Purifications without entanglement, State-reconstructions and Entropy

Chirag Dhara
St. Xavier’s College, Mumbai-400001, INDIA

N.D. Hari Das
The Institute of Mathematical Sciences, Chennai 600-113, INDIA

In this paper we have analysed in detail two different purification protocols. The first one, proposed by Sudarshan, is based on the preservation of probabilities. We have constructed a second protocol here based on optimization of fidelities. We have considered both complete and partial measurements and have established bounds and inequalities for various fidelities. For every type of measurement, we have analysed post-measurement states based on the Maximum Entropy principle as well as what we have proposed as unbiased states. We show that our purification protocol always leads to better state reconstruction. These schemes can be thought of as operations in the sense of Kraus and we have explicitly constructed the Kraus operators for these. We have also shown that the entropy either increases or remains the same depending on the choice of the purification basis.

I. INTRODUCTION

In the density matrix formulation the process of going from a mixed state to a pure state is called Purification. There is a vast literature on this topic. In the current literature ‘Purification’ is mostly understood to be the process of associating a suitable pure state of a larger system whose reduced density matrix is the mixed state one started with. This necessarily involves entanglement. We interpret purification in a larger sense to mean any protocol that produces a pure state from a mixed state. In this paper we consider purification schemes without entanglement.

We specifically consider two schemes for purification the first of which was introduced by Sudarshan based on preservation of probabilities(scheme A), and a second proposed by us here based on optimal fidelities(scheme B). We apply both these schemes to the problem of reconstructing the pure state before a measurement from the post-measurement mixed state. We do so for both partial as well as complete measurements. In all such cases there is still a lot of freedom in constructing the post-measurement state itself. We have focussed on two specific choices: The so-called maximum entropy state $\rho_{\text{max}}$ as well as an equal mixture $\rho_{\text{unb}}$ of the subensembles resulting from each measurement. We call the latter the ’unbiased state’. Operationally it is straightforward to realise as one has to simply divide the initial ensemble of pre-measurement state into equal parts for each measurement. These are discussed in Sec. IV.A.

We apply our considerations to qubits only. We separately consider the cases of a) complete measurement involving measurements of $S_x, S_y, S_z$, b) partial measurement involving measurements of $S_y, S_z$ and finally c) which involves measuring $S_z$ only. For each of these cases we apply the two purification schemes mentioned above to both $\rho_{\text{max}}, \rho_{\text{unb}}$. We show that in all cases scheme-B produces pure states with better overlap with the unknown initial pure state.

We then show that there exist well defined operations in the sense of Kraus which realise all these purification schemes. We explicitly construct the relevant Kraus operators. Though an ‘environment’ system is needed for realising these operations, our schemes do not require any entanglement between the qubit and the environment.

We end our paper with an explicit proof that the total entropy of the qubit and the environment either remains the same or increases, in accordance with the second law. Though this is expected, we thought it desirable to explicitly prove it.

II. SOME PURIFICATION SCHEMES

A. Purification Protocol - A

Consider some density matrix which can be regarded as the mixture of two orthogonal states

$$\rho = p_1\rho_1 + p_2\rho_2$$

(1)

where $p_1^2 = \rho_1^2 = \rho_2^2, tr(\rho_1\rho_2) = 0, tr\rho_1 = tr\rho_2 = 1$.

The Purification Protocol discussed here is based on the principle of preservation of probabilities. In [2] it was taken to mean that the overlap of the purified state $\rho^{(A)}$ with $\rho_{1,2}$ is $p_{1,2}$.

This Protocol leads to the family of pure states:

$$\rho^{(A)} = p_1\rho_1 + p_2\rho_2 + \frac{\rho_1\Pi\rho_2 + \rho_2\Pi\rho_1}{\sqrt{tr(\rho_1\Pi)tr(\rho_2\Pi)}}$$

