ON THE CHARACTERIZATION OF
CLASSICAL DYNAMICAL SYSTEMS USING
SUPERSYMMETRIC NONLINEAR $\sigma$-MODELS

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We construct a two dimensional nonlinear $\sigma$-model that describes the Hamiltonian flow in the loop space of a classical dynamical system. This model is obtained by equivariantizing the standard N=1 supersymmetric nonlinear $\sigma$-model by the Hamiltonian flow. We use localization methods to evaluate the corresponding partition function for a general class of integrable systems, and find relations that can be viewed as generalizations of standard relations in classical Morse theory.

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Originally quantum field theories were introduced to understand particle physics, but the techniques have also found extensive applications in statistical mechanics and condensed matter physics. Very recently, quantum field theoretical methods have also been applied to study aspects of classical Hamiltonian dynamics where many important problems remain unsolved, in particular in the case of chaotic systems.

In applications to classical dynamical systems, quantum field theories appear to be particularly effective when one investigates issues such as counting the number of certain classical trajectories. Such counting problems can be usually formulated by properly generalizing the Morse theory \[.] to infinite dimensions. These infinite dimensional versions are usually ill-defined, but difficulties can be at least partially cured using a field theory formulation.

One of the most actively studied problems in classical Hamiltonian dynamics is the Arnold conjecture \[2\], \[3\] that estimates a lower bound for the number of periodic solutions to Hamilton’s equations of motion. We consider a compact symplectic manifold \( \mathcal{M} \) with local coordinates \( \phi^\mu \) and Poisson bracket

\[
\{\phi^\mu, \phi^\nu\} = \omega^{\mu\nu}(\phi)
\]

We assume that \( H(t, \phi) : \mathbb{R} \times \mathcal{M} \to \mathbb{R} \) is a smooth, in general explicitly time dependent (i.e. energy non-conserving) and time periodic Hamiltonian,

\[
H(t, \phi) = H(t + T, \phi)
\]

and we are interested in \( T \)-periodic solutions to Hamilton’s equations of motion

\[
\partial_t \phi^\mu = \{\phi^\mu, H\} = \omega^{\mu\nu} \partial_\nu H \equiv \mathcal{X}^\mu_H
\]

(1)

Such periodic solutions are of fundamental importance in many problems: For example, they appear in the old Bohr quantization rule. Furthermore, the dynamics of a classically integrable system is entirely determined by periodic trajectories restricted on invariant torii. The Gutzwiller trace formula which is the only effective tool presently available to study quantum chaos, also approximates the energy spectrum using periodic classical trajectories.

If the solutions to (1) are non-degenerate, the Arnold conjecture states that their number is bounded from below by the sum of the Betti numbers \( B_k = \)
\[ \dim H^k(M), \; 0 \leq k \leq \dim M, \]

\[ \# \{ \text{contractible } T\text{-periodic solutions} \} \geq \sum_k \dim H^k(M) \tag{2} \]

Obviously, this conjecture can be viewed as a natural generalization of the familiar finite dimensional Morse inequality that estimates the number of critical points of a smooth real-valued function on \( M \). Until now it has only been proven in special cases, most notably for symplectic manifolds \( M \) such that the integral of the symplectic two-form \( \omega \) over every sphere \( S^2 \subset M \) vanishes,

\[ [\omega] \pi_2(M) = 0 \iff \int_{S^2} \omega = 0 \quad \text{for all } S^2 \subset M \tag{3} \]

In particular, for the symplectic one-form \( \vartheta \) such that \( \omega = d\vartheta \) this implies that

\[ \oint_{\partial D} \vartheta = \int_D \omega \tag{4} \]

is independent of the disc \( D \) which is bounded by the loop \( \partial D : \phi^\mu \to \phi^\mu(t) \).

The condition (3) is very restrictive: The volume-form of a compact \( 2n \)-dimensional symplectic manifold \( M \) is proportional to \( \omega^n \), hence the cohomology class \( [\omega] \) must be nontrivial in \( H^2(M, \mathbb{R}) \). However if \( \pi_1(M) = 0 \), the homomorphism \( \pi_2(M) \to H_2(M, \mathbb{Z}) \) is an isomorphism. For (3) to be valid, we should then consider manifolds with \( \pi_1(M) \neq 0 \) such as the torus \( T^2 = S^1 \times S^1 \). At the moment only partial results are known for manifolds with \( [\omega] \pi_2(M) \neq 0 \).

In a series of papers Floer (for a review see [3]) formulates the Arnold conjecture in terms of infinite dimensional Morse theory. He uses the classical action

\[ S_{cl}(\phi) = \int_0^T \{ \partial_\mu \dot{\phi}^\mu - H(t, \phi) \} \, dt \tag{5} \]

to define a gradient flow in the space \( \mathcal{L}M \) of closed loops \( \phi^\mu(0) = \phi^\mu(T) \),

\[ \frac{\partial \phi^\mu}{\partial s} = -g_{\mu\nu} \frac{\delta}{\delta \phi^\nu} S_{cl}(\phi) \tag{6} \]

Here \( s \) parametrizes the flow of the loops, and \( g_{\mu\nu} \) is a Riemannian metric which is consistent with an almost complex structure \( I^\mu_\nu \) on \( M \),

\[ g^{\mu\lambda} \omega_{\lambda\nu} = I^\mu_\nu \]
\[ I^\mu_\lambda I^\lambda_\nu = -\delta^\mu_\nu \]

Explicitly, (6) is
\[
\frac{\partial \phi^\mu}{\partial s} + I^\mu_\nu \frac{\partial \phi^\nu}{\partial t} + g^{\mu\nu} \partial_\nu H(t, \phi) = 0 \tag{7}
\]

and this equation is defined on the cylinder \( R \times S^1 \) with local coordinates \( s \) and \( t \).

