Explicit method to make shortened stabilizer EAQECC from stabilizer QECC

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Abstract
In the previous research by Grassl, Huber and Winter, they proved a theorem which can make entanglement-assisted quantum error-correcting codes (EAQECC) from general quantum error-correcting codes (QECC). In this paper, we prove that the shortened EAQECC is a stabilizer code if the original EAQECC is a stabilizer code.

1 Introduction
Quantum error correction is an important tool for realizing large-scale quantum computers [13]. Stabilizer codes [7][2][3] is a large class of quantum error-correcting codes (QECC), which allows efficient encoding of quantum information into quantum codewords, and relatively efficient decoding. On the other hand, the most general class of QECCs [4] allows neither efficient encoding nor decoding.

The stabilizer code is defined as a common eigenspace of mutually commuting complex unitary matrices (collectively called as stabilizer), which is a bit difficult to handle (for conventional coding theorists working over finite fields). Calderbank et al. [2][3] showed that most aspects of stabilizer codes can be studied through linear subspaces over finite fields that are self-orthogonal with respect to the symplectic inner product [9]. The self-orthogonality corresponds to the commutativity of unitary matrices, but construction and study of self-orthogonal linear spaces remain more difficult than those of the conventional linear codes without orthogonality requirements.

Later Brun et al. [1] made a break-through that enables construction of QECC from any linear spaces over finite fields, and shorten the code length by utilizing pre-shared entanglement between an encoder and a decoder while keeping the number of information symbols and the error correcting capability. It is called entanglement-assisted quantum error-correcting codes (EAQECCs). Brun et al.’s proposal was for qubit (binary case), and EAQECC was generalized to qudits (q-ary case) as well [15][10][12]. Galindo et al. [5] provided a description of q-ary EAQECCs by linear codes over finite fields. Those cited studies concerned with EAQECC constructed from a stabilizer. Recently, Grassl et al. showed the most general framework of EAQECC [8], and a method of constructing an EAQECC from any entanglement-unassisted QECC in Theorem 7 of [8]. A feature of [8] Theorem 7] is that it is unclear whether or not constructed EAQECC is based on stabilizer, even when the original QECC is stabilizer-based. There is an advantage of an EAQECC being stabilizer-based, for example, efficient encoding [11] and decoding [12] procedures are known for stabilizer-based EAQECCs. To this direction, Galindo et al. [6] Proposition 5] showed a construction of EAQECCs from stabilizer codes, and resulting EAQECCs are also stabilizer-based. Our contribution relative to [6] is that our theorem has a weaker assumption than [6] (see Remark 1], and it enables us to construct a wider set of EAQECCs from the same stabilizer code than [6].

We will show that the resulting EAQECC is stabilizer-based if the original QECC is stabilizer-based. Our tools are the conventional puncturing and shortening of linear
codes [14], tailored for self-orthogonal linear spaces with respect to the symplectic inner product.

This letter is organized as follows: First, in Section 2, we explain the necessary assumptions and definitions for the proof, and then in Section 3, we prove some lemmas necessary for main theorem and show its proof. Concluding remarks are given in Section 4.

2 Preparation

$F_q$ denotes a finite field of order $q$. For two vectors $\tilde{a}, \tilde{b} \in \mathbb{F}_q^n$, the Euclidean inner product is defined by

$$\langle \tilde{a}, \tilde{b} \rangle_E = a_1 b_1 + \cdots + a_n b_n.$$ 

$(\tilde{a}, \tilde{b}) \in \mathbb{F}_q^{2n}$ denotes the vector concatenating $\tilde{a}$ and $\tilde{b}$ in $\mathbb{F}_q^n$.

For two vectors $(\tilde{a}, \tilde{b}), (\tilde{c}, \tilde{d}) \in \mathbb{F}_q^{2n}$, the symplectic inner product is defined by

$$\langle (\tilde{a}, \tilde{b}), (\tilde{c}, \tilde{d}) \rangle_s = \langle \tilde{a}, \tilde{d} \rangle_E - \langle \tilde{b}, \tilde{c} \rangle_E.$$ 

For a linear code $V \subseteq \mathbb{F}_q^{2n}$, its symplectic dual code is defined by $V^\perp_s = \{ (\tilde{a}, \tilde{b}) \mid \langle \tilde{a}, \tilde{d} \rangle_E - \langle \tilde{b}, \tilde{c} \rangle_E = 0 \}$, and for $(\tilde{a}, \tilde{b}) \in \mathbb{F}_q^{2n}$, the symplectic weight is defined by

$$w(\tilde{a}, \tilde{b}) = \# \{ i \mid (a_i, b_i) \neq (0, 0) \},$$

where $\#$ denotes the number of elements in a set.

