A NEW METHOD FOR OBTAINING FIBONACCI IDENTITIES

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Abstract
For the Lucas sequence \( \{U_k(P, Q)\} \) we discuss the identities such as the well-known Fibonacci identities. For example, the generalizations of \( F_{2k} = F_{k+1}^2 - F_{k-1}^2 \) and \( F_{2k+1} = F_{k+1}^2 + F_k^2 \) are \( PU_{2k} = U_{k+1}^2 - QU_{k-1}^2 \) and \( U_{2k+1} = U_{k+1}^2 - QU_{k-1}^2 \), respectively. We propose a new simple method for obtaining identities involving any recurrences and we use it to obtain new identities involving the Fibonacci numbers such as \( F_{5k+3} - 60F_k^5 = 8(F_{5k+2}^2 + F_{5k-2}^2) + 40(F_{k+1}^5 - F_{k-1}^5) \).

1. Introduction
The Lucas sequences \( \{U_k(P, Q)\} \) and \( \{V_k(P, Q)\} \) are defined recursively by

\[ f_{k+2} = Pf_{k+1} - Qf_k, \]

with the initial values \( U_0 = 0, U_1 = 1, V_0 = 2, V_1 = P \). The characteristic equation of the recurrence relation \( x^2 - Px + Q = 0 \) is \( \Delta = P^2 - 4Q \). Its roots are \( \alpha = \frac{P + \sqrt{\Delta}}{2} \) and \( \bar{\alpha} = \frac{P - \sqrt{\Delta}}{2} \), where \( \Delta = P^2 - 4Q \). The Lucas sequences can be expressed in terms of \( \alpha \) and \( \bar{\alpha} \) according to Binet formulas

\[ U_k = \frac{\alpha^k - \bar{\alpha}^k}{\alpha - \bar{\alpha}}, \quad V_k = \alpha^k + \bar{\alpha}^k, \quad P = \alpha + \bar{\alpha}, \quad Q = \alpha\bar{\alpha}, \quad \sqrt{\Delta} = \alpha - \bar{\alpha}. \]

The sequence of the Fibonacci numbers \( \{F_k\} \) is defined by the recurrence relation \( F_{k+2} = F_{k+1} + F_k \) \( (F_0 = 0, F_1 = 1) \). From this definition follows that \( F_k = U_k(1, -1) \). There are many identities involving the Fibonacci numbers. Some of them derive from identities involving \( \{U_k(P, Q)\} \) if we put \( P = 1, Q = -1 \). In this paper we generalize some identities for \( \{F_k\} \) in terms of \( \{U_k(P, Q)\} \). Often such generalized identities have a form close to the initial one. We also present a new method for obtaining identities involving any recurrences. To show an efficiency of this method we obtain many new identities for the Fibonacci numbers.

\footnote{In this paper instead of \( U_k(P, Q) \) and \( V_k(P, Q) \) we will write \( U_k \) and \( V_k \), if it is not ambiguous.}
2. Generalizations of Fibonacci identities

It is clear that any identity which holds for \( \{U_k(P, Q)\} \) can be easily transformed into an identity for \( \{F_k\} \). But there exist identities involving \( \{F_k\} \) for which analogues in terms of \( \{U_k(P, Q)\} \) are unknown. In this section we present generalized analogues for some Fibonacci identities.

The problem of generalized Fibonacci identities has already been considered, see [2, 3, 5]. For example, Candido’s identity [4] was generalized in [2]. This result is as follows:

\[
(F.0) \quad 2(F^4_k + F^4_{k+1} + F^4_{k+2}) = (F^2_k + F^2_{k+1} + F^2_{k+2})^2,
\]

\[
(GF.0) \quad 2(Q^4W^4_k + P^4W^4_{k+1} + W^4_{k+2}) = (Q^2W^2_k + P^2W^2_{k+1} + W^2_{k+2})^2.
\]

Here, the sequence \( \{W_k\} \) is defined by the relation

\[
W_{k+2} = P W_{k+1} - Q W_{k},
\]

with \( W_0 = A, W_1 = B \). This sequence was intensively studied by Horadam [12, 13].

2.1. Two important identities for \( \{U_k(P, Q)\} \)

For the Fibonacci numbers the following holds:

\[
(F.1) \quad F_{2k+1} = F^2_{k+1} + F^2_k,
\]

\[
(F.2) \quad F_{2k} = F_k(2F_{k+1} - F_k).
\]

The generalizations of (F.2), (F.3) are:

\[
(GF.1) \quad U_{2k+1} = U^2_{k+1} - QU_k^2,
\]

\[
(GF.2) \quad U_{2k} = U_k(2U_{k+1} - PU_k).
\]

Lemma 1. Let \( k \) be a positive integer and \( \{U_k(P, Q)\} \) be the Lucas sequence with parameters \( P, Q \) in an arbitrary field \( \mathbb{F} \). Then (GF.1) and (GF.2) are valid.

