SYMMETRIC DIFFERENTIALS AND JETS EXTENSION OF $L^2$ HOLOMORPHIC FUNCTIONS

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CONTENTS

1. Introduction 1
2. Preliminaries 3
3. Raising operators on symmetric powers of the cotangent bundles 5
  3.1. Hodge type identities 5
  3.2. Hodge type identities over the complex hyperbolic space form 9
4. Vanishing Theorem on the symmetric power of the cotangent bundle 10
5. Construction of $L^2$ holomorphic functions on $\Omega$ 11
  5.1. Precomputations 11
  5.2. Necessary condition to be holomorphic functions 14
  5.3. Proof of Theorem 1.3 17
References 22

ABSTRACT. Let $\Sigma = B^n/\Gamma$ be a compact complex hyperbolic space with torsion-free lattice $\Gamma \subset SU(n,1)$ and $\Omega$ a quotient of $B^n \times B^n$ with respect to the diagonal action of $\Gamma$ which is a holomorphic $B^n$-fiber bundle over $\Sigma$. The goal of this article is to investigate the relation between symmetric differentials of $\Sigma$ and the weighted $L^2$ holomorphic functions on the exhaustions $\Omega_\epsilon$ of $\Omega$. If there exists a holomorphic function on $\Omega_\epsilon$ on some $\epsilon$ and it vanishes up to $k$-th order on the maximal compact complex variety in $\Omega$, then there exists a symmetric differential of degree $k+1$ on $\Sigma$. Using this property, we show that $\Sigma$ always has a symmetric differential of degree $N$ for any $N \geq n+1$. Moreover for each symmetric differential over $\Sigma$, we construct a weighted $L^2$ holomorphic function on $\Omega_{\sqrt{\epsilon}}$.

1. INTRODUCTION

Let $\Sigma$ be a compact complex hyperbolic space form, i.e. $\Sigma = B^n/\Gamma$ is the quotient of the complex unit ball $B^n = \{z \in \mathbb{C}^n : |z| < 1\}$ by a torsion-free cocompact lattice $\Gamma \subset \text{Aut}(B^n)$. Since the holomorphic cotangent bundle $T^*_\Sigma$ of $\Sigma$ is ample, its $m$-th symmetric power $S^mT^*_\Sigma$ is generated by global sections when $m$ is sufficiently large.

In general the existence of such global sections, also known as symmetric differentials, is expected to be related to the topological fundamental group of the base manifold. In the case Riemann surfaces, the existence of symmetric differentials is related to the degree of the cotangent bundle. In particular, when the cotangent bundle has positive degree, i.e. when the genus $g$ of the curve is greater than or equal to two, the dimension of the set of symmetric differentials...
differentials of degree $m$ equals to $(q - 1)(2m - 1)$. However if we consider higher dimensional complex manifolds, relation between topological properties and the symmetric differentials is not simple. We refer the interested reader to [2, 3] and the references therein.

Consider $\Omega := \mathbb{B}^n \times \mathbb{B}^n / \Gamma$ and $\mathbb{B}^n \times \mathbb{C}^n / \Gamma$ under the diagonal action
\[
\gamma(z, w) := (\gamma z, \gamma w)
\]
with $\gamma \in \Gamma$. Since there is a canonical embedding of $\text{Aut}(\mathbb{B}^n)$ into $\text{Aut}(\mathbb{C}^n)$, the diagonal action is well defined on $\mathbb{B}^n \times \mathbb{C}^n$. Let $D$ be the maximal compact complex variety in $\Omega$, i.e. $D = \{(z, z) \in \mathbb{B}^n \times \mathbb{B}^n\}/\Gamma$. Inside $\mathbb{B}^n \times \mathbb{C}^n / \Gamma$, the domain $\Omega$ is connected with real analytic boundary. Moreover it possesses a bounded plurisubharmonic exhaustion function $-\rho^2$ where $\rho$ is given by
\[
\rho(z, w) = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - w \cdot \bar{z}|^2}
\]
(cf. [3]). Since it is invariant under the diagonal action, $\rho$ is well defined on $\Omega$. Note that $\rho$ vanishes on $D$ and one may consider $\Omega$ as a holomorphic $\mathbb{B}^n$-fiber bundle over $\Sigma$. The case $n = 1$ was recently studied by Adachi. In [1], the set of weighted $L^2$ holomorphic functions on $\Omega$ was identified by determining how jets of holomorphic functions belonging to $H^0(D, \mathcal{I}_D^N / \mathcal{I}_D^{N+1})$ extend to holomorphic functions on the whole of $\Omega$. Here $\mathcal{I}_D$ denotes the ideal sheaf of $D$. Starting from a holomorphic section of the $k$-th tensor power of the canonical line bundle of the Riemann surface, he constructed a sequence of recursive $\bar{\partial}$-equations to obtain the higher order terms.

The aim of this article is to generalize the results of Adachi to the case of higher dimensional hyperbolic space forms. We describe the symmetric differentials on $\Sigma$ and weighted $L^2$ holomorphic functions on exhaustions of $\Omega$.

For $\epsilon \in (0, 1)$, let
\[
\Omega_\epsilon = \{[(z, w)] \in \Omega : 1 - \rho(z, w) < \epsilon^2\}
\]
be a subdomain in $\Omega$ which contains $D$. As $\epsilon$ tends to $0$, $\Omega_\epsilon$ becomes a smaller neighborhood of $D$ and as $\epsilon$ tends to $1$, $\Omega_\epsilon$ exhausts $\Omega$. Let $\mathcal{T}_{D, \epsilon}^k$ denote the subset of $\mathcal{O}(\Omega_\epsilon)$ whose elements vanish on $D$ up to the $k$-th order. Our main results are the following:

**Theorem 1.1.** Let $\Sigma = \mathbb{B}^n / \Gamma$ be a compact complex hyperbolic space form with a torsion-free cocompact lattice of the automorphism groups of the complex unit ball $\mathbb{B}^n$. Let $\Omega = \mathbb{B}^n \times \mathbb{B}^n / \Gamma$ be a holomorphic $\mathbb{B}^n$-fiber bundle under the diagonal action of $\Gamma$ on $\mathbb{B}^n \times \mathbb{B}^n$ and $D$ the subset $\{(z, z) \in \Omega : z \in \mathbb{B}^n\}$. Let $\Omega_\epsilon = \{[(z, w)] \in \Omega : 1 - \rho(z, w) < \epsilon^2\}$ with $\rho(z, w) = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - w \cdot \bar{z}|^2}$ for some $0 < \epsilon \leq 1$. Then there exists a map
\[
\Psi_\epsilon : \mathcal{O}(\Omega_\epsilon) \to \bigoplus_{m=0}^\infty H^0(\Sigma, S^m T^*_\Sigma)
\]
such that if $f \in \mathcal{O}(\Omega_\epsilon)$ vanishes on $D$ up to $k$-th order, then $\Psi_\epsilon(f)$ is a symmetric differential of degree $k+1$.

In particular, since we have Poincaré series on $\mathbb{B}^n \times \mathbb{B}^n$, we have the following.

**Corollary 1.2.** Let $\Gamma$ be a cocompact lattice of the automorphism groups of the unit ball $\mathbb{B}^n$. For any $N \geq n+2$, there exists a symmetric differential of degree $N$.

**Theorem 1.3.** Let $\Sigma = \mathbb{B}^n / \Gamma$ be a compact complex hyperbolic space form with a torsion-free cocompact lattice of the automorphism groups of the complex unit ball $\mathbb{B}^n$. Let $\Omega = \mathbb{B}^n \times \mathbb{B}^n / \Gamma$ be a holomorphic $\mathbb{B}^n$-fiber bundle under the diagonal action of $\Gamma$ on $\mathbb{B}^n \times \mathbb{B}^n$. Then we have an
injective linear map
\[\Phi: \bigoplus_{m=n+2}^{\infty} H^0(\Sigma, S^m T^*_{\Sigma}) \to \bigcap_{a>-1} A^2_{\alpha}(\Omega_{1\frac{\alpha}{n}}) \subset \mathcal{O}(\Omega_{1\frac{\alpha}{n}})\]

having a dense image in \(\mathcal{O}(\Omega_{1\frac{\alpha}{n}})/\mathcal{I}_{\frac{1}{n}}\) equipped with compact open topology on \(\mathcal{O}(\Omega_{1\frac{\alpha}{n}})\).

The map \(\Phi\) defined in Theorem 1.3 is constructed as follows. Let \(N \geq n + 2\). For a given \(\psi \in H^0(\Sigma, S^N T^*_{\Sigma})\), define a sequence \(\varphi_k \in C^\infty(\Sigma, S^{N+m} T^*_{\Sigma})\) with \(k = 1, 2, \ldots\) given by
\[
\left\{
\begin{aligned}
\varphi_k &= 0 & \text{if} & \ k < N, \\
\varphi_N &= \psi,
\end{aligned}
\right.
\]

and for \(m \geq 1\), the section \(\varphi_{N+m}\) is the \(L^2\) minimal solution of the following \(\overline{\partial}\)-equation:
\[\overline{\partial}\varphi_{N+m} = -(N + m - 1)\mathcal{R}_G (\varphi_{N+m-1}),\]

where \(\mathcal{R}_G\) is the raising operator given in Section 3 with respect to the Kähler form \(G\) of the induced metric from the Bergman metric on \(\mathbb{B}^n\). Denote \(\varphi_m = \sum_{|I|=m} f_I e^I\) with a unitary frame \(e\) of \(T^*_{\Sigma}\) given by (2.5) and define
\[\Phi(\psi)(z, w) = \sum_{|I|=0}^{\infty} f_I(z)(T_z w)^I\]

where \(T_z\) is an automorphism on \(\mathbb{B}^n\) so that \(T_z z = 0\) given in (2.1).

The proof of Theorem 1.3 relies crucially on the Hodge type identities for raising operators given in Section 3. In the case of a compact Kähler manifold \(X\) with Kähler form \(G\), there exists a collection of identities satisfied by the operators defined on \(\Lambda^{p,q} T^*_X\), the holomorphic vector bundle of exterior differentials. These play an important role in Hodge theory. One of them is given by \([L, \square] = 0\) where \(L\) is the Lefschetz operator given by \(L: u \mapsto u \wedge G\), \(u \in \Lambda^{p,q} T^*_X\) and \(\square = \overline{\partial}\partial + \partial^* \overline{\partial}\) is the \(\overline{\partial}\)-Laplacian. For the symmetric power of \(T^*_{\Sigma}\), we can define a raising operator analogous to \(L\), which we denote \(\mathcal{R}_G\) and in particular we obtain the identity
\[\{\square, \mathcal{R}_G\} = \mathcal{R}_{\Theta(S^m T^*_{\Sigma})},\]

where \(\mathcal{R}_{\Theta(S^m T^*_{\Sigma})}\) is the raising operator corresponding to the Chern curvature tensor \(\Theta\) of \(S^m T^*_{\Sigma}\). If we apply this identity to \(\Sigma\), then we obtain that \(\mathcal{R}_G\) preserves the eigenspaces of the Laplacian (Corollary 3.3).

Throughout this article, for a map \(F: U \to W\) where \(U\) and \(V\) are open subsets in the complex Euclidean spaces, \(F_k\) denotes the \(k\)-th component of the map \(F\). Moreover we will use the multi-index. For example, for \(I = (i_1, \ldots, i_n)\) we denote \(e^{i_1} \cdots e^{i_n} \) by \(e^I\), \((T_z w)_1^{i_1} \cdots (T_z w)_n^{i_n}\) by \((T_z w)^I\), \(i_1 + \cdots + i_n\) by \(|I|\) and \(i_1! \cdots i_n!\) by \(!I!\).

