RAMIFICATION THEORY FOR ARTIN-SCHREIER EXTENSIONS OF VALUATION RINGS

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Abstract

The goal of this paper is to generalize and refine the classical ramification theory of complete discrete valuation rings to more general valuation rings, in the case of Artin-Schreier extensions. We define refined versions of invariants of ramification in the classical ramification theory and compare them. Furthermore, we can treat the defect case.

0. INTRODUCTION

We present a generalization and refinement of the classical ramification theory of complete discrete valuation rings to valuation rings satisfying either (I) or (II) (as explained in 0.2), in the case of Artin-Schreier extensions. The classical theory considers the case of complete discrete valued field extension $L|K$ where the residue field extension is separable. In his paper [KK89], Kato gives a natural definition of the Swan conductor for complete discrete valuation rings without the separability assumption. He also defines the refined Swan conductor $r_{sw}$ in this case using differential 1-forms and powers of the maximal ideal $m_L$.

The generalization we present is a further refinement of this definition. Moreover, we can deal with the extensions with defect, a case which was not treated previously.

0.1. Invariants of Ramification Theory. Let $K$ be a valued field of characteristic $p > 0$ with henselian valuation ring $A$, additive valuation $v_K$ and residue field $k$. Let $L = K(\alpha)$ be the Artin-Schreier extension defined by $\alpha^p - \alpha = f$ for some $f \in K^\times$. Assume that $L|K$ is non-trivial, that is, $[L : K] = p$. Let $B$ be the integral closure of $A$ in $L$, it follows that $B$ is a valuation ring. Let $v_L$ be the valuation on $L$ that extends $v_K$ and let $l$ denote the residue field of $L$. $\Gamma := v_K(K^\times)$ denotes the value group of $K$. The Galois group $Gal(L|K) = G$ is cyclic of order $p$, generated by $\sigma : \alpha \mapsto \alpha + 1$.

Let $\mathfrak{A} = \{f \in K^\times \mid \text{the solutions of the equation } \alpha^p - \alpha = f \text{ generate } L \text{ over } K\}$. Consider the ideals $\mathcal{J}_\sigma$ and $H$, of $B$ and $A$ respectively, defined as below:

$$\mathcal{J}_\sigma = \left\{ \frac{\sigma(b)}{b} - 1 \mid b \in L^\times \right\} \subset B$$

$$H = \left\{ \frac{1}{f} \mid f \in \mathfrak{A} \right\} \subset A$$

Our first result compares these two invariants via the norm map $N_{L|K} = N$, by considering the ideal $N_{\sigma}$ of $A$ generated by the elements of $N(\mathcal{J}_\sigma)$. We also consider the ideal $\mathcal{I}_\sigma = (\{\sigma(b) - b \mid b \in B\})$ of $B$. The ideals $\mathcal{I}_\sigma$ and $\mathcal{J}_\sigma$ play the roles of $i(\sigma)$ and $j(\sigma)$ (the Lefschetz numbers in the classical case, as explained in 2.1), respectively, in the generalization.

0.2. Main Results. We did not make any assumptions regarding the rank or defect in these definitions. Now consider two special cases of the scenario described above:

(I) (Defectless) In this case, we assume that $L|K$ is defectless. For Artin-Schreier extensions $L|K$ considered in this paper, it means that either $v_L(L^\times)/v_L(K^\times)$ has order $p$ and the residue extension $l|k$ is trivial or the residue extension $l|k$ is of degree $p$ and $L$ has the same value group $\Gamma$ as $K$.

(II) (Rank 1) The value group $\Gamma$ of $K$ is isomorphic to a subgroup of $\mathbb{R}$ as an ordered group.

We will prove the following results:
Theorem 0.3. If $L|K$ satisfies (I) or (II), we have the following equality of ideals of $A$:

\[(0.4) \quad H = N_\sigma\]

Theorem 0.5. If $L|K$ satisfies (I) or (II), we consider the $A$-module $\omega^1_A$ of logarithmic differential 1-forms and the $B$-module $\omega^1_{B|A}$ of logarithmic differential 1-forms over $A$ (as defined in section 1.1). Then

(i) There exists a unique homomorphism of $A$-modules $rsw : H/H^2 \to \omega^1_A/(I_\sigma \cap A)\omega^1_A$ such that $1/f \mapsto d\log f$; for all $f \in A$.

(ii) There is a $B$-module isomorphism $\varphi_\sigma : \omega^1_{B|A}/J_\sigma \omega^1_{B|A} \to J_\sigma^2/J_\sigma^2$ such that $d\log x \mapsto \sigma(x)/x - 1$, for all $x \in L^\times$.

(iii) Furthermore, these maps induce the following commutative diagram:

\[
\begin{array}{ccc}
\omega^1_{B|A}/J_\sigma \omega^1_{B|A} & \xrightarrow{\varphi_\sigma} & J_\sigma^2/J_\sigma^2 \\
\downarrow{\Delta N} & & \downarrow{N} \\
\omega^1_A/(I_\sigma \cap A)\omega^1_A & \xleftarrow{rsw} & H/H^2
\end{array}
\]

where $\Delta N, N$ are induced by the norm map $N$.

The map $rsw$ in (i) is a refined generalization of the refined Swan conductor of Kato for complete discrete valuation rings [KK89].

Remark 0.6. It is worth noting that if $p = 2$, both the results are true without any assumptions regarding defect or rank, as seen in 5.1 and Proposition 7.4.

0.3. Outline of the Contents.

- Review, Small Results, Examples: In sections 1, 2 we present some preliminaries and the discrete valuation ring case. Section 3 contains some elementary results that help us understand the cases I and II. Section 4 explores an example of Artin-Schreier extension with defect in detail.
- Proofs of Main Results: In section 5, we focus on the defect case. 5.1 allows us to express the ring $B$ as a filtered union of rings $A[x]|A$, where elements $x \in L^\times$ are chosen very carefully. This enables us to treat the defect case of 0.5. We prove 0.3 and 0.5 for both cases I and II in sections 6 and 7, respectively.
- The Different Ideal and Further Results: Section 8 presents the description of the different ideal $D_{B|A}$ when $L|K$ satisfies (I) or (II). Section 9 contains some results in case(II) under special assumptions.

1. Preliminaries: Differential Forms, Defect, Cyclic Extensions, Trace

1.1. Definitions: Differential Forms and Different Ideal $D_{B|A}$

Definition 1.1. Differential 1-Forms

(i) Let $R$ be a commutative ring. The $R$-module $\Omega^1_R$ of differential 1-forms over $R$ is defined as follows: $\Omega^1_R$ is generated by

- The set $\{db \mid b \in R\}$ of generators.
- The relations being the usual rules of differentiation: For all $b, c \in R$,

  (a) (Additivity) $d(b + c) = db + dc$

  (b) (Leibniz rule) $d(bc) = cdb + bdc$

(ii) For an integral domain $A$ and a commutative $A$-algebra $B$, the $B$-module $\Omega^1_{B|A}$ of relative differential 1-forms over $A$ is defined to be the cokernel of the map $B \otimes_A \Omega^1_A \to \Omega^1_B$.

Definition 1.2. Logarithmic Differential 1-Forms
(i) For a valuation ring \( A \) with the field of fractions \( K \), we define the \( A \)-module \( \omega_A^1 \) of logarithmic differential 1-forms as follows: \( \omega_A^1 \) is generated by

- The set \( \{ db \mid b \in A \} \cup \{ d \log x \mid x \in K^\times \} \) of generators.
- The relations being the usual rules of differentiation and an additional rule: For all \( b, c \in A \) and for all \( x, y \in K^\times \),
  
  (a) (Additivity) \( d(b + c) = db + dc \)
  
  (b) (Leibniz rule) \( d(bc) = cdb + bdc \)
  
  (c) (Log 1) \( d \log(xy) = d \log x + d \log y \)
  
  (d) (Log 2) \( b \log b = db \) for all \( 0 \neq b \in A \)

(ii) Let \( L|K \) be an extension of henselian valued fields, \( B \) the integral closure of \( A \) in \( L \) and hence, a valuation ring. We define the \( B \)-module \( \omega_{B/A}^1 \) of logarithmic relative differential 1-forms over \( A \) to be the cokernel of the map \( B \otimes_A \omega_A^1 \to \omega_B^1 \).

**Definition 1.3. The Different Ideal \( \mathcal{D}_{B/A} \)**

Let \( A \) be an integrally closed integral domain with the field of fractions \( K \). Let \( L|K \) be a separable extension of fields, \( B \) the integral closure of \( A \) in \( L \). As in the classical case, we define the inverse different \( \mathcal{D}_{B/A}^{-1} \) by \( \mathcal{D}_{B/A}^{-1} := \{ x \in L \mid \text{Trace}_{L|K}(xB) \subset A \} \).

This is a fractional ideal of \( L \). The different \( \mathcal{D}_{B/A} \) of \( B \) with respect to \( A \) is defined as the inverse ideal of \( \mathcal{D}_{B/A}^{-1} \).

### 1.2. Valuation Rings and Differential 1-Forms.

**Definition 1.4.** Let \( A \) be a valuation ring with field fraction \( K \) and additive valuation \( v \). For any \( x \in K^\times \), we define an \( A \)-module homomorphism \( dx : M_x \to \omega_A^1 \) by \( h \mapsto hx \ d \log x \) where \( M_x := (\frac{1}{x}) \).

For \( x = 0 \), we define \( d0 \) to be the zero map: \( M_0 \to \omega_A^1 \) by \( h \mapsto 0 \) where \( M_0 := K \).

**Lemma 1.5.** Let \( A, K, v \) be as above and \( x, y \in K \). Then we have the following properties.

(i) (Additivity) The \( A \)-module homomorphisms \( dx, dy, d(x + y) : M \to \omega_A^1 \) satisfy \( d(x + y) = dx + dy \). Here, \( M = M_x \cap M_y \cap M_{x+y} \).

(ii) (Leibniz rule) The \( A \)-module homomorphisms \( dx, dy, d(xy) : M \to \omega_A^1 \) satisfy \( d(xy) = ydx + xdy \). Here, \( M = M_x \cap M_y \cap M_{xy} \).

**Proof.** (i) We may assume that \( v(x) \leq v(y) \) and write \( y = ax; \ a \in A \). Note that in \( \omega_A^1 \), \( da = ad \log a \) and \( d1 = d \log 1 = 0 \). Hence, \( (a + 1)d \log(a + 1) = d(a + 1) = da = ad \log a \).

For all \( h \in M \),

\[
\begin{align*}
  d(x + y)(h) &= h(x + y)d \log(x + y) \\
  &= hx(a + 1)d \log[x(a + 1)] \\
  &= hx(a + 1)[d \log x + d \log(a + 1)] \\
  &= hx(a + 1)d \log x + hx(a + 1)d \log(a + 1) \\
  &= hx d \log x + hxa d \log x + hxa d \log a \\
  &= hx d \log x + hxa d \log xa \\
  &= dx(h) + dy(h)
\end{align*}
\]

(ii) For all \( h \in M \),

\[
\begin{align*}
  d(xy)(h) &= hxy d \log(xy) \\
  &= hxy d \log x + hxy d \log y \\
  &= ydx(h) + xdy(h)
\end{align*}
\]

\[\square\]
Lemma 1.6. Let $L|K$ be as in 0.1. Then we have

(1) A surjective $B$-module homomorphism $\Phi_\sigma : \Omega^1_{B|A}/\mathcal{I}_\sigma \Omega^1_{B|A} \rightarrow \mathcal{I}_\sigma/\mathcal{I}_\sigma^2$ such that $\Phi_\sigma(db) = \sigma(b) - b$ for all $b \in B$.

(2) A surjective $B$-module homomorphism $\varphi_\sigma : \omega^1_{B|A}/\mathcal{J}_\sigma \omega^1_{B|A} \rightarrow \mathcal{J}_\sigma/\mathcal{J}_\sigma^2$ such that $\varphi_\sigma(d\log x) = \frac{\sigma(x)}{x} - 1$ for all $x \in L^\times$.

Proof. Since $\sigma$ fixes $K$, $\sigma(a) - a = 0$ for all $a \in A$ and $\frac{\sigma(x)}{x} - 1 = 0$ for all $x \in K^\times$. Let $b, c \in B$. (1) follows from $\sigma(b + c) - (b + c) = (\sigma(b) - b + \sigma(c) - c)$ and

$$\sigma(bc) - bc = (\sigma(b) - b)(\sigma(c) - c) + c(\sigma(b) - b) + b(\sigma(c) - c)$$

$$\equiv c(\sigma(b) - b) + b(\sigma(c) - c) \mod \mathcal{I}_\sigma^2.$$

Let $x, y \in L^\times$. (2) follows from

$$\frac{\sigma(xy)}{xy} - 1 = \left(\frac{\sigma(x)}{x} - 1\right)\left(\frac{\sigma(y)}{y} - 1\right) + \frac{\sigma(x)}{x} - 1 + \frac{\sigma(y)}{y} - 1$$

$$\equiv \frac{\sigma(x)}{x} - 1 + \frac{\sigma(y)}{y} - 1 \mod \mathcal{J}_\sigma^2. \quad \Box$$

1.3. Defect: Introduction.

Definition 1.7. Let $E|F$ be a finite algebraic extension of fields of degree $[E : F] = n$ and $v$ a non-trivial valuation on $F$. Denote the extensions of $v$ from $F$ to $E$ by $v_1, \ldots, v_g$. Let $F_v$ be the residue field and $v(F^\times)$ the value group for the valued field $(F,v)$. Similarly, define $E_{v_i}$ and $v_i(E^\times)$. For each $1 \leq i \leq g$, define:

- The ramification index $e_i = (v_i(E^\times) : v(F^\times))$
- The inertia degree $f_i = [E_{v_i} : F_v]$.

