Abstract

In this study, an algorithm for computing the inverse of periodic $k$ banded matrices, which are needed for solving the differential equations by using the finite differences, the solution of partial differential equations and the solution of boundary value problems is obtained and the inverses of periodic anti $k$ banded matrices are computed. In addition, the determinant of these type of matrices and the solution of linear systems having these coefficient matrices are investigated. When obtaining this algorithm, the $LU$ factorization is used. The algorithm is implementable to the CAS (Computer Algebra Systems) such as Maple and Mathematica.

Keywords: Periodic $k$ banded matrix, Periodic anti $k$ banded matrix, Inverse matrix of periodic $k$ banded and anti $k$ banded matrix, $LU$ factorization.
1 Introduction

The $n \times n$ periodic $k$ banded and anti $k$ banded matrices take the following form respectively:

$$M = \begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1,k+1} & 0 \\
    a_{21} & a_{22} & \cdots & a_{2,k+1} & a_{2,k+3} \\
    a_{31} & a_{32} & \cdots & a_{3,k+1} & a_{3,k+3} \\
    \vdots & \vdots & & \vdots & \vdots \\
    a_{k+1,1} & a_{k+1,2} & \cdots & a_{k+1,k+1} & a_{k+1,k+3} \\
    0 & a_{k+2,2} & \cdots & a_{k+2,k+1} & a_{k+2,k+3} \\
    \vdots & \vdots & & \vdots & \vdots \\
    0 & 0 & \cdots & a_{n-k-1,k+1} & a_{n-k-1,k+3} \\
    \vdots & \vdots & & \vdots & \vdots \\
    0 & 0 & \cdots & 0 & 0 \\
    a_{n1} & 0 & \cdots & 0 & 0
\end{bmatrix}$$

(1)
The periodic $k$ banded matrices are needed in science and engineering applications for example solving differential equations by using the finite differences, the solution of partial differential equations and the solution of boundary value problems.

In [1], the authors obtained an algorithm to find the inverse of the periodic tridiagonal matrix by using Doolittle LU factorization and the inverse of periodic anti-tridiagonal matrix is obtained when the inverse of periodic tridiagonal matrix exists. A new algorithm is obtained for the inverse of periodic pentadiagonal and anti-pentadiagonal matrix in [2]. This paper is an expansion of [1]. In [3] an algorithm for solving linear systems having periodic pentadiagonal coefficient matrices is obtained. It is presented that a new computational algorithm to evaluate the determinant of the tridiagonal matrix with its cost In [4]. In [5] Hadj and Elouafi obtained a fast numerical algorithm for the inverse of a tridiagonal and pentadiagonal matrix.

In this work we obtain an algorithm to find the inverse of periodic $k$ banded matrix when its inversion exists. When the algorithm is obtained the Doolittle LU factorization is used. After finding the inverse of periodic $k$ banded matrix, the periodic anti $k$ banded matrix is inverted by using the inversion of periodic $k$
banded matrix. Also, an algorithm is studied to solve the linear systems having these coefficient matrices.

## 2 Main Result

In this section, the LU factorization of the matrix $M$ is computed firstly where $L$ and $U$ are lower and upper triangular matrices, respectively. It is as in the following:

$$
L = \begin{bmatrix}
1 \\
l_{21} & \ddots \\
l_{31} & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
l_{n1} & \cdots & l_{n,n-k+1} & l_{n,n-k+3} & \cdots & l_{n,n-2} & 1
\end{bmatrix}
$$

and

$$
U = \begin{bmatrix}
u_{11} & u_{12} & u_{13} & \cdots & u_{1,k+1} & 0 & \cdots & 0 & u_{1n} \\
u_{22} & u_{23} & u_{24} & \cdots & u_{2,k+1} & u_{2,k+3} & \cdots & 0 & u_{2n} \\
u_{33} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
u_{k+1,k+1} & u_{k+1,k+3} & u_{k+1,k+5} & \cdots & 0 & u_{k+1,n} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
u_{n-1,n-1} & u_{n-1,n} \\
u_{nn}
\end{bmatrix}
$$

where

$$
u_{i,r} = a_{i,r} - \sum_{j=\frac{i-r+1}{2}}^{i-1} l_{ij} u_{jr}, \ i = 1, 2, \ldots n + i - r - 1 \quad (3)
$$

