The $L^p - L^p$ Estimate for the Schrödinger Equation on the Half-Line

Ricardo Weder †
Instituto Argentino de Matemática, CONICET
Saavedra 15, 1083-Buenos Aires, Argentina
E-Mail: weder@servidor.unam.mx

Abstract

In this paper we prove the $L^p - L^p$ estimate for the Schrödinger equation on the half-line and with homogeneous Dirichlet boundary condition at the origin.

---

*2000 AMS classification 35P25, 35R30 and 81U40. Research partially supported by proyecto PAPIIT, IN 105799, DGAPA-UNAM.
†On leave of absence from Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas, Universidad Nacional Autónoma de México, Apartado Postal 20-726, México D.F. 01000. Fellow, Sistema Nacional de Investigadores.
1 Introduction

We consider the following Schrödinger equation,

\[ i \frac{\partial}{\partial t} u(x, t) = H u(x, t), \quad u(0, t) = 0, \quad u(x, 0) = \phi(x), \]  

(1.1)

where \( x \in \mathbb{R}^+ := (0, \infty), \ t \in \mathbb{R} \). The Hamiltonian, \( H \), is the following operator,

\[ H := -\frac{d^2}{dx^2} + V(x), \]  

(1.2)

with domain,

\[ D(H) := \left\{ \phi \in L^2 : \phi, \frac{d}{dx} \phi \text{ are absolutely continuous in } (0, \infty), \left( -\frac{d^2}{dx^2} + V(x) \right) \phi \in L^2, \phi(0) = 0 \right\}, \]  

(1.3)

where \( L^2 \) denotes the Hilbert space of square-integrable functions on \( \mathbb{R}^+ \). The potential, \( V \), is real valued and it satisfies the following condition,

\[ \int_0^\infty x |V(x)| \, dx < \infty. \]  

(1.4)

The operator \( H \) is self-adjoint in \( L^2 \) (see Section 2), it is the self-adjoint realization of the differential expression \( -\frac{d^2}{dx^2} + V(x) \) with homogeneous Dirichlet boundary condition, \( \phi(0) = 0 \), at zero. The unique solution to the initial-boundary value problem (1.1) is given by,

\[ u = e^{-itH} \phi, \]  

(1.5)

where the strongly continuous one-parameter unitary group \( e^{-itH} \) is defined by functional calculus. By \( W_{l,p}, l = 0, 1, 2, \cdots, 1 \leq p \leq \infty, \) we denote the standard Sobolev spaces \( W \) in \( \mathbb{R}^+ \) and by \( W_{l,p}^{(0)}, 1 \leq p < \infty, \) the completion of \( C_0^\infty(\mathbb{R}^+) \) in the norm of \( W_{l,p} \). The functions in \( W_{l,p}^{(0)}, l \geq 1, \) satisfy the homogeneous Dirichlet boundary condition at zero, \( \frac{d^j}{dx^j} u(0) = 0, j = 0, 1, \cdots, l - 1. \) In the case \( l = 0 \) we use the standard notation, \( W_{0,p} = W_{0,p}^{(0)} = L^p, p < \infty. \) For \( l \geq 1 \) we use the notation, \( W_{l,\infty}^{(0)} := \{ \phi \in W_{l,\infty} : \frac{d^j}{dx^j} \phi(0) = 0, j = 0, 1, 2, \cdots, l - 1 \}. \) Let us denote by \( H_0 \) the self-adjoint realization of \( -\frac{d^2}{dx^2} \) with domain \( W_{2,2} \cap W_{1,2}^{(0)} \), i.e., the self-adjoint realization with homogeneous Dirichlet boundary condition at zero. It follows from a simple calculation using the Fourier transform that \( e^{-itH_0} \) is an integral operator,
\[ e^{-itH_0} \phi = \int_0^\infty k_{t,0}(x,y) \phi(y) \, dy, \quad (1.6) \]

with the kernel,
\[
k_{t,0}(x,y) := \frac{1}{2\pi} \int_{-\infty}^\infty e^{-itk^2} \left[ e^{ik(x-y)} - e^{ik(x+y)} \right] \, dk = \frac{1}{\sqrt{4\pi it}} \left[ e^{i(x-y)^2/4t} - e^{i(x+y)^2/4t} \right]. \quad (1.7)
\]