Fact I: For each $1 \leq i \leq g$, $e_i$ and $f_i$ are finite. Moreover, we have the fundamental inequality:

$$[E : F] = n \geq \sum_{i=1}^{g} e_if_i \quad (1.8)$$

If the equality holds, it is called the fundamental identity.

Fact II: When $(F,v)$ is henselian, $g = 1$ and we deal with a single ramification index $e_{E|F} = e$ and a single inertia degree $f_{E|F} = f$. Furthermore, in this case, $n$ is divisible by the product $ef$ and we can write

$$n = d_{E|F}e_{E|F}f_{E|F} \quad (1.9)$$

for some positive integer $d_{E|F}$.

Definition 1.10. The integer $d_{E|F}$ above is called the defect of the extension $(E|F,v)$. It is known that $d_{E|F}$ is a power of $q$; where $q = \max\{\text{char}(F_v), 1\}$.

1.4. Cyclic Extensions of Prime Degree. Let $E|F$ be a cyclic Galois extension of henselian valued fields, of prime degree $p$. Let $O_E$ and $O_F$ denote the valuation rings of $E$ and $F$ respectively. Let $E$ and $F$ be the respective residue fields.

Lemma 1.11. If $E|F$ is defectless, then we have two cases:

(a) Order of $v(E^\times)/v(F^\times)$ is $p$ and it is generated by $v_E(\mu)$ for some $\mu \in E^\times$.

(b) There is some $\mu \in E^\times$ such that the residue extension $E|F$ is purely inseparable of degree $p$, generated by the residue class of $\mu$. 

Lemma 1.12. Let \( E[F, \mu] \) be as in [1.11] and \( x_i \in F \) for all \( 0 \leq i \leq p - 1 \). Then
\[
\sum_{i=0}^{p-1} x_i \mu^i \in \mathcal{O}_E \text{ if and only if } x_i \mu^i \in \mathcal{O}_E \text{ for all } i.
\]

Proof. If \( x_i \mu^i \in \mathcal{O}_E \) for all \( i \), then clearly, \( \sum_{i=0}^{p-1} x_i \mu^i \in \mathcal{O}_E \). For the converse, we observe that if \( v_E(x_i \mu^i) \) are all distinct for \( 0 \leq i \leq p - 1 \), then \( 0 \leq v_E\left(\sum_{i=0}^{p-1} x_i \mu^i\right) = \min_{0 \leq i \leq p-1} v_E(x_i \mu^i) \). Hence, the converse is true in this case. Now let us break down the rest into two cases (a) and (b) as described in the lemma above.

(a) We claim that in this case, \( v_E(x_i \mu^i); 0 \leq i \leq p - 1, x_i \neq 0 \) all have to be distinct.

Assume to the contrary. Let \( 0 \leq i < j \leq p - 1 \) be such that \( v_E(x_i \mu^i) = v_E(x_j \mu^j) \); \( x_i, x_j \) are non-zero.

Then \( v_E(\mu^{j-i}) = (j-i)v_E(\mu) = v_E\left(\frac{x_i}{x_j}\right) \in v(\mathbb{F}^\times) \). This is impossible, since the order of \( v_E(\mu) \) in \( v(\mathbb{F}^\times)/v(\mathbb{F}^\times) \) is \( p \) and \( p \nmid j-i \).

(b) We observe that \( v(\mu) = 0 \). The only case we need to consider is when \( \min_{0 \leq i \leq p-1} v(x_i \mu^i) = v < 0 \) and the minimum is achieved by more than one \( x_i \mu^i \). Let \( 0 \leq i_1 < ... < i_r \leq p - 1 \); \( r \geq 2 \) integer such that \( v(x_{i_s} \mu^{i_s}) = v \) for all \( 1 \leq s \leq r \). Clearly, \( v(x_{i_s}) = v \) for all \( 1 \leq s \leq r \). In particular, \( x_{i_1} \neq 0 \). Since \( v\left(\sum_{s=1}^{r} x_{i_s} \mu^{i_s}\right) > v \), we see that \( v\left(\sum_{s=1}^{r} x_{i_s} \mu^{i_s}\right) > 0 \).

Equivalently, \( z = \sum_{s=1}^{r} \frac{x_{i_s}}{x_{i_1}} \mu^{i_s} \in \mathfrak{m}_E \); where \( \mathfrak{m}_E \) is the maximal ideal of \( \mathcal{O}_E \).

Since \( \mathfrak{m}^s \)'s are \( \mathbb{F}^\times \)-linearly independent ; \( 0 \leq i \leq p - 1 \), \( z \in \mathfrak{m}_E \iff \overline{z} = 0 \in \overline{\mathcal{O}_E} \iff \overline{(\frac{x_{i_s}}{x_{i_1}})} = 0 \in \overline{\mathbb{F}} \) for all \( s \). However, this is impossible since \( v(x_{i_s}) = v \) for all \( 1 \leq s \leq r \).

\[ \square \]

Corollary 1.13. Let \( \mu \) be as in [1.11]. Then \( d \log \mu \) generates the \( \mathcal{O}_E \)-module \( \omega^1_{\mathcal{O}_E[\mathcal{O}_F]} \).

Proof. By 1.12, it is enough to consider the elements \( d \log(x \mu^i); 0 \leq i \leq p - 1, x \in \mathbb{K} \).

\( d \log(x \mu^i) = d \log x + i d \log \mu \). The rest follows from the fact that \( d \log x = 0 \) in \( \omega^1_{\mathcal{O}_E[\mathcal{O}_F]} \).

\[ \square \]

1.5. Trace.

Lemma 1.14. Let \( R \) be an integrally closed integral domain with the field of fractions \( F \). \( E[F] \) be a separable extension of fields of degree \( n \). Suppose that \( \beta \in E \) is such that \( E = F(\beta) \). Let \( g(T) = \min_F(\beta) \), the minimal polynomial of \( \beta \) over \( F \). Then

1. \( \text{Tr}_{E[F]}(\frac{\beta^m}{g(\beta)}) \) is zero for all \( 1 \leq m \leq n - 2 \) and \( \text{Tr}_{E[F]}(\frac{\beta^{n-1}}{g(\beta)}) = 1 \).

2. Assume, in addition, that \( \beta \) is integral over \( R \). Then \( \{x \in E \mid \text{Tr}_{E[F]}(xR[\beta]) \subset R\} = \frac{1}{g'(\beta)} R[\beta] \)

Details can be found in [KKS].

2. Discrete Valuation Rings

2.1. Classical Theory: Complete Discrete Valuation Rings with Perfect Residue Fields. Let \( K \) be a complete discrete valued field of residue characteristic \( p > 0 \) with normalised additive valuation \( v_K \), valuation ring \( A \) and perfect residue field \( k \). Consider \( L|K \), a finite Galois extension of \( K \). Let \( e_L|K \) be the
ramification index of $L|K$ and $G = \text{Gal}(L|K)$. Let $v_L$ be the valuation on $L$ that extends $v_K$, $B$ the integral closure of $A$ in $L$ and $l$ the residue field of $L$. In this case, we have the following invariants of ramification theory:

- The Lefschetz number $i(\sigma)$ and the logarithmic Lefschetz number $j(\sigma)$ for $\sigma \in G \setminus \{1\}$ are defined as

\begin{align}
  i(\sigma) &= \min\{v_L(\sigma(a) - a) \mid a \in B\} \\
  j(\sigma) &= \min\{v_L\left(\frac{\sigma(a)}{a} - 1\right) \mid a \in L^*\}
\end{align}

(2.1) (2.2)

Both the numbers are non-negative integers.

- For a finite dimensional representation $\rho$ of $G$ over a field of characteristic zero, the Artin conductor $\text{Art}(\rho)$ and the Swan conductor $\text{Sw}(\rho)$ are defined as

\begin{align}
  \text{Art}(\rho) &= \frac{1}{e_{L|K}} \sum_{\sigma \in G \setminus \{1\}} i(\sigma)(\dim(\rho) - \text{Tr}(\rho(\sigma))) \\
  \text{Sw}(\rho) &= \frac{1}{e_{L|K}} \sum_{\sigma \in G \setminus \{1\}} j(\sigma)(\dim(\rho) - \text{Tr}(\rho(\sigma)))
\end{align}

(2.3) (2.4)

Both these conductors are integers. This is a consequence of the Hasse-Arf Theorem (see [S]).

The invariants $j(\sigma)$ and $\text{Sw}(\rho)$ are the parts of $i(\sigma)$ and $\text{Art}(\rho)$, respectively, which handle the wild ramification. We wish to generalize these to all valuation rings considered in this paper. Namely, the case where $L$ is a non-trivial Artin-Schreier extension of $K$, a valued field with henselian valuation ring, defined by $\alpha^p - \alpha = f$, where $f \in K$. Let us begin with the case of discrete valuation rings, possibly with imperfect residue fields.

2.2. Best $f$ and Swan Conductor. Let $K$ be a discrete valued field of residue characteristic $p > 0$ with normalized additive valuation $v_K$, valuation ring $A$ and residue field $k$. We do not assume that $k$ is perfect. Let $L = K(\alpha)$ be the (non-trivial) Artin-Schreier extension defined by $\alpha^p - \alpha = f$, where $f \in K$. Let $v_L$, $B$ and $l$ denote the valuation, valuation ring and the residue field of $L$, respectively. We define the Swan conductor of this extension as described below.

**Definition 2.5.** Let $\mathfrak{P} : K \to K$ denote the additive homomorphism $x \mapsto x^p - x$. Note that the extension $L$ does not change when $f$ is replaced by any element $g \in K$ such that $g \equiv f \mod \mathfrak{P}(K)$. Because, if $g = f + h^p - h$ for some $h \in K$, then the corresponding Artin-Schreier extension is generated by $\alpha + h$ over $K$.

1. If there is such $g \in A$, $L$ is unramified over $K$ and the Swan conductor is defined to be 0.
2. If there is no such $g \in A$, the Swan conductor is defined to be $\min\{-v_K(g) \mid g \equiv f \mod \mathfrak{P}(K)\}$.

An element $f$ of $K$ which attains this minimum will be referred to as “best $f$” throughout this paper. It is well-defined modulo $\mathfrak{P}(K)$.

This definition coincides with the classical definition of the Swan conductor when $k$ is perfect.

Existence of best $f$ relies on the existence of $\min\{-v_K(g) \mid g \equiv f \mod \mathfrak{P}(K)\}$. This is guaranteed in the case of discrete valuation rings, but not in the case of general valuation rings.

**Example 2.6.** Let $K = k((t))$ where $k$ is of characteristic $p > 0$. $t$ is a prime element of $K$. Let $n$ be a positive integer coprime to $p$. In this case, the Swan conductor of the extension given by $\alpha^p - \alpha = \frac{1}{t^n}$ is $n$.

More generally, let $m \geq 0$ be an integer and $n$ as above. Then the Swan conductor of the extension given by $\alpha^p - \alpha = \frac{1}{t^{np^m}}$ is also $n$. This follows from $\frac{1}{t^{np^m}} \equiv \frac{1}{t^n} \mod \mathfrak{P}(K)$.
A concrete description of the Swan conductor is given by the following lemma:

**Lemma 2.7.** By replacing \( f \) with an element of \( \{ g \in K \mid g \equiv f \ \text{mod} \ \mathcal{P}(K) \} \), we have \( f \) which satisfies exactly one of the following properties:

1. \( v_K(f) = -n \) where \( n \) is a positive integer relatively prime to \( p \).
2. \( f = at^{-n} \) where \( n > 0 \), \( p|n \), \( t \) is a prime element of \( K \) and \( a \in A^\times \) such that the residue class of \( a \) in \( k \) does not belong to \( k^p = \{ x^p \mid x \in k \} \).

In the case (i), the Swan conductor is 0. In the cases (ii) and (iii), the Swan conductor is \( n \).

### 2.3. Refined Swan Conductor \( rsw \)

**Definition 2.8.** Let \( K \) be a discrete valued field of residue characteristic \( p > 0 \) with normalised additive valuation \( v_K \), valuation ring \( A \) and residue field \( k \) (possibly imperfect). Let \( L = K(\alpha) \) be the Artin-Schreier extension defined by \( \alpha^p - \alpha = f \) where \( f \) is best. The refined Swan conductor \( rsw \) of this extension is defined to be the \( A \)-homomorphism \( df : \left( \frac{1}{f} \right) \to \omega_A^1 \) given by \( h \mapsto (hf) \cdot \log f \). We note that for \( h \in \left( \frac{1}{f} \right), hf \in A \) and hence, \((hf) \cdot \log f\) is indeed an element of \( \omega_A^1 \).

\( rsw \) is well-defined up to certain relations, as discussed below.

**Lemma 2.9.** Let \( L/K \) be as above, given by best \( f \), \( H = \left( \frac{1}{f} \right) \). Then \( rsw \) is well-defined as the \( A \)-homomorphism \( : H \to \omega_A^1/\omega_A^1; \) where \( I \) is the ideal \( \{ x \in K \mid v_K(x) \geq \left( \frac{p-1}{p} \right) v_K \left( \frac{1}{f} \right) \} \) of \( A \).