(2)

where $\Pi$ is a projection which is not orthogonal to either $\rho_1$ or $\rho_2$. If $\rho_1 = |0\rangle\langle 0|, \rho_2 = |1\rangle\langle 1|$ and $\Pi$ is of the form $|\chi\rangle\langle \chi|$ with $\chi = \mu|0\rangle + \nu|1\rangle$ ($\mu, \nu \neq 0$ since $|\chi\rangle$ is not orthogonal to either $|0\rangle$ or $|1\rangle$) then the purified state is given by ($\phi$ is the phase of $\mu\nu^*$)

$$\rho^{(A)} = |\psi\rangle\langle \psi|, \quad |\psi\rangle = \sqrt{p_1}|0\rangle + \sqrt{p_2}e^{-i\phi}|1\rangle$$

(3)
The reason that only this phase appears in the purified state is that preservation of probabilities leaves only a phase left unspecified in a pure state. Different choices of φ lead to different purified states. There is no principle that selects a particular value of φ. This protocol can only be implemented probabilistically.

B. Purification Protocol - B

Consider the state
\[ ρ = \left( \begin{array}{cc} a & p \\ p^* & 1-a \end{array} \right) \] (4)
which we wish to 'purify'. The principle we adopt to fix the purified state is: the purified state must have maximal overlap with the mixed state we started with. (The overlap between states ρ1 and ρ2 is \( F = tr(ρ_1 ρ_2) \).) This amounts to demanding that the purified state be as close as possible to the mixed state. In contrast to A this is a deterministic protocol.

It is elementary to show that the pure state ρψ with maximum overlap with ρ is the eigenstate of ρ corresponding to the maximum eigenvalue \( \lambda_+ \). To see this let \( λ_+ \) be the maximum eigenvalue (considered degenerate for simplicity) and let the others be \( λ_i, i = 1, ..., N-1 \) where N is the dimensionality of the Hilbert space, and let |λ⟩ be the corresponding eigenstates. Denoting the pure state by \( |ψ⟩ = α_+ |λ_+⟩ + \sum \alpha_i |λ_i⟩ \) we have
\[ tr \rho ρ_ψ = λ_+ + \sum |α_i|^2 (λ_i - λ_+) \] (5)
Since \( λ_+ - λ_+ \) is negative it is obvious that the overlap is maximised when \( α_i = 0 \) i.e \( |ψ⟩ = |λ_+⟩ \) and that the maximum value of the overlap equals \( λ_+ \). It is easy to show that
\[ λ_+ = a + Δ/2; \quad Δ = 1 - 2a + \sqrt{4|p|^2 + (1 - 2a)^2} \] (6)
and
\[ ρ_ψ = \frac{2}{Δ(Δ + 2a - 1)} \left( \begin{array}{cc} |p|^2 & pΔ/2 \\ p^*Δ/2 & Δ^2/4 \end{array} \right) \] (7)

III. KRAUS FORMALISM AND PURIFICATION.

Kraus et al \[ 8, 9 \] have given a formalism to study all possible changes of quantum states through the so-called operations. An Operation O is defined as follows: Consider a quantum system in the state ρsys with a Hilbert Space \( \mathcal{H} \) which is coupled to another quantum system, often called the environment, in the state ρE and which has a state space \( \mathcal{H}_E \). The system and the environment interact through a Unitary Evolution \( U \) which acts on the total Hilbert Space \( \mathcal{H} \otimes \mathcal{H}_E \). Some property of the environment is selectively measured (what this means is that one selects a particular measurement outcome), represented by a projection operator \( Q_E \) so that the combined state becomes:
\[ \hat{ρ} = (I \otimes Q_E)U(ρ_{sys} \otimes ρ_E)U^†(I \otimes Q_E) \] (8)
The system is then described by the reduced density matrix, \( \hat{ρ}_{sys} = Tr_E \hat{ρ} \); where the trace is taken over all possible states of the environment. The resulting state change \( O : ρ_{sys} \rightarrow \hat{ρ}_{sys} \) is called an Operation. This operation can be represented in terms of Kraus operators \( A_k \) acting on the state space of the system such that
\[ \hat{ρ}_{sys} = \sum_{k \in K} A_k ρ_{sys} A_k^† \] (9)
The operators \( A_k \) are defined by
\[ (f, A_k g) = ((f \otimes f^E), U(g \otimes g^E)) \] (10)
where \( f, g \) arbitrary vectors in the state space of the system, \( \{f_k \} \) are an orthonormal basis of \( Q_E H_E \) extended to \( H_E \) and \( g^E \) is the pure state in which the environment can be assumed to have started in. As the measurement \( Q_E \) is selective, the \( A_k \) operators satisfy the trace non-increasing condition \( \sum_{k \in K} A_k^† A_k \leq I \) where K is some indexing set.

