High Dimensional Spatial Rank Test for Two-Sample Location Problem

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Abstract

This article concerns tests for the two-sample location problem when the dimension is larger than the sample size. The traditional multivariate-rank-based procedures cannot be used in high dimensional settings because the sample scatter matrix is not available. We propose a novel high-dimensional spatial rank test in this article. The asymptotic normality is established. We can allow the dimension being almost the exponential rate of the sample sizes. Simulations demonstrate that it is very robust and efficient in a wide range of distributions.

keywords
High dimensional Tests; Spatial sign; Spatial rank

1 Introduction

Nowadays, high-dimensional data have been generated in many areas, such as microarray analysis, hyperspectral imagery. The traditional statistical methods, which assume the dimension is fixed, may not work in the high-dimensional settings. In the last decades, statistician devoted many new methods to deal with high dimensional data. Specially, many efforts have been devoted to high dimensional hypothesis testing problems. See Dempster (1958), Bai and Saranadasa (1996), Srivastava (2009), Chen and Qin (2010), Chen et al. (2011), Biswas and Ghosh (2014), and Feng and Sun (2015) for two-sample tests for means, Ledoit and Wolf (2002), Schott (2005), Chen, Zhang and Zhong (2010) and Zou et al. (2014) for testing a specific covariance structure, Goeman et al. (2006), Zhong and Chen (2011) and Feng et al. (2013) for high-dimensional regression coefficients.

In this paper, we consider the high dimensional two sample location problem. Bai and Saranadasa (1996) proposed a test statistic by replacing the Mahalanobis norm in Hotelling’s $T^2$ test statistic with Euclidian norm. To allow simultaneous testing for ultra-high dimensional data, Chen and Qin (2010) proposed a test statistic by removing the square term in Bai and Saranadasa (1996). However, both these two test statistics are not invariant under scalar transformation, $X \rightarrow BX$ where $B$ is a diagonal matrix. Recently, many scalar-invariant test statistics
have been constructed, such as Srivastava and Du (2008), Srivastava, Katayama and Kano (2013), Park and Ayyala (2013), Feng et al. (2015), Gregory et al. (2015). All these methods are based on the multivariate normal assumption or the diverging factor model (Bai and Saranadasa, 1996). These assumptions are a little restrictive for application. For example, the multivariate $t$-distribution or mixture of multivariate normal distribution do not belong to them. Moreover, the performance of these moment based tests would be degraded for heavy-tailed distributions.

In the traditional fixed dimension circumstance, multivariate sign or rank based methods are often used to construct robust tests. Those test statistics are very efficient and distribution free under mild assumptions, or asymptotic so. However, those classic multivariate sign or rank based tests also cannot be directly used in high dimensional data because the scatter matrix is not available. Recently, Wang, Peng and Li (2015) proposed a high-dimensional nonparametric multivariate test for one sample location problem based on spatial-signs. Paindaveine and Verdebout (2015) also proposed a high-dimensional spatial sign test for the one sample problem by restricting to spherical cases. Feng, Zou and Wang (2015) proposed a scalar invariant test statistic based on spatial sign for two sample location problem. They show that their proposed test is robust and efficient for a wide range of distributions. However, to estimate the location parameter, they only can allow the dimension being the square of the sample sizes at most. It is too restrictive for application. Thus, we need to propose a new robust test procedure to allow for ultra-high dimensional data.

Multivariate rank based methods also perform very efficient in constructing robust test procedures. In this article, we propose a high-dimensional spatial rank test for two sample location problem. First, we estimate the scale of each variable by spatial rank based procedures. Then, we propose our high dimensional spatial-rank test based on leave out method. The test statistic is scalar-invariant and treats all the variables in a “fair” way. Furthermore, unlike the spatial sign, we do not need to estimate the location parameters for spatial rank. Thus, there are no bias term in our test statistic when the dimension increases. So we can allow the dimension being almost exponential rate of the sample sizes. We also establish its asymptotic normality and propose the asymptotic relative efficiency with respect to Park and Ayyala (2013)’s test. Simulation studies show that our test performs better than other moment-based test procedures under heavy-tailed distributions. And when the dimension is ultra-high against the sample sizes, our test would be more powerful than Feng, Zou and Wang (2015)’s test because of its conservation. All the detailed proofs are given in the Supplementary Material.

2 High-Dimensional Spatial-rank test

2.1 The proposed test statistic

Assume $\{X_{i1}, \cdots, X_{in_i}\}$ for $i = 1, 2$ are two independently and identically distributed (i.i.d.) random samples from $p$-variate elliptical distribution with density functions

$$\det(\Sigma_i)^{-1/2}g(||\Sigma_i^{-1/2}(x - \mu_i)||)$$
where \( \mu_i \)'s are the symmetry centers and \( \Sigma_i \) is the positive definite symmetric \( p \times p \) scatter matrix. Here we consider the equal scatter matrix case, i.e. \( \Sigma = \Sigma_1 = \Sigma_2 \). We wish to test

\[
H_0 : \mu_1 = \mu_2 \quad \text{versus} \quad H_1 : \mu_1 \neq \mu_2. \tag{1}
\]

Hotelling's \( T^2 \) test statistic \( H_n = \frac{n_{12}}{n} (\bar{X}_1 - \bar{X}_2)^T S_n^{-1} (\bar{X}_1 - \bar{X}_2) \) is the classic method to deal with this two sample location problem, where \( n = n_1 + n_2 \), \( \bar{X}_i \) are the sample mean and \( S_n \) is the pooled sample covariance matrix. However, it is not very efficient for the heavy-tailed distributions. Multivariate sign or rank based methods are often used to construct robust test for these location testing problem (Oja, 2010). Define the spatial sign function \( U(x) = \frac{x}{||x||}I(x \neq 0) \). When the dimension \( p \) is fixed, the spatial rank test using inner standardization is

\[
Q^2 = np \sum_{i=1}^{n} \sum_{j=1}^{n_i} \frac{||\tilde{V}_{ij}||^2}{||\hat{V}_{ij}||^2},
\]

where \( \tilde{V}_i = n_i^{-1} \sum_{j=1}^{n_i} \hat{V}_{ij} \), \( \hat{V}_{ij} = n_i^{-1} \sum_{k=1}^{n_i} U(S^{-1/2}(X_{ij} - X_{ik})) \) and the full rank transformation matrix \( S^{-1/2} \) satisfy

\[
RCOV = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n_i} \hat{V}_{ij} \hat{V}_{ij} \propto I_p. \tag{2}
\]

In traditional fixed \( p \) circumstance, \( Q^2 \) is affine-invariant and very robust. When the distribution is heavy-tailed, \( Q^2 \) is more efficient than the classic Hotelling’s \( T^2 \) test. However, when the dimension \( p \) is larger than the sample size \( n \), the scatter matrix is not available and then \( Q^2 \) is not well defined. Alternatively, we could estimate the diagonal matrix of \( \Sigma \) which is able to treat all the variables in a “fair” way.

For the sample \( \{X_{ij}\}_{j=1}^{n_i} \), similar to [2], we suggest to find a diagonal matrix \( D_i \) satisfy

\[
\text{diag} \left\{ \frac{1}{n_i} \sum_{j=1}^{n_i} R(D_i^{-1/2}X_{ij})R(D_i^{-1/2}X_{ij}) \right\} \propto I_p,
\]

where

\[
R(D_i^{-1/2}X_{ij}) = \frac{1}{n_i} \sum_{k=1}^{n_i} U(D_i^{-1/2}(X_{ij} - X_{ik})).
\]

Thus, we adopt the following recursive algorithm

\[
D_i \leftarrow D_i^{1/2} \text{diag} \left\{ \frac{1}{n_i} \sum_{j=1}^{n_i} R(D_i^{-1/2}X_{ij})R(D_i^{-1/2}X_{ij}) \right\} D_i^{1/2}, \quad D_i \leftarrow \frac{p}{\text{tr}(D_i)} D_i.
\]

The resulting estimators of diagonal matrix are denoted as \( \hat{D}_i \). We may use the sample variances as the initial estimators. Note that we fixed \( \text{tr}(D_i) = p \) in this algorithm. Since
$U(\sigma \mathbf{x}) = U(\mathbf{x})$, without loss of generality, we assume tr($\Sigma$) = $p$ in the following. So $\hat{D}_i$ is a consistent estimator of the diagonal matrix of $\Sigma$, i.e. $D$ (Lemma ?? in the Appendix).

A natural idea is mimicking Chen and Qin (2010) and considering the following test statistic

$$G_n = \frac{\sum_{i \neq j}^n \hat{V}_{ij}^T \hat{V}_{ij}}{n_1(n_1 - 1) n_2(n_2 - 1)} + \frac{\sum_{i \neq j}^n \hat{V}_{2i}^T \hat{V}_{2j}}{n_2(n_2 - 1)} - 2 \frac{\sum_{i=1}^n \sum_{j=1}^n \hat{V}_{1i} \hat{V}_{2j}}{n_1 n_2},$$

where $\hat{V}_{ij} = n^{-1} \sum_{k=1}^n \sum_{l=1}^{n_k} R(\hat{D}^{-1/2}(X_{kl} - X_{ij}))$ is the sample “spatial-rank” of $X_{ij}$ in the total samples. And $\hat{D}$ is also the estimator of $D$ based on the total samples. However, there are two drawbacks. First, there would be a non-negligible bias term in $G_n$ with the growth of dimension (Feng, Zou and Wang, 2015). Second, those terms of $R(\hat{D}^{-1/2}(X_{il} - X_{ij}))$ are useless in detecting the difference between these two samples. Thus, based on leave out method and excluding $R(\hat{D}^{-1/2}(X_{il} - X_{ij}))$ terms, we propose the following high dimensional spatial rank test statistic

$$T_n = \frac{1}{n_1(n_1 - 1) n_2(n_2 - 1)} \sum_{i \neq j} \sum_{s \neq t} \sum_{i \neq j} U(\hat{D}^{-1/2}(X_{1i} - X_{2s}))U(\hat{D}^{-1/2}(X_{1j} - X_{2t}));$$

where $\hat{D}_{(i,j,s,t)} = \frac{n_i}{n} \hat{D}_{1(i,j)} + \frac{n_s}{n} \hat{D}_{2(s,t)}$, $\hat{D}_{1(i,j)}$ and $\hat{D}_{2(s,t)}$ are the corresponding estimators of the diagonal matrix with leave-two-out samples $\{X_{1k}\}_{k \neq i,j}$, $\{X_{2k}\}_{k \neq s,t}$, respectively. Obviously, $U(\hat{D}^{-1/2}(X_{1i} - X_{2s}))$ will be deviate from zero if $\mu_1 \neq \mu_2$ and then we will reject the null hypothesis with large values of $T_n$. As shown later, the expectation of $T_n$ is asymptotic negligible compared to its standard deviation under $H_0$. We do not need a bias correction procedure (Feng et al., 2015). Moreover, the value of $T_n$ remains unchanged for $\tilde{X}_{ij} = \hat{D}X_{ij} + c$, where $\hat{D} = \text{diag}\{d_1^2, \ldots, d_p^2\}$, $d_i$ are non-zero constants and $c$ is a constant vector. Our proposed test statistic is invariant under location shift and the group of scalar transformations.