**Proof.** Let \( g \) be best as well. Hence, there exists \( a \in K \) such that \( g = f + a^p - a \) and \( v_K(f + a^p - a) = v_K(f) \). Since \( v_K(a) \geq v_K(f), H \cap M_a = H \). By Lemma 1.5, \( df - da = -\log f \) on \( H \).

For \( h = \frac{b}{f} \in H, b \in A, da(h) = ha \cdot \log a = b \left( \frac{a}{f} \right) d\log a \in \left( \frac{a}{f} \right) \omega_A^1 \). It is enough to show that

\[
v_K \left( \frac{a}{f} \right) \geq \left( \frac{p-1}{p} \right) v_K \left( \frac{1}{f} \right).
\]

This is clear in the case \( a \in A \).

If \( a \in K \setminus A \), then \( v_K(a^p - a) = pv_K(a) \geq v_K(f) = v_K(f + a^p - a) \). Hence, proved.

**Remark 2.10.** We note that \( I = \{ x \in K \mid v_K(x) \geq \left( \frac{p-1}{p} \right) v_K \left( \frac{1}{f} \right) \} = \{ x \in K \mid v_L(x) \geq \left( \frac{p-1}{p} \right) v_L \left( \frac{1}{f} \right) \} \)

### 3. Small Results

In this section, we present some small results that help us understand the two special cases I and II better. First we extend the notion of “best \( f \)” to the general case.

#### 3.1. Best \( f \)

**Definition 3.1.** Let \( K \) be as in 0.1, \( \mathcal{P} : K \to K \) as before. We say that \( f \in K^\times \) is best if either \( f \in A^\times \) or if \( f \) satisfies \( -v(f) = \inf \{-v(g) \mid g \equiv f \ \text{mod} \ \mathcal{P}(K) \} \).

Since we cannot guarantee the existence of best \( f \) in general, as seen in the example below, we will reinterpret the notion of the refined Swan conductor using the logarithmic differential 1-forms over \( A \), as stated in 0.5.

**Example 3.2.** (Non-DVR)

Consider the extension \( L/K \) as described in section 4. The value group \( \Gamma \) is isomorphic to \( \mathbb{Z}[\frac{1}{p}] \). We have a sequence of elements \( f_i := f_{i,0} \in A \) for all integers \( i \geq 0 \), each better than the previous one, such that

\[
(1) \quad -v(f_i) = n - \sum_{j=1}^{i} \frac{1}{p^j}
\]
(2) The ideal $H$ of $A$ is generated by $\{ \frac{1}{f} \mid i \geq 0 \}$.

Since $\inf_{i \geq 0} -v(f_i) = n - \frac{1}{p - 1} = c \in \mathbb{R} \setminus \Gamma$, there is no best $f$.

**Corollary 3.3.** Let $L|K$ satisfy (I), given by $\alpha^p - \alpha = f$ where $f$ is best. Then

(i) $B$ is described as follows:

(a) If $e_{L|K} = p$, $B = \sum_{i=0}^{p-1} A_i \alpha^i$ where $A_0 := A$ and for all $1 \leq i \leq p - 1$,

$$A_i := \{ x \in K \mid v_L(x) \geq -iv_L(\alpha) \} = \{ x \in A \mid v_L(x) \geq -iv_L(\alpha) \}.$$

(b) If $e_{L|K} = 1$, $B = A[\alpha \gamma]$ where $\gamma \in A$ such that $\alpha \gamma \in B^\times$.

(ii) $d \log \alpha$ generates the $B$-module $\omega_B^1$.

**Proof.**

(i) We apply [1.12]

(a) $-v_0 := v_L(\alpha)$ generates the group $v_L(L^\times)/v_L(K^\times)$ of order $p$. In particular, for all $1 \leq i \leq p-1$,

$$iv_0 \notin v_L(K^\times).$$

For $x \in K^\times$, $x \alpha^i \in B$ if and only if $v_L(x) \geq iv_0$ if and only if $v_L(x) > iv_0$.

(b) Since $c = 1$, there exists $\gamma \in A$ such that $\alpha \gamma \in B^\times$. We just take $\mu = \alpha \gamma$.

(ii) This is a direct consequence of (i) and [1.13].

\[\square\]

3.2 **Fractional Ideals in a Valued Field:** Let $F$ be a valued field with additive valuation $v$, value group $\Gamma$, valuation ring $O := O_F$ and residue field $\overline{F}$. A subset $S$ of $F$ is a fractional ideal of $F$ if there exists $0 \neq b \in O$ such that $bS$ is an (integral) ideal of $O$.

We note that in such a case, $S = \{ x \in F \mid v(x) \geq v(s) \text{ for some } s \in S \} = \bigcup_{s \in S} sO$.

**Definition 3.4.** Consider the case (II), we can regard $\Gamma$ as an ordered subgroup of $\mathbb{R}$. Let $S$ be a fractional ideal of $F$ and $\inf_{s \in S} v(s) = t \in \mathbb{R}$. We define $F$-valuation of $S$ as follows:

(i) If $t \in \Gamma \subset \mathbb{R}$, $v(S) := t$.

(ii) If $t \in \mathbb{R} \setminus \Gamma$, $v(S) := t^+$

We can define the $F$-valuation of $S$ by (i) when $S$ is generated by a single element $s \in F$, even if $\Gamma$ is not isomorphic to an ordered subgroup of $\mathbb{R}$. In that case, $v(S) := v(s)$ and $S = s'O$ for any $s' \in F$ such that $v(s') = v(s)$.

3.3 **Defect and $\mathcal{J}_\sigma$.**

**Lemma 3.5.** The fractional ideals $\mathcal{J}_\sigma$ and $H$ are integral ideals of $L$ and $K$ respectively, that is,

(i) $\mathcal{J}_\sigma = \left\{ \frac{\sigma(b)}{b} - 1 \mid b \in L^\times \right\} \subset B$

(ii) $H = \left\{ \frac{1}{f} \mid f \in \mathfrak{A} \right\} \subset A$

**Proof.**

(i) For $b \in L^\times$, $v_L(\sigma(b) - b) \geq \min\{v_L(\sigma(b)), v_L(b)\} = v_L(\sigma(b)) = v_L(b)$. Hence, $\frac{\sigma(b)}{b} - 1 \in B$.

(ii) We need to show that for each $f \in \mathfrak{A}$, $\frac{1}{f} \in A$. Assume the contrary that there is some $f \in \mathfrak{m}_K \cap \mathfrak{A}$.

Since $K$ is henselian, roots of $\alpha^p - \alpha = f$ are already in $K$, contradicting our assumption that $L|K$ is non-trivial.

\[\square\]

**Lemma 3.6.** Let $L|K$ be as in 0.1 and $b \in B$ such that $\sigma(b) - b$ generates $\mathcal{J}_\sigma$. Define $A$-linear maps $D_i : L \to L$ inductively for $0 \leq i \leq p - 1$ by

$$(3.7) \quad D_0 := id_L : L \to L, \quad D_i(x) := \frac{(\sigma - 1)(D_{i-1}(x))}{(\sigma - 1)(D_{i-1}(b^i))}; 1 \leq i \leq p - 1$$

These maps have the following properties:
(1) \( D_i(b^i) = 1; 0 \leq i \leq p - 1 \)
(2) \( D_i(b^j) = 0; 0 \leq j \leq i - 1, 1 \leq i \leq p - 1 \)
(3) For \( x \in B \), \( D_i(xb^i) = \sigma^i(b)D_i(x) + D_{i-1}(x); 0 \leq i \leq p - 1 \) (If \( i = 0 \), we set \( D_{i-1}(x) = 0 \).)

(4) \( D_i(b^{i+1}) = \sum_{j=0}^i \sigma^j(b); 0 \leq i \leq p - 1 \)

(5) For each \( 0 \leq i \leq p - 2 \), \( (\sigma - 1)(D_i(b^{i+1})) = \sigma^{i+1}(b) - b \) and hence, is a generator of \( \mathcal{I}_\sigma \).

In particular, it is non-zero.

**Proof.** First we note that \( (\sigma - 1)(D_0(b^1)) = (\sigma - 1)(b) \neq 0 \) and hence, the definition of \( D_1 \) is valid. As we prove (1)-(5) inductively, validity of the definition of \( D_i; 1 \leq i \leq p - 1 \) will become clear. (1) Follows directly from the definition. For (2), clearly, \( D_1(1) = 0 \). If \( 0 \leq j \leq i - 1 \leq p - 2 \), \( (\sigma - 1)(D_{i-1}(b^j)) = (\sigma - 1)(1) \) or \( (\sigma - 1)(0) \Rightarrow D_i(b^j) = 0 \).

The \( i = 0 \) case of (3)-(5) follows directly from the definition. For \( 1 \leq i \leq p - 1 \), we have:

\[
D_i(xb) = \frac{(\sigma - 1)(D_{i-1}(xb))}{(\sigma - 1)(D_{i-1}(b^i))} = \frac{(\sigma - 1)(\sigma^{i-1}(b)D_{i-1}(x) + D_{i-2}(x))}{\sigma^i(b) - b} = \frac{(\sigma - 1)(\sigma^{i-1}(b))D_{i-1}(x) + (\sigma - 1)(D_{i-2}(x))}{\sigma^i(b) - b} = \sigma^i(b)D_i(x) + \frac{(\sigma - 1)(\sigma^{i-1}(b))D_{i-1}(x) + (\sigma - 1)(D_{i-2}(x))}{\sigma^i(b) - b} = \sigma^i(b)D_i(x) + D_{i-1}(x)(\sigma - 1)(\sigma^{i-1}(b)) + \sigma^{i-1}(b) - b \]

This proves (3) for \( i \). (4) follows from (3). For any fixed \( 0 \leq i \leq p - 2 \), \( (\sigma - 1)(D_i(b^{i+1})) \) has the same valuation as \( (\sigma - 1)(b) \) and hence generates \( \mathcal{I}_\sigma \).

**Corollary 3.8.** For all \( 0 \leq i \leq p - 1 \), \( D_i(B) \) is a subset of \( B \).

**Proof.** This is clearly true for \( i = 0 \). We proceed by induction. Fix some \( 1 \leq i \leq p - 1 \) and assume that the statement is true for \( i - 1 \). Hence, for all \( x \in B \), \( D_{i-1}(x) \in B \Rightarrow (\sigma - 1)(D_{i-1}(x)) \in \mathcal{I}_\sigma \). By 3.6(5), \( D_i(x) = \frac{(\sigma - 1)(D_{i-1}(x))}{(\sigma - 1)(D_{i-1}(b^i))} \in B \).

**Proposition 3.9.** Let \( L|K \) be as in 0.1. \( \mathcal{J}_\sigma \) is principal if and only if \( L|K \) is defectless.

**Proof.** If the extension is defectless, by 1.11, 1.13 and 1.6(a) \( \mathcal{J}_\sigma \) is principal. Now suppose that the extension is with defect and that \( \mathcal{J}_\sigma \) is principal. Hence, by 5.5 \( \mathcal{J}_\sigma = \mathcal{I}_\sigma \). Let \( b \in B \) such that \( \sigma(b) - b \) generates \( \mathcal{I}_\sigma \).

We claim that \( B = A[b] \).

Consider \( x_i \in K; 0 \leq i \leq p - 1 \) such that \( y = \sum_{i=0}^{p-1} x_i b^i \in B \). We must show that \( x_i \in A \); for all \( i \).

Define \( y_i := \sum_{j=0}^{p-i} x_j b^j; 1 \leq i \leq p \). We show that \( y_i \in B \) and consequently, \( D_{p-i}(y_i) = x_{p-i} \in B \).

Clearly, \( y_1 = y \in B \). Assume that \( y_i \in B \) for some \( 1 \leq i \leq p \). Since \( x_{p-i}, b \in B \), \( y_{i+1} = y_i - x_{p-i} b^{p-i} \in B \).
B. Thus, \( x_i \in B \cap K = A; \) for all \( i \) and hence, \( B = A[b] \).

Since the extension is with defect, \( f = 1 \) and \( b = a + b' \) for some \( a \in A \) and for some \( b' \in m_L. \) Therefore, we may assume \( b \in m_L. \) Also, due to the defect, \( e = 1 \) and \( b = ab' \) for some \( a \in m_K \) and for some unit \( b' \) of \( B. \) \( \sigma(b) - b = a(\sigma(b') - b'), \) \( \sigma(b') - b = c(\sigma(b) - b) \) for some \( c \in B. \) Hence, \( ac = 1. \) This is impossible since \( a \in m_K. \) Thus, the extension must be defectless if \( \mathcal{J}_\sigma \) is principal.

\[ \Box \]

4. DEFECT: EXAMPLE AND REMARKS

Example 4.1. Let \( k \) be a field of characteristic \( p > 0. \) Using blow-ups, we successively construct local rings \( A_{i,j}; 0 \leq j \leq p + 2, 0 \leq i, \) such that

(i) \( A_{0,0} = k[x, y]_{(x,y)} \subset k(x, y). \) Put \( x_{0,0} = x, y_0 = y. \)

(ii) \( A_{0,0} \subset \cdots \subset A_{i,0} \subset A_{i,1} \subset \cdots \subset A_{i,p+1} \subset A_{i,p+2} \subset A_{i+1,0} \subset \cdots \)

(iii) We can regard \( A_{i,j} = k(x, y) \)

Define \( A := \bigcup_{0 \leq j \leq p+2, 0 \leq i} A_{i,j}, \) \( K = frac(A). \) We will see that \( K \) is a valued field with valuation ring \( A. \)

Consider the Artin-Schreier extension \( L|K \) given by \( \alpha^p - \alpha = \frac{1}{x^n} = \frac{1}{x_{0,0}^n}; \) \( n \) is a fixed positive integer coprime to \( p. \) We claim that this is an extension of valued fields with defect.

