Femto-lensing due to a Cosmic String

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We consider the femto-lensing due to a cosmic string. If a cosmic string with the deficit angle \( \Delta \sim 100 \) [femto-arcsec] \( \sim 10^{-18} \) [rad] exists around the line of sight to a gamma-ray burst, we may observe characteristic interference patterns caused by gravitational lensing in the energy spectrum of the gamma-ray burst. This “femto-lensing” event was first proposed as a tool to probe small mass primordial black holes. In this paper, we propose use of the femto-lensing to probe cosmic strings with extremely small tension. Observability conditions and the event rate are discussed. Differences between the cases of a point mass and a cosmic string are presented.

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I. INTRODUCTION

Cosmic strings are line-like topological defects that are likely to emerge through phase transitions with spontaneous symmetry breaking [1–4]. Another possibility to produce cosmic strings has been pointed out in the context of super-string theory and their properties are quite similar to those of field theoretic cosmic strings, except for the fact that inter-commuting probability between strings can be much lower than 1 [5–9] (see [10–17] for recent reviews). Usually, cosmic strings are characterized by the string tension \( \mu \). The geometry around an infinite straight string is locally identical to that of flat spacetime, but globally it corresponds to a conical spacetime with a deficit angle [18],

\[
\Delta = \frac{8\pi G \mu}{c^4},
\]

where \( G \) is Newton’s gravitational constant and \( c \) is the speed of light. One typical effect of the deficit angle is the formation of a double image of a light source located behind the string [18,19]. We call all such effects which are caused by gravitational fields of a cosmic string gravitational lensing due to a cosmic string.

For ordinary field-theoretic strings, since the relevant energy scale is given by the symmetry breaking scale \( E_{SB} \), the string tension can be estimated as \( G \mu/c^4 \sim E_{SB}^2/E_{Pl}^2 \), where \( E_{Pl} \) is the Planck energy. For the Grand Unification scale, we have \( E_{SB} \sim 10^{16} \text{GeV} \) or equivalently \( \Delta \sim 10^{-6} \). By contrast, the tension of cosmic strings from symmetry breaking along a supersymmetric flat direction in the potential may have much smaller tension. In this scenario, there can exist two typical scales which determine the string tension. One is the supersymmetry breaking scale, which can be of the order of TeV, and the other is the cutoff scale \( \sim E_{Pl} \). Such strings may have the deficit angle given by \( 10^{-18} \lesssim \Delta \lesssim 10^{-6} \) [20–23]. The effective four-dimensional tension of cosmic super-strings strongly depends on the details of the compactification and the inflationary scenario. In the case of the KKLMMT scenario [2], the predicted tension of cosmic superstrings is expected to lie in the range \( 10^{-12} \lesssim \Delta \lesssim 10^{-6} \) [3, 6, 24].

So far, many attempts have been made to search for the signature of cosmic strings in various observations including the cosmic microwave background (CMB), gravitational waves and gravitational lensing. The CMB anisotropy spectrum has excluded cosmic strings with \( \Delta \gtrsim 10^{-6} \) from the dominant energy components of the universe [25–28]. Non-detection of gravitational waves from cosmic string loops also rules out the string tension \( \Delta \sim 10^{-5} \) [29, 30].

Gravitational lensing phenomena could in principle serve as more direct evidence for cosmic strings [31–34], although none have been detected yet [35–37]. A recent search for lensed galaxy pairs found no evidence for the presence of long straight cosmic strings with \( \Delta > 7.5 \times 10^{-6} \) out to redshifts greater than 0.6 [38]. Non-detection of characteristic variability of quasars due to the crossing of cosmic strings constrains lighter strings with \( 10^{-12} < \Delta < 10^{-8} \) down to the level of \( \Omega_{cs} = 0.01 \) [39], where \( \Omega_{cs} \) is the average density of cosmic strings in the units of the critical density. A more model-dependent but interesting bound on the local abundance of the strings with much lower tension \( 10^{-15} < \Delta < 10^{-9} \).
could be obtained by considering the variability of Galactic stars and pulsar timing \([40]\).

Forecasts for future constraints on the string tension include the B-mode polarisation of the CMB induced by straight strings with \(\Delta \lesssim 10^{-6}\) \([27, 28]\), weak lensing \([41, 42]\), gravitational wave bursts, the stochastic background from string loops with \(\Delta \lesssim 10^{-6}\) \([43]\), lensing at radio frequencies by loops with \(\Delta \sim 10^{-8}\) \([33]\) and the 21 cm radiation from strings with \(\Delta \gtrsim 10^{-9} - 10^{-11}\) \([44]\).

