Classical dynamics as observed quantum dynamics

Julián López, Laura Ares and Alfredo Luis∗
Departamento de Óptica, Facultad de Ciencias Físicas, Universidad Complutense, 28040 Madrid, Spain
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We examine a case study where classical evolution emerges when observing a quantum evolution. This a single-mode quantum Kerr evolution interrupted by measurement of the double-homodyne kind projecting the evolved field state into classical-like coherent states or quantum squeezed states. We show that irrespective of whether the measurement is classical or quantum there is no quantum Zeno effect and the evolution turns out to be classical. We provide a practical scheme to perform such measurements.

I. INTRODUCTION

The proper relation between the quantum and classical theories has been a subject of interest, research and debate from the very beginning of the quantum theory. And this refers in special to quantum measurement processes and their effects. We may say that at a fundamental level the issue is not solved yet.

Leaving aside mere formal mathematical limits such as $\hbar \to \infty$ that are unpractical and provide little understanding, the most popular account refers to decoherence as the practical mechanism by which quantum paradoxes disappears leading to the emergence of the classical world [1–4].

Decoherence is the result of the coupling of the system with a large enough environment. Both system and environment are modified by this coupling in different forms according to their different size and complexity. The key point for us is that this is the basis of measurement on its more pure form. Inspired by this reasoning, in this work we follow a promising avenue of research which may be formulated in this way: Classical mechanics is an observed quantum dynamics [4].

We prove this idea in a very specific arena. This is a nonlinear single-mode Kerr effect [5], which produces notable quantum phenomena, such as revivals and Schrödinger cat states, for example [6–8]. The evolution is observed via a complex-amplitude measurement of the kind of double-homodyne detection that, depending on its balanced or unbalanced setting, is governed by projection on classical-like coherent states or quantum squeezed states, respectively [9–11]. The idea is that these measurements may be fuzzy enough to respect the details of the evolution. In this regard, coherent and squeezed states form a variety isomorphic to the phase space, in our case a plane. Moreover we discuss the lack of Zeno effect in this model.

II. CLASSICAL DYNAMICS

The Hamiltonian describing the nonlinear part of a single-mode propagation through a Kerr medium is of the form [5]:

$$H_c = \chi |\alpha|^4,$$

where $\chi$ is the corresponding nonlinear susceptibility in appropriate units, and $\alpha$ is the dimensionless complex amplitude of the field mode. For definiteness we consider the interaction picture where we focus just on the effects caused by the nonlinear term $H_c$. The evolution can be described by the differential equation obtained via Poisson brackets as

$$\dot{\alpha} = \{\alpha, H_c\}, \quad \{A, B\} = -i(\partial_\alpha A \partial_{\alpha^*} B - \partial_{\alpha^*} A \partial_\alpha B),$$

after the usual relation between complex and real variables

$$\alpha = \frac{1}{\sqrt{2}} (q + ip).$$

This results in

$$\alpha(t) = e^{-\Omega t}, \quad \Omega = 2\chi |\alpha|^2,$$

where it must be noticed that $|\alpha|$ is a constant of the motion.

III. UNOBSERVED QUANTUM EVOLUTION

The quantum version of the Hamiltonian (2.1) is

$$H = \hbar \chi \hat{n}^2, \quad \hat{n} = a^\dagger a,$$

where $\hat{n}$ is the photon-number operator, and $a$ is the complex-amplitude operator satisfying the commutation relation $[a, a^\dagger] = 1$. In the quantum case we find advantageous to express the evolution via the action of the unitary operator

$$U(t) = e^{-it\chi \hat{n}^2}.$$

For the sake of illustration we may consider the evolution of the mean value of $a$ when the field is initially in

*Electronic address: alluis@fis.ucm.es URL: http://www.ucm.es/info/gioq
IV. OBSERVED QUANTUM EVOLUTION

