An invariant of rational homology 3-spheres via vector fields

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Abstract

We define an invariant of rational homology 3-spheres via vector fields. The construction of our invariant is a generalization of both that of the Kontsevich-Kuperberg-Thurston invariant and that of Watanabe’s Morse homotopy invariant, which implies the equivalence of these two invariants.

1 Introduction.

In this paper, we construct an invariant \( \tilde{z}_n \) of rational homology 3-spheres via vector fields. As an application, we prove that the Kontsevich-Kuperberg-Thurston invariant \( z_{KKT} = \{ z_{KKT}^n \}_{n \in \mathbb{N}} \) coincides with Watanabe’s Morse homotopy invariants \( z_{FW} = \{ z_{2n,3n}^{FW} \}_{n \in \mathbb{N}} \) for any rational homology 3-sphere. Note that both \( z_{KKT}^n \) and \( z_{n}^{FW} \) are topological invariants which take values in the real vector space \( A_n(\emptyset) \) of Jacobi diagrams.

M. Kontsevich [Kon94], S. Axelrod and I. M. Singer [AS92] proposed the Chern-Simons perturbation theory and gave a topological invariant of 3-manifolds. Based on Kontsevich’s work, G. Kuperberg and D. Thurston constructed in [KT99] a topological invariant \( z_{KKT} \) of rational homology 3-spheres. Kuperberg and Thurston proved that \( z_{KKT} \) is a universal finite type invariant for homology 3-spheres by showing surgery formulas. C. Lescop obtained surgery formulas of other types in [Les04b] and [Les09]. Lescop reviewed \( z_{KKT} \) and gave a more direct proof of well-definedness of this invariant in [Les04a].

K. Fukaya [Fuk96] constructed a topological invariant of 3-manifolds with local coefficients using Morse functions. Fukaya’s invariant is closely related to the theta graph \( \theta \). His invariant essentially takes values in \( A_1(\emptyset) \). M. Futaki [Fut06] pointed out that Fukaya’s invariant depends on the choice of Morse functions. T. Watanabe [Wat12] gave an invariant of rational homology 3-spheres without local coefficients using Morse functions. He also investigated higher loop graphs (and broken graphs) and then he defined a topological invariant \( z_{2n,3n}^{FW} \) of (rational) homology 3-spheres taking values in \( A_n(\emptyset) \) for each \( n \in \mathbb{N} \). The construction of \( z_{2n,3n}^{FW} \) is related to the construction of a Morse propagator constructed by Lescop [Les12a].

Fukaya’s construction is inspired by the construction of the 2-loop term of the Chern-Simons perturbation theory and he conjectured in §8 in [Fuk96] that his invariant is related to the 2-loop term of the Chern-Simons perturbation theory.
Watanabe also conjectured in Conjecture 1.2 in [Wat12] that his invariants is related to Axelrod and Singer’s invariant [AS92] or Kontsevich’s invariant [Kon94].

The main theorem of this paper is the following.

**Theorem 1.1.** $\tilde{z}_n^{\text{KKT}}(Y) = z_{2n,3n}^{\text{FW}}(Y)$ for any rational homology 3-sphere $Y$, for any $n \in \mathbb{N}$.

The idea of the proof of Theorem 1.1 is the following. We construct an invariant $\tilde{z}_n$ of rational homology 3-spheres using vector fields. Let $Y$ be a rational homology 3-sphere and let $\infty \in Y$ be a base point. $z_n^{\text{KKT}}(Y)$, $z_{2n,3n}^{\text{FW}}(Y)$ and $\tilde{z}_n$ are defined by using an extra information of $Y$. The extra information used in definition of $z_n^{\text{KKT}}$, $z_{2n,3n}^{\text{FW}}$ and $\tilde{z}_n$ are a framing of $Y \setminus \infty$, a family of Morse functions on $Y \setminus \infty$ and a family of vector fields on $Y \setminus \infty$, respectively. We prove that it is possible to regard the constructions of $z_{2n,3n}^{\text{FW}}$ and $z_n^{\text{KKT}}$ as special cases of the construction of $\tilde{z}_n$. In fact a framing gives us a non-vanishing vector field and a Morse function gives us a gradient vector field. The principal term of $\tilde{z}_1$ is related to Lescop’s invariant [Les12b] for rational homology 3-spheres with non-vanishing vector fields.

The organization of this paper is as follows. In Section 2 we prepare some notations. In Section 3 we review notions and facts about configuration spaces and graphs discussed by Lescop [Les04a] and Watanabe [Wat12]. In Section 4 we define the invariants $\tilde{z}_n$ using vector fields and prove the independence of the choice of vector fields. In Section 5 we review the construction in Lescop [Les04a] of $z_n^{\text{KKT}}$. In Section 6 we review the construction of $z_n^{\text{FW}}$ in Watanabe [Wat12] with a little modification. In Section 7 we prove Theorem 1.1. In Section 8 we prove some Lemmas for a compactification of the moduli space of flow graphs used in Sections 6 and 7. In Appendix A we give a more direct proof of Theorem 1.1 in the case of $n = 1$.

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**2 Notation and some remarks.**

In this article, all manifolds are smooth and oriented. Homology and cohomology are with rational coefficients. Let $c$ be a $\mathbb{Q}$-linear sum of finitely many maps from compact $k$-dimensional manifolds with corners to a topological space $X$. We consider $c$ as a $k$-chain of $X$ via appropriate (not unique) triangulations of each $k$-manifold. Let $Y$ be a submanifold of a manifold $X$. Let $c = \sum_i a_i(f_i : \Sigma_i \to X)$ be a chain of $X$, where $f_i : \Sigma_i \to X$ are smooth maps from compact manifolds with corners and
are rational numbers. If \( f_i \) is transverse to \( Y \) for each \( i \), then we say that \( c \) is transverse to \( f \).

When \( B \) is a submanifold of a manifold \( A \), we denote by \( A(B) \) the manifold given by real blowing up of \( A \) along \( B \). Namely \( A(B) = (A \setminus B) \cup S\nu_B \) where \( \nu_B \) is the normal bundle of \( B \subset A \) and \( S\nu_B \) is the sphere bundle of \( \nu_B \) (see [KT99] for more details of real blow up). Note that if a submanifold \( C \subset A \) is transverse to \( B \), then \( C(A \cap B) \) is a proper embedded submanifold of \( A(B) \).

Let us denote by \( \Delta \subset A \times \cdots \times A \) the fat diagonal of the direct sum of a manifold \( A \). Let us denote by \( R^k \) the trivial vector bundle over an appropriate base space with rank \( k \in \mathbb{N} \). For a real vector space \( X \), we denote by \( SX \) or \( S(X) \) the unit sphere of \( X \) and for a real vector bundle \( E \to B \) over a manifold \( B \), we denote by \( SE \) or \( S(E) \) the unit sphere bundle of \( E \).

2.1 Notations about 3-manifolds and Morse functions.

Let \( f : Y \to \mathbb{R} \) be a Morse function on a 3-dimensional manifold \( Y \) with a metric satisfying the Morse-Smale condition. Let \( \text{grad} f \) be the gradient vector field of \( f \) and the metric of \( Y \). Let us denote by \( \text{Crit}(f) \) the set of all critical points of \( f \). Let \( \{ \Phi_t \}_{t \in \mathbb{R}} : Y \to Y \) be the 1-parameter group of diffeomorphisms associated to \( -\text{grad} f \). We denote by

\[
A_p = \{ x \in Y \mid \lim_{t \to \infty} \Phi_t(x) = p \}
\]

\[
D_p = \{ x \in Y \mid \lim_{t \to -\infty} \Phi_t(x) = p \}
\]

the ascending manifold and descending manifold at \( p \in \text{Crit}(f) \) respectively.

2.2 Conventions on orientations.

Boundaries are oriented by the outward normal first convention. Products are oriented by the order of the factors. Let \( y \in B \) be a regular point of a smooth map \( f : A \to B \) between smooth manifolds \( A \) and \( B \). Let us orient \( f^{-1}(y) \) by the following rules: \( T_x f^{-1}(y) \oplus f^* T_{f(x)} B = T_x A \), for any \( x \in f^{-1}(C) \) where \( f^* : T_{f(x)} B \to T_x A \) is a linear map satisfying \( f_\ast \circ f^* = \text{id}_{T_{f(x)} B} \). We denote by \( -X \) the orientation reversed manifold of oriented manifold \( X \).

Suppose that \( Y, f \) and \( \text{grad} f \) are as above. Let us orient ascending manifolds and descending manifolds by imposing the condition: \( T_{p,A} A_p \oplus T_{p,D} D_p \cong T_p Y \) for any \( p \in \text{Crit}(f) \). Let \( p, q \in \text{Crit}(f) \) be the critical points of index 2 and 1 respectively. By the Morse-Smale condition, \( D_q \cap A_p \) is a 1-manifold. Let us orient \( D_q \cap A_p \) by the following rule:

\[
T_{q'}(D_q \cap A_p) \oplus T_{q'} D_q \cong T_{q'} D_p,
\]

where \( q' \in D_q \cap A_p \) is a point near \( q \).
3 Configuration space and Jacobi diagrams.

In this section, we introduce some notations about configuration spaces and Jacobi diagrams. Most of this section depends on Lescop [Les04a].

3.1 The configuration space $C_{2n}(Y)$.

The reference here is Lescop [Les04a §1.1,1.2.1]. Let $Y$ be a homology 3-sphere with a base point $\infty$. Let $N(\infty; Y)$ be a regular neighborhood (that is diffeomorphic to an open ball) of $\infty$ in $Y$ and let $N(\infty; S^3)$ be a regular neighborhood of $\infty$ in $S^3 = \mathbb{R}^3 \cup \infty$. We fix a diffeomorphism $\tau^\infty : (N(\infty; Y), \infty) \cong (N(\infty; S^3), \infty)$ between $N(\infty; Y)$ and $N(\infty; S^3)$. We identify $N(\infty; Y)$ with $N(\infty; S^3)$ under $\tau^\infty$.

Let $\tilde{C}_{2n}(Y) = (Y \setminus \infty)^{2n} \setminus \Delta = \{(1, \cdots, 2n) \mapsto Y \setminus \infty\}$ and let $C_{2n}(Y)$ the compactification of $\tilde{C}_{2n}(Y)$ given by Lescop [Les04a §3]. (This compactification is similar to Fulton-MacPherson compactification [FM94].) Roughly speaking, $C_{2n}(Y)$ is obtained from $Y^{2n}$ by real blowing up along all diagonals and $\{(x_1, \cdots, x_{2n} | \exists i \text{ such that } x_i = \infty\}$. See §3 in [Les04a] for the complete definition.) Note that $C_2(Y)$ is given by real blowing up $Y^2$ along $(\infty, \infty), (\infty, \infty) \times (\infty, \infty), (\infty, \infty) \times \Delta$, and $\Delta$ in turn. Let us denote by $q : C_2(Y) \to (Y \setminus \infty)^2$ the composition of the blow down maps. Then $\partial C_2(Y) = ST_{\infty}Y \times (Y \setminus \infty) \cup (Y \setminus \infty) \times ST_{\infty}Y \cup \nu_{\Delta (Y \setminus \infty)} \cup q^{-1}(\infty^2)$. We identify $\nu_{\Delta (Y \setminus \infty)}$ with $STY |_{Y \setminus \infty}$ by the canonical isomorphism $\nu_{\Delta Y} \cong STY$. The involution $Y^2 \to Y^2, (x,y) \mapsto (y,x)$ induces an involution of $C_2(Y)$. We denote by $\iota : C_2(Y) \to C_2(Y)$ this involution.

Let $p_1 : (\partial C_2(Y) \supset ST_{\infty}Y \times (Y \setminus \infty) \to ST_{\infty}Y \cong ST_{\infty}S^3 \cong S^2$ and $p_2 : (\partial C_2(Y) \supset (Y \setminus \infty) \times ST_{\infty}Y \to ST_{\infty}Y = ST_{\infty}S^3 = S^2$ be the projections. We denote by $\iota : S^2 \to S^2$ the involution induced by $\times (-1) : \mathbb{R}^3 \to \mathbb{R}^3$.

Let $p_c : C_2(S^3) \to S^2$ be the extension of the map $\text{int}C_2(S^3) = (\mathbb{R}^3 \times \mathbb{R}^3) \setminus \Delta \to S^2$, $(x,y) \mapsto (y-x)/\|y-x\|$. Since it is possible to identify $q^{-1}(N(\infty; Y)^2) \subset \partial C_2(Y)$ with $q^{-1}(N(\infty; S^3)^2) \subset \partial C_2(S^3)$ by $\tau^\infty$, we get a map $(\partial C_2(Y) \supset q^{-1}(N(\infty; Y) \setminus \infty)^2) \cong S^2$. Since $p_1, \iota \circ p_2$ and $p_c$ are compatible on boundary, these maps define
the map
\[ p_Y : \partial C_2(Y) \setminus S\nu_{\Lambda(Y \setminus N(\infty; Y))} \to S^2. \]
(Here we note that \( \partial C_2(Y) \setminus S\nu_{\Lambda(Y \setminus N(\infty; Y))} = ST_\infty Y \times (Y \setminus \infty) \cup (Y \setminus \infty) \times ST_\infty Y \cup q^{-1}(N(\infty; Y)^2) \).)

### 3.2 More on the boundary \( \partial C_{2n}(Y) \).

The reference here is Lescop [Les04a, §2.2, §3]. For \( B \subset \{1, \ldots, 2n\} \), we set
\[ F(\infty; B) = q^{-1}(\{(x_1, \ldots, x_{2n}) \mid x_i = \infty \text{ iff } i \in B, \text{ if } i, j \notin B \text{ then } x_i \neq x_j\}), \]
and for \( B \subset \{1, \ldots, 2n\}(\#B \geq 2) \), we set
\[ F(B) = q^{-1}(\{(x_1, \ldots, x_{2n}) \in (Y \setminus \infty)^{2n} \mid \exists y, x_i = y \text{ iff } i \in B, \text{ if } i, j \notin B \text{ then } x_i \neq x_j\}). \]
Under these notations, \( \partial C_{2n}(Y) = \bigcup_B F(\infty; B) \cup \bigcup_{\#B \geq 2} F(B) \). We remark that \( \partial C_{2n}(Y) \) has smooth structure (See [Les04a, §3]).

