A group-theoretic generalization of the $p$-adic local monodromy theorem

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Abstract. Let $G$ be a connected reductive group over a $p$-adic number field $F$. We propose and study the notions of $G$-$\phi$-modules and $G$-$(\phi, \nabla)$-modules over the Robba ring, which are exact faithful $F$-linear tensor functors from the category of $G$-representations on finite-dimensional $F$-vector spaces to the categories of $\phi$-modules and $(\phi, \nabla)$-modules over the Robba ring, respectively, commuting with the respective fiber functors. We study Kedlaya's slope filtration theorem in this context, and show that $G$-$(\phi, \nabla)$-modules over the Robba ring are "$G$-quasi-unipotent," which is a generalization of the $p$-adic local monodromy theorem proved independently by Y. André, K. S. Kedlaya, and Z. Mebkhout.

1 Introduction

Let $p$ be a prime number and $q$ a power of $p$. Let $K$ be a complete nonarchimedean discretely valued field of characteristic 0 equipped with an automorphism $\phi$, the Frobenius, inducing the $q$-power map on the residue field $\kappa \supseteq \mathbb{F}_q$. We also require $K$ to be unramified over the fixed subfield $F$ under $\phi$. See Hypothesis 2.1 for a concrete example.

The Robba ring $\mathcal{R} = \mathcal{R}(K, t)$ is the ring of bidirectional power series $\sum_{i \in \mathbb{Z}} c_i t^i$ in one variable $t$ with coefficients in $K$ which converge in an annulus $[\alpha, 1)$ for some series-dependent $0 < \alpha < 1$. The Robba ring $\mathcal{R}$ is endowed with an absolute Frobenius lift $\phi$ which extends the Frobenius on $K$ and lifts the $q$-power map on $\kappa((t))$, and with the derivation $\partial = d/dt$.

A $(\phi, \nabla)$-module over $\mathcal{R}$ is a triple $(M, \Phi, \nabla)$, where $M$ is a finite free $\mathcal{R}$-module, $\Phi$ is a Frobenius, i.e., a $\phi$-linear endomorphism of $M$ whose image spans $M$ over $\mathcal{R}$, and $\nabla : M \to M \otimes_{\mathcal{R}} \mathcal{R} dt$ is a connection. Moreover, $\Phi$ and $\nabla$ should satisfy the gauge compatibility condition, which says that, after choosing an $\mathcal{R}$-basis for $M$, the actions $\Phi$ and $\nabla$ are given by matrices $A$ and $N$, respectively, and these matrices should satisfy $N = \mu \cdot A(\phi(N))A^{-1} - \partial(A)A^{-1}$, where $\mu := \partial(\phi(t))$. 

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The $(\varphi, \nabla)$-modules, also known as the overconvergent $F$-isocrystals in the literature, are closely related to $p$-adic local systems on Spec $\kappa((t))$ (for a summary, we refer to [13]), for which the correct monodromy theorem is the $p$-adic local monodromy theorem (pLMT), conjectured by Crew [5], and proved independently by Andrê [1], Kedlaya [9], and Mebkhout [17]. It states that every $(\varphi, \nabla)$-module over $\mathcal{R}$ is quasi-unipotent. Concretely, a $(\varphi, \nabla)$-module $M$ over $\mathcal{R}$, after an étale extension to $\mathcal{R}_L$ (the Robba ring canonically associated to some finite separable extension $L$ of $\kappa((t))$), admits a filtration by $(\varphi, \nabla)$-submodules such that the connections induced on the gradation are trivial. A matricial description of the theorem is given as follows. Let $d$ be the rank of $M$ over $\mathcal{R}$, and let $A \in \text{GL}_d(\mathcal{R})$ (resp. $N \in \text{Mat}_{d,d}(\mathcal{R})$) be the matrix of $\Phi$ (resp. $\nabla$) in some basis. Then, there exists $B \in \text{GL}_d(\mathcal{R}_L)$ such that $BNB^{-1} - \partial(B)B^{-1}$ is an upper-triangular block matrix with zero blocks in the diagonal.

We mention two applications of the $p$LMT in $p$-adic Hodge theory.

- In [3], Berger associated to every $p$-adic de Rham representation $V$ a $(\varphi, \nabla)$-module $N_{\text{dR}}(V)$ over a Robba ring. He showed that $V$ is potentially semistable if and only if $N_{\text{dR}}(V)$ is quasi-unipotent. Using the $p$LMT, he could prove the $p$-adic monodromy theorem (previously a conjecture of Fontaine): every $p$-adic de Rham representation is potentially semistable.
- In [16], Marmora used the $p$LMT to construct a functor from the category of $(\varphi, \nabla)$-modules over $\mathcal{R}$ to that of $K^{\text{nr}}$-valued Weil–Deligne representations of the Weil group $\mathcal{W}_{\kappa((t))}$, where $K^{\text{nr}}$ is the maximal unramified extension of $K$ in a fixed algebraic closure of $K$.

Rather than the general linear group, a Galois representation may take value in some connected reductive group $G$, such as a special linear group or a symplectic group. In order to have appropriate formulations of the above results in this context, it is helpful to establish a $G$-version of the $p$LMT, which is the main motivation of our present paper.

In this paper, we introduce the notion of $G$-$\varphi$-modules over $\mathcal{R}$ (resp. $G$-$(\varphi, \nabla)$-modules over $\mathcal{R}$), which are exact faithful $F$-linear tensor functors from the category $\text{Rep}_F(G)$ of $G$-representations on finite-dimensional $F$-vector spaces to the category $\text{Mod}^G_{\mathcal{R}}$ of $\varphi$-modules over $\mathcal{R}$ (resp. to the category $\text{Mod}^{G,\nabla}_{\mathcal{R}}$ of $(\varphi, \nabla)$-modules over $\mathcal{R}$), commuting with the respective fiber functors. These constructions are inspired by that of $G$-isocrystals introduced in [6, Section IX.1].

Before coming to the main theorem, we first explain the group-theoretic gauge compatibility condition (Definition 4.6). Let $G$ be an affine algebraic $F$-group, and let $\mathfrak{g}$ be its Lie algebra. For any $y \in G(\mathcal{R})$ and $Y \in \mathfrak{g} \otimes_F \mathcal{R}$, we define $\Gamma_y(Y) := \text{Ad}(y)(Y) - d\text{log}(y)$, where $\text{Ad} : G \to \text{GL}_\mathfrak{g}$ is the adjoint representation, and $d\text{log}: G(\mathcal{R}) \to \mathfrak{g} \otimes_F \mathcal{R}$ is defined in Construction 4.4. We say $g \in G(\mathcal{R})$ and $X \in \mathfrak{g} \otimes_F \mathcal{R}$ satisfy the gauge compatibility condition if $X = \Gamma_y(\mu(\varphi(X)))$. When $G = \text{GL}_d$, we have $\text{Ad}(y)(Y) = yYy^{-1}$ and $d\text{log}(y) = \partial(y)y^{-1}$. In this context, the group-theoretic gauge compatibility condition coincides with the aforementioned matrical one.

Our main theorem is the following $G$-version of the $p$LMT.
Theorem 1.1 (Theorem 4.21) Let $G$ be a connected reductive $F$-group, and let $g$ be its Lie algebra. If $g \in G(\mathcal{R})$ and $X \in g \otimes \mathcal{R}$ satisfy $X = \Gamma_g(\mu \varphi(X))$, then there exists a finite separable extension $L$ over $\kappa((t))$ and an element $b \in G(\mathcal{R}_L)$ such that $\Gamma_b(X) \in \text{Lie}(U_{G_{\mathcal{R}}}(-\lambda_g)) \otimes_{\mathcal{R}} \mathcal{R}_L$.

Here, $\lambda_g: G_{m,\mathcal{R}} \to G_{\mathcal{R}}$ is a cocharacter associated to $g$ whose reciprocal is denoted by $-\lambda_g$, and $U_{G_{\mathcal{R}}}(-\lambda_g)$ denotes the unipotent radical of the parabolic subgroup of $G_{\mathcal{R}}$ associated to $-\lambda_g$. When $G = \text{GL}_d$, $g$ (resp. $X$) should be thought as the matrix of the Frobenius (resp. the matrix of the connection), and $\Gamma_b(\_)$ as the matrix of a connection under the change-of-basis via $b^{-1}$. Moreover, $\text{Lie}(U_{G_{\mathcal{R}}}(-\lambda_g)) \otimes_{\mathcal{R}} \mathcal{R}_L$ consists of upper-triangular matrices over $\mathcal{R}_L$ with zero blocks (of certain sizes) in the diagonal. As such, Theorem 1.1 recovers the matricial $p$LMT described above.

In Proposition 4.9, we show that $G_{\mathcal{R}}(\varphi, \nabla)$-modules over $\mathcal{R}$ are indeed pairs $(g, X)$ subject to the gauge compatibility condition in the theorem. In this sense, the theorem can be interpreted as saying that $G_{\mathcal{R}}(\varphi, \nabla)$-modules over $\mathcal{R}$ are “$G$-quasi-unipotent.” In Examples 4.10 and 4.11, we give examples of the existence of such pairs for $G$ a special linear group and a symplectic group, respectively.

More examples of $G_{\mathcal{R}}(\varphi, \nabla)$-modules are expected from Berger’s functor $N_{\mathcal{R}}$ mentioned previously. Explicitly, we hope to show in a future work that if a $p$-adic de Rham representation $V$ takes value in a connected reductive group $G$, then $N_{\mathcal{R}}(V)$ is a $G_{\mathcal{R}}(\varphi, \nabla)$-module. As another future work, we intend to use Theorem 1.1 to formulate a $G_{\mathcal{R}}$-version of Marmora’s functor, namely, to construct a functor from the category of $G_{\mathcal{R}}(\varphi, \nabla)$-modules over $\mathcal{R}$ to that of Weil–Deligne representations of the Weil group $W_{\kappa((t))}$ taking value in $G(K^{nr})$.

Our approach to the theorem closely follows that of the $p$LMT in [9] for absolute Frobenius lifts, wherein the author used his slope filtration theorem (along with applying the pushforward functor and twisting to each quotient of the filtration) to reduce the problem to the unit-root case, and then apply the unit-root $p$LMT attributed to Tsuzuki [23] to finish. More precisely, we use Kedlaya’s slope filtration theorem to construct a $Q$-filtered fiber functor $\text{HN}_g$ from $\text{Rep}_F(G)$ to $Q_{\text{Fil}}_{\mathcal{R}}$, the category of $Q$-filtered modules over $\mathcal{R}$ (see Theorem 3.4). We then reduce $\text{HN}_g$ to a $Z$-filtered fiber functor $\text{HN}^Z_g$ from $\text{Rep}_F(G)$ to $Z_{\text{Fil}}_{\mathcal{R}}$, the category of $Z$-filtered modules over $\mathcal{R}$ (see Lemma 3.10). Then, a result of Ziegler (Theorem 2.12) immediately implies that $\text{HN}^Z_g$ is splittable, i.e., factors through a $Z$-graded fiber functor (see Proposition 3.11).

In particular, for any splitting of $\text{HN}^Z_g$, we construct a morphism $\lambda_g: G_{m,\mathcal{R}} \to G_{\mathcal{R}}$ of $\mathcal{R}$-groups in Section 3.4, which is called the $Z$-slope morphism of $g$. With this, we can reduce the $G_{\mathcal{R}}(\varphi, \nabla)$-module $(g, X)$ over $\mathcal{R}$, involving the (generalized) pushforward functor and twisting, to a unit-root one (see Corollary 4.20). Theorem 1.1 then follows from the unit-root $p$LMT and a Tannakian argument.

The paper is organized as follows. In Section 2, we set up basic notation and conventions, and then recall some necessary background on the theory of slopes and Tannakian formalism. In Section 3, we study $G_{\mathcal{R}}(\varphi)$-modules over the Robba ring, and construct slope morphisms. In Section 4, we consider $G_{\mathcal{R}}(\varphi, \nabla)$-modules over the Robba ring, and prove our main result, Theorem 1.1, in the last subsection.
2 Preliminaries

2.1 Notation and conventions

When \( k \) is a field, we denote by \( \text{Vec}_k \) the category of finite-dimensional \( k \)-vector spaces. When \( R \) is a \( k \)-algebra,\(^1\) we denote by \( \text{Mod}_R \) the category of \( R \)-modules, and by \( \text{Alg}_R \) the category of \( R \)-algebras. When \( V, W \in \text{Vec}_k \), we write \( V_R \) for \( V \otimes_k R \), and write \( \alpha_R := \alpha \otimes R \), the \( R \)-linear extension of \( \alpha \), for all \( k \)-linear maps \( \alpha : V \to W \). When \( G \) is an affine algebraic \( k \)-group, we denote by \( k[G] \) the Hopf algebra of \( G \), by \( G_R := G \times_{\text{Spec} k} \text{Spec} R \) the base extension, by \( H^i(k, G) := H^i(\text{Gal}(k^{\text{sep}}/k), G(k^{\text{sep}})) \) the first Galois cohomology set, and by \( \text{Rep}_k(G) \) the category of representations of \( G \) on finite-dimensional \( k \)-vector spaces.

By a reductive \( k \)-group, we mean a (not necessarily connected) affine algebraic \( k \)-group \( G \) such that every smooth connected unipotent normal subgroup of \( G_k \) is trivial, where \( k \) is an algebraic closure of \( \kappa \).

For the rest of this paper, we work under the following hypothesis.

**Hypothesis 2.1** Let \( p \) be a prime number and \( q = p^f \) an integral power of \( p \). Let \( F \) be a finite extension of \( \mathbb{Q}_p \) with the ring of integers \( \mathcal{O}_F \), a fixed uniformizer \( \varpi_F \) and the residue field \( \mathbb{F}_q \) of \( q \) elements. Let \( \kappa \) be a perfect field containing \( \mathbb{F}_q \). Let \( \mathcal{O}_K = \mathcal{O}_F \otimes_{\mathbb{F}_q} W(\kappa) \), where \( W(\mathbb{F}_q) \) (resp. \( W(\kappa) \)) denotes the ring of Witt vectors with coefficients in \( \mathbb{F}_q \) (resp. in \( \kappa \)). Then, \( K := \text{Frac}(\mathcal{O}_K) \cong F \otimes_{\mathbb{F}_q} W(\kappa) \) is a complete discretely valued field with ring of integers \( \mathcal{O}_K \), a uniformizer \( \varpi := \varpi_F \otimes 1 \) and residue field \( \kappa \). Let Frob be the ring endomorphism of \( W(\kappa) \) induced by the \( p \)-power map on \( \kappa \), and let

\[
\varphi := 1_{\mathcal{O}_F} \otimes \text{Frob}^f : K \longrightarrow K
\]

be the Frobenius automorphism on \( K \) relative to \( F \). Then, \( \varphi \) reduces to the \( q \)-power map on \( \kappa \), and the fixed field of \( \varphi \) on \( K \) is \( F \otimes_{\mathbb{F}_q} W(\mathbb{F}_q) \cong F \).

2.2 The Robba ring and its variants

For \( \alpha \in (0,1) \), we put

\[
\mathcal{R}_\alpha := \left\{ \sum_{i \in \mathbb{Z}} c_i t^i \mid c_i \in K, \lim_{i \to \pm \infty} |c_i| p^i = 0, \ \forall \rho \in [\alpha, 1) \right\}.
\]

For any \( \rho \in [\alpha, 1) \), we define the \( \rho \)-Gauss norm on \( \mathcal{R}_\alpha \) by setting \( |\sum c_i t^i|_\rho := \sup_i |c_i| p^i \). The **Robba ring** is defined to be the union \( \mathcal{R} := \bigcup_{\alpha \in (0,1)} \mathcal{R}_\alpha \). For any \( \sum c_i t^i \in \mathcal{R} \), we define \( |\sum c_i t^i|_1 := \sup_i |c_i| \in \mathbb{R}_{\geq 0} \cup \{ \infty \} \), the 1-Gauss norm.

The **bounded Robba ring** \( \mathcal{E}^\dagger = \mathcal{E}^\dagger(K, t) \) is the subring of \( \mathcal{R} \) consisting of bounded elements (i.e., elements with finite 1-Gauss norm), which is actually a Henselian discretely valued field w.r.t. the 1-Gauss norm with residue field \( \kappa((t)) \).

\(^1\) By an algebra, we always mean a commutative algebra with 1.
Let $R \in \{\mathcal{R}, \mathcal{E}^\dagger\}$. An absolute $q$-power Frobenius lift on $R$ is a ring endomorphism $\varphi: R \to R$ given by $\sum_{i \in \mathbb{Z}} c_i t^i \mapsto \sum_{i \in \mathbb{Z}} \varphi(c_i) u^i$ for $u = \varphi(t) \in R$ such that $|u - t^q|_1 < 1$.

For any $\alpha \in (0, 1)$, we define $\mathcal{R}_\alpha$ to be the ring of formal sums $\sum_{i \in \mathbb{Q}} c_i t^i$ with $c_i \in K$, subject to the following properties.

- For any $c > 0$, the set $\{i \in \mathbb{Q} \mid |c_i| \geq c\}$ is well-ordered.
- For any $\rho \in [\alpha, 1)$, we have $\lim_{i \to \pm \infty} |c_i|^\rho = 0$.

For any $\rho \in [\alpha, 1)$, we define the $\rho$-Gauss norm on $\mathcal{R}_\alpha$ by setting

$$\left| \sum_i c_i t^i \right|_\rho = \sup_{i \in \mathbb{Q}} \{|c_i|^\rho\}.$$  

We define $\mathcal{R} := \mathcal{R}(K, t) := \bigcup_{\alpha \in (0, 1)} \mathcal{R}_\alpha$, the extended Robba ring. The absolute Frobenius lift on $\mathcal{R}$ is the ring automorphism on $\mathcal{R}$ given by $\sum_{i \in \mathbb{Q}} c_i t^i \mapsto \sum_{i \in \mathbb{Q}} \varphi(c_i) t^{iq}$. We denote by $\tilde{\mathcal{E}}^\dagger$ the subring of $\mathcal{R}$ consisting of bounded elements. By [11, Proposition 2.2.6], we have a $\varphi$-equivariant embedding $\psi: \mathcal{R} \to \tilde{\mathcal{R}}$ such that $|\psi(x)|_\rho = |x|_\rho$ for $\rho$ sufficiently close to 1.

### 2.3 The slope filtration theorem

We recall Kedlaya's theory of slopes. Let $R \in \{\mathcal{E}^\dagger, \mathcal{R}, \tilde{\mathcal{E}}^\dagger, \tilde{\mathcal{R}}\}$ equipped with a Frobenius lift $\varphi$. For the notions of $\varphi$-modules and $(\varphi, \nabla)$-modules over $R$, we refer to [9, Section 2.5]. We denote by $\text{Mod}^\varphi_R$ (resp. $\text{Mod}^{\varphi, \nabla}_R$) the category of $\varphi$-modules (resp. $(\varphi, \nabla)$-modules) over $R$.

Let $(M, \varphi) \in \text{Mod}^\varphi_R$, and let $n$ be a positive integer. Then, $(M, \varphi^n)$ is a $\varphi^n$-module over $(R, \varphi^n)$. The $n$-pushforward functor is given by

$$[n]: \text{Mod}^\varphi_R \to \text{Mod}^\varphi_{R^n}, \quad (M, \varphi) \mapsto (M, \varphi^n).$$

For any $s \in \mathbb{Z}$, we define the twist $M(s)$ of $(M, \varphi)$ by $s$ to be the $\varphi$-module $(M, \varphi^s \Phi)$.

Now, let $M$ be a $\varphi$-module over $R$ of rank $d$.