Puncturing in this paper refers to making a new linear code $C' \subset \mathbb{F}_q^{2n-2}$ from a linear code $C \subset \mathbb{F}_q^{2n}$ by eliminating the $i$th and the $(n + i)$th components ($1 \leq i \leq n$) of all vectors in $C$. Shortening in this paper refers to making a new linear code $C' \subset \mathbb{F}_q^{2n-2}$ from a linear code $C \subset \mathbb{F}_q^{2n}$ by selecting vectors in $\tilde{C}$ where the $i$th and the $(n + i)$th components ($1 \leq i \leq n$) are both zero and then eliminating the $i$th and the $(n + i)$th components of the selected vectors.

An $[n, k, d; c]_q$ stabilizer EAQECC is a linear code $C \subset \mathbb{F}_q^{2n}$ with $c = (\dim C - \dim C \cap C^{\perp_s})/2$, $k = c + n - \dim C$ and $d = \min \{ w(\tilde{x}) \mid \tilde{x} \in C^{\perp_s} \setminus C \}$. An $[n, k, d; 0]_q$ stabilizer QECC is a linear code $C \subset \mathbb{F}_q^{2n}$ with $0 = (\dim C - \dim C \cap C^{\perp_s})/2$, $k = n - \dim C$ and $d = \min \{ w(\tilde{x}) \mid \tilde{x} \in C^{\perp_s} \setminus \{0\} \}$.

3 Main Theorem and Proof

Theorem 1. For a linear code $C \subset \mathbb{F}_q^{2n}$ with $C \subset C^{\perp_s}$, $k = n - \dim C$, $d = \min \{ w(\tilde{x}) \mid \tilde{x} \in C^{\perp_s} \setminus \{0\} \}$ and for all natural numbers $\ell$ satisfying $1 \leq \ell \leq d - 1$, puncturing and shortening can create a new linear code $C^{(\ell)} \subset \mathbb{F}_q^{2(n-\ell)}$ with $(\dim C^{(\ell)} - \dim C^{(\ell)} \cap C^{(\ell)}^{\perp_s})/2 = \ell$, $\ell + (n - \ell) - \dim C^{(\ell)} = k$ (therefore $\dim C^{(\ell)} = \dim C$), $d \leq \min \{ w(\tilde{x}) \mid \tilde{x} \in C^{(\ell)} \setminus \{0\} \}$.

Remark 1. Galindo et al. [6, Proposition 5] assumed $2\ell$ is less than the minimum Hamming weight of $C^{\perp_s}$ regarded as an ordinary linear code of length $2n$, while the conclusion was the same as Theorem 1. Galindo et al.'s assumption implies our assumption $\ell < d$, and Theorem 7 has wider applicability.

Example 1. The following set is a basis of a $[5, 1, 3; 0]_2$ stabilizer QECC $A$

$$\begin{align*}
(10010|01100), \\
(01001|00110), \\
(10100|00011), \\
(01010|10001) \\
\end{align*}$$

Then the following set is a basis of its symplectic dual code $A^{\perp_s}$

$$\begin{align*}
(10010|01100), \\
(01001|00110), \\
(10100|00011), \\
(01010|10001), \\
(00001|10100), \\
(00000|11111) \\
\end{align*}$$

When we puncture $A$ at the 3rd and the 8th components, a basis of the punctured code $A^{(1)}$ is as follows

$$\begin{align*}
(1010|0010), \\
(0101|0010), \\
(1000|0011), \\
(0110|1001) \\
\end{align*}$$

When we shorten $A^{\perp_s}$ at the 3rd and the 8th components, a basis of the shortened code $A^{(1)-\perp_s}$ is as follows

$$\begin{align*}
(1010|1011), \\
(0101|1101), \\
(0110|1001), \\
(0001|1010) \\
\end{align*}$$
$A^{(1)}$ is a $[4,1,3;1]_2$ stabilizer EAQecc and $(A^{(1)})^{(1)}$ is a symplectic dual code of $A^{(1)}$.

In order to prove Theorem 1, we prove the following four lemmas.

**Lemma 1.1.** Let $C_{(p)} \subset F_q^{2(n-1)}$ be a linear code made from a linear code $C \subset F_q^{2n}$ with the minimum symplectic weight $w_s \geq 2$, by puncturing $C$ once. Then we have \[ \dim C = \dim C_{(p)}. \]

**Proof.** Let $\dim C = k'$. Suppose that the number of basis vectors in $C_{(p)}$ is less than $C$.

Let $\{\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_{k'}\}$ be a basis of $C$, and $\{\tilde{e}_1', \tilde{e}_2', \ldots, \tilde{e}_{k'}'\}$ be vectors made by puncturing it once. When the number of basis vectors in $C_{(p)}$ is less than $C$, there are two cases, namely (1) a zero vector exists in $\{\tilde{e}_1', \tilde{e}_2', \ldots, \tilde{e}_{k'}'\}$ and (2) $\{\tilde{e}_1', \tilde{e}_2', \ldots, \tilde{e}_{k'}'\}$ is linearly dependent.