Proof. Consider the well-known identity \( U_{n+m} = U_nU_{m+1} - QU_mU_{n-1} \). If we put \( n = k + 1 \) and \( m = k \), then we obtain (GF.1). If we put \( n = k \) and \( m = k \), then we obtain \( U_{2k} = U_k(U_{k+1} - QU_{k-1}) \). By the definition of \( \{U_k\} \) we use \( QU_{k-1} = PU_k - U_{k+1} \) in the previous equality, then we obtain (GF.2).

Despite the importance of these two identities, they are very rare in the literature: both identities are presented in [6]. (GF.1) is in [7, 8]. To show the importance of (GF.1) and (GF.2), we note that the following properties are widespread and are used to calculate the Lucas sequences [9].

\[
(p.1) \quad U_{2k+1} = U_{k+1}V_k - Q^k,
\]

\[
(p.2) \quad V_{2k+1} = V_{k+1}V_k - PQ^k,
\]

\[
(p.3) \quad U_{2k} = U_kV_k,
\]

\[
(p.4) \quad V_{2k} = V_k^2 - 2Q^k.
\]
2.2. Higher order Fibonacci identities

For the Fibonacci numbers the following holds:

\[(F.3)\quad F_{3k} = F_{k+1}^3 + F_k^3 - F_{k-1}^3,\]

\[(F.4)\quad F_{4k} = F_{k+1}^4 + 2F_k^4 - F_{k-1}^4 + 4F_k^3F_{k-1},\]

\[(F.5)\quad F_{5k} = F_{k+1}^5 + 4F_k^5 - F_{k-1}^5 + 10F_{k+1}F_k^3F_{k-1}.\]

The reader can find \((F.3), (F.4)\) in [10, 11]. The generalizations of the above identities are:

\[(GF.3)\quad U_{3k} = U_{k+1}^3/P + (U_3 - P^2)U_k^3 + Q^3U_{k-1}^3/P,\]

\[(GF.4)\quad U_{4k} = U_{k+1}^4/P + (U_4 - P^3)U_k^4 - Q^4U_{k-1}^4/P + 4Q^2U_k^3U_{k-1},\]

\[(GF.5)\quad U_{5k} = U_{k+1}^5/P + (U_5 - P^4)U_k^5 + Q^5U_{k-1}^5/P + 10Q^2U_{k+1}U_k^3U_{k-1}.\]

One way to prove \((GF.3)-(GF.5)\) is to use (2). We checked this by a computer. But it is very difficult to obtain such identities using Binet’s formulas. In the next section we discuss how we obtain similar identities. Moreover, we show that there exists an identity which relates \(U_{4k}\) to the fourth powers of \(U_{k+m}, U_{k+l}, U_{k-l}, U_{k+m}\). There also exists an identity which relates \(U_{5k}\) to the fifth powers of \(U_{k+m}, U_{k+l}, U_k, U_{k-l}, U_{k+m}\) and (we conjecture) so on.

2.3. A new method for obtaining identities for any recurrences

First we consider the matrix method that is often used to prove some identities concerning the generalized Fibonacci and Lucas numbers [6, 8]. For the Lucas sequences we have the following matrix formula:

\[
\begin{pmatrix}
U_{k+1} \\
V_{k+1}
\end{pmatrix} = M
\begin{pmatrix}
U_k \\
V_k
\end{pmatrix}, \quad \text{where } M = \begin{pmatrix} P & -Q \\ 1 & 0 \end{pmatrix}.
\]

(3)

Then

\[
\begin{pmatrix}
U_{k+1} \\
V_{k+1}
\end{pmatrix} = M^k
\begin{pmatrix} 1 & P \\ 0 & 2 \end{pmatrix}.
\]

(4)

Lemma 2. Let the sequence \(\{W_k(A, B; P, Q)\}\) be defined by the relation \(W_{k+2} = PW_{k+1} - QW_k\), with \(W_0 = A, W_1 = B\). Then \(W_k = BU_k - AQU_{k-1}\).

