Elliptic operators associated with groups of quantized canonical transformations

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1. Let $M$ be a compact closed smooth manifold and $G$ a discrete group. A $G$-operator on $M$ is an operator given by a finite sum

$$D = \sum_{g \in G} D_g \Phi_g : H^s(M) \to H^{s-m}(M),$$

(1)

where the $D_g$ are (pseudo)differential operators of order $m$ on $M$, and $g \mapsto \Phi_g$ defines a representation of $G$ in the operators acting on functions on $M$. $G$-operators arise naturally, for example, when $G$ acts on $M$ and the operators $\Phi_g$ are shift operators along the orbits of the group action: $\Phi_g u(x) = u(g^{-1}(x))$. Such operators have been studied successfully in the literature (for instance, see [1], [2], [4] and the references cited there), and they have interesting applications in non-commutative geometry, non-local problems of mathematical physics, and other areas of mathematics.

The aim of this paper is to study a new class of elliptic $G$-operators associated with representations of groups $G$ by quantized canonical transformations $\Phi_g$ (see [3], for instance). Such operators have arisen in several recent problems in index theory and non-commutative geometry. We note also that in the simplest case, where $D = \Phi_g$ is a single quantized canonical transformation, we recover the well-known Atiyah–Weinstein index problem.

Now we define the symbol of $G$-operators and state a Fredholm theorem. We apply methods of the theory of $C^*$-algebras and crossed products of them, and therefore we assume that in (1) we have $s = m = 0$ (that is, we consider operators acting in $L^2(M)$) and that the $\Phi_g$ are unitary operators. The general case can be reduced to this situation in a standard way.

For a $G$-operator (1), its coefficients $D_g$ are pseudodifferential operators, and hence their symbols, denoted by $\sigma(D_g)$, are smooth functions on the cosphere bundle $S^*M = T_0^*M/\mathbb{R}_+$ of the manifold, where $T_0^*M$ stands for the cotangent bundle with zero section deleted. The symbol of the $G$-operator (1) is the element

$$\sigma(D) = \{\sigma(D_g)\} \in C(S^*M) \rtimes G$$

(2)

of the maximal $C^*$-crossed product $C(S^*M) \rtimes G$ of the algebra $C(S^*M)$ of continuous functions on the cosphere bundle and the group $G$, acting on this algebra by automorphisms defined by canonical transformations $g: S^*M \to S^*M$.

We say that $D$ is elliptic if its symbol is invertible.

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Theorem 1. Elliptic $G$-operators are Fredholm operators.

Note that Theorem 1 is also valid under the weaker assumption that the correspondence $g \mapsto \Phi_g$ is an almost-representation, that is, a representation modulo compact operators.

2. From now on let $G$ be a Lie group of positive dimension and let $g \mapsto \Phi_g$ be its unitary representation in the quantized canonical representations on $L^2(M)$, which is continuous in the following sense: we are given a smooth action of $G$ on $T^*_0 M$ by homogeneous canonical transformations and a smooth family of amplitudes $\{a_g\}_{g \in G} \subset C^\infty_c(G \times S^* M)$, such that we have a decomposition $\Phi_g = \Phi(g, a_g) + K_g$, where $\Phi(g, a_g)$ is a fixed quantization of $g$ and $a_g$, while $K_g$ is a norm-continuous family of compact operators. One can show that under this continuity assumption the $G$-operator $D = \int_G D_g \Phi_g \, dg : L^2(M) \to L^2(M)$ (3) is a well-defined bounded operator provided that $\{D_g\}_{g \in G}$ is a compactly supported norm-continuous function on $G$ with values in the $C^*$-algebra of zero-order pseudodifferential operators with continuous symbols.

It turns out that $G$-operators for Lie groups are not Fredholm as a rule. For example, $G$-operators associated with group actions on $M$ are smoothing along the orbits. One therefore has to introduce the notion of symbol on the space transversal to the orbits. Let us define this space. Note that an element $H$ of the Lie algebra $G$ of $G$ defines a Hamiltonian vector field on $T^* M$, and we denote by $H(x, p)$ the corresponding Hamiltonian function on $T^*_0 M$. Then with the action of $G$ on $T^*_0 M$ by homogeneous canonical transformations we associate the following closed conical $G$-invariant subset of $T^*_0 M$:

$$T^*_G M = \{(x, p) \in T^*_0 M \mid H(x, p) = 0 \text{ for all } H \in \mathcal{G}\}.$$ (4)

Similarly to (2) we define the symbol of the $G$-operator (3) to be the element $\sigma(D) \in C(S^*_G M) \rtimes G$ of the maximal crossed product, where $S^*_G M = T^*_G M/\mathbb{R}_+$. The following theorem is the main result of this paper.

Theorem 2. Suppose that the $G$-operator $1 + D : L^2(M) \to L^2(M)$ is elliptic, that is, its symbol $1 + \sigma(D) \in (C(S^*_G M) \rtimes G)^+$ is invertible in the crossed product with adjoined unit. Then this $G$-operator is a Fredholm operator.

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