma\). In other words, it depends on the flag \([y, \sigma]\). Therefore when we have \(C^0\)-Finsler structure \(F\) such that the restriction of \(F^2\) to their tangent spaces aren’t inner products, it is natural to consider Riemannian
approximations of $F^2|_{T_yM}$ in a neighborhood of $y$. We will use this idea in this section.

Although the main idea to define limit flag curvature comes from Finsler geometry, the calculations made here are purely from Riemannian geometry since the approximation is made using Riemannian metrics.

Let $G$ be a Lie group endowed with a left invariant Riemannian metric $g$. Then the sectional curvatures are calculated in [19] as follows. Fix an oriented orthonormal basis $\{v_1, \cdots, v_n\}$ on the Lie algebra $\mathfrak{g}$ and define the structure constants $\alpha^g_{ijk}$ by $[v_i, v_j] = \sum_k \alpha^g_{ijk} v_k$. Observe that $\alpha^g_{ijk} = \langle [v_i, v_j], v_k \rangle$. The sectional curvature of $\text{span}\{v_1, v_2\}$ is given by

$$\kappa_g(v_1, v_2) = \sum_j \left( \frac{1}{2} \alpha^g_{12j} (-\alpha^g_{12j} + \alpha^g_{2j1} + \alpha^g_{j12}) ight) - \frac{1}{4} \left( \alpha^g_{12j} - \alpha^g_{2j1} + \alpha^g_{j12} - \alpha^g_{j11} \alpha^g_{j22} \right).$$ (15)

Now we introduce a type of asymptotic flag curvature for left invariant $p$-Finsler structures on $G$. Since we are using the Hamiltonian formalism on Lie groups, this curvature will depend on $a \in \mathfrak{g}^*\{0\}$.

Consider $a \in S_{F^p}$ and $v_1 \in S_{F^p}$, that maximizes $a$ on $S_{F^p}$. Let $\sigma$ be a two-dimensional subspace of $\mathfrak{g}^*$ containing $v_1$. Notice that $\ker a$ is transversal to $\text{span}\{v_1\}$ due to $a(v_1) = 1$. Then we can choose $v_2$ in the one-dimensional space $\sigma \cap \ker a$ and we have $\sigma = \text{span}\{v_1, v_2\}$. The vector $v_1$ is called the flagpole and $v_2$ is the transverse edge of the flag $\{v_1, \sigma\}$. For the sake of simplicity, we denote $(v_1, v_2) := \{v_1, \sigma\}$.

The face of $S_{F^p}$ that maximizes $a$ is contained in the affine subspace $\ker a + \{v_1\}$. We will introduce the asymptotic expansion of the flag curvature of $(v_1, v_2)$ with respect to $a$. Let $\{v_2, \ldots, v_n\}$ be a basis of $\ker a$. Then $B := \{v_1, \ldots, v_n\}$ is a basis of $\mathfrak{g}$ associated to a left invariant Riemannian metric $g$ on $G$ such that $\{v_1, \ldots, v_n\}$ is an orthonormal basis of $g|_{\mathfrak{g}}$. Now we are ready to introduce the curvatures associated to this setting.

**Definition 4.2.** Consider $B$ given in the previous paragraph. Define for each $k \in \mathbb{N}^*$ the basis $B^k := \{v_1, kv_2, \ldots, kv_n\}$ of $\mathfrak{g}$. Consider $g_k$ the unique left invariant Riemannian metric that makes $B^k$ an orthonormal basis of $\mathfrak{g}$. Define the function associated to the flag $(v_1, v_2)$

$$\kappa_{g_k}(v_1, kv_2) = \kappa_{g_k}(v_1, v_2)$$

which depends on $k$. We call it the asymptotic expansion of flag curvature of $(v_1, v_2)$ in $(\mathfrak{g}, F^p)$ with respect to $B$.

**Remark 4.3.** Notice that $\kappa_{g_k}(v_1, v_2)$ is calculated using (15), which is an object of Riemannian geometry. Therefore no element of Finsler geometry is required.

