A CLASSIFICATION OF THE AUTOMORPHISM GROUPS OF POLARIZED
ABELIAN THREEFOLDS OVER FINITE FIELDS

WONTAE HWANG, BO-HAE IM AND HANSOL KIM

Abstract. We give a classification of maximal elements of the set of finite groups that can be
realized as the automorphism groups of polarized abelian threefolds over finite fields.

1. Introduction

Let $k$ be a field, and let $X$ be an abelian threefold over $k$. We denote the endomorphism ring of
$X$ over $k$ by $\text{End}_k(X)$. It is a free $\mathbb{Z}$-module of rank $\leq 36$. We also let $\text{End}_k^0(X) = \text{End}_k(X) \otimes \mathbb{Q}$. This $\mathbb{Q}$-algebra $\text{End}_k^0(X)$ is called the endomorphism algebra of $X$ over $k$. Then $\text{End}_k^0(X)$ is a
finite dimensional semisimple algebra over $\mathbb{Q}$ with $6 \leq \dim_\mathbb{Q} \text{End}_k^0(X) \leq 36$. Moreover, if $X$ is $k$-simple, then $\text{End}_k^0(X)$ is a division algebra over $\mathbb{Q}$. Now, it is a well-known fact that $\text{End}_k(X)$ is
a $\mathbb{Z}$-order in $\text{End}_k^0(X)$, and the group $\text{Aut}_k(X)$ of the automorphisms of $X$ over $k$ is not finite, in
general. But if we fix a polarization $\mathcal{L}$ on $X$, then the group $\text{Aut}_k(X, \mathcal{L})$ of the automorphisms
of the polarized abelian threefold $(X, \mathcal{L})$ over $k$ is always finite. In this regard, it might be
interesting to consider the following problems.

Problem 1.1. (1) Classify all (finite) groups $G$ (up to isomorphism) such that there exist a field
$k$ and an abelian variety $X$ of dimension 3 over $k$ with $G \leq \text{Aut}_k(X)$.
(2) Classify all finite groups $G$ (up to isomorphism) such that there exist a field $k$ and an abelian
variety $X$ of dimension 3 over $k$ with $G = \text{Aut}_k(X, \mathcal{L})$ for some polarization $\mathcal{L}$ on $X$.

We note that the answer for Problem 1.1 (2) gives a (partial) answer for Problem 1.1 (1). For
the case when $k = \mathbb{C}$, Birkenhake, González and Lange [3] computed all finite automorphism
groups of complex tori of dimension 3, which are maximal in the isogeny class. The goal of
this paper is to give an almost complete answer for Problem 1.1 (2) for the case when $k$ is a
finite field, by classifying all finite groups $G$ that can be realized as the automorphism group
$\text{Aut}_k(X, \mathcal{L})$ of a polarized abelian threefold $(X, \mathcal{L})$ over a finite field $k$, which are maximal in
the following sense: there is no finite group $H$ such that $G$ is isomorphic to a proper subgroup of
$H$ and $H = \text{Aut}_k(Y, \mathcal{M})$ for some abelian threefold $Y$ over $k$ that is $k$-isogenous to $X$ with a
polarization $\mathcal{M}$.

Along this line, the first author gave a classification of maximal automorphism groups for
arbitrary polarized abelian surfaces over finite fields in [8], and provided a complete list of finite
groups that can be realized as the automorphism groups of simple polarized abelian varieties of
odd prime dimension over finite fields in [9]. In this paper, as the next step toward the goal of
completing the three dimensional case, we will give such a classification for arbitrary polarized
abelian threefolds over finite fields. Two main difficulties in achieving the goal are as follows: 1)
there are way more cases to be considered according to the possible decomposition of abelian

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threefolds (compared to those of [8]), and 2) we need to deal with finite subgroups of rings of matrices of larger dimension over real quadratic fields and over totally definite quaternion algebras over \( \mathbb{Q} \) (compared to those of [8, 9]).

Now, our main result is summarized in the following theorem.

**Theorem 1.2.** The possibilities for maximal automorphism groups of a polarized abelian threefold \((X, \mathcal{L})\) over a finite field are given by the lists in Theorems 6.2, 6.3, 6.5, 6.7, 6.9, and 6.10 in the case of \( X \) being simple, a product of a simple abelian surface and an elliptic curve, a product of three pairwise non-isogenous elliptic curves, a product of a power of an elliptic curve and an elliptic curve that are non-isogenous, a power of an ordinary elliptic curve, and a power of a supersingular elliptic curve, respectively.

To obtain such a classification, we need to combine various mathematical areas, including the theory of abelian varieties over finite fields, finite subgroups of division rings, and quaternionic representation theory. For more detailed statements and proofs, see Theorems 6.2, 6.3, 6.5, 6.7, 6.9, and 6.10 below. This kind of classification result might not only be interesting in its own sake, but also, have some applications to other areas of mathematics, such as group theory computing the exact values of Jordan constants of the automorphism groups of certain abelian varieties over fields of positive characteristic (see [10]), and (potentially) cryptography applying the classification to discuss the rationality of geometrically rational generalized Kummer surfaces over a finite field (see [11]).

This paper is organized as follows: In Section 2 we introduce several facts which are related to our desired classification. Explicitly, we will recall some facts about endomorphism algebras of abelian varieties (§2.1), the theorem of Tate (§2.2), Honda-Tate theory (§2.3), and maximal orders over a Dedekind domain (§2.4). In Section 3 we give a classification of all finite subgroups of the multiplicative subgroup of certain division algebras over \( \mathbb{Q} \). In Section 4 we describe all the maximal finite subgroups of \( GL_3(K) \) for the case when either \( K = \mathbb{Q} \) or \( K \) is a real quadratic field. In Section 5 we record some useful results on quaternionic matrix groups that are related to our classification, following a paper of Nebe [15]. In Section 6 we finally obtain the desired classification using the facts that were introduced in the previous sections.

In the sequel, let \( q = p^a \) for some prime number \( p \) and an integer \( a \geq 1 \), unless otherwise stated. Also, for an integer \( n \geq 1 \), let \( \varphi(n) \) \((C_n, D_n, \text{respectively})\) denote the number of integers that are smaller than or equal to \( n \) and relatively prime to \( n \) (a cyclic group of order \( n \), a dihedral group of order \( 2n \), respectively). Finally, for a matrix \( A \in GL_2(\mathbb{C}) \), \( \hat{A} \) denotes the block matrix

\[
\begin{pmatrix}
A & 0 \\
0 & \det(A)^{-1}
\end{pmatrix} \in GL_3(\mathbb{C}),
\]

and for an \( \alpha \in \text{Sym}_3 \), \( I_{3, \alpha} \in GL_3(\mathbb{C}) \) denotes the permutation matrix corresponding to \( \alpha \).

2. Preliminaries

In this section, we briefly recall some of the facts in the general theory of abelian varieties over a field and maximal orders over a Dedekind domain. Our main references are [5], [14], and [17].

2.1. Endomorphism algebras of abelian varieties. In this section, we give some basic facts about the endomorphism algebra of an abelian variety over a field. To this aim, let \( X \) be an abelian variety over a field \( k \). Then the set \( \text{End}_k(X) \) of endomorphisms of \( X \) over \( k \) is actually
a ring. Because it is more difficult to deal with \( \text{End}_k(X) \) itself, in general, if we want to work with \( \mathbb{Q} \)-algebras instead, then we define \( \text{End}_k^0(X) := \text{End}_k(X) \otimes_{\mathbb{Z}} \mathbb{Q} \). The \( \mathbb{Q} \)-algebra \( \text{End}_k^0(X) \) is called the endomorphism algebra of \( X \) over \( k \).

If \( X \) is a simple abelian variety over \( k \), then \( \text{End}_k^0(X) \) is a division algebra over \( \mathbb{Q} \). If \( X \) is an arbitrary abelian variety over \( k \), then it is well known that there exist simple abelian varieties \( Y_1, \ldots, Y_n \) over \( k \), no two of which are \( k \)-isogenous, and positive integers \( m_1, \ldots, m_n \) such that \( X \) is \( k \)-isogenous to \( Y_1^{m_1} \times \cdots \times Y_n^{m_n} \). In this situation, we have

\[
\text{End}_k^0(X) \cong M_{m_1}(D_1) \times \cdots \times M_{m_n}(D_n)
\]

where \( D_i := \text{End}_k^0(Y_i) \) for each \( i \). Note that each \( D_i \) is a division algebra over \( \mathbb{Q} \). Moreover, \( \text{End}_k^0(X) \) is a finite dimensional semisimple \( \mathbb{Q} \)-algebra of dimension at most \( 4 \cdot (\dim X)^2 \) (see [5 Corollary 12.11]).

2.2. The theorem of Tate. In this section, we recall a theorem of Tate and a description on the structure of the endomorphism algebra of simple abelian varieties over finite fields, which play an important role throughout the paper.

We first recall that an abelian variety \( X \) over a field \( k \) is called elementary if \( X \) is \( k \)-isogenous to a power of a simple abelian variety over \( k \). Then Tate [20] obtained the following result for the case when the base field \( k \) is finite.

**Theorem 2.1** ([20 Theorem 2]). Let \( X \) be an abelian variety of dimension \( g \) over a finite field \( k = \mathbb{F}_q \), and let \( \pi_X \) be the Frobenius endomorphism of \( X \). Then we have:

(a) The center of \( \text{End}_k^0(X) \) is the subalgebra \( \mathbb{Q}[[\pi_X]] \). In particular, \( X \) is elementary if and only if \( \mathbb{Q}[[\pi_X]] = \mathbb{Q}(\pi_X) \) is a field, and this occurs if and only if \( f_X \) is a power of an irreducible polynomial in \( \mathbb{Q}[t] \) where \( f_X \) denotes the characteristic polynomial of \( \pi_X \).

(b) Suppose that \( X \) is elementary. Let \( h = f_X^q \) be the minimal polynomial of \( \pi_X \) over \( \mathbb{Q} \). Further, let \( d = [\text{End}_k^0(X) : \mathbb{Q}(\pi_X)]^{\frac{1}{2}} \) and \( e = [\mathbb{Q}(\pi_X) : \mathbb{Q}] \). Then \( dh = 2g \) and \( f_X = h^d \).

(c) We have \( 2g \leq \dim_{\mathbb{Q}} \text{End}_k^0(X) \leq (2g)^2 \) and \( X \) is of CM-type.

(d) The following conditions are equivalent:

- \( \dim_{\mathbb{Q}} \text{End}_k^0(X) = 2g \);
- \( \text{End}_k^0(X) = \mathbb{Q}[[\pi_X]] \);
- \( \text{End}_k^0(X) \) is commutative;
- \( f_X \) has no multiple root.

(e) The following conditions are equivalent:

- \( \dim_{\mathbb{Q}} \text{End}_k^0(X) = (2g)^2 \);
- \( \mathbb{Q}[[\pi_X]] = \mathbb{Q} \);
- \( f_X \) is a power of a linear polynomial;
- \( \text{End}_k^0(X) \cong M_p(D_{p,\infty}) \) where \( D_{p,\infty} \) is the unique quaternion algebra over \( \mathbb{Q} \) that is ramified at \( p \) and \( \infty \), and split at all other primes;
- \( X \) is isogenous to \( E^g \) for a supersingular elliptic curve \( E \) over \( k \) all of whose endomorphisms are defined over \( k \).

Now, in view of Theorem 2.1 (a), if \( X \) is an elementary abelian variety over a finite field \( k \), then \( \text{End}_k^0(X) \) is a simple algebra over its center \( \mathbb{Q}[[\pi_X]] \). For a more precise description on \( \text{End}_k^0(X) \), we record the following two results.
Proposition 2.2 ([5 Corollary 16.30]). Let \( X \) be an elementary abelian variety over a finite field \( k = \mathbb{F}_q \). Let \( K = \mathbb{Q}([\pi_X]). \) If \( \nu \) is a place of \( K \), then the local invariant of \( \text{End}_k^0(X) \) in the Brauer group \( Br(K) \) is given by

\[
\text{inv}_\nu(\text{End}_k^0(X)) = \begin{cases} 
0 & \text{if } \nu \text{ is a finite place not above } p; \\
\frac{\text{ord}_\nu([\pi_X])}{\text{ord}_\nu(\mathbb{Q})} \cdot [K_\nu : \mathbb{Q}_p] & \text{if } \nu \text{ is a place above } p; \\
\frac{1}{2} & \text{if } \nu \text{ is a real place of } K; \\
0 & \text{if } \nu \text{ is a complex place of } K.
\end{cases}
\]

Proposition 2.3 ([5 Corollary 16.32]). Let \( X \) be a simple abelian variety over a finite field \( k \). Let \( d \) be the index of the division algebra \( D := \text{End}_k^0(X) \) over its center \( \mathbb{Q}(\pi_X) \) (so that \( d = [D : \mathbb{Q}(\pi_X)]^{\frac{1}{2}} \) and \( f_X = (f_{\pi_X}^X)^d \)). Then \( d \) is the least common denominator of the local invariants \( i_\nu = \text{inv}_\nu(D) \).

2.3. Abelian varieties up to isogeny and Weil numbers: Honda-Tate theory. In this section, we recall an important theorem of Honda and Tate with the following definition.

Definition 2.4. (a) A q-Weil number is an algebraic integer \( \pi \) such that \( |\iota(\pi)| = \sqrt{q} \) for all embeddings \( \iota: \mathbb{Q}[\pi] \to \mathbb{C} \).
(b) Two q-Weil numbers \( \pi \) and \( \pi' \) are said to be conjugate if they have the same minimal polynomial over \( \mathbb{Q} \), or equivalently, there is an isomorphism \( \mathbb{Q}[\pi] \to \mathbb{Q}[\pi'] \) sending \( \pi \) to \( \pi' \).

Theorem 2.5 ([7 Main Theorem] or [5 §16.5]). For every q-Weil number \( \pi \), there exists a simple abelian variety \( X \) over \( \mathbb{F}_q \) such that \( \pi_X \) is conjugate to \( \pi \), where \( \pi_X \) denotes the Frobenius endomorphism of \( X \). Moreover, we have a bijection between the set of isogeny classes of simple abelian varieties over \( \mathbb{F}_q \) and the set of conjugacy classes of q-Weil numbers given by \( X \mapsto \pi_X \).

The inverse of the map \( X \mapsto \pi_X \) associates to a q-Weil number \( \pi \) a simple abelian variety \( X \) such that \( f_X \) is a power of the minimal polynomial \( f_{\pi_X}^X \) of \( \pi \) over \( \mathbb{Q} \).

2.4. Maximal orders over a Dedekind domain. In this section, we review the general theory of maximal orders over a Dedekind domain that will be used later in this paper.

Throughout this section, let \( R \) be a noetherian integral domain with the quotient field \( K \), and let \( A \) be a finite dimensional \( K \)-algebra. Recall that a maximal \( R \)-order in \( A \) is an \( R \)-order which is not properly contained in any other \( R \)-order in \( A \). For our later use, we introduce several results about maximal orders.

Theorem 2.6 ([17 Theorem 8.7]). Let \( A \) be a finite dimensional \( K \)-algebra. If \( \Lambda \) is a maximal \( R \)-order in \( A \), then for each \( n \geq 1 \), \( M_n(\Lambda) \) is a maximal \( R \)-order in \( M_n(A) \). If \( R \) is integrally closed, then \( M_n(R) \) is a maximal \( R \)-order in \( M_n(K) \).

If we impose additional conditions that \( R \) is integrally closed and \( A \) is a separable \( K \)-algebra, then we have the following result saying that a decomposition of \( A \) into simple components yields a decomposition of maximal orders in \( A \).

Theorem 2.7 ([17 Theorem 10.5]). Let \( A \) be a separable \( K \)-algebra with simple components \( \{A_i\}_{1 \leq i \leq t} \) and let \( R_i \) be the integral closure of \( R \) in the center \( K_i \) of \( A_i \) for each \( i \). Then we have:

(a) For each maximal \( R \)-order \( \Lambda \) in \( A \), we have \( \Lambda = \bigoplus_{i=1}^t \Lambda e_i \) where \( \{e_i\}_{1 \leq i \leq t} \) are the central idempotents of \( A \) such that \( A_i = \Lambda e_i \) for each \( i \). Moreover, each \( \Lambda e_i \) is a maximal \( R \)-order in
\[ A_i = A e_i. \]

(b) If \( \Lambda_i \) is a maximal \( R \)-order in \( A_i \) for each \( i \), then \( \bigoplus_{i=1}^{t} \Lambda_i \) is a maximal \( R \)-order in \( A \).

(c) An \( R \)-order \( \Lambda_i \) in \( A_i \) is a maximal \( R \)-order if and only if \( \Lambda_i \) is a maximal \( R_i \)-order in \( A_i \).

(Here, the symbol \( \bigoplus \) denotes the (external) direct sum.)

Finally, we further assume that \( R \) is a Dedekind domain with its quotient field \( K \neq R \). Let \( A \) be a separable \( K \)-algebra (which is simple). In the following last theorem, we determine all maximal \( R \)-orders in \( A \).

**Theorem 2.8** ([17 Theorem 21.6]). Let \( A = \text{Hom}_D(V, V) \cong M_r(D) \) be a simple algebra, where \( V \) is a right vector space of dimension \( r \) over a division algebra \( D \) with center \( K \). Let \( \Delta \) be a fixed maximal \( R \)-order in \( D \), and let \( M \) be a full right \( \Delta \)-lattice in \( V \). Then \( \text{Hom}_{\Delta}(M, M) \) is a maximal \( R \)-order in \( A \). If \( N' \) is a maximal \( R \)-order in \( A \), then there is a full right \( \Delta \)-lattice \( N \) in \( V \) such that \( N' = \text{Hom}_{\Delta}(N, N) \).

### 3. Finite Subgroups of Division Algebras

In this section, we give a classification of all possible finite groups that can be embedded in the multiplicative subgroup of a division algebra over \( \mathbb{Q} \) with certain properties that are related to our situation later in Section [5]. Our main reference is a paper of Amitsur [11]. We start with the following notion.

**Definition 3.1.** Let \( m, r \) be two relatively prime positive integers, and we put \( s := \gcd(r - 1, m) \) and \( t := \frac{m}{s} \). Also, let \( n \) be the smallest integer such that \( r^n \equiv 1 \pmod{m} \). We denote by \( G_{m,r} \) the group generated by two elements \( a, b \) satisfying the relations

\[ a^m = 1, \ b^n = a^1, \ bab^{-1} = a^r. \]

This type of groups includes the dicyclic group of order \( mn \), in which case, we often write \( \text{Dic}_{mn} \) for \( G_{m,r} \). As a convention, if \( r = 1 \), then we put \( n = s = 1 \), and hence, \( G_{m,1} \) is a cyclic group of order \( m \).

Given \( m, r, s, t, n \), as above, we will consider the following two conditions in the sequel:

- (C1) \( \gcd(n, t) = \gcd(s, t) = 1 \).
- (C2) \( n = 2n', m = 2^a m', s = 2s' \) where \( \alpha \geq 2 \), and \( n', m', s' \) are all odd integers. Moreover, \( \gcd(n, t) = \gcd(s, t) = 2 \) and \( r \equiv -1 \pmod{2^a} \).

