The purpose of this article is to view the Penrose kite from the perspective of symplectic geometry.

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Introduction

The kite in a Penrose aperiodic tiling by kite and darts [8, 9] is an example of a simple convex polytope. By the Atiyah, Guillemin–Sternberg convexity theorem [1, 6], convex polytopes that are rational can be obtained as images of the moment mapping for Hamiltonian torus actions on compact symplectic manifolds. Moreover, the Delzant theorem [5] provides an exact correspondence between symplectic toric manifolds and simple rational polytopes that satisfy a special integrality condition; a crucial feature of this theorem is that it gives an explicit construction of the manifold that is associated to each polytope. The Penrose kite however is the most elementary and beautiful example of a simple convex polytope that is not rational. The purpose of this article is to apply to the kite a generalization of the Delzant construction for non–rational polytopes, which was introduced by the second–named author in [10]. We recall that this generalized construction allows to associate to any simple convex polytope $\Delta$ in $(\mathbb{R}^k)^*$ a $2k$–dimensional compact symplectic quasifold. Quasifolds are a natural generalization of manifolds and orbifolds: a local $n$–dimensional model is given by the quotient of an open connected subset of $\mathbb{R}^n$ by the action of a finitely generated group. In the generalized construction the lattice of the rational case is replaced by a quasilattice $Q$, which is the $\mathbb{Z}$–span of a set of generators of $\mathbb{R}^k$. The torus is replaced accordingly by a quasitorus, which is the quotient of $\mathbb{R}^k$ modulo $Q$. The action of the quasitorus on the quasifold is smooth, effective and Hamiltonian, and exactly as in the Delzant case, the image of the corresponding moment mapping is the polytope $\Delta$.

In order to apply the generalized Delzant construction to the kite we need to choose a suitable quasilattice $Q$, and a set of four vectors in $Q$ that are orthogonal to the edges of the kite and that point toward the interior of the polytope. The most natural choice is to consider, among the various inward–pointing orthogonal vectors, those four which have the same length as the longest edge of the kite, and then to choose $Q$ to be the quasilattice that they generate. We remark that these choices are justified by the geometry of the kite, and, more globally, by the geometry of any kite and dart tiling, in the following sense. Let us consider the quasilattice $R$ which is generated by the
vectors that are dual to the generators of $Q$; notice that the generators of $R$ are parallel to the edges of the kite. Then the quasilattice $R$ contains the four vertices of the kite. Moreover, given any kite and dart tiling, if we consider one of its kites and the associated quasilattice $R$, then all of the vertices of the tiling lie in $R$.

The four–dimensional quasifold that we obtain turns out to be a very nice example of a quasifold that is not a global quotient of a manifold modulo the action of a finitely generated group, as is the case instead with the symplectic quasifolds that have been associated to a Penrose rhombus tiling in [2].

Quasilattices arise naturally also in the study of the physics of quasicrystals. Quasicrystals are some very special alloys, which were discovered by Shechtman, Blech, Gratias and Cahn in 1982 [13], that have discrete but non–periodic diffraction patterns. We remark that the quasilattice $R$ describes the diffraction pattern of quasicrystals with pentagonal axial symmetry, which is prohibited for ordinary crystals (see [4]). Another symmetry that is forbidden for crystals but is allowed for quasicrystals is the icosahedral symmetry. In this case too there is a quasilattice underlying both the structure of the quasicrystals and the suitable analogues of Penrose tilings in dimension 3. A symplectic interpretation of this case will be given in the forthcoming paper [3].

The paper is structured as follows. In Section 1 we recall the classical construction of the Penrose kite and dart from the pentagram. In Section 2 we introduce the quasilattices $Q$ and $R$ and we discuss the relevant properties. In Section 3 we sketch the generalized Delzant procedure. In Section 4 we apply the procedure to the kite and we also describe a full atlas for the corresponding quasifold. Finally in the last section we show that each kite of any kite and dart tiling gives rise to the same symplectic quasifold. We can therefore think of this quasifold as being a global invariant of the tiling.

1 The Penrose Kite and Dart

Let us now recall the procedure for obtaining the Penrose kite and dart from the pentagram. For a proof of the facts that are needed we refer the reader to [2], and for additional historical remarks see [11]. Let us consider a regular pentagon whose edges have length 1 and let us consider the corresponding inscribed pentagram, as in Figure 1. It can be shown that the length of the diagonal of the pentagon is equal to the golden ratio, $\phi = \frac{1+\sqrt{5}}{2} = 2 \cos \frac{\pi}{5}$. The polygon having vertices $A$, $B$, $E$ and $G$ is a Penrose kite, the polygon having vertices $A$, $B$, $F$ and $G$ is a Penrose dart, and their union is the Penrose thick rhombus having vertices $E$, $B$, $F$ and $G$ (see Figure 2). Remark that the angles of the kite measure $2\pi/5$ at the vertices $B$, $E$ and $G$ and $4\pi/5$ at the vertex $A$, moreover the longest edges, $EG$ and $EB$, and the longest diagonal, $EA$, have the same length, which is 1, whilst the shortest edges $AG$ and $AB$ have length $1/\phi$. The angles of the dart measure $\pi/5$ at the vertices $G$ and $B$, $2\pi/5$ at the vertex $F$ and $6/5\pi$ at the vertex $A$. 
Figure 1: The pentagram

