Bregman circumcenters: applications

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Abstract

Recently, we systematically studied the basic theory of Bregman circumcenters in another paper. In this work, we aim to investigate the application of Bregman circumcenters. Here, we propose the forward Bregman monotonicity which is a generalization of the powerful Fejér monotonicity, and show a weak convergence result of the forward Bregman monotone sequence. Then we provide sufficient conditions for the sequence of iterations of the forward Bregman circumcenter mapping to be forward Bregman monotone. Furthermore, we prove that the sequence of iterations of the forward Bregman circumcenter mapping weakly converges to a point in the intersection of the fixed point sets of relevant operators, which reduces to the known weak convergence result of the circumcentered method under the Euclidean distance. In addition, particular examples are provided to illustrate the Bregman isometry and Browder’s demiclosedness principle, and our convergence result.

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1 Introduction

Throughout the work, we assume that $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\{m, n\} \subseteq \mathbb{N} \setminus \{0\}$, and that $\mathcal{H}$ is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. Projection methods, including the method of cyclic projections based on Bregman distances, are employed in many applications (see, e.g., [2], [3], [4], and [18] and the references therein). The convergence rate of certain circumcentered methods is no worse than that of the method of cyclic projections under the Euclidean distance, and, in some cases, some circumcentered methods converge much faster than the method of cyclic projections, for solving the best approximation problem or the feasibility problem (see, [9], [10], [11], [13], [14], [15], [16], [17], and [21] for details).

In our recent work [12], we presented multiple beautiful theoretical results on the Bregman circumcenter. Various examples under general Bregman distance were also provided to illustrate our main results.

In this work, we aim to introduce the circumcenter mappings and methods, and to investigate the convergence of circumcenter methods under general Bregman distances.

We present the main results in this work below.

R1: Theorem 3.3 characterizes the weak convergence of the forward Bregman monotone sequence.

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R2: Theorem 4.12 shows that the sequence of iterations of the forward Bregman circumcenter mapping converges weakly to a point in the intersection of the fixed point sets of operators inducing the mapping.

The remainder of the work is organized as follows.

In Section 2, we collect fundamental definitions and facts. In Section 3, we introduce the forward Bregman monotonicity and investigate the weak convergence of the forward Bregman monotone sequence. In Section 4, we introduce the forward Bregman circumcenter mapping induced by a finite set of operators, and specify sufficient conditions for the weak convergence of the forward Bregman circumcenter method to a point in the intersection of fixed point sets of the related operators. Moreover, we provide particular examples to illuminate our hypotheses and main result under general Bregman distance.

We now turn to the notation used in this paper. \( \Gamma_0(\mathcal{H}) \) is the set of proper closed convex functions from \( \mathcal{H} \) to \( ]-\infty, +\infty[ \). Let \( f : \mathcal{H} \to ]-\infty, +\infty[ \) be proper. The domain (conjugate function, gradient, subgradient, respectively) of \( f \) is denoted by \( \text{dom} f \) \( (f^*, \nabla f, \partial f, \text{respectively}) \). We say \( f \) is coercive if \( \lim_{\|x\| \to +\infty} f(x) = +\infty \). Let \( C \) be a nonempty subset of \( \mathcal{H} \). Its interior and boundary are abbreviated by \( \text{int} C \) and \( \text{bd} C \), respectively. \( C \) is an affine subspace of \( \mathcal{H} \) if \( C \neq \varnothing \) and \( (\forall \rho \in \mathbb{R}) \rho C + (1 - \rho)C = C \). The smallest affine subspace of \( \mathcal{H} \) containing \( C \) is denoted by \( \text{aff} C \) and called the affine hull of \( C \). The best approximation operator (or projector) onto \( C \) under the Euclidean distance is denoted by \( P_C \), that is, \( (\forall x \in \mathcal{H}) P_C x := \arg\min_{y \in C} \|x - y\| \). \( \iota_C \) is the indicator function of \( C \), that is, \( (\forall x \in \mathcal{C}) \iota_C(x) = 0 \) and \( (\forall x \in \mathcal{H} \setminus C) \iota_C(x) = +\infty \). Id stands for the identity mapping. For every \( x \in \mathcal{H} \) and \( \delta \in \mathbb{R}_{++}, B[x; \delta] \) is the closed ball with center at \( x \) and with radius \( \delta \). Let \( A : \mathcal{H} \to 2^\mathcal{H} \) and let \( x \in \mathcal{H} \). Then \( A \) is locally bounded at \( x \) if there exists \( \delta \in \mathbb{R}_{++} \) such that \( A(B[x; \delta]) \) is bounded. Denote by \( \text{Fix} A := \{ x \in \mathcal{H} : x \in A(x) \} \). For other notation not explicitly defined here, we refer the reader to [5].

## 2 Bregman distances and projections

In this section, we collect some essential definitions and facts to be used subsequently.

It is clear that if \( f = \frac{1}{2} \| \cdot \|^2 \) in the following Definition 2.1, we recover the Euclidean distance \( D : \mathcal{H} \times \mathcal{H} \to [0, +\infty[ : (x, y) \mapsto \frac{1}{2} \|x - y\|^2 \).

**Definition 2.1.** [2, Definitions 7.1 and 7.7] Suppose that \( f \in \Gamma_0(\mathcal{H}) \) with \( \text{int} \text{dom} f \neq \varnothing \) and that \( f \) is Gâteaux differentiable on \( \text{int} \text{dom} f \). The Bregman distance \( D_f \) associated with \( f \) is defined by

\[
D_f : \mathcal{H} \times \mathcal{H} \to [0, +\infty[ : (x, y) \mapsto \begin{cases} f(x) - f(y) - \langle \nabla f(y), x - y \rangle, & \text{if } y \in \text{int} \text{dom} f; \\ +\infty, & \text{otherwise}. \end{cases}
\]

Moreover, let \( C \) be a nonempty subset of \( \mathcal{H} \). For every \( (x, y) \in \text{dom} f \times \text{int} \text{dom} f \), define the backward Bregman projection (or simply Bregman projection) of \( y \) onto \( C \) and forward Bregman projection of \( x \) onto \( C \), respectively, as

\[
\overline{P}_C^f(y) := \{ u \in C \cap \text{dom} f : (\forall c \in C) D_f(u, y) \leq D_f(c, y) \},
\]

\[
\overline{P}_C^f(x) := \{ v \in C \cap \text{int} \text{dom} f : (\forall c \in C) D_f(x, v) \leq D_f(x, c) \}.
\]

Abusing notation slightly, we shall write \( \overline{P}_C^f(y) = u \) and \( \overline{P}_C^f(x) = v \), if \( \overline{P}_C^f(y) \) and \( \overline{P}_C^f(x) \) happen to be the singletons \( \overline{P}_C^f(y) = \{ u \} \) and \( \overline{P}_C^f(x) = \{ v \} \), respectively.

The following definitions are compatible with their classical counterparts as [1, Definitions 2.1, 2.3 and 2.8] in the finite-dimensional Euclidean space (see, [2, Theorem 5.11] for more details).

**Definition 2.2.** [2, Definition 5.2 and Theorem 5.6] Suppose that \( f \in \Gamma_0(\mathcal{H}) \). We say \( f \) is:
(i) essentially smooth, \( \text{dom} \partial f = \text{int dom } f \neq \emptyset \), \( f \) is Gâteaux differentiable on \( \text{int dom } f \), and \( \| \nabla f(x_k) \| \to +\infty \), for every sequence \( (x_k)_{k \in \mathbb{N}} \) in \( \text{int dom } f \) converging to some point in \( \text{bd dom } f \).

(ii) essentially strictly convex, if \( (\partial f)^{-1} \) is locally bounded on its domain and \( f \) is strictly convex on every convex subset of \( \text{dom } \partial f \).

(iii) Legendre (or a Legendre function or a convex function of Legendre type), if \( f \) is both essentially smooth and essentially strictly convex.

