LOCAL SMOOTH REPRESENTATION OF SOLUTION SETS IN PARAMETRIC LINEAR FRACTIONAL PROGRAMMING PROBLEMS

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Abstract. The purpose of this paper is to investigate the structure of the solution sets in parametric linear fractional programming problems. It is shown that the solution set of a parametric linear fractional programming problem with smooth data has a local smooth representation. As a consequence, the corresponding marginal function is differentiable and the solution map admits a differentiable selection. We also give an example to illustrate the result.

1. Introduction. The study of the structure of the optimal solution sets occupies an important position in mathematical programming problems. It is clear that every polyhedron is a finitely generated convex set [11]. This result is applied to study the optimal solution sets in the linear programming problems. In general, the mathematical programming problems with some linear structure have nice properties. In this paper, we consider the linear fractional programming whose objective function is the ratio of two linear functions and constraint functions are linear [1]. The references [3, 4, 10, 12, 14, 15] show more details about the linear fractional programming problems. Analogous to the linear case, the linear fractional programming problem has a polyhedral optimal solution set [16, 17]. For more results about the structure of the optimal solution sets in mathematical programming problems, we refer the readers to the references and therein [6, 13, 18, 19].

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Sensitivity analysis [2, 7] involves determining how much changes in the parametric optimization problems influence the optimal solution sets where the optimal value can be attained. By using the smooth representation technique for the parametric polyhedra, Luc [8], Luc and Dien [9] proved that a parametric linear programming problem with smooth data has a polyhedra optimal solution set which has a local smooth representation. Fang [5] further developed the smooth representation technique for parametric semiclosed polyhedra and applied it to prove that the solution set of a smooth parametric piecewise linear programming problem can be locally represented as a finite union of parametric semiclosed polyhedra generated by finite smooth functions. Following this line, in this paper, we further investigate the sensitivity analysis of the linear fractional programming problem. We shall prove that the solution set of a parametric linear fractional programming problem with smooth data has a local smooth representation. As a consequence, the corresponding marginal function is differentiable and the solution map admits a differentiable selection.

The paper is structured as follows. In Section 2, we recall some preliminary results of the linear fractional programming problem. In Section 3, we prove that the solution set of the parametric linear fractional programming problem with smooth data has a local smooth representation. And we give an example to illustrate our theorem.

2. Preliminaries. In this section, we recall some concepts and results of the linear fractional programming problems.

Definition 2.1. [1] The linear fractional programming problem (LFP) is formulated as follows:

$$\max f(x) := \frac{p^T x + \alpha}{q^T x + \beta},$$

s.t. $Ax \leq b$, $x \geq 0$,

where $p, q \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$. $A$ is an $m \times n$ matrix and $b \in \mathbb{R}^m$.

The feasible region of the LFP is denoted by $L$, that is,

$L := \{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\}$.

Throughout this paper, we always suppose that $L \neq \emptyset$ and $q^T x + \beta > 0$, $\forall x \in L$.

For further discussion, it is necessary to recall the following two lemmas on the LFP problem which play crucial roles in proving the main result.

Lemma 2.2. ([10]) For the LFP problem, if $f(x)$ has a finite maximum on the set $L$, the optimum is achieved on at least one vertex of the polyhedron $L$.

Lemma 2.3. ([16, 17]) For the LFP problem, assume that the optimal set $S \neq \emptyset$. Then there exist vertices and directions of the set $L$, $\{v_i : i \in I\}$ and $\{d_j : j \in J\}$, where $I, J$ are finite index sets, such that

$$S = \left\{ x \in \mathbb{R}^n : x = \sum_{i \in I} \lambda_i v_i + \sum_{j \in J} \mu_j d_j, \sum_{i \in I} \lambda_i = 1, \lambda_i \geq 0, \forall i \in I, \mu_j \geq 0, \forall j \in J \right\}.$$
Definition 2.4. The parametric linear fractional programming problem (LFP(Ω)) is formulated as follows:

\[
\max f(x, \omega) = \frac{p^T(\omega)x + \alpha(\omega)}{q^T(\omega)x + \beta(\omega)},
\]

s.t. \(A(\omega)x \leq b(\omega), \quad x \geq 0,\)

where \(p(\omega), q(\omega) \in C^r(\Omega, \mathbb{R}), \alpha(\omega), \beta(\omega) \in C^r(\Omega, \mathbb{R}), A(\omega) \in C^r(\Omega, \mathbb{R}^{n \times n}), b(\omega) \in C^r(\Omega, \mathbb{R}^n)\) and \(r\) is a positive integer. In addition, a parametric space \(\Omega\) is an open subset of a finite dimensional space.

