Continuous-variable quantum teleportation with non-Gaussian entangled states generated via multiple-photon subtraction and addition

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We investigate the Einstein-Podolsky-Rosen correlation (EPR), the quadrature squeezing and the continuous variable quantum teleportation when considering non-Gaussian entangled states generated by applying multiple-photon addition and multiple-photon subtraction to a two-mode squeezed vacuum state (TMSVs). Our results indicate that, in the case of symmetric multiple-photon-subtracted TMSVs, the corresponding EPR correlation, the two-mode squeezing, the sum squeezing and the fidelity of teleporting a coherent state and a squeezed vacuum state can be enhanced for any squeezing parameter $r$, and these enhancements increase with the number of the operations in small-squeezing regime. While asymmetric multiple-photon subtractions will generally reduce these quantities. For the multiple-photon added TMSVs, although it holds stronger entanglement, its EPR correlation, two-mode squeezing, sum squeezing and the fidelity of a coherent state are always smaller than that of the TMSVs. Only when considering teleporting a squeezed vacuum state, the symmetric photon addition makes somewhat improvement of the fidelity in large squeezing parameters. Furthermore, the optimal entanglement, the optimal EPR correlation, the optimal quadrature squeezing and the optimal fidelity, the four quantities always prefer to symmetrical arrangements of photon addition or subtraction on the two modes. Finally, we analytically prove that one-mode multiple-photon subtracted TMSVs is equivalent to that of the one-mode multiple-photon added one, and one-mode multiple-photon operations do not enhance the above four quantities at all, and even diminish them.

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I. INTRODUCTION

In recent years, non-Gaussian entangled states with continuous variables as communication resources have received more attention in quantum information and communication technologies. This is mainly because non-Gaussian states and non-Gaussian operations are indispensable for performing some certain continuous-variable quantum information tasks, such as quantum entanglement distillation [1–11], quantum error correction [12], and universal quantum computation [13].

Photon subtraction and addition are the typical non-Gaussian operations to generate non-Gaussian states with highly nonclassical properties. Agarwal and Tara [14] firstly studied the nonclassical properties of photon-added coherent states, which was implemented [15] via a nondegenerate parametric amplifier with small coupling strength. The photon-subtracted single-mode squeezed state which can be used to conditionally generate the Schrödinger cat state [16] was implemented with a beam splitter of high transmissivity [17]. The degree of entanglement and the fidelity of the quantum teleportation increase by subtracting one photon from two modes of a two-mode squeezed vacuum state (TMSVs) was first proposed by Opatrny et al [1]. Experimentally, it has been demonstrated [3, 9] that the TMSVs can be "de-gaussified" by photon subtraction, resulting in a mixed non-Gaussian state whose entanglement degree and teleportation fidelity are enhanced. Very recently, Kurochkin et al [15] also experimentally demonstrate that applying the photon subtraction operator to both modes of the TMSVs, they raise the fraction of the two-photon component in the state, resulting in the increasing of both squeezing and entanglement by about 50%. For a review on quantum-state engineering with photon addition and subtraction, we refer to Ref. [14, 20].

For a TMSVs as Gaussian entangled resource, larger squeezing leads to larger entanglement and higher teleportation fidelity. In experiments, the teleportation of Gaussian states in the standard continuous-variable Vaidman-Braunstein-Kimble (VBK) teleportation protocol [21, 22], including the coherent state and the squeezed vacuum state, has been reported [23–27]. And teleportation of a non-Gaussian state is carried out recently [28]. Although entanglement is indispensable in the quantum teleportation, stronger entanglement does not mean the higher fidelity of quantum teleportation when non-Gaussian states are considered as entanglement resources [1, 2, 29]. Dell’Anno et al [6, 7] studied the performance of squeezed Bell states and photon-added/photon-subtracted TMSVs as particular cases in the VPK teleportation protocol. And they found that subtracting/adding one photon from/to both modes of the TMSVs lead to the same entanglement. However, only the former can always enhance the teleportation fidelity when compared to the TMSVs. This is due to, in the VPK protocol of quantum teleportation of continuous variable, the quantum channel is based on the Einstein-Podolsky-Rosen (EPR) correlation and the fidelity of teleported states depends on the EPR correlation. Recently, for implement multiple-photon addition and subtraction on both modes of the
TMSVs, Navarrete-Benlloch et al \cite{30} demonstrate that the entanglement generally increases with the number of such operations. And multiple-photon addition typically provides a stronger entanglement enhancement than multiple-photon subtraction. In this paper, enlightened by these work in Ref.\cite{4,30}, we further investigate the EPR correlation and the quadrature squeezing effects of non-Gaussian states generated by multiple-photon-subtracted TMSVs (PS-TMSVs) and multiple-photon-added TMSVs (PA-TMSVs). Then, we consider these non-Gaussian entangled states as entangled resources to teleport a coherent state and a squeezed vacuum state in standard VBK protocol. Particularly, we show how multiple-photon operations affect the EPR correlation and the teleportation fidelity, as well as the relations among the EPR correlation, the two-mode squeezing and the teleportation fidelity.

The paper is organized as follows. In section II, we make a brief review about the PS-TMSVs and the PA-TMSVs, as well as their entanglement entropy. In section III, we extend Navarrete-Benlloch’s work, and further investigate the EPR correlation, the quadrature squeezing of non-Gaussian entangled states. In section IV, considering these non-Gaussian entangled states as entangled resources, we study the teleportation fidelity of a coherent state and a squeezed vacuum state in standard VBK protocol. Our results indicate that the optical EPR correlation, the optimal quadrature squeezing and the optimal fidelity are obtained when same number of operations is applied to both modes. For the PS-TMSVs as an entangled resource, the optimal EPR correlation, the optimal quadrature squeezing and the optimal fidelity are always larger than that of the TMSVs, and these improvement is obvious in the small-squeezing regime. The more symmetric photon-subtraction is applied to both modes the better the improvement is. Our main results are summarized in Sec.V.

