A NOTE ON FINITE HORIZON OPTIMAL INVESTMENT AND
CONSUMPTION WITH TRANSACTION COSTS

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Abstract. In this note, we remove the technical assumption $\gamma > 0$ imposed by Dai et al. [SIAM J. Control Optim., 48 (2009), pp. 1134-1154] who consider the optimal investment and consumption decision of a CRRA investor facing proportional transaction costs and finite time horizon. Moreover, we present an estimate on the resulting optimal consumption.

1. Introduction. In [6], Merton shows that in the absence of transaction costs, the optimal investment and consumption policy of a constant relative risk aversion (CRRA) investor is to keep a constant fraction of total wealth in each asset and to consume at a certain rate. In [5], Magill and Constantinides introduce proportional transaction costs to the Merton’s model, which leads to a singular stochastic control problem. Nowadays, there has been an extensive literature on this topic. See, for example, Davis and Norman [3] and Shreve and Soner [7] for the infinite horizon optimal investment and consumption, and Liu and Loewenstein [4] and Dai and Yi [2] for the finite horizon optimal investment without consumption.

Dai et al. [1] study the finite horizon investment and consumption decision with transaction costs. It turns out that the associated value function satisfies the following HJB equation:

$$\min \left\{ -\partial_t \varphi - \mathcal{L} \varphi, -(1 - \mu) \partial_x \varphi + \partial_y \varphi, (1 + \lambda) \partial_x \varphi - \partial_y \varphi \right\} = 0 \text{ in } \mathcal{S} \times [0, T]$$

with the terminal condition

$$\varphi(x, y, T) = \begin{cases} \frac{1}{\gamma} (x + (1 - \mu)y)^{\gamma} & \text{for } \gamma < 1, \gamma \neq 0, \\ \ln (x + (1 - \mu)y) & \text{for } \gamma = 0, \end{cases}$$

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where $\mathcal{S} = \{y > 0, x + (1 - \mu)y > 0\}$,

$$L \varphi = \begin{cases} \frac{1}{2} \sigma^2 y^2 \partial_{yy} \varphi + \alpha y \partial_y \varphi + x \partial_x \varphi - \beta \varphi + \frac{1}{\gamma} (\partial_t \varphi)^{\frac{1}{\gamma - 1}} \text{ for } \gamma < 1, \gamma \neq 0, \\ \frac{1}{\gamma^2} \sigma^2 y^2 \partial_{yy} \varphi + \alpha y \partial_y \varphi + x \partial_x \varphi - \beta \varphi - (1 + \log(\partial_x \varphi)) \text{ for } \gamma = 0, \end{cases}$$

and $\alpha > r > 0, \sigma > 0, \beta > 0, \lambda \geq 0, \mu \in [0,1) (\lambda + \mu > 0), \gamma$ are all constants.

It can be shown that $\varphi(x,y,t)$ has the homogeneity property: for any $\rho > 0$,

$$\varphi(\rho x, \rho y, t) = \left\{ \begin{array}{ll} \rho^\gamma \varphi(x, y, t) & \text{for } \gamma < 1, \gamma \neq 0, \\ \varphi(x, y, t) + g(t) \ln \rho & \text{for } \gamma = 0, \end{array} \right.$$ 

where

$$g(t) = \frac{1 - e^{-\beta(T-t)}}{\beta} + e^{-\beta(T-t)}. \quad (2)$$

This enables us to reduce the problem dimension by the following transformation:

$$z = \frac{x}{y}, \quad V(z,t) = \left\{ \begin{array}{ll} \frac{1}{y^\gamma} \varphi(x, y, t) & \text{for } \gamma < 1, \gamma \neq 0, \\ \varphi(x, y, t) - g(t) \ln y & \text{for } \gamma = 0. \end{array} \right.$$ 

Problem (1) gives rise to two free boundaries corresponding to the optimal buy and sell boundaries between which no trading takes place. Hence, one can characterize the optimal investment strategy by studying behaviors of the free boundaries. The optimal consumption strategy turns out to be

$$C = (\partial_x \varphi)^{1/(\gamma - 1)} \quad (3)$$

in the no trading region (NT, for short).

