Abstract. We give closed product formulas for the irreducible characters of the symmetric groups related to rectangular 'almost square' Young diagrams $p \times (p + \delta)$ for a fixed value of an integer $\delta$ and an arbitrary integer $p$.

1. Introduction

1.1. The main result. Let $\pi$ be a partition of an integer $k$ and let $\lambda$ be a partition of an integer $n$. Let $\text{Ch}_\pi(\lambda)$ denote the normalized character of the symmetric group $S_n$ defined by

\begin{equation}
\text{Ch}_\pi(\lambda) = \begin{cases} n^{\lambda} \cdot \frac{\chi_\mu(1^n)}{f_\mu}, & \text{if } n \geq k, \\ 0, & \text{otherwise}, \end{cases}
\end{equation}

where $\chi_\lambda$ is the usual character of the irreducible representation of the symmetric group $S_n$ associated with $\lambda$, evaluated on the conjugacy class associated with $\mu$. This choice of the normalization is quite natural, in particular in the context of the asymptotic representation theory, see for example [Bia03, IK99].

In this note we will concentrate on the case when $\pi = (k)$ consists of a single part, i.e., on the characters evaluated on a single cycle (augmented by a necessary number of fixpoints). Also, we will concentrate on the special case when

\[ \lambda = p \times q = (q, \ldots, q) \]

is a rectangular Young diagram, see Figure 1. We will give closed product formulas for such characters in the almost square setting when $q - p$ is a fixed integer and $p$ is arbitrary. The exact form of the formula depends on the parity of the length $k$ of the cycle, as well as on the parity of the difference $q - p$, so altogether there are four distinct formulas for such

2020 Mathematics Subject Classification. 20C30 (Primary), 05A15 (Secondary).

Key words and phrases. Characters of the symmetric groups, rectangular Young diagrams, Stanley polynomials.
characters. As a teaser, we start with the case when $k = 2j - 1$ is odd while $q - p = 2d$ is even.

**Theorem 1.1.** Let $j$ be a positive integer. Then

\[
\text{Ch}_{2j-1} ((e - d) \times (e + d)) = (-1)^{j-1} \text{Cat}(j-1) \sum_{k=0}^{j} f_k(j) \left( \prod_{r=0}^{k-1} (d^2 - r^2) \right) \left( \prod_{r=k}^{j-1} (e^2 - r^2) \right),
\]

where

\[
\text{Cat}(j-1) = \frac{(2j-2)!}{(j-1)! j!}
\]

is the Catalan number; furthermore $f_0(j) = 1$ and

\[
f_k(j) = (-1)^k j^{2k} (2j-1)^{\uparrow\uparrow k}
\]

for $1 \leq k \leq j$.

Above we used the double rising factorial $a^{\uparrow\uparrow k}$ (which is somewhat analogous to the double factorial $a!!$ in which the factors form an arithmetic progression with the step 2) which is defined by

\[
a^{\uparrow\uparrow k} = \prod_{r=0}^{k-1} (a + 2r) = 2^k \left( \frac{a}{2} \right)^{\uparrow k}
\]

for a complex number $a$ and a positive integer $k$, and by $a^{\uparrow\uparrow 0} := 1$. Thus, $f_k(j) \in \mathbb{Q}[j]$ is a polynomial in the variable $j$ of degree $2k$.

Note that in the above result there is no assumption that $e - d$ and $e + d$ are non-negative integers; in fact $e$ and $d$ can be arbitrary complex numbers.
The reader may feel uneasy about the case when \((e - d) \times (e + d)\) does not make sense as a Young diagram; later on in Corollary 2.1 we will explain why in this case the left-hand side of (1.2) still makes sense.

1.2. The product formula. In the special case when \(|d|\) is a small integer, the formula (1.2) takes a simpler form because each summand on the right-hand side which corresponds to \(k\) such that \(k > |d|\) is equal to zero; in this way the sum can be taken over \(k \in \{1, \ldots, j \wedge |d|\}\). This observation is especially convenient in the aforementioned almost square setting when we consider the character corresponding to a rectangular Young diagram \(\lambda = p \times q\) in the setup where \(q - p = 2d\) is a fixed even integer and \(p\) is arbitrary. In particular, we get the following closed product form for the character.

Corollary 1.2. Let \(j, p, q\) be positive integers; we denote by \(n = pq = |p\times q|\) the number of the boxes of the corresponding Young diagram. Suppose that \(q - p\) is an even integer which we denote by \(2d := q - p\). Then

\[
\text{Ch}_{2j-1}(p \times q) = (-1)^{j-1} \text{Cat}(j - 1) G_d(j, n) \prod_{r=0}^{j-|d|-1} \left( n - r(r + 2|d|) \right),
\]

where

\[
G_d(j, n) = \sum_{k=0}^{|d|} f_k(j) \left( \prod_{r=0}^{k-1} (d^2 - r^2) \right) \left( \prod_{r=k}^{|d|-1} (n + d^2 - r^2) \right)
\]

with \(f_k(j)\) given, as before, by (1.3).

