RECENT RESULTS ON LOWER BOUNDS OF EIGENVALUE PROBLEMS BY NONCONFORMING FINITE ELEMENT METHODS

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Abstract. A short survey of lower bounds of eigenvalue problems by nonconforming finite element methods is given. The class of eigenvalue problems considered covers Laplace, Steklov, biharmonic and Stokes eigenvalue problems.

1. Introduction. It is well known that the eigenvalue problems are very important, which appear in many fields, such as fluid mechanics, quantum mechanics, stochastic process and etc.. Thus, a fundamental task of computational mathematics is to find the eigenvalues of partial differential equations. Since last century, abundant works have been dedicated to this topic.

In the famous paper [18], Feng cites the pioneer work of Pólya in computing the upper bound of Laplace eigenvalue problem. It is known that based on the minimum-maximum principle discovered by Rayleigh, Poincaré, Courant and Fischer etc., any conforming finite element method will give the upper bound of the eigenvalue (see [40]). Nevertheless, for the lower bound, until 1979, Rannacher [39] first gives some numerical results for the biharmonic eigenvalue problem.

There is few work on analysis of the lower bound for a long time. Inspired by the minimum-maximum principle, people try to find the lower bound with the nonconforming element methods. Recently, a series of works make progress in this aspect, e.g. Lin and Lin [29] use the integral-identity technique to obtain the asymptotic expansion of the eigenvalue approximations by nonconforming finite element method; also see the numerical reports of Liu and Liu [34], Liu and Yan [35], Lin, Huang and Li [28], and the work of Yao and Qiao [47]. Another way by Armentano and Durán [3] is to use a general kind of expansion method to get the lower bound, which is of less restriction on the partition compared with the integral-identity skill. Then this idea has been extended and generalized to more

2010 Mathematics Subject Classification. Primary: 65N30, 65N15; Secondary: 35J25.

Key words and phrases. Lower bound, nonconforming finite element, ECR, $EQ^{rot}_1$, Morley, eigenvalue problem, Laplace, Steklov, biharmonic, Stokes.

This work is supported in part by the National Science Foundations of China (NSFC 11001259, 11031006, 2011CB309703 and 2010DFR00700) and Croucher Foundation of Hong Kong Baptist University, the national Center for Mathematics and Interdisciplinary Science, CAS and the President Foundation of AMSS-CAS.
and more nonconforming finite element methods and eigenvalue problems: Zhang, Yang and Chen [49], Yang and Bi [42], Li [26] considered the Wilson’s element, Li [25] analyzed \( EQ^{1h} \) element, etc.. For more details, please read papers: Zhang, Yang and Chen [49], Yang, Zhang and Lin [46], Hu, Huang and Lin [23], Lin, Xie, etc. [31], Luo, Lin and Xie [36]. The paper [49] surveys the topic of obtaining lower bound for the eigenvalues of second order elliptic and biharmonic operators by the corresponding nonconforming finite elements. In [23], the lower bounds of eigenvalue approximations have been analyzed for the Laplace, biharmonic and general \( 2m \)-order eigenvalue problems in a general framework. The lower bounds of eigenvalues of the Stokes, Steklov eigenvalue problems have been analyzed in [32] and [27], respectively. The lower-bound result of Laplace eigenvalue problem without the convergence-order assumption is given in Luo, Lin and Xie [36]. Hu, Huang and Shen [22] gets the lower bound of Laplace eigenvalue problem by conforming linear and bilinear elements together with the mass lumping method.

The aim of this paper is to introduce the current art of the lower bounds of the eigenvalue problems by nonconforming finite element methods. Furthermore, we only concentrate on the class of the nonconforming finite element methods which can produce the lower bounds without any additional regularity assumption of the eigenfunctions. For simplicity, we only discuss the problem in \( \mathbb{R}^2 \), but the methods and results here can be extended to the case \( \mathbb{R}^3 \). In this paper, we will use the standard notation of Sobolev spaces (see, e.g., [14, 16, 40]).

The outline of the paper will go as follows. In Section 2, some abstract results for the eigenvalue problem by nonconforming finite element methods are introduced. In Section 3, 4 and 5, we will give lower-bound results for the Laplace, Steklov, biharmonic eigenvalue problems, respectively. Section 6 is devoted to analyzing the lower-bound results of the Stokes eigenvalue problem by mixed finite element methods. Some concluding remarks are given in the last section.

2. The eigenvalue problem and finite element methods. In this section, we introduce some notation and error estimates of the nonconforming finite element approximation for eigenvalue problems. The letter \( C \) (with or without subscripts) denotes a generic positive constant which may be different at its different occurrences through the paper.

Let \((V, \| \cdot \|)\) be a real Hilbert space with inner product \((\cdot, \cdot)\) and norm \(\| \cdot \|\), respectively. Let \(a(\cdot, \cdot)\) be a symmetric bilinear form on \(V \times V\) satisfying

\[
\begin{align*}
    a(w, v) & \leq C\|w\|\|v\|, \quad \forall w \in V \text{ and } \forall v \in V, \\
    \|w\|^2 & \leq Ca(w, w), \quad \forall w \in V.
\end{align*}
\]

From (1) and (2), we know that \(\| \cdot \|_a := (a(\cdot, \cdot))^{1/2}\) and \(\| \cdot \|\) are two equivalent norms on \(V\) and we shall use \(a(\cdot, \cdot)\) and \(\| \cdot \|_a\) as the inner product and norm on \(V\) in the rest of this paper. Let \(W\) denote another Hilbert space and \(b(\cdot, \cdot)\) be a symmetric bilinear form on \(W \times W\) satisfying

\[
0 < b(w, w), \quad \forall w \in W \text{ and } w \neq 0.
\]

We assume \(b(\cdot, \cdot)\) is an inner product on \(W\) and the norm \(\| \cdot \|\) is relatively compact with respect to the norm \(\| \cdot \|_b = (b(\cdot, \cdot))^{1/2}\) in the sense that any sequence in \(V\) which is bounded in \(\| \cdot \|\), one can extract a subsequence which is Cauchy in \(W\) with respect to \(\| \cdot \|_b\).

In this paper we are concerned with the abstract eigenvalue problem:
Find \((\lambda, u) \in \mathcal{R} \times V\) such that \(b(u, u) = 1\) and
\[
(4) \quad a(u, v) = \lambda b(u, v), \quad \forall v \in V.
\]

From [5], we know that the eigenvalue problem (4) has a positive eigenvalue sequence \(\{\lambda_j\}\) with
\[
0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots, \quad \lim_{k \to \infty} \lambda_k = \infty,
\]
and the corresponding eigenfunction sequence \(\{u_j\}\)
\[
u_1, u_2, \cdots, u_k, \cdots,
\]
with the property \(b(u_i, u_j) = \delta_{ij}\).