A. Measurement

In the following we may find convenient the decomposition of $a$ in terms of quadrature operators $\hat{q}$, $\hat{p}$ as in Eq. (2.3) satisfying the typical position-linear momentum commutation relation

$$a = \frac{1}{\sqrt{2}}(\hat{q} + i\hat{p}), \quad [\hat{q}, \hat{p}] = i. \quad (4.1)$$

As a suitable measurement we will consider the projection on displaced states

$$|z\rangle = D(z)|\psi\rangle, \quad z = \left( \begin{array}{c} q \\ p \end{array} \right), \quad (4.2)$$

where $D(z)$ is the displacement operator

$$D(z) = e^{\alpha \sigma^1 - \alpha^* \sigma^2}, \quad (4.3)$$

with the relation between complex and real variables in Eq. (2.3), and $|\psi\rangle$ can be in principle any state. For the whole procedure it is crucial that the state-labels, either in the vector form $z$ or in the complex scalar $\alpha$, form a variety isomorphic to the phase space of the observed system. So the outcomes $z$ can be regarded as an observation of the phase space.

If $|\psi\rangle = |0\rangle$ is the vacuum state, then $|z\rangle$ are Glauber coherent states universally considered as the most classical-like pure states $[12, 13]$. If $|\psi\rangle$ is a squeezed vacuum state then $|z\rangle$ are squeezed coherent states, and so clearly nonclassical [16]. So, this simple model includes projection on classical or nonclassical states. Moreover some other possibilities may be considered for $|\psi\rangle$. In any case, for simplicity we will always assume that

$$\langle \psi|a|\psi\rangle = 0. \quad (4.4)$$

We recall that for any $|\psi\rangle$ the states $|z\rangle$ provide a resolution of identity [17]

$$\frac{1}{2\pi} \int dz |z\rangle\langle z| = I, \quad (4.5)$$

where $dz = dq dp$, and $I$ is the identity.

B. Process

For simplicity we consider that the initial state belongs to this same measurement family $|z\rangle$, and will be denoted by $|z_0\rangle$. The quantum evolution from $t = 0$ to $t$ is interrupted $N$ times at times $t_j = j\tau$, $j = 1, 2, \ldots, N$, to perform a measurement whose effect is the projection of the system state on some vector $|z_j\rangle$. So we have a series of $N$ continuous evolution during a time $\tau$ governed by the action of the unitary operator interrupted by $N$ sudden jumps from $U(\tau)|z_j\rangle$ to $|z_{j+1}\rangle$.

The fundamental quantity regarding the observed evolution is the conditional probability

$$p(z_{j+1}|z_j) = \frac{1}{2\pi} |\langle z_{j+1}|U(\tau)|z_j\rangle|^2. \quad (4.6)$$

This defines a quantum trajectory represented by a series of measurement results $z_0, z_1, z_2, \ldots$ that occur with probability

$$p(z_1, \ldots, z_N) = p(z_N|z_{N-1})p(z_{N-1}|z_{N-2})\cdots p(z_1|z_0). \quad (4.7)$$

This is a Markovian process as far as there is no memory, this is, the transition probability from $z_j$ to $z_{j+1}$ does not depend on the preceding results $z_{j-1}, z_{j-2}, \ldots$.

In most situations we will no interested in keeping track of the intermediate results, being just interested in the final distribution for the last outcome $z = z_N$ after a total evolution time $t = N\tau$, which is

$$p_N(z|z_0) = \int dz_1 \cdots dz_{N-1} p(z|z_{N-1})\cdots p(z_1|z_0). \quad (4.8)$$

Roughly speaking, we may imagine that the result is some random deviation from a mean drift caused by the Hamiltonian part of the evolution. The fundamental question to be addressed here is whether this drift resembles the classical evolution or not, and if so, which is the...
particular effect of the measurement. We will also pay attention to examine whether there is any difference between measurements projecting on classical or quantum states, i. e., between quantum or classical measurements \[18\].