Let \( X \) be a 3-dimensional real vector space. Let \( V \) be a finite set. we define \( \tilde{S}_V(X) \) to be the set of injective maps from \( V \) to \( X \) up to translations and dilations. Set \( k = \{1, \ldots, k\} \). Note that \( \tilde{S}_k(X) = S(X) \). For an \( \mathbb{R}^3 \) vector bundle \( E \to M \), we denote by \( \tilde{S}_V(E) \to M \) the fiber bundle where the fiber over \( x \in M \) is \( \tilde{S}_V(E_x) \). Under these notations, \( F(2n) = \tilde{S}_{2n}(T(Y \setminus \infty)) \).

We remark that \( F(B) \) has a fiber bundle structure where the typical fiber is \( \tilde{S}_B(\mathbb{R}^3) \).

Lescop gave a compactification \( S_V(X), S_V(E) \) of \( \tilde{S}_V(X), \tilde{S}_V(E) \) respectively in [Les04a]. Let \( f(B)(X) = \tilde{S}_B(X) \times \tilde{S}_{\{B\} \cup B}(X) \), for \( B \subset V \) with \( B \neq V \) and \( \#B \geq 2 \). Let \( f(B)(E) \to M \) be the fiber bundle where the fiber over \( x \in M \) is \( f(B)(E_x) \). Under this notation,
\[ \partial S_V(X) = \bigcup_{\#B \geq 2} f(B)(X), \partial S_V(E) = \bigcup_{\#B \geq 2} f(B)(E) \]
(See Proposition 2.8 in [Les04a]). We remark that \( f(B)(E) \) has a fiber bundle structure where the typical fiber is \( \tilde{S}_B(\mathbb{R}^3) \).

### 3.3 Jacobi diagrams.

The reference here is Lescop [Les04a, §1.3, 2.3]. A Jacobi diagram of degree \( n \) is defined to be a trivalent graph with \( 2n \) vertices and \( 3n \) edges without simple loops. For a Jacobi diagram \( \Gamma \), we denote by \( H(\Gamma), E(\Gamma) \) and \( V(\Gamma) \) the set of half edges, the set of edges and the set of vertices respectively. An orientation of a vertex of \( \Gamma \) is a cyclic order of the three half-edges that meet at the vertex. A Jacobi diagram is oriented if all its vertices are oriented. Let
\[ \mathcal{A}_n(\emptyset) = \{ \text{degree } n \text{ oriented Jacobi diagrams} \} \mathbb{R}/\text{AS, IHX}, \]
where the relations AS and IHX are locally represented by the following pictures. Let

\[ \mathcal{A}_n(\emptyset) = \{ \text{degree } n \text{ oriented Jacobi diagrams} \} \mathbb{R}/\text{AS, IHX}, \]
Here the orientation of each vertex is given by counterclockwise order of the half edges.

\[ \mathcal{E}_n = \{ \Gamma = (\overrightarrow{\mathcal{G}}, \varphi_E, \varphi_V, \text{ori}_E) \} \]

Here \( \overrightarrow{\mathcal{G}} \) is a Jacobi diagram of degree \( n \), \( \varphi_E : E(\overrightarrow{\mathcal{G}}) \cong \{ 1, 2, \ldots, 3n \} \) and \( \varphi_V : V(\overrightarrow{\mathcal{G}}) \cong \{ 1, 2, \ldots, 2n \} \) are labels of edges and vertices respectively, and \( \text{ori}_E \) is a collection of orientations of each edge. These data and an orientation of \( \overrightarrow{\mathcal{G}} \) induce two orientations of \( H(\overrightarrow{\mathcal{G}}) \). The first one is the edge-orientation induced by \( \varphi_E \) and \( \text{ori}_E \). The second one is the vertex-orientation induced by \( \varphi_V \) and orientation of \( \overrightarrow{\mathcal{G}} \). We choose the orientation of \( \overrightarrow{\mathcal{G}} \) so that the edge-orientation coincides with the vertex-orientation.

Let us denote by \([\Gamma] \in A_n(\emptyset)\) the oriented Jacobi diagram given by \( \Gamma \) in such a way.

Remark 3.1. The notation \( A_{2n,3n} \) used by Watanabe [Wat12] coincides with the notation \( A_n(\emptyset) \) used by Lescop [Les04a] as \( \mathbb{R} \)-vector spaces.

4 Construction of an invariant of rational homology 3-sphere via vector fields.

Let \( n \) be a natural number. In this section, we define an invariant \( \tilde{z}_n \) using vector fields. The idea of construction of \( \tilde{z}_n \) is based on Kuperberg, Thurston [KT99], Lescop [Les04a] and the construction of the anomaly part of Watanabe’s Morse homotopy invariant [Wat12]. Let \( Y \) be a rational homology 3-sphere with a base point \( \infty \). In Subsection 4.1, we will define the notion of admissible vector fields on \( T(Y \setminus \infty) \). In Subsections 4.2, 4.4, we will define \( \tilde{z}_n(Y; \vec{\gamma}) \) and \( \tilde{z}_n^{\text{anomaly}}(\vec{\gamma}) \) using a family of admissible vector fields \( \vec{\gamma} \). Thus we obtain a topological invariant \( \tilde{z}_n(Y) = \tilde{z}_n(Y; \vec{\gamma}) - \tilde{z}_n^{\text{anomaly}}(\vec{\gamma}) \) of \( Y \) in Subsection 4.5. We will prove well-definedness of \( \tilde{z}_n \) in Subsection 4.6.

4.1 Admissible vector fields on \( T(Y \setminus \infty) \).

For \( a \in S^2 \subset \mathbb{R}^3 \), the map \( q_a : \mathbb{R}^3 \to \mathbb{R} \) is defined by \( q_a(x) = \langle x, a \rangle \) where \( \langle , \rangle \) is the standard inner product on \( \mathbb{R}^3 \). Write \( \pm a = \{ a, -a \} \).

Definition 4.1. A vector field \( \gamma \in \Gamma T(Y \setminus \infty) \) is an admissible vector field (with respect to \( a \)) if the following conditions hold.

- \( \gamma|_{N(\infty; Y) \setminus \infty} = -\text{grad} \ q_a|_{N(\infty; S^3) \setminus \infty} \),
- \( \gamma \) is transverse to the zero section in \( T(Y \setminus \infty) \).

Example 4.2. We give two important examples of admissible vector fields with respect to \( a \).
(1) Let $\tau_{R^3} : T^*R^3 \cong R^3$ be the standard framing of $T^*R^3$. We regard $a \in R^3$ as a constant section of the trivial bundle $R^3$. For a framing $\tau : T(Y \setminus \infty) \cong R^3$ such that $\tau|_{N(\infty; Y) \setminus \infty} = \tau_{R^3}|_{N(\infty; S^3) \setminus \infty}$, the pull-back vector field $\tau^*a$ is an admissible vector field with respect to $-a$.

(2) For a Morse function $f : Y \setminus \infty \to R$ such that $f|_{N(\infty; Y) \setminus \infty} = q_0|_{N(\infty; S^3) \setminus \infty}$, $\nabla f$ is an admissible vector field with respect to $a$.

The following lemma plays an important role in the next subsection. For an admissible vector field $\gamma$, let

$$\tau_\gamma = \left\{ \frac{\gamma(x)}{\| \gamma(x) \|} \in ST_x Y \mid x \in Y \setminus (\infty \cup \gamma^{-1}(0)) \right\} \subset ST(Y \setminus \infty).$$

Here we choose the orientation of $\tau_\gamma$ such that the restriction of the projection $STY \to Y$ to $\tau_\gamma$ is orientation preserving.

**Lemma 4.3.**

$$c_0(\gamma) = \tau_\gamma \cup \tau_{-\gamma}$$

is a submanifold of $ST(Y \setminus \infty)$ without boundary.

To prove this lemma, we first remark the following lemma. Let $n, k \geq 0$ be integral numbers. Let $s : (R^{n+k}, 0) \to (R^n, 0)$ be a $C^\infty$ map which is transverse to the origin $0 \in R^n$.

**Lemma 4.4.** There is a diffeomorphism $\varphi : (R^{n+k}, 0) \to (R^{n+k}, 0)$ such that $s \circ \varphi$ coincides with $p_{R^n}$ as germs at $0 \in R^{n+k}$. Here $p_{R^n} : R^{n+k} = R^n \times R^k \to R^n$ is the orthogonal projection.

**Proof.** This is a consequence of the implicit function theorem. \qed

**Proof of Lemma 4.3.** It is sufficient to check this claim near $\gamma^{-1}(0)$. Let $x \in \gamma^{-1}(0)$. We fix a trivialization $\psi : T(Y \setminus \infty)|_{U_0} \cong U_0 \times R^3$ on a neighborhood $U_0$ of $x$ in $Y$. By the above Lemma 4.4, there is a neighborhood $U \subset U_0$ of $x$ and local coordinates $\varphi : R^3 \cong U$ (which is independent of $\psi$) such that $((\varphi^{-1} \times id) \circ \psi \circ \gamma \circ \varphi : R^3 \to R^3 \times R^3$ is represented by $(\varphi^{-1} \times id) \circ \psi \circ \gamma \circ \varphi(x) = (x, x)$). We fix these local trivialization and coordinates and we write $\gamma$ instead of $(\varphi^{-1} \times id) \circ \psi \circ \gamma \circ \varphi$.

We first show that $\partial \tau_\gamma \cap STU = - (\partial \tau_{-\gamma} \cap STU)$ as oriented manifolds. Let $D_+ = \tau_\gamma \cap STU$ and $D_- = \tau_{-\gamma} \cap STU$. Under the above local coordinates, $D_+ = \{ (tx, x/\| x \|) \mid x \in S^2, t \in [0, \infty) \} \subset (S^2 \times [0, \infty) / (S^2 \times 0)) \times S^2 = R^3 \times S^2$ and $D_- = \{ (tx, -x/\| x \|) \mid x \in S^2, t \in [0, \infty) \}$. Both projection $\pi : D_+ \to R^3$ and the projection $\pi : D_- \to R^3$ are orientation preserving (or reversing). Let $g : R^3 \times S^2 \to R^3 \times S^2$ be the bundle map defined by $(x, v) \mapsto (x, -v)$. So $g : \partial D_+ \cong \partial D_-$ is orientation preserving. On the other hand, $g|_{\{0\} \times S^2} : \{0\} \times S^2 \to \{0\} \times S^2$ is orientation reversing. Hence, the identity map $id : \{0\} \times S^2 \to \{0\} \times S^2$ is orientation reversing map as a map between $\partial D_+$ and $\partial D_-$. Therefore $\partial \tau_\gamma = \partial D_+ = - \partial D_- = - \partial \tau_{-\gamma}$.

We next prove that $c_0(\gamma) \cap STU$ is a submanifold of $STU \cong R^3 \times S^2$. Let $p_2 : R^3 \times S^2 \to S^2$ be the projection. For each $v \in S^2$, we have $(p_2|_{c_0(\gamma)})^{-1}(v) =$
Define \(γ\). By the definition of \(\tilde{γ}\), in this subsection, we define the principal term \(H\) in 4.2 The principal term. 

For any \(v_0 \in S^2\), let a diffeomorphism  \(\partial C \cap \{0\} \subset \mathbb{R}^3 \times S^2\). The set \(\mathbb{R}v \times \{v\} \subset \mathbb{R}^3 \times S^2\). In fact, for any \(v_0 \in S^2\) and for any sufficiently small neighborhood \(B_{v_0} \subset S^2\) of \(v_0\) we can take a diffeomorphism \(\Phi_{v_0} : (\mathbb{R}^3 \times B_{v_0}) \cup (\mathbb{R}v \times \{v\}) \rightarrow (\mathbb{R}^3 \times B_{v_0}, \mathbb{R}v \times \{v\})\) as follows. Here \(w_0 \in S^2 \subset \mathbb{R}^3\) is a point orthogonal to \(v_0\) in \(\mathbb{R}^3\) and \(\mathbb{R}w_0\) is the 1-dimensional vector subspace of \(\mathbb{R}^3\) spanned by \(w_0\). For each \(v \in B_{v_0}\), let \(m(v, w_0) \in S^2\) be the middle point of the geodesic segment from \(v\) to \(w_0\). Let \(\rho(v, w_0) \in SO(3)\) be the rotation with axis directed by \(m(v, w_0)\) and with angle \(\pi\). So \(\rho(v, w_0)\) exchanges \(v\) and \(w_0\). Then we can define \(\Phi_{v_0} : \mathbb{R}^3 \times B_{v_0} \rightarrow \mathbb{R}^3 \times B_{v_0}\) by \(\Phi_{v_0}(x, v) = (\rho(v, w_0)(x), v)\) for each \((x, v) \in \mathbb{R}^3 \times B_{v_0}\).

Therefore \(c_0(γ) \cap (\mathbb{R}^3 \times S^2) = \bigcup_{v \in S^2} \mathbb{R}v \times \{v\}\) is a submanifold of \(\mathbb{R}^3 \times S^2\). \(\square\)

4.2 The principal term \(\tilde{z}(Y; \tilde{γ})\).

In this subsection, we define the principal term \(\tilde{z}(Y; \tilde{γ})\) of the invariant \(\tilde{z}(Y)\). We define

\[c(γ) = p_r^{-1}(±a) \cup c_0(γ) \subset \partial C_2(Y)\]

By the definition of \(γ\) and Lemma 4.3, \(c(γ)\) is a closed 3-manifold. Therefore \([c(γ)] \in H_3(\partial C_2(Y); \mathbb{R})\).

Let \(ω^{a}_{S^2}\) be an anti-symmetric closed 2-form on \(S^2\) such that \(ω^{a}_{S^2}\) represents the Poincaré dual of \([±a]\) and the support of \(ω^{a}_{S^2}\) is concentrated in near \(±a\). Let \(ω_θ(γ)\) be a closed 2-form on \(\partial C_2(Y)\) satisfying the following conditions.

