Shape in an atom of space: exploring quantum geometry phenomenology

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Abstract
A phenomenology for the deep spatial geometry of loop quantum gravity is introduced. In the context of a simple model of an atom of space, it is shown how purely combinatorial structures can affect observations. The angle operator is used to develop a model of angular corrections to local, continuum flat-space 3-geometries. The physical effects involve neither breaking of local Lorentz invariance nor Planck-scale suppression, but rather only rely on the combinatorics of $SU(2)$ recoupling. Bhabha scattering is discussed as an example of how the effects might be observationally accessible.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Quantum gravity phenomenology has developed into a broad field encompassing many possible effects arising from potential fundamental descriptions of spacetime. From cosmological perturbations to quantum decoherence to TeV scale black holes to particle astrophysics to violations of spacetime symmetries, many, but not all, of the effects arise from the addition of a Planck scale. When the scale is at the naïve Planck scale the effects are only observationally accessible with huge ‘lever arms’. For instance, when local Lorentz symmetry is violated the lever arm of bringing the Planck scale within the reach of observation is the magnificent sensitivity of particle physics in the effective field theory framework to the breaking spacetime symmetries [1–4]. Cosmological distances can act as a lever arm to raise effects from the additional scale into the realm of the observable, even outside the effective field theory framework [5, 6]. When the Planck scale is shifted due to the affect of additional physics from large extra dimensions [7] or a hidden gravitational matter sector [8], the lever arm is more general, lowering the effective four-dimensional Planck energy scale toward the natural scales of the standard model.
This paper introduces a model in which the lever arm is intrinsic to the structure representing the discrete geometry of an atom of spatial geometry. As such it is an example of a phenomenology arising not from breaking local Lorentz invariance but rather from the structure of a possible fundamental description of spacetime. The model discussed here is based on the kinematic states of loop quantum gravity (LQG).

In LQG an atom of spatial geometry is a single node of the labeled graph, specifically the spin network, representing a quantum 3-geometry. In the addition of angular momenta the different ways of combining fundamental representations determine the form of higher dimensional representations. Likewise the combinatorics of the network at a node determine geometric quantities such as angle [10], which is the basis for the model discussed here.

The model is based on the interplay between the angle spectrum and the combinatorics of the node. From dimensional analysis the behavior of the angle spectrum does not depend on the Planck scale. Rather, the highly non-uniform spectrum gives rise to an asymmetric distribution characterized by a dimensionless ‘shape parameter’. The combinatorics of the node suggests the definition of this shape parameter, which serves as an expansion parameter for the corrections to classical, flat-space continuum geometry. This semi-classical limit is also characterized by very large spins. So while the shape involves no physical scales, the high spin determines an effective length via the quantum 3-volume. This length determines a mesoscopic scale above the Planck scale where the effects of the ‘shape of space’ become manifest. A simple analysis of Bhabha scattering is used as an example of how the effects might be accessible to observation.

In the next section, the angle operator is introduced within the combinatorial framework of LQG. This framework was introduced by Zapata [9] and recently used as the kinematic setting for spin foam models in the review [16] (see also [15, 17]). This framework has the advantage that the resulting kinematical Hilbert space is separable and is free of the physically mysterious continuous moduli that label equivalence classes of diffeomorphism invariant states in the embedded framework. (The angle operator [10] in the embedded spin network context is reviewed in appendix A.) In section 3 the details of the model are developed. The example of combinatoric corrections in the context of Bhabha scattering is discussed briefly in section 4. Comments on the model are collected in the final section.

2. Angle operator

The angle operator, originally defined in [10], may be conveniently cast into the combinatorial framework of LQG [9]. The kinematics of this framework, relevant for spatial geometric operators, describes the space of spatial quantum 3-geometries. The state space, the combinatorial \( \mathcal{H} \), is a separable Hilbert space, defined as equivalence classes of a direct sum of Hilbert spaces \( \mathcal{H}_\Gamma \), each supported on a non-embedded, or abstract, (directed) graph \( \Gamma \). The gravitational field operator \( L_\Gamma \) is the generator of the left SU(2) action in \( \mathcal{H}_\Gamma \) and has an interpretation as the flux of the inverse triad across the dual ‘surface’ of the link \( l \).

Spin networks form a convenient basis on \( \mathcal{H}_\Gamma \); they are the eigenspace of spatial geometric observables. States in this spin network basis, \( |j_j v_n\rangle \), are labeled by quantum ‘numbers’ consisting of the abstract graph \( \Gamma \), the SU(2) irreducible representations \( j_j \) on the links \( l \) and

\[ \text{As it may be confusing I note that the ‘combinatorics’ mentioned in this paper arise in two different contexts. In the ‘combinatorial framework’, the combinatorics refers to an approach to the topology of space while in the ‘combinatorics of the node’ the combinatorics refers to the combinatorics of SU(2) representations.} \]

\[ \text{For } L \text{ links and } N \text{ nodes the graph Hilbert space is } \mathcal{H}_\Gamma = L^2(SU(2)^L / SU(2)^{N-1}) \text{ where the Haar measure is used.} \]

\[ \text{Classical subgraph structure. See [16] for details.} \]
intertwiner labels \( v_n \) for each node \( n \) of the graph. The intertwiner label \( v_n \) has, in turn, an orthonormal basis that may be labeled by a trivalent graph decomposition with a number of external links equal to the valence of the node \( v \), and a set of \( SU(2) \) irreducible representations on the internal links.

Non-embedded spin networks were first used by Penrose as a combinatorial foundation for Euclidean 3-space [13]. Penrose [13] and Moussouris [14] constructed proofs that demonstrated that the angles of three-dimensional space could be modeled by operators on spin network states. The kinematics of the combinatorial framework bring (this version of) loop quantum gravity into essentially the same framework used by Penrose and Moussouris\(^3\).