Now, let \( p \) be a prime number that divides \( m \). We define:

- (i) \( \alpha_p \) is the largest integer such that \( p^{\alpha_p} \mid m \).
- (ii) \( n_p \) is the smallest integer satisfying \( r^{n_p} \equiv 1 \pmod{mp^{-\alpha_p}} \).
- (iii) \( \delta_p \) is the smallest integer satisfying \( p^{\delta_p} \equiv 1 \pmod{mp^{-\alpha_p}} \).

Then we have the following theorem that provides us with a useful criterion for a group \( G_{m,r} \) to be embedded in a division ring.

**Theorem 3.2** ([11 Theorem 3, Theorem 4, and Lemma 10]). A group \( G_{m,r} \) can be embedded in a division ring if and only if either (C1) or (C2) holds, and one of the following conditions holds:

1. \( n = s = 2 \) and \( r \equiv -1 \pmod{m} \).
2. For every prime number \( q \mid n \), there exists a prime number \( p \mid m \) such that \( q \nmid n_p \) and that either
(a) \( p \neq 2 \), and \( \gcd(q, (p^a - 1)/s) = 1 \), or

(b) \( p = q = 2 \), \((C2)\) holds, and \( m/4 \equiv \delta \equiv 1 \pmod{2} \).

Now, let \( G \) be a finite group. One of our main tools in this section is the following result.

**Theorem 3.3** ([1] Theorem 7). \( G \) can be embedded in a division ring if and only if \( G \) is of one of the following types:

1. Cyclic groups.
2. \( G_{m,r} \) where the integers \( m, r, \) etc., satisfy Theorem 3.2 (which is not cyclic).
3. \( \Sigma^* \times G_{m,r} \) where \( \Sigma^* \) is the binary tetrahedral group of order 24, and \( G_{m,r} \) is either cyclic of order \( m \) with \( \gcd(m, 6) = 1 \), or of type (2) with \( \gcd(|G_{m,r}|, 6) = 1 \) (where \( |G_{m,r}| \) denotes the order of the group \( G_{m,r} \)). In both cases, for all primes \( p \mid m \), the smallest integer \( \gamma \) satisfying \( 2^\gamma \equiv 1 \pmod{p} \) is odd.
4. \( \Sigma^* \), the binary octahedral group of order 48.
5. \( \Sigma^* \), the binary icosahedral group of order 120.

Having stated most of the necessary results, we can proceed to achieve the goal of this section. First, we give two important lemmas, both of whose proofs follow from Theorem 3.2 unless otherwise stated.

**Lemma 3.4.** Let \( m, r, s, t, n \) be as in Definition 3.1 satisfying (C1) with \( n = 2 \) and \( m \in \{2, 3, 4, 6, 7, 9, 14, 18\} \). Then the group \( G := G_{m,r} \) can be embedded in a division ring if and only if \( G \) is one of the following groups:

1. A cyclic group \( C_4 \).
2. \( Dic_{12} \), a dicyclic group of order 12;
3. \( Dic_{28} \), a dicyclic group of order 28;
4. \( Dic_{36} \), a dicyclic group of order 36.

**Proof.** Note that \( t \) is an odd integer since \( n = 2 \) and \( \gcd(n, t) = 1 \). Hence, we have the following three cases to consider:

(I) If \( m \in \{3, 7\} \), then \( |G| = 2m \) is square-free. Now, since \( n = 2 \) and \( m \) is an odd prime number, it is easy to see that \( r \equiv -1 \pmod{m} \) (by the definition of \( n \)), and then by looking at the presentation of \( G \), we can see that \( G \) is not cyclic. By [1] Corollary 5, \( G \) cannot be embedded in a division ring.

(II) Suppose that \( m = 9 \). Then since \( \gcd(s, t) = 1 \), we cannot have \( s = t = 3 \). Also, since \( n = 2 \), we cannot have \( s = 9, t = 1 \) by the definition of \( n \). Hence, the only possible case is that \( s = 1, t = 9 \). Since \( n = 2 \) and \( \gcd(r + 1, r - 1) \leq 2 \), we have \( r \equiv -1 \pmod{9} \) so that the order of \( \pi \) in the unit group \( U(\mathbb{Z}/9\mathbb{Z}) \) is exactly 2. By [12] Theorem 3.1, \( G \) cannot be embedded in a division ring.

(III) If \( m \in \{2, 4, 6, 14, 18\} \), then we have the following five subcases to consider:

(i) If \( m = 2 \), then since \( \gcd(n, t) = 1 \), we get \( s = 2, t = 1 \). In particular, we have \( n = s = 2 \) and \( r \equiv -1 \pmod{2} \). Hence, \( G \) can be \( G_{2,r} = C_4 \).

(ii) If \( m = 4 \), then since \( \gcd(n, t) = 1 \), we get \( s = 4, t = 1 \). By the definition of \( s \), we have \( m = 4 \mid r - 1 \), and hence, this case cannot occur because of our assumption that \( n = 2 \).

(iii) If \( m = 6 \), then since \( \gcd(n, t) = 1 \), either \( s = 6, t = 1 \) or \( s = 2, t = 3 \). If \( s = 6, t = 1 \), then by a similar argument as in (ii), this case cannot occur. Now, if \( s = 2, t = 3 \), then since \( n = 2 \), we

\[ \text{If we want } G_{m,r} \text{ to be non-cyclic as in Theorem 3.3 then we may exclude this case.} \]
have $r \equiv -1 \pmod{6}$. Hence, $G$ can be $G_{6,r} = \text{Dic}_{12}$.

(iv) Similarly, if $m = 14$, then we only need to consider the case when $s = 2, t = 7$. Since $n = 2$, we have $r \equiv -1 \pmod{14}$. Hence, $G$ can be $G_{14,r} = \text{Dic}_{28}$.

(v) If $m = 18$, then, since $n = 2$, we can see that $r \equiv -1 \pmod{18}$ by a direct computation. Then it follows that $s = \gcd(r - 1, 18) = 2$ and $t = 9$. As before, $G$ can be $G_{18,r} = \text{Dic}_{36}$.

This completes the proof.

Lemma 3.5. Let $m, r, s, t, n$ be as in Definition 3.4 satisfying (C2) with $n = 2$ and $m \in \{2, 3, 4, 6, 7, 9, 14, 18\}$. Then the group $G := G_{m,r}$ can be embedded in a division ring if and only if $G = Q_8$, a quaternion group of order 8.

Proof. Since $\alpha \geq 2$, we have that $4 \mid m$, and hence, we have only one case to consider, namely, the case when $m = 4$. In this case, we get that $n = s = 2$ and $r \equiv -1 \pmod{4}$, and hence, $G$ can be $G_{4,r} = Q_8$.

This completes the proof.

Now, let $D$ be a division algebra of degree 2 over its center $K$ (i.e. $[D : K] = 4$) where $K$ is an algebraic number field with $[K : \mathbb{Q}] = 3$. Then it follows that $\text{dim}_{\mathbb{Q}}D = 12$ so that the order of every element of finite order of $D^\times$ is at most 18. If the group $G := G_{m,r}$ is contained in $D^\times$, then $n \mid 2$ (see [11, §7]), and hence, we have either $n = 1$ or $n = 2$. Also, since $G$ contains an element of order $m$, we have $\varphi(m) \mid 6$. The last preliminary result that we need is the following theorem.

Theorem 3.6 ([11, Theorem 10]). Let $D$ be a division algebra of degree 2 over an algebraic number field $K$.

(a) If $D$ contains a binary octahedral group $\mathcal{O}^*$, then $\sqrt{2} \in K$.

(b) If $D$ contains a binary icosahedral group $I^*$, then $\sqrt{5} \in K$.

An immediate consequence of Theorem 3.6 is the following.

Corollary 3.7. Let $D$ be a division algebra of degree 2 over an algebraic number field $K$ with $[K : \mathbb{Q}] = 3$. Then $D$ does not contain $\mathcal{O}^*$ and $I^*$.

Summarizing, we have the following classification.

Theorem 3.8. Let $D$ be a division algebra of degree 2 over an algebraic number field $K$ with $[K : \mathbb{Q}] = 3$. A finite group $G$ (of even order\footnote{This assumption can be made based on the goal of this paper in the sense that the order of the automorphism group of a polarized abelian threefold over a finite field is even.}) can be embedded in $D^\times$ if and only if $G$ is one of the following groups:

(1) $C_2, C_4, C_6, C_{14}, C_{18}$;

(2) $Q_8, \text{Dic}_{12}, \text{Dic}_{28}, \text{Dic}_{36}$;

(3) $\mathcal{S}^*$, the binary tetrahedral group of order 24.

Proof. We refer the list of possible such groups to Theorem 3.3 and Corollary 3.4. Suppose that $G$ is cyclic. Then we can write $G = \langle f \rangle$ for some element $f$ of order $d$. Then according to the argument given before Theorem 3.6 we have $d \in \{2, 4, 6, 14, 18\}$. Hence, we obtain (1). If $G = G_{m,r}$ with $n = 2$ and $m \in \{2, 3, 4, 6, 7, 9, 14, 18\}$, then (2) follows from Lemmas 3.4 and 3.5 (If $n = 1$, then $s = m$ so that $t = 1$. Now, the presentation of the group tells us that the group is cyclic of order $m$ in this case.) Now, if $G = \mathcal{S}^* \times G_{m,r}$ is a general $T$-group, then the only
possible such groups are $\mathfrak{T}^*$ and $\mathfrak{T}^\ast \times C_7$. (If $n = 2$ so that $G_{m,r}$ is not cyclic, then $|G_{m,r}| = 2m$, and hence, we have $\gcd(|G_{m,r}|, 6) \neq 1$.) Also, clearly, both $\mathfrak{T}^*$ and $\mathfrak{T}^\ast \times C_7$ satisfy the condition of Theorem 3.3. On the other hand, if $G = \mathfrak{T}^\ast \times C_7$, then, in view of [2] Table 5, we must have that $K = \mathbb{Q}(\zeta_7)$, which contradicts the fact that $[K : \mathbb{Q}] = 3$.

Conversely, in view of Theorem 5.11 below, the two groups $\mathfrak{T}^*$ and $\text{Dic}_{12}$ are finite subgroups of $D^\times$ where $D = D_{3,\infty}$ and $D = D_{3,\infty}$, respectively. Let $K$ be a totally real cubic field. Then we can see that $\mathfrak{T}^*$ (resp. $\text{Dic}_{12}$) is a finite subgroup of $D^\times$ where $D = D_{2,\infty} \otimes \mathbb{Q} K$ (resp. $D = D_{3,\infty} \otimes \mathbb{Q} K$), both of which are quaternion division algebras over $K$. Also, by [13], Theorem 6.1 (c)], the two groups $\text{Dic}_{28}$ and $\text{Dic}_{36}$ are finite subgroups of $D^\times$ where $D = D_{\zeta_{18} + \zeta_{18}^{-1}, \infty, 3}$ respectively, where $D_{\zeta_m + \zeta_m^{-1}, \infty, p}$ denotes the quaternion division algebra over $\mathbb{Q}(\zeta_m + \zeta_m^{-1})$ that is ramified at the primes of $\mathbb{Q}(\zeta_m + \zeta_m^{-1})$ lying above $p$ and $\infty$. Then since $C_2 \leq C_4 \leq Q_8 \leq \mathfrak{T}^*$, $C_6 \leq \text{Dic}_{12}$, $C_{14} \leq \text{Dic}_{28}$, and $C_{18} \leq \text{Dic}_{36}$, the desired result follows.

This completes the proof. \(\square\)

Remark 3.9. Let $D$ be as in Theorem 3.3. It turns out that the group $Q_8$ cannot be a maximal (up to isomorphism) finite subgroup of $D^\times$. Indeed, if $G = Q_8$ is a finite subgroup of $D^\times$, then by [1] Theorem 9, we know that $D = D_{2,\infty} \otimes \mathbb{Q} K$ for some number field $K$ with $[K : \mathbb{Q}] = 3$. Then it follows that $\mathfrak{T}^* \leq D^\times$, and since $Q_8 \leq \mathfrak{T}^*$, we can see that $Q_8$ cannot be a maximal finite subgroup of $D^\times$.

4. MAXIMAL FINITE SUBGROUPS OF $GL_r(\mathbb{C})$ WHEN $K$ IS $\mathbb{Q}$ OR A REAL QUADRATIC FIELD

Throughout this section, either $K = \mathbb{Q}$ or $K$ is a real quadratic number field. Let $d > 0$ be a square-free integer such that $K = \mathbb{Q}(\sqrt{d})$. (If $K = \mathbb{Q}$, then we take $d = 1$.) Note that $\mu_K = \{\pm 1\}$, where $\mu_K$ denotes the set of all roots of unity in $K$.

We begin with the following lemma, in which, we deal with finite subgroups of $GL_r(\mathbb{C})$ consisting of diagonal matrices for some integer $r \geq 1$.

Lemma 4.1. Let $L/F$ be a finite Galois extension with $\Gamma = \text{Gal}(L/F)$, and let $H \subset (L^*)^\Gamma \subset GL_r(L)$ be a finite subgroup of diagonal matrices such that $\text{Tr}(A) \in F$ for all $A \in H$. Then there exists a group action $*$ of $\Gamma$ on the set $\{1, 2, \cdots, r\}$ such that $\sigma x_i = x_{\sigma i}$ for all $\sigma \in \Gamma$ and $A = \Delta_{(x_1, \cdots, x_r)} \in H$ (where $\Delta_{(x_1, \cdots, x_r)}$ denotes the diagonal matrix with entries $x_1, \cdots, x_r$, here and in the sequel).

Proof. The inclusion map $\rho : H \hookrightarrow GL_r(\mathbb{C})$ is a representation of $H$ whose associated character $\chi$ has values in $F$ by assumption. Since $H$ is abelian, $\rho$ is realizable over $F$ i.e. there is a matrix $P = (u_1 u_2 \cdots u_r) \in GL_r(L)$ such that $PHP^{-1} \subseteq GL_r(F)$. Hence $L^r$ decomposes uniquely into a direct sum $\bigoplus_{i \in I} W_i$ of maximal eigenspaces $W_i$ over $L$ of $PHP^{-1}$ for some index set $I$. In other words, each $W_i$ is a maximal subspace of $L^r$ such that $S|_{W_i}$ is a scalar multiplication for any $S \in PHP^{-1}$.

Now, we define a group action $\circ$ of $\Gamma$ on the index set $I$ as follows: for any $\sigma \in \Gamma$ and $i \in I$, note that $\sigma W_i$ is a maximal eigenspace of $L^r$, and hence, there exists a unique $j \in I$ such that $\sigma W_i = W_j$. We let $\sigma \circ i = j$. Then it is easy to see that $\circ$ is a group action of $\Gamma$ on $I$. Moreover, we can choose bases $B_i$ of $W_i$ for each $i \in I$ in such a way that $\sigma B_j = B_{\sigma i}$ for any $\sigma \in \Gamma$ and $j \in I$. Then $B = \bigsqcup_{i \in I} B_i$ is a basis of $L^r$ on which $\Gamma$ acts. Let $Q = (u_1 v_2 \cdots u_r) \in GL_r(L)$ be a square matrix consisting of column vectors in $B$ such that $u_i$ and $v_i$ are in the same eigenspace for each $1 \leq i \leq r$. Then we have $QAQ^{-1} = PAP^{-1}$ for all $A \in H$. 


The action of $\Gamma$ on $B$ induces an action $*$ of $\Gamma$ on the set $\{1, 2, \cdots, r\}$ such that $\sigma v_i = v_{\sigma v_i}$. Now, for any $A = \Delta_{(x_1, \cdots, x_r)} \in H$, let $S = QAQ^{-1} = PAP^{-1} \in GL_r(F)$. Then it follows that
\[ x_{\sigma v_i}v_{\sigma v_i} = Sv_{\sigma v_i} = \sigma(Sv_i) = \sigma(x_i v_i) = \sigma x_i v_{\sigma v_i}, \]
whence, $\sigma x_i = x_{\sigma v_i}$ for any $\sigma \in \Gamma$ and $i \in \{1, 2, \cdots, r\}$.

This completes the proof. \qed

For our own purpose of this paper, we focus on the case when $r = 3$.

**Lemma 4.2.** Let $H$ be a finite abelian subgroup of $GL_3(K)$ of exponent $N > 1$. Then we have:
(a) There is a finite subgroup $G$ of $GL_3(K)$ such that $G \cong H \times C_2$. Moreover, if $N > 2$, then there is a finite subgroup $G$ of $GL_3(K)$ that is not isomorphic to $H \times C_2$ (i.e. the conjugation action of such a $G$ on $H$ is not trivial).
(b) If there is a finite subgroup $G$ of $GL_3(K)$ such that $G \cong H \times C_3$, then $H$ is similar to $b = \langle \Delta_{(1,1,1)}, \Delta_{(1,1,1)} \rangle$, $\langle b, -I_3 \rangle$, or a subgroup of $a = \langle -I_3, \Delta_{(1,1,1)} \rangle$. Moreover, $H$ is similar to $b$ or $\langle b, -I_3 \rangle$ if and only if $G \not\cong H \times C_3$, and $H$ is similar to a subgroup of $a$ if and only if $G \cong H \times C_3$.

**Proof.** We first observe that $H$ is similar to a subgroup of $GL_3(K)$ that consists of diagonal matrices. Then some $P \in GL_3(K(\zeta_N))$ diagonalizes $H$ i.e. $H' := P^{-1}HP \subseteq (L^\times)^3$ where $L = K(\zeta_N)$. We find all possible $H'$ for each $N$.

If $N > 2$, then there exists an $A \in H'$ of order $N$, and we may write $A = \Delta(\zeta_N^a, \zeta_N^b, \zeta_N^c)$ for some $a, b, c \in \mathbb{Z}$. Since $|A| = N$, we get that $\gcd(a, b, c, N) = 1$ and $K(\zeta_N^a, \zeta_N^b, \zeta_N^c) = K(\zeta_N)$.

