Many-Body Separability of Warm Qubits

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We analyze the separability of the joint state of a collection of two-level systems at finite temperature $T$. The fact that only separable states are found in the neighborhood of their thermal equilibrium state guarantees that unimpeded thermal decoherence will destroy any initially arranged entanglement in a finite time.

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Significant advances have occurred in the last decade in both the mathematical characterization of entanglement and in its applications, but it remains an important open issue how entanglement responds to the influence of environmental noise [1]. The interaction of a quantum system with its surroundings is a generic phenomenon because real physical systems cannot be isolated completely from external noise.

So far, it has been remarkably challenging to find anything generically provable about the process of decoherence (i.e., disentanglement). For example, if there are more than two bodies involved it is still not known how to calculate whether a general many-body mixed state is separable or not. Here we provide many-body results concerning separability of a physically important class of mixed states. These arise from consideration of interactions with thermal reservoirs and properties of thermal equilibrium. In many respects these are the least specialized considerations that can be imagined, as every real physical system is exposed to a thermal bath at some non-zero temperature.

We suppose only that each sub-system in a many-body network of qubits relaxes over time to its thermal equilibrium state. We then demonstrate that the many-qubit thermal equilibrium state is special, in the sense that it is embedded in a finite neighborhood of completely separable states. So far as we know, the specification of a many-body state with a finite neighborhood of guaranteed separability has been established up to the present time only for the identity state (see Zyczkowski, et al. [2]). Our result is easily extended to a wide range of other physically accessible states, essentially filling all of separable state space. This is indicated in Fig. 1.

Preliminary Considerations:– Our general result is established without speculating about approximate or “reasonable” measures of many-body entanglement, and is first proven abstractly. We then provide new results regarding wide-scale entanglement decay in a network of two-level systems interacting with independent thermal reservoirs.

![Fig. 1: Possible routes of dynamical evolution of states are suggested by the trajectories sketched in an imaginary non-metrical state space where the interior solid line is the boundary between separable (S) and non-separable (NS) states. The arrow heads simply indicate a direction of evolution. Our first result is that evolution to thermal steady state for $T > 0$ must end at a position on the interior of S, as shown for the route labelled $T > 0$.](image)

Our results are ultimately based on the fact that the thermal equilibrium state itself is separable and diagonal in the energy representation. We may represent it by the density matrix

$$\rho_0 = diag(p_1, p_2, ..., )$$

with $p_j > 0$, ($j = 1, 2, ..., N$), for any finite temperature $T > 0$.

Although our main result is equally valid for an arbitrary number of subsystems, it is convenient in the beginning to divide the many-body state space $\mathcal{H}$ into two parts and consider a bipartite system: $\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B$. We assume that both $\mathcal{H}^A$ and $\mathcal{H}^B$ are finite many-dimensional spaces. An operator $\rho$ acting on $\mathcal{H}$ describes a state if (1) $tr(\rho) = 1$, and (2) $tr(\rho P) \geq 0$ for any projection operator $P$. That is, $\rho$ is a positive Hermitian operator.

We will work from the fact that a bipartite density matrix $\rho$ is non-separable (entangled) if and only if there exists a Hermitian “entanglement witness” matrix $W$ with the two properties [2]:

$$tr(W \rho) < 0, \quad \text{and}$$

$$tr(W \sigma) \geq 0, \quad \text{for all separable states } \sigma.$$  

**Finite Neighborhood of Separability:** We assert that for a density matrix with the thermal equilibrium form...
there exists a neighborhood \( U \) of \( \rho_0 \) such that all states in that neighborhood are separable. The result of Zyczkowski, et al. can be regarded as the limiting case of our result as \( T \to \infty \). We will prove our assertion by showing that the opposite case must be false - i.e., the assumption that an entangled state exists in every neighborhood of \( \rho_0 \) leads to a contradiction.

First suppose that a neighborhood \( U_n \) of \( \rho_0 \) has an entangled state \( \rho \) in it. Then consider an infinite sequence of smaller neighborhoods, \( U_{n+1} \subset U_n \), all having at least one entangled state. This implies a sequence of entangled states \( \rho_n \) converging to \( \rho_0 \) as \( n \to \infty \).