Since it is intractable to examine the behaviors of the free boundaries from problem (1), Dai et al. [1] adopt an indirect approach. For illustration, let us focus on the case $\gamma < 1, \gamma \neq 0$. Denote $w(z,t) = \frac{1}{\gamma} \ln(\gamma V)$ and let

$$v(z,t) = \partial_z w(z,t) = \frac{\partial_x \varphi(x, y, t)}{\gamma y \varphi(x, y, t)}. \quad (4)$$

Then it is shown that problem (1) can be reduced to an equivalent problem for $v(z,t)$:

$$\left\{ \begin{array}{l} \min \left\{ \max \left\{ -\partial_z v + L_w v, v - \frac{1}{\gamma + 1 - \lambda} \right\}, v - \frac{1}{\gamma + 1 - \lambda} \right\} = 0, \\
 v(z^*, t) = \frac{1}{\gamma + 1 - \lambda}, \quad v(z, T) = \frac{1}{\gamma + 1 - \lambda}, \end{array} \right. \quad (5)$$

in $\Omega_T \equiv (z^*, +\infty) \times (0, T)$, where $z^*$ is bigger than and sufficiently close to $-(1-\mu)$, and the differential operators $L_w$ and $L$ are given as follows:

$$L_w v = (e^{\gamma w v})^{-\frac{1}{\gamma - 1}} \left( v^2 + \partial_z v \right)$$

and

$$L v = \frac{1}{2} \sigma^2 z^2 \partial_{zz} v - \left( \alpha - r - (2 - \gamma) \sigma^2 \right) z \partial_z v - \left( \alpha - r - (1 - \gamma) \sigma^2 \right) v \right)$$

$$+ \gamma \sigma^2 \left( z^2 v \partial_z v + z v^2 \right).$$

$^1$ $y > 0$ means that short selling is not considered in this problem, that is because the optimal strategy is bound to immediately closeout stock if $y \leq 0$. $x + (1 - \mu)y > 0$ means the net wealth must be positive. We refer interested readers to Dai et al. [1] for model formulation and the implication of these conditions and parameters.

$^2$ It can be shown that $v(z,t) = \frac{1}{\gamma + 1 - \lambda}$ in $-(1-\mu) < z < z^*$.
As a consequence, they instead study (5) and entirely characterize the behaviors of the free boundaries.

In the equivalence proof of (1) and (5), the following inequalities play a critical role:

\[-\frac{K}{(z+1-\mu)^2} \leq \partial_z v \leq -v^2,\]

where \( K \) is a positive constant and \( v \) is the solution to problem (5) with \( w(z,t) \) belonging to some given class of functions.\(^3\) However, a technical assumption \( \gamma > 0 \) is imposed in Dai et al. [1] to obtain the left hand side inequality (see Remark 2.2 in Dai et al. [1]), which unfortunately precludes the application to the case of the commonly used utility function \( -\frac{1}{x} \) or \( -\frac{1}{2}x^2 \) (i.e., \( \gamma = -1, -2 \)).

The main purpose of this note is to fill the gap. As a by-product, we will also characterize the optimal consumption strategy.\(^4\) The rest of the paper is organized as follows. In section 2 we consider the power utility case (\( \gamma < 1, \gamma \neq 0 \)). We present a proposition to fill the only gap in Dai et al. [1]. The proposition is further used to study the optimal consumption strategy. In section 3 we consider the logarithm utility case (\( \gamma = 0 \)).

2. The power utility case. We will use comparison principle to prove the left hand side inequality of (6). However, a straightforward application of comparison principle to the differential operator for \( \partial_z v \) does not work for \( \gamma < 0 \). So we need to construct an auxiliary function with which comparison principle can work. Let us present an insight. The right hand side inequality of (6) is based on the concavity of the original value function \( \varphi(x,y,t) \), namely,

\[0 \geq (\partial_{xy} \varphi)^2 - \partial_{xx} \varphi \partial_{yy} \varphi = \gamma^2 (1 - \gamma) y^{2\gamma - 4} V^2 (\partial_z v + v^2).\]

To prove the left hand side inequality of (6), we shall also rely on financial implication or the properties of \( \varphi(x,y,t) \). From (3)-(4), we can observe \( \partial_z v \) has a connection to \( \partial_{xx} \varphi \), which inspires us to consider the change of the optimal consumption:

\[\partial_x C = \frac{1}{\gamma - 1} (\partial_x \varphi)^{1/(\gamma - 1)} - \partial_{xx} \varphi = \frac{C}{\gamma - 1} \partial_{zz} V \frac{1}{y(\gamma - 1)v} (\partial_z v + \gamma v^2).\]

This motivates us to estimate the lower bound of \( \partial_z v + \gamma v^2 \).

