COMPLETE KÄHLER-EINSTEIN METRIC ON STEIN MANIFOLDS WITH NEGATIVE CURVATURE

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Abstract. We show the existence of complete negative Kähler-Einstein metric on Stein manifolds with negatively pinched holomorphic sectional curvature. We prove that any Kähler metrics on such manifolds can be deformed to the complete negative Kähler-Einstein metric using the normalized Kähler-Ricci flow.

1. INTRODUCTION

In [22], Wu and Yau proved that if a complete noncompact Kähler manifold supports a complete bounded curvature Kähler metric with holomorphic sectional curvature bounded from above by a negative constant, then it supports a complete negative Kähler-Einstein metric with bounded curvature. This extended the previous work [23, 19] to the noncompact case. Using the Kähler-Ricci flow approach, this was recovered by Tong [20], see also the compact case in [14]. A natural question is to ask if the curvature boundedness assumption is necessary in order to obtain the existence of Kähler-Einstein without bounded curvature conclusion. In [9], the author together with the collaborators showed that the curvature boundedness assumption can be weakened to the existence of a exhaustion function with bounded complex hessian. The main ingredient is the construction of long-time Kähler-Ricci flow solution using Chern-Ricci flow approximation technique introduced by the author and Tam in [12].

In this note, we continue to study the existence problem using the Kähler-Ricci flow approach. We focus on the special case when $M$ is a Stein manifold. We show the existence of complete negative Kähler-Einstein on $M$ by establishing a long-time solution to the Kähler-Ricci flow on $M$. More precisely, we showed that on the negatively curved Stein manifold $M$, the Kähler-Ricci flow has a long-time solution starting from any Kähler metric which can be incomplete and have unbounded curvature.

Theorem 1.1. Suppose $M^n$ is a Stein manifold and $h$ is a complete Kähler metric on $M$ with holomorphic sectional curvature $H_h \leq -\kappa$ for some $\kappa > 0$. Then any Kähler metric $g_0$ on $M$ admits a longtime solution to the Kähler-Ricci flow on $M \times [0, +\infty)$ such that $g(t)$ is instantaneous complete for $t > 0$. 

Date: October 8, 2019.

2010 Mathematics Subject Classification. Primary 32Q15; Secondary 53C44.

The author was partially supported by NSF grant DMS-1709894.
On general Kähler manifold, the short-time existence of the Kähler-Ricci flow was first studied by Shi [17, 18] when the initial metric $g_0$ is complete with bounded curvature. In general without curvature condition, the existence theory was still being unclear except the surface case. In the case of surface, the existence of the Ricci flow starting from incomplete metric has been studied in details by Giesen and Topping [3, 4, 5]. This can be viewed as a partial generalization of their result in higher dimension.

By combining the existence with the convergence result in [9], we have the existence of the complete negative Kähler-Einstein metric.

Corollary 1.1. Suppose $M$ is a Stein manifold and $h$ is a complete Kähler metric on $M$ with holomorphic sectional curvature $H_h \leq -\kappa$ for some $\kappa > 0$, then $M$ admits a unique complete Kähler-Einstein metric with negative scalar curvature.

Using Corollary 1.1, we know that the following situation admit complete negative Kähler-Einstein metric.

Example 1.1. Using a result of Wu [21] and Corollary 1.1, any complete simply connected Kähler manifold $(M, g)$ with sectional curvature bounded from above by $-1$ admits a complete negative Kähler-Einstein metric.

Example 1.2. Let $\mathbb{B}^{n-1}$ be the unit ball in $\mathbb{C}^n$, then any proper embedded complex sub-manifold $\Sigma$ of $\mathbb{B}^{n-1}$ admits a complete negative Kähler-Einstein metric. To see this, $\mathbb{B}^{n-1}$ admits the Bergman metric $h$ with constant holomorphic sectional curvature. By the decreasing property of holomorphic curvature and properness, the pull-back metric of $h$ on $\Sigma$ is a complete Kähler metric with holomorphic sectional curvature bounded from above by negative value. Then the existence followed from Corollary 1.1.

Example 1.3. Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$ associated with the Poincaré metric. Let $\Sigma$ be a proper embedded complex sub-manifold $\Sigma$ of $\mathbb{D}^n = \mathbb{D} \times \cdots \times \mathbb{D}$. The product metric on $\mathbb{D}^n$ is complete with holomorphic sectional curvature bounded from above by negative value. Then $\Sigma$ admits a complete negative Kähler-Einstein metric using the argument in previous example.