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2. Preliminaries

In this section we collect some properties of the Bergman kernel, automorphisms and volume forms of the unit ball for the future use. For more details see, for example, Chapter 1 in [7].
Let $\mathbb{B}^n = \{ z \in \mathbb{C}^n : |z|^2 < 1 \}$ be the unit ball in the complex Euclidean space $\mathbb{C}^n$. Let $K : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{C}$ denote its Bergman kernel, i.e.

$$K(z, w) = \frac{1}{(1 - z \cdot \overline{w})^{n+1}}$$

where $z \cdot \overline{w} = \sum_{j=1}^{n} z_j \overline{w}_j$. Let

$$B(z) = (B_{\alpha \beta}) = \frac{1}{n+1} \begin{pmatrix} \frac{\partial^2}{\partial z_1 \partial z_1} \log K(z, z) & \cdots & \frac{\partial^2}{\partial z_1 \partial z_n} \log K(z, z) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2}{\partial z_n \partial z_1} \log K(z, z) & \cdots & \frac{\partial^2}{\partial z_n \partial z_n} \log K(z, z) \end{pmatrix}$$

and let $g(z) = \sum B_{\alpha \beta} dz_\alpha \otimes dz_\beta$ be the Bergman metric of $\mathbb{B}^n$. For a fixed point $z \in \mathbb{B}^n$, let $T_z$ be an automorphism of $\mathbb{B}^n$ defined by

$$T_z(w) = \frac{z - P_z(w) - s_z Q_z(w)}{1 - w \cdot \overline{z}}, \quad (2.1)$$

where $s_z = \sqrt{1 - |z|^2}$ with $|z|^2 = z \cdot \overline{z}$, $P_z$ is the orthogonal projection from $\mathbb{C}^n$ onto the one dimensional subspace $[z]$ generated by $z$, and $Q_z$ is the orthogonal projection from $\mathbb{C}^n$ onto $\mathbb{C}^n \ominus [z]$. One has

$$P_z(w) = \frac{w \cdot \overline{z}}{|z|^2} z \quad \text{and} \quad Q_z(w) = w - \frac{w \cdot \overline{z}}{|z|^2} z.$$ 

Moreover $T_z$ is an involution, i.e. $T_z \circ T_z(w) = w$ and it satisfies

$$1 - T_z(w) \cdot \overline{z} = \frac{1 - |z|^2}{1 - w \cdot \overline{z}} \quad (2.2)$$

for any $w \in \mathbb{B}^n$. Let $\delta_\epsilon$ be a real-valued function on $\mathbb{B}^n \times \mathbb{B}^n$ given by

$$\delta_\epsilon := 1 - \frac{|T_z w|^2}{\epsilon}. \quad (2.3)$$

Remark that

$$1 - |T_z w|^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - z \cdot \overline{w}|^2} \quad (2.4)$$

and $- \log \delta$ is a plurisubharmonic exhaustion on $\mathbb{B}^n \times \mathbb{B}^n$ which is a strictly plurisubharmonic off the diagonal and invariant under the diagonal action of Aut($\Omega$). This implies that the domain

$$\Omega_\epsilon = \{ (z, w) \in \Omega : |T_z w| < \epsilon \}$$

in $\Omega$ with $\epsilon < 1$ is well defined strongly pseudoconvex domain in $\Omega$ and $\mathbb{B}^n \times \mathbb{C}P^n / \Gamma$.

**Lemma 2.1**

1. $dT_z(0) = -(1 - |z|^2)P_z - \sqrt{1 - |z|^2} Q_z$,
2. $J_z T_z(0) = (1 - |z|^2)^{n+1}$,
3. $dT_z(z) = \frac{1}{1 - |z|^2} - \frac{Q_z}{\sqrt{1 - |z|^2}}$,
4. $B(z) = (dT_z(z))^* (dT_z(z))$.
5. Fix $\gamma \in \text{Aut}(\mathbb{B}^n)$. Then $B(z) = d\gamma(z) B(\gamma z) d\gamma(z)$ where

$$d\gamma = \begin{pmatrix} \frac{\partial \gamma_1}{\partial z_1} & \cdots & \frac{\partial \gamma_1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \gamma_n}{\partial z_1} & \cdots & \frac{\partial \gamma_n}{\partial z_n} \end{pmatrix}$$

Let $A = (A_{jk}) := dT_z(z)$ and define

$$e_j = \sum_k A_{jk} dz_k. \quad (2.5)$$
Lemma 2.2. \( \{ e_j \}_{j=1}^n \) is an orthonormal frame of \( T^*_{\mathbb{B}^n} \) with respect to the Bergman metric on \( \mathbb{B}^n \).

Proof. Since the matrix representation of the given metric on \( T^*_{\mathbb{B}^n} \) is \( (B(z)^{-1})^* \), by (4) of Lemma 2.1 and the orthonormality of \( dz_k \)'s with respect to the Euclidean metric on \( \mathbb{C}^n \), we have that \( (A(dz_k), A(dz_m))_g = \delta_{mk} \).

Let \( X_1, \ldots, X_n \) be the frame on \( T\mathbb{B}^n \) dual to \( e_1, \ldots, e_n \), i.e.

\[
X_j = \sum_k A^{kj} \frac{\partial}{\partial z_k}
\]

where \( (A^{kj}) \) is the inverse of \( (A_{jk}) \), i.e. \( \sum_j A^{kj} A_{jl} = \delta_{kl} \). Define

\[
\Gamma^{j\mu}_l := \sum_k A^{kj} \frac{\partial A_{l\mu}}{\partial z_k} A_{s\mu}.
\]

Then we have

\[
\sum_{i,j} A^{kj} \frac{\partial}{\partial z_k} (A_{is} dz_s) = \sum_{i,j} A^{kj} \frac{\partial A_{is}}{\partial z_k} A_{s\mu} \epsilon_{\mu} = \sum_{i,j} \Gamma^{j\mu}_l \epsilon_{\mu}.
\]

(2.6)

Let \( \Gamma \subset \text{Aut}(\mathbb{B}^n) \) be a cocompact lattice and let \( \Sigma := \mathbb{B}^n / \Gamma \). Without ambiguity we denote the induced metric by \( g \) on \( \Sigma \). Let \( \Omega \) be the quotient of \( \mathbb{B}^n \times \mathbb{B}^n \) under the diagonal action of \( \Gamma \). We will use the volume form \( dV_\omega \) on \( \Omega \) defined by \( \omega \wedge \overline{\omega} \) where \( \omega \) is given by

\[
\omega = \left( \frac{\sqrt{-1}}{2} \right)^n K(z,z) dz \wedge \overline{dz} + \left( \frac{\sqrt{-1}}{2} \right)^n \frac{|K(w,z)|^2}{K(z,z)} dw \wedge \overline{dw}
\]

with \( dz := dz_1 \wedge \cdots \wedge dz_n \) and \( dw := dw_1 \wedge \cdots \wedge dw_n \). Note that \( \omega \) is an invariant \((n,n)\)-form under the diagonal action of \( \text{Aut}(\mathbb{B}^n) \) on \( \mathbb{B}^n \times \mathbb{B}^n \). That is

\[
dV_\omega = \left( \frac{\sqrt{-1}}{2} \right)^n |K(w,z)|^2 dz \wedge d\overline{z} \wedge dw \wedge d\overline{w}
\]

and it is an \( \text{Aut}(\mathbb{B}^n) \)-invariant volume form on \( \Omega \). Now for a domain \( \Omega_\epsilon \subset \Omega \) and measurable functions \( f, h \) on \( \Omega_\epsilon \) we set

\[
\langle f, h \rangle_{\epsilon,\alpha} := c_\alpha \int_{\Omega'_{\epsilon}} f \overline{h} \delta_\alpha \ dV_\omega
\]

where \( c_\alpha = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)} \). Define the weighted \( L^2 \)-space by

\[
L^2_{\alpha}(\Omega_\epsilon) = \{ f : f \text{ is a measurable function on } \Omega_\epsilon, \| f \|_{\epsilon,\alpha}^2 := \langle f, f \rangle_{\epsilon,\alpha} < \infty \}
\]

and the weighted Bergman space by \( A^2_{\alpha}(\Omega_\epsilon) = L^2_{\alpha}(\Omega_\epsilon) \cap \mathcal{O}(\Omega_\epsilon) \).

3. Raising operators on symmetric powers of the cotangent bundles

3.1. Hodge type identities. Let \( X \) be a Kähler manifold of dimension \( n \) and \( g \) be its Kähler metric. Let \( S^m T^*_X \) be the \( m \)-th symmetric power of holomorphic cotangent bundle \( T^*_X \) of \( X \). For \( u \in S^m T^*_X \) and \( v \in S^t T^*_X \), we will denote by \( uv \) the symmetric product of \( u \) and \( v \). We will denote by \( \Lambda^p q T^*_X \) the vector bundle of complex-valued \((p,q)\)-forms over \( X \). For any \( \tau : C^\infty(X, S^m T^*_X) \to C^\infty(X, S^m T^*_X \otimes \Lambda^p q T^*_X) \), define a map

\[
\mathcal{R}^m_{\tau} : C^\infty(X, S^m T^*_X) \to C^\infty(X, S^{m+p} T^*_X \otimes \Lambda^0 q T^*_X)
\]

by

\[
\mathcal{R}^m_{\tau}(u) = \sum \tau_{PQ}(u) e^P \otimes e^Q
\]
where $v(u) = \sum_{|P|=p,|Q|=q} \tau_{PQ}(u) \otimes e^P \otimes \bar{e}^Q$ for $u \in C^\infty(\Sigma, S^m T_X^*)$. Note that $\mathcal{R}_\tau^m$ does not depend on the choice of frame $e$. Moreover it is globally well-defined on $X$. If there is no ambiguity, we will denote $\mathcal{R}_\tau^m$ by $\mathcal{R}_\tau$. 

**Example 3.1.** For the Kähler metric $g$, let $G \in C^\infty(X, \Lambda^{1,1} T_X^*)$ be its Kähler form. Then we may consider $G$ as a map from $C^\infty(X, S^m T_X^*)$ to $C^\infty(X, S^m T_X^* \otimes \Lambda^{1,1} T_X^*)$ so that $G(u) = u \otimes G$. Hence for the local frame $\{e_1, \ldots, e_n\}$ so that $g = \sum_{\ell} e_\ell \otimes \bar{e}_\ell$, the corresponding map $\mathcal{R}_G$ is defined as follows:

$$
\mathcal{R}_G : C^\infty(X, S^m T_X^* \otimes \Lambda^{0,q} T_X^*) \to C^\infty(X, S^{m+1} T_X^* \otimes \Lambda^{0,q+1} T_X^*)
$$

$$
u = \sum_{J,K} u_{JK} e^J \otimes \bar{e}^K \mapsto \sum_{J,K,l} (u_{JK} e^J \otimes (\bar{e}^K \wedge \bar{e}_l)). \quad (3.1)
$$

For a holomorphic coordinate system $(z_1, \ldots, z_n)$ on $X$, express $g = \sum g_{\alpha\beta} dz_\alpha \otimes d\bar{z}_\beta$ with $g_{\alpha\beta} = g \left( \frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial \bar{z}_\beta} \right)$. Then for any $u \in C^\infty(X, S^m T_X^*)$, $\mathcal{R}_G(u) = \sum_{\alpha,\beta} g_{\alpha\beta} u_{z_\alpha} \otimes d\bar{z}_\beta$ and hence we have the following lemma.

**Lemma 3.2.** Suppose that $u \in C^\infty(X, S^m T_X^*)$. Then

$$
\mathcal{R}_G(u) = \sum_{\alpha} u_{z_\alpha} \otimes d\bar{z}_\alpha + O(\lvert z \rvert^2)
$$

for a normal coordinate system $(z_1, \ldots, z_n)$ on $X$ with respect to $g$.

