Mean Field Behavior during the Big Bang for Coalescing Random Walk

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Coalescing Random Walk (CRW) on a graph $G$:

- Initially one walker at each vertex of graph $G$.
- Each walker performs independent continuous time random walk. Jump rate equals 1 along each edge.
- Whenever two walkers meet (collide), they merge into one walker. This walker continues to perform random walk.
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Can be defined for general Markov chain with jump rate $\{r_{x,y}\}$.

Common choices

- $r_{x,y} = 1[x \sim y]$ for general graph
- $r_{x,y} = 1[x \sim y]/d(x)$ for regular graph
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Motivation: duality with the voter model.
An example

Black = occupied, Green = vacant

Diagram of a triangle with three nodes, two of which are connected by a line.
An example

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Black = occupied, Green = vacant
CRW on the complete graph

$G$ is a complete graph (clique). $r_{x,y} = 1/(n - 1)$. 
$L_t$: # of walkers at time $t$. 
$L_0 = n$. $L_t \to L_t - 1$ at rate $L_t(L_t - 1)/(n - 1)$. 
$	au_{\text{coal}}$: (random) coalescence time (only one walker left) 
$	au_{\text{coal}} = \sum_{i=1}^{n} e_i (i - 1)/n$. 
$e_i, i \geq 1$ are i.i.d. with dist. $\text{Exp}(1)$. 
$e_i (i - 1)/n$ is the time it takes for the $n - i + 1$-th coalescence to occur (corresponding to $L_t$ from $i$ to $i - 1$). 
Related model: Kingman's coalescent. 
$L_0 = \infty$. $L_t \to L_t - 1$ at rate $L_t(L_t - 1)/2$. 
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Decay of density on the complete graph

Define the expected density (expected fraction of occupied sites)

\[ P_t = \frac{\mathbb{E}(L_t)}{n} \]

Determine \( L_t \): the time it takes to make \( h \) coalescences

\[
\sum_{i=n-h+1}^{n} \frac{e_i}{i(i-1)/(n-1)} \sim n \left( \frac{1}{n-h} - \frac{1}{n} \right)
\]

for \( 1 \ll h \ll n \). Set this expression to be \( t \), we get

\[ L_t = n - h \sim \frac{n}{t+1}. \]

Thus

\[ P_t \sim \frac{1}{t+1}. \]
Spatial structure

Often there is a spatial structure.

- $\mathbb{Z}^d$.
- $T^d$.
- General vertex transitive graphs.
- Random graphs (e.g., configuration model).
Heuristic argument [van den Berg-Kesten, 2000]

Consider $\mathbb{Z}^d$. $P_t = P_t(o)$: prob. that origin is occupied at time $t$. Take $1 \ll \Delta(t) \ll t$.

$$- \frac{dP_t}{dt} = P(o \text{ and } e_1 \text{ occupied at } t)$$

$$\sim \sum_{x,y} P(x \text{ and } y \text{ occupied at } t - \Delta(t)) \times$$

$$P(x + S_{\Delta(t)} = o, y + S'_{\Delta(t)} = e_1, x + S_r \neq y + S'_r, \forall r \leq \Delta(t))$$

$$\sim P_{t-\Delta(t)}^2 \alpha \Delta(t).$$

- $x$ and $y$ are the location of the walkers that later come to $o$ and $e_1$. $S, S'$: independent random walks starting from $o$.
- $\alpha \Delta(t)$: the probability that two time-reversed random walk starting from $o$ and $e_1$ don’t collide by time $\Delta(t)$. 
Results on $\mathbb{Z}^d$

Assuming $P_t \sim P_{t-\Delta(t)}$ and $\alpha_t \sim \alpha_{t-\Delta(t)}$. The heuristic suggests that $P_t \approx 1/(t\alpha_t)$ for moderately large $t$. This was known to be true for SRW on $\mathbb{Z}^d$, $d \geq 2$.

Theorem (Bramson-Griffeath, 1980)

Consider the CRW on $\mathbb{Z}^d$. We have, as $t \to \infty$,

$$P_t \sim \begin{cases} \frac{1}{\pi} \frac{\log t}{t} & d = 2 \\ (\gamma_d t)^{-1} & d \geq 3 \end{cases}$$

where $\gamma_d$ is the probability that a simple random walk in $\mathbb{Z}^d$ starting from origin never returns to it.
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where $\gamma_d$ is the probability that a simple random walk in $\mathbb{Z}^d$ starting from origin never returns to it.

By justifying previous heuristic argument, [van der Berg-Kesten, 2000] proved the same result for $d \geq 3$. 

