Time operator within projection evolution model

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We apply the projection evolution approach to the particle detection process and calculation of the detection moment. Influence of the essential system properties on the evolution process is discussed. It is shown, that using only the projection postulate in the evolution scheme allows to understand the time as a kind of observable.

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I. INTRODUCTION

The usual formulation of quantum mechanics leads to many conceptual problems. Most of the proposed interpretations concerning the measurement process is unsatisfactory. The great interest in quantum computers and wide range of obstacles to them, caused mainly by decoherence of the entangled states has contributed to retrospect the essentials of the quantum mechanics.

The main problem is that there are two quite different evolution laws. The Schrödinger equation describes the evolution of the quantum objects as long as it is not disturbed by the experiment. It is unitary and completely deterministic process. On the other hand there is the projection postulate which operates whenever the object is affected by the measurement. Here one can calculate only the probabilities for each possible outcome in the experiment, but the change of the object state is unpredictable. Moreover, there is no way to estimate the moment of time when the process occurs.

A measurement of time stands as the separate general problem of the quantum mechanics. Time has been treated as a parameter in the evolution law for decades, even it has been proven that the time operator can not be defined properly within standard quantum mechanics.

The Pauli’s theorem [1] states that the self-adjoint time operator implies an unbound continuous energy spectrum. That means, it is impossible to build a self-adjoint time operator canonically conjugate to a Hamiltonian bounded from below. However, some attempts to apply the concept of time as an observable were made, according to the statement, that any given observable uniquely characterized by the probability distribution of the measurement results in the different states accessible to the system.

Thus there were proposed several types of times corresponding to some types of measurements, e.g. the time of arrival introduced by Aharonov and Bohm [2], further investigated by e.g. [3, 4, 5, 6], also [3, 7], the tunneling time or the time of a quantum clock given by a phase variable.

Another concept of time operator was introduced by Olkhovsky, Recami and others [12], where the basic idea is to extract it from the average "presence time" relation \( \langle t \rangle \). More recent approach applies a positive operator valued measure (POVM) [4, 10, 11] to this problem. Some other publications presents the time-of-arrival problem using some kind of "screen observables" [13]. More detailed review can be found in e.g. [15]. An interesting idea is also the Event-Enhanced Quantum Theory which replaces the Schrödinger equation by a special deterministic algorithm [6], however it seems to be unsatisfactory introducing some "non quantum" elements.

But apart from the pure time operator the problem lies in the process of time measurement and evolution. The first model proposed by Allcock [8] consisted of a free particle and an interacting Hamiltonian in the form of the complex potential. The solutions, however, were in disagreement with the Heisenberg uncertainty relation. Aharonov and Bohm proposed to consider a system consisting of "a clock" among other quantum objects (particles, apparatus, etc.). The time of interaction is thus determined by a physical observable of this clock particle [2] where the corresponding time operator was obtained by a simple symmetrization of the classical expression \( t = \frac{mx}{px} \).

But what is that mysterious force described by a complex potential or "a clock"? What is its foundation? And what defines the physical process as an experiment which obeys the projection postulate rather than the unitary
transformation? The questions are still open until today.

Some recent work was made concerning quantum computation process where the evolution of a qubit is assumed to be the process of a succession of some measurements, e.g., as the result of some additional periodically interrupting influence \[16\]. Very close model was introduced by Nielsen \[17\], who showed that no coherent unitary dynamics is needed at all in order to simulate the quantum computer. Thus projective measurements are universal for quantum computation \[18, 19\].

We have made one step forward following the hypothesis of the projection evolution \[9\], which is a sequence of some measurements made permanently by the Nature. Each system behaves like it was affected by some kind of the "apparatus" but we postulate it is its natural inner property. A set of projection evolution operators responsible for the measurement at each moment of evolution depends on the essential properties of the system under consideration. Detailed description of the approach can be found in \[9\].

Within this idea many of the mentioned above problems is solved, in particular the Schrödinger equation is a special case of the projection evolution. Moreover, within this idea, there is no need to introduce an observer because the "collapse of states" is processed due to the postulated fundamental Law of the Nature. Further, this model lets to introduce a kind of the time operator, which can not be done within the standard quantum mechanics. More detailed information about the time observable is presented in the next section of this paper.

