Stable configurations in social networks

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Abstract

We present and analyze a model of opinion formation on an arbitrary network whose dynamics comes from a global energy function. We study the global and local minimizers of this energy, which we call stable opinion configurations, and describe the global minimizers under certain assumptions on the friendship graph. We show a surprising result that the number of stable configurations is not necessarily monotone in the strength of connection in the social network, i.e. the model sometimes supports more stable configurations when the interpersonal connections are made stronger.

Keywords: social network, balanced graph, graph Laplacian, Poincare–Miranda theorem, complex potential, bifurcation, opinion formation

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(Some figures may appear in colour only in the online journal)

1. Introduction

1.1. Social network models

Over the last couple of decades, there has been a large degree of interest in models of general dynamical systems defined on networks [Str01, NBW11, NG04, JGN01] and in particular models of social or biological dynamics on networks [WS98, NG04, CFL09]. Of course a wide variety of models and potential applications exist, but one simple context is the study of opinion formation on a network. The classical voter model [HL75, HL78, DL94, Lig99] and its generalizations [SAR08] were one of the first models considered for opinion formation, but a variety of more complicated models exist [SWS00, KLB09, YAO+28, Alt12, KSL+10, GS13, AJGA14, DPLM14, DGM14, BRG16].
In this paper, we consider a relatively simple dynamical model of social opinion formation whose dynamics are given by a single global potential function. In the model, there are \( n \) agents, each of which holds an opinion represented by a scalar quantity. In the absence of any interaction with the other individuals, each agent will relax to one of two opinions, each of which is the negative of each other—more specifically, each individual relaxes in a symmetric double well potential. We allow the agents to interact through a coupling which can be either attractive or repulsive. Thus we allow for both ‘friends’ and ‘enemies’ in this network, with the idea that one’s opinions move towards those of one’s friends, and away from those of one’s enemies.

In the model, all of the dynamics can be represented as a gradient flow in a potential; therefore, while the model might have multiple stable configurations, we can compare them energetically and determine which is the ‘most stable’ configuration, i.e. the one which globally minimizes the potential. In this sense, the model is quite reminiscent of both [ACTA17] and [BFG07a, BFG07b]; in fact the latter two papers are a study of the model with only friendly interactions of unit strength in certain graph topologies and a particular double well potentials.

The main results of this paper are twofold. In section 2 we describe the global minimizers of the energy functional whenever the graph topology is ‘balanced’—a condition on the graph summarized in two clauses ‘the enemy of my enemy is my friend’ and ‘the friend of my friend is my friend’. In section 3 we show that the dependence of the global system on the strength of the coupling can be quite complicated, and we show that increasing the coupling strength can both increase and decrease the number of minima.

1.2. Description of model

**Definition 1.1.** Let \( G \) be an undirected weighted graph with vertex set \( \{1, \ldots, n\} \) and edge set a subset of \( \{(i,j) : i \neq j\} \). To each edge \( \{i,j\} \) we associate the weight \( \gamma_{ij} \) so that \( \gamma_{ij} = \gamma_{ji} \). We let \( \mathcal{G}_n \) denote the set of such graphs.

**Definition 1.2.** Given a graph \( G \in \mathcal{G}_n \), a function \( W : \mathbb{R} \to \mathbb{R} \), and a parameter \( \kappa \in [0, \infty) \) we define the energy \( E_{W,G,\kappa} \) by

\[
E_{W,G,\kappa}(x) := \sum_{i=1}^{n} W(x_i) + \kappa \sum_{i,j=1}^{n} \gamma_{ij}(x_i - x_j)^2.
\]

The model we consider is the gradient flow for this energy, namely

\[
\frac{dx_i}{dt} = -\frac{\partial}{\partial x_i} E_{\kappa}(x) = \kappa (\mathcal{L}(G)x)_i - W'(x_i),
\]

where \( \mathcal{L}(G) \) is the graph Laplacian whose components are given by

\[
(\mathcal{L}(G))_{ij} = \begin{cases} 
\gamma_{ij}, & i \neq j, \\
-\sum_{k \neq i} \gamma_{ik}, & i = j.
\end{cases}
\]

Since we consider the gradient flow, the attracting fixed points of (1.2) are exactly the local minima of (1.1). These minima are the main object of study in the current paper. Here \( x_i \) represents the value of the opinion of the \( i \)-th person. A function \( W \) corresponds to the dynamics of an individual’s opinion when uninfluenced by others, which we choose to be the same for each member. Moreover, \( \mathcal{L}(G) \) represents how each member is influenced by the entire
network with $\gamma_{ij}$ as the strength of the interaction between individuals $i$ and $j$. By setting $\kappa = 0$, we can turn off the interaction, and we see that each individual independently moves to a local minimum of the function $W$. Setting $\kappa$ large makes the interactions between individuals dominate. We stress here that we do not assume $\gamma_{ij} \geq 0$, which would lead to only a friendly attracting force between individuals—we allow $\gamma_{ij} < 0$, so that individuals who are enemies will have repelling opinions. We refer to $W$ as the ‘individual potential’ and $\gamma_{ij}$ as the ‘interaction strengths’.

Now we define the set $\mathcal{W}$ of allowable potentials:

**Definition 1.3.** Let $\mathcal{W}$ be the set of twice continuously differentiable even functions $W: \mathbb{R} \to \mathbb{R}$ whose only critical points are $\pm m$ and zero which are local minima and a local maximum respectively. We further require that

$$
\lim_{x \to \pm\infty} \frac{W(x)}{x^2} = \infty
$$

so that the member’s opinions in equation (1.2) cannot diverge to infinity as $t \to \infty$.

One can easily check that $W(x) = \frac{1}{4}(1-x^2)^2$ is in the class $\mathcal{W}$ with $m = 1$, and we will refer to this in some cases as the ‘classical potential’.