- \(2ω_θ(γ)\) represents the Poincaré dual of \([c(γ)]\),
- The support of \(ω_θ(γ)\) is concentrated in near \(c(γ)\),
- \(i^*ω_θ(γ) = -ω_θ(γ)\) and
- \(ω_θ(γ)|_{\partial C_2(Y) \setminus S_ω(\Delta(Y), N(∞; Y))} = \frac{1}{2} p_r^2 Ω_{S_ω^2}\).

Since \(Y\) is a rational homology 3-sphere, the restriction \(H^2(C_2(Y); \mathbb{R}) \rightarrow H^2(\partial C_2(Y); \mathbb{R})\) is an isomorphism. Thus there is a closed 2-form \(ω(γ)\) on \(C_2(Y)\) satisfying the following conditions.

- \(ω(γ)|_{\partial C_2(Y)} = ω_θ(γ)\) and
- \(i^*ω(γ) = -ω(γ)\).

**Definition 4.5** (propagator). We call \(ω(γ)\) a propagator with respect to \(γ\). Take \(a_1, \ldots, a_{3n} \in S^2\) (we may take, for example, \(a_1 = \ldots = a_{3n}\)). Let \(γ_i\) be an admissible vector field with respect to \(a_i\) and let \(ω(γ_i)\) be a propagator with respect to \(γ_i\) for each \(i \in \{1, \ldots, 3n\}\). To simplify notation, we write \(\tilde{γ}\) instead of \((γ_1, \ldots, γ_{3n})\).
For each $\Gamma = (\mathbf{T}, \varphi_E, \varphi_V, \ori_E) \in \mathcal{E}_n$ and for each $\varphi_E^{-1}(i) \in E(\mathbf{T})$, let $s(i), t(i) \in \{1, \cdots, 2n\}$ denote the labels of the initial vertex and the terminal vertex of $\varphi_E^{-1}(i)$ respectively. The embedding $\{1, 2\} \cong \{s(i), t(i)\} \hookrightarrow \{1, \cdots, 2n\}$ induces the projection $\pi_{C_{2n}(Y)} : \bar{C}_{2n}(Y) \to \bar{C}_2(Y)$. Furthermore it is possible to extend $\pi_{C_{2n}(Y)}$ to $C_{2n}(Y)$ by the definition of $C_{2n}(Y)$. We denote by $P_i(\Gamma) : C_{2n}(Y) \to C_2(Y)$ such the extended map (see [Les04a] §2.3 for more detail).

Definition 4.6.

$$\bar{z}_n(Y; \bar{\gamma}) = \sum_{\Gamma \in \mathcal{E}_n} \left( \int_{C_{2n}(Y)} \bigwedge_i P_i(\Gamma)^* \omega(\gamma_i) \right) \left[ \Gamma \right] \in \mathcal{A}_n(\emptyset).$$

Remark 4.7. By the above definition, the value $\bar{z}_n(Y; \bar{\gamma})$ often depends on the choices of $\omega(\gamma_i)$ even if we fix $\bar{\gamma}$. We will prove in Subsection 4.6 that $\bar{z}_n(Y; \bar{\gamma})$, however, depends only on the choice of $\bar{\gamma}$ for generic $\bar{\gamma}$.

4.3 Alternative description of $\bar{z}_n(Y; \bar{\gamma})$.

In this subsection, we give an alternative description of $\bar{z}_n(Y; \bar{\gamma})$ using cohomologies of simplicial complexes with coefficients in $\mathbb{R}$. This description will be needed in Section 7. The admissible vector field $\gamma_i$ with respect to $a_i$ and the 3-cycle $c(\gamma_i) \subset \partial C_2(Y)$ are as above. Let $T_{C_2(Y)}$ be the simplicial decomposition of $C_2(Y)$ given by pulling back a simplicial decomposition of $C_2(Y)/\iota$. So the simplicial decomposition $T_{C_2(Y)}$ is compatible with the action of $\iota$. By replacing such a simplicial decomposition if necessary, we may assume that each simplex of $T_{C_2(Y)}$ is transverse to $c(\gamma_i)$. Let $\omega^s_\delta(\gamma_i) \in S^2(\partial C_2(Y))$ be the 2-cocycle defined by $\omega^s_\delta(\gamma_i)(\sigma) = \frac{1}{2}(\sigma \cap c(\gamma_i))$ for each 2-chain $\sigma$ in $T_{C_2(Y)} \cap \partial C_2(Y)$. Thus $\omega^s_\delta(\gamma_i)$ is anti-symmetric under the involution $\iota$. Let $\omega^s(\gamma_i)$ be an extension of $\omega^s_\delta(\gamma_i)$ to $C_2(Y) = |T_{C_2(Y)}|$ satisfying the following conditions.

- $\omega^s(\gamma_i)|_{\partial C_2(Y)} = \omega^s_\delta(\gamma_i)$ and
- $\iota^* \omega^s(\gamma_i) = -\omega^s(\gamma_i)$.

We call it a simplicial propagator. Take an appropriate simplicial decomposition of $C_{2n}(Y)$. Then we have the 2-cocycle $P_i(\Gamma)^* \omega^s(\gamma_i) \in S^{2n}(C_{2n}(Y))$. By the construction, $\bigwedge_i P_i(\Gamma)^* \omega^s(\gamma_i)$ is a cocycle in $(C_{2n}(Y), \partial C_{2n}(Y))$. If necessary we replace the simplicial decompositions with a smaller one, we have the following lemma via the intersection theory.

Lemma 4.8 (Alternative description of $\bar{z}_n(Y; \gamma)$). If $(\bigcap_i P_i(\Gamma)^{-1}\text{support}(\omega^s(\gamma_i))) \cap \partial C_{2n}(Y) = \emptyset$ for any $\Gamma$,

$$\bar{z}_n(Y; \bar{\gamma}) = \sum_{\Gamma \in \mathcal{E}_n} \langle \bigwedge_i P_i(\Gamma)^* \omega^s(\gamma_i), [C_{2n}(Y), \partial C_{2n}(Y)] \rangle \left[ \Gamma \right] \in \mathcal{A}_n(\emptyset).$$

Here $[C_{2n}(Y), \partial C_{2n}(Y)]$ denotes the fundamental homology class and $\langle, \rangle$ denotes the Kronecker product.
4.4 The anomaly term $z_{\gamma}^{\text{anomaly}}$.

In this subsection, we define the anomaly term $z_{\gamma}^{\text{anomaly}}(Y; \gamma)$ of the invariant $z_{\gamma}(Y)$. The idea of the construction of this anomaly term is based on the construction of the anomaly term of Watanabe’s invariant $\tilde{z}_{\gamma}(Y)$. Let $Y$, $\gamma_1, \ldots, \gamma_{3n}$ be admissible vector fields with respect to $a_1, \ldots, a_{3n}$ respectively and $\omega(\gamma_1), \ldots, \omega(\gamma_{3n})$ be the same as above. Let $X$ be a connected oriented 4-manifold with $\partial X = Y$ and $\chi(X) = 0$. For example, we can take $X = (T^4 \# \mathbb{CP}^2) \setminus B_4$ when $Y = S^3$. For a framing $\tau'$ of $TY$ or $\mathbb{R} \oplus TY$, we denote by $\sigma_Y(\tau') \in \mathbb{Z}$ the signature defect of $\tau'$. Let $\tau_{S^3}$ be a framing of $T S^3$ satisfying the following two conditions:

- $\sigma_{S^3}(\tau_{S^3}) = 2$,
- $\tau_{S^3}|_{S^3 \setminus N'(\infty; S^3)} = \tau_{\mathbb{R}^3}|_{S^3 \setminus N'(\infty; S^3)}$.

Here $N'(\infty; S^3)$ is a neighborhood of $\infty$ smaller than $N(\infty; S^3)$, i.e., $\infty \in N'(\infty; S^3) \subset N(\infty; S^3)$.

**Remark 4.9.** There is no special meaning in the number “2” in the condition $\sigma_{S^3}(\tau_{S^3}) = 2$. The anomaly term $z_{\gamma}^{\text{anomaly}}(\gamma)$ is independent of the choice of $\tau_{S^3}$ even if $\sigma_{S^3}(\tau_{S^3})$ not be 2. We remark that there is no framing $\tau$ on $S^3$ such that $\sigma_{S^3}(\tau) = 0$.

Let $\eta_Y$ be the outward unit vector field of $TY = T(\partial X) \subset TX|_Y$ in $TX$. Since $\chi(X) = 0$, it is possible to extend $\eta_Y$ to a unit vector field of $TX$. We denote by $\eta_X \in \Gamma TX$ such an extended vector field. Let $T^\nu X$ be the normal bundle of $\eta_X$. We remark that $T^\nu X|_Y = TY$.

The vector field $\tau_{S^3}^* a_i$ of $TY|_{N(\infty; Y)}$ is the pull-back of $a_i \in S^2 \subset \mathbb{R}^3$ along $\tau_{S^3}|_{N(\infty; Y)}$. Since $\gamma_i|_{Y \setminus N(\infty; Y)} \in \Gamma TY|_{N(\infty; Y)}$ and $\tau_{S^3}^* a_i|_{N(\infty; Y)} \in \Gamma TY|_{N(\infty; Y)}$ are compatible, these vector fields define the vector field $\gamma'_i \in \Gamma TY$. Let $\beta_i \in \Gamma T^\nu X$ be a vector field of $T^\nu X$ transverse to the zero section in $T^\nu X$ satisfying $\beta_i|_Y = \gamma'_i$. By a similar argument of Lemma 4.3

$$c_0(\beta_i) = \left\{ \begin{array}{l}
\beta(x) \\
\frac{-\beta(x)}{\|\beta(x)\|} \end{array} \right\} \in S(T^\nu X)_{x} \left| x \in X \setminus \beta^{-1}(0) \right\}$$

is a submanifold of $ST^\nu X$ satisfying $\partial c_0(\beta_i) \subset STY$. Hence $c_0(\beta_i)$ is a cycle of $(ST^\nu X, \partial ST^\nu X)$. Here we choose the orientation of $c_0(\beta_i)$ such that the restriction of the projection $ST^\nu X \to X$ to $c_0(\beta_i)$ is orientation preserving.

We note that $c_0(\beta_i)$ satisfies $c_0(\beta_i) \cap S T \nu (Y \setminus N(\infty; Y)) = c_0(\gamma_i)$. Let $W(\gamma_i)$ be a closed 2-form on $ST^\nu X$ satisfying the following conditions.

- $2W(\gamma_i)$ represents the Poincaré dual of $[c_0(\beta_i), \partial c_0(\beta_i)]$,
- The support of $W(\gamma_i)$ is concentrated in near $c_0(\beta_i)$,
- $W(\gamma_i)|_{ST(Y \setminus N(\infty; Y))} = \omega(\gamma_i)|_{ST \nu (Y \setminus N(\infty; Y))}$ and

\[\text{There is such a framing. For example, the Lie framing } \tau_{SU(2)} \text{ of } S^3 = SU(2) \text{ satisfies } \sigma_{S^3}(\tau_{SU(2)}) = 2. \text{ See R. Kirby and P. Melvin [KM99] for more details. We can get } \tau_{S^3} \text{ by modifying } \tau_{SU(2)}. \]
Lemma 4.10. There exists $\mu_n \in \mathcal{A}_n(\emptyset)$ such that

$$-\mu_n \text{Sign}X + \sum_{\Gamma \in \mathcal{E}_n} \int_{S^2n(T^vX)} \wedge_i \phi_i(\Gamma)^*W(\gamma_i)[\Gamma] \in \mathcal{A}_n(\emptyset)$$

does not depend on the choice of $X$, $\beta_i$, and $W(\gamma_i)$.

Proof of Lemma 4.10 Let $X$ be a closed 4-manifold with $\text{Sign}X = 0$ and $\chi(X) = 0$. When $X$ is not connected, we assume that the Euler number of each component of $X$ is zero. Let $\eta_X$ be an unit vector field of $TX$ and let $T^vX$ be the normal bundle of $\eta_X$ in $TX$. Let $\beta_1, \ldots, \beta_{3n}$ be a family of sections of $T^vX$ that are transverse to the zero section in $T^vX$. Let $W_i$ be a closed 2-form that represents the Poincaré dual of $\tau_0(\beta_i)$ in $ST^vX$, for $i = 1, \ldots, 3n$. By a cobordism argument, it is sufficient to show that $\sum_{\Gamma \in \mathcal{E}_n} \int_{S^2n(T^vX)} \wedge_i \phi_i(\Gamma)^*W_i[\Gamma] = 0$.

We first prove that there exist an oriented compact 5-manifold $Z$ and there exist unit vector fields $\eta^1_Z, \eta^2_Z \in \Gamma^2Z$ such that:

- $\partial Z = X \sqcup X$,
- $\eta^1_Z, \eta^2_Z$ are linearly independent at any point in $Z$, i.e., $(\eta^1_Z, \eta^2_Z)$ is a 2-framing of $TZ$,
- $\eta^1_Z|_{\partial Z}$ is the outward unit vector field of $X = \partial Z$, and
- $\eta^2_Z|_{\partial Z} = \eta_X \sqcup \eta_X$.

Since $\text{Sign}X = 0$, there exists a connected compact oriented 5-manifold $Z_0$ such that $\partial Z_0 = X$. Let $\eta_{Z_0} \in \Gamma^2Z_0|_X$ be the outward unit vector field of $X = \partial Z_0$. By attaching 2-handles along the knots generating $H_1(Z_0; \mathbb{Z}/2)$ if necessary, we may assume that $H_1(Z_0; \mathbb{Z}/2) \cong H^4(Z_0; \partial Z_0; \mathbb{Z}/2) = 0$. Thus the primary obstruction $o_{Z_0}$ to extending the 2-framing $(\eta_{Z_0}, \eta_X)$ of $TZ_0|_X$ into $Z_0$ is in $H^5(Z_0, \partial Z_0; \pi_4(V_{5,2})) = H^5(Z_0, \partial Z_0; \mathbb{Z}/2)$. Let $Z = Z_0 \sqcup Z_0$. Then the obstruction to extending the 2-framing $(\eta_{Z_0} \sqcup \eta_{Z_0}, \eta_X \sqcup \eta_X)$ of $TZ|_{X \sqcup X}$ into $Z$ is $o_{Z_0} + o_{Z_0} = 0 \in H^5(Z, \partial Z; \mathbb{Z}/2)$. So we can take $\eta^1_Z, \eta^2_Z$ satisfying the above conditions.

