Iwasawa Theory of Hilbert modular forms for anticyclotomic extension

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Abstract

Following Bertolini and Darmon’s method, with “Ihara’s lemma” among other conditions Longo and Wang proved one divisibility of Iwasawa main conjecture for Hilbert modular forms of weight 2 and general low parallel weight respectively. In this paper, we remove the “Ihara’s lemma” condition in their results.

Introduction

Iwasawa theory studies the mysterious relation between pure arithmetic objects and special values of complex $L$-functions. Its precise statement is usually called “main conjecture” that provides an equality between a quality measuring Selmer groups and a $p$-adic $L$-function (interpolating the special values of a complex $L$-function). Its proof is usually divided into two parts, one part proving one divisibility by Ribet’s method, the other proving the converse divisibility by Euler systems.

In [1] Bertolini and Darmon proved one divisibility of the Iwasawa main conjecture for elliptic curve over $\mathbb{Q}$ in the anticyclotomic setting. Note that Bertolini-Darmon assumed a $p$-isolated condition among other technical conditions. The $p$-isolated condition was removed by Pollack and Weston [12]. In [3] Chida and Hsieh generalized this one divisibility to low weight elliptic modular forms. Their results were generalized to the setting of Hilbert modular forms by Longo [9] for parallel weight 2, and by Wang [15] for general low parallel weight. There are other generalizations obtained by Fouquet [6] and Nekovar [11].

Their approach relies on a version of Ihara’s Lemma. In the case of elliptic modular forms, the needed Ihara’s Lemma is Theorem 12 in [5]. In the totally real case, [5, Theorem 12] is partially generalized by Javis [8]. It seems that in the unpublished paper [2] Ihara’s Lemma was proved under the conditions that the base totally real number field $F$ is sufficiently small, i.e. $[F : \mathbb{Q}] < p$, and that the level of the Hilbert modular form in question is sufficiently large. In their recent preprint [10] Manning and Shotton [10] proved Ihara’s lemma under a large image hypothesis. None of the result in [8], [2] or [10] covers what is need in [9, 15].

In this paper we remove the condition of Ihara’s Lemma, and thus obtain an uncondition result for all totally real number fields. We need to persist technical conditions in [9, 15] other than Ihara’s Lemma. Instead of proving Ihara’s Lemma, we take an approach of avoiding it.

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Let $F$ be a totally real field, $p$ an ideal of $F$ above $p$. Let $K$ be a totally imaginary quadratic extension of $F$. Then we form the anticyclotomic $\mathbb{Z}_p[[p]]$-extension $K_\infty$ of $K$. Put $\Gamma = \text{Gal}(K_\infty/K)$.

Let $f$ be a new Hilbert cusp form of parallel weight $k \geq 2$. Let us write the conductor $n$ of $f$ in the form $n = n^+n^-$, where $n^+$ is divided by primes that split in $K$, and $n^-$ is divided by primes that do not split in $K$. We assume that $n^-$ is the product of different primes whose cardinal number has the same parity of $[F : \mathbb{Q}]$. This condition ensures that $f$ comes from a modular form on a definite quaternion algebra with discriminant $n^-$. We also assume $p \nmid nD_{K/F}$ and $f$ is ordinary at $p$. Namely the Hecke eigenvalue of $f$ at each prime of $F$ above $p$ is a $p$-adic unit.

Let $\rho_f : G_F \to \text{GL}_2(E_f)$ be the $p$-adic Galois representation attached to $f$ (see [16, 14] among other references). Then $\det \rho_f = e^{k-1}$, where $e$ is the $p$-adic cyclotomic character of $G_F = \text{Gal}(\overline{F}/F)$. We consider the self-dual twist of $\rho_f$, namely $\rho_f^* = \rho_f \otimes e^{\frac{-k}{2}}$. Let $V_f$ be the underlying representation space for $\rho_f^*$. Fix a Galois stable lattice $T_f$ of $V_f$, and put $A_f = V_f/T_f$.

Let $\text{Sel}(K_\infty, A_f)$ be the minimal Selmer group of $A_f$. Put $\Lambda = \mathcal{O}_f[[\Gamma]]$, where $\mathcal{O}_f$ is the ring of integers in $E_f$. Then the Pontryagin dual $\text{Sel}(K_\infty, A_f)^\vee$ is a $\Lambda$-module. Let $\text{char}_\Lambda \text{Sel}(K_\infty, A_f)^\vee \in \Lambda$ be its characteristic.

On the other hand, one can attach to $f$ an anticyclotomic $p$-adic $L$-function $L_p(K_\infty, f) \in \Lambda$ that interpolates the special values $L(f/K, \chi, k/2)$ of the $L$-function of $f$ (where $\chi$ runs over anticyclotomic character).

**Conjecture 0.1.** (Iwasawa main conjecture). $\text{Sel}(K_\infty, A_f)$ is a cofinitely generated $\Lambda$-module, and 
\[\text{char}_\Lambda \text{Sel}(K_\infty, A_f)^\vee = (L_p(K_\infty, f)).\]

Our main result is the following

**Theorem 0.2.** Assume that $f$ satisfies the conditions (CR$^+$), (PO) and (n$^+$-DT) given in [15]. Then $\text{Sel}(K_\infty, A_f)$ is a cofinitely generated $\Lambda$-module, and 
\[\text{char}_\Lambda \text{Sel}(K_\infty, A_f)^\vee \mid (L_p(K_\infty, f)).\]

The strategy is using the Euler system of Heegner points to bound the Selmer groups. In [15] these Heegner points were showed to satisfy two properties called the first reciprocity law and the second reciprocity law. The second reciprocity law needs “Ihara Lemma”. Our input is to prove a weaker form of the second reciprocity law without “Ihara Lemma”. Our weaker version is sufficient for us to run through Bertolini and Darmon’s Euler system argument to prove Theorem 0.2. This is done in Section 3. See Proposition 2.1 and Corollary 2.7 for the precise statements of the first reciprocity law and the weaker version of the second reciprocity law.