According to (2) and (3), for each entangled state \( \rho_n \) there must exist a witness operator \( W_n \) that satisfies both

\[
tr(W_n \rho_n) < 0, \quad \text{and} \quad tr(W_n \sigma) \geq 0, \quad (4)
\]

for \( \rho_n \) and all separable states \( \sigma \). Therefore, for the sequence of states and witnesses, we get (as \( n \to \infty \)):

\[
tr(W_n \rho_n) \to tr(\bar{W} \rho_0),\; tr(W_n \sigma) \to tr(\bar{W} \sigma) \quad \text{for all separable states } \sigma .
\]

Moreover, by combining these expressions, and remembering the separable character of \( \rho_0 \), we have

\[
tr(\bar{W} \rho_0) \leq 0, \quad \text{and} \quad tr(\bar{W} \sigma) \geq 0. \quad (5)
\]

Then since \( \rho_0 \) is acceptable as a possible \( \sigma \), we conclude that satisfying both (3) and (4) requires

\[
tr(\bar{W} \rho_0) = p_1 \bar{W}_{11} + p_2 \bar{W}_{22} + \ldots = 0. \quad (7)
\]

At this point a useful observation is that the state represented by \( \sigma_1 = \text{diag}(1,0,\ldots,0)_{A} \otimes \text{diag}(1,0,\ldots,0)_{B} \) is obviously separable, and if \( \bar{W} \) is an entanglement witness, then \( tr(\bar{W} \sigma_1) \geq 0 \) requires \( \bar{W}_{11} \geq 0 \), and by extension \( \bar{W}_{jj} \geq 0 \) for all \( j \). Thus, since we are concentrating on the case in which the \( p_j \)s in (1) are all greater than zero, we conclude from (3) that all the diagonal elements \( \bar{W}_{jj} \) must vanish. Therefore, we have \( tr\bar{W} = tr(\bar{W} I) = 0 \), where \( I \) is the identity matrix. Since \( \bar{W} \) is not a zero matrix, there exists a product projection \( P \otimes Q \) such that \( tr(\bar{W} P \otimes Q) \neq 0 \) (see, e.g., [4]). By adding and subtracting \( I \) and using \( tr(\bar{W} I) = 0 \) again, this is sufficient to show that

\[
tr[\bar{W}(P \otimes Q)] = -tr[\bar{W}(I - P \otimes Q)] \neq 0. \quad (8)
\]

Both \( P \otimes Q \) and \( (I - P \otimes Q) \) are (at least proportional) to separable states, so (8) cannot hold for both. This is the needed contradiction that proves our main result.

An immediate implication of the existence of a separable neighborhood for a system consisting of \( M \) qubits, each coupled to a local thermal heat bath, is that unimpeded thermal decoherence must destroy the entanglement of every initial state in a finite time, an effect we discussed for zero-temperature two-atom spontaneous emission in a previous note ([8], referred to below as YE for short). This has been labelled ESD for early-stage decoherence or “entanglement sudden death”, indicating that in order to reach \( \rho_0 \) asymptotically the system must enter the separable neighborhood of \( \rho_0 \) after only a finite time.

\textbf{M-Qubit Systems Under Thermal Noise.—} The dynamics of pure thermal decoherence is completely determined by the reduced density matrix of the M-qubit system, obtained by tracing over the other qubits and the thermal reservoir’s variables. In the familiar Born-Markov approximation, when each qubit is in contact with a broadband harmonic reservoir at temperature \( T \), one finds a compact Lindblad master equation (e.g., see [3]) for \( M \) qubits (\( \hbar = 1 \)):

\[
\frac{d}{dt} \rho = -i[H_{\text{sys}}, \rho] + \mathcal{L}(\rho), \quad (9)
\]

where \( H_{\text{sys}} = \sum_{i=1}^{M} \frac{1}{2} \omega_i |i\rangle \langle i| \) and the Liouvillian superoperator is

\[
\mathcal{L}(\rho) = \sum_{i=1}^{M} \sum_{j=1}^{4} \left( C_{ij}^\dagger \rho C_{ij} - \frac{1}{2} \rho C_{ij} C_{ij}^\dagger - \frac{1}{2} C_{ij}^\dagger C_{ij} \rho \right)
\]

and the Lindblad operators are

\[
c_{11} = \sqrt{(n+1)\Gamma_{-}}, \quad c_{2} = \sqrt{n\Gamma_{+}}, \quad c_{3} = \sqrt{(n+1)\Gamma_{-}}, \quad c_{4} = \sqrt{n\Gamma_{+}}. \quad (10)
\]