Proof. We have:

\[
\begin{pmatrix} W_{k+1} \\
W_k
\end{pmatrix} = M^k
\begin{pmatrix} B \\
A
\end{pmatrix} = BM^k
\begin{pmatrix} 1 \\
0
\end{pmatrix} + AM^k
\begin{pmatrix} 0 \\
1
\end{pmatrix} = B \begin{pmatrix} U_{k+1} \\
U_k
\end{pmatrix} + AM^{k-1}
\begin{pmatrix} -Q \\
0
\end{pmatrix} = (BU_{k+1} - AQU_k, BU_k - AQU_{k-1}).
\]

(5)
In a sense, the sequence \( \{U_k\} \) is basic. This result is well-known, see \cite{12}. By Lemma 2 we get the following well-known identity:

\[
V_k = PU_k - 2QU_{k-1}. \tag{6}
\]

We will use the following notation:

\[
S = \begin{pmatrix} 1 & P \\ 0 & 2 \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} 1 & -P/2 \\ 0 & 1/2 \end{pmatrix}, \quad [MS]_k = \begin{pmatrix} U_{k+1} & V_{k+1} \\ U_k & V_k \end{pmatrix}, \quad [MSU]_k = \begin{pmatrix} U_{k+1} \\ U_k \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} 0 & 1 \\ -1/Q & P/Q \end{pmatrix}. \tag{7}
\]

Here, \( S \) denotes the initial values matrix.

**Theorem 1.** Let \( U_{mk} \) be a term of the Lucas sequence \( \{U_k(P, Q)\} \). Then there exists the following representation of \( U_{mk} \) via \( U_k, U_{k+1} \):

\[
U_{mk} = \sum_{i=0}^{m} a_i(P, Q)U_iU_{m-i}^{k+1}. \tag{8}
\]

Here, \( a_i(P, Q) \) are the functions of two variables, whose forms depend only on \( m \). In other words, \( a_i(P, Q) \) are fixed for any \( k \) if \( m \) is not changed. Moreover, such a representation is unique.

**Proof.** We have

\[
\begin{pmatrix} U_{mk+1} \\ U_m \end{pmatrix} = M^{mk} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \underbrace{M^kSS^{-1}M^kSS^{-1} \cdots M^kSS^{-1}}_{m-1 \text{ times}} M^k \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (M^kSS^{-1})^{m-1} \begin{pmatrix} U_{k+1} \\ U_k \end{pmatrix}. \tag{9}
\]

By (7) we get

\[
[MSU]_{mk} = ([MS]_kS^{-1})^{m-1}[MSU]_k. \tag{10}
\]

Now we calculate \( [MS]_kS^{-1} \):

\[
[MS]_kS^{-1} = \begin{pmatrix} U_{k+1} \\ U_k \end{pmatrix} \begin{pmatrix} -PU_{k+1} + V_{k+1}/2 \\ -PU_k + V_k/2 \end{pmatrix}. \tag{11}
\]

With the help of (6) we eliminate \( V_k, V_{k+1} \):

\[
[MS]_kS^{-1} = \begin{pmatrix} U_{k+1} \\ U_k \end{pmatrix} \begin{pmatrix} -QU_k \\ -QU_{k-1} \end{pmatrix}. \tag{12}
\]

Since \( -QU_{k-1} = U_{k+1} - PU_k \) by the definition of the Lucas sequences, we have

\[
[MS]_kS^{-1} = \begin{pmatrix} U_{k+1} \\ U_k \end{pmatrix} \begin{pmatrix} -QU_k \\ U_{k+1} - PU_k \end{pmatrix}. \tag{13}
\]
We note that the elements of \([MS]_{|S^{-1}}\) are linearly related to the terms \(U_k, U_{k+1}\). Finally, we can modify \((14)\) into

\[
\begin{pmatrix}
U_{mk+1} \\
U_{mk}
\end{pmatrix} = \begin{pmatrix}
U_{k+1} & -QU_k \\
U_k & U_{k+1} - PU_k
\end{pmatrix}^{m-1} \begin{pmatrix}
U_{k+1} \\
U_k
\end{pmatrix}.
\tag{14}
\]

We obtain the representation \((8)\), where the forms of \(a_i(P, Q)\) depend only on \(m\). Moreover, by \((14)\) we have proved that for \(U_{mk+1}\) there exists a similar representation. Now suppose that there is another representation of \(U_{mk}\) via \(U_k, U_{k+1}\). Let it be of the form:

\[
U_{mk} = \sum_{i=0}^{m} b_i(P, Q)U_k^iU_{k+1}^{m-i},
\tag{15}
\]

such that not all \(b_i\) are equal to \(a_i\). Then \(\sum_{i=0}^{m}(b_i - a_i)U_k^iU_{k+1}^{m-i} = 0\). From this we conclude that \(U_k(P, Q) | b_0(P, Q) - a_0(P, Q)\) for any \(k, P\) and \(Q\). But this is possible only if \(b_0 = a_0\). By analogy we prove that \(b_i = a_i\) for \(0 \leq i \leq m\). So the representation \((8)\) is unique.

\[\square\]

**Remark.** The matrix formula \((14)\) by itself is valuable, as it allows to quickly find the identity that relate \(U_{mk}\) to \(U_k, U_{k+1}\). So for \(m = 2\) we get \((GF.1), (GF.2)\).

