THE MEASURE-THEORETICAL ENTROPY OF A LINEAR CELLULAR AUTOMATA WITH RESPECT TO A MARKOV MEASURE

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Abstract. In this paper we study the measure-theoretical entropy of the one-dimensional linear cellular automata (CA hereafter) $T_{f[-l,r]}$, generated by local rule $f(x_{-l}, \ldots, x_r) = \sum_{i=-l}^{r} \lambda_i x_i (\text{mod } m)$, where $l$ and $r$ are positive integers, acting on the space of all doubly infinite sequences with values in a finite ring $\mathbb{Z}_m$, $m \geq 2$, with respect to a Markov measure. We prove that if the local rule $f$ is bipermutative, then the measure-theoretical entropy of linear CA $T_{f[-l,r]}$ with respect to a Markov measure $\mu_{\pi P}$ is

$$h_{\mu_{\pi P}}(T_{f[-l,r]}) = -(l+r) \sum_{i,j=0}^{m-1} p_i p_{ij} \log p_{ij}.$$ 

1. Introduction

Cellular automata (CA for short), begun by Ulam and von Neumann, has been systematically studied by Hedlund from purely mathematical point of view [9]. Hedlund's paper started investigation of current problems in symbolic dynamics. The study of such dynamics called CA from the point of view of the ergodic theory has received remarkable attention in the last few years ([1], [6], [10]), because CA have been widely investigated in a number of disciplines (e.g., mathematics, physics, computer sciences, etc.). In [11], Shereshevsky has defined $n$th iteration of a permutative CA and shown that if the local rule $f$ is right (left) permutative, then its $n$th iteration also is right (left) permutative.

In [10], Mass and Martinez have studied the dynamics of Markov measures by a particular linear cellular automata (LCA). They have reviewed some results on the evolution of probability measures under CA acting on a fullshift. In [5], the author has studied the ergodic properties of CA with respect to Markov measures.

Although the LCA theory and the entropy of this LCA have grown up somewhat independently, there are strong connections between entropy theory and CA theory. We give an introduction to LCA theory and then discuss the entropy of this LCA. For a definition and some properties of one-dimensional LCA we refer to [8, 9]. The study of the endomorphisms and the automorphisms (i.e., continuous shift commuting maps, invertible or non-invertible) of the full shift and its subshifts was initialed by Hedlund in [9].

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It is well known that there are several notions of entropy (i.e., measure entropy, topological entropy, directional entropy etc.) of measure-preserving transformation on probability space in ergodic theory. It is important to know how these notions are related with each other. The last decade (see, e.g [11], [8], [10]), a lot papers are devoted to this subject.

In [11], the author has compute the measure-theoretical entropy with respect to uniform Bernoulli measure for the case \( \lambda_i = 1 \), for all \( i \in \mathbb{Z}_m \). The author proved that the uniform Bernoulli measure is the maximal measure for these LCA. He has also posed the question whether the maximal measure is unique.

In [2], the author has investigated the measure-theoretical directional entropy of \( \mathbb{Z}^2 \)-actions generated by LCA and the shift map with respect to uniform Bernoulli measure. Let \( m, k \in \mathbb{N} \) and \( \lambda_k, \ldots, \lambda_1 \in \mathbb{Z} \). Let \( \phi: \mathbb{Z}_m^Z \rightarrow \mathbb{Z}_m^Z \) be a LCA with \( \mathbb{Z}_m = \{0, \ldots, m - 1\} \) and local rule \( \phi(x) = \sum_{j=1}^{k} \lambda_j x_{i+j} \pmod{m} \). Assume \( \phi \) is surjective and consider \( v = (v_1, v_2) \in \mathbb{Z}_m^2 \) with \( v_2 > v_1 \). In [2], the author has shown that the directional entropy of \( \phi \) with respect to the uniform Bernoulli measure is given by the formula \( h_v(\phi) = 2v_2 \log m \).

In this paper we compute the measure-theoretical entropy of the one-dimensional LCA \( T_{[l, r]} \), generated by a bipermutative local rule 

\[
f(x_{-l}, \ldots, x_r) = \sum_{i=-l}^{r} \lambda_i x_i \pmod{m},
\]

acting on the space of all doubly infinite sequences with values in a finite ring \( \mathbb{Z}_m \), \( m \geq 2 \), \( \lambda_i \in \mathbb{Z}_m \), with respect to a Markov measure \( \mu_{x,P} \). We show that if the local rule is bipermutative, then we have \( \mu_{x,P} \)

\[h_{\mu_{x,P}}(T_{[l, r]}) = -(l + r) \sum_{i,j=0}^{m-1} p_{ij} \log p_{ij}.
\]

Also, we prove that if the Markov measure \( \mu_{x,P} \) is uniform, then we get \( h_{\mu_{x,P}}(T_{[l, r]}) = (l + r) \log m \).