It is not hard to see that if we choose $\kappa = 0$, then there are $2^n$ minima of the energy given in equation (1.1) at the points $(\pm m, \pm m, \ldots, \pm m)$; in this case these are all also global minima. As we increase $\kappa$, we can expect several things to occur: for some range of $\kappa$ near zero, minima might move but will persist, but of course only some of them will remain global minima. We also expect that minima can disappear under bifurcations (and in fact it is shown in [BFG07a] that all bifurcations cause minima to disappear under certain conditions.

**Example 1.4.** Consider the graph on three vertices with edge weights 1, 1, and $-2$, where $W(x) = \frac{1}{4}(1-x^2)^2$, so that our entire energy function is

$$
E_n(x) = \frac{1}{4}(1-x_1^2)^2 + \frac{1}{4}(1-x_2^2)^2 + \frac{1}{4}(1-x_3^2)^2 + \frac{\kappa}{2}(x_1-x_2)^2 + \frac{\kappa}{2}(x_1-x_3)^2 - \kappa(x_2-x_3)^2.
$$

In figures 1 and 2 we give two plots showing how the minima evolve as $\kappa$ increases from zero to infinity. We now discuss the nature of the bifurcations in this example. First consider the first bifurcation at $x_0 = (0,0,0)$ and $\kappa_0 = 1/6$ and set $y = x - x_0$ and $\mu = \kappa - \kappa_0$. We can rewrite our system as

$$
\frac{dy}{dr} = \frac{1}{3} \begin{pmatrix} 4 & 1 & -2 \\ 1 & 1 & 1 \\ -2 & 1 & 4 \end{pmatrix} y + 2\mu \begin{pmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{pmatrix} y - y^3.
$$

The eigenvalues of the left most matrix are 0, 1, and 2. Let $V$ be the matrix whose rows are the corresponding eigenvectors and make the change of variables $w = V y$. Then our system becomes

$$
\frac{dw}{dr} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} w + 6\mu \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} w - V(V^T w)^3.
$$

From basic bifurcation theory we know that we have a center manifold which is locally represented by two functions $w_2(w_1, \mu)$ and $w_3(w_1, \mu)$ which vanish along with their first
partial derivatives at \((w_1, \mu) = (0, 0)\). Plugging these functions into the system yields the following
\[
\begin{align*}
w_2(w_1, \mu) &= O(\text{cubic terms}) \quad \text{and} \quad w_3(w_1, \mu) = O(\text{cubic terms}) \\
\text{hence} \quad \frac{d}{dt} w_1 &= -6 \mu w_1 - \frac{1}{2} w_3^2 + O(\text{sextic terms}).
\end{align*}
\]

This last equation however is locally topologically equivalent to
\[
\frac{dw_1}{dt} = -\mu w_1 - w_3^2
\]

which represents a pitchfork bifurcation, as can be seen from its phase portrait in figure 3. See [Wig03].

Note that this equation only governs the dynamics on the center manifold and that the system is unstable in the directions corresponding to the other variables \(w_2\) and \(w_3\) since their
corresponding eigenvalues are positive. Therefore when $\mu < 0$, namely, $\kappa < \kappa_0 = 1/6$ we have three fixed points two of which are 2-saddles and one of which is a maximum $w_1 = 0$ which corresponds to $x = (0, 0, 0)$. When $\mu > 0$, namely, $\kappa > \kappa_0 = 1/6$ we have only one fixed point, $x = (0, 0, 0)$, which is a 2-saddle.

A similar calculation can be used to understand the two other bifurcations. The Jacobian of the bifurcation at $x = (0, -\varphi, \varphi)$ and $\kappa = \varphi/6$ has eigenvalues $0$, $-3 - \sqrt{5}$, and $-3 - 2\sqrt{5}$. Since we have two negative eigenvalues this implies that we have two stable directions and the interesting dynamics occurs on a two dimensional center manifold. It is easy to see from figure 2 that this is a pitchfork bifurcation and therefore conclude that this bifurcation consists of two minima colliding with a 1-saddle to form a single minimum. This pitchfork bifurcation can be shown using the same methods as used above, but we do not give the computation since it is conceptually the same but much messier numerically. In the same way the Jacobian for the bifurcation at $x = (1, 1, 1)$ and $\kappa = 1/3$ has eigenvalues $0$, $-2$, and $-4$. Thus there are two stable directions. One can show again by a normal form argument that this a pitchfork bifurcation and conclude that it consists of a minima and two 1-saddles colliding to form a single 1-saddle.

2. Balanced graphs and global minima

Definition 2.1. A graph $G$ is said to be balanced if every cycle contains an even number of negative edges.

One way of describing this type of graph colloquially is with the two phrases ‘the enemy of my enemy is my friend’ and ‘the friend of my friend is my friend’. For example, if the graph contains a triangle, then by the definition above, the three vertices in this triangle must all be
friends, or exactly two of the pairs must be enemies. A major result of Cartwright and Harary [CH56, CH68] is that all balanced graphs have a type of signed bipartite structure, namely:

**Theorem 2.2 (Cartwright–Harary).** A graph G is balanced if and only if its vertex set V can be decomposed into two mutually exclusive subsets V_1 and V_2 (referred to as cliques) such that γ_ij ≥ 0 if i and j belong to the same subset and γ_ij ≤ 0 if i and j belong to different subsets.

**Theorem 2.3.** Suppose that G is a balanced graph, κ > 0, and x is a global minimum of the energy E_w. Then x_i ≠ 0 for all i and x_i and x_j have the same sign if and only if i and j belong to the same clique.