Now consider another way of obtaining the identity that involves \(U_{mk}, U_k, U_{k+1}\). By Theorem \(11\) we know that such identities exist. We can use the method similar to the partial fraction decomposition. For this by using the representation \((8)\) we get the system of \(m + 1\) equations whose variables and coefficients are functions of \(P, Q\).

\[
U_{mk} = \sum_{i=0}^{m} a_i(P, Q)U_k^iU_{k+1}^{m-i} (- \lfloor (m - 1)/2 \rfloor \leq k \leq \lceil m/2 \rceil).
\tag{16}
\]

Since \(k\) may be negative in \((16)\) we need the extension of the Lucas sequence to negative indices \((16)\), namely \(U_{-k} = \frac{-U_k}{Q^k}\). For \(k = 0\) we have \(a_0 = 0\) since \(U_0 = 0, U_1 = 1\). For \(k = -1\) we have \(a_m = (-1)^{m+1}U_m\).

**Example 1.** \(U_{3k} = a_0U_{k+1} + a_1U_kU_{k+1} + a_2U_{k+1}U_k + a_3U_k^3\). The system \((16)\) for this case is:

\[
\begin{aligned}
U_{-3} &= a_0U_0^3 + a_1U_0^2U_{-1} + a_2U_0U_{-1}^2 + a_3U_{-1}^3, \\
U_0 &= a_0U_0^3 + a_1U_0^2U_0 + a_2U_0U_0^2 + a_3U_0^3, \\
U_3 &= a_0U_3^3 + a_1U_3^2U_1 + a_2U_3U_1^2 + a_3U_1^3, \\
U_6 &= a_0U_6^3 + a_1U_6^2U_2 + a_2U_6U_2^2 + a_3U_2^3.
\end{aligned}
\tag{17}
\]

Using \(U_{-3} = -(P^2 - Q)/Q^3, U_{-1} = -1/Q, U_0 = 0, U_1 = 1, U_3 = P^2 - Q, U_6 = P^5 - 4P^3Q + 3PQ^2\) we get the solution \(a_0 = 0, a_1 = 3, a_2 = -3P, a_3 = P^2 - Q\). So \(U_{3k} = 3U_{k+1}U_k - 3PU_{k+1}U_k^2 + (P^2 - Q)U_k^3\). This is consistent with \((14)\).

**Example 2.** Consider the Fibonacci Pythagorean triples identity \((13)\).

\[
(F_{k-1}F_{k+2})^2 + (2F_kF_{k+1})^2 = F_{2k+1}^2.
\tag{F.6}
\]
Suppose that there exists an identity of the form
\[ c_1(U_{k-1}U_{k+2})^2 + c_2(U_kU_{k+1})^2 + c_3U_{2k+1}^2 = 0. \] (18)

To get the system for the variables \(c_i\) we put \(k = -1, 0, 1\).

\[
\begin{align*}
&c_1P^2/Q^4 + c_3/Q^2 = 0, \\
&c_1P^2/Q^2 + c_3 = 0, \\
&c_2P^2 + c_3(P^2 - Q)^2 = 0.
\end{align*}
\] (19)

If the system has no solutions (the rank equals 3), then we can state that the identity of the form (18) does not exist. In this case we may modify it by adding new terms and obtain a new system. In fact, the rank of the system (19) equals 2. We put \(c_3 = -1\), then the solution is \(c_1 = Q^2/P^2, c_2 = (P^2 - Q)^2/P^2, c_3 = -1\). But when we use (2) to check the obtained formula we conclude that it is not valid. If we consider the system which corresponds \(k = 0, 1, 2\), then the determinant of the system matrix equals \(-2P^4(P^2 - 2Q)(P^2 - Q)(P^2 + Q)/Q\). So the solution exists if one of the following holds: \(P = 0, P^2 = 2Q, P^2 = \pm Q\). If we add \(c_4U_{k-1}U_kU_{k+1}U_{k+2}\) to the left side of (18), then we find the generalization of (F.6) as follows:

\[ (GF.6) \quad (QU_{k-1}U_{k+2})^2 + ((P^2 - Q)U_kU_{k+1})^2 = (PU_{2k+1})^2 + 2Q(P^2 + Q)U_{k-1}U_kU_{k+1}U_{k+2}. \]

We see that Pythagorean triples can be found in the Lucas sequence if \(P^2 + Q = 0\).