Let $T^vZ$ be the normal bundle of $(\eta^1_Z, \eta^2_Z)$ in $TZ$. Then $T^vZ$ is a rank 3 sub-bundle of $TZ$ satisfying $T^vZ|_{X} = T^vX$. Let $\tilde{\beta}_i \in \Gamma T^vZ$ be a vector field transverse to the zero section in $T^vZ$ satisfying $T^vZ|_{X} = \beta_i$. Then $c_0(\tilde{\beta}_i) = \left\{ \frac{\tilde{\beta}_i(x)}{\|\tilde{\beta}_i(x)\|}, \frac{-\tilde{\beta}_i(x)}{\|\tilde{\beta}_i(x)\|} \mid x \in Z \setminus \tilde{\beta}_i^{-1}(0) \right\}'$ is a submanifold of $ST^vZ$ satisfying $\partial c_0(\tilde{\beta}_i) = c_0(\beta_i)$. Let $W(\tilde{\beta}_i)$ be a closed 2-form on $ST^vZ$ that represents...
the Poincaré dual of \([c_0(\bar{\beta}_i), \partial c_0(\bar{\beta}_i)]\) and satisfying \(W(\bar{\beta}_i)|_{ST^*X} = W_i\). By Stokes’ theorem, we have

\[
0 = \sum_{\Gamma \in \mathcal{E}_n} \int_{S_2n(T^*Z)} \frac{d}{\partial \Gamma} \left( \bigwedge_i \phi_i(\Gamma)^*W(\bar{\beta}_i) \right) \left[ \Gamma \right]
\]

\[
= 2 \sum_{\Gamma \in \mathcal{E}_n} \int_{S_2n(T^*X)} \bigwedge_i \phi_i(\Gamma)^*W_i[\Gamma] + \sum_{\Gamma \in \mathcal{E}_n} \int_{\partial S_2n(T^*Z)} \bigwedge_i \phi_i(\Gamma)^*W(\bar{\beta}_i)[\Gamma]
\]

\[
= 2 \sum_{\Gamma \in \mathcal{E}_n} \int_{S_2n(T^*X)} \bigwedge_i \phi_i(\Gamma)^*W_i[\Gamma] + \sum_{\Gamma \in \mathcal{E}_n} \sum_{2 \leq B < 2n} \int_{f(B)(T^*Z)} \bigwedge_i \phi_i(\Gamma)^*W(\bar{\beta}_i)[\Gamma]
\]

\[
= 2 \sum_{\Gamma \in \mathcal{E}_n} \int_{S_2n(T^*X)} \bigwedge_i \phi_i(\Gamma)^*W_i[\Gamma].
\]

The last equation is given by Lemma 4.20.

Let \(\tau_Y\) be a framing of \(T(Y \setminus \infty)\) satisfying \(\tau_Y|_{N(\infty;Y)\setminus \infty} = \tau_{\mathbb{R}^3}|_{N(\infty;\mathbb{R}^3)\setminus \infty}\). Then \(\tau^*_Y a = (\tau^*_Y a_1, \ldots, \tau^*_Y a_{3n})\) is a family of admissible vector fields. Let \(\tau^*_Y = \tau_Y|_{Y\setminus N(\infty;Y)\cup \tau^*_{\mathbb{R}^3}|_{N(\infty;\mathbb{R}^3)}\setminus \infty}\). So \(\tau^*_Y\) is a framing of \(TY\). Take \(W(\tau^*_Y a_i)|_{ST^*Y} = \frac{1}{2}(\tau^*_Y)^*\omega^i_{S^2}\). Let \(a_i\) be an alternative choice of \(a_1, \ldots, a_{3n}\). Let \(\tilde{\omega}^i_{S^2}\) be a closed 2-form on \(S^2 \times [0, 1]\) satisfying \(\tilde{\omega}^i_{S^2}|_{S^2 \times \{0\}} = \omega_{S^3}^i\) and \(\tilde{\omega}^i_{S^2}|_{S^2 \times \{1\}} = \omega_{S^2}^i\). Let \(ST^*X = ST^*X\) be the collar of \(ST^*Y\) such that \(ST^*Y \times \{0\} = \partial ST^*X\). We take \(W(\tau^*_Y a_i)|_{ST^*Y \times \{0\}} = \frac{1}{2}(\tau^*_Y)^*\tilde{\omega}^i_{S^2}\). Thus \(W(\tau^*_Y a_i)|_{ST^*Y \times \{1\}} = W(\tau^*_Y a_i)|_{ST^*Y \times \{1\}} = W(\tau^*_Y a_i)|_{ST^*Y \times \{1\}}\). Since Lemma 4.10 (1), we have

\[
\int_{S_2n(T^*X)} \bigwedge_i \phi_i(\Gamma)^*W(\tau^*_Y a_i) - \int_{S_2n(T^*X)} \bigwedge_i \phi_i(\Gamma)^*W(\tau^*_Y a_i)
\]

\[
= \int_{S_2n(T^*Y) \times [0, 1]} \bigwedge_i \phi_i(\Gamma)^*W(\tau^*_Y a_i)
\]

\[
= \frac{1}{2^{3n}} \int_{S_2n(T^*Y) \times [0, 1]} \bigwedge_i \phi_i(\Gamma)^*(\tau^*_Y \times \text{id})^*\tilde{\omega}^i_{S^2}
\]

The map \(S_2n(T^*Y) \times [0, 1] \xrightarrow{\Pi \phi_i(\Gamma)} (ST^*X \times [0, 1])^{3n} \xrightarrow{\tau^*_Y \times \text{id})^{3n}} \xrightarrow{\text{Im}(\Omega^{6n}(S_2n(\mathbb{R}^3) \times [0, 1]))} \text{dim}\left(\bigwedge_i \phi_i(\Gamma)\right)^*(\tau^*_Y \times \text{id})^*\tilde{\omega}^i_{S^2}\), we have \(\int_{S_2n(T^*Y) \times [0, 1]} \bigwedge_i \phi_i(\Gamma)^*(\tau^*_Y \times \text{id})^*\tilde{\omega}^i_{S^2} = 0\).

Because of the above two lemmas, \(\mu_n \text{Sign} X + \sum_{\Gamma \in \mathcal{E}_n} \int_{S_2n(T^*X)} \bigwedge_i \phi_i(\Gamma)^*W(\tau^*_Y a_i)[\Gamma]\) is independent of the choice of a 4-manifold \(X\) bounded by \(S^3\) and a family \(a_1, \ldots, a_{3n}\). We define

\[
c_n = -\mu_n \text{Sign} X + \sum_{\Gamma \in \mathcal{E}_n} \int_{S_2n(T^*X)} \bigwedge_i \phi_i(\Gamma)^*W(\tau^*_Y a_i)[\Gamma] \in \mathcal{A}_n(\emptyset).
\]
Definition 4.12.
\[ z_n^{\text{anomaly}}(\gamma) = -\mu_n \text{Sign} X + \sum \int_{\Gamma \in \mathcal{E}_n} \int_{S^2(T^\nu X)} \phi_i(\Gamma)^* W(\gamma_i)[\Gamma] - c_n \in \mathcal{A}_n(\emptyset). \]

Remark 4.13. We will show that \( \mu_n = \frac{3}{2}c_n \) in Lemma 4.7 and Lemma 7.8. We can show that \( \mu_1 = 72[\theta] \in \mathbb{Q}[\theta] = \mathcal{A}_1(\emptyset) \) by explicit computation (cf. the proof of Proposition A.1).

4.5 Definition of the invariant.

Theorem 4.14.
\[ \tilde{z}_n(Y) = \tilde{z}_n(Y; \gamma) - z_n^{\text{anomaly}}(\gamma) \in \mathcal{A}_n(\emptyset) \]
does not depend on the choice of \( \gamma \). Thus \( \tilde{z}_n(Y) \) is a topological invariant of \( Y \).

Definition 4.15.
\[ \tilde{z}_n(Y) = \tilde{z}_n(Y; \gamma) - z_n^{\text{anomaly}}(\gamma) \in \mathcal{A}_n(\emptyset). \]

4.6 Well-definedness of \( \tilde{z}_n(Y) \) (proof of Theorem 4.14).

In this section we give the sketch of the proof of well-definedness of \( \tilde{z}_n(Y) \), i.e., Theorem 4.14. The proof of well-definedness of \( \tilde{z}_n \) is almost parallel to that of \( \tilde{z}_n^{\text{KKT}} \) by Lescop [Les04a].

Fix \( i \in \{1, \ldots, 3n\} \). For any \( j \in \{1, \ldots, 3n\} \), let \( a_j', \gamma_j', \beta_j, \omega(\gamma_j') \) and \( W(\gamma_j') \) be alternative choices of \( a_j, \gamma_j, \beta_j, \omega(\gamma_j) \) and \( W(\gamma_j) \) respectively. Here \( a_j' = a_j, \gamma_j' = \gamma_j, \omega(\gamma_j') = \omega(\gamma_j), \beta_j' = \beta_j \) and \( W(\omega_j') = W(\omega_j) \) for \( j \neq i \). By the same argument of Proposition 2.15 in [Les04a], we have the following lemma.

Lemma 4.16. There exists a one-form \( \eta_{S^2} \in \Omega^1(S^2) \) such that \( d\eta_{S^2} = \omega_{S^2}^{a_j'} - \omega_{S^2}^{a_j}, \) and a one-form \( \eta \in \Omega^1(C_2(Y)) \) such that

- \( d\eta = \omega(\gamma_i') - \omega(\gamma_i), \)
- \( \eta|_{\partial C_2(Y)\setminus S^2_{\Delta(Y \setminus N(\infty; Y))}} = p_T^{s^*} \eta_{S^2}. \)

Similarly, the following lemma holds.

Lemma 4.17. There exists a one-form \( \eta_X \in \Omega^1(ST^\nu X) \) such that

- \( d\eta_X = W(\gamma_i') - W(\gamma_i), \)
- \( \eta_X|_{ST(Y \setminus N(\infty; Y))} = \eta|_{S^2_{\Delta(Y \setminus N(\infty; Y))}} \)
- \( \eta_X|_{ST^\nu X \setminus (\infty; Y)} = \tau_{S^2}^{s^*} \eta_{S^2}. \)

Proof. Set \( \eta_X^0 = \eta|_{ST(Y \setminus N(\infty; Y))} \cup \tau_{S^2}^* \eta_{S^2}. \) By the construction of \( c_0(\beta_i), c_0(\beta_i') \), we have \( [W(\gamma_i)] = [W(\gamma_i')] \in H^2(ST^\nu X) \) (cf. Lemma A.2). Thus there is a one-form \( \eta_X \in \Omega^1(ST^\nu Y) \) such that \( d\eta_X^1 = W(\gamma_i) - W(\gamma_i'). \) Since \( H^1(ST^\nu X) = 0 \), there is a function \( \mu_X \in \Omega^0(ST^\nu Y) \) such that \( d\mu_X = \eta_X^1|_{ST^\nu Y} - \eta_X^0. \) Let \( h : ST^\nu X \rightarrow \mathbb{R} \) be a \( C^\infty \) function such that \( h \equiv 1 \) near \( ST^\nu Y (= \partial ST^\nu X) \) and \( h \equiv 0 \) far from \( ST^\nu Y. \) We can take \( \eta_X = \eta_X^1 - d(h\mu_X) \) using collar of \( ST^\nu Y \) in \( ST^\nu X. \) \( \square \)
Set
\[ \tilde{\omega}_j = \begin{cases} \omega(\gamma_j)(= \omega'(\gamma_j)) & j \neq i, \\ \eta & j = i. \end{cases} \]

Set
\[ \tilde{W}_j = \begin{cases} W(\gamma_j)(= W'(\gamma_j)) & j \neq i, \\ \eta_X & j = i. \end{cases} \]

By Stokes’ theorem,
\[
\int_{C_{2n}(Y)} \bigwedge_j P_j(\Gamma)^* \omega(\gamma_j) - \int_{C_{2n}(Y)} \bigwedge_j P_j(\Gamma)^* \omega(\gamma'_j) = \int_{\partial C_{2n}(Y)} \bigwedge_j P_j(\Gamma)^* \tilde{\omega}_j.
\]

Lemma 4.18 (Lescop [Les04a, Lemma 2.17]). For any non-empty subset \( B \) of \( 2n = \{1, \ldots, 2n\} \), for any \( \Gamma \in \mathcal{E}_n \),
\[
\int_{F(\infty; B)} \bigwedge_j P_j(\Gamma)^* \tilde{\omega}_j = 0.
\]

Lemma 4.19 (Lescop [Les04a, Lemma 2.18, 2.19, 2.20, and 2.21]). For any \( B \subset \{1, \ldots, 2n\} \) with \( \#B \geq 2 \) and \( B \neq \{1, \ldots, 2n\} \)
\[
\sum_{\Gamma \in \mathcal{E}_n} \left( \int_{F(B)} \bigwedge_j P_j(\Gamma)^* \tilde{\omega}_j \right)[\Gamma] = 0.
\]

The proofs of these two lemmas are completely same as the proof in [Les04a]. The following lemma is proved as Lemma 2.18, 2.19, 2.20, and 2.21 in [Les04a] (See also the proof of Proposition 2.10 in [Les04a]).