The organization of the paper is as follows: In section 2 we give the basic formulation of problem to state our main result. In section 3 we state our main theorem and prove it. In section 4 we conclude by pointing some further problems.

2. Preliminaries

Let \( \mathbb{Z}_m = \{0, 1, \ldots, m - 1\} \) \( (m \geq 2) \) be a ring of the integers modulo \( m \) and \( \mathbb{Z}_m^\mathbb{Z} \) be the space of all doubly-infinite sequences \( x = (x_n)_{n=-\infty}^\infty \in \mathbb{Z}_m^\mathbb{Z} \) and \( x_n \in \mathbb{Z}_m \).

A CA can be defined as a homomorphism \( \mathbb{Z}_m^\mathbb{Z} \) of with product topology. The shift \( \sigma: \mathbb{Z}_m^\mathbb{Z} \rightarrow \mathbb{Z}_m^\mathbb{Z} \) defined by \( (\sigma x)_i = x_{i+1} \) is a homeomorphism of compact metric space \( \mathbb{Z}_m^\mathbb{Z} \).

A CA is a continuous map, which commutes with \( \sigma \), \( T: \mathbb{Z}_m^\mathbb{Z} \rightarrow \mathbb{Z}_m^\mathbb{Z} \) defined by \( (Tx)_i = f(x_{i+1}, \ldots, x_{i+r}) \), where \( f: \mathbb{Z}_m^{r+l+1} \rightarrow \mathbb{Z}_m \) is a given local rule or map.

Favati et al. [8] have defined a local rule \( f \), they have stated that a local rule \( f \) is linear (additive) if and only if it can be written as

\[
f(x_1, \ldots, x_r) = \sum_{i=1}^{r} \lambda_i x_i \pmod{m},
\]

where at least one between \( \lambda_1 \) and \( \lambda_r \) is nonzero. We consider 1-dimensional linear cellular automata (LCA) \( T_{[l, r]} \) determined by the local rule \( f \):

\[
(Tx) = (y_n)_{n=-\infty}^\infty, y_n = f(x_{n+l}, \ldots, x_{n+r}) = \sum_{i=l}^{r} \lambda_i x_{n+i} \pmod{m},
\]
where $\lambda_1, \ldots, \lambda_r \in \mathbb{Z}_m$.

We are going to use the notation $T_{f[l,r]}$ for LCA-map defined in (2.2) to emphasize the local rule $f$ and the numbers $l$ and $r$. If the local rule $f$ is given as Eq. (2.1), then the finite formal power series ($fps$ for brevity) $F$ associated with $f$ is defined as $F(X) = \sum_{i=l}^{r} \lambda_i X^{-i}$, where $\lambda_1, \ldots, \lambda_r \in \mathbb{Z}_m$ (see [3], [10] for details). The technique of $fps$ is well known for the study of these problems. In [3], the author has studied the topological entropy of $n$th iteration of a linear CA by using the $fps$.

The notion of permutative CA was first introduced by Hedlund in [9]. If the linear local rule $f : \mathbb{Z}_m^{-l+1} \rightarrow \mathbb{Z}_m$ is given in (2.1), then it is permutative in the $j$th variable if and only if $gcd(\lambda_j, m) = 1$, where $gcd$ denotes the greatest common divisor. A local rule $f$ is said to be right (respectively, left) permutative if $gcd(\lambda_r, m) = 1$ (respectively, $gcd(\lambda_l, m) = 1$). It is said that $f$ is bipermutative if it is both left and right permutative.

3. The measure entropy of the one-dimensional LCA

In this section we study the measure entropy of the LCA defined in Eq. (2.2) with respect to uniform Markov measure.

**Definition 3.1.** Let $(X, \mathcal{B}, \mu, T)$ be a measure-theoretical dynamical system and $\alpha$ be a partition of $X$. The partition $\alpha$ is called a generator if $\bigvee_{k=0}^{\infty} T^{-k} \alpha = \mathcal{B}$.