From the result in [5], the Rayleigh quotient is defined by
\[
R(v) := \frac{a(v, v)}{b(v, v)} \quad \text{and} \quad R(u) = \lambda.
\]

Let \(\mathcal{T}_h\) be a quasi-uniform decomposition of the polygonal domain \(\Omega\) into triangles or rectangles (c.f. [14, 16]). The mesh diameter \(h\) describes the maximum diameter of all cells \(K \in \mathcal{T}_h\). Let \(\mathcal{E}_h\) denote the edge set of \(\mathcal{T}_h\) and \(\mathcal{E}_h = \mathcal{E}_h^i \cup \mathcal{E}_h^b\), where \(\mathcal{E}_h^i\) denotes the interior edge set and \(\mathcal{E}_h^b\) denotes the edge set lying on the boundary \(\partial \Omega\). The finite element space \(V_h\) is the corresponding finite element space on the partition, i.e. \(V_h \subsetneq V\) as a nonconforming space and \(V_h \subset V\) a conforming space.

In the rest of this paper, we only concentrate on the nonconforming finite element case and assume the bilinear form \(b(\cdot, \cdot)\) can also be defined on \(V_h \times V_h\). The finite element approximation of (4) is defined as follows:

Find \((\lambda_h, u_h) \in \mathcal{R} \times V_h\) such that \(b(u_h, u_h) = 1\) and
\[
(5) \quad a_h(u_h, v_h) = \lambda_h b(u_h, v_h), \quad \forall v_h \in V_h,
\]
where the bilinear form \(a_h(\cdot, \cdot)\) coincides with an elementwise representation of \(a(\cdot, \cdot)\) in the nonconforming situation, e.g.
\[
a_h(u_h, v_h) = \sum_{K \in \mathcal{T}_h} a_K(u_h, v_h).
\]

We assume the bilinear form \(a_h(\cdot, \cdot)\) is \(V_h\)-elliptic on \(V + V_h\). Thus we define the norms on \(V_h + V\)
\[
\|v\|_{a,h}^2 = a_h(v, v) \quad \text{for} \quad v \in V + V_h.
\]

For the eigenvalue problem (5), the Rayleigh quotient holds for the eigenvalue \(\lambda_h\)
\[
R(v_h) = \frac{a_h(v_h, v_h)}{b(v_h, v_h)} \quad \text{and} \quad R(v_h) = \lambda_h.
\]

Similarly, the discrete eigenvalue problem (5) has also an eigenvalue sequence \(\{\lambda_{j,h}\}\) with
\[
0 < \lambda_{1,h} \leq \lambda_{2,h} \leq \cdots \leq \lambda_{N_h,h},
\]
and the corresponding discrete eigenfunction sequence \(\{u_{j,h}\}\)
\[
u_{1,h}, u_{2,h}, \cdots, u_{k,h}, \cdots, u_{N_h,h}
\]
with the property \(b(u_{i,h}, u_{j,h}) = \delta_{ij}, 1 \leq i,j \leq N_h\) (\(N_h\) is the dimension of \(V_h\)).
There is a classical result about eigenvalue: minimum-maximum principle. Let \( \lambda_j \) be the \( j \)-th eigenvalue of (4) and \( \lambda_{j,h} \) be the \( j \)-th eigenvalue of (5), respectively. Arranging them by increasing order, then we have (see e.g., [5])

\[
\lambda_j = \min_{v_j \in \mathcal{V}} \max_{v \in v_j} R(v), \quad \lambda_{j,h} = \min_{v_j \in \mathcal{V}_h} \max_{v \in v_j} R(v).
\]

From (6), it is obvious that the conforming element methods \((V_h \subset V)\) can only obtain the upper bounds of eigenvalue problems.

In order to give the error estimates of the eigenpair approximation by finite element methods, we define the following notation

\[
\lambda(T_f,v) = b(f,v), \quad \forall v \in V,
\]

for any \( f \in W \). As we know, the operator \( T : W \longrightarrow V \) by

\[
\lambda T u = u.
\]

We also define the corresponding discrete operator \( T_h : W \longrightarrow V_h \) by

\[
a_h(T_h f, v_h) = b(f,v_h), \quad \forall v_h \in V_h,
\]

for any \( f \in W \). Similarly the discrete eigenvalue problem (5) can be written as

\[
\lambda_h T_h u_h = u_h.
\]

Let \( M(\lambda_j) \) denote the eigenfunction set corresponding to the eigenvalue \( \lambda_j \) which is defined by

\[
M(\lambda_j) = \{ w \in V : w \text{ is an eigenfunction of (4) corresponding to } \lambda_j \}
\]

and \( \| w \|_b = 1 \} \).

Now we state the convergence result of the eigenvalue problems by nonconforming finite element methods. For this aim, we define the following notation

\[
\varepsilon_h(\lambda_j) = \|(T - T_h)\|_{M(\lambda_j)}\|a,h\|,
\]

\[
\rho_h(\lambda_j) = \|(T - T_h)\|_{M(\lambda_j)}\|b\|.
\]

**Lemma 2.1.** ([37, 39, 43]) Suppose that \( \|T_h - T\|_b \rightarrow 0 \) \((h \rightarrow 0)\). Let \((\lambda_{j,h}, u_{j,h}) \in \mathcal{R} \times V_h\) be the \( j \)-th nonconforming finite element eigenpair approximation satisfying (5). Then \( \lambda_{j,h} \rightarrow \lambda_j \) and there exist \( u_j \in M(\lambda_j) \) such that

\[
\|u_j - u_{j,h}\|_{a,h} \leq C(\lambda_j \varepsilon_h(\lambda_j) + \rho_h(\lambda_j)),
\]

\[
\|u_j - u_{j,h}\|_b \leq C\rho_h(\lambda_j),
\]

\[
|\lambda_j - \lambda_{j,h}| \leq C\rho_h(\lambda_j).
\]

For the eigenvalue problem, we have the following basic expansion which was introduced in [3, 49] and has been extensively used in [23, 31, 41, 42, 46]. For the convenience of reading, we give a simple proof.