So far, no way to probe cosmic strings with \(\Delta < 10^{-16}\) has been proposed. In this paper, we propose that femto-lensing events of gamma-ray bursts (GRBs) can be used to probe the cosmological abundance of cosmic strings with such small tensions. Though the expected image separation due to a cosmic string with \(\Delta < 10^{-16}\) is too small to be angularly resolved, the interference between the images could induce observable characteristic patterns in the energy spectrum of the lensed source objects \([45]\). This effect, called femto-lensing \([46]\), was first proposed as a method to probe light compact objects like primordial black holes (PBHs) \([46, 49]\) and GRBs were considered as the target sources. The interference is expected when the time delay induced by lensing is comparable to the inverse of the gamma-ray frequency and femto-lensing of GRBs is sensitive to compact objects with a mass range \(10^{17} \text{g} - 10^{20} \text{g}\). In the case of cosmic strings, as will be shown in the following sections, those with the deficit angle \(10^{-19} - 10^{-17}\) can be probed by GRBs.

Since the launch of the FERMI satellite, observational studies of GRBs have progressed significantly thanks to its unprecedented sensitivity. Recently, a constraint on the cosmological density of compact objects in the mass range \(5 \times 10^{17} \text{g} - 10^{20} \text{g}\) was derived from the non-detection of femto-lensing events by the FERMI data in Ref. \([50]\). Likewise, we can expect a constraint on cosmic strings with tiny deficit angle \(10^{-19} \lesssim \Delta \lesssim 10^{-17}\), which can hardly be probed by other observations.

The main purpose of this paper is to investigate the interference pattern in the energy spectrum of a GRB induced by a cosmic string and to determine how it differs from that for a compact object. The interference pattern for a compact object was discussed in Refs. \([47, 48]\). Though typical gravitational lensing effects are often understood by using the geometrical optics approximation, the wave nature of the light is important in the case of femto-lensing. In the following sections, we discuss the observability of femto-lensing events due to cosmic strings by investigating wave propagation in a spacetime with a cosmic string \([51]\).

This paper is organised as follows. In Sec. II setting a lens system composed of the observer, a source and a straight cosmic string, we present the lensed waveform. A derivation of the lensed waveform using the Kirchhoff integral theorem is given in Appendix A. The observability conditions and the event rate are discussed in Sec. III. In Sec. IV we point out some differences between the case of a cosmic string and that of a point mass. Sec. V is devoted to a summary.

II. LENSED WAVEFORM

In this section, we obtain the lensed waveform of a massless field in a cosmic string spacetime and see how the oscillatory behaviour in the energy spectrum can arise due to the
interference between the waves reaching the observer.

A. Configuration of the Lens System

We consider the spacetime with the deficit angle $\Delta$, where $\Delta$ is defined so that the total angle around the cosmic string is given by $2\pi - \Delta$. Let us consider the configuration of the lens system specified by the following quantities: the distances from the observer to the string $d_O$, from the source to the string $d_S$, from the observer to the source along the string direction $d_z$ and the angle $\varphi$ which specifies the source position as shown in Fig. 1. Hereafter, we assume $\varphi = \mathcal{O}(\Delta)$.

An important quantity characterising the wave propagation in the lens system is the optical path difference $l$ defined by

$$l := \frac{d_Od_S\Delta^2}{8D},$$

where

$$D := \sqrt{(d_O + d_S)^2 + d_z^2}.$$  \hspace{1cm} (2)

The meaning of $l$ is clarified by focusing on the $\varphi = 0$ case, see Fig. 2. Then the distance from the observer to the source is given by (see the left of Fig. 2)

$$\sqrt{d_z^2 + d_O^2 + d_S^2 + 2d_Od_S\cos\left(\frac{\Delta}{2}\right)} = D\left(1 - \frac{l}{D} + \mathcal{O}(\Delta^4)\right).$$  \hspace{1cm} (4)

It is understood from Eq. (4) that $l$ approximately measures the shortening of the distance due to the presence of the string.
For later use we also introduce an angle
\[
\beta := \frac{d_S \Delta}{2 (d_O + d_S)}.
\] (5)
Again the meaning of $\beta$ becomes clear by setting $\varphi = 0$: then the half of the separation angle of the two images projected on the plane perpendicular to the string (see the right of Fig. 2) is $\beta + O(\Delta^2)$.

B. Lensed Waveform

Let us analyse the waveform after propagation from a stationary source to the observer. If we neglect the tiny tilt of the polarisation vector caused by the lensing, the local propagation equation for electromagnetic waves can be reduced to the wave equation for the massless scalar field $\phi$:
\[
\left( \nabla^2 + \frac{\omega^2}{c^2} \right) \phi = \left( \frac{\partial^2}{\partial t^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\omega^2}{c^2} \right) \phi = 0,
\] (6)
where $\omega$ is the angular frequency of the mode and $(r, \theta, z)$ is a cylindrical coordinate system in which the cosmic string is on the $z$-axis. Since the spacetime with a deficit angle is locally flat, $\nabla^2$ is just the flat Laplacian, but the range of the azimuthal angle $\theta$ around the cosmic string is $[0, 2\pi - \Delta)$.