II. THE TIME OPERATOR

The projection evolution model as described in \[9\] defines the evolution as a process of causally related physical events, ordered by the evolution parameter \(\tau\), called in \[9\] the etime. The time-like variable parameter \(\tau\) has been already regarded by Caster and Reznik \[14\] as the state of the clock particle within the time-of-arrival problem, but in contradiction to that hypothesis, within the projection evolution approach each of the events is related to a measurement "made", according to the proposed procedure, by the Nature. That means, the state of a physical system at the moment (step of evolution) denoted by \(\tau\) is the projection of the previous state with respect to the essential system properties, as it is shown below (for simplicity we assume here a discrete structure of the evolution parameter (etime) \(\tau\):

\[
\rho(\tau_{n+1};\nu_0,\nu_1,\ldots,\nu_{n+1}) = \frac{\mathbb{E}(\tau_{n+1};\nu_{n+1})\rho(\tau_n;\nu_0,\nu_1,\ldots,\nu_n)\mathbb{E}(\tau_{n+1};\nu_{n+1})}{\text{Tr}[\mathbb{E}(\tau_{n+1};\nu_{n+1})\rho(\tau_n;\nu_0,\nu_1,\ldots,\nu_n)\mathbb{E}(\tau_{n+1};\nu_{n+1})]}.
\] (1)

Here \(\rho(\tau_n;\nu_0,\nu_1,\ldots,\nu_n)\) is the quantum density operator describing the state of the physical system at \(n\)-th step of the evolution. The next step is specified by the projection evolution operator \(\mathbb{E}(\tau_{n+1};\nu_{n+1})\) randomly chosen from a set of possible projections which constitute the orthogonal resolution of unity, each representing the properties of our physical system at the moment \(\tau_{n+1}\) of the etime. The orthogonal resolution of unity, for discrete quantum numbers \(\nu\) are defined by the conditions:

\[
\mathbb{E}(\tau;\nu)\mathbb{E}(\tau;\nu') = \delta_{\nu\nu'}\mathbb{E}(\tau;\nu)
\]

\(\sum_{\nu} \mathbb{E}(\tau;\nu) = \mathbb{I}.
\] (2)

and can be naturally generalized to arbitrary sets of quantum numbers.

The probability distribution of choosing the specified projection is given by a rather standard formula:

\[
\text{Prob}(\tau_{n+1};\nu_0,\nu_1,\ldots,\nu_{n+1}) = \text{Tr}[\mathbb{E}(\tau_{n+1};\nu_{n+1})\rho(\tau_n;\nu_0,\nu_1,\ldots,\nu_n)\mathbb{E}(\tau_{n+1};\nu_{n+1})].
\] (3)

This is, in principle, the conditional probability because the eq. \[8\] describes a probability of choosing next state, under assumption that the system is already in the given state \(\rho(\tau_n;\nu_0,\nu_1,\ldots,\nu_n)\).

It is shown in \[9\] that assuming the projection evolution operator in the form of resolution of unity shifted by an unitary operators like traditional unitary evolution operator for closed systems:

\[
\mathbb{E}(\tau;\nu) = U(\tau - \tau_0)\mathbb{E}(\tau_0;\nu)U^\dagger(\tau - \tau_0),
\] (4)

under some assumptions, such process while considering the continuous measurements leads to the unitary evolution of the Schrödinger’s type \(\rho(\tau) = U(\tau - \tau_0)\rho'(\tau_0)U^\dagger(\tau - \tau_0)\), where \(\rho'(\tau_0)\) is the state chosen by the Nature just before \(\tau_0\). One needs observe here that this approach allows to consider time on nearly the same foot as any other quantum observable because the time should be related to the parameter etime ordering causal events, but it is formally independent of it.

The projection evolution procedure suggest also a possibility to consider two types of observables.
The first type consists of physical observables related to real interactions with the system under considerations and reducing its state due to the interaction. This kind of observables can be a part of the projection evolution operator.

The second type of observables (information observables) are more theoretical ones because they allow only to investigate structure of states.

Let the decomposition of unity \( \{ M_A(a) \} \) be an information observable and \( \rho \) be a state of the system. In this case the expression \( \text{Tr} (M_A(a)\rho) \) should rather be interpreted as a "potential probability" which give us an information about the structure of the system state but not affect it. Such information should not have any influence into the real process of the evolution.

On the other hand, it is important to notice that probably most of possible observables can be used formally in both meanings.

However, an existence of the first type observables, for a given physical system, is determined by structure of the system and its interactions.

Assuming now a nonrelativistic four dimensional spacetime \( x = (x_0 = t, \vec{x}) \in \mathbb{R}^4 \), using the specified reference frame, there is now the possibility to build a well defined "time operator" \( M_T(t) \) projecting onto the subspace of the simultaneous events. As mentioned in the previous section that is impossible in the standard quantum mechanics where time is a parameter. For this simple model of space and time the states responsible for the event "to be in \( \vec{x} \) at the time \( t = x_0 \)" can be described by the Dirac \( \delta \)-type distribution \( \delta^4(y - x) \) corresponding to the appropriate state \( |x\rangle \). Making use this notation the appropriate projection operator (generalized resolution of unity) can be written as:

\[
M_T(t) = \int_{\mathbb{R}^3} d^3\vec{x} |x\rangle \langle x|,
\]

where \( d^3\vec{x} \) denotes an element of 3 dimensional volume of the coordinate space.