Since $[K : Q] = 2$, $K \cap Q(\zeta_N)$ equals either $K$ or $Q$. If $K \cap Q(\zeta_N) = K$, then $K \subseteq Q(\zeta_N)$ and $3 \ge [K(\zeta_N^a, \zeta_N^b, \zeta_N^c) : K] = [K(\zeta_N) : K] = [Q(\zeta_N) : Q].$

Again, since $[K : Q] = 2$, it follows that $\varphi(N) = [Q(\zeta_N) : Q] \in \{2, 4, 6\}$, and hence, we obtain $N \in \{3, 5, 8, 10, 12\} \subseteq \{3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 18\}$ because the unique quadratic number field in $Q(\zeta_N)$ is not real for $N \in \{3, 4, 6, 7, 9, 14, 18\}$. If $K \cap Q(\zeta_N) = Q$, then we have

\[ 3 \ge [K(\zeta_N^a, \zeta_N^b, \zeta_N^c) : K] = [K(\zeta_N^a, \zeta_N^b, \zeta_N^c) : Q] = [Q(\zeta_N) : Q] = \varphi(N) \]

so that $N \in \{3, 4, 6\}$. To sum up, $\text{Gal}(K(\zeta_N)/K)$ is of order 2 and the non-trivial element of $\text{Gal}(K(\zeta_N)/K)$ is the complex conjugation for any $K$. Because the diagonal entries are the solutions of the characteristic polynomial of $A$, which is defined over $K$, the group $\text{Gal}(K(\zeta_N)/K)$ permutes the diagonal entries of $A$. Since $|\text{Gal}(K(\zeta_N)/K)| = 2$, $\text{Gal}(K(\zeta_N)/K)$ fixes a diagonal entry of $A$. We may assume that the last diagonal entry is fixed by $\text{Gal}(K(\zeta_N)/K)$. Since $|A| > 2$, other diagonal entries of $A$ are not fixed. Thus we can say that $A = \Delta(\zeta_N^a, \zeta_N^b, \zeta_N^c)$ for some $\epsilon \in \mu_2 = \{\pm 1\}$. Also, by Lemma 4.1 any $B \in H'$ is of the form $\Delta_{(\zeta_N^a, \zeta_N^b, \zeta_N^c)}$ for some $b \in \mathbb{Z}$ and $w \in \{\pm 1\}$. Consequently, we see that $H' \leq \langle \Delta_{(\zeta_N^a, \zeta_N^b, \zeta_N^c)} \rangle.$

For $N = 2$, let $r = \dim_{\mathbb{F}_2} H' > 0$. If $r = 1$, then $H' = \langle \Delta_{(a,b,c)} \rangle$ for some $a, b, c \in \{\pm 1\}$. At least two of $a, b, c$ are the same. We may assume that $a = b$, and then $H'$ is a subgroup of $a$. If $r = 2$, then we consider the group homomorphism $\text{det}: H' \to \{\pm 1\}$. Considering the orders of the domain and codomain of $\text{det}$, we get that $|\text{ker}(\text{det})| \geq 2$. For a non-identity matrix $A \in \text{ker}(\text{det})$, we rearrange the diagonal entries of $A$ and may assume that $A = \Delta_{(1,1,1)}.$
For $B \in H' \setminus \langle A \rangle$, we may assume that $B = \Delta_{(1,a,b)} \neq I_3$ for some $a, b \in \{\pm 1\}$ by replacing $B$ by $AB$ if the first diagonal entry of $B$ equals $-1$. Hence we can see that $H'$ is equal to $\langle \Delta_{(-1,-1,1)}, \Delta_{(1,-1,1)} \rangle$ or $a$ or $b$. If $r = 3$, then $H' = \langle \Delta_{(-1,-1,1)}, \Delta_{(1,-1,1), \Delta_{(1,1,-1)}} \rangle = \langle b, -I_3 \rangle$.

(a) It is easy to show that the matrix $\Delta_{(1,1,-1)}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ acts on any $H'$ and we have $G' = \langle H', \Delta_{(1,1,-1)}' \rangle \cong H' \times C_2$. If $N > 2$, the conjugation action of $\Delta_{(1,1,-1)}'$ on $H'$ is not trivial.

(b) Assume that there is a finite group $G \leq GL_3(K)$ such that $G \cong H \times C_3$. Then for some $X \in G' := P^{-1}GP$, one has that $|X| = 3$ and $X$ acts on $H'$ by conjugation. As before, $X$ permutes the diagonal entries of matrices of $H'$. There is a homomorphism $f : \langle X \rangle \to Sym_3$ such that $XAX^{-1} = I_{3,a}A_{3,0}^{-1}$ for any $A \in H'$ with $\alpha = f(X)$. Note that $\alpha$ is the identity $1 \in Sym_3$ or a 3-cycle. We claim that $N = 2$. Indeed, if $N > 2$, then the form of the matrices of $H'$ says that $\alpha = 1$. Thus $G \cong G' := H' \times \langle X \rangle$ is abelian of exponent $\text{lcm} \ (N, 3)$. Since both $N$ and $\text{lcm} \ (N, 3)$ are contained in the set $\{3, 4, 6\}$, we know that $N = \text{lcm} \ (N, 3)$ is equal to 3 or 6. Then since there is no abelian subgroup of $GL_3(K)$ of exponent $N$ and order $18 = |G| = |G'|$, we conclude that $N = 2$, and hence, the first assertion follows.

If $G \cong H \times C_3$, then $G$ is an abelian group of exponent 6. Thus $G' := P^{-1}GP$ is a subgroup of $\langle \Delta_{(6,6,1)}, \Delta_{(1,1,-1)} \rangle$ and the 2-torsion part $H' = P^{-1}HP$ of $G'$ is a subgroup of $\langle a \rangle$. Conversely, if $H'$ is a subgroup of $\langle a \rangle$, then the conjugation action by any matrix $X$ of order 3 on $H'$ is trivial so that $G \cong H \times C_3$. Also, note that the form of the matrices of $H'$ says that $\alpha = 1$.

Finally, it suffices to show that $H$ is not similar to $\langle \Delta_{(-1,-1,1)}, \Delta_{(1,-1,1)} \rangle$ to prove that $H$ is similar to $b$ or $\langle b, -I_3 \rangle$ if and only if $G \ncong H \times C_3$. Indeed, if $H' = \langle \Delta_{(-1,-1,1)}, \Delta_{(1,-1,1)} \rangle$, then $\alpha = 1$ and $G'$ is an abelian group of exponent 6. However, $\langle \Delta_{(-1,-1,1)}, \Delta_{(1,-1,1)} \rangle$ is not a subgroup of $\langle a \rangle$, which contradicts the above argument.

This completes the proof. □

Lemma 4.3. Let $F$ be a number field and $n$ the order of the torsion part of $F^\times$ (i.e. we have $\mu_n = \mu_F$), and let $r > 0$ be an integer that is relatively prime to $n$. Then we have:

(a) If $G$ is a finite subgroup of $GL_r(F)$ containing $\mu_n \cdot I_r$, then $G = \langle G_0, \mu_n \cdot I_r \rangle \cong G_0 \times C_n$ where $G_0 = G \cap SL_r(F)$.

(b) Assume that a finite subgroup $H$ of $GL_r(F)$ contains $\mu_n \cdot I_r$. Then a group $G$ is a subgroup of $H$ if and only if $G_0$ is a subgroup of $H_0$, where $G_0 = G \cap SL_r(F)$ and $H_0 = H \cap SL_r(F)$.

Proof. Note first that for any $z \in \mu_n$, there is an $x \in \mu_n$ such that $\det (x \cdot I_r) = z$ because $\gcd (r, n) = 1$.

(a) Clearly, we have $\langle G_0, \mu_n \cdot I_r \rangle \subseteq G$ by our assumption. To prove the reverse inclusion, let $A \in G$. Then we have $\det (A) \in \mu_n$, and hence, there is an $x \in \mu_n$ such that $x^{-1}A \in G_0$ by the above observation. Thus we can see that $A = (x \cdot I_r) \cdot (x^{-1}A) \in \langle G_0, \mu_n \cdot I_r \rangle$, and hence, we have $A \subseteq \langle G_0, \mu_n \cdot I_r \rangle$. Now, since $\gcd (r, n) = 1$, we have $G_0 \cap (\mu_n \cdot I_r) = \{I_r\}$. Moreover, any element of $\mu_n \cdot I_r$ commutes with any element of $G_0$. Therefore, we can conclude that $G \cong G_0 \times C_n$.

(b) It is clear that if $G \leq H$, then $G_0 \leq H_0$ by definition. To prove the converse, we note that $G \leq \langle G_0, \mu_n \cdot I_r \rangle$ and $H = \langle H_0, \mu_n \cdot I_r \rangle$ by part (a). Since $G_0 \leq H_0$, it follows that $G \leq H$.

This completes the proof. □

As an immediate consequence of the above lemma, we get the following corollary.
Corollary 4.4. Let $F$, $n$, $r$, and $G$ be as in Lemma 4.3. If $G$ is a maximal finite subgroup of $GL_r(F)$, then $G = \langle G_0, \mu_n \cdot I_r \rangle \cong G_0 \times C_n$ where $G_0$ is a maximal finite subgroup of $SL_r(F)$.

Remark 4.5. In light of Corollary 4.4, we can reduce the problem of finding all maximal finite subgroups of $GL_3(K)$ to that of finding all maximal finite subgroups of $SL_3(K)$ because we have $\mu_K = \mu_2 = \{\pm 1\}$. In other words, there is a 1-1 correspondence between maximal finite subgroups of $GL_3(K)$ and maximal finite subgroups of $SL_3(K)$ given by $G \mapsto G_0 = G \cap SL_3(K)$. The inverse of the map is given by $G_0 \mapsto G = \langle G_0, -I_3 \rangle$.

In view of Remark 4.5, we classify all maximal finite subgroups $G_0$ of $SL_3(K)$ for some $K$ in the following subsequent lemmas.

Lemma 4.6. Let $G_0$ be a finite subgroup of $SL_3(\mathbb{C})$ of the type (E), (F), (G), (I), (J), (K), or (L) in [18, Definition 17]. Then there is no $K$ such that $G_0 \leq SL_3(K)$.

Proof. This follows immediately from [6, §5.3] by observing that there is no $K$ such that $|G_0|$ divides $S(3, K)$ for all of these types.

Lemma 4.7. Let $G_0$ be a finite subgroup of $SL_3(\mathbb{C})$ of the type (A) or (C) or (D) in [18, Definition 17]. If $G_0$ is a maximal (up to isomorphism) finite subgroup of $SL_3(K)$ for some $K$, then $G_0 = \text{Sym}_4$ and $K$ is arbitrary.

Proof. Let $H = \{A \in G_0 \mid A \text{ is diagonal}\} \leq G_0$. Then we need to consider the following three cases:

(i) If $G_0$ is of type (A), then we have $G_0 = H$.

(ii) If $G_0$ is of type (C), then we have $G_0 = \langle H, T \rangle$ where $T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

(iii) If $G_0$ is of type (D), then we have $G_0 = \langle H, T, Q \rangle$ where $Q = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

By Lemma 4.2, it follows that such a unique maximal $G_0$ is equal to $G_0 = \langle h, T, Q \rangle \cong \text{Sym}_4$ (among the groups of type (A), (C), or (D)). Now, we show that $G_0 = \langle h, T, Q \rangle$ is a maximal finite subgroup of $SL_3(K)$. Indeed, suppose on the contrary that there is a finite subgroup $G_0'$ of $SL_3(K)$ such that $G_0 \leq G_0'$. Then by the above observation, $G_0'$ cannot be of type (A), (C), or (D). Also, $G_0'$ cannot be of type (E), (F), (G), (I), (J), (K), or (L) by Lemma 4.6. Hence it suffices to consider the following two cases:

1. (Case 1): $G_0'$ is of type (B). In this case, we can show that $G_0'$ is a dihedral group (see Lemma 4.8 below). Then since $G_0 \leq G_0'$, $G_0$ must be either cyclic or a dihedral group, which is absurd.

2. (Case 2): $G_0'$ is of type (H). In this case, we have $|G_0'| = 60$ and this contradicts the fact that $G_0 = \langle h, T, Q \rangle \cong \text{Sym}_4$. Therefore, from (Case 1) and (Case 2), we can conclude that $G_0$ is a maximal finite subgroup of $SL_3(K)$, as desired.

This completes the proof.

Lemma 4.8. Let $G_0$ be a finite subgroup of $SL_3(\mathbb{C})$ of the type (B) in [18, Definition 17]. If $G_0$ is a maximal (up to isomorphism) finite subgroup of $SL_3(K)$ for some $K$, then $G_0 = D_n$ for some $n$, and $\zeta_n + \zeta_n^{-1} \in K$.

Proof. Note that $G_0 = \{\tilde{A} \mid A \in H\}$ for some finite subgroup $H$ of $GL_2(\mathbb{C})$. Let $H_0 = H \cap SL_2(\mathbb{C})$. Then by [18, §2], we have $H_0 \in \{\langle \Delta_n \rangle, \langle \Delta_n, R \rangle, \langle \Delta_4, R, C \rangle, \langle \Delta_8, R, C \rangle, \langle \Delta_{10}, R, D \rangle : n \text{ is even}\}$.
where \( \Delta_n = \Delta_{(\zeta_n^i, \zeta_n)} \), \( R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), \( C = \frac{1}{2} \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \), and \( D = \frac{1}{\sqrt{2}} \begin{pmatrix} -\zeta_n^i + \zeta_n^{-i} & \zeta_n^i - \zeta_n^{-i} \\ \zeta_n^i - \zeta_n^{-i} & -\zeta_n^i + \zeta_n^{-i} \end{pmatrix} \). We claim further that \( H_0 = \langle \Delta_n \rangle \) for an even integer \( n \). Indeed, suppose on the contrary that \( H_0 \) is similar to one of the groups \( \langle \Delta_n, R \rangle \) (\( n \) : even) or \( \langle \Delta_n, R, C \rangle \) for \( n \in \{4, 8\} \) or \( \langle \Delta_{10}, R, D \rangle \). Then it follows that \( G \) contains a subgroup \( \langle \tilde{\Delta}_n, \tilde{R} \rangle \), that is isomorphic to the diyclic group \( \text{Dic}_{2n} \), which is absurd, because of the well-known classification of finite subgroups of \( GL_3(\mathbb{R}) \) (see Example 4 in §[13]). Hence we can see that \( H_0 = \langle \Delta_n \rangle \) for some even integer \( n \).

Now, let \( \tilde{A} \in \tilde{H} = G_0 \). In view of the proof of Lemma 4.2 we can see that \( \tilde{A} \) commutes with \( \tilde{\Delta}_n \) if and only if \( A \in H_0 \). Let \( B \in H \setminus H_0 \). Then \( B\Delta_n B^{-1} \in H_0 \) and it is a diagonal matrix. Since the conjugation by the matrix \( B \) permutes the diagonal entries of \( \Delta_n \), we have \( B\Delta_n B^{-1} = \Delta_n^{-1} \). Hence it follows that \( B = \begin{pmatrix} 0 & y \\ 0 & 1 \end{pmatrix} \) for some \( x, y \), and \( \tilde{B}^2 = \Delta_{(xy, xy, 1/(xy)^2)} \in G_0 \).

Then since \( \langle \tilde{B}^2 \rangle \) is an abelian finite subgroup of \( GL_3(K) \), the proof of Lemma 4.2 shows that \( xy = \pm 1 \). Since \( B \notin H_0 \), we can see that \( B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \). Now, for any \( B' \in H \) which does not commute with \( \Delta_n \), we can see that \( BB' \) commutes with \( \Delta_n \) and \( BB' \in H_0 \). Hence it follows that \( G_0 \in \left\{ \langle \Delta_n \rangle, \langle \Delta_n, \tilde{B} \rangle \right\} \), and then, since \( G_0 \) is assumed to be maximal, we can conclude that \( G_0 = \langle \Delta_n, \tilde{B} \rangle \cong D_n \). The last assertion follows from the observation that the characteristic polynomial of \( \Delta_n \in G_0 \) is defined over \( K \) if and only if the characteristic polynomial of \( \Delta_n \in H_0 \) is defined over \( K \).

This completes the proof. □

**Lemma 4.9.** Let \( G_0 \) be a finite subgroup of \( SL_3(\mathbb{C}) \) of the type (H) in [18, Definition 17]. If \( G_0 \) is a maximal (up to isomorphism) finite subgroup of \( SL_3(K) \) for some \( K \), then \( G_0 = \text{Alt}_5 \) and \( K = \mathbb{Q}(\sqrt{5}) \).

**Proof.** This follows from a similar argument as in the proof of Lemma 4.7. □

Consequently, we can obtain all maximal finite subgroups of \( GL_3(K) \) for some \( K \) as follows.

**Corollary 4.10.** Let \( G \) be a maximal (up to isomorphism) finite subgroup of \( GL_3(K) \) for some \( K \). Then \( G \in \{ C_2 \lhd \text{Sym}_3, D_n \times C_2 \ (\text{some } n), \text{Alt}_5 \times C_2 \} \).

**Proof.** This follows from Remark 4.9 and Lemmas 4.6, 4.7, 4.8, 4.9 together with the fact that \( \text{Sym}_3 \times C_2 \cong C_2 \lhd \text{Sym}_3 \). □

The next lemma says that the group \( D_n \times C_2 \) (for \( n \geq 4 \) even) is isomorphic to a subgroup of \( GL_3(\mathbb{Q}(\zeta_n + \zeta_n^{-1})) \), which is not irreducible.

**Lemma 4.11.** Let \( n \geq 2 \) be an even integer and let \( \rho \) be a 3-dimensional representation of the group \( D_n \times C_2 \). Then \( \rho \) realizes over \( \mathbb{Q}(\zeta_n + \zeta_n^{-1}) \) and it is not irreducible.

**Proof.** Note that it suffices to show that all irreducible representations of \( D_n \times C_2 \) have dimension at most 2 and realize over \( \mathbb{Q}(\zeta_n + \zeta_n^{-1}) \). Then any subgroup of \( GL_3(\mathbb{Q}(\zeta_n + \zeta_n^{-1})) \) which is isomorphic to \( D_n \times C_2 \) is not irreducible. Indeed, we recall that the group \( D_n \) has the presentation \( D_n = \langle \alpha, \beta \mid \alpha^n = 1, \beta^2 = 1, \beta \alpha \beta^{-1} = \alpha^{-1} \rangle \), and \( D_n \) has four 1-dimensional irreducible representations and \((n/2 - 1)\) number of 2-dimensional irreducible representations. Any 1-dimensional irreducible character is given by \( \chi_{x,y} : \alpha^m \mapsto x^m, \alpha \beta \mapsto x^m y \) for some \((x, y) \in \{\pm 1\}^2 \). Each 2-dimensional irreducible character is given by \( \chi_{u} : \alpha^m \mapsto \zeta_n^m + \zeta_n^{-m}, \alpha^m \beta \mapsto \alpha^{-m} \) for each \( 1 \leq u \leq n/2 - 1 \). Since \( D_n \) can be embedded into \( GL_2(\mathbb{Q}(\zeta_n + \zeta_n^{-1})) \) and \( SL_3(\mathbb{Q}(\zeta_n + \zeta_n^{-1})) \), \( \chi_{u} \) realizes over \( \mathbb{Q}(\zeta_n + \zeta_n^{-1}) \) for all \( 1 \leq u \leq n/2 - 1 \). Also, any irreducible representation of \( C_2 \) realizes over \( \mathbb{Q}(\zeta_n + \zeta_n^{-1}) \). Hence, we can see that any irreducible representation of \( D_n \times C_2 \) has
dimension at most 2 and realizes over \( \mathbb{Q}(\zeta_n + \zeta_n^{-1}) \).