**Remark 4.4.** Observe that as $k \to \infty$, the unit sphere of $(\mathfrak{g}, g_k)$ converges locally to $\{v_1\} + \ker a$. In fact, consider the coordinate system $(x^1, \ldots, x^n)$ on $\mathfrak{g}$ with respect to the basis $(v_1, \ldots, v_n)$. The affine subspace $\{v_1\} + \ker a$ is the graph of the function $(x^2, \ldots, x^n) \to x^1$ given by $x^1 = 1$. The unit sphere of $(\mathfrak{g}, g_k)$ is the ellipsoid given by

$$(x^1)^2 + \sum_{i=2}^n (x^i)^2 \frac{k^2}{k^2} = 1,$$
which is the graph of the function

\[ x^1 = \sqrt{1 - \frac{\sum_{j=2}^{n}(x^j)^2}{k^2}} \]  

(16)

in a neighborhood of \( v_1 = (1, 0, \ldots, 0) \). Notice that if \( k \to \infty \), then \( 10 \) converges uniformly to \( x^1 = 1 \) in a neighborhood of \( 0 \in \text{span}\{v_2, \ldots, v_n\} \).

Let \( s, t, u \in \{2, \ldots, n\} \). Then it is straightforward that

\[ \alpha g_{s1}^q = k \alpha t_{s1}^q, \quad \alpha g_{st}^q = \alpha t_{st}^q, \quad \alpha g_{st1}^q = k^2 \alpha t_{st1}^q \text{ and } \alpha g_{stu}^q = k \alpha t_{stu}^q. \]  

(17)

We split the analysis of \( \kappa_{g_k} \) in two cases: \( n = 2 \) and \( n \geq 3 \).

If \( n = 2 \), then \( 17 \) and \( \alpha g_{jk} = -\alpha jk \) give

\[ \kappa_{g_k}(v_1, v_2) = -(\alpha g_{121}^q)^2 - (\alpha g_{122}^q)^2 = -k^2(\alpha g_{121}^q)^2 - (\alpha g_{122}^q)^2. \]  

(18)

For \( n \geq 3 \), if we split \( 15 \) in the cases \( j = 1 \) and \( j \geq 2 \) and consider \( 17 \), we get

\[ \kappa_{g_k}(v_1, v_2) = k^4 \sum_{j=3}^{n} \frac{1}{4}(\alpha g_{2j1}^q)^2 \]

\[ - k^2(\alpha g_{121}^q)^2 + k^2 \sum_{j=2}^{n} \left( \alpha g_{2j1}^q \left( \frac{1}{2} \alpha g_{12j}^q - \frac{1}{2} \alpha g_{j12}^q \right) - \alpha g_{1j1}^q \alpha g_{j22}^q \right) \]

\[ + \sum_{j=2}^{n} \alpha g_{12j}^q \left( - \frac{3}{4} \alpha g_{12j}^q + \frac{1}{2} \alpha g_{j12}^q \right) + \frac{1}{4} (\alpha g_{j12}^q)^2. \]  

(19)

In both cases \( n = 2 \) and \( n \geq 3 \), \( \kappa_{g_k}(v_1, v_2) \) is a polynomial in \( k \).

**Definition 4.5.** The coefficient of the highest degree term of \( \kappa_{g_k}(v_1, v_2) \) is denoted by \( K_B(a) \).

**Proposition 4.6.**

\[ K_B(a) = -(\alpha g_{121}^q)^2 \quad \text{if} \quad n = 2 \]

and

\[ K_B(a) = \sum_{j=3}^{n} \frac{1}{4}(\alpha g_{2j1}^q)^2 \quad \text{if} \quad n \geq 3. \]

**Definition 4.7.** Let \( S \) be a subset of \( g \). The normalizer of \( S \) in \( g \) is the subset \( N_g(S) = \{ b \in g \mid [b, S] \subset S \} \) of \( g \).

**Proposition 4.8.** Suppose that \( n \geq 3 \). Then \( K_B(a) = 0 \) iff \( v_2 \in \ker a \cap N_g(\ker a) \).

**Proof.**

\[ K_B(a) = \frac{1}{4} \sum_{j=3}^{n} (\alpha g_{2j1}^q)^2 = 0 \]

iff \([v_2, v_j] \subset \ker a \) for every \( j = 2, \ldots, n \). Therefore \( K_B(a) = 0 \) iff \( v_2 \in \ker a \cap N_g(\ker a) \).