Analogously, the feasible region of the LFP(\(\omega\)) is denoted by \(L(\omega)\), that is,

\[L(\omega) := \{x \in \mathbb{R}^n : A(\omega)x \leq b(\omega), \ x \geq 0\}.\]

We also suppose that \(L(\omega) \neq \emptyset, \ \forall \omega \in \Omega\) and \(q^T(\omega)x + \beta(\omega) > 0, \ \forall x \in L(\omega)\).

The marginal function of the LFP(\(\omega\)) is defined as

\[\varphi(\omega) := \max \{f(x, \omega) : x \in L(\omega)\},\]

and the solution map is defined as

\[S(\omega) := \{x \in L(\omega) : f(x, \omega) = \varphi(\omega)\}.\]

It is easy to calculate that

\[\nabla_x f(x, \omega) = \frac{(q^T(\omega)x + \beta(\omega))p(\omega) - (p^T(\omega)x + \alpha(\omega))q(\omega)}{(q^T(\omega)x + \beta(\omega))^2},\]

\[= \frac{p(\omega) - f(x, \omega)q(\omega)}{q^T(\omega)x + \beta(\omega)}.\]

The following lemma shows that the existence of local smooth representing vectors for the polyhedral convex set in \(\mathbb{R}^n\), thereby showing that \(L(\omega)\) admits a local smooth parametrization as well as a local smooth selection.

Lemma 2.5. (Proposition 2.1 of \([8]\)) For every open set \(U \subset \Omega\), there exists an open subset \(U_0 \subset U\) such that either \(L(\omega) = \emptyset, \ \forall \omega \in U_0\), or there exist two finite families \(\{v_i(\omega) : i \in I\}\), \(\{d_j(\omega) : j \in J\}\) in \(C^r(\Omega, \mathbb{R}^n)\) such that

\[L(\omega) = \left\{x \in \mathbb{R}^n : x = \sum_{i \in I} \lambda_i v_i(\omega) + \sum_{j \in J} \mu_j d_j(\omega), \ \sum_{i \in I} \lambda_i = 1, \lambda_i \geq 0, \ \forall i \in I, \mu_j \geq 0, \ \forall j \in J, \right\},\]

for every \(\omega \in U_0\).

3. Main Result. Now, we prove that the solution set of the parametric linear fractional programming problem with smooth data has a local smooth representation.

Theorem 3.1. Assume that \(f(x, \omega)\) has a finite maximum \(\forall \omega \in V\), where \(V\) be an open subset of \(\Omega\). Then there exists an open subset \(V_0 \subset V\) and two finite families
of \( C^r(V_0, \mathbb{R}^n) \), \( \{v_i(\omega) : i \in P\} \) and \( \{d_j(\omega) : j \in Q\} \) such that

\[
S(\omega) = \left\{ x \in \mathbb{R}^n : x = \sum_{i \in P} \lambda_i v_i(\omega) + \sum_{j \in Q} \mu_j d_j(\omega), \quad \sum_{i \in P} \lambda_i = 1, \lambda_i \geq 0, \forall i \in P, \mu_j \geq 0, \forall j \in Q \right\},
\]

\[
f(v_i(\omega), \omega) = \varphi(\omega), \forall i \in P,
\]

\[
\langle p(\omega) - \varphi(\omega)q(\omega), d_j(\omega) \rangle = 0, \forall j \in Q,
\]

for every \( \omega \in V_0 \). In particular, \( \varphi \) is of class \( C^r(V_0, \mathbb{R}^n) \).