As applications of Theorem 0.2, we have the following consequences.

**Corollary 0.3.** Let $A$ be a modular elliptic curve (or more generally modular abelian varieties of $\text{GL}_2$-type) over $F$. Assume that $F_p = \mathbb{Q}_p$ and modular form attached to $A$ satisfies the assumptions in Theorem 0.2. Then $A(K_\infty)$ is finitely generated.

In [7] Hung proved vanishing of the analytic $\mu$-invariant, generalizing the result of Chida and Hsieh [4]. Combining Theorem 0.2 and Hung’s result, we obtain the following

**Corollary 0.4.** Keep the assumption of Theorem 0.2. Then the algebraic $\mu$-invariant of the $\Lambda$-module $\text{Sel}(K_\infty, A_f)^\vee$ is zero.

Corollary 0.3 and Corollary 0.4 were already obtained by Longo [9] and Wang [15] respectively, under the assumption of “Ihara Lemma”.


Notations

Fix a prime number $p \nmid nD_{K/F}$. Let $p$ be a prime of $F$ above $p$. Let $K_m$ be the ring class field over $K$ of conductor $p^m$ and put $\Gamma_m = \text{Gal}(K_m/K)$. Put $K_\infty = \cup_m K_m$ and $\Gamma = \lim_{\leftarrow m} \Gamma_m$.

Let $B_\Delta$ be a quaternion algebra over $F$, $\Delta$ the discriminant of $B_\Delta$. Let $m^+$ be an ideal of $F$ which is prime to $\Delta$.

Let $\mathbb{T}_{B_\Delta}(m^+;p^n)$ be the Hecke algebra defined in [15, Chapter 3.1.2]. For an open compact subgroup $U$ of $B_\Delta^+$ let $S_{B_\Delta}^U(\mathfrak{m}, A)$ denote the space of $A$-valued modular forms of level $U$ [15, Chapter 3.3]. What we are interested in is the case when $U$ is of the form $U_{m^+,p^n}$. Set $Y = \hat{F}^\times$. Then there is an action of $Y$ on $S_{B_\Delta}^U(\mathfrak{m}, A)$; put $S_{B_\Delta}^U(Y, A) = S_{B_\Delta}^U(\mathfrak{m}, A)^Y$.

For a Hilbert modular form $f \in S_k(n,1)$ of level $n$ and weight $k$, one can attach to $f$ a Hecke character $\lambda_f : \Gamma_B(n^+, p^n) \otimes \mathcal{O}_f \to \mathcal{O}_f$ [15, Chapter 5].

Fix $n$. Put $\mathcal{O}_{f,n} = \mathcal{O}_f/\omega^n$, where $\omega$ is a uniformizer of $\mathcal{O}_f$. We set $T_{f,n} = T_f/\omega^n$ and $A_{f,n} = \ker(A_f \xrightarrow{\omega^n} A_f)$.

A prime $\mathfrak{p}$ of $F$ is said to be $n$-admissible for $f$ if the following conditions hold.

- $\mathfrak{p} \nmid n$.
- $\mathfrak{p}$ is inertia in $K$.
- $N(\mathfrak{p})^2 - 1$ is not divided by $p$.
- $\omega^n$ divides $N(\mathfrak{p})^2 + N(\mathfrak{p}) \xrightarrow{\mathfrak{p}} - \epsilon_1 a_1(f)$, where $\epsilon_1 = \pm 1$.

By an $n$-admissible form $\mathcal{D} = (\Delta, g)$ we mean a pair such that

- $\Delta$ is square free product of primes in $F$ inert in $K$; $n^-|\Delta$; $\Delta/n^-$ is a product of $n$-admissible primes.
- $g \in S_{B_\Delta}^U(Y, n, \mathcal{O}_{f,n})$ such that $\lambda_g \equiv \lambda_f \text{ mod } \omega^n$.

For an $n$-admissible form $\mathcal{D} = (\Delta, g)$ and each $\mathfrak{m}$ one can attach to them a theta element $\theta_{\mathfrak{m}}(g) \in \mathcal{O}_{f,n}[\Gamma_{\mathfrak{m}}]$ [15, Chapter 6.1]. These element $\theta_{\mathfrak{m}}(g)$ is compatible in the sense that $\pi_{\mathfrak{m}+1,\mathfrak{m}}(\theta_{\mathfrak{m}+1}(g)) = \theta_{\mathfrak{m}}(g)$ and thus define an element of $\theta_{\infty}(g) \in \mathcal{O}_{f,n}[[\Gamma]]$. Here, $\pi_{\mathfrak{m}+1,\mathfrak{m}}$ is the quotient map $\Gamma_{\mathfrak{m}+1} \to \Gamma_{\mathfrak{m}}$.

Let us state the conditions (CR$^+$), (n$^+$-DT) and (PO) in [15]. Let $\bar{\rho}_f$ denote the residue Galois representation of $\rho_f$.