### 2.2 Theoretical Results

We need the following conditions for asymptotic analysis: as $n, p \to \infty$,

(C1) $n_1/n \to \kappa \in (0, 1)$;

(C2) $\text{tr}(R^4) = o(\text{tr}^2(R^2))$ where $R = D^{-1/2} \Sigma D^{-1/2}$;

(C3) $\log(p) = o(n)$ and $\text{tr}(R^2) - p = o(n^{-1} p^2)$;

Condition (C2) is the same as the condition (4) in Park and Ayyala (2013). Condition (C3) is used to get the consistency of the diagonal matrix estimators. To reduce the the difference between $D^{-1/2}(X_{ij} - \mu_j)$ and $\varepsilon_{ij} \sim \Sigma^{-1/2}(X_{ij} - \mu_j)$, we require the correlation between those variables to be not very strong and the dimension to be higher enough.

First, we establish the asymptotic null distribution of $T_n$. 


Theorem 1. Under Conditions (C1)–(C3) and $H_0$, as $(p, n) \to \infty$,

$$T_n/\sigma_n \xrightarrow{L} N(0, 1),$$

where $\sigma_n^2 = \left(\frac{1}{2n_1(n_1-1)p^2} + \frac{1}{2n_2(n_2-1)p^2} + \frac{1}{n_1n_2p^2}\right) \text{tr}(R^2)$.

In order to construct the test procedure, we still need to estimate $\text{tr}(R^2)$. Here we proposed the following three ratio-consistent estimator of $\text{tr}(R^2)$, $s = 1, 2, 3$,

$$\text{tr}(R^2)_s = \frac{2p^2}{P_n^4} \sum_{i=1}^s U \left( \frac{D_{s(i_1,i_2,i_3,i_4)}^{-1}}{s} (X_{s_{i_1}} - X_{s_{i_2}}) \right)^T U \left( \frac{D_{s(i_1,i_2,i_3,i_4)}^{-1}}{s} (X_{s_{i_3}} - X_{s_{i_4}}) \right)$$

$$\times U \left( \frac{D_{s(i_1,i_2,i_4,i)}^{-1}}{s} (X_{s_{i_3}} - X_{s_{i_2}}) \right)^T U \left( \frac{D_{s(i_1,i_2,i_3,i_4)}^{-1}}{s} (X_{s_{i_1}} - X_{s_{i_4}}) \right),$$

$$\text{tr}(R^2) = \frac{p^2}{n_1n_2} \sum_{i_1 \neq i_2} \sum_{i_3 \neq i_4} \sum_{i_1 \neq i_2} \sum_{i_3 \neq i_4} \left( U \left( \frac{D_{s(i_1,i_2,i_3,i_4)}^{-1}}{s} (X_{1_{i_1}} - X_{1_{i_2}}) \right)^T U \left( \frac{D_{s(i_1,i_2,i_3,i_4)}^{-1}}{s} (X_{1_{i_3}} - X_{1_{i_4}}) \right) \right)^2.$$ 

where $\hat{D}_{s(i_1,i_2,i_3,i_4)}$ are the corresponding estimators of diagonal matrix with leave-four-out samples $\{X_{s_j}\}_{k \neq i_1,i_2,i_3,i_4}$. Throughout this article, we use $\sum_s$ to denote summations over distinct indexes. For example, in $\text{tr}(R^2)_1$, the summation is over the set $\{i_1 \neq i_2 \neq i_3 \neq i_4\}$, for all $i_1, i_2, i_3, i_4 \in \{1, \cdots, n_1\}$ and $P_n = n!/(n-m)!$.

Proposition 1. Under Condition (C1)–(C3), as $(p, n) \to \infty$, we have

$$\frac{\text{tr}(R^2)_s}{\sigma_n^2} p \xrightarrow{1}, \ s = 1, 2, 3.$$

Here we use the following ratio-consistent estimator for $\sigma_n^2$,

$$\hat{\sigma}_n^2 = \frac{1}{2n_1(n_1-1)p^2} \text{tr}(R^2)_1 + \frac{1}{2n_2(n_2-1)} \text{tr}(R^2)_2 + \frac{1}{n_1n_2p^2} \text{tr}(R^2)_3.$$

This result suggests rejecting $H_0$ with $\alpha$ level of significance if $T_n/\hat{\sigma}_n > z_\alpha$ where $z_\alpha$ is the upper $\alpha$ quantile of $N(0, 1)$.

Next, we consider the asymptotic distribution of $T_n$ under the alternative hypothesis (C4) $(\mu_1 - \mu_2)^T D^{-1/2} (\mu_1 - \mu_2) = o(c_0^{-2} \sigma_n)$, $(\mu_1 - \mu_2)^T D^{-1/2} D^{-1/2} (\mu_1 - \mu_2) = o(npc_0^{-2} \sigma_n)$ where $c_0 = E(||D^{-1/2} (X_{ij} - X_{ik})||^{-1})$.

Condition (C4) requires the difference between $\mu_1$ and $\mu_2$ is not large so that the variance of $T_n$ is still asymptotic $\sigma_n^2$. And then we can propose the explicit power expression of our test.

Theorem 2. Under Conditions (C1)–(C4), as $(p, n) \to \infty$, we have

$$\frac{T_n - c_0^2(\mu_1 - \mu_2)^T D^{-1/2} (\mu_1 - \mu_2)}{\sigma_n} \xrightarrow{L} N(0, 1).$$


As a consequence, the asymptotic power of our proposed test (abbreviated as SR hereafter) becomes

\[ \beta_{SR}(||\mu_1 - \mu_2||) = \Phi \left( -z_{\alpha} + \frac{2c_0^2 m \kappa (1 - \kappa)(\mu_1 - \mu_2)^T D^{-1}(\mu_1 - \mu_2)}{\sqrt{2 \text{tr}(R^2)}} \right). \]

In comparison, Park and Ayyala (2013) show that the asymptotic power of their proposed test (abbreviated as PA hereafter) is

\[ \beta_{PA}(||\mu_1 - \mu_2||) = \Phi \left( -z_{\alpha} + \frac{np \kappa (1 - \kappa)(\mu_1 - \mu_2)^T D^{-1}(\mu_1 - \mu_2)}{E(||\varepsilon_{ij}||^2) \sqrt{2 \text{tr}(R^2)}} \right). \]

The asymptotic relative efficiency (ARE) of SR with respect to PA is

\[ \text{ARE}(SR, PA) = 2c_0^2 E(||\varepsilon_{ij}||^2) \approx 2 \left\{ E(||\varepsilon_1 - \varepsilon_2||^{-1}) \right\}^2 \frac{E(||\varepsilon||^2)}{E(||\varepsilon_1 - \varepsilon_2||^2)} \geq 1. \]

by the Cauchy inequality and \( c_0 = E(||\varepsilon_1 - \varepsilon_2||^{-1}(1 + o(1)) \) under Condition (C4) (see the proof of Theorem 1). If \( ||\varepsilon_1 - \varepsilon_2||^2/E(||\varepsilon_1 - \varepsilon_2||^2) \overset{p}{\rightarrow} 1 \), SR is equivalent to PA. Otherwise, our SR test would be more efficient.

The ARE values for multivariate \( t \)-distribution with \( \nu = 3, 4, 5, 6 \) are 1.98, 1.48, 1.31, and 1.22, respectively. Clearly, the SR test is more powerful than PA when the distributions are heavy-tailed (\( \nu \) is small), which is also verified by simulation studies in Section 3.

### 3 Simulation

Here we report a simulation study designed to evaluate the performance of the proposed SR test. All the simulation results are based on 2,500 replications. We consider the following five scenarios:

(I) Multivariate normal distribution. \( X_{ij} \sim N(\mu_i, R) \).

(II) Multivariate normal distribution with different component variances. \( X_{ij} \sim N(\mu_i, \Sigma) \), where \( \Sigma = D^{1/2}RD^{1/2} \) and \( D = \text{diag}\{d_1^2, \ldots, d_p^2\}, d_j^2 = 3, j \leq p/2 \) and \( d_j^2 = 1, j > p/2 \).

(III) Multivariate \( t \)-distribution \( t_{p,3} \). \( X_{ij} \)'s are generated from \( t_{p,3} \) with \( \Sigma = R \).

(IV) Multivariate \( t \)-distribution with different component variances. \( X_{ij} \)'s are generated from \( t_{p,3} \) and \( d_j^2 \)'s are generated from \( \chi^2_3 \).