This completes the proof. \( \square \)

**Remark 4.12.** We compute the rational group algebra \( \mathbb{Q}G \) of \( G = \langle \mathbb{D}_n, \Delta_{(1,1,-1)} \rangle \cong D_n \times C_2 \) where \( \mathbb{D}_n = \langle \begin{pmatrix} 0 & 1 \\ 1 & \zeta_n + \zeta_n^{-1} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \zeta_n + \zeta_n^{-1} & -1 \end{pmatrix} \rangle \). More precisely, we show that

\[
\mathbb{Q}G = M_2(F) \bigoplus M_1(\mathbb{Q})
\]

where \( F = \mathbb{Q}(\zeta_n + \zeta_n^{-1}) \). Indeed, clearly, we have \( \mathbb{Q}G \subset M_2(F) \bigoplus M_1(\mathbb{Q}) \). Now, since \( \Delta_{(1,1,0)} \in \mathbb{Q} \langle \Delta_{(1,1,-1)} \rangle \) and \( \mathbb{Q}D_n = M_2(F) \), we have

\[
M_2(F) \bigoplus 0 = \mathbb{Q}D_n \Delta_{(1,1,0)} \subseteq \mathbb{Q}G,
\]

and since \( \Delta_{(0,0,1)} \in \mathbb{Q} \langle \Delta_{(1,1,-1)} \rangle \), we also have

\[
0 \bigoplus M_1(\mathbb{Q}) \subseteq \mathbb{Q}G.
\]

Hence it follows from dimension counting that \( \mathbb{Q}G = M_2(F) \oplus M_1(\mathbb{Q}) \), as desired.

In summary, we obtain the following useful result.

**Lemma 4.13.** Let \( G \) be an irreducible maximal (up to isomorphism) finite subgroup of \( GL_3(K) \) for some \( K = \mathbb{Q}(\sqrt{d}) \) where \( d > 0 \) is a square-free integer. Then \( G \) is one of the following groups:

\[
\begin{array}{cc}
G & d \\
\mathbb{Z}_1 & C_2 \wr \text{Sym}_3 \text{ any } d \\
\mathbb{Z}_2 & Alt_5 \times C_2 & 5 \\
\end{array}
\]

Table 1

**Proof.** In view of Corollary 4.10 and Lemma 4.11 we only need to consider those two groups in Table 1 above. The irreducibility of the two groups can be derived directly from the Schur orthogonality relations. To show that \( Alt_5 \times C_2 \cong \langle F_{60}, -I_3 \rangle \) is similar to a subgroup of \( GL_3(\mathbb{Q}(\sqrt{5})) \), we show that the Schur index of \( \langle F_{60}, -I_3 \rangle \) equals 1. (Here, \( F_{60} \) denotes the group generated by \( \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_3^{-1} & 0 \\ 0 & 0 & \zeta_3 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \text{ and } \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & \zeta_2^2 + \zeta_5 \zeta_5^{-1} & \zeta_2^2 + \zeta_5^{-1} \\ \zeta_2 + \zeta_5 \zeta_5^{-1} & 1 & \zeta_2 + \zeta_5 \\ \zeta_2^2 + \zeta_5^{-1} & \zeta_2 + \zeta_5^{-1} & 1 \end{pmatrix} ) \). To this aim, let \( G = \langle F_{60}, -I_3 \rangle \) and consider the inclusion representation \( \rho: G \hookrightarrow GL_3(\mathbb{C}) \). Let \( m \) be the Schur index of \( \rho \). Then it is well known that we have \( m^2 \leq |G| = 120 = 2^3 \cdot 3^1 \cdot 5^1 \) and \( m \mid \text{dim} \rho = 3 \), and hence, it follows that \( m = 1 \).

This completes the proof. \( \square \)

**Remark 4.14.** We compute the rational group algebra \( \mathbb{Q}G \) of the two groups \( G \) in Table 1.

(i) For \( C_2 \wr \text{Sym}_3 \cong G = \langle \Delta_{(-1,1,1)}, I_{3,\alpha} : \alpha \in \text{Sym}_3 \rangle \), we have \( \mathbb{Q}G = M_3(\mathbb{Q}) \). Indeed, clearly, we have \( \mathbb{Q}G \subseteq M_3(\mathbb{Q}) \). Also, it is easy to see that \( \{ d_i T_j : 1 \leq i, j \leq 3 \} \) are linearly independent where \( d_1 = \Delta_{(-1,1,1)}, d_2 = \Delta_{(1,-1,1)}, d_3 = \Delta_{(1,1,-1)} \), and \( T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \). Then it follows that \( 9 \leq \text{dim}_\mathbb{Q}(\mathbb{Q}G) \leq \text{dim}_\mathbb{Q}(M_3(\mathbb{Q})) = 9 \), and hence, we get \( \mathbb{Q}G = M_3(\mathbb{Q}) \).

(ii) Note that there is a \( P \in GL_3(\mathbb{C}) \) such that \( Alt_5 \times C_2 \cong G = P \cdot \langle F_{60}, -I_3 \rangle \cdot P^{-1} \subseteq GL_3(\mathbb{Q}(\sqrt{5})) \) according to the argument in the proof of Lemma 4.13. Then by a similar argument as in (i),
together with the observation that we have
\[
\mathbb{Q} \langle F_{60}, -I_3 \rangle \supseteq \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha \\ 0 & \alpha & 0 \end{pmatrix} : \alpha \in \mathbb{Q}(\zeta_5) \right\} \bigoplus \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha \\ \beta & 0 & 0 \end{pmatrix} : \alpha, \beta \in \mathbb{Q}(\sqrt{5}) \right\}
\]
we can also see that \( \mathbb{Q}G = M_3(\mathbb{Q}(\sqrt{5})) \).

5. Quaternionic Matrix Representations

In this section, we introduce some facts about finite quaternionic matrix groups that will be used later, following a paper of Nebe [15].

Throughout this section, let \( D \) be a definite quaternion algebra over a totally real number field \( K \) and let \( V = D^{1 \times n} \) be a right module over \( M_n(D) \). Endomorphisms of \( V \) are given by left multiplication by elements of \( D \). For computations, it is also convenient to let the endomorphisms act from the right. Then we have \( \text{End}_{M_n(D)}(V) \cong D^{op} \), which we identify with \( D \) since \( D \) is a quaternion algebra.

Now, we start with the following definition.

**Definition 5.1** ([15, Definition 2.1]). Let \( G \) be a finite group and \( \Delta : G \to GL_n(D) \) be a representation of \( G \).

(a) Let \( L \) be a subring of \( K \). Then the enveloping \( L \)-algebra \( \overline{L\Delta(G)} \) is defined as
\[
\overline{L\Delta(G)} = \left\{ \sum_{x \in \Delta(G)} l_x \cdot x \mid l_x \in L \right\} \subseteq M_n(D).
\]

(b) \( \Delta \) is called absolutely irreducible if the enveloping \( \mathbb{Q} \)-algebra \( \overline{\Delta(G)} := \overline{Q\Delta(G)} \) of \( \Delta(G) \) equals \( M_n(D) \).

(c) \( \Delta \) is irreducible if the commuting algebra \( C_{M_n(D)}(\Delta(G)) \) is a division algebra.

(d) A subgroup \( G \leq GL_n(D) \) is called irreducible (resp. absolutely irreducible) if the natural representation \( \text{id} : G \to GL_n(D) \) is irreducible (resp. absolutely irreducible).

**Remark 5.2** ([15, page 110]). (a) The irreducible maximal finite subgroups \( G \) of \( GL_n(D) \) are absolutely irreducible in their enveloping \( \mathbb{Q} \)-algebras \( \overline{\Delta(G)} \) where \( \overline{\Delta(G)} \cong M_m(D') \) for some integer \( m \geq 1 \) and division algebra \( D' \) with \( m^2 \cdot \dim_{\mathbb{Q}} D' \) dividing \( n^2 \cdot \dim_{\mathbb{Q}} D \).

(b) The reducible maximal finite subgroups of \( GL_n(D) \) can be built up from the irreducible maximal finite subgroups of \( GL_l(D) \) for \( l < m \).

For Lemmas 5.3, 5.5, 5.6, 5.7, and 5.8 below, we assume further that \( D = D_{p, \infty} \) for some prime number \( p \). Then in view of Remark 5.2 and the double centralizer theorem, we have the following result.

**Lemma 5.3.** Let \( G \) be an irreducible, but not absolutely irreducible, maximal finite subgroup of \( GL_3(D) \). Then \( \overline{\Delta} \) is one of the followings:
Example 5.4. Let $G = GL_3(\mathbb{F}_2) \times C_2$. Then it is known in [3] that $G$ is a maximal finite subgroup of $GL_3(\mathbb{Q}(\sqrt{−7}))$. Now, since 13 is inert in $\mathbb{Q}(\sqrt{−7})$ so that $\mathbb{Q}(\sqrt{−7}) \subset D_{13,\infty}$, we have $G \leq GL_3(D_{13,\infty})$. We claim that $G$ is a maximal finite subgroup of $GL_3(D_{13,\infty})$ (up to isomorphism). Indeed, suppose on the contrary that there is a finite subgroup $H$ of $GL_3(D_{13,\infty})$ such that $G$ is (isomorphic to) a proper subgroup of $H$. Since $G$ is an absolutely irreducible maximal finite subgroup of $GL_3(\mathbb{Q}(\sqrt{−7}))$, it follows that $H = M_3(D_{13,\infty})$ by dimension counting over $\mathbb{Q}$. This means that $H$ is an absolutely irreducible finite subgroup of $GL_3(D_{13,\infty})$, and this contradicts Theorems 5.12 and 5.13 below. Hence, $G$ is a maximal finite subgroup of $GL_3(D_{13,\infty})$. Also, by definition, $G$ is irreducible because we have $C_{M_3(D_{13,\infty})(G)} = \mathbb{Q}(\sqrt{−7})$, which is a division algebra. Similar arguments apply for cases 2, 3 of Table 3 below, with $p = 11$.

By a similar argument as in Example 5.3 together with the use of [3] Theorem 6.1, we can obtain the following lemma.

Lemma 5.5. Let $G$ be an irreducible maximal (up to isomorphism) finite subgroup of $GL_3(D)$ with $D = D_{p,\infty}$ for some prime $p$, whose enveloping $\mathbb{Q}$-algebra is $M_3(K')$ for some imaginary quadratic field $K'$. Then $G$ is one of the following groups:

| $\sharp$ | $G$ |
|---|---|
| 1 | $GL_3(\mathbb{F}_2) \times C_2$ |
| 2 | $C_2^3 \rtimes \text{Sym}_3 \cong C_4 \wr \text{Sym}_3$ |
| 3 | $C_2^3 \rtimes \text{Sym}_3 \cong C_6 \wr \text{Sym}_3$ |
| 4 | $[(C_3 \times C_6) \rtimes \text{Sym}_3 \cdot C_2] \cdot C_3 \cong (\text{He}_3 \times (-I_3)) \rtimes \text{SL}_2(\mathbb{F}_3)$ |

Table 3

If $G = M_3(K')$ for some totally real field with $[K' : \mathbb{Q}] \leq 2$, then in view of Lemma 4.13 we have the following two results.

Lemma 5.6. There is no irreducible maximal (up to isomorphism) finite subgroup $G$ of $GL_3(D)$ with $D = D_{p,\infty}$ for some prime $p$, whose enveloping $\mathbb{Q}$-algebra is $M_3(K')$ for some real quadratic field $K'$.

Proof. By Lemma 4.13 and Remark 4.14 we only need to consider the group $G := \text{Alt}_5 \times C_2$. Suppose on the contrary that $G$ is an irreducible maximal finite subgroup of $GL_3(D)$ with $D = D_{p,\infty}$ for some prime $p$. Recall that we have $\overline{G} = M_3(\mathbb{Q}(\sqrt{5}))$. Then since we have $M_3(\mathbb{Q}) \leq \overline{G} = M_3(\mathbb{Q}(\sqrt{5})) \leq M_3(D)$,

it follows that

$$C_{M_3(D)}(M_3(D)) \leq C_{M_3(D)}(M_3(\mathbb{Q}(\sqrt{5})) \leq C_{M_3(D)}(M_3(\mathbb{Q})).$$
or equivalently,
\[ \mathbb{Q} \subseteq \mathbb{Q}(\sqrt{5}) \subseteq \mathcal{D} = D_{p,\infty}. \]
But then, since the infinite place of \( \mathbb{Q} \) splits completely in \( \mathbb{Q}(\sqrt{5}) \), the last inclusion is absurd.

This completes the proof. \( \square \)

Lemma 5.7. Let \( G \) be an irreducible maximal (up to isomorphism) finite subgroup of \( GL_3(\mathcal{D}) \) with \( \mathcal{D} = D_{p,\infty} \) for some prime \( p \), whose enveloping \( \mathbb{Q} \)-algebra is \( M_3(\mathbb{Q}) \). Then \( G = C_2 \wr \text{Sym}_3 \).

Proof. This follows from Lemma 4.13 and Remark 4.14. In fact, \( G \) is an irreducible maximal finite subgroup of \( GL_3(\mathcal{D}) \) with \( \mathcal{D} = D_{p,\infty} \) for \( p = 109 \). \( \square \)

Now, we move to the case \( \#2 \) of Lemma 5.3.

Lemma 5.8. Let \( G \) be an irreducible maximal (up to isomorphism) finite subgroup of \( GL_3(\mathcal{D}) \) with \( \mathcal{D} = D_{p,\infty} \) for some prime \( p \), whose enveloping \( \mathbb{Q} \)-algebra is \( M_1(\mathcal{D}') \) for a quaternion division algebra \( \mathcal{D}' \) over a cubic field \( K' \). Then \( G = \text{Dic}_{28} \) with \( K' = \mathbb{Q}(\zeta_{14} + \zeta_{14}^{-1}) \).

Proof. In light of Theorem 3.8 together with the computation of \( \overline{G} \), we can see that all the possible candidates for such maximal finite subgroups are \( \text{Dic}_{28} \) and \( \text{Dic}_{36} \). Now, we consider those two groups one by one.

(1) Take \( G := \text{Dic}_{28} \). Let \( K = \mathbb{Q}(\zeta_{14} + \zeta_{14}^{-1}) \). By the first theorem of [13, page 407], we have \( K \subseteq M_3(\mathbb{Q}) \), which, in turn, implies that \( K \subseteq M_3(D_{7,\infty}) \). Then it follows that \( D_{7,\infty} \otimes_{\mathbb{Q}} K \subseteq D_{7,\infty} \otimes_{\mathbb{Q}} M_3(\mathbb{Q}) \cong M_3(D_{7,\infty}) \). Now, we recall that \( \overline{G} = D_{\zeta_{14} + \zeta_{14}^{-1}, \zeta_{14}, \zeta_{14}^{-1}} \cong D_{7,\infty} \otimes_{\mathbb{Q}} K \) by [13, Theorem 6.1]. Also, note that \( \text{Dic}_{28} \) is not isomorphic to a subgroup of \( \pm L_2(7).2 \), the latter being a primitive absolutely irreducible maximal finite subgroup of \( GL_3(D_{7,\infty}) \) (see Theorem 5.12 below), and hence, \( \text{Dic}_{28} \) is maximal.

(2) For the group \( G := \text{Dic}_{36} \), we take \( K := \mathbb{Q}(\zeta_{18} + \zeta_{18}^{-1}) \), and then, we can proceed as in the proof of (1) with the choices of \( D_{3,\infty} \otimes_{\mathbb{Q}} K \cong D_{\zeta_{18} + \zeta_{18}^{-1}, \zeta_{18}, \zeta_{18}^{-1}} \) and \( M_3(D_{3,\infty}) \). Here, note that we have \( \text{Dic}_{36} \leq \text{Dic}_{12} \wr \text{Sym}_3 \), the latter being an imprimitive absolutely irreducible maximal finite subgroup of \( GL_3(D_{3,\infty}) \) (see Theorem 5.13 below), and hence, \( \text{Dic}_{36} \) is not maximal.

This completes the proof. \( \square \)

Now, for the rest of this section, \( \mathcal{D} \) will denote a totally definite quaternion algebra over a totally real number field \( K \), unless otherwise specified. As in the case of \( GL_n(\mathbb{Q}) \), the notion of primitivity gives an important reduction in the determination of the maximal finite subgroups of \( GL_n(\mathcal{D}) \).

Definition 5.9 ([15, Definition 2.2]). Let \( G \) be an irreducible finite subgroup of \( GL_n(\mathcal{D}) \). Consider \( V := \mathcal{D}^{1 \times n} \) as a \( \mathcal{D} \)-G-bimodule. Then \( G \) is called imprimitive if there is a decomposition \( V = V_1 \oplus \cdots \oplus V_s \) (\( s > 1 \)) of \( V \) as a direct sum of nontrivial \( \mathcal{D} \)-left modules such that \( G \) permutes the \( V_i \) (i.e. for all \( x \in G \) and for all \( 1 \leq i \leq s \), there is a \( j \in \{1, \cdots, s\} \) such that \( V_i x \subseteq V_j \)). If \( G \) is not imprimitive, then \( G \) is called primitive.

The following remark gives a way to produce the imprimitive absolutely irreducible maximal finite subgroups of \( GL_n(\mathcal{D}) \) from the primitive absolutely irreducible maximal finite subgroups of smaller degree.

Remark 5.10 ([15, page 110]). Let \( G \) be an imprimitive absolutely irreducible maximal finite subgroup of \( GL_n(\mathcal{D}) \). Since \( G \) is maximal and finite, it follows that \( G \) is a wreath product of primitive absolutely irreducible maximal finite subgroups of \( GL_d(\mathcal{D}) \) with the symmetric group.
Sym\_n of degree \( \frac{n}{d} \) for divisors \( d \) of \( n \). In particular, every imprimitive absolutely irreducible maximal finite subgroup of \( GL_3(D) \) is a wreath product of primitive absolutely irreducible maximal finite subgroups of \( GL_1(D) = D^\times \) with the symmetric group \( Sym_3 \).

Now, we introduce two results about absolutely irreducible maximal finite subgroups of \( GL_n(D) \) with \( n \in \{1, 3\} \).

Suppose first that \( n = 1 \). Then we have the following fact.

**Theorem 5.11** ([15, Theorem 6.1]). Let \( G \) be an absolutely irreducible maximal finite subgroup of \( GL_1(D) = D^\times \) for some totally definite quaternion algebra \( D \) over a totally real field \( K \). Then:

(a) \( K \) is the maximal totally real subfield of a cyclotomic field.
(b) If \( [K : \mathbb{Q}] \leq 2 \), then \( G \) is one of the following groups (up to isomorphism):

| \( G \) | \( D \) |
|---|---|
| \( \mathbb{Q}^+ \cong SL_2(\mathbb{F}_3) \) | \( D_{2,\infty} \) |
| \( \mathbb{Q} \cong Dic_{12} \) | \( D_{3,\infty} \) |
| \( Q_{24} \cong Dic_{24} \) | \( D_{\sqrt{3},\infty} \) |
| \( \mathbb{Q}^{\pm} \cong SL_2(\mathbb{F}_5) \) | \( D_{\sqrt{5},\infty} \) |

**Table 4**

where \( D_{\alpha,\infty} \) (for \( \alpha \in \{\sqrt{2}, \sqrt{3}, \sqrt{5}\} \)) denotes the definite quaternion algebra over \( K \) such that \( K = \mathbb{Q}(\alpha) \) and \( D_{\alpha,\infty} \) is ramified only at the infinite places of \( K \).