**Example 3.3.** Let $g^{-m}$ be the metric on $S^m T_X^*$ induced from the Kähler metric $g$ on $X$. Let $D_m : C^\infty(X, S^m T_X^*) \to C^\infty(X, S^m T_X^* \otimes \Lambda^{1,0} T_X^*)$ be the $(1,0)$ part of the Chern connection on $(S^m T_X^*, g^{-m})$. Then in a local coordinate system $(z_1, \ldots, z_n)$ on $X$

$$
\mathcal{R}_{D_m} : C^\infty(X, S^m T_X^*) \to C^\infty(X, S^{m+1} T_X^*)
$$

is given by

$$
\mathcal{R}_{D_m} \left( \sum_{\alpha} u_{\alpha} h_{\alpha} \right) = \sum_{k,\alpha} \frac{\partial u_{\alpha}}{\partial z_k} h_{\alpha} d\bar{z}_k + \sum_{\alpha,\mu,\beta} u_{\alpha} \theta^\mu_{\alpha\beta} d\bar{z}_\beta h_{\mu}, \quad (3.2)
$$

where $\theta^\mu_{\alpha} = \sum_{\beta} \theta^\mu_{\alpha\beta} d\bar{z}_\beta$ is the $(1,0)$ part of the connection one form of $D_m$. \hfill \square

**Example 3.4.** The Chern curvature

$$
\Theta(S^m T_X^*) \in C^\infty(X, \text{Hom}(S^m T_X^*, S^m T_X^*) \otimes \Lambda^{1,1} T_X^*)
$$

with respect to $g^{-m}$ is given by

$$
\Theta(S^m T_X^*) s_1 s_2 \cdots s_m = \sum_{1 \leq s_1 \leq s_2 \cdots \leq s_m} s_1 s_2 \cdots \Theta(T_X^*) \cdot s_j \cdots s_m \quad (3.3)
$$

with $s_j \in C^\infty(X, T_X^*)$ for each $j = 1, \ldots, m$. If we denote $\Theta(S^m T_X^*) = \sum R^\beta_{\alpha k l} h^*_\alpha \otimes h^*_\beta \otimes d\bar{z}_k \wedge d\bar{z}_l$ with a holomorphic local frame $h_\alpha$ of $S^m T_X^*$, we have

$$
\mathcal{R}_{\Theta(S^m T_X^*)} u = \sum_{\alpha,\beta,k,l} u_{\alpha} R^\beta_{\alpha k l} h^*_\beta d\bar{z}_k \otimes d\bar{z}_l \quad (3.4)
$$

for $u = \sum_{\alpha} u_{\alpha} h_{\alpha}$. \hfill \square

For the Hermitian metric $g^{-m}$ on $S^m T_X^*$ and the Kähler metric $g$ on $X$, consider the Laplacian

$$
\Box_m : C^\infty(X, S^m T_X^* \otimes \Lambda^{0,s} T_X^*) \to C^\infty(X, S^m T_X^* \otimes \Lambda^{0,s} T_X^*)
$$

defined by

$$
\Box_m = \partial \partial^* + \partial^* \partial.
$$
If there is no confusion, we will omit subscript \(m\) and superscript \(s\) for brevity.

For given \(m \in \mathbb{N}, \ell \in \mathbb{N} \cup \{0\}\), and \(r \in \mathbb{Z}\), let \(\mathcal{A}_{r,t}^{m}\) be the set of operators which map

\[
S_{k}^{m}T_{X}^{s} \rightarrow S_{k+r}^{m+\ell}T_{X}^{s},
\]

with \(S_{k}^{m}T_{X}^{s} := S_{k}^{m}T_{X}^{s} \otimes \Lambda^{k}T^{*}X\). For fixed \(\ell, r, \) consider a family of operators \(A = \{A_{m}\}\) where \(A_{m} \in \mathcal{A}_{r,t}^{m}\), i.e. \(A \subset \bigoplus_{m} \mathcal{A}_{r,t}^{m}\). Let \(A \subset \bigoplus_{m} \mathcal{A}_{n,1}^{m}\) and \(B \subset \bigoplus_{m} \mathcal{A}_{b,0}^{m}\) be such families of operators. Define

\[
\{A, B\}_{m} := A_{m} \circ B_{m} - (-1)^{ab}B_{m+1} \circ A_{m}
\]

and denote the collection of \(\{A, B\}_{m}\) by \(\{A, B\}\). Note that \(\{A, B\} \in \bigoplus_{m} \mathcal{A}_{a+b,1}^{m}\). Also, one defines

\[
\{B, A\}_{m} = B_{m+1} \circ A_{m} - (-1)^{ba}A_{m} \circ B_{m}
\]

and it is easy to verify that \(\{A, B\}_{m} = (-1)^{ab+1}\{B, A\}_{m}\).

**Lemma 3.5.**

\[
\{\Box, \mathcal{R}_{G}\} = \Box \circ \mathcal{R}_{G} - \mathcal{R}_{G} \circ \Box = \{\bar{\partial}, \{\bar{\partial}^{*}, \mathcal{R}_{G}\}\} + \{\bar{\partial}^{*}, \{\mathcal{R}_{G}, \bar{\partial}\}\}
\]

**Proof.** Note that \(\mathcal{R}_{G} \in \bigoplus_{m} \mathcal{A}_{a+1}^{m}\) and \(\Box \in \bigoplus_{m} \mathcal{A}_{a+1}^{m}\). Since

\[
\{\bar{\partial}, \{\bar{\partial}^{*}, \mathcal{R}_{G}\}\} = \bar{\partial}_{m+1}\{\bar{\partial}^{*}, \mathcal{R}_{G}\} - (-1)^{1-0}\{\bar{\partial}^{*}, \mathcal{R}_{G}\}\bar{\partial}_{m}
\]

\[
= \bar{\partial}_{m+1}\left(\bar{\partial}^{*}_{m+1}\mathcal{R}_{G} - (-1)^{1}\mathcal{R}_{G}\bar{\partial}^{*}_{m}\right) - \left(\bar{\partial}^{*}_{m+1}\mathcal{R}_{G} - (-1)^{1}\mathcal{R}_{G}\bar{\partial}^{*}_{m}\right)\bar{\partial}_{m}
\]

\[
= \bar{\partial}_{m+1}\bar{\partial}^{*}_{m+1}\mathcal{R}_{G} + \bar{\partial}_{m+1}\mathcal{R}_{G}\bar{\partial}^{*}_{m} - \bar{\partial}^{*}_{m+1}\mathcal{R}_{G}\bar{\partial}^{*}_{m} - \mathcal{R}_{G}\bar{\partial}^{*}_{m}\bar{\partial}_{m}
\]

and

\[
\{\bar{\partial}^{*}, \{\mathcal{R}_{G}, \bar{\partial}\}\} = \bar{\partial}^{*}_{m+1}\{\mathcal{R}_{G}, \bar{\partial}\} - (-1)^{-1-2}\{\mathcal{R}_{G}, \bar{\partial}\}\bar{\partial}^{*}_{m}
\]

\[
= \bar{\partial}^{*}_{m+1}\left(\mathcal{R}_{G}\bar{\partial}_{m} - (-1)^{1}\bar{\partial}^{*}_{m+1}\mathcal{R}_{G}\right) - (\mathcal{R}_{G}\bar{\partial}_{m} - (-1)^{1}\bar{\partial}^{*}_{m+1}\mathcal{R}_{G}\) \bar{\partial}^{*}_{m}
\]

\[
= \bar{\partial}^{*}_{m+1}\mathcal{R}_{G}\bar{\partial}_{m} + \bar{\partial}^{*}_{m+1}\bar{\partial}^{*}_{m+1}\mathcal{R}_{G} - \mathcal{R}_{G}\bar{\partial}^{*}_{m} - \bar{\partial}^{*}_{m+1}\mathcal{R}_{G}\bar{\partial}^{*}_{m},
\]

it follows that

\[
\{\bar{\partial}, \{\bar{\partial}^{*}, \mathcal{R}_{G}\}\} + \{\bar{\partial}^{*}, \{\mathcal{R}_{G}, \bar{\partial}\}\} = \Box_{m+1}\mathcal{R}_{G} - \mathcal{R}_{G}\Box_{m} = \{\Box, \mathcal{R}_{G}\}.
\]

□

**Lemma 3.6.** Let \((X, g)\) be a Kähler manifold. Let \(S^{m}T_{X}^{*}\) and \(T_{X}^{*}\) equip the Hermitian metrics \(g^{-m}\) and \(g\). Then for any \(u \in C^{\infty}(X, S^{m}T_{X}^{*})\),

\[
\{\Box, \mathcal{R}_{G}\}u = \mathcal{R}_{\Theta(S^{m}T_{X}^{*})}u.
\]

**Proof.** Fix a point \(p \in X\). Let \((z_{1}, \ldots, z_{n})\) be a normal coordinate system on \(X\) at \(p\) with respect to \(g\). For any \(u = \sum_{a}u_{a}h_{\alpha} \in C^{\infty}(X, S^{m}T_{X}^{*})\) where \(h_{\alpha}\) is a holomorphic frame of \(S^{m}T_{X}^{*}\), we
have
\[
\{\bar{\partial}^*, R_G\} u = \bar{\partial}^* R_G(u) - (-1)^{|1|} R_G \bar{\partial}^*(u) = \bar{\partial}^* R_G(u)
\]
\[
= \bar{\partial}^* \left( \sum_{\alpha, \beta, \gamma} g_{\alpha \beta} u_{\gamma} d_{\alpha} \otimes d_{\beta} \right) = \sum_{\alpha, \beta, \gamma} h_{\gamma} d_{\alpha} \otimes \bar{\partial}_{\gamma}^* (u_{\gamma} g_{\alpha \beta} d_{\beta}) + O(|z|)
\]
\[
= \sum_{\alpha, \beta, \gamma} h_{\gamma} d_{\alpha} \otimes \left( -\sum_k \frac{\partial}{\partial z_k} \frac{\partial (u_{\gamma} g_{\alpha \beta})}{\partial z_k} d_{\beta} \right) + O(|z|) \tag{3.8}
\]
\[
= -\sum_{\alpha, \beta, \gamma} \left( h_{\gamma} d_{\alpha} \otimes \sum_k g_{\alpha \beta} \frac{\partial u_{\gamma}}{\partial z_k} \delta_k^{\beta} \right) + O(|z|)
\]
\[
= -\sum_{\alpha, \beta, \gamma} g_{\alpha \beta} \frac{\partial u_{\gamma}}{\partial z_{\beta}} h_{\gamma} d_{\alpha} + O(|z|).
\]

By evaluating (3.2), (3.8) at the point \( p \) we obtain
\[
\{\bar{\partial}^*, R_G\} u = -R_{D_m} u. \tag{3.9}
\]

Therefore by (3.2) and the relation \( \Theta(S^m T_X) = \partial \theta \), we have
\[
\bar{\partial} R_{D_m} u = \sum_{k, \alpha, \mu} \frac{\partial^2 u_{\alpha}}{\partial z_k \partial \bar{z}_{\mu}} d_{z_{k}} h_{\alpha} \otimes d_{\bar{z}_{\mu}} + \sum_{\alpha, \beta, \mu, \zeta} u_{\alpha} \frac{\partial \theta^\mu_{\alpha \beta}}{\partial \bar{z}_{\zeta}} d_{\bar{z}_{\beta}} h_{\mu} \otimes d_{\bar{z}_{\zeta}} + O(|z|)
\]
\[
= \sum_{k, \alpha, \mu} \frac{\partial^2 u_{\alpha}}{\partial z_k \partial \bar{z}_{\mu}} d_{z_k} h_{\alpha} \otimes d_{\bar{z}_{\mu}} - \sum_{\alpha, \beta, \mu, \zeta} u_{\alpha} R^\mu_{\alpha \beta \zeta} h_{\alpha} \otimes d_{z_{\beta}} \wedge d_{\bar{z}_{\zeta}} + O(|z|) \tag{3.10}
\]

with \( \Theta(S^m T_X) = \sum_{\alpha, \mu} \Omega^{\alpha \mu}_\alpha h_{\alpha} \otimes h_{\mu} = \sum_{\alpha, \beta, \mu, \zeta} R^\mu_{\alpha \beta \zeta} h_{\alpha} \otimes h_{\mu} \otimes d_{z_{\beta}} \wedge d_{\bar{z}_{\zeta}}. \)

On the other hand by Lemma 3.2 we have
\[
R_G \bar{\partial}^* \bar{\partial} u = -\sum_{\mu, \alpha, \beta, \gamma} \frac{\partial^2 u_{\alpha}}{\partial z_{\mu} \partial \bar{z}_{\bar{z}_{\beta}}} h_{\alpha} g_{\gamma \beta} d_{z_{\gamma}} \otimes d_{\bar{z}_{\beta}} + O(|z|)
\]
and
\[
\bar{\partial}^* R_G \bar{\partial} u = \bar{\partial}^* R_G \left( \sum_{\mu, \alpha} \frac{\partial u_{\alpha}}{\partial \bar{z}_{\mu}} h_{\alpha} \otimes d_{\bar{z}_{\mu}} \right) = \bar{\partial}^* \left( \sum_{\mu, \alpha, \beta, \gamma} g_{\gamma \beta} h_{\alpha} d_{z_{\gamma}} \otimes \frac{\partial u_{\alpha}}{\partial \bar{z}_{\beta}} d_{\bar{z}_{\beta}} \right)
\]
\[
= \sum_{\mu, \alpha, \beta, \gamma} g_{\gamma \beta} h_{\alpha} d_{z_{\gamma}} \otimes \left( -\frac{\partial^2 u_{\alpha}}{\partial z_{\mu} \partial z_{\gamma}} \delta_{\gamma}^{\mu} d_{z_{\beta}} + \frac{\partial^2 u_{\alpha}}{\partial z_{\beta} \partial z_{\mu}} \delta_{\beta}^{\mu} d_{z_{\alpha}} \right) + O(|z|)
\]
\[
= \sum_{\mu, \alpha, \beta, \gamma} g_{\gamma \beta} h_{\alpha} \left( -\frac{\partial^2 u_{\alpha}}{\partial z_{\mu} \partial z_{\beta}} d_{z_{\gamma}} \otimes d_{\bar{z}_{\beta}} + \frac{\partial^2 u_{\alpha}}{\partial z_{\beta} \partial z_{\mu}} d_{z_{\gamma}} \otimes d_{\bar{z}_{\mu}} \right) + O(|z|).
\]

