Backwards on Minkowski’s road. From $4D$ to $3D$
Maxwellian electromagnetism

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Abstract. Minkowski’s concept of a four-dimensional physical space is a central paradigm of modern physics. The three-dimensional Maxwellian electrodynamics is uniquely generalized to the covariant four-dimensional form. Is the $(1+3)$ decomposition of the covariant four-dimensional form unique? How do the different sign assumptions of electrodynamics emerge from this decomposition? Which of these assumptions are fundamental and which of them may be modified? How does the Minkowski space-time metric emerge from this preliminary metric-free construction? In this paper we are looking for answers to the problems mentioned. Our main result is the derivation of four different possible sets of electrodynamic equations which may occur in different types of isotropic electromagnetic media. The wave propagation in each of these media is described by the Minkowskian optical metrics. Moreover, the electric and magnetic energies are nonnegative in all cases. We also show that the correct directions of the Lorentz force (as a consequence of the Dufay and the Lenz rules) hold true for all these cases. However, the differences between these four types of media must have a physical meaning. In particular, the signs of the three electromagnetic invariants are different.

1. Introduction

One of the most famous result of Hermann Minkowski in physics is the four-dimensional Minkowski metric of space-time $[1]$. This concept generated the generally adopted form of special relativity and formed a basis of general relativity and quantum field theory. Apparently in mathematical literature this metric is referred to as the Lorentzian metric, i.e., after a physicist Hendrik Lorentz and not after a mathematician Hermann Minkowski. This strange situation is due to the fact that another principally different and not less famous Minkowski metric is a central concept in the theory of final normalized functional spaces.

Although the Minkowski metric is a fundamental concept of classical and quantum physics, its origin does not come from some general philosophical paradigm. The $4D$ Minkowski metric has its roots in Maxwell’s theory of the electromagnetic field. This fact is already clear from the original title of Einstein’s first special relativity paper: “Zur Elektrodynamik bewegter Körper” (“On the Electrodynamics of Moving Bodies”).

Consequently, the special choice of signs in the Minkowski metric, or, more generally, the signature of the curved spacetime metric, has to be in a correlation
with various sign conventions and rules used in classical three-dimensional Maxwellian electrodynamics. Recall some of these assumptions on which the signs are based:

i. The special sets of signs in every one of the four Maxwell equations when they are written in 3D-space;

ii. positivity of the electromagnetic energy;

iii. two possibilities of the signs of the electric charges;

iv. attraction between opposite charges and repulsion between charges of the same sign — Dufay’s rule;

v. pulling of a ferromagnetic core into a solenoid independently of the direction of the current — Lenz’s rule;

vi. positivity of the electric permittivity and the magnetic permeability constants for most of the natural dielectric materials.

The aim of the current paper is to study which of these sign assumptions are only customarily accepted conventions and which of them are consequences of the fundamental physical laws and cannot be modified without changing the laws. On the other hand, which of them can be modified without braking the fundamental 4D laws? Do such modified laws possibly describe some physical meaningful reality, electromagnetic behavior in manufactured materials, for instance?

In line of Minkowski’s idea that 3D physics has to be considered as a chapter of 4D physics, 3D Maxwellian electromagnetism becomes part of the covariant 4D electromagnetic theory. Differential forms and tensors are commonly used for such a generalization. In this paper, we follow the differential form approach.

In the standard form of Maxwell’s theory, all the sign assumptions listed above appear together and the Minkowski metric is postulated from the very beginning. Such a construction does not allow to investigate the specific meaning of the electromagnetic sign conventions and their relations to the Minkowski signature. In this paper, we base our consideration on the premetric formulation of the electrodynamics. Although the roots of such an approach can be found in the older literature [2], its final form was derived only recently in a series of papers [3]-[7] and in a book [8] of Hehl and Obukhov.

The current paper is a continuation of the paper [9] due to Hehl and the first-named author. Instead of using a general constitutive relation constrained only by the reciprocity relation, as it was made in [9], we restrict ourselves now to the electrodynamics in vacuum and in isotropic media. For such media we may assume that the metric in the “rest space hypersurfaces” is Euclidean. With this restriction we hope to go deeply into the study of the roots of the the sign conventions listed above. Our primary aim is to find the physically motivated conditions which enable the existence of non-ordinary materials with non-positive electric permittivity and/or magnetic permeability constants. We start with a four-dimensional electromagnetic theory based on the following assumptions:

i. Charge conservation;

ii. flux conservation;

iii. positivity of the electric and magnetic energy;

iv. energy-momentum conservation;

v. local linear constitutive law which we restrict to the isotropic form.
We show that under these assumptions a unique 4D electromagnetic system is reduced to four different types of 3D Maxwell-type systems. For each of these models, the electric and magnetic energies are positive and the Lorentz force has a standard direction (Dufay and Lenz rules). Moreover the optical metric is Minkowskian for all these models. However, they differ in the values of the four electromagnetic invariants. We also briefly discuss the possibility to identify our models with recently manufactured metamaterials.

The organization of the paper is as follows: In section 2, we present some mathematical preliminaries and notations which are used in the sequel. In section 3, we give a brief account of premetric electrodynamics. Here, our main aim is to introduce an (1 + 3) decomposition relative to an arbitrary observer. These sign factors in Maxwell equations, which have no physical meaning and are only subject of conventions, are determined here. Section 4 is devoted to the constitutive relation. We define a restricted constitutive map based on transformation properties of the fields and their different physical dimensions. In the (1 + 3) decomposition, we arrive at isotropic media with the two electromagnetic constants \( \varepsilon \) and \( \mu \). In Section 5, we consider the expression of the energy-momentum current in differential forms. The conservation of the energy-momentum current is required to be in conformity with the conservation of the charge current (the same orientation of the boundary is used). Consequently, we derive the sign of the Lenz rule. Moreover, we come to a list of four different types of electromagnetic media. This is our main result. The optical properties of the media are studied in Section 6 and the optical metric of Minkowskian signature is derived. In section 7, we consider the Lorentz force expression and derive the Dufay and Lenz rules for different media. In Section 8, we give a summary of our results. Section 9 is devoted to conclusions and a discussion.