We can see that, with \(d\) fixed, \(G_d(j, n) \in \mathbb{Q}[j, n]\) is a polynomial in the variables \(j, n\) of the total degree \(2|d|\) if we declare that the degrees of the variables \(j\) and \(n\) are given by \(\deg j = 1\) and \(\deg n = 2\). In fact, \(G_d(j, n) \in \mathbb{Z}[j, n]\) is a polynomial with integer coefficients, see Proposition 5.1.

Example 1.3.

\[
G_0(j, n) = 1,
G_{\pm 1}(j, n) = n + 1 - j(2j - 1),
G_{\pm 2}(j, n) = (n + 4)(n + 3) - 4j(2j - 1)(n + 3) + 2j(j - 1)(2j - 1)(2j + 1).
\]

1.3. Convention for products. Corollary 1.2 remains valid in the case when \(j \leq |d|\), however in this case the product on the right hand side of
should be understood using the following non-standard convention:

\[
\prod_{r=0}^{l} a_r = \begin{cases} 
    a_0 \cdots a_l & \text{if } l \geq 0, \\
    1 & \text{if } l = -1, \\
    \frac{1}{a_{l+1}a_{l+2} \cdots a_{-2}a_{-1}} & \text{if } l \leq -2.
\end{cases}
\]

This convention was chosen in such a way that the identity

\[
\prod_{r=0}^{l+1} a_r = \left[ \prod_{r=0}^{l} a_r \right] \cdot a_{l+1}
\]

holds for any (positive or negative) integer \( l \).

1.4. Collection of results. Below we present a collection of the results which cover the remaining choices for the parity for the length of the cycle and the difference \( q - p \) between the rectangle sides.

1.4.1. The length of the cycle is odd, the difference of the rectangle sides is odd. The following result is a counterpart of Theorem 1.1 which is particularly useful for a rectangular Young diagram \( \lambda = p \times q \) for which \( q - p \) is an odd integer.

**Theorem 1.4.** Let \( j \) be a positive integer. Then

\[
\text{Ch}_{2j-1} \left( (e - d) \times (e + d) \right) = (-1)^{j-1} \text{Cat}(j-1) \sum_{k=0}^{j} f_k(j) \left[ \prod_{r=0}^{k-1} \left( d^2 - (r + \frac{1}{2})^2 \right) \right] \left( \prod_{r=k}^{j-1} \left( e^2 - (r + \frac{1}{2})^2 \right) \right).
\]

**Corollary 1.5.** Let \( j, p, q \) be positive integers and set \( n = pq \). Suppose that \( q - p \) is an odd integer \( 2d := q - p \), where \( d \in \{ \pm \frac{1}{2}, \pm \frac{3}{2}, \ldots \} \). Then

\[
\text{Ch}_{2j-1} \left( p \times q \right) = (-1)^{j-1} \text{Cat}(j-1) H_d(j, n) \prod_{r=0}^{j-|d| - \frac{1}{2}} \left( n - r(r + 2|d|) \right),
\]

where

\[
H_d(j, n) = \sum_{k=0}^{|d| - \frac{1}{2}} (-1)^k j^{\downarrow k} (2j - 1)^{\uparrow k} \frac{1}{k! (2k - 1)!!} \prod_{r=0}^{k-1} \left( d^2 - (r + \frac{1}{2})^2 \right) \prod_{r=k}^{|d| - \frac{3}{2}} \left( n + d^2 - (r + \frac{1}{2})^2 \right).
\]
Example 1.6.
\[ H_{\pm \frac{1}{2}}(j, n) = 1, \]
\[ H_{\pm \frac{3}{2}}(j, n) = n + 2 - 2j(2j - 1), \]
\[ H_{\pm \frac{5}{2}}(j, n) = (n + 6)(n + 4) - 6j(2j - 1)(n + 4) + 4j(j - 1)(2j - 1)(2j + 1). \]

1.4.2. The length of the cycle is even, the difference of the rectangle sides is even. The following result is a direct counterpart of Theorem 1.1 for the even cycle.

Theorem 1.7. Let \( j \) be a positive integer. Then
\[
\text{Ch}_{2j} \left( (e - d) \times (e + d) \right)
= (-1)^{j-1} \binom{2j - 1}{j} \sum_{k=0}^{j} g_k(j) 2d \left( \prod_{r=1}^{k} (d^2 - r^2) \right) \left( \prod_{r=k+1}^{j} (e^2 - r^2) \right),
\]
where \( g_0(j) = 1 \) and
\[
g_k(j) = (-1)^k \frac{j^{2(k+1)}(2j+1)^{2k+1}}{k!(2k+1)!!}
\]
for \( 1 \leq k \leq j \).

