Generalized Umemura polynomials and Hirota–Miwa equations

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Abstract

We introduce and study generalized Umemura polynomials $U^{(k)}_{n,m}(z, w; a, b)$ which are the natural generalization of the Umemura polynomials $U_n(z, w; a, b)$ related to the Painlevé VI equation. We show that if either $a = b$, or $a = 0$, or $b = 0$, then polynomials $U^{(0)}_{n,m}(z, w; a, b)$ generate solutions to the Painlevé VI equation. We give new proof of Noumi–Okada–Okamoto–Umemura conjecture, and describe connections between polynomials $U^{(0)}_{n,m}(z, w; a, 0)$ and certain Umemura polynomials $U_k(z, w; \alpha, \beta)$. Finally we show that after appropriate rescaling, Umemura’s polynomials $U^{(k)}(z, w; a, b)$ satisfy the Hirota–Miwa bilinear equations.

§1. Introduction

There exists a vast body of literature about the Painlevé VI equation $P_{VI} := P_{VI}(\alpha, \beta, \gamma, \delta)$:

$$\frac{d^2 q}{dt^2} = \frac{1}{2} \left( \frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t} \right) \left( \frac{dq}{dt} \right)^2 - \left( \frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t} \right) \frac{dq}{dt} + \frac{q(q-1)(q-t)}{t^2(t-1)^2} \left( \alpha - \beta \frac{t}{q^2} + \gamma \frac{(t-1)}{(q-1)^2} + \delta \frac{t(t-1)}{(q-t)^2} \right)$$

where $t \in \mathbb{C}$, $q := q(t; \alpha, \beta, \gamma, \delta)$ is a function of $t$, and $\alpha, \beta, \gamma, \delta$ are arbitrary complex parameters, see e.g. [NOOU, OI-OIV, 4, 5] and the literature quoted therein. It is well–known and goes back to Painlevé that any solution $q(t)$ of the equation $P_{VI}$ satisfies the so–called Painlevé property:

- the critical points 0, 1 and $\infty$ of the equation (1.1) are the only fixed singularities of $q(t)$.
- any movable singularity of $q(t)$ (the position of which depends on integration constants) is a pole.

In this paper we introduce and initiate the study of certain special polynomials related to the Painlevé VI equation, namely, the generalized Umemura polynomials $U^{(k)}_{n,m}(z, w; a, b)$. 

These polynomials have many interesting combinatorial and algebraic properties and in the particular case \( n = 0 = k \) coincide with Umemura’s polynomials \( U_m(z^2, w^2; a, b) \), see e.g. \cite{U, NOOU}. The main goal of the present paper is to study certain recurrence relations between polynomials \( U_{n,m}^{(k)}(z, w; a, b) \). Our main result is Theorem 1 which gives a generalization of the recurrence relation between Umemura’s polynomials \cite{U}. In some particular cases the recurrence relation obtained in Theorem 1 coincides with that for Umemura’s polynomials. As a corollary, we obtain a new proof of the Noumi–Okada–Okamoto–Umemura conjecture \cite{NOOU}, and show that polynomials \( U_m^{(0)}(z, w; a, b) \) also generate solutions to the equation Painlevé VI. The main mean in our proofs is Lemma 2 from Section 4. For example, using this Lemma, we prove a new recurrence relation between Umemura’s polynomials (Theorem 2), describe explicitly connections between polynomials \( U_{n,m}(0, b) \) and Umemura’s polynomials \( U_m(b_1, b_2) \), see Lemma \( \square \), and prove that after appropriate rescaling Umemura’s polynomials \( U_n(z, w; a, b) \) satisfy the Hirota–Miwa bilinear equations, see Proposition 5. Finally, in Section 5, Proposition 6, we state and prove an analog of the Plücker relations between certain Umemura’s polynomials.

§2. Painlevé VI

In this section we collect together some basic results about the equation Painlevé VI. More detail and proofs may be found in familiar series of papers by K. Okamoto \cite{NOI-OIV}. We refer the reader to the Proceedings of Conference ”The Painlevé property. One century later” \cite{P}, where different aspects of the theory of Painlevé equations may be found.

2.1 Hamiltonian form

It is well–known and goes back to a paper by Okamoto \cite{NOI-OIV} that the sixth Painlevé equation (1.1) is equivalent to the following Hamiltonian system:

\[
\mathcal{H}_{VI}(b; t, q, p) : \begin{cases} 
\frac{dq}{dt} = \frac{\partial H}{\partial p}, \\
\frac{dp}{dt} = -\frac{\partial H}{\partial q},
\end{cases}
\]

(2.1)

with the Hamiltonian

\[
H := H_{VI}(b; t, q, p) = \frac{1}{t(t-1)} \left[ (q-1)(q-t)p^2 - \{(b_1 + b_2)(q-1)(q-t) + \right. \\
+ (b_1 - b_2)(q-t) + (b_3 + b_4)(q-1)\} p + (b_1 + b_3)(b_1 + b_4)(q-t)\left. \right],
\]

where \( b = (b_1, b_2, b_3, b_4) \) belongs to the parameters space \( \mathbb{C}^4 \); the parameters \( (\alpha, \beta, \gamma, \delta) \) and \( (b_1, b_2, b_3, b_4) \) are connected by the following relations

\[
\alpha = \frac{1}{2}(b_3 - b_4)^2, \quad \beta = -\frac{1}{2}(b_1 + b_2)^2, \quad \gamma = \frac{1}{2}(b_1 - b_2)^2, \quad \delta = -\frac{1}{2}(b_3 - b_4)(b_3 + b_4 - 2).
\]

(2.2)
Proposition 1 (K. Okamoto, [OI-OIV].) If \((q(t), p(t))\) is a solution to the Hamiltonian system (2.1), the function
\[
h(b, t) = t(t - 1)H_{VI}(b; t, q(t), p(t)) + e_2(b_1, b_3, b_4)t - \frac{1}{2}e_2(b_1, b_2, b_3, b_4)
\]
satisfies the equation \(E_{VI}(b)\) :
\[
\frac{dh}{dt} \left[ t(t - 1) \frac{d^2 h}{dt^2} \right]^2 + \left[ \frac{dh}{dt} \left\{ 2h - (2t - 1) \frac{dh}{dt} \right\} + b_1 b_2 b_3 b_4 \right] = \prod_{k=1}^{4} \left( \frac{dh}{dt} + b_k^2 \right), \tag{2.3}
\]
where \(e_2(x_1, \ldots, x_n) = \sum_{1 \leq i < j \leq n} x_i x_j\) denotes the degree 2 elementary symmetric polynomial.