Figure 2: The Penrose kite, dart and thick rhombus
2 The Quasilattice

First of all let us recall the definition of quasilattice:

**Definition 2.1 (Quasilattice)** Let $V$ be a real vector space. A quasilattice in $V$ is the span over $\mathbb{Z}$ of a set of $\mathbb{R}$-spanning vectors $V_1, \ldots, V_d$ of $V$.

Notice that $\text{Span}_{\mathbb{Z}}\{V_1, \ldots, V_d\}$ is a lattice if and only if it admits a set of generators which is a basis of $V$.

It is easy to see that, in a suitably chosen coordinate system, the unitary vectors

\[
\begin{align*}
Y_1 &= (\cos \frac{2\pi}{5}, \sin \frac{2\pi}{5}) = \frac{1}{2}(1, \sqrt{2} + \phi) \\
Y_2 &= (\cos \frac{4\pi}{5}, \sin \frac{4\pi}{5}) = \frac{1}{2}(-\phi, \frac{1}{\phi} \sqrt{2} + \phi) \\
Y_3 &= (\cos \frac{6\pi}{5}, \sin \frac{6\pi}{5}) = \frac{1}{2}(-\phi, -\frac{1}{\phi} \sqrt{2} + \phi) \\
Y_4 &= (\cos \frac{8\pi}{5}, \sin \frac{8\pi}{5}) = \frac{1}{2}(1, -\sqrt{2} + \phi)
\end{align*}
\]

are orthogonal to each of the four different edges of the kite (cf. Figure 2). Now notice that $Y_0 = (1, 0)$ is given by $Y_0 = -(Y_1 + Y_2 + Y_3 + Y_4)$ (see Figure 3). Therefore

\[\begin{align*}
Y_0^* &= (0, 1)
\end{align*}\]

The quasilattice $Q$ is not a lattice, it is dense in $\mathbb{R}^2$ and a minimal set of generators of $Q$ is made of four vectors.

The quasilattice $Q$ is naturally linked to the kite in the following sense. Consider the vectors dual to $Y_1, Y_2, Y_3, Y_4$; they are given by

\[
\begin{align*}
Y_1^* &= \frac{1}{2}(-\sqrt{2} + \phi, \frac{1}{\phi}) \\
Y_2^* &= \frac{1}{2}(1, \phi \sqrt{2} + \phi) \\
Y_3^* &= \frac{1}{2}(\frac{1}{\phi} \sqrt{2} + \phi, -\phi) \\
Y_4^* &= \frac{1}{2}(\sqrt{2} + \phi, \frac{1}{\phi})
\end{align*}
\]

Notice that, in the same coordinate system as above, the four edges of the kite are parallel to these four vectors, and that its vertices are contained in the quasilattice $R$ that they generate. Notice that

\[Y_0^* = (0, 1)\]
is given by

$$Y_0^* = -(Y_1^* + Y_2^* + Y_3^* + Y_4^*).$$

Therefore \( \{Y_1^*, Y_2^*, Y_3^*, Y_4^*\} \) and \( \{Y_0^*, Y_1^*, Y_2^*, Y_3^*, Y_4^*\} \) generate the same quasilattice.

We show the star of five vectors \( Y_0^*, Y_1^*, Y_2^*, Y_3^*, Y_4^* \) in Figure 4.

Let us now show that this connection between the quasilattice \( R \) and the kite miraculously extends to any kite and dart tiling. We recall that a kite and dart tiling is a tiling of the plane by kites and darts that obey the matching rules shown in Figure 5 (cf. [8, 9] and the book by Senechal [12] for a review on quasilattices and aperiodic tilings). There are uncountably many such tilings and each of them is non-periodic. Notice that the kite and dart can never be joined to yield a thick rhombus, namely the configuration in Figure 2 is not allowed. Consider now the vectors \( Y_k^* \) and their opposites \( -Y_k^* \), with \( k = 0, \ldots, 4 \). For each vector \( Y_k^* \) let \( \Delta_k^+ \) be the kite such that, in the notations of Figure 2, \( E \) coincides with the origin and \( A - E = Y_k^* \). We obtain in this way a star of five kites. Analogously denote by \( \Delta_k^- \) the kites corresponding to the vectors \( -Y_k^* \), thus obtaining a star of kites rotated by \( \pi/5 \) with respect to the first one.