**Fact 2.3.** [2, Lemma 7.3] Suppose that \( f \in \Gamma_0(\mathcal{H}) \) with \( \text{int dom } f \neq \emptyset \) and that \( f \) is Gâteaux differentiable on \( \text{int dom } f \). Let \( x \in \mathcal{H} \) and \( y \in \text{int dom } f \). Then the following statements hold.

(i) If \( f \) is essentially strictly convex, then \( D_f(x,y) \) is coercive.

(ii) If \( f \) is essentially strictly convex, then \( D_f(x,y) = 0 \iff x = y \).

The following Facts 2.4 and 2.6 on the existence and uniqueness of backward and forward Bregman projections are fundamental to some results in Section 4 below.

**Fact 2.4.** [2, Corollary 7.9] Suppose that \( f \in \Gamma_0(\mathcal{H}) \) is Legendre, that \( C \) is a closed convex subset of \( \mathcal{H} \) with \( C \cap \text{int dom } f \neq \emptyset \), and that \( y \in \text{int dom } f \). Then \( \overrightarrow{P}^f_C(y) \) is a singleton contained in \( C \cap \text{int dom } f \).

Not surprisingly, not every Legendre function allows forward Bregman projections and the backward and forward Bregman projections are different notions (see, e.g., [1], [4] and [6] for details). In particular, we can find well-known functions satisfying the hypotheses of the following Fact 2.6 in [4, Examples 2.1 and 2.7]. We refer the interested readers to [1], [4] and [6] for details on the backward and forward Bregman projections.

**Definition 2.5.** [4, Definition 2.4] Suppose that \( f \in \Gamma_0(\mathcal{H}) \) is Legendre such that \( \text{dom } f^* \) is open. We say the function \( f \) allows forward Bregman projections if it satisfies the following properties.

(i) \( \nabla^2 f \) exists and is continuous on \( \text{int dom } f \).

(ii) \( D_f \) is convex on \( (\text{int dom } f)^2 \).

(iii) For every \( x \in \text{int dom } f \), \( D_f(x, \cdot) \) is strictly convex on \( \text{int dom } f \).

**Fact 2.6.** [4, Fact 2.6] Suppose that \( \mathcal{H} = \mathbb{R}^n \) and \( f \in \Gamma_0(\mathcal{H}) \) is Legendre such that \( \text{dom } f^* \) is open, that \( f \) allows forward Bregman projections, and that \( C \) is a closed convex subset of \( \mathcal{H} \) with \( C \cap \text{int dom } f \neq \emptyset \). Then \( (\forall x \in \text{int dom } f) \overrightarrow{P}^f_C(x) \) is a singleton contained in \( C \cap \text{int dom } f \).

### 3 Forward Bregman monotonicity

In this section, we show the weak convergence of the forward Bregman monotone sequence, which plays a critical role in the proof of our main result in the next section.

The following is a variant of the Bregman monotonicity defined in [3, Definition 1.2]. Note that both [3, Definition 1.2] and the following **Definition 3.1** are natural generalizations of the classical Fejér monotonicity. In view of the broad applications of Fejér monotonicity and Bregman monotonicity in the proof of the convergence of iterative algorithms (see, e.g., [5], [3], [20] and the references therein), we assume the forward Bregman monotonicity defined in **Definition 3.1** below is interesting on its own, and probably can be used in many other iterative algorithms under general Bregman distances.

**Definition 3.1.** A sequence \( (x_k)_{k \in \mathbb{N}} \) in \( \mathcal{H} \) is forward Bregman monotone with respect to a set \( C \subseteq \mathcal{H} \) if \( C \cap \text{int dom } f \neq \emptyset \), \( (x_k)_{k \in \mathbb{N}} \) lies in \( \text{int dom } f \), and

\[
(\forall c \in C \cap \text{int dom } f) \ (\forall k \in \mathbb{N}) \ D_f(x_{k+1}, c) \leq D_f(x_k, c).
\]
Fact 3.2. Suppose that \( f \in \Gamma_0(\mathcal{H}) \) with \( \text{int dom } f \neq \varnothing \), that \( f \) is Gâteaux differentiable on \( \text{int dom } f \), and that \( f \) is essentially strictly convex. Then \( f \) is strictly convex on \( \text{int dom } f \), which is equivalent to that

\[
(\forall x \in \text{int dom } f)(\forall y \in \text{int dom } f) \quad x \neq y \Rightarrow \langle x - y, \nabla f(x) - \nabla f(y) \rangle > 0.
\]

Proof. Because \( f \) is convex, by [5, Propositions 3.45(ii), 8.2 and 16.27], \( \text{int dom } f \) is a convex subset of \( \text{dom } \partial f \). Then employ Definition 2.2(ii) to see that \( f \) is strictly convex on \( \text{int dom } f \).

We obtain the last equivalence by applying [5, Proposition 17.10] to the function \( f + i_{\text{int dom } f} \).

Theorem 3.3. Suppose that \( f \in \Gamma_0(\mathcal{H}) \) with \( \text{int dom } f \neq \varnothing \) and that \( f \) is Gâteaux differentiable on \( \text{int dom } f \). Let \( C \) be a closed convex set in \( \mathcal{H} \) with \( C \cap \text{int dom } f \neq \varnothing \), and let \( (x_k)_{k \in \mathbb{N}} \) be in \( \text{int dom } f \) and be forward Bregman monotone with respect to \( C \). Then the following statements hold.

(i) \( (\forall c \in C \cap \text{int dom } f)(D_f(x_k, c))_{k \in \mathbb{N}} \) is decreasing, nonnegative and convergent.

(ii) Let \( \{y, z\} \subseteq C \cap \text{int dom } f \). Then the limit \( \lim_{k \to \infty} \langle \nabla f(y) - \nabla f(z), x_k \rangle \) exists.

(iii) Suppose that \( f \) is essentially strictly convex. Then \( (x_k)_{k \in \mathbb{N}} \) is bounded.

(iv) Suppose that \( f \) is essentially strictly convex. Then \( (x_k)_{k \in \mathbb{N}} \) weakly converges to some point in \( C \cap \text{int dom } f \) if and only if all weak sequential cluster points of \( (x_k)_{k \in \mathbb{N}} \) are in \( C \cap \text{int dom } f \).

Proof. (i): This is clear from Definitions 2.1 and 3.1.

(ii): According to Definition 2.1, for every \( k \in \mathbb{N} \),

\[
D_f(x_k, y) - D_f(x_k, z) = f(x_k) - f(y) - \langle \nabla f(y), x_k - y \rangle - \left( f(x_k) - f(z) - \langle \nabla f(z), x_k - z \rangle \right)
\]

\[
\Leftrightarrow \langle \nabla f(z) - \nabla f(y), x_k \rangle = D_f(x_k, y) - D_f(x_k, z) + f(y) - f(z) - \langle \nabla f(y), y \rangle + \langle \nabla f(z), z \rangle.
\]

which, via (i), yields the desired result.

(iii): Let \( c \in C \cap \text{int dom } f \). Then the boundedness of \( (D_f(x_k, c))_{k \in \mathbb{N}} \) and Fact 2.3(i) imply that \( (x_k)_{k \in \mathbb{N}} \) is bounded.

(iv): “\( \Rightarrow \)” This is trivial.