(V) Multivariate mixture normal distribution \( MN_{p,\gamma,9} \). \( X_{ij} \)'s are generated from \( \gamma f_p(\mu_i, R) + (1 - \gamma)f_p(\mu_i, 9R) \), denoted by \( MN_{p,\gamma,9} \), where \( f_p(\cdot, \cdot) \) is the density function of \( p \)-variate multivariate normal distribution. \( \gamma \) is chosen to be 0.8.
Table 1: Empirical Size and power comparison at 5% significance when $p < n$.

| Scenario | (30,24) | (40,32) | (30,24) | (40,32) | (30,24) | (40,32) |
|----------|---------|---------|---------|---------|---------|---------|
| (I)      | 1.6     | 6.3     | 1.3     | 5.4     | 27      | 63      | 41      | 96      | 42      | 58      | 63      | 96      |
| (II)     | 0.7     | 6.5     | 1.8     | 5.2     | 27      | 64      | 43      | 98      | 42      | 58      | 69      | 96      |
| (III)    | 1.4     | 6.4     | 0.8     | 5.7     | 19      | 50      | 31      | 82      | 29      | 45      | 46      | 81      |
| (IV)     | 1.2     | 4.9     | 1.1     | 5.3     | 51      | 60      | 72      | 91      | 29      | 54      | 54      | 59      |
| (V)      | 1.2     | 5.3     | 0.9     | 4.5     | 18      | 45      | 25      | 78      | 24      | 44      | 43      | 76      |

First, we consider the low dimensional case $p < n$ and compare the SR test with the traditional spatial-rank-based test $Q^2$ (abbreviated as TR). The common correlation matrix is $R = (0.5^{j-k})$. For power comparison, we consider the same configurations of $H_1$: $\eta_1 = ||D^{-1/2}(\mu_1 - \mu_2)||^2 / \sqrt{\text{tr}(R^2)} = 0.5$. Without loss of generality, under $H_1$, we fix $\mu_1 = 0$ and choose $\mu_2$ as follows. The percentage of $\mu_{2l} = \mu_{2l}$ for $l = 1, \ldots, p$ are chosen to be 95% (Sparse Case) and 50% (Dense Case), respectively. At each percentage level, all the nonzero $\mu_{2l}$ are equal. Two combinations of $(n, p)$ are considered: (30, 24) and (40, 32). Table 1 reports the empirical sizes and power of these two tests. The empirical sizes of TR is significantly smaller than the nominal level. However, our SR test can control the empirical sizes in most cases. In addition, our SR test is more powerful than the TR test in all cases. This findings are consistent with the results in Bai and Saranadasa (1996). Classical Mahalanobis distance may lose efficiency because of the contamination bias in estimating the covariance matrix with large $p$. When $p/n \to c \in (0, 1)$, having the inverse of the estimate of the scatter matrix in constructing tests would be no longer beneficial.

Next, we consider the high-dimensional cases, $p > n$, and compare the SR with the tests proposed by Chen and Qin (2010) (abbreviated as CQ hereafter), Park and Ayyala (2013) and Feng, Zou and Wang (2015) (abbreviated as SS hereafter). The sample size $n_i$ is chosen as $n_1 = n_2 = 20$. Four dimensions $p = 100, 200, 400, 800$ are considered. The other settings are all the same as low dimensional cases. Table 2 reports the empirical sizes and power of these four tests under normal and non-normal cases. The sizes of CQ, PA and SR tests are generally close to the nominal level under all the scenarios. In contrast, the sizes of the SS test are a little smaller than 5%, i.e., too conservative when $p/n^2$ is large. It is not strange because SS test can only allow the dimension $p$ being the square of the sample size $n$. As shown in Feng, Zou and Wang (2015), when $p/n^2$ is larger, there would be a non-negligible bias term in SS test statistic because of the estimation of location parameters. However, our SR test can allow the dimension being almost the exponential rate of the sample sizes. Consequently, our SR will be more powerful than the SS test when $p/n^2$ is large. Moreover, under the normal cases (Scenarios (I) and (II)), SS and SR tests perform similar to PA test. However, under the non-normal cases (Scenarios (III)-(V)), both SS and SR tests are clearly
more efficient than PA and CQ tests. It is consistent with the theoretical results in Section 2. Our SR test is more robust and efficient when the distribution is heavy-tailed. Finally, SS, PA and SR tests are more powerful than CQ test when the variance of each variables are not equal (Scenario (II) and (IV)), which demonstrates that a scalar-invariant test is needed.

Next, we also consider the unequal scatter matrix case, i.e. \( \Sigma_1 \neq \Sigma_2 \). Now, we consider \( \mathbf{R}_1 = (0.5^{i-j}) \) and \( \mathbf{R}_2 = \mathbf{I}_p \) in this study. The other settings are all the same as the above equal scatter matrix cases except that \( \eta =: ||\mathbf{D}^{-1/2}(\mu_1 - \mu_2)||^2 / \sqrt{\text{tr}(\mathbf{R}_1^2)} = 0.5. \) Here we only consider two combinations of \((n_i, p)\): (20, 200), (20, 800). We report the simulation results in Table 3. Our SR test can also control the empirical sizes under this unequal scatter matrix assumption. The other results are all similar to Table 2. SS is still a little conservative. Our SR still performs better than the other tests in most cases. It shows that our SR test can also be used in the unequal scatter matrix cases.

Finally, to study the effect of correlation matrix on the proposed test and to further discuss the application scope of our method, we explore another four scenarios with different correlations and distributions. The following moving average model is used:
\[
X_{ijk} = ||\rho_i||^{-1}(\rho_{i1}Z_{ij} + \rho_{i2}Z_{ij+1} + \cdots + \rho_{iT_i}Z_{ij+T_i-1}) + \mu_{ij}
\]
for \( i = 1, 2, j = 1, \cdots, n_i \) and \( k = 1, \cdots, p \) where \( \rho_i = (\rho_{i1}, \ldots, \rho_{iT_i})^T \) and \( \{Z_{ijk}\} \) are i.i.d. random variables. Consider four scenarios for the innovation \( \{Z_{ijk}\} \):

(VI) All the \( \{Z_{ijk}\} \)'s are from \( N(0, 1) \);

(VII) the first \( p/2 \) components of \( \{Z_{ijk}\}_{k=1}^{p} \) are from centralized Gamma(8,1), and the others are from \( N(0, 1) \).

(VIII) All the \( \{Z_{ijk}\} \)'s are from \( t_3 \);

(IX) All the \( \{Z_{ijk}\} \)'s are from \( 0.8N(0, 1) + 0.2N(0, 9) \).

The coefficients \( \{\rho_{il}\}_{l=1}^{T_i} \) are generated independently from \( U(2, 3) \) and are kept fixed once generated through our simulations. The correlations among \( X_{ijk} \) and \( X_{ijl} \) are determined by \( |k-l| \) and \( T_i \). We consider the “full dependence” for the first sample and the “2-dependence” for the second sample, i.e. \( T_1 = p \) and \( T_2 = 3 \), to generate different covariances of \( X_{ij} \). For simplicity, set \( \eta =: ||\mu_1 - \mu_2||^2 / \sqrt{\text{tr}(\Lambda_1^2) + \text{tr}(\Lambda_2^2)} = 0.1. \) where \( \Lambda_i \) is the covariance matrix of \( X_{ij} \) and \( (n_i, p) = (20, 200), (20, 800) \). Table 4 reports the simulation results under these four non-elliptical distributions. Our SR test also performs well in these cases. The empirical sizes of SR are close to the nominal level. The power of SR test is still a little larger than PA and SS in most cases. It also shows the robustness of our SR test.

All the above simulation results show that our SR test is very robust and efficient test procedure in a wide range of distributions. SR performs better than the other tests based on the direct observations when the distribution is heavy-tailed. In addition, when the dimension is larger than the square of sample sizes, our SR test is more efficient than SS test because of the conservation of SS test in this case.
4 Discussion

In this article, we propose a new test for high dimensional two sample location problem based on spatial rank. Compared with the other $L_2$-norm-based tests, our proposed test is very robust and efficient, especially for heavy-tailed or skewed distributions. In another direction, Cai, Liu and Xia (2014) proposed a test based on max-norm of marginal $t$-statistics. Zhong, Chen and Xu (2013) also proposed a $L_2$-thresholding statistic. Both these two tests can detect more sparse and stronger signals whereas the $L_2$-norm-based tests is for denser but fainter signals. Developing a spatial-rank-based test for sparse signals is very interest and deserves further study.

In the case of elliptical distributions, Hallin and Paindaveine (2002) propose a class of tests based on interdirections and pseudo-Mahalanobis ranks when the dimension is fixed, which are distribution-free, affine-invariant, and achieve semiparametric efficiency at given reference densities. The Hallin-Paindaveine signs and ranks have been successful in many problems involving elliptical densities (one and two-sample location; scatter; homogeneity of scatter; regression; VARMA dependence; principal components, etc.). How to construct a test base on Hallin-Paindaveine signs and ranks for high dimensional data deserves further studies.