This operator should give us all information about an overlap of our system over the subspace of states corresponding to any position at time \( t \). We suppose also that the operator can be used in non-relativistic case (it is not covariant in respect to the Lorentz transformations) as an approximation of the physical observable which detect objects at a given time \( t \). In this case it must be related to a specific physical structure, like a detector, which is able to interact with our system at this given time.

Within the projection evolution model, as was already mentioned above, the whole process of evolution is ordered by the special evolution parameter \( \tau \). Thus the sequence of all events is independent from the observer’s configuration space. Using the time projection operator we can now determine the potential probability density, that the state described by \( \rho(\tau) \) can be found at the specified moment \( t \), as follows:

\[
\text{Prob} (t; \rho(\tau)) = \text{Tr} (M_T(t)\rho(\tau)M_T(t)).
\]

As mentioned previously the operator \( M_T(t) \) can be also used as a kind of "time trigger". Then every event enumerated with the \( \tau \) parameter can be related with the real world, and can occur only at the specified moments of time with well defined probability.

In this way we can describe the system evolution with respect to the evolution parameter \( \tau \) as well as the time \( t \).

The following example of the particle detection presented in the next sections shows how we can describe the evolution, its duration and the moment of the measurement using only the projection evolution scheme.

### III. MEASUREMENT IN THE PROJECTION EVOLUTION SCHEME

Let us consider a closed system of a single particle "moving" towards a measurement device. To simplify considerations let us assume the device is a kind of detector that registers a particle when it comes across a definite region of the space. Physical mechanisms related to the process of absorption and detection are beyond of our scope.

Due to the projection evolution model we need to construct an appropriate set of projection operators \( \mathbb{E}(\tau; \nu) \), responsible for time evolution. Here \( \tau \) is a real c-number evolution parameter that enumerates the subsequent evolution steps, and \( \nu \) represents any set of quantum numbers describing the system properties.

Because the particle position and its linear momentum (due to the Heisenberg uncertainty principle) cannot be well determined simultaneously, we assume, in general, that the free particle can evolve in a form of some wave packets \( |\nu\rangle \) which constitute the orthonormal basis within the state space of the particle, as written below:

\[
|\nu\rangle = \int_{\mathbb{R}^4} d^4k \; \alpha_\nu(k) \; |k\rangle,
\]
where \( k \equiv (k_0, \vec{k}) \) denotes a wave vector with \( k_0 \) proportional to the particle energy and \( \vec{k} \) represents its linear momentum. The vector \( k \) can be thought as kind of energy–momentum four-vector in \( R^4 \).

The shape of the wave packet is fully determined by \( \alpha_\nu(k) \) that should fulfill the normalization condition \( \int_{R^4} d^4k \ |\alpha_\nu(k)|^2 = 1 \). That means the free particle can be in one of the possible states \(|\nu\rangle\) at every moment (step) of its evolution.

Our considerations allow to define the family of projections which are an orthogonal resolution of unity and can determine a motion of free particle within the projection evolution approach

\[
M_{wp}(\nu) = |\nu\rangle\langle\nu|.
\]

In particular \( \nu \) are related to some system (particle) properties under consideration. In particular the free particle evolution, one can expect, could be described by a set of Gauss-like wave packets or the coherent-like states in order to minimize the Heisenberg uncertainty principle.

Another set of projection operators we have to define, is related to the detector as a measurement device. In the simplest case the appropriate operators can be written as follows:

\[
M_D(\Delta) = \int_\Delta d^3\vec{x} \ |\vec{x}\rangle\langle\vec{x}|,
\]

where \(|\vec{x}\rangle\) denotes the generalized eigenvectors of the position operators and \( \Delta \) describes the detector shape, that is the region where a particle is registered when arriving. This operator ”projects” onto the coordinate states belonging to the detector. The complementary operator \( M_D^\prime = M_D(\nu = 0) = 1 - M_D(\nu = \Delta) \) takes care of the situation when the particle is outside the detection area.

First we consider the evolution with the detector without the ”time trigger”. In the following we take into account the wave packets which are \( \delta \)-localized in time and spread out over the coordinate space.

Under this assumption we can construct the evolution operators as follows:

\[
E(\tau; \nu) = \begin{cases} 
U(\tau - \tau_0) M_{wp}(\nu) U^\dagger(\tau - \tau_0); & \text{for } \tau_0 \leq \tau < \tau_D \text{ and } \nu \text{ of the form } \mathcal{E}, \\
M_D(\nu); & \text{for } \tau = \tau_D \text{ and } \nu = \Delta \text{ or } \nu = 0,
\end{cases}
\]

where \( U(\tau) \) is the unitary evolution operator of the Schrödinger like form, very close to the traditional one for free particle, generated by the kinetic energy:

\[
U(\tau) = e^{i\beta_0 k^0 \tau} e^{-i\vec{k} \vec{x} \tau},
\]

where \( \beta_0 \) and \( \beta \) are some coefficients dependent on physical system and \( \vec{k}^0, \vec{k} \) denotes here the ”four-momentum operator” for which the vectors \(|k\rangle\) are generalized eigenvectors i.e., \( \vec{k}^\mu |k\rangle = k^\mu |k\rangle \).

