ANALOGUE OF SYLVESTER-CAYLEY FORMULA FOR INVARIANTS OF
n-ARY FORM

LEONID BEDRATYUK

ABSTRACT. The number $\nu_{n,d}(k)$ of linearly independent homogeneous invariants of degree $k$ for the $n$-ary form of degree $d$ is calculated. The following formula holds:

$$\nu_{n,d}(k) = \sum_{s \in W} (-1)^{|s|} c_{n,d}(k, (\rho - s(\rho))^*) ,$$

here $W$ is Weyl group of Lie algebra $\mathfrak{sl}_n$, $(-1)^{|s|}$ is the sign of the element $s \in W$, $\rho = (1, 1, \ldots, 1)$ is half the sum of the positive roots of $\mathfrak{sl}_n$, the weight $\lambda^*$ means the unique dominant weight on the orbit $W(\lambda)$ and $c_{n,d}(k, (m_1, m_2, \ldots, m_{n-1}))$ is the number of nonnegative integer solutions of the system of equations

$$\begin{align*}
2\omega_1(\alpha) + \omega_2(\alpha) + \cdots + \omega_{n-1}(\alpha) &= dk - m_1 , \\
\omega_1(\alpha) - \omega_2(\alpha) &= m_2 , \\
\vdots \\
\omega_{n-2}(\alpha) - \omega_{n-1}(\alpha) &= m_{n-1} , \\
|\alpha| &= k .
\end{align*}$$

Here $\omega_r(\alpha) = \sum_{i \in I_{n,d}} i_r \alpha_i$, $I_{n,d} := \{ i = (i_1, i_2, \ldots, i_{n-1}) \in \mathbb{Z}^{n-1}_+, |i| \leq d \}$, $|i| = i_1 + \cdots + i_{n-1}$.

1. Let be $F_{d,n}$ the $\mathbb{C}$-space of $n$-ary forms of degree $d$:

$$\sum_{i \in I_{n,d}} a_i \binom{d}{i} x_1^{d-(i_1+\cdots+i_{n-1})} x_2^{i_1} \cdots x_n^{i_{n-1}} ,$$

where $I_{n,d} := \{ i = (i_1, i_2, \ldots, i_{n-1}) \in \mathbb{Z}^{n-1}_+, |i| \leq d \}$, $|i| = i_1 + \cdots + i_{n-1}$, $a_i \in \mathbb{C}$ and

$$\binom{d}{i} := \frac{d!}{i_1! i_2! \cdots i_{n-1}!(d - (i_1 + \cdots + i_{n-1}))!} .$$

Let us identify the algebra of polynomial function $\mathbb{C}[F_{d,n}]$ with the polynomial $\mathbb{C}$-algebra $A_{d,n}$ of the variables set $\{ a_i, i \in I_{n,d} \}$. The natural action of the group $SL_n$ on $F_{d,n}$ induces the actions of $SL_n$ ( and $\mathfrak{sl}_n$) on the algebra $A_{d,n}$. The corresponding ring of invariants $A^{SL_n}_{d,n} = A^{\mathfrak{sl}_n}_{d,n}$ is called the ring of invariants for the $n$-ary form of degree $d$.

The ring $A^{\mathfrak{sl}_n}_{d,n}$ is graded ring

$$A^{\mathfrak{sl}_n}_{d,n} = (A^{\mathfrak{sl}_n}_{d,n})_0 + (A^{\mathfrak{sl}_n}_{d,n})_1 + \cdots + (A^{\mathfrak{sl}_n}_{d,n})_k + \cdots ,$$

here $(A^{\mathfrak{sl}_n}_{d,n})_k$ is the vector subspace generated by homogeneous invariants of degree $k$.

Denote $\nu_{n,d}(k) := \dim(A^{SL_n}_{d,n})_k$. For the binary form the number $\nu_{2,d}(k)$ is calculated by well-known Sylvester-Cayley formula, see [1]. For the tenary form the number $\nu_{3,d}(k)$ is calculated in the paper of the present author, see [2]. In this paper we generalize those formulas for the case of $n$-ary form.

2. In the Lie algebra $\mathfrak{sl}_n$ denote by $E_{ij}$ the matrix unitaries. The matrices $H_1 := E_{2,2} - E_{1,1}$, $H_2 := E_{3,3} - E_{2,2}$, $\ldots$, $H_{n-1} := E_{n-1,n-1} - E_{n-2,n-2}$ generate the Cartan subalgebra in $\mathfrak{sl}_n$.