Hence we have
\[
\{\bar{\partial}^*, R_G\} \bar{\partial} u = \bar{\partial}^* R_G \bar{\partial} u + R_G \bar{\partial}^* \bar{\partial} u
\]
\[
= -2 \sum_{\mu, \alpha, \beta, \gamma} \frac{\partial^2 u_{\alpha}}{\partial z_{\mu} \partial \bar{z}_{\beta}} h_{\alpha} g_{\gamma \beta} d_{z_{\gamma}} \otimes d_{\bar{z}_{\beta}} + \sum_{\mu, \alpha, \beta, \gamma} g_{\gamma \beta} h_{\alpha} \frac{\partial^2 u_{\alpha}}{\partial z_{\beta} \partial \bar{z}_{\mu}} d_{z_{\gamma}} \otimes d_{\bar{z}_{\mu}} + O(|z|). \tag{3.11}
\]
By (3.11), (3.9) and (3.10) we obtain
\[
\{\bar{\partial}, \{\bar{\partial}^*, R_G\}\}u = \bar{\partial}\{\bar{\partial}^*, R_G\}u - \{\bar{\partial}^*, R_G\}\bar{\partial}u = -\bar{\partial}R_{D_m}u - \{\bar{\partial}^*, R_G\}\bar{\partial}u
\]
\[
= -2 \sum_{k,\alpha,\mu} \frac{\partial^2 u_\alpha}{\partial z_k \partial \bar{z}_\mu} dz_k h_\alpha \otimes d\bar{z}_\mu + \sum_{\alpha,\beta,\mu,\zeta} u_\alpha R^\mu_{\alpha\beta\zeta} dz_\mu \otimes d\bar{z}_\zeta
\]
\[
+ 2 \sum_{\mu, l, \alpha} \frac{\partial^2 u_\alpha}{\partial z_\mu \partial \bar{z}_l} h_\alpha dz_l \otimes d\bar{z}_l
\]
at the point \(p\). Similarly at \(p\) we obtain
\[
\{\bar{\partial}^*, \{R_G, \bar{\partial}\}\}u = \bar{\partial}^*\{R_G, \bar{\partial}\}u - \{R_G, \bar{\partial}\}\bar{\partial}^*u = \bar{\partial}^*R_G\bar{\partial}u + \bar{\partial}^*\bar{\partial}R_Gu
\]
\[
= -2 \sum_{\mu, l, \alpha} h_\alpha dz_l \otimes \frac{\partial^2 u_\alpha}{\partial \bar{z}_\mu \partial z_\mu} dz_\mu + 2 \sum_{\mu, l, \alpha} h_\alpha dz_l \otimes \frac{\partial^2 u_\alpha}{\partial z_\mu \partial \bar{z}_\mu} d\bar{z}_\mu
\]
and hence by Lemma 3.5 we obtain
\[
\{\square, R_G\}u = \{\bar{\partial}, \{\bar{\partial}^*, R_G\}\}u + \{\bar{\partial}^*, \{R_G, \bar{\partial}\}\}u = \sum_{\alpha, \beta, \mu, \zeta} u_\alpha R^\mu_{\alpha\beta\zeta} dz_\mu \otimes d\bar{z}_\zeta
\]
at \(p\) which implies the lemma. \(\square\)

3.2. Hodge type identities over the complex hyperbolic space form. Let \(\Sigma = \mathbb{B}^n / \Gamma\) and \(g\) be the metric induced from the Bergman metric on \(\mathbb{B}^n\). Then \(S^m T^*_\Sigma\) has the induced metric \(g^{-m}\).

For any measurable section \(\phi\) of \(S^m T^*_\Sigma \otimes \Lambda^{p, q}(\Sigma)\) define an \(L^2\)-norm by
\[
||\phi||^2 = \int_\Sigma \langle \phi, \phi \rangle dV_\Sigma
\]
where \(\langle \cdot, \cdot \rangle\) and \(dV_\Sigma\) are induced by \(g\) on \(\Sigma\).

Suppose \(m\) is sufficiently large and let \(v\) be an element of \(C^\infty(\Sigma, S^{N+m} T^*_\Sigma \otimes \Lambda^{0,1} T^*_\Sigma)\) satisfying \(\bar{\partial}v = 0\). For the Green operator \(G^1\) of \(S^{N+m} T^*_\Sigma\)-valued \((0,1)\)-forms, we have
\[
\square^1 \circ G^1 v = v.
\]
if \(\ker \square^1 = 0\). In particular, \(u := \bar{\partial}^* G^1 v\) is the \(L^2\) minimal solution for \(\bar{\partial}\) -equation \(\bar{\partial}u = v\) when \(\bar{\partial}v = 0\) and \(\ker \square^1 = 0\).

**Proposition 3.7.** \(R_G\) is a linear injective map and for any \(u \in C^\infty(\Sigma, S^m T^*_\Sigma)\)
\[
||R_G(u)||^2 = n||u||^2
\]
and
\[
\{\square, R_G\}(u) = 2m R_G(u).
\]

**Proof.** It is clear that the map \(R_G\) is an injective linear map. The equation (3.12) can be induced by the following:
\[
||R_G(u)||^2 = \int_\Sigma \sum_{\ell, s} \langle ue_\ell \otimes \bar{e}_\ell, ue_s \otimes \bar{e}_s \rangle g^{-m-1} \otimes g dV
\]
\[
= \int_\Sigma \sum_{\ell, s} \langle ue_\ell, ue_s \rangle g^{-m-1} \langle \bar{e}_\ell, \bar{e}_s \rangle g dV
\]
\[
= \int_\Sigma \sum_{\ell} \langle ue_\ell, ue_\ell \rangle g^{-m-1} dV = n||u||^2
\]
By (3.3) we have

\[
\mathcal{R}_{(S^mT^*_\Sigma)} \left( \sum_{|I|=m} f_I e_I^I \right) = \sum_{|I|=m} f_I \sum_{j} i_j e_1^{i_1} \cdots e_j^{i_j-1} \cdots e_n^{i_n} \mathcal{R}_{T^*_\Sigma}(e_j).
\] (3.14)

Since the Chern curvature tensor of \( T^*_\Sigma \) with respect to \( g \) is given by

\[
(\Theta T^*_\Sigma)^{a}_{b} = -(e_a \wedge \bar{e}_b + \delta_{ab} \sum_r e_r \wedge \bar{e}_r),
\]

one has

\[
(\Theta T^*_\Sigma)^{a}_{b} = (e_b \wedge \bar{e}_a + \delta_{ab} \sum_r e_r \wedge \bar{e}_r).
\]

This implies

\[
\Theta T^*_\Sigma(e_j) = \sum_a e_a \otimes \Theta(T^*_\Sigma)^{a}_{} = \sum_a e_a \otimes \left( e_j \wedge \bar{e}_a + \delta_{ja} \sum_r e_r \wedge \bar{e}_r \right)
\]

\[
= \sum_a e_a \otimes e_j \wedge \bar{e}_a + \sum_r e_j \otimes e_r \wedge \bar{e}_r
\]

and hence by (3.14) we obtain

\[
\mathcal{R}_{(S^mT^*_\Sigma)} \left( \sum_{|I|=m} f_I e_I^I \right) = \sum_{|I|=m} f_I \left( \sum_{j} i_j e_1^{i_1} \cdots e_j^{i_j-1} \cdots e_n^{i_n} \left( 2 \sum_a e_j e_a \otimes \bar{e}_a \right) \right)
\]

\[
= 2 \sum_{|I|=m} \sum_{j} i_j f_I e_I^I \sum_a e_a \otimes \bar{e}_a = 2m \mathcal{R}_G u.
\]

By Lemma 3.6 we obtain the lemma.

\[ \square \]

**Corollary 3.8.**

\[ \mathcal{R}^m_G(\ker(\Box_m^0 - \lambda I)) \subset \ker \left( \Box^1_m - (\lambda + 2m) I \right). \]

**Proof.** Let \( \lambda \) is an eigenvalue of \( u \) for \( \Box^0_m \). Then

\[ \{ \Box, \mathcal{R}^m_G \} u = \Box \mathcal{R}^m_G(u) - \mathcal{R}^m_G(\lambda u) = 2m \mathcal{R}^m_G(u). \]

Therefore, \( \Box \mathcal{R}^m_G(u) = (2m + \lambda) \mathcal{R}^m_G(u) \) follows. \[ \square \]

4. **Vanishing Theorem on the Symmetric Power of the Cotangent Bundle**

The goal of this section is to establish a vanishing theorem for \( H_{\overline{\partial}}^{0,1}(\Sigma, S^mT^*_\Sigma) \).

**Definition 4.1.** A line bundle \( L \) on a Kähler manifold \( X \) is said to be positive if there exists a hermitian metric \( h \) on \( L \) with the Chern curvature form \( -\Theta(L) \) is a positive \((1,1)\) form.

**Theorem 4.2** (Kodaira-Nakano vanishing theorem). If \( (E,h) \) is a positive line bundle on a compact Kähler manifold \( (X,\omega) \), then

\[ H_{\overline{\partial}}^{p,q}(X,E) = 0 \]

for \( p + q \geq n + 1 \).

**Proposition 4.3.** Let \( \Sigma = \mathbb{B}^n/\Gamma \) be a compact complex hyperbolic space form. For \( m \geq n+2 \),

\[ H_{\overline{\partial}}^{0,1}(\Sigma, S^mT^*_\Sigma) = 0. \]
Proof. The following proof is influenced by the argument in [4]. Let \( \mathbb{P}T_\Sigma \) be the projectivization of the holomorphic tangent bundle \((T_\Sigma, g) \to \Sigma\). Then one has the associated line bundle \((\mathcal{O}_{\mathbb{P}T_\Sigma}(-1), \hat{g}) \to \mathbb{P}T_\Sigma\), which is obtained by the fiberwise Hopf blow-up process. Now we impose a Kähler form \( \omega \) on \( \mathbb{P}T_\Sigma \) by \(-c_1(\mathcal{O}_{\mathbb{P}T_\Sigma}(-1), \hat{g})\), where \(c_1\) denotes the first Chern class.

Let \( K_{\mathbb{P}T_\Sigma} \) be the canonical line bundle over \( \mathbb{P}T_\Sigma \). Then a direct calculation yields that \( K_{\mathbb{P}T_\Sigma}^{-1} \otimes \mathcal{O}_{\mathbb{P}T_\Sigma}(m) \) is positive if \( m \geq n + 2 \) (see [3][6]). Therefore the Kodaira-Nakano vanishing theorem guarantees that \( H^1(\mathbb{P}T_\Sigma, \mathcal{O}_{\mathbb{P}T_\Sigma}(m)) \cong H^{n,1}_{\hat{g}}(\mathbb{P}T_\Sigma, K_{\mathbb{P}T_\Sigma}^{-1} \otimes \mathcal{O}_{\mathbb{P}T_\Sigma}(m)) = 0 \). Since \( H^1(\Sigma, S^m T^*_\Sigma) \cong H^1(\mathbb{P}T_\Sigma, \mathcal{O}_{\mathbb{P}T_\Sigma}(m)) \), if \( m \geq n + 2 \) then \( H^0,1(\Sigma, S^m T^*_\Sigma) \) vanishes.

5. Construction of \( L^2 \) holomorphic functions on \( \Omega \)

5.1. Precomputations.

Lemma 5.1. Let

\[
\left. \frac{\partial (T_z w)_k}{\partial \bar{z}_j} \right|_{w=T_z t} = \sum_l B_l(z) t^l \quad \text{and} \quad \left. \frac{\partial (T_z w)_k}{\partial z_j} \right|_{w=T_z t} = \sum_l C_l(z) t^l
\]

be the expansion in \( t \) variable. Then \( B_l = 0 \) if \(|l| \geq 3\) and \( C_l = 0 \) if \(|l| \geq 2\).