Let us now consider any kite and dart tiling \( T \) with kites having longest edge of length 1. Denote by \( AB \) one edge of the tiling \( T \). From now on we will choose our coordinates so that \( A = O \) and so that \( B - A \) is parallel to \( Y_0^* \) with the same orientation.
**Proposition 2.2** Let $T$ be a kite and dart tiling with kites having longest edge of length 1. Then each kite of the tiling is the translate of either a $\Delta_k^+$, $k = 0, \ldots, 4$ or a $\Delta_k^-$, $k = 0, \ldots, 4$. Moreover each vertex of the tiling lies in the quasilattice $R$.

**Proof.** The argument is very simple. Let $C$ be a vertex of the tiling that is different from 0 and the above vertex $B$. We can join $B$ to $C$ with a broken line made of subsequent edges of the tiling. We denote the vertices of the broken line thus obtained by $T_0 = A, T_1 = B, \ldots, T_j, \ldots, T_m = C$. The angle of the broken line at each vertex $T_j$ is necessarily a multiple of $\pi/5$ (see Section 1). Therefore each vector $V_j = T_j - T_{j-1}$ is either one of the vectors $\pm Y_k^*$, $k = 0, \ldots, 4$, to account for the edges of length 1, or one of the vectors $\pm(Y_k^* + Y_{k+2}^*)$, $k = 0, \ldots, 4$ (here $Y_0^* = Y_0^+$ and $Y_6^* = Y_1^+$), to account for the edges of length $1/\phi$. Since $C - A = T_m - T_0 = V_m + \cdots + V_1$ our assertion is proved: the vertex $C$ lies in $R$ and each kite having $C$ as vertex is the translate of one of the ten kites $\Delta^+_k$ and $\Delta^-_k$, $k = 0, \ldots, 4$. □

**Remark 2.3** Recall from [12, Chapter 6, Section 1] that a kite and dart tiling gives rise to a rhombus tiling and vice versa. This can be done by bisecting the tiles into isosceles triangles and then by composing these triangles into thin and thick rhombuses. A subdivided kite will become part either of a thick rhombus and a thin rhombus, as shown in the local configuration (a) in Figure 6 or of two thick rhombuses, as shown in the local configuration (b). It can never become part of two thin rhombuses, this is forbidden by the matching rules. All vertices of a rhombus tiling are contained in the quasilattice $R$ (cf. [2, Proposition 1.3]).

![Figure 6: Passing from a kite and dart tiling to a rhombus tiling](image)

**3 The Generalized Delzant Procedure**

We now outline the generalization of the Delzant procedure to nonrational simple convex polytopes. We refer the reader to [10] for the proof, for the definition of quasitorus,
quasifold and their geometry. The article [2] contains an updated version of the definition of quasifold.

Let us now recall what is a simple convex polytope.

**Definition 3.1 (Simple polytope)** A dimension $n$ convex polytope $\Delta \subset (\mathbb{R}^n)^*$ is said to be simple if there are exactly $n$ edges stemming from each vertex.

Let us now consider a dimension $n$ convex polytope $\Delta \subset (\mathbb{R}^n)^*$. If $d$ is the number of facets of $\Delta$, then there exist elements $X_1, \ldots, X_d$ in $\mathbb{R}^n$ and $\lambda_1, \ldots, \lambda_d$ in $\mathbb{R}$ such that

$$\Delta = \bigcap_{j=1}^{d} \{ \mu \in (\mathbb{R}^n)^* \mid \langle \mu, X_j \rangle \geq \lambda_j \}. \quad (3)$$

**Definition 3.2 (Q–rational polytope)** Let $Q$ be a quasilattice in $\mathbb{R}^n$. A convex polytope $\Delta \subset (\mathbb{R}^n)^*$ is said to be $Q$–rational, if the vectors $X_1, \ldots, X_d$ can be chosen in $Q$.

All polytopes in $(\mathbb{R}^n)^*$ are $Q$–rational with respect to some quasilattice $Q$; it is enough to consider the quasilattice that is generated by the elements $X_1, \ldots, X_d$ in (3). Notice that if the quasilattice is a honest lattice then the polytope is rational.