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Table 2: Empirical Size and power comparison at 5% significance with equal scatter matrix

| (nᵢ, p) | Size | Dense Case | Sparse Case |
|---------|------|------------|-------------|
|         | CQ   | SS         | PA          | SR |
|         | CQ   | SS         | PA          | SR |
|         | CQ   | SS         | PA          | SR |

Scenario (I)

| (20,100) | 6.9  | 4.8  | 5.0  | 6.3  | 83   | 77   | 77   | 82   | 79   | 73   | 74   | 79   |
|-----------|------|------|------|------|------|------|------|------|------|------|------|------|
| (20,200)  | 5.3  | 3.4  | 3.3  | 5.0  | 87   | 81   | 83   | 86   | 86   | 81   | 82   | 87   |
| (20,400)  | 5.5  | 2.5  | 4.1  | 4.7  | 90   | 80   | 85   | 90   | 89   | 80   | 85   | 90   |
| (20,800)  | 5.4  | 1.2  | 4.0  | 4.5  | 92   | 81   | 89   | 92   | 91   | 79   | 88   | 91   |

Scenario (II)

| (20,100) | 7.5  | 5.0  | 4.3  | 6.0  | 44   | 78   | 78   | 82   | 40   | 73   | 73   | 78   |
|-----------|------|------|------|------|------|------|------|------|------|------|------|------|
| (20,200)  | 4.5  | 3.1  | 2.5  | 4.4  | 47   | 81   | 82   | 87   | 44   | 81   | 82   | 87   |
| (20,400)  | 6.0  | 2.5  | 4.1  | 5.7  | 46   | 81   | 86   | 91   | 42   | 80   | 85   | 90   |
| (20,800)  | 6.6  | 1.2  | 3.8  | 5.6  | 42   | 80   | 85   | 90   | 45   | 79   | 88   | 91   |

Scenario (III)

| (20,100) | 5.8  | 4.3  | 3.3  | 6.2  | 42   | 64   | 29   | 62   | 37   | 59   | 29   | 58   |
|-----------|------|------|------|------|------|------|------|------|------|------|------|------|
| (20,200)  | 4.2  | 3.0  | 1.5  | 4.7  | 44   | 67   | 34   | 69   | 41   | 63   | 27   | 63   |
| (20,400)  | 6.1  | 2.0  | 3.9  | 5.7  | 44   | 63   | 32   | 67   | 41   | 63   | 31   | 66   |
| (20,800)  | 4.4  | 0.7  | 3.7  | 5.3  | 44   | 61   | 32   | 72   | 44   | 60   | 32   | 71   |

Scenario (IV)

| (20,100) | 5.8  | 4.3  | 3.3  | 6.3  | 11   | 68   | 33   | 67   | 70   | 70   | 33   | 69   |
|-----------|------|------|------|------|------|------|------|------|------|------|------|------|
| (20,200)  | 7.5  | 3.0  | 1.7  | 4.4  | 85   | 73   | 33   | 73   | 13   | 65   | 28   | 65   |
| (20,400)  | 6.4  | 2.0  | 3.9  | 5.7  | 10   | 64   | 30   | 69   | 90   | 62   | 29   | 66   |
| (20,800)  | 5.7  | 0.7  | 3.7  | 5.2  | 9.6  | 60   | 32   | 73   | 8.6  | 62   | 32   | 72   |

Scenario (IV)

| (20,100) | 7.5  | 4.0  | 5.5  | 5.2  | 42   | 59   | 35   | 60   | 35   | 55   | 30   | 57   |
|-----------|------|------|------|------|------|------|------|------|------|------|------|------|
| (20,200)  | 6.5  | 3.0  | 4.5  | 5.2  | 43   | 60   | 32   | 63   | 41   | 56   | 32   | 61   |
| (20,400)  | 6.1  | 2.2  | 3.7  | 5.3  | 41   | 57   | 30   | 64   | 39   | 57   | 28   | 64   |
| (20,800)  | 5.5  | 0.8  | 4.0  | 5.6  | 41   | 54   | 30   | 68   | 39   | 52   | 28   | 66   |

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Table 3: Empirical Size and power comparison at 5% significance with unequal scatter matrix

| Scenario | CQ | SS | PA | SR | Dense Case | CQ | SS | PA | SR | Sparse Case | CQ | SS | PA | SR |
|----------|----|----|----|----|------------|----|----|----|----|--------------|----|----|----|----|
|          |    |    |    |    | $(n_i, p) = (20, 200)$ |    |    |    |    | $(n_i, p) = (20, 800)$ |    |    |    |    |
| (I)      | 5.8| 2.8| 4.2| 6.5| 95          | 89 | 93 | 95 |    | 92           | 87 | 88 | 93 |    |
| (II)     | 6.5| 2.5| 3.4| 5.5| 58          | 91 | 93 | 95 |    | 52           | 87 | 88 | 93 |    |
| (III)    | 4.4| 2.2| 2.5| 3.9| 55          | 79 | 43 | 77 |    | 49           | 74 | 36 | 72 |    |
| (IV)     | 6.6| 2.1| 2.6| 4.6| 65          | 80 | 45 | 79 |    | 11           | 75 | 40 | 74 |    |
| (V)      | 6.1| 2.3| 5.2| 5.1| 53          | 73 | 39 | 74 |    | 49           | 72 | 38 | 75 |    |

Table 4: Empirical Size and power comparison at 5% significance with MV model

| Scenario | CQ | SS | PA | SR | Dense Case | CQ | SS | PA | SR | Sparse Case | CQ | SS | PA | SR |
|----------|----|----|----|----|------------|----|----|----|----|--------------|----|----|----|----|
|          |    |    |    |    | $(n_i, p) = (20, 200)$ |    |    |    |    | $(n_i, p) = (20, 800)$ |    |    |    |    |
| (VI)     | 6.7| 5.8| 6.4| 5.9| 32          | 28 | 26 | 31 |    | 43           | 42 | 36 | 50 |    |
| (VII)    | 3.9| 4.1| 6.1| 4.8| 23          | 32 | 31 | 33 |    | 38           | 72 | 69 | 76 |    |
| (VIII)   | 6.3| 3.7| 7.1| 5.1| 41          | 34 | 35 | 41 |    | 42           | 44 | 34 | 53 |    |
| (IX)     | 6.5| 5.4| 6.3| 5.7| 29          | 27 | 26 | 33 |    | 34           | 36 | 28 | 47 |    |
| (VI)     | 6.6| 4.8| 6.7| 5.8| 32          | 27 | 26 | 28 |    | 33           | 35 | 22 | 40 |    |
| (VII)    | 4.1| 4.0| 6.3| 4.7| 37          | 43 | 38 | 46 |    | 38           | 72 | 68 | 74 |    |
| (VIII)   | 5.9| 3.9| 7.8| 5.4| 39          | 32 | 31 | 42 |    | 41           | 29 | 31 | 49 |    |
| (IX)     | 6.4| 3.4| 6.2| 5.1| 34          | 32 | 28 | 34 |    | 31           | 31 | 23 | 35 |    |
Supplementary Material of “High Dimensional Spatial Rank Test for Two-Sample Location Problem”

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1 Proofs of Theorems

1.1 Proof of Theorem 1

Define \( u_{1i} = E(U(\varepsilon_{1i} - \varepsilon_{2j})|\varepsilon_{1i}) \) and \( u_{2j} = E(U(\varepsilon_{1i} - \varepsilon_{2j})|\varepsilon_{2j}) \). Obviously, \( u_{1i}, u_{2j} \) have the same distribution. And \( E(u_{1i}u_{1i}^T) = \tau_F p^{-1}I_p \) where the constant \( \tau_F \) depend on the background distribution \( F \). Define

\[
Y_{ij} = D^{-1/2}(X_{ij} - \mu_i), \quad V_{2i} = -E(U(Y_{1j} - Y_{2i})|Y_{2i}),
\]

\[
V_{1i} = E(U(Y_{1i} - Y_{2j})|Y_{1i}), \quad P_{is} = U(Y_{1i} - Y_{2s}) - V_{1i} + V_{2s},
\]

\[
W_{ij} = U(Y_{1i} - Y_{1j}) - V_{1i} - V_{1j},
\]

\[
A = E(V_{1i}V_{1i}^T) = E(V_{2j}V_{2j}^T).
\]

Let \( D_i = \text{diag}\{d_{i1}, \cdots, d_{ip}\} \) be the diagonal matrix of \( \Sigma_i \) and \( \hat{D}_i = \text{diag}\{\hat{d}_{i1}, \cdots, \hat{d}_{ip}\} \).

First, we restate Lemma 4 in [Zou et al. (2014)] and propose some useful Lemmas. The proof of these Lemmas are given in Appendix B.

**Lemma 1** Suppose \( u \) are independent identically distributed uniform on the unit \( p \) sphere. For any \( p \times p \) symmetric matrix \( M \), we have

\[
E(u^T Mu)^2 = \{\text{tr}^2(M) + 2\text{tr}(M^2)\}/(p^2 + 2p),
\]

\[
E(u^T Mu)^4 = \{3\text{tr}^2(M^2) + 6\text{tr}(M^4)\}/\{p(p + 2)(p + 4)(p + 6)\}.
\]

**Lemma 2** Under Condition \((C4)\), we have \( \max_{1 \leq j \leq p}(\hat{d}_{ij} - d_{ij}) = O_p(n_i^{-1/2}(\log p)^{1/2}) \).

**Lemma 3** \( \tau_F \to 0.5 \) as \( p \to \infty \).
Lemma 4 Suppose the conditions given in Theorem 1 all hold, we have $T_n = Z_n + o_p(\sigma_n)$, where

$$Z_n = \frac{1}{n_1(n_1 - 1)} \sum_{i \neq j}^{n_1} \sum_{j}^{n_1} V_{1i}^T V_{1j} + \frac{1}{n_2(n_2 - 1)} \sum_{i \neq j}^{n_2} \sum_{j}^{n_2} V_{2i}^T V_{2j} + \frac{2}{n_1n_2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} V_{1i} V_{2j}.$$ 

Lemma 5 Suppose the conditions given in Theorem 1 all hold. Then, $Z_n/\sigma_n \xrightarrow{d} N(0,1)$.