Conversely, for a solution \(h := h(b, t)\) to the equation \(E_{VI}(b)\) such that \(\frac{d^2 h}{dt^2} \neq 0\), there exists a solution \((q(t), p(t))\) to the Hamiltonian system (2.1). Furthermore, the function \(q := q(t)\) is a solution to the Painlevé equation (1.1), where parameters \((\alpha, \beta, \gamma, \delta)\) are determined by the relations (2.2).

We will call the equation \(E_{VI}(b)\) by the Painlevé–Okamoto equation.

2.2 Bäcklund transformation

Consider the following linear transformation of the parameters space \(\mathbb{C}^4\):
\[
s_1 := (b_1, b_2, b_3, b_4) \longrightarrow (b_2, b_1, b_3, b_4),
\]
\[
s_2 := (b_1, b_2, b_3, b_4) \longrightarrow (b_1, b_3, b_2, b_4),
\]
\[
s_3 := (b_1, b_2, b_3, b_4) \longrightarrow (b_1, b_2, b_4, b_3),
\]
\[
s_0 := (b_1, b_2, b_3, b_4) \longrightarrow (b_1, b_2, -b_3, -b_4),
\]
\[
l_3 := (b_1, b_2, b_3, b_4) \longrightarrow (b_1, b_2, b_3 + 1, b_4).
\]

Denote by \(W = < s_0, s_1, s_2, s_3, l_3 >\) the subgroup of \(\text{Aut}\mathbb{C}^4\) generated by these transformation. It is not difficult to see, that \(W \cong W(D_4^{(1)})\), i.e. \(W\) is isomorphic to the affine Weyl group of type \(D_4^{(1)}\).

Proposition 2 (K. Okamoto, [OI-OIV].) For each \(w \in W\), there exists a birational transformation
\[
L_w : \{\text{solutions to } \mathcal{H}_{VI}(b)\} \longrightarrow \{\text{solutions to } \mathcal{H}_{VI}(w(b))\}.
\]
The birational transformations $L_w$, $w \in W(D_4^{(1)})$ are called by Bäcklund transformations associated to the equation Painlevé $VI$.

### 2.3 $\tau$–function

Let $(q(t), p(t))$ be a solution to the Hamiltonian system (2.1), the $\tau$–function $\tau(t)$ corresponding to the solution $(q(t), p(t))$ is defined by the following equation

$$\frac{d}{dt} \log \tau(t) = H_{VI}(b; t, q(t), p(t));$$

in other words,

$$\tau(t) = (\text{constant}) \exp \left( \int H_{VI}(b; t, q(t), p(t)) dt \right).$$

### 2.4 Umemura polynomials

Suppose that $b_3 = -\frac{1}{2}$, $b_4 = 0$, then it is well–known and goes back to Umemura’s paper [U], that the pair

$$(q_0, p_0) = \left( \frac{(b_1 + b_2)^2 - (b_1^2 - b_2^2) \sqrt{t(1-t)}}{(b_1 - b_2)^2 + 4b_1b_2t}, \frac{b_1 q_0 - \frac{1}{2}(b_1 + b_2)}{q_0(q_0 - 1)} \right)$$

defines a solution to the Hamiltonian system (2.1) with parameters $b = (b_1, b_2, -\frac{1}{2}, 0)$. Note, see e.g. [U], that

$$H_0(t) = H_{VI} \left( (b_1, b_2, -\frac{1}{2}, 0), t, q_0(t), p_0(t) \right)$$

$$= \frac{1}{t(t-1)} \left\{ b_1(b_1 - 1)(1 - 2t) + 2b_1^2 \sqrt{t(t-1)} + 2b_2(b_1 - t) + b_2(b_2 - 1)(1 - 2t) - 2b_2^2 \sqrt{t(t-1)} \right\},$$

and

$$\tau_0(t) = \exp \left\{ \int H_0(t) dt \right\}.$$

To introduce Umemura’s polynomials, let $(q_m, p_m)$ be a solution to the Hamiltonian system $H_{VI}(b_1, b_2, -\frac{1}{2} + m, 0) = H_{VI}(b_1, b_2, -\frac{1}{2}, 0)$ obtained from the solution $(q_0, p_0)$ by applying $m$ times the Bäcklund transformation $l_3$. Consider the corresponding $\tau$–function $\tau_m$:

$$\frac{d}{dt} \log \tau_m = H_{VI}((b_1, b_2, -\frac{1}{2} + m, 0); t, q_m(t), p_m(t)).$$
It follows from Proposition 1, see e.g. [OI-OIV, U], that \( \tau \)-functions \( \tau_n := \tau_n(t) \) satisfy the Toda equation
\[
\frac{\tau_{n-1} \tau_{n+1}}{\tau_n^2} = \frac{d}{dt} \left( t(t-1) \frac{d}{dt} (\log \tau_n) \right) + (b_1 + b_2 + n)(b_3 + b_4 + n). \tag{2.4}
\]

Follow H. Umemura [U], define a family of functions \( T_n(t) \), \( n = 0, 1, 2, \ldots \), by
\[
\tau_n(t) = T_n(t) \exp \left( \int \left( H_0(t) - \frac{n(b_1 t - \frac{1}{2}(b_1 + b_2))}{t(t-1)} \right) dt \right).
\]

**Proposition 3** (H. Umemura, [U]) \( T_n(t) \) is a polynomial with rational coefficients in the variable \( v := \sqrt{\frac{t}{t-1}} + \sqrt{\frac{t-1}{t}} \).

For example, \( T_0 = 1, T_1 = 1, T_2 = \frac{1}{2}(-4b_1^2 + 1)(2 - v)/4 + (-4b_2^2 + 1)(2 + v)/4 \).

It follows from the Toda equation (2.4) that polynomials \( T_n := T_n(v) \) satisfy the following recurrence relation [U]:
\[
T_{n-1}T_{n+1} = \left\{ \frac{1}{4}(-2b_1^2 - 2b_2^2 + (b_1^2 - b_2^2)v) + (n - \frac{1}{2})^2 \right\} T_n^2 \tag{2.5}
\]
\[+ \frac{1}{4}(v^2 - 4)^2 \left\{ \frac{d^2 T_n}{dv^2} - \left( \frac{dT_n}{dv} \right)^2 \right\} \]
\[+ \frac{1}{4}(v^2 - 4)vT_n \frac{dT_n}{dv} \]
with initial conditions \( T_0 = T_1 = 1 \).