It follows from (1.6) and (1.7), integrating by parts, and as \( \|\phi\|_{L^\infty} \leq \|\phi\|_{W^{1,1}} \), that \( e^{-itH_0} \) satisfies the \( L^1 - L^\infty \) estimate,
\[
\|e^{-itH_0}\|_{\mathcal{B}(W^{l,1}_1,W^{(0)}_{1,\infty})} \leq C \frac{1}{\sqrt{|t|}}, \quad l = 0, 1, \quad (1.8)
\]

and the \( L^2 - L^2 \) estimate,
\[
\|e^{-itH_0}\|_{\mathcal{B}(W^{l,2}_{1,2},W^{(0)}_{1,2})} = 1, \quad l = 0, 1, \quad (1.9)
\]

where for any pair of Banach spaces, \( X, Y \), we denote by \( \mathcal{B}(X,Y) \) the Banach space of bounded operators from \( X \) into \( Y \) and with \( W_{0,\infty}^{(0)} := L^\infty \).

Interpolating between (1.8) and (1.9) \cite{13} we obtain the \( L^p - L^{\hat{p}} \) estimate in the free case, \( V \equiv 0 \),
\[
\|e^{-itH_0}\|_{\mathcal{B}(W^{l,p}_{1,p},W^{(0)}_{1,\hat{p}})} \leq C \frac{1}{|t|^{(1/p-1/2)}}, \quad 1/p + 1/\hat{p} = 1, \quad 1 \leq p \leq 2, \quad l = 0, 1. \quad (1.10)
\]

It is clear that \( e^{-itH} \) will not satisfy an estimate as (1.10) if \( H \) has eigenvectors. However, as we prove below it satisfies the \( L^p - L^{\hat{p}} \) estimate when restricted to the subspace of continuity of \( H, \mathcal{H}_c \), i.e., to the subspace of \( L^2 \) orthogonal to all eigenvectors of \( H \). We denote by \( P_c \) the orthogonal projector onto \( \mathcal{H}_c \).

**THEOREM 1.1.** (The \( L^p - L^{\hat{p}} \) estimate). Suppose that \( V \) satisfies (1.4) then,
\[
\|e^{-itH_0}P_c\|_{\mathcal{B}(L^p,L^{\hat{p}})} \leq C \frac{1}{|t|^{(1/p-1/2)}}, \quad 1/p + 1/\hat{p} = 1, \quad 1 \leq p \leq 2. \quad (1.11)
\]

If, furthermore, \( V \in L^1 \),
\[
\|e^{-itH}P_c\|_{\mathcal{B}(W^{l,p}_{1,p},W^{(0)}_{1,\hat{p}})} \leq C \frac{1}{|t|^{(1/p-1/2)}}, \quad 1/p + 1/\hat{p} = 1, \quad 1 \leq p \leq 2. \quad (1.12)
\]

We prove this result in Section 2 using the the Jost solution (the scattering solution) to the stationary Schrödinger equation. For the proof of the \( L^p - L^{\hat{p}} \) estimate for the Schrödinger equation on the line see
for the case of $\mathbb{R}^n, n \geq 3$, see [7] and [24], [25]. See [26], [6] for the problem in $\mathbb{R}^2$, with $1 < p \leq 2$. The $L^p - L^{\hat{p}}$ estimate expresses the *smoothing properties* of the linear Schrödinger equation with a potential (1.1) in a quantitative way, and it exhibits the dispersive nature of this equation. In fact, this estimate is of independent interest. As is well known, $L^p - L^{\hat{p}}$ estimates play an important role in the study of non-linear initial value problems [14], [4] and [2]. In particular, the $L^p - L^{\hat{p}}$ estimate implies the famous Strichartz’s estimates for the linear Schrödinger equation with a potential, see [15], [8], [9] and Proposition 2.4 below. It is also the key issue in scattering and inverse scattering for non-linear Schrödinger equations [14], [4], [2], [8], [7], [17], [18], [20] and [21]. The $L^p - L^{\hat{p}}$ estimate is also important in the construction of center manifolds for non-linear Schrödinger equations with potential, see [16], [12] and [19]. In [22] we apply our $L^p - L^{\hat{p}}$ estimate to the solution of the direct and inverse scattering problems for the forced non-linear Schrödinger equation with a potential on the half-line.

