Optimal kernel estimates for a Schrödinger type operator

Anna Canale, Cristian Tacelli

Abstract

In the paper the principal result obtained is the estimate for the heat kernel associated to the Schrödinger type operator \((1 + |x|^\alpha)\Delta - |x|^{\beta}\)

\[ k(t, x, y) \leq Ct^{-\frac{\theta}{\alpha}} \frac{\varphi(x)\varphi(y)}{1 + |x|^\alpha}, \]

where \(\varphi = (1 + |x|^\alpha)^{\frac{\beta - \alpha}{2}} \), \(\theta \geq N\) and \(0 < t < 1\), provided that \(N > 2\), \(\alpha > 2\) and \(\beta > \alpha - 2\). This estimate improves a similar estimate in [3] with respect to the dependence on spatial component.

1 Introduction

In this paper we consider the elliptic operator defined by

\[ Au(x) = a(x)\Delta u(x) - V(x)u(x), \quad x \in \mathbb{R}^N, \]

where \(a(x) = 1 + |x|^\alpha\), \(\alpha > 2\) and \(V(x) = |x|^{\beta}\), \(\beta > \alpha - 2\). Our aim is to give better estimate for the associated heat kernel than those obtained in [3].

Recently elliptic operators with unbounded coefficients have been studied in several paper (see for example [12], [13], [14], [15], [10], [7], [4], [9], [8], [5], [6]). In [10] and [4] it is proved that \(A\) endowed with domain

\[ D_p(A) = \{ u \in W^{2,p}(\mathbb{R}^N) \mid (1 + |x|^\alpha)D^2u, (1 + |x|^\alpha)^{1/2}\nabla u, |x|^{\beta} u \in L^p(\mathbb{R}^N) \} \]

(2)
generates a strongly continuous and analytic semigroup \(T(\cdot)\) in \(L^p(\mathbb{R}^N)\) for \(1 < p < \infty\), for \(\alpha > 2\) and \(\beta > \alpha - 2\). This semigroup is also consistent, irreducible and ultracontractive. As regards the case \(\beta = 0\) we refer to [7] and [12].

Due to the regularity of the coefficients of the operator \(A\), the semigroup \(T(t)\) can be represented in the following integral form through a heat kernel \(k(t, x, y)\)

\[ T(t)f(x) = \int_{\mathbb{R}^N} k(t, x, y)f(y)dy, \quad t > 0, \quad x \in \mathbb{R}^N, \]

for any \(f \in L^p\) (see [2], [11]).

In [3] was obtained the heat kernel estimate provided that \(N > 2\), \(\alpha \geq 2\) and \(\beta > \alpha - 2\)
\[ k(t, x, y) \leq c_1 e^{\lambda_0 t} e^{c_2 t - b} \psi(x) \psi(y) \frac{1}{1 + |y|^\alpha} , \quad t > 0, \; x, y \in \mathbb{R}^N, \]

where \( c_1, c_2 \) are positive constant, \( b = \frac{\beta - \alpha + 2}{\beta + \alpha - 2} \) and \( \psi(x) \) is the eigenfunction associated to the first eigenvalue, which is equivalent to the function

\[ |x|^{- \frac{N-1}{2}} |y|^{\frac{\beta - \alpha}{2}} e^{- \int_1^{|x|} s^{\beta/2} \sqrt{1 + s^\alpha} ds}. \]

A better estimate with respect to the time variable was also obtained for small values of \( t \)

\[ k(t, x, y) \leq C t^{- \frac{N}{2}} (1 + |x|^\alpha)^{\frac{2-N}{2}} (1 + |y|^\alpha)^{\frac{2-N}{2}} - 1, \quad 0 < t \leq 1. \]

Comparing (3) and (4) ones can see that improving the dependence on \( t \) involves worsening in the dependence on the space component. Conversely, if the space component is improved the other worsens.