Lemma 4.20. For any \( B \subset \{1, \ldots, 2n\} \) with \( 2 \leq \#B < 2n \),

1. \[ \sum_{\Gamma \in \mathcal{E}_n} \int_{f(B)(\mathbf{T}^X)} \bigwedge_j \phi_j(\Gamma)^* \tilde{W}_j[\Gamma] = 0, \]

2. \[ \sum_{\Gamma \in \mathcal{E}_n} \int_{f(B)(\mathbf{T}^Z)} \bigwedge_j \phi_j(\Gamma)^* W(\tilde{\beta}_j)[\Gamma] = 0 \] (See the proof of Lemma 4.10 for the notation \( Z, W(\tilde{\beta}_j) \)).

By Lemma 4.18 and Lemma 4.19,
\[
\tilde{z}_n(Y; \tilde{\gamma}) - \tilde{z}_n(Y; \tilde{\gamma}') = \sum_{\Gamma \in \mathcal{E}_n} \left( \int_{C_{2n}(Y)} \bigwedge_j P_j(\Gamma)^* \omega(\gamma_j) \right)[\Gamma] - \sum_{\Gamma \in \mathcal{E}_n} \left( \int_{C_{2n}(Y)} \bigwedge_j P_j(\Gamma)^* \omega(\gamma'_j) \right)[\Gamma]
\]
\[ = \sum_{\Gamma \in \mathcal{E}_n} \left( \int_{F(2n)} \bigwedge_j P_j(\Gamma)^* \tilde{\omega}_j \right)[\Gamma]. \]
Since $F(2n) = \bar{S}(T(Y \setminus \infty))$, the restriction of $P_j(\Gamma)$ to $F(2n)$ coincides with $\phi_j^0(\Gamma) : \bar{S}_{2n}(T(Y \setminus \infty)) \to SU_\Delta(Y(\infty)) \subset \partial C_2(Y)$. Therefore

$$\sum_{\Gamma \in \mathcal{E}_n} \int_{F(2n)} \bigwedge_j P_j(\Gamma)^* \bar{\omega}_j[\Gamma]$$

$$= \sum_{\Gamma \in \mathcal{E}_n} \int_{\bar{S}_{2n}(T(Y \setminus \infty))} \bigwedge_j \phi_j^0(\Gamma)^* \bar{\omega}_j[\Gamma]$$

$$= \sum_{\Gamma \in \mathcal{E}_n} \int_{\bar{S}_{2n}(T(Y \setminus N(\infty;Y)))} \bigwedge_j \phi_j^0(\Gamma)^* \bar{\omega}_j[\Gamma] + \sum_{\Gamma \in \mathcal{E}_n} \int_{\bar{S}_{2n}(T(N(\infty;Y) \setminus \infty))} \bigwedge_j \phi_j^0(\Gamma)^* \bar{\omega}_j[\Gamma]$$

$$= \sum_{\Gamma \in \mathcal{E}_n} \int_{\bar{S}_{2n}(T(Y \setminus N(\infty;Y)))} \bigwedge_j \phi_j^0(\Gamma)^* \bar{\omega}_j[\Gamma].$$

The last equation comes from the following lemma.

**Lemma 4.21.** $\sum_{\Gamma \in \mathcal{E}_n} \int_{\bar{S}_{2n}(T(N(\infty;Y) \setminus \infty))} \bigwedge_j \phi_j^0(\Gamma)^* \bar{\omega}_j[\Gamma] = 0.$

**Proof.** Since $\bar{S}_{2n}(T(N(\infty;Y) \setminus \infty)) = (N(\infty;Y) \setminus \infty) \times \bar{S}_{2n}(\mathbb{R}^3)$ and $\bar{\omega}_j|_{ST(N(\infty;Y) \setminus \infty)} = \tau_{S^1} \omega_{S^2}$ (or $\tau_{S^3} \eta_{S^2}$), the form $\bigwedge_j \phi_j^0(\Gamma)^* \bar{\omega}_j[\Gamma_{2n}(T(N(\infty;Y) \setminus \infty))$ is in the image of the map $(\tau_{S^3})^{3n} \circ \prod_j \phi_j^0(\Gamma)$. The map $(\tau_{S^3})^{3n} \circ \prod_j \phi_j^0(\Gamma)|_{\bar{S}_{2n}(T(N(\infty;Y) \setminus \infty))} : \bar{S}_{2n}(T(N(\infty;Y) \setminus \infty)) \to (ST(N(\infty;Y) \setminus \infty))^{3n} \to (S^2)^{3n}$ factors through $\bar{S}_{2n}(\mathbb{R}^3)$. Since dim $\bar{S}_{2n}(\mathbb{R}^3) = 6n - 4 < 6n - 1 = \dim \bigwedge_j \phi_j^0(\Gamma)^* \bar{\omega}_j$, we have $\sum_{\Gamma \in \mathcal{E}_n} \int_{\bar{S}_{2n}(T^*Y|_{N(\infty;Y)})} \bigwedge_j \phi_j^0(\Gamma)^* \bar{\omega}_j[\Gamma] = 0.$

On the other hand, by Stokes’ theorem,

$$\sum_{\Gamma \in \mathcal{E}_n} \int_{\bar{S}_{2n}(T^*Y)} \bigwedge_j \phi_j(\Gamma)^* \bar{W}_j[\Gamma] + \sum_{\Gamma \in \mathcal{E}_n} \int_{\partial \bar{S}_{2n}(T^*X)} \bigwedge_j \phi_j(\Gamma)^* \bar{W}_j[\Gamma]$$

$$\Rightarrow \sum_{\Gamma \in \mathcal{E}_n} \int_{\bar{S}_{2n}(T^*Y)} \bigwedge_j \phi_j(\Gamma)^* \bar{W}_j[\Gamma]$$

$$= \sum_{\Gamma \in \mathcal{E}_n} \int_{\bar{S}_{2n}(T(Y \setminus N(\infty;Y)))} \bigwedge_j \phi_j(\Gamma)^* \bar{W}_j[\Gamma] + \sum_{\Gamma \in \mathcal{E}_n} \int_{\bar{S}_{2n}(T^*Y|_{N(\infty;Y)})} \bigwedge_j \phi_j(\Gamma)^* \bar{W}_j[\Gamma]$$

$$= \sum_{\Gamma \in \mathcal{E}_n} \int_{\bar{S}_{2n}(T(Y \setminus N(\infty;Y)))} \bigwedge_j \phi_j(\Gamma)^* \bar{W}_j[\Gamma].$$

The equation (*) is given by Lemma 4.20(1) and the last equation comes from the following lemma.

**Lemma 4.22.** $\sum_{\Gamma \in \mathcal{E}_n} \int_{\bar{S}_{2n}(T^*Y|_{N(\infty;Y)})} \bigwedge_j \phi_j(\Gamma)^* \bar{W}(\gamma_j')[\Gamma] = 0.$

The proof of this lemma is parallel to the proof of Lemma 4.21.
Since \( \tilde{W}_j|_{\nu_{\Delta(N(\infty;Y))}} = \tilde{w}_j|_{\nu_{\Delta(N(\infty;Y))}} \) for any \( j \), we have
\[
\tilde{z}(Y; \gamma) - \tilde{z}(Y; \gamma') = \sum_{\Gamma \in E_n} \int_{\tilde{S}_2n(T(Y\setminus N(\infty;Y)))} \bigwedge_j \phi_j(\Gamma)^* \tilde{w}_j[\Gamma] \\
= \sum_{\Gamma \in E_n} \int_{S_2n(T(Y\setminus N(\infty;Y)))} \bigwedge_j \phi_j(\Gamma)^* \tilde{W}_j[\Gamma] = \tilde{z}^{\text{anomaly}}(\gamma) - \tilde{z}^{\text{anomaly}}(\gamma').
\]
Now we finish the proof of Theorem 4.14.

5 Review of \( z_{n,\text{KKT}} \).

In this section, we review the construction of \( z_{n,\text{KKT}} \) for rational homology 3-spheres. This section is based on Lescop [Les04a].

Let \( \tau_Y : T(Y \setminus \infty) \cong \mathbb{R}^3 \) be a framing satisfying \( \tau_Y|_{N(\infty;Y)\setminus \infty} = \tau_{\mathbb{R}^3} \). \( \tau_Y|_{N(\infty;Y) \cup \tau_{3}(N(\infty;S^3))} \) is a framing of \( TY \) by the assumption of \( \tau_Y \). We define
\[
\sigma_{Y\setminus \infty}(\tau_Y) = \sigma_Y(\tau_Y|_{N(\infty;Y) \cup \tau_{3}(N(\infty;S^3)))} - \sigma_{3}(\tau_{3})
\]
and call it the signature defect of \( \tau_Y \) of a framing of \( Y\setminus \infty \). For example \( \sigma_{\mathbb{R}^3}(\tau_{\mathbb{R}^3}) = 0 \).

The canonical isomorphism \( \nu_{\Delta(Y\setminus \infty)} \cong T(Y\setminus \infty) \) and the framing \( \tau_Y \) induce the map \( p_{\Delta}(\tau_Y) : \nu_{\Delta(Y\setminus \infty)} \rightarrow S^2 \). Since the assumption of \( \tau_Y \), maps \( p_{\Delta}(\tau_Y) \) and \( p_Y : \partial C_2(Y) \setminus \nu \rightarrow S^2 \) are compatible. So we get the map \( p(\tau_Y) = p_Y \cup p_{\Delta}(\tau_Y) : \partial C_2(Y) \rightarrow S^2 \). Let \( \omega_{S^2} \in \Omega^2(S^2) \) be an anti-symmetric 2-form satisfying \( \int_{S^2} \omega_{S^2} = 1 \). Let \( \omega(\tau_Y) \) be an anti-symmetric closed 2-form on \( C_2(Y) \) satisfying \( \omega(\tau_Y)|_{\partial C_2(Y)} = p(\tau_Y)^*\omega_{S^2} \in \Omega^2(\partial C_2(Y)) \).

**Proposition 5.1** (Lescop [Les04a, Theorem 1.9 and Proposition 2.11]). There exists constants \( \delta_n \in A_n(\emptyset) \) such that
\[
\sum_{\Gamma \in E_n} \int_{C_2n(Y)} \left( \bigwedge_i P_i(\Gamma)^* \omega(\tau_Y) \right) [\Gamma] - \frac{\sigma_{Y\setminus \infty}(\tau_Y)}{4} \delta_n \in A_n(\emptyset)
\]
does not depend on the choice of \( \tau_Y \).

**Definition 5.2** (Kuperberg and Thurston [KT99], Lescop [Les04a]).
\[
z_{n,\text{KKT}}(Y; \tau_Y) = \sum_{\Gamma \in E_n} \int_{C_2n(Y)} \left( \bigwedge_i P_i(\Gamma)^* \omega(\tau_Y) \right) [\Gamma],
\]
\[
z_{n,\text{KKT}}(Y) = z_{n,\text{KKT}}(Y; \tau_Y) - \frac{\sigma_{Y\setminus \infty}(\tau_Y)}{4} \delta_n \in A_n(\emptyset).
\]

We remark that \( \delta_n \) is given by explicit formula in Proposition 2.10 in [Les04a].

**Remark 5.3.** The universal finite type invariant \( z_{n,\text{KKT}} \) described in [Les04a] equals to the degree \( n \) part of \( \exp(\sum_n \frac{1}{2n!}(3n)! z_{n,\text{KKT}}) \). See before Lemma 2.12 in [Les04a] for more detail.

**Remark 5.4.** We will show that \( \delta_n = \frac{1}{2} \mu_n \) in Lemma 7.7.
6 Review of Watanabe’s Morse homotopy invariants $z_{n}^{FW}$.

In this section we give a modified construction of Watanabe’s Morse homotopy invariant $z_{2n,3n}^{FW}$ for rational homology 3-spheres. We will remark the differences between our modified construction and Watanabe’s original construction after the definition of $z_{2n,3n}^{FW}(Y)$. The invariant $z_{2n,3n}^{FW}(Y)$ is a sum of the principal term $\tilde{z}_{2n,3n}(Y; \vec{f})$ and the anomaly term $\tilde{z}_{2n,3n}^{anomaly}(\vec{f})$ of $\vec{f}$ where $\vec{f} = (f_1, f_2, \cdots, f_{3n})$ is a family of Morse functions on $Y \setminus \infty$.

Fix a point $a \in S^2$.

Definition 6.1. A Morse function $f : Y \setminus \infty \to \mathbb{R}$ is an admissible Morse function with respect to $a$ if it satisfies the following conditions.

- $f|_{N(\infty; Y)\setminus \infty} = q_a|_{N(\infty; S^3)\setminus \infty}$ and
- $f$ has no critical point of index 0 or 3.

Let $\text{Crit}(f) = \{p_1, \cdots, p_k, q_1, \cdots, q_k\}$ be the set of critical points of $f$ where $\text{ind}(p_i) = 2, \text{ind}(q_i) = 1$. Let

$$0 \to C_2(Y \setminus \infty; f) \xrightarrow{\partial} C_1(Y \setminus \infty; f) \to 0$$

be the Morse complex of $f$ with rational coefficients. Let $g : C_1(Y \setminus \infty; f) \to C_2(Y \setminus \infty; f), g([q_i]) = \sum_j g_{ij}[p_j]$ be the inverse map of the boundary map $\partial : C_2(Y \setminus \infty; f) \to C_1(Y \setminus \infty; f), \partial[p_i] = \sum_j \partial_{ij}[q_j]$. ($g$ is called a combinatorial propagator in [Wat12].)

We now construct $\mathcal{M}(f)$ which is the weighted sum of (non-compact) 4-manifold in $Y^2 \setminus \Delta$. Let $M_\pm = \text{pr}(\varphi^{-1}(\Delta))$ where $\varphi : Y \times Y \times (0, \infty) \to Y \times Y$ is the map defined by $(x, y) \mapsto (y, \Phi_f(x))$ and $\text{pr} : Y \times Y \times (0, \infty) \to Y \times Y$ is the projection. We choose the orientation of $M_- (f)$ such that the inclusion $Y \times (0, \varepsilon) \hookrightarrow M_- (f), (x, t) \mapsto (x, \Phi_f(x))$ preserves orientations. We define

$$\mathcal{M}(f) = M_- (f) - \sum_{i,j} g_{ij} (A_{q_i} \times D_{p_j}) \setminus \Delta.$$

We remark that the orientation of $\mathcal{M}(f)$ does not depend on the choice of orientations of $A_{q_i}, D_{p_j}$.