(i) We say that $M$ is a unit-root $\varphi$-module if there exists a basis $v_1, \ldots, v_d$ of $M$ over $R$ in which $\varphi$ acts via an invertible matrix in $\text{GL}_d(\mathbb{O}_\mathcal{E}^\dagger)$ if $R \in \{\mathcal{E}^\dagger, \mathcal{R}\}$, or $\text{GL}_d(\mathbb{O}_{\tilde{\mathcal{E}}^\dagger})$ if $R \in \{\tilde{\mathcal{E}}^\dagger, \tilde{\mathcal{R}}\}$.

(ii) Let $\mu = s/r \in \mathbb{Q}$ with $r > 0$ and $(s, r) = 1$. We say that $M$ is pure of slope $\mu$ if $([r]_s M) (= s)$ is unit-root.

Let $M \in \text{Mod}^\varphi_R$. We have a canonical filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_l = M$ of $\varphi$-submodules over $R$ such that each quotient $M_i/M_{i-1}$ is pure of some slope $\mu_i$ with $\mu_1 < \cdots < \mu_l$, by [11, Theorem 1.71] if $R = \mathcal{R}$ or [11, Proposition 1.4.15 and Theorem 2.1.8]] if $R = \tilde{\mathcal{R}}$. This is called the slope filtration of $M$. We call $\mu_1, \ldots, \mu_l$ the jumps of the slope filtration. The (uniquely determined, not necessarily strictly) increasing sequence $(\mu_1, \ldots, \mu_1, \ldots, \mu_1, \ldots, \mu_l)$, with each $\mu_i$ appearing $\text{rk}_R(M_i/M_{i-1})$ times, is said to be the slope sequence for $M$. We call $\text{rk}_R(M_i/M_{i-1})$ the multiplicity of $\mu_i$ for all $1 \leq i \leq l$. 

Moreover, if $M$ is a $(\phi, \nabla)$-module over $\mathcal{R}$, then the slope filtration can be refined to a filtration of $(\phi, \nabla)$-submodules. This is [9, Theorem 6.12], and is referred to as the slope filtration theorem for $(\phi, \nabla)$-modules over $\mathcal{R}$.

To continue, we need to recall some notions introduced in [12, Section 14]. A difference ring (resp. difference field) is a ring (resp. field) $R$ equipped with an endomorphism $\phi$. A difference module over $R$ is an $R$-module equipped with a $\phi$-linear endomorphism $\Phi$. A finite free difference module $M$ over $R$ is said to be dualizable (resp. trivial) if $M$ admits a basis over $R$ such that $\Phi$ acts via an invertible matrix (resp. the identity matrix). For example, a $\phi$-module over $\mathcal{R}$ is a dualizable difference module over $\mathcal{R}$ where $\mathcal{R}$ is any of the rings constructed in Section 2.2. Admissible difference module $M$ over $R$ is said to be standard if it admits a basis $e_1, \ldots, e_d$ such that $e_i = \Phi(e_{i-1})$ for $2 \leq i \leq d$ and $\Phi(e_d) = \lambda e_1$ for some $\lambda \in R^\times$. A difference field $(k, \phi_k)$ is called strongly difference-closed if $\phi_k$ is an automorphism and any dualizable difference module over $k$ is trivial.

Let $k$ be a complete nonarchimedean valued field and $(k, \phi_k)$ is a difference field in which $\phi_k$ is bijective. An admissible extension of $(k, \phi_k)$ is a difference field $(\ell, \phi_\ell)$, where $\ell$ is a field extension of $k$ complete for the valuation extending the one on $k$ with the same value group, and $\phi_\ell$ is an automorphism of $\ell$ extending $\phi_k$. (See [11, Definition 3.2.1].)

**Lemma 2.2** [15, Lemma 1.5.3] The field $K$ admits an admissible extension $E$ such that the residue field $\kappa_E$ of $E$ is strongly difference-closed.

The following lemma is a recollection of some results which will be used in the sequel.

**Lemma 2.3** Let $E$ be an admissible extension of $K$ such that $\kappa_E$ is strongly difference-closed.

(i) Let $M \in \text{Mod}_E^\phi$. Then, tensoring the slope filtration of $M$ with $\mathcal{R}(E, t)$ gives the slope filtration of $M \otimes_{\mathcal{R}} \mathcal{R}(E, t)$.

(ii) Let $0 \rightarrow M_1 \rightarrow M \rightarrow M_2 \rightarrow 0$ be a short exact sequence of $\phi$-modules over $\mathcal{R}(E, t)$ such that the slopes of $M_1$ are all less than the smallest slope of $M_2$. Then, the sequence splits.

(iii) Every $\phi$-module over $\mathcal{R}(E, t)$ admits a Dieudonné–Manin decomposition, i.e., it is a direct sum of standard $\phi$-submodules.

(iv) Let $M$ and $N$ be $\phi$-modules over $\mathcal{R}(E, t)$. If the slopes of $M$ are all less than the smallest slope of $N$, then no nonzero morphism from $M$ to $N$ exists.

Assertion (i) is [15, Proposition 1.5.6]. Assertion (ii) is [15, Proposition 1.5.11], and assertion (iii) is Proposition 1.5.12 in loc. cit. Assertion (iv) is [11, Proposition 1.4.18].

### 2.4 The Tannakian duality

In this subsection, $k$ denotes a field unless otherwise specified. We follow the definitions and notations in [7]. We denote by $\omega^G$ the forgetful functor $\text{Rep}_k(G) \rightarrow \text{Vec}_k$, which is called the fiber functor.

The following Tannakian duality will be repeatedly used in this paper, whose proof can be found, e.g., in [18, Theorem 9.2].
\textbf{Theorem 2.4} Let $G$ be an affine algebraic $k$-group, and let $R \in \text{Alg}_k$. Suppose that for any $(V, \rho_V) \in \text{Rep}_k(G)$, we are given an $R$-linear map $\lambda_V : V_R \to V_R$. If the family $\{\lambda_V : (V, \rho_V) \in \text{Rep}_k(G)\}$ satisfies

(i) $\lambda_V \otimes W = \lambda_V \otimes \lambda_W$ for all $V, W \in \text{Rep}_k(G)$;

(ii) $\lambda_1$ is the identity map where $1$ is the trivial representation on $k$;

(iii) for all $G$-equivariant maps $\alpha : V \to W$, we have $\lambda_W \circ \alpha = \alpha_R \circ \lambda_V$.

Then, there exists a unique $g \in G(R)$ such that $\lambda_V = \rho_V(g)$ for all $(V, \rho_V) \in \text{Rep}_k(G)$.

\textbf{Corollary 2.5} Let $G$ be an affine algebraic $k$-group. We have an isomorphism $G \cong \text{Aut}^\otimes(\omega^G)$ of affine algebraic $k$-groups.

\textbf{Corollary 2.6} Let $G$ be a smooth affine algebraic $k$-group. Let $\ell/k$ be a field extension, and let $\eta : \text{Rep}_k(G) \to \text{Vec}_\ell$ be a fiber functor over $\ell$. Then, $\text{Hom}^\otimes(\omega^G, \eta)$ is a $G$-torsor over $\ell$. In particular, if $H^1(\ell, G) = \{1\}$ and $G(\ell) \neq \emptyset$, then $\omega^G$ is isomorphic to $\eta$ over $\ell$.

\textbf{Proof} Notice that we have an action

$$\text{Hom}^\otimes(\omega^G, \eta) \times \text{Aut}^\otimes(\omega^G) \to \text{Hom}^\otimes(\omega^G, \eta)$$

by precomposition. By [7, Theorem 3.2(i)], $\text{Hom}^\otimes(\omega^G, \eta)$ is an $\text{Aut}^\otimes(\omega^G)$-torsor. In particular, it is a $G$-torsor over $\ell$ by Corollary 2.5.

Because $G$ is a $\ell$-group variety, $G$-torsors over $\ell$ are $\ell$-varieties by [18, Proposition 2.69], whose isomorphism classes are classified by $H^1(\ell, G)$. It follows from the triviality of $H^1(\ell, G)$ that $\text{Hom}^\otimes(\omega^G, \eta)(\ell) \cong G(\ell)$; hence, $\text{Hom}^\otimes(\omega^G, \eta)(\ell) \neq \emptyset$. [7, Proposition 1.13] then implies the second assertion.

To end this subsection, we give a Lie algebra version of Theorem 2.4. We start with recalling the notion of the Lie algebra of a $k$-group functor. (See [8, Chapitre II, Section 4] for more details. Notice that $k$ denotes a ring in loc. cit.)

For any $R \in \text{Alg}_k$, we define the $R$-algebra of dual numbers $R[\varepsilon] := R[X]/(X^2)$. Put $\varepsilon := X + (X^2)$; we then have the canonical projection $\pi_R : R[\varepsilon] \to R$, $\varepsilon \mapsto 0$. Let $G$ be a $k$-group functor. We define

$$\text{Lie}(G)(R) := \text{Ker } G(\pi_R).$$

Let $f : G \to H$ be a morphism of $k$-group functors. The commutative diagram

$$\begin{array}{ccc}
\text{Lie}(G)(R) = \text{Ker } G(\pi_R) & \xrightarrow{\iota_G} & \text{Lie}(H)(R) = \text{Ker } H(\pi_R) \\
\downarrow{\iota_G} & & \downarrow{\iota_H} \\
G(R[\varepsilon]) & \xrightarrow{f(R[\varepsilon])} & H(R[\varepsilon]) \\
\downarrow{G(\pi_R)} & & \downarrow{H(\pi_R)} \\
G(R) & \xrightarrow{f(R)} & H(R)
\end{array}$$

implies that $f(R[\varepsilon]) \circ \iota_G(X) \in \text{Lie}(H)(R)$ for all $X \in \text{Lie}(G)(R)$. We define $\text{Lie}(f) := f(R[\varepsilon]) \circ \iota_G : \text{Lie}(G)(R) \to \text{Lie}(H)(R)$. Hence, $\text{Lie}(\_)(R)$ is a functor from the category of $k$-group functors to that of abelian groups.
For an affine algebraic \( k \)-group \( G \), we write \( I \) for the kernel of the counit \( \varepsilon_G : k[G] \to \mathcal{R} \). Because \( k[G] \) is Noetherian, \( I/I^2 \) is a finite-dimensional vector space over \( k \cong k[G]/I \). We then have \( \text{Hom}_k(I/I^2, \mathcal{R}) \cong \text{Hom}_k(I/I^2, k) \otimes_k \mathcal{R} \). By [8, Corollaire II.3.3], we have canonical group isomorphisms \( \text{Lie}(\mathcal{R})(\mathcal{R}) \cong \text{Hom}_k(I/I^2, \mathcal{R}) \) and \( g = \text{Lie}(\mathcal{R})(k) \cong \text{Hom}_k(I/I^2, k) \), whence \( \text{Lie}(\mathcal{R})(\mathcal{R}) \cong \mathcal{R}_k \). The Lie structure on \( \mathcal{R}_k \) then canonically gives a Lie structure on \( \mathcal{R}_k \) and hence on \( \text{Lie}(\mathcal{R})(\mathcal{R}) \). We call \( \text{Lie}(\mathcal{R})(\mathcal{R}) \) the \textit{Lie algebra} of \( \mathcal{R} \) over \( k \), and will identify it with \( \mathcal{R}_k \). Moreover, \( \text{Lie}(\mathcal{R})(\mathcal{R}) \) is a functor from the category of affine algebraic \( k \)-groups to that of Lie algebras over \( k \).

**Remark 2.7** More generally, let \( k \) be a commutative ring with 1 and let \( \mathcal{R} \) be a smooth \( k \)-group scheme. For any \( k \)-algebra \( \mathcal{R} \), we can similarly define \( \text{Lie}(\mathcal{R})(\mathcal{R}) \) as above. Because the \( \mathcal{O}_G \)-module \( \Omega^1_{G/k} \) is finite locally free, we have \( \text{Lie}(\mathcal{R})(\mathcal{R}) \cong \text{Lie}(\mathcal{R})(\mathcal{R}) \otimes_k \mathcal{R} \) by [8, Proposition II.4.8].

**Remark 2.8** For any \( d \)-dimensional \( G \)-representation \( (V, \rho_V) \), we write \( \text{gl}_V := \text{Lie}(\text{GL}_V)(k) \). Then we have \( \text{gl}_V, \mathcal{R} = \{ I_d + \varepsilon B \mid B \in \text{Mat}_{d,d}(\mathcal{R}) \} \), after choosing a \( k \)-basis for \( V \). Then, \( I_d + \varepsilon B \mapsto B \) gives a group isomorphism from \( \text{gl}_V, \mathcal{R} \) to \( \text{End}_\mathcal{R}(V_k) \). Henceforth, we will identify \( \text{Lie}(\rho_V)(X) \) as an endomorphism of \( V_k \), for all \( X \in \mathcal{R}_k \).

Replacing \( H \) with \( \text{GL}_V \) and \( f \) with \( \rho_V \) in diagram (1), we obtain a morphism \( \text{Lie}(\rho_V) = \rho_V(\mathcal{R}[\varepsilon]) \circ \iota_G : \mathcal{R}_k \to \text{gl}_V, \mathcal{R} \) of Lie algebras over \( k \). Let \( (W, \rho_W) \in \text{Rep}_k(G) \), and let \( \alpha \in \text{Hom}_G(V, W) \). We then have \( \alpha_R \circ \text{Lie}(\rho_V)(X) = \text{Lie}(\rho_W)(X) \circ \alpha_R \) for all \( X \in \mathcal{R}_k \).

Applying the functor \( \text{Lie}(\mathcal{R})(\mathcal{R}) \) on both sides of the isomorphism in Corollary 2.5 gives us an isomorphism \( \mathcal{R}_k \cong \text{Lie}(\text{Aut}_R(\omega^G))(\mathcal{R}) \) of Lie algebras over \( k \). The following corollary indicates that elements in \( \text{Lie}(\text{Aut}_R(\omega^G))(\mathcal{R}) \) are exactly the derivatives (in the sense of taking derivations of conditions (i–iii) in Theorem 2.4) of elements in \( \text{Aut}_R(\omega^G)(\mathcal{R}) \).

**Corollary 2.9** Let \( G \) be an affine algebraic \( k \)-group, and let \( \mathcal{R} \) be a \( k \)-algebra. Suppose that for any \( (V, \rho_V) \in \text{Rep}_k(G) \), we are given an \( \mathcal{R} \)-linear endomorphism \( \theta_V \) of \( V_k \) subject to the conditions:

(i) \( \theta_V \otimes \mathcal{R} = \theta_V \otimes \text{Id}_{\mathcal{R}_k} + \text{Id}_{\mathcal{R}_k} \otimes \theta_V \) for all \( V \in \text{Rep}_k(G) \);

(ii) \( \theta_V = 0 \), where \( 1 = k \) is the trivial \( G \)-representation;

(iii) \( \theta_V \circ \alpha_R = \alpha_R \circ \theta_V \) for all \( \alpha \in \text{Hom}_G(V, W) \).

Then, there exists a unique element \( X \in \mathcal{R}_k \) such that \( \theta_V = \text{Lie}(\rho_V)(X) \) for all \( (V, \rho_V) \in \text{Rep}_k(G) \).

**Proof** For any \( (V, \rho_V) \in \text{Rep}_k(G) \) and \( \theta_V : V_k \to V_k \), we define an \( \mathcal{R}[\varepsilon] \)-linear map

\( \varepsilon \theta_V : V_k[\varepsilon] \to V_k[\varepsilon], \quad \nu \otimes (x + y\varepsilon) \longmapsto \theta_V(v \otimes x) \otimes \varepsilon. \)

We define an \( \mathcal{R}[\varepsilon] \)-linear endomorphism

\( \tilde{\theta}_V := \text{Id}_{V_k[\varepsilon]} + \varepsilon \theta_V : V_k[\varepsilon] \to V_k[\varepsilon]. \)

Then, \( \tilde{\theta}_V \in \text{Lie}(\text{GL}_V)(\mathcal{R}) \subseteq \text{GL}_V(\mathcal{R}[\varepsilon]) \), because \( \pi_V(\tilde{\theta}_V) = \text{Id}_{V_k} \).

We claim that the family

\( \{ \tilde{\theta}_V : V_k[\varepsilon] \to V_k[\varepsilon] \mid (V, \rho_V) \in \text{Rep}_k(G) \} \)
of $R[\varepsilon]$-linear endomorphisms satisfies conditions (i–iii) in Theorem 2.4. Granting this claim for a moment, we conclude that $\hat{\theta} \in \text{Aut}^\otimes(\omega^G)(R[\varepsilon])$. In particular, there exists a unique element $X \in G(R[\varepsilon])$ such that $\hat{\theta}_V = \rho_V(X)$ for all $(V, \rho_V) \in \text{Rep}_k(G)$. Because $\pi_R(\hat{\theta}) = \text{Id} \in \text{Aut}^\otimes(\omega^G)(R)$, we have $\hat{\theta} \in \text{Lie}(\text{Aut}^\otimes(\omega^G))(R)$. The isomorphism $g_R \cong \text{Lie}(\text{Aut}^\otimes(\omega^G))(R)$ then implies that $X \in g_R$. Furthermore, it follows from the construction that $\hat{\theta}_V = \text{Lie}(\rho_V)(X)$ for all $(V, \rho_V) \in \text{Rep}_k(G)$, and the corollary follows.

It remains to prove the claim. Condition (ii) is clear from the construction. Given $(W, \rho_W) \in \text{Rep}_k(G)$, we compute

\[
\hat{\theta}_V \otimes W = \text{Id}(V \otimes W)_R + \varepsilon \theta_V \otimes W \\
= \text{Id}(V \otimes W)_R + \varepsilon(\theta_V \otimes \text{Id}_{W_R} + \text{Id}_{V_R} \otimes \theta_V) \\
= (\text{Id}_{V_R} + \varepsilon \theta_V) \otimes (\text{Id}_{W_R} + \varepsilon \theta_V) \\
= \hat{\theta}_V \otimes \hat{\theta}_W.
\]

Hence, (2) satisfies condition (i). It remains to show that Theorem 2.4 satisfies condition (iii). Let $\alpha \in \text{Hom}_G(V, W)$. For any $v \otimes (x + y \varepsilon) \in V_{R[\varepsilon]}$, we compute

\[
\alpha_{R[\varepsilon]} \circ \theta_V(v \otimes (x + y \varepsilon)) = \alpha_{R[\varepsilon]}(\theta_V(v \otimes x) \otimes \varepsilon) = (\alpha_R \circ \theta_V)(v \otimes x) \otimes \varepsilon \\
= (\theta_W \circ \alpha_R(v \otimes x) \otimes \varepsilon = \theta_W(\alpha(v) \otimes x) \otimes \varepsilon \\
= \varepsilon \theta_W(\alpha(v) \otimes (x + y \varepsilon)) = \varepsilon \theta_W \circ \alpha_{R[\varepsilon]}(v \otimes (x + y \varepsilon)).
\]

It follows that

\[
\alpha_{R[\varepsilon]} \circ \hat{\theta}_V = \alpha_{R[\varepsilon]} \circ (\text{Id}_{V_R[\varepsilon]} + \varepsilon \theta_V) = \alpha_{R[\varepsilon]} + \alpha_{R[\varepsilon]} \circ \varepsilon \theta_V \\
= \alpha_{R[\varepsilon]} + \varepsilon \theta_W \circ \alpha_{R[\varepsilon]} = (\text{Id}_{W_R[\varepsilon]} + \varepsilon \theta_W) \circ \alpha_{R[\varepsilon]} \\
= \hat{\theta}_W \circ \alpha_{R[\varepsilon]},
\]

as desired.