Approximation for coalescence time

\( \pi \): stationary distribution.
Mean meeting time (the time it take for two indep. walkers to meet)

\[ t_{\text{meet}} = \mathbb{E}_{\pi^2}(\tau_{\text{meet}}). \]

For complete graph \( t_{\text{meet}} = (n - 1)/2. \)
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Aldous and Fill conjectured that for finite transitive graph (transitivity means the graph looks the same from every vertex)

$$\frac{\tau_{\text{coal}}}{t_{\text{meet}}} \sim \sum_{i=2}^{\infty} \frac{e_i}{i(i - 1)/2}.$$

Equality holds for complete graph (replacing $\infty$ by $n$). $e_i \sim \text{Exp}(1)$. 
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The factor \( i(i - 1)/2 \) counts the number of pairs

- Why exponential?
Aldous-Brown approximation

Lemma (Aldous-Brown, 1992)

For an irreducible reversible Markov chain on a finite state set $V$ with stationary distribution $\pi$ and $A \subset V$, if we denote the hitting time of $A$ by $T_A$ and its density function w.r.t. the stationary chain by $f_{T_A}$, then

$$\left| \mathbb{P}_\pi (T_A > t) - \exp \left( - \frac{t}{\mathbb{E}_\pi (T_A)} \right) \right| \leq \frac{t_{\text{rel}}}{\mathbb{E}_\pi (T_A)},$$

and

$$\frac{1}{\mathbb{E}_\pi (T_A)} \left( 1 - \frac{2t_{\text{rel}} + t}{\mathbb{E}_\pi (T_A)} \right) \leq f_{T_A}(t) \leq \frac{1}{\mathbb{E}_\pi (T_A)} \left( 1 + \frac{t_{\text{rel}}}{2t} \right).$$

Consider the product chain and take $A$ to be the diagonal set. We have $E_\pi (T_A) = t_{\text{meet}}$. 
Second Prediction

[Oliveira, 2013] proved the Aldous-Fill conjecture under the condition $t_{\text{mix}} \ll t_{\text{meet}}$ (equivalent to $t_{\text{rel}} \ll t_{\text{meet}}$ due to Hermon). $t_{\text{mix}}$ and $t_{\text{rel}}$ quantify the rate of convergence to stationary distribution (See Markov Chains and Mixing Times).
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The time it takes to make $h$ coalescences is about

$$t_{\text{meet}} \sum_{i \geq n-h+1} \frac{e_i}{i(i-1)/2} \sim \frac{2t_{\text{meet}}}{n-h}.$$ 

$$\frac{2t_{\text{meet}}}{n-h} = t \Rightarrow n - h = \frac{2t_{\text{meet}}}{t}.$$ 

Hence we have another prediction

$$P_t = \frac{E(L_t)}{n} = \frac{n - h}{n} \sim \frac{2t_{\text{meet}}}{nt}.$$
Equivalence of the two predictions

Two predictions for $P_t$

$P_t \sim \frac{1}{t \alpha_t}$

where $\alpha_t = r(o) \mathbb{P}_{o, \nu_o}(\tau_{\text{meet}} > t)$ ($\nu_o$ is a random neighbor of $o$)

$P_t \sim \frac{2t_{\text{meet}}}{nt}$ for finite graphs

They are equivalent to each other for many graphs by Kac’s formula (in continuous time) and Aldous-Brown approximation:

$\frac{1}{t_{\text{meet}}} \sim f_{T_A}(t) = \frac{2}{n} \mathbb{P}_{o, \nu_o}(\tau_{\text{meet}} > t)$ for $r(o) = 1$. 
Main Results: finite graphs

Theorem (Hermon-Li-Yao-Zhang, 2021)

Two predictions holds as long as $1 \ll t \ll t_{\text{coal}}$ (called the Big Bang regime since the number of particles is evolving rapidly in this regime) for

- transitive graphs $G_n$ such that $\text{diam}(G_n)^2 \ll n/\log n$,
- Configuration Model $\text{CM}(n, D)$ with $3 \leq D < M$.

