Robust data-driven fixed-order controller synthesis: Model matching approach

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Abstract
In this paper, a new data-driven fixed-order $H_\infty$ controller design method based on convex optimisation is proposed for linear single input single output systems. The proposed non-parametric frequency domain data based approach renders the need for a mathematical model of the controlled plant unnecessary. First, a semi-definite convex optimisation algorithm is proposed to simultaneously compute a minimal uncertainty model and an optimal nominal model from the experimental data. Then the $H_\infty$ robust performance condition, control input constraints and the closed-loop model matching objective are described by convex functions with respect to the parameters of the controller. The usefulness and efficiency of the proposed approach are verified experimentally with application to the control of a thrust vector control system.

1 | INTRODUCTION

Classical methods related to the modelling, simulation and control of dynamical plants require accurate mathematical description of system dynamics. The performance of these methods is largely dependent on the accuracy of the model. Therefore, mathematical model of the real-time system to be controlled is usually obtained using laws of physics or system identification theory at the beginning of the automatic control design technique. This representation is a practical approximation of the real plant; therefore, modelling errors, which are due to unmodelled dynamics, process nonlinearities, truncation of high frequency dynamics and accuracy of models, are almost unavoidable and may alter controller performance because of contradictory features of performance and robustness. In addition, generation of an accurate mathematical model is sometimes quite complex, time-consuming and requires prior knowledge about the system dynamics. For these reasons, model based methods may become inefficient to meet the increasingly high performance requirements of today’s complex dynamical systems.

Data-driven control methods were developed to deal with above mentioned problems by synthesising controllers using only a set of online [1, 2], off-line [3, 4] or hybrid (both online and off-line) [5, 6] time domain or frequency domain measurements [7] instead of a parametric model. This advantage has made these methods a popular research topic within the automatic control community in recent years [8]. The fundamental goal of such methods is to directly synthesise controllers through experimentally obtained data, particularly for high precision control applications and high order dynamical systems with unavailable mathematical models.

In model-based robust $H_\infty$ control theory, the controller is synthesised by using the plant model, user-defined weighting functions and worst-case condition of bounded model uncertainties. The order of the resulting full-order robust controller is equal to the order of the augmented plant, that is, the sum of the order of the three above mentioned function. In practical industrial control applications, fixed-order (low-order) controllers such as lead-lag compensators and proportional-integral-derivative (PID) controller are preferred because of their easily adjustable structures, practicality and low processing requirements on embedded system [9]. The $H_\infty$ control problem leads to an NP-hard non-convex problem in controller parameter space if a fixed-order controller structure is considered instead of a full-order controller in state space or linear matrix inequalities (LMI) based solution algorithms [10]. Recently, non-smooth optimisation, [11, 12], meta-heuristic approach [13], Kalman–Yakubovich–Popov (KYP) lemma [14, 15], inner convex approximation [16], convex-concave optimisation [17] and regional pole assignment method [18] have
been adopted in fixed-order controller design problem with $H_\infty$ criterion. The lead-lag compensator synthesis methodology in fixed-order $H_\infty$ framework using LMI techniques is proposed in [19] more recently. However, these methods are model based and introduce some conservatism into the designed closed-loop control systems.

The model-free feature of the non-parametric frequency domain data-driven control method and the practicality of the fixed-order method can be combined within the $H_\infty$ control framework. In such approaches, the controller design problem is generally converted into a convex or non-convex optimisation problem with objective functions, which are either to be minimised or maximised, and $H_\infty$ norm constraints in frequency domain. In [20], a linear programming approach for robust linearly parameterised fixed-order controller synthesis is proposed. Robustness margins such as gain, phase and modulus margin are imposed as constraints in the Nyquist diagram. Although the method is also suitable for multiple models, performance requirements are limited to selection of an interval for crossover frequency. This approach was later improved to synthesise a data-driven linearly parameterised robust controller via convex optimisation for an uncertain model in [7] by introduction of a desired loop gain model. However, this method leads to a conservative solution and narrowing of the solution space of non-convex control problem due to the convexification approach. Another contribution to frequency domain data-driven synthesis of fixed-order controller for non-parametric systems is [21]. In their work, the authors used a line to constrain the critical point of Nyquist diagram for nominal stability or nominal performance requirement. The effectiveness of the obtained controller largely depends on the selection of this constraint line. A non-linear optimisation based solution approach for tuning fixed-order controllers is presented in [22, 23]. This approach is based on frequency domain data that is obtained from closed-loop tests. The most important reason limiting the use of this method is the need for three different initial controllers to obtain frequency response of controlled plant. A convex–concave procedure for robust PID controller design with low-pass measurement filter is proposed in [24], but this paper does not consider unstructured uncertainty. In [25], a robust data-driven digital controller design method for two degree of freedom RST controller is presented, where only measurement process based uncertainty is taken into account. Another non-convex optimisation algorithm based frequency domain data-driven fixed-order controller synthesis approach is introduced in [26] to solve $H_\infty$ control problem for linear systems. However, in this study, model uncertainties and robust performance constraints are not considered in the optimisation problem.

In the present study, a fixed-order $H_\infty$ controller synthesis algorithm for non-parametric single input single output (SISO) systems is introduced by using linearly parameterised Laguerre basis functions in the frequency domain via convex optimisation. In order to make the data-driven structured $H_\infty$ approach more applicable and to reduce the conservatism of the method, this paper addresses the aforementioned limitations of the available methods. First, unstructured multiplicative model uncertainty bound is minimised by selecting the nominal model and uncertainty circle via the concept of Chebyshev center of a set of points at the corresponding frequency points on the Nyquist plot. Therefore, this algorithm reduces the conservatism and improves the robust performance of the proposed method. Thanks to this approximation, multiple measurements can be considered in the robust control design algorithm instead of one set of measurement with minimal uncertainty bound. Second, this paper applies the robust data-driven fixed-order $H_\infty$ control design methodology for linear systems with control input constraints. Physical systems usually have input signal limit, because the power supplies cannot provide infinitely large control input. Two inequality constraints are added to the optimisation based control design problem to account for this limit in the convex optimisation framework. Third, in this paper the objective function of the convex optimisation problem is formulated as a closed-loop model matching problem. Due to the fact that the model matching does not guarantee internal stability, a sufficient condition for closed-loop stability is derived and represented by a convex constraint on the Nyquist plot. Therefore, designed closed-loop control system is stable and experimental closed-loop frequency response matches a frequency response of predefined reference model in the $H_\infty$ norm sense. Finally, the proposed algorithm is verified experimentally with application to the control of an electromechanical flexible nozzle thrust vector control (TVC) system that is used to direct the course of an air vehicle. For comparison purpose, the performance of the presented method is compared with the available frequency-domain robust control toolbox (FDRC) [27] on the experimental test setup.

The remainder of this paper is organised as follows: The general preliminaries, estimation of the frequency response of plant and definition of the structure of linearly parameterised controllers are presented in Section 2. In Section 3, the formulation and solution of the constrained convex optimisation problem are addressed. In Section 4, a real-time experimental implementation is presented to validate the achievement of the presented study. Some final conclusions and directions for future works are provided in Section 5.

2 | PRELIMINARIES

In this section, we assume that the system is a stable, strictly proper, linear time-invariant (LTI) controllable SISO system with bounded infinity norm. On the other hand, the set of multiple LTI SISO stable, strictly proper frequency domain models belonging to uncertain systems with bounded uncertainties is considered in the next sections.