**Proof.** Notice that E_κ(0) = nW(0) > nW(±m) = E_κ(±m1), so that 0 is never a global minimum. Thus fix x ≠ 0, and without loss of generality assume x_1 > 0 and 1 ∈ V_1. Define ˜x by

\[ ˜x_i = \begin{cases} |x_i| & \text{if } i ∈ V_1, \\ -|x_i| & \text{if } i ∈ V_2. \end{cases} \]

We first show that γ_0 ˜x_i ˜x_j ≥ 0 for all i and j. If i and j belong to the same clique, then by theorem 2.2, γ_0 ≥ 0 and ˜x_i and ˜x_j have the same sign. However, if i and j belong to different cliques, then γ_0 ≤ 0 and ˜x_i and ˜x_j have different signs. Since W is even, the transformation x → ˜x does not change the W terms in (1.1) and can only make the quadratic terms more negative, so E_κ(˜x) ≤ E_κ(x).

Now let us suppose that x is a global minimum and x_i = 0 for some i. Reusing the argument above gives us sign-definiteness of γ_0 ˜x_i ˜x_j: if i ∈ V_1, then γ_0 ˜x_j ≥ 0 for all j, and if i ∈ V_2, then γ_0 ˜x_j ≤ 0 for all j. We compute

\[ \sum_{j \neq i} γ_0 ˜x_j = \frac{1}{2κ} (W'( ˜x_i) − ( ∇ E_κ( ˜x_i) ) ) = 0, \]

but since all γ_0 ˜x_i have the same sign, this implies γ_0 ˜x_j = 0 for all j. Therefore for every j either γ_0 = 0 or x_j = ˜x_j = 0. This implies that x_j = 0 for every j that is a neighbor of i in the graph. Proceeding by induction, this means that x_j = 0 for any j path-connected to i. Since we assume that G is connected, this implies that x_j = 0 for all j, but we showed above that 0 is not a global minimum. This is a contradiction, and thus we conclude that x_i ≠ 0 for all i whenever x is a global minimum.

Finally we show that if x ≠ ˜x for any nonzero x, then E_κ( ˜x) < E_κ(x). Let us first consider the quantity γ_0( ˜x_i ˜x_j − x_i x_j). Note that this is either exactly zero, or equal to 2γ_0 ˜x_i ˜x_j > 0. (Moreover, this is positive whenever exactly one of the x_i, x_j changes parity when x → ˜x.) We then note that

\[ E_κ(x) − E_κ( ˜x) = \frac{κ}{2} \sum_{i,j=1}^n γ_0 ( ˜x_i ˜x_j − x_i x_j), \]

and since each term is nonnegative the sum is positive as long as at least one term is positive. Choose i so that x_i = ˜x_i. Since G is connected, there is a path from 1 to i, i.e. there is a sequence of vertices n_1, n_2, ..., n_k such that n_1 = 1, n_k = i, and γ_{n_k n_{k+1}} ≠ 0 for ℓ = 1, ..., k − 1. Since x_1 = ˜x_1 and x_i ≠ ˜x_i, there exists an ℓ such that x_{n_ℓ} = ˜x_{n_ℓ} and x_{n_{ℓ+1}} ≠ ˜x_{n_{ℓ+1}}, which implies

\[ γ_{n_{ℓ} n_{ℓ+1}} ( ˜x_{n_ℓ} ˜x_{n_{ℓ+1}} − x_{n_ℓ} x_{n_{ℓ+1}} ) = 2γ_{n_{ℓ} n_{ℓ+1}} ˜x_{n_ℓ} ˜x_{n_{ℓ+1}} > 0, \]

and therefore E_κ( ˜x) < E_κ(x). ❑
3. Non-monotone potentials

3.1. Overview

It was shown in [BFG07a] that in the case where the underlying graph is a ring, and all of the connections are friendly, that increasing the interaction can only decrease the number of minima, i.e. the number of minima of (1.1) is a nonincreasing function of \( \kappa \). In this section, we study pairs \((W, G)\) that do not have this monotonicity property. We will show that such pairs exist, and, in fact, we can construct pairs that have arbitrarily more minima for some positive \( \kappa \) than for \( \kappa = 0 \).

**Definition 3.1.** For any pair \((W, G)\) \( \in \mathcal{W} \times G_n \) define \( E_{W,G,\kappa} \) as above, and let \( m_{W,G}(\kappa) \) be its number of local minima. We say that the pair \((W, G)\) is monotone (resp. non-monotone) if the function \( m_{W,G}(\kappa) \) is monotone non-increasing (resp. ever increases) as a function of \( \kappa \).

3.2. Main results

We have several results that we prove in this section. The main result, theorem 3.7, shows that we can construct a potential \( W \) such that the pair \((W, G)\) is always nonmonotone, and, in fact, we can get a uniform bound on how many new minima are created as we increase the coupling. This theorem has a few corollaries that allow us to bound the size of the coupling in the graph \( G \) that would lead to nonmonotonocities.

**Definition 3.2.** Define the functional \( \|\cdot\| : \mathbb{R}^{N \times N} \to \mathbb{R} \) by

\[
\|M\| = \max_{1 \leq i \leq n} \sum_{j \neq i} |M_{ij}|
\]

**Remark 3.3.** For symmetric zero-row-sum matrices this functional has the following properties:

1. \( \|M\| \geq 0 \) with equality if and only if \( M = 0 \)
2. \( \|\alpha M\| = |\alpha| \|M\| \)
3. \( \|M_1 + M_2\| \leq \|M_1\| + \|M_2\| \)

Thus \( \|\cdot\| \) is a norm on the space of all zero-row-sum matrices (but is only a seminorm on matrices). In fact it is comparable to the norm \( \|\cdot\|_{1 \to 1} \). Note that every Laplacian matrix, \( L(G) \), is a symmetric zero-row-sum matrix.