Although we will not take cosmological expansion into account below, since the Maxwell equations in vacuum are conformally invariant, we can easily apply the results to cosmological situations replacing Euclidean distances and the angular frequency $\omega$ with angular diameter distances and the redshifted angular frequency $(1 + z) \omega$, respectively, where $z$ is the redshift at the intersection of the string and the line of sight and $\omega$ is the angular frequency at the observer.
Wave propagation in a locally flat spacetime with the deficit angle $\Delta$ has been fully studied in Ref. [51]. Using appropriate approximations (see Ref. [51] and Appendix A), we may solve the wave propagation to express the lensed waveform at the observer as

$$\phi(d_O, d_S, d_z, \varphi; \Delta, \omega) = F(w, y) \phi_0(d_O, d_S, d_z, \varphi; \omega),$$

where $\phi_0$ is the unlensed waveform (i.e. when $\Delta = 0$) and $F$ is the amplification factor, respectively, given by

$$\phi_0(d_O, d_S, d_z, \varphi; \omega) := \frac{A}{D} \exp \left[ \frac{i \omega D}{c} \left( 1 - \frac{d_O d_S}{2D^2} \varphi^2 \right) \right],$$

$$F(w, y) := \exp \left[ \frac{w}{2i} (1 + 2y) \right] \left\{ 1 - \frac{1}{2} \text{Erfc} \left[ \sqrt{\frac{w}{2i}} (1 + y) \right] \right\}$$

$$+ \exp \left[ \frac{w}{2i} (1 - 2y) \right] \left\{ 1 - \frac{1}{2} \text{Erfc} \left[ \sqrt{\frac{w}{2i}} (1 - y) \right] \right\}$$

with $A$ being an arbitrary amplitude, Erfc the complementary error function defined by

$$\text{Erfc}(x) := \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt,$$

and $w$ and $y$ the non-dimensional variables defined by

$$w := \frac{2\omega l}{c} = \frac{\omega d_O d_S \Delta^2}{4cD}, \quad y := \frac{\varphi d_s}{\beta (d_O + d_S)} = \frac{2\varphi}{\Delta}.$$  

It is worth noting that $w$ gives roughly the ratio between the path difference and the wavelength, and $y$ gives the source position $\varphi d_S$ on the source plane in the units of $\beta (d_O + d_S)$.

We are interested in the absolute square of $F$, which is observable as magnification or demagnification of the flux. It is depicted as a function of $w$ for several values of $y$ in Fig. 3.

The oscillatory behaviour of $|F|^2$ in the frequency domain is of most importance for our purpose. In Fig. 3, the geometrical optics limit ($w \to \infty$) of $|F|^2$,

$$|F_{\text{geo}}|^2 := \lim_{w \to \infty} |F|^2 = \begin{cases} 1 & \text{for } |y| > 1 \\ 2 + 2\cos(2wy) & \text{for } |y| < 1 \end{cases}$$

is also depicted. Comparing $|F|^2$ with $|F_{\text{geo}}|^2$ reveals that for most cases when $y > 0$ the dominant mode in $|F|^2$ is the one that oscillates with period $\delta w = \pi/y$. This is a consequence of the fact that the error function parts (those inside the curly brackets of the expression for $F$ in Eq. (9)) only weakly depend on $w$.

To summarise this section, we have analytically obtained the lensed waveform of the massless field in the cosmic string spacetime, Eq. (7). The numerical plots in Fig. 3 show

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1 We note that notations in this paper are totally different from Ref. [51].
FIG. 3: The absolute square of the amplification factor $|F|^2$ and its geometrical optics limit $|F_{geo}|^2$ as functions of $w$ for different values of $y$. 
that the absolute square of the amplification factor $|F|^2$ oscillates in the frequency domain with the period $\delta \omega$ given by

$$\delta \omega = \frac{\pi c}{2y l} = \frac{4\pi D}{d_O d_S \Delta^2 y}. \quad (13)$$

Using the wavelength $\lambda$, we obtain

$$\frac{\delta \omega}{\omega} = \frac{\lambda}{4y l} = \frac{2D\lambda}{d_O d_S \Delta^2 y}. \quad (14)$$

### III. OBSERVABILITY CONDITIONS

In this section, we apply the above analysis to gamma-ray bursts from cosmological distances to determine the conditions under which the lensing effect due to a string can be observable.