In this paper our aim is to explain how this happens. In order to state the relationship between the dependence on the time and the space components we will state an estimate which depend on a parameter \( \theta \). In particular we will prove the following estimate for small values of \( t \)

\[ k(t, x, y) \leq C t^{- \frac{\theta}{2}} \frac{\varphi(x) \varphi(y)}{1 + |y|^\alpha}, \]

where \( \varphi = (1 + |x|^\alpha)^{-\frac{2-N}{N} + \frac{\theta}{2}} \), \( \theta \geq N \). We observe that (4) is a particular case of (5) obtained for \( \theta = N \).

2 Weighted spaces and Weighted Nash Inequalities

Let \( T(t) \) be the semigroup generated by the operator \( (A, D_p(A)) \), where \( A \) and \( D_p(A) \) are defined by (1) and (2). First we show that \( T(t) \) can be seen as a suitable semigroup \( T(t) \) on a weighted space. So, we can deduce heat kernel estimates of \( T(t) \) by heat kernel estimates of \( T(t) \). Then, in order to obtain kernel estimates of \( T(t) \) we prove a weighted ultracontractivity of the semigroup obtained by a weighted Nash Inequality.

Let us introduce the measure \( d\mu(x) = (1 + |x|^\alpha)^{-1} dx \) and the Hilbert space \( L_\mu^2 \) endowed with its canonical inner product.

\[ H = \{ u \in L_\mu^2 \cap W^{1,2}_{\text{loc}} : V^{1/2} u \in L_\mu^2, \nabla u \in L^2 \} \]

be the Sobolev space endowed with the inner product

\[ (u, v)_H = \int_{\mathbb{R}^N} (1 + V) u \bar{v} \, d\mu + \int_{\mathbb{R}^N} \nabla u \cdot \nabla \bar{v} \, dx. \]

We consider the close and accretive symmetric form so defined

\[ a(u, v) = \int_{\mathbb{R}^N} \nabla u \cdot \nabla \bar{v} \, dx + \int_{\mathbb{R}^N} u \bar{v} \, d\mu \]

for \( u, v \) belonging to the closure \( V \) of \( C_0^\infty \) in \( H \), with respect to the norm of \( H \). Then we can associate to \( a \) the self-adjoint operator

\[ Au = f \]
with domain

\( D(A) = \{ u \in \mathcal{V} : \text{there exists } f \in L^2_{\mu} \text{ s.t. } a(u, v) = -\int_{\mathbb{R}^N} f \bar{v} d\mu \text{ for any } v \in \mathcal{V} \} \).

By classical results the operator \( A \) generates an analytic semigroup of contractions \( T(t) \) in \( L^2_{\mu} \) which is a positive, symmetric and \( L^\infty \)-contractive Markov semigroup.

The Lemma below (see [3]) shows that the semigroup \( T(t) \) coincides in \( L^p \cap L^2_{\mu} \) with the semigroup \( T(t) \) generated by \( (A, D_p(A)) \) in \( L^p(\mathbb{R}^N) \).

**Lemma 2.1** We get

\( D(A) \subset \{ u \in \mathcal{V} \cap W^{2,2}_{loc} : (1 + |x|^\alpha)\Delta u - V(x)u \in L^2_{\mu} \} \)

and \( Au = (1 + |x|^\alpha)\Delta u - V(x)u \) for \( u \in D(A) \). If \( \lambda > 0 \) and \( f \in L^p \cap L^2_{\mu} \), then

\[ (\lambda - A)^{-1}f = (\lambda - A)^{-1}f. \]

Denoting by \( k(t, x, y) \) and \( k_{\mu}(t, x, y) \) respectively the heat kernel associated to \( T(t) \) and \( T(t) \), by the previous lemma we can deduce that

\[ k_{\mu}(t, x, y) = (1 + |y|^\alpha)k(t, x, y). \]

In the following we describe how estimates of the kernel of a symmetric Markov semigroup can be obtained by using the equivalence between a weighted Nash inequality and a “weighted” ultracontractivity of the semigroup. The equivalence was stated in [10] Theorem 3.3 and was reformulated in [1] Theorem 2.5. The equivalence is obtained by means of a suitable Lyapunov functions for the generator of the semigroup.

