We give a geometric derivation of Schottky’s equation in genus four for the period matrices of Riemann surfaces among all period matrices. The equation arises naturally from the singularity theory of the Gauss map on the theta divisor, and thus generalizes for any genus $g \geq 4$ to a certain ideal of Siegel modular forms vanishing on period matrices of Riemann surfaces. This ideal is generated by modular forms associated to the invariants of cubic forms in $g-1$ variables which vanish on the Fermat cubic.

For each $g$-dimensional principally polarized abelian variety $(A, \Theta)$ together with a marked odd theta characteristic $\xi$, consider the theta function $\vartheta[\xi](z, \Omega)$ and its Taylor expansion with respect to $z$ about the origin,

$$\vartheta[\xi](z, \Omega) = \ell(z, \Omega) + m(z, \Omega) + \ldots,$$

where $\ell = \ell_\xi$ and $m = m_\xi$ are, respectively, linear and cubic forms in $z$. Classical transformation formulas show that all of the ideals $(\ell)$, $(\ell, m)$, $\ldots$, transform by the same automorphy factor under the action on $\Omega$ of a suitable finite index subgroup of $\text{Sp}(2g, \mathbb{Z})$. The main idea—suggested by the singularity theory of the Gauss map—is simply to use the ideal $(\ell, m) \subset \mathbb{C}[z]$ as an invariant of $(A, \Theta, \xi)$. When the linear form $\ell$ is not identically zero, we can consider the restriction $m$ of the cubic form $m$ to the hyperplane $\ell(z) = 0$. This cubic form $m$ is the key object studied in this paper. Our main observation is that if $(A, \Theta)$ is the Jacobian of a generic curve of genus $g$, then for any choice of odd theta characteristic, $m(z)$ is a Fermat cubic, the sum of $g-1$ cubes.

Geometrically, the marking of an odd theta characteristic $\xi$ of a principally polarized abelian variety $(A, \Theta)$ determines a symmetric theta divisor $\Theta[\xi]$ on which the origin of $A$ has odd multiplicity. When $\Theta[\xi]$ is smooth at 0, its local tangent hyperplane section at 0 is singular and of odd multiplicity. In particular, if the cubic form $m(z)$ is not identically zero, then it defines the projectivized tangent cone $V$ of this singularity. The projectivized tangent space $E = \mathbb{P}T_0(\Theta[\xi])$ is a hyperplane in the canonical projective space $\mathbb{P}^{g-1} = \mathbb{P}T_0(A)$. Thus we associate to $(A, \Theta, \xi)$ the cubic hypersurface $V \subset E \cong \mathbb{P}^{g-2}$, and for a generic Jacobian, $V$ is projectively equivalent to the standard Fermat cubic.

The main result of this paper is a precise way to convert invariants vanishing on the Fermat cubic into modular forms vanishing on Jacobians. If $\varphi$ is any homogeneous polynomial invariant of cubic forms in $g-1$ variables, then there is a natural globalization $G_{\varphi}$ so that the equation $\varphi(m) = 0$ is expressed by the vanishing of a Siegel modular form $G_{\varphi}(m)$. Therefore, if the invariant $\varphi$ vanishes on the Fermat
cubic, then $G_\varphi(\overline{m})$ vanishes on Jacobians. The Fermat ideal is the ideal generated by the Siegel modular forms resulting from all invariants vanishing on the Fermat cubic.

We prove in genus $g = 4$ that the zero set of the Fermat ideal (which is principal in this case) is exactly the closure of the locus of period matrices of genus four Riemann surfaces, and we deduce that the generator is Schottky’s equation (up to a constant multiple). (References for genus four Schottky are [I4], [Fr1], [E].) The Fermat ideal consists entirely of cusp forms, and hence for $g \geq 5$ it cannot coincide with the ideal of Schottky-Jung equations. (For Schottky-Jung, cf. [F1], [F2], [vG], [Do3], [C], [M2], [R-F].) In general we obtain two loci in $A_g$, the level 1 moduli space for principally polarized abelian varieties, corresponding to the vanishing of the Fermat ideal for some odd theta characteristic or for all odd theta characteristics. These loci are algebraic subsets of $A_g$ containing the Jacobians; we call them the big and small Fermat loci, by analogy with the Schottky-Jung theory [Do1].

It is well known that interesting invariants of a principally polarized abelian variety $(A, \Theta)$ can be obtained from the common zeros of various terms in the Taylor expansions in $z$ of theta functions; cf. [ACGH, Cor. p. 232], [K1]. As far as we know, the particular combination $\ell, m$ (defined by an odd theta characteristic $\xi$) used here has not been investigated before (apart from our previous work [AMSV2]). In fact, the condition that $\overline{m}$ be a Fermat cubic can be viewed as a necessary algebraic condition for a theta divisor to be of translation type (cf. Prop. (2.3.4), [B, p. 105 (c)], [I5, p. 167], [L], [M2]).

In §1 we define the various moduli spaces of abelian varieties that we will use, and we recall the transformation laws for the theta function $\vartheta(\xi)(z, \Omega)$. In particular we recall the space $A_g^{(4,8)}$, its fundamental positive line bundle $P$ (which is well known in the theory of theta constants) and the dual line bundle $N$, with $P^2 \cong \Lambda = \det H$, where $H$ is the Hodge vector bundle. The transformation law for $\vartheta(\xi)$ shows that the automorphy factor for the line bundle $P$ governs every term, modulo all the previous terms, in the Taylor expansion of $\vartheta(\xi)$ with respect to $z$. Thus, for each odd theta characteristic $\xi$, the linear term of $\vartheta(\xi)$ defines a homomorphism $\ell_\xi : N \to H$.

In §2 we introduce an open set $U_\xi$ over which the quotient sheaf $H/\ell_\xi(N)$ is a rank $g - 1$ vector bundle $E_\xi^*$, so that the cubic term of $\vartheta(\xi)$ defines a homomorphism $\overline{m}_\xi : N \to S^3E_\xi^*$ to a vector bundle of cubic forms in $g - 1$ variables, and we check that the complement of $U_\xi$ is Zariski-closed of codimension at least 2. We observe that, for every odd theta characteristic $\xi$ of the Jacobian of a generic curve of genus $g$, the cubic form $\overline{m}_\xi(z)$ is a Fermat cubic (extending the genus three and four cases treated in [MSV1], [AMSV2]).

In §3 we show that a degree $d$ polynomial invariant $\varphi$ of cubic forms in $g - 1$ variables yields, for every rank $g - 1$ vector bundle $E$, a degree $d$ polynomial mapping $G_\varphi : S^3E^* \to (\Lambda^{g-1}E^*)^w$, where $3d = (g - 1)w$. In §4 we conclude that each degree $d$ polynomial mapping $G_\varphi(\overline{m}_\xi) : N \to (\Lambda^{g-1}E^*)^w$ can be represented (in a canonical way) by a Siegel modular form of a certain weight for the group $\Gamma(4,8)$. Combining this general construction with the property that for Jacobians the cubic form $\overline{m}_\xi(z)$ is a Fermat cubic, we obtain a simple way to construct Siegel modular forms vanishing on period matrices of Jacobians. We recover the Schottky equation in genus $g = 4$ and we show that these modular forms also vanish on products for every genus $g \geq 4$. In §5 we indicate some open questions.
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1. Theta transformation laws

1.1 The moduli space $\mathcal{A}_g(4,8)$.

Let $\mathcal{A}_g$ be the moduli space of $g$-dimensional principally polarized abelian varieties $(A, \Theta)$. Here the theta divisor $\Theta$ is specified only up to translation in $A$; a symmetric representative of $\Theta$ is specified only up to translation by a point of order 2. The moduli space of primary interest in this paper is $\tilde{\mathcal{A}}_g$, the isomorphism classes of $(A, \Theta, \xi)$, $g$-dimensional principally polarized abelian varieties together with an odd theta characteristic $\xi$. Recall that the $2^{2g}$ theta characteristics of a principally polarized abelian variety correspond precisely to the symmetric representatives of the theta divisor, and that the parity of a characteristic $\xi$ is the same as the parity of $\text{mult}_0(\Theta[\xi])$, the multiplicity of the corresponding symmetric theta divisor at the origin of the abelian variety. Thus, a point $(A, \Theta, \xi)$ of $\tilde{\mathcal{A}}_g$ is simply a $g$-dimensional abelian variety $A$ with a distinguished symmetric theta divisor $\Theta[\xi]$ on which the origin has odd multiplicity.

Faced with the usual technical problems (related to coarse moduli and fixed points for the action of the modular group $\text{Sp}(2g, \mathbb{Z})$ on the Siegel upper half space $\mathcal{H}_g$), we will do most of our calculations on a finite cover $\mathcal{A}'_g \to \tilde{\mathcal{A}}_g$, where $\mathcal{A}'_g = \mathcal{H}_g/\Gamma'$ for a suitable subgroup $\Gamma'$ of $\text{Sp}(2g, \mathbb{Z})$. In fact, among the spaces that will serve our purposes, there is a distinguished one in the theory of theta transformation laws. This particularly useful space is $\mathcal{A}_g^{(4,8)} = \mathcal{H}_g/\Gamma(4,8)$, introduced by Igusa [I1]. We abbreviate $\mathcal{A}_g^{(4,8)}$ to $\mathcal{A}'_g$ and use $(A, \Theta, \xi')$ to denote a point of $\mathcal{A}'_g$ with image $(A, \Theta, \xi)$ in $\tilde{\mathcal{A}}_g$. We will not need the precise moduli interpretation of $\xi'$, a level $(4,8)$ structure (cf. [M-F, app. B to Ch. 7, pp. 191-199, esp. p. 195]); it would be sufficient to work on the level 8 moduli space $\mathcal{A}_g^{(8)} = \mathcal{H}_g/\Gamma(8)$, or with any principal congruence level $n$ such that $8|n$. We will use primarily the following properties: $\mathcal{A}'_g$ is smooth, maps finitely onto $\tilde{\mathcal{A}}_g$, and carries a family of abelian varieties together with a family of theta divisors corresponding to the marked odd theta characteristic $\xi$. (It is convenient that also the transformation law for the theta function $\vartheta[\xi]$ becomes quite simple, as we will see below.)

In summary, the relevant moduli spaces and maps are:

$$\mathcal{H}_g \to \mathcal{A}'_g \to \tilde{\mathcal{A}}_g \to \mathcal{A}_g.$$ 

We must choose a map $\pi_\xi : \mathcal{A}'_g \to \tilde{\mathcal{A}}_g$ factoring the Galois covering $\mathcal{A}'_g \to \mathcal{A}_g$ through the canonical (but non-Galois) covering $\tilde{\mathcal{A}}_g \to \mathcal{A}_g$. The level structure $\xi'$ which is specified on each element $(A, \Theta, \xi') \in \mathcal{A}'_g$ determines a “framing” of the set of theta characteristics of $(A, \Theta)$, so it makes sense to speak of fixing (universally) an odd theta characteristic $\xi$ on every $(A, \Theta, \xi') \in \mathcal{A}'_g$, and such a choice defines a map $\pi_\xi$ as above. To be more precise, we first pass from $\mathcal{A}'_g$ to the level 2 moduli space $\mathcal{A}_g^{(2)}$, which is Galois over $\mathcal{A}_g$ with group $\text{Sp}(2g, \mathbb{Z})/\Gamma(2) \cong \text{Sp}(2g, \mathbb{Z}/2)$. A level 2 structure of a principally polarized abelian variety $(A, \Theta)$ corresponds to a symplectic isomorphism $(\mathbb{Z}/2)^{2g} \to H_1(A, \mathbb{Z}/2)$, where $(\mathbb{Z}/2)^{2g}$ has the standard...
\[ Z/2 \text{-symplectic form, and } H_1(A, Z/2) \text{ has the mod 2 reduction of the alternating Riemann form on } H_1(A, Z) \text{ determined by the polarization. Thus a level 2 structure allows the specification of an odd theta characteristic of } (A, \Theta) \text{ simply by a suitable } Z/2 \text{-quadratic form on } (Z/2)^2 \text{ (cf. [M1, \S2], [I3, pp. 209-210]).} \]

1.2 Automorphy factors.

We recall the basic theory of automorphy factors (cf. [G, \S5.1, esp. (5.1.1) p. 198, p. 199], [L-B, App. B, pp. 412-414]). Suppose that a group \( \Gamma \) acts properly discontinuously (as in [G, (D) p. 203], [M3, (c), p. 180]) on a topological space \( X \). Suppose that a group \( \Gamma \) acts properly discontinuously on \( X \), \( (\gamma, x) \mapsto \gamma \cdot x \), and let \( p : X \to Y = X/\Gamma \) be the quotient map. Let \( E \) be a rank \( r \) complex vector bundle on \( Y \) and \( \tilde{E} = p^*(E) \) the pullback to \( X \). Then \( \Gamma \) acts in a natural way on the vector bundle \( \tilde{E} \) and its space of sections \( H^0(X, \tilde{E}) \) and \( H^0(Y, E) \cong (H^0(X, \tilde{E}))^\Gamma \); i.e., sections of \( E \) over \( Y \) correspond to invariant sections of \( \tilde{E} \) over \( X \). Now suppose that \( \tilde{E} \) is trivial on \( X \) and let \( \varphi : \tilde{E} \to X \times \mathbb{C}^r \) be a trivialization. Then via the trivialization there is an action of \( \Gamma \) on \( X \times \mathbb{C}^r, \gamma \cdot (x, w) = \varphi(\gamma \cdot (\varphi^{-1}(x, w))) = (x', w') \), where \( x' = \gamma \cdot x \) and \( w' = \mu(\gamma, x)w \). The mapping

\[ \mu : \Gamma \times X \to \text{GL}_r(\mathbb{C}) \]

is called the automorphy factor for \( E \) (associated to the trivialization \( \varphi \) of \( \tilde{E} \) on \( X \)) and satisfies the cocycle condition

\[ \mu(\gamma' \gamma, x) = \mu(\gamma', \gamma \cdot x) \mu(\gamma, x) \]

for all \( \gamma, \gamma' \in \Gamma \), \( x \in X \). Under the trivialization \( \varphi \), a section \( s \) of \( \tilde{E} \) over \( X \) corresponds to a \( \mathbb{C}^r \)-valued function \( f \) on \( X \) by the formula

\[ \varphi(s(x)) = (x, f(x)). \]

The action of \( \Gamma \) on sections of \( \tilde{E} \) can then be written

\[ (\gamma \cdot f)(x) = \mu(\gamma, \gamma^{-1} \cdot x) f(\gamma^{-1} \cdot x). \]

It follows that \( f \) is invariant under this action of \( \Gamma \) if and only if for every \( \gamma \in \Gamma \) and \( x \in X \), \( f(\gamma \cdot x) = \mu(\gamma, x) f(x) \).