“\( \Leftarrow \)” Suppose that all weak sequential cluster points of \( (x_k)_{k \in \mathbb{N}} \) are in \( C \cap \text{int dom } f \). Because the boundedness of \( (x_k)_{k \in \mathbb{N}} \) is proved in (iii) above, bearing [5, Lemma 2.46] in mind, we know that it remains to prove that \( (x_k)_{k \in \mathbb{N}} \) has at most one weak sequential cluster point in \( C \cap \text{int dom } f \). Assume that \( \bar{x} \) and \( \hat{x} \) are two weak sequential cluster points of \( (x_k)_{k \in \mathbb{N}} \) in \( C \cap \text{int dom } f \). Then there exist subsequences \( (x_{k_n})_{n \in \mathbb{N}} \) and \( (x_{k_t})_{t \in \mathbb{N}} \) of \( (x_k)_{k \in \mathbb{N}} \) such that \( x_{k_n} \rightharpoonup \bar{x} \) and \( x_{k_t} \rightharpoonup \hat{x} \). Then

\[
\langle \nabla f(\bar{x}) - \nabla f(\hat{x}), x_{k_n} \rangle \to \langle \nabla f(\bar{x}) - \nabla f(\hat{x}), \bar{x} \rangle; \quad (3.1a)
\]

\[
\langle \nabla f(\bar{x}) - \nabla f(\hat{x}), x_{k_t} \rangle \to \langle \nabla f(\bar{x}) - \nabla f(\hat{x}), \hat{x} \rangle. \quad (3.1b)
\]

On the other hand, apply (ii) with \( y = \bar{x} \) and \( z = \hat{x} \) to deduce that \( \lim_{k \to \infty} \langle \nabla f(\bar{x}) - \nabla f(\hat{x}), x_k \rangle \) exists. This combined with (3.1) implies that \( \langle \nabla f(\bar{x}) - \nabla f(\hat{x}), \bar{x} \rangle = \langle \nabla f(\bar{x}) - \nabla f(\hat{x}), \hat{x} \rangle \), that is,

\[
\langle \nabla f(\bar{x}) - \nabla f(\hat{x}), \bar{x} - \hat{x} \rangle = 0,
\]

which, combining with Fact 3.2, yields \( \hat{x} = \bar{x} \). Altogether, the proof is complete.
4 Bregman circumcenter methods

As we mentioned in the introduction, projection methods based on Bregman distances have broad applications and some circumcentered methods accelerate the method of cyclic projections with Bregman projections for finding the best approximation point onto or a feasibility point in the intersection of finitely many closed convex sets. In this section, we introduce backward and forward Bregman (pseudo-)circumcenter mappings and methods. We shall also investigate the convergence of the forward Bregman circumcenter method.

Throughout this section, suppose that \( f \in \Gamma_0(\mathcal{H}) \) with \( \text{int dom } f \neq \emptyset \) and that \( f \) is Gâteaux differentiable on \( \text{int dom } f \). Set \( I := \{1, \ldots, m\} \).

**Bregman circumcenter mappings**

Henceforth, for every \( D \subseteq \mathcal{H} \), denote by \( \mathcal{P}(D) \) the set of all nonempty subsets of \( D \) containing finitely many elements.

Suppose that \( K \in \mathcal{P}(\text{dom } f) \). Set

\[
\overrightarrow{E}_f(K) := \{ q \in \text{int dom } f : (\forall x \in K) \ D_f(x, q) \text{ is a singleton} \}. 
\]

Suppose additionally \( K \in \mathcal{P}(\text{int dom } f) \). Denote by

\[
\overleftarrow{E}_f(K) := \{ p \in \text{dom } f : (\forall y \in K) \ D_f(p, y) \text{ is a singleton} \}. 
\]

**Definition 4.1.** [12, Definition 3.5] Let \( K \in \mathcal{P}(\text{int dom } f) \).

(i) Define the *backward Bregman circumcenter operator* \( \overleftarrow{CC} \) w.r.t. \( f \) as

\[
\overleftarrow{CC} : \mathcal{P}(\text{int dom } f) \to 2^\mathcal{H} : K \mapsto \text{aff}(K) \cap \overleftarrow{E}_f(K).
\]

(ii) Define the *backward Bregman pseudo-circumcenter operator* \( \overleftarrow{CC}^{ps} \) w.r.t. \( f \) as

\[
\overleftarrow{CC}^{ps} : \mathcal{P}(\text{int dom } f) \to 2^\mathcal{H} : K \mapsto \text{aff}(\nabla f(K)) \cap \overleftarrow{E}_f(K).
\]

In particular, for every \( K \in \mathcal{P}(\text{int dom } f) \), we call the element in \( \overleftarrow{CC}(K) \) and \( \overleftarrow{CC}^{ps}(K) \) Backward Bregman circumcenter and Backward Bregman pseudo-circumcenter of \( K \), respectively.

**Definition 4.2.** [12, Definition 4.5] Let \( K \in \mathcal{P}(\text{dom } f) \).

(i) Define the *forward Bregman circumcenter operator* w.r.t. \( f \) as

\[
\overrightarrow{CC} : \mathcal{P}(\text{dom } f) \to 2^\mathcal{H} : K \mapsto \text{aff}(K) \cap \overrightarrow{E}_f(K).
\]

(ii) Define the *forward Bregman pseudo-circumcenter operator* w.r.t. \( f \) as

\[
\overrightarrow{CC}^{ps} : \mathcal{P}(\text{dom } f) \to 2^\mathcal{H} : K \mapsto (\nabla f^\ast(\text{aff } K)) \cap \overrightarrow{E}_f(K).
\]

In particular, for every \( K \in \mathcal{P}(\text{dom } f) \), we call the element in \( \overrightarrow{CC}(K) \) and \( \overrightarrow{CC}^{ps}(K) \) forward Bregman circumcenter and forward Bregman pseudo-circumcenter of \( K \), respectively.
Suppose that $D_f$ is symmetric, that is, $\text{dom } f = \text{int } \text{dom } f$ and $(\forall \{x,y\} \subseteq \text{dom } f) \ D_f(x,y) = D_f(y,x)$. Then the backward Bregman circumcenter and forward Bregman circumcenter are consistent. In particular, if $f := \frac{1}{2} \| \cdot \|^2$, then $\nabla f = \text{Id}$; and hence, the notions backward Bregman circumcenter, backward Bregman pseudo-circumcenter, forward Bregman circumcenter and forward Bregman pseudo-circumcenter are all the same and reduce to the circumcenter defined in [7, Definition 3.4] under the Euclidean distance.

From now on, for every set-valued operator $A : \mathcal{H} \to 2^\mathcal{H}$, if $A(x)$ is a singleton for some $x \in \mathcal{H}$, we sometimes by slight abuse of notation allow $A(x)$ to stand for its unique element. The intended meaning should be clear from the context.

**Fact 4.3.** [12, Corollary 5.1] Suppose that

$$K := \{q_0, q_1, \ldots, q_m\} \subseteq \text{dom } f \text{ is nonempty.}$$

Then the following assertions hold.

1. Suppose that $\mathcal{H} = \mathbb{R}^n$, that $f$ is Legendre such that $\text{dom } f^* \text{ is open, that } \left(\mathbb{E}_f(K) \neq \emptyset, \text{ that } f \text{ allows forward Bregman projections, that } \text{aff } (K) \subseteq \text{int } \text{dom } f, \text{ and that } \nabla f(\text{aff } (K)) \text{ is a closed affine subspace. Then } (\forall z \in \mathbb{E}_f(K)) \ P^f_{\text{aff } (K)}(z) \in \mathbb{C}(K).$

2. Suppose that $\mathbb{E}_f(K) \neq \emptyset$ and that $K \subseteq \text{int } \text{dom } f$ and $\text{aff } (\nabla f(K)) \subseteq \text{dom } f$. Then $(\forall z \in \mathbb{E}_f(K)) \ \mathbb{C}(K) = P_{\text{aff } (\nabla f(K))}(z)$.

3. Suppose that $K \subseteq \text{int } \text{dom } f$ and $\text{aff } (\nabla f(K)) \subseteq \text{dom } f$, and that $\nabla f(q_0), \nabla f(q_1), \ldots, \nabla f(q_m)$ are affinely independent. Then $\mathbb{C}(K)$ uniquely exists and has an explicit formula.