II. MULTIPLE-PHOTON-ADDED AND MULTIPLE-PHOTON-SUBTRACTED TWO-MODE SQUEEZED VACUUM STATES

In this section, we firstly make a brief review about the PS-TMSVs and PA-TMSVs, as well as their entanglement entropy. We consider a TMSVs as an input state, which is typical Gaussian entangled state produced in experiment. Theoretically, the TMSVs can be obtained by adding the two-mode squeezed operator $S_2 (r) = \exp \left[r \left(a^\dagger b^\dagger - ab\right)\right]$ with squeezing parameter $r$ to a vacuum state with modes A and B, that is $|r\rangle = S_2 (r) |0,0\rangle = \text{sech} \sum_{n=0}^\infty \tanh^n r |n, n\rangle$.

By performing the multiple-photon-added operation on the TMSVs, one obtains the normalized output state as \cite{30}

$$|r\rangle_{\text{pa}} = C_{k,l}^{-1/2} a^k b^l S_2 (r) |0,0\rangle = \sum_{n=0}^\infty \frac{(n+k)! (n+l)!}{(n!)^2 C_{k,l} \cosh^2 r} \tanh^n r |n+k, n+l\rangle$$

where $C_{k,l} = \text{Tr} \left(a^k b^l S_2 (r) |0,0\rangle \langle 0,0| S_2 (-r) a^k b^l\right)$ is the normalized factor of the PA-TMSVs. Different from that in Ref.\cite{30}, here for our purpose, we first calculate the expectation value of a general product of operators $a^pa^qb^hb^j$ in the TMSVs. After straightforward calculation, we have [See Appendix A]

$$C_{p,q,h,j} = \text{Tr} \left(|r\rangle \langle r| a^p a^q b^h b^j\right) = \sum_{m=\text{min}[p,k]}^{\min[p+k]} \frac{plq!j! \cosh 2r \sinh h \cosh 2r}{2^{p+2h} m! (p-m)! (h-m)!} \times \frac{\tanh^{2m} r \delta_{p+j+g+h}}{(j-h+m)!}.$$  \hspace{1cm} (2)

Obviously, when $p = q = k$ and $h = j = l$, $C_{p,q,h,j}$ reduces to the normalization factor $C_{k,l}$. Note that the four quantities $n$, $n+\alpha$, $n+\beta$, and $n+\alpha+\beta$ are nonnegative integers, the Jacobi polynomial can be written as

$$P_n(\alpha,\beta) (x) = \frac{1}{2^n n!} \sum_{k=0}^{n} \binom{n+\alpha}{k} \binom{n+\beta}{n-k} \times (x-1)^{n-k} (x+1)^k.$$  \hspace{1cm} (3)

Thus, the normalization factor $C_{k,l}$ (without loss of generality assuming $k \geq l$) can be written as

$$C_{k,l} = k! \cosh 2k \cosh 2l P_{l-k}(0, k-l) (\cosh 2r).$$  \hspace{1cm} (4)

When only one of the modes undergoes photon addition while the other is unchanged (for example $l = 0$), Eq.\hspace{1cm} (4) reduces to

$$C_{k,0} = k! \cosh 2k r.$$  \hspace{1cm} (5)

Then, noting that the relation $S_2 (-r) a^l S_2 (r) = a^l \cosh r + b \sinh r$, the normalized one-mode added TMSVs can be written as

$$\frac{1}{\sqrt{k! \cosh 2k r}} a^k S_2 (\xi) |0,0\rangle = S_2 (\xi) |k,0\rangle,$$  \hspace{1cm} (6)

which is a two-mode squeezed number state.

On the other hand, subtracting multiple photons from two modes of the TMSVs, one obtains the PS-TMSVs state. Theoretically, the normalized PS-TMSVs can be written as \cite{31}

$$|r\rangle_{\text{ps}} = N_{k,l}^{-1/2} a^k b^l S_2 (r) |0,0\rangle = \sum_{n=0}^\infty \frac{(n)!^2 N_{k,l}^{-1} \text{sech} r}{(n-k)! (n-l)!} \tanh^n r |n-k, n-l\rangle.$$  \hspace{1cm} (7)
where $N_{k,l}$ is the normalized factor of the PS-TMSVs. Similarly, we firstly derive the expectation value of a general product of operators $a^\dagger p a^\dagger q b^\dagger j b^\dagger h$ in the TMSVs [Also see Appendix A].

\[
N_{p,q,h,j} = \text{Tr} \left( |r \rangle \langle r | a^\dagger p a^\dagger q b^\dagger j b^\dagger h \right) = \sum_{m, p, q, h, j} \frac{p! q! h! j! \sinh r \sinh r \sinh r \sinh r}{2^{r} \cosh r (m! (p-m)! (h-m)!)} \times \coth r \times \frac{\sinh r \sinh r \sinh r \sinh r}{(j-h+m)!}.
\]

Thus, in the case of $p = q = k$ and $h = j = l$, $N_{p,q,h,j}$ reduces to the normalization factor $N_{k,l}$, i.e.,

\[
N_{k,l} = k! \sinh 2k r P_{l}^{(k-0,0)} (\cosh 2r).
\]

For the case $l = 0$, we have

\[
N_{k,0} = k! \sinh 2k r
\]

Then, the normalized one-mode subtracted TMSVs is

\[
\frac{1}{\sqrt{k! \sinh 2k r}} a^k S_2 (\xi) |0 \rangle = S_2 (\xi) |0, k\rangle,
\]

which is another two-mode squeezed number state. Since the symmetry of Eqs. (6) and (10), we can see adding $k$ photons to the first mode is equivalent to subtracting them from the same mode. In addition, adding $k$ photons to the first mode has also the same effect as subtracting them from the second mode [39]. Therefore, one-mode photon-subtracted TMSVs and one-mode photon-added TMSVs have same quantum statistical effects.