In our situation we only need to consider the special case of simple convex polytopes in 2-dimensional space. Let $Q$ be a quasilattice in $\mathbb{R}^2$ and let $\Delta$ be a simple convex polytope in the space $(\mathbb{R}^2)^*$ that is $Q$–rational. Consider the space $\mathbb{C}^d$ endowed with the standard symplectic form $\omega_0 = \frac{1}{2\pi i} \sum_{j=1}^{d} dz_j \wedge d\bar{z}_j$ and the standard action of the torus $T^d = \mathbb{R}^d/\mathbb{Z}^d$:

$$\tau : \quad T^d \times \mathbb{C}^d \longrightarrow \mathbb{C}^d$$

$$((e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_d}), z) \longmapsto (e^{2\pi i \theta_1} z_1, \ldots, e^{2\pi i \theta_d} z_d).$$

This action is effective and Hamiltonian and its moment mapping is given by

$$J : \quad \mathbb{C}^d \longrightarrow (\mathbb{R}^d)^*$$

$$z \longmapsto \sum_{j=1}^{d} |z_j|^2 e_j^* + \lambda, \quad \lambda \in (\mathbb{R}^d)^* \text{ constant.}$$

The mapping $J$ is proper and its image is the cone $C_\lambda = \lambda + 0$, where 0 denotes the positive orthant in the space $(\mathbb{R}^d)^*$. Now consider the surjective linear mapping

$$\pi : \quad \mathbb{R}^d \longrightarrow \mathbb{R}^2,$$

$$e_j \longmapsto X_j.$$

Consider the dimension 2 quasitorus $D = \mathbb{R}^2/Q$. Then the linear mapping $\pi$ induces a quasitorus epimorphism $\Pi : T^d \longrightarrow D$. Define now $N$ to be the kernel of the mapping $\Pi$ and choose $\lambda = \sum_{j=1}^{d} \lambda_j e_j^*$. Denote by $i$ the Lie algebra inclusion $\text{Lie}(N) \rightarrow \mathbb{R}^d$ and notice that $\Psi = i^* \circ J$ is a moment mapping for the induced action of $N$ on $\mathbb{C}^d$. Then the quasitorus $T^d/N$ acts in a Hamiltonian fashion on the compact symplectic quasifold $M = \Psi^{-1}(0)/N$. If we identify the quasitori $D$ and $T^d/N$ using the epimorphism $\Pi$, we get a Hamiltonian action of the quasitorus $D$ whose moment mapping has image equal
to \((\pi^*)^{-1}(C_{\lambda} \cap \ker i^*) = (\pi^*)^{-1}(C_{\lambda} \cap \operatorname{im} \pi^*) = (\pi^*)^{-1}(\pi^*(\Delta))\) which is exactly \(\Delta\). This action is effective since the level set \(\Psi^{-1}(0)\) contains points of the form \(z \in \mathbb{C}^d, z_j \neq 0, j = 1, \ldots, d\), where the \(T^d\)-action is free. Notice finally that \(\dim M = 2d - 2 \dim N = 2d - 2(d - 2) = 4 = 2 \dim D\).

Let us remark that this construction depends on two arbitrary choices: the choice of the quasilattice \(Q\) with respect to which the polytope is \(Q\)-rational, and the choice of the inward-pointing vectors \(X_1, \ldots, X_d\) in \(Q\).

**Remark 3.3** It is easy to show that if the vectors \(X_1, \ldots, X_d\) are generators of the quasilattice \(Q\), then \(N = \exp(\mathfrak{n})\) and is therefore connected.

## 4 The Kite from a Symplectic Viewpoint

Let us consider the kite \(\Delta^+_0\) and let us label its edges with the numbers 1, 2, 3, 4, as in Figure 7. Our choice of inward-pointing vectors is given by \(X_1 = -Y_1, X_2 = Y_2, X_3 = -Y_3, X_4 = Y_4\). Remark that the vectors \(X_1, X_2, X_3, X_4\) generate the quasilattice \(Q\) defined by (2), and that the kite is \(Q\)-rational. It is easy to see that \(\lambda_1 = \lambda_4 = -\frac{1}{2}\sqrt{2 + \phi}\), and \(\lambda_2 = \lambda_3 = 0\). Let us consider the linear mapping defined by

\[
p: \mathbb{R}^4 \rightarrow \mathbb{R}^2
\]

\[
e_i \mapsto X_i.
\]

Consider now the subgroup \(N = \{ \exp(X) \in T^4 \mid X \in \mathbb{R}^4, \pi(X) \in Q \}\). It is easy to see using the relations

\[
Y_2 = -Y_1 - \phi Y_4
\]
\[ Y_3 = -\phi Y_1 - Y_4 \]

that its Lie algebra is given by
\[
\mathfrak{n} = \left\{ X \in \mathbb{R}^4 \mid X = (-s + \phi t, s, t, -t + \phi s) \right\}, s, t \in \mathbb{R}.
\]

Therefore by Remark 3.3 we have that
\[
N = \exp(\mathfrak{n}) = \left\{ \exp(X) \in T^4 \mid X = (-s + \phi t, s, t, -t + \phi s) \right\}, s, t \in \mathbb{R}.
\]