Thus if \( E \) is a rank \( r \) complex vector bundle on \( Y = X/\Gamma \), then a trivialization of the pullback \( \tilde{E} \) on \( X \) induces a canonical isomorphism

\[ H^0(Y, E) \cong \{ f : X \to \mathbb{C}^r \mid f(\gamma \cdot x) = \mu(\gamma, x) f(x) \text{ for all } \gamma \in \Gamma, x \in X \}, \]

where \( \mu : \Gamma \times X \to \text{GL}_r(\mathbb{C}) \) is the automorphy factor for \( E \). By the same reasoning for each open subset \( V \) of \( Y \), we get a description of the sheaf of sections of \( E \) over \( Y \).

We will work with various vector bundles on \( \mathcal{A}_g' \); their natural descriptions are by automorphy factors over \( \mathcal{H}_g \), so we can carry out the constructions in terms of sheaves, i.e., sections of the bundles over variable open subsets of \( \mathcal{A}_g' \). Thus we work mostly in the holomorphic category, with the analytic structure sheaf \( \mathcal{O} \) and analytic \( \mathcal{O} \)-modules.
1.3 The Hodge vector bundle.

We describe the modular group action on Siegel upper half-space $H_\mathfrak{g}$ and on the product $\mathbb{C}^g \times H_\mathfrak{g}$. The space $H_\mathfrak{g}$ of all $g \times g$ period matrices is the set of all $g \times g$ complex matrices $\Omega$ such that $^t\Omega = \Omega$ and $\text{Im}(\Omega) > 0$. For $\gamma \in \text{Sp}(2g, \mathbb{Z})$, we write

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with $g \times g$ blocks $A, B, C$ and $D$ satisfying

$$^t \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Then for $\Omega \in H_\mathfrak{g}$, let

$$\gamma \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1},$$

and for $(z, \Omega) \in \mathbb{C}^g \times H_\mathfrak{g}$, let

$$\gamma \cdot (z, \Omega) = (^t(C\Omega + D)^{-1}z, (A\Omega + B)(C\Omega + D)^{-1})$$

(cf. [I2, p. 226], [M2, pp. 71-72], [M3, p. 177], [R-F, pp. 86-87]).

For the moduli-theoretic meaning of these actions of $\text{Sp}(2g, \mathbb{Z})$, see [M3, pp. 171-177, 184-185]; $H_\mathfrak{g}$ can be regarded as a fine moduli space for principally polarized abelian varieties $(A, \Theta)$ with “level $\infty$ structure,” i.e., a symplectic isomorphism $\alpha : \mathbb{Z}^{2g} \to H_1(A, \mathbb{Z})$. Then $\gamma \in \text{Sp}(2g, \mathbb{Z}) \subset \text{GL}_{2g}(\mathbb{Z})$ acts by $\gamma \cdot (A, \Theta, \alpha) = (A, \Theta, \alpha^\gamma \gamma^{-1})$, and one obtains the standard formula above for $\gamma \cdot \Omega$ after an automorphism of $\text{Sp}(2g, \mathbb{Z})$ [M3, p. 174].

Over $H_\mathfrak{g}$ there is the universal family of abelian varieties $\{A_\Omega = \mathbb{C}^g/(\mathbb{Z}^g + \Omega \mathbb{Z}^g)\}$, and the action of $\text{Sp}(2g, \mathbb{Z})$ on $H_\mathfrak{g}$ lifts canonically to the total space of this family. The family of abelian varieties defines a vector bundle $\tilde{V} \to H_\mathfrak{g}$ of tangent spaces $\{T_0 A_\Omega\}$, with an induced action of $\text{Sp}(2g, \mathbb{Z})$. This vector bundle is trivial; i.e., $\tilde{V} \cong \mathbb{C}^g \times H_\mathfrak{g}$ over $H_\mathfrak{g}$, and the action of $\text{Sp}(2g, \mathbb{Z})$ on the bundle corresponds to the automorphy factor $\mu(\gamma, \Omega) = ^t(C\Omega + D)^{-1}$; that is, $\gamma \cdot (z, \Omega) = (\mu(\gamma, \Omega)z, \gamma \cdot \Omega)$, for $\gamma \in \text{Sp}(2g, \mathbb{Z})$ and $(z, \Omega) \in \mathbb{C}^g \times H_\mathfrak{g}$.

Let $\mathcal{V} = (\mathbb{C}^g \times H_\mathfrak{g})/\Gamma'$, a rank $g$ vector bundle on $A'_\mathfrak{g} = H_\mathfrak{g}/\Gamma'$. The fibre of $\mathcal{V}$ at $y = (A, \Theta, \xi) \in A'_\mathfrak{g}$ is $V_y = T_0 A$. Thus $V_y^* = T_0^* A$, so $\mathcal{V} = H^*$, the dual of the Hodge vector bundle $H$ on $A'_\mathfrak{g}$. For $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2g, \mathbb{Z})$ and $\Omega \in H_\mathfrak{g}$, let $\eta(\gamma, \Omega) = C\Omega + D$. Then $\eta$ is the automorphy factor for $H$ and $\text{det} \eta(\gamma, \Omega) = \text{det}(C\Omega + D)$ is the automorphy factor for the Hodge line bundle $\Lambda = \text{det} H$.

1.4 Transformation laws.

We consider transformation laws for theta functions $\vartheta[\xi](z, \Omega)$ and regard the relevant automorphy factor over $H_\mathfrak{g}$ as the description of a fundamental line bundle on the moduli space $A'_\mathfrak{g} = A_{g}^{(4,8)}$. First we recall the classical transformation theory for theta functions $\vartheta[\xi](z, \Omega)$ with characteristics. The characteristic $[\xi]$ is specified by two $g$-entry column vectors $a, b$ of zeros and ones. As a function of $z \in \mathbb{C}^g$ for fixed $\Omega \in H_\mathfrak{g}$ and $\xi \in \{0,1\}^{2g}$,

$$\vartheta[\xi](z, \Omega) = \sum_{n \in \mathbb{Z}^g} \exp\left(\pi i \left( ^t(n + \frac{a}{2})\Omega(n + \frac{b}{2}) + 2^t(n + \frac{a}{2})(z + \frac{b}{2}) \right) \right),$$
where \( \exp(w) = e^w \) for \( w \in \mathbb{C} \). The theta function is holomorphic and quasi-periodic with respect to the lattice \( \mathbb{Z}^g + \Omega \mathbb{Z}^g \subset \mathbb{C}^g \) and it is either even or odd according to the parity of \( \xi \). By definition the parity of \( \xi \) is the parity of \( q(\xi) = ^t a \cdot b \), and \( \vartheta[\xi](z, \Omega) = (-1)^{(\xi, \Omega)} \vartheta[\xi](z, \Omega) \). In particular, if \( \Theta[\xi] \subset A = \mathbb{C}^g / (\mathbb{Z}^g + \Omega \mathbb{Z}^g) \) denotes the zero divisor of \( \vartheta[\xi](z, \Omega) \), then \( \Theta[\xi] \) is a symmetric theta divisor, and \( \text{mult}_0(\Theta[\xi]) \equiv q(\xi) \) (mod 2).

The more difficult part of the theory describes the transformation of \( \vartheta[\xi](z, \Omega) \) under the natural action of \( \text{Sp}(2g, \mathbb{Z}) \) on \( \{0, 1\}^{2g} \times \mathbb{C}^g \times \mathcal{H}_g \). Our notation for the action of \( \gamma \in \text{Sp}(2g, \mathbb{Z}) \) on \( (\xi, z, \Omega) \) is \( (\xi, z, \Omega) = \gamma \cdot (\xi, z, \Omega) \); the formula for \( [\xi] = \gamma \cdot [\xi] \) is given in [I1, p. 226]. The transformation law then relates \( \vartheta[\hat{\xi}](\hat{\z}, \hat{\Omega}) \) to \( \vartheta[\xi](z, \Omega) \); i.e., there exists a function \( \sigma = \sigma(\gamma, \xi, z, \Omega) \) such that

\[
\vartheta[\hat{\xi}](\hat{\z}, \hat{\Omega}) = \sigma(\gamma, \xi, z, \Omega) \vartheta[\xi](z, \Omega),
\]

\[
\sigma(\gamma, \xi, z, \Omega) = e^{2\pi i \phi_\xi(\gamma)} \det(C\Omega + D)^{1/2} \cdot e^{-\pi i Q_{\gamma,\Omega}(z)}.
\]

(There may be a choice of sign in \( \det(C\Omega + D)^{1/2} \) and in \( \kappa(\gamma) \) separately, but the product \( \kappa(\gamma) \) \( \det(C\Omega + D)^{1/2} \) is a well-defined function of \( (\gamma, \Omega) \). For the properties of \( \kappa(\gamma) \) and the formula for \( \phi_\xi(\gamma) \), see [I1], [I2]; the formula for \( Q_{\gamma,\Omega}(z) \) will be given below.)

For any theta characteristic \( \xi \), define

\[
\Gamma[\xi] = \{ \gamma \in \text{Sp}(2g, \mathbb{Z}) \mid \gamma \cdot [\xi] = [\xi] \}.
\]

Then in particular, putting \( \sigma_\xi(\gamma, z, \Omega) = \sigma(\gamma, \xi, z, \Omega) \), there exists an automorphy factor \( \sigma_\xi : \Gamma[\xi] \times (\mathbb{C}^g \times \mathcal{H}_g) \to \mathbb{C}^* \) for the action of \( \Gamma[\xi] \) on \( \mathbb{C}^g \times \mathcal{H}_g \) such that \( \vartheta[\xi](\hat{\z}, \hat{\Omega}) = \sigma_\xi(\gamma, z, \Omega) \vartheta[\xi](z, \Omega) \) for every \( \gamma \in \Gamma[\xi] \) and \( (z, \Omega) \in \mathbb{C}^g \times \mathcal{H}_g \).

Note that \( \Gamma(2) \subset \Gamma[\xi] \) for each theta characteristic \( \xi \). In fact, if \( q_\xi \) denotes the \( \mathbb{Z}/2\)-quadratic form corresponding to the theta characteristic \( \xi \) and \( O(q_\xi) \subset \text{Sp}(2g, \mathbb{Z}/2) \) denotes the orthogonal group defined by \( q_\xi \), then \( \Gamma[\xi] \) is the preimage of \( O(q_\xi) \) under the canonical surjection \( \text{Sp}(2g, \mathbb{Z}) \to \text{Sp}(2g, \mathbb{Z}/2) \). Since \( \text{Sp}(2g, \mathbb{Z}) \) acts transitively on the set of odd theta characteristics [I3, Cor. p. 213], the subgroups \( \Gamma[\xi], \xi \) odd, are all conjugate in \( \text{Sp}(2g, \mathbb{Z}) \), and for any one of them, \( \mathcal{H}_g / \Gamma[\xi] \simeq \hat{A}_g \).

To render the transformation law for \( \vartheta[\xi] \) as simple as possible, we pass to the important subgroup

\[
\Gamma' = \Gamma(4, 8) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2g, \mathbb{Z}) \mid \begin{pmatrix} A & B \\ C & D \end{pmatrix} \equiv I_{2g} \pmod{4}, \text{ and diagonal entries of } A^t B \text{ and } C^t D \equiv 0 \pmod{8} \right\}
\]

(from [I1, p. 220]); it lies between the congruence subgroups \( \Gamma(8) \) and \( \Gamma(4) \) and is normal in \( \text{Sp}(2g, \mathbb{Z}) \) [I1, p. 222]. Thus we have the containments

\[
\Gamma(8) \subset \Gamma(4, 8) \subset \Gamma(4) \subset \Gamma(2) \subset \Gamma[\xi] \subset \text{Sp}(2g, \mathbb{Z}).
\]

1.4.1 The transformation law for \( \vartheta[\xi](z, \Omega) \) with respect to \( \Gamma' \). For \( \gamma \in \Gamma' \) and \( (z, \Omega) \in \mathbb{C}^g \times \mathcal{H}_g \), if we write

\[
(\hat{\z}, \hat{\Omega}) = \gamma \cdot (z, \Omega) = (C\Omega + D)^{-1}z, (A\Omega + B)(C\Omega + D)^{-1},
\]
then

\[ \vartheta[\xi](\hat{z}, \Omega) = \det((C\Omega + D)^{1/2}e^{\pi i Q_{\gamma,\Omega}(z)}\vartheta[\xi](z, \Omega)), \]

where \( Q_{\gamma,\Omega}(z) = \tau z(C\Omega + D)^{-1}Cz \) [I1, p. 227].