**Hypothesis (CR$^+$).** 1. $p > k + 1$ and $\#((F^\times_p)^{k-1}) > 5$.

2. The restriction of $\bar{\rho}_f$ to $G_F(\sqrt{p^*})$ is irreducible, where $p^* = (-1)^{\frac{k-1}{2}} p$.

3. $\bar{\rho}_f$ is ramified at $\mathfrak{p}$ if $|n^-\text{ and } N(\mathfrak{p})^2 \equiv 1 \text{ (mod } p)$. $\mathfrak{p}$.

4. If $n_p$ denotes the Artin conductor of $\bar{\rho}_f$, then $\mathfrak{n}/n_p$ is coprime to $n_p$.

**Hypothesis (n$^+$-DT).** If $|n^+\text{ and } N(\mathfrak{p}) \equiv 1 \text{ (mod } p)$, then $\bar{\rho}_f$ is ramified at $\mathfrak{p}$.

**Hypothesis (PO).** $a_\mathfrak{p}(f) \equiv 1 \text{ (mod } p)$ for all $\mathfrak{p}$ if $k = 2$. Here, $a_\mathfrak{p}(f)$ is the Hecke eigenvalue of $f$ at the prime $\mathfrak{p}$.

We also need an auxiliary condition (n$^+$-min).

**Hypothesis (n$^+$-min).** If $|n^+$, then $\bar{\rho}_f$ is ramified at $\mathfrak{p}$.

1 Selmer groups

For reader’s convenience, we recall the definition of Selmer groups. See [1, 3, 9, 15] for more details.
Let $L$ be a finite extension of $F$. For each prime ideal $\mathfrak{l}$ of $F$ and each discrete $G_F$-module $M$, we put
\[
H^1(L_\mathfrak{l}, M) = \bigoplus_{\lambda|\mathfrak{l}} H^1(L_\lambda, M), \quad H^1(I_{L_\mathfrak{l}}, M) = \bigoplus_{\lambda|\mathfrak{l}} H^1(I_{L_\lambda}, M),
\]
where $\lambda$ runs through all primes of $L$ above $\mathfrak{l}$. Denote by
\[
\text{res}_\mathfrak{l} : H^1(L, M) \to H^1(L_\mathfrak{l}, M)
\]
the restriction map at $\mathfrak{l}$.

We define the fine part $H^1(L_\mathfrak{l}, M)$ as
\[
H^1_{\text{fin}}(L_\mathfrak{l}, M) = \ker(H^1(L_\mathfrak{l}, M) \to H^1(I_{L_\mathfrak{l}}, M))
\]
and the singular quotient as
\[
H^1_{\text{sing}}(L_\mathfrak{l}, M) = H^1(L_\mathfrak{l}, M)/H^1_{\text{fin}}(L_\mathfrak{l}, M).
\]

From the Hochschild-Serre spectral sequence
\[
E^{ij}_2 = H^i(G_{L_\mathfrak{l}}/I_{L_\mathfrak{l}}, H^j(I_{L_\mathfrak{l}}, M)) \Rightarrow H^{i+j}(G_{L_\mathfrak{l}}, M)
\]
we obtain the following exact sequence
\[
\bigoplus_{\lambda|\mathfrak{l}} H^1(G_{L_\lambda}/I_{L_\lambda}, M^{I_{L_\lambda}}) \to H^1(L_\mathfrak{l}, M) \to \bigoplus_{\lambda|\mathfrak{l}} H^1(I_{L_\lambda}, M)^{G_{L_\lambda}/I_{L_\lambda}}.
\]
Then $H^1_{\text{fin}}(L_\mathfrak{l}, M)$ coincides with the image of the map
\[
\bigoplus_{\lambda|\mathfrak{l}} H^1(G_{L_\lambda}/I_{L_\lambda}, M^{I_{L_\lambda}}) \to H^1(L_\mathfrak{l}, M),
\]
and $H^1_{\text{sing}}(L_\mathfrak{l}, M)$ is naturally isomorphic to the image of the residue map
\[
\partial_\mathfrak{l} : H^1(L_\mathfrak{l}, M) \to \bigoplus_{\lambda|\mathfrak{l}} H^1(I_{L_\lambda}, M)^{G_{L_\lambda}/I_{L_\lambda}}.
\]
By abuse of notation, the composition map $\partial_\mathfrak{l} \circ \text{res}_\mathfrak{l}$ is also denoted by $\partial_\mathfrak{l}$. If an element $s \in H^1(G_{L_\mathfrak{l}}, M)$ satisfies $\partial_\mathfrak{l}(s) = 0$, then $\text{res}_\mathfrak{l}(s)$ is in $H^1_{\text{fin}}(L_\mathfrak{l}, M)$ and we will denote it as $v_\mathfrak{l}(s)$.

If $\mathfrak{l} \mid \mathfrak{n}^-$ or $\mathfrak{l} \mid p$, then the restriction $\rho^{ij}_F|_{G_{F_{\mathfrak{l}}}}$ of $V_F$ to $G_{F_{\mathfrak{l}}}$ sits in a short exact sequence
\[
0 \to F^+_1 V_F \to V_F \to F^-_1 V_F \to 0,
\]
where $G_{F_{\mathfrak{l}}}$ acts on $F^+_1 V_F$ by $\pm \epsilon$ (resp. $\chi^{-1} \epsilon^{k/2}$) if $\mathfrak{l} \mid \mathfrak{n}^-$ (resp. $\mathfrak{l} \mid p$). Here, $\chi$ is the unramified character of $G_{F_{\mathfrak{l}}}$, $\langle \mathfrak{l} \rangle$ such that $\chi(\text{Frob}) = \alpha_{\mathfrak{l}}$, where $\alpha_{\mathfrak{l}}$ is the unit root of the Hecke polynomial $x^2 - a_1(f)x + N(\mathfrak{l})^{k-1}$. Then for $\mathfrak{l} \mid \mathfrak{n}^-$ we define the ordinary part of $H^1_{\text{ord}}(L_\mathfrak{l}, A_{f,n})$ to be the image of
\[
H^1(G_{L_\mathfrak{l}}, F^+_1 A_{f,n}) \to H^1(G_{L_\mathfrak{l}}, A_{f,n}).
\]
We define $H^1_{\text{ord}}(T_{f,n})$ similarly.