1. **Detectable range of the deficit angle**

First, we evaluate the range of wavelength $\lambda$ within which a lensing event is detectable through the observation of photons with a fixed value of the deficit angle $\Delta$. At least one whole phase of sinusoidal oscillation should be detectable in the spectrum. Hence, let us focus on one period of the sinusoidal oscillation around the angular frequency $\omega = c w/2l$. Obviously, $w$ should satisfy $2w y \gtrsim \pi$ so that at least one period can be included in the spectrum. Thus, a necessary condition for an observable effect is given by

$$\lambda \lesssim 8y l. \quad (15)$$

On the other hand, a lower bound for the wavelength is given by the energy resolution of the detector. The resolution of the detector must be sufficient to resolve the oscillation in the spectrum. We assume that the frequency resolution limit is given by $\alpha \omega$ with the typical value of $\alpha$ being $\mathcal{O}(0.1)$. Then, we obtain the following inequality as another necessary condition:

$$\alpha \lesssim \frac{\delta \omega}{\omega} = \frac{\lambda}{4y l}. \quad (16)$$

Eventually, we find

$$\lambda \lesssim 4y l \lesssim \alpha^{-1} \lambda \iff 4\alpha y l \lesssim \lambda \lesssim 8y l. \quad (17)$$

This implies that only a narrow range of wavelength, with the fractional width of order $\alpha$, is usable in probing a given configuration of the lens system. We note that we are interested in the case $\varphi \lesssim \Delta/2$, or equivalently $y \lesssim 1$.

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2 As can be seen from Fig. 3, the gravitational lensing effect becomes only significant when $w$ is larger than 1, so we need $w \gtrsim 1 \iff \lambda \lesssim 4\pi l$. This inequality gives another upper bound for $\lambda$. 
If we take $\lambda = 10^{-9}$ cm and $2d_O = 2d_S = D = 10^{28}$ cm with the observations of GRBs in mind, we have

$$\alpha y \left( \frac{4d_O d_S / D}{10^{28} \text{ cm}} \right) \left( \frac{\Delta}{10^{-18}} \right)^2 \lesssim \frac{\lambda}{10^{-9} \text{ cm}} \lesssim 2y \left( \frac{4d_O d_S / D}{10^{28} \text{ cm}} \right) \left( \frac{\Delta}{10^{-18}} \right)^2 .$$

Thus in the case of GRBs, the typical separation angle is evaluated as $\Delta \sim 100$ femto arcsec, confirming the origin of the name “femto-lensing.” Correspondingly, the detectable energy scale of the string tension is estimated as

$$\left( \frac{G\mu}{c^4} \right)^{1/2} E_{\text{Pl}} = \left( \frac{\Delta}{8\pi} \right)^{1/2} E_{\text{Pl}} = \left( \frac{E_{\text{Pl}}}{\pi^2 d_O d_S} \right)^{1/4} \left( \frac{4d_O d_S / D}{10^{28} \text{ cm}} \right)^{-1/4},$$

where we have assumed $l \sim \lambda$.

In reality, the wavelength of GRB photons typically spans the finite range $[\lambda_{\text{min}}, \lambda_{\text{max}}] = [10^{-10} \text{ cm}, 10^{-7} \text{ cm}]$, which roughly corresponds to the energy scale from 1 keV to 1 MeV. Then the detectable range of the deficit angle is found to be

$$2 \times 10^{-19} y^{-1/2} \left( \frac{\lambda_{\text{min}}}{10^{-10} \text{ cm}} \right)^{1/2} \left( \frac{4d_O d_S / D}{10^{28} \text{ cm}} \right)^{-1/2} \lesssim \Delta$$

$$\lesssim 3 \times 10^{-17} y^{-1/2} \left( \frac{\alpha}{0.1} \right)^{-1/2} \left( \frac{\lambda_{\text{max}}}{10^{-7} \text{ cm}} \right)^{1/2} \left( \frac{4d_O d_S / D}{10^{28} \text{ cm}} \right)^{-1/2} .$$

2. **Source radius**

So far, it has been assumed that a GRB can be treated as a point source. However, if the source radius is sufficiently large, the interference pattern will be smeared out. The upper limit for the source radius can be estimated by considering the $y$ dependence of the amplification factor. For a fixed value of $w$, the square of the amplification factor oscillates with the period $\delta y = \pi / w$ as a function of $y$. The corresponding length scale on the source plane to this period is given by $\delta y \beta (d_O + d_S) = \pi \Delta d_S / 2w$. If the source radius is larger than $\pi \Delta d_S / 2w$, the interference effect is smeared out and the oscillation in the spectrum cannot be observed. Since we are mainly interested in the situation $l \sim \lambda$, we obtain $w \sim 4\pi$ and find that the source radius $r_s$ must satisfy

$$r_s \lesssim \frac{\pi \Delta d_S}{2w} \approx 6 \times 10^8 \text{ cm} \left( \frac{w}{4\pi} \right)^{-1} \left( \frac{\Delta}{10^{-18}} \right) \left( \frac{2d_S}{10^{28} \text{ cm}} \right).$$

We estimate the source radius $r_s$ following Ref. [48]. The appropriate linear source size is given by $\Gamma c \Delta t$ with $\Gamma$ and $\Delta t$ being the bulk Lorentz factor and the smallest variability time scale detected, respectively. Then we obtain

$$r_s \sim 3 \times 10^8 \text{ cm} \left( \frac{\Gamma}{100} \right) \left( \frac{\Delta t}{0.1 \text{ ms}} \right).$$

(22)
Once we fix the source radius $r_s$, the inequality (21) represents the lower bound for the wavelength $\lambda$ as follows:

$$\lambda \gtrsim 1.5 \times 10^{-10} \text{cm} \left( \frac{r_s}{3 \times 10^8 \text{cm}} \right) \left( \frac{2d_O}{D} \right) \left( \frac{\Delta}{10^{-18}} \right).$$

Comparing this inequality with the range of wavelength (18), we see that, although the finite source effect might be significant in some cases, the inequality (21) may be satisfied in many cases. We do not discuss the finite source effect in this paper. In a practical analysis, however, this effect may have to be taken into account (see Ref. [47] for the point mass case).