The angle operator is defined at a node. Links incident to \( n \) are partitioned into three sets \( C_1, C_2 \) and \( C_3 \). (One may visualize the partitioning as arising from three regions in the surface dual to the node, as represented in figure A1. However, the combinatorics only requires a tripartite partition.) Three gravitational field operators \( L_1^j, L_2^j \) and \( L_3^j \) are associated with these partitions. For instance, if there are \( s_l \) links \( l_m, m = 1, \ldots, s_l \), in the partition \( C_l \) then \( L_i^j = \sum_{m=1}^{s_l} (L_{lm})^j \). In terms of these field operators the quantum angle operator between dual surfaces corresponding to partitions \( C_1 \) and \( C_2 \) is

\[
\hat{\theta}_{(12)} := \arccos \frac{L_1^j L_2^j}{|L_1^j||L_2^j|}, \tag{1}
\]

in which \(|L| = \sqrt{L^2} \). Because the partitions are exhaustive and because of gauge invariance, \( \sum_{k=1}^{3} L_k^j = 0 \). The partitioning of links incident to \( n \) gives a preferred (class of) intertwiners \( v_n \). These are given by three, trivalent tree graphs that connect in a trivalent node, an ‘intertwiner core’\(^6\). It is convenient to label the links of the intertwiner core with irreducible representations \( j_k \) (and, later, with \( n_k = 2 j_k \)) and the basis of the intertwiner core by \(|j_1 j_2 j_3\rangle \) (later by \(|\vec{n} \rangle \)).

For the purposes of the angle spectrum, the remaining labels on the internal edges are not important. However, they do play a role in the phenomenology discussed in the next section.

Deriving the spectrum of the angle operator of equation (1) is a simple exercise in angular momentum algebra [10]

\[
\begin{align*}
\hat{\theta}_{(12)}|j_1 j_2 j_3\rangle &= \theta_{(12)}|j_1 j_2 j_3\rangle \\
\theta_{(12)} &= \arccos \left( \frac{j_3(j_3+1) - j_1(j_1+1) - j_2(j_2+1)}{2j_1(j_1+1)j_2(j_2+1)^{1/2}} \right). \tag{2}
\end{align*}
\]

As is clear from a glance at the spectrum shown in figure 1 there are two aspects of the continuum angular distribution that are difficult to model. First, small angles are sparse. Second, the distribution of values is asymmetric and weighted toward large angles. As discussed in [11, 12] the asymmetry persists even when the spins are large. The effects discussed in this paper are due to this asymmetric bias\(^4\).

The original idea of Penrose was to determine the angle via correlations between two disjoint sets of links [13]. In the combinatorial framework these sets are dual to faces in the simplicial complex. The closed dual surface of the node, topologically \( S^2 \), is partitioned into three regions, \( S_1, S_2 \) and \( S_3 \), such that the annular region \( S_2 \) separates \( S_1 \) and \( S_3 \), as shown in figure A1. (One may also use a definition of the three surfaces shown in figure A1(a).) From this picture it is clear that the angle defined above represents the zenith or polar angle \( \theta \) of spherical coordinates. As the partitions, or selection of regions \( S_k \), vary, the possible core intertwiner labels vary changing the spectrum of the angle. Although this picture is

\(^3\) The spin networks of [14] were all trivalent. While this is also the case in the LQG context if one includes the sub-graphs of the intertwiners, to model spatial geometric quantities like angle and volume in LQG it is critical that the abstract graphs \( \Gamma \) may contain higher valence nodes.

\(^4\) The striking fan-like structure in the spectrum is discussed in [11].
Figure 1. Angle operator spectrum for increasing flux at a node or ‘total spin’ of the vertex $n = 2 \sum_j j_k$. The complete spectrum is plotted for total spins from 3 to 100. With few eigenvalues at small angle and the non-uniform spacing, the spectral distribution differs strongly from the continuum distribution. Adapted from [11, 12].

convenient this construction is not necessary for the combinatorics. All that is required in the combinatorial setting is the partition of the incident links.

In sum, the angle operator may be simply defined in the combinatorial framework. The angle operator of equation (1) acts on nodes and the spectrum may be expressed in terms of the $SU(2)$ representations of the intertwiner core at a single node, determined by a partition of the links incident to the node. This angle operator is closely related to the combinatorial operator discussed in the works of Penrose [13]. As was done originally in [10] the angle operator may also be defined in terms of the electric flux variables and the usual embedded graphs of LQG. This is discussed in appendix A, along with some further comments on the definition of the angle operator in that framework.

The notation for the remainder of the paper is as follows. Twice the sum of the representations on the links incident to the node in partition $C_k$ is denoted by the ‘flux’ $s_k$ also denoted as $\vec{s}$. In the dual surface picture this is the flux of spin through the respective surfaces. In the literature this is also sometimes called ‘area’ (see, for instance, [18]). The quantities $n_k = 2j_k$ uniquely specify the intertwiner core that ‘collect’ the fluxes $s_k$ from each of the three partitions $C_k$, or dual surfaces. The relevant orthonormal states are $| n_1 n_2 n_3 \rangle$, denoted by $| \vec{n} \rangle$. The fluxes $s_k$ and core labels $n_k$ are distinct and satisfy $n_k \leq s_k$. In terms of labels $n_k$ the angle becomes

$$\theta(\vec{n}) = \arccos \left( \frac{n_1(n_3 + 2) - n_1(n_1 + 2) - n_2(n_2 + 2)}{2\sqrt{n_1(n_1 + 2)n_2(n_2 + 2)}} \right).$$

(3)
3. Combinatorial phenomenology

The semi-classical, or continuum, limit of the angle operator was numerically investigated in [11, 12]. To model an atom of 3-geometry we made the ansatz that the probability measure on the space of intertwiners was uniform, and that every possible set of labels on internal links of the node was equally likely. We further made a simplifying assumption that all incident links to the node were spin-$\frac{1}{2}$, i.e. ‘monochromatic’ and ‘simple’. The assumptions made the combinatorial problem quite tractable.

