Some Examples of Algebraic Geodesics on Quadrics

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SOME EXAMPLES OF ALGEBRAIC GEODESICS ON QUADRICS

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In this note we give the conditions for the existence of algebraic geodesics on some two-dimensional quadrics, namely, on hyperbolic paraboloids and elliptic paraboloids. It appears that in some cases, such geodesics are rational space curves.

Keywords: Integrable systems; two-dimensional quadrics; algebraic geodesics.

1. Introduction

The problem of geodesics on the second order surfaces (quadrics) is a classical one. For a two-dimensional ellipsoid, an explicit description of geodesics was given by Jacobi [4] and Weierstrass [8]. For other quadrics, this problem was considered by Halphen [2] and Hadamard [3] (for the modern exposition of this topic, see [5–7]).

It is well known that the generic geodesic on a two-dimensional quadric is a transcendental space curve. However, in some cases, this geodesic becomes an algebraic space curve. Hence, such geodesics may be considered as the complete intersection (or a connected component of the intersection) of the two-dimensional quadric with the algebraic surface in the space \( \mathbb{R}^3 \).

In the paper [1], an approach was proposed for the description of such surfaces in the case of two-dimensional ellipsoid and some of them were described explicitly.

In this note we give the conditions for the existence of algebraic geodesics on other two-dimensional quadrics, namely, on hyperbolic paraboloids and elliptic paraboloids. It appears that in some cases, such geodesics are rational space curves.

2. Hyperbolic Paraboloid

The equation for the hyperbolic paraboloid in the three-dimensional Euclidean space \( \mathbb{R}^3 \) can be expressed in the form

\[
\frac{x^2}{a} - \frac{y^2}{b} - 2z = 0, \quad a > 0, \quad b > 0.
\]
Following [3] let us express coordinates $x, y, z$ in terms of elliptic coordinates $\lambda, \mu$:

\begin{align}
  x^2 &= -\frac{a(\lambda + a)(\mu + a)}{a + b}, \\
  y^2 &= -\frac{b(\lambda - b)(\mu - b)}{a + b}, \\
  2z &= -(\lambda + \mu + a - b), \quad \lambda < -a, \quad \mu > b.
\end{align}

In these coordinates, the element of length takes the form

\begin{equation}
  ds^2 = \frac{\lambda - \mu}{4} \left[ \frac{\lambda d\lambda^2}{(\lambda + a)(\lambda - b)} - \frac{\mu d\mu^2}{(\mu + a)(\mu - b)} \right],
\end{equation}

and the geodesic is given by the equation

\begin{equation}
  \int d\lambda \sqrt{\frac{\lambda}{(\lambda + c)(\lambda + a)(\lambda - b)}} = \int d\mu \sqrt{\frac{\mu}{(\mu + c)(\mu + a)(\mu - b)}},
\end{equation}

where a constant $c$ characterizes the geodesic.

Recall that on the hyperbolic paraboloid there are two families of straight lines, and any such line is geodesic. Moreover, two principal parabolas (the intersection of planes $x = 0$ and $y = 0$ with paraboloid) are also geodesics.

As it was shown in [3], if $c \neq a$ or $c \neq -b$, then $\mu \to +\infty$ as $t \to +\infty$ or $t \to -\infty$ and any geodesic tends to the straight line given by the formulae

\begin{align}
  x &\sim \pm \sqrt{\frac{qa}{a + b}}, \\
  y &\sim \pm \sqrt{\frac{qb}{a + b}}, \\
  2z &\sim (1 - q)\mu, \quad q = \text{const} = \lim_{t \to +\infty} \frac{|\lambda|}{\mu}.
\end{align}

The main result of this note is the following theorem.

**Theorem 1.** If $c = a$ and $\sqrt{\frac{a + b}{a}}$ is a rational number, the geodesic defined by Eq. (6) is an algebraic curve.