### 3. Cosmic string motion

In the previous discussions, we have assumed a static configuration of the lens system. However, if the relative velocity between the source-observer system and the cosmic string is too large, the string would pass through the region in which the lensing effect is significant before a detector collects enough photons.

A straight cosmic string with relativistic vertical velocity $v$ would pass through the region of interest in the time

$$\frac{2d_O \beta}{v} \sim 0.3 \left( \frac{\Delta}{10^{-18}} \right) \left( \frac{4d_O d_S / (d_O + d_S)}{10^{28} \text{cm}} \right) \left( \frac{10^{10} \text{cm/s}}{v} \right).$$

(24)

A clear spectrum must be obtained within a time much shorter than this, otherwise the lens system cannot be regarded as static. For a large value of $v$, only a limited number of bright burst events would be usable to detect femto-lensing events. This demand might be compared to those in the case of compact objects, which are believed to have a non-relativistic velocity dispersion.

### 4. Cosmic string density

Finally, we estimate how many strings in between the source and the observer are necessary in order that femto-lensing events can actually take place for a given number of GRBs. Here, we assume that the string network can be represented by a collection of straight string segments.

Suppose a spherical volume of radius $D$, centered at the observer, contains $N_{cs}$ straight strings of length $L$. Assuming the distances to the strings are $\sim D/2$, we obtain the angular scale of the length of a string as $2L/D$. Therefore, the solid angle in which the lensing effect becomes significant is given by $\Delta \times 2L/D$. If there are no overlapping regions between the solid angles of neighbouring strings, the total solid angle given by all the strings is $2N_{cs}\Delta L/D$. Dividing this total solid angle by $4\pi$, we obtain the event probability $P$ for a single source as

$$P = \frac{2N_{cs} \Delta L}{D} \times \frac{1}{4\pi}.$$  

(25)
Meanwhile, the average mass density is given in terms of $N_{cs}$ by

$$\rho_{cs} = \frac{N_{cs} L \mu / c^2}{4\pi D^3 / 3}. \tag{26}$$

Introducing the density parameter of the straight strings $\Omega_{cs} \equiv 8\pi G \rho_{cs} / (3H_0^2)$, where $H_0$ is the current Hubble parameter, we find

$$P = \frac{2\Omega_{cs} H_0^2 D^2}{c^2}. \tag{27}$$

When we consider a source and a lens separated by cosmological distances, since $D \sim c H_0^{-1}$, we find $P \sim \Omega_{cs}$.

In reality, a GRB consists of many spike emissions, and cosmic strings are moving with the velocity $v$. Even if a femto-lensing event cannot be observed at the first spike emission, it might be observed at a subsequent spike emission during a single GRB event. This would in principle be possible if the GRB as a whole lasts longer than the crossing time scale given by [22] while there are spikes with timescales shorter than it. Still, those spikes may not individually contain enough photons to provide a clear spectrum; then we would need to collect a bunch of spikes. Let $n_b$ denote the mean number of such qualifying bunches within one GRB. Then, the event probability $P$ for a single GRB has to be multiplied by the factor $n_b$. Assuming $\Omega_{cs} \sim 0.001$ and $n_b \sim 10$, we can expect roughly one femto-lensing event among 100 available gamma-ray burst events. This roughly corresponds to one detection per year with FERMI satellite if most of the observed gamma-ray bursts can be used for the femto-lensing search. Even if we do not observe such an event, we may obtain an observational limit on $\Omega_{cs}$. Further detailed analysis of GRB spectra is needed for a more precise estimation of the event rate and giving observational limits on $\Omega_{cs}$.

IV. DISTINCTION BETWEEN COSMIC STRING AND POINT MASS LENS

Cosmic strings are not the only candidates for a femto-lensing object, as point masses would also cause a similar effect. It is necessary to study the differences between these two cases in order to determine how they can be distinguished observationally. Below, we shall review the femto-lensing caused by a point mass and highlight the difference between this and the case of a string.