In the simple monochromatic case the dimension of the Hilbert space of the node with fixed flux $\vec{s}$ and fixed intertwiner core labels $\vec{n}$, $\text{dim} \mathcal{H}_{\vec{s}, \vec{n}}$, is simply the number of intertwiners with $\vec{s}$ and $\vec{n}$. This equals the product of the distinct ways of labeling the three branches of the intertwiner graph. For each branch $i$ the dimension of the intertwiner space for fixed flux $s_i$ and core label $n_i$ is equivalent to a well-known path-counting problem [11]. The result for one intertwiner branch $i$ is [11]

$$Q(s_i, n_i) = n_i + 1 \left( \frac{s_i + 1}{2} + 1 \right).$$

With the assumptions of uniform probability and simple monochromatic nodes, the combinatorics gives a probability distribution. Since each branch contributes a factor as in equation (4), $Q(\vec{s}, \vec{n}) := \prod_{i=1}^{3} Q(s_i, n_i)$. The probability distribution is then $p(\vec{s}, \vec{n}) = Q(\vec{s}, \vec{n})/|Q(\vec{s}, \vec{n})|$ where the norm $|Q(\vec{s}, \vec{n})| = \sum_{\vec{n}} Q(\vec{s}, \vec{n})$, i.e. the dimension of the invariant intertwiner with fluxes $\vec{s}$.

It was apparent in the numerical studies of [11] that the non-uniformity in the spectrum shifted the probability distribution $p(\vec{s}, \vec{n})$ away from the usual $\sin \theta$ distribution of angles in three-dimensional flat space. To recover this it was necessary to take large fluxes, corresponding to a very high valence node, and, in particular, $1 \ll s_j \ll s_3$, $j = 1, 2$. Fluxes $\vec{s}$ that satisfy these relations are called ‘semi-classical fluxes’.

There is another reason why the we might wish to consider nodes with large spin. Most physical processes we currently consider, such as scattering events, are ‘local’ on the scale of the theory being tested. But in terms of the quantum geometry the scales are very large, typically many orders of magnitude above the Planck scale. In the volume operator likely to be relevant for the combinatorial framework, the volume scales as $(s_T)^{3/2}$, where $s_T$ is the total flux, $s_T = \sum_{i=1}^{3} s_i$. The scaling with volume can be used to define an effective length $\ell = \sqrt{s_T} \ell_P$ and an effective energy $M = M_P / \sqrt{s_T}$. A surprising result in [11, 12] suggests—but does not predict—that to model the correct distribution of angles in 3-space, the total flux is $\sim 10^{32}$ giving an effective length scale of about $10^{-19}$ m, a perhaps not altogether hopeless scale.

These initial results suggest that states with semi-classical fluxes are a promising source for phenomenology. The remainder of the paper focuses on this model of the atom of quantum 3-geometries: simple monochromatic nodes with uniform probability distribution on the intertwiners and semi-classical fluxes. While it is expected that the simple monochromatic node will dominate the sum, relaxing this assumption will change the quantitative results reported here. The following analysis shows that the degree to which the semi-classical flux relations are satisfied determines the size of the combinatorial corrections investigated here.

5 See section 5 for a discussion on the volume on LQG.
6 It seems likely that the generalization of the methods of [18] can give the general case.
The combinatorics of the model can be solved analytically for semi-classical fluxes. For large flux \( s_i \) the normalized probability distribution of \( Q(s_i, n_i) \) is given by [11]

\[
Q(s_i, n_i) \simeq \frac{n_i + 1}{s_i + 1} \exp \left[ -\frac{n_i^2 + 2n_i}{2(s_i + 1)} \right] \simeq \frac{n_i}{s_i} \exp \left( -\frac{n_i^2}{2s_i} \right) =: P_s(n_i).
\] (5)

Interestingly, the distribution \( P_s(n_i) \) is the Rayleigh distribution for the distance \( n_i \) covered in \( 2s_i \) steps in an isotropic random walk with unit step size in two spatial dimensions. Since each branch of the intertwiner is independent, the distribution for the whole intertwiner is simply the product

\[
p_s(\hat{n}) \simeq \prod_{i=1}^{3} \frac{n_i}{s_i} \exp \left( -\frac{n_i^2}{2s_i} \right).
\] (6)

The distribution \( P_s(n_i) \) is peaked at \( \sim \sqrt{s_i} \) and has a width ('FWHM') of approximately \( 2\sqrt{s_i \ln(2)} \). For large \( s_i \) the likely values of \( n_i \) are also large and we can approximate the \( \hat{n} \) by continuous values. In this case \( \cos \theta \) becomes, from equation (3),

\[
\cos \theta \simeq \frac{n_1^2 - n_2^2 - n_3^2}{2n_1n_2} \quad \text{or} \quad \theta(\hat{n}) = \arccos \left( \frac{n_1^2 - n_2^2 - n_3^2}{2n_1n_2} \right).
\] (7)

To study the effects of the combinatorics it is useful to work with the exact, discrete quantum states before the continuum approximation. The states of the spatial atom are labeled by the full intertwiner \( v_n \). However, accessible measurements of the atom include 3-volume, (roughly) determined by the total flux, and angle, determined by the states \( |\hat{n}\rangle \) of the intertwiner core. In this model the fluxes \( s \) determine a mixed state

\[
\rho_s = \sum_{\hat{n}} p_s(\hat{n}) |\hat{n}\rangle \langle \hat{n}|,
\] (8)

where \( P_s[\hat{n}] \) is the projector on the orthonormal basis of the intertwiner core. The sum is over the admissible 3-tuple of integers \( \hat{n} \) such that \( n_i \leq s_i \). In the discrete case the projector is \( P_s[\hat{n}] = \theta(1)|\theta(1)\rangle \), as usual, where the orthonormal \( |\theta(1)\rangle = \sum_{\hat{n}} c_\theta(\hat{n}) |\hat{n}\rangle \). At a fixed angle the amplitudes \( c_\theta(\hat{n}) \) vanish except when \( \hat{n} \) gives \( \theta(1) \). Due to the symmetry of the angle operator, angles enjoy a degeneracy under the exchange of \( n_1 \) and \( n_2 \). It would be interesting to explore possible effects of the relative phases in \( c_\theta(\hat{n}) \), but they will play no role in the following.

The probability of finding the angle eigenvalue \( \theta(1) \) in the mixed state \( \rho_s \) is

\[
\text{Prob}(\theta = \theta(1); \rho_s) = \text{tr}(\rho_s P_{\theta(1)}) = \sum_{\hat{n}} p_s(\hat{n}) |\theta(1)\rangle \langle \theta(1)| \equiv p_s(\theta).
\] (9)

This procedure can be used to calculate \( p_s(\theta) \) in the continuum approximation.