Recall that the ordered set of integer numbers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n-1})$ is called the weight of $\mathfrak{sl}_n$-module $V$, if there exists $v \in V$ such that $v$ is common eigenvector of the operators $H_s$ and
$H_s(v) = \lambda_s v$, $s = 1, \ldots, n - 1$. A weight is said to be dominant weight if all $\lambda_s \geq 0$. Note, our definition of weight is slightly different than the standard weight definition as function on the Cartan subalgebra. Denote by $\Lambda_V$ the set of all weight of $\mathfrak{sl}_n$-module $V$ and denote by $\Lambda^+_V$ the set of dominant weight of $V$. Also, denote by $\Gamma_\lambda$ the unique irreducible $\mathfrak{sl}_n$-module with highest weight $\lambda$.

Let $A$ be the vector subspace of $A_{d, n,d}$ generated by all elements $\{a_i, i \in I_{n,d}\}$ of degree 1. The operators $H_i, i = 1 \ldots n - 1$ act on the basis elements $a_i, i \in I_{n,d}$ of the space $A$ in the following way, see [3]:

$$H_1(a_i) = (d - (2i_1 + i_2 + \cdots + i_{n-1}))a_i, H_2(a_i) = (i_2 - i_3)a_i, \ldots H_{n-1}(a_i) = (i_{n-2} - i_{n-1})a_i.$$

Therefore, the basis elements $a_i, i \in I_{n,d}$ of $\mathfrak{sl}_n$-module $A$ are the common eigenvectors of the operators $H_s$ with the weights

$$\varepsilon_i = (d - (2i_1 + i_2 + \cdots + i_{n-1}), i_2 - i_3, \ldots, i_{n-2} - i_{n-1}).$$

It is clear that $A$ is irreducible $\mathfrak{sl}_n$-module with the highest vector $a_{(0, 0, \ldots, 0)}$ and with the highest weight $(d, 0, \ldots, 0)$. Thus, $A \cong \Gamma_{(d, 0, \ldots, 0)}$. The number $\nu_{n,d}(k)$ is the multiplicities of trivial $\mathfrak{sl}_n$-module $\Gamma_{(0, 0, \ldots, 0)}$ in the decomposition the symmetrical power $S^k(A)$ on irreducible $\mathfrak{sl}_n$-modules, i.e. $\nu_{n,d}(k) = \gamma_{d,n}(k, (0, 0, \ldots, 0))$. Here $\gamma_{d,n}(k, (0, 0, \ldots, 0))$ is determined from the decomposition:

$$S^k(A) \cong \sum_{\lambda} \gamma_{d,n}(k, \lambda)\Gamma_\lambda, \lambda \in \Lambda^+_S(A).$$

Since the multiplicities of all weights are equal to 1 we may write down the the character of $A$:

$$\text{Char}(A) = \sum_{i \in I_{n,d}} e(\varepsilon_i).$$

Here $e(\varepsilon_i)$ are generating elements of the group ring of the weight lattice $\mathbb{Z}(\Lambda_A)$.

We need the following technical lemma

**Lemma 1.** The monomial $\prod_{i \in I_{n,d}} a_i^{\alpha_i}$ of total degree $k$ is the weight vector of $\mathfrak{sl}_n$-module $S^k(A)$ with the weight

$$(n, d - 2\omega_1(\alpha) + \omega_2(\alpha) + \cdots + \omega_{n-1}(\alpha), \omega_1(\alpha) - \omega_2(\alpha), \ldots, \omega_{n-2}(\alpha) - \omega_{n-1}(\alpha)),$$

$$\partial \omega_s(\alpha) = \sum_{i \in I_{n,d}} i_s \alpha_i, |\alpha| := \sum_{i \in I_{n,d}} \alpha_i = k.$$

**Proof.** Direct calculations. \qed

The character of $\mathfrak{sl}_n$-module $S^k(A)$ is the complete symmetrical polynomial $H_k$ of the variable set $e(\varepsilon_i), i \in I_{n,d}$, see [4]. Therefore, we have

$$\text{Char}(S^k(A)) = \sum_{|\alpha| = k} \prod_{i \in I_{n,d}} e(\varepsilon_i)^{\alpha_i}$$