Operations can connect any given pair of density matrices \( \{ρ_1, ρ_2\} \), and in particular, a mixed state and a pure state. The entropy aspects of this are discussed in section V.

A. Kraus Operators for Qubits

We shall relax the condition of selectivity in measurements and consider all possible outcomes for measurements. We shall also restrict ourselves to qubits only. Then one needs two Kraus operators for a general operation.

Consider some orthonormal basis (called purification basis) \( \{0\}, \{1\} \); and any pair of operators
\[ A_0 = |ψ⟩⟨0|, \quad A_1 = |ψ⟩⟨1| \] (11)
satisfying \( A_0 A_0^† + A_1 A_1^† = I \). For any arbitrary density matrix \( ρ_{in} \), these operators produce
\[ ρ_{out} = A_0 ρ_{in} A_0^† + A_1 ρ_{in} A_1^† = |ψ⟩⟨ψ| \] (12)
Clearly, \( ρ_{out} \) is a pure state and it is independent of the initial state \( ρ_{in} \). By eqn (11) the Kraus operators here are of the form,
\[ A_0 = ⟨0_E|U|0_E⟩, \quad A_1 = ⟨1_E|U|0_E⟩ \] (13)

\[ \text{We thank the referee for pointing this out to us.} \]
where the initial state of the environment is the pure state $|0_E\rangle$. It is straightforward to check that the unitary operator $U$ that generates the Kraus operators of equation (11) is (with the notation $|\psi, \chi\rangle = |\psi\rangle |\chi\rangle_E, \langle \xi |\psi\rangle = 0$):

$$U = \sum_{s=0}^{1} (|\psi, s\rangle\langle s, 0| + |\xi, s\rangle\langle s, 1|)$$ (14)

Kraus Operators relevant for the purification schemes of Sec II can easily be worked out.

IV. STATE RECONSTRUCTION FOR QUBITS AND EFFICIENCY OF PURIFICATIONS.

Now we apply these considerations to address the issue of optimal reconstruction of unknown quantum states through both complete as well as partial measurements. The question we answer is: 'which of these schemes produces a pure state that is closest to the pre-measurement pure state?'

A. Post measurement state

Clearly there are many ways of performing both complete as well as partial measurements leading to different post-measurement (mixed) states. Here we consider two particular ways of obtaining the post-measurement state: i) first is based on the well known maximum entropy principle [3] and we shall denote the corresponding post-measurement state by $\rho_{\text{max}}$. Such states have been studied extensively in [3]; ii) second is a proposal wherein the mixed states arising from different measurements are equally mixed to give the post-measurement density matrices to give $\rho_{\text{unb},3} = (1/3)(\rho_1 + \rho_2 + \rho_3)$. With the notation $A_j = 2p_j - 1; j = 1, 2, 3$ we can write,

$$\rho_{\text{unb},3} = \frac{1}{6} \begin{pmatrix} A_1 + 3 & A_3 - iA_2 \\ A_3 + iA_2 & 3 - A_1 \end{pmatrix}$$ (16)

where '3' denotes the number of orthogonal spin components measured. Knowing $p_1, p_2, p_3$ amounts to a complete measurement and the initial density matrix has to be

$$\rho_{\text{ini}} = \frac{1}{2} \begin{pmatrix} 1 + A_1 & A_3 - iA_2 \\ A_3 + iA_2 & 1 - A_1 \end{pmatrix}$$ (17)

Clearly, the relation between $\rho_{\text{unb},3}$ and $\rho_{\text{ini}}$ is,

$$\rho_{\text{unb},3} = (1/3)(I + \rho_{\text{ini}})$$ (18)