for $r = i + 1, i + 2, \ldots i + \frac{k-1}{2},$

$$
l_{i,r} = \frac{1}{u_{rr}} \left( a_{ir} - \sum_{j=i-\frac{k-1}{2}}^{r-1} l_{ij} u_{jr} \right), \ i = 1 + i - r, \ldots n - 1 \quad (4)
$$
for \( r = i - 1, i - 2, \ldots i - \frac{k}{2} \) and

\[
u_{ii} = \begin{cases} 
a_{ii} - \sum_{j=i-\frac{k+1}{2}}^{i-1} l_{ij} u_{ji}, & i = 1, 2, \ldots, n - 1 \\
a_{nn} - \sum_{j=1}^{i-1} l_{nj} u_{jn}, & i = n \end{cases}
\]

(5)

\[
u_{i,n} = \begin{cases} 
a_{1n} - \sum_{j=i-\frac{k+1}{2}}^{i-1} l_{ij} u_{jn}, & i = 1 \\
a_{i,n} - \sum_{j=i-\frac{k+1}{2}}^{i-1} l_{ij} u_{jn}, & i = n + \frac{1}{2} \left( \frac{k+1}{2} \right), \ldots, n - 1 \\
\frac{a_{nn}}{u_{ii}} - \left( -\frac{1}{u_{ii}} \sum_{j=i-\frac{k+1}{2}}^{i-1} l_{nj} u_{ji} \right), & i = 2, 3, \ldots, n - \frac{k}{2}
\end{cases}
\]

(6)

\[
l_{n,i} = \begin{cases} 
\frac{1}{u_{ii}} \left( a_{ii} - \sum_{j=i-\frac{k+1}{2}}^{i-1} l_{nj} u_{ji} \right), & i = n + \frac{1}{2} \left( \frac{k+1}{2} \right), \ldots, n - 1
\end{cases}
\]

(7)

for \( i, j \leq 0 \) then \( l_{ij} = 0 \) and \( u_{ij} = 0 \).

From here, it is clear that

\[
\det(M) = \prod_{i=1}^{n} u_{ii}.
\]

The inverse matrix is computed as in the following, if the matrix \( M \) is nonsingular:

Let \( C_r \) be the \( r \)th column of \( M^{-1} \) for \( r = 1, 2, \ldots, n \) then

\[
M^{-1} = (S_{i,j})_{1 \leq i,j \leq n} = (C_1, C_2, \ldots, C_r, \ldots, C_n). C_r = (S_{1,r}, S_{2,r}, \ldots, S_{n,r})
\]

and it can be written as in the following:

\[
C_r = (C_1, C_2, \ldots, C_r, \ldots, C_n) E_r
\]

where \( E_r \) is the Kronecker symbol \((E_r = (\delta_{1r}, \delta_{2r}, \ldots, \delta_{nr})^T, r = 1, 2, \ldots, n.)\).

Now, the algorithm for the inverse of the periodic \( k \) banded matrix can be developed. By using the \( LU \) factorization, the entries of the last \( \frac{k-1}{2} \) columns of \( M^{-1} \) are computed as follows:
For $r = n, n - 1, \ldots, m + 1, m$ ($m = n - \frac{k+1}{2}$)

$S_{i,r} = \begin{cases} \frac{1}{u_{ii}} \left( t_{ir} - \sum_{j=i+1}^{n} u_{ij} S_{jr} \right), & i = n, n - 1, \ldots, r + 1 \\ \frac{1}{u_{ii}} \left( 1 - \sum_{j=i+1}^{n} u_{ij} S_{jr} \right), & i = r \\ -\frac{1}{u_{ii}} \left( \sum_{j=i+1}^{n} u_{ij} S_{jr} \right), & i = r - 1, r - 2, \ldots, 2, 1 \end{cases}$ (8)

where $t_{ij}$s are the entries of the inverse of $L$ and for $r = 1, 2, \ldots, n - 1$ they are computed with the following recurrence relation:

$t_{ir} = -l_{ir} - \sum_{j=r+1}^{i-1} l_{ij} t_{jr} , i = r + 1, r + 2, \ldots, n$. (9)

Up to now, the entries of the last $k+1$ columns are obtained. Entries of the remaining $n - \frac{k-1}{2}$ columns are computed by using the following equation

$M^{-1}M = I_n$.