2. a-priori estimates for the Kähler-Ricci flow

Let $M^n$ be a complex manifold. The Kähler-Ricci flow on $M$ starting from initial metric $g_0$ is a family of Kähler metric $g(t)$ which satisfies

$$
\begin{align*}
\frac{\partial}{\partial t} g_{ij} &= -R_{ij}; \\
g_{ij}(0) &= g_{ij},
\end{align*}
$$

on $M \times [0, T]$ for some $T > 0$. We say that $g(t)$ is a complete solution if $g(t)$ is a complete metric on $M$ for each $t \in [0, T]$. 

Lemma 2.1. Let \( g(t) \) be a complete solution to the Kähler-Ricci flow on \( \Omega \times [0, T] \) for \( T \leq +\infty \). Suppose \( h \) is a Kähler metric on \( \Omega \) which can possibly be incomplete such that its holomorphic sectional curvature is bounded from above by \(-\kappa\) for some \( \kappa > 0 \). Then on \( \Omega \times (0, T] \), \( g(t) \) satisfies

\[
\frac{(n+1)\kappa t}{2n} h \leq g(t).
\]

Proof. Using parabolic Schwarz Lemma gives

\[
(\partial_t - \Delta_t) \text{tr}_g h \leq g^{ij} g^{kl} \tilde{R}_{ijkl},
\]

where \( \tilde{R} \) denotes the curvature tensor of metric \( h \). By using a trick of Royden \[6, Lemma, p.552\], the bisectional curvature quantities is bounded from above,

\[
g^{ij} g^{kl} \tilde{R}_{ijkl} \leq -\frac{n+1}{2n} \kappa (\text{tr}_g h)^2.
\]

Noted that the inequality also holds when \( h \) is not complete. Hence, we have

\[
(\partial_t - \Delta_t) \text{tr}_g h \leq -\frac{n+1}{2n} \kappa (\text{tr}_g h)^2.
\]

If \( T < +\infty \), applying \[9, Lemma 5.1\] with \( Q = \text{tr}_g h \) gives

\[
\text{tr}_g h \leq \frac{2n}{\kappa(n+1)T}
\]
on \( \Omega \times [0, T] \). If \( T = +\infty \), then apply it on \([0, L]\) and followed by letting \( L \to +\infty \).

We also have an instantaneous bound on the scalar curvature.

Lemma 2.2. Let \( g(t) \) be a complete solution to the Kähler-Ricci flow on \( \Omega \times [0, T] \) for \( T \leq +\infty \), then the scalar curvature satisfies

\[
R_{g(t)} \geq -\frac{n}{t}.
\]

Proof. Noted that the scalar curvature \( R \) satisfies \( (\partial_t - \Delta_t) R \geq \frac{1}{n} R^2 \). The proof is similar to the that in Lemma 2.1 by applying \[9, Lemma 5.1\] with \( Q = -R \).

We also need the following version of pseudolocality of Ricci flow from \[7\].

Proposition 2.1. For any \( m \in \mathbb{N} \) and \( v > 0 \), there is \( \sigma(m, v), \Lambda(m, v) > 0 \) such that if \( g(t) \) is a complete Ricci flow solution on \( M^m \times [0, T] \) with \( \sup_{[\tau, T]} |\text{Rm}(g(t))| < +\infty \) for all \( \tau > 0 \) and satisfies

1. \( |\text{Rm}(g(0))| \leq r^{-2} \text{ on } B_{g_0}(p, r) \);
2. \( V_{g_0}(B_{g_0}(p, r)) \geq r^m v \),
for some \( p \in M \) and \( r > 0 \), then we have
\[
|R_m(x, t)| \leq \Lambda r^{-2}
\]
on \( B_g(t)(p, \sigma r), \ t \leq \min\{T, \sigma^2 r^2\} \).

Proof. The follows by applying \[7, \text{Corollary 3.1}\] to the rescaled Ricci flow solution \( r^{-2}g(r^2t) \) and then rescale it back. \( \square \)

3. APPROXIMATION USING PSEUDOCONVEX DOMAIN

In this section, we will construct local Kähler-Ricci flow solution using the Steinness of \( M \). By a result of Grauert \[10\], there exists a smooth strictly pluri-subharmonic function \( \rho \) on \( M \) which is a exhaustion function. For \( R > 1 \) large, denote its sublevel set to be
\[
U_R = \{ x \in M : \rho(x) < R \}.
\]
This is a bounded strictly pseudo-convex domain in \( M \) as defined in \[2\]. By Sard’s Theorem, we may find a sequence of \( R_m \to +\infty \) such that \( d\rho \neq 0 \) on \( \partial U_{R_m} \). By the result in \[2\], there exists a complete Kähler-Einstein metric \( \tilde{\omega}_m = -\text{Ric}(\tilde{\omega}_m) \) with bounded curvature defined on \( U_{R_m} \). Now we will construct approximating Kähler-Ricci flow using \( \{(U_{R_m}, \tilde{\omega}_m)\}_{m \in \mathbb{N}} \). Let \( \omega_0 \) be the Kähler form of the initial metric \( g_0 \). On \( U_{R_m} \), define \( \omega_{0,m,\epsilon} = \omega_0 + \epsilon \tilde{\omega}_m \). We will use \( g_{0,m,\epsilon} \) to denote the corresponding Kähler metric.