2. Mathematical preliminaries

2.1. Local observer on a premetric manifold

We start with a bare four-dimensional differential manifold \( M \) as a preliminary model of the physical space-time. As mentioned by Minkowski [1]: “We should have in the world no longer space, but an infinite number of spaces, analogously as in three-dimensional space an infinite number of planes. Three-dimensional geometry becomes a chapter in four-dimensional physics”. This suggests to introduce a foliation of the manifold \( M \) by a set of smooth non-intersecting hypersurfaces representing the rest space.

Although on \( M \) an invariant meaning can be given only to 4D tensorial quantities, their real physical nature emerges when a notion of a local observer is introduced. As Minkowski write in [1]: “The separation of the field produced by the electron into electric and magnetic force is relative with regard to the underlying time axis”. This is similar to the notion of an observer used in modern literature.

On a manifold endowed with a metric of Minkowski signature, two different, but locally equivalent, descriptions of an observer are in use [10]. Due to the “congruence point of view”, the observer traveling on a wordline \( z^i = z^i(\tau) \) is described locally by a timelike tangent vector field \( n^i = \partial z^i / \partial \tau \). The orthogonal complement to this local field defines the local rest space. In the second “slicing approach”, one starts with a foliation of the total space by spacelike hypersurfaces \( \Sigma_\tau \) and reinstates (in a locally unique manner) the timelike field orthogonal to \( \Sigma_\tau \). It is clear that in both approaches
the metric tensor plays a crucial role.

Since we are dealing with a manifold without predefined metric structure, the notion of a local observer must be modified to involve both of these approaches.

A local observer on $M$ is defined by

(i) a foliation of the manifold $M$ generated by a set of smooth non-intersecting hypersurfaces $\Sigma_\tau$, each of which is diffeomorphic to $\mathbb{R}^3$. The hypersurfaces are continuously numbered by a real parameter $\tau \in \mathbb{R}$ and represent the rest space;

(ii) a field of directed curves $z^i = z^i(\tau)$ every one of which is diffeomorphic to $\mathbb{R}$ and parametrized by the same real parameter $\tau$ with $z^i(\tau) \in \Sigma_\tau$. The tangent vector

$$n^i = \frac{\partial z^i}{\partial \tau}$$

is assumed to be defined and differs from zero for every value of $\tau$;

(iii) every curve $z^i(\tau)$ is is assumed to be transversal to the hypersurfaces $\Sigma_\tau$ in the following sense: Locally, $\Sigma_\tau$ is described by the equation $d\tau = 0$, such that $n|d\tau = 1$. Here, and in the sequel, $\|$ denotes the interior product operation.

The parameter $\tau$ will serve as a prototype of a time coordinate while the folio $\Sigma_\tau$ is a prototype of the rest space for a chosen observer.

In this paper, we will deal with the local properties of the manifold, so we will restrict to a bounded region $R \subset M$. Without any restriction on the topological nature of $M$, this region can be considered to be orientable. Moreover, we assume that a defined orientation, i.e., a positive non degenerated volume element $(4) vol$ is chosen on $R$.

Relative to the chosen observer, the positive volume element $(4) vol$ is decomposed as

$$(4) vol = d\tau \wedge (3) vol.$$ 

This equation defines uniquely a positive volume element $(3) vol$ on a hypersurface $\Sigma_\tau$. Hence, a chosen observer necessary transforms the positive orientation of the total 4D space $M$ into a unique positive orientation of a 3D hypersurface $\Sigma_\tau$.

The introduction of a local observer on $M$ has some implications for 4D differential forms. Since an integration of a differential form over the oriented submanifold $\Sigma_\tau$ (or over a domains in it) must yield a non-trivial invariant scalar, the integrand must be a twisted 3-form, i.e., such a form that changes its sign under basis transformations with negative determinant. For instance, such a twisted behavior has to be prescribed for the electromagnetic current $J$. Since the electromagnetic excitation 2-form $H$ is connected to $J$, it must be a twisted form, too.

Every twisted 3-form lying in a folio has only one component which is proportional to the 3D-volume element. Consequently, the notion of a sign can also be rigorously ascribed to such special 3-forms. Observe that a general 3-form in four dimensional space has four independent components and the positive and negative forms cannot be defined rigorously.

On the other hand, the two- and one-dimensional submanifolds of $M$ cannot be given a preferred orientation only by a chosen local observer. Consequently, an integral over such lower dimensional submanifolds must involve untwisted differential forms, i.e., such ones that are not changed under any transformations of the basis. The electromagnetic field strength $F$ is an example of such a form.

This simple observation can serve as an additional justification of the premetric electrodynamics construction which is based on two integral conservation laws — a
conservation law for the twisted 3-form $J$ and a conservation law for an untwisted 2-form $F$. In this sense, the premetric approach is preferable to the ordinary electrodynamics construction which does not provide us with any motivation of the different behavior of the two basic electromagnetic fields.

2.2. $(1+3)$-splitting of differential forms and their differentials

Consider an arbitrary differential $p$-form $\alpha$ defined on $M$. Relative to a chosen local observer, the decomposition of this form is usually given by

$$\alpha = d\tau \wedge \alpha_\perp + \alpha_\parallel,$$  \hspace{1cm} (2.3)

where the $(p - 1)$-form $\alpha_\perp$ is a transversal component of $\alpha$ while the $p$-form $\alpha_\parallel$ is its longitudinal component.

Since we are interested not only in a formal decomposition of the form $\alpha$, but mainly in a physical interpretation that can be given to its parts, we will use a slightly different $(1+3)$-splitting:

$$\alpha = s_T d\tau \wedge \beta + s_S \gamma.$$  \hspace{1cm} (2.4)

Here we introduced the sign factors $s_T$ and $s_S$ for the time and the spatial components, respectively. These sign factors have values from the set $\{-1, +1\}$. Any additional positive scalar factor is assumed to be absorbed in the corresponding form. The actual rigorous values of the factors $s_T, s_S$ will to be given in correspondence with the physical interpretation of the forms $\beta$ and $\gamma$ and the physical laws they obey.