For $j = n - \frac{k+1}{2}, n - \frac{k+3}{2}, \ldots, 1$

$C_j = \frac{1}{a_{j,j+\frac{k-1}{2}}} \left( E_{j+\frac{k-1}{2}} - \sum_{r=j+1}^{j+\frac{k-1}{2}} a_{r,j+\frac{k-1}{2}} C_r \right)$ (10)

where if $j = 1, 2, \ldots, n - \frac{k+1}{2}$ then $a_{j,j+\frac{k-1}{2}} \neq 0$. Here, $i > n$, $j > n$ then $a_{ij} = 0$, if $i > n$ then $C_i = 0$.

Algorithm 1:

INPUT: $n$ is the order of the periodic $k$ banded matrix, $k$ is the bandwidth of the matrix.

OUTPUT: The inverse matrix $M^{-1} = (S_{i,j})_{1\leq i,j \leq n}$.

Step1: For $i = 1, 2, \ldots, n - \frac{k+1}{2}$, if $a_{i,i+\frac{k-1}{2}} = 0$, then $a_{i,i+\frac{k-1}{2}} = \lambda$.

Step2: For $i = \frac{k+3}{2}, \frac{k+5}{2}, \ldots, n$, if $a_{i,i-\frac{k-1}{2}} = 0$, then $a_{i,i-\frac{k-1}{2}} = \lambda$.

Step3: By using (3)-(7), compute the elements of the matrices $L$ and $U$.

For $i = 1, 2, \ldots, n$, if $u_{ii} = 0$, then $u_{ii} = \lambda$.

Step 4: Compute $\det(M) = \left( \sum_{i=1}^{n} u_{ii} \right)_{\lambda=0}$. If $M$ is singular, then the output is "Singular Matrix".

Step 5: Compute the elements $t_{i,r}$ by using (9).

Step6: Compute the elements of the last $\frac{k+1}{2}$ columns by using (8).

Step7: Compute the elements of the remaining $n - \frac{k+1}{2}$ columns by using (10).
**Step 8:** Substitute the actual value of \( \lambda \) in all elements of the inverse matrix \( M^{-1} \).

The inverse matrix of the periodic anti \( k \) banded matrix \( N \) can be obtained by using the inverse of periodic \( k \) banded matrix \( M \) easily.

Let \( R \) be an \( n \times n \) matrix as in the following form:

\[
R = \begin{bmatrix}
0 & \cdots & \cdots & 0 & 1 \\
\vdots & \ddots & 1 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots & \ddots \\
1 & 0 & \cdots & \cdots & 0
\end{bmatrix}
\]

It is clear that \( R \) is nonsingular and its inversion is itself \( R \). The following relation is true for (\( \square \)) and (2).

\[
N = MR
\]

Thus, the inverse of (2) is obtained as in the following:

\[
N^{-1} = RM^{-1}
\]

Also, linear systems having periodic \( k \) banded coefficient matrix, \( Mx = y \), can be solved by using the Dolittle \( LU \) factorization of this type of matrix. Here the component \( u_{ii} \) is important. Because, the system has a unique solution if \( u_{ii} \neq 0 \), \( i = 1, 2, \ldots, n \).

**Algorithm 2:**

**Step 1:** By using the Step1-Step3 of Algorithm 1, compute the \( LU \) factorization of the coefficient matrix \( M \).

**Step 2:** For \( i = 1, 2, \ldots, n \)

\[
z_i = y_i - \sum_{j=1}^{i-1} l_{ij} z_j.
\]

**Step 3:** For \( i = n, n-1, \ldots, 2, 1 \)

\[
x_i = \begin{cases}
\frac{1}{u_{ii}} z_i , & i = n \\
\frac{1}{u_{ii}} \left( z_i - \sum_{j=i+1}^{i+\frac{k-1}{2}} u_{ij} x_j \right) , & i = n-1, \ldots, n - \frac{k-1}{2} \\
\frac{1}{u_{ii}} \left( z_i - \sum_{j=i+1}^{n} u_{ij} x_j - u_{in} x_n \right) , & i = n - \frac{k+1}{2}, \ldots, 1
\end{cases}
\]
3 Numerical Example