**Lemma 3.1.** For any \( \epsilon > 0 \), \( \omega_{0,m,\epsilon} = \omega_0 + \epsilon \tilde{\omega}_m \) is a complete Kähler metric on \( U_{R_m} \) which is uniformly equivalent to \( \tilde{\omega}_m \).

**Proof.** The completeness is clear. It remains to establish the upper bound of \( \omega_{0,m,\epsilon} \) with respect to \( \tilde{\omega}_m \). Since \( \rho_{ij} > 0 \) and \( U_{R_m} \) is pre-compact in \( M \), there is \( C_m >> 1 \) such that
\[
(3.1) \quad \omega_0 \leq C_m \sqrt{-1} \partial \bar{\partial} \rho
\]
on \( U_{R_m} \). On the other hand by \[2\], \( \tilde{\omega}_m \) is uniformly equivalent to the standard metric \( -\sqrt{-1} \partial \bar{\partial} \log(R - \rho) \) on \( U_{R_m} \). Therefore,
\[
(3.2) \quad \omega_0 \leq -C_m \sqrt{-1} \partial \bar{\partial} \log(R - \rho) \leq C \tilde{\omega}_m.
\]
This completes the proof. \( \square \)

Using the uniform equivalence of metrics, we can deform the Kähler metric using the Kähler-Ricci flow on each \( U_{R_m} \) with long-time existence.

**Proposition 3.1.** There exists a complete long-time solution \( \omega_{m,\epsilon}(t) \) to the Kähler-Ricci flow with initial metric \( \omega_{0,m,\epsilon} \) on \( U_{R_m} \times [0, +\infty) \) such that \( \omega_{m,\epsilon}(t) \) is uniformly equivalent to \( \tilde{\omega}_m \) on any \( [a, b] \subset [0, +\infty) \).
Proof. From Lemma 3.1 there exists $\delta > 0$ such that
\[
\begin{align*}
\omega_{0,m,\epsilon} &\leq \delta^{-1}\bar{\omega}_m; \\
\omega_{0,m,\epsilon} - s\text{Ric}(\bar{\omega}_m) &> \delta\bar{\omega}_m.
\end{align*}
\]
for any $s > 0$. The proposition follows from [1, Theorem 1.4].

\section{Proof of main theorem}

In this section, we will construct a global longtime solution $g(t)$ to the Kähler-Ricci flow with initial metric $g(0) = g_0$ using the idea in [4] adapted in higher dimensional Kähler case. The main goal is to show that $g(t) = \lim_{m \to +\infty} \lim_{\epsilon \to 0} g_{m,\epsilon}(t)$ exists. In [8, 1], it was showed that $g_{m}(t) = \lim_{\epsilon_i \to 0} g_{m,\epsilon_i}(t)$ exists smoothly up to $t = 0$ for some subsequence $\epsilon_i \to 0$ using estimates from parabolic Monge-Ampère equation. Since the information of Ric$(h)$ on $M$ is missing, we need to take an alternative approach to obtain the estimates for compactness when $m \to +\infty$.

Proof of Theorem 1.1. From the discussion in section 3, there is a sequence of bounded strictly pseudo-convex domain $U_{R_m}$ exhausting $M$. Moreover, there is a sequence of complete Kähler-Ricci flow $g_{m,\epsilon}(t)$ defined on $U_{R_m} \times [0, +\infty)$ with $g_{m,\epsilon}(0) = g_{0,m,\epsilon}$. Because of completeness, we may apply Lemma 2.1 and Lemma 2.2 to deduce that for all $t \in [0, +\infty)$,
\[
\begin{align*}
(n+1)\kappa t h &\leq g_{m,\epsilon}(t); \\
2nR_{g_{m,\epsilon}(t)} &\geq -nt^{-1}.
\end{align*}
\]

Here we regard $h$ as an incomplete metric on the pre-compact set $U_{R_m}$. Moreover by uniform equivalence with a metric with bounded curvature, the Kähler-Ricci flow $g_{m,\epsilon}(t)$ has bounded curvature when $t > 0$, for example see [11, 16, 13].