In addition, it can be seen that Eq. (2) (or Eq. (1)) and Eq. (5) (or Eq. (13)) substantially differ for the exchange of the hyperbolic coefficients. And they are important for further studying nonclassical properties of both PA-TMSVs and PS-TMSVs. Particularly, with the help of Eqs. (2), (4), and (5), it is convenient to explore some quantum optical nonclassicality that are expressed in terms of expectation values of field operators, such as non-Poissonian statistics, the cross-correlation, anti-bunching effects [32], quadrature squeezing properties (including sum squeezing and different squeezing) [33], as well as the entanglement characterized by some inseparability criteria [34–37].

For our purpose, let us review the Neumann entropy of both PS-TMSVs and PA-TMSVs [39]. For a pure state in Schmidt form, $|\psi\rangle_{AB} = \sum_{n=1} c_n \alpha_n |\alpha_n\rangle_A \beta_n |\beta_n\rangle_B$ ($c_n$: real positive) with the orthonormal states $|\alpha_n\rangle_A$ and $|\beta_n\rangle_B$, the quantum entanglement is quantified by the partial von Neumann entropy of the reduced density operator [39]

\[
E(|\psi\rangle_{AB}) = -\text{Tr} (\rho_A \ln \rho_A) = -\sum_{n=1} c_n^2 \ln c_n^2 c_n^2,
\]

where the local state is given by $\rho_A = \text{Tr}_B |\psi\rangle_{AB} \langle \psi|$. Note that Eqs. (1) and (7) are in Schmidt forms, thus the entanglement of the PA-TMSVs and the PS-TMSVs can be directly obtained [30], respectively,

\[
E_{pa}^{k,l} = -\sum_{n=0}^{\infty} \frac{(n+k)!(n+l)!}{(n!)^2 C_{k,l} \cosh r} \tanh 2n r \times \log_2 \frac{(n+k)!(n+l)!}{(n!)^2 C_{k,l} \cosh r} \tanh 2n r,
\]

and

\[
E_{ps}^{k,l} = -\sum_{n=0}^{\infty} \frac{(n!)^2 N_{k,l}^{-1} \cosh r \sinh r}{(n-k)!(n-l)!} \tanh 2n r \times \log_2 \frac{(n!)^2 N_{k,l}^{-1} \cosh r \sinh r}{(n-k)!(n-l)!} \tanh 2n r.
\]

The amount of entanglement for the TMSVs (in the case of $k = l = 0$) is analytically given by $E = \cosh r \log_2 (\cosh r) - \sinh r \log_2 (\sinh r)$, and those for other states can be evaluated numerically by their Schmidt coefficients. When $k = l$ (symmetric operation), one can prove analytically $E_{pa} = E_{ps}$. Actually, Eq. (1) can be rewritten as

\[
|\xi\rangle_{ps} = \sum_{n=0}^{\infty} \frac{(n+k)!(n+k)!}{(n!)^2 N_{k,n} \tanh 2k r \cosh 2r} \tanh 2n r |n, n\rangle.
\]

From Eq. (8), we can derive

\[
N_{k,k} \tanh 2k r = \sum_{m} \frac{k!^4 \cosh r \sinh r \sinh r \cosh 2m r}{[m! (k-m)!]^2} \cosh 2m r.
\]

If setting $k - m = m'$, one can immediately obtain

\[
N_{k,k} \tanh 2k r = C_{k,k} = \sum_{m} \frac{k!^4 \cosh r \tanh 2m r}{[m! (k-m)!]^2},
\]

Therefore, for symmetric operation, both PA-TMSVs and PS-TMSVs hold the same set of Schmidt coefficients which leads to the exact same quantum entanglement, a result noted in Ref. [6]. For multiple-photon added and multiple-photon subtracted operations, as pointed out in Ref. [30], the optimal entanglement enhancement is obtained when the same number of operations is applied to both modes as shown Fig. 1, where addition and subtraction give the same entanglement enhancement. And the entanglement increases with the number of operations. For asymmetric, their numerical analysis shows that it is always better to perform addition rather than subtraction in order to increase the entanglement, i.e., $E_{pa}^{k,l} > E_{ps}^{k,l}$. For the detail discussion about the entanglement of these non-Gaussian state, please see the work in Ref. [30].
FIG. 1. (Color online) Entanglement entropy as function of the squeezing parameter \( r \) for PS-TMSVs (or PA-TMSVs) with different values of \((k, l)\). (a) PS-TMSVs or PA-TMSVs, from top to bottom lines correspond to \((5, 5), (5, 4), (5, 3), (5, 2), (5, 1)\) and \((5, 0)\). The black dashed curve corresponds to the TMSVs.

III. EPR CORRELATION AND SQUEEZING PROPERTIES

In this section, we will further investigate the EPR correlation and the quadrature squeezing effects (including the two-mode squeezing and the sum squeezing) of non-Gaussian entangled states generated by multiple-photon addition and multiple-photon subtraction.

A. Einstein-Podolsky-Rosen correlation

Besides the degree of entanglement, non-Gaussian states expressed by Eqs. 11 and 17 can be characterized by second-order EPR correlations between quadrature-phase components of the two modes. As counterparts of position and momentum operators of a massive particle, the quadrature-phase operator of each mode are defined as \( X_j = \left( a_j + a_j^\dagger \right) / \sqrt{2} \) and \( P_j = \left( a_j - a_j^\dagger \right) / (i \sqrt{2}) \) \((j = A, B)\). Historically, continuous-variable entanglement originated in the paper of Einstein, Podolsky and Rosen, arguing on the incompleteness of quantum mechanics [39]. They proposed an ideal state which is the common eigenstate of a pair of EPR-like operator, \( X_A - X_B \) (the relative position) and \( P_A + P_B \) (the total momentum). And its explicit form is given by \[ | \eta \rangle = \exp \left[ -\frac{1}{2} | \eta \rangle^2 + | \eta \rangle a^\dagger - \eta^* b^\dagger a + a^\dagger b \right] | 00 \rangle , \tag{17} \]