**Proof.** In this case, the elliptic integral in (6) reduces to a more simple form:

\begin{equation}
  \int d\lambda \sqrt{\frac{\lambda}{(\lambda + a)(\lambda - b)}} = \int d\mu \sqrt{\frac{\mu}{(\mu + a)(\mu - b)}}.
\end{equation}

The integrand $I$ has two poles at $\lambda = -a$ and $\lambda = \infty$:

\begin{align*}
  I &\sim \frac{1}{(\lambda + a)} \quad \text{as} \quad \lambda \to -a, \\
  I &\sim \frac{1}{\lambda} \quad \text{as} \quad \lambda \to \infty.
\end{align*}

Here

\begin{equation}
  r = \sqrt{\frac{a + b}{a}}.
\end{equation}

It is easy to see that if $r$ is a rational number $r = p/q$ (for $p$ and $q$ integer), then the integral in (8) is the logarithm of an algebraic function.
In fact, the integral in (8) can be calculated explicitly by means of the standard change of variables \((\lambda, \mu) \to (\xi, \eta)\):

\[
\lambda = \frac{b}{1 - \xi^2}, \quad \mu = \frac{b}{1 - \eta^2}
\]  

(10)

In this way we come to the algebraic equation

\[
(1 - \xi)^p(1 - \eta)^q(p + \eta)^r - c_2(1 + \xi)^p(1 + \eta)^q(r - \xi)^q(r - \eta)^q = 0,
\]

(11)

where \(c_2\) is a constant of integration. Changing variables \((\xi, \eta)\) for \((x, y)\) and using Eqs. (2) and (3) we get the equation

\[
F(x, y) = 0,
\]

(12)

where \(F(x, y)\) is a polynomial in \((x, y)\). Hence, this equation defines an algebraic plane curve. The variable \(z\) may be found now from (1) or (4). Then we obtain an algebraic space curve.

So, we proved that our geodesic is an algebraic space curve.

Note that our geodesic asymptotically approaches the principal parabola \((x = 0)\) as \(t \to \infty\) and a straight line as \(t \to -\infty\). It also has the property: \(x > 0, y(t_0) = 0, y > 0, t > t_0; \ y < 0, \ t < t_0\).

Note also that \((\lambda + a) \sim \mu^{-r}\) as \(\lambda \to -a, \mu \to \infty\).

**Example 1.** Let us consider in more detail the case \(a = 1, b = 3, r = 2, p = 2, q = 1\). Taking \(c_2 = 1\) in (11) we get the algebraic equation for new variables \(\xi, \eta\)

\[
(1 - \xi)^2(1 - \eta)^2(2 + \xi)(2 + \eta) - (1 + \xi)^2(1 + \eta)^2(2 - \xi)(2 - \eta) = 0.
\]

(13)

Expanding the left-hand side we get

\[
(\xi + \eta)(\xi^2 + \eta^2 - \xi - 3) = 0.
\]

(14)

Using Eqs. (2), (3), and (14) we obtain simple expressions for \(x\) and \(y\):

\[
x^2 = \frac{1}{4} \left( 2 - \frac{\zeta}{\zeta + 1} \right), \quad y^2 = \frac{9}{4} \frac{\zeta^2}{(2 - \zeta)(\zeta + 1)}.
\]

(15)

where

\[
\zeta = \xi \eta, \quad -1 < \zeta < 2.
\]

(16)

Eliminating \(\zeta\) from these equations we get a relation between \(x\) and \(y\)

\[
x \left( x - \frac{y}{\sqrt{3}} \right) = \frac{1}{2}, \quad \text{or} \quad x \left( x + \frac{y}{\sqrt{3}} \right) = \frac{1}{2}.
\]

(17)

From this we obtain a parametrization of our geodesic:

\[
x = \frac{\tau^2 + 1}{2}, \quad y = \sqrt{3} \frac{\tau^2 - 2}{2\tau}, \quad z = \frac{\tau^2 - 1}{2\tau^2}.
\]

(18)