The absolute physical dimensions of the $p$-forms $\alpha$ and $\gamma$ are the same. The $(p - 1)$-form $\beta$ has the absolute dimension of $\alpha$ divided by the dimension of time. The forms $\beta$ and $\gamma$ satisfy the relations

$$n|\beta = n|\gamma = 0,$$ \hspace{1cm} (2.5)

i.e., they lie in the folio $\Sigma_\tau$. Thus the decomposition (2.4) is unique and the forms $\beta$ and $\gamma$ can be derived from $\alpha$ as

$$\beta = s_T n|\alpha,$$

$$\gamma = s_S \left(\alpha - s_T d\tau \wedge (n|\alpha)\right).$$  \hspace{1cm} (2.6)

The 4-dimensional exterior derivative operator can be decomposed as

$$d = d\tau \wedge \frac{\partial}{\partial \tau} + d,$$  \hspace{1cm} (2.7)

where the spatial exterior derivative $d$ refers to local coordinates on the hypersurface $\Sigma_\tau$. Using the standard formula for the exterior derivative of a wedge product, we find

$$d\alpha = d\tau \wedge (-s_T d\beta + s_S \gamma) + s_S d\gamma.$$  \hspace{1cm} (2.8)

Here and in the sequel, the partial derivative $\partial/\partial \tau$ is abbreviated by a dot on top of the corresponding quantity.
2.3. Conservation laws in \((1+3)\)-splitting

For a given \(p\)-form \(\alpha\), a conservation law is described by the vanishing of the integral

\[
\int_{\partial C_{p+1}} \alpha = 0 \tag{2.9}
\]

over the boundary of any closed connected \((p+1)\)-dimensional region \(C_{p+1}\). Under this condition, the Stokes theorem implies

\[
\int_{\partial C_{p+1}} \alpha = \int_{C_{p+1}} d\alpha = 0. \tag{2.10}
\]

Applying the \((1+3)\)-decomposition \(\eqref{2.8}\) of \(d\alpha\), we find

\[
\int_{C_{p+1}} d\alpha = s_T \int_{C_3} d\tau \wedge \left( -\frac{s_T}{s_S} d\beta + \dot{\gamma} \right) + s_S \int_{C_{p+1}} d\gamma = 0. \tag{2.11}
\]

This equation is simplified when \(\alpha\) is a 3-form on a 4-dimensional manifold \(M\). In this case, the spatial exterior derivative of its longitudinal component is equal to zero, \(d\gamma = 0\). Consequently \(\eqref{2.11}\) takes the form

\[
\frac{s_T}{s_S} \int_{C_4} d\tau \wedge d\beta = \int_{C_3} d\tau \wedge \dot{\gamma}. \tag{2.12}
\]

Let a region be of the form of a tube \(C_4 = C_3 \times [\tau_1, \tau_2]\) with a bounded hypersurface \(C_3\) independent of \(\tau\). When integration over the coordinate \(\tau\) is applied on both sides of \(\eqref{2.12}\), it can be rewritten as

\[
\frac{s_T}{s_S} \int_{C_3} d\beta = \frac{\partial}{\partial \tau} \int_{C_3} \gamma. \tag{2.13}
\]

Using once more the Stokes theorem, we can rewrite it as

\[
\frac{s_T}{s_S} \int_{C_2} \beta = \frac{\partial}{\partial \tau} \int_{C_3} \gamma, \tag{2.14}
\]

where \(C_2 = \partial C_3\) is the boundary of \(C_3\).

The integrals on both sides of this equation have a clear physical meaning. On the right hand side we have the time derivative of the total charge (or of some other physical quantity) denoted by \(\gamma\) contained in the closed region \(C_3\). The integral on the left hand side represents the flux of the same charge through the boundary \(C_2\) of \(C_3\). The sign factor \(\frac{s_T}{s_S}\) depends now only on the choice of the orientation of the boundary \(C_2\) relative to a given orientation of the region \(C_3\). With a customary choice of the boundary orientation (the vectors and 1-forms transversal to the boundary are directed outward the region) we have

\[
\frac{s_T}{s_S} = -1. \tag{2.15}
\]

Under these circumstances, the \((1+3)\)-decomposition of a conserved 3-form \(\alpha\) has necessarily to be of the form

\[
\alpha = -s_T (-d\tau \wedge \beta + \gamma) = s_S (-d\tau \wedge \beta + \gamma). \tag{2.16}
\]

Moreover, since the 3-form \(\gamma\) lies in a folio, it can be considered as a positive form (proportional to the volume element with a positive factor). Hence, the parameter \(s_T\) supplies a sign to the 3D-charge \(\gamma\) and to the 4D-current \(\alpha\). Two physically different possibilities are acceptable:
(i) The charges which can carry only positive sign (like the energy). In this case we have to choose $s_\mathcal{S} = 1$.

(ii) The charges that can carry both signs (like the electric charge). In this case the parameter $s_\mathcal{S}$ can be absorbed into the form $\gamma$.

In both cases, the decomposition of the form is given by

$$\alpha = -d\tau \wedge \beta + \gamma.$$  \hspace{1cm} (2.17)

If the integral conservation law of a $p$-form $\alpha$ holds for an arbitrary closed submanifold $\partial C^{p+1}$, the relation (2.10) is equivalent to $d\alpha = 0$. By using the (1+3)-splitting (2.8) of this form, we get

$$d\alpha = 0 \quad \iff \quad \begin{cases} 
 d\beta = -\dot{\gamma}, \\
 d\gamma = 0.
\end{cases} \hspace{1cm} (2.18)$$

3. Premetric electrodynamics in (1+3)-splitting

3.1. Electric charge conservation

On a 4D differential manifold $M$ with a chosen local observer, the electric charge current density $J$ allows to compute the total charge $Q$. For this, we have to integrate over a closed oriented 3D manifold which is transversal to the wordline of the observer. Consequently $J$ must be given by a twisted 3-form. The decomposition (2.4) of a current $J$ relative to the foliation $\Sigma_\tau$ can be written as

$$J = i_\mathcal{T} d\tau \wedge j + i_\mathcal{S} \rho.$$  \hspace{1cm} (3.1)

Here $\rho$ is a 3-form of the electric charge density with the same absolute dimension as $J$. In the absolute dimensions approach [11], we have $[J] = [\rho] = \text{charge}$. The 2-form $j$ is the electric current density with the absolute dimension of $[j] = \text{charge/time}$. Observe that both forms lie in the 3D-folio $\Sigma_\tau$, thus the 3-form $\rho$ has only one component, while the 2-form $j$ has only 3 independent components. Consequently the decomposition (3.1) is equivalent to the ordinary textbooks description.