**Lemma 2.**

$$\text{Char}(S^k(A)) = \sum_{\mu} c_{n,d}(k, \mu)e(\mu), \mu \in \Lambda_{S^k(A)}$$
here \( c_{n,d}(k, \mu) := c_{n,d}(k, (\mu_1, \mu_2, \ldots, \mu_{n-1})) \) is the number of nonnegative integer solutions of the system of equations
\[
\begin{align*}
2\omega_1(\alpha) + \omega_2(\alpha) + \cdots + \omega_{n-1}(\alpha) &= kd - \mu_1, \\
\omega_1(\alpha) - \omega_2(\alpha) &= \mu_2, \\
&\cdots \\
\omega_{n-2}(\alpha) - \omega_{n-1}(\alpha) &= \mu_{n-1}, \\
|\alpha| &= k.
\end{align*}
\]

**Proof.** Direct calculations, using Lemma 1. \(\square\)

For arbitrary \( \mu \in \Lambda_{\Gamma_{\lambda}} \) denote by \( \mu^* \) the unique dominant weight on the orbit \( W(\mu) \) of the Weyl group \( W \). Such dominant weight exists and unique, see \([3]\).

On the \( \mathfrak{sl}_n \)-module \( \Gamma_{\lambda} \) let us define the value \( E_{\lambda} \) in the following way
\[
E_{\lambda} = \sum_{s \in W} (-1)^{|s|} n_{\lambda}((\rho - s(\rho))^*).
\]
Here \( \rho \) is half the sum of the positive roots of Lie algebra \( \mathfrak{sl}_n \), \( n_{\lambda}(\mu) \) is the multiplicities of the weight \( \mu \) in \( \Gamma_{\lambda} \) and \( |s| \) is the sign of the element \( s \in W \). Note, \( n_{\lambda}(\mu) = 0 \) if \( \mu \not\in \Gamma_{\lambda} \).

The following lemma plays crucial role in the calculation.

**Lemma 3.**
\[
E_{\lambda} = \begin{cases} 
1, & \lambda = (0, \ldots, 0), \\
0, & \lambda \neq (0, \ldots, 0).
\end{cases}
\]

**Proof.** For \( \lambda = (0, \ldots, 0) \), there is nothing to prove – the multiplicities of trivial weight in the trivial representation is equal to 1.

Suppose now \( \lambda \neq (0, \ldots, 0) \). We use the following recurrence formula, see \([6]\), for the multiplicities \( n_{\lambda}(\mu) \) of the weight \( \mu \) in the \( \mathfrak{sl}_n \)-module \( \Gamma_{\lambda} \):
\[
\sum_{s \in W} (-1)^{|s|} n_{\lambda}(\mu + \rho - s(\rho)) = 0.
\]
Substituting \( \mu = (0, \ldots, 0) \) and taking into account that the multiplicities of the weights \( \rho - s(\rho) \) and \( (\rho - s(\rho))^* \) coincides we obtain the formula. \(\square\)

Now we are ready to calculate the value \( \nu_{n,d}(k) \)

**Theorem 1.** The number \( \nu_{n,d}(k) \) of linearly independent homogeneous invariants of degree \( k \) \( n \)-ary form of degree \( d \) is calculated by the formula
\[
\nu_{n,d}(k) = \sum_{s \in W} (-1)^{|s|} c_{n,d}(k, (\rho - s(\rho))^*). 
\]