Proof. By differentiating \((T_z w)_k\) with respect to \(\frac{\partial}{\partial z_j}\) and \(\frac{\partial}{\partial \bar{z}_j}\) respectively, we obtain

\[
\left. \frac{\partial (T_z w)_k}{\partial \bar{z}_j} \right|_{w=T_z t} = -\frac{\partial}{\partial \bar{z}_j} \left( P_z(w) + s_z Q_z(w) \right)_k \frac{1}{1 - w \cdot \bar{z}} + \frac{(z - P_z(w) - s_z Q_z(w))_k w_j}{(1 - w \cdot \bar{z})^2}
\]

and

\[
\left. \frac{\partial (T_z w)_k}{\partial z_j} \right|_{w=T_z t} = \frac{\partial}{\partial z_j} \left( z - P_z(w) - s_z Q_z(w) \right)_k \frac{1}{1 - w \cdot \bar{z}} - \frac{\partial}{\partial \bar{z}_j} \left( P_z(w) + s_z Q_z(w) \right)_k \frac{1}{1 - w \cdot \bar{z}}.
\]

Note that \(-\frac{\partial}{\partial \bar{z}_j} \left( P_z(w) + s_z Q_z(w) \right)_k\) and \(-\frac{\partial}{\partial z_j} \left( P_z(w) + s_z Q_z(w) \right)_k\) are linear operators and holomorphic in the \( w \) variable. We will denote these by \( L_z(w) \) and \( \bar{L}_z(w) \), respectively.

By virtue of (2.2)

\[
\left. \frac{\partial (T_z w)_k}{\partial \bar{z}_j} \right|_{w=T_z t} = \frac{L_z(T_z t)_k}{1 - T_z t \cdot \bar{z}} + \frac{t_k(T_z t)_j}{1 - T_z t \cdot \bar{z}}
\]

\[
= \frac{L_z \left( z - P_z(t) - s_z Q_z(t) \right)_k}{1 - t \cdot \bar{z}} + \frac{t_k \left( z - P_z(t) - s_z Q_z(t) \right)_j}{1 - |z|^2}
\]

and

\[
\left. \frac{\partial (T_z w)_k}{\partial z_j} \right|_{w=T_z t} = \frac{\delta_{jk}}{1 - |z|^2} + \frac{L_z(T_z t)_k}{1 - T_z t \cdot \bar{z}}
\]

\[
= \frac{\delta_{jk}}{1 - |z|^2} + \frac{L_z \left( z - P_z(t) - s_z Q_z(t) \right)_k}{1 - t \cdot \bar{z}} + \frac{t_k \left( z - P_z(t) - s_z Q_z(t) \right)_j}{1 - |z|^2}
\]

(5.1)
Since $z - P_z(t) - s_z Q_z(t)$ is holomorphic and linear in $t$ variable, $\frac{\partial(T_z w)_k}{\partial z_j}|_{w=T_z t}$ is a polynomial of degree two and $\frac{\partial(T_z w)_k}{\partial z_j}|_{w=T_z t}$ is a polynomial of degree one in $t$ variable.

\textbf{Lemma 5.2.}

\[
\sum_{k=1}^{n} \left( \frac{\partial(T_z w)_k}{\partial z_j} \right)_{w=T_z t} z_k = \left( -1 + \frac{s_z}{1 - |z|^2} \right) t_j + \left( \frac{1 - s_z}{1 - |z|^2} \right) \frac{t \cdot \bar{z}}{|z|^2} - \frac{1}{1 - |z|^2} \left( 1 - s_z \right) (t \cdot z)^2 z_j.
\]

\textbf{Proof.} Differentiating (2.2) with respect to $\frac{\partial}{\partial z_j}$, we have

\[
\sum_{k=1}^{n} \left( \frac{\partial(T_z w)_k}{\partial z_j} \right)_{w=T_z t} = -(T_z w)_j - \frac{\partial}{\partial z_j} \left( \frac{1 - |z|^2}{1 - w \cdot z} \right).
\]

Since

\[
\frac{\partial}{\partial z_j} \left( \frac{1 - |z|^2}{1 - w \cdot z} \right) = \frac{-z_j}{1 - w \cdot z} + \frac{w_j (1 - |z|^2)}{(1 - w \cdot z)^2}
\]

and

\[
(T_z t)_j = \frac{z_j - t \cdot z}{1 - t \cdot \bar{z}},
\]

by (2.2) we obtain

\[
\frac{\partial}{\partial z_j} \left( \frac{1 - |z|^2}{1 - w \cdot z} \right)_{w=T_z t} = -\frac{1 - t \cdot \bar{z}}{1 - |z|^2} \left( \frac{\bar{z}}{1 - |z|^2} \bar{z}_j + s_z \left( t_j - \frac{t}{|z|^2} \bar{z}_j \right) \right)
\]

and hence we obtain the lemma.

\textbf{Lemma 5.3.}

\[
\sum_{k=1}^{n} \left( \frac{\partial(T_z w)_k}{\partial z_j} \right)_{w=T_z t} \left( t_k + \frac{\partial(T_z w)_k}{\partial \bar{z}_j} \right)_{w=T_z t} t_k \right) = -\sum_{l} \frac{z_l \bar{z}_l (s_z - 1)}{|z|^2 (1 - |z|^2)} t_l + \frac{s_z}{1 - |z|^2} t_j - \sum_{l} \frac{s_z}{1 - |z|^2} t_l t_j \bar{t}_l + \sum_{l,m} \frac{z_l \bar{z}_m (s_z - 1)}{|z|^2 (1 - |z|^2)} t_l \bar{t}_m.
\]

\textbf{Proof.} By differentiating (2.4) with respect to $-\frac{\partial}{\partial z_j}$ we obtain

\[
\sum_{k=1}^{n} \left( \frac{\partial(T_z w)_k}{\partial z_j} \right)_{w=T_z t} \left( t_k + \frac{\partial(T_z w)_k}{\partial \bar{z}_j} \right)_{w=T_z t} t_k \right) = -\frac{\partial}{\partial z_j} \left( \frac{|z|^2 (1 - |w|^2)}{1 - w \cdot z} \right)_{w=T_z t}.
\]

Since

\[
\frac{\partial}{\partial z_j} \left( (1 - |z|^2)(1 - |w|^2) \right) = -(1 - |w|^2) z_j
\]

and

\[
\frac{\partial}{\partial z_j} \frac{1}{1 - w \cdot z} = \frac{w_j (1 - z \cdot \bar{w})}{|1 - w \cdot z|^2},
\]

one obtains

\[
\frac{\partial}{\partial z_j} \frac{|z|^2 (1 - |w|^2)}{1 - w \cdot z} = \frac{1 - |w|^2 z_j}{1 - |z|^2} + \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - w \cdot z|^2} \frac{w_j (1 - z \cdot \bar{w})}{|1 - w \cdot z|^4}.
\]

Since one obtains by (2.2) and (2.4)

\[
-\frac{(1 - |w|^2) z_j}{1 - |z|^2} = \frac{z_j}{1 - |z|^2}
\]
and

\[(1 - |z|^2)(1 - |w|^2) \frac{w_j(1 - z \cdot \bar{w})}{|1 - w \cdot z|^4} \bigg|_{w = T_z t} = \frac{1 - |t|^2}{1 - |z|^2} \left( \frac{t_j - z_j}{|z|^2} - s_z \left( \frac{t_j - z_j}{|z|^2} \right) \right),\]

by (5.7) and (5.5).

Lemma 5.5. In the expression

\[\sum_{k=1}^{n} \left( \frac{\partial (T_z w)_k}{\partial ar{z}_j} \right) \bigg|_{w = T_z t} = \frac{1 - |t|^2}{1 - |z|^2} \left( \frac{t_j - z_j}{|z|^2} + s_z \left( \frac{t_j - z_j}{|z|^2} \right) \right),\]

and hence the lemma is proved.

\[\square\]

Lemma 5.4.

\[\sum_{k=1}^{n} \frac{\partial (T_z w)_k}{\partial z_j} \bigg|_{w = T_z t} \bar{z}_k = \frac{\bar{z}_j}{1 - |z|^2} - \frac{\bar{z}_j}{1 - |z|^2} t^l \cdot \bar{z}. \tag{5.6}\]

Proof. Since

\[\sum_{k=1}^{n} \frac{\partial (T_z w)_k}{\partial z_j} \bar{z}_k = - \frac{\partial}{\partial z_j} \left( \frac{1 - |z|^2}{1 - w \cdot \bar{z}} \right) = \frac{\bar{z}_j}{1 - w \cdot \bar{z}} \tag{5.7}\]

by differentiating the formula (2.2) with respect to \( \frac{\partial}{\partial z_j} \), we obtain the lemma by substituting \( w = T_z t \) in (5.7) and by applying (2.2).

\[\square\]

By Lemma 5.1 we may express \( \frac{\partial (T_z w)_k}{\partial z_j} \bigg|_{w = T_z t} \) and \( \frac{\partial (T_z w)_k}{\partial z_j} \bigg|_{w = T_z t} \) as the following:

\[\frac{\partial (T_z w)_k}{\partial z_j} \bigg|_{w = T_z t} = \sum_{l=1}^{n} B^{jk}_{l}(z) t_l + \sum_{l > m}^{n} B^{jk}_{lm}(z) t_l t_m + \sum_{l=1}^{n} B^{jk}_{l}(z) t_l t_l \tag{5.8}\]

Lemma 5.5. In the expression (5.8), we have

\[B^{jk}_{l} = 0 \quad \text{if } k, l, j \text{ are not identical,}\]
\[B^{jk}_{l} + C^{jl}_{k} = 0,\]
\[C^{jl}_{k} = 0 \quad \text{if } l \neq k,\]

and

\[B^{jk}_{k} = -\frac{s_z}{1 - |z|^2} \delta_{\alpha j} + \frac{z_j(s_z - 1)}{|z|^2(1 - |z|^2)} \bar{z}_\alpha \quad \text{if } k \geq \alpha,\]
\[B^{jk}_{k} = -\frac{s_z}{1 - |z|^2} \delta_{\alpha j} + \frac{z_j(s_z - 1)}{|z|^2(1 - |z|^2)} \bar{z}_\alpha \quad \text{if } k \leq \alpha.\]

Proof. By substituting (5.8) in the equation of Lemma 5.2 we see that \( B^{jk}_{l} = 0 \) if \( l, j, k \) are not identical. By substituting (5.8) in the equation (5.1) we obtain

\[\sum_{k=1}^{n} \left( C^{jk}(z) t_k + \sum_{l=1}^{n} \left( B^{jk}_{l} + C^{jl}_{k} \right) t_l t_k \right) + \sum_{l > m}^{n} \sum_{l=1}^{n} B^{jk}_{lm}(z) t_l t_m t_k + \sum_{l=1}^{n} B^{jk}_{l}(z) t_l t_l t_k \]

\[= - \sum_{l} \frac{z_j(s_z - 1)}{|z|^2(1 - |z|^2)} \bar{z}_l t_l + \frac{s_z}{1 - |z|^2} t_j - \frac{s_z}{1 - |z|^2} t_j \bar{z}_\alpha + \frac{z_j(s_z - 1)}{|z|^2(1 - |z|^2)} \bar{z}_l t_l t_m.\]

By comparing the coefficients of \( t \) variable, we obtain the lemma.

\[\square\]

Lemma 5.6.  