That is, for \( \gamma \in \Gamma' \) there is a canonical choice of square root of \( \det(C\Omega + D) \) so that \( \sigma(\gamma, z, \Omega) = \det((C\Omega + D)^{1/2}e^{\pi i Q_{\gamma,\Omega}(z)}) \) is the transformation factor, which is now independent of \( \xi \). We let \( \rho : \Gamma' \times H_g \to \mathbb{C}^* \) denote the fundamental automorphy factor \( \rho(\gamma, \Omega) = \det((C\Omega + D)^{1/2}). \)

For us the main point will be following. For any fixed odd characteristic \( \xi \), consider the Taylor expansion of the theta function \( \vartheta[\xi](z, \Omega) \) with respect to \( z \) about the origin:

\[ \vartheta[\xi](z, \Omega) = \ell(z, \Omega) + m(z, \Omega) + \cdots, \]

where \( \ell = \ell_\xi, m = m_\xi, \ldots \), are respectively the linear, cubic and higher odd order terms in the variable \( z \). Then the transformation law (1.4.1) shows that all of the ideals \( (\ell), (\ell, m), \ldots \), transform by the same automorphy factor \( \rho \). In other words, by general principles on jet prolongation of sections of bundles, the automorphy factor \( \rho \) governs every homogeneous Taylor term of \( \vartheta[\xi] \) modulo all the previous terms.

Let \( P \) denote the fundamental (positive) line bundle on \( \mathcal{A}'_g \) with automorphy factor \( \rho \). As we see from the transformation law, the classical theta nulls \( \vartheta[\xi](0, \Omega) \) for even \( \xi \), define sections of \( P \) (and an embedding theorem is proved in [I1], [I2], [I3]; cf. [M2, p. 73]), but we will employ this line bundle in connection with the odd theta functions. Clearly, \( P^2 \cong \det H = \Lambda \), since \( \rho^2 = \det(C\Omega + D) \) and \( \eta(\gamma, \Omega) = C\Omega + D \) is the automorphy factor for \( H \). Let \( N = P^* \) be the fundamental (negative) line bundle on \( \mathcal{A}'_g \) with automorphy factor \( \nu(\gamma, \Omega) = \det((C\Omega + D)^{-1/2}) \) with respect to \( \Gamma' \).

1.4.2 Remark. Apparently, the line bundle \( N \) corresponds to Kempf’s \( Q(A) \) in [K2, p. 70], and his Cor. 8.7 [K2, p. 72] reflects the progressively simpler transformation laws for \( \vartheta \).

1.5 The homomorphism \( \ell_\xi \).

Now we consider the linear term (in \( z \)) of an odd theta function \( \vartheta[\xi](z, \Omega) \) at the origin of \( \mathbb{C}^g \), as \( \Omega \) varies over \( H_g \). We will show that this linear term can be viewed as a homomorphism \( N \to H \) over \( \mathcal{A}'_g \), with the geometric interpretation that for each \( y = (A, \Theta, \xi') \in \mathcal{A}'_g \), the image of the fibre \( N_y \) in \( H_y = T^*_{\Theta}A \) is the conormal line at 0 to \( \Theta[\xi] \) in \( A \) when the theta divisor \( \Theta[\xi] \) is nonsingular at 0 (and the image is 0 when \( \Theta[\xi] \) is singular at 0).

1.5.1 Proposition. For each odd theta characteristic \( \xi \), \( \ell(z, \Omega) = \nabla_z |_0(\vartheta[\xi](z, \Omega)) \) defines a section of \( P \otimes H \) over \( \mathcal{A}'_g \), or equivalently, a homomorphism

\[ \ell_\xi : N \to H \]

of bundles on \( \mathcal{A}'_g \).

Proof. Here \( \nabla_z |_0(\vartheta[\xi](z, \Omega)) \) is the column vector, with entries \( a_j(\Omega) = \frac{\partial}{\partial z_j} |_0(\vartheta[\xi](z, \Omega)), \)

\[ 1 \leq j \leq g, \]

which represents the linear form \( \sum a_j(\Omega)z_j \), the differential of \( \vartheta[\xi](z, \Omega) \).
By the chain rule, where

\[ \eta_h \]

Thus we have shown that

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\[ \mu \]

\[ \gamma \]

\[ t \]

\[ \mathcal{H}_g \]

\[ P \otimes H \]

\[ (\vartheta[\xi])(z, \gamma \cdot \Omega) = \det((C\Omega + D)^{1/2}e^{\pi i \hat{Q}_{\gamma,\Omega}(z)}) \vartheta[\xi](\ell((C\Omega + D)z, \Omega)), \]

where \( \hat{Q}_{\gamma,\Omega}(z) = C_{\gamma,\Omega}(t((C\Omega + D)z) = t_z C \ell((C\Omega + D)z) \) is a homogeneous quadratic function of \( z \).

Now we compute \( \nabla_z |_0(\vartheta[\xi](z, \gamma \cdot \Omega)) \):

\[
\frac{\partial}{\partial z_j}(\vartheta[\xi](z, \gamma \cdot \Omega)) = \frac{\partial}{\partial z_j} \left( \det((C\Omega + D)^{1/2}e^{\pi i \hat{Q}_{\gamma,\Omega}(z)}) \vartheta[\xi](\ell((C\Omega + D)z, \Omega)) \right) \\
= \det((C\Omega + D)^{1/2} \frac{\partial}{\partial z_j}((1 + h_{\gamma,\Omega}(z))) \vartheta[\xi](\ell((C\Omega + D)z, \Omega))) + \\
(1 + h_{\gamma,\Omega}(z)) \frac{\partial}{\partial z_j}(\vartheta[\xi](\ell((C\Omega + D)z, \Omega))) \right),
\]

where \( h_{\gamma,\Omega} \) has order 2 and higher in \( z \). Thus

\[
\nabla_z |_0(\vartheta[\xi](z, \gamma \cdot \Omega)) = \det((C\Omega + D)^{1/2}\nabla_z |_0(\vartheta[\xi](\ell((C\Omega + D)z, \Omega))).
\]

By the chain rule,

\[
\nabla_z |_0 \left( \vartheta[\xi](\ell((C\Omega + D)z, \Omega)) \right) = \eta^{(1)}(\gamma, \Omega) \left( \nabla_z |_0(\vartheta[\xi](z, \Omega)) \right) \\
= (C\Omega + D)\nabla_z |_0(\vartheta[\xi](z, \Omega)),
\]

where \( \eta^{(1)}(\gamma, \Omega) \) is the appropriate cotangent map. Therefore,

\[
\nabla_z |_0(\vartheta[\xi](z, \gamma \cdot \Omega)) = \det((C\Omega + D)^{1/2}(C\Omega + D)\nabla_z |_0(\vartheta[\xi](z, \Omega))).
\]

Thus we have shown that \( \ell(z, \Omega) = \nabla_z |_0(\vartheta[\xi](z, \Omega)) \) satisfies the transformation rule

\[
\ell(z, \gamma \cdot \Omega) = \det((C\Omega + D)^{1/2}(C\Omega + D)\ell(z, \Omega),
\]

for all \( \gamma \in \Gamma' \) and \( \Omega \in \mathcal{H}_g \), where \( \det((C\Omega + D)^{1/2}(C\Omega + D) \) is the automorphy factor for \( P \otimes H \). Q.E.D.

For \( \gamma \in \text{Sp}(2g, \mathbb{Z}) \), we may regard the map \( L = L(\gamma, \Omega) : \mathbb{C}^g \rightarrow \mathbb{C}^g, z \mapsto t((C\Omega + D)^{-1}z) \), as an isomorphism from the universal cover of the abelian variety \( A_{\Omega} = \mathbb{C}^g/\Gamma(\mathbb{Z}^g + \Omega \mathbb{Z}^g) \) to that of the abelian variety \( A_{\hat{\Omega}} = \mathbb{C}^g/\Gamma(\mathbb{Z}^g + \hat{\Omega} \mathbb{Z}^g) \), where \( \hat{\Omega} = \gamma \cdot \hat{\Omega}. \) Then the derivative \( \eta^{(1)}(\gamma, \Omega) \) of \( L(\gamma, \Omega) \) at 0 is the map \( z \mapsto \mu(\gamma, \Omega)z \) given by the automorphy factor \( \mu(\gamma, \Omega) = t((C\Omega + D)^{-1} \), and the cotangent map \( \eta^{(1)}(\gamma, \Omega) \) of \( L(\gamma, \Omega) \) at 0 is the dual map \( z \mapsto \eta(\gamma, \Omega)z \) given by the automorphy factor \( \eta(\gamma, \Omega) = C\Omega + D. \)

Note that, although the homomorphism \( \ell_\xi : N \rightarrow H \) is injective as a map on sheaves of sections, it induces the 0-map on the fibres on \( N \) over points of \( A'_g \), for which the theta divisor \( \Theta[\xi] \) is singular at the origin.
2. The cubic hypersurface associated to an odd theta characteristic

2.1 The open set $U_\xi$.

Recall that there exists, in addition to a universal family $\mathfrak{A} \to \mathcal{A}_g'$ of abelian varieties, a universal family $\Theta \to \mathcal{A}_g'$ of theta divisors $\{\Theta_y = \Theta[\xi] \subset A_g \mid y = (A, \Theta, \xi') \in \mathcal{A}_g'\}$. In particular, for each $(A, \Theta, \xi') \in \mathcal{A}_g'$, the marked symmetric theta divisor $\Theta[\xi] \subset A$ has odd multiplicity at the origin $0$ of $A$. Since we want to look at the tangent hyperplanes of these theta divisors at the origin, we will pass to the open subset of $\mathcal{A}_g'$ over which the theta divisors have multiplicity exactly one at the origin. Let

$$U_\xi = \{y \in \mathcal{A}_g' \mid \Theta_y \text{ is nonsingular at } 0\}.$$  

Then $U_\xi \subset \mathcal{A}_g'$ is an open subset and on $U = U_\xi$ we have a rank $g - 1$ vector bundle $E = E_\xi$ whose fibre at each point $y \in U$ is $E_y = T_y \Theta_y$. Note that $E$ is naturally a subbundle of the restriction to $U$ of the rank $g$ vector bundle $V = H^*$ whose fibre at $y$ is $T_0(A)$. Recall that on $U$ we have the line subbundle $L_\xi = \ell_\xi(N)$ of $H = V^*$ defined by the conormal lines to the theta divisors $\Theta[\xi]$ at the origin of the abelian varieties. Thus, by construction, $E$ is the rank $g - 1$ subbundle $L_\xi^\perp$ of $V$ over the open set $U$ and we have the exact sequence

$$0 \to L_\xi \to H \to E_\xi^* \to 0$$

of vector bundles on $U$.

The next step is to use the cubic term (in $z$) of the theta function $\vartheta[\xi](z, \Omega)$ to construct cubic forms on the fibres of the vector bundle $E$ over $U$. It will be quite important in our applications that in passing from $\mathcal{A}_g'$ to $U$ we have not lost information about divisors of $\mathcal{A}_g'$.

2.1.1 Lemma. For each $g \geq 4$, $Z = \mathcal{A}_g' - U$ is a closed algebraic subset of $\mathcal{A}_g'$ of codimension $\geq 2$.

Proof. First we check that $Z \subset \mathcal{A}_g'$ is algebraic. Now $\Theta \to \mathcal{A}_g'$ is a proper morphism of algebraic varieties. The critical locus $\Sigma \subset \Theta$ is an algebraic subset of $\Theta$ with the property that, for each $y = (A, \Theta, \xi') \in \mathcal{A}_g'$, the fibre of $\Sigma$ over $y$ is $\Sigma_y = \text{Sing}(\Theta_y)$, the singular locus of the theta divisor $\Theta_y \subset A_g$. Also the zero section $E$ of the family of abelian varieties $\mathfrak{A} \to \mathcal{A}_g'$ is an algebraic subset of $\Theta$. Therefore, the intersection $E \cap \Sigma = \{(y, 0) \mid 0 \text{ is singular on } \Theta_y\} \subset \Theta$ is an algebraic subset of $E$, and hence under the isomorphism $E \to \mathcal{A}_g'$, the image $Z = \{y \mid 0 \text{ is singular on } \Theta_y\}$ of $E \cap \Sigma$ is algebraic in $\mathcal{A}_g'$.

Now we check that the codimension of $Z$ in $\mathcal{A}_g'$ is at least 2. Denote by $\mathfrak{g}_g'$ the preimage in $\mathcal{A}_g'$ of the Jacobian locus $\mathfrak{g}_g \subset \mathcal{A}_g$. Using the fact that for any $g \geq 4$ the Picard number of $\mathcal{A}_g'$ is equal to one, it suffices to check that $Z \cap \mathfrak{g}_g'$ has codimension $\geq 2$ in $\mathfrak{g}_g'$. (Cf. [S-V, Cor. (0.7), p. 350],) (For $g \geq 4$ and for any arithmetic subgroup $\Gamma$ of $\text{Sp}(2g, \mathbb{R}), \text{Pic}(\mathfrak{H}_g/\Gamma) \otimes \mathbb{Q} \cong \mathbb{Q}$, by the result of [Bo] that $H^2(\Gamma, \mathbb{Q}) \cong \mathbb{Q}$ (cf. [M4, pp. 354-355], [S-V, Part 0, §C]). For our purposes, it suffices to work in $\mathcal{A}_g$ and use the result of [Fr1] that for $g \geq 3$, $H^2(\text{Sp}(2g, \mathbb{Z}), \mathbb{Z}) \cong \mathbb{Z}$.) Now we pass to level 1 moduli, i.e., from $Z \cap \mathfrak{g}_g' \subset \mathcal{A}_g'$ to its image $\mathfrak{g}_g \subset \mathcal{A}_g$. 


Consider the locus of Jacobians which have at least one odd theta characteristic $\xi$ such that the corresponding theta divisor $\Theta[\xi]$ is singular at 0; this type of locus has been studied by Harris and Teixidor. In the moduli space of curves $M_g$, the locus in question is denoted $M^2_\xi$; it is the locus of genus $g$ curves having an odd theta characteristic with $h^0 \geq 3$. From [T, Thm. 2.13, p. 109] we see that every component of $M^2_\xi$ has codimension at least 3 in $M_g$. Q.E.D.

2.1.2 Remark. Here are two possible alternative arguments, both based on the use of the codimension one boundary of the Igusa-Mumford compactification $\bar{A}_g$ of $A_g$ (as in [M4], [D, Lemme 2.1, p. 698]).