The general solutions of the master equation (9) can be expressed compactly by the Kraus operators \( K_j \):

\[
\rho(t) = \sum_{i_1,..,i_M=1}^{4} K_{i_1} \otimes \ldots \otimes K_{i_M} \rho(0) [K_{i_1} \otimes \ldots \otimes K_{i_M}]^\dagger, \quad (10)
\]

where the elementary Kraus operators \( K_j \) with \( \sum_{j=1}^{4} K_j^\dagger K_j = I \) are the same for each qubit, given equal temperatures for all the reservoirs, and are explicitly given by

\[
K_1 = \sqrt{\frac{n+1}{2n+1}} \begin{pmatrix} 0 & \gamma(t) & 0 \\ \gamma(t) & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (11)
\]

\[
K_2 = \sqrt{\frac{n+1}{2n+1}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (12)
\]

\[
K_3 = \sqrt{\frac{n}{2n+1}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad (13)
\]

\[
K_4 = \sqrt{\frac{n}{2n+1}} \begin{pmatrix} 0 & \omega(t) & 0 \\ \omega(t) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (14)
\]

where the time-dependent Kraus matrix elements are \( \gamma(t) = \exp[-\frac{1}{2} \Gamma (2n+1)] \), \( \omega(t) = \sqrt{1 - \gamma^2(t)} \). The operators \( K_1, K_2 \) provide the transitions from the excited state \( |+\rangle \) to the ground state \(-\rangle \) caused by stimulated and spontaneous emission whereas the operators \( K_3, K_4 \) account for absorptive transitions from the ground state \(-\rangle \) to the excited state \(+\rangle \).

For the \( M \)-qubit system described by the master equation above, the steady state will be given by (11). Let \( \rho(t) \) denote the state of \( M \) qubits at \( t \). Then \( \rho(t) \to \rho_0 \) as \( t \to \infty \). Our generic result implies that a finite ESD time, \( t_{\text{esd}} \), exists for an arbitrary initial state. A simple expression for \( t_{\text{esd}} \) is given in YE for two-atom spontaneous emission.
Pair Entanglement in Thermal States:—For a two-qubit subsystem, our generic results established above can be realized in a more concrete way by explicitly solving the time evolution. Bipartite dynamics under thermal noise for $T > 0$ has been treated previously in considering various aspects of entanglement, e.g., generation, fragility, influence of squeezed reservoirs, Brownian particle diffusion, and universality of ESD. We now show explicitly that when each qubit relaxes to its equilibrium state asymptotically, the entanglement between any pair of qubits will always terminate in a finite time irrespective of the initial states of the qubits.

For any qubit pair, the steady state will be the diagonal matrix with $N = 4$:

$$\rho_{st} = \text{diag}(p_1, p_2, p_3, p_4),$$  \hspace{1cm} (15)

in the standard basis $|+\rangle, |+\rangle, |-\rangle, |-\rangle$, where the thermal probabilities are $p_1 = \bar{n}^2/(2\bar{n} + 1)^2$, $p_2 = \bar{n}^2/(2\bar{n} + 1)^2$, and $p_4 = (\bar{n} + 1)^2/(2\bar{n} + 1)^2$.

A standard measure of entanglement for our two-qubit system is Wooters' concurrence, denoted $C$. By construction, the concurrence varies from $C = 0$ for a separable state to $C = 1$ for a maximally entangled state.

For the two qubits $A$ and $B$ we have:

$$C^{AB}(\rho) = \max \{0, \Lambda(\rho)\},$$ \hspace{1cm} (16)

$$\Lambda(\rho) = \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4},$$  \hspace{1cm} (17)

and the $\lambda_i$ are the eigenvalues in decreasing order of the matrix $\zeta_\rho \equiv \rho(\sigma_y^A \otimes \sigma_y^B)\rho^* (\sigma_y^A \otimes \sigma_y^B)$. Here $\rho^*$ denotes the complex conjugate of $\rho$ in the standard basis given above and $\sigma_y^{A,B}$ are the Pauli matrices expressed in the same basis.

Formulated in this way, a necessary and sufficient condition for $\rho$ to be separable (zero entanglement) is $\Lambda(\rho) < 0$. We have noted previously that the negative values of $\Lambda$ possess quantum state information not contained in the concurrence $C(\rho)$. Both $\Lambda = 0$ and $\Lambda < 0$ signal that the two-qubit density matrix $\rho$ is separable, and when $\Lambda(\rho) < 0$, we call the state $\rho$ a super-separable state. The distinction for quantum state trajectories is shown in Fig. 4 where some trajectories terminate inside the S zone ($\Lambda < 0$) and some terminate exactly at the edge of the S zone ($\Lambda = 0$). For the thermal steady state $\rho_{st}$, straightforward calculations yield $\Lambda(\rho_{st}) = -2\sqrt{p_2 p_3} = -2\bar{n}(\bar{n} + 1)/(2\bar{n} + 1)^2 \leq 0$, so the steady state is separable for any reservoir temperature including zero, but super-separable for all $T > 0$.