Corollary 1.8. Let \( j, p, q \) be positive integers and set \( n = pq \). Suppose that \( q - p \) is an even integer \( 2d = q - p \), where \( d \in \{0, \pm 1, \pm 2, \ldots \} \). Then
\[
\text{Ch}_{2j} \left( p \times q \right) = (-1)^{j-1} \binom{2j - 1}{j} I_d(j, n) \prod_{r=0}^{j-|d|} (n - r(r + 2|d|)), \]
where \( I_0(j, n) = 0 \) and, for \( d \in \{ \pm 1, \pm 2, \ldots \} \)
\[
I_d(j, n) = d \sum_{k=0}^{|d|-1} (-1)^k \frac{j^{2(k+1)}(2j+1)^{2k+1}}{k!(2k+1)!!} \left( \prod_{r=1}^{k} (d^2 - r^2) \right) \left( \prod_{r=k+1}^{|d|-1} (n + d^2 - r^2) \right).
\]

With \( d \) fixed, \( I_d(j, n) \in \mathbb{Q}[j, n] \) is a polynomial in the variables \( j, n \) of the total degree \( 2|d| - 1 \) if we give \( \deg j = 1 \) and \( \deg n = 2 \).

Example 1.9.
\( I_0(j, n) = 0, \)
\( I_1(j, n) = 1, \)
\( I_2(j, n) = 2(n + 3) - 2j(2j + 1) = 2(n - (j - 1)(2j + 3)), \)
\( I_3(j, n) = 3(n + 8)(n + 5) - 8j(2j + 1)(n + 5) + 4j(j - 1)(2j + 1)(2j + 3). \)
For negative integers \( d \), we have \( I_d(j, n) = -I_{-d}(j, n) \).
1.4.3. *The length of the cycle is even, the difference of the rectangle sides is odd.* The following result is a direct counterpart of Theorem 1.4 for an even cycle.

**Theorem 1.10.** Let \( j \) be a positive integer. Then

\[
\text{Ch}_{2j} \left( (e - d) \times (e + d) \right) = (-1)^{j-1} \binom{2j - 1}{j} \sum_{k=0}^{j} g_k(j) 2d \left( \prod_{r=1}^{k} \left( d^2 - (r - \frac{1}{2})^2 \right) \right) \left( \prod_{r=k+1}^{j} \left( e^2 - (r - \frac{1}{2})^2 \right) \right).
\]

**Corollary 1.11.** Let \( j, p, q \) be positive integers and set \( n = pq \). Suppose that \( q - p \) is an odd integer \( 2d := q - p \), where \( d \in \{ \pm \frac{1}{2}, \pm \frac{3}{2}, \ldots \} \). Then

\[
\text{Ch}_{2j} \left( p \times q \right) = (-1)^{j-1} \binom{2j - 1}{j} J_{d}(j, n) \prod_{r=0}^{j-|d|+\frac{1}{2}} (n - r(r + 2|d|)),
\]

where

\[
J_{d}(j, n) = 2d \sum_{k=0}^{|d| - \frac{1}{2}} (-1)^{k} \binom{j}{k} (2j + 1)^{\uparrow k} \frac{(2j + 1)!}{k! (2k + 1)!} \left( \prod_{r=1}^{k} \left( d^2 - (r - \frac{1}{2})^2 \right) \right) \times \left( \prod_{r=k+1}^{|d| - \frac{1}{2}} \left( n + a^2 - (r - \frac{1}{2})^2 \right) \right).
\]

**Example 1.12.**

\[
J_{\frac{1}{2}}(j, n) = 1,
\]

\[
J_{\frac{3}{2}}(j, n) = 3(n + 2) - 2j(2j + 1) = 3n - 2(j - 1)(2j + 3),
\]

\[
J_{\frac{5}{2}}(j, n) = 5(n + 6)(n + 4) - 10j(2j + 1)(n + 4) + 4j(j - 1)(2j + 1)(2j + 3).
\]

For negative half integers \( d \), we have \( J_{d}(j, n) = -J_{-d}(j, n) \).

1.5. *Vanishing of some special characters.* The special case of Corollary 1.5 for \( d = \frac{3}{2} \) and \( p = 2j - 2 \), \( q = 2j + 1 \) gives rise to the following somewhat surprising corollary for which we failed to find an alternative simple proof.