**Lemma 3.4.** If \( \lambda \) is an eigenvalue of \( L(G) \), then \( |\lambda| \leq 2 \|L(G)\| \).

**Proof.** Gershgorin’s theorem states that

\[ \lambda \in \bigcup_{i=1}^{N} \left[ \gamma_i - \sum_{j \neq i} |\gamma_j|, \gamma_i + \sum_{j \neq i} |\gamma_j| \right], \]

and since

\[ \bigcup_{i=1}^{N} \left[ \gamma_i - \sum_{j \neq i} |\gamma_j|, \gamma_i + \sum_{j \neq i} |\gamma_j| \right] \subseteq \left[ -2 \max_{1 \leq i \leq n} \sum_{j \neq i} |\gamma_j|, 2 \max_{1 \leq i \leq n} \sum_{j \neq i} |\gamma_j| \right], \]

we have that \( |\lambda| \leq 2 \|L(G)\| \).

\[ \blacksquare \]
Lemma 3.5. Fix $0 < \ell < m < r$ and $M > 0$, and let $W \in \mathcal{W}$ be such that $W'' \geq M$ on $[\ell, r]$. Further let $G \in \mathcal{G}_n$ and choose $\kappa$ so that
\[
\kappa \|\mathcal{L}(G)\| = \frac{M \min\{r - m, m - \ell\}}{2r}.
\]
Then there exists continuous functions $f_p : [0, \kappa] \to (-r, -\ell) \cup [\ell, r]^n$ for $p \in \{-m, m\}^n$ satisfying the following conditions:
1. $f_p(0) = p$.
2. $\nabla E_\kappa(f_p(\kappa'), \kappa') = 0$ for all $\kappa' \in [0, \kappa]$.
3. $\nabla^2 E_\kappa(f_p(\kappa'), \kappa')$ is positive definite for all $\kappa' \in [0, \kappa]$.

Remark 3.6. The lemma gives a lower bound on $\kappa$ so that the minima of the energy $E_\kappa$ evolving according to the implicit function theorem starting from the various minima of $E_0$ remain in the region $([-r, -\ell] \cup [\ell, r]^n)$.

Proof. By lemma 3.4, all of the eigenvalues of $\mathcal{L}(G)$ are in the range $[-2 \|\mathcal{L}(G)\|, 2 \|\mathcal{L}(G)\|]$, and by assumption, $W'' \geq M$. From this it follows that $\nabla^2 E_\kappa(x, \kappa)$ is positive definite on $([-r, -\ell] \cup [\ell, r]^n)$ if $0 \leq \kappa < \frac{M}{2 \|\mathcal{L}(G)\|}$. Fix $p \in \{-m, m\}^n$. Then the implicit function theorem gives us a function $f_p : [0, \kappa] \to U_p$ for some $\kappa$, where $U_p$ is some deleted neighborhood of $p \in \mathbb{R}^n$.

Now since for any $i$
\[
\kappa_p := \frac{-W''(f_p(\kappa_i))}{(\mathcal{L}(G)f_p(\kappa_i))} = \frac{|W'(f_p(\kappa_i))|}{|\langle (\mathcal{L}(G)f_p(\kappa_i)) \rangle|} \geq \frac{\min\{|W'(\ell)|, |W'(r)|\}}{2r \sum_{j \neq i} \gamma_j} \geq \frac{M \min\{r - m, m - \ell\}}{2r \|\mathcal{L}(G)\|} > 0
\]
we can extend our implicit function to $f_p : [0, \kappa_p] \to (-r, -\ell) \cup [\ell, r]^n$. Now since $p$ was arbitrary in $\{-m, m\}^n$, we obtain the desired result.

Theorem 3.7. Let $d \geq 1$. Then there exists a potential $W \in \mathcal{W}$ such that for any $n \geq 2$ and $G \in \mathcal{G}_n$, there exists a $\kappa > 0$ such that the difference of the number of minima of $E_\kappa$ and $E_0$ is at least $d$. In particular, $(W, G)$ is non-monotone for every $G$.

Proof. First we consider the cases of $d \leq 2$. Fix $0 < \ell' < \ell < m < r$ and $M > 0$. Choose a $W \in \mathcal{W}$ satisfying $W'' \geq M$ on $[\ell', r'] \cup [\ell, r]$ and
\[
W'(\ell') < -\frac{M(r + r') \min\{r - m, m - \ell\}}{2r}, \quad W'(r') > -\frac{M(r - r) \min\{r - m, m - \ell\}}{2r}
\]
(3.1)
\[
W'(\ell) < -M \min\{r - m, m - \ell\}, \quad W'(r) > M \min\{r - m, m - \ell\}.
\]
(3.2)

Fix $G \in \mathcal{G}_n$ and choose $\kappa$ according to lemma 3.5. We will show that $E_\kappa(x)$ has a non-zero fixed point $x_0 \notin (-r, -\ell) \cup [\ell, r]^n$ and therefore conclude that $E_\kappa(x)$ has at least $2^n + 2$ fixed points, namely, each $f_p(\kappa)$ and $\pm x_0$.

To find $x_0$, choose a vertex $i$ such that $\|\mathcal{L}(G)\| = \sum_{j \neq i} \gamma_j$ and let $j$ denote a generic vertex not equal to $i$. Define $R$ to be the rectangular region consisting of all $x \in \mathbb{R}^n$ such that $x_i \in [\ell', r']$, $x_j \in [-r, -\ell]$ if $\gamma_j > 0$, and $x_j \in [\ell, r]$ if $\gamma_j < 0$. We will find our $x_0$ in $R$. To do this we first note that
\[ M(\ell - r') \min \{ r - m, m - \ell \} \leq \kappa(\mathcal{L}(G)x)_i \leq M(r + r') \min \{ r - m, m - \ell \}, \]

\[ -M \min \{ r - m, m - \ell \} \leq \kappa(\mathcal{L}(G)x)_j \leq M \min \{ r - m, m - \ell \}, \]

for all \( x \in R \) and therefore conclude that

\[ \nabla E_\kappa(x) \bigg|_{x_i = \ell'} < 0 < \nabla E_\kappa(x) \bigg|_{x_i = r'} \quad \text{and} \quad \nabla E_\kappa(x) \bigg|_{x_j = \ell} < 0 < \nabla E_\kappa(x) \bigg|_{x_j = r}, \]

for all \( x \) in the indicated faces of \( R \). Therefore by the Poincare–Miranda theorem [Maw13] there exists a critical point \( x_0 \) in \( R \). Finally since we have the same lower bound on \( W'' \) as in lemma 3.5 we see that \( \nabla^2 E_\kappa(x) \) is positive definite for all \( x \in R \) and therefore conclude that our \( x_0 \) is in fact a local minimum.