We will be needing the following bases for \( \mathfrak{n} \)
\[
B_{14} = \{(-1, 1, 0, \phi), (\phi, 0, 1, -1)\}
\]
\[
B_{12} = \{(1, 1/\phi, 1, 0), (-1/\phi, 1/\phi, 0, 1)\}
\]
\[
B_{23} = \{(1, 1/\phi, 1, 0), (0, 1, 1/\phi, 1)\}
\]
\[
B_{34} = \{(1, 0, 1/\phi, -1/\phi), (0, 1, 1/\phi, 1)\}
\]

and the following identity for the golden ratio
\[
\phi = 1 + \frac{1}{\phi}.
\]

Let us consider \( \psi \), the moment mapping of the induced \( N \)-action, and let us write it down in four different ways, relatively to the four different bases above:
\[
\psi(z_1, z_2, z_3, z_4) = \left(-|z_1|^2 + |z_2|^2 + \phi|z_4|^2 - \sigma, \phi|z_1|^2 + |z_3|^2 - |z_4|^2 - \sigma\right) \quad (4)
\]
\[
= \left(|z_1|^2 + \frac{1}{\phi}|z_2|^2 + |z_3|^2 - \phi \sigma, -\frac{1}{\phi}|z_1|^2 + \frac{1}{\phi}|z_2|^2 + |z_4|^2 - \frac{\sigma}{\phi}\right) \quad (5)
\]
\[
= \left(|z_1|^2 + \frac{1}{\phi}|z_2|^2 + |z_3|^2 - \phi \sigma, |z_2|^2 + \frac{1}{\phi}|z_3|^2 + |z_4|^2 - \frac{\sigma}{\phi}\right) \quad (6)
\]
\[
= \left(|z_1|^2 + \frac{1}{\phi}|z_3|^2 - \frac{1}{\phi}|z_4|^2 - \frac{\sigma}{\phi}, |z_2|^2 + \frac{1}{\phi}|z_3|^2 + |z_4|^2 - \frac{\sigma}{\phi}\right) \quad (7)
\]

where we wrote \( \sigma = \frac{1}{2\phi} \sqrt{2 + \phi} \) for brevity.

We recall that to define a quasifold structure on the topological space \( M_0^+ = \Psi^{-1}(0)/N \) we need to give an atlas of charts, each of which is homeomorphic to a quotient of an open connected subset of \( \mathbb{R}^4 \) by the smooth action of a finitely generated group. Overlapping charts are then required to be compatible.

The atlas here is given by four charts, each of which corresponds to a vertex of the kite. Let us begin by considering the vertex which is given by the intersection of the edges numbered 1 and 4. Consider the open subset of \( \mathbb{C}^2 \) given by
\[
\tilde{U}_{14} = \left\{ (z_1, z_4) \in \mathbb{C}^2 \mid -|z_1|^2 + \phi|z_4|^2 < \sigma, \phi|z_1|^2 - |z_4|^2 < \sigma \right\}.
\]

Then, it is easy to show using (4) that the following mapping gives a slice of \( \Psi^{-1}(0) \) transversal to the \( N \)-orbits
\[
\tilde{U}_{14} \xrightarrow{\pi_{14}} (z_1, z_4) \mapsto (z_1, \sqrt{\sigma + |z_1|^2 - \phi|z_4|^2}, \sqrt{\sigma - \phi|z_1|^2 + |z_4|^2}, z_4)
\]
which induces the homeomorphism

\[ \tilde{U}_{14}/\Gamma_{14} \xrightarrow{\tilde{\tau}_{14}} U_{14} \]

where the open subset \( U_{14} \) of \( M_0^+ \) is the quotient

\[ \{ \tilde{z} \in \Psi^{-1}(0) \mid z_2 \neq 0, z_3 \neq 0 \} / N \]

and the finitely generated group

\[ \Gamma_{14} = \{ (e^{2\pi ih\phi}, e^{2\pi ik\phi}) \in T^2 \mid h, k \in \mathbb{Z} \} \]

acts smoothly on \( \tilde{U}_{14} \) and freely on the connected and dense subset given by

\[ \{ (z_1, z_4) \in C^2 \mid z_1 \neq 0, z_4 \neq 0 \} \]

The triple \( (U_{14}, \tau_{14}, \tilde{U}_{14}/\Gamma_{14}) \) is our first chart for \( M_0^+ \).