**Definition 1** Polynomials \( U_n := U_n(z, w, b_1, b_2) := 2^{n(n-1)}T_n(v) \), where \( z = \frac{2 - v}{4} \), \( w = \frac{2 + v}{4} \), are called by Umemura polynomials.

The formula (2.6) below was stated as a conjecture by M. Noumi, S. Okada, K. Okamoto and H. Umemura [NOOU] and has been proved recently by M. Taneda, and A.N. Kirillov (independently):
\[
2^{n(n-1)}T_n(v) := U_n(z, w, b_1, b_2) = \sum_{I \subseteq [n-1]} d_n(I)c_1d_{[n-1]\setminus I} z^{|I|} w^{|I'|}, \tag{2.6}
\]
where

(i) \( [n-1] = \{1, 2, \ldots, n-1\} \); for any subset \( I = \{i_1 > i_2 > \cdots > i_p\} \subset [n-1] \), \( d_n(I) := \dim_{GL(n)} \) stands for the dimension of irreducible representation of the general linear group \( GL(n) \) corresponding to the highest weight \( \lambda(I) \) with the Frobenius’ symbol \( \lambda(I) = (i_1, i_2, \ldots, i_p| i_1 - 1, i_2 - 1, \ldots, i_p - 1) \).
(ii) \( c = -4b_1^2, \quad d = -4b_2^2, \quad z = \frac{2-v}{4}, \quad w = \frac{2+v}{4}; \)

(iii) \( \tilde{c}_k = c + (2k-1)^2, \quad \tilde{d}_k = d + (2k-1)^2, \quad c_k = \tilde{c}_1\tilde{c}_2\cdots\tilde{c}_k, \quad d_k = \tilde{d}_1\tilde{d}_2\cdots\tilde{d}_k; \)

(iv) \(|I| = i_1 + i_2 + \cdots + i_p.\)

Recall that Frobenius’ symbol \((a_1, a_2, \ldots, a_p|b_1, b_2, \ldots, b_p)\) denotes the partition which corresponds to the following diagram

\[ \begin{array}{c}
\text{§3. Generalized Umemura polynomials} \\
\end{array} \]

Let \( n, m, k \) be fixed nonnegative integers, \( k \leq n \). Denote by \([n;m]\) the set of integers \( \{1, 2, \ldots, n, n+2, n+4, \ldots, n+2m\} \). Let \( I \) be a subset of the set \([n;m]\). Follow [DK], define the numbers

\[
d_{n,m}(I) = \prod_{i \in I, j \in [n;m]\setminus I} \frac{|i+j|}{i-j}, \quad c(I) = \sum_{i \in I, i > n} \frac{i-n}{2} \tag{3.1}
\]

It has been shown in [DK], that in fact \( d_{n,m}(I) \) are integers for any subset \( I \subset [n;m] \). Now we are going to introduce the generalized Umemura polynomials

\[
U_{n,m}^{(k)} := U_{n,m}^{(k)}(z, w; a, b) = \sum_{|k| \subset I \subset [n;m]} \prod_{i \in I \setminus |k|, j \in |k|} \left( \frac{i+j}{i-j} \right) d_{n,m}(I)(-1)^{c(I)} e_I^{(n,m,k)}(z, w),
\]

where

(i) \([k]\) stands for the set \( \{1, 2, \ldots, k\} \);

(ii) \( \bar{a}_k = a + (k-1)^2, \quad \bar{b}_k = b + (k-1)^2 \) and \( a_{2k} = \bar{a}_2\bar{a}_4\cdots\bar{a}_{2k}, \quad a_{2k+1} = \bar{a}_1\bar{a}_3\cdots\bar{a}_{2k+1}; \)

\( b_{2k} = b_2b_4\cdots b_{2k}, \quad b_{2k+1} = b_1b_3\cdots b_{2k+1}; \)

(iii) for any subset \( I \subset [n;m] \), we set \( a_I = \prod_{i \in I} a_i, \quad b_I = \prod_{i \in I} b_i; \)

(iv) \( e_I^{(n,m,k)}(z, w) = a_{I\setminus |k|} b_{[n;m]\setminus I} z^{I\setminus |k|} w^{[n;m]\setminus I}. \)
Note that the polynomial $U_{0,m}^{(0)}$ coincides with Umemura’s polynomial $T_m(z^2, w^2; a, b)$. The formula for generalized Umemura polynomials stated below follows from the Cauchy identity, and was used by J.F. van Diejen and A.N. Kirillov [DK] in their study of $q$–spherical functions.

**Lemma 1** The generalized Umemura polynomials $U_{n,m}^{(k)}(a, b; z, w)$ admit the following determinantal expression

$$U_{n,m}^{(k)}(a, b; z, w) = \det \begin{vmatrix} a_i w^i & \prod_{s \in [k]} \left( \frac{i + s}{i - s} \right) \delta_{i,j} + \frac{2i}{i + j} (-1)^{c(i)} \prod_{s \in [n; m], s \neq i} \left| \frac{i + s}{i - s} \right| b_i z^i \\ i, j \in [n; m] \setminus [k] \end{vmatrix},$$

where $c(i) = i$ if $i \leq n$, and $c(i) = (i - n)/2$ if $i > n$.

In the particular case $k = 0$, $n = 0$ this formula gives a determinantal representation for Umemura’s polynomials and has many applications.

§4. Main result

Let us introduce notation $U_{n,m} := U_{n,m}^{(0)}(z, w; a, b)$. The main result of our paper describes a recurrence relation between polynomials $U_{n,m}$.

**Theorem 1**

$$U_{n,m-1} U_{n,m+1} = (-\bar{a}_{n+2m+2} z^2 + \bar{b}_{n+2m+2} w^2) U_{n,m}^2 + 8 z^2 w^2 D_x^2 U_{n,m} \circ U_{n,m},$$

where for any two functions $f = f(x)$ and $g = g(x)$

$$D_x^2 f \circ g = f'' g - 2 f' g' + f g''$$

denotes the second Hirota derivative, and $\frac{d}{dx}$: here variables $z, w$ and $x$ are connected by the relations $z = \frac{1}{2} (e^x + e^{-x} - 2)^{1/2}$, $w = \frac{1}{2} (e^x + e^{-x} + 2)^{1/2}$.

The main step of our proof is to establish the following algebraic identity which appears to have an independent interest.