Let $\Delta$ be a square free product of primes in $F$ such that $\Delta/\mathfrak{n}^-$ is a product of $n$-admissible primes. Let $S$ be a finite (maybe empty) set of places of $F$ that are prime to $p\Delta \mathfrak{n}$. 

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Definition 1.1. For \( M = A_{f,n} \) or \( T_{f,n} \) we define the Selmer group \( \text{Sel}^S_m(G_L, M) \) to be the the group of elements \( s \in H^1(G_L, M) \) such that

- \( \text{res}_l(s) \in H^1_{\text{fin}}(L_l, M) \) if \( l \nmid p\Delta \) and \( l \notin S \); 
- \( \text{res}_l(s) \in H^1_{\text{ord}}(L_l, M) \) for all \( l | p\Delta \); 
- \( \text{res}_l(s) \) is arbitrary if \( l \in S \).

We put

\[
H^1(K_\infty, A_{f,n}) = \lim_{r \rightarrow \infty} H^1(K_r, A_{f,n}), \quad \hat{H}^1(K_\infty, T_{f,n}) = \lim_{r \rightarrow \infty} H^1(K_r, T_{f,n})
\]

\[
H^1(K_{\infty,l}, A_{f,n}) = \lim_{r \rightarrow \infty} H^1(K_{l,r}, A_{f,n}), \quad \hat{H}^1(K_{\infty,l}, T_{f,n}) = \lim_{r \rightarrow \infty} H^1(K_{l,r}, T_{f,n})
\]

The finite parts and the singular quotients \( H^1_{\text{fin}}(K_{\infty,l}, A_{f,n}) \) and \( \hat{H}^1_{\text{sing}}(K_{\infty,l}, T_{f,n}) \) for \( \ell \in \{ \text{fin}, \text{sing} \} \) are defined similarly. We define

\[
\text{Sel}^S_m(K_\infty, A_{f,n}) = \lim_{r \rightarrow \infty} \text{Sel}^S_m(K_r, A_{f,n}), \quad \hat{\text{Sel}}^S_m(K_\infty, T_{f,n}) = \lim_{r \rightarrow \infty} \hat{\text{Sel}}^S_m(K_r, T_{f,n}).
\]

If \( S \) is empty, we drop \( S \) from the above notations. When \( S = \emptyset \) and \( \Delta = \mathfrak{n}^- \), we drop both \( S \) and \( \Delta \) from the notations; the Selmer group in Theorem 0.2 are in this case.

2 Euler system of Heegner points

Fix \( n \geq 1 \). Let \((\Delta, g)\) be an \( n\)-admissible form. Let \( l \nmid \Delta \) be an \( n\)-admissible prime of \( f \) \[15, Definition 2.2.1\]. In \[15, Section 5.1\] it is attached to \( g \) a character of Hecke algebra \( \lambda_g[l] : T_B(Y_{\mathfrak{n}}^+, \mathfrak{p}^n) \otimes \mathcal{O}_{f,n} \rightarrow \mathcal{O}_{f,n} \), and let \( \mathcal{Z}_g[l] \) be the kernel of \( \lambda_g[l] \).

Let \( B' \) be the indefinite quaternion algebra with discriminant \( 1,\Delta \). Let \( M_n[0] \) be the Shimura curve associated to \( B' \) defined in \[15, 4.1.1\], \( J_n[0] \) the Jacobian of \( M_n[0] \).

In \[15\] a system of cohomology classes in \( H^1(K_m, T_{f,n}) \) denoted by \( \kappa_\varnothing(0)_m \) (\( l \nmid \Delta \) \( n\)-admissible for \( f \)) is constructed by using Heegner points on \( M_n[0] \). We omit the precise construct of \( \kappa_\varnothing(0)_m \) since we will not use it.

When \( m \) varies, \( \kappa_\varnothing(0)_m \) is compatible for the corestriction map \[15, Lemma 5.4.1\], and thus defines an element \( \kappa_\varnothing(0) \) of \( \hat{H}^1(K_\infty, T_{f,n}) \). By \[15, Proposition 5.4.2\] \( \kappa_\varnothing(0) \) belongs to \( \hat{\text{Sel}}_{\Delta l}(K_\infty, T_{f,n}) \).

Proposition 2.1. (First reciprocity law, \[15, Theorem 6.1.2\]) Let \( m \geq n \geq 0 \). For each \( n\)-admissible form \( \varnothing = (\Delta, g) \) and each \( n\)-admissible \( l \nmid \Delta \), we have

\[
\partial_l(\kappa_\varnothing(0)_m) = \theta_m(g) \in \mathcal{O}_{f,n}[\Gamma_m]
\]

up to multiplication by elements of \( \mathcal{O}_f^\times \) and \( \Gamma_m \).