**Remark 4.9.** Notice that the equality \( K_B(a) = 0 \) doesn’t depend on \( v_1 \) or on the choice of \( B \) due to Proposition 4.8. It only depends on \( a \) and \( v_2 \). From now on, we denote \( K_B(a) = 0 \) by \( K(a, v_2) = 0 \).
5. Uniqueness of Solutions of the Extended Geodesic Field for Lie groups

For $p$-Finsler structures, the control $u \in S_F$ that maximizes $a \in g^*$ is not necessarily unique, as it is in the strictly convex case. If $a(t)$ is the vertical part of a Pontryagin extremal, then the uniqueness of $u(t)$ such that $(u(t), a(t))$ is a Pontryagin extremal is important because $a(t)$ would be enough to define an extremal. This section is dedicated to finding relationships between this problem and the measure of

$$I = \{ t \in I; \text{ there exist } v_2(t) \in \ker a(t) \setminus \{0\} \text{ such that } \mathcal{K}(a(t), \cdot) = 0 \}.$$

Remark 5.1. Notice that since $r = M(a(t)) = \max_{u \in S_F} a(u)$ is constant due to Pontryagin Maximum Principle, then $a(t) \in S_F, [0, r] \subset g^*$.

Definition 5.2. A vertical part of a Pontryagin extremal $a(t)$ admits infinitely many $u(t)$ such that $(u(t), a(t))$ is a Pontryagin extremal if there are infinitely many admissible controls $u(t) \in C(a(t))$ such that $a'(t) = -\text{ad}^*(u(t))(a(t))$ for $a.e.$ $t$. Likewise, we can define when $a(t)$ admits a unique solution $u(t)$ for $a'(t) = -\text{ad}^*(u(t))(a(t))$. Here we consider two controls $u(t) = u(t)$ a.e. as the same.

Remark 5.3. Notice that $a(t)$ admits infinitely many solutions $u(t)$ for $a'(t) = -\text{ad}^*(u(t))a(t)$ iff $a(t)$ admits at least two different solutions for $a'(t) = -\text{ad}^*(u(t))a(t)$.

In fact, if there are two different admissible controls $u(t)$ and $\tilde{u}(t)$ such that

$$a'(t) = -\text{ad}^*(u(t))a(t)$$

and

$$\tilde{a}'(t) = -\text{ad}^*(\tilde{u}(t))a(t),$$

then $\tilde{u}(t) = su(t) + (1 - s)\tilde{u}(t)$ also maximizes $a(t)$ for every $s \in [0, 1]$ and $a'(t) = -\text{ad}^*(\tilde{u}(t))a(t)$.

Remark 5.4. If $(u(t), a(t))$ and $(\tilde{u}(t), a(t))$ are two different Pontryagin extremals, then the corresponding extremals $x_u(t)$ and $x_{\tilde{u}}(t)$ in $G$ satisfying $x_u(t_0) = x_{\tilde{u}}(t_0)$ and $\mathbb{1}_u$ are different. Although we have multiple controls for a given $a(t)$, Definition 5.1 doesn’t have any direct relationship with singular extremals.

Let us present this problem for $n = 2$. Here $g$ can be the abelian Lie algebra or else the unique non-abelian two-dimensional Lie algebra.

If $g$ is the abelian algebra, then $\kappa_{g_0}$ is trivially zero. If $(u(t), a(t))$ is a solution of (12), then $a(t) = a_0$ for some $a_0 \in g^* \setminus \{0\}$. It is straightforward that there are infinitely many solutions $(u(t), a_0)$ with vertical part $a_0$ iff $a_0$ admits infinitely many $u \in S_F$ that maximizes $a_0$, that is, $a_0$ is maximized by an edge in $S_F$.

If $g$ is the unique non-abelian two-dimensional Lie algebra, then choose a basis $\{ e_1, e_2 \}$ of $g$ such that $[e_1, e_2] = -e_1$. Direct calculations show that $K(a, v_2) = 0$ iff $v_2$ is proportional to $e_1$, that is, $a$ is a non-zero multiple of $dx^2$. In this case, $a'(t) = 0$ and $a(t) = a$. If $S_F$ admit an edge $E$ parallel to the $x$-axis that maximizes $a$, then any measurable control $u(t) \in E$ is such that $(u(t), a)$ is a Pontryagin extremal (compare with [13]). Therefore the two dimensional case shows evidences that given a vertical part of a Pontryagin extremal $a(t)$, the existence of infinitely many controls $u(t)$ such that $(u(t), a(t))$ is a Pontryagin extremal is related to the equation $K(a(t), v_2(t)) = 0$.