In general choose \( s \) so that \( 2s \geq d \) and choose numbers \( 0 < \ell'_1 < r'_1 < \ldots < \ell'_{s-1} < r'_{s-1} < \ell < m < r \) and impose the same restrictions to \( W' \) and \( W'' \) as above for each pair \( \ell'_t \) and \( r'_t \) for \( t \in \{1, \ldots, s - 1\} \). This results in \( 2s \geq d \) fixed points all of which are distinct since the \( i \)th component lies in the interval \( \pm (\ell'_t, r'_t) \) which is disjoint from the others.

**Remark 3.8.** The portion of the potential \( W \) with \( x \in (-r', \ell') \cup (\ell, r) \) forms something like a “shelf” on which a new fixed point can sit. In our proof we show that such a shelf is guaranteed to produce at least two new fixed points. However it appears that a shelf tends to produce substantially more new fixed points as demonstrated by the table in figure 4.

### 3.3. Examples

**Example 3.9.** In this example we show that we may actually achieve more minima than predicted by our theorem. Let \( \kappa = 1 \), \( \epsilon = 0.01 \), and \( W \) be the classical \( W \) potential on the set

| \( \kappa \) | \( m_{W,G}(\kappa) \) |
|---|---|
| 0 | 8 |
| 0.02 | 23 |
| 0.04 | 32 |
| 0.06 | 32 |
| 0.08 | 25 |
| 0.1 | 21 |
| 0.12 | 18 |
| 0.14 | 18 |
| 0.16 | 16 |
| 0.18 | 16 |
| 0.2 | 16 |

**Figure 4.** A plot of the potential \( W \) defined in equation (3.3) for example 3.10 and a table of the function \( m_{W,G} \) for the same \( W \) and graph on three vertices with unit edge weights. As mentioned in remark 3.8 the increase in the number of minima can be substantially larger than the number of shelves, namely, four. Note that the function \( m_{W,G} \) appears to have a local maximum near \( \kappa = 0.05 \).
(-\infty, -1) \cup (-1/4 + 2\epsilon, 1/4 - 2\epsilon) \cup (1, \infty), \text{ decreasing and smooth on } (1/4 - 2\epsilon, 1), \text{ increasing and smooth on } (-1, 1/4 + 2\epsilon), \text{ and even. Additionally, suppose that } W'(x) = \begin{cases} 
 & \text{ if } x = -1/4, -1/2, \text{ and } -3/4, 
 & 10 \text{ if } x = -1/4 + \epsilon, -1/2 + \epsilon, -3/4 + \epsilon, \text{ and } -1 + \epsilon, \n & -10 \text{ if } x = 1/4 - \epsilon, 1/2 - \epsilon, 3/4 - \epsilon, \text{ and } 1 - \epsilon, 
 & -\epsilon \text{ if } x = 1/4, 1/2, \text{ and } 3/4. 
\end{cases}

Now consider the graph with two nodes and edge of weight 1. Let L be the corresponding graph Laplacian. Then the gradient of the energy is
\[ \nabla E_n(x) = \begin{pmatrix} W'(x_1) + x_1 - x_2 \\
W'(x_2) + x_1 - x_2 \end{pmatrix}. \]

Let \( f_1(x_1, x_2) = W'(x_1) + x_1 - x_2 \) and \( f_2(x_1, x_2) = W'(x_2) - x_1 + x_2 \). Consider the sets \( A_n = [-n/4, -n/4 + \epsilon] \) and \( B_n = [n/4 + \epsilon, n/4] \) for \( n = 1, 2, 3, 4 \). There are 16 Cartesian products of the form \( A_i \times B_j \). Consider \( (x_1, x_2) \in A_i \times B_j \) for some \( 1 \leq i, j \leq 4 \). Then certainly \( x_1 - x_2 \in [-2, -2/4 + 2\epsilon] \) and \( -x_1 + x_2 \in [2/4 - 2\epsilon, 2] \). So we have that
\[ f_1(x_1, x_2) = \begin{cases} 
W'(x_1) + x_1 - x_2 \leq -2/4 + 3\epsilon < 0 & \text{ for } x_1 = -1/4, -1/2, -3/4, \text{ and } -1, 
W'(x_1) + x_1 - x_2 \geq 10 - 2 > 0 & \text{ for } x_1 = -1/4 + \epsilon, -1/2 + \epsilon, -3/4 + \epsilon, \text{ and } -1 + \epsilon, 
\end{cases} \]
and
\[ f_2(x_1, x_2) = \begin{cases} 
W'(x_2) - x_1 + x_2 \geq 2/4 - 3\epsilon > 0 & \text{ for } x_2 = 1/4, 1/2, 3/4, \text{ and } 1, 
W'(x_2) - x_1 + x_2 \leq -10 + 2 < 0 & \text{ for } x_2 = 1/4 - \epsilon, 1/2 - \epsilon, 3/4 - \epsilon, \text{ and } -1 + \epsilon, 
\end{cases} \]

Applying the Poincaré–Miranda theorem we get 16 local extrema, one for each of the 16 sets, of the form \( x_1 < 0 < x_2 \). By symmetry there are another 16 of the form \( x_2 < 0 < x_1 \). It is left to the reader that these extrema happen in regions with a positive Hessian. There are an additional 2 local minima at \((1, 1)\) and \((-1, -1)\), giving a total number of 34 minima at \( \kappa = 1 \).