Proposition 2.2. For any two different \( n\)-admissible primes \( l_1 \) and \( l_2 \) \( (l_1, l_2 \nmid \Delta) \), there exists a nonnegative integer \( n_0 \leq n \) and an \( (n - n_0)\)-admissible form \( \varnothing'' = (\Delta l_1 l_2, g'') \) such that

\[
v_{l_2}(\kappa_\varnothing(l_1)_m) = \omega^{n_0} \theta_m(g'') \in \mathcal{O}_{f,n}[\Gamma_m]
\]

up to multiplication by elements of \( \mathcal{O}_f^\times \) and \( \Gamma_m \).
Proof. Let \( \mathcal{S}_{l_2} \) be the set of supersingular points of \( M_{n}^{[l_1]} \) at the place \( l_2 \). Note that the image of the map
\[
j_{l_1}^i(K_n)/J_{g_1} \to H^1(K_{n,t_2}, T_{f,n})
\]
belongs to \( H^1_{fin}(K_{n,t_2}, T_{f,n}) \cong \mathcal{O}_{f,n} \). Thus we obtain a map
\[
\gamma: \mathcal{S}_{l_2} \to \mathcal{O}_{f,n}.
\]
Let \( n_0 \) be the smallest integer such that \( \text{Im}(\gamma) \in \omega^{n_0} \mathcal{O}_{f,n} \). Then by [9, Lemma 7.17] we obtain the existence of \( g'' = (\Delta_{l_1}(l_2, g') \). Running the argument in the proof of [9, Theorem 7.23] or the computation in the proof of [15, Theorem 6.3.3], we obtain the desired relation \( v_{l_2}(\kappa(g)(l_1)_m) = \omega^{n_0} \theta_m(g'') \).

Remark 2.3. It is expected that \( n_0 = 0 \). This is proved by Longo [9] (for the weight 2 case) and Wang [15] under the assumption that a version of Ihara’s lemma holds.

One expects to show that
\[
v_{l_2}(\kappa(g)(l_1)_m) = v_{l_1}(\kappa(g)(l_2)_m)
\]
without Ihara’s lemma. But we can only prove a weaker result below.

For any homomorphism
\[
\varphi: \mathcal{O}_{f,n}[\Gamma_m] \to \mathcal{O}_{f,n}
\]
and any element \( \theta \in \mathcal{O}_{f,n}[\Gamma_m] \) we write \( \text{ord}_{\varphi}(\theta) \) for the integer \( \text{val}_{\varphi}(\varphi(\theta)) \).

Theorem 2.4. Assume (CR+) and (n+DT) hold. If there exists a homomorphism \( \varphi: \mathcal{O}_{f,n}[\Gamma_m] \to \mathcal{O}_{f,n} \) such that
\[
\text{ord}_{\varphi}(\partial_{l_1}(\kappa(g)(l_1)_m)) + \text{ord}_{\varphi}(v_{l_2}(\kappa(g)(l_1)_m)) < n,
\]
then
\[
v_{l_2}(\kappa(g)(l_1)_m) = v_{l_1}(\kappa(g)(l_2)_m)
\]
up to multiplication by elements of \( \mathcal{O}_{f,n}^{\times} \) and \( \Gamma_m \).

Let \( \tau \) be a complex conjugacy which depends on a choice of embedding of the algebraic closure of \( \overline{\mathbb{F}} \) in \( \mathbb{C} \). Both the complex conjugacy \( \tau \) and \( \Gamma_m \) act on \( H^1(K_m, T_{f,n}) \). For \( \sigma \in \Gamma_m \) we have \( \tau \sigma = \sigma^{-1} \tau \). The homomorphism \( \sigma \mapsto \sigma^{-1} \) of \( \Gamma_m \) induces an involution \( \iota \) on \( \mathcal{O}_{f,n}[\Gamma_m] \). Thus \( \tau \) acts as \( \iota \).

For each \( i \in \{1, 2\} \), as \( l_i \) splits completely in \( K_m \) [15, Lemma 2.4.2], the number of primes of \( K_m \) above \( l_i \) is \( [K_m : K] \). Thus there exists a prime \( l'_i \) of \( K_m \) above \( l_i \) that is fixed by \( \tau \). Then all primes above \( l'_i \) are \( \{ \sigma l'_i : \sigma \in \text{Gal}(K_m/K) \} \).

Both \( H^1_{fin}(K_{m,l'_i}, T_{f,n}) \) and \( H^1_{sing}(K_{m,l'_i}, T_{f,n}) \) are isomorphic to \( \mathcal{O}_{f,n} \). We choose generators \( c_{l'_i} \) and \( d_{l'_i} \) of \( H^1_{fin}(K_{m,l'_i}, T_{f,n}) \) and \( H^1_{sing}(K_{m,l'_i}, T_{f,n}) \) such that \( (c_{l'_i}, d_{l'_i}) = 1 \). Note that \( \tau c_{l'_i} = \epsilon_i c_{l'_i} \) and \( \tau d_{l'_i} = \epsilon_i' d_{l'_i} \) with \( \epsilon_i, \epsilon_i' = \pm 1 \).

To avoid confusion we write \( n_{1,2} \) and \( g''_{1,2} \) for \( n_0 \) and \( g'' \) in Proposition 2.2 respectively. So, when \( l_1 \) and \( l_2 \) are exchanged, we have \( n_{2,1} \) and \( g''_{2,1} \).