2 The $L^p - L^{\hat{p}}$ Estimate

We first state a number of results on the linear Schrödinger equation,

$$
\left(-\frac{d^2}{dx^2} + V(x)\right)\phi(x) = k^2 \phi(x), k \in \mathbb{C}.
$$

(2.1)

Let us denote by $f(k, x), k \in \mathbb{C}, \text{Im}k \geq 0$, the Jost solution to (2.1) (see [3], [10] and [11]). It is the solution to (2.1) that satisfies $f(k, x) \sim e^{ikx}, x \to \infty$. The Jost solution is not required to satisfy the homogeneous Dirichlet boundary condition at zero. It only satisfies it if $k^2$ is an eigenvalue of $H$. Let us denote, $\sigma(x) := \int_x^\infty |V(y)| dy$ and $\sigma_1(x) := \int_x^\infty \sigma(y) dy$. Condition (1.4) is equivalent to $\sigma_1(0) < \infty$. The Jost solution can be represented as follows [10],

$$
f(k, x) = e^{ikx} + \int_x^\infty K(x, y) e^{iky} dy,
$$

(2.2)

where the kernel $K(x, y)$ is real valued and it satisfies the inequality,

$$
|K(x, y)| \leq \frac{1}{2} \sigma \left(\frac{x + y}{2}\right) \exp \left(\sigma_1(x) - \sigma_1 \left(\frac{x + y}{2}\right)\right),
$$

(2.3)

and $K(x, y) = 0, y < x$. Note that $\overline{f(k, x)} = f(-k, x), k \in \mathbb{R}$. For any $k \in \mathbb{C}$ equation (2.1) has also the regular solution, $\phi(k, x)$, that satisfies $\phi(k, 0) = 0, \phi'(k, 0) = 1$. Then, it follows from Theorem 5.8 of [23]
that \( H \) is self-adjoint in the domain \( \mathbb{D}(H) \). By the Parseval identity (see equation (3.2.4) in page 201 of [10]) we have that for all \( \phi \in L^2 \),

\[
e^{-itH} P_c \phi = \int k_t(x,y) \phi(y) \, dy,
\]

(2.4)

where,

\[
k_t(x,y) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itk^2} \left[ f(k,x) \overline{f(k,y)} - f(k,x) f(k,y) S(k) \right] \, dk,
\]

(2.5)

with \( S(k) \) the "scattering matrix",

\[
S(k) := \frac{f(-k,0)}{f(k,0)}.
\]

(2.6)

We decompose \( k_t(x,y) \) as follows,

\[
k_t(x,y) := \sum_{j=0}^{\infty} k_{t,j}(x,y),
\]

(2.7)

where

\[
k_{t,0}(x,y) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itk^2} \left[ e^{ik(x-y)} - e^{ik(x+y)} \right] \, dk = \frac{1}{\sqrt{4\pi it}} \left[ e^{i(x-y)^2/4t} - e^{i(x+y)^2/4t} \right],
\]

(2.8)

corresponds to the free evolution with \( V \equiv 0 \) and, denoting,

\[
d(k,x) := f(k,x) - e^{ikx}, \quad T(k) := S(k) - 1,
\]

(2.9)

we have that,

\[
k_{t,1}(x,y) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itk^2} \left[ d(k,x) e^{-iky} + e^{ikx} \overline{d(k,y)} + d(k,x) \overline{d(k,y)} - d(k,x) e^{iky} - e^{ikx} d(k,y) - d(k,x) d(k,y) - T(k) \left( e^{ik(x+y)} + d(k,x) e^{iky} + e^{ikx} d(k,y) + d(k,x) d(k,y) \right) \right] \, dk.
\]

(2.10)

As indicated in the introduction (see (1.8)-(1.10)) the term with \( k_{t,0}(x,y) \) satisfies the \( L^p - L^p \) estimate corresponding to the free case, \( V \equiv 0 \). We prove below a similar statement for the term with \( k_{t,1}(x,y) \).