The integral closure of \( A \) in \( L \) is \( B = \bigcup_{0 \leq j \leq p+2, 0 \leq i} B_{i,j}, \) where \( B_{i,j} \) be the integral closure of \( A_{i,j} \) in \( L|K. \)

We will study \( B \) using \( B_{i,j}'s \) by considering \( L|K \) as the extension given by \( \alpha^p - \alpha = f_{i,j}. \) The elements \( f_{i,j}; 0 \leq j \leq p + 2, 0 \leq i \) are obtained during each blow-up and they all give rise to the same extension \( L|K. \)

4.1. Construction. We consider the rings \( A_{i,j} = k[x_{i,j}, y_{i}]_{(x_{i,j}, y_{i})}; 0 \leq j \leq p + 2, j \neq p, 0 \leq i \) and \( A_{i,p} = k[x_{i,p}, y_{i}]_{(x_{i,p}, y_{i})}; 0 \leq i \) where the local parameters \( x_{i,j}, y_{i} \) are obtained as described below.

We begin with \( x_{0,0}, y_0. \) Let \( n_0 = n > 0 \) be an integer coprime to \( p \) and \( f_{0,0} = \frac{1}{x_{0,0}^n}. \) Write \( x_{0,0} = y_0 x_{0,1} \)

to obtain \( f_{0,1} = \frac{1}{y_0^n x_{0,1}^n}. \) Similarly, for \( 1 \leq j \leq p, \) write \( x_{0,j-1} = y_0 x_{0,j} \) to obtain \( f_{0,j} = \frac{1}{y_0^n x_{0,j}^n}. \) Choose \( x_{0,p}, \) to be near 1 and write \( \frac{1}{x_{0,p}} = 1 + x_{0,p+1}. \) The Artin Schreier extensions arising from

\[
\begin{align*}
    f_{0,p+1} &= \frac{1 + x_{0,p+1}}{y_0^{np}} \\
    f_{0,p+1} &= \frac{1 + x_{0,p+1}}{y_0^{np}} - \left( \frac{1}{y_0^n} \right)^p + \frac{1}{y_0^n}\frac{x_{0,p+1}}{y_0^{np}} + \frac{1}{y_0^n} \quad \text{are the same. We define “new”}
\end{align*}
\]

Write \( x_{0,p+1} = y_0 x_{0,p+2} \) to obtain \( f_{0,p+2} = \frac{x_{0,p+2}}{y_0^{np-1}} + \frac{1}{y_0^{np-1}} = \frac{x_{0,p+2} + y_0^{np-1-n}}{y_0^{np-1}}. \) Here, \( x_{0,p+2} \) is a unit, \( y_0^{np-1-n} \) is in the maximal ideal, and consequently, \( x_{0,p+2} + y_0^{np-1-n} \) is also a unit. Since \( np - 1 \) is coprime to \( p, \) there exists a unit \( u \) such that \( u^{np-1} = x_{0,p+2} + y_0^{np-1-n}. \)

Put \( x_{1,0} = \frac{y_0}{u} \) and \( f_{1,0} = \frac{1}{x_{1,0}^{n_1}}. \) We note that since \( np - 1 \) is coprime to \( p, f_{1,0} \) and \( f_{0,0} \) have a similar form. Iterate this process with \( x_{i,0}, y_{i} \) by using \( n_i = n_{i-1}p - 1 \) and \( f_{i,0} = \frac{1}{x_{i,0}^{n_i}} \) for \( i \geq 0. \)
4.2. **Valuation on \(A\) and \(B\):** Due to their construction using successive blow ups we note that \(A\) and \(B\) are valuation rings. Let \(v_K = v\) be the valuation on \(K\). We see from the calculations below that the value group of \(K\) is \(\Gamma \equiv \mathbb{Z}[\frac{1}{p}]\); \(v(x_{0,0}) \mapsto 1\).

For all \(1 \leq j \leq p, 0 \leq i\), we have \(-n_iv(x_{i,0}) = v(f_{i,0}) = v(f_{i,j})\) and \(v(f_{i,p+1}) = v(f_{i,p+2}) = v(f_{i+1,0}) = -(n_ip - 1)v(x_{i+1,0})\).

Thus, the only change in the valuation of \(f_{i,j}\)'s occurs during the construction of the “new” \(f_{i,p+1}\). Let us compute valuations of the elements involved.

\[
1 \approx x_i = \frac{x_i,p - 1}{y_i} = \cdots = \frac{x_i,0}{y_i} \Rightarrow v(y_i) = \frac{1}{p}v(x_{i,0}). \quad \text{Since } u \text{ is a unit, } v(x_{i+1,0}) = v(y_i).
\]

Combining the two, we have \(v(x_{i+1,0}) = \frac{1}{p}v(x_{i,0}) = \frac{1}{p^{i+1}}v(x_{0,0})\).

Hence, \(v(f_{i+1,0}) = -(n_ip - 1)v(x_{i+1,0}) = -\frac{1}{p}(n_ip - 1)v(x_{i,0}) = -(n_{i} - \frac{1}{p})v(x_{i,0}) = v(f_{i,0}) + \frac{1}{p}v(x_{i,0})\).

Since \(\Gamma\) is \(p\)-divisible, \(L|K\) has defect. We will use \(v\) to denote \(v_L\) as well.

**Notation 4.2.** Due to (\(\ast\)), we only consider the corresponding \(A_{i,j}\)'s and \(B_{i,j}\)'s. Let \(A_{i,p+1} := A_i\) and \(B_{i,p+1} := B_i\). \(\alpha\)'s are also unaffected until we reach stages (\(\ast\)). Let us denote them by \(\alpha_i\) and we have for all \(i \geq 0\):

\[
\alpha_i^p - \alpha_i = \frac{1}{x_i^p} - \frac{1}{x_i} = \frac{1 + x_i}{x_i} \quad \text{and} \quad \alpha_{i+1} = \alpha_i - \frac{1}{y_i}.
\]

Note that \(\alpha_{i+1}\) (and not \(\alpha_i\)) corresponds to \(B_i\). It satisfies the equation \(\alpha_{i+1}^p - \alpha_{i+1} = \frac{x_i + y_i}{x_i} + \frac{1}{y_i}\).

4.3. **Special Ideals and Differential Modules.** Due to the defect, we have \(\mathcal{I}_\sigma = \mathcal{J}_\sigma\). Also, it is enough to look at \(\Omega^1\)'s instead of \(\omega^1\)'s.

The elements \(\frac{1}{\alpha_i}\) for \(i \geq 0\) generate the ideal \(\mathcal{J}_\sigma\) of \(B\) and the elements \(\frac{1}{f_{i,0}}\) for \(i \geq 0\) generate the ideal \(H\) of \(A\). Hence,

- \(\mathcal{I}_\sigma = \mathcal{J}_\sigma = \{b \in B \mid v(b) > \frac{1}{p}(n - 1)v(x_{0,0}) = v_0\}\), and consequently,
- \(\mathcal{N}_\sigma = \{a \in A \mid v(a) > (n - 1)v(x_{0,0}) = pv_0\}\).
- \(H = \{a \in A \mid v(a) > (n - 1)v(x_{0,0})\}\).

Thus, \(\mathcal{O}_3\) is clearly true in this case. Next, consider the differential modules \(\Omega^1_{B_i\mid A_i}\) and \(\Omega^1_{B_{i+1}\mid A_i}\)'s.

\(B_i = A_i[\beta_i]\), since \(\beta_i := \alpha_{i+1}y_i^p\) has the minimal polynomial \(F_i(T) = T^p - T\cdot y_i^{n_i(p-1)} - x_i + y_i^{n_i(p-1)}\) over \(A_i\); \(F'_i(T) = -y_i^{n_i(p-1)}\). We have an isomorphism of \(B_i\)-modules : \(A_i[\beta_i]/F'_i(\beta_i) \rightarrow \Omega^1_{B_i\mid A_i}\) via the \(A_i\)-linear map \(a \mapsto ad\beta_i\) for all \(a \in A_i\). Consider the fractional ideals \(\Theta\) and \(\Theta'\) of \(L\) given by

\(\Theta = \{x \in L \mid v(x) > \frac{-v(x_{0,0})}{p(p-1)} =: v_1\}\) and \(\Theta' = \{x \in L \mid v(x) > \left(\frac{n(p-1)}{p}\right)v(x_{0,0}) + v_1 =: v_2\}\).

Then we have:

- \(\Omega^1_{B_i\mid A_i} \cong \Theta/\Theta'\)
- \(\Theta/\mathcal{J}_\sigma \Theta \cong \mathcal{J}_\sigma/\mathcal{J}_\sigma^2\)

We can also verify that

- \(\mathcal{D}^{-1}_{B_i\mid A_i} = \cap_{i \geq 0} \mathcal{D}^{-1}_{B_{i+1}\mid A_i}\)
- \(\mathcal{D}^1_{B_i\mid A_i} = \{x \in L \mid v(x) > -(p-1)v_0\}\)
- \(\mathcal{D}^1_{B_i\mid A_i} = \mathcal{J}^p_{\sigma-1}\) is the annihilator of \(\Omega^1_{B_i\mid A_i}\).
5. Filtered Union in the Defect Case

In the previous sections, we developed ramification theory for defectless Artin-Schreier extensions. The case with defect is more difficult to deal with. To generalize the results to the defect case, we write the ring $B$ as a filtered union of rings $A[x]$, where the elements $x$ are chosen very carefully. Although these are not valuation rings, each ring is generated by a single element (over $A$). This makes the extensions $K(x)|K$ and the corresponding differential modules, special ideals easier to understand.

**Theorem 5.1.** Consider $\mathcal{I} = \{\alpha \in L \mid \alpha^p - \alpha = f; f \in K, and \alpha generates L|K\}$. For each $\alpha \in \mathcal{I}$, we can find $\alpha' \in B^\times \cap \alpha K^\times$ such that $B = \cup_{\alpha \in \mathcal{I}} A[\alpha']$ is a filtered union, that is, the following are true:

(i) For any $\alpha_1, \alpha_2 \in \mathcal{I}$, either $A[\alpha_1'] \subset A[\alpha_2']$ or $A[\alpha_2'] \subset A[\alpha_1']$.

(ii) Given any $\beta \in B$, there exists $\alpha \in \mathcal{I}$ such that $\beta \in A[\alpha]$.

5.1. $p = 2$. We already have 0.3 and 0.5 for the defect case when $p = 2$. However, these methods don’t work when $p > 2$. We would like to consider the form of the filtered union in the $p = 2$ case, in the hope that it will give us an idea about the general construction when $p > 2$.

**Proposition 5.2.** For $p = 2$, $B = \cup_{\alpha \in L\setminus K} A[\frac{T_\alpha}{\alpha}]$ is a filtered union.

**Proof.** We are dealing with the defect case, so $v_L = v_K = v$. Let $\alpha_1, \alpha_2 \in L\setminus K$. $\frac{T_\alpha}{\alpha} = \beta_i \in B$.

$$\sigma(\frac{\alpha_1}{T_\alpha}) = \frac{\sigma(\alpha_1)}{\sigma(T_\alpha)} = \frac{\alpha_1}{T_\alpha}, \beta_1 \in B.$$ $$\sigma(\frac{\alpha_2}{T_\alpha}) = \frac{\sigma(\alpha_2)}{\sigma(T_\alpha)} = \frac{\alpha_2}{T_\alpha}, \beta_2 \in B.$$ Therefore, $\frac{\alpha_1}{T_\alpha} < \frac{\alpha_2}{T_\alpha}$ in $B$.

**Claim:** If $v(c_1) > v(c_2)$, then $A[\beta_1]$ is a subset of $A[\beta_2]$. 

**Proof of Claim:** It is enough to show that $\beta_1 \in A[\beta_2]$.

Since $\frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2} + c_2$, it is an element of $A[\beta_2]$. Consequently, $\frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2}, c_2 \in A[\beta_2]$.

Claim: $v(c_1) = v(c_2)$ implies $A[\beta_1] = A[\beta_2]$.

This can be shown by following steps:

- $\beta_1 = \frac{c_1}{\beta_1} - c_1, c_1 \in A$. 
- $\frac{c_1}{\beta_1} = c_1(\frac{1}{\beta_1} + g) = \frac{c_2}{\beta_2} + c_1 g$.

Now $-v(c_1) = v(\frac{1}{c_1}) = v(\frac{1}{c_2} + g^2 - g) < -v(c_2) = v(\frac{1}{c_2}) \leq 0$

$\Rightarrow -v(c_1) = v(\frac{1}{c_2} + g^2 - g) = v(g^2 - g) = 2v(g)$ (the last equality follows from $0 > -v(c_1)$.)

$\Rightarrow v(c_1 g) = -2v(g) + v(g) = -v(g) > 0$

$\Rightarrow c_1 g \in A$ (since $c_1 g$ is already in $K$.)

This proves the claim. We just need to deal with the case $v(c_2) = v(c_1)$ now.