where \( \eta = \eta_1 + i \eta_2 \) is a complex number. In order to quantify how well two-mode states approximate to the EPR state of Eq. (17), one can define the EPR correlation parameter for a generic state \( \rho \) as \[ \Upsilon (\rho ) = \Delta^2 \langle X_A - X_B \rangle + \Delta^2 \langle P_A + P_B \rangle = 2 \left( \langle a^\dagger a \rangle + \langle b^\dagger b \rangle - \langle ab \rangle - \langle a^\dagger b^\dagger \rangle \right) + 1 \]

\[ -2 \left( \langle a \rangle - \langle b \rangle \right)^2 \left( \langle a^\dagger \rangle - \langle b^\dagger \rangle \right) , \tag{18} \]

which is the total variance of EPR-like operations \( X_A - X_B \) and \( P_A + P_B \). In the EPR state [39–41] expressed by Eq. (17), one can easily prove that the EPR correlation \( \Upsilon (\rho ) \) equals to zero. For separable two-mode states or any classical two-mode states, the total variance is larger than or equal to 2. The condition \[ \Upsilon (\rho ) < 2 \] indicates quantum entanglement, which can be a crucial resource for quantum protocols using continuous variables [21, 22].

Based on Eqs. (2) and (18), we can prove that \[ \langle a \rangle = \langle a^\dagger \rangle = \langle b \rangle = \langle b^\dagger \rangle = 0 \text{ and } \langle a b \rangle = \langle a^\dagger b^\dagger \rangle , \]

as well as \[ \langle a^2 b^2 \rangle = \langle a^2 b^2 \rangle + 1 \]. As a matter of convenience, we derive the expectation values of operators \( a^\dagger a, b^\dagger b, ab, a^2 b^2 \) in the PS-TMSVs as follows:

\[ \langle a^\dagger a \rangle_{\text{ps}} = N_{k,l+1} N_{k,l} , \langle b^\dagger b \rangle_{\text{ps}} = N_{k,l} N_{k,l+1} , \]

\[ \langle a b \rangle_{\text{ps}} = \frac{N_{k,l+1} N_{k,l+1}}{N_{k,l}} , \langle a^2 b^2 \rangle_{\text{ps}} = \frac{N_{k,l+2} N_{k,l+2}}{N_{k,l}} , \tag{20} \]

and

\[ \langle a^\dagger a b^\dagger b \rangle_{\text{ps}} = \frac{N_{k,l+1} N_{k,l+1}}{N_{k,l}} . \tag{21} \]

For the PA-TMSVs, we have

\[ \langle a^\dagger a \rangle_{\text{pa}} = \frac{C_{k,l+1}}{C_{k,l}} - 1 , \langle b^\dagger b \rangle_{\text{pa}} = \frac{C_{k,l+1}}{C_{k,l}} - 1 , \]

\[ \langle a b \rangle_{\text{pa}} = \frac{C_{k,l+1} C_{k,l+1}}{C_{k,l}} , \langle a^2 b^2 \rangle_{\text{pa}} = \frac{C_{k,l+2} C_{k,l+2}}{C_{k,l}} , \tag{22} \]

and

\[ \langle a^\dagger a b^\dagger b \rangle_{\text{pa}} = \frac{C_{k,l+1} - C_{k,l+1} - C_{k,l+1}}{C_{k,l}} + 1 . \tag{23} \]

Thus, the EPR correlation for PS-TMSVs is

\[ \Upsilon (\rho_{\text{ps}}) = 2 \left( \frac{N_{k,l+1} + N_{k,l+1} - 2 N_{k,l+1} k_{l+1} l_{l+1} + N_{k,l}}{N_{k,l}} \right) . \tag{24} \]
In the case of the PA-TMSVs, the EPR correlation is described in this form

$$\Upsilon (\rho_{pa}) = 2C_{k+1,l} + C_{k,l+1} - 2C_{k+1,k,l+1,l} - C_{k,l}.$$  (25)

Because adding $k$ photons to the first mode has the same effect as subtracting them from the same mode, when $l = 0$ (or $k = 0$), the PS-TMSVs and the PA-TMSVs have the same EPR correlation, and Eqs. (24) and (25) reduce to a simple form

$$\Upsilon (\rho_{pa}) |_{l=0} = \Upsilon (\rho_{pa}) |_{l=0} = (2k + 2)e^{-2r}.$$  (26)

Obviously, the EPR correlation of the non-Gaussian squeezed state also depends on the degree of the squeezing. In the case of $k = l = 0$, i.e., Eq. (26) reduces to

$$\Upsilon (\rho_0) = 2e^{-2r},$$  (27)

which tends to zero (the ideal EPR value) asymptotically for the squeezing parameter $r \to \infty$. From Eq. (26), it clearly shows that non-Gaussian entangled states generated by one-mode operations have lower the EPR correlation than that of the TMSVs. Therefore, one-mode photon-addition or photon-subtraction operations will diminish the EPR correlation.

In order to see clearly the EPR correlation of both PS-TMSVs and PA-TMSVs with different values of $(k,l)$, the figure is plotted in Fig. 2 for symmetric operations $(k = l)$. From Fig. 2, it shows that the EPR correlation of the PS-TMSVs is always larger than that of the TMSVs. And the EPR correlation increases with the number of subtracting photons. While, the EPR correlation of the PA-TMSVs is always smaller than that of the input state for any values of $(k,l)$. For asymmetric operations, both photon addition and subtraction generally reduce the EPR correlation as shown in Fig. 3, which is different from that of the entanglement entropy (both photon addition and subtraction operations lead to larger entanglement). In addition, our results indicate that the EPR correlation of both PS-TMSVs and PA-TMSVs is optimized at symmetric operations $(k = l)$, which is similar to their entanglement properties. In addition, when $r \to \infty$, the EPR correlation approaches to zero, photon addition or subtraction cannot improve much on this. Because when $r \to \infty$, the EPR correlation is already very approaching to the ideal EPR state.