**Corollary 1.13.** For each integer \( j \geq 2 \) the irreducible character related to the rectangular diagram \( (2j - 2) \times (2j + 1) \) vanishes on the cycle of length \( 2j - 1 \), i.e.

\[
\chi_{2j-1, (2j-3)(2j-1)}^{(2j-2) \times (2j+1)} = 0.
\]
1.6. The link with spin characters related to the staircase strict partition. One of the motivations for the current paper was the recent progress related to the spin characters of the symmetric groups \cite{Sch11}. On one hand, De Stavola \cite[Proposition 4.18, page 91]{DS17} gave an explicit formula for the spin character related to the staircase strict partition 
\[ \Delta_p = (p, p - 1, p - 2, \ldots, 2, 1) \]
which has the property that its double 
\[ D(\Delta_p) = (p + 1, p + 1, \ldots, p + 1) = p \times (p + 1) \]
is a rectangular Young diagram which is almost square. On the other hand, in our recent paper \cite{MS20} we found an identity which gives a link between the spin characters and their usual (linear) counterparts 
\[ 2 \text{Ch}^{\text{spin}}_{2j-1}(\xi) = \text{Ch}_{2j-1}(D(\xi)) \]
which holds true for any strict partition \( \xi \).

By combining these two results one gets a closed product formula for the linear character \( \text{Ch}_{2j-1} \left( p \times (p + 1) \right) \) corresponding to a rectangular Young diagram which is almost square; this closed formula coincides with the special case of Corollary \[1.5\] for \( d = \frac{1}{2} \) (in fact the formula in the original paper of De Stavola has an incorrect sign). In his proof, De Stavola employed some computations in Maple; by turning the argument around our Corollary \[1.5\] gives a purely algebraic proof of his product formula for \( \text{Ch}^{\text{spin}}_{2j-1}(\Delta_p) \).

**Problem 1.14.** It would be interesting to extend this result to the almost staircase strict partition of the form 
\[ \Delta_{p,d} = (p, \ldots, p, p - 1, p - 2, \ldots, 2, 1) \]
and the corresponding spin character \( \text{Ch}^{\text{spin}}_{2j-1}(\Delta_{p,d}) \) in the setting when \( d \geq 0 \) is a small integer and \( p \) varies. The above question is equivalent to finding closed formulas for the linear characters \( \text{Ch}_{2j-1} \left( D(\Delta_{p,d}) \right) \); note that the double \( D(\Delta_{p,d}) \) can be viewed as a large rectangular Young diagram \( (p + d) \times (p + d + 1) \) of almost square shape from which another, small rectangle \( d \times (d + 1) \) of almost square shape was removed.

1.7. Sketch of the proof. We will start in Section 2 by collecting some formulas for the irreducible characters related to rectangular shapes. Our strategy towards the proof of Theorem \[1.1\] is threefold.

Firstly, we will fix the value of an integer \( j \geq 1 \) and we shall investigate the function 
\[ (d, e) \mapsto \text{Ch}_{2j-1} \left( (e - d) \times (e + d) \right). \]
We will show that it is a polynomial in the variables $d, e$ of the total degree $2j$.

Secondly, we will show that this polynomial is of the form

$$\text{Ch}_{2j-1}((e - d) \times (e + d)) = \sum_{k=0}^{j} c_k(j) \left( \prod_{r=0}^{k-1} (d^2 - r^2) \right) \left( \prod_{r=k}^{j-1} (e^2 - r^2) \right)$$

with certain coefficients $c_k(j)$ independent of $d, e$.

Thirdly, by finding explicitly the value $\text{Ch}_{2j-1}((-1) \times (2k - 1))$, we will determine the coefficients $c_k(j)$.

The proofs of Theorems 1.4, 1.7 and 1.10 are fully analogous to the proof of Theorem 1.1 and we skip them, see Section 4 for some additional comments.

2. Characters on rectangular diagrams

For any partition $\pi$ of $k$, Stanley’s character formula for rectangular shapes \cite{Sta03} is given by

\begin{equation}
\text{Ch}_\pi(p \times q) = (-1)^k \sum_{\sigma_1, \sigma_2 \in S_k \atop \sigma_1 \sigma_2 = \pi} (-q)^{\kappa(\sigma_1)} p^{\kappa(\sigma_2)},
\end{equation}

where $\kappa(\sigma)$ is the number of cycles in $\sigma$ and $w_\pi$ is a fixed permutation of the cycle type $\pi$.

**Corollary 2.1.** For each partition $\pi$ the corresponding character

$$\text{Ch}_\pi(p \times q) \in \mathbb{Z}[p, q]$$

can be identified with a polynomial in the variables $p$ and $q$. This polynomial is of degree $|\pi| + \ell(\pi)$ and fulfills the equality

\begin{equation}
\text{Ch}_\pi(p \times q) = (-1)^{|\pi| - \ell(\pi)} \text{Ch}_\pi(q \times p).
\end{equation}

**Proof.** It is easy to show that if two polynomial functions from $\mathbb{Q}[p, q]$ take equal values on each lattice point $(p, q)$ with $p, q \in \mathbb{N}$ then they are equal as polynomials; it follows that the polynomial given by the right-hand side of (2.1) is unique.