Floer uses the fact that in the nondegenerate case, the bounded orbits of (7) tend asymptotically to the \( T \)-periodic solutions of Hamilton’s equations of motion (1). He applies Fredholm theory to construct a chain complex generated by these \( T \)-periodic solutions, and shows that the homology of this complex is independent of the Hamiltonian \( H(t, \phi) \). This implies in particular, that it defines an invariant of the underlying manifold. By continuing his complex to the classical Morse complex defined by the gradient flow
\[
\dot{\phi}^\mu = -g^{\mu\nu} \partial_\nu H
\]

Floer then proves the non-degenerate Arnold conjecture (3) on a symplectic manifold \( M \) subject to the condition (3), by showing that his homology group is a model of the singular homology of the manifold \( M \).

Subsequently, Witten [6] formulates Floer homology in terms of a relativistic, 1+1-dimensional topological nonlinear \( \sigma \)-model. He uses the fact that since the Floer homology is independent of the Hamiltonian \( H \), one should be able to describe it by entirely ignoring the Hamiltonian. Instead of (7), we then have its \( H = 0 \) version
\[
\frac{\partial \phi^\mu}{\partial s} + I^\mu_\nu \frac{\partial \phi^\nu}{\partial t} = 0 \tag{8}
\]

Witten constructs his topological \( \sigma \)-model so that the corresponding path integral localizes to the solutions of this equation. He then relates the number of these solutions to the Floer homology, by showing that the space of quantum ground states of his \( \sigma \)-model coincides with the Floer group.

Obviously, it would be very interesting to develop quantum field theory techniques of [6] further, so that one could actually analyze the \( H \)-dependent properties of the space of solutions to (1). Such a quantum field theory approach to classical dynamical systems could prove most useful in a number of problems. In particular, it could develop into a new application of field theory techniques that might be useful in the investigation of classically chaotic dynamical systems.
An attempt to formulate classical mechanics using quantum mechanical path integrals has been discussed in [7]. However, the approach used there is quite different from that developed in [6].

Recently, Sadov [8] has proposed a very interesting approach to extend the methods in [6] to include a nontrivial Hamiltonian flow. He constructs a nonrelativistic 1 + 1 dimensional σ-model based on the topological model introduced in [6]. However, the model he constructs is quite elaborated, and until now there has been very little progress in analyzing its consequences to classical dynamical systems.

The purpose of the present Letter is to investigate how path integral techniques applied to 1 + 1 dimensional σ-models could be used to study the space of classical solutions of a Hamiltonian system. Instead of the topological σ-model discussed in [6], [8], we shall here consider a somewhat simpler construction which is based on the conventional $N = 1$ supersymmetric non-linear σ-model. In this way we find results that extend aspects of standard Morse theory to the loop space of classical solutions.

In order to apply the $N = 1$ supersymmetric σ-model to describe the space of solutions to (1), we first need to present an appropriate construction of its action [9]. For this, we introduce two commuting fields $\phi^\mu(x^\pm)$, $F_\mu(x^\pm)$ and two anticommuting fields $\eta^\mu(x^\pm)$, $\bar{\eta}_\mu(x^\pm)$, where $x^\pm = s \pm t$ are coordinates on the cylinder $S^1 \times R^1$. We interpret $\phi^\mu$ and $\bar{\eta}_\mu$ as coordinates on a supertangent bundle $S^*M$, and identify the variables $\eta^\mu$ and $F_\mu$ as the corresponding basic one-forms, $\eta^\mu \sim d\phi^\mu$ and $F_\mu \sim d\bar{\eta}_\mu$.

The nilpotent exterior derivative $d$ on the exterior algebra in the space of maps from the cylinder to $S^*M$ is

$$d = \int dx^+ dx^- (\eta^\mu \frac{\partial}{\partial \phi^\mu} + F_\mu \frac{\partial}{\partial \bar{\eta}_\mu})$$

(In the following we will usually not write explicitly integrals over $x^\pm$.) We also introduce the interior multiplication operator along the vector field $(\partial_+ \phi^\mu, \partial_- \bar{\eta}_\mu)$,

$$i_- = \int \partial_- \phi^\mu i_\mu + \partial_- \bar{\eta}_\mu \pi^\mu$$

where $i_\mu$ and $\pi^\mu$ is the basis of contractions dual to $\eta^\mu \sim d\phi^\mu$ and $F_\mu \sim d\bar{\eta}_\mu$ respectively i.e. in the space of two-dimensional fields

$$i_\mu \eta^\nu = \delta_\mu^\nu \delta(s-s') \delta(t-t')$$
\[ \pi^\mu F_\nu = \delta^\mu_\nu \delta(s - s') \delta(t - t') \]

We define the following equivariant exterior derivative
\[ Q_- = d + i_- = \eta^\mu \partial_\mu + F_\mu \frac{\partial}{\partial \bar{\eta}_\mu} + \partial_- \phi^\mu i_\mu + \partial_- \bar{\eta}_\mu \pi^\mu \] (9)

(Notice that integration over the cylinder \( x^\pm \) is implicit here.) The corresponding (functional) Lie-derivative is
\[ \mathcal{L}_- = Q^2_- = \partial_- \phi^\mu \partial_\mu + \partial_- \eta^\mu i_\mu + \partial_- F_\mu \pi^\mu + \partial_- \bar{\eta}_\mu \frac{\partial}{\partial \bar{\eta}_\mu} \equiv \partial_- \] (10)

In order to relate this to the \( \sigma \)-model, we introduce the functional
\[ \Phi = \Gamma^\rho_{\mu\nu} \pi^\mu \eta^\nu \bar{\eta}_\rho \] (11)

where \( \Gamma^\rho_{\mu\nu} \) are components of a (metric) connection on \( \mathcal{M} \) i.e. with \( g_{\mu\nu} \) a Riemannian metric on \( \mathcal{M} \),
\[ \Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \]

We then conjugate (9) as follows,
\[ Q_- \to e^{-\Phi} Q_- e^{\Phi} = Q_- - \{Q_-, \Phi\} + \frac{1}{2!} \{\{Q_-, \Phi\}, \Phi\} + ... \]

This gives for the conjugated \( Q_- \)
\[ Q_- = \eta^\mu \partial_\mu + (F_\mu + \Gamma^\rho_{\mu\nu} \eta^\nu \bar{\eta}_\rho) \frac{\partial}{\partial \bar{\eta}_\mu} + \partial_- \phi^\mu i_\mu \]