**Lemma 2** For any two subsets $I$, $J$ of the set $[n; m]$, we have

$$\prod_{\lambda \in I} \left( \frac{x + 2 + \lambda}{x + 2 - \lambda} \right) \prod_{\lambda \in J} \left( \frac{x - \lambda}{x + \lambda} \right) + \prod_{\lambda \in I} \left( \frac{x - \lambda}{x + \lambda} \right) \prod_{\lambda \in J} \left( \frac{x + 2 + \lambda}{x + 2 - \lambda} \right)$$

$$= 2 + \sum_{\lambda \in I \cup J} \frac{b^{I, J}_{\lambda}}{(x + 2 - \lambda)(x + \lambda)},$$

where the coefficients $b^{I, J}_{\lambda}$ have the following expressions:
(i) If \( \lambda \neq 1 \) and \( \lambda \in I \cap J \), then \( b^{I,J}_{\lambda} = \)

\[
4\lambda(\lambda - 1) \left\{ \prod_{\lambda' \in I \setminus \{\lambda\}} \left( \frac{\lambda + \lambda'}{\lambda - \lambda'} \right) \prod_{\lambda' \in J} \left( \frac{\lambda - 2 - \lambda'}{\lambda - 2 + \lambda'} \right) + \prod_{\lambda' \in I} \left( \frac{\lambda - 2 - \lambda'}{\lambda - 2 + \lambda'} \right) \prod_{\lambda' \in J \setminus \{\lambda\}} \left( \frac{\lambda + \lambda'}{\lambda - \lambda'} \right) \right\};
\]

(ii) If \( \lambda \neq 1 \), \( \lambda \in I \) and \( \lambda \notin J \), then

\[
\lambda^{I,J}_{\lambda} = 4\lambda(\lambda - 1) \left\{ \prod_{\lambda' \in I \setminus \{\lambda\}} \left( \frac{\lambda + \lambda'}{\lambda - \lambda'} \right) \prod_{\lambda' \in J} \left( \frac{\lambda - 2 - \lambda'}{\lambda - 2 + \lambda'} \right) \right\};
\]

(iii) If \( 1 \in I \cap J \), then

\[
b^{I,J}_{1} = -8 \prod_{\lambda \in I \setminus \{1\}} \left( \frac{1 + \lambda}{1 - \lambda} \right) \prod_{\lambda \in J \setminus \{1\}} \left( \frac{1 + \lambda}{1 - \lambda} \right).
\]

**Proof.** Using the partial fraction expansion, we have

\[
\prod_{\lambda \in I} \left( \frac{x + 2 + \lambda}{x + 2 - \lambda} \right) \prod_{\lambda \in J} \left( \frac{x - \lambda}{x + \lambda} \right) + \prod_{\lambda \in I} \left( \frac{x - \lambda}{x + \lambda} \right) \prod_{\lambda \in J} \left( \frac{x + 2 + \lambda}{x + 2 - \lambda} \right) = 2 + \frac{A}{x + 1} + \frac{B}{(x + 1)^2} + \sum_{\lambda \in I, \lambda \neq 1} \frac{C_{\lambda}}{x + 2 + \lambda} + \sum_{\lambda \in I, \lambda \neq 1} \frac{D_{\lambda}}{x - \lambda} + \sum_{\lambda \in J, \lambda \neq 1} \frac{E_{\lambda}}{x + 2 + \lambda} + \sum_{\lambda \in J, \lambda \neq 1} \frac{F_{\lambda}}{x - \lambda},
\]

where \( A, B, C_{\lambda}, D_{\lambda}, E_{\lambda} \) and \( F_{\lambda} \) are some constants. It follows from the residue theorem that we have

\[
A = 0 \text{ and } C_{\lambda} = -D_{\lambda} = 2\lambda \left\{ \prod_{\lambda' \in I \setminus \{\lambda\}} \left( \frac{\lambda + \lambda'}{\lambda - \lambda'} \right) \prod_{\lambda' \in J} \left( \frac{\lambda - 2 - \lambda'}{\lambda - 2 + \lambda'} \right) \right\}.
\]

Similarly, we get expressions for \( E_{\lambda} \) and \( F_{\lambda} \). Moreover, If \( 1 \in I \cap J \), then

\[
B = -8 \prod_{\lambda \in I \setminus \{1\}} \left( \frac{1 + \lambda}{1 - \lambda} \right) \prod_{\lambda \in J \setminus \{1\}} \left( \frac{1 + \lambda}{1 - \lambda} \right).
\]

All the statements of Lemma follow from the above expressions for coefficients \( C_{\lambda}, D_{\lambda} \) and \( B \) by direct calculations.
Lemma 3  For any two subsets $I$, $J$ of the set $[n;m]$, we have
\[ \sum_{\lambda \in I \cup J} b_{\lambda}^{I,J} = 4(|I| - |J|)^2 - 4(|I| + |J|). \]  \hspace{1cm} (4.2)