Case (1) happens when puncturing of $C$ eliminates a non-zero component of a vector with $w_s = 1$ in a basis of $C$, which contradicts to $w_s \geq 2$.

Case (2) happens when there exists $\tilde{e}_m' (1 \leq m \leq k')$ with

$$\tilde{e}_m' = a_1\tilde{e}_1' + \cdots + a_{m-1}\tilde{e}_{m-1}' + a_{m+1}\tilde{e}_{m+1}' + \cdots + a_{k'}\tilde{e}_{k'}',$$

where $a_i \in F_q (1 \leq i \leq k')$. $C$ has a vector

$$\tilde{e}_m - (a_1\tilde{e}_1 + \cdots + a_{m-1}\tilde{e}_{m-1} + a_{m+1}\tilde{e}_{m+1} + \cdots + a_{k'}\tilde{e}_{k'}$$

of symplectic weight 1, so the assumption $w_s \geq 2$ does not hold.

From Lemma 1.1, we have,

**Corollary 1.1.1.** For a linear code $C \subset F_q^{2n}$ with the minimum symplectic weight $w_s \geq 2$ and for an integer $\ell$ $(1 \leq \ell \leq w_s - 1)$, let $C_{(p)}^{(\ell)} \subset F_q^{2(n-\ell)}$ be a linear code made from $C$ by puncturing it $\ell$ times, then we have

$$\dim C_{(p)}^{(\ell)} = \dim C.$$

**Example 2.** According to Example 2, minimum symplectic weight $w_s$ of $A$ is 3. We have $\dim A^{(1)}_{(p)} = \dim A = 4$.

For a linear code $C \subset F_q^{2n}$ with the minimum symplectic weight $w_s \geq 2$, let $C_{(p)}^{(1)}$ be its symplectic dual code. Let $\dim C_{(p)}^{(1)} = k_{(p)}^{(1)}$, where $\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_{k_{(p)}^{(1)}}$ are row vectors. Define a matrix $M$ by

$$M = \begin{pmatrix} \tilde{e}_1' \\ \vdots \\ \tilde{e}_{k_{(p)}^{(1)}}' \end{pmatrix} = \begin{pmatrix} e_{1(1)}' & \cdots & e_{1(2n)}' \\ \vdots & \ddots & \vdots \\ e_{k_{(p)}^{(1)}(1)}' & \cdots & e_{k_{(p)}^{(1)}(2n)}' \end{pmatrix}.$$ 

Then we have the following lemma.

**Lemma 1.2.** If a column in the matrix $M$ is the zero vector or there is an index $(1 \leq i \leq n)$ such that the $i$th column in $M$ is a scalar multiple of the $(n+i)$th column in $M$, then the minimum symplectic weight of $C$ is 1.

By taking the contraposition of Lemma 1.2, we have,

**Corollary 1.2.1.** If the minimum symplectic weight of $C$ is 2 or larger, then there is neither column vector in the matrix $M$ whose all components are zero, nor an index $(1 \leq i \leq n)$ such that the $i$th column vector in $M$ is a scalar multiple of the $(n+i)$th column vector in $M$.

**Lemma 1.3.** For a linear code $C \subset F_q^{2n}$ with the minimum symplectic weight $w_s \geq 2$, let $C_{(q)}^{(1)}$ be its symplectic dual code, then shortening of $C_{(q)}^{(1)}$ reduces the dimension of $C_{(q)}^{(1)}$ by 2.

**Proof.** From Corollary 1.2.1 and the minimum symplectic weight of $C$ being $\geq 2$, the matrix $M$ can be transformed by elementary row operations as follows

$$M = \begin{pmatrix} e_{1(1)}' & \cdots & 1 & \cdots & 0 & \cdots & e_{1(2n)}' \\ e_{2(1)}' & \cdots & 0 & \cdots & 1 & \cdots & e_{2(2n)}' \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ e_{k_{(p)}^{(1)}(1)}' & \cdots & 0 & \cdots & 0 & \cdots & e_{k_{(p)}^{(1)}(2n)}' \end{pmatrix}.$$ 

All row vectors after the third row in $M$ is linearly independent, so when shortening the $i$th and the $(n+i)$th columns of $C_{(p)}^{(1)}$, a new linear code has $(\dim C_{(p)}^{(1)} - 2)$ row vectors as its basis.