4. Suppose that $f$ is Legendre, that $\text{aff } (K) \cap \text{int } \text{dom } f \neq \emptyset$, and that $\mathbb{E}_f(K) \neq \emptyset$. Then $(\forall z \in \mathbb{E}_f(K)) \ P^f_{\text{aff } (K)}(z) \in \mathbb{C}(K).$

5. Suppose that $f$ is Legendre, and that $\text{aff } (K) \subseteq \text{int } \text{dom } f^*$ and $\mathbb{E}_f(K) \neq \emptyset$. Then $(\forall z \in \mathbb{E}_f(K)) \ \mathbb{C}(K) = \mathbb{C}^f(\text{aff } (\nabla f(z)))$.

6. Suppose that $f$ is Legendre, that $\text{aff } (K) \subseteq \text{int } \text{dom } f^*$, and that $q_0, q_1, \ldots, q_m$ are affinely independent. Then $\mathbb{C}(K)$ uniquely exists and has an explicit formula.

Note that we have particular examples in [12] with functions $f \neq \frac{1}{2} \| \cdot \|^2$ and sets $K$ such that the hypotheses of each item of Fact 4.3 above hold.

**Definition 4.4.** Let $t \in \mathbb{N} \setminus \{0\}$. Suppose that $G_1, \ldots, G_t$ are operators from $\mathcal{H}$ to $\mathcal{H}$. Set

$$S := \{G_1, \ldots, G_t\} \text{ and } (\forall x \in \mathcal{H}) \ S(x) := \{G_1x, \ldots, G_tx\}.$$

The forward Bregman circumcenter mapping induced by $S$ is

$$\mathbb{C}_S : \mathcal{H} \to 2^{\mathcal{H}} : y \mapsto \mathbb{C}(S(y)),$$

that is, for every $y \in \mathcal{H}$, if the forward Bregman circumcenter of the set $S(y)$ defined in Definition 4.2(i) does not exist, then $\mathbb{C}_S(y) = \emptyset$. Otherwise, $\mathbb{C}_S(y)$ is the set of points $v$ satisfying the two conditions below:

1. $v \in \text{aff } (S(x)) \cap \text{int } \text{dom } f$, and
(ii) $D_f(G_1x, v) = \cdots = D_f(G_1x, v)$.

Let $C$ be a subset of $H$. If $(\forall y \in C) \ C^S y$ contains at most one element, we say that $C^S$ is at most single-valued on $C$. Naturally, if $C = H$, then we omit the phrase “on $C$”.

Analogously, we define the forward Bregman pseudo-circumcenter mapping $C^S$ induced by $S$, backward Bregman circumcenter mapping $C^S$ induced by $S$, and backward Bregman pseudo-circumcenter mapping $C^pS$ induced by $S$ with replacing the forward Bregman circumcenter operator $C^S$ above by the corresponding operator, $C^pS$, $C^p$, and $C^p$, respectively.

We can also directly deduce the following result by [9, Theorem 3.3(ii)].

**Corollary 4.5.** Suppose that $f := \frac{1}{2}||z||^2$. Let $(\forall i \in I) T_i : H \rightarrow H$ be a linear isometry. Set $S := \{T_1, \ldots, T_m\}$ and $(\forall x \in H) S(x) = \{T_1x, \ldots, T_mx\}$. Then $\forall z \in S(x)$, $\forall z \in \text{Int dom } f$ and $D_f(z, z) = D_f(T_1z, z) = \cdots = D_f(T_mz, z)$.

**Proof.** Notice that $\nabla f = \text{Id}$ and $\text{dom } f = H$ in this case. Because, due to [12, Corollary 5.6(ii)], $0 \in \bigcap_{i=1}^n \text{Fix } T_j \subseteq T^* f(K) = T^* f(K)$, the required result follows easily from Fact 4.3(ii) or Fact 4.3(v).

Henceforth, suppose $T_1, \ldots, T_m$ are operators from $H$ to $H$ with $\text{Int dom } f \cap (\bigcap_{i=1}^n \text{Fix } T_i) \neq \emptyset$. Set

$$S := \{\text{Id}, T_1, \ldots, T_m\} \quad \text{and} \quad (\forall x \in H) S(x) := \{x, T_1x, \ldots, T_mx\}.$$  

**Lemma 4.6.** Suppose that $f$ is essentially strictly convex. Then the following statements hold.

(i) $\text{Fix } C^S = \text{Int dom } f \cap (\bigcap_{i=1}^n \text{Fix } T_i)$ and $\text{Fix } C^pS = \text{Int dom } f \cap (\bigcap_{i=1}^n \text{Fix } T_i) \cap \text{Fix } \nabla f$.

(ii) $\text{Fix } C^S = \text{Int dom } f \cap (\bigcap_{i=1}^n \text{Fix } T_i) \cap \text{Fix } \nabla f$.

**Proof.** (i): Let $z \in H$.

$$z \in \text{Fix } C^S \iff z \in \text{Int dom } f \cap (\bigcap_{i=1}^n \text{Fix } T_i) \cap \text{Fix } \nabla f$$

which guarantees that $\text{Fix } C^S = \text{Int dom } f \cap (\bigcap_{i=1}^n \text{Fix } T_i)$.

Similarly,

$$z \in \text{Fix } C^pS \iff z \in \text{Int dom } f \cap (\bigcap_{i=1}^n \text{Fix } T_i) \cap \text{Fix } \nabla f$$

which deduces that $\text{Fix } C^pS = \text{Int dom } f \cap (\bigcap_{i=1}^n \text{Fix } T_i) \cap \text{Fix } \nabla f$.

In addition, clearly, $\text{Id} \in S$ entails that $\text{Fix } \nabla f \subseteq \text{Fix } (\nabla f)$ and $\text{Fix } \nabla f \subseteq (\bigcap_{i=1}^n \text{Fix } T_i) \cap \text{Fix } (\nabla f)$.

On the other hand, $\forall y \in (\bigcap_{i=1}^n \text{Fix } T_i) \cap \text{Fix } (\nabla f)$, $y \in \text{Fix } (\nabla f)$ and $\forall y \in (\bigcap_{i=1}^n \text{Fix } T_i) \cap \text{Fix } (\nabla f)$.

Altogether, $\text{Fix } C^pS = \text{Int dom } f \cap (\bigcap_{i=1}^n \text{Fix } T_i) \cap \text{Fix } \nabla f$.

(ii): The proof is similar to the proof of (i) above.

---

1A mapping $T : H \rightarrow H$ is said to be isometric or an isometry if $(\forall x \in H) \ 0 \in (\bigcap_{i=1}^n \text{Fix } T_i) \cap \text{Fix } \nabla f$. Therefore, $\forall y \in (\bigcap_{i=1}^n \text{Fix } T_i) \cap \text{Fix } (\nabla f)$ and $\forall y \in (\bigcap_{i=1}^n \text{Fix } T_i) \cap \text{Fix } (\nabla f)$.
Bregman isometry and Browder’s Demiclosedness Principle

In this subsection, we generalize the traditional isometry and Browder’s Demiclosedness Principle from the Euclidean distance to general Bregman distances, and investigate the Bregman isometry and Browder’s Demiclosedness Principle which play critical roles in the main result in this section. The Bregman isometry and Browder’s Demiclosedness Principle are interesting in its own right.

Definition 4.7. Let $C$ be a nonempty subset of $\mathcal{H}$ and let $T : C \to \mathcal{H}$. We say $T$ is Bregman isometric (or a Bregman isometry) w.r.t. $f$, if $(\forall \{x, y\} \subseteq \text{dom } f \times \text{int dom } f)$ $(Tx, Ty) \in \text{dom } f \times \text{int dom } f,$ and

$$D_f(Tx, Ty) = D_f(x, y). \quad (4.1)$$

In view of the definition, it is trivial that given an arbitrary function $g$, the identity operator $\text{Id}$ is Bregman isometric w.r.t. $g$.

Notice that if $f = \frac{1}{2}\|\cdot\|^2$, then the Bregman isometry deduces the traditional isometric (see, e.g., [9, Lemma 2.23] for examples of isometries under the Euclidean distance).