Proof. By Lemma 2, we have
\[
\sum_{\lambda \in I \cup J} b_{\lambda}^{I,J} = \lim_{x \to \infty} \left\{ (\text{L.H.S of Lemma 3 (4.2)}) - 2 \right\} x^2
\]
\[
= \lim_{x \to \infty} x^2 \left\{ \prod_{\lambda \in I} \left( \frac{x + 2 + \lambda}{x + 2 - \lambda} \right) \cdot \prod_{\lambda \in J} \left( \frac{x - \lambda}{x + \lambda} \right) + \prod_{\lambda \in I} \left( \frac{x - \lambda}{x + \lambda} \right) \cdot \prod_{\lambda \in J} \left( \frac{x + 2 + \lambda}{x + 2 - \lambda} \right) - 2 \right\}
\]
\[
= \lim_{x \to \infty} \frac{A(x)}{\prod_{\lambda \in I} (x + 2 - \lambda) \cdot \prod_{\lambda \in I} (x + \lambda) \cdot \prod_{\lambda \in J} (x + 2 - \lambda) \cdot \prod_{\lambda \in J} (x + \lambda)},
\]
where
\[
A(x) = x^2 \left\{ \prod_{\lambda \in I} (x + 2 + \lambda) \cdot \prod_{\lambda \in I} (x + \lambda) \cdot \prod_{\lambda \in J} (x + 2 - \lambda) \cdot \prod_{\lambda \in J} (x - \lambda) + \prod_{\lambda \in I} (x + 2 - \lambda) \cdot \prod_{\lambda \in I} (x - \lambda) \cdot \prod_{\lambda \in J} (x + 2 + \lambda) \cdot \prod_{\lambda \in J} (x + \lambda) + 2 \cdot \prod_{\lambda \in I} (x + 2 - \lambda) \cdot \prod_{\lambda \in I} (x + \lambda) \cdot \prod_{\lambda \in J} (x + 2 - \lambda) \cdot \prod_{\lambda \in J} (x + \lambda) \right\}.
\]
It is easy to see that coefficients of $x^{|I|+|J|+2}$ and $x^{|I|+|J|+1}$ in $A(x)$ are disappear. Now let us compute the coefficient of $x^{|I|+|J|}$ in $A(x)$:
\[
\sum_{\lambda \in I} \{(2 + \lambda)(\lambda) + (2 - \lambda)(-\lambda) - 2(2 - \lambda)(\lambda)\}
\]
\[
+ \sum_{\lambda_1, \lambda_2 \in I, \lambda_1 < \lambda_2} \{(2 + \lambda_1)(2 + \lambda_2) + (2 + \lambda_1)(\lambda_2) + (2 + \lambda_2)(\lambda_1) + (\lambda_1)(\lambda_2) + (2 - \lambda_1)(2 - \lambda_2) + (2 - \lambda_1)(-\lambda_2) + (2 - \lambda_2)(-\lambda_1) + (-\lambda_1)(-\lambda_2) - 2(2 - \lambda_1)(2 - \lambda_2) - 2(2 - \lambda_1)(\lambda_2) - 2(2 - \lambda_2)(\lambda_1) - 2(\lambda_1)(\lambda_2)\}
\]
\[
+ \sum_{\lambda_1 \in I, \lambda_2 \in J} \{(2 + 2\lambda_1)(2 - 2\lambda_2) + (2 - 2\lambda_1)(2 + 2\lambda_2) - 8\}
\]
\[
+ \sum_{\lambda \in J} \{(2 + \lambda)(\lambda) + (2 - \lambda)(-\lambda) - 2(2 - \lambda)(\lambda)\}\]
Lemma 5 This lemma follows from Lemma 2 by direct calculation.

Easy proof by direct computation.

Lemma 4 For an element \( \lambda \in I \cap J \), we have \( b_\lambda^{I,J} = 0 \) if and only if \( \lambda - 2 \in I \cap J \). For an element \( \lambda \in I \setminus (I \cap J) \), we have \( b_\lambda^{I,J} = 0 \) if and only if \( \lambda - 2 \in J \).

This lemma follows from Lemma 2 by direct calculation.

Lemma 5 If \( n_1 + m_1 = n_2 + m_2 \), then

\[
4z^2 w^2 D_x^2 z^{n_1} w^{m_1} \circ z^{n_2} w^{m_2} = \left[ -\left\{ (n_1 + n_2) - (n_1 - n_2)^2 \right\} w^2 + \left\{ (m_1 + m_2) - (m_1 - m_2)^2 \right\} z^2 \right] z^{n_1 + n_2} w^{m_1 + m_2}.
\]

Easy proof by direct computation.

Now we are ready to prove our main theorem.

Since \( D_x^2 \) is a bilinear operator, using the above identity (4.3), we have

\[
\left( -\tilde{a}_{n+2m+2} z^2 + \tilde{b}_{n+2m+2} w^2 \right) U_{n,m}^2 + 8 z^2 w^2 D_x^2 U_{n,m} \circ U_{n,m}
\]

\[
= \sum_{I,J \subseteq [n:m]} P_b(I, J) w^2 d_{n,m}(I) d_{n,m}(J)(-1)^{c(I)+c(J)} e_{I}^{(m,0)}(z, w) e_{J}^{(m,0)}(z, w)
\]

\[
- \sum_{I,J \subseteq [n:m]} P_a(I, J) z^2 d_{n,m}(I) d_{n,m}(J)(-1)^{c(I)+c(J)} e_{[n:m]\setminus I}^{(m,0)}(z, w) e_{[n:m]\setminus J}^{(m,0)}(z, w),
\]

The latter expression coincide with the RHS (4.2), and therefore the proof of Lemma 3 is finished.
where we write

\[ P_a(I, J) = \bar{a}_{n+2m+2} - 2 \left\{ (|I| + |J|) - (|I| - |J|)^2 \right\}, \]

\[ P_b(I, J) = \bar{b}_{n+2m+2} - 2 \left\{ (|I| + |J|) - (|I| - |J|)^2 \right\}. \]

By Lemma 3 we have

\[
P_a(I, J) = \bar{a}_{n+2m+2} + \frac{1}{2} \sum_{\lambda \in I \cup J} b^{I,J}_{\lambda} \]

\[
= \bar{a}_{n+2m+2} \left( 1 + \sum_{\lambda \in I \cup J} \frac{b^{I,J}_{\lambda}}{2(n + 2m + 2 - \lambda)(n + 2m + \lambda)} \right) \]

\[
- \sum_{\lambda \in I \cup J} \frac{\bar{a}_{n+2m+2} - (n + 2m + 2 - \lambda)(n + 2m + \lambda)}{2(n + 2m + 2 - \lambda)(n + 2m + \lambda)} b^{I,J}_{\lambda}. \]

Using Lemma 3, the latter expression can be transformed to the following form:

\[
P_a(I, J) = \frac{\bar{a}_{n+2m+2}}{2} \left( \prod_{\lambda \in I} \frac{n + 2m + 2 + \lambda}{n + 2m + 2 - \lambda} \prod_{\lambda \in J} \frac{n + 2m - \lambda}{n + 2m + \lambda} \right) \]

\[
+ \sum_{\lambda \in I \cup J} \frac{\bar{a}_{n+2m+2} - (n + 2m + 2 - \lambda)(n + 2m + \lambda)}{2(n + 2m + 2 - \lambda)(n + 2m + \lambda)} b^{I,J}_{\lambda}. \]

Now if \( n + 2m \in J \), then it is not difficult to check that

\[
\frac{\bar{b}_{n+2m+2}}{2} \prod_{\lambda \in I} \frac{n + 2m + 2 + \lambda}{n + 2m + 2 - \lambda} \prod_{\lambda \in J} \frac{n + 2m - \lambda}{n + 2m + \lambda} \]

\[
\times w^2 d_{n,m}(I) d_{n,m}(J) (-1)^{e(I)+e(J)} e^{(n,m,0)}_I(z, w) e^{(n,m,0)}_J(z, w) \]

\[
= \frac{1}{2} d_{n,m+1}(I) d_{n,m-1}(J) (-1)^{e(I)+e(J)} e^{(n,m+1,0)}_I(z, w) e^{(n,m-1,0)}_J(z, w). \]