In an earlier publication [4] we have established the result analogous to eqn (18) for arbitrary systems with finite dimensional $\mathcal{H}$. The Fidelity of $\rho_{\text{unb},3}$ with the initial state is:

$$\mathcal{F}(\rho_{\text{ini}}, \rho_{\text{unb},3}) = tr(\rho_{\text{ini}}\rho_{\text{unb},3}) = 2/3$$ (19)

independent of $\rho_{\text{ini}}$. Since, $\rho_{\text{ini}}$ is pure, its eigen-values are 0, 1. Hence, the eigenvalues of $\rho_{\text{unb},3}$ are, from (18), 1/3 and 2/3. Therefore,

$$\rho_{\text{unb},3} = \frac{2}{3}|l\rangle\langle l| + \frac{1}{3}|s\rangle\langle s|$$ (20)

Substituting in eqn (18) and using the completeness relation $I = |l\rangle\langle l| + |s\rangle\langle s|$, one finds

$$\rho_{\text{ini}} = |l\rangle\langle l|$$ (21)

The eigenvector of $\rho_{\text{unb},3}$ corresponding to the largest eigenvalue is the initial state. The Purification of (20) by Protocol - A is (see eqn (3))

$$\rho_{\text{unb},3}^{(A)} = (2/3)|l\rangle\langle l| + (1/3)|s\rangle\langle s| + \sqrt{2/3}e^{i\phi}|l\rangle\langle s| + e^{-i\phi}|s\rangle\langle l|$$ (22)

This is clearly not the initial state. In fact, the fidelity of this state with the initial pure state is still only,

$$\mathcal{F}(\rho_{\text{ini}}, \rho_{\text{unb},3}^{(A)}) = 2/3$$ (23)

On the other hand, the purification of $\rho_{\text{unb},3}$ by protocol- B is simply to select the state with the largest eigenvalue i.e.

$$\rho_{\text{unb},3}^{(B)} = |l\rangle\langle l|$$
which is indeed the initial pure state by virtue of eqn (21). The initial state is the closest pure state to the mixed state \( \rho_{\text{unb},3} \). Protocol-B has purified the post-measurement state \( \rho_{\text{unb},3} \) to maximum fidelity (in this case 100%) with the initial state.

State reconstruction by the MaxEnt Principle gives:

\[
\rho_{\text{max},3} = \frac{1}{2} (I + A_1 \sigma_z + A_2 \sigma_y + A_3 \sigma_z) = \rho_{\text{ini}} \tag{24}
\]

Clearly, in the case of a complete measurement state reconstruction by the MaxEnt Principle and purification by protocol-B both yield the initial state.

It should also be emphasised that even if one did not know that \( p_i \) were the probabilities of outcome from complete measurements, but only knew the form of \( \rho_{\text{unb},3} \), B would nevertheless reconstruct the initial state. The power of B is clearly evident in the case of partial measurements, discussed next.

C. Partial Measurements: \( S_y, S_z \) measured.

In the case of partial measurements the initial state can never be unambiguously reconstructed. In this section we establish the following two results: (i) the purified state under protocol-B always has a greater fidelity with the initial state than does the post-measurement state, (ii) the fidelity of the purified state under protocol-B with the initial state is always greater than that of the purified state under protocol-A (in an phase-average sense as protocol-A does not favour any single pure state) except in some singular cases where the fidelities are the same. Thus protocol-B is the better when trying to reconstruct the initial state from the post-measurement state.

Suppose two measurements are made. Let \( p_1 \) and \( p_2 \) by the probabilities of outcome for \( |+\rangle_z \) and \( |+\rangle_y \) respectively. Then post-measurement state (in the notation defined in Sec.IV.B) is,

\[
\rho_{\text{unb},2} = \frac{1}{4} \begin{pmatrix}
A_1 + 2 & -i A_2 \\
i A_2 & 2 - A_1
\end{pmatrix} \tag{25}
\]

Let, the initial state be \( \psi_{\text{ini}} = \alpha |+\rangle_z + \beta |\rangle_z \). Then, \( (1 + A_1)/2 = |\alpha|^2, Im(\alpha \beta^*) = A_2/2 \). From these relations we can compute the fidelity:

\[
F(\rho_{\text{ini}}, \rho_{\text{unb},2}) = \frac{1}{4} (A_1^2 + A_2^2 + 2) \tag{26}
\]

where \( \rho_{\text{ini}} = |\psi_{\text{ini}}\rangle \langle \psi_{\text{ini}}| \).