**Definition 3.2.** Let $\alpha$ be a partition of $X$. The quantity

$$H_\mu(\alpha) = -\sum_{A \in \alpha} \mu(A) \log \mu(A)$$

is called the entropy of the partition $\alpha$. Let $\alpha$ be a partition with finite entropy. Then the quantity

$$h_\mu(\alpha, T) = \lim_{n \to \infty} H_\mu\left(\bigvee_{i=0}^{n-1} T^{-i} \alpha\right)$$

is called the entropy of $\alpha$ with respect to $T$. One often writes $h(\alpha, T)$ instead of $h_\mu(\alpha, T)$. The quantity

$$h_\mu(T) = \sup_\alpha \{h_\mu(\alpha, T) : \alpha \text{ is a partition with } H_\mu(\alpha) < \infty\}$$

is called the measure-theoretical entropy of $(X, \mathcal{B}, \mu, T)$, the entropy of $T$ (with respect to $\mu$).

**Theorem 3.3.** (Kolmogorov-Sinai Theorem) Let $(X, \mathcal{B}, \mu, T)$ be a measure-theoretical dynamical system and $\alpha$ a generator with $H_\mu(\alpha) < \infty$. Then $h_\mu(T) = h_\mu(\alpha, T)$.

In order to apply entropy theory to the one-dimensional LCA over ring $\mathbb{Z}_m$ ($m \geq 2$) we must define $\sigma$-algebra $\mathcal{B}$ and the Markov measure $\mu : \mathcal{B} \to [0, 1]$. In
symbolic dynamical system, it is well known that this $\sigma$-algebra $B$ is generated by thin cylinder sets

$$C =_{a} [j_{0}, j_{1}, \ldots, j_{s}]_{s+a} = \{ x \in \mathbb{Z}_{m}^{Z} : x_{a} = j_{0}, \ldots, x_{a+s} = j_{s} \},$$

where $j_{0}, j_{1}, \ldots, j_{s} \in \mathbb{Z}_{m}$.

Recall that a subshift of finite type $\sigma : X \to X$ defined on a space

$$X = \{ x \in \mathbb{Z}_{m}^{Z} : M(x_{n}, x_{n+1}) = 1, n \in \mathbb{Z} \}$$

for some $m \times m$ matrix $M$ with entries either zero or unity. Let $P = (p_{(i,j)})$ denote a $m \times m$ stochastic matrix ($p_{(i,j)} \geq 0$, $\sum_{j=0}^{m-1} p_{(i,j)} = 1$) with entries $p_{(i,j)} = 0$ iff $M_{(i,j)} = 0$, let $\pi = \{ \pi_{0}, \pi_{1}, \ldots, \pi_{m-1} \}$ be its left eigenvector. It is well known that $\pi P = \pi$. A pair $\pi, P$ defines a set function $\mu_{\pi} P$ on the cylinders of $\mathbb{Z}_{m}^{Z}$. Recall that the associated Markov measure is defined as follows:

$$\mu_{\pi} P(a[i_{0}, \ldots, i_{k}]_{k}) = \pi_{i_{0}} p_{(i_{0},i_{1})} \cdots p_{(i_{k-1},i_{k})}.$$  

See [7], [10] and [12] for the properties of the Markov measure.

Let $\xi$ be the zero-time partition of $\mathbb{Z}_{m}^{Z}$: $\xi = \{ 0[i] : 0 \leq i < m \}$, where $0[i] = \{ x \in \mathbb{Z}_{m}^{Z} : x_{0} = i \}$ is a cylinder set for all $i, 0 \leq i < m$. So, we can state the partition $\xi$ as follows:

$$(3.2) \quad \xi = \{ 0[0], 0[1], \ldots, 0[m - 1] \}.$$

Denote by $A(\xi)$ sub-$\sigma$-algebra of $B$ generated by the zero-time partition $\xi$ of $\mathbb{Z}_{m}^{Z}$.

Let us consider a particular case. Assume that the local rule $f$ is bipermutative, so, we have the following Lemma.

**Lemma 3.4.** ([1], Lemma)) Suppose that $f(x_{-l}, \ldots, x_{r}) = \sum_{i=-l}^{r} \lambda_{i} x_{i} (mod m)$ is a bipermutative local rule, and $\xi$ is a partition of $\mathbb{Z}_{m}^{Z}$ given in Eq. (3.2), then the partition $\xi$ is a generator for one-dimensional LCA generated by $f$.