Hence we have

\[
(-\bar{a}_{n+2m+2} z^2 + \bar{b}_{n+2m+2} w^2) U^2_{n,m} + 8z^2 w^2 D_z^2 U_{n,m} = U_{n,m} \]

\[
= \sum_{I \subseteq [n,m+1], J \subseteq [n,m-1]} d_{n,m+1}(I) d_{n,m-1}(J) (-1)^{e(I)+e(J)} e^{(n,m+1,0)}_I(z, w) e^{(n,m-1,0)}_J(z, w). \]
Let us observe that if \( \lambda \) and \( n \lambda \in \lambda \) and 

Similarly, if 

To continue our proof, for \( \lambda \neq 1, \lambda \in I \), we define the function \( \text{Split}(b_\lambda)(I, J) \) by 

Let us observe that if \( \lambda \in I \cap J \) and \( \lambda \neq 1 \), then 

Similarly, if \( \lambda \in I, \lambda \neq 1 \) and \( \lambda \notin J \), then \( b^{I,J}_\lambda = \text{Split}(b_\lambda)(I, J) \).

Finally, if \( \lambda \in I, \lambda \neq 1 \) and \( \lambda - 2 \notin J \), then 

\[
\prod_{\lambda' \in I \setminus \{\lambda\}} \left( \frac{\lambda + \lambda'}{\lambda - \lambda'} \right) d_{n,m}(I) = (-1)^{\#[n;m] > \lambda} \prod_{\lambda \in [n;m] \setminus (I \setminus \{\lambda\})} \left( \frac{\lambda + \lambda'}{\lambda - \lambda'} \right) d_{n,m}([n;m] \setminus (I \setminus \{\lambda\}))
\]

and 

\[
\prod_{\lambda' \in I \setminus \{\lambda\}} \left( \frac{\lambda - 2 - \lambda'}{\lambda - 2 + \lambda'} \right) d_{n,m}(I) = (-1)^{\#[n;m] > \lambda - 2} \prod_{\lambda \in [n;m] \setminus (I \cup (\lambda - 2))} \left( \frac{\lambda - 2 - \lambda'}{\lambda - 2 + \lambda'} \right) d_{n,m}([n;m] \setminus (I \cup \{\lambda - 2\})),
\]

where \( \#[n;m] > \lambda = \# \{ i \in [n;m] | i > \lambda \} \).

Let us summarize the results of above calculations as an auxiliary lemma.
Lemma 6 If $\lambda \in I$, $\lambda \neq 1$ and $\lambda - 2 \notin J$, then

$$\text{Split}(b_\lambda(I, J) d_{n, m}(I) d_{n, m}(J)) = (-1)^A \text{Split}(b_\lambda(I', J') d_{n, m}(I') d_{n, m}(J')),$$

where $I' = [n; m] \setminus \{I \setminus \{\lambda\}\}$, $J' = [n; m] \setminus \{J \cup \{\lambda - 2\}\}$ and $A = 1$ if $\lambda \leq n$, and $A = -1$ if $\lambda > n$.

Note that under the assumption of Lemma 6, we have $\overline{\lambda} e_{I}^{(n, m, 0)} e_{J}^{(n, m, 0)} = \overline{\lambda} e_{I'}^{(n, m, 0)} e_{J'}^{(n, m, 0)}$ and $\overline{a} w e_{I}^{(n, m, 0)} e_{J}^{(n, m, 0)} = \overline{a} w e_{I'}^{(n, m, 0)} e_{J'}^{(n, m, 0)}$. Hence, using Lemma 4, we obtain the following equality

$$(-\overline{a} n + 2 m + 2 z^2 + \overline{b} n + 2 m + 2 w^2) U_{n, m} + 8 z^2 w^2 D_{2} U_{n, m} \circ U_{n, m}$$

$$= U_{n, m + 1} U_{n, m - 1}$$

$$+ \sum_{I, J \in [n; m], 1 \leq \ell \cap J} \frac{b_{I, J}^{I, J} d_{n, m}(I) d_{n, m}(J) (-1)^{c(I) + c(J)}}{2(n + 2 m + 1)^2} \overline{b}_1 w^2 e_{J}^{(n, m, 0)}(z, w) e_{J}^{(n, m, 0)}(z, w)$$

$$- \sum_{I, J \in [n; m], 1 \leq \ell \cap J} \frac{b_{I, J}^{I, J} d_{n, m}(I) d_{n, m}(J) (-1)^{c(I) + c(J)}}{2(n + 2 m + 1)^2} \overline{a} z^2 e_{[n; m] \setminus J}^{(n, m, 0)}(z, w) e_{[n; m] \setminus J}^{(n, m, 0)}(z, w).$$

To simplify the right hand side of the latter equality we are going to use the following expression for the coefficients $b_{I, J}^{I, J}$ which is an easy consequence of Lemma 2:

$$b_{I, J}^{I, J} = -8 \prod_{\lambda' \in I \setminus \{1\}} \left(\frac{1 + \lambda'}{1 - \lambda'}\right) \prod_{\lambda' \in J \setminus \{1\}} \left(\frac{1 + \lambda'}{1 - \lambda'}\right).$$

After substituting the above expressions for the coefficients $b_{I, J}^{I, J}$ to the both sums which appear in the right hand side of the equality under consideration, it remains to observe that if $1 \in I \subset [n; m]$, then

$$\prod_{\lambda' \in I \setminus \{1\}} \left(\frac{1 + \lambda'}{1 - \lambda'}\right) d_{n, m}(I) = \prod_{\lambda' \in [n; m] \setminus I} \left(\frac{1 + \lambda'}{1 - \lambda'}\right) d_{n, m}(I \setminus \{1\}),$$

$$a^2 b w^2 z^2 e_{I}^{(n, m, 1)}(z, w) e_{J}^{(n, m, 1)}(z, w) = \overline{b}_1 w^2 e_{I}^{(n, m, 0)}(z, w) e_{J}^{(n, m, 0)}(z, w)$$

and

$$a b^2 w^2 z^2 e_{I}^{(n, m, 1)}(z, w) e_{J}^{(n, m, 1)}(z, w) = \overline{a} z^2 e_{[n; m] \setminus J}^{(n, m, 0)}(z, w) e_{[n; m] \setminus J}^{(n, m, 0)}(z, w).$$

The proof of Theorem 1 is finished.
Remarks 1. If $n = 0$, then $U_{0,m} = U_{m+1}(z^2, w^2; a, b)$ coincides with the Umemura polyno-
mial, and $U_{0,m}^{(1)} = 0$. In this case the recurrence relation (4.1) has been used by M. Taneda in his proof of Noumi–Okada–Okamoto–Umemura’s Conjecture (2.6).