**Lemma 2.2.** ([3, 49]) Suppose \((\lambda_j, u_j)\) is the \( j \)-th eigenpair of the original problem (4), \((\lambda_{j,h}, u_{j,h}) \in \mathcal{R} \times V_h\) is the corresponding \( j \)-th eigenpair of the discrete problem (5). We have the following expansion

\[
\lambda_j - \lambda_{j,h} = \|u_j - u_{j,h}\|_{a,h}^2 - \lambda_{j,h}\|v_{h} - u_{j,h}\|_b^2
\]

\[
+ \lambda_{j,h}(\|u_{h}\|_b^2 - \|u_j\|_b^2) + 2a_h(u_j - v_h, u_{j,h}), \quad \forall v_h \in V_h.
\]
Proof. Since \( \|u_j\|_b = \|u_j,h\|_b = 1 \), \( a_h(u_j, u_j) = \lambda_j \) and \( a_h(u_j,h, u_j,h) = \lambda_j,h \), we have
\[
\lambda_j + \lambda_j,h = a_h(u_j - u_j,h, u_j - u_j,h) + 2a_h(u_j, u_j,h)
\]
\[
= \|u_j - u_j,h\|_{a,h}^2 + 2a_h(v_h, u_j,h) + 2a_h(u_j - v_h, u_j,h)
\]
\[
= \|u_j - u_j,h\|_{a,h}^2 - \lambda_j,h \|v_h - u_j,h\|_b^2 + \lambda_j,h \|u_j,h\|_b^2
\]
\[
+ \lambda_j,h \|u_h\|_b^2 + 2a_h(u_j - v_h, u_j,h)
\]
\[
= \|u_j - u_j,h\|_{a,h}^2 - \lambda_j,h \|v_h - u_j,h\|_b^2 + 2\lambda_j,h
\]
\[
+ \lambda_j,h(\|v_h\|_b^2 - \|u_j\|_b^2) + 2a_h(u_j - v_h, u_j,h), \quad \forall v_h \in V_h.
\]
Then (17) can be obtained and we complete the proof. \( \square \)

3. Laplace eigenvalue problem. In this section, we consider the lower-bound results of the Laplace eigenvalue problem by nonconforming finite element methods. The concerned Laplace eigenvalue problem can be stated as follows:

Find \( (\lambda, u) \) such that

\[
\begin{align*}
-\Delta u &= \lambda u, \quad \text{in } \Omega, \\
\int_\Omega u^2 d\Omega &= 1,
\end{align*}
\]

(18)

where \( \Omega \) is a bounded POLYGONAL domain in \( \mathbb{R}^2 \) with continuous Lipschitz boundary \( \partial \Omega \). In order to give the error estimates of the finite element method, we assume the eigenfunction has the regularity

\[
u \in H^{1+\gamma}(\Omega),
\]

where \( 0 < \gamma \leq 1 \) depends on the maximum angle of the boundary \( \partial \Omega \) and \( \gamma = 1 \) when the domain \( \Omega \) is convex (c.f. [20]).

In this case, the weak form (4) for the eigenvalue problem (18) can be defined with \( V = H^1_0(\Omega), \ W = L^2(\Omega), \ | | \ |_b = | | \ |_0 \) and

\[
a(u, v) = \int_\Omega \nabla u \cdot \nabla v d\Omega, \quad | | v | |_a = \left( a(v, v) \right)^{\frac{1}{2}}, \quad \text{and} \quad b(u, v) = \int_\Omega uv d\Omega.
\]

From [36], we are concerned with two types of nonconforming finite elements: Enriched Crouzeix-Raviart (ECR) (c.f. [23, 32]) and Extended \( Q_1^{rot} \) (\( EQ_1^{rot} \)) (c.f. [30]) for triangular and rectangular partitions, respectively.

- ECR element is defined on the triangular partition and

\[
V_h := \left\{ v \in L^2(\Omega) : v|_K \in \text{span}\{1, x, y, x^2 + y^2\}, \int_\ell v|_{K_1} ds = \int_\ell v|_{K_2} ds, \right\}
\]

(20)

where \( K, K_1, K_2 \in T_h \).

- \( EQ_1^{rot} \) element is defined on the rectangular partition and

\[
V_h := \left\{ v \in L^2(\Omega) : v|_K \in \text{span}\{1, x, y, x^2, y^2\}, \int_\ell v|_{K_1} ds = \int_\ell v|_{K_2} ds, \right\}
\]

(21)

where \( K, K_1, K_2 \in T_h \).
Based on the ECR and $EQ_1^{rot}$ elements, the discrete eigenvalue problem (5) for the eigenvalue problem (18) can be define with

$$a_h(u_h, v_h) = \sum_{K \in T_h} \int_K \nabla u_h \cdot \nabla v_h \, dK$$

and

$$\|v_h\|_{a,h} = \left( a_h(v_h, v_h) \right)^{\frac{1}{2}}.$$  

The interpolation operator $\Pi_h : V \mapsto V_h$ corresponding to ECR and $EQ_1^{rot}$ elements can be defined in the same way (c.f. [23, 30, 32]):

$$\int_{\ell} (u - \Pi_h u) \, ds = 0, \quad \forall \ell \in E_h,$$

$$\int_K (u - \Pi_h u) \, dK = 0, \quad \forall K \in T_h.$$  

**Lemma 3.1.** ([36]) For any $u \in V$, the interpolation defined in (22)-(23) has the following results

$$a_h(u - \Pi_h u, v_h) = 0, \quad \forall v_h \in V_h,$$

$$\|\Pi_h u\|_{a,h} \leq \|u\|_a,$$

$$\|u - \Pi_h u\|_{b,h} \leq C h \|u - \Pi_h u\|_{a,h},$$

$$\|u - \Pi_h u\|_{b,h} \leq C h \|u - u_h\|_{a,h}.$$  

Furthermore, the interpolation operator has error estimates

$$\|u - \Pi_h u\|_b + h \|u - \Pi_h u\|_{a,h} \leq C h^{1+\gamma} \|u\|_{1+\gamma},$$

for any $u \in H^{1+\gamma}(\Omega)$.

Now we state a lower bound of the convergence rate for the eigenfunction approximation by finite element methods which will be used in the analysis of the lower bound of eigenvalue approximation.

**Lemma 3.2.** ([33, Section 3]) If we solve the eigenvalue problem (4) by $EQ_1^{rot}$ or ECR, the following lower bound of the convergence rate holds

$$\|u - u_h\|_{a,h} \geq C h.$$  

For analyzing the lower bounds of the eigenvalue problems, we need to show that $\|(T - T_h)\|_{M(\lambda)}$ is a higher order term in terms of $\|(T - T_h)\|_{M(\lambda)}_{a,h}$.

**Theorem 3.3.** ([31]) Assume the nonconforming finite element owns a type of interpolation operator $\Pi_h$ satisfying the orthogonal property (24). We have the following error estimate

$$\|(T - T_h)\|_{M(\lambda)} \leq C h^\gamma \|(T - T_h)\|_{M(\lambda)}_{a,h}.$$  

**Proof.** For any $f \in M(\lambda)$, let $u = Tf$ and $u_h = T_h f$. For any $\psi \in W$, we have

$$\|u - u_h\|_b = b(u - u_h, \psi) = a_h(u, \varphi_\psi) - a_h(u_h, \varphi_h) = a_h(u - u_h, \varphi_\psi - \varphi_h) + a_h(u_h, \varphi_\psi - \varphi_h) + a_h(u - u_h, \varphi_h) = a_h(u - u_h, \varphi_\psi - \varphi_h) - a_h(u - u_h, \varphi_\psi - b(u - u_h, \psi))$$

$$- \left[ a_h(u, \varphi_\psi - \varphi_h) - b(f, \varphi_\psi - \varphi_h) \right],$$