To purify the state by protocol-A, we can adopt the following procedure: From (25) we find the eigenvalues of \( \rho_{\text{unb},2} \) to be \( p_1 = (2 + |A|)/4 \) and \( p_2 = (2 - |A|)/4 \), where \( |A| = (A_1^2 + A_2^2)^{1/2} \). Using the notation \( \mathbf{A} = (A_1, A_2) \) and \( \sigma = (\sigma_z, \sigma_y) \) we can write \( \rho_{\text{unb},2} \) as a mixture of two orthogonal pure states:

\[
\rho_{\text{unb},2} = p_1 \left( \frac{1}{2} + \frac{\hat{n}_\sigma}{2} \right) + p_2 \left( \frac{1}{2} - \frac{\hat{n}_\sigma}{2} \right) \tag{27}
\]

where \( \mathbf{A} = |A| \hat{n} \). With \( \rho_1 = (\frac{1}{2} + \frac{\hat{n}_\sigma}{2}) \) and \( \rho_2 = (\frac{1}{2} - \frac{\hat{n}_\sigma}{2}) \), this is of the form (1).

The initial state from (17) is,

\[
\rho_{\text{ini}} = \frac{1}{2} + \frac{\mathbf{A} \sigma}{2} \pm \frac{1}{2} \sqrt{1 - |A|^2 \sigma_x} \tag{28}
\]

Denoting \( \rho_1 = |+\rangle \langle +| \) and \( \rho_2 = |\rangle \langle | \) the purification of \( \rho_{\text{unb},2} \) by protocol-A is,

\[
\rho_{\text{unb},2}^{(A)} = \left( \frac{1}{2} + \frac{|A|}{4} \right) |+\rangle \langle +| + \left( \frac{1}{2} - \frac{|A|}{4} \right) |\rangle \langle | \right.
\]

\[
+ \left. \frac{1}{\sqrt{4 - |A|^2}} \left( |+\rangle \langle -| e^{i \delta} + |\rangle \langle +| e^{-i \delta} \right) \right) \tag{29}
\]

Then

\[
tr(\rho_{\text{unb},2}; \rho_{\text{unb},2}^{(A)}) = \frac{1}{2} + |A|^2 \tag{30}
\]

The eigenstates of \( \rho_{\text{unb},2} \) are:

\[
|\pm\rangle^T = N_{\pm} (iA_2, A_1 \mp |A|) \tag{31}
\]

where \( N_{\pm} = 2(|A|^2 + |A|A_1) \) are the normalization factors. Using \( N_+, N_- = 2|A|A_2 \) and \( \langle +| \sigma_z |+| \rangle = -i e^{i \delta} \) (\( \delta \) is the phase of \( A_2 \)) and \( \langle +| \sigma_z |\rangle = 0 \) we can compute \( F(\rho_{\text{ini}}, \rho_{\text{unb},2}^{(A)}) \). This is a function of the arbitrary phase \( \beta \) (from protocol-A). We therefore average the fidelity over the \( \beta \). This can be done in two ways: i) Average over all possible values of the \( \beta \) and ii) Average over only those values of the \( \beta \) (two such values) which gives optimal fidelity with the two possibilities for the initial state (eqn (28)). After some tedious work one gets

\[
F_{av}(\rho_{\text{ini}}, \rho_{\text{unb},2}^{(A)}) = F(\rho_{\text{ini}}, \rho_{\text{unb},2}) \tag{32}
\]

Using eqn (30) and \( |A| \leq 1 \) one sees that this fidelity can never exceed 5/8 for any initial state. Now, it can be verified that purification by protocol-B (7) gives,

\[
\rho_{\text{unb},2}^{(B)} = \left( \frac{ie^{-i \delta}}{\hat{p} \sqrt{\hat{p} (1 - \hat{p})}} \right) \left( \frac{\hat{p} e^{i \delta}}{1 - \hat{p}} \right) \tag{33}
\]

where

\[
\hat{p} = \frac{1}{2} (1 + \hat{n}_1) \tag{34}
\]