In the continuum the mixed states for nodes with fixed semi-classical fluxes have the density matrix

\[
\hat{\rho}_s = \int d^3n P_s(\hat{n}) \hat{P}_{[\hat{n}]},
\] (10)

where \( \hat{P}_{[\hat{n}]} \) is the projector on the states \( |\hat{n}\rangle \). So for large fluxes and a value of the measured angle \( \theta \), now taking continuous values, within an interval \( \Delta \theta = (\theta - \delta \theta, \theta + \delta \theta) \) the geometric probability distribution is

\[
\text{Prob}(\theta \in \Delta \theta; \hat{\rho}_s) = \text{tr}(\hat{\rho}_s \hat{E}_{\Delta \theta}) = \int d^3n P_s(\hat{n}) |\theta\rangle \langle \theta| \hat{E}_{\Delta \theta} |\theta\rangle.
\] (11)

where \( \hat{E}_{\Delta \theta} \) is the projector onto the interval \( \Delta \theta \). Geometrically it projects the state onto a (thickened) surface in \( \hat{n} \)-space given by \( \theta(\hat{n}) \in \Delta \theta \). Taking the limit \( \delta \theta \to 0 \) gives, heuristically, the geometric probability distribution

\[
P_s(\theta) := \int d^3n p_s(\hat{n}) |c_\theta(\hat{n})|^2 \delta \left( \theta - \theta(\hat{n}) \right).
\] (12)
This is the continuum approximation to equation (9). The normalization of the continuum approximation $|\theta\rangle$ states is determined by the area of the surface $\theta = \theta(\vec{n})$. This gives $|c_{\theta}(\vec{n})|^2 = |c_{\theta}(\vec{s})|^2$ and so $|c_{\theta}(\vec{n})|^2$ becomes an overall factor in the above integration.

The integration of equation (12) is straightforward and done in appendix B. The key step in the calculation is the identification of the ‘shape parameter’ $\epsilon = \sqrt{s_1 s_2}/s_3$ that measures the asymmetry in the distribution of angles. Small for semi-classical fluxes, $\epsilon$ is the parameter used for the expansion of the combinatorial corrections.

The result gives a modified distribution of polar angles $\theta$ given by $\rho_{\epsilon}(\theta) = P_{\epsilon}(\theta)/N$. As shown in equation (B.6) the normalization $N$ is determined by the requirement of recovering the continuum distribution in the limit of vanishing $\epsilon$. The resulting distribution or measure, when expressed in terms of Legendre polynomials and to $O(\epsilon^3)$, is from equation (B.7):

$$\rho_{\epsilon}(\theta) = \sin(\theta) Q_{\epsilon}(\theta) \simeq \sin(\theta) \left( 1 - \frac{8}{\pi} P_1(\cos(\theta))\epsilon + \frac{3}{2} P_2(\cos(\theta))\epsilon^2 \right). \quad (13)$$

The affect of the modified distribution of polar angles is that the ‘shape’ of space is altered by the combinatorics of the vertex; the local angular geometry differs from flat Euclidean 3-space. For instance, the expectation value of an angular quantity $f(\theta)$ in the mixed state $\rho_{\epsilon}$ is corrected

$$\langle f(\hat{\theta})\rangle_{\epsilon} = \int d\theta \text{ tr}(\hat{\rho}_{\epsilon} \hat{E}_{\Delta\hat{\theta}}) = \int d\theta f(\theta) \rho_{\epsilon}(\theta). \quad (14)$$

The distribution (13) reproduces the usual distribution of angles in the limit of vanishing shape parameter. This is an analytic expression of what was found numerically in [11] and is a manifestation of the spin geometry theorem. The $\rho_{\epsilon}(\theta)$ distribution is compared to the usual one in figure 2. The angular corrections are shown in figure 3.

In the semi-classical flux limit, the values of the fluxes $\vec{s}$ enter into the distribution only through the shape parameter. One can average over semi-classical fluxes, and thus $\epsilon$, which effectively determines an average shape parameter $\epsilon$.

As a result of the angle spectrum and the uniform probability measure on the intertwiner space, combinatorial effects of the toy model are parameterized by a single dimensionless shape parameter $\epsilon = \sqrt{s_1 s_2}/s_3$. While these effects would be in principle observable at any
flux, the results here are valid for the semi-classical flux, \( 1 \ll s_j \ll s_3 \) for \( j = 1, 2 \). In this model the total flux \( s_T \) determines the 3-volume of the spatial atom and thus an effective mesoscopic length scale, \( \ell = \sqrt{s_T \ell_P} \), greater than the fundamental discreteness scale of \( \ell_P \).

So while the shape parameter \( \epsilon \) is free of the Planck scale, the effective length scale, determined by total flux \( s \), is tied to the discreteness scale of the theory. The mesoscopic scale corresponds to an energy scale of \( M = 1/\ell \).

4. Example: Scattering

If the scale \( \ell \) of the spatial atom is large enough then the underlying geometry would be accessible to observations of particle scattering. To see how the combinatorial effects might be manifest I will briefly discuss combinatoric corrections to Bhabha scattering. This process is convenient because the \( e^+ e^- \) scattering process involves ‘point-like’ fundamental particles and for the practical reason that data are readily available [20]. This serves as an example of possible combinatorial corrections and will not yield constraints on the model parameters. In the experiment reported on in [20] the center-of-mass energy was 29 GeV, corresponding to a rough length scale of \( 10^{-17} \) m in the center-of-mass frame. This length scale corresponds to a flux of roughly \( 10^{36} \) so combinatorial corrections should be negligible. Nevertheless the data serve as a simple example of how \( \epsilon \) might be constrained using a more complete analysis.

The pure QED differential cross section for the process at lowest order is [20]

\[
\alpha^2 \left( \frac{3 + \cos^2 \theta}{4 \left(1 - \cos \theta\right)^2} \right)
\]

where \( s \) is the square of the center of mass energy. The differential cross section at 29 GeV is affected by electroweak effects. To lowest order the effect is roughly to reduce the differential cross section by 1–2% [20]. As discussed below the combinatorial corrections mimic this correction.