**Proposition 1.** Assume \( \gamma < 0 \). Let \( v(z,t) \) be the solution to problem (5), in which \( w(z,t) \) possesses the following properties:

\[|w(z,t) - \ln(z+1-\mu)| \leq M_T,\]

\[\frac{1}{z+1+\lambda} \leq \partial_z w(z,t) \leq \frac{1}{z+1-\mu},\]

where \( M_T \) is a positive constant. Then there is a positive constant \( K \) such that

\[\frac{\partial_z v + \gamma v^2}{v} \geq -\frac{K}{z+1-\mu} \text{ in } \Omega_T,\]

where \( K \) depends on \( M_T \), but not on \( w \).

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\(^3\)Here we cannot directly assume \( \partial_x w = v \) because we need to use a fixed-point theorem to show the equivalence of (1) and (5).

\(^4\)Dai et. al. [1] only characterize the optimal investment strategy.
Proof. As in Dai et al. [1], we can restrict attention to the following approximation problem of (5) in a bounded domain $\Omega^R_T = (z^*, R) \times (0, T)$: for sufficiently small $\delta > 0$,\[
\begin{aligned}
-\partial_t v_\delta - L_\delta v_\delta + L_w v_\delta &= 0, & \text{if } & \frac{1}{z + 1 + \lambda} < v_\delta < \frac{1}{z + 1 - \mu}, \\
-\partial_t v_\delta - L_\delta v_\delta + L_w v_\delta &\geq 0, & \text{if } & v_\delta = \frac{1}{z + 1 + \lambda}, \\
-\partial_t v_\delta - L_\delta v_\delta + L_w v_\delta &\leq 0, & \text{if } & v_\delta = \frac{1}{z + 1 - \mu}, \\
\partial_z v_\delta(z^*, \tau) = -\frac{1}{(z^* + 1 - \mu)^2} \partial_z v_\delta(R, t) + v_\delta^2(R, t) = 0, & v_\delta(z, T) = \frac{1}{z + 1 - \mu}.
\end{aligned}
\] (11)

where $L_\delta v_\delta = L v_\delta + \delta \partial_z v_\delta$. We only need to show that there exists a positive constant $K$ independent of $\delta$, $R$ and $w$ such that\[
\frac{\partial_z v_\delta + \gamma v_\delta^2}{v_\delta} = \partial_z (\ln v_\delta) + \gamma v_\delta \geq -\frac{K}{z + 1 - \mu} \quad \text{in } \Omega^R_T. (12)
\]

Owing to $z > z^* > \mu - 1$ and (8), there exist two positive constants $C$, $c$ such that

\[
\gamma \ln(z + 1 - \mu) - |\gamma| M_T \leq \gamma w \leq \gamma \ln(z + 1 - \mu) + |\gamma| M_T,
\]

\[
C(z + 1 - \mu) \geq \frac{1}{1 - \gamma} (e^{\gamma w} v_\delta)^{-\frac{1}{1 - \gamma}} \geq \tilde{c}(z + 1 - \mu) \geq c.
\]

Without loss of generality, we confine ourselves to the region

\[
\mathcal{M} \equiv \left\{(z, t) \in \Omega^R_T : \frac{1}{z + 1 + \lambda} < v_\delta < \frac{1}{z + 1 - \mu}\right\}.
\]

It is clear that $v_\delta$ is governed by the following equation

\[
-\partial_t v_\delta - L^* v_\delta - \gamma \sigma^2 z v_\delta (\alpha - r - (2 - \gamma) \sigma^2) z \partial_z v_\delta - (\alpha - r - (1 - \gamma) \sigma^2) v_\delta = 0 \quad \text{in } \mathcal{M},
\]

where $L^*$ is the linear part of the operate $L_\delta$, i.e.,

\[
L^* v_\delta = \left(\frac{1}{2} \sigma^2 z^2 + \delta\right) \partial_z v_\delta - \left(\alpha - r - (2 - \gamma) \sigma^2\right) z \partial_z v_\delta - \left(\alpha - r - (1 - \gamma) \sigma^2\right) v_\delta.
\]

