NORMAL MODES IN SYMPLECTIC STRATIFIED SPACES

EUGENE LERMAN

Abstract. We generalize the Weinstein-Moser theorem on the existence of nonlinear normal modes (i.e., periodic orbits) near an equilibrium in a Hamiltonian system to a theorem on the existence of relative periodic orbits near a relative equilibrium in a Hamiltonian system with continuous symmetries.

More specifically we significantly improve a result proved earlier jointly with Tokieda: we remove a strong technical hypothesis on the symmetry group.

1. Introduction

The goal of the paper is to generalize the Weinstein-Moser theorem ([W1, Mq, W2, MnRS, B] and references therein) on the nonlinear normal modes (i.e., periodic orbits) near an equilibrium in a Hamiltonian system to a theorem on the existence of relative periodic orbits near a symmetric Hamiltonian system.

More specifically let \((M, \omega_M)\) be a symplectic manifold with a proper Hamiltonian action of a Lie group \(G\) and a corresponding moment map \(\Phi : M \to \mathfrak{g}^*\). Assume that the moment map is equivariant. Let \(h \in C^\infty(M)^G\) be a \(G\)-invariant Hamiltonian. We will refer to the tuple \((M, \omega_M, \Phi : M \to \mathfrak{g}^*, h \in C^\infty(M)^G)\) as a symmetric Hamiltonian system.

The main result of the paper is the following theorem (the terms used in the statement are explained below):

**Theorem 1.** Let \((M, \omega_M, \Phi : M \to \mathfrak{g}^*, h \in C^\infty(M)^G)\) be a symmetric Hamiltonian system. Suppose \(m \in M\) is a relative equilibrium for the system such that the coadjoint orbit through \(\mu = \Phi(m)\) is locally closed. Suppose further that there exists a symplectic slice \(\Sigma \hookrightarrow M\) through \(m\) such that the Hessian \(d^2(h|\Sigma)(m)\) is positive definite.

Then for every sufficiently small \(E > 0\) the set \(\{h = E + h(m)\} \cap \Phi^{-1}(\mu)\) (if nonempty) contains a relative periodic orbit of \(h\).

More precisely, let \(h_\mu\) denotes the reduced Hamiltonian on the reduced space \(M_\mu = \Phi^{-1}(G \cdot \mu)/G\). Then for every sufficiently small \(E > 0\) the set \(\{h_\mu = E + h(m)\}\) contains \(N\) or more periodic orbits of the reduced Hamiltonian \(h_\mu\), where \(N\) is the Ljusternik-Schnirelman category of the union of closed strata in the symplectic link of the point \(\bar{m} \in M_\mu\) corresponding to \(m \in M\).

Recall that given a symmetric Hamiltonian system \((M, \omega_M, \Phi : M \to \mathfrak{g}^*, h \in C^\infty(M)^G)\), a point \(m \in M\) is a relative equilibrium of the Hamiltonian vector field \(X_h\) of \(h\) (a relative equilibrium of \(h\) for short), if the trajectory of \(X_h\) through \(m\) lies on the \(G\) orbit through \(m\). Equivalently, since the flow of \(X_h\) is \(G\) equivariant, it descends to a flow on the quotient \(M/G\); \(m\) is a relative equilibrium if the corresponding point \(\bar{m} \in M/G\) is fixed by the induced flow. Thus if \(m\) is a relative equilibrium, then the whole \(G\) orbit \(G \cdot m\) consists of relative equilibria. Similarly, we say that a trajectory \(\gamma(t)\) a \(G\) invariant Hamiltonian vector field \(X_h\) is a relative periodic orbit...
(r.p.o.) if there is $T > 0$ and $g \in G$ such that $\gamma(T) = g \cdot \gamma(0)$. Equivalently $\gamma(t)$ is relatively periodic if the corresponding trajectory $\tilde{\gamma}(t)$ in $M/G$ is periodic.

Recall next that if a Lie group $G$ acts properly on a manifold $M$, then at every point $x \in M$ there is a slice for the action of $G$, i.e., there is a submanifold $S$ passing through $x$, which is invariant under the action of the isotropy group $G_x$ of $x$, is transverse to the orbit $G \cdot x$ and such that the open set $G \cdot S$ is diffeomorphic to the associated bundle $G \times_{G_x} S$.

If additionally the manifold $M$ has a $G$ invariant symplectic form $\omega_M$, then the local normal form theorem of Marle and of Guillemin and Sternberg \cite{Ma, GS} guarantees that we can find a symplectic submanifold $\Sigma$ passing through $x$ which is invariant with the property that the tangent space $T_x \Sigma$ is a maximal symplectic subspace of the tangent space to the slice $T_x S$. Moreover, $\Sigma$ can be chosen to be $G_x$ equivariantly symplectomorphic to a ball about the origin in a linear symplectic representation of $G_x$ on $T_x \Sigma$. Such a submanifold is called a symplectic slice.