Let $a_1, \cdots, a_{3n} \in S^2 \subset \mathbb{R}^3$ be the points such that any different three points of them are linearly independent in $\mathbb{R}^3$. Let $f_i : Y \setminus \infty \to \mathbb{R}$ be a sufficiently generic admissible Morse function with respect to $a_i$ for each $i = 1, \cdots, 3n$. We write $\vec{f} = (f_1, \cdots, f_{3n})$ to simplify notation. We replace a metric of $Y$ such that the Morse-Smale condition holds for each $f_i$ if necessary.

Set $\mathcal{M}(\pm f_i) = \mathcal{M}(f_i) + \mathcal{M}(-f_i)$.

Definition 6.2. For generic $\vec{f}$,

$$z_{2n,3n}^{FW}(Y; \vec{f}) = \sum_{\Gamma \in \mathcal{E}_n} \frac{1}{2^{3n}} \left( \prod_{i=1}^{3n} P_i(\Gamma)|_{Y^2 \setminus \Delta} \mathcal{M}(\pm f_i) \right) [\Gamma] \in \mathcal{A}_n(0).$$
We next define the anomaly part. Set $\text{grad } \vec{f} = (\text{grad } f_1, \cdots, \text{grad } f_{3n})$.

**Definition 6.3.**

$$z_{2n, 3n}^{\text{anomaly}}(\vec{f}) = z_n^{\text{anomaly}}(\text{grad } \vec{f}).$$

**Definition 6.4** (Watanabe [Wat12]).

$$z_{2n, 3n}^{FW}(Y) = z_{2n, 3n}^{FW}(Y; \vec{f}) - z_{2n, 3n}^{\text{anomaly}}(\vec{f}).$$

**Remark 6.5.** A difference between our modified construction of $z_{2n, 3n}^{FW}$ and Watanabe’s original construction in [Wat12] is the conditions for Morse functions. Our Morse function is on $Y \setminus \infty$ and explicitly written on $N(\infty; Y) \setminus \infty$. On the other hand, Watanabe uses any Morse functions on $Y$. We note that $Y \setminus \infty \subset Y \sharp S^3$ where $Y \sharp S^3$ is the connected sum of $Y$ and $S^3$ at $\infty \in Y$ and $0 \in S^3$. Then it is possible to extend $f : Y \setminus \infty \to \mathbb{R}$ to $Y \sharp S^3 \cong Y$ in standard way. Then we can show that the difference between the value $z_{2n, 3n}^{FW}(Y)$ described in this Section and the value of Watanabe’s original invariant of $Y$ is a constant which is independent of $Y$.

We must prove that $\sharp \left( \bigcap_i P_i(\Gamma)^{-1}_{(Y \setminus \infty)^{2n} \setminus \Delta}(\mathcal{M}(\pm f_i)) \right)$ is well defined for generic $\vec{f}$, because Morse functions used in the above definition differ from Morse functions used in the original definition in [Wat12] near $N(\infty; Y) \setminus \infty$ (See Remark 6.5 for more details).

**Lemma 6.6.** $P_1(\Gamma)^{-1}_{(Y \setminus \infty)^{2n} \setminus \Delta}(\mathcal{M}(\pm f_i))$, $\cdots$, $P_{3n}(\Gamma)^{-1}_{(Y \setminus \infty)^{2n} \setminus \Delta}(\mathcal{M}(\pm f_i))$ transversally intersect at finitely many points, for generic $f_1, \cdots, f_{3n}$ and $a_1, \cdots, a_{3n}$, for any $\Gamma \in \mathcal{E}_n$.

**Proof.** Let $x = (x_1, \cdots, x_{2n}) \in \bigcap_i P_i(\Gamma)^{-1}_{(Y \setminus \infty)^{2n} \setminus \Delta}(\mathcal{M}(\pm f_i)) \subset (Y \setminus \infty)^{2n} \setminus \Delta$.

The case of $x \in (Y \setminus N(\infty; Y))^{2n}$.

Thanks to §2.4 of [Wat12], the transversality at $x$ is given by generic $\vec{f}$.

The case of $x \notin (Y \setminus N(\infty; Y))^{2n}$.

We show that for generic $a_1, \cdots, a_{3n}$, there are no such $x$. (Then, in particular,
\[ \bigcap_i P_i(\Gamma)_{|Y \setminus \infty|^{2n+1}}^1(M(\pm f_i)) \text{ is a 0-dimensional compact manifold}. \] Let \( B = \{ i \in \{1, \cdots, 2n\} \mid x_i \in Y \setminus N(\infty; Y) \} \). Let
\[ E_B = \{ i \in \{1, \cdots, 3n\} \cong E(\Gamma) \mid \{ s(i), t(i) \} \subset B \}, \]
\[ E_B^0 = \{ i \in \{1, \cdots, 3n\} \cong E(\Gamma) \mid \{ s(i), t(i) \} \cap B \neq \emptyset \} \setminus E_B. \]

Let \( \Gamma/B \) be the labelled graph obtained from \( \Gamma \) by collapsing \( B \) to a point \( b_0 \) and removing all edges in \( E_B \). Here the label of edges and vertices of \( \Gamma/B \) are \( \{1, \cdots, 3n\} \setminus E_B, \{0, 1, \cdots, 2n\} \setminus B \) respectively (the label of \( b_0 \) is 0). Note that \( \sharp(V(\Gamma/B) - \{b_0\}) = 2n - 4B \) and \( \sharp E(\Gamma/B) \geq 3n - \frac{36B}{2} \).

Let \( \pi : Y \setminus \infty \to Y/(Y \setminus N(\infty; Y)) \cong \mathbb{R}^3 \) be the map obtained by collapsing \( Y \setminus N(\infty; Y) \) to the point \( 0 \in \mathbb{R}^3 \). Let \( \pi'_t : \mathbb{R} \to \mathbb{R} \) be the map obtained by collapsing \( \text{Im}(f_i) : Y \setminus N(\infty; Y) \to \mathbb{R} \) to 0. Then \( \pi'_t \circ f_i = q_{a_i} \circ \pi : Y \setminus \infty \to \mathbb{R} \). Let \( x' : V(\Gamma/B) - \{b_0\} \hookrightarrow \mathbb{R}^3 \) be the restriction of \( \pi \circ x : V(\Gamma) \hookrightarrow \mathbb{R}^3 \) to \( V(\Gamma/B) - \{b_0\} \subset V(\Gamma) \). Let \( a' \in (S^2)^{E(\Gamma/B)} \) be the points obtained from \( a = (a_1, \cdots, a_{3n}) \) removing all \( a_i, i \in E_B \). We define the map
\[ \varphi : (\mathbb{R}^3)^{V(\Gamma/B) - \{b_0\}} \setminus \Delta \to (S^2)^{E(\Gamma/B)} \]
as
\[ \varphi(y) = \left( \frac{y_{s(i)} - y_{t(i)}}{\|y_{s(i)} - y_{t(i)}\|} \right)_{i \in E(\Gamma/B)}. \]

Here if \( i \in E_B^0 \) then either \( s(i) \) or \( t(i) \) is 0. Then \( x' \in \varphi^{-1}(a') \). By the following lemma, there is no \( x' \) for a generic \( a' \). Therefore there is no \( x \) for a generic \( a \).

**Lemma 6.7.** For a generic \( a' \) we have \( \varphi^{-1}(a') = \emptyset \).

**Proof.** For any \( y \in \varphi^{-1}(a') \) and for any \( t \in (0, \infty) \), we have \( ty \in \varphi^{-1}(a') \). Thus if \( \varphi^t(a') \neq \emptyset \), we have \( \dim \varphi^{-1}(a') \geq 1 \). On the other hand, \( \dim((\mathbb{R}^3)^{V(\Gamma/B) - \{b_0\}}) = 6n - 3\sharp B \leq 2\sharp E(\Gamma/B) = \dim((S^2)^{E(\Gamma/B)}) \). Hence we have \( \dim \varphi^{-1}(a') \leq 0 \) for a generic \( a' \). This is contradiction. \( \square \)

7. Proof of Theorem 1.1

In this section we prove Theorem 1.1 in Section 1

**7.1 Proof of** \( \tilde{z}_n(Y) = z_n^{\text{KKT}}(Y) \).

We follow the notations used in Section 5. For example, \( Y \) is a rational homology 3-sphere and \( \infty \in Y \) is a base point, and so on. Let \( \tau_Y : T(Y \setminus \infty) \cong \mathbb{R}^3 \) be a framing of \( Y \setminus \infty \) satisfying \( \tau_{N(\infty; Y) \setminus \infty} = \tau_{N(\infty; S^3) \setminus \infty} \). We denote \( \tau^a = (\tau^a_Y a, \cdots, \tau^a_Y a) \) for \( a \in S^2 \). We take \( \omega_{S^2} = \frac{1}{2} \omega^a_{S^2} \) in the definition of \( z_n^{\text{KKT}}(Y; \tau_Y) \), and we take \( \omega(\tau^a_Y a) = \omega(\tau_Y a) \) in the definition of \( \tilde{z}_n(Y; \tau^a_Y a) \). Thus
\[ \tilde{z}_n(Y; \tau^a_Y a) = \sum_{\Gamma \in \mathcal{E}_n} \int_{C_{2n}(Y)} P_i(\Gamma) \omega(\tau_Y | \Gamma|) = z_n^{\text{KKT}}(Y; \tau_Y). \]
Then we only need show that
\[ \tilde{z}_n^{\text{anomaly}}(Y; \tau_Y^*a) = \frac{1}{4}\sigma_{Y\setminus\infty}(\tau_Y)\delta_n \]
in this condition.

The idea of the proof of \( \tilde{z}_n^{\text{anomaly}}(Y; \tau_Y^*a) = \frac{1}{4}\sigma_{Y\setminus\infty}(\tau_Y)\delta_n \) is as follows. We first prove this equation in the case of \( Y = S^3 \). The well-definedness of \( \tilde{z}_n^{\text{anomaly}}(Y) \) implies that \( \tilde{z}_n^{\text{anomaly}}(S^3; \tau^*a) = \frac{1}{4}\sigma_{S^3}(\tau)\delta_n \) for any framing \( \tau \) of \( S^3 \setminus \infty \). The general case is reduced to the case of \( Y = S^3 \) by a cobordism argument.

We introduce notation. For a compact 4-manifold \( X \) such that \( \partial X = Y \) and \( \chi(X) = 0 \), we denote \( \tilde{z}_n^{\text{anomaly}}(\tilde{\gamma}; X) = \sum_{n \in \mathbb{E}_n} \int_{S_2n(T^{-}X)} \bigwedge_i \phi_i(\Gamma^*)W(\gamma_i)[\Gamma] = \tilde{z}_n^{\text{anomaly}}(\tilde{\gamma}) + \mu_n \text{Sign}X + c_n \). Then \( \tilde{z}_n^{\text{anomaly}}(\tilde{\gamma}; X) = \tilde{z}_n^{\text{anomaly}}(\tilde{\gamma}; X) + \mu_n \text{Sign}X - c_n \) by the definition.

**Lemma 7.1.** \( \tilde{z}_n(S^3) = z^{\text{KKT}}(S^3) \).

*Proof.* Let \( X \) be a compact 4-manifold with \( \partial X = S^3 \) and \( \chi(X) = 0 \).

\[ \tilde{z}_n(S^3) = \tilde{z}_n(S^3; \tau_{R^3}^*a) - \tilde{z}_n^{\text{anomaly}}(\tau_{R^3}^*a; X) + \mu_n \text{Sign}X + c_n \]
\[ = \tilde{z}_n(S^3; \tau_{R^3}^*a) = z^{\text{KKT}}(S^3; \tau_{R^3}) = z^{\text{KKT}}(S^3) \]

Therefore \( \tilde{z}_n(S^3) = z^{\text{KKT}}(S^3; \tau_{R^3}) = z^{\text{KKT}}(S^3) \). \( \square \)

Since \( \tilde{z}_n^{\text{anomaly}}(S^3) \) is independent of the choice of framing on \( \mathbb{R}^3 = S^3 \setminus \infty \), we have the following corollary.

**Corollary 7.2.** For any framing \( \tau \) on \( \mathbb{R}^3 = S^3 \setminus \infty \) such that \( \tau|_{N(\infty;S^3)\setminus\infty} = \tau_{R^3}|_{N(\infty;S^3)\setminus\infty} \), the equation \( \tilde{z}_n^{\text{anomaly}}(S^3; \tau_{R^3}^*a) = \frac{1}{4}\sigma_{R^3}(\tau)\delta_n \) holds.

Recall that the framing \( \tau_Y \) of \( T(Y \setminus \infty) \) gives the framing \( \tau_Y \cup \tau_{S^3} = \tau_Y|_{Y\setminus N(\infty;Y)} \cup \tau_{S^3}|_{N(\infty;S^3)} \) of \( TY \) and \( \sigma_{Y\setminus\infty}(\tau_Y) = \sigma_Y|_{\tau_Y \cup \tau_{S^3}} - \sigma_{\tau_{S^3}} = \sigma_Y|_{\tau_Y \cup \tau_{S^3}} - 2 \). We give the spin structure on \( Y \) using \( \tau_Y \cup \tau_{S^3} \).

**Lemma 7.3.** There exists a positive integer \( k \) and a spin 4-manifold \( X_0 \) such that \( \chi(X_0) = 0 \) and \( \partial X_0 = Y \sqcup k(-S^3) \) as spin manifolds. Here \(-S^3 \) is \( S^3 \) with the opposite orientation.