We define the length $\|\sigma\|$ of a permutation $\sigma \in S_k$ as the minimal number of factors necessary to write it as a product of transpositions. It is a classical result that

$$\|\sigma\| = k - \kappa(\sigma).$$
In this way the exponent on the right-hand side of (2.1) is bounded from above by
\[ \kappa(\sigma_1) + \kappa(\sigma_2) = 2k - \|\sigma_1\| - \|\sigma_2\| \]
\[ \leq 2k - \|\sigma_1\|\sigma_2\| = k + (k - \|\sigma_1\|) = |\pi| + \ell(\pi), \]
as required.

Equation (2.2) is a consequence of the general formula for the character which corresponds to the transposed Young diagram
\[ \text{Ch}_\pi(\lambda^T) = (-1)^{|\pi| - \ell(\pi)} \text{Ch}_\pi(\lambda). \]

Corollary 2.2. For each integer \( k \geq 1 \)
\[ \text{Ch}_k\left((-1) \times q\right) = (-1) q(q + 1) \cdots (q + k - 1), \]
\[ \text{Ch}_k\left(p \times (-1)\right) = (-1)^k p(p + 1) \cdots (p + k - 1). \]

Proof. We start with the following equality which holds in the symmetric group ring \( \mathbb{Z}[S_k] \)
\[ \sum_{\pi \in S_k} \pi = (1 + J_1)(1 + J_2) \cdots (1 + J_k), \]
where
\[ J_i = (1, i) + \cdots + (i - 1, i) \in \mathbb{Z}[S_k] \]
denotes the Jucys–Murphy element. By investigating how the length of the permutations changes after multiplying by consecutive factors on the right-hand side of (2.3) it follows from the Stanley formula (2.1) that
\[ \text{Ch}_k\left(p \times (-1)\right) = (-1)^k \sum_{\sigma \in S_k} p^{\kappa(\sigma)} = (-1)^k p(p + 1) \cdots (p + k - 1), \]
as required.

The other identity follows first one and (2.2). \( \square \)

3. PROOF OF THEOREM 1.1

We fix a positive integer \( j \); note that the following notation depends on \( j \) implicitly. We denote by
\[ P(d) = \text{Ch}_{2j-1}\left((e - d) \times (e + d)\right) \]
the left-hand side of (1.2) viewed as a polynomial in the variable \( d \) with the coefficients in the polynomial ring \( \mathbb{Q}[e] \). From the Stanley character formula (2.1) and Corollary 2.1 it follows that the degree of \( P(d) \) is at most \( 2j \).
Equation (2.2) implies that
\[ Ch_{2j-1}((e-d) \times (e+d)) = Ch_{2j-1}((e+d) \times (e-d)); \]
in other words the polynomial \( P(d) \) is even.

The linear space of even polynomials in the variable \( d \) has a linear basis
\[ 1, \ d^2, \ d^2(d^2-1^2), \ d^2(d^2-1^2)(d^2-2^2), \ldots; \]
and it follows that there exist polynomials \( P_0, \ldots, P_j \in \mathbb{Q}[e] \) with the property that
\[ (3.1) \quad Ch_{2j-1}((e-d) \times (e+d)) = P_0(e) + P_1(e) \ d^2 + P_2(e) \ d^2(d^2-1^2) + \cdots = \sum_{k=0}^{j} P_k(e) \prod_{r=0}^{k-1}(d^2-r^2). \]

Additionally, from Corollary 2.1 it follows that the degree of the polynomial \( P_k(e) \) is at most \( 2(j-k) \).

The parity of the total degree of each monomial on the right-hand side of the Stanley formula (2.1) is the same as the parity of \( |\pi| - \ell(\pi) \). In our case \( \pi = (2j-1) \) this parity is even; it follows that
\[ Ch_{2j-1}((e-d) \times (e+d)) = Ch_{2j-1}((-e+d) \times (-e-d)) = Ch_{2j-1}((-e-d) \times (-e+d)), \]
where the second equality is the consequence of the above observation that the polynomial \( P(d) \) is even. We proved in this way that \( P(d) \) is invariant under the involutive automorphism of the polynomial ring \( \mathbb{Q}[e] \) which is given by the substitution \( e \mapsto -e \). It follows that each coefficient \( P_k(e) \) is an even polynomial in the variable \( e \).

**Lemma 3.1.** For each \( k \in \{0, \ldots, j\} \) there exists some constant \( c_k \) with the property that
\[ P_k(e) = c_k \prod_{r=k}^{j-1}(e^2 - r^2). \]

**Proof.** We will use induction over the variable \( k \). For the induction step let \( k_0 \in \{0, \ldots, j\} \); we assume that (3.2) holds true for each integer \( k \in \{0, \ldots, k_0 - 1\} \).