C. Linear approximation

Any progress along this line involves the computation of the transition probability \( p(z_{j+1}|z_j) \) in Eq. (4.6). This is in general rather awkward unless some suitable approximations might be made. Here we are going to consider that when \( \tau \to 0 \) we can approximate \( U(\tau) \) by a linear transformation, this is

\[
U(\tau) \approx M(\tau) \hat{z}, \quad \hat{z} = \begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix},
\]

(4.9)

where \( M(\tau) \) is a symplectic matrix,

\[
M^T \Omega M = \Omega, \quad \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]

(4.10)

and the superscript \( T \) denotes matrix transposition. This condition on \( M \) grants the preservation of the commutation relations, or, equivalently, the Poisson brackets in the classical domain.

For our Kerr evolution this approximation is valid when we consider not too quantum and intense enough fields, so that \( \bar{n} \gg 1 \) and \( \Delta n/\bar{n} \ll 1 \). This allows us to approximate \( \hat{n}^2 \) by

\[
\hat{n}^2 \approx \bar{n}^2 + 2\bar{n}(\hat{n} - \bar{n}) + \ldots \approx 2\bar{n}\hat{n} - \bar{n}^2,
\]

(4.11)

where \( \bar{n} \) is the mean number of photons of the initial state, that is a constant of the motion of the Kerr evolution. So, for our purposes we have

\[
U(\tau) \approx e^{-2i\chi \bar{n} \tau \hat{n}},
\]

(4.12)

and the associated symplectic matrix is

\[
M(\tau) = \begin{pmatrix} \cos(\Omega \tau) & \sin(\Omega \tau) \\ -\sin(\Omega \tau) & \cos(\Omega \tau) \end{pmatrix},
\]

(4.13)

where

\[
\Omega = 2\chi \bar{n}.
\]

(4.14)

Note that Eqs. (4.9), (4.13), and (4.14) are fully equivalent to the \( \Omega \) in the classical evolution \[2.4\] after Eq. \[2.3\].

D. Observed evolution via measurement on classical and nonclassical states

Next we analyze the effect of the measurements. The basic transition probability \( p(\psi | \psi) \) can be computed using the Wigner-function representation \[16, 21\]. This is because of two key properties of Wigner functions: i) Under linear transformations Wigner functions transform as a classical probability distribution by a simple transformation of arguments, say that for the transformation \( 4.9 \) we have

\[
W(z; t) = W[M^{-1}(\tau) z; 0].
\]

(4.15)

This includes as well displacements \( D(z) \) so the Wigner functions of the vectors \( |z_{j+1}\rangle \) and \( U(\tau) |z_j\rangle \) in Eq. (4.6) are, respectively,

\[
W(z - z_{j+1}), \quad W[M^{-1}(\tau) z - z_j],
\]

(4.16)

where \( W(z) \) is the Wigner function of the state \( |\psi\rangle \). ii) The scalar product of vectors can be computed by the overlap of their Wigner functions, i. e.,

\[
|\langle \varphi | \psi \rangle|^2 = 2\pi \int dz W_\varphi(z) W_\psi(z).
\]

(4.17)

Therefore the transition probability (4.6) can be expressed as

\[
p(z_{j+1} | z_j) = \int dz W(z - z_{j+1}) W[M^{-1}(\tau) z - z_j],
\]

(4.18)

so we are ready to compute \( p(z | z_0) \) via the chain in Eq. (4.8). To this end we prove by induction the following theorem:

\[
p_N(z | z_0) = W_N[M^{-N}(\tau) z - z_0],
\]

(4.19)

where \( W_N(z) \) are functions to be determined. The key point of the theorem is that the complete dependence on the initial \( z_0 \) and final \( z \) phase-space-like coordinates is exclusively of the form \( M^{-N}(\tau) z - z_0 \), which is precisely the classical evolution \( 4.15 \) as far as

\[
M^N(\tau) = M(N\tau) = M(t),
\]

(4.20)

which is specially clear regarding the Kerr case in Eq. \[2.4\]. The difference with the classical case is the change of the functional form \( W_N(z) \) with the number of measurements \( N \).