Many authors have pointed out that the larger amount of entanglement does not always mean stronger EPR correlations [4, 8, 22]. And those states holding large entanglement may not be useful to improve quantum information processing. In VBK protocol of quantum teleportation [21, 22], the quantum channel is based on the EPR correlations and the fidelity of teleported states depends on the EPR correlations. Hence, we expect that the quality of quantum teleportation of continuous-variables can be improved by use of multiple-photon subtraction, and the fidelity can increase with the number of photon subtractions.

**B. Two-mode squeezing**

In quantum optics, squeezing is an earliest studied nonclassical phenomenon. There are various types of squeezing, i.e., single-mode squeezing, two-mode squeezing, sum squeezing and different squeezing [14, 45]. We study here the two-mode squeezing in which the correlation between modes starts to play a role. For a two-mode system, the quadrature-phase amplitudes can be expressed as $X = (X_A + X_B)/\sqrt{2}$ and $P = (P_A + P_B)/\sqrt{2}$ ($(X, P) = 1$), respectively. Based on Eqs. (2) and (8), it is easy to see that $\langle a \rangle = \langle a\rangle^\dagger = \langle b \rangle = \langle b\rangle^\dagger = 0$, and $\langle a^2 \rangle = \langle a^2\rangle^\dagger = \langle b^2 \rangle = \langle b^2\rangle^\dagger = 0$ as well as $\langle ab\rangle = \langle a\rangle^\dagger b\rangle = 0$, which leads to $\langle X \rangle = 0$ and $\langle P \rangle = 0$. Thus, the covariances of operators $X$ and $P$ in the PA-TMSVs or PS-TMSVs can be respectively written as

$$\langle \Delta X \rangle^2 = \frac{\langle a^2 \rangle + \langle b^2 \rangle + 2\langle ab \rangle + 1}{2},$$  (28)

and

$$\langle \Delta P \rangle^2 = \frac{\langle a^2 \rangle + \langle b^2 \rangle - 2\langle ab \rangle + 1}{2}.$$  (29)

According to quantum mechanics, operators $X$ and $P$ satisfy the uncertainty relation $\langle \Delta X \rangle^2 \langle \Delta P \rangle^2 \geq 1/4$. When $\langle \Delta X \rangle^2 < 1/2$ or $\langle \Delta P \rangle^2 < 1/2$, we can say there exists squeezing in the "$q$" or "$p$" direction. Compared with Eqs. (13), (24) and (25), we can see that the EPR
correlation of non-Gaussian entangled states is four times as much as that of the corresponding quantum fluctuation of \((\Delta P)^2\), i.e.,

\[
\mathcal{T} (\rho) = 4 (\Delta P)^2,
\]

which indicates the the conditions of squeezing and entanglement become identical, an interesting result. Therefore, the variations of the two-mode squeezing effect for both non-Gaussian states with different values of \((k, l)\) yield the same law, as shown in Figs. (2) and (3). The subtraction operation can enhance the degree of the two-mode squeezing, particularly in the case of the symmetric operation. However, photon additions always weaken the squeezing of the TMSVs. According to Eqs. (19) and (26), we can see that one-mode photon-addition and photon-subtraction operations will diminish the two-mode squeezing and the EPR correlation of the TMSVs.

In the limit of infinite squeezing, since TMSVs tends to become the ideal EPR state with perfect EPR correlations, the fidelity of teleportation also approaches unity. Up to present, the largest achievable two-mode squeezing in a stable optical configuration is about \(r \approx 1.15\) (i.e., about 10 dB) [46]. Hence, techniques that improve the performance of teleportation without demanding higher magnitudes of squeezing are vital commodities in quantum information. In this regard, for a given two-mode squeezing parameter \(r\), TMSVs which is engineered by symmetric multiple-photon subtraction, contains more squeezing (or higher EPR correlation). And the enhancement arisen from symmetric multiple-photon subtraction is more significant in small squeezing parameter \(r\). Thus, Non-Gaussian entangled states generated by symmetric photon subtraction should be better suited for teleportation [48, 29].

**C. Sum squeezing**

Sum and difference squeezing are both higher-order, two-mode squeezing effects [47, 48]. For two arbitrary modes A and B, the sum squeezing is associated with a so-called two-mode quadrature operator \(V_\varphi\) of the form [47]

\[
V_\varphi = \frac{1}{2} (e^{i\varphi} a^\dagger b^\dagger + e^{-i\varphi} ab)
\]

where \(\varphi\) is an angle made by \(V_\varphi\) with the real axis in the complex plane. A state is said to be sum squeezed for a \(\varphi\) if

\[
\langle (\Delta V_\varphi)^2 \rangle < \frac{1}{4} \langle a^\dagger a + b^\dagger b + 1 \rangle.
\]

From Eq. (32), one can define the degree of sum squeezing \(S\) in the form of normally ordered operators as following

\[
S = \frac{4 \langle (\Delta V_\varphi)^2 \rangle - \langle a^\dagger a + b^\dagger b + 1 \rangle}{\langle a^\dagger a + b^\dagger b + 1 \rangle}.
\]

Substituting Eq. (31) into Eq. (33), we obtain \(S\) as

\[
S = \frac{2 \langle a^\dagger ab^\dagger b \rangle + 2 \text{Re} (e^{-2i\varphi} \langle a^2 b^2 \rangle) - 4 \text{Re}^2 (e^{-i\varphi} \langle ab \rangle)}{\langle a^\dagger a + b^\dagger b + 1 \rangle},
\]

Then its negative value in the range \([-1, 0]\) indicates sum squeezing (or higher-order nonclassicality). It is clear that \(S\) has a lower bound equal to \(-1\). Hence, the closer the value of \(S\) to \(-1\) the higher the degree of sum squeezing. When \(l = 0\) (or \(k = 0\)), the optimal degree of the sum squeezing of both PS-TMSVs and PA-TMSVs reduces to that of the TMSVs at \(\varphi = \pi/2\),

\[
S_{\text{opt}} = \frac{(e^{i\varphi} - 1)^2}{(e^{i\varphi} + 1)^2},
\]

which is another interesting result.