Denote $f = \ln v_\delta$, $p = \partial_z f$, $q = \gamma v_\delta$, then

\[
-\partial_t f - L^* f - \left(\frac{1}{2} \sigma^2 z^2 + \delta\right) p^2 + \left(\alpha - r - (1 - \gamma) \sigma^2\right) (1 - f)
\]

\[
- \gamma q \sigma^2 z v_\delta (zp + 1) + e^{-\frac{\gamma w + f}{1 - \gamma}} (v_\delta + p) = 0 \quad \text{in } \mathcal{M}.
\]

Differentiating the above equation with respect to $z$, and denoting $L = p + q$, we have

\[
-\partial_t p - L^* p - (\sigma^2 z - 2 \delta q) \partial_z L + (\sigma^2 z q^2 - \sigma^2 - 4 \delta q^2) L - (\sigma^2 z^2 + 2 \delta) L \partial_z L
\]

\[
+ (2 \delta q - \sigma^2 z^2) L^2 + (e^{\gamma w} v_\delta)^{-\frac{1}{1 - \gamma}} \left[(v_\delta p + \partial_z p) - \frac{1}{1 - \gamma} (\gamma \partial_z w + p)(v_\delta + p)\right]
\]

\[
+ \sigma^2 z^2 q^2 - \sigma^2 z^2 q^3 + 2 \delta q^3 = 0,
\]

\[
5\text{By the same method in Dai et al. [1], we can show that } v_\delta \text{ weakly converges to } v \text{ in } W^{2,1}_p \text{ space with any fixed } p > 1 \text{ if (12) is obtained.}
\]

\[
6\text{In Dai et al. [1], } \partial_z v_\delta \text{ is continuous in } \Omega^R_T, \text{ and the only gap is to prove the inequality (12).}
\]
where we have used $\partial_z p = \partial_z L - qL + q^2$. On the other hand, $q$ satisfies
\[-\partial_t q - L^* q - \sigma^2 z^2 q^2 + \sigma_2^2 + \sigma^2 z q^2 + (e^\gamma w) - \frac{1}{1 - e^\gamma} q(v_3 + p) = 0.\]
So, $L$ is governed by
\[
-\partial_t L - L^* L - (\sigma^2 z^2 - 2\delta q)\partial_z L - (\sigma^2 + 4\delta q^2) L - (\sigma^2 z^2 + 2\delta) L \partial_z L + (2\delta q - \sigma^2 z) L^2 + 2\delta q^3 + (e^\gamma w) - \frac{1}{1 - e^\gamma} q(v_3 - \partial_z w)L - \frac{L^2}{1 - e^\gamma}
\]
\[-q(\partial_z w - v_3) \triangleq T L = 0 \quad \text{in } M.\]
Let
\[
W(z, t) = e^{M(T-t)} \Phi(z),
\]
\[
\Phi(z) = \begin{cases}
\frac{C_1}{z}, & z \leq z \leq z_1, \\
\frac{C_2}{(1 - \mu)^2} + \frac{C_2 z}{(1 - \mu)^2} - \left( \frac{C_2}{(1 - \mu)^2} + \frac{C_1}{z_1^2} \right) \frac{z^2}{2z_1}, & z_1 \leq z \leq 0, \\
\frac{C_2}{z + 1 - \mu}, & z \geq 0,
\end{cases}
\]
where $C_1, M$ to be determined,
\[
C_2 = \frac{-3(1 - \mu)^2 C_1}{2z_1}, \quad z_1 \geq 0.
\]
with $z_1 \in (z^*, 0)$.

It is not difficult to check $W \in W^2_{p, \text{loc}} (M) \cap C(\overline{M})$. Moreover, in the case $z_1 < z < 0$, we calculate that $\partial_z W(z, t) \geq 0$, and
\[
-\frac{W(z_1, t)}{z_1} \leq \partial_z W(z, t) \leq -\frac{3W(z_1, t)}{2(1 - \mu)}, \quad 3W(z_1, t) \leq W(z, t) \leq W(z_1, t),
\]
and
\[
TW \leq W(z_1, t)(M - C) + W^2(z_1, t) \left( \bar{C}(z_1^2 - z_1 + \delta) - c \right) + \bar{C},
\]
where $C, \bar{C}$ are some positive constants. By choosing $\delta, z_1$ small enough such that $\bar{C}(z_1^2 - z_1 + \delta) - c \leq 0$, and choosing $M$ large enough, we deduce $TW < 0$ when $z_1 < z < 0$.

