New Consistent Limits of M-theory

Michael Faux
Humboldt-Universität zu Berlin, Institut für Physik
D-10115 Berlin, Germany

Abstract

The construction of effective field theories describing M-theory compactified on $S^1/Z_2$ is revisited, and new insights into the parameters of the theory are explained. Particularly, the web of constraints which follow from supersymmetry and anomaly cancelation is argued to be more rich than previously understood. In contradistinction to the lore on the subject, a consistent classical theory describing the coupling of eleven dimensional supergravity to super Yang-Mills theory constrained to the orbifold fixed points is suggested to exist.
1 Introduction

In recent years M-theory has challenged the weakly-coupled heterotic string as the most phenomenologically promising potentially-ultimate fundamental description of nature. As yet, the rudiments of M-theory remain mostly obscure. Matrix theory and brane dynamics are two endeavors which seem likely to bear fruit on this issue. Regardless of the precise microscopic description of M-theory, of fundamental importance are the predictions made by the theory on low-energy phenomena. Fortunately, despite our present shortcomings in understanding all microscopic aspects, we have solid insights into the low-energy regime of the theory. It is the purpose of this paper to describe some new observations in this direction.

Since M-theory, appropriately compactified, coincides with the strongly-coupled limit of the type IIA or the heterotic $E_8 \times E_8$ string theories, there exist ample clues to the low-energy description of the theory. As forcefully demonstrated in [1] and [2], of central importance to this construction is eleven-dimensional supergravity. The effective theory is then further determined by a web of constraints, involving issues from supersymmetry to anomaly freedom, which collectively suffice to unambiguously determine at least certain leading terms in an effective action. This construction has had encouraging success in explaining away some phenomenological shortfalls which have plagued weakly-coupled perturbative string phenomenology. Notable is the fact that the four-dimensional Newton constant can take plausibly relevant magnitudes when derived from M-theory $^1$.

Effective M-theories are interesting from a purely field-theoretical standpoint. This is so particularly with regard to consistency. Exotic anomaly cancellation devices, like the Green-Schwarz mechanism crucial for weakly-coupled string phenomenology, are rare. An analogous device is required for a viable M-theory phenomenology. This issue was examined in [3], where the consistent coupling of eleven-dimensional supergravity to ten-dimensional super Yang-Mills theories propagating on orbifold fixed-points was addressed. Central to both the weakly-coupled string effective theories, and also to the effective M-theories is the coupling of Chern-Simons forms to higher-degree tensors. Such couplings are generically manifest in Bianchi identities. For the case of weakly-coupled heterotic string theory, the celebrated result
\[ dH \sim F \wedge F - R \wedge R \]

is characteristic. The proper implementation of an analogous relation is central to M-theory phenomenology. In this paper we demonstrate a generalization of the construction presented in [3], describe how this generalization is important for the proper implementation of anomaly cancelation, and explain how it gives rise to additional couplings relevant to phenomenology.

It was asserted in the original detailed work on this subject, particularly [3], that there exists a consistent coupling of eleven-dimensional supergravity to ten-dimensional super Yang-Mills theory propagating on orbifold fixed-points, but only if quantum effects are included. The impossibility of a consistent classical coupling of the sort described above, has become part of M-theory lore. In this paper we challenge that assertion, and suggest that a consistent classical theory may indeed exist. This fact is directly related to the generalizations mentioned in the previous paragraph.

In the weakly-coupled heterotic string effective theory, a central ingredient is the necessity for the two-form gauge potential in ten-dimensional $N = 1$ supergravity to transform nontrivially under gauge transformations associated with minimally coupled Yang-Mills supermultiplets. In the classical theory this poses no obstruction to consistency because the simultaneous inclusion of Chern-Simons forms, properly coupled to the tensor fields, fully compensates for any violation of gauge symmetry. In the quantum theory, this feature is crucial for the implementation of gauge anomaly cancelation through the addition of counterterms not needed at the classical

$^1$There do, however, exist other solutions to this problem, which do not require M-theory. See [4] for a comprehensive review of this issue.
level.

In effective M-theories, it would seem plausible that the three-form \( C_{IJK} \) in eleven-dimensional supergravity is likewise forced to transform nontrivially under transformations \( \delta_\theta \) associated with minimally coupled Yang-Mills supermultiplets. In the quantum theory this indeed turns out to be so. In [3] it was assumed that even the minimal classical theory would require \( \delta_\theta C_{IJK} \neq 0 \). In that case, the \( C \wedge G \wedge G \) interaction, which is unavoidable in eleven-dimensional supergravity, would violate gauge invariance. In other words, were the above assumptions to be true, there would be an obstruction to constructing a classical theory which simultaneously respects both local supersymmetry and Yang-Mills gauge invariance. In the quantum theory the classical obstruction dissipates because the \( C \wedge G \wedge G \) interaction becomes a counterterm whose variation exactly cancels another anomalous variation arising as a loop effect. All of this has been argued as evidence that M-theory exists only as a quantum theory. Although this suggestion would be interesting, there is a loophole which needs to be properly examined.

The minimal coupling of eleven-dimensional supergravity to Yang-Mills supermultiplets propagating on orbifold fixed-points unavoidably requires a modification to a Bianchi identity satisfied by the four-form field strength \( G_{IJKL} \), to include a contribution from the Yang-Mills field strengths \( F_{AB} \). This modification follows from minimally implementing supersymmetry. However, the modified relation does not completely fix the dependence of \( G_{IJKL} \) on the Yang-Mills potentials \( A_\alpha^A \). The general solution to the modified Bianchi identity still allows a freedom to manipulate features from the purely ten-dimensional components \( G_{ABCD} \) to the mixed components \( G_{(11)ABC} \). Thus, supersymmetry alone does not completely determine \( G_{IJKL} \). However, this additional freedom is fixed by the additional requirements of gauge invariance and local Lorentz invariance. This consideration may allow for a consistent classical coupling because in the absence of quantum anomalies it does become possible to organize the theory to contain simultaneously a nontrivially modified Bianchi identity and a three-form \( C_{IJK} \) which remains a Yang-Mills invariant. The effective theory of Hořava and Witten [3] does not take into account this possible interplay between the two types of components of \( G_{IJKL} \).