Proof of Theorem 1: According to Lemma 4 and 5, we can easily obtain the result. □

1.2 Proof of Proposition 1

Taking the same procedure as Theorem 1, we can obtain

$$\operatorname{tr}(\mathbf{R}^2)^2 = \frac{2p^2}{P_4} \sum_{n_1}^{*} U(\mathbf{Y}_{1i1} - \mathbf{Y}_{1i2})^T U(\mathbf{Y}_{1i1} - \mathbf{Y}_{1i2}) U(\mathbf{Y}_{1i3} - \mathbf{Y}_{1i4}) U(\mathbf{Y}_{1i1} - \mathbf{Y}_{1i4}) + o_p(\operatorname{tr}(\mathbf{R}^2))$$

$$= \frac{2p^2}{P_4} \sum_{n_1}^{*} (\mathbf{V}_{1i1} - \mathbf{V}_{1i2})^T (\mathbf{V}_{1i1} - \mathbf{V}_{1i2}) (\mathbf{V}_{1i3} - \mathbf{V}_{1i4}) (\mathbf{V}_{1i1} - \mathbf{V}_{1i4}) + o_p(\operatorname{tr}(\mathbf{R}^2))$$

$$= \frac{2p^2}{P_4} \sum_{n_1}^{*} (\mathbf{V}_{1i1} - \mathbf{V}_{1i2})^2 - \frac{4p^2}{P_4} \sum_{n_1}^{*} \mathbf{V}_{1i1}^T \mathbf{V}_{1i2} \mathbf{V}_{1i3} \mathbf{V}_{1i4} + o_p(\operatorname{tr}(\mathbf{R}^2))$$

$$\triangleq J_1 + J_2 + J_3 + o_p(\operatorname{tr}(\mathbf{R}^2)).$$

Obviously, $E(J_1) = 4p^2 \operatorname{tr}(\mathbf{A}^2)$. And

$$\operatorname{tr}(\mathbf{A}^2) = E\{(\mathbf{V}_{1i1}^T \mathbf{V}_{1i2})^2\} = E\left[\frac{(\mathbf{u}_{1i1}^T \mathbf{R} \mathbf{u}_{1i2})^2}{\{1 + (\mathbf{u}_{1i1}^T (\mathbf{R} - \mathbf{I}_p) \mathbf{u}_{1i2}\}^2}\right]$$

$$= E\{(\mathbf{u}_{1i1}^T \mathbf{R} \mathbf{u}_{1i2})^2\} \{1 + o_p(1)\}$$

$$= \tau^2 p^{-2} \operatorname{tr}(\mathbf{R}^2) \{1 + o(1)\} = 4^{-1} p^{-2} \operatorname{tr}(\mathbf{R}^2) \{1 + o(1)\},$$

because $E\{(\mathbf{u}_{1i1}^T (\mathbf{R} - \mathbf{I}_p) \mathbf{u}_{1i2}\}^2\} = \tau^2 p^{-2} \{\operatorname{tr}(\mathbf{R}^2) - I_p\} = o(1)$ by Condition (C3). So $E(J_1) = \operatorname{tr}(\mathbf{R}^2) \{1 + o(1)\}$. Taking the same procedure as above, we have

$$\operatorname{var}\left(\frac{2p^2}{n_1(n_1 - 1)} \sum_{i \neq j}^{n_1} (\mathbf{V}_{1i1}^T \mathbf{V}_{1i2})^2\right)$$

$$= O(n_1^{-2} p^4) E\{(\mathbf{V}_{1i1}^T \mathbf{V}_{1i2})^4\} + O(n_1^{-1} p^4) [E\{(\mathbf{V}_{1i1}^T \mathbf{V}_{1i2})^2\}]^2$$

$$= O(n_1^{-2} \operatorname{tr}(\mathbf{R}^4) + n_1^{-1} \operatorname{tr}(\mathbf{R}^2)) = o(\operatorname{tr}(\mathbf{R}^2)).$$
by Lemma 4 and Condition (C2). Similarly, we can show that

\[ E(J_2^2) = O(n_1^{-3} p^4) E((V_1^T A V_i)^2) + O(n_1^{-2} p^4) \text{tr}(A^4) = o(tr^2(R^2)), \]

\[ E(J_3^2) = O(n_1^{-4} p^4 \text{tr}(A^4)) = o(tr^2(R^2)). \]

Thus, \( \text{tr}(R^2)_t = \text{tr}(R^2)(1 + o_p(1)) \). We can also show the ratio-consistency of the other estimators. □

1.3 Proof of Theorem 2

Define \( U_{is} = U(Y_{1i} - Y_{2s}) \) and \( r_{is} = ||Y_{1i} - Y_{2s}|| \). Firstly, taking the same procedure as Lemma 4, we can show that

\[
T_n = \frac{1}{n(n-1)} \frac{1}{n(n-1)} \frac{1}{n(n-1)} \sum_{i \neq j} \sum_{s \neq t} U(D^{-1/2}(X_{1i} - X_{2s}))^T U(D^{-1/2}(X_{1j} - X_{2t})) + o_p(\sigma_n)
\]

\[
= \frac{1}{n(n-1)} \frac{1}{n(n-1)} \frac{1}{n(n-1)} \sum_{i \neq j} \sum_{s \neq t} U(Y_{1i} - Y_{2s})^T U(Y_{1j} - Y_{2t})
\]

\[
+ \frac{1}{n(n-1)} \frac{1}{n(n-1)} \frac{1}{n(n-1)} \sum_{i \neq j} \sum_{s \neq t} \sum_{j \neq t} \sum_{s \neq t} r_{is}^{-1} U_{jt} [I_p - U_{is} U_{is}^T] D^{-1/2}(\mu_1 - \mu_2)
\]

\[
+ \frac{1}{n(n-1)} \frac{1}{n(n-1)} \frac{1}{n(n-1)} \sum_{i \neq j} \sum_{s \neq t} \sum_{j \neq t} \sum_{s \neq t} r_{is}^{-1} r_{jt}^{-1} (\mu_1 - \mu_2) D^{-1/2} [I_p - U_{jt} U_{jt}^T]
\]

\[
\times [I_p - U_{is} U_{is}^T] D^{-1/2}(\mu_1 - \mu_2) + o_p(\sigma_n).
\]

According to the same arguments as Theorem 1, we have

\[
\frac{2}{n(n-1)} \frac{1}{n(n-1)} \frac{1}{n(n-1)} \sum_{i \neq j} \sum_{s \neq t} \sum_{j \neq t} \sum_{s \neq t} r_{is}^{-1} U_{jt} [I_p - U_{is} U_{is}^T] D^{-1/2}(\mu_1 - \mu_2)
\]

\[
= \frac{2c_0}{n} \sum_{i=1}^{n} V_{1i}^T D^{-1/2}(\mu_1 - \mu_2) - \frac{2c_0}{n} \sum_{i=1}^{n} V_{2i}^T D^{-1/2}(\mu_1 - \mu_2) + o_p(\sigma_n),
\]

\[
\frac{2}{n(n-1)} \frac{1}{n(n-1)} \frac{1}{n(n-1)} \sum_{i \neq j} \sum_{s \neq t} \sum_{j \neq t} \sum_{s \neq t} r_{is}^{-1} r_{jt}^{-1} (\mu_1 - \mu_2) D^{-1/2} [I_p - U_{jt} U_{jt}^T]
\]

\[
\times [I_p - U_{is} U_{is}^T] D^{-1/2}(\mu_1 - \mu_2)
\]

\[
= c_0^2 (\mu_1 - \mu_2)^T D^{-1}(\mu_1 - \mu_2) + o_p(\sigma_n).
\]
where \( c_0 = E(r_{i,i}^{-1}) \). Thus,

\[
T_n = Z_n + \frac{2c_0}{n_1} \sum_{i=1}^{n_1} V_{1i}^T D^{-1/2}(\mu_1 - \mu_2) - \frac{2c_0}{n_2} \sum_{i=1}^{n_2} V_{2i}^T D^{-1/2}(\mu_1 - \mu_2) + c_0^2(\mu_1 - \mu_2)^T D^{-1}(\mu_1 - \mu_2) + o_p(\sigma_n).
\]

And

\[
E \left( \frac{2c_0}{n_1} \sum_{i=1}^{n_1} V_{1i}^T D^{-1/2}(\mu_1 - \mu_2) \right)^2 = O(n^{-1}p^{-1}c_0^2(\mu_1 - \mu_2)^T D^{-1/2}RD^{-1/2}(\mu_1 - \mu_2)),
\]

\[
E \left( \frac{2c_0}{n_2} \sum_{i=1}^{n_2} V_{2i}^T D^{-1/2}(\mu_1 - \mu_2) \right)^2 = O(n^{-1}p^{-1}c_0^2(\mu_1 - \mu_2)^T D^{-1/2}RD^{-1/2}(\mu_1 - \mu_2)).
\]

Under Condition (C4), the second and third parts of \( T_n \) are all \( o_p(\sigma_n) \). Then, by Theorem 1, we have

\[
\frac{T_n - c_0^2(\mu_1 - \mu_2)^T D^{-1}(\mu_1 - \mu_2)}{\sigma_n} \xrightarrow{D} N(0, 1).
\]

Here we complete the proof. \( \square \)

## 2 Proof of Lemmas

### 2.1 Proof of Lemma 2

**Proof.**

\[
R(D^{-1/2}X_{1i}) = \frac{1}{n_1} \sum_{j=1}^{n_1} U(Y_{1i} - Y_{1j}) = \frac{1}{n_1} \sum_{j=1}^{n_1} (V_{1i} - V_{1j} + W_{ij})
\]

\[
= V_{1i} - \frac{1}{n_1} \sum_{j=1}^{n_1} V_{2j} + \frac{1}{n_1} \sum_{j=1}^{n_1} W_{ij}.
\]

Obviously, \( \frac{1}{n_1} \sum_{j=1}^{n_1} V_{2j} = O_P(\sqrt{n^{-1}tr(A^2)}) \) and \( \frac{1}{n_1} \sum_{j=1}^{n_1} W_{ij} = O_P(\sqrt{n^{-1}tr(A^2)}) \). Thus,

\[
E(R(D^{-1/2}X_{1i})R(D^{-1/2}X_{1i})^T)
\]

\[
= E \left( V_{1i}V_{1i}^T \right) (1 + o(1))
\]

\[
= E(\epsilon_{1i}^2)U(R^{1/2}(\epsilon_{1i} - \epsilon_{1j}) U(R^{1/2}(\epsilon_{1i} - \epsilon_{1j})^T | R^{1/2} \epsilon_{1i})) (1 + o(1))
\]

\[
= E \left( \frac{R^{1/2}u_{1i} | R^{1/2}u_{1i}^T R^{1/2}}{1 + u_{1i}^T(R - I_p)u_{1i}} \right) (1 + o(1)).
\]