+ \{\partial_- \bar{\eta}_\mu + \Gamma^\rho_{\mu\nu} F_\rho \eta^\nu - \partial_- \phi^\nu \Gamma^\rho_{\mu\nu} \eta^\nu \bar{\eta}_\rho + \frac{1}{2} R^\rho_{\mu\lambda\nu} \eta^\nu \eta^\lambda \bar{\eta}_\rho \} \pi^\mu \] (12)

where \( R^\rho_{\mu\lambda\nu} \) is the Riemann curvature tensor on \( \mathcal{M} \). Notice that this conjugation leaves the Lie-derivative (10) invariant,
\[ \mathcal{L}_- \to e^{-\Phi} \mathcal{L}_- e^{\Phi} = \mathcal{L}_- = \partial_- \] (13)

Using the metric \( g_{\mu\nu} \) we define
\[ \xi_1 = g^{\mu\nu} F_\mu \bar{\eta}_\nu \]
\[ \xi_2 = g_{\mu\nu} \partial_+ \phi^\mu \eta^\nu \] (14)
and introduce

\[ S_\sigma = Q_-(\xi_1 + \xi_2) = \int \left\{ g_{\mu\nu} \partial_+ \phi^\mu \partial_- \phi^\nu + g_{\mu\nu} F^\mu F^\nu \right\} \]

\[ + \frac{1}{2} R_{\mu\nu\rho\sigma} \eta^\sigma \eta^\rho \eta^\mu \eta^\nu - \eta^\mu D^{(+)}_{\mu\nu} \eta^\nu - \bar{\eta}^\mu D^{(-)}_{\mu\nu} \bar{\eta}^\nu \} \]

(15)

where we have defined

\[ D^{(\pm)}_{\mu\nu} = g_{\mu\nu} \partial_{\pm} + \partial_{\pm} \phi^\rho g_{\rho\sigma} \Gamma^\sigma_{\mu\nu} \]

We identify (15) as the action of the N=1 supersymmetric non-linear \(\sigma\)-model. In particular, the present construction can be viewed as an infinite dimensional version \([10]\) of the Mathai-Quillen formalism \([11], [12]\). This formalism has been previously applied in the investigation of Witten’s topological \(\sigma\)-model \([13], [14]\), however in a manner different from the present approach. Notice, that as a \(Q_-\) exact quantity the action has the standard form of a topological action \([15]\).

The action (15) has an obvious \(\pm\)-symmetry. Indeed, if in (12) we exchange \(x^+ \leftrightarrow x^-, \eta \leftrightarrow \bar{\eta}, F \leftrightarrow -F\) so that

\[ Q_- \rightarrow Q_+ = \bar{\eta}^\mu \partial_\mu + (-F^\mu + \Gamma^\rho_{\mu\nu} \bar{\eta}^\nu \eta^\rho) \frac{\partial}{\partial \eta^\mu} + \partial_+ \phi^\mu j_\mu \]

\[ + \left\{ -\partial_+ \eta_\mu + \Gamma^\rho_{\mu\nu} F^\rho \bar{\eta}^\nu - \partial_+ \phi^\rho \Gamma^\rho_{\mu\nu} \eta_\rho + \frac{1}{2} R^\rho_{\mu\lambda\nu} \bar{\eta}^\rho \bar{\eta}^\lambda \eta^\rho \right\} \pi^\mu \]

(16)

where \(j_\mu\) denotes contraction dual to the one-form \(\bar{\eta}^\mu\), and instead of (14) we define

\[ \tilde{\xi}_1 + \tilde{\xi}_2 = - \eta^\mu F^\mu \eta_\nu + g_{\mu\nu} \partial_- \phi^\mu \bar{\eta}^\nu \]

we find that we can also represent (13) as

\[ S_\sigma = Q_+(\tilde{\xi}_1 + \tilde{\xi}_2) \]

Instead of (13) we now find for the corresponding Lie derivative

\[ Q^2_+ = \mathcal{L}_+ = \partial_+ \]

(17)

and we can identify \(Q_-\) and \(Q_+\) as the left and right chiral generators in the (1, 1) world-sheet supersymmetry algebra of the nonlinear \(\sigma\)-model. This supersymmetry algebra is defined by the relations (13) and (17) with the addition of

\[ Q_- Q_+ + Q_+ Q_- = 0 \]
We shall now proceed to generalize the previous construction so, that it can be used to investigate the solutions of (1). For this we first observe that the zeroes of the \( i_\mu, j_\mu \) components of the vector fields in (12), (16) are solutions to

\[
\partial_\pm \phi^\mu = 0 \tag{18}
\]

Consequently we expect the action (15) to describe the properties of these field configurations. In particular, if we compare (18) with the \( H = 0 \) version (8) of (7), we conclude that (18) can be interpreted as the \( H = 0 \) version of the following Hamiltonian flow equation in the loop space,

\[
\frac{\partial \phi^\mu}{\partial s} = \pm \omega^{\mu\nu} \frac{\delta}{\delta \phi^\nu} S_{cl}(\phi) \tag{19}
\]

This is a natural loop space generalization of (1), with the classical action (3) as a loop space Hamiltonian function(al): Just like (3), the equation (19) is also a geometric equation that can be associated with the classical action (3). While the equation (3) describes flow between critical loops of \( S_{cl} \), the equation (19) describes flow that circulates around such (isolated) critical loops.

In component form, (19) reads

\[
\frac{\partial \phi^\mu}{\partial s} - \frac{\partial \phi^\mu}{\partial t} + \omega^{\mu\nu} \partial_\nu H(t, \phi) \equiv \partial_- \phi^\mu + \mathcal{X}^\mu_H = 0 \tag{20}
\]

where we have specified the + sign in the r.h.s. of (19). In order to construct the generalization of the supersymmetric \( \sigma \)-model that describes the properties of the classical trajectories (1), we use (20) to generalize (9) to

\[
Q_- \rightarrow Q_S = \eta^\mu \partial_\mu + F_\mu \frac{\partial}{\partial \eta_\mu} + (\partial_- \phi^\mu - \mathcal{X}_H^\mu) i_\mu + (\partial_- \bar{\eta}_\mu + \partial_\mu \mathcal{X}_H^\nu \bar{\eta}_\nu) \pi^\mu \tag{21}
\]

so that the \( i_\mu \)-component vanishes on (21) while the \( \pi^\mu \)-component vanishes if \( \bar{\eta}_\mu \) is a Jacobi field of (20).