where $\varphi_\psi = T \psi$, $\varphi_h = T_h \psi$ and

$$\|\varphi_\psi\|_{1+\gamma} \leq C \|\psi\|_b.$$
First for the first term in the right hand side of (31), we have the estimate
\begin{equation}
|a(u - u_h, \varphi - \varphi_h)| \leq Ch^2 |u - u_h|_{a,h}.
\end{equation}
From the standard error estimate theory of the nonconforming finite element method, the following estimate holds
\begin{align*}
&\left| a_h(u - u_h, \varphi - \varphi_h) - b(u - u_h, \psi) \right| \\
&= \left| \sum_{K \in T_h} \int_K \nabla(u - u_h) \nabla \varphi_h dK - \int_{\Omega} (u - u_h) \psi d\Omega \right| \\
&\leq \left| \sum_{K \in T_h} \int_{\partial K} (u - u_h) \partial_{\nu} \varphi_h ds \right| \\
&\leq Ch^\gamma |u - u_h|_{a,h} \| \varphi - \varphi_h \|_{1+\gamma}.
\end{align*}
From the orthogonal property (24), we have the estimate for the third term in the right hand side of (31)
\begin{align*}
&\left| a_h(u, \varphi - \varphi_h) - b(f, \varphi - \varphi_h) \right| \\
&\leq \left| a_h(u, \Pi_h \varphi - \varphi_h) - b(f, \Pi_h \varphi - \varphi_h) \right| + \left| a_h(u, \varphi - \Pi_h \varphi) \right| + \left| b(f, \varphi - \Pi_h \varphi) \right| \\
&\leq C(|u - u_h|_{a,h} \| \Pi_h \varphi - \varphi_h \|_{a,h} + \| \varphi - \Pi_h \varphi \|_{a,h}) + \| f \|_b \| \varphi - \Pi_h \varphi \|_b \\
&\leq Ch^\gamma \| \varphi - \varphi_h \|_{1+\gamma} |u - u_h|_{a,h} + Ch^{1+\gamma} \| f \|_b \| \varphi - \varphi_h \|_{1+\gamma} \\
&\leq Ch^\gamma \| \varphi - \varphi_h \|_{1+\gamma} |u - u_h|_{a,h} + Ch^{1+\gamma} \| f \|_b.
\end{align*}
Combining (31), (32), (33), (34) and (35), we have
\begin{equation}
\| u - u_h \|_b = \sup_{0 \neq \psi \in W} \frac{b(u - u_h, \psi)}{\| \psi \|_b} \leq Ch^\gamma \sup_{0 \neq \psi \in W} \| \varphi - \varphi_h \|_{1+\gamma} \| u - u_h \|_{a,h}
\end{equation}
\begin{equation}
\leq Ch^\gamma |u - u_h|_{a,h} + Ch^{1+\gamma} \| f \|_b.
\end{equation}
The desired inequality (30) can be obtained by combining (29), (36) and \( \| f \|_b = 1 \) and we complete the proof.

From [4, 5, 23, 31, 41], the following basic error estimates for the two nonconforming finite elements hold
\begin{align}
\| \lambda_j - \lambda_{j,h} \|_b &\leq Ch^{2\gamma} \| u_j \|_{1+\gamma}^2, \\
\| u_j - u_{j,h} \|_{a,h} &\leq Ch^\gamma \| u_j \|_{1+\gamma}, \\
\| u_j - u_{j,h} \|_b &\leq Ch^\gamma \| u_j - u_{j,h} \|_{a,h} \leq Ch^{2\gamma} \| u_j \|_{1+\gamma}.
\end{align}
The following lower-bound result of the eigenvalue approximation by ECR or \( EQ_1^{\text{rot}} \) element has been discussed in [23, 25, 31, 36]. Here we state the result and give a simple proof.

**Theorem 3.4.** ([36]) Let \( \lambda_j \) and \( \lambda_{j,h} \) be the \( j \)-th exact eigenvalue and its corresponding numerical approximation by ECR or \( EQ_1^{\text{rot}} \) element. When \( h \) is small enough, we have
\begin{equation}
0 \leq \lambda_j - \lambda_{j,h} \leq Ch^{2\gamma} \| u_j \|_{1+\gamma}^2.
\end{equation}
Proof. The result for ECR and $EQ^1_{rot}$ elements can be proved in the uniform way. We choose $v_h = \Pi_h u_j$ in (17). For the second term in (17), from (39) and Lemma 3.1, we have
\[
\|u_{j,h} - \Pi_h u_j\|_b^2 \leq 2\|u_{j,h} - u_j\|_b^2 + 2\|u_j - \Pi_h u_j\|_b^2 
\leq Ch^{2\gamma}\|u_j - u_{j,h}\|_{a,h}^2 + Ch^{2+2\gamma}\|u\|_{1+\gamma}^2.
\]
For the third term in (17), from (27), we have
\[
\|v_h\|_b^2 - \|u_j\|_b^2 = (\Pi_h u_j - u_j, \Pi_h u_j + u_j) 
= (\Pi_h u_j - u_j, (\Pi_h u_j + u_j) - \Pi_0(\Pi_h u_j + u_j)) 
\leq Ch\|\Pi_h u_j - u_j\|_b\|\Pi_h u_j + u_j\|_{a,h} 
\leq Ch^2\|u_j - u_{j,h}\|_{a,h}\|\Pi_h u_j + u_j\|_{a,h}.
\]
Here and hereafter in this paper $\Pi_0$ denotes the piecewise constant interpolation. Together with (24) and Lemma 3.2, the first positive term is dominant in (17) and then (40) can be derived with (37).

4. Steklov eigenvalue problem. In this section, we are concerned with the lower-bound results of the Steklov eigenvalue problem. Steklov eigenvalue problem arises in a number of applications such as surface waves (see [9]), stability of mechanical oscillators immersed in a viscous fluid (see [17]), the vibration modes of a structure in contact with an incompressible fluid (see [10]), the antiplane shearing on a system of collinear faults under slip-dependent friction law (see [13]), vibrations of a pendulum (see [1]), eigen oscillations of mechanical systems with boundary conditions containing the frequency (see [21]).

The analysis of the conforming finite element methods for the Steklov eigenvalue problems has been given by Bramble and Osborn [11], Andreev and Todorov [2]. Recently, the nonconforming finite element methods for the Steklov eigenvalue problems have also been analyzed by Yang, Li and Li [44] on the convex domain and by Li, Lin and Xie [27] on both convex and concave domains.

In this section, we are concerned with the following Steklov eigenvalue problem
\[
\begin{cases}
-\Delta u + u = 0, & \text{in } \Omega, \\
\partial_{\nu} u = \lambda u, & \text{on } \partial \Omega, \\
\int_{\partial \Omega} u^2 ds = 1,
\end{cases}
\]
where $\Omega \subset \mathbb{R}^2$ is a bounded polygonal domain and $\partial_{\nu}$ denote the outward normal derivative on $\partial \Omega$.