Claim: $v(c_1) = v(c_2) \Rightarrow A[\beta_1] = A[\beta_2]$.

Proof of Claim: The proof is very similar to the one above. We just need to show that $v(c_1 g) \geq 0$. If $v(g) \geq 0$, this is clearly true. Let $v(g) < 0, v(c_1) = v(c_2) = v \geq 0$.

$v(\frac{1}{c_2} + g^2 - g) = v(\frac{1}{c_2}) \Rightarrow 2v(g) = v(g^2 - g) \geq v(\frac{1}{c_2}) = -v \Rightarrow v(c_1 g) \geq v - \frac{v}{2} = \frac{v}{2} \geq 0$

**Remark 5.4.** This particular construction in the case $p = 2$ doesn’t appear to have an easy generalization to the case $p > 2$. We use a different approach.
5.2. Some Elementary Results for \( p > 2 \). Due to the defect, given any \( \alpha \in \mathcal{I} \) there exists \( \gamma_\alpha = \gamma \in A \) such that \( v(\gamma) = -v(\alpha) = -p v(f) \). Define \( \alpha' = \alpha \gamma \in B^\times \). We claim that this choice of \( \alpha' \) satisfies the conditions of Proposition 5.9. We note that the ring \( A[\alpha'] \) does not depend on the choice of \( \gamma \).

Lemma 5.5. In the defect case, we have

(i) \( (\sigma(b) - b \mid b \in B^\times) = \mathcal{I}_\sigma = \mathcal{J}_\sigma = (\sigma(b) - b \mid b \in B^\times) \)

(ii) \( \Omega^1_{B|A} = \omega^1_{B|A} \)

Proof. Given any \( b \in L \), there are elements \( a \in K, b' \in B^\times \) such that \( b = ab' \).

(i) For \( b \in B, a \in A \) and \( \sigma(b) - b = a(\sigma(b') - b') \).

(ii) Let \( b \in L^x \). By Proposition 5.3, \( \log b = d \log a + d \log b' = d \log b' \) since \( \log a = 0 \) in \( \omega^1_{B|A} \).

Lemma 5.6. If \( \alpha_1, \alpha_2 \in \mathcal{I} \) such that \( v(\alpha_1) \leq v(\alpha_2) \), then \( A[\alpha'_1] \subset A[\alpha'_2] \).

Proof. We have (by choosing appropriate conjugates) \( \sigma(\alpha_2 - \alpha_1) = (\alpha_2 + 1) - (\alpha_1 + 1) = \alpha_2 - \alpha_1 \).

Hence, \( \alpha_2 - \alpha_1 = h \in K \).

\( v(\alpha_1) \leq v(\alpha_2) \Rightarrow v(\gamma_1) \geq v(\gamma_2) \) and \( v(h) \geq v(\alpha_1) = -v(\gamma_1) \). Therefore, \( \frac{\alpha_1'h}{\gamma_2}, \gamma_1 h \in A \).

Consequently, \( \alpha'_1 = \gamma_1 (\alpha_2 - h) = \frac{\alpha_1'h}{\gamma_2} - \gamma_1 h \in A[\alpha'_2] \).

Lemma 5.7. Given any \( \beta \in B \), there exists \( \alpha \in \mathcal{I} \) such that \( (\sigma(\beta) - \beta) \subset (\sigma(\alpha') - \alpha') \).

Proof. Let \( v := v(\sigma(\beta) - \beta), v_0 := \inf_{b \in B^\times} v(\sigma(b) - b) \in \mathbb{R} \). Hence, \( \mathcal{I}_\sigma = \mathcal{J}_\sigma = \{ x \in L^x \mid v(x) > v_0 \} \)

and \( \mathcal{N}_\sigma = \{ x \in K^x \mid v(x) > pv_0 \} \). Since this is the defect case, by Proposition 5.3, \( \mathcal{I}_\sigma \) is not a principal ideal. We need to show that \( v > c \), where \( c \in \mathbb{R} \) is defined by

\[ c := \inf_{\alpha \in \mathcal{I}} v(\sigma(\alpha') - \alpha') = \inf_{\alpha \in \mathcal{I}} v(\gamma_\alpha) = \inf_{\alpha \in \mathcal{I}} -v(\alpha) = \inf_{f \in \mathbb{A}} -\frac{1}{p} v(f). \]

Note that \( H = \{ x \in K^x \mid v(x) > pv_0 \} \). By Proposition 5.3, \( H = \mathcal{N}_\sigma \) and hence, \( c = v_0 \). To conclude the proof, we observe that \( \sigma(\beta) - \beta \in \mathcal{I}_\sigma \Rightarrow v > v_0 = c \).

Lemma 5.8. For \( x, y \in L \), we have \( (\sigma - 1)^n(xy) = \sum_{k=0}^{n} \binom{n}{k} (\sigma - 1)^{n-k}(x)(\sigma - 1)^{k}(y) \)

In particular, for \( n = 1, (\sigma - 1)(xy) = (\sigma - 1)(x)(\sigma(y) + x(\sigma - 1)(y)) \).

Proof. This can be proved by using induction on \( n \) and the binomial identity \( \binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k} \).

5.3. Filtered Union for \( p > 2 \).

Proposition 5.9. Given any \( \beta \in B^\times \), there exists \( \alpha \in \mathcal{I} \) such that \( (\sigma - 1)^{p-1}(\frac{1}{F'(\alpha')}) A[\alpha', \beta] \subset B \). Here, \( F \) denotes the minimal polynomial of \( \alpha' \) over \( K \).

Proof. We compute valuation of these elements and show that it is non-negative.

For all \( \alpha \in \mathcal{I}, 1 \leq k \leq p - 1, \sigma^k(\alpha') - \alpha' = (\sigma^k(\alpha) - \alpha)\gamma = k\gamma \). Therefore, \( F'(\alpha') = \gamma^{p-1}(p-1)! \).

In particular, it is an element of \( K \) and hence, fixed by \( \sigma \).

We wish to select \( \alpha \) such that for all \( i, j \geq 0, \)

\[ v((\sigma - 1)^{p-1}(\alpha^i\beta^j)) \geq v(F'(\alpha')) = (p-1)v(\gamma) \]
(Step 1) **Construction of the special $\alpha'$**

We begin with an $\alpha_0$ satisfying $(\sigma(\beta) - \beta) \subset (\sigma(\alpha_0') - \alpha_0')$. Let $(\sigma - 1)(\beta) = b_1\gamma_0; b_1 \in B$. Therefore, $(\sigma - 1)^2(\beta) = (\sigma - 1)(b_1)\gamma_0$. We don’t know much about the valuation of $(\sigma - 1)(b_1)$, however. Let $\alpha_1$ be such that $((\sigma - 1)(b_1)) \subset ((\sigma - 1)(\alpha_1'))$. Write $(\sigma - 1)(b_1) = b_2\gamma_1; b_2 \in B$. Now we can write $(\sigma - 1)^2(\beta) = b_2\gamma_1\gamma_0$. Using this process, we can find $b_i$’s and $\alpha_i$’s such that $(\sigma - 1)^i(\beta) = b_i\gamma_i...\gamma_0$; where $b_i \in B$.

Let $\gamma$ be the $\gamma_j$ with smallest valuation involved in the expression for $i = p - 1$. Let $\alpha$ denote the corresponding $\alpha_j$. We will show that this $\alpha$ satisfies the required property (5.10).

(Step 2) **Proof for $\beta$**

$(\sigma(\beta) - \beta) \subset (\sigma(\alpha_0') - \alpha_0') \subset (\sigma(\alpha') - \alpha') = (\gamma)$, since $v(\gamma) \leq v(\gamma_0)$. Due to the choice of $\gamma$, we also have $v((\sigma - 1)^{t}(\beta)) \geq tv(\gamma)$ for all $1 \leq t \leq p - 1$. In particular, this is true for $t = p - 1$, proving the statement (5.10) for the case $i = 0, j = 1$.

(Step 3) **Terms $\alpha^i\beta^j$**

For the terms of the form $\beta^j$, we use induction on $j$ and 5.8. Valuation of each term in the expansion is at least $(p - 1)v(\gamma)$. In fact, by a similar argument, $v((\sigma - 1)^{k}(\beta^j)) \geq kv(\gamma)$ for all $1 \leq k \leq p - 1$.

For the general terms $\alpha^i\beta^j$, first note that $(\sigma - 1)^k(\alpha') = (\sigma - 1)^{k-1}(\gamma) = 0$ for all $k > 1$.

Therefore, (again using the identity), we have

$$(\sigma - 1)^{p-1}(\alpha^i\beta^j) = \alpha^i(\sigma - 1)^{p-1}(\beta^j) + (p - 1)(\sigma - 1)(\alpha^i)(\sigma - 1)^{p-2}(\sigma(\beta^j))$$

Once again, both these terms have valuation $\geq (p - 1)v(\gamma)$.

This concludes the proof of the proposition.

5.4. **Proof of 5.1**

Let $\beta$ and corresponding special $\alpha'$ be as described above in (Step 1). We recall that for an $A$-module $R \subset L$, $R^* := \{ x \in L \mid Trace_{L/K}(xR) \subset A \}$.

1. $A[\alpha', \beta]^* = A[\alpha']^*$

   **Proof.** Clearly, $A[\alpha', \beta]^* \subset A[\alpha']^* = \frac{1}{\mu(\alpha')}A[\alpha']$. We proved that $(\sigma - 1)^{p-1}(\frac{1}{\mu(\alpha')}A[\alpha', \beta]) \subset B$.

   Since $(\sigma - 1)^{p-1} = Trace_{L/K}$ has image in $K$, $Trace_{L/K}(\frac{1}{\mu(\alpha')}A[\alpha', \beta]) \subset B \cap K = A$ and we have the reverse inclusion.

2. $R := A[\alpha', \beta], S := A[\alpha']$ are finitely generated free $A$-modules.

   **Proof.** Since $\beta, \alpha'$ are integral over $A$, $R$ and $S$ are finitely generated $A$-modules. $A$ is a valuation ring and $R, S$ are finitely generated torsion-free $A$-modules. Therefore, $R, S$ are free $A$-modules (of finite ranks).

3. $A[\alpha', \beta] = A[\alpha']$

   **Proof.** $R$ is a free $A$-module of finite rank. Hence, $R^{**} = (R^*)^* = R$. Similarly, $S^{**} = S$. By (1), $R^* = S^*$ and hence, $R = S$.

These statements, in combination with 5.8 prove part (ii) of 5.1. Part (i) was already proved in Lemma 5.6. This concludes the proof.

6. **Proof of 0.3**

We prove that $H = N_\sigma$. Let $f \in A$. Then $(-1)^p N(\alpha) = - f$. Equivalently, $\frac{1}{f} = N(\frac{1}{\alpha}) = N(\frac{\alpha}{\alpha}) - 1$. From this, it follows that $H$ is a subset of $N_\sigma$, without any assumptions regarding defect or the value group $\Gamma_K$. Next, we prove the reverse inclusion $N_\sigma \subset H$. If $L|K$ is defectless, this follows directly from results in section 3. $H$ is generated by $\frac{1}{f}$, where $f$ is best. Since $J_\sigma = (\frac{1}{\alpha}), N_\sigma = (N(\frac{1}{\alpha})) = H$. Proof in the defect case, however, requires some work.
Let $L|K$ satisfy (II) and have defect. The value group $\Gamma = \Gamma_K$ can be regarded as an ordered subgroup of $\mathbb{R}$. Let $v$ denote the valuation on $L$ and also on $K$. We analyze a special case first.

6.1. Case $p = 2$. For any $x \in L$, $\sigma(x) - x = x - \sigma(x) = \sigma(x) - x$, since the characteristic is 2. Hence, $\sigma(x) - x = \sigma(x) + x = \text{Tr}_{L|K}(x) \in K$; for all $x \in L$. For any fixed $x \in L$, let $y = \sigma(x) - x$. $\sigma(y) - \frac{y}{2} = 1$ if $y$ is non-zero, that is, if $x$ does not belong to $K$. Let $z = \frac{x}{2}$, $z + \sigma(z) = 2z + 1 = 1$ and $N(z) = z(z + 1) = z^2 + z = z^2 - z = f \in K$. Thus, $\frac{x}{2}$ is a solution of an Artin-Schreier extension $\alpha^2 - \alpha = f; f \in K$. All Artin-Schreier extensions over $K$ having solution in $L$ are obtained in this way.

Any generator of $\mathcal{J}_\sigma$ has the form $\frac{\sigma(x) - x}{x}$. Letting $\frac{1}{f} = N(\frac{\sigma(x) - x}{x})$ we get the corresponding Artin-Schreier extension.

Remark 6.1. We don’t need $\Gamma$ to be an ordered subgroup of $\mathbb{R}$ for this case, the argument is true for any value group.

6.2. Case $p > 2$. We wish to show $N_\sigma \subset H$, equivalently, for each $\beta \in L^\times \setminus K^\times$ the ideal of $A$ generated by $N(\frac{\beta}{\sigma(\beta)} - 1)$ is a subset of $H$.

Let us begin with some elementary observations:

(01) We may assume $\beta \in B \setminus A$: $\frac{\sigma(1/\beta)}{1/\beta} - 1 = (\frac{\sigma(\beta)}{\beta} - 1)(-\frac{\beta}{\sigma(\beta)})$. Since $(-\frac{\beta}{\sigma(\beta)}) \in B^\times$, norms of elements $\frac{\sigma(1/\beta)}{1/\beta} - 1$ and $\frac{\sigma(\beta)}{\beta} - 1$ generate the same ideal of $A$.