2.1 | Notation

The set of all real matrices, complex matrices and non-negative real matrices are denoted as $\mathbb{R}^{p \times q}$, $\mathbb{C}^{p \times q}$, $\mathbb{R}^+ \times \mathbb{C}^q$, respectively. The set of all real and all complex numbers are represented without the superscript $p$ and $q$. The notation $G(s)$, $G(z)$ and
**Frequency response identification**

The complex value \(G(jw)\), which is \(G(\zeta)\big|_{\zeta=e^{jw}}\), gives full information about the system in steady-state case if the system is stable and the input is sinusoidal. Therefore, \(G(jw)\) is called the frequency response function (FRF) of the plant. This approach may not be feasible in all situations because we can not sequentially implement all possible sinusoidal inputs. An extension of the direct FRF method is the empirical transfer function estimate (ETFE) [29] that estimates FRF from normal inputs instead of sinusoidal ones. Assume that we sample the signal of the system output \(y[k]\) to the periodogram of the system output \(y[k]\) as follows:

\[
\hat{G}_N(jw) = \left[ \frac{1}{\sqrt{N}} \sum_{k=1}^{N} y[k] e^{-jwk} \right] \left[ \frac{1}{\sqrt{N}} \sum_{k=1}^{N} u[k] e^{-jwk} \right]^{-1},
\]

where \(N\) is the number of available measurement samples for each experiment, \(k\) is the time instant, \(w = 2\pi n/N\) for \(n = 1, 2, \ldots, N\).

On the other hand, experimental systems often contain measurement noise. An experimental test setup could produce an output to applied input as

\[
y[k] = \sum_{m=1}^{\infty} g[k] u[k - m] + v[k],
\]

where \(g[k]\) is the discrete-time unit-pulse response of the system, \(v[k]\) is the random measurement noise, and \(u[k]\) and \(y[k]\) are uncorrelated. Since the \(u[k]\) signal is not entirely random, \(y[k]\) is a quasi-stationary signal. Estimation of the auto-correlation function of \(u[k]\) can be defined as

\[
\hat{R}_u^N(\tau) := \frac{1}{N} \sum_{k=0}^{N-1} u[k] u[k - \tau], |\tau| \leq N - 1.
\]

Similarly, estimated cross-correlation function between \(u[k]\) and \(y[k]\) is given by

\[
\hat{R}_{yu}(\tau) := \frac{1}{N} \sum_{k=0}^{N-1} u[k] y[k - \tau], |\tau| \leq N - 1.
\]

The spectral density of \(u[k]\) and cross spectral density between \(u[k]\) and \(y[k]\) are defined by

\[
\hat{\phi}_u(jw) = \sum_{\tau=-\infty}^{\infty} \hat{R}_u^N(\tau) e^{-jw\tau},
\]

\[
\hat{\phi}_{yu}(jw) = \sum_{\tau=-\infty}^{\infty} \hat{R}_{yu}^N(\tau) e^{-jw\tau},
\]

where \(\hat{R}_{yu}^N(\tau)\) is the lag window that is using for smoothing the estimated FRF. Finally, \(\hat{G}_N(jw)\) can be estimated from these spectral density functions as

\[
\hat{G}_N(jw) = \frac{\hat{\phi}_{yu}(jw)}{\hat{\phi}_u(jw)}.
\]

**Linearly parameterised controllers**

Linearly parameterised fixed-order controllers can be modelled with stable orthogonal basis functions as

\[
K(s, k) = k\psi(i),
\]

where \(k = [k_1 \ k_2 \ \ldots \ k_L] \in \mathbb{R}^{1 \times n}\) is the gain matrix of the controller and \(\psi(i)^T = [\psi_1(i) \ \ldots \ \psi_n(i)] \in \mathbb{R}^{n \times 1}\) is the matrix of transfer functions. These basis functions can be formed by using Laguerre functions, Kautz functions or generalised orthonormal basis functions [30]. In this study, we consider the Laguerre basis functions, also known as Laguerre filters, given by

\[
\psi_i(i) = \frac{\sqrt{\xi}}{(i + \xi)^{n-1}} (i - \xi)^{i-1},
\]

for \(i = 1, \ldots, n\) with \(\xi > 0\) which is called the time scaling factor of the Laguerre functions. The poles of these functions are all at the same location \(\xi\). All functions except \(\psi_1(i)\), which is a low-pass filter, are formed with the all-pass filter in series with a first order filter [31]. The block diagram of the Laplace domain structure of the \(n^{th}\)-order Laguerre model based controller is shown in Figure 1.
2.4 Closed-loop transfer functions

The frequency domain system and controller are connected in the one degree-of-freedom feedback control structure given in Figure 2, defined by

\[
\begin{align*}
E(s) & \rightarrow \psi_1(s) \rightarrow K_0 + k_1 \psi_1(s) + \cdots + k_n \psi_n(s) \rightarrow \psi_n(s) \\
U(s) & \rightarrow \psi_1(s) \rightarrow K_1 + k_2 \psi_1(s) + \cdots + k_n \psi_n(s) \\
\vdots & \quad \vdots \\
U(s) & \rightarrow \psi_1(s) \rightarrow k_n \psi_1(s) \\
U(s) & \rightarrow \psi_1(s) \\
\end{align*}
\]

**FIGURE 1** Structure of the linearly parameterised controller

The controller input signal \(E(s)\) is filtered by Laguerre functions. Then, the control signal of the plant \(U(s)\) is obtained by multiplication of each filtered outputs by their respective controller parameters \(k_i\), as

\[
U(s) = \left[ k_0 + k_1 \psi_1(s) + \cdots + k_n \psi_n(s) \right] E(s). \quad (12)
\]

The continuous-time PID controller also can be written in linearly parameterised form as

\[
K_{PID}(s) = k \psi(s) = \begin{bmatrix} K_p & K_i & K_d \end{bmatrix} \begin{bmatrix} 1 \\ 1/s \\ 1/(\tau_d s + 1) \end{bmatrix}, \quad (13)
\]

where \(k = [K_p, K_i, K_d] \in \mathbb{R}_+^{1 \times 3}\) are the proportional gain, integral gain and derivative gain, respectively. The \(\tau_d \in \mathbb{R}_+\) parameter of the PID controller is the derivative time constant, which is assumed to be fixed.

The basic idea behind using linearly parameterised type controllers is to represent the frequency response of closed-loop transfer functions as an affine function of the design parameter \(k\).

2.4.1 Sensitivity function

The sensitivity function is the transfer function from the output disturbance to the plant output and is defined as

\[
S(jw, k) = \frac{y}{d_o} = \frac{1}{1 + G(jw)K(jw)}, \quad (15)
\]

where \(L(jw) = G(jw)K(jw)\) is the loop transfer function [10].

2.4.2 Complementary sensitivity function

The complementary sensitivity function is the transfer function from the reference input to the plant output and is defined as

\[
T(jw, k) = \frac{y}{r} = \frac{G(jw)K(jw)}{1 + G(jw)K(jw)}, \quad (16)
\]

also, one have \(S(jw) + T(jw) = I\).

2.4.3 \(Q\)-parameter

The \(Q\)-parameter function is the transfer function from the reference input to the control input and is defined as

\[
Q(jw, k) = \frac{u}{r} = \frac{K(jw)}{1 + G(jw)K(jw)}, \quad (17)
\]

which is used as an indicator of the actuator effort [17].