It is useful to consider the analogous situation in weakly-coupled heterotic string theory by way of contrast. In that case, the dependence of the ten-dimensional three-form field strength \( H_{ABC} \) on the Yang-Mills gauge fields is completely fixed by supersymmetry. Because the Yang-Mills fields and the three-form each have only ten-dimensional components, there is no flexibility analogous to the case in M-theory. But, in the case of the weakly-coupled heterotic string one can nevertheless construct an effective theory consistent even in the classical limit because supersymmetry alone does not require the counterterms necessary for quantum anomaly cancelation. Such terms are therefore omitted in that limit.

In addition to the considerations described above, the effective M-theory construction also requires modifications to the transformation rules associated with eleven-dimensional supergravity and ten-dimensional super Yang-Mills theory. In particular, the supersymmetry transformation rule for the three-form \( C_{IJK} \) obtains contributions from the Yang-Mills gauge fields \( A_\alpha^A \). The modification in the rule for the mixed components \( \delta_\theta' C_{AB(11)} \) was described in [3]. The consistent implementation of the mechanism described above requires as well a modification to the supersymmetry transformation rule for the purely ten-dimensional components \( C_{ABC} \). This is described in this paper.

Upon publication of [3], much attention was immediately focussed on the phenomenological consequences of the effective theory described in that paper. Notable efforts have described M-theoretic modifications to string threshold effects [6, 8], and possible M-theoretic explanations of supersymmetry breaking [4, 10, 11]. But scant attention has been paid to various relevant

---

2In this paper \( I, J, K \) are eleven-dimensional world indices taking the values 1, ..., 11, while indices \( A, B, C \) refer to the ten-dimensional subset 1, ..., 10.
issues at the heart of the theory, and no complete analysis has been paid to the problem of consistency. For instance, of crucial importance is the cancellation of gravitational and mixed anomalies. Hoˇ rava and Witten [3] have described the cancellation of gauge anomalies in some detail, and have strongly motivated the cancellation of gravitational and mixed anomalies. But a proof that the coefficients work precisely is lacking. We argue in this paper that a detailed analysis of gravitational and mixed anomalies in M-theory, which is crucial to justify effective theories used as the basis for M-theory phenomenology, is incomplete. We argue as well that the proper implementation of anomaly cancelation necessitates the new generalizations described several times already in this introduction.

To clarify this last point, as described previously, one of the Green-Schwarz-like counterterms in the low-energy description of M-theory is precisely the erstwhile-enigmatic $C \wedge G \wedge G$ interaction, which is present even in the minimal classical theory due to supersymmetry. Additional counter terms, crudely of the form $C \wedge R^4$ where $R$ is the Ricci two-form, are necessary for the cancellation of gravitational and mixed anomalies. The cancellation of gravitational and mixed anomalies places constraints on the coefficients of these terms. Importantly, these constraints must be proven consistent with other constraints posed by supersymmetry and the cancellation of pure gauge anomalies. Consistency amongst all of these constraints requires the new modifications which we present in this paper.

A mechanism similar to the one discussed in this paper was presented in [5] and also in [6]. In those papers, as in this one, an important ingredient is to include the general solution to the modified Bianchi identity for $G_{IJKL}$, as opposed to the specific, restricted solution used in the original work [3].

When comparing expressions in this paper or expressions in most recent literature on M-theory to most supergravity literature, one often finds discrepancies in factors of the gravitational coupling $\kappa$. These discrepancies can be reconciled with appropriate rescaling of fields. To be clear, we list the dimensionality of relevant objects in the table.

With the conventions listed in the table, both the gravitational coupling constant $\kappa$ and the Yang-Mills coupling constant $\lambda$ completely factor out of the respective zeroth order (uncoupled) actions. See equations (2.5) and (2.7) below to clarify this statement. All modifications necessary for a consistent coupling between these two sectors can be classified in terms of ratios of these parameters. Of particular interest is the dimensionless combination $\lambda^6/\kappa^4$, which serves as an important parameter in the theory. One finds that its value is determined by consistency.

Throughout this paper we work in the so-called “upstairs” picture, which means that our

| object | $\kappa$ | $\lambda$ | $\epsilon$ | $e_I^a$ | $C_{IJK}$ | $\psi_I$ | $A_A^a$ | $\chi^a$ |
|--------|----------|----------|----------|--------|-----------|---------|--------|--------|
| dimension | -9/2 | -3 | -1/2 | -1 | -3 | -1/2 | 0 | 3/2 |

Table: The dimensions of objects relevant to the low-energy description of M-theory, in units of length. For example, the gravitational coupling constant $\kappa$ has dimension $(\text{length})^{-9/2}$ and the $E_8$ coupling constant $\lambda$ has dimension $(\text{length})^{-3}$. 
eleven-dimensional spacetime is always assumed to be the orbifold $R^{10} \times S^1/\mathbb{Z}_2$, described in more detail below. In [3] it is advocated that one can alternately describe the spacetime in a “downstairs” picture, as an eleven-dimensional manifold with two ten-dimensional boundaries. We avoid this second interpretation in this paper.

We should clarify the relationship of the ten-dimensional metric $g_{AB}$, which we use to raise and lower ten-dimensional indices on fields constrained to orbifold fixed-points, to the eleven-dimensional metric $g_{IJ}$. At the orbifold fixed-points the elfbein $e_I^a$ can be taken block-diagonal and is defined to be

$$e_I^a \bigg| = \begin{pmatrix} e_A^m \\ \phi \end{pmatrix}, \quad (1.1)$$

where the vertical bar indicates that the expression is being evaluated at an orbifold fixed point. Thus, we do not include the physically irrelevant conformal factor of $\phi^{-1/8}$ which would multiply $e_A^m$ in this decomposition to ensure an Einstein-normalized gravitational kinetic action in ten-dimensions if we were performing a dimensional reduction. Since we are not performing a dimensional reduction, we absorb this conformal factor into our definition of $e_A^m$. This is most natural in this context since it optimally simplifies all expressions. Furthermore it allows for the direct identification of the ten-dimensional supersymmetry parameter with the restriction to the fixed hyperplanes of the eleven-dimensional supersymmetry parameter.