If $D$ is a constant $d$ then $\text{CM}(n, D)$ is random $d$-regular graph.
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Remarks:

- For such graphs $t_{\text{coal}}$ and $t_{\text{meet}}$ both have order $n$.
- By [Tessera and Tointon, 2019], $\text{diam}(G_n)^2 \ll n/\log n$ implies

$$\lim_{s \to \infty} \lim_{n \to \infty} \sup_{x,y} \int_{s \land t_{\text{rel}}}^{t_{\text{rel}}} p_s(x, y) \, ds = 0.$$
Configuration model

Construction of the configuration model $\mathcal{CM}_n(D)$

- Let $D$ be a probability measure on $\mathbb{Z}_+$, and $n \in \mathbb{Z}_+$.
- We take $n$ vertices labeled $1, \ldots, n$, and $d_1, \ldots, d_n$ i.i.d. sampled from $D$.
- For each vertex $i$ we attach $d_i$ half edges to it. Then we get $G_n$ by uniformly matching all half edges, conditioned on $\sum_{i=1}^n d_i$ being even.
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The local weak limit $\mathcal{UGT}(D)$ of $\mathcal{CM}_n(D)$ is a unimodular Galton-Watson tree where

- the root has offspring distribution $D$
- later generations have offspring distribution $D^*$:

$$
P(D^* = k) := \frac{(k + 1)P(D = k + 1)}{\sum_{i=0}^{\infty} iP(D = i)}$$
Main Results: infinite Graphs

Theorem (Hermon-Li-Yao-Zhang, 2021)

The prediction $P_t(o) \sim 1/(t^\alpha)$ as $t \to \infty$ where

$$\alpha = \mathbb{E}(r(o)P_{o,\nu_o}(\tau_{meet} = \infty))$$

holds for

- all transient transitive unimodular graphs, including
  - Cayley graphs
  - amenable graphs (= graphs with subexponential decay of return probability)
- unimodular Galton-Watson tree $UGT(D)$. If $D$ is a constant $d$ then $UGT(D) = \mathbb{T}^d$. 
Duality with voter model

Voter model: at rate $r_{x,y}$, $x$ adopts the opinion of $y$. 
A site is occupied in CRW $\iff$ the opinion is not lost in VM.

Figure: Left panel: CRW; right panel: voter model
Proof Sketch of [Bramson-Griffeath, 1980]

\( n_t \): \# walkers that end up at origin at time \( t \).

\( \eta_t \): the voter model starting from different opinions at every site.

\( \hat{N}_t \) := \{ \( x \): \( \eta_t(x) = \eta_t(o) \} \). [Kelly, 1977] gives

\[
\mathbb{P}(\hat{N}_t = j) = j\mathbb{P}(n_t = j), \quad j \geq 0, \quad (\text{i.e., size-biased version of } n_t)
\]

\[
P_t = \mathbb{P}(n_t > 0) = \mathbb{E}(\hat{N}_t^{-1}) = \mathbb{E} \left[ \left( \frac{\hat{N}_t}{\mathbb{E}(\hat{N}_t)} \right)^{-1} \right] \mathbb{E}(\hat{N}_t)^{-1}.
\]

\( \mathbb{E}(\hat{N}_t) \) is equal to \( \mathbb{E}(R_{2t}) \) where \( R \) is the range of a random walk.
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P_t = P(n_t > 0) = \mathbb{E}(\hat{N}_t^{-1}) = \mathbb{E} \left[ \left( \frac{\hat{N}_t}{\mathbb{E}(\hat{N}_t)} \right)^{-1} \right] \mathbb{E}(\hat{N}_t)^{-1}.
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**Theorem (Sawyer, 1979)**

Consider CRW on \( \mathbb{Z}^d, d \geq 2 \).

\[
\lim_{t \to \infty} \mathbb{E} \left[ \left( \frac{\hat{N}_t}{\mathbb{E}(\hat{N}_t)} \right)^k \right] = \frac{(k + 1)!}{2^k}.
\]
A remark from [Bramson-Griffeath,1980]: Sawyer’s theorem comes tantalizingly close to determining the asymptotics of $P_t$. Gap: the function $f(x) = x^{-1}$ is unbounded near $x = 0$. 
Proof Sketch of [Bramson-Griffeath, 1980]-cont’d

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Gap: the function $f(x) = x^{-1}$ is unbounded near $x = 0$.

Theorem (Bramson-Griffeath, 1980)

$$P_t = \begin{cases} O \left( \frac{\log t}{t} \right) & d = 2, \\ O \left( \frac{1}{t} \right) & d \geq 3. \end{cases}$$

Lemma (Bramson-Griffeath, 1980)

Sawyer’s Theorem + upper bound on $P_t$ gives the asymptotics of $P_t$.