In this kinematic model the observation of the particle shower, and subsequent reconstruction of the processes, is a measurement of angle. There are (at least) two effects from the discrete local geometry; both the angular distribution and the averaging over angles are modified on small scales. The former is dominant.
The probability distribution of angles effectively alters the local angular geometry and leads to an angle-dependent rescaling of the cross section. The scattering data are binned in terms of the solid angle. The number of events $N_i$ counted in an interval of $\theta$ is proportional to the differential scattering cross section

$$ s \left( \frac{d\sigma}{d\Omega} \right) \propto \frac{N_i}{\Delta \Omega_i}. \quad (16) $$

To account for the asymmetry in the combinatorics, or equivalently the modification of the local angular geometry of space, the angular normalization must be adjusted. The solid angle is modified as $d\Omega \rightarrow \rho_i(\theta) d\theta d\phi \equiv Q_i(\theta) d\Omega$. This is the first and dominant effect.

The second effect arises in averaging an angular quantity $f(\theta)$ such as the differential cross section. The angle is only measured to some finite precision so the quantities are averaged over an interval $\Delta \theta = (\theta_0 - \delta\theta, \theta_0 + \delta\theta)$:

$$ f(\theta) = \frac{\int_{\Delta \theta} \rho_i(\theta) f(\theta) d\theta}{\int_{\Delta \theta} \rho_i(\theta) d\theta}. \quad (17) $$

Expanding the function gives a weighted Taylor series for the average

$$ f(\theta) \simeq f(\theta) + f'(\theta) w_1(\theta, \delta\theta, \epsilon) + \frac{1}{2} f''(\theta) w_2(\theta, \delta\theta, \epsilon), \quad (18) $$

where the weights are given in appendix B (B.9). As the leading corrections are $O(\delta\theta^2)$ or $O(\delta\theta^2 \epsilon)$, the effects are negligible for an experiment in which $\delta\theta$ is about 0.002 rad. Thus, the comparison will be only from effects arising from the asymmetry in the local geometry arising from the combinatorics of the angle operator.

Short-distance modifications to QED may be usefully expressed in the Drell parameterization [20, 21]:

$$ \left( \frac{d\sigma}{d\Omega} \right) / \left( \frac{d\sigma}{d\Omega} \right)_{\text{QED}} = 1 \mp \left( \frac{3s}{\Lambda_{\perp}^2} \right) \left( \frac{\sin^2 \theta}{3 + \cos^2 \theta} \right). \quad (19) $$

These modifications correspond to a short-range potential added to the Coulomb potential. The local, discrete geometry manifests itself through the combinatorial corrections to the local geometry. The shape corrections would be evident above the energy scale $M$ and give an angle-dependent scaling of the cross section

$$ \frac{d\sigma}{d\Omega} \rightarrow \frac{d\sigma}{d\Omega} Q^{-1}(\theta) = \frac{d\sigma}{d\Omega} \left( 1 + \frac{8}{\pi} \cos(\theta) \epsilon + \cdots \right). \quad (20) $$

The short distance, shape-corrected cross section has a correction of the form

$$ \left( \frac{d\sigma}{d\Omega} \right) / \left( \frac{d\sigma}{d\Omega} \right)_{\text{QED}} = 1 \mp \left( \frac{3s}{\Lambda_{\perp}^2} \right) \left( \frac{\sin^2 \theta}{3 + \cos^2 \theta} \right) \left( 1 + \frac{8}{\pi} \cos(\theta) \epsilon + \cdots \right). \quad (21) $$

It remains to be seen whether a full derivation along the lines of [21] that incorporates shape corrections would yield this form of the correction. Nevertheless, this example serves to show that scattering processes could constrain the model parameters.

The comparison between the model and the data is shown in figure 4. The data from [20], the pure QED prediction at lowest order, and the short-distance, shape-corrected prediction ($\epsilon = 0.005$) are plotted. The second plot contains the differential cross section normalized by the electroweak differential cross section at lowest order, as reported in [20]. The pure QED result is shown and it clearly deviates from the data and the electroweak result. Adding in the shape-correction effectively adds positive tilt to the results. The top and bottom curves show the approximate 95% confidence limits for the QED cutoff parameters $\Lambda_{\perp}$ [20], with $\epsilon = 0.005$ shape corrections. The effect is to tilt the Drell parameterization
Figure 4. The $e^+e^-$ scattering differential cross section $d\sigma/d\Omega$ as a function of $\cos \theta$ at 29 GeV [20]. (a) The pure QED cross section, the shape-corrected cross section, with $\epsilon = 0.005$, and the data [20] are plotted. The deviation between the two curves is slight and the curves agree with the data. (b) The ratio of the experimentally determined differential cross section and the predicted electroweak cross section at lowest order clearly shows the small deviations between the data [20] (filled squares with error bars) and the electroweak cross section. The pure QED prediction (empty squares) also deviates from the lowest order electroweak result [20]. The shape-corrected pure QED prediction (solid triangles) more closely matches the electroweak result. The positive tilt in the shape-corrected Drell parameterization (top and bottom curves) is due to the correction for $\epsilon = 0.005$. The pair of curves is due to the two energy scales of the Drell parameterization [20], the $\Lambda_{\pm}$ of equation (21).
As reported in [20] the scale of $\Lambda_1$ indicates point-like scattering down to a length scale of approximately $10^{-18}$ m. The shape correction reduces power at small angles and increases power at large angles, as is clear it should from the distribution of angles shown in figure 2.

This example exhibits the potential accessibility of the shape corrections to observational tests. The shape correction would appear as the tilt shown in figure 4 that would seem to be an additional systematic error. Any quantitative constraints on $\epsilon$; however, require a more complete analysis of the corrections, such as the one starting with the QED interaction.

5. Discussion

This paper introduces a new quantum gravity phenomenology that explores effects arising from combinatorial structures in the deep spatial quantum geometry of LQG. The example model relies on the combinatorics of a single atom of spatial geometry, the spin network node. This model shows that potentially observable effects of quantum geometry need not be tied to (obvious) violations of local Lorentz symmetry and that a scale above the fundamental scale of the theory can arise out of combinatorial effects. This is demonstrated using the angle operator and the combinatorics at the node.