1. \( B^{jk}_{l} = \sum_{\alpha} \frac{\partial A_{l\alpha}}{\partial z_j} A_{l\alpha} = 0 \quad \text{if } l \neq k.\)
(2) \[ C^j_k = \sum \alpha \frac{\partial A_{\alpha}}{\partial z_j} A^{\alpha l} - \sum \alpha \frac{\partial^2 (T_z w)^k}{\partial w_\alpha \partial w_\gamma} \bigg|_{z=w} A^{\alpha l} \]

Proof. Note that
\[ B^j_k(z) = \left. \frac{\partial}{\partial t} \right|_{t=0} \left( \frac{\partial (T_z w)^k}{\partial z_j} \bigg|_{w=T_z t} \right) = \sum \alpha \frac{\partial^2 (T_z w)^k}{\partial w_\alpha \partial z_j} \bigg|_{w=z} \frac{\partial (T_z t)^\alpha}{\partial t} \bigg|_{t=0} = \sum \alpha \frac{\partial A_{\alpha}}{\partial z_j} A^{\alpha l}. \quad (5.9) \]

The second equality of (5.9) follows by
\[ \frac{\partial}{\partial z_j} \left( \frac{\partial (T_z w)^k}{\partial w_\alpha} \bigg|_{z=w} \right) = \frac{\partial (T_z w)^k}{\partial w_\alpha \partial z_j} \bigg|_{z=w}, \]

by the chain rule and the fact that \( \frac{\partial (T_z w)^k}{\partial w_\alpha} \) is holomorphic in \( w \) variable. Hence, (1) is followed by Lemma 3.6. To prove (2) note that
\[ C^j_k = \left. \frac{\partial}{\partial t} \right|_{t=0} \left( \frac{\partial (T_z w)^k}{\partial z_j} \bigg|_{w=T_z t} \right) = \sum \alpha \frac{\partial^2 (T_z w)^k}{\partial z_j \partial w_\alpha} \bigg|_{w=z} \frac{\partial (T_z w)^\alpha}{\partial t} \bigg|_{t=0} \]

But
\[ \frac{\partial}{\partial z_j} \left( \frac{\partial (T_z w)^k}{\partial w_\alpha} \bigg|_{w=z} \right) = \sum \alpha \left( \frac{\partial^2 (T_z w)^k}{\partial z_j \partial w_\alpha} \bigg|_{w=z} + \frac{\partial (T_z w)^k}{\partial w_j \partial w_\alpha} \bigg|_{w=z} \right) \]

Therefore (2) follows. \( \square \)

In particular we have
\[ \left. \frac{\partial (T_z w)^k}{\partial z_j} \right|_{w=T_z t} = B^j_k(z) t_k + \sum_{m=1}^{k-1} B^j_k(z) T_z t_m + \sum_{t=k+1}^n B^j_k(z) T_z t_k + B^j_k(z) T_k t_k. \quad (5.10) \]

Corollary 5.7.
\[ \left. \frac{\partial (T_z w)^k}{\partial z_j} \right|_{w=z} = 0. \]

Proof. Since \( T_z(0) = z \), it is followed by (5.10). \( \square \)

Lemma 5.8.
\[ \sum_j A^{j \mu} B^j_k = \delta_{\mu \alpha}. \]

Proof. Since one has
\[ A^{j \mu} = -\frac{s_z^2 - \frac{\mu z_j}{|z|^2} - s_z \left( \delta_{j \mu} - \frac{\mu z_j}{|z|^2} \right)}{s_z}, \]

by straightforward calculation using Lemma 5.5 we obtain the lemma. \( \square \)

5.2. Necessary condition to be holomorphic functions. Let \( f \) be a holomorphic function on \( \Omega_\epsilon \) for some \( \epsilon < 1 \). Then we may consider \( f \) as a \( \Gamma \)-invariant holomorphic function on \( \{ (z, w) \in \mathbb{B}^n \times \mathbb{B}^n : |T_z w| < \epsilon \} \). By putting \( t = T_z w \), there exists \( \tilde{f} \) so that
\[ f(z, w) = f(z, T_z t) = \tilde{f}(z, t). \]

Note that \( \tilde{f}(z, t) \) is holomorphic in \( t \) but not in \( z \). Express
\[ \tilde{f}(z, t) := \sum_{|\ell| = 0}^\infty f_\ell(z) t^\ell \quad \text{with} \quad f_\ell(z) = \frac{1}{\ell!} \left. \frac{\partial^{|\ell|} \tilde{f}}{\partial t^{|\ell|}} \right|_{z=0}. \]

Note that
\[ f(z, w) = \sum_{|\ell| = 0}^\infty f_\ell(z) (T_z w)^\ell. \]
Proposition 5.9. Suppose that $f$ is a holomorphic function on $\Omega_{\epsilon}$ for some $\epsilon < 1$. Then

$$\overline{X}_\mu f_I + \sum_k i_k f_I \Gamma_k^{jk} + (|I| - 1) f_{i_1 \ldots i_{|I| - 1} i_n} = 0$$

for each $I = (i_1, \ldots, i_n)$.

Proof. Since $f(z, w)$ is holomorphic on $\Omega_{\epsilon}$, we have

$$0 = \frac{\partial}{\partial z_j} f(z, w) = \frac{\partial}{\partial z_j} f(z, w(t)) = \frac{\partial}{\partial z_j} \bar{f}(z, w(t))$$

$$= \frac{\partial \bar{f}}{\partial z_j}(z, t) + \sum_k \frac{\partial \bar{f}}{\partial t_k}(z, t) \frac{\partial (T_k w)_k}{\partial z_j} |_{w=T_z t}. \tag{5.11}$$

By (5.11), we obtain

$$0 = \frac{1}{L!} \frac{\partial^m}{\partial t^l} \bigg|_{t=0} \left( \frac{\partial \bar{f}}{\partial z_j}(z, t) + \sum_k \frac{\partial \bar{f}}{\partial t_k}(z, t) \frac{\partial (T_k w)_k}{\partial z_j} \bigg|_{w=T_z t} \right)$$

$$= \frac{\partial f_I}{\partial z_j}(z, t) + \frac{1}{L!} \sum_k \frac{\partial^m}{\partial t^l} \bigg|_{t=0} \left( \frac{\partial \bar{f}}{\partial t_k}(z, t) \frac{\partial (T_k w)_k}{\partial z_j} \bigg|_{w=T_z t} \right). \tag{5.12}$$

Since

$$\frac{\partial \bar{f}}{\partial t_k} = \sum_{|I|=1, i_k \neq 0}^{\infty} i_k f_{i_1 \ldots i_{|I| - 1} i_k} t_1^{i_1} \ldots t_{n+1}^{i_{n+1}},$$

by Lemma 5.1 we have

$$\frac{\partial \bar{f}}{\partial t_k} \frac{\partial (T_k w)_k}{\partial z_j} \bigg|_{w=T_z t} = \sum_{|I|=1, i_k \neq 0}^{\infty} i_k f_{i_1 \ldots i_{|I| - 1} i_k} t_1^{i_1} \ldots t_{n+1}^{i_{n+1}} \left( B_k^{j_k k} t_k \right)$$

$$+ \sum_{|I|=1, i_k \neq 0}^{\infty} i_k f_{i_1 \ldots i_{|I| - 1} i_k} t_1^{i_1} \ldots t_{n+1}^{i_{n+1}} \sum_{m=1}^{k-1} B_k^{l_m} (z) t_k t_m$$

$$+ \sum_{|I|=1, i_k \neq 0}^{\infty} i_k f_{i_1 \ldots i_{|I| - 1} i_k} t_1^{i_1} \ldots t_{n+1}^{i_{n+1}} \left( \sum_{l=k+1}^{n} B_k^{j_k l} (z) t_l t_k + B_k^{j_k k} (z) t_k^2 \right).$$

Therefore we have

$$\frac{1}{L!} \frac{\partial^m}{\partial t^l} \bigg|_{t=0} \left( \frac{\partial \bar{f}}{\partial t_k} \frac{\partial (T_k w)_k}{\partial z_j} \bigg|_{w=T_z t} \right) = i_k f_{i_1 \ldots i_n} B_k^{j_k} + i_k \sum_{m=1}^{k-1} f_{i_1 \ldots i_{|I| - 1} i_n} B_k^{l_m}$$

$$+ \sum_{l=k+1}^{n} f_{i_1 \ldots i_{|I| - 1} i_n} B_k^{j_k l} + (i_k - 1) f_{i_1 \ldots i_{|I| - 1} i_n} B_k^{j_k k}.$$
and hence by (5.12), Lemma 5.8 implies that
\[
0 = \sum_j \mathcal{X}^\mu_j \left( \frac{\partial f_{i_1 \ldots i_n}}{\partial z_j}(z) + \frac{1}{1!} \sum_k \frac{\partial^m}{\partial t^m} \left. \frac{\partial (T_z w)_k}{\partial z_j} \right|_{t=0} \right)
\]
\[
= \mathcal{X}_\mu f_{i_1 \ldots i_n} + \sum_k \left( i_k f_{i_1 \ldots i_n} \sum_j \mathcal{X}^\mu_j B^j_k + i_k \sum_{m=1}^{k-1} f_{i_1 \ldots i_{m-1} \ldots i_n} \sum_j \mathcal{X}^\mu_j B^j_{km} 
\right.
\]
\[
+ \left. i_k \sum_{l=k+1}^n f_{i_1 \ldots i_{l-1} \ldots i_n} \sum_j \mathcal{X}^\mu_j B^j_{lk} + (i_k - 1) f_{i_1 \ldots i_k \ldots i_n} \sum_j \mathcal{X}^\mu_j B^j_{kk} \right)
\]
\[
= \mathcal{X}_\mu f_{i_1 \ldots i_n} + \sum_k i_k f_{i_1 \ldots i_n} \Gamma^\mu_k + \sum_k \left( i_k \sum_{m=1}^{k-1} f_{i_1 \ldots i_{m-1} \ldots i_n} \delta_{\mu m} 
\right.
\]
\[
+ \left. i_k \sum_{l=k+1}^n f_{i_1 \ldots i_{l-1} \ldots i_n} \delta_{\mu l} + (i_k - 1) f_{i_1 \ldots i_k \ldots i_n} \delta_{\mu k} \right)
\]
\[
= \mathcal{X}_\mu f_{i_1 \ldots i_n} + \sum_k i_k f_{i_1 \ldots i_n} \Gamma^\mu_k + (|I| - 1) f_{i_1 \ldots i_{\mu - 1} \ldots i_n}.
\]
\]

\[\square\]

For a \(\Gamma\)-invariant holomorphic function \(\sum_{|I|=f} f_I(z)(T_z w)^I\) on \(\{(z, w) \in \mathbb{B}^n \times \mathbb{B}^n : |T_z w| < \epsilon\}\) for some \(\epsilon < 1\), define
\[
\varphi_I := \begin{cases} f_I(z) e^I & \text{when } |I| = m, \ m \geq 1 \\
0 & \text{when } |I| = 0 \end{cases},
\]
\[
\varphi_k := \sum_{|I|=k} \varphi_I,
\]
and
\[
\varphi(z) := \sum_{|I|=0}^{\infty} \varphi_I \in \bigoplus_{m=0}^{\infty} C^\infty(\Sigma, S^m T^*_{\Sigma}).
\]

(5.13)

We will call \(\varphi\) the associated differential of \(f\). For fixed \(m\) and \(I = (i_1, \ldots, i_n)\) with \(|I| = m\), we have
\[
\partial \varphi_{i_1 \ldots i_n} = \sum_{\mu} \left( \mathcal{X}_\mu f_{i_1 \ldots i_n} + \sum_k i_k f_{i_1 \ldots i_n} \Gamma^\mu_k \right) e^{i_1} \cdots e^{i_n} \otimes \bar{e}_\mu
\]
\[
= - \sum_{\mu} (|I| - 1) f_{i_1 \ldots i_{\mu - 1} \ldots i_n} e^{i_1} \cdots e^{i_n} \otimes \bar{e}_\mu
\]
\[
= - (|I| - 1) \sum_{\mu} \varphi_{i_1 \ldots i_{\mu - 1} \ldots i_n} \otimes \bar{e}_\mu,
\]
which implies
\[
\partial \varphi_k = -(k-1) \mathcal{R}_G (\varphi_{k-1}),
\]
(5.14)
since for fixed \(\mu\), we have \(\sum_{|I|=k} \varphi_{i_1 \ldots i_{\mu - 1} \ldots i_n} = \sum_{|I|=k-1} \varphi_{i_1 \ldots i_n} \).

\[\text{Proof of Theorem 1.1}\] Let \(f \in O(\Omega_\epsilon)\) which vanishes up to \(k\)-th order on \(D\). Then \(\varphi_m \equiv 0\) for any \(m \leq k\) but \(\varphi_{k+1} \not\equiv 0\) and hence we have \(\varphi_{k+1} \in H^0(\Sigma, S^{k+1} T^*_{\Sigma})\) by the equations (5.14). Define \(\Psi_\epsilon(f) = \varphi_{k+1}\). Then the proof is completed. \[\square\]
Proof of Corollary 5.10. For a bounded domain $D$ in $\mathbb{C}^n$ if $\Gamma$ is a discrete subgroup of $\text{Aut}(\mathbb{B}^n)$, it is known by Poincaré that $\sum_{\gamma \in \Gamma} |\mathcal{J}_C \gamma(z)|^2$ locally uniformly converges to a smooth function on $D$ where $\mathcal{J}_C \gamma$ denotes the determinant of complex Jacobian matrix of $\gamma$. By straightforward calculation, for any $\gamma \in \text{Aut}(\mathbb{B}^n)$, we obtain

$$(1 - |\gamma^{-1}(0)|^2)|\gamma(z) - \gamma(w)|^2 \leq |\mathcal{J}_C \gamma(z)|^{\frac{2}{n+1}} |\mathcal{J}_C \gamma(w)|^{\frac{2}{n+1}} |z - w|^2.$$ 

Hence

$$\sum_{\gamma \in \Gamma} (1 - |\gamma^{-1}(0)|^2)^{N/2} \sum_{j=1}^{n} (\gamma_j(z) - \gamma_j(w))^N$$

is a $\Gamma$-invariant holomorphic function on $\mathbb{B}^n \times \mathbb{B}^n$ for any $N \geq n + 1$ with respect to the diagonal action. By Theorem 1.11 there exists a symmetric differential of degree $N$ for any $N \geq n + 2$. □

Corollary 5.10. Let $\Gamma \subset \text{Aut}(\mathbb{B}^n)$ be a torsion-free Kottwitz lattice, and $\Omega = \mathbb{B}^n \times \mathbb{B}^n/\Gamma$ the corresponding holomorphic $\mathbb{B}^n$-fiber bundle. Assume $n+1$ is prime. Then there is no holomorphic function which vanishes on $D$ up to order $k$ for $1 < k \leq n - 1$ on $\Omega$, for any $0 < \epsilon \leq 1$.