Basically, the upper bound on $P_t$ implies the $\hat{N}_t/E(\hat{N}_t)$ doesn’t have too much mass near 0.
Transform to coalescence probability

Let $N_t$ be the number of walkers that collide with the walker starting at $U$. $N_0 = 1$.

$$N_t = \sum_x 1[\text{the particle starting at } x \text{ coalesced with } U \text{ by time } t]$$

$$P_t = \mathbb{E}(N_t^{-1}) = [\mathbb{E}(N_t)]^{-1} \mathbb{E} \left[ \left( \frac{N_t}{\mathbb{E}(N_t)} \right)^{-1} \right].$$

(A graph rooted at a uniform vertex is unimodular.)
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(A graph rooted at a uniform vertex is unimodular.)

$C$: coalescence time for $k+1$ walkers.

$$\mathbb{E}(N_t^k) = \frac{1}{n} \sum_{x_1, \ldots, x_{k+1} \in V} \mathbb{E} \left[ 1[X_i(0) = x_i, \forall 1 \leq i \leq k + 1] \right]$$

$$\times 1[C(X_1, \ldots, X_{k+1}) \leq t]$$

$$= n^k \mathbb{P}_{\pi \otimes k+1}(C(X_1, \ldots, X_{k+1}) \leq t),$$
Ingredients of the proof

Using the machinery in the proof of $\mathbb{Z}^d$ case by Braomson-Griffeath, it suffices to

- give an upper bound of $P_t$ that differs from the ‘true value’ of $P_t$ by a multiplicative constant,
- show that the coalescence probability

$$P_{\pi^{k+1}}(\mathcal{C}(X_1, \ldots, X_{k+1}) \leq t) \sim (k+1)! \left(\frac{t}{t_{\text{meet}}}\right)^k.$$  

Another indication of mean field! B-G proof heavily relies on the specific geometric structure of $\mathbb{Z}^d$. 
Solution

• For the first part, we show that for any $t > 0$,

$$
\inf_{x \in G} \frac{\int_0^t p_s(x, x) ds}{t} \leq P_t \leq \sup_{x \in G} \frac{\int_0^t p_s(x, x) ds}{t}.
$$

where $c$ and $C$ are universal constants.

• For the second part, we use the reversibility of random walk to transform collision probability to non-colliding probability. If two forward paths collide at $t$ then (after reversing time) the backward paths don’t collide in $[0, t]$. 
We want to estimate $P_{\pi \otimes k+1}(C(X_1, \ldots, X_{k+1}) \leq t)$.

Consider the case $k = 1$. The probability that $X_1$ and $X_2$ collide within time interval $[t, t + dt]$ is about

$$2 \sum_{u, v} P(X_1(t) = u, X_2 \text{ jumps from } u \text{ to } v \text{ in } [t, t + dt])$$

$$\sim 2 \sum_{u, v} P(X_1(t) = u, X_2(t) = v, \text{ no collisions in } [0, t]) r_{v, u} dt$$

$$\sim 2 \sum_u P(\gamma_1(0) = u) r(u) \times$$

$$\sum_v \frac{r_{u, v}}{r(u)} P(\gamma_2(0) = v) P_{u, v}(\gamma_1(s) \neq \gamma_2(s), \forall 0 \leq s \leq t) dt,$$

where $\gamma_1$ and $\gamma_2$ are the time-reversals of $X_1, X_2$ on $[0, t]$. 
Collision Pattern and Branching Structure

We can imagine $\gamma_1$ is the parent of $\gamma_2$ and interpret the term $r_{u,v}/r(u)$ as the probability of the particle at $u$ giving birth to a particle at location $v$. Can be generalized to $k \geq 3$ paths.
If two walkers don't collide in time $O(t_{\text{rel}})$, then they will also not collide before time $t$.

**Lemma**

For any $x \neq y$ and $0 < s < t$, the probability that two walkers starting from $x$ and $y$ collide between time $s$ and $t$ is bounded by

$$2 \exp\left(-\frac{s}{t_{\text{rel}}}\right) \frac{\max_z \int_0^{2s} p_s(z, z) \, ds}{\min_z \int_0^{2s} p_s(z, z) \, ds} + \frac{8t}{n} \left(s^{-1} \lor r_{\text{max}}\right).$$

$r_{\text{max}} = \max_x r(x)$. The error is small for $t_{\text{rel}} \ll s \leq t \ll n$. 
Open Question

For finite graphs our results (the expectation of the number of occupied sites) can be upgraded to a weak law of large numbers using negative correlation

\[ P(\text{both } x \text{ and } y \text{ occupied at } t) \leq P(x \text{ occupied at } t)P(y \text{ occupied at } t). \]

What about fluctuations? Do we have a Gaussian limit as in the mean field case ([Aldous, 1999])?
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Thanks!