We again introduce the conjugation by (11) which gives for (21)

\[
Q_S = \eta^\mu \partial_\mu + (F_\mu + \Gamma^\rho_\mu \eta^\nu \bar{\eta}_\nu) \frac{\partial}{\partial \bar{\eta}_\mu} + (\partial_- \phi^\mu - \mathcal{X}_H^\mu) i_\mu + \{ \partial_- \bar{\eta}_\mu + \partial_\mu \mathcal{X}_H^\nu \bar{\eta}_\nu + \Gamma^\rho_\mu F_\rho \eta^\nu - \partial_- \phi^\nu \Gamma^\rho_\mu \bar{\eta}_\rho + \Gamma^\nu_\mu \mathcal{X}_H^\rho \bar{\eta}_\rho + \frac{1}{2} R^\rho_{\mu\lambda\nu} \eta^\lambda \bar{\eta}_\rho \} \pi^\mu \tag{22}
\]
If we now introduce
\[ S_0 = \int \partial_- \phi^\mu - H + \frac{1}{2} \eta^\mu \omega_{\mu\nu} \eta^\nu \] (23)
we find that this is a closed quantity with respect to (22),
\[ Q_S S_0 = 0 \]
The desired generalization of (15) is then
\[ S = S_0 + Q_S (\xi_1 + \xi_2) = \int \{ g_{\mu\nu} \partial_+ \phi^\mu \partial_+ \phi^\nu + g_{\mu\nu} F^\mu F^\nu + \partial_+ \phi^\mu - g_{\mu\nu} X_H^\nu \partial_+ \phi^\mu - H - \eta^\mu D^{(+)\mu} \eta^\nu - \bar{\eta}^\mu D^{(-)\mu} \bar{\eta}^\nu + \frac{1}{2} \eta^\mu \omega_{\mu\nu} \eta^\nu - \frac{1}{2} \bar{\eta}^\mu \Omega_{\mu\nu} \bar{\eta}^\nu + \frac{1}{2} R_{\mu\nu\rho\sigma} \eta^\rho \bar{\eta}^\sigma \bar{\eta}^\nu \} \] (24)
where
\[ \Omega_{\mu\nu} = \partial_\mu (g_{\nu\rho} X_H^\rho) - \partial_\nu (g_{\mu\rho} X_H^\rho) \] (25)

In the rest of the present Letter we shall discuss the properties of this action. In particular, we argue that this action describes the space of classical trajectories (1) in the sense of (infinite dimensional) Morse theory. For this, we consider a special class of Hamiltonians $H$ with the property, that at each value of $t$ the metric tensor $g_{\mu\nu}$ is Lie conserved by the Hamiltonian vector field $X_H^\mu$,
\[ L_H g = X_H^\mu \partial_\mu g_{\rho\sigma} + g_{\mu\rho} \partial_\nu X_H^\nu + g_{\nu\rho} \partial_\mu X_H^\mu = 0 \] (26)
The global existence of such an invariant metric means that $H$ - at each time $t$ - is a generator in the Lie algebra of an isometry group $G$ of $g_{\mu\nu}$ that acts canonically on $\mathcal{M}$. Without loss of generality, we may view $H$ as a ($t$-dependent) $U(1)$-generator in the Cartan subalgebra of this isometry group.

For this general class of Hamiltonians, the action (24) admits a $Q_S$-supersymmetry, as a direct consequence of the fact that with (26) the Lie derivative
\[ L_H = \frac{\partial}{\partial \phi^\mu} + \eta^\mu \partial_\nu X_H^\nu \eta^\nu - \partial_\mu X_H^\nu \bar{\eta}^\nu \frac{\partial}{\partial \bar{\eta}^\nu} - F^\nu \partial_\mu X_H^\nu \pi^\mu \] (27)
transforms the variables $\phi^\mu$, $\eta^\mu$, $F_\mu$ and $\bar{\eta}_\mu$ in a generally covariant manner. Hence
\[ Q_S S = Q_S S_0 + Q_S^2 (\xi_1 + \xi_2) = Q_S S_0 + (\partial_- + L_H)(\xi_1 + \xi_2) = 0 \] (28)
We observe that if we attempt to construct the corresponding generalization of the other supersymmetry generator (16) we find that the action (24) fails to be invariant under a full \((1,1)\) supersymmetry: The inclusion of a nontrivial Hamiltonian flow subject to (26) explicitly breaks the \((1,1)\) supersymmetry of the action (15) down to a chiral \((1,0)\) supersymmetry of (24). This breaking of the \((1,1)\) supersymmetry can be understood as follows. The \((+\eta)\) chiral sector of the action (24) contains the symplectic one-form \(\vartheta_{\mu}\) and the corresponding symplectic two-form \(\omega_{\mu\nu} = \partial_{\mu}\vartheta_{\nu} - \partial_{\nu}\vartheta_{\mu}\). The \((-\bar{\eta})\) sector of the action contains the one-form \(\theta_{\mu} = g_{\mu\nu}\chi_H^{\nu}\) and the corresponding two-form \(\Omega_{\mu\nu}\) in a similar fashion. If we define the Hamiltonian

\[ K = \frac{1}{2}g_{\mu\nu}\chi_H^{\mu}\chi_H^{\nu} \equiv \frac{1}{2}g^{\mu\nu}\theta_{\mu}\theta_{\nu} \]

we find that the pair \((H, \omega)\) and \((K, \Omega)\) determines a bi-hamiltonian pair in the sense that the corresponding classical equations (1) coincide,

\[ \Omega_{\mu\nu}\dot{\phi}^\nu = \partial_{\mu}K = \Omega_{\mu\nu}\omega_{\rho}^\nu\partial_{\rho}H \]

However, since only the Hamiltonian \(H\) of the \((+\eta)\) chiral sector appears in (24) we also understand why the original \((1,1)\) supersymmetry of (15) must be broken down to a chiral \((1,0)\) supersymmetry.