Let us now consider the vertex that is the intersection of the edges labeled 1 and 2. Consider the open subset of \( \mathbb{C}^2 \) given by

\[ \tilde{U}_{12} = \left\{ (z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + \frac{1}{\phi}|z_2|^2 < \phi \sigma, -|z_1|^2 + |z_2|^2 < \sigma \right\} \]

We now use (5) to construct the following slice

\[ \tilde{U}_{12} \xrightarrow{\tilde{\tau}_{12}} \]

which induces the homeomorphism

\[ \tilde{U}_{12}/\Gamma_{12} \xrightarrow{\tilde{\tau}_{12}} U_{12} \]

where the open subset \( U_{12} \) of \( M_0^+ \) is the quotient

\[ \{ \tilde{z} \in \Psi^{-1}(0) \mid z_3 \neq 0, z_4 \neq 0 \} / N \]

and the finitely generated group \( \Gamma_{12} \) is given by

\[ \Gamma_{12} = \{ (e^{-2\pi ih\phi}, e^{2\pi i(h+k)\phi}) \in T^2 \mid h, k \in \mathbb{Z} \} \]

The triple \( (U_{12}, \tau_{12}, \tilde{U}_{12}/\Gamma_{12}) \) is our second chart for \( M_0^+ \).

We construct now a third chart by considering the vertex given by the intersection of the edges 2 and 3. As we did above we consider the open subset of \( \mathbb{C}^2 \) given by

\[ \tilde{U}_{23} = \left\{ (z_2, z_3) \in \mathbb{C}^2 \mid \frac{1}{\phi}|z_2|^2 + |z_3|^2 < \phi \sigma, |z_2|^2 + \frac{1}{\phi}|z_3|^2 < \frac{\sigma}{\phi} \right\} \]
We now use (6) to construct the following slice
\[
\tilde{U}_{23} \xrightarrow{\tilde{\tau}_{23}} \left\{ \mathbf{z} \in \Psi^{-1}(0) \mid z_1 \neq 0, z_4 \neq 0 \right\}
\]
\[
(z_2, z_3) \longmapsto \left( \sqrt{\phi \sigma - \frac{1}{\phi} |z_2|^2} - |z_3|^2, z_2, z_3, \sqrt{\frac{\sigma}{\phi} - \frac{1}{\phi} |z_2|^2} - \frac{1}{\phi} |z_3|^2 \right)
\]
which induces the homeomorphism
\[
\tilde{U}_{23}/\Gamma_{23} \xrightarrow{\tilde{\tau}_{23}} U_{23}
\]
\[
[(z_2, z_3)] \longmapsto [\tilde{\tau}_{23}(z_2, z_3)]
\]
where the open subset \( U_{23} \) of \( M_0^+ \) is the quotient
\[
\left\{ \mathbf{z} \in \Psi^{-1}(0) \mid z_1 \neq 0, z_4 \neq 0 \right\}/N
\]
and the finitely generated group \( \Gamma_{23} \) is given by
\[
\Gamma_{23} = \left\{ (e^{2\pi i h \phi}, e^{-2\pi i k \phi}) \in T^2 \mid h, k \in \mathbb{Z} \right\}.
\]
This yields our third chart \((U_{23}, \tau_{23}, \tilde{U}_{23}/\Gamma_{23})\).

To construct our fourth and last chart we consider the last vertex, given by the intersection of the edges 3 and 4. Consider the open subset of \( \mathbb{C}^2 \) given by
\[
\tilde{U}_{34} = \left\{ (z_3, z_4) \in \mathbb{C}^2 \mid |z_3|^2 - |z_4|^2 < \sigma, \frac{1}{\phi} |z_3|^2 + |z_4|^2 < \sigma \right\}.
\]
We now use (7) to construct the following slice
\[
\tilde{U}_{34} \xrightarrow{\tilde{\tau}_{34}} \left\{ \mathbf{z} \in \Psi^{-1}(0) \mid z_1 \neq 0, z_2 \neq 0 \right\}
\]
\[
(z_3, z_4) \longmapsto \left( \sqrt{\frac{\sigma}{\phi} - |z_3|^2 + |z_4|^2}, \sqrt{\sigma - \frac{1}{\phi} |z_3|^2 - |z_4|^2}, z_3, z_4 \right)
\]
which induces the homeomorphism
\[
\tilde{U}_{34}/\Gamma_{34} \xrightarrow{\tilde{\tau}_{34}} U_{34}
\]
\[
[(z_3, z_4)] \longmapsto [\tilde{\tau}_{34}(z_3, z_4)]
\]
where the open subset \( U_{34} \) of \( M_0^+ \) is the quotient
\[
\left\{ \mathbf{z} \in \Psi^{-1}(0) \mid z_1 \neq 0, z_2 \neq 0 \right\}/N
\]
and the finitely generated group \( \Gamma_{34} \) is given by
\[
\Gamma_{34} = \left\{ (e^{2\pi i (h+k) \phi}, e^{-2\pi i h \phi}) \in T^2 \mid h, k \in \mathbb{Z} \right\}.
\]
This yields our fourth chart \((U_{34}, \tau_{34}, \tilde{U}_{34}/\Gamma_{34})\).