Spacetime is taken to be eleven-dimensional. The eleventh dimension is compact, and takes values on the interval $[-\pi, \pi]$, with endpoints identified. Additionally a $\mathbb{Z}_2$ projection, which defines the orbifolding, enforces invariance under $x^{11} \rightarrow -x^{11}$. There are two ten-dimensional hyperplanes which are fixed under this projection, defined by $x^{11} = 0$ and $x^{11} = \pi$. As a consequence of the projection, conditions are placed on the behavior of the eleven-dimensional fields at the $\mathbb{Z}_2$ fixed-points. The elfbein is constrained as indicated in (1.1). Additionally, the components of the three-form $C_{IJK}$ which have purely ten-dimensional indices are constrained to vanish at these points. Thus, $C_{ABC} \bigg| = 0$. The components of the gravitino field $\psi_I$ with ten-dimensional indices are constrained to a specific chirality from the ten-dimensional point of view. Thus,

$$\Gamma_{11} \psi_A \bigg| = \pm \psi_A \bigg| \quad (1.2)$$

The remaining component $\psi_{11}$ has the opposite chirality at the $\mathbb{Z}_2$ fixed points. Half of the supersymmetry is broken at the fixed points by the projection; the supersymmetry parameter satisfies $\Gamma_{11} \epsilon \bigg| = \pm \epsilon \bigg|$. This paper is organized as follows.

In section 2 we describe a systematic method for deriving the consistent coupling of eleven-dimensional supergravity to ten-dimensional super Yang-Mills theory propagating on the fixed hyperplane defined by $x^{11} = 0$. An identical derivation would pertain to the couplings at the other fixed hyperplane defined by $x^{11} = \pi$. This derivation, and indeed this entire paper, is inspired by the analogous derivation in [3]. But the systematics which we employ differ somewhat, and our results include an important generalization to the results of that previous work. We demonstrate how a consistent classical coupling may exist, which does not require the three-form $C_{IJK}$ to transform nontrivially under Yang-Mills transformations.

In section 3 we explain how quantum effects modify the constraints posed by supersymmetry and by Yang-Mills gauge invariance. Unlike the classical case discussed in section 2, in the quantum theory the three-form is shown to necessarily transform nontrivially under Yang-Mills transformations. We indicate that the implementation of supersymmetry and Yang-Mills gauge invariance is not sufficient to fix all couplings in the quantum theory. There remains a singe real parameter which is constrained to be a real root of a particular cubic equation defined by the order parameter $\lambda^6/\kappa^4$. This freedom is fixed, however, by requiring the absence of gravitational and mixed anomalies.
In section 4 we describe how the results of sections 2 and 3 are relevant to M-theory phenomenology, and we make some concluding remarks.

## 2 The Classical Limit

As a first step in constructing the low-energy limit of M-theory, we study the coupling of eleven-dimensional supergravity to ten-dimensional super Yang-Mills theory propagating on the fixed points of the orbifolding described in the introduction. We first attempt to construct this as a purely classical theory. It is shown that such a coupling may indeed exist.

A systematic approach to solving this problem is to first glean as much information about the theory and its couplings from a study of the gauge superalgebra. If we were working with off-shell representations of supersymmetry it would be possible to derive all of the relevant transformation rules without recourse to an action. The construction of an invariant action would then constitute an independent venture. Unfortunately, given the state of the art of supergravity theory, we are constrained to work with on-shell representations in eleven and ten dimensions. So the issues of determining the transformation rules and obtaining the invariant action become intimate with each other. Nevertheless, a useful observation is that even for on-shell theories the gauge superalgebra closes when acting on bosonic fields. We can use this fact to our advantage, as we describe below.

Although, a-priori, we know the transformation rules and the invariant action for eleven-dimensional supergravity theory and similarly for the globally supersymmetric ten-dimensional Yang-Mills theory, we do not know the modifications which are necessary to describe a consistent coupling between these theories. In effect, we need to resolve self-consistently both the complete superalgebra and its representation in terms of specific transformation rules. For instance, a pair of supersymmetry transformations should commute into a particular combination of transformations in the full gauge algebra of the theory. For eleven-dimensional supergravity these constitute general coordinate transformations $\delta_{\text{g.c.t.}}(\zeta)$, supersymmetry transformations $\delta_Q(\epsilon)$, Lorentz transformations $\delta_L(\varepsilon^{ab})$, and also tensor transformations $\delta_{\Sigma}(\Sigma)$ which act on the three-form as $\delta_{\Sigma}(\Sigma)C_{IJK} = 3 \partial_I\Sigma_{JK}$. For the case at hand, we must include the Yang-Mills transformations which act on the gauge fields propagating on the orbifold fixed-points, as $\delta_A A^a_A = D_A \theta^a$. A closed gauge algebra necessarily includes the following commutation relation,

$$[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] = \delta_{\text{g.c.t.}}(\zeta) + \delta_L(\epsilon) + \delta_Q(\epsilon_3) + \delta_{\Sigma}(\Sigma) + \delta_\theta(\theta), \quad (2.3)$$

where $\zeta^I, \epsilon^{ab}, \epsilon_3, \Sigma_{IJ}$ and $\theta^a$ are combinations of $\epsilon_1$ and $\epsilon_2$ which parameterize covariant coordinate transformations, Lorentz transformations, supersymmetry transformations, tensor gauge transformations and Yang-Mills gauge transformations respectively.

The case of eleven-dimensional supergravity and the case of ten-dimensional Yang-Mills theory represent independent solutions to the problem, with different expressions for the dependent parameters on the right-hand-side of (2.3). We seek a solution which includes the complete set of fields from each of these theories, but which satisfies (2.3) with a common and unique set of parameters. The independent solutions just mentioned comprise the zeroth-order solution to our problem. So we begin by reiterating these known results.
**D=11 Supergravity:**

One solution to (2.3) is the case of eleven-dimensional supergravity \[13\], which involves the elfbein \(e^I_{ a} \), a three-form gauge potential \(C_{ IJK} \) and a spin 3/2 gravitino field \(\psi_I \). The supersymmetry transformation rules are given by

\[
\begin{align*}
\delta e^I_{ a} &= \frac{1}{2} \epsilon \Gamma^a \psi_I \\
\delta C_{ IJK} &= -\frac{\sqrt{2}}{8} \epsilon \Gamma_{ [IJ} \psi_K] \\
\delta \psi_I &= D_I(\hat{\omega}) \epsilon + \frac{\sqrt{2}}{288} (\Gamma^J_{ IJKL} - 8 \delta^J_{ IJ} \Gamma^KLM) \epsilon \hat{G}_{ JKL} ,
\end{align*}
\]