3 ROBUST DATA-DRIVEN FIXED-ORDER CONTROLLER SYNTHESIS

In order to account for model uncertainty, we will assume that system dynamics of plant is represented by a set \(\mathbf{M}\) of possible models. The number of frequency-domain models in the set \(\mathbf{M}\) is \(m\) and the number of frequency points is \(N\); therefore, the multiple model set \(\mathbf{M}\) can be represented with unstructured multiplicative uncertainty frequency response function \(\tilde{W}_2(j\omega)\) or unstructured additive uncertainty frequency response function \(W_2^{\alpha}\) by

\[
\mathbf{M} : \tilde{G}(j\omega) := G_i(j\omega)(1 + \tilde{W}_2(j\omega)\Delta), \quad (18)
\]

\[
\mathbf{M} : G^+(j\omega) := G_i(j\omega) + W_2^{\alpha}(j\omega)\Delta,
\]
respectively, for $\Delta \in \mathbb{R}^{H}_{\infty}$, $\|\Delta(j\omega)\|_{\infty} \leq 1 \ \forall \omega, \ i = 1, \ldots, m; \ n = 1, \ldots, N$. In these equations, $G_{i}(j\omega)$ denotes the perturbed real plant dynamics. The functions $W_{2,i}(j\omega_{n})$ and $W^{-1}_{2,i}(j\omega_{n})$ are stable, strictly proper frequency response functions which define the magnitude of the uncertainty at each interested frequency point.

Note that, throughout the paper we assume that the multiple model set $\mathbf{M}$ contains same frequency points for all family of perturbed plants and uncertainty functions. However, the proposed controller synthesis approach is suitable for the models with different frequency points as well since only one data is sufficient for each frequency values to define a corresponding nominal model and uncertainty magnitude.

Modelling of the uncertainties is an essential part of the robust $H_{\infty}$ control theory. A set of frequency domain data measured from experimental plant at different operating conditions is used to define the uncertainty model of the system. Because of the contradictory features of performance and robustness, it is desirable that magnitude of the uncertainty weighting function be as small as possible. A classical way to define $W_{2}(j\omega)$ is given in [32] as

$$
W_{2}(j\omega) = \left| \frac{G_{i}(j\omega_{n}) - G_{\text{nom}}(j\omega_{n})}{G_{\text{nom}}(j\omega_{n})} \right| \leq W_{2}(j\omega_{n}), \quad (19)
$$

where $G_{\text{nom}}(j\omega_{n})$ denotes the nominal model. One simple method for computing $G_{\text{nom}}(j\omega_{n})$ is to calculate the average of the experimental data at each frequency point as

$$
G_{\text{avg}}^{m}(j\omega_{n}) = \frac{1}{m} \sum_{i=1}^{m} G_{i}(j\omega_{n}), \quad (20)
$$

for $n = 1, \ldots, N$.

### 3.1 Computing optimal multiplicative uncertainty models

In this section, a convex optimisation method that is based on the concept of Chebyshev center of a set of points for the computation of optimal non-parametric uncertainty models is proposed. This approach simultaneously defines the optimal nominal frequency domain plant dynamics and minimal, least conservative uncertainty weighting function such that all $G_{i}(j\omega_{n})$ exist within the uncertainty tube around the nominal model $G_{\text{nom}}(j\omega_{n})$. The main objective is to find smallest uncertainty weighting function magnitude, which covers all of the available experimental data, and optimal nominal model at each frequency point.

Multiplicative uncertainty optimisation tries to minimise the magnitude of $W_{2}(j\omega_{n})G_{\text{nom}}(j\omega_{n})$. For this cost function, the nominal model and multiplicative uncertainty function appear as products. Hence, this is a non-convex objective function with respect to $W_{2}(j\omega_{n})$ and $G_{\text{nom}}(j\omega_{n})$. However, in the additive uncertainty case the objective function is convex because the optimisation algorithm attempts to minimise the magnitude of

$$
W_{2}^{A}(j\omega_{n}) \text{ only. Therefore, one way to calculate a multiplicative uncertainty with minimum amplitude for a SISO system by convex semi-definite programming (SDP) is to solve convex optimisation problem for additive uncertainty and then calculate the equivalent multiplicative weighting function using}
$$

$$
W_{2}(j\omega_{n}) = G_{\text{nom}}(j\omega_{n})^{-1}W_{2}^{A}(j\omega_{n}), \quad (21)
$$

which is evident from Equation (18). Note that, Equation (21) is true if and only if $G_{\text{nom}}(j\omega_{n})^{-1}$ exists. In case of $G_{\text{nom}}(j\omega_{n})^{-1} = 0$, the $G_{\text{nom}}$ value can be replaced with a parameter $\epsilon, \epsilon$ being a sufficiently small number.

An optimal multiplicative uncertainty model $W_{2}^{\text{opt}}(j\omega_{n})$, which covers the data $G_{i}(j\omega_{n})$ at frequency point $\omega_{n}$, and the optimal nominal model $G_{\text{nom}}^{\text{opt}}(j\omega_{n})$ can be computed by forming a Chebyshev ball in the complex plane.

**Definition 1.** The set of points

$$
\mathcal{B}(x_{n}, r_{i}) = \{ \bar{x} \mid \| \bar{x} - x_{n} \|_{2} \leq r_{i} \},
$$

forms a Chebyshev ball with respect to the Euclidean norm around Chebyshev center $x_{n}$ with radius $r_{i}$ [33, 34].

**Proposition 1.** An optimal multiplicative uncertainty model $W_{2}^{\text{opt}}(j\omega_{n})$ which contains the experimental data around the optimal nominal model $G_{\text{nom}}^{\text{opt}}(j\omega_{n})$ can be calculated by the solution of the following convex optimisation problem for optimal additive uncertainty model at each frequency point of interest:

$$
\begin{align*}
\min_{G_{\text{nom}}(j\omega_{n}), \|W_{2}^{A}(j\omega_{n})\|} & \|W_{2}^{A}(j\omega_{n})\| \\
\text{s.t.} & \begin{bmatrix} \Re (G_{i}(j\omega_{n})) \\ \Im (G_{i}(j\omega_{n})) \end{bmatrix} \in \mathcal{B} \begin{bmatrix} \Re (G_{\text{nom}}(j\omega_{n})) \\ \Im (G_{\text{nom}}(j\omega_{n})) \end{bmatrix}, \|W_{2}^{A}(j\omega_{n})\| \end{align*}, \quad (23)
$$

for $i = 1, \ldots, m; \ n = 1, \ldots, N$ and using Equation (21) as

$$
\|W_{2}^{\text{opt}}(j\omega_{n})\| = \|W_{2}^{A}(j\omega_{n})\|/\|G_{\text{nom}}^{\text{opt}}(j\omega_{n})\|.
$$

**Proof.** Frequency response of $G_{i}(j\omega_{n})$ can be represented by a complex number as

$$
G_{i}(j\omega_{n}) = \Re (G_{i}(j\omega_{n})) + j\Im (G_{i}(j\omega_{n})),
$$

where the vector $x_{\delta}$ defined as $[x_{1,\delta}, x_{2,\delta}]^{T}$. Using the definition of Chebyshev ball, optimal additive uncertainty modelling problem can be formulated as:

$$
\begin{align*}
\min_{\bar{x}, \delta} & \|\bar{x} - x_{\delta}\|_{2} - \delta \\
\text{s.t.} & \|\bar{x} - x_{\delta}\|_{2} - \delta \leq 0,
\end{align*} \quad (25)
$$

$$\text{subject to: } W_{2}^{A}(j\omega_{n}) = G_{\text{nom}}(j\omega_{n})^{-1}W_{2}(j\omega_{n}) = \frac{1}{m} \sum_{i=1}^{m} G_{i}(j\omega_{n}). \quad (26)$$