Let \( T(t) \) be a symmetric Markov semigroup generated by a self-adjoint operator \( A \) associated to an accretive, closed, symmetric form defined on a domain \( \mathcal{V} \) in \( L^2_{\mu} \). Let \( k_{\mu} \) its associated heat kernel. We define Lyapunov function in the following way (see also [13], [14])

**Definition 2.2** A Lyapunov function is a positive function \( \varphi \in L^2_{\mu} \) such that

\[ T(t)\varphi(x) = \int_{\mathbb{R}^N} k_{\mu}(t, x, y)\varphi(y)d\mu(y) \leq e^{\kappa t}\varphi(x) \]

for any \( x \in \mathbb{R}^N \), \( t > 0 \), and for some real constant \( \kappa \), called Lyapunov constant.

Now, we define the weighted Nash inequality.

**Definition 2.3** Let \( \varphi \) be a positive function on \( \mathbb{R}^N \) and \( \psi \) be a positive function defined on \( (0, \infty) \) with \( \frac{\psi(x)}{x} \) non decreasing. The form \( a \) on \( L^2_{\mu} \) satisfies a weighted Nash inequality with weight \( \varphi \) and rate function \( \psi \) if

\[ \psi\left( \frac{\|u\|_{L^2_{\mu}}^2}{\|u\varphi\|_{L^1_{\mu}}^2} \right) \leq \frac{a(u, u)}{\|u\varphi\|_{L^1_{\mu}}^2} \]

for any functions \( u \in \mathcal{V} \) such that \( \|u\|_{L^2_{\mu}}^2 > 0 \) and \( \|u\varphi\|_{L^1_{\mu}}^2 < \infty \).
Theorem 2.4 Let $T(t)$ be a Markov semigroup with generator $A$ symmetric in $L^2_w(\mathbb{R}^N)$. Let us assume that there exists a Lyapunov function $\varphi$ with Lyapunov constant $\kappa \geq 0$ and that the associated form $a$ satisfies a weighted Nash inequality with weight $\varphi$ and rate function $\psi$ such that

$$\int_\infty^\infty \frac{1}{\psi(x)} dx < \infty.$$  

Then

$$\|T(t)f\|_{L^2_w} \leq K(2t)e^{\kappa t}\|f\varphi\|_{L^1_w}$$

for any functions $f \in L^2_w$ such that $f\varphi \in L^1_w$. The function $K$ is defined by

$$K(t) = \sqrt{U^{-1}(t)},$$

where

$$U(t) = \int_t^\infty \frac{1}{\psi(u)} du.$$  

Finally from [1, Corollary 2.8] we get the estimate of $k_\mu$

Corollary 2.5 If the Markov semigroup $T(t)$ satisfies the assumptions of Theorem 2.4 then the kernel $k_\mu$ satisfies

$$k_\mu(2t, x, y) \leq K(2t)^2 e^{2\kappa t} \varphi(x)\varphi(y)$$

for any $t > 0, (x, y) \in \mathbb{R}^N \times \mathbb{R}^N$.

3 Heat kernel estimates

In this section we will prove upper bound estimates for the kernel $k$. First we prove that the function $\varphi(x) = (1 + |x|^\alpha)^{\frac{\gamma}{2}}$ is a Lyapunov function if $\gamma < -\frac{N}{2} + \frac{\alpha}{2}$.

Lemma 3.1 Let $\gamma < -\frac{N}{2} + \frac{\alpha}{2}$ be a real constant. Then the function $\varphi(x) = (1 + |x|^\alpha)^{\frac{\gamma}{2}} \in L^2_w(\mathbb{R}^N)$ and satisfies the inequality $A\varphi(x) \leq \kappa \varphi$ for some $\kappa > 0$.