(2.4)

where \(\hat{G}_{ IJKL} \) is a supercovariant field strength given by \(\hat{G}_{ IJKL} = 24 \partial_{ [I} C_{ JKL]} + \frac{3}{\sqrt{2}} \bar{\psi}_{ [I} \Gamma_{ JKL} \psi_{ L]} \), and where \(\hat{\omega}_{ I}^{ ab} \) is a supercovariantized spin connection. In this solution we do not include Yang-Mills fields, and so the transformation \(\delta \theta \) which appears in (2.3) identically vanishes. The invariant action is given by

\[
S^{(11)}_{ SG} = \kappa^{-2} \int_{M^{11}} \left( -\frac{1}{4} e^{(11)} \mathcal{R}^{(11)}(e,\omega) - \frac{1}{2} e^{(11)} \bar{\psi}_{I} \Gamma^{IJK} D_{J}(\omega) \psi_{K} - \frac{1}{384} e^{(11)} \hat{G}^{2}_{ IJKL} \\
- \frac{\sqrt{2}}{192} e^{(11)} (\bar{\psi}_{I} \Gamma^{IJKLM} \psi_{J} + 12 \psi^{K} \Gamma^{LM} \psi^{N}) G_{ KLMN} \\
- \sqrt{2} C \wedge G \wedge G + (4 - \text{fermi}) \right).
\]

(2.5)

The four-form field strength is given by \(G_{ IJKL} = 24 \partial_{ [I} C_{ JKL]} \), while the supercovariant field strength \(\hat{G}_{ IJKL} \) is given above. In the sequel, when we discuss modifications to \(G_{ IJKL} \), we refer to the unhatted object. The analogous hatted object can be then be inferred from the ultimate transformation rules. The four-fermi terms in the action can be completely absorbed by replacing \(\omega \rightarrow \frac{1}{2}(\omega + \hat{\omega}) \) in the gravitino kinetic term, and by replacing \(G \rightarrow \frac{1}{2}(G + \hat{G}) \) in the \(\psi^2 G \) interaction term. The parameter \(\kappa \) is the gravitational coupling constant, which has dimensions of \((\text{length})^{-9/2} \). We have chosen the dimensionalities of our fields so that this parameter acts as an overall multiplicative factor.

**D=10 Super Yang-Mills Theory:**

Another solution to (2.3) is the case of ten-dimensional super Yang-Mills theory, which involves gauge potentials \(A^a_{ A} \) and spin 1/2 Majorana-Weyl gaugino fields \(\chi^a \). For our purposes, these are taken to transform in the adjoint representation of the gauge group, which is ultimately shown to be \(E_8 \), necessarily. The supersymmetry transformation rules are given by

\[
\begin{align*}
\delta A^a_{ A} &= \frac{1}{2} \epsilon \Gamma^a \chi^A \\
\delta \chi^a &= -\frac{1}{4} \Gamma^{AB} \epsilon F^a_{ AB} ,
\end{align*}
\]

(2.6)

where \(F^a_{ AB} \) is the field strength associated with \(A^a_{ A} \). The invariant action is given by

\[
S^{(10)}_{ YM} = \lambda^{-2} \int_{M^{10}} \left( -\frac{1}{4} e^{(10)} F^2_{ AB} - \frac{1}{2} e^{(10)} \bar{\chi} \mathcal{D}(\omega) \chi \right).
\]

(2.7)

The parameter \(\lambda \) is the gauge coupling constant, which has dimensions of \((\text{length})^{-3} \). We have chosen the dimensionalities of our fields so that this parameter acts as an overall multiplicative factor.
Starting with the zeroth order theory, consisting of the transformation rules given above and combined action $S^{(11)}_{SG} + S^{(10)}_{YM}$, we seek to modify the transformation rules, and add appropriate interaction terms in order that the resulting action is invariant under an arbitrary local supersymmetry transformation. An equivalent requirement is that the transformation rules represent a common algebra when acting on all fields in the combined theory. As described above, were we working with an off-shell representation, we could solve the problem of obtaining transformation rules independently from the problem of constructing an invariant action. Since we work with an on-shell representation, we must be more careful. In this case, the algebra need only close up to equations of motion. However, the algebra does close on bosons. We choose to exploit this fact to obtain as much information as possible before considering the action.

We adopt the following philosophy. First we solve for the most general solution to (2.3), in terms of transformation rules, that results in a closed algebra when acting on the bosonic fields, $e_I^a, C_{IJK}$ and $A^a_A$. Second, we demand an invariant action. This second constraint automatically ensures that the algebra closes up to equations of motion on all fermionic fields. Finally, we enforce that the action is invariant under Yang-Mills transformations. This final requirement is nontrivial in both the classical case and in the quantum case, but for different reasons. In the classical case, one has to be careful that the $C \land G \land G$ interaction is inert, which requires that $C_{IJK}$ not transform under the Yang-Mills transformations. In the quantum case, we must require that this term does transform, but in such a way as to cancel the gauge anomaly. In the quantum case there are further constraints from gravitational and mixed anomalies.

### 2.1 Close Algebra on Bosons

In this subsection, we determine the set of modifications to the transformation rules which permit a closed gauge algebra when acting on bosonic fields. A crucial part of the analysis is to generalize to the Yang-Mills transformations to act, not only on the fields of the Yang-Mills multiplet, but also on certain components of the three-form $C_{IJK}$. This curious modification is anticipated from a related well-known result pertaining to effective theories of the weakly-coupled heterotic string.

The historic attempt to describe weakly-coupled heterotic string phenomenology involved a supergravity puzzle completely analogous to the one at hand. In that case, the task was to couple the same ten-dimensional super Yang-Mills theory which we are considering to ten-dimensional $N = 1$ supergravity. To consistently describe that coupling, it proved necessary to generalize the Yang-Mills transformation to act on the two-form potential in that supergravity multiplet. This was first described in the abelian case in [14, 15], and later generalized to the nonabelian case in [16]. In addition to being necessary for a consistent classical theory, this modification also enabled the cancelation all gauge and gravitational anomalies [17].