**Proof.** The number \( \nu_{n,d}(k) \) is equal to the multiplicities \( \gamma_{n,d}(k, (0,0, \ldots, 0)) \) of trivial representation \( \Gamma_{(0,0, \ldots, 0)} \) in the symmetrical power \( S_k(A) \). The decomposition
\[
S_k(A) \cong \sum_{\lambda} \gamma_{n,d}(k, \lambda) \Gamma_{\lambda}, \lambda \in \Lambda^+_S(A),
\]
implies the following characters decompositions
\[
\text{Char}(S_k(A)) = \sum_{\lambda \in \Lambda^+_S(A)} \gamma_{n,d}(k, \lambda) \text{Char}(\Gamma_{\lambda}).
\]
Taking into account
\[
\text{Char}(\Gamma_{\lambda}) = \sum_{\mu \in \Lambda_{\lambda}} n_{\lambda}(\mu)e(\mu),
\]
we get
\[
\text{Char}(S^k(A)) = \sum_{\lambda \in \Lambda^+_{S^k(A)}} \gamma_{n,d}(k, \lambda) \sum_{\mu \in \Lambda_{\lambda}} n_{\lambda}(\mu)e(\mu) = \sum_{\lambda \in \Lambda^+_{S^k(A)}} \left( \sum_{\mu \in \Lambda_{\lambda}} \gamma_{n,d}(k, \lambda)n_{\lambda}(\mu) \right)e(\mu).
\]
By using Lemma 2 we get
\[
\text{Char}(S^k(A)) = \sum_{\mu \in \Lambda^+_{S^k(A)}} c_{n,d}(k, \mu)e(\mu).
\]
Therefore
\[
\sum_{\mu \in \Lambda^+_{S^k(A)}} c_{n,d}(k, \mu)e(\mu) = \sum_{\lambda \in \Lambda^+_{S^k(A)}} \left( \sum_{\mu \in \Lambda_{\lambda}} \gamma_{n,d}(k, \lambda)n_{\lambda}(\mu) \right)e(\mu).
\]
By equating the coefficients of \(e(\mu)\), we obtain
\[
c_{n,d}(k, \mu) = \sum_{\lambda \in \Lambda^+_{S^k(A)}} \gamma_{n,d}(k, \lambda)n_{\lambda}(\mu).
\]
Then, by using previous lemma we have
\[
\sum_{s \in W} (-1)^{|s|} c_{n,d}(k, (\rho - s(\rho))^s) = \sum_{s \in W} (-1)^{|s|} \sum_{\lambda \in \Lambda^+_{S^k(A)}} \gamma_{n,d}(k, \lambda)n_{\lambda}( (\rho - s(\rho))^s ) =
\]
\[
= \sum_{\lambda \in \Lambda^+_{S^k(A)}} \gamma_{n,d}(k, \lambda) \left( \sum_{s \in W} (-1)^{|s|} n_{\lambda}( (\rho - s(\rho))^s ) \right) = \sum_{\lambda \in \Lambda^+_{S^k(A)}} \gamma_{n,d}(k, \lambda) E_\lambda = \gamma_{n,d}(k, (0, 0, \ldots, 0)).
\]
Thus,
\[
\nu_{n,d}(k) = \sum_{s \in W} (-1)^{|s|} c_{n,d}(k, (\rho - s(\rho))^s).
\]
This concludes the proof. \(\Box\)

3. It is easy to see that the multiplicities \(\gamma_{n,d}(k, \lambda)\) is equal to number of linearly independent highest vectors of \(\mathfrak{sl}_n\)-module \(S^k(A)\) with highest weight \(\lambda\). Any highest vector is the invariant of the subalgebra of upper triangular unipotent matrices of the Lie algebra \(\mathfrak{sl}_n\). Such elements is called semi-invariants of \(n\)-ary form. In the same way we can prove the theorem

**Theorem 2.**

\[
\gamma_{n,d}(k, \lambda) = \sum_{s \in W} (-1)^{|s|} c_{n,d}(k, (\lambda + \rho - s(\rho))^s), \lambda \in \Lambda^+_{S^k(A)}.
\]

Let us introduce the concept of covariants for \(n\)-ary form. Denote by \(C_{n,d}\) the algebra of polynomial functions of the following \(\mathfrak{sl}_n\)-module

\[
A_{n,d} \oplus \Gamma_{(1,0,\ldots,0)} \oplus \Gamma_{(0,1,\ldots,0)} \cdot \cdot \cdot \oplus \Gamma_{(0,0,\ldots,1)}.
\]

The subalgebra \(C_{n,d}^{\mathfrak{sl}_n}\) of \(\mathfrak{sl}_n\)-invariants is called the algebra of covariants of \(n\)-ary form. For \(n = 2\) the classical Roberts’ theorem states that the algebras of semi-invariants and covariants are isomorphic. The similar result for ternary form proved by the author in the paper. 

The following statement seems to hold

**Conjecture.** The algebras of semi-invariants and covariants of \(n\)-ary form are isomorphic.
If it is true, then Theorem 2 defines the formula for calculation of the number linearly independent covariants for \(n\)-ary form of the weight \(\lambda\) and degree \(k\).