**Example 3.10.** We construct an explicit example of a potential whose existence is guaranteed by theorem 3.7. We show how to combine multiple shelves to guarantee at least an increase of four minima. For simplicity choose \( 0 < \ell'_1 = \epsilon < \ell'_2 = 1 - \epsilon < \ell'_3 = 1 + \epsilon < \ell'_4 = 2 < \ell = 3 < m = 4 < r = 5 \) where \( \epsilon > 0 \) is chosen sufficiently small. We then choose our potential to satisfy
\[ W(x) = \begin{cases} 
S(x) & \text{if } \ell'_1 \leq x \leq \ell'_2, 
S(1) + S(x - 1) & \text{if } \ell'_2 \leq x \leq \ell'_3, 
2S(1) + T(x - 4) & \text{if } x \geq \ell, 
\end{cases} \]
where \( S(x) = \frac{1}{2}x^2 - \frac{1}{4}x^{1/2} x \) and \( T(x) = x^2 - 4 \). An example of such a potential is given in figure 4 where we have chosen \( \epsilon = 0.1 \). We have filled the gaps in the definition of \( W \) using interpolating polynomials so that the resulting potential is twice continuously differentiable. With this definition we have that our inequalities \( W''(x) \geq M \) for \( x \in [\ell'_1, \ell'_2] \cup [\ell'_2, \ell'_3] \cup [\ell, r] \) and
\[ W'(\ell'_1) < -\frac{M(r + r'_1) \min\{r - m, m - \ell\}}{2r}, \quad W'(r'_4) > -\frac{M(\ell - r'_1) \min\{r - m, m - \ell\}}{2r}, \]

\[ (3.4) \]
\[ W'(\ell_2') < -\frac{M(r + r'_2)}{2r} \min\{r - m, m - \ell\}, \quad W'(r_2') > -\frac{M(\ell - r_2')}{2r} \min\{r - m, m - \ell\}, \]  
(3.5)

\[ W'(\ell) < -M \min\{r - m, m - \ell\}, \quad W'(r) > M \min\{r - m, m - \ell\}, \]  
(3.6)

are all satisfied with \( M = 1 \). By construction the intervals \((-r'_2, -\ell_2'), (-r'_1, \ell'_1), (\ell'_1, r'_1)\), and \((\ell'_2, r'_2)\) each contribute at least one new fixed point at \( \kappa = \frac{1}{10\|\mathcal{L}(G)\|} \) which results in an increase of four fixed points as desired. We give a plot of our potential in figure 4. Furthermore let us choose \( G \) to be the complete graph on three vertices with unit edge weights. Then \( \|\mathcal{L}(G)\| = 1 \) and we are guaranteed to have four new fixed points at \( \kappa = \frac{1}{m} \). We give a table of the number of minima of \( E_\kappa \) for our potential and graph at certain values of \( \kappa \) in figure 4. We see that we obtain over twenty more new minima. Also we notice that \( m_{G,K}(\kappa) \) appears to peak before we reach the value of \( \kappa \) used in the proof of theorem 3.7. We obtained numerical estimates for the number of minima at each value of \( \kappa \) by using a Monte Carlo method evolving each point under our gradient flow until we determine that we are sufficiently close to a minimum. The values in the table are therefore lower bounds for the actual values of \( m_{G,K}(\kappa) \).

**Example 3.11.** The main drivers in the increase of minima in the proofs are the inflection points in the potential which gave rise to ‘shelves’ that would separate out the different points. In this example, we show that these shelves are useful but not necessary for non-monotonicity. Let \( \kappa = 0.1 \). Let \( W \) be a smooth, even function with only two inflection points so that \( W(x) = (|x| - 1)^4 \) on \((-\infty, -1/2] \cup [1/2, \infty)\). Now consider the graph with two nodes and edge of weight 1. Then the gradient of the energy is

\[ \nabla E_\kappa(x) = \begin{pmatrix} W'(x_1) - .1x_1 + .1x_2 \\ W'(x_2) + .1x_1 - .1x_2 \end{pmatrix}. \]

Let \( f_1(x_1, x_2) = W'(x_1) - .1x_1 + .1x_2 \) and \( f_2(x_1, x_2) = W'(x_2) + .1x_1 - .1x_2 \). We have that

\[ f_1(x_1, x_2) = \begin{cases} W'(x_1) - .1x_1 + .1x_2 < 0 - .1 < 0 & \text{for } x_1 = 1, x_2 \in (-2, -1) \\ W'(x_1) - .1x_1 + .1x_2 > 4 - .2 > 0 & \text{for } x_1 = 2, x_2 \in (-2, -1) \end{cases} \]

and

\[ f_2(x_1, x_2) = \begin{cases} W'(x_2) + .1x_1 - .1x_2 < -4 + .2 + .2 < 0 & \text{for } x_2 = -2, x_1 \in (1, 2) \\ W'(x_2) + .1x_1 - .1x_2 > 0 + .1 + .1 > 0 & \text{for } x_2 = -1, x_1 \in (1, 2). \end{cases} \]

By the Poincare–Miranda theorem we have an extrema in the square \((-2, -1) \times (1, 2)\), and by symmetry a second extrema. Also we have

\[ f_1(x_1, x_2) = \begin{cases} W'(x_1) - .1x_1 + .1x_2 < -1/2 - .05 - .1 < 0 & \text{for } x_1 = 1/2, x_2 \in (1, 2) \\ W'(x_1) - .1x_1 + .1x_2 > 0 - .1 + .1 > 0 & \text{for } x_1 = 1, x_2 \in (1, 2) \end{cases} \]

and

\[ f_2(x_1, x_2) = \begin{cases} W'(x_2) + .1x_1 - .1x_2 < 0 + .1 - .1 = 0 & \text{for } x_2 = 1, x_1 \in (1/2, 1) \\ W'(x_2) + .1x_1 - .1x_2 > 4 + .05 - .1 > 0 & \text{for } x_2 = 2, x_1 \in (1/2, 1). \end{cases} \]
By the Poincare–Miranda theorem we have an extrema in the square $(1/2, 1) \times (1, 2)$. By symmetry there are three more extrema in $(1, 2) \times (1/2, 1)$, $(-1, -1/2) \times (-2, -1)$, and $(-2, -1) \times (-1, -1/2)$. It is left to the reader that these extrema happen in regions with a positive Hessian. Thus we have a total of 6 minima at $\kappa = 0.1$. Based on this example, one may conjecture that for any $W$ potential there exists a graph with negative edge weights so that minima increase locally.