Note that, throughout the paper we assume that the multiple model set $\mathbf{M}$ contains same frequency points for all family of perturbed plants and uncertainty functions.
for \( i = 1, \ldots, m; \, n = 1, \ldots, N \). Solution algorithm of this convex optimisation problem gives:

\[
G_{\text{nom}}(j\omega) = \Re(x_n^\ast) + j\Im(x_n^\ast) ,
\]

\[
\begin{bmatrix}
W^A_{G_{\text{nom}}}(j\omega) \\
W^B_{G_{\text{nom}}}(j\omega)
\end{bmatrix} = \delta^*,
\]

which concludes the proof with Equation (21). \( \Box \)

Remark 1. The optimisation Equation (23) is a SDP convex optimisation problem since both of its objective function and its inequality constraint function are convex.

Remark 2. We assume that the uncertainty structure is in multiplicative form for the optimal uncertainty modelling approach in rest of the study. Obviously, the same convex optimisation method can be applied to other uncertainty models (such as inverse additive uncertainty structure, ...).

Note that, the robust data-driven fixed-order controller synthesis problem is designed using the optimal nominal model and minimal multiplicative uncertainty weighting function in the next chapters where \( G(j\omega) = G_{\text{nom}}(j\omega) \) and \( W_m(j\omega) = W_{G}^A(j\omega). \)

### 3.2 Model matching problem

A closed-loop model matching problem is an objective function based optimal control problem, that is, it concerns the synthesis of the controller such that obtained closed-loop system is stable and matches as closely possible a chosen reference stable model. This predefined reference model \( T_{d}(j\omega) \) is generally a low-order model that includes the desired dynamic behaviour of the controlled plant [35]. Using the FRF of the system and the linearly parameterised controller, the closed-loop model matching problem can be defined as

\[
\min_k \left\| \begin{bmatrix}
W_m(j\omega) \\
W_m(j\omega)
\end{bmatrix} \begin{bmatrix}
T(j\omega,k) - T_d(j\omega)
\end{bmatrix} \right\|_{\infty},
\]

\[
= \min_k \left\| \begin{bmatrix}
W_m(j\omega) \\
W_m(j\omega)
\end{bmatrix} \begin{bmatrix}
G(j\omega)K(j\omega) - T_d(j\omega)
\end{bmatrix} \right\|_{\infty},
\]

\[
= \min_k \left\| \begin{bmatrix}
W_m(j\omega) \\
W_m(j\omega)
\end{bmatrix} \begin{bmatrix}
G(j\omega)k\Psi(j\omega)(1 - T_d(j\omega)) - T_d(j\omega)
\end{bmatrix} \right\|_{\infty},
\]

in the \( H_{\infty} \) sense where \( W_m(j\omega) \) is the FRF of a stable penalty function weighting the frequency domain requirements. A block diagram representation of the closed-loop model matching problem is given in Figure 3.

The objective function of the optimisation problem Equation (27) is not a convex function with respect to the controller parameters \( k \), because the denominator of this function includes design parameters. In order to approximate this non-convex optimisation problem to a convex optimisation problem, one approach is to replace the term \( G(j\omega)k\Psi(j\omega) \) in the denominator with the desired loop gain \( L_{d}(j\omega) \) and formulate the sub-optimal control problem as

\[
\min_k \left\| \begin{bmatrix}
W_m(j\omega) \\
W_m(j\omega)
\end{bmatrix} \begin{bmatrix}
G(j\omega)k\Psi(j\omega)(1 - T_d(j\omega)) - T_d(j\omega)
\end{bmatrix} \right\|_{\infty},
\]

\[
= \min_k \left\| \begin{bmatrix}
W_m(j\omega) \\
W_m(j\omega)
\end{bmatrix} \begin{bmatrix}
G(j\omega)k\Psi(j\omega)(1 - T_d(j\omega)) - T_d(j\omega)
\end{bmatrix} \right\|_{\infty},
\]

where the desired loop-gain \( L_{d}(j\omega) \) is given by

\[
L_{d}(j\omega) = \frac{T_d(j\omega)}{1 - T_d(j\omega)}. \quad (29)
\]

Note that objective function Equation (28) \( f : \mathbb{R}^{1 \times n} \to \mathbb{C} \) is affine with respect to the \( k \) that is, linear function plus a constant term:

\[
f(k) = \frac{W_m(j\omega)G(j\omega)\Psi(j\omega)(1 - T_d(j\omega)) - W_m(j\omega)T_d(j\omega)}{1 + L_{d}(j\omega)k}. \quad (30)
\]

Therefore, it can be considered as convex function.

### 3.3 Robust performance constraints

Robust performance (RP) constraints can be derived based on a block diagram representation of the proposed control system which is shown in Figure 4, where \( W_1(j\omega) \) is the performance weighting function, \( \omega \) is the exogenous input, \( \zeta \) is the performance variable of interest and \( \Delta_p(j\omega) \) is a fictitious block defined through the performance channel such that \( \|\Delta_p(j\omega)\|_{\infty} \leq 1 \). This configuration is similar to the structured singular value, which is denoted by the \( \mu \) or SSV, based analysis technique of robust performance criterion with fictitious block \( \Delta_p(j\omega) \). Note that, for notation purposes, the dependence in \( j\omega \) will be omitted throughout the rest of the thesis and it will be used only if necessary.

The augmented plant \( P \) is constructed by separating the controller \( K(j\omega) \), \( \Delta_p(j\omega) \) and \( \Delta(j\omega) \) from the control system in
Figure 4. $P$ matrix can be partitioned in matrix form as

$$
[z \Delta \xi e] = \begin{bmatrix}
P_{11}^{(11)} & P_{11}^{(12)} & P_{12}^{(11)} \\
P_{11}^{(21)} & P_{11}^{(22)} & P_{12}^{(21)} \\
P_{21}^{(11)} & P_{21}^{(12)} & P_{22}^{(11)}
\end{bmatrix} \begin{bmatrix}
\omega \Delta \\
\\
\omega
\end{bmatrix}
$$

(31)

where $P$ represents the transfer function from $[\omega \Delta \xi e]^T$ to

$$
[z \Delta \xi e]^T.
$$

With the augmented plant $P$, the control system given by Figure 4 can be transformed to an equivalent configuration given in Figure 5.

**Lemma 1.** The closed-loop SISO system given by Figures 4 and 5 satisfies the robust performance condition for a given internally stable plant $G$ if and only if Nyquist plot of

$$
Q_{RP}(k, \Delta, \Delta_p, j\omega) = I - \Delta P \Delta + \Delta W_2 GK + GK
$$

function does not encircle the origin of the complex plane for $\forall \omega \in \mathbb{R} \cup \{\infty\}, \Delta_p, \Delta \in RH_{\infty}, \|\Delta P\|_{\infty}, \|\Delta\|_{\infty} \leq 1$.