Example 1 Consider the $6 \times 6$ matrices $M$ and $N$ as in the following

$$M = \begin{bmatrix}
2 & 1 & 0 & 0 & 0 & 1 \\
1 & -1 & 2 & 0 & 0 & 0 \\
0 & 2 & -2 & 3 & 0 & 0 \\
0 & 0 & -1 & 1 & 1 & 0 \\
0 & 0 & 0 & 2 & -3 & -2 \\
2 & 0 & 0 & 0 & 1 & 5
\end{bmatrix}$$

and

$$N = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 2 & -1 & 1 \\
0 & 0 & 3 & -2 & 2 & 0 \\
0 & 1 & 1 & -1 & 0 & 0 \\
-2 & -3 & 2 & 0 & 0 & 0 \\
5 & 1 & 0 & 0 & 0 & 2
\end{bmatrix}.$$  

We apply the Algorithm 1 to the matrix $M$ and we have

- For $i = 1, 2, 3, 4$ \( u_{i,i+1} = (u_{12}, u_{23}, u_{34}, u_{45}) = (1, 2, 3, 1) \)
- For $i = 2, 3, 4, 5$ \( l_{i,i-1} = (l_{21}, l_{32}, l_{43}, l_{54}) = (\frac{1}{2}, -\frac{4}{3}, -\frac{3}{2}, \frac{3}{4}) \)
- For $i = 1, 2, 3, 4, 5, 6$ \( u_{i} = (u_{11}, u_{22}, u_{33}, u_{44}, u_{55}, u_{66}) = (2, -\frac{3}{2}, \frac{11}{2}, -\frac{37}{11}, \frac{153}{37}) \)
- For $i = 1, 2, 3, 4, 5$ \( u_{i,6} = (u_{16}, u_{26}, u_{36}, u_{46}, u_{56}) = (1, -\frac{1}{2}, -\frac{2}{3}, -1, -\frac{14}{17}) \)
- For $i = 1, 2, 3, 4, 5$ \( t_{6,i} = (t_{61}, t_{62}, t_{63}, t_{64}, t_{65}) = (1, \frac{2}{3}, -2, \frac{22}{11}, \frac{1}{3}) \)
- \( \text{det}(M) = 153 \)
- For $i = 2, 3, 4, 5, 6$ \( t_{i,1} = (t_{21}, t_{31}, t_{41}, t_{51}, t_{61}) = (-\frac{1}{2}, -\frac{2}{3}, -1, \frac{4}{11}, -\frac{34}{37}) \)
- For $i = 3, 4, 5, 6$ \( t_{i,2} = (t_{32}, t_{42}, t_{52}, t_{62}) = (\frac{4}{3}, 2, -\frac{8}{11}, -\frac{6}{37}) \)
- For $i = 4, 5, 6$ \( t_{i,3} = (t_{43}, t_{53}, t_{63}) = (\frac{3}{2}, -\frac{6}{11}, \frac{14}{37}) \)
- For $i = 5, 6$ \( t_{i,4} = (t_{54}, t_{64}) = (-\frac{4}{11}, -\frac{40}{37}) \)
- For $i = 6$ \( t_{i,5} = (t_{65}) = (-\frac{1}{37}) \)
- \( C_6 = (S_{66}, S_{56}, S_{46}, S_{36}, S_{26}, S_{16}) = (\frac{37}{153}, \frac{2}{11}, -\frac{10}{11}, \frac{8}{37}, \frac{23}{133}, -\frac{7}{133}) \)
- \( C_5 = (S_{65}, S_{55}, S_{45}, S_{35}, S_{25}, S_{15}) = (-\frac{1}{133}, -\frac{1}{11}, \frac{8}{133}, -\frac{37}{153}, \frac{42}{133}, \frac{25}{133}) \)
- \( C_4 = (S_{64}, S_{54}, S_{44}, S_{34}, S_{24}, S_{14}) = (-\frac{40}{133}, \frac{14}{11}, -\frac{103}{133}, -\frac{124}{133}, \frac{82}{133}) \)
- \( C_3 = (S_{63}, S_{53}, S_{43}, S_{33}, S_{23}, S_{13}) = (\frac{14}{133}, \frac{2}{11}, \frac{41}{133}, -\frac{59}{133}, -\frac{74}{133}, \frac{44}{133}) \)
- \( C_2 = (S_{62}, S_{52}, S_{42}, S_{32}, S_{22}, S_{12}) = (-\frac{2}{51}, \frac{4}{11}, \frac{41}{51}, \frac{28}{51}, \frac{4}{51}, -\frac{1}{51}) \)
• \( C_1 = (S_{61}, S_{51}, S_{41}, S_{31}, S_{21}, S_{11}) = (-\frac{2}{3}, 0, -\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}, \frac{5}{3}) \)