In order to show that the four charts given above are compatible we need to show that the changes of charts are well defined for each pair of overlapping charts. Let us
see this in detail for the pair of charts $U_{14}$ and $U_{34}$. Observe first that $\tau_{14}^{-1}(U_{14} \cap U_{34})$ is a submodel of the model $\tilde{U}_{14}/\Gamma_{14}$ since

$$
\tau_{14}^{-1}(U_{14} \cap U_{34}) = \{(z_1, z_4) \in \tilde{U}_{14} | z_1 \neq 0\}/\Gamma_{14}
$$

in the same way $\tau_{34}^{-1}(U_{14} \cap U_{34})$ is a submodel of the model $\tilde{U}_{34}/\Gamma_{34}$ since

$$
\tau_{34}^{-1}(U_{14} \cap U_{34}) = \{(z_3, z_4) \in \tilde{U}_{34} | z_3 \neq 0\}/\Gamma_{34}
$$

Let us introduce

$$
\tilde{V}_{14} = \{(z_1, z_4) \in \tilde{U}_{14} | z_1 \neq 0\}
$$

and

$$
\tilde{V}_{34} = \{(z_3, z_4) \in \tilde{U}_{34} | z_3 \neq 0\}.
$$

We need to prove that the mapping

$$
\tau_{34}^{-1} \circ \tau_{14} : \tilde{V}_{14}/\Gamma_{14} \longrightarrow \tilde{V}_{34}/\Gamma_{34}
$$

is a diffeomorphism of models. This means, by definition of model diffeomorphism, that $\tau_{34}^{-1} \circ \tau_{14}$ lifts to an equivariant diffeomorphism between appropriate open subsets of $\mathbb{R}^4$. In order to have a well defined lift we are allowed to consider covering models. The set $\tilde{V}_{14}$ has first fundamental group $\Pi_{14} \simeq \mathbb{Z}$, its universal covering is the open subset

$$
V_{14}^\sharp = \left\{ (\rho_1, \theta_1, z_4) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{C} | -\rho_1^2 + \phi |z_4|^2 < \sigma, \phi \rho_1^2 - |z_4|^2 < \sigma \right\}.
$$

The finitely generated group

$$
\Gamma_{14}^\sharp \simeq \mathbb{Z}^3
$$

acts on $V_{14}^\sharp$ in the following way:

$$
\begin{align*}
\Gamma_{14}^\sharp \times V_{14}^\sharp & \longrightarrow V_{14}^\sharp \\
(\rho_1, \theta_1, z_4) & \longmapsto (\rho_1, \theta_1 + \phi h + m, e^{2\pi i k} z_4)
\end{align*}
$$

This action is smooth on $V_{14}^\sharp$ and is free on a connected and dense subset. The group $\Gamma_{14}^\sharp$ is an extension of the group $\Gamma_{14}$ by the group $\Pi_{14}$

$$
1 \longrightarrow \Pi_{14} \longrightarrow \Gamma_{14}^\sharp \longrightarrow \Gamma_{14} \longrightarrow 1.
$$

Therefore the quotient $V_{14}^\sharp/\Gamma_{14}^\sharp$ is a quasifold model, homeomorphic to the model $\tilde{V}_{14}/\Gamma_{14}$. We say that $V_{14}^\sharp/\Gamma_{14}^\sharp$ is a covering model of $\tilde{V}_{14}/\Gamma_{14}$. Analogously we consider the covering of the model $\tilde{V}_{34}/\Gamma_{34}$ given by the quotient of the open set

$$
V_{34}^\sharp = \left\{ (\rho_3, \theta_3, z_4) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{C} | \rho_3^2 - |z_4|^2 < \sigma, \frac{1}{\phi} \rho_3^2 + |z_4|^2 < \frac{\sigma}{\phi} \right\}.
$$
Symplectic Kite

by the following action of the group $\Gamma_{34}^\sharp \simeq \mathbb{Z}^3$:

$$
\begin{align*}
\Gamma_{34}^\sharp \times V_{34}^\sharp & \longrightarrow V_{34}^\sharp \\
(m, h, k) \times (\rho_3, \theta_3, z_4) & \longmapsto (\rho_3, \theta_3 + \phi h + m, e^{2\pi i k}z_4)
\end{align*}
$$

The mapping

$$
(\tau_{34}^{-1} \circ \tau_{14})^\sharp : V_{14}^\sharp \longrightarrow V_{34}^\sharp \\
(\rho_1, \theta_1, z_4) \longmapsto \left(\sqrt{\sigma - \phi \rho_1^2 + |z_4|^2}, -\frac{1}{\phi} \theta_1, e^{2\pi i \phi \theta_1}z_4\right)
$$

is an equivariant diffeomorphism and it is a lift of $\tau_{34}^{-1} \circ \tau_{14}$, namely the following diagram is commutative:

$$
\begin{array}{ccc}
V_{14}^\sharp & \xrightarrow{(\tau_{34}^{-1} \circ \tau_{14})^\sharp} & V_{34}^\sharp \\
\downarrow & & \downarrow \\
V_{14}^\sharp/\Gamma_{14}^\sharp & \sim & V_{34}^\sharp/\Gamma_{34}^\sharp \\
\end{array}
$$

We proceed in the same way for the other pairs of overlapping charts. The four charts given above turn out to be compatible, thus defining on $M_0^+$ a quasifold structure.