The sum squeezing of the PS-TMSVs is also optimized at \(\varphi = \pi/2\). Our numerical analysis shows that the sum squeezing of the PS-TMSVs is always larger than that of the TMSVs, and the optimal sum squeezing is obtained for symmetric operations as shown in Fig.4. Thus, the degree of the sum squeezing can be improved by photon subtraction operation, particularly in the case of symmetric operations. On the other hand, photon addition weakens the sum squeezing.
The state can be described through the teleportation fidelity, which is a measure of how close between the initial input state and the final (mixed) output quantum state. The fidelity can be written as

\[ F = \int \frac{d^2 \alpha}{\pi} \chi_{in}(-\alpha) \chi_{out}(\alpha) \]  

where \( \chi_{out}(\alpha) = \chi_{in}(\alpha) \chi_E(\alpha^*, \alpha) \) is the characteristic function of the output teleported state. Here, \( \chi_E(\alpha^*, \alpha) = \text{Tr}[D_1(\alpha) D_2(\beta) \rho_E] \) is the characteristic function of an entangled resource, and \( \chi_{in}(\alpha) = \text{Tr}[D(\alpha) \rho_{in}] \) is that of the input state. In the following, we consider the non-Gaussian entangled states as entangled resources to teleport a coherent state and a squeezed vacuum state in the standard VKB teleportation protocol, respectively.

### A. Teleporting a coherent state

Let us first consider the behaviour of the fidelity for input a coherent state \( |\beta\rangle \), whose characteristic function is

\[ \chi_{coh}(\alpha) = \exp \left[ \frac{1}{2} |\alpha|^2 + 2 \text{Im}[\alpha^* \beta] \right]. \]  

Using the same approach as deriving Eq. (2), the characteristic functions of the PA-TMSVs and the PS-TMSVs read, respectively

\[ \chi_{pa}(\alpha, \beta) = \frac{\chi_e(\alpha, \beta) \partial^{2k+2} \partial^f \partial^r \partial^t}{C_{k,l} \partial^f \partial^r \partial^t} e^{(f + s + r)t} \sinh^2 r + (f + s + r) \sinh 2r \]

\[ \times e^{f(\alpha \cosh^2 r - \beta \sinh 2r)} - s(\alpha^* \cosh^2 r - \beta \sinh 2r) \]

\[ \times e^{t(\beta \cosh^2 r - \alpha^* \sinh 2r)} - r(\beta^* \cosh^2 r - \alpha \sinh 2r), \]  

and

\[ \chi_{ps}(\alpha, \beta) = \frac{\chi_e(\alpha, \beta) \partial^{2k+2} \partial^f \partial^r \partial^t}{N_{k,l} \partial^f \partial^r \partial^t} e^{(f + s + r)t} \sinh^2 r + (f + s + r) \sinh 2r \]

\[ \times e^{f(\alpha \sinh^2 r - \beta \sinh 2r)} - s(\alpha^* \sinh^2 r - \beta \sinh 2r) \]

\[ \times e^{t(\beta \sinh^2 r - \alpha^* \sinh 2r)} - r(\beta^* \sinh^2 r - \alpha \sinh 2r), \]  

where \( \chi_e(\alpha, \beta) \) is the characteristic function of the TMSVS

\[ \chi(\alpha, \beta) = e^{-\frac{\sinh 2r}{2} (|\alpha|^2 + |\beta|^2) + (\alpha^* \alpha + \beta^* \beta) \sinh 2r}. \]  

Upon substituting these characteristic functions into Eq. (29), we can work out the fidelity for teleporting a coherent state by using non-Gaussian states as entangled resource. For the PS-TMSVs, we have the fidelity of teleportation of a coherent state

\[ F_{ps} = \frac{N_{k,l}^{-1} 2^k k!!}{e^{-2r t} + 1} \left( \frac{e^{4r} + 4 e^{2r} + 5}{4 e^{2r} + 4} \right)^{k-l}, \]  

where \( F_{n(\alpha, \beta)}(2) \) is the Jacobi polynomial. Different from work of Ref. [52], here we obtain the general expression of the fidelity for teleporting a coherent state. For the PA-TMSVs, the fidelity of teleportation of a coherent state can be written in a simple form

\[ F_{pa} = \frac{(k + l)! (e^{2r} + 1)^{k+l}}{4^{k+l} C_{k,l}} e^{-2r t}, \]  

where \( C_{k,l} = \binom{k+l}{k} \binom{k}{l} \) is the binomial coefficient.
which is a special case of that in Ref. 54. It can be seen that the fidelity depends on the squeezing parameter \( r \) and the number of operations \((k,l)\). Note that the fidelity is independent of amplitude of the coherent state, thus Eqs. (41) and (42) are just the fidelity for teleporting a vacuum state. In the case of one-mode operations (for example \( l = 0 \)), Eqs. (41) and (42) reduces to

\[
F' = \left( \frac{1}{e^{-2r} + 1} \right)^{k+1},
\]

which is always smaller than that of the TMSVs, for any values of the squeezing parameter \( r \). Therefore, one-mode operations will also diminish the fidelity of teleporting a coherent state.