In the case $z^* < z < z_1$, we have
\[
TW \leq C_1 e^{M(T-t)} \left( \frac{M - C}{z} + \frac{2\delta(C_1 e^{M(T-t)} - 1)}{z^3} \right) + \frac{C_2^2 e^{2M(T-t)}}{z^2} (2\delta q - c) + \bar{C} < 0,
\]
for $\delta$ small enough and $C_1, M$ large enough. If $z > 0$, we find
\[
TW \leq -\frac{C_2 e^{M(T-t)}}{z + 1 - \mu} \left( M - C \frac{1 + z^2}{(z + 1 - \mu)^2} \right) + \frac{C_2^2 e^{2M(T-t)}}{(z + 1 - \mu)^2} \left( \frac{\sigma^2 z^2 + 2\delta}{z + 1 - \mu} + 2\delta q - \sigma^2 z - c \right)
\]
\[
\leq C_2^2 e^{2M(T-t)} \frac{2\delta + 2\delta \gamma - \sigma^2 (1 - \mu) z}{z + 1 - \mu} - c < 0,
\]
for $\delta$ small enough and $M$ large enough.
Note that we can choose $C_1$ large enough such that $L \geq W$ on the parabolic boundary of $M$. Applying comparison principle leads to $L \geq W$. It is easy to verify that

$$-(z + 1 - \mu)W \leq e^{MT} \left( \frac{C_1(z_1 + 1 - \mu)}{-z_1} + C_2 - \frac{3C_2z_1}{2(1 - \mu)} - \frac{C_1(1 - \mu)}{2z_1} \right) := K.$$ 

Then (12) follows.

By Proposition 2.1, we immediately get the left hand side inequality of (6) in the case $\gamma < 0$, which fills the gap of Dai et al. [1]. Hence, all results in Dai et al. [1] hold for $\gamma < 0$. In particular, $w(z,t) = \frac{1}{\gamma} \ln(V)$ satisfies (8) for any $\gamma \neq 0$, which is useful to estimate the optimal consumption.

Now let us present an estimate on the optimal consumption. In the absence of transaction costs, Merton [6] shows that the optimal consumption takes the form of

$$C = \frac{b}{1 - (1 - b) e^{-b(T-t)}}(x + y),$$

where $b = \frac{1}{1-\gamma} \left[ \beta - \gamma r - \frac{\gamma(a-r)^2}{2(1-\gamma)\sigma^2} \right]$. Then the fraction of wealth in consumption and the change rate of consumption are

$$\frac{C}{x+y} = \frac{b}{1 - (1 - b) e^{-b(T-t)}},$$

and

$$\frac{\partial_x C}{C} = \frac{1}{x+y},$$

respectively. In the presence of transaction costs, we only need to consider the consumption in the no trading region

$$NT = \{(z,t) \in \Omega_T : \frac{1}{z + 1 - \mu} < v(z,t) < \frac{1}{z + 1 - \mu} \}.$$ 

We have the following results.

**Proposition 2.** Let $C$ be the optimal consumption as given in (3). Then

i) There are two positive constants $k_1$ and $k_2$ such that

$$k_1 \leq \frac{C}{x+y} \leq k_2$$

for any $(\xi, \eta, t) \in NT$.

ii) There is a constant $k_3$, such that

$$\frac{1}{x + (1 + \lambda) \eta} \leq \frac{\partial_x C}{C} \leq \frac{k_3}{x+y}$$

for any $(\xi, \eta, t) \in NT$.