**Proof.** By Lemma 2.5 and \( L(\omega) \neq \emptyset \), there exist an open set \( V_1 \subset V \) and two finite families \( v_i(\omega), d_j(\omega) \in C^r(V_1, \mathbb{R}^n) \) such that

\[
L(\omega) = \left\{ x \in \mathbb{R}^n : x = \sum_{i \in I} \lambda_i v_i(\omega) + \sum_{j \in J} \mu_j d_j(\omega), \quad \sum_{i \in I} \lambda_i = 1, \lambda_i \geq 0, \forall i \in I, \mu_j \geq 0, \forall j \in J \right\}.
\]

Since \( \varphi(\omega) < +\infty \), we have \( \langle p(\omega) - \varphi(\omega)q(\omega), d_j(\omega) \rangle \leq 0, \forall \omega \in V_1, \forall j \in J \). From Lemma 2.2, we have \( S(\omega) \neq \emptyset \).

Now, define

\[
P(\omega) := \left\{ i \in I : \varphi(\omega) = f(v_i(\omega), \omega) \right\}, \forall \omega \in V_1,
\]

\[
Q(\omega) := \left\{ j \in J : \langle p(\omega) - \varphi(\omega)q(\omega), d_j(\omega) \rangle = 0 \right\}, \forall \omega \in V_1.
\]

By Lemma 2.2, we can know \( P(\omega) \neq \emptyset \). It follows that

\[
\varphi(\omega)(q^T(\omega)v_i(\omega) + \beta(\omega)) = p^T(\omega)v_i(\omega) + \alpha(\omega), \forall i \in P(\omega),
\]

\[
\varphi(\omega)q^T(\omega)d_j(\omega) = p^T(\omega)d_j(\omega), \forall j \in Q(\omega),
\]

and

\[
\varphi(\omega) = f(v_i(\omega), \omega) > f(v_i(\omega), \omega), \forall i \in P(\omega), \forall i' \in I \setminus P(\omega),
\]

\[
0 = \langle p(\omega) - \varphi(\omega)q(\omega), d_j(\omega) \rangle
\]

\[
> \langle p(\omega) - \varphi(\omega)q(\omega), d_j(\omega) \rangle, \forall j \in Q(\omega), \forall j' \in J \setminus Q(\omega).
\]

Therefore, we have

\[
\varphi(\omega)(q^T(\omega)v_i(\omega) + \beta(\omega)) > p^T(\omega)v_i(\omega) + \alpha(\omega), \forall i' \in I \setminus P(\omega),
\]

\[
\varphi(\omega)q^T(\omega)d_j(\omega) > p^T(\omega)d_j(\omega), \forall j' \in J \setminus Q(\omega).
\]

Choose \( \bar{\omega} \in V_1 \) such that \( P(\bar{\omega}) := \min\{|P(\omega)| : \omega \in V_1\} \). Now, we prove that \( P(\bar{\omega}) = P(\omega) \), for any \( \omega \) in an open neighborhood of \( \bar{\omega} \). Since \( f(x, \omega) \) is continuous and \( v_i(\omega) \in C^r(V_1, \mathbb{R}^n) \), there exists a sufficiently small open neighborhood \( V_2 \) of \( \bar{\omega} \) such that for all \( \omega \in V_2 \),

\[
f(v_i(\omega), \omega) > f(v_i(\omega), \omega), \forall i \in P(\omega), \forall i' \in I \setminus P(\bar{\omega}).
\]

By the definition of \( \varphi(\omega) \), we have

\[
\varphi(\omega) \geq f(v_i(\omega), \omega), \forall i \in P(\bar{\omega}).
\]
Thus,
\[ \varphi(\omega) > f(v_i(\omega), \omega), \forall i \in I \setminus P(\bar{\omega}). \]

We get \( I \setminus P(\bar{\omega}) \subset I \setminus P(\omega) \), that is, \( P(\omega) \subset P(\bar{\omega}) \). By the definition of \( P(\bar{\omega}) \), \( P(\omega) = P(\bar{\omega}), \forall \omega \in V_2 \). As a consequence, \( \varphi(\omega) \) is continuous on \( V_2 \).