**THEOREM 2.1. (The \( L^1 - L^\infty \) estimate).** Suppose that \( V \) satisfies (1.4). Then,

\[
\| e^{-itH} P_c \|_{\mathcal{B}(L^1,L^\infty)} \leq C \frac{1}{\sqrt{|t|}}.
\]

(2.11)

If moreover, \( V \in L^1 \), then,

\[
\| e^{-itH} P_c \|_{\mathcal{B}(W_1^1,W_1^\infty)} \leq C \frac{1}{\sqrt{|t|}}.
\]

(2.12)
Proof: We first prepare some results. Let us denote,

\[ h(u, v) := K(u - v, u + v), u, v \geq 0, u \geq v. \] (2.13)

Then, (see [10], page 176), \( h(u, v) \) is the unique solution to the following equation,

\[ h(u, v) = \frac{1}{2} \int_u^\infty V(y) \, dy + \int_u^\infty dx \int_0^v V(x - y) \, h(x, y) \, dy. \] (2.14)

We denote,

\[ q(u, v) := \frac{1}{2} \sigma(u) \exp \left( \sigma_1(u - v) - \sigma_1(u) \right), u \geq v. \] (2.15)

We have that [10],

\[ |h(u, v)| \leq q(u, v). \] (2.16)

Since \( q(u, v) \) is a non-increasing function of \( u \), and as \( q(u, v) \) is non-decreasing on \( v \), we have that,

\[ \left| \frac{\partial}{\partial u} h(u, v) \right| \leq \frac{1}{2} |V(u)| + \sigma(u - v) \, q(u, v), \] (2.17)

and,

\[ \left| \frac{\partial}{\partial v} h(u, v) \right| \leq \sigma(u - v) \, q(u, v). \] (2.18)

We decompose \( k_{t,1} \) as follows,

\[ k_{t,1}(x, y) = k_{t,2}(x, y) + k_{t,3}(x, y), \] (2.19)

where,

\[ k_{t,2}(x, y) := \frac{1}{2\pi} \int_{-\infty}^\infty e^{-itk^2} \left[ d(k, x) e^{-iky} + e^{ikx} d(k, y) + d(k, x) \overline{d(k, y)} - d(k, x) e^{iky} - e^{ikx} d(k, y) - d(k, x) \overline{d(k, y)} - T(k) \left( d(k, x) e^{iky} + e^{ikx} d(k, y) + d(k, x) \overline{d(k, y)} \right) \right] \, dk. \] (2.20)

and

\[ k_{t,3}(x, y) := -\frac{1}{2\pi} \int_{-\infty}^\infty e^{-itk^2} T(k) \, e^{ik(x+y)}. \] (2.21)

Recall that,

\[ f_t(z) := \frac{1}{2\pi} \int_{-\infty}^\infty e^{-itk^2} e^{-ikz} \, dk = \frac{1}{\sqrt{4\pi it}} e^{iz^2/4t}, \] (2.22)
\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d(k, x) e^{-ikz} dk = \sqrt{2\pi} K(x, z). \quad (2.23)
\]

By (2.22), (2.23) and the convolution theorem for the Fourier transform,

\[
b_t(x, y) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itk^2} d(k, x) e^{-iky} dk = \int_x^{\infty} f_t(y - z) K(x, z) dz. \quad (2.24)
\]