• \( M^{-1} = \)

\[
\begin{bmatrix}
\frac{5}{9} & \frac{1}{9} & -\frac{144}{9} & \frac{82}{9} & \frac{25}{9} & \frac{7}{9} \\
\frac{2}{9} & \frac{7}{9} & \frac{28}{9} & \frac{14}{9} & \frac{4}{9} & \frac{2}{9} \\
0 & \frac{2}{9} & \frac{11}{9} & \frac{5}{9} & \frac{1}{9} & \frac{0}{9} \\
-\frac{2}{9} & \frac{1}{9} & -\frac{13}{9} & \frac{5}{9} & \frac{0}{9} & \frac{2}{9} \\
0 & \frac{2}{9} & \frac{14}{9} & \frac{40}{9} & \frac{13}{9} & \frac{5}{9} \\
-\frac{2}{9} & \frac{1}{9} & -\frac{16}{9} & \frac{12}{9} & \frac{5}{9} & \frac{0}{9}
\end{bmatrix}
\]

• \( N^{-1} = \)

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{5}{9} & \frac{1}{9} & -\frac{144}{9} & \frac{82}{9} & \frac{25}{9} & \frac{7}{9} \\
\frac{2}{9} & \frac{7}{9} & \frac{28}{9} & \frac{14}{9} & \frac{4}{9} & \frac{2}{9} \\
0 & \frac{2}{9} & \frac{11}{9} & \frac{5}{9} & \frac{1}{9} & \frac{0}{9} \\
-\frac{2}{9} & \frac{1}{9} & -\frac{13}{9} & \frac{5}{9} & \frac{0}{9} & \frac{2}{9} \\
0 & \frac{2}{9} & \frac{14}{9} & \frac{40}{9} & \frac{13}{9} & \frac{5}{9} \\
-\frac{2}{9} & \frac{1}{9} & -\frac{16}{9} & \frac{12}{9} & \frac{5}{9} & \frac{0}{9}
\end{bmatrix}
\]

Example 2 Consider the 10 \( \times \) 10 matrices \( M \) and \( N \) as in the following

\[
M = \begin{bmatrix}
1 & -1 & 2 & 2 & -1 & 0 & 0 & 0 & 0 & 1 \\
2 & -1 & 3 & 1 & 1 & 2 & 0 & 0 & 0 & 0 \\
1 & -1 & 1 & 2 & 1 & -2 & -1 & 0 & 0 & 0 \\
-3 & 1 & -1 & 1 & -3 & 1 & 1 & -3 & 0 & 0 \\
2 & -1 & 1 & 0 & -3 & 2 & 1 & -1 & -1 & 0 \\
0 & 1 & 2 & 0 & -1 & 0 & -2 & 1 & 0 & 1 \\
0 & 0 & -2 & 0 & 1 & -1 & 1 & -2 & 1 & -1 \\
0 & 0 & 0 & 1 & 3 & 2 & -1 & 1 & 2 & 1 \\
0 & 0 & 0 & -1 & 0 & 2 & 1 & -2 & 1 & 0 \\
2 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & -1 & 2
\end{bmatrix}
\]

and

\[
N = \begin{bmatrix}
2 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & -1 & 2 \\
0 & 0 & 0 & 0 & -1 & 0 & 2 & 1 & -2 & 1 \\
0 & 0 & 0 & 1 & 3 & 2 & -1 & 1 & 2 & 1 \\
0 & 0 & -2 & 0 & 1 & -1 & 1 & -2 & 1 & -1 \\
0 & 1 & 2 & 0 & -1 & 0 & -2 & 1 & 0 & 1 \\
2 & -1 & 1 & 0 & -3 & 2 & 1 & -1 & -1 & 0 \\
-3 & 1 & -1 & 1 & -3 & 1 & 1 & -3 & 0 & 0 \\
1 & -1 & 1 & 2 & 1 & -2 & -1 & 0 & 0 & 0 \\
2 & -1 & 3 & 1 & 1 & 2 & 0 & 0 & 0 & 0 \\
1 & -1 & 2 & 2 & -1 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