From now on suppose that $n \geq 3$. Let $a(t)$ be a vertical part of a Pontryagin extremal. Theorem 5.5 gives a necessary condition in order to $a(t)$ admit infinitely
many $u(t)$ such that $(u(t), a(t))$ is a Pontryagin extremal and Theorem 5.6 gives a sufficient condition.

**Theorem 5.5.** Let $a(t) \in S_F$ be the vertical part of a Pontryagin extremal. Suppose that $u(t)$ and $\tilde{u}(t)$ are different measurable functions such that $(u(t), a(t))$ and $(\tilde{u}(t), a(t))$ are Pontryagin extremals. Then the set

$$I := \{ t \in I; \text{ there exist } v_2(t) \in \ker a(t) \setminus \{0\} \text{ such that } \mathcal{K}(a(t), v_2(t)) = 0 \}$$

has positive measure.

**Proof.** Denote

$$I' = \{ t \in I; u(t) - \tilde{u}(t) \neq 0 \},$$

which has positive measure by hypothesis. Notice that

$$(u(t) - \tilde{u}(t), \cdot) = a'(t) - a'(t) = 0, \quad (20)$$

which implies that $u(t) - \tilde{u}(t) \in \ker a(t) \cap N_g(a(t))$ and $\mathcal{K}(a(t), u(t) - \tilde{u}(t)) = 0$ for every $t \in I$ due to Proposition 4.8. Moreover $v_2(t) := u(t) - \tilde{u}(t) \neq 0$ for every $t \in I'$. Therefore $I \supset I'$ also has positive measure. \qed

**Theorem 5.6.** Let $L$ be a $k$-dimensional face of $S_F$ which maximizes the relative interior of the $(n - k - 1)$-dimensional face $L^*$ in $S_F$. (see Remark 2.23). Let $(u(t), a(t))$ be a Pontryagin extremal of $(G, F)$ such that $u(t) \in L$ and $a(t) \in L^*$ for every $t \in I$. Suppose that there exist a measurable function $w(t)$ parallel to $L$ such that $u(t) + w(t) \in L$ and a set of positive measure $\mathcal{I} \subset I$ such that for every $t \in \mathcal{I}$,

- $w(t) \neq 0$;
- $\mathcal{K}(a(t), w(t)) = 0$.

Then $(u(t) + w(t), a(t))$ is also a Pontryagin extremal of $(G, F)$ and $a(t)$ admits infinitely many controls $\tilde{u}(t)$ such that $(\tilde{u}(t), a(t))$ is a Pontryagin extremal.

**Proof.** We claim that $a(t)((u(t), \cdot)) = 0$. In fact, for each $t \in I$, an arbitrary $v(t) \in g$ can be written as $a(t)u(t) + b(t)z(t)$, where $a(t), b(t) \in \mathbb{R}$ and $z(t) \in \ker a(t)$. Then

$$a(t)((u(t), v(t))) = -a(t).a(t)((u(t), w(t))) + b(t).a(t)((w(t), z(t)))$$

$$= -a(t).a'(t)(w(t)) + b(t).a(t)((w(t), z(t))). \quad (21)$$

The first term of the right-hand-side of (21) is zero because $a'(t)$ is parallel to $L^*$, $w(t)$ is parallel to $L$ and $b(\tilde{v}) = 1$ for every $b \in L^*$ and $\tilde{v} \in L$.

The second term of the right-hand-side of (21) is zero because $\mathcal{K}(a(t), w(t)) = 0$, which is equivalent to the condition $w(t) \in \ker a(t) \cap N_g(\ker a)$ due to Proposition 4.8. Then $[w(t), z(t)] \in \ker a$ and the claim holds. Therefore

$$a'(t) = a(t)((u(t), \cdot)) = a(t)((u(t) + w(t), \cdot)),$$

and $(u(t) + w(t), a(t))$ is also a Pontryagin extremal. Finally $a(t)$ admits infinitely many $\tilde{u}(t)$ such that $(\tilde{u}(t), a(t))$ is a Pontryagin extremal due to Remark 5.3. \qed

From now on, we restrict our study to the case of three-dimensional Lie groups endowed with a left invariant $p$-Finsler structure $F$. We call two-dimensional faces of $S_F \subset g$ (or else of $S_{F^*} \subset g^*$) simply by face, the one-dimensional faces by edges and the zero-dimensional faces by vertices.