The simple model is defined by three assumptions. First, the spin network node is simple and monochromatic; all the links incident to the node are in the representation $j = \frac{1}{2}$. Second, the probability measure on the intertwiners is uniform; for fixed flux $\vec{s}$, all possible intertwiners in the basis of the angle operator are equally likely. Then, if the angle spectrum is to resemble the angles of continuum, flat 3-geometry, the fluxes must be very large, resulting in an effectively continuous set of intertwiner core labels $\vec{n}$, which determine the angle $\theta$. Third, a measurement of angle, when it may be traced to a suitably small scale, is a measurement of the underlying quantum geometry at that scale. This is not so radical an assumption for it amounts to the observation that the scalar product of two vectors $\vec{u} \cdot \vec{v} = h_{ij} u^i v^j$ depends on the effective local geometry when the physical process determining the scalar product is sufficiently localized.

To the extent that the spatial geometry is described by the assumptions of this model, the model predicts modifications to microscopic angular geometry. A key step lies in identifying the small parameter depending on the state of the atom of geometry in the spin network model. This asymmetry or shape parameter, $\epsilon = \sqrt{s_1 s_2 / s_3}$, is specific to the model and is a measure of the asymmetry of the angular flux (or ‘area’) relative to the background flux (or ‘area’) of the surfaces $S_k$ in the dual complex of the node. In the mixed state given by (8) the probability distribution of polar angles is modified. The usual flat-space $\sin \theta$ distribution is recovered for small values of the parameter $\epsilon$ and large fluxes.

By analyzing angular correlation data constraints can be placed on the shape parameter. Scattering of ‘point-like’ particles such as in Bhabha scattering could place constraints on the shape parameter $\epsilon$ at the scale $M$ set by the center of mass energy. The analysis in section 4 is too simplistic to reach definite constraints, but it does show that such effects can be potentially constrained using high energy scattering data. Constraining the deviations from the usual cross sections would specify properties of a ‘generic atom of space’ via constraints on the parameters. Note that the effects do not arise from fluctuations of geometry. In this model the combinatorics of space is fixed; the excitations of the geometry only occur at much higher energy scales.

It might seem that a discrete geometry would violate energy–momentum conservation of particles but it does not obviously do so. Even if the granularity of the angle operator were taken into account, say by taking much smaller fluxes, the combinatorics do not suggest a
breakdown in energy–momentum conservation since the effects result only in the modification of the distribution of angles, not the loss of angular correlation of outgoing particles. In the context of scattering in the lab frame for instance, an angle of $\pi$ between the products of a $2 \to 2$ scattering event is possible. Of course, since the distribution here is derived from the kinematics of LQG, i.e. from discrete spatial geometry, it is not able to show that the geometric state does not break Lorentz symmetry in this framework. For more discussion on the issue of Lorentz symmetry in LQG, see [22].

There are two obvious developments needed before constraints can be placed on this model of an atom of space. First, although the monochromatic assumption is perhaps not too restrictive since these labels are likely generic, it seems possible to generalize the counting arguments of [18] to include the general case of arbitrary spin. The continuum approximation used could also be checked numerically in the exact, discrete model.

Second, matter couplings should be modeled. One way to do this is through local metric corrections to the QED interaction by smoothing out the electron current $\bar{\psi}(x)\gamma_\mu \psi(x)$ and the photon field $A_\mu$ along the lines of [21]. One choice is

$$L'\delta(x) = -e \int d^4 x \bar{\psi}(x)\gamma_\mu \psi(x) g^{\mu\nu}_{\text{eff}}(x) A_\nu(z) F_{\delta((x-z)^2)}.$$  (22)

By expressing the effective metric in terms of combinatoric corrections the expected form of the cross sections could be determined.

Finally, three comments are given. First, the current model is built on assumptions about the atom of 3-geometry. First, the dependence of effective scale on total flux is via the volume operator. However our knowledge of the spatial volume operator in LQG remains incomplete. (For a recent work see [25–29].) There are two volume operator definitions, the Rovelli–Smolin (RS) volume [23, 24] and the Ashtekar–Lewandowski (AL) volume [19].

The key difference in the analysis of the spectrum is the treatment of embedding information. The RS operator does not depend on the embedding of the node. The AL volume has an embedding-dependent sign factor which turns out to strongly affect both the spectrum and the complexity of the analysis. For instance, one interesting result is that the AL volume has no non-vanishing minimum eigenvalue [27]. Due to the embedding information, the spectrum of the AL volume is not known for high valence vertices. In [28] the AL volume spectrum is investigated up to valence 7 and shows that large volume does not scale with maximum spin. As the role of the embedding information in LQG kinematics is still under debate there are a variety of perspectives on the volume operator including the consistency of the definition for nodes of different valence [25, 26]. In this paper I used the scaling property of the RS volume, that the largest eigenvalue scales as $(s T)^{3/2}$ for volumes large compared to the Planck scale [11].

Second, the physical determination of angle may occur over a larger subgraph of the network. When an angular measurement is taken, such as in the context of a scattering event, it is not clear that it is possible to distinguish a fundamental spin network from a coarse-grained or effective spin network. If that is the case then the fundamental graph could be a lower valence graph and the coarse-grained sub-graph would be a high valence node. As the effective length scale of the measurement was increased the graining would become more coarse, the total flux would increase and the averaged shape parameter would tend to zero. Scattering (or other) data give limits on the shape as a function of scale. If the measurement process inherently involved a coarse-graining then the study would be one of a ‘molecule’ of quantum geometry rather than an ‘atom’.

Third, one might suspect that given the large fluxes, the distribution on the space of intertwiner cores $\vec{n}$ would be purely ‘statistical’ in that it should be given by the distribution of points $\vec{n}$ determined from the sum over unit vectors with random orientations and fixed
length. Such a distribution may be seen to be equivalent to a random walk in 3-space. The Rayleigh distribution or a ‘radial’ distance $|\vec{n}|$ covered in $r$ steps of equal length in three spatial dimensions is

\[ P_r^{(3)}(\vec{n}) = \frac{n^2}{r^{3/2}} e^{-\bar{n}^2/r} \neq p_r(\vec{n}). \]

Comparing this distribution to the one used in the analysis, equation (6), one may see that it is not equivalent. As the expressions differ in form, due to different effective spatial dimension, we can see from this that the resulting combinatorial corrections are not ‘statistical’.