Proof. Let us consider $\varphi(x) = (1 + |x|^\alpha)^{\frac{\gamma}{2}}$. It is easy to see that $\varphi \in L^2_w$ if $2\gamma - \alpha < -N$. Furthermore we get

$$A\varphi = \gamma (\gamma - \alpha)(1 + |x|^{\alpha})^{\frac{\gamma - 1}{2}} |x|^{2\alpha - 2} + (\alpha - 2 + N)(1 + |x|^{\alpha})^{\frac{\gamma}{2}} |x|^{\alpha - 2} - |x|^\beta (1 + |x|^{\alpha})^{\frac{\gamma}{2}}$$

$$= \gamma (\gamma - \alpha) \frac{|x|^{2\alpha - 2}}{1 + |x|^{\alpha}} \varphi(x) + \gamma (\alpha - 2 + N)|x|^{\alpha - 2} \varphi(x) - |x|^\beta \varphi(x)$$

$$= \left[ |x|^{\alpha - 2} \left( \gamma (\gamma - \alpha) \frac{|x|^{\alpha}}{1 + |x|^{\alpha}} + \gamma (\alpha - 2 + N) \right) - |x|^\beta \right] \varphi(x).$$

Then, since $\beta > \alpha - 2$, one can see that there exists a positive constant $k$ such that

$$A\varphi(x) \leq k\varphi(x)$$

Lemma 3.1
Arguing as in [12, Section 2] we have that $\varphi$ is actually a Lyapunov function.

**Theorem 3.2** The function $\varphi$ is a Lyapunov function with constant $\kappa_0$ for any $\kappa_0 > \kappa$.

**Proof.** Let us observe that $\varphi \in C_0(\mathbb{R}^N)$. Then we can consider $u = R(\lambda, A_{\min})\varphi = (\lambda - A_{\min})^{-1}\varphi \in C_0(\mathbb{R}^N)$ (see [4, Section 2]). Let $\kappa_0 > \kappa$, $\lambda \geq \frac{\kappa_0}{\kappa_0 - \kappa}$ and $w = (1 + \frac{\kappa_0}{\kappa_0 - \kappa})\varphi - \lambda u$. Since $Au = \lambda u - \varphi$, we have

$$Aw - \lambda w = \frac{\kappa_0 + \lambda}{\lambda}A\varphi - \lambda(\lambda u - \varphi) - \lambda \left(\frac{\kappa_0 + \lambda}{\lambda} - \varphi - \lambda u\right)$$

$$= \frac{\kappa_0 + \lambda}{\lambda}A\varphi - \lambda^2 u + \lambda\varphi - (\kappa_0 + \lambda)\varphi + \lambda^2 u$$

$$\leq \frac{\kappa_0 + \lambda}{\lambda}\kappa_0 - \kappa_0\varphi = \frac{1}{\lambda}(\kappa_0\kappa + \lambda\kappa - \kappa_0\kappa) \leq 0.$$

By the maximum principle we have $w > 0$ in $\mathbb{R}^N$. Then

$$\left(1 + \frac{\kappa_0}{\lambda}\right)\varphi \geq \lambda R(\lambda, A)\varphi.$$  

Iterating the last inequality we get

$$\left(1 + \frac{\kappa_0}{\lambda}\right)^n \varphi \geq \lambda^n R^n(\lambda, A)\varphi.$$  

So, we obtain

$$T(t)\varphi = \lim_{n \to \infty} \left[\frac{n}{t}R\left(\frac{n}{t}, A\right)\right]^n \varphi \leq \lim_{n \to \infty} \left(1 + \frac{\kappa_0 t}{n}\right)^n \varphi = e^{\kappa_0 t}\varphi.$$  

Finally to get kernel estimates we will prove the weighted Nash inequality (see Definition 2.3) with Lyapunov function

$$\varphi = (1 + |x|^\alpha)^{\frac{2n}{p} + \frac{N}{2} - \frac{\alpha}{2}}$$

and rate functions

$$\psi(t) = t^{1 + \frac{N}{p}}$$

for $\theta \geq N$. We observe that $\varphi$ satisfies hypothesis Lemma 3.1 if $\alpha > 2$.

In order to prove the weighted Nash inequality we use a weighted Sobolev inequality which we recall for reader’s convenience (see [13, Proposition 3.5]).