The corresponding weak form (4) for the eigenvalue problem (41) can be defined with $V = H^1(\Omega), \| \cdot \|_a = \| \cdot \|_1, W = L^2(\partial \Omega)$ and
\[
a(u,v) = \int_{\Omega} (\nabla u \nabla v + uv) d\Omega, 
\]
\[
b(u,v) = \int_{\partial \Omega} uv ds, \|v\|_b = \left( b(u,v) \right)^{\frac{1}{2}}.
\]

In this section, we consider the lower bounds of the Steklov eigenvalue problem by ECR and $EQ^1_{rot}$ elements defined in (20) and (21), respectively. Then the nonconforming finite element approximation of (41) can be defined by (5) with
\[
a_h(u_h, v_h) = \sum_{K \in T_h} \int_K (\nabla u_h \nabla v_h + u_h v_h) dK \text{ and } \|v_h\|_{a,h} = \left( a_h(v_h, v_h) \right)^{\frac{1}{2}}.
\]
Lemma 4.1. ([11, (4.10)], [10, Proposition 4.4]) For the Steklov eigenvalue problem (41), the corresponding operator $T$ defined by (7) has the regularity results
\begin{align}
\|Tf\|_{1+\frac{\gamma}{2}} &\leq C\|f\|_{b}, \\
\|Tf\|_{1+\gamma} &\leq C\|f\|_{\frac{1}{2},\partial\Omega},
\end{align}
where $0 < \gamma \leq 1$ depends on the maximum angle of the boundary $\partial\Omega$.

Theorem 4.2. ([27, 44]) For the Steklov eigenvalue problem, the following error estimates hold for the nonconforming finite element approximation
\begin{align}
|\lambda_j - \lambda_{j,h}| &\leq C\lambda_j^2 h^{2+\gamma}\|u_j\|_{b}, \\
\|u_j - u_{j,h}\|_b &\leq C h^{2+\gamma}\|u_j - u_{j,h}\|_{a,h} \leq C\lambda_j^2 h^{2+\gamma}\|u_j\|_{\frac{1}{2},\partial\Omega}, \\
\|u_j - u_{j,h}\|_{a,h} &\leq C\lambda_j h^\gamma\|u_j\|_{\frac{1}{2},\partial\Omega},
\end{align}
where $C$ is a constant independent of $h$ and $\lambda_j$.

Proof. The estimates (44) and (46) can be obtained from Theorem 3.1 in [27]. The estimate (45) can be proved by the same process in the proof of Theorem 3.3. □

In order to derive the lower bounds of the Steklov eigenvalue problem, we need the following estimates.

Lemma 4.3. Let $u \in H^{1+\gamma}(\Omega)$. The following estimate holds
\begin{equation}
\|a_h(u - \Pi_h u, v_h)\| \leq C h^{2+\gamma}\|u\|_{1+\gamma}\|v_h\|_{a,h}, \quad \forall v_h \in V_h,
\end{equation}
for ECR and $EQ^{1st}$ elements.

Proof. From (24), we have
\[\sum_{K \in T_h} \int_K \nabla(u - \Pi_h u)\nabla v_h dK = 0, \quad \forall v_h \in V_h.\]
Thus
\begin{align}
a_h(u - \Pi_h u, v_h) &= \sum_{K \in T_h} \int_K \nabla(u - \Pi_h u)\nabla v_h dK + \int_{\Omega} (u - \Pi_h u)v_h d\Omega \\
&= \int_{\Omega} (u - \Pi_h u)v_h d\Omega, \quad \forall v_h \in V_h.
\end{align}

With the help of (23), the following estimate holds
\begin{align*}
&\left|\int_{\Omega} (u - \Pi_h u)v_h d\Omega\right| \\
&= \left|\int_{\Omega} (u - \Pi_h u)\Pi_0 v_h d\Omega + \int_{\Omega} (u - \Pi_h u)(v_h - \Pi_0 v_h) d\Omega\right| \\
&\leq C h^{2+\gamma}\|u\|_{1+\gamma}\|v_h\|_{a,h}.
\end{align*}
This is the desired result (47) and we complete the proof. □

Lemma 4.4. Let $u \in H^{1+\gamma}(\Omega)$ be an eigenfunction of (41). Then the following estimate holds
\begin{equation}
\|u - \Pi_h u\|_b \leq C h^{\frac{1}{2}+\gamma}\|u\|_{1+\gamma}.
\end{equation}
Proof. The interpolation error in the \( \| \cdot \|_b \) can be derived from the following trace inequality
\[
\| v \|_{0, \partial K} \leq C h^{-\frac{1}{2}} \| v \|_{0, K} + h^{\frac{1}{2}} \| \nabla v \|_{0, K}
\]
and the interpolation error estimate (28).

**Theorem 4.5.** When \( h \) is sufficiently small, we have
\[
\lambda_j - \lambda_{j,h} = \| u_j - u_{j,h} \|_{a,h}^2 + 2 \int_{\Omega} (u_j - \Pi_h u_j) u_{j,h} d\Omega + R,
\]
where \( |R| \leq C h^{1+2\gamma} \).

**Proof.** First from (43), we have \( u_j \in H^{1+\gamma}(\Omega) \). Taking \( v_h = \Pi_h u_j \) in (17), we estimate the second, third and fourth terms on the right-hand side of (17). From (45) and (49), we have
\[
\| \Pi_h u_j - u_{j,h} \|_b \leq \| \Pi_h u_j - u_j \|_b + \| u_j - u_{j,h} \|_b \leq C h^{\frac{1}{2} + \gamma} \| u \|_{1+\gamma} + C h^{\frac{1}{2} + \gamma} \| u_j - u_{j,h} \|_{a,h}.
\]
In addition, we introduce the piecewise constant interpolation operator \( J_0 \) on \( \partial \Omega \). Then, from (49), we have
\[
\| \Pi_h u_j \|_b^2 - \| u_j \|_b^2 = 2b(\Pi_h u_j - u_j, u_j) - b(\Pi_h u_j - u_j, u_j - \Pi_h u_j) \leq 2b(\Pi_h u_j - u_j, u_j - J_0 u_j) + C h^{1+2\gamma} \| u_j \|_{1+\gamma} + C h^{1+2\gamma} \| u_j - u_{j,h} \|_{a,h} \leq C h^{\frac{1}{2} + \gamma} \| u_j \|_{1+\gamma}^2 + C h^{\frac{1}{2} + \gamma} \| u_j - u_{j,h} \|_{a,h}^2 \leq C h^{1+2\gamma} \| u_j \|_{1+\gamma}^2 + C h^{1+2\gamma} \| u_j - u_{j,h} \|_{a,h}^2.
\]
Thus from (47), (51) and (52), we obtain (50).