Anticipating a similar necessity for our problem, we make as our first anzatz a generalization of the Yang-Mills gauge transformation rules. The analysis of [3] describes such a modification as well. In this paper we wish to reexplore the analysis of that paper in an independently systematic way. We therefore modify the Yang-Mills transformation rule to include the following action on the three-form

$$\delta_\theta C_{(11)} = \beta(x^{11}) \theta^a F^a_{AB}, \quad (2.8)$$

where $\beta(x^{11})$ is an as-yet unspecified function of $x^{11}$. The remaining components $C_{ABC}$ remain inert under the Yang-Mills transformations.
We also alter the four-form field strength by the following modifications,

\[ G_{ABCD} = 24 \partial_{[A} C_{BCD]} + \gamma(x^{11}) F^a_{[AB} F^a_{CD]} \]

\[ G_{(11)ABC} = 24 \partial_{[11} C_{ABC]} + 6 \beta(x^{11}) \omega_{ABC}, \]  

(2.9)

where \( \gamma(x^{11}) \) is another unspecified function of \( x^{11} \), and \( \omega_{ABC} \) is the Chern-Simons form associated with the gauge potential \( A_A \), which has the property that \( \delta \omega_{ABC} = 3 \partial_A (\theta^a F^a_{BC}] \). This modification uniquely preserves the gauge invariance of \( G_{IJKL} \). The first expression in (2.9) is gauge invariant for any choice of \( \gamma(x^{11}) \). But the coefficient of the Chern-Simons term in the second expression of (2.9) is fixed by (2.8) and the transformation property of the Chern-Simons form.

Further generalizations are necessary in the quantum theory due to the presence of gravitational anomalies. These necessitate the inclusion of factors of \( R_{[AB} R_{CD]} \) and the Lorentz Chern-Simons form into the definition of \( G_{IJKL} \). We avoid these concerns at this point in our analysis. We will return to this issue in the following section, where we discuss the consistent quantum theory.

Using the ansatz (2.8), and the definitions (2.9), we compute the commutator (2.3) on all bosonic fields \( e^I \), \( C_{IJK} \) and \( A^a_A \) using the transformation rules (2.4) and (2.6), supplemented with sufficiently general modifications consistent with Lorentz covariance and gauge covariance. After some work, we find the unique solution to (2.3), which tells us the set of modifications necessary for the algebra to close. In this way, we determine the following new terms in the supersymmetry transformation rules,

\[ \delta' Q C_{ABC} = -\frac{1}{12} \gamma(x^{11}) \bar{\epsilon} \Gamma_{[A} \chi^a F^a_{BC]} \]

\[ \delta' Q C_{AB(11)} = \beta(x^{11}) \bar{\epsilon} \Gamma_{[A} \chi^a A^a_{B]} , \]  

(2.10)

Note that the functions \( \beta(x^{11}) \) and \( \gamma(x^{11}) \) are conceivably related to auxiliary fields.

Of particular interest are Bianchi identities satisfied by \( G_{IJKL} \) defined by (2.9). The contribution to \( dG \) with five ten-dimensional indices vanishes. Thus, \( 5 \partial_A G_{BCDE} = 0 \). But the components of \( dG \) with one eleven index become nontrivial. We find

\[ 5 \partial_{[11} G_{ABCD]} = \left( \gamma'(x^{11}) - 36 \beta(x^{11}) \right) F^a_{[AB} F^a_{CD]} . \]  

(2.11)

This expression is crucial for the analysis which follows.

In the following we restrict our analysis to the coupling of the super Yang-Mills theory propagating on the fixed-hyperplane at \( x^{11} = 0 \). A complementary analysis applies to the other fixed hyperplane at \( x^{11} = \pi \). We omit a discussion of the complementary coupling for reasons of economy.

### 2.2 Close Algebra on Fermions (Construct Invariant Action)

As described above, since we are working with on-shell representations, the superalgebra closes on fermions only if one invokes equations of motion derived from an invariant action. Therefore, we may not straightforwardly employ the techniques used in the previous subsection. But it is sufficient to determine transformation rules which leave invariant an action. If the action is invariant under the transformation rules, then the rules must close up to equations of motion. One can force closure of the action by first including the general modifications to the transformation rules obtained in the previous subsection, and then by adding suitable interaction terms to the combined action. Demanding invariance of the action under supersymmetry further constrains the modifications involving the functions \( \beta(x^{11}) \) and \( \gamma(x^{11}) \) introduced above, but not
completely. All remaining ambiguities are then removed by demanding invariance as well under local Yang-Mills transformations and local Lorentz transformations.

The logical first step in modifying the action is to add a coupling between the gravitino field and the supercurrent obtained by varying the Yang-Mills action (2.7). This “Noether” term is given by

$$S_{\text{Noether}} = \lambda^{-2} \int_{M^{10}} \left( - \frac{1}{4} e^{(10)} \bar{\psi}_A \Gamma^{BC} \Gamma^A \chi^a F^a_{BC} \right).$$

As described in [3], adding this term does not yet yield invariance. Assorted further interactions are also needed. Crucial is a necessary modification to the Bianchi identity for $G_{IJKL}$ found to be

$$5 \partial_{11} G_{ABCD} = \mp 3 \sqrt{2} \kappa^2 \lambda^2 \delta(x^{11}) F^a_{[AB} F^a_{CD]},$$

where $\delta(x^{11})$ is a delta function, and where the $\mp$ depends on whether $\Gamma_{11} \psi_A = \pm \psi_A$ at the orbifold fixed-point. The cancellation which results with the modification (2.13) when verifying invariance of the action requires an integration by parts. In the orbifold picture, which we use throughout this paper, there is no boundary, and therefore the integration by parts does not involve a nontrivial boundary contribution. The delta function in (2.13) restricts the required new coupling to the fixed hyperplane at $x^{11} = 0$, where the Yang-Mills fields are constrained to propagate.