Using the relations for \( |\alpha|^2, A_1 \) and \( A_2 \),

\[
F(\rho_{\text{ini}}, \rho_{\text{unb},2}^{(B)}) = \frac{1}{2} (1 + |A|) \tag{35}
\]

This can indeed reach unity for a large class of initial states. From (25) (26) it is clearly seen that,

\[
F(\rho_{\text{ini}}, \rho_{\text{unb},2}^{(B)}) \geq F(\rho_{\text{ini}}, \rho_{\text{unb},2}) \tag{36}
\]
State reconstruction by the MaxEnt Principle \[7\] leads to:
\[
\rho_{\text{max},2} = \frac{1}{2} \left( \begin{array}{cc} 1 + A_1 & -iA_2 \\ iA_2 & 1 - A_1 \end{array} \right) \tag{37}
\]
and we have the relation:
\[
\rho_{\text{unb},2} = \frac{1}{4} (I + 2\rho_{\text{max},2}) \tag{38}
\]
Using this relation we get the fidelity:
\[
\mathcal{F}(\rho_{\text{ini}},\rho_{\text{max},2}) = \frac{1 + |A|^2}{2} \tag{39}
\]
Relation \[38\] implies that the eigenstates of \(\rho_{\text{max},2}\) are the same as that for \(\rho_{\text{unb},2}\). Further, the eigenvalues can also be calculated directly from \[38\]. They are \(q_1 = (1 + |A|)/2\) and \(q_2 = (1 - |A|)/2\). Purification of \(\rho_{\text{max},2}\) by protocol-A gives,
\[
\rho^{(A)}_{\text{max},2} = \left(1 + \frac{|A|}{2}\right)|+\rangle\langle +| + \left(\frac{1 + |A|}{2}\right)|-\rangle\langle -|
+ \sqrt{\frac{1 - |A|^2}{4}}(|+\rangle\langle -|e^{i\beta} + \text{c.c.}) \tag{40}
\]
Again, using similar arguments as earlier in this section, we can calculate:
\[
\mathcal{F}_{\text{av}}(\rho_{\text{ini}},\rho^{(A)}_{\text{max},2}) = \mathcal{F}(\rho_{\text{ini}},\rho_{\text{max},2}) \tag{41}
\]
Since the largest eigenvalue eigenstate of \(\rho_{\text{max},2}\) is the same as that for \(\rho_{\text{unb},2}\), we have \(\rho^{(B)}_{\text{max},2} = \rho^{(B)}_{\text{unb},2}\). Finally we have,
\[
\mathcal{F}(\rho_{\text{ini}},\rho^{(B)}_{\text{max},2}) = \mathcal{F}(\rho_{\text{ini}},\rho^{(B)}_{\text{unb},2}) \geq \mathcal{F}(\rho_{\text{ini}},\rho_{\text{max},2}) = \mathcal{F}_{\text{av}}(\rho_{\text{ini}},\rho^{(A)}_{\text{max},2}) \geq \mathcal{F}(\rho_{\text{ini}},\rho_{\text{unb},2}) = \mathcal{F}_{\text{av}}(\rho_{\text{ini}},\rho^{(B)}_{\text{unb},2}) \tag{42}
\]