(02) Trace and $\langle \sigma - 1 \rangle$:

We have the formal expression $(\sigma - 1)^{p-1} = \frac{(\sigma - 1)}{\sigma - 1} = \frac{\sigma - 1}{\sigma - 1} = \sigma^{p-1} + \sigma^{p-2} + \ldots + \sigma + 1$.

Thus, for any $x \in L$, $(\sigma - 1)^{p-1}(x) = \text{Trace}_{L|K}(x)$.

(03) Reduction: If we can find an element $x_\beta = x \in L \setminus K$ such that $v(\frac{(\sigma - 1)(x)}{x}) \leq v(\frac{\sigma(\beta)}{\beta} - 1)$, then we have:

$$0 \leq v(N(\frac{(\sigma - 1)(x)}{x})) = t_1 \leq v(N(\frac{\sigma(\beta)}{\beta} - 1)) = t_2$$

After this, it is sufficient to show that the ideal of $A$ generated by $N(\frac{\sigma(x)}{x} - 1)$ is a subset of $H$.

(04) $\sigma - 1$ and changes in valuation: Let $b \in L^\times$.

- $\frac{\sigma(b) - b}{b} \in B \Rightarrow v(\sigma(b) - b) = v(b) + s_b$ for some $s_b \geq 0$.
- For $1 \leq i \leq p - 1$,

$$v(\sigma^i(b) - b) = v(\sum_{1 \leq j \leq i} \sigma^j(b) - \sigma^{j-1}(b)) \geq \min_{1 \leq j \leq i} \{v(\sigma^j(b) - \sigma^{j-1}(b))\} = v(\sigma(b) - b)$$

- By the same argument, applied to $\tau = \sigma^i$, $(\tau = \sigma^m$ for some $1 \leq m \leq p - 1$) we have

$$v(\sigma(b) - b) \geq v(\sigma^i(b) - b)$$

and thus, the following equality:

For all $1 \leq i \leq p - 1$,

$$v(\sigma^i(b) - b) = v(b) + s_b$$

Proof. For given $\beta$ as above, let $g(T) = \min_K(\beta)$ and $x_\beta = x = (\sigma - 1)^{p-2}(\frac{\sigma^{p-1}}{g'(\beta)})$.

Put $y = \sigma(x) - x = (\sigma - 1)(x)$. By (02) and $\frac{x}{y} = \text{Trace}_{L|K}(x) = 1$. In particular, $y \neq 0$ and we may divide by $y$. As in the case $p = 2$, we have $\sigma(\frac{x}{y}) - \frac{x}{y} = 1/\sigma - 1)(x) = 1$.

$z = \frac{x}{y} = (\sigma - 1)^{p-2}(\frac{\sigma^{p-1}}{g'(\beta)}) \in L \setminus K$ satisfies $\sigma(z) = z + 1$ and hence, the Artin Schreier equation $\alpha^p - \alpha = N(z)$. Thus, we have

$$\frac{1}{N(z)} = N\left(\frac{(\sigma - 1)(x)}{x}\right) = N\left(1/(\sigma - 1)^{p-2}(\frac{\sigma^{p-1}}{g'(\beta)})\right) \in H.$$
If $s' \leq s$, then $N(\sigma(\beta) - 1) \in H$ and hence, $\left( N\left(\frac{\sigma(\beta)}{\beta} - 1\right) \right) = N\left(\frac{\sigma(\beta)}{\beta} - 1\right) A \subset H$.

Now suppose that $s' > s$. Put $r = \frac{\beta_{p-1}}{\beta(\beta)}$, $g'(\beta) = \prod_{1 \leq i \leq p-1}(\beta - \sigma^i(\beta))$. Hence, by (O4),

$$v(r) = -(p-1)s$$

For $1 \leq i \leq p-1$, let $v((\sigma - 1)^i(r)) = v((\sigma - 1)^{i-1}(r)) + c_i; c_i \geq 0$. $c_{p-1} = s'$ by definition. Since $v((\sigma - 1)^{p-1}(r)) = v(1) = 0$, from (6.3), we see that

$$\sum_{i=1}^{p-1} c_i = -(p-1)s$$

Let $c := \inf\{v(\sigma(b)/b - 1) | b \in L^x\} = \inf\{s_b | b \in L^x\} \in \mathbb{R}$, where $s_b$ is as described in (O4).

Hence, by (O4),

We observe $(p-1)s = \sum_{i=1}^{p-2} c_i + s' \geq (p-2)c + s' \geq (p-2)c + s \geq (p-1)c \geq 0$.

In particular, $(p-2)(s - c) \geq s' - s > 0$. By the definition of $c$, we can take $s$ very close to $c$ such that $s' \leq s$ for this new $s$.

This concludes the proof.

\[\Box\]

Corollary 6.5. Under the assumptions of [6.3], the following statements are equivalent:

1. Best $f$ exists.
2. $H$ is a principal ideal of $A$.
3. $J_\sigma$ is a principal ideal of $B$.
4. $L/K$ is defectless.

7. PROOF OF [0.5]

Lemma 7.1. $N_{L/K} : N : B \rightarrow A/(I_\sigma \cap A)$ is a surjective ring homomorphism.

Proof. We just need to check the additive property of $N : B \rightarrow A/(I_\sigma \cap A)$ in order to prove that it is a ring homomorphism. For $x \in B$, $N(x) = x \prod_{i=1}^{p-1}\sigma^i(x)$.

For each $1 \leq i \leq p-1$, $\sigma^i(x) \equiv x \mod I_\sigma$.

Thus, $N : B \rightarrow B/I_\sigma$ is just the $p$-power map, that is, $x \mapsto x^p \mod I_\sigma$ and hence, additive. This makes $N : B \rightarrow A/(I_\sigma \cap A)$ additive as well. \[\Box\]

Remark 7.2. We don’t need any assumptions regarding defect or rank here.

7.1. Case I: Relation between the ideals $H, I, I_\sigma, J_\sigma$.

Notation 7.3. Case I is the defectless case, so best $f$ exists and we can define the ideal $I$ of $A$ by

$I := \left\{ \frac{a}{f} \in K \mid v_K(f + a^p - a) = v_K(f) \right\}$. It is worth noting that this definition coincides with the one in Lemma 2.9. Let $v_L(\alpha) = -v_0 \leq 0$. Hence, $v_L(f) = -pv_0$, $H = \{x \in K \mid v_L(x) \geq pv_0\}$, $I = \{x \in K \mid v_L(x) \geq (p-1)v_0\}$ and $J_\sigma = \{x \in L \mid v_L(x) \geq v_0\}$

Proposition 7.4. $H \subset I \subset I_\sigma \cap A$.

Proof. Comparing valuations mentioned above, it is clear that $H \subset I$.

We break down the rest of the argument into several cases:

- If $e_{L/K} = 1$, $I_\sigma = J_\sigma = \{x \in L \mid v_L(x) \geq v_0\}$ and the result follows.
- Let $e_{L/K} = p$.
  - (i) $p > 2$
    \[\frac{1}{a} \in B \Rightarrow \sigma \left( \frac{1}{a} \right) - \frac{1}{a} = \frac{1}{a(a+1)} \in I_\sigma.\]
    Hence, $\{x \in L \mid v_L(x) \geq 2v_0\} \subset I_\sigma$. Since $p > 2$, $p - 1 \geq 2$ and hence, $I \subset I_\sigma \cap A$. We cannot use this argument for $p = 2$, since in that case, $p - 1 = 1 < 2$. 

\[\text{16}\]
and the ratio \( \gamma_0/\gamma_0 = a_0 \in A \). Then we have the following commutative diagram:

\[
\begin{array}{ccc}
\Omega^1_{B|A}[a_0] & \xrightarrow{\sim} & A[a_0]/(c_0) \\
\downarrow \rho_{a_0} & & \downarrow \iota_a \\
\Omega^1_{B|A}[a'] & \xrightarrow{\sim} & A[a']/(c_a)
\end{array}
\]
Here, \( a_0 = \gamma_0/\gamma_0 = 1 \in A \) and the isomorphisms are given by \( b_0 da_0' \mapsto b_0 \mapsto \frac{b_0}{a_0} \); for all \( b_0 \in A[\alpha_0'] \) and \( bda' \mapsto b \mapsto \frac{b}{\alpha_0} \); for all \( b \in A[\alpha'] \). The vertical maps are described as follows. We look at the relationship between the generators \( \alpha_0' \), \( \alpha' \) and similarly, between \( da_0' \), \( da' \). Since \( \alpha \) and \( \alpha_0 \) give rise to the same extension \( L/K, \alpha_0 - \alpha =: h \in K \). Comparing the valuations, we see that \( v(\alpha_0) = v(h) < v(\alpha) \) and hence, \( u = h \gamma_0 \in A^\times \).

\[
(7.6) \quad \alpha_0' = (\alpha + h) \gamma_0 = (\alpha + h) \gamma_0 
\]

Since \( \alpha' \in B \) and \( \alpha, u \in A \), \( \alpha' da_\alpha = 0 = du \) in the differential module \( \Omega^1_{A[\alpha']/A} \). Therefore, we have

\[
(7.7) \quad da_0' = a_\alpha da_\alpha' + \alpha' da_\alpha + du = a_\alpha da_\alpha'
\]

Thus, \( \rho_\alpha', \tau_\alpha \) are given by multiplication by \( a_\alpha \). The map \( j_\alpha \) is also multiplication by \( a_\alpha \) and rises from the inclusions

\[
(7.8) \quad \frac{1}{a_0'} A[\alpha_0'] \subset \frac{1}{a_\alpha} A[\alpha'] \mapsto \frac{1}{a_\alpha} a_\alpha = \frac{1}{a_\alpha} a_\alpha^2
\]

and

\[
(7.9) \quad \frac{c_\alpha}{a_0} A[\alpha_0'] \subset \frac{c_\alpha}{a_\alpha} A[\alpha'] \mapsto \frac{c_\alpha}{a_\alpha} a_\alpha = \frac{c_\alpha}{a_\alpha} a_\alpha^p
\]

**Lemma 7.10.** Consider the fractional ideals \( \Theta \) and \( \Theta' \) of \( L \) given by \( \Theta = \{ x \in L \mid v(x) > v_0 - \mu \} \) and \( \Theta' = \{ x \in L \mid v(x) > pv_0 - \mu \} \). Then we have:

(a) \( \Omega^1_{B/L} \cong \Theta/\Theta' \)

(b) \( \Theta/\mathcal{J}_\sigma \Theta \cong \mathcal{J}_\sigma / \mathcal{J}_\sigma^2 \)

**Proof:** (a) Let \( I \) be the fractional ideal of \( L \) generated by the elements \( (1/a_\alpha) \). Let \( I' \) be the fractional ideal of \( L \) generated by the elements \( (\alpha_0/a_\alpha) \). Under the isomorphisms described in the preceding discussion, we can identify each \( \Omega^1_{A[\alpha'/A]} \) with \( (1/a_\alpha) A[\alpha']/(\alpha_0/a_\alpha) A[\alpha'] \). Taking limit over \( \alpha \)'s, we can identify \( \Omega^1_{B/L} \) with \( I/I' \).

Since \( -v_0(a_\alpha) = v_0(\gamma_0) = v(\gamma_0) - \mu \), \( I = \{ x \in L \mid v(x) > \inf v_\alpha(\gamma_0) - \mu \} = \Theta \). Similarly, \( v(c_\alpha) = (p - 1)v_\alpha(\gamma_0) \Rightarrow v_\alpha(c_\alpha) = v(\gamma_0) - \mu \Rightarrow I' = \Theta' \).

(b) This follows from the fact that \( \Theta \cong \mathcal{J}_\sigma \) as \( B \)-modules, via the map \( x \gamma_0 : x \mapsto x \gamma_0 \).

\[\square\]

**7.4. Proof of 0.5 in Case II.** Due to the defect, we consider \( \Omega^1_{B/L} \) and \( \Omega^1_A \) instead:

\[
\Omega^1_{B/L}/\mathcal{J}_\sigma \Omega^1_{B/L} \xrightarrow{\varphi_\sigma} \mathcal{J}_\sigma / \mathcal{J}_\sigma^2 \xrightarrow{\Delta_N} \Omega^1_A/(I_\sigma \cap A) \Omega^1_A \xleftarrow{\text{rs}} \bar{H}/H^2
\]

As discussed in **7.10**, we can write \( \Omega^1_{B/L} = \lim_{\alpha \in \mathcal{J}_0} \Omega^1_{A[\alpha'/A]} \) and it is enough to consider the diagram for each \( \alpha \in \mathcal{J}_0 \):

\[
\Omega^1_{A[\alpha'/A]}/(\frac{1}{\alpha}) A[\alpha'] \Omega^1_{A[\alpha'/A]} \xrightarrow{\varphi_\sigma} (\frac{1}{\alpha}) A[\alpha']/(\frac{1}{\alpha})^2 A[\alpha'] \xrightarrow{\Delta_N} \Omega^1_A/(I_\sigma \cap A) \Omega^1_A \xleftarrow{\text{rs}} (\frac{1}{\alpha}) A/(\frac{1}{\alpha})^2 A
\]

(7.11)
where the maps are given by

\[ \varphi_\alpha : (\alpha')_{\text{ideal}} \to (\alpha')_{\text{ideal}} \]

\[ \Delta_N(\alpha') \to N(\alpha') \]

We note that in \( \omega_{B|A}^1, d \log \alpha = d \log \alpha' + d \log \gamma = d \log \alpha' = \frac{d \alpha'}{\alpha'} \text{ and } \frac{\sigma(\alpha')}{\alpha'} - 1 = \frac{1}{\alpha} \).