From the previous observation, we in particular conclude that the action (24) naturally incorporates the bi-hamiltonian structure which is characteristic to integrable models.

If in addition of (26) we also assume that the symplectic one-form \(\vartheta\) is \(H\)-invariant

\[ \mathcal{L}_H \vartheta = 0 \quad \Rightarrow \quad H = \chi_H^{\mu}\partial_{\mu} = g^{\mu\nu}\partial_{\mu}\theta_{\nu} \]  

we can write (24) in a functional form which is entirely \(\pm\)-symmetric,

\[ S = \int \left\{ g_{\mu\nu}\partial_{\nu}\phi^\mu\partial_{-}\phi^\nu + g_{\mu\nu}F^\mu F^\nu + \partial_{\mu}\partial_{-}\phi^\mu - \theta_{\mu}\partial_{\nu}\phi^\mu - g^{\mu\nu}\partial_{\mu}\theta_{\nu} \right\} \]

\[ - \eta^{\mu}D_{\nu}^{+(\mu)}\eta^\nu - \bar{\eta}^{\mu}D_{\nu}^{-(\mu)}\bar{\eta}^\nu + \frac{1}{2}\eta^{\mu}\omega_{\mu\nu}\eta^\nu - \frac{1}{2}\bar{\eta}^{\mu}\Omega_{\mu\nu}\bar{\eta}^\nu + \frac{1}{2}R_{\mu\nu\rho\sigma}\eta^\rho\eta^\sigma\bar{\eta}^\mu\bar{\eta}^\nu \}

In this form, we can readily generalize it to any pair of symplectic structures \((\vartheta, \omega)\) and \((\theta, \Omega)\). This could be useful in the investigation of bi-Hamiltonian structures.
In particular, we can also write this action in the form of a topological quantum field theory of the cohomological type \cite{13},

\[ S = Q S (\vartheta_{\mu} \eta^\mu + \xi_1 + \xi_2) \]

Consider a world-sheet Lorentz transformation

\[ x^\pm \to e^{\pm \theta} x^\pm \]

in (24). If we implement this transformation in the action (24), we find that its only effect is to scale the symplectic two-form

\[ \omega_{\mu \nu} \to e^{-\theta} \omega_{\mu \nu} \quad (31) \]

In the quantum theory, consistency demands that exponential of \((i\text{ times})\) the integral of the symplectic one-form over a closed loop \(\partial D\) in \(\mathcal{M}\) must be independent of a disk \(D\) which is bounded by the loop; see (4). This implies in particular, that the integral of \(\omega\) over any \(S^2 \subset \mathcal{M}\) must be an integer multiplet of \(2\pi\). Consequently (31) implies that unless the additional condition (3) is satisfied, the world-sheet Lorentz invariance will be broken. Moreover, even if the condition (3) is satisfied only expectation values of operators that are independent of \(\omega\) can be invariant under world-sheet Lorentz transformations.

Indeed, the action (30) can be represented in a world-sheet Lorentz-covariant form in a very suggestive manner: For this we introduce 1+1 dimensional \(\gamma\)-matrices in terms of the Pauli matrices by \(\gamma^0 = \sigma^2\), \(\gamma^1 = i\sigma^1\) and define a two component spinor by

\[ \psi^\mu = \begin{pmatrix} \gamma^\mu \\ \bar{\eta}^\mu \end{pmatrix} \]

and as usual,

\[ \bar{\psi}_\mu = \psi_\mu T \gamma^0 \]

We also define the left and right \(U(1)\) gauge fields

\[ A_{+\mu} = \vartheta_\mu \]

\[ A_{-\mu} = -g_{\mu \nu} \chi_H^\nu \]

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and corresponding field strength tensors

\[ F_{+\mu\nu} = \partial_\mu A_{+\nu} - \partial_\nu A_{+\mu} = \omega_{\mu\nu} \]
\[ F_{-\mu\nu} = \partial_\mu A_{-\nu} - \partial_\nu A_{-\mu} = \Omega_{\mu\nu} \]

With \( h^{ij} \) the world-sheet Lorentz metric in a light-cone basis \((i, j = \pm)\) we can then write the action (30) in the Lorentz-covariant form

\[ S = \int \left\{ g_{\mu\nu} h^{ij} \partial_i \phi^\mu \partial_j \phi^\nu + g_{\mu\nu} F^\mu F^\nu + h^{ij} A_{i\mu} \partial_j \phi^\mu - h^{ij} g^{\mu\nu} A_{i\mu} A_{j\nu} \right\} \]
\[ + \frac{1}{8} R_{\mu\nu\rho\sigma} \bar{\psi}^\rho \gamma^i D_i \psi^\nu + \frac{1}{2} \bar{\psi}^\mu \gamma^i F_{i\mu\nu} \psi^\nu \}

Notice that this is also generally covariant on the target manifold \( \mathcal{M} \).

A similar construction also applies to the original action (24) with an arbitrary \( H \), except that now we can not identify \( H = h^{ij} g^{\mu\nu} A_{i\mu} A_{j\nu} \).