According to the projection evolution model \( \mathcal{E} \), the system can follow any path of the evolution. In particular it can travel as a free particle from a source in the form of wave packet \( \mathcal{E} \) denoted by \(|\mu\rangle\), and after the specified evolution steps \( \tau_D \) hit the detector. It can also miss the detector of course, but this is less interesting now. The probability of the first situation can be calculated using the formula \( \mathcal{O} \):

\[
\text{Prob}(\tau = \tau_D; \nu_1 = \mu, \nu_2 = \Delta, t) = \text{Tr} \left[ M_D(\Delta) U(\tau_D - \tau_0) M_{wp}(\mu) \rho_0 M_{wp}(\mu) U^\dagger(\tau_D - \tau_0) M_D(\Delta) \right],
\]

where \( \rho_0 \) is a quantum density operator describing the particle initial state. It depends on the source properties and physical mechanisms associated with the particle creation process.

Using as a basis the generalized eigenstates \(|x\rangle = |x^0, \vec{x}\rangle\) of formal positions operators within the spacetime \( R^4 \), together with \( \mathcal{S} \) and \( \mathcal{O} \) we can rewrite the expression for the probability \( \mathcal{P} \) as:

\[
\text{Prob}(\tau = \tau_D; \nu_1 = \mu, \nu_2 = \Delta) = \int_{R^4} d^4x \int_\Delta d^3\vec{x} \int_\Delta d^3\vec{x} \ |x\rangle \langle x| U(\tau_D - \tau_0) |\mu\rangle\langle\mu| \rho_0 |\mu\rangle
\]

\[
\int_\Delta d^3\vec{x} \int_\Delta d^3\vec{x} \int_\Delta d^3\vec{x} \int_\Delta d^3\vec{x} \ \langle x| U(\tau_D - \tau_0) |\mu\rangle
\]

As one can see the first part of \( \mathcal{P} \), that is \( |\langle \mu | \rho_0 | \mu \rangle| \) is the probability distribution of finding the particle in the state \(|\mu\rangle\). In particular, it can be equal 1 if only the initial state created from the source is just a pure vector \(|\mu\rangle\).

More interesting is the second part of \( \mathcal{P} \) describing the probability of finding the particle within the detector after the specified steps of evolution \( \mathcal{P} \). Let us denote this probability by:

\[
\pi_D(\nu) = \int_{R^4} d^4x \left( \int d^3\vec{x} \ |x| U(\tau_D - \tau_0) |\mu\rangle \right)^2 = \int_{R^4} d^4x \left( \int d^3\vec{x} \right) \int_{R^4} d^4k \left( \int d^3\vec{k} \right) \ |x| U(\tau_D - \tau_0) \alpha_\mu(k) |k\rangle^2
\]
The statement given above becomes much simpler if we consider the states for which the coefficients \( \alpha_\mu(k) \) can be factorized into the energy (time) and linear momentum (space) parts. Thus, we assume, the wave packet \( |\mu\rangle \) defined by (7) fulfills the following condition:

\[
\alpha_\mu(k) \equiv \kappa_\mu(k^0) \alpha_\mu(\tilde{k}),
\]

with the appropriate normalization \( \int_R dk^0 \kappa_\mu(k^0) = 1 \) and \( \int_{R^3} d\tilde{k}^3 |\alpha_\mu(\tilde{k})|^2 = 1 \). This is the particular case that strictly corresponds to the non-relativistic description of the common quantum mechanics when time is treated as the parameter.

According to (15) we can now rewrite (14) as follows:

\[
\pi_D(\mu) = \int_R d\tilde{k}^0 \int_R d\tilde{k} \langle \tilde{k}^0 \rangle e^{i\beta \tilde{k}^0 \tau_D} \kappa_\mu(k^0) |k^0\rangle \left| \int_R d^3k \langle \tilde{k}^0 \rangle e^{-i\beta \tilde{k}^0 \tau_D} \alpha_\mu(\tilde{k}) |\tilde{k}\rangle \right|^2
\]

The mentioned separation of the time and space coordinates makes further calculations easier to process. Moreover, making use of the orthogonal resolution of unity \( \int_R dx^0 |x^0\rangle \langle x^0| = 1 \) it is easy to show that the time dependent part of the (16) is equal to one:

\[
\int_R d\tilde{k}^0 \int_R d\tilde{k} \langle \tilde{k}^0 \rangle e^{i\beta \tilde{k}^0 \tau_D} \kappa_\mu(k^0) |k^0\rangle \left| \int_R d^3k \langle \tilde{k}^0 \rangle e^{-i\beta \tilde{k}^0 \tau_D} \alpha_\mu(\tilde{k}) |\tilde{k}\rangle \right|^2 = 1,
\]

which means, it does not affect anything in the process (measurement) probability.