At each \( \alpha \)-level, we observe the following:

(i) The map \( \varphi_\alpha : (\alpha')_{\text{ideal}} / (\frac{1}{\alpha}) \to (\frac{1}{\alpha})^2 \) is same as the one obtained from 7.10.

Proof. By 7.10, \( (\alpha')_{\text{ideal}} / (\frac{1}{\alpha}) \to (\frac{1}{\alpha})^2 \) under the composition \( d \alpha' \to \frac{1}{\alpha} \). On the other hand, \( \varphi_\alpha(d \alpha') = \alpha' \left( \frac{\sigma(\alpha')}{\alpha'} - 1 \right) = \frac{\alpha'}{\alpha} \). \( \square \)

(ii) The map \( rsw \) is well-defined.

Proof. Define the ideal \( I_{\alpha} \) of \( A \) by \( I_{\alpha} := \left\{ \frac{a}{\alpha} \in K \mid v_K(f_\alpha + a^p - a) = v_K(f_\alpha) \right\} \). As in case (I), we have \( (\frac{1}{\alpha})A \subset I_{\alpha} \subset \sigma(\alpha') \cap A \). Since \( (\frac{1}{\alpha})A \cap A \subset J_\alpha \cap A = I_\alpha \cap A \), the map \( rsw \) is well-defined. \( \square \)

8. The Different Ideal \( D_{B|A} \)

8.1. Basic Properties. We recall that \( D_{B|A}^{-1} := \{ x \in L \mid \text{Trace}_{L|K}(xB) \subset A \} = B^* \) and the different ideal \( D_{B|A} \) is defined to be its inverse ideal.

Lemma 8.1. (Lemma 6.76 of [KKS]) If \( \mu \in B \), \( L = K(\mu) \) has the minimal polynomial \( F(T) \in K[T] \) over \( K \), then \( A[\mu]^* = \frac{1}{F'(\mu)}A[\mu] \)

We consider the following three sub-cases:

(i) \( e_{L|K} = 1, f_{L|K} = p \)

Let \( v \) denote both \( v_L \) and \( v_K \). Assume that \( L|K \) is generated by \( \alpha^p - \alpha = f \) where \( f \) is best. There exists \( \gamma \in A \) such that \( \alpha' := \alpha \gamma \in B^x \) and \( l|k \) is purely inseparable, generated by the residue class of \( \alpha' \). Let \( v(\alpha) = -v_0 \). Hence, \( v(f) = -pv_0, v(\gamma) = v_0 \). Since \( f \gamma^p \in A^x, F(T) = T^p - T \gamma^p \) is the minimal polynomial of \( \mu \) over \( A \). Therefore, \( F'(T) = pT^{p-1} - (p-1)\gamma^{p-1} = \gamma^{p-1} \). By 1.11, we see that \( B = A[\alpha'] \). Hence, \( D_{B|A}^{-1} = B^* = A[\alpha']^* = \frac{1}{F'(\alpha')}A[\alpha'] \) is clearly a fractional ideal of \( L \), generated by a single element \( \frac{1}{F'(\alpha')} \).

(ii) \( e_{L|K} = p, f_{L|K} = 1 \)

Let \( f \) be best, \( v_L(\alpha) = -v_0 \). Recall that \( B = \sum_{i=0}^{p-1} A_i \alpha^i \) where \( A_0 := A \) and \( A_i := \{ x \in K \mid v(x) \geq iv_0 \} = \{ x \in A \mid v(x) > iv_0 \} \) for all \( 1 \leq i \leq p-1 \). Let \( y \in L \). Then for all \( 0 \leq i \leq p-1 \),

\[ y = \sum_{j=0}^{p-1} y_j \alpha^j \in D_{B|A}; y_j \in K \iff \text{Trace}_{L|K}(y \alpha^i A_i) \subset A \]

\( \alpha \) has the minimal polynomial \( F(T) = T^p - T - f \). Hence, \( F'(\alpha) = -1 \). Also, for \( 1 \leq i \leq p-1, \alpha^{i+p-1} = \alpha^i + f \alpha^{i-1} \). By 1.14, we have
Proof. Lemma 8.3.

Let $x_i \in A_i$. Then

\[
\text{Trace}_{L|K}(\alpha^i) = \begin{cases} 
0; & 0 \leq i \leq p - 2 \\
-1; & i = p - 1, 2(p - 1) \\
0; & p \leq i \leq 2(p - 1) - 1
\end{cases}
\]

Hence, $y \in D_{B|A}^{-1} \iff A_0y_{p-1}, A_{p-1}(y_0 + y_{p-1}), A_iy_{p-1-i} \subset A$ (for all $1 \leq i \leq p - 2$).

(iii) Rank 1 and $e_{L|K} = 1$, $f_{L|K} = 1$

Let $\Gamma \subset \mathbb{R}$ and let $v$ denote both $v_L, v_K$. By 5.1, we can write $B = \bigcup_{\alpha \in \mathcal{S}} A[\alpha']$; $\alpha' = \alpha \gamma \in B^\times$, $\gamma \in A$. Recall that $v_0 := \inf_{\alpha \in \mathcal{S}} v(\gamma \alpha) \in \mathbb{R} \setminus \Gamma$. By an argument similar to (i) above, we have

\[
D_{A[\alpha']|A}^{-1} = \{ x \in L \mid v(x) \geq (p - 1)v(\alpha) = -(p - 1)v(\gamma \alpha) \},
\]

Since all the $A[\alpha']$'s and $B$ have the same fraction field $L$, $D_{B|A}^{-1} \subset D_{A[\alpha']|A}$ for all $\alpha \in \mathcal{S}$. Hence,

\[
\gamma^{-p-1}_{\alpha}D_{B|A}^{-1} \subset \gamma^{-p-1}_{\alpha}D_{A[\alpha']|A} \subset A[\alpha'] \subset B \iff D_{B|A}^{-1}
\]

is a fractional ideal of $L$ described by

\[
D_{B|A}^{-1} = \bigcap_{\alpha \in \mathcal{S}} D_{A[\alpha']|A}^{-1}
\]

\[
= \bigcap_{\alpha \in \mathcal{S}} \{ x \in L \mid v(x) \geq (p - 1)v(\alpha) \}
\]

\[
= \{ x \in L \mid v(x) \geq (p - 1)v(\alpha) \forall \alpha \in \mathcal{S} \}
\]

\[
= \{ x \in L \mid v(x) \geq -(p - 1)v_0 \}
\]

8.2. Results in the case $e_{L|K} = 1$. Let $L|K$ satisfy (I) or (II) and assume further that $e_{L|K} = 1$.

Lemma 8.3. \{ $x \in L \mid \text{Trace}_{L|K}(xB) \subset H$ \} = $\mathcal{J}_\sigma$.

Proof. Since $e_{L|K} = 1$, given any $x \in L$, there exists $x' \in B^\times, a \in K$ such that $x = x'a$. In particular, $B = B^\times A$. Hence,

\[
\text{Trace}(xB) \subset H \iff \text{Trace}(xB^\times A) \subset H
\]

\[
\iff \text{Trace}(xB^\times) \subset H
\]

\[
\iff a\text{Trace}(a'b^\times) \subset H
\]

\[
\iff a\text{Trace}(B^\times) \subset H
\]

(i) We note that $\text{Trace}(a) = \frac{1}{\alpha}$. Hence, $\mathcal{J}_\sigma = \{ \frac{1}{\alpha} \} B \subset \{ x \in L \mid \text{Trace}(xB) \subset H \}$. Conversely, suppose that $\text{Trace}(xB) \subset H = \{ \frac{1}{\gamma} \} A$. In particular, $a\text{Trace}(\frac{1}{\alpha}) = a\text{Trace}(\frac{1}{\alpha}) = \frac{\alpha}{\gamma} \text{Trace}(\frac{1}{\alpha}) = \frac{\alpha}{\gamma} \left( \frac{1}{\gamma} \right) \in H$. Hence, $\frac{\alpha}{\gamma} \in A \Rightarrow \alpha \gamma \in B \Rightarrow x \in \mathcal{J}_\sigma$.

(iii) The argument is very similar to (i). Again, $\mathcal{J}_\sigma \subset \{ x \in L \mid \text{Trace}(xB) \subset H \}$. Conversely, suppose that $\text{Trace}(xB) \subset H$.

Hence, for all $\alpha \in \mathcal{S}$,

\[
\frac{\alpha}{\gamma} \left( \frac{1}{\alpha} \right) \in H
\]

\[
\Rightarrow v(\alpha) - v(\gamma) - v(f_\alpha) > pv_0
\]

\[
\Rightarrow v(\alpha) > (p - 1)(v_0 - v(\gamma)) + v_0.
\]

Since this is true for all $\alpha \in \mathcal{S}$, we have $v(\alpha) \geq v_0$.

But $v_0 \notin \Gamma \Rightarrow v(\alpha) > v_0 \Rightarrow x \in \mathcal{J}_\sigma$.

\[\Box\]
Corollary 8.4. In particular, if \( L \mid K \) satisfies (II) and \( e_{L \mid K} = 1 \), then \( \{ x \in L \mid Tr(xJ_\sigma) \subset H \} = B \).

Proof. By the preceding lemma, \( \{ x \in L \mid Tr(xJ_\sigma) \subset H \} = \{ x \in L \mid xJ_\sigma \subset J_\sigma \} \) and hence, clearly contains \( B \). The reverse inclusion is due to the rank 1 assumption.

Proposition 8.5. In the cases (i) and (iii), \( \mathcal{D}^{-1}_{B \mid A} \) is described by:

(i) \( \mathcal{D}^{-1}_{B \mid A} = J_\sigma^{-1} \) and

(iii) \( \mathcal{D}^{-1}_{B \mid A} = \{ x \in L \mid x BH \subset J_\sigma \} \).

Proof. (i) \( v(F'(\mu)) = (p - 1)v(\gamma) = (p - 1)v_0 \Rightarrow \mathcal{D}^{-1}_{B \mid A} = \{ x \in L \mid v(x) \geq -(p - 1)v_0 \} \). The rest follows from \( J_\sigma = I_\sigma = (\frac{1}{\alpha}) B \).

(iii) We use the lemma above.

\[ Tr(xB) \subset A \iff Tr(xB^\times)A \subset A \]
\[ \iff Tr(xB^\times)H \subset H \]
\[ \iff Tr(xBH) \subset H \]
\[ \iff xBH \subset J_\sigma \]

8.3. Results in the case \( e_{L \mid K} = p \).

Lemma 8.6. Let \( S \) be a fractional ideal of \( L \). Then for \( y = \sum_{j=0}^{p-1} y_j \alpha^j \); \( y_j \in K \), \( y \in S \iff y_i \alpha^i \in S \) for all 0 \( \leq i \leq p - 1 \).

Proof. Since \( e_{L \mid K} = p \), \( v_L(y_i \alpha^i) \) are all distinct and \( v_L(y) = \min_{0 \leq i \leq p-1} v_L(y_i \alpha^i) = v_L(y_j \alpha^j) \), for some \( 0 \leq j \leq p - 1 \). \( y \in S \iff v_L(y) = v_L(y_j \alpha^j) \geq v_L(s) \) for some \( s \in S \). Thus, \( y \in S \Rightarrow v_L(y_i \alpha^i) \geq v_L(s) \) for all \( 0 \leq i \leq p - 1 \Rightarrow y_i \alpha^i \in S \) for all \( 0 \leq i \leq p - 1 \). The converse is clearly true.

Applications:

- \( \left[ \mathcal{D}^{-1}_{B \mid A} \right] \); \( y \in \mathcal{D}^{-1}_{B \mid A} \iff Tr(y_i \alpha^i b) \in A \) for all \( b \in B \) for all \( 0 \leq i \leq p - 1 \).

  Hence, \( \mathcal{D}^{-1}_{B \mid A} = \cup_{0 \leq i \leq p-1} D_i \), where \( D_i := \{ y \alpha^i \mid y \in K, y \alpha^i \in \mathcal{D}^{-1}_{B \mid A} \} \).

Fix some \( i \), let \( y \in K \). Write \( b = \sum_{j=0}^{p-1} x_j \alpha^j ; x_j \in A_j \). \( Tr(y \alpha^i b) \in A \iff \sum_{j=0}^{p-1} y x_j Tr(\alpha^{i+j}) \in A \).

Thus, if \( i = p - 1 \), then \( y \alpha^p \in \mathcal{D}^{-1}_{B \mid A} \iff v_L(y) + v_L(x_0 + x_{p-1}) \geq 0 \) for all \( x_0 \in A \), for all \( x_{p-1} \in A_{p-1} \).

\( \iff v_L(y) \geq 0 \) and hence, \( D_{p-1} B = A \alpha^{p-1} B = \alpha^{p-1} B = J_\sigma^{-(p-1)} \).

If \( 0 \leq i \leq p - 2 \), \( y \alpha^i \in \mathcal{D}^{-1}_{B \mid A} \iff v_L(y) + v_L(x_{p-1-i}) \geq 0 \) for all \( x_{p-1-i} \in A_{p-1-i} \).