Theorem 2.1 and Theorem 2.2 say that there exists \( u_1, u_2, w_{1,2}, u_{2,1} \in \mathcal{O}_{f,n}^{\times} \) and \( \sigma_1, \sigma_2, \sigma_{1,2}, \sigma_{2,1} \in \Gamma_m \) such that
\[
\partial_{l_1}(\kappa(g)(l_1)_m) = u_1 \sigma_1 \cdot \theta_m(g) \cdot d_{l'_i},
\]
\[
v_{l_2}(\kappa(g)(l_1)_m) = u_{1,2} \omega^{n_{1,2}} \sigma_{1,2} \cdot \theta_m(g''_{1,2}) \cdot c_{l'_i},
\]
\[
v_{l_1}(\kappa(g)(l_2)_m) = u_{2,1} \omega^{n_{2,1}} \sigma_{2,1} \cdot \theta_m(g''_{2,1}) \cdot c_{l'_i}.
\]
Put 
\[ \theta_i = u_i \sigma \theta_m(g), \quad \theta_{1,2} = u_{1,2} \sigma_1 \sigma_2 \theta_m(g''_{1,2}), \quad \theta_{2,1} = u_{2,1} \sigma_2 \sigma_1 \theta_m(g''_{2,1}). \]

**Proposition 2.5.** We have
\[ \epsilon_1 \omega^{n_2,1} \theta_1 \theta_{2,1} + \epsilon_2 \omega^{n_1,2} \theta_2 \theta_{1,2} = 0 \tag{2.1} \]
in \( O_{f,n}[\Gamma_m] \).

**Proof.** Note that \( T_{f,n} \) is self dual, so we can form the local Tate pairing \( \langle \cdot , \cdot \rangle_v \) on \( H^1(K_m, T_{f,n}) \) for each place \( v \) of \( K_m \).

For any \( c_1, c_2 \in H^1(K_m, T_{f,n}) \) and each place \( v \) of \( K_m \), we write \( \langle c_1, c_2 \rangle_v = \langle \text{res}_v(c_1), \text{res}_v(c_2) \rangle_v \).

Then \( \sum_v \langle c_1, c_2 \rangle_v = 0 \). We apply this to \( c_1 = \tau(\kappa_{\mathcal{D}}(1_1)) \) and \( c_2 = \kappa_{\mathcal{D}}(1_2) \). When \( v \) is not above \( I_1 \) or \( I_2 \), both \( \tau \kappa_{\mathcal{D}}(1_1) \) and \( \kappa_{\mathcal{D}}(1_2) \) are fine or ordinary, so \( \langle \tau \kappa_{\mathcal{D}}(1_1), \kappa_{\mathcal{D}}(1_2) \rangle_v = 0 \). Hence we have
\[ \sum_{\sigma \in \Gamma_m} \langle \tau \kappa_{\mathcal{D}}(1_1), \kappa_{\mathcal{D}}(1_2) \rangle_{\sigma \Gamma_1} + \langle \tau \kappa_{\mathcal{D}}(1_1), \kappa_{\mathcal{D}}(1_2) \rangle_{\sigma \Gamma_2} = 0. \]

Note that this sum is exactly the coefficient of \( 1 \in \Gamma_m \) in the left hand of (2.1), and thus this coefficient is 0. Repeating the above argument for \( c_1 = \tau \kappa_{\mathcal{D}}(1_1) \) and \( c_2 = \kappa_{\mathcal{D}}(1_2) \), we obtain that the coefficient of \( \sigma \in \Gamma_m \) in the left hand of (2.1) is zero. \( \square \)

Proposition 2.5 has the following two direct corollaries.

**Corollary 2.6.** For any homomorphism \( \varphi : O_{f,n}[\Gamma_m] \to O_{f,n} \) we have
\[ \epsilon_1 \omega^{n_2,1} \varphi(\theta_1) \varphi(\theta_{2,1}) + \epsilon_2 \omega^{n_1,2} \varphi(\theta_2) \varphi(\theta_{1,2}) = 0. \]

**Corollary 2.7.** If \( \varphi : O_{f,n}[\Gamma_m] \to O_{f,n} \) is a homomorphism such that
\[ \text{ord}_\varphi(\partial_1(\kappa_{\mathcal{D}}(1_1)_m)) + \text{ord}_\varphi(\nu_1(\kappa_{\mathcal{D}}(1_1)_m)) < n, \]
then
\[ \text{ord}_\varphi(\nu_1(\kappa_{\mathcal{D}}(1_1)_m)) = \text{ord}_\varphi(\nu_1(\kappa_{\mathcal{D}}(1_1)_m)). \]

Now, Theorem 2.4 follows from Corollary 2.7 and the multiplicity one properties ensured by the conditions (CR\(^+\)) and (n\(^+\)-DT) \cite[Theorem 9.1.1]{15}.

**Remark 2.8.** Note that (CR\(^+\)) and (n\(^+\)-DT) are not needed in Proposition 2.5, Corollary 2.6 and Corollary 2.7.

### 3 Proof of the main theorem

Let \( \varphi : O_f[[\Gamma]] \to O \) be a homomorphism from \( O_f[[\Gamma]] \) to the ring of integers of a finite extension of \( \mathbb{Q}_p \). Enlarging \( O_f \) if necessary we may assume that \( O = O_f \). Let \( \varphi_m \) be the composition \( O_f[[\Gamma]] \to O \to O/(\omega^m) \), where \( \omega \) is a uniformizing element of \( O \).

If \( \mathcal{D} = (\Delta, g) \) is an \( m \)-admissible form, we put \( t_{\varphi, g} := \text{ord}(\varphi_m(\theta_g)) \). Here we write \text{ord} for the valuation of \( O \) whose value on \( \omega \) is 1.