Now, we can numerically study the behavior of the fidelity for teleporting a coherent state by making use of the PS-TMSVs and the PA-TMSVs. In Fig. 5(a), we plot the fidelity for input a coherent state in the case of \( k = l \). From Fig. 5(a), we can see that the fidelity for the PS-TMSVs is always larger than that of the TMSVs when teleporting a coherent state. While, the fidelity for the PA-TMSVs is always smaller than that of the TMSVs, even smaller than 1/2 in the small-squeezing regime. For asymmetric operation \((k \neq l)\), our numerical analysis shows that both photon addition and subtraction generally weaken the fidelity as shown in Fig. 6. Thus, the optimal fidelity is arrived for symmetric operations.

In VBK protocol of quantum teleportation of continuous variables, the fidelity of teleported states depends on the EPR correlations. Thus, for teleporting a coherent state, the higher EPR correlation means the higher fidelity. Comparing the teleportation fidelity with the EPR correlation in Figs. 2 and 4, one can observe that the EPR correlation and the fidelity can be enhanced by symmetric photon-subtraction operation in the whole region of the squeezing parameter \( r \). In addition, the teleportation fidelity for both PS-TMSVs and PA-TMSVs could be beyond the classical limit of 1/2 without the EPR correlation as shown in Figs. 2 and 4. Thus, the teleportation fidelity, which is larger than 1/2, does not guarantee the existence of the EPR correlation.

B. Teleporting a squeezed vacuum state

The characteristic functions of the squeezed Gaussian input state, \( |\varepsilon\rangle = S(\varepsilon)|0\rangle \) with \( S(\varepsilon) = \exp[\varepsilon(\alpha^2 + a^2)/2] \), reads

\[
\chi_{\text{in}}(\alpha) = \exp \left[ \frac{-\cosh 2\varepsilon}{2} |\alpha|^2 + (\alpha^2 + a^2) \frac{\sinh 2\varepsilon}{4} \right].
\]

(44)

With the help of the integral formula

\[
\int d^2z \frac{\exp(\xi z^2 + \eta z^* + f z^2 + g z^*)}{\sqrt{\xi^2 - 4fg}} = \frac{1}{\sqrt{\xi^2 - 4fg}} e^{-\frac{\xi \eta + \xi^* g + g^* \eta}{\xi^2 - 4fg}},
\]

(45)

whose convergent condition is \( \text{Re}(\xi \pm f \pm g) < 0 \), \( \text{Re}(\xi^2/4 + f^2/4) < 0 \), we can work out the fidelity for teleporting a squeezed state by using the PS-TMSVs

\[
F_{ps} = \frac{F_0}{N_{k,l}} \frac{\partial^{2r+2l}}{\partial f^k \partial s^l \partial t^k \partial \tau^l} \exp \left\{ (fs + tr) \sinh^2 r + (ft + s\tau) \frac{\sinh 2r}{2} + \frac{(f-t)(t-s)(e^{-2r} - 1)^2(e^{-2r} + \cosh 2\varepsilon)}{4(2e^{-2r}\cosh 2\varepsilon + e^{-4r} + 1)} + \left[ (f-t)^2 + (t-s)^2 \right] (e^{-2r} - 1)^2 \sinh 2\varepsilon \right\}_{f,s,t}
\]

(46)

where \( F_0 \) is the fidelity for the TMSVs,

\[
F_0 = \sqrt{\frac{1}{2e^{-2r}\cosh 2\varepsilon + e^{-4r} + 1}}.
\]

(47)

It can be seen that the fidelity not only dependents on the squeezing parameter \( r \) and the number of addition operations \((k,l)\), but also weaken the fidelity as shown in Fig. 6. Thus, the optimal fidelity is arrived for symmetric operations.
In the case of expressed by Eq. (42). 

The squeezing parameter for different states with given the single-mode k, l curve corresponds to the TMSVs.

FIG. 6. (Color online) Fidelity as function of the squeezing parameters for different states. For the PA-TMSVs, we obtain the fidelity of Eq. (29) reduces to Eq. (41), i.e., the fidelity of a coherent state. Obviously, when ε = 0, Eq. (51) reduces to Eq. (13), the fidelity of teleporting a coherent state.

Compared with the coherent state, the fidelity of teleporting a squeezed vacuum state decreases with its squeezing parameter ε, as shown in Figs. (5b) and (7). For symmetric operation (k = l), the fidelity for the PS-TMSVs is always larger than that of the TMSVs when teleporting a squeezed vacuum state. While, the fidelity for the PA-TMSVs is generally smaller than that of the TMSVs, even smaller than 1/2 in the small-squeezing regime. Only in the large-squeezing regime, the fidelity for the PA-TMSVs can be larger than that of the TMSVs as shown in Fig. 5(b). For asymmetric operation (k ≠ l), both photon addition and subtraction generally weaken the fidelity, which is similar to that behavior of teleporting a coherent state as shown in Fig. 6. Thus, the optimal fidelity of teleporting a squeezed vacuum state is also arrived for symmetric operations.

V. CONCLUSIONS

In summary, we have shown that symmetric multiple-photon subtraction (k = l) can enhance the EPR correlation, the quadrature squeezing and the teleportation fidelity of a two-mode squeezed vacuum state (TMSVs) in the whole region of the squeezing r. And those enhancements are more distinct in the small-squeezing regime, and increase with the number of symmetric operations. For asymmetric operations (k ≠ l), compared with the TMSVs, the multiple-photon subtraction generally diminishes the EPR correlation, the two-mode squeezing, the sum squeezing and the teleportation fidelity, although it can enhance the entanglement. On the other hand, for any values of (k, l), the multiple-photon addition can better increase the degree of entanglement while diminishes the EPR correlation, the two-mode squeezing, the

\[ F' = \frac{F_0^{2k+1}}{(e^{-2r} \cosh 2\varepsilon + 1)} \sum_{m=0}^{[k/2]} k! \left( \frac{\sinh 2\varepsilon}{(\cosh 2\varepsilon + e^{2r})} \right)^{2m} \]

Noting that the new expression of Legendre polynomials

\[ P_k(x) = x^k \sum_{m=0}^{[k/2]} \frac{k!}{2^m (m!)^2} \frac{(1 - \frac{1}{2})^m}{(k - 2m)!} \]