In order to calculate the measure-theoretical entropy of the one-dimensional LCA $T_{f[-l,r]}$ with respect to uniform Markov measure we must prove whether $T_{f[-l,r]}$ is a measure-preserving transformation.

**Proposition 3.5.** Let $T_{f[-l,r]}$ be an one-dimensional LCA over $\mathbb{Z}_{m}$, and $f[-l, r]$ be bipermutative local rule. Then $T_{f[-l,r]}$ is the uniform Markov measure-preserving transformation.

**Proof.** Consider a cylinder set

$$C =_{a} [j_{0}, j_{1}, \ldots, j_{s}]_{s+a} = \{ x \in \mathbb{Z}_{m}^{Z} : x_{a}^{(0)} = j_{0}, \ldots, x_{a+s}^{(0)} = j_{s} \}.$$  

Then the first preimage of $C$ under $T_{f[-l,r]}$ is the follows:

$$T_{f[-l,r]}^{-1}(C) = T_{f[-l,r]}^{-1}(\{ x \in \mathbb{Z}_{m}^{Z} : x_{a}^{(0)} = j_{0}, \ldots, x_{a+s}^{(0)} = j_{s} \})$$

$$= \bigcup_{(i_{0}, i_{1}, \ldots, i_{2r+s}) \in \mathbb{Z}_{m}^{Z}^{2r+s+1}} (i_{0}, i_{1}, \ldots, i_{2r+s})_{a+s+r},$$
From the definition of entropy we have
\[ x_a^{(0)} = \sum_{k=0}^{r} \lambda_k x_{a+k}^{(1)} \pmod{m} \]
and
\[ x_{a+s}^{(0)} = \sum_{k=0}^{r} \lambda_k x_{s+a+k}^{(1)} \pmod{m}. \]
It is clear that,
\[ (a-l)[i_0, i_1, \ldots, i_{l+r+s}]a+s+r = \{ x \in \mathbb{Z}_m^r : x_{a-l}^{(1)} = i_0, \ldots, x_{a+s+r}^{(1)} = i_{l+r+s} \}. \]
Then we have
\[ \mu \pi p(a[j_0, j_1, \ldots, j_s]s+a) = \mu_0 \pi p(j_0, j_1) \cdots p(j_{s-1}, j_s). \]

**Theorem 3.6.** Let \( \mu \pi p \) be a Markov measure given by the stochastic matrix \( P = (p_{ij}) \) and the probability vector \( \pi = (p_i) \). Assume that \( l \) and \( r \) are positive integers and \( \gcd(\lambda_r, m) = 1 \), \( \gcd(\lambda_s, m) = 1 \). Then we have
\[ h_{\mu \pi p}(T_{l-r, r}) = -(l + r) \sum_{i,j=0}^{m-1} p_{ij} \log p_{ij}. \]

**Proof.** Now we can calculate the measure entropy of the one-dimensional LCA by means of the Kolmogorov-Sinai Theorem ([12], p. 95), namely,
\[ h_{\mu \pi p}(T_{l-r, r}) = h_{\mu \pi p}(T_{l-r, r}^n) \]
Let \( \xi \) be the zero-time partition of \( \mathbb{Z}_m^r \): \( \xi = \{ o[i] : 0 \leq i < m \} \),
where \( o[i] = \{ x \in \mathbb{Z}_m^r : x_0 = i \} \) is a cylinder set for all \( i, 0 \leq i < m \). So, we can state the partition \( \xi \) as follows:
\[ \xi = \{ o[0], o[1], \ldots, o[m-1] \}. \]
Denote by \( \mathcal{A}(\xi) \) sub-\( \sigma \)-algebra of \( \mathcal{B} \) generated by the zero-time partition \( \xi \) of \( \mathbb{Z}_m^r \).
From the definition of entropy we have
\[ H_{\mu \pi p}(\xi) = -m \mu \pi p(o[i]) \log \mu(o[i]) = \log m. \]