The symplectic link of a point $\bar{m}$ in a reduced space $M_\mu$ is a symplectic stratified space which is an invariant of a singularity of $M_\mu$ at $\bar{m}$. A precise definition is given later. If the space $M_\mu$ is smooth near $\bar{m}$ then the symplectic link of $\bar{m}$ is smooth: it is the complex projective space $\mathbb{CP}^{n-1}$ where $n = \frac{1}{2} \dim M_\mu$.

If the isotropy group of the point $m$ is trivial, then the reduced space $M_\mu$ is smooth near the corresponding point $\bar{m}$. Moreover the symplectic slice $\Sigma$ through $m$ is symplectomorphic to a neighborhood of $\bar{m}$ in $M_\mu$, and under the identification of $\Sigma$ with an open subset of the reduced space the restriction $h|_\Sigma$ is the reduced Hamiltonian $h_\mu$. In this case Theorem 1 reduces to a theorem of Weinstein on the existence of nonlinear normal modes of a Hamiltonian system near an equilibrium:

**Theorem 2** (Weinstein, \cite{W1}). Let $h$ be a Hamiltonian on a symplectic vector space $V$ such that the differential of $h$ at the origin $dh(0)$ is zero and the Hessian at the origin $d^2 h(0)$ is positive definite. Then for every small $\varepsilon > 0$, the energy level $h^{-1}(h(0) + \varepsilon)$ carries at least $\frac{1}{2} \dim V$ periodic orbits of (the Hamiltonian vector field of) $h$.

On the other hand, if $m$ is a singular point of the moment map $\Phi$, then the reduced space $M_\mu$ at $\mu$ is a stratified space, and the reduced dynamics preserves the stratification \cite{AMM, SL, BL}. Unless the stratum through $\bar{m}$ is an isolated point, we have again $\frac{1}{2} \dim(\text{stratum})$ families of r.p.o.'s, provided appropriate conditions hold on the Hessian of the restriction of the reduced Hamiltonian to the stratum. But what if the stratum through $\bar{m}$ is a point?

Recently Tokieda and I showed that in the singular case as in the regular case there are relative periodic orbits near the relative equilibrium provided a certain quadratic form is definite and the isotropy group $G_\mu$ of $\mu$ is a torus \cite{LTk}. Here as above $\mu$ is the value of the moment map on the relative equilibrium. The proof amounted to a reduction to the case where the full symmetry group is a torus followed by computation in ‘good coordinates.’ The computation allowed us to reduce our problem to Weinstein’s nonlinear normal mode theorem.\footnote{The assumption that $G_\mu$ is a torus is not essential. For the proof to work it is sufficient to assume that $\mu$ is split in the sense of \cite{GLS} and that $G_\mu$ splits up to a finite cover: $G_\mu = (G_m \times H)/\Gamma$ where $G_m$ is the isotropy group of the relative equilibrium, $H$ a complementary subgroup and $\Gamma$ is a finite group.} This left open a question:

- Can the assumption on the isotropy group of the value of the moment map at the relative equilibrium be removed?

Theorem 1 answers the question affirmatively. The guiding principle comes from \cite{SL}: a symplectic quotient is locally modeled on a symplectic quotient of a symplectic slice (cf. Theorem 15 in \cite{BL}).
We end the paper with Proposition [1] which provides a practicable method for checking the existence of a symplectic slice $\Sigma$ through a relative equilibrium $m$ so that the Hessian $d^2(h|_{\Sigma})(m)$ is positive definite.

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As I was writing up this paper I discovered that Ortega and Ratiu have independently obtained a similar result [OR2].

### 2. Proof of Theorem 3

Our first step is to reduce the proof to a special case where the manifold $M$ is a symplectic vector space and the relative equilibrium is an equilibrium. Let $(M,\omega_M,\Phi : M \to \mathfrak{g}^*, h \in C^\infty(M)^G)$ be a symmetric Hamiltonian system and suppose $m \in M$ is a relative equilibrium for the system. Then the restriction $h_\Sigma$ of $h$ to the symplectic slice $\Sigma$ through $m$ satisfies $dh_\Sigma(m) = 0$. This is because the Hamiltonian vector field of $h$ points along the the group orbit $G \cdot m$ and the tangent space to the orbit $T_m(G \cdot m)$ lies in the symplectic perpendicular to the symplectic slice directions. Consequently the Hessian $d^2h_\Sigma(m) = d^2(h|_{\Sigma})(m)$ is well-defined.

Recall next that if the coadjoint orbit through $\mu = \Phi(m)$ is locally closed, then the reduced space $M_\mu := \Phi^{-1}(G \cdot \mu)/G$ is locally isomorphic, as a symplectic stratified space, to the reduction at zero of a symplectic slice $\Sigma$ through $m$ by the action of the isotropy group $G_m$; see, for example [BL, Theorem 15]. Moreover, it follows from the proof of Theorem 15, op. cit., that if $h$ is any $G$ invariant Hamiltonian on $M$ then the corresponding reduced Hamiltonian $h_\mu$ on $M_\mu$ near $\bar{m} = G \cdot m$ can also be obtained by first restricting $h$ to $\Sigma$ and then carrying out the reduction by the group $G_m$.