Conservation of the electromagnetic charge means that the 3-form $J$ vanishes, if integrated over an arbitrary closed oriented 3D submanifold $\partial C_4 \in M$, i.e.,

$$\oint_{\partial C_4} J = 0.$$  \hspace{1cm} (3.2)

Since the electric charge can carry two opposite signs, the factor $i_\mathcal{S}$ can be absorbed into the charge form $\rho$ and consequently into $j$. Thus, we assume that $\rho$ can carry two possible signs. Then (2.17) yields

$$J = -d\tau \wedge j + \rho,$$  \hspace{1cm} (3.3)

which does not involve any undefined sign factors.

3.2. Inhomogeneous Maxwell equation

In premetric electrodynamics, the inhomogeneous Maxwell equation is treated as a consequence of the electric charge conservation law. Due to de Rham’s theorem, if a 3-form is closed in a region that does not admit non-trivial 3-cycles (cycles which are not boundaries of 4D regions), then it is exact. Under this condition, the conservation of
the electric current results in the existence of a twisted 2-form $H$ of the electromagnetic excitation

$$dJ = 0 \implies J = dH.$$  \hfill (3.4)

Due to this equation, the 2-form $H$ has the same absolute dimension as the current $J$, i.e., $[H] = \text{charge}$.

We write the $(1 + 3)$-decomposition (2.4) of $H$ as

$$H = h_H d\tau \wedge \mathcal{H} + h_S D.$$  \hfill (3.5)

Here $\mathcal{H}$ is the 1-form of the \textit{magnetic excitation} with an absolute dimension $[\mathcal{H}] = \text{charge/time}$, while $D$ is the 2-form of the \textit{electric excitation} with an absolute dimension $[D] = \text{charge}$. Since both these forms lie in the folio, they have 3 independent components, which is consistent with the textbooks description. We introduced again the sign factors $h_T$ and $h_S$ with values from the set $\{+1, -1\}$.

By applying the exterior derivative operator to the differential form $H$, we get

$$dH = d\tau \wedge (-h_T \mathcal{H} + h_S \dot{D}) + h_S dD.$$  \hfill (3.6)

Using (3.3), we obtain the $1 + 3$ decomposition of the equation (3.4)

$$dH = J \iff \begin{cases} h_T \mathcal{H} - h_S \dot{D} = j, \\ h_S dD = \rho. \end{cases}$$  \hfill (3.7)

The second equation, which is a scalar one in 3D, can be used to define the sign of the electric excitation $D$. Here the right hand side has a definite sign, which does not change under coordinate transformation. We assume

$$h_S = 1,$$  \hfill (3.8)

implying

$$H = h_T d\tau \wedge \mathcal{H} + D.$$  \hfill (3.9)

Such a choice implies that, for a positive charge, the form $D$ (similar to a corresponding vector) points out of the charge. Since the first equation is of a vector type, no preferred sign could be associated with $h_T$ by a choice of a sign of $\mathcal{H}$. This sign factor remains undefined.

Thus the inhomogeneous Maxwell pair of equations takes the form

$$\begin{cases} h_T \mathcal{H} - \dot{D} = j, \\ dD = \rho. \end{cases}$$  \hfill (3.10)

### 3.3. Homogeneous Maxwell equation

The homogeneous Maxwell equation is dealing with an untwisted 2-form $F$ of the \textit{electromagnetic field strength}. This field is defined by means of the Lorentz force density which expresses how the electromagnetic field $F$ acts on a test charge with charge density $J$. The differential form expression for this force density is

$$F_\alpha = (e_{\alpha}) F \wedge J,$$  \hfill (3.11)

where $e_{\alpha}$ describes the frame. It follows from (3.11) that the absolute dimension of the field $F$ is equal to $[F] = \text{action/charge}$, which differs from the dimension of the electric excitation $[H] = \text{charge}$, see [8].
The law of magnetic flux conservation is given by
\[ \oint_{C_2} F = 0 , \] (3.12)
for an arbitrary closed submanifold \( C_2 \). Since a two-dimensional submanifold of the \( 4D \)-manifold \( M \) cannot be oriented uniquely by a choice of a local observer, the integrand \( F \) has to be an untwisted 2-form. The equation (3.12) is assumed to hold for an arbitrary submanifold \( C_2 \subset M \). Thus, it is equivalent to the closure of the form \( F \):
\[ \oint_{C_2} F = 0 \iff dF = 0 . \] (3.13)

According to (3.13), the field strength \( F \) is determined only up to an arbitrary closed 1-form. However, the expression for the Lorentz force removes this ambiguity.

The \((1+3)\)-decomposition (2.4) of the electromagnetic field strength reads
\[ F = f_T d\tau \wedge E + f_S B , \] (3.14)
where \( E \) is the 1-form of the electric field strength while \( B \) is the 2-form of the magnetic field strength. Both forms lie in the folio. Thus, \( E \) as well as \( B \) has 3 independent components, which confirms with the standard description. Also here we introduced the sign factors \( f_T \) and \( f_S \) with values from \( \{ +1, -1 \} \).

By applying (2.8), the homogeneous Maxwell equation \( dF = 0 \) decomposes as
\[ dF = d\tau \wedge \left( -f_T dE + f_S \dot{B} \right) + f_S dB = 0 . \] (3.15)
Thus,
\[ dF = 0 \iff \begin{cases} f_T dE - f_S \dot{B} = 0 , \\ dB = 0 . \end{cases} \] (3.16)

Due to the homogeneity of these equations, only the quotient \( f_T/f_S \) can play a role. Thus one of the factors \( f_T, f_S \) is conventional. We choose
\[ f_S = 1 , \] (3.17)
implying
\[ F = f_T d\tau \wedge E + B . \] (3.18)

As we will see in the following, such a choice leads to the usual form of the Lorentz force. In this case, in vacuum or in an ordinary dielectric, the force acting on a positive charge is directed as the field \( E \). We will return to this point below, when the \((1+3)\) decomposition of the Lorentz force will be considered.

Left over is the homogeneous Maxwell pair
\[ \begin{cases} f_T dE - \dot{B} = 0 , \\ dB = 0 , \end{cases} \] (3.19)
with one undefined factor \( f_T \).
4. Constitutive relation

4.1. Linear relation and its restrictions

So far, the electromagnetic fields $F$ and $H$ describe two different and, at this stage of the construction, completely independent physical aspects. The electromagnetic excitation $H$ describes the field generated by the source. The field strength $F$ represent the force acting on a test current. The relation between these two fields is a necessary additional element of the formalism.