By (1.4), (2.3), (2.23) and (2.24)

\[
|b_t(x, y)| \leq C \sqrt{|t|}, \quad (2.25)
\]

and moreover, if \( V \in L^1 \), it follows from (2.21), (2.24), and since \( K(x, y) = h(\frac{x+y}{2}, \frac{y-x}{2}) \) that,

\[
\left| \frac{\partial}{\partial x} b_t(x, y) \right| \leq \frac{C}{\sqrt{|t|}}. \quad (2.26)
\]

In the same way we prove that,

\[
c_t(x, y) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itk^2} e^{ikx} d(k, y) dk = \int_y^{\infty} f_t(x - z) K(y, z) dz, \quad (2.27)
\]

and that,

\[
|c_t(x, y)| \leq C \sqrt{|t|}. \quad (2.28)
\]

Moreover, integrating by parts in \( z \) we prove that,

\[
\frac{\partial}{\partial x} c_t(x, y) = f_t(x - y) K(y, y) + \int_y^{\infty} f_t(x - z) \frac{\partial}{\partial z} K(y, z) dz, \quad (2.29)
\]

and it follows that,

\[
\left| \frac{\partial}{\partial x} c_t(x, y) \right| \leq \frac{C}{\sqrt{|t|}}. \quad (2.30)
\]

Similarly,

\[
e_t(x, y) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itk^2} d(k, x) d(k, y) dk = \sqrt{2\pi} \int f_t(z_1) K(x, z_1 - z_2) K(y, -z_2) dz_1 dz_2, \quad (2.31)
\]

and we prove as above that if (1.4) is satisfied then,
\[ |e_t(x, y)| \leq \frac{C}{\sqrt{|t|}}, \]  
(2.32)

and, if \( V \in L^1 \),

\[ \left| \frac{\partial}{\partial x} e_t(x, y) \right| \leq \frac{C}{\sqrt{|t|}}, \]  
(2.33)

We estimate the remaining terms in the right hand side of (2.20) in the same way. For this purpose, note that it follows from the extension of Wiener’s theorem to the Fourier transform (see the Corollary to Theorem 4.204 in page 154 of [5]) and the argument given in pages 212 and 213 of [10] that the Fourier transform of \( T(k) \) is integrable on \((-\infty, \infty)\). Then, if (1.4) holds,

\[ |k_{t,2}(x, y)| \leq \frac{C}{\sqrt{|t|}}, \]  
(2.34)

and, if \( V \in L^1 \),

\[ \left| \frac{\partial}{\partial x} k_{t,2}(x, y) \right| \leq \frac{C}{\sqrt{|t|}}. \]  
(2.35)

Furthermore, let us denote by \( T_{t,3} \) the integral operator with kernel \( k_{t,3}(x, y) \). Then denoting by \( \tilde{T} \) the inverse Fourier transform of \( T \), we have that,

\[ T_{t,3} \phi = \int_{-\infty}^{\infty} dz \int_{0}^{\infty} f_i(x + y - z) \phi(y) dy. \]  
(2.36)

Hence,

\[ \|T_{t,3}\|_{B(L^1, L^\infty)} \leq C \frac{1}{\sqrt{|t|}}, \]  
(2.37)

and integrating by parts in \( y \) and as \( \|\phi\|_{L^\infty} \leq \|\phi\|_{W^{1,1}} \), we prove that,

\[ \|T_{t,3}\|_{B(W^{1,1}, W^{1,\infty})} \leq C \frac{1}{\sqrt{|t|}}. \]  
(2.38)

The Theorem follows from (1.8), (2.4), (2.7), (2.19), (2.31), (2.35), (2.37) and (2.38).