Since the symplectic slice is equivariantly symplectomorphic to a ball in a symplectic vector space with a linear symplectic action of a compact Lie group, it follows that in order to prove Theorem 3, it suffices to prove the following special case:

**Theorem 3.** Let $(V,\omega_V)$ be a symplectic vector space with a linear symplectic action of a compact Lie group $K$, and let $\Phi_V : V \to \mathfrak{k}^*$ denotes the corresponding homogeneous moment map. Assume that $\Phi_V^{-1}(0) \setminus \{0\}$ is nonempty, i.e., assume that the reduced space $V_0 = \Phi_V^{-1}(0)/K$ is not one point. Suppose $h_V \in C^\infty(V)^K$ is a $K$ invariant Hamiltonian with $dh_V(0) = 0$, and suppose the Hessian $d^2h_V(0)$ is positive definite. Then for every $E > h_V(0)$ sufficiently close to $h_V(0)$ the set $\{h_0 = E\}$ in the reduced space $V_0$ contains $N$ or more periodic orbit of the reduced Hamiltonian $h_0$. Here $N$ is the Liusternik-Schnirelman category of the union of closed strata in the symplectic link of the point $* \in V_0$ corresponding to $0 \in V$.

The idea of the proof of Theorem 3 is straightforward. Consider the quadratic part $q(x)$ of the Hamiltonian $h_V$ at zero, that is, let $q(x) = d^2h(0)(x,x)$. Let $q_0$ denote the corresponding reduced Hamiltonian on reduced space $V_0$. We will see that for every sufficiently small $E > 0$ and for certain strata $\mathcal{T}$ of $V_0$ the manifolds

$$\{q_0 = E\} \cap \mathcal{T}$$

contain weakly nondegenerate periodic manifolds $C \subset \{q_0 = E\} \cap \mathcal{T}$ of $q_0$. Then by a theorem of Weinstein [W2, p. 247], every such compact manifold $C$ gives rise to $\text{Cat}(C/S^1)$ periodic orbits of the reduced Hamiltonian $h_0$ in the stratum $\mathcal{T}$, where $\text{Cat}$ denotes the Liusternik-Schnirelman category.

Recall a characterization of weakly nondegenerate periodic manifolds given in a corollary on p. 246 of [W2], which we take as our definition.
Definition 4. Let \((N, \omega_N, h)\) be a Hamiltonian system. A submanifold \(C \subset N\) consisting of periodic orbits of the Hamiltonian vector field \(X_h\) of \(h\) is weakly nondegenerate iff for each orbit \(c\) in \(C\)

(i) \(X_h(c) \neq 0\)

(ii) the space \(\{ x \in T_{c(0)}(h^{-1}(E)) \mid x - Px \text{ is a multiple of } X_h(c(0)) \}\) has the same dimension as \(C\). Here \(E = h(c(0))\), and \(P : T_{c(0)}(h^{-1}(E)) \to T_{c(0)}(h^{-1}(E))\) denotes the linearization of the Poincaré map along the periodic orbit \(c\).

Thus to prove Theorem 3 it is enough to establish the existence of compact weakly nondegenerate periodic manifolds of the reduced Hessian \(q_0\) and to estimate the Liusternik-Schnirelman category of the quotients of these manifolds by \(S^1\).

We now proceed with a proof of Theorem 3. Since the function \(q\) is quadratic, its Hamiltonian vector field \(X_q\) is linear, hence of the form \(X_q(x) = \xi x\) for some linear map \(\xi \in \mathfrak{sp}(V, \omega)\), the Lie algebra of the symplectic group \(\text{Sp}(V, \omega)\). Since \(q\) is definite, \(\xi\) must lie in a compact Lie subalgebra of \(\mathfrak{sp}(V, \omega)\); in particular the closure of \(\{ \exp t\xi \mid t \in \mathbb{R} \}\) is a torus \(T \subset \text{Sp}(V, \omega)\). Since \(q\) is \(K\)-invariant, the groups \(K\) and \(T\) commute in \(\text{Sp}(V, \omega)\). Since both groups are compact, we may assume that \(V = \mathbb{C}^n\) (\(n = \frac{1}{2} \dim V\)), that \(K\) is a subgroup of \(U(n)\) and that \(T\) is contained in the standard maximal torus of \(U(n)\).

Lemma 5. Let \((V, \omega, K, \Phi : V \to \mathfrak{k}^*\), \(q(x) \in C^\infty(V)^K\)) be as above. Then the Hamiltonian \(q\) has no relative equilibria in the set \(\Phi^{-1}(0) \setminus \{0\}\). Consequently the action of the torus \(T\) generated by \(q\) on the reduced space \(V_0\) has no fixed points in \(V_0 \setminus \{0\}\).