\section{Filtered and graded fiber functors}

We recall the notions of filtered and graded fiber functors on Tannakian categories following [25]. Let $\Gamma$ be a totally ordered abelian group (written additively), and let $R \in \text{Alg}_k$. A $\Gamma$-graded $R$-module is an $R$-module $M$ together with a direct sum decomposition $M = \bigoplus_{\gamma \in \Gamma} M_{\gamma}$. A morphism between two $\Gamma$-graded $R$-modules $M$ and $N$ is an $R$-linear map $f : M \to N$ such that $f(M_{\gamma}) \subseteq N_{\gamma}$ for all $\gamma \in \Gamma$. We denote by $\Gamma$-$\text{Grad}_R$ the category of $\Gamma$-graded modules over $R$. For $M, N \in \Gamma$-$\text{Grad}_R$, we define the tensor product $(M \otimes_R N)_{\gamma} = \bigoplus_{\gamma' + \gamma'' = \gamma} (M_{\gamma'} \otimes_R N_{\gamma''}).$

Let $M$ be an $R$-module. A $\Gamma$-filtration on $M$ is an increasing map

\[\mathcal{F} : \Gamma \longrightarrow \{R\text{-submodules of } M\}, \quad \gamma \longmapsto \mathcal{F}^\gamma M,\]

such that $\mathcal{F}^\gamma M = 0$ for $\gamma \ll 0$ and $\mathcal{F}^\gamma M = M$ for $\gamma \gg 0$, which is increasing in the sense that $\mathcal{F}^\gamma M \subseteq \mathcal{F}^{\gamma'} M$ whenever $\gamma \leq \gamma'$. A $\Gamma$-filtered $R$-module is an $R$-module $M$ with a $\Gamma$-filtration. To abbreviate notations, we sometimes denote $\mathcal{F}^\gamma M$ by $M^\gamma$ if no confusion
A group-theoretic generalization of the $p$-adic local monodromy theorem

shall arise. A morphism between two $\Gamma$-filtered $R$-modules $M$ and $N$ is an $R$-linear map $f: M \to N$ such that $f(M^\gamma) \subseteq N^\gamma$ for all $\gamma \in \Gamma$. We denote by $\Gamma$-Fil$_R$ the category of $\Gamma$-filtered modules over $R$.

Let $M$ be a $\Gamma$-filtered module over $R$. For any $\gamma \in \Gamma$, we put $F_\gamma^{-} M := \sum_{\gamma' < \gamma} F_{\gamma'} M$. We define

$$\text{gr}_\gamma F M := F_{\gamma} M / F_{\gamma}^{-} M.$$ 

Then, $\text{gr}_\gamma F M := \bigoplus_{\gamma \in \Gamma} \text{gr}_\gamma F M$ is a $\Gamma$-graded $R$ module, and is called the $\Gamma$-graded $R$-module associated to $F$. We thus have a functor

$$\text{gr}: \Gamma - \text{Fil}_R \to \Gamma - \text{Grad}_R.$$ 

Elements $\gamma \in \Gamma$ such that $\text{gr}_\gamma F M \neq 0$ are said to be the $\Gamma$-jumps (or simply jumps) of $F$.

The tensor product structure in $\Gamma$-Fil$_R$ is defined by

$$F_\gamma (M \otimes_R N) = \sum_{\gamma' + \gamma'' = \gamma} F_{\gamma'} M \otimes_F_{\gamma''} N,$$

for all $\Gamma$-filtered modules $M$ and $N$ over $R$.

A morphism $f: M \to N$ in $\Gamma$-Fil$_R$ is said to be admissible (or strict) if

$$f(M^\gamma) = f(M) \cap N^\gamma, \quad \forall \gamma \in \Gamma.$$ 

Following [25, Section 4.1], we say that a short sequence $0 \longrightarrow M' \xrightarrow{f'} M \xrightarrow{f''} M'' \longrightarrow 0$ in $\Gamma$-Fil$_R$ is exact if both of $f'$ and $f''$ are admissible, and the underlying short sequence in $\text{Mod}_R$ is exact.

Let $\mathcal{T}$ be a Tannakian category over $k$, and let $R$ be a $k$-algebra.

(i) A $\Gamma$-graded fiber functor on $\mathcal{T}$ over $R$ is an exact faithful $k$-linear tensor functor $\tau: \mathcal{T} \to \Gamma$-Grad$_R$.

(ii) A $\Gamma$-filtered fiber functor on $\mathcal{T}$ over $R$ is an exact faithful $k$-linear tensor functor $\eta: \mathcal{T} \to \Gamma$-Fil$_R$.

(iii) Given an object $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$ in $\Gamma$-Grad$_R$, we put $\mathcal{T}^\gamma(M) := \bigoplus_{\gamma' \leq \gamma} M_\gamma$. This gives rise to a functor $\text{fil}: \Gamma$-Grad$_R \to \Gamma$-Fil$_R$.

(iv) A $\Gamma$-filtered fiber functor $\eta$ is called splittable if there exists a $\Gamma$-graded fiber functor $\tau$ such that $\eta = \text{fil} \circ \tau$, and $\tau$ is called a splitting of $\eta$.

**Remark 2.10** More concretely, a $\Gamma$-filtered fiber functor is a $k$-linear functor $\eta: \mathcal{T} \to \Gamma$-Fil$_R$ satisfying the following properties (cf. [6, Definition 4.2.6 and Remark 4.2.7]).

(i) It is admissibly (or strictly) functorial, i.e., for any morphism $\alpha: X \to Y$ in $\mathcal{T}$, we have $\eta(\alpha)(\mathcal{T}^\gamma(X)) = \eta(\alpha)(\mathcal{T}^\gamma(Y)) \cap \mathcal{T}^\gamma(\eta(Y))$ for all $\gamma \in \Gamma$.

(ii) It is compatible with tensor products, i.e., we have

$$\mathcal{T}^\gamma(\eta(X \otimes Y)) = \sum_{\gamma' + \gamma'' = \gamma} \mathcal{T}^{\gamma'}(\eta(X)) \otimes \mathcal{T}^{\gamma''}(\eta(Y)),$$

for all $X, Y \in \text{Ob}(\mathcal{T})$ and $\gamma \in \Gamma$. 
(iii) \[
\mathcal{F}^y \eta(\mathbb{I}) = \begin{cases} 
R & \text{for } y \geq 0, \\
0 & \text{for } y < 0,
\end{cases}
\]

where \(\mathbb{I}\) is the identity object in \(\mathcal{T}\). Note that \(\eta(\mathbb{I}), \gamma \mapsto \mathcal{F}^y \eta(\mathbb{I})\) is the identity object in \(\Gamma\text{-Fil}_R\).

Construction 2.11 Let \((M,F) \in Z\text{-Fil}_R\) be a \(Z\)-filtered module with \(Z\)-jumps \(j_1 < \cdots < j_n\). For any \(y \in \mathbb{Q}_{>0}\), we define a \(\mathbb{Q}\)-filtered module \((M,[\gamma]\cdot F)\) by

\[
([\gamma]\cdot F)^y M := \begin{cases} 
0 & \text{for } x < j_1y, \\
M^{j_i} & \text{for } j_iy \leq x < j_{i+1}y, \ 1 \leq i \leq n - 1, \\
M & \text{for } x \geq j_ny.
\end{cases}
\]

We then have a fully faithful embedding \([\gamma]\cdot : Z\text{-Fil}_R \to Q\text{-Fil}_R\). Similarly, we have a fully faithful embedding \([\gamma]\cdot : Z\text{-Grad}_R \to Q\text{-Grad}_R\) by defining \([\gamma]\cdot := gr \circ [\gamma]\cdot \circ \text{fil}.

To end this subsection, we exhibit the following theorem for latter use. (Be aware that in [25], the author only considers \(\Gamma\)-gradings and \(\Gamma\)-filtrations for \(\Gamma = \mathbb{Z}\).)

**Theorem 2.12** [25, Theorem 4.15] Let \(\mathcal{T}\) be a Tannakian category over a field \(k\), and let \(R\) be a \(k\)-algebra. Let \(\eta : \mathcal{T} \to Z\text{-Fil}_R\) be a \(Z\)-filtered fiber functor. If \(\text{Aut}_{\text{\circ}}^\otimes (\text{forg} \circ \eta)\) is prosmooth (i.e., a limit of smooth algebraic group schemes) over \(R\), where \(\text{forg} : Z\text{-Fil}_R \to \text{Mod}_R\) is the forgetful functor, then \(\eta\) is splitable.

### 3 \(G\)-\(\varphi\)-modules over the Robba ring

We fix an affine algebraic \(F\)-group \(G\) in this section.

#### 3.1 Definition

Let \(R \in \{\mathbb{E}^\dagger, \mathbb{R}, \mathbb{E}^\dagger, \mathbb{R}\}\) equipped with an absolute Frobenius lift \(\varphi\). The following definition is motivated by that of \(G\)-isocrystals introduced in [6, Section IX.1].

**Definition 3.1** A \(G\)-\(\varphi\) -module over \(R\) is an exact faithful \(F\)-linear tensor functor

\(I: \text{Rep}_F(G) \to \text{Mod}_R^\varphi\),

which satisfies \(\text{forg} \circ I = \omega^G \otimes R\), where \(\text{forg}: \text{Mod}_R^\varphi \to \text{Mod}_R\) is the forgetful functor. The category of \(G\)-\(\varphi\) -modules over \(R\) is denoted by \(G\text{-Mod}_R^\varphi\), whose morphisms are morphisms of tensor functors.

Let \((V, \rho) \in \text{Rep}_F(G)\), and let \(g \in G(R)\). We define \(I(g)(V) := (V_R, g\varphi)\), where

\[g\varphi : V_R \to V_R, \quad v \otimes f \mapsto \rho(g)(v \otimes 1)\varphi(f)\].

Let \(V, W \in \text{Rep}_F(G)\). We have a canonical isomorphism \((V \otimes W)_R \cong V_R \otimes_R W_R\), and we will henceforth identify them. Given any \(\alpha \in \text{Hom}_G(V, W)\), we define
I(g)(\alpha) := \alpha_R. We thus have the following $G$-$\phi$-module over $R$ (associated to $g$).
\[
I(g) \colon \text{Rep}_F(G) \longrightarrow \text{Mod}_R^g, \quad V \longmapsto (V_R, g\phi).
\]
We call $I(g)(V) = (V_R, g\phi)$ the $G$-$\phi$-module over $R$ associated to $g$.

For any $g \in G(R)$, we sometimes write $\Phi_g = \Phi_{g,V}$ for the $\phi$-linear action $g\phi$ on $V_R$. Both notations have their own advantages in practice.

**Remark 3.2** For any $g \in G(R)$, we define $\Phi(g) := G(\phi)(g)$. For any $(V, \rho) \in \text{Rep}_F(G)$, we have a commutative diagram
\[
\begin{array}{ccc}
G(R) & \xrightarrow{\rho(R)} & GL_V(R) \\
G(\phi) \downarrow & & \downarrow GL_V(\phi) \\
G(R) & \xrightarrow{\rho(R)} & GL_V(R)
\end{array}
\]
Hence, $\rho(\phi(g)) = \phi(\rho(g))$. For any $h \in G(R)$ and $n, m \geq 0$, we have the following formula in $G(R) \rtimes \langle \phi \rangle$:
\[
(h\phi^n) \circ (g\phi^m) = (h\phi^n(g))\phi^{n+m}.
\]

### 3.2 The $\mathbb{Q}$-filtered fiber functor $HN_g$

We fix an element $g \in G(R)$.

**Construction 3.3** For any $V \in \text{Rep}_F(G)$, we have a $\phi$-module $(V_R, g\phi)$ over $\mathcal{R}$. Kedlaya’s slope filtration theorem [9, Theorem 6.10] then provides a filtration
\[
0 \subseteq V_{\mathcal{R}}^{\mu_1} \subseteq \cdots \subseteq V_{\mathcal{R}}^{\mu_l} = V_{\mathcal{R}},
\]
satisfying
- $V_{\mathcal{R}}^{\mu_i}$ is pure of some slope $\mu_1 \in \mathbb{Q}$ and each $V_{\mathcal{R}}^{\mu_i}/V_{\mathcal{R}}^{\mu_{i-1}}$ is pure of some slope $\mu_i \in \mathbb{Q}$ for $2 \leq i \leq l$;
- $\mu_1 < \cdots < \mu_l$.

We thus have an increasing map
\[
HN_g : \mathbb{Q} \longrightarrow \{\mathcal{R}\text{-modules of } V_{\mathcal{R}}\}
\]
\[
x \longmapsto \mathcal{H}\mathcal{N}_g^x(V_{\mathcal{R}}),
\]
where
\[
\mathcal{H}\mathcal{N}_g^x(V_{\mathcal{R}}) = \begin{cases} 
0 & \text{for } x < \mu_1, \\
V_{\mathcal{R}}^{\mu_i} & \text{for } \mu_i \leq x < \mu_{i+1}, 1 \leq i \leq l-1, \\
V_{\mathcal{R}} & \text{for } x \geq \mu_1.
\end{cases}
\]

Then, $(V_{\mathcal{R}}, \mathcal{H}\mathcal{N}_g)$ is a $\mathbb{Q}$-filtered module over $\mathcal{R}$ with $\mathbb{Q}$-jumps $\mu_1 < \cdots < \mu_l$. We will denote $\mathcal{H}\mathcal{N}_g^x(V_{\mathcal{R}})$ by $V_x$ when $\mathcal{H}\mathcal{N}_g$ is clear in the context.

**Theorem 3.4** The assignments
\[
V \longmapsto (V_{\mathcal{R}}, \mathcal{H}\mathcal{N}_g) \quad \text{and} \quad \alpha \longmapsto \alpha_{\mathcal{R}},
\]
for all $\alpha \in \text{Hom}_G(V, W)$, define a $Q$-filtered fiber functor 
$$H_N^g: \text{Rep}_F(G) \to Q\text{-Fil}_R.$$ 

**Proof** This is Propositions 3.5 and 3.6 below. 

For any admissible extension $E$ of $K$, we first remark that the $\varphi$-equivariant embedding $\psi: \mathcal{R} \to \tilde{\mathcal{R}}(E, t)$ is faithfully flat (see [11, Remark 3.5.3]). We also remark that, if $M_1$ and $M_2$ are pure $\varphi$-modules over $\mathcal{R}$ of slopes $\mu_1$ and $\mu_2$, respectively, then $M_1 \otimes_{\mathcal{R}} M_2$ is pure of slope $\mu_1 + \mu_2$ (cf. [11, Corollary 1.6.4]). These facts will be repeatedly used in the sequel.

**Proposition 3.5** The assignments in Theorem 3.4 yield a faithful $F$-linear tensor functor $H_N^g: \text{Rep}_F(G) \to Q\text{-Fil}_R$.

**Proof** Let $\mathbb{I} = F$ be the trivial $G$-representation. Then, $\mathbb{I} \otimes_F \mathcal{R} = \mathcal{R}$ is of rank 1 with slope 0, proving that $H_N^g$ preserves identity objects.

We claim that $H_N^g$ is functorial. Let $\alpha \in \text{Hom}_G(V, W)$ be a morphism of finite-dimensional $G$-modules. We need to show that $\alpha_{\mathcal{R}}(V^x_{\mathcal{R}}) \subseteq W^x_{\mathcal{R}}$ for all $x \in Q$. Choose by Lemma 2.2 an admissible extension $E$ of $K$ such that $\kappa_E$ is strongly difference-closed. For any fixed $x \in Q$, we set $V^x_{\tilde{\mathcal{R}}(E, t)} := V^x_0 \otimes_{\mathcal{R}} \tilde{\mathcal{R}}(E, t)$, and $W^x_{\tilde{\mathcal{R}}(E, t)} := W^x_0 \otimes_{\mathcal{R}} \tilde{\mathcal{R}}(E, t)$. By Lemma 2.3(iii), we have a decomposition $W^x_{\tilde{\mathcal{R}}(E, t)} = W^x_{\tilde{\mathcal{R}}(E, t)} \oplus W^x_{\tilde{\mathcal{R}}(E, t)}$ of $\varphi$-modules over $\tilde{\mathcal{R}}(E, t)$, where $W^x_{\tilde{\mathcal{R}}(E, t)}$ (resp. $W^x_{\tilde{\mathcal{R}}(E, t)}$) has slopes less or equal to $x$ (resp. greater than $x$). By Lemma 2.3(iv), the induced morphism $V^x_{\tilde{\mathcal{R}}(E, t)} \to W^x_{\tilde{\mathcal{R}}(E, t)}$ of $\varphi$-modules is zero. We thus have $\alpha_{\tilde{\mathcal{R}}(E, t)}(V^x_{\tilde{\mathcal{R}}(E, t)}) \subseteq W^x_{\tilde{\mathcal{R}}(E, t)}$. Given any $v \in V^x_{\tilde{\mathcal{R}}(E, t)}$, we may write $\alpha_{\tilde{\mathcal{R}}(E, t)}(v \otimes 1) = \alpha_{\tilde{\mathcal{R}}}(v) \otimes 1 = \sum_{i \in I} w_i \otimes s_i$ for some finite set $I$, with $w_i \in W^x_i$ and $s_i \in \tilde{\mathcal{R}}(E, t)$ for all $i \in I$. Let $M$ be the $\mathcal{R}$-submodule of $W^x_{\mathcal{R}}$ generated by $\alpha_{\mathcal{R}}(v)$ and the $w_i$, and let $N$ be the $\mathcal{R}$-submodule of $W^x_{\mathcal{R}}$ generated by the $w_i$. We then have $(M/N) \otimes_{\mathcal{R}} \tilde{\mathcal{R}}(E, t) \cong (M \otimes_{\mathcal{R}} \tilde{\mathcal{R}}(E, t))/(N \otimes_{\mathcal{R}} \tilde{\mathcal{R}}(E, t)) = 0$. It follows that $M/N = 0$ as $\mathcal{R} \to \tilde{\mathcal{R}}(E, t)$ is faithfully flat. We thus have $\alpha_{\mathcal{R}}(v) \in N \subseteq W^x_{\mathcal{R}}$, as desired.

It remains to show that $H_N^g$ preserves tensor products (in the sense of Remark 2.10(ii)). Let $V$ and $W$ be two finite-dimensional $G$-modules, and suppose that the slope filtration of $(V_{\mathcal{R}}, g\varphi)$ (resp. $(W_{\mathcal{R}}, g\varphi)$) has jumps $\mu_1 < \cdots < \mu_r$ (resp. $v_1 < \cdots < v_{l_w}$). By [12, Lemma 16.4.3], $((V \otimes_F W)_{\mathcal{R}}, g\varphi)$ has jumps $\{\mu_i + v_j | 1 \leq i \leq l_V, 1 \leq j \leq l_W\}$. Fix any $1 \leq l \leq l_V$ and $1 \leq s \leq l_W$; we need to show

$$((V \otimes_F W)^{\mu_l + v_s}_{\mathcal{R}}) = \sum_{x, y \in \mathbb{Q}} \frac{V^x_{\mathcal{R}} \otimes_{\mathcal{R}} W^y_{\mathcal{R}}}{x + y = \mu_l + v_s},$$

and we will do so in the remainder of the proof.