From Theorem 3.3 one has
\[ h_{\mu \pi p}(T_{l-r, r}) = - \lim_{n \to \infty} \frac{1}{n} H_{\mu \pi p}(\bigvee_{k=0}^{n-r} T_{l-r, r}^{i_k} \xi) \]
\[ = - \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{m-1} p_{i-n} p_{i-n+1} \cdots p_{i+n} \]
\[ = - \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{m-1} p_{i-n} p_{i-n+1} \cdots p_{i+n} \]
It is easy to prove by induction that the sum on the right hand side is equal to
\[ (nl+r+1) \sum_{i,j=0}^{m-1} p_{ij} \log p_{ij} + \sum_{i=0}^{m-1} p_i \log p_i. \]
This implies the result. □

A Markov measure on $\mathbb{Z}_m^m$ is uniform, if measure of any one-dimensional cylinder is equal to $\frac{1}{m}$, where $m$ is a cardinality of $\mathbb{Z}_m$. A doubly stochastic matrix is a matrix $P$ such that $P$ and $P^\text{tr}$ (transpose) are both stochastic. If a matrix $P$ is a doubly stochastic then corresponding Markov measure is a uniform measure. Cardinality of $\mathbb{Z}_m$ is equal to $m$, so that any doubly stochastic matrix $P$ of $m \times m$ size will generate uniform Markov measure.

**Corollary 3.7.** Let $\mu_{\pi P}$ be the uniform Markov measure on $\mathbb{Z}_m^m$ and $f(x_{-r}, \ldots, x_r) = \sum_{i=-r}^{r} \lambda_i x_i \pmod{m}$, where $f[-r, r]$ is bipermutative. Then measure-theoretic entropy of the one-dimensional LCA $T_f[-r, r]$ with respect to $\mu_{\pi P}$ is equal to $2r \log m$.

**Proof.** It is clear that the partition $\xi \vee T_f[-r, r]^{-1}(\xi)$ is the following:

$$\xi \vee T_f[-r, r]^{-1}(\xi) = \{ -r[i_{-r}, \ldots, i_r], \ldots, i_{r} \in \mathbb{Z}_m \}.$$  

Because of the uniform Markov measure we get

$$H_{\mu_{\pi P}}(\xi \vee T_f[-r, r]^{-1}(\xi)) = -m(2r+1) \mu_{\pi P}(-r[i_{-r}, \ldots, i_r]_r) \log \mu_{\pi P}(-r[i_{-r}, \ldots, i_r]_r)$$

$$= -m(2r+1) p(i_{-r}, i_{-1, i_r}) \cdots p(i_{r-1, i_r}) \cdots p(i_{r-1, i_r})$$

$$= -m(2r+1) m^{-2r+1} \log m^{-2r+1} = (2r+1) \log m.$$

If we continue, from Lemma 3.4 we have the following results:

$$H_{\mu_{\pi P}}(\bigvee_{k=0}^{n} T_f[-r, r]^{-k}\xi) = -m(2nr+1) \mu_{\pi P}(-nr[i_{-nr}, \ldots, i_{nr}]_{nr}) \times$$

$$\log \mu_{\pi P}(-nr[i_{-nr}, \ldots, i_{nr}]_{nr})$$

$$= -m(2nr+1) m^{-(2nr+1)} \log m^{-(2nr+1)}$$

$$= (2nr+1) \log m.$$

From Theorem 3.3 and Lemma 3.4 we have

$$h_{\mu_{\pi P}}(T_f[-r, r]) = \lim_{n \to \infty} \frac{1}{n} H_{\mu_{\pi P}}(\bigvee_{k=0}^{n} T_f[-r, r]^{-k}\xi) = 2r \log m.$$  

□

4. Conclusion

This paper contains the following results: We have found a generating partition for the one-dimensional LCA generated by a bipermutative local rule (Lemma 3.4). We have calculated the measure-theoretical entropy of the one-dimensional LCA with respect to any Markov measure (Theorem 3.6). This is the first step toward arbitrary Markov measure classification of multi-dimensional CA defined on alphabets of composite cardinality. In [1] the author has compute the measure-theoretical entropy with respect to uniform Bernoulli measure for the case $\lambda_i = 1$, for all $i \in \mathbb{Z}_m$. The author proved that the uniform Bernoulli measure is the maximal measure for these LCA. He also posed the question whether the maximal measure is unique.
Thus, where a question raises:
Using the Theorem (12, Theorem 7.13. (ii)), can one calculate the topological entropy of the LCA $T_{[f^{-r},r]} : X \to X$, where $X$ is defined as in Eq. (3.1)? Also it is open question whether the uniform Markov measure is maximal measure for $T_{[f^{-r},r]} : X \to X$.

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