Also from the mathematical point of view the system is undefined. Indeed, the fields $F$ and $H$ have together 12 components, which are related by 8 independent field equations.

So a constitutive relation between the fields $F$ and $H$ is necessary. This relation can be of a rather involved form. For instance, in media with a complicated interior structure (such as a ferromagnet) non-linear and non-local constitutive relations are in use. Even in the linear case the corresponding tensor has 36 independent components, which can be decomposed into three irreducible pieces. In this paper, we will restrict the constitutive relation to its simplest principal part. First we require the operator $\kappa$ to be local and linear. In this case, a functional constitutive relation takes the form of a tensorial equation. In order to relate an untwisted 2-form to a twisted one, a constitutive pseudotensor $\kappa$ has to be involved

$$H = \kappa(F).$$ \hspace{1cm} (4.1)

Due to the linearity of the map $\kappa$, the $(1+3)$-decomposition of (4.1) is given by

$$h_T d\tau \wedge H + D = f_T \kappa(d\tau \wedge E) + \kappa(B).$$ \hspace{1cm} (4.2)

This linear relation is the most general one. In particular, the constitutive pseudotensor $\kappa$ has all its 36 independent components. In an $(1+3)$-decomposition, these components are arranged in four $3 \times 3$-matrices. Two of these matrices describe relations between electric fields and between magnetic fields. The two remaining matrices relate an electric field to a magnetic one and vice versa. We make now a further principal restriction: we assume that the operator $\kappa$ links separately the electric excitation $D$ to the electric field strength $E$ and the magnetic field strength $B$ to the magnetic excitation $H$. Thus we neglect the magnetoelectric cross effects (like the Faraday effect or optical activity). In fact, it is known that such effects can destroy the light cone structure. Recall, however, that in this paper we are interested only in the signature of the metric, not in its explicit form.

With these restrictions, equation (4.2) splits into

$$h_T d\tau \wedge H = \kappa(B), \quad D = f_T \kappa(d\tau \wedge E).$$ \hspace{1cm} (4.3)

Therefore we have to find an operator $\kappa$ that fulfills the following conditions

$$\kappa(d\tau \wedge E) = f_T D, \quad \kappa(B) = h_T d\tau \wedge H.$$ \hspace{1cm} (4.4)

Recall that all the fields involved here lie in one folio and that both undefined factors are equal to $\pm 1$.

4.2. Constitutive pseudotensor and Hodge map

Roughly speaking, in (4.4) the operator $\kappa$ has to perform a sequence of operations:

(i) To remove the timelike element $d\tau$, i.e., to apply the interior product with the vector $e_0 = \partial/\partial \tau$;
(ii) to transform a 1-form $E$ into a 2-form $\mathcal{D}$. Since both these form lie in the same folio, it can be done by applying the three-dimensional Euclidean Hodge map which we will denote by $\ast$.

(iii) we must also take into account that the physical dimensions of the fields $B$ and $\mathcal{D}$ are different. Thus we need a dimensional factor, which will be denoted by $\varphi$.

With these preparations, we can define the action of the operator $\kappa$ on an arbitrary 2-forms of the type $d\tau \wedge \alpha$ (where $\alpha$ is an arbitrary 1-form lying in the folio) by the relation

$$\kappa(d\tau \wedge \alpha) = \varphi \ast \alpha. \quad (4.5)$$

We must also define the action of the operator $\kappa$ on a 2-forms lying in the folio. Now the operator $\kappa$ has to transform in the folio a 2-form into a 1-form and to multiply the result by the time element $d\tau$. The dimensional factor also has to be involved. Consequently, for a spacelike 2-form $\beta$, we define

$$\kappa(\beta) = \psi d\tau \wedge \ast \beta. \quad (4.6)$$

This way we have constructed a special constitutive relation for an arbitrary 2-form in a 4D-manifold. Certainly this procedure is not the only possible one. A much more involved operator can be introduced [8] even in the local linear case. We assume that the operator defined by (4.5,4.6) is a principal ingredient of every generic constitutive map.

Since the 3-dimensional Euclidean Hodge map satisfies the relation $\ast^2 = 1$, we can derive from (4.5) and (4.6)

$$\kappa^2(d\tau \wedge \alpha) = \varphi \kappa(\ast \alpha) = \varphi \psi(d\tau \wedge \ast^2 \alpha) = \varphi \psi(d\tau \wedge \alpha) \quad (4.7)$$

and

$$\kappa^2(\beta) = \psi \kappa(d\tau \wedge \ast \beta) = \varphi \psi(\ast^2 \beta) = \varphi \psi(\beta). \quad (4.8)$$

Consequently the operator $\kappa^2$ acts on an arbitrary form only by multiplication by the scalar factor $\varphi \psi$. Thus, we find the reciprocity relation

$$\kappa^2 = (\varphi \psi) \text{id}; \quad (4.9)$$

for a complete discussion of the physical meaning that can be given to the reciprocity relation, see [8].

### 4.3. Isotropic media

We apply the definitions (4.5) and (4.6) to the electromagnetic fields appearing in (4.4),

$$\kappa(d\tau \wedge E) = \varphi \ast E, \quad \kappa(B) = \psi d\tau \wedge \ast B. \quad (4.10)$$

Consequently (4.4) yields

$$\mathcal{D} = f T \varphi \ast E, \quad \mathcal{H} = h T \psi \ast B. \quad (4.11)$$

These equations are reminiscent of the standard constitutive relations for isotropic media. Thus we introduce the electric permittivity and the magnetic permeability constants, respectively,

$$\varepsilon = f T \varphi, \quad \mu = h T \frac{1}{\psi}. \quad (4.12)$$
We can rewrite now the field equations (3.10) and (3.16) as

\[
\begin{align*}
\psi d(*B) - f_T \varphi \dot{E} &= j, \\
f_T d(*E) - \dot{B} &= 0, \\
f_T \varphi d(*E) &= \rho, \\
d B &= 0.
\end{align*}
\]

(4.13)

Observe that only one undefined factor appears in these equations.