**Proof.** Let us first prove part ii). Thanks to (7), we have

$$\frac{\partial_x C}{C} = \frac{1}{y(\gamma-1)v} (\partial_x v + \gamma v^2).$$

In virtue of the right hand side inequality of (6), we infer

$$\frac{\partial_x C}{C} \geq \frac{1}{y(\gamma-1)v} (-v^2 + \gamma v^2) = \frac{v}{y} \geq \frac{1}{y \xi + (1 + \lambda)} = \frac{1}{x + (1 + \lambda) \eta},$$

where $\eta = \eta(t)$.
On the other hand, applying Proposition 2.1 gives
\[
\frac{\partial x}{C} \leq -\frac{K}{y(\gamma - 1)v} \frac{1}{(z + 1 - \mu)^2} \leq \frac{K}{y(1 - \gamma)} \frac{z + 1 + \lambda}{(z + 1 - \mu)^2}
\]
\[
\leq \frac{K_1}{y(1 - \gamma)} \frac{1}{z + 1} = \frac{K_1}{1 - \gamma} \frac{1}{x + y},
\]
where the last inequality is due to \(z > z^* > -(1 - \mu)\). We then get the desired result by taking \(k_3 = -K_1/(1 - \gamma)\).

It remains to prove part i). Notice
\[
\frac{\partial x}{\phi} = y^{\gamma - 1} \frac{\partial z}{V(z,t)} = y^{\gamma - 1} v(z,t) \exp(\gamma w(z,t)).
\]

Due to (8), there are two positive constants \(k_1\) and \(k_2\) such that
\[
\frac{1}{z + 1 + \lambda} \leq \exp(\gamma w(z,t)) \leq \frac{1}{z + 1 - \mu}, \quad z > z^* \quad \text{and} \quad (3),
\]
gives the desired result.

### 3. The logarithm utility case.

In the logarithm utility case, we make the following transformation
\[
v(z,t) = \frac{\partial_z V(z,t)}{g(t)}
\]
and obtain
\[
\begin{cases}
\min \left\{ \max \left\{ -\partial_z v - \bar{L} v + \frac{1}{g(t)} \left( v + \frac{\partial_z v}{v} \right), v - \frac{1}{z + 1 - \mu} \right\}, v - \frac{1}{z + 1 + \lambda} \right\} = 0 \quad \text{in } \Omega_T, \\
v(z^*,t) = \frac{1}{z + 1 - \mu}, \quad v(z,T) = \frac{1}{z + 1 - \mu},
\end{cases}
\]
where \(g(t)\) is as given in (2), and
\[
\bar{L} v = \frac{1}{2} \sigma^2 z^2 \partial_{zz} v - (\alpha - r - 2\sigma^2) z \partial_z v - (\alpha - r - \sigma^2) v.
\]

As before, the concavity of \(\phi\) implies that we should have
\[
(\partial_{xy} \phi)^2 - \partial_{xx} \phi \partial_{yy} \phi = \frac{g^2(t)(\partial_z v + v^2)}{y^4} \leq 0.
\]

And the change of consumption rate w.r.t. \(x\) is
\[
\frac{\partial_x C}{C} = \frac{-\partial_{xx} \phi}{\partial_x \phi} = \frac{-\partial_z V}{y \partial_z V} = \frac{-\partial_z v}{y v}.
\]
Hence, we shall first establish the following proposition:

**Proposition 3.** Let \(v(z,t)\) be the solution to problem (13). Then there is a positive constant \(K\) such that
\[
\frac{\partial_z v + v^2}{v} \leq 0 \quad \text{in } \Omega_T,
\]
\[
\frac{\partial_z v}{v} \geq -\frac{K}{z + 1 - \mu} \quad \text{in } \Omega_T.
\]

**Remark 1.** It should be pointed out that different from the power utility case, Eq. (13) is a self-contained problem. We can directly use the same technique developed by Dai and Yi [1] to obtain the equivalence between Eq. (13) and Eq. (1) as long as (15) is established.
Proof of Proposition 3. As before, we restrict attention to the following approximation problem of (13) in a bounded domain \( \Omega_R^T \), for sufficiently small \( \delta > 0 \),