Proof. If \(v \in \Phi^{-1}(0) \setminus \{0\}\) is a relative equilibrium then

\[
d(\Phi, \eta)(v) = dq(v)
\]

for some \(\eta \in \mathfrak{k}\). Since \(\Phi\) is quadratic homogeneous, we have \(tv \in \Phi^{-1}(0)\) for all \(t > 0\). Hence \(v \in \ker d(\Phi, \eta)(v)\). On the other hand, since \(q\) is definite, the ray \(\{tv \mid t > 0\}\) is transverse to the level set \(\{ q = q(v) \}\), hence \(v \not\in \ker dq(v)\). Contradiction.

The structure of the symplectic quotient \(V_0\). Next we tersely recall a number of results explained in [SL]. Suppose as above that \(T\) is a subtorus of the maximal torus of \(U(n)\) and that \(K\) is a closed subgroup of \(U(n)\) which commutes with \(T\). Let \(U\) denote the central circle subgroup of \(U(n)\). Then the actions of \(U\) and \(K\) on \(\mathbb{C}^n\) commute, the action of \(U\) is Hamiltonian and a moment map \(f\) for the action of \(U\) on \(\mathbb{C}^n\) can be taken to be \(f(z) = ||z||^2\).

Since by assumption \(\Phi^{-1}(0) \setminus \{0\} \neq \emptyset\), the group \(U\) is not contained in \(K\). In fact the Lie algebras of \(U\) and \(K\) intersect trivially. Let us prove this. If \(u \cap \mathfrak{k} \neq \emptyset\) then, since \(\dim u = 1\) we would have \(u \subset \mathfrak{k}\). But then \(||z||^2\) would be a component of the \(K\)-moment map \(\Phi\) and so \(\Phi^{-1}(0)\) would only contain zero.

Since \(\Phi\) is homogeneous, the level set \(\Phi^{-1}(0)\) is a cone on \(\Phi^{-1}(0) \cap S^{2n-1}\) where \(S^{2n-1}\) is the standard round sphere in \(\mathbb{C}^n\) of radius 1, \(S^{2n-1} = \{ z \mid ||z||^2 = 1\}\). Hence the reduced space \(V_0\) is a cone on the set \(L := (\Phi^{-1}(0) \cap S^{2n-1})/K\), i.e., \(V_0 = \check{c}(L) := (L \times [0, \infty))/\sim\) where \((x, 0) \sim (x', 0)\) for all \(x, x' \in L\). The vertex \(*\) of the cone corresponds to \(0 \in V\). Moreover, (see [SL, Corollary 6.12]) the set \(L\) is a stratified space; it is the link of the singularity of \(V_0\) at \(*\). The stratifications of \(L\) and of \(V_0\) are related: given a stratum \(S\) of \(L\), the set \(S \times (0, \infty) \subset \check{c}(L)\) is a stratum of \(V_0\).

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\(^2\)Calling \(L\) the link is slightly nonstandard. Strictly speaking we should call \(L\) the link only if the set \(*\) is a stratum; that is, if the set of fixed points \(V^K\) is only the origin.

\(^3\)There is one exception: if \(V^K \neq \{0\}\) then \((V^K \cap S^{2n-1})/K = V^K \cap S^{2n-1}\) is a stratum of \(L\), but \((V^K \cap S^{2n-1}) \times (0, \infty) = V^K \setminus \{0\}\), rather than \(V^K\) which is a stratum of \(V_0\). See previous footnote.
By [SL, Theorem 5.3] the action of the circle $U$ on $L$ is locally free and preserves the stratification of $L$. The quotient $L := L/U$ is again a stratified space. In fact, $L$ is a symplectic stratified space since it is a reduction of $\mathbb{C}^n$ by the action of $K \times U$. The space $\tilde{L}$ is called the **symplectic link** of $*$ in the symplectic stratified space $V_0$.

**Remark 6.** The symplectic link $L$ has two natural decompositions. There is a stratification of $L$ into manifolds as a symplectic stratified space. There is also a coarser decomposition: since the action of $U$ on $L$ is locally free and preserves the stratification of $L$, the quotients of the strata of $L$ form a decomposition of the symplectic link $L$ into symplectic orbifolds: $L = \bigsqcup_{S \subset L} \pi(S)$ where $S$ are strata of $L$ and $\pi : L \rightarrow L$ is the $U$-orbit map. We will use the coarser decomposition.

**Remark 7.** Note that since the Hamiltonian action of $T$ on $V$ commutes with the action of $U \times K$, it descends to a Hamiltonian action on the symplectic link $L$.

Finally recall a description of the symplectic structure on the strata of the reduced space $V_0$ [SL, Theorem 5.3]: For each stratum $S$ of $L$ there exists a connection one-form $A_S$ on the Seifert fibration $S \rightarrow S/U$ such that the curvature of $A_S$ is a symplectic form. The reduced symplectic form on $S \times (0, \infty)$ is $d(sA_S)$, where $s$ denotes the natural coordinate on $(0, \infty)$. There is no loss of generality in assuming that the connections $A_S$ are $T$-invariant.