**Example 3.12.** This example gives an increase in minima with only two inflection points and without negative edge weights. Let $\kappa = 3/40$ and $\epsilon = 0.01$. Let $W$ be a smooth, even function with only two inflection points so that $W$ is the classical potential on $(-\infty, -1] \cup [1, \infty)$, is smooth, and has second derivative $W''(x) > 0$ on $(-1, -1 + \epsilon) \cup (1 - \epsilon, 1)$. Also suppose the following about $W'$,

$$W'(x) = \begin{cases} 
10 & \text{for } x \in [-1 + \epsilon, -1/2 - \epsilon] \\
1/10 & \text{for } x \in [-1/2 + \epsilon, -\epsilon] \\
-1/10 & \text{for } x \in [\epsilon, 1/2 - \epsilon] \\
-10 & \text{for } x \in [1/2 + \epsilon, 1 - \epsilon].
\end{cases}$$

Now consider the graph with two nodes and edge of weight 1. Consider the gradient of the energy

$$\nabla E_\epsilon(x) = \left( W'(x_1) + \frac{3}{40} x_1 - \frac{3}{40} x_2 \right) \left( W'(x_2) - \frac{3}{40} x_1 + \frac{3}{40} x_2 \right).$$

Let $f_1(x_1, x_2) = W'(x_1) + \frac{3}{40} x_1 - \frac{3}{40} x_2$ and $f_2(x_1, x_2) = W'(x_2) - \frac{3}{40} x_1 + \frac{3}{40} x_2$. We have that

$$f_1(x_1, x_2) = \begin{cases} 
W'(x_1) + \frac{3}{40} x_1 - \frac{3}{40} x_2 < 0 + \frac{3}{40}(-1 - \epsilon) < 0 & \text{for } x_1 = -1, x_2 \in (\epsilon, 1/2 - \epsilon) \\
W'(x_1) + \frac{3}{40} x_1 - \frac{3}{40} x_2 > 10 + \frac{3}{40}(-1 + \epsilon - (1/2 - \epsilon)) > 0 & \text{for } x_1 = -1 + \epsilon, x_2 \in (\epsilon, 1/2 - \epsilon)
\end{cases}$$

and

$$f_2(x_1, x_2) = \begin{cases} 
W'(x_2) - \frac{3}{40} x_1 + \frac{3}{40} x_2 < -1/10 + \frac{3}{40}(-1) + \epsilon < 0 & \text{for } x_2 = \epsilon, x_1 \in (-1, -1 + \epsilon) \\
W'(x_2) - \frac{3}{40} x_1 + \frac{3}{40} x_2 > -1/10 + \frac{3}{40}(-1 + \epsilon + 1/2 - \epsilon) > 0 & \text{for } x_2 = 1/2 - \epsilon, x_1 \in (-1, -1 + \epsilon)
\end{cases}.$$

By the Poincare–Miranda theorem we have an extrema in the square $(-1, -1 + \epsilon) \times (\epsilon, 1/2 - \epsilon)$, and by symmetry a second extrema. Now for any point in the open set $(-1, -1 + \epsilon) \times (\epsilon, 1/2 - \epsilon)$ there exists a $\delta > 0$ so that

$$D^2 E \geq \begin{pmatrix} \delta & 0 \\
0 & 1 - \delta
\end{pmatrix} \begin{pmatrix} 0 & 1 \\
-1 & 1
\end{pmatrix} = \begin{pmatrix} 1 + \delta & -1 \\
-1 & 1
\end{pmatrix}$$

which is positive.

Now we also have that

$$f_1(x_1, x_2) = \begin{cases} 
W'(x_1) + \frac{3}{40} x_1 - \frac{3}{40} x_2 < 0 + \frac{3}{40}(-1 - \epsilon) < 0 & \text{for } x_1 = 1, x_2 \in (1 - \epsilon, 1) \\
W'(x_1) + \frac{3}{40} x_1 - \frac{3}{40} x_2 > 10 + \frac{3}{40}(-1 + \epsilon - (1/2 - \epsilon)) > 0 & \text{for } x_1 = -1 + \epsilon, x_2 \in (1 - \epsilon, 1)
\end{cases}$$

By the Poincare–Miranda theorem we have an extrema in the square $(-1, 1 + \epsilon) \times (1 - \epsilon, 1)$. By symmetry there is another extrema. Here the Hessian is easier to bound so it is left to the reader. There are 2 more minima at $(1, 1)$ and $(-1, -1)$. Thus there are at least 6 minima at
\( \kappa = 3/40 \). This example shows that minima may increase locally on some \( W \) potentials with graphs of all positive edge weights.

4. Conclusions

There were two directions explored in this paper: in section 2 we considered the global minima of the energy functional whenever the graph topology is balanced, and in section 3 we studied the nonmonotonicity of the number of minima. Some interesting directions were not addressed in our results. For example, the boundedness of the number of stable points for arbitrarily large \( \kappa \) in terms of the number of stable points when \( \kappa = 0 \), given some potential and network. Such a bound would assert that too much significance on the social network in our model guarantees a relative size of the number of stable configurations. The existence of a \( W \) potential so that the number of stable points is monotonic in \( \kappa \) for any network would assert that our potentials, described in theorem 3.7, are special.