Next, we suppose that \( n = 3 \). For the purpose of this paper, we only consider the case when \( K = \mathbb{Q} \). Then the following important result is obtained.

**Theorem 5.12** ([15, Theorem 13.1]). Let \( G \) be a primitive absolutely irreducible maximal finite subgroup of \( GL_3(D) \) for some totally definite quaternion algebra \( D \) over \( \mathbb{Q} \). Then \( D \) is one of \( D_{2,\infty}, D_{3,\infty}, \) or \( D_{7,\infty} \) and \( G \) is (conjugate to) one of the following groups:

| \( G \) | \( D \) |
|---|---|
| \( SL_2(\mathbb{F}_5) \) | \( D_{2,\infty} \) |
| \( \pm U_3(3) \) | \( D_{3,\infty} \) |
| \( \pm D_{1+2, GL_2(\mathbb{F}_3)} \) | \( D_{3,\infty} \) |
| \( \pm L_2(7).2 \) | \( D_{7,\infty} \) |

**Table 5**

where \( \pm H \) (for a finite matrix group \( H \)) denotes the group generated by \( H \) and the negative identity matrix.

Finally, in view of Remark 5.10 and Theorem 5.11 we also have the following result.

**Theorem 5.13.** Let \( G \) be an imprimitive absolutely irreducible maximal finite subgroup of \( GL_3(D) \) for some totally definite quaternion algebra \( D \) over \( \mathbb{Q} \). Then \( D \) is either \( D_{2,\infty} \) or \( D_{3,\infty} \) and \( G \) is one of the following groups:

| \( G \) | \( D \) |
|---|---|
| \( SL_2(\mathbb{F}_3) \) i Sym_3 | \( D_{2,\infty} \) |
| \( Dic_{12} \) i Sym_3 | \( D_{3,\infty} \) |

**Table 6**
We conclude this section by introducing reducible maximal finite subgroups of $GL_3(D)$ with $D = D_{p,\infty}$ for some prime $p$. To this aim, let $G \leq GL_3(D)$ be a reducible maximal finite subgroup. Then in view of Remark 5.2 (together with [13, Remark II.4]), we have the following two cases:

(Type I) $G$ is conjugate to a diagonal block matrix $\text{Diag}(G_1,G_2)$ where $G_1$ (resp. $G_2$) is an irreducible maximal finite subgroup of $GL_2(D)$ (resp. $D^\times$).

(Type II) $G$ is conjugate to a diagonal block matrix $\text{Diag}(G_1,G_2,G_3)$ where $G_1,G_2,G_3$ are irreducible maximal finite subgroups of $D^\times$.

We examine each case in the following subsequent theorems.

**Theorem 5.14.** Let $G$ be a reducible maximal finite subgroup of $GL_3(D)$ of (Type I) above with $D = D_{p,\infty}$ for some prime $p$. Then $G = G_1 \times G_2$ (up to isomorphism) with the following specified groups $G_1$ and $G_2$:

1. $G_1 \in \{2_+\cdot \text{Alt}_5, SL_2(\mathbb{F}_3) \times \text{Sym}_3\}$ and $G_2 = D^\ast$;
2. $G_1 \in \{SL_2(\mathbb{F}_9), C_3 : (SL_2(\mathbb{F}_3),2), \mathfrak{T}^\ast \rtimes C_4\}$ and $G_2 = Dic_{12}$;
3. $G_1 \in \{SL_2(\mathbb{F}_5),2, SL_2(\mathbb{F}_5) : 2, \mathfrak{T}^\ast \rtimes C_3\}$ and $G_2 = C_6$;
4. $G_1 \in \{GL_2(\mathbb{F}_3),2, \mathfrak{T}^\ast, Dic_{24}, \mathfrak{T}^\ast\}$ and $G_2 = C_2$;
5. $G_1 \in \{GL_2(\mathbb{F}_3), C_{12} \rtimes C_2, \mathfrak{T}^\ast \rtimes C_4, Dic_{24}, \mathfrak{T}^\ast\}$ and $G_2 = C_4$;
6. $G_1 \in \{GL_2(\mathbb{F}_3), \mathfrak{T}^\ast, Dic_{24}, \mathfrak{T}^\ast\}$ and $G_2 = C_6$;
7. $G_1 = \mathfrak{T}^\ast \rtimes C_4$ and $G_2 = C_6$;
8. $G_1 \in \{\mathfrak{T}^\ast \rtimes C_3, Dic_{12} \rtimes C_6\}$ and $G_2 = C_4$;
9. $G_1 = \mathfrak{T}^\ast$ and $G_2 = C_4$;
10. $G_1 \in \{D_6, \mathfrak{T}^\ast, Dic_{12}\}$ and $G_2 = C_2$.

**Proof.** We provide a detailed proof for (1), and then, we can proceed in a similar fashion with the specified choices for (2)-(10).

(1) Take $p = 2$ so that $D = D_{2,\infty}$. Among all irreducible finite subgroups of $GL_2(D_{2,\infty})$, only the three groups $2_+\cdot \text{Alt}_5, SL_2(\mathbb{F}_3) \times \text{Sym}_3$, and $\mathfrak{T}^\ast \rtimes \text{Sym}_2$ are maximal (up to isomorphism). (For the list of such irreducible finite subgroups, see [8, §4].) Also, recall from [13, Theorem 6.1] that the group $\mathfrak{T}^\ast$ is an absolutely irreducible maximal finite subgroup of $D_{2,\infty}^\times$. Then case (1) follows from the observation that $(\mathfrak{T}^\ast \rtimes \text{Sym}_2) \times \mathfrak{T}^\ast \leq \mathfrak{T}^\ast \rtimes \text{Sym}_2$, the latter being an imprimitive absolutely irreducible maximal finite subgroup of $GL_3(D_{2,\infty})$ (see Theorem 5.13).

(2) Take $p = 3$. Then among all irreducible finite subgroups of $GL_2(D_{3,\infty})$, only the four groups $SL_2(\mathbb{F}_9), C_3 : (SL_2(\mathbb{F}_3),2), \text{Dic}_{12} \rtimes \text{Sym}_2$, and $\mathfrak{T}^\ast \rtimes C_4$ are maximal, and $\text{Dic}_{12}$ is an absolutely irreducible maximal finite subgroup of $D_{3,\infty}^\times$. (In fact, we have $(\text{Dic}_{12} \rtimes \text{Sym}_2) \times \text{Dic}_{12} \leq \text{Dic}_{12} \rtimes \text{Sym}_3$, while the other three groups are maximal finite subgroups of $GL_3(D_{3,\infty})$.)

(3) Take $p = 5$. Then among all irreducible finite subgroups of $GL_2(D_{5,\infty})$, only the three groups $SL_2(\mathbb{F}_5) : 2, SL_2(\mathbb{F}_5) : 2$, and $\mathfrak{T}^\ast \rtimes C_3$ are maximal, and $C_6$ is an irreducible maximal finite subgroup of $D_{5,\infty}^\times$.

(4) Take $p = 37$. Then among all irreducible finite subgroups of $GL_2(D_{37,\infty})$, only the three groups $GL_2(\mathbb{F}_3), \mathfrak{T}^\ast$, and $\mathfrak{D}^\ast$ are maximal, and $C_2$ is an irreducible maximal finite subgroup of $D_{37,\infty}^\times$.

(5) For the groups $G_1 = GL_2(\mathbb{F}_3), C_{12} \rtimes C_2, \mathfrak{T}^\ast \rtimes C_4$, or $\text{Dic}_{24}$, take $p = 31$. Then all those four groups are irreducible finite subgroups of $GL_2(D_{31,\infty})$, which are (the only) maximal, and $C_6$ is an irreducible maximal finite subgroup of $D_{31,\infty}^\times$.

For the group $G_1 = \mathfrak{D}^\ast$, take $p = 19$, and note that $\mathfrak{D}^\ast$ is an irreducible finite subgroup of $GL_2(D_{19,\infty})$, that is maximal (along with three other maximal finite subgroups $C_{12} \rtimes C_2, \mathfrak{T}^\ast \rtimes$
Take $p = 53$. Then among all irreducible finite subgroups of $GL_2(D_{53,\infty})$, the six groups $\mathfrak{T}^* \times C_3, \text{Dic}_{12} \times C_6, GL_2(\mathbb{F}_3), \mathfrak{D}^*, \text{Dic}_{24}, \text{and } \mathfrak{T}^*$ are maximal, and $C_6$ is an irreducible maximal finite subgroup of $D_{53,\infty}^\times$.

(7)-(8) Take $p = 71$. Then among all irreducible finite subgroups of $GL_2(D_{71,\infty})$, the four groups $C_{12} \times C_2, \mathfrak{T}^* \times C_4, \mathfrak{T}^* \times C_3$, and $\text{Dic}_{12} \times C_6$ are maximal, and $C_4$ and $C_6$ are irreducible maximal finite subgroups of $D_{71,\infty}^\times$. (In fact, if we fix $G_2 = C_6$, then only two groups $(\mathfrak{T}^* \times C_4) \times C_6$ and $(\mathfrak{T}^* \times C_3) \times C_6$ are maximal finite subgroups of $GL_3(D_{71,\infty})$. (Note that the latter group is in case (3).) Also, if we fix $G_2 = C_4$, then the two groups $(\mathfrak{T}^* \times C_3) \times C_4$ and $(\text{Dic}_{12} \times C_6) \times C_4$ are maximal finite subgroups of $GL_3(D_{71,\infty})$, while the other two groups are in case (5).)

(9) Take $p = 43$. Then among all irreducible finite subgroups of $GL_2(D_{43,\infty})$, the five groups $C_{12} \times C_2, \mathfrak{T}^* \times C_4, \mathfrak{D}^*, \text{Dic}_{24}, \text{and } \mathfrak{T}^*$ are maximal, and $C_4$ is an irreducible maximal finite subgroup of $D_{43,\infty}^\times$. (In fact, the group $\mathfrak{T}^* \times C_4$ is a maximal finite subgroup of $GL_3(D_{43,\infty})$, while the other four groups are in case (5).)

(10) Take $p = 241$. Then all the four irreducible finite subgroups $D_4, D_6, \text{Dic}_{12}$, and $\mathfrak{T}^*$ of $GL_2(D_{241,\infty})$ are maximal, and $C_2$ is an irreducible maximal finite subgroup of $D_{241,\infty}^\times$. (In fact, we have $D_4 \times C_2 \leq C_2 \wr \text{Sym}_3$, the latter being an irreducible finite subgroup of $GL_3(D_{241,\infty})$, while the other three groups are maximal finite subgroups of $GL_3(D_{241,\infty})$.)

This completes the proof. \hfill \□

**Theorem 5.15.** There exists no reducible maximal finite subgroup of $GL_3(D)$ of (Type II) above with $D = D_{p,\infty}$ for some prime $p$.

**Proof.** In view of Theorem 5.11 and the observation that the cyclic groups $C_6, C_4$, and $C_2$ are irreducible maximal finite subgroups of $D_{p,\infty}^\times$ for some corresponding prime $p \geq 5$, we can see that the possible maximal reducible finite subgroups $G$ of $GL_3(D_{p,\infty})$ of (Type II) are of the form $G = G_1^3$ for $G_1 \in \{\mathfrak{T}^*, \text{Dic}_{12}, C_6, C_4, C_2\}$ or $G = G_1 \times G_2 \times G_3$ for $G_1, G_2, G_3 \in \{C_4, C_6\}$ (depending on $p$). If $G = (\mathfrak{T}^*)^3$ (resp. $G = \text{Dic}_{12}^3$) so that $p = 2$ (resp. $p = 3$), then $G$ is isomorphic to a subgroup of $SL_2(\mathbb{F}_3) \wr \text{Sym}_3$ (resp. $\text{Dic}_{12} \wr \text{Sym}_3$), which is the imprimitive absolutely irreducible maximal finite subgroup of $GL_3(D_{2,\infty})$ (resp. $GL_3(D_{3,\infty})$). If $G = C_6^3$ so that $C_6 \leq D_{p,\infty}$ then $G \leq C_6 \wr \text{Sym}_3$, the latter being an irreducible finite subgroup of $GL_3(D_{p,\infty})$. Similar arguments apply for the cases when $G = C_4^3$ or $G = C_2^3$. Finally, if $G = C_6^2 \times C_4$ (resp. $G = C_6 \times C_4^2$) so that $C_4, C_6 \leq D_{p,\infty}$ then $G \leq (\text{Dic}_{12} \times C_6) \times C_4$ (resp. $G \leq (\mathfrak{T}^* \times C_4) \times C_6$), which is a reducible finite subgroup of $GL_3(D_{p,\infty})$ of (Type I).

This completes the proof. \hfill \□

## 6. Main Result

In this section, we give a classification of finite groups that can be realized as the automorphism group of a polarized abelian threefold over a finite field which is maximal in its isogeny class in the following sense.

**Definition 6.1.** Let $X$ be an abelian variety over a field $k$, and let $G$ be a finite group. Suppose that the following two conditions hold:

(i) (realizability) there exists an abelian variety $X'$ over $k$ that is $k$-isogenous to $X$ with a polarization $\mathcal{L}$ such that $G = \text{Aut}_k(X', \mathcal{L})$, and

(ii) (maximality) there is no finite group $H$ such that $G$ is isomorphic to a proper subgroup of $H$ and $H = \text{Aut}_k(Y, \mathcal{M})$ for some abelian variety $Y$ over $k$ that is $k$-isogenous to $X$ with a
polarization $\mathcal{M}$.

In this case, $G$ is said to be realizable maximally (or maximal, in short) in the isogeny class of $X$ as the full automorphism group of a polarized abelian variety over $k$.

The resulting classification is much more complicated than that of polarized abelian surfaces case [8, §6]. In the sequel, $G$ will always denote a finite group. Following [9], we are ready to introduce one of the main results of this section.

**Theorem 6.2 ([9] Theorem 4.1).** There exists a finite field $k$ and a simple abelian threefold $X$ over $k$ such that $G$ is the automorphism group of a polarized abelian threefold over $k$, which is maximal in the isogeny class of $X$ if and only if $G$ is one of the cyclic groups $C_n$ for $n \in \{2, 4, 6, 14, 18\}$ (up to isomorphism).

For the rest of this section, we take care of various cases of non-simple abelian threefolds in view of the decomposition of abelian varieties as the product of powers of simple abelian varieties, up to $k$-isogeny (see §2.1 above). The next theorem takes care of the cases in which our abelian threefold is isogenous to the product of a simple abelian surface and an elliptic curve.

**Theorem 6.3.** There exist a finite field $k$, a simple abelian surface $Y$, and an elliptic curve $E$ over $k$ such that $G$ is the automorphism group of a polarized abelian threefold over $k$, which is maximal in the isogeny class of $X := Y \times E$ if and only if $G = G_1 \times G_2$ is one of the following groups (up to isomorphism) with the specified groups $G_1$ and $G_2$:

1. $G_1 = \mathfrak{A}^* \text{ and } G_2 \in \{C_2, C_4\}$;
2. $G_1 = \mathfrak{D}^* \text{ and } G_2 \in \{C_2, C_4\}$;
3. $G_1 = \text{Dic}_2 \text{ and } G_2 \in \{C_2, C_6\}$;
4. $G_1 = \mathfrak{T}^* \text{ and } G_2 \in \{C_2, C_4, C_6\}$;
5. $G_1 = \text{Dic}_2 \text{ and } G_2 \in \{C_2, C_4\}$;
6. $G_1 \in \{C_8, C_{10}, C_{12}\} \text{ and } G_2 \in \{C_2, C_4, C_6, \text{Dic}_2, \mathfrak{T}^*\}$;
7. $G_1 = C_6 \text{ and } G_2 \in \{C_2, C_4, C_6\}$;
8. $G_1 = C_4 \text{ and } G_2 \in \{C_2, C_4\}$;
9. $G_1 = C_2 \text{ and } G_2 = C_2$.

**Proof.** Suppose first that there exist a finite field $k$, a simple abelian surface $Y$, and an elliptic curve $E$ over $k$ such that $G$ is the automorphism group of a polarized abelian threefold over $k$, which is maximal in the isogeny class of $X := Y \times E$. In particular, we have $\text{End}_k^0(X) = \text{End}_k^0(Y) \oplus \text{End}_k^0(E)$. Since $G$ is a maximal finite subgroup of $\text{End}_k^0(X)$ by assumption, it follows from Goursat’s Lemma, [8, Theorem 6.5 and Corollary 3.4], and Remark 6.4 below that $G$ must be one of the 32 groups (up to isomorphism) in the statement of the theorem. Hence, it suffices to show the converse. We prove the converse by considering them one by one. First, we provide a detailed proof for the groups in case (1).

1. Take $G = \mathfrak{A}^* \times C_2$. Let $\pi$ be a zero of the quadratic polynomial $h := t^2 - 5 \in \mathbb{Z}[t]$ so that there is a simple abelian surface $Y$ over $k = \mathbb{F}_5$ such that $\text{End}_k^0(Y) = D_{2,\infty} \otimes \mathbb{Q}(\sqrt{5})$. (For the existence of such a $Y$, see the proof of [8, Theorem 6.5].) Also, note that there is an ordinary elliptic curve $E$ over $k$ such that $\text{End}_k^0(E) = \mathbb{Q}(\sqrt{-19})$ by [21, Theorem 4.1]. Let $X = Y \times E$. Then we have $D := \text{End}_k^0(X) = (D_{2,\infty} \otimes \mathbb{Q}(\sqrt{5})) \oplus \mathbb{Q}(\sqrt{-19})$. Let $\mathcal{O}_1$ be a maximal $\mathbb{Z}$-order in $D_{2,\infty} \otimes \mathbb{Q}(\sqrt{5})$ with $\mathfrak{A}^* \leq \mathcal{O}_1^*$, and let $\mathcal{O}_2 = \mathbb{Z}\left[\frac{1+i\sqrt{-19}}{2}\right]$. Then $\mathcal{O} := \mathcal{O}_1 \oplus \mathcal{O}_2$ is a maximal $\mathbb{Z}$-order in $D$ by Theorem [27,4] and hence, there exists an abelian threefold $X'$ over $k$ such that $X'$ is $k$-isogenous to $X$ and $\text{End}_k(X') = \mathcal{O}$ by [21, Theorem 3.13]. By Goursat’s lemma, $G$ is a
maximal finite subgroup of $O^\times$.

Now, let $L$ be an ample line bundle on $X'$, and put $L' := \bigotimes_{f \in G} f^* L$. Then $L'$ is also an ample line bundle on $X'$ that is preserved under the action of $G$ so that $G \leq \text{Aut}_k(X', L')$. Since $\text{Aut}_k(X', L')$ is a finite subgroup of $\text{Aut}_k(X') = O^\times$, it follows from the maximality of $G$ that $G = \text{Aut}_k(X', L')$.