We now describe explicitly the symplectic structure on $M_0^+$ induced by the reduction procedure. To define a symplectic structure on a quasifold we first need to define a symplectic structure on each chart, and then require that the different structures behave well under coordinate changes. Consider for example the chart

$$U_{14} \simeq \tilde{U}_{14}/\Gamma_{14}.$$

A symplectic form here is given by a symplectic form on $\tilde{U}_{14}$ which is invariant under the action of $\Gamma_{14}$. We take the restriction to $\tilde{U}_{14}$ of the standard symplectic form on $\mathbb{C}^2$, denoted by $\tilde{\omega}_{14}$. We do the same for the three other charts. Consider now the changes of charts, for example the one described above: let $\omega_{14}^\sharp$ and $\tilde{\omega}_{34}^\sharp$ be the pullbacks of the forms $\tilde{\omega}_{14}$ and $\tilde{\omega}_{34}$ to $V_{14}^\sharp$ and $V_{34}^\sharp$ respectively; it is easy to check that the pullback of $\omega_{34}^\sharp$ via the mapping $(\tau_{34}^{-1} \circ \tau_{14})^\sharp$ is exactly the form $\omega_{14}^\sharp$. The same argument applies to the other changes of charts. This shows that the local forms $\tilde{\omega}_{14}$, $\tilde{\omega}_{12}$, $\tilde{\omega}_{23}$ and $\tilde{\omega}_{34}$ behave well under the changes of charts, thus defining on the quasifold $M_0^+$ a symplectic structure $\omega$. This symplectic structure is the one that is induced by the reduction procedure, namely:

$$
\begin{array}{ccc}
\Psi^{-1}(0) & \longrightarrow & \mathbb{C}^4 \\
\downarrow & & \\
\Psi^{-1}(0)/\mathbb{N}
\end{array}
$$
the pullback of $\omega$ via the projection to the quotient coincides with the pullback of the standard form on $\mathbb{C}^4$ via the inclusion mapping.

5 Global Symplectic Interpretation of a Kite and Dart Tiling

Recall that we denoted by $M_0^+$ the symplectic quasifold associated to kite $\Delta^+_0$. Consider the ten distinguished kites $\Delta^+_k$ and $\Delta^-_k$, $k = 0, \ldots, 4$. Notice that each of these kites has a natural choice of inward–pointing orthogonal vectors, these are $-Y_{k+1}, Y_{k+2}, -Y_{k+3}, Y_{k+4}$ for $\Delta^+_k$, and $Y_{k+1}, -Y_{k+2}, Y_{k+3}, -Y_{k+4}$ for $\Delta^-_k$. Consider now a kite and dart tiling with longest edges of length 1. Remark that, by Proposition 2.2, in our choice of coordinates, each of its kites can be obtained by translation from one of the 10 kites $\Delta^+_k$ and $\Delta^-_k$. We can then prove the following

**Theorem 5.1** The compact symplectic quasifold corresponding to each kite of a kite and dart tiling with longest edges of length 1 is given by $M_0^+$.

**Proof.** Observe that, for each $k = 1, \ldots, 4$, there exists an orthogonal transformation $P$ of $\mathbb{R}^2$ that leaves the quasilattice $Q$ invariant, that sends the orthogonal vectors relative to the kite $\Delta^+_k$ to the orthogonal vectors relative to the kite $\Delta^+_0$, and such that the dual transformation $P^*$ sends the kite $\Delta^+_0$ to the kite $\Delta^+_k$. The same is true for the kites $\Delta^-_k$, with $k = 1, \ldots, 4$. This implies that the reduced space corresponding to each of the 10 kites $\Delta^+_k$ and $\Delta^-_k$, with the choice of orthogonal vectors and quasilattice specified above, is exactly $M_0^+$. This yields a unique symplectic quasifold, $M_0^+$, for all the kites considered. Finally it is straightforward to check that translating the kites $\Delta^+_k$ and $\Delta^-_k$ does not produce any change in the corresponding quotient spaces, therefore, by Proposition 2.2 we are done. □

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Dipartimento di Matematica Applicata "G. Sansone", Università di Firenze, Via S. Marta 3, 50139 Firenze, ITALY, fiammetta.battaglia@unifi.it

and

Dipartimento di Matematica e Applicazioni per l’Architettura, Università di Firenze, Piazza Ghiberti 27, 50122 Firenze, ITALY, elisa.prato@unifi.it