**PROPOSITION 2.2.** Suppose that

\[ \sup_{x \in \mathbb{R}} \int_{x}^{x+1} |V(y)| dy < \infty. \]  
(2.39)
Then, for any \( \epsilon > 0 \) there is a constant, \( K_\epsilon \), such that,

\[
\int_0^\infty |V(x)||\phi(x)|^2 \, dx \leq \epsilon \|\dot{\phi}\|^2_{L^2} + K_\epsilon \|\phi\|^2_{L^2}, \phi \in W_{1,2}.
\] (2.40)

**Proof:** If \( \phi \in W_{1,2} \), for any \( n = 0, 1, \cdots \), any \( x, y \in [n, n+1] \) and any \( \delta > 0 \), we have that,

\[
|\phi(x)|^2 - |\phi(y)|^2 = 2 \text{Re} \int_y^x \phi(z)\overline{\phi(z)} \, dz \leq \delta \int_n^{n+1} |\phi(z)|^2 \, dz + \frac{1}{\delta} \int_n^{n+1} |\phi(z)|^2 \, dz.
\] (2.41)

By the mean value theorem we can choose \( y \) such that,

\[
|\phi(y)|^2 = \int_n^{n+1} |\phi(z)|^2 \, dz,
\]

and it follows that,

\[
|\phi(x)|^2 \leq \delta \int_n^{n+1} |\phi(z)|^2 \, dz + \left(1 + \frac{1}{\delta}\right) \int_n^{n+1} |\phi(z)|^2 \, dz.
\] (2.42)

Let \( C \) be the finite quantity in the left-hand side of (2.39). Then,

\[
\int_n^{n+1} |V(x)||\phi(x)|^2 \, dx \leq C\delta \int_n^{n+1} |\phi(z)|^2 \, dz + \left(1 + \frac{1}{\delta}\right) \int_n^{n+1} |\phi(z)|^2 \, dz.
\] (2.43)

Taking \( \delta \) so small that \( \epsilon = \delta C \), and adding over \( n \) we obtain (2.40).

\[\blacksquare\]

If \( V \) satisfies (2.39) - in particular if \( V \in L^1 \) - it follows from (2.40) that the quadratic form, \( \mathcal{H}(\phi, \psi) := (\dot{\phi}, \dot{\psi})_{L^2} + (V\phi, \psi)_{L^2} \) with domain \( W_{1,2}^{(0)} \) is closed and bounded below. Furthermore, \( H \) is the associated self-adjoint operator (cf Theorem X.17 of [13]). Hence, the form domain of \( H \) is \( W_{1,2}^{(0)} \), and, \( D(\sqrt{H + N}) = W_{1,2}^{(0)} \), where \( N \) is so large that, \( H + N > 0 \). By (2.40) and as \( (H + N)^{-1/2} \) is bounded from \( L^2 \) into \( W_{1,2}^{(0)} \) the norm \( \|\sqrt{H + N}\phi\|_{L^2} \) is equivalent to the norm of \( W_{1,2}^{(0)} \). We use this equivalence below without further comment. We denote by \( P_{pp} \) the projector onto the pure point subspace of \( H \).

**THEOREM 2.3. (The \( L^2 - L^2 \) estimate).** Suppose that \( V \) satisfies (1.4) and let us denote, \( P_1 := I \), \( P_2 := P_{pp} \) and \( P_3 := P_c \). Then,

\[
\|e^{-itH} P_n\|_{\mathcal{B}(L^2, L^2)} \leq C, n = 1, 2, 3.
\] (2.44)

If furthermore, \( V \in L^1 \),

\[
\|e^{-itH} P_n\|_{\mathcal{B}(W_{1,2}^{(0)}, W_{1,2}^{(0)})} \leq C.
\] (2.45)
Proof: Equation (2.44) is just the unitarity of \( e^{-itH} \) in \( L^2 \). Equation (2.45) with \( n = 1 \) follows from 
\[
\sqrt{H + M} e^{-itH} = e^{-itH} \sqrt{H + M}.
\]
It is proven in [10] that \( H \) has a finite number of eigenvalues, \( E_1, E_2, \ldots, E_Q \), with \( E_j = (ik_j)^2 \), and where \( 0 < k_1 < k_2, \ldots, k_Q \) are the zeros of \( f(k, 0) \) i.e., \( f(k_j, 0) = 0, 1 \leq j \leq M \).
We use the notation, \( \hat{f}_j(x) := f(ik_j, x)/\|f(ik_j, \cdot)\|_{L^2} \). Then, the projector onto the pure point subspace of \( H, P_{pp} \), is given by,
\[
P_{pp} \phi = \sum_{j=1}^{Q} \hat{f}_j(x) (\phi, \hat{f}_j).
\]
(2.46)
As \( \hat{f}_j \in W_{1,2}^{(0)} \), we have that \( P_{pp} \in B(L^2, W_{1,2}^{(0)}) \). Hence, (2.45) with \( n = 2 \) holds because \( P_{pp} \) commutes with \( e^{-itH} \). The case \( n = 3 \) follows since \( \hat{P} = I - P_{pp} \).