**Proof.** Loop gain of the positive feedback control system shown in Figure 5 is given by

$$
L_{pf} = \begin{bmatrix}
P_{11}^{(11)} & P_{11}^{(12)} & P_{12}^{(11)} \\
P_{11}^{(21)} & P_{11}^{(22)} & P_{12}^{(21)} \\
P_{21}^{(11)} & P_{21}^{(12)} & P_{22}^{(11)}
\end{bmatrix} \begin{bmatrix}
\Delta & 0 & 0 \\
0 & \Delta_p & 0 \\
0 & 0 & K
\end{bmatrix}
$$

(33)

Generalised Nyquist stability theorem for given positive feedback system with stable plant can be written as

$$
\det(I - L_{pf}) \neq 0,
$$

$$
\iff 1 - P_{11}^{(11)} \Delta
$$

$$
+ (P_{11}^{(12)} P_{12}^{(22)} - P_{12}^{(12)} P_{11}^{(22)}) \Delta_p
$$

$$
+ (P_{11}^{(22)} P_{22}^{(11)} - P_{22}^{(12)} P_{11}^{(21)}) K
$$

$$
+ (P_{11}^{(21)} P_{21}^{(11)} - P_{21}^{(12)} P_{11}^{(22)}) \Delta
$$

$$
+ (P_{12}^{(11)} P_{22}^{(11)} - P_{22}^{(12)} P_{12}^{(12)}) \Delta_p
$$

$$
+ (P_{12}^{(21)} P_{21}^{(21)} - P_{21}^{(22)} P_{12}^{(12)}) K
$$

$$
+ (P_{21}^{(11)} P_{22}^{(12)} - P_{22}^{(11)} P_{21}^{(12)}) \Delta
$$

(34)

Then, substituting the components of matrix $P$ into Equation (34) we obtain:

$$
I - \Delta_p W_1 + \Delta W_2 GK + GK \neq 0,
$$

(35)

which is the statement of the Lemma.

Perturbed sensitivity function $\tilde{S}$ can be written as

$$
\tilde{S} = \frac{1}{1 + GK(1 + W_2 \Delta)}.
$$

(36)

with multiplicative uncertainty. The maximum magnitude of the $\tilde{S}$ occurs if $\Delta = 1$ and the phase angle of the terms $(W_2 GK)$ and $(1 + GK)$ have opposite signs. Therefore, $\tilde{S}$ with the possible maximum magnitude is given by

$$
\tilde{S}_{\text{max}} = \frac{1}{|1 + GK(1 + W_2 \Delta)|}.
$$

(37)

It can be shown that $RP$ condition Equation (32) is equivalent to the

$$
\|\|W_1 \tilde{S} + W_2 T\|\|_{\infty} < 1,
$$

$$
\|\tilde{S}_{\text{max}}\|_{\infty} < \frac{1}{|W_1|},
$$

(38)

conditions which are the same robust performance constraints [32].

In Figure 5, $\omega \Delta \rightarrow z \Delta$ is the uncertainty channel and $\omega \rightarrow z$ is the performance channel. Using these channels, robust stability (RS), nominal performance (NP) and nominal stability (NS) conditions can be defined.

Constraint functions of the fixed-order $H_{\infty}$ control problem can be derived using Nyquist plot based on the robust performance condition given by Equation (32). The fact that frequency dependent $Q_{RP}(k, \Delta, \Delta_p, j\omega)$ polynomial does not encircle the origin of the Nyquist plot constitutes the main constraint function of optimisation problem. This robust performance condition inequality can be modified as

$$
\Delta W_2 GK + GK \neq -I + \Delta_p W_1,
$$

(39)
where $\Delta$ and $\Delta_\infty$ blocks represent two different blocks in complex plane such that $\|\Delta\|_\infty, \|\Delta_\infty\|_\infty \leq 1$. Note that, when $\|\Delta_\infty\|_\infty = 1$ (worst-case) right hand side of Equation (39) defines a circle, which is called performance circle, with radius $|W|_\infty$ and center $(-1, 0)$ in the Nyquist diagram. Similarly, if $\Delta$ block satisfies the worst-case condition ($\|\Delta\|_\infty = 1$), left hand side of Equation (39) defines another circle, which is called robustness circle, with radius $|W_2GK|$ and center $(\Re(GK), \Im(GK))$ (Figure 6). Therefore, robust performance condition given by Equation (32) is satisfied if and only if the performance circle and the robustness circle do not intersect each other in complex plane for all frequencies $w$. This statement holds if and only if the performance circle and robustness circle does not have intersection. Hence, robust performance constraint can be adapted to the robust $H_\infty$ control problem by preventing the intersection of these circles via a frequency dependent line.

The robust performance condition given by Equation (32) is satisfied if robustness circle lies below the line $y_n(W_1, L_d) = a_nx + b_n$ in Nyquist diagram as shown in Figure 6.

In order to represent robust performance constraint as a convex constraint in optimisation problem, parameters of the line can be defined with respect to $L_d$. Constructed line tangent to the performance circle and orthogonal to the line from the $(-1, 0)$ point to $L_d$ as shown in Figure 6. The frequency dependent parameters of this line can be defined using geometrical relationships in Figure 6 as

\[
\begin{align*}
a_n &= \frac{1 - \Re(L_d)}{\Im(L_d)} , \\
b_n &= a_n \left( \sin(\theta_n) - |W|_\infty \right) \left( \sin(\theta_n) \right)\cdot \sin(\theta_n) ,
\end{align*}
\]

where $\theta_n$ is the slope of the line and given by

\[
\theta_n = \tan^{-1} \left( \frac{1 - \Re(L_d)}{\Im(L_d)} \right) .
\]

Now, consider the nearest critical point $C_p(jw_n)$ from the robustness circle to the line $y_n(W_1, L_d) = a_nx + b_n$ in Figure 6. Then, the robustness circle lies below the line if and only if $C_p(jw_n)$ exists below the line for all frequencies $w$. Therefore, the representation of the robust performance constraint in $H_\infty$ controller synthesis problem with a sufficient condition is given by following proposition:

**Proposition 2.** Closed-loop control system given by Figure 4, satisfies the robust performance condition Equation (32) if

\[
\begin{align*}
\Im(k\psi G) - a_n \Re(k\psi G) \\
+ |W_2k\psi G| (a_n \sin(\theta_n) + \cos(\theta_n)) - b_n \leq 0 ,
\end{align*}
\]

for $\forall w \in \mathbb{R} \cup \{\infty\}$.

**Proof.** $C_p(jw_n)$ point lies below the line if

\[
C_p(jw_n) \leq y_n = a_nx + b_n ,
\]

in the Nyquist plot. Real and imaginary parts of the the critical point $C_p(jw_n)$ in Figure 6 can be defined with respect to the origin of the Nyquist plot as

\[
\Re(C_p(jw_n)) = \Re(k\psi G) - |W_2k\psi G| \sin(\theta_n) , \\
\Im(C_p(jw_n)) = \Im(k\psi G) + |W_2k\psi G| \cos(\theta_n) ,
\]

respectively. Then, substituting real part of this equation into Equation (44) yields:

\[
\begin{align*}
\Im(k\psi G) - a_n \Re(k\psi G) \\
+ |W_2k\psi G| (a_n \sin(\theta_n) + \cos(\theta_n)) - b_n \leq 0 ,
\end{align*}
\]

for $\forall w \in \mathbb{R} \cup \{\infty\}$, which is the statement of the proposition. \qed

**Remark 3.** We consider only the RP condition throughout the rest of the paper. However, in some control applications, synthesis or analysis of the control system may be required according to the RS, NP or NS conditions. Since RS, NP and NS conditions are specific forms of RP condition, these conditions can be derived by making the necessary arrangements in Equation (43).