We claim that

$$\sum_{x, y \in \mathbb{Q}} \frac{V^x_{\mathcal{R}} \otimes_{\mathcal{R}} W^y_{\mathcal{R}}}{x + y = \mu_l + v_s} = \sum_{1 \leq i \leq l_V, 1 \leq j \leq l_W} V^{\mu_i}_{\mathcal{R}} \otimes_{\mathcal{R}} W^{v_j}_{\mathcal{R}}.$$
It is clear that the RHS is contained in the LHS; we now show the reverse inclusion. Let \( x, y \in \mathbb{Q} \) such that \( x + y = \mu_1 + v_1 \). If \( x < \mu_1 \) or \( y < v_1 \), then \( V_{\mathbb{R}}^x \otimes \mathbb{R} W_{\mathbb{R}}^y = 0 \) which is contained in the RHS. Otherwise, there exists the largest integer \( 1 \leq i \leq l_v \) (resp. \( 1 \leq j \leq l_w \)) with the property that \( \mu_i \leq x \) (resp. \( v_j \leq y \)). We then have \( V_{\mathbb{R}}^x \otimes \mathbb{R} W_{\mathbb{R}}^y = V_{\mathbb{R}}^{\mu_i} \otimes \mathbb{R} W_{\mathbb{R}}^{v_j} \) and \( \mu_i + v_j \leq \mu_1 + v_1 \). The claim is thus proved.

From Lemma 2.3(iii), we see that
\[
(V \otimes W)^{\mu_i + v_j}_{\hat{R}(E, t)} = \left( \sum_{\mu_i + v_j \leq \mu_1 + v_1} V_{\mathbb{R}}^{\mu_i} \otimes W_{\mathbb{R}}^{v_j} \right) \otimes \hat{R}(E, t).
\]
Therefore, we have
\[
(V \otimes W)^{\mu_i + v_j}_{\mathbb{R}} = \sum_{\mu_i + v_j \leq \mu_1 + v_1} V_{\mathbb{R}}^{\mu_i} \otimes W_{\mathbb{R}}^{v_j}
\]
by Lemma 2.3(i) and the fact that \( \mathbb{R} \to \hat{R}(E, t) \) is faithfully flat. The desired equality (i) then follows from the preceding claim. \( \blacksquare \)

Let \( (M, \varphi) \) be a \( \varphi \)-module over \( \hat{R} \) of rank \( d \). Then, \( \Phi \) is invertible, because the Frobenius lift on \( \hat{R} \) is bijective, and \( (M, \varphi^{-1}) \) is a \( \varphi^{-1} \)-module over \( \hat{R} \). More explicitly, let \( A \in \text{GL}_d(\hat{R}) \) be the matrix of action of \( \varphi \) in some basis for \( M \) over \( \hat{R} \). Then, in the same basis, the matrix of action of \( \varphi^{-1} \) is \( \varphi^{-1}(A^{-1}) \). For example, if \( M = V_{\mathbb{R}}^x \) for some \( x \in \text{Rep}_F(G) \), and \( \Phi = \psi(g) \varphi \) where \( \psi \) denotes (by abuse of notation) the group morphism \( G(\mathbb{R}) \to G(\hat{R}) \) induced by the embedding \( \psi: \mathbb{R} \to \hat{R} \) recalled above Proposition 3.5, then
\[
(\psi(g) \varphi) \cdot (\varphi^{-1}(\psi(g^{-1})) \varphi^{-1}) = 1
\]
in \( G(\hat{R}) \times (\varphi) \) (cf. Remark 3.2), which implies that \( \varphi^{-1} = \varphi^{-1}(\psi(g^{-1})) \varphi^{-1} \).

Let \( M \) be a standard \( \varphi \)-module over \( \hat{R} \) of slope \( \mu = s/r \) with \( r > 0 \) and \( (s, r) = 1 \). Namely, we have a standard basis \( e_1, \ldots, e_r \) in which \( \varphi \) acts via
\[
A = \begin{pmatrix}
0 & & & \varphi'^{r-1} \\
1 & & & 0 \\
& \ddots & \ddots & \ddots \\
& & 1 & 0
\end{pmatrix}.
\]
Then,
\[
\varphi^{-1}(A^{-1}) = \begin{pmatrix}
0 & & & 1 \\
1 & & & 0 \\
& \ddots & \ddots & \ddots \\
& & 1 & 0
\end{pmatrix},
\]
which implies that \( (M, \varphi^{-1}) \) is a standard \( \varphi^{-1} \)-module pure of slope \(-\mu\).

**Proposition 3.6**  The functor \( \text{HN}_{G}: \text{Rep}_F(G) \to \mathbb{Q}-\text{Fil}_R \) is exact.

**Proof**  Let \( \alpha \in \text{Hom}_G(V, W) \) be a morphism of finite-dimensional \( G \)-modules. We need to show that \( \alpha_{\mathbb{R}}(V_{\mathbb{R}}^x) = \alpha_{\mathbb{R}}(V_{\mathbb{R}}^y) \cap W_{\mathbb{R}}^x \) for all \( x \in \mathbb{Q} \). For any fixed \( x \in \mathbb{Q} \), the functoriality in Proposition 3.5 already implies that \( \alpha_{\mathbb{R}}(V_{\mathbb{R}}^x) \subseteq \alpha_{\mathbb{R}}(V_{\mathbb{R}}^y) \cap W_{\mathbb{R}}^x \). Thus, it suffices to show that for any nonzero element \( v \in V_{\mathbb{R}}^y \) such that \( \alpha_{\mathbb{R}}(v) \in W_{\mathbb{R}}^x \), there exists \( v' \in V_{\mathbb{R}}^x \) with \( \alpha_{\mathbb{R}}(v') = \alpha_{\mathbb{R}}(v) \).
By Lemma 2.3(iii), we have decompositions
\[
(4) \quad V_{\mathcal{R}(E,t)} = V_{\mathcal{R}(E,t)}^x \oplus V_{\mathcal{R}(E,t)}^r \quad \text{and} \quad W_{\mathcal{R}(E,t)} = W_{\mathcal{R}(E,t)}^x \oplus W_{\mathcal{R}(E,t)}^r
\]
of \(\phi\)-modules over \(\mathcal{R}(E,t)\), in which \(V_{\mathcal{R}(E,t)}^x\) and \(W_{\mathcal{R}(E,t)}^x\) have slopes less or equal to \(x\), while \(V_{\mathcal{R}(E,t)}^r\) and \(W_{\mathcal{R}(E,t)}^r\) have slopes greater than \(x\). Notice that the composition
\[
\xi: V'_{\mathcal{R}(E,t)} \longrightarrow V_{\mathcal{R}(E,t)}^x \oplus V_{\mathcal{R}(E,t)}^r \stackrel{a_{\mathcal{R}(E,t)}}{\longrightarrow} W_{\mathcal{R}(E,t)}^x \oplus W_{\mathcal{R}(E,t)}^r \longrightarrow W_{\mathcal{R}(E,t)}^x
\]
is a morphism of \(\phi\)-modules. We claim that \(\xi = 0\). We write \(\phi = \psi(g)\phi\), then \(\phi^{-1} = \phi^{-1}(\psi(g^{-1}))\phi^{-1}\). Because \(\alpha\) is \(G\)-equivariant and \(\phi^{-1}(\psi(g^{-1})) \in G(\mathcal{R}(E,t))\), we have that \(\alpha: (V_{\mathcal{R}(E,t)}, \phi^{-1}) \rightarrow (W_{\mathcal{R}(E,t)}, \phi^{-1})\) is a morphism of \(\phi^{-1}\)-modules. On the other hand, we also have decompositions of \(\phi^{-1}\)-modules as in (2), together with the induced morphism \(\xi: V'_{\mathcal{R}(E,t)} \rightarrow W_{\mathcal{R}(E,t)}^x\) of \(\phi^{-1}\)-modules. But in this case, \(V'_{\mathcal{R}(E,t)}\) has slopes less than \(x\), while \(W_{\mathcal{R}(E,t)}^x\) has slopes greater or equal to \(x\). It then follows from Lemma 2.3(iv) that \(\xi = 0\), as claimed.

Therefore, we find \(v_1, \ldots, v_n \in V_{\mathcal{R}}^x\) and \(s_1, \ldots, s_n \in \mathcal{R}(E,t)\) such that
\[
\alpha_{\mathcal{R}(E,t)}(v \otimes 1) = \alpha_{\mathcal{R}}(v) \otimes 1 = \sum_{i=1}^n \alpha_{\mathcal{R}}(v_i) \otimes s_i.
\]
Let \(M\) be the submodule of \(W_{\mathcal{R}}\) generated by \(\alpha_{\mathcal{R}}(v)\) and the \(\alpha_{\mathcal{R}}(v_i)\), and let \(N\) be the submodule generated by the \(\alpha_{\mathcal{R}}(v_i)\). We then have
\[
(M/N) \otimes_{\mathcal{R}} \mathcal{R}(E,t) \cong (M \otimes_{\mathcal{R}} \mathcal{R}(E,t))/(N \otimes_{\mathcal{R}} \mathcal{R}(E,t)) = 0.
\]
It follows that \(M/N = 0\) as \(\mathcal{R} \rightarrow \mathcal{R}(E,t)\) is faithfully flat, and hence, \(\alpha_{\mathcal{R}}(v) = \sum_{i=1}^n r_i \alpha_{\mathcal{R}}(v_i) \in W_{\mathcal{R}}^x\) for some \(r_i \in \mathcal{R}\). Put \(v' := \sum_{i=1}^n r_i v_i \in V_{\mathcal{R}}^x\), we then have \(\alpha_{\mathcal{R}}(v') = \alpha_{\mathcal{R}}(v)\), as desired.

### 3.3 Splittings of \(\text{HN}_g\)

As before, we fix an element \(g \in G(\mathcal{R})\). In Section 3.2, we have constructed a \(Q\)-filtered fiber functor \(\text{HN}_g : \text{Rep}_F(G) \rightarrow \text{Q-Fil}_\mathcal{R}\). In this subsection, we show that \(\text{HN}_g\) is splittable whenever \(G\) is smooth. Our strategy goes as follows. We first use Lemma 3.10 reducing \(\text{HN}_g\) to a \(Z\)-filtered fiber functor \(\text{HN}_g^Z\) to which Theorem 2.12 is applicable. This \(\text{HN}_g^Z\) then admits a \(Z\)-splitting. Finally, in Theorem 3.12, we lift such a \(Z\)-splitting to a \(Q\)-splitting of \(\text{HN}_g\).

**Definition 3.7** We define the **support** of \(\text{HN}_g\) by
\[
\text{Supp}(\text{HN}_g) := \{ x \in Q \mid \text{gr}_{\text{HN}_g}^x(V) \neq 0 \text{ for some } V \in \text{Rep}_F(G) \}.
\]
Notice that \(\text{Supp}(\text{HN}_g)\) is the set of jumps of the slope filtrations of \((V_{\mathcal{R}}, g\phi)\) for all \(V \in \text{Rep}_F(G)\).

The general idea of the following construction was addressed in [2], after Definition 2.5 in loc. cit.; we will make it more explicit in our case.
A group-theoretic generalization of the $p$-adic local monodromy theorem

Construction 3.8 Let $W \in \text{Rep}_F(G)$ be a faithful representation. Because $G$ is algebraic, $W$ is a tensor generator for $\text{Rep}_F(G)$, i.e., any representation $V$ of $G$ is a subquotient of a direct sum of representations $\bigotimes^m (W \oplus W^\vee)$ for various $m \in \mathbb{N}$. (See [18, Theorem 4.14].) Therefore, $\text{Supp}(HN_g)$ is the additive subgroup of $\mathbb{Q}$ finitely generated by the $\mathbb{Q}$-jumps $v_1, \ldots, v_n$ of $(W_\mathcal{R}, g\varphi)$. We write $v_i = s_i/d_i$ with $d_i > 0$ and $(s_i, d_i) = 1$ for $1 \leq i \leq n$. Let $d_g \in \mathbb{N}$ be the least common multiple of the $d_i$. We then have $d_g v_i \in \mathbb{Z}$ for $1 \leq i \leq n$. In particular, we have
\[ d_g = \min\{m \in \mathbb{N} \mid mx \in \mathbb{Z}, \forall x \in \text{Supp}(HN_g)\}. \]

Therefore, $d_g$ is uniquely determined by $g$. We call $d_g$ the least common denominator of $g$.

Remark 3.9 We conclude from Construction 3.8 that $\text{Supp}(HN_g)$ is isomorphic to $\mathbb{Z}$ or $0$. In fact, if $(W_\mathcal{R}, g\varphi)$ has only one jump $0$, then $\text{Supp}(HN_g) = 0$. Otherwise, the choice of $d_g$ implies that $\gcd(d_g v_1, \ldots, d_g v_n) = 1$. We then have $d_g \text{Supp}(HN_g) = \mathbb{Z}$, because the $d_g v_i$ generate $\mathbb{Z}$ as a $\mathbb{Z}$-module. Therefore, $x \mapsto d_g x$ gives an isomorphism $\text{Supp}(HN_g) \cong \mathbb{Z}$.

Lemma 3.10 $HN_g$ factors through a $\mathbb{Z}$-filtered fiber functor $HN^Z_g : \text{Rep}_F(G) \to Z\text{-Fil}_\mathcal{R}$, which makes the diagram
\[
\begin{array}{ccc}
\text{Rep}_F(G) & \xrightarrow{HN_g} & \text{Q\text{-Fil}_\mathcal{R}} \\
\downarrow{HN^Z_g} & & \downarrow{[d_g^{-1}]_s} \\
\text{Z\text{-Fil}_\mathcal{R}} & & \end{array}
\]
commute.

We remark that the functor $[d_g^{-1}]_s$ (see Construction 2.11) is nothing but relabeling the jumps by multiplying all jumps with $d_g^{-1}$. In particular, this lemma implies that $\text{gr}_{HN_g}^x (V) = \text{gr}_{HN^Z_g}^{d_g^{-1}x} (V)$ for all $x \in \mathbb{Q}$ and $V \in \text{Rep}_F(G)$.

Proof of Lemma 3.10 Let $V \in \text{Rep}_F(G)$, and let $\mu_1, \ldots, \mu_l$ be the $\mathbb{Q}$-jumps of $(V_\mathcal{R}, g\varphi)$. We then have $d_g \mu_i \in \mathbb{Z}$ for all $i$. We have an increasing map
\[ \mathcal{F}_g : \mathbb{Z} \to \{\mathcal{R}\text{-submodules of } V_\mathcal{R}\}, \]
\[ x \mapsto \mathcal{F}_g^x (V_\mathcal{R}), \]
where
\[ \mathcal{F}_g^x (V_\mathcal{R}) := \begin{cases} 0 & \text{for } x < d_g \mu_1, \\ \text{HN}^\mu_i (V_\mathcal{R}) & \text{for } d_g \mu_i \leq x < d_g \mu_{i+1}, 1 \leq i \leq l - 1, \\ V_\mathcal{R} & \text{for } x \geq d_g \mu_1. \end{cases} \]

Then, $(V_\mathcal{R}, \mathcal{F}_g)$ is a $\mathbb{Z}$-filtered module over $\mathcal{R}$ with $\mathbb{Z}$-jumps $d_g \mu_1 < \cdots < d_g \mu_l$. We thus have a $\mathbb{Z}$-filtered fiber functor
\[ HN^Z_g : \text{Rep}_F(G) \to Z\text{-Fil}_\mathcal{R}, \]
\[ V \mapsto (V_\mathcal{R}, \mathcal{F}_g), \]
satisfying $HN_g = [d_g^{-1}]_s \circ HN^Z_g$. 

\[ \square \]
By the definition of $\text{Aut}^\otimes$ and Corollary 2.5, we have $\text{Aut}^\otimes(\omega^G_R)(R) = \text{Aut}^\otimes(\omega^G_R) \cong G(R)$ for all $R \in \text{Alg}_k$. For any $R$-algebra $S$, we then have

$$\text{Aut}^\otimes(\omega^G_R)(S) = \text{Aut}^\otimes(\omega^G_R \otimes S) = \text{Aut}^\otimes(\omega^G_S) \cong G_R(S).$$

**Proposition 3.11** Let $G$ be a smooth $F$-group. Then, $HN^Z_g$ is splittable.

**Proof** Because $\text{forg} \circ HN^Z_g = \omega^G_R \otimes R$, we have

$$\text{Aut}^\otimes(\text{forg} \circ HN^Z_g) = \text{Aut}^\otimes(\omega^G_R) \cong G_R.$$ Notice that $G_R$ is smooth over $R$; the proposition then follows from Theorem 2.12. 

**Theorem 3.12** Let $G$ be a smooth $F$-group. Then, the $Q$-filtered fiber functor $HN_g$ is splittable.

**Proof** Choose a splitting $\tau_g : \text{Rep}_F(G) \rightarrow Z\text{-Grad}_R$ of $HN^Z_g$ by Proposition 3.11, we then have a $Q$-graded fiber functor $[d_{g}^{-1}]_* \circ \tau_g : \text{Rep}_F(G) \rightarrow Q\text{-Grad}_R$. On the other hand, we have the diagram

(5) $\begin{array}{ccc}
Z\text{-Grad}_R & \xrightarrow{\text{fil}} & Z\text{-Fil}_R \\
\downarrow \tau_g & & \downarrow [d_{g}^{-1}]_* \\
\text{Rep}_F(G) & \xrightarrow{HN^Z_g} & Q\text{-Fil}_R \\
\downarrow \text{fil} & & \\
Q\text{-Grad}_R & & \\
\end{array}$

with the upper-left, the upper-right, and the bottom triangles commutative. Here, the commutativity of the upper-left (resp. the upper-right) triangle follows from Proposition 3.11 (resp. Lemma 3.10); for the bottom one, we note that $[d_{g}^{-1}]_* \circ \text{fil} = \text{fil} \circ [d_{g}^{-1}]_*$. Hence, the outer diagram also commutes, which implies that $HN^Z_g$ factors through the $Q$-graded fiber functor $[d_{g}^{-1}]_* \circ \tau_g$, as desired.

### 3.4 The slope morphism

Let $R$ be a commutative ring with 1, and let $\Gamma$ be an abelian group (not necessarily finitely generated). We first continue the discussions in Section 2.5 to see how $\Gamma$-gradings over $R$ are related to $D_R(\Gamma)$-modules, for some affine group scheme $D_R(\Gamma)$ which will be defined as follows.

The group algebra $R[\Gamma] := \bigoplus_{\gamma \in \Gamma} Re_\gamma$ carries a Hopf algebra structure, where the comultiplication is given by $\Delta(e_\gamma) = e_\gamma \otimes e_\gamma$, the counit is given by $\varepsilon(e_\gamma) = 1$, and the antipode is given by $S(e_\gamma) = e_{-\gamma}$, for all $\gamma \in \Gamma$. We denote by $D_R(\Gamma)$ the affine $R$-group scheme represented by $R[\Gamma]$. For any $\gamma \in \Gamma$, the Hopf algebra morphism $R[\mathbb{Z}] \rightarrow R[\Gamma], e_1 \mapsto e_\gamma$ gives rise to a character $\chi_\gamma : D_R(\Gamma) \rightarrow G_{m, R}$ of $D_R(\Gamma)$. For the remainder of this paper, we denote by $D_R$ the $R$-group scheme $D_R(Q)$.

Let $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$ be a $\Gamma$-graded module over $R$. Then, $M$ becomes a $D_R(\Gamma)$-module where $D_R(\Gamma)$ acts on each $M_\gamma$ via $\chi_\gamma$. The following lemma shows that this assignment gives an equivalence of categories.
Lemma 3.13 [8, Proposition II.2.5] $\Gamma$-Grad$_R$ is equivalent to the category of $D_R(\Gamma)$-modules.

Corollary 3.14 For any $y \in \mathbb{Q}_{>0}$, the functor $[y]_*: Z$-Grad$_R \to \mathbb{Q}$-Grad$_R$ corresponds to the character $\chi_y: D_R \to G_{m, R}$.