This is the case when the system properties do not change their character during the evolution process. For example, if the detector could register the particles only for the specified period, the time dependent factor would be significant in the probability result.

We assumed, here, the detector is a kind of stationary device which register a particle immediately after it reaches the detection area.

Describing the particle evolution we are interested if it is able to come across the detector area. For this purpose we specify the state \( |\mu\rangle \) as the packet of states \( |k\rangle \) equally distributed between \( \tilde{k} - 2\Delta \tilde{k} \) and \( \tilde{k} + 2\Delta \tilde{k} \) as defined by (7) with (15), where

\[
\kappa_\mu(\tilde{k}) = \begin{cases} 
\alpha = \text{const}; & \text{for } \tilde{k} \in (\tilde{k}_0 - \Delta \tilde{k}, \tilde{k}_0 + \Delta \tilde{k}) \\
0; & \text{for } \tilde{k} \notin (\tilde{k}_0 - \Delta \tilde{k}, \tilde{k}_0 + \Delta \tilde{k})
\end{cases}
\]

and \( \tilde{k}_0 \) is here a constant vector of linear momentum centering the wave packet.

Using the normalization condition \( \langle \mu |\mu\rangle = 1 \) we can calculate \( \alpha = (1/2\Delta k_x)^1/2 (1/2\Delta k_y)^1/2 (1/2\Delta k_z)^1/2 \). Thus the linear momentum of the particle, as it can be shown in a simple way, is represented by a spectrum of values between \( h(\tilde{k}_0 - \tilde{\Delta}k) \) and \( h(\tilde{k}_0 + \tilde{\Delta}k) \) with the average value equal \( \tilde{h}\tilde{k}_0 \).

Now after the reduction of the time coordinate integrals and using the specified form of the wave packet (17) we can rewrite (16) as follows:

\[
\pi_D(\mu) = \alpha^2 \int_\Delta d^3\tilde{x} \left| \int_{\tilde{k}_0 - \Delta \tilde{k}}^{\tilde{k}_0 + \Delta \tilde{k}} d^3k \langle \tilde{k}^0 \rangle \tilde{k} |\tilde{k}\rangle e^{-i\beta \tilde{k}^0 \tau_D} \right|^2
\]

\[
= \alpha^2 (\frac{1}{2\pi})^3 \int_\Delta d^3\tilde{x} \left| \int_{\tilde{k}_0 - \Delta \tilde{k}}^{\tilde{k}_0 + \Delta \tilde{k}} d^3k e^{i\tilde{k}\tilde{x}} e^{-i\beta \tilde{k}^0 \tau_D} \right|^2
\]

According to the regarding system properties we are tending to minimize the uncertainty of the particle location and its linear momentum but both strongly depend on the wave packet width.

Let us consider an approximation which corresponds to the small linear momentum spread around \( \tilde{k}_0 \). Using the linear expansion to the first order in the small deviations of \( \tilde{k} \) from \( \tilde{k}_0 \) we have below:

\[
\tilde{k}^2 = (\tilde{k}_0 + \delta \tilde{k})^2 \approx 2\tilde{k}\delta \tilde{k} - \tilde{k}_0^2
\]

This approximation prevents us from using any numerical receipts in order to calculate the inner integral of (18), which is presented below:

\[
\int_{\tilde{k}_0 - \Delta \tilde{k}}^{\tilde{k}_0 + \Delta \tilde{k}} d^3k e^{i\tilde{k}\tilde{x}} e^{-i\beta \tilde{k}^0 \tau_D} = e^{-i\beta \tilde{k}_0^0 \tau_D} \int_{\tilde{k}_0 - \Delta \tilde{k}}^{\tilde{k}_0 + \Delta \tilde{k}} d^3k e^{i\tilde{k}(\tilde{x} - 2\beta \tilde{k}\tau_D)}
\]

\[
= e^{-i\beta \tilde{k}_0^0 \tau_D} e^{i(\tilde{x} - 2\beta \tilde{k}\tau_D)} \left[ \frac{2\sin[(x - 2\beta k_{0x}\tau_D)\Delta k_x]}{x - 2\beta k_{0x}\tau_D} \frac{2\sin[(y - 2\beta k_{0y}\tau_D)\Delta k_y]}{y - 2\beta k_{0y}\tau_D} \frac{2\sin[(z - 2\beta k_{0z}\tau_D)\Delta k_z]}{z - 2\beta k_{0z}\tau_D} \right]
\]
The second integral of the \(\pi_D\) depends on the shape of the detector. The rectangular box seems to be the simplest one, so let us choose that shape. The probability of detection of the particle can be now calculated from (18), with respect to the previous assumptions [13], as follows:

\[
\pi_D(\mu) = \alpha^2 \frac{1}{(2\pi)^3} \int_{a_1}^{a_2} \int_{b_1}^{b_2} \int_{c_1}^{c_2} \left| e^{-i\beta \hat{K}_D e^{i(x - 2\beta \hat{K}_0 \tau_D)k_0}} \right|^2 d^3z = 2\sin[(x - 2\beta \hat{K}_0 \tau_D)\Delta k_x] 2\sin[(y - 2\beta \hat{K}_0 \tau_D)\Delta k_y] 2\sin[(z - 2\beta \hat{K}_0 \tau_D)\Delta k_z] \]

\[
\int_{a_1}^{a_2} \frac{1}{\pi \Delta k_x} \int_{b_1}^{b_2} \frac{1}{\pi \Delta k_y} \int_{c_1}^{c_2} \frac{1}{\pi \Delta k_z} dx dy dz \sin^2[(x - 2\beta \hat{K}_0 \tau_D)\Delta k_x] \sin^2[(y - 2\beta \hat{K}_0 \tau_D)\Delta k_y] \sin^2[(z - 2\beta \hat{K}_0 \tau_D)\Delta k_z]
\]

(21)

Here \((a_1, a_2), (b_1, b_2), (c_1, c_2)\) denote the measurement device corner points on the defined reference frame.

As one can see the integral is easily separable into three one-dimensional integrals, each of them representing the probability of the particle detection in the \(k_x, k_y, k_z\) direction, respectively.

After a bit of algebra we can derive the following:

\[
\pi_D(\mu_x) = \frac{1}{\pi \Delta k_x} \int_{a_1}^{a_2} \frac{1}{(x - 2\beta \hat{K}_0 \tau_D)^2} dx \sin^2[(x - v_{gx} \tau_D)\Delta k_x] = \frac{\sin^2[(a_1 - v_{gx} \tau_D)\Delta k_x]}{\pi(a_1 - v_{gx} \tau_D)\Delta k_x} - \frac{1}{\pi} \left\{ \sin[2\Delta k_x(a_1 - v_{gx} \tau_D)] - \sin[2\Delta k_x(a_2 - v_{gx} \tau_D)] \right\}
\]

(22)

where \(\vec{v}_g \equiv 2\beta \hat{K}_0\) can be understood as the wave packet group velocity, well defined if only \(\Delta k \ll k_0\) according to the assumption [13].

Some example results of the above are presented in the section \(\text{V}\).

**IV. TIME OF THE MEASUREMENT**

The results we have obtained above give us the information about the probability of the particle detection after the specified number of the evolution steps \(\tau_D\). But how can we bound this parameter with time? Some thoughts on this problem have been already presented in section \(\text{III}\), and now we are to make some calculations using the time operator \(M_T(t)\) defined in [14].

The time operator can be used here either as an information observable allowing to investigate the temporal structure of states or as a kind of "time trigger", an "additional part" of the detector which is responsible for counting particles at given time. In both cases the calculated probabilities will be different.

In the first case the state of the particle entering the detector region is given by

\[
\rho(\tau_D; \nu_1 = \mu, \nu_2 = \Delta) = \frac{M_D(\Delta) U(\tau_D) M_{wp}(\mu) \rho_0 M_{wp}(\mu) U^\dagger(\tau_D) M_D(\Delta)}{\text{Tr} \left[ M_D(\Delta) U(\tau_D) M_{wp}(\mu) \rho_0 M_{wp}(\mu) U^\dagger(\tau_D) M_D(\Delta) \right]}
\]

(23)

Using now the same analysis as presented in sec. \(\text{III}\) and after some rather simple algebra, one can find that the equation (22) can be rewritten as:

\[
\text{Prob}(t; \tau_D; \mu, \Delta) \equiv \text{Tr} [M_T(t) \rho(\tau_D; \nu_1 = \mu, \nu_2 = \Delta) = \frac{\int_\Delta d^3\vec{x} \parallel \langle t, \vec{x} | U(\tau_D) | \mu \rangle \parallel^2}{\int_R d^4x \int_\Delta d^3\vec{x} \parallel \langle x | U(\tau_D) | \mu \rangle \parallel^2}
\]

(24)

where \(\text{Prob}(t; \tau_D; \mu, \Delta)\) can be interpreted as conditional probability (or probability density) of registration of our particle in the detector at time \(t\) when the particle is already in the state (23).

Using (11) and taking into account the form of the coefficients (15), and after separating the space and time dependent factors the trace (24) can be further rewritten as:

\[
\text{Tr} [M_T(t) \rho(\tau_D; \nu_1 = \mu, \nu_2 = \Delta) = \frac{\parallel \langle t | \hat{e}^{i\beta \hat{K}_0 \tau_D} | \mu \rangle \parallel^2}{\int_R d^4x \parallel \langle x | \hat{e}^{i\beta \hat{K}_0 \tau_D} | \mu \rangle \parallel^2}
\]

(25)
Here $|\mu^0\rangle = \int_R dk^0 \kappa_\nu(k^0)|k^0\rangle$ represents the time coordinate part of the particle wave packet.