### 3.4 Control input constraints

Real time control systems usually have control input limits, because the power sources cannot provide infinitely large control input. In order to take into account available input limits in the convex optimisation problem, in this subsection we derive the control input constraints for fixed-order $H_\infty$ control synthesis scheme.

The block diagram representation of the closed loop control system with control signal weighting function $W_u$ is shown in Figure 7.

![Graphical representation of the robust performance constraint](image-url)

**FIGURE 6** Graphical representation of the robust performance constraint.
performance weighting function

By using this figure, $Q$-parameter transfer function from the reference input to the control input with multiplicative type model uncertainty of the plant given by

$$Q = \frac{\mathbf{1}}{1 + Gk\mathbf{1} + W_2\Delta}.$$  \hspace{1cm} (47)

Then, control input constraint can be written as

$$|W_2Q| - \bar{n}_{\text{max}} \leq 0,$$

$$\Leftrightarrow \left| \frac{W_2k\mathbf{1}}{1 + Gk\mathbf{1} + W_2\Delta} \right| \leq \bar{n}_{\text{max}},$$

$$\Leftrightarrow -\bar{n}_{\text{max}} \leq \frac{W_2k\mathbf{1}}{1 + Gk\mathbf{1} + W_2\Delta} \leq \bar{n}_{\text{max}},$$

for $\forall w \in \mathbb{R} \cup \{\infty\}$, where $\bar{n}_{\text{max}}$ is the upper bound of the weighted control input signal in frequency domain. The maximum control input occurs at minimum loop-gain condition; therefore, the worst-case control input generated when $\Delta = 1$ and the phase angle of the terms ($W_2k\mathbf{1}$) and ($1 + Gk\mathbf{1}$) have opposite signs. Hence,

$$|W_2Q_{\text{max}}| - \bar{n}_{\text{max}} \leq 0,$$

$$\Leftrightarrow -\bar{n}_{\text{max}} \leq \frac{|W_2k\mathbf{1}|}{1 + Gk\mathbf{1}} - |W_2k\mathbf{1}| \leq \bar{n}_{\text{max}},$$

$$\Leftrightarrow \begin{bmatrix} -\bar{n}_{\text{max}} \left| 1 + Gk\mathbf{1} \right| - |W_2k\mathbf{1}| - |W_2k\mathbf{1}| \\ -\bar{n}_{\text{max}} \left| 1 + Gk\mathbf{1} \right| - |W_2k\mathbf{1}| + |W_2k\mathbf{1}| \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which are the control input constraint functions of the fixed-order $H_\infty$ control problem. Notice that these constraint functions are convex (affine) with respect to the controller parameters $k$.

### 3.5 Optimisation problem

In order to satisfy the robust performance condition, the control input constraints and the closed-loop model matching objective, the fixed-order $H_\infty$ controller design problem can be formulated as constrained convex optimisation problem with respect to the controller parameters. According to these requirements, a convex optimisation problem is arranged for the optimal synthesis of the fixed-order $H_\infty$ controller formulated as follows:

$$\min_k \left\| \frac{W_mG\mathbf{1}(1 - T_d)}{1 + L_d} - \frac{W_mT_d}{1 + L_d} \right\|_\infty$$

s.t. $\mathfrak{I}(k\mathbf{1}G) - a_m\mathfrak{R}(k\mathbf{1}G) - b_n \leq 0, $  \hspace{1cm} (50)$

$$+|W_2k\mathbf{1}| \left(a_m\sin(\omega_n) + \mathfrak{R}(\omega_n)\right) \leq 0,$$

$$-\bar{n}_{\text{max}} \left| 1 + Gk\mathbf{1} \right| - |W_2k\mathbf{1}| - |W_2k\mathbf{1}| \leq 0,$$

$$-\bar{n}_{\text{max}} \left| 1 + Gk\mathbf{1} \right| - |W_2k\mathbf{1}| + |W_2k\mathbf{1}| \leq 0,$$

for $\forall w \in \mathbb{R} \cup \{\infty\}$.

This optimisation problem involves an infinite number of constraints; therefore, it is a convex semi-infinite programming (SIP) problem. In order to transform this SIP problem into SDP problem, which can be solved numerically using available convex optimisation techniques and solvers, finite number of frequencies. A randomised scenario approach [36] can be used to compute the minimum number of frequency point to guarantee the constraints with a chosen probability level. According to the scenario approach, if the number of scenarios $N$ satisfies

$$N \geq \frac{2}{\varepsilon} \left( d_p - 1 + \ln \frac{1}{\beta} \right),$$  \hspace{1cm} (51)

condition for $d_p$, number of optimisation variables, risk parameter $\varepsilon \in (0, 1)$, and confidence parameter $\beta \in (0, 1)$, then, constraints hold with a probability level $\geq 1 - \beta$.

### 3.6 Performance weighting function selection

Deviations from the predefined reference model are inevitable due to the uncertainties in the system dynamics. Therefore, it is useful to determine the performance weighting function with respect to worst-case reference model matching requirement. Since the objective function of the optimisation problem related to the complementary sensitivity function, the worst-case desired closed-loop transfer function $T_d^1$ can be considered as

$$T_d^1 = \frac{\omega_n}{\omega_n^2 + 2\zeta\omega_n + \omega_n^2},$$  \hspace{1cm} (52)

where $\omega_n$ is the natural frequency and $\zeta$ is the damping ratio. In order to choose the performance weighting function $W_1$, we
consider the loop-gain of a standard second-order system as

$$ I_{d} = \frac{T_{d}^{-1}}{1 - T_{d}^{-1}} = \frac{w_{mp}^{2}}{s^{2} + 2\zeta w_{mp} s}, $$ (53)

then, ideal sensitivity function can be defined as

$$ S_{d} = \frac{1}{1 + I_{d}} = \frac{s^{2} + 2\zeta w_{mp} s + w_{mp}^{2}}{s^{2} + 2\zeta w_{mp} s + w_{mp}^{2}}. $$ (54)

Note that

$$ |S_{d}(jw_{mp}/\sqrt{2})| = 1, $$

$$ M_{s} := \|S_{d}\|_{\infty} = \frac{\beta \sqrt{\beta^{2} + 4\zeta^{2}}}{\sqrt{(1 - \beta^{2})^{2} + 4\zeta^{2} \beta^{2}}}, $$ (55)

where $\beta = \sqrt{0.5 + 0.5 \sqrt{1 + 8\zeta^{2}}}$, $w_{mp}$ is the cut-off frequency of $S_{d}$ and $M_{s}$ is the peak gain of $S_{d}$ at $w_{mp} = \beta w_{mp}$ frequency point [28]. Sensitivity function $S$ is a good indicator of control performance. Therefore, performance weighting function can be defined with respect to $S$. A possible choice of performance weight $W_{1}$ is given as

$$ W_{1} = \left( \frac{s\|M_{1}\|_{\nu} + w_{mp}}{s + w_{mp} \nu} \right)^{\nu}, $$ (56)

where $\nu$ bounds the steady-state error for $\nu \geq 1$ [28].

Note that, predefined reference model in Equation (27) can be chosen as given in Equation (52) as well. However, for a feasible choice of performance weighting function $W_{1}$, the natural frequency of $T_{d}$ should be smaller than the natural frequency of reference model $T_{d}$ in Equation (27). Otherwise, the inverse of performance weighting function can not limit the desired sensitivity function in some frequency range and there will be a contradiction between robust performance and model matching achievement.