Proof Let $M \in Z$-Grad$_R$. By Lemma 3.13, we may write $M = \bigoplus_{n \in \mathbb{Z}} M_n$ as a direct sum of eigenmodules. By construction, we have $[y]_*(M) = \bigoplus_{n \in \mathbb{Z}} ([y]_*(M))_n$ with $([y]_*(M))_n = M_n$ for all $n$, which is also a decomposition into eigenmodules. Therefore, giving $[y]_*$ is equivalent to giving the commutative diagram

$$
\begin{array}{ccc}
M_n & \longrightarrow & ([y]_*(M))_n \\
\downarrow & & \downarrow \\
M_n \otimes_R R[Z] & \longrightarrow & ([y]_*(M))_n \otimes_R R[Q]
\end{array}
$$

of $R$-modules for all $n \in \mathbb{Z}$ such that $M_n \neq 0$. Here, the left (resp. the right) vertical arrow is given by $m \mapsto m \otimes e_n$ (resp. $m \mapsto m \otimes e_{yn}$). The diagram then corresponds to $R[Z] \rightarrow R[Q]$, $e_1 \mapsto e_y$, as desired. $\blacksquare$

We now apply the preceding discussions to the functors constructed in Section 3.3, following [14, Section 4].

Construction 3.15 Let $g \in G(\mathcal{R})$; we fix a splitting $\tau_g$ of $HN^Z_g$ given by Proposition 3.11. For any $(V, \rho) \in \text{Rep}_F(G)$, $\tau_g$ gives a decomposition of $V_{\mathcal{R}}$, which induces a morphism $\lambda_{\rho, g}: G_{m, \mathcal{R}} \rightarrow \text{GL}_{V_{\mathcal{R}}}$ by Lemma 3.13. Let $S$ be an $\mathcal{R}$-algebra, and let $a \in G_{m, \mathcal{R}}(S)$. We then have a family

$$
\{ \lambda_{\rho, g}(a): V_S \rightarrow V_S \mid (V, \rho) \in \text{Rep}_F(G) \}
$$

of $S$-linear maps. Because $\tau_g$ is a tensor functor, this family satisfies conditions (i–iii) in Theorem 2.4. We thus find a unique element $b \in G_{\mathcal{R}}(S)$ such that $\lambda_{\rho, g}(a) = \rho(b)$ for all $(V, \rho) \in \text{Rep}_F(G)$. The assignment $a \mapsto b$ is functorial in $S$, because both $\lambda_{\rho, g}$ and $\rho$ are functorial. We then have a morphism of $\mathcal{R}$-groups

$$
\lambda_g: G_{m, \mathcal{R}} \rightarrow G_{\mathcal{R}},
$$

which is said to be the $\mathbb{Z}$-slope morphism of $g$.

By Corollary 3.14, $[d_{g}^{-1}]_*$ gives a unique morphism $\chi_{d_{g}^{-1}}: D_{\mathcal{R}} \rightarrow G_{m, \mathcal{R}}$. We define

$$
v_g := \lambda_g \circ \chi_{d_{g}^{-1}}: D_{\mathcal{R}} \rightarrow G_{\mathcal{R}},
$$

which is said to be the $\mathbb{Q}$-slope morphism of $g$.

The following example demonstrates explicitly how $\lambda_g$ and $v_g$ are related to the splittings constructed in Section 3.3 (see Diagram 3).

Example 3.16 Let $(V, \rho) \in \text{Rep}_F(G)$ and suppose that the slope filtration of $(V_{\mathcal{R}}, g\varphi)$ is

$$
0 \subseteq V_{\mathcal{R}}^{\mu_1} \subseteq \cdots \subseteq V_{\mathcal{R}}^{\mu_i} = V_{\mathcal{R}}
$$
with jumps $\mu_1 < \cdots < \mu_l$. By Theorem 3.12, the functor $[d_{g}^{-1}]_{\ast} \circ \tau_g : \text{Rep}_F(G) \to \mathbb{Q}$. \text{Grad}_\mathbb{R}$ gives a splitting

$$V_\mathbb{R} = V_{\mathbb{R},\mu_1} \oplus \cdots \oplus V_{\mathbb{R},\mu_l}$$

of $\text{HN}_g(V)$, i.e., we have $\bigoplus_{i=1}^{j} V_{\mathbb{R},\mu_i} = V_{\mathbb{R},\nu}$ for all $1 \leq j \leq l$.

First, we fix $1 \leq i \leq l$. Let $S \in \text{Alg}_\mathbb{R}$ and $a \in D_\mathbb{R}(S)$, then $\rho \circ v_g(a)$ acts on $V_{\mathbb{R},\mu_i} \otimes_\mathbb{R} S$ via multiplication by $\chi_{\mu_i}(a)$. By Lemma 3.10, $\rho \circ \lambda_g(b)$ acts on $V_{\mathbb{R},\mu_i}$ via multiplication by $b^{d_{\mu_i}}$, for all $b \in G_{m,\mathbb{R}}(S)$. Notice that for any $\frac{m}{n} \in \mathbb{Q}$, we have $\varepsilon_\frac{m}{n} = (\varepsilon_\frac{1}{n})^m \in \mathbb{R}[Q]$, and hence, $\chi_{\varepsilon_\frac{m}{n}} = (\chi_{\varepsilon_\frac{1}{n}})^m$. In particular, we have $\chi_{\mu_i} = \chi_{d_{\mu_i}} = (\chi_{d_{\mu_i}})^{d_{\mu_i}}$. Then, on $V_{\mathbb{R},\mu_i} \otimes_\mathbb{R} S$, we have

$$\rho \circ v_g(a) = \chi_{\mu_i}(a) = (\chi_{d_{\mu_i}}(a))^{d_{\mu_i}} = \rho \circ \lambda_g \circ \chi_{d_{\mu_i}}(a).$$

We next apply this result to all $1 \leq i \leq l$. Because $V_{\mathbb{R}} = \bigoplus_{i=1}^{l} V_{\mathbb{R},\mu_i}$, we conclude that $\rho \circ v_g = \rho \circ \lambda_g \circ \chi_{d_{\mu_i}}$. It follows that $v_g = \lambda_g \circ \chi_{d_{\mu_i}}$ once we choose a faithful representation, as is expected from the definition of $v_g$.

If $G = \text{GL}_V$ for some $V \in \text{Vec}_F$, we consider the standard representation $(V, \rho)$ of $G$ where $\rho$ is the identity. The discussion in the above example then indicates that the image of $\lambda_g$ is contained in a split maximal torus in $G_\mathbb{R}$; we conjecture that this property holds true for an arbitrary split reductive $F$-group $G$, and we shall give one more evidence as follows.

**Example 3.17** Fix a $d$-dimensional $F$-vector space $V$. For any $R \in \text{Alg}_F$, we define $\text{SL}_V(R) := \{ g \in \text{GL}_V(R) \mid \det(g) = 1 \}$. The affine algebraic $F$-group $\text{SL}_V$ is called the **special linear group** (associated to $V$).

Fix an arbitrary $g \in \text{SL}_V(\mathbb{R})$. With the notation as in Construction 4.14, we suppose the jumps of the slope filtration of $(V_\mathbb{R}, \Phi_g)$ are $\mu_1, \ldots, \mu_l$ and $\xi_g(V) = \bigoplus_{i=1}^{l} V_{\mathbb{R},\mu_i}$. For each $i$, we write $r_i = \text{rk}_\mathbb{R}(V_{\mathbb{R},\mu_i})$, then the $r_i$-th exterior product $\Lambda^{r_i}(V_{\mathbb{R},\mu_i})$ is of rank 1. We choose a generator $m_i$, then $\Lambda^{r_i}(\Phi_g(m_i)) = f_i m_i$ for some $f_i \in \mathbb{R}^\times = (\mathbb{E}^\times)^\times$. Let $\nu$ be the valuation of the 1-Gauss norm on $\mathbb{E}^\times$. We then have $\mu_i = \frac{\nu(f_i)}{r_i}$ by [11, Definition 1.4.4].

Let $e_1, \ldots, e_d$ be a basis for $V$ over $F$, and let $A \in \text{SL}_d(\mathbb{R})$ be the matrix of action of $\Phi_g$ in $e_1 \otimes 1, \ldots, e_d \otimes 1$. Let $B \in \text{GL}_d(\mathbb{R})$ represent a change-of-basis over $\mathbb{R}$. Then, in the new basis, the matrix of action of $\Phi_g$ is $B^{-1} A \varphi(B)$. Notice that $\det(B) \in (\mathbb{E}^\times)^\times$ and $\varphi$ preserves $\nu$, we then have

$$\nu(\det(B^{-1} A \varphi(B))) = \nu(\det(B^{-1}) \det(A) \varphi(\det(B))) = \nu(\det(A)),$$

which implies that the valuation of the determinant of the matrix of action of $\Phi_g$ is invariant under change-of-basis. We denote by $\nu(\det(\Phi_g))$ this invariant. In particular, we have $\nu(\det(\Phi_g)) = 0$, because $\det(A) = 1$ by assumption. Put $\Phi_g' := \bigoplus_{i=1}^{l} \Phi_{g,\mu_i}$.
where each $\Phi_{g,\mu_i}$ is the projection of $\Phi_g$ to the $\mu_i$-th graded piece of $\xi_g(V)$ (cf. Construction 4.14 below). We thus have

$$0 = \nu(\det(\Phi_g)) = \nu(\det(\Phi'_g)) = \nu(f_1) + \cdots + \nu(f_1) = r_1\mu_1 + \cdots + r_i\mu_i.$$ 

Let $S \in \text{Alg}_R$ and $t \in \text{Gr}_m(S)$. Because $\lambda_g(t)$ acts on each $V_R,\mu_i \otimes_R S$ via multiplication by $t^{d_g\mu_i}$ where $d_g$ is the least common denominator of $g$, we then have

$$\det(\lambda_g(t)) = t^{d_g(r_1\mu_1 + \cdots + r_i\mu_i)} = 1.$$ 

Therefore, the image of $\lambda_g$ is contained in a split maximal torus in $SL_{V,R}$.

4 \quad $G-(\varphi, \nabla)$-modules over the Robba ring

In this section, we fix an affine algebraic group $F$-group $G$.

4.1 Definition and an identification

Let $R \in \{ E^\dagger, R \}$ equipped with an absolute Frobenius lift $\varphi$ and the usual derivation $\partial = \partial_1 = d/dt$ on $R$.

**Definition 4.1** A $G-(\varphi, \nabla)$-module over $R$ is an exact faithful $F$-linear tensor functor

$$\mathcal{I}: \text{Rep}_F(G) \rightarrow \text{Mod}_R^{\varphi, \nabla},$$

which satisfies for $G \otimes_R I = \varphi G \otimes_R R$, where $\text{forg}: \text{Mod}_R^G \rightarrow \text{Mod}_R$ is the forgetful functor. The category of $G-(\varphi, \nabla)$-modules over $R$ is denoted by $G\text{-Mod}_R^{\varphi, \nabla}$, whose morphisms are morphisms of tensor functors. A $G-(\varphi, \nabla)$-module $I$ over $R$ is called unit-root if $I(V, \rho)$ is a unit-root $(\varphi, \nabla)$-module over $R$ for all $(V, \rho) \in \text{Rep}_F(G)$.

**Remark 4.2** We remark that $G\text{-Mod}_R^{\varphi, \nabla}$ is a groupoid, because both $\text{Rep}_F(G)$ and $\text{Mod}_R^{\varphi, \nabla}$ are rigid tensor categories over $F$, and any morphism of tensor functors between rigid tensor categories is an isomorphism by [7, Proposition 1.13]. Note that tensor products and duals in $\text{Mod}_R^{\varphi, \nabla}$ are defined as in [22, Section 3.1], and the identity object is $(R, \varphi, \partial)$.

We put

$$\mu := \mu(\varphi, t) := \partial(\varphi(t)).$$

Let $\Omega^1_R := \Omega_{R/K}^1$ be the free $R$-module generated by the symbol $dt$, with the $K$-linear derivation $d: R \rightarrow \Omega^1_R$, $f \mapsto \partial(f)dt$. We also define a $\varphi$-linear endomorphism

$$d\varphi: \Omega^1_R \rightarrow \Omega^1_R, \quad fdt \mapsto \mu \varphi(f)dt.$$

Given a finite-dimensional representation $\rho: G \rightarrow \text{GL}_V$, we have a morphism $\text{Lie}(\rho): g \rightarrow \text{gl}_V$ of Lie algebras, and hence a morphism $g_R \rightarrow \text{gl}_V \otimes R \cong \text{End}_R(V_R)$ of Lie algebras over $R$ (which is injective if $\rho$ is a closed embedding). For any $X \in g_R$, we denote by $X$ the action of $\text{Lie}(\rho)(X)$ on $V_R$ (see Remark 2.8). We define the connection $\nabla_X$ of $V_R$ associated to $X$ by

$$\nabla_X := \nabla_{X,V}: V_R \rightarrow V_R \otimes_R \Omega^1_R, \quad v \otimes f \mapsto (v \otimes 1) \otimes d(f) + X(v \otimes f) \otimes dt.$$
Because $fdt \mapsto f$ gives an isomorphism $\Omega^1_R \cong R$, we have an isomorphism $i: V_R \otimes_R \Omega^1_R \to V_R$. Let $\Theta_X := \Theta_{X,V}$ be the differential operator associated to $\nabla_X$ given by the following composition:

$$V_R \xrightarrow{\nabla_X} V_R \otimes_R \Omega^1_R \xrightarrow{i} V_R.$$ 

We have that $\Theta_X(v \otimes f) = v \otimes \partial(f) + X(v \otimes f)$ for all $v \otimes f \in V_R$.

When $G = \text{GL}_V$ for some $V \in \text{Vec}_F$, we may canonically associate to any $G$-(\varphi, \nabla)-module $I$ over $R$ a $(\varphi, \nabla)$-module $(V_R, \Phi, \nabla)$ over $R$, where $(V_R, \Phi, \nabla) := I(V, \rho)$ and $\rho: G \to G$ is the identity. Choose a basis $e_1, \ldots, e_d$ of $V$, we define elements $g \in G(R)$ and $X \in \mathfrak{g}_R$ by setting $g(e_i \otimes 1) := \Phi(e_i \otimes 1)$ and $X(e_i \otimes 1) := i \circ \nabla(e_i \otimes 1)$, respectively. We then have $\Phi = g \Phi$ and $\nabla = \nabla_X$.

**Lemma 4.3** Let $V, W \in \text{Rep}_F(G)$, and let $\alpha \in \text{Hom}_G(V, W)$. We then have

$$\alpha_R \circ \Theta_{X,V} = \Theta_{X,W} \circ \alpha_R, \quad \text{and} \quad \Theta_{X,V \otimes W} = \Theta_{X,V} \otimes \text{Id}_{W_R} + \text{Id}_{V_R} \otimes \Theta_{X,W}.$$ 

**Proof** The first equality holds, because $\alpha_R$ commutes with $X$ (see Remark 2.8), and the second one follows from a direct computation. $\blacksquare$

**Construction 4.4** We consider the $R$-algebra morphism

$$\hat{\partial}: R \to R[\varepsilon], \quad r \mapsto r + \partial(r)\varepsilon,$$

which induces a morphism $G(\hat{\partial}): G(R) \to G(R[\varepsilon])$. Notice that $\pi_R \circ \hat{\partial} = \text{Id}_R$; we then have $G(\pi_R) \circ G(\hat{\partial}) = \text{Id}_{G(R)}$, in particular, $G(\pi_R)(G(\hat{\partial})(g)) = g$. Identifying $g$ with its image in $G(R[\varepsilon])$ induced by the inclusion $R \to R[\varepsilon], r \mapsto r$, we then have

$$G(\hat{\partial})(g)g^{-1} \in \text{Ker} G(\pi_R) = \mathfrak{g}_R.$$ 

For $g \in G(R)$, we define $\partial(g) := G(\hat{\partial})(g) \in G(R[\varepsilon])$, and put

$$\text{dlog}(g) := \partial(g)g^{-1} \in \mathfrak{g}_R.$$ 

**Example 4.5** Let $G = \text{GL}_d$ for some $d \in \mathbb{N}$, and let $B \in G(R)$. We have that $\text{dlog}(B) = I_d + \varepsilon \partial(B)B^{-1}$, where $I_d$ is the $d \times d$ identity matrix and $\partial$ acts on $B$ entrywise. Using the isomorphism $\text{Lie}(G)(R) = \{I_d + \varepsilon B | B \in \text{Mat}_{d,d}(R)\} \cong \{B | B \in \text{Mat}_{d,d}(R)\}$, we may identify $\text{dlog}(B)$ with $\partial(B)B^{-1}$.

**Definition 4.6**

(i) We define the gauge transformation

$$\Gamma_g: \mathfrak{g}_R \to \mathfrak{g}_R, \quad X \mapsto \text{Ad}(g)(X) - \text{dlog}(g),$$

where $\text{Ad}: G \to \text{GL}_d$ is the adjoint representation.

(ii) We define $\mathcal{B}^{\varphi, \nabla}(G, R)$ to be the groupoid whose objects are $(g, X) \in G(R) \times \mathfrak{g}_R$ satisfying $X = \Gamma_g(\mu \varphi(X))$, and whose morphisms $(g, X) \to (g', X')$ are elements $x \in G(R)$ such that $g' = xg\varphi(x^{-1})$ and $X' = \Gamma_x(X)$.

**Lemma 4.7** Let $(g, X) \in \mathcal{B}^{\varphi, \nabla}(G, R)$. Then, $(V_R, g\varphi, \nabla_X)$ is a $(\varphi, \nabla)$-module over $R$ for all $V \in \text{Rep}_F(G)$. 
A group-theoretic generalization of the p-adic local monodromy theorem

Proof Choose a basis \( e_1, \ldots, e_d \) for \( V \) over \( F \) where \( d = \dim_F V \). Let \( A = (a_{ij})_{i,j} \in \text{GL}_d(R) \) (resp. \( N = (n_{ij})_{i,j} \in \text{Mat}_{n,n}(R) \)) be the representing matrix of \( \rho(g) \) (resp. \( X \)).

For any \( v = \sum_{i=1}^{d} e_i \otimes f_i \in V_R \), we compute

\[
g\varphi(\Theta_X(v)) = g\varphi\left( \sum_{i=1}^{d} e_i \otimes \partial(f_i) + \sum_{j=1}^{d} e_j \otimes \sum_{i=1}^{d} n_{ji}f_i \right)
\]

\[
= \sum_{j=1}^{d} e_j \otimes \sum_{i=1}^{d} a_{ji} \varphi(\partial(f_i)) + \sum_{k=1}^{d} e_k \otimes \sum_{i=1}^{d} \sum_{j=1}^{d} a_{kj} \varphi(n_{ji}f_i),
\]

and

\[
\Theta_X(g\varphi(v)) = \Theta_X\left( \sum_{j=1}^{d} e_j \otimes \sum_{i=1}^{d} a_{ji} \varphi(f_i) \right)
\]

\[
= \sum_{j=1}^{d} e_j \otimes \sum_{i=1}^{d} \partial(a_{ji}) \varphi(f_i) + \sum_{j=1}^{d} e_j \otimes \sum_{i=1}^{d} a_{ji} \partial(\varphi(f_i))
\]

\[
+ \sum_{k=1}^{d} e_k \otimes \sum_{i=1}^{d} \sum_{j=1}^{d} n_{ki} a_{ji} \varphi(f_i).
\]

Because \( \mu \cdot \sum_{j=1}^{d} e_j \otimes \sum_{i=1}^{d} a_{ji} \varphi(\partial(f_i)) = \sum_{j=1}^{d} e_j \otimes \sum_{i=1}^{d} a_{ji} \partial(\varphi(f_i)) \), we have that \( \mu \cdot g\varphi \circ \Theta_X = \Theta_X \circ g\varphi \) if and only if \( \mu A \varphi(N) = \partial(A) + NA \), i.e., \( N = \mu A \varphi(N)A^{-1} - \partial(A)A^{-1} \). The last equality holds because of the assumption \( X = \Gamma_g(\mu \varphi(X)) \), which completes the proof.