\( y \alpha^i, x_{p-1-i} \alpha^{p-1-i} \in \alpha^{p-1} B \).

- \( \left[ J_\sigma \right] \); \( J_\sigma \) is generated by \( \{(\sigma - 1)(x_i \alpha^i) \mid x_i \in A_i, 1 \leq i \leq p - 1 \} \). For a fixed \( i \), \( (\sigma - 1)(A_i \alpha^i)B = A_i \alpha^i [(1 + \frac{1}{\alpha})^i - 1]B = A_i \alpha^{i \frac{1}{\alpha}} B = A_i \alpha^i J_\sigma \). Thus, \( J_\sigma = \cup_{1 \leq i \leq p-1} A_i \alpha^i B \).

Definition 8.7. We consider the \( B \)-sub-module \( \Omega^1_{B \mid A} \) of \( \Omega^1_{B \mid A} \) generated by the set \( \{ db \mid b \in m_L \} \) of generators (and the relations described for \( \Omega^1_{B \mid A} \)).
**Lemma 8.8.** \( \Omega^1_{B|A} \cong \Omega^1_{B|A} \) as \( B \)-modules.

**Proof.** \( \Omega^1_{B|A} \rightarrow \Omega^1_{B|A} \) is the map \( db \mapsto db \). Consider the map \( \pi : \Omega^1_{B|A} \rightarrow \Omega^1_{B|A} \) described below.

For \( b \in B \), there exists \( x \in A \) such that \( b - x \in \mathfrak{m}_L \). We define \( \pi(db) = d(b - x) \). Note that this definition is independent of the choice of \( x \). It is enough to show that \( \pi \) preserves the relations.

Let \( b, c \in B, x, y \in A \) such that \( b - x, c - y \in \mathfrak{m}_K \).

Additivity is preserved, since \( \pi(d(b + c)) = d(b + c - x - y) = d(b - x) + d(c - y) = \pi(db) + \pi(dc) \).

Since \( dx = 0, dy = 0 \) and \( bc - xy = c(b - x) + x(c - y) \in \mathfrak{m}_L \),

\[
\begin{align*}
\pi(d(bc)) &= \pi(dbc - dx) = \pi(dbc) - \pi(dx) = \pi(dbc) - \pi(d) \pi(c) = \pi(dbc) - \pi(dc) \\
&= \pi(dbc) - \pi(dc).
\end{align*}
\]

Hence, \( \pi(d(bc)) = c\pi(db) + b\pi(dc) \). \( \square \)

---

**9. SOME RESULTS IN CASE (II)**

**Notation 9.1.** Let \( L|K \) satisfy (II). Assume further that \( e_{L|K} = p \) and the value group \( \Gamma_K \) of \( K \) (as an ordered subgroup of \( \mathbb{R} \)) is not isomorphic to \( \mathbb{Z} \). Thus, \( L|K \) is a defectless, rank 1 Artin-Schreier extension such that \( \Gamma_K \) is a dense ordered subgroup of \( \mathbb{R} \).

**Lemma 9.2.** Under the assumptions of this section,

(a) For \( 1 \leq i \leq p - 1 \), \( A_i B = \mathcal{J}_i^\dagger \mathfrak{m}_L \).
(b) \( \mathcal{I}_\sigma = \mathcal{J}_\sigma \mathfrak{m}_L \).

**Proof.** (a) For \( 1 \leq i \leq p - 1 \),

\[
A_i B = \{ x \in K \mid v_L(x) > iv_0 \} B = \{ x \in L \mid v_L(x) > iv_0 \}. \quad \text{Hence,} \quad A_i B = \frac{1}{x^i} \mathfrak{m}_L = \mathcal{J}_i^\dagger \mathfrak{m}_L.
\]

(b) By (a), for \( 1 \leq i \leq p - 1 \), \( A_i \alpha^i B = \frac{1}{x^i} \alpha^i \mathfrak{m}_L = \mathfrak{m}_L \). Hence,

\[
\mathcal{I}_\sigma = \bigcup_{1 \leq i \leq p - 1} A_i \alpha^i B \mathcal{J}_\sigma = \mathcal{J}_\sigma \mathfrak{m}_L.
\]

**Remark 9.3.** In the general case when \( e_{L|K} = p \), \( 1 \leq i \leq p - 1 \), we have \( A_i B \subset \mathcal{J}_i^\dagger \mathfrak{m}_L \) and \( \mathcal{I}_\sigma \subset \mathcal{J}_\sigma \mathfrak{m}_L \).

**Proposition 9.4.** Under the assumptions of this section,

(a) \( \mathcal{D}_{B|A}^{-1} = \mathcal{J}_\sigma^{-p-1} \)
(b) \( \Omega^1_{B|A} \cong \omega^1_{B|A} \otimes_B \mathfrak{m}_L \cong \mathcal{I}_\sigma \mathcal{I}_\sigma^{-1} \mathfrak{m}_L \)

**Proof.** (a) We recall that \( \mathcal{D}_{p-1} = \mathcal{J}_\sigma^{-p-1} \) and hence, \( \mathcal{J}_\sigma^{-p-1} \subset \mathcal{D}_{B|A}^{-1} \).

If \( 0 \leq i \leq p - 2 \),

\[
\begin{align*}
\text{\( y_0 \alpha^i \in \mathcal{D}_{B|A}^{-1} \) & \iff v_L(y_0 \alpha^i) \geq 0 \quad \forall \ x_{p-1-i} \in A_{p-1-i} \\
& \iff v_L(y_0) + (p - 1 - i)v_0 \geq 0 \quad (\text{since } \Gamma_K \text{ is dense}) \\
& \iff v_L(y_0^{\alpha^i}) \geq -(p - 1)v_0 \iff v_0 \in \mathcal{J}_\sigma^{-p-1}.
\end{align*}
\]

Hence, \( \mathcal{D}_{B|A}^{-1} \subset \mathcal{J}_\sigma^{-p-1} \Rightarrow \mathcal{D}_{B|A}^{-1} = \mathcal{J}_\sigma^{-p-1} \).

(b) For convenience, let \( \Omega := \Omega^1_{B|A}; \Omega' := \Omega^1_{B|A} \). Consider the following maps:

\[
\xi : \Omega' \rightarrow \omega^1_{B|A} \otimes_B \mathfrak{m}_L ; \quad \xi(db) = d \log b \otimes b
\]
Thus, \( \psi \): \( \omega_{B|A}^1 \otimes_B m_L \to \Omega \);
\[
\psi(d \log b \otimes c) = \frac{c}{ab} d(ab)
\]
where \( b \in L^x, c \in m_L, a \in K^x \); \( 0 \leq v_L(ab) \leq v_L(c) \). Such an \( a \) exists since \( \Gamma_K \) is dense in \( \mathbb{R} \).
* Let \( 0 \neq b, c \in m_L \), \( 0 < v_L(c) \leq v_L(b) \). We can write \( b = ch; h \in B \).
\[
d \log(b + c) \otimes (b + c) = d \log c(1 + h) \otimes c(1 + h)
\]
\[
= (1 + h)d \log c \otimes c + (1 + h)d \log(1 + h) \otimes c
\]
\[
= d \log c \otimes c + hd \log c \otimes c + hd \log h \otimes c
\]
\[
= d \log c \otimes c + hd \log ch \otimes c
\]
\[
= d \log c \otimes c + d \log ch \otimes ch
\]
\[
= d \log c \otimes c + d \log b \otimes b
\]
* Let \( 0 \neq b, c \in m_L \)
\[
d \log(bc) \otimes (bc) = d \log b \otimes bc + d \log c \otimes bc
\]
\[
= cd \log b \otimes b + bd \log c \otimes c
\]
Thus, \( \xi \) is well-defined.
* Let \( b \in L^x, c \in m_L, a, a' \in K^x \) such that \( 0 \leq v_L(ab) \), \( v_L(a'b') \leq v_L(c) \). Since \( da = 0 = da' \),
\[
\frac{c}{ab} d(ab) = \frac{c}{ab} (adb + bda) = \frac{c}{a} db = \frac{c}{a} d(a'b').
\]
Thus, \( \psi \) is independent of choice of \( a \).
* Let \( 0 \neq b \in B, c \in m_L, a \in K^x \) as described in the definition of \( \psi \). Since \( da = 0 \), we have
\[
\psi(db \otimes c) = b \frac{c}{ab} d(ab) = b \frac{c}{a} (adb + bda) = cdb.
\]
Hence, \( \psi \) preserves additivity and Leibniz rule.
* Let \( b, b' \in L, c, c' \in m_L, a, a' \in K^x \) such that \( 0 \leq v_L(ab) \leq v_L(c) \) and \( 0 \leq v_L(a'b') \leq v_L(c') \).
Furthermore, since \( \Gamma_K \) is dense in \( \mathbb{R} \), we can choose \( a, a' \) such that \( 0 \leq v_L(ab') \leq v_L(c) \).
\[
\frac{c}{aa'bb'} d(aa'b'b') = \frac{c}{aa'bb'} [a'b'd(ab) + abd(a'b')] = \frac{c}{ab} d(ab) + \frac{c}{a'b'} d(a'b')
\]
Thus, \( \psi \) is well-defined.
Next, we consider the maps \( \xi \circ \pi \circ \psi : \omega_{B|A}^1 \otimes_B m_L \to \omega_{B|A}^1 \otimes_B m_L \) and \( \psi \circ \xi \circ \pi : \Omega \to \Omega \).
* Let \( b \in L^x, c \in m_L, a \in K^x, x \in A \) such that \( 0 \leq v_L(ab) \leq v_L(c) \) and \( ab - x \in m_L \).
\[
\xi \circ \pi \circ \psi(d \log b \otimes c) = \frac{c}{ab} d \log(ab - x) \otimes (ab - x)
\]
\[
= \frac{ab - x}{ab} d \log(ab - x) \otimes c
\]
\[
= \frac{ab}{ab} d \log(ab) \otimes c
\]
\[
= d \log a \otimes c + d \log b \otimes c = d \log b \otimes c
\]
* Let \( 0 \neq b \in B, x \in A \) such that \( b - x \in m_L \).
\[
\psi \circ \xi \circ \pi(db) = \psi(d \log(b - x) \otimes (b - x))
\]
\[
= \left( \frac{b - x}{b - x} \right) d(b - x)
\]
\[
= d(b - x) = db
\]
This proves the first isomorphism.
Next, we prove that
\( \omega_{B|A}^1 \cong B/J_p^{p-1} \)

\( \omega_{B|A}^1 \) is generated by \( d \log \alpha = -d \log \left( \frac{1}{\alpha} \right) \). In \( \omega_{B|A}^1 \), we have

\[
0 = - \left( 1 - \frac{1}{\alpha^{p-1}} \right) d \log \left( \frac{1}{f} \right) = \left( 1 - \frac{1}{\alpha^{p-1}} \right) d \log f
\]

\[
= \left( 1 - \frac{1}{\alpha^{p-1}} \right) d \log(\alpha^p) + \left( 1 - \frac{1}{\alpha^{p-1}} \right) d \log \left( 1 - \frac{1}{\alpha^{p-1}} \right)
\]

\[
= d \left( 1 - \frac{1}{\alpha^{p-1}} \right) = d \left( \frac{1}{\alpha^{p-1}} \right)
\]

\[
= -(p-1) \left( \frac{1}{\alpha^{p-1}} \right) d \log \left( \frac{1}{\alpha} \right)
\]

Therefore, \( J_p^{p-1} = \left( \frac{1}{\alpha^{p-1}} \right) \) annihilates \( \omega_{B|A}^1 \).

Conversely, let \( 0 \neq b \in B \) such that \( b \omega_{B|A}^1 = 0 \). Hence, for all \( 1 \leq i \leq p - 1, x_i \in A_i, bd(x_i \alpha^i) = 0 \)

\( \Rightarrow b \in \cap_{i,x_i} G'_{i,x_i}(x_i \alpha^i)B \), where \( G_{i,x_i} \) is the minimal polynomial of \( x_i \alpha^i \) over \( K \). Let \( G := G_{i,x_i} \) for fixed \((i,x_i)\). Then

\[
G'(x_i \alpha^i) = \prod_{1 \leq j \leq p-1} x_i \alpha^i \left( 1 - \left( \frac{\alpha + j}{\alpha} \right)^i \right)
\]

\[
= (x_i \alpha^i)^{p-1} \prod_{1 \leq j \leq p-1} \left( 1 - \left( \frac{\alpha + j}{\alpha} \right)^i \right)
\]

\[
= (x_i \alpha^i)^{p-1} \prod_{1 \leq j \leq p-1} \left( 1 - \left( \frac{\alpha + j}{\alpha} \right)^i \right) u; \ u \in B^x
\]

\[
= (x_i \alpha^i)^{p-1} \left( \frac{-1}{\alpha} \right)^{p-1} (p-1)! u
\]

Thus, \( b \in \cap_{i,x_i} G'_{i,x_i}(x_i \alpha^i)B = \cap_{i,x_i} (x_i \alpha^i)^{p-1} J_p^{p-1} \Rightarrow b \in J_p^{p-1} \)

By Equation (9.7) and 9.2

\[
\omega_{B|A}^1 \otimes m_L \cong B/J_p^{p-1} \otimes m_L \cong m_L/J_p^{p-1} m_L \cong \mathcal{J}_\sigma m_L/\mathcal{J}_p^\sigma m_L = \mathcal{I}_\sigma/\mathcal{I}_p^\sigma.
\]

\[\square\]
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