# EXPERIMENTAL IMPLEMENTATION

In this section, proposed controller design methodology is applied to the position control of an electromechanical TVC system.

## 4.1 Thrust vector control system

Several guided air vehicle platforms generally need steering mechanism in order to direct their course especially during the exoatmospheric flight conditions. TVC system is used to control the flight of the vehicle by changing the direction of main thrust vector.

In this study, a flexible joint nozzle type TVC system is used as an experimental test bench that consists of two electromechanical actuators (EMA) (Figure 8).

The EMA configuration, which is composed of a brushless DC (BLDC) electric motor, planetary gear train, a ballscrew and a digital position sensor, is presented in this figure.

A nominal dynamical equation of the EMA can be obtained based on the BLDC electric motor dynamics. The separate voltage equations of the $q$ and $d$ axis of the three-phase, two-pole BLDC motor are given by

$$ v_{q} = r_{s}i_{q} + w_{r}\lambda_{dq} + \frac{d}{dt}\lambda_{q}, $$ (57)

$$ v_{d} = r_{s}i_{d} - w_{r}\lambda_{dq} + \frac{d}{dt}\lambda_{d}, $$ (58)

in which $\lambda_{dq} = L_{q}i_{q} + \lambda_{d}, \lambda_{dq} = L_{q}i_{q} + \lambda_{d}$, where $\lambda_{m}$ is the flux linkage amplitude generated by permanent magnets, $r_{s}$ is the stator resistance, $i_{q}$ is the $q$-axis current, $i_{d}$ is the $d$-axis current, $w_{r}$ is the electrical angular velocity, $\lambda_{dq}$ is the $q$-axis flux linkage, $\lambda_{dq}$ is the $d$-axis flux linkage, $L_{q}$ is the $q$-axis inductance and $L_{d}$ is the $d$-axis inductance. By using these voltage equations, the current equation of the $q$ and $d$ axis can be derived as

$$ \frac{d}{dt}i_{q} = \frac{1}{L_{q}}(v_{q} - w_{r}L_{dq}i_{d} - w_{r}\lambda_{d} - r_{s}i_{q}), $$ (59)

$$ \frac{d}{dt}i_{d} = \frac{1}{L_{d}}(v_{d} + w_{r}L_{dq}i_{q} - r_{s}i_{d}), $$ (60)

respectively [37].
The electromagnetic torque equation of the BLDC motor is given by

\[ T_e = \left( \frac{3}{2} \right) \left( \frac{p}{2} \right) \lambda_0 i_{qs} = K_t i_{qs} \quad (61) \]

in which \( p \) and \( K_t \) are the number of poles and torque constant of motor, respectively. The equation of the motion of the TVC system in terms of equivalent moments of inertia \( J_e \), equivalent viscous damping \( B_e \), equivalent Coulomb friction \( F_e \), load torque \( T_L \) and mechanical reduction ratio \( N_g \) can be written as

\[ J_e \ddot{\theta} + B_e \dot{\theta} + F_e \text{sign} (\dot{\theta}) + T_L = K_t i_{qs} N_g \quad (62) \]

where \( \theta \) is the deflection angle of TVC system. The control goal of this sub-system is to provide required thrust vector rotation angle despite disturbances, high frequency flexible dynamics and vibration. Therefore, the input of the system is \( i_{qs} \) and the output is \( \theta \).

### 4.2 Experimental test setup

An experimental test bench has been built up to obtain FRF’s of real-time plant and validate the closed-loop control performance of the proposed robust data-driven fixed-order controller. Signal flows between the TVC system and other items of test setup are shown in Figure 9.

![Signal flows for experimental testing of the TVC system](image)

An NI 6221 data acquisition (DAQ) board is used to receive the measurement data and to send the control signal. NI SCB-68 shielded input/output (I/O) connector block is used to connect to NI 6221 DAQ cards with 68 screw terminals. The connector block also has a signal conditioning capability for filtering the signals. The EMAs are mounted on the nozzle to provide two degree of freedom rotation to TVC system. These EMAs are controlled by two separate ESCON 50/5 servo amplifier which is a commercial product of Maxon company. Real-time implementation of the control algorithm is provided by using xPC target toolbox of MATLAB software. This toolbox includes discrete time controller matrix, communication protocols and signal type converters. The synthesised fixed-order controllers are tested on the real-time hardware via xPC target toolbox. A host computer is used for off-line programming of the closed-loop control algorithm. The transfer function of the obtained controller is digitalised using bilinear transformation method. Digital closed-loop position control loop of experimental TVC system is operated at 1 kHz frequency.

#### 4.3 Frequency response identification of TVC system

The FRF of the TVC system varies depending on the variable environmental conditions, unmodelled system dynamics, non-linear system behaviour, material life and aging. These uncertainties may cause unpredictable TVC system performance which may lead, in some cases, to the system instability. It is not always possible to guarantee required system performance under these adverse conditions with a controller synthesised using a single nominal model [38]. Therefore, in this study, frequency response identification is carried out under different working conditions in order to incorporate model uncertainties into the control system design process. Due to the schedule of environmental testing system and long temperature conditioning process, only six open-loop system identification experiment could be performed.

In order to obtain six different FRF’s of the TVC system, multiple tests were carried out in the temperature range of...
$-20^\circ C$ to $80^\circ C$ by increasing the temperature by $20^\circ C$ degrees steps at each test condition. A pseudo-random binary sequence (PRBS) signal was used as the $q$-axis current reference of the open-loop TVC system in the experiments to obtain the time domain response of the plant. The input $q$-axis current and output angle $\theta$ signals acquired from the frequency response identification experiments are shown in Figure 10.

Frequency domain experimental data was obtained with 400 logarithmically spaced frequency points, a value which is approximately calculated using Equation (51) where $\varepsilon = 0.05$, $\beta = 0.001$ and $d_0 = 4$, between $w_l = 1 \text{ rad/s}$ and $w_u = 100 \text{ rad/s}$. FRF's of the real-time system, which are obtained with Equation (9), are given in Figure 11.

The optimum multiplicative uncertainty weighting function and corresponding nominal model are computed using the measured multiple FRF's of the TVC system. Then, these optimised frequency domain models are used to design a fixed-order controller in next chapter via convex optimisation.

4.4 Controller synthesis for TVC system

The reference model for desired closed-loop control system was selected as

$$T_d = \frac{(30\pi)^2}{s^2 + 1.4(30\pi)s + (30\pi)^2},$$

where $\xi = 0.7$ and $w_n = 30\pi \text{ rad/s}$. Similarly, the worst-case reference model $T_{d1}$ for the selection of the performance weight was chosen as

$$T_{d1} = \frac{(6\pi)^2}{s^2 + 1.4(6\pi)s + (6\pi)^2}.$$

Therefore, performance weighing function $W_1$ used to design robust controller is given by

$$W_1 = \frac{0.749s^2 + 23.081s + 177.661}{s^2 + 4.741s + 5.619},$$

which is obtained by using Equation (56) with $\nu = 2$, $\varphi = 0.001$. While designing the robust controller, $\tilde{a}_{\text{max}}$, $W_u$ and $W_m$ were taken as 1 for the sake of simplicity and to make a fair comparison with FDRC toolbox.