**Proposition 3.3** Let $\beta'$, $\gamma'$, $\nu, p, q$ real values such that

$$1 < p \leq q < \infty \quad \gamma' - 1 \leq \beta' \leq \gamma',$$

$$0 \leq 1 - \frac{1}{p} = \frac{1 - \gamma' + \beta'}{N}, \quad N + p(\gamma' - 1) \neq 0, \quad p \leq q \leq p^*, \quad p < N.$$

Then there exists a positive constant $C$ such that for any $u \in C_0^\infty(\mathbb{R}^N)$

$$\left(\int_{\mathbb{R}^N} (1 + |x|)^{\frac{\beta'}{q}} |u(x)|^q dx\right)^\frac{1}{q} \leq C \left(\int_{\mathbb{R}^N} (1 + |x|)^{\gamma'} |\nabla u(x)|^p dx\right)^\frac{1}{p}$$

$$+ C \left(\int_{\mathbb{R}^N} (1 + |x|)^\nu |u(x)|^p dx\right)^\frac{1}{p}.$$  

5
Theorem 3.4 If $\alpha > 2$ and $\beta > \alpha - 2$, then the kernel $k_\mu$ of the semigroup generated by $A$ satisfies the inequality

$$k_\mu(t, x, y) \leq \frac{C}{t^\theta} \varphi(x)\varphi(y)$$

for every $0 < t \leq 1$, $x$, $y \in \mathbb{R}^N$.

**Proof.** Let $u \in V$ such that $\|u\varphi\|_{L^1_\mu} < \infty$.

Applying Hölder’s inequality with $p = \frac{\theta + 2}{\theta - 2}$ we get

$$\int_{\mathbb{R}^N} |u|^2 d\mu = \int_{\mathbb{R}^N} |u| \varphi \varphi d\mu$$

$$\leq \left( \int_{\mathbb{R}^N} |u|^{\frac{2\theta}{\theta - 2}} \varphi \varphi d\mu \right)^{\frac{\theta - 2}{\theta}} \left( \int_{\mathbb{R}^N} |u| \varphi d\mu \right)^{\frac{\theta}{\theta}}$$

$$\leq \left( \int_{\mathbb{R}^N} |u|^{\frac{2\theta}{\theta - 2}} (1 + |x|)^{2 \frac{\alpha - N}{2q - \beta}} dx \right)^{\frac{\theta - 2}{\theta}} \left( \int_{\mathbb{R}^N} |u| \varphi d\mu \right)^{\frac{\theta}{\theta}}$$

$$\leq \left( \int_{\mathbb{R}^N} |u|^{\frac{2\theta}{\theta - 2}} (1 + |x|)^{2 \frac{\alpha - N}{2q - \beta}} dx \right)^{\frac{\theta - 2}{\theta}} \left( \int_{\mathbb{R}^N} |u| \varphi d\mu \right)^{\frac{\theta}{\theta}}.$$ 

Then

$$\left( \frac{\|u\|_{L^2_{\mu}}}{\|u\varphi\|_{L^1_\mu}} \right)^{1 + \frac{\theta}{2}} \leq \left( \int_{\mathbb{R}^N} |u|^{\frac{2\theta}{\theta - 2}} (1 + |x|)^{2 \frac{\alpha - N}{2q - \beta}} dx \right)^{\frac{\theta - 2}{\theta}} \|u\varphi\|_{L^1_\mu}^\frac{\theta}{\theta},$$

from which

$$\psi\left( \frac{\|u\|_{L^2_{\mu}}}{\|u\varphi\|_{L^1_\mu}} \right) \leq \frac{\left( \int_{\mathbb{R}^N} |u|^{\frac{2\theta}{\theta - 2}} (1 + |x|)^{2 \frac{\alpha - N}{2q - \beta}} dx \right)^{\frac{\theta - 2}{\theta}}}{\|u\varphi\|_{L^1_\mu}^\frac{\theta}{\theta}} \|u\varphi\|_{L^1_\mu}.$$ 