It is easy to show that the denominator of the \((25)\) is equal 1, so finally we get:

$$\text{Prob} \left( t; \tau_D; \mu, \Delta \right) = \left| \langle t | e^{i\beta_0 \kappa^0 \tau D} | \mu^0 \rangle \right|^2 \quad (26)$$

The calculation of this conditional probability lets us to predict the evolution in terms of the time corresponding to the specified observer and his reference frame. Thus we will able to verify if our assumptions are correct. In particular the free particle evolution process should be very close to the Schrödinger’s one and in classical limits it should represent any observed or measured quantity values. Time of the detection of the particle moving towards the detector is one of these quantities.

The physical conditions require the probability \((26)\) should be a function with well pronounced maximum, i.e. the function well localized in time. For this purpose the amplitudes $\kappa_\nu(k^0)$ in the vector $|\mu^0\rangle$ should be nearly a "plain-wave" type function, because one can show that the matrix element:

$$\langle t | e^{i\beta_0 \kappa^0 \tau D} | 0 \rangle = \delta(t_0 + \beta_0 \tau D - t) \quad (27)$$

Here $t_0$ can be understood as the initial time the wave packet starts to move.

Thus using only the time operator $M_T(t)$ we have got the formula that bounds the evolution parameter $\tau_D$ with the time $t$. As one can see in this very simple case they are proportional one to another and if only $\beta_0 = 1$ and $t_0 = 0$ these two quantities are the same.

According to the idea of projection evolution the above procedure gives us only an information about temporal structure of our state.

Another way is to "rebuild" the detector making use of the "time trigger". In this case we have to reconstruct the evolution operator \((10)\) adding the resolution of unity $M_T(t)$, $t \in R^3$ as follows:

$$\mathbb{E}(\tau; \nu) = \begin{cases} U(\tau - \tau_0) \, M_{wp}(\nu)\, U^\dagger(\tau - \tau_0); & \text{for } \tau_0 \leq \tau < \tau_D \text{ and } \nu \text{ of the form } \nu, \\ M_D(\nu); & \text{for } \tau = \tau_D \text{ and } \nu = \Delta \text{ or } \nu = 0, \\ M_T(t); & \text{for } \tau = \tau_D + \epsilon \text{ and } t \in R, \end{cases} \quad (28)$$

where $\epsilon$ is an arbitrary small positive number which is only formally needed but all results are independent of it.

Now the probability (density) $\text{Prob}(\tau_D + \epsilon; \mu, \Delta, t)$ of the situation that $\rho(\tau_D + \epsilon)$ is the density operator describing the system state existing in time $t$ can be calculated multiplying the conditional probability \((24)\) and the probability of finding our particle in the state \((23)\) which is equal to the denominator of the r.h.s. of \((23)\), for details see \((9)\).

According to our rules the state $\rho(\tau_D + \epsilon)$ can be written as:

$$\rho(\tau_D + \epsilon; \nu_1 = \mu, \nu_2 = \Delta) = \frac{M_T(t) M_D(\Delta) \, U(\tau_D) \, M_{wp}(\mu) \, \rho_0 \, M_{wp}(\mu) \, U^\dagger(\tau_D) \, M_D(\Delta) \, M_T(t)}{\text{Tr} \left[ \rho_0 \, M_{wp}(\mu) \, U(\tau_D) \, M_{wp}(\mu) \, M_T(t) \right]}. \quad (29)$$

The trace in the denominator is the required probability $\text{Prob}(\tau_D + \epsilon; \mu, \Delta, t)$ of a particle detection after the etime $\tau_D$ at time $t$. Making use of the equation \((24)\) the probability can be easily expressed as:

$$\text{Prob}(\tau_D + \epsilon; \mu, \Delta, t) = \int_{\Delta} d^3 \vec{x} \left| \langle t, \vec{x} \rangle U(\tau_D) | \mu \rangle \right|^2 \quad (30)$$

which after separating space and time dependent factors can be further rewritten as:

$$\text{Prob}(\tau_D + \epsilon; \mu, \Delta, t) = \left| \langle t | e^{i\beta_0 \kappa^0 \tau D} | \mu^0 \rangle \right|^2 \int_{\Delta} d^3 \vec{x} \left| \langle \vec{x} | e^{-i\beta_0 \kappa^0 \tau D} | \vec{\mu} \rangle \right|^2, \quad (31)$$

where $|\vec{\mu}\rangle$ denotes the spatial part of the wave packet.

We see that, under the same assumptions as for "information observable" case we have obtained again a $\delta$ type dependence in the numerator of \((31)\). The time dependence is exactly the same as in previous case. This makes the analysis easier and more instructive.