2. Note that $U_{0,m} = U_{2,m-1}^{(1)}/(2m + 1)$, and more generally

$$U_{k,m}^{(k)} = U_{k+2,m-1}^{(k+1)}(2k + 1)!!/(2m - 1)!!/(2k + 2m + 1)!!,$$

where $(2n + 1)!! = 1 \cdot 3 \cdot 5 \cdots (2n + 1)$.

3. ”Unwanted term” in (4.1) which contains $\left(U_{n,m}^{(1)}\right)^2$ vanishes if either $a = 0$, or $b = 0$, or $a = b$.

In the case $a = b$ and $k = 0$ the expression $e^{(n,m,k)}(z, w)$ doesn’t depend on a subset $I \subset [n; m]$ and is equal to $a_{[n;m]}^2|I|w^{[n;m]\setminus I}$. Hence, in this case

$$U_{n,m}^{(0)}(z, w; a, a) = a_{[n;m]} \sum_{I \subset [n; m]} d_{n,m}(I)(-1)^{|I|}z^{(|I|\setminus [n;m]\setminus I)}$$

$$= a_{[n;m]}(z + w)^{\left(n + m + 1\right)}(z - w)^{\left(m + 1\right)}.$$

The last equality in (4.4) has been proved for the first time by J.F. van Diejen and A.N. Kirillov [DK]. On the other hand, we can show that polynomials

$$X_{n,m}(z, w; a) = a_{[n;m]}(z + w)^{\left(n + m + 1\right)}(z - w)^{\left(m + 1\right)}$$

also satisfy the recurrence relation (4.1) and coincide with polynomials $U_{n,m}^{(0)}(z, w; a, a)$ if $m = 0$. From this observation we can deduce the equality $X_{n,m}(z, w; a) = U_{n,m}^{(0)}(z, w; a, a)$, which is equivalent to the main identity from [DK]. Another case when ”unwanted term” in (4.1) vanishes is the case when either $a = 0$, or $b = 0$. In this case we have

**Corollary 1** Assume that $b = 0$, then polynomial $U_{n,m}(z, w; a, 0)$ defines a solution to the equation Painlevé VI.

Finally, we are going to compare polynomials $U_{n,m}(z, w; a, 0)$ and $U_m(z, w; \alpha, \beta)$. For this goal, let us consider functions

$$h_0 := h_0(t) = \left\{ b_1^2 \left( \sqrt{t} - \sqrt{t - 1} \right)^2 + b_2^2 \left( \sqrt{t} + \sqrt{t + 1} \right)^2 \right\}/4,$$

and

$$h_{n,m} := h_{n,m}(b_1, b_2) = t(t - 1) \log(U_{n,m})' - h_0.$$
Proposition 4 (i) $h_{0,m}$ satisfies the Painlevé–Okamoto equation $E_{VI}(b_1, b_2, m + 1/2, 0)$;

(ii) $h_{1,m} = -(2t - 1)(m + 1)^2/2$ satisfies the equation $E_{VI}(0, m + 1, b_3, b_4)$;

(iii) $h_{n,m}(0, b_2)$ satisfies the equation $E_{VI}(0, b_2, n + 2m + 1/2, m + 1/2)$.

Proposition 4 follows from Lemma 7 and Lemma 8 below.

Let us define $U_{n,m}(b_1, b_2) := U_{n,m}(z, w; -4b_1^2, -4b_2^2)$, then

Lemma 7

$$U_{n,m}(0, b_2) = \begin{cases} b_{[n;m]_{odd}}(n/2)^2 U_{0,m+n/2}(n/2, b_2), & \text{if } n \text{ is even,} \\ b_{[n;m]_{odd}}(n^2/2)^2 U_{0,m+n/2}(m + n + 1/2, b_2), & \text{if } n \text{ is odd,} \end{cases}$$

where $[n;m]_{odd} = \{i \in [n;m] | i \text{ is odd} \}$.

Proof. By the definition of generalized Umemura polynomials, one can see

$$U_{n,m}(0, b_2) = \sum_{I \subset [n;m]_{even}} d_{n,m} I (-1)^{e(I)} e_I^{(n,m,0)}(z, w).$$

Assume first that both $n$ and $i$ are even, then we have

$$a_i|_{b_1=0} = ((i-1)!!)^2,$$

$$a_i|_{b_1=n/2} = \prod_{j=1}^{i/2} \left(-n^2 + (2j - 1)^2\right) a_i|_{b_1=0}.$$ 

Now if $i \leq n$, then

$$\prod_{j \in [n;m]_{odd}} \left|\frac{i+j}{i-j}\right| a_i|_{b_1=0} = \frac{(i+n-1)!!}{(n-1-i)!!} = (n-i+1)(n-i+3) \cdots (n-i-1)$$

$$= \prod_{j=1}^{i} (n^2 - (2j-1)^2) = (-1)^{i/2} a_i|_{b_1=n/2},$$

and if $i > n$, then

$$\prod_{j \in [n;m]_{odd}} \left|\frac{i+j}{i-j}\right| a_i|_{b_1=0} (-1)^{i+n/2} = (-1)^{i+n/2} (i+n-1)!!(i-n-1)!!$$

$$= (-1)^{i+n/2} (n-i+1)(n-i+3) \cdots (n-i-1)$$

$$= \prod_{j=1}^{i} (n^2 - (2j-1)^2) = (-1)^{i/2} a_i|_{b_1=n/2}. $$

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Finally, let \( n \) be odd and \( i \) be even, then
\[
\prod_{j \in \left[ n; m \right] \text{odd}} \frac{|i + j|}{|i - j|} a_i |_{b_1 = 0} = \frac{(i + n + 2m)!!}{(n + 2m - i)!!}
\]
\[
= (n + 2m - 2 - i)(n + 2m + 4 - i) \cdots (n + 2m + i)
\]
\[
= \prod_{j=1}^{i} ((n + 2m + 1)^2 - (2j - 1)^2) = (-1)^{i/2} a_i |_{b_1 = (n + 2m + 1)/2}.
\]

\[\square\]

From Lemma 7 we can deduce the following

**Lemma 8**

\[
h_{n,m}(0, b_2) = \begin{cases} 
  h_{0,m+\frac{1}{2}} \left( \frac{n}{2}, b_2 \right), & \text{if } n \text{ is even,} \\
  h_{0,n-\frac{1}{2}} \left( m + \frac{n + 1}{2}, b_2 \right), & \text{if } n \text{ is odd.}
\end{cases}
\]

It follows from Lemma 7 (4.5) and Theorem 1, that Umemura’s polynomials \( U_m(b_1, b_2) \) satisfy a new recurrence relation with respect to the first argument \( b_1 \).