As a consequence, we may define a functor

\[
B^{\varphi,\nabla}(G, R) \longrightarrow \text{G-Mod}_{R}^{\varphi,\nabla}, \quad (g, X) \longmapsto \text{I}(g, X),
\]

where \( \text{I}(g, X)(V) := (V_R, g\varphi, \nabla_X) \). We next show that this functor is an isomorphism. To do this, we need the following elementary descent result.

Lemma 4.8 Fix a field \( k \), and let \( A \) and \( B \) be finitely generated \( k \)-algebras. Let \( \rho: X \rightarrow Y \) be a closed embedding of affine algebraic \( k \)-schemes for \( X = \text{Spec} A \) and \( Y = \text{Spec} B \). Let \( \iota: S \rightarrow \hat{S} \) be an embedding in \( \text{Alg}_k \). Suppose that we are given an element \( \hat{z} \in X(\hat{S}) \) such that \( \rho(\hat{z}) \in Y(\iota(S)) \), then there exists a unique element \( z \in X(S) \) such that \( \hat{z} = X(\iota)(z) \).

Proof We have a diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\rho^*} & B \\
\downarrow{\exists \alpha} & & \downarrow{\beta} \\
\hat{S} & \xrightarrow{i} & S
\end{array}
\]

with the outer triangle commutative in which \( \rho^* \) is surjective. We prove the lemma by constructing a unique \( k \)-algebra morphism \( \alpha: A \rightarrow S \) such that \( \hat{z} = i \circ \alpha \), as follows. For any \( a \in A \), the surjectivity of \( \rho^* \) gives us some \( b \in B \) such that \( \rho^*(b) = a \). We define \( \alpha(a) := \beta(b) \). Because \( i \) is injective, we have \( \text{Ker} \rho^* \subseteq \text{Ker} \beta \), which implies that \( \alpha \) is well-defined. We then have \( \hat{z} \circ \rho^* = i \circ \beta = i \circ \alpha \circ \rho^* \), which implies that \( \hat{z} = \iota \circ \alpha \).
because $\rho^*$ is surjective. Moreover, $\alpha$ is a $k$-algebra morphism, because $\iota$ is injective and both $\iota$ and $\tilde{\zeta} = \iota \circ \alpha$ are $k$-algebra morphisms. Finally, we see that $\alpha$ is unique, again because $\iota$ is injective. 

**Proposition 4.9** The functor $B^{\rho, \nabla}(G, R) \to G \text{-Mod}^{\rho, \nabla}_R$ defined in (7) is an isomorphism of categories.

**Proof** The proof is similar to that of [6, Lemma 9.1.4]. We first show that the functor is fully faithful. Let $(g, X), (g', X') \in B^{\rho, \nabla}(G, R)$. Then, any morphism $\eta: I(g, X) \to I(g', X')$ is an isomorphism according to [7, Proposition 1.13] (see also Remark 4.2). By composing $\eta$ with the forgetful functor, we then have an automorphism of the fiber functor $\omega^G \otimes R$. By Corollary 2.5, this automorphism is given by a unique element $x \in G(R)$, which then gives an isomorphism between $(g, X)$ and $(g', X')$, as desired.

It remains to show that, for any $I \in G \text{-Mod}^{\rho, \nabla}_R$, there exists a unique $(g, X) \in B^{\rho, \nabla}(G, R)$ such that $I = I(g, X)$. For any $(V, \rho_V) \in \text{Rep}_F(G)$, we write $I(V, \rho_V) = (V_R, \Phi_V, \nabla_V)$ for a $\varphi$-linear map $\Phi_V$ and a connection $\nabla_V$ on $V_R$. The proof consists of two steps.

**Step 1:** There exists a unique $X \in g_R$ such that $\nabla_V = \nabla_X$. We write $\Theta_V$ for the composition of

$$V_R \xrightarrow{\nabla_V} V_R \otimes \Omega^1_R \xrightarrow{\iota} V_R,$$

where $\iota$ is induced by $f dt \mapsto f$, and put $\theta_V := \Theta_V - \text{Id}_V \otimes \partial$. It is clear that $\theta_{V|_I} = 0$, where $I$ denotes the trivial representation. Lemma 4.3 then implies that the family

$$\{ \theta_V: V_R \rightarrow V_R \mid (V, \rho_V) \in \text{Rep}_F(G) \}$$

of $R$-linear endomorphisms satisfies conditions (i–iii) in Corollary 2.9. We thus find a unique $X \in g_R$ such that $\theta_V = \text{Lie}(\rho_V)(X)$ for all $(V, \rho_V) \in \text{Rep}_F(G)$, which implies that $\nabla_V = \nabla_X$.

**Step 2:** There exists a unique $g \in G(R)$ such that $\Phi_V = g \varphi$. We put $\tilde{\Phi}_V := \Phi_V \otimes \varphi$, where $\varphi$ is the Frobenius lift on $\mathcal{R}$ (in particular, $\mathcal{R}$ is viewed as an $\mathcal{R}$-module via the $\varphi$-equivariant embedding $\psi$ described in Section 2.3). The family

$$\{ \lambda_V := \tilde{\Phi}_V \circ (\text{Id}_V \otimes \varphi^{-1}): V_{\mathcal{R}} \rightarrow V_{\mathcal{R}} \mid V \in \text{Rep}_F(G) \}$$

of $\mathcal{R}$-linear endomorphisms satisfies conditions (i–) in Theorem 2.4, which provides a unique element $\tilde{g} \in G(\mathcal{R})$ such that $\lambda_V = \rho_V(\tilde{g})$ for all $(V, \rho_V) \in \text{Rep}_F(G)$. We next reduce $\tilde{g}$ to an element in $G(\mathcal{R})$. We compute

$$\tilde{\Phi}_V \circ (\text{Id}_V \otimes \varphi^{-1})(v \otimes f) = \tilde{\Phi}_V(v \otimes \varphi^{-1}(f)) = \rho_V(\tilde{g})(v \otimes f),$$

which implies that $\tilde{\Phi}_V(v \otimes f) = \rho_V(\tilde{g})(v \otimes \varphi(f))$, and hence, $\tilde{\Phi}_V = \tilde{g} \varphi$. We now fix a $d$-dimensional faithful representation $(V, \rho_V)$, and an $F$-basis $e_1, \ldots, e_d$ for $V$.

Suppose that $\tilde{\Phi}_V(e_i) = \sum_{j=1}^d a_{ij} e_j$, where $a_{ij} \in R$ for all $1 \leq i, j \leq d$. Put $A = (a_{ij})_{i,j} \in \text{GL}_d(R)$. Then, $\psi(A) = (\psi(a_{ij}))_{i,j} \in \text{GL}_d(\mathcal{R})$ describes the $\varphi$-linear action of $\tilde{\Phi}_V$ as well as the $\mathcal{R}$-linear action $\rho(\tilde{g})$ in the basis $e_1 \otimes 1, \ldots, e_d \otimes 1$. By replacing $X$ with $G$, $Y$ with $\text{GL}_d$, $S$ with $R, \tilde{S}$ with $\mathcal{R}$, and $i$ with $\psi$ in Lemma 4.8, we find a unique element $g \in G(R)$ such that $\psi(g) = \tilde{g}$. It follows that $\Phi_V = g \varphi$, as desired. 

\[ \blacksquare \]
Example 4.10  Let $d \in \mathbb{N}$. The affine algebraic $F$-group $\text{SL}_d$ is defined by

$$\text{SL}_d(S) = \{ A \in \text{GL}_d(S) \mid \det(A) = 1 \}$$

for all $S \in \text{Alg}_F$, whose Lie algebra $\mathfrak{sl}_d$ consists of $d \times d$ matrices with entries in $F$ and with trace zero.

(i) We claim that any pair $(A, N) \in \text{SL}_d(\mathcal{R}) \times \text{Mat}_{d, d}(\mathcal{E})$ satisfying

$$N = \mu A \varphi(N) A^{-1} - \partial(A) A^{-1}$$

is already an object in $B^{p, \nu}(\text{SL}_d, \mathcal{R})$. It is equivalent to showing that the trace $\text{Tr}(N)$ of $N$ is zero. Recall that the Frobenius lift $\varphi$ on $\mathcal{E}$ is given by $\varphi\left( \sum_{i \in \mathbb{Z}} c_i t^i \right) = \sum_{i \in \mathbb{Z}} \varphi(c_i) u_i$, where $u = \varphi(t)$ satisfies $|u - t^q|_1 < 1$. If we write $u = \sum_{i \in \mathbb{Z}} u_i t^i$, $u_i \in K$, we then have $|u_j|_1 < 1$ for all $j \neq q$ and $|u_q|_1 = 1$. It follows that $|\mu|_1 = |\partial(u)|_1 = | \sum_{i \in \mathbb{Z}} iu_i t^{i-1}|_1 < 1$. On the other hand, we have $\text{Tr}(\partial(A) A^{-1}) = 0$, because $\partial(A) A^{-1} \in \mathfrak{sl}_d(\mathcal{R})$ (see Construction 4.4). Assume, to the contrary, that $\text{Tr}(N) \neq 0$, we have

$$|\text{Tr}(N)|_1 = |\mu \text{Tr}(\varphi(N))|_1 = |\mu \varphi(\text{Tr}(N))|_1 < |\varphi(\text{Tr}(N))|_1 = |\text{Tr}(N)|_1,$$

a contradiction (we have the last equality, because $\varphi$ preserves the 1-Gauss norm on $\mathcal{E}$).

(ii) We use the Bessel isocrystal as described in [12, Example 20.2.1] (see also [9, Section 1.5] and [24, Example 6.2.6]) to construct an object in $B^{p, \nu}(\text{SL}_2, \mathcal{R})$. We first briefly recall the Bessel isocrystal. In Hypothesis 2.1, we let $q = p$ be an odd prime, $\kappa = \mathbb{F}_p$, and $F = Q_p(\pi)$, where $\pi = (p - 1)^{st}$ root of $-p$ in $\hat{Q}_p$. Then, the $(p$-power) Frobenius on $K = F$ is the identity. Let $\varphi$ be the Frobenius lift on $\mathcal{R}$ given by $\varphi(t) = t^p$. Then, [12, Example 20.2.1] gives rise to a pair $(A_0, N_0) \in \text{GL}_2(\mathcal{R}) \times \text{Mat}_{2, 2}(\mathcal{E})$ with $\det(A_0) = p$ satisfying the gauge compatibility condition, in which $N_0 = \left( \begin{smallmatrix} 0 & 1 \\ \pi^{-1} \alpha & 0 \end{smallmatrix} \right) \in \mathfrak{sl}_2(\mathcal{E})$. We now assume that $p \equiv 1(\text{mod } 4)$, and $i$ is a square root of $-1$ in $\hat{Q}_p$. Because $p - 1$ is even, we may set $\alpha := \frac{1}{\pi^{(p-1)/2}} \in F^\times$. We then have $\alpha^2 = p^{-1} = \det(A_0)^{-1}$. Put $D_0 = \left( \begin{smallmatrix} 0 & 1 \\ \alpha & 0 \end{smallmatrix} \right) \in \text{GL}_2(F)$. Then, $D_0 A_0 D_0 \in \text{SL}_2(\mathcal{R})$. Moreover, we see that $D_0 N_0 D_0^{-1} = D_0^{-1} N_0 D_0 \in \mathfrak{sl}_2(\mathcal{E})$. Put $A := D_0 A_0 D_0$ and $N := D_0 N_0 D_0^{-1}$. Then, a straightforward verification shows $N = \mu A \varphi(N) A^{-1} - \partial(A) A^{-1}$ (noting that $\varphi(D_0) = D_0$ and $\partial(D_0)$, as desired. We thus have $(A, N) \in B^{p, \nu}(\text{SL}_2, \mathcal{R})$, as desired.

(iii) Let $(A, N) \in B^{p, \nu}(\text{SL}_d, \mathcal{R})$. We show that $(A, N)$ is“$\text{SL}_d$-quasi-unipotent” (as described in the introduction) by modifying the classical monodromy as follows. By the classical $p$LMT, we find a finite separable extension $L$ of $\kappa((t))$ and $B \in \text{GL}_d(\mathcal{R}_L)$ such that $B N B^{-1} - \partial(B) B^{-1}$ has trace zero being an upper-triangular block matrix with zero blocks in the diagonal. We wish to replace $B$ with an element in $\text{SL}_d(\mathcal{R}_L)$. To this end, we deduce first that $\text{Tr}(\partial(B) B^{-1}) = \text{Tr}(B N B^{-1} - \partial(B) B^{-1}) = \text{Tr}(N) = 0$. It then follows from Jacobi’s formula that $\partial(\det(B)) = \det(B) \cdot \text{Tr}(B^{-1} \partial(B)) = 0$. Put $D := \text{Diag}(\det(B)^{-1}, 1, \ldots, 1)$. Then, $DB \in \text{SL}_d(\mathcal{R}_L)$ and $\partial(D) = 0$. We then have

$$(DB) N(DB)^{-1} - \partial(DB)(DB)^{-1} = D \left( B N B^{-1} - \partial(B) B^{-1} \right) D^{-1},$$
which is an upper-triangular block matrix with zero blocks, and the sizes of the blocks are the same as those in $BNB^{-1} - \partial(B)B^{-1}$ (the said properties are preserved under conjugation by a diagonal matrix). Hence, $DB$ is a desired replacement of $B$ and we are done.

**Example 4.11**  For any matrix $X$, we denote by $X^T$ its transpose, and by $X^{-T}$ the inverse of transpose if $X$ is invertible.

We fix the skew-symmetric matrix $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The affine algebraic $F$-group $\text{Sp}_4$ is defined by

$$\text{Sp}_4(S) := \{ A \in \text{GL}_4(S) \mid A^{-1} = J^{-1}A^T \},$$

for all $S \in \text{Alg}_F$. We denote by $\mathfrak{sp}_4$ the Lie algebra of $\text{Sp}_4$. For any $S \in \text{Alg}_F$, we then have $\mathfrak{sp}_{4,S} = \{ X \in \text{Mat}_{4,4}(S) \mid X = JX^T \}$. We remark that the specific choice of $J$ preserves Borel subgroups under conjugation, which will be useful in the monodromy considered below.

Given any $(\varphi, \nabla)$-module over $\mathcal{R}$ of rank 2, e.g., the Bessel isocrystal described above, we obtain a pair $(A_0, N_0) \in \text{GL}_2(\mathcal{R}) \times \text{Mat}_{2,2}(\mathcal{R})$ satisfying $N_0 = \mu A_0 \varphi(N_0) A_0^{-1} - \partial(A_0) A_0^{-1}$. Put

$$A := \begin{pmatrix} A_0 & 0 \\ 0 & (-1)^{-1} A_0^{-T} (-1) \end{pmatrix} \quad \text{and} \quad N := \begin{pmatrix} N_0 & 0 \\ 0 & (-1) N_0^{-T} (-1) \end{pmatrix}.$$

A straightforward verification shows that $A \in \text{Sp}_4(\mathcal{R}), N \in \mathfrak{sp}_{4,\mathcal{R}}$, and, moreover, $N = \mu A \varphi(N) A^{-1} - \partial(A) A^{-1}$ (noting that $(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix})^{-1} = -(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}))$, which implies that $(A, N) \in B^{\varphi, \nabla}(\text{Sp}_4, \mathcal{R})$.

We next show that $(A, N)$ is “$\text{Sp}_4$-quasi-unipotent.” By the classical $p$LMT, we find a finite separable extension $L$ of $\kappa((t))$ and $B_0 \in \text{GL}_2(\mathcal{R}_L)$ such that

$$B_0 N_0 B_0^{-1} - \partial(B_0) B_0^{-1} = \begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix},$$

for some $n \in \mathcal{R}_L$ ($n$ could be 0). Put

$$B := \begin{pmatrix} B_0 & 0 \\ 0 & (-1)^{-1} B_0^{-T} (-1) \end{pmatrix}.$$

We then have $B \in \text{Sp}_4(\mathcal{R}_L)$, and

$$BNB^{-1} - \partial(B) B^{-1} = \begin{pmatrix} 0 & n & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & n \end{pmatrix},$$

again by straightforward computations.

### 4.2 The pushforward functor

Let $R \in \{ E^+, \mathcal{R} \}$. For any $g \in G(R)$ and $n \in \mathbb{N}$, we define

$$[n]_*(g) := g \varphi(g) \cdots \varphi^{n-1}(g) \in G(R),$$

the $n$-pushforward of $g$. Notice that $[n]_*(g) \varphi^n = (g \varphi)^n \in G(R) \times \{ \varphi \}$ for all $n \in \mathbb{N}$.
We define the $n$-pushforward functor by

$$[n]_*: B^{\varphi, \nabla}(G, R) \to B^{\varphi^n, \nabla}(G, R), \quad (g, X) \mapsto ([n]_*(g), X),$$

and $[n]_*(x) = x$ for all morphisms $x \in B^{\varphi, \nabla}(G, R)$. The following lemma shows that this functor is well-defined (in particular, faithful).

**Lemma 4.12** Let $(g, X) \in B^{\varphi, \nabla}(G, R)$. We then have $([n]_*(g), X) \in B^{\varphi^n, \nabla}(G, R)$ for all $n \in \mathbb{N}$.

**Proof** We show by induction on $n$ that

$$X + \text{dlog} \left( [n]_*(g) \right) = \mu(\varphi^n, t) \text{Ad} \left( [n]_*(g) \right) (\varphi^n(X)).$$

There is nothing to show when $n = 1$. We now assume by the induction hypothesis that

$$X + \text{dlog} \left( [n-1]_*(g) \right) = \mu(\varphi^{n-1}, t) \text{Ad} \left( [n-1]_*(g) \right) (\varphi^{n-1}(X)),$$

We notice that $\mu(\varphi^{n-1}, t) = \mu(\varphi(\mu)) \cdots (\varphi^{n-2}(\mu))$, and hence,

$$\partial(\varphi^{n-1}(f)) = \mu(\varphi(\mu)) \cdots (\varphi^{n-2}(\mu)) \varphi^{n-1}(\partial(f)) = \mu(\varphi^{n-1}, t) \varphi^{n-1}(\partial(f)), \quad \forall f \in R,$$

which implies that

$$\text{dlog}(\varphi^{n-1}(g)) = \mu(\varphi^{n-1}, t) \varphi^{n-1}(\text{dlog}(g)).$$

On the other hand, because $X + \text{dlog}(g) = \mu \text{Ad}(g)(\varphi(X))$, we have

$$\varphi^{n-1}(X) + \varphi^{n-1}(\text{dlog}(g)) = \varphi^{n-1}(\mu) \text{Ad}(\varphi^{n-1}(g))(\varphi^n(X)).$$

We now compute

$$X + \text{dlog} \left( [n]_*(g) \right) = X + \text{dlog} \left( [n-1]_*(g) \right) + \text{Ad} \left( [n-1]_*(g) \right) \left( \text{dlog}(\varphi^{n-1}(g)) \right)$$

$$= \mu(\varphi^{n-1}, t) \text{Ad} \left( [n-1]_*(g) \right) (\varphi^{n-1}(X))$$

$$+ \mu(\varphi^{n-1}, t) \text{Ad} \left( [n-1]_*(g) \right) (\varphi^{n-1}(\text{dlog}(g)))$$

$$= \mu(\varphi^{n-1}, t) \text{Ad} \left( [n-1]_*(g) \right) (\varphi^{n-1}(X) + \varphi^{n-1}(\text{dlog}(g)))$$

$$= \mu(\varphi^{n-1}, t) \text{Ad} \left( [n-1]_*(g) \right) (\varphi^{n-1}(\mu) \text{Ad}(\varphi^{n-1}(g))(\varphi^n(X)))$$

$$= \mu(\varphi^{n}, t) \text{Ad} \left( [n]_*(g) \right) (\varphi^n(X)),$$

which proves the lemma.