4. Example. Let \(n = 3\). Then half the sum of the positive roots \(\rho\) is equal to \((1, 1)\). The Weyl group of Lie algebra \(\mathfrak{sl}_3\) is generated by the three reflections \(s_{\alpha_1}, s_{\alpha_2}, s_{\alpha_3}\); here \(\alpha_1 = (2, -1)\), \(\alpha_2 = (-1, 2)\) i \(\alpha_3 = (1, 1)\) are all positive roots. The orbit \(W(\rho)\) consists of 6 weights \(−(1, 1)\) and

\[
\begin{align*}
    s_{\alpha_1}(1, 1) &= (−1, 2), & (−1)^{|s_{\alpha_1}|} &= −1, \\
    s_{\alpha_2}(1, 1) &= (2, −1), & (−1)^{|s_{\alpha_2}|} &= −1, \\
    s_{\alpha_3}(1, 1) &= (−1, −1), & (−1)^{|s_{\alpha_3}|} &= −1, \\
    s_{\alpha_1}s_{\alpha_3}(1, 1) &= (1, −2), & (−1)^{|s_{\alpha_1}s_{\alpha_3}|} &= 1, \\
    s_{\alpha_3}s_{\alpha_1}(1, 1) &= (−2, 1), & (−1)^{|s_{\alpha_3}s_{\alpha_1}|} &= 1,
\end{align*}
\]

Therefore

\[
\rho - W(\rho) = \{(0, 0), (2, −1), (−1, 2), (2, 2), (0, 3), (3, 0)\}.
\]

Thus, taking into account the signs of the corresponding elements of the group \(W\) we get the following identity for any dominant weight \(\lambda\) of the standard irreducible \(\mathfrak{sl}_3\)-module \(\Gamma_\lambda\):

\[
n_\lambda(0, 0) - n_\lambda(2, -1) - n_\lambda(-1, 2) - n_\lambda(2, 2) + n_\lambda(0, 3) + n_\lambda(3, 0) = 0.
\]

Since the weights \((1, 1), (2, -1), (-1, 2)\) lies on the same orbit it is implies \((2, -1)^* = (−1, 2)^* = (1, 1)\) and

\[
n_\lambda(1, 1) = n_\lambda(2, -1) = n_\lambda(-1, 2).
\]

Thus

\[
n_\lambda(0, 0) - 2n_\lambda(1, 1) - n_\lambda(2, 2) + n_\lambda(0, 3) + n_\lambda(3, 0) = 0.
\]

Therefore, using Theorem 1, we obtain

\[
\nu_{3,d}(k) = c_{3,d}(k, (0, 0)) - 2c_{3,d}(k, (1, 1)) - c_{3,d}(k, (2, 2)) + c_{3,d}(k, (0, 3)) + c_{3,d}(k, (3, 0)).
\]

It coincides completely with result of the paper \cite{2}.

5. Let us derive the formula for calculation of \(\nu_{n,d}(k)\). Solving the system of equations

\[
\begin{align*}
    2\omega_1(\alpha) + \omega_2(\alpha) + \cdots + \omega_n−1(\alpha) &= dn - \mu_1, \\
    \omega_1(\alpha) - \omega_2(\alpha) &= \mu_2, \\
    \cdots \\
    \omega_{n−2}(\alpha) - \omega_{n−1}(\alpha) &= \mu_{n−1}, \\
|\alpha| &= k.
\end{align*}
\]

for \(\omega_1(\alpha), \omega_2(\alpha), \ldots, \omega_{n−1}(\alpha)\) we get

\[
\begin{align*}
    \omega_1(\alpha) &= \frac{k}{n}d \left(\frac{1}{n} (\mu_1 + 2 \mu_2 + \cdots + (n-1) \mu_{n-1}) - (\mu_2 + \mu_3 + \cdots + \mu_{n-1})\right) \\
    \omega_2(\alpha) &= \frac{k}{n}d \left(\frac{1}{n} (\mu_1 + 2 \mu_2 + \cdots + (n-1) \mu_{n-1}) - (\mu_3 + \mu_4 + \cdots + \mu_{n-1})\right) \\
    \cdots \\
    \omega_s(\alpha) &= \frac{k}{n}d \left(\frac{1}{n} (\mu_1 + 2 \mu_2 + \cdots + (n-1) \mu_{n-1}) - (\mu_{s+1} + \mu_{s+2} + \cdots + \mu_{n-1})\right) \\
    \cdots \\
    \omega_{n-1}(\alpha) &= \frac{k}{n}d \left(\frac{1}{n} (\mu_1 + 2 \mu_2 + \cdots + (n-1) \mu_{n-1})\right)
\end{align*}
\]
It is not hard to prove that the number \( c_{n,d}(k, (0, 0, \ldots, 0)) \) of nonnegative integer solutions of the following system