**Proposition 5.7.** Let $G$ be a three-dimensional Lie group endowed with a $p$-Finsler structure $F$ and $a \in S_{F^*}$. The following statements are equivalent:

1. $\mathcal{K}(a, v_2) = 0$ for some $v_2 \in \ker a$;
(2) \( \ker a \) is a subalgebra of \( g \);
(3) \( \mathcal{K}(a, v) = 0 \) for every \( v \in \ker a \);
(4) \( [v, \ker a] \subset \ker a \) for every \( v \in \ker a \).
(5) There exists \( v \in \ker a \setminus \{0\} \) such that \( [v, \ker a] \subset \ker a \).

Proof. First of all we choose a basis \( B = (v_1, v_2, v_3) \) such that \( v_1 \) maximizes \( a \) on \( S_{F_c} \) and \( \ker a = \text{span}\{v_2, v_3\} \). Then

\[
\mathcal{K}(a, v_2) = \frac{1}{4} (a_{231}^g)^2,
\]

where

\[
[v_2, v_3] = a_{231}^g v_1 + a_{232}^g v_2 + a_{233}^g v_3
\]

and Item 1 is equivalent to Item 2. The equivalence between Item 2 and Item 3 follows from the fact that Item 2 doesn’t depend on \( v_2 \in \ker a \). The equivalence between Item 2 and Item 4 is straightforward. Item 4 \( \Rightarrow \) Item 5 is trivial. Finally, if Item 5 holds, consider \( w \in \ker a \) such that \( \{w, v\} \) is a basis of \( \ker a \). We have that \( [w, v] \in \ker a \) by Item 5, which implies that \( \ker a \) is a subalgebra of \( g \) and consequently Item 4 holds. \( \square \)

Remark 5.8. Proposition [7] states that the equality \( \mathcal{K}(a, v_2) = 0 \) doesn’t depend on \( v_2 \in \ker a \). Therefore in the three-dimensional case we can simply denote \( \mathcal{K}(a) = \mathcal{K}(a, v_2) \).

Remark 5.9. If \( (u(t), a(t)) \) is a Pontryagin extremal and \( t_0 \) is such that \( a(t_0) \) lies in the relative interior of a face of \( S_{F_c} \), then there exists \( \varepsilon > 0 \) such that \( \mathcal{C}(a(t)) = \{u_0\} \) is a vertex of \( S_{F} \) for \( t \in (t_0 - \varepsilon, t_0 + \varepsilon) \). Therefore the problem of uniqueness of \( u(t) \) is trivial in this case. We will suppose from here that \( a(t) \) lies on an edge of \( S_{F_c} \).

Theorem 5.10. Let \( G \) be a three dimensional connected Lie group endowed with a left invariant polyhedral Finsler structure \( F \). Let \( L \) be a edge of \( S_{F_c} \) that maximizes the relative interior of the edge \( L^* \) of \( S_{F_c} \). Let \( a : I \to L^* \) be the vertical part of a Pontryagin extremal \( (u(t), a(t)) \), where \( u(t) \in L \). Then \( a(t) \) admits infinitely many \( \tilde{u}(t) \in L \) such that \( (\tilde{u}(t), a(t)) \) is a Pontryagin extremal iff \( \mathcal{I} := \{t \in I; \mathcal{K}(a(t)) = 0\} \) has positive measure.

Proof. If \( a(t) \) admits infinitely many \( u(t) \in L \) such that \( (u(t), a(t)) \) is a Pontryagin extremal, then \( \mathcal{I} \) has positive measure due to Theorem [5]. Reciprocally, let \( w \in g \) be a non-zero vector parallel to \( L \). Recall that \( \mathcal{I} = \{t \in I; \mathcal{K}(a(t), w) = 0\} \) due to Proposition [5]. If \( \mathcal{I} \) has positive measure, then consider a measurable function \( \lambda : I \to \mathbb{R} \) such that:

\[\begin{align*}
\lambda(t) &\neq 0 \quad \text{for } t \in \mathcal{I}; \\
\lambda(t) & = 0 \quad \text{for } t \notin \mathcal{I}; \\
\lambda(t) + \lambda(t) w \in L.
\end{align*}\]