Fix a compact set $\Omega \subset M$, since $g_{0,m,\epsilon}$ converges to $g_0$ uniformly in any $C^k$ as $\epsilon \to 0$ and $m \to +\infty$, we can find $r > 0$ and $v > 0$ small enough such that for all $x \in \Omega$, $m$ large and $\epsilon_0(n, m, g_0, \Omega) > \epsilon > 0$, we have
\begin{itemize}
  \item[(a)] $B_{g_{0,m,\epsilon}}(x, r) \subset U_{R_m};$
  \item[(b)] $|\text{Rm}(g_{0,m,\epsilon})| \leq r^{-2}$ on $B_{g_{0,m,\epsilon}}(x, r);$
  \item[(c)] $V_{g_{0,m,\epsilon}}(B_{g_{0,m,\epsilon}}(x, r)) \geq vr^{2n}.$
\end{itemize}

By Proposition 2.1 for all $m$ large and $\epsilon < \epsilon_0(n, m, g_0, \Omega)$, we have
\[
\sup_{\Omega} |\text{Rm}(g_{m,\epsilon})(x, t)| \leq \Lambda(n, g_0, \Omega)
\]
for $0 \leq t \leq \sigma(n, g_0, \Omega)$. In particular, for all $(x, t) \in \Omega \times [0, \sigma]$, \begin{equation}
\tag{4.3}
e^{-\Lambda\sigma}g_{0,m,\epsilon} \leq g_{m,\epsilon}(t) \leq e^{\Lambda\sigma}g_{0,m,\epsilon}.
\end{equation}

For $t > \sigma$, we make use of (4.1). Using the evolution of volume form with the second inequality in (4.1), we have
\[
\frac{\partial}{\partial s} \left( \log \frac{\det g_{m,\epsilon}(s)}{\det h} \right) = -R_{g_{m,\epsilon}(t)} \leq \frac{n}{s}.
\]

\[
\tag{4.4}
\]
By integrating it from $\sigma$ to $t$, we have
\[
\det g_{m,\epsilon}(t) \leq \left(\frac{t}{\sigma}\right)^n \det g_{m,\epsilon}(\sigma)
\]
\[
\leq e^{n\Lambda} \left(\frac{t}{\sigma}\right)^n \det g_{0,m,\epsilon}.
\]
(4.5)

Combines (4.5), (4.1) and the elementary inequality $\text{tr}_h g \leq \frac{\det g}{\det h} (\text{tr}_g h)^{n-1}$, we deduce that for all $(x, t) \in \Omega \times [\sigma, +\infty)$,
\[
\frac{(n + 1)\kappa t}{2n} h \leq g_{m,\epsilon}(t) \leq \left(\frac{e^{\Lambda} t}{\sigma}\right)^n \left[\frac{2n}{(n + 1)\kappa t}\right]^{n-1} \left(\frac{\det g_{0,m,\epsilon}}{\det h}\right) h.
\]
(4.6)

Using (4.3), (4.6) and the fact that $g_{0,m,\epsilon}$ converges to $g_0$ locally uniformly, in conclusion we have shown that for any $T < \infty$ and compact set $\Omega \subset M$, there is $\lambda > 1$ such that for all $m$ sufficiently large, $\epsilon$ sufficiently small and $t \in [0, T]$,
\[
\lambda^{-1} h \leq g_{m,\epsilon}(t) \leq \lambda h.
\]

By applying higher order estimate of the Kähler-Ricci flow \[16\] on local chart, we have all higher order estimates of $g_{m,\epsilon}(t)$ on $\Omega \times [0, T]$ which is independent of $m >> 1$ and $\epsilon < \epsilon_0(n, m, g_0, \Omega)$.

By Ascoli-Arzelà Theorem, for each $m$ large we may first let $\epsilon_i \to 0$ for some subsequence $\epsilon_i$ to obtain a Kähler-Ricci flow solution $g_m(t), t \in [0, T]$ on each $U_{R_m}$. Then by diagonal subsequence argument and Ascoli-Arzelà Theorem again, we may let $m_i \to +\infty$ and followed by $T \to +\infty$ to obtain a global solution $g(t)$ of the Kähler-Ricci flow on $M \times [0, +\infty)$. The instantaneous completeness of $g(t)$ followed by passing (1.1) to the limiting solution $g(t)$. $\square$

Proof of Corollary 1.1. By Theorem 1.1, there is a long-time solution to the Kähler-Ricci flow starting from $h$. The existence of complete negative Kähler-Einstein followed from [9, Theorem 5.1]. $\square$

Remark 4.1. It is clear from the proof of Theorem 1.1 and Corollary 1.1 that the reference metric $h$ can be an incomplete metric. Then the resulting Kähler-Ricci flow solution and Kähler-Einstein metric maybe a incomplete metric.