Now, suppose that $Z$ is an abelian threefold over $k$ which is $k$-isogenous to $X$. In particular, we have $\text{End}_k^0(Z) = (D_{2,\infty} \otimes \mathbb{Q}(\sqrt{5})) \oplus \mathbb{Q}(\sqrt{-19})$. Suppose that there is a finite group $H$ such that $H = \text{Aut}_k(Z, \mathcal{M})$ for a polarization $\mathcal{M}$ on $Z$, and $G$ is isomorphic to a proper subgroup of $H$. Then $H$ is a finite subgroup of $D^\times$, and hence, it follows from the maximality of $G$ as a finite subgroup of $D^\times$ that $H = G$, which is a contradiction. Thus, we can conclude that $G$ is maximal in the isogeny class of $X$.

For the group $G = \mathfrak{T} \times C_4$, we can proceed as above with the choices of $h = t^2 - 5 \in \mathbb{Z}[t]$ and an ordinary elliptic curve $E$ over $k = \mathbb{F}_5$ with $\text{End}_k^0(E) = \mathbb{Q}(\sqrt{-1})$.

Then, for the groups $G$ in (2)-(9), we proceed in a similar argument as in (1) with the following choices for a simple abelian surface $Y$ over a finite field $k$, and an elliptic curve $E$ over $k$ satisfying the endomorphism conditions specified below.

| $G$ | $h \in \mathbb{Z}[t]$ | $k$ | $\text{End}_k^0(Y)$ | $\text{End}_k^0(E)$ |
|-----|---------------------|-----|---------------------|---------------------|
| (2) | $D^* \times C_2$ | $t^2 - 2$ | $\mathbb{F}_2$ | $D_{2,\infty} \otimes \mathbb{Q}(\sqrt{2})$ | $\mathbb{Q}(\sqrt{-1})$ |
| (3) | $\text{Dic}_{24} \times C_2$ | $t^2 - 3$ | $\mathbb{F}_3$ | $D_{3,\infty} \otimes \mathbb{Q}(\sqrt{3})$ | $\mathbb{Q}(\sqrt{-2})$ |
| (4) | $\mathfrak{T}^* \times C_2$ | $t^2 - 3$ | $\mathbb{F}_3$ | $D_{3,\infty} \otimes \mathbb{Q}(\sqrt{3})$ | $\mathbb{Q}(\sqrt{-3})$ |
| (5) | $\text{Dic}_{12} \times C_2$ | $t^2 - 11$ | $\mathbb{F}_{11}$ | $D_{11,\infty} \otimes \mathbb{Q}(\sqrt{11})$ | $\mathbb{Q}(\sqrt{-1})$ |
| | $\text{Dic}_{12} \times C_4$ | $t^2 - 17$ | $\mathbb{F}_{17}$ | $D_{17,\infty} \otimes \mathbb{Q}(\sqrt{17})$ | $\mathbb{Q}(\sqrt{-1})$ |
| (6) | $C_8 \times C_2$ | $t^4 + 16$ | $\mathbb{F}_4$ | $\mathbb{Q}(\zeta_8)$ | $\mathbb{Q}(\sqrt{-16})$ |
| | $C_8 \times C_4$ | $t^4 + 81$ | $\mathbb{F}_9$ | $\mathbb{Q}(\zeta_8)$ | $\mathbb{D}_{3,\infty}$ |
| | $C_8 \times C_6$ | $t^4 - 5t^3 + 25t^2 - 125t + 625$ | $\mathbb{F}_{25}$ | $\mathbb{Q}(\zeta_{10})$ | $\mathbb{Q}(\sqrt{-11})$ |
| | $C_8 \times \mathfrak{T}$ | $t^4 - 3t^3 + 9t^2 - 27t + 81$ | $\mathbb{F}_9$ | $\mathbb{Q}(\zeta_{10})$ | $\mathbb{Q}(\sqrt{-3})$ |
| | $C_8 \times \text{Dic}_{12}$ | $t^4 - 2t^3 + 4t^2 - 8t + 16$ | $\mathbb{F}_4$ | $\mathbb{Q}(\zeta_{10})$ | $\mathbb{D}_{3,\infty}$ |
| (7) | $C_6 \times C_2$ | $t^4 + 4t^2 + 16$ | $\mathbb{F}_4$ | $\mathbb{Q}(\zeta_{12})$ | $\mathbb{D}_{3,\infty}$ |
| | $C_6 \times C_4$ | $t^4 + 2t^2 + 25$ | $\mathbb{F}_5$ | $\mathbb{Q}(\sqrt{2} + \sqrt{-3})$ | $\mathbb{Q}(\sqrt{-19})$ |
be one of the 26 groups (up to isomorphism) in the statement of the theorem. Hence, it suffices by assumption, it follows from Goursat’s Lemma, \([8, Corollary 3.4]\), and Remark 6.6 that 

\[
\begin{array}{|c|c|c|c|}
\hline
 & C_6 \times C_4 & t^4 + 10t^2 + 361 & \mathbb{Q}(\sqrt{7} + 2\sqrt{-3}) \\
& C_6 \times C_6 & & \mathbb{Q}(\sqrt{-3}) \\
& C_4 \times C_2 & t^4 - 10t^2 + 49 & \mathbb{F}_7 \\
& C_4 \times C_4 & t^4 - 18t^2 + 289 & \mathbb{F}_{17} \\
\hline
(8) & C_4 \times C_2 & t^4 - 6t^2 + 49 & \mathbb{Q}(\sqrt{5} + \sqrt{-2}) \\
(9) & C_4 \times C_2 & & \mathbb{Q}(\sqrt{-6}) \\
\hline
\end{array}
\]

We remark that \(D_{3,\infty} \otimes \mathbb{Q} \mathbb{Q}(\sqrt{3}) \cong D_{2,\infty} \otimes \mathbb{Q} \mathbb{Q}(\sqrt{3}), D_{13,\infty} \otimes \mathbb{Q} \mathbb{Q}(\sqrt{13}) \cong D_{2,\infty} \otimes \mathbb{Q} \mathbb{Q}(\sqrt{13}), D_{11,\infty} \otimes \mathbb{Q} \mathbb{Q}(\sqrt{11}) \cong D_{2,\infty} \otimes \mathbb{Q} \mathbb{Q}(\sqrt{11})\), and \(D_{17,\infty} \otimes \mathbb{Q} \mathbb{Q}(\sqrt{17}) \cong D_{3,\infty} \otimes \mathbb{Q} \mathbb{Q}(\sqrt{17})\).

This completes the proof. \(\square\)

**Remark 6.4.** Among all possible 45 combinations of \(G := G_1 \times G_2\) with 

\[
G_1 \in \{C_2, C_4, C_6, C_{10}, C_{12}, Dic_{12}, T^*, Dic_{24}, D^*, \bar{T}^*\}
\]

and \(G_2 \in \{C_2, C_4, C_6, Dic_{12}, T^*\}\), up to isomorphism, there are 13 groups which cannot be indeed realized as the automorphism group of a polarized abelian threefold over a finite field \(k\), which is maximal in the isogeny class of the product a simple abelian surface and an elliptic curve over \(k\). For example, the groups \(G := G_1 \times Dic_{12}\) with \(G_1 \in \{T^*, D^*\}\) cannot occur due to the issue of the characteristic of the base field \(k\). On the other hand, the group \(G := Dic_{24} \times Dic_{12}\) cannot be realized due to the fact that the latter group can occur only when the cardinality of the base field \(k\) is an even power of \(p = 3\), while the former group occurs only when the cardinality of \(k\) is an odd power of its characteristic.

If our abelian threefold happens to be \(k\)-isogenous to the product of three non-isogenous elliptic curves over a finite field \(k\), then we have the following result.

**Theorem 6.5.** There exist a finite field \(k\) and three non-isogenous elliptic curves \(E_1, E_2, E_3\) over \(k\) such that \(G\) is the automorphism group of a polarized abelian threefold over \(k\), which is maximal in the isogeny class of \(X := E_1 \times E_2 \times E_3\) if and only if \(G = G_1 \times G_2 \times G_3\) is one of the following groups (up to isomorphism) with the specified groups \(G_1, G_2,\) and \(G_3\):

1. \(G_1 = G_2 = C_2\) and \(G_3 \in \{C_2, C_4, C_6, Dic_{12}, T^*\}\);
2. \(G_1 = C_2, G_2 = C_4,\) and \(G_3 \in \{C_4, C_6, Dic_{12}, T^*\}\);
3. \(G_1 = C_2, G_2 = C_6,\) and \(G_3 \in \{C_6, Dic_{12}, T^*\}\);
4. \(G_1 \in \{C_2, C_4, C_6\}\) and \(G_2 = G_3 = Dic_{12};\)
5. \(G_1 \in \{C_2, C_4, C_6\}\) and \(G_2 = G_3 = T^*;\)
6. \(G_1 = G_2 = C_4\) and \(G_3 \in \{C_4, C_6\};\)
7. \(G_1 = C_4, G_2 = C_6,\) and \(G_3 \in \{C_6, Dic_{12}, T^*\};\)
8. \(G_1 = G_2 = C_6\) and \(G_2 \in \{C_6, Dic_{12}, T^*\}.\)

**Proof.** Suppose first that there exist a finite field \(k\) and three non-isogenous elliptic curves \(E_1, E_2, E_3\) over \(k\) such that \(G\) is the automorphism group of a polarized abelian threefold over \(k\), which is maximal in the isogeny class of \(X := E_1 \times E_2 \times E_3\). In particular, we have \(\text{End}_k(X) = \text{End}_k^0(E_1) \oplus \text{End}_k^0(E_2) \oplus \text{End}_k^0(E_3).\) Since \(G\) is a maximal finite subgroup of \(\text{End}_k(X)\) by assumption, it follows from Goursat’s Lemma, \([8, Corollary 3.4]\), and Remark 6.6 that \(G\) must be one of the 26 groups (up to isomorphism) in the statement of the theorem. Hence, it suffices to show the converse. We prove the converse by considering them one by one. First, we provide a detailed proof for the cases in (1).

1. Take \(G = C_2 \times C_2 \times C_2.\) Let \(k = \mathbb{F}_3\). Then there are three non-isogenous elliptic curves \(E_1, E_2,\) and \(E_3\) over \(k\) such that \(\text{End}_k^0(E_1) = \mathbb{Q}(\sqrt{-2})\) and \(\text{End}_k^0(E_2) = \text{End}_k^0(E_3) = \mathbb{Q}(\sqrt{-11})\)
by [21] Theorem 4.1. Let \( X = E_1 \times E_2 \times E_3 \) so that \( \operatorname{End}^0_k(X) = D := \mathbb{Q} \sqrt{-2} \oplus \mathbb{Q} \sqrt{-11} \oplus \mathbb{Q} \sqrt{-11} \). By Theorem 27, \( \mathcal{O} := \mathbb{Z} \left[ \sqrt{-2} \right] \oplus \mathbb{Z} \left[ \frac{1 + \sqrt{-2}}{2} \right] \oplus \mathbb{Z} \left[ \frac{1 + \sqrt{-2}}{2} \right] \) is a maximal \( \mathbb{Z} \)-order in \( D \), and hence, there exists an abelian threefold \( X' \) over \( k \) such that \( X' \) is \( k \)-isogenous to \( X \) and \( \operatorname{End}^0_k(X') = \mathcal{O} \) by [21] Theorem 3.13]. Then it follows that \( G \cong \mathcal{O}^* = \operatorname{Aut}_k(X') \) (with the principal product polarization). It is also easy to see that \( G \) is maximal in the isogeny class of \( X \).

For the group \( G = C_2 \times C_2 \times C_4 \), we can proceed as above with the choices \( k = \mathbb{F}_5 \) and three non-isogenous elliptic curves \( E_1, E_2, E_3 \) with \( \operatorname{End}^0_k(E_1) = \mathbb{Q} \sqrt{-5} \), \( \operatorname{End}^0_k(E_2) = \mathbb{Q} \sqrt{-19} \), and \( \operatorname{End}^0_k(E_3) = \mathbb{Q} \sqrt{-11} \).

Similarly:

For the group \( G = C_2 \times C_2 \times C_6 \), we take \( k = \mathbb{F}_7 \) and three non-isogenous elliptic curves \( E_1, E_2, E_3 \) with \( \operatorname{End}^0_k(E_1) = \mathbb{Q} \sqrt{-7} \), \( \operatorname{End}^0_k(E_2) = \mathbb{Q} \sqrt{-3} \), and \( \operatorname{End}^0_k(E_3) = \mathbb{Q} \sqrt{-11} \).

For the group \( G = C_2 \times C_2 \times \mathbb{Z}^* \), we take \( k = \mathbb{F}_9 \) and three non-isogenous elliptic curves \( E_1, E_2, E_3 \) with \( \operatorname{End}^0_k(E_1) = \mathbb{Q} \sqrt{-3} \), \( \operatorname{End}^0_k(E_2) = \mathbb{Q} \sqrt{-5} \), and \( \operatorname{End}^0_k(E_3) = \mathbb{Q} \sqrt{-15} \).

Finally, for the group \( G = C_2 \times C_2 \times \mathbb{Z}^* \), we take \( k = \mathbb{F}_4 \) and three non-isogenous elliptic curves \( E_1, E_2, E_3 \) with \( \operatorname{End}^0_k(E_1) = \mathbb{Q} \sqrt{-3} \), \( \operatorname{End}^0_k(E_2) = \mathbb{Q} \sqrt{-7} \), and \( \operatorname{End}^0_k(E_3) = \mathbb{Q} \sqrt{-1} \).

Then, for (2)-(8), we can proceed in a similar argument as in (1) with the following specified choices for the groups \( G \), the finite field \( k \), and three non-isogenous elliptic curves \( E_1, E_2, E_3 \) over \( k \) with the endomorphism conditions given below.

|   | \( G \) | \( k \) | \( \operatorname{End}^0_k(E_1), \operatorname{End}^0_k(E_2), \operatorname{End}^0_k(E_3) \) |
|---|---|---|---|
| (2) | \( C_2 \times C_4 \times C_4 \) | \( \mathbb{F}_5 \) | \( \mathbb{Q} \sqrt{-5}, \mathbb{Q} \sqrt{-1}, \mathbb{Q} \sqrt{-1} \) |
|     | \( C_2 \times C_4 \times C_6 \) | \( \mathbb{F}_4 \) | \( \mathbb{Q} \sqrt{-7}, \mathbb{Q} \sqrt{-1}, \mathbb{Q} \sqrt{-1} \) |
|     | \( C_2 \times C_4 \times \mathbb{Z}^* \) | \( \mathbb{F}_4 \) | \( \mathbb{Q} \sqrt{-3}, \mathbb{Q} \sqrt{-1}, \mathbb{Q} \sqrt{-1} \) |
|     | \( C_2 \times C_6 \times C_6 \) | \( \mathbb{F}_4 \) | \( \mathbb{Q} \sqrt{-7}, \mathbb{Q} \sqrt{-1}, \mathbb{Q} \sqrt{-1} \) |
|     | \( C_2 \times C_6 \times \mathbb{Z}^* \) | \( \mathbb{F}_4 \) | \( \mathbb{Q} \sqrt{-3}, \mathbb{Q} \sqrt{-1}, \mathbb{Q} \sqrt{-1} \) |
| (4) | \( C_2 \times \mathbb{Z}^* \times \mathbb{Z}^* \) | \( \mathbb{F}_4 \) | \( \mathbb{Q} \sqrt{-15}, \mathbb{Q} \sqrt{-3}, \mathbb{Q} \sqrt{-3} \) |
|     | \( C_4 \times \mathbb{Z}^* \times \mathbb{Z}^* \) | \( \mathbb{F}_4 \) | \( \mathbb{Q} \sqrt{-15}, \mathbb{Q} \sqrt{-3}, \mathbb{Q} \sqrt{-3} \) |
|     | \( C_6 \times \mathbb{Z}^* \times \mathbb{Z}^* \) | \( \mathbb{F}_4 \) | \( \mathbb{Q} \sqrt{-15}, \mathbb{Q} \sqrt{-3}, \mathbb{Q} \sqrt{-3} \) |
| (6) | \( C_4 \times C_4 \times C_4 \) | \( \mathbb{F}_5 \) | \( \mathbb{Q} \sqrt{-1}, \mathbb{Q} \sqrt{-1}, \mathbb{Q} \sqrt{-1} \) |
|     | \( C_4 \times C_4 \times C_6 \) | \( \mathbb{F}_{13} \) | \( \mathbb{Q} \sqrt{-1}, \mathbb{Q} \sqrt{-1}, \mathbb{Q} \sqrt{-1} \) |
| (7) | \( C_4 \times C_6 \times C_6 \) | \( \mathbb{F}_4 \) | \( \mathbb{Q} \sqrt{-1}, \mathbb{Q} \sqrt{-1}, \mathbb{Q} \sqrt{-1} \) |
|     | \( C_4 \times C_6 \times \mathbb{Z}^* \) | \( \mathbb{F}_4 \) | \( \mathbb{Q} \sqrt{-1}, \mathbb{Q} \sqrt{-1}, \mathbb{Q} \sqrt{-1} \) |
|     | \( C_6 \times C_6 \times C_6 \) | \( \mathbb{F}_7 \) | \( \mathbb{Q} \sqrt{-3}, \mathbb{Q} \sqrt{-3}, \mathbb{Q} \sqrt{-3} \) |
|     | \( C_6 \times C_6 \times \mathbb{Z}^* \) | \( \mathbb{F}_4 \) | \( \mathbb{Q} \sqrt{-3}, \mathbb{Q} \sqrt{-3}, \mathbb{Q} \sqrt{-3} \) |
| (8) | \( C_6 \times C_6 \times C_6 \) | \( \mathbb{F}_7 \) | \( \mathbb{Q} \sqrt{-3}, \mathbb{Q} \sqrt{-3}, \mathbb{Q} \sqrt{-3} \) |
|     | \( C_6 \times C_6 \times \mathbb{Z}^* \) | \( \mathbb{F}_4 \) | \( \mathbb{Q} \sqrt{-3}, \mathbb{Q} \sqrt{-3}, \mathbb{Q} \sqrt{-3} \) |

This completes the proof. \( \square \)
Remark 6.6. Among all possible 35 combinations of $G := G_1 \times G_2 \times G_3$ with $G_1, G_2, G_3 \in \{C_2, C_4, C_6, Dic_{12}, \Sigma^*\}$, up to isomorphism, there are 9 groups which cannot be indeed realized as the automorphism group of a polarized abelian threefold over a finite field $k$, which is maximal in the isogeny class of the product of three non-isogenous elliptic curves over $k$. For example, the groups $G := G_1 \times Dic_{12} \times \Sigma^*$ with $G_1 \in \{C_2, C_4, C_6, Dic_{12}, \Sigma^*\}$ cannot occur due to the issue of the characteristic of the base field $k$. On the other hand, the groups $G := Dic_{12} \times Dic_{12} \times Dic_{12}$ and $G := \Sigma^* \times \Sigma^* \times \Sigma^*$ cannot be realized due to the fact that any two of those three elliptic curves $E_1, E_2, E_3$ are necessarily isogenous to each other.