Applying the weighted Sobolev inequality with $p = 2$, $q = \frac{2\theta}{\theta - 2}$, $\gamma' = 0$, $q\beta' = 2\frac{\alpha - N}{2q - \beta}$ and $\nu = -\alpha$ we obtain

$$\left( \int_{\mathbb{R}^N} |u|^{\frac{2\theta}{\theta - 2}} (1 + |x|)^{2 \frac{\alpha - N}{2q - \beta}} dx \right)^{\frac{\theta - 2}{\theta}} \leq C \left( \|\nabla u\|_{L^2}^2 + \int_{\mathbb{R}^N} (1 + V) u^2 d\mu \right) = C\tilde{a}(u, u),$$

where $\tilde{a}(u, u) = a(u, u) + \int u^2 d\mu$ is the quadratic form associated with the operator $A + I$. Since $\varphi$ is a Lyapunov function with constant $\kappa + 1$ for the operator $A + I$, applying Corollary 2.5 we get

$$\tilde{k}_\mu(t, x, y) \leq \frac{C e^{(\kappa + 1)t}}{t^\frac{\theta}{2}} \varphi(x)\varphi(y),$$

where $\tilde{k}_\mu = e^t k_\mu$ is the kernel associated with $A + I$. This gives the result.

**Theorem 3.5** Let us assume $\alpha > 2$, $\beta > \alpha - 2$. Then the kernel $k$ of the semigroup generated by $A$ for every $0 \leq t \leq 1$ satisfies the bound

$$k(t, x, y) \leq C \frac{e^{\kappa t}}{t^\frac{\theta}{2}} \frac{\varphi(x)\varphi(y)}{1 + |x|^\alpha}. \quad (7)$$
References

[1] D. Bakry, F. Bolley, I. Gentil, and P. Maheux, *Weighted nash inequalities*, arXiv: 1004.3456.

[2] M. Bertoldi and L. Lorenzi, *Analytical methods for markov semigroups*, Chapman & Hall/CRC, 2007.

[3] A. Canale, A. Rhandi, and C. Tacelli, *Kernel estimates for schrödinger operators with unbounded diffusion and potential terms*, Preprint.

[4] ______, *Schrödinger type operators with unbounded diffusion and potential terms*, to appear in Ann. Sc. Norm. Super. Pisa Cl. Sci.

[5] A. Canale and C. Tacelli, *Optimal kernel estimates for a schrödinger type operator*, Preprint.

[6] T. Durante, R. Manzo, and C. Tacelli, *Kernel estimates for schrödinger type operators with unbounded coefficients and singular potential terms*, Preprint.

[7] S. Fornaro and L. Lorenzi, *Generation results for elliptic operators with unbounded diffusion coefficients in L^p- and C_b-spaces*, Discr. Cont. Dyn. Syst. A 18 (2007), 747–772.

[8] M. Kunze, L. Lorenzi, and A. Rhandi, *Kernel estimates for nonautonomous kolmogorov equations with potential term*, New prospects in direct, inverse and control problems for evolution equations *Springer INdAM Ser.*, 10 (2014), 229–251.

[9] ______, *Kernel estimates for nonautonomous kolmogorov equations*, Adv. Math. 287 (2016), 600–639.

[10] L. Lorenzi and A. Rhandi, *On Schrödinger type operators with unbounded coefficients: generation and heat kernel estimates*, J. Evol. Equ. 15 (2015), 53–88.

[11] G. Metafune, D. Pallara, and M. Wacker, *Feller semigroups on \( \mathbb{R}^n \)*, Semigroup Forum 65 (2002), 159–205.

[12] G. Metafune and C. Spina, *Elliptic operators with unbounded coefficients in L^p spaces*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 11 (2012), no. 2, 303–340.

[13] ______, *Kernel estimates for some elliptic operators with unbounded coefficients*, DCDS-A 32 (2012), 2285–2299.

[14] G. Metafune, C. Spina, and C. Tacelli, *Elliptic operators with unbounded diffusion and drift coefficients in L^p spaces*, Adv. Diff. Equat 19 (2012), no. 5-6, 473–526.

[15] ______, *On a class of elliptic operators with unbounded diffusion coefficients*, Evol. Equ. Control Theory 3 (2014), no. 4, 671–680.

[16] F. Y. Wang, *Functional inequalities and spectrum estimates: the infinite measure case*, J. Funct. Anal. 194 (2002), 288–310.