We apply the Algorithm1 to the matrix \( M \) and we have
For $i = 1, 2, 3, 4, 5, 6, 7, 8$

$u_{i,i+1} = (u_{12}, u_{23}, u_{34}, u_{45}, u_{56}, u_{67}, u_{78}, u_{89}) = (-1, -1, 0, 6, 3, -7, -\frac{53}{29}, \frac{175}{84})$

For $i = 1, 2, 3, 4, 5, 6, 7$

$u_{i,i+2} = (u_{13}, u_{24}, u_{35}, u_{46}, u_{57}, u_{68}, u_{79}) = (2, -3, 2, -1, 1, 42, 1218)$

For $i = 1, 2, 3, 4, 5, 6$

$u_{i,i+3} = (u_{14}, u_{25}, u_{36}, u_{47}, u_{58}, u_{69}) = (2, 3, -2, -2, -4, 8)$

For $i = 1, 2, 3, 4, 5$

$u_{i,i+4} = (u_{15}, u_{26}, u_{37}, u_{48}, u_{59}) = (-1, 2, -1, -3, -1)$

For $i = 2, 3, 4, 5, 6, 7, 8, 9$

$l_{i,i-1} = (l_{21}, l_{32}, l_{43}, l_{54}, l_{65}, l_{76}, l_{87}, l_{98}) = (2, 0, -3, 1, 3, \frac{3}{5}, -\frac{2}{27}, -\frac{27}{209})$

For $i = 3, 4, 5, 6, 7, 8, 9$

$l_{i,i-2} = (l_{31}, l_{42}, l_{53}, l_{64}, l_{75}, l_{86}, l_{97}) = (1, -2, 2, 3, \frac{3}{5}, -1)$

For $i = 4, 5, 6, 7, 8, 9$

$l_{i,i-3} = (l_{41}, l_{52}, l_{63}, l_{74}, l_{85}, l_{96}) = (-3, 1, -3, 0, \frac{2}{7}, \frac{3}{58})$

For $i = 5, 6, 7, 8, 9$

$l_{i,i-4} = (l_{51}, l_{62}, l_{73}, l_{84}, l_{95}) = (2, 1, 2, 1, \frac{7}{2})$

For $i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$

$u_{ii} = (u_{11}, u_{22}, u_{33}, u_{44}, u_{55}, u_{66}, u_{77}, u_{88}, u_{99}, u_{1010})$

$= (1, 1, -1, 1, -2, -29, \frac{54}{29}, \frac{215}{27}, -\frac{309}{86}, \frac{944}{1545})$

For $i = 1, 2, 3, 4, 5, 6, 7, 8, 9$

$u_{i,10} = (u_{110}, u_{210}, u_{310}, u_{410}, u_{510}, u_{610}, u_{710}, u_{810}, u_{910}, u_{1010})$

$= (1, -2, -1, -4, -2, 28, \frac{74}{29}, \frac{182}{27}, -\frac{248}{215})$

For $i = 1, 2, 3, 4, 5, 6, 7, 8, 9$

$l_{10,i} = (l_{101}, l_{102}, l_{103}, l_{104}, l_{105}, l_{106}, l_{107}, l_{108}, l_{109})$

$= (2, 2, 2, 10, \frac{26}{29}, \frac{95}{54}, \frac{331}{439}, \frac{473}{309})$

$\det(M) = 1888$

For $i = 2, 3, 4, 5, 6, 7, 8, 9, 10$

$t_{i,1} = (t_{21}, t_{31}, t_{41}, t_{51}, t_{61}, t_{71}, t_{81}, t_{91}, t_{101})$

$= (-2, -1, -4, -2, 27, \frac{58}{29}, \frac{417}{27}, -\frac{1493}{209}, -\frac{2699}{209})$

For $i = 3, 4, 5, 6, 7, 8, 9, 10$

$t_{i,2} = (t_{32}, t_{42}, t_{52}, t_{62}, t_{72}, t_{82}, t_{92}, t_{102}) = (0, 2, 1, -15, -\frac{21}{29}, -\frac{25}{27}, \frac{28}{19}, \frac{33}{29})$