In order for the system to reach the zone of super-separable states, $\Lambda$ must go from positive to negative values. Since $\Lambda$ is a real number, this means it must cross the value $\Lambda = 0$, and this suggests a different graphical representation of trajectories with metrical elements that can assist interpretation, as in Fig. 2. Whenever the long-time steady state of a two-qubit system is super-separable, the system must experience ESD. This means that it must become separable after only a finite time, as in the lower two curves in the figure, where the dots identify two locations for $t_{\text{esd}}$. This possibility was apparently first noticed in several different physical contexts independently (see Refs. [13, 18, 19, 20]). A two-qubit ESD time was evaluated in YE, and ESD was first reported experimentally by the Davidovich group and also recently by the Klimov group.

In some cases of interest the ESD times $t_{\text{esd}}$ can be determined explicitly. It is instructive to follow entanglement evolution for the two Bell states $|\Psi_\pm\rangle = 1/(\sqrt{2})(|+\rangle \pm |-\rangle)$. From (10), the evolving Bell density matrix at $t$ is given by

$$\rho(t) = \begin{bmatrix} a(t) & 0 & 0 & 0 \\ 0 & b(t) & z(t) & 0 \\ 0 & z(t) & c(t) & 0 \\ 0 & 0 & 0 & d(t) \end{bmatrix},$$  \hspace{1cm} (18)

where the time-dependent matrix elements are given by the following:

$$a(t) = p_2 \omega_1^2 + p_1 \gamma^2 \omega_2, b(t) = c(t) = p_1 \gamma^2 + p_2 \gamma^2 + p_3 (\gamma^4 + 1 + \omega_4),$$

$$d(t) = p_1 \omega_2^2 + p_3 \gamma^2 \omega_2, z(t) = \pm \frac{\gamma}{2}.\$$

Then it is straightforward to compute $\Lambda(\rho(t)) = 2|z(t)| - 2\sqrt{a(t)d(t)}$. Thus, we conclude that when

$$t \geq t_{\text{esd}} = \frac{1}{\Gamma} \ln \left[ \frac{1 + 2\sqrt{p_2 p_3}}{2\sqrt{p_1 p_2}} \right],$$  \hspace{1cm} (19)

the initial Bell states $|\Psi_\pm\rangle$ become completely disentangled. As suggested by the two solid curves in Fig. 2, the higher the temperature the shorter the disentanglement decay time $t_{\text{esd}}$. The upper solid curve is consistent with YE, showing no Bell disentanglement in a finite time when the environments are at zero temperature.

Conclusion:—The central element of this note is our demonstration of a finite neighborhood of separable states surrounding the thermal equilibrium state, but the proof could equally well have been given for any state in a finite dimensional space having diagonal form and positive definite elements. In fact continuously many other
states of this type are also physically realizable, in some cases easily. Any completely incoherent partially excited state would suit, and an obvious example is the state associated in spin resonance with finite but “negative” temperature or in laser physics with a positive partial inversion.

All of these also have finite neighborhoods of separability and all such neighborhoods are clearly linked to each other. It is an open and interesting question whether the finite number $M$ of qubits under consideration, and so the existence of a finite separable neighborhood is independent of $M$. The result is topological, and does not conflict either with the remark of Zyczkowski, et al., [2] that the volume of separable states decreases with system size, or the proof by Eisert, et al., [23] that non-separable states are dense in continuously infinite Hilbert spaces.

Not surprisingly, ESD and other forms of dissipative entanglement dynamics and their control are of current interest, with examinations reported of many different dynamical processes and realizations of qubits, including coupled mechanical oscillators [18], qubits in a spin chain and coupled to an Ising chain [24, 25], multi-cavity QED [26, 27], spin ensembles coupled via lossy photonic channels [28], qubit-qutrit combination [29], two-qubit decoherence dynamics [30, 31, 32, 33, 34] and multiple noises [35], to name a few.

To summarize, the main purpose of this note is to extend the study of entanglement in thermal and thermal-like environments to general multipartite systems with arbitrary initial states. We have achieved this goal by exploring topological properties of the state space of a many-body system. As an illustration, we have demonstrated the necessary thermal decoherence of two-qubit systems. The suppression of thermal decoherence and treatment of interacting qubits will be considered in future publications.

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