We now study one (connected) stratum $P$ of the link $L$. Denote by $B$ the quotient of $P$ by the action of $U$: $B = P/U$. Then $U \rightarrow P \xrightarrow{\pi} B$ is a Seifert fiber bundle. Denote the connection one form on $P$ by $A$, so that the symplectic form on $P \times (0, \infty)$ is $d(sA)$. By assumption $B$ is a symplectic orbifold.

**Lemma 8.** Let $S^1 \rightarrow P \xrightarrow{\pi} B$ be a Seifert fibration over an even dimensional closed orbifold $B$. Suppose there exists a connection one-form $A$ on $P$ so that the form $d(sA)$ on $P \times (0, \infty)$ is symplectic ($s$ is the coordinate on $(0, \infty)$). Suppose further that a torus $T$ acts on $P$ without fixed points, and that the action of $T$ commutes with the action of $S^1$ and preserves the connection $A$. Then the action of $T$ on $(P \times (0, \infty), d(sA))$ is Hamiltonian. Let $F$ denotes a corresponding moment map.

Then for a generic vector $Y$ in the Lie algebra $\mathfrak{t}$ of $T$ and for any $s_0 \in (0, \infty)$ the set $\pi^{-1}(B^T) \times \{s_0\}$ consists of nondegenerate periodic manifolds of the Hamiltonian $H = \langle F, Y \rangle$ on the energy surfaces $\{H = E\}$ for appropriate $E$’s.

**Proof.** Since the action of $T$ preserves the one-form $sA$, the action of $T$ on $(P \times (0, \infty), d(sA))$ is Hamiltonian. The action of $S^1$ on $(P \times (0, \infty), d(sA))$ is also Hamiltonian: $f(p,s) = s$ is a corresponding moment map. Consequently $f^{-1}(1)/S^1$ is a symplectic orbifold diffeomorphic to $B$; from now on we identify $B$ and $f^{-1}(1)/S^1$.

The Hamiltonian action of $T$ on $(P \times (0, \infty), d(sA))$ descends to Hamiltonian action on $B$. Since $B$ is compact and the action of $T$ is Hamiltonian, the set of fixed points $B^T$ is nonempty. In fact $B^T$ is a disjoint union of connected symplectic suborbifolds of $B$ (see for example [LT] for more details on Hamiltonian group actions on orbifolds). For any point $x \in \pi^{-1}(B^T)$, the $T$ orbit $T \cdot x$ is contained in the $S^1$ orbit $S^1 \cdot x$. Since $T$ acts on $P$ without fixed points we in fact have that $T \cdot x = S^1 \cdot x$ for any $x \in \pi^{-1}(B^T)$. Consequently the union of manifolds $\pi^{-1}(B^T) \times (0, \infty)$ consists of periodic manifolds of the Hamiltonian $H$.

It remains to check that for a connected component $\Sigma$ of $\pi^{-1}(B^T)$, the manifold $(\Sigma \times (0, \infty)) \cap \{H = E\}$ is a nondegenerate periodic manifold of $H$. Now the time $t$ map of the flow of the Hamiltonian vector field of $H$ on $P \times (0, \infty)$ is given by $(p,s) \mapsto ((\exp tY) \cdot p, s)$ where $\exp : t \rightarrow T$ is the exponential map.
So let \((p, s)\) be a point in \((\Sigma \times (0, \infty)) \cap \{H = E\}\). Then \((p, s)\) is a relative \(S^1\) equilibrium of \(H\). Hence the differential \(dH\) at \((p, s)\) is proportional to the differential of the \(S^1\) moment map, which is \(ds\). Hence \(T_{(p,s)}\{H = E\} = T_p P\). Therefore it’s enough to compute the differential at \(p\) of the “Poincaré map” \(P \to P, q \mapsto \exp(\tau Y) \cdot q\), where \(\tau\) is the smallest positive number with \(\exp(\tau Y) \cdot p = p\).

Since the \(T\) orbit of \(p\) is a circle, the isotropy group of \(p\) is of the form \(\Gamma \times T_2\), where \(\Gamma\) is a finite abelian subgroup of \(T\) and \(T_2\) is a subtorus of \(T\) of codimension one. Moreover, we can split \(T\) as \(T = T_1 \times T_2\) where \(T_1\) is isomorphic to \(S^1\) and contains \(\Gamma\).