The following definition is a generalization of the well-known Browder’s Demiclosedness Principle [19, Theorem 3(a)]. In view of [5, Corollary 4.25], when $f = \frac{1}{2}\|\cdot\|^2$, the Bregman Browder’s Demiclosedness Principle holds for all nonexpansive operators.

Definition 4.8. Let $C$ be a nonempty subset of $\mathcal{H}$, let $T : C \to \mathcal{H}$ and let $x \in \text{int dom } f$. We say the Bregman Browder’s demiclosedness principle associated with $f$ holds at $x \in C \cap \text{int dom } f$ for $T$ if for every sequence $(x_k)_{k \in \mathbb{N}}$ in $\text{int dom } f$, $(Tx_k)_{k \in \mathbb{N}}$ in $\text{int dom } f$ and

$$x_k \to x \quad D_f(x_k, Tx_k) \to 0 \quad \Rightarrow \quad x \in \text{Fix } T. \quad (4.2)$$

In addition, we say $T$ is $f$-demiclosed, if the Bregman Browder’s demiclosedness principle associated with $f$ holds for every $x \in C \cap \text{int dom } f$.

Because $\text{Fix } \text{Id} = \mathcal{H}$, given an arbitrary function $g$, $\text{Id}$ is $g$-demiclosed.

The following examples are used to illustrate the two new concepts above.

Example 4.9. Suppose that $\mathcal{H} = \mathbb{R}^n$. Denote by $(\forall x \in \mathbb{R}^n) x = (x_i)_{i=1}^n$. Define $f : x \mapsto -\sum_{i=1}^n \ln(x_i)$, with $\text{dom } f = [0, +\infty)^n$, which is the Burg entropy.

Then the following statements hold.

(i) $f$ is Legendre such that $\text{dom } f$ and $\text{dom } f^*$ are open.

(ii) Define $T : \mathbb{R}^n \to \mathbb{R}^n : (x_i)_{i \in J} \mapsto (c_i x_i)_{i \in J}$. Set $J := \{1, \ldots, n\}$. Let $(c_i)_{i \in J} \in \mathbb{R}^n$ with $(\forall i \in J)$ $c_i \in \mathbb{R}_{++}$. Then:

(a) $T$ is Bregman isometric w.r.t. $f$.

(b) Suppose that there exists $j \in J$ such that $c_j \neq 1$. Then $T$ is $f$-demiclosed.

(iii) Suppose that $\mathcal{H} = \mathbb{R}$. Let $c \in \mathbb{R}_{++}$ and $\mu \in \mathbb{R}_{++} \setminus \{1\}$. Define $T : \mathbb{R} \to \mathbb{R} : t \mapsto ct^\mu$. Then:

(a) $\text{int dom } f \cap \text{Fix } T = \{c^{-\frac{1}{\mu-1}}\}$, which is nonempty closed and convex.

(b) $T$ is $f$-demiclosed.

Proof. (i): This is immediately form [4, Examples 2.1].

(ii)(a): This is clear from Definition 2.1 and the definitions of $f$ and $T$. 

Proof. (ii)(a): This is clear from Definition 2.1 and the definitions of $f$ and $T$. 

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(ii)(b): Let \( (x^k)_{k \in \mathbb{N}} \) be in \( \text{dom} \ f \). Because \((\forall x \in \text{dom} \ f) \ Tx \in \text{dom} \ f\), we have \((Tx^k)_{k \in \mathbb{N}} \) is a sequence in \( \text{dom} \ f \).

Suppose that \( x^k \to \bar{x} \) with \( \bar{x} := \left( \bar{x}_i \right)_{i \in J} \in ]0, +\infty[^n \). Denote by \((\forall k \in \mathbb{N}) \ x^k := \left( x^k_i \right)_{i \in J} \). Now,

\[
D_f(x^k, Tx^k) = f(x^k) - f(Tx^k) - \langle \nabla f(Tx^k), x^k - Tx^k \rangle
\]

\[
= - \sum_{i=1}^{n} \ln(x_i^k) + \sum_{i=1}^{n} \ln(c_i x_i^k) + \sum_{i=1}^{n} \frac{x_i^k - c_i x_i^k}{c_i x_i^k}
\]

\[
= \sum_{i=1}^{n} \ln c_i x_i^k + \sum_{i=1}^{n} \frac{x_i^k - c_i x_i^k}{c_i x_i^k},
\]

which implies that

\[
D_f(x^k, Tx^k) \to \sum_{i=1}^{n} \left( \ln(c_i) + \frac{1}{c_i} - 1 \right).
\]

On the other hand, consider the function \( g : \mathbb{R}_{++} \to \mathbb{R} : t \mapsto \ln(t) + \frac{1}{t} - 1 \). By some easy calculus, \( (\forall t \in \mathbb{R}_{++} \setminus \{1\}) \ g(t) < g(1) = 0 \). Because there exists \( j \in J \) such that \( c_j \neq 1 \), we know that 

\[
\sum_{i=1}^{n} \left( \ln(c_i) + \frac{1}{c_i} - 1 \right) < 0.
\]

Hence, the hypothesis of (4.2) is always false, which deduces that (4.2) is true. Therefore, due to Definition 4.8, \( T \) is \( f \)-demiclosed.

(iii)(a): This is trivial.

(iii)(b): Let \( a \in ]0, +\infty[ \). Let \( (a_k)_{k \in \mathbb{N}} \) be in \( \text{int} \text{ dom} \ f \) such that \( a_k \to a \). Then clearly \((Ta_k)_{k \in \mathbb{N}} \) is a sequence in \( ]0, +\infty[ \). Suppose that \( D_f(a_k, Ta_k) \to 0 \). Note that

\[
D_f(a_k, Ta_k) = - \ln(a_k) + \ln(ca_k^\mu) + \frac{a_k - ca_k^\mu}{ca_k^\mu} \to \ln(ca^\mu) - \ln(a) + \frac{a^{1-\mu}}{c} - 1.
\]

Denote by \( h : \mathbb{R}_{++} \to \mathbb{R} : t \mapsto \ln(ct^\mu) - \ln(t) + \frac{\mu t^{-\mu}}{\mu - 1} - 1 \). Then \( (\forall t \in \mathbb{R}_{++} \setminus \{1\}) \ h'(t) = (\mu - 1) \left( \frac{1}{t} - \frac{1}{c} \right) \). Then by considering the two cases \( \mu > 1 \) and \( \mu < 1 \) separately, we easily get that \( \ln(ca^\mu) - \ln(a) + \frac{a^{1-\mu}}{c} - 1 \) implies that \( a = ca^\mu \), that is, \( a = c^{-\frac{1}{\mu - 1}} \in \text{Fix} \ T \). Altogether, by Definition 4.8, \( T \) is \( f \)-demiclosed.

Consider our examples \( f \) and \( T \) in Example 4.9(ii). Notice that if there exists \( j \in J \) such that \( c_j \neq 1 \), then \( \text{int} \text{ dom} \ f \cap \left( \cap_{i \in J} \text{Fix} \ T_i \right) = \emptyset \).

The following example of \( f \) and \( T \) satisfies all requirements in our main result Theorem 4.12 below.

Example 4.10. Suppose that \( \mathcal{H} = \mathbb{R}^n \). Denote by \((\forall x \in \mathbb{R}^n) \ x = (x_i)_{i=1}^n \). Define \( f : x \mapsto \sum_{i=1}^{n} x_i \ln(x_i) + (1 - x_i) \ln(1 - x_i) \), with \( \text{dom} \ f = [0, 1]^n \), which is the Fermi-Dirac entropy. Set \( J := \{1, \ldots, n\} \). Then the following statements hold.

(i) \( f \) is Legendre with \( \text{dom} f^* \) open, and allows forward Bregman projections.

(ii) \( \text{Id} \) is \( f \)-demiclosed.