For instance, if \( \bar{u} \) is the midpoint of \( L \) and \( r_L > 0 \) is such that \( L \) is the line segment \([\bar{u} - r_L w, \bar{u} + r_L w]\), then we can define \( \lambda(t) \) as

\[
\lambda(t) = \begin{cases} 
0 & \text{if } t \notin \mathcal{I}; \\
r_L & \text{if } u(t) \in \mathcal{I} \cap [\bar{u} - r_L w, \bar{u}] \\
-r_L & \text{if } u(t) \in \mathcal{I} \cap [\bar{u}, \bar{u} + r_L w].
\end{cases}
\]
Therefore $u_\lambda(t) = u(t) + \lambda(t)w$ is another control such that $(u_\lambda(t), a(t))$ is a Pontryagin extremal due to Theorem 5.6 and there are infinitely many controls $\tilde{u}(t) \in L$ such that $(\tilde{u}(t), a(t))$ is a Pontryagin extremal due to Remark 5.3. □

**Theorem 5.11.** Let $G$ be a three dimensional connected Lie group endowed with a left invariant polyhedral Finsler structure $F$. Suppose that $a_0$ is a vertex of $S_F$, and that the constant map $a(t) = a_0$ is the vertical part of a Pontryagin extremal of $(G, F)$. Then $a_0$ admits infinitely many $u(t)$ such that $(u(t), a_0)$ is a Pontryagin extremal iff $K(a_0) = 0$.

**Proof.** Suppose that $a_0$ is maximized by the face $L = \Psi^{-1}(a) \subset S_F$. If $(u(t), a_0)$ is a Pontryagin extremal of $(G, F)$, then we have necessarily that $u(t) \in L$. If $a(t) = a_0$ admits infinitely many $u(t)$ such that $(u(t), a(t))$ is a Pontryagin extremal, then $K(a_0) = 0$ due to Theorem 5.5.

Reciprocally, if $K(a_0) = 0$, then any measurable function $\tilde{u}(t) \in L$ such that $\tilde{u}(t) - u(t) \neq 0$ can be written as $\tilde{u}(t) = u(t) + (\tilde{u}(t) - u(t))$, where $w(t) = \tilde{u}(t) - u(t)$ satisfies all conditions for $w(t)$ of Theorem 5.6 for $I = I$. Thus $a_0$ admits infinitely many controls $\tilde{u}(t)$ such that $(\tilde{u}(t), a_0)$ is a Pontryagin extremal. □

6. Final remarks

In this section, we make remarks about this work and give suggestions for future works.

Theorems 5.10 and 5.11 can be used in order to try to classify the left invariant $p$-Finsler structures on three-dimensional Lie groups such that for every vertical part $a(t)$ of a Pontryagin extremal, there exist a unique $u(t)$ such that $(u(t), a(t))$ is a Pontryagin extremal.

In Finsler geometry, if we consider a point $p \in M^n$ and a geodesic $\gamma$ such that $\gamma(0) = p$, then the first conjugate point on $\gamma$ can be estimated by the flag curvatures along $\gamma$. Moreover the geodesic is no longer minimizing beyond this conjugate point.

In fact, Ricci curvature can be defined for Finsler manifolds in a similar way that it is defined for Riemannian manifolds, that is, as a sum of $(n - 1)$ flag curvatures (see [7] Section 7.6). If $(n - 1)\rho$ is a positive lower bound for the Ricci curvature, then the distance of two conjugate points in $M$ is less than or equal to $\pi/\sqrt{\rho}$ (see [7] Section 7.7). Eventually such an analysis can be tried on Lie groups with a left invariant $p$-Finsler structure with an approximation of $F_e$ by Riemannian metrics. Another question is whether the property of $\ker a$ being a subalgebra in higher dimensions has something to do with geodesic properties of $G$. For instance, if $(u(t), a(t))$ is a Pontryagin extremal of $G$ such that $\ker a(t)$ is a subalgebra of $g$ for every $t$ and $x(t)$ is the extremal in $G$ corresponding to $(u(t), a(t))$, does it implies that $x(t)$ is a geodesic?.

Finally we hope that Theorems 5.10 and 5.11 would help us to understand and eventually classify the dynamics of the control system $a'_t(t) = -\text{ad}^*(u(t))(a(t))$ on three-dimensional Lie groups endowed with a left invariant $p$-Finsler structure.

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EMAILS: JBUZATTOPGMAIL.COM; RFUKUOKA@UEM.BR