So we begin with the first link in the chain (4.8), this is (4.6),

\[
p(z_1 | z_0) = \int dz W(z - z_1) W(M^{-1} z - z_0),
\]

(4.21)
and for simplicity we skip the dependence on \( \tau \). We perform the unit-Jacobian change of variables \( z' = z - z_1 \) to get

\[
p(z_1|z_0) = \int dz' W(z') W \left( M^{-1}z_1 - z_0 + M^{-1}z' \right),
\]

which proves that \( p(z_1|z_0) \) depends on \( z_1 \) and \( z_0 \) just on the form \( M^{-1}z_1 - z_0 \). For definiteness let us define the function

\[
W_1(z) = \int dz' W(z') W \left( z + M^{-1}z' \right)
\]

so that

\[
p(z_1|z_0) = W_1 \left( M^{-1}z_1 - z_0 \right).
\]

To proceed via induction now we assume that after \( j \) measurements \( p_j(z_j|z_0) \) is of the form

\[
p_j(z_j|z_0) = W_j(M^{-j}z_j - z_0),
\]

and we have to demonstrate that \( p_{j+1}(z_{j+1}|z_0) \) fulfills the theorem. We begin with

\[
p_{j+1}(z_{j+1}|z_0) = \int dz_j p(z_j+1|z_j)p_j(z_j|z_0),
\]

so that

\[
p_{j+1}(z_{j+1}|z_0) = \int dz_j W_1(M^{-1}z_{j+1} - z_j) W_j(M^{-j}z_j - z_0),
\]

that after the unit-Jacobian change of variables

\[
z' = M^{-1}z_{j+1} - z_j, \quad z_j = M^{-1}z_{j+1} - z',
\]

becomes

\[
p_{j+1}(z_{j+1}|z_0) = \int dz' W_1(z') W_j \left[ M^{-j} \left( M^{-1}z_{j+1} - z' \right) - z_0 \right],
\]

which clearly shows that \( p_{j+1}(z_{j+1}|z_0) \) depends on \( z_{j+1} \) and \( z_0 \) just on the form \( M^{-j-1}z_{j+1} - z_0 \)

\[
p_{j+1}(z_{j+1}|z_0) = W_{j+1} \left( M^{-j-1}z_{j+1} - z_0 \right).
\]

This satisfies the theorem simply defining

\[
W_{j+1}(z) = \int dz' W_1(z') W_j \left( z - M^{-j}z' \right).
\]

This completes the proof.

This includes classical as well as nonclassical measurements since the states \( |z\rangle \) can represent classical-like coherent states, as well as clearly nonclassical states such as arbitrary squeezed states, or even displaced number states or many other sophisticated nonclassical states.

The differences regarding particular choices of \( \{|\psi\rangle\} \) means a different structure for the fluctuations around the classical trajectory represented by \( W_N(z) \). This is the factor that includes the nonclassical features of the measurement, as far as such fluctuations are of quantum origin.

### E. Gaussian states

Next we may proceed computing explicitly the function \( W_N(z) \) in the case of squeezed coherent states, where we can take advantage of the fact that the Wigner function of \( |\psi\rangle \) is a Gaussian, which by hypothesis is centered at the origin, say

\[
W(z) = \frac{1}{2\pi\sqrt{\det C}} e^{-z^T C^{-1}z/2}, \quad C = \frac{1}{2} \begin{pmatrix} e^{2r} & 0 \\ 0 & e^{-2r} \end{pmatrix},
\]

where \( r \) is a compression parameter. After the convolutions in Eqs. (4.23) and (4.31) it is clear that all \( W_j(z) \) are Gaussians centered at the origin, say

\[
W_j(z) = \frac{1}{2\pi\sqrt{\det C_j}} e^{-z^T C_j^{-1}z/2},
\]

where \( C_j \) is the corresponding covariance matrix

\[
C_j = \int dz \ z z^T W_j(z).
\]