Let us next assume, to make the exposition simpler, that \(\Gamma\) is trivial. Then it follows from the slice theorem that a neighborhood of \(p\) in \(P\) is \(T\) equivariantly diffeomorphic to a neighborhood of \((1, 0, 0)\) in \(T_1 \times \mathbb{C}^m \times \mathbb{C}^k\) where \(m = \dim \Sigma - 1\). Here \(T = T_1 \times T_2\) acts on \(T_1 \times \mathbb{C}^m \times \mathbb{C}^k\) by

\[
(\lambda, t) \cdot (\mu, w_1, \ldots, w_m, z_1, \ldots z_k) = (\lambda \mu, w_1, \ldots, w_m, \chi_1(t) z_1, \ldots \chi_k(t) z_k),
\]

where \(\chi_1, \ldots, \chi_k : T_2 \to U(1)\) are nontrivial characters of \(T_2\).

Let \(pr_a : T \to T_\alpha\), \(\alpha = 1, 2\) denote the projections. Then \(pr_1(\exp(\tau Y))\) = 1. We claim that for all \(r\) between 1 and \(k\), \(\chi_a(pr_2(\exp(\tau Y)))\) is of the form \(e^{2\pi ir}\), where \(y_r\) are irrational numbers. Note that the claim implies immediately that the algebraic multiplicity of the eigenvalue 1 of the differential of the “Poincaré map” \(q \mapsto \exp(\tau Y) \cdot q\) is \(\dim \Sigma\), hence that \((\Sigma \times (0, \infty)) \cap \{H = E\}\) is a nondegenerate periodic manifold of \(\bar{H}\).

The claim holds because the one-parameter subgroup \(\{\exp(tY) \mid t \in \mathbb{R}\}\) is dense in \(T\). More specifically, let \(e_1, \ldots, e_s\) be a basis of the integral lattice of the torus \(T\) which is compatible with the splitting \(T = T_1 \times T_2\) so that \(\{\exp(te_1) \mid t \in \mathbb{R}\} = T_1\) and \(e_2, \ldots, e_s\) is a basis of the integral lattice of \(T_2\). Then \(Y = a_1 e_1 + \sum_{j=2}^s a_j e_j\) for some \(a_j \in \mathbb{R}\). Moreover, since the one-parameter subgroup defined by \(Y\) is dense in \(T\), the sum \(\sum_{j=1}^s q_j a_j\) is not a rational number for any \(s\) tuple of rational numbers \((q_1, \ldots, q_s)\). Since \(\exp(pr_1(\tau Y)) = 1\), \(a_1 = \pm \frac{1}{\tau}\). Consequently

\[
\chi_r(pr_2(\exp(\tau Y))) = \chi_r(\pm \sum_{j=2}^s \frac{a_j}{a_1} e_j) = e^{2\pi i(\pm \sum_{j=2}^s \frac{d\chi_r(e_j)}{a_1} a_j)}
\]

and the claim follows (note that \(d\chi_r(e_j)\) are integers).

If the group \(\Gamma\) is not trivial, then a neighborhood of \(p\) in \(P\) is modeled by the quotient \((T_1 \times \mathbb{C}^m \times \mathbb{C}^k)/\Gamma\), where \(\Gamma\) acts on \(T^1\) by multiplication and on \(\mathbb{C}^m \times \mathbb{C}^k\) linearly by \(m + k\) characters, so that the actions of \(T\) and \(\Gamma\) commute. The argument as above still works: for the Poincaré map on the quotient to have an eigenvector in \(\mathbb{C}^k\) with eigenvalue 1 it is necessary for \(\chi_r(pr_2(\exp(\tau Y)))\) to be a root of unity. But this is impossible as we have seen. This proves Lemma 8. \(\square\)

In fact in proving Lemma 8, we have proved more:

**Proposition 9.** Let \((V, \omega)\) be a symplectic vector space with a linear action of a compact Lie group \(K\) and a corresponding homogeneous moment map \(\Phi : V \to \mathfrak{k}^*\). Let \(q \in C^\infty(V)^K\) be a \(K\) invariant positive definite quadratic Hamiltonian; let \(q_0\) be the corresponding reduced Hamiltonian on \(V_0 = \Phi^{-1}(0)/K\). Let \(T \subset Sp(V, \omega)\) be the torus generated by \(q\). Let \(L\) be the link of the point \(* \in V_0\) corresponding to 0, let \(\mathcal{L}\) be the symplectic link and let \(\pi : L \to \mathcal{L}\) be the orbit map.

Then for every stratum \(S\) of the link \(L\) such that the fixed point set \(\pi(S)^T\) is nonempty and for every \(E > 0\) the manifold

\[
\{q_0 = E\} \cap (S \times (0, \infty))
\]

contains a weakly nondegenerate periodic manifold \(C\) of \(q_0 : C = \pi^{-1}(\pi(S)^T)\). If the orbifold \(\pi(S)^T\) is compact, then the manifold \(C\) is compact as well.
Note that by construction the circle action on the periodic manifolds $C$ in Proposition 3 is simply the action of the circle $U$. Hence $C/S^1 = \pi^{-1}(\pi(S)^T)/U = \pi(S)^T$. It follows from a result of Weinstein [W2, p. 247] that in Theorem 3 the number of periodic orbits of the Hamiltonian $h_0$ on a given energy surface $\{h_0 = E\}$ is bounded below by

$$
(2.1) \quad N_1 = \sum \text{Cat}(\pi(S)^T)
$$

where the sum is taken over all strata $S$ of the link $L$ such that the sets $\pi(S)^T$ are compact. Since the link $L$ is compact, the closed strata of $L$ must be compact. It follows that the number $N_1$ in equation (2.1) is positive.