\[
\begin{cases}
-\partial_t v^\delta - \mathcal{L}_\delta v^\delta + \frac{1}{g(t)} \left( v^\delta + \frac{\partial_z v^\delta}{v^\delta} \right) = 0, & \text{if } \frac{1}{z+1+\lambda} < v^\delta < \frac{1}{z+1-\mu}, \\
-\partial_t v^\delta - \mathcal{L}_\delta v^\delta + \frac{1}{g(t)} \left( v^\delta + \frac{\partial_z v^\delta}{v^\delta} \right) \geq 0, & \text{if } v^\delta = \frac{1}{z+1+\lambda}, \\
-\partial_t v^\delta - \mathcal{L}_\delta v^\delta + \frac{1}{g(t)} \left( v^\delta + \frac{\partial_z v^\delta}{v^\delta} \right) \leq 0, & \text{if } v^\delta = \frac{1}{z+1-\mu}, \\
\partial_z v^\delta(z^*, \tau) = -\frac{1}{(z^*+1-\mu)^2}, \quad \partial_z v^\delta(R, t) + v^\delta_0(R, t) = 0, \quad v^\delta(z, T) = \frac{1}{z+1-\mu},
\end{cases}
\]

(17)

where \( \mathcal{L}_\delta v^\delta = \mathcal{L} v^\delta + \delta \partial_z v^\delta \). And we only need to show that there exists a positive constant \( K \) independent of \( \delta, R \), such that

\[
\frac{\partial_z v^\delta + v^\delta_0}{v^\delta} = \partial_z (\ln v^\delta) + v^\delta \leq 0 \quad \text{in } \Omega^R_T,
\]

(18)

\[
\frac{\partial_z v^\delta}{v^\delta} = \partial_z (\ln v^\delta) \geq \frac{-K}{(z + 1 - \mu)^2} \quad \text{in } \Omega^R_T.
\]

(19)

Without loss of generality, we confine ourselves to the region

\[
\tilde{\mathcal{M}} \equiv \left\{ (z, t) \in \Omega^R_T : \frac{1}{z+1+\lambda} < v^\delta < \frac{1}{z+1-\mu} \right\}.
\]

Clearly \( v^\delta \) is governed by the following equation

\[
-\partial_t v^\delta - \mathcal{L}_\delta v^\delta + \frac{1}{g(t)} \left( v^\delta + \frac{\partial_z v^\delta}{v^\delta} \right) = 0 \quad \text{in } \mathcal{M}.
\]

Denote \( f = \ln v^\delta, \quad p = \partial_z f \). It follows

\[
-\partial_t f - \mathcal{L}_\delta f - (\alpha - r - 2\sigma^2) f
\]

\[
= -\frac{1}{g(t)} \left( 1 + \frac{p}{v^\delta} \right) + \left( \frac{\sigma^2 z^2}{2} + \delta \right) p^2 - (\alpha - r - \sigma^2) + \sigma^2 f \quad \text{in } \mathcal{M}.
\]

Differentiating the above equation with respect to \( z \), we have

\[
-\partial_t p - \mathcal{L}_\delta p - \sigma^2 z \partial_z p = -\frac{\partial_z p}{g(t)v^\delta} + \frac{p \partial_z v^\delta}{g(t)v^\delta} + \sigma^2 z p^2 + (\sigma^2 z^2 + 2\delta) p \partial_z p + \sigma^2 p \quad \text{in } \mathcal{M}.
\]

(20)

Denoting \( L = p + v^\delta \), we have

\[
-\partial_t L - \mathcal{L}_\delta L - \sigma^2 z \partial_z L + \frac{\partial_z L}{g(t)v^\delta} - \frac{L \partial_z v^\delta}{g(t)v^\delta} - \sigma^2 z p L
\]

\[
= (\sigma^2 z^2 + 2\delta) \frac{p \partial_z L}{g(t)} - \sigma^2 L + \frac{L}{g(t)}
\]

\[
= -2\sigma^2 z \partial_z v^\delta - (\sigma^2 z^2 + 2\delta) \left( \frac{\partial_z v^\delta}{v^\delta} \right)^2 - \sigma^2 v^\delta
\]

\[
= -\frac{\sigma^2 (z \partial_z v^\delta + v^\delta)^2 + 2\delta (\partial_z v^\delta)^2}{v^\delta} \leq 0 \quad \text{in } \mathcal{M}.
\]