Let us derive a recursive relation for \( C_j \). Starting from Eq. (4.31), and after a tricky change of variables of the form \( z' = z - M^{-j}z' \) we get in few steps to

\[
C_{j+1} = C_j + M^{-j}C_1 (M^{-j})^T,
\]

that leads to

\[
C_N = \sum_{j=0}^{N-1} M^{-j}C_1 (M^{-j})^T.
\]

To simplify calculus we may take into account that we are interested in the limit \( \tau \to 0 \) so that the above sum might be approximated by an integral

\[
C_N = \sum_{j=0}^{N-1} M^{-j}C_1 (M^{-j})^T \approx \frac{1}{\tau} \int_0^\tau dt' M(-t')C_1M^T(-t'),
\]

where we have used that \( M^{-1}(t) = M(-t) \). Since \( t = N\tau \), we have \( C_N \propto N \) and uncertainty around the classical trajectory grows with the number of measurements.

It might be interesting to examine the properties of the noise added by the observation. This may be estimated from \( C_N \) via \( \det C_N \) which is the Robertson–Schrödinger form of phase-space uncertainty relations, being \( \det C_N = 1/4 \) the quantum limit satisfied equally by coherent and squeezed states as minimum uncertainty states.

In the case of a classical-like measurement given by projection on Glauber coherent states, this is \( r = 0 \) in
produce a transformation \( \hat{V} \) in the joint system and auxiliary variables of the form:

\[
\hat{V}\phi_s|\phi_a\rangle = e^{-i(aP^1+a^\dagger P^1)}|\phi_s\rangle|\phi_a\rangle,
\]

with\( \langle \phi_s|\phi_a\rangle \) and \( |\phi_a\rangle \) are the pre-measurement states of the system and auxiliary variables. We assume that \( [P,P^\dagger] = 0 \) so we can use the disentangling formula of Baker, Campbell and Hausdorff:

\[
e^{-i(aP^1+a^\dagger P)} = e^{-iaP^1}e^{-ia^\dagger P}e^{P^1P^1/2},
\]

and then

\[
\hat{V}|\phi_s\rangle|\phi_a\rangle = \frac{1}{2\pi} \int dz e^{-iaP^1}|z\rangle e^{-ia^\dagger P|\phi_s\rangle e^{P^1P^1/2}|\phi_a\rangle}.
\]

We suitably insert the resolution of the identity (5.5)

\[
\hat{V}|\phi_s\rangle|\phi_a\rangle = \frac{1}{2\pi} \int dz\langle z|\phi_s\rangle e^{-i(aP^1+a^\dagger P)}e^{P^1P^1/2}|\phi_a\rangle,
\]

and then we use that \( a|z\rangle = \alpha|z\rangle \) and its Hermitian conjugate to get after rearranging factors

\[
\hat{V}|\phi_s\rangle|\phi_a\rangle = \frac{1}{2\pi} \int dze^{-i\alpha P^1}|z\rangle e^{-i\alpha^\dagger P}|\phi_s\rangle e^{P^1P^1/2}|\phi_a\rangle,
\]

that can be also expressed as

\[
\hat{V}|\phi_s\rangle|\phi_a\rangle = \frac{1}{2\pi} \int dze^{-i\alpha P^1|z\rangle e^{-i\alpha^\dagger P}|\phi_s\rangle e^{P^1P^1/2}|\phi_a\rangle}.
\]
being
\[ |\phi_a(z)\rangle = e^{-i(\alpha P^1 + \alpha^* P)} e^{P^1 P/2} |\phi_a\rangle. \] (5.9)

If the measurement is good then the states \(|\phi_a(z)\rangle\) will be clearly distinguishable for different values of \(z\) so that a suitable measurement on the auxiliary degrees of freedom will reveal a particular \(z\) so the system state will be projected to the corresponding \(|z\rangle\) with probability proportional to \(|\langle z|\phi_a\rangle|^2\). So our proposal works and we just finish by presenting the measurement to be performed on the auxiliary space.