Next, we choose \( \bar{\omega} \in V_2 \) such that \( Q(\bar{\omega}) := \min \{ |Q(\omega)| : \omega \in V_2 \} \). By the continuity of \( \varphi(\omega) \) for any \( \omega \in V_2 \) and \( d_j(\omega) \in C^r(V_1, \mathbb{R}^n) \), there exists a sufficiently small open neighborhood \( V_0 \subset V_2 \) of \( \bar{\omega} \) such that for all \( \omega \in V_0 \),
\[ \langle p(\omega) - \varphi(\omega)q(\omega), d_j(\omega) \rangle > \langle p(\omega) - \varphi(\omega)q(\omega), d_j(\omega) \rangle, \forall j \in Q(\bar{\omega}), \forall j' \in J \setminus Q(\bar{\omega}). \]
Since \( \varphi(\omega) < +\infty \), we have \( \langle p(\omega) - \varphi(\omega)q(\omega), d_j(\omega) \rangle \leq 0 \). It follows that
\[ 0 > \langle p(\omega) - \varphi(\omega)q(\omega), d_j(\omega) \rangle, \forall j' \in J \setminus Q(\bar{\omega}). \]

Thus, \( Q(\omega) \subset Q(\bar{\omega}) \) for all \( \omega \in V_0 \). By the definition of \( Q(\bar{\omega}) \), we get \( Q(\omega) = Q(\bar{\omega}) \), \( \forall \omega \in V_0 \).

Let \( P := P(\bar{\omega}) \), and \( Q := Q(\bar{\omega}) \). For every \( \omega \in V_0 \), we have
\[ \varphi(\omega) = f(v_i(\omega), \omega), \forall i \in P, \]
\[ \langle p(\omega) - \varphi(\omega)q(\omega), d_j(\omega) \rangle = 0, \forall j \in Q. \]

As a consequence, \( \varphi \) is of class \( C^r \) on \( V_0 \).

Now, we prove
\[
S(\omega) = \left\{ x \in \mathbb{R}^n : x = \sum_{i \in P} \lambda_i v_i(\omega) + \sum_{j \in Q} \mu_j d_j(\omega), \right. \\
\left. \sum_{i \in P} \lambda_i = 1, \lambda_i \geq 0, \forall i \in P, \mu_j \geq 0, \forall j \in Q \right\}, \forall \omega \in V_0. \tag{5}
\]

Let \( \omega \in V_0 \) and
\[ h(\omega) := \sum_{i \in P} \lambda_i v_i(\omega) + \sum_{j \in Q} \mu_j d_j(\omega) \]
with
\[ \sum_{i \in P} \lambda_i = 1, \lambda_i \geq 0, \forall i \in P, \mu_j \geq 0, \forall j \in Q. \]

Define
\[ \lambda_i := 0, \forall i \in I \setminus P \text{ and } \mu_j := 0, \forall j \in J \setminus Q, \]
then
\[ h(\omega) = \sum_{i \in I} \lambda_i v_i(\omega) + \sum_{j \in J} \mu_j d_j(\omega) \]
with
\[ \sum_{i \in I} \lambda_i = 1, \lambda_i \geq 0, \forall i \in I, \mu_j \geq 0, \forall j \in J. \]

Thus \( h(\omega) \in L(\omega) \). And now, we prove \( h(\omega) \in S(\omega) \).
\[
f(h(\omega), \omega) = \frac{p^T(\omega)h(\omega) + \alpha(\omega)}{q^T(\omega)h(\omega) + \beta(\omega)}
\]
By (1) and (2), we have

\[
\frac{p^T(\omega) \left( \sum_{i \in P} \lambda_i v_i(\omega) + \sum_{j \in Q} \mu_j d_j(\omega) \right) + \alpha(\omega)}{q^T(\omega) \left( \sum_{i \in P} \lambda_i v_i(\omega) + \sum_{j \in Q} \mu_j d_j(\omega) \right) + \beta(\omega)}
\]

Suppose \( \varphi_i : \omega \rightarrow \mathbb{R} \).

Conversely, let \( \omega \in V_0 \) and \( g(\omega) \in S(\omega) \). Since \( g(\omega) \) is \( S(\omega) \subseteq L(\omega) \), there exist \( \{\lambda_i : i \in I\} \) and \( \{\mu_j : j \in J\} \) such that

\[
g(\omega) = \sum_{i \in I} \lambda_i v_i(\omega) + \sum_{j \in J} \mu_j d_j(\omega)
\]

with

\[
\sum_{i \in I} \lambda_i = 1, \lambda_i \geq 0, \forall i \in I, \mu_j \geq 0, \forall j \in J.
\]

Now, we shall prove that

\[
\sum_{i \in I \setminus P} \lambda_i + \sum_{j \in J \setminus Q} \mu_j = 0.
\]

Suppose \( \sum_{i \in I \setminus P} \lambda_i + \sum_{j \in J \setminus Q} \mu_j > 0 \). From (1)-(4), and \( q^T(\omega)x + \beta(\omega) > 0 \), it follows that

\[
\varphi(\omega) = f(g(\omega), \omega)
\]

Therefore, \( h(\omega) \in S(\omega), \forall \omega \in V_0 \).