In order to relate our action and the space of classical solutions to (1), we now explicitly evaluate the path integral

\[ Z = \int [d\phi] [dF] [d\eta] [d\bar{\eta}] \exp \{ iS \} \] (32)

for the general class of Hamiltonians (26) using localization techniques [14] with \( S \) the \( Q_S \)-supersymmetric action (30). For this we use the fact [17] that (32) remains invariant under

\[ S \rightarrow S + Q_S \xi \]

provided \( \mathcal{L}_S \xi = 0 \). Using this invariance, we replace in (24)

\[ \xi_2 \rightarrow \lambda \xi_2 = \lambda g_{\mu\nu} \partial_+ \phi^\mu \eta^\nu \]

where \( \lambda \) is a parameter. For the action (30) this gives

\[ S + Q_S (\xi_1 + \lambda \xi_2) = \int \left\{ g_{\mu\nu} \partial_+ \phi^\mu \partial_- \phi^\nu + g_{\mu\nu} F^\mu F^\nu + \partial_\mu \partial_- \phi^\mu - \lambda g_{\mu\nu} X^\nu \partial_+ \phi^\mu - H \right. \]
\[ - \lambda \eta^\mu D^{(+)\mu} \eta^\nu - \bar{\eta}^\mu D^{(-)\mu}_{\bar{\eta}^\nu} + \frac{1}{2} \eta^\mu \omega_{\mu\nu} \eta^\nu - \frac{1}{2} \bar{\eta}^\mu \Omega_{\mu\nu} \bar{\eta}^\nu + \frac{1}{2} R_{\mu\nu\rho\sigma} \eta^\rho \eta^\sigma \bar{\eta}^\mu \bar{\eta}^\nu \} \] (33)
In order to evaluate the path integral we first expand the fields \( \phi^\mu(x^+, x^-) \) and \( \eta^\mu(x^+, x^-) \) as follows,

\[
\phi^\mu(x^+, x^-) = \phi^\mu_-(x^-) + \delta \phi^\mu(x^+, x^-)
\]

\[
\eta^\mu(x^+, x^-) = \eta^\mu_-(x^-) + \delta \eta^\mu(x^+, x^-)
\]

That is, we separate the part which is independent of \( x^+ \). The path integral measure in (32) is now defined by

\[
[d\phi][d\eta] = [d\phi_-][d\eta_-][d\delta \phi][d\delta \eta]
\]

We introduce the following change of variables which has unit Jacobian in the path integral measure

\[
\delta \phi^\mu(x^+, x^-) \rightarrow \frac{1}{\sqrt{\lambda}} \delta \phi^\mu(x^+, x^-)
\]

\[
\delta \eta^\mu(x^+, x^-) \rightarrow \frac{1}{\sqrt{\lambda}} \delta \eta^\mu(x^+, x^-)
\]

and set \( \lambda \rightarrow \infty \). In this limit the only surviving terms involving \( \delta \phi \) and \( \delta \eta \) are

\[
\delta \phi^\mu (-D^\mu_{\nu} - \frac{1}{2} \Omega^\mu_{\nu} + \frac{1}{2} R^\mu_{\rho \mu \nu} \eta^\rho \eta^\nu) \partial_+ \delta \phi^\nu + \delta \eta^\mu (-g_{\mu \nu} \partial_+) \delta \eta^\nu
\]

We evaluate the corresponding functional integral which gives

\[
\text{det}^{-\frac{1}{2}} \| - D(-)^{\mu \nu} - \frac{1}{2} \Omega^\mu_{\nu} + \frac{1}{2} R^\mu_{\nu \rho \sigma} \eta^\rho \eta^\sigma \|
\]

Next, we consider the integral over \( \bar{\eta}^\mu \) and \( F^\mu \): We again introduce

\[
\bar{\eta}^\mu(x^+, x^-) = \bar{\eta}^\mu_-(x^-) + \delta \bar{\eta}^\mu(x^+, x^-)
\]

\[
F^\mu(x^+, x^-) = F^\mu_-(x^-) + \delta F^\mu(x^+, x^-)
\]

and define the path integral measures accordingly. We then find that the integral over \( \delta \bar{\eta}^\mu \) and \( \delta F^\mu \) cancels (33). As a consequence the original \( D = 2 \) path integral (32) reduces to a one dimensional path integral with action

\[
S_{D=1} = \int d\tau \left\{ g_{\mu \nu} F^\mu F^\nu + \partial_\tau \delta \phi^\mu - H(\tau, \phi) + \frac{1}{2} \eta^\mu \omega_{\mu \nu} \eta^\nu \right\}
\]
\[ + \bar{\eta}^\nu (-D_{\tau \mu \nu} - \frac{1}{2} \Omega_{\mu \nu} + \frac{1}{2} R_{\mu \nu \rho \sigma} \eta^\rho \eta^\sigma) \bar{\eta}^\nu \] (36)

where the fields now depend on the variable \( \tau \equiv x^- \) only, and in particular \( H(\tau, \phi) \) is the average of \( H(t, \phi) \) over \( x^+ \).

The action (36) is equivariantly closed with respect to the one-dimensional version of (22),

\[ Q_D = 1 = \eta^\mu \frac{\partial}{\partial \phi^\mu} + (F^\rho_{\mu \nu} \eta^\nu \bar{\eta}_\rho) \frac{\partial}{\partial \bar{\eta}_\mu} + (\partial_\tau \phi^\mu - X^\mu_H) \bar{\eta}_\mu \]

\[ + \{ \Gamma^\sigma_{\mu \nu} F^\rho_{\sigma \nu} \eta^\nu \eta^\sigma \bar{\eta}_\rho - (\partial_\tau \phi^\nu - X^\nu_H) \Gamma^\rho_{\mu \nu} \bar{\eta}_\rho + (\delta^\rho_{\mu} \partial_\tau + \partial_\mu X^\rho_H) \bar{\eta}^\rho \} \pi^\mu \]

\[ Q_{D=1} S_{D=1} = 0 \]

Consequently, if we consider the following one-dimensional path integral

\[ Z_{D=1} = \int [d\phi] [dF] [d\eta] [d\bar{\eta}] \exp \{ iS_{D=1} + Q_{D=1} \xi \} \]

(37)

we conclude that it is independent of \( \xi \) whenever

\[ Q_{D=1}^2 \xi = \mathcal{L}_{D=1} \xi = 0 \]

(38)

Hence it coincides with the path integral for the action (36), obtained by setting \( \xi = 0 \).