For example let us decompose \(P\) as \(P = (P_x + iP_y)/\sqrt{2}\) being \(P_{x,y}\) two independent linear momentum operators acting on independent Hilbert spaces with the associated position operators \(X, Y\) such that \([X, P_x] = [Y, P_y] = i\) and \([P_x, P_y] = [X, Y] = [X, P_y] = [Y, P_x] = 0\), being \(|x, y\rangle\), \(|p_x, p_y\rangle\) the corresponding eigenstates with
\[ \langle x, y|p_x, p_y\rangle = \frac{1}{2\pi} e^{i(p_x x + p_y y)}. \] (5.10)

Since linear momentum generates position translations the proper measurement in the auxiliary space is the measurement of the pair \(X, Y\). So let us compute \(|x, y|\phi_a(z)\rangle\) to find the reduced state in the system state and its probability. Besides \(\alpha = (q + ip)/\sqrt{2}\) we will use that
\[ \alpha P^1 + \alpha^* P = qP_x + pP_y, \quad P^1 P = \frac{1}{2} \left( P_x^2 + P_y^2 \right). \] (5.11)

In \(|x, y|\phi_a(z)\rangle\) we insert a suitable resolution of the identity in momentum representation
\[ \langle x, y|\phi_a(z)\rangle = \int dp_x dp_y \langle x, y|p_x, p_y\rangle e^{-i(qp_x + pq_y)} \times e^{(p_x^2 + p_y^2)/4} (p_x, p_y|\phi_a). \] (5.12)

Finally, as auxiliary state \(|\phi_a\rangle\) let us consider a Gaussian state with the following wave function in momentum representation
\[ (p_x, p_y|\phi_a) = \frac{1}{\sqrt{2\pi}} e^{-(p_x^2 + p_y^2)/4}, \] (5.13)

that after a little algebra leads to
\[ \langle x, y|\phi_a(z)\rangle = \sqrt{2\pi} \delta(q - x) \delta(p - y). \] (5.14)

Therefore, the reduced state in the system variables conditioned to the outcome \(x, y\) in the measurement of \(X\) and \(Y\) on the auxiliary system is
\[ \langle x, y|\tilde{V}|\phi_a\rangle|\phi_a\rangle = \frac{1}{\sqrt{2\pi}} \langle z|\phi_a\rangle|z\rangle, \quad z = \begin{pmatrix} x \\ y \end{pmatrix}. \] (5.15)

VI. CONTEXTUAL ZENO EFFECT

The situation where a dynamics is frequently interrupted to detect whether the state remains in the initial state typically leads to the Zeno effect \([22, 23]\). Our monitoring of the evolution falls within the category since each measurement checks whether the evolved state is one member of the family of states \(|z\rangle\). Since the initial state \(|z_0\rangle\) belongs to this family we are actually checking whether the evolved state continues in the initial state \(|z_0\rangle\). This is the usual scenario that leads to Zeno effect in the form of a complete stop of the evolution freezing the system in the initial state \(|z_0\rangle\). But this does not occur in our case, the evolution is no stopped. We think this may deserve a brief analysis.

The key point is the nature of the family of states where the measurement projects. For example, let us consider a dichotomic measurement with just two projectors
\[ \Delta(0) = |z_0\rangle\langle z_0|, \quad \Delta(-0) = I - |z_0\rangle\langle z_0|, \] (6.1)

where \(|z_0\rangle\) is the initial state. The standard Zeno analysis leads to a survival probability \(P_0\) that tends to one as measurement tends to be more frequent, i.e.,
\[ P_0 \gtrsim e^{-\Delta^2 \hat{n}^2 (\chi^2)/N}, \] (6.2)
so \(P_0 \to 1\) as \(N \to \infty\), where the right-hand side is computed assuming that all measurement results confirm that the system is in state \(|z_0\rangle\).