5. Energy-momentum current

5.1. Premetric energy-momentum current

On a manifold endowed with a Minkowski metric, one is often dealing with a symmetric energy-momentum tensor. This tensor includes the products of the components of the fields \( F \) and \( H \) contracted by the metric tensor. Evidently such a construction is not suitable for the premetric approach which is dealing with electromagnetic fields on a bare differential manifold without a predefined metric structure. Moreover, the integration cannot be applied directly to the tensor components due to invariance argument. Since we need an integral over a region of a three-dimensional submanifold (the rest space), the integrand has to be a twisted differential 3-form.

In the differential forms formalism, the energy-momentum current is considered as a covector-valued 3-form \( \Sigma_\alpha \). In general, such a quantity has 16 independent components. Already, in order to extract from \( \Sigma_\alpha \) a symmetric energy-momentum tensor of 10 independent components, one needs a metric tensor. Indeed \cite{12}, if a metric tensor \( \eta^{\alpha\beta} \) is available, one can define a 2-form \( \eta^{\alpha\beta} e_\alpha \Sigma_\beta \) of 6 independent components. In the case when this 2-form vanishes, the covector-valued 3-form \( \Sigma_\alpha \) is equivalent to a symmetric tensor. In the axiomatic electrodynamics formalism \cite{8}, the energy-momentum current of the electromagnetic field is postulated as

\[
\Sigma_\alpha := \frac{1}{2} [(e_\alpha \lrcorner F) \wedge H - (e_\alpha \lrcorner H) \wedge F].
\]

(5.1)

Here \( e_\alpha \) is an arbitrary frame, not necessary a holonomic one. It is straightforward that for a coframe \( \vartheta^\alpha \), which is dual to \( e_\alpha \), the relation \( \vartheta^\alpha \wedge \Sigma_\alpha = 0 \) holds. This fact is equivalent to the tracelessness of the current \( \Sigma_\alpha \). Consequently, the current (5.1) has 15 independent components in general.

5.2. \((1 + 3)\) decomposition of the energy-momentum

Relative to a chosen observer with a tangential vector \( n \), one can define an energy-momentum current

\[
\Sigma := \frac{1}{2} [(n \lrcorner F) \wedge H - (n \lrcorner H) \wedge F].
\]

(5.2)

Evidently, \( \Sigma = n^\alpha \Sigma_\alpha \). Since we are interested in the energy of the electromagnetic field, it is sufficient to discuss the current \( \Sigma \).

Let us decompose \( \Sigma \) into time and space pieces. Because of \( n \lrcorner dt = 1 \) and \( n \lrcorner \mathcal{H} = n \lrcorner \mathcal{D} = n \lrcorner E = n \lrcorner B = 0 \) (forms lie in the folio \( \tau \)), we find, by using (3.9) and (3.18),

\[
\Sigma = \frac{1}{2} f_T E \wedge (h_T dt \wedge \mathcal{H} + \mathcal{D}) - \frac{1}{2} h_T \mathcal{H} \wedge (f_T dt \wedge E + B)
\]

\[
= - h_T f_T dt \wedge E \wedge \mathcal{H} + \frac{1}{2} f_T \mathcal{D} \wedge E - \frac{1}{2} h_T \mathcal{H} \wedge B.
\]

(5.3)
From the energy conservation law it follows that the 3-form $\Sigma_0$ must have the $(1 + 3)$-decomposition of the type (2.17)

$$\Sigma = -d\tau \wedge \sigma + u,$$

(5.4)

where the 3-form $u$ represents the electromagnetic energy, while the 2-form $\sigma$ is the electromagnetic energy flux. Both forms lie in the folio $\Sigma_{\tau}$. Thus, the electromagnetic energy flux is

$$\sigma = h_{\tau} f_{\tau} E \wedge \mathcal{H}$$

(5.5)

and the energy is

$$u = \frac{1}{2} f_{\tau} E \wedge \mathcal{D} - \frac{1}{2} h_{\tau} \mathcal{H} \wedge B.$$  

(5.6)

There is a strong physical requirement: The energy of the electromagnetic field has to be positive. The signs of the 3-forms $\mathcal{D} \wedge E$ and $\mathcal{H} \wedge B$ are determined when the constitutive relations (4.11) are used. Indeed,

$$u_{el} = \frac{1}{2} f_{\tau} E \wedge \mathcal{D} - \frac{1}{2} \phi E \wedge \ast E$$

(5.7)

and

$$u_{mag} = -\frac{1}{2} h_{\tau} \mathcal{H} \wedge B = -\frac{1}{2} \phi \ast B \wedge B.$$  

(5.8)

On a folio, the Euclidean metric is assumed. Thus the 3-forms $E \wedge \ast E$ and $\ast B \wedge B$ are positive. Consequently, both energy expressions (5.7) and (5.8) are positive if and only if

$$\phi > 0, \quad \psi < 0.$$  

(5.9)

The energy conservation law takes the usual form, which is in correspondence to the ordinary orientation of the boundary of a region, provided

$$d\Sigma_0 = 0 \iff d\sigma + \dot{u} = 0.$$  

(5.10)

From the Maxwell system (4.13) and the constitutive relations (4.11) we have in the source-free case

$$d \ast B = f_{\tau} \frac{\phi}{\psi} \ast \dot{E}, \quad d E = f_{\tau} \dot{B}.$$  

(5.11)

Thus,

$$d \sigma = h_{\tau} f_{\tau} d( E \wedge \mathcal{H}) = \psi f_{\tau} d( E \wedge \ast B) = \psi f_{\tau} (d E \wedge \ast B - E \wedge d \ast B)$$

$$= \psi \dot{B} \wedge \ast B - \phi \dot{E} \wedge \ast E = -\frac{1}{2} \frac{d}{dt} (\psi E \wedge \ast E - \psi B \wedge \ast B) = (5.12)$$

Consequently, the conservation law is in correspondence with the orientation of the boundary for arbitrary values of the sign factor $h_{\tau} f_{\tau}$. 