**Corollary 1.** Let \( \lambda_j \) and \( \lambda_{j,h} \) be the \( j \)-th exact eigenvalue and its corresponding numerical approximation by ECR or \( \text{EQ}_1^{rot} \) element. Under the condition of \( \gamma > 1/2 \), we have
\[
\lambda_{j,h} \leq \lambda_j,
\]
when \( h \) is small enough.

**Proof.** From Lemma 3.2, (47) and \( |R| \leq C h^{1+2\gamma} \), we know that the second and the third terms on the right-hand side of (50) are infinitesimals of higher order than the first term \( \| u_j - u_{j,h} \|_{a,h}^2 \). So the sign of the right-hand side of (50) is determined by the first positive term. Thus (53) holds.

5. **Biharmonic eigenvalue problem.** In this section, we analyze the lower-bound results of the biharmonic eigenvalue problem:

Find \((\lambda, u)\) such that
\[
\begin{align*}
\Delta^2 u &= \lambda u, & \text{in } \Omega, \\
u &= \partial_{\nu} u &= 0, & \text{on } \partial\Omega, \\
\int_{\Omega} u^2 d\Omega &= 1,
\end{align*}
\]
where \( \Omega \subset \mathbb{R}^2 \) is a bounded polygonal domain with Lipschitz continuous boundary \( \partial \Omega \), \( \partial_{\nu} \) denotes the outward normal derivative on \( \partial\Omega \).
The corresponding weak form (4) for the eigenvalue problem (54) can be defined with $V = H_0^2(\Omega)$, $W = L^2(\Omega)$ and
\[
\begin{align*}
  a(u, v) &= \int_{\Omega} \sum_{1 \leq i, j \leq 2} \partial_{ij} u \partial_{ij} v d\Omega, \quad \|v\|_a = \left( a(v, v) \right)^{\frac{1}{2}}, \\
  b(u, v) &= \int_{\Omega} u v d\Omega, \quad \|v\|_b = \left( b(v, v) \right)^{\frac{1}{2}}.
\end{align*}
\]
Evidently the bilinear form $a(\cdot, \cdot)$ is symmetric, continuous and coercive over the product space $V \times V$.

Here we analyze the triangular Morley element which is defined by
\[
V_h := \left\{ v \in L^2(\Omega) : v|_K \in \mathcal{P}_2, \forall K \in \mathcal{T}_h, \ v \text{ is continuous at the vertices}, \right. \\
\left. \quad \text{and} \int_{\ell} \partial_{\nu} v|_{K_1} ds = \int_{\ell} \partial_{\nu} v|_{K_2} ds, \ \text{where} \ \ell = \partial K_1 \cap \partial K_2 \right\}
\]
(55)
where $\mathcal{P}_2 = \text{span}\{1, x, y, x^2, xy, y^2\}$.

Based on the nonconforming finite element space $V_h$ defined in (55), the discrete eigenvalue problem (5) for the eigenvalue problem (54) can be defined with
\[
a_h(u_h, v_h) = \sum_{K \in \mathcal{T}_h} \sum_{1 \leq i, j \leq 2} \partial_{ij} u_h \partial_{ij} v_h dK \quad \text{and} \quad \|v_h\|_{a,h} = \left( a_h(v_h, v_h) \right)^{\frac{1}{2}}.
\]
The eigenfunction of (54) has the following regularity
\[
\text{(56)} \quad u \in H^{2+\gamma}(\Omega),
\]
where $0 < \gamma \leq 1$ depends on the maximum angle of the boundary $\partial \Omega$. It is well known that $\gamma = 1$ when $\Omega$ is convex.

Here, we give the analysis of the lower-bound results of eigenvalues by the Morley element method. First, we define the corresponding interpolation operator $\Pi_h : H^2(\Omega) \rightarrow V_h$ as
\[
\text{(57)} \quad \left\{ \begin{array}{l}
  \Pi_h u(Z) = u(Z), \quad \forall z \in N_h, \\
  \int_{\ell} \partial_{\nu} \Pi_h u ds = \int_{\ell} \partial_{\nu} u ds, \quad \forall \ell \in E_h.
\end{array} \right.
\]
From [4, 5, 31, 39, 41], the eigenpair approximation $(\lambda_j, u_{j,h})$ obtained by the Morley element method defined (55) has the following error estimates
\[
\text{(58)} \quad |\lambda_j - \lambda_{j,h}| \leq C h^{2\gamma} ||u_j||_{2+\gamma},
\]
\[
\text{(59)} \quad ||u_j - u_{j,h}||_{a,h} \leq C h^{\gamma} ||u_j||_{2+\gamma},
\]
\[
\text{(60)} \quad ||u_j - u_{j,h}||_{b} \leq C h^{\gamma} ||u_j - u_{j,h}||_{a,h}.
\]

**Lemma 5.1.** The interpolation operator $\Pi_h$ defined by (57) has the following error estimates
\[
\text{(61)} \quad ||u - \Pi_h u||_b + h^2 ||u - \Pi_h u||_{a,h} \leq C h^{2+\gamma} ||u||_{2+\gamma},
\]
where $u \in H^{2+\gamma}(\Omega)$. Furthermore, the following orthogonal property holds (c.f. [23, 31, 45])
\[
\text{(62)} \quad a_h(u - \Pi_h u, v_h) = 0, \quad \forall v_h \in V_h.
\]

Similarly, we have the following lower-bound result of the convergence rate by the Morley element method.
Lemma 5.2. ([33, Section 4]) The eigenfunction approximation $u_h$ of the biharmonic eigenvalue problem by the Morley element method has the following lower bound of the convergence rate

$$\|u - u_h\|_{a,h} \geq Ch.$$  

(63)

Theorem 5.3. ([23, 31, 45]) Assume $\lambda_j$ and $\lambda_{j,h}$ are the $j$-th eigenvalues of (54) and its approximation by Morley element, respectively. We have the following lower-bound result

$$\lambda_{j,h} \leq \lambda_j,$$

(64)

when $h$ is small enough.

Proof. Let $v_h = \Pi_h u_j$. We estimate the terms of (17).

First, we have the following estimates for the second and third terms

$$\|\Pi_h u_j - u_{j,h}\|_b^2 \leq \|\Pi_h u_j - u_j\|_b^2 + \|u_j - u_{j,h}\|_b^2,$$

(65)

$$\left|\|\Pi_h u_j\|_b^2 - \|u_j\|_b^2\right| = \left|\int_\Omega (u_j - \Pi_h u_j)(u_j + \Pi_h u_j) d\Omega\right| \leq Ch^2 + \gamma \|u_j\|_b \|u_j\|_b^2,$$

(66)

Combining (62), (63), (65), (66) and Lemma 2.2, we know that the first positive term is dominant. This means we obtain the desired result (64) and complete the proof. 