(1) R. Salvati Manni (oral communication) has calculated that the closure of $Z$ in $\bar{A}_g$ has the expected codimension $g$ on the boundary; in particular, there is no codimension 1 component of $Z$.

(2) By definition, $Z \subset N_0 = \{(A, \Theta, \xi') \in A'_g | \Theta \text{ is singular}\}$. Debarre [D] has shown that in the level 1 moduli space, $N_0$ has exactly two irreducible components, both of codimension one, and over the generic point of each of these two irreducible components, the only singularities of the theta divisor are double points. On the other hand, over every point $y \in Z$, there is a singularity of $\Theta_y$ that is not a double point, and hence $Z$ cannot contain an irreducible component of $N_0$. Therefore, $\text{codim}(Z) > \text{codim}(N_0) = 1$.

2.1.3 Corollary. For any holomorphic line bundle $\mathcal{M}$ on $A'_g$, $H^0(A'_g, \mathcal{M}) \to H^0(U, \mathcal{M})$ is an isomorphism.

Proof. It suffices to establish the isomorphism of sections locally on $A'_g$; since the line bundle $\mathcal{M}$ is locally trivial on $A'_g$, it is enough to know that for all sufficiently small open sets $U \subset A'_g$, $H^0(U, \mathcal{O}) \to H^0(U - Z, \mathcal{O})$ is an isomorphism. Since $Z$ has complex codimension $\geq 2$, $\mathcal{O}(U) \to \mathcal{O}(U - Z)$ is an isomorphism by Hartogs’ Theorem [G-R, p. 132]. Q.E.D.

2.1.4 Remark. Rather than simply removing $Z$ to define the bundle $\mathcal{E}$, one can utilize a natural Nash blow-up construction that yields a (proper, birational) modification $\sigma : Y \to A'_g$ such that the bundle inclusion $\mathcal{L} = \mathcal{L}_\xi \subset H$ on $U$ extends to an inclusion $\hat{\mathcal{L}} \subset \sigma^*(H)$ on $Y$, and hence the vector bundle $\mathcal{E}$ on $U \subset A'_g$ extends to a vector subbundle $\hat{\mathcal{E}} = \hat{\mathcal{L}}^\perp \subset \sigma^*(\mathcal{V})$ on all of $Y$.

2.2 The homomorphism $m_\xi$.

We now consider the cubic Taylor term of an odd theta function $\vartheta[\xi](z, \Omega)$, modulo the linear term.

2.2.1 Proposition. For each odd theta characteristic $\xi$,

$$\frac{1}{3!} \nabla_z^3|_0 (\vartheta[\xi](z, \Omega)) \quad \text{(mod } \nabla_z|_0 (\vartheta[\xi](z, \Omega)))$$

defines a section of $P \otimes S^3\mathcal{E}_\xi^*$ over $U \subset A'_g$, or, equivalently, a homomorphism

$$m_\xi : N \to S^3\mathcal{E}_\xi^*$$

of bundles on $\mathcal{U}$.

Proof. Here $\nabla_z^3|_0 (\vartheta[\xi](z, \Omega))$ is the cubic form $\sum_{1 \leq j, k, l \leq g} a_{jkl}(\Omega)z_jz_kz_l$, where $a_{jkl}(\Omega) = \frac{\partial^3}{\partial z_j \partial z_k \partial z_l}|_0 (\vartheta[\xi](z, \Omega))$. Let $p : \mathcal{H}_g \to \mathcal{H}_g/\Gamma' = A'_g$ be the quotient map, and assume
In other words, where \(\eta\) represents a section of \(P \otimes S^3E^*_\xi\) over \(U\). Consider the Taylor expansion \(\vartheta[\xi](z, \Omega) = \ell(z, \Omega) + m(z, \Omega) + \ldots\) We apply the transformation law (1.4.1), expanding \(e^{\pi i Q_{\gamma, \Omega}(z)} = 1 + q_{\gamma, \Omega}(z) + \ldots\), where \(q_{\gamma, \Omega}(z) = \pi i Q_{\gamma, \Omega}(z)\) is homogeneous quadratic in \(z\). Then

\[
\ell(\hat{z}, \hat{\Omega}) + m(\hat{z}, \hat{\Omega}) + \ldots = \det(C\Omega + D)^{1/2}(1 + q_{\gamma, \Omega}(z) + \ldots 0)(\ell(z, \Omega) + m(z, \Omega) + \ldots)
\]

\[
= \det(C\Omega + D)^{1/2}(\ell(z, \Omega) + (m(z, \Omega) + q_{\gamma, \Omega}(z)\ell(z, \Omega)) + \ldots),
\]

so we obtain \(\ell(\hat{z}, \hat{\Omega}) = \det(C\Omega + D)^{1/2}\ell(z, \Omega)\) (as before) and

\[
m(\hat{z}, \hat{\Omega}) = \det(C\Omega + D)^{1/2}(m(z, \Omega) + q_{\gamma, \Omega}(z)\ell(z, \Omega)).
\]

Now we replace \(\hat{z}\) on the left by \(z\) and hence \(z\) on the right by \(\bar{z} = \iota(C\Omega + D)z\). Then we have

\[
m(z, \gamma \cdot \Omega) = \det(C\Omega + D)^{1/2}(m(\bar{z}, \Omega) + q_{\gamma, \Omega}(\bar{z})\ell(\bar{z}, \Omega)).
\]

In other words,

\[
m(z, \gamma \cdot \Omega) = \det(C\Omega + D)^{1/2}\eta^{(3)}(\gamma, \Omega)(m(z, \Omega) + q_{\gamma, \Omega}(z)\ell(z, \Omega)),
\]

where \(\eta^{(3)}(\gamma, \Omega)\) is the natural map on cubic forms on \(\mathbb{C}^g\) induced by the cotangent map \(\eta^{(1)}(\gamma, \Omega)\).

Now we consider the cubic form \(m(z, \Omega)\) modulo multiples of \(\ell(z, \Omega)\) (by quadratic forms). Equivalently, we restrict the cubic form to the hyperplane \(\{z \in \mathbb{C}^g \mid \ell(z, \Omega) = 0\}\). Note that this hyperplane is exactly the fibre \(E_\Omega \subset \mathcal{V}_\Omega = \mathbb{C}^g\) of \(p^*(\mathcal{E})\) at \(\Omega\). Thus, the resulting cubic form \(\overline{m}(z, \Omega)\) on the fibres of \(p^*(\mathcal{E})\) transforms as follows:

\[
\overline{m}(z, \gamma \cdot \Omega) = \det(C\Omega + D)^{1/2}\overline{\eta}^{(3)}(\gamma, \Omega)(\overline{m}(z, \Omega)),
\]

where \(\overline{\eta}^{(3)}(\gamma, \Omega)\) is the induced map on cubic forms mod \(\ell(z, \Omega)\). This proves that \(\overline{m}(z, \Omega)\) transforms as a section of \(P \otimes S^3E^*_\xi\). Q.E.D.

2.2.2 Remark. After the modification \(\sigma : \mathcal{Y} \to \mathcal{A}'_g\) indicated in Remark (2.1.4), the homomorphism \(\overline{m} : N \to S^3E^*\) over \(U \subset \mathcal{A}'_g\) extends to a homomorphism \(\hat{m} : \sigma^*(N) \to S^3E^*\) over all of \(\mathcal{Y}\).

Thus, for each point \(y = (A, \Theta, \xi') \in U \subset \mathcal{A}'_g\), if the cubic form \(\overline{m}_\xi\) is nonzero, then it defines a distinguished cubic hypersurface \(V\) in \(E_\xi = \mathbb{P}T_0(\Theta[\xi]) \cong \mathbb{P}^{g-2}\), and \(V \subset \mathbb{P}^{g-2}\) depends (up to projective equivalence) only on the image point \((A, \Theta, \xi) \in \mathcal{A}_g\). For \((A, \Theta, \xi) \in \mathcal{A}_g\), let \(V_\xi\) denote the cubic hypersurface of \(E_\xi = \mathbb{P}(\ell_\xi = 0)\) defined in \(\mathbb{P}^{g-1}\) by the ideal \((\ell_\xi, m_\xi)\) (provided that \(\ell_\xi, m_\xi\) is a regular sequence).
2.3 Jacobians.

Recall the correspondence between the theta characteristics of a curve and those of its Jacobian. A theta characteristic of a genus \( g \) curve \( C \) is a degree \( g - 1 \) line bundle \( \xi \) (up to isomorphism) such that \( \xi^\otimes 2 \cong \Omega_C \), the canonical bundle of \( C \). The parity of the theta characteristic \( \xi \) is the parity of \( h^0(\xi) \). The one-to-one correspondence (preserving parity) between the theta characteristics of a curve \( C \) and those of its Jacobian can be found in [ACGH, pp. 281-294], [C, pp. 140-143], [R-F, pp. 176-177], [M3, pp. 162-170]. Let \( \mathcal{M}_g \) be the moduli space of genus \( g \) curves over \( \mathbb{C} \) and let \( \mathcal{J}_g \) be its image in \( \mathcal{A}_g \). Thus \( \mathcal{M}_g \) is the set of isomorphism classes of smooth, connected, complete curves of genus \( g \) over \( \mathbb{C} \), and \( \mathcal{J}_g \) is the (Zariski-locally-closed) subset of \( \mathcal{A}_g \) consisting of isomorphism classes of principally polarized Jacobians of genus \( g \) curves.

An odd theta characteristic \( \xi \) of a genus \( g \) curve \( C \) is nondegenerate if the complete linear series \( |\xi| = \{D\} \) for a single degree \( g - 1 \) divisor \( D = p_1 + \cdots + p_{g-1} \) consisting of \( g - 1 \) distinct points \( p_1, \ldots, p_{g-1} \). Recall that an odd theta characteristic \( \xi \) is nonsingular if \( h^0(\xi) = 1 \); i.e., \( |\xi| = \{D\} \) for a single effective degree \( g - 1 \) divisor \( D \). (By the Riemann singularities theorem, \( h^0(\xi) = \mu_{s0}(\mathcal{O}[\xi]) \), so an odd theta characteristic \( \xi \) of \( C \) is nonsingular if and only if the origin 0 is a nonsingular point of \( \mathcal{O}[\xi] \).) Note that by geometric Riemann-Roch (cf. [ACGH, p. 12]), if \( D \) is any effective divisor of degree \( g - 1 \) then \( h^0(\mathcal{O}(D)) = 1 \) if and only if \( D \) spans a hyperplane of \( \mathbb{P}^{g-1} \).

2.3.1 Examples. (i) Let \( C \subset \mathbb{P}^2 \) be a nonsingular plane quartic (\( g = 3 \)) with a higher flex \( p \) (i.e., a weight 2 Weierstrass point). Then \( \xi = \mathcal{O}(2p) \) is a degenerate nonsingular odd theta characteristic. Indeed, \( |\xi| = |K - 2p| \) is cut out by the lines that are tangent to \( C \) at \( p \), and there is only one such line.

(ii) Let \( C \) be a hyperelliptic curve of genus \( g = 4k + 1 \), \( k \geq 1 \), and let \( \Gamma \subset \mathbb{P}^{4k} \) be its canonical image. Then \( \xi = \mathcal{O}(2k \cdot g_1^k) \) is a singular odd theta characteristic. For if \( D \) consists of the sum of the \( g_1^k \) divisors over \( 2k \) of the branch points on \( \Gamma \), then \( |K - \xi| = |K - D| \) is cut out by the hyperplanes through the \( 2k \) points in \( \mathbb{P}^{4k} \), so this linear system has projective dimension \( 2k \); hence \( h^0(\xi) = h^0(K - \xi) = 2k + 1 \).

(iii) In genus 4, all the odd theta characteristics of a hyperelliptic curve are nondegenerate. By choosing any 3 of the 10 Weierstrass points, we get 120 nondegenerate odd theta characteristics, so these are all the odd theta characteristics.

2.3.2 Proposition. For \( g \geq 3 \), let \( N\mathcal{M}_g = \{C \in \mathcal{M}_g \mid \text{every odd theta characteristic of } C \text{ is nondegenerate} \} \) and let \( N\mathcal{J}_g \) be the image of \( N\mathcal{M}_g \) in \( \mathcal{J}_g \).

(i) \( N\mathcal{M}_g \) is a nonempty Zariski-open subset of \( \mathcal{M}_g \), and hence \( N\mathcal{J}_g \) is Zariski-dense in \( \mathcal{J}_g \).

(ii) For each \( (J, \Theta) \in N\mathcal{J}_g \) and each odd theta characteristic \( \xi \) of \( (J, \Theta) \), let \( \Omega \in \mathcal{H}_g \) be a period matrix for \( (J, \Theta) \), and let \( \ell_\xi \) and \( m_\xi \) be the linear and cubic terms, respectively, of the theta function \( \vartheta[\xi](\cdot, \Omega) \). Let \( E_\xi \) be the hyperplane of canonical space \( \mathbb{P}^{g-1} = \mathbb{P}T_0(J) \) defined by the linear form \( \ell_\xi \), and let \( V_\xi \) be the cubic hypersurface of \( E_\xi \) defined in \( \mathbb{P}^{g-1} \) by the ideal \( (\ell_\xi, m_\xi) \). Then \( V_\xi \) is a Fermat cubic.

Proof of (i). First we prove that \( N\mathcal{M}_g \) is a Zariski-open subset of \( \mathcal{M}_g \). Let \( \{C\} \) be an irreducible family of genus \( g \) curves that dominates \( \mathcal{M}_g \) by a finite proper map. Let \( S = \{C \mid \text{some odd theta characteristic of } C \text{ is singular} \} \), and let \( D = \)
\{C \mid \text{some odd theta characteristic of } C \text{ is degenerate}\}. By upper semicontinuity of \( h^0 \), \( S \) is Zariski-closed, so it suffices to show that \( D - S \) is Zariski-closed in the complement of \( S \). To show this we pass to the family \( \{(C, \xi)\} \), where \( \xi \) is a nonsingular odd theta characteristic, so that \( |\xi| \) has a unique representative \( p_1 + \cdots + p_{g-1} \in C^{(g-1)} \), and use the fact that the discriminant is Zariski-closed in the total space of the family \( \{C^{(g-1)}\} \).