In connection with the pushforward functor on $\varphi$-modules as recalled in Section 2.3, we state the following lemma resulting from [11, Lemma 1.6.3 and Remark 1.7.2], which will not be explicitly used in the sequel.

**Lemma 4.13** Let $g \in G(R)$. Then, $(V_R, g\varphi)$ is pure of slope $\mu$ if and only if $(V_R, [n]_*(g)\varphi^n)$ is pure of slope $n\mu$ for all $n \in \mathbb{N}$. Moreover, if $(V_R, g\varphi)$ has jumps $\mu_1, \ldots, \mu_j$, then $(V_R, [n]_*(g)\varphi^n)$ has jumps $n\mu_1, \ldots, n\mu_j$.

### 4.3 $G$-$\varphi$-modules attached to splittings

Let $g \in G(\mathcal{R})$. We fix a splitting $\xi_g$ of $HN_g$ by Theorem 3.12.
Construction 4.14 Let \((V_\mathcal{R}, g\varphi, \nabla_X)\) be a \((\varphi, \nabla)\)-module over \(\mathcal{R}\) with the slope filtration

\[0 \subseteq V_{\mathcal{R}}^{\mu_1} \subseteq \cdots \subseteq V_{\mathcal{R}}^{\mu_l} = V_{\mathcal{R}},\]

with jumps \(\mu_1 < \cdots < \mu_l\). Then, \(\xi_g(V)\) is the decomposition

\[V_{\mathcal{R}} = \bigoplus_{i=1}^l V_{\mathcal{R}, \mu_i}\]

of \(\mathcal{R}\)-modules such that \(\bigoplus_{i=1}^j V_{\mathcal{R}, \mu_i} = V_{\mathcal{R}}^{\mu_j}\) for \(j = 1, \ldots, l\).

(i) For any \(1 \leq j \leq l\) and \(v \in V_{\mathcal{R}, \mu_j}\), we have \(\Phi_g(v) \in V_{\mathcal{R}}^{\mu_j}\), whence a unique expression

\[\Phi_g(v) = \sum_{i=1}^j v_i\text{ with } v_i \in V_{\mathcal{R}, \mu_i}.\]

We thus have a \(\varphi\)-linear map

\[\Phi_g,_{\mu_j} : V_{\mathcal{R}, \mu_j} \rightarrow V_{\mathcal{R}, \mu_j}, \quad v \mapsto v_j.\]

We then define \(\Phi_g' := \bigoplus_{i=1}^l \Phi_g,_{\mu_i}\). We define

\[I'(g)(V) := (V_{\mathcal{R}}, \Phi_g').\]

For a morphism \(\alpha : V \rightarrow W\) of finite-dimensional \(G\)-modules, we define \(I'(g)(\alpha) := \alpha_{\mathcal{R}}\).

(ii) Similarly, for any \(1 \leq j \leq l\) and \(v \in V_{\mathcal{R}, \mu_j}\), we have \(\Theta_X(v) \in V_{\mathcal{R}}^{\mu_j}\), whence a unique expression

\[\Theta_X(v) = \sum_{i=1}^j v_i\text{ with } v_i \in V_{\mathcal{R}, \mu_i}.\]

We thus have a \(K\)-linear differential operator

\[\Theta_{X,_{\mu_j}} : V_{\mathcal{R}, \mu_j} \rightarrow V_{\mathcal{R}, \mu_j}, \quad v \mapsto v_j.\]

We then define \(\Theta_X' := \bigoplus_{i=1}^l \Theta_{X,_{\mu_i}}\).

Notice that \((V_{\mathcal{R}, \mu_i}, \Phi_g,_{\mu_i}) = (V_{\mathcal{R}}^{\mu_i}, \Phi_g|_{V_{\mathcal{R}}^{\mu_i}})\), and \((V_{\mathcal{R}, \mu_i}, \Phi_g,_{\mu_i})\) is isomorphic to \(V_{\mathcal{R}}^{\mu_i}/V_{\mathcal{R}}^{\mu_{i-1}}\) as \(\varphi\)-modules for \(2 \leq i \leq l\). Similarly, we have \((V_{\mathcal{R}, \mu_i}, \Theta_X,_{\mu_i}) = (V_{\mathcal{R}}^{\mu_i}, \Theta_X|_{V_{\mathcal{R}}^{\mu_i}})\), and \((V_{\mathcal{R}, \mu_i}, \Theta_X,_{\mu_i})\) is isomorphic to \(V_{\mathcal{R}}^{\mu_i}/V_{\mathcal{R}}^{\mu_{i-1}}\) as a differential module for \(2 \leq i \leq l\).

The remainder of this subsection is devoted to the consequences of Construction 4.14 (i). We will continue to discuss (ii) in Section 4.4; we will show, in particular, that both constructions assemble to give a \(G-(\varphi, \nabla)\)-module over \(\mathcal{R}\).

Lemma 4.15 \(I'(g) : \text{Rep}_F(G) \rightarrow \varphi-\text{Mod}_\mathcal{R}\) is a \(G\)-\(\varphi\)-module over \(\mathcal{R}\).

Proof By Definition 3.1, it amounts to show that \(I'(g)\) is an exact faithful \(F\)-linear tensor functor. In this proof, we fix \(V, W \in \text{Rep}_F(G)\), and suppose the slope filtration of \((V_{\mathcal{R}}, g\varphi)\) (resp. of \((W_{\mathcal{R}}, g\varphi)\)) has jumps \(\mu_1 < \cdots < \mu_{l_v}\) (resp. \(v_1 < \cdots < v_{l_w}\)).
We first check the functoriality of $I'(g)$ (the exactness, faithfulness, and $F$-linearity will follow immediately). Given $\alpha \in \text{Hom}_R(V, W)$, we need to show that

$$\alpha_R \circ \Phi'_g = \Phi'_g \circ \alpha_R.$$ 

For any fixed $1 \leq l \leq l_V$, we have that $\alpha_R(V_{R, \mu_l}) \subseteq W_{R, \mu_l}$ by Theorem 3.12. Notice that $W_{R, \mu_l} = W_{R, \nu_r}$ if $\mu_l = \nu_r$ for some $1 \leq s \leq l_W$, and $W_{R, \mu_l} = 0$ otherwise. In the latter case, it is clear that $\alpha_R \circ \Phi'_g = \Phi'_g \circ \alpha_R = 0$ on $V_{R, \mu_l}$, and we are done. Suppose now we are in the former case. Let $v$ be a nonzero element in $V_{R, \mu_l}$. We then have $\Phi'_g(v) \in V_{R, \mu_l}^i$ and $\alpha_R(v) \in W_{R, \nu_r}$. We have unique expressions

$$\Phi'_g(v) = \sum_{i=1}^l v_i, \quad v_i \in V_{R, \mu_l},$$

and

$$\alpha_R \circ \Phi'_g(v) = \sum_{i=1}^s w_{i}, \quad w_{i} \in W_{R, \nu_r};$$

therefore $\alpha_R(v_i) = w_i$. We also write

$$\Phi'_g \circ \alpha_R(v) = \sum_{i=1}^s w'_i, \quad w'_i \in W_{R, \nu_r};$$

we then have $w_i = w'_i$ for $i = 1, \ldots, s$, as $\alpha_R \circ \Phi'_g = \Phi'_g \circ \alpha_R$. We thus have $\alpha_R \circ \Phi'_g \circ \alpha_R = \alpha_R \circ \Phi'_g \circ \alpha_R$, as desired.

It remains to show that $I'(g)$ preserves tensor products. Because $\tau_g$ is a tensor functor, the $(\mu_i + \nu_s)$th graded piece of $\tau_g(V \otimes W)$ is then

$$\left( \bigotimes_{l \leq l_V, 1 \leq j \leq l_W} V_{R, \mu_i} \otimes W_{R, \nu_j} \right),$$

for all $1 \leq l \leq l_V$ and $1 \leq s \leq l_W$. It then follows from Construction 4.14(i) that

$$\Phi'_{g, \mu_i + \nu_s} = \bigotimes_{l \leq l_V, 1 \leq j \leq l_W} \left( \Phi'_{g, \mu_i} \otimes \Phi'_{g, \nu_j} \right),$$

which implies that $I'(g)(V \otimes W)$ coincides with $I'(g)(V) \otimes I'(g)(W)$ on all $(V \otimes W)_{R, \mu_i + \nu_s}$, whence on $(V \otimes W)_R$. This completes the proof. \hfill \qed

With Lemma 4.15, we imitate Step 2 in the proof of Proposition 4.9 and have the following proposition.

**Proposition 4.16** There exists a unique element $z \in G(R)$ such that $I'(g) = I(z)$.

### 4.4 $G(\varphi, \nabla)$-modules attached to splittings

We fix $(g, X) \in B^e, V(G, R)$. We also fix a splitting $\xi_g$ of $H_N g$ given by Theorem 3.12.
We now look back at Construction 4.14(ii). We claim that $\Theta'_X - \text{Id}_V \otimes \partial : V_R \to V_R$ is $R$-linear for all $(V, \rho_V) \in \text{Rep}_F(G)$. Let $1 \leq j \leq l$ and let $\nu \otimes f \in V_{R, \mu_V}$. Suppose that $\Theta_X(v \otimes f) = \sum_{i=1}^l v_i$ with $v_i \in V_{R, \mu_i}$. Then, $\Theta'_X(v \otimes f) = v_j$ by construction. Let $f' \in R$. We compute

$$\Theta'_X(v \otimes ff') = v \otimes \partial(f)f' + \nu \otimes f\partial(f') + X(v \otimes ff')$$

$$= (v \otimes \partial(f) + X(v \otimes f))f' + \nu \otimes f\partial(f')$$

$$= \Theta'_X(v \otimes f)f' + \nu \otimes f\partial(f')$$

$$= f' \sum_{i=1}^j v_i + \nu \otimes f\partial(f'),$$

which implies that $\Theta'_X(v \otimes ff') = f'v_j + \nu \otimes f\partial(f')$. We thus have

$$(\Theta'_X - \text{Id}_V \otimes \partial)(v \otimes ff') = f'v_j + \nu \otimes f\partial(f') - \nu \otimes \partial(ff')$$

$$= f'v_j + \nu \otimes f\partial(f') - \nu \otimes \partial(f)f' - \nu \otimes f\partial(f')$$

$$= f'(v_j - \nu \otimes f\partial(f))$$

$$= f'((\Theta'_X - \text{Id}_V \otimes \partial)(v \otimes f)),$$

as desired.

The following proposition (and its proof) is analogous to Lemma 4.15.

**Proposition 4.17** There exists a unique element $X_0 \in \mathfrak{g}_R$ such that $\Theta'_X = \Theta_{X_0}$.

**Proof** For any $(V, \rho_V) \in \text{Rep}_F(G)$, we define $\theta_V := \Theta'_X - \text{Id}_V \otimes \partial$. We claim that the family

$$\{ \theta_V : V_R \to V_R \mid (V, \rho_V) \in \text{Rep}_F(G) \}$$

of $R$-linear endomorphisms satisfies conditions (i–iii) in Corollary 2.9. The lemma will follow immediately.

It is clear that $\theta_V = 0$ if $V = F$ is the trivial $G$-representation. For the remainder of the proof, we fix $(V, \rho_V) \in \text{Rep}_F(G)$, and suppose the slope filtration of $(V_R, g\varphi)$ (resp. of $(W_{R, g\varphi})$) has jumps $\mu_1 < \cdots < \mu_{l_V}$ (resp. $\nu_1 < \cdots < \nu_{l_W}$). Let $\alpha \in \text{Hom}_G(V, W)$. To show that $\theta_V \circ \alpha_R = \alpha_R \circ \theta_W$, it suffices to show that $\Theta'_X \circ \alpha_R = \alpha_R \circ \Theta'_X$. Notice that $\alpha_R$ respects gradings. Replacing $\Theta$ with $\Theta_X$ (possibly with proper decorations) in the second paragraph of the proof of Lemma 4.15, we have the desired result.

It remains to show that

$$\theta_V \otimes W = \theta_V \otimes \text{Id}_{W_R} + \text{Id}_{V_R} \otimes \theta_W.$$
for all $1 \leq l \leq l_V$ and $1 \leq s \leq l_W$. It follows from Lemma 4.3 and Construction 4.14 that
\[
\Theta'_{X,\mu_l+v_s} = \bigoplus_{1 \leq i \leq l_V, 1 \leq j \leq l_W} \left( \Theta'_{X,\mu_i} \otimes \text{Id}_{W_{x,v_j}} + \text{Id}_{V_{x,v_i}} \otimes \Theta'_{X,v_j} \right).
\]
Let $v \otimes f \otimes w \otimes f' \in V_{R,\mu_l} \otimes_{\mathbb{R}} W_{R,v_j}$. We compute
\[
\left( \theta_v \otimes \text{Id}_{W_{x,v_i}} + \text{Id}_{V_{x,v_i}} \otimes \theta_w \right)(v \otimes f \otimes w \otimes f')
\]
\[
= \left( \Theta'_{X,\mu_l}(v \otimes f) - v \otimes \partial(f) \right) \otimes w \otimes f' + v \otimes f \otimes \left( \Theta'_{X,v_j}(w \otimes f') - w \otimes \partial(f') \right)
\]
\[
= \left( \Theta'_{X,\mu_l} \otimes \text{Id}_{W_{x,v_j}} + \text{Id}_{V_{x,v_i}} \otimes \Theta'_{X,v_j} \right)(v \otimes f \otimes w \otimes f') - v \otimes 1 \otimes w \otimes \partial(ff')
\]
\[
= \theta_v \otimes w(v \otimes w \otimes ff'),
\]
which completes the proof.

We now summarize what we have shown thus far. The splitting $\xi_g$ of $\text{HN}_g$ gives a unique element $z \in G(\mathbb{R})$ such that $\nu'(g) = \nu(z)$ by Proposition 4.16, and a unique element $X_0 \in \mathfrak{g}_R$ such that $\Theta'_X = \Theta_{X_0}$ by Proposition 4.17. These two elements are related as in Proposition 4.19 below.

We next recall some notions from [4, Section 2.1]. Let $k$ be a commutative ring with 1, and let $\mathfrak{G}$ be a reductive $k$-group. Hereupon, we denote by $\kappa(s)$ the residue field of $s$ and $\bar{\kappa}(s)$ an algebraic closure of $\kappa(s)$, for all $s \in \text{Spec } k$. A subgroup $\mathfrak{V}$ of $\mathfrak{G}$ is a parabolic (resp. Borel) subgroup if $\mathfrak{V}$ is smooth and $\mathfrak{V}_{\bar{\kappa}(s)}$ is a parabolic (resp. Borel) subgroup of $\mathfrak{G}_{\bar{\kappa}(s)}$, for all $s \in \text{Spec } k$.

Suppose we have a cocharacter $\lambda: G_m \to \mathfrak{G}$ over $k$. For any $k$-algebra $R$, we let $G_{m,R}$ act on $\mathfrak{G}_R$ via the conjugation
\[
G_{m,R}(S) \times \mathfrak{G}_R(S) \to \mathfrak{G}_R(S), \quad (t, x) \mapsto t.x := \lambda(t)x\lambda(t)^{-1}
\]
for all $R$-algebra $S$. For any $x \in \mathfrak{G}(R)$, we have an orbit map $\alpha_x: G_{m,R} \to \mathfrak{G}_R$ given by
\[
\alpha_x: G_{m,R}(S) \to \mathfrak{G}_R(S), \quad t \mapsto t.x
\]
for all $R$-algebras $S$. Let $A^1$ be the affine $k$-line. We say that the limit $\lim_{t \to 0} t.x$ exists if $\alpha_x$ extends (necessarily uniquely) to a morphism $\tilde{\alpha}_x: A^1_R \to \mathfrak{G}_R$ of affine $R$-schemes, and put $\lim_{t \to 0} t.x := \tilde{\alpha}_x(0) \in \mathfrak{G}_R(R)$. We define
\[
P_{\mathfrak{G}}(\lambda)(R) := \{ x \in \mathfrak{G}(R) \mid \lim_{t \to 0} t.x \text{ exists} \},
\]
\[
U_{\mathfrak{G}}(\lambda)(R) := \{ x \in \mathfrak{G}(R) \mid \lim_{t \to 0} t.x = 1 \},
\]
and
\[
Z_{\mathfrak{G}}(\lambda)(R) := P_{\mathfrak{G}}(\lambda)(R) \cap P_{\mathfrak{G}}(-\lambda)(R),
\]
where $-\lambda$ is the reciprocal of $\lambda$. Then, $P_{\mathfrak{G}}(\lambda)$ is a closed $k$-subgroup of $\mathfrak{G}$ [4, Lemma 2.1.4], $U_{\mathfrak{G}}(\lambda)$ is an affine algebraic $k$-normal subgroup of $P_{\mathfrak{G}}(\lambda)$, and $Z_{\mathfrak{G}}(\lambda)$ is the
centralizer of the $G_m$-action in $\mathfrak{G}$ [4, Lemma 2.1.5]. By [4, Proposition 2.1.8(3)], these subgroups are smooth, because $\mathfrak{G}$ is smooth.

It follows from the definitions that the formations of $P_{\mathfrak{G}}(\lambda)$, $U_{\mathfrak{G}}(\lambda)$, and $Z_{\mathfrak{G}}(\lambda)$ commute with any base extension on $k$. In particular, for every $s \in \text{Spec } k$, we have $P_{\mathfrak{G}}(\lambda_{\kappa(s)}) = P_{\mathfrak{G}_{\kappa(s)}}(\lambda_{\kappa(s)})$, which is a parabolic subgroup of $\mathfrak{G}_{\kappa(s)}$ by [20, Proposition 8.4.5]. Hence, $P_{\mathfrak{G}}(\lambda)$ is a parabolic $k$-group.

By [4, Proposition 2.1.8(2)], the multiplication map gives an isomorphism

$$U_{\mathfrak{G}}(\lambda) \times Z_{\mathfrak{G}}(\lambda) \longrightarrow P_{\mathfrak{G}}(\lambda)$$

of affine algebraic $k$-groups.

Now, let $G_m$ act on $g = \text{Lie}(\mathfrak{G})(k)$ through the adjoint representation. We then have $g = \bigoplus_{n \in \mathbb{Z}} g_n$, where $g_n = \{X \in g \mid t.X = t^n X, \forall t \in G_m\}$ for all $n \in \mathbb{Z}$. We have $\text{Lie} \left( Z_{\mathfrak{G}}(\lambda) \right) = g_0$ (which is the centralizer of the $G_m$-action on $g$), $\text{Lie} \left( U_{\mathfrak{G}}(\lambda) \right) = \bigoplus_{n \geq 0} g_n$, and $\text{Lie} \left( P_{\mathfrak{G}}(\lambda) \right) = \bigoplus_{n \geq 0} g_n$. In particular, we have the following decomposition:

$$\text{(8)} \quad \text{Lie} \left( P_{\mathfrak{G}}(\lambda) \right) = \text{Lie} \left( Z_{\mathfrak{G}}(\lambda) \right) \oplus \text{Lie} \left( U_{\mathfrak{G}}(\lambda) \right).$$

**Lemma 4.18** With the notion above, we have

$$Z - \text{Ad}(u)(Z) \in \text{Lie} \left( U_{\mathfrak{G}}(\lambda) \right),$$

for all $u \in U_{\mathfrak{G}}(\lambda)(k)$ and $Z \in \text{Lie} \left( Z_{\mathfrak{G}}(\lambda) \right)$.