(iii) Define \( T : \mathbb{R}^n \to \mathbb{R}^n : (x_i)_{i=1}^n \mapsto (1 - x_i)_{i=1}^n \). Then:

(a) \( T \) is Bregman isometric w.r.t. \( f \).

(b) \( T \) is \( f \)-demiclosed.

(c) \( \text{Fix} T \cap \text{int} \text{ dom} \ f = \{ (\frac{1}{2})_{i \in J} \} \) is nonempty, closed and convex.
(iv) Let \( \Lambda \) be a subset of \( J \). Define \( T : \mathbb{R}^n \to \mathbb{R}^n \) by \( (\forall x \in \mathbb{R}^n) \) \( T x := (y_i)_{i=1}^n \) where \( (\forall i \in J \setminus \Lambda) y_i = x_i \) and \( (\forall i \in \Lambda) y_i = (1 - x_i) \).

(a) Fix \( T = \{ x \in \mathbb{R}^n : (\forall i \in \Lambda) x_i = \frac{1}{2} \} \).
(b) \( T \) is Bregman isometric w.r.t. \( f \).
(c) \( T \) is \( f \)-demiclosed.

Proof. (i): This follows immediately form [4, Examples 2.1 and 2.7].

(ii): This is trivial.

(iii)(a): This follows easily from Definition 2.1 and the definitions of \( f \) and \( T \).

(iii)(b): Let \( \bar{x} \in [0,1]^n \). Denote by \( \bar{x} = (x_i)_{i=1}^n \). Let \( (x^{(k)})_{k \in \mathbb{N}} \) be a sequence in \([0,1]^n\) such that \( x^{(k)} \to \bar{x} \). Denote by \( (\forall k \in \mathbb{N}) x^{(k)} = (x^{(k)}_i)_{i \in \mathbb{1}} \). In view of the definition of \( T \), \( (T x_k)_{k \in \mathbb{N}} \) is also a sequence in \([0,1]^n\). Suppose that \( D_f(x_k, T x_k) \to 0 \). Notice that

\[
D_f(x_k, T x_k) = \sum_{i=1}^n \left( x^{(k)}_i \ln(x^{(k)}_i) + (1 - x^{(k)}_i) \ln(1 - x^{(k)}_i) \right) - \sum_{i=1}^n \left( (1 - x^{(k)}_i) \ln(1 - x^{(k)}_i) + x^{(k)}_i \ln(x^{(k)}_i) \right)
\]

\[= - \sum_{i=1}^n \left( x^{(k)}_i - (1 - x^{(k)}_i) \right) \ln \left( \frac{1 - x^{(k)}_i}{x^{(k)}_i} \right) \]

Consider the function \( g : \mathbb{R} \to \mathbb{R} : t \mapsto - (t - (1 - t)) \ln(\frac{1-t}{t}) \). Because \( (\forall t \in [0,1[) g''(t) > 0 \) and \( g'(\frac{1}{2}) = 0 \), we know that \( (\forall t \in [0,1[ \setminus \{ \frac{1}{2} \}) g(t) > 0 \). Hence, clearly, \( - \sum_{i=1}^n \left( \bar{x}_i - (1 - \bar{x}_i) \right) \ln \left( \frac{1 - \bar{x}_i}{\bar{x}_i} \right) \)

0 implies that \( \bar{x} = (\frac{1}{2})_{i \in \mathbb{1}} \in \text{Fix } T \). Hence, the assertion is true.

(iii)(c): The required results are trivial.

(iv): According to the definitions of \( T \) and \( f \), it is easy to see that

\[
(\forall x \in \text{dom } f) (\forall y \in \text{dom } f) \quad f(x) - f(T x) = 0 \quad \text{and} \quad \langle \nabla f(T y), T x - T y \rangle = \langle \nabla f(y), x - y \rangle.
\]

(iv)(a): This is clear.

(iv)(b): This follows immediately from Definition 4.7 and Definition 2.1.

(iv)(c): Applying a proof similar to that of (iii)(b) and invoking (iv)(a), we obtain the required result.

Convergence of forward Bregman circumcenter methods

Motivated by the Lemma 4.6, to find a point in the intersection \( \cap_{i \in \mathbb{1}} \text{Fix } T_i \), we prefer the backward and forward Bregman circumcenter mappings induced by \( S \) rather than the backward and forward Bregman pseudo-circumcenter mappings induced by \( S \) with smaller fixed point sets. Moreover, notice that comparing with the hypothesis of Fact 4.3(i) on the existence of the backward Bregman circumcenter, we don’t have strong requirement for \( f \) in the Fact 4.3(iv) on the existence of the forward Bregman circumcenter.

Therefore, we consider only on the convergence of sequences of iterations generated by forward Bregman circumcenter mappings in this work.

The forward Bregman circumcenter method generates the sequence of iterations of the forward Bregman circumcenter mapping.

According to [8, Theorem 4.3(ii)] and [9, Theorem 3.3(ii)], the circumcenter mappings induced by finite sets of isometries (see [9, Definition 2.27 for the exact definition), are special operators \( G \) satisfying conditions in Theorem 4.11 below.
Theorem 4.11. Suppose that $f$ is Legendre. Let $C$ be a nonempty subset of $\text{int} \text{ dom} \ f$. Suppose that $G : C \to C$ satisfies that $\text{Fix} \ G \cap \text{int} \text{ dom} \ f \neq \emptyset$ and $\text{Fix} \ G$ is a closed convex subset of $H$, and that $(\forall x \in C \cap \text{int} \text{ dom} \ f)$ \((\forall z \in \text{Fix} \ G \cap \text{int} \text{ dom} \ f) G(x) = \overline{P}^{f}_{\text{aff}(S(x))}z\). Let $x \in C \cap \text{int} \text{ dom} \ f$. Then the following statements hold.

(i) \((G^{k}x)_{k \in \mathbb{N}}\) is a well-defined sequence in \(C \cap \text{int} \text{ dom} \ f\).

(ii) \((G^{k}x)_{k \in \mathbb{N}}\) is forward Bregman monotone with respect to $\text{Fix} \ G$. Consequently, $(\forall z \in \text{Fix} \ G \cap \text{int} \text{ dom} \ f)$ $(D_{f}(G^{k}x, z))_{k \in \mathbb{N}}$ converges.

(iii) $\lim_{k \to \infty} D_{f}(G^{k}x, G^{k+1}x) = 0$.

(iv) Suppose that the Bregman Browder’s Demiclosedness Principle associated with $f$ holds for $G$, and that all weak sequential cluster points of $(G^{k}x)_{k \in \mathbb{N}}$ lie in $\text{int} \text{ dom} \ f$. Then $(G^{k}x)_{k \in \mathbb{N}}$ weakly converges to a point in $\text{int} \text{ dom} \ f \cap \text{Fix} \ G$.

Proof. Invoking the definition of the operator $G : C \to C$ and Fact 2.4,

\[
(\forall y \in C \cap \text{int} \text{ dom} \ f) \quad (\forall z \in \text{Fix} \ G \cap \text{int} \text{ dom} \ f) \quad G(y) = \overline{P}^{f}_{\text{aff}(S(y))}z \in C \cap \text{int} \text{ dom} \ f. \tag{4.3}
\]

Hence, use $y \in \text{aff}(S(y))$ and apply [12, Theorem 2.9(i)] with $U = \text{aff}(S(x))$ to yield that

\[
(\forall y \in C \cap \text{int} \text{ dom} \ f) \quad (\forall z \in \text{Fix} \ G \cap \text{int} \text{ dom} \ f) \quad D_{f}(y, z) = D_{f}(y, G(y)) + D_{f}(G(y), z). \tag{4.4}
\]

(i): This is clear from (4.3) by induction.