Next we show that \( N_M g \) is nonempty. (1) There exists a genus \( g \) curve with a nondegenerate odd theta characteristic. For example, a genus \( g \) hyperelliptic curve \( C \) has \( 2g+2 \) distinct Weierstrass points \( p_1, \ldots, p_{2g+2} \), and if \( D = p_1 + \cdots + p_{g-1} \), then \( \xi = \mathcal{O}(D) \) is a nondegenerate odd theta characteristic. (2) There exists a family of genus \( g \) curves over a connected, smooth base \( B \) such that \( B \) dominates \( M \) and the monodromy of the family is transitive on the set of odd theta characteristics. An example is the universal family of curves over the nonempty open subset of \( M \) of automorphism-free curves (cf. [ACGH, p. 294]). Now (1) and (2) imply that all the odd theta characteristics of a generic genus \( g \) curve are nondegenerate.

Finally, since \( N_M g \) is nonempty Zariski-open in \( M \) and \( M \) is irreducible, \( N_M g \) is Zariski-dense in \( M \), so its image \( N_\mathcal{J} g \) in \( \mathcal{J} g \) is Zariski-dense under the surjection \( M \to \mathcal{J} g \). Q.E.D.

2.3.3 Remark. That \( N_M g \) is nonempty is a special case of a general result of J. McKernan [Mc] on the structure of the loci of hyperplane sections of a generic canonical curve that have a given type (set of multiplicities); in our case the type is \( (2, \ldots, 2) \).

**Proof of (ii).** For \((J, \Theta) \in N_\mathcal{J} g \) and any odd theta characteristic \( \xi \) of \((J, \Theta)\), let \( C \) be a genus \( g \) curve with (polarized) Jacobian \( J(C) \cong (J, \Theta) \), such that \( \xi \) is a nondegenerate odd theta characteristic of \( C \). We will use \( C \) to determine the cubic hypersurface \( V_\xi \). Let \( D = p_1 + \cdots + p_{g-1} \) be the unique divisor of \( |\xi| \) on \( C \). The points \( p_1, \ldots, p_{g-1} \) are distinct and their images \( \varphi(p_1), \ldots, \varphi(p_{g-1}) \) under the canonical map \( \varphi : C \to \mathbb{P}^{g-1} \) span a (unique) hyperplane \( H \). In particular, the points \( \varphi(p_i), i = 1, \ldots, g-1 \), are linearly independent, and at each of them the hyperplane \( H \) is simply tangent to \( C \) (since locally \( H \) cuts \( 2p_i \) on \( C \)). Now one can just compute (following Andreotti) the Gauss map on the theta divisor \( \Theta[\xi] \) of \( J \) around the origin by parametrizing \( C \) around each of the points \( p_i \). In appropriate coordinates the Gauss map is the gradient of a function whose leading term cuts out \( V_\xi \) (see the proof of 2.3.4). Cf. [MSV1, p. 735 (ii)], [AMSV1, p. 21, 3.6(iv)], [AMSV2, tables 1, 2 (P_8)]. Q.E.D.

In fact, instead of just citing our previous calculations, we prove the following more general result. Let \( A \) be a \( g \)-dimensional abelian variety. A hypersurface germ \((M, 0) \subset A\) is of translation type if there exist \( g-1 \) germs \( \Gamma_1, \ldots, \Gamma_{g-1} \) of smooth curves through \( 0 \) such that the addition map \( \mu : \Gamma_1 \times \cdots \times \Gamma_{g-1} \to A \) has image \((M, 0)\). Such a germ is of nondegenerate translation type if the curves \( \Gamma_1, \ldots, \Gamma_{g-1} \) can be chosen so that their tangent lines at \( 0 \) are linearly independent.

The theta divisor \( \Theta[\xi] \subset J(C) \) is of nondegenerate translation type at \( 0 \) for all odd theta characteristics \( \xi \) of a generic curve \( C \) of genus \( g \geq 3 \). (For discussion of theta divisors’ being (singly or doubly) of translation type, cf. [L], [M2, pp. 81-85].)

**2.3.4 Proposition.** For \((A, \Theta, \xi) \in \tilde{A}_g, g \geq 4\), if the theta divisor \( \Theta[\xi] \subset A \) is of nondegenerate translation type at \( 0 \), then \( V_\xi \subset E_\xi \cong \mathbb{P}^{g-2} \) is a Fermat cubic
degree $\phi$ element $E$ show how to express a classical invariant globally, so that it applies to degree $E$ change of coordinates we may arrange that $\sigma(g) \in \mathbb{C}^g$ where $f$ parametrization of the $i$th curve germ (by a disc), $i = 1, \ldots, g - 1$. Then by a linear change of coordinates we may arrange that $\sigma_i(t_i)$ has $t_i$ in the $i$th coordinate position and higher order terms in the other coordinates. Let $f_i$ denote the last component function (i.e., the $g_i$th coordinate) of $\sigma$. Now if $F(z_1, \ldots, z_g) = 0$ is a local analytic equation for $\Theta[\xi]$, then we may uniquely write $F(z_1, \ldots, z_g) = z_g - f(z_1, \ldots, z_{g-1})$, where $f(0) = 0$ and $f$ has no linear term. Therefore, $f(z_1, \ldots, z_{g-1}) = f_1(t_1) + \cdots + f_{g-1}(t_{g-1}) + (\text{higher order terms in } z_1, \ldots, z_{g-1})$. (The Gauss map of $\Theta[\xi] \subset (\mathbb{C}^g, 0)$ has the local form $\gamma(z_1, \ldots, z_{g-1}) = \nabla f(z_1, \ldots, z_{g-1})$.) Since there exists an equation $F$ for $\Theta[\xi]$ at 0 that is odd, $f(z_1, \ldots, z_{g-1})$ is an odd function of $z_1, \ldots, z_{g-1}$, with no linear term. Therefore the Taylor expansion of $F$ at 0 is

$$F(z_1, \ldots, z_g) = z_g - (\lambda_1 z_1^3 + \cdots + \lambda_{g-1} z_{g-1}^3) + \ldots,$$

so $\overline{m} = -(\lambda_1 z_1^3 + \cdots + \lambda_{g-1} z_{g-1}^3)$. Q.E.D.

3. Classical invariant theory

3.1 Globalization of invariants.

We review the classical invariant theory of degree $k$ forms in $n$ variables and we show how to express a classical invariant globally, so that it applies to degree $k$ forms on a vector bundle. Let $E$ be an $n$-dimensional vector space over $\mathbb{C}$. Then the general linear group $GL(E)$ acts on $F = S^k E^*$, the vector space of degree $k$ forms (homogeneous polynomials) on $E$, and hence on $P = S^d F^*$, the forms of degree $d$ on $F$. (For $f \in F$, $x \in E$, $(g \cdot f)(x) = f(g^{-1} x)$; for $\varphi \in P$, $f \in F$, $(g \cdot \varphi)(f) = \varphi(g^{-1} \cdot f)$. A degree $d$ invariant of the degree $k$ forms on $E$ is an element $\varphi \in P$ such that $g \cdot \varphi = \varphi$ for all $g \in SL(E)$, the special linear group. It is easy to determine the action of a general element of $GL(E)$ on such $\varphi$. Namely, for $g \in GL(E)$, write $g = (cI)h$, where $c$ is a scalar, $I$ is the identity and $h \in SL(E)$; of course, $\det(g) = c^n$. Then $g \cdot \varphi = c^{kd} \varphi$. In particular, $GL(E)$ acts on the vector space $\mathbb{C}\varphi$, so there exists a character $\chi$ of $GL(E)$ such that, for all $g \in GL(E)$, $g \cdot \varphi = \chi(g) \varphi$. Any character $\chi$ of $GL(E)$ (as an algebraic group over $\mathbb{C}$) has the form $\chi(g) = \det(g)^w$ for some $w \in \mathbb{Z}$. Therefore, if the invariant $\varphi$ is nonzero, then there exists a unique integer $w$ such that, whenever $g = (cI)h$ as above, we have $c^{kd} = \det(g)^w$. It follows that $n|kd$. Thus, in order for nontrivial degree $d$ invariants of degree $k$ forms to exist, we must have $kd = nw$ for a nonnegative integer $w$ (cf. [G-Y, Ch. II, §31, p. 28, Ch. XII, §199, p. 246]).

Let $kd = nw$ for a nonnegative integer $w$ and let

$$P_{(w)} = \{ \varphi \in P \mid g \cdot \varphi = \det(g)^w \varphi \text{ for all } g \in GL(E) \}$$

be the set of relative invariants of weight $w$. To create absolute invariants from relative ones, we must eliminate the determinant factor. Thus, we let $W = \left(\Lambda^n E\right)^w$ and consider $P' = P \otimes W^*$ with the natural action of $GL(E)$. Now $P_{(w)} \otimes W^* \subset P'$ consists of elements that transform trivially under $GL(E)$, and we can regard an element $\varphi \in P'$ as a homogeneous polynomial of degree $d$ on $F = S^k E^*$ with values in $W^* = \left(\Lambda^n E^*\right)^w$. 
Now suppose that our $n$-dimensional vector space $E$ is $\mathbb{C}^n$ (i.e., that we have a distinguished basis for $E$). Then $\Lambda^n \mathbb{C}^n$ and hence $W = (\Lambda^n \mathbb{C}^n)^w$ are canonically trivial (as $\mathbb{C}$-vector spaces, not as $\text{GL}_n(\mathbb{C})$-modules). Let $R_d$ denote the space of degree $d$ invariants of degree $k$ forms in $n$ variables,

$$R_d = \{ \varphi \in S^d(\mathbb{C}^n)^\ast \mid g \cdot \varphi = \varphi \text{ for all } g \in \text{SL}_n(\mathbb{C}) \}.$$

### 3.1.1 Proposition. Let $kd = nw$ for a nonnegative integer $w$. For any rank $n$ complex vector bundle $\mathcal{E}$, there exists a canonical injection $G$ from $R_d$ to the set of degree $d$ homogeneous polynomial vector bundle mappings from $S^k \mathcal{E}^\ast$ to $(\Lambda^n \mathcal{E}^\ast)^w$.

**Proof.** Let $U$ be an open subset of the base space over which $\mathcal{E}$ is trivial and choose a trivializing frame $\sigma = \{ \sigma_1, \ldots, \sigma_n \}$ for $\mathcal{E}^\ast$ over $U$. Consider a section $m = \sum a_{k1 \ldots k_n} \sigma_1^{k_1} \ldots \sigma_n^{k_n}$ of $S^k \mathcal{E}^\ast$ over $U$, where $a_{k1 \ldots k_n}$ are regular functions on $U$; let $M = \sum a_{k1 \ldots k_n} x_1^{k_1} \ldots x_n^{k_n}$ be the corresponding polynomial of degree $k$ (with variable coefficients). Then set $G_\varphi(m) = \varphi(M)(\sigma_1 \wedge \cdots \wedge \sigma_n)^\otimes w$. Now consider another choice $\tilde{\sigma} = \{ \tilde{\sigma}_1, \ldots, \tilde{\sigma}_n \}$ of frame for $\mathcal{E}^\ast$ over $U$. Treating $\sigma$ and $\tilde{\sigma}$ as $n$-entry column vectors, we express $\tilde{g} = gg_\psi$, where $g = (g_{ij}) : U \to \text{GL}(n, \mathbb{C})$ is an invertible matrix of regular functions. Then we use the relative invariance of $\varphi$ to compute that $\varphi(M)(\tilde{\sigma}_1 \wedge \cdots \wedge \tilde{\sigma}_n)^\otimes w = \varphi(M)(\sigma_1 \wedge \cdots \wedge \sigma_n)^\otimes w$, so there is everywhere a unique local realization of $G_\varphi$, hence a unique global realization. Q.E.D.

$G$ is actually a ring homomorphism. Let $R = \oplus R_d$ be the graded ring of invariants. Then a degree $d$ element $\varphi \in R_d$ gives a polynomial vector bundle map $G_\varphi : S^k \mathcal{E}^\ast \to (\Lambda^n \mathcal{E}^\ast)^w$, where $kd = nw$. The function $G$ is multiplicative in the sense that for $\varphi \in R_d$ and $\psi \in R_d'$, $G_{\varphi \psi} = G_\varphi \cdot G_\psi$ as maps from $S^k \mathcal{E}^\ast$ to $(\Lambda^n \mathcal{E}^\ast)^{(w+w')}$. The product $G_{\varphi \cdot G_\psi}$ is defined by $(G_{\varphi \cdot G_\psi})(m) = G_\varphi(m) \cdot G_\psi(m)$, the natural multiplication of sections being obtained from the tensor product of line bundles

$$(\Lambda^n \mathcal{E}^\ast)^w \otimes (\Lambda^n \mathcal{E}^\ast)^{w'} \cong (\Lambda^n \mathcal{E}^\ast)^{(w+w')}.$$ 

In terms of a choice of local frame $\sigma$,

$$G_{\varphi \psi}(m) = (\varphi(\psi))(M)(\sigma_1 \wedge \cdots \wedge \sigma_n)^\otimes (w+w')$$

$$= (\varphi(M)\psi(M))((\sigma_1 \wedge \cdots \wedge \sigma_n)^\otimes w \cdot (\sigma_1 \wedge \cdots \wedge \sigma_n)^\otimes w')$$

$$= \varphi(M)(\sigma_1 \wedge \cdots \wedge \sigma_n)^\otimes w \cdot \psi(M)(\sigma_1 \wedge \cdots \wedge \sigma_n)^\otimes w'$$

$$= G_\varphi(m) \cdot G_\psi(m).$$

### 3.1.2 Examples. (i) The discriminant $\delta$ is an invariant of degree $n(k-1)^{(n-1)}$, as a polynomial in the coefficients of degree $k$ forms in $n$ variables. It is defined by $\delta(f) = R(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$, where $R$ is the $n$-variable resultant [vW, p. 15], and it has the property that $\delta(f) = 0$ if and only if the projective hypersurface $f = 0$ is singular (as a subscheme of $\mathbb{P}^{n-1}$) (cf. [M-F, Prop. 4.2, p. 79]).