The situation is completely different if the projection on the initial state \(|z_0\rangle\) is embedded on a continuous family of nonorthogonal projectors, such as in the measurement we are considering in this work, this is
\[ \Delta(z) = \frac{1}{2\pi} |z\rangle\langle z|, \] (6.3)
and naturally a key point is the factor \(1/(2\pi)\). Let us compute in this case the survival probability in the best possible scenario \(\Omega = 2m\pi\) for integer \(m\) so that \(M^{-N} z_0 - z_0 = 0\). After Eqs. (4.19), (4.33) and (4.38) for a measurement projecting on coherent states
\[ P_0 = p_N(z = z_0|z_0) = \frac{1}{2\pi \sqrt{\text{det}\ C_N}} \propto \frac{1}{N}, \] (6.4)
so that \(P_0 \to 0\) as \(N \to \infty\).

During an infinitesimal evolution the state moves from \(|z_0\rangle\) to an infinitesimally close state \(U(\tau)|z_0\rangle\), and it turns out that within the family \(|z\rangle\) there may be a neighbour state \(|z\rangle\) different from \(|z_0\rangle\) which turns out to be closer to the infinitesimally evolved state \(U(\tau)|z_0\rangle\) than \(|z_0\rangle\). This never happens if the measurement states are not so close enough, say as in Eq. (6.1). This to say that the Zeno effect is very sensible to the way in which the projection on the original state \(|z_0\rangle\) is embedded. In this way we may say that the Zeno effect is contextual.
A. Overlapping kills the Zeno effect

Let us try to elucidate further the reasons explaining the lack of Zeno effect. Regarding the measurement basis \( |z \rangle \) there are multiple characteristics that might contribute, such as continuity, overcompleteness, and overlapping between different \( |z \rangle \). We can present an extremely simple scenario with discrete outcomes and without overcompleteness, where the only explanation for the lack of Zeno effect is the overlap between POVMs elements.

Let us consider a two-dimensional space spanned by two orthogonal states \( |1 \rangle, |2 \rangle \) and the following POVM, in such basis

\[
\Delta_1 = \cos^2 \alpha \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Delta_2 = \begin{pmatrix} \sin^2 \alpha & 0 \\ 0 & 1 \end{pmatrix},
\]

assuming that the outcome-dependent reduced states after the measurement are the corresponding normalized states

\[
\rho_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \rho_2 = \frac{1}{1 + \sin^2 \alpha} \begin{pmatrix} \sin^2 \alpha & 0 \\ 0 & 1 \end{pmatrix},
\]

being \( \rho_1 \) always the initial state. Note that the POVM is discrete, with just two only outcomes, so there is discreteness and no overcompleteness, while there is a clear overlap depending on the free parameter \( \alpha \):

\[
\text{tr} (\Delta_1 \Delta_2) = \frac{1}{4} \sin^2(2\alpha).
\]

Let the evolution be governed by the Hamiltonian

\[
H = \hbar \omega \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

so that

\[
U(\tau) = e^{-iH\tau/\hbar} = \begin{pmatrix} \cos(\omega \tau) & -i \sin(\omega \tau) \\ -i \sin(\omega \tau) & \cos(\omega \tau) \end{pmatrix}.
\]

The fundamental conditional probability reads in this case

\[
p(j|k) = \text{tr} [\Delta_j U(\tau) \rho_k U^\dagger(\tau)],
\]

which can be suitably arranged in a \( 2 \times 2 \) matrix as

\[
T = \begin{pmatrix} p(1|1) & p(1|2) \\ p(2|1) & p(2|2) \end{pmatrix},
\]

with

\[
p(1|1) = \cos^2(\alpha) \cos^2(\omega \tau), \quad p(2|1) = \sin^2(\alpha) \cos^2(\omega \tau) + \sin^2(\omega \tau), \quad p(1|2) = \frac{\cos^2(\alpha) \cos(\omega \tau) \sin^4(\alpha) + \sin^4(\omega \tau)}{1 + \sin^2(\alpha)}, \\
p(2|2) = \frac{\cos^2(\omega \tau) [1 + \sin^4(\alpha) + 2 \sin^2(\alpha) \sin^2(\omega \tau)]}{1 + \sin^2(\alpha)}.
\]