(ii): Taking (4.4) and Definition 2.1 into account, we deduce that

\[
(\forall y \in C \cap \text{int} \text{ dom} \ f) \quad (\forall z \in \text{int} \text{ dom} \ f \cap \text{Fix} \ G) \quad D_{f}(y, z) \geq D_{f}(Gy, z). \tag{4.5}
\]

Employing (i) above and Definition 3.1, we know that $(G^{k}x)_{k \in \mathbb{N}}$ is forward Bregman monotone with respect to $\text{Fix} \ G$, which, combining with Theorem 3.3(i), implies the last assertion.

(iii): Let $z \in \text{Fix} \ G \cap \text{int} \text{ dom} \ f$. For every $k \in \mathbb{N}$, substitute $y$ in (4.4) by $G^{k}x$ to deduce that $D_{f}(G^{k}x, z) = D_{f}(G^{k}x, G^{k+1}x) + D_{f}(G^{k+1}x, z)$ which implies that

\[
\sum_{k \in \mathbb{N}} D_{f}(G^{k}x, G^{k+1}x) = D_{f}(x, z) - \lim_{t \to \infty} D_{f}(G^{t}x, z) < \infty,
\]

where the existence of the limit $\lim_{t \to \infty} D_{f}(G^{t}x, z)$ is from (ii). Therefore, using the nonnegativity of the Bregman distance, we yield (iii).

(iv): Let $z$ be a weak sequential cluster point of $(G^{k}x)_{k \in \mathbb{N}}$. Then there exists a subsequence $(G^{k_{j}}x)_{j \in \mathbb{N}}$ of $(G^{k}x)_{k \in \mathbb{N}}$ such that $G^{k_{j}}x \to z$. Notice that, in view of (iii) above, $D_{f}(G^{k}x, G^{k+1}x) \to 0$. Due to the assumption, $z \in \text{int} \text{ dom} \ f$. Hence, utilizing the assumption that the Bregman Browder’s Demiclosedness Principle holds for $G$, we observe that

\[
G^{k_{j}}x \to z \quad D_{f}(G^{k_{j}}x, G(G^{k_{j}}x)) \to 0 \quad \Rightarrow z \in \text{Fix} \ G
\]

which implies that all weak sequential cluster points of $(G^{k}x)_{k \in \mathbb{N}}$ lie in $\text{Fix} \ G \cap \text{int} \text{ dom} \ f$, since $z$ is an arbitrary cluster point of $(G^{k}x)_{k \in \mathbb{N}}$. Therefore, via Theorem 3.3(iv), $(G^{k}x)_{k \in \mathbb{N}}$ weakly converges to some point in $\text{int} \text{ dom} \ f \cap \text{Fix} \ G$. \(\blacksquare\)
Theorem 4.12(vi)(a) and Theorem 4.12(vi)(b) reduce to the beautiful [8, Theorem 3.17] and [9, Theorem 4.7], respectively, when \( f = \frac{1}{2}\|\cdot\|^{2} \). Notice that the circumcenter mappings studied in [9, 10, 11], and [21] under the Euclidean distance are all single-valued operators, and that all of the examples of backward and forward Bregman (pseudo)-circumcenters presented in [12] are singletons. So, our assumption “\( \overline{CC}_S \) is at most single-valued on \( \text{int dom } f \)” in the following Theorem 4.12 is not too weird. In addition, it is clear that if \( f = \frac{1}{2}\|\cdot\|^{2} \), then the following condition (4.6) is a direct result from the triangle inequality.

**Theorem 4.12.** Suppose that \( f \) is Legendre, that \( \text{int dom } f \cap (\bigcap_{i \in I} \text{Fix } T_i) \) is nonempty closed and convex, that \( (\forall i \in I) \ T_i \) is Bregman isometric w.r.t. \( f \), and that \( \overline{CC}_S \) is at most single-valued on \( \text{int dom } f \). Let \( y \in \text{int dom } f \). Then the following statements hold.

(i) \( (\forall x \in \text{dom } f) \) \( \text{int dom } f \cap (\bigcap_{i \in I} \text{Fix } T_i) \subseteq \overline{E}_{f}(S(x)) \).

(ii) \( (\forall z \in \text{int dom } f \cap (\bigcap_{i \in I} \text{Fix } T_i)) \overline{CC}_S y = \overline{P}_{\text{aff}(S(y))}(z) \in \text{int dom } f. \)

(iii) \( (\overline{CC}_S^k y)_{k \in \mathbb{N}} \) is a well-defined sequence in \( \text{int dom } f \).

(iv) \( (\overline{CC}_S^k y)_{k \in \mathbb{N}} \) is forward Bregman monotone with respect to \( \bigcap_{i \in I} \text{Fix } T_i. \) Consequently, \( (\forall z \in \text{int dom } f \cap (\bigcap_{i \in I} \text{Fix } T_i)) \left(D_f \left(\overline{CC}_S^k y, z\right)\right)_{k \in \mathbb{N}} \) converges.

(v) \( \lim_{k \to \infty} D_f \left(\overline{CC}_S^k y, \overline{CC}_S^{k+1} y\right) = 0. \)

(vi) Suppose that \( (\forall i \in I) \ T_i \) is \( f \)-demiclosed, and that for every sequence \( (u_k)_{k \in \mathbb{N}} \) in \( \text{int dom } f \),

\[
D_f(u_k, \overline{CC}_S u_k) \to 0 \quad (\forall i \in I) \quad D_f(T_i u_k, \overline{CC}_S u_k) \to 0 \quad \Rightarrow \quad (\forall i \in I) \quad D_f(u_k, T_i u_k) \to 0. \tag{4.6}
\]

Then the following hold.

(a) \( \overline{CC}_S \) is \( f \)-demiclosed.

(b) Suppose that all weak sequential cluster points of \( (\overline{CC}_S^k y)_{k \in \mathbb{N}} \) lie in \( \text{int dom } f \). Then \( (\overline{CC}_S^k y)_{k \in \mathbb{N}} \) weakly converges to some point in \( \text{int dom } f \cap (\bigcap_{i \in I} \text{Fix } T_i) \).

**Proof.** Because \( (\forall i \in I) \ T_i \) is Bregman isometric, we know that \( (\forall i \in I) \ (\forall (x, y) \in \text{dom } f \times \text{int dom } f) \ (T_i x, T_i y) \in \text{dom } f \times \text{int dom } f \), and that

\[
(\forall i \in I) \quad (\forall (x, y) \subseteq \text{dom } f \times \text{int dom } f) \quad D_f(T_i x, T_i y) = D_f(x, y). \tag{4.7}
\]

(i): Let \( x \in \text{dom } f \) and \( z \in \text{int dom } f \cap (\bigcap_{i \in I} \text{Fix } T_i) \). Then

\[
(\forall i \in I) \quad D_f(x, z) = D_f(T_i x, z) \quad \Leftrightarrow \quad (\forall i \in I) \quad D_f(x, z) = D_f(T_i x, T_i z) \quad \text{by } z \in \bigcap_{i \in I} \text{Fix } T_i \tag{4.7}
\]

\[
\Rightarrow \quad (\forall i \in I) \quad D_f(x, z) = D_f(x, z).
\]

Hence, \( z \in \overline{E}_{f}(S(x)) \) by the definition of \( \overline{E}_{f}(S(x)) \) presented in Definition 4.2.

(ii): Let \( z \in \text{int dom } f \cap (\bigcap_{i \in I} \text{Fix } T_i) \). Because \( y \in \text{int dom } f \), we see that \( y \in \text{aff } (S(y)) \cap \text{int dom } f \neq \emptyset \), and that, by (i) above, \( z \in \overline{E}_{f}(S(y)) \). Applying Fact 4.3(iv) with \( K \) replaced by \( S(y) \), we deduce that \( \overline{P}_{\text{aff}(S(y))}(z) \in \overline{CC}(S(y)) \). Combine this with Definition 4.4, Fact 2.4, and the assumption that \( \overline{CC}_S \) is at most single-valued, to yield that \( \overline{CC}_S y = \overline{CC}(S(y)) \). Therefore, \( \overline{CC}_S y \in \text{int dom } f \).
(iii)&(iv)&(v): By Lemma 4.6(i), \( \text{Fix } \overrightarrow{C}_S^z = \text{int dom } f \cap (\cap_{i \in I} \text{Fix } T_i) \). Hence, (ii) implies that

\[
(\forall x \in \text{int dom } f) \ (\forall z \in \text{Fix } \overrightarrow{C}_S \cap \text{int dom } f) \quad \overrightarrow{C}_S f x = \overrightarrow{f}_{\text{aff}(S(x))}(z) \in \text{int dom } f.
\]

Therefore, the required results follow directly from Theorem 4.11(i)&(ii)&(iii), respectively, with \( \mathcal{C} = \text{int dom } f \) and \( G = \overrightarrow{C}_S^z \).