Of course, this is not the rule and in general the dependence $\tau \leftrightarrow t$ can be much more complicated.
V. DISCUSSION ON THE DETECTION MOMENT PROBABILITY

One of the advantages of the projection evolution scheme is that there is a possibility to estimate the moment of occurring the measurement. The formula (13) gives the probability of the particle detection after the specified evolution steps $\tau_D$. We have assumed the particle is represented by a wave packet of the form (7) with (15), which corresponds to the particle moving with the average velocity $\vec{v}_g$ as introduced in (22).

Let us consider as an example the electron moving towards the cubic detector standing some distance far from the initial particle location. To simplify considerations we have calculated the probability of the particle detection in one dimension only, choosing the axis parallel to the electron linear momentum vector. The figure (1) presents the particle detection probability calculated using (22) for several various final etimes $\tau_D$.

![Particle detection probability with respect to the final etime $\tau_D$.](image)

The results, of course, depend on some essential system properties, like the particle speed and the region $\Delta$ of the space occupied by the detector. However, from this example one can draw quite general conclusions.

In the figure, the source of the particle is at the origin of the coordinate system. The group velocity of the wave packet $v_g = 0.55$ and our one dimensional detector occupies the region between $x = 5$ and $x = 10$ on the "Ox" axis. All the physical quantity are in some arbitrary units.

As one can see, there is a non-zero probability that the particle will be detected at the beginning of its evolution and that probability raises until it achieves the maximum value at the etime $\tau_D = 15$. According to the previous section we can bound the evolution step parameter $\tau_D$ with the real time $t$, which in the simplest case are equal one to another. In that way the maximum of the detection probability corresponds to the classical time $t = s/v_g$, where $s$ is the distance between the source of the particles and the middle point of the detector $x_D = 7.5$.

The figure suggests that the particle will be detected with large probability if only it touches the detection area but there is also possible it will not be registered at all during its evolution.

Further analysis shows that the detection probability is strongly dependent on the detector shape and the uncertainty of the particle localization. Especially in case of well defined linear momentum of the particle ($\Delta k = 0$) its localization is spread over a large region of the space and if the detector does not cover a sufficient piece of the space the particle can omit it.

The figure (2) shows the particle detection probability for specified $\tau_D$ with respect to the wave packet width $\Delta k$, that is for various values of the localization and linear momentum uncertainty. As one can see if the linear momentum of the electron is well known, i.e. $\Delta k \approx 0$ and the position is not well determined, detection of the particle is very difficult – almost impossible. The more precise are both the quantities, that is when minimizing the uncertainty due to the Heisenberg uncertainty principle, the more sharp and effective is the measurement; that means the probability of the measurement is almost equal 0 outside the detector and nearly equal 1 for the time corresponding to the detection area arrival.

Although the linear approximation (19) do not allow us to use (22) when the distribution of linear momentum is not narrow, some numerical calculations shows that in that case the particle detection has small probability with the maximum value not greater than 0.2. This means the particle will come across the detector and probably will not be
registered. According to the figure (3) one can find that the maximum of the measurement probability depends on

the detector shape, i.e. the more larger is the detector the more probably it registers the particles, even if they are not well localized in space ($\Delta k \approx 0.2$). The detector can scan larger area and more wave packet is overlapped.

The problem is very similar if we consider the wave packet described by the coherent state, described i.e. by
with (15) and

\[ \alpha_{\mu}(\vec{k}) = \left( \frac{\sigma}{\pi} \right)^{3/4} e^{-\frac{\sigma(\vec{R} - \vec{a})^2}{2}} e^{i(\vec{R} - \vec{a})a} \]

where \( \vec{R} \) denotes the linear momentum vector of the particle, \( a \) is the initial particle position and \( \sigma \) denotes the wave packet width. The detection probability, however, is in this case very alike the presented in the figure (1). There is no qualitative differences in the evolution or measurement description, only the calculations are more difficult and have to be made using some numerical recipes.

Another interesting situation is when the particle is moving not toward the detector, but in quite opposite direction. The probability of the detection, assuming the same parameters as previously, but with the group velocity \( \vec{v}_g' = -\vec{v}_g \), is presented on the figure (4). The far the particle is from the detection area the lower is the probability of its detection, and the probability, even if exist is very small, nearly equal zero. That means there is a little chance to detect such a particle even if it is far from the measurement device.

VI. SUMMARY

In this paper we have presented a problem of time relations for the particle detection process. A closed system of the particle moving in the neighborhood of the measurement device has been described in terms of projection evolution, according to the postulated new Law of Nature. The example shows how to introduce in quantum mechanics the notion of time as an observable.

The idea gives an opportunity to build a time operator as the observable projecting on the subspace of simultaneous events. Thus we are able to obtain the probability of the particle measurement as a function of time which was not possible within the standard quantum mechanics.

Some calculations within a schematic model has been presented. The results are very close to our intuition about the process giving the proper dependences which comparable with the experiments.

It is also important to note that the projection evolution postulate let us to describe the decoherence as the inner property of quantum objects.

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