It is reassuring that the form of this new coupling is consistent with the result obtained in the previous subsection based on a study of the gauge algebra. Now, we can reconcile this new result with our previous results to obtain further constraints on the functions $\gamma(x^{11})$ and $\beta(x^{11})$. Comparing (2.13) with (2.11) we determine that

$$\gamma'(x^{11}) - 36 \beta(x^{11}) = \mp 3 \sqrt{2} \kappa^2 \lambda^2 \delta(x^{11}).$$

A minimal solution to this equation, which restricts all nontrivial features of the functions $\beta(x^{11})$ and $\gamma(x^{11})$ to the fixed hyperplane at $x^{11} = 0$ is the following,

$$\gamma(x^{11}) = b \frac{\kappa^2}{\lambda^2} \theta(x^{11})$$

$$\beta(x^{11}) = \frac{1}{36} (2b \pm 3 \sqrt{2}) \frac{\kappa^2}{\lambda^2} \delta(x^{11}),$$

where $\theta(x^{11})$ is the Heavyside (or step) function, which has the property that $\theta'(x^{11}) = 2 \delta(x^{11})$, and $b$ is a real dimensionless constant which cancels in (2.14). We strongly emphasise that the constant $b$ is not fixed by supersymmetry. However, it is fixed by the requirements of gauge invariance and local Lorentz invariance.

To enforce invariance of the action, it is also necessary to add some higher fermi terms to the action, and also to further modify the supersymmetry transformation rules for the gravitino $\psi_I$ and the gaugino $\chi^a$. These modifications are all higher-fermi. They are straightforward to compute, and they do not pose any obstruction to the results which we have already derived. Most significantly they do not pose any additional restrictions on the value of the constant $b$. We therefore suppress these extra modifications in this paper.

---

3We could also add an arbitrary smooth function $36 f(x^{11})$ to $\gamma(x^{11})$. If we simultaneously added $f'(x^{11})$ to $\beta(x^{11})$, this extra function would drop out of (2.14). But we ignore such a possibility in this analysis. However, it has been suggested that it may be fruitful to consider Yang-Mills fields not rigidly constrained to the fixed hyperplanes, but rather somehow inhabiting a boundary layer. In such a scenario one could envision the delta function smearing out. In that case, such a modification could prove useful.
It is useful to summarize the results which we have derived so far. First, the four-form field strength must have the following form.

\[
G_{ABCD} = 24 \partial_{[A} C_{BCD]} + b \frac{k^2}{\lambda^2} \theta(x^{11}) F^a_{[AB} F^b_{CD]}.
\]

\[
G_{(11)ABC} = 24 \partial_{[11} C_{ABC]} + \frac{1}{6} (2b \pm 3 \sqrt{2}) \frac{k^2}{\lambda^2} \delta(x^{11}) \omega_{ABC}.
\] (2.16)

Second, supersymmetry transformations of the three-form should include the following modifications to the rule exhibited in (2.4),

\[
\delta Q' C_{ABC} = - \frac{1}{12} b \frac{k^2}{\lambda^2} \theta(x^{11}) \bar{\epsilon} \Gamma_{[A \chi^a} F^a_{BC]}.
\]

\[
\delta Q' C_{AB(11)} = \frac{1}{36} (2b \pm 3 \sqrt{2}) \frac{k^2}{\lambda^2} \delta(x^{11}) \bar{\epsilon} \Gamma_{[A \chi^a A^a B]}.
\] (2.17)

Third, and most importantly, the Yang-Mills gauge transformation should act on the mixed components of the three-form as follows,

\[
\delta \theta C_{AB(11)} = \frac{1}{36} (2b \pm 3 \sqrt{2}) \frac{k^2}{\lambda^2} \delta(x^{11}) \theta^a F^a_{AB}.
\] (2.18)

Aside from the higher fermi terms which we have suppressed, the results (2.16), (2.17) and (2.18) represent the general solution to the constraints obtained from implementing supersymmetry. There still remains the unfixed constant \(b\). This is determined by imposing gauge invariance. We consider this issue in the following subsection.

Since the step function \(\theta(x^{11})\) is discontinuous at \(x^{11} = 0\), the precise boundary value of \(G_{ABCD}\), as defined in (2.16), is not well-defined. However, the following product does have a well-defined behavior,

\[
G_{[ABCD} G_{CDEF]} = b^2 \frac{k^4}{\lambda^4} F^a_{[AB} F^b_{CD} F^c_{EF} F^d_{GH]}.
\] (2.19)

This relation is necessary for proving the gauge invariance of the theory. We examine this issue presently.

### 2.3 Implement Gauge Invariance

Above, we considered the possibility that the three-form \(C_{IJK}\) may transform nontrivially under a Yang-Mills gauge transformation. We therefore draw special attention to the following term in the action,

\[
W = - \frac{\sqrt{2}}{k^2} \int_{M^{11}} C \wedge G \wedge G.
\] (2.20)

Under a Yang-Mills variation, this term may also transform nontrivially due to the nonvanishing of \(\delta \theta C_{AB(11)}\). In the classical case, such behavior would obstruct gauge invariance and spoil the consistency of the theory. In the quantum case, on the other hand, this behavior might be welcome, as \(W\) may provide a counterterm for gauge anomalies. This possibility was introduced in [3]. Using the transformation rule (2.18) as well as the boundary condition (2.19) it follows that \(W\) transforms as follows,

\[
\delta \theta W = -(2b \pm 3 \sqrt{2}) \frac{\sqrt{2}}{12} \frac{k^4}{\lambda^6} \int_{M^{10}} \theta^a F^a \wedge F^b \wedge F^b \wedge F^c \wedge F^c.
\] (2.21)
In the classical theory, we can retain gauge invariance only if this term vanishes. Therefore, in the classical theory, gauge invariance requires the following condition,

\[(2b \pm 3\sqrt{2})b^2 = 0.\]  \hspace{1cm} (2.22)

There are exactly two solutions to this condition, \(b = 0\) or \(b = \mp 3/\sqrt{2}\). From the expressions in the previous subsection, particularly (2.16), we see that the first of these choices, \(b = 0\), amounts to concentrating all new features into components of \(G_{IJKL}\) with mixed indices. The second choice, \(b = \mp 3/\sqrt{2}\) amounts to concentrating all new features into the complementary components of \(G_{IJKL}\), or those which have purely ten-dimensional indices. From equation (2.18) we also see that the first choice, \(b = 0\), requires a nontrivial Yang-Mills transformation law for \(C_{AB(11)}\), whereas the second choice, \(b = \mp 3/\sqrt{2}\) allows \(C_{IJK}\) to remain completely inert under Yang-Mills transformations.