Hence, the geodesic tends to the straight line in the plane \(z = 1/2\) as \(\tau \to \infty\) (in accordance with [3]), and to the principal hyperbola \((x = 0)\) as \(\tau \to 0\).
Observe that if
\[
\frac{x^2}{1} - \frac{y^2}{3} = 2z
\]
we have
\[
(\lambda + 1) \sim -\frac{1}{\mu^2} \quad \text{as} \quad \mu \to \infty.
\]
In a more general case
\[
\frac{x^2}{1} - \frac{y^2}{b} = 2z, \quad b = n^2 - 1, \quad n \text{ is integer},
\]
we have
\[
(\lambda + 1) \sim -\frac{\alpha_n}{\mu^n} \quad \text{where} \quad \alpha_n \text{ is constant}.
\]
From this it follows that
\[
x^2 \sim \frac{1}{\mu^{n+1}}, \quad y^2 \sim (n^2 - 1)\mu, \quad xy^{n-1} \to \text{const}.
\]

3. Elliptic Paraboloid

The basic formulae for this case are similar to the formulae of previous section, so we give just few ones. The equation for elliptic non-degenerate paraboloid in the three-dimensional Euclidean space $\mathbb{R}^3$ has the form
\[
\frac{x^2}{a} + \frac{y^2}{b} - 2z = 0, \quad a > b > 0.
\]
In elliptic coordinates $\lambda$ and $\mu$, we have
\[
x^2 = -\frac{d(\lambda + a)(\mu + a)}{a - b},
\]
\[
y^2 = -\frac{b(\lambda + b)(\mu + b)}{b - a},
\]
\[
2z = -(\lambda + \mu + a + b), \quad \lambda < -a, \quad -a < \mu < -b.
\]
The expression for the element of length is of the form
\[
ds^2 = \frac{\lambda - \mu}{4} \left[ \frac{\lambda d\lambda^2}{(\lambda + a)(\lambda + b)} - \frac{\mu d\mu^2}{(\mu + a)(\mu + b)} \right]
\]
and the geodesic is represented by the equation
\[
\int d\lambda \sqrt{\frac{\lambda}{(\lambda + c)(\lambda + a)(\lambda + b)}} = \int d\mu \sqrt{\frac{\mu}{(\mu + c)(\mu + a)(\mu + b)}},
\]
where a constant $c$ characterizes the geodesic.

Theorem 2. If $c = a, \quad r = \sqrt{(a - b)/a}$ is a rational number, the geodesic defined by Eq. (28) is an algebraic curve.

Proof. It is completely analogous to the proof of Theorem 1.  \qed
Example 2. In the simplest case $r = 1/2$, we have
\[ \frac{x^2}{3} + \frac{y^2}{4} - 2z = 0. \]  
(29)

Taking $c_2 = 1$ in (11) we come to the equation
\[ (1 - \xi)(1 - \eta) \left( \frac{1}{2} + \xi \right)^2 \left( \frac{1}{2} + \eta \right)^2 - (1 + \xi)(1 + \eta) \left( \frac{1}{2} - \xi \right)^2 \left( \frac{1}{2} - \eta \right)^2 = 0. \]  
(30)

After simplifications we get
\[ (\xi + \eta) \left( \xi^2 + \eta^2 - \xi \eta - \frac{3}{4} \right) = 0. \]  
(31)

This equation implies the following equations for coordinates $x$ and $y$:
\[ \frac{x^2}{4} = 10 \frac{\xi + \frac{1}{2}}{\zeta}, \quad \frac{y^2}{3} = 9 \frac{\zeta^2}{(\zeta - \frac{1}{2})^2}, \]  
(32)

where
\[ \zeta = \xi \eta, \quad -\frac{1}{4} \leq \zeta < \frac{1}{2}. \]

Eliminating $\zeta$ from these equations we obtain a simple relation between $x$ and $y$:
\[ \frac{y}{\sqrt{3}} = \frac{x^2}{32} - 1. \]  
(33)

So, our geodesic is the intersection of the elliptic paraboloid (29) with the parabolic cylinder (33).

Note that in both above examples the geodesic is a rational space curve. It would be interesting to find other examples of rational geodesics on hyperbolic and elliptic paraboloids.

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