**Theorem 2**

\[
U_m(b_1 - 1, b_2) U_m(b_1 + 1, b_2) (b_1^2 - b_2^2) = (b_1^2 - b_2^2) U_m^2(b_1, b_2) + 2z^2 D_x^2 U_m(b_1, b_2) \circ U_m(b_1, b_2).
\]

Recall that \( D_x^2 \) denotes the second Hirota derivative.

**Proof.** It follows from Lemma 7 (4.5) and Theorem 1 that we have

\[
U_{0, \frac{n+1}{2}}(m + 1 + \frac{n + 1}{2}, b_2) U_{0, \frac{n+1}{2}}(m + 1 + \frac{n + 1}{2}, b_2) w^2 b_{n+2m+2} = \bar{b}_{n+2m+2} w^2 \left( U_{0, \frac{n+1}{2}}(m + \frac{n + 1}{2}, b_2) \right)^2 \\
+ 8z^2 w^2 D_x^2 U_{0, \frac{n+1}{2}}(m + \frac{n + 1}{2}, b_2) \circ U_{0, \frac{n+1}{2}}(m + \frac{n + 1}{2}, b_2).
\]

We regard the recurrence relation of Theorem 2 as an algebraic equation with respect to the variable \( b_1 \). By the above identity (4.8), this algebraic equation has infinitely many solutions. The proof of Theorem 2 is finished. 

\[\square\]
Corollary 2

\[ U_{m-1}(b_1, b_2)U_{m+1}(b_1, b_2) - 4w^2(b_1^2 - b_2^2)U_m(b_1 - 1, b_2)U_m(b_1 + 1, b_2) \quad (4.9) \]

\[ = (\bar{a}_{2m+2}) U_m^2(b_1, b_2). \]

\[ U_{m-1}(b_1, b_2)U_{m+1}(b_1, b_2) - 4z^2(b_1^2 - b_2^2)U_m(b_1, b_2 - 1)U_m(b_1, b_2 + 1) \quad (4.10) \]

\[ = (\bar{b}_{2m+2}) U_m^2(b_1, b_2), \]

\[ U_{m-1}(b_1, b_2)U_{m+1}(b_1, b_2) - \bar{b}_{2m+2}w^2U_m(b_1 - 1, b_2)U_m(b_1 + 1, b_2) \quad (4.11) \]

\[ + \bar{a}_{2m+2}z^2U_m(b_1, b_2 - 1)U_m(b_1, b_2 + 1) = 0. \]

Let us define functions \( X(k, l, m) \) by the following recurrence relations:

\[ X(k, l, m - 1)X(k, l, m + 1) = X(k - 1, l, m)X(k + 1, l, m)(4(b_2 + l)^2 - (2m + 1)^2)w^2 \]

\[ = X(k, l - 1, m)X(k, l + 1, m)(-4(b_1 + k)^2 + (2m + 1)^2)z^2, \]

with initial conditions \( X(0, 0, m) = X(0, 1, m) = X(1, 0, m) = X(1, 1, m) = 1. \) To solve these recurrence relations, let us introduce the following functions.

\[ Y_{l,m}(n) = \prod_{j=1}^{n} Y_{l,m-n+1+2j}, \]

\[ Z_{l,m}(n) = \prod_{j=1}^{n} Z_{l,m-n+1+2j}. \]

With these notation the explicit formula for \( X(k, l, m) \) looks as follows

\[ X(k, l, m) = 1 / \left( \prod_{j=1}^{k-1} Y_{l,m}(j) \prod_{j=1}^{l-1} Z_{l,m}(j) \right). \]

Finally let us introduce the function \( T_{(k,l,m)} \) to be

\[ T_{(k,l,m)} := T_{(k,l,m)}(b_1, b_2; z, w) = U_m(b_1 + k, b_2 + l)X(k, l, m) \quad (4.12) \]

Proposition 5 Functions \( T_{(k,l,m)} \) satisfy the Hirota–Miwa equation

\[ T_{(k-1,l,m)}T_{(k+1,l,m)} + T_{(k,l-1,m)}T_{(k,l+1,m)} + T_{(k,l,m-1)}T_{(k,l,m+1)} = 0. \quad (4.13) \]

This is a direct consequence of Corollary 2.
§5. Example

Let us define function \( q_m := q_m(t) \) by the following formula

\[
q_m - t = 4U_m^2 \left\{ \left( m + \frac{1}{2} \right) t(t - 1) \frac{d}{dt} \log U_{m+1} - \left( m + \frac{3}{2} \right) t(t - 1) \frac{d}{dt} \log U_m \right. \\
- \frac{1}{2} b_1 b_2 + \frac{1}{4} \left( b_1^2 \frac{z}{w} + b_2^2 \frac{w}{z} \right) \right\} / (U_{m+1} U_{m-1} - (2m + 1)^2 U_m^2).
\]

One can check that the function \( q_m \) is a solution to both equations \( P_{VI}(b_1, b_2, m + \frac{1}{2}, 0) \) and \( P_{VI}(b_1, b_2, 0, m + \frac{1}{2}) \). It follows from Okamoto’s theory \[OLOV\] that the function

\[
\bar{h}_{1,m} = t(t - 1) \frac{d}{dt} \log U_{m+1} - \frac{1}{4} \left( b_1^2 \frac{z}{w} + b_2^2 \frac{w}{z} \right) + \left( m + \frac{1}{2} \right) q_m - \frac{1}{2} \left( m + \frac{1}{2} \right)
\]

is also a solution to \( E_{VI}(b_1, b_2, m + \frac{1}{2}, 1) \). Based on the latter expression for the function \( \bar{h}_{1,m} \), and using Proposition 4 (iii), we come to the following

**Proposition 6** If \( b_1 = 0 \), then we have

\[
U_{m+1} U_{m-1} - (2m + 1)^2 U_m^2 = \frac{1}{4b_2^2} U_{2,m-1}^2,
\]

where \( U_m := U_m(0, b_2) \) is a special case of Umemura’s polynomial, and

\[
U_{2,m-1} = U_{2,m-1}(0, b_2).
\]

Proof follows from Corollary 2, (4.10).
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