Does the theory make a unique choice between the two possibilities \(b = 0\) or \(b = \mp 3/\sqrt{2}\)? Substituting the expressions from the previous subsection into the action formula, one finds that the choice \(b = 0\) gives rise to factors of \(\delta(0)\) in the action, whereas the other choice \(b = \mp 3/\sqrt{2}\) does not. In the quantum theory there exist conceivable ways to regulate these factors. But in the classical theory there is no obvious way to make sense of them. In order to describe a well-defined classical theory, it is apparent that only the choice \(b = \mp 3/\sqrt{2}\) is permissible.

Note that we have described a coupling in which the tensor field remains inert under the Yang-Mills transformation. Note also the contrast with the familiar ten-dimensional coupling necessary for the low-energy description of the heterotic string. In that case, it was impossible to find such a coupling; the two-form was required to transform even in the classical theory. In the weakly-coupled heterotic string case, however, there is no classical obstruction since one can omit the counter terms if one also avoids loop effects. In the case of M-theory, we require the Green-Schwarz-like interaction even in the classical theory, but we find that we may nevertheless formulate a consistent classical theory which leaves the three-form inert under Yang-Mills transformations.

3 The Quantum Limit

When we include quantum modifications to the effective action, new effects, attributable to loop diagrams, contribute to gauge, gravitational and mixed anomalies. Particularly, there exists an anomalous contribution to Yang-Mills gauge transformations of the type \(\int M^{10} \theta F^5\). The group theoretic factors in this expression organize into the form of (2.21) precisely and uniquely for the choice of gauge group \(E_8\). When we compute the relevant coefficient, and combine the result with (2.21), gauge anomaly cancellation is found to require

\[(2b + 3\sqrt{2})b^2 = \nu,\]  \hspace{1cm} (3.23)

where we have specialized to the case where \(\Gamma_{11}\psi_A| = \psi_A|\). In equation (3.23), the real parameter \(\nu\) arises from an anomalous contribution to the variation of the effective action. The precise value of \(\nu\) can be computed using the techniques explained in [18]. We find

\[\nu = \frac{\sqrt{2}}{2(4\pi)^5} \frac{\lambda^6}{\kappa^4}.\]  \hspace{1cm} (3.24)

In the classical theory, described above, we do not involve loop effects, and so \(\nu = 0\). In the quantum theory \(\nu \neq 0\), and Yang-Mills invariance requires \(b\) to be a real root of the cubic equation (3.23).
(4π)υ = (4π)4λκ

Figure: Curve indicating the value(s) of the parameter b per-
missible for gauge anomaly cancelation in M-theory, given
the values of the coupling constants λ and κ, or vice-versa.

For any value of ν there is at least one such solution, given by

\[ b = 4^{-1/3} \left( \nu - \sqrt{2} + \sqrt{\left( \nu - 2\sqrt{2} \right) \nu} \right)^{1/3} + 4^{-1/3} \left( \nu - \sqrt{2} - \sqrt{\left( \nu - 2\sqrt{2} \right) \nu} \right)^{1/3} - \frac{1}{\sqrt{2}}. \]  

(3.25)

This equation straightforwardly gives the required value of b for any given value of \( \nu \geq 2\sqrt{2} \). The equation is also valid in the regime \( 0 \leq \nu < 2\sqrt{2} \), but in this case more care must be used. In this second case the first two terms of (3.25) are complex, but the imaginary parts cancel. The remaining two roots of (3.23) are not real and are therefore irrelevant to us (since b is necessarily real) when \( \nu > 2\sqrt{2} \). However, in the regime \( 0 \leq \nu < 2\sqrt{2} \), all three roots of (3.23) are real. The locus of permitted values of the pair \((\nu, b)\) are shown in the figure.

At two special points, corresponding to the turning points evident in the figure, the two complex roots of (3.23) become real and degenerate. The first of these is at the point \( \nu = 0 \), which coincides with the classical theory, as described above. In that case, in addition to the solution \( b = -\frac{3}{2}\sqrt{2} \), there is another solution \( b = 0 \). This degeneracy is described above, and is removed by the fact that the \( b = 0 \) solution gives rise to inscrutable factors of \( \delta(0) \) in the action which cannot be removed by a regulator in the classical case. In the quantum case, these \( \delta(0) \) factors are unavoidable, since we cannot choose \( b = -\frac{3}{2}\sqrt{2} \), as this would not be compatible with (3.23). But in the quantum case this is not as problematic, since then we expect short distance curiosities, and we expect that a mechanism exists to regulate these factors. An important point, however, is that the existence of ultraviolet problems, like the factors of \( \delta(0) \), are entirely distinct from the issue of anomaly cancellation. All gauge, gravitational and mixed anomalies should cancel irrespective of short-distance oddities, such as the factors of \( \delta(0) \).

There is one unique point in the quantum regime (where \( \nu \neq 0 \)) where the two complex roots become real and degenerate. At this point \( \nu = 2\sqrt{2} \), and one root tells us that \( b = \frac{1}{2}\sqrt{2} \). The additional degenerate roots tell us that we could also take \( b = -\sqrt{2} \). This constitutes the second special point mentioned above. As yet we have no reason to believe that the theory selects this point.
The cancelation of all anomalies requires a further modification to the Bianchi identity for $G_{IJKL}$ discussed above, so that it involves factors of $R_{[IJ}R_{KL]}$ and/or the Lorenz Chern-Simons form. The probability that the gravitational anomalies can be eliminated depends on nontrivial factorization properties required on the precise structure of the anomalies. Hořava and Witten have demonstrated in \cite{2} and \cite{3} that these properties are indeed satisfied. The mechanism requires more Green-Schwarz-like counterterms, crudely of the form $C \wedge R^4$. Since the three-form has the Yang-Mills transformation property determined above by the elimination of pure gauge anomalies, and since this rule involves the yet-undetermined parameter $b$, it is clear that the value of $b$ is selected only from a knowledge of the precise coefficients describing the gravitational or mixed anomalies. As described in \cite{3}, it is not a simple matter to determine these numbers. We will discuss this issue in detail in a forthcoming paper.