For $i = 4, 5, 6, 7, 8, 9, 10$

$t_{i,3} = (t_{43}, t_{53}, t_{63}, t_{73}, t_{83}, t_{93}, t_{103}) = (3, 1, -14, -\frac{161}{58}, -\frac{215}{54}, \frac{5}{2}, -\frac{160}{309})$

For $i = 5, 6, 7, 8, 9, 10$

$t_{i,4} = (t_{54}, t_{64}, t_{74}, t_{84}, t_{94}, t_{104}) = (1, -11, -\frac{27}{29}, -2, \frac{161}{215}, \frac{419}{309})$

For $i = 6, 7, 8, 9, 10$

$t_{i,5} = (t_{65}, t_{75}, t_{85}, t_{95}, t_{105}) = (-8, -\frac{63}{29}, -\frac{7}{6}, \frac{307}{199}, -\frac{3241}{299})$
• For $i = 7, 8, 9, 10$
  \[ t_{i,6} = (t_{76}, t_{86}, t_{96}, t_{10,6}) = \left( -\frac{3}{58}, -\frac{1}{18}, -\frac{3}{430}, \frac{777}{1030} \right) \]
• For $i = 8, 9, 10$  \[ t_{i,7} = (t_{87}, t_{97}, t_{10,7}) = \left( \frac{2}{27}, -\frac{213}{215}, -\frac{639}{1030} \right) \]
• For $i = 9, 10$  \[ t_{i,8} = (t_{98}, t_{10,8}) = \left( \frac{27}{215}, -\frac{949}{1990} \right) \]
• For $i = 10$  \[ t_{i,9} = (t_{10,9}) = \left( -\frac{373}{979} \right) \]

• $C_{10} = (S_{10,10}, S_{9,10}, S_{8,10}, S_{7,10}, S_{6,10}, S_{5,10}, S_{4,10}, S_{3,10}, S_{2,10}, S_{1,10})$
  \[ = \left( \frac{451}{1030}, \frac{390}{979}, \frac{330}{649}, \frac{270}{419}, \frac{210}{165}, \frac{150}{109}, \frac{90}{65}, \frac{30}{23}, \frac{5}{2}, \frac{1}{1} \right) \]
• $C_9 = (S_{9,9}, S_{8,9}, S_{7,9}, S_{6,9}, S_{5,9}, S_{4,9}, S_{3,9}, S_{2,9}, S_{1,9})$
  \[ = \left( \frac{1865}{1030}, \frac{1569}{979}, \frac{1349}{649}, \frac{1113}{419}, \frac{909}{419}, \frac{697}{165}, \frac{509}{109}, \frac{309}{65}, \frac{159}{23}, \frac{79}{1} \right) \]
• $C_8 = (S_{8,8}, S_{7,8}, S_{6,8}, S_{5,8}, S_{4,8}, S_{3,8}, S_{2,8}, S_{1,8})$
  \[ = \left( \frac{1888}{1030}, \frac{1545}{979}, \frac{1272}{649}, \frac{1055}{419}, \frac{888}{419}, \frac{777}{165}, \frac{676}{109}, \frac{576}{65}, \frac{416}{23}, \frac{27}{1} \right) \]
• $C_7 = (S_{7,7}, S_{6,7}, S_{5,7}, S_{4,7}, S_{3,7}, S_{2,7}, S_{1,7})$
  \[ = \left( \frac{1888}{1030}, \frac{1545}{979}, \frac{1272}{649}, \frac{1055}{419}, \frac{888}{419}, \frac{777}{165}, \frac{676}{109}, \frac{576}{65}, \frac{416}{23}, \frac{27}{1} \right) \]
• $C_6 = (S_{6,6}, S_{5,6}, S_{4,6}, S_{3,6}, S_{2,6}, S_{1,6})$
  \[ = \left( \frac{1888}{1030}, \frac{1545}{979}, \frac{1272}{649}, \frac{1055}{419}, \frac{888}{419}, \frac{777}{165}, \frac{676}{109}, \frac{576}{65}, \frac{416}{23}, \frac{27}{1} \right) \]
• $C_5 = (S_{5,5}, S_{4,5}, S_{3,5}, S_{2,5}, S_{1,5})$
  \[ = \left( \frac{1888}{1030}, \frac{1545}{979}, \frac{1272}{649}, \frac{1055}{419}, \frac{888}{419}, \frac{777}{165}, \frac{676}{109}, \frac{576}{65}, \frac{416}{23}, \frac{27}{1} \right) \]
• $C_4 = (S_{4,4}, S_{3,4}, S_{2,4}, S_{1,4})$
  \[ = \left( \frac{1888}{1030}, \frac{1545}{979}, \frac{1272}{649}, \frac{1055}{419}, \frac{888}{419}, \frac{777}{165}, \frac{676}{109}, \frac{576}{65}, \frac{416}{23}, \frac{27}{1} \right) \]
• $C_3 = (S_{3,3}, S_{2,3}, S_{1,3})$
  \[ = \left( \frac{1888}{1030}, \frac{1545}{979}, \frac{1272}{649}, \frac{1055}{419}, \frac{888}{419}, \frac{777}{165}, \frac{676}{109}, \frac{576}{65}, \frac{416}{23}, \frac{27}{1} \right) \]
• $C_2 = (S_{2,2}, S_{1,2})$
  \[ = \left( \frac{1888}{1030}, \frac{1545}{979}, \frac{1272}{649}, \frac{1055}{419}, \frac{888}{419}, \frac{777}{165}, \frac{676}{109}, \frac{576}{65}, \frac{416}{23}, \frac{27}{1} \right) \]
• $C_1 = (S_{1,1})$
  \[ = \left( \frac{1888}{1030}, \frac{1545}{979}, \frac{1272}{649}, \frac{1055}{419}, \frac{888}{419}, \frac{777}{165}, \frac{676}{109}, \frac{576}{65}, \frac{416}{23}, \frac{27}{1} \right) \]