Let us consider the point mass lens system with distances from the observer to a lens $D_L$, from the lens to a source $D_{LS}$ and the observer to the source $D_S = D_L + D_{LS}$. Let $\eta_p$ denote the distance between the source position and the intersection of the source plane and the line which connects the observer and the lens. For the point mass case, we have the following expression for the amplification factor [52, 54]:

$$|F| = \left| e^{\pi w_p / 4 \Gamma} \left(1 - \frac{i w_p}{2}\right)_1 F_1 \left(\frac{i w_p}{2}, 1; \frac{i w_p y_p}{2}\right)\right|, \tag{28}$$
where $\Gamma$ and $\mathbf{1}_F_1$ are the gamma function and the confluent hyper-geometric function, respectively, and

$$w_p = \frac{4 \omega GM}{c^3}, \quad y_p = \eta_p \sqrt{\frac{c^2 D_L}{4GMD_{LS}D_S}}. \quad (29)$$

Considering the $\eta_p = 0$ case, we can find the following expressions for the path difference $l_p$ and half of the separation angle $\beta_p$:

$$l_p = \frac{2GM}{c^2}, \quad \beta_p = \sqrt{\frac{4GMD_{LS}}{c^2D_LD_S}}. \quad (30)$$

These $l_p$ and $\beta_p$ correspond to $l$ and $\beta$ in the cosmic string case. Using these quantities, we can express $y_p$ and $w_p$ as

$$w_p = \frac{2\omega l_p}{c}, \quad y_p = \frac{\eta_p}{\beta_p (D_L + D_{LS})}. \quad (31)$$

One can see that these expressions for $y_p$ and $w_p$ are similar to those for $y$ and $w$ in the cosmic string case $^{11}$. Therefore, we identify $y_p$ and $w_p$ as $y$ and $w$, respectively, and omit the subscript p hereafter.

In the geometrical optics approximation ($w \to \infty$), we obtain

$$|F|^2 \to |F_{\text{geo}}|^2 := |\mu_+| + |\mu_-| + 2\sqrt{|\mu_+\mu_-|} \cos(\theta_- - \theta_+), \quad (32)$$

where

$$\mu_\pm = \pm \frac{1}{4} \left[ \frac{y}{\sqrt{y^2 + 4}} + \frac{\sqrt{y^2 + 4}}{y} \right] \pm 2, \quad (33)$$

$$\theta_+ = w \left( \frac{1}{2x_+^2} - \ln|x_+| \right), \quad (34)$$

$$\theta_- = w \left( \frac{1}{2x_-^2} - \ln|x_-| \right) - \frac{\pi}{2}, \quad (35)$$

$$x_\pm = \frac{1}{2} \left( y \pm \sqrt{y^2 + 4} \right). \quad (36)$$

$\theta_- - \theta_+$ can be rewritten as

$$\theta_- - \theta_+ = w \left[ \frac{1}{2} y \sqrt{y^2 + 4} + \ln \frac{\sqrt{y^2 + 4} + y}{\sqrt{y^2 + 4} - y} \right] \mp \frac{\pi}{2} =: w\tau(y) - \frac{\pi}{2}. \quad (37)$$

Therefore we obtain

$$\cos(\theta_- - \theta_+) = \sin(w\tau(y)). \quad (38)$$

It is meaningful to compare $\tau(y)$ with $2y$ because we have Eq. $^{12}$ for the cosmic string case in the geometrical optics approximation. As is shown in Fig. $^{4}$, $\tau(y)$ is close to $2y$ for $y \lesssim 1$.

Before comparing the cosmic string case with the point mass case, we compare the amplification factor with that of the geometrical optics approximation for each case. As is
shown in Fig. 5, the spectrum quickly gets closer to the geometrical optics approximation in the point mass case. Though we show only the result for $y = 1/2$, this behaviour is a common feature for any value of $y$. On the other hand, this is not the case when we consider the cosmic string, as is shown in Fig. 6. While the phase of the oscillation in the spectrum seems to be well approximated by the geometrical optics approximation, there are remarkable deviations in the amplitude. In this sense, we can conclude that the wave effect
is more important for the cosmic string case.

Finally, let us look at the differences between the cosmic string case and the point mass case. In order to highlight the differences, we focus on the following quantity:

\[ \frac{|F|^2}{\langle |F_{\text{geo}}|^2 \rangle} - 1, \]  

(39)

where \( \langle |F_{\text{geo}}|^2 \rangle \) is the 1 cycle average of the square of the amplification factor in the geometrical optics approximation. Using this procedure, we can extract the oscillating part of the spectrum. This quantity is plotted as a function of \( w \) for each case in Fig. 7.

We find two remarkable differences between the two cases. One is the \( \pi/2 \) phase difference, the origin of which is the \( -\pi/2 \) shift in Eq. (35). This shift is caused by the caustic of one of the ray bundles in the case of the point mass lens. This does not happen in the cosmic string case. The other difference is that the oscillation amplitude of \( |F|^2/\langle |F_{\text{geo}}|^2 \rangle \) can exceed 1 in the cosmic string case, but cannot in the point mass case. The reason why it does not occur in the point mass case is obvious from Eq. (32) and the goodness of the geometrical optics approximation. These differences might be used to identify the lens object if we succeed to detect the oscillatory behaviour in the spectrum of a GRB.