**Lemma 1.4.** For a linear code $C \subset F_q^{2n}$ with the minimum symplectic weight $w_s \geq 2$, let $C_{(p)} \subset F_q^{2(n-1)}$ be a linear
code made from $C$ by puncturing $C$ once, and $C^\perp_{(s)} \subseteq F_q^{2(n-1)}$ be a linear code made from $C^\perp_{(s)}$ by shortening $C^\perp_{(s)}$ once, $C^\perp_{(s)}$ is the symplectic dual code of $C_{(p)}$.

**Example 3.** According to Example 2 (A-$\perp_{(1)}$) is a symplectic dual code of $A^\perp$.

**Proof.** Let $C^\perp \subseteq F_q^{2n}$ be the symplectic dual code of a linear code $C \subseteq F_q^{2n}$. Let $H$ be the set of vectors in $C^\perp$ whose $i$th and $(n+i)$th components $(1 \leq i \leq n)$ of are $0$. Then, we have $C \subseteq H^\perp$. Also, let $G$ be the set of vectors in $C$ whose $i$th and $(n+i)$th components of are changed to $0$. Here, we have $G \subseteq H^\perp$ because the $i$th and the $(n+i)$th components of vectors in $C$ are multiplied by components of vectors in $H$ which are $0$.

On the other hand, we have $C_{(p)} \subseteq C^\perp_{(s)}$ because $C_{(p)}$ is the set of vectors in $G$ whose $i$th and $(n+i)$th components of are eliminated, and $C^\perp_{(s)}$ is the set of vectors in $H$ which the $i$th and the $(n+i)$th components of are eliminated.

From Corollary 1.1.1 and Lemma 1.3, we have

$$\dim C_{(p)} + \dim C^\perp_{(s)} = \dim C + (\dim C^\perp_{(s)} - 2) = 2(n-1).$$

From the above and $C_{(p)} \subseteq C^\perp_{(s)}$, $C^\perp_{(s)}$ is the symplectic dual code of $C_{(p)}$.

From Lemma 1.3 and 1.4, we have,

**Corollary 1.4.1.** For a linear code $C \subseteq F_q^{2n}$ with the minimum symplectic weight $w_s \geq 2$ and $\ell$ $(1 \leq \ell \leq w_s - 1)$, let $C^\perp_{(p)} \subseteq F_q^{2(n-\ell)}$ be a linear code made from $C$ by puncturing it $\ell$ times, and let $(C^\perp_{(s)})^\perp \subseteq F_q^{2(n-\ell)}$ be a linear code made from $C^\perp_{(s)}$ by shortening it $\ell$ times, $(C^\perp_{(s)})^\perp$ is the symplectic dual code of $C^\perp_{(p)}$. 

From the above lemmas, we can now prove Theorem 1.

**Proof of Theorem 1.** First, from Corollary 1.1.1 we have

$$\dim C^\perp_{(p)} = \dim C.$$

According to Section 4.1 of the reference [5], For a linear code $C \subseteq F_q^{2n}$ with $C \subseteq C^\perp$, $C^\perp_{(p)} \cap (C^\perp_{(p)})^\perp = C^\perp_{(s)}$ is satisfied. Therefore,

$$\dim C^\perp_{(p)} \cap (C^\perp_{(p)})^\perp = \dim C^\perp_{(s)} = \dim C - 2\ell.$$

So, from Corollary 1.4.1 we have

$$\ell = c = (\dim C^\perp_{(p)} - \dim C^\perp_{(p)} \cap (C^\perp_{(s)})^\perp)/2$$

is satisfied.

Second, we will prove

$$d \leq \min \{w(\tilde{x}) \mid \tilde{x} \in (C^\perp_{(s)})^\perp \setminus \tilde{0}\}.$$  

In other words, we will prove that the minimum symplectic weight of $(C^\perp_{(s)})^\perp \setminus C^\perp_{(p)}$ does not decrease from $d$.

The minimum symplectic weight of $(C^\perp_{(s)})^\perp \setminus C^\perp_{(p)}$ does not decrease from $d$ because $(C^\perp_{(s)})^\perp$ is created by shortening the subset of $(C^\perp_{(s)})$ and shortening is to eliminate components of vectors in $(C^\perp_{(s)})$ which are zero. 

4 Concluding remarks

In this letter, we consider an extra assumption to [8, Theorem 7] that an original QECC is stabilizer-based. Our extra assumption makes resulting EAQECC being also stabilizer-based, which facilitates efficient encoding and decoding. It is unclear to the authors if stabilizer-based EAQECCs can be constructed from a class of QECCs wider than the stabilizer codes.

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