4 Phenomenological Ramifications and Conclusions

The essential modification which enables the coupling between eleven-dimensional supergravity and fixed-point ten-dimensional super Yang-Mills theory is exhibited in (2.16). From that equation we see that the fundamental parameter governing the coupling is $\kappa^2/\lambda^2$. However, we have seen that consistency requires that $\lambda^2 \sim \kappa^{4/3}$, with a precise numerical coefficient determined by the requirements described above. Thus, the fundamental expansion parameter is $\kappa^{2/3}$. Without presenting details of higher-fermi modifications, we have described effects occurring at lowest-order in this parameter which are required by supersymmetry and by anomaly cancelation. These are summarized by equation (2.16). In the quantum theory, the parameter $b$ is a real root of the cubic equation (3.23). Since the value of $\nu$ depends on the ratio $\lambda^6/k^4$, further constraints are required to completely fix its value. These are provided by gravitational or mixed anomaly cancelation.

To leading order in $\kappa^{2/3}$ all of the purely bosonic contributions to the action are obtained by substituting (2.16) into (2.5). In the classical case, there are no terms involving $\delta(0)$ because the choice $b = \mp 3/\sqrt{2}$ removes them. In that case, terms involving $\theta(x^{11})$ appear either squared, and are therefore relatively well defined, or vanish due to the fixed-point conditions on the various fields. It is apparent that considerations similar to those already discussed will apply at higher orders in $\kappa^{2/3}$. Thus, in the classical limit we only retain modifications which are relatively well defined (ie: $\theta(x^{11})$ appears raised only to even powers.). The complete justification of this statement of course necessitates that one computes these higher-order modifications.

To date, all attempts at describing M-theory phenomenology have had an endemic problem with factors of $\delta(0)$. This issue is dealt with at levels of rigor ranging from a (low) of sweeping the issue completely under the carpet to a (high) of demonstrating some purely formal cancelations or by systematically classifying these factors as an effect occurring at higher order in the expansion parameter $\kappa^{2/3}$. In \cite{12}, this issue was addressed in an analogous situation in five-dimensions, where the factors of $\delta(0)$ were shown to derive from the elimination of an auxiliary field. In the quantum theory, the proper regulation of the terms formally proportional to $\delta(0)$ has yet to be fully explained.

In the current analysis, we find that we can at least move the issue of the delta functions one rung down the ladder of relevance, by suggesting a classical limit completely devoid of these factors. This fact opens the door to a more “honest” phenomenology.

There is enough information about the low-energy behavior of M-theory to make concrete predictions about nature. At the same time, it should be possible to discuss a classical limit. That is, we should be able to take $\hbar \to 0$, and obtain meaningful leading-order results from these predictions, without having to rationalize the neglect of otherwise inscrutable terms such as those containing formal factors of $\delta(0)$. In this paper we have described a coherent way of
understanding this classical limit.

At the same time we have discussed a new ingredient necessary for implementing proper anomaly cancellation in these theories. We have argued that the web of constraints which include all gauge, gravitational and mixed anomalies is more rich than previously understood. We argue that the new parameter, called $b$ in this paper, is necessary to ensure that the full complement of consistency requirements in the theory not be overconstrained. A detailed analysis of precise anomaly coefficients is necessary to reconcile this issue. This will be discussed in a forthcoming paper.

Acknowledgement

I acknowledge useful discussions with Dieter Lüst, Christian Prietchopf, and André Lukas. I especially thank Burt Ovrut for his hospitality at the University of Pennsylvania, where some of this work was accomplished.

References

[1] String Theory Dynamics in Various Dimensions
   E.Witten, hep-th/9503124

[2] Heterotic and Type I String Dynamics from Eleven Dimensions
   P.Hořava and E.Witten, hep-th/9510209

[3] Eleven-Dimensional Supergravity on a Manifold with Boundary
   P.Hořava and E.Witten, hep-th/9603142

[4] String Theory and the Path to Unification: A Review of Recent Developments
   K.R.Dienes, Phys. Rep. 287 (1997) 447, hep-th/9602045

[5] On the Strongly Coupled Heterotic String
   E.Dudas and J.Mourad, Phys. Lett. 400B (1997) 71, hep-th/9701048

[6] Remarks on M-theory Coupling Constants and M-Brane Tension Quantizations
   J.X.Lu, hep-th/9711014

[7] String-Unification, Universal One-Loop Corrections and Strongly Coupled Heterotic String Theory
   H.P.Nilles and S.Stieberger, hep-th/9702110

[8] On the Four-Dimensional Effective Action of Strongly Coupled Heterotic String Theory
   A.Lukas, B.A.Ovrut and D.Waldram, hep-th/9710208

[9] Four-Dimensional M-theory and supersymmetry breaking
   E.Dudas and C.Grojean, hep-th/9704177

[10] Supersymmetry Breaking and Soft Terms in M-Theory
    H.P.Nilles, M.Olechowski and M.Yamaguchi, hep-th/9707143

[11] Supersymmetry breaking in M-theory
    I.Antoniadis and M.Quirós, hep-th/9709023

[12] Transmission of Supersymmetry Breaking from a 4-dimensional Boundary
    E.A.Mirabelli and M.E.Peskin, hep-th/9712214.
[13] *Supergravity Theory in 11 Dimensions*
   E.Cremmer, B.Julia and J.Scherk, Phys. Lett. 76B (1978), 409-412

[14] *N = 4 Supergravity Coupled to N = 4 Matter*
   A.Chamseddine, Nucl. Phys. B185 (1981) 403.

[15] *Ten-Dimensional Maxwell-Einstein Supergravity, its Currents and the Issue of its Auxiliary Fields*
   E.Bergshoeff, M.de Roo, B.de Wit and P.van Nieuwenhuizen, Nucl. Phys. B195 (1982) 97.

[16] *Unification of Yang-Mills Theory and Supergravity in Ten Dimensions*
   G.F.Chapline and N.S.Manton, Phys. Lett. 120B (1983) 105-109

[17] *Anomaly Cancellations in Supersymmetric D = 10 Gauge Theory and Superstring Theory*
   M.B.Green and J.H.Schwarz, Phys. Lett. 149B (1984), 117-122.

[18] *Chiral Anomalies, Higher Dimensions, and Differential Geometry*
   B.Zumino, Y.S.Wu and A.Zee, Nucl. Phys. B239 (1984), 477-507.