*Proof.* Since the 3-dimensional spin cobordism group equals to zero, there exists a spin 4-manifold \( \tilde{X} \) such that \( \partial \tilde{X} = Y \). Let \( k = \chi(\tilde{X}) \). We may assume that \( k \geq 0 \), by replacing \( X \) by \( \tilde{X} \sharp nK3 \) for sufficiently large integer \( n \) if necessary. Let \( X_0 \) be the spin 4-manifold obtained by removing \( k \) disjoint 4-balls, i.e., \( X_0 = \tilde{X} \setminus kB^4 \). Then \( \chi(X_0) = 0 \) and \( \partial X_0 = Y \sqcup k(-S^3) \). \( \square \)

**Remark 7.4.** Since \( \chi(X_0\sharp T^4) = \chi(X_0) - 2, \chi(X_0\sharp K3) = \chi(X_0) + 22 \) and \( T^4, K3 \) are spin, it is possible to choose \( k + 2n \) instead of \( k \) for any \( n \in \mathbb{Z} \).

**Remark 7.5.** Since the Euler number of a closed spin 4-manifold is even, the number \( k(Y) = k \mod 2 \in \mathbb{Z}/2 \) is an invariant of a spin 3-manifold \( Y \). It is known that \( k(Y) = rkH_1(Y; \mathbb{Z}/2) + 1 \) (See Theorem 2.6 in [KMc99]). We also remark that \( k(Y) \equiv \sigma_{Y\setminus\infty}(\tau_Y) + 1 \mod 2 \).
Let $X_0$ be a spin 4-manifold such that $\chi(X_0) = 0$ and $\partial X_0 = Y \cup k(-S^3)$ for some $k \geq 1$. We denote $S^3_i$ the $i$-th $S^3$-boundary of $X_0$. Then $\partial X_0 = Y - S^3_1 - \cdots - S^3_n$. By the obstruction theory, it is possible to extend the framing $\eta_Y \oplus (\tau_Y \cup \tau_{S^3})$ of $TX_0|_Y$ to $X_0$ where $\eta_Y$ is the outward unit vector field on $Y \subset \partial X_0$ (see [KM99] for more details). We choose such a extended framing $\tilde{\tau}_X$ such that $\tilde{\tau}_X^i(1, 0, 0, 0)|_{k(-S^3)}$ is the inward unit vector field on $k(-S^3) \subset \partial X_0 \subset X_0$. If necessary we modify $\tilde{\tau}_X$ by using homotopy, we may assume that there exists a framing $\tau_i$ of $S^3_i \setminus \infty$ such that $\tau_i|_{N(\infty; S^3_i) \setminus \infty} = \tau_{\mathbb{R}^3}|_{N(\infty; S^3_i) \setminus \infty}$ and $-\eta_i \oplus (\tau_i \cup \tau_{S^3}) = \tilde{\tau}_X|_{-S^3_i}$. Here $-\eta_i$ is the inward unit vector field on $-S^3_i \subset X_0$.

Let $X'$ be a compact oriented 4-manifold with $\chi(X') = 0$ and $\partial X' = S^3$. Then $X_0 \cup kX'$ is a compact 4-manifold with $\chi(X_0 \cup kX') = 0$ and $\partial(X_0 \cup kX') = Y$.

**Lemma 7.6.** The following three equations hold.

1. $\tilde{z}_{\text{anomaly}}^\gamma(\tau_Y^* a; X_0 \cup kX') = \sum_{i=1}^k \tilde{z}_{\text{anomaly}}^\gamma(\tau_i^* a; X')$.

2. $\sigma_{Y \setminus \infty}(\tau_Y) = \sum_{i=1}^k \sigma_{\mathbb{R}^3}(\tau_i) + 2(k - 1) - 3\text{Sign}X_0$.

3. $\tilde{z}_{\text{anomaly}}^\gamma(\tau_Y^* a) = \frac{1}{4}\sigma_{Y \setminus \infty}(\tau_Y)\delta_n + \left(\frac{3}{4}\delta_n - \mu_n\right)\text{Sign}X_0 + \frac{k-1}{2}\delta_n + (k - 1)c_n$.

**Proof.** (1) We take a 3-bundle $T^v(X_0 \cup kX') \subset T(X_0 \cup kX')$ over $X_0 \cup kX'$ such that $T^v(X_0 \cup kX')|_{X_0}$ is the normal bundle of $\tilde{\tau}_X^i(1, 0, 0, 0)$. We denote $T^vX_0 = T^v(X_0 \cup kX')|_{X_0}$, $T^v(kX') = T^v(X_0 \cup kX')|_{kX'}$. Let $\beta$ be a section of $T^v(X_0 \cup kX')$ such that $\beta|_{X_0} = \tilde{\tau}_X^i a$ and $\beta$ is transverse to the zero section in $T^v(X_0 \cup kX')$. In this setting, we can take $W(\tau_Y^* a)|_{ST^vX_0} = \tilde{\tau}_X^i \omega_{S^2}$. Then $\tilde{z}_{\text{anomaly}}^\gamma(\tau_Y^* a; X_0 \cup kX') = \sum_{i} \int_{S_{2n}(T^vX_0)} \Lambda_i \phi_i(\Gamma)^* W(\tau_Y^* a)[\Gamma] = \sum_{i} \int_{S_{2n}(T^vX_0)} \Lambda_i \phi_i(\Gamma)^* \tilde{\tau}_X^i \omega_{S^2} [\Gamma]$

We show that $\int_{S_{2n}(T^vX_0)} \Lambda_i \phi_i(\Gamma)^* \tilde{\tau}_X^i \omega_{S^2} = 0$ for any $\Gamma \in \mathcal{E}_n$. The map $(\tilde{\tau}_X^i)^3n \circ (\prod_i \phi_i(\Gamma)) : S_{2n}(T^vX_0) \to (S^2)^{3n}$ factors through $S_{2n}(\mathbb{R}^3)$:

\[
\begin{array}{ccc}
S_{2n}(T^vX_0) & \xrightarrow{\prod_i \phi_i(\Gamma)} & (ST^vX_0)^{3n} \\
\tilde{\tau}_X & \circ & (\tilde{\tau}_X)^{3n} \\
S_{2n}(\mathbb{R}^3) & \xrightarrow{\prod_i \phi_i(\Gamma)^* \tilde{\tau}_X^i \omega_{S^2}} & (S^2)^{3n}.
\end{array}
\]

Hence we have $\Lambda_i \phi_i(\Gamma)^* \tilde{\tau}_X^i \omega_{S^2}|_{ST^vX_0} = ((\prod \tilde{\tau}_X)^{3n} \circ \Lambda_i \phi_i(\Gamma)^* (\omega_{S^2}))^{3n} \in \text{Im}(\Omega^{6n}(S_{2n}(\mathbb{R}^3)) \to \Omega^{6n}(S_{2n}(T^vX_0)))$. Since $\dim \mathbb{R}^3 = 6n - 4$, we have $\Lambda_i \phi_i(\Gamma)^* \tilde{\tau}_X^i \omega_{S^2} = 0$. Therefore

\[
\tilde{z}_{\text{anomaly}}^\gamma(\tau_Y^* a; X_0 \cup kX') = \sum_{i} \int_{S_{2n}(T^v kX')} \Lambda_i \phi_i(\Gamma)^* W(\tau_Y^* a)[\Gamma] = \sum_{i=1}^k \tilde{z}_{\text{anomaly}}^\gamma(\tau_i^* a; X').
\]

(2) By the obstruction theory and the definition of the signature defect, we have $\sigma_Y(\tau_Y \cup \tau_{S^3}) + 3\text{Sign}X_0 = \sum_{i=1}^k \sigma_{S^3}(\tau_i \cup \tau_{S^3})$. Since $\sigma_{Y \setminus \infty}(\tau_Y) = \sigma_Y(\tau_Y \cup \tau_{S^3}) - 2$ and $\sigma_{\mathbb{R}^3}(\tau_i) = \sigma_{S^3}(\tau_i \cup \tau_{S^3}) - 2$, the equation (2) holds.

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Proof of Lemma 7.8.

Let $\lambda = \lambda'$ and $\mu = \mu'$. Then we have $0 = \lambda_n(\tau_i^* \bar{a}) - c_n \sum_{\mu} \delta_n - \mu_n \text{Sign}(X_0 + (k-1)c_n)

\text{Lemma 7.7.} \: \mu_n = \frac{3}{4} \delta_n.

Proof. Let $X_0 = K3\#11T^4 \setminus (B^4 \cup B^4)$. Then $X_0$ is a spin 4-manifold satisfying $\chi(X_0) = 0$ and $\text{Sign}(X_0) = 16$. It is possible to deal with $\partial X_0 = S^3 \cup -S^3$. By Lemma 7.6 (3), we have $0 = \lambda_n(\tau_i^* \bar{a}) = (\frac{3}{4} \delta_n - \mu_n) \text{Sign}(X_0)$. Since $\text{Sign}(X_0) = 16 \neq 0$, we have $\mu_n = \frac{3}{4} \delta_n$.

\text{Lemma 7.8.} \: c_n = \frac{1}{2} \delta_n.

Proof. Let $X_0 = K3\#10T^4 \setminus (B^4 \cup 3B^4)$. Then $X_0$ is a spin 4-manifold satisfying $\chi(X_0) = 0$ and $\text{Sign}(X_0) = 16$. It is possible to deal with $\partial X_0 = S^3 \cup 3(-S^3)$. By Lemma 7.6 (3) and Lemma 7.7, we have $0 = \lambda_n(\tau_i^* \bar{a}) = -\delta_n + 2c_n$. Then $c_n = \frac{1}{2} \delta_n$.

\text{Proposition 7.9.} \: \lambda_n(\tau_i^* \bar{a}) = \frac{1}{4} \sigma_{Y \setminus \infty}(\tau_Y) \delta_n.

Proof. Take $X_0, k, \tau_X$ as in Lemma 7.6. By Lemma 7.6 (3), Lemma 7.7 and Lemma 7.8, we have $\lambda_n(\tau_i^* \bar{a}) = \frac{1}{4} \sigma_{Y \setminus \infty}(\tau_Y) \delta_n - \frac{k-1}{2} \delta_n + (k-1)c_n = \frac{1}{4} \sigma_{Y \setminus \infty}(\tau_Y) \delta_n$.

7.2 Proof of $\lambda_n(Y) = z_{\text{FW}} (2n, 3n)(Y)$.

Let $f$ be an admissible Morse function with respect to $a \in S^2$. The weighted sum $\mathcal{M}(f) + \mathcal{M}(-f)$ consists of weighted pairs of two distinct points on a gradient trajectory. There is a compactification $\mathcal{M}_S(\pm f)$ of $\mathcal{M}(f) + \mathcal{M}(-f)$ by adding pairs of points on broken trajectories as the Morse theory. Then $\mathcal{M}_S(\pm f)$ becomes a 4-cycle in $(C_2(Y), \partial C_2(Y))$ (Lemma 8.4). See for Section 8 for the detail of the above argument.

\text{Lemma 7.10.} \: \partial \mathcal{M}_S(\pm f) = c(\text{grad} f) \text{ for any admissible Morse function } f.
Proof. Since $\text{grad} f|_{N(\infty; Y)} = \text{grad} q_0$, if $(x, u) \in \partial \mathcal{M}_S(\pm f) \cap ((Y \setminus \infty) \times S T_\infty Y)$ then $u = \pm a$. On the other hand, $\partial \mathcal{M}_S(\pm f) \cap \{(x) \times S T_\infty Y\} = \{(x, a), (x, -a)\}$ for any $x \notin \text{Crit}(f)$. Since $\partial \mathcal{M}_S(\pm f)$ is a 3-cycle, we have $\partial \mathcal{M}_S(f) \cap ((Y \setminus \infty) \times S T_\infty Y) = (Y \setminus \infty) \times (\pm a)$. With a similar argument, we have $\partial C_2(Y) \setminus S v_{\Delta Y \setminus \infty} = p_{\gamma}^{-1}(\pm a)$. Since this fact and Lemma 8.3 we conclude the proof. \qed

We follow the notations $a_1, \ldots, a_3n, f_1, \ldots, f_3n$ as in Section 6. In the following proposition, the notion "generic $\vec{f}$" means that $\bigcap_i P_i(\Gamma)^{-1} \mathcal{M}_S(\pm f_i) = \emptyset$ for any $\Gamma \in \mathcal{E}_n$. We remark that there exists such a $\vec{f}$ (See Remark 7.12).

Proposition 7.11. For generic $\vec{f}$, $z_{2n, 3n}^{FW}(Y; \vec{f}) = \tilde{z}_n(Y; \text{grad} \vec{f})$.

Proof. We define the 2-cocycle $\omega^\ast(\text{grad} f_i) \in S^2([\mathcal{T}_{C_2}(Y)])$ by $\omega^\ast(\text{grad} f_i)(\sigma) = \frac{1}{2}\omega(\sigma \cap \mathcal{M}_S(f_i))$ for each 2-cycle $\sigma$ of $\mathcal{T}_{C_2}(Y)$.