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Appendix A. Angle operator in the embedded framework

In the embedded spin network framework the angle operator is defined using a partition of a closed surface around a node. Using the surfaces $S_1, S_2$ shown in figure A1, the flux variables $E_i^j = \int_S d^2\sigma_n a_i^{\mu} E_{\mu}^{\nu}$ and the area operator of the surfaces $A_S$, the angle operator is defined as

\[ \theta_{(12)}^{(12)} := \arccos \frac{E_i^j \cdot E_{k}^{l}}{A_{S_1} A_{S_2}}. \] (A.1)

All these operators commute. Since the surfaces depend on the graph the operator is explicitly graph dependent. The spectrum is the same as in equation (2) if all the links are oriented in the same direction. For arbitrary orientations the operator may be written as

\[ \frac{L^2_{(1+)}}{2\sqrt{L_{(1)^2}^2}} \left( \frac{L^2_{(1)}}{2\sqrt{L_{(1)^2}^2}} \right) \] (A.2)

in which $1+, 1- (2+, 2-)$ label the oriented links oriented outward or inward through the surfaces $S_1 (S_2)$, respectively.

It is clear from the construction that the visualization in terms of the surfaces is heuristic. While continuum angles are well approximated by fixed semi-classical fluxes, the picture of figure A1(b) suggests that by varying over the regions $S_i$ we could obtain a distribution that is peaked on the appropriate continuum angle $\theta$. But this is not the case. The continuum angular distribution is obtained at fixed semi-classical fluxes $\vec{s}$. Of course, in a model without the uniform probability assumption the situation would be different.

The additional definitions of [10] for the scalar product and cosine operators are based on intersecting surfaces. While these operator definitions are not graph dependent, the operators have ordering ambiguities.

Appendix B. Integration of the distribution $P_s(\theta)$

For large $s$ the limits on the $n_i$ integrations $(1, s_i)$ may be extended to $(0, \infty)$; the error is $O(1/s_i)$. Re-expressing the delta function in terms of $n_3$ the distribution defined in equation (12) becomes

\[ P_s(\theta) = \int_0^\infty d^3n \frac{\delta(n_3 - n_3^*)}{\delta g(\vec{n}, \theta)/\delta n_3} \prod_{i=1}^3 \frac{n_i}{s_i} \exp \left( -\frac{n_i^2}{2s_i} \right), \] (B.1)
Figure A1. The three regions $S_i$ in the surface dual to the node. (a) The symmetric identification of surfaces $S_1$ and $S_2$ as done in [10]. (b) The polar angle identification of surfaces, with an annular region $S_2$.

where $g(n, \theta) := \theta - \arccos \left( \frac{n_1^2 - n_2^2}{2n_1n_2} \right)$ and $n_3^2 = \sqrt{n_1^2 + n_2^2 + 2xn_1n_2}$ are the roots of $g$, with the usual definition $x = \cos \theta$. Performing the trivial $n_3$ integration gives

$$P(\vec{s}(\theta)) = \sin \theta \left[ \frac{1}{s_1s_2s_3} \exp \left( - \left( \frac{n_1^2}{2s_1} + \frac{n_2^2}{2s_2} + \frac{n_1n_2x}{s_3} \right) \right) \right]$$

(B.2)

in which $s_i := \frac{s_i}{(1+s_i/s_3)} = s_i\delta_i$, $i = 1, 2$. For the moment I set $\delta_i = 1$ but will comment on these azimuthal asymmetry factors shortly. The next integration is a straightforward quadratic

$$P(\vec{s}(\theta)) = \sin \theta \left[ \frac{1}{s_1s_2s_3} \exp \left( - \left( \frac{n_1^2}{2s_1} \frac{n_2^2}{2s_2} \frac{n_1n_2x}{s_3} \right) \right) \right]$$

(B.3)

where $\Phi$ is the complementary error function. I have introduced the shape parameter $\epsilon := \sqrt{s_1s_2/s_3}$. While the first term of the integrand is again a simple quadratic integration, the second is a bit more involved but still is quadratic. The result is ($x = \cos \theta$)

$$P(\vec{s}(\theta)) = \sin \theta \left[ \epsilon \arccot(\epsilon x\sqrt{1 - \epsilon^2x^2})(1 + 2\epsilon^2x^2) - 3\epsilon x\sqrt{1 - \epsilon^2x^2} - 29\epsilon^3x^4 + 75\epsilon^4x^4 + O(\epsilon^5) \right].$$

(B.4)

At fourth order this expands to

$$P(\vec{s}(\theta)) = \sin \theta \left[ \frac{\pi}{2} - 4\epsilon x + \frac{9\pi}{4} \epsilon^2x^2 - \frac{29}{3} \epsilon^3x^3 + \frac{75\pi}{16} \epsilon^4x^4 + O(\epsilon^5) \right].$$

(B.5)

The actual distribution of angles space $\rho(\theta) := N P(\vec{s}(\theta))$ must be normalized such that the distribution recovers the usual $4\pi$ solid angle of three-dimensional spatial geometry in the limit of vanishing $\epsilon$. Hence, the norm $N$ is fixed by

$$2 = N \int_0^\pi P(\vec{s}(\theta)) d\theta.$$
Using the resulting norm and rewriting in terms of Legendre polynomials, one finds
\[
\rho_\epsilon(\theta) = \sin \theta \left[ 1 - \frac{8}{\pi} P_1(\cos \theta) \epsilon + \frac{3}{2} P_2(\cos \theta) \epsilon^2 
- \frac{2}{5\pi} \left( P_1(\cos \theta) - \frac{58}{3} P_3(\cos \theta) \right) \epsilon^3 + O(\epsilon^4) \right].
\] (B.7)

This is the distribution used in the body of the paper.