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Thus, by the Cauchy inequality and Tyler expansion,

$$E \left( \text{diag} \left\{ E \left( R(D^{-1/2}X_{1i})R(D^{-1/2}X_{1i}^T) \right) - \tau_F p^{-1}I_p \right\} \right) \leq C_4 \{ E(u_i^T(R - I_p)u_i) E(\text{diag}\{R^{1/2}u_i^TR^{1/2}\} - \tau_F p^{-1}I_p) \}^{1/2} = O(p^{-1} \sqrt{\text{tr}(R^2) - p}) = o(n^{-1/2}).$$

by Condition (C3). The above equation define the functional equation for each component of $d_i = (d_{i1}, \cdots, d_{ip})$,

$$T_{ij}(F, d_{ij}) = o_p(n^{-1/2}), \quad (1)$$

where $F_i$ is the distribution function of $X_{ij}$, $i = 1, 2$. Similar to $\mathfrak{?}$, the linearisation of this equation produces

$$\sqrt{n_i}(\hat{d}_{ij} - d_{ij}) = -H_{ij}^{-1}(\sqrt{n_i}(T_{ij}(F_{ni}, d_{ij}) - T_{ij}(F, d_{ij}))) + o_p(1),$$

where $F_{ni}$ is the empirical distribution function of $X_{ij}$, $j = 1, \cdots, n_i$, $H_{ij}$ is the corresponding Hessian matrix of the functional defined in (1), and

$$T_i(F_{ni}, d_i) = \left( \text{vec} \left( \text{diag} \left( n_i^{-1} \sum_{j=1}^{n_i} R(D^{-1/2}X_{1i})R(D^{-1/2}X_{1i})^T - \tau_F p^{-1}I_p \right) \right) \right),$$

where $T_i(F_{ni}, d_i) = (T_{i1}(F_{ni}, d_{i1}), \cdots, T_{ip}(F_{ni}, d_{ip}))$ and $\text{vec}(B)$ means the vector of the diagonal matrix of $B$. For each variance estimator $d_{ij}$, we have

$$\sqrt{n_i}(\hat{d}_{ij} - d_{ij}) \overset{p}{\rightarrow} N(0, \zeta_{ij}^2),$$

where $\zeta_{ij}^2$ is the corresponding asymptotic variance. Define $\zeta_{\text{max}} = \max_{1 \leq i \leq 2, 1 \leq j \leq p} \zeta_{ij}$.

$$P \left( \max_{1 \leq j \leq p} (\hat{d}_{ij} - d_{ij}) > \sqrt{2} \zeta_{\text{max}} n_i^{-1/2} (\log p)^{1/2} \right)$$

$$\leq \sum_{j=1}^{p} P \left( \sqrt{n_i}(\hat{d}_{ij} - d_{ij}) > \sqrt{2} \zeta_{\text{max}} (\log p)^{1/2} \right)$$

$$= \sum_{i=1}^{p} \left( 1 - \Phi(\sqrt{2} \zeta_{\text{max}} \sigma_{ij}^{-1}(\log p)^{1/2}) \right) \leq p \left( 1 - \Phi((2 \log p)^{1/2}) \right)$$

$$\leq \frac{p}{\sqrt{4\pi \log p}} e^{-\log p} = (4\pi)^{-1/2}(\log p)^{-1/2} \rightarrow 0.$$
2.2 Proof of Lemma 3

Proof. 

\[ E(\varepsilon_i^T \varepsilon_i) = E((\varepsilon_i - \varepsilon_j)^T(\varepsilon_i - \varepsilon_k)) \]

\[ = E(E((\varepsilon_i - \varepsilon_j)^T(\varepsilon_i - \varepsilon_k) | \varepsilon_i)) \]

\[ = E(E(||\varepsilon_i - \varepsilon_j|| | \varepsilon_i)U(\varepsilon_i - \varepsilon_k) | \varepsilon_i)) \]

\[ = E(||\varepsilon_i - \varepsilon_j||^2)E(U(\varepsilon_i - \varepsilon_k) | \varepsilon_i)) \]

\[ = E(||\varepsilon_i - \varepsilon_j||^2)E(u_i^TU_i) = \tau \mu E(||\varepsilon_i - \varepsilon_j||^2) \]

In addition, \( E(||\varepsilon_i||^2) = 0.5E(||\varepsilon_i - \varepsilon_j||^2) \). Thus, we only need to show that

\[ \frac{E(||\varepsilon_i - \varepsilon_j||^2)}{E(||\varepsilon_i||^2)} \to 1. \]

Because \( \varepsilon_i \) has the elliptical distribution, \( \varepsilon_i - \varepsilon_j \) also has the elliptical distribution. Define the density function of \( ||\varepsilon_i - \varepsilon_j|| \) as \( f(t) = c_p t^{p-1}g(t) \) where \( c_p = \frac{2\pi^{p/2}}{\Gamma(p/2)} \). Thus,

\[ \frac{E(||\varepsilon_i - \varepsilon_j||^2)}{E(||\varepsilon_i - \varepsilon_j||^2)} = \frac{\int c_p t^{p-1}g(t)dt}{\int c_p t^{p+1}g(t)dt} \]

\[ = \frac{c_p^{2p+1}}{c_p^{p+2}} = \frac{\Gamma^2((p+1)/2)}{\Gamma(p/2)\Gamma((p+2)/2)} \]

By the Stirling’s formula,

\[ \lim_{x \to \infty} \frac{\Gamma(x+1)}{(x/e)^x(2\pi x)^{1/2}} = 1, \]

as \( p \to \infty \), we have

\[ \frac{c_p^{2p+1}}{c_p^{p+2}} \to \frac{(p-1)^{p-1}}{p^{p/2}(p-2)^{(p-2)/2}} = (1 - p^{-1})^{p/2}(1 + (p-2)^{-1})^{(p-2)/2} \to 1. \]

Here we complete the proof. \( \square \)

2.3 Proof of Lemma 4

Proof. By the Taylor’s expansion, we have

\[ U(D^{-1/2}_{(i,j,s,t)}(X_{1i} - X_{2s}))^TU(D^{-1/2}_{(i,j,s,t)}(X_{1j} - X_{2t})) \]

\[ = U(D^{-1/2}(X_{1i} - X_{2s}))^TU(D^{-1/2}(X_{1j} - X_{2t})) \]

\[ + U(D^{-1/2}(X_{1i} - X_{2s}))^T(I_p - U(D^{-1/2}(X_{1i} - X_{2s}))U(D^{-1/2}(X_{1i} - X_{2s}))^T) \]

\[ \times (\hat{D}^{-1/2}_{(i,j,s,t)}D^{1/2} - I_p) \]

\[ + U(D^{-1/2}(X_{1i} - X_{2s}))^T(I_p - U(D^{-1/2}(X_{1j} - X_{2t}))U(D^{-1/2}(X_{1j} - X_{2t}))^T) \]

\[ \times (\hat{D}^{-1/2}_{(i,j,s,t)}D^{1/2} - I_p) \]

\[ + o_p(\sigma_n). \]
Next, we only show that
\[
\frac{1}{n_1(n_1-1)} \frac{1}{n_2(n_2-1)} \sum_{i \neq j} \sum_{n_1} \sum_{n_2} \sum_{n_2} U(D^{-1/2}(X_{1j} - X_{2})^T(D^{-1/2} - I_p) \\
\times U(D^{-1/2}(X_{1i} - X_{2s})) = o_p(n).
\]
Similarly, by \(U(D^{-1/2}(X_{1i} - X_{2s})) = V_{1i} - V_{2s} + P_{is},\) we have
\[
\frac{1}{n_1(n_1-1)} \frac{1}{n_2(n_2-1)} \sum_{i \neq j} \sum_{n_1} \sum_{n_2} \sum_{n_2} U(D^{-1/2}(X_{1j} - X_{2})^T(D^{-1/2} - I_p) \\
\times U(D^{-1/2}(X_{1i} - X_{2s})) \\
= \frac{1}{n_1(n_1-1)} \sum_{i \neq j} \sum_{n_1} V_{1i}^T(D^{-1/2} - I_p) V_{1j} \\
+ \frac{1}{n_2(n_2-1)} \sum_{i \neq j} \sum_{n_2} V_{2i}^T(D^{-1/2} - I_p) V_{2j} \\
+ \frac{2}{n_1 n_2} \sum_{i=1} \sum_{j=1} n_1 \sum_{n_2} P_{is}^T(D^{-1/2} - I_p) V_{1j} \\
+ \frac{2}{n_1 n_2(n_1-1)} \sum_{i \neq j} \sum_{n_1} \sum_{n_2} P_{is}^T(D^{-1/2} - I_p) V_{2j} \\
+ \frac{1}{n_1(n_1-1)} \frac{1}{n_2(n_2-1)} \sum_{i \neq j} \sum_{n_1} \sum_{n_2} \sum_{n_2} P_{is}^T(D^{-1/2} - I_p) P_{jt} \\
= Q_1 + Q_2 + Q_3 + Q_4 + Q_5 + Q_6.
\]
Here we only proof \(E(Q_2^n) = o(\sigma_n^2).\) The other parts \(Q_2, Q_3\) are similar to \(Q_1.\) And the last three past are similar to the following proof of \(T_1.\) Taking the same arguments as Lemma 2
\[
E(Q_2^n) = \frac{1}{n_1(n_1-1)} E \left( V_{1i}^T(D^{-1/2} - I_p) V_{1j} \right)^2 \\
= \frac{1}{n_1(n_1-1)} E \left( u_{1i}^T R^{1/2}(D^{-1/2} - I_p) R^{1/2} u_{1j} \right)^2 (1 + o(1)) \\
= \frac{1}{n_1(n_1-1)} \text{tr}(A^2)(\log(p)/n)(1 + o(1)) = o(\sigma_n^2).
\]
Thus, under $H_0$, we have

$$T_n = \frac{1}{n_1(n_1-1)} \frac{1}{n_2(n_2-1)} \sum_{i \neq j} \sum_{s \neq t} (V_{1i} + V_{2s} + P_{is})^T (V_{1j} + V_{2t} + P_{jt}) + o_p(\sigma_n)$$

$$= \frac{1}{n_1(n_1-1)} \sum_{i \neq j} \sum_{s \neq t} V_{1i}^T V_{1j} + \frac{1}{n_1(n_1-1)} \sum_{i \neq j} \sum_{s \neq t} V_{2s}^T V_{2t}$$

$${} + 2 \frac{n_1}{n_1 n_2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} V_{1i}^T V_{2j} + 2 \frac{n_1}{n_1 n_2} \sum_{i \neq j} \sum_{s=1}^{n_2} P_{is}^T V_{1j}$$

$${} + \frac{2}{n_2 n_1(n_2-1)} \sum_{s \neq t} \sum_{i=1}^{n_1} P_{is}^T V_{2t} + \frac{1}{n_1(n_1-1)} \frac{1}{n_2(n_2-1)} \sum_{i \neq j} \sum_{s \neq t} \sum_{i \neq j} \sum_{s \neq t} P_{is}^T P_{jt} + o_p(\sigma_n)$$

$$= Z_n + T_1 + T_2 + T_3 + o_p(\sigma_n).$$

Next, we only show that $T_1 = o_p(\sigma_n)$. $T_2$ and $T_3$ are similar to $T_1$.