**Proof** Recall that $Z \in Z_{\mathfrak{G}}(\lambda)(k[\varepsilon])$ by definition; we may also view $u$ as an element in $U_{\mathfrak{G}}(\lambda)(k[\varepsilon])$ via the inclusion $i: k \hookrightarrow k[\varepsilon]$. By the definition of the adjoint representation, we have

$$Z - \text{Ad}(u)(Z) = Z(uZu^{-1})^{-1} = ZuZ^{-1}u^{-1} \in P_{\mathfrak{G}}(\lambda)(k[\varepsilon]).$$

Because $U_{\mathfrak{G}}(\lambda)$ is normal in $P_{\mathfrak{G}}(\lambda)$, we have that $ZuZ^{-1} \in U_{\mathfrak{G}}(\lambda)(k[\varepsilon])$, and so is $ZuZ^{-1}u^{-1}$. Consider the following commutative diagram:

$$\begin{array}{ccc}
U_{\mathfrak{G}}(\lambda)(k[\varepsilon]) & \rightarrow & P_{\mathfrak{G}}(\lambda)(k[\varepsilon]) \\
\downarrow & & \downarrow \\
U_{\mathfrak{G}}(\lambda)(k) & \rightarrow & P_{\mathfrak{G}}(\lambda)(k)
\end{array}$$

Because both $Z$ and $uZ^{-1}u^{-1}$ lie in the kernel of the right vertical map, so does their product $ZuZ^{-1}u^{-1}$. Hence, $ZuZ^{-1}u^{-1} \in U_{\mathfrak{G}}(\lambda)(k[\varepsilon])$ lies in the kernel of the left vertical map. The lemma then follows.

**Proposition 4.19** Let $z \in G(\mathcal{R})$ and $X_0 \in g_{\mathcal{R}}$ be the unique elements given by Propositions 4.16 and 4.17, respectively. We have $X_0 = \Gamma_z \cdot (\mu \varphi(X_0))$. In particular, $I(z, X_0)$ is a $G(\varphi, \nabla)$-module over $\mathcal{R}$.

**Proof** The second assertion follows from the first assertion and Lemma 4.7. For the first assertion, we need to show

$$\text{(9)} \quad X_0 = \mu \cdot \text{Ad}(z) \left( \varphi(X_0) \right) - \text{dlog}(z).$$
It suffices to show (3) with both sides understood as elements in $\text{End}_R(V_R)$ for some faithful representation $(V, \rho) \in \text{Rep}_F(G)$. Suppose that $\dim F V = d$, and suppose that $v_g(V)$ is the decomposition $V_R = \bigoplus_{i=1}^d V_{R,i}$. We choose for each graded-piece $V_{R,i}$ a basis. They altogether give a basis $v_1, \ldots, v_d$ of $V_R$, in which $\Phi_X$ acts via a block-upper-triangular matrix

$$A = \begin{pmatrix} A_1 & \ast & \cdots & \ast \\ & A_2 & \cdots & \ast \\ & & \ddots & \ast \\ & & & A_d \end{pmatrix} \in \text{GL}_d(R),$$

where each $A_i$ is an $m_i$ by $m_i$ invertible matrix with $m_i$ the multiplicity of $\mu_i$. Then, $\Phi_X$ acts in this basis via $Z := \text{Diag}(A_1, \ldots, A_d)$. Likewise, $\Theta_X$ acts in the basis $v_1, \ldots, v_d$ via a block-upper-triangular matrix

$$N = \begin{pmatrix} N_1 & \ast & \cdots & \ast \\ & N_2 & \cdots & \ast \\ & & \ddots & \ast \\ & & & N_d \end{pmatrix} \in \text{Mat}_{d,d}(R),$$

where each $N_i$ is an $m_i$ by $m_i$ matrix, and $\Theta_X$ acts via $\overline{N} := \text{Diag}(N_1, \ldots, N_d)$. Write $A = ZU$ for $U \in \text{GL}_d(R)$, and $N = \overline{N} + N_+$ for $N_+ \in \text{Mat}_{d,d}(R)$. Because $X = \Gamma_Z(\mu \varphi(X))$, we have $N = \mu \cdot \varphi(N) A^{-1} - \partial(A) A^{-1}$, and then

$$\overline{N} + N_+ = \mu \cdot (UZ)(\varphi(\overline{N} + N_+))(UZ)^{-1} - \partial(UZ)(UZ)^{-1} = \mu \cdot (UZ)\varphi(\overline{N})Z^{-1}U^{-1} + \mu \cdot (UZ)\varphi(N_+)Z^{-1}U^{-1} - \partial(U)U^{-1} - U \partial(Z)Z^{-1}U^{-1}.$$

Applying $\text{Ad}(U^{-1})$ on both sides, we then have

$$\mu \cdot Z\varphi(\overline{N})Z^{-1} - \partial(Z)Z^{-1} + \mu \cdot Z\varphi(N_+)Z^{-1} - U^{-1}\partial(U) = U^{-1}\overline{N}U + U^{-1}N_+U = \overline{N} - (\overline{N} - U^{-1}\overline{N}U - U^{-1}N_+U).$$

We claim that $\mu \cdot Z\varphi(\overline{N})Z^{-1} - \partial(Z)Z^{-1} = \overline{N}$. Put $\lambda_{p,g} := \rho \circ \lambda_g : G_m \rightarrow \text{GL}_V$, where $\lambda_g : G_m \rightarrow G_R$ is the slope morphism defined in Construction 3.15. Identifying $\text{GL}_V$ with $\text{GL}_d(R)$ via the basis $v_1, \ldots, v_d$ given in the preceding paragraph, and letting $\mathfrak{g} = \text{GL}_d(R)$, we then have an isomorphism

$$U_{\mathfrak{g}}(-\lambda_{p,g}) \rtimes Z_{\mathfrak{g}}(-\lambda_{p,g}) \cong P_{\mathfrak{g}}(-\lambda_{p,g})$$

of affine algebraic $R$-groups. Because $\mu_1 < \cdots < \mu_i$, we have

$$A \in P_{\mathfrak{g}}(-\lambda_{p,g})(R), \quad U \in U_{\mathfrak{g}}(-\lambda_{p,g})(R), \quad Z \in Z_{\mathfrak{g}}(-\lambda_{p,g})(R);$$

$$N \in \text{Lie}(P_{\mathfrak{g}}(-\lambda_{p,g})), \quad N_+ \in \text{Lie}(U_{\mathfrak{g}}(-\lambda_{p,g})), \quad \overline{N} \in \text{Lie}(Z_{\mathfrak{g}}(-\lambda_{p,g})).$$

It follows from Lemma 4.18 that $\overline{N} - U^{-1}\overline{N}U \in \text{Lie}(U_{\mathfrak{g}}(-\lambda_{p,g}))$. In particular, we have $\overline{N} - U^{-1}\overline{N}U - U^{-1}N_+U \in \text{Lie}(P_{\mathfrak{g}}(-\lambda_{p,g}))$. On the other hand, it is clear that $\mu \cdot Z\varphi(\overline{N})Z^{-1} - \partial(Z)Z^{-1} \in \text{Lie}(Z_{\mathfrak{g}}(-\lambda_{p,g}))$ and $\mu \cdot Z\varphi(N_+)Z^{-1} - U^{-1}\partial(U) \in \text{Lie}(U_{\mathfrak{g}}(-\lambda_{p,g}))$. By decomposition (2), we have $\mu \cdot Z\varphi(\overline{N})Z^{-1} - \partial(Z)Z^{-1} = \overline{N}$, and the desired equality (3) follows. \[\square\]

Recall that the least common denominator $d_g$ of $g$ is constructed in Construction 3.8, and $\lambda_g : G_m \rightarrow G_R$ is the slope morphism (see Construction 3.15). We next
reduce the $G\cdot (\varphi, \nabla)$-module $(z, X_0)$ over $\mathcal{R}$ to a unit-root one by applying the pushforward functor $[d_g]_*$ and twisting by $\lambda(g(\varphi^{-1}))$.

**Corollary 4.20**  
$I \big( \lambda(g(\varphi^{-1}))[d_g]_*(z), X_0 \big)$ is a unit-root $G\cdot (\varphi^{d_g}, \nabla)$-module over $\mathcal{R}$. 

**Proof**  
For any $V \in \text{Rep}_F(G)$, it suffices to show that $(V_{\mathcal{R}}, [d_g]_*(z)\varphi^{d_g}, \nabla_{X_0})$ is unit-root. By Proposition 4.19 and Lemma 4.12, $(V_{\mathcal{R}}, [d_g]_*(z)\varphi^{d_g}, \nabla_{X_0})$ is a $(\varphi^{d_g}, \nabla)$-module over $\mathcal{R}$. Equivalently, we have $\Theta_{X_0} \circ \Phi_{d_g} = \mu \cdot \phi \circ \Theta_{X_0}$.

We put $\Theta_{X_0} \circ \rho(\lambda(g(\varphi^{-1}))) \circ \Phi_{d_g} = \rho(\lambda(g(\varphi^{-1}))) \circ \Theta_{X_0} \circ \Phi_{d_g} = \mu \cdot \rho(\lambda(g(\varphi^{-1}))) \circ \Phi_{d_g} \circ \Theta_{X_0}$, which completes the proof. 

\[ \tag{4.5} \]

**A $G$-version of the $p$-adic local monodromy theorem**

Let $L$ be a finite separable extension of $\kappa((t))$, and let $E^\dagger_L$ be the unique unramified extension of $E^\dagger$ with residue field $L$. We put $\mathcal{R}_L := \mathcal{R} \otimes_{E^\dagger} E^\dagger_L$.

We put 
\[ E^{\dagger, \text{nr}} := \lim_{\substack{\rightarrow \cr \mathcal{L}}} E^\dagger_L, \quad \text{and} \quad \mathcal{B}_0 := \lim_{\substack{\rightarrow \cr \mathcal{L}}} \mathcal{R}_L \cong \mathcal{R} \otimes_{E^\dagger} E^{\dagger, \text{nr}}, \]

where $L$ runs through all finite separable extensions of $\kappa((t))$. In fact, $E^{\dagger, \text{nr}}$ is the maximal unramified extension of $E^\dagger$ with residue field $\kappa((t))^{\text{sep}}$, the separable closure of $\kappa((t))$.

The main result of this paper is the following theorem.

**Theorem 4.21**  
Let $G$ be a connected reductive $F$-group, and let $(g, X) \in B^{\varphi, \nabla}(G, \mathcal{R})$. Then, there exist a finite separable extension $L$ of $\kappa((t))$ and an element $b \in G(\mathcal{R}_L)$ such that $\Gamma_b(X) \in \text{Lie} \big( U_{G, L}(-\lambda_g) \big)_{\mathcal{R}_L}$.

We will make use of the following lemma, which is often mentioned as Steinberg’s theorem. The theory of fields of cohomological dimension $\leq 1$ can be found in, e.g., [19, Chapter II, Section 3]; for us, the most important example will be a Henselian discretely valued field of characteristic 0 with algebraically closed residue field (see [19, Chapter II, Section 3.3]).

**Lemma 4.22** ([21, Theorem 1.9])  
Suppose that $k$ is a field of cohomological dimension $\leq 1$ and $G$ is a connected reductive $k$-group, then $H^1(k, G) = 1$.

**Proof of Theorem 4.21**  
Let $z \in G(\mathcal{R})$ and $X_0 \in g_{\mathcal{R}}$ be the unique elements given by Propositions 4.16 and 4.17, respectively.

Let $(V, \rho)$ be a $d$-dimensional $G$-representation (not necessarily faithful). Suppose the slope filtration of $(V_{\mathcal{R}}, \varphi)$ has jumps $\mu_1, \ldots, \mu_l$. Suppose that $\xi_g(V) = \bigoplus_{i=1}^l V_{\mathcal{R}, \mu_i}$, we put $d_i := \text{rk}_{\mathcal{R}}(V_{\mathcal{R}, \mu_i})$ for all $i$. In the proof of Corollary 4.20, we see
that \((\mathcal{V}_{\mathbb{R},\mu_1}, \lambda_{\mathbb{R}}(\alpha^{-1}))[d_{\mathbb{R}}]_*(z) \varphi^d, \nabla_{\mathbb{R}})\) is a unit-root \((\varphi, \nabla)\)-module over \(\mathbb{R}\) for all \(1 \leq i \leq l\). Let \(\Phi_x = z\varphi\), and let \(\Theta_x: \mathbb{R} \to \mathbb{R}\) be the differential operator associated to \(\nabla_{\mathbb{R}}\). Then, \(\Phi_x\) (resp. \(\Theta_x\)) may be extended to \(V \otimes_F \mathcal{B}_0\), which is still denoted by \(\Phi_x\) (resp. \(\Theta_x\)). By the unit-root pLMT [9, Theorem 6.11], we find:

(i) a finite separable extension \(L(V)\) of \(K((t))\);
(ii) for each \(1 \leq i \leq l\), a basis \(\omega_i = \omega_i^{(1)}, \ldots, \omega_i^{(d_i)}\) for \(V_{\mathbb{R},\mu_1} \otimes_{\mathbb{R}} \mathcal{R}_{L(V)}\) over \(\mathcal{R}_{L(V)}\) such that \(\Theta_x(\omega_i^{(j)}) = 0\), for all \(1 \leq j \leq d_i\).

Then, for each \(1 \leq i \leq l\), we have that

\[
W_i := (\mathcal{V}_{\mathbb{R},\mu_1} \otimes_{\mathcal{R}} \mathcal{B}_0)_{\Theta_x = 0} = \{x \in \mathcal{V}_{\mathbb{R},\mu_1} \otimes_{\mathcal{R}} \mathcal{B}_0 \mid \Theta_x(x) = 0\}
\]

is a \(d_i\)-dimensional \(K^{nr}\)-vector space spanned by \(\omega_i^{(1)}, \ldots, \omega_i^{(d_i)}\). In particular, we have

\[
(\mathcal{V}_{\mathcal{B}_0})_{\Theta_x = 0} = \{x \in \mathcal{V}_{\mathcal{B}_0} \mid \Theta_x(x) = 0\} = \bigoplus_{i=1}^l W_i,
\]

which is a \(d_i\)-dimensional \(K^{nr}\)-vector space.

We now have two \(K^{nr}\)-valued fiber functors

\[
\omega_1: \text{Rep}_F(G) \to \text{Vec}_{K^{nr}}, \quad V \mapsto V \otimes K^{nr},
\]

and

\[
\omega_2: \text{Rep}_F(G) \to \text{Vec}_{K^{nr}}, \quad V \mapsto (\mathcal{V}_{\mathcal{B}_0})_{\Theta_x = 0}.
\]

Moreover, we have an action

\[
\text{Isom}^\otimes(\omega_1, \omega_2) \times \text{Aut}^\otimes(\omega_1) \to \text{Isom}^\otimes(\omega_1, \omega_2)
\]

of \(\text{Aut}^\otimes(\omega_1)\) on \(\text{Isom}^\otimes(\omega_1, \omega_2)\), given by precomposition. We note that \(\text{Aut}^\otimes(\omega_1) = \text{Aut}^\otimes(\omega_0^G \otimes K^{nr}) \simeq G_{K^{nr}}\), so \(\text{Isom}^\otimes(\omega_1, \omega_2)\) may be viewed as a \(G_{K^{nr}}\)-torsor over \(K^{nr}\).

By Lemma 4.22, we have \(H^1(K^{nr}, G_{K^{nr}}) = 1\). Thus, \(\text{Isom}^\otimes(\omega_1, \omega_2)\) is isomorphic to the trivial \(G_{K^{nr}}\)-torsor over \(K^{nr}\), i.e., we have \(\text{Isom}^\otimes(\omega_1, \omega_2)_{K^{nr}} \simeq G_{K^{nr}}\).

On the other hand, we have an isomorphism \(\gamma: \omega_2 \otimes \mathcal{B}_0 \to \omega_1 \otimes \mathcal{B}_0\) of tensor functors, induced by the \(\mathcal{B}_0\)-linear extension of the inclusion

\[
(\mathcal{V}_{\mathcal{B}_0})_{\Theta_x = 0} \hookrightarrow (\mathcal{V}_{\mathcal{B}_0})_{\Theta_x = 0}
\]

for all \((V, \rho) \in \text{Rep}_F(G)\). We now fix \(\tilde{\beta} \in \text{Isom}^\otimes(\omega_1, \omega_2)(K^{nr})\); we then have an element \(\tilde{\beta} := \gamma \circ \tilde{\beta}_{\mathcal{B}_0} \in \text{Aut}^\otimes(\omega_1 \otimes \mathcal{B}_0)(\mathcal{B}_0) = G(\mathcal{B}_0)\). Let \(b \in G(\mathcal{B}_0)\) be the inverse of the image of \(\tilde{\beta}\) under the following isomorphism:

\[
\text{Aut}^\otimes(\omega_1 \otimes \mathcal{B}_0)(\mathcal{B}_0) \to G(\mathcal{B}_0).
\]

Because \(F[G]\) is finitely presented over \(F\), the functor \(\text{Hom}_{\text{Alg}_F}(F[G], -)\) commutes with colimits. We have

\[
G(\mathcal{B}_0) = G(\varprojlim_{\mathcal{R}_L}) = \varinjlim_{\mathcal{R}_L} G(\mathcal{R}_L),
\]

\[\text{Footnote:}\]

\[\text{Footnote:}\]
where $L$ runs over all finite separable extensions of $\kappa((t))$; we thus find a finite separable extension $L$ of $\kappa((t))$ such that $b \in G(R_L)$.

For any $(V, \rho) \in \text{Rep}_F(G)$, it follows from the construction of $b$ that the automorphism $\rho(b^{-1}) : V_{B_0} \to V_{B_0}$ factors through $(V_{B_0})^{\Theta_{X_0}} \otimes B_0$. Notice that $\Theta_{X_0}$ and $X_0$ agree on $\omega_1(V) = V_{K^w}$. Therefore, we have

\begin{equation}
\rho(b)X_0\rho(b^{-1}) - \partial(\rho(b))\rho(b^{-1}) = 0.
\end{equation}

We now fix a faithful representation $(V, \rho)$. The equality (4) then implies

$$\Gamma_b(X_0) = 0.$$ 

Put $X_1 := X - X_0 \in g_R$; we then have

$$\Gamma_b(X) = \text{Ad}(b)(X_0 + X_1) - d\log(b)$$
$$= \text{Ad}(b)(X_0) - d\log(b) + \text{Ad}(b)(X_1)$$
$$= \Gamma_b(X_0) + \text{Ad}(b)(X_1)$$
$$= \text{Ad}(b)(X_1).$$

Conserving the notation as in the second paragraph, $\Theta_X = \rho(b)X_1\rho(b^{-1})$ acts in the basis $w_{1_1}^{(1)}, \ldots, w_{d_1}^{(1)}, \ldots, w_{1_l}^{(l)}, \ldots, w_{d_l}^{(l)}$ via a matrix of the form

$$\begin{pmatrix}
0 & 0 & * \\
0 & \ddots & 0
\end{pmatrix} \in \text{Mat}_{d_1 \times d_1}(R_L).$$

Here, the $i$th 0 in the diagonal denotes the zero matrix of size $d_i \times d_i$. Hence, $\Gamma_b(X) \in \text{Lie} \left( \text{U}_{G_{R_L}}(-\lambda_g, R_L) \right) = \text{Lie} \left( \text{U}_{G_{R_L}}(-\lambda_g) \right)_{R_L} = \text{Lie} \left( \text{U}_{G_{R_L}}(-\lambda_g) \right)_{R_L}$, as desired. 

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