V. SUMMARY

Femto-lensing events of gamma-ray bursts (GRBs) due to cosmic strings have been studied. The detectable range of the deficit angle \( \Delta \) was derived taking the wave nature of gamma-rays and the energy resolution of a detector into account. Assuming the range of observable wavelengths to be \( 10^{-10}\text{cm} - 10^{-7}\text{cm} \), we obtain the detectable range as

FIG. 6: \( |F|^2 \) and \( |F_{\text{geo}}|^2 \) as functions of \( w \) for the cosmic string case with \( y = 1/2 \).
$10^{-19} \lesssim \Delta \lesssim 10^{-17}$. Observability conditions associated with the source radius and the relative motion of the lensing system have also been discussed. The relative motion of a cosmic string gives a limitation on the duration of photon counting. This limitation may be tighter than that in the case of a compact lens object because of the relativistic motion of a cosmic string. A limited number of bright burst events can be candidates for a femto-lensing event if the relative velocity of a cosmic string is comparable to the speed of light. The event probability $P$ for a single GRB event is roughly estimated as $P \sim n_b \Omega_{cs}$, where $n_b$ and $\Omega_{cs}$ are the mean number of available bunches of spike emissions in a GRB and the average density of cosmic strings in units of the critical density, respectively.

Two typical differences between the lensed spectrum in the case of a point mass lens and that of a cosmic string have been pointed out. One of these is a phase shift in the spectrum oscillation. In the point mass case, one of two ray trajectories experiences a caustic which causes a $-\pi/2$ phase shift, and this shift appears in the lensed spectrum. The same does not occur in the cosmic string case. The other difference is in the oscillation amplitude of the spectrum. In the point mass lens case, the oscillation amplitude does not exceed the mean value, while it can exceed the mean value in the cosmic string case because of the wave effect. These differences might be used to identify the lens object if we succeed to detect the oscillating behaviour in the spectrum of a GRB.
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Appendix A: Derivation of the Lensed Waveform

We give an overview of the derivation of Eq. (7) using the Kirchhoff integral theorem\cite{kirchhoff}. This approach for the point mass lens case can be seen in Ref. \cite{point_mass}. A different approach is used in Ref. \cite{different} to derive Eq. (7). We use the unit \( c = 1 \) in this Appendix for notational simplicity. First, let us introduce the coordinate system \( \xi = (\xi, \zeta, z) \) where the observer and string are located at \((0, -d_O, 0)\) and \(\xi = \zeta = 0\), respectively. For convenience, we remove the region of the deficit angle so that the source is just split in two as shown in the left of Fig. 8. In this coordinate system, the source position \( \xi_s \) can be expressed in two ways. Let

\[
(\xi_s^+, \zeta_s^+, d_z) \quad \text{and} \quad (\xi_s^-, \zeta_s^-, d_z)
\]
denote the two expressions for the source position, where \( \xi_s^+ > \xi_s^- \). Then, we
find
\[ \xi_s^\pm = \pm d_s \sin \left( \frac{\Delta}{2} \pm \varphi \right), \quad (A1) \]
\[ \zeta_s^\pm = d_s \cos \left( \frac{\Delta}{2} \pm \varphi \right). \quad (A2) \]

Our purpose is to obtain an approximate solution for the wave equation given by
\[ (\nabla^2 + \omega^2) \phi = -4\pi A \delta(\xi - \xi_s). \quad (A3) \]

The strategy is summarised as follows. First, we assume that the geometrical optics approximation is valid in the domain \( \zeta > 0 \). Then, we calculate the waveform at the observer by applying the Kirchhoff integral theorem to the domain \( \zeta < 0 \), where the waveform of the geometrical optics approximation on the \( \zeta = 0 \) plane is used as a boundary condition. To evaluate the Kirchhoff integral, we use appropriate approximations associated with the small quantities \( \varphi \sim \Delta \sim \epsilon \) and \( 1/(\omega D) \).

Using the geometrical optics approximation, the waveform around the \( \zeta = 0 \) plane is given by
\[ \phi |_{\zeta = 0} = \frac{A}{D_1^\pm} \exp(i\omega D_1^\pm) \quad \text{for} \quad \pm \xi > 0, \quad (A4) \]
where
\[ D_1^\pm = \sqrt{(\xi - \xi_s^\pm)^2 + (\zeta - \zeta_s^\pm)^2 + (z - d_z)^2}. \quad (A5) \]

Applying the Kirchhoff integral theorem to the domain \( \zeta < 0 \) and neglecting the contribution from the boundary at infinity, we obtain the following integral for the waveform at the observer:
\[ \phi_O := \phi(0, -d_O, 0) = -\frac{1}{4\pi} \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} d\xi \left\{ \phi \frac{\partial}{\partial \zeta} \left( \frac{e^{i\omega D_2}}{D_2} \right) - e^{i\omega D_2} \left( \frac{\partial}{\partial \zeta} \phi \right) \right\}_{\zeta = 0}, \quad (A6) \]
where
\[ D_2 = \sqrt{\xi^2 + (\zeta + d_O)^2 + z^2}. \quad (A7) \]