Proof of Theorem 1.1: Equation (1.11) follows from (2.11), (2.44) with \( n = 3 \) and interpolation [13]. By (2.12), (2.45) with \( n = 3 \) and approximating \( \phi \in W_{1,1}^{(0)} \) by a sequence \( \phi_n \in W_{1,1}^{(0)} \cap W_{1,2}^{(0)} \) we prove that
\[
\|e^{-itH} P_{c}\|_{B(W_{1,1}^{(0)}, W_{1,\infty}^{(0)})} \leq C \frac{1}{\sqrt{|t|}}.
\]
(2.47)
By (2.45) with \( n = 3 \), (2.47) and interpolation, (1.12) holds.

The \( L^p - L^{\dot{p}} \) estimate implies that \( e^{-itH} P_{c} \) has smoothing effects that are expressed by its action between certain function spaces. Let us denote,
\[
L^{p,r} := L^r(I, L^p), \quad 1 \leq r, p \leq \infty, \quad I := [0, T], T > 0.
\]
(2.48)
In the case \( T = \infty \) we take \( I := [0, \infty) \). We find it convenient to represent the pair \( p, r \) by the point \( P := (1/p, 1/r) \) in the square, \([0, 1] \times [0, 1]\). Let us denote, \( L(P) := L^{p,r} \). Let \( s \) be the closed segment connecting \( B := (1/2, 0) \) and \( C := (0, 1/4) \). The equation for \( s \) is: \( 1/p + 2/r = 1/2, 0 \leq 1/p \leq 1/2 \). For any \( P = (1/p, 1/r) \in s \) the dual point \( \dot{P} \) is defined as \( \dot{P} := (1/\dot{p}, 1/\dot{r}) \) where, \( 1/p + 1/\dot{p} = 1, 1/r + 1/\dot{r} = 1 \). \( \dot{P} \) is on the dual segment, \( \dot{s} \), connecting \( \dot{B} = (1/2, 1) \) to \( \dot{C} = (1, 3/4) \). Let us define the following linear operators,
\[
(\Gamma \phi)(t) := e^{-itH} P_{c} \phi, t \in I,
\]
(2.49)
and
\[(Gf)(t) := \int_0^t e^{-i(t-\tau)H} P_c f(\tau) d\tau, \quad t \in I.\] (2.50)

**PROPOSITION 2.4. (Strichartz’s estimate)** Suppose that (1.4) holds. Then \(\Gamma\) is a bounded operator from \(L^2\) into every \(L(P), P \in s\) and \(\Gamma^*\) is bounded from every \(L(Q), Q \in \dot{s}\) into \(L^2\) with operator norm independent of \(T\). Moreover, \(G\) is bounded from any \(L(Q), Q \in \dot{s}\) to any \(L(P), P \in s\) with operator norm independent of \(T\).

Proof: the proposition follows from (1.11) and Theorem 1.2 of [9].

Note that in Proposition 2.4 we can replace \(L(B)\) by the space, \(\overline{L}(B)\), of the bounded and continuous functions from \(I\) into \(L^2\).

**References**

[1] R.A. Adams, ”Sobolev Spaces”, Academic Press, New York, 1970.

[2] J. Bourgain, ”Global Solutions of Nonlinear Schrödinger Equations”, Colloquium Publications 46, Amer. Math. Soc., Providence, RI, 1999.