For (38), it is sufficient that \( \xi \) is generally covariant. If we select

\[ \xi = g^{\mu \nu} F^\mu_{\nu \rho} \bar{\eta}_\rho + \frac{\lambda}{2} g_{\mu \nu} (\partial_\tau \phi^\mu - X^\mu_H) \eta^\nu \]

(39)

where \( \lambda \) is a parameter, we conclude that (37) is independent of \( \lambda \) and we can take \( \lambda \to \infty \). If we assume that the \( T \)-periodic solutions to (1) (where \( T \) now denotes a period in \( \tau = x^- \)) are non-degenerate. With \( S_{cl}(\phi) \) the corresponding classical action (5), we find that (37) localizes to critical points of \( S_{cl} \),

\[ Z = Z_{D=1} = \sum_{\delta S_{cl}=0} \text{sign}(\det |\frac{\delta^2 S_{cl}}{\delta \phi^\mu \delta \phi^\nu}|) \exp \{ iS_{cl} \} \]

(40)

Obviously, we can view this as an equivariant loop space generalization of the quantity that appears in the Gauss-Bonnet-Chern theorem in classical Morse theory [1].

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Hence we conclude in particular, that our field theory (24) describes the space of classical solutions to (1) in the sense of equivariant loop space Morse theory.

We can relate (40) to the topology of $\mathcal{M}$ as follows: If we select

$$\xi = g^{\mu\nu} F_{\mu\bar{\nu}} + \frac{\lambda}{2} g_{\mu\nu} \partial_{\tau} \phi^\mu \eta^\nu$$

we find in the $\lambda \to \infty$ limit a finite dimensional integral over $\mathcal{M}$,

$$Z = Z_{D=1} = \int_{\mathcal{M}} \exp\left\{-iT(H + \frac{1}{2} \eta^{\mu} \omega_{\mu\nu} \eta^{\nu})\right\} \text{Pf}\left[\frac{1}{2} (\Omega^{\mu}_{\nu} + R^{\mu}_{\nu\rho\sigma} \eta^\rho \eta^\sigma)\right]$$  (41)

Here we recognize a combination of the equivariant Chern character and the equivariant Euler class for the Hamiltonian $H$ [12]. Consequently we identify (41) as an equivariant version of the quantity that appears in the Poincaré-Hopf theorem in classical Morse theory [1], [4], [12]. Notice in particular that contrary to (40), this result can also be used if the set of critical trajectories of $S_{cl}$ is degenerate.

We observe that if we set $H = \vartheta = 0$ so that our action (24) reduces to the action (15) of the standard N=1 supersymmetric nonlinear $\sigma$-model, (41) reduces to the Euler characteristic of the manifold $\mathcal{M}$ consistent with the results in [19].

Combining (40) and (41) we find

$$\sum_{\delta S_{cl}=0} \text{sign}(\det ||\delta^2 S_{cl}/\delta \phi^\mu \delta \phi^\nu||)) \exp\{i S_{cl}\}$$

$$= \int_{\mathcal{M}} \exp\left\{-iT(H + \frac{1}{2} \eta^{\mu} \omega_{\mu\nu} \eta^{\nu})\right\} \text{Pf}\left[\frac{1}{2} (\Omega^{\mu}_{\nu} + R^{\mu}_{\nu\rho\sigma} \eta^\rho \eta^\sigma)\right]$$  (42)

and in particular our generalization of the supersymmetric $\sigma$-model describes the space of solutions to (1) in a manner which generalizes the familiar Gauss-Bonnet-Chern and Poincaré-Hopf theorems of classical Morse theory to equivariant loop space context. Indeed, if we take $T \to 0$, we obtain the standard result

$$\sum_{dH=0} \text{sign}(\det ||\partial^2 H/\partial \phi^\mu \partial \phi^\nu||))$$

$$= \int_{\mathcal{M}} \text{Pf}[ R^{\mu}_{\nu\rho\sigma} \eta^\rho \eta^\sigma ] \equiv \sum_{k} (-)^{k} \dim H^{k}(\mathcal{M})$$  (43)
The Hamiltonians (26) are examples of perfect Morse functions [18] which means that their Morse index \( i.e. \) the number of negative eigenvalues of \((\partial^2 H)_{\mu \nu}\) is even for every critical point \(dH = 0\). Consequently the \(l.h.s.\) of (43) counts the number of these critical points. In particular, in the \(T \to 0\) limit (40) counts the number of solutions to (1). This is the quantity that appear in the \(l.h.s.\) of the Arnold conjecture (2). Furthermore, since \(\mathcal{M}\) admits perfect Morse functions only if all \(H^{2k+1}(\mathcal{M}) = 0\), we conclude from the \(r.h.s.\) of (43) that in the \(T \to 0\) limit (11) yields the \(r.h.s.\) in (2). Consequently (42) is consistent with the Arnold conjecture as \(T \to 0\).

This result means in particular, that in some sense the classical actions with (26) can be viewed as “perfect Morse functionals” \(i.e.\) as functionals that in a natural fashion saturate the lower bound of the inequality (2). However, since the number of classical solutions to (1) depends on \(T\), the definition of a perfect Morse functional is not necessarily very useful.

We note that as a consequence of the homogeneity of the Pfaffian, (11) is manifestly Lorentz-invariant in the sense that if we introduce the world-sheet Lorentz transformation \(x^\pm \to e^{-\theta} x^\pm\) so that \(\omega\) scales according to (31), the partition function (11) is independent of \(\theta\) since all \(\theta\)-dependence from \(\omega\) is cancelled by \(\theta\)-dependence from \(\Omega\). Moreover, if instead of \(\mathcal{M}\) we integrate (11) over nontrivial lower dimensional cycles in \(\mathcal{M}\), the \(\theta\)-dependence appears only as an overall normalization factor. Hence we may view such nontrivial cycles as Lorentz-invariant “observables” in our theory, for manifolds that satisfy the condition (3). However, we also note that if all odd Betti numbers \(B_{2k+1}\) vanish and in addition \(\omega\) satisfies (3), we obtain further conditions from the ensuing requirement that \(\pi_1(\mathcal{M}) \not= 0\): Since \(H_1(\mathcal{M})\) is the Abelianization of \(\pi_1(\mathcal{M})\) [21], we conclude that the Abelian part of \(\pi_1(\mathcal{M})\) must vanish. Recently [21], examples of four dimensional symplectic manifolds with \(\pi_1 \sim G\) where \(G\) is any finitely presented group, have been constructed. For \(G\) that does not admit an Abelian part, we then have \(B_1 = B_3 = 0\) and \(\pi_2(\mathcal{M}) \not= H_2(\mathcal{M}, \mathbb{Z})\). If in addition \(\omega\) satisfies (3), it might become possible to construct examples of supersymmetric Lorentz invariant theories based on an equivariant generalization of the standard supersymmetry algebra. However, we note that from the point of view
of classical dynamical systems, world sheet Lorentz invariance is not an important property.