6. Stokes eigenvalue problem. In this section, we are concerned with the lower-bound results of the Stokes eigenvalue problems by nonconforming mixed finite element methods. The study of Stokes eigenmodes is required when the dynamics behaviors governed by the Navier-Stokes equations resulting from the way this nonlinear dynamics is controlled by diffusion. For the other reasons to study the Stokes eigenmodes, please read the papers [8, 24].

Osborn [38], Mercier, Rappaz and Raviart [37] give an abstract analysis for the eigenpair approximations by mixed/hybrid finite element methods based on the general theory of compact operators (c.f. [15]). Recently, Xie, Yin and Gao in [48] obtains asymptotic error expansions of the Stokes eigenvalue approximations and gives extrapolation schemes to improve the convergence order for the eigenvalue approximations. Their numerical results show the lower-bound phenomena.

For simplicity, we consider the Stokes eigenvalue problem with the homogenous Dirichlet boundary condition:

Find $(\lambda, u, p)$ such that

$$\begin{cases}
-\Delta u + \nabla p &= \lambda u, &\text{in } \Omega, \\
\nabla \cdot u &= 0, &\text{in } \Omega, \\
u &= 0, &\text{on } \partial\Omega, \\
\int_\Omega \|u\|^2 d\Omega &= 1,
\end{cases}$$

(67)

where $\Omega \subset R^2$ is a bounded polygonal domain with Lipschitz boundary $\partial\Omega$ and $\Delta$, $\nabla$, $\nabla \cdot$ denote the Laplacian, gradient and divergence operators, respectively. In addition, denoted by $L_0^2(\Omega)$ the subspace of $L^2(\Omega)$ that consists of functions on $L^2(\Omega)$ having mean value zero. We use the vector valued functions $(H^m(\Omega))^2$ just as [12, 19]. In this section, we use new definitions for bilinear forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$.

The corresponding weak form of (67) is:
Find $(\lambda, \mathbf{u}, p) \in \mathcal{R} \times \mathbf{V} \times W$ such that $s(\mathbf{u}, \mathbf{u}) = 1$ and
\begin{equation}
\lambda = \frac{a(\mathbf{u}, \mathbf{v})}{s(\mathbf{u}, \mathbf{u})},
\end{equation}
where $\mathbf{V} = (H_0^1(\Omega))^2$, $W = L_0^2(\Omega)$ and
\begin{align*}
a(\mathbf{u}, \mathbf{v}) &= \int_\Omega \nabla \mathbf{u} \nabla \mathbf{v} d\Omega, \quad b(\mathbf{v}, p) = -\int_\Omega \nabla \cdot \mathbf{v} d\Omega, \quad s(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{u} v d\Omega.
\end{align*}

For the eigenvalue, there exists the following Rayleigh quotient expression
\begin{equation}
\lambda = \frac{a(\mathbf{u}, \mathbf{u})}{s(\mathbf{u}, \mathbf{u})},
\end{equation}
where $\mathbf{V}$ following norm on $\mathbf{V}$.

The eigenfunction of (67) has the following regularity (c.f. [6, 7])
\begin{equation}
\|\mathbf{u}\|_{1+\gamma} + \|p\|_{\gamma} \leq C,
\end{equation}
where $0 < \gamma \leq 1$ depends on the maximum angle of the boundary $\partial \Omega$. It is well known that $\gamma = 1$ when $\Omega$ is convex.

Associated with the partition $\mathcal{T}_h$, we define the nonconforming finite element spaces $\mathbf{V}_h \not\subset \mathbf{V}$ and $W_h \subset W$ (c.f. [12, 19]). The nonconforming finite element spaces $\mathbf{V}_h = (V_h)^2$ with $V_h$ is defined by (20) or (21) for ECR or $EQ_1^{rot}$, respectively. The finite element space $W_h$ is defined as
\begin{equation}
W_h := \{ w \in L_0^2(\Omega) : w|_K \in \text{span}\{1\}, \text{ for any } K \in \mathcal{T}_h \}.
\end{equation}

Now, let us define the approximation of the eigenpair $(\lambda, \mathbf{u}, p)$ by the nonconforming mixed finite element method:

Find $(\lambda_h, \mathbf{u}_h, p_h) \in \mathcal{R} \times \mathbf{V}_h \times W_h$ such that $s(\mathbf{u}_h, \mathbf{u}_h) = 1$ and
\begin{equation}
\begin{cases}
a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_h) &= \lambda_h s(\mathbf{u}_h, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \\
b_h(\mathbf{u}_h, q_h) &= 0, \quad \forall q_h \in W_h,
\end{cases}
\end{equation}
where the bilinear forms $a_h(\cdot, \cdot)$ and $b(\cdot, \cdot)$ are defined by
\begin{align*}
a_h(\mathbf{u}_h, \mathbf{v}_h) &= \sum_{K \in \mathcal{T}_h} \int_K \nabla \mathbf{u}_h \nabla \mathbf{v}_h dK, \\
b_h(\mathbf{v}_h, q_h) &= \sum_{K \in \mathcal{T}_h} \int_K \nabla \cdot \mathbf{v}_h q_h dK.
\end{align*}

Since the bilinear $a_h(\cdot, \cdot)$ is symmetric and elliptic on $\mathbf{V} + \mathbf{V}_h$, we can define the following norm on $\mathbf{V} + \mathbf{V}_h$
\begin{equation}
\|\mathbf{v}_h\|_{a,h} = \left( a_h(\mathbf{v}_h, \mathbf{v}_h) \right)^{\frac{1}{2}}, \quad \forall \mathbf{v}_h \in \mathbf{V} + \mathbf{V}_h.
\end{equation}

From (72), we can know the following Rayleigh quotient for $\lambda_h$ also holds
\begin{equation}
\lambda_h = \frac{a_h(\mathbf{u}_h, \mathbf{u}_h)}{s(\mathbf{u}_h, \mathbf{u}_h)}.
\end{equation}

We know from [5] the Stokes eigenvalue problem (72) has an eigenvalue sequence \(\{\lambda_{j,h}\}\) with
\begin{align*}
0 < \lambda_{1,h} \leq \lambda_{2,h} \leq \cdots \leq \lambda_{k,h} \leq \cdots \leq \lambda_{N_h,h},
\end{align*}
and the corresponding eigenfunction sequence \(\{\mathbf{u}_{j,h}, p_{j,h}\}\)
\begin{align*}
(\mathbf{u}_{1,h}, p_{1,h}), \ldots, (\mathbf{u}_{k,h}, p_{k,h}), \ldots, (\mathbf{u}_{N_h,h}, p_{N_h,h}),
\end{align*}
where $s(\mathbf{u}_{i,h}, \mathbf{u}_{j,h}) = \delta_{ij}$, $1 \leq i, j \leq N_h$ ($N_h$ is the dimension of $V_h \times W_h$).
The well-posedness (no spurious eigenvalues) of the discrete eigenvalue problem (72) can be guaranteed by the fact that the corresponding approximation spaces $V_h$ and $W_h$ satisfy the Babuška-Brezzi condition (c.f. [12, 19])

\[
\inf_{0 \neq q_h \in W_h} \sup_{0 \neq v_h \in V_h} \frac{b_h(v_h, q_h)}{\|v_h\|_{a,h}\|q_h\|_0} \geq C > 0.
\]