The bound given by (2.1) is somewhat unsatisfactory — it ultimately depends on the Hamiltonian, while no such dependence is present in Weinstein’s nonlinear normal modes theorem (Theorem 2 above). We will see in Lemma 10 below that $\text{Cat}(\pi(S)^T) \geq \text{Cat}(\pi(S))$ for any closed stratum $S$ of the link $L$. Consequently

$$
N_1 \geq \sum \text{Cat}(\pi(S)),
$$

where the sum is taken over all closed strata $S$ of the link $L$. This will finish our proof of Theorem 3 hence of Theorem 4.

**Lemma 10.** Let $B$ be a closed symplectic orbifold with a Hamiltonian action of a torus $T$. Then the Liusternik-Schnirelman category $\text{Cat}(B^T)$ of the set of $T$-fixed points is bounded below by the Liusternik-Schnirelman category of $B$: $\text{Cat}(B^T) \geq \text{Cat}(B)$.

**Proof.** We use open sets in our definition of the category. Let $f : B \to \mathbb{R}$ be a generic component of the moment map for the action of $T$ on $B$ so that $B^T$ is precisely the set of critical points of $f$. The function $f$ is Bott-Morse. Therefore $B$ decomposes as a disjoint union of the unstable orbifolds $W_1, \ldots, W_k$ of the gradient flow of $f$. Clearly $\text{Cat}(\coprod W_k) = \text{Cat}(B^T)$. Now “thicken” $W_j$’s by replacing them with their tubular neighborhoods $\tilde{W}_j$ inside $B$. Then $\text{Cat}(\tilde{W}_j) = \text{Cat}(W_j)$, the sets $\tilde{W}_j$’s are open and $\cup_j \tilde{W}_j = B$. Hence $\text{Cat}(B) \leq \sum \text{Cat}(\tilde{W}_j) = \text{Cat}(B^T)$.

We end the paper by describing a practicable method for checking the existence of a symplectic slice $\Sigma$ through a relative equilibrium $m$ of a symmetric Hamiltonian system $(M, \omega_M, \Phi : M \to g^*, h \in C^\infty(M)^G)$ so that the Hessian $d^2(h|_\Sigma)(m)$ is positive definite.

Recall that if a point $m$ is a relative equilibrium of a symmetric Hamiltonian system, then there exists a vector $\eta \in g$ so that

$$
(2.2) \quad d(h - \langle \Phi, \eta \rangle)(m) = 0.
$$

Then the Hessian $d^2(h - \langle \Phi, \eta \rangle)(m)$ is a well-defined quadratic form, which we will use shortly. The vector $\eta$ is not unique: for every $\zeta$ in the Lie algebra $g_m$ of the isotropy group of $m$, the vector $\eta + \zeta$ also satisfies $d(h - \langle \Phi, \eta + \zeta \rangle)(m) = 0$. It is not hard to show that $\eta$ has to lie in the isotropy Lie algebra $g_\mu$ where $\mu = \Phi(m)$.

Since by assumption the action of $G$ on $M$ is proper, the isotropy group $G_m$ is compact. Hence we can choose a $G_m$ invariant inner product on the Lie algebra $g$ and use it to define an orthogonal complement $m$ of $g_m$ in $g_\mu$. There exists a unique vector $\eta \in m$ so that (2.2) holds. The vector $\eta$ is called the **orthogonal velocity** of the relative equilibrium $m$.

**Proposition 11.** Let $m$ be a relative equilibrium of a symmetric Hamiltonian system $(M, \omega_M, \Phi : M \to g^*, h \in C^\infty(M)^G)$ and let $\eta \in g_\mu$ be the orthogonal velocity of $m$ with respect to some $G_m$ invariant inner product on $g$ (where $\mu = \Phi(m)$). Suppose that the quadratic form $d^2(h - \langle \Phi, \eta \rangle)(m)|_{\ker d\Phi(m)}$ is semi-definite of maximal possible rank (the dimension of a symplectic slice at $m$). Then there exists a symplectic slice $\Sigma$ through $m$ such that the form $d^2(h|_\Sigma)(m)$ is positive definite.
Proof. The proof is a standard computation that uses the local normal form of the moment map of Marle and of Guillemin and Sternberg [Ma, GS]. Similar computations are carried out in [LS, p. 1643] and in [OR1]. We use the version of the normal form theorem described in [BL, pp. 211–215] which we now recall without proofs:

Let the symbols \((M,\omega), G, \Phi : M \to g^*\), \(\mu = \Phi(m)\), \(g_m\), \(g\mu\) and \(m\) have the same meaning as above.