Our strategy is to evaluate (3.1) for \( d = k_0 \) and \( e \in \{k_0 - 1, \ldots, j - 1\} \). Each summand on the right-hand side which corresponds to \( k > k_0 \) vanishes as it contains the factor \( (d^2 - r^2) \) for \( r = k_0 \). On the other hand, each summand on the right-hand side which corresponds to \( k < k_0 \) vanishes
because either (a) $k_0 = 0$ and there are no such summands, or (b) by the inductive hypothesis $P_k(e)$ contains the factor $(e^2 - r^2)$ for $r = e$. We proved in this way that for $e \in \{k_0 - 1, \ldots, j - 1\}$

\[(3.3) \quad Ch_{2j-1} \left((e - k_0) \times (e + k_0)\right) = P_{k_0}(e) \prod_{r=0}^{k_0-1} (k_0^2 - r^2).\]

In fact, in the special case when $k_0 = 0$ it is easy to check that the above equality holds true for an arbitrary choice of $e \in \mathbb{C}$ and

\[(3.4) \quad Ch_{2j-1}(e \times e) = P_0(e).\]

The formula \((3.3)\) has twofold consequences.

Firstly, in the special case when $e \in \{k_0, \ldots, j - 1\}$ the rectangular Young diagram $(e - k_0) \times (e + k_0)$ is well-defined and the defining formula \((1.1)\) can be used. Furthermore, this Young diagram does not contain any rim hooks of length $2j - 1$; from the Murnaghan–Nakayama rule it follows that the left-hand side of \((3.3)\) is equal to zero; as a consequence $P_{k_0}(e) = 0$.

We proved in this way that $P_{k_0}$ is an even polynomial which has roots in $k_0, k_0 + 1, \ldots, j - 1$; it follows that the polynomial $P_{k_0}(e)$ is divisible by the product

\[
\prod_{r=k_0}^{j-1} (e^2 - r^2).
\]

Since the degree of $P_{k_0}$ is at most $2(j - k_0)$, this determines the polynomial $P_{k_0}$ up to a scalar multiple and shows that \((3.2)\) holds true for $k := k_0$. This completes the proof of the inductive step of Lemma \(3.1\). \(\square\)

As an extra bonus, for $k_0 \geq 1$ the special case of \((3.3)\) and \((3.2)\) for $e = k_0 - 1$ gives (in order to keep the notation lightweight we write $k = k_0$)

\[
Ch_{2j-1}\left((-1) \times (2k - 1)\right) = c_k \prod_{r=k}^{j-1} ((k - 1)^2 - r^2) \prod_{r=0}^{k-1} (k^2 - r^2).
\]

The left-hand side can be evaluated thanks to Corollary \(2.2\) which gives an explicit product formula for the constant $c_k$ for $k \geq 1$. Note that this argument cannot be applied in the special case $k = 0$ because for $e = -1$ the information about the value of the polynomial $P_0(-1)$ is not linearly independent from the information about $P_0(1) = P_0(-1)$.

A combination of \((3.2)\) and \((3.4)\) gives

\[
Ch_{2j-1}(e \times e) = c_0 \prod_{r=0}^{j-1} (e^2 - r^2).
\]
In order to evaluate the constant $c_0$ we need some additional piece of information about the polynomial on the left-hand side. One possible approach is to evaluate its value for $e := j$; in this special case the Murnaghan–Nakayama rule has only one summand therefore value of the normalized character is given a product formula based on the hook-length formula. An alternative approach is based on calculating the leading coefficient $[e^{2j}] \operatorname{Ch}_{2j-1}(e \times e)$ based on the ideas of the asymptotic representation theory, see Remark 3.3.

Thanks to these explicit values of the constants $c_k$, Theorem 1.1 follows by a straightforward algebra and its proof is now complete.

Theorem 1.1 gives a new proof of the following result.

**Corollary 3.2.** For each integer $j \geq 1$

$$[e^{2j}] \operatorname{Ch}_{2j-1}(e \times e) = (-1)^{j-1} \operatorname{Cat}(j - 1).$$

**Remark 3.3.** Corollary 3.2 is not new; in the following we only give a rough sketch of an alternative proof based on existing results. The work of Biane ([Bia98, Theorem 1.3] or [Bia03]) implies that

$$\lim_{e \to \infty} \frac{1}{e^{2j}} \operatorname{Ch}_{2j-1}(e \times e) = R_{2j}(\Box),$$

where $R_{2j}(\Box)$ denotes the free cumulant of the one-box Young diagram $\Box = (1)$. More specifically, $R_{2j}(\Box)$ is the free cumulant of the Kerov transition measure of $\Box$ which is equal to the Bernoulli measure

$$\frac{1}{2} (\delta_{-1} + \delta_1).$$

Standard combinatorial tools of free probability [MS17] give a closed formula for such a free cumulant in terms of Catalan numbers.

4. **Comments about the proof of Theorems 1.4, 1.7 and 1.10**

As we already mentioned, the proofs of Theorems 1.4, 1.7 and 1.10 are analogous to the proof Theorem 1.1. Below we revisit only some key places which require an adjustment.