Unfortunately, at the moment there is not very much we can say about (24) for a general \( t \)-dependent Hamiltonian \( H(t, \phi) \): The path integral (32) is too complicated to be explicitly evaluated in the general case. However, it is natural to expect that also in the general case our theory describes the space of classical trajectories in the sense of an infinite dimensional Morse theory. Indeed, we can still identify a local supersymmetric structure:

If \( H \) does not satisfy the condition (26), the action (24) fails to be (globally) supersymmetric. However, for any \( H \) we can still find a local supersymmetry in any neighborhood in \( \mathcal{M} \) that does not include critical points of \( H \). Using canonical transformations, we can always identify \( H \) e.g. with some component \( p_i \) of the canonical momentum in such (Darboux) neighborhoods. If in these neighborhoods we set \( g_{\mu \nu} = \delta_{\mu \nu} \), we find that the condition (26) is satisfied. Consequently, locally the construction of (24) reflects supersymmetric structure and the supersymmetry is broken only due to the fact that \( g_{\mu \nu} \) can not be globally defined.

Indeed, even in the general case the action (24) is constructed using a \((1,0)\) supersymmetry algebra: Instead of the supersymmetry algebra of (30),

\[
Q_S^2 = \partial_- + \mathcal{L}_H \\
Q_S(\partial_- + \mathcal{L}_H) - (\partial_- + \mathcal{L}_H)Q_S = 0
\]

in the general case the construction of the action (24) uses the isomorphic algebra

\[
Q_S^2 = \partial_- + \mathcal{L}_H \\
Q_S(\partial_- + \mathcal{L}_H) - (\partial_- + \mathcal{L}_H)Q_S = 0
\]

where

\[
\mathcal{L}_H = \mathcal{L}_H + \mathcal{L}_{\Gamma^\rho_{\mu \nu}} \bar{\eta}_\rho \gamma^\nu \pi^\mu
\]

and \( \mathcal{L}_{\Gamma^\rho_{\mu \nu}} \) denotes Lie derivative of the connection \( \Gamma^\rho_{\mu \nu} \) in the original manifold \( \mathcal{M} \),

\[
\mathcal{L}_{\Gamma^\rho_{\mu \nu}} = \mathcal{X}^\sigma_H \partial_\sigma \Gamma^\rho_{\mu \nu} + \partial_\nu \mathcal{X}^\sigma_H \Gamma^\rho_{\mu \sigma} + \partial_\mu \mathcal{X}^\sigma_H \Gamma^\rho_{\sigma \nu} - \Gamma^\rho_{\mu \sigma} \partial_\sigma \mathcal{X}^\rho_H + \partial_{\mu \nu} \mathcal{X}^\rho_H
\]
which vanishes if (26) is satisfied.

In conclusion, we have investigated a generalization of the standard $N = 1$ supersymmetric nonlinear $\sigma$-model, obtained by equivariantizing the $N = 1$ model by a nontrivial loop space Hamiltonian flow. By explicitly evaluating the corresponding path integral for a class of Hamiltonians we have found, that our generalization describes properties of a classical dynamical system in the sense of classical Morse theory. Our approach suggest that quantum field theoretical methods could be very effective in analyzing the structure of classical dynamical systems, and obviously a generalization of our results to models that do not obey the condition (27) would be very interesting.

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References

[1] J. Milnor, Morse Theory (Princeton University Press, 1973).

[2] V.I. Arnold, C.R. Acad. Paris 261 (1965) 3719; and Uspeki Math. Nauk. 18 (1963) 91.

[3] H. Hofer and E. Zehnder, Symplectic Invariants and Hamiltonian Dynamics (Birkhäuser Verlag, 1994).

[4] E. Witten, J. Diff. Geom. 17 (1982) 661.

[5] A. Floer, Duke Math. J. 53 (1986) 1; and Comm. Math. Phys. 118 (1988) 215; and J. Diff. Geom. 18 (1988) 513; and Comm. Pure Appl. Math. 41 (1988) 775; and ibid. 42 (1989) 775.

[6] E. Witten, Comm. Math. Phys. 118 (1988) 411.
[7] E. Gozzi, M. Reuter, W.D. Thacker, Phys. Rev. D40 (1989) 3363; E. Gozzi, M. Reuter, Phys. Lett. B240 (1990) 137; E. Gozzi, preprint INFN-AE-93-18.

[8] V. Sadov, hep-th/9310153

[9] K. Palo, Phys. Lett. B321 (1994) 61.

[10] A.J. Niemi and K. Palo, hep-th/9406068

[11] V. Mathai and D. Quillen, Topology 25 (1986) 85; for a review, see M. Blau, J. Geom. Phys. 11 (1993) 95

[12] N. Berline, E. Getzler and M. Vergne, Heat Kernels and Dirac Operators (Springer Verlag, Berlin, 1991)

[13] S. Wu, hep-th/9406013

[14] for a review, see M. Blau and G. Thompson, Localization and Diagonalization (to appear)

[15] D. Birmingham, M. Blau, M. Rakowski and G. Thompson, Phys. Repts. 209 (1991) 129

[16] M. Blau, Int. J. Mod. Phys. A (1991) 365

[17] M. Blau, E. Keski-Vakkuri and A.J. Niemi, Phys. Lett. B246 (1990) 92

[18] F. Kirwan, Topology 26 (1987) 37

[19] E. Witten, Nucl. Phys. B185 (1981) 513; ibid. B202 (1982) 253

[20] M. Greenberg, Lectures on Algebraic Topology (Benjamin Publishing Company, New York 1966)

[21] R.E. Gompf, Some New Symplectic 4-Manifolds (preprint MPI/92-47)