**Remark:** In general, we cannot assert that the method leads to the final result which holds for all \(k\) since a supposed formula similar to (18) is checked by a system only for some values of \(k\). So we need to use (2) to prove that the final result is valid for all \(k\). In Example 1 this check is not necessary, since by Theorem 1 we know that the identity which involves \(U_{mk}, U_k, U_{k+1}\) exists. But we obtained the unique solution using the method.

**Example 3.** We want to get an identity of the form
\[ c_1U_{k+1}^2 + c_2U_k^2 + c_3U_{k-1}^2 = 0. \] (20)

We put \(k = -1, 0, 1\) and obtain the following system

\[
\begin{align*}
&c_2/Q^2 + c_3P^2/Q^4 = 0, \\
&c_1 + c_3/Q^2 = 0, \\
&c_1P^2 + c_2 = 0.
\end{align*}
\] (21)

Since the determinant of the system matrix equals \(2P^2/Q^4\), we conclude that for nonzero \(P\) there is no identity which contains only squares of three consecutive terms of \(\{U_k\}\). But if we try find an identity of the form
\[ c_1U_{k+1}^2 + c_2U_k^2 + c_3U_{k-1}^2 + c_4U_{k-2}^2 = 0, \] (22)
then we obtain

\[(GF.7)\quad U_{k+1}^2 - Q^2 U_{k-2}^2 = (P^2 - Q)(U_k^2 - QU_{k-1}^2).\]

If \(P = 1\) and \(Q = -1\), then we get the identity for the Fibonacci numbers:

\[(F.7)\quad F_{k+1}^2 + F_{k-2}^2 = 2(F_k^2 + F_{k-1}^2).\]

As is seen the method is good not only for generalizing Fibonacci identities, but also for finding new identities. Below we present some results that we obtained using this method.

\[(F.8)\quad F_{2k} = F_{k+1}^2 - F_{k-1}^2.\]

\[(GF.8)\quad PU_{2k} = U_{k+1}^2 - Q^2 U_{k-1}^2.\]

\[(F.9)\quad F_{k+1}^2 = 4F_k F_{k-1} + F_{k-2}^2.\]

\[(GF.9)\quad U_{k+1}^2 = 2P(P^2 - Q)U_k U_{k-1} + (Q^2 - P^4)U_{k-1}^2 + P^2 Q^2 U_{k-2}^2.\]

\[(GF.10)\quad U_{k+2}^2 - Q^3 U_{k-2}^2 = (P^2 - 2Q)(U_{k+1}^2 - Q^2 U_{k-1}^2).\]

\[(F.10)\quad F_{k+2}^2 - F_{k-2}^2 = 3(F_{k+1}^2 - F_{k-1}^2).\]

The identity \((GF.10)\) is given before \((F.10)\) since \((F.10)\) was not known earlier. Note that \((F.7)\) and \((F.10)\) have similar forms. They can be generalized. To generalize \((F.10)\) consider \(c_1 U_{k+m}^2 + c_2 U_{k+l}^2 + c_3 U_{k-l}^2 + c_4 U_{k-m}^2 = 0\). With the help of the method we get the following system:

\[
\begin{align*}
    c_1 U_{m+1}^2 & + c_2 U_{l+1}^2 + c_3 U_{l-1}^2 / Q^{2(l+1)} + c_4 U_{m+1}^2 / Q^{2(m+1)} = 0, \\
    c_1 U_{m}^2 & + c_2 U_{l}^2 + c_3 U_{l-1}^2 / Q^{2l} + c_4 U_{m}^2 / Q^{2m} = 0, \\
    c_1 U_{m+1}^2 & + c_2 U_{l+1}^2 + c_3 U_{l-1}^2 / Q^{2(l-1)} + c_4 U_{m-1}^2 / Q^{2(m-1)} = 0.
\end{align*}
\]

The rank equals 2. Let \(c_1 = U_{l+1}^2 - Q^2 U_{l-1}^2\), then \(c_2 = -(U_{m+1}^2 - Q^2 U_{m-1}^2)\), \(c_3 = Q^{2l} (U_{m+1}^2 - Q^2 U_{m-1}^2)\), \(c_4 = -Q^{2m} (U_{l+1}^2 - Q^2 U_{l-1}^2)\). Now we use \((GF.8)\) and obtain

\[(GF.11)\quad U_{2l}(U_{k+m}^2 - Q^{2m} U_{k-m}^2) = U_{2m}(U_{k+l}^2 - Q^{2l} U_{k-l}^2).\]

\[(F.11)\quad F_{2l}(F_{k+m}^2 - F_{k-m}^2) = F_{2m}(F_{k+l}^2 - F_{k-l}^2).\]