Conversely, let \( \omega \in V_0 \) and \( g(\omega) \in S(\omega) \). Since \( g(\omega) \in S(\omega) \subseteq L(\omega) \), there exist \( \{\lambda_i : i \in I\} \) and \( \{\mu_j : j \in J\} \) such that

\[
g(\omega) = \sum_{i \in I} \lambda_i v_i(\omega) + \sum_{j \in J} \mu_j d_j(\omega)
\]

with

\[
\sum_{i \in I} \lambda_i = 1, \lambda_i \geq 0, \forall i \in I, \mu_j \geq 0, \forall j \in J.
\]

Now, we shall prove that

\[
\sum_{i \in I \setminus P} \lambda_i + \sum_{j \in J \setminus Q} \mu_j = 0.
\]

Suppose \( \sum_{i \in I \setminus P} \lambda_i + \sum_{j \in J \setminus Q} \mu_j > 0 \). From (1)-(4), and \( q^T(\omega)x + \beta(\omega) > 0 \), it follows that

\[
\varphi(\omega) = f(g(\omega), \omega)
\]
Theorem 3.1 generalizes the Proposition 3.1 of Luc [8] which establishes local smooth representation of the solution set of the parametric linear programming problem with smooth data.

Now we illustrate Theorem 3.1 by an example. Let us consider the following parametric linear fractional programming problem.

Example 3.2.

\[ \max f(x_1, x_2, \omega) = \frac{x_1 + x_2 + 5 + \omega}{2x_1 + 3x_2 + 14 + 3\omega}, \]

\[ \text{s.t. } 3x_1 - x_2 \geq -3 + \omega, \]
\[ (2 + \omega)x_1 - x_2 \leq 2, \]
\[ x_1 \geq 0, \ x_2 \geq 0. \]

The vertices of the feasible region \( L(\omega) \) are \( v_1(\omega) = (\frac{2}{2+\omega}, 0), v_2(\omega) = (0, 3 - \omega), v_3(\omega) = (0, 0), \) and the extreme directions of the \( L(\omega) \) are \( d_1(\omega) = (\omega + 4, \omega^2 + 6\omega + 8), d_2(\omega) = (1, 3). \)

Thus, the polyhedral convex set \( L(\omega) \) of this problem can be represented as

\[ L(\omega) = \{ x \in \mathbb{R}^n : x = \lambda_1 v_1(\omega) + \lambda_2 v_2(\omega) + \lambda_3 v_3(\omega) + \mu_1 d_1(\omega) + \mu_2 d_2(\omega), \]
\[ \lambda_1 + \lambda_2 + \lambda_3 = 1, \lambda_i \geq 0, i = 1, 2, 3, \mu_j \geq 0, j = 1, 2 \}. \]

It is easy to verify that the optimal vertex is \( v_1(\omega) = (\frac{2}{2+\omega}, 0), \) and the optimal extreme direction is \( d_1(\omega) = (\omega + 4, \omega^2 + 6\omega + 8). \)
According to the Theorem 3.1, the optimal solution set of this problem can be represented as follows:

\[ S(\omega) = \{ x \in \mathbb{R}^n : x = v_1(\omega) + \mu d_1(\omega), \mu \geq 0 \} \].

4. Conclusion. Luc [8], Luc and Dien [9] proved that a parametric polyhedron with smooth data has a local smooth representation. As a consequence, they proposed the smooth representation technique to investigate the sensitivity of the linear programming problem by proving that the solution set of a parametric linear programming problem with smooth data has a local smooth representation. In this paper, we further apply the smooth representation technique to investigate the sensitivity analysis of the linear fractional programming problem. Our result generalizes the main result of Luc [8].

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