$$E(T_1^2) = E \left( \frac{2}{n_1 n_2(n_1-1)} \sum_{i \neq j} \sum_{s=1}^{n_2} P_{is}^T V_{1j} \right)^2$$

$$= \frac{4 n_1(n_1-1) n_2(n_2-1)}{n_1^2 n_2^2(n_1-1)^2} E(P_{is}^T V_{1j} P_{is}^T V_{1j})$$

$${} + \frac{4 n_1(n_1-1) n_2(n_2-1)}{n_1^2 n_2^2(n_1-1)^2} E(P_{is}^T V_{1j} P_{is}^T V_{1i})$$

$${} + \frac{4 n_1(n_1-1)(n_1-2) n_2}{n_1^2 n_2^2(n_1-1)^2} E(P_{is}^T V_{1j} P_{is}^T V_{1j})$$

$${} + \frac{4 n_1(n_1-1)(n_1-2) n_2}{n_1^2 n_2^2(n_1-1)^2} E(P_{is}^T V_{1j} P_{is}^T V_{1i})$$

$${} + \frac{4}{n_1 n_2(n_1-1)} E(P_{is}^T V_{1j} V_{1j}^T P_{is}) + \frac{4}{n_1 n_2(n_1-1)} E(P_{is}^T V_{1j} P_{is}^T V_{1i})$$

$$= \frac{4}{n_1 n_2(n_1-1)} E(P_{is}^T V_{1j} V_{1j}^T P_{is}) + \frac{4}{n_1 n_2(n_1-1)} E(P_{is}^T V_{1j} P_{is}^T V_{1i}),$$

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because

\[ E(P_{is}^{T}V_{1j}P_{is}^{T}V_{1j}) = E(P_{is}^{T}AP_{is}) \]
\[ = E((U(Y_{1i} - Y_{2s}) - V_{1i} - V_{2s})^T A U(Y_{1i} - Y_{2s}) - E(U(Y_{1i} - Y_{2s})^T A V_{1i})) \]
\[ = E(U(Y_{1i} - Y_{2s})^T A U(Y_{1i} - Y_{2s})) - E(U(Y_{1i} - Y_{2s})^T A V_{1i}) \]
\[ - E(U(Y_{1i} - Y_{2s})^T A V_{1i}) + E(V_{1i}^T A V_{1i}) \]
\[ = E(E(U(Y_{1i} - Y_{2s})^T A U(Y_{1i} - Y_{2s} | Y_{1i})) - E(E(U(Y_{1i} - Y_{2s})^T A V_{1i}) | Y_{1i}) \]
\[ - E(E(U(Y_{1i} - Y_{2s})^T A V_{1i}) | Y_{1i}) + tr(A^2) \]
\[ = E(V_{1i}^T A V_{1i}) - E(V_{1i}^T A V_{1i}) - E(V_{1i}^T A V_{1i}) + tr(A^2) \]
\[ = 0, \]
\[ E(P_{is}^{T}V_{1j}P_{ks}^{T}V_{1j}) = E(P_{is}^{T}AP_{ks}) = 0, \]
\[ E(P_{is}^{T}V_{1j}P_{js}^{T}V_{1i}) = tr(E(V_{1i}P_{js}^T))^2, \]
\[ E(P(Y_{1i})P_{is}^{T}) = E(V_{1i} U(Y_{1i} - Y_{2s}) - V_{1i} - V_{2s})) \]
\[ = E(V_{1i} U(Y_{1i} - Y_{2s})^T) - E(V_{1i} V_{1i}^T) \]
\[ = E(E(V_{1i} U(Y_{1i} - Y_{2s})^T | Y_{1i})) - E(V_{1i} V_{1i}^T) \]
\[ = E(V_{1i} V_{1i}^T) - E(V_{1i} V_{1i}^T) = 0. \]

Next, we will show that \( E(P_{is}^{T}V_{1j}P_{is}^{T}P_{is}) = E(P_{is}^{T}AP_{is}) = O(tr(A^2)) \). In fact, we only need to show that \( E(U(Y_{1i} - Y_{2j})^T A U(Y_{1i} - Y_{2j})) = O(tr(A^2)) \).

\[
U(Y_{1i} - Y_{2j})^T A U(Y_{1i} - Y_{2j}) = \]
\[
= \frac{Y_{1i} - Y_{2j}}{||Y_{1i} - Y_{2j}||} A U(Y_{1i} - Y_{2j}) \]
\[
= \frac{Y_{1i} - Y_{0} + Y_{0} - Y_{2j}}{||Y_{1i} - Y_{2j}||} A U(Y_{1i} - Y_{2j}) \]
\[
= \frac{Y_{1i} - Y_{0} A U(Y_{1i} - Y_{2j})}{||Y_{1i} - Y_{2j}||} + \frac{Y_{0} - Y_{2j} A U(Y_{1i} - Y_{2j})}{||Y_{1i} - Y_{2j}||} \]
\[
= U(Y_{1i} - Y_{0})^T A U(Y_{1i} - Y_{2j}) \frac{||Y_{1i} - Y_{0}||}{||Y_{1i} - Y_{2j}||} \]
\[
+ U(Y_{0} - Y_{2j})^T A U(Y_{1i} - Y_{2j}) \frac{||Y_{0} - Y_{2j}||}{||Y_{1i} - Y_{2j}||}. \]
Additionally,
\[
E \left( U(Y_{1i} - Y_0)A(UY_{1i} - Y_{2j}) \frac{\|Y_{1i} - Y_0\|}{\|Y_{1i} - Y_{2j}\|} \right) 
\]
\[
= E \left( U(\varepsilon_{1i} - \varepsilon_0)^T R^{1/2} A R^{1/2} U(\varepsilon_{1i} - \varepsilon_{2j}) \frac{\|\varepsilon_{1i} - \varepsilon_0\|}{\|\varepsilon_{1i} - \varepsilon_{2j}\|} \right) (1 + o(1)) 
\]
\[
= E \left( U(\varepsilon_{1i} - \varepsilon_0)^T R^{1/2} A R^{1/2} U(\varepsilon_{1i} - \varepsilon_{2j}) \frac{\|\varepsilon_{1i} - \varepsilon_0\|}{\|\varepsilon_{1i} - \varepsilon_{2j}\|} \varepsilon_{1i} \right) (1 + o(1)) 
\]
\[
= E \left( u_{1i}^T R^{1/2} A R^{1/2} u_{1i} \right) E \left( \frac{\|\varepsilon_{1i} - \varepsilon_0\|}{\|\varepsilon_{1i} - \varepsilon_{2j}\|} \right) (1 + o(1)) 
\]
\[
= O(\text{tr}^2(A^2)).
\]

Similarly, we can show another part is also $O(\text{tr}(A^2))$. Thus, $E(U(Y_{1i} - Y_{2j})^T A U(Y_{1i} - Y_{2j})) = O(\text{tr}(A^2))$. Then, we obtain that $T_n = Z_n + o_p(\sigma_n)$. 

\[ \square \]

2.4 Proof of Lemma [5]

**Proof.** Let $U_i = V_{1i}$ for $i = 1, \ldots, n_1$ and $U_{j+n_1} = V_{2j}$ for $j = 1, \ldots, n_2$ and for $i \neq j$, 
\[
\phi_{ij} = \begin{cases} 
  n_1^{-1}(n_1 - 1)^{-1} U_i^T U_j, & i, j \in \{1, 2, \ldots, n_1\}, \\
  -n_1^{-1} n_2^{-1} U_i^T U_j, & i \in \{1, 2, \ldots, n_1\}, j \in \{n_1 + 1, \ldots, n\}, \\
  n_2^{-1}(n_2 - 1)^{-1} U_i^T U_j, & i, j \in \{n_1 + 1, n_1 + 2, \ldots, n\}.
\end{cases}
\]

Define $Z_{nj} = \sum_{i=1}^{j-1} \phi_{ij}$ for $j = 2, 3, \ldots, n$, $S_{nm} = \sum_{j=1}^{m} Z_{nj}$ and $F_{nm} = \sigma\{U_1, U_2, \ldots, U_m\}$ which is the $\sigma$-algebra generated by $\{U_1, U_2, \ldots, U_m\}$. Now
\[
Z_n = 2 \sum_{j=2}^{n} Z_{nj}.
\]

We can verify that for each $n$, $\{S_{nm}, F_{nm}\}_{m=1}^{n}$ is the sequence of zero mean and a square integrable martingale. In order to prove the normality of $Z_n$, according to Hall and Heyde (1980), it suffices to show the following two results:

\[
\sum_{j=2}^{n} E[Z_{nj}^2 | F_{n,j-1}] \overset{p}{\rightarrow} \frac{1}{4}, \tag{2}
\]
\[
\sigma_n^{-2} \sum_{j=2}^{n} E[Z_{nj}^2 I(|Z_{nj}| > \epsilon \sigma_n) | F_{n,j-1}] \overset{p}{\rightarrow} 0. \tag{3}
\]
First, we proof result (2). Note that

\[
E[Z_{n j}^2 | \mathcal{F}_{n,j-1}] = \frac{1}{n_j^2(n_j - 1)^2} E \left\{ \left( \sum_{i=1}^{j-1} U_i^T U_j \right)^2 | \mathcal{F}_{n,j-1} \right\}
\]

\[
= \frac{1}{n_j^2(n_j - 1)^2} E \left\{ \sum_{i_1,i_2=1}^{j-1} U_{i_1}^T U_j U_j^T U_{i_2} | \mathcal{F}_{n,j-1} \right\}
\]

\[
= \frac{1}{n_j^2(n_j - 1)^2} \sum_{i_1,i_2=1}^{j-1} U_{i_1}^T E(U_j U_j^T | \mathcal{F}_{n,j-1}) U_{i_2}
\]

\[
= \frac{1}{n_j^2(n_j - 1)^2} \sum_{i_1,i_2=1}^{j-1} U_{i_1}^T A U_{i_2},
\]

where \( \tilde{n}_j = n_1 \), for \( j \in [1, n_1] \) and \( \tilde{n}_j = n_2 \), for \( j \in [n_1, n] \). Define \( \eta_n = \sum_{j=2}^{n} E[Z_{n j}^2 | \mathcal{F}_{n,j-1}] \).