Eq. (19) can be rewritten as

\[ F' = F_0^{k+1} P_k \left( \frac{(e^{-2r} \cosh 2\varepsilon + 1)}{\sqrt{e^{-2r} \cosh 2\varepsilon + e^{-4r} + 1}} \right) . \]
sum squeezing and the fidelity for teleporting a coherent state at the same time. When considering teleporting a squeezed vacuum state $|\varepsilon\rangle$, the symmetric photon addition makes somewhat improvement of the fidelity in the case of large squeezing parameters $r$ and $\varepsilon$. Therefore, both PA-TMSVs and PS-TMSVs, the optimal entanglement, the optimal EPR correlation, the optimal quadrature squeezing and the optimal fidelity, the four quantities always prefer to symmetrical arrangements of photon addition or subtraction on the two modes. Our results indicate that symmetrical multiple-photon subtractions may more useful than photon addition in continuous-variable quantum information processing.

In addition, we have analytically proved that one-mode multiple-photon-subtracted TMSVs is equivalent to that of the one-mode multiple-photon-added one. For the one-mode operation, we have derived analytical expressions of the EPR correlation, two-mode squeezing, sum squeezing and the teleportation fidelity, respectively. These analytical expressions clearly represent that one-mode multiple-photon operations do not enhance them at all, and even diminish them. In addition, the teleportation fidelity for both PS-TMSVs and PA-TMSVs could be beyond the classical limit of 1/2 without the EPR correlation. Thus, the teleportation fidelity, which is larger than 1/2, does not guarantee the existence of the EPR correlation. Finally, we have proved that the EPR correlation of PA-TMSVs and PS-TMSVs is four times as much as that of the corresponding quantum fluctuation of $(\Delta P)^2$, which indicates the the conditions of squeezing and entanglement become identical, an interesting result.

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APPENDIX A: DERIVATION OF Eqs. (2) AND (3)

In the Fock space, the TMSVs can be written as

$$S_2 (\xi) |00\rangle = \frac{1}{\cosh r} \exp \left[a^\dagger b^\dagger \tanh r \right] |00\rangle.$$  \hfill (A1)

The expectation value of a general product of operators $a^{p}a^{q}b^{h}b^{j}$ in the TMSVs reads

$$C_{p,q,h,j} = Tr \left(a^{p}a^{q}b^{h}b^{j} S_2 (\xi) |00\rangle \langle 00| S_2^\dagger (\xi) a^{p}b^{j}\right)$$  \hfill (A2)

Substituting Eq.(A1) into Eq.(A2) and inserting the completeness relation of the two-mode coherent state, as well as using the integral formulae

$$\int \frac{d^2z}{\pi} \exp[|\xi|^2 + \xi z + \eta z^*] = \frac{1}{\zeta} \exp \left[-\frac{\xi \eta}{\zeta}\right],$$  \hfill (A3)

whose convergent condition is $\text{Re}(\zeta) < 0$, after doing straightforward calculation, we obtain

$$C_{p,q,h,j} = \frac{1}{\cosh^2 r} \int \frac{d^2z_1 d^2z_2}{\pi^2} \langle z_1^p z_2^q | z_1z_2 | z_1^s z_2^t \rangle \exp \left[\tanh \left(z_1^s z_2^t + z_1z_2\right) - \left(|z_1|^2 + |z_2|^2\right)\right]$$

$$= \frac{1}{\cosh^2 r} \int \frac{d^2z_1 d^2z_2}{\pi^2} \frac{\partial^{p+q+h+j}}{\partial f \partial s \partial h \partial j} \exp \left[-\left(|z_1|^2 + |z_2|^2\right) + f z_1 + s z_2^* + t z_2 + \tau z_2^* + \tanh \left(z_1^s z_2^t + z_1z_2\right)\right] |f,s,t,\tau = 0$$

$$= \frac{\partial^{p+q+h+j}}{\partial f \partial s \partial h \partial j} \exp \left[(ft + s\tau) \cosh^2 r + \left(s\sinh 2r + t \cosh 2r\right)\right] |f,s,t,\tau = 0.$$  \hfill (A4)

By the binomial theorem, we further obtain Eq. (2)

$$C_{p,q,h,j} = \frac{\partial^{p+h}}{\partial s \partial h^p} \left(s \cosh^2 r + t \sinh 2r \right)^p$$

$$\times \left(s \sinh 2r + t \cosh 2r\right)^j$$

$$= \min[p,h] \frac{p!q!h!}{m! \delta_{p+j,q+h}} \frac{(cosh^2 r)^{p+h-2m}}{(p-m)! (h-m)!}$$

$$\times \frac{(s \sinh 2r)^{j-h-2m}}{(j-h+m)!}.$$  \hfill (A5)
Next, the expectation value of a general product of operators $a^\dagger a p b^\dagger b j$ in the TMSVS reads

$$N_{p,q,h,j} = Tr \left( a^p b^q S_2 (\xi) |00\rangle \langle 00| S_2^\dagger (\xi) a^q b^j \right). \quad (A6)$$

Using the same approach as deriving Eq.(2), substituting Eq.(A1) into Eq.(A6) and inserting the completeness relation of two-mode coherent states for two times, we have

$$N_{p,q,h,j} = \frac{\partial^{p+q+h+j}}{\partial f \partial q \partial h \partial j} \exp \left[ (fs + t\tau) \sinh^2 r \right]$$
$$+ (fs + t\tau) \frac{\sin 2r}{2} \bigg|_{f,s,t,\tau=0} \quad (A7)$$

Similarly, we finally obtain Eq.(6).

$$N_{p,q,h,j} = \sum_{m} \frac{\min[p,h]}{m! (p-m)! (h-m)!}$$
$$\times \left( \frac{\sinh 2r}{2} \right)^{j-h+2m} \delta_{p+j,q+h}, \quad (A8)$$
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