\[
\begin{align*}
\omega_1(\alpha) &= \frac{k d}{n} \\
\omega_2(\alpha) &= \frac{k d}{n} \\
\vdots & \\
\omega_{n-1}(\alpha) &= \frac{k d}{n} \\
|\alpha| &= k
\end{align*}
\]

is equal to coefficient of \( t^k(q_1 q_2 \cdots q_{n-1})^{\frac{k d}{n}} \) of the expansion of the series

\[
R_{n,d} = \left( \prod_{|\mu| \leq d} (1 - tq_1^{\mu_1} q_2^{\mu_2} \cdots q_{n-1}^{\mu_{n-1}}) \right)^{-1}.
\]

Denote it in such way:

\[
c_{n,d}(k, (0, 0, \ldots, 0)) = \left( R_{n,d} \right)^{t^k(q_1 q_2 \cdots q_{n-1})^{\frac{k d}{n}}}. \]

Then, for a set of integer numbers \((\mu_1, \mu_2, \ldots, \mu_{n-1})\) the number \( c_{n,d}(k, (i_1, i_2, \ldots, i_{n-1})) \) of integer nonnegative solutions of the system of equations

\[
\begin{align*}
\omega_1(\alpha) &= \frac{k d}{n} - \mu_1 \\
\omega_2(\alpha) &= \frac{k d}{n} - \mu_2 \\
\vdots & \\
\omega_{n-1}(\alpha) &= \frac{k d}{n} - \mu_{n-1} \\
|\alpha| &= k
\end{align*}
\]

is equals

\[
c_{n,d}(k, (\mu_1, \mu_2, \ldots, \mu_{n-1})) = \left( q_1^{\mu_1} q_2^{\mu_2} \cdots q_{n-1}^{\mu_{n-1}} R_{n,d} \right)^{t^k(q_1 q_2 \cdots q_{n-1})^{\frac{k d}{n}}}. \]

By using the multi-index notation rewrite the last expression in the form

\[
c_{n,d}(k, \mu) = \left( q^\mu R_{n,d} \right)^{t^k(q)^{\frac{k d}{n}}}. \]

To each \( \mu \in I_{n,d} \) assing the following vector

\[
\mu_\omega = \left( \frac{1}{n} \left( \sum_{s=1}^{n-1} s \mu_s \right) - \sum_{s=2}^{n-1} \mu_s, \frac{1}{n} \left( \sum_{s=1}^{n-1} s \mu_s \right) - \sum_{s=3}^{n-1} \mu_s, \ldots, \frac{1}{n} \left( \sum_{s=1}^{n-1} s \mu_s \right) - \sum_{s=n-1}^{n-1} \mu_s \right).
\]

Then the following formula holds

\[
\nu_{n,d}(k) = \left( \sum_{s \in W} (-1)^{|s|} c_{n,d}(k, (\rho - s(\rho))^*) q^{(\rho-s(\rho))^*} R_{n,d} \right)^{t^k(q)^{\frac{k d}{n}}}. \]
References

[1] Hilbert, D., Theory of algebraic invariants. Lectures. Cambridge University Press, 1993.
[2] Bedratyuk L.P., Analogue of Sylvester-Cayley formula for invariants of ternary form, submitted to the Michigan Mathematical Journal, arXiv:0806.1920
[3] Bedratyuk L., The Roberts’ theorem for the ternary forms, Nauk. Visn. Chernivets’kogo Univ., Mat., 349, 5–13, 2007.
[4] Fulton W., Harris J., Representation theory: a first course, 1991.
[5] Humphreys J., Introduction to Lie Algebras and Representation Theory, 1978.
[6] Naimark, M.A., Stern, A.I., Theory of group representations. Grundlehren der Mathematischen Wissenschaften, 246, New York -Heidelberg - Berlin: Springer-Verlag, 1982.
[7] Roberts M. The covariants of a binary quantic of the n-th degree, Quarterly J. Math., 4 P.168–178, 1861

Khmel’nyts’ky National University, Instytuts’ka st. 11, Khmel’nyts’ky, 29016, Ukraine
E-mail address: bedratyuk@ief.tup.km.ua