5.3. Four types of electromagnetic media

As a result of the consideration above, we remain with the undefined sign factors \( h_T \) and \( f_T \). Consequently, we have four possibilities for the signs,

\[
\begin{align*}
    h_T &= \pm 1, \quad f_T = \pm 1.
\end{align*}
\]  

(5.13)

Recall that positivity of the electromagnetic energy requires the condition \( (5.9) \) for the parameters \( \varphi \) and \( \psi \). Using the definition of the of the electric permittivity and the magnetic permeability constants, respectively,

\[
\varepsilon = f_T \varphi, \quad \mu = h_T \frac{1}{\psi},
\]

(5.14)

we can derive the following list for the possible signs for the electromagnetic constants:

| \( f_T \) | \( h_T \) | \( \varepsilon < 0 \) | \( \mu < 0 \) |
|---------|---------|----------------|----------------|
| +1      | -1      | -              | -              |
| -1      | +1      | +              | -              |
| -1      | +1      | +              | +              |
| +1      | -1      | +              | +              |

Using the inequalities \( (5.9) \), we are able to express the parameters \( \varphi \) and \( \psi \) and the sign factors via the electric permittivity and the magnetic permeability constants:

\[
\varphi = ||\varepsilon||, \quad \psi = -\frac{1}{||\mu||},
\]

(5.15)

and

\[
 f_T = \frac{\varepsilon}{||\varepsilon||}, \quad h_T = -\frac{\mu}{||\mu||}.
\]

(5.16)

6. Electromagnetic waves

One of the principal facts of electromagnetic phenomena is that the free electromagnetic field propagates by waves. Consider the free Maxwell equations

\[
\begin{align*}
    \psi \varepsilon d(B) - f_T \varphi \varepsilon \dot{E} &= 0, \\
    f_T \varepsilon \varphi d(B) &= 0, \\
    \mu d(E) &= 0, \\
    \mu B &= 0.
\end{align*}
\]

(6.1)

Applying to the first equation the Hodge dual and the time derivative and using the commutativity of these operations we obtain

\[
\psi \star \varepsilon \dot{\star B} - f_T \varphi \varepsilon \dot{E} = 0.
\]

(6.2)

From the first equation of the second system of \( (6.1) \), we have

\[
\dot{B} = f_T \varepsilon \dot{E}.
\]

(6.3)

Substituting it into \( (6.2) \), we can rewrite it as

\[
\dot{E} - \frac{\psi}{\varphi} \star \varepsilon \star d\varepsilon E = 0.
\]

(6.4)

Taking into account the equation \( d\varepsilon E = 0 \), we rewrite \( (6.3) \) as

\[
\dot{E} + \frac{\psi}{\varphi} \varepsilon \Delta E = 0.
\]

(6.5)
We introduced here the Laplace operator $\triangle = -dd^\dagger - d^\dagger d$ that in a three-dimensional Euclidean manifold acts on 1-forms as
\[ \triangle = -d\star d + \star d\star d \]  \tag{6.6}
and on 2-forms as
\[ \triangle = d\star d - \star d\star d. \]  \tag{6.7}

Taking into account (6.1), we are able to derive that all four fields $E, B, D, \text{ and } H$ satisfy the same type of equation
\[ \ddot{M} + \frac{\psi}{\varphi} \triangle M = 0. \]  \tag{6.8}

Since we assumed a Euclidean metric in the folio $\Sigma_\tau$, we recognize that in the case $\psi \varphi < 0$ (6.9) the equation (6.8) is hyperbolic and represents a wave equation. Thus, the corresponding four-dimensional optical metric is Minkowskian.

Observe that this result is independent on the sign parameters $f_T$ and $h_T$. Thus, it holds for any medium independently of the signs of the electromagnetic constants $\varepsilon$ and $\mu$. Indeed, due to (5.15), we can rewrite (6.8) as
\[ \ddot{M} - \frac{1}{|\varepsilon\mu|} \triangle M = 0. \]  \tag{6.10}

7. Lorentz force density

In this section, we turn to an additional and independent ingredient of electromagnetic theory — the Lorentz force. In the formalism of differential forms, the Lorentz force density is treated as a twisted covector-valued 4-form. Observe that this quantity has four independent components as in the ordinary vector description. The Lorentz force is postulated \cite{8} as
\[ F_\alpha = (e_\alpha \lrcorner F) \wedge J, \]  \tag{7.1}
where $e_\alpha$ denotes frame. When the $(1+3)$-decomposition of the field strength $F$ is substituted, the Lorentz force density (7.1) can be decomposed according to
\[ F_0 = -f_T E \wedge j \wedge d\tau \]  \tag{7.2}
and
\[ F_\mu = \left[ f_T (e_\mu | E) \rho + (e_\mu | B) \wedge j \right] \wedge d\tau, \quad \mu = 1, 2, 3. \]  \tag{7.3}

Recall that we already have chosen the sign factor $f_3 = +1$.

In particular, the Lorentz force governs the law of attraction and repulsion between charged particles. In order to understand how the sign parameters are related to this law, we consider a system of two static charged particles. In this case, the Maxwell field equations (4.13) show that the field $E$ is static and satisfies the equations
\[ dE = 0, \quad d\star E = \frac{f_T}{\varphi} \rho_0. \]  \tag{7.4}
Here $\rho_0$ is the charge density of the source. The sign of the solution of these equations, the sign of the field $E$, is proportional to the sign of the factor $\frac{f}{\varphi}$. When the solution is substituted into the electrostatic part of the Lorentz force,

$$\mathcal{F}_\mu = f_T(\epsilon_\mu |E|)\rho \land d\tau, \quad \mu = 1, 2, 3,$$

(7.5)

the force density is generated in a special direction. We derive that the sign of the field $\mathcal{F}_\mu$ is proportional to the sign of the product $f^2 T \varphi = \varphi$. Comparing to ordinary dielectric materials (the first row in Table 1), we derive that in all media with a positive factor $\varphi$ the ordinary Dufay rule of attraction between opposite charges and repulsion between charges of the same sign holds true:

$$\varphi > 0 \iff \text{Dufay rule.}$$

(7.6)

Consider now the pure magnetic contribution to the Lorentz force. Due to (4.13), the magnetostatic case can be described by the field $B$ satisfying the equations

$$\psi d(\ast B) = j, \quad dB = 0.$$  

(7.7)

Consequently, the direction of the field $B$ is determined by the sign of the parameter $\psi$. A solution of (7.7) is substituted now into the magnetic part of the Lorentz force,

$$\mathcal{F}_\mu = (\epsilon_\mu |B|) \land j \land d\tau, \quad \mu = 1, 2, 3.$$

(7.8)