In order to obtain an approximate form for this integral, we keep terms up to the order of \( \epsilon^2 \) in the phase part and up to the leading order in the amplitude. It is intuitively obvious that only the region \( \xi \lesssim \epsilon D \) can give significant contribution to the integral. This fact can be justified by applying the stationary phase approximation to the integral over \( \xi \) with \( 1/(\omega D) \ll 1 \). Taking the above discussion into account, the integral can be approximated as
\[
\begin{align*}
\phi_O & \simeq \frac{-i\omega A}{4\pi} \int_{-\infty}^{\infty} dz \left( \frac{d_O}{D_3 D_5^2} + \frac{d_s}{D_3^2 D_5} \right) \exp \left[ i\omega \left( D_3 + D_5 \right) \right] \\
& \times \left[ \exp \left[ -i\omega \left( \frac{D_5 \xi_s^2}{2D_3 (D_3 + D_5)} \right) \right] \right] \int_{0}^{\infty} d\xi \exp \left[ \frac{i\omega(D_3 + D_5)}{2D_3 D_5} \left( \xi - \frac{D_5 \xi_s^2}{D_3 + D_5} \right)^2 \right] \\
& + \left[ \exp \left[ -i\omega \left( \frac{D_5 \xi_s^2}{2D_3 (D_3 + D_5)} \right) \right] \right] \int_{-\infty}^{0} d\xi \exp \left[ \frac{i\omega(D_3 + D_5)}{2D_3 D_5} \left( \xi - \frac{D_5 \xi_s^2}{D_3 + D_5} \right)^2 \right]. \quad (A8)
\end{align*}
\]
where

\[ D_3 = \sqrt{d_S^2 + (z - d_z)^2}, \quad (A9) \]
\[ D_5 = \sqrt{d_O^2 + z^2}. \quad (A10) \]

Defining \( a \) and \( b^\pm \) by

\[ a = \frac{\omega(D_3 + D_5)}{2D_3 D_5}, \quad (A11) \]
\[ b^\pm = \frac{D_5 \xi_s^\pm}{D_3 + D_5}, \quad (A12) \]

we can evaluate the integral of \( \xi \) as

\[ \int_0^\infty d\xi \exp \left[ i a (\pm \xi - b^\pm)^2 \right] = \sqrt{\frac{\pi i}{a}} \left( 1 - \frac{1}{2} \text{Erfc} \left( \pm \sqrt{-iab}^\pm \right) \right). \quad (A13) \]

The integral with respect to \( z \) can be evaluated by using the stationary phase approximation. Since the dominant contribution in the phase part comes from \( i\omega(D_3 + D_5) \), we write the Eq. (A8) in the following form

\[ \phi_O = \int_{-\infty}^\infty dz X(z) \exp[i\omega(D_3 + D_5)], \quad (A14) \]

where

\[
\begin{align*}
X(z) &= \frac{-i\omega A}{4\pi} \left( \frac{d_O}{D_3 D_5} + \frac{d_S}{D_5 D_5} \right) \\
&\times \left[ \exp \left[ -i\omega \left( \frac{D_5 \xi_s^2}{2D_3 (D_3 + D_5)} \right) \right] \sqrt{\frac{\pi i}{a}} \left( 1 - \frac{1}{2} \text{Erfc} \left( \sqrt{-iab}^\pm \right) \right) \\
&+ \exp \left[ -i\omega \left( \frac{D_5 \xi_s^{-2}}{2D_3 (D_3 + D_5)} \right) \right] \sqrt{\frac{\pi i}{a}} \left( 1 - \frac{1}{2} \text{Erfc} \left( \sqrt{-iab}^\mp \right) \right) \right]. \quad (A15)
\end{align*}
\]

Then, we consider the stationary phase approximation with the phase \( i\omega(D_3 + D_5) \). Since the stationary point is given by

\[ \frac{d}{dz}(D_3 + D_5) = 0 \Leftrightarrow z = z' := \frac{d_O}{d_O + d_S} d_z, \quad (A16) \]

Eq. (A14) can be approximated by

\[
\begin{align*}
\int_{-\infty}^\infty dz X(z) \exp[i\omega(D_3 + D_5)] &\simeq X(z') \int_{-\infty}^\infty dz \exp \left[ i\omega \left\{ D + \frac{(d_O + d_S)^4}{2d_O d_S D^3} (z - z')^2 \right\} \right] \\
&= X(z') \sqrt{\frac{2\pi id_O d_S D^3}{\omega (d_O + d_S)^4}} \exp[i\omega D]. \quad (A17)
\end{align*}
\]

Calculating \( X(z') \) and simplifying the expression, we finally get Eq. (7).

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