Regarding the survival probability we are interested in quantum trajectories of the form

\[
p(1, j, k, \ldots, m, 1) = p(1|j)p(j|k) \cdots p(m|1).
\]

When we are not interested in the intermediate results the survival probability becomes

\[
p_0 = \sum_{j, k, \ldots, m = 1, 2} p(1|j)p(j|k) \cdots p(m|1).
\]

This is actually the matrix element \((T^N)_{1,1}\) of the matrix \( M^N \), where \( N \) is the number of measurements.

Let us first focus on the typical Zeno effect, in which all outcomes confirm that the state remains always at the initial state. In this case, after \( N \) measurements, this occurs with probability

\[
p(1, 1, \ldots, 1) = \cos^{2N}(\alpha) \cos^{2N}(\omega \tau),
\]

that in the limit \( N \gg 1 \) and small enough \( \tau = t/N \), where \( t \) is the total evolution time, becomes

\[
p(1, 1, \ldots, 1) \rightarrow \cos^{2N} \alpha e^{-\omega^2 t^2/N}.
\]

This clearly tends to zero as \( N \) increases for the overlapping case \( \sin(2\alpha) \neq 0 \).

On the other hand, when we are not interested in the intermediate results, we have that, after some little algebra, the survival probability (6.14) becomes

\[
p_0 = \frac{\cos^2 \alpha}{2} + \left( 1 - \frac{\cos^2 \alpha}{2} \right) \left[ \frac{\cos^2 \alpha \cos(2\omega \tau)}{2 - \cos^2 \alpha} \right]^N.
\]

When there is overlap, i. e., \( \sin(2\alpha) \neq 0 \), we get lack of Zeno effect, as far as for \( N \rightarrow \infty \) only the first term survives:

\[
p_0 \rightarrow \frac{\cos^2 \alpha}{2},
\]

pointing to the overlap between POVM elements as the key feature inhibiting Zeno effect, the larger the overlap the lesser the survival probability. We also note that there is no classical-like evolution because the observation provides no enough density of states.

To further investigate the interplay between Zeno and lack of Zeno effect, we may consider the situation in which the overlap may depend on the number of measurements \( N \) through \( \alpha \) in such a way that \( \alpha \rightarrow 0 \) as \( N \rightarrow \infty \). When \( N \gg 1 \) and \( \alpha \ll 1 \) we get that (6.17) can be approximated as

\[
p_0 \simeq \frac{1}{2} \left( 1 + e^{-2N\alpha^2} e^{-2\omega^2 t^2/N} \right).
\]

So if \( \alpha \) tends to 0 faster that \( 1/\sqrt{N} \) Zeno effect occurs, while otherwise there is no Zeno effect.
VII. CONCLUSIONS

We have presented a simple model where the observation of quantum dynamics leads to a fully classical evolution. We think there are some interesting points worth to be followed. We have obtained the same classical trajectory for classical as well as for quantum measurements. Nevertheless in general there are differences between both classes of observation in the structure and the amount of the uncertainty around the classical trajectory. This is interesting as far as such noise is of pure quantum origin purely introduced by the observation.

Seemingly there are two basic features that might be further pursued. On the one hand the manifold of measurement outcomes is isomorphic to the phase space of the problem. On the other hand the possibility of approximating infinitesimal transformations by linear transformations. Both points can be strongly dependent on the basic variables and operators used to parametrize both the measurement and the phase space. So we cannot exclude that these results might be universal under a suitable choice of variables adapted to the problem at hand.

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