The null directions of the restriction \(\omega(m)|_{\ker d\Phi(m)}\) is \(T_m(G_m \cdot m)\). Hence \(V = \ker d\Phi(m)/T_m(G_m \cdot m)\) is naturally a symplectic vector space. Denote the corresponding symplectic form by \(\omega_V\). Moreover, the linear action of the isotropy group \(G_m\) on \(\ker d\Phi(m)\) descends to a linear symplectic action on \((V,\omega_V)\). Denote the corresponding homogeneous moment map by \(\Phi_V\).

The \(G_m\) invariant inner product on \(g\) chosen above defines \(G_m\) equivariant embeddings: \(i : g^*_m \to g^*\) and \(j : m^* \to g^*\). Note that by construction of \(i\) and \(m\) we have that \(i(\ell,\eta) = 0\) for any \(\ell \in g^*_m\) and any \(\eta \in m\).

There exists a closed two-from \(\sigma\) on the associated bundle \(Y = G \times G_m (m^* \times V)\) and an open \(G\) equivariant embedding \(\psi\) of a neighborhood the zero section of \(Y \to G/G_m\) into \(M\) with the following properties.

1. \(\psi([1,0,0]) = m\).
2. \(\psi^*\sigma = \sigma\).
3. \((\psi^*\Phi)((g,\lambda,v)) = Ad^\ell(g)(\mu + j(\lambda) + i(\Phi_V(v)))\) for all \([g,\lambda,v] \in G \times G_m (m^* \times V)\).
4. The embedding \(\iota : (V,\omega_V) \to (G \times G_m (m^* \times V),\sigma)\), \(\iota(v) = [1,0,v]\) is symplectic. Consequently for a small enough neighborhood \(U\) of \(0\) in \(V\), \(\psi(\iota(U))\) is a symplectic slice through \(m\).

We now prove that \(\Sigma = \psi(\iota(U))\) is the desired symplectic slice. Since \(d(h - \langle \Phi,\eta \rangle)(m) = 0\), the Hessian \(d^2(h - \langle \Phi,\eta \rangle)(m)\) is well-defined and behaves well under restrictions. In particular, \(d^2(h - \langle \Phi,\eta \rangle)(m)|_{T_m(G_m \cdot m)} = d^2(h - \langle \Phi,\eta \rangle)|_{G_m \cdot m}(m)\). Since \(h\) is \(G\) invariant and since for any \(a \in G_m\) we have \(\langle \Phi,\eta\rangle(a \cdot m) = \langle Ad^\ell(a)\Phi(m),\eta\rangle = \langle Ad^\ell(a)\mu,\eta\rangle = \langle \mu,\eta \rangle = \langle \Phi,\eta \rangle(m)\). Therefore \((h - \langle \Phi,\eta \rangle)|_{G_m \cdot m}\) is constant and hence \(d^2(h - \langle \Phi,\eta \rangle)|_{T_m(G_m \cdot m)} = 0\).

Since the null directions of \(\omega(m)|_{\ker d\Phi(m)}\) is \(T_m(G_m \cdot m)\), it follows that for any symplectic slice \(\Sigma'\) through \(m\) which is tangent to \(\ker d\Phi(m)\), we have \(T_m\Sigma' \oplus T_m(G_m \cdot m) = \ker d\Phi(m)\).

Combining this with the previous computation we see that the rank of \(d^2(h - \langle \Phi,\eta \rangle)(m)|_{\ker d\Phi(m)}\) is at most \(\dim \Sigma'\). Thus by assumption \(d^2(h - \langle \Phi,\eta \rangle)(m)|_{T_m\Sigma'}\) is positive definite for any symplectic slice which is tangent to \(\ker d\Phi(m)\). It is easy to check that the manifold \(\psi(\iota(U))\) is such a slice. We finally show that

\[
d^2(h - \langle \Phi,\eta \rangle)(m)|_{\psi(\iota(U))} = d^2(h|_{\psi(\iota(U))})(m).
\]

Now for any \(u \in U\) we have \(\langle \Phi,\eta(\psi(\iota(u))) = (\psi^*\Phi,\eta([1,0,u]) = \langle \mu + j(0) + i(\Phi_V(u)),\eta \rangle = \langle \mu,\eta \rangle\) (since \(i(g^*_m,\eta) = 0\) by construction of \(i\) and \(\eta\)). Therefore \(d^2(h - \langle \Phi,\eta \rangle)(m)|_{\psi(\iota(U))} = d^2((h - \langle \Phi,\eta \rangle)|_{\psi(\iota(U)))}(m) = d^2(h|_{\psi(\iota(U))} - \langle \mu,\eta \rangle)(m) = d^2(h|_{\psi(\iota(U))})(m).\)

\[\square\]

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Department of Mathematics, University of Illinois, Urbana, IL 61801

E-mail address: lerman@math.uiuc.edu