Retaining the azimuthal asymmetry factors \(\delta_i\) introduced above it is still possible to integrate the distribution \(P_\mathbf{s}(\theta)\) exactly with the result
\[
P_\mathbf{s}(\theta) = N \epsilon \sin \theta \left\{ \left( \delta_1 \delta_2 \right)^{3/2} (1 - \delta_1 \delta_2 \epsilon^2 x^2)^{-1/2} \left( 1 - \epsilon^2 x^2 \delta_1 (1 - 3 + \delta_2) \right) \arctan \left( \frac{1 - \delta_1 \delta_2 \epsilon^2 x^2}{\delta_1 \delta_2 x} \right) 
- 3 \epsilon x \delta_1^2 \delta_2^2 \left( 1 - \frac{5}{3} \epsilon^2 x^2 \delta_1 (-1 + \delta_2) + \frac{2}{3} \epsilon^4 x^4 \delta_1^2 \delta_2 (-1 + \delta_2) \right) \right\} (1 - \delta_1 \delta_2 \epsilon^2 x^2)^{-2}.
\] (B.8)

However upon expanding in \(\epsilon\) the azimuthal asymmetry factors cancel and the result is the same as equation (B.7), as might be expected given the symmetry of the angle operator; there is no parameterization of the azimuthal angle.

The weights for the averages in equation (18) are
\[
w_1(\theta_0, \delta \theta, \epsilon) = \frac{1}{(\Delta \theta)^3} \int_{\Delta \theta} \rho_\epsilon(\theta) (\theta - \theta_0) \, d\theta 
\approx \frac{\delta \theta^2}{3} \cot(\theta_0) + \delta \theta \epsilon \left( \frac{4 \sin(2 \theta_0) \cot(\theta_0) \csc(\theta_0)}{3\pi} - \frac{8 \cos(2 \theta_0) \csc(\theta_0)}{3\pi} \right).
\]
w_2(\theta_0, \delta \theta, \epsilon) = \frac{1}{(\Delta \theta)^3} \int_{\Delta \theta} \rho_\epsilon(\theta) (\theta - \theta_0)^2 \, d\theta 
\approx \frac{\delta \theta^2}{3}. \quad (B.9)

References

[1] Jacobson T, Liberati S and Mattingly D 2001 arXiv:hep-ph/0110094
[2] Konopka T and Major S 2002 New J. Phys. 4 57 (arXiv:hep-ph/0201184)
[3] Jacobson T, Liberati S and Mattingly D 2006 Ann. Phys. 321 150–96 (arXiv:astro-ph/0505267v2)
[4] Liberati S and Maccione L 2009 Annu. Rev. Nucl. Part. Sci. 59 245–67 (arXiv:0906.0681)
[5] Amelino-Camilia G et al 1998 Nature 393 763
[6] Fermi LAT and GBM Collaborations 2009 Science 323 5922 (http://www.sciencemag.org/cgi/content/abstract/1169101)
[7] Abd Al Hamed N, Dimopoulos S and Dvali G 1998 Phys. Lett. B 429 263 (arXiv:hep-ph/9803315)
[8] Dvali G, Gabadadze G, Kolanovic M and Nitti F 2002 Phys. Rev. D 65 024031 (arXiv:hep-th/0106058)
[9] Dvali G 2007 arXiv:0706.2050
Dvali G and Redi M 2008 Phys. Rev. D 77 045027 (arXiv:0710.4344)
[10] Major S 1999 Class. Quantum Grav. 16 3859 (arXiv:gr-qc/9905019)
[11] Major S 2001 arXiv:gr-qc/0101032
[12] Major S and Seifert M 2002 Class. Quantum Grav. 19 2211–222 (arXiv:gr-qc/0108047)
[13] Penrose R 1971 Angular momentum: an approach to combinatorial space-time *Quantum Theory and Beyond* ed T Bastin (Cambridge: Cambridge University Press)
Penrose R 1979 Combinatorial quantum theory and quantized directions *Advances in Twistor Theory (Research Notes in Mathematics* vol 37) ed L P Houghton and R S Ward (San Francisco, CA: Pitman) pp 301–7
Penrose Roger 1971 *Combinatorial Mathematics and Its Application* ed D J A Welsh (London: Academic)
Penrose R Theory of quantized directions (unpublished notes)

[14] Moussouris J P 1983 Quantum models as space-time based on recoupling theory *D.Phil. Thesis* Oxford University, Oxford

[15] Bianchi E 2009 *Nucl. Phys. B* **807** 591–624 (arXiv:0806.4710)

[16] Rovelli C 2010 A new look at loop quantum gravity arXiv:1004.1780

[17] Rovelli C 2004 *Quantum Gravity* (Cambridge, UK: Cambridge University Press) sections 1.2.2 and 6.7

[18] Freidel L and Livine E 2009 arXiv:0911.3553

[19] Ashtekar A and Lewandowski J 1997 *Adv. Theor. Math. Phys.* **1** 388 (arXiv:gr-qc/9711031)

[20] Derrick M et al 1986 *Phys. Rev. D* **34** 3286

[21] Drell S 1958 *Ann. Phys. (NY)* **4** 75

[22] Rovelli C and Speziale S 2003 *Phys. Rev. D* **67** 064019 (arXiv:gr-qc/0205108)

[23] Rovelli C and Smolin L 1995 *Nucl. Phys. B* **442** 593

[24] Rovelli C and Smolin L 1995 *Nucl. Phys. B* **456** 753 (erratum)

[25] Rovelli C and Smolin L 1995 *Nucl. Phys. B* **442** 593 (erratum)

[26] Loll R 1995 *Phys. Rev. Lett.* **75** 3048

[27] Flori C and Thiemann T 2008 Semiclassical analysis of the loop quantum gravity volume operator: I. Flux coherent states arXiv:0812.1537

[28] Flori C 2009 Semiclassical analysis of the loop quantum gravity volume operator: area coherent states arXiv:0904.1303

[29] Brunnemann J and Rideout D 2008 *Class. Quantum Grav.* **25** 065001 (arXiv:0706.0469)

[30] Brunnemann J and Rideout D 2008 *Class. Quantum Grav.* **25** 065002

[31] Brunnemann J and Rideout D 2010 Oriented matroids—combinatorial structures underlying loop quantum gravity arXiv:1003.2348

[32] Ding Y and Rovelli C 2009 The volume operator in covariant quantum gravity arXiv:0911.0543

[33] Ding Y and Rovelli C 2010 Physical boundary Hilbert space and volume operator in the Lorentzian new spin-foam theory arXiv:1006.1294