Another way to prove \((GF.11)\) is to use Catalan’s identity for \(\{U_k\}\). We know that \(U_i^2 - Q^{-j} U_j^2 = U_{i+j} U_{i-j}\). From this it follows \(U_{k+m}^2 - Q^{2m} U_{k-m}^2 = U_{2k} U_{2m}, U_{k+l}^2 - Q^{2l} U_{k-l}^2 = U_{2k} U_{2l}\). This completes the proof.

To generalize \((F.7)\) we consider the most general formula which involves only four squares of sequence terms: \(U_k^2 + c_1 U_{k+m}^2 + c_2 U_{k+l}^2 + c_3 U_{k+n}^2 = 0\). Using the method we get

\[(GF.12)\quad U_{k}^2 - U_{k+m}^2 U_{l}^2 / (Q^{2m} U_{l-m} U_{s-m}) - U_{k+l}^2 U_{m}^2 / (Q^{2l} U_{s-l} U_{m-l}) - U_{k+n}^2 U_{l}^2 / (Q^{2n} U_{l-n} U_{s-n}) = 0,\]

\[(F.12)\quad F_{k}^2 - F_{k+m}^2 F_{l}^2 / (F_{l-m} F_{s-m}) - F_{k+l}^2 F_{m}^2 / (F_{s-l} F_{m-l}) - F_{k+n}^2 F_{l}^2 / (F_{l-n} F_{s-n}) = 0.\]
In these identities we mean that $l, m, s$ are distinct integers. If $k = 0$ in (F.12), then we get the identity:

(F.12.0) \[ F_m^2 F_i F_s / (F_{-m} F_{s-m}) + F_l^2 F_s F_n / (F_{-l} F_{n-l}) + F_s^2 F_m F_i / (F_{-s} F_{i-s}) = 0. \]

The following identities involve five cubes:

(GF.13) \[ U_{2l} U_l \left( U_{k+m}^3 + Q^{lm} U_{k-m}^3 \right) - U_{2m} U_m \left( U_{k+l}^3 + Q^{ml} U_{k-l}^3 \right) + Q^l V_m V_l U_{m+l} U_{m-l} U_{k}^3 = 0, \]

(F.13) \[ F_{2l} F_l \left( F_{k+m}^3 + (-1)^{-3m} U_{k-m}^3 \right) - F_{2m} F_m \left( F_{k+l}^3 + (-1)^{-3l} U_{k-l}^3 \right) + (-1)^l L_m L_l F_{m+l} F_{m-l} F_{k}^3 = 0. \]

Here $L_m, L_n$ are the Lucas numbers. If we put $m = 2$ and $l = 1$, then we obtain $F_{k+2}^3 F_{k-2}^3 = 3 (F_{k+1}^3 - F_{k-1}^3) + 6F_k^3$. By (F.3) we get $3F_{3n} = F_{n+2}^3 - 3F_n^3 + 3F_{n-2}^3$.

This cubic relation is known [14, 15]. To get something new we put $m = 3$ and $l = 1$, then we obtain $F_{k+3}^3 - F_{k-3}^3 = 16(F_{k+1}^3 - F_{k-1}^3) + 12F_k^3$. Using (F.3) we get $16F_{3k} = F_{k+3}^3 + 4F_k^3 - F_{k-3}^3$.

The following identities involve six biquadratics:

(GF.14) \[ U_{2l} U_{2n} U_{i+n} U_{i-n} \left( U_{k+m}^4 - Q^{4m} U_{k-m}^4 \right) - U_{2m} U_{2n} U_{m+n} U_{m-n} \left( U_{k+l}^4 - Q^{4l} U_{k-l}^4 \right) + Q^{l-n} U_{2m} U_{2n} U_{m+l} U_{m-l} \left( U_{k+n}^4 - Q^{4n} U_{k-n}^4 \right) = 0, \]

(F.14) \[ F_{2l} F_{2n} F_{i+n} F_{i-n} \left( F_{k+m}^4 - F_{k-m}^4 \right) - F_{2m} F_{2n} F_{m+n} F_{m-n} \left( F_{k+l}^4 - F_{k-l}^4 \right) + (-1)^{l-n} F_{2m} F_{2n} F_{m+l} F_{m-l} \left( F_{k+n}^4 - F_{k-n}^4 \right) = 0. \]

We put $m = 3, l = 2, n = 1$, then $F_{k+3}^4 - F_{k-3}^4 = 4(F_{k+2}^4 - F_{k-2}^4) + 20(F_{k+1}^4 - F_{k-1}^4)$.