D. Partial Measurement: \(S_z\) measured.

When only one component of spin is measured, say, \(S_z\) we have
\[
\rho_{\text{unb},1} = p_1 |+\rangle\langle +| + (1 - p_1)|-\rangle\langle -|
\]
Again, the initial state is of the form: \(\psi_{\text{ini}} = \alpha |+\rangle_z + \beta |\rangle_z\). Now, we know only that \(|\alpha|^2 = p_1\). The Fidelity of the post-measurement state with the initial state is:
\[
\mathcal{F}(\rho_{\text{ini}},\rho_{\text{unb},1}) = p_1^2 + (1 - p_1)^2
\]
Now, if we purify \(\rho_{\text{unb},1}\) by protocol- A, then
\[
\rho^{(A)}_{\text{unb},1} = \left(\frac{p_1}{\sqrt{p_1(1 - p_1)}} e^{-i\phi} \sqrt{p_1(1 - p_1)} e^{i\phi}\right) \tag{43}
\]
\(\rho_{\text{ini}}\) is of the form:
\[
\rho_{\text{ini}} = \left(\frac{p_1}{\sqrt{p_1(1 - p_1)}} e^{i\phi} \sqrt{p_1(1 - p_1)} e^{-i\phi}\right) \tag{44}
\]
where \(p_1\) is known from measurement, but the phase \(\theta\) is unknown. The average Fidelity with equal weightage for all \(\theta\) is,
\[
\mathcal{F}_{\text{av}}(\rho_{\text{ini}},\rho^{(A)}_{\text{unb},1}) = p_1^2 + (1 - p_1)^2 \tag{45}
\]
Now, if \(p_1 \geq 1/2\), then the purified state according to protocol-B is,
\[
\rho^{(B)}_{\text{unb}} = |+\rangle z z \langle +| \tag{46}
\]
Then,
\[
\mathcal{F}(\rho_{\text{ini}},\rho^{(B)}_{\text{unb},1}) = p_1 \tag{47}
\]
Since, \(p_1 \geq 1/2\), it can be verified that
\[
\mathcal{F}(\rho_{\text{ini}},\rho^{(B)}_{\text{unb},1}) \geq \mathcal{F}(\rho_{\text{ini}},\rho^{(A)}_{\text{unb},1}) = \mathcal{F}_{\text{av}}(\rho_{\text{ini}},\rho^{(A)}_{\text{unb},1}) \tag{48}
\]
It can be verified that even for \(p_1 < 1/2\) protocol-B always leads to an improvement in fidelity. Thus, the fidelity offered by protocol-B is better than the average fidelity offered by protocol-A.

In this case the state reconstruction by the MaxEnt Principle is:
\[
\rho_{\text{max},1} = \begin{pmatrix} p_1 & 0 \\ 0 & 1 - p_1 \end{pmatrix} = \rho_{\text{unb},1} \tag{49}
\]
Hence, the fidelity calculations remain the same as for \(\rho_{\text{unb},1}\).

V. ENTROPY OF PURIFICATION.

As mentioned before, the purification process leads to interesting questions regarding entropy. Associated with every quantum state is the von Neumann entropy
\[
S_v = -tr \rho \ln \rho \tag{50}
\]
In terms of the eigenvalues \(\{\lambda_i\}\) of the density matrix, one has
\[
S_v = - \sum_i \lambda_i \ln \lambda_i \tag{51}
\]
Hence the entropy \(S_v\) is zero for pure states. On the other hand, for mixed states this entropy is always positive. Thus purification reduces the entropy of the system. But the real issue is what purification, understood as an Operation on the composite of the system and the environment, does to the total entropy.

To address this, let us first consider the action of the unitary operator \(U\) in eqn[8] which is given explicitly
in eqn (14). The initial state of the composite is \( \rho^{(0)} \otimes \langle 0_E | 0_E \rangle \). It is easy to see that

\[
U(\rho^{(0)} \otimes \langle 0_E | 0_E \rangle)U^\dagger = |\psi\rangle \langle \psi| \otimes \rho'_E
\]

where \( \rho'_E \) is given by

\[
\rho'_E = \rho^{(0)}(0,0)|0_E\rangle\langle 0_E| + \rho^{(0)}(1,0)|1_E\rangle\langle 1_E| + \rho^{(0)}(1,1)|1_E\rangle\langle 1_E|
\]

(53)

There are two remarkable properties of eqns (12 and 23): i) the unitary transformation maintains the disentanglement of the system and environment and ii) the von Neumann entropy of the environment, which was zero to start with, has become equal to the system entropy we started with. So at this stage, not surprisingly, the total entropy has not changed (unitary transformations cannot change this!). This is an example of entropy swapping.

Now we consider the effect of the projections \( Q \). Remembering that for a total and not selective measurement we need two of them, the resulting environment density matrix is given by

\[
\rho_{E}^{fin} = \rho^{(0)}(0,0)|0_E\rangle\langle 0_E| + \rho^{(0)}(1,1)|1_E\rangle\langle 1_E|
\]

(54)

If the purifying basis \(|0\rangle, |1\rangle\) is the same as the eigenbasis of \( \rho^{(0)} \), it is immediately obvious that the entropy of \( \rho_{E}^{fin} \) is the same as that of \( \rho^{(0)} \) and the purification process does not change the total entropy.