In section 2 we only considered a particular class of graphs that had a natural structure that led to our being able to describe the global minimum. Note also that the configuration which globally minimized the energy was independent of the coupling strength \( \kappa \) (although of course its energy changes as \( \kappa \) changes). We conjecture that this property is held (with some trivial exceptions) only by balanced graphs, i.e. if a graph is not balanced, then the minimum-energy configuration changes as a function of \( \kappa \).

The results of section 3 are a bit technical, but they show a surprising fact, if we consider the thermalization of such potentials. For example, we could add small white noise to any of these ODEs, and we know that all of the (local) minima identified above now become metastable. An observer who could only observe the nonequilibrium behavior of our potentials would not be able to detect the ‘shelves’ in the one-dimensional potentials, but these shelves play a huge role when these potentials are coupled together.

In terms of social dynamics, this increase of minima illustrates the complexities of social network interactions. A slight increase in the significance of the social interactions can arbitrarily increase the number of stable configurations thus making the network’s values more densely distributed and more difficult to locate, while too much significance often causes a reduction of configurations due to bifurcations. Too much significance on the social network could cause loss in individuality and Likewise too little significance could cause an excess of individual bias. In a stable configuration each member has found a balance between social pressures and their own natural response. Having more stable configurations may allow a member to explore the ‘opinion dimension’ with less energy exerted, balancing one’s own bias with social pressure.

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References

[ACTA17] Ashwin P, Creaser J and Tsaneva-Atanasova K 2017 Fast and slow domino effects in transient network dynamics (arXiv:1701.06148)

[AJGA14] Aouay S, Jamoussi S, Gargouri F and Abraham A 2014 Modeling dynamics of social networks: a survey 6th Int. Conf. on Computational Aspects of Social Networks (Piscataway, NJ: IEEE) pp 49–54
[Alt12] Altafini C 2012 Dynamics of opinion forming in structurally balanced social networks PLoS One 7 e38135

[BFG07a] Berglund N, Fernandez B and Gentz B 2007 Metastability in interacting nonlinear stochastic differential equations: I. From weak coupling to synchronization Nonlinearity 20 2551

[BFG07b] Berglund N, Fernandez B and Gentz B 2007 Metastability in interacting nonlinear stochastic differential equations: II. Large-n behaviour Nonlinearity 20 2583

[BRG16] Burghardt K, Rand W and Girvan M 2016 Competing opinions and stubborness: connecting models to data Phys. Rev. E 93 032305

[CFL09] Castellano C, Fortunato S and Loreto V 2009 Statistical physics of social dynamics Rev. Mod. Phys. 81 591

[CH56] Cartwright D and Harary F 1956 Structural balance: a generalization of heider’s theory Psychol. Rev. 63 277

[CH68] Cartwright D and Harary F 1968 On the coloring of signed graphs Elemente Math. 23 85–9

[DGM14] Das A, Gollapudi S and Munagala K 2014 Modeling opinion dynamics in social networks Proc. of the 7th ACM Int. Conf. on Web Search and Data Mining (New York: ACM) pp 403–12

[DL94] Durrett R and Levin S A 1994 Stochastic spatial models: a user’s guide to ecological applications Phil. Trans. R. Soc. B 343 329–50

[DPLM14] Dai Pra P, Louis P-Y and Minelli I G 2014 Synchronization via interacting reinforcement J. Appl. Probab. 51 556–68

[GS13] Ghaderi J and Srikant R 2013 Opinion dynamics in social networks: a local interaction game with stubborn agents American Control Conf. (Piscataway, NJ: IEEE) pp 1982–7

[HL75] Holley R A and Liggett T M 1975 Ergodic theorems for weakly interacting infinite systems and the voter model Ann. Probab. 3 643–63

[HL78] Holley R and Liggett T M 1978 The survival of contact processes Ann. Probab. 6 198–206

[JGN01] Jin E M, Girvan M and Newman M E 2001 Structure of growing social networks Phys. Rev. E 64 046132

[KLB09] Kunegis J, Lommatzsch A and Bauckhage C 2009 The slashdot zoo: mining a social network with negative edges Proc. of the 18th Int. Conf. on World Wide Web (New York: ACM) pp 741–50

[KSL+10] Kunegis J, Schmidt S, Lommatzsch A, Lerner J, De Luca E W and Albayrak S 2010 Spectral analysis of signed graphs for clustering, prediction and visualization Proc. of the 2010 SIAM Int. Conf. on Data Mining (Philadelphia, PA: SIAM) pp 559–70

[Lig99] Liggett T M 1999 Stochastic Interacting Systems: Contact, Voter and Exclusion Processes (Grundlehen der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences) vol 324) (Berlin: Springer)

[Maw13] Mawhin J 2013 Variations on poincaré-miranda’s theorem Adv. Nonlinear Stud. 13 209–17

[NBW11] Newman M, Barabasi A-L and Watts D J 2011 The Structure and Dynamics of Networks (Princeton, NJ: Princeton University Press)

[NG04] Newman M E and Girvan M 2004 Finding and evaluating community structure in networks Phys. Rev. E 69 026113

[SAR08] Sood V, Antal T and Redner S 2008 Voter models on heterogeneous networks Phys. Rev. E 77 041121

[Str01] Strogatz S H 2001 Exploring complex networks Nature 410 268–76

[SWS00] Sznajd-Weron K and Sznajd J 2000 Opinion evolution in closed community Int. J. Mod. Phys. C 11 1157–65

[Wig03] Wiggins S 2003 Introduction to Applied Nonlinear Dynamical Systems and Chaos (Texts in Applied Mathematics vol 2) 2nd edn (New York: Springer)

[WS98] Watts D J and Strogatz S H 1998 Collective dynamics of small-world networks Nature 393 440–2

[YAO+11] Yildiz E, Acemoglu D, Ozdaglar A E, Saberi A and Scaglione A 2011 Discrete opinion dynamics with stubborn agents