Finally, we give the identities involve seven terms whose exponents are equal 5:

(GF.15) \[ U_{2l} U_{2n} U_{i+n} U_{i-n} U_{i} U_{n} \left( U_{k+m}^5 + Q^{5m} U_{k-m}^5 \right) - U_{2m} U_{2n} U_{m+n} U_{m-n} U_{m} U_{n} \left( U_{k+l}^5 + Q^{5l} U_{k-l}^5 \right) + Q^{l-n} U_{2m} U_{2n} U_{m+l} U_{m-l} U_{i} U_{n} \left( U_{k+n}^5 + Q^{5n} U_{k-n}^5 \right) - Q^{l-n} V_m V_n U_{m+n} U_{m-n} U_{i+n} U_{i-n} U_{i} U_{n} U_{k} = 0, \]

(F.15) \[ F_{2l} F_{2n} F_{i+n} F_{i-n} F_{i} F_{n} \left( F_{k+m}^5 + (-1)^{5m} F_{k-m}^5 \right) - F_{2m} F_{2n} F_{m+n} F_{m-n} F_{m} F_{n} \left( F_{k+l}^5 + (-1)^{5l} F_{k-l}^5 \right) + (-1)^{l-n} F_{2m} F_{2n} F_{m+l} F_{m-l} F_{i} F_{n} \left( F_{k+n}^5 + F_{k-n}^5 \right) - (-1)^{l-n} L_m L_n L_{i+n} L_{i-n} L_{i} L_{n} F_{m+n} F_{m-n} F_{m+i} F_{m-l} F_{i} F_{n} F_{i+n} F_{i-n} = 0. \]

If $m = 3, l = 2, n = 1$, then $F_{k+3}^5 - F_{k-3}^5 + 60F_k^5 = 8(F_{k+2}^5 + F_{k-2}^5) + 40(F_{k+1}^5 - F_{k-1}^5)$.

2.4. $U_{bk}$ as $b$ terms whose exponents are equal to $b$

Let us consider $U_{bk} = c_1 U_{k+m}^2 + c_2 U_{k-m}^2$. For $k = 0, 1$ we have:

\[
\begin{align*}
    c_1 U_{m}^2 + c_2 U_{-m}^2 &= 0, \\
    c_1 U_{m+1}^2 + c_2 U_{1-m}^2 &= P.
\end{align*}
\]
The determinant of the system matrix is $U^2_{m-1}/Q^{2(m-1)} - U^2_{m+1}/Q^{2m} = U_m^2(Q^2 U_{m-1} - U_{m+1})/Q^{2m}$. With the help of (GF.8) we conclude that it is equal to $-P U^2_m U_{2m}/Q^{2m}$. By the Cramer rule we get $c_1 = 1/U_{2m}$ and $c_2 = -Q^{2m}/U_{2m}$. So $U_{2m} U_{2k} = U^2_{k+m} - Q^{2m} U^2_{k-m}$. The check of the obtained formula by \( \Box \) approves it. Note that we obtain the particular case of the generalized Catalan identity $U_{i+j} = U^2_i - Q^{i-j} U^2_j$.

If we consider $U_{3k} = c_1 U^3_{k+m} + c_2 U^3_k + c_3 U^3_{k-m}$, then we can get the following:

\[
\begin{align*}
(GF.16) & \quad U_m U_{2m} U_{3k} = U^3_{k+m} - Q^m V_m U^3_k + Q^{3m} U^3_{k-m}, \\
(F.16) & \quad F_m F_{2m} F_{3k} = F^3_{k+m} - (-1)^m L_m F^3_k + (-1)^{3m} F^3_{k-m}.
\end{align*}
\]

If $m = 4$, then we have $63 F_{3k} = F^3_{k+4} - 7 F^3_k + F^3_{k-4}$.

Let us consider $U_{4k} = c_1 U^4_{k+m} + c_2 U^4_{k+l} + c_3 U^4_{k-l} + c_4 U^4_{k-m}$, then

\[
\begin{align*}
(GF.17) & \quad U_{m-l} U_{m+l} U_{2m} U_{2l} U_{4k} = U_{2l} (U^4_{k+m} - Q^{4m} U^4_{k-m}) - Q^{m-l} U_{2m} (U^4_{k+l} - Q^{4l} U^4_{k-l}), \\
(F.17) & \quad F_{m-l} F_{m+l} F_{2m} F_{2l} F_{4k} = F_{2l} (F^4_{k+m} - F^4_{k-m}) - (-1)^{m-l} F_{2m} (F^4_{k+l} - F^4_{k-l}).
\end{align*}
\]

If $m = 2$ and $l = 1$, then we have $6 F_{4k} = F^4_{k+2} - F^4_{k-2} + 3 (F^4_{k+1} - F^4_{k-1})$.