If that is not so, the entropy of \( \rho_{E}^{fin} \) is still the entropy of a density matrix \( \rho^{(2)} \) which is diagonal and whose diagonal elements are the same as that of \( \rho^{(0)} \). It therefore follows that the determinant of \( \rho^{(2)} \) is greater than that of \( \rho^{(0)} \). Now we establish an important result: the von Neumann entropy of a two level system is an increasing function of the determinant of the density matrix.

To prove this, note that the eigenvalues of the density matrix are given by \( \lambda_{\pm} = (1 \pm \sqrt{1 - 4\Delta_{\rho}})/2 \) where \( \Delta_{\rho} \) is the determinant \( (0 \leq \Delta_{\rho} \leq 1/4) \). It is then straightforward to show that

\[
\frac{\partial S_{E}}{\partial \Delta_{\rho}} = \frac{1}{\sqrt{1 - 4\Delta_{\rho}}} \ln \frac{1 + \sqrt{1 - 4\Delta_{\rho}}}{1 - \sqrt{1 - 4\Delta_{\rho}}} \geq 0
\]

(55)

In this case the total entropy increases.

VI. CONCLUSION

In this paper we have addressed schemes for producing pure states from mixed states. Though our schemes necessarily involve an environment, what is novel is that no entanglement between the system and environment is necessary. In this respect our purification schemes differ from those that are mostly dealt with in the current literature. We give explicit expressions for the Kraus operators that are required.

We have considered two purification schemes A and B as well as two post-measurement states, one of which is based on the maximum entropy principle. In all cases considered we find that scheme B purifies more efficiently than scheme A. The unbiased mixture state and the MaxEnt states perform equally well in all cases in producing a pure state as close as possible to the pre-measurement state. Operationally, unbiased mixture states are straightforward to realize compared to MaxEnt states.

Finally we show that the total entropy either increases or remains the same. This is possible because we are able to achieve purifications without entanglement resulting in a case of entropy swapping. This along with a theorem we prove on entropy of two-state systems leads to the result.

VII. ACKNOWLEDGEMENTS

CD thanks Prof. Ajay Patwardhan of St. Xavier’s College, Mumbai, for his continuous encouragement and guidance. CD thanks The Institute of Mathematical Sciences for a Summer Student Fellowship during 2004 and the Indian Academy of Sciences for a fellowship in 2006.

[1] A. Peres, Quantum Theory: Concepts and Methods, Kluwer Academic Publishers, 1993.
[2] E.C.G Sudarshan, “Quantum Computers and Computing” V.4, No.1, 2003.
[3] V. Buzek, G. Drobný, R. Derka, G. Adam, and H. Wiedemann, Chaos, Solitons Fractals 10, 981 (1999).
[4] N.D. Hari Dass and Chirag Dhara, Phys. Rev. A 72, 024102 (2005), quant-ph/0406160
[5] K.-E. Hellwig and K. Kraus, Commun. Math. Phys., 11(1969), 214; ibid 16(1970), 142.
[6] K. Kraus, Ann. of Phys., 64 (1971), 311.
[7] E.T. Jaynes, Phys. Rev. 108, 171 (1957); ibid. 108, 620 (1957) 620; Am. J. Phys. 31 66 (1963).
[8] J. Fiurasek, quant-ph/0403165; M. Ricci et al, quant-ph/0403118; Kurt Jacobs, Phys. Rev. A68, 054302(BR)(2003); A. Bassi and G.C. Ghirardi, Phys. Lett. A309(2003)24; A.S. Holevo, quant-ph/0204077; T.A. Brun, C.M. Caves, R. Schack, Phys. Rev. A63, 042309(2001); H. Mack et al, Phys. Rev. A62, 042301(2000); J.I. Cirac, A.K. Ekert and C. Macchiavello, Phys.Rev.Lett 82 (1999) 4344; J. Dittmann and A. Uhlmann, J. Math.Phys. 40(1999)3246; V. Vedral and M.B. Plenio, Phys.Rev. A57(1998)1619; S. Bose, V. Vedral and P.L. Knight, Phys. Rev. A60(1999) 194.