By some tedious algebra, we can obtain that \( E(\eta_n) = \frac{1}{4} \sigma^2_n (1 + o(1)) \).

Now write \( E(\eta_n^2) \) as

\[
E(\eta_n^2) = E \left\{ \sum_{j=2}^{n} \frac{1}{n_j^2(n_j - 1)^2} \sum_{i_1,i_2=1}^{j-1} U_{i_1}^T A U_{i_2} \right\}^2
= 2E \left\{ \sum_{2 \leq j_1 < j_2} \frac{1}{n_{j_1}^2(n_{j_1} - 1)^2} \frac{1}{n_{j_2}^2(n_{j_2} - 1)^2} \sum_{i_1,i_2=1}^{j_1-1} \sum_{i_3,i_4=1}^{j_2-1} U_{i_1}^T A U_{i_2} U_{i_3}^T A U_{i_4} \right\}
\]

\[
+ E \left\{ \sum_{j=2}^{n} \frac{1}{n_j^4(n_j - 1)^4} \sum_{i_1,i_2=1}^{j-1} \sum_{i_3,i_4=1}^{j-1} U_{i_1}^T A U_{i_2} U_{i_3}^T A U_{i_4} \right\}
\]

\[
\approx L_1 + L_2.
\]

Consider the first part \( L_1 \).

\[
E \left\{ \sum_{2 \leq j_1 < j_2} \frac{1}{n_{j_1}^2(n_{j_1} - 1)^2} \frac{1}{n_{j_2}^2(n_{j_2} - 1)^2} \sum_{i_1,i_2=1}^{j_1-1} \sum_{i_3,i_4=1}^{j_2-1} U_{i_1}^T A U_{i_2} U_{i_3}^T A U_{i_4} \right\}
\]

\[
= E \left\{ \sum_{2 \leq j_1 < j_2} \frac{1}{n_{j_1}^2(n_{j_1} - 1)^2} \frac{1}{n_{j_2}^2(n_{j_2} - 1)^2} \sum_{i_1,i_2=1}^{j_1-1} U_{i_1}^T A U_{i_2} U_{i_1}^T A U_{i_2} \right\}
\]

\[
+ E \left\{ \sum_{2 \leq j_1 < j_2} \frac{1}{n_{j_1}^2(n_{j_1} - 1)^2} \frac{1}{n_{j_2}^2(n_{j_2} - 1)^2} \sum_{i_1,i_2=1}^{j_1-1} \sum_{i_2=1}^{j_2-1} U_{i_1}^T A U_{i_2} U_{i_2}^T A U_{i_2} \right\}
\]

\[
+ E \left\{ \sum_{2 \leq j_1 < j_2} \frac{1}{n_{j_1}^2(n_{j_1} - 1)^2} \frac{1}{n_{j_2}^2(n_{j_2} - 1)^2} \sum_{i_1,i_2=1}^{j_1-1} \sum_{i_1=1}^{j_2-1} U_{i_1}^T A U_{i_1} U_{i_2}^T A U_{i_2} \right\}
\]

\[
\approx L_{11} + L_{12} + L_{13}.
\]
Taking the same procedure as Lemma 5 and some tedious calculations, we can verify that
\( L_{11} = o(\sigma_n^4), \) \( L_{12} + L_{13} = E^2(\eta_n) \) and \( E(L_2^2) = o(\sigma_n^4). \) So, \( \text{var}(\eta_n) = E(\eta_n^2) - E^2(\eta_n) = o(\sigma_n^4). \) This completes the proof of (3).

Next, we prove result (3). First of all, we note that
\[
\sigma^{-2} \sum_{j=2}^{n} E[Z_{nj}^2 I(|Z_{nj}| > \epsilon \sigma_n)] |\mathcal{F}_{n,j-1}] \leq \sigma^{-4} \sum_{j=2}^{n} E[Z_{nj}^4 |\mathcal{F}_{n,j-1}].
\]
Accordingly, the assertion of this lemma is true if we can show
\[
E \left\{ \sum_{j=2}^{n} E[Z_{nj}^4 |\mathcal{F}_{n,j-1}] \right\} = o(\sigma_n^4).
\]
Notice that
\[
E \left\{ \sum_{j=2}^{n} E[Z_{nj}^4 |\mathcal{F}_{n,j-1}] \right\} = \sum_{j=2}^{n} E(Z_{nj}^4) = O(n^{-8}) \sum_{j=2}^{n} E \left( \sum_{i=1}^{j-1} \phi_{ij} \right)^4.
\]
Similar to Chen and Qin (2010), the last term can be decomposed as \( 3Q + P, \) where
\[
Q = O(n^{-8}) \sum_{j=2}^{n} \sum_{s \neq t} E(U_j^T U_s U_s^T U_j U_j^T U_t U_t^T U_t),
\]
\[
P = O(n^{-8}) \sum_{j=2}^{n} \sum_{s=1}^{j-1} E(U_j^T U_j)^4.
\]
Note that
\[
Q = O(n^{-8}) \sum_{j=2}^{n} \sum_{s \neq t} E(U_j^T U_s U_s^T U_j U_j^T U_t U_t^T U_t)
\]
\[
= O(n^{-8}) \sum_{j=2}^{n} \sum_{s \neq t} E(U_j^T A U_j U_j^T A U_j)
\]
\[
= O(n^{-8}) \sigma_n^4,
\]
where the last equality is followed by \( E((U_j^T A U_j)^2) = O(\text{tr}(A^2)). \) Here we will show it.
\[
V_{1i} = E(U(Y_{1i} - Y_{2j}) | Y_{1i}) = E(U(R^{1/2}(\epsilon_{1i} - \epsilon_{2j})) | R^{1/2} \epsilon_{1i})
\]
\[
= E(U(R^{1/2}(\epsilon_{1i} - \epsilon_{2j})) | \epsilon_{1i})
\]
\[
= E \left( \frac{R^{1/2}(\epsilon_{1i} - \epsilon_{2j})}{\|R^{1/2}(\epsilon_{1i} - \epsilon_{2j})\|} | \epsilon_{1i} \right).
\]
Because of \[ \| (R^{1/2} - I_p)(e_{1i} - e_{2j}) \|^2 = O(\text{tr}(R^{1/2} - I_p)^2) = o(n^{-1}p^2), \]
then
\[ \| R^{1/2}(e_{1i} - e_{2j}) \| = \| (e_{1i} - e_{2j}) + (R^{1/2} - I_p)(e_{1i} - e_{2j}) \| = \| e_{1i} - e_{2j} \| (1 + o_p(1)), \]
and \( V_{1i} = R^{1/2}u_{1i}(1 + o_p(1)). \) Thus
\[ E((U_j^T A U_j)^2) = E((u_{1i}^T R^{1/2} A R^{1/2} u_{1i})^2)(1 + o(1)) = O(\text{tr}^2(A^2)). \]

Accordingly, we can verify that \( Q = o(\sigma^4_n). \) In addition,
\[ P = O(n^{-8}) \sum_{j=2}^{n} \sum_{s=1}^{j-1} E(U_s^T U_j)^4 \]
\[ = O(n^{-8}) \left\{ \sum_{j=1}^{n_1} \sum_{s=1}^{j-1} E(U_s^T U_j)^4 + \sum_{j=n_1+1}^{n} \sum_{s=1}^{j-1} E(U_s^T U_j)^4 + \sum_{j=n_1+1}^{n} \sum_{s=n_1+1}^{j-1} E(U_s^T U_j)^4 \right\} \]
\[ \leq O(n^{-8})(P_1 + P_2 + P_3). \]

As the procedures for handling \( P_1, P_2, P_3 \) are similar, let us only consider \( P_2. \) By Lemma \([\text{1}]\) \( E((V_{1i}^T V_{2j})^4) = O(\text{tr}^2(A^2) + \text{tr}(A^4)), \) and then \( O(n^{-8}P_2) = o(\sigma^4_n). \) Similarly, \( O(n^{-8}P_1) = o(\sigma^4_n) \) and \( O(n^{-8}P_3) = o(\sigma^4_n). \) This completes the proof of (4). Thus, according to the martingale central limit theorem \([\text{Hall and Hyde, 1980}]\), we have
\[ \frac{Z_n}{\sqrt{\text{var}(Z_n)}} \xrightarrow{\mathcal{L}} N(0,1). \]

Obviously,
\[ \text{var}(Z_n) = \frac{2}{n_1(n_1-1)}E((V_{1i}^T V_{1j})^2) + \frac{2}{n_2(n_2-1)}E((V_{2i}^T V_{2j})^2) + \frac{4}{n_1 n_2}E((V_{1i}^T V_{2j})^2). \]
So \( \text{var}(Z_n) = \sigma^2_n(1 + o(1)). \) Then we complete the proof of this lemma. \( \Box \)

**References**

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