The next theorem deals with the case when exactly two of those three elliptic curves in the above are isogenous to each other.

Theorem 6.7. There exist a finite field $k$ and two non-isogenous elliptic curves $E_1, E_2$ over $k$ such that $G$ is the automorphism group of a polarized abelian threefold over $k$, which is maximal in the isogeny class of $X := E_1^2 \times E_2$ if and only if $G = G_1 \times G_2$ is one of the following groups (up to isomorphism) with the specified groups $G_1$ and $G_2$:

1. $G_1 \in \{D_4, D_6\}$ and $G_2 \in \{C_2, C_4, C_6, Dic_{12}, \Sigma^*\}$;
2. $G_1 = Dic_{12}$ and $G_2 \in \{C_2, C_4, C_6, \Sigma^*\}$;
3. $G_1 = SL_2(\mathbb{F}_3)$ and $G_2 \in \{C_2, C_4, C_6\}$;
4. $G_1 \in \{C_{12} \times C_2, \Sigma^* \times C_4\}$ and $G_2 \in \{C_2, C_4, C_6, Dic_{12}, \Sigma^*\}$;
5. $G_1 \in \{\{C_6 \times C_6\} \times C_2, \Sigma^* \times C_3\}$ and $G_2 \in \{C_2, C_4, C_6, Dic_{12}, \Sigma^*\}$;
6. $G_1 \in \{\{\Sigma^*, Dic_{24}, \Sigma^*\}\}$ and $G_2 \in \{C_2, C_4, C_6\}$;
7. $G_1 \in \{\mathbb{Z}^{1+1}, Alt_5, SL_2(\mathbb{F}_3) \times Sym_3, (SL_2(\mathbb{F}_3))^2 \times Sym_2\}$ and $G_2 \in \{C_2, C_4, C_6, \Sigma^*\}$;
8. $G_1 \in \{\mathbb{Z}^{1+1}, Alt_5, SL_2(\mathbb{F}_3) \times Sym_3, (SL_2(\mathbb{F}_3))^2 \times Sym_2\}$ and $G_2 \in \{C_2, C_4, C_6, Dic_{12}\}$;
9. $G_1 \in \{SL_2(\mathbb{F}_9), C_3 \times (SL_2(\mathbb{F}_3), 2) \times Dic_{12}^2 \times Sym_2\}$ and $G_2 \in \{C_2, C_4, C_6, Dic_{12}\}$;
10. $G_1 \in \{\mathbb{Z}^{1+1}, Alt_5, SL_2(\mathbb{F}_3) \times Sym_3, (SL_2(\mathbb{F}_3))^2 \times Sym_2\}$ and $G_2 \in \{C_2, C_4, C_6\}$.

Proof. Suppose first that there exist a finite field $k$ and two non-isogenous elliptic curves $E_1, E_2$ over $k$ such that $G$ is the automorphism group of a polarized abelian threefold over $k$, which is maximal in the isogeny class of $X := E_1^2 \times E_2$. In particular, we have $\text{End}^0_k(X) = M_2(\text{End}^0_k(E_1)) \oplus \text{End}^0_k(E_2)$. Since $G$ is a maximal finite subgroup of $\text{End}^0_k(X)$ by assumption, it follows from Goursat’s Lemma, [8 Theorems 6.8 and 6.9], and Remark 6.8 below that $G$ must be one of the 80 groups (up to isomorphism) in the statement of the theorem. Hence, it suffices to show the converse. We prove the converse by considering them one by one. First, we provide a detailed proof for the groups in case (1).

1. Take $G = D_4 \times C_2$ (resp. $G = D_6 \times C_2$). Let $k = \mathbb{F}_4$. Then there are two non-isogenous elliptic curves $E_1$ and $E_2$ over $k$ such that $\text{End}^0_k(E_1) = \text{End}^0_k(E_2) = \mathbb{Q}(\sqrt{-15})$ by [21 Theorem 4.1]. Let $X = E_1^2 \times E_2$ so that $\text{End}^0_k(X) = D := M_2(\mathbb{Q}(\sqrt{-15})) \oplus \mathbb{Q}(\sqrt{-15})$. By Theorems 2.6 and 2.7, $\mathcal{O} := M_2\left(\mathbb{Z}\left[\frac{1+\sqrt{-15}}{2}\right]\right) \oplus \mathbb{Z}\left[\frac{1+\sqrt{-15}}{2}\right]^{\times}$ is a maximal $\mathbb{Z}$-order in $D$, and hence, there exists an abelian threefold $X'$ over $k$ such that $X'$ is $k$-isogenous to $X$ and $\text{End}_k(X') = \mathcal{O}$ by [21 Theorem 3.13]. Also, it is known [11 Table 2] that $D_4$ (resp. $D_6$) is a maximal finite subgroup of $GL_2\left(\mathbb{Z}\left[\frac{1+\sqrt{-15}}{2}\right]\right)$, and hence, $G$ is a maximal finite subgroup of $\mathcal{O}^{\times}$ by Goursat’s Lemma.

Now, let $\mathcal{L}$ be an ample line bundle on $X'$, and put $\mathcal{L}' := \bigotimes_{f \in G} f^*\mathcal{L}$. Then $\mathcal{L}'$ is also an ample line bundle on $X'$ that is preserved under the action of $G$ so that $G \leq \text{Aut}_k(X', \mathcal{L}')$. Since $\text{Aut}_k(X', \mathcal{L}')$ is a finite subgroup of $\text{Aut}_k(X') = \mathcal{O}^{\times}$, it follows from the maximality of $G$ that $G = \text{Aut}_k(X', \mathcal{L}')$. Furthermore, by a similar argument, we can see that $G$ is maximal in the
isogeny class of \( X \).

For the groups \( G = D_4 \times C_4 \) or \( G = D_6 \times C_4 \), we can proceed as above with the choices \( k = \mathbb{F}_4 \) and two non-isogenous elliptic curves \( E_1, E_2 \) with \( \text{End}_k^0(E_1) = \mathbb{Q}(\sqrt{-15}) \) and \( \text{End}_k^0(E_2) = \mathbb{Q}(\sqrt{-1}) \). Similarly:

For the groups \( G = D_4 \times C_6 \) or \( G = D_6 \times C_6 \), we take \( k = \mathbb{F}_7 \) and two non-isogenous elliptic curves \( E_1, E_2 \) with \( \text{End}_k^0(E_1) = \mathbb{Q}(\sqrt{-6}) \) and \( \text{End}_k^0(E_2) = \mathbb{Q}(\sqrt{-3}) \).

For the groups \( G = D_4 \times \text{Dic}_{12} \) or \( G = D_6 \times \text{Dic}_{12} \), we take \( k = \mathbb{F}_9 \) and two non-isogenous elliptic curves \( E_1, E_2 \) with \( \text{End}_k^0(E_1) = \mathbb{Q}(\sqrt{-5}) \) and \( \text{End}_k^0(E_2) = D_{3,\infty} \).

Finally, for the groups \( G = D_4 \times \mathbb{T}^* \) or \( G = D_6 \times \mathbb{T}^* \), we take \( k = \mathbb{F}_4 \) and two non-isogenous elliptic curves \( E_1, E_2 \) with \( \text{End}_k^0(E_1) = \mathbb{Q}(\sqrt{-15}) \) and \( \text{End}_k^0(E_2) = D_{2,\infty} \).

Then, for (2)-(10), we can proceed in a similar argument as in (1) with the following specified choices for the groups \( G \), the finite field \( k \), and two non-isogenous elliptic curves \( E_1, E_2 \) over \( k \) with the endomorphism conditions given below.

| \( G \) | \( k \) | \( \text{End}_k^0(E_1), \text{End}_k^0(E_2) \) |
|---|---|---|
| \( \text{Dic}_{12} \times C_2 \) | \( \mathbb{F}_{25} \) | \( \mathbb{Q}(\sqrt{-21}), \mathbb{Q}(\sqrt{-6}) \) |
| \( \text{Dic}_{12} \times C_4 \) | \( \mathbb{F}_{25} \) | \( \mathbb{Q}(\sqrt{-21}), \mathbb{Q}(\sqrt{-1}) \) |
| \( \text{Dic}_{12} \times C_6 \) | \( \mathbb{F}_{25} \) | \( \mathbb{Q}(\sqrt{-21}), \mathbb{Q}(\sqrt{-3}) \) |
| \( \text{Dic}_{12} \times \mathbb{T}^* \) | \( \mathbb{F}_4 \) | \( \mathbb{Q}(\sqrt{-7}), D_{2,\infty} \) |
| \( \text{SL}_2(\mathbb{F}_3) \times C_2 \) | \( \mathbb{F}_{25} \) | \( \mathbb{Q}(\sqrt{-6}), \mathbb{Q}(\sqrt{-6}) \) |
| \( \text{SL}_2(\mathbb{F}_3) \times C_4 \) | \( \mathbb{F}_{25} \) | \( \mathbb{Q}(\sqrt{-6}), \mathbb{Q}(\sqrt{-1}) \) |
| \( \text{SL}_2(\mathbb{F}_3) \times C_6 \) | \( \mathbb{F}_{25} \) | \( \mathbb{Q}(\sqrt{-6}), \mathbb{Q}(\sqrt{-3}) \) |
| \( (C_{12} \times C_2) \times C_2, (\mathbb{T}^* \times C_4) \times C_2 \) | \( \mathbb{F}_5 \) | \( \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-1}) \) |
| \( (C_{12} \times C_2) \times C_4, (\mathbb{T}^* \times C_4) \times C_4 \) | \( \mathbb{F}_5 \) | \( \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-1}) \) |
| \( (C_{12} \times C_2) \times C_6, (\mathbb{T}^* \times C_4) \times C_6 \) | \( \mathbb{F}_5 \) | \( \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-1}) \) |
| \( (C_{12} \times C_2) \times \text{Dic}_{12}, (\mathbb{T}^* \times C_4) \times \text{Dic}_{12} \) | \( \mathbb{F}_9 \) | \( \mathbb{Q}(\sqrt{-1}), D_{3,\infty} \) |
| \( (C_{12} \times C_2) \times \mathbb{T}^*, (\mathbb{T}^* \times C_4) \times \mathbb{T}^* \) | \( \mathbb{F}_4 \) | \( \mathbb{Q}(\sqrt{-7}), D_{2,\infty} \) |
| \( \text{GL}_2(\mathbb{F}_3) \times C_2 \) | \( \mathbb{F}_{17} \) | \( \mathbb{Q}(\sqrt{-2}), \mathbb{Q}(\sqrt{-3}) \) |
| \( \text{GL}_2(\mathbb{F}_3) \times C_4 \) | \( \mathbb{F}_{17} \) | \( \mathbb{Q}(\sqrt{-2}), \mathbb{Q}(\sqrt{-1}) \) |
| \( \text{GL}_2(\mathbb{F}_3) \times C_6 \) | \( \mathbb{F}_9 \) | \( \mathbb{Q}(\sqrt{-2}), \mathbb{Q}(\sqrt{-3}) \) |
| \( \text{GL}_2(\mathbb{F}_3) \times \text{Dic}_{12} \) | \( \mathbb{F}_9 \) | \( \mathbb{Q}(\sqrt{-2}), D_{3,\infty} \) |
| \( ((C_6 \times C_6) \times C_2) \times C_2, (\mathbb{T}^* \times C_4) \times C_2 \) | \( \mathbb{F}_7 \) | \( \mathbb{Q}(\sqrt{-3}), \mathbb{Q}(\sqrt{-6}) \) |
| \( ((C_6 \times C_6) \times C_2) \times C_6, (\mathbb{T}^* \times C_4) \times C_6 \) | \( \mathbb{F}_7 \) | \( \mathbb{Q}(\sqrt{-3}), \mathbb{Q}(\sqrt{-3}) \) |
| \( ((C_6 \times C_6) \times C_2) \times C_4, (\mathbb{T}^* \times C_4) \times C_4 \) | \( \mathbb{F}_{25} \) | \( \mathbb{Q}(\sqrt{-3}), \mathbb{Q}(\sqrt{-1}) \) |
| \( ((C_6 \times C_6) \times C_2) \times \text{Dic}_{12}, (\mathbb{T}^* \times C_4) \times \text{Dic}_{12} \) | \( \mathbb{F}_9 \) | \( \mathbb{Q}(\sqrt{-3}), D_{3,\infty} \) |
| \( ((C_6 \times C_6) \times C_2) \times \mathbb{T}^*, (\mathbb{T}^* \times C_4) \times \mathbb{T}^* \) | \( \mathbb{F}_4 \) | \( \mathbb{Q}(\sqrt{-3}), D_{2,\infty} \) |

for \( G_1 \in \{ 3^*, \text{Dic}_{24} \} \)

\[
\begin{align*}
G_1 \times C_2 \\
G_1 \times C_4 \\
G_1 \times C_6 \\
\mathbb{D}^* \times C_2 \\
\mathbb{D}^* \times C_4 \\
\mathbb{D}^* \times C_6 \\
\end{align*}
\]

\( D_{7,\infty} \cdot \mathbb{Q}(\sqrt{-5}) \)

\( D_{7,\infty} \cdot \mathbb{Q}(\sqrt{-1}) \)

\( D_{7,\infty} \cdot \mathbb{Q}(\sqrt{-3}) \)

\( D_{11,\infty} \cdot \mathbb{Q}(\sqrt{-2}) \)

\( D_{11,\infty} \cdot \mathbb{Q}(\sqrt{-1}) \)

\( D_{11,\infty} \cdot \mathbb{Q}(\sqrt{-3}) \)
Suppose first that there exists a finite field $G := G_1 \times G_2$ with $G_1, G_2$ being as above, up to isomorphism, there are 19 groups which cannot be indeed realized as the automorphism group of a polarized abelian threefold over a finite field $k$, which is maximal in the isogeny class of the product of a power of an elliptic curve and a non-isogenous elliptic curve over $k$. For example, the groups $G := G_1 \times \text{Dic}_{12}$ with $G_1 \in \{SL_2(\mathbb{F}_9), 3, (\text{Dic}_{12})^2 \times \text{Sym}_2\}$ cannot occur due to the issue of the characteristic of the base field $k$.

For the case when the abelian threefold is a power of an ordinary elliptic curve over a finite field, we may use a result of [3] to obtain the following theorem.

**Theorem 6.9.** There exists a finite field $k$ and an ordinary elliptic curve $E$ over $k$ such that $G$ is the automorphism group of a polarized abelian threefold over $k$, which is maximal in the isogeny class of $X := E^3$ if and only if $G$ is one of the following groups (up to isomorphism):

1. $D_6 \times C_2$;
2. $C_2^3 \times \text{Sym}_3$;
3. $GL_3(\mathbb{F}_2) \times C_2$;
4. $[(Q_8 \times C_3) \times C_2] \times C_2$;
5. $(C_{12} \times C_2) \times C_4$;
6. $C_4^3 \times \text{Sym}_3$;
7. $[(Q_8 \times C_3) \times C_4] \times C_4$;
8. $[(Q_8 \times C_3) \times C_3] \times C_6$;
9. $C_6^3 \times \text{Sym}_3$;
10. $[((C_3 \times C_6) \times \text{Sym}_3) \cdot C_2] \cdot C_3$.

**Proof.** Suppose first that there exists a finite field $k$ and an ordinary elliptic curve $E$ over $k$ such that $G$ is the automorphism group of a polarized abelian threefold over $k$, which is maximal in the isogeny class of $X := E^3$. In particular, we have $\text{End}_k^0(X) = M_3(\mathbb{Q}(\sqrt{-d}))$ for some square-free positive integer $d$. Thus, by assumption, $G$ is a maximal finite subgroup of $GL_3(\mathbb{Q}(\sqrt{-d}))$, and then, by [3] Theorems 6.1 and 7.1], we can see that $G$ must be one of the 10 groups in the statement of the theorem. Hence, it suffices to show the converse. We prove the converse by considering them one by one. We provide a detailed proof for (1), and then, we can proceed in...
a similar fashion with the specified choices for (2)-(10).

(1) Take \( G = D_6 \times C_2 \). Let \( k = \mathbb{F}_4 \). Then there is an ordinary elliptic curve \( E \) over \( k \) such that \( \text{End}_0^k(E) = \mathbb{Q}(\sqrt{-15}) \) by [21] Theorem 4.1. Let \( X = E^3 \). Then we have \( D := \text{End}_0^k(X) = M_3(\mathbb{Q}(\sqrt{-15})) \). Let \( \mathcal{O} = M_3 \left( \mathbb{Z}[\sqrt{15}/2] \right) \). By Theorem 2.3, \( \mathcal{O} \) is a maximal \( \mathbb{Z} \)-order in \( D \), and hence, there exists an abelian threefold \( X' \) over \( k \) such that \( X' \) is \( k \)-isogenous to \( X \) and \( \text{End}_k(X') = \mathcal{O} \) by [21] Theorem 3.13. In view of [3] Theorem 7.1, \( G \) is a maximal finite subgroup of \( GL_3(\mathbb{Q}(\sqrt{-15})) \), and hence, \( G \) is a maximal finite subgroup of \( \mathcal{O}^x \), too.

Now, let \( L \) be an ample line bundle on \( X' \), and put \( L' := \bigotimes_{f \in G} f^* L \). Then \( L' \) is also an ample line bundle on \( X' \) that is preserved under the action of \( G \) so that \( G \leq \text{Aut}_k(X', L') \). Since \( \text{Aut}_k(X', L') \) is a finite subgroup of \( \text{Aut}_k(X') = \mathcal{O}^x \), it follows from the maximality of \( G \) that \( G = \text{Aut}_k(X', L') \).

Now, suppose that \( Y \) is an abelian threefold over \( k \) which is \( k \)-isogenous to \( X \). In particular, we have \( \text{End}_0^k(Y) = M_3(\mathbb{Q}(\sqrt{-15})) \). Suppose that there is a finite group \( H \) such that \( H = \text{Aut}_k(Y, \mathcal{M}) \) for a polarization \( \mathcal{M} \) on \( Y \), and \( G \) is isomorphic to a proper subgroup of \( H \). Then \( H \) is a finite subgroup of \( GL_3(\mathbb{Q}(\sqrt{-15})) \), and hence, it follows from the maximality of \( G \) as a finite subgroup of \( GL_3(\mathbb{Q}(\sqrt{-15})) \) that \( H = G \), which is a contradiction. Thus, we can conclude that \( G \) is maximal in the isogeny class of \( X \).

(2) For \( G = C_3^3 \rtimes \text{Sym}_3 \), we take \( k = \mathbb{F}_4 \) and an ordinary elliptic curve \( E \) over \( k \) with \( \text{End}_0^k(E) = \mathbb{Q}(\sqrt{-15}) \).

(3) For \( G = GL_3(\mathbb{F}_2) \times C_2 \), we take \( k = \mathbb{F}_4 \) and an ordinary elliptic curve \( E \) over \( k \) with \( \text{End}_0^k(E) = \mathbb{Q}(\sqrt{-7}) \).