[3] K. Chadan and P.C. Sabatier, ”Inverse Problems in Quantum Scattering Theory. Second Edition”, Springer, Berlin,1989.

[4] J. Ginibre, ”Introduction aux Équations de Schrödinger nonlinéares”, Onze Editions, Paris, 1998.

[5] E. Hille and R.S. Phillips, ”Functional Analysis and Semigroups”, Colloquium Publications XXI, Amer. Math. Soc., Providence, 1957.

[6] A. Jensen and K. Yajima, A remark on \(L^p\)-boundedness of wave operators for two dimensional Schrödinger operators, Comm. Math. Phys. 225 (2002), 633-637.

[7] J.L. Journé, A. Soffer and C.D. Sogge, Decay estimates for Schrödinger operators, Comm. Pure Appl. Math. 44 (1991), 573-604.
8. T. Kato, Nonlinear Schrödinger equations, Lecture Notes in Physics 345, (1989), 218-263, Springer, Berlin.

9. M. Keel, and T. Tao, Endpoint Strichartz estimates, Amer. J. Math. 120 (1998), 955-980.

10. V.A. Marchenko, "Sturm-Liouville Operators and Applications", Birkhäuser, Basel, 1986.

11. R.G. Newton, "Scattering Theory of Waves and Particles. Second Edition", Springer, Berlin, 1982.

12. C.-A. Pillet and C.E. Wayne, Invariant manifolds for a class of dispersive, hamiltonian, partial differential equations, J. Differential Equations 141 (1997), 310-326.

13. M. Reed and B. Simon, "Methods of Modern Mathematical Physics II Fourier Analysis, Self-Adjointness", Academic Press, New York, 1975.

14. W.A. Strauss, "Nonlinear Wave Equations", CBMS-RCNM 73, Amer. Math. Soc., Providence, R.I., 1989.

15. R.S. Strichartz, Restriction of Fourier transform to quadratic surfaces and decay of solutions of wave equations, Duke Math. J. 44 (1977), 705-774.

16. A. Soffer and M.I. Weinstein, Multichannel nonlinear scattering for nonintegrable equations II. The case of anisotropic potentials and data, J. Differential Equations 98 (1992), 376-390.

17. R. Weder, Inverse scattering for the nonlinear Schrödinger equation, Comm. Partial Differential Equations 22 (1997), 2089-2103.

18. R. Weder, $L^p - L^q$ estimates for the Schrödinger equation on the line and inverse scattering for the nonlinear Schrödinger equation with a potential, J. Funct. Anal. 170 (2000), 37-68.

19. R. Weder, Center manifold for nonintegrable nonlinear Schrödinger equations on the line, Comm. Math. Phys. 215 (2000), 343-356.

20. R. Weder, Inverse scattering for the non-linear Schrödinger equation: reconstruction of the potential and the nonlinearity, Math. Methods Appl. Sci. 24 (2001), 245-254.
[21] R. Weder, Inverse scattering for the nonlinear Schrödinger equation II. Reconstruction of the potential and the nonlinearity in the multidimensional case, Proc. Amer. Math. Soc. 129 (2001), 3637-3645.

[22] R. Weder, Scattering for the forced non-linear Schrödinger equation with a potential on the half-line, preprint, 2002.

[23] J. Weidmann, "Spectral Theory of Ordinary Differential Operators", Lecture Notes in Math. 1258, Springer, Berlin, 1987.

[24] K. Yajima, The $W^{k,p}$-continuity of wave operators for Schrödinger operators, J. Math. Soc. Japan 47 (1995), 551-581.

[25] K. Yajima, The $W^{k,p}$-continuity of the wave operators for Schrödinger operators. III. Even dimensional cases $m \geq 4$, J. Math. Sci. Univ. Tokyo 2 (1995), 311-346.

[26] K. Yajima, $L^p$-boundedness of wave operators for two dimensional Schrödinger operators, Commun. Math. Phys. 208 (1999), 125-152.