(ii) All invariants of positive degree vanish on forms which are missing a variable. That is, suppose that $f$ is a degree $k$ form in $n$ variables such that, after some linear change of coordinates, $x_n$ can be expressed in terms of fewer variables; geometrically, the projective hypersurface $f = 0$ in $\mathbb{P}^{n-1}$ is a cone. Then any invariant $\varphi \in R_d$, $d > 0$, must vanish on $f$; i.e., such an $f$ is a nullform (or is unstable (cf. [M-F, Thm. 2.1, p. 49]). Indeed, assuming that the variable $x_n$ is missing, consider the
1-parameter group $\lambda : \mathbb{C}^* \to \text{SL}(S^k \mathbb{C}^n)$ induced on degree $k$ forms by the action $t \cdot (x_1, \ldots, x_{n-1}, x_n) = (tx_1, \ldots, tx_{n-1}, t^{-(n-1)}x_n)$, $t \in \mathbb{C}^*$. Then $\lambda(t)(f) = t^k f$, so $\varphi(f) = \varphi(\lambda(t)(f)) = t^{kd} \varphi(f)$. Hence, letting $t \to 0$, we get $\varphi(f) = 0$.

### 3.2 Invariants of cubic forms.

In our applications we will use the invariant theory of cubic forms in $g - 1$ variables; thus $k = 3$ and $n = g - 1$, so the fundamental equation becomes $3d = (g - 1)w$. Now $E = \mathbb{C}^{g-1}$ and $\text{GL}(E)$ acts on $F = S^3 E^*$, the vector space of cubic forms on $E$, and hence on $S^d F^*$, the homogeneous polynomials of degree $d$ on $F$. In this situation, the discriminant $\delta$ has degree $(g - 1)^2$.

The standard Fermat cubic form in $g - 1$ variables is $f_0 = x_1^3 + \cdots + x_{g-1}^3 \in F$ and the zero set of $f_0$ in $\mathbb{P}^{g-2}$ is the standard Fermat cubic hypersurface. The $\text{GL}_{g-1}(\mathbb{C})$-orbit of $f_0$ consists of all cubic forms that can be expressed as the sum of cubes of $g - 1$ linearly independent linear forms, and we will refer to any element of this orbit as a Fermat cubic in $g - 1$ variables. Let $I \subset R$ be the homogeneous ideal of all invariants vanishing on $f_0$ (or equivalently on the orbit $\text{GL}(E) \cdot f_0$, or on the closure of this orbit).

We summarize the well known structure of the ring of invariants of ternary cubics and the (principal) ideal of all invariants vanishing on the Fermat cubic $x^3 + y^3 + z^3$. For $E = \mathbb{C}^3$, $F = S^3 E^*$ is 10-dimensional, and we are interested in the graded ring $R$ of $\text{SL}_3(\mathbb{C})$-invariant polynomial functions on $F$. The structure theorem for invariants of ternary cubics states that there are two algebraically independent invariants $S$ and $T$, of degrees 4 and 6 respectively, such that $R = \mathbb{C}[S, T]$ ([E1, §§291-293, 295, pp. 381-389], [J, p. 4]).

We now describe the explicit forms for the discriminant $\delta$ and the ideal $I$ of invariants vanishing on the Fermat cubic. Using Hesse’s canonical form for cubics (cf. [E1, §229], [J, p. 6]), any nonsingular cubic in $x$, $y$ and $z$ can be put, by a linear change of coordinates, in the form $x^3 + y^3 + z^3 + 6mxyz$; in particular, any invariant of cubics is determined by its values on such normalized cubics. Now $S = m - m^4$ and $T = 1 - 20m^3 - 8m^6$ [E1, pp. 384, 386], [J, p. 7]. (For further information, see [D-K, p. 250], [J, p. 6], [E1, p. 384].) The discriminant $\delta$ is $T^2 + 64S^3$ [J, p. 27] and the ideal $I$ is $(S)$ (cf. [D-K, p. 251], [G-Y, §248, pp. 312-313]). All members of the closure of the Fermat orbit $\text{GL}_3(\mathbb{C}) \cdot (x^3 + y^3 + z^3)$ are described geometrically in [D-K, Prop. 5.13.2 (ii), p. 251]. Finally, the $j$-invariant, the fundamental rational invariant of cubic curves in $\mathbb{P}^2$, is given by $j = (\text{constant}) S^3/\delta$. 
4. The construction of certain Siegel modular forms

4.1 The ring homomorphism $h_\xi$.

We assemble the results of the previous sections to construct a ring homomorphism from invariants of cubic forms to Siegel modular forms. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$, the additive semigroup of (extended) natural numbers. Let $R = \oplus R_d$, $d \in \mathbb{N}$, be the graded ring of invariants of cubic forms in $g - 1$ variables. Let $S = \oplus S_k$, $k \in \frac{1}{2}\mathbb{N}$, be the graded ring of genus $g$ Siegel modular forms with respect to $\Gamma(4, 8)$.

A Siegel modular form of genus $g > 1$, weight $k \in \frac{1}{2}\mathbb{N}$ and level $\Gamma(4, 8)$ (with trivial character) is a holomorphic function $f : \mathcal{H}_g \to \mathbb{C}$ such that for every $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma(4, 8)$ and $\Omega \in \mathcal{H}_g$, $f(\gamma \cdot \Omega) = \det(C \Omega + D)^k f(\Omega)$. Thus, for $k \in \frac{1}{2}\mathbb{N}$, $S_k = H^0(\mathcal{A}_g', P^{2k})$.

4.1.1 Theorem. For every genus $g \geq 4$ and each choice of odd theta characteristic $\xi$, there exists a ring homomorphism

$$h_\xi : R \to S$$

with the following properties

1. $h_\xi(R_d) \subset S_{md}$, where $m = \frac{g+8}{2(g-1)}$.
2. If $\delta \in R$ is the discriminant for cubic forms, then $h_\xi(\delta) \neq 0$.
3. If $I \subset R$ is the homogeneous ideal of all invariants vanishing on the Fermat cubic, then all elements of $h_\xi(I) \subset S$ vanish on period matrices of genus $g$ Riemann surfaces.

4.1.2 Remarks. (i) Since $3d = (g - 1)w$ for some $w \in \mathbb{N}$ in order for nontrivial invariants to exist, it follows that if $R_d \neq 0$, then $md \in \frac{1}{2}\mathbb{N}$.
(ii) $h_\xi(\delta) \in S_k$ for $k = (g + 8)2^{g-3}$.

Proof. From (2.1) we have an open subset $\mathcal{U} = \mathcal{U}_\xi$ of the moduli space $\mathcal{A}_g'$, and over $\mathcal{U}$ a rank $g - 1$ vector bundle $\mathcal{E} = \mathcal{E}_\xi \subset \mathcal{V}$, where $\mathcal{V} = H^*$ is the dual of the Hodge vector bundle $H$. In (2.2) we constructed a homomorphism $\mathcal{m}_\xi : N \to S^3 \mathcal{E}^*$ from the fundamental negative line bundle $N$ to the bundle of cubic forms on $\mathcal{E}$. Now let $\varphi$ be a degree $d$ invariant of cubic forms in $g - 1$ variables, i.e., $\varphi \in R_d = S^d((S^3 \mathcal{C}^{g-1} \mathcal{E}^*))_{(w)}$ where $w = 3d/(g - 1)$. By (3.1.1), $\varphi$ determines a degree $d$ homogeneous polynomial mapping $G_\varphi : S^3 \mathcal{E}^* \to (\Lambda^{g-1} \mathcal{E}^*)^w$. Therefore we obtain (on $\mathcal{U}$) a degree $d$ homogeneous polynomial mapping $G_\varphi \circ \mathcal{m}_\xi : N \to (\Lambda^{g-1} \mathcal{E}^*)^w$, hence a (linear) homomorphism from $N^d$ to $(\Lambda^{g-1} \mathcal{E}^*)^w$, and so a section $s_\varphi$ of the line bundle $N - d \otimes (\Lambda^{g-1} \mathcal{E}^*)^w$ on $\mathcal{U}$. That is, $\varphi \in R_d$ determines $s_\varphi \in H^0(\mathcal{U}, N - d \otimes (\Lambda^{g-1} \mathcal{E}^*)^w)$.

From the exact sequence $0 \to \mathcal{L}_\xi \to H \to \mathcal{E}_\xi^* \to 0$ on $\mathcal{U}$, we have $\Lambda^g H \cong \mathcal{L}_\xi \otimes \Lambda^{g-1} \mathcal{E}_\xi^*$. Thus $\Lambda^{g-1} \mathcal{E}_\xi^* \cong \mathcal{L}_\xi^{-1} \otimes \Lambda^g H \cong N^{-1} \otimes \Lambda^g H \cong P^3$. Thus $N - d \otimes (\Lambda^{g-1} \mathcal{E}^*)^w \cong P^{(d+3)w}$. Since $w = 3d/(g - 1)$, we have $d + 3w = \mu d \in \mathbb{N}$, where $\mu = (g + 8)/(g - 1)$. Thus, from a degree $d$ invariant $\varphi$ we get a section $s_\varphi$ of $N - \mu d = P^\mu d$ on $\mathcal{U}$. By (2.1.3), $H^0(\mathcal{U}, P^\mu d) \cong H^0(\mathcal{A}_g', P^\mu d)$, and we let $h_\xi(\varphi)$ denote the unique extension of $s_\varphi$ to a section of $P^\mu d$ over all of $\mathcal{A}_g'$. Therefore $h_\xi(\varphi)$ is a Siegel modular form of weight $k = \mu d/2 = md$, where $m$ is as stated in (0).
Next we check that \( h_\xi(\varphi \psi) = h_\xi(\varphi)h_\xi(\psi) \) for \( \varphi \in R_d, \psi \in R_{d'} \). Let \( \tilde{G}_\varphi \) denote the homomorphism \( S^d(S^3E^*) \to (\Lambda^{g-1}E^*)^w \) corresponding to the degree \( d \) homogeneous polynomial map \( G_\varphi : S^3E^* \to (\Lambda^{g-1}E^*)^w \) defined by \( \varphi \). Then \( h_\xi(\varphi) \) is the section of \( N^{-2md} \cong N^{-d} \otimes (\Lambda^{g-1}E^*)^w \) which corresponds to the composite homomorphism \( \tilde{G}_\varphi \circ (\overline{m}_\xi)^d : N^d \to S^d(S^3E^*) \to (\Lambda^{g-1}E^*)^w \). Thus the equality of the product \( h_\xi(\varphi)h_\xi(\psi) \) in \( (N^{-d} \otimes (\Lambda^{g-1}E^*)^w) \otimes (N^{-d} \otimes (\Lambda^{g-1}E^*)^{w'}) \) with \( h_\xi(\varphi \psi) \) in \( N^{-(d+d')} \otimes (\Lambda^{g-1}E^*)^{(w+w')} \) follows from the equality of the product homomorphism

\[
(\tilde{G}_\varphi \circ (\overline{m}_\xi)^d)(\tilde{G}_\psi \circ (\overline{m}_\xi)^{d'}) : N^d \otimes N^{d'} \to (\Lambda^{g-1}E^*)^w \otimes (\Lambda^{g-1}E^*)^{w'}
\]

with \( \tilde{G}_{\varphi \psi} \), where \( \tilde{G}_{\varphi \psi} \) and \( \tilde{G}_\varphi \cdot \tilde{G}_\psi \) agree since they are the homomorphisms corresponding to the degree \( d + d' \) homogeneous polynomial maps \( G_\varphi \cdot G_\psi \) and \( G_{\varphi \psi} : S^3E^* \to (\Lambda^{g-1}E^*)^{(w+w')} \), respectively, and we know from \( \S 3 \) that \( G_\varphi \cdot G_\psi = G_{\varphi \psi} \).

To establish properties (1) and (2), it suffices to recall from (2.3.2) (ii) that for a generic genus \( g \) Riemann surface and any choice of odd theta characteristic \( \xi \), the associated cubic form \( \overline{m}_\xi(z) \) is a Fermat cubic in \( g-1 \) variables. Indeed, if \( \Omega_0 \in H_g \) is a period matrix over a generic Jacobian \( (J, \Theta, \ell') \in J' \), then (1) \( h_\xi(\delta)(\overline{\Omega}_0) = \delta(\overline{m}_\xi(z)) \neq 0 \) since the Fermat cubic hypersurface is nonsingular, and (2) holds by construction. Q.E.D.

4.1.3 Remark. The construction of the graded ring homomorphism \( h_\xi \) and the calculation of its degree is simplified slightly by the introduction of a formal cube root \( P^{1/3} \) of \( P \) (in analogy with the squaring principle of [H-T, p. 77]). Then \( P \otimes S^3E^* = S^3(P^{1/3} \otimes E^*) \), so \( G_\varphi \) takes values in \( \Lambda^{g-1}(P^{1/3} \otimes E^*)^w \cong P^{(d+3w)} \).