Proof. Suppose there is a holomorphic function which vanishes on $D$ of order $k$ with $0 < k \leq n - 1$ on $\Omega$, for some $\epsilon$. Then by Theorem 1.11 there exists a symmetric differential $\psi \in H^0(\Sigma, S^k T^*_\Sigma)$. However by Theorem 1.11 in [3], there is no symmetric differential of degree $1, \ldots, n-1$. □

5.3. Proof of Theorem 1.3. Let $N \geq n + 2$. For a given $\psi \in H^0(\Sigma, S^N T^*_\Sigma)$, define $\varphi(z) := \sum_{k=0}^{\infty} \varphi_k \in \bigoplus_{k=0}^{\infty} C^\infty(\Sigma, S^m T^*_\Sigma)$ by

$$\begin{cases} 
\varphi_k = 0 & \text{if } k < N, \\
\varphi_N = \psi,
\end{cases} \quad (5.15)$$

and for $m \geq 1$, $\varphi_{N+m}$ is the solution of the following $\overline{\partial}$-equation:

$$\overline{\partial} \varphi_{N+m} = -(N + m - 1) \mathcal{R}_G (\varphi_{N+m-1}) \quad (5.16)$$

By the following lemma and Proposition 1.3 the $L^2$ minimal solution of (5.16)

$$\overline{\partial}^* G^1 (- (N + m - 1) \mathcal{R}_G (\varphi_{N+m-1})) \quad (5.17)$$

exists.

Lemma 5.11. $\mathcal{R}_G (\varphi_{N+m-1})$ is closed.

Proof. Let $\{dz^j\}$ be a local holomorphic frame on $S^{N+m-2} T^*_\Sigma$ where $dz^I$ denotes $dz_1^I \ldots dz_n^I$. We may express $\varphi_{N+m-1}$ by

$$\varphi_{N+m-1} = \sum_{|I| = N+m-1} \varphi_I dz^I.$$ 

By (5.16) we have

$$\overline{\partial} \varphi_{N+m-1} = \sum_{|I| = N+m-1} \sum_j \overline{\mathcal{X}}_j \varphi_I dz^I \otimes \bar{e}_j = -(N + m - 1) \sum_l \varphi_{N+m-2} e_l \otimes \bar{e}_l$$

implying

$$\overline{\mathcal{X}}_j \varphi_I dz^I = -(N + m - 1) \varphi_{N+m-2} e_j \quad (5.18)$$

for any $j$. Therefore we have

$$\overline{\partial} \left( \sum_l \varphi_{N+m-2} e_l \otimes \bar{e}_l \right) = \sum_{l,j,I} \overline{\mathcal{X}}_j \varphi_I dz^I e_l \otimes \bar{e}_l \wedge \bar{e}_j + \sum_{m,j,l} \varphi_{N+m-1} dz_m \otimes (\overline{\partial} (A_{lm} \overline{\mathcal{X}}_{lj}) \wedge d\bar{z}_j).$$

Since by (5.18)

$$\sum_{l,j,I} \overline{\mathcal{X}}_j \varphi_I dz^I e_l \otimes \bar{e}_l \wedge \bar{e}_j = - \sum_{j,l} (N + m - 1) \varphi_{N+m-2} e_j e_l \otimes \bar{e}_l \wedge \bar{e}_j = 0,$$
and by Lemma 5.1 and Lemma 5.5 we have
\[
\sum_{j,l} \partial(A_{lm}A_{lj}) \wedge d\bar{z}_j = \sum_{j,l,k} \left( \frac{\partial A_{lm}}{\partial z_k} A_{lj} + A_{lm} \frac{\partial A_{lj}}{\partial z_k} \right) \overline{d\bar{z}_k} \wedge d\bar{z}_j
\]
\[
= \sum_{j,l,k} \left( \sum_s B_{s}^{kl} A_{sm} A_{lj} + A_{lm} \left( \sum_s A_{sj} C_{s}^{kl} + \frac{\partial^2 (T_z w)_l}{\partial w_k \partial w_j} \right) \right) \overline{d\bar{z}_k} \wedge d\bar{z}_j = 0.
\]

\[\square \]

**Lemma 5.12.** Let \( \{\varphi_k\}_{k=0}^{\infty} \in \bigoplus_{k=0}^{\infty} C^\infty(\Sigma, S^k T_*^*_\Sigma) \) be the sequence defined by (5.15) and (5.16). Then
\[
\|\varphi_{N+m}\|^2 = \left( \frac{(2N-1)! \cdot \{(N+m-1)!\}^2}{((N-1)!)^2 \cdot (2N+m-1)!} \right) \frac{n^m}{m!} \|\psi\|^2
\]
for any \( m \geq 1 \).

**Proof.** First we will show that \( \varphi_{N+m} \) is an eigenvector of \( \Box^0 \). Let \( E_{N,m} \) be its eigenvalue. Since \( \varphi_N = \psi \) is a holomorphic section, \( \varphi_N \) is an eigenvector of \( \Box^0 \) with eigenvalue \( E_{N,0} = 0 \). Suppose that \( \varphi_{N+m} \) is an eigenvector of \( \Box^0 \) for some \( m \geq 0 \). By Corollary 9.3 we have
\[
\mathcal{R}_G(\varphi_{N+m}) = \Box^1 G^1 (\mathcal{R}_G(\varphi_{N+m})) = G^1 ((2N+m) + E_{N,m}) \mathcal{R}_G(\varphi_{N+m}),
\]
and by \( \Box^0 \partial^* = \partial^* \Box^1 \) and \( \Box^1 G^1 = G^1 \Box^1 \) we have
\[
\Box^0 \varphi_{N+m+1} = \Box^0 \partial^* G^1 (- (N + m) \mathcal{R}_G(\varphi_{N+m}))
\]
\[
= (2N+m) + E_{N,m} \partial^* G^1 (- (N + m) \mathcal{R}_G(\varphi_{N+m}))
\]
\[
= (2N+m) + E_{N,m} \varphi_{N+m+1}
\]
which implies that \( \varphi_{N+m+1} \) is an eigenvector of \( \Box^0 \) and \( E_{N+m+1} = 2(N+m) + E_{N,m} \). Moreover we have
\[
E_{N,m} = 2 (0 + N + (N+1) + \cdots + (N+m-1)) = m(2N+m-1).
\]

By (5.16), (5.17), (5.19) and (3.12) we have
\[
\|\varphi_{N,m}\|^2 = \frac{(N + m - 1)^2 \langle \partial^* G^1 \mathcal{R}_G(\varphi_{N,m-1}), \partial^* G^1 \mathcal{R}_G(\varphi_{N,m-1}) \rangle}{\langle G^1 \mathcal{R}_G(\varphi_{N,m-1}), \mathcal{R}_G(\varphi_{N,m-1}) \rangle}
\]
\[
= \frac{(N + m - 1)^2}{2(N+m-1) + E_{N,m-1}} \|\mathcal{R}_G(\varphi_{N,m-1})\|^2
\]
\[
= \frac{n^m}{E_{N,m}} \|\varphi_{N+m-1}\|^2.
\]
Continuing this process we obtain
\[
\|\varphi_{N+m}\|^2 = \left( \prod_{j=1}^{m} \frac{n^m (N+m-j)^2}{E_{N,m-j+1}} \right) \|\psi\|^2 = \left( \prod_{j=1}^{m} \frac{n^m (N+m-j)^2}{(m-j+1)(2N+m-j)} \right) \|\psi\|^2
\]
\[
= \frac{n^m (2N-1)! \cdot \{(N+m-1)!\}^2}{\{(N-1)\}^2 m!(2N+m-1)!} \|\psi\|^2.
\]

\[\square \]

Let us express
\[
\varphi_{\ell} = \sum_{|I| = \ell} f_I e^I
\]
and define a formal sum $f$ on $\Omega$ by

$$f(z, w) = \sum_{|I|=0}^{\infty} f_I(z)(T_z w)^I. \quad (5.20)$$

**Lemma 5.13.** $f(z, w)$ is $\Gamma$-invariant.

**Proof.** Fix $\gamma \in \text{Aut}(\mathbb{B}^n)$. Since

$$T_{\gamma z}\gamma w = U_z T_z w \quad (5.21)$$

for some unitary matrix $U_z$ depending only on $z$, we have $dT_{\gamma z}|_{\gamma w}d\gamma|_w = U_z dT_z|_w$ and in particular

$$U_z = dT_{\gamma z}|_{\gamma w}d\gamma|_z dT_z|_0.$$

Since $\phi \in \bigoplus_{N=0}^{\infty} H^0(\Sigma, S^k T^\Sigma)$, we have $\gamma^*\phi_k = \phi_k$. Note that

$$\gamma^* e_j = \sum_k A_{jk}(\gamma z) d\gamma_k = \sum_{k,m,l} A_{jk}(\gamma z) \frac{\partial \gamma_k}{\partial z^l} A^{lm} e_m$$

where $(A^{lm})$ denotes the inverse matrix of $A$, i.e.

$$\gamma^* e = A(\gamma z) d\gamma(z) A^{-1} e = U_z e.$$

This implies

$$\sum_{|I|=0}^{\infty} f_I(z)e^I = \sum_{|I|=0}^{\infty} f_I(\gamma z)(\gamma^* e)^I = \sum_{|I|=0}^{\infty} f_I(\gamma z)(U_z e)^I \quad (5.22)$$

and hence by (5.21) and (5.22) we have

$$f(\gamma z, \gamma w) = \sum_{|I|=0}^{\infty} f_I(\gamma z)(T_{\gamma z}\gamma w)^I = \sum_{|I|=0}^{\infty} f_I(\gamma z)(U_z T_z w)^I = \sum_{|I|=0}^{\infty} f_I(\gamma z)(T_z w)^I = f(z, w).$$

□

**Lemma 5.14.** Let $f$ be a formal sum given in (5.20). Then

$$\|f\|^2_{\tilde{c}, \alpha} \leq \pi^n \sum_{|I|=0}^{\infty} c_0^{2|I|+2n||\varphi||} \left( \frac{|I|! \Gamma(n + \alpha + 1)}{\Gamma(n + |I| + \alpha + 1)} \right) \quad (5.23)$$

**Proof.** Let $\tilde{\Sigma}$ denote the fundamental domain of $\Sigma$ in $\mathbb{B}^n$ and $\tilde{\Omega}_x$ denote the corresponding domain of $\Omega_x$ in $\tilde{\Sigma} \times \mathbb{B}^n \subset \Omega$. Then

$$\|f\|^2_{\tilde{c}, \alpha} = c_\alpha \int_{\tilde{\Omega}_x} \left| \sum_{|I|=0}^{\infty} f_I(z)(T_z w)^I \right|^2 \left( 1 - \left| \frac{T_z w}{\epsilon} \right| \right)^\alpha |K(z, w)|^2 d\lambda_z d\lambda_w. \quad (5.23)$$