\[
M^{-1} = \begin{bmatrix}
\frac{501}{1888} & \frac{53}{1030} & \frac{23}{979} & -\frac{189}{649} & \frac{759}{419} & \frac{709}{419} & \frac{707}{419} & \frac{561}{419} & \frac{367}{419} & \frac{199}{1}
\end{bmatrix}
\]
Example 3 Solve the periodic pentadiagonal system given as follows:

\[
\begin{bmatrix}
1 & 2 & -1 & 0 & 0 & 1 \\
2 & -1 & -3 & 1 & 0 & 0 \\
1 & 1 & -1 & 1 & 2 & 0 \\
0 & 2 & 1 & 1 & -1 & -2 \\
0 & 0 & -1 & -2 & 1 & 3 \\
1 & 0 & 0 & 1 & 1 & 1
\end{bmatrix}
= \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6
\end{bmatrix}
= \begin{bmatrix}
3 \\
-1 \\
4 \\
1 \\
1 \\
4
\end{bmatrix}
\]

We apply the Algorithm 2:

- For \( i = 1, 2, 3, 4 \) \( u_{i,i+1} = (u_{12}, u_{23}, u_{34}, u_{45}) = (2, -1, \frac{4}{5}, -7) \)
- For \( i = 1, 2, 3 \) \( u_{i,i+2} = (u_{13}, u_{24}, u_{35}) = (-1, 1, 2) \)
For $i = 2, 3, 4, 5$ \( l_{i,i-1} = (l_{21}, l_{32}, l_{43}, l_{54}) = (2, \frac{1}{3}, 3, -2) \)

• For $i = 3, 4, 5$ \( l_{i,i-2} = (l_{31}, l_{42}, l_{53}) = (1, -\frac{2}{3}, -5) \)

• For $i = 1, 2, 3, 4, 5, 6$ \( u_{ii} = (u_{11}, u_{22}, u_{33}, u_{44}, u_{55}, u_{66}) = (1, -5, \frac{1}{5}, -1, -3, -\frac{14}{3}) \)

• For $i = 1, 2, 3, 4, 5$ \( u_{i,6} = (u_{16}, u_{26}, u_{36}, u_{46}, u_{56}) = (1, -2, -\frac{2}{3}, -1, -2) \)

• For $i = 1, 2, 3, 4, 5$ \( l_{6,i} = (l_{61}, l_{62}, l_{63}, l_{64}, l_{65}) = (1, \frac{2}{5}, 7, 5, -\frac{22}{5}) \)

• \( \det(M) = 14 \)

• \( (z_1, z_2, z_3, z_4, z_5, z_6) = (3, -7, \frac{12}{5}, -9, -5, -\frac{4}{3}) \)

• \( (x_1, x_2, x_3, x_4, x_5, x_6) = (1, 1, 1, 1, 1, 1) \)

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