4.1.4 Reprise. Given \( \Omega \in H_g \), let us try to actually get a number from the above abstract construction. Recall that we have a linear form \( \ell \) and a cubic form \( m \) on the standard vector space \( \mathbb{C}^g \). Let \( \varphi \) be a classical invariant of cubic forms in \( g-1 \) variables. Assuming the linear form \( \ell \) is not identically 0, let \( E \subset \mathbb{C}^g \) be the hyperplane \( \ell(z) = 0 \) and let \( \overline{m} \) be the restriction of the cubic form to \( E \). Then if we take any basis \( B = \{v_2, \ldots, v_g\} \) for \( E \), we get a cubic form \( M_B(x) = \overline{m}(x_2v_2 + \cdots + x_gv_g) \) in \( g-1 \) variables, so we can apply \( \varphi \) to get a number \( \varphi(M_B) \). If we change the basis of \( E \) by \( \alpha \in \text{GL}(E) \), then \( \varphi(M_{\alpha B}) = \det(\alpha)^p \cdot (M_B) \) for some fixed integer \( p \). Now suppose we construct the basis for \( E \) only in the following way. We take a basis \( B \) for \( \mathbb{C}^{g*} \) of the form \( \{\ell_1 = \ell, \ell_2, \ldots, \ell_g\} \). Then the dual basis \( v_1, v_2, \ldots, v_g \) for \( \mathbb{C}^g \) has the property that \( v_2, \ldots, v_g \) form a basis \( B \) for \( E \) and we consider, instead of \( \varphi(M_B) \), the number \( \det(B)^p \cdot \varphi(M_B) \). It is easy to compute that if we change to a new basis \( \tilde{B} = \{\ell_1 = \ell, \ell_2, \ldots, \ell_g\} \) then \( \det(\tilde{B})^p \cdot \varphi(M_B) = \det(B)^p \cdot \varphi(M_B) \). This gives the numerical value \( \det(B)^p \cdot \varphi(M_B) \) does not depend on the choice of (special) basis \( B \) of \( \mathbb{C}^{g*} \); hence this formula can be used to evaluate the modular form \( h_\xi(\varphi) \) at a period matrix \( \Omega \) directly in terms of the Taylor expansion of the theta function about \( z = 0 \).
4.2 The genus 4 case.

Let \( J'_4 \) be the preimage in \( A'_4 \) of the locus of genus 4 Jacobians \( J_4 \subset A_4 \) and let \( \bar{J}'_4 \) be the closure of \( J'_4 \) in \( A'_4 \).

4.2.1 Proposition. For \( g = 4 \), let \( h \) be the homomorphism determined by a marked odd theta characteristic, let \( S \) be the classical quartic invariant of ternary cubics, and set \( f = h(S) \). Then \( f \) is a Siegel modular form of weight 8 with respect to \( \Gamma' = \Gamma(4,8) \) and \( \bar{J}'_4 \subset A'_4 \) is the zero divisor of \( f \) (with multiplicity 1).

Proof. When \( g = 4 \) we have \( m = 2 \) in (4.1.1), and hence if \( \varphi = S \) is the classical quartic invariant of cubic forms on \( \mathbb{C}^3 \), then \( f = h(S) \) is a Siegel modular form of degree 8. From our previous indirect argument [AMSV2, pp. 5-6], we know that this modular form \( f \) is not identically zero. (We also have [MSV2] a direct local argument that the local modulus \( a(\Omega) \) in the cubic form \( m = x^3 + y^3 + z^3 + a(\Omega)xyz \), is not identically 0. Presumably it could also be verified by numerical calculation that \( f(\Omega) \) is nonzero for a specific period matrix \( \Omega \).) Therefore, it remains only to use the following.

4.2.2 Lemma. The divisor class of \( J'_4 \) in \( \text{Pic}(A'_4) \otimes \mathbb{Q} \) is equal to \( 8\lambda \), where \( \lambda = c_1(\Lambda) \) is the class of the Hodge line bundle.

Proof (Mumford). We use adjunction and Mumford’s formula for the canonical bundle of \( M_g \). Note that \( A'_4 \) is smooth, \( J'_4 \) is a smooth Zariski-locally-closed divisor of \( A'_4 \), and \( \bar{J}'_4 \) is closed in \( A'_4 - R' \), where the subset \( R' \) of decomposables is Zariski-closed in \( A'_4 \) of codimension \( \geq 2 \) (and is actually contained in \( \bar{J}'_4 \)). Now we consider the conormal bundle sequence for \( J = J'_4 \subset A'_4 - R' = \mathbb{A} : \)

\[
0 \to N^*(\mathcal{J}/\mathbb{A}) \to T^*(\mathbb{A})|\mathcal{J} \to T^*(\mathcal{J}) \to 0.
\]

Hence, \( K_{\mathcal{A}|\mathcal{J}} = K_{\mathcal{J}} + [N^*] \). By the standard formula \( K_{\mathcal{A}_g} = (g + 1)\lambda \) (cf. [Fr 2. III §2]), we have \( K_{\mathcal{A}} = 5\lambda \), and by Mumford’s formula \( K_{M_g} = 13\lambda \) (cf. [H-M, §2]), we have \( K_{\mathcal{J}} = 13\lambda \), where \( \lambda \) is the restriction of \( \Lambda \) to \( \mathcal{J} \). Therefore \([N^*] = 5\lambda - 13\lambda = -8\lambda \), so \([N] = 8\lambda \). By Borel’s results [Bo], \( A'_g \) has Picard number one for \( g \geq 4 \), so \( \mathcal{J} = r\lambda \) in \( \text{Pic}(\mathbb{A}) \otimes \mathbb{Q} \) for some \( r \in \mathbb{Q} \). Thus, if \( i : \mathcal{J} \hookrightarrow \mathbb{A} \) is the inclusion, we get \( i^*[\mathcal{J}] = i^*(r\lambda) = r\lambda \) in \( \text{Pic}(\mathcal{J}) \otimes \mathbb{Q} \). But of course \( i^*[\mathcal{J}] = i^*(\mathcal{O}_\mathbb{A}(\mathcal{J})) = i^*\mathcal{O}_\mathbb{A}(\mathcal{J}) = [N] \). Thus we have both \([N] = 8\lambda \) and \( r\lambda = [N] \), whence \( r = 8 \). Therefore, \([\bar{J}'_4] = 8\lambda \) in \( \text{Pic}(A'_4) \otimes \mathbb{Q} \). Q.E.D. (4.2.2)

We can conclude the genus 4 argument by again using Picard number one. Let \( D \) be the divisor of \( f \) in \( A'_4 \). Since \( f \) vanishes on \( \bar{J}'_4 \) and hence on \( \bar{J}'_4 \), we have \( D = \bar{J}'_4 + E \) with \( E \geq 0 \). Now take classes in \( \text{Pic} \otimes \mathbb{Q} \). We have \( 8\lambda = [D] = [\bar{J}'_4] + [E] = 8\lambda + [E] \), so \([E] = 0 \). But the divisor class (in \( \text{Pic} \) of a nonzero effective divisor on \( A'_4 \) cannot be 0 (by the existence of the Satake compactification \( (A'_4)^* \), which has small boundary), and hence a nonzero effective divisor on \( A'_4 \) cannot be torsion either. Therefore the divisor \( E \) is 0 and hence \( D = \bar{J}'_4 \), as desired. Q.E.D. (4.2.1)

We can reformulate the previous result in the following way.

4.2.3 Corollary. Let \( j : \tilde{A}_4 \dashrightarrow \mathbb{P}^1 \) be the rational function defined by \( j(A, \Theta, \xi) = j(a) \), where \( (\text{after a suitable linear change of coordinates, if possible}) \overline{m}\xi(x, y, z) = x^3 + y^3 + z^3 + a \cdot xyz \) for some \( a \in \mathbb{C} \), and \( j(a) \) is the classical \( j \)-invariant of \( \overline{m}\xi \). The zero divisor of \( j \) in \( \tilde{A}_4 \) is the closure of \( \bar{J}'_4 \) (with multiplicity 3).
4.2.4 Corollary. The modular form $f$ of (4.2.1) is a nonzero scalar multiple of Schottky's equation. A fortiori, $f$ has level one; i.e., $f$ is a modular form of weight 8 with respect to the entire group $\text{Sp}(2g, \mathbb{Z})$.

Proof. Let $J$ be the Schottky equation, which is a genus 4 Siegel modular form of weight 8 and level 1, and which has zero divisor $\mathfrak{J}_4 \subset \mathcal{A}_4$ (with multiplicity one) ([I4], [Fr1]). In particular, $J$ can be viewed on $\mathcal{A}'_4$ as a section of $P^{16} = \Lambda^8$ with zero divisor $\mathfrak{J}_4'$. Thus, $f$ and $J$ are sections of the same line bundle and have the same zero divisor on $\mathcal{A}'_4$, and hence $f/J$ is a nowhere vanishing holomorphic function $c$ on $\mathcal{A}'_4$. But $H^0(\mathcal{A}'_4, \mathcal{O}) = \mathbb{C}$ (by the existence of the Satake compactification $(\mathcal{A}'_4)^*$, with boundary $(\mathcal{A}'_4)^* - \mathcal{A}'_4$ of codimension $\geq 2$), so $c$ is a nonzero scalar. Then, since $f = cJ$, we see that in fact $f$ has level 1. Q.E.D.

Here is a self-contained argument for level one, based directly on Proposition (4.2.1).

4.2.5 Lemma. Let $f$ be a modular form on $\mathcal{H}_g$, $g \geq 3$, of integral weight $k$ and level $\Gamma$, a finite index subgroup of $\text{Sp}(2g, \mathbb{Z})$. If the zero divisor of $f$ is invariant under $\text{Sp}(2g, \mathbb{Z})$, then $f$ is modular with respect to all of $\text{Sp}(2g, \mathbb{Z})$.

Proof. Define an action of $\text{Sp}(2g, \mathbb{Z})$ on $\mathcal{O}(\mathcal{H}_g)$ as follows (cf. [Fr2, p. 53]). For each integer $k$, $f \in \mathcal{O}(\mathcal{H}_g)$ and $\gamma = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \in \text{Sp}(2g, \mathbb{Z})$, define $(f|_{k\gamma})(\Omega) = \lambda(\gamma, \Omega)^{-k} f(\gamma \cdot \Omega)$, where $\lambda(\gamma, \Omega) = \det(C\Omega + D)$ and $\gamma \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}$.

If the zero divisor of $f$ is invariant under $\gamma \in \text{Sp}(2g, \mathbb{Z})$, there is a nowhere zero holomorphic function $u_\gamma$ on $\mathcal{H}_g$ such that $f|_{k\gamma} = u_\gamma \cdot f$. In this way, from $f$ we get a 1-cocycle $\{u_\gamma\}$ of $\text{Sp}(2g, \mathbb{Z})$ with coefficients in $\mathcal{O}^*(\mathcal{H}_g)$ (a multiplicative group on which $\text{Sp}(2g, \mathbb{Z})$ acts nontrivially). Now if $f$ is a modular form on $\mathcal{H}_g$ (of weight $k$) with respect to $\Gamma$, then the functions $u_\gamma$ are holomorphic on $\mathcal{H}_g$ and invariant under $\Gamma$, and the only modular forms of weight 0 with respect to $\Gamma$ are the constants. Thus $\{u_\gamma\} \in Z^1(\text{Sp}(2g, \mathbb{Z}), \mathbb{C}^*) = \text{Hom}(\text{Sp}(2g, \mathbb{Z}), \mathbb{C}^*)$ since $\text{Sp}(2g, \mathbb{Z})$ acts trivially on the coefficients $\mathbb{C}^*$. But $\text{Hom}(\text{Sp}(2g, \mathbb{Z}), \mathbb{C}^*)$ is trivial, since by [Ma] the commutator subgroup of $\text{Sp}(2g, \mathbb{Z})$ is all of $\text{Sp}(2g, \mathbb{Z})$ if $g \geq 3$. Therefore $u_\gamma = 1$ for all $\gamma \in \text{Sp}(2g, \mathbb{Z})$; i.e., $f|_{k\gamma} = f$ for all $\gamma \in \text{Sp}(2g, \mathbb{Z})$, so $f$ is modular with respect to all of $\text{Sp}(2g, \mathbb{Z})$. Q.E.D.

4.3 The Fermat locus.

Let $I_\xi$ denote the ideal $(h_\xi(I)) \subset S$. Every homogeneous element of $S$, as a holomorphic function on $\mathcal{H}_g$, has a zero set which is invariant under $\Gamma' = \Gamma(4, 8)$, so the zero set of the ideal $I_\xi$ is well defined in $\mathcal{A}'_4 = \mathcal{H}_g/\Gamma'$. Now this zero set is actually well defined in $\tilde{\mathcal{A}}_g$, i.e., modulo $\Gamma[\xi]$. Note that $\mathcal{A}'_g \to \mathcal{A}_g$ is Galois (since $\Gamma'$ is a normal subgroup of $\text{Sp}(2g, \mathbb{Z})$ [I1, Lemma 1, p. 222]), hence the finite cover $\pi_\xi : \mathcal{A}'_g \to \tilde{\mathcal{A}}_g$ is also Galois and the Galois group of $\pi_\xi$ is $\Gamma[\xi]/\Gamma'$ (modulo the subgroup $\{\pm I\}$). In particular, $\Gamma[\xi]$ acts on $\mathcal{A}'_g$ and preserves the open set $U \subset \mathcal{A}'_g$.

4.3.1 Lemma. The zero set in $\mathcal{H}_g$ of each homogeneous element of $h_\xi(I)$ is invariant under the action of the group $\Gamma[\xi]$, i.e., is well-defined in $\tilde{\mathcal{A}}_g = \mathcal{H}_g/\Gamma[\xi]$.

Proof. Let $I$ be the ideal of all invariants of cubic forms in $g - 1$ variables which vanish on the Fermat cubic. Under an element of $\Gamma[\xi]$, the theta function $\vartheta[\xi](z, \Omega)$ transforms to a multiple of itself. Consequently the action of $\Gamma[\xi]$ preserves the condition that $I$ vanishes on the cubic $\overline{c}$ defined by $\vartheta[\xi](z, \Omega)$. Q.E.D.
For each $g \geq 4$, the Fermat locus is the zero locus $F_g = V(I_ξ) \subset \tilde{A}_g$. If $\tilde{J}_g \subset \tilde{A}_g$ is the preimage of the locus of Jacobians $J_g \subset A_g$, then $\tilde{J}_g \subset F_g$ by part (2) of Theorem (4.1.1).