The eigenpair approximation $(\lambda_{j,h}, u_{j,h}, p_{j,h})$ by the mixed finite element space $V_h \times W_h$ has the following error estimates (c.f. [12, 19, 37, 38]):

\[
|\lambda_j - \lambda_{j,h}| \leq Ch^{2\gamma}(\|u_j\|_{1+\gamma} + \|p_j\|_\gamma)^2,
\]

\[
\|u_j - u_{j,h}\|_{a,h} + \|p_j - p_{j,h}\|_0 \leq Ch^\gamma(\|u_j\|_{1+\gamma} + \|p_j\|_\gamma),
\]

\[
\|u_j - u_{j,h}\|_0 \leq Ch^{\gamma}(\|u_j\|_{1+\gamma} + \|p_j\|_\gamma).
\]

Similarly, we have the following lower-bound result of the convergence rate for the Stokes eigenvalue problem.

**Lemma 6.1.** ([33]) The eigenfunction approximation $u_h$ of the Stokes eigenvalue problem by ECR or $EQ_1^{\text{rot}}$ elements has the following lower-bound result of the convergence rate

\[
\|u - u_h\|_{a,h} \geq Ch.
\]

The following lemma is similar to Lemma 2.2 but with an additional term.

**Lemma 6.2.** ([32]) Assume that $(\lambda_j, u_j, p_j) \in R \times V \times W$ is an eigenpair of (67) and $(\lambda_{j,h}, u_{j,h}, p_{j,h}) \in R \times V_h \times W_h$ is the corresponding eigenpair approximation and $W_h \subset W$. We have the following expansion

\[
\lambda_j - \lambda_{j,h} = \|u_j - u_{j,h}\|_{a,h}^2 - \lambda_{j,h}\|v_h - u_{j,h}\|_0^2 + \lambda_{j,h}(\|v_h\|_0^2 - \|u_j\|_0^2)
+ 2b_h(u_j - v_h, u_{j,h}) - 2b_h(v_h - u_{j,h}, p_{j,h}), \quad \forall v_h \in V_h.
\]

**Proof.** Since $s(u_j, u_j) = 1$, $s(u_{j,h}, u_{j,h}) = 1$, $a_h(u_j, u_j) = \lambda_j$ and $a_h(u_{j,h}, u_{j,h}) = \lambda_{j,h}$, we have

\[
\|u_j - u_{j,h}\|_{a,h}^2 = \|u_j\|_{a,h}^2 + \|u_{j,h}\|_{a,h}^2 - 2a_h(u_j, u_{j,h})
= \lambda_j + \lambda_{j,h} - 2a_h(u_j - v_h, u_{j,h}) - 2a_h(v_h, u_{j,h}), \quad \forall v_h \in V_h.
\]

From (72), we have

\[
-2a_h(v_h, u_{j,h}) - 2b_h(v_h, p_{j,h}) = -2\lambda_{j,h}s(v_h, u_{j,h})
= \lambda_{j,h}\|v_h - u_{j,h}\|_0^2 - \lambda_{j,h}\|v_h\|_0^2 - \lambda_{j,h}\|u_{j,h}\|_0^2
\]

\[
= \lambda_{j,h}\|v_h - u_{j,h}\|_0^2 - \lambda_{j,h}(\|v_h\|_0^2 - \|u_{j,h}\|_0^2) - 2\lambda_{j,h}.
\]

Adding $-2b_h(v_h, p_{j,h})$ to both sides of (80) and combining the fact that $b_h(u_j, p_{j,h}) = 0$ and (81) leads to

\[
\|u_j - u_{j,h}\|_{a,h}^2 - 2b_h(v_h - u_{j,h}, p_{j,h})
= \lambda_j - \lambda_{j,h} - 2a_h(u_j - v_h, u_{j,h}) + \lambda_{j,h}\|v_h - u_{j,h}\|_0^2 - \lambda_{h}(\|v_h\|_0^2 - \|u_{j,h}\|_0^2).
\]

This is the desired result (79) and we complete the proof.

Applying Lemma 6.2 to ECR and $EQ_1^{\text{rot}}$ elements, we have the following lower-bound result.
Theorem 6.3. ([32]) Let $\lambda_j$ and $\lambda_{j,h}$ be the $j$-th exact eigenvalue and its corresponding numerical approximation by ECR/$P_0$ or EQ$_1^{rot}/P_0$, respectively. Then, when $h$ is small enough, we have

$$\lambda_{j,h} \leq \lambda_j.$$  

(82)

Proof. Setting $v_h = \Pi_h u_j$, we estimate all the five terms in the eigenvalue expansion (79).

First, we have the following estimates for the second and third terms

$$\|\Pi_h u_j - u_{j,h}\|_0^2 \leq \|\Pi_h u_j - u_j\|_0^2 + \|u_j - u_{j,h}\|_0^2 \leq C h^{2+2\gamma} \|u_j\|_1^{1+\gamma} + C h^{2\gamma} \|u_j\|_{a,h},$$

(83)

$$\|\Pi_h u_j\|_0^2 - \|u_j\|_0^2 = \left| \int_\Omega (u_j - \Pi_h u_j)(u_j + \Pi_h u_j) dx \right| \leq C h^{2+\gamma} \|u_j\|_1^{1+\gamma} \|u_j\|_{a,h}.$$  

Similarly, the following orthogonal property holds

$$a_h(u_j - \Pi_h u_j, v_h) = 0, \quad \forall v_h \in V_h.$$  

(85)

Combining the integration by parts and (22), we have

$$b_h(\Pi_h u_j - u_j, p_{j,h}) = \sum_{K \in T_h} \int_{\partial K} (\Pi_h u_j - u_j) \cdot n_{p_{j,h}} ds = 0.$$  

(86)

Combining (78), Lemma 6.2, (83), (84), (85) and (86), we know the first positive term in the right hand side of (79) is dominant. Then we obtain the desired result (82) and complete the proof.

7. Concluding remarks. In this paper, lower-bound results of the eigenvalue problems by nonconforming finite element methods are introduced. Here, we only concentrate on the class of nonconforming elements which can produce the lower bounds of eigenvalues without any additional eigenfunction regularity assumption. From the analysis given in this paper, we find that the main ingredients of producing the lower-bound results is the eigenvalue expansion (17), the lower bound of the convergence rate (29) by finite element methods and the properties of the associated nonconforming finite element methods. For the numerical experiments, please read the corresponding references. Of course, the idea and results can be extended to more general eigenvalue problems and other nonconforming finite element methods.

Acknowledgments. We would like to thank the referees very much for their valuable comments and suggestions.

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