Repeating the argument in the proof of \cite[Theorem 7.4.3]{15} one deduces Theorem 0.2 follows from the following auxiliary Theorem 3.1.
Theorem 3.1. If \( \mathcal{D} = (\Delta, g) \) is an \((n + t_{\varphi, g})\)-form, and if \( t_{\varphi, g} \leq n \), then we have

\[
\text{length}_\mathcal{O}(\text{Sel}_\Delta(K_\infty, A_{f,n})^\vee \otimes_\varphi \mathcal{O}) \leq 2t_{\varphi, g}.
\] (3.1)

We prove (3.1) by induction on \( t_{\varphi, g} \). Write \( N = n + t_{\varphi, g} \).

First we assume (CR\(^+\)), (PO) and \((n^+-\text{min})\) hold. In [15, Chapter 7] it is showed that under these conditions there exists an auxiliary finite set \( S \) of \( N \)-admissible primes such that \( \text{Sel}_\Delta^S(K_\infty, T_{f,N}) \otimes_\varphi \mathcal{O} \) is free over \( \mathcal{O}/(\omega^N) \). For any \( N \)-admissible prime \( l \notin S \), considering \( \kappa_\varphi(l) = \varphi(\kappa_\varphi(l)) \) as an element of \( \text{Sel}_\Delta^S(K_\infty, T_{f,N}) \otimes_\varphi \mathcal{O} \), we put \( e_l = \text{ord}\kappa_\varphi(l) \). Then there exists \( \tilde{\kappa}'(l) \in \text{Sel}_\Delta^S(K_\infty, T_{f,N}) \otimes_\varphi \mathcal{O} \) such that \( \omega^l\tilde{\kappa}'(l) = \kappa_\varphi(l) \). By [15, Theorem 7.1.2] one has

\[
e_l = \min_{l'} \text{ord}(v_l \kappa_\varphi(l)),
\]

where \( l' \) run over all \( N \)-admissible primes that do not divide \( l\Delta \) and are not in \( S \).

The quotient map \( T_{f,N} \to T_{f,n} \) induces a homomorphism

\[
\text{Sel}_\Delta^S(K_\infty, T_{f,N}) \otimes_\varphi \mathcal{O} \to \hat{H}^1(K_\infty, T_{f,n}) \otimes_\varphi \mathcal{O}.
\]

Lemma 3.2. Let \( \kappa'(l) \) be the image of \( \tilde{\kappa}'(l) \) in \( \hat{H}^1(K_\infty, T_{f,n}) \otimes_\varphi \mathcal{O} \).

(a) \( \text{ord}\kappa'(l) = 0 \)

(b) \( \text{ord} \partial_l \kappa'(l) = t_{\varphi, g} - e_l \).

(c) \( \partial_q \kappa'(l) = 0 \) for \( q \nmid l\Delta \).

(d) \( \text{res}_q \kappa'(l) \) in \( \hat{H}^1_{\text{ord}}(K_\infty, T_{f,n}) \otimes_\varphi \mathcal{O} \) for \( q \mid l\Delta \).

Proof. Assertions (a) and (b) follow from the definition of \( \kappa'(l) \) and the first reciprocity law. The latter two assertions for \( q \notin S \) also follow from the definition of \( \kappa'(l) \). These two assertions for \( q \in S \) follow from the fact \( \kappa_\varphi(l) \in \text{Sel}_\Delta^S(K_\infty, T_{f,N}) \otimes_\varphi \mathcal{O} \) and the fact that the fine (resp. ordinary) part of \( \hat{H}^1(K_\infty, T_{f,N}) \otimes_\varphi \mathcal{O} \) and that of \( H^1(K_\infty, A_{f,N}) \) are annihilators of each other under the local Tate pairing [15, Lemma 2.4.1, Proposition 2.4.4].

\( \square \)

Corollary 3.3. If \( l \notin S \) is \( N \)-admissible, then \( e_l \leq t_{\varphi, g} \).

Choose an \( N \)-admissible prime \( l_1 \notin S \) such that

\[
e_{l_1} = \min_{l \notin S:N\text{-admissible}} e_l.
\]

Lemma 3.4. (a) If \( t_{\varphi, g} = 0 \), then \( \text{Sel}_\Delta(K_\infty, A_{f,n})^\vee \otimes_\varphi \mathcal{O} \) is trivial.

(b) If \( t_{\varphi, g} > 0 \), then \( e_{l_1} < t_{\varphi, g} \).

Proof. Assertion (a) is [15, Lemma 7.3.5], and (b) is [15, Lemma 7.3.6].

\( \square \)
By Lemma 3.4 (a), (3.1) holds when \( t_{\varphi, g} = 0 \). So we assume that \( t_{\varphi, g} > 0 \).

Let \( I_2 (I_2 \nmid \Delta_1 \Delta \text{ and } I_2 \notin S) \) be an \( N \)-admissible prime such that \( \text{ord } v_{I_2}(\kappa_\varphi(I_1)) = e_{I_1} \). By the choice of \( I_2 \) and the minimality of \( e_{I_1} \), we have

\[
\text{ord } v_{I_2}(\kappa_\varphi(I_1)) = e_{I_1} \leq e_{I_2} \leq \text{ord } v_{I_1}(\kappa_\varphi(I_2)).
\]

As

\[
\text{ord}_v v_{I_2}(\kappa_{g'}(I_1)) + \text{ord}_v \partial_{I_1}(\kappa_{g'}(I_1)) = e_{I_1} + t_{\varphi, g} < n + t_{\varphi, g} < N,
\]

by Proposition 2.2 and Corollary 2.7 there exists an integer \( n_0 \) and an \((N - n_0)\)-admissible form \((\Delta I_2, g''')\) such that

\[
n_0 + t_{\varphi, g'''} = e_{I_1} = \text{ord } v_{I_2}(\kappa_\varphi(I_1)) = \text{ord } v_{I_1}(\kappa_\varphi(I_2)) = e_{I_2}.
\]

The reason for the last equality is that we always have

\[
\text{ord } v_{I_1}(\kappa_\varphi(I_2)) \geq e_{I_2} \geq e_{I_1}.
\]

It follows that

\[
\text{ord } v_{I_1}(\kappa'(I_2)) = \text{ord } v_{I_2}(\kappa'(I_1)) = 0. \tag{3.2}
\]

Note that

\[
N - n_0 = n + t_{\varphi, g} - n_0 > n + e_{I_1} - n_0 \geq n + t_{\varphi, g'''}.
\]

So \((\Delta I_2, g''')\) is an \((n + t_{\varphi, g''})\)-admissible form.