Thus, also the direction of this force is determined by the sign of the parameter $\psi$. Comparing to ordinary media which are described by a negative parameter $\psi$, we derive that all cases with $\psi < 0$ have the same behavior. In particular, the Lenz rule (pulling of a ferromagnetic core into a solenoid independently on the direction of the current) holds true:

$$\psi < 0 \iff \text{Lenz rule.}$$

(7.9)

### 8. Summary: Four types of media

In this section we give a brief summary of the physical properties of the four types of media derived above. Recall that, in the $4D$ formalism, all these media are described by the same system of the field equations

$$dF = 0, \quad dH = J.$$  

(8.1)

Moreover, we are dealing with the same isotropic constitutive relation. Also the $4D$ expressions for the energy-momentum

$$\Sigma_\alpha := \frac{1}{2} [ (\epsilon_\alpha |F|) \land H - (\epsilon_\alpha |H|) \land F ]$$

(8.2)

and for the Lorentz force density

$$\mathcal{F}_\alpha = (\epsilon_\alpha |F|) \land J,$$

(8.3)

are the same. In all cases, the energy-momentum is conserved and its electric and magnetic energy parts are positive. These parts can be written, using the absolute values of the electromagnetic parameters of the media, as

$$u_{el} = \frac{1}{2} |\varepsilon| E \land \ast E, \quad u_{mag} = \frac{1}{2|\mu|} B \land \ast B.$$  

(8.4)

The standard Dufay and Lenz rules are satisfied. As we have shown, also here the absolute values of the electromagnetic parameters of the media appear.
The difference between the four types of media originates from different expansions of the four-dimensional fields relative to a chosen observer. Is it possible that all of these types present the same laws and transform one into another by redefinitions? In order to answer this question, let us consider the 4D invariants of the electromagnetic field. They are defined as follows:

\[ I_1 = F \wedge F = 2 f T \epsilon \wedge E \wedge B, \quad (8.5) \]

\[ I_2 = H \wedge H = 2 h T \mu \wedge E \wedge B, \quad (8.6) \]

and

\[ I_3 = F \wedge H = f T \epsilon \wedge E \wedge B \quad \text{plus terms involving} \quad h T \mu \]

Since the expressions \( f T \epsilon \) and \( h T \mu \) are the same in all four models, the invariant \( I_3 \) is also the same. However, the invariants \( I_1 \) and \( I_2 \) are different. Thus, they can be used for the separation of the physical features of our models.

|       | \( f_T = +1 \) | \( f_T = -1 \) | \( f_T = +1 \) | \( f_T = -1 \) |
|-------|----------------|----------------|----------------|----------------|
| \( h_T = -1 \) | \( \epsilon > 0, \mu > 0 \) | \( \epsilon < 0, \mu < 0 \) | \( \epsilon < 0, \mu > 0 \) | \( \epsilon > 0, \mu < 0 \) |
| \( H = \) | \( -d\tau \wedge H + D \) | \( d\tau \wedge H + D \) | \( d\tau \wedge H + D \) | \( -d\tau \wedge H + D \) |
| \( F = \) | \( d\tau \wedge E + B \) | \( -d\tau \wedge E + B \) | \( d\tau \wedge E + B \) | \( -d\tau \wedge E + B \) |
| Maxwell-1 | \( d\mathcal{H} - \dot{D} = j \) | \( d\mathcal{H} - \dot{D} = j \) | \( d\mathcal{H} - \dot{D} = j \) | \( -d\mathcal{H} - \dot{D} = j \) |
| Maxwell-2 | \( dD = \rho \) | \( dD = \rho \) | \( dD = \rho \) | \( dD = \rho \) |
| Maxwell-3 | \( dE - \dot{B} = 0 \) | \( dE + \dot{B} = 0 \) | \( dE - \dot{B} = 0 \) | \( dE + \dot{B} = 0 \) |
| Maxwell-4 | \( dB = 0 \) | \( dB = 0 \) | \( dB = 0 \) | \( dB = 0 \) |
| invariants | \( I_1, I_2 \) | \( -I_1, -I_2 \) | \( I_1, -I_2 \) | \( -I_1, I_2 \) |

In Table 1 we present the main features of the four types of isotropic electromagnetic media. Notice that the sign in front of \( d\mathcal{H} \) in the first Maxwell equation is opposite to the sign ordinarily used. This is due to our form of the 1 + 3 decomposition of \( H \).
9. Conclusions and discussion

Minkowski’s idea on the four-dimensional space as a proper physical reality includes a 4D reformulation of Maxwell’s electrodynamics. We start with a premetric 4D construction which is explicitly covariant and does not even involve a metric tensor. Alternatively, the metric with Minkowskian signature may be incorporated into the structure of the field equations and the constitutive relations. This structure is unique, provided the following physically meaningful conditions are assumed:

i. Charge conservation;
ii. Flux conservation;
iii. Positivity of the electric and magnetic energy;
iv. Energy-momentum conservation;

iv. Local linear constitutive law which we restrict to the isotropic form.

Does this mean that the backward (1 + 3)-splitting of the general covariant Maxwell system is unique? We show in this paper that the answer is negative. In the framework of the conditions listed above, we derive that four different sets of the three-dimensional Maxwell equations correspond to the same four-dimensional system. These sets are different because of the signs of the values of the electromagnetic constants. So we can treat them as four different electromagnetic media. The main features of standard electrodynamics remain true for all these media. In particular, the electric and magnetic energies are positive and the standard direction of the Lorentz force (Dufay’s and Lenz’s rules) is preserved. However, these models differ from one another in particular by the 4D electromagnetic invariants.

Recently media with non-positive electromagnetic parameters are studied intensively [13]–[16]. Moreover, such materials (called metamaterials, or left-handed materials) are already manufactured and even proposed to be useful in technology. However, the description of such materials is sophisticated. In particular, these metamaterials are treated sometimes as electromagnetic media in which the energy of the electromagnetic field is negative. Note that the theoretical approach used in metamaterials is completely different from ours. In particular, the Maxwell equations which were developed and tested only for $\varepsilon > 0, \mu > 0$ are used in the situations where such assumptions do not hold. It is not surprising that contradictions to the basic laws of physics emerge.

Alternatively, in our approach the same type of non-ordinary media are derived from the unique 4D system and the basic physical laws are preserved.

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