(4) For \( G = ([Q_8 \times C_3] \times C_2) \times C_2 \), we take \( k = \mathbb{F}_{17} \) and an ordinary elliptic curve \( E \) over \( k \) with \( \text{End}_0^k(E) = \mathbb{Q}(\sqrt{-2}) \).

(5)-(7) For the groups \( G = (C_{12} \times C_2) \times C_4, C_4^3 \rtimes \text{Sym}_3 \), or \( ([Q_8 \times C_3] \cdot C_4) \times C_4 \), we take \( k = \mathbb{F}_5 \) and an ordinary elliptic curve \( E \) over \( k \) with \( \text{End}_0^k(E) = \mathbb{Q}(\sqrt{-1}) \).

(8)-(10) For the groups \( G = ([Q_8 \times C_3] \times C_3) \times C_6, C_6 \rtimes \text{Sym}_3 \), or \( ([(C_3 \times C_6) \rtimes \text{Sym}_3] \cdot C_2) \cdot C_2 \), we take \( k = \mathbb{F}_7 \) and an ordinary elliptic curve \( E \) over \( k \) with \( \text{End}_0^k(E) = \mathbb{Q}(\sqrt{-3}) \).

This completes the proof. \( \square \)

Finally, in the next theorem, we consider the case when the abelian threefold is a power of a supersingular elliptic curve over a finite field.

**Theorem 6.10.** There exists a finite field \( k \) and a supersingular elliptic curve \( E \) over \( k \) (all of whose endomorphisms are defined over \( k \)) such that \( G \) is the automorphism group of a polarized abelian threefold over \( k \), which is maximal in the isogeny class of \( X := E^3 \) if and only if \( G \) is one of the following groups (up to isomorphism):

1. \( SL_2(\mathbb{F}_5) \);
2. \( (SL_2(\mathbb{F}_3))^3 \rtimes \text{Sym}_3 \);
3. \( \pm U_3(3) \);
4. \( \pm 3^{1+2} GL_2(\mathbb{F}_3) \);
5. \( (\text{Dic}_{12})^3 \rtimes \text{Sym}_3 \);
6. \( \pm L_2(7).2 \);
7. \( GL_3(\mathbb{F}_2) \times C_2 \);
8. \( C_3^3 \rtimes \text{Sym}_3 \cong C_4 \rtimes \text{Sym}_3 \);
9. \( C_6^3 \rtimes \text{Sym}_3 \cong C_6 \rtimes \text{Sym}_3 \).
\((10)\) \([(\{C_3 \times C_6\} \times \text{Sym}_3) \cdot C_2) \cdot C_2] \cdot C_3 \cong (\text{He}_3 \times \langle -I_3 \rangle) \times SL_2(\mathbb{F}_3);

\((11)\) \text{Dic}_{28};

\((12)\) \(C^3_3 \times \text{Sym}_3 \cong C_4 \times \text{Sym}_3\);

\((13)\) \(2^{1+4}.\text{Alt}_5 \times \mathbb{T}^*\) and \((\text{SL}_2(\mathbb{F}_9) \times \text{Dic}_{12}, C_3) : (\text{SL}_2(\mathbb{F}_3), 2) \times \text{Dic}_{12}\), and \((\mathbb{T}^* \times C_4) \times \text{Dic}_{12}\);

\((14)\) \(\text{SL}_2(\mathbb{F}_9) \times \text{Dic}_{12}, C_3) : (\text{SL}_2(\mathbb{F}_3), 2) \times \text{Dic}_{12},\) and \((\mathbb{T}^* \times C_4) \times \text{Dic}_{12}\);

\((15)\) \(\text{SL}_2(\mathbb{F}_9) \times C_2, \mathbb{T}^* \times C_2,\) and \(\mathbb{D}^* \times C_2\);

\((16)\) \(\text{SL}_2(\mathbb{F}_9) \times C_2, \mathbb{T}^* \times C_2,\) and \(\mathbb{D}^* \times C_2;\)

\((17)\) \(\text{SL}_2(\mathbb{F}_9) \times C_2, \mathbb{T}^* \times C_2,\) and \(\mathbb{D}^* \times C_2,\) and \(\mathbb{D}^* \times C_2;\)

\((18)\) \(\text{SL}_2(\mathbb{F}_9) \times C_2, \mathbb{T}^* \times C_2,\) and \(\mathbb{D}^* \times C_2;\)

\((19)\) \((\mathbb{T}^* \times C_4) \times C_6;\)

\((20)\) \((\mathbb{T}^* \times C_3) \times C_4\) and \((\text{Dic}_{12} \times C_6) \times C_4;\)

\((21)\) \(\mathbb{T}^* \times C_4;\)

\((22)\) \(D_6 \times C_2, \mathbb{T}^* \times C_2,\) and \(\text{Dic}_{12} \times C_2.\)

\textbf{Proof.} Suppose first that there exists a finite field \(k = \mathbb{F}_q (q = p^n)\) and a supersingular elliptic curve \(E\) over \(k\) (all of whose endomorphisms are defined over \(k\)) such that \(G\) is the automorphism group of a polarized abelian threefold over \(k\), which is maximal in the isogeny class of \(X := E^3\). In particular, we have \(\text{End}_k^0(\mathcal{E}) = M_3(D_{p,\infty})\). Thus, by assumption, \(G\) is a maximal finite subgroup of \(\text{GL}_3(D_{p,\infty})\) (up to isomorphism). If \(G\) is absolutely irreducible, then \(G\) must be one of the 6 groups \((1) \sim (6)\) in the above list by Theorems \(5.12\) and \(5.13\). If \(G\) is reducible, but not absolutely irreducible, then, as we have seen in Section 5, \(G\) must be one of the 6 groups \((7) \sim (12)\) in the above list. If \(G\) is reducible, then by Theorems \(5.14\) and \(5.15\), \(G\) is one of the groups in cases \((13) \sim (22)\). Hence, it suffices to show the converse. We prove the converse by considering them one by one.

Groups \((1)-(6)\): We provide a detailed proof for \((1)\), and then, we can proceed in a similar fashion with the specified choices for \((2)-(6)\).

\((1)\) Let \(k = \mathbb{F}_4\). Then there is a supersingular elliptic curve \(E\) over \(k\) such that \(\text{End}_k^0(\mathcal{E}) = D_{2,\infty}\) by [21, Theorem 4.1]. Let \(X = E^3\). Then we have \(D := \text{End}_k^0(\mathcal{E}) = M_3(D_{2,\infty})\). Let \(V = D^3_{2,\infty};\mathcal{O} = \mathbb{Z} \{i, j, ij, 1+i+j+ij\},\) and \(G = \text{SL}_2(\mathbb{F}_3)\). By Theorem \(6.12\), \(G\) is a primitive absolutely irreducible maximal finite subgroup of \(\text{GL}_3(D_{2,\infty})\). Recall also that \(\mathcal{O}\) is a maximal \(\mathbb{Z}\)-order in \(D_{2,\infty}\). By [15, Definition and Lemma 2.6], there is a \(G\)-invariant \(\mathcal{O}\)-lattice \(L\) in \(V\), and then, by Theorem \(2.8\) it follows that \(\mathcal{O}' := \text{Hom}_G(L, L)\) is a maximal \(\mathbb{Z}\)-order in \(\text{Hom}_{D_{2,\infty}}(V, V) = M_3(D_{2,\infty})\). By the choice of \(L\), \(G\) can be regarded as a subgroup of \((\mathcal{O}')^\times\).

Now, let \(\mathcal{E}\) be an ample line bundle on \(X\), and put \(\mathcal{L}' := \bigotimes_{f \in G} f^*\mathcal{L}\). Then \(\mathcal{L}'\) is also an ample line bundle on \(X\) that is preserved under the action of \(G\) so that \(G \leq \text{Aut}_k(X', \mathcal{L}')\). Since \(\text{Aut}_k(X', \mathcal{L}')\) is a finite subgroup of \(\text{Aut}_k(X') = (\mathcal{O}')^\times\), it follows from the maximality of \(G\) that \(G = \text{Aut}_k(X', \mathcal{L}')\). Furthermore, by a similar argument, we can see that \(G\) is maximal in the isogeny class of \(X\).

\((2)\) For the group \(G = (\text{SL}_2(\mathbb{F}_3))^3 \times \text{Sym}_3\), we can proceed as above with the choices \(k = \mathbb{F}_4\) and a supersingular elliptic curve \(E\) over \(k\) with \(\text{End}_k^0(\mathcal{E}) = D_{2,\infty}\). Here, we recall that \(G\) is an imprimitive absolutely irreducible maximal finite subgroup of \(\text{GL}_3(D_{2,\infty})\). Similarly:

\((3)-(5)\) For the groups \(G = \pm U_3(3)\) or \(G = \pm 3^{1+2}.\text{GL}_2(\mathbb{F}_3)\) or \(G = (\text{Dic}_{12})^3 \times \text{Sym}_3\), we take...
$k = \mathbb{F}_9$, a supersingular elliptic curve $E$ over $k$ with $\text{End}_k^0(E) = D_{3,\infty}$, and a (unique) maximal $\mathbb{Z}$-order $\mathcal{O}$ in $D_{3,\infty}$. Here, we recall that the first two groups (resp. the last group) are primitive (resp. imprimitive) absolutely irreducible maximal finite subgroups of $GL_3(D_{3,\infty})$.

(6) For the group $G = \pm L_2(7).2$, we take $k = \mathbb{F}_{49}$, a supersingular elliptic curve $E$ over $k$ with $\text{End}_k^0(E) = D_{7,\infty}$, and a (unique) maximal $\mathbb{Z}$-order $\mathcal{O}$ in $D_{7,\infty}$. Here, we recall that $G$ is a primitive absolutely irreducible maximal finite subgroup of $GL_3(D_{7,\infty})$.

Groups (7)-(12): We provide a detailed proof for (7), and then, we can proceed in a similar fashion with the specified choices for (8)-(12).

(7) Let $k = \mathbb{F}_{169}$. Then there is a supersingular elliptic curve $E$ over $k$ such that $\text{End}_k^0(E) = D_{13,\infty}$ by [21] Theorem 4.1. Let $X = E^3$. Then we have $D := \text{End}_k^0(X) = M_3(D_{13,\infty})$. Let $V = D_{13,\infty}, \mathcal{O}$ a maximal $\mathbb{Z}$-order in $D_{13,\infty}$, and $G = GL_3(\mathbb{F}_2) \times C_2$. By Lemma 5.3 and Example 5.4, $G$ is an irreducible maximal finite subgroup of $GL_3(D_{13,\infty})$. By [15] Definition and Lemma 2.6, there is a $G$-invariant $\mathcal{O}$-lattice $L$ in $V$, and then, by Theorem 2.8, it follows that $\mathcal{O}' := \text{Hom}_\mathcal{O}(L, L)$ is a maximal $\mathbb{Z}$-order in $\text{Hom}_{D_{13,\infty}}(V, V) = M_3(D_{13,\infty})$. Then by a similar argument as in the proof of (1), we can see that there exists an abelian threefold $X'$ over $k$ (being $k$-isogenous to $X$) with a polarization $\mathcal{L}'$ such that $G = \text{Aut}_k(X', \mathcal{L}')$ and $G$ is maximal in the isogeny class of $X$.

(8)-(10) For the groups $G = C_3^3 \times \text{Sym}_3$ or $G = C_3^3 \times \text{Sym}_3$ or $G = \left[[((C_3 \times C_3) \times \text{Sym}_3) \cdot C_2] \cdot C_2\right] \cdot C_3$, we take $k = \mathbb{F}_{121}$, a supersingular elliptic curve $E$ over $k$ with $\text{End}_k^0(E) = D_{11,\infty}$, and a maximal $\mathbb{Z}$-order $\mathcal{O}$ in $D_{11,\infty}$. Here, we recall that those groups are irreducible maximal finite subgroups of $GL_3(D_{11,\infty})$ (see Example 5.3).

(11) For the group $G = \text{Dic}_{28}$, we take $k = \mathbb{F}_{49}$, a supersingular elliptic curve $E$ over $k$ with $\text{End}_k^0(E) = D_{7,\infty}$, and a maximal $\mathbb{Z}$-order $\mathcal{O}$ in $D_{7,\infty}$. Here, we recall that $G$ is an irreducible maximal finite subgroup of $GL_3(D_{7,\infty})$ (see Lemma 5.3).

(12) For the group $G = C_2 \times \text{Sym}_3$, we take $k = \mathbb{F}_{11881}$, a supersingular elliptic curve $E$ over $k$ with $\text{End}_k^0(E) = D_{109,\infty}$, and a maximal $\mathbb{Z}$-order $\mathcal{O}$ in $D_{109,\infty}$. Here, we recall that $G$ is an irreducible maximal finite subgroup of $GL_3(D_{109,\infty})$ (see Lemma 5.4).

Groups in (13)-(22): We provide a detailed proof for the groups in case (13), and then, we can proceed in a similar fashion with the specified choices for the groups in cases (14)-(22).

(13) Let $k = \mathbb{F}_4$. Then there is a supersingular elliptic curve $E$ over $k$ such that $\text{End}_k^0(E) = D_{2,\infty}$ by [21] Theorem 4.1. Let $X = E^3$. Then we have $D := \text{End}_k^0(X) = M_3(D_{2,\infty})$. Let $V = D_{2,\infty}, \mathcal{O} = \mathbb{Z} \left[i, j, ij, \frac{1+i+j+i+j}{2}\right]$, and $G = 2_1^{1+4}.\text{Alt}_5 \times \mathbb{T}^*$ or $G = (SL_2(\mathbb{F}_3) \times \text{Sym}_3) \times \mathbb{T}^*$. By Theorem 5.14, $G$ is a reducible maximal finite subgroup of $GL_3(D_{2,\infty})$. Recall also that $\mathcal{O}$ is a maximal $\mathbb{Z}$-order in $D_{2,\infty}$. By [15] Definition and Lemma 2.6, there is a $G$-invariant $\mathcal{O}$-lattice $L$ in $V$, and then, by Theorem 2.8, it follows that $\mathcal{O}' := \text{Hom}_\mathcal{O}(L, L)$ is a maximal $\mathbb{Z}$-order in $\text{Hom}_{D_{2,\infty}}(V, V) = M_3(D_{2,\infty})$. Then by a similar argument as in the proof of (1), we can see that there exists an abelian threefold $X'$ over $k$ (being $k$-isogenous to $X$) with a polarization $\mathcal{L}'$ such that $G = \text{Aut}_k(X', \mathcal{L}')$ and $G$ is maximal in the isogeny class of $X$.

(14) For the groups in (14), we take $k = \mathbb{F}_9$, a supersingular elliptic curve $E$ over $k$ with $\text{End}_k^0(E) = D_{3,\infty}$, and a maximal $\mathbb{Z}$-order $\mathcal{O}$ in $D_{3,\infty}$. Here, we recall that those groups are reducible maximal finite subgroups of $GL_3(D_{3,\infty})$.

(15) For the groups in (15), we take $k = \mathbb{F}_{25}$, a supersingular elliptic curve $E$ over $k$ with $\text{End}_k^0(E) = D_{5,\infty}$, and a maximal $\mathbb{Z}$-order $\mathcal{O}$ in $D_{5,\infty}$. Here, we recall that those groups are
reducible maximal finite subgroups of $GL_3(D_{5,\infty})$.

(16) For the groups in (16), we take $k = \mathbb{F}_{1369}$, a supersingular elliptic curve $E$ over $k$ with $\text{End}_k^0(E) = D_{37,\infty}$, and a maximal $\mathbb{Z}$-order $\mathcal{O}$ in $D_{37,\infty}$. Here, we recall that those groups are reducible maximal finite subgroups of $GL_3(D_{37,\infty})$.

(17) For the groups $G = GL_2(\mathbb{F}_3) \times C_4$, $G = (C_{12} \times C_2) \times C_4$, $G = (\mathbb{Z}^+ \times C_4) \times C_4$, or $G = \text{Dic}_{24} \times C_4$, we take $k = \mathbb{F}_{961}$, a supersingular elliptic curve $E$ over $k$ with $\text{End}_k^0(E) = D_{31,\infty}$, and a maximal $\mathbb{Z}$-order $\mathcal{O}$ in $D_{31,\infty}$. Here, we recall that those groups are reducible maximal finite subgroups of $GL_3(D_{31,\infty})$.

For the group $G = \mathcal{O}^* \times C_4$, we take $k = \mathbb{F}_{361}$, a supersingular elliptic curve $E$ over $k$ with $\text{End}_k^0(E) = D_{19,\infty}$, and a maximal $\mathbb{Z}$-order $\mathcal{O}$ in $D_{19,\infty}$. Here, we recall that $G$ is a reducible maximal finite subgroup of $GL_3(D_{19,\infty})$.

(18) For the groups in (18), we take $k = \mathbb{F}_{2809}$, a supersingular elliptic curve $E$ over $k$ with $\text{End}_k^0(E) = D_{53,\infty}$, and a maximal $\mathbb{Z}$-order $\mathcal{O}$ in $D_{53,\infty}$. Here, we recall that those groups are reducible maximal finite subgroups of $GL_3(D_{53,\infty})$.

(19)-(20) For the groups in (19) and (20), we take $k = \mathbb{F}_{5041}$, a supersingular elliptic curve $E$ over $k$ with $\text{End}_k^0(E) = D_{71,\infty}$, and a maximal $\mathbb{Z}$-order $\mathcal{O}$ in $D_{71,\infty}$. Here, we recall that those groups are reducible maximal finite subgroups of $GL_3(D_{71,\infty})$.

(21) For the group $G = \mathbb{Z}^+ \times C_4$, we take $k = \mathbb{F}_{1849}$, a supersingular elliptic curve $E$ over $k$ with $\text{End}_k^0(E) = D_{43,\infty}$, and a maximal $\mathbb{Z}$-order $\mathcal{O}$ in $D_{43,\infty}$. Here, we recall that $G$ is a reducible maximal finite subgroup of $GL_3(D_{43,\infty})$.

(22) For the groups in (22), we take $k = \mathbb{F}_{58081}$, a supersingular elliptic curve $E$ over $k$ with $\text{End}_k^0(E) = D_{241,\infty}$, and a maximal $\mathbb{Z}$-order $\mathcal{O}$ in $D_{241,\infty}$. Here, we recall that those groups are reducible maximal finite subgroups of $GL_3(D_{241,\infty})$.

This completes the proof. \hfill \Box

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School of Mathematics, Korea Institute for Advanced Study, 85 Hoegiro, Dongdaemun-gu, Seoul 02455, South Korea

Email address: hwangwon@kias.re.kr

Dept. of Mathematical Sciences, KAIST, 291 Daehak-ro, Yuseong-gu, Daejeon, South Korea, 34141

Email address: bhim@kaist.ac.kr

Dept. of Mathematical Sciences, KAIST, 291 Daehak-ro, Yuseong-gu, Daejeon, South Korea, 34141

Email address: jawlang@kaist.ac.kr