For example, in order to prove Theorem 1.4 we need to write

$$\operatorname{Ch}_{2j-1}((e - d) \times (e + d)) = \sum_{k=0}^{j} P_k'(e) \prod_{r=0}^{k-1} \left( d^2 - (r + \frac{1}{2})^2 \right).$$

and then to show the following analogue of Lemma 3.1: for each $k \in \{0, \ldots, j\}$ there exists some constant $c_k'$ with the property that

$$P_k'(e) = c_k' \prod_{r=k}^{j-1} \left( e^2 - (r + \frac{1}{2})^2 \right).$$
In order to achieve this goal, the strategy of the induction step is to fix 
\( d = k_0 + \frac{1}{2} \) and to consider \( e \in \{k_0 - \frac{1}{2}, \ldots, j - \frac{1}{2}\} \).

The calculation of the constants \( c'_k \) is particularly easy now because both \( c'_k \) as well as the constants \( c_k \) from Equations (3.2) and (3.1) coincide with the coefficient of a specific monomial in the Stanley polynomial

\[
c'_k = \left[ d^{2k} e^{2(j-k)} \right] \text{Ch}_{2j-1} \left( (e - d) \times (e + d) \right),
\]

hence they are equal.

Theorems [1.7] and [1.10] concern the character \( \text{Ch}_{2j} \) on an even cycle. In this case the corresponding polynomial

\[
P(d) = \text{Ch}_{2j} \left( (e - d) \times (e + d) \right)
\]
is odd and its degree is at most \( 2j + 1 \), therefore we may write

\[
\text{Ch}_{2j} \left( (e - d) \times (e + d) \right) = \sum_{k=0}^{j} P''_k(e) \ d \prod_{r=1}^{k} (d^2 - r^2)
\]

for some even polynomials \( P''_k(e), P''_k(e) \in \mathbb{Q}[e] \) which are of order at most \( 2(j-k), \) where \( k \in \{0, \ldots, j\} \).

The proof of Theorem [1.7] involves analysis of the polynomials \( P''_k \) which is analogous to the one from the proof of Theorem [1.1]; in particular an analogue of Lemma [3.1] says that

\[
P''_k(e) = c''_k \prod_{r=k+1}^{j} (e^2 - r^2).
\]

The proof of its inductive step is based on fixing \( d = k_0 + 1 \) and considering \( e \in \{k_0, \ldots, j\} \). The values \( e \in \{k_0 + 1, \ldots, j\} \) are the positive roots of the even polynomial \( P''_k \); the polynomial is therefore determined up to a multiplicative constant. The special case \( e = k_0 \) allows to find explicitly the value of \( c''_k \); interestingly (opposite to the case in the proof of Theorem [1.1]) the case \( k = 0 \) does not require a separate proof.

5. Integrality of the Coefficients

**Proposition 5.1.** Let \( d \) be an integer. Each coefficient of the polynomial \( G_d(j, n) \) (defined in Corollary [1.2]) is an integer.

**Proof.** We will show a stronger result that for each integer \( k \geq 1 \)

\[
\frac{1}{2^{k-1}} \prod_{r=0}^{k-1} \frac{(d^2 - r^2)}{k! (2k - 1)!!} = 2d \cdot \prod_{r=1}^{k-1} \frac{(d + r)}{(2k)!}
\]

(5.1)
is an integer. We will do it by proving that for each prime number \( p \) the exponent by which it contributes to the factorization of the numerator is at least its counterpart for the denominator. In the case when \( p \neq 2 \) these exponents are equal, respectively, to

\[
\sum_{c \geq 1} \left( [p^c \mid d] + \#\{i \in \{d - k + 1, \ldots, d + k - 1\} : p^c \mid i\} \right)
\]

and

\[
\sum_{c \geq 1} \#\{i \in \{d - k + 1, \ldots, d + k\} : p^c \mid i\}.
\]

Here and in the following we use the notation

\[
[\text{condition}] = \begin{cases} 
1 & \text{if condition holds true,} \\
0 & \text{otherwise.}
\end{cases}
\]

We will show that for each \( c \geq 1 \) the corresponding summand on the right-hand side of (5.2) is greater or equal to its counterpart in (5.3).

We start with the observation that in any collection of \( p^c \) consecutive integers there is exactly one which is divisible by \( p^c \); it follows that a collection of \( 2k \) consecutive integers contains at least \( \left\lfloor \frac{2k}{p^c} \right\rfloor \) such numbers divisible by \( p^c \). As a consequence we get the following lower bound for the summand on the right-hand side of (5.2):

\[
[p^c \mid d] - [p^c \mid d + k] + \#\{i \in \{d - k + 1, \ldots, d + k\} : p^c \mid i\} \geq [p^c \mid d] - [p^c \mid d + k] + \left\lfloor \frac{2k}{p^c} \right\rfloor.
\]