Since $t = T_z w$, $J_{T_z}(0) = (1 - |z|^2)^{n+1}$, $d\lambda_w = |J_{T_z}t|^2 d\lambda_t$ and

$$K(z, w) = K(T_z 0, T_z t) = \frac{K(0, t)}{J_{T_z}(0) J_{T_z}(t)} = \frac{1}{J_{T_z}(0) J_{T_z}(t)},$$

where $K(0, t)$ is the symmetric differential on $\Omega$ by

$$f(z, w) = \sum_{|I|=0}^{\infty} f_I(z)(T_z w)^I. \quad (5.20)$$
by (5.23) we obtain
\[
\|f\|_{e,\alpha}^2 = c_\alpha \int_{\mathbb{C}^n} \frac{1}{(1 - |z|^2)^{n+1}} d\lambda_z \int_{\mathcal{R}_+} \left| \sum_{|l|=0}^\infty f_l(z) t^l \left( 1 - \frac{|t|}{\epsilon} \right)^\alpha \right|^2 d\lambda_t
\]
\[
= c_\alpha \epsilon^{2n} \int_{\mathbb{C}^n} \frac{1}{(1 - |z|^2)^{n+1}} d\lambda_z \int_{\mathcal{R}_+} \left| \sum_{|l|=0}^\infty f_l(z) e^{l|t|} \left( 1 - |t|^2 \right)^\alpha \right|^2 d\lambda_t
\]
\[
= c_\alpha \epsilon^{2n} \int_{\mathbb{C}^n} \frac{1}{(1 - |z|^2)^{n+1}} d\lambda_z \int_{\mathcal{R}_+} \sum_{|l|=0}^\infty \left| f_l(z) e^{l|t|} \right|^2 \left( 1 - |t|^2 \right)^\alpha d\lambda_t
\]  
(5.24)
where \( \mathbb{B}^n = \{ z \in \mathbb{C}^n : |z| < \epsilon \} \). The second line of the equation can be induced by the orthogonality of polynomials with respect to the inner product \( \int_{\mathbb{C}^n} f \overline{g}(1 - |t|^2)^\alpha d\lambda_t \) (See [7]). Since we have
\[
dV \Sigma = \det(B(z)) = K(z, z) d\lambda_z
\]
and
\[
\|\varphi_l\|^2 = \int_{\mathbb{C}^n} \sum_{|l|=1}^\infty \langle f_l e^l, f_l e^l \rangle dV \Sigma = \sum_{|l|=1}^\infty \int_{\mathbb{C}^n} \frac{|f_l|^2}{(1 - |z|^2)^{n+1}} d\lambda_z,
\]
by (5.24) one has
\[
\|f\|_{e,\alpha}^2 = \sum_{|l|=0}^\infty c_\alpha \epsilon^{2|l|+2n} \int_{\mathbb{C}^n} \frac{|f_l(z)|^2}{(1 - |z|^2)^{n+1}} d\lambda_z \int_{\mathcal{R}_+} |t|^2 \left( 1 - |t|^2 \right)^\alpha d\lambda_t
\]
\[
\leq \pi^n \sum_{|l|=0}^\infty 2^{2|l|+2n} \|\varphi_l\|^2 \frac{|l|! \Gamma(n + \alpha + 1)^2}{\Gamma(n + |l| + \alpha + 1)}. \]

\[\square\]

**Corollary 5.15.** Suppose \( \epsilon \leq \frac{1}{\sqrt{n}} \). Then the formal sum (5.20) converges in \( L^2 \) sense on \( \Omega_\epsilon \).

**Proof.** The partial sums
\[
F_{N+m} := \sum_{|l|=0}^{N+m} f_l(z)(T_z w)^l
\]  
(5.25)
satisfy
\[
\|F_{N+m}\|_{e,\alpha}^2 \leq \sum_{l=0}^m \|\varphi_{N+l}\|^2 \frac{\pi^n (N + l)! \epsilon^{2(N+l)+2n} \Gamma(n + \alpha + 1)}{\Gamma(n + N + l + \alpha + 1)}
\]
\[= \pi^n \sum_{l=0}^m \frac{(N + l)! 2^{(N+l)+2n} \Gamma(n + \alpha + 1)}{\Gamma(n + N + l + \alpha + 1)} \frac{(2N - 1)! \{(N + l - 1)\}^2 n^l}{\{(N - 1)\}^2} \frac{\|\psi\|^2}{N!}
\]
\[\leq \pi^n \epsilon^{2N+2n} \frac{\Gamma(n + \alpha + 1) \Gamma(N + 1)}{\Gamma(N + n + \alpha + 1)} 3F_2 (N + 1, N, N; 2N, N + n + 1; 1)
\]
by Lemma 5.14 and Lemma 5.12 where \( _3F_2 \) is a generalized hypergeometric function and it is known that this series converges in our case. In particular, \( F_{N+m} \) converges in \( L^2 \) sense as \( m \to \infty \) on \( \Omega_\epsilon \) with \( \epsilon^n \leq 1 \).

\[\square\]

**Lemma 5.16.** Let \( \epsilon \leq \frac{1}{\sqrt{n}} \) and \( f \) be the \( L^2 \)-limit of the partial sums (5.25) on \( \Omega_\epsilon \). Then \( f \) is holomorphic on \( \Omega \).
Proof. Let \( F_m(z, w) := \sum_{|I|=0}^m f_I(z)(T_z w)^I \) and \( \varphi_I := f_I t^I \). It suffices to show that
\[
\|\bar{\partial} F_m\|_{c, \alpha}^2 \to 0
\] (5.26)
as \( m \to \infty \) for \( \alpha = 1 \).

Since
\[
\frac{\partial F_m}{\partial z_j}(z, w) = \frac{\partial \tilde{F}_m}{\partial z_j}(z, T_z w) + \sum_k \frac{\partial \tilde{F}_m}{\partial t_k}(z, T_z w) \frac{\partial (T_z w)_k}{\partial z_j}
\]
with \( \tilde{F}_m(z, t) := \sum_{|I|=0}^m f^I(z) t^I \) by a similar way when we induced the equation (5.11), we obtain
\[
\sum_j \bar{\partial} \varphi_I = \sum_{|I|=0}^m \left( \sum_j A^{j \mu} \left( \frac{\partial \tilde{F}_m}{\partial z_j}(z, T_z w) + \sum_k \frac{\partial \tilde{F}_m}{\partial t_k}(z, T_z w) \frac{\partial (T_z w)_k}{\partial z_j} \right) \right) e^I \otimes \bar{\partial} \varphi_I
\]
by (5.8) and Lemma 5.8. If we express \( \varphi_I = \sum_{|I|=l} f_I t^I \), we have
\[
\bar{\partial} \varphi_I = \sum_{|I|=l} \sum \left( \sum_j A^{j \mu} \left( \frac{\partial \tilde{F}_m}{\partial z_j}(z, T_z w) + \sum_k \frac{\partial \tilde{F}_m}{\partial t_k}(z, T_z w) \frac{\partial (T_z w)_k}{\partial z_j} \right) \right) e^I \otimes \bar{\partial} \varphi_I
\]
(5.27)
On the other hand one has
\[
\bar{\partial} \varphi_I = -(l - 1) \mathcal{R}_G(\varphi_{I-1}) = -(l - 1) \sum_{|I|=l-1} \sum \frac{1}{|J|=l-1} f_J t^J e_\mu \otimes \bar{\partial} \varphi_I.
\]
(5.28)
Hence by comparing (5.27) and (5.28) one obtains
\[
\sum_{|I|=l} \sum \left( \sum_j A^{j \mu} f_I + \sum_k i_k f_I \Gamma^\mu_k \right) t^I = -(l - 1) \sum_{|I|=l-1} \sum \frac{1}{|J|=l-1} f_J t^J t_\mu
\]
Therefore we obtain
\[
\sum_{|I|=l} \sum \left( \sum_j A^{j \mu} f_I + \sum_k i_k f_I \Gamma^\mu_k \right) t^I = -(l - 1) \sum_{|I|=l-1} \sum \frac{1}{|J|=l-1} f_J t^J t_\mu
\]
If \( g \) and \( h \) are monomials in \( t \) with \( g \neq ch \) for any \( c \in \mathbb{R} \), we have \( \int_{\mathbb{R}^n} g h (1 - |t|^2)^\alpha d\lambda_t = 0 \). Hence by Lemma 5.12, one obtains
\[
\|\bar{\partial} F_m\|_{c, 1} = m^2 \left\| \sum_{|I|=m} \sum_k f_I(T_z w)^I(T_z w)_k \right\|_{c, 1}^2
\]
\[
= c_1 m^2 \left( \sum_{|I|=m} \int_{\mathbb{R}^n} \frac{f_I(z)^2}{(1 - |z|^2)^{n+1}} d\lambda_z \left( \sum_k \int_{\mathbb{R}^n} |t^I t_k|^2 (1 - |t|^2) d\lambda_t \right) \right)
\]
\[
\lesssim m^2 \left\| \varphi_m \right\|^2 \pi \frac{(m+1)! \Gamma(n+2)}{\Gamma(n+m+3)} \epsilon^{2(m+1)+2n} \leq \frac{(m+1)! \Gamma(n+2)}{\Gamma(n+m+3)} \left\| \varphi_m \right\|^2 \epsilon^{2m} m^2
\]
\[
\lesssim \frac{1}{(n/m + 1 + 2/m)(n/m + 1 + 1/m)(m+n)!} \left\| \varphi_m \right\|^2 \epsilon^{2m}.
\]
Since $\epsilon^2 n \leq 1$ and
\[
\|\varphi_{N+1}\|_{L^2}^2 = \frac{(2N-1)!((N+m-1))}{((N-1))!2(2N+m-1)!} (\epsilon^2 n)^m \|\psi\|^2 = O\left(\frac{1}{m}\right)
\]
by the Stirling’s formula, the Lemma is proved. \hfill \Box

Now define $\Phi: \bigoplus_{k=n+2}^\infty H^0(\Sigma, S^k T^*_{\Sigma}) \to \mathcal{O}(\Omega_{\frac{1}{\sqrt{m}}})$ so that $\Phi(\psi)$ to be a $L^2$ holomorphic function \( \sum_{|i|=0}^\infty f_i(z)(T_{z,w})^i \) for $\psi \in \bigoplus_{k=n+2}^\infty H^0(\Sigma, S^k T^*_{\Sigma})$. The following lemma completes the proof of Theorem 1.3.

**Lemma 5.17.** Suppose $\epsilon \leq \frac{1}{\sqrt{n}}$. Then the image of $\Phi$ is dense on $(\mathcal{O}(\Omega_{\epsilon})/T^*_{D,\epsilon})$ for the compact open topology.

**Proof.** The proof is similar to that given in \[\Pi\]. We give a sketch of the proof for the completion of the article. First, we note that $(\mathcal{O}(\Omega_{\epsilon})/T^*_{D,\epsilon})$ is a closed subspace in $A^2(\Omega_{\epsilon})$. Let $f$ be a function which is orthogonal to $\Phi\left(\bigoplus_{j=n+2}^\infty H^0(\Sigma, S^j T^*_{\Sigma})\right)$ in $(\mathcal{O}(\Omega_{\epsilon})/T^*_{D,\epsilon})$ and let $\{\varphi_j\} \in C^\infty(\Sigma, S^m T^*_{\Sigma})$ be its associated differentials. Let $N \geq n+2$ be the least integer such that $\varphi_j = 0$ if $j < N$, but $\varphi_N \neq 0$. In particular $\varphi_N \in H^0(\Sigma, S^N T^*_{\Sigma})$. Let
\[
\Pi^0_{N+m, E_{N+m}}: L^2(\Sigma, S^{N+m} T^*_{\Sigma} \otimes \Lambda^2 T^*_\Sigma) \to \ker(\Box(0) - E_{N+m} I),
\]
be the projection. Then $\Phi(\varphi_N)$ is spanned by $\{\Pi^0_{N+m, E_{N+m}}(\varphi_m)\}$, $m \geq 0$ since we choose the $L^2$-minimal solution to construct $\Phi$ and hence $\langle \langle f, \Phi(\varphi_N) \rangle \rangle = \langle \langle \varphi_N, \varphi_N \rangle \rangle_0 > 0$. This contradiction induces the lemma. \hfill \Box

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