By the construction, $\omega^\ast(\text{grad} f_i)$ is simplicial propagator for each $i$. By the intersection theory and Lemma 4.8 we have

$$
z_{2n, 3n}^{FW}(Y; \vec{f}) = \langle \prod_i P_i(\Gamma)^\ast \omega^\ast(\text{grad} f_i), [C_{2n}(Y), \partial C_{2n}(Y)] \rangle = \frac{1}{2^{3n}} \left( \prod_i P_i(\Gamma)^{-1} \mathcal{M}_S(f_i) \right)
$$

for any $\Gamma \in \mathcal{E}_n$. \qed

Remark 7.12. We can show that $\partial C_{2n}(Y) \cap (\bigcap_i P_i(\Gamma)^{-1} \mathcal{M}_S(\pm f_i)) = \emptyset$ for generic $\vec{f}$ by an argument similar to Lemma 2.7 in Watanabe [Wat12]. For example, we take the following $\Phi'_\Gamma$ instead of $\Phi$ in Lemma 2.7 in [Wat12] when we prove $F(\{1, 2, 4\}) \cap (\bigcap_{i=1}^4 P_i(\text{Smooth}(\Gamma))^\ast \mathcal{M}_S(\pm f_i)) = \emptyset$ for the graph $\Gamma$ in the picture (2.2) in [Wat12] (See Example 2.6 in [Wat12] and see §3.4 of [Wat12] for the definition of the operator Smooth).

$$
\phi'_{\Gamma} : F(\{1, 2, 4\}) \times \left( \bigcup_{f_1 \in U_1} \mathcal{A}_p(f_1) \cap \mathcal{D}_q(f_1) \right) \times (\mathbb{R}_{>0})^3 \times \prod_{i=2}^4 U_i \rightarrow Y^3 \times (TY)^2 \times (TY)^2 \times Y^3;
$$

$$
\Phi'_{\Gamma}\((x_1, [w_1, w_2, w_3]), x_3, u, t_2, t_3, t_4, f_2, f_3, f_4) = ((x_1, u, \Phi_{f_0}^\ast(x_3)), (\text{grad}_x f_2, \frac{w_2 - w_1}{||w_2 - w_1||}), (\text{grad}_x f_3, \frac{w_4 - w_2}{||w_4 - w_2||}), (x_3, \Phi^t_{f_4}(x_1), \Phi^t_{f_3}(x_3))).
$$

Here $x_1 \in Y \setminus \infty$, $[w_1, w_2, w_3] \in \tilde{S}_{[1,2,4]} T_{x_1} Y$, $x_3 \in Y \setminus \{x_1, \infty\}$. Let

$$
\Delta'_{\Gamma} = \{(y_1, y_2, y_3), (y_2, s_2 v_2), (y_2, t_2 v_2), (y_3, s_3 v_3), (y_3, t_3 v_3), (y_4, s_4 v_4), (y_4, t_4 v_4)\}
$$

$$
| (y_1, y_2, y_3, y_4) \in (Y \setminus \infty)^4, t_i, s_i \geq 0, v_i \in T_{y_i} Y \}. \}
$$

Then $\Phi'_{\Gamma}$ is transverse to $\Delta'_{\Gamma}$ as Lemma 2.7 in [Wat12].

It is obvious that $z_{2n, 3n}^{\text{anomaly}}(Y; \vec{f}) = z_{n}^{\text{anomaly}}(Y; \text{grad} \vec{f})$ by the definitions of the anomaly parts.
8 Compactification of moduli space $\mathcal{M}(f)$

In this section we give a compactification $\mathcal{M}_S(\pm f)$ of $\mathcal{M}(f) \cup \mathcal{M}(-f)$ and then show that $\mathcal{M}_S(\pm f)$ is a 4-cycle in $(C_2(Y), \partial C_2(Y))$. Let $M_\to(f) = \varphi^{-1}|_{Y^2 \times (0, \infty)}(\Delta)$ where $\varphi: Y^2 \times (-\infty, \infty) \to Y^2, (x, y) \mapsto (y, \Phi_i(x))$.

**Lemma 8.1** (Watanabe [Wat12, Proposition 2.12] (cf. [BH01])). There is a manifold with corners $\overline{M}_\to(f)$ satisfying the following conditions.

1. $\overline{M}_\to(f) = \{ g: I \to Y \mid I \subset -\mathbb{R},
   g \text{ is a piecewise smooth map, } f(g(t)) = t, \frac{dg(t)}{dt} = \frac{\nabla f(t)}{\|\nabla f(t)\|}$ for any $t \}$ as sets,
2. $\text{int}\overline{M}_\to(f) = M_\to(f)$, and
3. $\partial \overline{M}_\to(f) = \sum_i A_{p_i} \times D_{p_i} + \sum_j A_{q_j} \times D_{q_j}.$

Note that $\text{int}(\overline{M}_\to(f) + \overline{M}_\to(-f)) = \varphi^{-1}(\Delta)$. We denote by $\overline{M}_\to(f) \to (Y \setminus \infty)^2$ the continuous map that is the extension of the embedding $M_\to(f) \to (Y \setminus \infty)^2$ to $\overline{M}_\to(f)$. For simplicity of notation, we write $\overline{M}_\to(f)$ instead of $\overline{M}_\to(f) \to (Y \setminus \infty)^2$.

Similarly we denote by $\overline{A}_{p_i} \to Y$ the extension of $B^1(1) \cong A_{p_i} \to Y$ to $B^1(1)$ and we write $\overline{A}_{q_j}$ instead of $\overline{A}_{p_i} \to Y$ (We remark that $A_{p_i}$ is diffeomorphic to $B^1(1)$ the interior of unit disk in $\mathbb{R}^3$). We also define $\overline{D}_{p_i}, \overline{A}_{q_j}$, and so on.

**Lemma 8.2.** (1) $\overline{M}_\to(f) + \overline{M}_\to(-f)$ is transverse to $\Delta$.

2. $\overline{A}_{q_j} \times \overline{D}_{p_i}$ is transverse to $\Delta$.

**Proof.** (1) $\nabla f$ (which is the section of $\nu_{\Delta(Y \setminus \infty)}$) is transverse to the zero section in $\nu_{\Delta(Y \setminus \infty)}$. $A_p \times D_p \subset Y^2$ is transverse to $\Delta$ for any critical point $p \in \text{Crit}(f) = \text{Crit}(-f)$. Thanks to Lemma 8.1 (2),(3), this finishes the proof of (1).

(2) is immediate from the Morse-Smale condition. \(\square\)

By this Lemma, $(\overline{M}_\to(f) + \overline{M}_\to(-f))(\Delta)$ and $(\overline{A}_{q_j} \times \overline{D}_{p_i})(\Delta)$ are well-defined.

It is clear that $(\overline{M}_\to(f) + \overline{M}_\to(-f))(\Delta) = (\overline{M}_\to(f) + \overline{M}_\to(-f))(\Delta) \cup \{ (x, \frac{\pm \nabla f}{\|\nabla f\|}) \mid x \in Y \setminus (\infty \cup \text{Crit}(f)) \}$ by the construction.

**Definition 8.3.** $\mathcal{M}_S^0(\pm f) = (\overline{M}_\to(f) + \overline{M}_\to(-f))(\Delta) + \sum_{i,j} g_{ij} (\overline{A}_{q_j} \times \overline{D}_{p_i})(\Delta) + \sum_{i,j} (-g_{ij})(\overline{D}_{p_j} \times \overline{A}_{q_i})(\Delta)$.

Let $\mathcal{M}_S(\pm f)$ be the extension of $\mathcal{M}_S^0(\pm f)$ to $C_2(Y)$.

**Lemma 8.4.** $\mathcal{M}_S(\pm f)$ is a 4-cycle in $(C_2(Y), \partial C_2(Y))$.
Proof. Since \( \text{Im}(\partial(\mathbb{A}_{q_i} \times \mathbb{D}_{p_j}) \to Y^2) = \sum_k \partial_{k,i} \mathbb{A}_{p_k} \times \mathbb{D}_{p_j} + \sum_k \partial_{j,k} \mathbb{A}_{q_i} \times \mathbb{D}_{q_k}, \)
\[
\text{Im}(\sum_{i,j} g_{ij} \partial(\mathbb{A}_{q_i} \times \mathbb{D}_{p_j} \to Y^2))
= \sum_{i,j,k} g_{ij} \partial_{k,i} \mathbb{A}_{p_k} \times \mathbb{D}_{p_j} + \sum_{i,j,k} g_{ij} \partial_{j,k} \mathbb{A}_{q_i} \times \mathbb{D}_{q_k}
= \sum_{i,j,k} \delta_{kj} \mathbb{A}_{p_k} \times \mathbb{D}_{p_j} + \sum_{i,j,k} \delta_{ik} \mathbb{A}_{q_i} \times \mathbb{D}_{q_k}
= \sum_j \mathbb{A}_{p_j} \times \mathbb{D}_{p_j} + \sum_j \mathbb{A}_{q_j} \times \mathbb{D}_{q_j}
= \partial \mathcal{M}_s(f) \setminus \Delta.
\]
Therefore \( \partial \mathcal{M}_s(\pm f) \setminus \partial C_2(Y) = \emptyset. \)

Under the identification \( S\nu_{\Delta(Y \setminus \infty)} \cong ST(Y \setminus \infty), \) we have the following description.

**Lemma 8.5.** \( \partial \mathcal{M}_s(\pm f) \cap ST(Y \setminus \infty) = \{(x, \frac{\pm \text{grad}_xf}{||\text{grad}_xf||}) | x \in Y \setminus (\infty \cup \text{Crit}(f))\}. \)

**Proof.** Note that \( (\mathbb{A}_{qi} \times \mathbb{D}_{p_j}) \cap \Delta = \mathbb{A}_{qi} \cap D_{p_j}. \) By the definition of blow up, we have \( \partial \mathcal{M}_s(\pm f) \cap S\nu_{\Delta(Y \setminus \infty)} \)
\[
= \left\{ (x, \frac{\pm \text{grad}_xf}{||\text{grad}_xf||}) \right\} + \sum_{i,j} g_{ij} \pi^{-1}(\mathbb{A}_{qi} \cap D_{p_j}) + \sum_{i,j} (-g_{ij}) \pi^{-1}(D_{p_j} \cap \mathbb{A}_{qi})
\]
where \( \pi : STY \to Y \) is the projection.

Since \( \sum_{i,j} g_{ij} \pi^{-1}(\mathbb{A}_{qi} \cap D_{p_j}) + \sum_{i,j} (-g_{ij}) \pi^{-1}(D_{p_j} \cap \mathbb{A}_{qi}) = 0 \) as chains, we conclude the proof. \( \square \)

## A Another proof of \( \tilde{z}_1(Y) = z_1^{\text{KKT}}(Y). \)

In this section we give a more direct proof of Preposition [7.9] in the case of \( n = 1. \) Remark that \( \mathcal{A}_1(\emptyset) = \mathbb{Q}[\theta] \) and \( \sharp \mathcal{E}_1 = 96. \)

**Proposition A.1** (Preposition [7.9] in the case of \( n = 1). \) \( \tilde{z}_1^{\text{anomaly}}(\tau_Y \vec{a}) = \frac{1}{2} \sigma_{Y \setminus \infty}(\tau_Y) \delta_1. \)

To show this proposition we first prepare some notations and lemmas. Let \( \tau_1 : F_X \to X \) be the tangent bundle along the fiber of \( \tau_2 : ST^vX \to X. \) Let \( ST^vX/STY \) be the real vector bundle over \( X/Y \) obtained by collapsing \( STY \) to a point using the framing \( \tau_Y \cup \tau_{S^3} = \tau_Y|_{Y \setminus (\infty \cup \text{Crit}(f))} \cup \tau_{S^3}|_{N(\infty \cup \text{Crit}(f))}. \) We define \( F_{X/Y}, ST^vX/STY \) as same way.

Let \( e(F_X; \tau_Y) \in H^2(ST^vX/STY) = H^2(ST^vX, STY) \) be the Euler class of \( F_{X/Y} \) and let \( p_1(F_X; \tau_Y) \in H^2(ST^vX/STY) = H^2(ST^vX, STY) \) be the 1st Pontrjagin class of \( F_{X/Y}. \) By a standard argument, for example the Chern-Weil theory, we have \( p_1(F_X; \tau_Y) = e_1(F_X; \tau_Y)^2. \)

**Lemma A.2.** \( 2[W(\tau_Y \vec{a})] = e(F_X; \tau_Y) \in H^2(ST^vX/STY). \)
Proof. Let $\beta$ be the section of $T^vX$ such that $\beta|_{\partial X} = (\tau_Y \cup \tau_{S^3})^*a$ as Subsection 4.4. We define the map $f : ST^vX \to \mathbb{R}$ by

$$f(x) = \langle u, \beta(x) \rangle_{(T^vX)_x}$$

where $\langle ., . \rangle_{(T^vX)_x}$ is the standard inner product on $(T^vX)_x(\cong \mathbb{R}^3)$. We define the vector field $V \in \Gamma F_X$ by $V|_{(ST^vX)_x} = \text{grad}(f|_{(ST^vX)_x})$ for any $x \in X$. Thus $V$ is transverse to the zero section in $F_X$ and $V^{-1}(0) = c_0(\beta)$. Thus the Poincaré dual of $(c_0(\beta), \partial c_0(\beta))$ represents $e(F_X; \tau_Y)$. Since the closed 2-form $2W(\tau_Y^*a)$ represents the Poincaré dual of $(c_0(\beta), \partial c_0(\beta))$ and $W(\tau_Y^*a)|_{STY} = (\tau_Y \cup \tau_{S^3})^*\omega_{S^2}$, we conclude the proof.

proof of Proposition A.1. By the Lemma A.2, we have

$$\int_{S_2(T^vX)} W(\tau_Y^*a)^3 = \frac{1}{8} \int_{S_2(T^vX)} e(F_X; \tau_Y)^3$$

$$= \frac{1}{8} \int_{S_2(T^vX)} e(F_X; \tau_Y)p_1(F_X; \tau_Y)$$

$$\overset{(*)}{=} \frac{1}{8} \int_{S_2(T^vX)} e(F_X; \tau_Y)p_2^*(TX; \tau_Y)$$

$$= \frac{1}{4} \int_X p_1(TX; \tau_Y)$$

$$= \frac{1}{4} \sigma_Y(\tau_Y \cup \tau_{S^3}) + \frac{3}{4} \text{Sign}X$$

$$= \frac{1}{4} \sigma_Y(\tau_Y) + \frac{3}{4} \text{Sign}X + \frac{1}{2}.$$ 

The equation (*) is given by the following two relations: $\mathbb{R} \oplus F_X = \pi^*T^vX$ and $\mathbb{R} \oplus T^vX = TX$. Then we have

$$\tilde{z}_1^\text{anomaly}(\tau_Y^*a) = 96 \int_{S_2(T^vX)} W(\tau_Y^*a)^3[\theta] - \mu_1 \text{Sign}X - c_1$$

$$= \frac{96}{4} [\theta] \sigma_Y(\tau_Y) + (72[\theta] - \mu_1) \text{Sign}X - (c_1 - 48[\theta]).$$

Since this equation holds for any $\tau_Y$ and $X$, then we have $\mu_1 = 72[\theta]$, $c_1 = 48[\theta]$, $\delta_1 = 96[\theta]$. Thus $\tilde{z}_1^\text{anomaly}(\tau_Y^*a) = \frac{1}{4} \sigma_Y(\tau_Y)\delta_1$.

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