If we consider $U_{5k} = c_1 U^5_{k+m} + c_2 U^5_{k+l} + c_3 U^5_k + c_4 U^5_{k-l} + c_5 U^5_{k-m}$, then we can get

\[
\begin{align*}
(GF.18) & \quad U_{m-l} U_{m+l} U_m U_{2m} U_{2l} U_{5k} = U_l U_{2l} (U^5_{k+m} + Q^{5m} U^5_{k-m}) - Q^{m-l} U_m U_{2m} (U^5_{k+l} + Q^{5l} U^5_{k-l}) + Q^{m+l} V_m U_{m-l} U_{m+l} U^5_k, \\
(G.18) & \quad F_{m-l} F_{m+l} F_m F_{2m} F_{2l} F_{5k} = F_l F_{2l} (F^5_{k+m} + (-1)^{5m} F^5_{k-m}) - (-1)^{m-l} F_m F_{2m} (F^5_{k+l} + (-1)^{5l} F^5_{k-l}) + (-1)^{m+l} L_m F_{m-l} F_{m+l} F^5_k.
\end{align*}
\]

If $m = 2$ and $l = 1$, then we have $6 F_{5k} = F^5_{k+2} + F^5_{k-2} - 6 F^5_k + 3 (F^5_{k+1} - F^5_{k-1})$.

Note that the above generalized identities involve $U^r_{k+m} = (-1)^r Q^{rm} U^r_{k-m}$. We denote such expression by $\tilde{U}_{k,m}(r)$. Using the extension of the Lucas sequence to negative indices we get $(-1)^r Q^{rm} U^r_{k-m} = Q^{rk} U^r_{m-k}$. Then we obtain the following property:

\[
\tilde{U}_{k,m}(r) - \tilde{U}_{m,k}(r) = \begin{cases} 0, & \text{if } r \text{ is an even,} \\ 2 (Q^m U_{m+k})^r, & \text{if } r \text{ is an odd.} \end{cases} \tag{25}
\]

From \( \Box \) follows that if $r$ is an even, then $\tilde{U}_{k,m}(r)$ is symmetric under interchange
of the indices. We give $\bar{U}_{k,m}(r)$ for $1 \leq r \leq 5$. The results are as follows:

\begin{align*}
\bar{U}_{k,m}(1) &= U_k V_m. \\
\bar{U}_{k,m}(2) &= U_{2k} V_{2m}. \\
\bar{U}_{k,m}(3) &= \left(U_{3k} V_{3m} - 3Q^{k+m}U_k V_m\right) / \Delta. \\
\bar{U}_{k,m}(4) &= U_{2k} U_{2m} \left(V_{2k} V_{2m} - 4Q^{k+m}\right) / \Delta. \\
\bar{U}_{k,m}(5) &= \left(U_{5k} V_{5m} - 5Q^{k+m}U_{3k} V_{3m} + 10Q^{2(k+m)}U_k V_m\right) / \Delta^2.
\end{align*}

These formulas and the above identities suggest that there is a general formula involving a sum of $\bar{U}_{k,m}(r)$ which is valid for any exponents.

In conclusion we present the following:

\begin{align*}
U_k^n &= U_{k+m} U_l U_p U_s / (Q^{3m}U_{l-m} U_{p-m} U_{s-m}) + \\
&+ U_{k+l} U_m U_p U_s / (Q^{3l}U_{m-l} U_{p-l} U_{s-l}) + U_{k+p} U_l U_m U_s / (Q^{3p}U_{l-p} U_{m-p} U_{s-p}) + \\
&+ U_{k+s} U_l U_m U_p / (Q^{3s}U_{l-s} U_{m-s} U_{p-s}).
\end{align*}

Since this is similar to (GF.12), we conjecture that for any distinct integers $d_i$ ($0 \leq i \leq n$)

$$U_k^n = \sum_{i=0}^{n} \left( \frac{U_{k+d_i}^{n}}{Q^{nd_i}} \prod_{j=0}^{n} \frac{U_{d_j}}{U_{d_j-d_i}} \right).$$

Although the new method was only used for obtaining identities involving the Lucas sequences. It can also be applied to any recurrence sequence.

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