Remark 4.2. Since $h$ only served as a reference metric in the Schwarz Lemma, the existence of Kähler metric $h$ can be replaced by the existence of a Hermitian metric with negatively pinched real bisectional curvature, see [24, 11].

References

[1] A. Chau; Lee, M.-C., The Kähler Ricci flow around complete bounded curvature Kähler metrics, arXiv:1904.02233. To appear in Trans. Amer. Math. Soc
[2] Cheng, S.-Y.; Yau, S.-T., On the existence of a complete Kähler Metric on noncompact complex manifolds and the regularity of Fefferman’s equation, Communication of Pure and Applied Mathematics, 33 (1980), no. 4, 507-544.
[3] Giesen, G.; Topping, P.-M., Ricci flow of negatively curved incomplete surfaces, Calc. Var. and PDE, 38 (2010), 357-367.
[4] Giesen, G.: Topping, P.-M., Existence of Ricci flows of incomplete surfaces, Comm. Partial Differential Equations, 36 (2011), 1860–1880.
[5] Giesen, G.: Topping, P.-M., Ricci flows with unbounded curvature, Math. Zeit., 273 (2013), 449–460.
[6] H.L. Royden, The Ahlfors-Schwarz lemma in several complex variables, Comment. Math. Helv. 55 (1980), no. 4, 547–558.
[7] He, F.; Lee, M.-C., Weakly PIC1 manifolds with maximal volume growth, arXiv:1811.03318
[8] Huang, S.-C.; Lee, M.-C.; Tam, L.-F., Instantaneously complete Chern-Ricci flow and Kähler-Einstein metrics, Calculus of Variations and PDE(2019) 58: 161.
[9] Huang, S.-C; Lee, M.-C.; Tam L.-F.; Tong, F., Longtime existence of Kähler-Ricci flow and holomorphic sectional curvature, arXiv:1805.12328
[10] H. Grauert, On Levi's problem and the imbedding of real analytic manifolds, Ann. of Math. 68 (1958) 460–472, MR 0098847, Zbl 0108.07804.
[11] K. Tang, On real bisectional curvature and Kähler-Ricci flow. Proc. Amer. Math. Soc., 147 (2019), no. 2, 793–798.
[12] Lee, M.-C.; Tam L.-F., Chern-Ricci flows on noncompact complex manifolds, To appear in J. Differential Geometry.
[13] Lott, J.; Zhang, Z., Ricci flow on quasiprojective manifolds II, J. Eur. Math. Soc. (JEMS) 18 (2016), no. 8, 1813–1854, MR3519542, Zbl 1351.53081.
[14] Nomura, R., Kähler Manifolds with Negative Holomorphic Sectional Curvature, Kähler-Ricci Flow Approach, International Mathematics Research Notices, Vol. 2017, No. 00, pp. 16.
[15] Perelman, G., The entropy formula for the Ricci flow and its geometric applications, arXiv:math.DG/0211159
[16] Sherman, M.; Weinkove, B., Interior derivative estimates for the Kähler-Ricci flow, Pacific J. Math. 257 (2012), no. 2, 491–501, MR2972475, Zbl 1262.53056.
[17] Shi, W.-X., Deforming the metric on complete Riemannian manifolds. J. Differ. Geom. 30, 223–301 (1989).
[18] Shi, W.-X., Ricci flow and the uniformization on complete noncompact Kähler manifolds, J. Differential Geom. 45 (1997), no. 1, 942–20.
[19] Tosatti, V.; Yang, X.-K., An extension of a theorem of Wu-Yau, to appear in J. Differential Geom. arXiv:1506.01145.
[20] Tong, F., The Kähler-Ricci flow on manifolds with negative holomorphic curvature, arXiv:1805.03562
[21] H. Wu, Negatively curved Kähler manifolds, Notices Amer. Math. Soc., 14 (1967), 515.
[22] Wu, D.-M.; Yau, S.-T., Invariant metrics on negatively pinched complete Kähler manifolds, arXiv:1711.09476
[23] Wu, D.-M.; Yau, S.-T., Negative Holomorphic curvature and positive canonical and positive canonical bundle, Invent. Math. 204 (2016), no. 2, 595–604.
[24] Yang, X.; Zheng, F., On real bisectional curvature for Hermitian manifolds, Trans. Amer. Math. Soc., 371(4):2703–2718, 2019.

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