(vi)(a): Suppose that \( x \in \text{int dom } f \) and that \( (x_k)_{k \in \mathbb{N}} \subseteq \text{int dom } f \) such that \( x_k \to x \). Notice that, due to (ii) above, \( \{ \overrightarrow{C}_S^z x_k \}_{k \in \mathbb{N}} \subseteq \text{int dom } f \). Assume that \( D_f(x_k, \overrightarrow{C}_S^z x_k) \to 0 \). In view of Definition 4.4,

\[
(\forall i \in I) \ (\forall k \in \mathbb{N}) \quad D_f \left( x_k, \overrightarrow{C}_S^z x_k \right) = D_f \left( T_i x_k, \overrightarrow{C}_S^z x_k \right),
\]

which, connecting with \( D_f(x_k, \overrightarrow{C}_S^z x_k) \to 0 \) and applying (4.6) with \( (\forall k \in \mathbb{N}) \ u_k = x_k \), implies that \( (\forall i \in I) \ D_f(x_k, T_i x_k) \to 0 \). Let \( i \in I \). Note that \( T_i \) is \( f \)-demiclosed. So the results \( x_k \to x \) and \( D_f(x_k, T_i x_k) \to 0 \) imply that \( x \in \text{Fix } T_i \). Because \( i \in I \) is chosen arbitrarily, by Lemma 4.6(i), \( x \in \text{int dom } f \cap (\cap_{i \in I} \text{Fix } T_i) = \text{Fix } \overrightarrow{C}_S^z \).

(vi)(b): Bearing (vi)(a) in mind, we observe that the desired convergence follows from Theorem 4.11(iv) with \( \mathcal{C} = \text{int dom } f \) and \( G = \overrightarrow{C}_S^z \).

To end this work, we revisit the operator \( T \) and the function \( f \) in Example 4.10, which satisfy all requirements in Theorem 4.12. In view of Example 4.10, the following example illustrates Theorem 4.12, and demonstrates that the forward Bregman circumercenter method finds the desired fixed point by one iterate. In particular, the following Example 4.13(v)&(vi) suggest the one step convergence of the forward Bregman circumercenter method.

**Example 4.13.** Suppose that \( \mathcal{H} = \mathbb{R}^n \). Define \( f : x \mapsto \sum_{i=1}^n x_i \ln(x_i) + (1 - x_i) \ln(1 - x_i) \), with dom \( f = [0, 1]^n \), which is the Fermi-Dirac entropy. Let \( \Lambda \) be a subset of \( J \). Define \( T : \mathbb{R}^n \to \mathbb{R}^n \) by \( (\forall x \in \mathbb{R}^n) \ T x := (y_i)_{i=1}^n \) where \( (\forall i \in J \setminus \Lambda) \ y_i = x_i \) and \( (\forall i \in \Lambda) \ y_i = (1 - x_i) \). Let \( x := (x_i)_{i \in J} \in [0, 1]^n \). Denote by \( \Phi := \{ i \in J : x_i \neq \frac{1}{2} \} \). Then the following statements hold.

(i) \( \overrightarrow{E}_f(S(x)) = \left\{ (p_i)_{i \in J} : 0 = \sum_{i \in \Lambda} (2x_i - 1) \ln \left( \frac{p_i}{1 - p_i} \right) \right\} \):

(ii) \( \overrightarrow{C}_S^z(x) = \left\{ (p_i)_{i \in J} : 0 = \sum_{i \in \Lambda} (2x_i - 1) \ln \left( \frac{p_i}{1 - p_i} \right) \right\} \) Moreover,

\[
\left\{ (p_i)_{i \in J} : 0 = \sum_{i \in \Lambda} (2x_i - 1) \ln \left( \frac{p_i}{1 - p_i} \right) \right\} \subseteq \overrightarrow{C}_S^z(x);
\]

\[
\emptyset \neq \left\{ (p_i)_{i \in J} : 0 = \sum_{i \in \Lambda} (2x_i - 1) \ln \left( \frac{p_i}{1 - p_i} \right) \right\} \cap \text{Fix } T \cap \text{int dom } f \cap \overrightarrow{C}_S^z(x).
\]

(iii) Suppose that \( (\forall i \in \Lambda \setminus \Phi) \ x_i > \frac{1}{2} \) or \( (\forall i \in \Lambda \setminus \Phi) \ x_i < \frac{1}{2} \). Then

\[
\overrightarrow{C}_S^z(x) = \left\{ (p_i)_{i \in J} : 0 = \sum_{i \in \Lambda} (2x_i - 1) \ln \left( \frac{p_i}{1 - p_i} \right) \right\} \cap \text{Fix } T \cap \text{int dom } f \cap \overrightarrow{C}_S^z(x).
\]

(iv) Suppose that \( (\forall i \in \Lambda \setminus \Phi) \ x_i > \frac{1}{2} \) or \( (\forall i \in \Lambda \setminus \Phi) \ x_i < \frac{1}{2} \), and that \( \Lambda \cap \Phi = \emptyset \). Then

\[
\overrightarrow{C}_S^z(x) = \left\{ (p_i)_{i \in J} : 0 = \sum_{i \in \Lambda} (2x_i - 1) \ln \left( \frac{p_i}{1 - p_i} \right) \right\} \cap \text{Fix } T \cap \text{int dom } f.
\]
(v) Suppose that $\Lambda$ is a singleton, say $\Lambda := \{i_0\}$. Then if $x_{i_0} = \frac{1}{2}$, then $\bar{\mathcal{C}_S}(x) = ]0, 1[^n \cap \text{aff} \{x, 1 - x\}$; if $x_{i_0} \neq \frac{1}{2}$, then

$$\bar{\mathcal{C}_S}(x) = \left\{ (p_i)_{i \in J} \in ]0, 1[^n \cap \text{aff} \{x, 1 - x\} : p_{i_0} = \frac{1}{2} \right\} \subseteq \text{Fix } T \cap \text{int dom } f.$$

(vi) Suppose that $\mathcal{H} = \mathbb{R}$. Then

$$\left( \forall x \in [0, 1) \setminus \left\{ \frac{1}{2} \right\} \right) \bar{\mathcal{C}_S}(x) = \left\{ \frac{1}{2} \right\} = \text{int dom } f \cap \text{Fix } T.$$

Consequently, the forward Bregman circumcenter method converges to the desired point in $\text{int dom } f \cap \text{Fix } T$ in one iterate if the initial point is not the desired solution.

Proof. (i): This is easy from the related definitions. Alternatively, we can deduce the required result by applying [12, Example 4.8(ii)(a)] with $S = \{x, Tx\}$.

(ii): Based on Example 4.10(iii)(b),

$$\text{Fix } T \cap \text{int dom } f = \left\{ x \in ]0, 1[^n : (\forall i \in \Lambda)x_i = \frac{1}{2} \right\}.$$  

Hence, this is clear from Definition 4.4.

(iii): This follows immediately form (ii).

(iv)&(v)&(vi): The required results follows easily from (iii).

We shall explore more particular examples of $T$ satisfying the requirements in Theorem 4.12 under general Bregman distances associated with $f \neq \|\cdot\|^2$ in future work.

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