Let \( S_{I_1, I_2} \) be the subgroup of \( \text{Sel}_\Delta(K_\infty, A_{f,n}) \) consisting of elements that are locally trivial at the primes dividing \( I_1 \) or \( I_2 \). By definition of Selmer groups, we have the following two exact sequences

\[
\tilde{H}^1_{\text{sing}}(K_{\infty, I_1}, T_f, n) \oplus \tilde{H}^1_{\text{sing}}(K_{\infty, I_2}, T_f, n) \xrightarrow{\eta_s} \text{Sel}_\Delta(K_{\infty, A_{f,n}})^\vee \xrightarrow{} S_{I_1, I_2}^\vee \xrightarrow{} 0
\]

and

\[
\tilde{H}^1_{\text{fin}}(K_{\infty, I_1}, T_f, n) \oplus \tilde{H}^1_{\text{fin}}(K_{\infty, I_2}, T_f, n) \xrightarrow{\eta_f} \text{Sel}_{\Delta I_1 I_2}(K_{\infty, A_{f,n}})^\vee \xrightarrow{} S_{I_1, I_2}^\vee \xrightarrow{} 0,
\]

where \( \eta_s \) and \( \eta_f \) are induced by the local Tate pairing \( \langle \cdot, \cdot \rangle_{I_1} \oplus \langle \cdot, \cdot \rangle_{I_2} \).

**Lemma 3.5.** The kernel of \( \eta_f^\vee \) contains \( (v_{I_1}(\kappa'(I_2)), 0) \) and \( (0, v_{I_2}(\kappa'(I_1))) \).

**Proof.** Let \( s \) be in \( \text{Sel}_{\Delta I_1 I_2}(K_{\infty, A_{f,n}}) \). By Lemma 3.2 (c), for any \( q \nmid \Delta_1 I_2 \),

\[
\langle \kappa'(I_1), s \rangle_q = \langle \partial_q \kappa'(I_1), s \rangle_q = 0.
\]

By Lemma 3.2 (d), for any \( q \mid \Delta_1 I_1 \),

\[
\langle \kappa'(I_1), s \rangle_q = 0.
\]

Thus by the global reciprocity law we have \( (v_{I_1} \kappa'(I_1), s)_{I_2} = \langle \kappa'(I_1), s \rangle_{I_2} = 0 \). The same argument shows that \( (v_{I_1} \kappa'(I_2), s)_{I_1} = 0 \).

\[\square\]

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By (3.2) and Lemma 3.5 we obtain that
\[ \text{Sel}_{\Delta l_1 l_2}(K_\infty, A_{f,n})^\vee \otimes \varphi O \cong S^\vee_{l_1 l_2} \otimes \varphi O. \]

As \( t_{\varphi,g''} \leq e_{l_1} < t_{\varphi,g} \), by the inductive assumption we have
\[ \text{length}_O S^\vee_{l_1 l_2} \otimes \varphi O = \text{length}_O \text{Sel}_{\Delta l_1 l_2}(K_\infty, A_{f,n})^\vee \otimes \varphi O \leq 2t_{\varphi,g''}. \]

By [15, (7.3.6)] \( \eta^\xi \) factors through the quotient
\[ O/\langle \partial_{l_1} \kappa'(l_1) \rangle \oplus O/\langle \partial_{l_2} \kappa'(l_2) \rangle. \]

Therefore,
\[ \text{length}_O \text{Sel}_{\Delta}(K_\infty, A_{f,n})^\vee \otimes \varphi O \leq \text{ord}_{\partial_{l_1} \kappa'(l_1)} + \text{ord}_{\partial_{l_2} \kappa'(l_2)} + \text{length}_O S^\vee_{l_1 l_2} \leq (t_{\varphi,g} - e_{l_1}) + (t_{\varphi,g} - e_{l_2}) + 2t_{\varphi,g''} \leq 2t_{\varphi,g}. \]

This finishes the inductive argument.

Next, we relax the condition \((n^+-\text{min})\) to \((n^+\text{-DT})\). Using what we have proved under the stronger condition \((n^+\text{-min})\), and following the method of Pollack-Weston (cf.[13]), Wang [15, Chapter 10] proved that there exists an auxiliary finite set \( S \) of \( N \)-admissible primes such that \( \hat{\text{Sel}}_{\Delta}^{n^+ S}(K_\infty, T_{f,N}) \otimes \varphi O \) is free over \( O/\omega^N \). Note that under our assumption on \( n^+ \), \( \hat{\text{Sel}}_{\Delta}^{n S}(K_\infty, T_{f,N}) \) coincides with \( \hat{\text{Sel}}_{\Delta}^{S}(K_\infty, T_{f,N}) \) [15, Lemma 2.4.2(1)]. Then we repeat our above argument to finish the proof of Theorem 3.1.

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