Moreover, it is not hard to deduce that \( L = 0 \) on the parabolic boundary of \( \mathcal{M} \). Then applying comparison principle yields (18).
On the other hand, (20) can be rewritten as
\[ \tilde{T}_p = 0 \text{ in } M, \]
where
\[ \tilde{T}_p = -\partial_t p - \tilde{L} p - \left( \sigma^2 z - \frac{1}{g(t)v_0} \right) \partial_x p - \sigma^2 p - \left( \frac{1}{v_0} + \sigma^2 z \right) \frac{p^2}{g(t)} - \left( \sigma^2 z^2 + 2\delta \right) \frac{p \partial_z p}{g(t)}. \]

Using the same \( W \) as in Section 2, we have for \( z_1 < z < 0 \),
\[ \tilde{T} W \leq W(z_1, t)(M - C) + \frac{W^2(z_1, t)}{g(t)} \left[ -\frac{1}{v_0} + \tilde{C}(z_1^2 + z_1 + \delta) \right]. \]

It follows \( \tilde{T} W < 0 \) for \( z_1 < z < 0 \), by choosing \( \delta, z_1 \) small enough such that \( \tilde{C}(z_1^2 + z_1 + \delta) - 1/v_0 \leq 0 \) and by choosing \( M \) large enough.

In the case \( z^* < z < z_1 \),
\[ \tilde{T} W \leq C_1 e^{M(T-t)} \left( \frac{M - C}{z} - \frac{2\delta}{z^2} \right) \leq C_1 e^{M(T-t)} \left( \frac{M - C}{z^*} - \frac{2\delta}{z_1^2} \right) < 0 \]
for \( M \) large enough. If \( z > 0 \), then we find
\[ \tilde{T} W \leq -C_2 e^{M(T-t)} \left( \frac{M - C}{z} - \frac{1 + z^2}{(z + 1 - \mu)^2} \right) \]
\[ + \frac{C_2 e^{2M(T-t)}}{(z + 1 - \mu)^2} \left( \frac{1}{v_0} - \sigma^2 z + \frac{\sigma^2 z^2 + 2\delta}{z + 1 - \mu} \right) \]
\[ \leq \frac{C_2 e^{2M(T-t)}}{(z + 1 - \mu)^2} \left( -(1 - \mu) + \frac{2\delta}{z + 1 - \mu} \right) < 0 \]
for \( \delta \) small enough and \( M \) large enough.

Moreover, we can choose \( C_1 \) large enough such that \( W \leq L \) on the parabolic boundary of \( M \). Applying comparison principle gives \( L \geq W \). We then obtain (19).

Thank to Proposition 3, we have the following result.

**Proposition 4.** Let \( C \) be the optimal consumption in the problem with logarithm. Then

i) 
\[ 1 - \frac{\mu y}{x + y} \leq \frac{g(t)C}{x + y} \leq 1 + \frac{\lambda y}{x + y} \]
for any \((\tilde{z}, t) \in NT\).

ii) There is a constant \( k_5 \), such that
\[ \frac{1}{x + (1 + \lambda) y} \leq \frac{\partial_x C}{C} \leq \frac{k_5}{x + y} \]
for any \((\tilde{z}, t) \in NT\).

**Proof.** Note that
\[ \frac{C}{x + y} = \frac{1}{(x + y) \partial_x \phi} = \frac{1}{(x + y) \partial_x (V + g(t) \ln y)} = \frac{y}{(x + y)v} = \frac{1}{(z + 1)g(t)v}. \]
which yields part i) in terms of $\frac{1}{x+1+\lambda} \leq v \leq \frac{1}{z+1-\mu}$. Now let us prove part ii). In view (14) and (15), we have
\[
\frac{\partial_x C}{C} = -\frac{\partial_z v}{yv} \geq \frac{v^2}{yv} = v \geq \frac{1}{y \left( z + (1 + \lambda) \right)} = \frac{1}{x + (1 + \lambda) y}.
\]
On the other hand, applying Proposition 1 gives
\[
\frac{\partial_x C}{C} = -\frac{\partial_z v}{yv} \leq \frac{1}{y \left( z + 1 - \mu \right)} \leq \frac{k_5}{y(z+1)} = \frac{k_5}{x+y},
\]
where the last inequality is due to $z > z^* > -(1 - \mu)$.

**Remark 2.** It is worthwhile pointing out that part i) in the above proposition is sharp in the sense that $\frac{\partial(CC)}{x+y} = 1$ in the absence of transaction costs.

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