4.3.2 Proposition. Let $R_g = \bigcup A_{ij} \subset A_g$ be the locus of all products of (positive dimensional) principally polarized abelian varieties and let $\tilde{R}_g$ be its preimage in $\tilde{A}_g$. Then for each $g \geq 4$, $\tilde{R}_g \subset F_g$.

Proof. For a product principally polarized abelian variety $(A, \Theta) = (A_1, \Theta_1) \times (A_2, \Theta_2)$, every odd theta characteristic has the form $ξ = ξ_1 + ξ_2$, where $ξ_1$ and $ξ_2$ are theta characteristics of $(A_1, \Theta_1)$ and $(A_2, \Theta_2)$, respectively, of opposite parity. Then $\vartheta[ξ](z) = \vartheta[ξ_1](z_1) \cdot \vartheta[ξ_2](z_2)$ where $z = z_1 + z_2$, so we can obtain the Taylor expansion of $\vartheta[ξ](z)$ from the product of those of $\vartheta[ξ_1](z_1)$ and $\vartheta[ξ_2](z_2)$. Say that $ξ_1$ is even and $ξ_2$ is odd. Then

$$\vartheta[ξ](z) = (c + q(z_1) + \ldots) \cdot (ℓ(z_2) + m(z_2) + \ldots)$$

Thus we have $ℓ_ξ(z) = c \cdot ℓ(z_2)$ and $m_ξ(z) = c \cdot m(z_2) + q(z_1) \cdot ℓ(z_2)$. Hence, assuming $c \neq 0$ and $ℓ \neq 0$ (say that $(A_i, \Theta_i)$ is generic in $A_{g_i}$), we see $m_ξ(z)$ is a scalar multiple of $m(z_2)$ on the hyperplane $ℓ(z_2) = 0$ in $(z_1, z_2)$-space. In particular, the cubic hypersurface $m_ξ(z) = 0$ in $P(ℓ(z_2) = 0)$ is a cone. But by (3.1.2)(ii), all (positive degree) invariants of cubic forms in $g - 1$ variables vanish on a cubic cone. Therefore, for the product $(A, Θ, ξ) = ((A_1, Θ_1) \times (A_2, Θ_2), ξ_1 + ξ_2) \in \tilde{A}_g$ with cubic form $m_ξ$ and any invariant $ϕ \in R_d$, $d > 0$, we have $ϕ(m_ξ) = 0$, so $(A, Θ, ξ) \in F_g$. Q.E.D.

4.3.3 Corollary. The ideal $I_ξ = (h_ξ(I)) \subset S$ consists of cusp forms, i.e., elements in the kernel of the Siegel operators for $Γ'$ (cf. [Fr2, Ch. II, §6, esp. Def. 6.9, p. 129]).

Proof. Let $(R'_g)^*$ be the closure of the preimage of $R_g$ in the Satake compactification $(A'_g)^* \cong \text{Proj}(S)$ of $A_g$. From the fact that $R^*_g$ contains the (single component) Satake boundary $B = A'_g - A_g$ of $A_g$, it follows that $(R'_g)^*$ contains the Satake boundary $B' = (A'_g)^* - A'_g$. By (4.3.1) and (4.3.2), we know that $R'_g \subset V(I_ξ)$ in $A_g$, hence $(R'_g)^* \subset V(I_ξ)$ in $(A'_g)^*$. Therefore $B' \subset V(I_ξ)$ in $(A'_g)^*$; that is, all the modular forms in $I_ξ$ vanish on the Satake boundary $B'$, so they are cusp forms for $Γ'$, as claimed. Q.E.D.

4.3.4 Remark. In fact, if $R_+ = \bigoplus_{d>0} R_d$, then the ideal $(h_ξ(R_+)) \subset S$ consists of cusp forms: $h_ξ(δ)$ is a cusp form, by the same argument as given for (4.3.2), and $R_+$ is the radical of the ideal $(I, δ)$ of $R$.

4.3.5 Remark. The boundary of the Satake compactification $\tilde{A}_g^*$ of $\tilde{A}_g$ has two irreducible components $(g \geq 2)$, and the closure of $F_g$ in $\tilde{A}_g^*$ contains both of them by (4.3.3).

5. Discussion

If $π : \tilde{A}_g \rightarrow A_g$ is the map to level 1 moduli, we define the big Fermat locus $F_g^{(\text{big})} = π(F_g) \subset A_g$ and the small Fermat locus $F_g^{(\text{small})} = \{y | π^{-1}(y) \subset F_g\} \subset$
Thus we have $\mathcal{J}_g \cup \mathcal{R}_g \subset \mathcal{F}_g^{(\text{small})}$ for every $g \geq 4$, and $\mathcal{J}_4 = \mathcal{F}_4^{(\text{big})}$. As a focal point for further work, the following seems reasonable.

**5.1 Conjecture.** For every genus $g \geq 4$, $\mathcal{J}_g$ is an irreducible component of $\mathcal{F}_g^{(\text{big})} \subset \mathcal{A}_g$.

**5.2 Remark.** Let $\mathcal{C} \subset \mathcal{A}_5$ be the locus of intermediate Jacobians of cubic 3-folds, and let $\tilde{\mathcal{C}} \subset \tilde{\mathcal{A}}_5$ denote the canonical lifting of $\mathcal{C}$ defined by the distinguished odd theta characteristic of the intermediate Jacobian of a cubic 3-fold (cf. [Do2, p. 211]). Then it seems to be true, in analogy with [Do2], that $\tilde{\mathcal{C}} \subset \mathcal{F}_5^{(\text{big})} \subset \mathcal{A}_5$.

One uses the Nash construction (2.1.4) and the fact that the Fermat cubic appears in the linear system of hyperplane sections of a generic cubic 3-fold in $\mathbb{P}^4$.) From the structure of the Gauss map [C-G, §§12-13] one could presumably also infer that $\mathcal{C} \not\subset \mathcal{F}_5^{(\text{small})}$.

**5.3 Questions.**

1. To what extent can the nondegeneracy assumption be removed from Proposition (2.3.4) on translation hypersurfaces? In our work on the Gauss map in genus $g$ (cf. [MSV2]), we can allow the curve germs $(\Gamma_i, 0)$ in $\mathbb{C}^g$ to coincide in subcollections, say $(\Gamma_i, 0)$ appears $m_i$ times, and still conclude that $(A, \Theta, \xi) \in \mathcal{F}_g$. But we assume that the Gauss images $(\Gamma_i, p_i)$ in $\mathbb{P}^{g-1}$ are smooth disjoint curve germs and the divisor $D = \sum m_i p_i$ spans a hyperplane of $\mathbb{P}^{g-1}$. For example, this situation arises for the theta divisor of the Jacobian of a smooth genus 4 canonical curve in $\mathbb{P}^3$ that has a plane section of the form $2p + 4q$. Then $D = p + 2q$, and the cubic form $m_\xi(x, y, z)$ attached to the odd theta characteristic $\xi = \mathcal{O}(D)$ looks like $x^3$, a degenerate Fermat cubic.

2. What is the codimension 1 part of the boundary of the closure of $\mathcal{F}_g$ in a toroidal compactification of $\tilde{\mathcal{A}}_g$?

3. Let $\Delta$ be the level 1 modular form $\prod_{\xi \text{ odd}} h_\xi(\delta)$ and let $D \subset \mathcal{A}_g$ be the divisor defined by $\Delta = 0$. Is $D$ irreducible?

4. Is $\mathcal{J}_g = \mathcal{F}_g^{(\text{small})} - \mathcal{R}_g$?

### References

[ACGH] E. Arbarello, M. Cornalba, P. A. Griffiths, J. Harris, *Geometry of Algebraic Curves I*, Springer-Verlag, 1985.

[AMSV1] M. Adams, C. McCrory, T. Shifrin, R. Varley, *Symmetric Lagrangian singularities and Gauss maps of theta divisors*, Lecture Notes in Math. 1462 (1991), Springer-Verlag, 1–26.

[AMSV2] **Invariants of Gauss maps of theta divisors**, Proc. Symp. Pure Math. **54**, (1993), Amer. Math. Soc., Providence, 1–8.

[B] A. Beauville, *Le problème de Schottky et la conjecture de Novikov*, Séminaire Bourbaki 1986/87, Astérisque **152–153** (1987), 101–112.

[Bo] A. Borel, *Stable real cohomology of arithmetic groups II*, Manifolds and Lie Groups, Birhäuser, Boston, 1981, pp. 21–55.

[C] C. H. Clemens, *A Scrapbook of Complex Curve Theory*, Plenum Press, New York, 1980.

[C-G] C. H. Clemens and P. Griffiths, *The intermediate Jacobian of the cubic threefold*, Annals of Math. **95** (1972), 281–356.

[D] O. Debarre, *Le lieu des variétés abéliennes dont le diviseur thête est singulier à deux composantes*, Ann. Scient. Éc. Norm. Sup. (ser. 4) **25** (1992), 687–708.

[D-K] I. Dolgachev and V. Kaney, *Polar covariants of plane cubics and quartics*, Advances in Math. **98** (1993), 216-301.

[Do1] R. Donagi, *Big Schottky*, Inventiones Math. **89** (1987), 569–599.
[Do2] Non-Jacobians in the Schottky loci, Annals of Math. 146 (1987), 193–217.
[Do3] The Schottky problem, Theory of Moduli, Lecture Notes in Math. 1337 (1988), Springer-Verlag, 84–137.

[E] L. Ehrenpreis, The Schottky relation in genus 4, Contemporary Mathematics 136 (1992), Amer. Math. Soc., Providence, 139–160.

[El] E. B. Elliot, An Introduction to the Algebra of Quantics, Clarendon Press, Oxford, 1895.

[F1] H. M. Farkas, On the Schottky relation and its generalization to arbitrary genus, Annals of Math. 92 (1970), 56–81.

[F2] Schottky-Jung theory, Proc. Symp. Pure Math. 49 (1989), Amer. Math. Soc., Providence, 459–483.

[Fr1] E. Freitag, Die Irreduzibilitä t der Schottky-Relation (Bemerkung zu einem Satz von J. Igusa), Archiv der Mathematik 40 (1983), 255–259.

[Fr2] Siegelsche Modulfunktionen, Springer-Verlag, 1983.

[vG] B. van Geemen, Siegel modular forms vanishing on the moduli space of curves, Inventiones Math. 78 (1984), 329–349.

[G-Y] J. H. Grace and A. Young, The Algebra of Invariants, Cambridge Univ. Press, 1903.

[G-R] H. Grauert and R. Remmert, Coherent Analytic Sheaves, Springer-Verlag, 1984.

[G] A. Grothendieck, Sur quelques points d’algèbre homologique, Tohoku Math. Jour. 9 (1957), 119–221.

[H-M] J. Harris and D. Mumford, On the Kodaira dimension of the moduli space of curves, Inventiones Math. 67 (1982), 23–86.

[H-T] J. Harris and L. Tu, On symmetric and skew-symmetric determinantal varieties, Topology 23 (1984), 71–84.

[I1] J. Igusa, On the graded ring of theta-constants, Amer. Jour. Math. 86 (1964), 219–246.

[I2] On the graded ring of theta-constants (II), Amer. Jour. Math. 88 (1966), 221–236.

[I3] Theta Functions, Springer-Verlag, 1972.

[I4] On the irreducibility of Schottky’s divisor, J. Fac. Sci. Tokyo 28 (1981), 531–545.

[I5] Problems on Abelian functions at the time of Poincaré and some at present, Bull. Amer. Math. Soc. 6 (1982), 161–174.

[J] T. Johnsen, A classification of covariants and contravariants of plane cubics, Univ. of Oslo preprint series no. 10 (1982).

[K1] G. Kempf, The equations defining a curve of genus 4, Proc. Amer. Math. Soc. 97 (1986), 219–225.

[K2] G. Kempf, Complex Abelian Varieties and Theta Functions, Springer-Verlag, 1991.

[L-B] H. Lange and Ch. Birkenhake, Complex Abelian Varieties, Springer-Verlag, 1992.

[L] J. Little, Translation manifolds and the Schottky problems, Proc. Symp. Pure Math. 49 (1989), Amer. Math. Soc., Providence, 517–529.

[Ma] H. Maas, Die Multiplikatorsysteme zur Siegelsche Modulgruppe, Nachr. Akad. Wiss. Göttingen 11 (1964), 125–135.

[MSV1] C. McCrory, T. Shifrin and R. Varley, The Gauss map of a genus three theta divisor, Trans. Amer. Math. Soc. 331 (1992), 727–750.

[MSV2] The Gauss map of a genus four theta divisor (to appear).

[Mc] J. McKernan, Versality for canonical curves and complete intersections, preprint, University of Texas, 1993.

[M1] D. Mumford, On the equations defining abelian varieties I, Inventiones Math. 1 (1966), 287–354.

[M2] Curves and Their Jacobians, U. Michigan Press, Ann Arbor, 1975.

[M3] Tata Lectures on Theta I, Birkhäuser, Boston, 1983.

[M4] On the Kodaira dimension of the Siegel modular variety, Lecture Notes in Math. 997 (1983), Springer-Verlag, 348–375.

[M-F] D. Mumford and J. Fogarty, Geometric Invariant Theory, 2nd ed., Springer-Verlag, 1982.

[R-F] H. Rauch and H. Farkas, Theta Functions with Applications to Riemann Surfaces, Williams and Wilkins, 1974.
[S-V] R. Smith and R. Varley, *Components of the locus of singular theta divisors of genus five*, Lecture Notes in Math. **1124** (1985), Springer-Verlag, 338–416.

[T] M. Teixidor, *Half-canonical series on algebraic curves*, Trans. Amer. Math. Soc. **302** (1987), 99–115.

[vW] B. L. van der Waerden, *Modern Algebra II*, Ungar, New York, 1950.

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