The optimal nominal model ($G_{\text{nom}}^{opt}$) and optimal multiplicative uncertainty function ($W_2^{opt}$) were calculated using the semi-definite convex optimisation method given by Proposition 1. This convex optimisation problem was solved using CVX solver [39] which is a MATLAB-based package for convex optimisation problems. For comparison purpose, another nominal model ($G_{\text{nom}}^{avg}$) was calculated by average method, which is given in Equation (20). Additionally, corresponding multiplicative uncertainty weighing ($W_2^{avg}$) function was constructed by classical method, which is given in Equation (19).

Obtained Nyquist plot of $G_{\text{nom}}^{opt}(jw_n)$ and $G_{\text{nom}}^{avg}(jw_n)$ are shown in Figure 12 with corresponding uncertainty models.

The magnitude plots of the corresponding multiplicative uncertainty weighting functions are given in Figure 13 for both methods.
The obtained Chebyshev center and Chebyshev radius of a set of frequency response data points to cover all of the data at a sample point $\omega = 4\pi$ rad/s are shown in Figure 14. The classical multiplicative uncertainty weighting function and average model also shown in this figure. Classical uncertainty modelling approach produces a considerably more conservative weighting function magnitude model than the proposed optimal uncertainty modelling method as shown in Figures 12, 13 and 14. Since several data points have relatively large gain at $\omega = 4\pi$ rad/s frequency point, classical method generate larger uncertainty magnitude than the radius of optimal uncertainty function as shown in Figure 14. These results demonstrate that proposed convex optimisation based algorithm reduces the conservatism of uncertainty; therefore, improves the robustness of the closed-loop control system.

It is possible to increase controller order such that the $H_\infty$ robust performance condition Equation (32) is satisfied. Therefore, a third-order linearly parametrised controller transfer
The resulting robust controller satisfies the robust performance condition (38) such that $\|\tilde{S}_{\text{max}}(j\omega)\|_\infty = 0.96 < 1$. This result proves that the worst-case sensitivity function remains smaller than inverse of the frequency dependent performance weighting function $W_1(j\omega_n)$, such that

$$\|\tilde{S}_{\text{max}}(j\omega_n)\|_\infty < \frac{1}{\|W_1(j\omega_n)\|}.$$  

For $n = 1 \ldots 400$. Moreover, the nominal sensitivity function $S_{\text{nom}}(j\omega, K G_{\text{nom}})$ matches the desired sensitivity function $S_d$ as shown in Figure 15.

For comparison purposes, the control problem was also solved by using FDRC toolbox to design another third-order controller. The parameters of the fixed-order controller were obtained as $k = \left[85.02 -669.69 -231.29 -12.5\right]$ and the transfer function of the obtained controller is given as

$$K_c(s) = \frac{84.98 s^3 + 12104 s^2 + 539140 s + 15204000}{s^3 + 279 s^2 + 25950 s + 8044000}. \quad (69)$$

The robust performance achievement of this controller is $\|\tilde{S}_{\text{FDRC}}(j\omega_n)\|_\infty = 1.23$. Obtained model matching achievement by the FDRC toolbox is similar to those obtained with the proposed method as shown in Figure 15.

In order to investigate the robustness improvement of the proposed uncertainty modelling approach, same controller design problem was solved by using proposed method and FDRC toolbox with average nominal model Equation (20) and corresponding uncertainty model Equation (19). The pole of Laguerre basis function was chosen as $\xi = 66$ by a linear search for $\xi$ between $\xi = 1$ and $\xi = 100$. The transfer functions of the obtained fixed-order robust $H_\infty$ controllers are given by

$$K_{c^{\text{pro}}}(s) = \frac{50.79 s^3 + 11150 s^2 + 173700 s + 8535000}{s^3 + 198 s^2 + 13070 s + 287500}, \quad (70)$$

$$K_{c^{\text{FDRC}}}(s) = \frac{102 s^3 + 16310 s^2 + 436300 s + 11860000}{s^3 + 198 s^2 + 13070 s + 287500}, \quad (71)$$

for proposed method and FDRC toolbox, respectively. The robust performance achievement of these controllers are $\|\tilde{S}_{\text{max}}(j\omega_n)\|_\infty = 1.21, \|\tilde{S}_{\text{FDRC}}(j\omega_n)\|_\infty = 1.39$, respectively (Figure 16).

Therefore, robust performance achievement of the obtained controller with classical uncertainty modelling method is worse.
than the robustness value obtained with proposed data-driven fixed-order controller, which can be observed from Figures 15 and 16. From these figures, it can be observed that the optimal uncertainty modelling and the optimal choice of the nominal model used for a real-time system can considerably improve the desired robust performance specifications.

The controllers in Equations (67) and (69) satisfy the control input constraints Equation (48), where \( \bar{\tilde{u}}_{\text{max}} = 1 \), \( W_u = 1 \), as shown in Figure 17. Therefore, obtained worst-case \( \tilde{Q}_{\text{max}} \)-parameter transfer function with the FDRC toolbox is similar to those obtained with the proposed method as shown in Figure 17.

Real-time hardware in the loop tests were carried out to verify the performance of the synthesised fixed-order \( H_\infty \) controller. This controller was applied to the experimental system in the real-time hardware in the loop tests. The step and sinusoidal input responses of the system in time-domain is given in Figure 18. In this test filtered step function was applied to the system to prevent sudden current consumption. As can be seen from this figure, the synthesised data-driven fixed-order \( H_\infty \) position controller satisfies the defined model matching objective. As can be seen from the position control of TVC system, which requires precise positioning, the objective of designing a data-driven, fixed-order, low-order controller in frequency domain could be achieved via proposed approach.

### 5 Conclusions and Future Works

This paper presents a novel data-driven method to synthesise robust fixed-order \( H_\infty \) controllers by simultaneously computing minimal uncertainty bound and assigning optimal nominal model from experimental data. The proposed controller design algorithm consists of two steps: First, the non-parametric frequency response of system models with minimal unstructured uncertainty model is identified from the multiple measurement data. Therefore, variations in the system dynamics are represented by minimal uncertainty circle around the optimal nominal model for the corresponding frequency points on the Nyquist diagram. In the second step, a fixed-order \( H_\infty \) controller design algorithm is introduced by using linearly parameterised Laguerre basis functions for identified non-parametric perturbed model in the frequency domain. In this algorithm, \( H_\infty \) robust performance condition, control input constraints and closed-loop model matching objective are described by con-
convex functions with respect to the parameters of the controller. Then the control design problem is formulated as a constrained convex optimisation problem, which can be solved efficiently using convex optimisation techniques to compute the parameters of the structured controllers. Moreover, the proposed method can be applied to any linearly parameterised controller structure such as PID with any convex objective function and constraint functions. An experimental flexible nozzle type electromechanical TVC system is used to validate proposed control design algorithm. The obtained results show the practicality and efficiency of the approach to synthesise fixed-order $H_\infty$ controllers for non-parametric frequency domain perturbed plants. Furthermore, the closed-loop measurements confirm that data-driven control method with the optimal uncertainty modelling approach considerably reduces the uncertainty bound and consequently improves the robust performance.

Future research work directions include extension of the presented approach to multiple input multiple output (MIMO) systems, as well as optimal choice of the performance weighting function and basis functions.

ACKNOWLEDGEMENTS

The authors would like to thank TÜBİTAK SAGE for their support in this study.

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How to cite this article: Daş E, Başlamış Selahattin Ç. Robust data-driven fixed-order controller synthesis: Model matching approach. IET Control Theory Appl. 2021;15:179–194. https://doi.org/10.1049/cth2.12024