Since (5.2) can be alternatively written as

\[
\sum_{c \geq 1} \left( [p^c \mid d] + \#\{i \in \{d - k + 1, \ldots, d + k - 1\} : p^c \mid i\} \right)
\]

and

\[
\sum_{c \geq 1} \#\{i \in \{1, \ldots, 2k\} : p^c \mid i\}.
\]
an analogous reasoning to the one above gives the following alternative lower bound for the summand on the left-hand side of (5.2)

\[(5.6) \quad \left[ p^c | d \right] - \left[ p^c | d + k \right] + \#\left\{ i \in \{d - k + 1, \ldots, d + k\} : p^c | i \right\} \geq \left[ p^c | d \right] - \left[ p^c | d - k \right] + \left\lfloor \frac{2k}{p^c} \right\rfloor.\]

On the other hand, for the corresponding summand in (5.3) we get the following exact expression

\[(5.7) \quad \#\left\{ i \in \{1, \ldots, 2k\} : p^c | i \right\} = \left\lfloor \frac{2k}{p^c} \right\rfloor.\]

If one of the following two conditions holds true: (a) the right-hand side of (5.4) is greater or equal to the right-hand side of (5.7), or (b) the right-hand side of (5.6) is greater or equal to the right-hand side of (5.7), then the desired inequality holds true. The opposite case, i.e., when both \(d - k\) and \(d + k\) are divisible by \(p^c\) and \(d\) is not divisible by \(p^c\), is clearly not possible since \((d + k) + (d - k) = 2d\).

For the case when \(p = 2\) let us consider half of the expression (5.1); then (5.2) and (5.3) still provide the exponents in the numerator and the denominator. The proof proceeds without any modifications until after Equation (5.7). It might happen that none of the conditions (a) and (b) holds true; if this is indeed the case then both \(d - k\) and \(d + k\) are divisible by \(2^c\) (hence \(d\) is divisible by \(2^{c-1}\)) and \(d\) is not divisible by \(2^c\). There exists at most one index \(c\) with this property. It follows that the sum (5.2) is at least (5.3) less one. Since we considered half of the expression (5.1), this completes the proof. \(\square\)

**Proposition 5.2.** Let \(d \in \{\pm \frac{1}{2}, \pm \frac{3}{2}, \ldots\}\). Each coefficient of the polynomial \(H_d(j, n)\) (defined in Corollary 1.5) is an integer.

**Proof.** By the symmetry \(H_d(j, n) = H_{-d}(j, n)\), we may suppose that \(d\) is positive and we write it as \(d = d' + \frac{1}{2}\), where \(d'\) is a non-negative integer. For each integer \(k \in \{0, 1, 2, \ldots, d'\}\),

\[\prod_{r=0}^{k-1} \left( d^2 - (r + \frac{1}{2})^2 \right) = \frac{2^k}{k!} \frac{\prod_{r=-k+1}^{k} (d' + r)}{(2k)!} = 2^k \binom{d' + k}{2k}.\]
is clearly an integer. Thus, we see that
\[
H_{d^2 + \frac{1}{2}}(j, n) = \sum_{k=0}^{d'} (-1)^k 2^k \left( \frac{d' + k}{2k} \right)^{j+k} (2j - 1)^{k+1} k!
\times \prod_{r=k}^{d' - 1} (n + d'(d' + 1) - r(r + 1))
\]
has integer coefficient, as required. \[\square\]

**Proposition 5.3.** Let $d$ be an integer. Each coefficient of the polynomial $I_d(j, n)$ (defined in Corollary 1.8) is an integer.

**Proof.** By the symmetry $I_d(j, n) = -I_{-d}(j, n)$, we may suppose that $d$ is a positive integer. For each integer $k \in \{1, 2, \ldots, d - 1\}$,
\[
\frac{d! \prod_{r=1}^{k} (d^2 - r^2)}{k!(2k + 1)!!} = \frac{2^k \prod_{r=-k}^{k} (d + r)(d - r)}{(2k + 1)!} = 2^k \left( \frac{d + k}{2k + 1} \right)
\]
is clearly an integer which completes the proof. \[\square\]

**Proposition 5.4.** Let $d$ be a half integer. Each coefficient of the polynomial $J_d(j, n)$ (defined in Corollary 1.11) is an integer.

**Proof.** We may write $d = d' - \frac{1}{2}$, where $d'$ is an integer. We will show a stronger result that for each integer $k \geq 1$
\[
\frac{1}{2^k} \frac{2d \prod_{r=1}^{k} (d^2 - (r - \frac{1}{2})^2)}{k!(2k + 1)!!} = \frac{(2d' - 1) \prod_{r=-k}^{k-1} (d' + r)}{(2k + 1)!}
\]
is an integer. The proof is analogous to the proof of Proposition 5.1. \[\square\]

**Acknowledgments**

Sho Matsumoto was supported by JSPS KAKENHI, grant number 17K05281.
Piotr Śniady was supported by Narodowe Centrum Nauki, grant number 2017/26/A/ST1/00189.

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